We consider the setting of an oblivious adversary. We study the online discrepancy minimization problem for vectors in $\mathbb{R}^d$ in the oblivious setting where an adversary is allowed fix the vectors $x_1, x_2, \ldots, x_n$ in arbitrary order ahead of time. We give an algorithm that maintains $O(\sqrt{\log(nd/\delta)})$ discrepancy with probability $1 - \delta$, matching the lower bound given in [Bansal et al., 2020b] up to a $O(\sqrt{\log \log n})$ factor in the high-probability regime. We also provide results for the weighted and multi-color versions of the problem.

1 Introduction

The problem of online discrepancy minimization, first proposed by [Spencer, 1977], is as follows: given a set of vectors $x_1, x_2, \ldots, x_n \in [-1, 1]^d$ observed one at a time, assign $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ to each vector such that the partial discrepancy, $\left\| \sum_{i=1}^t \epsilon_i x_i \right\|_\infty$ is as small as possible for all $1 \leq t \leq n$. Unless otherwise stated, we will assume the Komlós setting where the norm of each $x$ is bounded from above by one, i.e., $\|x\|_2^2 \leq 1$.

In the most general, fully adversarial setting, where the values of subsequent $x$ are allowed to change in response to the assignment of $\epsilon$, the best possible bound is $\Omega(\sqrt{n})$ since an adversary can always choose $x_{t+1}$ orthogonal to the current partial sum $w_t = \sum_{i=1}^t \epsilon_i x_i$ at each step.

We consider the setting of an oblivious adversary, where the sequence of $x$ can be arbitrarily set by the adversary, but is not allowed to change in response to the choice of $\epsilon$ (as is the case in the fully adversarial setting). Randomly selecting $\epsilon$ with probability 1/2 results in a discrepancy of $O(\sqrt{n})$. Recently, [Alweiss et al., 2020] gave an algorithm which obtains a discrepancy of $O(\log(nd/\delta))$ with probability $1 - \delta$. In addition, they conjectured that a variant of their algorithm can achieve $O(\sqrt{\log(nd/\delta)})$. We show this is indeed the case.

A more constrained setting considered recently was the stochastic setting, where the incoming vectors are sampled i.i.d. from a known distribution. [Bansal and Spencer, 2019] considered the special case where the distribution is the uniform distribution on all $\{-1, 1\}^d$ vectors and gave an $O(\sqrt{d \log n})$ bound for the $\ell_\infty$ norm with high probability. In a more general setting, where the vectors are sampled from a distribution supported in $[-1, 1]^d$, [Bansal et al., 2020b] achieved $\ell_\infty$ bound of $O(d^2 \log(nt))$ with high probability, and [Aru et al., 2018] achieved a bound of $O_d(\sqrt{\log(n)})$ with an implicit exponential dependence on $d$. Finally, for distributions supported on vectors with Euclidean norm at most 1, [Bansal et al., 2020a] achieved $\ell_\infty$ bound of $O(\log^4(nt))$ with high probability.

Finally, we also provide results for two variants of the online discrepancy minimization problem:

**Weighted Online Discrepancy Minimization.** A variant of the online Komlós problem, where vectors $x_1, \ldots, x_n$ arrive one by one and must be immediately assigned a weighted sign of either $-q$ or $1 - q$, for $0 < q < 1$, such that the weighted discrepancy $\left\| \sum_{i=1}^t \epsilon_i x_i \right\|_\infty$, where $\epsilon_i$ is the weighted sign given to $x_i$, is minimized for all $1 \leq t \leq n$. 

**Multi-Color Online Discrepancy Minimization.** A variant of the online Komlós problem, where vectors $x_1, \ldots, x_n$ arrive one by one and must be immediately assigned a color from $\{1, \ldots, q\}$, for $1 \leq q \leq n$, such that the discrepancy $\left\| \sum_{i=1}^t \epsilon_i x_i \right\|_\infty$, where $\epsilon_i$ is the color assigned to $x_i$, is minimized for all $1 \leq t \leq n$.
Multicolor Weighted Online Discrepancy Minimization. The multi-color version of the online discrepancy problem, where each vector is assigned to a color in \( M = \{ m_1, \ldots, m_k \} \) immediately on arrival. For each \( 1 \leq i \leq k \), color \( m_i \) is associated with a weight \( \alpha_i \in [1, \eta] \). The goal is to minimize the multi-coloring discrepancy:

\[
\max_{m_1, m_2 \in M} \text{disc}(m_1, m_2) = \max_{m_1, m_2 \in M} \left\| \frac{1}{\alpha_1} s_t(m_1) - \frac{1}{\alpha_2} s_t(m_2) \right\|_\infty
\]

for all \( 1 \leq t \leq n \), where \( s_t(m) \) is the sum of all vectors assigned to color \( m \) at step \( t \). The weighted online discrepancy minimization problem is a special case of the multi-color discrepancy problem with two colors. \cite{Alweiss2020} study the multicolor problem in the stochastic arrival setting and achieved an \( \ell_\infty \) bound for problem (1) of \( O \left( \log^2(\eta \eta) \cdot \log^4(dn) \right) \).

1.1 High-Level Idea and Results

We first give an overview of the algorithm given in \cite{Alweiss2020}, on which our algorithm is based. Let \( x_1, \ldots, x_n \) be the input vectors, where \( \|x_i\|_2 \leq 1 \) for all \( 0 \leq i \leq n \), and let \( \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\} \) be the signs to be chosen. At each step \( t \), the algorithm chooses a sign \( \epsilon_t \) based on the current partial sum \( w_{t-1} = \sum_{i=1}^{t-1} \epsilon_i x_i \), and the incoming vector \( x_t \). In particular, \( \epsilon_t \) is chosen to be 1 with probability \( p_t = \frac{1}{2} \left( 1 - \frac{\langle w_{t-1}, x_t \rangle}{c} \right) \) and \(-1\) with probability \( 1 - p_t \), where \( c = 30 \log(nd/\delta) \) is an upper bound on \( \langle w_{t-1}, x_t \rangle \) for all \( t \) with high probability. Since the analysis heavily depends on the fact that \( p_t \) is linear in \( w_{t-1} \), the constant \( c \) is needed to ensure that \( p_t \) always lies within 0 and 1.

There are two drawbacks of this design. First, \( c \) depends on the number of vectors \( n \), and hence \( n \) needs to be known ahead of time. Second, \( c \) needs to be large enough so keep \( p_t \) in \([0, 1]\) for large \( w \). As a result the algorithm assigns nearly randomly in the beginning of the process. Each of these characteristics are particularly undesirable in empirical applications where finite-sample performance is especially crucial.

We introduce a simple idea to overcome these issues, and obtain an improved bound for discrepancy. In particular, when \( p_t \) falls outside of \([0, 1]\), we clamp it to either 0 if \( \langle w_{t-1}, x_t \rangle > c \) or 1 if \( \langle w_{t-1}, x_t \rangle < -c \). As we show, most of the analysis of \cite{Alweiss2020} remains valid under this modification. However, this modification allows \( c \) to be a much smaller number, which results in improved discrepancy bounds. In fact, \( c \) can be set to 1 and excluded from our algorithm, which makes it very simple and robust.

We then consider the problem of discrepancy minimization in a more generalized setting, instead of choosing a sign in \([-1, 1]\), we were tasked with assigning weighted signs in \([-q, 1 - q]\), for \( 0 \leq q \leq 1 \). In this setting, we give an algorithm that achieves \( \tilde{O}(\sqrt{\log(dn/\delta)}) \) discrepancy with probability \( 1 - \delta \) (Theorem 1). For \( n \geq d \) and \( \delta = 1/\text{poly}(n) \), we achieve a discrepancy of \( O(\sqrt{\log(dn)}) \) which matches the lower bound \( O(\sqrt{\log n} / \log \log n) \) up to a \( O(\sqrt{\log \log n}) \) factor.

Using the aforementioned result, we develop a solution for the online multi-color discrepancy problem by building a binary tree where each leaf corresponds to a different color. A similar structure was used in \cite{Bansal2020a}. For each internal node of the tree with two children, we run an independent oracle which minimizes weighted two-color discrepancy, where the weights are obtained from the weights of the leaves under its two children. When a new vector comes, we repeatedly feed it to the oracles, starting from the root to a leaf, where it can be assigned a color. Our algorithm maintains discrepancy of \( O(\log k \sqrt{\log(dnk/\delta)}) \) with probability \( 1 - \delta \) (Theorem 3). Setting \( \delta = 1/\text{poly}(n) \) improves on the discrepancy of \( O(\log^2(\eta \eta) \cdot \log^4(dn)) \) achieved by \cite{Bansal2020a}.

2 Preliminaries

2.1 Mean Preserving Spreads

Our analysis closely resembles that of \cite{Alweiss2020}. We will make extensive use of mean-preserving spreads \cite{Rothschild1970} and their properties throughout our proofs, which we introduce here with the following definitions and lemmas. These appear in \cite{Alweiss2020} and are given here for completeness.

Definition 1. Given two random variables \( Y \) and \( X \), we say that \( Y \) is a spread of \( X \) if a coupling can be defined such that the expectation of \( Y \) given \( X \) is equal to \( X \), i.e. \( \mathbb{E}[Y | X] = X \).
An alternative, and perhaps more intuitive, definition of mean-preserving spreads is that $Y$ is a mean-preserving spread of $X$ if realizations $y \sim Y$ can be generated by first drawing from $X$ and then adding zero mean noise, i.e. $y = x + \epsilon, x \sim X$.

**Lemma 1.** If $Z$ is a spread of $Y$ and $Y$ is a spread of $X$ then $Z$ is also a spread of $X$.

**Lemma 2.** If $Y$ is a spread of $X$, then for any linear transformation $M$ on $\mathbb{R}^d$, $MY$ is a spread of $MX$.

**Lemma 3.** Let $X$ be a real-valued random variable with $E[X] = 0$ and $|X| \leq C$. Then $G = \mathcal{N}(0, \pi C^2 / 2)$ is a spread of $X$.

**Lemma 4.** Consider random variables $X, Y, W, Z$. Suppose that $W$ is a spread of $X$ and $Z$ is a fixed random variable such that $Z$ is a spread of the conditional distribution of $Y$ given $X = x$ for any value $x$. Then $W + Z$, where $W$ and $Z$ are sampled independently, is a spread of $Y$.

**Lemma 5.** Let $Y$ be a spread of $X$ and $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Then $E_{x\sim X}[\Phi(x)] \leq E_{y\sim Y}[\Phi(y)]$.

### 3 Weighted Discrepancy

We consider an online weighted discrepancy problem, where each vector is assigned a weighted sign of either $-1$ or $0$. Corollary 1. Note that by Lemma 6 and Definition 2, we immediately have the following corollary:

**Corollary 1.** $0 \leq M_i \leq LI$ for all $i$. 

#### 3.1 Weighted Spread

**Definition 2.** Fix $L = \pi / 2$ and define $M_0 = 0$. $M_i$ is given inductively as:

\[
M_i = \begin{cases} 
(I - x_i x_i^T) M_{i-1} & \text{if } (w_{i-1}, x_i) \in [-(1 - q), q], \\
\left(I - \frac{1 - q}{|\langle w_{i-1}, x_i \rangle|} x_i x_i^T\right) M_{i-1} & \text{if } (w_{i-1}, x_i) < -(1 - q), \\
\left(I - \frac{q}{|\langle w_{i-1}, x_i \rangle|} x_i x_i^T\right) M_{i-1} & \text{if } (w_{i-1}, x_i) > q.
\end{cases}
\]

**Lemma 6.** Let $0 \leq M \leq LI, c \geq 1, L \geq 0$, and $x$ be any vector in $\mathbb{R}^n$ with $\|x\|_2 \leq 1$. Let

\[
M' = \left(I - \frac{1}{c} xx^T\right) M \left(I - \frac{1}{c} xx^T\right).
\]

and

\[
M'' = (I - xx^T) M (I - xx^T) + Lxx^T.
\]

Then $0 \leq M' \leq LI$ and $0 \leq M'' \leq LI$.

**Proof.** Since $M \leq LI$, we have

\[
M' \leq \left(I - \frac{1}{c} xx^T\right) LI \left(I - \frac{1}{c} xx^T\right) = LI - \frac{L}{c} \left(2 - \frac{\|x\|^2}{c}\right) xx^T \leq LI
\]

Similarly,

\[
M'' \leq (I - xx^T) LI (I - xx^T) + Lxx^T = LI - L \left(1 - \frac{\|x\|^2}{c}\right) xx^T \leq LI
\]

Note that by Lemma 6 and Definition 2, we immediately have the following corollary:

**Corollary 1.** $0 \leq M_i \leq LI$ for all $i$. 

---

3
Algorithm 1: takes each input vector $x_i$ and assigns it $\{-q, 1-q\}$ signs online to maintain low weighted discrepancy with probability $1-\delta$.

**Input:** $x, q$

**begin**

for $i$ from 1 to $n$ do

$p_i \left\{ \begin{array}{ll} 0, & \text{if } \langle w_{i-1}, x_i \rangle > q \\ 1, & \text{if } \langle w_{i-1}, x_i \rangle < q - 1 \\ q - \langle w_{i-1}, x_i \rangle, & \text{otherwise} \end{array} \right.$

$\epsilon_i \left\{ \begin{array}{ll} 1 - q, & \text{with probability } p_i \\ -q, & \text{with probability } 1 - p_i \end{array} \right.$

$w_i \leftarrow w_{i-1} + \epsilon_i x_i$

end

**Output:** $\epsilon, w$

Let $\mathcal{D}_i$ be the distribution of $w_i$. Lemma\[7\] which says that the distribution of $w$ at each step is spread by a normal distribution with covariance given in Definition\[2\] is important in the analysis. Notice that by Corollary\[11\] all covariances are upperbounded by $LI$.

**Lemma 7.** $\mathcal{N}(0, M_i)$ is a spread of $\mathcal{D}_i$ for all $1 \leq i \leq n$.

**Proof.** At step $i$, consider 3 cases: $-(1-q) \leq \langle w_{i-1}, x_i \rangle \leq q$, $\langle w_{i-1}, x_i \rangle < -(1-q)$ and $\langle w_{i-1}, x_i \rangle > q$.

If $-(1-q) \leq \langle w_{i-1}, x_i \rangle \leq q$, we have

$$E[w_i|w_{i-1}] = w_{i-1} + ((1-q)p_i - q(1-p_i)) x_i = w_{i-1} + (p_i - q)x_i$$

$$w_{i-1} - \langle w_{i-1}, x_i \rangle x_i = (I - x_i x_i^T) w_{i-1}$$

Therefore,

$$w_i = (I - x_i x_i^T) w_{i-1} + Rx_i,$$

where

$$R = \begin{cases} 1 - p_i & \text{with probability } p_i \\ -p_i & \text{with probability } 1 - p_i. \end{cases}$$

By induction, $\mathcal{N}(0, M_{i-1})$ is a spread of the distribution of $w_{i-1}$. By Lemma\[2\]

$$\mathcal{N}(0, (I - x_i x_i^T) M_{i-1} (I - x_i x_i^T))$$

is a spread of the distribution of $(I - x_i x_i^T) w_{i-1}$. Since $R$ is supported in $[-1, 1]$, by Lemma\[3\] $R$ is spread by a zero-mean Gaussian with variance $\pi/2$. Therefore, $\mathcal{N}(0, M_i)$ is a spread of the distribution of $w_i$ by Definition\[2\] and Lemma\[4\].

Now consider the case where $\langle w_{i-1}, x_i \rangle > q > 0$. We have a deterministic update step:

$$w_i = w_{i-1} - qx_i$$

$$= w_{i-1} - q \frac{\langle w_{i-1}, x_i \rangle}{\langle w_{i-1}, x_i \rangle} x_i$$

$$= (I - \frac{qx_i x_i^T}{\langle w_{i-1}, x_i \rangle}) w_{i-1}$$

Therefore, $\mathcal{N}(0, M_i)$ is a spread of the distribution of $w_i$ by Definition\[2\] and Lemma\[2\].

Similarly, for $\langle w_{i-1}, x_i \rangle < -(1-q) < 0$, the update step becomes

$$w_i = w_{i-1} + (1-q)x_i$$

$$= w_{i-1} - (1-q) \frac{\langle w_{i-1}, x_i \rangle}{\langle w_{i-1}, x_i \rangle} x_i$$

$$= (I - \frac{(1-q)x_i x_i^T}{\langle w_{i-1}, x_i \rangle}) w_{i-1}$$

$\square$
Again, $\mathcal{N}(0, M_i)$ is a spread of the distribution of $w_i$ by Definition 2 and Lemma 2.

**Lemma 8.** For any vector $u$ with $\|u\|_2 \leq 1,

$$E_{w \sim \mathcal{N}(0, M_i)} \left[ \exp \left( \frac{\langle w, u \rangle^2}{4L} \right) \right] \leq \sqrt{2}

**Proof.** Let $y = \langle w, u \rangle$. Then $y$ is distributed as $\mathcal{N}(0, u^T M_i u)$. Since $M_i \preceq LI$ (Corollary 1),

$$u^T M_i u \leq u^T (LI) u = L \|u\|_2^2 \leq L.

Therefore, $\mathcal{N}(0, L)$ is a spread of $\mathcal{N}(0, u^T M_i u)$. By Lemma 8,

$$E_y \sim \mathcal{N}(0, u^T M_i u) \left[ \exp \left( \frac{y^2}{4L} \right) \right] \leq E_{y \sim \mathcal{N}(0, L)} \left[ \exp \left( \frac{y^2}{4L} \right) \right] = \sqrt{2}.

**Theorem 1.** Algorithm 7 maintains discrepancy $\|w_i\|_\infty = O \left( \sqrt{\log(dn/\delta)} \right)$ for all $0 \leq i \leq n$, with probability $1 - \delta$.

**Proof.** For any fixed vector $u$ with $\|u\|_2 < 1$, we have:

$$\Pr_{w_i \sim D_i} \left( |\langle w_i, u \rangle| \geq 4L \log \left( \frac{2dn}{\delta} \right) \right) \leq \exp \left( -\frac{4L \log(2dn/\delta)}{4L} \right) E_{w_i \sim D_i} \left[ \exp \left( \frac{\langle w_i, u \rangle^2}{4L} \right) \right] \leq \frac{\sqrt{2} \delta}{2dn} < \frac{\delta}{dn},

where the first inequality is Markov’s inequality and the second inequality follows from Lemma 8. Union bounding over $d$ basis vectors and $n$ rounds gives the result.

Notice that if our goal is minimizing the final discrepancy instead of maintaining low discrepancy for all rounds, we can check the failure condition only at the end. Hence, the number of failure cases in the union bound step in the proof of Theorem 1 can be reduced to only $d$ instead of $dn$. We have the following theorem:

**Theorem 2.** Algorithm 7 obtains final discrepancy $\|w_n\|_\infty = O \left( \sqrt{\log(d/\delta)} \right)$ with probability $1 - \delta$.

### 4 Weighted Multi-Color Discrepancy

In this section, we consider the multi-color version of the online discrepancy problem, where each vector is assigned to a color in $M = \{m_1, m_2, \ldots, m_k\}$ immediately on arrival. For each $1 \leq i \leq k$, color $m_i$ is associated with a weight $\alpha_i$ in $[1, \eta]$. The goal is to minimize the multi-coloring discrepancy:

$$\max_{m_1, m_2 \in M} \text{disc}_t(m_1, m_2) = \max_{m_1, m_2 \in M} \left\| \frac{1}{\alpha_1} s_t(m_1) - \frac{1}{\alpha_2} s_t(m_2) \right\|_\infty

for all $1 \leq t \leq n$, where $s_t(m)$ is the sum of all vectors assigned to color $m$ at step $t$.

The same notation was used in [Bansal et al., 2020b]. Notice that by setting $\alpha_1 = \frac{1}{q}$ and $\alpha_2 = \frac{1}{q}$ in $\text{disc}_t(m_1, m_2)$, we can recover the definition given in Section 3 for the weighted discrepancy between two colors $m_1$ and $m_2$.

Similar to the two-color case, we also give two versions of the result generated by our algorithm (Algorithm 2), one for maintaining low discrepancy in every single round, and the other for obtaining a low final discrepancy. Our algorithm can leverage any oracle (we call it 2-COLOR in Algorithm 2) that minimizes the weighted two-color discrepancy. In particular, our results are obtained by setting 2-COLOR to be Algorithm 1.

We first build a binary tree where each leaf of the tree corresponds to one of the $k$ colors in $M$. Let $h$ be the smallest integer such that $2^h \geq k$. We start with a complete binary tree of height $h$, and then remove $2^h - k$ leaves from the tree such that no two siblings are removed. Note that this is possible by the definition of $h$. We further contract each internal node with only one child to its child. This process does not change the number of leaves in the tree.

Let $T$ be the obtained tree. By construction, all internal nodes of $T$ have 2 children and $T$ has exactly $k$ leaves. We then associate each leaf of $T$ with a color in $M$. For each vector assigned to color $m_i$, we also say that it is assigned to the leaf corresponding to $m_i, \forall 1 \leq i \leq k$. For each node $v \in T$, denote by $s(v)$ the sum of all vectors assigned to
Algorithm 2: takes each input vector $x_i$ and assigns it to one of the colorings online to maintain low discrepancy with probability $1 - \delta$.

**Input:** $x, k, \alpha$

**begin**

1. $h \leftarrow$ smallest integer such that $2^h \geq k$
2. $T \leftarrow$ complete binary tree with height $h$. Remove $2^h - k$ leaves from $T$ such that no two siblings are removed. Associate each color to a leaf of $T$. Contract each internal node in $T$ with one child to its child.

**for node $v$ in $T$ do**

1. $\alpha(v) \leftarrow$ sum of all weights of colorings associating with leaves under $v$.

**for internal node $v$ of $T$ do**

1. Instantiate 2-COLOR($v$) $\leftarrow$ oracle for weighted discrepancy problem at $v$ with weighted signs $\frac{\alpha(v_s)}{\alpha(v_l) + \alpha(v_r)}$ and $-\frac{\alpha(v_r)}{\alpha(v_l) + \alpha(v_r)}$.

**for $i$ from 1 to $n$ do**

1. $v \leftarrow$ root of $T$
2. For each internal node $v$ of $T$, the weighted discrepancy vector at $v$ is defined as:

$$w(v) = \frac{\alpha(v_r)}{\alpha(v_l) + \alpha(v_r)} s(v_l) - \frac{\alpha(v_l)}{\alpha(v_l) + \alpha(v_r)} s(v_r)$$

$$= \frac{1}{\alpha(v_l)} s(v_l) - \frac{1}{\alpha(v_r)} s(v_r),$$

where $v_l$ and $v_r$ are the left and right child of $v$ respectively.

For each internal node $v$ in $T$, we maintain an independent run of a two-coloring algorithm that minimizes $\|w(v)\|_{\infty}$. At a high level, we minimize the weighted discrepancies at all internal nodes simultaneously. When a new vector $x$ arrives, we first feed it to the algorithm at root $r_0$. If the result is $+$, we continue with the left sub-tree of $r_0$. Otherwise, we go to the right sub-tree. We continue in that manner until we reach a leaf $l$. $x$ will then be assigned to $l$ (and the color associated with $l$).

**Theorem 3.** Let 2-COLOR be Algorithm 1. Then Algorithm 2 maintains $O(\log k \sqrt{\log(dnk/\delta)})$ discrepancy at all rounds with probability $1 - \delta$ and obtains $O(\log k \sqrt{\log(dk/\delta)})$ final discrepancy with probability $1 - \delta$.

**Proof.** Let $D(\delta)$ be the discrepancy obtained by 2-COLOR as a function of the failure probability $\delta$. We will show that $(2 \log k)D(\delta/k)$ is the corresponding discrepancy obtained by Algorithm 2. Theorem 3 will then follow from Theorem 1 and Theorem 2. Notice that with probability $\delta/k$, each run of 2-COLOR at an internal node of $T$ has discrepancy $D(\delta/k)$. By union bounding over $O(k)$ internal nodes, we have that with probability $1 - \delta$, all of the discrepancies are bounded by $D(\delta/k)$.

Assume all the discrepancies at the internal nodes in $T$ are bounded, we show how to bound the discrepancy between any two colors. Let $l$ and $l'$ be any two leaves in $T$. The goal is to show that

$$\left\| \frac{\alpha(l')}{\alpha(l') + \alpha(l)} s(l) - \frac{\alpha(l)}{\alpha(l') + \alpha(l)} s(l') \right\|_{\infty}$$

is small. First we relate $s(v)$ to $s(v_l)$ and $s(v_r)$ where $v_l, v_r$ are the left and right children of $v$. By definition,

$$w(v) = \frac{\alpha(v_r)}{\alpha(v_l) + \alpha(v_r)} s(v_l) - \frac{\alpha(v_l)}{\alpha(v_l) + \alpha(v_r)} s(v_r)$$

and

$$s(v) = s(v_l) + s(v_r).$$
Therefore,

$$w(v) = s(v_1) - \frac{\alpha(v_1)}{\alpha(v_1) + \alpha(v_r)} s(v),$$

and

$$-w(v) = s(v_r) - \frac{\alpha(v_r)}{\alpha(v_1) + \alpha(v_r)} s(v).$$

Hence, both

$$\left\| s(v_1) - \frac{\alpha(v_1)}{\alpha(v)} s(v) \right\|_\infty$$

and

$$\left\| s(v_r) - \frac{\alpha(v_r)}{\alpha(v)} s(v) \right\|_\infty$$

are bounded by $D(\delta/k)$.

Now consider $v_1, v_2$ and $v_3$ in $T$ such that $v_1$ is a child of $v_2$ and $v_2$ is a child of $v_3$. We have, by triangle inequality,

$$\left\| s(v_1) - \frac{\alpha(v_1)}{\alpha(v_3)} s(v_3) \right\|_\infty \leq \left\| s(v_1) - \frac{\alpha(v_1)}{\alpha(v_2)} s(v_2) \right\|_\infty + \frac{\alpha(v_1)}{\alpha(v_2)} \left\| s(v_2) - \frac{\alpha(v_2)}{\alpha(v_3)} s(v_3) \right\|_\infty \leq \left( 1 + \frac{\alpha(v_1)}{\alpha(v_2)} \right) D(\delta/k).$$

Let $l$ be a leaf in $T$ and let $lv_1v_2\ldots r$ be the path from $l$ to the root $r$. Repeatedly applying the above relation along the path gives

$$\left\| s(l) - \frac{\alpha(l)}{\alpha(r)} s(r) \right\|_\infty \leq \left( 1 + \frac{\alpha(l)}{\alpha(v_1)} + \frac{\alpha(l)}{\alpha(v_2)} + \ldots + \frac{\alpha(l)}{\alpha(r)} \right) D(\delta/k).$$

(2)

Since there are at most $\log k$ nodes in the path from $l$ to $r$,

$$\left\| s(l) - \frac{\alpha(l)}{\alpha(r)} s(r) \right\|_\infty \leq (\log k) D(\delta/k).$$

Finally, for any two leaves $l$ and $l'$,

$$\left\| \frac{s(l)/\alpha(l) - s(l')/\alpha(l')}{1/\alpha(l) + 1/\alpha(l')} \right\|_\infty \leq \left\| \frac{s(l)/\alpha(l) - s(r)/\alpha(r)}{1/\alpha(l) + 1/\alpha(l')} \right\|_\infty + \left\| \frac{s(l')/\alpha(l') - s(r)/\alpha(r)}{1/\alpha(l) + 1/\alpha(l')} \right\|_\infty \leq (2 \log k) D(\delta/k).$$

\[ \square \]

**Remark 1.** If all weights are uniform, the summation in (2) becomes a geometric series and can be bounded by a constant. Therefore, we can remove the factor $\log k$ in Theorem 3.

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