Testing the tetraquark structure for the $X$ resonances in low-lying region

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Assuming four-quark structure for the $X$ resonances in low-lying region, we calculate their masses using the color-spin interaction. In specific, the hyperfine masses of the color-spin interaction are calculated for the possible states in spin-0, spin-1, spin-2 channels. The two states in spin-0 channel as well as the two states in spin-1 channel are diagonalized in order to generate the physical hyperfine masses. By matching the difference in hyperfine masses with the splitting in corresponding hadron masses and using the $X(3872)$ mass as an input, we estimate the masses corresponding to the states $J^{PC}=0^{+}, 1^{++}, 2^{++}$. We find that the masses of two states in $1^{++}$ are close to those of $X(3823)$, $X(3900)$, and the mass of the $2^{++}$ state is close to that of $X(3940)$. For them, the discrepancies are about $\sim 10$ MeV. This may suggest that the quantum numbers of the controversial states are $X(3823) = 1^{++}$, $X(3900) = 1^{++}$, $X(3940) = 2^{++}$. In this work, we use the same inputs parameters, the constituent quark masses and the strength of the color-spin interaction, that have been adopted in the previous work on the $D$ or $B$-meson excited states. Therefore, it was shown that the four-quark structure can be manifested in their excited states. Thus, our results in this work provide a consistent treatment on open- and hidden-charm mesons as far as the four-quark model is concerned.

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I. INTRODUCTION

The exotic resonances, commonly refereed to $XYZ$ particles, are interesting subjects in hadron physics in recent years. This interest was triggered by the measurement of $X(3823)$ back in 2003 by the Belle collaboration\textsuperscript{1} whose existence was further confirmed in the later experiments\textsuperscript{2,3}.

The theoretical issues on $X(3823)$ are mainly on its structure. The traditional picture would be the quarkonium where it is viewed as heavy quark-antiquark bound state, $cc$\textsuperscript{4}. The other picture for $X(3823)$ is a hadronic molecule like $DD^*$\textsuperscript{5,6}, where the two mesons are bounded weakly by hadronic interactions. There is also a hybrid model like the type $ccg$\textsuperscript{7} containing a gluon. This type of the hybrid model is also applied\textsuperscript{8} to the other resonance with higher mass, $Y(4260)$, whose existence is experimentally reported in Ref.\textsuperscript{10}. Another exciting scenario for $X(3823)$ would be a long-sought tetraquark. Along this line, the tetraquark possibility was investigated under a traditional picture based on a diquark-antidiquark form\textsuperscript{11,12} as well as under a more extensive treatment\textsuperscript{13,14}. As far as we know, there is no consensus on the structure among all these possibilities. The recent review given in Ref.\textsuperscript{15} might be useful for current status of $XYZ$ spectroscopy and possible future direction.

Additional resonances reported from later experiments, which have similar masses with $X(3872)$, escalate the interest further because of the expectation that they might share a similar structure. These include $X(3823)$\textsuperscript{10}, $X(3900)$\textsuperscript{8,17,18}, $X(3940)$\textsuperscript{9}, $X(3940)$\textsuperscript{21}. These resonances, including $X(3872)$, constitute low-lying resonances in this $X$ spectrum. Of course, there are other resonances with higher masses, $Y(4260)$\textsuperscript{10}, $Z(4430)$\textsuperscript{21} etc, which provide other exciting areas that should be investigated in future.

One practical problem, in dealing with these resonances especially in low-lying region, is that their quantum numbers are scarcely known. According to the Particle Data Group (PDG)\textsuperscript{22}, the only resonance whose quantum numbers are completely determined is $X(3872)$. The quantum numbers of others in this low-lying region are quite unknown as one can see from Table I. This precisely makes it difficult to construct a realistic model for them. In particular, for $X(3823)^0$, most of the quantum numbers are unknown except its charge conjugation being odd. One interpretation for this state is an accompanying member of $\psi(3770)$ when $\psi(3770)$ is viewed as a $^3D_1$ of $cc$. Namely, $X(3823)^0$ could be a $^3D_2$ state but, without confirming its quantum numbers, the issue is not settled yet. The exotic possibility is still open for this resonance. Therefore, at this stage, the available data on these resonances which can be used to test a model construction are their measured masses and decay modes.

In this work, we investigate the tetraquark possibility for the $X$ resonances in low-lying region. Especially we look for such a possibility from the electrically neutral members, $X(3823)^0$, $X(3872)^0$, $X(3900)^0$, $X(3940)^0$.
by assuming that they belong to the same isospin multiplet. Since $X(3872)$ is isoscalar ($I = 0$), this assumption is equivalent to the statement that the other resonances, $X(3823)^0$, $X(3900)^0$, $X(3940)^0$, are also isoscalar. Of course, according to the $SU(3)_f$ symmetry, one can expect their charged partners (isovector) in the same $SU(3)_f$ multiplet whose existence can support our isoscalar framework more. Currently there is only one candidate for them in the similar mass region, $X(3900)^\pm$, which is not enough to support our isoscalar tetraquark fully. As more or more resonances are updating in this scarcely explored region, one can expect that more isovector resonances might appear in future. One possible explanation for the lack of the isovector resonances in the current PDG might due to their large width, which makes them difficult to be measured. In particular, the isovector resonances can decay to $\bar{c}c + \pi$ which is however hindered by the isospin conservation for the isoscalar $X$ resonances.

We introduce the four-quark states consisting of diquark-antidiquark, $cq\bar{c}\bar{q}$, ($q = u, d$) and test their justification by calculating the masses using the color-spin interaction. The construction of diquark-antidiquark and its application in light-quark sectors can be found in Ref. [22–23]. This type of four-quark model for the $X$ resonances was also studied by Maiani et al. [11, 12]. They introduced this type of wave function and calculated the masses using the color-spin interaction. In their approach, the constituent quark masses as well as the strength of the color-spin interaction are all fitted to the low-lying baryon and meson spectra. What we want to do in this work is to upgrade this approach by implementing a few ingredients.

Firstly, we notice that the color-spin interaction alone is only useful to calculate the mass splittings through the hyperfine mass splittings. The color-spin interaction does not represent the full interaction between two quarks and, therefore, may not be enough to calculate the hadron mass itself. In specific, there are additional potentials such as the color-electric term and the constant shift as will be discussed below. What makes the color-spin interaction special is that, in the mass splittings, the color-spin interaction survives while the additional potentials cancel. Secondly, as we will discuss below, there is a physical issue like mixing in the possible spin states. If one constructs spin states from $cq\bar{c}\bar{q}$, $(q = u, d)$, there are two states in spin-0 and three states in spin-1. Some of them must mix in generating the mass eigenstates, which constitutes another ingredient in this development.

Another thing, which is more important to us, is that this type of approach is closely related to the earlier work [20]. Namely, in Ref. [20] where two authors (H.Kim and M.K.Cheoun) in the present paper are involved, we suggested that most of the $D$ or $B$-meson excited states currently listed in PDG, especially their mass spectrum, can be understood if they are viewed as tetraquarks with the diquark-antidiquark form, $cq\bar{c}\bar{q}$, $(q = u, d, s)$. Using the strength of the color-spin interaction fixed from the mass difference of $D^0_2(2463) - D_0^0(2318)$, we were able to reproduce the masses of other resonances in the excited states of $D$ and $B$ mesons quite successfully. Also our four-quark model provides interesting phenomenology related to decays of spin-1 mesons, which seems to fit nicely with experimental observation. Based on its phenomenological success, we made some predictions for the $D$ and $B$ mesons to be found in future. This approach in Ref. [26] can be straightforwardly applied to the hidden-charm case here. Through this application, one can test the parameters fixed in Ref. [26] which can help to construct a consistent four-quark picture that might be relevant for both, open- and hidden-charm mesons.

The paper is organized as follows. In Sec. III we present four-quark wave functions that could be relevant for the $X$ resonances. The color-spin interaction and its application to the hadron spectroscopy is discussed in Sec. IV. In Sec. IV we present our calculations of the hyperfine masses and discuss their implication in the $X$ spectroscopy. We summarize in Sec. V.

| Meson | $I^C(J^{PC})_f$ | Mass (MeV) | $\Gamma$ (MeV) |
|-------|----------------|------------|---------------|
| $X(3823)^0$ | $? (?^-)$ | 3823.1 | < 24 |
| $X(3872)^0$ | $0^+(1^{++})$ | 3871.7 | < 1.2 |
| $X(3000)^\pm$ | $? (1^+)$ | 3888.7 | 35 |
| $X(3900)^0$ | $? (?)$ | 3904 | - |
| $X(3940)^0$ | $? (?)^*$ | 3942 | 37 |

TABLE I. The $X$ resonances collected from PDG [22] in low-lying regions. The charge, denoted by a superscript in the name of each resonance, is assigned based on its decaying channel. Note, the measured masses in the third column are slightly different from the numbers appearing in the resonance names in the first column. The measured masses in the third column will be used in our analysis.
forms a bound state.

For the other color configuration for the diquark $cq$, namely $6_c$, we do not consider this possibility firstly because this is less compact than the $3_c$ $cq$ in the spin-0 channel. Secondly, including the $6_c$ scenario through the mixing with the $3_c$ doubles the possible states, which do not seem to fit the current $X$ phenomenology. More importantly, the similar tetraquark framework applied to the excited states of $D, D_s, B, B_s$ mesons in Ref. [20], where the $3_c$ diquark is adopted only, provides nice phenomenological consequences, such as mass splittings. Moreover, this approach predicts two spin-1 resonances which fits very well with $D_{s1}^*(2460), D_{s1}^*(2535)$. It is our interest to test whether the similar framework works for the other hadrons like the $X$ resonances.

Instead of this type of diquark-antidiquark picture, one can also consider a four-quark structure of the form $q\bar{q} - c\bar{c}$ because of the expectation that the color-spin interaction can be more attractive for $q\bar{q}$ than for the $3_c$ $cq$. Indeed, the color-spin interaction is most attractive when $q\bar{q}$ is in a state with color singlet and spin zero, i.e., $(1_c, J=0)$. With this $q\bar{q}, q\bar{q}$ and $c\bar{c}$ are both colorless. This leads to a hadronic molecule of meson-meson bound state, which tends to be separate with a long-range interaction.

Now with the other possible color-spin configurations for $q\bar{q}$, $(8_c, J=0), (\bar{8}_c, J=1)$, one can still construct tetraquarks of the form, $q\bar{q} - c\bar{c}$, because now $q\bar{q}$ and $c\bar{c}$ are both colorful. Then the question is whether the $q\bar{q}$ with $(8_c, J=0), (\bar{8}_c, J=1)$ configurations are more compact than the $3_c$ $cq$. One can check this explicitly by calculating the expectation value of the color-spin interaction with respect to the $q\bar{q}$ and the $3_c$ $cq$ states respectively. Using the constituent quark masses adopted in our work, we find that the $3_c$ $cq$ with $J=0$ is most attractive among them. Thus, the diquark $cq$ with the color $3_c$ and spin $J=0$ is appropriate for the tetraquark study.

For the $J=1$ diquark $cq$ with $3_c$, its color-spin interaction is stronger than the $q\bar{q}$ with $(8_c, J=0)$ but weaker than the $q\bar{q}$ with $(\bar{8}_c, J=1)$. In this sense, it is not clear whether the $J=1$ diquark is appropriate for the tetraquark study. However, at the same time, since $c\bar{c}$ is less compact than $q\bar{q}$, it is also not clear whether the composite system, $q\bar{q} - c\bar{c}$, is easier to form than the $cq - q\bar{q}$ when the $J=1$ diquark is involved.

As we stated above, we take the approximate spin independence of the heavy quark limit to assume that the $J=1$ diquark forms a bound state. Note, however, that the calculation based on the diquark-antidiquark basis, $cq\bar{q}\bar{c}$, does not rule out the $q\bar{q} - c\bar{c}$ configuration entirely. Since we calculate the color-spin interactions among all the quark pairs [see Eq. (6) below], the $q\bar{q}$ components with the color-octet as well as the color-singlet are also included in our framework.

In our work, we assume that the $X$ resonances, $X(3823)^0, X(3872)^0, X(3900)^0, X(3940)^0$, are isoscalar. Then the flavor structure would be like $[cu\bar{c}u + cd\bar{c}d]/\sqrt{2}$.

Technically, this is equivalent to considering the configuration of the form, $cu\bar{c}u$, within our framework facilitating the color-spin interaction.

For illustration purpose, we label the four quarks in the state, $cu\bar{c}u$, with the numerical indices 1234. So [12] refers the two quarks in the diquark $cu$ and [34] refers to the two quarks in the antidiquark $\bar{c}\bar{u}$. Using this labeling, we can explain our formalism more clearly on the one hand, and on the other hand, the general expressions involving the indices can be easily extended to other situations in future.

Assuming that all the quarks are in an $S$-wave state, the possible spins for the four-quark states are $J = 0, 1, 2$. If we denote these spin states in terms of the total spin $J$, the diquark spin $J_{12}$, the antidiquark spin $J_{34}$, i.e., $(J, J_{12}, J_{34})$, then we come up with the possible spin configurations for each spin state as follow,

$$
J = 0: |000\rangle, |011\rangle,
J = 1: |111\rangle, |110\rangle, |101\rangle,
J = 2: |211\rangle,
$$

(1)

where all the states have the positive parity by their construction. From these spin configurations, one can investigate how they behave under charge conjugation $[11][12]$. The states in $J = 0$ and $J = 2$ are even under charge conjugation $C$. $|111\rangle$ is odd under $C$. And $|110\rangle$ and $|101\rangle$ transform among each other under $C$. Using these properties, one can classify the spin configurations in terms of the states based on the quantum numbers $J^{PC}$, which then can be identified with physical states easily.

For $J = 2$, there is only one spin configuration which can be denoted as

$$|2^{++}\rangle = |211\rangle. \tag{2}$$

For $J = 0$, we have two spin configurations and these have the same quantum numbers $J^{PC} = 0^{++}$. To distinguish the two states in $|J^{PC}\rangle$ notation, we further label them, using $a$ and $b$, as

$$|0^+_a\rangle = |000\rangle, \quad |0^+_b\rangle = |011\rangle. \tag{3}$$

What we want to point out in this work is that these two are not the mass eigenstates. Indeed, as we will show in Sec. IV the mixing element of the color-spin interaction, $(0^+_a |V|0^+_b)$, is not zero. Thus, the mass eigenstates, which can be recognized as the physical states, should be linear combinations of the two in Eq. (3).

For $J = 1$, we have three spin configurations. We can assign one of them or a combination of them to $X(3872)$. $X(3872)$ has well-determined quantum numbers, i.e., $J^{PC} = 1^{++}$. Only possible way to generate this state with $1^{++}$ from the three configurations is the following symmetric combination $[11][12]$.

$$|1^{++}\rangle = \frac{1}{\sqrt{2}}(|110\rangle + |101\rangle). \tag{4}$$

The antisymmetric combination of $|110\rangle$ and $|101\rangle$ generates the state belonging to $J^{PC} = 1^{-+}$ and the other
spin-1 state, $|111\rangle$, also belongs to $J^{PC} = 1^{-+}$. Thus, we have two states in $J^{PC} = 1^{-+}$ given by

$$|1^+_a\rangle = \frac{1}{\sqrt{2}} (|110\rangle - |101\rangle) , \quad |1^+_b\rangle = |111\rangle ,$$ (5)

where another labeling, $a$ and $b$, has been introduced to distinguish the two. Here in $J = 1$ channel again, these two states are not the mass eigenstates as there is mixing among them through the color-spin interaction. Their linear combinations which diagonalize the hyperfine masses are the mass eigenstates and, therefore, can be identified as the physical states.

### III. COLOR-SPIN INTERACTIONS

The color-spin interaction provides a simple way to calculate the mass splittings among hadrons with the same flavor content, once the wave functions are constructed in terms of quark fields. But, in its application to the hadron masses instead of the splittings, one may need to consider additional terms like color-electric and constant terms which are often neglected in common practices. In this section, we want to look at this aspect more carefully in order to explain why the color-spin interaction works well for the mass splittings but not for the masses themselves.

The color-spin interaction takes the following simple form

$$V = \sum_{i<j} v_0 \lambda_i \cdot \lambda_j \frac{J_i \cdot J_j}{m_i m_j} .$$ (6)

Here $\lambda_i$ denotes the Gell-Mann matrix for the color SU(3), $J_i$ the spin, and $m_i$ the constituent mass of the $i$-th quark. The parameter $v_0$ represents the strength of the color-spin interaction to be fitted from the hadron masses. It is common practice to write down the hadron masses formally by

$$M_H \sim \sum_i m_i + \langle V \rangle ,$$ (7)

where the hyperfine mass $\langle V \rangle$ is the expectation value of the color-spin interaction evaluated with respect to an appropriate hadron wave function of concern. Phenomenologically, the successful aspect of the color-spin interaction is that the mass splitting among hadrons (with the same flavor content) can be estimated simply by the difference in the hyperfine masses

$$\Delta M_H \sim \Delta \langle V \rangle .$$ (8)

This relation can be tested, for instance, in the baryon sector. In this case, one can fix the strength, $v_0$ in Eq. (6), for example, from the $\Delta - N$ mass difference, and using the standard values for the constituent quark masses, $m_u = 330$ MeV, $m_s = 500$ MeV, $m_c = 1500$ MeV, one can calculate the mass differences among other hadrons like $\Sigma - \Lambda$, etc., through Eq. (8). The agreement with actual mass splittings is quite good, with only $\sim 10$ MeV error. (See Table VI in Ref. [26].)

But, if one calculates the actual masses using the formal relation, Eq.(7), the results do not agree with the experimental masses as one can see from the third column in Table II. The disagreement is about 300 MeV for charmed baryons, but for the rest, it is about 100 MeV. So, the disagreement even has a flavor dependence also. If one sticks to the relation Eq.(7) firmly, one may attribute this disagreement to the parameters involved, $v_0$ and the quark masses. One can then adjust these parameters to reproduce the baryon masses and use them to calculate the masses of other hadrons [11]. But our standpoint is that there are physical reasons why Eq.(7) fails in reproducing the masses.

Physically, the interaction between two quarks inside a hadron can have two different sources, one-gluon-exchange potential $V_{OGE}$ and instanton-induced potential $V_{ins}$ [33,34]. The color-spin interaction is a common ingredient in these potentials but there are additional terms, the color-electric type, $\sim \lambda_i \cdot \lambda_j$, and the constant term. Therefore, it may be more natural to think that the disagreement in the masses above may come from these types of interactions instead of adjustment of the parameters involved in Eq.(7).

Effectively, the additional potentials, being the color-electric type and the constant shift, can be parameterized as

$$V_{add} = \sum_{i<j} v_1 \frac{\lambda_i \cdot \lambda_j}{m_i m_j} + v_2 ,$$ (9)

with the additional parameters $v_1$ and $v_2$ to be fitted from some of the baryon masses. Then the baryon mass is given by

$$M_H \sim \sum_i m_i + \langle V \rangle + \langle V_{add} \rangle .$$ (10)

Now by fitting the additional parameters $v_1$ and $v_2$ to the masses of $N$ and $\Lambda$, for example, we obtain other masses given in the fourth column of Table II. The results are better than the ones determined from the color-spin interaction only, though there are still disagreement in the charmed baryons more than 160 MeV or so. One can adjust the parameters at this stage to reproduce the charmed baryon masses using Eq. (10). In particular, by changing $m_c$ from 1500 MeV to 1670 MeV, we come out with much better fits to the charmed baryon masses, namely, $M_{\Sigma_c} = 2450$ MeV, $M_{\Lambda_c} = 2294$ MeV, $M_{\Xi_c^+} = 2508$ MeV. Although the results are nice but, at the same time, we are facing the results that are somewhat sensitive to the input parameters. Then it is questionable whether this new set of parameters can be consistently applied to other hadrons altogether.

What is interesting is that the additional potentials $\langle V_{add} \rangle$ in Eq.(10) cancel in the mass splittings among hadrons with the same flavor content. Since two quarks
inside a baryon are always in the color state $\mathbf{3}_c$, the expectation value of the $\lambda_i \cdot \lambda_j$ term is the same for all the baryons considered in Table III. The constant term is also the same for all the baryons. The flavor dependence comes from the quark masses only. Hence, the relation for the mass splitting, Eq. (9), still holds regardless of the sector as well as in the meson sector, this method relying on the mass splittings can determine the masses quite reliably.

### IV. RESULTS AND DISCUSSION

Now we present our calculation for the hyperfine masses which can be used to estimate the mass splittings among the $X$ resonances. The hyperfine mass is the expectation value of Eq. (4) with respect to the corresponding state that we have introduced in Sec. II. Then the color and spin parts need to be calculated separately. For the color part, since the diquark and antidiquark are in $\mathbf{3}_c, \mathbf{3}_c$ respectively, one can easily evaluate the expectation value of the terms like $\lambda_1 \cdot \lambda_2$ and $\lambda_3 \cdot \lambda_4$. For the other combinations like $\lambda_1 \cdot \lambda_3$ and $\lambda_2 \cdot \lambda_3$, one can rearrange the wave functions from the diquark-antidiquark (|12⟩|34⟩) basis into the |13⟩|24⟩ basis or the (|14⟩|23⟩) basis and evaluate the expectation values straightforwardly. One can take the similar steps to calculate the spin parts also. (See Ref. 24 for technical details.)

Table IV shows our results for the hyperfine masses evaluated for the states, $|0_{a_1}^{++}⟩, |0_{b_1}^{++}⟩, |1^{++}⟩, |1_{a_1}^{--}⟩, |1_{b_1}^{--}⟩, |1_{a_1}^{++}⟩$, as well as the mixing terms appearing in spin-0, spin-1 channels through $⟨0_{a_1}^{++}|V|0_{b_1}^{++}⟩$, $⟨1_{a_1}^{--}|V|1_{b_1}^{--}⟩$. The origin of the mixing is that, even though the diquark (|12⟩) and antidiquark (|34⟩) are in definite color and spin states, the quark pairs like |13⟩, |14⟩, |23⟩, |24⟩, are not in definite color and spin states. The color-spin interaction between two quarks in such a pair gives nonzero contribution in the matrix elements of $⟨0_{a_1}^{++}|V|0_{b_1}^{++}⟩$, $⟨1_{a_1}^{--}|V|1_{b_1}^{--}⟩$. One can explicitly see this from our formulas for the mixing terms in Table IV where only the terms related to such a pair appear. Note, the state $|1^{++}⟩$ can not mix with $|1_{a_1}^{--}⟩$ or $|1_{b_1}^{--}⟩$ because they are in the different states in charge conjugation. Or
TABLE III. The formulas for the hyperfine masses are provided here, which are obtained from expectation values of the color-spin interaction with respect to the states indicated in the first column. The formulas presented here are for a general flavor, that one can explicitly check this in our approach by showing

$$\langle 1^{++}|V|1_{a}^{a^-}\rangle = 0.$$  

In our numerical calculations, the input parameters are $m_c, m_u$, and $v_0$ in Eq. (10). For a consistent treatment with our earlier work on the $D$ and $B$ meson excited states, we take the same parameters, namely $m_c = 1500$ MeV, $m_u = 330$ MeV, $v_0 \sim (-193)^3$ MeV$^3$. In Ref. [26], the strength $v_0$ was fixed by the mass difference $D_2^0(2463) - D_0^0(2318)$ in the four-quark formalism for the $D$ and $B$ meson excited states. This value is slightly different from the one fitted from the $\Delta N$ mass difference in the baryon sector. But $v_0$ becomes larger when it is fitted from the $\rho - \pi$ mass difference. Thus, the extracted value for $v_0$ has some dependence on the number of quarks consisting the hadrons and this is a limitation of the color-spin interaction at the moment. Then the question is which $v_0$ to use in our work. It may be quite realistic to use the $v_0$ determined from the $D$ meson spectrum in Ref. [26] because both formalisms are commonly for four-quark systems. Now, the only step left in getting actual numbers for the hyperfine masses is to identify $m_1 = m_c, m_2 = m_u, m_3 = m_c, m_4 = m_u$ in Table III.

In the spin-0 channel, the hyperfine masses for the two possible states, $|0_{a}^{++}\rangle, |0_{b}^{++}\rangle$ are obtained from the corresponding formulas in Table III. As there are the mixing terms between the two states, one has to diagonalize the matrix to calculate the physical hyperfine masses which then correspond to the physical states. We denote the physical states with the capital letters in the subscripts as $|0_{A}^{++}\rangle, |0_{B}^{++}\rangle$.

| States, $|J^{PC}\rangle$ | Hyperfine mass $\langle J^{PC}| V |J^{PC}\rangle_{q_1q_2q_3q_4}$ |
|------------------------|------------------------------------------------|
| $|0_{a}^{++}\rangle$    | $2v_0 \left( \frac{1}{m_1 m_2} + \frac{1}{m_3 m_4} \right)$ |
| $|0_{b}^{++}\rangle$    | $-\frac{2}{3}v_0 \left( \frac{1}{m_1 m_2} + \frac{1}{m_3 m_4} + \frac{1}{m_1 m_3} - \frac{1}{m_2 m_3} \right)$ |
| Mixing $[|0_{a}^{++}, |0_{b}^{++}\rangle]$ | $\frac{1}{\sqrt{3}}v_0 \left( \frac{1}{m_1 m_3} - \frac{1}{m_2 m_3} + \frac{1}{m_1 m_4} \right)$ |
| $|1^{++}\rangle$        | $\frac{3}{2}v_0 \left( \frac{2}{m_1 m_2} + \frac{2}{m_3 m_4} - \frac{1}{m_1 m_3} + \frac{1}{m_2 m_3} - \frac{1}{m_1 m_4} \right)$ |
| $|1_{a}^{++}\rangle$    | $\frac{3}{2}v_0 \left( \frac{2}{m_1 m_3} + \frac{2}{m_3 m_4} - \frac{1}{m_1 m_3} + \frac{1}{m_2 m_3} + \frac{1}{m_1 m_4} \right)$ |
| $|1_{b}^{++}\rangle$    | $-\frac{2}{3}v_0 \left( \frac{1}{m_1 m_2} + \frac{1}{m_3 m_4} + \frac{1}{m_1 m_3} - \frac{1}{m_2 m_3} - \frac{1}{m_1 m_4} \right)$ |
| Mixing $[|1_{a}^{++}, |1_{b}^{++}\rangle]$ | $\frac{2}{3}v_0 \left( \frac{1}{m_1 m_2} - \frac{1}{m_2 m_3} + \frac{1}{m_1 m_4} \right)$ |
| $|2^{++}\rangle$        | $\frac{2}{3}v_0 \left( \frac{1}{m_1 m_2} + \frac{1}{m_3 m_4} + \frac{1}{m_1 m_3} + \frac{1}{m_1 m_4} \right)$ |

Thus, in this spin-0 channel, the hyperfine masses for physical states, $|0_{A}^{++}\rangle$ and $|0_{B}^{++}\rangle$, are the followings,

$$|0_{A}^{++}\rangle: -58.0 \rightarrow |0_{B}^{++}\rangle: -28.1 \rightarrow |0_{A}^{++}\rangle: -23.1 \rightarrow |0_{B}^{++}\rangle: 0.00 \rightarrow -75.9.$$  

Similarly, using corresponding formulas for the spin-1 channel in Table III we can calculate the hyperfine masses for the two possible configurations, $|1_{a}^{++}\rangle, |1_{b}^{++}\rangle$, as well as their mixing terms. Then after the diagonalization, we obtain the followings.

$$|1_{a}^{++}\rangle: -32.7 \rightarrow |1_{b}^{++}\rangle: -19.9 \rightarrow |1_{a}^{++}\rangle: 19.9 \rightarrow |1_{b}^{++}\rangle: 0.00 \rightarrow -65.9.$$  

Again, we have denoted the physical states with the capital letters in the subscripts, $|1_{A}^{++}\rangle, |1_{B}^{++}\rangle$. Thus, in this spin-1 channel, the hyperfine masses for physical states are the followings,

$$|1_{A}^{++}\rangle: 19.9 \rightarrow |1_{B}^{++}\rangle: -65.9 \rightarrow |1_{A}^{++}\rangle: -19.9 \rightarrow |1_{B}^{++}\rangle: 0.00 \rightarrow -65.9.$$  

There is one more state in spin-1, $|1^{++}\rangle$, and the spin-2 state, $|2^{++}\rangle$, which do not mix with the other states. Their hyperfine masses are calculated to be

$$|1^{++}\rangle: -5.96 \rightarrow |2^{++}\rangle: 52.0 \rightarrow |1^{++}\rangle: -5.96 \rightarrow |2^{++}\rangle: 52.0.$$  

(13)
Now using Eq. (8), the mass splittings among these resonances can be estimated from the differences in the hyperfine masses given in Eqs. (11–13). For example, the mass difference between the states, \( |2^{++}\rangle, |1^{++}\rangle \) is given by \( (2^{++}|V|2^{++}) - (1^{++}|V|1^{++}) = 57.96 \text{ MeV} \). As we have emphasized in Sec. III, the mass splittings calculated this way are quite insensitive to the input parameters and do not suffer from the additional potentials.

Since all these resonances have the same flavor content, we can take the mass of one resonance as an input in order to estimate the other masses. The well-known resonance \( X(3872) \), which can be identified with the state \( |1^{++}\rangle \) because of its quantum numbers, would be the best candidate for the reference mass. Then, combining with the mass splittings, one can estimate all the masses of the states, \( |0_A^{++}\rangle, |0_B^{++}\rangle, |1_A^{+-}\rangle, |1_B^{-+}\rangle, |1^{++}\rangle, |2^{++}\rangle \). For example, the mass of the spin-2 state can be calculated through

\[
M(J = 2) = M_X(38722) + (2^{++}|V|2^{++}) - (1^{++}|V|1^{++}) = 3871.7 + 52 - (5.96) = 3929.7 \text{ MeV}. \quad (14)
\]

Similarly we can estimate the masses of the other states. The states corresponding to the quantum numbers and their calculated masses are summarized as follows,

\[
\begin{align*}
|0_A^{++}\rangle & : 3849.5 \text{ MeV}, \\
|0_B^{++}\rangle & : 3801.8 \text{ MeV}, \\
|1^{++}\rangle & : 3871.7 \text{ MeV} \text{ (input)}, \\
|1_A^{+-}\rangle & : 3897.7 \text{ MeV}, \\
|1_B^{-+}\rangle & : 3811.7 \text{ MeV}, \\
|2^{++}\rangle & : 3929.7 \text{ MeV}. \quad (15)
\end{align*}
\]

If we compare these masses with the \( X \) resonances in Table I, we find that the calculated mass of \( |1_A^{+-}\rangle \) is close to \( X(3890)^0 \) with only 7 MeV difference, and the mass of \( |1_B^{-+}\rangle \), is 11 MeV lower than \( X(3823)^0 \). Also the mass of \( |2^{++}\rangle \) is found to be only 12 MeV lower than the mass of the resonance \( X(3940) \). Thus, if our framework really works, we can identify \( X(3823)^0, X(3900)^0, X(3940) \) with the quantum numbers \( J^{PC} = 1^{-+}, 1^{-+}, 2^{++} \). Of course, future measurements on the quantum numbers can settle the identifying issues raised in this work.

In particular, the quantum number for \( X(3823) \) is certainly not settled. It could be \( ^3D_2 \) state of \( cc \) which can explain a narrow decay width observed experimentally. Our four-quark assignment with \( J^{PC} = 1^{-+} \) leads to a narrow width also based on observation that its fall-apart decay to \( D^*\bar{D} \) is prohibited kinematically. In addition, there is other approach based on a hadronic molecule [33] suggesting that its quantum number \( J^{PC} = 1^{--} \). Eventually it will be important to confirm the quantum number of \( X(3823) \) by measuring for example the angular distribution of \( X(3823) \) and clarify its nature.

As we have already mentioned, in this calculation, we use the same values for the input parameters as in our previous works [28], where we develop the four-quark wave functions for the \( D \) and \( B \) meson excited states. There, we found some of those resonances fit very nicely with our picture. To us, it is quite interesting to see that the masses of the \( X \) resonances come out very close to the experimental values without any parameter tuning at least in the spin-1 and spin-2 channels. This may provide a consistent treatment on open- and hidden-charm mesons within a four-quark framework.

Currently in PDG, there are no resonances corresponding to the states in the spin-0 channel, \( |0_A^{++}\rangle \) and \( |0_B^{++}\rangle \). There could be various reasons for this. One possibility may be due to their broad widths. If they decay through a fall-apart mechanism, the decay channel with low-invariant mass like \( D\bar{D} \) can be open and their decay width becomes broad. If so, these resonances are hard to be measured experimentally. A similar situation can be seen also in the \( B \) meson excited states where there are no resonances observed in the spin-0 channel. (See Table III in Ref. [28].) This was explained in our four-quark model where the spin-0 mesons are expected to have broad widths due to the presence of the decay modes kinematically favorable. But this scenario does not work for the spin-1 and spin-2 channels and there are indeed some sharp resonances in PDG in these channels. It is quite likely that the similar explanation can be applied in the \( X \) resonances. Or alternatively one can look for some other QCD dynamics which might impede the formation of resonances in the spin-0 channel. Certainly more works need to be done in future to clarify this issue.

V. SUMMARY

In this work, we have calculated the masses of the \( X \) resonances in low-lying region by considering them as tetraquark states. The color-spin interaction is used to calculate the mass splittings and, using \( X(3872) \) as an input, we calculate the masses corresponding to the quantum numbers \( J^{PC} = 0^{++}, 1^{-+}, 2^{++} \). We found that masses of the two states in \( 1^{-+} \) and one state in \( 2^{++} \) fit very nicely to \( X(3823), X(3900), X(3940) \) within \( \sim 10 \text{ MeV} \). Future experiments can settle the issues of identifying the resonances by measuring the quantum numbers of the controversial states. We stressed that the color-spin interaction is powerful in generating the mass splittings but not for the mass themselves. To directly calculate the masses of hadrons of concern, the color-spin interaction may not fully represent the potential among two quarks and one needs the additional potentials which, however, cancel in the mass splittings. The input parameters that have been used in the model prediction were taken to be the same as the ones used in the \( D \) and \( B \) meson excited states in the four-quark formalism. Therefore, we believe that our results, if they are confirmed by the measurement of the \( X \) quantum numbers, can provide a consistent picture for the open- and hidden-charm mesons in this four-quark formalism.
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