Common Fixed Point Theorems for Self Mappings on a Complete Vector S-metric Space

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Abstract

S-metric space was introduced by Sedghi et al. [6] in 2012. We derive some common fixed point results for self-mappings on vector valued complete S-metric space. In support of our results, we also give some examples.

Keywords: Vector lattice, Vector metric space, Vector S-metric space.

1 Introduction

Fixed point theory is amongst the crucial mathematical theory with applications in various branches of science. Banach contraction principle was derived first by S. Banach [2] in 1922. It has a vital role in fixed point theory and became very famous due to iterations used in the theorem. The evaluation of fixed points of mappings satisfying many contractive conditions is at the center of research work and several vital results have been established by many authors. Over the last few years, several researchers have devoted themselves to define many variations of metric space. We give below some definitions and

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results which will help in proving our main results for vector $S$-metric spaces.

**Definition 1.1** [5] On a set $\mathcal{C}$, a relation $\preceq$ is a partial order if it follows the conditions stated below:

(a) $\eta_1 \preceq \eta_1$ (reflexive)

(b) $\eta_1 \preceq \eta_2$ and $\eta_2 \preceq \eta_1$ implies $\eta_1 = \eta_2$ (anti-symmetry)

(c) $\eta_1 \preceq \eta_2$ and $\eta_2 \preceq \eta_3$ implies $\eta_1 \preceq \eta_3$ (transitivity)

\forall \eta_1, \eta_2, \eta_3 \in \mathcal{C}$. The set $\mathcal{C}$ with partial order $\preceq$ is known as partially ordered set (poset).

A partially ordered set $(\mathcal{C}, \preceq)$ is called linearly ordered if for $\eta_1, \eta_2 \in \mathcal{C}$, we have either $\eta_1 \preceq \eta_2$ or $\eta_2 \preceq \eta_1$.

**Definition 1.2** [5] Let $\mathcal{C}$ be linear space which is real and $(\mathcal{C}, \preceq)$ be a poset. Then the poset $(\mathcal{C}, \preceq)$ is said to be an ordered linear space if it follows the properties mentioned below:

(a) $p_1 \preceq p_2 \implies p_1 + p_3 \preceq p_2 + p_3$

(b) $p_1 \preceq p_2 \implies \omega p_1 \preceq \omega p_2 \quad \forall p_1, p_2, p_3 \in \mathcal{C}$ and $\omega > 0$

**Definition 1.3** [5] A poset is called lattice if each set with two elements has an infimum and a supremum.

**Definition 1.4** [5] An ordered linear space where the ordering is lattice is called vector lattice. This is also called Riesz space.

**Definition 1.5** [5] A vector lattice $V$ is called Archimedean if $\inf \left\{ \frac{1}{m} \vartheta \right\} = 0$ for every $\vartheta \in V^+$ where

$$V^+ = \{ \vartheta \in V : \vartheta \geq 0 \}.$$

**Definition 1.6** [4] Let $V$ be a vector lattice and $\mathcal{R}$ be a nonvoid set. A function $d : \mathcal{R} \times \mathcal{R} \to V$ is called vector metric on $\mathcal{R}$ if it follows the conditions stated below:

(a) $d(h_1, h_2) = 0$ iff $h_1 = h_2$

(b) $d(h_1, h_2) \preceq d(h_1, h_3) + d(h_3, h_2) \quad \forall h_1, h_2, h_3 \in \mathcal{R}$
The triplet $(\mathcal{R}, d, V)$ is called vector metric space.

**Definition 1.7** Let $\mathcal{R}$ be a nonvoid set. A function $S : \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to [0, \infty)$ is called $S$-metric on $\mathcal{R}$ if it follows the conditions stated below:

(a) $S(\flat_1, \flat_2, \flat_3) \geq 0,$

(b) $S(\flat_1, \flat_2, \flat_3) = 0$ iff $\flat_1 = \flat_2 = \flat_3,$

(c) $S(\flat_1, \flat_2, \flat_3) \leq S(\flat_1, \flat_2, \alpha) + S(\flat_2, \flat_2, \alpha) + S(\flat_3, \flat_3, \alpha),$

for all $\flat_1, \flat_2, \flat_3, \alpha \in \mathcal{R}.$

The pair $(\mathcal{R}, S)$ is called $S$-metric space.

Now, we define vector valued $S$-metric space as follows:

**Definition 1.8** Let $V$ be a vector lattice and $\mathcal{R}$ be a nonvoid set. A function $S : \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to V$ is called vector $S$-metric on $\mathcal{R}$ that satisfies the conditions mentioned below:

(a) $S(\flat_1, \flat_2, \flat_3) \geq 0,$

(b) $S(\flat_1, \flat_2, \flat_3) = 0$ iff $\flat_1 = \flat_2 = \flat_3,$

(c) $S(\flat_1, \flat_2, \flat_3) \leq S(\flat_1, \flat_2, \alpha) + S(\flat_2, \flat_2, \alpha) + S(\flat_3, \flat_3, \alpha),$

for all $\flat_1, \flat_2, \flat_3, \alpha \in \mathcal{R}.$

The triplet $(\mathcal{R}, S, V)$ is called vector $S$-metric space.

**Example 1.9** Let $\mathcal{R}$ be a nonvoid set and $V$ be a vector lattice. A function $S : V \times V \times V \to V$ is defined by

$$S(\flat_1, \flat_2, \flat_3) = |(\flat_1, \flat_3)| + |(\flat_2, \flat_3)| \quad \forall \flat_1, \flat_2, \flat_3 \in \mathcal{R}$$

then the triplet $(\mathcal{R}, S, V)$ is vector $S$-metric space.

**Definition 1.10** A sequence $\langle \flat_n \rangle$ in a vector $S$-metric space $(\mathcal{R}, S, V)$ is called $V$-convergent to some $\flat \in V$ if there is a sequence $\langle \mu_n \rangle$ in $V$ satisfying $\mu_n \downarrow 0$ and $S(\flat_n, \flat_n, \flat) \leq \mu_n$ and denote it by $\flat_n \overset{S,V}{\longrightarrow} \flat.$

**Definition 1.11** A sequence $\langle \flat_n \rangle$ in a vector $S$-metric space $(\mathcal{R}, S, V)$ is known as $V$-Cauchy sequence if $\exists \langle \mu_n \rangle \in V$ satisfying $\mu_n \downarrow 0$ and $S(\flat_n, \flat_n, \flat_{n+q}) \leq \mu_n \quad \forall q$ and $n.$

**Definition 1.12** A vector $S$-metric space $(\mathcal{R}, S, V)$ is called $V$-complete if
all V-Cauchy sequence is V-convergent to a limit in \( \mathbb{R} \).

**Lemma 1.13** For a vector \( S \)-metric space \((\mathbb{R}, S, V)\),

\[
S(\vartheta, \vartheta, \mu) = S(\mu, \mu, \vartheta) \quad \forall \mu, \vartheta \in \mathbb{R}.
\]

**Proof.** Using the condition (c) of definition (1.8), we have

\[
S(\vartheta, \vartheta, \mu) \leq S(\vartheta, \vartheta, \vartheta) + S(\vartheta, \vartheta, \mu) + S(\mu, \mu, \vartheta) \quad (1)
\]

By (1) and (2), we get

\[
S(\vartheta, \vartheta, \mu) = S(\mu, \mu, \vartheta).
\]

**2 Main Results**

**Lemma 2.1** Let \((\mathbb{R}, S, V)\) be a vector \( S \)-metric space which is complete and \( V \)-Archimedean. Let a sequence \( \langle h_\vartheta \rangle \) be in \( \mathbb{R} \) such that

\[
S(h_\vartheta, h_\vartheta, h_{\vartheta+1}) \leq \alpha S(h_{\vartheta-1}, h_{\vartheta-1}, h_\vartheta) \quad \forall \vartheta \in \mathbb{N} \quad (1)
\]

where \( \alpha \in [0, 1) \). Then \( \langle h_\vartheta \rangle \) is a \( V \)-Cauchy sequence in \( \mathbb{R} \).

**Proof.** Using (1), we get

\[
S(h_\vartheta, h_\vartheta, h_{\vartheta+1}) \leq \alpha S(h_{\vartheta-1}, h_{\vartheta-1}, h_\vartheta) \leq \alpha^2 S(h_{\vartheta-2}, h_{\vartheta-2}, h_{\vartheta-1}) \leq \cdots \leq \alpha^\vartheta S(h_0, h_0, h_1)
\]

So, for \( \vartheta > \ell \), we have

\[
S(h_\ell, h_\ell, h_\ell) \leq 2S(h_\ell, h_\ell, h_{\ell+1}) + S(h_\ell, h_\ell, h_{\ell+1})
\]

\[
= 2S(h_\ell, h_\ell, h_{\ell+1}) + S(h_{\ell+1}, h_{\ell+1}, h_\ell)
\]

\[
\leq 2S(h_\ell, h_\ell, h_{\ell+1}) + 2S(h_{\ell+1}, h_{\ell+1}, h_{\ell+2}) + S(h_\ell, h_\ell, h_{\ell+2})
\]

\[
= 2S(h_\ell, h_\ell, h_{\ell+1}) + 2S(h_{\ell+1}, h_{\ell+1}, h_{\ell+2}) + S(h_{\ell+2}, h_{\ell+2}, h_\ell)
\]

\[
\leq 2S(h_\ell, h_\ell, h_{\ell+1}) + 2S(h_{\ell+1}, h_{\ell+1}, h_{\ell+2}) + \cdots + S(h_{\vartheta-1}, h_{\vartheta-1}, h_\vartheta)
\]

\[
< \alpha^\ell + \alpha^{\ell+1} + \cdots + \alpha^{\vartheta-1} S(h_0, h_0, h_1)
\]

\[
< 2\alpha^\ell (1 + \alpha + \alpha^2 + \ldots) S(h_0, h_0, h_1)
\]

\[
< \frac{2\alpha^\ell}{1 - \alpha} S(h_0, h_0, h_1) \downarrow 0 \quad \ell \to \infty.
\]
Thus \( \langle h_0 \rangle \) is a \( V \)-Cauchy sequence.

**Theorem 2.2** Let \((\mathcal{R}, S, V)\) be a vector \(S\)-metric space which is complete and \(V\)-Archimedean. Let \( K : \mathcal{R} \to \mathcal{R} \) be a continuous mapping and a map \( f : \mathcal{R} \to \mathcal{R} \) which commutes with \( K \). Suppose the conditions given below are satisfied:

(a) \( f(\mathcal{R}) \subseteq K(\mathcal{R}) \)

(b) \( S(fh, fh, f\vartheta) \leq qU(h, h, \vartheta) \) for all \( h, \vartheta \in \mathcal{R} \) where \( q \in \left[0, \frac{1}{3}\right) \) is a constant and

\[
U(h, h, \vartheta) \in \{S(Kh, Kh, K\vartheta), S(Kh, Kh, fh), S(K\vartheta, K\vartheta, f\vartheta), S(Kh, Kh, f\vartheta), S(K\vartheta, K\vartheta, fh)\}
\]

(c) \( K(\mathcal{R}) \) or \( f(\mathcal{R}) \) is \( V \)-complete as a subspace of \( \mathcal{R} \). Then, prove that \( K \) and \( f \) have a common fixed point which is unique.

**Proof** Fix arbitrary \( \vartheta_0 \in \mathcal{R} \), so we can take sequence \( \langle h_0 \rangle \) in \( \mathcal{R} \) such that

\[ h_b = f\vartheta = K\vartheta_{b+1} \quad b \geq 0. \]

Then

\[
S(h_b, h_b, h_{b+1}) = S(f\vartheta_b, f\vartheta_b, f\vartheta_{b+1}) \leq qU(\vartheta_b, \vartheta_b, \vartheta_{b+1}) \tag{2}
\]

where

\[
U(\vartheta_b, \vartheta_b, \vartheta_{b+1}) \in \{S(K\vartheta_b, K\vartheta_b, K\vartheta_{b+1}), S(K\vartheta_b, K\vartheta_b, f\vartheta_b), S(K\vartheta_{b+1}, K\vartheta_b, f\vartheta_b), S(K\vartheta_{b+1}, K\vartheta_b, f\vartheta_b)\}
\]

\[
= \{S(h_{b-1}, h_{b-1}, h_b), S(h_{b-1}, h_{b-1}, h_b), S(h_b, h_b, h_{b+1}), S(h_{b-1}, h_{b-1}, h_b)\}
\]

The possible four cases are:

(i) \( S(h_b, h_b, h_{b+1}) \leq qS(h_{b-1}, h_{b-1}, h_b) \)

(ii) \( S(h_b, h_b, h_{b+1}) \leq qS(h_{b-1}, h_{b-1}, h_{b+1}) \)

and so

\[ S(h_b, h_b, h_{b+1}) = 0. \]
(iii) \( S(h_0, h_0, h_{b+1}) \leq qS(h_{b-1}, h_{b-1}, h_{b+1}) \leq 2qS(h_{b-1}, h_{b-1}, h_b) + qS(h_{b+1}, h_{b+1}, h_b) \).

and

\[
S(h_0, h_0, h_{b+1}) \leq \frac{2q}{(1-q)} S(h_{b-1}, h_{b-1}, h_b)
\]

(iv) \( S(h_0, h_0, h_{b+1}) \leq q, 0 = 0 \)

and so

\[
S(h_0, h_0, h_{b+1}) = 0.
\]

Thus \( S(h_0, h_0, h_{b+1}) \leq \sigma S(h_{b-1}, h_{b-1}, h_b) \) where \( \sigma \in \{q, \frac{2q}{(1-q)}\} < 1 \)

Since \( V \) is Archimedean, by lemma (2.1) \( \langle h_0 \rangle \) is a \( V \)-Cauchy sequence and range of \( f \) is contained in the range of \( K \) and atleast one range is \( V \)-complete, there exist \( h \in K(\mathbb{R}) \) such that \( K\vartheta_{b+1} \to h \). Hence there exist a sequence \( \langle \alpha_b \rangle \) in \( V \) such that \( \alpha_b \downarrow 0 \) and

\[
S(K\vartheta_b, K\vartheta_b, h) \leq \alpha_b.
\]

Thus

\[
h_b = f\vartheta_b = K\vartheta_{b+1} \to h.
\]

We prove that

\[
K\overline{h} = f\overline{h} = \overline{h}
\]

Now,

\[
S(K\overline{h}, K\overline{h}, f\overline{h}) \leq 2S(K\overline{h}, K\overline{h}, fK\vartheta_b) + S(fh, fh, fK\vartheta_b)
\]

\[
S(K\overline{h}, K\overline{h}, f\overline{h}) \leq 2S(K\overline{h}, K\overline{h}, fK\vartheta_b) + S(fK\vartheta_b, fK\vartheta_b, f\overline{h})
\]

(4)

Also, we have

\[
S(fK\vartheta_b, fK\vartheta_b, f\overline{h}) \leq qU(K\vartheta_b, K\vartheta_b, h)
\]

Then (3) becomes

\[
S(K\overline{h}, K\overline{h}, f\overline{h}) \leq 2S(K\overline{h}, K\overline{h}, fK\vartheta_b) + qU(K\vartheta_b, K\vartheta_b, h)
\]

(5)
where

\[ U(K\vartheta, K\vartheta, \hbar) \in \{ S(K^2\vartheta, K^2\vartheta, K\hbar), S(K^2\vartheta, K\vartheta, fK\vartheta), S(K\hbar, K\hbar, fh), S(K^2\vartheta, K^2\vartheta, fh), S(K\hbar, K\hbar, fK\vartheta) \} \]

Since \( f \) commutes with \( K \) and by using continuity of \( K \), we get

\[ fK\vartheta = Kf\vartheta \xrightarrow{SV} K\hbar \]

and by using (3)

\[ K^2\vartheta_{b+1} \xrightarrow{SV} K\hbar, \]

then there exist a sequence \( \langle \alpha_b \rangle \) and \( \langle \beta_b \rangle \) in \( V \) such that \( \alpha_b \downarrow 0 \) and \( \beta_b \downarrow 0 \), then we have

\[ S(K\hbar, K\hbar, fK\vartheta) \leq \alpha_b \]

and

\[ S(K^2\vartheta, K^2\vartheta, K\hbar) \leq \beta_b. \]

By (5) and (6), we have the following cases:

(i) \[ S(K\hbar, K\hbar, fh) \leq 2S(K\hbar, K\hbar, fK\vartheta) + qS(K^2\vartheta, K^2\vartheta, K\hbar) \leq 2\alpha_b + q\beta_b \]

(ii) \[ S(K\hbar, K\hbar, fh) \leq 2S(K\hbar, K\hbar, fK\vartheta) + qS(K^2\vartheta, K^2\vartheta, fK\vartheta) \leq 2S(K\hbar, K\hbar, fK\vartheta) + q[2S(K^2\vartheta, K^2\vartheta, K\hbar) + S(fK\vartheta, fK\vartheta, K\hbar)] = 2S(K\hbar, K\hbar, fK\vartheta) + 2qS(K^2\vartheta, K^2\vartheta, K\hbar) + qS(K\hbar, K\hbar, fK\vartheta) \leq (2 + q)\alpha_b + 2q\beta_b \]

(iii) \[ S(K\hbar, K\hbar, fh) \leq 2S(K\hbar, K\hbar, fK\vartheta) + qS(K\hbar, K\hbar, fh) \leq 2\alpha_b \]

\[ (1 - q)S(K\hbar, K\hbar, fh) \leq \frac{2\alpha_b}{(1 - q)} \]
(iv) \[ S(K\hbar, K\hbar, f\hbar) \leq 2S(K\hbar, K\hbar, fK\partial_\phi) + qS(K^2\partial_\phi, K^2\partial_\phi, f\hbar) \]
\[ \leq 2S(K\hbar, K\hbar, fK\partial_\phi) + 2qS(K^2\partial_\phi, K^2\partial_\phi, K\hbar) + qS(f\hbar, f\hbar, K\hbar) \]
\[ = 2S(K\hbar, K\hbar, fK\partial_\phi) + 2qS(K^2\partial_\phi, K^2\partial_\phi, K\hbar) + qS(K\hbar, K\hbar, f\hbar) \]
\[ (1 - q)S(K\hbar, K\hbar, f\hbar) \leq 2\alpha + 2q\beta \]
\[ S(K\hbar, K\hbar, f\hbar) \leq \frac{2\alpha + 2q\beta}{1 - q} \]

(v) \[ S(K\hbar, K\hbar, f\hbar) \leq 2S(K\hbar, K\hbar, fK\partial_\phi) + qS(K\hbar, K\hbar, fK\partial_\phi) \]
\[ \leq (2 + q)S(K\hbar, K\hbar, fK\partial_\phi) \]
\[ \leq (2 + q)\alpha \]
\[ \leq 3\alpha \]

In the last inequality of each case, the infimum on the right hand side is 0. So we get

\[ S(K\hbar, K\hbar, f\hbar) = 0. \]

This implies

\[ K\hbar = f\hbar \]

(7)

So

\[ S(K\hbar, K\hbar, h) \leq 2S(K\hbar, K\hbar, f\partial_\phi) + S(h, h, f\partial_\phi) \]
\[ = 2S(K\hbar, K\hbar, f\partial_\phi) + S(f\partial_\phi, f\partial_\phi, h) \]
\[ \leq S(f\partial_\phi, f\partial_\phi, h) + 2S(f\hbar, f\hbar, f\partial_\phi) \]
\[ = S(f\partial_\phi, f\partial_\phi, h) + 2S(f\partial_\phi, f\partial_\phi, f\hbar) \]
\[ S(K\hbar, K\hbar, h) \leq S(f\partial_\phi, f\partial_\phi, h) + 2qU(\partial_\phi, \partial_\phi, h) \]

(8)

where

\[ U(\partial_\phi, \partial_\phi, h) \in \{S(K\partial_\phi, K\partial_\phi, K\hbar), S(K\partial_\phi, K\partial_\phi, f\partial_\phi), S(K\hbar, K\hbar, f\hbar), S(K\partial_\phi, K\partial_\phi, f\hbar), S(K\hbar, K\hbar, f\partial_\phi)\} \]
\[ = \{S(K\partial_\phi, K\partial_\phi, K\hbar), S(K\partial_\phi, K\partial_\phi, f\partial_\phi), 0, S(K\hbar, K\hbar, f\partial_\phi)\} \]

(9)
there exist a sequence \( \langle d_n \rangle \) and \( \langle g_n \rangle \) in \( V \) such that \( d_n \downarrow 0 \) and \( g_n \downarrow 0 \), then we have

\[
S(K\vartheta, K\vartheta, \hbar) \leq d_n
\]

and

\[
S(f\vartheta, f\vartheta, \hbar) \leq g_n.
\]

Using (8) and (9), we have the following cases:

(i) \[ S(K\hbar, K\hbar, \hbar) \leq S(f\vartheta, f\vartheta, \hbar) + 2qS(K\vartheta, K\vartheta, \hbar) + S(K\hbar, K\hbar, \hbar) \]

\[ (1 - 2q)S(K\hbar, K\hbar, \hbar) \leq g_n + 4qd_n \]

(ii) \[ S(K\hbar, K\hbar, \hbar) \leq S(f\vartheta, f\vartheta, \hbar) + 2qS(K\vartheta, K\vartheta, f\vartheta) \]

\[ = S(f\vartheta, f\vartheta, \hbar) + 2q[S(K\vartheta, K\vartheta, \hbar) + S(\hbar, \hbar, f\vartheta)] \]

\[ S(K\hbar, K\hbar, \hbar) \leq g_n + 2q(2d_n + g_n) \]

\[ \leq g_n(1 + 2q) + 4qd_n \]

(iii) \[ S(K\hbar, K\hbar, \hbar) \leq S(f\vartheta, f\vartheta, \hbar) + 2q0 \]

\[ S(K\hbar, K\hbar, \hbar) \leq g_n \]

(iv) \[ S(K\hbar, K\hbar, \hbar) \leq S(f\vartheta, f\vartheta, \hbar) + 2qS(K\hbar, K\hbar, f\vartheta) \]

\[ = S(f\vartheta, f\vartheta, \hbar) + 2q[S(K\hbar, K\hbar, \hbar) + S(f\vartheta, f\vartheta, \hbar)] \]

\[ (1 - 4q)S(K\hbar, K\hbar, \hbar) \leq g_n + 2qg_n \]

\[ S(K\hbar, K\hbar, \hbar) \leq \frac{(1 + 2q)g_n}{(1 - 4q)}. \]

In the last inequality of each case, the infimum on the right hand side is 0. So we get
\[ S(Kh, Kh, h) = 0 \]

So
\[ Kh = h. \]

From (7)
\[ Kh = fh = h \]

So, \( f \) and \( K \) have common fixed point \( h \).

If \( K \) and \( f \) have another common fixed point \( \mu_1 \) then
\[ K\mu_1 = f\mu_1 = \mu_1. \]

From hypothesis (b)
\[ S(h, h, \mu_1) = S(fh, fh, f\mu_1) \preceq qU(h, h, \mu_1) \]

where
\[
\begin{align*}
U(h, h, \mu_1) & \in \{ S(Kh, Kh, K\mu_1), S(Kh, Kh, fh), S(K\mu_1, K\mu_1, f\mu_1), S(Kh, Kh, f\mu_1), S(K\mu_1, K\mu_1, fh) \} \\
& = \{ 0, S(h, h, \mu_1) \}
\end{align*}
\]

Thus
\[ S(h, h, \mu_1) = 0. \]

So
\[ h = \mu_1. \]

Hence \( f \) and \( K \) have a common fixed point \( h \) that is unique.

**Corollary 2.3** Let \((\mathbb{R}, S, V)\) be a vector \( S \)-metric space which is complete and \( V \)-Archimedean. Let \( K : \mathbb{R} \to \mathbb{R} \) be a map which is continuous and \( f : \mathbb{R} \to \mathbb{R} \) be a map which commutes with \( K \). Also let \( f \) and \( K \) satisfy
\[ f(\mathbb{R}) \subseteq K(\mathbb{R}) \]
and
\[ S(fh, fh, f\vartheta) \preceq qS(Kh, Kh, K\vartheta) \]
for all \( q \in \left[ 0, \frac{1}{3} \right] \) and \( h, \vartheta \in \mathbb{R} \). Then \( K \) and \( f \) have common fixed point which is unique.

**Theorem 2.4** Let \((\mathbb{R}, S, V)\) be a vector \( S \)-metric space which is complete and \( V \)-Archimedean. Suppose a map \( K^2 : \mathbb{R} \to \mathbb{R} \) is continuous and a map \( f : \mathbb{R} \to \mathbb{R} \) which commutes with \( K \). Suppose the conditions given below are satisfied:

(i) \( fK(\mathbb{R}) \subseteq K^2(\mathbb{R}) \)
(ii) \( S(fh, fh, f\vartheta) \leq qU(h, h, \vartheta) \) for all \( h, \vartheta \in \mathbb{R} \) where \( q \in \left[ 0, \frac{1}{3} \right] \) is a constant and

\[
U(h, h, \vartheta) \in \{ S(Kh, Kh, K\vartheta), S(Kh, Kh, fh), S(K\vartheta, K\vartheta, f\vartheta), \frac{1}{3}[S(Kh, Kh, f\vartheta) + S(K\vartheta, K\vartheta, fh)] \}
\]

(iii) \( K(\mathbb{R}) \) or \( f(\mathbb{R}) \) as a subspace of \( \mathbb{R} \) which is \( V \)-complete.

Then prove that \( K \) and \( f \) have a common fixed point which is unique.

**Proof.** Fix arbitrary \( \vartheta_0 \in K(\mathbb{R}) \), so we can take sequence \( \langle h_b \rangle \) in \( K(\mathbb{R}) \) such that

\[
h_b = f\vartheta_b = K\vartheta_{b+1} \quad b \geq 0.
\]

Now

\[
Kh_{b+1} = Kf\vartheta_{b+1} = fK\vartheta_{b+1} = fh_b = p_b \quad b \geq 0
\]

It can be shown as in theorem(2.2) that \( \langle p_b \rangle \) is a \( V \)-Cauchy sequence and converge to \( p \in \mathbb{R} \) and

\[
K^2p = fKp
\]

\[
\lim_{b \to \infty} Kf\vartheta_b = \lim_{b \to \infty} Kh_b = \lim_{b \to \infty} p_{b-1} = p. \tag{10}
\]

It follows that

\[
\lim_{b \to \infty} K^4\vartheta_b = \lim_{b \to \infty} K^3(K\vartheta_{b-1})
\]

\[
= \lim_{b \to \infty} K^3(h_{b-1})
\]

\[
= \lim_{b \to \infty} K^2(Kh_{b-1})
\]

\[
= \lim_{b \to \infty} K^2(p_{b-2})
\]

\[
= K^2p, \quad \text{because } K^2 \text{ is continuous}. \tag{11}
\]

So we get

\[
S(K^2p, K^2p, fKp) \leq 2S(K^2p, K^2p, fK^3\vartheta) + S(fKp, fKp, fK^3\vartheta)
\]

\[
= 2S(K^2p, K^2p, fK^3\vartheta) + S(fK^3\vartheta, fK^3\vartheta, fKp)
\]

\[
S(K^2p, K^2p, fKp) \leq 2S(K^2p, K^2p, fK^3\vartheta) + qU(K^3\vartheta, K^3\vartheta, Kp) \tag{12}
\]
where

\[ U(K^3\vartheta, K^3\vartheta, Kp) \subseteq \{S(K^4\vartheta, K^4\vartheta, K^2p), S(K^4\vartheta, K^4\vartheta, fK^3\vartheta), S(K^2p, K^2p, fKp), \frac{1}{3}\{S(K^4\vartheta, K^4\vartheta, fKp) + S(K^2p, K^2p, fK^3\vartheta)\}\} \]  

(13)

Since \( K^2 \) is continuous and by using (10), we get

\[ K^3 f\vartheta \xrightarrow{S,V} K^2p \]

and

\[ K^4\vartheta \xrightarrow{S,V} K^2p \]

then there exist a sequence \( \langle \alpha_0 \rangle \) and \( \langle \beta_0 \rangle \) in \( V \) such that \( \alpha_0 \downarrow 0 \) and \( \beta_0 \downarrow 0 \), then we have

\[ S(K^2p, K^2p, fK^3\vartheta) \leq \alpha_0 \]
\[ S(K^4\vartheta, K^4\vartheta, K^2p) \leq \beta_0 \]

Using (12) and (13), then we have the following:

**Case (i)** \( S(K^2p, K^2p, fKp) \)

\[ \leq 2S(K^2p, K^2p, fK^3\vartheta) + qS(K^4\vartheta, K^4\vartheta, K^2p) \]
\[ \leq 2\alpha_0 + q\beta_0 \]

**Case (ii)** \( S(K^2p, K^2p, fKp) \)

\[ \leq 2S(K^2p, K^2p, fK^3\vartheta) + qS(K^4\vartheta, K^4\vartheta, fK^3\vartheta) \]
\[ \leq 2S(K^2p, K^2p, fK^3\vartheta) + q[2S(K^4\vartheta, K^4\vartheta, K^2p) + S(fK^3\vartheta, fK^3\vartheta, K^2p)] \]
\[ = 2S(K^2p, K^2p, fK^3\vartheta) + q[2S(K^4\vartheta, K^4\vartheta, K^2p) + S(K^2p, K^2p, K^3f\vartheta)] \]
\[ \leq 2\alpha_0 + q(2\beta_0 + \alpha_0) \]
\[ \leq (2 + q)\alpha_0 + 2q\beta_0 \]

**Case (iii)** \( S(K^2p, K^2p, fKp) \)

\[ (1 - q)S(K^2p, K^2p, fKp) \]
\[ \leq 2\alpha_0 \]
\[ S(K^2p, K^2p, fKp) \]
\[ \leq \frac{2\alpha_0}{(1 - q)} \]

**Case (iv)** \( S(K^2p, K^2p, fKp) \)

\[ \leq 2S(K^2p, K^2p, fK^3\vartheta) + \frac{q}{3}[S(K^4\vartheta, K^4\vartheta, fKp) + S(K^2p, K^2p, fK^3\vartheta)] \]
\[ S(K^2p, K^2p, fKp) \]
\[ \leq 2S(K^2p, K^2p, fK^3\vartheta) + \frac{q}{3}[2S(K^4\vartheta, K^4\vartheta, K^2p) + S(fKp, fKp, K^2p) + S(K^2p, K^2p, fK^3\vartheta)] \]
\[
S(K^2p, K^2p, fK^3\vartheta) + \frac{q}{3} S(K^4\vartheta, K^4\vartheta, K^2p) + S(K^2p, K^2p, fKp) + S(K^2p, K^2p, fK^3\vartheta)
\]

\[
(1 - \frac{q}{3}) S(K^2p, K^2p, fKp) \leq 2\alpha + \frac{q}{3} [2\beta + \alpha]
\]

\[
(3 - q) S(K^2p, K^2p, fKp) \leq 6\alpha + q[2\beta + \alpha]
\]

\[
S(K^2p, K^2p, fKp) \leq \frac{(6 + q)\alpha + 2q\beta}{(3 - q)}
\]

In the last inequality of each case, the infimum on the right hand side is 0. So

\[
S(K^2p, K^2p, fKp) = 0
\]

that is

\[
K^2p = fKp
\]

Putting in \( S(fh, fh, f\vartheta) \leq qU(h, h, \vartheta) \), \( h = fKp, \vartheta = Kp \) then we get \( f(fKp) = fK \), we get \( f(fKp) = fKp \). Because \( K^2p = fKp \) i.e. \( K(Kp) = f(Kp) \), we have \( K(fKp) = fK^2p = f(fKp) = fKp \). So \( f \) and \( K \) have a common fixed point \( fKp \).

If \( f \) and \( K \) have a common fixed point \( f_1K_1p_1 \) then

\[
K(f_1K_1p_1) = f(f_1K_1p_1) = f_1K_1p_1
\]

Now

\[
S(fKp, fKp, f_1K_1p_1) = S(f(fKp), f(fKp), f(f_1K_1p_1)) \leq qU(fKp, fKp, f_1K_1p_1)
\]

where

\[
U(fKp, fKp, f_1K_1p_1) \in \{ S(K(fKp), K(fKp), K(f_1K_1p_1)), S(K(fKp), K(fKp), f(fKp)), S(K(f_1K_1p_1), K(f_1K_1p_1), f(f_1K_1p_1)), S(K(fKp), K(fKp), f(f_1K_1p_1)), S(K(f_1K_1p_1), K(f_1K_1p_1), f(fKp)) \}
\]

\[
\in \{ 0, S(fKp, fKp, f_1K_1p_1) \}
\]

Thus

\[
S(fKp, fKp, f_1K_1p_1) = 0.
\]

So

\[
fKp = f_1K_1p_1.
\]
Hence \( f \) and \( K \) have a common fixed point \( fKp \) which is unique.

**Example 2.5** Let \( V = \mathbb{R}^2_+ \) with coordinatewise ordering and \( \mathbb{R} = \mathbb{R} \)

\[
S(h, \vartheta, \eta) = (\rho|h - \eta|, \sigma|\vartheta - \eta|)
\]

where \( \rho, \sigma > 0, \) and \( h, \vartheta, \eta \in \mathbb{R} \). Then

\[
fh = h^2 + 5
\]

and

\[
Kh = 2h^2.
\]

We have

\[
S(fh, f\vartheta, f\eta) = (\rho|h^2 - \eta^2|, \sigma|\vartheta^2 - \eta^2|) = \frac{1}{2}S(Kh, K\vartheta, K\eta) \leq qS(Kh, K\vartheta, K\eta)
\]

for \( q \in [\frac{1}{2}, 1) \), \( f(\mathbb{R}) = [5, \infty) \subseteq [0, \infty) = K(\mathbb{R}) \) and a self map \( K \) is continuous on \( \mathbb{R} \) and \( f(\mathbb{R}) \) is \( V \)-complete subspace of \( \mathbb{R} \). Hence \( K \) and \( f \) have common fixed point that is unique.

**Example 2.6** Let \( V = \mathbb{R} \) with coordinatewise ordering and \( \mathbb{R} = [0, 1] \)

\[
S(h, \vartheta, \eta) = |h - \eta| + |\vartheta - \eta|
\]

where \( h, \vartheta, \eta \in \mathbb{R} \). Then

\[
fh = \frac{h}{4}
\]

and

\[
Kh = \frac{h}{2}
\]

We have

\[
S(fh, f\vartheta, f\eta) = \left| \frac{h}{4} - \frac{\eta}{4} \right| + \left| \frac{\vartheta}{4} - \frac{\eta}{4} \right| = \frac{1}{2}S(Kh, K\vartheta, K\eta) \leq qS(Kh, K\vartheta, K\eta)
\]

for \( q \in [\frac{1}{2}, 1) \), \( f(\mathbb{R}) = \left[ 0, \frac{1}{4} \right] \subseteq \left[ 0, \frac{1}{2} \right] = K(\mathbb{R}) \) and a self map \( K \) is continuous on \( \mathbb{R} \) and \( f(\mathbb{R}) \) is \( V \)-complete subspace of \( \mathbb{R} \). Hence \( K \) and \( f \) have common fixed point that is unique.
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