THE FOURIER DIMENSION IS NOT FINITELY STABLE

FREDRIK EKSTRÖM

Abstract. The Fourier dimension is not in general stable under finite unions of sets. Moreover, the stability of the Fourier dimension on particular pairs of sets is independent from the stability of the compact Fourier dimension.

1. Introduction

The present note gives an example to show that the Fourier dimension is not stable under finite unions of sets. This improves one of the results in [1], where it was shown that the Fourier dimension is not countably stable. For a more detailed introduction to the Fourier dimension than is given here, see for example [1] and references therein.

The Fourier transform of a finite Borel measure $\mu$ on $\mathbb{R}^d$ is defined as

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} d\mu(x),$$

where $\xi \in \mathbb{R}^d$ and $\cdot$ denotes the Euclidean inner product. The Fourier dimension of $\mu$ measures how quickly $\hat{\mu}$ decays at infinity, and is defined by

$$\dim_F \mu = \sup \left\{ s \in [0, d]; \exists C \in \mathbb{R} \text{ such that } |\hat{\mu}(\xi)| \leq C|\xi|^{-s/2} \text{ for all } \xi \right\}.$$ 

If $A$ is a Borel subset of $\mathbb{R}^d$ then the Fourier dimension of $A$ is defined to be

$$\dim_F A = \sup \{ \dim_F \mu; \mu \in \mathcal{P}(A) \},$$

where $\mathcal{P}(A)$ denotes the set of Borel probability measures on $\mathbb{R}^d$ that give full measure to $A$. It can be shown (see [2] Lemma 12.12) that if $\mu$ has compact support and $0 < \dim_F \mu < d$ then

$$I_s(\mu) := \iint |x - y|^{-s} d\mu(x) d\mu(y)$$

is finite for all $s < \dim_F \mu$, and from this it follows that $\dim_F A \leq \dim_H A$ for any Borel set $A$.

As an alternative way to define the Fourier dimension, one could require that the measure in the definition should give full measure to some compact subset of $A$ and not just to $A$ itself. This compact Fourier dimension is thus defined as

$$\dim_{FC} A = \sup \{ \dim_F \mu; \mu \in \mathcal{P}(K), K \subset A \text{ is compact} \}.$$ 

One of the examples in [1] shows that the compact Fourier dimension is not finitely stable, but that example is not a counterexample to finite stability of the Fourier dimension. Example [2] below produces the opposite situation, namely, sets $A'$ and
$B'$ such that
\[
\dim_F(A' \cup B') > \max(\dim_F A', \dim_F B'),
\]
\[
\dim_{FC}(A' \cup B') = \max(\dim_{FC} A', \dim_{FC} B').
\]

2. **The Example**

The following lemma is used in the example. It previously appeared in [1], but is included here for completeness.

**Lemma 1.** For any $\varepsilon \in (0, 1]$, 
\[
\inf_{\mu} \sup_{j \geq 1} |\hat{\mu}(j)| \geq \frac{\pi \varepsilon}{8 + 2\pi \varepsilon} \left( \geq \frac{\varepsilon}{5} \right),
\]
where the infimum is over all $\mu \in \mathcal{P}([\varepsilon, 1])$ and the supremum is over all positive integers $j$.

**Proof.** Fix $\varepsilon > 0$ and take any $\mu \in \mathcal{P}([\varepsilon, 1])$. If $\varphi$ is a real-valued continuous function supported on $[0, \varepsilon]$ such that
\[
\int \varphi(x) \, dx = 1 \quad \text{and} \quad \sum_{k=-\infty}^{\infty} |\hat{\varphi}(k)| < \infty
\]
then
\[
0 = \mu(\varphi) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)\hat{\mu}(k) = 1 + 2 \Re \left( \sum_{k=1}^{\infty} \hat{\varphi}(k)\hat{\mu}(k) \right),
\]
and thus
\[
\frac{1}{2} \leq \sum_{k=1}^{\infty} |\hat{\varphi}(k)||\hat{\mu}(k)| \leq \left( \sum_{k=1}^{\infty} |\hat{\varphi}(k)| \right) \left( \sup_{j \geq 1} |\hat{\mu}(j)| \right).
\]

Now let $\chi$ be the indicator function of $[0, \varepsilon/2]$ and take $\varphi$ to be the triangle pulse 
\[
\varphi(x) = \left( \frac{2\chi}{\varepsilon} \right) \ast \left( \frac{2\chi}{\varepsilon} \right),
\]
where $\ast$ denotes convolution. Then
\[
|\hat{\varphi}(k)| = \left| \frac{2\chi(k)}{\varepsilon} \right|^2 = \text{sinc}^2 \left( \frac{k\pi \varepsilon}{2} \right) \leq \min \left( 1, \frac{4}{k^2 \pi^2 \varepsilon^2} \right),
\]
so that
\[
\sum_{k=1}^{\infty} |\hat{\varphi}(k)| \leq \left[ \frac{2}{\pi \varepsilon} \right] + \frac{4}{\pi^2 \varepsilon^2} \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2 \varepsilon^2} \leq \frac{2}{\pi \varepsilon} + 1 + \frac{4}{\pi^2 \varepsilon^2} \int_{\frac{\pi \varepsilon}{2}}^{\infty} \frac{1}{x^2} \, dx = \frac{4 + \pi \varepsilon}{\pi \varepsilon}.
\]

It follows that
\[
\sup_{j \geq 1} |\hat{\mu}(j)| \geq \frac{1}{2} \cdot \frac{\pi \varepsilon}{4 + \pi \varepsilon}.
\]

\[\square\]

**Example 2.** Let $s \in (\sqrt{3} - 1, 1)$ and choose $b$ such that
\[
\frac{1 - s}{s} < b < \frac{s}{2}
\]
(this is possible since \( s > \sqrt{3} - 1 \)). Let \((l_k)_{k=1}^{\infty}\) be a sequence of natural numbers such that
\[
\lim_{k \to \infty} \frac{l_{k+1}}{l_k} = \infty,
\]
and set \(m_k = [b l_k]\). Whenever \(x \in [0, 1]\) is not a dyadic rational, and thus has a unique binary decimal expansion \(x = 0.x_1 x_2 \ldots\), let
\[
f(x) = \sup \{\{k; x_{l_k+1} \ldots x_{l_k+m_k} = 0^m\} \cup \{0\}\}.
\]
Then for each \(k\)
\[
\lambda \{x; f(x) = \infty \} \leq \lambda \left( \bigcup_{j=k}^{\infty} \{x; x_{l_j+1} \ldots x_{l_j+m_j} = 0^m\} \right) \leq \sum_{j=k}^{\infty} 2^{-m_j},
\]
where \(\lambda\) denotes Lebesgue measure. The sum converges since \((m_k)\) is eventually strictly increasing, and thus \(f(x)\) is finite for \(\lambda\)-a.e. \(x \in [0, 1]\). Let
\[
A = \{x \in [0, 1]; f(x) \text{ is even}\}, \quad B = \{x \in [0, 1]; f(x) \text{ is odd}\};
\]
then \(\dim_F (A \cup B) = 1\) since \(\lambda(A \cup B) = 1\).

To see that \(\dim_F A \leq s\), take any \(\mu \in \mathcal{P}(A)\) and define for odd \(k\)
\[
A_k = \{x \in A; x_{l_k+1} \ldots x_{l_k+m_k} = 0^m\}, \quad A^j_k = \{x \in A_k; f(x) = j\}.
\]
If \(x \in A_k\) then \(f(x) \geq k\), so there must be some even number \(j \geq k + 1\) such that \(f(x) = j\). Hence for each \(k\)
\[
A_k = \bigcup_{j \geq k+1} A^j_k.
\]
Let
\[
\alpha_k = \mu(A_k), \quad \alpha^j_k = \mu(A^j_k),
\]
and let \(P\) be the set of natural numbers \(k\) such that
\[
\alpha^j_k \leq \frac{2^{-(m_k + j - k)}}{6} \quad \text{for all even } j \geq k + 1.
\]

Suppose first that the set \(P\) is infinite. Let \(\mu_k = \mu|_{A \setminus A_k}\) and let \(\nu_k\) be the image of \(\mu_k\) under the map \(x \mapsto 2^k x \pmod{1}\). The measure \(\nu_k\) is concentrated on \([2^{-m_k}, 1]\) and has total mass \(1 - \alpha_k\), so by Lemma II there is some \(r_k \geq 1\) such that
\[
\hat{\mu}_k (2^k r_k) = \hat{\nu}_k (r_k) \geq (1 - \alpha_k) \frac{2^{-m_k}}{5}.
\]
For \(k \in P\),
\[
\alpha_k = \sum_{j \geq k+1} \alpha^j_k \leq \frac{2^{-m_k}}{6},
\]
and hence
\[
(2^k r_k)^{s/2} \left| \hat{\mu} (2^k r_k) \right| \geq 2^{s/2} \left( (\hat{\mu}_k (2^k r_k)) - \alpha_k \right) \geq 2^{s/2} 2^{-m_k} \left( \frac{1 - \alpha_k}{5} - \frac{1}{6} \right),
\]
where the exponent is positive for large \( k \) since \( b < s/2 \). Thus if \( (k_i)_{i=1}^{\infty} \) is an enumeration of \( P \) then

\[
\limsup_{|\xi| \to \infty} |\xi|^{s/2} |\hat{\mu}(\xi)| \geq \lim_{i \to \infty} \left( 2^{k_i} r_{k_i} \right)^{s/2} |\hat{\mu}(2^{k_i} r_{k_i})| = \infty,
\]

so \( \dim_F \mu \leq s \).

Suppose on the other hand that \( P \) is finite. If \( k \) is odd and \( j \geq k + 1 \) is even, then
\[
A_{k}^j \subset \{ x \in [0, 1]; x_{l_k+1} \ldots x_{l_k+m_k} = 0^{m_k} \text{ and } x_{l_j+1} \ldots x_{l_j+m_j} = 0^{m_j} \} \subset \bigcup_p I_p,
\]
where \( \{ I_p \} \) are \( 2^{l_j-m_k} \) intervals, each of length \( 2^{-l_j+m_j} \). Thus
\[
I_s(\mu) \geq \sum_{p=1}^{2^{l_j-m_k}} \int_{I_p} |x-y|^{-s} d\mu(x) d\mu(y) \geq 2^{s(l_j+m_j)} \sum_{p=1}^{2^{l_j-m_k}} \mu(I_p)^2 \geq 2^{s(l_j+m_j)} \left( \alpha_{l_j}^2 \right)^2 / 2^{l_j-m_k},
\]
where the inequality \( \| \cdot \|_2 \geq d^{-1} \| \cdot \|_1 \) for norms in \( \mathbb{R}^d \) is used in the last step. If \( k \not\in P \) then \( j = j(k) \) can be chosen such that the last expression is greater than or equal to
\[
\frac{1}{36} \cdot 2^{-((s(1+b)-1)l_j - 2j - m_k)}.
\]
Because \( b > (1-s)/s \) and \( l_j \) grows exponentially with \( j \), there is some \( \varepsilon > 0 \) such that the exponent is at least
\[
\varepsilon l_j - m_k \geq \varepsilon l_{k+1} - \lfloor bl_k \rfloor
\]
whenever \( k \) (and hence \( j \)) is large enough. Since \( P \) contains arbitrarily large odd \( k \) it follows that \( I_s(\mu) = \infty \), and thus \( \dim_F \mu \leq s \) in this case too.

This shows that \( \dim_F A \leq s \), and similar computations show that \( \dim_F B \leq s \) as well. Thus the Fourier dimension is not finitely stable.

Consider now the \( F_\alpha \)-sets
\[
A' = \bigcup_{k \text{ even}} \overline{f^{-1}(k)}, \quad B' = \bigcup_{k \text{ odd}} \overline{f^{-1}(k)},
\]
which are slightly larger than \( A \) and \( B \). From the inclusion
\[
\overline{f^{-1}(k)} \subset \{ x; x_{l_k+1} \ldots x_{l_k+m_k} = 0^{m_k} \} \cap \bigcap_{j=k+1}^{\infty} \{ x; x_{l_j+1} \ldots x_{l_j+m_j} \neq 0^{m_j} \},
\]
it follows that the difference sets \( A' \setminus A \) and \( B' \setminus B \) only contain dyadic rationals, and in particular they are countable. Thus \( A' \) and \( B' \) have the same Fourier dimensions as \( A \) and \( B \) respectively (using that any measure that gives positive mass to a countable set has Fourier dimension 0), and in particular
\[
\dim_F (A' \cup B') > \max(\dim_F A', \dim_F B').
\]
On the other hand, Proposition 5 in [P] implies that the compact Fourier dimension is finitely stable on \( F_\alpha \)-sets, and thus
\[
\dim_{FC} (A' \cup B') = \max(\dim_{FC} A', \dim_{FC} B').
\]
References

[1] Fredrik Ekström, Tomas Persson, and Jörg Schmeling. On the Fourier dimension and a modification. Preprint: arXiv:1406.1480v3 [math.FA]. To appear in Journal of Fractal Geometry.

[2] Pertti Mattila. Geometry of Sets and Measures in Euclidean Spaces — Fractals and Rectifiability. Cambridge University Press, 1995.

Centre for Mathematical Sciences, Lund University, Box 118, 22 100 Lund, Sweden
E-mail address: fredrike@maths.lth.se