The existence of G-invariant constant mean curvature hypersurfaces

Tongrui Wang1 · Zhiang Wu2

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Abstract
In this paper, we consider a closed Riemannian manifold \( M^{n+1} \) with dimension \( 3 \leq n+1 \leq 7 \), and a compact Lie group \( G \) acting as isometries on \( M \) with cohomogeneity at least 3. Suppose the union of non-principal orbits \( M \setminus M^\text{reg} \) is a smooth embedded submanifold of \( M \) with dimension at most \( n-2 \). Then for any \( c \in \mathbb{R} \), we show the existence of a nontrivial, smooth, closed, almost embedded, \( G \)-invariant hypersurface \( \Sigma^n \) of constant mean curvature \( c \).

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1 Introduction
Given an \((n + 1)\)-dimensional closed, oriented, smooth Riemannian manifold \((M^{n+1}, g_M)\), constant mean curvature (CMC) hypersurfaces are critical points of the area functional amongst variations preserving the enclosed volume. CMC hypersurfaces constitute a classical and important topic in differential geometry. The existence problem for CMC hypersurfaces has been studied from a number of perspectives.

In the case \( n = 2 \), Heinz [12, 13], Hildebrandt [14], Struwe [24–26], Brezis–Coron [5], etc, studied the boundary value problems of CMC hypersurfaces using the mapping method. For the case of \( n \geq 3 \), Duzaar–Steffen [8], Duzaar–Grotowski [9] studied the boundary value problems of CMC hypersurfaces using the geometric measure theory, while both methods can only produce CMC hypersurfaces whose mean curvatures satisfy a certain upper bound.

For the case of closed CMC hypersurfaces, the classical approach is to solve the isoperimetric problem for a given volume. It is well-known that there exists a smooth minimizer for
each fixed volume except for a set of Hausdorff dimension at most \( n - 7 \), see for instance [19]. However, this approach does not yield any control on the value of the mean curvature. Zhou and Zhu developed a min–max theory for the construction of constant mean curvature hypersurfaces of prescribed mean curvature \( c \) in [31] with the Lagrange-multiplier functional

\[
\mathcal{A}^c = \text{Area} - c \cdot \text{Vol}.
\]  

(1.1)

It is obvious that a hypersurface has constant mean curvature \( c \) if it is a critical point of the \( \mathcal{A}^c \)-functional. Recently, they showed in [32] that the CMC min–max theory can be extended to construct min–max prescribed mean curvature hypersurfaces for certain classes of prescription functions, including a generic set of smooth functions, and all nonzero analytic functions.

We want to consider when a Lie group \( G \) acts by isometries on \((M^{n+1}, g_M)\), can the CMC min–max theory generate a CMC hypersurface invariant under the action of \( G \)? A similar problem for minimal hypersurfaces has been studied by many people. To the author’s knowledge, J. Pitts and J. H. Rubinstein firstly announced a version of the equivariant min–max theory for finite groups acting on 3-dimensional manifolds in [21, 22]. Inspired by Ketover’s work, a more general version of \( G \)-equivariant min–max theory under the smooth sweepouts settings (see [6, 7]) was built by Liu [16] to prove the existence of \( G \)-invariant smooth minimal hypersurfaces on manifolds with dimension \( 3 \leq n + 1 \leq 7 \). After that, the first author in [29] built the \( G \)-equivariant min–max theory under the Almgren-Pitts setting to show that there are infinitely many minimal hypersurfaces on a positive Ricci curvature Riemannian manifold \((M^{n+1}, g_M)\) with dimension \( 3 \leq n + 1 \leq 7 \) and \( \dim(M \setminus M^{\text{reg}}) \leq n - 2 \).

Based on the min–max theory developed in [31] for CMC hypersurfaces, we resolve the existence of closed \( G \)-invariant CMC hypersurfaces. In particular, we prove that:

**Theorem 1.1** Let \( 2 \leq n \leq 6 \), and \( (M^{n+1}, g_M) \) be an \((n + 1)\)-dimensional smooth, closed Riemannian manifold with a compact Lie group \( G \) acting as isometries of cohomogeneity \( \text{Cohom}(G) \geq 3 \). Suppose the union of non-principal orbits \( M \setminus M^{\text{reg}} \) is a smooth embedded submanifold of \( M \) with \( \dim(M \setminus M^{\text{reg}}) \leq n - 2 \). Then for any \( c \in \mathbb{R} \), there exists a nontrivial, smooth, closed, almost embedded, \( G \)-invariant hypersurface \( \Sigma^n \) of constant mean curvature \( c \).

Our work is inspired by the approach of Zhou-Zhu and Liu. However, there are some difficulties when dealing with \( G \)-invariant elements. Above all, since we only consider the Caccioppoli sets that are invariant under the action of \( G \), the variations, in this case, should keep being \( G \)-invariant. Hence, due to the \( G \)-invariant restrictions, it seems that our variation result is weaker than the classical one which takes all kinds of variations into account. There are already some approaches dealing with this kind of difficulties, see for example [15, 16, 29]. The idea is using an averaging argument to make the usual vector field or current to be \( G \)-invariant one. Combining the approach in [31] with the averaging arguments, we show the existence of a nontrivial \( G \)-invariant integral varifold as a weak solution of \( G \)-invariant prescribed constant mean curvature \( c \)-hypersurfaces.

Since we are dealing with tubes around orbits rather than balls, there are also some technical problems coming up in the proof of the regularity result. In the paper [31], Zhou-Zhu showed the \( c \)-min–max varifolds have replacements in small annuli, which meets the property similar to but different from the ‘good replacement property’ in [7]. Nevertheless, the good replacement property can be obtained through a blowup procedure, since small annuli in \( T_pM \) are closed to those in \( M \). Thus, the regularity of tangent cones, as well as various blowups, of the \( c \)-min–max varifolds was built by the regularity result for varifolds.
with good replacement property [31, Appendix C]. However, under \( G \)-invariant restrictions, we only have replacements in \( G \)-sets for \((G, c)\)-min–max varifolds, and \( G \)-annuli in \( M \) can be vastly different to annuli in the tangent space \( T_p M \) (note \( T_p M \) is only a \( G_p \)-space not a \( G \)-space).

To overcome this problem, we show the splitting property for the tangent cones and the various blowups of \((G, c)\)-min–max varifolds, which has been used in [16, Appendix A] and [29, Lemma 6.5]. Using the splitting property, we only need to show the regularity of blowups in the normal space \( N_p(G \cdot p) \). Firstly, for the normal part of tangent cones, we observe it is invariant under the action of the isotropy group \( G_p \). Therefore, we can apply the argument in [31, Proposition 5.11] with [31, Appendix C] being replaced by the results in [16, Proposition 6.2]. When it comes to the normal part of various blowups, the assumption on \( M \setminus M^{reg} \) the union of non-principal orbits makes us only need to consider the case that \( p_i \) and \( p \) in the various blowups are all contained in principal orbits by regarding \( M \setminus M^{reg} \) as a whole. Consequently, the argument in the proof of [31, Lemma 5.10] would carry over with [16, Proposition 6.2], which indicates the regularity of various blowups.

1.1 Outline

In Sect. 2, we describe some basic notations and some useful propositions of \( G \)-invariant currents and varifolds including the compactness theorem and the equivalence between \( c \)-bounded first variation and \((G, c)\)-bounded first variation for \( G \)-varifolds.

In Sect. 3, we define the \( G \)-equivariant min–max theory under the Almgren-Pitts setting dealing with \( G \)-invariant objects. We also prove the existence of nontrivial \( G \)-sweepouts (Theorem 3.8). Then for any critical sequence, we extract a min–max sequence \( \{\partial\Omega_{x_i}\} \) that converges in the measure-theoretic sense to a nontrivial \( G \)-varifold \( V \).

In Sect. 4, we first recall the tightening map adapted to the \( A^c \) functional constructed in [31]. In the remainder of Sect. 4, we prove that after applying the modified tightening map to a critical sequence \( S = \{\varphi_i\}_{i \in \mathbb{N}} \), every element \( V \) in the critical set \( C(S) \) has \( c \)-bounded first variation.

In Sect. 5, we introduce the notion of \((G, c)\)-almost minimizing varifolds, and show the existence of \( V \in C(S) \) which is \((G, c)\)-almost minimizing in small regular annuli with \( c \)-bounded first variation.

In Sect. 6, we show the existence and the regularity of \((G, c)\)-replacements for \((G, c)\)-almost minimizing varifolds. Finally, we obtain the regularity result for \((G, c)\)-min–max varifolds (Theorem 6.10) similarly to [31, Section 6].

2 Preliminary

In this paper, we always assume \((M^{n+1}, g_M)\) to be a closed, oriented, connected, smooth Riemannian manifold of dimension \( 3 \leq (n + 1) \leq 7 \). Assume \( G \) is a compact Lie group acting on \( M \) as isometries. Then there is a unique orbit type which is open and dense in \( M \) [27, Theorem 3.5.5]. Orbits of this type are called principal orbits and let \( M^{reg} \) denote the union of all principal orbits in \( M \). The cohomogeneity \( \text{Cohom}(G) \) is the codimension of principal orbits with respect to \( M \). In this paper, we always assume \( \text{Cohom}(G) = l \geq 3 \). Let \( \mu \) be a bi-invariant Haar measure on \( G \) normalized to \( \mu(G) = 1 \). By the main theorem of [18], there is an orthogonal representation of \( G \) on some Euclidean space \( \mathbb{R}^L, L \in \mathbb{N} \), and a \( G \)-equivariant isometric embedding from \( M \) into \( \mathbb{R}^L \).
2.1 Basic notation

In this section, we collect some definitions and notions from the geometric measure theory following from [16, 20, 31]. Since we are constantly dealing with G-invariant objects in our paper, we add $G$-invariant objects meaning they are $G$-invariant:

- a $G$-varifold $V$ satisfies $g_\# V = V$ for all $g \in G$;
- a $G$-current $T$ satisfies $g_\# T = T$ for all $g \in G$;
- a $G$-vector field $X$ satisfies $g_\* X = X$ for all $g \in G$;
- a $G$-map $F$ satisfies $g^{-1} \circ F \circ g = F$, $\forall g \in G$, (i.e. $F$ is $G$-equivariant);
- a $G$-set ($G$-neighborhood) is an (open) set which is a union of orbits.

We denote $\mathcal{H}^k$ to be the $k$-dimensional Hausdorff measure, and $\mathcal{X}(M)$ to be the space of smooth vector fields in $M$. Denote $B_r(p), \text{Clos}(B_r(p))$ as the open and closed Euclidean ball of $\mathbb{R}^L$, and $\hat{B}_r(p), \text{Clos}(\hat{B}_r(p))$ as the open and closed geodesic ball of $(M, g_M)$. We also sometimes add a subscript or superscript $G$ to signify $G$-invariance just like in [16]:

- $\pi$: the projection $\pi : M \mapsto M/G$ defined by $p \mapsto [p]$.
- $G \cdot p$: the orbit of $p \in M$, i.e $G \cdot p = \{g \cdot p : g \in G\}$.
- $G_p$: the isotropy group of $p$, i.e $G_p = \{g \in G : g \cdot p = p\}$.
- $B^G_r(p)$: open tubes with radius $\rho$ around the orbit $G \cdot p$ in $\mathbb{R}^L$.
- $\hat{B}^G_r(p)$: open tubes with radius $\rho$ around the orbit $G \cdot p$ in $M$.
- $\mathcal{X}^G(M)$: the space of smooth $G$-vector fields on $M$.
- $\text{An}^G(p, s, t)$: the open $G$-annulus $B^G_r(p) \setminus \text{Clos}(B^G_r(p))$.
- $\mathcal{AN}^G_r(p)$: the set $\{\text{An}^G(p, s, t) : 0 < s < t < r\}$.
- $T_q(G \cdot p)$: the tangent space of the orbit $G \cdot p$ at some point $q \in G \cdot p$.
- $N_q(G \cdot p)$: the normal space of the orbit $G \cdot p$ at some point $q \in G \cdot p$.

The spaces we work with in this paper are:

- $I^G_k(M; \mathbb{Z}_2)$ ($I^G_k(M; \mathbb{Z}_2)$) the space of $k$-dimensional ($G$-invariant) mod-2 flat chains in $\mathbb{R}^L$ with support contained in $M$ (see [10, 4.2.26] for more details).
- $Z^G_k(M; \mathbb{Z}_2)$ ($Z^G_k(M; \mathbb{Z}_2)$) the space of ($G$-invariant) mod-2 flat chains $T \in I^G_k(M; \mathbb{Z}_2)$ ($T \in I^G_k(M; \mathbb{Z}_2)$) with $\partial T = 0$.
- $\mathcal{V}_k(M)$ ($\mathcal{V}^G_k(M)$) the closure, in the weak topology, of the space of $k$-dimensional ($G$-invariant) rectifiable varifolds in $\mathbb{R}^L$ with support contained in $M$. The space of integral ($G$-invariant) rectifiable $k$-dimensional varifolds with support contained in $M$ is denoted by $\mathcal{T} \mathcal{V}_k(M)$ ($\mathcal{T} \mathcal{V}^G_k(M)$).
- $C(M)$ ($C^G(M)$) the space of ($G$-invariant) sets $\Omega \subset M$ with finite perimeter, which are also known as Caccioppoli sets, [23, §14].

Just like in [31] we also utilize the following definitions:

(a) Given $T \in I^G_k(M; \mathbb{Z}_2)$, $|T|$ and $\|T\|$ denote respectively the integral varifold and the Radon measure in $M$ associated with $T$.

(b) $\mathcal{F}$ and $\mathcal{M}$ are the flat norm [23, §31] and mass norm [23, 26.4] on $I^G_k(M; \mathbb{Z}_2)$; The varifold $\mathbf{F}$-metric on $\mathcal{V}_k(M)$ is defined in Pitts’s book [20, P.66], which induces the weak topology on $\mathcal{V}_k(M) \cap \{V : \|V\| \leq C\}$ for all $C > 0$. The current $\mathbf{F}$-metric on $I^G_k(M; \mathbb{Z}_2)$ is defined by

$$\mathbf{F}(S, T) = \mathcal{F}(S - T) + \mathbf{F}(|S|, |T|).$$
Note the mass of currents is continuous in the **F-metric** and lower semi-continuous in the flat topology. Additionally, the current **F-metric** satisfies

\[ \mathcal{F}(S - T) \leq \mathcal{F}(S, T) \leq 2\mathcal{M}(S - T), \quad \forall S, T \in I_{k}(M; \mathbb{Z}_{2}). \]

(c) Given \( c > 0 \), a varifold \( V \in \mathcal{V}_{k}(M) \) is said to have \emph{c-bounded first variation in an open subset} \( U \subset M \), if

\[ |\delta V(X)| \leq c \int_{M} |X|d\mu_{V}, \]

for all \( X \in \mathcal{X}(U) \), where \( \delta V(X) = \int_{G_{k}(M)} \text{div}_{S} X(x)dV(x, S) \) is the first variation of \( V \) along \( X \).

(d) Given a set \( \Omega \subseteq C(M) \) \((C^{G}(M))\) with finite perimeter, \([\Omega]\) denotes the corresponding \((G\text{-invariant})\) integral currents with \( \mathbb{Z}_{2}\)-coefficient. \( \partial \Omega \) denotes the reduced-boundary of \([\Omega]\) as a \((G\text{-invariant})\) integral current with \( \mathbb{Z}_{2}\)-coefficient. When the boundary \( \Sigma = \partial \Omega \) is a smooth immersed hypersurface, we have

\[ \text{div}_{\Sigma} X = H \cdot \tilde{\nu}, \]

where \( \tilde{\nu} = \tilde{\nu}_{\partial \Omega} \) is the outward pointing unit normal of \( \partial \Omega \) \([23, 14.2]\), and \( H \) is the mean curvature of \( \Sigma \) with respect to \( \tilde{\nu} \).

In [31], Xin Zhou and Jonathan J. Zhu developed a min–max theory for constant mean curvature hypersurfaces of prescribed mean curvature \( c \) in \((M, g_{M})\). They studied the following \emph{\( A^{c}\text{-functional} \)} defined on \( C(M) \) for \( c > 0 \):

\[ A^{c}(\Omega) = \mathcal{H}^{n}(\partial \Omega) - c\mathcal{H}^{n+1}(\Omega). \quad (2.1) \]

The \emph{first variation formula} for \( A^{c} \) along \( X \in \mathcal{X}(M) \) is

\[ \delta A^{c}|_{\Omega}(X) = \int_{\partial \Omega} \text{div}_{\partial \Omega} Xd\mu_{\partial \Omega} - c \int_{\partial \Omega} X \cdot \tilde{\nu} d\mu_{\partial \Omega}, \quad (2.2) \]

When \( \Omega \) is a critical point of \( A^{c} \) and \( \Sigma = \partial \Omega \) is a smooth immersed hypersurface, the first variation formula \((2.2)\) directly implies that \( \Sigma \) has constant mean curvature \( c \) with respect to the outward unit normal \( \tilde{\nu} \). Furthermore, the \emph{second variation formula} for \( A^{c} \) along a normal vector field \( X \in \mathcal{X}(M) \) is (see [3, Proposition 2.5]):

\[ \delta^{2}A^{c}|_{\Omega}(X, X) = II_{\Sigma}(X, X) = \int_{\Sigma} \left( |\nabla_{\Sigma} \varphi|^{2} - \left( \text{Ric}^{M}(\tilde{\nu}, \tilde{\nu}) + |A^{\Sigma}|^{2} \right) \varphi^{2} \right) d\mu_{\Sigma}, \quad (2.3) \]

where \( X = \varphi \tilde{\nu} \) on \( \Sigma \) for some \( \varphi \in C^{\infty}(\Sigma) \), \( \nabla_{\Sigma} \varphi \) is the gradient of \( \varphi \) on \( \Sigma \), \( \text{Ric}^{M} \) is the Ricci curvature of \( M \), and \( A^{\Sigma} \) is the second fundamental form of \( \Sigma \).

### 2.2 G-invariant currents and varifolds

To deal with \( G \)-invariant objects, some useful propositions for \( G \)-currents and \( G \)-varifolds are collected in this subsection from [29].

**Proposition 2.1** (Compactness Theorem for \( I_{G}^{c}(M; \mathbb{Z}_{2}) \)) \emph{For any} \( C > 0 \), \emph{the set:}

\[ \{ T \in I_{G}^{c}(M; \mathbb{Z}_{2}) : \mathcal{M}(T) + \mathcal{M}(\partial T) \leq C \} \]

\emph{is compact under the flat metric} \( \mathcal{F} \).
Lemma 2.9 For any $V$ bounded first variation and $c$ for $G$, follows from [16, Lemma 3.2] that for any vector field $U$ if and only if $V$ has $c$-bounded first variation in $U$.

**Proposition 2.2** (Compactness Theorem for $\mathcal{V}_k^G(M)$) For any $C > 0$, the set:

$$\{ V \in \mathcal{V}_k^G(M) : ||V||(M) \leq C \}$$

is compact under the $F$-metric of varifolds.

**Proposition 2.3** (Restrict to $G$-sets) For any $V \in \mathcal{V}_k^G(M)$ and Borel $G$-set $U \subset M$, we have $V \subset U \in \mathcal{V}_k^G(U)$.

**Proposition 2.4** ($G$-equivariant pushing forward) Let $\Phi$ be a $G$-equivariant diffeomorphism on $M$, then for any $T \in I_k^G(M; \mathbb{Z}_2)$ and $V \in \mathcal{V}_k^G(M)$, the pushing forward $\Phi_#T \in I_k^G(M; \mathbb{Z}_2)$, and $\Phi_#V \in \mathcal{V}_k^G(M)$.

**Proposition 2.5** ($G$-invariant slice) Let $f$ be a $G$-invariant Lipschitz function on $M$ and $T \in I_k^G(M; \mathbb{Z}_2)$, then for almost all $t \in \mathbb{R}$, the slice of $T$ by $f$ at $t$ exists and $(T, f, t) \in I_k^G(M; \mathbb{Z}_2)$.

**Proposition 2.6** ($G$-invariant Isoperimetric Lemma) There is a positive constant $v_M$ such that for any $T_1, T_2 \in Z_{n}^{G}(M; \mathbb{Z}_2)$ with

$$F(T_1 - T_2) < v_M,$$

there exists a unique $Q \in I_{n+1}^G(M; \mathbb{Z}_2)$ such that:

$$\delta Q = T_1 - T_2, \quad M(Q) = F(T_1 - T_2).$$

**Remark 2.7** An important case for $G$-equivariant diffeomorphisms is the one-parameter group of diffeomorphisms generated by some $X \in \mathfrak{X}^G(M)$. A simple example for $G$-invariant slice is taking $f$ to be the distance function $\text{dist}(G \cdot p, \cdot)$ to an orbit $G \cdot p$.

**Definition 2.8** ($(G, c)$-bounded first variation) Given $c > 0$, a varifold $V \in \mathcal{V}_k^G(M)$ is said to have $(G, c)$-bounded first variation in an open $G$-subset $U \subset M$, if

$$|\delta V(X)| \leq c \int_M |X|d\mu_V,$$

for any $X \in \mathfrak{X}^G(U)$.

In [16, Lemma 3.2], Z. Liu has shown the equivalence between $G$-stationary and stationary for $G$-varifolds. We notice that Liu’s arguments also imply the equivalence between $(G, c)$-bounded first variation and $c$-bounded first variation for $G$-varifolds. Indeed, we have:

**Lemma 2.9** For any $V \in \mathcal{V}_k^G(M)$ and open $G$-set $U \subset M$, $V$ has $(G, c)$-bounded first variation in $U$ if and only if $V$ has $c$-bounded first variation in $U$.

**Proof** If $V$ has $c$-bounded first variation in $U$, it’s clear that $V$ has $(G, c)$-bounded first variation in $U$ by Definition 2.8. Suppose now $V$ has $(G, c)$-bounded first variation in $U$. It follows from [16, Lemma 3.2] that for any vector field $X \in \mathfrak{X}(U)$ with $|X| \leq 1$, there exists a $G$-vector field $X_G$ supported in $U$, such that

$$\delta V(X) = \delta V(X_G),$$

and $\int_M |X_G|d\mu_V \leq \int_M |X|d\mu_V$. Since $V$ has $(G, c)$-bounded first variation in $U$, we have

$$|\delta V(X_G)| \leq c \int_M |X_G|d\mu_V \leq c \int_M |X|d\mu_V.$$

Hence,

$$|\delta V(X)| = |\delta V(X_G)| \leq c \int_M |X|d\mu_V.$$

This implies that $V$ has $c$-bounded first variation in $U$. $\square$

\[ Springer \]
2.3 Compactness of G-stable CMC G-hypersurfaces

**Definition 2.10 (G-stable (G,c)-hypersurfaces)** Let Σ be a smooth, immersed, two-sided, hypersurface with the unit normal vector field \( \vec{v} \). We say that Σ is a **G-stable (G,c)-hypersurface** in an open G-subset \( U \subset M \) if

- Σ is G-invariant;
- the mean curvature \( H \) of Σ ∩ U with respect to \( \vec{v} \) equals to c;
- \( II_\Sigma(X, X) \geq 0 \) for all G-invariant vector field \( X \in \mathfrak{X}^{G,\perp}(\Sigma \cap U) \).

In the above definition, the prefix ‘G’ of ‘stable’ means that only G-invariant vector fields are considered for variations. While the prefix ‘G’ of ‘hypersurface’ means the hypersurface is invariant under the action of G. Clearly, the prefix ‘c’ of ‘hypersurface’ implies that the hypersurface has constant mean curvature which equals c with respect to the given unit normal vector field. With these in mind, one can similarly define stable (G,c)-hypersurfaces as well as stable c-hypersurfaces.

In [29, Lemma 2.9], the first author has shown the equivalence between the G-stability and stability for minimal G-hypersurfaces of boundary type. Note the second variation formula of \( \mathcal{A}^c \) functional is the same as that of Area functional. Hence, Wang’s argument can also imply the equivalence between the stability and G-stability of boundary type G-invariant c-hypersurfaces:

**Lemma 2.11** Let \( U \subset M \) be an open G-set, and \( \Sigma = \partial \Omega \) be the boundary of \( \Omega \in C^G(M) \). If \( \Sigma \) is a c-hypersurface, then \( \Sigma \) is G-stable in \( U \) if and only if it is stable in \( U \).

Noting the Lie group G is not assumed to be connected, so a connected component of a G-hypersurface may not be G-invariant. Hence, we introduce the following G-connectivity analogs of the usual notions of connectivity.

**Definition 2.12 (G-connectivity)** Suppose \( \Sigma \) is a G-hypersurface, and \( \{\Sigma_i\}_{i=1}^k \) are connected components of \( \Sigma \). We say \( \Sigma \) is G-connected, if for any \( i, j \in \{1, \ldots, k\} \), there exists \( g \in G \) such that \( \Sigma_i = g \cdot \Sigma_j \).

Similarly, we say a G-set U is G-connected if for any two connected components of U there exists \( g \in G \) making them transferable from one to the other.

The following definition comes from [31, Definition 2.2].

**Definition 2.13** Let \( \Sigma_i, i = 1, 2 \), be two connected embedded two-sided hypersurfaces in a connected open set \( U \subset M \), with unit normals \( \vec{v}_i \) and \( \partial \Sigma_i \cap U = \emptyset \). We say that \( \Sigma_2 \) lies on one side of \( \Sigma_1 \) in U if \( \Sigma_1 \) divides U into two connected components \( U_1 \cup U_2 = U \setminus \Sigma_1 \), where \( \vec{v}_1 \) points into \( U_1 \), and either:

- \( \Sigma_2 \subset \text{Clos}(U_1) \), which we write as \( \Sigma_1 \leq \Sigma_2 \) or that \( \Sigma_2 \) lies on the positive side of \( \Sigma_1 \);
- or
- \( \Sigma_2 \subset \text{Clos}(U_2) \), which we write as \( \Sigma_1 \geq \Sigma_2 \) or that \( \Sigma_2 \) lies on the negative side of \( \Sigma_1 \).

A manifold \( \Sigma \) is said to be a G-manifold if \( G \) acts on \( N \) by diffeomorphisms. Therefore, we can generalize the definition of almost embedding [31, Definition 2.3] into the following form:

**Definition 2.14 (G-equivariant almost embedding)** Let \( U \subset M^{n+1} \) be an open G-set, and \( \Sigma' \) be an n-dimensional G-manifold. A smooth G-equivariant immersion \( \phi : \Sigma' \to U \) is said to be a G-equivariant almost embedding if at any point \( p \in \phi(\Sigma') \) where \( \phi(\Sigma') \) fails to be embedded, there exists a small G-connected G-neighborhood \( W \subset U \) of \( p \), such that
\[ \Sigma' \cap \phi^{-1}(W) \text{ is a disjoint union of connected components } \bigcup_{i=1}^l \Sigma'_i; \]

\[ \phi(\Sigma'_i) \text{ is an embedding for each } i = 1, \ldots, l; \]

\[ \text{for each } i, \text{ suppose } \phi(\Sigma'_i) \subset W_i, \text{ where } W_i \text{ is a connected component of } W, \text{ then any other component } \phi(\Sigma'_j), j \neq i, \text{ either lies on one side of } \phi(\Sigma'_i) \text{ in } W_i, \text{ or is contained in a different connected component of } W. \]

Denote \( \phi(\Sigma') \) and \( \phi(\Sigma'_i) \) by \( \Sigma \) and \( \Sigma_i \) for simplicity. The subset of points in \( \Sigma \) failing to be embedded will be called the **touching set** and denoted by \( \mathcal{S}(\Sigma) \). Additionally, we call \( \Sigma_i \mathcal{S}(\Sigma) \) the regular set, and denote it by \( \mathcal{R}(\Sigma) \). It is clear that the collection of components \( \Sigma_i \) meet tangentially along \( \mathcal{S}(\Sigma) \), and the touching set \( \mathcal{S}(\Sigma) \) as well as the regular set \( \mathcal{R}(\Sigma) \) of \( \Sigma \) is a \( G \)-set.

Furthermore, by [31, Proposition 2.9] and the isometric actions of \( G \), one verifies that, for any \( G \)-invariant \( c \)-hypersurface \( \Sigma \subset M \), \( \Sigma \) is \( G \)-equivariant almost embedded if and only if it is almost embedded in the sense of [31, Definition 2.3].

In Definition 2.14, since \( \phi \) is \( G \)-equivariant, it’s clear that \( \Sigma' \cap \phi^{-1}(W) = \bigcup_{i=1}^l \Sigma'_i \) is \( G \)-invariant. However, we generally do not know whether each \( \Sigma'_i \) is \( G \)-invariant or not. For example, consider the \( G = \mathbb{Z}_2 \) action on \( \mathbb{R}^2 \) as \( e \cdot (x, y) = (x, y), g \cdot (x, y) = (-x, y). \)

Define \( \Sigma_i = (S^1 + ((-1)^i, 0)), i = 1, 2, \) to be two unit 1-spheres with center points \((-1, 0)\) and \((1, 0)\). Then \( \Sigma = \Sigma_1 \cup \Sigma_2 \) is an equivariant almost embedded \( G \)-hypersurface with one touching point \( \mathcal{S}(\Sigma) = \{(0, 0)\} \). But none of \( \Sigma_i \) is \( G \)-invariant. Nevertheless, if \( G \) is connected, then each \( \Sigma_i = \phi(\Sigma'_i) \) must be \( G \)-invariant. Indeed, for any \( g \in G \), there exists a curve \( g(t) : [0, 1] \to G \) with \( g(0) = e, g(1) = g \) by the connectivity of \( G \). Since \( \Sigma' \cap \phi^{-1}(W) = \bigcup_{i=1}^l \Sigma'_i \) is \( G \)-invariant, we have \( g \cdot \Sigma'_i \subset \bigcup_{i=1}^l \Sigma'_i, \forall j \in \{1, \ldots, l\} \).

Meanwhile, for any \( x \in \Sigma'_j \), there is a curve \( \gamma(t) = g(t) \cdot x \) from \( x \) to \( g \cdot x \) implying \( g \cdot \Sigma'_j \subset \Sigma'_j \) by the connectivity of \( \Sigma'_j \). Hence, \( g \cdot \Sigma'_j = \Sigma'_j \) for all \( g \in G \), and each \( \Sigma_j \) is \( G \)-invariant.

**Definition 2.15** (**G-equivariant almost embedded** (**G,c**)-boundary) Suppose \( U \subset M \) is an open \( G \)-set.

(1) A \( G \)-equivariant almost embedded \( G \)-hypersurface \( \Sigma \subset U \) is said to be a **G-equivariant almost embedded G-boundary** if there is an open \( G \)-set \( \Omega \in C^G(U) \) such that \( \Sigma \) is equal to the boundary of \( \partial \Omega \) in \( U \) in the sense of currents.

(2) The **outer unit normal** \( \nu/\Sigma \) of \( \Sigma \) is the choice of the unit normal of \( \Sigma \) which points outside of \( \Omega \) along the regular part \( \mathcal{R}(\Sigma) \).

(3) \( \Sigma \) is called a **stable** (**G,c**)-boundary if \( \Sigma \) is a **G-boundary** as well as a stable immersed (**G,c**)-hypersurface.

**Remark 2.16** By Lemma 2.11, \( \Sigma \) is a stable (**G,c**)-boundary if and only if \( \Sigma \) is **G-stable** (**G,c**)-boundary.

We need the following compactness theorem for **G-stable** (**G,c**)-hypersurfaces, which essentially follows from [31, Theorem 2.11].

**Theorem 2.17** (**Compactness theorem for G-stable** (**G,c**)-boundary) Let \( 2 \leq n \leq 6 \). Suppose \( \Sigma_k \subset U \) is a sequence of smooth, \( G \)-equivariant almost embedded, \( G \)-stable, (**G,c**)-boundaries in an open \( G \)-set \( U \), with \( \sup_k \text{Area}(\Sigma_k) < \infty \) and \( \sup_k c_k < \infty \). Then the following hold:

(i) If \( \inf_k c_k > 0 \), then up to a subsequence, \( \{\Sigma_k\} \) converges locally smoothly to some **G-equivariant almost embedded, G-stable, (G,c)-boundary** \( \Sigma_\infty \) in \( U \). The density of \( \Sigma_\infty \) is 1 along \( \mathcal{R}(\Sigma_\infty) \) and 2 along \( \mathcal{S}(\Sigma_\infty) \).
(ii) If $c_k \to 0$, then up to a subsequence, $\{\Sigma_k\}$ converges locally smoothly (with multiplicity) to some smooth, embedded, stable, $G$-invariant minimal hypersurface $\Sigma_\infty$ in $U$.

**Proof** By Remark 2.16, each $\Sigma_k$ is an almost embedded, stable, $G$-invariant $c_k$-boundary. Hence, we can apply the Compactness Theorem [31, Theorem 2.11(i)(ii)] to get a subsequence converging locally smoothly to an almost embedded, stable, $c$-boundary $\Sigma_\infty$ satisfying the density requirements. Denote $\Sigma_k = \partial \Omega_k$ for some $\Omega_k \in C^G(M)$. Combining Proposition 2.1, 2.2 with the proof of [31, Theorem 2.11(ii)], we have $\partial \Omega_k$ converges weakly as $G$-currents to some $\partial \Omega_\infty$ with $\Omega_\infty \in C^G(M)$ and $\Sigma_\infty = \partial \Omega_\infty$ as $G$-varifolds. Thus $\Sigma_\infty$ is a $G$-equivariant almost embedded $G$-boundary. Moreover, the second statement follows directly from [31, Theorem 2.11(iii)]. \qed

### 3 G-invariant min–max theory for CMC hypersurfaces

This section is parallel to [31, Section 3] with $C^G(M)$ in place of $C(M)$. Our min–max construction is introduced in the scheme developed by Almgren and Pitts [1, 2, 20]. And the final purpose of this section is to obtain a non-trivial $G$-equivariant sweepout with a positive $A^\mathcal{F}$-min–max value.

#### 3.1 Cell complex

For any $j \in \mathbb{N}$, we denote $I(1, j)$ to be the cell complex on $I = [0, 1]$ with 1-cells

$$[0, 3^{-j}], [3^{-j}, 2 \cdot 3^{-j}], \ldots, [1 − 3^{-j}, 1],$$

and 0-cells (vertices)

$$[0], [3^{-j}], \ldots, [1 − 3^{-j}], [1].$$

Given a 1-cell $\alpha \in I(1, j)_1$, and $k \in \mathbb{N}$, we use the following notations:

- $I(1, j)_p$: the set of all $p$-cells in $I(1, j)$;
- $I_0(1, j)_0$: the set $\{[0], [1]\}$;
- $\alpha(k)$: the sub-complex of $I(1, j + k)$ formed by all cells contained in $\alpha$;

We also utilize the following definitions:

- The boundary homeomorphism $\partial : I(1, j) \to I(1, j)$ is given by $\partial[a, b] = [b] − [a]$ if $[a, b] \in I(1, j)_1$, and $\partial[a] = 0$ if $[a] \in I(1, j)_0$.
- The distance function $d : I(1, j)_0 \times I(1, j)_0 \to \mathbb{N}$ is defined as $d(x, y) = 3^j|x − y|$. We say $x, y \in I(1, j)_0$ are adjacent if $d(x, y) = 1$.
- The map $n(i, j) : I(1, i)_0 \to I(1, j)_0$ is defined as: $n(i, j)(x) \in I(1, j)_0$ is the unique element of $I(1, j)_0$ such that

$$d(x, n(i, j)(x)) = \inf \{d(x, y) : y \in I(1, j)_0\}.$$

- For any map $\phi : I(1, j)_0 \to C^G(M)$, the fineness of $\phi$ with respect to the $\mathbf{M}$ norm is defined as:

$$f(\phi) = \sup \left\{ \left. \frac{\mathbf{M}(\partial \phi(x) − \partial \phi(y))}{d(x, y)} \right| x, y \in I(1, j)_0, x \neq y \right\}.$$

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It was observed by Fernando C. Marques and André Neves that $f(\phi) < \delta$ if and only if $M(\partial \phi(x) - \partial \phi(y)) < \delta$ whenever $d(x, y) = 1$. Similarly, we can define the fineness of $\phi$ with respect to the $F$-norm and $F$-metric.

- Denote $\phi : I(1, j)_0 \to (C^G(M), \{0\})$ as a map such that $\phi(I(1, j)_0) \subset C^G(M)$ and $\partial \phi|_{I_0(1, j)_0} = 0$, i.e. $\phi([0]), \phi([1]) = \emptyset$ or $M$.

### 3.2 Homotopy sequences

**Definition 3.1** Given $\delta > 0$ and $\phi_i : I(1, k_i)_0 \to (C^G(M), \{0\})$, $i = 1, 2$, we say $\phi_1$ is 1-homotopic to $\phi_2$ in $(C^G(M), \{0\})$ with fineness $\delta$, if we can find a map

$$\psi : I(1, k_3)_0 \times I(1, k_3)_0 \to C^G(M),$$

for some $k_3 \geq \max\{k_1, k_2\}$ such that

- $f(\psi) \leq \delta$;
- $\psi([i - 1], x) = \phi_i(n(k_3, k_i)(x)), i = 1, 2$, for all $x \in I(1, k_3)_0$;
- $\partial \psi(y) = 0$, for all $y \in I(1, k_3)_0 \times I_0(1, k_3)_0$.

**Remark 3.2** Note that the first and the second conditions imply that $f(\phi_i) \leq \delta, i = 1, 2$.

**Definition 3.3** A $(1, M)$-homotopy sequence of mappings into $(C^G(M), \{0\})$ is a sequence of mappings $\{\phi_i\}_{i \in \mathbb{N}}$,

$$\phi_i : I(1, k_i)_0 \to (C^G(M), \{0\}),$$

such that $\phi_i$ is 1-homotopic to $\phi_{i+1}$ in $(C^G(M), \{0\})$ with fineness $\delta_i > 0$, and

- $\lim_{i \to \infty} \delta_i = 0$;
- $\sup_i \{M(\partial \phi_i(x)) : x \in I(1, k_i)_0\} < +\infty$.

**Remark 3.4** Note that the second condition implies that $\sup_i \{A^c(\phi_i(x)) : x \in I(1, k_i)_0\} < +\infty$.

**Definition 3.5** Given two $(1, M)$-homotopy sequences of mappings $S_1 = \{\phi^1_i\}_{i \in \mathbb{N}}$ and $S_2 = \{\phi^2_i\}_{i \in \mathbb{N}}$ into $(C^G(M), \{0\})$, $S_1$ is $G$-homotopic to $S_2$ if there exists $\{\delta_i > 0\}_{i \in \mathbb{N}}$, such that

- $\phi^1_i$ is 1-homotopic to $\phi^2_i$ in $(C^G(M), \{0\})$ with fineness $\delta_i$;
- $\lim_{i \to \infty} \delta_i = 0$.

It is easy to see that the $G$-homotopic relation is an equivalence relation on the space of $(1, M)$-homotopy sequences of mappings into $(C^G(M), \{0\})$. An equivalence class is named as a $(1, M)$-homotopy class of mappings into $(C^G(M), \{0\})$. The set of all such equivalence classes is denoted by $\pi^G_1(C^G(M, M), \{0\})$.

### 3.3 min–max construction

**Definition 3.6** (min–max definition) Given $\Pi \in \pi^G_1(C^G(M, M), \{0\})$, define $L^c : \Pi \to \mathbb{R}^+$ as a function given by:

$$L^c(S) = L^c(\{\phi_i\}_{i \in \mathbb{N}}) = \lim_{i \to \infty} \sup_{i \in \mathbb{N}} \{A^c(\phi_i(x)) : x \text{ lies in the domain of } \phi_i\}.$$
The \( A^c \)-min–max value of \( \Pi \) is defined as

\[
L^c(\Pi) = \inf \{ L^c(S) : S \in \Pi \}. \tag{3.1}
\]

A sequence \( S = \{\phi_i\}_{i \in \mathbb{N}} \in \Pi \) is called a critical sequence if \( L^c(S) = L^c(\Pi) \). The image set of \( S \) is the compact subset \( K(S) \subset \mathcal{V}_n^G(M^{n+1}) \) given by

\[
K(S) = \{ V = \lim_{j \to \infty} |\partial \phi_i(x_j)| : x_j \text{lies in the domain of } \phi_i \}. 
\]

The critical set of \( S \) is the subset \( C(S) \subset K(S) \subset \mathcal{V}_n^G(M) \) defined by

\[
C(S) = \{ V = \lim_{j \to \infty} |\partial \phi_i(x_j)| : \lim_{j \to \infty} A^c(\phi_i(x_j)) = L^c(S) \}. \tag{3.2}
\]

By a diagonal sequence argument as in [20, 4.1(4)], we immediately have:

**Lemma 3.7** Given any \( \Pi \in \pi^1 (C^G(M, M), \{0\}) \), there exists a critical sequence \( S \in \Pi \).

### 3.4 Existence of nontrivial G-sweepouts

**Theorem 3.8** There exists \( \Pi \in \pi^1 (C^G(M, M), \{0\}) \), such that for any \( c > 0 \), we have \( L^c(\Pi) > 0 \).

**Proof** Firstly, we can take a \( G \)-equivariant Morse function \( \phi : M \to [0, 1] \), and consider the sub-level sets \( \Phi : [0, 1] \to C^G(M) \), given by \( \Phi(t) = \{ x \in M : \phi(x) < t \} \) just like in [16, Lemma 2.1] and [31, Theorem 3.9]. Then the map \( \partial \Phi : [0, 1] \to (\mathcal{Z}_n^G(M; \mathbb{Z}_2), \mathcal{F}) \) is continuous and \( \mathcal{M}(\partial \Phi(t)) \) is continuous in \( t \in [0, 1] \). It follows that the map \( \partial \Phi : [0, 1] \to (\mathcal{Z}_n^G(M; \mathbb{Z}_2), \mathcal{F}) \) is continuous. By [29, Lemma 4.2], \( \partial \Phi \) has no concentration of mass on orbits.

Secondly, using Propositions 2.1, 2.2, 2.3, 2.5 and 2.6, the discretization theorem given by Xin Zhou [30, Theorem 5.1] can be adapted to a \( G \)-invariant version. In fact, we only need to consider the case 1 of [30, Lemma 5.5] just like in [17, Section 13], under the condition that \( \partial \Phi \) has no concentration of mass on orbits. We can replace \( B_r(p) \) with \( B_r^G(p) \) in [30, Lemma 5.6], then [30, Lemma 5.5] and [30, Proposition 5.3] follow directly from [30, Lemma 5.6]. Therefore, by the \( G \)-invariant discretization/interpolation theorem, \( \Phi \) can be discretized to a \((1, M)\)-homotopy sequences \( S = \{\phi_i\}_{i \in \mathbb{N}} \) where \( \phi_i : I(1, k_i)_{0} \to (C^G(M), \{0\}) \).

Finally, consider \( \Pi = \{S\} \), then the proof for \( \Pi \in \pi^1 (C^G(M, M), \{0\}) \) with \( L^c(\Pi) > 0 \) proceeds as in the CMC setting [31, Theorem 3.9]. \( \square \)

### 4 Tightening

In this section, we recall the tightening map in [31, Section 4] adapted for the \( A^c \)-functional. After generalizing the tightening procedure to the equivariant case, we make every element in the critical set has \( c \)-bounded first variation.

#### 4.1 Review of constructions in [31, §4]

We first recall several key ingredients obtained in [31, §4] for the tightening process. Given \( L > 0 \) and \( c > 0 \), we use the following notations:

- \( A^L = \{ V \in \mathcal{V}_n^G(M) : \|V\|_1(M) \leq 2L \} \);
• $A^c_\infty = \{ V \in A^L : |\delta V(X)| \leq c \int_M |X| d\mu_X, \forall X \in \mathcal{X}^G(M) \};$
• $A_j = \{ V \in A^L : \frac{1}{2^j} \leq F(V, A^c_\infty) \leq \frac{1}{2^{j-1}} \}, j \in \mathbb{N};$
• $\gamma(V) = F(V, A^c_\infty)$, for any $V \in A^L$;
• $\Phi_X : \mathbb{R} \times M \to M$ denotes the one parameter group of $G$-equivariant diffeomorphisms generated by $X \in \mathcal{X}^G(M)$.

Clearly, for any $V \in A^c_\infty$, $V$ has $(G, c)$-bounded first variation. By Proposition 2.2 and the fact that $(G, c)$-bounded first variation is a closed condition, we have $A^L$, $A_j$, $A^c_\infty$ are compact subsets of $\mathcal{V}_G^G(M)$ under the varifold $F$-metric for all $j \in \mathbb{N}$.

We need the following lemma which can be used to construct a continuous map from $A^L$ to $\mathcal{X}^G(M)$.

**Lemma 4.1** For any $V \in A_j$, there exists $X_V \in \mathcal{X}^G(M)$, such that

$$
\|X_V\|_{C^1(M)} \leq 1, \quad \delta V(X_V) + c \int_M |X_V| d\mu_X \leq -c_j < 0,
$$

(4.1)

where $c_j > 0$ depends only on $j$.

**Proof** This follows from a contradiction argument by the compactness of $A_j$. \qed

Therefore, we have the following lemma, which is a straightforward consequence of Lemma 4.1 and the construction in [31, §4.2].

**Lemma 4.2** There exists a continuous map $X : A^L \to \mathcal{X}^G(M)$ with respect to the $C^1$ topology on $\mathcal{X}^G(M)$, so that $X(V)$ satisfies (4.1) for all $V \in A^L$.

Our goal is to show that given $\Omega \in \mathcal{C}^G(M)$ with $|\partial \Omega| \in A^L \setminus A^c_\infty$, $A^c(\Omega)$ can be deformed by a fixed amount depending only on $\gamma(|\partial \Omega|) = F(|\partial \Omega|, A^c_\infty)$. The following results can be obtained as in [31, §4.3]:

**Proposition 4.3** ($G$-equivariant deformations)

1. There are two continuous functions $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\rho(0) = 0$ and

$$
\delta W(X(V)) + c \int_M |X(V)| d\mu_W \leq -g(\gamma(V)),
$$

(4.2)

if $W \in A^L$ and $F(W, V) \leq \rho(\gamma(V))$;

2. There exists a continuous time function $T : [0, +\infty) \to [0, +\infty)$, such that

• $\lim_{t \to 0} T(t) = 0$, and $T(t) > 0$, if $t \neq 0$;
• for any $V \in A^L$ and $0 \leq t \leq T(\gamma(V))$, we have

$$
F((\Phi_V(t) # V, V) \leq \rho(\gamma(V)),
$$

(4.3)

where $\Phi_V = \Phi_{X(V)}$;

3. For any $V \in A^L \setminus A^c_\infty$, define $\Psi_V(t) = \Phi_V(T(\gamma(V))t, \cdot)$ and $V_t = (\Psi_V(t) # V$. Then we have

• $F(V_t, V) \leq \rho(\gamma(V))$ for all $t \in [0, 1]$;
• the map $(t, V) \to V_t$ is continuous under the $F$-metric;
• the flow $\Psi_V(t) = \Phi_V(T(\gamma(V))t, \cdot)$ is generated by the vector field

$$
\tilde{X}(V) = T(\gamma(V))X(V).
$$

(4.4)
When $\Omega \in C^G(M)$ satisfies $|\partial \Omega| \in A^L \setminus A_{\infty}$, Proposition 4.3(1) implies
\[
\delta V(X(|\partial \Omega|)) + c \int_M |X(|\partial \Omega|)|d\mu_V \leq -g(\gamma(|\partial \Omega|)),
\]
for all $V \in A^L$ with $F(V, |\partial \Omega|) \leq \rho(\gamma(|\partial \Omega|))$. Noting $\tilde{X}(\gamma(|\partial \Omega|)) \in \mathcal{X}^G(M)$, we can deform $\Omega$ by $\Psi_{|\partial \Omega|}(t)$ to get a $1$-parameter family $\gamma_t = (\Psi_{|\partial \Omega|}(t))_{#}(\Omega) \in C^G(M)$. Hence, by the first variation formula (2.2) and Proposition 4.3,
\[
A^c(\Omega_t) - A^c(\Omega) \leq \int_0^T (\gamma(|\partial \Omega|)) [\delta A^c|_{\Omega_t}](X(|\partial \Omega|))dt
\leq -T(\gamma(|\partial \Omega|))g(\gamma(|\partial \Omega|)) = -L(\gamma(|\partial \Omega|)) < 0,
\]
where $L(\gamma(V)) = T(\gamma(V))g(\gamma(V))$ satisfies $L(0) = 0$ and $L(\gamma(V)) > 0$ whenever $\gamma(V) > 0$.

4.2 Deforming sweepouts by the tightening map

We now apply our tightening map to the critical sequence provided by Lemma 3.7. As in the usual min–max theory for the CMC setting, we confirm the existence of a critical sequence $S$ where each varifold in $C(S)$ has $c$-bounded first variation. Indeed, the proof proceeds essentially unchanged with $X \in \mathcal{X}^G(M)$ in replace of $X \in \mathcal{X}(M)$.

**Proposition 4.4** (Tightening) Let $\Pi \in \pi^\#_1(C^G(M, M), \{0\})$ with $L^c(\Pi) > 0$. For any critical sequence $S^*$ for $\Pi$, there exists another critical sequence $S$ for $\Pi$ such that $C(S) \subset C(S^*)$, and each $V \in C(S)$ has $c$-bounded first variation.

**Proof** Take $S^* = \{\varphi_i^*\}_{i \in \mathbb{N}}$, where $\varphi_i^* : I(1, k_i)_0 \rightarrow (C^G(M), \{0\})$, and $\varphi_i^*$ is $(1, M)$-homotopic to $\varphi_{i+1}^*$ in $(C^G(M), \{0\})$ with fineness $\delta_i \searrow 0$. Let $\Xi_i : I(1, k_i)_0 \times [0, 1] \rightarrow C^G(M)$ be defined as
\[
\Xi_i(x, t) = \Psi_{|\varphi_i^*(x)|}(t)(\varphi_i^*(x)),
\]
where $\Psi_{|\varphi_i^*(x)|}$ is the tightening map defined in Proposition 4.3.

Denote $\tilde{\varphi_i^*}(x) = \Xi_i(x, t)$. The idea is to extend $\varphi_i^*(x)$ to an $F$-continuous map $\tilde{Q}_i(x)$ defining on $[0, 1]$, and apply the discretization-interpolation theorem [30, Theorem 5.1] to $\tilde{Q}_i(x)$. To be exact, let $\tilde{X}_i(x) = (1 - x)\tilde{X}_i(0) + x\tilde{X}_i(1)$, and $\tilde{Q}_i(x) = (\Phi_{\tilde{X}_i(x)}(1))_{#}\varphi_i^*(0)$, where $\tilde{X}_i(0) = \tilde{X}(\mid \varphi_i^*(0))$ and $\tilde{X}_i(1) = \tilde{X}(\mid \varphi_i^*(1))$ are defined in Proposition 4.3(4.4). Note $\tilde{X}_i(x) \in \mathcal{X}^G(M)$, for each $x \in [0, 1]$. By Proposition 2.4, Lemma 4.2 and Proposition 4.3, both $\tilde{Q}_i(x) : [0, 1] \rightarrow (C^G(M, M)$ and $\partial \tilde{Q}_i(x) : [0, 1] \rightarrow (\mathcal{Z}^G(M; \mathbb{Z}_2), F)$ are continuous maps. Thus we can apply the $G$-invariant discretization/interpolation theorem [30,Theorem 5.1] just as in the proof of Theorem 3.8 to get the desired $(1, M)$-homotopy sequence $S = \{\tilde{Q}_i\}_{i \in \mathbb{N}}$, which is $G$-homotopic to $S^* = \{\varphi_i^*\}_{i \in \mathbb{N}}$. The proof proceeds essentially unchanged as in [31, Proposition 4.4, Appendix B].

5. $(G, c)$-almost minimizing varifolds

In this section, we introduce the notion of $(G, c)$-almost minimizing varifolds, and show the existence of such varifolds from min–max constructions.
Definition 5.1 ((G,c)-almost minimizing varifolds) Let $v$ be the $\mathcal{F}$-norm, or the $\mathbf{M}$-norm, or the $\mathbf{F}$-metric. For any given $\epsilon, \delta > 0$ and an open $G$-set $U \subset M$, we define $\alpha^{G,\epsilon,\delta}(U; \epsilon, \delta; v)$ to be the set of all $\Omega \in \mathcal{C}^{G}(M)$ such that if $\Omega = \Omega_{0}, \Omega_{1}, \Omega_{2}, \ldots, \Omega_{m} \in \mathcal{C}^{G}(M)$ is a sequence with:

(i) $\operatorname{spt}(\Omega_{i} - \Omega) \subset U$,
(ii) $v(\partial \Omega_{i+1} - \partial \Omega_{i}) \leq \delta$,
(iii) $\mathcal{A}^{c}(\Omega_{i}) \leq \mathcal{A}^{c}(\Omega) + \delta$, for $i = 1, \ldots, m$,

then $\mathcal{A}^{c}(\Omega_{m}) \geq \mathcal{A}^{c}(\Omega) - \epsilon$.

A varifold $V \in \mathcal{V}_{n}^{G}(M)$ is said to be $(G, c)$-almost minimizing in $U$ if there exist sequences $\epsilon_{i} \to 0, \delta_{i} \to 0, \Omega_{i} \in \alpha^{G,\epsilon,\delta}(U; \epsilon_{i}, \delta_{i}; \mathcal{F})$, such that $\mathbf{F}(|\partial \Omega_{i}|, V) \leq \epsilon_{i}$.

A simple consequence of the definition is the following lemma.

Lemma 5.2 Let $V \in \mathcal{V}_{n}^{G}(M)$ be $(G, c)$-almost minimizing in $U$, then $V$ has $c$-bounded first variation in $U$.

Proof This lemma follows similarly to [31, Lemma 5.2]. We include the details for completeness. By Lemma 2.9, $V$ has $c$-bounded first variation in $U$ if and only if $V$ has $(G, c)$-bounded first variation in $U$. Suppose by contradiction that $V$ does not have $(G, c)$-bounded first variation in $U$, then there exist $\epsilon_{0} > 0$ and a smooth vector field $X \in \mathcal{X}^{G}(U)$, such that

$$
\int_{G_{n}(M)} \operatorname{div} S X(x) dV(x, S) \leq -(c + \epsilon_{0}) \int_{M} |X| d\mu_{V}.
$$

By the continuity and the first variation formula (2.2) for $\mathcal{A}^{c}$, we can find $\epsilon_{1} > 0$ small enough depending only on $\epsilon_{0}, V,$ and $X$, such that if $\Omega \in \mathcal{C}^{G}(M)$ with $\mathbf{F}(|\partial \Omega|, V) < 2\epsilon_{1}$, then

$$
\delta \mathcal{A}^{c}|_{\Omega}(X) \leq \int_{\partial \Omega} \operatorname{div} S X d\mu_{\partial \Omega} + c \int_{\partial \Omega} |X| d\mu_{\partial \Omega} \leq -\frac{\epsilon_{0}}{2} \int_{M} |X| d\mu_{V} < 0.
$$

If $\mathbf{F}(|\partial \Omega|, V) < \epsilon_{1}$, we then consider the $G$-equivariant deformation of $\Omega$ along the flow $\{\Phi_{t}(X) : t \in [0, T]\}$ generated by the $G$-vector field $X$ for a uniform short time $T > 0$. Just as in Proposition 4.3, we can get a 1-parameter family $\{\Omega_{t} \in \mathcal{C}^{G}(M) : t \in [0, T]\}$, such that $t \mapsto \partial \Omega_{t}$ is continuous under the $\mathbf{F}$-metric, with $\operatorname{spt}(\Omega_{t} - \Omega) \subset U$, $\mathbf{F}(|\partial \Omega_{t}|, V) < 2\epsilon_{1}$, and $\mathcal{A}^{c}(\Omega_{t}) \leq \mathcal{A}^{c}(\Omega_{0}) = \mathcal{A}^{c}(\Omega)$ for $t \in [0, T]$. Additionally, $\mathcal{A}^{c}(\Omega_{T}) \leq \mathcal{A}^{c}(\Omega) - \epsilon_{2}$ for some $\epsilon_{2} > 0$ depending only on $\epsilon_{0}, \epsilon_{1}, V,$ and $X$.

This implies that if $\Omega \in \mathcal{C}^{G}(M)$ and $\mathbf{F}(|\partial \Omega|, V) < \epsilon = \frac{1}{2} \min\{\epsilon_{1}, \epsilon_{2}\}$, then $\Omega \notin \alpha^{G,\epsilon,\delta}(U; \epsilon, \delta; \mathcal{F})$ for any $\delta > 0$, which contradicts to the assumption that $V \in \mathcal{V}_{n}^{G}(M)$ is $(G, c)$-almost minimizing in $U$.

Definition 5.3 A varifold $V \in \mathcal{V}_{n}^{G}(M)$ is said to be $(G, c)$-almost minimizing in small annuli, if for each $p \in M$, there exists $r = r(G \cdot p) > 0$ such that $V$ is $(G, c)$-almost minimizing in $\mathcal{A}^{G}(p, s, t) \cap M$ for any $\mathcal{A}^{G}(p, s, t) \in \mathcal{A}^{G}(r)(p)$.

Note $f(\phi)$ the fineness of discrete homotopy sequence of mappings is defined under the $\mathbf{M}$-norm, while the $(G, c)$-almost minimizing varifolds are defined under the $\mathcal{F}$-metric. We need the following equivalence results of $(G, c)$-almost minimizing concepts using different topologies.

Theorem 5.4 Let $V \in \mathcal{V}_{n}^{G}(M)$. The following statements satisfy $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$:
(a) $V$ is $(G, c)$-almost minimizing in $U$;
(b) for any $\epsilon > 0$, there exists $\delta > 0$ and $\Omega \in \mathcal{A}_n^{c, G}(U; \epsilon, \delta; F)$ such that $F(V, |\partial \Omega|) < \epsilon$;
(c) for any $\epsilon > 0$, there exists $\delta > 0$ and $\Omega \in \mathcal{A}_n^{c, G}(U; \epsilon, \delta; M)$ such that $F(V, |\partial \Omega|) < \epsilon$;
(d) $V$ is $(G, c)$-almost minimizing in $W$ for any relative open $G$-subset $W \subset U$ with $\text{Clos}(W) \subset M^{\text{reg}}$, where $M^{\text{reg}}$ is the union of all principal orbits.

**Proof** It’s clear that $(a) \Rightarrow (b) \Rightarrow (c)$ by definitions.

To show $(c) \Rightarrow (d)$, we first need an equivariant version of boundary type interpolation lemma (i.e. using boundaries of $G$-invariant Caccioppoli sets, rather than integral cycles $Z_\epsilon(M; \mathbb{Z}_2)$, as interpolating sequences in [20, Lemma 3.8]). In fact, [29, Appendix B] has shown how to get interpolating sequences consisting integral $G$-cycles $T \in Z_n^G(M; \mathbb{Z}_2)$. In the arguments of [29, Appendix B], the assumption that every orbit in $\text{Clos}(W)$ is principal plays a necessary role in getting a positive lower bound of the injective radius of orbits in $W$ and the Weyl’s tube formula [29,(18)]. Additionally, [30, Lemma 5.5] has essentially the same idea as [20, Lemma 3.7, 3.8] except using Caccioppoli sets, rather than integral cycles $Z_\epsilon(M; \mathbb{Z}_2)$, as interpolating sequences. Thus, we can combine the arguments in [29, Appendix B] and [30, Section 5.1] to deduce an equivariant version of [30, Proposition 5.3]. With the equivariant version of [30, Proposition 5.3] ($G$-boundary type interpolation lemma), the proof of [31, Lemma A.1] would carry over for $G$-invariant Caccioppoli sets. □

**Corollary 5.5** Let $V \in \mathcal{V}_n^G(M)$. Suppose every non-principal orbit is isolated. Then $V$ is $(G, c)$-almost minimizing in small annuli if and only if for any $p \in M$ there exists $r = r(G \cdot p) > 0$ so that for any $0 < s < t < r$ and $\epsilon > 0$, there exists $\delta > 0$ and $T \in \mathcal{A}_n^{c, G}\left(\text{An}^G(p, s, t) \cap M; \epsilon, \delta; M\right)$ with $F(V, |T|) < \epsilon$.

**Proof** The necessity part comes from Theorem 5.4 (a)⇒(c). Since every non-principal orbit is isolated and $M^{\text{reg}} \subset M$ is open, there exists a $G$-neighborhood $B^G_r(p)$ of $G \cdot p$ for all $p \in M$, so that $(B^G_r(p) \setminus G \cdot p) \cap M \subset M^{\text{reg}}$. Hence, there exists $r = r(G \cdot p) > 0$ for every $p \in M$ satisfying $\text{Clos}(\text{An}^G(p, s, t) \cap M) \subset M^{\text{reg}}, 0 < s < t < r$. Therefore, the sufficiency comes from Theorem 5.4 (c)⇒(d). □

To weaken the constraints on non-principal orbits, we take the following notations and definitions as in [29, Section 5]. Firstly, we denote

$$P_p := \begin{cases} p & \text{if } p \in M^{\text{reg}} \\ M_p & \text{if } p \in M \setminus M^{\text{reg}}, \end{cases}$$

where $M_p$ is the connected component of $M \setminus M^{\text{reg}}$ containing $p$. Hence, for any $p \in M$, there exists $r_1 = r_1(G \cdot P_p) > 0$ so that

$$\text{Clos}\left(\text{An}^G(P_p, s, t) \cap M\right) \subset M^{\text{reg}}, \forall \text{An}^G(P_p, s, t) \in \mathcal{A}_n^{G}_{r_1}(P_p),$$

by regarding $M \setminus M^{\text{reg}}$ as a whole. We call such $G$-annulus $\text{An}^G(P_p, s, t)$ a regular annulus since its closure is contained in $M^{\text{reg}}$. It is obvious that $G \cdot P_p = G \cdot p$ whenever $G \cdot p$ is a principal orbit or an isolated non-principal orbit.

**Definition 5.6** A $G$-varifold $V \in \mathcal{V}_n^G(M)$ is said to be $(G, c)$-almost minimizing in regular annuli if for each $p \in M$, there exists $r = r(G \cdot P_p) \in (0, r_1)$ such that $V$ is $(G, c)$-almost minimizing in $\text{An}^G(P_p, s, t) \cap M$ for any $\text{An}^G(P_p, s, t) \in \mathcal{A}_n^{G}_{r}(P_p)$, where $r_1$ is given by (5.2).

Combining (5.2) with Theorem 5.4, it’s easy to get the following corollary:
Proof The necessity part comes from Theorem 5.4 (a)⇒(c). As for the sufficiency part, one notices that Clos \((\text{An}^G(P_p, s, t) \cap M; \epsilon, \delta; M)\) with \(F(V, |T|) < \epsilon\).

Theorem 5.8 (Existence of \((G, c)\)-almost minimizing varifolds) Let \(\Pi \in \pi^\#_i((C^G(M, M), \{0\}) with \(L^c(\Pi) > 0\). There exists a nontrivial \(G\)-invariant varifold \(V \in \mathcal{V}_n^G(M)\) called \((G, c)\)-min–max varifold, such that

(i) \(V \in C(S)\) for some critical sequence \(S\) of \(\Pi\);

(ii) \(V\) has \(c\)-bounded first variation in \(M\);

(iii) \(V\) is \((G, c)\)-almost minimizing in regular annuli.

Proof It is easy to see that (i), (ii) come from Theorem 3.8 and Proposition 4.4. Moreover, the proof of [31, Theorem 5.6(iii)] would carry over under \(G\)-invariant restrictions with Theorem 5.4 and [29, Theorem 5.9] in places of [31, Proposition 5.3] and [20, Theorem 4.10].

6 Regularity for \((G, c)\)-min–max varifold

In this section, we show the regularity result for \((G, c)\)-min–max varifolds. First of all, let’s begin with the existence of \((G, c)\)-replacements for \((G, c)\)-almost minimizing varifolds.

6.1 Good \((G, c)\)-replacement property

Lemma 6.1 (A constrained minimization problem-I) Given \(\epsilon, \delta > 0\), an open \(G\)-set \(U \subset M\), and any \(\Omega \in a_n^G(U; \epsilon, \delta; \mathcal{F})\), fix a compact \(G\)-subset \(K \subset U\). Let \(C^G_\Omega\) be the set of all \(\Lambda \in C^G(M)\) such that there exists a sequence \(\Omega^* = \Omega_0, \Omega_1, \ldots, \Omega_m = \Lambda\) in \(C^G(M)\) satisfying:

(a) \(\text{spt}(\Omega_i - \Omega) \subset K\);

(b) \(\mathcal{F}(\delta \Omega_i - \delta \Omega_{i+1}) \leq \delta\);

(c) \(\mathcal{A}^c(\Omega_i) \leq \mathcal{A}^c(\Omega) + \delta\), for \(i = 1, \ldots, m\).

Then there exists \(\Omega^* \in C^G(M)\) such that:

(i) \(\Omega^* \in C^G_\Omega\), and

\(\mathcal{A}^c(\Omega^*) = \inf\{\mathcal{A}^c(\Lambda) : \Lambda \in C^G_\Omega\};\)

(ii) \(\Omega^*\) is locally \((G, \mathcal{A}^c)\)-minimizing in \(\text{int}(K)\);

(iii) \(\Omega^* \in a_n^G(U; \epsilon, \delta; \mathcal{F})\).

Here we say \(\Omega^*\) is locally \((G, \mathcal{A}^c)\)-minimizing in \(\text{int}(K)\) if for any \(p \in \text{int}(K)\), there exists \(\rho > 0\) so that

\(\mathcal{A}^c(\Omega^*) \leq \mathcal{A}^c(\Lambda),\)

for all \(\Lambda \in C^G(M)\) with \(\text{spt}(\Lambda - \Omega^*) \subset \tilde{B}^G_f(p) \subset \text{int}(K)\).
Proof Proof of (i): Let \( \{ \Lambda_j \} \subset C_G^2 \) be a minimizing sequence such that
\[
\lim_{j \to \infty} A^c(\Lambda_j) = \inf \{ A^c(\Lambda) : \Lambda \in C_G^2 \}.
\]
Since \( \text{spt}(\Omega - \Lambda_j) \subset K \) and \( A^c(\Lambda_j) \leq A^c(\Omega) + \delta \), we can apply the Compactness Theorem [23, Theorem 6.3] and Proposition 2.1 to get a subsequence \( \partial \Lambda_j \) (without changing notations) converges weakly to \( \partial \Omega^* \) for some \( \Omega^* \in C_G^2(M) \) with \( \text{spt}(\Omega^* - \Omega) \subset K \). Hence, by the weakly convergence, we have
\[
\bullet \ M(\partial \Omega^*) \leq \lim_{j \to \infty} M(\partial \Lambda_j);
\]
\[
\bullet \ cH^{n+1}(\partial \Omega^*) = \lim_{j \to \infty} cH^{n+1}(\partial \Lambda_j).
\]
This implies that \( A^c(\Omega^*) \leq \inf \{ A^c(\Lambda) : \Lambda \in C_G^2 \} \leq A^c(\Omega) + \delta \). Moreover, for \( j \) large enough, we have \( F(\partial \Lambda_j - \partial \Omega^*) < \delta \). Consider the sequence which makes \( \Lambda_j \) in \( C_G^2 \) and add one more element \( \Omega^* \) into its end. It’s clear that the new sequence satisfies all requirements in the definition of \( C_G^2 \). Therefore, \( \Omega^* \in C_G^2 \) and \( A^c(\Omega^*) = \inf \{ A^c(\Lambda) : \Lambda \in C_G^2 \} \).

Proof of (ii): For any \( p \in \text{int}(K) \), we first choose \( \rho > 0 \) small enough such that \( \tilde{B}^G_\rho(p) \subset \text{int}(K) \). Since \( |\partial \Omega^*| \) is rectifiable, one can take \( \rho \) even smaller so that
\[
\bullet \ cH^{n+1}(\tilde{B}^G_\rho(p)) \leq \delta/4;
\]
\[
\bullet \ ||\partial \Omega^*||(\tilde{B}^G_\rho(p)) \leq \delta/4.
\]
Suppose \( \Lambda \in C_G^2(M) \), \( \text{spt}(\Lambda - \Omega^*) \subset \tilde{B}^G_\rho(p) \), and \( A^c(\Omega^*) > A^c(\Lambda) \). Then
\[
||\partial \Lambda||(\tilde{B}^G_\rho(p)) < ||\partial \Omega^*||(\tilde{B}^G_\rho(p)) - cH^{n+1}(\Omega^*) + cH^{n+1}(\Lambda)
\leq ||\partial \Omega^*||(\tilde{B}^G_\rho(p)) + 2cH^{n+1}(\tilde{B}^G_\rho(p))
\leq 3\delta/4.
\]
This implies that
\[
F(\partial \Lambda - \partial \Omega^*) \leq M(\partial \Lambda - \partial \Omega^*)
= ||\partial \Lambda - \partial \Omega^*||(\tilde{B}^G_\rho(p))
\leq (||\partial \Lambda|| + ||\partial \Omega^*||)(\tilde{B}^G_\rho(p)) < \delta.
\]

Similarly, we can add \( \Lambda \) into the end of the finite sequence making \( \Omega^* \) in \( C_G^2 \). Hence, \( \Lambda \in C_G^2 \) and \( A^c(\Omega^*) > A^c(\Lambda) \), which is a contradiction to (i).

Proof of (iii): Suppose that the claim is false. Then there is a sequence \( \Omega^* = \Omega^*_0, \Omega^*_1, \ldots, \Omega^*_q \) satisfies the conditions of Definition 5.1 (i)(ii)(iii), but \( A^c(\Omega^*_q) < A^c(\Omega^*) - \epsilon \). Since \( \Omega^* \in C_G^2 \), there is a sequence \( \Omega = \Omega_0, \Omega_1, \ldots, \Omega_m = \Omega^* \) in \( C_G^2(M) \), which satisfies the conditions (a)(b)(c). By Definition 5.1, we can construct a new \( G \)-invariant \( (\epsilon, \delta) \)-deformation \{\( \Omega_0, \ldots, \Omega_m, \Omega^*_0, \ldots, \Omega^*_q \)\} of \( \Omega = \Omega_0 \). Therefore, we have \( A^c(\Omega^*_q) < A^c(\Omega^*) - \epsilon \leq A^c(\Omega) - \epsilon \). This contradicts to the choice of \( \Omega \in \partial c_G^2(U; \epsilon, \delta, F) \).

Note \( \Omega^* \) in the above lemma can only minimize the \( A^c \)-function under locally \( G \)-invariant variations. Nevertheless, by an averaging procedure, we get the following lemma, which indicates that \( \Omega^* \) is locally \( A^c \)-minimizing.

Lemma 6.2 (A constrained minimization problem-II) The \( G \)-set \( \Omega^* \in C_G^2(M) \) obtained in Lemma 6.1 is locally \( A^c \)-minimizing in \( \text{int}(K) \).

Proof For any \( p \in \text{int}(K) \), let \( \rho > 0 \) such that \( \tilde{B}^G_\rho(p) \subset \text{int}(K) \) and
\begin{itemize}
  \item $c\mathcal{H}^{n+1}(\widetilde{B}^{G}_{\rho}(p)) \leq \delta/4$,
  \item $||\partial \Omega^*||(\widetilde{B}^{G}_{\rho}(p)) \leq \delta/4$.
\end{itemize}

Then $\Omega^*$ is $(G, A^c)$-minimizing in $\widetilde{B}^{G}_{\rho}(p)$ by the proof of Lemma 6.1. Let $B = \widetilde{B}^{G}_{\rho}(p)$ and suppose $\Lambda^* \in \mathcal{C}(M)$ satisfies $\text{spt}(\Omega^* - \Lambda^*) \subset B$. We now going to show that $A^c(\Lambda^*) \leq A^c(\Omega^*)$.

Define a $G$-invariant function $f$ on $M$ as:

$$f(x) = \int_{G} 1_{\text{Clos}(\Lambda^*)}(g \cdot x) \, d\mu(g).$$

Since $\text{Clos}(\Lambda^*)$ is closed, function $1_{\text{Clos}(\Lambda^*)}$ is upper-semicontinuous. By Fatou’s Lemma, $f$ is also upper-semicontinuous. Hence,

$$\Lambda^* = f^{-1}[\lambda, 1] = M \setminus f^{-1}[0, \lambda)$$

are $G$-invariant closed sets in $M$.

Define then $E_f = f \cdot [M]$, where $[M]$ is the integral current induced by $M$ with $\mathbb{Z}_2$-coefficient. For any $n$-form $\omega$ on $M$, we have

$$\partial E_f(\omega) = \int_{M} \int_{G} \langle d\omega, \xi \rangle 1_{\text{Clos}(\Lambda^*)}(g \cdot x) \, d\mu(g) \, d\mathcal{H}^{n+1}(x)$$

(6.1)

$$= \int_{G} \int_{M} \langle d\omega, \xi \rangle 1_{g^{-1}(\text{Clos}(\Lambda^*))}(x) \, d\mathcal{H}^{n+1}(x) \, d\mu(g)$$

$$= \int_{G} \partial((g^{-1})_\# \Lambda^*)(\omega) \, d\mu(g).$$

Hence, using the lower semi-continuity of mass and the fact that $G$ acts as isometries on $(M, g_M)$ (mass is invariant under $g_\#$), we have

$$\mathbf{M}(\partial E_f) \leq \int_{G} \mathbf{M}(\partial((g^{-1})_\# \Lambda^*)) \, d\mu(g) = \mathbf{M}(\partial \Lambda^*).$$

(6.2)

Combining this with $\mathbf{M}(E_f) \leq \mathbf{M}([M])$, it’s clear that $E_f$ is a normal current. By [10, 4.5.9(12)], $\partial(f^{-1}[\lambda, 1])$ is rectifiable for almost all $\lambda \in [0, 1]$, which implies $\Lambda^*_\lambda \in C^G(M)$ for almost all $\lambda \in [0, 1]$. Therefore, by [10, 4.5.9(13)] and (6.2), we have

$$\int_{0}^{1} \mathbf{M}(\partial([\Lambda^*_\lambda])) \, d\lambda = \mathbf{M}(\partial E_f) \leq \mathbf{M}(\partial \Lambda^*).$$

(6.3)

Moreover,

$$\mathbf{M}(E_f) = \int_{M} f(x) \, d\mathcal{H}^{n+1}(x) = \int_{0}^{1} \mathcal{H}^{n+1}(f^{-1}[\lambda, 1]) \, d\lambda.$$

$$= \int_{0}^{1} \mathcal{H}^{n+1}(\Lambda^*_\lambda) \, d\lambda;$$

$$\mathbf{M}(E_f) = \int_{M} \int_{G} 1_{\text{Clos}(\Lambda^*)}(g \cdot x) \, d\mu(g) \, d\mathcal{H}^{n+1}(x)$$

$$= \int_{G} \int_{M} 1_{g^{-1}(\text{Clos}(\Lambda^*))}(x) \, d\mathcal{H}^{n+1}(x) \, d\mu(g)$$

$$= \int_{G} \mathcal{H}^{n+1}(g^{-1} \cdot \text{Clos}(\Lambda^*)) \, d\mu(g)$$
where the last equality used the fact that every $g \in G$ is an isometry. Thus,

$$\int_0^1 \mathcal{H}^{n+1}(\Lambda_\lambda) \, d\lambda = M(E_f) = \mathcal{H}^{n+1}(\Lambda^*) \tag{6.4}$$

Additionally, (6.3) together with (6.4) imply:

$$\int_0^1 A^c(\Lambda_\lambda) \, d\lambda = A^c(E_f) \leq A^c(\Lambda^*) \tag{6.5}$$

Since the closed set $\text{spt}(\Omega^* - \Lambda^*) \subset B$, there exists $0 < r < \rho$ such that $\Omega^* = \Lambda^*$ on $\tilde{B}^G_p(p) \setminus B^G_r(p)$. Thus for any $\lambda \in (0, 1)$, we have $f = 1_{\text{Clos}(\Lambda^* \lambda)} = 1_{\Lambda_\lambda}$ on $\tilde{B}^G_p(p) \setminus B^G_r(p)$, and $\Omega^* = \Lambda^* = \lambda \lambda = E_f$ on $\tilde{B}^G_p(p) \setminus B^G_r(p)$, which implies $\text{spt}(\Omega^* - \Lambda_\lambda) \subset B$. Since $\Omega^*$ is $(G, A^c)$-minimizing in $\tilde{B}^G_p(p)$, we have

$$A^c(\Omega^*) \leq \int_0^1 A^c(\Lambda_\lambda) \, d\lambda \leq A^c(\Lambda^*)$$

Hence, $\Omega^*$ is $A^c$-minimizing in $\tilde{B}^G_p(p)$.

\textbf{Proposition 6.3} (Existence and properties of $(G, c)$-replacements) Let $V \in \mathcal{V}_n^G(M)$ be $(G, c)$-almost minimizing in an open $G$-set $U \subset M$, and $K \subset U$ be a compact $G$-subset, then there exists $V^* \in \mathcal{V}_n^G(M)$, called a $(G, c)$-replacement of $V$ in $K$, such that

\begin{enumerate}[leftmargin=*, label=(i), itemsep=0pt, parsep=0pt]
  \item $V \subset (M \setminus K) = V^* \subset (M \setminus K)$;
  \item $-c \text{Vol}(K) \leq \|V\|(M) - \|V^*\|(M) \leq c \text{Vol}(K)$;
  \item $V^*$ is $(G, c)$-almost minimizing in $U$;
  \item $V^* = \lim_{i \to \infty} \|\partial \Omega_i^*\|$ as varifolds for some $\Omega_i^* \in \mathcal{C}_G(M)$ such that $\Omega_i^* \in a_n^cG(U; \epsilon_i, \delta_i; F)$ with $\epsilon_i, \delta_i \to 0$, and $\Omega_i^*$ locally minimizes $A^c$ in $\text{int}(K)$;
  \item if $V$ has $c$-bounded first variation in $M$, then so does $V^*$.
\end{enumerate}

\textbf{Proof} Since $V \in \mathcal{V}_n^G(M)$ is $(G, c)$-almost minimizing in $U$, there exists a sequence $\Omega_i \in a_n^cG(U; \epsilon_i, \delta_i; F)$ with $\epsilon_i, \delta_i \to 0$ such that $V = \lim_{i \to \infty} \|\partial \Omega_i\|$. By Lemma 6.1, there is a $G$-invariant $A^c$-minimizer $\Omega_i^* \in \mathcal{C}_G$ for each $i$. Since $M(\partial \Omega_i^*)$ is uniformly bounded, by compactness theorem, $|\partial \Omega_i^*|$ converge as varifolds to some $V^* \in \mathcal{V}_n^G(M)$ after passing to a subsequence, i.e., $V^* = \lim_{i \to \infty} \|\partial \Omega_i^*\|$. We claim this $V^* = \lim_{i \to \infty} \|\partial \Omega_i^*\|$ is exactly what we need.

Indeed, since $\text{spt}(\Omega_i^* - \Omega_i) \subset K$, we have (i) $V \subset (M \setminus K) = V^* \subset (M \setminus K)$. Additionally, due to the fact that $\Omega_i \in a_n^cG(U; \epsilon_i, \delta_i; F)$, and $\Omega_i^*$ is an $A^c$-minimizer in $\mathcal{C}_G$, the following inequalities hold:

\begin{align*}
M(\partial \Omega_i) - \epsilon_i - c \text{Vol}(K) & \leq M(\partial \Omega_i) - \epsilon_i - c \mathcal{H}^{n+1}(\Omega_i) + c \mathcal{H}^{n+1}(\Omega_i^*) \\
& = A^c(\Omega_i) - \epsilon_i + c \mathcal{H}^{n+1}(\Omega_i^*) \\
& \leq A^c(\Omega_i^*) + c \mathcal{H}^{n+1}(\Omega_i^*) \\
& = M(\partial \Omega_i^*) \\
& \leq A^c(\Omega_i) + c \mathcal{H}^{n+1}(\Omega_i^*) \\
& = M(\partial \Omega_i) - c \mathcal{H}^{n+1}(\Omega_i) + c \mathcal{H}^{n+1}(\Omega_i^*) \\
& \leq M(\partial \Omega_i) + c \text{Vol}(K).
\end{align*}
Hence, (ii) \(-c \text{Vol}(K) \leq \|V\|(M) - \|V^*\|(M) \leq c \text{Vol}(K)\) is true. As for (iii) and (iv), they clearly follow from the facts that \(\Omega_i^q \in \alpha_n^G(U; \epsilon_i, \delta_i; \mathcal{F})\) and \(\Omega_i^q\) is locally \(\mathcal{A}^c\)-minimizing in \(\text{int}(K)\) by Lemmas 6.1 and 6.2. Finally by (iii) and Lemma 5.2, \(V^*\) has \(c\)-bounded first variation in \(U\). Combining (i) with a standard cutoff trick, it is easy to show that \(V^*\) has \(c\)-bounded first variation in \(M\) whenever \(V\) does. 

\[ \square \]

**Lemma 6.4** (Regularity of \((G, c)\)-replacements) Let \(2 \leq n \leq 6\). Under the same hypotheses as Proposition 6.3, if \(\Sigma = \text{spt} \|V^*\| \cap \text{int}(K)\), then

1. \(\Sigma\) is a smooth, \(G\)-equivariant almost embedded, stable \((G, c)\)-boundary;
2. the density of \(V^*\) is \(1\) along \(\mathcal{R}(\Sigma)\) and \(2\) along \(\mathcal{S}(\Sigma)\);
3. the restriction of the \((G, c)\)-replacement \(V^*\| \cap \text{int}(K) = \Sigma\).

**Proof** By Proposition 6.3(iv) and the regularity for local minimizers of the \(\mathcal{A}^c\) functional ([31, Theorem 2.14]), we know that each \(\partial \Omega_i^q\) is a smooth, embedded, \((G, c)\)-boundary in \(\text{int}(K)\). Moreover, if \(\partial \Omega_i^q\) is not \(G\)-stable in \(\text{int}(K)\), then there exists \(X \in \mathcal{X}^G(\text{int}(K))\) such that

\[ \mathcal{A}^c((\Phi_X(t))\#(\Omega_i^q)) < \mathcal{A}^c(\Omega_i^q), \quad t \in (0, \tau), \]

where \(\Phi_X(t)\) are \(G\)-equivariant diffeomorphisms generated by \(X\). For \(t\) small enough, we have \(\mathcal{F}(\partial((\Phi_X(t))\#(\Omega_i^q)), \partial \Omega_i^ q) < \delta_i\) implying \((\Phi_X(t))\#(\Omega_i^q)) \in C^G_{\Omega_i},\) which is a contradiction to the \(\mathcal{A}^c\)-minimizing property of \(\Omega_i^q\) in \(C^G_{\Omega_i}\). Thus, \(\partial \Omega_i^q\) is \(G\)-stable in \(\text{int}(K)\), and additionally stable in \(\text{int}(K)\) by Lemma 2.11. The lemma then follows from the Compactness Theorem 2.17. 

\[ \square \]

### 6.2 Tangent cones and blowups

By the main theorem of [18], there is an orthogonal representation of \(G\) on some Euclidean space \(\mathbb{R}^L\) and an isometric embedding from \(M\) into \(\mathbb{R}^L\) which is \(G\)-equivariant. Thus we can regard \(g \in G\) as an orthogonal \(L\)-matrix. Given any \(p \in M\) and \(r > 0\), let \(\eta_{p, r} : \mathbb{R}^L \to \mathbb{R}^L\) be the dilation defined by \(\eta_{p, r}(x) = \frac{x - p}{r}\). The following lemma shows the splitting property of blowups.

**Lemma 6.5** Let \(2 \leq n \leq 6\) and \(\mathcal{H}^{n-1}(M \setminus M^{\text{reg}}) = 0\). Suppose \(V \in \mathcal{V}^G_n(M)\) has \(c\)-bounded first variation in \(M\) and is \((G, c)\)-almost minimizing in regular annuli. Let \(\overline{V} = \lim_{i \to \infty}(\eta_{p_i, r_i})_\# V\), where \(p_i \to p\) in \(\text{spt} \|V\| \cap M^{\text{reg}}\), and \(r_i > 0\) with \(r_i \to 0\). Then \(\overline{V} = T_p(G \cdot p) \times W\) for some rectifiable varifold \(W \in \mathcal{V}_n-\dim(G \cdot p)(\mathcal{N}_p(G \cdot p))\).

**Proof** Firstly, since \(V\) has \(c\)-bounded first variation in \(M\) and is \((G, c)\)-almost minimizing in regular annuli, one can use Proposition 6.3 to get a volume ratio bound as [29, Lemma 6.4], which implies that \(V\) as well as \(\overline{V}\) is rectifiable.

For any \(w \in T_p(G \cdot p)\), there exists a curve \(g(t)\) in \(G\) such that \(g(0) = e\) and \(w = \frac{d}{dt} \big|_{t=0} g(t) \cdot p\). By the orthogonal representation of \(G\) on \(\mathbb{R}^L\), we can write \(g(t) = I_L + tA(t) \subset O(\mathbb{R}^L)\), where \(I_L\) is the identity map in \(\mathbb{R}^L\) and \(A(t)\) is a continuous curve in the \(L\)-matrix space such that \(A(0) \cdot p = w\). A straightforward computation shows that

\[ (\eta_{p_i, r_i} \circ g(r_i))(x) = \frac{g(r_i) \cdot x - p_i}{r_i} = \frac{g(r_i) \cdot x - g(r_i) \cdot p_i}{r_i} + \frac{g(r_i) \cdot p_i - p_i}{r_i} = (\tau_{A(r_i) \cdot p_i} \circ g(r_i) \circ \eta_{p_i, r_i})(x), \]

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where \( \tau_q(y) = y + q \). Because \( V \) is \( G \)-invariant, \( A(r_i) \cdot p_i \to w \), and \( g(r_i) \to e \),

\[
\nabla = \lim_{i \to \infty} (\eta_{p_i,r_i})_\# V \\
= \lim_{i \to \infty} (\eta_{p_i,r_i})_\# (g(r_i))_\# V \\
= \lim_{i \to \infty} (\tau_{A(r_i) \cdot p_i} \circ g(r_i) \circ \eta_{p_i,r_i})_\# V \\
= (\tau_w)_\# \nabla.
\]

Thus \( \nabla \) is invariant under the translation along \( T_p(G \cdot p) \). Since \( T_p M = T_p(G \cdot p) \times N_p(G \cdot p) \) and \( \nabla \) is rectifiable, we have \( \nabla = T_p(G \cdot p) \times W \) for some \( W \in \mathcal{V}_{n-\dim(G \cdot p)}(N_p(G \cdot p)) \).

**Remark 6.6** Suppose \( V_i \) is a \((G, c)\)-replacement of \( V \) in a small \( G \)-annulus \( A_{n_i}^G \) such that

\[
\lim_{i \to \infty} \eta_{p_i,r_i}(A_{n_i}^G) = T_p(G \cdot p) \times A_n \subset T_p M \text{ for some } G_p \text{-annulus } A_n \text{ in } N_p(G \cdot p).
\]

We also have the splitting property for \( \lim_{i \to \infty} (\eta_{p_i,r_i})_\# V_i \) by the argument above.

Since the center of the blowup procedure is a point rather than an orbit, we cannot expect to preserve the \( G \)-invariance of a \( G \)-varifold during the blowup. Nevertheless, noting \( T_p M \) is a \( G_p \)-space for any \( p \in M \), we can show the blowup procedure at \( p \) is \( G_p \)-equivariant, and thus the tangent varifold \( C \) of \( V \in \mathcal{V}_c^G(M) \) at \( p \) is \( G_p \)-invariant. Combining the splitting property and the \((G, c)\)-replacement, we show the following proposition classifying the tangent cones of \( c \)-min–max varifolds.

**Proposition 6.7** (Tangent cones) Let \( 2 \leq n \leq 6 \) and \( \mathcal{H}^{n-1}(M \setminus M^{reg}) = 0 \). Suppose \( V \in \mathcal{V}_c^G(M) \) has \( c \)-bounded first variation in \( M \) and is \((G, c)\)-almost minimizing in regular annuli. Then \( V \) is integer rectifiable. Moreover, for any \( C \in \text{VarTan}(V, p) \) with \( p \in \text{spt} \parallel V \parallel \cap M^{reg} \),

\[
C = \Theta^n(\parallel V \parallel, p)S \text{ for some } n\text{-plane } S \subset T_p M, \text{ where } \Theta^n(\parallel V \parallel, p) \in \mathbb{N}.
\]

**Proof** Let \( r_i \to 0 \) be a sequence such that \( C \) is the varifold limit:

\[
C = \lim_{i \to \infty} (\eta_{p_i,r_i})_\# V.
\]

Clearly, we know \( C \) is stationary in \( T_p M \). By Lemma 6.5, \( C \) has the form \( C = T_p(G \cdot p) \times W \) for some \( W \in \mathcal{V}_{n-\dim(G \cdot p)}(N_p(G \cdot p)) \). Moreover, if \( g \in G_p \) (i.e. \( g \cdot p = p \)), then

\[
(g \circ \eta_{p_i,r_i})(x) = \frac{g \cdot x - g \cdot p}{r_i} = (\eta_{p_i,r_i} \circ g)(x),
\]

which implies \( C = g#C, \forall g \in G_p \). Thus \( W \in \mathcal{V}_{n-\dim(G \cdot p)}(N_p(G \cdot p)) \).

To prove the proposition, we only need to show that \( W \) is an integer multiple of some \((n - \dim(G \cdot p))\)-plane in \( N_p(G \cdot p) \), which shares essentially the same idea as [31, Lemma 5.10]. First, \( W \) is stationary in \( N_p(G \cdot p) \) by the product structure of \( T_p M \) and the fact that \( C \) is stationary in \( T_p M \).

Next, we claim that \( W \) has the good \( G_p \)-replacement property ([16, Proposition 6.11]) in any open \( G_p \)-set \( D \subset N_p(G \cdot p) \). Indeed, fix a bounded open \( G_p \)-set \( D \subset N_p(G \cdot p) \) and an arbitrary \( x \in D \). Take any \( G_p \)-annulus \( A_n = A_{n_i}^{G_p}(x, s, t) \subset D \) with \( t \leq 1 \), and denote \( A_{n_i} = (\eta_{p_i,r_i})^{-1}(A_n) \). As in [31, Lemma 5.10], we identify \( \eta_{p_i,r_i}(M) \) with \( T_p M \) on compact sets for \( i \) large enough by the locally uniformly convergence \( \eta_{p_i,r_i}(M) \to T_p M \). Moreover, since \( \eta_{p_i,r_i} \) is \( G_p \)-equivariant, one can regard \( A_{n_i} \) as a \( G \)-annulus in \( B_p \) for sufficiently large \( i \), where \( B_p \) is a slice of \( G \cdot p \) at \( p \). Let \( A_{n_i}^G = G \cdot A_{n_i} \) be a \( G \)-annulus in \( M \) with \( A_{n_i}^G \cap B_p = A_{n_i} \). Therefore, for every \( i \) large enough, we can apply Proposition 6.3 to get a
(G, c)-replacement $V^*_i$ of $V$ in $\text{Clos}(\text{An}^G)$. Denote $\overline{V}^*_i = (\eta_{p, r_i})_{\#} V^*_i$ and $\overline{V}_i = (\eta_{p, r_i})_{\#} V$. After passing to a subsequence, we have

$$C' = \lim_{i \to \infty} \overline{V}^*_i \in \mathcal{V}^G_n (T_p M).$$

On the other hand, we have $\lim_{i \to \infty} \eta_{p, r_i} (\text{An}_i^G) = T_p (G \cdot p) \times \text{An} \subset T_p M$. By Remark 6.6, $C'$ also has the splitting property:

$$C' = T_p (G \cdot p) \times W',$$

for some $W' \in \mathcal{V}^G_{n-\dim(G \cdot p)} (N_p (G \cdot p))$. Combining the Proposition 6.3, Lemma 6.4 and the Compactness Theorem 2.17, the argument in the proof of [31, Lemma 5.10] indicates that $W'$ is a good $G_p$-replacement of $W$ in $\text{An}$. Moreover, by Proposition 6.3(iii), we can repeat the procedure above for a finite times and produce a good $G_p$-replacement $W^{(k)}$ of $W^{(k-1)}$ in any other $G_p$-annulus $\text{An}^{(k)} \subset D$ with outer radius no more than 1.

Finally, $W$, as well as $C$, is a multiple of some complete, smooth, embedded minimal hypersurface by the regularity result [16, Proposition 6.2] and the Euclidean volume growth of $C$, which is a direct corollary of the monotonicity formula. (We mention that the regularity results in [16, Section 5, 6] do not need $G$ to be connected by using Lemma 2.11 in the proof of [16, Lemma 5.3].) Furthermore, $C$ is a cone by [23, Theorem 19.3], and hence, $\text{spt} \| C \|$ must be an $n$-plane. \hfill $\square$

**Remark 6.8** Suppose $V$ is a $(G, c)$-min–max varifold and $C \in \text{VarTan}(V, p)$. By Proposition 6.7, the tangent cone $C$ is only $G_p$-invariant. When it comes to the regularity of blowup varifolds, the good $G_p$-replacement property plays an important role.

By the splitting property, we also have the following lemma which is a modified version of [31, Lemma 5.10].

**Lemma 6.9** (Various blowups) Let $2 \leq n \leq 6$ and $\mathcal{H}^{n-1} (M \setminus M^{reg}) = 0$. Suppose $U \subset M$ is an open $G$-set, and $V \in \mathcal{V}^G_n (M)$ is a $(G, c)$-almost minimizing varifold in $U$. Given a sequence $p_i \in U$ with $p_i \to p \in U \cap M^{reg}$, and a sequence $r_i > 0$ with $r_i \to 0$, let $\overline{V} = \lim (\eta_{p_i, r_i})_{\#} V$ be the varifold limit. Then $\overline{V}$ is an integer multiple of some complete, smooth, embedded minimal hypersurface $\Sigma$ in $T_p M$, and moreover, $\Sigma$ is proper.

**Proof** By the openness of $M^{reg}$ and $p_i \to p$, we can suppose $p_i \in M^{reg}$, i.e. both $\{G \cdot p_i\}_{i \in \mathbb{N}}$ and $G \cdot p$ are principal orbits. Hence, there exists a sequence $\{g_i\}_{i \in \mathbb{N}} \subset G$ such that $G_{p_i} = g_i \cdot G \cdot p \cdot g_i^{-1}$, where $G_{p_i}$ is the isotropy group of $p_i$. By the compactness of $G$, we can suppose $g_i \to g_0 \in G$ after passing to a subsequence. Moreover, it’s clear that $g_0 \cdot G_p \cdot g_0^{-1} = G_p$ since $p_i \to p$ and conjugate transformations are isomorphisms. Note $\eta_{p_i, r_i}$ is $G_{p_i}$-equivariant:

$$h \circ \eta_{p_i, r_i} (x) = \frac{h \cdot x - h \cdot p_i}{r_i} = \frac{h \cdot x - p_i}{r_i} = \eta_{p_i, r_i} \circ h (x), \quad \forall h \in G_{p_i}.$$

Therefore, for any $g \in G_p$ and $h_i = g_i \cdot g_0^{-1} \cdot g \cdot g_0 \cdot g_i^{-1} \in G_{p_i}$, it’s clear that $h_i \to g$, and hence,

$$g_{\#} \overline{V} = \lim_{i \to \infty} (h_i)_{\#} (\eta_{p_i, r_i})_{\#} V = \lim_{i \to \infty} (\eta_{p_i, r_i})_{\#} (h_i)_{\#} V = \lim_{i \to \infty} (\eta_{p_i, r_i})_{\#} V = \overline{V},$$
which implies that $\overline{V} \in \mathcal{V}_n^G(T_pM)$.

Since $V$ is $(G, c)$-almost minimizing in $U$, $V$ has $c$-bounded first variation in $U$ by Lemma 5.2. Thus we can apply Lemma 6.5 to get the splitting property of $\overline{V}$:

$$\overline{V} = T_p(G \cdot p) \times W,$$

where $W \in \mathcal{V}_n - \dim(G) - (N_p(G \cdot p))$. Since $T_p(G \cdot p)$ is $G_p$-invariant as a varifold, it follows that $W \in \mathcal{V}_n - \dim(G) - (N_p(G \cdot p))$. To prove the lemma, we only need to show that $W$ is an integer multiple of some complete, smooth, embedded minimal hypersurface in $N_p(G \cdot p)$.

The rest of the proof is essentially the same as Proposition 6.7. One just need to notice that, $V$ has $c$-bounded first variation in $U$ implies that the blowup $\overline{V} = \lim(\eta_{p_i, r_i})# V$ is stationary in $T_pM$. Thus, $W$ is stationary in $N_p(G \cdot p)$ by the product structure of $T_pM$. Moreover, since $\eta_{p_i, r_i} = G_{p_i}$-equivariant and $G_{p_i} \to G_p$, one can regard $A_{p_i} = (\eta_{p_i, r_i})^{-1}(A_n)$ as a $G_{p_i}$-annulus in $B_{p_i}$ for $i$ large enough, where $B_{p_i}$ is a slice of $G \cdot p_i$ at $p_i$.

$\square$

### 6.3 Main regularity

Now we have the following regularity result for $(G, c)$-min–max varifolds.

**Theorem 6.10** (Main regularity) Let $2 \leq n \leq 6$, and $(M^{n+1}, g_M)$ be an $(n+1)$-dimensional smooth, closed Riemannian manifold with a compact Lie group $G$ acting as isometries of cohomogeneity $\text{Cohom}(G) \geq 3$. Suppose the union of non-principal orbits $M \setminus M^{reg}$ is a smooth embedded submanifold of $M$ with dimension at most $n - 2$. If $V \in \mathcal{V}_n(M)$ is a varifold which

- has $c$-bounded first variation in $M$, and
- is $(G, c)$-almost minimizing in regular annuli,

Then $V$ is induced by $\Sigma \subset M$, where

1. $\Sigma$ is a closed, almost embedded, $(G, c)$-hypersurface (possibly not $G$-connected);
2. the density of $V$ is exactly $1$ at the regular set $R(\Sigma)$ and $2$ at the touching set $S(\Sigma)$.

**Remark 6.11** We only focus on the case of $c > 0$ in Theorem 6.10. Indeed the work of Liu [16] under the smooth sweepouts settings and Wang [29] under the Almgren-Pitts settings completely resolved the case of $c = 0$ in Theorem 1.1.

We mention that $M \setminus M^{reg}$ is a submanifold without boundary under the assumption in Theorem 6.10. Indeed, if $p \in \partial(M \setminus M^{reg})$, then $G \cdot p \subset \partial(M \setminus M^{reg})$ is a non-principal orbit by the openness of $M^{reg}$. Let $\overrightarrow{v}$ be the unit normal vector field along $G \cdot p$ pointing outside $M \setminus M^{reg}$, which is clearly a $G$-invariant vector field. Hence, for $\epsilon > 0$ small enough, we have $G \cdot \exp_{\overrightarrow{v}}(\epsilon \overrightarrow{v}(p))$ is a principal orbit. However, since $\overrightarrow{v}$ is $G$-invariant and $\exp_{\overrightarrow{v}}$ is $G$-equivariant, the map $f(g \cdot p) = g \cdot \exp_{\overrightarrow{v}}(\epsilon \overrightarrow{v}(p))$ gives a diffeomorphism from a non-principal orbit $G \cdot p$ to a principal orbit $G \cdot \exp_{\overrightarrow{v}}(\epsilon \overrightarrow{v}(p))$, which is a contradiction.

**Proof** Let $p \in \spt\|V\|$. By the assumption of $\text{Cohom}(G) \geq 3$ and $M \setminus M^{reg}$ is a smooth embedded submanifold of $M$ with $\dim(M \setminus M^{reg}) \leq n - 2$, there exists $0 < r_0 < r(G \cdot P_p)$ such that for any $0 < r < r_0$, the mean curvature of $\partial B^G_r(P_p) \cap M$ is greater than $c$. Here $r(G \cdot P_p)$ is as in Definition 5.6.

By the maximum principal [28, Theorem 5] (see also [31, Proposition 2.13]) and the convexity of $B^G_r(P_p)$, if $W \in \mathcal{V}_n(M)$ has $c$-bounded first variation in $B^G_r(P_p) \cap M$ and...
\[ W \subset B_r^G(P_p) \neq 0, \]

\[ \emptyset \neq \text{spt}(W) \cap \partial B_r^G(P_p) = \text{Clos}[\text{spt}(W) \setminus \text{Clos}(B_r^G(P_p))] \cap \partial B_r^G(P_p). \quad (6.7) \]

**Step 1.** After adding \( G \)-in front of relevant objects, the first step in the proof of [31, Theorem 6.1] would carry over.

**Step 2.** We only need to modify [31, Theorem 6.1 Step 2] in a few places.

First, note every small \( G \)-annuli around \( P_p \) is contained in \( M^\text{reg} \) and the intersection set \( \Gamma \subset \text{An}^G(P_p, s, t) \cap M \subset M^\text{reg} \) (see (5.2)). Hence, Proposition 6.7, Lemma 6.9 and all the results in the previous subsection can be applied to every \( q \in \text{An}^G(P_p, s, t) \in \mathcal{A}_G^G(P_p) \), by \( \text{Cohom}(G) \geq 3 \) and \( \text{dim}(M \setminus M^\text{reg}) \leq n - 2 \). We can also show that \( \Sigma_1 \) glues together continuously with \( \Sigma_2 \) and the \((G, c)\)-replacement \( V^{**} = |\partial \Omega^{**}| \) in \( \text{An}^G(P_p, s_1, t) \cap M \), where \( \Omega^{**} \) is a \( G \)-invariant Caccioppoli set.

As for the sub-case (A), we can use Lemma 6.9 in the place of [31, Lemma 5.10], and get the following result as \([31, \text{P.475 Claim 3(A)}]:\)

Fix any \( x \in \mathcal{R}(\Gamma) \). Then for any sequence of \( x_i \to x \) with \( x_i \in \mathcal{R}(\Gamma) \) and \( r_i \to 0 \), we have

\[ \lim_{i \to \infty} (\eta_{x_i, r_i})_\#V^{**} = T_x \Sigma_1 \text{ as varifolds}. \]

Therefore, replacing the blowups \( \eta_{x_i, r_i}(\Sigma_2 \cap B_{r_i/2}(z_i)) \) with \( \eta_{x_i, r_i}(\Sigma_2 \cap B_{r_i/2}(z_i)) \) in the proof of [31, P.476 Claim 4(A)], we can show the gluing of the \((G, c)\)-replacements is smooth on the overlap for the sub-case (A).

As for the sub-case (B), we can also get an equivariant version of [31, P.477 Claim 3(B)] by using Lemma 6.9 and considering the various blowups \( (\eta_{x_i, r_i})_\#V^{**}, \eta_{x_i, r_i}(\Sigma_1, 1), \eta_{x_i, r_i}(\Sigma_1, 2) \), as well as the vector \( \tilde{v} = \lim_{i \to \infty} x_i - x_i / r_i \).

To sum up, we can construct successive \( G \)-replacements \( V^* \) and \( V^{**} \) on two overlapping \( G \)-regular annuli \( \text{An}^G(P_p, s, t) \), \( \text{An}^G(P_p, s_1, s_2) \), \( 0 < s_1 < s < s_2 < t \), and glue them smoothly across \( \partial B_{r-}^G(P_p) \).

**Step 3.** As the third step in the proof of [31, Theorem 6.1], we denote \( V^{**} \) by \( V^{**} \) and \( \Sigma_2 = \text{spt}(V^{**}) \) by \( \Sigma_1 \) to indicate the dependence on \( s_1 \). Since the unique continuation holds for immersed CMC hypersurfaces, we have \( \Sigma_1 = \text{spt}(V^*) \in \text{An}^G(P_p, s, s_2) \) by Step 2, and moreover, we have \( \Sigma_1' = \Sigma_1 \) in \( \text{An}^G(P_p, s_1, s_2) \), for all \( s_1' < s_1 < s \). Hence,

\[
\Sigma := \bigcup_{0 < s_1 < s} \Sigma_{s_1}
\]

is a smooth, almost embedded, stable \((G, c)\)-hypersurface in \( (B_{r-}^G(P_p) \setminus G \cdot P_p) \cap M \). By Proposition 6.3, \( V^{**}_{s_1} \) has \( c \)-bounded first variation and uniformly bounded mass for all \( 0 < s_1 < s \). Combining the covering argument [16, Lemma A.1] with the monotonicity formula [23, Theorem 40.2], we have

\[
\| V^{**}_{s_1} \| (B_r^G(P_p)) \leq C r^{n - \dim(G \cdot P_p)} \leq C r^2,
\]

for some \( C > 0 \) independent of \( s_1 \). Therefore, as \( s_1 \to 0 \), the family \( V^{**}_{s_1} \) converges to a varifold \( \tilde{V} \in V^G_n(M) \), which satisfies \( \| \tilde{V} \| (G \cdot P_p) = 0 \), and

\[
\tilde{V} = \begin{cases} 
\Sigma & \text{in } (B_{r-}^G(P_p) \setminus G \cdot P_p) \cap M, \\
V^* & \text{in } M \setminus B_r^G(P_p).
\end{cases}
\]
Since $p \in spt \|V\|_{1}$ and $\tilde{V}$ is G-Invariant, we have $p \in spt \|\tilde{V}\|$ by the upper semi-continuity of density function for varifolds with bounded first variation.

**Step 4.** The regularity of $\tilde{V}$ at $G \cdot p_{p}$ comes from [4, Theorem 1.1]. First, we have $\tilde{V} = \lim_{s_{1} \to 0} V_{s_{1}}^{**} \|\tilde{V}\|/(G \cdot p_{p}) = 0$, and $\tilde{V} = \Sigma$ in $(B_{2}^{G}(p_{p}) \cdot G \cdot p_{p}) \cap M$. Since $V^{**}$ has c-bounded first variation in $M$ and c-bounded first variation is a closed condition, we know that $\tilde{V}$ has c-bounded first variation in $M$. After modifying [31, Theorem 6.1 Claim 1] to an equivariant version, we can show that $\tilde{V} = |\tilde{\Omega}|$ in $(B_{2}^{G}(p_{p}) \cdot G \cdot p_{p}) \cap M$, where $\tilde{\Omega} \subseteq C^{G}(M)$ and $\tilde{\Omega}$ is a smooth, almost embedded, stable $(G, c)$-hypersurface in $(B_{2}^{G}(p_{p}) \cdot G \cdot p_{p}) \cap M$.

Now we use the regularity result in [4, Theorem 4.1] (see “Appendix A”) to remove the singularity at $G \cdot p_{p}$. If $(G \cdot p_{p} \cap spt |\tilde{\Omega}|) \subseteq \text{gen-reg}|\tilde{\Omega}|$ is satisfied, then we are fine. Assume $(G \cdot p_{p} \cap spt |\tilde{\Omega}|) \subseteq (spt |\tilde{\Omega}| \backslash (\text{gen-reg}|\tilde{\Omega}|))$. Note the center $G \cdot p_{p}$ of $B_{2}^{G}(p_{p})$ is a principle orbit if $p \in M^{reg}$, or a smooth embedded submanifold of $M$ without boundary if $p \in M \backslash M^{reg}$. The extension argument (c.f. the proof in [11, Theorem 4.1]) is still valid by choosing $A = G \cdot p_{p}$, and thus $|\tilde{\Omega}|$ has c-bounded first variation in $B_{2}^{G}(p_{p}) \cap M$. Therefore the condition $(a_{1})$ in Theorem A.1 is satisfied. Since Cohom$(G) \geq 3$ and dim$(M^{reg}) \leq n - 2$, we have dim$(G \cdot p_{p}) \leq n - 2$. It follows that $|\tilde{\Omega}|$ has no classical singularities in $B_{2}^{G}(p_{p}) \cap M$, i.e the conditions $(a_{2})$ in Theorem A.1 is satisfied. By [31,Proposition 2.9], we know $H^{d}(\text{sing}_{G}|\tilde{\Omega}|)$ is satisfied. It follows that the condition $(a_{3})$ in Theorem A.1 is satisfied. We note that $|\tilde{\Omega}| = \Sigma$ is a smooth, almost embedded, stable $(G, c)$-hypersurface in $(B_{2}^{G}(p_{p}) \cdot G \cdot p_{p}) \cap M$. Therefore the conditions $(c)$ and $(d)$ in Theorem A.1 are satisfied.

Hence, the singularity of $|\tilde{\Omega}|$ as well as $\tilde{V}$ at $G \cdot p_{p}$ is removable by Theorem A.1.

**Step 5.** Same with the final step in the proof of [31, Theorem 6.1] except using $G$-invariant objects.

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### Appendix A. Regularity for Stable Codimension 1 CMC Varifolds

Here we record the necessary results about the regularity theorem for stable codimension 1 CMC varifolds in [4]. We summarize the notations and definitions from the paper [4] that will be used in our paper as follows. Let $V \in TV_{n}(M)$ be an integral varifold.

- ([4, Definition 1.3]) A point $p \in spt \|V\|$ is a regular point of $V$ if there exists $\sigma > 0$ such that $spt \|V\| \cap B_{\sigma}(p)$ is an embedded $C^{2}$ hypersurface of $B_{\sigma}(p)$. The regular set of $V$, denoted $\text{reg}V$, is the set of all regular points of $V$. The singular set of $V$, denoted $\text{sing}V$, is $spt \|V\| \backslash \text{reg}V$.
- ([4, Definition 1.5]) A point $p \in \text{sing}V$ is a classical singularity of $V$ if there exists $\sigma > 0$ such that $spt \|V\| \cap B_{\sigma}(p)$ is the union of three or more embedded $C^{1,\alpha}$ hypersurfaces (with boundary) meeting pairwise only along their common $C^{1,\alpha}$ boundary $\gamma$ containing $p$ and such that at least one pair of the hypersurfaces meet transversely everywhere along $\gamma$. The set of all classical singularities of $V$ is denoted by $\text{sing}_{C}V$.
- ([4, Definition 1.6]) A point $p \in \text{sing}_{C}V \backslash \text{sing}V$ is a touching singularity of $V$ if there exists $\sigma > 0$ such that $spt \|V\| \cap B_{\sigma}(p) = \Sigma_{1} \cup \Sigma_{2}$, where $\Sigma_{j}$ is an embedded $C^{1,\alpha}$ hypersurface of $B_{\sigma}(p)$ with $\partial \Sigma_{j} \cap B_{\sigma}(p) = \emptyset$ for $j = 1, 2$. The set of all touching singularities of $V$ is denoted by $\text{sing}_{T}V$.
• ([4, Definition 1.7]) A point \( p \in \text{spt} \| V \| \) is a generalized regular point if either (i) \( p \in \text{reg} \ V \) or (ii) \( p \in \text{sing}_T V \) and \( \Sigma_1, \Sigma_2 \) are embedded \( C^2 \) hypersurface and \( \text{spt} \| V \| \cap B_\sigma (p) = \Sigma_1 \cup \Sigma_2 \). The set of all generalized regular points is denoted by \( \text{gen-reg} \ V \).

In [4, Theorem 1.1], Bellettini and Wickramasekera developed a regularity theory for stable prescribed mean curvature integral varifolds of codimension 1. Here, we only focus on CMC varifolds.

**Theorem A.1** ([4, Theorem 1.1]: special case \( g \equiv c \)) Suppose \( 2 \leq n \leq 6 \), and \( (M^{n+1}, g_M) \) is a smooth, closed Riemannian manifold of dimension \( n+1 \). Let \( V \) be an integral \( n \)-varifold in \( M \) such that

(a1) the first variation of \( V \) is in \( L^p_{\text{loc}} (\| V \|) \) for some \( p > n \);

(a2) \( \text{sing}_C V = \emptyset \);

(a3) \( \mathcal{H}^n (\text{sing}_T V) = 0 \);

Suppose moreover that the following variational assumptions are satisfied:

(c) the embedded hypersurface \( S = \text{reg} \ V \) is a CMC hypersurface;

(d) for each orientable open set \( U \subset M \setminus (\text{spt} \| V \| \setminus \text{gen-reg} \ V) \), the hypersurface \( \text{gen-reg} \ V \cap U \) is a stable CMC hypersurface, [4, Remark 1.3].

Then \( \text{spt} \| V \| \) is a smooth, almost embedded CMC hypersurface.

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