Yangian Symmetry and Quantum Inverse Scattering Method for the One-Dimensional Hubbard Model

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Abstract

We develop the quantum inverse scattering method for the one-dimensional Hubbard model on the infinite interval at zero density. $R$-matrix and monodromy matrix are obtained as limits from their known counterparts on the finite interval. The $R$-matrix greatly simplifies in the considered limit. The new $R$-matrix contains a submatrix which turns into the rational $R$-matrix of the XXX-chain by an appropriate reparametrization. The corresponding submatrix of the monodromy matrix thus provides a representation of the $Y(su(2))$ Yangian. From its quantum determinant we obtain an infinite series of mutually commuting Yangian invariant operators which includes the Hamiltonian.

Keywords Hubbard model, quantum inverse scattering, Yangian
1 Introduction

Among the exactly solvable 1d quantum systems the Hubbard model has probably the most interesting applications in solid state physics. Its elementary excitations, their dispersion relations and $S$-matrix at half filling [1] have been calculated exactly by use of the coordinate Bethe Ansatz in conjunction with the SO(4) symmetry of the model [2, 3]. The story of this development is long and originates in the seminal paper [4] of Lieb and Wu. A recent overview is offered by the reprint volume [5].

From the point of view of coordinate Bethe Ansatz the one-dimensional Hubbard model appears similar to the fermionic nonlinear Schrödinger model. In fact, Lieb and Wu in their article used this analogy to obtain the Bethe Ansatz equations from Yang’s earlier result [6, 7]. Algebraically, however, the model seems to be more complicated. Nearly 20 years passed before the basic tools of quantum inverse scattering method (QISM), $R$-matrix and $L$-matrix were derived by Shastry [8, 9] and by Olmedilla et al. [10, 11, 12], and it was shown only recently that the $R$-matrix satisfies the Yang-Baxter equation (YBE) [13]. The $R$-matrix and monodromy matrix of the Hubbard model have unusual features. The monodromy matrix is $4 \times 4$ rather than $3 \times 3$, as one might have guessed naively from the fact that there are two levels of Bethe Ansatz equations or from the analogy with the fermionic nonlinear Schrödinger model. It seems to be impossible to find a parametrization of the $R$-matrix, such that it becomes a function of the difference of the spectral parameters. For these reasons an algebraic Bethe Ansatz is difficult and was performed only recently by Ramos and Martins [14].

There is another algebraic structure related to the Hubbard model on the infinite line. As was discovered by Uglov and Korepin [15] the Hubbard Hamiltonian commutes with two independent and mutually commuting representations of the Yangian $Y(su(2))$. Yangians are the quantum groups connected to rational solutions of the YBE [16, 17, 18]. The Yangian invariance of the Hubbard Hamiltonian became likely after the observation that the $S$-matrix of elementary excitations at half filling is essentially a direct sum of two rational solutions of the YBE, each corresponding to the XXX spin chain [19].

There is some hope that the Yangian symmetry might be used to obtain excitation spectrum and $n$-point correlators of the Hubbard model in a way similar to the calculation of these quantities for the XXZ-chain by usage of its $U_q(sl_2)$ symmetry [20]. Such kind of approach
might also be applicable to an extended class of non-nearest-neighbour Hubbard models [21], which have recently been shown to be Yangian symmetric, too [22], and for which a QISM approach is unlikely to exist.

The main concern of this Letter is to show how QISM and Yangian symmetry of the Hubbard model are connected. We benefit from the experience of one of the authors with the fermionic nonlinear Schrödinger model [23, 24]. It turns out that the situation in case of the Hubbard model is to a large extent analogous. The Yangian symmetry reveals, when the model is considered on the infinite interval. Below $R$-matrix and monodromy matrix are obtained as limits from their known counterparts on the finite interval. The $R$-matrix greatly simplifies in the considered limit. The new $R$-matrix contains a submatrix which turns into the rational $R$-matrix of the XXX-chain by an appropriate reparametrization. The corresponding submatrix of the monodromy matrix thus provides a representation of the $Y(\text{su}(2))$ Yangian. This representation is identified as the Yangian representation constructed earlier by Uglov and Korepin using ad hoc methods. From the quantum determinant of the considered submatrix of the monodromy matrix we obtain an infinite series of mutually commuting Yangian invariant operators which is including the Hamiltonian.

2 Quantum Inverse Scattering Method on the Infinite Interval

The Hamiltonian of the one-dimensional Hubbard model is

$$\hat{H} = - \sum_{j,\sigma=\uparrow,\downarrow} (c_{j+1,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{j+1,\sigma}) + U \sum_{j} \left[ (n_{j\uparrow} - \frac{1}{2}) (n_{j\downarrow} - \frac{1}{2}) - \frac{1}{4} \right], \quad (2.1)$$

where $c_{j,\sigma}$ and $c_{j,\sigma}^\dagger$ are annihilation and creation operators of electrons of spin $\sigma$ at site $j$ of a 1d lattice, and $n_{j\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma}$ is the particle number operator. Since we want to study finite excitations over the zero density vacuum $|0\rangle$ of the infinite interval, we normalized the Hamiltonian such that $\hat{H} |0\rangle = 0$.

Starting point for the QISM for the Hubbard model is the exchange relation [11]

$$\mathcal{R}(\lambda, \mu) \mathcal{L}_{m}(\lambda) \otimes_s \mathcal{L}_{m}(\mu) = [\mathcal{L}_{m}(\mu) \otimes_s \mathcal{L}_{m}(\lambda)] \mathcal{R}(\lambda, \mu), \quad (2.2)$$

where $\otimes_s$ denotes the Grassmann direct product

$$[A \otimes_s B]_{\alpha_\gamma, \beta_\delta} = (-1)^{[P(\alpha)+P(\beta)]P(\gamma)} A_{\alpha\beta} B_{\gamma\delta} \quad (2.3)$$
with grading $P(1) = P(4) = 0$, $P(2) = P(3) = 1$. We adopt the expressions for the matrices $\mathcal{R}$ and $\mathcal{L}$ in terms of two parametrizing functions $\alpha(\lambda)$, $\gamma(\lambda)$ from ref. [11]. For later convenience, however, we shift the arguments of $\alpha(\lambda)$ and $\gamma(\lambda)$ by $\frac{\pi}{4}$, such that we simply have $\alpha(\lambda) = \cos \lambda$, $\gamma(\lambda) = \sin \lambda$. The $L$-matrix is

$$\mathcal{L}_m(\lambda) = \begin{pmatrix} -e^{h(\lambda)}m_\uparrow m_\downarrow & -f_{m\uparrow}c_{m\downarrow} & ic_{m\uparrow}f_{m\downarrow} & ic_{m\uparrow}c_{m\downarrow}e^{h(\lambda)} \\ -f_{m\uparrow}c_{m\downarrow} & e^{-h(\lambda)}f_{m\downarrow}m_\downarrow & e^{-h(\lambda)}c_{m\downarrow}c_{m\downarrow} & ic_{m\downarrow}g_{m\downarrow} \\ ic_{m\uparrow}f_{m\downarrow} & e^{-h(\lambda)}c_{m\downarrow}c_{m\downarrow} & e^{-h(\lambda)}g_{m\downarrow}f_{m\downarrow} & g_{m\uparrow}c_{m\downarrow} \\ -ic_{m\uparrow}c_{m\downarrow} & ic_{m\downarrow}g_{m\downarrow} & g_{m\uparrow}c_{m\downarrow} & -g_{m\downarrow}g_{m\downarrow}e^{h(\lambda)} \end{pmatrix}$$

(2.4)

where $f_{m\sigma}(\lambda) = \gamma(\lambda)(1 - n_{m\sigma}) + i\alpha(\lambda)n_{m\sigma}$, $g_{m\sigma}(\lambda) = \alpha(\lambda)(1 - n_{m\sigma}) - i\gamma(\lambda)n_{m\sigma}$, and $h(\lambda)$ is defined as

$$\frac{\sinh 2h(\lambda)}{\sin 2\lambda} = \frac{U}{4}.$$  

(2.5)

Due to space limitations we do not reproduce the $R$-matrix here. It is $16 \times 16$ and contains 36 nonvanishing entries, only ten of which are different modulo signs. The ten different entries are denoted by $\rho_i$, $i = 1, \ldots, 10$, in ref. [11]. They are rational functions of $\alpha(\lambda)$, $\gamma(\lambda)$ and $e^{h(\lambda)}$. We provide a list and some basic formulae which have been used in our calculations in Appendix A. The $(m - n)$-site Hubbard model ($m > n$) is characterized by the monodromy matrix

$$\mathcal{T}_{mn}(\lambda) = \mathcal{L}_{m-1}(\lambda)\mathcal{L}_{m-2}(\lambda) \cdots \mathcal{L}_n(\lambda).$$

(2.6)

It has been shown in ref. [3, 11] that the logarithmic derivative of the graded trace of $\mathcal{T}_{mn}(\lambda)$ at $\lambda = 0$ reproduces the Hamiltonian (2.1) under periodic boundary conditions. Like $\mathcal{L}_m(\lambda)$ the monodromy matrix $\mathcal{T}_{mn}(\lambda)$ satisfies (2.2). In contrast to the classical case [25] a formulation of QISM on the infinite interval [26, 27] has so far only been possible for zero density of elementary particles. This is due to the complicated structure of the finite density (finite band filling) vacua formed by an infinite number of interacting particles. For the Hubbard model, there are four simple vacua, the empty band, the completely filled band and the half filled band with all spins up or all spins down. In the following we will consider the empty band $|0\rangle$ ($c_{m\sigma}|0\rangle = 0$) as reference state. Zero particle density means to consider only states with a finite number of particles in the empty band. Expectation values of the $L$-matrix with respect to this space have a finite limit for $|m| \to \infty$, which formally can be obtained by setting normal ordered products of operators equal to zero [26, 27],

$$\mathcal{L}(\lambda) \to V(\lambda) = \text{diag}(-\gamma(\lambda)^2e^{h(\lambda)}, \alpha(\lambda)\gamma(\lambda)e^{-h(\lambda)}, \alpha(\lambda)\gamma(\lambda)e^{-h(\lambda)}, -\alpha(\lambda)^2e^{h(\lambda)}).$$

(2.7)
$V(\lambda)$ can be used to split off the asymptotics of $\mathcal{T}_{mn}(\lambda)$. We expect the matrix

$$\tilde{T}_{mn}(\lambda) = V(\lambda)^{-m}\mathcal{T}_{mn}(\lambda)V(\lambda)^n$$

(2.8)

to have a finite limit for $m \to \infty$, $n \to -\infty$. This limit,

$$\tilde{T}(\lambda) = \lim_{m,-n \to \infty} \tilde{T}_{mn}(\lambda),$$

(2.9)

will be the monodromy matrix on the infinite interval. To derive an exchange relation for $\tilde{T}(\lambda)$, consider the asymptotics $W(\lambda, \mu)$ of $\mathcal{L}_m(\lambda) \otimes \mathcal{L}_m(\mu)$ for large $|m|$ again by omitting all normal ordered products of Fermi operators. Then $W(\lambda, \mu)^{-m}(\mathcal{T}_{mn}(\lambda) \otimes \mathcal{T}_{mn}(\mu))W(\lambda, \mu)^n$ is expected to have a finite limit. $W(\lambda, \mu)$ is not just the tensor product $V(\lambda) \otimes V(\mu)$. Due to normal ordering there appear additional off-diagonal elements,

$$W(\lambda, \mu)_{12,21} = W(\lambda, \mu)_{13,31} = -i\gamma(\lambda)\gamma(\mu), \quad W(\lambda, \mu)_{14,23} = -W(\lambda, \mu)_{14,32} = -i\gamma(\lambda)\alpha(\mu),$$

$$W(\lambda, \mu)_{24,42} = W(\lambda, \mu)_{34,43} = i\alpha(\lambda)\alpha(\mu), \quad W(\lambda, \mu)_{23,41} = -W(\lambda, \mu)_{32,41} = -i\alpha(\lambda)\gamma(\mu),$$

$$W(\lambda, \mu)_{14,41} = -e^{h(\lambda)+h(\mu)}.$$  

It follows from the exchange relation for the monodromy matrix that

$$\mathcal{R}(\lambda, \mu)W(\lambda, \mu) = W(\mu, \lambda)\mathcal{R}(\lambda, \mu).$$

(2.10)

Using once more the exchange relation and the definition (2.8) of $\tilde{T}_{mn}(\lambda)$ we obtain

$$U_m(\mu, \lambda)^{-1}\mathcal{R}(\lambda, \mu)U_m(\lambda, \mu)[\tilde{T}_{mn}(\lambda) \otimes \tilde{T}_{mn}(\mu)]$$

$$= [\tilde{T}_{nm}(\mu) \otimes \tilde{T}_{nm}(\lambda)]U_n(\mu, \lambda)^{-1}\mathcal{R}(\lambda, \mu)U_n(\lambda, \mu),$$

(2.11)

where

$$U_m(\lambda, \mu) = W(\lambda, \mu)^{-m}[V(\lambda)^m \otimes V(\mu)^m].$$

(2.12)

Postponing the discussion of convergence for while we formally take the limits $m, -n \to \infty$ in (2.11). Then by use of the definitions

$$U_{\pm}(\lambda, \mu) = \lim_{m \to \pm\infty} U_m(\lambda, \mu), \quad \mathcal{R}^{(\pm)}(\lambda, \mu) = U_{\pm}(\mu, \lambda)^{-1}\mathcal{R}(\lambda, \mu)U_{\pm}(\lambda, \mu),$$

(2.13)

we arrive at the exchange relation for the monodromy matrix $\tilde{T}(\lambda)$ on the infinite interval,

$$\mathcal{R}^{(+)}(\lambda, \mu) [\tilde{T}(\lambda) \otimes \tilde{T}(\mu)] = [\tilde{T}(\mu) \otimes \tilde{T}(\lambda)] \mathcal{R}^{(-)}(\lambda, \mu).$$

(2.14)
The matrices \( U_{\pm}(\lambda, \mu) \), their inverses and the \( R \)-matrices \( \tilde{R}^{(\pm)}(\lambda, \mu) \) can be calculated as functions of the \( \rho_i \)’s by utilizing the formulae provided in Appendix A. The non-zero matrix elements of \( U_{\pm}(\lambda, \mu) \) follow as

\[
U_{\pm}(\lambda, \mu)_{\alpha\beta, \alpha'\beta'} = 1, \quad U_{\pm}(\lambda, \mu)_{14,41} = \frac{-\rho_5}{\rho_5 - \rho_4},
\]

\[
U_{\pm}(\lambda, \mu)_{12,21} = U_{\pm}(\lambda, \mu)_{13,31} = \frac{-i\rho_2}{\rho_{10}}, \quad U_{\pm}(\lambda, \mu)_{14,23} = -U_{\pm}(\lambda, \mu)_{14,32} = \frac{i\rho_6}{\rho_5 - \rho_4},
\]

\[
U_{\pm}(\lambda, \mu)_{23,41} = -U_{\pm}(\lambda, \mu)_{32,41} = \frac{i\rho_6}{\rho_5 - \rho_4}, \quad U_{\pm}(\lambda, \mu)_{24,42} = U_{\pm}(\lambda, \mu)_{34,43} = \frac{i\rho_2}{\rho_9},
\]

where \( \rho_i = \rho_i(\lambda, \mu) \). The corresponding matrices \( \tilde{R}^{(\pm)} \) are

\[
\tilde{R}^{(+)}(\lambda, \mu) = \tilde{R}^{(-)}(\lambda, \mu) = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(2.15)

The reader is urged to compare this expression with the \( R \)-matrix on the finite interval \([11]\). Instead of the 36 nonvanishing elements of the original \( R \)-matrix we have only 18 nonvanishing elements here, which brings about simpler commutation relations between elements of the monodromy matrix.

From our experience with the finite interval case we know that the monodromy matrix is of the following block form,

\[
\tilde{T}(\lambda) = \begin{pmatrix}
D_{11}(\lambda) & C_{11}(\lambda) & C_{12}(\lambda) & D_{12}(\lambda) \\
B_{11}(\lambda) & A_{11}(\lambda) & A_{12}(\lambda) & B_{12}(\lambda) \\
B_{21}(\lambda) & A_{21}(\lambda) & A_{22}(\lambda) & B_{22}(\lambda) \\
D_{21}(\lambda) & C_{21}(\lambda) & C_{22}(\lambda) & D_{22}(\lambda)
\end{pmatrix},
\]

(2.16)
where on the finite interval $A(\lambda)$ corresponds to the su(2) Lie algebra of rotations, $D(\lambda)$ corresponds to the $\eta$-pairing su(2) Lie algebra and the blocks $B(\lambda)$ and $C(\lambda)$ are connected to each other by particle-hole and gauge transformations [28]. Using the explicit form of the matrices $\tilde{R}(\pm)$ the exchange relation (2.16) implies
\[
\rho_9(\lambda, \mu)\rho_{10}(\lambda, \mu) - \rho_2(\lambda, \mu)^2 [A_{\alpha\beta}(\lambda), A_{\gamma\delta}(\mu)] = A_{\gamma\delta}(\mu)A_{\alpha\beta}(\lambda) - A_{\gamma\beta}(\lambda)A_{\alpha\delta}(\mu),
\] (2.17)
i.e. the commutation relations between the matrix elements of $A(\lambda)$ are decoupled from the rest of the algebra. This fact will be crucial for the derivation of the $Y(\text{su}(2))$ Yangian representation below.

So far we avoided to comment on the analytic structure of the exchange relation (2.14). This is indeed a delicate point. Along with the calculation of the limits $U_\pm(\lambda, \mu)$ we obtain convergence conditions. The convergence conditions for $U_+(\lambda, \mu)$ and $U_-(\lambda, \mu)$ are complementary. This situation is typical for the QISM on the infinite interval. Thus the first equation in (2.17) is only formal for the present time. We conjecture, however, that analytic continuation in $\lambda$ and $\mu$ respects the exchange relation (2.14) with the possible exception of $\lambda = \mu$ (modulo periods), where singular terms (like $\delta(\lambda - \mu)$) may destroy the first equality in (2.13). We know from our experience with the fermionic nonlinear Schrödinger model [23, 24] that such singular terms are irrelevant for the derivation of a Yangian representation from the exchange relation (2.14).

3 Yangian Symmetry

The $Y(\text{su}(2))$ Yangian [16, 17, 18] algebra is generated by six generators $Q_n^a$ ($n = 0, 1; a = 1, 2, 3$), satisfying the following relations;
\[
\left[Q_0^a, Q_0^b\right] = f^{abc}Q_0^c, \quad (3.1) \\
\left[Q_0^a, Q_1^b\right] = f^{abc}Q_1^c, \quad (3.2) \\
\left[\left[Q_0^a, Q_1^b\right], \left[Q_0^c, Q_0^d\right]\right] + \left[\left[Q_0^c, Q_1^d\right], \left[Q_0^a, Q_0^b\right]\right] = \kappa^2 (A_{abcdef}f^{cdk} + A_{cdkfg}f^{abk})\{Q_0^0, Q_0^1, Q_0^0\}, \quad (3.3)
\]
where $\kappa$ is a nonzero constant, $\sigma^a$ ($a = 1, 2, 3$) are the Pauli matrices, $f^{abc} = i\varepsilon^{abc}$ is the antisymmetric tensor of structure constants of su(2), and $A_{abcdef} = f^{adk}f^{bel}f^{cfm}f^{klm}$. Here and in the following we are using implicit summation over doubly occurring indices. The
The bracket { } denotes the symmetrized product
\[
\{x_1, \ldots, x_m\} = \frac{1}{m!} \sum_{\sigma \in S_m} x_{\sigma 1} \cdots x_{\sigma m}.
\]

Now we will show how to obtain a representation of \(Y(\text{su}(2))\) from the submatrix \(A(\lambda)\) of the monodromy matrix (2.16). Introducing the reparametrization
\[
v(\lambda) = -2i \cot 2\lambda \cosh 2h(\lambda),
\]
the prefactor on the lhs of (2.17) becomes
\[
\frac{\rho_9(\lambda, \mu)\rho_{10}(\lambda, \mu)}{\rho_3(\lambda, \mu)\rho_4(\lambda, \mu) - \rho_2(\lambda, \mu)^2} = \frac{v(\lambda) - v(\mu)}{iU},
\]
and we can write (2.17) in matrix form as
\[
(iU + \{v(\lambda) - v(\mu)\} P) [A(\lambda) \otimes A(\mu)] = [A(\mu) \otimes A(\lambda)] (iU + \{v(\lambda) - v(\mu)\} P).
\]
Here \(P\) is a 4 \times 4 permutation matrix. (3.7) implies that \(A_{\alpha\beta}(\lambda)\) is a generating function of the \(Y(\text{su}(2))\) Yangian. The Yangian generators \(Q_0^a\) and \(Q_1^a\) are the first coefficients in the asymptotic expansion (3.8).

In order to obtain compact expression for \(Q_0^a\) and \(Q_1^a\) we introduced the abbreviations \(\tilde{\sigma}^1 = -\sigma^2, \tilde{\sigma}^2 = \sigma^1, \tilde{\sigma}^3 = \sigma^3\)\]. There are several possibilities to perform the limit \(v(\lambda) \to \infty\). However, we found that only one of these yields finite results for \(Q_0^a\) and \(Q_1^a\). We have to take \(\text{Im}(\lambda) \to \infty\) and further have to choose the proper branch of solution in eqn.(2.5) which determines \(h(\lambda)\) as a function of \(\lambda\). Some of the details of the calculation are given in Appendix B. The final result is
\[
Q_0^a = \frac{1}{2} \sum_j \sigma_{\alpha\beta}^a c_{j,\alpha} c_{j,\beta},
\]
\[
Q_1^a = -i \sum_j \sigma_{\alpha\beta}^a c_{j,\alpha} c_{j,\beta} - \frac{iU}{4} \sum_{i,j} \text{sgn}(j-i) \sigma_{\alpha\beta}^a c_{i,\alpha} c_{j,\beta} c_{i,\gamma} c_{j,\beta}. \tag{3.10}
\]

* The reader should not worry about this notation. It results from our choice of the \(L\)-matrix, which we took from ref. \[11\] to facilitate comparison with earlier work. It is easy to introduce a slight change of the \(L\)-matrix, compatible with the exchange relation, such that \(\tilde{\sigma}^a\) is replaced by \(\sigma^a\) in (3.8).
In our conventions the constant $\kappa$ in (3.3) is equal to $i U$. Comparing (3.9), (3.10) with the result of Uglov and Korepin [15], we find complete equivalence.

$$E_0 = Q_0^1 + i Q_0^2, \quad F_0 = Q_0^1 - i Q_0^2, \quad H_0 = 2Q_0^3,$$

$$E_1 = Q_1^1 + i Q_1^2, \quad F_1 = Q_1^1 - i Q_1^2, \quad H_1 = 2Q_1^3,$$  

(3.11)

(3.12)

where the expressions on the lhs are taken from the paper of Uglov and Korepin.

Associated with the exchange relation (3.7) we can consider the quantum determinant [30, 31],

$$\text{Det}_q A(\lambda) = A_{11}(v(\lambda))A_{22}(v(\lambda) - iU) - A_{12}(v(\lambda))A_{21}(v(\lambda) - iU),$$  

(3.13)

which is in the center of the Yangian and provides a generating function of mutually commuting operators,

$$[\text{Det}_q A(\lambda), A_{\alpha\beta}(\mu)] = 0, \quad [\text{Det}_q A(\lambda), \text{Det}_q A(\mu)] = 0.$$  

(3.14)

The second of these equations is of course a consequence of the first one. Performing again the asymptotic expression in terms of $v(\lambda)$,

$$\text{Det}_q A(\lambda) = 1 + iU \sum_{n=0}^{\infty} \frac{1}{v(\lambda)^{n+1}} I_n,$$  

(3.15)

we obtain $I_0 = 0$, $I_1 = i\hat{H}$, i.e. the Hamiltonian is among the commuting operators. All the conserved operators are Yangian invariant by construction. It will be interesting to investigate their relation to the formerly known conserved quantities [8, 9, 12, 32], which were obtained for the finite periodic model.

In closing this section we shall add a comment. The Hubbard Hamiltonian on the infinite interval is invariant under the transformation

$$c_{j,\downarrow} \rightarrow c_{j,\downarrow}, \quad c_{j,\uparrow} \rightarrow (-1)^j c_{j,\uparrow}, \quad U \rightarrow -U.$$  

(3.16)

The Yangian generators $Q_0^a$ and $Q_1^a$, however, are transformed into a pair of generators $Q_0^{a'}$ and $Q_1^{a'}$ of a second, independent representation of $Y(\text{su}(2))$ [15]. These two representations mutually commute. Therefore they can be combined to a direct sum $Y(\text{su}(2)) \oplus Y(\text{su}(2))$. The reason why we get only one of these representations from our QISM approach is that, in order to perform the passage to the infinite interval, we refer to the zero density vacuum $|0\rangle$. This vacuum has lower symmetry than the Hamiltonian. It is invariant under the $\text{su}(2)$ Lie
algebra of rotations, but does not respect the $\eta$-pairing $\text{su}(2)$ symmetry of the Hamiltonian. A fully $\text{su}(2) \oplus \text{su}(2)$ invariant vacuum would be the singlet ground state at half filling \[19\]. It seems to be yet a formidable task to formulate the QISM with respect to this state.

4 Concluding Remarks and Discussion

We have developed the QISM for the Hubbard model on the infinite interval with respect to the zero density vacuum. The $R$-matrix \[2.15\] thus obtained is greatly simplified in comparison with the $R$-matrix of the finite periodic model. Particularly, it reveals a hidden rational structure, which arises from a certain combination \[3.6\] of the functions $\rho_i$. This structure was discovered earlier by Ramos and Martins \[14\] as part of the exchange relation for the Hubbard model on the finite interval. Note, however, that our reparametrization \[3.6\], which is essentially unique, differs from that given in \[14\]. A comparison is obstructed by the fact that the authors do not expose their parameters $\alpha_j$. Along with the simplified $R$-matrix we obtained the asymptotic expansion \[3.8\] of the submatrix $A(\lambda)$ of the monodromy matrix, which naturally provides a representation of $Y(\text{su}(2))$ and generates an infinite series of mutually commuting Yangian invariant operators including the Hamiltonian.

There is a number of interesting open problems related to the QISM on the infinite interval. The analytic properties of the $R$-matrices $\tilde{R}^{(\pm)}$ \[2.15\] and of the monodromy matrix \[2.9\] deserve further investigations. Only the submatrix $A(\lambda)$ of the monodromy matrix has a limit for $\text{Im}(\lambda) \to \infty$. All other matrix elements diverge. It is therefore not clear at the present stage of investigation how to obtain creation operators of elementary excitations that are compatible with the Yangian generators $Q_0^a$ and $Q_1^a$. Creation operators are indispensable for the discussion of irreducible Yangian representations \[23, 24\].

Another interesting task will be the construction of Dunkl operators \[33\] associated with the Yangian representation discussed in this Letter. Dunkl operators are building blocks of Yangian generators \[33, 33, 23, 24\]. They are useful for the investigation of eigenstates. For the one-dimensional Hubbard model, although several attempts \[34, 36\] have been made, no satisfactory Dunkl operator is known.


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Appendix A Relations Between the Elements of the $R$-Matrix

In this appendix we collect functional relations among the elements of the $R$-matrix, which have been used in the calculation of the matrix $U_{\pm}(\lambda, \mu)$. We begin with the defining relations of the matrix elements, which are

\[
\frac{\rho_1(\lambda, \mu)}{\rho_2(\lambda, \mu)} = e^{i} \alpha(\lambda) \alpha(\mu) + e^{-i} \gamma(\lambda) \gamma(\mu), \quad (A.1)
\]

\[
\frac{\rho_4(\lambda, \mu)}{\rho_2(\lambda, \mu)} = e^{i} \gamma(\lambda) \gamma(\mu) + e^{-i} \alpha(\lambda) \alpha(\mu), \quad (A.2)
\]

\[
\frac{\rho_9(\lambda, \mu)}{\rho_2(\lambda, \mu)} = -e^{i} \alpha(\lambda) \gamma(\mu) + e^{-i} \gamma(\lambda) \alpha(\mu), \quad (A.3)
\]

\[
\frac{\rho_{10}(\lambda, \mu)}{\rho_2(\lambda, \mu)} = e^{i} \gamma(\lambda) \alpha(\mu) - e^{-i} \alpha(\lambda) \gamma(\mu), \quad (A.4)
\]

\[
\frac{\rho_3(\lambda, \mu)}{\rho_2(\lambda, \mu)} = \frac{e^{i} \alpha(\lambda) \alpha(\mu) - e^{-i} \gamma(\lambda) \gamma(\mu)}{\alpha^2(\lambda) - \gamma^2(\mu)}, \quad (A.5)
\]

\[
\frac{\rho_5(\lambda, \mu)}{\rho_2(\lambda, \mu)} = \frac{-e^{i} \gamma(\lambda) \gamma(\mu) + e^{-i} \alpha(\lambda) \alpha(\mu)}{\alpha^2(\lambda) - \gamma^2(\mu)}, \quad (A.6)
\]

\[
\frac{\rho_6(\lambda, \mu)}{\rho_2(\lambda, \mu)} = \frac{e^{-h}[e^{i} \alpha(\lambda) \gamma(\lambda) - e^{-i} \alpha(\mu) \gamma(\mu)]}{\alpha^2(\lambda) - \gamma^2(\mu)}, \quad (A.7)
\]

where $h = h(\lambda) + h(\mu)$, $l = h(\lambda) - h(\mu)$. There are relations among the $\rho_i$ functions;

\[
\rho_1(\lambda, \mu) \rho_4(\lambda, \mu) + \rho_9(\lambda, \mu) \rho_{10}(\lambda, \mu) = \rho_2(\lambda, \mu)^2, \quad (A.8)
\]

\[
\rho_3(\lambda, \mu) \rho_5(\lambda, \mu) - \rho_6(\lambda, \mu)^2 = \rho_2(\lambda, \mu)^2, \quad (A.9)
\]

\[
\rho_1(\lambda, \mu) \rho_5(\lambda, \mu) + \rho_3(\lambda, \mu) \rho_4(\lambda, \mu) = 2\rho_2(\lambda, \mu)^2. \quad (A.10)
\]

We found that there is a set of relations "dual" to (A.1)-(A.7). Introducing a transformation $\Phi$, which keeps $\lambda$ unchanged and substitutes $\mu + \pi$ for $\mu$, we get the following transformation

\[
\frac{\rho_1(\lambda, \mu)}{\rho_2(\lambda, \mu)} = \frac{\rho_1(\lambda, \mu)}{\rho_2(\lambda, \mu)}, \quad (A.11)
\]

\[
\frac{\rho_4(\lambda, \mu)}{\rho_2(\lambda, \mu)} = \frac{\rho_4(\lambda, \mu)}{\rho_2(\lambda, \mu)}, \quad (A.12)
\]

\[
\frac{\rho_9(\lambda, \mu)}{\rho_2(\lambda, \mu)} = \frac{\rho_9(\lambda, \mu)}{\rho_2(\lambda, \mu)}, \quad (A.13)
\]

\[
\frac{\rho_{10}(\lambda, \mu)}{\rho_2(\lambda, \mu)} = \frac{\rho_{10}(\lambda, \mu)}{\rho_2(\lambda, \mu)}, \quad (A.14)
\]

\[
\frac{\rho_3(\lambda, \mu)}{\rho_2(\lambda, \mu)} = \frac{\rho_3(\lambda, \mu)}{\rho_2(\lambda, \mu)}, \quad (A.15)
\]

\[
\frac{\rho_5(\lambda, \mu)}{\rho_2(\lambda, \mu)} = \frac{\rho_5(\lambda, \mu)}{\rho_2(\lambda, \mu)}, \quad (A.16)
\]

\[
\frac{\rho_6(\lambda, \mu)}{\rho_2(\lambda, \mu)} = \frac{\rho_6(\lambda, \mu)}{\rho_2(\lambda, \mu)}, \quad (A.17)
\]
rules;

\[
\begin{align*}
\rho_1 & \rightarrow \rho_5 - \rho_4, \\
\rho_2 & \rightarrow \rho_6, \\
\rho_3 & \rightarrow \rho_5, \\
\rho_4 & \rightarrow \rho_3 - \rho_1, \\
\rho_5 & \rightarrow \rho_2, \\
\rho_6 & \rightarrow \rho_2, \\
\rho_9 & \rightarrow -\rho_{10}, \\
\rho_{10} & \rightarrow -\rho_9,
\end{align*}
\]

Explicitly, we obtain the relations

\[
\frac{\rho_5(\lambda, \mu) - \rho_4(\lambda, \mu)}{\rho_6(\lambda, \mu)} = -e^{h} \alpha(\lambda) \gamma(\mu) + e^{-h} \gamma(\lambda) \alpha(\mu), 
\]

(A.11)

\[
\frac{\rho_3(\lambda, \mu) - \rho_1(\lambda, \mu)}{\rho_6(\lambda, \mu)} = e^{h} \gamma(\lambda) \alpha(\mu) - e^{-h} \alpha(\lambda) \gamma(\mu), 
\]

(A.12)

\[-\frac{\rho_{10}(\lambda, \mu)}{\rho_6(\lambda, \mu)} = -e^{h} \alpha(\lambda) \alpha(\mu) - e^{-h} \gamma(\lambda) \gamma(\mu),
\]

(A.13)

\[-\frac{\rho_9(\lambda, \mu)}{\rho_6(\lambda, \mu)} = -e^{h} \gamma(\lambda) \gamma(\mu) - e^{-h} \alpha(\lambda) \alpha(\mu),
\]

(A.14)

\[-\frac{\rho_5(\lambda, \mu)}{\rho_6(\lambda, \mu)} = \frac{e^{h} \alpha(\lambda) \gamma(\mu) - e^{-h} \gamma(\lambda) \alpha(\mu)}{\alpha^2(\lambda) - \alpha^2(\mu)},
\]

(A.15)

\[-\frac{\rho_3(\lambda, \mu)}{\rho_6(\lambda, \mu)} = \frac{e^{h} \gamma(\lambda) \alpha(\mu) - e^{-h} \alpha(\lambda) \gamma(\mu)}{\alpha^2(\lambda) - \alpha^2(\mu)},
\]

(A.16)

\[-\frac{\rho_2(\lambda, \mu)}{\rho_6(\lambda, \mu)} = \frac{e^{-i}[e^{h} \alpha(\lambda) \gamma(\lambda) + e^{-h} \alpha(\mu) \gamma(\mu)]}{\alpha^2(\lambda) - \alpha^2(\mu)},
\]

(A.17)

which are shown by direct calculation. The relations (A.8)-(A.10) are invariant under \(\Phi\).

**Appendix B  Asymptotic Expansion of the Elements of the Monodromy Matrix**

We shall explain below details of expansion of the monodromy matrix \(\tilde{T}(\lambda)\) in terms of \(v(\lambda)^{-1}\). \(\tilde{T}_{mn}(\lambda)\) satisfies the recursion relation

\[
\tilde{T}_{m+1,n}(\lambda) = \tilde{L}_m(\lambda)\tilde{T}_{m,n}(\lambda), \quad \tilde{T}_{m,m}(\lambda) = 1,
\]

(B.1)

where

\[
\tilde{L}_m(\lambda) = V(\lambda)^{-m-1}L_m(\lambda)V(\lambda)^{m}
\]

\[
= \left( \begin{array}{cc}
(i \cot \lambda)^{n_{m\dagger}} c_{m\dagger} & (i \cot \lambda)^{n_{m}} c_{m} e^{-h(\lambda)} e^{im\mu(\lambda)} \\
-i (i \cot \lambda)^{n_{m\dagger}} & c_{m}^{\dagger} e^{-h(\lambda)} e^{-im\mu(\lambda)}
\end{array} \right)
\]

\[
= \left( \begin{array}{cc}
(i \cot \lambda)^{n_{m\dagger}} c_{m\dagger} & (i \cot \lambda)^{n_{m}} c_{m} e^{-h(\lambda)} e^{im\mu(\lambda)} \\
-i (i \cot \lambda)^{n_{m\dagger}} & c_{m}^{\dagger} e^{-h(\lambda)} e^{-im\mu(\lambda)}
\end{array} \right)
\]

11
\[-i c_{m\uparrow} (i \cot \lambda)^{n_{m\uparrow}} e^{-h(\lambda)/\sin \lambda} e^{imp(\lambda)} \]
\[c_{m\downarrow} c_{m\downarrow} e^{i \lambda \cos \lambda} \]
\[-i (i \cot \lambda)^{-n_{m\uparrow} + n_{m\downarrow}} \]
\[-i (i \cot \lambda)^{-n_{m\uparrow} - n_{m\downarrow}} c_{m\downarrow} e^{-h(\lambda)/\cos \lambda} e^{-imk(\lambda)} \]
\[-i c_{m\uparrow} c_{m\downarrow} e^{i \lambda \cos \lambda} \cot 2m \lambda \]
\[i c_{m\uparrow} (i \cot \lambda)^{-n_{m\downarrow}} c_{m\downarrow} e^{imk(\lambda)} \]
\[-i (i \cot \lambda)^{-n_{m\uparrow} - n_{m\downarrow}} c_{m\downarrow} e^{-h(\lambda)/\sin \lambda} e^{imk(\lambda)} \]
\[(i \cot \lambda)^{-n_{m\uparrow} + n_{m\downarrow}} \]
\[(B.7) \]

Here we introduced new functions
\[e^{ik(\lambda)} = -e^{2h(\lambda)} \cot \lambda, \quad e^{ip(\lambda)} = -e^{-2h(\lambda)} \cot \lambda, \quad (B.3)\]

which we adopted from the recent analytic Bethe Ansatz for the Hubbard model by Yue and Deguchi [36]. It follows from (3.5) and (B.3) that
\[
\sin k(\lambda) = -\frac{v(\lambda)}{2} + \frac{i U}{4}, \quad \sin p(\lambda) = -\frac{v(\lambda)}{2} - \frac{i U}{4}. \quad (B.4)
\]

In the limit \(|m| \rightarrow \infty\), the above matrix \(\tilde{L}_m(\lambda)\) converges in the weak sense to the identity matrix. Solving (B.1) iteratively we obtain \(\tilde{T}(\lambda)\) as
\[
\tilde{T}(\lambda) = \cdots \tilde{L}_{m+1}(\lambda) \tilde{L}_m(\lambda) \tilde{L}_{m-1}(\lambda) \cdots = 1 + \sum_k (\tilde{L}_k(\lambda) - 1) + \sum_{k<l} (\tilde{L}_k(\lambda) - 1)(\tilde{L}_l(\lambda) - 1) + \cdots. \quad (B.5)
\]

To expand \(\tilde{L}_m(\lambda)\) in terms of \(v(\lambda)^{-1}\), we consider the limit \(\text{Im}(\lambda) \rightarrow \infty\) and, to begin with, expand each function in terms of \(e^{2i\lambda}\). For \(e^{-2h(\lambda)}\) there are two possible choices of branch,
\[e^{-2h(\lambda)} = -\frac{U}{4} \sin 2\lambda \pm \sqrt{1 + \left(\frac{U}{4} \sin 2\lambda\right)^2}. \quad (B.6)\]

To achieve convergence of the matrix elements \(\tilde{T}_{\alpha\beta}\) (\(\alpha, \beta = 2, 3\)) we have to take the lower sign in (B.6). For this choice we get
\[e^{2h(\lambda)} = \frac{4i}{U} e^{2i\lambda} + O(e^{6i\lambda}), \quad e^{ik(\lambda)} = -\frac{4}{U} \{e^{2i\lambda} + 2e^{4i\lambda} + O(e^{6i\lambda})\}, \quad e^{-ip(\lambda)} = \frac{4}{U} \{e^{2i\lambda} - 2e^{4i\lambda} + O(e^{6i\lambda})\}, \quad (B.7)\]

Now the leading terms in the sums in (B.3) are of order \(e^{2i\lambda}, e^{4i\lambda}, \cdots\). Thus, from the first two sums in (B.3), we get the expansion of the matrix \(A(\lambda)\) up to order \(e^{4i\lambda}\). Then the last equation in (B.7) yields the required expansion in \((v(\lambda))^{-1}\) up to second order.

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