On spectral cover equations in Simpson integrable systems

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Abstract

Abstract. Following Simpson we consider the integrable system structure on the moduli spaces of Higgs bundles on a compact Kähler manifold $X$. We propose a description of the corresponding spectral cover of $X$ as the fiberwise projective dual to a hypersurface in the projectivization $\mathbb{P}(T_X \oplus O_X)$ of the tangent bundle $T_X$ to $X$. The defining equation of the hypersurface dual to the Simpson spectral cover is explicitly constructed in terms of the Higgs fields.

1 Introduction

The notion of a spectral curve, or, more precisely, a family of spectral curves plays an important role in the theory of integrable systems. Thus the phase space of an integrable system is interpreted as a total space of a family of the Lagrangian abelian varieties identified with the Jacobian varieties of the corresponding spectral curves. This provides effective means to explicitly solve the integrable system via geometry of divisors on spectral curves. The standard construction of the families of spectral curves is via characteristic polynomials of the Lax operators depending on the spectral parameter.

In original constructions the spectral parameter is a coordinate function on an underlying elliptic curve or its degeneration, and the spectral curve is a finite cover of the underlying curve realized as a subvariety of its cotangent bundle. More general construction was proposed by Hitchin [3]. The phase space of the integrable system is identified with the moduli space of Higgs bundles, a certain degeneration of the moduli space of complex $G$-bundles over an arbitrary algebraic curve $\Sigma$. The corresponding spectral curve is given by a finite cover of $\Sigma$ realized as a hypersurface in the cotangent bundle $T^*\Sigma$, defined by the characteristic polynomial of the Higgs field on $\Sigma$. 

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One of the important applications of the spectral curve construction is to the method of the quantum separation of variables [Sk1], [Sk2]. In this approach the equation defining spectral curve as a hypersurface in $T^*\Sigma$ transforms into the defining equation for corresponding quantum integrable system eigenfunctions in the separated variables. The Hitchin integrable systems, both classical and quantum, are much studied and have found many applications in representation theory, quantum field theory and string theory.

A higher-dimensional generalisation of the integrable systems was proposed by Simpson [S1]. The phase space of the corresponding integrable system is the moduli space of Higgs bundles on a Kähler manifold $X$ and the role of the spectral curve is played by a spectral cover of $X$ realized as a subvariety of the total space of some vector bundle over $X$. A particularly interesting choice of the vector bundle is given by the cotangent bundle $T^*X$ to $X$. By the analogy with the Hitchin description of the spectral curves, it is possible to provide a set of polynomial equations describing the spectral cover of $X$ as a subvariety of $T^*X$. However, in the higher-dimensional case the corresponding set of equations turns out to be highly overdetermined. This might potentially be a problem in the constructive approach to finding explicit solutions of the corresponding integrable system.

In this note we point out a curious property of the Simpson spectral cover. It is projectively dual to a hypersurface defined by a single explicit equation. More precisely - consider the projective compactification $\mathbb{P}(\mathcal{T}_X^\vee \oplus \mathcal{O})$ of the holomorphic cotangent bundle $\mathcal{T}_X^\vee$. We propose a description of the Simpson spectral cover in $\mathbb{P}(\mathcal{T}_X^\vee \oplus \mathcal{O})$ via fiberwise projective dual to a hypersurface in the projective compactification $\mathbb{P}(\mathcal{T}_X \oplus \mathcal{O})$ of the tangent bundle $\mathcal{T}_X$. This hypersurface is defined by a single fiberwise polynomial function on $\mathbb{P}(\mathcal{T}_X \oplus \mathcal{O})$ (see equation (4.8)). In the case of $X$ being an algebraic curve this equation essentially coincides with the Hitchin equation of the spectral curve. The formulation in terms of the tangent bundle instead of cotangent bundle bears a resemblance with the Legendre transform between the Hamiltonian and the Lagrangian formulations of classical mechanics. One might hope that the proposed description of the Simpson spectral cover via projective duality will be useful in various applications of the theory of integrable systems associated with higher dimensional algebraic varieties.

The theory of the Simpson integrable systems is an exciting area of research expanding the traditional horizons of the theory of integrable systems. For instance, the Simpson integrable systems, similar to the Hitchin systems, allow a non-commutative deformation akin to the Knizhnik-Zamolodchikov connection leading to quantum versions of the spectral covers. The corresponding higher-dimensional analog of the quantum spectral curve shall be an important part of the higher-dimensional holomorphic/chiral quantum field theory, which is an interesting direction to pursue. Note in this regard that the phase spaces of the Simpson integrable systems appear naturally in the description of $D$-brane string backgrounds and thus via dualities appear in various related problems. One might also expect applications of the Simpson integrable systems to some kind of higher dimensional version of the geometric Langlands correspondence.

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2 Hitchin integrable systems

We start by briefly recalling the construction of the integrable systems associated with the holomorphic vector bundles over algebraic curves due to Hitchin [H].

Let $E$ be a rank $n$ complex $C^\infty$ vector bundle over a Riemann surface $\Sigma$. Let $\Omega^{p,q}_\Sigma$ be complex vector bundles of differential forms of the type $(p,q)$ and $\Omega^{p,q}_\Sigma(E) = \Omega^{p,q}_\Sigma \otimes E$. To define a holomorphic vector bundle $E$ associated with $E$ it is enough to pick a differential operator $\partial E : \Omega^{0,0}_\Sigma(E) \to \Omega^{0,1}_\Sigma(E)$ such that $\partial E(s) = (\partial f)s + f\partial E s$, for $s \in \Omega^{0,0}_\Sigma(E)$ and $f \in C^\infty(\Sigma)$. Then the holomorphic sections of $E$ satisfy the equation $\partial E s = 0$. The space of complex structures on the $C^\infty$-bundle $E$ is an infinite-dimensional affine space modeled on $\Omega^{0,1}_\Sigma(\text{End}(E))$. The group $G$ of automorphisms of the vector bundle $E$ acts on the operator $\partial E$ via conjugation $\partial E \to g^{-1}\partial E g$. The moduli space $\mathcal{N}$ of the stable complex structures on a complex $C^\infty$ vector bundle $E$ is obtained by taking the quotient of the space $\mathcal{A}^s$ of the stable complex structures on $E$ by the action of the automorphism group $G$. Recall that the holomorphic vector bundle $E$ is stable iff for every proper subbundle $F \subset E$ one has $\frac{\deg(F)}{\text{rank}(F)} > \frac{\deg(E)}{\text{rank}(E)}$. In general the quotient space $\mathcal{N} = \mathcal{A}^s/G$ is a smooth non-compact space, but for the rank $n$ and the first Chern class $c_1(E)$ being mutually prime the space $\mathcal{N}$ is compact.

The cotangent bundle $T^*\mathcal{N}$ to the moduli space $\mathcal{N}$ can be succinctly described via Hamiltonian reduction of the space of the Higgs bundles with respect to the action of the group of automorphisms $G$. For a rank $n$ complex vector bundle $E$ let us consider the cotangent space $T^*\mathcal{A}^s$ naturally identified with the space of pairs $(\partial E, \Phi)$ of complex structures on $E$ defined by $\partial E$ and $\Phi \in \Omega^{1,0}_\Sigma \otimes \text{End}(E)$. This space has a canonical holomorphic symplectic structure of the cotangent bundle. The Hamiltonian reduction over zero set of moment map with respect to the action of the automorphism group $G$ provides a realization of the cotangent bundle $T^*\mathcal{N}$. Points of $T^*\mathcal{N}$ parametrize the Higgs bundles, i.e. the pairs $(\partial E, \Phi)$ satisfying the equation

$$\partial E \Phi = 0.$$  \hspace{1cm} (2.1)

There exists a more relaxed stability condition of the Higgs bundles (only $\Phi$-stable subbundles are considered) leading to the Hitchin moduli space $\mathcal{M}$ of stable Higgs bundles such that $T^*\mathcal{N} \subset \mathcal{M}$ [H].

The total space of the cotangent bundle $T^*\mathcal{N}$ is naturally a holomorphic symplectic manifold with the holomorphic symplectic structure descending from the $G$-invariant holomorphic symplectic structure

$$\Omega = \int_{\Sigma_0} \text{Tr} \, \delta \tilde{A} \wedge \delta \Phi,$$  \hspace{1cm} (2.2)

on $T^*\mathcal{A}^s$. Here $\tilde{A} \in \Omega^{0,1}_\Sigma(\text{End}(\mathcal{E}))$ are affine coordinates on the space of complex structures on $\mathcal{E}$, i.e. for two complex structures we have $\overline{\partial}_E^2 = \overline{\partial}_E^{(1)} + \tilde{A}^{(2)} - \tilde{A}^{(1)}$ and thus the two-form (2.2) is well-defined. The holomorphic symplectic structure allows an extension on the moduli space $\mathcal{M}$ of stable Higgs bundles.
According to Hitchin [H], the symplectic space $\mathcal{M}$ is an algebraically completely integrable system. The corresponding set of mutually commuting algebraic functions is provided by the Hitchin map

$$\mathcal{M} \longrightarrow \bigoplus_{i=1}^{n} H^0(\Sigma, \text{Sym}^i(\Omega^{1,0}_\Sigma)), \quad (2.3)$$

sending the pair $\left( \partial \mathcal{E}, \Phi \right)$ to the vector $(\text{Tr} \Phi, \text{Tr} \Phi^2, \cdots, \text{Tr} \Phi^n)$. To construct the explicit basis of the algebraic Hamiltonian functions one shall fix a bases $\{\mu_{i,j}\}$, $i = 1, \ldots, n$, $j = 1, \ldots, n_i$, $n_i = \text{dim} H^1(\Sigma, \text{Sym}^{i-1}T)$ in the space $\bigoplus_{i=1}^{n} H^1(\Sigma, \text{Sym}^{i-1}T)$ where $T$ is the holomorphic tangent bundle to $\Sigma$. Then the ring of integrable Hamiltonians is generated by the elements

$$H_{i,j} = \int_{\Sigma} \mu_{i,j} \text{Tr} \Phi^j, \quad i = 1, \ldots, n_i, \quad j = 1, \ldots, n_i. \quad (2.4)$$

Mutual commutativity of the Hamiltonians $(2.4)$ follows from their invariance under the action of the automorphism group $G$ and obvious mutual commutativity considered as functions on the space $T^*\mathcal{A}^s$.

Generic fibers of the Hitchin map $(2.3)$ are compact Lagrangian abelian varieties identified with the Jacobi varieties of the algebraic curves parametrized by the target space of the map $(2.4)$. This family of curves, known as a family of spectral curves of the integrable system, allows the following construction. One constructs a coherent sheaf $\mathcal{E}$ on the total space of $T^*\Sigma$ of the cotangent bundle to $\Sigma$ so that the spectral curve is given by the support of $\mathcal{E}$. To define a coherent sheaf on $T^*\Sigma$ is the same as to define a sheaf of $\pi_*\mathcal{O}_{T^*\Sigma}$-module on $\Sigma$. Here the direct image $\pi_*\mathcal{O}_{T^*\Sigma}$ is the sheaf of the polynomial algebras in one variable over $\mathcal{O}_\Sigma$ with the stalk over a point $z \in \Sigma$ given by the algebra of the polynomial functions over the fiber $T^*_z\Sigma$. Using the Higgs field $\Phi$ we can supply $\mathcal{E}$ with the structure of $\pi_*\mathcal{O}_{T^*\Sigma}$-module. For this it is enough to allow the generator of the polynomial algebra to act as multiplication by the matrix $\Phi_z(z)$ where we locally have $\Phi = \Phi_z(z)dz$. Now the spectral curve associated with $\left( \partial \mathcal{E}, \Phi \right)$ is given by the support of the coherent sheaf $\mathcal{E}$ on $T^*\Sigma$. Let $U \subset \Sigma$ be an open subset over which the Higgs field can be diagonalised with non-coincident eigenvalues. Then the construction above realises an open part of the spectral curve as a non-ramified $n$-sheet cover of $U$. The spectral curve then provides the $n$-sheet cover of $\Sigma$ but, in general, ramified over some points. It is clear that the spectral curve, constructed this way, can be conveniently defined as the space of solutions of the characteristic equation of the Higgs field

$$\det(w - \Phi(z)) = 0, \quad (2.5)$$

where $w$ is a linear coordinate on the fibers of the line bundle $T^*\Sigma$. The coefficients of the spectral cover equation

$$\det(w - \Phi(z)) = \sum_{m=0}^{n} (-1)^m w^{n-m} \text{Tr}_{\wedge^m} \Phi, \quad (2.6)$$

provide another set of generators of the ring of integrable Hamiltonians

$$\tilde{H}_{m,j} = \int_{\Sigma} \mu_{m,j} \text{Tr}_{\wedge^m} \Phi. \quad (2.7)$$
Here we denote $\text{Tr} \wedge^m \Phi$ the trace of $\Phi$ in the $m$-th fundamental representation of $GL_n$ realised in the space of $m$-th exterior powers of the standard representation $\mathbb{C}^n$.

At the end of this short review of the Hitchin construction let us note that the explicit description (2.5) of the spectral curve via the characteristic equation is widely used in the theory of integrable systems and its applications.

3 Simpson integrable systems

A generalization of the Hitchin integrable systems for higher-dimensional base was introduced by Simpson [S1] using the notion of the Higgs bundles in arbitrary dimensions. In the following we consider a particular case of the Simpson’s construction based on Higgs pairs associated with the cotangent bundle of the underlying manifold.

Let $X$ be a $d$-dimensional Kähler manifold $X$ with the Kähelr form $\omega$. Let $E$ be a rank $n$ complex vector bundle over $X$. The integrable complex structure on $E$ can be defined via differential operator $\overline{\partial}_E : \Omega_{X}^{0,0}(E) \rightarrow \Omega_{X}^{0,1}(E)$ such that $\overline{\partial}_E(fs) = (\overline{\partial}_f)s + f\overline{\partial}_E s$, for $s \in \Omega_{X}^{0,0} (\mathcal{E})$ and $f \in C^\infty (X)$ and the integrability condition $\overline{\partial}_E^2 = 0$ holds. Let $\mathcal{E}$ be the corresponding holomorphic vector bundle over $X$. The moduli space of stable integrable complex structures modulo action of the automorphism group $\mathcal{G}$ of the vector bundle is a non-singular and in general a non-compact Kähelr manifold.

The generalized Higgs pair $(\overline{\partial}_E, \Phi)$ is a pair of: 1. an operator $\overline{\partial}_E$ defining integrable complex structure on $E$, and 2. a holomorphic section $\Phi$ of $\Omega_{X}^{1,0} \otimes \text{End}(\mathcal{E})$ which is squared to zero. Thus we have the following defining equations for the Higgs pair $(\overline{\partial}_E, \Phi)$

$$\overline{\partial}_E^2 = 0, \quad \overline{\partial}_E \Phi = 0, \quad \Phi \wedge \Phi = 0.$$  \hspace{1cm} (3.1)

Note that the last equation can be written in components as the commutativity conditions

$$[\Phi_i, \Phi_j] = 0, \quad \Phi = \sum_{i=1}^{d} \Phi_i(z)dz^i.$$  \hspace{1cm} (3.2)

In the following we consider the case of $\mathcal{E}$ having trivial rational Chern classes. In this case to there is one-to-one correspondence between isomorphism classes of the Higgs bundles defined above and the rank $n$ complex local systems on $X$ [S1].

Let $\mathcal{M}$ be the moduli space of stable Higgs bundles considered modulo action of the automorphism group $\mathcal{G}$. The stability condition on the Higgs bundles is given by the standard stability condition for vector bundles with the exception that only $\Phi$-stable subbundles are considered [S1]. The space $\mathcal{M}$ is a non-singular holomorphic symplectic manifold with the holomorphic symplectic structure obtained by the reduction from the holomorphic symplectic structure

$$\Omega = \int_{X} \omega^{d-1} \text{Tr} \delta \bar{A} \wedge \delta \Phi,$$  \hspace{1cm} (3.3)

on the space of pairs $(\overline{\partial}_E, \Phi)$ consisting of not necessary integrable complex structures $\overline{\partial}_E$ and $\Phi \in \Omega_{X}^{1,0} \otimes \text{End}(\mathcal{E})$. Here $\omega$ is the Kähler structure on $X$ and $\bar{A}$ is the affine coordinate
on the space of not necessary integrable complex structures. The reason for the existence of the holomorphic symplectic structure (and moreover the hyperkähler structure) is the non-linear version of the Hodge decomposition on the first non-abelian complex cohomology of X (see [S2] and references therein). The precise construction is analogous to the construction of the holomorphic symplectic structure (and more generally hyperkähler structure) on the moduli space of rank \( n \) complex local systems on \( X \) (see [F] and references in [S2]); moreover, the holomorphic symplectic structure on the moduli space of Higgs bundles can be obtained from the holomorphic structure on the moduli space of complex local systems by degeneration using the identification of two moduli space [S1], [S2]. For a different approach see also [DM].

The moduli space \( \mathcal{M} \) is an algebraically completely integrable system. The analog of the Hitchin map (2.3)

\[
\mathcal{M} \longrightarrow \bigoplus_{i=1}^{n} H^0(X, \text{Sym}^i(\Omega^1_X)),
\]

sends the Higgs field \( \Phi \) to the array \((\text{Tr} \Phi, \text{Tr} \Phi^2, \cdots, \text{Tr} \Phi^n)\) of symmetric \( GL_n \)-invariant polynomial functions. Precisely, let us contract the matrix valued one form \( \Phi \) with a vector field \( v \) on \( X \) and consider traces \( \text{Tr} (\iota_v \Phi)^i \) of its power. This gives degree \( i \) symmetric function of \( v \) and thus an element of \( H^0(X, \text{Sym}^i(\Omega^1_X)) \). The explicit set of Hamiltonians can be described using basis \( \{ \mu_{i,j} \} \), \( i = 1, \ldots, n \), \( j = 1, \ldots, n \) in each vector space \( H^d(X, \text{Sym}^{i-1} T \otimes \Omega^d_X) \) where \( T \) is the tangent bundle to \( X \) (this is similar to Hitchin case, see Section 2). The generators of the ring of integrable Hamiltonians are then given by

\[
H_{i,j} = \int_X \mu_{i,j} \text{Tr} \Phi^i.
\]

The fibers of the projection (3.4) are Lagrangian abelian varieties that are Lagrangian with respect to the symplectic structure on \( \mathcal{M} \). Simpson in [S1] defined an analog of the Hitchin spectral curve given by a covering space \( \tilde{X} \to X \). In good cases these spectral covers are non-singular, and the abelian varieties in the fibration (3.4) are identified with moduli spaces of line bundles on \( \tilde{X} \).

These spectral covers can be represented as subvarieties of the cotangent space \( T^*X \). Precisely they can be realized via supports of coherent sheaf \( \hat{\mathcal{E}} \) on the total space of \( T^*X \) associated with the holomorphic vector bundle \( \mathcal{E} \) corresponding to the holomorphic structure on \( E \) defined by \( \bar{\partial}_E \). This construction straightforwardly generalises the corresponding construction of the spectral curve for the Hitchin integrable system described in the previous Section. To provide a description of \( \hat{\mathcal{E}} \) we shall construct a sheaf of \( \pi_*O_{T^*X} \)-modules over \( X \). Note that \( \pi_*O_{T^*X} \) is a sheaf of polynomial algebras of \( d \)-variables over \( O_X \). Using the Higgs field \( \Phi \) we supply \( \mathcal{E} \) with the structure of \( \pi_*O_{T^*X} \)-module by allowing the generators of the polynomial algebra act as multiplications by the matrices \( \Phi_k(z) \) where we locally have \( \Phi = \sum_{k=1}^{d} \Phi_k(z)dz^k \). Now the spectral cover \( \tilde{X} \) associated with \( (\bar{\partial}_E, \Phi) \) is given by the support of the coherent sheaf \( \hat{\mathcal{E}} \) on \( T^*X \). An open part of the spectral cover is realised as a non-ramified \( n \)-sheet cover of an open subset of \( X \) over which at least one component \( \Phi_k \) of the the Higgs field can be diagonalised with non-coincident eigenvalues. Let us give an explicit local construction of the spectral covering locally over \( X \). Let \( (z^1, \ldots, z^d) \) be local coordinates over a small open subset \( U \) of \( X \) and \( (w_1, \ldots, w_d) \) be local linear coordinates in
the fiber of the cotangent bundle $T^*U$. Suppose that over $U$ the Higgs field can be chosen in the diagonal form

$$
\Phi = \sum_{k=1}^{d} \sum_{a=1}^{n} E_{aa} \Phi_k^a dz^k,
$$

where $E_{ab}$ are elementary $(n \times n)$-matrices $(E_{ab})_{\alpha \beta} = \delta_{aa} \delta_{b\beta}$. Then the support of $\hat{\mathcal{E}}$ is locally a collection of $n$ copies of $U_0$ defined by union of $n$ embeddings of $U$ into $T^*U$

$$(z^1, \ldots, z^d) \mapsto (z^1, \ldots, z^d, \Phi^a_1(z), \ldots, \Phi^a_d(z)), \quad a = 1, \ldots, n. \tag{3.7}$$

This provides a structure of degree $n$ non-ramified covering of $U$.

The above construction of the spectral cover can be easily reformulated in terms of a set of defining characteristic equations (see e.g. [KOP]) generalizing the spectral curve description via characteristic equation (2.5) for the Hitchin integrable systems. Let $(z^1, \ldots, z^d, w_1, \ldots, w_d)$ be local holomorphic coordinates on $T^*X$ and let $v$ be a section of the holomorphic tangent bundle $\mathcal{T}$. Then by construction the linear operator $(\iota_v \Phi - \iota_v w)$ has non-trivial kernel when $(z, w) \in \tilde{X}$ and thus $\det(\iota_v \Phi - \iota_v w) = 0$ on $\tilde{X}$. This shall hold for any vector field $v$. Thus points of the spectral cover shall satisfy the following equation

$$
\det(\Phi - w) = 0, \tag{3.8}
$$

as an element of $n$-th symmetric power of the cotangent bundle. This is a direct analog of (2.5).

The set of equation is overdetermined as the spectral cover is a codimension $d$ subspace in $T^*X$ while the condition (3.8) provides $\dim(S^n \mathbb{C}^d) = \binom{n+d-1}{n}$-equations on $T^*X$. This renders the description (3.8) of the Simpson spectral cover less explicit than the analogous equation (2.5) of the Hitchin spectral cover. In the following we demonstrate that there is a convenient description of the Simpson spectral cover by a single equation but in another space. This construction uses fiberwise projective duality.

4 Spectral cover via projective duality

Let us start by recalling the standard projective duality relation between linear subspaces in complex projective spaces. Let $\mathbb{P}(V)$, $\dim V = d + 1$ be a projective space and $\mathbb{P}(V^*)$ be its dual. We fix dual homogeneous coordinates $(\zeta^0, \ldots, \zeta^d)$ and $(\eta_0, \ldots, \eta_d)$ in $V$ and $V^*$ correspondingly. A linear $k$-dimensional subspace $W_k \subset \mathbb{P}(V)$ can be represented by a $(k+1)$-dimensional subspace of $V$ and thus be described by a set of $d - k$ linear homogeneous equations

$$
\sum_{j=0}^{d} A_{ij}^j \zeta^j = 0, \quad i = 1, \ldots, d - k. \tag{4.1}
$$
For instance the hyperplane in $\mathbb{P}(V)$ is a space of solutions of a linear homogeneous equation
\[
\sum_{j=0}^{d} \alpha_j \xi^j = 0. \tag{4.2}
\]

Linear subspaces generically intersect according to their dimensions, i.e. a linear $k$-dimensional subspace $W_k$ in $\mathbb{P}(V)$ intersects a linear $m$-dimensional subspace $W_m$ over a linear subspace of the dimension $k + m - d$ for $k + m \geq d$.

The projective duality interchanges dimension $k$ linear space in $\mathbb{P}(V)$, $V = \mathbb{C}^{d+1}$, and dimension $(d-k-1)$ linear subspaces in $\mathbb{P}(V^*)$. Let us parametrize points of the $k$-dimensional space $W_k \subset \mathbb{P}(V)$ by $(k+1)$-tuple $(s^0, \ldots, s^k) \in \mathbb{C}^{k+1}$ of variables as follows
\[
\xi^i = \sum_{j=0}^{k} A_{ij} s^j, \quad i = 0, \ldots, d. \tag{4.3}
\]

The corresponding dual $(d - k - 1)$-linear subspace in $\mathbb{P}(V^*)$ is defined by the set of linear homogeneous equations
\[
\sum_{i=0}^{d} \eta_i A_{ij} = 0, \quad j = 0, \ldots, k. \tag{4.4}
\]

In particular to the hyperplanes $H$ in $\mathbb{P}^d(V)$, defined by the linear equation (4.2), corresponds the dual point in $\mathbb{P}(V^*)$ given by the line in $V^*$
\[
L_H = \{(\lambda \alpha_0, \ldots, \lambda \alpha_d) | \lambda \in \mathbb{C}^*\}. \tag{4.5}
\]

This line can be also characterized as a solution of a system of linear homogeneous equations
\[
\sum_{j=0}^{d} \beta_j \eta_j = 0, \quad i = 1, \ldots, d, \tag{4.6}
\]

and (4.2) and (4.6) are related by the consistency condition
\[
\sum_{j=0}^{d} \beta_j \alpha_j = 0, \quad i = 1, \ldots, d. \tag{4.7}
\]

The projective duality respects incidence relations between various linear subspaces. For instance two points belong to the same hyperplane in $\mathbb{P}(V)$ are dual to two hyperplanes in $\mathbb{P}(V^*)$ intersecting at the point dual to the hyperplane in $\mathbb{P}(V)$.

This classical projective duality picture allows various generalisations (see e.g. [GKZ]). In particular, one can define projective duals for non-linear subvarieties in $\mathbb{P}(V)$. Precisely, the projective dual to a subvariety $Z \in \mathbb{P}(V)$ is a subvariety $Z^*$ in $\mathbb{P}(V^*)$ given by the union of duals to all tangent planes to $X$. More generally given a non-linear bundle of projective spaces over $X$ e.g. the projectivisation $\mathbb{P}(\mathcal{E}_X \oplus \mathcal{O})$ of a vector bundle $\mathcal{E}_X$ over the base $X$ one
can consider the bundle of the dual projective spaces \( \mathbb{P}(\mathcal{E}_X^* \oplus \mathcal{O}) \). In this case the classical projective duality holds fiberwise.

The main point of this note is the following statement:

Let \( \mathbb{P}(T_X \oplus \mathcal{O}_X) \) be the projectivization of the tangent bundle \( T_X \) to a compact complex manifold \( X \), \( \dim X = d \), \((z^1, \ldots, z^d)\) be local coordinates on \( X \) and \((\xi^0, \ldots, \xi^d)\) be homogeneous coordinates in the fibers of the projective bundle \( \mathbb{P}(T_X \oplus \mathcal{O}_X) \). Let \((\overline{\partial}_\mathcal{E}, \Phi)\) be a Higgs pair associated with a rank \( n \) holomorphic vector bundle \( \mathcal{E} \) on \( X \) and let the Higgs field be locally written as \( \Phi = \sum_{j=1}^d \Phi_j(z)dz^j \). Consider the hypersurface \( \tilde{X} \) in \( \mathbb{P}(T_X \oplus \mathcal{O}_X) \) defined by the equation

\[
\det(\sum_{j=1}^d \xi_j \Phi_j(z) + \xi^0) = 0. \tag{4.8}
\]

Then the fiberwise projective dual to this hypersurface is the projective compactification of the Simpson spectral cover \( \tilde{\Sigma} \) associated with the Higgs pair \((\overline{\partial}_\mathcal{E}, \Phi)\).

Assume that a subset \( U \subset X \) is such that the Higgs field can be diagonalized

\[
\Phi = \sum_{a=1}^n \sum_{j=1}^d E_{aa} \Phi^a_j(z)dz^j, \tag{4.9}
\]

where \( E_{ab} \) are elementary \((n \times n)\)-matrices \((E_{ab})_{\alpha\beta} = \delta_{\alpha a}\delta_{\beta b}\). Due to (3.2) this is equivalent to the condition that at least one of the Higgs field component is diagonalizable. Then the corresponding hypersurface (4.8) in \( \mathbb{P}(T_X \oplus \mathcal{O}_X) \) is given by the equation

\[
\prod_{a=1}^n (\sum_{j=1}^d \Phi^a_j(z)\xi^j + \xi^0) = 0, \tag{4.10}
\]

and thus is a family of unions of \( n \) hyperplanes

\[
\sum_{j=1}^d \Phi^a_j(z)\xi^j + \xi^0 = 0, \quad a = 1, \ldots, n, \tag{4.11}
\]

in the projective fibers of \( \mathbb{P}(T_X \oplus \mathcal{O}_X) \) parametrized by \( U \subset X \). Applying the projective duality correspondence (4.2), (4.5) between hyperplanes and points we obtain the family of \( n \) points \( a = 1, \ldots, n \)

\[
(\eta_0, \eta_1, \ldots, \eta_d) = (\lambda, \lambda \Phi^a_1(z), \ldots, \lambda \Phi^a_d(z)), \quad \lambda \in \mathbb{C}^*, \tag{4.12}
\]

in the fibers of the dual projective family \( \mathbb{P}(T_X^* \oplus \mathcal{O}_X) \). This indeed defines the spectral cover of \( U \subset X \) as a subset of \( \mathbb{P}(T_X^* \oplus \mathcal{O}_X) \). We expect that this description can be compactified by adding components where the Higgs field is not diagonalizable. This then provides a concise projective dual description (4.8) of the Simpson spectral cover.

Let us stress that the proposed construction (4.8) is trivially connected with the Hitchin spectral cover equation (2.5). Following (4.8) the spectral cover \( \tilde{\Sigma} \) of the algebraic curve \( \Sigma \)
associated with the Higgs pair $\overline{E}_\Phi$ is defined by the fiberwise homogeneous equation in $\mathbb{P}(T_\Sigma \oplus O_\Sigma)$
\begin{equation}
\det(\xi^1 \Phi(z) + \xi^0) = 0, \quad \Phi = \Phi(z)dz,
\end{equation}
(4.13)
and let us consider the diagonalized Higgs field (4.9) over a subspace $U \subset \Sigma$. The dimension of the projective fiber is equal to one and thus the projective duality maps points in the fibers of the projective bundle $\mathbb{P}(T_\Sigma \oplus O_\Sigma)$ into the points of the projective fibers of $\mathbb{P}(T_{\Sigma_0} \oplus O_{\Sigma_0})$ according to the incidence relation (4.7). The incidence relation (4.6) is easily solved so that the equation for the dual curve in $\mathbb{P}(T_{\Sigma_0} \oplus O_{\Sigma_0})$ is given by
\begin{equation}
\det(\eta_1 - \eta_0 \Phi(z)) = 0,
\end{equation}
(4.14)
which is indeed the projective compactification of the Hitchin curve
\begin{equation}
\det(w - \Phi(z)) = 0.
\end{equation}
(4.15)
The Hitchin description (2.5) is recovered in the chart $\eta_0 = 1, \eta_1 = w$.

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