A filtration question on Belyĭ pairs and dessins

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Abstract

A Belyĭ pair is a holomorphic map from a Riemann surface to $S^2$ with additional properties. A dessin d’enfants is a bipartite graph with additional structure. It is well known that there is a bijection between Belyĭ pairs and dessins d’enfants.

Vassiliev has defined a filtration on formal sums of isotopy classes of knots. Motivated by this, we define a filtration on formal sums of Belyĭ pairs, and another on dessin d’enfants. We ask if the two definitions give the same filtration.

1 Introduction

First, we recall some definitions [2, 3]. A Belyĭ pair is a Riemann surface $C$ together with a holomorphic map $f : C \to S^2 = \mathbb{C} \cup \{\infty\}$ to the Riemann sphere, such that $f'(p)$ is non-zero provided $f(p)$ is not 0, 1 or $\infty$. (Belyĭ proved that given $C$ such an $f$ can be found iff $C$ can be defined as an algebraic curve over the algebraic numbers.)

A dessin d’enfants, or dessin for short, is a graph $G$ together with a cyclic order of the edges at each vertex, and also a partition of the vertices $V$ into two sets $V_0$ and $V_1$ such that every edge joins $V_0$ to $V_1$. Necessarily, $G$ must be a bipartite graph. Traditionally, the vertices in $V_0$ and $V_1$ are coloured black and white respectively.

It is easy to see that a Belyĭ pair gives rise to a dessin, where $V_0 = f^{-1}(0)$, $V_1 = f^{-1}(1)$, and the edges are the components of the inverse image $f^{-1}([0,1])$ of the unit interval in $\mathbb{C}$. The cyclic order arise from local monodromy around the vertices.

A much harder result, upon which our definitions rely, is that up to isomorphism every dessin arises from exactly one Belyĭ pair, or in other words that there is a bijection between isomorphism classes of Belyĭ pairs and dessins.
2 Definitions

Definition 1 (Belyi object). A Belyi object $B$ consists of $((B_C, B_f), B_D)$ where $(B_C, B_f)$ is a Belyi pair and $B_D$ is the associated dessin (or vice versa for the dessin and the pair).

Definition 2 (Vassiliev space). The Vassiliev space $V = V_C$ (for Belyi objects) is the vector space over $\mathbb{C}$ which has as basis the isomorphism classes of Belyi objects.

Clearly, when an edge is removed from a dessin then it is still a dessin. Suppose $D$ is a dessin, and $T$ is a subset of its edges. We will use $D \setminus T$ to denote the dessin so obtained. This same operation can also be applied to a Belyi object $B$, even though computing the associated curve $(B \setminus T)_C$ from $B_D$ and $T$ might be hard.

We will now define one or two filtrations of $V$.

Definition 3 (Dessin with $d$ optional edges). Let $D$ be dessin and $S$ a $d$-element subset of $D$. Each subset $T$ of $S$ determines a dessin $S \setminus T$ and hence a Belyi object $B_{S \setminus T}$. Let $|T|$ denote the number of edges in $T$. Use

$$B_S = \sum_{T \subseteq S} (-1)^{|T|} B_{S \setminus T}$$

to define a vector $B_S$ in $V$, which we call the expansion of a dessin with $d$ optional edges.

Definition 4 (Dessin filtration). Let $V_{D,d}$ be the span of the expansions of all dessins with $d$ optional edges. The sequence

$$V = V_{D,0} \supseteq V_{D,1} \supseteq V_{D,2} \supseteq V_{D,3} \ldots$$

is the dessin filtration of $V$.

We can also think of a Belyi object as a map $f : C \to S^2$ (with special properties). Let $(C_1, f_1)$ and $(C_2, f_2)$ be Belyi pairs. Then there is of course a map

$$g : C_1 \times C_2 \to S^2 \times S^2.$$

Let $\Delta \subset S^2 \times S^2$ denote the diagonal, and let $C$ denote $g^{-1}(\Delta)$, and $f$ the restriction of $g$ to $C$. In general

$$f : C \to \Delta \cong S^2$$

will not be a Belyi pair. There are two possible problems. The first is that $C \subset C_1 \times C_2$ might have self intersections or be otherwise singular. If this happens, we replace $C$ by its resolution, which is unique.

The second problem is more interesting. It might be that $f$ has critical points not lying above the special points $0, 1$ and $\infty$. This problem cannot be avoided. However, the above discussion does show that there is product, which we will denote by ‘$\cdot$’, on holomorphic branched covers of $S^2$. 

2
**Definition 5** (Product filtration). Let $W$ be the vector space with basis isomorphism classes of branched covers of $S^2$. We set $W_n$ to be the span of all products of the form

$$(A_1 - B_1) \circ (A_2 - B_2) \circ \ldots \circ (A_n - B_n)$$

for $A_i$ and $B_i$ basis vectors of $W$. Clearly, the $W_n$ provide a filtration of $W$.

**Definition 6** (Bely˘ı filtration). The induced filtration of $V$ defined by $V_{B,n} = W_n \cap V$ is called the Bely˘ı filtration of $V$.

### 3 Questions

**Question 1.** Are the two filtrations $V_D$ and $V_B$ equal?

If so, then we have also answered the next two questions.

**Question 2.** The absolute Galois group acts on Bely˘ı pairs, and preserves the Bely˘ı filtration. Does this action also preserve the dessin filtration?

**Question 3.** Because the dessins with $d$ edges, all of which are optional, span $V_d/V_{d+1}$, the dessin filtration has finite dimensional quotients. Does the Bely˘ı filtration have finite dimensional quotients?

Investigating the last two questions might help us answer the first. They might also be of interest in their own right.

### References

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