GEOMETRIC SEN THEORY OVER RIGID ANALYTIC SPACES

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Abstract. In this work we develop geometric Sen theory for rigid analytic spaces, generalizing the previous work of Pan for curves. We also extend the axiomatic Sen-Tate formalism of Berger-Colmez to a certain class of locally analytic representations.

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1. Introduction

Let $p$ be a prime number, $\text{Gal}_{\mathbb{Q}_p}$ the absolute Galois group of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the $p$-adic completion of an algebraic closure of $\mathbb{Q}_p$. In the recent work [Pan22], Lue Pan has introduced a new and powerful tool to compute proétale cohomology of $\mathcal{O}$-modules over rigid analytic spaces based on the axiomatic framework of Sen theory of Berger-Colmez [BC08, BC16]. The objective of this paper is to generalize Pan’s method to log smooth adic spaces over non-archimedean fields; our motivation
to develop such a theory is the study of the Hodge-Tate structure of completed cohomology of Shimura varieties, this is carried out in the author’s work [RC22].

Let \((C, C^+)\) be a non-archimedean algebraically closed extension of \(\mathbb{Q}_p\). Let \(X\) be a log smooth fs locally noetherian adic space over \((C, C^+)\) with reduced normal crossing divisors, we let \(\mathcal{O}_X^\text{pro-Kummer-étale}\) and \(\mathcal{O}_X^\text{Kummer-étale}\) denote the structural sheaves of the pro-Kummer-étale and Kummer-étale sites of \(X\) respectively, finally let \(\Omega^1_X(\log)\) be the sheaf of log differential forms of \(X\), we refer to [DLLZ19] for the theory of log geometry on adic spaces.

In order to state the main theorem in geometric Sen theory we need the following definition.

Definition 1.0.1. A pro-Kummer-étale \(\mathcal{O}_X\)-module \(\mathcal{F}\) over \(X\) is said relative locally analytic ON Banach if there is a Kummer-étale cover \(\{U_i\}_{i \in I}\) of \(X\) such that, for all \(i\), the restriction \(\mathcal{F}|_{U_i}\) admits a \(p\)-adically complete \(\mathcal{O}_X^\text{pro-Kummer-étale}\)-lattice \(\mathcal{F}_i^0\), and there is \(\varepsilon > 0\) (depending on \(i\)) such that \(\mathcal{F}_i^0 / p^\varepsilon = ae \bigoplus I \mathcal{O}_X^\text{Kummer-étale} / p^\varepsilon\).

Remark 1.0.2. The relative locally analytic condition can be considered as a “smallness” condition on the sheaf \(\mathcal{F}\) in the sense of Faltings [Fal05]. Examples of ON relative locally analytic \(\mathcal{O}_X\)-modules are \(\mathcal{O}_X\)-vector bundles, this can be deduced from \(v\)-descent of vector bundles on perfectoid spaces [SW20, Lemma 17.1.8].

Theorem 1.0.3 (Theorem 3.3.2). Let \(\mathcal{F}\) be a relative locally analytic ON Banach \(\mathcal{O}_X\)-sheaf over \(X\). Then there is a geometric Sen operator

\[ \theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X(\log)(-1) \]

satisfying the following properties:

1. The formation of \(\theta_{\mathcal{F}}\) is functorial in \(\mathcal{F}\).
2. Let \(\nu : X_{\text{proét}} \to X_{\text{ét}}\) be the projection from the pro-Kummer-étale site to the Kummer-étale site, then there is a natural equivalence

\[ R^i\nu_*\mathcal{F} = \nu_* H^i(\theta_{\mathcal{F}}, \mathcal{F}), \]

where \(H^i(\theta_{\mathcal{F}}, \mathcal{F})\) is the cohomology of the Higgs complex

\[ 0 \to \mathcal{F} \xrightarrow{\theta_{\mathcal{F}}} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X(\log)(-1) \to \cdots \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^d_X(\log)(-d) \to 0. \]

3. Suppose that \(\theta_{\mathcal{F}} = 0\), then \(\nu_* \mathcal{F}\) is locally on the Kummer-étale topology of \(X\) an ON Banach \(\mathcal{O}_X\)-module and \(\mathcal{F} = \mathcal{O}_X \otimes_{\mathcal{O}_X} \nu_* \mathcal{F}\). Conversely, for any locally ON Banach \(\mathcal{O}_X\)-module \(\mathcal{G}\) the pullback \(\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{G}\) has trivial Sen operator.

4. If \(X\) has a form \(X'\) over a discretely valued field with perfect residue field \((K, K^+)\), then \(\theta_{\mathcal{F}}\) is Galois equivariant. In particular, we recover the natural splitting

\[ R\nu_* \mathcal{O}_X \cong \bigoplus_{i=0}^d \Omega^i_X(\log)(-i)[-i]. \]
Let $Y$ be another fs log smooth adic space over $(C, C^+)$, and let $f : Y \to X$ be a morphism. Then there is a commutative diagram

\[
\begin{array}{ccc}
 f^* \mathcal{F} & \xrightarrow{f^* \theta} & f^* \Omega^1_X(\log)(-1) \\
 \downarrow \theta_f & & \downarrow \text{id} \otimes f^* \\
 f^* \mathcal{F} \otimes \mathcal{O}_Y & & f^* \Omega^1_Y(\log)(-1).
\end{array}
\]

The property of being a relative locally analytic $\widehat{\mathcal{O}}_X$-module might look a bit mysterious. Nevertheless, these sheaves arise naturally when studying locally analytic vectors of proétale cohomology. Let us explain in which context they appear.

Let $G$ be a compact $p$-adic Lie group and $\tilde{X} \to X$ a $G$-torsor (e.g. take $X$ a finite level modular curve and $\tilde{X}$ the perfectoid modular curve). Let $V$ be a $\mathbb{Q}_p$-Banach locally analytic representation of $G$, for example, we can take $V = \mathcal{O}(\mathbb{G})$, for some group affinity neighbourhood $\mathbb{G}$ of $G$, endowed with the left regular action. Then $V$ defines a pro-Kummer-étale sheaf $V_{\kett}$ over $X$ by descending the $G$-representation $V$ along the torsor $\tilde{X} \to X$. By Lemma 2.1.4 down below, there is a lattice $V_0 \subset V$, $\epsilon > 0$, and an open subgroup $G_0 \subset G$, such that $G_0$ stabilizes $V_0$, and that the action of $G_0$ on $V_0 / p^\epsilon$ is trivial. Therefore, the pro-Kummer-étale $\widehat{\mathcal{O}}_X$-module $V_{\kett} \otimes \mathbb{Q}_p \widehat{\mathcal{O}}_X$ is a relative locally analytic sheaf. Indeed, the restriction of $V_{\kett} \otimes \mathbb{Q}_p \widehat{\mathcal{O}}_X$ to $X_{G_0} := \tilde{X} / G_0$ satisfies the conditions of Definition 1.0.1. Furthermore, in this situation we have a more refined result.

**Theorem 1.0.4 (Theorem 3.3.4).** Let $X$ be an fs log smooth adic space over $\text{Spa}(C, C^+)$ with log structure given by normal crossing divisors. Let $G$ a $p$-adic Lie group and $\tilde{X} \to X$ a pro-Kummer-étale $G$-torsor. Then there is a geometric Sen operator of the torsor $\tilde{X}$

\[
\theta_{\tilde{X}} : \widehat{\mathcal{O}}_X \otimes \mathbb{Q}_p (\text{Lie } G)_{\kett}^\vee \to \widehat{\mathcal{O}}_X(-1) \otimes \mathcal{O}_X \Omega^1_X(\log)
\]

such that $\theta_{\tilde{X}} \wedge \theta_{\tilde{X}} = 0$. Moreover, for any locally analytic Banach representation $V$ of $G$, we have a commutative diagram

\[
\begin{array}{ccc}
 V_{\kett} \otimes \mathbb{Q}_p \widehat{\mathcal{O}}_X & \xrightarrow{d_V \otimes \text{id}} & (V_{\kett} \otimes \mathbb{Q}_p \widehat{\mathcal{O}}_X) \otimes \mathbb{Q}_p (\text{Lie } G)_{\kett}^\vee \\
 \downarrow \theta_V & & \downarrow \text{id} \otimes \theta_{\tilde{X}} \\
 (V_{\kett} \otimes \mathbb{Q}_p \widehat{\mathcal{O}}_X(-1)) \otimes \mathcal{O}_X \Omega^1_X(\log)
\end{array}
\]

such that $d_V : V \to V \otimes \mathbb{Q}_p (\text{Lie } G)^\vee$ is induced by derivations, and $\theta_V$ is the geometric Sen operator of $V_{\kett} \otimes \mathbb{Q}_p \widehat{\mathcal{O}}_X$ of Theorem 1.0.3.

Moreover, let $H \to G$ be a morphism of $p$-adic Lie groups, let $Y$ an fs log smooth adic space over $(C, C^+)$ and let $\tilde{Y} \to Y'$ be an $H$-torsor. Suppose we are given with
a commutative diagram compatible with the group actions

\[
\begin{array}{ccc}
Y & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
Y & \overset{f}{\longrightarrow} & X
\end{array}
\]

Then the following square is commutative

\[
\begin{array}{ccc}
f^*(\text{Lie } G)_{k\text{et}} \otimes_{Q_p} \mathcal{O}_Y & \overset{f^*\theta_Y}{\longrightarrow} & f^*\Omega^1_X(\log) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-1) \\
\downarrow & & \downarrow \\
(\text{Lie } H)_{k\text{et}} \otimes_{Q_p} \mathcal{O}_Y & \overset{\theta_Y}{\longrightarrow} & \Omega^1_Y(\log) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-1)
\end{array}
\]

Let Sen_{\tilde{X}} : \Omega^1_X(\log) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \to (\text{Lie } G)_{k\text{et}} \otimes_{Q_p} \mathcal{O}_X be the dual of the Sen operator \( \theta_{\tilde{X}} \). As a corollary of the previous theorem we deduce that the locally analytic vectors of the torsor \( \tilde{X} \) satisfy some differential equations.

**Corollary 1.0.5** (Corollary 3.4.6). Let \( \mathcal{O}_{\tilde{X}} \subset \mathcal{O}_X|_{\tilde{X}} \) be the subsheaf of \( \tilde{X} \) consisting on locally analytic sections for the action of \( G \). Then, the action of \( \mathcal{O}_{\tilde{X}} \otimes_{Q_p} \text{Lie } G \) on \( \mathcal{O}_{\tilde{X}} \) by derivations vanishes when restricted to the image of the Sen map \( \text{Sen}_{\tilde{X}} : \Omega^1_X(\log) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \to \text{Lie } G \otimes_{Q_p} \mathcal{O}_{\tilde{X}} \).

To motivate the construction of the geometric Sen operator over rigid spaces we first briefly recall how the arithmetic Sen operator is defined. Let \( Q_p(\zeta_p^\infty) \) be the cyclotomic extension and \( \Gamma = \text{Gal}(Q_p^\text{cyc}/Q_p) \). Let \( V \) be a finite dimensional representation of \( \text{Gal}_p \), the Sen module \( \text{Sen}(V) \) attached to \( V \) is the finite dimensional \( Q_p^\text{cyc} \)-vector space consisting of the finite \( \Gamma \)-vectors of \( (V \otimes_{Q_p} C_p)^H \). It turns out that the dimension of \( \text{Sen}(V) \) over \( Q_p(\zeta_p^\infty) \) is equal to \( \dim_{Q_p} V \). Indeed, one has a \( \text{Gal}_p \)-equivariant isomorphism

\[
\text{Sen}(V) \otimes_{Q_p(\zeta_p^\infty)} C_p = V \otimes_{Q_p} C_p.
\]

As it is explained in [BC16], an equivalent way to construct \( \text{Sen}(V) \) is by taking the \( \Gamma \)-locally analytic vectors of \( (V \otimes_{Q_p} C_p)^H \), in particular \( \text{Sen}(V) \) admits an action of \( \theta_{Q_p} = \text{Lie } \Gamma \). Moreover, since \( \Gamma \) acts smoothly on \( Q_p(\zeta_p^\infty) \), \( \theta_{Q_p} \) is a \( Q_p(\zeta_p^\infty) \)-linear operator. Then, one defines the Sen operator of \( V \otimes_{Q_p} C_p \) to be the \( C_p \)-extension of scalars of \( \theta_{Q_p} \).

Summarizing, one can consider \( \text{Sen}(V) \) as a decompletion of the semilinear representation \( V \otimes_{Q_p} C_p \) by taking locally analytic vectors along the “perfectoid cyclotomic coordinate” \( Q_p^\text{cyc}/Q_p \). The Sen operator is then a “differential operator” obtained from the cyclotomic extension.

Let us now sketch the definition of the geometric Sen operator. For simplicity, we assume that \( X = \mathbb{T} = \text{Spa}(C(T^{\pm 1}), C^+(T^{\pm 1})) \) is a one dimensional torus. Let \( \mathcal{F} \) be a relative locally analytic sheaf over \( \mathbb{T} \), suppose in addition that there is a
Moreover, we have that $R^0 \subset \mathcal{F}$, and $\epsilon > 0$, such that $\mathcal{F}^0/p^\epsilon = \bigoplus_j \mathcal{O}_X^+ / p^\epsilon$. We want to compute the (geometric) pro-étale cohomology $R\Gamma_{\text{pro-ét}}(\mathbb{T}, \mathcal{F})$. Let

$$T_n = \text{Spa}(C(T^{\pm 1}/p^n), C^+(T^{\pm 1}/p^n)),$$

and let $T_\infty = \lim_n T_n$ be the perfectoid torus. The perfectoid torus $T_\infty$ is a Galois cover of $T$ with group $\Gamma = \mathbb{Z}_p(1)$. By Scholze’s almost acyclicity of $\mathcal{O}_X^+ / p$ in affinoid perfectoid spaces [Sch12, Proposition 7.13], one deduces that

$$R\Gamma_{\text{pro-ét}}(\mathbb{T}, \mathcal{F}) = R\Gamma(\Gamma, \mathcal{F}(T_\infty)).$$

By the comparison theorem between continuous and locally analytic cohomology (Theorem 2.12), one has that

$$R\Gamma_{\text{pro-ét}}(\mathbb{T}, \mathcal{F}) = R\Gamma(\Gamma, \mathcal{F}(T_\infty))^{R\Gamma-\text{la}},$$

where $V^{R\Gamma-\text{la}}$ are the derived locally analytic vectors of a $\Gamma$-representation $V$, $R\Gamma(\text{Lie} \Gamma, -)$ is Lie algebra cohomology, and $R\Gamma(\Gamma^\text{an}, -)$ is smooth cohomology, see [2.7]. In other words, we have separated the problem of computing pro-étale cohomology in three steps: first, we need to compute the derived locally analytic vectors of $\mathcal{F}(T_\infty)$. Second, we take the Lie algebra cohomology of $\mathcal{F}(T_\infty)$, and finally, we take the $\Gamma$-invariants of a smooth representation.

Let us focus in the first step which seems to be the more subtle. For $n \geq m$ there are normalized traces

$$R^\sigma_m : C(T^{\pm 1}/p^n) \to C(T^{\pm 1}/p^m) (i.e. \frac{1}{p^m} \sum_{\sigma \in \rho^m \Gamma} \sigma \sigma).$$

These extend to Tate traces

$$R_m : C(T^{\pm 1}/p^m) \to C(T^{\pm 1}/p^m)$$

such that, for any $f \in C(T^{\pm 1}/p^m)$, the sequence $(R_m(f))_m$ converges to $f$. Furthermore, the tuple $(C(T^{\pm 1}/p^m), \Gamma)$ satisfies the Colmez-Sen-Tate axioms of [BC08], see [2.7] for a generalization.

Let $(\zeta^{p^n})_n$ be a compatible system of primitive $p$-th power roots of unity, and let $\psi : \mathbb{Z}_p \to \Gamma$ be the induced isomorphism. Using $\psi$ we define the affinoid group $G_n$ which is a copy of the additive group of radius $p^{-n}$ (i.e. $G_n(\mathbb{Q}_p) = p^n\mathbb{Z}_p$). We will keep using the expression “$p^n\Gamma$-analytic” instead of $G_n$-analytic. The following theorem is a generalization of [BC08, Proposition 3.3.1] to relative $\epsilon$-analytic representations, it can be seen as a decomposition theorem à la Kedlaya-Liu [KL19] using locally analytic vectors.

**Theorem 1.0.6 (Theorem 2.4.3).** There exists $n \gg 0$ depending on $\epsilon$ such that

$$\mathcal{F}(T_\infty) = C(T^{\pm 1}/p^\infty) \otimes_{C(T^{\pm 1}/p^\infty)} \mathcal{F}(T_\infty)^{p^n\Gamma-\text{an}}.$$

Moreover, we have that

$$\mathcal{F}(T_\infty)^{R\Gamma-\text{la}} = \mathcal{F}(T_\infty)^{\Gamma-\text{la}} = \lim_m C(T^{\pm 1}/p^m) \otimes_{C(T^{\pm 1}/p^m)} \mathcal{F}(T_\infty)^{p^n\Gamma-\text{an}}.$$
The previous theorem shows that, under certain conditions on \( F \), the derived locally analytic vectors of \( F(\mathcal{T}_\infty) \) are concentrated in degree 0, and that all the relevant information is already encoded in the \( p^m\Gamma \)-analytic vectors for some \( m \gg 0 \). In particular, we have that

\[
R \Gamma_{\text{proé}}(\mathcal{T}_\infty, F) = R \Gamma(\Gamma^{\text{an}}, R \Gamma(\text{Lie } \Gamma, F(\mathcal{T}_\infty)^{p^m\Gamma})).
\]

Thus, the problem of computing proétale cohomology has been reduced to a problem of computing Lie algebra cohomology. The module \( F(\mathcal{T}_\infty)^{p^m\Gamma} \) is not mysterious at all, it is the Higgs bundle attached to \( F \) via the local \( p \)-adic Simpson correspondence of \( \mathcal{T} \).

Now, the action of \( \text{Lie } \Gamma \) is \( C(\mathbb{T}_{\pm 1/p^n}) \)-linear and \( \Gamma \)-equivariant. By extending scalars, it induces a \( \Gamma \)-equivariant \( C(\mathbb{T}_{\pm 1/p^n}) \)-linear action on \( F(\mathcal{T}_\infty) \). Moreover, if \( X \) has a form over \((K, K^+)\), this action is \( \text{Gal}_K \)-equivariant, where \( \text{Gal}_K \) acts on \( \text{Lie } \Gamma \) via the cyclotomic character. On the other hand, we can naturally identify \( \text{Lie } \Gamma \otimes C(\mathbb{T}_{\pm 1/p^n}) = \Omega^1_X \otimes_{\mathcal{O}_X} F(\mathcal{T}_{-1}) \), see Proposition 3.1.2. This shows that the action of \( \text{Lie } \Gamma \) defines an \( \hat{\mathcal{O}}_X \)-linear map of proétale sheaves over \( \mathcal{T} \)

\[
\text{Sen}_F : \Omega^1_X \otimes_{\mathcal{O}_X} F(1) \to F,
\]
or equivalently, it defines the Sen operator

\[
\theta_F : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_X \Omega^1(-1).
\]

In general, we shall construct the Sen operators locally on the Kummer-étale topology of \( X \) by taking charts to products of tori and closed affinoid discs. The fact that \( \theta_F \) does not depend on the charts and that it is functorial with respect to the sheaf and the space will be proved in §3.3.

An overview of the paper. In §2.1 we briefly review some basic facts of the theory of solid locally analytic representations of the author and Rodrigues Jacinto [RJRC22, RJRC23], particularly we focus on the cohomology comparison theorems. Then, in §2.2-2.5 we extend the abstract formalism of Sen theory of Berger-Colmez [BC08] that will be used in the main applications of the paper.

In §3.1 we extend the Hodge-Tate decomposition of Scholze [Sch13b] from smooth to log smooth adic spaces over an algebraically closed field; this follows formally from Scholze's proof using the formalism of log adic spaces. In §3.2 we construct the geometric Sen operator in local coordinates, proving local versions of Theorems 1.0.3 and 1.0.4; then, in §3.3 we prove the global versions of the theorems. In §3.4 we study locally analytic vectors of torsors of \( p \)-adic Lie groups, proving Corollary 1.0.5. We end the section in §3.5 by discussing the relation of geometric Sen theory and the \( p \)-adic Simpson correspondence of [LZ17, DLLZ23, Wan23].

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2. Abstract Sen theory after Berger-Colmez

Sen theory has shown to be a powerful tool in the Galois theory of $p$-adic fields. For example, it is used to compute Galois cohomology over period rings:

**Proposition 2.0.1** ([Tat67, Proposition 8]). Let $\mathbb{C}_p$ be the $p$-adic completion of an algebraic closure of $\mathbb{Q}_p$, and let $G_{\mathbb{Q}_p}$ be the absolute Galois group. For $i \in \mathbb{Z}$ we let $\mathbb{C}_p(i)$ denote the $i$-th Tate twist. Then

$$H^k(G_{\mathbb{Q}_p}, \mathbb{C}_p(i)) = \begin{cases} 0 & \text{if } i \neq 0 \\ \mathbb{Q}_p & \text{if } i = 0 \text{ and } k = 0 \\ \mathbb{Q}_p \log \chi_{\text{cyc}} & \text{if } i = 0 \text{ and } k = 1. \end{cases}$$

In [BC08], Berger-Colmez defined an axiomatic framework where Sen theory can be applied. Using this formalism different constructions attached to finite dimensional Galois representations become formally the same: the Sen module (relative to $\mathbb{Q}_p^{\text{cyc}}$), the overconvergent ($\varphi, \Gamma$)-module (relative to $\tilde{\mathbb{B}}(\mathbb{Q}_p^{\text{cyc}})$), the module $D_{\text{diff}}$ of differential equations (relative to $\mathbb{D}_{\text{dR}}(\mathbb{Q}_p^{\text{cyc}})$). Moreover, using Sen theory, Berger-Colmez describe in [BC16] the locally analytic vectors of completed Galois extensions of $\mathbb{Q}_p$ with group isomorphic to a $p$-adic Lie group.

The work of Lue Pan [Pan22] is an excellent application of this tool. Inspired from the work of Berger-Colmez, Pan describes the $p$-adic Simpson correspondence of the $\text{GL}_2(\mathbb{Q}_p)$-equivariant local systems of the modular curves in terms of sheaves over the flag variety. Furthermore, based on the strategy of [BC16], he manages to use the axiomatic Sen theory to describe the Hodge-Tate structure of locally analytic “interpolations” of finite rank local systems.

The main goal of this section is to provide a more conceptual understanding of the work of Pan and Berger-Colmez via the theory of (solid) locally analytic representations. More precisely, we will prove that the construction of the Sen module holds not only for finite rank representations of the profinite group $\Pi$ (which is $G_{\mathbb{Q}_p}$ in classical Sen theory), but for a larger class of locally analytic representations, and that it can be described as taking locally analytic vectors for a suitable quotient of $\Pi$. 

2. Abstrakt Sen Theorie nach Berger-Colmez

Sen-Theorie hat sich als mächtige Werkzeug in der Galois-Theorie der $p$-adischen Körper erwiesen. Zum Beispiel wird es verwendet, um Galoiskohomologie über Perioderinge zu berechnen:

**Satz 2.0.1** ([Tat67, Proposition 8]). Sei $\mathbb{C}_p$ der $p$-adische Abschluss eines algebraischen Schließungsfeldes von $\mathbb{Q}_p$, und sei $G_{\mathbb{Q}_p}$ die absolute Galoisgruppe. Für $i \in \mathbb{Z}$ bezeichnen wir mit $\mathbb{C}_p(i)$ die $i$-te Tate-Drehung. Dann gilt:

$$H^k(G_{\mathbb{Q}_p}, \mathbb{C}_p(i)) = \begin{cases} 0 & \text{falls } i \neq 0 \\ \mathbb{Q}_p & \text{falls } i = 0 \text{ und } k = 0 \\ \mathbb{Q}_p \log \chi_{\text{cyc}} & \text{falls } i = 0 \text{ und } k = 1. \end{cases}$$

In [BC08], Berger-Colmez definierten ein axiomatisches Framework, in dem Sen-Theorie anwendbar ist. Verwenden sie diese Formalismen, werden unterschiedliche Konstruktionen an endlich dimensionalen Galois-repräsentationen formal derselben Art: der Sen-Modul (relativ zu $\mathbb{Q}_p^{\text{cyc}}$), der überkonvergente ($\varphi, \Gamma$)-Modul (relativ zu $\tilde{\mathbb{B}}(\mathbb{Q}_p^{\text{cyc}})$), der Modul $D_{\text{diff}}$ der Differentialgleichungen (relativ zu $\mathbb{D}_{\text{dR}}(\mathbb{Q}_p^{\text{cyc}})$). Zudem, mittels Sen-Theorie, beschreiben sie in [BC16] die lokal-analytischen Vektoren von abgeschlossenen Galois-Erweiterungen von $\mathbb{Q}_p$ mit Gruppe isomorph zu einem $p$-adischen Lie-Gruppe.

Die Arbeit von Lue Pan [Pan22] ist eine hervorragende Anwendung dieses Werkzeugs. Inspiriert von der Arbeit von Berger-Colmez, beschreibt Pan die $p$-adische Simpson-Korrespondenz der $\text{GL}_2(\mathbb{Q}_p)$-gekoppelten lokalen Systeme der Modularkurven in Form von Schichten über der Flaggebiet. Ferner, basierend auf der Strategie von [BC16], erfasst er die Sen-Korrespondenz der Hodge-Tate-Struktur von lokal-analytischen "Interpolations" von endlichem Rang lokalen Systemen.

Das Hauptziel dieser Abschnitt ist es, eine konzeptionelleren Verständnis der Arbeit von Pan und Berger-Colmez via die Theorie von (solid) lokal-analytischen Darstellungen zu geben. Genauer gesagt, werden wir beweisen, dass die Konstruktion des Sen-Moduls nicht nur für endlich rechteckige Darstellungen der profinierten Gruppe $\Pi$ (welche $G_{\mathbb{Q}_p}$ in der klassischen Sen-Theorie ist), sondern für eine größere Klasse von lokal-analytischen Darstellungen, und dass es sich als lokale-analytische Vektoren für einen geeigneten Quotienten von $\Pi$ beschreiben lässt.
2.1. **Recollections in locally analytic representations.** In this small subsection we recall some tools from the theory of locally analytic representations of compact $p$-adic Lie groups. We will be only concerned in continuous representations on (colimits of) Banach spaces, so everything can be stated in the classical language of Schneider–Teitelbaum [ST02,ST03] and Emerton [Eme17]. However, we will use some technology from the theory of solid locally analytic representations as developed in [RJRC22] and [RJRC23], more precisely, we will use the general comparison theorem between continuous and locally analytic cohomology.

Let $G$ be a compact $p$-adic and let $C^{\text{la}}(G, \mathbb{Q}_p)$ be the LB space of locally analytic functions of $G$. We let $C^{\text{la}}(G, \mathbb{Q}_p)_{\star 1}$ denote the left regular action (resp. $\star_2$ for the right regular action). Let $\mathbb{Q}_p[[G]] = \mathbb{Z}_p[[G]][[\frac{1}{p}]]$ denote the rational Iwasawa algebra of $G$, seen as a solid $\mathbb{Q}_p$-algebra. Let $\text{Solid}(\mathbb{Q}_p[[G]])$ be the abelian category of solid $\mathbb{Q}_p[[G]]$-representations, or equivalently the category of solid $G$-representations. We let $\mathcal{D}_{\text{la}}(\mathbb{Q}_p[[G]])$ be the derived category of $\text{Solid}(\mathbb{Q}_p[[G]])$.

**Definition 2.1.1.** Let $V \in \mathcal{D}_{\text{la}}(\mathbb{Q}_p[[G]])$.

1. The solid group cohomology of $V$ is given by
   
   $R_{\text{la}}^\Gamma(G, V) := R\text{Hom}_{\mathbb{Q}_p[[G]]}(\mathbb{Q}_p, V)$,
   
   where $\text{Hom}$ is the internal Hom as solid $\mathbb{Q}_p[[G]]$-module. We denote by $R_{\text{la}}^\Gamma(G, V) := R\Gamma(G, V)(\ast)$ the underlying $\mathbb{Q}_p$-vector space.

2. The derived locally analytic vectors of $V$ are defined as
   
   $V^{\text{RG-la}} := R_{\text{la}}^\Gamma(G, (V \otimes_{\mathbb{Q}_p} C^{\text{la}}(G, \mathbb{Q}_p))_{\star 1,3})$,
   
   where $\otimes_{\mathbb{Q}_p}$ is the solid tensor product, and the $\star_{1,3}$ is the diagonal action on $V$ and $C^{\text{la}}(G, \mathbb{Q}_p)_{\star 1}$. If $V \in \text{Solid}(\mathbb{Q}_p[[G]])$ we denote $V^{\text{G-la}} := H^0(V^{\text{RG-la}})$.

3. We say that a solid $G$-representation $V$ is locally analytic if the natural map
   
   $V^{\text{RG-la}} \to V$ is a quasi-isomorphism.

The following theorem summarizes the main features of the theory of solid locally analytic representations.

**Theorem 2.1.2.** Let $\text{Rep}_{\mathbb{Q}_p}^{LC}(G)$ be the category of $\mathbb{Q}_p$-linear compactly generated complete locally compact continuous representations of $G$. The following hold:

1. There is a fully faithful inclusion
   
   $(-) : \text{Rep}_{\mathbb{Q}_p}^{LC}(G) \hookrightarrow \text{Solid}(\mathbb{Q}_p[[G]])$.

2. Under the inclusion of (1), there is a natural quasi-isomorphism of $\mathbb{Q}_p$-vector spaces
   
   $R\Gamma_{\text{la}}(G, V) = R\Gamma(G, V)$
   
   between solid and continuous cohomology.

3. The functor $(-)^{\text{RG-la}}$ is idempotent. Furthermore, let $\mathcal{D}_{\text{la}}(G^{\text{la}}) \subset \mathcal{D}_{\text{la}}(\mathbb{Q}_p[[G]])$ be the full subcategory of locally analytic representations. Then $\mathcal{D}_{\text{la}}(G^{\text{la}})$ is stable under colimits and tensor products. Moreover, the inclusion has by right adjoint $(-)^{\text{RG-la}}$ the functor of derived locally analytic vectors.
Let $V \in \mathcal{D}^o_{\mathbb{Q}_p[[G]]}$, then there is a natural equivalence

$$R\Gamma_v(G, V) = R\Gamma(G^{\text{sm}}, R\Gamma(\text{Lie } G, V^{\text{RG-la}})),$$

where $R\Gamma(\text{Lie } G, -)$ is Lie algebra cohomology, and $R\Gamma(G^{\text{sm}}, -)$ is smooth group cohomology.

Proof. Parts (1) and (2) follow from Proposition 3.7 and Lemma 5.2 of [RJRC22] respectively. Part (3) follows from Proposition 3.1.11 and Proposition 3.2.3 of [RJRC23]. Finally, part (4) follows from Theorems 5.3 and 5.5 of [RJRC22], or Theorem 6.3.4 of [RJRC23].

An immediate consequence of the theorem is the following corollary:

**Corollary 2.1.3.** Let $V \in \text{Solid}(\mathbb{Q}_p[[G]])$ be solid $G$-representation. Suppose that $V^{\text{RG-la}} = V^{G-la}$, i.e., that the higher derived locally analytic vectors vanish. Then there are isomorphisms

$$H^i(G, V) = H^i(\text{Lie } G, V^{G-la})^G.$$

In particular, if $V$ arises from a compactly generated locally convex $\mathbb{Q}_p$-vector space, we have isomorphisms

$$H^i(G, V) = H^i(\text{Lie } G, V^{G-la})^G.$$

Proof. This follows from parts (2) and (4) of Theorem 2.1.2.

The property of being a locally analytic representation roughly means that the operators $[g] - [1]$ for $g \in G$ have norm $< 1$. An example of this phenomena appears in the following lemma.

**Lemma 2.1.4.** Let $V$ be a Banach representaion of $G$. Then $V$ is locally analytic if and only if there is an open compact subgroup $G^0$, a $G^0$-stable lattice $V^+ \subset V$ and $\epsilon > 0$ such that the action of $G^0$ on $V^+/p^\epsilon$ is trivial. Furthermore, if this is the case, then the action of $G$ on $V$ is analytic for some affinoid neighborhood $G \subset \mathbb{G}$.

Proof. This is a particular case of [RJRC23, Proposition 3.3.3].

Finally, we will need a projection formula for the functor of locally analytic vectors.

**Lemma 2.1.5.** Let $A$ be a Banach $\mathbb{Q}_p$-algebra endowed with a locally analytic action of $G$. Let $X$ and $M$ be $G$-equivariant Banach $A$-modules such that $M$ has a topological basis over $A$, and such that the action of $G$ on $M$ is locally analytic. Then there is an equivalence of solid vector spaces

$$(\hat{X} \otimes_A M)^{\text{RG-la}} = \hat{X}^{\text{RG-la}} \otimes_A^L M.$$
Then, since Banach spaces are flat solid $\mathbb{Q}_p$-vector spaces (see [RJRC22] Lemma 3.21), the Barr resolution for the tensor gives rise a $G$-equivariant long exact sequence

$$\cdots \to X \otimes_{\mathbb{Q}_p} A^n \otimes_{\mathbb{Q}_p} M \to \cdots \to X \otimes_{\mathbb{Q}_p} M \to X \otimes_{\mathbb{A}_p}^L M \to 0.$$  

Then, by the projection formula of [RJRC23] Corollary 3.15 (3), we have that

$$(X \otimes_{\mathbb{Q}_p} A^n \otimes_{\mathbb{Q}_p} M)^{\mathbb{R}G-\text{la}} = X^{\mathbb{R}G-\text{la}} \otimes_{\mathbb{Q}_p} A^n \otimes_{\mathbb{Q}_p} M.$$  

Since $(-)^{\mathbb{R}G-\text{la}}$ has finite cohomological dimension ([RJRC23] Lemma 3.2.2), one deduces that

$$(X \otimes_{\mathbb{A}_p}^L M)^{\mathbb{R}G-\text{la}} = X^{\mathbb{R}G-\text{la}} \otimes_{\mathbb{A}_p}^L M$$

as wanted. \(\square\)

### 2.2. Colmez-Sen-Tate axioms

Let us introduce the terminology. Let $(A, A^+)$ be a uniform affinoid $\mathbb{Q}_p$-algebra, given $B \subset A$ a closed subalgebra we denote $B^+ := B \cap A^+$. Let $d \geq 1$ be an integer, $\Pi$ a profinite group and $\chi : \Pi \to \mathbb{Z}_p^d$ a surjective continuous group homomorphism with kernel $H$, we will refer to $\chi$ as the “character”. Given $\Pi' \subset \Pi$ an open subgroup and $H' := H \cap \Pi'$ we define the following objects:

- Let $N_{H'}$ be the normalizer of $H'$ in $\Pi$.
- Let $\Gamma_{H'} = N_{H'}/H'$ and $C_{H'} \subset \Gamma_{H'}$ its center. By Lemma 3.1.1 of [BC08] the group $C_{H'}$ is open in $\Gamma_{H'}$.
- We let $n_1(H') \in \mathbb{N}$ be the smallest integer $n$ such that $\chi(C_{H'})$ contains $p^n\mathbb{Z}_p^d$.
- More generally, given an open subgroup $C' \subset C_{H'}$ such that $\ker \chi \cap C' = 1$, we denote by $n(C') \in \mathbb{N}$ the smallest integer such that $p^n\mathbb{Z}_p^d \subset \chi(C')$.
- Let $C'$ be as above. Let $e_1, \ldots, e_d \in \mathbb{Z}_p^d$ be the standard basis, for $n \geq n(C')$ we let $\gamma_1^{(n)}, \ldots, \gamma_d^{(n)}$ denote the inverse image of $p^ne_1, \ldots, p^ne_d$ in $C'$. Thus, if $k \geq 0$, we have that $\gamma_1^{(n+k)} = (\gamma_1^{(n)}, \ldots, \gamma_d^{(n)})$. We shall write $\gamma^{(n)} = (\gamma_1^{(n)}, \ldots, \gamma_d^{(n)})$ and let $\langle \gamma^{(n)} \rangle \subset C'$ be the generated subgroup.

Let us suppose that $\Pi$ acts continuously on $(A, A^+)$. The action of $\Pi$ on $A^+/p^s$ is smooth for any $s \geq 1$ since this last ring is discrete, we suppose in addition that $A^+/p^s = \lim_{H' \subset H} A^{H',+}/p^s$ where $H'$ runs over all the open subgroups of $H$. Let $I \subset \mathbb{Q}_{\geq 0}$ be a dense additive submonoid containing $\mathbb{N}$, suppose that there are topologically nilpotent units $\{\sigma^f\}_{f \in I_{<0}}$ in $A^H$ such that:

1. For any $x \in \text{Spa}(A, A^+)$ we have $|\sigma^{f+\delta}|_x = |\sigma^{f}|_x |\sigma^{\delta}|_x$.
2. Let $\| \cdot \| : A \to \mathbb{R}_{\geq 0}$ be the norm making $A^+$ the unit ball and $\sigma^f A^+$ the ball of radius $p^{-\epsilon}$. Then $\| \cdot \|$ is a submultiplicative non-archimedean norm, i.e. it satisfies
   - $\|xy\| \leq \|x\| |y|.$
   - $\|x + y\| \leq \sup\{\|x\|, |y|\}.$
3. $\Pi$ acts by isometries on $(A, \| \cdot \|)$.

Condition (1) implies that the elements $\sigma^f$ are multiplicative units for the norm $\| \cdot \|$. Moreover, the ideal of topologically nilpotent elements of $A$ is equal to $A^{00} = \cdots \to X \otimes_{\mathbb{Q}_p} A^n \otimes_{\mathbb{Q}_p} M \to \cdots \to X \otimes_{\mathbb{Q}_p} M \to X \otimes_{\mathbb{A}_p}^L M \to 0.$
$\bigcup_{\iota \in I_0} \sigma^\iota A^+$. From now on we always take $\epsilon \in I$. In the following we consider almost mathematics with respect to the sequence $\{\sigma^\iota\}_{\iota > 0}$.

**Remark 2.2.1.** In the main application of the paper the units $\sigma^\iota$ will be algebraic numbers over $\mathbb{Q}_p$ with $p$-adic valuation $|\sigma^\iota| = |p|^\iota$. We have decided to develop the theory in this slightly more general situation where $\sigma^\iota$ might not be algebraic over $\mathbb{Q}_p$ in order to include the framework of overconvergent $(\varphi, \Gamma)$-modules, where the elements $\sigma^\iota$ arise as Teichmüller lifts $[p^h, e]$ of $p$-power roots of $\pi = e - 1 \in \mathbb{Q}_p^{\text{cycl}}$, and $e = (\zeta_p^h)_{h \geq 0}$ is a compatible sequence of $p$-th power roots of unity, see [CC98].

**Definition 2.2.2** (Colmez-Sen-Tate axioms). We define the following axioms for the triple $(A, \Pi, \chi)$.

(CST0) **Almost purity.** For $H_1 \subset H_2 \subset H$ open subgroups, the trace map $\text{Tr}_{H_2/H_1} : A^{+, H_1} \to A^{+, H_2}$ is almost surjective.

(CST1) **Tate’s normalized traces.** There is $c_2 > 0$, for all open subgroup $H' \subset H$ an integer $n(H') \geq n_1(H')$, a sequence of closed subalgebras $(A_{H', n})_{n \geq n(H')}$ of $A^{H'}$, and for $n \geq n(H')$ $\mathbb{Q}_p$-linear maps $R_{H', n} : A^{H'} \to A_{H', n}$, satisfying the following conditions:

1. We can write $I = \bigcup_{n \in \mathbb{N}} I_n$ as colimit of additive submonoids such that for any $H'$ and $n \geq n(H')$ we have $\{\sigma^\iota\}_{\iota \in I_n} \subset A_{H', n}$.
2. $R_{H', n}$ is an $A_{H', n}$-linear projection onto $A_{H', n}$. We let $X_{H', n}$ denote the kernel of $R_{H', n}$ and $X_{H', n}^+ := X_{H', n} \cap A^+$.
3. $gA_{H', n} = A_{gH', g^{-1}n}$ and $gR_{H', n}(x) = R_{gH', g^{-1}n}(gx)$ for all $g \in \Pi$ and $x \in A^{H'}$.
4. For all $n \geq n(H')$ and $x \in A^{+, H'}$, we have $R_{H', n}(x) \in \sigma^{-c_2} A_{H', n}$. In other words $\|R_{H', n}(x)\| \leq |\sigma^{-c_2}||x||$ for $x \in A^{H'}$.
5. Given $x \in A^{H'}$ we have $\lim_{n \to \infty} R_{H', n}(x) = x$.
6. The action of $\Gamma_{H'}$ on the Banach algebra $A_{H', n}$ is locally analytic.

Equivalently, by Lemma 2.1.4 there is an open subgroup $\Pi' \subset \Pi$ with $\Pi' \cap H = H'$ such that the action of $\Pi'$ on $A_{H', n}/\sigma^\iota$ is trivial.

(CST2) **Bounds for the vanishing of cohomology.** There exists $c_3 > 0$, and for an open subgroup $\Pi' \subset \Pi$ an integer $n(\Pi') \geq n_1(\Pi')$ such that if $n \geq n(\Pi')$, and $C' \subset C_H$ is an open subgroup satisfying $C' \cap \ker \chi = 1$ and $n(C') \leq n$, then for all $n(C') \leq m \leq n$ the cohomology groups $H^i((\varphi^{(m)}), \sigma^\iota X_{H,n})$ are $\sigma^\iota$-torsion for $i = 0, 1, 2$ and $\epsilon \in \pm 1$.

In the application to rigid spaces we will have the following stronger axioms

(CST1*) **Decomposable traces.** Let $\Pi' \subset \Pi$ be an open subgroup and $H' = \Pi' \cap H$. There exists $c_2$ and an integer $n(H') \geq n_1(H')$ satisfying:

1. For $n \geq n(H')$ and $i = 1, \ldots, d$, there are closed $\mathbb{Q}_p$-subalgebras $A^n_{H', n}$ of $A^{H'}$, and $A^{i, n}$-linear projections $R^n_{H', n} : A^{H'} \to A^n_{H', n}$. We let $X^{i, +}_{H', n}$ denote the kernel of $R^n_{H', n}$ and $X^{i, +}_{H', n} = X^n_{H', n} \cap A^+$. 

2. For \( g \in C_H \), we have that \( gA^i_{H,n} = A^i_{H,n} \) and \( gR_{H,n}^i(x) = R_{H,n}^i(gx) \) for all \( x \in A^H \) and all \( i = 1, \ldots, d \).

3. For \( x \in A^+H \) we have \( R_{H,n}^i(x) \in \sigma^{-c_2}A_{H,n}^1 \). In other words,

\[ \|R_{H,n}^i(x)\| \leq |\sigma^{-c_2}||x||. \]

4. Given a fixed \( H' \) and \( n \), the maps \( R_{H',n}^i \) commute for \( i = 1, \ldots, d \), and their composition \( R_{H',n}^d := R_{H',n}^d \circ \cdots \circ R_{H',n}^1 \) satisfies the axiom (CST1).

\( \text{(CST2*) Strong bounds for the vanishing of cohomology.} \) There exists \( c_3 > 0 \), and for an open subgroup \( \Pi' \subset \Pi \) an integer \( n(\Pi') \geq n_1(H') \), such that if \( n \geq n(\Pi') \), and \( C' \subset C_H \) is an open subgroup with \( C' \cap \ker \chi = 1 \) and \( n(C') \leq n \), then for all \( n(C') \leq m \leq n \) we have:

- The multiplication map \( \gamma_i^{(m)} - 1 : X_{H,n}^i \rightarrow X_{H,n}^i \) is invertible with \( \|\gamma_i^{(m)} - 1\| \leq |\sigma^{-c_3}| \), i.e. \( (\gamma_i^{(m)} - 1)^{-1}(x) \in \sigma^{-c_3}X_{H,n}^{i+} \) for \( x \in X_{H,n}^{i+} \).

**Remark 2.2.3.** The \( \sigma^{-c_3} \)-torsion in (CST2) means the following: let \( \gamma^{(m)} = (\gamma_1^{(m)}, \ldots, \gamma_d^{(m)}) \), the cohomology \( R\Gamma((\gamma^{(m)}), \sigma^eX_{H,n}^+) \) is represented by a Koszul complex \( \text{Kos}(\gamma^{(m)}, \sigma^eX_{H,n}^+) \).

Then \( H'(\gamma^{(m)}, X_{H,n}) = 0 \) for \( i = 0, 1, 2 \) and if \( \beta \) is an \( i \)-coycle for \( \sigma^eX_{H,n}^+ \) for \( i = 0, 1, 2 \), there exists a \((i - 1)\)-cochain \( \beta' \) of \( \sigma^{-c_3}X_{H,n}^+ \) such that \( d(\beta') = \beta \). The condition for \( i = 2 \) guarantees that we can lift a \( 1 \)-coycle \( \beta \) of \( X_{H,n}^+ / \sigma^e \) to a \( 1 \)-coycle \( \beta' \) of \( X_{H,n}^+ \) which agrees with \( \beta \) modulo \( \sigma^{-c_3} \).

**Remark 2.2.4.** The axioms (CST0), (CST1) and (CST2) above are generalizations of the axioms (TS1), (TS2) and (TS3) of [BC08] respectively. There is a subtle difference between (CST1) and (TS2), which are the additional properties (0) and (6). Condition (6) holds in the context of classical arithmetic Sen theory and for overconvergent \( \phi, \Gamma \)-modules, it arises from the intuition that one is decompleting the algebra \( A^H \) by its locally analytic vectors for the action of \( \Gamma^H \). The condition (0) says that the topologically nilpotent units for which the almost setting is defined are locally analytic. The axioms (CST1*) and (CST2*) are stronger generalizations of (TS2) and (TS3) which we will encounter in the geometric applications, see Example 2.2.7.

**Lemma 2.2.5.** Suppose that (CST1*) holds. Then (CST2*) implies (CST2). Moreover, under (CST2*) the group cohomology \( R\Gamma((\gamma^{(m)}), \sigma^eX_{H,n}^+) \) is \( \sigma^{-c_3} \)-torsion for \( n(C') \leq m \leq n \).

**Proof.** Without loss of generality let us take \( \epsilon = 0 \), the argument for arbitrary \( \epsilon \) is the same. For \( a_1, \ldots, a_d \in \{\pm\} \) set \( \alpha = (a_1, \ldots, a_d) \). We define

\[ X_{H,n}^\alpha := \left( \bigcap_{a_i = +} R_{H,n}^i \right) \left( \bigcap_{a_i = -} (1 - R_{H,n}^i) \right) A^H. \]

Let \( + = (+, \ldots, +) \), by (CST1*) (4) we have \( A_{H,n} = X_{H,n}^{(+)} \), and \( X_{H,n}^\alpha = \bigoplus_{\alpha \neq (+)} X_{H,n}^\alpha \) as \( C_{H,n}^\alpha \)-modules. Notice that if \( \alpha_i = - \), then \( X_{H,n}^{\alpha_i} \subset X_{H,n}^i \) and the restriction of
\( \gamma^{(m)} - 1 \) to \( X^H_{\alpha^m} \) is still an isomorphism. Without loss of generality we can take \( \alpha_1 = -1 \). Define the following maps for \( 0 \leq i \leq d - 1 \)

\[
h^i : \bigwedge^i (X^\alpha_{H^\delta})^{\otimes d} \to \bigwedge^{i+1} (X^\alpha_{H^\delta})^{\otimes d}
\]

\( (x_{i_1, \ldots, i_d})_{i_1 < \ldots < i_d} \mapsto \begin{cases} 0 & \text{if } i_1 = 1 \\ ((\gamma - 1)^{-1}x_{i_1, \ldots, i_d})_{i_1 < \ldots < i_d} & \text{otherwise.} \end{cases} \)

The group cohomology \( R^i(\langle \gamma^{(m)} \rangle, X^\alpha_{H^\delta}) \) is represented by the Koszul complex \( \text{Kos}(\langle \gamma^{(m)} \rangle, X^\alpha_{H^\delta}) \). A direct computation shows that the map \( h^d \) is a chain homotopy between the identity and 0 on \( \text{Kos}(\langle \gamma^{(m)} \rangle, X^\alpha_{H^\delta}) \), in particular \( R^i(\langle \gamma^{(m)} \rangle, X^\alpha_{H^\delta}) = 0 \). To see that \( R^i(\langle \gamma^{(m)} \rangle, X^\alpha_{H^\delta}) \) is \( \sigma^{-1} \) torsion, notice that the homotopy \( h^* \) is bounded by \( |\sigma^{-1}| \) thanks to (CST2*). This proves the lemma.

**Definition 2.2.6.** A Sen theory is a triple \((A, \Pi, \chi)\) as above satisfying (CST0)-(CST2). If in addition we can take \( c_2 \) and \( c_3 \) arbitrarily small as \( n \to \infty \), and if \((A, \Pi, \chi)\) satisfies (CST1*) and (CST2*), we say that \((A, \Pi, \chi)\) is a strongly decomposable Sen theory.

**Example 2.2.7.** The most important example for this paper is given by products of perfectoid torus and discs. Let \( T_{C_p} := \text{Spa}(C_p(T^{e+1}), O_{C_p}(T^{e+1})) \) and \( D_{C_p} := \text{Spa}(C_p(S), O_{C_p}(S)) \), we denote \( S^{(e,d-e)} := C_p \times D^{d-e} \). Let \( T_{C_p,n} \) and \( D_{C_p,n} \) be the finite (Kummer-)étale covers of the torus and the disc defined by taking a \( p^n \)-th root of \( T \) and \( S \) respectively. Let \( T_{C_p,\infty} = \lim_{n} T_{C_p,n} \) and \( D_{C_p,\infty} = \lim_{n} D_{C_p,n} \) denote the perfectoid torus and perfectoid unit disc. We denote \( S^{(e,d-e)}_{C_p,n} = T_{C_p,n} \times D^{d-e}_{C_p,n} \) and \( S^{(e,d-e)}_{C_p,\infty} = \lim_{n} S^{(e,d-e)}_{C_p,n} = T_{C_p,\infty} \times D^{d-e}_{C_p,\infty} \).

We set \( \Pi = \mathbb{Z}_p(1)^d \), \( A = \mathcal{O}(S^{(e,d-e)}_{\infty}) \), and let \( \chi : \Pi \to \mathcal{Z}_{p}^{d} \) be the isomorphism provided by a compatible system of \( p \)-th power roots of unity \( (\zeta_{p^n})_{n \in \mathbb{N}} \). Let \( f_1, \ldots, f_d \), be the standard basis of \( \mathbb{Z}_p^{d} \) so that

\[
f_i T_j^{p^n} = \zeta_{p^n}^{\delta_{ij}} T_j^{p^n} \]

\[
f_i S_j^{p^n} = \zeta_{p^n}^{\delta_{ij}} S_j^{p^n}
\]

where \( \delta_{ij} = 1 \) if \( i = j \) and 0 otherwise. Then \((A, \Pi, \chi)\) is a strongly decomposable Sen theory. Indeed, for \( n \in \mathbb{N} \) define

\[
A_{n} = \begin{cases} C_p(T_1^{\pm p^n}, \ldots, T_i^{\pm p^n}, \ldots, T_e^{\pm p^n}, S^{\pm p^n}) & \text{if } 1 \leq i \leq e \\
C_p(T_1^{\pm p^n}, S_{e+1}^{\pm p^n}, \ldots, S_j^{\pm p^n}, \ldots, S_d^{\pm p^n}) & \text{otherwise.} 
\end{cases}
\]

One has normalized Sen traces \( R^i_{n,m} : A \to A_{n} \) which are given as the unique continuous extension to \( A \) of the normalized traces

\[
R^i_{n,m} = \frac{1}{p^{n-m}} \text{Tr}_{A_{n}/A_{m}} : A_{n} \to A_{m}.
\]
Therefore, if $n > m$ and $1 \leq i \leq e$ one gets
\[(f_i^{m})^{*} \cdot T_i^{\frac{1}{m^{v_p}}} - T_i^{\frac{1}{p^{v_p}}} = (\zeta_{p^{v_p}}^{1} - 1)T_i^{\frac{1}{p^{v_p}}},\]
whence $f_i^{m} - 1$ is invertible on $X_n^i := \ker R^i_n$ and bounded by $|\zeta_{p^{v_p}}^{1} - 1|$. A similar property holds for the $S_j$'s. Now, defining $A_n = \mathcal{O}(\mathbb{S}_{\mathbb{C}_p}^{(e,d-e)})$, and $R_n = R_{d-e}^1 \circ \cdots \circ R_{d}^1$, one verifies (CST1*) and (CST2*).

**Example 2.2.8.** Let us keep the notation of Example [2.2.7] and let’s consider $\mathbb{S}_{\mathbb{C}_p}^{(e,d-e)}$ endowed with the log structure induced by the normal crossing divisors $S_{e+1} \cdots S_{d} = 0$. For $i \in \{e+1, \ldots, d\}$ let $D_i \subset \mathbb{S}_{\mathbb{C}_p}^{(e,d-e)}$ be the divisor defined by $S_i = 0$, and given $J \subset \{e+1, \ldots, d\}$ we let $D_J = \bigcap_{i \in J} D_i$. We endow $D_J$ with the log structure $\mathcal{M}$ given by pullback along the map $D_J \to \mathbb{S}_{\mathbb{C}_p}^{(e,d-e)}$. For $n \in \mathbb{N} \cup \{\infty\}$ let $D_{J,n} = \mathbb{S}_{\mathbb{C}_p}^{(e,d-e)} \times_{\mathbb{S}_{\mathbb{C}_p}^{(e,d-e)}} \mathcal{O}_{D_J}$ where the fiber product is an fs log adic space, see [DCLZ19] Proposition 2.3.27. Without loss of generality let us assume that $J = \{i+1, \ldots, d\}$ for some $i \geq e$, the underlying adic space of $D_{J,\infty}$ is the perfectoid space
\[D_J = \text{Spa} \mathbb{C}_p(\frac{1}{x_i}, S_i^{1/p^{v_p}}, S_i^{1/p^{v_p}}, \ldots, S_i^{1/p^{v_p}}) = \mathbb{S}_{\mathbb{C}_p}^{(e,d-e)} \times_{\mathbb{S}_{\mathbb{C}_p}^{(e,d-e)}} \mathcal{O}_{D_J}.
\]
On the other hand, the log structure $\mathcal{M}$ of $D_{\infty}$ is induced by the map of monoids
\[\delta : (\mathbb{N}[\frac{1}{p}])^{d} \to A_{\infty,J} := \mathbb{C}_p(\frac{1}{x_i}, S_i^{1/p^{v_p}}, S_i^{1/p^{v_p}}, \ldots, S_i^{1/p^{v_p}})
\]
mapping the $j$-th component to $S_j$ if $j \leq i$, and to 0 if $j > i$. In particular, $(D_{J,\infty}, \mathcal{M}_{\infty}) \to (D_J, \mathcal{M})$ is a $\Gamma = \mathbb{Z}_p(1)^{d}$-torsor in the pro-Kummer-étale site of $(D_J, \mathcal{M})$. Let $\Gamma_J \subset \Gamma$ be the subgroup generated by the $J$-th components, where $J^e = \{e+1, \ldots, d\} \setminus J$, and let $\text{pr}_J : \Gamma \to \Gamma_J$ be the projection onto the $J^e$-components. Then the triple $(A_{\infty,J}, \text{pr}_J)$ is not a geometric Sen theory on the nose because the almost purity axiom (CST0) is not satisfied. However, the action of $\Gamma_J$ is trivial on the ring $A_{\infty,J}$ and the triple $(A_{\infty,J}/\Gamma_J, \text{pr}_J)$ with $\text{pr}_J : \Gamma/\Gamma_J \cong \Gamma_J$ is a decomposable Sen theory by Example [2.2.7].

### 2.3. Relative locally analytic representations

We keep the conventions of the triple $(A, \Pi, \chi)$ as in the beginning of the previous section. In the next paragraph we will give an ad-hoc definition of a relative locally analytic representation over a Sen theory. The motivation is provided by Lemma [2.1.4] saying that, a continuous action of a compact $p$-adic Lie group $G$ on a Banach space $V$ is locally analytic if and only if there is a $G$-stable lattice $V^0 \subset V$ such that $G$ acts through a finite quotient on $V^0/p$.

To adapt the devisage process in [BC08], we need to consider continuous 1-cocycles of infinite rank $A$-modules. In other words, we want to consider continuous maps from $\Pi$ to some $\text{Aut}_A(V)$, where $V$ is an ON Banach $A$-module. In order to endow $\text{Aut}_A(V)$ with the correct topology let us show the following lemma:
Lemma 2.3.1. Let $B$ be a $\mathbb{Q}_p$-Banach algebra and $B^0 \subset B$ a bounded $p$-adically complete subring. Then

\[
\text{End}_{B}(\bigoplus_i B) = \text{Hom}_{B}(\bigoplus_i B, \bigoplus_j B) = \left( \prod_i \hat{B}(\bigoplus_j B) \right)^{\frac{1}{p}},
\]

(2.1)

\[
\text{End}_{B^0}(\bigoplus_i B^0) = \text{Hom}_{B^0}(\bigoplus_i B^0, \bigoplus_j B^0) = \prod_i \hat{B}(\bigoplus_j B^0)
\]

Where the internal $\text{Hom}$ is as continuous $B$ or $B^0$-modules. We endow the $\text{Hom}_{B^0}$ space with its natural product topology, and the $\text{Hom}_B$ space with the locally convex topology making the $\text{Hom}_{B^0}$ space a bounded subspace. Equivalently, we endow $\text{Hom}_{B^0}$ and $\text{Hom}_B$ with the compact open topology (these two topologies agree thanks to [CST9, Proposition 4.2]).

Proof. We have that $\text{Hom}_B(\bigoplus_i B, \bigoplus_j B) = \text{Hom}_{B^0}(\bigoplus_i B^0, \bigoplus_j B^0)[\frac{1}{p}]$, so it suffices to prove the second equality. One has that

\[
\text{Hom}_{B^0}(\bigoplus_i B^0, \bigoplus_j B^0) = \lim_{\substack{\to \\ \ \ n}} \text{Hom}_{B^0/p^n}(\bigoplus_i B^0/p^n, \bigoplus_j B^0/p^n) = \lim_{\substack{\to \ \ n}} \prod_i \bigoplus_j B^0/p^n = \prod_i \bigoplus_j B^0.
\]

Definition 2.3.2. Let $V$ be an ON Banach $A$-module. We define the topological group $\text{Aut}_A(V)$ to be the pullback

\[
\begin{array}{ccc}
\text{Aut}_A(V) & \longrightarrow & \text{End}_A(V) \times \text{End}_A(V) \\
\downarrow & & \downarrow \phi \\
\{\text{id}_V\} \times \{\text{id}_V\} & \longrightarrow & \text{End}_A(V) \times \text{End}_A(V)
\end{array}
\]

where $\phi(f, g) = (f \circ g, g \circ f)$. Equivalently, it is the closed subspace of $\text{End}_A(V) \times \text{End}_A(V)$ consisting on those pairs $(f, g)$ such that $f \circ g = \text{id}_V = g \circ f$. If $V^0$ is an ON $A^+$-lattice on $V$ we define $\text{Aut}_{A^+}(V^0)$ in a similar way.

The following lemma will be used to construct invertible elements in $\text{Aut}_A(V)$.

Lemma 2.3.3. Let $V$ be an ON Banach $A$-module. Let $M \in \text{End}_A(V)$ be an endomorphism whose norm operator satisfies $||M|| \leq ||\sigma^f||$ for some $\epsilon > 0$ and some lattice $V^0 \subset V$. Then $1 - M \in \text{Aut}_A(V)$ and its inverse is given by the convergent series $(1 - M)^{-1} = \sum_{n=0}^{\infty} M^n$.

Proof. Write $V = \bigoplus_i A$ so that $\text{End}_A(V) \equiv \left( \prod_i \bigoplus_j A^+ \right)[\frac{1}{p}]$. It is enough to show that $\sum_{n=0}^{\infty} M^n$ converges in $\text{End}_A(V)$, and that the sequence $((1 - M) \sum_{n=0}^{m} M^n)_{m \in \mathbb{N}}$ converges to $\text{id}_V$. By hypothesis $M' = \frac{1}{\sigma^f} M$ is an operator of $V^0$, thus $\sum_{n=0}^{\infty} M^n = \sum_{m=0}^{\infty} \sigma^f M M^n$ converges as $\text{End}_{A^+}(V^0) \equiv \prod_i \bigoplus_j A^+$ is $p$-adically complete, and both $\sigma$ and $p$ are topologically nilpotent units of $A$. One shows in a similar way that the sequence $((1 - M) \sum_{n=0}^{m} M^n)_{m \in \mathbb{N}}$ converges to $\text{id}_V$ finishing the proof. □
Given an index set $I$ let us denote $\GL_{I}(A) = \Aut_{A}(\bigoplus_{I} A)$ (resp. for $\GL_{I}(A^{+})$). The groups $\GL_{I}(A)$ and $\GL_{I}(A^{+})$ have a natural continuous action of $\Pi$ on the coefficients. Then, ON Banach $A$-semilinear representations of “rank $I$” are equivalent to continuous 1-cocycles of $\Pi$ on $\GL_{I}(A)$. We denote by $e_{i}$ the standard basis of $\bigoplus_{I} A$.

**Definition 2.3.4.** An ON Banach $A$-semilinear representation $\rho : \Pi \times V \to V$ is said relative locally analytic if there exists a basis $\{v_{i}\}_{i \in I}$ generating a lattice $V^{0}$ such that:

- There is $\Pi' \subset \Pi$ an open subgroup stabilizing $V^{0}$ and $\epsilon > 0$ such that the action of $\Pi'$ on $\{v_{i} \mod \sigma_{v}^{\epsilon}\}_{i \in I}$ is trivial.

We say that $\{v_{i}\}_{i \in I}$ is a relative locally analytic basis of $V$.

The previous definition can be rewritten in terms of 1-cocycles.

**Definition 2.3.5.** Let $V$ be an ON Banach $A$-module and $\rho$ an $A$-semilinear action of $\Pi$ on $V$. Let $v = \{v_{i}\}_{i \in I}$ be an ON basis of $V$, let $\Upsilon : \bigoplus_{i \in I} A \to V$ denote the $A$-linear isomorphism provided by the basis $v$, and let $\sigma_{v}$ be the $A$-semilinear action of $\Pi$ fixing $v$. We define the 1-cocycle of $\rho$ attached to $(V, v)$ to be the continuous map $U : \Pi \to \GL_{I}(A)$ given by

$$g \mapsto \Upsilon^{-1} \circ \rho(g) \circ \sigma_{v}(g)^{-1} \circ \Upsilon.$$  

An ON basis $v$ of $V$ is relative locally analytic if and only if there exists $\epsilon > 0$ and $\Pi' \subset \Pi$ an open subgroup such that the 1-cocycle $U|_{\Pi'}$ has values in $\GL_{I}(A^{+})$ and is trivial modulo $\sigma_{v}^{\epsilon}$. We say that $U$ is a locally analytic 1-cocycle.

The following lemma says that composing with matrices in $\Aut_{A}(V)$ that are close enough to $\text{id}_{V}$ preserves relative locally analytic basis.

**Lemma 2.3.6.** Let $V$ be an ON locally analytic representation of $\Pi$ and $v = \{v_{i}\}_{i \in I}$ a relative locally analytic basis, let $V^{0}$ be the lattice spanned by $\{v_{i}\}$. Let $\psi \in \End_{A}(V)$ be an operator such that $\|1 - \psi\| \leq |\sigma_{v}^{\epsilon}|$ for some $\epsilon > 0$. Then $\psi(v) = \{\psi(v_{i})\}_{i \in I}$ is a relative locally analytic basis of $V$.

**Proof.** Let $\Pi' \subset \Pi$ be an open subgroup stabilizing $V^{0}$, and let $\epsilon' > 0$ such that the action of $\Pi'$ on $\{v_{i} \mod \sigma_{v}^{\epsilon'}\}_{i \in I}$ is trivial. Let $\epsilon'' = \min\{\epsilon, \epsilon'\}$, then $\psi(v_{i}) \equiv v_{i} \mod \sigma_{v}^{\epsilon''}$ and $\Pi'$ acts on $\{\psi(v_{i}) \mod \sigma_{v}^{\epsilon''}\}_{i \in I}$ trivially. This proves the lemma.

**Example 2.3.7.**

1. Let $\Pi = G$ be a compact $p$-adic Lie group and $W$ be a Banach locally analytic representation over $\mathbb{Q}_{p}$. Then, by Lemma 2.1.4, $W \otimes_{\mathbb{Q}_{p}} A$ is a relative locally analytic representation of $\Pi$.

2. Slightly more generally, suppose that $\Pi$ admits by quotient $\Pi \to G$ a compact $p$-adic Lie group. Let $W$ be a Banach locally analytic representation of $G$ over $\mathbb{Q}_{p}$. Then $W \otimes_{\mathbb{Q}_{p}} A$ is a relative locally analytic representation of $\Pi$. This is the situation we will face in the application to rigid spaces.
2.4. The Sen functor. Let \((A, \Pi, \chi)\) be a Sen theory. Our next goal is to define the Sen functor which is nothing but a derived functor of locally analytic vectors. We will show that the Sen functor has a very good behaviour for relative locally analytic \(A\)-Banach representations of \(\Pi\). The strategy is to generalize the devisage of \([BC08]\) from finite rank \(A\) modules to \(A\)-modules.

**Definition 2.4.1** (The Sen functor). Let \(V\) be a relative locally analytic \(A\)-Banach representation of \(\Pi\), and \(H' \subset H\) an open subgroup. We define the Sen module of \(V\) to be

\[
S_{H'}(V) := (V^{H'})^{\Gamma_{H'} - la}.
\]

We also define the derived Sen functor to be

\[
RS_{H'}(V) := R\Gamma(H', V)^{RT_{W} - la}.
\]

**Remark 2.4.2.** In Definition 2.4.1(2) we see \(V\) as a solid \(\mathbb{Q}_p\)-vector space, \(R\Gamma(H', V)\) is the solid cohomology of \(V\), and \((-)^{RT_{W} - la}\) is the functor of derived locally analytic vectors. Note that, by Theorem 2.1.2 (2) and Lemma 2.4.4 down below, we actually have \(R\Gamma(H', V) = V^{H'}\), so \(RS_{H'}(V)\) is just the derived locally analytic vectors of \(V^{H'}\) as in \([Pan22, Definition 2.2.1]\). Moreover, the (derived) Sen module can be constructed for any \(A\)-semilinear \(\Pi\)-representation on solid vector spaces.

We can state the main theorem of this section regarding the good behaviour of the Sen functor for relative locally analytic \(A\)-Banach representations, cf. Proposition 3.3.1 of \([BC08]\).

**Theorem 2.4.3.** Let \((A, \Pi, \chi)\) be a Sen theory. Let \(V\) be a relative locally analytic \(A\)-Banach representation of \(\Pi\) and \(v = \{v_i\}_{i \in I}\) a relative locally analytic basis. Let \(U : \Pi \to GL_j(A)\) be the locally analytic 1-cocycle induced by \(v\) (see Definition 2.3.3). Let \(s > 2c_2 + 2c_3\) and let \(\Pi' \subset \Pi\) be an open normal subgroup such that \(U_{\Pi'}\equiv 1 \mod \varpi^s\). Let \(H' = \Pi' \cap H\) and \(n \geq n(H')\). The following hold

1. \(V\) contains a unique \(A\)-Banach \(A_{H',n}\)-submodule \(S_{H',n}(V)\), and there is a basis \(v' = \{v'_i\}_{i \in I}\) of \(S_{H',n}(V)\) such that:
   - (a) The \(A_{H',n}\)-module \(S_{H',n}(V)\) is fixed by \(H'\) and stable by \(\Pi\). Moreover, \(S_{H',n}(V)\) is a locally analytic representation of \(\Gamma_{H'} = \Pi/H'\).
   - (b) We have \(\hat{A}_{A_{H',n}}S_{H',n}(V) = V\) as an \(A\)-semilinear representation of \(\Pi\).
     The matrix \(M\) of base change from \(v\) to \(v'\) is trivial modulo \(\varpi^{c_1 + c_2}\).
   - (c) Let \(U'\) denote the 1-cocycle with respect to the basis \(v'\). For \(\gamma \in \Pi'/H'\), the matrix \(U'_{\gamma} \in GL_j(A_{H',n})\) is trivial modulo \(\varpi^{c_1 + c_2}\).

2. Suppose in addition that \((A, \Pi, \chi)\) is strongly decomposable. Then

\[
RS_{H'}(V) = S_{H'}(V) = \lim_{n \to \infty} S_{H',n}(V),
\]

in other words, the derived Sen functor is concentrated in degree 0.

From now on we suppose that \((A, \Pi, \chi)\) is a Sen theory, i.e., that it satisfies the axioms (CST0)-(CST2) of Definition 2.2.2. In order to prove Theorem 2.4.3 we need a series of technical lemmas. We first start with a devisage which is nothing but almost étale descent.
Lemma 2.4.4. Let $H' \subset H$ be an open subgroup and $V$ a relative locally analytic ON $A$-Banach representation of $\Pi$. Let $v = \{v_i\}_{i \in I}$ be a relative locally analytic basis generating a lattice $V^0$. Let $r > 0$ and $\Pi' \subset \Pi$ an open subgroup with $H' = \Pi' \cap H$, suppose that $\Pi'$ acts trivially on $v$ modulo $\sigma'$. The following hold.

(1) Let $0 < a < r$, there is a basis $\{v'_i\}_{i \in I}$ of $V^{H'}$ contained in $V^{0,H'}$ such that $v_i = v'_i \mod \sigma'^{-a}$ and $V^{0,H'}/\sigma'^{-a} = a \bigoplus_{i \in I} (A^{+,H'}/\sigma'^{-a})v'_i$ as $\Pi'/H'$-module.

(2) For all $s \geq 0$ we have $R\Gamma(H', V^0/\sigma^s) = ae (V^0/\sigma^s)^{H'} = ae V^{0,H'}/\sigma^s$. Taking derived inverse limits we have $R\Gamma(H', V^0) = ae V^{0,H'}$.

Proof. First, we claim that $R\Gamma(H', V^0/\sigma^s) = ae (V^0/\sigma^s)^{H'}$ for all $s > 0$. By taking short exact sequences

$$0 \to V^0/\sigma^s \xrightarrow{x \sigma^s} V^0/\sigma^{s(r+1)} \to V^0/\sigma^r \to 0$$

it is enough to take $s = r$. By hypothesis, we have an isomorphism of semilinear $H'$-representations provided by the basis $\{v_i\}_{i \in I}$

$$V^0/\sigma^r \cong \bigoplus_I A^{+,H'}/\sigma^r.$$ 

Then, it suffices to show that $R\Gamma(H', A^{+,\Pi}/\sigma^r) = ae A^{+,H'}/\sigma^r$. By hypothesis we can write $A^{+,\Pi}/\sigma^r = \lim_{H'' \subset H'} A^{+,H''}/\sigma^r$ where $H''$ runs over all the open normal subgroups of $H'$. Then

$$R\Gamma(H', A^{+,\Pi}/\sigma^r) = \lim_{H'' \subset H'} R\Gamma(H'/H'', A^{+,H''}/\sigma^r).$$

Let $A^{+,H''}[H'/H'']$ be the semilinear group ring of $H'/H''$ and $\varepsilon : A^{+,H''}[H'/H''] \to A^{+,H''}$ the augmentation map. We can compute group cohomology as the internal Hom space

$$(2.2) \quad R\Gamma(H'/H'', A^{+,H''}/\sigma^r) = R\text{Hom}_{A^{+,H''}[H'/H'']}(A^{+,H''}, A^{+,H''}/\sigma^r).$$

Let $\varepsilon > 0$, by (CTS0) there exists $\alpha \in A^{+,H''}$ such that $\overline{\text{Tr}}_{H''}^H(\alpha) = \varepsilon^\sigma$. Then $\sigma^r \varepsilon$ admits a section $h_{\varepsilon} : A^{+,H''} \to A^{+,H''}[H'/H'']$ given by $x \mapsto \sum_{g \in H'/H''} xg(\alpha) \cdot g$. This proves that $A^{+,H''}$ is almost $A^{+,H''}[H'/H'']$-projective, which implies that the RHS of (2.2) is almost concentrated in degree 0. Taking inverse limits one gets that $R\Gamma(H', A^{+})$ is almost equal to $A^{+,H'}$. Taking the $H'$-cohomology of the short exact sequence $0 \to A^{+} \xrightarrow{\overline{\sigma}^r} A^{+} \to A^{+}/\sigma^r \to 0$ one deduces that

$$R\Gamma(H', A^{+,H'}/\sigma^r) = ae A^{+,H'}/\sigma^r.$$ 

showing part (2).

To prove (1), let $0 < a < r$ and let $\overline{\sigma}^a v'_i \in V^{0,H'}$ be a lift of $(\sigma^a v_i \mod \sigma^r) \in V^{0,H'}/\sigma^r$. Then $v'_i \in V^{0,H'}$ and $v' = \{v'_i\}_{i \in I}$ is an ON basis of $V^{H'}$ such that $v'_i \equiv v_i \mod \sigma^r$. This proves the lemma. \square

Lemma 2.4.5 ([BC08 Léemme 3.2.3]). Let $\delta, a, b \in \mathbb{R}_{>0}$ such that $a \geq c_2 + c_3 + \delta$ and $b \geq \sup(a + c_2, 2c_2 + 2c_3 + \delta)$. Let $H' \subset H$ be an open subgroup, $n \geq n(H)$, and $\gamma = (\gamma_1, \ldots, \gamma_d)$ a sequence of linearly independent elements in $C_{H'}$, let $\langle \gamma \rangle$
be the subgroup generated by the $\gamma_i$'s. Let $(U_1, \ldots, U_d)$ be a 1-cocycle of $\langle \gamma \rangle$ in $GL_I(A^{+,H'})$ satisfying

i. $U_i = 1 + U_{i,1} + U_{i,2}$ where $U_{i,1} \in \prod_I \bigoplus_I A_{H',n}^+$ and $U_{i,2} \in \prod_I \bigoplus_I A^{+,H'}$.

ii. $U_{i,1} \equiv 0 \mod \varpi^a$ and $U_{i,2} \equiv 0 \mod \varpi^b$.

Then there exists $M \in GL_I(A^{+,H'})$ with $M \equiv 1 \mod \varpi^{b-c_2-c_3}$ such that

i. $M^{-1} U_j \gamma_j (M) = 1 + V_{i,1} + V_{i,2}$ with $V_{i,1} \in \prod_I \bigoplus_I A_{H',n}^+$ and $V_{i,2} \in \prod_I \bigoplus_I A^{+,H'}$.

ii. We have $V_{i,1} \equiv 0 \mod \varpi^a$ and $V_{i,2} \equiv 0 \mod \varpi^{b+\delta}$.

Proof. Let $R_{H',n} : A^{H'} \to A_{H',n}$ be the projection map and $X_{H',n}$ its kernel. Since we have the decomposition $A^{H'} = A_{H',n}^+ \oplus X_{H',n}$, the following space decomposes via $R_{H',n}$:

$$ \left( \prod_I \bigoplus_I A_i^+ \right)[1/p] = \left( \prod_I \bigoplus_I A_{H',i}^+ \right)[1/p] \oplus \left( \prod_I \bigoplus_I X_{H',i}^+ \right)[1/p]. $$

Then, using the bound of (CST1), we can write $U_{i,2} = R_{H',n}(U_{i,2}) + W_i$ with $W_i \in \prod_I \bigoplus I X_{H',i}^+$ and $W_i \equiv 0 \mod \varpi^{b-c_2}$. The cocycle condition of $(U_j^d)_{j=1}$ is equivalent to the equality

$$ 0 = U_j \gamma_j (U_i) - U_i \gamma_j (U_j) $$

$$ = U_{j,1} + U_{j,2} + \gamma_j (U_{i,1}) + \gamma_j (U_{i,2}) - U_{i,1} - U_{i,2} - \gamma_i (U_{j,1}) - \gamma_i (U_{j,2}) + Q_1 + Q_2 $$

for all $1 \leq i, j \leq d$, with $Q_1 \in \prod_I \bigoplus I A_{H',i}^+ [1/p]$, and $Q_2 \in \varpi^a \prod_I \bigoplus I A^{+,H'}$. Applying $1 - R_{H',n}$ we find

$$ 0 = W_j + \gamma_j (W_i) - W_i - \gamma_i (W_j) + (1 - R_{H',n})(Q_2) $$

where $(1 - R_{H',n})(Q_2) \equiv 0 \mod \varpi^{a+b-c_2}$. Therefore, $(W_j^d)_{j=1}$ defines a 1-cocycle of $\prod_I \bigoplus I (X_{H',i}^+ \varpi^{a+b-c_2})$. By (CST2), there exists a 1-cocycle $(W_j^d)_{j=1}$ in $\prod_I \bigoplus I X_{H',i}^+$ such that $(W_j^d)_{j=1} \equiv (W_j^d)_{j=1} \mod \varpi^{a+b-c_2-c_3}$. In particular, $(W_j^d)_{j=1} \equiv (W_j^d)_{j=1} \equiv 0 \mod \varpi^{b-c_2}$. Again by (CST2), there exists $M_0 \in \prod_I \bigoplus I X_{H',i}^+$ such that $W_j^d = M_0 - \gamma_j M_0$ for $j = 1, \ldots, d$ and $M_0 \equiv 0 \mod \varpi^{b-c_2-c_3}$. Taking $M = 1 + M_0$ we get the lemma. \hfill \square

Corollary 2.4.6. Let $\delta > 0$ and $b \geq 2c_2 + 2c_3 + \delta$. Let $H' \subset H$ be an open subgroup and $U_1, \ldots, U_d \in GL_I(A^{+,H'})$ a 1-cocycle verifying $U_j \equiv 1 \mod \varpi^b$ for $j = 1, \ldots, d$. Then there exists $M \in GL_I(A^{+,H'})$ with $M \equiv 1 \mod \varpi^{b-c_3-c_2}$ such that

$$ M^{-1} U_j \gamma_j (M) \in GL_I(A_{H',n}^+) $$

for $j = 1, \ldots, d$.

Proof. By the previous lemma there exists $M^{(1)} \in GL_I(A^{+,H'})$ with $M^{(1)} \equiv 1 \mod \varpi^{b-c_2-c_3}$ such that

$$ M^{(1)^{-1}} U_j \gamma_j (M^{(1)}) \in GL_I(A_{H',n}^+) \mod \varpi^{b+\delta}. $$
Let $k \in \mathbb{N}_{>1}$, by induction we can find matrices $M^{(k)} \in \text{GL}_f(A^{+,H'})$ with $M^{(k)} \equiv 1 \mod \varpi^{k+\delta - (k-1) - 2 \cdot c - 3 \cdot c}$ with
\[
U_i^{(k)} := M^{(k),-1}U_i^{(k-1)}\gamma_i(M^{(k)}) \in \text{GL}_f(A^{+,H',n}) \mod \varpi^{k+\delta}.
\]
Taking $k \to \infty$, and $M := M^{(1)}M^{(2)} \cdots$ one sees that the 1-cocycle $(U_i^{(d)}|_{i=1} := (M^{-1}U_{\gamma_i}(M))$ takes values in $\text{GL}_f(A^{+,H',n})$, and that $(U_i^{(d)}|_{i=1} \equiv 1 \mod \varpi^{2\cdot c - 2 \cdot c - 3}$. □

**Lemma 2.4.7** ([BC08 Lemme 3.2.5]). Let $H' \subset H$ be an open subgroup, $n \geq n(H')$, $\gamma = (\gamma_1, \ldots, \gamma_d)$ a sequence of linearly independent elements of $C_{H'}$, and $B \in \text{GL}_f(A^{H'})$. Suppose that we are given with $V_{1,j}, V_{2,j} \in \text{GL}_f(A^{+,H',n})$ with $V_{1,j} \equiv V_{2,j} \equiv 1 \mod \varpi^{e+\delta}$ for some $e > 0$, and that $\gamma_j(B) = V_{j,1}BV_{j,2}$. Then $B \in \text{GL}_f(A^{H',n})$.

**Proof.** Consider $C = B - R_{H',n}(B)$, then $\gamma_j(C) = V_{j,1}CV_{j,2}$. We have
\[
\gamma_j(C) - C = (V_{j,1} - 1)CV_{j,2} + V_{j,1}C(V_{j,2} - 1) - (V_{j,1} - 1)C(V_{j,2} - 1).
\]
Then $C \in \varpi^e \prod_{I} \bigoplus_{r} A^{+,H'}$ implies $\gamma_j(C) - C \in \varpi^{e+c_3+e} \prod_{I} \bigoplus_{r} A^{+,H'}$ for $i = 1, \ldots, d$. On the other hand, (CST2) provides an isomorphism $\iota$ between $X_{H',n}$ and the 1-cocycles $Z'(X_{H',n}) \subset X^{d}_{H',n}$ such that $\iota^{-1}(Z'(\varpi^{e+c_3+e}X^{+}_{H',n})) \subset \varpi^{2e+X^{+}_{H',n}}$. Therefore, $\gamma_j(C) - C \in \varpi^{e+c_3+e} \prod_{I} \bigoplus_{r} A^{+,H'}$ for $j = 1, \ldots, d$ implies $C \in \varpi^{e+c_3+e} \prod_{I} \bigoplus_{r} A^{+,H'}$. On deduces that $C = 0$ and that $B = R_{H',n}(B) \in \text{GL}_f(A^{H',n})$. □

**Proof of Theorem 2.4.3 (I).** Let $U : \Pi \to \text{GL}_f(A^+)$ be the 1-cocycle defined by the basis $\{v_i\}_I$. By hypothesis $U|_{\Pi} \equiv 1 \mod \varpi^e$ with $e > 2c_2 + 2c_3$. Let $\epsilon > 0$ such that $s' := s - \epsilon > 2c_2 + 2c_3$. By Lemma 2.4.4 we have $R\Gamma(H', V) = V^{H'}$, and there exists a matrix $M' \in \text{GL}_f(A^+)$ with $M' \equiv 1 \mod \varpi^\epsilon$ such that the cocycle $U'_g := M'^{-1}U_g(M')$ is trivial over $H'$.

Then, $U'$ is a 1-cocycle over $\text{GL}_f(A^{+,H'})$ satisfying $U'|_{\Pi} \equiv 1 \mod \varpi^\epsilon$. Let $n(H') \leq m \leq n$ and $\gamma = (\gamma_1^{(m)}, \ldots, \gamma_d^{(m)})$ be a pre-image of $(p^n e_{i=1}^{d})$ via $\chi : C_{H'} \to \mathbb{Z}_p^n$. Let $\delta > 0$ be such that $s' \geq 2c_2 + 2c_3 + \delta$, by Corollary 2.4.6 there exists $M' \in \text{GL}_f(A^{+,H'})$ with $M' \equiv 1 \mod \varpi^{s'-c_3-2}$ such that $M'^{-1}U'_g \gamma_j(M') \in \text{GL}_f(A^{+,H',n})$ for $j = 1, \ldots, d$.

Define $U''_g := M'^{-1}U'_g g(M')$, and let us show that $U''$ is a 1-cocycle of $\Pi$ in $\text{GL}_f(A^{+,H',n})$. Let $g \in \Pi/H'$, as $\gamma_j \in C_{H'}$ for all $j = 1, \ldots, d$ we see that
\[
U''_{g \gamma_j} = U''_{\gamma_j} U''_g \quad \text{ and } \quad U''_{g \gamma_j} = U''_{\gamma_j} g(U''_{\gamma_j}).
\]
Thus, $\gamma_j(U''_g) = U''_{\gamma_j}^{-1}U''_g g(U''_{\gamma_j})$. But $U''_{\gamma_j}^{-1}, g(U''_{\gamma_j}) \in \text{GL}_f(A^{+,H',n})$ are congruent to 1 modulo $\varpi^{s'-c_3-c_2}$, and $s' - c_3 - c_2 > c_3 + c_2$. By Lemma 2.4.7 we have $U''_g \in \text{GL}_f(A^{+,H',n})$ proving that $U''$ is a 1-cocycle in $\text{GL}_f(A^{+,H',n})$ whose restriction to $\Pi'$ is congruent to 1 modulo $\varpi^{s'+c_2}$. Setting $M := M'M''$ and $\psi : V \to V$ the associated isomorphism of $A$-modules, let $\{v'_i\}_I := \psi([v_i])$, $S_{H',n}(V)$ be the ON $A_{H',n}$-module spanned by $\{v'_i\}_I$ and $S_{H',n}(V^0)$ the lattice generated. Then $S_{H',n}(V) \subset V$ is stable
by $\Pi$ and the action factors through $\Gamma_{H'} = \Pi/H'$. Furthermore, by construction we have an isomorphism of semilinear $A$-representations of $\Pi$

$$A \otimes_{A_{H',n}} S_{H',n}(V) = V.$$ 

It is left to show that $S_{H',n}(V)$ is a locally analytic representation of $\Gamma_{H'}$. But this follows from Lemma 2.1.4, the fact that the elements $\gamma_1, \ldots, \gamma_d$ act trivially on the basis $\{v'_\ell\}_I \mod \mathfrak{m}^{c_3+c_2}$, and that the action on the Banach algebra $A_{H',n}$ is already locally analytic. \hfill $\square$

We still need some additional technical lemmas for proving part (2). Roughly speaking, we want to show that the Koszul complexes of the spaces $X_{H',n}$ for the action of $C_{H'}$ kill the locally analytic representations. In the rest of the section we suppose that $(A, \Pi, \chi)$ is a strongly decomposable Sen theory.

**Lemma 2.4.8.** Let $X_{H',\infty} = \lim_{\to \infty} X_{H',n}$, then

$$X_{H',\infty}^{RT,-la} = 0.$$ 

Moreover, for any locally analytic ON $A_{H',n}$-Banach representation $W$ of $\Gamma_{H'}$, we have that

$$(X_{H',\infty} \otimes_{A_{H',n}} W)^{RT,-la} = 0.$$ 

**Proof.** The second claim follows from the first and Lemma 2.1.5. To prove the first claim, we need to show that

$$RT(\Gamma_{H'}, (X_{H',\infty} \otimes_{\mathbb{Q}_p} C_{la}(\Gamma_{H'}, \mathbb{Q}_p))_{*,1,3}) = 0.$$ 

To simplify notation, let us write $A_m = A_{H',m}$. Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \{\pm\}^d$, consider the decomposition of Lemma 2.2.5

$$A_{H'} = \bigoplus_{\alpha \in \{\pm\}^d} X_{H',m}^\alpha$$

so that

$$A_m = X_{H',m}^{(+)} \text{ and } X_{H',m} = \bigoplus_{\alpha \neq (+)} X_{H',m}^\alpha.$$ 

Let us write $X_{H',\infty}^\alpha = \lim_{\to m} X_{H',m}^\alpha$ and consider $\Gamma_1$, the group generated by the basis $\gamma_i \in \chi^{-1}(\langle e_i \rangle)$, where $e_i \in \mathbb{Z}_p^d$ is the $i$-th basis element. Then, it suffices to show that for all $\alpha \neq (+)$ and $i$ such that $\alpha_i = -$, one has

$$(2.3) \quad RT(\langle \gamma_i \rangle, (X_{H',\infty}^\alpha \otimes_{\mathbb{Q}_p} C_{ha}(\langle \gamma_i \rangle, \mathbb{Q}_p))_{*,1,3}) = 0,$$

where $\langle \gamma_i \rangle \subset \Gamma_{H'}$ is the 1-parameter subgroup generated by $\gamma_i$. Therefore, we are reduced to prove the statement for a geometric Sen theory of dimension one, namely $\chi : \Pi \to \mathbb{Z}_p$. We will keep writing $X_m = X_{H',m}$ and $\Gamma = \Gamma_{H'} = \langle \gamma \rangle$. Then, since $(2.3)$ is represented by a Koszul-complex, it suffices to show that the following Ind-system is Ind-zero:

$$(RT(\langle \gamma \rangle, (X_{H',m} \otimes_{\mathbb{Q}_p} C_{ha}(\Gamma, \mathbb{Q}_p))_{*,1,3}))_{m \to \infty}.$$
where $C^b(\Gamma, \mathbb{Q}_p)$ is the Banach space of $p$-adic analytic functions of $\Gamma \equiv \mathbb{Z}_p$. Let us fix $h$. Since $\Gamma^{p'} \subset \Gamma$ has finite index, it suffices to show that the Ind-system
\begin{equation}
(2.4) \quad (\text{R} \Gamma(\langle \gamma^{p'} \rangle, (X_{H',m} \hat{\otimes}_{\mathbb{Q}_p} C^b(\Gamma, \mathbb{Q}_p))_{*,1,3}))_{s,m,h \to \infty},
\end{equation}
is Ind-zero. Take $s$ such that $\gamma^{p'}$ acts trivially on $C^b(\Gamma, \mathbb{Z}_p)/p$, and find $m \gg s$ such that $(\gamma^{p'} - 1)^{-1}$ acting on $X_m$ is bounded by $\sigma^\epsilon$ with $0 < \epsilon < 1$ (this is satisfied by (CST2*)). Then the formula
\begin{equation}
(\gamma^{p'} - 1)^{-1} = -\sum_{i=0}^{\infty} \gamma^{-p'}(\gamma^{-p'} - 1)^{-1} \otimes (\gamma^{p'} - 1)^i
\end{equation}
converges to an operator on $X_{H',m} \hat{\otimes}_{\mathbb{Q}_p} C^b(\Gamma, \mathbb{Q}_p)$, and defines an inverse for $\gamma^{p'} - 1$, proving that the Ind-system $(2.4)$ is zero as wanted. □

Proof of Theorem 2.4.3(2). We use the notation of Lemma 2.4.8. Write $G = \langle \gamma \rangle$. First, let $V$ be an ON relative locally analytic representation of $\Pi$ over $A$. By part (1) we already know that $V^{H'} = \text{R} \Gamma(H', V)$, and that $V^H = A^H \hat{\otimes}_{H'} S_{H',m}(V)$ for some $m \gg 0$ depending only on the analyticity condition of $V$. Thus, we can write
\begin{equation}
V^{H'} = \lim_{n} X_{H',m} \hat{\otimes}_{A^H} S_{H',m}(V) \oplus \lim_{n} A^H \hat{\otimes}_{A^H} S_{H',m}(V).
\end{equation}
Since $S_{H',m}(V)$ and $A^H$ are locally analytic representations of $G$, Theorem 2.1.2(3) implies that
\begin{equation}
(\lim_{n} A^H \hat{\otimes}_{A^H} S_{H',m}(V))^{\text{R} \Gamma(H', \text{la})} = \lim_{n} A^H \hat{\otimes}_{A^H} S_{H',m}(V).
\end{equation}
On the other hand, Lemma 2.4.8 implies that
\begin{equation}
(\lim_{n} X_{H',m} \hat{\otimes}_{A^H} S_{H',m}(V))^{\text{R} \Gamma(H', \text{la})} = 0,
\end{equation}
finishing the proof. □

2.5. Group cohomology via Sen theory. Throughout this section $(A, \Pi, \chi)$ will denote a strongly decomposable Sen theory. We finish with some formal consequences of Theorems 2.1.2 and 2.4.3 regarding the group cohomology of relative locally analytic representations of $\Pi$.

Corollary 2.5.1. Let $V$ be a relative locally analytic ON $A$-Banach representation of $\Pi$. Let $\Pi' \subset \Pi$ be an open subgroup, $H' = \Pi' \cap H$ and $\Gamma_{H'} = \Pi'/H'$. Then
\begin{equation}
\text{R} \Gamma(\Pi', V) = \text{R} \Gamma(\Gamma_{H'}^{\text{sm}}, \text{R} \Gamma(\text{Lie} \Gamma_{H'}, S_{H'}(V))).
\end{equation}
In particular,
\begin{equation}
H^1(\Pi', V) = H^1(\text{Lie} \Gamma_{H'}, S_{H'}(V))^{\Gamma_{H'}}.
\end{equation}

Proof. By Hochschild-Serre, almost purity (Lemma 2.4.4), and Theorems 2.1.2(4) and 2.4.3 we have that
\begin{align*}
\text{R} \Gamma(\Pi', V) &= \text{R} \Gamma(\Pi'/H', V^{H'}) \quad = \text{R} \Gamma(\Pi'/H', S_{H'}(V)) \quad = \text{R} \Gamma(\Gamma_{H'}^{\text{sm}}, \text{R} \Gamma(\text{Lie} \Gamma_{H'}, S_{H'}(V))).
\end{align*}
The last claim follows by taking cohomology groups, see Corollary 2.1.3.

**Remark 2.5.2.** In Example 2.2.8 we saw a natural framework where a triple \((A, \Pi, \chi)\) was not a geometric Sen theory, but where the only defect was an additional trivial action on \(A\). Suppose that \(\Pi = G \times \Pi_0\) with \(G\) a compact \(p\)-adic Lie group acting trivially on \(A\), and such that \(\chi\) factors as \(\chi : \Pi \to \Pi_0 \to \Gamma\) making the triple \((A, \Pi_0, \chi)\) a strongly decomposable Sen theory. Let us take \(H' = \ker(\Pi_0 \to \Gamma)\).

Then, Corollary 2.5.1 still applies after a suitable modification. Indeed, let \(V\) be a relative locally analytic ON \(A\)-Banach representation of \(\Pi\). We formally have that

\[
R\Gamma(\Pi, V) = R\Gamma(G, R\Gamma(\Pi_0, V)).
\]

By Corollary 2.5.1, we know that

\[
R\Gamma(G, R\Gamma(\Pi_0, V)) = R\Gamma(G, R\Gamma(\Gamma^{sm}_H, R\Gamma(\text{Lie} \, \Gamma_{H'}, S_{H'}(V)))).
\]

On the other hand, since the action of \(G\) and \(\Pi_0\)-commute, \(S_{H'}(V)\) has a natural action of \(G\) and (2.6) is equal to

\[
R\Gamma(\Gamma^{sm}_H, R\Gamma(\text{Lie} \, \Gamma_{H'}, R\Gamma(G, S_{H'}(V)))).
\]

Moreover, since \(G\) acts trivially on \(A\), Lemma 2.1.4 implies that the action of \(G\) on \(V\) is locally analytic, and so it is on the Sen module \(S_{H'}(V)\). We deduce by Theorem 2.1.2 (4) that

\[
R\Gamma(\Pi, V) = R\Gamma((G \times \Gamma_{H'})^{sm}, R\Gamma(\text{Lie}(G \times \Gamma_{H'}), S_{H'}(V))).
\]

We will consider a last hypothesis that holds in the main geometric application of this paper. In the arithmetic case over \(\mathbb{Q}_p\), this hypothesis is key in the proof of the Ax-Sen-Tate theorem.

(\text{AST}) Let \(\Pi' \subset \Pi'\) be an open subgroup and \(H' = \Pi' \cap H\). A Sen theory \((A, \Pi, \chi)\) satisfies the Ax-Sen-Tate property if the following conditions hold:

i. \(A_{H', n} = A_{\Pi', n}^\Pi\), where \(\Pi_{H', n}\) is the inverse image of \(p^n\mathbb{Z}_p\) via \(\chi : \Pi' \to \mathbb{Z}_p^d\) for all \(n \geq 0\).

ii. The traces \(R_{H', n} : A_{H', n}^{H'} \to A_{H', n}\) are constructed from normalized traces

\[
R_{H', n}^{H'} : A_{H', n} \to A_{H', n}
\]

\[
x \mapsto \frac{1}{p^{n-m}} \sum_{g \in \Pi_{H', m} / \Pi_{H', n}} g(x).
\]

A Sen theory satisfying the Ax-Sen-Tate axiom can be endowed with a Sen operator as follows.

**Definition 2.5.3.** Suppose that \((A, \Pi, \chi)\) satisfies (AST). Let \(V\) be a relative locally analytic ON \(A\)-Banach representation of \(\Pi\). Let \(H' \subset H\) be an open subgroup. The Sen operator of \(V\) is the \(A\)-linear map

\[
\theta_V : V \to V \otimes_{\mathbb{Q}_p} (\text{Lie} \, \Gamma_{H'})^\vee
\]

given by the \(A\)-extension of scalars of the connection

\[
S_{H'}(V) \to S_{H'}(V) \otimes_{\mathbb{Q}_p} (\text{Lie} \, \Gamma_{H'})^\vee.
\]
Equivalently, we define the Sen operator
\[ \Sen_V : \text{Lie} \Gamma_H \otimes V \to V \]
to be the extension of scalars of the derivation
\[ \text{Lie} \Gamma_H \otimes S_H(V) \to S_H'(V). \]

The Higgs cohomology \( R\Gamma(\theta_V, V) \) of \( V \) is defined as the complex
\[ 0 \to V \xrightarrow{\theta_V} V \otimes (\text{Lie} \Gamma_H)'^\vee \to \cdots \to V \otimes \bigwedge^d (\text{Lie} \Gamma_H)'^\vee \to 0, \]
we also denote \( H^i(\theta_V, V) := H^i(R\Gamma(\theta_V, V)). \)

In the rest of the section we suppose that \((A, \Pi, \chi)\) satisfies the Ax-Sen-Tate axiom, we also fix \( \Pi' \subset \Pi \) an open subgroup and set \( H' = \Pi' \cap H \). The next result describes some cohomological properties of \( A \)-semilinear \( \Pi \)-representations in terms of their Sen operators.

**Proposition 2.5.4.** Let \( V \) and \( W \) be a relative locally analytic ON \( A \)-Banach representations of \( \Pi \), and \( W \to V \) an \( A \)-linear \( \Pi \)-equivariant map. Consider the exact sequence in solid \( \mathbb{Q}_p \)-vector spaces
\[ 0 \to K \to W \to V \to Q \to 0 \]
Then the derived Sen module \( R\Sen H'(\cdot) \) of \( K \) and \( Q \) are in degree 0, and we have an exact sequence
\[ 0 \to S_H'(K) \to S_H'(W) \to S_H'(V) \to S_H'(Q) \to 0. \]
In particular, we have isomorphisms
\[ H^i(\Pi', V) = H^i(\theta_V, V)^{\Gamma_H'}. \]

**Remark 2.5.5.** One of the reasons we need to see \( W \) and \( V \) as solid \( \mathbb{Q}_p \)-vector spaces in the previous proposition is for the Sen module of \( Q \) to be well defined. Otherwise, \( Q \) could be a non-Hausdorff topological space and there would not be a good theory of locally analytic vectors.

We thank Lue Pan for the explanation of the following argument.

**Proof.** Let us assume the first statement. By Corollary 2.5.1, we know that
\[ H^i(\Pi', V) = H^i(\n_H, S_H(V))^{\Gamma_H'}, \]
by assumption we have
\[ H^i(\text{Lie} \Gamma_H', V^H')^{R\Gamma_H'-\text{la}} = H^i(\text{Lie} \Gamma_H', S_H(V)), \]
and the second statement follows from Theorem 2.1.2(4) and the degeneracy of the two hypercohomology spectral sequences computing
\[ R\Gamma(\Pi, V) = R\Gamma(\text{Lie} \Gamma_H', (V^H')^{R\Gamma_H'-\text{la}}). \]

Let us now prove the first assertion. Suppose that it holds for a Sen theory of dimension 1, by decomposing the trace \( R^d_{H', n} = R^d_{H', n} \circ \cdots \circ R^1_{H', n} \) and considering the algebras \( A^{[d]}_{H', n} \subset A^{[d-1]}_{H', n} \subset A^{[1]}_{H', n} = A^H \) with \( A^{[i]}_{H', n} = R^i_{H', n} \circ \cdots \circ R^1_{H', n}(A_H), \)
we can deduce the proposition by an induction from the one dimensional case. Indeed, the successive Sen modules \( S_{H'}^{[i]}(V) = \lim_n S_{H',n}^{[i]}(V) \) (resp. for \( W \)) are filtered colimits of locally analytic ON \( A_{H',n}^{[i]} \)-Banach representations, for which the one dimensional case can be applied.

Thus, without loss of generality, we can assume that \( \Gamma_{H'} \cong \mathbb{Z}_p \). Then, by taking the canonical truncation of the complex \( [W \to V] \), there is an hypercohomology spectral sequence with \( E_2 \)-page concentrated in \( [0,1]^2 \)

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & R^1 S_{H'}(K) & R^1 S_{H'}(Q) & 0 & 0 & 0 \\
0 & S_{H'}(K) & S_{H'}(Q) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

converging to the cohomology of the complex \( S_{H'}(W) \to S_{H'}(V) \). Then, the spectral sequence degenerates at the \( E_2 \)-page and we deduce that \( RS_{H'}(Q) = S_{H'}(Q) \).

Consider the short exact sequences

\[
0 \to B \to V \to Q \to 0
\]

and

\[
0 \to K \to W \to B \to 0.
\]

Since \( RS_{H'}(V) = S_{H'}(V) \) (resp. for \( W \)) by Theorem 2.4.3 (2), we deduce that \( RS_{H'}(B) = S_{H'}(B) \) and and therefore that \( RS_{H'}(K) = S_{H'}(K) \) as wanted. \( \square \)

**Corollary 2.5.6.** Keep the notation of Proposition 2.5.4. Suppose that \( \theta_V = 0 \), then there is an equivalence

\[
R\Gamma(\Pi, V) = \bigoplus_{i=0}^d V^{[i]} \otimes \bigwedge^i \text{Lie } \Gamma_{H'}[-i].
\]

Moreover, for \( n \) big enough we have that \( S_{H',n}(V) = (V^{H'})_{p^{d_1} \Gamma_{H'}} \) and

\[
V = A_{\text{Ad},n} \circ S_{H',n}(V).
\]

**Proof.** By hypothesis \( \theta_V = 0 \), this means that the action of \( \text{Lie } \Gamma_{H'} \) on \( S_{H'}(V) = \left(V^{H'}\right)^{\text{Ad}_{H'}} \) is zero, so that \( S_{H'}(V) \) is a smooth representation of \( \Gamma_{H'} \). Therefore, there is \( n >> 0 \) such that \( S_{H',n}(V) = (V^{H'})_{p^{d_1} \Gamma_{H'}} \). The first claim of the Corollary follows from Proposition 2.5.4 since the Higgs complex of \( V \) is split, the second holds since \( V = A_{\text{Ad},n} \circ S_{H',n}(V) \) by Theorem 2.4.3. \( \square \)

**Remark 2.5.7.** The equivalence of Corollary 2.5.6 depends on the Lie algebra \( \text{Lie } \Gamma_{H'} \) and so in the group \( \Gamma \). In applications we will allow \( \Gamma \) to vary, so to guarantee that the splitting is independent of \( \Gamma \) we shall need some additional structure (eg. Hodge-Tate weights arising from an arithmetic Galois action).
Remark 2.5.8. Proposition 2.5.4 can be slightly extended by allowing some additional trivial action. Indeed, let us keep the notation of Remark 2.5.2. We find that

\[ R\Gamma(\Pi, V) = R\Gamma((G \times \Gamma_H)^{sm}, R\Gamma(\text{Lie}(G \times \Gamma_H), S_{H'}(V))). \]

Then, Proposition 2.5.4 implies that

\[ H^i(\Pi, V) = H^i(\text{Lie } G \times \theta_V, V)^\Pi. \]

3. Geometric Sen theory

Let \((C, C^+)\) be an algebraically closed perfectoid field over \(\mathbb{Q}_p\), and let \(X\) be an fs log smooth adic space over \((C, C^+)\) with log structure given by reduced normal crossing divisors. For \(? \in \{\text{an, ét, két, proét, prokét}\}\) we let \(X_?\) denote the corresponding site over \(X\) (see [DLLZ19] Example 2.3.17 and §4 and §5). We let \(\mathcal{O}^{(+)}_X\) denote the (bounded) complete structural sheaf over \(X_{\text{prokét}}\), and for \(? \in \{\text{an, ét, két}\}\) we let \(\mathcal{O}^{\text{log}}_X\) be the (bounded) structural sheaf on \(X_?\). We also let \(\nu : X_{\text{prokét}} \to X_{\text{két}}\) and \(\eta : X_{\text{prokét}} \to X_{\text{an}}\) be the projection of sites, if \(X\) is clear from the context we write \(\nu\) and \(\eta\) instead. Suppose that \((C, C^+)\) is the algebraic closure of a discretely valued field with perfect residue field \((K, K^+)\), and that \(X\) has a form \(X'\) over \((K, K^+)\), we shall write \(\mathcal{O}_{\text{prokét}}^{(\dR, \text{log})} \text{ and } \mathcal{O}_{\text{log}} := \mathcal{O}_{\text{prokét}}^{(\dR, \text{log})} \) for the log Rham and Hodge-Tate period sheaves over \(X'_{\text{prokét}}\), see [DLLZ23].

The main goal of this section is to use the abstract Sen theory formalism of §2 to study the Hodge-Tate cohomology of \(X\), obtaining Theorems 1.0.3 and 1.0.4 of the introduction. In §3.1 we review the log-Kummer exact sequence and prove a log-analogue of [Sch13b] Proposition 3.23] computing \(R^1\nu_*\mathcal{O}_X(1) \cong \Omega^1_\text{v}(\log)\), where \(\nu : X_{\text{prokét}} \to X_{\text{két}}\) is the projection of sites. In §3.2 we construct the geometric Sen operator of \(X\) locally on toric coordinates, for which we are essentially reduced to the Sen theory of a product of tori and discs as in Example 2.2.7. Then, in §3.3 we show that these local constructions of the Sen operator glue, following an argument suggested by Lue Pan using that \(R^1\nu_*\mathcal{O}_X = \Omega^1_\nu(\log)\). Finally in §3.4 we apply the previous results to study the locally analytic vectors of the completed structural sheaf of pro-Kummer-étale torsors of \(p\)-adic Lie groups. We end the section by explaining the relation of geometric Sen theory with the works of [LZ17, DLLZ23, Wan23].

All the fiber products considered in the next sections are as fs log adic spaces in the sense of [DLLZ19] Proposition 2.3.27, in particular they might differ from the fiber products of usual adic spaces.

3.1. Log-Kummer exact sequence. Let \(X\) be an fs log smooth adic space over \((C, C^+)\) with log structure given by normal crossing divisors. Equivalently, locally in the étale topology, \(X\) admits an étale map towards \(S_{C}^{(e,d-e)} := \mathcal{T}_{C}^e \times \mathbb{D}_{C}^{d-e}\) with

\[ \mathcal{T}_{C}^e := \text{Spa}(C(T_1^{\pm 1}, \ldots, T_e^{\pm 1}), C^+(T_1^{\pm 1}, \ldots, T_d^{\pm 1})) \]

and

\[ \mathbb{D}_{C}^{d-e} := \text{Spa}(C(S_{e+1}, \ldots, S_d), C^+(S_{e+1}, \ldots, S_d)). \]
such that the log structure of $X$ is the pullback of the log structure on $S_{C}^{(e,d-e)}$ defined by $S_{e+1} \cdots S_{d} = 0$. An étale map $X \to S_{C}^{(e,d-e)}$ factoring as composite of finite étale maps and rational localizations is called a toric chart of $X$.

The log-Kummer exact sequence is constructed as follows.

**Lemma 3.1.1.** Let $M_{X}$ be the Kummer-étale sheaf of monoids defining the log structure of $X$, and let $M_{X}^{gr}$ be its group of fractions. We have a short exact sequence of pro-Kummer-étale sheaves over $X$

$$0 \to \hat{\mathbb{Z}}_{p}(1) \to \lim_{\leftarrow p} M_{X}^{gr} \to M_{X}^{gr} \to 0.$$ 

**Proof.** Consider the usual Kummer short exact sequence

$$0 \to \hat{\mathbb{Z}}_{p}(1) \to \lim_{\leftarrow p} \mathcal{O}_{X}^{\times} \to \mathcal{O}_{X}^{\times} \to 0.$$ 

To prove the lemma it suffices to see that the quotient

$$\overline{M}_{X}^{p} = M_{X}^{p} / \mathcal{O}_{X}^{\times}$$

is a $\mathbb{Z}[(1/p)]$-module. This property can be checked at the level of geometric points $\overline{x}$ of $X_{\kappa \text{et}}$. If $\overline{x}$ is disjoint to the divisor $D$ defining the log structure then $\overline{M}_{X,\overline{x}}^{p} = 0$ and we are done. Otherwise, we can assume that $X$ has a toric chart $X \to S_{C}^{(e,d-e)}$, which implies that

$$\overline{M}_{X,\overline{x}} = \mathbb{Q}_{\geq 0}^{k}$$

with $0 \leq k \leq d - e$. Then, the group of fractions of $\overline{M}_{X,\overline{x}}$ is a $\mathbb{Q}$-vector space proving the claim. \qed

**Proposition 3.1.2** ([Sch13b, Proposition 3.23] and [Heu20, Proposition 2.25]). Let $X$ be as before, and let $\nu : X_{\pro \kappa \text{et}} \to X_{\kappa \text{et}}$ be the projection of sites. Then there is a natural isomorphism

$$R^{1} \nu_{*} \hat{\mathcal{O}}_{X}(1) \cong \Omega_{X}^{1}(\log)$$

making the following diagram commute

$$\begin{array}{ccc}
\mathcal{M}_{X}^{gr} & \longrightarrow & R^{1} \nu_{*} \hat{\mathcal{O}}_{p}(1) \\
\downarrow{\delta} & & \downarrow \\
\Omega_{X}^{1}(\log) & \longrightarrow & R^{1} \nu_{*} \hat{\mathcal{O}}_{X}(1)
\end{array}$$

obtained by the log-Kummer-étale sequence and the log-differential map $\delta$.

**Remark 3.1.3.** We prefer to keep the Tate twist even if $C$ is algebraically closed; in the case where $X$ has a form $X'$ over $(K, K^+)$, the isomorphism of Proposition 3.1.2 is Galois equivariant. Later in §3.4 we will show that $R \nu_{*} \hat{\mathcal{O}}_{X} = \bigoplus_{i=1}^{d} \Omega_{X}(\log)(-i)[-i]$, i.e. that the pro-Kummer-étale cohomology of $\hat{\mathcal{O}}_{X}$ naturally splits.
Proof. If $X$ as trivial log structure this is [Sch13b Proposition 3.23], let us show that the same argument holds in the log-smooth situation. First, the image of the map $\delta$ generates $\Omega^1_X(\log)$ as $\mathcal{O}_X$-module, so if such an equivalence exists it must be unique. We can then argue locally in the étale topology of $X$ and assume it has toric coordinates $X \to \mathcal{S}_C^{(e,d-e)}$. Then, using the approximation argument of [Sch13b Lemma 3.24] (which only requires that the map $X \to \mathcal{S}_C^{(e,d-e)}$ factors as composites of finite étale maps and rational localizations), we can assume that $X \to \mathcal{S}_C^{(e,d-e)}$ arises from base change of an étale map $X' \to \mathcal{S}_Y^{(e,d-e)} = Y \times_{\Spa(C)} \mathcal{S}_p^{(e,d-e)}$ via a map $\Spa(C, C^+) \to Y$, where $Y$ is smooth of finite type over $\mathbb{Q}_p$. We endow $X'$ with the log structure arising from the normal crossing divisors of $\mathcal{S}_p^{(e,d-e)}$. Finally, the same argument of loc. cit. holds by using instead the log-Faltings extension of $\mathcal{S}_p^{(e,d-e)}$. we left the details to the reader. $\square$

Corollary 3.1.4. Let $f : Y \to X$ be a map of fs log smooth adic spaces over $(C, C^+)$ with normal crossing divisors. The following diagram is commutative

$$
\begin{array}{ccc}
\ 
\end{array}
$$

Proof. This follows from Proposition [3.1.2] the commutative diagram

$$
\begin{array}{ccc}
f^*\Omega^1_X(\log) & \xrightarrow{\sim} & f^*R^1\mu_X, \mathcal{O}_X(1) \\
\downarrow & & \downarrow \\
\Omega^1_Y(\log) & \xrightarrow{\sim} & f^*R^1\mu_Y, \mathcal{O}_Y(1).
\end{array}
$$

and the fact that the image of $\delta$ generates the sheaf of differentials as vector bundles. $\square$

Remark 3.1.5. The equivalence $\Omega^1_X(\log) \cong R^1\mu_X, \mathcal{O}_X$ can be made explicit for the space $X = \mathcal{S}_C^{(e,d-e)}$. Indeed, let $\mathcal{S}_C^{(e,d-e)}$ the perfectoid space obtained by taking $n$-th roots of $T_i$ and $S_i$ for all $n \in \mathbb{N}$. Then $\mathcal{S}_C^{(e,d-e)}$ is a perfectoid log adic space in the sense of [DLLZ19] Definition 5.3.1. Then, we can compute

$$
R\Gamma_{\text{prok}^{\mathcal{U}}}(\mathcal{S}_C^{(e,d-e)}, \mathcal{O}_X) = R\Gamma(\mathbb{Z}(1)^d, \mathcal{O}_X(\mathcal{S}_C^{(e,d-e)})).
$$

Writing $\mathbb{Z}(1) = \mathbb{Z}_p \times \mathbb{Z}^{p}(1)$, since $\mathbb{Z}^{p}(1)$ has no pro-$p$-Sylow subgroup, we get that (see Proposition [5.2.2] down below)

$$
R\Gamma_{\text{prok}^{\mathcal{U}}}(\mathcal{S}_C^{(e,d-e)}, \mathcal{O}_X) = R\Gamma(\mathbb{Z}_p(1)^d, \mathcal{O}_X(\mathcal{S}_C^{(e,d-e)})).
$$

Moreover, Example [2.2.7] and Proposition [2.5.4] imply that

$$
R\Gamma_{\text{prok}^{\mathcal{U}}}(\mathcal{S}_C^{(e,d-e)}, \mathcal{O}_X) = R\Gamma(\text{Lie}(\mathbb{Z}_p(1)^d), \mathbb{Q}_p)\otimes_{\mathbb{Q}_p} \mathcal{O}_X(\mathcal{S}_C^{(e,d-e)}).
$$
Now, the element $T_i \in M$ (resp. $S_i \in M$) is mapped to the cocycle defined by the $p$-th power roots $[T_i^p] = (T_i^p)_n$ (resp. to $[S_i^p] = (S_i^p)_n$). More concretely, it sends $T_i$ to the 1-cocycle

$$g \mapsto \frac{g \cdot [T_i^p]}{[T_i^p]}$$

(resp. for $S_i$). Therefore, taking a basis $e_j$ of $\mathbb{Z}_p(1)^d$ given by a compatible system of $p$-th power roots of 1, the isomorphism $\Omega^1_X(\log) \cong R^1\nu_*\mathcal{O}_X(1)$ maps $d\log T_i$ (resp. $d\log S_i$) to the 1-cocycle $\sigma_i$ in $R\Gamma(\mathbb{Z}_p(1)^d, \mathcal{O}_X(\xi_{d-e}))$ given by

$$\sigma_i(e_j) = \delta_{i,j}.$$

3.2. The geometric Sen operator: local computation. The goal of this section is to prove a local version of Theorems 1.0.3 and 1.0.4. Recall the definition of relative locally analytic sheaf of the introduction.

Definition 3.2.1. A proétale $\mathcal{O}_X$-sheaf $\mathcal{F}$ over $X$ is an ON relative locally analytic if there is a Kummer-étale cover $\{U_i\}_{i \in I}$ of $X$ such that, for all $i$, the restriction $\mathcal{F}|_{U_i}$ admits a $p$-adically complete $\mathcal{O}_X$-lattice $\mathcal{F}_i^0$, and there is $\varepsilon > 0$ (depending on $i$) such that $\mathcal{F}_i^0/p^\varepsilon = \bigoplus J \mathcal{O}_X^+/p^\varepsilon$ for some index set $J$.

3.2.1. The set-up. We keep the notation of §3.1 in particular $(C, C^+)$ an algebraically closed non-archimedean extension of $\mathbb{Q}_p$, $X/\text{Spa}(C, C^+)$ is an fs log smooth adic space with normal crossing divisors, $G$ is a compact $p$-adic Lie group, and $\overline{\mathbb{X}} \to X$ is a pro-Kummer-étale $G$-torsor. For a toric chart of $X$ we mean an étale map $\psi : X \to S^{(e_{d-e})} = T^\vee C \times D^{d-e}$ that factors as a composite of finite étale maps and rational localizations, cf. Example 2.2.7.

In the following, we will suppose that $X$ is affinoid and that it has a toric chart $\psi$. We let $S^{(e_{d-e})}_C = \text{Spa}(C(T^\vee C), C^+(T^\vee C), S^{1/e}_{d-e})$ be the perfectoid product of tori and polydiscs over $S^{(e_{d-e})}_C$, and let $\Gamma \equiv \mathbb{Z}_p(1)^d$ denote the Galois group of $S^{(e_{d-e})}_C$ over $S^{(e_{d-e})}_C$. We let $\Gamma_n = p^n\Gamma$ and $S^{(e_{d-e})}_C,_{n} \equiv S^{(e_{d-e})}_C/\Gamma_n$.

Given an open subgroup $G' \subset G$ we let $X_{G'} = \overline{X}/G'$. The space $X_{G'}$ is finite Kummer-étale over $X$, if $G'$ is normal then $X_{G'}$ is Galois over $X$ with group $G/G'$. We have a presentation as objects in $X_{\text{proét}}$

$$\overline{X} = \lim_{\longrightarrow} X_{G'_{\infty}}.$$

Given $n \in \mathbb{N} \cup \{\infty\}$ and $G' \subset G$ we let $X_n = X_{\psi, S^{(e_{d-e})}_C,_{n}}$ and $X_{G',n} = X_{G'_{\infty}} \times X_n$. We also denote $\overline{X}_n = \overline{X}_{\times X_n}$ and $\overline{X}_{\infty} = \overline{X}_{\times X_{\infty}}$. The following diagram illustrates...
the relation between these spaces as profinite-Kummer-étale objects in $X_{\text{prokét}}$

\begin{align*}
\tilde{X}_{\infty} & \xrightarrow{\Gamma_n} X_{G',\infty} \\
\tilde{X}_n & \xrightarrow{G'/\Gamma_n} X_{G',n} \\
\tilde{X} & \xrightarrow{G'/\Gamma_n} X_{G'} \\
X & \xrightarrow{G'/\Gamma_n} X_{G'}
\end{align*}

(3.1)

We fix the following notation for the global functions of the previous spaces for $G' \subset G$ and $n \in \mathbb{N} \cup \{\infty\}$:

\[ \tilde{B}_n = \tilde{\mathcal{O}}_X(\tilde{X}_n), \quad B_{G',n} := \tilde{\mathcal{O}}_X(X_{G',n}), \quad B_{G'} := \tilde{\mathcal{O}}_X(X_{G'}), \]

similarly for the spaces of bounded functions. Note that, since $\tilde{\mathcal{O}}_X$ is a pro-Kummer-étale sheaf, we have

\[ (\tilde{B}_{\infty})^\Gamma_n = \tilde{B}_n \text{ and } (\tilde{B}_{\infty})^{G'} = B_{G',\infty}. \]

If $G' \subset G$ is open and $n \in \mathbb{N}$ we also have

\[ B_{G',n} = \tilde{\mathcal{O}}_X(X_{G',n}). \]

3.2.2. Construction of abstract Sen theories. The following proposition implies that we can compute pro-Kummer-étale cohomology using the perfectoid toric coordinates.

**Proposition 3.2.2.** Let $Y \to \mathbb{S}^{(e,d-\epsilon)}_{C,\infty}$ be a Kummer-étale map that factors as composite of rational localizations and finite-Kummer-étale maps.

1. $\tilde{\mathcal{O}}_Y(Y)$ is a perfectoid $C$-algebra.
2. Let $\mathcal{F}$ be an ON $\tilde{\mathcal{O}}_Y$-sheaf over $Y_{\text{prokét}}$ admitting a $\tilde{\mathcal{O}}_Y^+$-lattice $\mathcal{F}^0$ such that $\mathcal{F}^+ / p^\epsilon = \bigoplus_j \tilde{\mathcal{O}}_Y^+ / p^j$ for some $\epsilon > 0$. Then $R\Gamma_{\text{prokét}}(Y, \mathcal{F}) = \mathcal{F}(Y)$ is an ON Banach $\tilde{\mathcal{O}}_Y(Y)$-module.

**Proof.** (1). Let $\mathbb{S}^{(e,d-\epsilon)}_{C,\infty} \to (\mathbb{S}^{(e,d-\epsilon)}_C)_{\mathbb{Z}_{\ell}}$ be the $\mathbb{Z}(1)^d$-torsor obtained by adding all the $n$-th powers of $T_i$ and $S_i$ respectively. The space $\mathbb{S}^{(e,d-\epsilon)}_{C,\infty}$ is log affinoid perfectoid in the sense of [DLLZ19, Definition 5.3.1], and a Galois cover over $\mathbb{S}^{(e,d-\epsilon)}_{C,\infty}$ with Galois group $\mathbb{Z}(1)^d$. Then, by [DLLZ19] Lemma 5.3.8 the pullback $\tilde{Y} = Y \times_{\mathbb{S}^{(e,d-\epsilon)}_{C,\infty}} \mathbb{S}^{(e,d-\epsilon)}_{C,\infty}$ is étale over $\mathbb{S}^{(e,d-\epsilon)}_{C,\infty}$ written as a composition of finite étale maps and rational localizations. One deduces that $\tilde{\mathcal{O}}_Y(\tilde{Y})$ is an affinoid perfectoid $C$-algebra. But then, $\tilde{\mathcal{O}}_Y(Y)$ is the $\mathbb{Z}_{\ell}(1)$-invariants of $\tilde{\mathcal{O}}_Y(\tilde{Y})$, this ring is clearly...
uniform and contains $C$, to show that it is perfectoid it suffices to prove that the Frobenius map

$$\varphi : \mathcal{O}_Y^+(Y) / p^f/p \to \mathcal{O}_Y^+(Y) / p^f$$

is an almost isomorphism. But taking $\widehat{\mathbb{Z}}_{\ell \neq p}(1)$-invariants is exact in $p$-power torsion $\mathbb{Z}_p$-modules, e.g. since we have a Haar measure as $N$ is invertible in $\mathbb{Z}_p$ for $N \in \mathbb{N}$ prime to $p$. Taking invariants of $\mathcal{O}_Y^+(\widehat{Y}) / p^f/p \cong \% ae \mathcal{O}_Y^+(\widehat{Y}) / p^f$ one deduces that (3.2) is an almost isomorphism.

(2). We can write $\mathcal{F}^0 = \varprojlim_n \mathcal{F}^0 / p^n$. Then, it suffices to show that

$$R \Gamma_{\text{prokét}}(Y, \mathcal{O}_Y^+ / p^f) = \% ae \mathcal{O}_Y^+(Y) / p^f.$$

By [DLLZ19, Lemma 5.3.8] and acyclicity of affinoid perfectoids in the proétale site, we have that $R \Gamma_{\text{prokét}}(\widehat{Y}, \mathcal{O}_Y^+ / p^f) = \% ae \mathcal{O}_Y^+(\widehat{Y}) / p^f$. But again, taking $\widehat{\mathbb{Z}}_{\ell \neq p}(1)$-cohomology is exact in $p$-power torsion modules, one deduces that

$$R \Gamma_{\text{proét}}(Y, \mathcal{O}_Y^+ / p^f) = \% ae \ R \Gamma(\widehat{\mathbb{Z}}_{\ell \neq p}(1), \mathcal{O}_Y^+(\widehat{Y}) / p^f) = \% ae \mathcal{O}_Y^+(Y) / p^f.$$

We saw in Example [22.2.7] that the tower $\{S_{C_n}^{(e,d,e)}\}_{n \in \mathbb{N}}$ of products of tori and polydiscs gives rise a strongly decomposable Sen theory. Our next task is to show that its pullback to $X_{\text{prokét}}$ also satisfies the Colmez-Sen-Tate axioms.

**Proposition 3.2.3.** The triple $(\widehat{B}_\infty, G \times \Gamma, \text{pr}_2)$ is a strongly decomposable Sen theory.

**Proof.** We have to show that the triple $(\widehat{B}_\infty, G \times \Gamma, \text{pr}_2)$ satisfies the Colmez-Sen-Tate axioms (CST0), (CST1*) and (CST2*) of Definition [2.2.2]. The almost purity axiom (CST0) follows from Proposition [3.2.7] and the almost purity for étale maps of perfectoid spaces [Sch12, Theorem 7.9]. We are left to construct the partial Sen traces $R_{H_{n'}}^\text{Sen}$ for (CST1*), and prove the bounds for the vanishing of cohomology for (CST2*).

Throughout the rest of the proof we slightly change the notation of the spaces and algebras. For $n = N' p^k$ with $N' \in \mathbb{N}$ prime to $p$ and $k \in \mathbb{N} \cup \{\infty\}$, we let $S_{C_n}^{(e,d,e)}$ denote the finite-Kummer-étale extension of $S_{C}^{(e,d,e)}$ obtained by adding $n$-th power roots of $T_i$ and $S_i$. For $G' \subset G$ and $n$ as before we let $\widetilde{X}_{G',n} = \widetilde{X} \times_{\mathcal{O}_k^{(e,d,e)}} S_{C_n}^{(e,d,e)}$, and let $\widetilde{B}_{G',n} = \mathcal{O}_{\widetilde{X}_{G',n}}$. Then, the ring $\widehat{B}_\infty$ in the statement of the proposition is now denoted by $\widehat{B}^{proe}$.

Given $G' \subset G$ an open subgroup we let $n(G')$ be the smallest integer $n$ such that $\widetilde{X}_{G',n} \to S_{C_n}^{(e,d,e)}$ is étale for all $m \geq n$, by [DLLZ19, Lemma 4.2.5] we know that $n(G') < \infty$. Write $n(G') = N' p^{s(G')}$ with $N'$ prime to $p$, and let $1 \leq i \leq d$. Recall
the rings of Example 2.2.7 (adapted to the new notation)

\[ A_n = \mathcal{O}(\mathbb{S}^{(e,d-e)}_{C,n}) = C(T^{\frac{z}{N^{p^n}}}, S^{\frac{\gamma^i}{N^{p^n}}}) \]

\[ A^i_{N^p} = \mathcal{O}(\mathbb{S}^{(e,d-e)}_{C,N^p}) = C(T^{\frac{z}{N^{p^n}}}, S^{\frac{\gamma^i}{N^{p^n}}}) \]

\[ A^i_{N^p} = \begin{cases} C(T^{\frac{z}{N^{p^n}}}, \ldots, T_i^{\frac{z}{N^{p^n}}}, \ldots, T_{e}^{\frac{z}{N^{p^n}}}, S^{\frac{\gamma^i}{N^{p^n}}}) & \text{if } 1 \leq i \leq e \\
C(T^{\frac{z}{N^{p^n}}}, S^{\frac{1}{N^{p^n}}}, \ldots, S_i^{\frac{1}{N^{p^n}}}, \ldots, S_d^{\frac{1}{N^{p^n}}}) & \text{otherwise.} \end{cases} \]

Let \((\gamma_i)_{i=1}^d\) be the standard basis of \(\widehat{\mathbb{Z}}\) given by fixing a compatible sequence of \(n\)-th power roots of unity, and let \(\gamma_i = (\gamma_i,0)\) be its decomposition in products of basis over \(\mathbb{Z}_\ell\) for all \(\ell\). For \(k \geq s\) let us define

\[ B^i_{G^{n},N^p} := (B_{G^{n},N^p} \otimes A^i_{N^p})^{\wedge-u} \]

where the \(u\)-completion is nothing but the \(p\)-adic completion with respect to the integral closure of \(B^i_{G^{n},N^p} \otimes A^i_{N^p}\).

We claim that the traces \(R^i_{N^p} : A^i_{N^p} \to A^i_{N^p}\) extend to \(R^i_{G^{n},N^p} : B^i_{G^{n},N^p} \to B^i_{G^{n},N^p}\) with \(|R_{G^{n},\cdot}| \leq |p^{-c_2}|\), for \(c_2\) arbitrarily small as \(k \to \infty\). Indeed, we have that

\[ B^i_{G^{n},N^p} = (B_{G^{n},N^p} \otimes A^i_{N^p})^{\wedge-u}, \]

and \(R^i_{N^p}\) extends to \(B^i_{G^{n},N^p}\) linear maps

\[ B_{G^{n},N^p} \otimes A_{N^p}^{i} \xrightarrow{R^i_{G^{n},N^p}} B_{G^{n},N^p} \otimes A_{N^p}^{i} \xrightarrow{A^i_{N^p}} B^i_{G^{n},N^p} \]

such that \(|R^i_{G^{n},N^p}| \leq |p^{-c_2}|\). We have to show that the image of \((B_{G^{n},N^p} \otimes A^i_{N^p})^{\wedge-u}\) is bounded in \(B^i_{G^{n},N^p}\). Lemma 4.5 of [Sch13a] implies that, given \(e > 0\), there exists \(s \geq s(G')\) such that for all \(k \geq s\), the cokernel of the map \(B^i_{G^{n},N^p} \otimes A_{N^p}^{i} \to B^i_{G^{n},N^p}\) is killed by \(p^e\), this implies the claim and that \(c_2\) can be taken arbitrary small as \(k \to \infty\). Next, to go down from level \(N'\) to level 1 prime to \(p\), we take \(\mathbb{Z}/N'\mathbb{Z}(1)\)-invariants of the trace maps \(R^i_{G^{n},N^p} : B^i_{G^{n},N^p} \to B^i_{G^{n},N^p}\) obtaining trace maps \(R^i_{G^{n},N^p} : B^i_{G^{n},N^p} \to B^i_{G^{n},N^p}\). This last operation preserves the norm of \(R^i_{G^{n},N^p}\) as \(N'\) is invertible in \(\mathbb{Z}_p\).

Finally, we have to show that for \(s(G) \leq s \leq k\) the map \(\gamma^i_{l(p)}(1) \xrightarrow{R^i_{l(p)}} 1\) over \(X^i_{G^{n},N^p} := \ker R_{G^{n},N^p}\) is invertible, with inverse bounded by \(|p^{-c_3}|\), and that we can take \(c_3\) arbitrarily small when \(k \to \infty\). Indeed, this follows by the same argument as before and the analogous property for the Sen theory of \(\mathbb{S}^{(e,d-e)}_{C,N'}\) with \(N'\) prime to \(p\).

\[ \square \]

**Remark 3.2.4.** We keep the notations of Examples 2.2.7 and 2.2.8. Let \(\psi : X \to \mathbb{Z}^{(e,d-e)}\) be the a toric chart, let \(J \subset \{e+1, \ldots, d\}\) be a finite subset and let \(D_J \subset X\) be the intersection of the divisors \(D_i = \{s_i = 0\}\) for \(i \in J\). For \(n \in \mathbb{N} \cup \{\infty\}\) let \(D_{J,n} = \mathbb{S}^{(e,d-e)}_{C,n} \times \mathbb{Z}^{(e,d-e)}_{C} D_J\), and denote \(B_{J,n} = \mathcal{O}_{D_{J,n}}(D_{J,n})\). The triple \((B_{J,n}, \Gamma, \id)\)
is not a Sen theory unless $J = \emptyset$, namely, the subgroup $\Gamma_J \subset \Gamma$ acts trivially on $D_{J,\infty}$. When $D_J$ is endowed with the log structure arising from the divisor $\prod_{i \in J} S_i = 0$, $D_J$ becomes an fs log smooth adic space $D'_J$ and Proposition 3.2.3 shows that $(B_J, \Gamma_J, \text{id})$ is a Sen theory. However, we still can compute the cohomology of relative locally analytic $O_{\mathcal{B}_{J,\infty}}$-Banach $\Gamma$-representations using Sen theory thanks to Remark 2.5.8. Moreover, by taking $\tilde{D}_J$ and letting $\tilde{\mathcal{B}}_{J,\infty}$, we can use loc. cit. to compute cohomology of relative locally analytic $O_{\tilde{\mathcal{B}}_{J,\infty}}$-Banach $G \times \Gamma$-representations.

3.2.3. Local version of main theorems. A first consequence of abstract Sen theory is a local version of Theorem 1.0.3.

**Proposition 3.2.5.** Let $\mathcal{F}$ be a relative locally analytic $O_{\mathcal{B}_\infty}$-module over $X$ admitting a lattice $\mathcal{F}_0$ such that $\mathcal{F}_0/p^\epsilon \cong \bigoplus O_{\mathcal{X}}/p^\epsilon$ for some $\epsilon > 0$. Then $R\Gamma_{\text{proket}}(\tilde{\mathcal{X}}_{\infty}, \mathcal{F}) = \mathcal{F}(\tilde{\mathcal{X}}_{\infty})$ is a relative locally analytic $O_{\tilde{\mathcal{B}}_{\infty}}$-representation of $G \times \Gamma$. In particular,

$$R\Gamma(G, \mathcal{F}(\tilde{\mathcal{X}}_{\infty}))^{R\Gamma-\text{la}} = \mathcal{F}(\tilde{\mathcal{X}}_{\infty})^{\Gamma-\text{la}}$$

is concentrated in degree 0. Furthermore, we have a natural action of a (local) geometric Sen operator

$$\theta_\mathcal{F} : \mathcal{F} \to \mathcal{F} \otimes_{\mathbb{Q}_p} (\text{Lie } \Gamma)^\vee$$

such that:

1. $\theta_\mathcal{F}$ is a Higgs field, i.e, $\theta_\mathcal{F} \wedge \theta_\mathcal{F} = 0$.
2. Higgs cohomology computes pro-Kummer-étale cohomology, namely, if $\nu : X_{\text{proket}} \to X_{\text{ket}}$ and $\eta : X_{\text{proket}} \to X_{\text{an}}$ are the projection of sites, then

$$R^i\nu_*\mathcal{F} = \nu_*H^i(\theta_\mathcal{F}, \mathcal{F}) \text{ and } R^i\eta_*\mathcal{F} = \eta_*H^i(\theta_\mathcal{F}, \mathcal{F}),$$

where $H^i(\theta_\mathcal{F}, \mathcal{F})$ is the cohomology of the Higgs complex

$$0 \to \mathcal{F} \to \mathcal{F} \otimes_{\mathbb{Q}_p} (\text{Lie } \Gamma)^\vee \to \cdots \to \mathcal{F} \otimes_{\mathbb{Q}_p}^{\bigwedge^d} (\text{Lie } \Gamma)^\vee \to 0.$$ (3) Suppose that $\theta_\mathcal{F} = 0$, then there is a natural equivalence

$$R\nu_*\mathcal{F} = \bigoplus_{i=0}^d \nu_*\mathcal{F} \otimes_{\mathbb{Q}_p} (\text{Lie } \Gamma)^\vee [-i] \text{ and } R\eta_*\mathcal{F} = \bigoplus_{i=0}^d \eta_*\mathcal{F} \otimes_{\mathbb{Q}_p} (\text{Lie } \Gamma)^\vee [-i]$$

depending on the toric chart. Moreover, $\nu_*\mathcal{F}$ is an $O_X$-Banach sheaf locally finite Kummer-étale on $X$, and we have

$$\mathcal{F} = \mathcal{O}_X \otimes_{O_X} \nu_*\mathcal{F} = \mathcal{O}_X \otimes_{O_X} \eta_*\mathcal{F}.$$ Conversely, if $\mathcal{G}$ is a locally $O_{\text{an}}$-Banach $O_X$-module in the Kummer-étale topology, then the geometric Sen operator of $\mathcal{O}_X \otimes_{O_X} \mathcal{G}$ vanishes.

We write $\text{Sen}_\mathcal{F} : \text{Lie } \Gamma \otimes_{\mathbb{Q}_p} \mathcal{F} \to \mathcal{F}$ for the adjoint of $\theta_\mathcal{F}$.
Proof. By Proposition 3.2.2 we know that for any compact open subgroup $G' \subset G$, we have that

$$R\Gamma_{\text{proK}}(X_{G',\infty}, \mathcal{F}) = R\Gamma(G', \mathcal{F}(\mathcal{X}_\infty)) = \mathcal{F}(X_{G',\infty}).$$

Then, Proposition 3.2.3 and Theorem 2.4.3 (2) imply that the derived $\Gamma$-locally analytic vectors of $\mathcal{F}(X_{G',\infty})$ are in degree 0, proving the first claim.

The Sen theory $(\mathcal{B}_\infty, G \times \Gamma, \text{pr}_2)$ satisfies the (AST) axiom by construction, namely, the Tate traces are obtained by normalized traces of actions of finite groups. Then, by Definition 2.5.3 we get a natural $G \times \Gamma$-equivariant Sen operator

$$\theta_{\mathcal{F}} : \mathcal{F}(\mathcal{X}_\infty) \rightarrow \mathcal{F}(\mathcal{X}_\infty) \otimes_{\mathbb{Q}_p} (\text{Lie } \Gamma)^\vee,$$

that defines a map of pro-Kummer-étale sheaves satisfying (1) in the Corollary. Next for (2), the equality

(3.3) $$R^i \eta_* \mathcal{F} = \eta_* H^i(\theta_{\mathcal{F}}, \mathcal{F})$$

follows from the fact that

$$R\eta_* \mathcal{F}(X) = R\Gamma(G \times \Gamma, \mathcal{F}(\mathcal{X}_\infty))$$

which holds by the acyclicity of Proposition 3.2.2 and by applying Proposition 2.5.4. The analogue statement for $\nu_*$ follows from (3.3) and the fact that the pull-back of $X_{\infty} \rightarrow X$ by any Kummer-étale map $X' \rightarrow X$ factoring as a composite of rational localizations and finite Kummer-étale maps still defines an abstract Sen theory as in Proposition 3.2.3. Finally, part (3) follows from Corollary 2.5.6, see also Remark 2.5.7. □

Next, we prove a local version of Theorem 1.0.4.

Proposition 3.2.6. Let $V$ be a locally analytic Banach representation of $G$, and let $V_{\text{Keb}}$ be the pro-Kummer-étale sheaf over $X$ constructed by $V$ via the $G$-torsor $\widetilde{X} \rightarrow X$. Then there is a natural geometric Sen operator

$$\theta_{\widetilde{X}} : \mathcal{O}_X \otimes_{\mathbb{Q}_p} (\text{Lie } G)_{\text{Keb}}^\vee \rightarrow \mathcal{O}_X \otimes_{\mathbb{Q}_p} (\text{Lie } \Gamma)^\vee,$$

or dually a map

$$\text{Sen}_{\widetilde{X}} : \text{Lie } \Gamma \otimes_{\mathbb{Q}_p} \mathcal{O}_X \rightarrow (\text{Lie } G)_{\text{Keb}} \otimes_{\mathbb{Q}_p} \mathcal{O}_X,$$

where $\text{Lie } G$ is endowed with the adjoint action, and such that we have a commutative diagram:

$$
\begin{array}{ccc}
V_{\text{Keb}} \otimes_{\mathbb{Q}_p} \mathcal{O}_X & \overset{d_V \otimes \text{id}_{\mathcal{O}_X}}{\longrightarrow} & (V_{\text{Keb}} \otimes_{\mathbb{Q}_p} \mathcal{O}_X) \otimes_{\mathbb{Q}_p} (\text{Lie } G)_{\text{Keb}}^\vee \\
\downarrow_{\theta_V} & & \downarrow_{\text{id}_V \otimes \theta_{\mathcal{F}}} \\
(V_{\text{Keb}} \otimes_{\mathbb{Q}_p} \mathcal{O}_X) \otimes_{\mathbb{Q}_p} (\text{Lie } \Gamma)^\vee & &
\end{array}
$$

such that $d_V : V \rightarrow V \otimes_{\mathbb{Q}_p} (\text{Lie } G)^\vee$ is induced by the derivations, and $\theta_V$ is the geometric Sen operator of $V_{\text{Keb}} \otimes_{\mathbb{Q}_p} \mathcal{O}_X$. 

Proof. The orbit map \( V \to V \otimes_{Q_p} C^{\text{an}}(G, Q_p) \) is \( G \)-equivariant, where the tensor product is endowed with the right regular action of \( G \) and the trivial action on \( V \). Then, it suffices to prove the statement when \( V = C^{\text{an}}(G, Q_p) \) is the space of \( G \)-analytic functions of \( G \) endowed with the right regular action, for a suitable radius \( p^{-h} \) of analyticity depending on a fixed local chart of \( G \). Then, Theorem 2.4.3 shows that

\[
C^{\text{an}}(G, Q_p) \otimes_{Q_p} \tilde{B}_\infty = \tilde{B}_\infty \otimes_{B_{\ell^\infty}} S_{G', \eta},
\]

with

\[
S_{G', \eta} = \tilde{B}_\infty^{G_{\text{an}}', \eta-an}
\]

the space of \( G_{\text{an}}' \times \Gamma_{\eta} \)-analytic functions (for \( n \) large enough). The action \( \text{Sen}_V \) of the \( \text{Sen} \) operators \( \text{Lie}_\Gamma \) on \( V \otimes_{Q_p} \tilde{B}_\infty \) provided by Proposition 3.2.5 is by construction the \( \tilde{B}_\infty \)-linear extension of the derivations on \( \tilde{B}_\infty^{G_{\text{an}}', \eta-an} \). Therefore, the action of \( \text{Lie}_\Gamma \) on \( V \otimes_{Q_p} \tilde{B}_\infty \) is via \( \tilde{B}_\infty \)-linear derivations.

On the other hand, the left regular action of \( G \) on \( V \otimes_{Q_p} \tilde{B}_\infty \) is \( \tilde{B}_\infty \)-linear and commutes with the right regular \( G \)-action. Therefore, \( \text{Lie}_\Gamma \) acts via left \( G \)-invariant \( \tilde{B}_\infty \)-linear derivations, so it must come from the natural action of right derivations of a \( G \times \Gamma \)-equivariant \( \tilde{B}_\infty \)-linear map

\[
\tilde{B}_\infty \otimes_{Q_p} \text{Lie}_\Gamma \to \tilde{B}_\infty \otimes_{Q_p} \text{Lie}_G,
\]

or equivalently, by a map of pro-Kummer-étale sheaves

\[
\text{Sen}_X : \tilde{\Omega}_X \otimes_{Q_p} \text{Lie}_\Gamma \to \tilde{\Omega}_X \otimes_{Q_p} (\text{Lie}_G)_{\text{ét}}.
\]

Taking duals we find the map \( \theta_X \) satisfying the conclusion of the proposition. \( \square \)

Remark 3.2.7. Proposition 3.2.6 shows that, in order to compute the \( \text{Sen} \) operator of a torsor it suffices to consider a faithful representation of \( g \).

3.3. The geometric \( \text{Sen} \) operator: globalization. In Propositions 3.2.5 and 3.2.6 we proved local versions of Theorems 1.0.3 and 1.0.4 respectively. In order to obtain the full version we need to show that the \( \text{Sen} \) operators glue in toric charts according to \( \Omega_\chi^1(\log) \).

3.3.1. Key case. Let \( X \) and \( Y \) be affinoid fs log smooth adic spaces over \((C, C^+)\), and suppose we have two toric charts \( \psi_X : X \to \mathbb{G}_m^{(e,d-e)} \) and \( \psi_Y : Y \to \mathbb{G}_m^{(g,h-g)} \) respectively. Let \( X_\infty \) and \( Y_\infty \) be the pro-Kummer-étale torsors over \( X \) and \( Y \) obtained by taking \( p \)-th power roots of the perfectoid charts, and let \( \Gamma_X \) and \( \Gamma_Y \) denote the Galois groups of \( X_\infty \to X \) and \( Y_\infty \to Y \) respectively. Let \( f : Y \to X \) be a morphism, and let \( f^*X_\infty = Y \times_X X_\infty \). The following is a direct generalization of [Pan22, Lemma 3.4.3], we thank Lue Pan for the simplifications of a previous proof.
Proposition 3.3.1. Let \( \nu_X : X_{\text{prokét}} \to X_{\text{két}} \) and \( \nu_Y : Y_{\text{prokét}} \to Y_{\text{két}} \) be the projection of sites. We have a commutative diagram of pro-Kummer-étale sheaves over \( Y \)

\[
\begin{array}{c}
(Lie \Gamma_\nu)^\wedge \otimes_{Q_p} \widehat{\mathcal{O}}_Y & \xleftarrow{\theta_{f \cdot \nu_{\infty}}} & f^*((Lie \Gamma_X)^\wedge \otimes_{Q_p} \widehat{\mathcal{O}}_X) \\
\downarrow & & \downarrow \\
(R^1 \nu_{\infty}, \widehat{\mathcal{O}}_Y) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y & \xleftarrow{f^*} & f^*((R^1 \nu_X, \widehat{\mathcal{O}}_X) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X) \\
\uparrow & & \uparrow \\
\Omega^1_{\nu}(log) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y(-1) & \xleftarrow{f^*} & f^*(\Omega^1_X(log) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(-1))
\end{array}
\]

where the top vertical maps are the isomorphisms obtained by the explicit cocycles if Remark 3.1.5 and the lower vertical maps are obtained by Proposition 3.1.2.

**Proof.** Let \( \gamma_{tX} \in \Gamma_X \) and \( \gamma_{tY} \in \Gamma_Y \) be the canonical basis, and let \( \log(\gamma_{tX}) \in Lie \Gamma_X \) and \( \log(\gamma_{tY}) \in Lie \Gamma_Y \) be the induced basis in the Lie algebras. Let \( \sigma_{tX} \) be the 1-cocycle

\[
\sigma_{tX}(log \gamma_{tX}) = \delta_{t,j},
\]

resp. for \( Y \). Then, by Remark 3.1.5, the composite of the top vertical arrow with the inverse of the lower vertical arrow sends \( \sigma_{tX} \) to the natural basis \( d_{tX} \) of \( \Omega^1_X(log) \) provided by the toric coordinates (resp. for \( Y \)). Let \( (a_{k,l}) \) be the base change matrix on \( \mathcal{O}_Y(Y) \) such that

\[
f^*d_{k,X} = \sum_l a_{k,l}d_{l,Y},
\]

we want to show that

\[
(3.4) \quad \theta_{f \cdot \nu_{\infty}}(\sigma_{t,X}) = \sum_l a_{k,l}\sigma_{t,Y}.
\]

Consider \( Y_{\infty,\infty} := Y_\infty \times_Y f^*X_\infty \), it is a \( \Gamma_Y \times \Gamma_X \)-torsor over \( Y \). By Proposition 3.1.4, we have that

\[
R^1 \nu_{Y,*}(\widehat{\mathcal{O}})(Y) = H^1(\Gamma_Y \times \Gamma_X, \widehat{\mathcal{O}}(Y_{\infty,\infty})) = H^1(\text{Lie}(\Gamma_Y \times \Gamma_X), \widehat{\mathcal{O}}(Y_{\infty,\infty})^{\Gamma_X \times \Gamma_Y - Ia})^{\Gamma_X \times \Gamma_Y}.
\]

Then, by Corollary 3.1.4, we know that

\[
f^*\sigma_{t,X} - \sum_l a_{k,l}\sigma_{t,Y} = 0.
\]

In particular, there are locally analytic vectors \( z_k \in \widehat{\mathcal{O}}(Y_{\infty,\infty})^{\Gamma_X \times \Gamma_Y - Ia} \) such that

\[
\log \gamma_{t,X} \cdot z_k = \delta_{k,l} \quad \text{and} \quad \log \gamma_{t,Y} \cdot z_k = -a_{k,l}.
\]

But then, let \( V \) be the algebraic representation of \( \Gamma_X \) fitting in a short exact sequence

\[
0 \to Q_{\mu}e \to V \to Q_{\mu}(\text{Lie} \Gamma_X)^\wedge \to 0
\]

such that the action of \( \text{Lie} \Gamma_X \) is trivial on \( Q_{\mu}e \) and induces the natural pairing

\[
\text{Lie} \Gamma_X \otimes_{Q_{\mu}} (\text{Lie} \Gamma_X)^\wedge \to Q_{\mu} \cong Q_{\mu}e.
\]
Let $V_{\kappa}$ be the associated pro-Kummer-étale sheaf over $Y$ obtained via the $\Gamma_X$-torsor $f^*X_{\kappa}$. Since $V$ is a faithful representation of Lie $\Gamma_X$, it suffices to compute the geometric Sen operators of $V_{\kappa} \otimes_{\hat{O}_Y} \hat{O}_Y$. We have a trivialization

$$(V_{\kappa} \otimes_{\hat{O}_Y} \hat{O}_Y)(Y_{\infty, \infty}) \cong \hat{O}_Y(Y_{\infty, \infty}) \oplus \bigoplus_{k=1}^h v_k \hat{O}_Y(Y_{\infty, \infty})$$

with $v_k = \sigma_{k,X} - z_k e$. Then $\text{Lie} \Gamma_X$ acts trivially on $e$ and $v_k$, and the action of the Sen operators $\text{Lie} \Gamma_Y$ on $V_{\kappa} \otimes_{\hat{O}_Y} \hat{O}_Y$ is the extension of scalars of the action on $\{e, v_1, \ldots, v_h\}$. By construction $\text{Lie} \Gamma_Y$ acts trivially on $e$, and satisfies

$$\log \gamma_{Y} \cdot v_k = a_k e.$$ 

This proves the equation (3.4) which gives the proposition. □

As a first consequence of Proposition 3.3.1 we deduce Theorem 1.0.3.

**Theorem 3.3.2.** Let $\mathcal{F}$ be an relative locally analytic ON Banach $\hat{O}_X$-sheaf over $X$. Then the local geometric Sen operators of Proposition 3.2.5 glue to a global geometric Sen operator

$$\theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\hat{O}_X} \Omega^1_X(\log)(-1).$$

Furthermore, the following properties hold:

1. The formation of $\theta_{\mathcal{F}}$ is functorial in $\mathcal{F}$.
2. Let $\nu : X_{\prok} \to X_{\kappa}$ be the projection from the pro-Kummer-étale site to the Kummer-étale site, then there is a natural equivalence

$$R^i \nu_* \mathcal{F} = \nu_* H^i(\theta_{\mathcal{F}}, \mathcal{F}),$$

where $H^i(\theta_{\mathcal{F}}, \mathcal{F})$ is the cohomology of the Higgs complex

$$0 \to \mathcal{F} \xrightarrow{\theta_{\mathcal{F}}} \mathcal{F} \otimes_{\hat{O}_X} \Omega^1_X(\log)(-1) \to \cdots \to \mathcal{F} \otimes_{\hat{O}_X} \Omega^d_X(\log)(-d) \to 0.$$ 

3. Suppose that $\theta_{\mathcal{F}} = 0$, then $\nu_* \mathcal{F}$ is locally on the Kummer-étale topology of $X$ an ON Banach $\hat{O}_X$-module and $\mathcal{F} = \hat{O}_X \otimes_{\hat{O}_X} \nu_* \mathcal{F}$. Conversely, for any locally ON Banach $\hat{O}_X$-module $\mathcal{G}$ the pullback $\hat{O}_X \otimes_{\hat{O}_X} \mathcal{G}$ has trivial Sen operator.

4. If $X$ has a form $X'$ over a discretely valued field with perfect residue field $(K, K^+)$, then $\theta_{\mathcal{F}}$ is Galois equivariant. In particular, we recover the natural splitting

$$Rv_* \hat{O}_X = \bigoplus_{i=0}^d \Omega^i_X(\log)(-d)[i].$$

5. Let $Y$ be another fs log smooth adic space over $(C, C^+)$, and let $f : Y \to X$ be a morphism. Then there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{f \theta_{\mathcal{F}}} & f^* \mathcal{F} \otimes_{\hat{O}_Y} f^* \Omega^1_X(\log)(-1) \\
\theta_{f^*} \mathcal{F} & \xrightarrow{id \otimes f^*} & f^* \mathcal{F} \otimes_{\hat{O}_Y} \Omega^1_Y(\log)(-1).
\end{array}
$$
Proof. The gluing of the local Sen operators of Proposition 3.2.5 follows from Proposition 3.3.1. Indeed, we can suppose that $X$ is affinoid and that has two charts $\psi_1, \psi_2 : X \to \mathcal{O}^\circ_{C}(d,e)$. Let $X_{\infty,0}$ and $X_{0,\infty}$ be the pro-Kummer-étales $\Gamma$-torsors obtained from the perfectoid toric charts $\psi_1$ and $\psi_2$, respectively, let $X_{\infty,0} = X_{\infty,0} \times X_{0,\infty}$. Let us write $\Gamma_1$ and $\Gamma_2$ for the Galois groups of $X_{\infty,0}$ and $X_{0,\infty}$, respectively. Then, by Proposition 3.2.3 and Theorem 2.4.3, there is $n \gg 0$ such that

$$\hat{\mathcal{O}}_X(X_{\infty,0}) \otimes \hat{\mathcal{O}}_X(X_{0,\infty})^{\mathbb{G}_{\mathbb{Q}}^\circ_{\infty,1}} \mathcal{F}(X_{\infty,0})^{\mathbb{G}_{\mathbb{Q}}^\circ_{\infty,1}}, \mathcal{F}(X_{0,\infty})^{\mathbb{G}_{\mathbb{Q}}^\circ_{0,\infty}}.$$

Therefore, by Proposition 3.2.6, the action of the Sen operators $\text{Lie} \Gamma_1$ on $\mathcal{F}$ can be either computed as the extension of the scalars via the natural derivations on $\mathcal{F}(X_{\infty,0})^{\mathbb{G}_{\mathbb{Q}}^\circ_{\infty,1}}, \mathcal{F}(X_{0,\infty})^{\mathbb{G}_{\mathbb{Q}}^\circ_{0,\infty}}$ or as the extension of the scalars of the action of $\text{Lie} \Gamma_2$ on $\mathcal{F}(X_{\infty,0})^{\mathbb{G}_{\mathbb{Q}}^\circ_{\infty,1}}, \mathcal{F}(X_{0,\infty})^{\mathbb{G}_{\mathbb{Q}}^\circ_{0,\infty}}$ by precomposing with the Sen map

$$\text{Sen} : (\text{Lie} \Gamma_1) \otimes_{\mathbb{Q}_p} \hat{\mathcal{O}}_X \to (\text{Lie} \Gamma_2) \otimes_{\mathbb{Q}_p} \hat{\mathcal{O}}_X.$$

We obtain the gluing thanks to Proposition 3.3.1.

Finally, part (1) follows from the gluing and the functoriality of Proposition 3.2.5. Parts (2) and (3) are consequences of parts (2) and (3) of loc. cit. respectively. For part (4), note that the Galois action of $\text{Gal}_k$ on the Lie algebra $\text{Lie} \Gamma$ induced by some local toric coordinate $\psi : X \to \mathcal{O}^\circ_{C}(d,e)$ is given by the cyclotomic character $\chi$. Finally, part (5) follows from Proposition 3.3.1 by using an analogue argument as in the beginning of the proof.

Under further assumptions on the space $X$ and the sheaf $\mathcal{F}$ we can even compute the projection to the analytic site:

**Corollary 3.3.3.** Let $\eta : X_{\text{proK}} \to X_{\text{an}}$ be the projection of sites. Suppose that $X$ has strict normal crossing divisors, namely, that $X$ admits toric charts locally in the analytic topology, and that $\mathcal{F}$ admits a lattice $\mathcal{F}^+$ such that $\mathcal{F}^+/p^\epsilon = \bigoplus \mathcal{O}_X^+/p^\epsilon$ for some $\epsilon > 0$ and some index set $I$. Then there are natural isomorphisms

$$R^i\eta_*\mathcal{F} = \eta_*H^i(\theta, \mathcal{F}).$$

Moreover, if $\theta = 0$ then $\mathcal{F} = \hat{\mathcal{O}}_X \otimes_{\mathcal{O}_X} \eta_*\mathcal{F}$.

**Proof.** This follows from Theorem 3.3.2 and Corollary 2.5.6 by applying Sen theory to toric coordinates arising locally in the analytic topology of $X$.

**3.3.2. Gluing for general $G$.** We have made all the preparations to show Theorem 1.0.4.

**Theorem 3.3.4.** Let $X$ be an fs log smooth adic space over $\text{Spa}(C, C^+)$ with log structure given by normal crossing divisors. Let $G$ a $p$-adic Lie group and $\tilde{X} \to X$ a pro-Kummer-étales $G$-torsor. Then the geometric Sen operators of Proposition 3.2.6 given by local charts of $X$ glue to a morphism of $\hat{\mathcal{O}}_X$-vector bundles over $\tilde{X}_{\text{proK}}$

$$\theta_X : \hat{\mathcal{O}}_X \otimes_{\mathbb{Q}_p} (\text{Lie} G)_{\text{ét}}^\vee \to \hat{\mathcal{O}}_X(-1) \otimes_{\mathcal{O}_X} \Omega^1_X(\log)$$
such that \( \theta_X \wedge \theta_X = 0 \). In particular, for any locally analytic Banach representation \( V \) of \( G \), we have a commutative diagram

\[
\begin{array}{ccc}
V_{\mathrm{két}} \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_X & \xrightarrow{\theta_V} & V_{\mathrm{két}} \otimes_{\mathcal{O}_X} (\mathrm{Lie} \ G)^\vee_{\mathrm{két}} \\
\downarrow^d & & \downarrow^\theta \otimes \hat{\mathcal{O}}_X \\
(V_{\mathrm{két}} \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_X(-1)) \otimes_{\mathcal{O}_X} \Omega_X^1(\log) & \xrightarrow{\theta_V} & (V_{\mathrm{két}} \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_X(-1)) \otimes_{\mathcal{O}_X} \Omega_X^1(\log)
\end{array}
\]

such that \( d_V : V \to V \otimes (\mathrm{Lie} \ G)^\vee \) is induced by derivations, and \( \theta_V \) is the geometric Sen operator of \( V_{\mathrm{két}} \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_X \).

Moreover, let \( H \to G \) be a morphism of \( p \)-adic Lie groups, let \( Y \) an fs log smooth adic space over \( (\mathbb{C}, \mathbb{C}^+) \) and let \( \tilde{Y} \to Y' \) be an \( H \)-torsor. Suppose we are given with a commutative diagram compatible with the group actions

(3.5)

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{f} & \tilde{X} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & X
\end{array}
\]

Then the following square is commutative

(3.6)

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{f^*} & \tilde{X} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & X
\end{array}
\]

Proof: We first construct the geometric Sen operator \( \theta_{\tilde{X}} \) for the \( G \)-torsor \( \tilde{X} \to X \).

Let \( V \) be a locally analytic representation of \( G \) on a Banach space, and consider the inclusion of the orbit map \( V \hookrightarrow V_{\mathrm{két}} \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_X \).

By Proposition 3.2.6, locally in the Kummer-étale topology of \( X \), \( \mathrm{Sen}_{\mathcal{F}} \) acts by \( \hat{\mathcal{O}}_X \)-linear derivations on \( \mathcal{F} \) that are in addition left \( G \)-equivariant. Then, the local maps of Proposition 3.2.6 glue to an \( \hat{\mathcal{O}}_X \)-linear morphism

\[
\mathrm{Sen}_{\mathcal{F}} : \Omega_X^1(\log)(1) \otimes_{\hat{\mathcal{O}}_X} \mathcal{F} \to \mathcal{F}.
\]

By Proposition 3.2.6 locally in the Kummer-étale topology of \( X \), \( \mathrm{Sen}_{\mathcal{F}} \) acts by \( \hat{\mathcal{O}}_X \)-linear derivations on \( \mathcal{F} \) that are in addition left \( G \)-equivariant. Then, the local maps of Proposition 3.2.6 glue to an \( \hat{\mathcal{O}}_X \)-linear morphism

\[
\mathrm{Sen}_{\mathcal{F}} : \Omega_X^1(\log)(1) \otimes_{\hat{\mathcal{O}}_X} \mathcal{F} \to \mathcal{F}.
\]
or equivalently by taking adjoints to a map

$$\theta_X : (\text{Lie } G)_{k \hat{\otimes} Q} \otimes_{Q_p} \hat{\mathcal{O}}_X \to \Omega^1_X(\log) \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_X(-1)$$

satisfying the conclusion of the theorem.

Next, we prove functoriality for commutative squares as in (3.5). We have a commutative diagram

$$\begin{array}{c}
\bar{Y} \\
\downarrow \\
Y
\end{array} \longrightarrow \begin{array}{c}
Y \times_X \bar{X} \\
\downarrow \\
Y
\end{array} \longrightarrow \begin{array}{c}
\bar{X} \\
\downarrow \\
X
\end{array} \quad \text{as wanted.}
$$

Then, it suffices to prove the functoriality of the geometric Sen operator for both outer squares.

**Case** \( f = \text{id}_X \). Consider the space \( C^{\text{la}}(G, Q_p) = \lim_{\rightarrow h} C^b(G, Q_p) \) of locally analytic functions of \( G \), written as colimit of \( h \)-analytic functions. We can find a cofinal system of \( h \)'s such that \( C^b(G, Q_p) \) also carries an action of \( H \) via the group homomorphism \( H \to G \). Let

$$\mathcal{F} = C^b(G, Q_p) \star_{2, k \hat{\otimes} Q} \hat{\mathcal{O}}_X,$$

Then, the sheaf \( \mathcal{F} \) can be obtained via a locally analytic representation of both \( G \) and \( H \), and by the first part of the theorem, the geometric Sen operator of \( \mathcal{F} \) arises from the geometric Sen operator of both the \( G \) and \( H \)-torsors. Since the action of \( \text{Lie } G \) is faithful on \( \mathcal{F} \), one has the commutativity of the diagram

$$\begin{array}{c}
(Lie G)_{k \hat{\otimes} Q} \otimes_{Q_p} \hat{\mathcal{O}}_X \\
\downarrow \\
(Lie H)_{k \hat{\otimes} Q} \otimes_{Q_p} \hat{\mathcal{O}}_X
\end{array} \longrightarrow \begin{array}{c}
\Omega^1_X(\log) \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_X(-1) \\
\downarrow \\
\Omega^1_X(\log) \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_Y(-1)
\end{array}$$

as wanted.

**Case** \( \bar{Y} = f^* \bar{X} \). We need to show that the following diagram is commutative

$$\begin{array}{c}
(Lie G)_{k \hat{\otimes} Q} \otimes_{Q_p} \hat{\mathcal{O}}_Y \\
\downarrow \\
(Lie G)_{k \hat{\otimes} Q} \otimes_{Q_p} \hat{\mathcal{O}}_X
\end{array} \longrightarrow \begin{array}{c}
\Omega^1_Y(\log) \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_Y(-1) \\
\downarrow \\
\Omega^1_Y(\log) \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_X(-1)
\end{array}$$

(3.7)
Taking \( \mathcal{F} = \mathcal{C}^h(G, \mathbb{Q}_p) \), by Theorem 3.3.2 we have a commutative diagram

\[
\begin{array}{ccc}
 f^* \mathcal{F} & \xrightarrow{f^* \theta_\mathcal{F}} & f^* \mathcal{F} \otimes \Omega^1_{Y} (\log)(-1) \\
 \theta_{f^* \mathcal{F}} & \downarrow \text{id} \otimes f^* & \\
 f^* \mathcal{F} \otimes \Omega^1_Y (\log)(-1) & & \\
\end{array}
\]

But the action of \( \text{Lie } G \) on \( \mathcal{C}^h(G, \mathbb{Q}_p) \) is faithful, this implies the commutativity of (3.7) as wanted.

The Sen morphism should encode the directions of perfectoidness of \( X \). We have the following conjecture, which is a generalization of a theorem of Sen saying that a \( p \)-adic Galois representation of \( p \)-adic field has vanishing Sen operator if and only if it is potentially unramified, see Corollary 3.32 of [F O].

**Conjecture 3.3.5.** Let \( X \) be an fs log smooth adic space over \((\mathbb{C}, \mathbb{C}^+)\) with normal crossing divisors, let \( G \) be a compact \( p \)-adic Lie group and \( \widetilde{X} \to X \) a pro-Kummer-étale \( G \)-torsor. Then the geometric Sen operator \( \theta_{\widetilde{X}} : (\text{Lie } G)^\vee \otimes \mathbb{Q}_p \otimes_{\mathcal{O}_X} \Omega^1_{\widetilde{X}} (\log)(-1) \) is surjective if and only if \( \widetilde{X} \) is a perfectoid space.

**Remark 3.3.6.** In the work [RC22], we show that the pro-(Kummer-)étale torsors defining the infinite level Shimura varieties satisfy the hypothesis of the Conjecture 3.3.5. The proof of this fact never uses the perfectoidness of the Shimura variety, only the \( p \)-adic Riemann-Hilbert correspondence of [DLLZ23]. Moreover, in [RC22] we use the explicit construction of the geometric Sen operators for Shimura varieties to prove the vanishing part of the rational Calegari-Emerton conjectures, see [CE12]. A confirmation of the Conjecture 3.3.5 would imply the vanishing of the Calegari-Emerton conjectures integrally.[1]

### 3.4. Locally analytic vectors of pro-Kummer-étale towers

We keep the previous notations, i.e., \((\mathbb{C}, \mathbb{C}^+)\) is an algebraically closed field, \( X \) is an fs log smooth adic space over \((\mathbb{C}, \mathbb{C}^+)\) with log structure given by normal crossing divisors, \( G \) a compact \( p \)-adic Lie group, and \( \widetilde{X} \) a pro-Kummer-étale \( G \)-torsor over \( X \). In this last section we apply Theorem 3.3.2 to study the locally analytic vectors of \( \widehat{\mathcal{O}}_{\widetilde{X}} \) for the action of \( G \), we also extend the cohomology computations of the theorem to log adic spaces arising from the boundary of \( X \). We finish by explaining the relation between the geometric Sen operator and the \( p \)-adic Simpson correspondence of [LZ17, DLLZ23, Wan23].

#### 3.4.1. Pro-Kummer-étale cohomology of the boundary

Let \( X \) and \( \widetilde{X} \to X \) be as before. Let \( D \subset X \) be the boundary divisor. Étale locally on \( X \), \( D \) can be written

---

1 Actually, by taking stalks, it suffices to show the analogue conjecture for the points of \( X \) in order to deduce the Calegari-Emerton conjectures.
as a disjoint union of irreducible components $D = \bigcup_{d \in I} D_d$, where the finite intersections of the $D_d$’s are smooth. If this holds locally in the analytic topology of $X$ we say that $D$ is a strict normal crossing divisor, for simplicity let us assume that this is the case. Given $J \subset I$ a finite subset we set $D_J = \bigcap_{d \in J} D_d$ endowed with the log structure pulled back from $X$, and write $t_J : D_J \subset X$ for the inclusion map. We denote by $\bar{O}_{D_J}$ the sheaf $t_{J*} \bar{O}_{D_J}$ over $X_{\text{pro\acute{e}t}}$.

In this section shall write $\bar{O}_{X,\text{k\acute{e}t}}$ and $\bar{O}_{X,\text{an}}$ for the structural sheaves on the Kummer-\acute{e}tale site and the analytic site respectively. We also denote $\bar{O}_{D_J,\text{k\acute{e}t}}$ and $\bar{O}_{D_J,\text{an}}$ for the sheaves defined by the boundary divisor $D_J$.

We start with a partial extension of Theorem 3.3.2 to the boundary.

**Theorem 3.4.1.** Let $\mathcal{F}$ be a relative locally analytic $\bar{O}_X$-module over $X_{\text{pro\acute{e}t}}$ and let $\theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\bar{O}_X} \Omega^1_X((log))(-1)$ be the geometric Sen operator of $\mathcal{F}$. Let $\nu_n : X_{\text{pro\acute{e}t}} \to X_{\text{an}}$ be the projection of sites, and let $J \subset I$ be a finite subset. Then there are natural (quasi-)isomorphisms

$$R\nu_{t_J*} \otimes_{\bar{O}_{X,\text{k\acute{e}t}}} \mathcal{F} \cong (R\nu_n \mathcal{F}) \otimes_{\bar{O}_{X,\text{an}}} \bar{O}_{D_J,\text{k\acute{e}t}}$$

and

$$R^i\nu_{t_J*} \otimes_{\bar{O}_{X,\text{k\acute{e}t}}} \mathcal{F} = \nu_n(H^i(\theta_{\mathcal{F}}, t_{t_J*}\mathcal{F})).$$

Moreover, let $\eta : X_{\text{pro\acute{e}t}} \to X_{\text{an}}$ be the projection of sites and suppose that $\mathcal{F}$ has a lattice $\mathcal{F}^+$ such that $\mathcal{F}^+ / \mathcal{F}^0 \cong \bigoplus_{\epsilon > 0} \bar{O}_X / \mathcal{F}^0 \mathcal{F}^\epsilon$ for some $\epsilon > 0$. Then

$$R\eta_{t_J*} \otimes_{\bar{O}_{X,\text{k\acute{e}t}}} \mathcal{F} \cong (R\eta_n \mathcal{F}) \otimes_{\bar{O}_{X,\text{an}}} \bar{O}_{D_J,\text{an}}$$

and

$$R^i\eta_{t_J*} \otimes_{\bar{O}_{X,\text{k\acute{e}t}}} \mathcal{F} = \eta_n(H^i(\theta_{\mathcal{F}}, t_{t_J*}\mathcal{F})).$$

**Proof.** All the statements are local on $X$ for the Kummer-\acute{e}tale or analytic topology, so we can assume that we have a toric chart $\psi : X \to \mathbb{C}^{d-e}$, and take $D = D_J$ to be the divisor $S_{e+1} \cdots S_d = 0$. Let $X_\infty$ be the $\Gamma \equiv \mathbb{Z}_{p^d}(1)$-torsor over $X$ obtained via $\psi$, let $D_\infty = X_\infty \times_X D$, and for $n \geq 0$ let $X_n = X_\infty / p^n \Gamma$ and $D_n = D_\infty / p^n \Gamma$. We write $\text{pr}_D : \Gamma \to \Gamma_D$ (resp. $\text{pr}_{D'} : \Gamma \to \Gamma_{D'}$) for the projection to the last $d-e$ components (resp. the projection to the first $e$ components). We can compute

$$R\nu_{t_J*} \otimes_{\bar{O}_{X,\text{k\acute{e}t}}} \mathcal{F} = R\Gamma(\Gamma, \mathcal{F}(X_\infty) \otimes_{\bar{O}_{X(X_\infty)}} \bar{O}_X(D_\infty)).$$

Since $\Gamma_D$ acts trivially on $D_\infty$, the action of $\Gamma_D$ is locally analytic on $\mathcal{F}(X_\infty)$ by Lemma 2.1.4. On the other hand, Remark 3.2.4 says that $(\bar{O}_D(D_\infty), \Gamma_D, \text{id})$ is a Sen theory. Moreover, Theorem 2.4.3(1) implies that the Sen module of $\mathcal{F} \otimes_{\bar{O}_X} \bar{O}_D$ is just the base change to $\varphi_{t_J}^{-1}$ of the Sen module of $\mathcal{F}$. Then, by Remarks 2.5.2 and 2.5.8 we can compute

$$R\Gamma(\Gamma, \mathcal{F}(X_\infty) \otimes_{\bar{O}_{X(X_\infty)}} \bar{O}_X(D_\infty)) = R\Gamma(\Gamma_\infty, \Gamma(\text{Lie} \Gamma, \mathcal{F}(X_\infty) \otimes_{\bar{O}_X(D_n)} \bar{O}_{D(n)}))$$

for some $n$ large enough. This translates to the equivalences (3.8) and (3.10). Finally, the equivalences (3.9) and (3.11) follow from the previous discussion and Remark 2.5.8.

$\square$
3.4.2. Locally analytic vectors of $\widehat{\mathcal{O}}_X$. Let $(C, C^+)$ be an algebraically closed non-Archimedean field, $X$ an fs log smooth adic space over $(C, C^+)$ with normal crossing divisors, and $\tilde{X} \to X$ a pro-Kummer-étale torsor for a compact $p$-adic Lie group $G$. Throughout the next paragraph we make the following hypothesis for the Sen operators

(BUN) The geometric Sen operator $\theta_{\tilde{X}} : (\text{Lie } G)^\vee_{\text{et}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{\tilde{X}} \to \Omega^1_X(\log) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(-1)$ of Theorem 3.3.4 is surjective.

Definition 3.4.2. Let $\widehat{\mathcal{O}}_{\tilde{X}}$ be the restriction of the completed structural sheaf of $X_{\text{proké}}$ to the analytic site of $\tilde{X}$. We let $\mathcal{O}^{la}_{\tilde{X}} \subset \widehat{\mathcal{O}}_{\tilde{X}}$ be the presheaf mapping a qcqs subspace $\tilde{U} \subset \tilde{X}$ to the space $\mathcal{O}^{la}_{\tilde{X}}(\tilde{U}) := \widehat{\mathcal{O}}_{\tilde{X}}(\tilde{U})^{G_{\tilde{U}}-la}$ of $G_{\tilde{U}}$-locally analytic sections of $\widehat{\mathcal{O}}_{\tilde{X}}(\tilde{U})$, where $G_{\tilde{U}}$ is the stabilizer of $\tilde{U}$.

Lemma 3.4.3. The presheaf $\mathcal{O}^{G-\text{la}}_{\tilde{X}}$ is a sheaf for the analytic topology of $\tilde{X}$.

Proof. Note that for any qcqs open subspace $\tilde{U} \subset \tilde{X}$, the ring $\widehat{\mathcal{O}}_{\tilde{X}}(\tilde{U})$ is a $\mathbb{Q}_p$-Banach space. Let $\{\tilde{V}_i\}_{i=1}^r$ be an open cover of $\tilde{U}$ by qcqs open subspaces. We have a short exact sequence

$$0 \to \widehat{\mathcal{O}}_{\tilde{X}}(\tilde{U}) \to \prod_i \widehat{\mathcal{O}}_{\tilde{X}}(\tilde{V}_i) \to \prod_{i,j} \widehat{\mathcal{O}}_{\tilde{X}}(\tilde{V}_i \cap \tilde{V}_j).$$

(3.12)

Let $G_0 \subset G$ be an open compact subgroup stabilizing all the $\tilde{V}_i$’s, then $G_0$ stabilizes $\tilde{U}$ and the intersections $\tilde{V}_i \cap \tilde{V}_j$, and (3.12) is a $G_0$-equivariant exact sequence of Banach spaces. Then, tensoring with $C^{la}(G_0, \mathbb{Q}_p)_{\ast,1}$ and taking $G_0$-invariant vectors we get an exact sequence

$$0 \to \mathcal{O}^{la}_{\tilde{X}}(\tilde{U}) \to \prod_i \mathcal{O}^{la}_{\tilde{X}}(\tilde{V}_i) \to \prod_{i,j} \mathcal{O}^{la}_{\tilde{X}}(\tilde{V}_i \cap \tilde{V}_j).$$

proving what we wanted. \qed

The following definition is useful to construct sheaves over $\tilde{X}$.

Definition 3.4.4. Let $G_0 \subset G$ be an open compact subgroup and let $X_{G_0} = \tilde{X}/G_0$, denote by $\eta_{G_0} : X_{G_0, \text{proké}} \to X_{G_0, \text{an}}$ the projection of sites. Let $\mathcal{F} := (\mathcal{F}_{G_0})_{G_0 \subset G}$ be a compatible sequence of pro-Kummer-étale sheaves on $X_{G_0}$ in the sense that if $G' \subset G_0$ we have a map $\psi_{G_0}^{G'} : \mathcal{F}_{G_0} |_{X_{G_0}} \to \mathcal{F}_{G'}$, and that for $G'' \subset G' \subset G_0$ we have $\psi_{G_0}^{G''} \circ \psi_{G_0}^{G'} = \psi_{G_0}^{G''}$. We define

$$\text{R} \eta_{G_0, \ast} \mathcal{F} := \lim_{\longrightarrow \limits_{G_0}} \text{R} \eta_{G_0, \ast} \mathcal{F}_{G_0}$$

seen as an object over $\tilde{X}_{\text{an}}$. 
Let $D \subset X$ be the boundary divisor and for simplicity let us suppose that $D = \bigcup_{i \in I} D_i$ is written as a disjoint union of irreducible divisors with smooth finite intersections. Given $J \subset I$ a finite subset we let $D_J = \bigcap_{i \in J} D_i$ (we declare $D_\emptyset = X$).

**Theorem 3.4.5.** Let $\eta : X_{\text{proKummer-étale}} \to X_{\text{an}}$ be the projection of sites, and let

$$
C^{la}(\text{Lie } G, \mathbb{Q})_{\text{1, ket}} := (C^{la}(G_0, \mathbb{Q})_{\text{1, ket}})_{G \in G}
$$

be the ind-sequence of pro-Kummer-étale sheaves over the tower $(X_{G_0})_{G \in G}$ as in Definition 3.4.4. For any finite subset $J \subset I$ we have that

$$
R\eta_{\infty,*}(C^{la}(\text{Lie } G, \mathbb{Q})_{\text{1, ket}} \otimes_\mathbb{Q}_p \widehat{\mathcal{O}}_{D_J}) = \left[\begin{array}{c} \theta_{X} \end{array} \right]_{X=\text{Lie } G_{\text{ket}}} \otimes_\mathbb{Q}_p \widehat{\mathcal{O}}_{D_J},
$$

where $\theta_{X} = \lim_{G \in G} \mathcal{O}_{X_{G_{\text{ann}}} \otimes \mathbb{Q}_p}^G \mathcal{O}_{D_J}$ and $\mathcal{O}_{D_J}$ are the $G$-smooth functions of $\tilde{X}$ and $\tilde{D}_J := \tilde{X} \times_X D_J$ respectively, and the completed tensor product is as ind-Banach sheaves.

**Proof.** We want to show that the natural map

$$
\left[\begin{array}{c} \theta_{X} \end{array} \right]_{X=\text{Lie } G_{\text{ket}}} \otimes_\mathbb{Q}_p \widehat{\mathcal{O}}_{D_J} \to R\eta_{\infty,*}(C^{la}(\text{Lie } G, \mathbb{Q})_{\text{1, ket}} \otimes_\mathbb{Q}_p \widehat{\mathcal{O}}_{D_J})
$$

is an equivalence, it is clearly an isomorphism on $H^0$, so it suffices to prove the vanishing of higher cohomology groups. By Theorem 3.4.1 it suffices to prove the case $J = \emptyset$, namely $D_J = X$. By hypothesis the map

$$(3.13) \quad \theta_X : (\text{Lie } G_{\text{ket}})_{\mathbb{Q}_p} \otimes_\mathbb{Q}_p \widehat{\mathcal{O}}_X \to \Omega_1^X(\log) \otimes_\mathbb{Q}_p \widehat{\mathcal{O}}_X(-1)$$

is surjective. Then, by localizing $X$ if necessary, we can assume that $\Omega_1^X(\log)^\vee(1)$ has a basis $\underline{x}_1, \ldots, \underline{x}_d$, and that there is a complementary basis $\underline{\tau}_{d+1}, \ldots, \underline{\tau}_g$ of $(\text{Lie } G_{\text{ket}})_{\mathbb{Q}_p} \otimes_\mathbb{Q}_p \widehat{\mathcal{O}}_X$ via the inclusion of the dual of the (3.13). Then, we have a basis $\{\underline{x}_1, \ldots, \underline{x}_d, \underline{\tau}_{d+1}, \ldots, \underline{\tau}_g\}$ of $(\text{Lie } G_{\text{ket}})_{\mathbb{Q}_p} \otimes_\mathbb{Q}_p \widehat{\mathcal{O}}_X$, and the algebra $\mathcal{F} := C^{la}(\text{Lie } G, \mathbb{Q})_{\text{1, ket}} \otimes_\mathbb{Q}_p \widehat{\mathcal{O}}_X$ can be written as the colimit

$$(3.14) \quad \lim_{h \to \infty} \left[\begin{array}{c} \underline{x}_1^\vee \\ p^h \\
\vdots \\
\underline{x}_d^\vee \\ p^h \\
\underline{\tau}_{d+1}^\vee \\ p^h \\
\vdots \\
\underline{\tau}_g^\vee \\ p^h \end{array} \right]$$

where $\{\underline{x}_i^\vee, \underline{\tau}_i^\vee\}$ is the dual basis. In the presentation (3.14) the Sen operators $\underline{x}_1, \ldots, \underline{x}_d$ act by left derivations, we deduce by the Poincaré lemma that

$$R\Gamma(\theta_\mathcal{F}, \mathcal{F}) = \lim_{h \to \infty} \left[\begin{array}{c} \underline{x}_1^\vee \\ p^h \\
\vdots \\
\underline{x}_d^\vee \\ p^h \\
\underline{\tau}_{d+1}^\vee \\ p^h \\
\vdots \\
\underline{\tau}_g^\vee \\ p^h \end{array} \right],$$

and so by Theorem 3.3.2

$$R\eta_{\infty,*} C^{la}(\text{Lie } G, \mathbb{Q})_{\text{1, ket}} \otimes_\mathbb{Q}_p \widehat{\mathcal{O}}_X = v_\ast \mathcal{F}^{0, \mathcal{F} = 0} = \left[\begin{array}{c} \theta_{X} \end{array} \right]_{X=\text{Lie } G_{\text{ket}}} \otimes_\mathbb{Q}_p \widehat{\mathcal{O}}_{D_J},$$

as wanted. $\square$

We deduce that the action of $\text{Lie } G$ on the locally analytic vectors of the tower vanishes on the geometric Sen operators, obtaining Corollary 1.0.5.
Corollary 3.4.6. The action of $\mathcal{O}_{\hat{\mathcal{X}}}^{la} \otimes \operatorname{Lie} G$ on $\mathcal{O}_{\hat{\mathcal{X}}}^{la}$ by derivations vanishes when restricted to the image of the Sen map $\operatorname{Sen}_{\hat{\mathcal{X}}}$. In other words, the locally analytic vectors of the completed structural sheaf over $\hat{\mathcal{X}}$ are killed by the differential operators defined by the Sen map $\operatorname{Sen}_{\hat{\mathcal{X}}}: \Omega^1_{\hat{\mathcal{X}}}(\log)^\vee \otimes \mathcal{O}_{\hat{\mathcal{X}}}^{la}(1) \to \mathcal{O}_{\hat{\mathcal{X}}}^{la} \otimes \operatorname{Lie} G$.

Proof. By Theorem 3.4.5 we have that \( v \) vectors of the completed structural sheaf over $G$ where the action of Lie $G$ has a section given by evaluation at $g$. By Theorem 3.5.1 we have that $\hat{\mathcal{X}}$ is a Galois $K$-torsor over $\mathcal{X}$ over $X$, namely, the hypothesis BUN is only used to show the concentration in degree 0 of the derived locally analytic vectors of $\mathcal{O}_{\hat{\mathcal{X}}}$.

Remark 3.4.7. Corollary 3.4.6 holds for any $G$-torsor $\hat{\mathcal{X}}$ over $X$, namely, the hypothesis BUN is only used to show the concentration in degree 0 of the derived locally analytic vectors of $\mathcal{O}_{\hat{\mathcal{X}}}$.

3.5. Relation with the $p$-adic Simpson correspondence. We finish this section with the relation between the geometric Sen operator and the $p$-adic Simpson correspondence. Let us recall the following result.

Theorem 3.5.1. ( [LZ17] Theorem 2.1 and [DLLZ23] Theorem 3.2.4). Let $X$ be an fs log smooth adic space over $\operatorname{Spa}(K, K^+)$ and let $\mathbb{L}$ be a pro-Kummer-étale $\mathbb{Q}_p$-local system admitting a lisse lattice $\mathbb{L}^0 \subset \mathbb{L}$. Let $\mathcal{O}_{\log} = \mathfrak{g}^0 \mathcal{P}_{\log}$ be the Hodge-Tate period sheaf, and $\nu: X_{\mathbb{C}, \prokét} \to X_{\mathbb{C}, \két}$ be the projection of sites. Then $\mathcal{H}(\mathbb{L}) := R\nu_!(\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\log})$ is a Galois-$K$-equivariant log Higgs bundle concentrated in degree 0. Let $\theta$ denote the Higgs field of $\mathcal{H}(\mathbb{L})$. Then one has

$$R\nu_! \mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\hat{\mathcal{X}}} = R\Gamma(\theta, \mathcal{H}(\mathbb{L})).$$

To reprove this theorem with our theory we first need to compute the geometric Sen operator of $\mathcal{O}_{\log}$.

Proposition 3.5.2. The geometric Sen operator of

$$\mathcal{O}_{\log} = \mathfrak{g}^0 (\mathcal{P}_{\log}) = \varprojlim_n \operatorname{Sym}^n(\mathfrak{g}^1 \mathcal{P}_{\log})$$

is given by

$$\mathcal{O}_{\log} \xrightarrow{\nabla} \mathcal{O}_{\log}(-1) \otimes_{\mathcal{O}_{\hat{\mathcal{X}}}} \Omega^1_{\log},$$
where $\nabla$ is the residual connection of $\nabla : \mathcal{O}_{\text{dR,log}} \to \mathcal{O}_{\text{dR,log}} \otimes_{\partial_X} \Omega^1(\log)$.

Proof. It suffices to identify $\theta_{\mathcal{O}_{\text{log}}}$ with $-\Delta$ locally on $X$, we can then assume that $X$ has toric coordinates $\psi : X \to \mathbb{S}^{(d-e)}$. Let $X_{C,\infty} = X \times_{\mathcal{O}(\mathbb{S}^{(d-e)},\mathbb{S}^{(d-e)})} \mathbb{S}^{(d-e)}$. Then, by [DLLZ19, Proposition 2.3.15] we have a presentation

$$\mathcal{O}_{\text{log}}|_{X_{C,\infty}} = \mathcal{O}_{X_{C,\infty}} \left[ \frac{\log T^{-1}[T^b]}{t}, \frac{\log S^{-1}[S^b]}{t} \right],$$

where $T^b = (T^b)^{\infty}_{n \in \mathbb{N}}$, $S^b = (S^b)^{\infty}_{n \in \mathbb{N}}$, and $t = \log(\epsilon)$ with $\epsilon = (\zeta^p)^{n \in \mathbb{N}}$. We see from this presentation that

$$\nabla (\log T^{-1}[T^b]) = -\frac{dT}{T} \text{ and } \nabla (\log S^{-1}[S^b]) = -\frac{dS}{S}.$$

On the other hand, a direct computation shows that

$$\mathcal{O}_{\text{log}} = C^\text{pol}(\Gamma, \mathbb{Q}_p) \ast_2 \kappa_{\mathcal{O}_p} \hat{\partial}_X,$$

where $C^\text{pol}(\Gamma, \mathbb{Q}_p)$ is the space of polynomial functions of $\Gamma$ endowed with the right regular action. This implies that

$$\theta_{\mathcal{O}_{\text{log}}} (\log T^{-1}[T^b]) = \frac{dT}{T} \text{ and } \theta_{\mathcal{O}_{\text{log}}} (\log S^{-1}[S^b]) = \frac{dS}{S},$$

proving that $\theta_{\mathcal{O}_{\text{log}}} = -\nabla$ as wanted. \hfill \Box

Proof of Theorem 3.5.1. We first need to make a construction. Let us suppose without loss of generality that $L^0$ is of rank $n$. Define the $\text{GL}_n(\mathbb{Z}_p)$-torsor

$$\widetilde{X} := \text{Isom}(\mathbb{Z}_p^n, L^0).$$

Thus, $L$ is constructed from the standard representation of $\text{GL}_n$ via the torsor $\widetilde{X}$. In particular, by Theorem 3.3.4 $L \otimes_{\mathbb{Q}_p} \hat{\partial}_X$ has a Sen operator $\theta_L$ arising from a map of pro-Kummer-étale sheaves

$$\theta_{\widetilde{X}} : (\mathcal{G}_L)^{\kappa_{\mathcal{O}_p}} \otimes_{\mathcal{O}_p} \hat{\partial}_X \to \Omega^1_{\text{log}}(\log) \otimes_{\partial_X} \hat{\partial}_X(-1),$$

or equivalently a map

$$\text{Sen}_{\widetilde{X}} : \Omega^1_{\text{log}}(\log)^{\vee} \otimes_{\partial_X} \hat{\partial}_X(1) \to \mathfrak{g}_L \otimes_{\mathcal{O}_p} \hat{\partial}_X.$$

On the other hand, by Proposition 3.5.2 the Sen operator of $\mathcal{O}_{\text{log}}$ is $-\nabla$. Thus, by Theorem 3.3.2 (2) one gets that

$$R^1\nu_* (L \otimes_{\mathcal{O}_p} \mathcal{O}_{\text{log}}) = \nu_* H^1(-\theta_\mathcal{L} \otimes \mathcal{O}_{\text{log}}, L \otimes_{\mathcal{O}_p} \mathcal{O}_{\text{log}}),$$

where $\theta_\mathcal{L} \otimes \mathcal{O}_{\text{log}} = \theta_\mathcal{L} \otimes \text{id}_{\mathcal{O}_{\text{log}}} - \text{id}_L \otimes \mathcal{O}_{\text{log}}$. We want to show that $R^i\nu_* (L \otimes \mathcal{O}_{\text{log}}) = 0$ for $i > 0$ and that it is a vector bundle of rank $n$ for $i = 0$. We need the following lemma, which is the Lie algebra incarnation of [LZ17, Lemma 2.15].

Lemma 3.5.3. The image of $\text{Sen}_{\widetilde{X}} : \Omega^1_{\text{log}}(\log)^{\vee} \otimes_{\partial_X} \hat{\partial}_X(1) \to \mathfrak{g}_L \otimes_{\mathcal{O}_p} \hat{\partial}_X$ is contained in a nilpotent subalgebra.
Proof. Since \( \mathbb{L} \) is obtained by the standard representation of \( \mathfrak{gl}_n \), it is enough to prove that the action of \( \text{Sen}_{\tilde{X}} \) on \( \mathbb{L} \otimes_{\mathbb{Q}_p} \tilde{\mathcal{O}}_X \) is nilpotent. The coefficients of the characteristic polynomial of \( \text{Sen}_{\tilde{X}} \) are given by Galois equivariant maps

\[
\sigma_i : \Omega^i_X(\log)^\vee \otimes_{\mathcal{O}_X} \tilde{\mathcal{O}}_X(1) \to \operatorname{End}_{\mathcal{O}_X}(\bigwedge^i \mathbb{L} \otimes_{\mathbb{Q}_p} \tilde{\mathcal{O}}_X) \xrightarrow{\text{Tr}} \tilde{\mathcal{O}}_X.
\]

This forces \( \sigma_i = 0 \) for all \( i = 1, \ldots, n \), proving that \( \text{Sen}_{\tilde{X}} \) is nilpotent.

Now, knowing that the action of \( \text{Sen}_{\tilde{X}} \) is nilpotent on \( \mathbb{L} \otimes_{\mathbb{Q}_p} \tilde{\mathcal{O}}_X \), one can show by taking local perfectoid toric coordinates \( \psi \) with Galois group \( \Gamma \cong \mathbb{Z}_p(1)^d \), that the action of a basis of Lie \( \Gamma \) on \( \mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}} \) can be integrated, proving that \( R\Gamma(\theta_{\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}}}, \mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}}) = (\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}})^{\theta_{\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}}}=0} \), so that

\[
\mathcal{H}(\mathbb{L}) = \nu_*(\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}})^{\theta_{\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}}}=0}.
\]

The fact that \( \mathcal{H}(\mathbb{L}) \) is a vector bundle of rank \( n \) can be deduced by a more careful study of the Sen module \( S_\theta(\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}}) \) in the local coordinates \( \psi \). Indeed, in the local coordinates we can reconstruct \( \mathcal{O}_{\text{log}} \) as \( C^\text{pol}(\Gamma, \mathbb{Q}_p)_\ast \otimes_{\mathbb{Q}_p} \tilde{\mathcal{O}}_X \), where \( C^\text{pol}(\Gamma, \mathbb{Q}_p)_\ast \) is the space of polynomial functions arising from the variables of \( \Gamma \cong \mathbb{Z}_p(1)^d \), endowed with the right regular action. Then, we have that

\[
S_\theta(\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}}) = S_\theta(\mathbb{L} \otimes_{\mathbb{Q}_p} \tilde{\mathcal{O}}_X) \otimes_{\mathbb{Q}_p} C^\text{pol}(\Gamma, \mathbb{Q}_p)_\ast \tilde{\mathcal{O}}_X,
\]

and by Lemma 3.5.3 the action of \( \Gamma \) on \( S_\theta(\mathbb{L} \otimes_{\mathbb{Q}_p} \tilde{\mathcal{O}}_X) \) is nilpotent. This implies that the action of \( \Gamma \) on \( S_\theta(\mathbb{L} \otimes_{\mathbb{Q}_p} \tilde{\mathcal{O}}_X) \) is locally unipotent, and therefore that after some finite Kummer-étale cover it is actually unipotent. But having an unipotent action is the same as the \( \Gamma \)-invariants of \( S_\theta(\mathbb{L} \otimes_{\mathbb{Q}_p} \tilde{\mathcal{O}}_X) \otimes_{\mathbb{Q}_p} C^\text{pol}(\Gamma, \mathbb{Q}_p)_\ast \), to be of rank \( n \). This shows that \( \mathcal{H}(\mathbb{L}) \) is, after a finite Kummer-étale cover, a vector bundle of rank \( n \). By Kummer-étale descent, \( \mathcal{H}(\mathbb{L}) \) arises from a vector bundle of rank \( n \) over \( X \).

On the other hand, the Higgs field \( \theta \) of \( \mathcal{H}(\mathbb{L}) \) is defined as the projection by \( \nu_* \) of the Higgs field \( \text{id}_\mathbb{L} \otimes \nabla : \mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}} \to \mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}} \otimes_{\mathcal{O}_X} \Omega^1_X(\log)(-1) \). Finally, the natural equivalence

\[
R\nu_*(\mathbb{L} \otimes_{\mathbb{Q}_p} \tilde{\mathcal{O}}_X) = \mathcal{H}(\mathbb{L})
\]

arises from the projection by \( \nu_* \) of the natural equivalence

\[
\mathbb{L} \otimes \tilde{\mathcal{O}}_X \cong [\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}} \to \mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}} \otimes \Omega^1_X(\log)(-1) \to \cdots \to \mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{log}} \otimes \Omega^d_X(\log)(-d)]
\]

\( \square \)

Remark 3.5.4. The previous argument can be extended to any \( \tilde{\mathcal{O}}_X \)-vector bundle \( \mathcal{F} \). Indeed, we can always find locally Kummer-étale on \( X \) a lattice \( \mathcal{F}^+ \subset \mathcal{F} \) such that \( \mathcal{F}^+/p = \mathbb{Q}_\ell \bigoplus_{\ell \mid v} \mathcal{O}_X^+ / p \) as pro-Kummer-étale sheaves. By Theorem 3.3.4, the sheaf \( \mathcal{F} \) admits a geometric Sen operator \( \theta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}(-1) \otimes \Omega^1_X(\log) \) computing its cohomology. Then, since \( X \) is defined over a discretely valued field, the proof of Lemma 3.5.3 shows that \( \theta_{\mathcal{F}} \) is actually a nilpotent operator. This would prove in particular that Theorem 3.5.1 holds not just for local systems but for any \( \mathcal{O}_X \)-vector
bundle in the case when \( X \) is defined over a discretely valued field. We thank Ben Heuer for pointing this out.

Finally, let us mention the relation with the work of Wang [Wan23]. Let \( X \) be a rigid analytic space over \( \mathbb{C}_p \) admitting a liftable good reduction \( \tilde{X} \) over \( \mathcal{O}_{\mathbb{C}_p} \) (this means that \( \tilde{X} \) admits a lifting over \( A_{\text{inf}}/\xi^2 \) where \( \xi = (|\epsilon| - 1)/(|\epsilon^p| - 1) \)). Recall the following theorem of Wang

**Theorem 3.5.5** ([Wan23 Theorem 5.3]). Let \( \mathcal{O}_{\mathbb{C}_p}^{\dagger \text{log}} \) denote the overconvergent Hodge-Tate period sheaf of Wang. Let \( a \geq 1/(p - 1) \) and \( \nu : X_{\text{prokét}} \to X_{\text{két}} \) be the projection of sites. Then the functor

\[
\mathcal{H}(L) := \nu_*(L \otimes_{\tilde{O}_X} \mathcal{O}_{\mathbb{C}_p}^{\dagger \text{log}})
\]

induces an equivalence from the category of \( a \)-small generalized representations to the category of \( a \)-small Higgs bundles.

We do not pretend to give a new proof of this statement, instead let us translate some of the main players in terms of the language used in this paper. An \( a \)-small generalized representation of rank \( l \) is a locally free \( \tilde{O}_X \)-module \( L \) admitting a lattice \( L_0 \) such that there is \( b > a + \text{val}(\rho_K) \) with \( L^0/p^b = \phi \oplus \Gamma \) (\( \rho_K \) being an element in \( m_{\mathbb{C}_p} \) depending on the ramification of a discretely valued subfield).

In particular, this is a relative locally analytic \( \tilde{O}_X \)-sheaf as in Definition 3.2.1. The way how Wang constructs the sheaf \( \mathcal{O}_{\mathbb{C}_p}^{\dagger \text{log}} \) is by considering a particular lattice of the Faltings extension provided by the lifting of \( \tilde{X} \) to \( A_2 \), cf. [Wan23 Corollary 2.19]. Locally on coordinates, the ring \( \mathcal{O}_{\mathbb{C}_p}^{\dagger \text{log}} \) is nothing but a suitable completion of a polynomial algebra into an overconvergent polydisc of radius \( |\rho_K| \) (cf. [Wan23 Theorem 2.27]). The \( a \)-smallness condition is a finite rank version of the relative locally analytic condition, where one imposes a fixed radius of analyticity. Finally, the decompletion used by Wang in [Wan23, §3.1] is the integral version of the decompletion provided by Berger-Colmez axiomatic Sen theory [BC08].

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