Power counting of various Dirac covariants in hadronic Bethe-Salpeter wave functions for decay constant calculations of pseudoscalar mesons

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Abstract

We have employed the framework of Bethe-Salpeter equation under covariant instantaneous ansatz to calculate leptonic decay constants of unequal mass pseudoscalar mesons like $\pi^{\pm}$, $K$, $D$, $D_s$ and $B$ and radiative decay constants of neutral pseudoscalar mesons like $\pi^0$ and $\eta_c$ in two photons. In the Dirac structure of hadronic Bethe-Salpeter wave function, the covariants are incorporated from their complete set in accordance with a recently proposed power counting rule. The decay constants are calculated with the incorporation of both Leading order and Next-to-leading order Dirac covariants. The results validate the power counting rule which provides a practical means of incorporating Dirac covariants in the Bethe-Salpeter wave function for a hadron.

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I. INTRODUCTION

Quantum Chromodynamics (QCD) is the theory to describe strong interactions. However, the large gauge coupling at low energies (long distances) destroys the perturbative expansion. As a result, many non-perturbative approaches have been proposed to deal with this long distance properties of QCD, such as QCD sum rules, Lattice QCD, dynamical-equation-based approaches like Schwinger-Dyson equation and Bethe-Salpeter equation (BSE), and potential models. Since the task of calculating hadron structures from QCD itself is very difficult, as can be seen from various Lattice QCD approaches, one generally relies on specific models to gain some understanding of QCD at low energies. BSE is a conventional approach in dealing with relativistic bound state problems. From the solutions we can obtain useful information about the inner structure of hadrons, which is also crucial in treating high energy hadronic scatterings. The BSE framework is firmly rooted in field theory, and provides a realistic description for analyzing hadrons as composite objects. Despite its drawback of having to input model-dependent kernel, these studies have become an interesting topic in recent years, since calculations have shown that BSE framework using phenomenological potentials can give satisfactory results on more and more data being accumulated.

In this paper we study leptonic decays of pseudoscalar mesons (P-mesons) such as $\pi$, $K$, $D$, $D_S$ and $B$, which proceed through the coupling of quark-antiquark loop to the axial vector current and also the two-photon decays of neutral pseudoscalar mesons such as $\pi^0$ and $\eta_c$ which proceed through the famous quark-triangle diagrams. We employ QCD motivated BSE under Covariant Instantaneous Ansatz (CIA) in this paper \cite{1,2}. CIA is a Lorentz-invariant generalization of Instantaneous Ansatz. For a $q\bar{q}$ system, the CIA formulation ensures an exact interconnection between 3D and 4D forms of BSE \cite{2,3}. The 3D form of BSE serves for making contact with the mass spectrum, whereas the 4D form provides the Hadron-quark vertex function for evaluation of various hadronic transition amplitudes through quark loop diagrams. In these studies one of the main ingredients is the Dirac structure of the Bethe-Salpeter wave function (BSW). The copious Dirac structure of BSW was already studied by Llewlyn Smith \cite{4} much earlier. Recent studies \cite{5,6} have revealed that various covariant structures in BSWs of various hadrons is necessary to obtain quantitatively accurate observables. It has been further noticed that all covariants do not contribute equally for calculation of meson observables. So it is interesting to investigate
how to arrange these covariants. In a recent work \[2\], we developed a power counting rule for incorporating various Dirac structures in BSW, order-by-order in powers of inverse of meson mass. We have outlined the Dirac covaiants and expanded the coefficients to the leading order (LO), and calculated the leptonic decay constants of vector mesons (\(\rho, \omega, \phi, \psi\)) \[2\] as well as pseudoscalar mesons (\(\pi, K, D, D_S\) and \(B\)) \[3\] at this order. The results agree with data well.

However, common to all the perturbative theories, it is better to calculate the next order(s) to the leading one and make sure it is (they are) really smaller w.r.t. the LO, before claiming the validation of the perturbation. At the same time, as more and preciser data accumulated, it is useful to arrange more available parameters inherent in our framework to accommodate better fits to gain more precise information of the structure of hadron. So the study of next-to-leading order (NLO) is natural and essential. For all the mesons, the pseudoscalar is the simplest in Dirac structure. As the first step, we collect the data of leptonic decay constants \(f_P\)'s for pseudoscalar mesons (\(\pi, K, D, D_S\) and \(B\)), to fit three parameters \(B_i\) s in our framework at NLO. We found: a) NLO works better than LO. b) NLO corrections are smaller than those of LO (\(\pi\) is exceptional for its small mass, to be discussed later in this paper). Then with the fitted parameters we calculate the radiative decay constants \(F_P\) of neutral pseudoscalar mesons, \(\pi^0\) and \(\eta_c\) at NLO. We also found satisfying agreement with data, and fair improvement w.r.t. LO. Thus the fact that three parameters can give a good fit not only for 5 different cases of \(f_P\), but also giving satisfactory results for two cases of \(F_P\), demonstrates the validity and robustness of this framework. These results indicate that our power counting scheme \[2\] provides a practical means of incorporating various Dirac structures from their complete set into the BS wave function.

In what follows, we give a detailed discussion of the fit and calculation at NLO, after a brief review of our framework. The paper is organized as follows: In section II we discuss the structure of BS wave function for P-mesons in BSE under CIA using the power counting rule. In section III we introduce the fitting to \(f_P\) for pseudoscalar mesons. The radiative decay constants \(F_P\) for \(\pi^0\) and \(\eta_c\) mesons are calculated in section 4, while we conclude with Discussion in section V.
II. THE BSW UNDER CIA

A. BSE under CIA

We first outline the BSE framework under CIA. We have employed for the case of scalar quarks for simplicity. For a \( q\bar{q} \) system with an effective kernel \( K \) and 4D wave function \( \Phi(P,q) \), the 4D BSE takes the form,

\[
i(2\pi)^4 \Delta_1 \Delta_2 \Phi(P,q) = \int d^4q K(q,q')\Phi(P,q'),
\]

where \( \Delta_1, \Delta_2 = m_1^2 + p_1^2, m_2^2 + p_2^2 \) are the inverse propagators of two scalar quarks, and \( m_{1,2} \) are (effective) constituent masses of quarks. The 4-momenta of the quark and anti-quark, \( p_{1,2} \), are related to the internal 4-momentum \( q_\mu \) and total momentum \( P_\mu \) of hadron of mass \( M \) as

\[
p_{1,2\mu} = \hat{m}_{1,2} P_\mu \pm q_\mu,
\]

where \( \hat{m}_{1,2} = \frac{[1 \pm (m_1^2 - m_2^2)/M^2]}{2} \) are the Wightman-Garding (WG) definitions of masses of individual quarks. Now it is convenient to express the internal momentum of the hadron \( q \) as the sum of two parts, the transverse component, \( \hat{q}_\mu = q_\mu - \frac{q \cdot P}{P^2} \) which is orthogonal to total hadron momentum \( P \) (ie. \( \hat{q} \cdot P = 0 \) regardless of whether the individual quarks are on-shell or off-shell), and the longitudinal component, \( \sigma P_\mu = (q \cdot P/P^2) P_\mu \), which is parallel to \( P \). We now use an Ansatz on the BS kernel \( K \) in Eq. (1) which is assumed to depend on the 3D variables \( \hat{q}_\mu, \hat{q}'_\mu \) i.e.

\[
K(q,q') = K(\hat{q},\hat{q}'),
\]

A similar form of the BS kernel was also earlier suggested in ref. [8]). Hence, the longitudinal component, \( \sigma P_\mu \) of \( q_\mu \), does not appear in the form \( K(\hat{q},\hat{q}') \) of the kernel. For reducing Eq.(1) to the 3D form, we define a 3D wave function \( \phi(\hat{q}) \) as

\[
\phi(\hat{q}) = \int_{-\infty}^{+\infty} M d\sigma \Phi(P,q).
\]
Substituting Eq. (4) in Eq. (1), with definition of kernel in Eq. (3), we get a covariant version of Salpeter equation,

$$(2\pi)^3 D(\hat{q}) \phi(\hat{q}) = \int d^3 \hat{q}' K(\hat{q}, \hat{q}') \phi(\hat{q}'),$$

where $D(\hat{q})$ is the 3D denominator function defined by

$$\frac{1}{D(\hat{q})} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{M d\sigma}{\Delta_1 \Delta_2},$$

whose value can be easily worked out by contour integration by noting positions of poles in the complex $\sigma$-plane (shown in detail in [9]) as,

$$D(\hat{q}) = \frac{(\omega_1 + \omega_2)^2 - M^2}{2\omega_1} + \frac{1}{2\omega_2}, \quad \omega_{1,2}^2 = m_{1,2}^2 + \hat{q}^2.$$  

We can see that RHS of Eq. (5) is identical to RHS of Eq. (1) by virtue of Equations (3) and (4). We thus have an exact interconnection between 3D wave function $\phi(\hat{q})$ and 4D wave function $\Phi(P,q)$:

$$\Delta_1 \Delta_2 \Phi(P,q) = \frac{D(\hat{q}) \phi(\hat{q})}{2\pi i} \equiv \Gamma(\hat{q}).$$

We also get the $H q \bar{q}$ vertex function $\Gamma(\hat{q})$ under CIA for case of scalar quarks. Further in the process, an exact interconnection between 3D and 4D BSE is thus brought out where the 3D form serves for making contact with the mass spectrum of hadrons, whereas the 4D form provides the vertex $H q \bar{q}$ function $\Gamma(\hat{q})$ which satisfies a 4D BSE with a natural off-shell extension over the entire 4D space (due to the positive definiteness of the quantity $\hat{q}^2 = q^2 - (q \cdot P)^2/P^2$ throughout the entire 4D space) and thus provides a fully Lorentz-invariant basis for evaluation of various transition amplitudes through various quark loop diagrams.
B. Dirac structure of Hadron-quark vertex function for P-mesons in BSE with power counting scheme

To obtain the form of Hadron-quark vertex function for the case of fermionic quarks constituting a particular meson, we first replace the scalar propagators $\Delta_i^{-1}$ in Eq. (7) by the proper fermionic propagators $S_F$. The $Hq\bar{q}$ vertex function $\Gamma(\hat{q})$ now is a $4 \times 4$ matrix in spinor space. For incorporation of the relevant Dirac structures in $\Gamma(\hat{q})$, we make use of the power counting rule we developed in [2], order-by-order in powers of inverse of meson mass [2]. Our aim of developing the power counting rule was to find a “criterion” so as to systematically choose among various Dirac covariants from their complete set to write wave functions for different mesons (vector mesons, pseudoscalar mesons etc.).

As far as a pseudoscalar meson is concerned, its hadron-quark vertex function which has a certain dimensionality of mass can be expressed as a linear combination of four Dirac covariants [4], each multiplying a Lorentz scalar amplitude, as function of $q \cdot P$. We note that in the expression for CIA vertex function in equation (7), the factor $D(\hat{q})\phi(\hat{q})$ is nothing but the Lorentz-invariant momentum dependent scalar which depends on $q^2$, $P^2$ and $q \cdot P$ and has a certain dimensionality of mass. However the Lorentz-scalar amplitudes multiplying various Dirac structures in [5] have different dimensionalities of mass. For adapting this decomposition to write the structure of $Hq\bar{q}$ vertex function $\Gamma(\hat{q})$ for a particular meson, we re-express this function by making these scalar amplitudes dimensionless by weighing each covariant by an appropriate power $M$, the meson mass. Thus each term in the expansion of $\Gamma(\hat{q})$ is associated with a certain power of $M$ and hence in detail we can express the hadron-quark vertex, $\Gamma(\hat{q})$ as a polynomial in various powers of $1/M$:

$$\Gamma^P(\hat{q}) = \Omega^P \frac{1}{2\pi i} N_P D(\hat{q})\phi(\hat{q}), \quad (9)$$

with

$$\Omega^P = \gamma_5 B_0 - i\gamma_5(\gamma \cdot P) \frac{B_1}{M} - i\gamma_5(\gamma \cdot q) \frac{B_2}{M} - \gamma_5[(\gamma \cdot P)(\gamma \cdot q) - (\gamma \cdot q)(\gamma \cdot P)] \frac{B_3}{M^2}, \quad (10)$$

where $B_i$ ($i = 0, \ldots, 3$) are four dimensionless coefficients to be determined. Since we use constituent quark masses, where quark mass $m$ is approximately half of the hadron mass
we can use the ansatz

\[ q << P \sim M \]  

(11)

in the rest frame of the hadron (however we wish to mention that among all the pseudoscalar mesons, pion enjoys the special status in view of its unusually small mass \( M < \Lambda_{QCD} \) and its case should be considered separately). Then each of the four terms in Eq. (9,10) would again receive suppression by different powers of \( 1/M \). Thus we can arrange these terms as an expansion in powers of \( O(1/M) \). We can then see in the expansion of \( \Omega^P \), that the structures associated with the coefficients \( B_0, B_1 \) have magnitudes \( O(1/M^0) \) and are of leading order, while those with \( B_2, B_3 \) are \( O(1/M^1) \) and are next-to-leading-order. This naïve power counting rule suggests that the maximum contribution to the calculation of any pseudoscalar meson observable should come from the Dirac structures \( \gamma_5 \) and \( i\gamma_5 (\gamma \cdot P)/M \) associated with the constant coefficients \( B_0 \) and \( B_1 \) respectively, followed by the other two higher order covariants associated with coefficients \( B_2 \) and \( B_3 \). In general, the coefficients \( B_i \) of the Dirac structures could be functions of \( q \cdot P \), and hence can be written as a Taylor series in powers of \( q \cdot P \). However the coefficients used here are dimensionless on lines of [2]. So they are in fact function of \( q \cdot P/M^2 \). Then the leading order contribution of the coefficients are the case when the \( B_i \)'s are constant. In this paper, we assume the coefficients are smooth functions of \( q \cdot P/M^2 \), so to NLO, we only consider the terms of eq.(10), with the coefficients \( B_i \) constant. Because the normalization of the BSW can be fixed (see below), \( B_0 \) here can be taken to be 1. So we totally have 3 parameters to be fitted at NLO, comparing to one parameter at LO. In a similar manner one can express the full hadron-quark vertex function for a scalar and axial vector meson also in BSE under CIA. At the same time, the restriction by charge parity on wave function of eigenstate should also be respected. Further, to get the complete set of the Dirac structures for a certain kind of meson, the restriction by the (space) Parity have been employed; and it is easy to see that the requirements of the space Parity and the charge Parity are the same for the vertex as well as the full wavefunction [10]. In this work to calculate the leptonic and radiative decay constants, we take the form of hadron-quark vertex as in Eqs. (9) and (10) which incorporates LO as well as NLO covariants and see the relative importance of various covariants.
C. BSE Kernel and the scalar wave function

From the above analysis of the structure of vertex function $Hq\bar{q}$, we notice that the structure of 3D wave function $\phi(\hat{q})$ as well as the form of the 3D BSE are left untouched and have the same form as in our previous works which justifies the usage of the same form of the input kernel we used earlier [2]. Now we briefly mention some features of the BS formulation employed. The structure of BSE is characterized by a single effective kernel arising out of a four-fermion lagrangian in the Nambu-Jonalasino [11, 12] sense. The formalism is fully consistent with Nambu-Jona-Lasino [11] picture of chiral symmetry breaking but is additionally Lorentz-invariant because of the unique properties of the quantity $\hat{q}^2$, which is positive definite throughout the entire 4D space. The input kernel $K(q, q')$ in BSE is taken as one-gluon-exchange like as regards color $[(\lambda^{(1)}/2) \cdot (\lambda^{(2)}/2)]$ and spin $(\gamma^{(1)}_\mu \gamma^{(2)}_\mu)$ dependence.

The scalar function $V(q - q')$ is a sum of one-gluon exchange $V_{OGE}$ and a confining term $V_{conf}$. Thus we can write the interaction kernel as [2, 12]:

$$K(q, q') = \left(\frac{1}{2}\lambda^{(1)}\right) \cdot \left(\frac{1}{2}\lambda^{(2)}\right) V^{(1)}_\mu V^{(2)}_\mu V(q - q');$$

$$V^{(1,2)}_\mu = \pm 2m_{1,2}\gamma^{(1,2)}_\mu;$$

$$V(q - q') = \frac{4\pi\alpha_S(Q^2)}{(\hat{q} - \hat{q}')^2} + \frac{3}{4} \omega^2_{q\bar{q}} \int d^3r \left[r^2(1 + 4a_0\hat{m}_1\hat{m}_2M^2r^2)^{-1/2} - \frac{C_0}{\omega^2_0}\right] e^{i(\hat{q} - \hat{q}') \cdot r};$$

$$\alpha_S(Q^2) = \frac{12\pi}{33 - 2f} \left(\ln \frac{M^2_\geq}{\Lambda^2}\right)^{-1}; \quad M_\geq = \text{Max}(M, m_1 + m_2).$$

The Ansatz employed for the spring constant $\omega^2_{q\bar{q}}$ in Eq. (12) is [2, 12],

$$\omega^2_{q\bar{q}} = 4\hat{m}_1\hat{m}_2M_\geq\omega^2_0\alpha_S(M^2_\geq),$$

where $\hat{m}_1, \hat{m}_2$ are the Wightman-Garding definitions of masses of constituent quarks defined earlier. Here the proportionality of $\omega^2_{q\bar{q}}$ on $\alpha_S(Q^2)$ is needed to provide a more direct QCD motivation to confinement. This assumption further facilitates a flavour variation in $\omega^2_{q\bar{q}}$. And $\omega^2_0$ in Eq. (12) and Eq. (13) is postulated as a universal spring constant which is
common to all flavours. Here in the expression for $V(\hat{q} - \hat{q}')$, as far as the integrand of the confining term $V_{conf}$ is concerned, the constant term $C_0/\omega_0^2$ is designed to take account of the correct zero point energies, while $a_0$ term ($a_0 \ll 1$) simulates an effect of an almost linear confinement for heavy quark sectors (large $m_1, m_2$), while retaining the harmonic form for light quark sectors (small $m_1, m_2$) as is believed to be true for QCD. Hence the term $r^2(1 + 4a_0\hat{m}_1\hat{m}_2M^2\gamma^2r^2)^{-1/2}$ in the above expression is responsible for effecting a smooth transition from harmonic ($q\bar{q}$) to linear ($Q\bar{Q}$) confinement. The basic input parameters in the kernel are just four i.e. $a_0 = 0.028$, $C_0 = 0.29$, $\omega_0 = 0.158$ GeV and QCD length scale $\Lambda = 0.20$ GeV and quark masses, $m_{u,d} = 0.265$ GeV, $m_s = 0.415$ GeV, $m_c = 1.530$ GeV and $m_b = 4.900$ GeV which have been earlier fit to the mass spectrum of $q\bar{q}$ mesons obtained by solving the 3D BSE under Null-Plane Ansatz (NPA). However due to the fact that the 3D BSE under CIA has a structure which is formally equivalent to the 3D BSE under NPA, near the surface $P\cdot q = 0$, the $q\bar{q}$ mass spectral results in CIA formalism are exactly the same as the corresponding results under NPA formalism. The details of BS model under CIA in respect of spectroscopy are thus directly taken over from NPA formalism (see [2, 9, 12]). Now comes to the problem of the 3D BS wave function. The ground state wave function $\phi(\hat{q})$ satisfies the 3D BSE on the surface $P\cdot q = 0$, which is appropriate for making contact with O(3)-like mass spectrum (see [12]). Its fuller structure is reducible to that of a 3D harmonic oscillator with coefficients dependent on the hadron mass $M$ and the total quantum number $N$. The ground state wave function $\phi(\hat{q})$ deducible from this equation thus has a gaussian structure and is expressible as:

$$\phi(\hat{q}) \sim e^{-\hat{q}^2/2\beta^2}. \quad (14)$$

In the structure of $\phi(\hat{q})$ in (14), the parameter $\beta$ is the inverse range parameter which incorporates the content of BS dynamics and is dependent on the input kernel $K(q, q')$. The structure of the parameter $\beta$ in $\phi(\hat{q})$ is taken as:

$$\beta^2 = (2\hat{m}_1\hat{m}_2M\omega_{qq}'/\gamma)^{1/2}, \gamma^2 = 1 - \frac{2\omega_{qq}'C_0}{M^2\omega_0'^2}. \quad (15)$$

We now give the calculation of leptonic decays constants of pseudoscalar mesons employing both LO and NLO Dirac covariants according to our power counting scheme in the
framework discussed in next section.

III. CALCULATIONS AND RESULTS FOR $f_P$

A. Leptonic decays of pseudoscalar mesons to NLO

Decay constants $f_P$ can be evaluated through the loop diagram which gives the coupling of the two-quark loop to the axial vector current and can be evaluated as:

$$f_P P_\mu = \langle 0 | \bar{Q} i \gamma_\mu \gamma_5 Q | P(P) \rangle,$$

which can in turn be expressed as a loop integral,

$$f_P P_\mu = \sqrt{3} \int d^4q \, Tr [\Psi_P(P,q) i \gamma_\mu \gamma_5].$$

Bethe-Salpeter wave function $\Psi(P,q)$ for a P-meson is expressed as,

$$\Psi(P,q) = S_F(p_1) \Gamma(\hat{q}) S_F(-p_2),$$

which is expressed as the quark and anti-quark propagators flanking the Hadron-quark vertex $\Gamma(\hat{q})$ function which is in turn expressed by Eq. (9).

Using $\Psi(P,q)$ from Eq. (18), and incorporating $Hq\bar{q}$ vertex function $\Gamma(\hat{q})$ from Eq. (9) in Eq. (17), evaluating trace over the gamma matrices and multiplying both sides of Eq. (17) by $P_\mu/(-M^2))$, we can express the leptonic decay constant $f_P$ as,

$$f_P = f_P^{(0)} + f_P^{(1)} + f_P^{(2)} + f_P^{(3)},$$

where $f_P^{(0)}$, $f_P^{(1)}$, $f_P^{(2)}$, $f_P^{(3)}$, are the contributions to $f_P$ from the four Dirac covariants asso-
cated with coefficients $B_i$ ($i = 0, 1, 2, 3$), and are expressed as:

$$
\begin{align*}
\bar{f}_{P(0)} &= \sqrt{3N} N P_0 \int d^3 \bar{q} \bar{D}(\bar{q}) \phi(\bar{q}) \int_{-\infty}^\infty \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2} \left[-2m_1 + \frac{2m_1^3}{M^2} - 2m_2 - \frac{2m_1^2m_2}{M^2} - \frac{2m_1m_2^2}{M^2}\right] \\
+ & \frac{2m_2^2}{M^2} + 4(m_1 - m_2)\sigma \right],
\end{align*}
$$

$$
\begin{align*}
\bar{f}_{P(1)} &= \sqrt{3N} N P_1 \int d^3 \bar{q} \bar{D}(\bar{q}) \phi(\bar{q}) \int_{-\infty}^\infty \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2} \left[M - \frac{m_1^4}{M^3} + 4 \frac{m_1m_2}{M^3} + \frac{2m_1^2m_2}{M^3} - \frac{m_2}{M^3} - 4 \sigma \frac{\bar{q}^2}{M}\right] \\
+ & (m_2^2 - m_1^2)\sigma \frac{4}{M} - 4M\sigma^2 \right],
\end{align*}
$$

$$
\begin{align*}
\bar{f}_{P(2)} &= \sqrt{3N} N P_2 \int d^3 \bar{q} \bar{D}(\bar{q}) \phi(\bar{q}) \int_{-\infty}^\infty \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2} \left[\frac{4}{M^2} (m_1^2 - m_2^2)\bar{q}^2\right] \\
+ & \left(M - \frac{m_1^4}{M^3} + 4 \frac{m_1m_2}{M^3} + \frac{2m_1^2m_2}{M^3} - \frac{m_2}{M^3} \right)\sigma + 4 \frac{\bar{q}^2}{M}\sigma^2 + 4 \frac{M}{M^2} (m_2^2 - m_1^2)\sigma^2 - 4M\sigma^3 \right],
\end{align*}
$$

$$
\begin{align*}
\bar{f}_{P(3)} &= \sqrt{3N} N P_3 \int d^3 \bar{q} \bar{D}(\bar{q}) \phi(\bar{q}) \int_{-\infty}^\infty \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2} \left(-8 \frac{m_1 + m_2}{M^2}\bar{q}^2\right).
\end{align*}
$$

(20)

In deriving the above expressions, we had made use of the scalar products of various momenta expressed in terms of integration variables $\bar{q}$ and $\sigma$ as,

$$
\begin{align*}
p_1 \cdot p_2 &= -M^2 (\bar{m}_1 + \sigma)(\bar{m}_2 - \sigma) - \bar{q}^3, \\
p_1 \cdot P &= -M^2 (\bar{m}_1 + \sigma), \\
p_2 \cdot P &= -M^2 (\bar{m}_2 - \sigma), \\
P \cdot q &= -M^2 \sigma, \\
p_1^2 &= -M^2 (\bar{m}_1 + \sigma)^2 + \bar{q}^2, \\
p_2^2 &= -M^2 (\bar{m}_2 - \sigma)^2 + \bar{q}^2, \\
p_1 \cdot q &= \frac{1}{2} \left\{2\bar{q}^2 - \sigma[m_1^2 - m_2^2 + M^2(1 + 2\sigma)]\right\}, \\
p_2 \cdot q &= \frac{1}{2} \left\{-2\bar{q}^2 + \sigma[m_1^2 - m_2^2 + M^2(-1 + 2\sigma)]\right\}.
\end{align*}
$$

(21)

We see that on the right hand side of the expression for $f_P$, each of the expressions multiplying the constant parameters $B_0$ and $B_1$ consist of two parts, of which only the second part explicitly involves the off-shell parameter $\sigma$. It is can be seen that the off-shell contribution which vanishes for $m_1 = m_2$ in case of using only the leading covariant $\gamma_5$, would no longer vanish for $m_1 = m_2$ in the above calculation for $f_P$ (when other covariants are
incorporated in $Hq\bar{q}$ vertex function besides the leading covariant $\gamma_5$) due to the terms like $4M$ and $4q^2/M$ multiplying $\sigma^2$ in $f_P^{(2)}$ and $f_P^{(3)}$ respectively. This possibly implies that when other covariants besides $\gamma_5$ are incorporated into the vertex function, the off-shell part of $f_P$ does not arise from unequal mass kinematics alone (which is in complete contrast to the earlier CIA calculation of $f_P$ employing only $\gamma_5$). This may be a pointer to the fact that Dirac covariants other than $\gamma_5$ might also be important for the study of processes involving large $q^2$ (off-shell). Carrying out integration over $d\sigma$ by method of contour integration by noting the pole positions in the complex $\sigma$-plane:

$$\Delta_1 = 0 \Rightarrow \sigma_1^\pm = \pm \frac{\omega_1}{M} - \hat{m}_1 \mp i\varepsilon, \quad \omega_1^2 = m_1^2 + q^2,$$

$$\Delta_2 = 0 \Rightarrow \sigma_2^\mp = \mp \frac{\omega_2}{M} + \hat{m}_2 \pm i\varepsilon, \quad \omega_2^2 = m_2^2 + q^2,$$

(22)
we can again express $f_P$ as $f_P = f_P^{(0)} + f_P^{(1)} + f_P^{(2)} + f_P^{(3)}$, where now

\[ f_P^{(0)} = \sqrt{3} N_P B_0 \int d^3 \hat{q} D(\hat{q}) \phi(\hat{q}) \left[ (-2m_1 + 2 \frac{m_1^3}{M^2} - 2m_2 - 2 \frac{m_1^2 m_2}{M^2} - 2 \frac{m_1 m_2^2}{M^2}) + 2 \frac{m_1^2}{M^2} \right] \frac{1}{D(\hat{q})} + 4(m_1 - m_2)R_1, \]

\[ f_P^{(1)} = \sqrt{3} N_P B_1 \int d^3 \hat{q} D(\hat{q}) \phi(\hat{q}) \left[ (M - \frac{m_1^4}{M^3} + 4 \frac{m_1 m_2}{M} + 2 \frac{m_1^2 m_2}{M^3} - \frac{m_2^4}{M^3}) \frac{1}{D(\hat{q})} \right. \]

\[-4 \frac{\hat{q}^2}{M} D(\hat{q}) + \left. \frac{4}{M} (m_2^2 - m_1^2) R_1 - 4MR_2 \right], \]

\[ f_P^{(2)} = \sqrt{3} N_P B_2 \int d^3 \hat{q} D(\hat{q}) \phi(\hat{q}) \left[ \frac{4}{M^3} (m_1^2 - m_2^2) \hat{q}^2 \frac{1}{D(\hat{q})} \right. \]

\[ + \left. \left( M - \frac{m_1^4}{M^3} + 4 \frac{m_1 m_2}{M} + 2 \frac{m_1^2 m_2}{M^3} - \frac{m_2^4}{M^3} \right) R_1 \right] \]

\[ +4 \frac{\hat{q}^2}{M} R_1 + \frac{4}{M} (m_2^2 - m_1^2) R_2 \right], \]

\[ f_P^{(3)} = \sqrt{3} N_P B_3 \int d^3 \hat{q} D(\hat{q}) \phi(\hat{q}) \left[ -8 \frac{1}{M^2} (m_1 + m_2) \hat{q}^2 \frac{1}{D(\hat{q})} \right], \]

and $D(\hat{q})$ is given in Eq. (6), and the results of $\sigma$-integration in the complex $\sigma$-plane, on whether the contour is closed from above or below the real $\sigma$-axis is:

\[ R_1 = \int_{-\infty}^{+\infty} \frac{Md\sigma}{2\pi i D_1 D_2} \sigma = \frac{M^2 (-\omega_1 + \omega_2) + (m_1^2 - m_2^2)(\omega_1 + \omega_2)}{4M^2 \omega_1 \omega_2 [M^2 - (\omega_1 + \omega_2)^2]}, \]

\[ R_2 = \int_{-\infty}^{+\infty} \frac{M\sigma^3}{2\pi i D_1 D_2} \sigma^2 \]

\[ = \frac{(-M^4 - m_1^2 \delta m^2 + 4M^2 \omega_1 \omega_2)(\omega_1 + \omega_2) + 2M^2 m_1 \delta m (\omega_2 - \omega_1)}{8M^4 \omega_1 \omega_2 [M^2 - (\omega_1 + \omega_2)^2]} \]

To calculate BS normalizer $N_P$ for a pseudoscalar meson in the expression for $f_P$ in Eq.
\[2iP_{\mu} = (2\pi)^4 \int d^4q \, Tr \left[ \overline{\Psi}(P, q) \left( \frac{\partial}{\partial P_{\mu}} S_F^{-1}(p_1) \right) \Psi(P, q) S_F^{-1}(-p_2) \right] + (1 \leftrightarrow 2). \tag{25}\]

Putting BS wave function \(\Psi(P, q)\) from Eq. (18) in the above equation, carrying out derivatives of inverse of propagators of constituent quarks with respect to total momentum of hadron \(P_{\mu}\), evaluating trace over the gamma matrices, following usual steps and multiplying both sides of equation by \(P_{\mu}/(-M^2)\) to extract out the normalizer \(N_P\) from the above expression, we then express the above expression in terms of integration variables \(\hat{q}\) and \(\sigma\). Noting that the four dimensional volume element \(d^4q = d^3\hat{q} Md\sigma\), we then perform pole integration over \(d\sigma\) in complex \(\sigma\)-plane, making use of the pole positions in Eq. (22). The calculation of normalizer is extremely complex due to unequal mass kinematics. We thus give here a general expression for the normalizer integral of the form,

\[N_P^{-1} = -(2\pi)^2 i \int d^3\hat{q} D^2(\hat{q}) \phi^2(\hat{q}) [g_1(B, \hat{q}) I_1 + g_2(B, \hat{q}) I_2 + g_3(B, \hat{q}) I_3 + g_4(B, \hat{q}) I_4], \tag{26}\]

where \(B \equiv (B_0, B_1, B_2, B_3)\) and \(g_1, ... g_4\) are extremely complicated functions of \(B\) and \(\hat{q}\) and are extremely lengthy expressions, and hence we do not present their actual forms here, whereas \(I_1, ... I_4\) are analytic results of pole integration over the off-shell variable \(\sigma\) in the complex \(\sigma\)-plane and are expressed as:

\[I_1 = \int_{-\infty}^{+\infty} \frac{Md\sigma}{\Delta_1^2 \Delta_2^2} = 2\pi i \left[ \frac{2\omega_1^3 - M^2\omega_2 + 5\omega_1^2\omega_2 + 4\omega_1\omega_2^2 + \omega_2^3}{4\omega_1^3\omega_2(M^2 - (\omega_1 + \omega_2)^2)^2} \right], \]

\[I_2 = \int_{-\infty}^{+\infty} \frac{Md\sigma}{\Delta_1^2 \Delta_2 \sigma} = 2\pi i \frac{-M^4\omega_2 + (m_1^2 - m_3^2)(\omega_1 + \omega_2)^2(2\omega_1 + \omega_2)M^2[6\omega_1^3 + 9\omega_1^2\omega_2 + 4\omega_1\omega_2^2 + \omega_2(-m_1^2 + m_3^2 + \omega_2^2)]}{8M^2\omega_1^3\omega_2[M^2 - (\omega_1 + \omega_2)^2]^2}.
\]
\[\begin{align*}
I_3 &= \int_{-\infty}^{+\infty} \frac{M d\sigma}{\Delta_1^2 \Delta_2} \sigma^2 \\
&= 2\pi i \frac{1}{16M^4 \omega_1^2 \omega_2 (-M^2 + (\omega_1 + \omega_2)^2)^2} \\
&\quad \times \left\{ -M^4 (2\omega_1^3 - 2m_1^2 \omega_2 + 2m_2^2 \omega_2 + \omega_1^2 \omega_2 + 4\omega_1 \omega_2^2 + \omega_2^3) \\
&\quad -M^2 [m_1^4 \omega_2 + m_2^4 \omega_2 + 4\omega_1^2 \omega_2 (\omega_1 + \omega_2)^2 + 2m_2^2 (-2\omega_1^3 + \omega_1^2 \omega_2 + 4\omega_1 \omega_2^2 + \omega_2^3) \\
&\quad -2m_2^2 (-2\omega_1^3 + m_2^2 \omega_2 + \omega_1^2 \omega_2 + 4\omega_1 \omega_2^2 + \omega_2^3)] \right\} \\
I_4 &= \int_{-\infty}^{+\infty} \frac{M d\sigma}{\Delta_1^2 \Delta_2} \sigma^3 \\
&= 2\pi i \left\{ \frac{(M^2 - m_1^2 + m_2^2 + 2M \omega_2)^3}{8M^6 \omega_2 (M^2 - \omega_1^2 + 2M \omega_2 + \omega_2^2)^2} \\
&\quad + \frac{(M^2 + m_1^2 - m_2^2 - 2M \omega_1)^2 [M^4 + M^2 (m_1^2 - m_2^2 - \omega_1^2 - \omega_2^2) + (m_1^2 - m_2^2)(3\omega_1^2 - \omega_2^2)]}{16M^6 \omega_1^3 (M^2 - 2M \omega_1 + \omega_1^2 - \omega_2^2)^2} \right\} \\
&\quad + \frac{(M^2 + m_1^2 - m_2^2 - 2M \omega_1)^2 [-4M \omega_1 (m_1^2 - m_2^2 + \omega_2^2)]}{16M^6 \omega_1^3 (M^2 - 2M \omega_1 + \omega_1^2 - \omega_2^2)^2} \right\}.
\end{align*}\]

After this, numerical integration over the 3-D variable \(d^3 \hat{q}\) in Eq. (26) is performed to evaluate \(N_P\).

We have thus evaluated the expressions for \(f_P\) and \(N_P\) in framework of BSE under CIA, with Dirac structures of eq. (10), introduced in the \(H \bar{q} \bar{q}\) vertex function besides \(\gamma_5\) according to our power counting rule. We see that so far the results are independent of any model for \(\phi(\hat{q})\). However, for calculating the numerical values of these decay constants one needs to know the constant coefficients \(B_0, B_1, B_2, B_3\) which are associated with the above Dirac structures. Because of the normalization condition, we take \(B_0 = 1\), and then there are 3 parameters \(B_1/B_0, B_2/B_0, B_3/B_0\), which will still be denoted as \(B_1, B_2, B_3\) for simplicity. To
see the contribution of various Dirac covariants on the calculation of meson decay constants, we first discuss the numerical procedure adopted to fit these coefficients

B. Numerical Calculation

Eq. (23) which expresses decay constants \( f_P \) of pseudo-scalar mesons in terms of the parameters \( B_0, B_1, B_2, B_3 \) is a highly non linear function of the \( B_i \)’s. This obviously implies that numerical methods must be applied to solve the problem.

We used a simple Mathematica procedure for calculating the numerical integrals and searching for accurate values of the \( B_i \) (\( i = 0, \ldots, 3 \)). We defined the following auxiliary function \( W(B) \) which is positive definite as,

\[
W(B) = \sum_P [f_P(B) - f_P(exp.)]^2,
\]

where \( B \equiv (B_0, B_1, B_2, B_3) \), and summation in the above equation runs over five pseudoscalar mesons \( \pi, K, D, D_S \) and \( B \) mesons studied in this work, and \( f_P(exp.) \) are the central values of experimental data on decay constants \[13, 14\] (indicated in Table II).

From the numerical point of view the problem reduces to finding values of \( B_i \)’s such that \( W(B) \) has a minimum. We used Mathematica package which has some useful functions for minimizing. Those functions start from a point and search for a minimum near to that initial point. We constrained all the \( B_i \)'s to lie within the interval \( [0,1] \). We generated in a random way values of the \( B_i \) in this interval. Starting from those values, the Mathematica minimization function finds a minimum. Then it is checked if this minimum is “sufficiently near to zero”. This check is done by evaluating the percent average of the absolute values of the differences between the predicted \( f_P \) values from the experimental value \( f_P(exp.) \).

Using this method we found that the values of coefficients \( B_0, \ldots, B_3 \) (with average error with respect to the experimental data less than 3.5%) respectively are: \( B_0 = 1, B_1/B_0 = 0.3727, B_2/B_0 = 0.2234, B_3/B_0 = 0.0821 \) to give the decay constant values, \( f_\pi = 0.130 \) GeV, \( f_K = 0.164 \) GeV, \( f_D = 0.194 \) GeV, \( f_{DS} = 0.296 \) GeV. and \( f_B = 0.228 \) GeV which are within the error bars of experimental data \[13, 14\] depicted in Table II for these five pseudoscalar mesons. These values of \( f_P \) along with the contributions from various covariants and comparison with various models and experimental results are listed in Tables I and II.
There is one important point which needs to be clarified: The experimental data have different error bar, e.g., the data of $\pi$ has a very high precision to the order of 0.1%, while for the case of $B$, the relative error is more than 16%. So in the fitting, we should take into account the difference, e.g., assign different weight for these data. However, we only give our formulation at NLO. From the above discussions, it is straightforward to recognize, the smaller the meson mass, the larger the contributions of higher orders. For the case of pion, we even can perspect that higher order contributions (coming from higher order terms of Taylor series of $B_i$'s as powers of $\frac{q^2}{M^2}$) could be also very important. So, it is not reasonable to expect the NLO formulae can fit the data of pion to the precision of order of 0.1%. This is the reason why we fit the central value of data equally, as described above.

**IV. RADIATIVE DECAY CONSTANTS OF NEUTRAL P–MESONS**

In this section we calculate the radiative decays of a neutral pseudoscalar meson such as $\pi^0$ or $\eta_c$ proceeding through the process $P \rightarrow \gamma\gamma$ which proceed through the famous quark-triangle diagrams in the above framework using both the leading order and the next-to-leading order covariants in the Hadron-quark vertex function, taking the values of parameters $B_0 = 1, B_1/B_0 = 0.3727, B_2/B_0 = 0.2234, B_3/B_0 = 0.0821$ fixed above in the calculation of $f_P$ values of $\pi, K, D, D_S$ and $B$ mesons. The invariant amplitude for the decay of a neutral P-meson into two photons can be expressed as summation over the two triangle diagrams corresponding to the Direct and Exchange processes as:

$$A(P \rightarrow 2\gamma) = \frac{e^2}{\sqrt{6}} \int d^4q Tr[\Psi(P,q)i\gamma.\epsilon_1 S_F(q-Q)i\gamma.\epsilon_2] + \frac{e^2}{\sqrt{6}} \int d^4q Tr[\Psi(P,q)i\gamma.\epsilon_2 S_F(q+Q)i\gamma.\epsilon_1]$$ (29)

where $\Psi(P,q)$ is the BS wave function of a neutral P-meson given explicitly in Eq.(20) and Eq.(12)-(13), $S_F(q \pm Q)$ are the propagators of the third quark in the Direct and Exchange diagrams respectively, where $Q = k_1 - k_2$ is the the difference in momenta of the two emitted photons with momenta $k_1$ and $k_2$ respectively, while $\epsilon_{1,2}$ are the polarization vectors of the two emitted photons in the above diagrams which differ from each other in the interchange $1 \leftrightarrow 2$. Evaluating traces over the gamma- matrices, combining various terms and then performing pole-integrations in the complex $\sigma-$plane, we
can express amplitude for the above process as:

\[
A(P \rightarrow 2\gamma) = [F_P] \epsilon_{\mu\nu\rho\sigma} P_\mu \epsilon_{2\nu} Q_\rho \epsilon_{1\sigma}, \tag{30}
\]

where \( P = p_1 + p_2 \) is the total hadron momentum, where \( p_{1,2} \) are the momenta of the quarks constituting the hadron, and the radiative decay constant, \( F_P \) is given as (the \( B_2 \) term vanishes because of wrong charge parity),

\[
F_P = \frac{e^2 N_P}{\sqrt{6}} \int d^3\hat{q}D(\hat{q})\phi(\hat{q}) \left[ B_0[8mS_1] + B_1[-\frac{16m^2}{M}S_1 + \frac{4}{M}S_2 + \frac{4}{M}S_3] + B_3[\frac{8m}{M^2}(S_2 + S_3 + S_4 - S_5)] \right], \tag{31}
\]

where \( S_{1,2,3,4,5} \) are the analytical results of integrals over the off-shell parameter \( \sigma \):

\[
S_1 = \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2 \Delta_3} = \frac{12}{M^4 \omega - 20M^2 \omega^3 + 64 \omega^5}; \tag{32}
\]

\[
S_2 = \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_2 \Delta_3} = \frac{4}{-M^2 \omega + 16 \omega^3}; \tag{33}
\]

\[
S_3 = \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_3} = \frac{4}{-M^2 \omega + 16 \omega^3}; \tag{34}
\]

\[
S_4 = \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_3} \sigma = \frac{-1}{-M^2 \omega + 16 \omega^3}; \tag{35}
\]

\[
S_5 = \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_2 \Delta_3} \sigma = \frac{1}{-M^2 \omega + 16 \omega^3}
\]

evaluated by the method of contour integrations by noting the various pole positions in the complex \( \sigma \)-plane:
\[ \Delta_1 = 0 \Rightarrow \sigma_1^\pm = \pm \frac{\omega}{M} - \frac{1}{2} \mp i\varepsilon; \]
\[ \Delta_2 = 0 \Rightarrow \sigma_2^\pm = \pm \frac{\omega}{M} + \frac{1}{2} \pm i\varepsilon; \]
\[ \Delta_3 = 0 \Rightarrow \sigma_3^\pm = \pm \frac{\omega}{M} \mp i\varepsilon; \]
\[ \omega^2 = m^2 + \hat{q}^2; \] (36)

From Eq.(33), it can be noticed that the contribution to radiative decay constant \( F_P \) from one of the next-to-leading order covariants associated with the parameter \( B_2 \) completely vanishes after trace evaluation. Numerical evaluation of \( F_P \) for \( \pi^0 \) and \( \eta_c \) using the same set of parameters, \( B_i / B_0 \) fixed from the calculation of leptonic decay constant \( f_P \) values of \( \pi, K, D, D_S \) and \( B \) mesons above gives \( F_\pi = 0.031 GeV^{-1} \), \( F_{\eta_c} = 0.006 GeV^{-1} \). These are very close to the experimental numbers \( F_\pi(Exp.) = 0.025 GeV^{-1} \), and \( F_{\eta_c}(Exp.) = 0.0074 GeV^{-1} \) which are arrived at through the expression, \( \Gamma(P \rightarrow 2\gamma) = \frac{F_P^2 M_P^3}{64\pi} \) connecting the decay width \( \Gamma \) with radiative decay constants, \( F_P \), using the central values of experimental data on decay widths for \( \pi \) and \( \eta_c \) mesons as \( \Gamma(\pi^0 \rightarrow 2\gamma) = 8.5 eV \) and \( \Gamma(\eta_c \rightarrow 2\gamma) = 7.4 KeV \) respectively.
V. DISCUSSION

In this paper we have calculated the decay constants $f_P$ of pseudoscalar mesons $\pi$, $K$, $D$, $D_S$ and $B$ and radiative decay constants $F_P$ for neutral pseudoscalar mesons $\pi^0$ and $\eta_c$ proceeding through the process $P \to 2\gamma$ in BSE under CIA. The Hadron-quark vertex function incorporates various Dirac covariants order-by-order in powers of inverse of meson mass within its structure in accordance with a power counting rule from their complete set. This power counting rule suggests that the maximum contribution to any meson observable should come from Dirac structures associated with Leading order terms alone, followed by Dirac structures associated with Next-to-Leading Order terms in the vertex function. Incorporation of all these covariants is found to bring calculated $f_P$ values much closer to results of experimental data \cite{13, 14} and some recent calculations \cite{3, 6, 15, 16} for $\pi$, $K$, $D$, $D_S$ and $B$ mesons. The $f_P$ are within the error bars of experimental data for each one of these five mesons by fitting three parameters. The calculation of radiative decay constants of $\pi^0$ and $\eta_c$ is again close to the experimental data\cite{17, 18}.

The results for $\pi$, $K$, $D$, $D_S$ and $B$ mesons with parameter set: $B_0 = 1$, $B_1/B_0 = 0.3727$, $B_2/B_0 = 0.2234$, $B_3/B_0 = 0.0821$ (giving $f_P$ values with average error with respect to experimental data less than 3.5%) are presented in Table I. In Fig. 1 we are plotting functions $I_P^i(\hat{q})$ ($i = 0, \ldots, 3$) vs $\hat{q}$, where $I_P^i(\hat{q})$ is the integrand of $f_P^{(i)}$ in equations (23). The plots of variations of $I_P^0(\hat{q}), \ldots I_P^3(\hat{q})$ with $\hat{q}$ for $\pi$, $K$, $D$, $D_S$ and $B$ mesons, along with the results in Table I, show that the contribution to $f_P$ from NLO covariants is much smaller than the contribution from LO covariants for $K$, $D$, $D_S$ and $B$ mesons. Comparison with experimental data and other models is shown in Table II. It is seen from Table I that as far as the various contributions to decay constants $f_P$ are concerned, for $K$ mesons, the LO terms contribute 60\%, while NLO terms 40\%. However for heavy-light meson $D$, the LO contribution increases to 90\%, while NLO contribution is 10\%. For $D_S$ meson, LO contribution is 91\%, while NLO contribution is 9\%. But for $B$ meson, the LO contribution is 96\%, while NLO contribution reduces to just 4\%. This is in conformity with the power counting rule according to which the leading order covariants, $\gamma_5$ and $i\gamma_5(\gamma \cdot P)(1/M)$ (associated with coefficients $B_0$ and $B_1$) should contribute maximum to decay constants followed by the next-to-leading order covariants, $-i\gamma_5(\gamma \cdot q)(1/M)$ and $-\gamma_5[(\gamma \cdot P)(\gamma \cdot q) - (\gamma \cdot q)(\gamma \cdot P)](1/M^2)$ (associated with coefficients $B_2$ and $B_3$) in the BS wave function, Eq. (9)-(10).
However the situation is different for the lightest meson \( \pi \) which enjoys a unique status due to the fact that mass of a pion, \( M \) is unusually small \( (\ll \Lambda_{QCD}) \), and the large difference between the sum of two constituent quark masses and the pion mass shows that the quarks are far off shell and the internal momentum \( q \) should be the same order as the pion mass and the approximation \( q \ll P \sim M \) breaks down for pion. Hence the contribution of NLO covariants in pion case is even larger than the contribution of LO covariants. Thus, the NLO covariants in pion should play a more dominant role in contrast to heavier mesons \( K, D, D_S \) and \( B \). However the sum of LO and NLO contributions adds up to the experimental value for pion \( f_\pi (=0.130 \text{ GeV}) \). Further investigations on higher order terms can show even more details of the pion structure.

To check the validity of our calculation, we then do numerical evaluation of radiative decay constants \( F_\pi \) for \( \pi^0 \) and \( \eta_c \) using the same set of parameters, \( B_i/B_0 \) fixed above from the calculation of leptonic decay constant \( f_\pi \) values of \( \pi, K, D, D_S \) and \( B \) mesons. This gives \( F_\pi = 0.031 \text{ GeV}^{-1} \), \( F_{\eta_c} = 0.006 \text{ GeV}^{-1} \). These are very close to the experimental numbers \( F_\pi (\text{Exp.}) = 0.025 \text{ GeV}^{-1} \), and \( F_{\eta_c} (\text{Exp.}) = 0.0074 \text{ GeV}^{-1} \) which are arrived at through the expression, \( \Gamma(P \to 2\gamma) = \frac{F_\pi^2 M^3}{64 \pi} \) connecting the decay width \( \Gamma \) with radiative decay constants, \( F_\pi \), using the central values of experimental data on decay widths for \( \pi \) and \( \eta_c \) mesons as \( \Gamma(\pi^0 \to 2\gamma) = 8.5 eV \) and \( \Gamma(\eta_c \to 2\gamma) = 7.4 \) \([17, 18]\) respectively.

The numerical results for leptonic decay constants, \( f_\pi \) and radiative decay constants, \( F_\pi \) obtained in our framework upto the next to leading order covariants demonstrates the validity of our power counting rule, which also provides a practical means of incorporating various Dirac covariants in the BS wave function of a hadron. By this rule, we also get to understand the relative importance of various covariants to calculate various meson observables. This would in turn help in obtaining a better understanding of the hadron structure. Here would would like mention the robustness of our framework: On one hand, at lower order(s), with limited number of parameters, we can globally reproduce almost all the decay constants of certain kinds of meson. On the other hand, by introducing higher order corrections, we can accommodate enough parameters to fit the data as precise as possible, so than to get a good parameterization of the structure of certain special hadron for further investigations.
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TABLE I: Decay constant $f_P$ values (in GeV) for $\pi$, $K$, $D$, $D_S$ and $B$ mesons in BSE with the individual contributions $f^0_P$, $f^1_P$, $f^2_P$, $f^3_P$ from various Dirac covariants along with the contributions from LO and NLO covariants and also their % contributions for parameter set: $B_0 = 1$, $B_1/B_0 = 0.3727$, $B_2/B_0 = 0.2234$, $B_3/B_0 = 0.0821$ (with average error with respect to the experimental data less than 3.5%)

|      | $f^0_P$ | $f^1_P$ | $f^2_P$ | $f^3_P$ | $f^P_{LO}$ | $f^P_{NLO}$ | $f^P_{PLO}(\%)$ | $f^P_{PLO}(\%)$ | $f_P = f^P_{LO} + f^P_{NLO}$ |
|------|---------|---------|---------|---------|------------|------------|----------------|----------------|-------------------------------|
| $\pi$ | 0.110   | -0.154  | 0.000   | 0.175   | 0.044      | 0.175      | 25%            | 75%            | 0.130                         |
| $K$  | 0.202   | -0.104  | 0.025   | 0.039   | 0.098      | 0.064      | 60%            | 40%            | 0.164                         |
| $D$  | 0.271   | -0.097  | 0.010   | 0.009   | 0.174      | 0.019      | 90%            | 10%            | 0.194                         |
| $D_S$| 0.426   | -0.156  | 0.013   | 0.013   | 0.270      | 0.026      | 91%            | 9%             | 0.296                         |
| $B$  | 0.345   | -0.125  | 0.005   | 0.003   | 0.220      | 0.008      | 96%            | 4%             | 0.228                         |

TABLE II: Comparison of results of $f_P$ (in GeV) for $\pi$, $K$, $D$, $D_S$ and $B$ in BSE with the parameter set $B_0 = 0.7045$, $B_1 = 0.2626$, $B_2 = 0.1574$, $B_3 = 0.0579$ (with average error 3.5%) with those of other models and experimental data.

|      | $f_\pi$ | $f_K$   | $f_D$   | $f_{D_S}$ | $f_B$   |
|------|---------|---------|---------|-----------|---------|
| BSE (3.5% average error) |         |         |         |          |         |
| present paper              | 0.130   | 0.164   | 0.194   | 0.296    | 0.228   |
| BSE [5]                     |         |         |         |          | 0.248   |
| SDE [6]                     |         | 0.164   |         |          |         |
| Lattice [15]                |         |         | 0.208±0.004 | 0.241±0.003 |
| QCD-SR [16]                 |         |         | 0.20±0.02 | 0.23±0.02 |
| Exp. Results [13]           | 0.1300±0.0001 | 0.159±0.001 | 0.22±0.02 | 0.29±0.03 |
| Babar+Belle Collaboration [14] |         |         |         |          | 0.24±0.04 |
$l_P^1(\hat{q})$

- Pion
- Kaon
- D Meson
- Ds Meson
- B Meson
$I_p^2(\hat{q})$

- Pion
- Kaon
- D Meson
- Ds Meson
- B Meson
$I_P^3(\hat{q})$

- Pion
- Kaon
- D Meson
- Ds Meson
- B Meson