Gauge fixing, families index theory, and topological features of the space of lattice gauge fields

David H. Adams

Physics dept., National Taiwan University, Taipei, Taiwan 106, R.O.C.
and
Physics dept., Duke University, Durham, NC 27708, U.S.A.

email: adams@phy.duke.edu

Abstract

The families index theory for the overlap lattice Dirac operator is applied to derive topological features of the space of SU($N$) lattice gauge fields on the 4-torus: The topological sectors, specified by the fermionic topological charge, are shown to contain noncontractible even-dimensional spheres when $N \geq 3$, and noncontractible circles in the $N=2$ case. We describe how certain obstructions to the existence of gauge fixings without the Gribov problem in the continuum setting correspond on the lattice to obstructions to the contractibility of these spheres and circles. We also point out a canonical connection on the space of lattice gauge fields with monopole-like singularities associated with the spheres.

1 Introduction

Lattice gauge fields on the 4-torus $T^4$ have a fermionic topological charge which arises in the overlap formalism \cite{1} and can be expressed as the index of the overlap Dirac operator \cite{2}. It determines a decomposition of the space $U$ of lattice gauge fields (with a given unitary gauge group) into topological sectors after excluding a measure-zero subspace on which the fermionic topological charge is ill-defined. In this

\footnote{Current address}
paper we consider the following natural question: what do these topological sectors look like? Their topology has been worked out in the U(1) case (with a certain admissibility condition imposed on the gauge fields) by Lüscher in [3]. This was a key part of the existence proof in [3] for gauge invariant abelian chiral gauge theory on the lattice when anomalies cancel. In this paper we derive first results on the topology of the sectors of U in the nonabelian SU(N) case. Besides copies of SU(N), which are present because the lattice gauge fields take values in the gauge group, we show that the sectors of U contain certain noncontractible even-dimensional spheres in the N ≥ 3 case, and noncontractible circles in the N = 2 case. These are a direct consequence of excluding the lattice gauge fields for which the fermionic topological charge is ill-defined.

The presence of the noncontractible 2n-spheres in U is derived from results on the orbit space of U obtained via the families index theory for the overlap Dirac operator in [11, 12]. The classical continuum limit result Theorem 2 of [11] reveals the presence of certain noncontractible 2n-spheres in the orbit space in the N ≥ 3 case. Similarly, the classical continuum limit result of Ref. [13] on the lattice version of Witten’s global anomaly implies the presence of noncontractible circles in the orbit space of U in the SU(2) case. We show that these give rise to noncontractible 2n-spheres/circles in U itself by exploiting the known fact that, in contrast to the continuum situation, gauge fixings without the Gribov problem exist on the lattice (e.g., the maximal tree gauges [14, 15]). This is done in Section 2. The argument is quite implicit though, and our aim in the rest of the paper is to give a more explicit and illuminating construction of noncontractible 2n-spheres/circles in U, and explain how the obstructions to their contractibility are the lattice counterparts of certain obstructions to the existence of

---

2This result, which has also been established by different means in the noncompact U(1) case [1], generalises to U(1) chiral gauge theory on arbitrary even-dimensional torus [3], and to the electroweak U(1)×SU(2) case [3]. In the general nonabelian case the existence of chiral gauge theory with exact gauge invariance on the lattice has been shown at the perturbative level in [7, 8]. However, despite the progress in [9], there is at present no nonperturbative existence proof (except for the special case where the fermion representation is real [11]).
gauge fixings without the Gribov problem in the continuum setting. We begin in Section 3 by discussing how the gauge fixing issue and topology of the gauge field sectors get related, both in the continuum and on the lattice, when considering the gauge invariance issue for the chiral fermion determinant. In Section 4, we give a fermionic description of certain obstructions to the existence of Gribov problem-free gauge fixings in the continuum SU($N$) gauge theory, based on families index theory for the Dirac operator, and Witten’s global anomaly, in the $N \geq 3$ and $N = 2$ cases, respectively. The lattice version of these considerations in Section 5 leads instead to obstructions to the contractibility of $2n$-spheres/circles in $U$. We also discuss a connection on the space of lattice gauge fields, with monopole-like singularities associated with the $2n$-spheres, which arises naturally in this context. The results of the paper are summarised in Section 6. A property of $G_0$ gauge transformations used in the text is derived in an appendix.

2 Topology of the space of lattice gauge fields

We assume familiarity with the lattice formulation of SU($N$) gauge theory on $T^4$ as summarised in [11] and refer to that paper for the definitions and notations used in the following. To begin with, the space of lattice gauge fields is

$$U_{\text{initial}} = \text{SU}(N) \times \text{SU}(N) \times \cdots \times \text{SU}(N)$$  \hspace{1cm} (2.1)

(one copy for each lattice link). This space is connected, but decomposes into disconnected sectors labeled by the fermionic topological charge after excluding the codimension 1 submanifold of gauge fields for which the charge is ill-defined. The fermionic topological charge is given by the index of the overlap Dirac operator \cite{2} and the excluded fields $U$ are those for which this operator is ill-defined, i.e. the ones for which the Hermitian Wilson-Dirac operator $H^U$ (with suitable negative mass term) has zero-modes. We denote the resulting space of lattice gauge fields by $U$. A sufficient (although not necessary) condition for a lattice gauge field $U$ to lie in $U$ is

\footnote{It reduces to the continuum topological charge in the classical continuum limit \cite{16}.}

It reduces to the continuum topological charge in the classical continuum limit [16].
that its plaquette variables satisfy the “admissibility condition”,

$$\|1 - U(p)\| < \epsilon$$

(2.2)

where $U(p)$ is the product of the $U_\mu(x)$'s around the lattice plaquette $p$. For sufficiently small $\epsilon$, this condition guarantees the absence of zero-modes for $H^U$ [17, 18]. When $U$ is the lattice transcript of a continuum field $A$, and $p$ is the plaquette specified by a lattice site $x$ and directions $\mu$ and $\nu$, then $1 - U(p) = a^2 F_{\mu\nu}(x) + O(a^3)$. Hence the lattice transcript of a smooth continuum field (or family of continuum fields) is guaranteed to lie in $\mathcal{U}$ when the lattice is sufficiently fine (see [16] for further discussion of this point).

Since $\mathcal{U}_{\text{initial}}$ is a product of copies of SU($N$), its topology is known. The topology of $\mathcal{U}$ is more complicated and has not yet been determined in the present SU($N$) case. (It has so far only been worked out in the U(1) case with the admissibility condition (2.2) imposed [3]; we discuss this case further below.) In the following we apply results from the families index theory for the overlap Dirac operator [11] to derive first results on the topology of $\mathcal{U}$ in the SU($N$) case:

**Theorem.** When the lattice is sufficiently fine, the topological sectors of $\mathcal{U}$ contain noncontractible $2n$-dimensional spheres for $1 \leq n \leq N-2$ in the $N \geq 3$ case, and noncontractible circles in the SU(2) case.

**Remark.** Note that, by (2.1), $\pi_{2n}(\mathcal{U}_{\text{initial}}) = 0$ for $n \leq N-1$ and $\pi_1(\mathcal{U}_{\text{initial}}) = 0$, since the same is true for $\pi_{2n}$ and $\pi_1$ of SU($N$). In contrast, by the theorem, $\pi_{2n}(\mathcal{U}) \neq 0$ for $1 \leq n \leq N-2$ in the $N \geq 3$ case, and $\pi_1(\mathcal{U}) \neq 0$ in the SU(2) case. Hence the noncontractible $2n$-spheres and circles mentioned in the theorem are all contractible in $\mathcal{U}_{\text{initial}}$; their noncontractibility in $\mathcal{U}$ reflects the topological consequences of excluding from $\mathcal{U}_{\text{initial}}$ the fields $U$ for which the fermionic topological charge is ill-defined.

We will give two proofs of the Theorem in this paper. Both involve studying $\mathcal{U}$ as a $G_0$-bundle over the orbit space $\mathcal{U}/G_0$, where $G_0$ is the subgroup of gauge transformations $\phi : \{\text{lattice sites}\} \rightarrow \text{SU}(N)$ satisfying the condition $\phi(x_0) = 1$ for some arbitrarily chosen basepoint $x_0$ in $T^4$. The reason for this condition is to exclude
the (nontrivial) constant gauge transformations; consequently, \( G_0 \) acts freely on \( U \) (unlike the full group \( G \)). This is crucial for our arguments in the following. The first, most direct proof of the theorem is given in the remainder of this Section. It makes use of the fact that \( G_0 \) gauge fixings without the Gribov problem exist on the lattice. Examples of these are the maximal tree gauges \([14, 15]\) which we review further below. Gauge fixings which do not have the Gribov problem \([19]\) are referred to as ‘good’ in the following. A good \( G_0 \) gauge fixing picks out a submanifold \( U_f \) of \( U \) which intersects each \( G_0 \) orbit precisely once. Since \( G_0 \) acts freely on \( U \), this determines a decomposition

\[
U \simeq U_f \times G_0 \tag{2.3}
\]

The one-to-one correspondence between elements of \( U_f \) and \( U/G_0 \) then gives

\[
U \simeq U/G_0 \times G_0 \tag{2.4}
\]

i.e. a trivialisation of \( U \) as a \( G_0 \)-bundle over \( U/G_0 \). Hence the topology of \( U \) is determined by that of \( U/G_0 \) and \( G_0 \). The \( N \geq 3 \) part of the Theorem now follows from results obtained via the families index theory for the overlap Dirac operator in \([11]\): It was shown there that the topological sectors of \( U/G_0 \) contain noncontractible \( 2n \)-spheres for \( 1 \leq n \leq N-2 \); these arise as the lattice transcripts of certain \( 2n \)-spheres in the continuum orbit space \( A/G_0 \) over which the topological charge (integrated Chern character) of the index bundle of the continuum Dirac operator is nonvanishing. By Theorem 2 of \([11]\), the topological charges of the lattice index bundle over the lattice transcripts of these \( 2n \)-spheres coincide with the continuum topological charges when the lattice is sufficiently fine. Since the charges are constant under smooth deformations of the spheres, their nonvanishing implies noncontractibility of these \( 2n \)-spheres in \( U/G_0 \). This together with the decomposition (2.4) implies the presence of the noncontractible \( 2n \)-spheres in \( U \) itself, as claimed in the Theorem.

The presence of noncontractible circles in \( U \) in the SU(2) case can be derived in a similar way, using the results of Ref. \([13]\) on the lattice version of Witten’s global

\[\text{For example, the trivial gauge field } U = 1 \text{ is invariant under all constant gauge transformations, and constant } \phi \text{ with values in the center of SU}(N) \text{ act trivially on all gauge fields.}\]
The lattice version of the global anomaly can be expressed as $\mathcal{N} \pmod{2}$, where $\mathcal{N}$ is the net number of crossings of the origin (counted with sign) by the eigenvalues of the overlap Dirac operator as the background gauge field is smoothly varied along a path $U_t$ from an initial field $U_0$ to a final field $U_1 = \phi \cdot U_0$. The results of \cite{13} show that, when $U_t$ and $\phi$ are the lattice transcripts of a continuum path $A_t$ and a topologically nontrivial map $\phi : T^4 \to SU(2)$, then the lattice anomaly reproduces the continuum one, i.e. $\mathcal{N} \pmod{2} = 1$ when the lattice is sufficiently fine (for a single fermion species in the fundamental representation of $SU(2)$).\footnote{The presence of the global anomaly on the lattice had earlier been verified numerically in \cite{20,21}.} The path $U_t$ is a circle in $U/G_0$, and clearly $\mathcal{N}$ can only change by even integers under deformations of this circle. Hence the nonvanishing of $\mathcal{N} \pmod{2}$ implies noncontractibility of the circle in $U/G_0$. This together with (2.4) implies the presence of noncontractible circles in $U$ itself. This completes our first proof of the Theorem.

The preceding argument gives more information on $\pi_{2n}(U)$ besides the fact that it is nonvanishing for $1 \leq n \leq N - 2$: The topological charges of the $2n$-spheres in the continuum orbit space $A/G_0$ discussed above can have arbitrary integer values; hence for any given integer $p$, $U/G_0$ will contain a $2n$-sphere with topological charge $p$ when the lattice is sufficiently fine. Since $2n$-spheres in $U/G_0$ with different topological charges represent different elements in $\pi_{2n}(U/G_0)$, it follows that the number of different elements in $\pi_{2n}(U/G_0)$ can be made arbitrarily large by taking the lattice to be sufficiently fine, and becomes infinite in the continuum limit. By (2.4), the same is true for $\pi_{2n}(U)$. On the other hand, the argument in the $SU(2)$ case does not indicate more than one nontrivial element in $\pi_1(U/G_0)$ or $\pi_1(U)$ since there is only one equivalence class of topologically nontrivial maps $\phi : T^4 \to SU(2)$.

We remark that imposing the admissibility condition (2.2) does not change the situation regarding the topological features of $U$ derived above. The $2n$-spheres/circles in $U/G_0$, from which the $2n$-spheres/circles in $U$ arise, are the lattice transcripts of certain $2n$-spheres/circles in $A/G_0$ which come from $2n$-dimensional balls/line seg-

\footnote{The requirement that $\phi(x_0) = 1$ is inconsequential here since we can replace $\phi \to \phi(x_0)^{-1}\phi$ to enforce this condition without affecting the topological properties of $\phi$.}
ments in $\mathcal{A}$ (see [11]), and the lattice transcripts of such families of continuum gauge fields are guaranteed to satisfy the admissibility condition for any $\epsilon > 0$ when the lattice is sufficiently fine, cf. the discussion following (2.2) above. Then the arguments above go through unchanged.

The decomposition (2.3), which was crucial for the above proof of the Theorem, relies on the fact that good $G_0$ gauge fixings exist on the lattice. Such gauge fixings can be obtained from maximal trees [14, 15] as we now discuss. A tree in the lattice is a collection of lattice links from which no closed loops can be formed. The tree is called maximal if it is not possible to add another lattice link without getting a closed loop. An example of a maximal tree in a 2-dimensional lattice is given in Fig. 1. It was pointed out by Creutz [14] (see also [15]) that, given a maximal tree, any lattice gauge field can be gauge transformed into a field whose link variables along the tree are all equal to 1 in a way that is unique up to constant gauge transformations. Furthermore, any two gauge fields in the same gauge orbit get transformed into fields which coincide up to a constant gauge transformation. Thus a maximal tree determines a lattice gauge fixing which, modulo constant gauge transformations, is free of Gribov ambiguities.\footnote{As discussed in [22], the residual symmetry under constant gauge transformations is easily handled. (General methods for dealing with theories with a residual global symmetry have also been discussed in [23].) However, finding physically acceptable lattice gauge fixings which can be implemented in Monte Carlo simulations without Gribov copies arising is a nontrivial problem which has yet to be completely resolved. For reviews of this issue see, e.g., [24, 25].} We now point out that maximal trees determine good $G_0$ gauge
fixings. Indeed, for each lattice gauge field there is a unique gauge transformation satisfying \( \phi(x_0) = 1 \) which transforms it to a field whose link variables along the tree are all equal to 1. The key observation now is that any two gauge fields in the same \( G_0 \) orbit get sent to the same gauge-fixed field under this procedure. This is seen as follows. For each link not contained in the maximal tree, the link variable of the gauge-fixed field is the product of the original link variables around a closed loop, starting and ending at \( x_0 \), which is formed by adding the link to the tree. (We leave the straightforward verification of this fact to the interested reader.) Consequently, under a \( G_0 \) gauge transformation of the original field, the nontrivial link variables of the gauge-fixed field just get conjugated by \( \phi(x_0) = 1 \), i.e. they are unchanged. Thus the procedure picks out precisely one lattice gauge field in each \( G_0 \) orbit, i.e. we have a good \( G_0 \) gauge fixing.

We note in passing the topological structure of \( U_{\text{initial}}/G_0 \), which can be determined from a maximal tree gauge fixing as follows. The “gauge-fixed” submanifold \( U_{f(\text{initial})} \) of \( U_{\text{initial}} \) picked out by a maximal tree \( G_0 \) gauge fixing consists of the lattice gauge fields whose link variables are 1 on the links of the tree. I.e. \( U_{f(\text{initial})} \) is a product of copies of \( SU(N) \) with one copy for each lattice link not contained in the tree. A general property of maximal trees (whose straightforward verification we again leave to the reader) is that the number of links making up the tree is \( s - 1 \), where \( s \) is the number of lattice sites. Thus \( U_{f(\text{initial})} \) is the product of \( 4s - (s - 1) = 3s + 1 \) copies of \( SU(N) \). Since \( U_{f(\text{initial})} \simeq U_{\text{initial}}/G_0 \), this reveals the topology of \( U_{\text{initial}}/G_0 \). In particular, it follows that the noncontractible \( 2n \)-spheres/circles in \( U/G_0 \) discussed above are all contractible in \( U_{\text{initial}}/G_0 \). As a small consistency check, we also note that the decomposition

\[
U_{\text{initial}} \simeq U_{\text{initial}}/G_0 \times G_0
\]

reproduces (2.1), since \( G_0 \) is the product of \( s - 1 \) copies of \( SU(N) \) (one for each lattice site except \( x_0 \)).

In the U(1) case, the topology of \( U \) when the admissibility condition (2.2) is imposed was completely worked out by Lüscher in [3]. In this case, the topological
sectors are labeled by topological fluxes $m = \{m_{\mu\nu}\}$ rather than a topological charge. Using a good $G_0$ gauge fixing (different from the maximal tree gauges; it is a lattice version of the Landau gauge supplemented with an additional condition to exclude Gribov copies), Lüscher showed that the topological structure of each topological sector is

$$U_{[m]} \simeq U(1)^4 \times U^\text{cr}_{[m]} \times G_0$$

where $U^\text{cr}_{[m]}$ is a contractible submanifold in $U_{[m]}$. The $U(1)^4$ factor in the gauge-fixed submanifold $U(1)^4 \times U^\text{cr}_{[m]}$ is a remnant of the product of copies of $U(1)$ making up $U_{\text{initial}}$. Thus, in the $U(1)$ case, decomposing $U_{\text{initial}}$ into topological sectors by imposing the admissibility condition (2.2) has the effect of removing some of the initial topological structure (by “breaking open” some of the copies of $U(1)$ to get the contractible subspace $U^\text{cr}_{[m]}$), but does not produce any new nontrivial topological structure. In contrast, the result of this section shows that, in the $SU(N)$ cases, new nontrivial structures are in fact produced (i.e. the noncontractible $2n$-spheres/circles).

The existence proof for the noncontractible $2n$-spheres/circles in $U$ given in this section was rather implicit. In Section 5 we give an alternative, more illuminating proof, involving a different, more explicit construction of $2n$-spheres/circles. In that approach, the obstructions to contractibility are seen to be the lattice counterparts of certain obstructions to the existence of good $G_0$ gauge fixings in the smooth continuum setting (discussed in Section 4). But first, in the next section, we describe a simpler version of this correspondence that arises naturally when considering the gauge invariance issue for the chiral fermion determinant.

3 Relating gauge fixing and topology via the chiral fermion determinant

An important issue in lattice chiral gauge theory is whether a smooth, gauge invariant phase choice exists for the (left- or right-handed) overlap chiral fermion determinant \[1\]. The latter can be expressed as $\det(D_{\pm}^U)$, where $D$ is the overlap Dirac operator.
Existence of good \( G_0 \) gauge fixings has implications for this: If a smooth phase choice for \( \det(D_{\pm}^U) \) exists on the gauge-fixed submanifold \( U_f \) picked out by the gauge fixing, then a smooth *gauge invariant* (under \( G_0 \)) phase choice on the whole of \( U \) is obtained by simply setting \( \det(D_{\pm}^\phi U_f) = \det(D_{\pm}^{U_f}) \), where \( \phi \cdot U_f \) denotes the action of a gauge transformation \( \phi \in G_0 \) on a gauge field \( U_f \in U_f \).

On the other hand it is known, both in the smooth continuum setting \cite{27} and in the lattice setting \cite{28, 29, 30}, that there are topological obstructions to gauge invariance of the chiral fermion determinant in the U(1) and SU(\( N \)) cases (with fermion in the fundamental representation, or, more generally, when anomalies don’t cancel).\(^8\) It follows that, in these cases, either no good \( G_0 \) gauge fixing exists, or, if one does exist, then no smooth phase choice for the chiral fermion determinant exists on the submanifold \( U_f \) picked out by the gauge fixing.

Recall that the chiral fermion determinant is really a section in a U(1) determinant line bundle over the space of gauge fields (cf. the final section of \cite{27} in the continuum, and \cite{1, 28, 3, 29} in the lattice setting). A smooth phase choice for the determinant on a submanifold of the space of gauge fields is equivalent to a trivialisation of the determinant line bundle over the submanifold. Therefore, the question of whether such a phase choice exists is intimately related to the topology of the space of gauge fields. In the continuum setting, the topological sectors of the space \( \mathcal{A} \) of smooth continuum gauge fields have no nontrivial topological structure – they are just infinite-dimensional affine vectorspaces. Consequently, by a standard mathematical fact, the determinant line bundle is trivialisable over \( \mathcal{A} \) (and any submanifold of \( \mathcal{A} \)). Combining this with the preceding observations, we conclude that, in the continuum setting, obstructions to \( G_0 \) gauge invariance of the chiral fermion determinant are also obstructions to the existence of good \( G_0 \) gauge fixings. In particular, such gauge fixings cannot exist in the U(1) and SU(\( N \)) continuum gauge theories.\(^9\)

\(^8\)The “smoothness” parts of these statements break down if \( G_0 \) is replaced by \( G \), since the latter does not act freely on the space of gauge fields.

\(^9\)The fact that we are restricting the gauge transformations to \( G_0 \) does not change this situation, cf. the Appendix.

\(^{10}\)There are more direct ways to see these obstructions to \( G_0 \) gauge fixings, which do not involve
On the other hand, in the lattice setting we already know that good $G_0$ gauge fixings exist. Then the preceding observations lead to the conclusion that, when obstructions to gauge invariance are present (e.g., in the U(1) and SU($N$) cases), the submanifold $U_f$ picked out by the gauge fixing is noncontractible in $U$. For if $U_f$ was contractible, then, by a standard mathematical fact, the determinant line bundle would be trivialisable over $U_f$, i.e. a smooth phase choice for the chiral fermion determinant would exist on $U_f$, and we could then obtain a $G_0$-gauge invariant phase choice on the whole of $U$ in the way described earlier.

The preceding gives a first demonstration of how obstructions to the existence of good $G_0$ gauge fixings in the continuum correspond on the lattice to obstructions to contractibility of certain submanifolds in $U$. Actually, the arguments above only establish this correspondence for the trivial topological sector, since in the other sectors the chiral fermion determinant vanishes. The obstructions to gauge invariance of the chiral fermion determinant have a natural description in the context of families index theory: they are the topological charges of the Dirac index bundle over 2-spheres in the gauge field orbit space (cf. [11] and the last section of [27] for the lattice and continuum settings, respectively). In the following sections we use the families index theory to to show more precisely the correspondence between gauge fixing obstructions in the continuum and topological structure in the sectors of the space $U$ of lattice gauge fields; the derived results hold in general and not just for the topologically trivial sector.

Finally, we remark that the noncontractibility of $U_f$ in the U(1) case, derived above, is in agreement with Lüscher’s explicit determination of $U_f$ for a particular good $G_0$ gauge fixing in [3]: From (2.6) we see that each sector of that $U_f$ is noncontractible since it contains the noncontractible factor $U(1)^4 \simeq T^4$. 

\[\text{the chiral fermion determinant; these are described in the SU($N$) case in Section 3.}\]
4 Continuum considerations

In this section we describe obstructions to good $G_0$ gauge fixings in the space $\mathcal{A}$ of smooth continuum $SU(N)$ gauge fields on $T^4$. For simplicity we restrict our considerations to the topologically trivial sector. Then $\mathcal{G}$ consists of the smooth maps $\phi : T^4 \rightarrow SU(N)$ and $G_0$ is the subgroup with $\phi(x_0) = 1$. The most direct way to see the obstructions is via the approach of I. Singer in [31] (see also [32], where Singer’s argument for $SU(2)$ gauge fields on $S^3$ is generalised to gauge groups including general $SU(N)$ and spacetimes including $T^4$):\footnote{Note that our setup is different from that of [31, 32]: There the considerations are restricted to the irreducible gauge fields, which are acted freely upon by $\mathcal{G}/Z_N$ (where $Z_N$ is the center of $SU(N)$). The obstructions are then given by $\pi_n(\mathcal{G}/Z_N)$ and are different from the ones in our case. A drawback of that setup is that the trivial gauge field $A = 0$ is excluded; hence one cannot consider, e.g., the free gluon propagator. Our setup, where the space of gauge fields is unrestricted and the gauge transformations are instead restricted to $G_0$, avoids this.} Regarding $\mathcal{A}$ as a $G_0$-bundle over $\mathcal{A}/G_0$, a good $G_0$ gauge fixing is equivalent to a trivialisation

$$\mathcal{A} \cong \mathcal{A}/G_0 \times G_0$$

(4.1)

Since $\pi_n(\mathcal{A}) = 0$, existence of a trivialisation implies that $\pi_n(\mathcal{A}/G_0) = 0$ and $\pi_n(G_0) = 0$ for all $n = 0, 1, 2, \ldots$. Nonexistence of good $G_0$ gauge fixings now follows from the fact that there are always nonvanishing $\pi_n(G_0)$’s, which can be seen as follows. Since the topological structure of $SU(N)$ is essentially $S^3 \times S^5 \times \cdots \times S^{2N-1}$ modulo a finite set of equivalence relations, there are smooth maps $\Phi : S^{2n-1} \times T^4 \rightarrow SU(N)$ with nonvanishing degree for $1 \leq n \leq N-2$. Such maps still exist when a condition $\Phi(\theta, x_0) = 1 \ \forall \theta \in S^{2n-1}$ is imposed (see the Appendix). In the latter case, $\Phi$ corresponds to a map $S^{2n-1} \rightarrow G_0$, $\theta \mapsto \phi_\theta$ where $\phi_\theta(x) = \Phi(\theta, x)$, i.e. $\Phi$ can be regarded as an element in $\pi_{2n-1}(G_0)$. This element is clearly nonzero since $\Phi$ has nonvanishing degree. It follows that $\pi_{2n-1}(G_0) \neq 0$ for $1 \leq n \leq N-2$, which in turn implies that good $G_0$ gauge fixings do not exist when $N \geq 3$. They don’t exist in the $N = 2$ case either, since in that case $\pi_0(G_0) \neq 0$ due to the existence of maps $T^4 \rightarrow SU(2)$ which cannot be continuously deformed to the identity map.
These considerations do not have an immediate lattice counterpart because \( \pi_n(U) \neq 0 \) for the space \( U \) of lattice gauge fields. Therefore, we now present another perspective on the above obstructions to good \( G_0 \) gauge fixings by showing that they are the obstructions to trivialising the \( G_0 \)-bundle \( A \) over certain \( 2n \)-spheres in \( A/G_0 \) in the \( N \geq 3 \) case, and the obstruction to trivialising \( A \) over certain circles in \( A/G_0 \) in the \( N = 2 \) case.

A smooth map \( \Phi : S^{m-1} \times T^4 \to SU(N) \) with \( \Phi(\theta, x_0) = 1 \) \( \forall \theta \in S^{m-1} \), together with a gauge field \( A \in A \), determines an \( m \)-sphere in \( A/G_0 \) as follows. Define the \( S^{m-1} \) family \( \phi_\theta \) in \( G_0 \) by \( \phi_\theta(x) = \Phi(\theta, x) \); then an \( S^{m-1} \)-family in \( A \) is obtained by setting \( A^\theta = \phi_\theta \cdot A \). Extend this to a \( B^m \) family by setting \( A^{(\theta,t)} = tA^\theta \). Here \( B^m \) denotes the \( m \)-dimensional unit ball and \( t \) is the radial coordinate. Since the \( A^{(\theta,1)} \)'s are all gauge equivalent, the \( B^m \) family descends to an \( S^m \) family in the orbit space, i.e. an \( m \)-sphere in \( A/G_0 \) which we denote by \( S^m \). In the \( m = 1 \) case, \( S^{m-1} = S^0 \) is to be regarded as the boundary of \( B^1 \), i.e. the disjoint union of two points. In this case, \( \Phi \) consists of two maps \( \phi, \phi' : T^4 \to SU(N) \).

**Proposition.** The \( G_0 \)-bundle \( A \) is trivialisable over \( S^m \) if and only if \( \Phi \) can be extended to a smooth map \( \tilde{\Phi} : B^m \times T^4 \to SU(N) \) with \( \tilde{\Phi}(\theta, t, x_0) = 1 \) \( \forall (\theta, t) \).

**Corollary.** (i) \( A \) is not trivialisable over \( S^m \) when the degree of \( \Phi \) is nonzero. Thus for \( N \geq 3 \) there are \( 2n \)-spheres in \( A/G_0 \) over which \( A \) is nontrivialisable.

(ii) In the SU(2) case there are circles in \( A/G_0 \) over which \( A \) is nontrivialisable.

**Proof.** Part (i) of the corollary follows from the proposition by noting that an extension \( \tilde{\Phi} \) of \( \Phi \) corresponds to a smooth family \( \Phi_t : S^{m-1} \times T^4 \to SU(N) \), given by \( \Phi_t(\theta, x) = \tilde{\Phi}(\theta, t, x) \), which describes a smooth deformation of \( \Phi = \Phi_1 \) to a map \( \Phi_0 : S^{m-1} \times T^4 \to SU(N) \) which is independent of \( \theta \in S^{m-1} \). The degree of such \( \Phi_0 \) vanishes; it follows that the same must be true for all \( \Phi_t \) and in particular for \( \Phi \). But we have already noted above that \( \Phi \)'s with nonvanishing degree exist when \( m = 2n \) with \( 1 \leq n \leq N-2 \). Part (ii) of the corollary follows from the \( m = 1 \) case of

\[12\text{We are assuming that no two interior points in the } B^m \text{ family } A^{(\theta,t)} \text{ are gauge equivalent, which will be true in the generic case.}\]
the proposition and the fact that there are maps $\phi : T^4 \to SU(2)$ which cannot be continuously deformed to the identity map: If $\Phi$ is taken to consist of such a $\phi$ and the identity map then no extension $\tilde{\Phi}$ connecting these exists.

To prove the Proposition we first show that trivialisability of $\mathcal{A}$ over $S^m$ implies that an extension $\tilde{\Phi}$ of $\Phi$ exists. A trivialisation of $\mathcal{A}$ over $S^m$,

$$\mathcal{A}|_{S^m} \simeq S^m \times G_0,$$

(4.2)

determines a “gauge-fixed” $m$-sphere $S^m_f$ in $\mathcal{A}$, defined as the image of $S^m \times \{1\}$ under the trivialisation map (4.2). Let $A_{(\theta,t)}^f$ denote the unique element of $S^m_f$ lying in the $G_0$ gauge orbit through $A_{(\theta,t)}^f$, and let $\rho_{(\theta,t)}$ denote the unique $G_0$ gauge transformation relating these by

$$A_{(\theta,t)}^f = \rho_{(\theta,t)} \cdot A_{(\theta,t)}^f$$

(4.3)

Since $A_{(\theta,1)}^f \equiv A_f$ is independent of $\theta$, we have $A_f = \rho_{(\theta,1)} \cdot A_{(\theta,1)}^f = \rho_{(\theta,1)} \phi_\theta \cdot A$, hence $\rho = \rho_{(\theta,1)} \phi_\theta$ is independent of $\theta$. Set $\tilde{\Phi}(\theta,t,x) = \rho_{(\theta,t)}(x)^{-1} \rho(x)$, then $\tilde{\Phi}(\theta,1,x) = \phi_\theta(x) = \Phi(\theta,x)$ so $\tilde{\Phi}$ is an extension of $\Phi$ with $\tilde{\Phi}(\theta,t,x_0) = 1 \ \forall (\theta,t)$. Conversely, given an extension $\tilde{\Phi}$ of $\Phi$ with $\tilde{\Phi}(\theta,t,x_0) = 1 \ \forall (\theta,t)$, define the smooth $B^m$ family $\rho_{(\theta,t)}$ in $G_0$ by $\rho_{(\theta,t)}(x) = \tilde{\Phi}(\theta,t,x)^{-1}$, and define $A_{(\theta,t)}^f$ by (4.3). From the definitions, $\rho_{(\theta,1)}(x) \phi_\theta(x) = 1$ which implies that $A_{(\theta,1)}^f$ is independent of $\theta$ and therefore that the family $A_{(\theta,t)}^f$ is an $m$-sphere $S^m_f$ in $\mathcal{A}$. It is clear from the constructions that the $G_0$ orbit through $A_{(\theta,t)}^f$ intersects $S^m_f$ at precisely one point, namely $A_{(\theta,t)}^f$, so $S^m_f$ is a “gauge-fixed” submanifold for $\mathcal{A}|_{S^m}$ determining a trivialisation (4.2).

In preparation for the lattice considerations in the next section we conclude this section with an alternative “fermionic” proof of the Corollary of the Proposition above, based on families index theory for the Dirac operator coupled to gauge fields [33] and Witten’s global SU(2) anomaly [34]. A well-known fact, following from the results of [33], is that the topological charge (integral Chern character) of the index bundle of the Dirac operator over the above $2n$-sphere $S^{2n}$ in $\mathcal{A}/G_0$ equals the degree of $\Phi$. If $S^{2n}$ is contractible in $\mathcal{A}/G_0$ then the topological charge must vanish, so the $\Phi$’s with nonvanishing degree determine noncontractible $2n$-spheres in $\mathcal{A}/G_0$, in
which case \( \pi_{2n}(\mathcal{A}/\mathcal{G}_0) \neq 0 \). On the other hand, as noted earlier, if a good \( \mathcal{G}_0 \) gauge fixing exists then \( \pi_m(\mathcal{A}/\mathcal{G}_0) = 0 \) for all \( m \). Thus we see again that the degree of \( \Phi \) is an obstruction to the existence of good \( \mathcal{G}_0 \) gauge fixings.

The fermionic proof that the degree of \( \Phi \) is an obstruction to trivialisability of \( \mathcal{A} \) over \( S^{2n} \) (part (i) of the Corollary) is as follows. If such a trivialisation exists then the corresponding “gauge fixed” \( 2n \)-sphere \( S^f_j \) in \( \mathcal{A} \) is related to \( S^{2n} \) as in [13] by a \( B^{2n} \) family \( \rho_{(\theta,t)} \) of \( \mathcal{G}_0 \) gauge transformations. Recalling that the index bundle over \( \mathcal{A}/\mathcal{G}_0 \) is obtained from the index bundle over \( \mathcal{A} \) by identifying the fiber over \( \mathcal{A} \) with the fibre over \( \phi \cdot \mathcal{A} \) via the action of \( \phi \) on the space of spinor fields, it is easy to see that the family \( \rho_{(\theta,t)} \) gives an isomorphism between the restriction of the index bundle over \( \mathcal{A} \) to \( S^{2n} \) and the restriction of the index bundle over \( \mathcal{A}/\mathcal{G}_0 \) to \( S^{2n} \). Hence the topological charges of these restricted bundles coincide, so the topological charge of the index bundle over \( S^{2n} \) is \( \deg(\Phi) \), implying that \( S^{2n} \) is noncontractible in \( \mathcal{A} \) if \( \deg(\Phi) \neq 0 \). Since all \( 2n \)-spheres in \( \mathcal{A} \) are contractible, this is a fermionic way to see that \( \mathcal{A} \) is not trivialisable over \( S^{2n} \) when the degree of \( \Phi \) is nonzero.

Turning now to the SU(2) case (part (ii) of the Corollary), we give a fermionic proof that \( \mathcal{A} \) is nontrivialisable over the circle \( S^1 \) in \( \mathcal{A}/\mathcal{G}_0 \) coming from the \( B^1 \)-family \( \mathcal{A}^t = (1-t)\mathcal{A} + t\phi \cdot \mathcal{A} \) in \( \mathcal{A} \) when \( \phi : T^4 \to \text{SU}(2) \) is a topologically nontrivial element in \( \mathcal{G}_0 \). Let \( \{\lambda_j\} \) denote the positive eigenvalues of the Dirac operator \( \mathcal{D} \mathcal{A} \), and \( \{\lambda_j(t)\} \) the flows of these eigenvalues when the Dirac operator is coupled to \( \mathcal{A}^t \). Let \( N \) denote the net number of crossings of the origin by these eigenvalues (counted with sign) as \( t \) increases from 0 to 1. A trivialisation of \( \mathcal{A} \) over \( S^1 \) determines a “gauge-fixed” circle \( S^f_j \) in \( \mathcal{A} \), related to \( S^1 \) similarly to [13] by \( \mathcal{A}^f_j = \rho(t) \cdot \mathcal{A}^f \) for some family \( \rho(t) \) of \( \mathcal{G}_0 \) gauge transformations. Then the Dirac operators coupled to \( \mathcal{A}^f \) and \( \mathcal{A}^f_j \) have the same eigenvalues, hence the flows \( \{\tilde{\lambda}_j(t)\} \) of the positive eigenvalues \( \{\tilde{\lambda}_j = \bar{\lambda}_j(0)\} \) of the Dirac operator coupled to \( \mathcal{A} = \mathcal{A}^f_j \) coincide with \( \{\lambda_j(t)\} \), and the net number \( N_f \) of crossings of the origin by \( \{\tilde{\lambda}_j(t)\} \) coincides with \( N \). Clearly \( N_f \) can only change.

\[13\] If necessary we can replace \( \phi \to \phi(x_0)^{-1} \phi \) to get a topologically nontrivial element in \( \mathcal{G}_0 \).

\[14\] The argument requires \( \mathcal{A} \) to be chosen such that the Dirac operator coupled to \( \mathcal{A} \) doesn’t have any accidental zero-modes.
by an even integer under a deformation of the circle $S^1_f$ in $A$. Therefore, if $S^1_f$ is contractible then $N_f = 0 \pmod{2}$, while on the other hand $N_f = 1 \pmod{2}$ implies $S^1$ is noncontractible in $A$. Since all circles in $A$ are contractible, the former must hold. It follows that $A$ is not trivialisable over $S^1$ when $\phi$ is topologically nontrivial, since in this case it is known that $N = 1 \pmod{2}$ —this is Witten’s global SU(2) anomaly [34].

5 Lattice considerations

In this section we describe a lattice version of the continuum considerations of Section 4. A good $\mathcal{G}_0$ gauge fixing in the SU($N$) lattice gauge theory is equivalent to a trivialisation

$$\mathcal{U} \cong \mathcal{U} / \mathcal{G}_0 \times \mathcal{G}_0 \quad (5.1)$$

The $\pi_n(\mathcal{U})$’s have not been determined at present, and can be nonvanishing (as we already know from Section 2). Hence the initial considerations of Section 4 do not have an immediate lattice counterpart. The subsequent parts of Section 4 do have lattice counterparts though, as we now discuss. A map $\Phi: S^{m-1} \times \{\text{lattice sites}\} \to \text{SU}(N)$ with $\Phi(\theta, x_0) = 1 \ \forall \theta \in S^{m-1}$, together with a gauge field $U \in \mathcal{U}$, determine an $S^{m-1}$-family $\phi_\theta$ in $\mathcal{G}_0$ where $\phi_\theta(x) = \Phi(\theta, x)$, and an $S^{m-1}$-family $U^\theta = \phi_\theta \cdot U$ in $\mathcal{U}$. An extension of $U^\theta$ to a $B^{2n}$-family $U^{(\theta, t)}$ then gives an $m$-sphere $S^m$ in $\mathcal{U} / \mathcal{G}_0$. Such an extension need not exist in general; nevertheless it is expected that generic $m$-spheres in $\mathcal{U} / \mathcal{G}_0$ will arise in this way. Examples of such $S^m$ are given by the lattice transcripts of the $m$-spheres in $\mathcal{A} / \mathcal{G}_0$ discussed in Section 4, i.e. $\Phi$, $U$ and $U^{(\theta, t)}$ are the lattice transcripts of $\Phi$, $A$ and $A^{(\theta, t)}$; the resulting family $U^{(\theta, t)}$ is guaranteed to lie in $\mathcal{U}$ when the lattice is sufficiently fine, cf. the discussion following (2.2) above.

Now, an argument completely analogous to the proof of the Proposition of Section 4 shows that the $\mathcal{G}_0$-bundle $\mathcal{U}$ is trivialisable over $S^m$ if and only if $\Phi$ can be smoothly extended to a map $\bar{\Phi}: B^m \times \{\text{lattice sites}\} \to \text{SU}(N)$ with $\bar{\Phi}(\theta, t, x_0) = 1 \ \forall (\theta, t)$.

Since trivialisations of $\mathcal{U}$ (i.e. good $\mathcal{G}_0$ gauge fixings) are already known to exist,
this implies that the extension \( \tilde{\Phi} \) is always guaranteed to exist. This is in contrast to the continuum case; the difference is due to the fact that on the lattice the is no requirement that \( \Phi(\theta, x) \) and \( \tilde{\Phi}(\theta, t, x) \) be smooth in the \( x \)-variable, since the lattice sites are discrete. (Note that the requirement \( \tilde{\Phi}(\theta, t, x_0) = 1 \) can always be satisfied in the lattice case by making this part of the definition of \( \tilde{\Phi} \).)

The preceding does not mean that every map \( \Phi : S^{m-1} \times \{\text{lattice sites}\} \to SU(N) \) can by extended to a map \( \tilde{\Phi} : B^m \times \{\text{lattice sites}\} \to SU(N) \). Clearly this is not possible, since, in cases where \( m \) is even, there are maps \( S^{m-1} \to SU(N), \theta \to \Phi(\theta, x) \) which wrap \( S^{m-1} \) nontrivially around an \((m-1)\)-sphere in \( SU(N) \), and therefore cannot be extended to maps \( B^m \to SU(N) \). What the preceding implies is that a necessary condition for the family \( U^{(\theta,t)} \) in \( U \), constructed using \( \Phi \), to admit an extension \( U^{(\theta,t)} \) is that \( \Phi \) admits an extension \( \tilde{\Phi} \). I.e. the \( \Phi \) cannot produce an \( m \)-sphere \( S^m \) in \( U/G_0 \) in the way described above if it doesn’t admit an extension \( \tilde{\Phi} \).

When \( S^m \) is the lattice transcript of an \( m \)-sphere in \( A/G_0 \) of the kind discussed in Section 4, i.e. \( U^{(\theta,t)} \) is the lattice transcript of \( A^{(\theta,t)} \), the existence of an extension \( \tilde{\Phi} \) for the lattice transcript \( \Phi : S^{m-1} \times \{\text{lattice sites}\} \to SU(N) \) can be seen explicitly as follows. Consider the element in \( \pi_{m-1}(SU(N)) \) represented by the map \( \theta \mapsto \Phi(\theta, x) \). Since the continuum \( \Phi(\theta, x) \) depends smoothly on \( x \), this element is independent of \( x \) (since \( T^4 \) is connected). It is zero when \( x = x_0 \) (since \( \Phi(\theta, x_0) = 1 \ \forall \theta \)), and is therefore zero for all \( x \). Hence an extension \( B^m \to SU(N), (\theta, t) \mapsto \tilde{\Phi}(\theta, t, x) \) exists for each lattice site \( x \).

We now give the promised second proof of the Theorem of Section 2. Unlike the first proof, it does not rely on the existence of of good \( G_0 \) gauge fixings on the lattice, and provides a more explicit description of noncontractible \( 2n \)-spheres/circles in \( U \). Let \( S^{2n} \) be the lattice transcript of a \( 2n \)-sphere in \( A/G_0 \) of the kind considered in Section 4. For each lattice site \( x \) choose a smooth extension of the map \( S^{2n-1} \to SU(N), \theta \mapsto \Phi(\theta, x) \) to a map \( B^{2n} \to SU(N), (\theta, t) \mapsto \tilde{\Phi}(\theta, t, x) \) (as discussed in the preceding). In particular, set \( \tilde{\Phi}(\theta, t, x_0) = 1 \). This determines a smooth map \( \tilde{\Phi} : S^{2n-1} \times \{\text{lattice sites}\} \to SU(N) \) with \( \tilde{\Phi}(\theta, 1, x) = \Phi(\theta, x) \) and \( \tilde{\Phi}(\theta, t, x_0) = 1 \). Now
define a $B^{2n}$-family $\rho(\theta,t)$ of $G_0$ lattice gauge transformations by $\rho(\theta,t)(x) = \tilde{\Phi}(\theta,t,x)^{-1}$ and set

$$U_f^{(\theta,t)} = \rho(\theta,t) \cdot U^{(\theta,t)}$$

(5.2)

An easy consequence of the definitions is that $U_f^{(\theta,1)} = U$ independent of $\theta \in S^{2n-1}$. Hence the family $U_f^{(\theta,t)}$ is actually a $2n$-sphere $S^{2n}_f$. Since $U_f^{(\theta,t)}$ is gauge equivalent to $U^{(\theta,t)}$, this $2n$-sphere is guaranteed to be contained in $U$, and to satisfy the admissibility condition (2.2) for any $\epsilon > 0$, when the lattice spacing is sufficiently small.

In the following we apply the families index theory for the overlap Dirac operator to show that $S^{2n}_f$ is noncontractible in $U$ when $\text{deg}(\Phi)$ is nonvanishing and the lattice is sufficiently fine. Since $\Phi$'s with nonzero degree exist when $1 \leq n \leq N-2$, this will establish the first part of the Theorem.

A formula for the topological charge $Q_{2n}$ of the index bundle of the overlap Dirac operator over the $2n$-sphere $S^{2n}$ in $U/G_0$ was derived in [11]. It reduces in the classical continuum limit to $\text{deg}(\Phi)$, the topological charge of the corresponding $2n$-sphere in $A/G_0$ [11, 12]. Thus $Q_{2n} = \text{deg}(\Phi)$ when the lattice spacing is sufficiently small. We now note that, just as in the continuum situation in Section 4, the relation (5.2) between $S^{2n}$ and the $2n$-sphere $S^{2n}_f$ in $U$ implies that the index bundle over $S^{2n}$ is isomorphic to the index bundle over $S^{2n}_f$, which in turn implies that their topological charges are the same. To see this explicitly, recall from [11] that, for $n \geq 1$, the topological charges of the index bundle over $S^{2n}$ and $S^{2n}_f$ coincide with those of the vectorbundle $\hat{C}$, given as follows: The fibre of $\hat{C}$ over $U \in U$ is $\tilde{C}^U = P^U(C)$ where $P^U$ is a projection operator acting on the space $C$ of lattice spinor fields, given by

$$P^U = \frac{1}{2}(1 + \hat{\gamma}_5^U), \quad \hat{\gamma}_5^U = \gamma_5(1 - aD^U)$$

(5.3)

where $D^U$ is the overlap Dirac operator coupled to $U$. Set $\tilde{C}_f^{(\theta,t)} := \tilde{C}^{U_f^{(\theta,t)}}$, then

$$\tilde{C}_{S^{2n}_f} = \{ \tilde{C}_f^{(\theta,t)} \}$$

(5.4)

15 This bundle was denoted "$\hat{C}^+$" in [11] but we omit the "+" subscript here.
Gauge covariance of \( P^U \) implies \( \phi(\hat{C}^U) = \hat{C}^{\phi \cdot U} \) for all \( \phi \in G_0, U \in \mathcal{U} \), so \( \hat{C} \) descends to a vector bundle over \( \mathcal{U}/G_0 \). The restriction of this bundle to \( S^{2n} \) is

\[
\hat{C}|_{S^{2n}} = \{ \hat{C}^{(\theta,t)} \} / \sim
\]

(5.5)

where \( \hat{C}^{(\theta,t)} := \hat{C}^{U^{(\theta,t)}} \) and the equivalence relation \( \sim \) means identify each \( \hat{C}^{(\theta,1)} \) with \( \hat{C}^{(0,1)} \) via the isomorphism

\[
\phi_\theta : \hat{C}^{(0,1)} \cong \hat{C}^{(\theta,1)}
\]

(5.6)

For \( t < 1 \) the fibres \( \hat{C}^{(\theta,t)} \) and \( \hat{C}^{(\theta,t)} \) are isomorphic via the gauge transformation \( \rho(\theta,t) \) in (5.2). This extends to a well-defined isomorphism between the fibres at \( t=1 \) since, as an easy consequence of the definitions, \( \rho(\theta,1) \phi_\theta = 1 \) independent of \( \theta \), i.e. \( \rho(\theta,1) \) respects the equivalence relation in (5.5). Thus \( \hat{C}|_{S^{2n}} \) and \( \hat{C}|_{S^{2n}_f} \) are isomorphic, and therefore have the same topological charge \( Q_{2n} \) as claimed. It follows that if \( S^{2n}_f \) is contractible in \( \mathcal{U} \) then \( Q_{2n} = 0 \). Therefore, if \( \text{deg}(\Phi) \neq 0 \) then the \( 2n \)-sphere \( S^{2n}_f \) constructed above must be noncontractible when the lattice is sufficiently fine. This proves the first part of the theorem.

The proof of the remaining part of the theorem is as follows. Let \( S^1 \) be the lattice transcript of a circle in \( \mathcal{A}/G_0 \) of the kind discussed in Section 3, i.e. start with a \( \phi \in G_0 \), pick \( A \in \mathcal{A} \), then the lattice transcript \( U^t \) of the family \( A^t = (1-t)A + t\phi \cdot A \) determines a circle \( S^1 \) in \( \mathcal{U}/G_0 \). For each lattice site \( x \) choose a smooth path \( \phi_t(x) \) in \( \text{SU}(2) \) connecting the identity element \( \phi_0(x) = 1 \) to \( \phi(x) \). In particular, set \( \phi_t(x_0) = 1 \) \( \forall t \). Define the family \( \rho_t \) in \( G_0 \) by \( \rho_t(x) = \phi_t(x)^{-1} \) and set

\[
U^t_f = \rho_t \cdot U^t.
\]

(5.7)

Since \( \rho_1 \phi = 1 = \rho_0 \), implying \( U^0_f = U^1_f \), the family \( U^t_f \) is a circle \( S^1_f \). Just as in the case of \( S^{2n}_f \) it is guaranteed to lie in \( \mathcal{U} \) when the lattice is sufficiently fine. The fermionic argument of Section 3 in the \( \text{SU}(2) \) case now has the following lattice version. The overlap Dirac operator \( D \) is normal and hence has a complete set of eigenvectors. The eigenvalues lie on a circle in the complex plane which passes through the origin,
and the nonreal eigenvalues come in complex conjugate pairs. Let \( \{ \lambda_j \} \) be the eigenvalues of \( D \) coupled to \( U^0 \) which have \( \text{Im}(\lambda) > 0 \), and \( \{ \lambda_j(t) \} \) the flows of these eigenvalues when \( D \) is coupled to \( U^t \). Let \( \mathcal{N} \) denote the net number of crossings of the origin (counted with sign) as \( t \) increases from 0 to 1. Let \( \mathcal{N}_f \) denote the analogous number for \( D \) coupled to \( U_t^f \). Then, just as in the continuum setting, the gauge covariance of \( D \) and the relation (5.7) imply \( \mathcal{N}_f = \mathcal{N} \). Clearly \( \mathcal{N}_f \) can only change by an even integer under a deformation of the circle \( S_1^f \) in \( \mathcal{U} \). Therefore, if \( S_1^f \) is contractible in \( \mathcal{U} \) then \( \mathcal{N}_f \) (mod 2) = 0. On the other hand, the results of [13] show that \( \mathcal{N} \) (mod 2) reproduces Witten’s global SU(2) anomaly in the classical continuum limit. It follows that \( S_1^f \) is noncontractible \( \mathcal{U} \) when \( \phi \) is topologically nontrivial and the lattice is sufficiently fine. This completes our second proof of the Theorem of Section 2.

Remarks. (i) The noncontractible 2\( n \)-spheres \( S_2^{2n}_f \) and circles \( S_1^f \) in \( \mathcal{U} \) constructed in the preceding are “rough”: they cannot arise as lattice transcripts of smooth continuum 2\( n \)-spheres and circles in \( \mathcal{A} \) since the contractibility of the latter implies contractibility of their lattice transcripts when the lattice is sufficiently fine. The roughness of \( S_2^{2n}_f \) and \( S_1^f \) originates from the roughness of the of the extensions \( \tilde{\Phi} : B^{2n} \times \{ \text{lattice sites} \} \to \text{SU}(N) \) and \( \tilde{\phi} : B^1 \times \{ \text{lattice sites} \} \to \text{SU}(2) \) (where \( \tilde{\phi}(t, x) = \phi_t(x) \)); these do not have smooth continuum versions when \( \text{deg}(\Phi) \neq 0 \), and \( \phi \) is topologically nontrivial, respectively.

(ii) Since the continuum gauge field \( A \) used as part of the starting point for constructing \( S_2^{2n}_f \) and \( S_1^f \) in the preceding is smooth and continuous on \( T^4 \) and therefore has vanishing topological charge, \( S_2^{2n}_f \) and \( S_2^{2n} \) lie in the trivial topological sector of \( \mathcal{U} \). However, noncontractible 2\( n \)-spheres and circles in the other topological sectors are readily constructed along the same lines as above by starting with a topologically nontrivial \( A \) in a singular gauge, such that \( A \) is still continuous on \( T^4 \) and smooth away

---

16 The convention used in defining the overlap Dirac operator is that the \( \gamma^\mu \) matrices are hermitian. This corresponds in the continuum to antihermitian Dirac operator with purely imaginary eigenvalues, with the nonzero ones coming in complex conjugate pairs.

17 The presence of the Witten anomaly on the lattice has also been verified numerically in [20, 21].
from the singularity, and the lattices are restricted to those for which the singularity of $A$ doesn’t lie on a lattice link.

Finally, we point out that the topological charge $Q_{2n}$ of the overlap Dirac index bundle over the $2n$-spheres $S_{f}^{2n}$ is associated with a monopole interpretation for a certain canonical connection on $U$ with values in the bundle $\tilde{C}$. The bundle $\tilde{C}$ arises as $\tilde{C} = \mathcal{P}(C)$ where $C$ is to be interpreted as the trivial bundle over $U$ with fibre $C$ (the space of lattice spinor fields on $T^4$) and $\mathcal{P} : C \to C$ is an orthogonal projection map whose action on the fibres is given by $P^{U}$, defined in (5.3). Then $\tilde{C}$ has the canonical connection $\nabla = \mathcal{P}d$, where $d$ is the exterior derivative on $U$. On $U_{\text{initial}}$ we see from (5.3) that $\nabla$ is singular at the points $U$ for which the overlap Dirac operator $D^{U}$, and hence the fermionic topological charge, are ill-defined. Such singularities are present in the interior of any $(2n+1)$-ball in $U_{\text{initial}}$ with $S_{f}^{2n}$ as its boundary when $Q_{2n}$ is nonvanishing. (Such balls always exist since $S_{f}^{2n}$ is contractible in $U_{\text{initial}}$.) These singularities of $\nabla$ are monopole-like: the topological charge of $\nabla$ on $S_{f}^{2n}$, given by integrating the Chern character of $\nabla$ over $S_{f}^{2n}$, equals $Q_{2n}$. Indeed, the Chern charater of $\nabla$ is a representative for the Chern character of the bundle $\tilde{C}$, and it was shown in [11] that the nonzero degree parts of the latter coincide with those of the overlap Dirac index bundle on $U$. (In fact the connection $\nabla$ was used to derive the formula (“lattice families index theorem”) for the Chern character of the lattice index bundle in [11].)

6 Summary

When the decomposition of the space of lattice gauge fields into topological sectors is specified by the fermionic topological charge, it is to be expected that fermionic techniques will be required to determine the (topological) structure of the sectors. In this paper we have seen that the lattice families index theory developed in [11, 12]...
is a useful tool in this regard. Using it, we obtained first results on the topology of the sectors of SU(N) lattice gauge fields on $T^4$: For sufficiently fine lattices, the sectors were shown to contain noncontractible $2n$-spheres when $1 \leq n \leq N-2$, and noncontractible circles in the SU(2) case. These are new topological features that arise as a consequence of excluding the lattice gauge fields for which the fermionic topological charge is ill-defined.

Two proofs of this result were given. The first of these, in Section 2, was a short, rather implicit argument using results on the topological structure of the orbit space $\mathcal{U}/\mathcal{G}_0$ obtained previously from the families index theory for the overlap Dirac operator in [11], and exploiting the fact that $\mathcal{G}_0$ gauge fixings without the Gribov problem exist on the lattice. The second argument, in Section 5, used the lattice families index theory in a more direct way, without relying on the existence of good $\mathcal{G}_0$ gauge fixings. It led to noncontractible $2n$-spheres $S^{2n}_f$ in the $N \geq 3$ case, and noncontractible circles $S^1_f$, in the $N=2$ case, obtained by a quite explicit prescription as follows: Start with a continuum gauge field $A$ and smooth map $\Phi : S^{2n-1} \times T^4 \to SU(N)$ with $\Phi(\theta, x_0) = 1$ $\forall \theta$ and $\text{deg}(\Phi) \neq 0$. Then, for each lattice site $x$, extend the map $S^{2n-1} \to SU(N), \theta \mapsto \Phi(\theta, x)$ to a map $B^{2n} \to SU(N), (\theta, t) \mapsto \tilde{\Phi}(\theta, t, x)$. This determines $S^{2n}_f$ via (5.2). $S^1_f$ is similarly determined via (5.7) after starting with a topologically nontrivial map $\phi : T^4 \to SU(2)$ and choosing for each lattice site $x$ a path $\phi_t(x)$ from 1 to $\phi(x)$ in SU(2). The noncontractible $S^{2n}_f$’s and $S^1_f$’s are rough – they do not have smooth continuum versions. Nevertheless, the gauge fields contained in them satisfy the admissibility condition (2.2) for any given $\epsilon$ when the lattice is sufficiently fine.

The arguments furthermore show that the number of elements in $\pi_{2n}(\mathcal{U})$ for $1 \leq n \leq N-2$ becomes infinite in the continuum limit, and that the topological charges of the index bundle over the $S^{2n}_f$’s are associated with a monopole interpretation for a certain canonical connection on the space of lattice gauge fields.

It would be interesting to see if further results on the topology of $\mathcal{U}$ can be extracted via the lattice families index theory or other fermionic techniques. A number of basic questions remain to be answered, for example: Is a given topological sector
of \( \mathcal{U} \) path-connected, or can there be more than one connected component? (The decomposition (2.6) shows that the answer is affirmative in the \( U(1) \) case, at least when the admissibility condition is imposed.) Determining the topology of \( \mathcal{U} \) would be necessary for extending Lüscher’s existence proof for gauge invariant abelian chiral gauge theory on the lattice \([3]\) to the \( SU(N) \) case.

In the continuum setting, the topological charge (integrated Chern character) of the Dirac index bundle over \( 2n \)-spheres in the orbit space \( \mathcal{A}/\mathcal{G}_0 \) was seen to be an obstruction to the existence of good \( \mathcal{G}_0 \) gauge fixings. Explicit examples were described where the obstruction is given by the degree of maps \( \Phi : S^{2n-1} \times T^4 \to SU(N) \) \((1 \leq n \leq N-2)\) in the \( N \geq 3 \) case. The Witten global anomaly was seen to be an obstruction to the existence of good \( \mathcal{G}_0 \) gauge fixings in the \( SU(2) \) case. In the lattice setting the situation is quite different: Good \( \mathcal{G}_0 \) gauge fixings exist, and the topological charge of the overlap Dirac index bundle over the \( 2n \)-spheres \( S^{2n} \) in \( \mathcal{U}/\mathcal{G}_0 \) (the lattice transcripts of the above-mentioned \( 2n \)-spheres in \( \mathcal{A}/\mathcal{G}_0 \)) was seen to be an obstruction to the contractibility of the \( 2n \)-spheres \( S^{2n} \) in \( \mathcal{U} \) obtained from \( S^{2n} \) via a trivialisation of the \( \mathcal{G}_0 \)-bundle \( \mathcal{U} \) over \( S^{2n} \). Similarly, the lattice version of the Witten global anomaly was seen to be an obstruction to contractibility of the circles \( S^1 \) in \( \mathcal{U} \) obtained from \( S^1 \) in the \( SU(2) \) case. Thus the obstructions to the contractibility of the \( 2n \)-spheres/circles in \( \mathcal{U} \) are the lattice counterparts of the obstructions to good \( \mathcal{G}_0 \) gauge fixings in the continuum setting. The correspondence is made even stronger by the fact that the lattice obstructions coincide with the continuum ones for sufficiently fine lattices by the classical continuum limit results of \([11]\) and \([13]\).

Acknowledgements. I thank Ting-Wai Chiu and his students for many stimulating discussions and kind hospitality at NTU. I would also like to thank Kazuo Fujikawa, Masato Ishibashi and Yoshio Kikukawa for discussions and kind hospitality during a visit to Japan, and Herbert Neuberger for his encouraging interest in the preliminary version of this work at the Cairns workshop. At NTU the author was supported by the Taiwan NSC (grant numbers NSC89-2112-M-002-079 and NSC90-2811-M-002-001).
Appendix

A remark on the condition $\Phi(\theta, x_0) = 1 \ \forall \theta$

For $1 \leq n \leq N-2$ there are maps $\Phi : S^{2n-1} \times T^4 \rightarrow SU(N)$ with arbitrary nonvanishing degree. In the following we point out that such maps continue to exist when the condition $\Phi(\theta, x_0) = 1 \ \forall \theta \in S^{2n-1}$ is imposed, with $x_0$ being some arbitrary basepoint in $T^4$. Start with a map $\Phi : S^{2n+3} \rightarrow SU(N)$ with arbitrary degree $d$, obtained by wrapping the $S^{2n+3}$ $d$ times around the $2n+3$-sphere in $SU(N)$. Choose a point $p_0$ in $S^{2n+3}$ and redefine $\Phi(p) \rightarrow \Phi(p_0)^{-1}\Phi(p)$ so that $\Phi(p_0) = 1$. Now view $S^{2n+3}$ as the box $[0, 1]^{2n+3}$ with all boundary points identified with $p_0$. Then $\Phi$ can be viewed as a map $\Phi : [0, 1]^{2n+3} \rightarrow SU(N)$ which maps all boundary points of the box to 1. View $[0, 1]^{2n+3}$ as $[0, 1]^{2n-1} \times [0, 1]^4$. If $x_0$ is a boundary point in $[0, 1]^4$ then $(u, x_0)$ is a boundary point in $[0, 1]^{2n+3}$ for all $u \in [0, 1]^{2n-1}$, hence $\Phi(u, x_0) = 1 \ \forall u$. Now impose periodic boundary conditions on $[0, 1]^4$ to get $T^4$, and identify all boundary points in $[0, 1]^{2n-1}$ to get $S^{2n-1}$. The map $\Phi : [0, 1]^{2n-1} \times [0, 1]^4 \rightarrow SU(N)$ respects these identifications (since it maps all boundary points to 1) and can therefore be regarded as a map $\Phi : S^{2n-1} \times T^4 \rightarrow SU(N)$. This map satisfies $\Phi(\theta, x_0) = 1 \ \forall \theta$ and has the same degree $d$ as the original $\Phi$.

In light of this, the obstructions to gauge invariance of the chiral fermion determinant mentioned in Section 2, which are given by the degree of some $\Phi$, continue to be present when $G$ is restricted to $G_0$.

References

[1] R. Narayanan and H. Neuberger, Phys. Lett. B 302, (1993) 62; Phys. Rev. Lett 71, (1993) 3251; Nucl. Phys. B 412, (1994) 574; Nucl. Phys. B 443, (1995) 305

[2] H. Neuberger, Phys. Lett. B 417, (1998) 141

[3] M. Lüscher, Nucl. Phys. B 549, (1999) 295
[4] H. Neuberger, Phys. Rev. D 63: 014503 (2001)

[5] T. Fujiwara, H. Suzuki and K. Wu, Phys. Lett. B 463 (1999) 63; Nucl. Phys. B 569 (2000) 643

[6] Y. Kikukawa and Y. Nakayama, Nucl. Phys. B 597 (2001) 519

[7] H. Suzuki, Nucl. Phys. B 585 (2000) 471

[8] M. Lüscher, JHEP 0006:028 (2000)

[9] M. Lüscher, Nucl. Phys. B 568, (2000) 162

[10] H. Suzuki, JHEP 0010:039 (2000)

[11] D.H. Adams, Nucl. Phys. B 624 (2002) 469

[12] D.H. Adams, Families index theory for Overlap lattice Dirac operator. II (in preparation)

[13] O. Bär and I. Campos, Nucl. Phys. B 581, (2000) 499

[14] M. Creutz, Phys. Rev. D 15, (1977) 1128

[15] C. DeTar, J.E. King, S.P. Li and L. McLerran, Nucl. Phys. B 249 (1985) 621

[16] D.H. Adams, J. Math. Phys. 42 (2001) 5522

[17] P. Hernández, K. Jansen and M. Lüscher, Nucl. Phys. B 552, (1999) 363

[18] H. Neuberger, Phys. Rev. D 61, 085015 (2000)

[19] V.N. Gribov, Nucl. Phys. B 139 (1978) 1

[20] H. Neuberger, Phys. Lett. B 437, (1998) 117

[21] O. Bär and I. Campos, Nucl. Phys. (Proc. Suppl.) 83, (2000) 594

[22] J.E. Mandula and M.C. Ogilvie, Phys. Rev. D 41, (1990) 2586
[23] F. Lenz, J.W. Negele, L. O’Raifeartaigh and M. Thies, Ann Phys. 258 (2000) 25

[24] L. Giusti, M.L. Paciello, C. Parrinello and S. Petrarca, Int. J. Mod. Phys. A 16 (2001) 3487

[25] P. van Baal, in Nato Advanced Study Institute on Confinement, duality, and nonperturbative aspects of QCD (Cambridge, 1997), ed: P. van Baal (Plenum Press, New York, 1998), hep-th/9711070

[26] R. Narayanan, Phys. Rev. D 58 (1998) 097501; Y. Kikukawa and A. Yamada, hep-lat/9810024

[27] L. Alvarez-Gaumé and P. Ginsparg, Nucl. Phys. B 243, (1984) 449

[28] H. Neuberger, Phys. Rev. D 59, 085006 (1999)

[29] D.H. Adams, Nucl. Phys. B 589, (2000) 633

[30] D.H. Adams, Phys. Rev. Lett 86, (2001) 200

[31] I.M. Singer, Comm. Math. Phys. 60 (1978) 7

[32] G. Jungman, Mod. Phys. Lett. A 7 (1992) 849

[33] M.F. Atiyah and I.M. Singer, Proc. Natl. Acad. Sci. U.S.A. 81, (1984) 2597

[34] E. Witten, Phys. Lett. B 117, (1982) 324

[35] M. Karoubi, Algebraic topology via differential geometry, LMS Lecture Note Series 99 (Cambridge Univ. Press, 1987)