Persistence probability of a random polynomial arising from evolutionary game theory

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Abstract

In this paper, we obtain an asymptotic formula for the persistence probability in the half nonnegative axis of a random polynomial arising from evolutionary game theory. It corresponds to the probability that a multi-player two-strategy random evolutionary game has no internal equilibria. The key ingredient is to approximate the sequence of random polynomials indexed by their degrees by an appropriate centered stationary Gaussian process.

1 Introduction

In this paper, we study the persistence probability, that is the probability of not changing sign, in the half nonnegative axis of the following random polynomial

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} a_i x^i,$$  (1)

where the coefficients $a_i$’s are real and independent identically distributed (i.i.d.) standard normal random variables.

Our first motivation comes from evolutionary game theory [MSP73, HS98]. The polynomial $f_n$ originates from the study of equilibrium points in random evolutionary game
theory: finding an internal equilibrium point in a symmetric $n$-player two-strategy random game is equivalent to finding a positive zero of $f_n$, see the derivation in Section 2. In particular, the persistence probability of $f_n$ in the half nonnegative axis corresponds to the probability that the random game has no internal equilibria. Random evolutionary games have been used widely and successfully in the mathematical modelling of social and biological systems where limited information is available or where the environment changes so rapidly and frequently that one cannot predict the payoffs of their inhabitants. Such scenario often arises from numerous fields such as biology, ecology, population genetics, economics and social sciences [May01, FH92, HTG12, GRLD09]. In these situations, due to randomness, characterizing the statistical properties of equilibrium points becomes essential and has attracted considerable interest in recent years. In [GT10, HTG12, GT14], the authors provide analytical and simulation results for random games with small number of players ($n \leq 4$) focusing on the probability of attaining the maximal number of equilibrium points. In [DH15, DH16, DTH17], the authors derive a closed formula for the expected number of internal equilibria, characterize its asymptotic behaviour and study the effect of correlations. Related work on the expected number of equilibrium points of random large complex systems arising from physics and ecology are presented in [Fyo04, FN12, FK16], see also references therein. More recently, [DTH17a] offers, among other things, an analytical formula for the probability that a multi-player two strategy game has a certain number of internal equilibria. Although the analytical formula is theoretically interesting, it involves complicated multiple integrals and is computationally intractable when the number of player becomes large. The present paper provides an asymptotic formula, as the number of players tends to infinity, for the probability that the game has no internal equilibria. Biologically this probability corresponds to the two extreme cases when the whole population consists of only one species/strategy while the other extincts.

Our second motivation is from random polynomial theory in which the study of zeros of a random polynomial has been studied extensively since the seminal paper of Block and Pólya [BP32]. We review here relevant work on the persistene probability and refer the reader to standard monographs [BRS86, Far98] and recent articles [TV15, NNV16, DVT17, BZ17] and references therein for information on other aspects of random polynomials such as the expected number, central limit theorem and large deviations. A random polynomial can be generally expressed by

$$P_n(x) = \sum_{i=0}^{n} c_i \xi_i x^i,$$

where $c_i$ are deterministic coefficients which may depend on both $n$ and $i$ and $\xi_i$ are random variables. The most popular random polynomials studied in the literature are:

(i) Kac polynomials (denoted by $P_n^K$): $c_i := 1$,

(ii) Weyl (or flat) polynomials ($P_n^W$): $c_i := \frac{1}{n}$,

(iii) Elliptic (or binomial) polynomials ($P_n^E$): $c_i := \sqrt{\binom{n}{i}}$. 
For Kac polynomials, it is shown in [LO39, LO48] that $P(K_n = 0) = O(1/\log n)$ where $N^K_n$ is the number of real zeros of the Kac polynomial $P^K_n$. This result is extended in [DPSZ02] to the case where $\xi_i$ are i.i.d. random variables with the common distribution having finite moments of all orders as

$$\mathbb{P}(P^K_n(x) > 0, \forall x \in \mathbb{R}) = n^{-b_0 + o(1)},$$

where the constant $b_0$ above is given by

$$b_0 = -\lim_{t \to \infty} t^{-1} \log \mathbb{P}(X > 0, \forall s \in [0, t])$$

with $X$ the centered stationary Gaussian process with correlation $\mathbb{E}(X_0X_t) = 1/\cosh(t/2)$. In [SM08], the authors develop a mean-field approximation to re-derive the persistence probability of (generalized) Kac polynomials relating it to zero crossing properties of the diffusion equation with random initial conditions. Moreover, using this method, they predict and numerically verify the following asymptotic formulas for elliptic and Weyl models:

(i) For elliptic polynomials:

$$\lim_{n \to \infty} \frac{\log \mathbb{P}(P^E_n(x) > 0, \forall x \in \mathbb{R})}{\sqrt{n}} = -2\pi b,$$

where $b$ is a positive constant defined as

$$b = -\lim_{T \to \infty} \frac{\log \mathbb{P}(\inf_{0 \leq t \leq T} Y(t))}{T},$$

where $Y(t)$ is a centered stationary Gaussian process with correlation $\mathbb{E}(Y_0Y_t) = e^{-t^2/2}$.

(ii) For Weyl polynomials

$$\lim_{n \to \infty} \frac{\log \mathbb{P}(P^W_n(x) > 0, \forall x \in \mathbb{R})}{\sqrt{n}} = -2b,$$

with the same constant $b$ as in (4).

The statement (3) for elliptic polynomial is settled in [DM15]. In addition, this work also shows (5) for the interval intervals $I_n = [0, \sqrt{n} - \alpha_n]$ with $n^{1/2}\alpha_n \to 0$ obtaining the persistence exponent $-b$. More recently, by extending the method of [DM15], the authors of [CP17] prove (3) for Weyl polynomials. Inspired by this development, in this paper we study the asymptotic behaviour of the persistence probability in the half nonnegative axis of the random polynomial $f_n$ in (1). As discussed in the first motivation, this corresponds to the probability that a $n$-player two-strategy random evolutionary game has no internal equilibria. Our main result is the following theorem.
Theorem 1.1. Let \( f_n \) be defined in (1) where the coefficients \( a_i \)'s are i.i.d. standard normal random variables. Then we have

\[
\lim_{n \to \infty} \frac{\log \mathbb{P}(f_n(x) > 0, \forall x \in (0, \infty))}{\pi \sqrt{n}} = -b,
\]

where \( b \) is the persistent exponent defined by

\[
b = -\lim_{T \to \infty} \frac{\log \mathbb{P}(\inf_{0 \leq t < T} Z(t) > 0)}{T},
\]

with \( Z(t) \) a centered stationary Gaussian process with correlation \( \mathbb{E}(Z_0Z_t) = e^{-t^2/4} \).

The idea of the proof is as follows. We first show that the contributions to the persistent exponent of intervals \((0, n^{-1/6})\) and \((n^{1/6}, \infty)\) are negligible. We then apply the method of [DM15] to prove that the main contribution from the interval \((n^{-1/6}, n^{1/6})\) can be calculated approximately from that of a centered stationary Gaussian process with covariance function \( R(t) = e^{-t^2/4} \).

The rest of paper is organized as follows. In Section 2 we review the derivation of the random polynomial \( f_n \) from the replicator dynamics for a symmetric multi-player two-strategy random evolutionary game. Section 3 contains technical lemmas. The proof of the main theorem is presented in Section 4. Finally, we provide further discussion on future work in Section 5.

2 Derivation of \( f_n \) from evolutionary game theory

In this section, we recall the derivation of the random polynomial \( f_n \) in (1) from the replicator dynamics for multi-player two-strategy games in evolutionary game theory. The replicator equation for multi-player two-strategy games has already been derived in previous works [HS98, Sig10, GT10]. For the sake of completeness, we rederive it here.

Let us consider an infinitely large population consists of individuals using two strategies, \( A \) and \( B \). Let \( y, 0 \leq y \leq 1 \), be the frequency of strategy \( A \) in the population. The frequency of strategy \( B \) is thus \((1 - y)\). The interaction of the individuals in the population is in randomly selected groups of \( n \) participants, that is, they interact and obtain their fitness from \( n \)-player games. In this paper, we consider symmetric games where the payoffs do not depend on the ordering of the players. Let \( a_k \) (respectively, \( b_k \)) be the payoff of that an \( A \)-strategist (respectively, \( B \)) achieves when interacting with a group containing \( k \) \( A \) strategists (and \( n - k \) \( B \) strategists). In symmetric games, the probability that an \( A \) strategist interacts with \( k \) other \( A \) strategists in a group of size \( n \) is

\[
\binom{n-1}{k} y^k (1-y)^{n-1-k}.
\]

We note that this probability depends only on the number \( k \) of \( A \) strategist but not on the particular order of the group. The average payoffs of \( A \) and \( B \) are, respectively

\[
\pi_A = \sum_{k=0}^{n-1} a_k \binom{n-1}{k} y^k (1-y)^{n-1-k}, \quad \pi_B = \sum_{k=0}^{n-1} b_k \binom{n-1}{k} y^k (1-y)^{n-1-k}.
\]
The replicator equation of a \( d \)-player two-strategy game is given by [HS98, Sig10, GT10]

\[
\dot{y} = y(\pi_A - \bar{\pi}) = y(1 - y)\left(\pi_A - \pi_B\right),
\]

where \( \bar{\pi} := y\pi_A + (1 - y)\pi_B \) is the average payoff of the population. The replicator equation reflects the natural selection. In fact, if \( \pi_A \geq \bar{\pi} \) then \( x \) increases, that is \( A \) spreads in the population; vice versa, if \( \pi_A < \bar{\pi} \) then \( y \) decreases and \( A \) declines. Equilibrium points of the dynamics satisfy that \( y(1 - y)(\pi_A - \pi_B) = 0 \). Since \( y = 0 \) and \( y = 1 \) are two trivial equilibrium points, we focus only on internal ones, i.e. \( 0 < y < 1 \). They satisfy the condition that the fitnesses of both strategies are the same \( \pi_A = \pi_B \), which gives rise to

\[
\sum_{k=0}^{n-1} \beta_k \binom{d-1}{k} y^k(1-y)^{n-1-k} = 0,
\]

where \( \beta_k = a_k - b_k \). Using the transformation \( x = \frac{y}{1-y} \), with \( 0 < x < +\infty \), dividing the left hand side of the above equation by \( (1-y)^{n-1} \) we obtain the following polynomial equation for \( x \)

\[
f_n(x) := \sum_{k=0}^{n-1} \beta_k \binom{n-1}{k} x^k = 0. \tag{8}
\]

Note that this equation can also be derived from the definition of an evolutionary stable strategy, see e.g., [BCV97]. In complex large systems, information about the interaction between participants is rarely available at the level of detail sufficient for the exact computation of the payoff matrix; therefore, it is necessary to suppose that the payoff matrix, \( a_k \) and \( b_k \) (thus \( \beta_k \)), for \( 0 \leq k \leq n-1 \), are random variables. We then obtain random games and the polynomial \( f_n \) becomes a random polynomial. It is exactly the random polynomial \( f_n \) in (11) that we start with.

\section{Preliminaries}

In this section, we prove some technical results that will be used in the proof of the main theorem presented in Section 4.

Since \( \{a_i\} \) are i.i.d. random variables of standard normal distribution, the random polynomial \( f_n(x) \) is a Gaussian process with autocorrelation function

\[
A_n(x, y) = \frac{M_n(\sqrt{xy})}{\sqrt{M_n(x)} \sqrt{M_n(y)}}, \tag{9}
\]

where

\[
M_n(x) = \sum_{i=0}^{n} \binom{n}{i}^2 x^{2i}. \tag{10}
\]

We prove here a key lemma on the behavior of \( M_n(x) \) as \( n \to \infty \).
Lemma 3.1. For $x \in (0, 1]$, we define

$$i_x = \left\lfloor \frac{nx}{x + 1} \right\rfloor.$$

(i) For $\frac{\log n}{6n} \leq x \leq 1$, we have

$$\left(\frac{n}{i_x}\right)^2 x^{2i_x} \leq M_n(x) \leq 3x^{3/4} \left(\frac{n}{i_x}\right)^2 x^{2i_x}.$$

(ii) For $n^{-1/6} \leq x \leq 1$, we have

$$M_n(x) = (1 + O(n^{-1/24})) \sqrt{\frac{\pi n x}{(x + 1)}}.$$ 

Proof. Let us start with Stirling formula that

$$i! = \sqrt{2\pi i}(1 + O(i^{-1})) \left(\frac{i}{e}\right)^i.$$ 

Therefore,

$$\left(\frac{n}{i}\right) = \sqrt{\frac{n}{2\pi i(n-i)}} (1 + O(i^{-1})) \left(\frac{n}{i}\right)^i \left(\frac{n}{n-i}\right)^{n-i}$$

$$= \sqrt{\frac{n}{2\pi i(n-i)}} (1 + O(i^{-1})) \exp\left(nI\left(\frac{i}{n}\right)\right),$$

where $I(0) = I(1) = 0$ and for $t \in (0, 1)$,

$$I(t) = (t - 1) \log(1 - t) - t \log t.$$ 

Hence

$$\left(\frac{n}{i}\right)^2 x^{2i} = \frac{n}{2\pi i(n-i)} (1 + O(i^{-1})) \exp\left(2nJ_x\left(\frac{i}{n}\right)\right)$$

$$\tag{11}$$

where

$$J_x(t) = I(t) + t \log x.$$ 

We notice that

$$J_x\left(\frac{x}{x+1}\right) = \log(x + 1), \quad J_x'\left(\frac{x}{x+1}\right) = 0, \quad J_x''(t) = \frac{-1}{t(1-t)} \quad \forall \ t \in (0, 1). \quad \tag{12}$$
Therefore, using Taylor expansion, we get

\[
J_x \left( \frac{i_n}{n} \right) = J_x \left( \frac{x}{n+1} \right) + J'_x \left( \frac{x}{n+1} \right) \left( \frac{i_n}{n} - \frac{x}{n+1} \right) + \frac{J''_x(\eta_x)}{2} \left( \frac{i_n}{n} - \frac{x}{n+1} \right)^2 
\]

with some \( \eta_x \in \left( \frac{i_n}{n}, \frac{x}{n+1} \right) \). We have

\[
\left| \frac{i_n}{n} - \frac{x}{n+1} \right| \leq \frac{1}{n}, \quad |J''(\eta_x)| = \frac{1}{\eta_x(1-\eta_x)} \leq \frac{2n}{i_n}.
\]

Hence

\[
2n \left( J_x \left( \frac{i_n}{n} \right) - \log(x+1) \right) \leq \frac{2}{i_n}.
\]

Therefore,

\[
\left( \begin{array}{c} n \\ i_n \end{array} \right)^2 x^{2i_n} = \frac{n}{2\pi i_n(n-i_n)} \left( 1 + O(i_n^{-1}) \right) \exp \left( 2n J_x \left( \frac{i_n}{n} \right) \right) 
\]

\[
= \frac{(x+1)^2}{2\pi n x} \left( 1 + O(i_n^{-1}) \right) (x+1)^{2n}
\]

\[
= \left( 1 + O(i_n^{-1}) \right) \frac{(x+1)^{2n+2}}{2\pi n x}.
\]

We now estimate \( M_n(x) \). Observe that

\[
M_n(x) = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right)^2 x^{2i} = \left( \begin{array}{c} n \\ i_n \end{array} \right)^2 x^{2i_n} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i_n \end{array} \right)^2 x^{2i}. \tag{14}
\]

We notice that for any \( i = 0, \ldots, n \)

\[
\left( \begin{array}{c} n \\ i \end{array} \right) \leq e^{n J(i/n)}.
\]

Thus for any \( i = 0, \ldots, n \)

\[
\left( \begin{array}{c} n \\ i \end{array} \right)^2 x^{2i} \leq e^{2n J_x(i/n)} \tag{15}.
\]

*Observation (O1).* By (12), the function \( J_x(t) \) is a concave in \((0,1)\) and it attains the maximum at \( t = \frac{x}{x+1} \). Thus for any closed interval \( A \subset (0,1) \),

\[
\max_{t \in A} J_x(t) = \max_{t \in A_x} J_x(t),
\]

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with
\[ A_x = \{ t : |t - \frac{x}{x+1}| = \min_{s \in A} |s - \frac{x}{x+1}| \}. \]

**Case 1.** \( i \leq i_x - \frac{i_x}{4} \), or \( i_x + \frac{i_x}{4} \leq i \leq 60i_x \). Then by (13), (15) and observation (O1), we have
\[
\left( \frac{n}{i_x} \right)^2 \frac{2i}{x^{2i}} \leq 4\pi i_x \exp \left( 2n \left[ J_x \left( \frac{i}{n} \right) - J_x \left( \frac{i_x}{n} \right) \right] \right) \\
\leq 4\pi i_x \exp \left( 2n \left[ J_x \left( \frac{i_x \pm \frac{i_x}{4}}{n} \right) - J_x \left( \frac{i_x}{n} \right) \right] \right).
\]

By Taylor expansion,
\[
J_x \left( \frac{i_x \pm \frac{i_x}{4}}{n} \right) - J_x \left( \frac{i_x}{n} \right) = \pm \frac{\frac{i_x}{4}}{n} J_x' \left( \frac{i_x}{n} \right) + J_x''(\nu_x) \frac{\frac{i_x}{4}^2}{2n^2},
\]
for some \( \nu_x \in \left( \frac{i_x - \frac{i_x}{4}}{n}, \frac{i_x + \frac{i_x}{4}}{n} \right) \). Notice that
\[
\left| J_x \left( \frac{i_x}{n} \right) \right| = \left| J_x' \left( \frac{i_x}{n} \right) \right| \left| \frac{x}{n-x+1} \right| = \sup_{y \in \left( \frac{i_x}{n}, \frac{x}{n-x+1} \right)} |J_x''(y)| \left| \frac{i_x}{n} - \frac{x}{n-x+1} \right| \\
\leq \frac{4}{nx}, \tag{16}
\]
by using (12) and \( |i_x - \frac{x}{n-x+1}| \leq \frac{1}{n} \). On the other hand,
\[
J_x''(\nu_x) = \frac{-1}{\nu_x(1-\nu_x)} \leq \frac{-1}{\nu_x} \leq \frac{-n}{i_x + \frac{i_x}{4}} \leq \frac{-1}{2x}.
\]
Combining above estimates, we get
\[
\left( \frac{n}{i_x} \right)^2 \frac{2i}{x^{2i}} \leq 4\pi i_x \exp \left( \frac{8\frac{i_x}{4}}{nx} - \frac{\frac{i_x}{4}}{2nx} \right) \leq 4\pi i_x \exp \left( -\frac{\frac{i_x}{4}}{4nx} \right) \leq 4\pi i_x e^{-\frac{\sqrt{2nx}}{16}}.
\]
Therefore, when \( nx \) is large enough,
\[
M_{1,n}(x) = \sum_{i \leq i_x - \frac{i_x}{4}} \left( \frac{n}{i_x} \right)^2 \frac{2i}{x^{2i}} + \sum_{i_x + \frac{i_x}{4} \leq i \leq 60i_x} \left( \frac{n}{i_x} \right)^2 \frac{2i}{x^{2i}} \\
\leq 61i_x \times 4\pi i_x e^{-\frac{\sqrt{2nx}}{16}} \leq 24\pi(nx)^2 e^{-\frac{\sqrt{2nx}}{16}} \leq \frac{1}{nx}. \tag{17}
\]
Case 2. $i > 60i_x$. Using the same arguments as in Case 1, we can show that

$$
\frac{(n_i)^2 x^{2i}}{(n_{i_x})^2 x^{2i_x}} \leq 4\pi i_x \exp \left( 2n \left[ J_x \left( \frac{60i_x}{n} \right) - J_x \left( \frac{i_x}{n} \right) \right] \right)
$$

$$
\leq 4\pi i_x \exp \left( 2n \left[ \frac{4}{nx} - \frac{1}{120i_x} \frac{(59i_x)^2}{2n^2} \right] \right)
$$

$$
\leq 4\pi i_x \exp \left( 472 - \frac{592n_x}{240} \right)
$$

$$
\leq 4\pi i_x \exp \left( 472 - \frac{592 \log n}{1440} \right) \leq 4\pi i_x e^{-12 \log n/5} = 4\pi i_x n^{-12/5}.
$$

Notice that for the last line, we assume that $n$ is large enough and $nx \geq \log n/6$. Therefore,

$$
M_{2,n}(x) = \sum_{i > 60i_x} \frac{(n_i)^2 x^{2i}}{(n_{i_x})^2 x^{2i_x}} \leq n \times 4\pi i_x n^{-12/5} \leq n^{-1/5}.
$$

(18)

Case 3. $|i - i_x| \leq i_x^{3/4}$. By (11), we have

$$
\frac{(n_i)^2 x^{2i}}{(n_{i_x})^2 x^{2i_x}} = (1 + O(i_x^{-1})) i_x (n - i_x) \exp \left( 2n \left[ \frac{i_x}{n} - J_x \left( \frac{i_x}{n} \right) \right] \right).
$$

(19)

Since $J_x(t)$ is a concave function,

$$
J_x \left( \frac{i}{n} \right) - J_x \left( \frac{i_x}{n} \right) \leq J'_x \left( \frac{i_x}{n} \right) \frac{(i - i_x)}{n}.
$$

(20)

On the other hand, by (16)

$$
|J'_x \left( \frac{i_x}{n} \right) (i - i_x)| \leq \frac{4|i - i_x|}{nx} \leq \frac{4i_x^{3/4}}{nx} \leq \frac{4}{(i_x^{1/4}).
$$

(21)

We are now in the position to prove (i). Indeed, combining (19), (20), (21) gives that

$$
\frac{(n_i)^2 x^{2i}}{(n_{i_x})^2 x^{2i_x}} \leq (1 + i_x^{-1/2}) i_x (n - i_x) \exp \left( 8i_x^{-1/4} \right) \leq (1 + i_x^{-1/8}).
$$

Hence,

$$
1 \leq M_{3,n}(x) = \sum_{|i - i_x| \leq i_x^{3/4}} \frac{(n_i)^2 x^{2i}}{(n_{i_x})^2 x^{2i_x}} \leq 2i_x^{3/4}(1 + i_x^{-1/8}).
$$

Combining this estimate with (17) and (18), we obtain (i).
We now prove (ii). Assume that \( x \in (n^{1/6}, 1) \). Then using (19) and Taylor expansion,
\[
\frac{(n)!}{(i)!} x^{2i} = (1 + O(i^{-1/4})) \exp \left( 2n \left[ J_x \left( \frac{i}{n} \right) - J_x \left( \frac{i}{n} \right) \right] \right)
\]
\[
= \left( 1 + O(i^{-1/4}) \right) \exp \left( 2n \left[ J_x' \left( \frac{i}{n} \right) \left( i - \frac{i}{n} \right) + J_x'' \left( \frac{i}{n} \right) \frac{(i - \frac{i}{n})^2}{2n^2} + J_x''' \left( \nu_{i,x} \right) \frac{(i - \frac{i}{n})^3}{6n^3} \right] \right),
\]
for some \( \nu_{i,x} \in \left( \frac{i}{n}, \frac{i}{n} \right) \). We notice that
\[
J_x'''(y) = O(y^{-2}).
\]
Therefore, by Taylor expansion,
\[
J_x'' \left( \frac{i}{n} \right) = J_x'' \left( \frac{x}{x + 1} \right) + O \left( \frac{1}{x^2} \right) \left( \frac{i}{n} - \frac{x}{x + 1} \right) = J_x'' \left( \frac{x}{x + 1} \right) + O \left( \frac{1}{nx^2} \right),
\]
and
\[
J_x''' \left( \nu_{i,x} \right) = O \left( \frac{1}{x^2} \right).
\]
Combining these estimates with (21), we get
\[
\frac{(n)!}{(i)!} x^{2i} = \exp \left( O(n^{-1/4}) \right) \sum_{|i - j| \leq 3/4} \exp \left( \frac{-j^2}{2n^2} + O(n^{-1/24}) \right),
\]
since \( x \in (n^{-1/6}, 1) \) and \(|i - j| \leq 3/4 = O((nx)^{3/4})\). Therefore, by using \( J_x'' \left( \frac{x}{x + 1} \right) = -\frac{(x+1)^2}{x} \) and integral approximations, we can prove that
\[
M_{3,n}(x) = \sum_{|i - j| \leq 3/4} \exp \left( \frac{-j^2}{2n^2} + O(n^{-1/24}) \right)
\]
\[
\approx (1 + O(n^{-5/24})) \sum_{|j| \leq 3/4} \exp \left( -\left( j \sqrt{\frac{(x+1)^2}{x} + O(n^{-1/24})} \right)^2 \right)
\]
\[
\approx (1 + O(n^{-5/24})) \frac{\sqrt{\pi} \sqrt{x}}{x^2 + O(n^{-1/24})} \left( \int_{-\infty}^\infty e^{-t^2} dt + O(e^{-((nx)^{1/4})}) \right)
\]
\[
\approx (1 + O(n^{-1/24})) \sqrt{\frac{\pi n x}{x + 1}}.
\]
In conclusion,

\[
\sum_{i=0}^{n} \binom{n}{i}^2 x^{2i} = \binom{n}{i_x}^2 x^{2i_x} (M_{1,n}(x) + M_{2,n}(x) + M_{3,n}(x))
\]

\[
= (1 + O(n^{-1/24}))(\binom{n}{i_x}^2 x^{2i_x} \frac{\sqrt{\pi n x}}{x + 1})
\]

\[
= (1 + O(n^{-1/24}))(x + 1)^{2n+1} \frac{2}{2\sqrt{\pi n x}}.
\]

The part (ii) follows. \hfill \Box

**Remark 3.2** (Asymptotic behaviour of \(M_n\) via Legendre polynomial). The asymptotic formula of \(M_n\) can also be calculated using Legendre polynomials as follows. Legendre polynomials, denoted by \(L_n(x)\), are solutions to Legendre’s differential equation

\[
\frac{d}{dx} \left[ (1 - x^2) \frac{d}{dx} L_n(x) \right] + n(n + 1)L_n(x) = 0,
\]

with initial data \(L_0(x) = 1, \quad L_1(x) = x\). They have been used widely in physics and engineering and have many interesting properties, see [BO99] for more information. For instance, \(L_n\) has the following explicit representation

\[
L_n(x) = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i}^2 (x - 1)^{n-i}(x + 1)^i.
\]

According to [DH16, Lemma 3], the polynomial \(M_n\) defined in (10) and Legendre polynomial \(L_n\) satisfy the following relation

\[
M_n(x) = (1 - x^2)^n L_n \left( \frac{1 + x^2}{1 - x^2} \right).
\] (22)

According to [BO99, Example 2, page 229] (see also [WWW12]), the Legendre polynomial \(L_n\) satisfies the following asymptotic behaviour as \(n \to \infty\) for any \(x > 1\),

\[
L_n(x) \sim \frac{1}{\sqrt{2\pi n}} \left( \frac{x + \sqrt{x^2 - 1}}{x^2 - 1} \right)^{n+\frac{1}{2}}, \quad \text{for } x > 1.
\] (23)

From (22) and (23), we obtain the following asymptotic behaviour for \(M_n(x)\) as \(n \to \infty\) for any \(0 < x < 1\)

\[
M_n(x) \sim \frac{(x + 1)^{2n+1}}{2\sqrt{\pi nx}}.
\]

This is the result obtained in part (ii) of Lemma [3.3]. However, that part provided a stronger statement offering quantitative estimate.
By transforming $x = \tan^2(t/2\sqrt{n})$ and $y = \tan^2(s/2\sqrt{n})$, we will show that for $x, y \in [n^{-1/6}, n^{1/6}]$, the autocorrelation $A_n(x, y)$ is close to $e^{-(t-s)^2/4}$. It means that the sequence of random polynomials $(f_n(x))$ converges weakly to the centered stationary Gaussian process $Z(t)$ with covariance function $R(t) = e^{-t^2/4}$. Then by heuristic arguments, the persistence probability of $f_n$ should tend to the corresponding one of $Z(t)$. To ensure the continuity of persistence exponents, we need some restrictive conditions on the autocorrelation function. The following result which is combination of Theorem 1.6, Remark 1.7 and Lemma 1.8 in [DM15] gives us such conditions.

**Lemma 3.3.** Let $S_+$ be the class of all non-negative autocorrelation functions. Then the following statements hold.

(a) For centered stationary Gaussian process $\{Z_t\}_{t \geq 0}$ of autocorrelation $A(s, t) = A(0, t - s) \in S_+$, the nonnegative limit

$$b(A) = -\lim_{T \to \infty} \frac{\log \mathbb{P}(\inf_{0 \leq t \leq T} Z(t) > 0)}{T},$$

exists.

(b) Let $\{Z_t^{(k)}\}_{t \geq 0}$, $1 \leq k \leq \infty$ be a sequence of centered Gaussian processes of unit variance and nonnegative autocorrelation functions $A_k(s, t)$, such that $A_\infty(s, t) \in S_+$. We consider the following conditions on the sequence of autocorrelation functions.

(b1) We have

$$\limsup_{k, \tau \to \infty} \sup_{s \geq 0} \left\{ \frac{\log A_k(s, s + \tau)}{\log \tau} \right\} < -1.$$

(b2) There exists a nonnegative autocorrelation function $D$ corresponding to some stationary Gaussian process such that for any finite $M$, there exist positive $\epsilon_k \to 0$ satisfying

$$(1 - \epsilon_k)A_\infty(0, \tau) + \epsilon_k D(0, \tau) \leq A_k(s, s + \tau) \leq (1 - \epsilon_k)A_\infty(0, \tau) + \epsilon_k,$$

for all $s, \tau$ such that $\tau \in [0, M]$ and both $s, s + \tau$ belong to the considering interval.

(b3) We have $A_k(s, s + \tau) \to A_\infty(0, \tau)$ pointwise and for some $\eta > 1$,

$$\limsup_{u \downarrow 0} \sup_{1 \leq k \leq \infty} \left| \log u \right|^\eta p_k^2(u) < \infty,$$

where $p_k^2(u) := 2 - 2 \inf_{s \geq 0, \tau \in [0, u]} A_k(s, s + \tau)$.

Assume that either (b1) and (b2) hold or (b1) and (b3) hold. Then

$$\lim_{k, T \to \infty} \frac{1}{T} \log \mathbb{P}\left( Z_t^{(k)} > 0, \forall t \in [0, T] \right) = -b(A_\infty).$$
While Lemma 3.3 shows the convergence of persistence exponent of general Gaussian process under strict conditions of autocorrelation function, Lemma 3.4 below provides a lower bound on the persistence probability of a differentiable Gaussian process $Z(t)$, assuming a simple condition that the variances of $Z(t)$ and $Z'(t)$ are comparable.

**Lemma 3.4.** [DM15, Lemma 4.1] There is a universal constant $\mu \in (0,1)$, such that the following statements hold.

(i) If $(Z_t)_{t \in [a,b]}$ is a $C^1$ process satisfying

$$2(b-a)^2 \sup_{t \in [a,b]} \mathbb{E}(Z_t'^2) \leq \sup_{t \in [a,b]} \mathbb{E}(Z_t^2),$$

then

$$\mathbb{P} \left( \inf_{t \in [a,b]} Z_t > 0 \right) \geq \mu.$$

(ii) If $(Z_t)_{t \in [0,\beta_n]}$ is a $C^1$ Gaussian process with nonnegative autocorrelation satisfying for all $t \leq \beta_n$

$$2 \Delta^2 \mathbb{E}(Z_t'^2) \leq \mathbb{E}(Z_t^2),$$

for some positive constant $\Delta$, then

$$\mathbb{P} \left( \inf_{t \in [0,\beta_n]} Z_t > 0 \right) \geq \mu^\lceil \frac{\beta_n}{\Delta} \rceil.$$

**Proof.** The part (i) is exactly Lemma 4.1 in [DM15]. The part (ii) is a direct consequence of (i). Indeed, we divide the interval $[0, \beta_n]$ into $\lceil \frac{\beta_n}{\Delta} \rceil$ small intervals of length $\Delta$. Then the condition of (i) is verified in each small interval. Thus using Slepian lemma and (i), we get (ii). \qed

## 4 Proof of Theorem 1.1

In this section, we prove the main theorem, Theorem 1.1 using preliminary lemmas in Section 3. The proof consists of three steps. In Subsection 4.1, we show that the contribution to the persistent probability of two intervals $(0, n^{-1/6})$ and $(n^{1/6}, \infty)$ is negligible. Then in Subsection 4.2, we compute the persistent exponent from the main interval $(n^{-1/6}, n^{1/6})$. Finally, by bringing two previous steps together, we conclude the proof in Subsection 4.3.

### 4.1 Negligible intervals

In this part, we show that the contribution of intervals $(0, n^{-1/6})$ and $(n^{1/6}, \infty)$ to the persistent exponent is negligible.

**Proposition 4.1.** We have
\[(i) \lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} \left( f_n(x) > 0 \quad \forall x \in (0, n^{-1/6}) \right) = 0,
\]

\[(ii) \lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} \left( f_n(x) > 0 \quad \forall x \in (n^{1/6}, \infty) \right) = 0.
\]

**Proof.** The part (ii) is a consequence of (i). Indeed, we have

\[
\mathbb{P} \left( f_n(x) > 0 \quad \forall x \in (n^{1/6}, \infty) \right) = \mathbb{P} \left( f_n(x) > 0 \quad \forall x \in (0, n^{-1/6}) \right),
\]

since for \( x > 0, \)

\[
f_n \left( \frac{1}{x} \right) = \frac{1}{x^n} \sum_{i=0}^{n} \binom{n}{i} a_i x^{n-i} \leq \frac{f_n(x)}{x^n}.
\]

Now it remains to prove (i). Since the upper bound that

\[
\log \mathbb{P} \left( f_n(x) > 0 \quad \forall x \in (0, n^{-1/6}) \right) \leq \log 1 = 0
\]

is trivial, we only need to show the lower bound

\[
\liminf_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} \left( f_n(x) > 0 \quad \forall x \in (0, n^{-1/6}) \right) \geq 0. \tag{24}
\]

By Slepian lemma, (24) follows from the following lower bounds,

\[
\liminf_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} \left( f_n(x) > 0 \quad \forall x \in (0, \frac{\log n}{6n}) \right) \geq 0. \tag{25}
\]

\[
\liminf_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} \left( f_n(x) > 0 \quad \forall x \in \left( \frac{\log n}{6n}, n^{-1/6} \right) \right) \geq 0. \tag{26}
\]

We first prove (25). We observe that

\[
\mathbb{P} \left( f_n(x) > 0 \quad \forall x \in \left( 0, \frac{\log n}{6n} \right) \right) \geq \mathbb{P} \left( a_0 > \left| \sum_{i=1}^{n} \binom{n}{i} a_i x^i \right| \quad \forall x \in \left( 0, \frac{\log n}{6n} \right) \right)
\]

\[
\geq \mathbb{P} \left( a_0 > \max_{1 \leq i \leq n} |a_i| \times \sum_{i=1}^{n} \binom{n}{i} x^i \quad \forall x \in \left( 0, \frac{\log n}{6n} \right) \right)
\]

\[
\geq \mathbb{P} \left( a_0 > \max_{1 \leq i \leq n} |a_i| \times \left( 1 + \frac{\log n}{6n} \right)^n \right)
\]

\[
\geq \mathbb{P} \left( a_0 > \log n \times \left( 1 + \frac{\log n}{6n} \right)^n \right) \times \mathbb{P} \left( \max_{1 \leq i \leq n} |a_i| \leq \log n \right)
\]

\[
= (1 - \Phi(\xi_n)) \times (\Phi(\log n))^n, \tag{27}
\]

where \( \Phi(x) \) is the normal distribution function, and

\[
\xi_n = \log n \times \left( 1 + \frac{\log n}{6n} \right)^n.
\]
We notice that \( \log(1 - \Phi(x)) = (\frac{1}{2} + o(1))x^2 \) as \( x \to \infty \). Therefore, for \( n \) large enough,\
\[
(\Phi(\log n))^n \geq \left(1 - e^{-\log^2 n/4}\right)^n \geq 1/2,
\]
(28)
and\
\[
\liminf_{n \to \infty} \frac{\log(1 - \Phi(\xi_n))}{\sqrt{n}} \geq \liminf_{n \to \infty} -\frac{\xi_n^2}{\sqrt{n}} \geq \liminf_{n \to \infty} -\frac{\log^2 n \times e^{\log n/3}}{\sqrt{n}} = 0.
\]
(29)
Combining (27), (28) and (29), we get (25). We now prove (26). Let us define\
\[ g_n(x) = (x + 1)^{-n} f_n(x). \]
Then\
\[
g'_n(x) = (x + 1)^{-n} \left( f'_n(x) - \frac{n}{x + 1} f_n(x) \right)
= (x + 1)^{-n} \sum_{i=0}^{n} \binom{n}{i} a_i x^i \left( \frac{i}{x} - \frac{n}{x + 1} \right).
\]
Using Lemma 3.1 we have\
\[
E(g_n(x)^2) = (x + 1)^{-2n} M_n(x) \geq (x + 1)^{-2n} \left( \binom{n}{i_x} \right)^2 x^{2i_x},
\]
with\
\[ i_x = \left[ \frac{nx}{x + 1} \right]. \]
Using the same arguments as in Lemma 3.1 we can also prove that\
\[
E(g'_n(x)^2) = (x + 1)^{-2n} \sum_{i=0}^{n} \left( \binom{n}{i} \right)^2 x^{2i} \left( \frac{i}{x} - \frac{n}{x + 1} \right)^2
\leq (x + 1)^{-2n} \left( 3i_x^{3/4} \binom{n}{i_x} \right)^2 x^{2i_x} \left( \frac{i}{x} - \frac{n}{x + 1} \right)^2.
\]
(31)
Combining (30) and (31), we obtain\
\[
E(g'_n(x)^2) \leq 3i_x^{3/4} \left( \frac{i_x}{x} - \frac{n}{x + 1} \right)^2 E(g_n(x)^2) \leq \frac{3n^{3/4}}{x^{5/4}} E(g_n(x)^2).
\]
(32)
Thus for \( x \in (\frac{\log n}{6n}, \frac{1}{\sqrt{n}}) \),\
\[
2 \Delta_{1,n}^2 E(g'_n(x)^2) \leq E(g_n(x)^2),
\]
15
with
\[ \Delta_{1,n} = \frac{(\log n)^{5/8}}{8n}. \]

Applying Lemma 3.4, we have
\[ \mathbb{P} \left( f_n(x) > 0 \quad \forall \, x \in \left( \frac{\log n}{6n}, \frac{1}{\sqrt{n}} \right) \right) = \mathbb{P} \left( g_n(x) > 0 \quad \forall \, x \in \left( \frac{\log n}{6n}, \frac{1}{\sqrt{n}} \right) \right) \geq \mu^{\frac{1}{n^{1/6}}} \Delta_{1,n}. \]

Therefore,
\[ \liminf_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} \left( f_n(x) > 0 \quad \forall \, x \in \left( \frac{\log n}{6n}, \frac{1}{\sqrt{n}} \right) \right) \geq \liminf_{n \to \infty} \frac{\log \mu}{n \Delta_{1,n}} = 0. \]

Using (32) for \( x \in (n^{-1/2}, n^{-1/6}) \), we get
\[ 2 \Delta_{2,n}^2 \mathbb{E}(g_n'(x)^2) \leq \mathbb{E}(g_n(x)^2), \]
with
\[ \Delta_{2,n} = \frac{1}{3n^{1/16}}. \]

Applying Lemma 3.4, we have
\[ \mathbb{P} \left( f_n(x) > 0 \quad \forall \, x \in (n^{-1/2}, n^{-1/6}) \right) = \mathbb{P} \left( g_n(x) > 0 \quad \forall \, x \in (n^{-1/2}, n^{-1/6}) \right) \geq \mu^{\frac{1}{n^{1/6}}} \Delta_{2,n}^2. \]

Therefore,
\[ \liminf_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P} \left( f_n(x) > 0 \quad \forall \, x \in (n^{-1/2}, n^{-1/6}) \right) \geq \liminf_{n \to \infty} \frac{\log \mu}{n^{2/3} \Delta_{2,n}} = 0. \]

Using (33), (34) and Slepian lemma, we get (26). \( \square \)

### 4.2 The main interval

We make a transformation
\[ x = \tan^2 \left( \frac{t}{2\sqrt{n}} \right). \]

Then \( x \in (n^{-1/6}, n^{1/6}) \) is equivalent to \( t \in (\alpha_n, \pi \sqrt{n} - \alpha_n) \), with
\[ \alpha_n = 2\sqrt{n} \tan^{-1}(n^{-1/12}) = (2 + o(1))n^{5/12}. \]

Let us define for \( t \in (\alpha_n, \pi \sqrt{n} - \alpha_n) \),
\[ h_n(t) = f_n(\tan^2(t/2\sqrt{n})). \]

Then
\[ \mathbb{P} \left( f_n(x) > 0 \quad \forall \, x \in (n^{-1/6}, n^{1/6}) \right) = \mathbb{P} \left( h_n(t) > 0 \quad \forall \, t \in (\alpha_n, \pi \sqrt{n} - \alpha_n) \right). \]
Moreover, the autocorrelation of $h_n(t)$ is

$$B_n(t, s) = A_n \left( \tan^2 \left( \frac{t}{2\sqrt{n}} \right), \tan^2 \left( \frac{s}{2\sqrt{n}} \right) \right) = \frac{M_n \left( \tan \left( \frac{t}{2\sqrt{n}} \right) \tan \left( \frac{s}{2\sqrt{n}} \right) \right)}{\sqrt{M_n(\tan^2 \left( \frac{t}{2\sqrt{n}} \right))} \sqrt{M_n(\tan^2 \left( \frac{s}{2\sqrt{n}} \right))}}. \quad (36)$$

We recall the approximation on $M_n(u)$. For $u \in (n^{-1/6}, 1)$,

$$M_n(u) = (1 + O(n^{-1/24})) \frac{(u + 1)^{2n+1}}{\sqrt{\pi nu}}. \quad (37)$$

For $u \in (1, \infty)$, we remark that

$$M_n(u) = u^{2n} M_n \left( \frac{1}{u} \right).$$

Therefore, the estimate (37) holds for all $u \in (n^{-1/6}, n^{1/6})$. Hence,

$$B_n(t, s) = \frac{M_n \left( \tan \left( \frac{t}{2\sqrt{n}} \right) \tan \left( \frac{s}{2\sqrt{n}} \right) \right)}{\sqrt{M_n(\tan^2 \left( \frac{t}{2\sqrt{n}} \right))} \sqrt{M_n(\tan^2 \left( \frac{s}{2\sqrt{n}} \right))}}^{2n+1 \over 2}$$

We shift the interval $(\alpha_n, \pi \sqrt{n} - \alpha_n)$ to the interval $(0, \pi \sqrt{n} - 2\alpha_n)$ by changing variable

$$u = t - \alpha_n,$$

and define

$$\bar{h}_n(u) = h_n(u + \alpha_n).$$

Then the autocorrelation of $\bar{h}_n(u)$ is

$$\bar{B}_n(u, v) = B_n(u + \alpha_n, v + \alpha_n) = (1 + O(n^{-1/24})) \left[ \cos \left( \frac{u - v}{2\sqrt{n}} \right) \right]^{2n+1}. \quad (38)$$

Observe that for fixed $u, v$,

$$\bar{B}_n(u, v) \to e^{-(u-v)^2/4}.$$

This fact suggests us to consider the conditions (b1) and (b3) of Lemma 3.3.

**Verification of the condition (b1).** As $\tau \to \infty$,

$$\frac{\log \bar{B}_n(u, u + \tau)}{\log \tau} = \frac{(2n + 1) \log \left( \cos \left( \frac{\tau}{2\sqrt{n}} \right) \right)}{\log \tau} + o(1) = \frac{-\tau^2}{4 \log \tau} + o(1) \to -\infty.$$
Thus the condition (b1) is verified.

Verification of the condition (b3). Using (38), we have for \( n \) large enough

\[
\bar{B}_n(t, t + \tau) \geq (1 - n^{-1/30}) \left[ \cos \left( \frac{\tau}{\sqrt{n}} \right) \right]^{2n+1} \geq (1 - n^{-1/30})(1 - \tau^2).
\]

Therefore,

\[
\bar{p}_n^2(w) = 2 - 2 \inf_{0 \leq \tau \leq w} \bar{B}_n(t, t + \tau) \leq 2w^2 + 2n^{-1/30}.
\]

Hence, for any \( \delta > 0 \), as \( w \to 0 \)

\[
|\log w|^2 \sup_{n \geq w^{-\delta}} \bar{p}_n^2(w) \to 0.
\]

Thus, to verify (b3), it suffices to show that

\[
\lim_{w \to 0^+} |\log w|^2 \sup_{n \geq w^{-\delta}} \bar{p}_n^2(w) < \infty. \tag{39}
\]

To show (39) holds, it is sufficient to prove that

\[
\lim_{u \to 0^+} |\log u|^2 \sup_{n \leq u^{-\delta}} p_n^2(u) < \infty, \tag{40}
\]

where

\[
p_n^2(u) = 2 - 2 \inf_{0 \leq y-x \leq u} A_n(x, y).
\]

Recall that

\[
A_n(x, y) = \frac{M_n(\sqrt{xy})}{\sqrt{M_n(x)} \sqrt{M_n(y)}},
\]

where

\[
M_n(x) = \sum_{i=0}^{n} \binom{n}{i}^2 x^{2i}.
\]

We have

\[
\varepsilon_n(x, y) := M_n(\sqrt{xy}) - M_n(x) = \sum_{i=0}^{n} \binom{n}{i}^2 x^i (y^i - x^i) \geq 0,
\]

since \( y \geq x \), and

\[
\tilde{\varepsilon}_n(x, y) := M_n(y) - M_n(x) - 2\varepsilon_n(x, y) = \sum_{i=0}^{n} \binom{n}{i}^2 (y^i - x^i)^2 \geq 0.
\]
Therefore,

\[ 0 \leq 1 - A_n(x, y) = 1 - \frac{M_n(x) + \varepsilon_n(x, y)}{\sqrt{M_n(x) (M_n(x) + 2\varepsilon_n(x, y) + \bar{\varepsilon}_n(x, y))}} \]

\[ \leq \frac{\sqrt{M_n(x) (M_n(x) + 2\varepsilon_n(x, y) + \bar{\varepsilon}_n(x, y))}}{M_n(x)} \]

\[ = (y - x)^2 \frac{M_n(x) \varepsilon_{1,n}(x, y) - \varepsilon_{1,n}^2(x, y)}{M_n(x)}, \quad (41) \]

with

\[ \varepsilon_{1,n}(x, y) = \sum_{i=0}^{n} \binom{n}{i} x^i \left( \frac{y - x}{y - x} \right), \]

and

\[ \bar{\varepsilon}_{1,n}(x, y) = \sum_{i=0}^{n} \binom{n}{i} \left( \frac{y - x}{y - x} \right)^2. \]

We notice that

\[ \sum_{i=0}^{n} a_i^2 \sum_{i=0}^{n} b_i^2 - \left( \sum_{i=0}^{n} a_i b_i \right)^2 = \sum_{i,j} (a_i - b_j)^2 \leq n \left( \sum_{i=0}^{n} a_i^2 + \sum_{i=0}^{n} b_i^2 \right). \]

Using this inequality, we get

\[ M_n(x) \varepsilon_{1,n}(x, y) - \varepsilon_{1,n}^2(x, y) \leq n (M_n(x) + \bar{\varepsilon}_{1,n}(x, y)). \quad (42) \]

Observe that

\[ \bar{\varepsilon}_{1,n}(x, y) \leq \sum_{i=1}^{n} \binom{n}{i} i^2 y^{2i-2} \leq \frac{n^2}{y^2} M_n(y). \quad (43) \]

Therefore, using Lemma 3.1 (ii) and (41), (42), (43) we have

\[ 0 \leq 1 - A_n(x, y) \leq (y - x)^2 \left( 1 + \frac{n}{y^2} \frac{M_n(y)}{M_n(x)} \right) \leq (y - x)^2 \left( 1 + \frac{2n}{y^2} \left( \frac{y + 1}{x + 1} \right)^{2n+1} \right) \]

\[ \leq (y - x)^2 \left( 1 + \frac{2n}{y^2} (1 + (y - x)^{2n+1}) \right). \]

Note that

\[ 0 \leq y - x \leq \tau \leq u, \quad n \leq u^{-\delta}, \quad x \geq n^{-1/6} \geq u^{\delta/6}. \quad (44) \]
Hence,
\[
0 \leq 1 - A_n(x, y) \leq u^2 \left[ 1 + \frac{2u^{-\delta}}{(u + u^{\delta/6})^2} (1 + u)^{2u^{-\delta/6}+1} \right] \\
\leq u^2 \left[ 1 + 2u^{-\delta-\delta/3} \times 10 \right] \leq 40u^{2-\delta/3}.
\]

As consequence, (40) holds.

By the validity of the conditions (b1) and (b3), we deduce from Lemma 3.3 the following proposition.

**Proposition 4.2.** We have
\[
\lim \frac{1}{\pi \sqrt{n}} \log \mathbb{P} \left( f_n(x) > 0 \quad \forall x \in (n^{-1/6}, n^{1/6}) \right) = -b,
\]
with $b$ as in the statement of Theorem 1.1.

### 4.3 Conclusion

Thanks to Slepian inequality, using Propositions 4.1 and 4.2 we get
\[
\liminf_{n \to \infty} \frac{1}{\pi \sqrt{n}} \log \mathbb{P} \left( f_n(x) > 0 \quad \forall x \in (-\infty, \infty) \right) \geq -b.
\]

On the other hand, it follows directly from Proposition 4.2 that
\[
\limsup_{n \to \infty} \frac{1}{\pi \sqrt{n}} \log \mathbb{P} \left( f_n(x) > 0 \quad \forall x \in (-\infty, \infty) \right) \leq -b.
\]

Combining these two inequalities we get Theorem 1.1.

### 5 Summary and future work

In this paper, we have obtained an asymptotic formula for the persistent probability of the random polynomial $f_n$ in (1) that arises from evolutionary game theory. The persistence probability corresponds to the probability that a symmetric $n$-player two-strategy random game has no internal equilibria. We note that $f_n$ forms a different class of random polynomials that have been studied extensively in the literature particularly in random polynomial theory. There are several open problems that are of interest for both evolutionary game theory and random polynomial theory that we do not address in this paper such as proving a central limit theorem and a large deviation principle for the empirical measures of the real zeros of $f_n$ as well as studying universality phenomena for this class of random polynomials. We leave these problems for future research.

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