THE ETA INVARIANT IN THE KÄHLERIAN CONFORMALLY COMPACT EINSTEIN CASE

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ABSTRACT. A formula for the eta invariant of a conformal structure on a 3-manifold is given, in case the latter is the boundary of a 4-manifold $M$, and the former is induced from an Einstein metric which has a conformal compactification $(g, \tau)$ with $g$ Kähler, not (anti-)selfdual, and with the pair satisfying two additional requirements. This result is analogous to a former result of Hitchin in the Einstein selfdual case. Using earlier work of Derdzinski and Maschler, it is shown that the 3-manifolds to which the result applies are all circle bundles over compact Riemann surfaces. Special cases related to Hirzebruch surfaces are also discussed.

1. Introduction

In [H], Hitchin computed an obstruction, in terms of a bound on the eta invariant for a conformal structure on the 3-sphere to be induced from a complete self-dual Einstein metric on the 4-ball. Following LeBrun [L], such conformal structures are said to have positive frequency, drawing on an analogy between this case and the classical obstruction for a smooth function on the circle to be the boundary value of a holomorphic function on the disk. Hitchin also computed the eta invariant in various examples.

In this paper we give a formula for the eta invariant of a conformal structure induced from another type of asymptotically hyperbolic Einstein metric in dimension four, namely one that is conformal to a (conformally compact) Kähler metric, and satisfies additional technical assumptions. The fact that one has a formula and not a bound suggests that such conformal structures should satisfy certain uniqueness properties. This conforms with the known uniqueness of these types of Einstein conformally Kähler structures, and in fact the relation is known in some cases (see [A] Remark 3.4) (but note that there can be two non-isometric Kähler structures conformal to isometric Einstein metrics).

Applying the results of [DM1, DM2], it is shown that such conformal structures must live on circle bundles over Riemann surfaces (so there is an intersection with the examples of Hitchin). Our formula is thus computed using only geometric premises, and from these one deduces to which spaces the formula applies. In terms of the notion of frequency, it may be said that what characterizes this case is a natural way of measuring, in terms of scalar curvature of appropriate Kähler metrics, the...
balance between the “amount” of positive and negative frequency present. Finally, as is natural for this case, well-known symplectic techniques are employed in the computation.

We end by noting that in the last section of this article, we consider the application of the formula to conformal compactifications which extend to closed manifolds, such as Hirzebruch surfaces.

2. Conformal compactifications of Einstein metrics

In the following let \((M, g_E)\) be conformally compact Einstein, i.e., \(M\) is a compact manifold-with-boundary, \(g_E\) a complete Einstein metric in the interior of \(M\), and there exists a smooth defining function \(\tau\) for the boundary \(\partial M\) (so that \(\tau \geq 0, \tau|_{\partial M} = 0, d\tau|_{\partial M} \neq 0\)) such that
\[
g = \tau^2 g_E\]
is smooth on \(M\).

The pair \((g, \tau)\) is called a conformal compactification for \(g_E\). The restriction \(\gamma = g|_{\partial M}\) varies with \(\tau\) within a fixed conformal class \([\gamma]\) on \(\partial M\).

Later we will consider such \((M, g_E)\) with a Kählerian (conformal) compactification, i.e. satisfying the main extra assumption:

(1) A defining function \(\tau\) exists for which \(g\) is Kähler in the interior of \(M\).

In explicit computations, this will be augmented by the assumption

(2) \[d\tau \wedge d\Delta \tau = 0,\]

where \(\Delta\) stands for the \(g\)-Laplacian. The importance of equations (1) and (2) lies in the fact that they imply that the function \(Q := |\nabla \tau|^2\) is constant on each level set of \(\tau\). This type of compactification may be called uniform. It is easier to achieve than the more well-known geodesic one, where \(Q\) is constant in a collar neighborhood of the boundary. From a given compactification, one must solve a PDE to obtain a geodesic compactification, but only an ODE to obtain a uniform one. On the other hand, (1) and (2) imply uniformity of the compactification throughout the open and dense set of non-critical \(\tau\) values, and not just in a collar neighborhood of the boundary.

3. Hitchin’s argument

We recall here Hitchin’s result, emphasizing its applicability, in addition to the four-ball, also to other four-manifolds.

**Theorem 1** (Hitchin). Suppose \((g, \tau)\) is a conformal compactification of \((M, g_E)\) as above, with \(M\) of real dimension four. Then

\[
\frac{1}{12\pi^2} \int_M (|W_+|_g^2 - |W_-|_g^2) \, vol_g + \sigma(M) = -\eta_\gamma(\partial M),
\]
where $W_\pm$ are the (anti-)self dual parts of the Weyl tensor of a compactification $g$, the quantity $\sigma(M)$ denoting the signature of $M$ and $\eta_\gamma(\partial M)$ the eta invariant of the boundary metric $\gamma$.

Here the eta invariant is defined as the value at $s = 0$ of the series $\sum_{\lambda \neq 0} (\text{sgn}\lambda)|\lambda|^{-s}$ (which is holomorphic for $\text{Re } s > -1/2$), where the summation is over nonzero eigenvalues of the self-adjoint operator on even forms of $\partial M$, given by

$$B(\alpha) = (-1)^p(*d - d*)\alpha, \quad \alpha \in \Omega^{2p}.$$  

This invariant depends only on the conformal class of $\gamma$ in $\partial M$.

**Proof.** The Atiyah, Patodi and Singer [APS] signature formula for a manifold with boundary reads [EGH]

$$\sigma(M) = -\frac{1}{24\pi^2} \int_M \text{tr}(R \wedge R) + \frac{1}{24\pi^2} \int_{\partial M} \text{tr}(\Pi \wedge R) - \eta_\gamma(\partial M),$$

where $R$ is the curvature tensor of $M$ and $\Pi$ is the second fundamental form of $\partial M$, considered as a one-form valued in (restrictions to $\partial M$ of) endomorphisms of the tangent bundle of $M$. It follows that this boundary term actually vanishes. In fact, as $g$ is conformal to the Einstein metric $g_E$, it satisfies a Ricci-Hessian equation $\nabla d\tau + (\tau / 2)r = \sigma g$, with $r$ the Ricci curvature of $g$, for some function $\sigma$. As $\partial M = \{\tau = 0\}$, its second fundamental form is the Hessian $\nabla d\tau$, and the Ricci-Hessian equation it satisfies yields that on the boundary this Hessian is a multiple of the metric, i.e. that $\partial M$ is totally umbilical. Thus, in an adapted orthonormal frame $\{e_i\}_{i=1}^4$ with $\{e_i\}_{i=1}^3$ tangent to, and $e_4$ a unit normal to $\partial M$, $\Pi = c \sum_{i=1}^3 e_i \otimes e_i$, which implies $\text{tr}(\Pi \wedge R) = c(e_1 \wedge R_{14} + e_2 \wedge R_{24} + e_3 \wedge R_{34}) + c(R_{1423} + R_{2431} + R_{3412})e_1 \wedge e_2 \wedge e_3 = 0$ by the Bianchi identity. The result now follows as $\text{tr}(R \wedge R) = 2 (|W_+|^2_g - |W_-|^2_g) \text{vol}_g$. □

For the four ball, $\sigma(M) = 0$, and so Hitchin concludes from (3) that the eta invariant of $S^3$ is nonpositive for conformal structures induced from self-dual Einstein metrics on $B^4$, and zero exactly for the standard structure. Our purpose here is to draw further conclusions, especially in the case where (1) holds. For example,

**Corollary 3.1.**

\begin{equation}
-\left(\sigma(M) + \frac{1}{12\pi^2} \int_M |W_+|^2_g \text{vol}_g\right) \leq \eta_\gamma(\partial M) \leq -\left(\sigma(M) + \frac{1}{12\pi^2} \int_M |W_-|^2_g \text{vol}_g\right)
\end{equation}

with equality holding on the left-hand side in the self-dual case, and on the right-hand side in the anti-selfdual case. Moreover, if (1) also holds, and $g$ is not anti-selfdual, then the left hand inequality of (4) can be rewritten as

\begin{equation}
-\left(\sigma(M) + \frac{1}{288\pi^2} \int_M s^2 g \text{vol}_g\right) \leq \eta_\gamma(\partial M).
\end{equation}
Proof. Only (5) needs to be justified, and for this see Derdzinski [D]. □

4. Geometry of compactifications satisfying (1) and (2)

If \((M^4, g_E)\) admits a Kählerian compactification \((g, \tau)\) as in (1), and also satisfies (2), then it falls under the local classification of [DM1] see §23, which we will now recall. Let \(\pi : (L, \langle \cdot, \cdot \rangle) \to (N, h)\) be a Hermitian holomorphic line bundle over a Riemann surface, the latter equipped with a constant curvature metric \(h\). Assume that the curvature of \(\langle \cdot, \cdot \rangle\) is a multiple, say \(p\), of the Kähler form \(\omega_h\) of \(h\). Consider, on \(L \setminus N\) (the total space of \(L\) excluding the zero section), a family of metrics \(g\) given by

\[
(6) \quad i] \ g|_H = \pi^* h \text{ or } ii] \ g|_H = 2|\tau - c| \pi^* h, \quad \text{and} \quad g|_V = \left(\frac{Q(\tau)}{(ar)^2}\right) \text{Re} \langle \cdot, \cdot \rangle,
\]

where

- \(\mathcal{V}, \mathcal{H}\) are the vertical/horizontal distributions of \(L\), respectively, the latter determined via the Chern connection of \(\langle \cdot, \cdot \rangle\),
- \(c, a \neq 0\) are constants,
- \(r\) is the norm induced by \(\langle \cdot, \cdot \rangle\),
- \(\tau\) is a function on \(L \setminus N\), obtained by composing with \(r\) another function, denoted via abuse of notation by \(\tau(r)\), and obtained as follows: one fixes an open interval \(I\) and a positive \(C^\infty\) function \(Q(\tau)\) on \(I\), solves the differential equation

\[
\left(\frac{a}{Q}\right) d\tau = d(\log r)
\]

to obtain a diffeomorphism \(r(\tau)\) from \(I\) onto an interval contained in \((0, \infty)\), and defines \(\tau(r)\) as the inverse of this diffeomorphism.

Note that in case i] of (6), \(L\) is a trivial bundle, and \(g\) is locally reducible with Kähler factors \(g|_\mathcal{H}, g|_\mathcal{V}\).

For the pair \((g, \tau)\), with \(\tau = \tau(r)\), the function \(Q(\tau(r))\) equals \(\nabla^2 \tau\) in case i] of (6). By [DM1], the metric is a Kählerian compactification of an Einstein metric if \(Q(\tau)\) is a member of one of the following three families of rational functions, in which \(K, \alpha, \beta, A, B\) and \(C\) denote constants (the precise choice of which makes \(Q(\tau)\) positive in \(I\) and also determines the Einstein constant of \(h\)):

a] \(Q = -K\tau^2 + (\alpha\tau^3 - \beta/2)/3\) in case i] of (6).

b] \(Q = -K\tau/2 + \alpha\tau^3 - \beta/3\) in case ii] of (6) with \(c = 0\).

c] \(Q = (\tau/c - 1)(AE(\tau/c) + BF(\tau/c) + C)\) in case ii] of (6) with \(c \neq 0\), where

\[
F(x) = (x - 2)x^3/(x - 1)^2, \quad E(x) = x^2 - 1.
\]

The four-dimensional version of the main result of [DM1] is that for any conformally Einstein Kähler metric \((M^4, J, g)\) with conformal factor \(\tau^2\) satisfying (2), any point in the complement \(M_\tau\) of the critical set of \(\tau\) has a neighborhood biholomorphically isometric to an open set in some triple \((L \setminus N, g, \tau(r))\) with \(Q(\tau)\) one of the above [DM1, Theorem 24.1]. This biholomorphic isometry identifies span \((\nabla \tau, J\nabla \tau)\) and its orthogonal complement, with \(\mathcal{V}\) and, respectively, \(\mathcal{H}\). Moreover, \(J\tau\) is a holomorphic Killing field generating a circle action.
Suppose now that the value \( \tau_0 \neq 0 \) which corresponds to \( r = 0 \) is an endpoint of \( I \), and the function \( Q(\tau) \) has at \( \tau_0 \) a simple zero as a limiting value. Then \( (g, \tau(r)) \) extends to the zero section of \( L \), and a biholomorphic isometry as in the previous paragraph can also be defined on neighborhoods of points in \( M \setminus M_\tau \) [DM2, special case of Remark 16.4]. In this case, if \( I \) contains the value \( \tau = 0 \) (so that \( Q(0) \neq 0 \)), then \( 0 \) is a regular value of the Morse-Bott function \( \tau \). It follows using the same methods leading to [DM2, Theorem 16.3] that this biholomorphic isometry can be extended to the interior of a manifold \( M \) whose boundary \( \partial M \) is identified with \( \{ \tau = 0 \} \). The metric \( g \) is thus a Kählerian conformal compactification of an Einstein metric \( g_E \) on the interior of \( M \). In particular we see that \( M \) is diffeomorphic to a closed disk bundle over a Riemann surface \( N \), while its boundary is a circle bundle over \( N \).

5. The signature

For a \( D^2 \) bundle \( M \) over a fixed Riemann surface \( N \), contractibility of the disk implies that \( H_2(M) = H_2(N) = \mathbb{Z} \), so that the intersection form is a number, namely the Euler number of the bundle \( M \), and the signature is either \( \pm 1 \) or 0. The group \( H_2(M) \) is generated by a section, i.e. a copy of \( N \), and the intersection matrix is determined by the self-intersection number of such a section. If \( M \) is the trivial bundle, the intersection matrix, as well as the signature both vanish.

6. The dual Kähler metric and formulas for \( \eta \)

On a four-manifold \( M \), let \( (g, \tau) \) be a compactification of \( g_E \) which is both Kählerian, i.e. (1) holds, and satisfies (2), while \( g \) is of type ii] in (6). Then there is another metric \( \widehat{g} \) in the conformal class of \( g_E \) which is Kähler with respect to an oppositely oriented complex structure. This metric is \( \widehat{g} := g/(\tau - c)^2 \), with \( c \) as in (6) (see [DM3, Remark 28.4]) and [M, Proposition 5.1]). Since \( W_- \) is, as a \((3,1)\) tensor, a conformal invariant, it follows that

\[
\widehat{g} W_+ = \frac{1}{(\tau - c)^4} W_+ = \frac{(\tau - c)^{-4}s^2}{24}
\]

\[
= (\tau - c)^{-4} \left( (s(\tau - c)^2 + 6(\tau - c)\Delta \tau - 12|\nabla \tau|^2)^2 / 24
\right)
\]

\[
= (\tau - c)^{-4} \left( s(\tau - c)^2 + 6(\tau - c)\Delta \tau - 12Q \right)^2 / 24,
\]

where hatted quantities are those of \( \widehat{g} \), and we have also used the conformal change formula for the scalar curvature. We therefore have

**Theorem 2.** Let \( M^4 \) be a four-manifold with boundary, with an Einstein metric \( g_E \) on its interior, which admits a conformal compactification \( (g, \tau) \) satisfying (7) and (2), where \( g \) is not (anti-)selfdual and is of type ii] in (6). Suppose \( I_0 = [0, \tau_0] \) is the interval of \( \tau \)-values as in \( \S 4 \) and assume \( c \notin I_0 \), with \( c \) the constant appearing...
in the expression (6) for \( g \). Then the eta invariant of the boundary metric \( \gamma = g|_{\partial M} \) is given by

(8) \[
\eta_\gamma(\partial M) = -\sigma(M) + \frac{1}{288\pi^2} \int_M \left[ s^2 - (\tau - c)^{-4} \tilde{s}^2 \right] \text{vol}_g \\
= -\sigma(M) + \frac{1}{288\pi^2} \int_M \left[ s^2 - (\tau - c)^{-4} \left( s(\tau - c)^2 + 6(\tau - c)\Delta\tau - 12Q \right) \right] \text{vol}_g
\]

which can be rewritten as

(9) \[
\eta_\gamma(\partial M) = -\sigma(M) + \frac{1}{288\pi^2} \int_M \left[ (a^2 - b^2(\tau - c)^{-6})t^2 \right] \frac{\omega}{2} \\
= -\sigma(M) + \frac{1}{288\pi^2} \int_{I_0} \left[ (a^2 - b^2(t - c)^{-6})t^2 \right] \left[ \omega_0 + t p \omega^h \right] \text{Vol}_h(N) \ dt \\
= -\sigma(M) + \frac{1}{288\pi^2} \int_{I_0} \left[ (a^2 - b^2(t - c)^{-6})t^2 \right] \left[ t + p t \right] \ dt
\]

for some nonzero constants \( a \) and \( b \) and a constant \( p \), with \( \omega \) the Kähler form of \( g \) on \( M^4 \), which is reduced to \( \omega_0 \) on \( N \) at the level \( \tau = 0 \), and with \( l \) standing for the constant \( 1 \) in case i] and \( 2|c| \) in case ii] of §4.

**Proof.** In view of (7), only the integrals in the (9) need to be explained. It is known (cf. [DM1, Proposition 6.5]) that in dimension four, the scalar curvature of a Kähler metric which is conformal to an Einstein metric is a constant multiple of the square root of the conformal factor. That square root is \( \tau \) for \( g \), and \( \tau / (\tau - c) \) for \( \hat{g} \). The first equality in (9) now follows directly from the first expression for \( \eta_\gamma(\partial M) \) in (8). The second is a consequence of the Duistermaat-Heckman Theorem [DH], which states that the measure derivative on the image of the moment map is the function \( \omega_0 + t p \omega^h \), as in this case \( p \omega^h \) is the curvature of \( L \), hence also of the circle bundle \( \partial M \). The third follows from (6), which gives

\[ \omega_0 = \omega^h \text{ in case i], while } \omega_0 = 2|c|\omega^h \text{ in case ii].} \]

\[ \square \]

**Remark 6.1.** There exist Einstein metrics \((M^4, g_E)\) with a conformal compactification \((g, \tau)\) satisfying (11) but not (2) for which nonetheless there is a second conformal compactification which is Kählerian with respect to an oppositely oriented complex structure (see [ACG]). For these, the first equality in (8) still holds. This case uses the moment map for a toric surface, and will be treated elsewhere.

**7. Hirzebruch surfaces and related special cases**

The final equation in (9) is quite straightforward to evaluate, though the result is messy. In the cases examined below, \( a, b, \) and \( \tau_0 \) can be determined topologically.
It turns out that they are, in those cases, all related to $p$. We indicate the main considerations involved.

An explicit parametrization of all metrics $(M, g)$ of the type appearing in §4 is not known. However, such a parametrization has been obtained in [DM3] for the case where $g$ extends to a closed manifold $\overline{M}$. The manifold $\overline{M}$ is in fact, a 2-sphere bundle over $N$, with $p$ now, as a ratio of Chern numbers of closed manifolds, a rational number. In dimension four, corresponding metrics $g$ come either from family $a]$ in §4, in which case they satisfy $s = 0$ and so are anti-selfdual, or from family $c]$ of that section. In the latter case, the most important examples are the Hirzebruch surfaces (with $N = S^2$). A Hirzebruch surface $\overline{M}$ with Chern number $2p \geq 3$ does have a corresponding $\tau$-interval containing $\tau = 0$ (see [DM3, Theorem 2.4ii] and §45], so that it does in fact extend a manifold with boundary $M$ of the type considered in §4. Also, the constant $c$ lies outside the image interval of $\tau$, as required in Theorem 2. The constants $a$ and $b$ can be determined in that case by topological means. For example,

$$2c_1[M] \cup [\omega] = \int_M s \omega^2/2 = a \int_M \tau \omega^2/2 = a \int_I (2|c| + pt)\, dt,$$

where $I$ is the image of the moment map $\tau$ on $\overline{M}$. If that interval is $I = [\tau_1, \tau_0]$, then it is known from [DM3] that $\tau_0 = c\tau_1/(\tau_1 - c)$. Also, a direct relation exists between $p$ and either $\tau_0$ or $\tau_1$, see [DM3, Equation (34.1)]. Finally, $\sigma(M)$ vanishes for even $p$, and is 1 for odd $p$. With these ingredients one obtains an explicitly topological expression for the eta invariant. While for these particular examples, this approach is more cumbersome than that of Hitchin, it applies uniformly to other base Riemann surfaces.

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