Modelling of a wake from cylinders row

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Abstract. Model of intermittent near wake from a row of parallel cylinders is constructing. The aim of the investigation includes besides neutral wakes modelling the interpretation and “physical modelling” of the glow-discharge switching effect on a cylinders wake discovered by us earlier. The form of this reduced-order model of multiple wake is nonlinear-interacting von Karman streets each resulting from one cylinder of the row and each represented by one-dimensional oscillator. Here the correlation between known Landau-Stuart, Van der Pole wake models and ours is considered. Various global modes of such multiple wake are discovered within the framework of the model. Among them a two-frequency asymmetric global mode of the wake was discovered in accordance with well-known observations for relatively close cylinders arrangements. For each global mode an analytical approximation or exact expression for the wake configuration is presented that may be useful in further analysis. Matching of these approximations with the results of numerical computing within the framework of the model is considered.

1. Introduction

A wake from cylinders group is typical example of a highly intensive intermittent turbulent flow. Such wake is an evolutionary flow with exact large-scale preserved structure. The form of this structure is well-known and considerably simple; it is quasi two-dimensional von Karman streets behind each cylinder. Stability of the wake partial structure is a key feature for investigation of wakes. It is accepted to extract substructures similar to von Karman streets in observed complex wake, and to classify the wake global modes in accordance with relative phase and intensity of oscillations in these partial von Karman streets [1−4]. As a result a map of the regimes of complex wake in parameters manifold is usually constructed. This information used for a verification of general conjectures on the interaction between vortex streets in their formation zones. In other words, such information used to introduce models of complex wakes. From the other hand, such complex wakes yields to numerical computing within the framework of Navier-Stokes equations for not so large values of Reynolds number. In accordance with the abovementioned consideration one may conclude that the wake from cylinders rows is a suitable object for complex study of turbulence dynamics under comparatively well-controlled conditions in laboratory experiment and in computing.

Previously a transformation of cylinders-pair wake due to glow discharge switching was observed [5]. That was a nitrogen flow of complex wake from two closely located circular cylinders at Reynolds number value of Re ~ 1000 and Mach number of M ~ 0.15 , based on unperturbed velocity and the cylinder diameter. The dimensionless distance between the cylinder axes was approximately
$L/D \approx 2.1 \pm 2.2$ and the cylinders height was low enough - $H/D \approx 3.5$. The plane containing the cylinder axes was normal to the unperturbed velocity direction in rectangular duct. Signal of oscillation of transverse component of the flow velocity was observed by means of special sensor of Siddon-Ribner type [6]. Its position was $x \approx 8D$ downstream the axis of one cylinder of the pair. In this arrangement the velocity spectrum with one peak was observed in the case of gas flow, when for the case of appropriate plasma wake the spectrum converts to two-peak shape [5, 7]. For treatment of this discharge effect one must use a model of this wake, moreover, such model in combination with the discharge model [8] then the hydrodynamic model of wake have to be mostly simple. Here it should be mentioned that the interaction between gas turbulent flow and glow discharge in that is of great interest for such applications as fast-flow lasers [9, 10].

2. Landau-Stuart and Van der Pole models of a wake
It is accepted to use evolution models of Landau-Stuart type for reduced-order modeling of wakes from rigidly mounted cylinders (solitary one or a group of them) for example see earlier paper [2]. The equation of Landau-Stuart model of laminar near wake from solitary cylinder is of the form (1).

$$\frac{dA}{dt} = \varepsilon \left[ \lambda - \mu |A|^2 \right] A$$

(1)

Here the wake governing parameter $A$ is the oscillation amplitude of transverse component of medium velocity in the wake while the spatial distribution of the velocity assumed to be equivalent to one of equilibrium von Karman street behind a cylinder. Since the parameter $A$ is a complex quantity it takes into account not only amplitude of velocity oscillations in the street but its phase both. Real model parameter $|\varepsilon| < 1$ have to be small in Landau-Stuart model; that provides slowness of the wake evolution in time scale of the wake oscillations period. The constants $\lambda$ and $\mu$ are complex parameters of the model.

On the other hand, when studying aero- hydro-elasticity problems it is common practice to use wake model of Van der Pole type [11, 12]:

$$\frac{d^2X}{dt^2} + X = \varepsilon \left[ 1-X^2 \right] \frac{dX}{dt}$$

(2)

Equation (2) prescribes the evolution of the wake governing parameter $X(t)$. All quantities in equation (2) are real. Here, model parameter $\varepsilon < 1$ is also small what leads to slowness of the wake evolution. Physical meaning of two wake models is very close. Similarly to previous case of Landau-Stuart model (1), in this case governing parameter $X(t)$ is identified with oscillation amplitude of transverse component of medium velocity at the some characteristic point of the wake $v'(t,r) = X(t)f(r)$. Spatial distribution of the velocity $f(r)$ assumed to be in accordance with the distribution in equilibrium von Karman street, than an interpretation of quantity $X$ is the same as a role of the quantity $|A|$ in Landau-Stuart model. Hence, from the form of both models it follows that they are models of near wake. Peculiarity of standard Van der Pole models as compared to Landau-Stuart models is a neglecting of the oscillation frequency-amplitude dependence but may be more precise account for other nonlinear effects.

Great advantage of the Landau-Stuart type models is an appropriate theoretical basis for the case of laminar wake from solitary cylinder. Asymptotic expansion with the small parameter $\varepsilon = (Re_{cr})^{-1} - (Re)^{-1}$, $Re_{cr} \approx 47$ of the solution of relevant two-dimensional boundary problem for Navier-Stokes equations for incompressible flow leads to the form of Landau-Stuart equation (1). Moreover, numerical estimates of the parameters of this Landau-Stuart model $\nu$ and $\mu$ are in agreement with appropriate experimental estimates for laminar regime of flow $47 < Re < 180$ [13].
Accordingly, the main problems of the wakes reduced-order modelling are the model generalization for account of interaction of partial wakes from each cylinder within complex wake and the model extension for account of turbulent 3D regimes. To solve the first problem of a cylinders group wake, it is accepted to use a system of coupled equations in the form of (1) or (2) where the dependent variable of each equation governs partial wake of corresponding cylinder in the group. Coupling of these partial wakes-streets is modeled by an additional term in each equation of the system. It is accepted to use simple interplay term, precisely, a linear form in governing parameters either $X$ or their time derivatives [e.g. 2, 11].

3. Model of nonlinearly interacting von Karman streets

With the aim to treat an electric discharge effect discovered [5, 7] the Van der Pole type model was built for cylinder pairs wake [14]. Here, instead of linear shape a nonlinear form of additional coupling term in the model equations was used with the aim to extend the validity domain of advanced model. The coupling term used is maximal agreement to classical Van der Pole and Landau equations. Within the framework of this variant of the model the set of global wake modes turned out to be rich enough to describe the intermittence observed in turbulent wakes. Unfortunately, the set of corresponding oscillation frequencies of these modes was fully degenerate one [15] in contradiction with experimental data [5, 7]. Accordingly, the model was generalized for direct account of oscillation frequency dependence on oscillation amplitude. Thus, frequency set degeneration was entirely removed [16]. Moreover, after the above model modification it has accepted a two-frequency global mode of cylinders pair wake [17]. This mode consists of a pair of adjacent von Karman streets of different intensity and different oscillation frequency where one street leaves each cylinder. Such mode is usually visualized in experiments with relatively closely located cylinders $1.5 / 2LD < L / D < 2$. Below is given slightly modified variant of the model described in [18]. This modification accounts for a smallness of the model parameter $|\Delta|<1$, moreover, it follows from data in [13] that the parameter $\Delta < 0$.

Now the equation set of our simple model of a wake from cylinders pair is as follows.

$$\frac{d^2X}{dt^2} + \left(1-\Delta(\rho^2-4)\right)X - \varepsilon \left[ 1 - X^2 - \lambda Y^2 - l XY \right] \frac{dX}{dt} = h_x $$

(3)

$$\frac{d^2Y}{dt^2} + \left(1-\Delta(r^2-4)\right)Y - \varepsilon \left[ 1 - Y^2 - \lambda X^2 - l XY \right] \frac{dY}{dt} = h_y$$

(4)

Here the wake governing parameters $X$ and $Y$ describe oscillations at characteristic points of partial vortex streets, respectively. Both equations of the system are modified Van der Pole equations. The modification is as follows: 1) Oscillation frequency-amplitude dependence is accounted for by term in parenthesis, where quantities $\rho$ and $r$ are slowly varying amplitudes of oscillating $X$ and $Y$, respectively, herein real quantity $\Delta$ is the model parameter. 2) Interplay of partial streets ($X$ and $Y$) is accounted for by terms in brackets parameterized by quantities $\lambda$ and $l$. 3) Turbulent, intermittent character of the wake considered by means of right-hand parts of inhomogeneous equations (3), (4). These terms $h_x$ and $h_y$ are random processes analogous to Langevin force; for the case of laminar wake they are zero.

Accordingly to the model parameter smallness $\varepsilon, |\Delta|<1$ the Krylov-Bogolyubov method of slow varying amplitudes $\rho, r$ and phases $P_x, P_y$ of oscillations have to be used for analytical solution of the system (3), (4) [19].

$$X = \rho \cos(t + P_x)$$

(5)

$$Y = r \cos(t + P_y)$$

(6)
Then the set of corresponding truncated equations of the wake model is:

\[
\frac{d\rho^2}{dt} - \epsilon \rho^2 \left[ 1 - \rho^2 + \lambda r^2 \left( 2 - \cos 2P \right) + l \rho r \cos P \right] = h_\rho, \tag{7}
\]

\[
\frac{dr^2}{dt} - \epsilon r^2 \left[ 1 - r^2 + \rho r^2 \left( 2 - \cos 2P \right) + l \rho r \cos P \right] = h_r, \tag{8}
\]

\[
\frac{dP_\rho}{dt} + \frac{\Delta}{2} \left( \rho^2 - 4 \right) + \frac{\epsilon \lambda r^2}{8} \sin 2 \left( P_\rho - P_r \right) + \frac{\epsilon l \rho r}{8} \sin \left( P_\rho - P_r \right) = h_\rho, \tag{9}
\]

\[
\frac{dP_r}{dt} + \frac{\Delta}{2} \left( r^2 - 4 \right) - \frac{\epsilon \lambda \rho^2}{8} \sin 2 \left( P_\rho - P_r \right) - \frac{\epsilon l \rho r}{8} \sin \left( P_\rho - P_r \right) = h_r \tag{10}
\]

Remind here that in the case of laminar wake from solitary cylinder \((\lambda = l = 0, h_i = 0)\) the forms (7)–(10) of our model are completely the same as that of Landau-Stuart model (1), the latter being wrote separately for real and imaginary parts of equation (1). Accordingly, the advanced model of complex wake, firstly, has considerable theoretical and experimental ground; secondly, simultaneously possess considerable degree of freedom in actual realization in terms of its parameters variation.

For further analysis of equations (7)–(10) a transformation to collective variables of the wake is useful. Appropriate collective variables are: summarized intensity of oscillation in both streets

\[ R^2 = \rho^2 + r^2 \tag{11} \]

the wake asymmetry

\[ N = \frac{\rho^2 - r^2}{2(\rho^2 + r^2)} \tag{12} \]

and relative phase of oscillation in vortex streets

\[ P = P_\rho - P_r \tag{13} \]

By means of transform (11), (12), (13) the form of our wake model arrives at following one

\[
\frac{dR^2}{dt} - \epsilon R^2 \left[ 1 - \frac{R^2}{4} + \left( \frac{1}{4} - N^2 \right) \cos P + 2 \left( \frac{1}{4} - N^2 \right) \left( \lambda (2 - \cos 2P) - 1 \right) \right] = h_R \tag{14}
\]

\[
\frac{dN}{dt} - \frac{\epsilon R^2}{2} \left[ \lambda (2 - \cos 2P) - 1 \right] N \left( \frac{1}{4} - N^2 \right) = h_N \tag{15}
\]

\[
\frac{dP}{dt} + R^2 \left[ \Delta N + \frac{\epsilon \lambda r^2}{8} \sin 2P + \frac{\epsilon l \rho r}{4} \sqrt{1 - N^2} \sin P \right] = h_P \tag{16}
\]

Remark: Instead of equations (3) and (4) one may start from slightly modified set of equations

\[
\frac{d^2 X}{dt^2} + \left( 1 - \delta \left( X^2 - 4 \right) \right) X - \epsilon \left[ 1 - X^2 - \lambda Y^2 - l XY \right] \frac{dX}{dt} = h_x \tag{1'}
\]

\[
\frac{d^2 Y}{dt^2} + \left( 1 - \delta \left( Y^2 - 4 \right) \right) Y - \epsilon \left[ 1 - Y^2 - \lambda X^2 - l XY \right] \frac{dY}{dt} = h_y \tag{2'}
\]
By means of the same transformations (11) – (13) this equation set leads to slightly modified system for collective variables. Precisely, in the case (3`) and (4`) the dynamic of collective variables is govern by equations (14), (15) and (16`).

\[
\frac{dP}{dt} + R^2 \left[ \frac{3\delta}{8} N + \frac{\varepsilon \lambda}{8} \sin 2P + \frac{\varepsilon l}{4} \sqrt{1 - N^2 \sin P} \right] = h_p
\]  

(16’)

One can see that equation (16) may be transformed to equation (16’) by means of the substitution \( \Delta = 3\delta/8 \). Consequently, there are no quality differences between first variant of model (14), (15), and (16) and the second variant (14), (15), and (16’). The differences may be only quantitative.

4. Global laminar modes of a wake from two cylinders

Having dynamic wake model at disposal one can accept the configuration of the wake modes. For definiteness let us study the first variant of the model where the parameter \( \Delta \) occurs: (14), (15), (16). Configuration of quasi-laminar one-frequency modes of wake may be determined as stationary points of equation system (14) – (16) on the case of zero right-hand parts of these equations. Moreover, observable modes have to be stable with respect to small perturbations that may be tested with the help of Routh-Hurwitz criterion. It was recognized in such manner that the model (14) – (16) contains three symmetric wake modes (for which the parameter \( N \) is zero).

4.1. Symmetric modes of cylinders pair wake:

1) Mode of sin-phase synchronized von Karman streets \( S_0 \). Its configuration is as \( N = 0, R^2 = 8/(1 + \lambda + l) \), \( P = 0 \) and square oscillation frequency is

\[
\omega = \frac{1 + (1 + 2\Delta)(\lambda + l)}{1 + \lambda + l}
\]

The domain of existence and stability of this mode is as: \( \lambda < 1, l > -2 \lambda \).

2) Mode of anti-phase synchronized streets \( S_\pi \) with the configuration: \( N = 0, R^2 = 8/(1 + \lambda - l) \), \( P = \pi \). Square oscillation frequency is

\[
\omega = \frac{1 + (1 + 2\Delta)(\lambda - l)}{1 + \lambda - l}
\]

The domain of existence and stability of this mode is as: \( \lambda < 1, l < 2 \lambda \).

3) Mode \( S_\phi \) of vortex streets synchronized at a variable angle differed from 0 or \( \pi \):

\( N = 0, P = \arccos(-l/2\lambda), R^2 = 8/(1 + 3\lambda - l^2/\lambda) \). Its square oscillation frequency is

\[
\omega = \frac{1 + (1 + 2\Delta)(3\lambda - l^2/\lambda)}{1 + 3\lambda - l^2/\lambda}
\]

This mode exists and is stable in the domain \(-1/3 < \lambda < 0, |l| < -2\lambda \) and \( \lambda < -1/3, \sqrt{\lambda(1 + 3\lambda)} < |l| < -2\lambda \).

4.2. Asymmetric one-frequency modes

4) Mode \( AS_\omega \) with total quenching of von Karman street behind one of the cylinders. Its configuration is \( N = 0.5, R^2 = 4 \), square oscillation frequency is as \( \omega^2 = 1 \). The domain of existence and stability of this mode is \( \lambda > 0.5 \).
5) The fifth mode \( AS_{\rho r} \) of two streets with different amplitudes exists but is unstable with respect to infinitesimal perturbations in entire domain of its formal existence within the framework of the model. Its configuration contains four branches:

\[
P'^r = \arccos \sqrt{1.5 - 0.5/\lambda}, \quad P'^\pi = \pi - \arccos \sqrt{1.5 - 0.5/\lambda},
\]

\[
P'^{III} = -\pi + \arccos \sqrt{1.5 - 0.5/\lambda}, \quad P'^{IV} = -\arcsin \sqrt{1.5 - 0.5/\lambda},
\]

\[
N'^{I,II} = \frac{8\left(\frac{\Delta}{\varepsilon}\right) \lambda \sin 2P + \lambda}{4 \lambda^2 - 1} \left[ -2(3\lambda - 1)(1 - \lambda) + 32 \left(\frac{\Delta}{\varepsilon}\right)^2 \right],
\]

\[
N'^{III,IV} = \frac{8\left(\frac{\Delta}{\varepsilon}\right) \lambda \sin 2P - \lambda}{4 \lambda^2 - 1} \left[ -2(3\lambda - 1)(1 - \lambda) + 32 \left(\frac{\Delta}{\varepsilon}\right)^2 \right]
\]

in the case \( \Delta < 0 \).

\[
R^2 = 4 \left( 1 + l \sqrt{0.25 - N^2 \cos P} \right)
\]

Remark: For somewhat different wake model [7, 16] this mode \( AS_{\rho r} \) turns out to be stable in narrow domains of the configuration and the model parameters spaces.

4.3. Asymmetric two-frequency mode of the wake

Besides one-frequency modes the wake model (14)–(16) contains two-frequency (asymmetric) global mode \( AS_\Omega \) that is usually visualized for enough close location of the cylinders.

6) Mode \( AS_\Omega \) consists of two vortex streets of different intensity and different oscillation frequency. One example of mode \( AS_\Omega \) configuration is adduced at figure 1, which represents time histories of the wake global parameters. The configuration was accepted by computing the equation set (14)–(16) within time interval enough to get stabilization of the mode.

One can see from the data at figure 1(a) and 1(b) that for the first approximation the asymmetry of the wake is constant for this mode \( AS_\Omega \) while that is not the case for the summarized intensity of oscillations. The latter undergo considerable modulation. Accordingly, for the first approximation the configuration of the mode may be characterized by long time averaged asymmetry \( \langle N \rangle \), intensity \( \langle R^2 \rangle \) and modulation depth of the intensity \( m \)

\[
\langle N \rangle \approx \frac{\text{sign}(l)}{2 \sqrt{1 + W \left( 1 - S \lambda^2 / l^2 (1 + \beta) \right)}},
\]

\[
W = \frac{16 \Delta \Delta^2}{\varepsilon l^2 (1 + \beta)}, \quad S = \frac{4 \lambda (1 - 2 \lambda)}{(1 - \lambda)^2}, \quad \beta = \left( \frac{\lambda \varepsilon}{4 \Delta} \right)^2
\]

(17)
Figure 1. Exact configuration of two-frequency wake mode $AS_{12}$ computed within the framework of the equation set (14) – (16) in the case $\varepsilon = 0.15; \Delta = -0.15; \lambda = 0.49; l = 1.0$:
(a) - summarized intensity $R^2$ of the oscillations;
(b) – asymmetry $N$ of the wake;
(c) – phase difference $P$ of oscillations in two partial von Karman streets forming the wake.

$$\langle R^2 \rangle \approx 4(1-m)\left[1 - \frac{2m\sqrt{1-c^2-b^2}}{(1+m)(1+c)} \right] \left[1 - \frac{8(1-m)\sqrt{1-c^2-b^2}}{\pi^2(1+m)(1+c)} \right] B,$$

$$B = 1 + L \frac{bc}{2} \left(1 + \frac{c^2 - b^2}{2}\right) + (1 + \alpha) \left[ \frac{fc}{2} \left(1 + \frac{c^2 - 2b^2}{4}\right) + \frac{gbc}{2} \left(1 + \frac{c^2 - b^2}{2}\right) \right] +$$

$$+ L \left[ \frac{fb}{4} \left(1 + c^2 + \frac{b^2}{4}\right) + \frac{g}{2} \left(1 - \frac{c^2}{4} \left(1 + b^2 + \frac{c^2}{4}\right) \right) \right] +$$

$$+ \lambda \left(\frac{1}{4} - \langle N \rangle^2\right) \left[ \frac{fc}{8} (c^2 - 3b^2) + \frac{fc}{2} \left(1 + \frac{c^2 - b^2}{2}\right) + \frac{gbc}{4} (c^2 + 3b^2) \right], \quad L = l \sqrt{0.25 - \langle N \rangle^2},$$

$$b = -\lambda \varepsilon / (8\Delta \langle N \rangle), \quad c = -L \varepsilon / (4\Delta \langle N \rangle), \quad \alpha = 2(2\lambda - 1)(0.25 - \langle N \rangle^2).$$

$$f = \frac{4\Delta \langle N \rangle L}{\varepsilon D}, \quad g = -\left(1 + \alpha - \lambda \langle N \rangle^2\right)\frac{L}{D}, \quad D = 16 \left(\frac{\Delta \langle N \rangle}{\varepsilon}\right)^2 + (1 + \alpha)^2 - \lambda^2 \langle N \rangle^4$$

$$m \approx \left|\frac{L}{D}\right| \sqrt{16 \left(\frac{\Delta \langle N \rangle}{\varepsilon}\right)^2 + (1 + \alpha - \lambda \langle N \rangle^2)^2},$$

The fourth and fives characteristics of the mode - oscillations main frequencies in the von Karman streets pair were defined accordingly the definition $\langle \omega \rangle = 2\pi \langle P \rangle t$; then mean frequencies splitting is as $\Omega = \langle \omega_1 \rangle - \langle \omega_2 \rangle$. Accordingly, the streets oscillations mean frequencies were approximated as:
By way, instantaneous splitting frequency $\Omega$ and instantaneous frequencies $\omega$ undergoes substantial oscillation as may be seen from data adduced at figure 1(c). Accordingly, all these mean quantities approximates (17) – (21) were accepted using an average of the equations set (14) – (16).

Dependence of main properties of two-frequency asymmetric wake mode $AS_{\Omega}$ on the model parameters $\lambda,l$ describing the streets interplay may be evaluated from the data collected in tables 1– 4 below.

**Table 1.** The asymmetry $\langle N \rangle$ of the wake at quasi-laminar mode $AS_{\Omega}$ calculated numerically from the set (14)-(16) with zero right-hand parts and from the estimate (17). All data are for the case $\varepsilon = 0.15; \Delta = -0.15$.

| $\lambda$ | $l$ | $\langle N \rangle_{est}$ | $\langle N \rangle_{num}$ | Error |
|----------|-----|-----------------|-----------------|-------|
| 0.49     | 2.5 | 0.426           | 0.425            | + 0.002 |
| 0.49     | 1.5 | 0.350           | 0.350            | 0.000  |
| 0.49     | 1.0 | 0.276           | 0.275            | + 0.004 |
| 0.48     | 1.5 | 0.292           | 0.288            | 0.01   |
| 0.47     | 1.5 | 0.260           | 0.250            | 0.04   |

**Table 2.** The oscillations summarized intensity $\langle R^2 \rangle$ of the wake at quasi-laminar mode $AS_{\Omega}$ calculated numerically from the set of equations (14)-(16) with zero right-hand parts and from the estimate (18). All data are for the case $\varepsilon = 0.15; \Delta = -0.15$.

| $\lambda$ | $l$ | $\langle R^2 \rangle_{est}$ | $\langle R^2 \rangle_{num}$ | Error |
|----------|-----|-----------------|-----------------|-------|
| 0.49     | 2.5 | 3.92            | 3.84            | + 0.02 |
| 0.49     | 1.5 | 3.92            | 3.84            | + 0.02 |
| 0.49     | 1.0 | 3.90            | 3.83            | + 0.03 |
| 0.48     | 1.5 | 3.80            | 3.65            | 0.04   |
| 0.47     | 1.5 | 3.64            | 3.42            | 0.06   |
Table 3. The modulation depth $m$ of the wake oscillations instantaneous intensity $R^2$ at quasi-laminar mode $AS_\Omega$ calculated numerically from the set (14)-(16) with zero right-hand parts and from the estimate (19). All data are for the case $\varepsilon = 0.15; \Delta = -0.15$.

| $\lambda$ | $l$ | $m_{est}$ | $\left(\frac{\max R^2 - \min R^2}{\max R^2 + \min R^2}\right)_{num}$ | The estimate (19) error |
|----------|-----|-----------|--------------------------------------------------|---------------------|
| 0.49     | 2.5 | 0.325     | 0.33                                              | -0.02               |
|          | 1.5 | 0.306     | 0.31                                              | -0.02               |
| 0.49     | 1.0 | 0.277     | 0.29                                              | -0.04               |
| 0.48     | 1.5 | 0.392     | 0.394                                             | 0.00                |
| 0.47     | 1.5 | 0.443     | 0.434                                             | 0.02                |

Table 4. The oscillation mean frequencies of two von Karman streets forming the wake at quasi-laminar mode $AS_\Omega$ calculated numerically from the set (14)-(16) with zero right-hand parts and from the estimate (20), (21). All data are for the case $\varepsilon = 0.15; \Delta = -0.15$.

| $\lambda$ | $l$ | $\langle \omega_1 \rangle_{est}$ | $\langle \omega_1 \rangle_{num}$ | The estimate (20) error |
|----------|-----|----------------------------------|----------------------------------|---------------------|
| 0.49     | 2.5 | 0.933                            | 0.956                            | -0.02               |
|          | 1.5 | 0.910                            | 0.935                            | -0.03               |
| 0.49     | 1.0 | 0.886                            | 0.914                            | -0.03               |
| 0.48     | 1.5 | 0.851                            | 0.901                            | -0.06               |
| 0.47     | 1.5 | 0.801                            | 0.873                            | -0.08               |

| $\lambda$ | $l$ | $\langle \omega_2 \rangle_{est}$ | $\langle \omega_2 \rangle_{num}$ | The estimate (21) error |
|----------|-----|----------------------------------|----------------------------------|---------------------|
| 0.49     | 2.5 | 0.728                            | 0.735                            | 0.01                |
|          | 1.5 | 0.753                            | 0.755                            | 0.00                |
| 0.49     | 1.0 | 0.780                            | 0.776                            | 0.01                |
| 0.48     | 1.5 | 0.773                            | 0.777                            | -0.01               |
| 0.47     | 1.5 | 0.784                            | 0.790                            | -0.01               |

It is seen from the above adduced data, exact or estimated, that an increasing of the model parameters $\lambda$ and $|l|$ values implies a growth of wake asymmetry $\langle N \rangle$ and a growth of the frequency splitting $\langle \Omega \rangle$ of the von Karman streets pair. Simultaneously, summarized intensity of the wake oscillation $\langle R^2 \rangle$ increases with growth of the model parameter $\lambda$ and is practically independent on $|l|$ value. The wake modulation depth $m$ undergoes substantial growth with the parameter $\lambda$ decreasing and weaker growth with value $|l|$ increasing.

Inherent question concerning the accuracy of estimates (17)–(21) may be treated on the basis of data collected in tables 1–4, as well. By the way, there are no analogous questions on the properties expressions for the other modes as the latter follow from exact stationary solutions of the system (14)–(16) (for the case: $\varepsilon, |\Delta| < 1$). In the whole, the accuracy of the formulas (17)–(21) is good in the investigated domain of the model parameters, but the error of estimate (20) increases rapidly with the parameter $\lambda$ decreasing.
5. Brief resume
1) The first step of simple model constructing for complex turbulent wake from cylinders row is fulfilling. Agreement between the results of our model and that of Landau-Stuart wake model is established for the case of laminar wake from a solitary cylinder. An approximates for the main characteristics of the wake quasi-laminar modes has been constructed. The next step of the model parameters evaluation lies ahead.

2) Within the framework of the model developed only two asymmetric global modes of complex wake exist and are stable with respect to small perturbations. They are, firstly, mode of total quenching of von Karman street behind one of the cylinders pair and, secondly, two-frequency mode with wake consisted from two von Karman streets of different oscillation frequency and different oscillation intensity. Asymmetric two-frequency wake mode $AS_{\Omega}$ exists in comparatively narrow domain $\lambda < 0.5$ for enough large values of the model parameter $|l|$. An increasing of values of model parameters $\lambda$ and $|l|$ implies considerable growth of wake asymmetry $\langle N \rangle$ and of the frequency splitting $\langle \Omega \rangle$ for the von Karman streets forming the wake at mode $AS_{\Omega}$. It should be mentioned here that asymmetric one-frequency mode of wake (consisted of two von Karman streets of different oscillation intensities but equal oscillation frequencies) appears to exist formally but to be unstable everywhere within the framework of actual form of the model.

3) Besides asymmetric modes, three symmetric one-frequency modes exist within the framework of the model, as well. Two of them may intermit with an asymmetric mode in appropriate domain of the model parameters, ore with each other. Algebraic formulas for the characteristics of these symmetric modes are adduced as well.

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