Horizons, singularities and causal structure of the generalized McVittie space-times

Malcolm Anderson
Mathematics, Faculty of Science, Universiti Brunei Darussalam
Jalan Tungku Link, Gadong BE 1410, Brunei Darussalam
Email: malcolm.anderson@ubd.edu.bn

Abstract. The generalized McVittie space-times, which are the natural extensions of a class of metrics introduced by McVittie in 1933 to model the space-time outside a spherical object embedded in an expanding universe, have recently been proposed by Faraoni and Jacques as candidate space-times for cosmological black holes. In this paper I analyze the singularities and horizon structure of the generalized McVittie space-times, and show that any expanding space-time of this type that satisfies the null energy condition and develops apparent horizons must asymptote to a standard McVittie solution. Furthermore, if the scale factor is asymptotically exponential then the space-time is future-incomplete and can be joined smoothly to an eternally-inflating Kottler (Schwarzschild-de Sitter) solution. I argue that no generalized McVittie space-time, apart from the Schwarzschild solution, can adequately represent a black hole, because all singular points are surrounded by anti-trapped regions rather than trapped regions.

1. Introduction
The (spatially-flat) McVittie space-times are a family of spherically symmetric, non-vacuum space-times with the general line element

\[ ds^2 = \frac{1}{2} [1 - M/(2ra)]^2 [1 + M/(2ra)]^{-2} dt^2 + a^2 [1 + M/(2ra)]^4 (dr^2 + r^2 d\Omega^2) \]  

(1)

where \( d\Omega^2 \) is the metric on the unit 2-sphere, \( M \) is a positive constant, and \( a \equiv a(t) \) is a positive function to be specified.

McVittie [1] introduced the line element (1) in 1933, in an attempt to model the space-time outside a non-rotating black hole embedded in an asymptotically Friedmann-Lemaître-Robertson-Walker (FLRW) universe. The McVittie line element (1) reduces to the Schwarzschild metric in isotropic coordinates if \( a \equiv 1 \), and to the spatially-flat FLRW line element if \( M = 0 \).

The physical interpretation of the McVittie space-times is controversial even today. Sussmann [2] and Ferraris, Francaviglia and Spallicci [3] have rejected the view that the McVittie line element describes a black hole in an otherwise homogeneous universe because in almost all cases there is a

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1 Geometrized units are used throughout, so that \( G = c = 1 \).

2 McVittie [1] also considered line elements that asymptote to universes with constant positive or negative spatial curvature, but attention will be restricted to the case of zero asymptotic curvature here.
curvature singularity at \( r = M/(2a) \), and not at \( r = 0 \) as McVittie [1] originally claimed.\(^3\) Moreover, the pressure of the underlying matter source is unbounded near this singularity while the density remains finite there, and such behavior is difficult to incorporate into a realistic equation of state.

The one case in which the surface \( r = M/(2a) \) is regular occurs when the scale factor \( a \) is an exponential function of \( t \). If \( a(t) = a_0 \exp[(\Lambda/3)^{1/2}t] \) with \( a_0 \) and \( \Lambda \) positive constants, a transformation from \((t, r)\) to coordinates \((T, R)\) defined by

\[
R = a_0 r \exp[(\Lambda/3)^{1/2}] \left[ 1 + (2a_0 r)^{-1} M \exp[(\Lambda/3)^{1/2} t] \right]^2
\]

(2)

and

\[
dT = dt + (\Lambda/3)^{1/2} (1 - 2M/R)^{-1/2} (1 - 2M/R - \frac{1}{3} \Lambda R^2)^{1/2} R dR
\]

(3)

casts the McVittie line element (1) in the static form

\[
 ds^2 = -(1 - 2M/R) (R - \frac{1}{3} \Lambda R^2) dT^2 + (1 - 2M/R) (R - \frac{1}{3} \Lambda R^2)^{-1} dR^2 + R^2 d\Omega^2,
\]

(4)

which is commonly referred to as the Schwarzschild-de Sitter solution, although it was first discovered by Kottler [5] in 1918. The Kottler solution, and the Schwarzschild solution to which it reduces when \( \Lambda = 0 \), are unique amongst the McVittie space-times in having a regular surface at \( r = M/(2a) \) (or equivalently at \( R = 2M \)), and finite underlying density and pressure.

In the first of a series of papers on the McVittie space-times, Nolan [6] initially defended the black hole interpretation by arguing that, in the spatially-flat case, the McVittie space-times are the unique spherically symmetric space-times with a shear-free fluid source and homogeneous energy density that asymptote to an FLRW metric at space-like infinity. In a second paper, Nolan [4] rewrote the line element (1) in terms of the geometric radius

\[
 R = ar[1 + M/(2ar)]^2
\]

(5)

and characterized the resulting space-time as a "point mass at \( R = 0 \) surrounded by a singularity [at \( R = 2M \)]."\(^4\) Nonetheless, he also demonstrated that, if the universe is expanding (so that \( \frac{d}{dt} a(t) > 0 \)), then the singular surface at \( R = 2M \) is surrounded by an anti-trapped region – the signature of a white hole – rather than the trapped region expected of a black hole.

More recently, Kaloper, Kleban and Martin [8] have criticized Nolan’s stipulation that the energy density near a black hole in a cosmological background be homogeneous as “very restrictive and not very well-motivated physically”, and have pointed out that it would almost inevitably induce an unbounded pressure gradient to balance the strong gravitational frame-dragging effects present at the horizon of a black hole. Despite this, they have registered their support for the black hole interpretation by demonstrating that, for some choices of the scale factor \( a \), ingoing radial null geodesics that asymptote to the surface \( r = M/(2a) \) are future-incomplete, and arguing that in these cases the completion of the McVittie space-time could contain a black hole region as well as a white hole region.

A natural generalization of the McVittie space-time that allows for an inhomogeneous density profile is the line element

\[
ds^2 = g_{\alpha \beta} dx^\alpha dx^\beta = -(1 - m/(2r))^2[1 + m/(2r)]^{-2} dt^2 + a^2[1 + m/(2r)]^4 (dr^2 + r^2 d\Omega^2),
\]

(6)

\(^3\) In fact, the effect of replacing \( r \) in the McVittie line element (1) with \( r^* = M^2/(4r) \) is to recover the same line element with \( a^{-1} \) in place of \( a \) [4], and so the limit \( r \to 0 \) corresponds to the asymptotically FLRW region of a second McVittie space-time. A closely analogous situation occurs in the Schwarzschild metric, where isotropic coordinates are known to provide a double cover of the region outside the event horizon.

\(^4\) However, Nolan [7] in a later paper showed that it is not possible to extend a McVittie space-time down to \( R = 0 \) unless the source fluid is tachyonic in the region \( R < 2M \).
where now \( a \equiv a(t) \) and \( m \equiv m(t) \) are positive functions to be specified. Clearly, the McVittie line element (1) is recovered from (6) if \( m \) is chosen to be \( M/a \). Faraoni and Jacques [9] have claimed that replacing \( t \) with a proper time coordinate in (6) can eliminate any singularities in the fluid pressure or density at the boundary surface \( r = m/2 \), but Carrera and Giulini [10] have correctly dismissed this claim and shown that the Ricci scalar diverges at \( r = m/2 \) in all cases except the Kottler solution.\(^5\)

In what follows I will refer to any line element of the form (6) as a generalized McVittie (or gMcV) space-time. The analog of the geometric radius (5) for the gMcV space-times is the quantity
\[
R = r (g_{rr})^{1/2} = ar[1+m/(2r)]^2.
\]
(7) Although it is known that all gMcV space-times bar the Kottler solution are singular on a surface with non-zero geometric radius \( R = 2am \), it might nonetheless be possible that some members of the family avoid other undesirable features of the McVittie space-times. It is conceivable, for example, that there exist expanding gMcV space-times in which the singular surface is surrounded by a trapped region rather than an anti-trapped region. Furthermore, the suggestion by Kaloper, Keban and Martin [8] that those McVittie (or gMcV) space-times that are known to be future-incomplete might contain black hole regions once they are completed has not yet been fully explored.

It is the purpose of this note to report the results of an extensive analysis of the salient features of the gMcV space-times, particularly the apparent horizons and causal structure. The overall conclusion is that the behavior of the gMcV space-times does not differ markedly from that of the McVittie space-times. In all cases the singular surface at \( r = m/2 \) is surrounded by an anti-trapped region if the space-time is expanding, and any future-incomplete McVittie or gMcV space-time that has \( \lim_{a \to \infty} (\dot{a}/a) > 0 \) can be completed by joining it smoothly to a Kottler extension, which also contains no trapped regions.

Indeed, the connections between the gMcV and McVittie space-times are even more intimate than might first appear, as any gMcV space-time that forms an apparent horizon inevitably asymptotes to a McVittie solution as \( t \to \infty \) if the null energy condition is imposed.

2. Overview of the generalized McVittie space-times
In what follows, the functions \( a \) and \( m \) appearing in the gMcV line element (6) are assumed to be smooth, positive functions of the time coordinate \( t \) for all \( t > 0 \). If the space-time supports a cosmological singularity – a moment of time at which \( a = 0 \) – then this is assumed to occur in the limit as \( t \to 0^+ \). Further restrictions on the behavior of \( a \) and \( m \) will be imposed by the null energy condition, which is discussed in some detail below.

The radial coordinate \( r \) is in all cases taken to satisfy the restriction \( r > m(t)/2 \), as the region inside \( r = m/2 \) is simply a copy of a second exterior gMcV region. This can be seen by replacing \( r \) with \( r^\ast \equiv m_0^2/(4r) \) in (6), where \( m_0 \) is any constant mass scale. The line element that results then has the same form as (6), save with \( m_0^2/m \) in place of \( m \) and \( am_0^2/m_0^2 \) in place of \( a \). The isotropic coordinates of (6) therefore break the range \((0, \infty)\) of \( r \) into the exterior regions of two distinct gMcV space-times, in much the same way as happens with the McVittie and Schwarzschild space-times.

\(^5\) Carrera and Giulini [10] have refuted a further statement made by Faraoni and Jacques [9], and assumed extensively by Faraoni in [11], that the line element of the Sultana-Dyer solution [12] can be transformed into the generalized McVittie form (6). This error has also been noticed by Sun [13].
2.1 The Einstein tensor and curvature singularities

With \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \), the non-zero components of the Einstein tensor generated by the gMcV line element (6) are:

\[
G^\mu_\nu = 3a^{-2} \Delta^{-2} (\Delta \dot{\Delta} + 2\dot{\xi})^2, \\
-\frac{1}{16} r^{-4} \Delta^{-2} a^2 (2r + m)^6 G^\mu_\nu = 8a^{-1} \Delta^{-3} (2r + m) \dot{\xi} \\
and \\
G^\rho_\sigma = G^\rho_\sigma = 3a^{-2} \Delta^{-2} (\Delta \dot{\Delta} + 2\dot{\xi})^2 + 2\Delta^{-1} (2r + m) \frac{\partial}{\partial \tau} [a^{-1} \Delta^{-1} (\Delta \dot{\Delta} + 2\dot{\xi})],
\]
where an overdot denotes \( \frac{d}{d\tau} \), while \( \xi = \frac{d}{d\tau} (am) \) and \( \Delta = 2r - m \). Carrera and Giulini [10] refer to the notable feature that \( G^\rho_\sigma = G^\rho_\sigma \) as “spatial Ricci-isotropy”.

An immediate consequence of (9) is that the space-time carries a non-zero radial energy flux unless \( \dot{\xi} \equiv 0 \), or equivalently \( m \propto a^{-1} \). It was with the objective of ruling out any transfer of energy between the ambient medium and the central object that McVittie [1] restricted his attention to the line element (1) that bears his name.

Also, the Ricci scalar of the gMcV space-times is:

\[
R^a_b = -6a^{-2} \Delta^{-3} \{(aa + a^2) \Delta^3 + 2[3\dot{\xi} + a\ddot{\xi} + m(\Delta \dot{\Delta} + 2\dot{\xi}))\Delta^2 + 2(5\dot{\xi}^2 - 3\dot{m}\dot{\xi} + 2am\ddot{\xi})\Delta + 4am\dot{\xi}\}}.
\]

(11)

It is clear that the Ricci scalar diverges at \( r = m/2 \) unless \( \dot{m}\dot{\xi} \equiv 0 \). If \( \xi \equiv 0 \) the standard McVittie space-time is recovered, but \( R^a_b \) still diverges at \( r = m/2 \) unless \( aa \equiv a^{-2} \), in which case \( a \) is constant or an exponential function of \( \tau \). On the other hand, if \( \dot{m} \equiv 0 \) then \( \dot{\xi} = \dot{a}m \) and \( \ddot{\xi} = \ddot{a}m \), and the Ricci scalar diverges at \( r = m/2 \) unless \( \dot{a}a + a^2 \equiv 0 \), or equivalently \( a \propto t^{1/2} \). If \( a = a_0 t^{1/2} \) for some constant \( a_0 \), and \( m \) is constant, then in fact \( R^a_b \) is zero everywhere, but the square of the Ricci scalar,

\[
R^{ab} R_{ab} = r^{-4} \Delta^{-4} [\frac{3}{4} (2r + m)^4 - 512a_0^{-2} (2r + m)^4 m^2 r^4 t^4],
\]
remains divergent at \( r = m/2 \). Thus, the Schwarzschild and Kottler solutions are the only members of the gMcV family that are non-singular at \( r = m/2 \), as Carrera and Giulini [10] claim.

The question of whether the Ricci scalar diverges at a cosmological singularity, where \( a(t) \to 0 \), is more delicate. \( R^a_b \) will be divergent in this limit unless all the coefficients of the various powers of \( \Delta \) in (11) go to zero at least as rapidly as \( a^2 \). As will be seen below, \( \lim_{t \to 0^+} m(t) = 0 \) is inconsistent with the null energy condition in an expanding universe, which requires that \( \dot{m} \leq 0 \), while the cosmological singularity is inaccessible to past-directed null geodesics if \( \lim_{t \to 0^+} m(t) = \infty \). So it is sufficient to consider only the case where \( \lim_{t \to 0^+} m(t) \) is a positive constant. The right-hand side

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\(6\) The convention for the Riemann tensor used here is that

\[
R^{\alpha \beta \gamma \delta} = \frac{1}{2} (\partial_\alpha \partial_\gamma G_{\beta \delta} + \partial_\beta \partial_\delta G_{\alpha \gamma} - \partial_\alpha \partial_\delta G_{\beta \gamma} - \partial_\beta \partial_\gamma G_{\alpha \delta}) + g_{mn} (\Gamma^n_{\alpha \beta} \Gamma^m_{\delta \gamma} - \Gamma^n_{\alpha \gamma} \Gamma^m_{\beta \delta})
\]

where \( \Gamma^m_{\alpha \beta} \) is the Christoffel symbol. Equivalently, \( \nabla_a \nabla_b V^b - \nabla_b \nabla_a V^b = R_{abcd} V^d \) for any smooth vector field \( V^d \), and in geometrized units the Einstein equation reads \( G^a_b = -8\pi T^a_b \).
of (11), if bounded, then imposes four independent constraints on \( \dot{a}, \ddot{a}, \xi \) and \( \dot{\xi} \) which are satisfied iff \( \dot{a}/a, \ddot{a}/a, \xi/a \) and \( \dot{\xi}/a \) (or equivalently \( \dot{m} \) and \( \ddot{m} \)) remain finite as \( a \to 0 \). But \( a \) cannot go to zero in a finite coordinate time \( t \) if \( \dot{a}/a \) is bounded, so the Ricci scalar as guaranteed to diverge at a cosmological singularity in an expanding universe if the null energy condition holds.

### 2.2 The null energy condition

In view of the wide range of choices available for the metric functions \( a \) and \( m \), it is useful to isolate a subset of the gMcV space-times that are potentially of physical interest by imposing a suitable condition on the stress-energy tensor \( T^a_b \) that the space-time inherits from the Einstein equations. Given that almost all the standard McVittie space-times violate the dominant energy condition — the requirement that \( \rho + p \) be a future-directed timelike or null vector for all future-directed timelike vectors \( \nu^a \) — I will instead follow Kaloper, Kleban and Martin [8] in imposing the null energy condition, which specifies that

\[
0 \leq \rho + p \leq \frac{\dot{a}/a}{a} + \frac{\dot{m}/m}{m}
\]

for all (future-directed) null vectors \( \nu^a \).

In the case of the gMcV space-times, the null energy condition reads simply

\[
G^t_t - G^r_r \geq 8a^{-1}r^2(2r + m)^{-3} |G^t_t|.
\] (13)

In particular, it follows that \( G^t_t - G^r_r \geq 0 \), which in view of (8) and (10) is equivalent to:

\[
2m(\xi/a) + 2\Delta \frac{d}{dt}(\xi/a) + \Delta^2 \frac{d}{dt}(\dot{a}/a) \leq 0
\] (14)

This in turn is true only if \( \dot{m} \xi \leq 0 \) and \( \frac{d}{dt}(\dot{a}/a) \leq 0 \). Given that \( \xi = am + \dot{a}m \), while \( a \) and \( m \) are positive by assumption and \( \dot{a} \geq 0 \) in an expanding universe, an immediate consequence is that \( \dot{m} \leq 0 \) and \( \xi \geq 0 \).

So the function \( m \) and the Hubble parameter \( \dot{a}/a \) are non-increasing, and the function \( am \) is non-decreasing, in any expanding gMcV space-time that satisfies the null energy condition. In addition, since now \( G^t_t \geq 0 \) the null energy condition (13) imposes the residual constraint

\[
G^t_t - G^r_r \geq 8a^{-1}r^2(2r + m)^{-3}G^t_t
\] (15)

or equivalently

\[
a^2(2r + m)^3(2m(\xi/a) + 2\Delta \frac{d}{dt}(\xi/a) + \Delta^2 \frac{d}{dt}(\dot{a}/a)) + 8r^2\Delta \xi \leq 0
\] (16)

This inequality needs considerable work to analyze fully, and will be examined in more detail in Section 3.2.

### 2.3 Possible matter sources

Faraoni and Jacques [9] and Carrera and Giulini [10] have already considered a number of possible matter sources for the gMcV space-times, and I will add little that is new here.

The simplest realistic matter source is a perfect fluid, with stress-energy tensor

\[
T^a_b = (\rho + p)u^a u_b + p \delta^a_b,
\] (17)

where

\[
u^a = (-g_{tt})^{-1/2} \cosh \beta \delta^a_t + (g_{rr})^{1/2} \sinh \beta \delta^a_r
\] (18)

is a radial 4-velocity field with rapidity \( \beta(t,r) \). In particular,

\[
T^r_r = \rho \sinh^2 \beta + p \cosh^2 \beta \quad \text{and} \quad T^0_0 = p.
\] (19)

Hence, the metric requirement \( G^t_t = G^0_0 \) can be satisfied by a perfect-fluid source only if

\[(\rho + p) \sinh \beta = 0. \]

But then \( T^t_t \propto (\rho + p) \sinh \beta \cosh \beta \) is also zero, and so from (9) \( \xi = 0 \).
Those gMcV space-times that do not belong to the standard McVittie class therefore require a more elaborate source model. Carrera and Giulini [10] have suggested adding a term representing a radial heat flux vector

\[ q^a = q\left((-g_{rr})^{-1/2}\sinh \beta \delta^a_r + (g_{rr})^{1/2}\cosh \beta \delta^a_r \right) \]  

(20)

orthogonal to \( u^a \), so that the stress-energy tensor becomes\(^7\)

\[ T^a_\beta = (\rho + p)u^a u_\beta + \rho \delta^a_\beta + q^a u_\beta + u^a q_\beta. \]

(21)

Then

\[ T^r_r = \rho \sinh^2 \beta + p \cosh^2 \beta + 2q \sinh \beta \cosh \beta \quad \text{and} \quad T^\theta_\theta = p, \]

(22)

and the metric condition \( G^r_r = G^\theta_\theta \) is satisfied if

\[ q = -\frac{1}{2}(\rho + p)\tanh \beta. \]

(23)

Combining the stress-energy tensor (21) with the components (8)-(10) of the Einstein tensor gives the following expressions for the phenomenological variables:

\[ 8\pi \rho = 3a^{-2}(a + 2\Delta^{-1}\xi)^2, \]

(24)

\[ 8\pi p = -3a^{-2}(a + 2\Delta^{-1}\xi)^2 - 2\Delta^{-3}(2r + m)[2m(a/a) + 2\Delta \frac{d}{dt}(\xi / a) + \Delta^2 \frac{d}{dt}(a / a)], \]

(25)

\[ 8\pi q = 32r^2a^{-2}(2r + m)^{-2}\Delta^2 \xi. \]

(26)

and

\[ \tanh \beta = 32r^2a^{-2}(2r + m)^{-3}[2m(a/a) + 2\Delta \frac{d}{dt}(\xi / a) + \Delta^2 \frac{d}{dt}(a / a)]^{-1}\Delta \xi. \]

(27)

Note that for generic choices of the metric functions \( a \) and \( m \) the right-hand side of (27) is not necessarily consistent with the constraint \(|\tanh \beta| \leq 1\). However, if source fluid satisfies the null energy condition (13) then

\[ |\tanh \beta| \leq 8a^{-1}r^{-2}\Delta(2r + m)^{-3}(T^r_r - T^\theta_\theta)^{-1}|T^r_r| \]

(28)

is automatically less than or equal to 1. So the heat flux model (21) is in some sense a natural choice for the source fluid underlying the subclass of gMcV space-times that satisfy the null energy condition.

From (24)-(27), the fluid density in the heat flux model is guaranteed to be positive or zero, and both \( \rho \) and \( q \) diverge as \( \Delta^{-2} \) in the limit \( r \rightarrow m/2 \) unless \( \xi = 0 \), in which case \( \rho \) remains bounded at \( r = m/2 \) and of course \( q = 0 \) everywhere. Also, the pressure \( p \) diverges as \( \Delta^{-3} \) unless \( m\xi = 0 \). If \( \xi = 0 \) it diverges as \( \Delta^{-1} \) unless \( a\ddot{a} - \dot{a}^2 = 0 \), in which case the solution is Kottler or Schwarzschild and the pressure remains finite at \( r = m/2 \); if \( \dot{m} = 0 \) the pressure diverges as \( \Delta^{-2} \) unless \( 2a\ddot{a} + \dot{a}^2 = 0 \), or equivalently \( a \propto r^{3/2} \), in which case \( p \) is everywhere zero. Finally, the rapidity \( \beta \) is undefined if \( \xi = 0 \), \( \dot{m}(\xi / a) = 0 \) and \( a \) is an exponential function of time, and is otherwise zero if \( \xi = 0 \). Leaving these special cases aside, \( \beta \) goes to zero linearly in \( \Delta \) as \( r \rightarrow m/2 \) unless \( \dot{m} = 0 \), in which case

\[ \lim_{r \rightarrow m/2} \tanh \beta = \frac{1}{2}(am)^{-1} \left[ \frac{d}{dt}\ln(a / a) \right]. \]

(29)

Furthermore, if the universe is expanding (\( \dot{a} \geq 0 \)) and the null energy condition is satisfied then \( \rho + p \geq 0 \), the heat flux is outwards (\( q \geq 0 \)) and the bulk flow is inwards (\( \beta \leq 0 \)).

\(^7\) Faraoni and Jacques [9] were the first to consider the addition of a heat flux vector, but chose \( q^a \) to be a purely spatial vector \( qa^{-1}[1 + m/(2r)]^{-2}\delta^a_r \).
2.4 The mass of the central object

A quasi-local mass function that is commonly used to measure the energy of spherically-symmetric
space-times is the Misner-Sharp mass $M_{\text{MS}}$, which is defined by [14]

$$1 - 2M_{\text{MS}}/r = g^{ab} \partial_a R \partial_b R,$$  

(30)

where $R$ is the geometric radius given explicitly for the gMcV space-times in (7), and $g^{ab}$ is of course
the inverse metric tensor. In the case of the gMcV space-times,

$$M_{\text{MS}} = am + \frac{1}{128} ar^{-3} (2r + m)^6 (\dot{a} + 2\Delta^{-1} \xi).$$  

(31)

It is evident from (31) that the Misner-Sharp mass is positive everywhere outside $r = m/2$ but is bounded in the vicinity of $r = m/2$ only if the space-time is McVittie (i.e. $\xi = 0$).

An alternative characterization of the Misner-Sharp mass is

$$M_{\text{MS}} = -\frac{1}{2} R^{3} R_{\theta \phi}^{\theta \phi},$$  

(32)

where the Riemann tensor component $R_{\theta \phi}^{\theta \phi}$ can be interpreted as the extrinsic curvature of the
surfaces of constant $r$ and $t$. By decomposing $R_{\theta \phi}^{\theta \phi}$ in the form

$$R_{\theta \phi}^{\theta \phi} = \frac{1}{2} (R_0^0 + R_0^0) - \frac{1}{6} R_a^a + C_{\theta \phi}^{\theta \phi},$$  

(33)

where $C_{cd}^{ab}$ is the Weyl tensor, it is possible to isolate the contributions to $M_{\text{MS}}$ of the fluid source
(the Ricci terms) and the gravitational field of the putative central object (the Weyl term), which
Carrera and Giulini [10] denote by $E_R$ and $E_W$ respectively. A straightforward calculation gives:

$$E_W = am$$  

and  

$$E_R = \frac{1}{128} ar^{-3} (2r + m)^6 (\dot{a} + 2\Delta^{-1} \xi).$$  

(34)

Thus, if the universe is expanding and the null energy condition holds, the mass $E_W$ of the central object – which Carrera and Giulini [10] call the “Weyl energy” and Nolan [4] calls “Hawking’s
renormalized quasi-local mass” – is positive and non-decreasing, as $\dot{\xi} \equiv \frac{d}{dt} (am) \geq 0$. Similarly, the energy $E_R$ of the fluid source is everywhere positive or zero, and is zero everywhere only if the
space-time is McVittie with $\dot{a} = 0$ (the Schwarzschild solution). In fact, in view of (24), $E_R = \frac{4\pi}{3} \rho R^3$
[10].

2.5 Radial null geodesics and apparent horizons

The causal structure of the gMcV space-times is largely determined by the behavior of the radial null
geodesics. These are the integral curves of the equations

$$\frac{dr}{dt} = \pm (g_{rr})^{-1/2} (-g_{tt})^{1/2} = \pm 4a^{-1} r^2 (2r + m)^{-3} (2r - m),$$  

(35)

where the upper and lower signs refer to outgoing and ingoing geodesics respectively. The corresponding null tangent vectors $u_+^a = dx^a/d\lambda$ are parallel to the null vector fields

$$v_+^a = (-g_n)^{-1/2} \sigma_n^a \pm (g_{rr})^{1/2} \sigma_r^a.$$  

(36)

That is, $u_+^a = \kappa_\pm v_+^a$ where, because $\lambda$ is an affine parameter, the conformal factors $\kappa_\pm$ are fixed (up to a positive multiplicative factor) by the equations

$$u_\pm^a \nabla_b u_\pm^a = 0.$$  

(37)

Taking the inner product of (37) with $v_{+b}$ and making use of (36) gives

$$\kappa_\pm^{-2} u_\pm^a \nabla_b u_\pm^a = -(v_\pm^a \nabla_b v_\pm^b / (v_\pm^c v_\pm^c),$$  

(38)

or equivalently
\[
\frac{d}{d\lambda} \kappa^\pm = -\left(g_{tt} g_{rr}\right)^{-1/2} \left[ \frac{\pm \hat{\partial}_r}{2} \left( g_{tt} \right)^{1/2} + \hat{\partial}_r \left( g_{rr} \right)^{1/2} \right] = a^{-1} (2r - m)^{-1} \left[ \pm 16 m r^2 (2r + m)^{-3} + (2r - m) \alpha + 2 \xi \right].
\]

(39)

Also, the expansions \( \theta^\pm = \nabla_a u^a \) of the two sets of null geodesic congruences are given by

\[
\theta^\pm = \kappa^\pm \left[ \nabla_a u^a - \left( v^a u^b \nabla_b v^a \right) / \left( v^a v^a \right) \right],
\]

(40)

or, in terms of the metric functions,

\[
\theta^\pm = 2\kappa^\pm a^{-1} (2r - m)^{-1} \left[ (2r - m) \alpha + 2 \xi \right] + 4r (2r + m)^{-3} (2r - m)^{-2}
\]

(41)

In particular, the expansions vanish whenever the term in square brackets in the first line of (41) is zero. Note that the expression \( g^{ab} \hat{\partial}_a R \hat{\partial}_b R \equiv (g_{tt})^{-1} (\hat{\partial}_t R)^2 + (g_{rr})^{-1} (\hat{\partial}_r R)^2 \) appearing in the definition (30) of the Misner-Sharp mass is proportional to \( \theta^+ \theta^- \), so the geometric radius \( R \) changes from space-like to time-like, or vice versa, whenever one of the expansions changes sign.

The apparent horizons of the gMcV space-times are defined to be the surfaces on which either \( \theta^+ \) or \( \theta^- \) changes sign. Inside a “normal” region of space-time \( \theta^+ > 0 \) and \( \theta^- < 0 \), while \( R \) is a space-like coordinate with \( \frac{d}{dt} R > 0 \) (\( \frac{d}{dt} R < 0 \)) on outgoing (ingoing) radial null geodesics. I will call any region of space-time in which both \( \theta^+ \) or \( \theta^- \) are negative “trapped”, and any region in which both expansions are positive “anti-trapped”. It is evident from (41) that \( \theta^+ > 0 \) at all points outside the surface \( r = m/2 \) in an expanding gMcV space-time that satisfies the null energy condition, and so there are no trapped regions in such space-times (or at least, not in the coordinate patch covered by the gMcV line element (6)). Nonetheless, expanding gMcV space-times do support anti-trapped regions, and the apparent horizons which form their boundaries are important for the causal structure of the space-times. Inside the anti-trapped regions, \( R \) is a time-like coordinate and \( \frac{d}{dt} R < 0 \) on all ingoing and outgoing radial null geodesics.

The behaviour of the radial null geodesics is fully described by Equations (35), (39) and (41), plus the relationship between the affine parameter \( \lambda \) and the coordinate time \( t \) on any particular geodesic, which reads

\[
\frac{dt}{d\lambda} = \left( g_{tt} \right)^{-1/2} \kappa^\pm = (2r - m)^{-1} (2r + m)^{-1} \kappa^\pm.
\]

(42)

or equivalently

\[
\lambda = \int (2r + m)^{-1} (2r - m)^{-1} dt.
\]

(43)

A radial null geodesic is future-complete if, starting from a point \((t_*, r_*)\) with \( t_* > 0 \) and \( r_* > m(t_*)/2 \), the total forward lapse in \( \lambda \) (the limiting value as \( t \to t_* \)) diverges. Otherwise, the geodesic is future-incomplete. Similarly, a radial null geodesic is past-complete if, starting from the same point \((t_*, r_*)\), the total backwards lapse in \( \lambda \) (the limiting value as \( t \to t_* \)) diverges, and is past-incomplete otherwise.

3. A phase-space description of the generalized McVittie space-times

It turns out that the analysis of the long-term behaviour and null geodesic structure of the gMcV space-times can be greatly simplified by introducing a new radial coordinate

---

8 This is in contrast to Hayward [15], who calls the two types of regions “future-trapped” and “past-trapped” respectively.
Further, the time evolution of any gMcV space-time can be represented by a trajectory in a
two-dimensional phase-space spanned by the functions \( \xi(t) = \frac{d}{dt}(am) \) and \( \sigma(t) = -a \frac{d}{dt}m \). Note
that if the space-time is expanding and satisfies the null energy condition then – because \( \xi \geq 0 \) and
\( \dot{m} \leq 0 \) – the trajectory lies at all times inside or on the boundary of the first quadrant in the
\( \sigma-\xi \) plane.

Now, since \( \Delta = mx \) and \( \dot{a}/a = (\sigma + \xi)/(am) \), the residual null energy condition (16) can be
rewritten in terms of \( \sigma, \xi \) and \( x \) as a polynomial constraint \( Q(x,t) \leq 0 \), where
\[
Q(x,t) = (am)(x(2 + x)^3[x\sigma + (2 + x)\xi] - \{2(2 + x)^3[x(2 + x)\xi + (2 + 2x + x^2)\sigma]\} - 8x(1 + x^2)\xi, \quad (45)
\]
or equivalently
\[
\quad (am)(x\sigma + (2 + x)\xi) \leq x^{-1}(2 + x)^{-3}q(x,t), \quad (46)
\]
where
\[
q(x,t) = 16\sigma + 8(5\sigma + 2\xi - 1)x + 4(11\sigma + 8\xi - 4)x^2 + 2(13\sigma + 12\xi - 4)x^3 + 8(\sigma + \xi)x^4 + (\sigma + \xi)x^5.
\]
(47)

It should be stressed that, at any given time \( t > 0 \), the inequality (46) must be satisfied for all values of
\( x \in (0, \infty) \). The interesting and non-trivial constraints this places on the evolution of the gMcV space-
times will be examined shortly.

Similarly, since \( dx/dt = m^{-1}[2dr/dt - (x+1)m] \) the radial null geodesic equation (35) becomes:
\[
dx/dt = (am)^{-1}(2 + x)^{-3}(1 + x)n_\pm(x,t), \quad (48)
\]
where
\[
n_\pm(x,t) = 8\sigma + 2(6\sigma \pm 1)x + 2(3\sigma \pm 1)x^2 + \sigma x^3. \quad (49)
\]

Also, the equation (39) for the conformal factor \( \kappa_\pm \) – once (42) has been used to convert the \( \lambda- \)
derivative to a \( t \)-derivative – reads:
\[
\frac{d}{dt}(\ln \kappa^{-1}_\pm) = (am)^{-1}(2 + x)^{-4}[p_\pm(x,t) \pm 2(1 + x)(2 + 2x - x^2)], \quad (50)
\]
while the equation (41) for the expansions becomes:
\[
\theta_\pm/\kappa^{-1}_\pm = 2(am)^{-1}x^{-1}(2 + x)^{-3}p_\pm(x,t), \quad (51)
\]
where
\[
p_\pm(x,t) = 16\xi + 8(4\xi + \sigma)x + 2(12\xi + 6\sigma \pm 1)x^2 + 2(4\xi + 3\sigma \pm 1)x^3 + (\xi + \sigma)x^4.
\]
(52)

Finally, the equation (43) for the affine parameter can be recast as:
\[
\lambda = \int (2 + x)^{-1}x \kappa^{-1}_\pm dt. \quad (53)
\]

3.1 Emergence and development of apparent horizons
It has already been remarked that expanding gMcV space-times do not contain trapped regions. This is
evident from the fact that the polynomial \( p_+(x,t) \), defined in (52), is positive definite for all \( x > 0 \) if
\( \sigma \) and \( \xi \) are non-negative.

However, the polynomial \( p_-(x,t) \) is not necessarily positive definite, as the coefficients
\( 2(12\xi + 6\sigma - 1) \) and \( 2(4\xi + 3\sigma - 1) \) can be negative. In fact, if degeneracies are ignored, either the two
coefficients are both positive or both negative, or \( 4\xi + 3\sigma - 1 \) is negative and
\( 12\xi + 6\sigma - 1 \geq 4\xi + 3\sigma - 1 \) is positive. In the first case \( p_- \) has no positive roots, and in the remaining
cases the coefficients of \( p_- \) change sign twice and so there are either no positive roots or two positive roots.

The boundary in the \( \sigma \)-\( \xi \) plane between the regimes where \( p_- \) has no positive roots and \( p_- \) has two positive roots can be found by solving the equations \( p_- = 0 \) and \( \frac{\partial}{\partial x} p_- = 0 \) simultaneously for \( \sigma \) and \( \xi \). This gives the parametric relations

\[
\sigma = (2 + x_h)^{-3} x_h^2 (4 + 4 x_h - x_h^2) \quad \text{and} \quad \xi = (2 + x_h)^{-4} x_h^2 (x_h^2 - 2x_h - 2),
\]

where \( x_h \) is the value of \( x \) at which the double root of \( p_-(x,t) \) occurs. A plot of \( \xi \) against \( \sigma \) on this boundary curve – which is labeled \( \Gamma_h \) – is shown in Figure 1. If \( (\sigma, \xi) \) lies inside \( \Gamma_h \) the space-time is “normal” over some compact, connected range of \( x \) on the corresponding constant-\( t \) hypersurface, while if \( (\sigma, \xi) \) lies outside \( \Gamma_h \) the space-time is anti-trapped at all points on the corresponding hypersurface.

Note that the value of \( x_h \) always lies between the positive root of \( x^2 - 2x - 2 \) (which corresponds to \( \xi = 0 \)) and the positive root of \( x^2 - 4x - 4 \) (corresponding to \( \sigma = 0 \)). That is, \( 1 + \sqrt{3} \leq x \leq 2(1 + \sqrt{2}) \) at any point where a “normal” region of space-time first appears. Also, the maximum values of \( \sigma \) and \( \xi \) on the boundary curve \( \Gamma_h \) are \( \frac{1}{8} \) and \( \sqrt{3}/9 \approx 0.19245 \) respectively.

![Figure 1](image.png)

**Figure 1.** Regions in the \( \sigma \)-\( \xi \) plane that prove to be important for the long-term evolution of the gMcV space-times

In the McVittie case, \( \xi = 0 \) and \( \dot{\sigma} = M \frac{dt}{dt} (\dot{a}/a) \leq 0 \) according to the null energy condition, so all McVittie space-times can be represented by a point moving leftwards along the \( \sigma \)-axis in Figure 1. Thus, once a McVittie space-time enters the parametric regime where \( p_- \) has two positive roots (inside \( \Gamma_h \)) it will never subsequently leave it. The region of “normal” space-time in a McVittie space-time (if one forms at all) therefore appears with the bifurcation of two apparent horizons at \( x = 1 + \sqrt{3} \) at a time \( t = t_h \) for which \( \sigma(t_h) = \sqrt{3}/9 \). As will be seen shortly, the inner horizon in a McVittie space-time moves monotonically inwards (that is, with \( \frac{dt}{dt} x \leq 0 \)) and the outer horizon monotonically outwards for all \( t > t_h \).
The persistence and expansion of the “normal” region of space-time in the McVittie case is just an example of a well-known theorem, due to Hayward [15], which states that the area of an inner (outer) apparent horizon decreases (increases) monotonically if the null energy condition is satisfied. It should therefore come as no surprise that analogous results exist for the generalized McVittie space-times.

In fact, if the parametric equations (54) for $h^\Gamma$ are substituted into the polynomial $q(x,t)$ defined by (47), it is easily checked that $q(x_h,t) = 0$ and so the null energy condition (46) at $x = x_h$ becomes

$$\dot{\xi} \leq -(2 + x_h)^{-1} x_h \dot{\sigma}$$

(55)
on $\Gamma_h$. Furthermore, it can be seen by differentiating the two parametric equations in (54) that the direction tangent to $\Gamma_h$ has

$$\frac{d}{dt} x = -(2 + x_h)^{3/2} [x \dot{\sigma} + (2 + x) \dot{\xi}] / \partial_{x} p_-$

(57)

Combining (57) and (58) gives:

$$\frac{d}{dt} R = -\frac{1}{2}(1 + x)^{3/2} (2 + x) [Q(x,t) - (3x + 4) \dot{\xi} p_-(x,t)] / \partial_{x} p_- ,$$

(59)

where $Q(x,t) \leq 0$ is defined in (45). Since $p_- = 0$ at an apparent horizon, and it is clear from the equation (52) for $p_-$ that $\partial_{x} p_- < 0$ at the inner horizon and $\partial_{x} p_+ > 0$ at the outer horizon, it follows that the area of the inner horizon must be decreasing ($\frac{d}{dt} R \leq 0$) and the area of the outer horizon must be increasing ($\frac{d}{dt} R \geq 0$), as expected.

Equation (57) also gives some clues to the behavior of $\frac{d}{dt} x$ on the two horizons. Since $\frac{d}{dt} R \leq 0$ on the inner horizon, $\frac{d}{dt} x$ must be strictly negative there in a gMcV space-time with $\xi > 0$. However, if $\xi > 0$ then $\frac{d}{dt} x$ can in principle be either positive or negative (or zero) on the outer horizon. In the McVittie case ($\xi = 0$) it is clear from (57) that $\frac{d}{dt} R$ and $\frac{d}{dt} x$ everywhere have the same sign, and so $\frac{d}{dt} x \geq 0$ ($\leq 0$) on the outer (inner) horizon, as mentioned above.

3.2 Long-term evolution of the generalized McVittie space-times

Consider now the constraints that the residual null energy condition (46) places on the time-development of the gMcV space-times. In the McVittie case (46) reduces to the inequality $\dot{\sigma} \leq 0$ and so adds nothing new to the problem, but if $\xi > 0$ it does place a non-trivial restriction on the direction of the vector $(\dot{\sigma}, \dot{\xi})$ if the polynomial $q(x,t)$ is negative or zero for at least one positive value of $x$.

From (47), it is evident that $q(x,t)$ – like $p_-(x,t)$ – has either two or no positive roots if $\sigma$ and $\xi$ are non-negative. By solving the equations $q = 0$ and $\frac{\partial}{\partial x} q = 0$ simultaneously for $\sigma$ and $\xi$, it can be
seen that the boundary between the parametric regimes where \( q \) has no positive roots and \( q \) has two positive roots has the form
\[
\sigma = 4(2 + x_n)^{-3}x_n^3 \quad \text{and} \quad \xi = 4(2 + x_n)^{-4}(2 + 4x_n^2 - x_n^4),
\]
where \( x_n \) is the value of \( x \) at which the double root of \( q \) occurs.

This boundary is plotted as the curve labeled \( \Gamma_n \) in Figure 1. It cuts the \( \xi \)-axis at \( \xi = \frac{1}{2} \), and the \( \sigma \)-axis at \( \sigma = \frac{4\sqrt{5}}{3}(2\sqrt{3} - 3)^{3/2} \approx 0.24339 \). The polynomial \( q(x,t) \) is positive definite outside \( \Gamma_n \), but is negative for some range of positive values of \( x \) at all points inside \( \Gamma_n \). Note that the first equation in (60) can be solved to give:
\[
x_n(\sigma) = 2(4^{1/3} - \sigma^{1/3})^{-1}\sigma^{1/3},
\]
and if this is substituted into the second equation in (60) the boundary curve \( \Gamma_n \) can be represented explicitly as
\[
\xi = \frac{1}{8}[4 - 12(2^{-1/6}\sigma^{1/3})^2 - 3(2^{-1/6}\sigma^{1/3})^4] = \xi_n(\sigma).
\]
[It is from this equation that the ordinate \( \sigma = \frac{4\sqrt{5}}{3}(2\sqrt{3} - 3)^{3/2} \) is taken.]

The actual form of the restrictions that apply inside \( \Gamma_n \) can be analyzed by returning to the inequality \( Q(x,t) \leq 0 \) and solving the polynomial equations \( Q = 0 \) and \( \frac{d}{dx}Q = 0 \) for \( \sigma \) and \( \dot{\xi} \). On defining \( S = (am)\sigma/\xi \) and \( T = (am)^{2/3}\xi - \xi \), this gives:
\[
S = 8(2 + x)^{-3}x(1 + x) - 2x^{-2}(1 + x)^2\sigma
\]
and
\[
T = -8(2 + x)^{-4}(1 + x)(1 + x + x^2) + x^{-1}(2 + x)^2\sigma.
\]

For each fixed value of \( \sigma \), equations (63) and (64) describe a curve in the plane of values of \( S \) and \( T \). Since
\[
\frac{1}{2}x^3(2 + x)^4 \frac{d}{dx}S = -\frac{1}{2}x^2(2 + x)^5 \frac{d}{dx}T
\]
\[
= 32\sigma + 80\sigma x + 80\sigma x^2 + 8(1 + 5\sigma)x^3 + 2(4 + 5\sigma)x^4 + (\sigma - 4)x^5,
\]
each curve starts with \( S \ll -1 \) and \( T \gg 1 \) for small values of \( x \) (provided that \( \sigma > 0 \)), approaches and crosses the \( T \)-axis with \( \frac{d}{dx}S > 0 \) and \( \frac{d}{dx}T < 0 \), until (if \( \sigma < 4 \)) there is a simultaneous turning point in \( S \) and \( T \) at some point on a locus that connects \( (S, T) = \left( \frac{4\sqrt{5}}{9}, -\frac{2}{3} \right) \) (for \( \sigma \to 0 \)) to \( (S, T) = (0, 4) \) (for \( \sigma \to 4 \)), after which the curve continues with \( \frac{d}{dx}S < 0 \) and \( \frac{d}{dx}T > 0 \) to \( (S, T) = (0, \sigma) \).

Now, unlike the polynomials \( p_- \) and \( q \) considered earlier, the number of positive roots of \( Q \) can increase or decrease by 1, so new positive roots need not appear in pairs. This is because the coefficient of the highest power (\( x^5 \)) in \( Q \), which is \( am(\sigma + \dot{\xi}) - \xi(\sigma + \dot{\xi}) \), has no fixed sign, and single positive roots can appear or disappear (at infinity) whenever this coefficient is zero. The locus of points in the \( S-T \) plane on which this occurs is the straight line
\[
S + T = \sigma,
\]
which joins the curve (63) and (64) for the double roots of $Q$ at $(S,T)=(0,\sigma)$, but also intersects the curve before the latter reaches its turning point.\(^9\)

For $\sigma > 0$, the union of the curve (63) and (64) for the double roots with the straight line $S + T = \sigma$ covers all the points in the $S$-$T$ plane where positive roots of $Q$ appear or disappear, and has a characteristic “swallow-tail” shape.\(^10\) The boundary curves are plotted for the cases $\sigma = 0$ and $\sigma = \sqrt[9]{3}$ in Figure 2. For each fixed value of $\sigma$, the polynomial $Q$ is negative definite (for $x > 0$) only in the domain that lies entirely on or below this set of points, and so is bounded above by the curve for the double roots as far as the point $C$ where it intersects the line $S + T = \sigma$, and then above by the straight line as it goes off to infinity in the fourth quadrant.

**Figure 2.** The boundary curves defined by (63) and (64), or (66), in the cases $\sigma = 0$ and $\sigma = \sqrt[9]{3}$. For each value of $\sigma$, the allowable region in the $S$-$T$ plane lies on or below the thickened parts of the curve.

The constraints that the null energy condition places on the phase-space evolution of the gMcV space-times can now be calculated as follows. From the definitions of $S$ and $T$, the zero point of $(\sigma, \dot{\xi})$ lies at $(S,T) = (0,-\xi)$, which for $\xi \geq 0$ is a point on the $T$-axis on or below the $S$-axis in Figure 2. If, \(^9\) This can be seen by substituting the expression (72) for $\sigma(x)$ at the intersection point $C$ into the right-hand side of (65), and verifying that the polynomial that results, $4x^2(4x^2 + x - 2)$, is positive definite for $|x| \geq 1$ (the range of values for which $\sigma(x) \geq 0$).

\(^10\) The limiting case when $\sigma = 0$ is slightly more complex. Here, the curve for the double roots does not start at infinity in the second quadrant, but instead at $(S,T) = (0,-\frac{1}{2})$ with $dT/dS = 0$. It initially descends to a simultaneous turning point at $(S,T) = (\frac{4\sqrt[9]{3}}{9}, -\frac{2}{3})$, then asymptotes to the origin. Single positive roots appear or disappear on the straight line $S + T = 0$, which of course goes through the origin and cuts the first curve to form a “swallow-tail”. However, because the constant term in $Q$ is zero when $\sigma = 0$, further single positive roots can appear or disappear when the coefficient of $x$, which is $8[2(am)\xi - 2\xi^2 - \xi]$, vanishes. This corresponds to a second, horizontal, line $T = -\frac{1}{2}$, which completes the boundary of the region where $Q$ is negative definite in the third quadrant.
for given values of $\sigma$ and $\xi$, the point $(0, -\xi)$ lies inside the region where $Q$ is negative definite, all possible values of $d\xi/d\sigma$ are permitted by the null energy condition, or equivalently $(\sigma, \xi)$ lies outside the curve $\Gamma_n$ in Figure 1.

On the other hand, if $(0, -\xi)$ lies outside the region where $Q$ is negative definite, the permissible values of $d\xi/d\sigma$ are the gradients of those lines in Figure 2 which start at $(0, -\xi)$ and subsequently pass through at least one point in the region where $Q$ is negative definite. So if $\sigma < 0$ then $d\xi/d\sigma$ must be greater than or equal to the gradient of the tangent line from $(0, -\xi)$ to the boundary curve (63) and (64) in the half-plane $S < 0$, while if $\sigma > 0$ then $d\xi/d\sigma$ must be less than or equal to the gradient of either the tangent line from $(0, -\xi)$ to the boundary curve in the half-plane $S > 0$ or of the line segment from $(0, -\xi)$ to the intersection point $C$. That there is a tangent line from $(0, -\xi)$ to the boundary curve on the left, and that the line from $(0, -\xi)$ to $C$ does not otherwise cut the boundary curve on the right, follows from the fact that the curve (63) and (64) is concave downwards and has a shallower gradient than the line $S + T = \sigma$ everywhere to the left of $C$. This can in turn be seen by noting that

$$
\frac{dT}{dS} = -(2 + x)^{-1} x > -1 \quad \text{and so} \quad \frac{dT^2}{dS^2} = -2(2 + x)^{-2}(dS/dx)^{-1} < 0 \quad (67)
$$
on the curve (63) and (64) (before the turning point), while of course $dT/dS = -1$ on the line $S + T = \sigma$.

If it is assumed that $d\xi/d\sigma$ takes on either of its two extreme possible values, then from the definitions of $S$ and $T$

$$
d\xi/d\sigma = (T + x)/S, \quad (68)
$$
with the right-hand side evaluated at the point where the line from $(0, -\xi)$ touches the boundary curve.

If the point of contact is a tangent point then, from (67), $d\xi/d\sigma = -(2 + x)^{-1} x$. This equation is equivalent to the constraint $q(x, t) = 0$, as can be seen by combining it with (68) and the expressions (63) and (64) for $S$ and $T$.

The equation $q(x) = 0$ can now be used to solve for $\xi$ and $d\xi/dx$ in terms of $x$, $\sigma$ and $d\sigma/dx$, and eliminating $\xi$ and $d\xi/dx$ from (68) results in a linear ODE for $\sigma$

$$
\frac{d\sigma}{dx} - 2x^{-1}(2 + x)^{-1}\sigma = -8(2 + x)^{-4} x^2, \quad (69)
$$
whose general solution is

$$
\sigma = (2 + x)^{-3} x(1 + x) + K (2 + x)^{-1} x, \quad (70)
$$

where $K$ is a constant of integration. Substituting this equation back into $q(x, t) = 0$ and solving for $\xi$ then gives

$$
\xi = -8(2 + x)^{-4}(1 + x)(1 + x + x^2) - K(2 + x)^{-2}(2 + 2x + x^2). \quad (71)
$$

On the other hand, if the point of contact is the intersection point $C$ then substituting the expressions (63) and (64) for $S$ and $T$ into the equation (66) for the straight-line boundary gives a polynomial equation which can be solved directly to give:

$$
\sigma = 4(2 + x)^{-4} x^2(x^2 - 1). \quad (72)
$$

If this equation is used to eliminate $\sigma$ and $d\sigma/dx$ from (68) then a linear equation for $\xi$ results:

$$
(4x^2 + x - 2)^{-1}(1 + 2x) \frac{d\xi}{dx} - (1 + x)^{-1}(2 + x)^{-1} x \xi = (2 + x)^{-5} x(1 + x)(x^2 - 2x - 2), \quad (73)
$$

which solves to give:

$$
\xi = -4(2 + x)^{-4} x^2(x^2 - 1) + \frac{661}{2\sqrt{3}} K(2 + x)^{-8} (1 + x)(1 + 2x)^{1/2} e^{2(x-1)}, \quad (74)
$$
where \( \overline{K} \) is a second constant of integration.

Together, equations (70) and (71) describe a family of curves that doubly foliate the region inside \( \Gamma_n \) in Figure 1. Each curve starts on the \( \xi \)-axis with \( x = 0 \) and \( \xi = -\frac{1}{2}(K+1) \). As \( x \) increases from 0, the point \( (\sigma, \xi) \) follows the shallower of the two sets of curves in Figure 1, with \( \sigma \) increasing and \( \xi \) decreasing. Given that \( 0 \leq \xi < \frac{1}{2} \) inside \( \Gamma_n \), the only relevant values of \( K \) on the shallower curves are \(-2 < K < -1\).

When \( x \) is equal to the positive root of the equation \( (K+2)(1+x) + \frac{1}{4}K x^2 = 0 \), the point \( (\sigma, \xi) \) lies on \( \Gamma_n \), and here crosses over to the steeper of the two sets of curves. It then follows these curves, with \( \sigma \) decreasing and \( \xi \) increasing, until it returns to the \( \xi \)-axis when \( x \) has reached the positive root of \( (K+2)(1+x) + \frac{1}{4}K x^2 = 0 \).\(^{11}\) The relevant values of \( K \) on the steeper set of curves are \(-2 < K < 0\). Note that the curve \( \Gamma_h \) bounding the solutions that contain apparent horizons corresponds to the steep curve with \( K = -1 \).

Throughout much of the region inside \( \Gamma_n \), the family of curves described by (72) and (74) are almost indistinguishable from the steeper set of curves described by (70) and (71). They start on the \( \xi \)-axis with \( x = 1 \) and \( \xi = \overline{K} \), and provided that \( \overline{K} > 0.1685 \) pass through a horizontal turning point before ever reaching the \( \sigma \)-axis. Nonetheless, the turning points are of little consequence for the evolution of the gMeV space-times, as the curves described by (72) and (74) constrain the vector \( (\sigma, \dot{\xi}) \) only inside the curve marked \( \Gamma_C \) in Figure 1, and all the turning points on the curves (72) and (74) lie outside \( \Gamma_C \). Between \( \Gamma_C \) and \( \Gamma_n \), the vector \( (\dot{\sigma}, \dot{\xi}) \) is instead constrained (for phase curves with \( \dot{\sigma} > 0 \)) by the steeper set of curves described by (70) and (71).

The curve \( \Gamma_C \) itself is defined by the condition that the tangent line from the point \( (0, -\xi) \) in Figure 2 touches the boundary curve in the right-half plane at the point where this curve intersects the straight line \( S + T = \sigma \). Substituting (70) and (71) into (63) and (64) and imposing (66) gives an equation for \( \lambda \) as a function of \( x \) which reduces (70) and (71) to

\[
\sigma = 4(2 + x)^{-4} x^2 (x^2 - 1) \quad \text{and} \quad \dot{\xi} = -4(2 + x)^{-5} (1 + x)(x^4 + x^3 - 2x^2 - 8x - 4).
\]

The corresponding curve \( \Gamma_C \) cuts the \( \xi \)-axis in Figure 1 at \( \xi = \frac{32}{81} \approx 0.3951 \), and the \( \sigma \)-axis at \( \sigma \approx 0.2196 \). As a result, the only physically relevant values of \( \overline{K} \) are \( 0 < \overline{K} < \frac{32}{81} \).

To summarize the constraints that the null energy condition places on the evolution of gMeV space-times inside \( \Gamma_n \) in Figure 1, phase curves with \( \dot{\sigma} < 0 \) must lie on or below the shallower set of curves described by (70) and (71), and phase curves with \( \dot{\sigma} > 0 \) must either lie either lie or on below the steeper set of curves described by (70) and (71) (between \( \Gamma_C \) and \( \Gamma_n \)) or lie or on below the curves described by (72) and (74) (inside \( \Gamma_C \)).

It is evident from the picture presented in Figure 1, therefore, that region of phase space from which the gMeV space-times cannot escape includes not only the region inside the curve \( \Gamma_h \), but a slightly larger region bounded by the steep curve, labeled \( \Gamma^* \), that passes through the point \( (\sigma, \xi) \approx (0.24339, 0) \) where \( \Gamma_n \) cuts the \( \sigma \)-axis. This is the steeper curve from the set defined by (70).

\(^{11}\) However, these steep curves are shown in Figure 1 only as far as the curve marked \( \Gamma_C \). Inside \( \Gamma_C \) the steeper set of constraining curves, shown in Figure 1, satisfy (72) and (74).
and (71) that has \( K = 4\sqrt{3} - 8 \approx -1.072 \), and its continuation inside \( \Gamma_c \) from the set defined by (72) and (74), which has \( \bar{K} \approx 0.1501 \).

By contrast, the evolution of gMcV space-times represented by phase points outside \( \Gamma_n \) is virtually unconstrained. The phase curves can arbitrary form in this region, although the null energy condition continues to place restrictions on the magnitudes of \( \dot{\sigma} \) and \( \dot{\xi} \). In particular, any space-time with \( \dot{\sigma} \) and \( \dot{\xi} \) constant satisfies the null energy condition, provided that the point \((\sigma_1, \xi_1)\) lies on or outside \( \Gamma_n \).

Other possible solutions of interest include space-times that cycle endlessly between the regions inside and outside \( \Gamma_n \). As an example, consider a phase curve that starts at a point \((\sigma_1, \xi_1)\) on \( \Gamma_n \) with \( 0 < \xi_1 < \xi_n \), traces the shallow curve left from \((\sigma_1, \xi_1)\), then crosses over onto one of the steep curves before reaching the steep curve that passes through \((\sigma, \xi) \approx (0.2196, 0)\), and follows the steep curve down and to the right until it returns to \((\sigma_1, \xi_1)\) on \( \Gamma_n \). In view of (46), the two-straight line segments will satisfy the null energy condition provided that (77) at each point on the respective segments. The phase loop described here can clearly be repeated indefinitely.

### 3.3 Further results

Two further results that can be established on the basis of the phase-plane analysis illustrated in Figures 1 and 2, but are too technical to be worthy of a detailed proof here, are:

1. For any expanding gMcV space-time that satisfies the null energy condition and whose phase curve enters the region inside the curve \( \Gamma^* \), the Weyl energy \( \text{am} \) must tend to a finite positive limit \( \lim_{t \to \infty} \text{am} \) as \( t \to \infty \). This follows again from the null energy condition (46), which can be recast as the requirement that

\[
(\text{am})\frac{\dot{\xi}}{\xi} \leq \min_{x > 0} [x^{-1}(2 + x)^{-4} q(x, t)] \quad \text{and} \quad (\text{am})\frac{\dot{\sigma}}{\sigma} \leq \min_{x > 0} [x^{-2}(2 + x)^{-3} q(x, t)]
\]

(76)

at each point on the respective segments. The phase loop described here can clearly be repeated indefinitely.

2. Any expanding gMcV space-time that satisfies the null energy condition and whose phase curve enters the region inside the curve \( \Gamma^* \) must have \( \lim_{t \to \infty} \xi(t) = 0 \).
and technical to be worth reproducing here, but hinges on the fact that if \( \xi \) does not tend to zero but is also not bounded away from zero (to avoid \( am \to \infty \)) then there must be a positive constant \( \xi^* \) and an infinite sequence of disjoint intervals \( \{I_k = [t_k, t_k + \epsilon_k]\} \) with finite total measure and \( \xi(t_k) = \frac{1}{2} \xi^* \leq \xi(t) \leq \xi(t_k + \epsilon_k) \) on \( I_k \). What this means in effect is that \( \hat{\xi} \) is unbounded above, and in view of the parametric equations (63) and (64) for the boundary of the allowable region in \( S-T \) space if \( (\sigma, \xi) \) is inside \( \Gamma_n \), which imply that \( \dot{\sigma} \propto -\dot{\xi}^2 \) in the limit of large \( \dot{\xi} \), it can be shown that \( \hat{\sigma} \) diverges so rapidly that \( \sigma \) must eventually become negative.

3. For any expanding gMcV space-time that satisfies the null energy condition and whose phase curve enters the region inside the curve \( \Gamma^* \), the function \( \sigma \) must tend to a finite limit \( \sigma_{\infty} \) as \( t \to \infty \), with \( 0 \leq \sigma_{\infty} < \sigma^* \) where \( \sigma^* = \frac{4\sqrt{3}}{9} (2\sqrt{3} - 3)^{3/2} \approx 0.24339 \) is the value of \( \sigma \) where \( \Gamma^* \) meets the axis \( \xi = 0 \). To verify this result, note first that those steep curves described by equations (70) and (71) or (72) and (74) that cut the axis \( \xi = 0 \) all have gradients \( |d\xi/d\sigma| \) bounded below by

\[
L = \sqrt{2/\sqrt{3}} - 1 \approx 0.39332, \text{which is the gradient of curve } \Gamma^* \text{ at the point where it touches the axis } \xi = 0. \text{ If } \sigma \text{ does not converge to a limit } \sigma_{\infty} \text{ on a particular trajectory inside } \Gamma^* \text{ then there must exist two constants } 0 \leq \sigma_1 < \sigma_2 \leq \sigma^* \text{ with the property that, for all times } \tilde{t} > 0, \inf_{t \geq \tilde{t}} \sigma(t) \leq \sigma_1 \text{ and } \sup_{t \geq \tilde{t}} \sigma(t) \geq \sigma_2 \text{. Since } \xi \to 0 \text{ from Result 2, for every } \varepsilon > 0 \text{ there exists a time } t_\varepsilon > 0 \text{ for which } \xi(t_\varepsilon) < \varepsilon \text{ and } \sigma(t_\varepsilon) - \sigma_1 < \varepsilon \text{. But whenever } \sigma \text{ is increasing with } t, \text{ the phase curve of the space-time must lie on or below the steeper family of curves in Figure 1, and so } \sigma(t) \leq \sigma_1 + \varepsilon + L^{-1} \varepsilon \text{ for all } t > t_\varepsilon \text{. Fixing } \varepsilon < (1 + 2L)^{-1}L(\sigma_2 - \sigma_1) \text{ therefore guarantees that } \sigma(t) < \sigma_2 - \varepsilon \text{ for all } t > t_\varepsilon \text{, which contradicts the assumption that } \sup_{t \geq \tilde{t}} \sigma(t) \geq \sigma_2 \text{.}

Together, these results imply that any gMcV space-time that develops an apparent horizon (and thereby enters the region inside \( \Gamma_n \)) inevitably asymptotes to a standard McVittie space-time with \( \xi = 0 \) and \( \sigma \) constant as \( t \to \infty \).

4. Radial null geodesics

4.1 Existence and uniqueness

In contrast to the null energy condition, the equation (48) for radial null geodesics in the gMcV space-times, which can be written as

\[
dx/dt = (am)^{-1}[(1 + x)\sigma \pm 2(2 + x)^{-3}(1 + x)^2 x], \tag{80}
\]

is relatively easy to analyze. Apart from the pre-factor \( (am)^{-1} \) – which is positive and constant (in the McVittie case) or decreasing – the right-hand side of (80) depends on \( t \) only through \( \sigma \). As a result, there is very little difference between the behavior of the radial null geodesics in a gMcV space-time and in the corresponding McVittie space-time (with \( \xi = 0 \) but \( \sigma \) unchanged).

Nolan [4] has shown that, with \( a \) assumed to have a power-law dependence on \( t \) (so that \( m = M/a \) has the inverse dependence, and \( \sigma \propto t^{-1} \)), there are unique ingoing and outgoing radial null geodesics through each point \( (t, r) = (t_*, r_*) \) with \( t_* > 0 \) and \( r_* > m/2 \), and that every point in this family has an initial point on the surface \( r = m/2 \) at some time \( t > 0 \). The analogs of these results for the gMcV space-times – with no specific functional forms assumed for \( a \) or \( m \) – can be established as follows.

Note first that the form of the term in (80) proportional to \( \sigma \) suggests replacing \( x \) as the dependent variable with \( u = \ln(1 + x) \). The equation then becomes
\[
\frac{du}{dt} = (am)^{-1}v_\pm(u,t) \quad \text{with} \quad v_\pm(u,t) = \sigma \pm 2(1 + e^u)^{-3}(e^u - 1)e^u. \tag{81}
\]

Furthermore, at each fixed value of \( t > 0 \), both \( v_\pm \) and \( \partial v_\pm/\partial u \) are bounded:
\[
|v_\pm(u,t)| \leq \sigma(t) + \sqrt{3}/9 \quad \text{and} \quad \left|\frac{\partial}{\partial u}v_\pm(u,t)\right| \leq \frac{1}{4} \quad \text{for all} \quad u \in (-\infty, \infty),
\tag{82}
\]

while the functions \( a \) and \( m \) are assumed to be \( C^\infty \) for all \( t \in (0, \infty) \). Hence, by the Picard-Lindelöf Theorem, unique ingoing and outgoing solutions to the equation (81) through each point \((t, u) = (t_*, u_*)\) with \( t_* > 0 \) and \( u_* \geq 0 \), and each solution can be extended to all \( t > t_* \). With \( u \) replaced by \( x \) and \( u_* \) by \( x_* - \) noting that \( u_* = 0 \) corresponds to \( x_* = 0 \) – the same statement is true of the original equation (80).

### 4.2 Outgoing radial null geodesics

In order to analyze the asymptotic properties of the radial null geodesics (both outgoing and ingoing) through a given point \((t, u) = (t_*, u_*)\), it is useful to introduce the following characteristic integrals:
\[
\tau_\infty \equiv \int_{t_*}^\infty (am)^{-1} dt, \quad \mu_\infty \equiv \int_{t_*}^\infty \sigma^{-1} dt = -\lim_{t \to \infty} \ln[m(t)/m_*],
\]
\[
\tau_0 \equiv \int_{t_*}^{0} (am)^{-1} dt \quad \text{and} \quad \mu_0 \equiv \int_{t_*}^{0} \sigma^{-1} dt = -\lim_{t \to 0^+} \ln[m(t)/m_*],
\tag{83}
\]

where \( m_* \equiv m(t_*) \).

Because \( \frac{dt}{d\tau}(am) \geq 0 \) and \( m \leq 0 \), in the limit \( t \to \infty \) the Weyl energy \( am \) remains finite and positive or goes to \( \infty \), while \( m \) remains finite and positive or goes to \( 0 \); and in the limit \( t \to 0^+ \) the reverse is true: \( am \) remains finite and positive or goes to \( 0 \), while \( m \) remains finite and positive or goes to \( \infty \). So the integrals \( \tau_\infty \) and \( \mu_\infty \) are either finite and positive or diverge to \( \infty \), while \( \tau_0 \) and \( \mu_0 \) are either finite and negative or diverge to \( -\infty \). Although all four integrals depend on the geodesic in question through the value of \( t_* \), their convergence or divergence are characteristics of the space-time only. The McVittie space-times, for example, all have \( \tau_\infty = \infty \) and \( \tau_0 \) finite, while – in view of Result 1 in Section 3.3 – any gMcV space-time that enters the region inside \( \Gamma^* \) in Figure 1 must also have \( \tau_\infty = \infty \).

It is now possible to determine the fate of the outgoing radial null geodesics at early and late times. From (81),
\[
(\text{am})^{-1} \sigma \leq \frac{du}{dt} \leq (\text{am})^{-1}(\sigma + \frac{\sqrt{3}}{9}) \tag{84}
\]
along the geodesics, and so integrating this inequality forwards and backwards from \( t = t_* \) gives
\[
\mu_{\infty} \leq \lim_{t \to \infty} (u - u_*) \leq \mu_\infty + \frac{\sqrt{3}}{9} \tau_\infty \quad \text{and} \quad \mu_0 \geq \lim_{t \to 0^+} (u - u_*) \geq \mu_0 + \frac{\sqrt{3}}{9} \tau_0 \tag{85}
\]
in the respective cases.

It follows from the first inequality in (85) that if \( \mu_\infty \) and \( \tau_\infty \) are both finite then \( u \) (and so \( x \)) remains bounded as \( t \to \infty \), while if \( \mu_\infty \) diverges then \( u \) and \( x \) also go to \( \infty \) as \( t \to \infty \). The one indeterminate possibility, when \( \mu_\infty \) is finite and \( \tau_\infty \) diverges, can be dealt with by noting from (81) that
\[
\frac{du}{dt} \geq 2(\text{am})^{-1}(1 + e^u)^{-3}(e^u - 1)e^u \tag{86}
\]
on the outgoing geodesics, which after integrating forwards from \( t = t_* \) gives:
\[
[cosh u + 4\ln(e^u - 1) - 2u]u_{t_*} \geq \int_{t_*}^{t} (\text{am})^{-1} dt. \tag{87}
\]
If \( \tau_\infty \) diverges this inequality can be satisfied only if \( u \) and \( x \) diverges. So \( x \to \infty \) as \( t \to \infty \) in all cases except when \( \mu_\infty \) and \( \tau_\infty \) are both finite.

When the geodesics are traced backwards towards \( t \to 0 \), there are two possible outcomes. Either \( \lim_{t \to 0^-} (u - u*) > -u* \), in which case the geodesic begins at \( t = 0 \) which may or may not mark the location of a cosmological singularity – at some value of \( x > 0 \), or \( \lim_{t \to 0^+} (u - u*) \leq -u* \) and the geodesic emerges from the singularity at \( x = 0 \) (the surface \( r = m/2 \)) at some time \( t \geq 0 \). From the second inequality in (85) it follows that both possibilities remain open if \( \mu_0 \) and \( \tau_0 \) are finite, but that all outgoing geodesics emerge from \( x = 0 \) if \( \mu_0 = -\infty \). The one remaining case, when \( \mu_0 \) is finite and \( \tau_0 \) diverges, can be treated by integrating (86) backwards from \( t = t* \) to give:

\[
[cosh u + 4ln(e^u - 1) - 2u]^\frac{1}{2} \leq \int_{t*}^{t} (am)^{-1}dt , \tag{88}
\]

from which it follows that \( u \) and \( x \) must go to 0 at some positive time if \( \tau_0 = -\infty \).

The coordinate \( x \) has, of course, no intrinsic physical significance, so it is instructive also to examine the behavior on the outgoing radial geodesics of the geometric radius \( R \), defined in equation (56). Because \( x \) is monotonically increasing along the outgoing geodesics, as is evident from (80), it is clear from the equation (57) for \( \frac{d}{dt} R \) that the same is true of \( R \). Furthermore, since \( am \) is a positive increasing function of \( t \), it follows from (56) that if \( x \to \infty \) then so does \( R \). On the other hand, if \( x \) converges to a finite limit as \( t \to \infty \) along an outgoing geodesic, then from the discussion above \( \tau_\infty = \int_{t*}^{\infty} (am)^{-1}dt \) must also converge. But this is possible only if \( am \) increases without bound, and hence \( R \) again diverges. In all possible cases, therefore, the geometric radius goes to infinity along the outgoing radial null geodesics.

If an outgoing geodesic is traced backwards towards \( t = 0 \), it is evident from (56) that \( R \to 0 \) along the geodesic in question if and only if \( am \to 0 \). Since \( m \leq 0 \), this in turn is possible if and only if \( a \to 0 \) along the geodesic, and so the geodesic emerges from a cosmological singularity. In all other situations, the limiting of \( R \) (as \( t \to 0 \) or \( x \to 0 \), whichever is encountered first) is positive.

### 4.3 Ingoing radial null geodesics

Ingoing radial null geodesics are solutions of the geodesic equation (48) with \( n_\infty(x,t) \) on the right. Despite their name, ingoing geodesics have \( \frac{d}{dt} R > 0 \) at all points \((t,x)\) except those on or inside the apparent horizons (if any form), as was explained in Section 2.5. In particular, in view of the equation (52) for \( p_\infty(x,t) \), the geometric radius \( R \) is increasing along the ingoing geodesics for both small and large values of \( x \), unless \( \sigma \) and \( \xi \) are identically zero (and so the metric is Schwarzschild).

In principle, the analysis of Section 4.2 can be repeated to explore the asymptotic behaviour of the ingoing radial null geodesics, but unfortunately the results are not as clear-cut as for the outgoing geodesics. The equivalent of the inequality (84) for the ingoing geodesics is

\[
(\alpha m)^{-1}(\sigma - \sqrt{\frac{\Upsilon}{9}}) \leq du/dt \leq (am)^{-1}\sigma , \tag{89}
\]

which when integrated forwards and backwards from \((t,x) = (t*,x*)\) gives

\[
\mu_\infty - \sqrt{\frac{\Upsilon}{9}} \tau_\infty \leq \lim_{t \to \infty} (u - u*) \leq \mu_\infty \quad \text{and} \quad \mu_0 - \sqrt{\frac{\Upsilon}{9}} \tau_0 \geq \lim_{t \to 0^+} (u - u*) \geq \mu_0 . \tag{90}
\]

All that can be inferred from the second inequality in (90) about the provenance of the ingoing geodesics is that they emerge from the singularity \( x = 0 \) at some time \( t > 0 \) if \( \tau_0 \) is finite and \( \mu_0 \) diverges. In connection with this, it should be noted that Nolan [4] has already shown that the
singularity at $r = m/2$ effectively hides the cosmological singularity from view in the restricted class of McVittie space-times with $\sigma \propto t^{-1}$ (which do have $\tau_0$ finite and $\mu_0 = -\infty$).

However, the problem of the fate of the ingoing geodesics can be simplified by appealing to a result due to Hayward [15], which states that if apparent horizons form then no ingoing geodesic can cross the horizons from the normal region into the surrounding anti-trapped region. This can be seen directly by differentiating the equation (52) for $p_-(x,t)$ along an ingoing geodesic. In view of (48) and (49), this gives:

$$\frac{dx}{dt} p_-(x,t) = (2 + x)^3 \left[x \sigma + (2 + x) \xi \right] + \frac{dp}{dx} p_-(x,t) \frac{dx}{dt} = (am)^{-1} x^{-1} [Q(x,t) + (2 + x)^{-3} p(x,t)p_-(x,t)],$$

(91)

where $p$ is a fourth-order polynomial in $x$ whose exact form is unimportant. So, given that $p_- = 0$ on the apparent horizons and $Q \leq 0$ if the null energy condition holds, it follows that $\frac{dx}{dt} p_- \leq 0$ whenever an ingoing geodesic crosses a horizon, as claimed.

Another important technical question is whether the ingoing geodesics can reach the singularity at $x = 0$ in a finite time $t$. It should be evident from (48) and (49) that this is not possible, as $\frac{dx}{dt} \to (am)^{-1} \sigma \geq 0$ in the limit as $x \to 0$, and in the marginal case $\sigma = 0$ equation (48) becomes $\frac{dx}{dt} \approx \frac{1}{a} (am)^{-1} x$ for small values of $x$. It follows therefore that $x$ can tend to zero on an ingoing geodesic only in the asymptotic limit $t \to \infty$, and this only if $\sigma$ is not eventually bounded away from zero and the integral $\tau_\infty$ diverges.

The analysis of the asymptotic behavior of the ingoing geodesics can now be divided into a small number of manageable cases.

(i) The phase curve remains on or outside $\Gamma^*$. Substituting (48) into (57) gives

$$\frac{dx}{dt} R = \frac{1}{2} (1 + x)^{-1} (2 + x)^{-2} p_+(x,t),$$

(92)

on each of the radial null geodesics. If the phase curve of the space-time remains in the region on or outside the contour $\Gamma^*$ in Figure 1 then $p_-(x,t) > 0$ for all positive $x$ and $t$, and $R$ is monotonic increasing on each ingoing geodesic. If $x$ is unbounded above on the geodesic, it follows from (56) that $R$ itself is unbounded above and so $R \to \infty$ as $t \to \infty$. On the other hand, if $x$ is bounded above by some number $x_*$ and $p_-(x,t)$ is eventually bounded below by some positive constant $\epsilon$ then eventually $\frac{dx}{dt} \geq \frac{1}{2} (1 + x_*)^{-1} (2 + x_*)^{-2} \epsilon$ is bounded below, and again $R \to \infty$ as $t \to \infty$.

The only points in the region on or outside the contour $\Gamma^*$ at which $p_-(x,t)$ is not bounded away from zero for $x \in (0, \infty)$ are those on the horizontal axis $\xi = 0$, as $\lim_{x \to 0} p_-(x,t) = 0$ if $\xi = 0$. It is easily seen from (49) that the function $n_-(x,t)$ is positive definite for positive $x$ whenever $\sigma > \sqrt{3}/9$. Let $\xi^*$ be the value of $\xi$ at the point $\Gamma^*$ in Figure 1 where $\Gamma^*$ intersects the line $\sigma = \sqrt{3}/9$. Then $\frac{dx}{dt} \geq 0$ on all ingoing geodesics whenever $\xi \leq \xi^*$.

Consider now a particular geodesic that passes through the point $(t,x) = (t_*, x^*)$. If the time interval $[t_*, \infty)$ is written as the union $S_+ \cup S_-$ of two closed subsets, with $\xi \geq \xi^*$ on $S_+$ and $\xi \leq \xi^*$ on $S_-$, then there are two possible cases. If $S_+$ has infinite measure then, since $\xi$ is bounded away from zero on $I_+$, it again follows from (92) that $R \to \infty$. Alternatively, if $S_+$ has finite measure $M_+$ it can be further divided into two disjoint subsets $T_+$ and $T_-$, with $\frac{dx}{dt} \geq 0$ on $T_+$ and $\frac{dx}{dt} \leq 0$ on $T_-$. $T_-$ can then be written as the union $\bigcup_I I_k$ of disjoint intervals $\{I_k = [a_k, b_k]\}$ with $x(b_k) \leq x(a_k)$ and $x(a_{k+1}) \geq x(b_k)$ for all $k$. The geodesic equation (48) can be recast in the form
where when integrated over $\bigcup_{k=1}^{N} I_k$ for any $N \geq 1$ gives
\[ \sum_{k=1}^{N} \ln[x(b_k)/x(a_k)] \geq -2(am)^{-1}M_\varepsilon. \] (94)
But the left-hand side of (94) satisfies the inequality
\[ \sum_{k=1}^{N} \ln[x(b_k)/x(a_k)] = \ln[x(b_N)/x(a_1)] + \sum_{k=1}^{N-1} \ln[x(b_k)/x(a_{k+1})] \leq \ln[x(b_N)/x(a_1)], \] (95)
and so (94) implies that the sequence $\{x(b_k)\}$ is bounded away from zero. Hence, $x$ itself and $p_-(x,t)$ are also bounded away from zero on the geodesic, and once again $R \to \infty$.

So in all possible cases $R \to \infty$ as $t \to \infty$ on the ingoing geodesics if the phase curve remains in the region on or outside $\Gamma^\ast$.

(ii) The phase curve enters the region inside $\Gamma^\ast$. If the phase curve of the space-time enters the region inside $\Gamma^\ast$ in Figure 1 then according to the result 1 of Section 3.3 the function $am$ tends to a finite positive limit $(am)_{\infty}$ as $t \to \infty$. Given that the geometric radius $R$ is bounded below by $2am > 0$, then $dR/dt$ along an ingoing geodesic is positive in an anti-trapped region and negative in a normal region (i.e. between any apparent horizons), and that no ingoing geodesic once inside the normal region can ever cross the apparent horizons into the anti-trapped region, it follows that $R$ is eventually a monotonic function of $t$, and so either $R \to \infty$ or $R$ converges to a finite value $R_\varepsilon \geq 2(am)_{\infty}$.

In the first case it is evident from (56) that $x \to \infty$, while in the second case $x$ converges to a finite value $x_\infty \geq 0$. Since $\xi \to 0$ and $\sigma \to \sigma_\infty \geq 0$ in the limit as $t \to \infty$, it follows that the right-hand side of (92) also converges, and therefore must converge to zero because $R$ converges. But $p_-(x,t) = 0$ only at an apparent horizon or (if $\xi = 0$) at $x = 0$. If $x_\infty = 0$ then $R_\varepsilon = 2(am)_{\infty}$ and $-\sigma$ as was mentioned above $-\sigma$ cannot eventually be bounded away from zero, so it follows that $\sigma_\infty = 0$, in which case (52) indicates that the inner horizon asymptotes to $x \to 0$ and the outer horizon has $x \to \infty$.

In summary, therefore, if the phase curve enters the region inside $\Gamma^\ast$ then every ingoing geodesic will either diverge to $R = \infty$ (if apparent horizons never form, or the geodesic is early enough to avoid the horizons) or asymptote to the inner or outer horizon. Furthermore, since $dR/dt < 0$ between the horizons and $dR/dt > 0$ outside the horizons, it is clear that any ingoing geodesics that asymptote to the outer horizon will be unstable to small perturbations, so geodesics of this type form a set of zero measure.

The fate of the ingoing geodesics as they are traced backwards from a point $(t, x) = (t_\ast, x_\ast)$ is a matter of considerably less importance, but it should be clear from the foregoing discussion what the main possibilities are. If the phase curve of the space-time lies outside $\Gamma^\ast$, initially then all ingoing geodesics have $dR/dt \geq 0$ at sufficiently small values of $t$, and so the geodesic either emerges from the singularity at $x = 0$ at some $t > 0$, or tends in the limit $t \to 0^+$ to a finite positive value of $R$ when $\lim_{t \to 0^+} (am) > 0$, or to $R = 0$ when $\lim_{t \to 0^+} (am) = 0$. If the phase curve lies on or inside $\Gamma^\ast$ initially then inner and outer horizons exist for all $t > 0$. If $(t_\ast, x_\ast)$ lies on or inside the inner horizon then $dR/dt \geq 0$ on the geodesic at sufficiently small values of $t$, and the alternatives mentioned in the previous paragraph apply, with the geodesic inside the inner horizon for all $t < t_\ast$. If $(t_\ast, x_\ast)$ lies between the horizons the same alternatives again apply, but the
point of emergence of the geodesic in the limit $t \to 0^+$ can lie on or inside the inner horizon, between the horizons, or on or outside the outer horizon. Finally, if $(t_\ast, x_\ast)$ lies on or outside the outer horizon then the geodesic remains outside the outer horizon for all $t < t_\ast$, and tends to a finite non-negative value of $R$ as $t \to 0^+$.

4.4 Geodesic completeness
A radial null geodesic passing through a point $(t, x) = (t_\ast, x_\ast)$ with $t_\ast > 0$ and $x_\ast > 0$ is future-complete if the total forward lapse in the affine parameter

$$\Delta \lambda_\infty = \int_{t_\ast}^{\infty} (2 + x)^{-1} x \kappa_\pm^{-1} dt$$

(96)
diverges, and is future-incomplete if $\Delta \lambda_\infty$ is finite. Here, the conformal factor $\kappa_\pm$ can be chosen to have an arbitrary positive value $\kappa_\ast$ at $(t_\ast, x_\ast)$, but its subsequent evolution is governed by equation (50), which can be rewritten in the form

$$\frac{d}{dt}(\ln \kappa_\pm^{-1}) = (am)(2 + x)^{-4} k_\pm(x, t),$$

(97)

where

$$k_\pm(x, t) = 16\xi + 8(4\xi + \sigma \pm 1)x + 4(6\xi + 3\sigma \pm 1)x^2 + 2(4\xi + 3\sigma)x^3 + (\xi + \sigma)x^4.$$  

(98)

Similarly, the geodesic is past-complete if the total backwards lapse in the affine parameter

$$\Delta \lambda_0 = \int_{t_\ast}^{0} (2 + x)^{-1} x \kappa_\pm^{-1} dt$$

(99)
diverges – where the bounding value $t_0$ here is either 0 or the value of $t$ at which the geodesic emerges from $x = 0$, whichever is larger – and is past-incomplete if $\Delta \lambda_0$ is finite.

Now, the polynomial $k_+(x, t)$ is clearly positive definite (for positive $x$) if $\xi \geq 0$ and $\sigma \geq 0$, so it is evident from (97) that $\kappa_+^{-1}$ is monotonically increasing with $t$, and is therefore bounded below by its value $\kappa_\ast^{-1}$ at $(t_\ast, x_\ast)$. The weighting factor $(2 + x)^{-1} x$ in (96) is also bounded below by $(2 + x_\ast)^{-1} x_\ast$, so it follows that $\Delta \lambda_\infty$ diverges for all outgoing radial null geodesics, which are therefore guaranteed to be future-complete.

Conversely, for all $t \leq t_\ast$ the function $\kappa_+^{-1}$ is bounded above by $\kappa_\ast^{-1}$, while the factor $(2 + x)^{-1} x$ is bounded above by 1, so $\Delta \lambda_0$ is finite for all outgoing radial geodesics. The outgoing geodesics are therefore past-incomplete, and their terminal points on the surface $x = 0$ or – if there is a cosmological singularity – at $t = 0$ are true singular boundaries of the space-time.

In the case of the ingoing geodesics, a useful consequence of (98) is that

$$k_-(x, t) = p_-(x, t) + 2(x + 1)(x^2 - x - 2).$$

(100)

The function $x^2 - x - 2$ is positive for all $x > 1 + \sqrt{5}$, and so if $p_-(x, t)$ remains positive or zero and $x$ is eventually bounded below by $1 + \sqrt{3}$ on an ingoing geodesic then, from (97), $\kappa_-^{-1}$ is eventually monotonically increasing with $t$, and the geodesic is future-complete.

In particular, if the phase curve of space-time enters the region inside $\Gamma \ast$ in Figure 1 then any ingoing geodesic on which $R \to \infty$ is future-complete, because the geodesic has $\frac{dR}{dt} R \geq 0$ and $R \to \infty$.

Similarly, although it is not generally true that $\frac{d}{dt} x \geq 0$ on the outer horizon, it can be seen by solving the equation $p_- = 0$ for $\sigma$ and thereby eliminating $\sigma$ from $\frac{d}{dx} p_-$ that

$$\frac{d}{dx} p_-(x, t) = -2x^{-1} \xi (2 + x)^3 + 2(2 + x)^{-1} x(x^2 - 2x - 2)$$

(101)
on the apparent horizons, and so \( \frac{\partial}{\partial x} p_-(x,t) < 0 \) on the horizon if \( x < 1 + \sqrt{3} \). This in turn is possible only if the horizon is an inner horizon, and it follows that \( x \geq 1 + \sqrt{3} \) on any outer horizon. Furthermore, an ingoing geodesic that asymptotes to the outer horizon must approach the horizon from outside the horizon, as \( \frac{d}{dt} R \geq 0 \) on the horizon. Hence, any ingoing geodesic that asymptotes to the outer horizon is also future-complete.

If the phase curve remains on or outside the contour \( \Gamma^* \), then any ingoing geodesic on which \( x \to \infty \) is also future-complete, as \( \frac{d}{dt} R \geq 0 \) on all geodesics in this case. So the only ingoing geodesics that are possibly future-incomplete are those that asymptote to the inner horizon (if horizons form) and those on which \( R \to \infty \) and \( am \to \infty \) but \( x \) is not eventually bounded below by \( 1 + \sqrt{3} \) (if the phase curve remains outside \( \Gamma^* \)).

To demonstrate that geodesics of the last type are always future-complete requires another long technical argument, and I will only sketch the proof here. Combining equations (48) and (50) gives

\[
\frac{d}{dt} \ln \{(2 + x)^{-1} x \} = (am)^{-1} x^{-1} (2 + x)^{-1} q(x,t) \tag{102}
\]

along an ingoing geodesic, where the function \( q(x,t) \) – defined in (47) – is positive definite (for \( x > 0 \)) if the phase curve lies outside \( \Gamma_n \). As was mentioned in Section 3.3, the minimum of the function on the right-hand side of (102) is occurs when \( x = 2(4^{1/3} - \sigma^{1/3})^{-1} \sigma^{1/3} \) if \( \sigma < 4 \), and in the limit as \( x \to \infty \) if \( \sigma \geq 4 \).

So

\[
\frac{d}{dt} \ln \{(2 + x)^{-1} x \} \geq \frac{1}{2} \left( \frac{1}{2} \right)^{1/6} \sigma^{1/3} + \frac{x}{2} \quad \text{if } \sigma < 4
\]

\[
\frac{d}{dt} \ln \{(2 + x)^{-1} x \} \geq \frac{1}{2} \left( \frac{1}{2} \right)^{1/6} \sigma^{1/3} + \frac{x}{2} \quad \text{if } \sigma \geq 4
\]

Now, on defining the function

\[\chi(t) = \begin{cases} 2 \left( \frac{1}{2} \right)^{1/6} \sigma^{1/3} + \frac{x}{2} & \text{if } \sigma < 4 \\ \sigma + \frac{x}{2} & \text{if } \sigma \geq 4 \end{cases}\]

(104)

it is evident that (78) together with its extension to \( \sigma \geq 4 \) read simply \( \frac{d}{dt} \chi \leq \psi \). It therefore follows that \( \frac{d}{dt} \ln \{(2 + x)^{-1} x \} \geq 0 \) outside \( \Gamma_n \), that \( \frac{d}{dt} \chi \leq 0 \) inside \( \Gamma_n \), and that

\[
\frac{d}{dt} \ln \{(2 + x)^{-1} x \} \geq \frac{1}{2} \frac{d}{dt} \chi \tag{105}
\]

everywhere.

If the geodesic is future-incomplete, then according to (96) \( (2 + x)^{-1} x \) cannot be bounded away from zero in the limit of late time, and so \( \ln \{(2 + x)^{-1} x \} \) cannot be bounded below on any unbounded interval \([t_1, \infty)\). Furthermore, because \( \frac{d}{dt} \ln \{(2 + x)^{-1} x \} \geq 0 \) outside \( \Gamma_n \) the phase curve cannot lie entirely on or outside \( \Gamma_n \) on any interval \([t_2, \infty)\). Also, \( \frac{1}{2} > \chi > 7 - 4 \sqrt{3} \approx 0.07180 \) in the region between \( \Gamma_n \) and \( \Gamma^* \), so the net change \( \Delta \chi \leq 0 \) in \( \chi \) along any segment of the phase curve that lies entirely in this region is bounded below.

An immediate consequence of (105) is that the net change in \( \ln \{(2 + x)^{-1} x \} \) on the time interval \([t_n, t]\) satisfies

\[
\Delta \ln \{(2 + x)^{-1} x \} \geq \sum_k \int_{t_k}^{t} \frac{d}{dt} \chi \ dt , \tag{106}
\]

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where \( \{I_k\} \) is the set of sub-intervals in \([\alpha, \tau]\) on which the phase curve lies on or inside \( \Gamma_n \). If \( \chi \) is monotonically decreasing on \([\alpha, \infty)\) then, because \( \Delta \chi \leq 0 \) is bounded below, \( \Delta[\ln(\mathcal{L})] \) will also be bounded below unless \( \xi \) is arbitrarily close to zero on \( \{I_k\} \).

In the latter case, the phase curve must eventually enter every neighborhood of the point \( (\sigma, \xi) \) on the axis \( \xi = 0 \) where \( \Gamma_n \) and \( \Gamma^* \) meet. At this point, \( \chi = 7 - 4\sqrt{3} \) and since \( \chi \) is monotonically decreasing the phase curve will eventually be trapped on or below the curve \( \chi = 7 - 4\sqrt{3} + \epsilon \) for any positive \( \epsilon \). However, if \( \epsilon \) is less than approximately 0.0055569 the curve \( \chi = 7 - 4\sqrt{3} + \epsilon \) intersects \( \Gamma^* \) at a point \( (\sigma, \xi) \) with \( \sigma > \sqrt{3}/9 \) and the phase curve is then trapped in the region \( \sigma > \sqrt{3}/9 \) [see Figure 1]. As mentioned in the previous section, \( dx/dt \) is then positive and bounded away from zero at all later times, contradicting the assumption that \( x \) is not eventually bounded below by \( 1 + \sqrt{3} \).

On the hand, if the supremum of \( \chi \) on \([\alpha, \infty)\) does not tend to its minimum possible value \( 7 - 4\sqrt{3} \) as \( t \to \infty \) then for each sufficiently late time interval \( I_k \) it is possible to find a finite union \( O_k = \bigcup_{j=1}^{n_{\text{max}}} s^{(j)}_k \) of later sub-intervals with the property that the phase curve lies outside \( \Gamma_n \) on \( O_k \), \( d\chi/dt > 0 \) on each \( s^{(j)}_k \), \( \chi \) is continuous from \( s^{(j)}_k \) to \( s^{(j+1)}_k \) for all \( j \leq n_k - 1 \), and \( \int_{O_k} d\chi/dt dt = -\int_{I_k} d\chi/dt dt \). The monotonic variation of \( \chi \) on \( I_k \) and \( O_k \) allows \( \xi \) to be written as functions of \( \chi \) on the two sets, which I will refer to as \( \xi_k^- \) and \( \xi_k^+ \) respectively. Because each curve \( \chi = \text{constant} \) has gradient \( d\xi/d\sigma < 0 \), and the point defined by \( (\chi, \xi_k^-) \) lies inside \( \Gamma_n \) while \( (\chi, \xi_k^+) \) lies outside \( \Gamma_n \), it follows that \( \xi_k^- > \xi_k^+ \) for all \( \chi \).

Hence, the net change in \( \ln[(2 + x)^{-1} x \kappa^{-1}] \) on \( I_k \cup O_k \) is
\[
\Delta[\ln(\mathcal{L})] \geq \int_{I_k} (\xi_k^-)^{-1} d\chi/dt dt + \int_{O_k} (\xi_k^+)^{-1} d\chi/dt dt
\]
\[
= \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} [(\xi_k^+)^{-1} - (\xi_k^-)^{-1}] d\chi \geq 0 \quad ,
\]
and the net variation of the integrand in (96) is eventually positive.

By contrast, it is easily verified that all ingoing radial null geodesics that asymptote to the inner horizon are future-incomplete. As was mentioned in Section 3.3, if apparent horizons appear and the null energy condition is satisfied then \( am \) tends to a finite positive limit \( (am)_{\infty} \) as \( t \to \infty \), while \( \xi \to 0 \) and \( \sigma \to \sigma_{\infty} \), where \( 0 \leq \sigma_{\infty} < \sqrt{3}/9 \). In addition, \( R \) and \( x \) tend to finite non-negative limits on any ingoing geodesic that asymptotes to the inner horizon, with the limiting value \( x_{\infty} \) of \( x \) fixed by the value of \( \sigma_{\infty} \) and lying in the range \([0, 1 + \sqrt{3}]\).

The geodesic equation (48) for an ingoing geodesic in this class can be solved in the limit of large \( t \) to give
\[
x(t) \approx x_{\infty} + C_1 e^{-\alpha t} \quad \text{with} \quad \alpha \equiv 2(1 + \xi_{\infty}^2)(2 + x_{\infty})^{-4} (1 + x_{\infty})(2 + 2x_{\infty} - x_{\infty}^2) > 0 \quad ,
\]
then gives, for large values of \( t \).

---

Note that, at each point in the region in Figure 1 inside \( \Gamma_n \), the curves \( \chi = \text{constant} \) have a gradient \( d\xi/d\sigma \) that lies between the gradients of the steeper and shallower dotted curves through the same point.
\[(2 + x)^{-1} \alpha^{-1} \approx C_2 e^{-\alpha t}\]  

(109)

where \(C_2\) is an arbitrary positive constant. It is evident now from (96) that \(\Delta \lambda_x\) converges and the geodesic is future-complete, as claimed.

The causal structure of gMcV space-times in which apparent horizons appear is therefore no different from the causal structure of the standard McVittie space-times with horizons. As was mentioned in Section 1, the causal structure of the McVittie solutions has been explored by Kaloper, Kleban and Martin [8], who have demonstrated that, if the space-time has the property that \(\lim_{x \to \infty} (\dot{a}/a)\) is a finite positive constant, all ingoing radial geodesics that emerge from the surface \(x = 0\) at sufficiently late times (as so cannot avoid the inner horizon) are future-incomplete.

Since \(\dot{a}/a = (am)^{-1}(\xi + \sigma)\), the requirement that \(\lim_{x \to \infty} (\dot{a}/a)\) be a finite positive constant is equivalent to stipulating that \((am)^{-1}\sigma_x\) be positive. Kaloper, Kleban and Martin [8] exclude the case \(\sigma_x = 0\) because it entails that \(x_\infty = 0\), and so the incomplete ingoing radial geodesics terminate at the singularity \(x = 0\).

Figure 3 shows the conformal diagram for a gMcV (or McVittie) space-time that initially has a cosmological singularity and no apparent horizons, but later develops horizons and has \(\sigma_\infty > 0\). The ingoing geodesics are represented by the 45° lines inclined up and to the left, and the outgoing geodesics by the lines inclined up and to the right.

**Figure 3.** Conformal diagram for a gMcV space-time in which apparent horizons first appear at some time \(t > 0\) and \(\sigma\) tends to a non-zero limit. The future-incomplete ingoing radial null geodesics emerge from the singular surface \(x = 0\) and pass through the broken line at the left of the diagram.

### 5. Conclusions

The principal conclusion of this paper is that the long-term behavior and causal structure of the generalized McVittie space-times are no different from those of the standard McVittie space-times. In particular, if a gMcV space-time satisfies the null energy condition and develops apparent horizons then it inevitably asymptotes to a McVittie space-time, with \(\xi \to 0\), \(am \to (am)_\infty > 0\) and \(\sigma \to \sigma_\infty \geq 0\) as \(t \to \infty\). Also, the only future-incomplete radial null geodesics in both the gMcV and McVittie cases are ingoing geodesics that asymptote to the inner apparent horizon.
It is tempting to conclude further that, if $\sigma_\infty > 0$, the space-time actually converges to a Kottler solution, which—as was mentioned in Section 1—has $\xi = 0$ and $\sigma$ constant. This is not strictly speaking true, as the Kottler solution is regular at $x = 0$ while all other gMcV space-times except the Schwarzschild solution are singular at $x = 0$, and typically remain singular there in the limit as $t \to \infty$, as can be seen from (11).

Nonetheless, Kaloper, Kleban and Martin [8] have chosen to ignore this technicality, and argue that “in the case $H \to H_0 > 0$ it [the surface $x = x_\infty$, $t = \infty$] is a regular event horizon, and the McVittie solution really is a black hole, which asymptotes the Schwarzschild-de Sitter [Kottler] solution". As evidence in support of this claim, they demonstrate that the ingoing radial null geodesics can be continued across the surface $x = x_\infty$, $t = \infty$, and that the limiting values of all the components of the Riemann tensor are finite at the surface, coinciding in fact with the corresponding Kottler values if $\lim_{t \to \infty} \dot{H} = 0$.

Indeed, there is a quicker way to show that a gMcV space-time that develops apparent horizons can always be matched onto the corresponding Kottler solution across the surface $x = x_\infty$, $t = \infty$, by calculating the first and second fundamental forms of the surfaces of constant $t$ in both the gMcV and Kottler space-times, and verifying that they coincide in the limit as $t \to \infty$. If equation (44), which solves to give $r = \frac{1}{2}(1 + x)m$, is used to replace $r$ with $x$ as the radial coordinate in (6), the gMcV line element becomes:

$$ds^2 = \frac{1}{4}(am)^2(1 + x)^{-4}(2 + x)^4$$

$$\times [dx - (am)^{-1}(2 + x)^{-3}(1 + x)n_+(x,t)dt] [dx - (am)^{-1}(2 + x)^{-3}(1 + x)n_-(x,t)dt]$$

$$+ \frac{1}{4}(am)^2(1 + x)^{-2}(2 + x)^4 d\Omega^2,$$

(110)

where the functions $n_\pm(x,t)$ are defined in (49), and depend on $t$ only through $\sigma$.

Now, the future-pointing unit vector orthogonal to a given surface of constant $t$ is:

$$n^a = x^{-1}(2 + x)[\delta^a_1 + (am)^{-1}(1 + x)\sigma \delta^a_\sigma].$$

(111)

Hence, the first fundamental form $h_{ab} = g_{ab} + n_an_b$ of the surface has non-zero components

$$h_{tt} = \frac{1}{4}(1 + x)^{-2}(2 + x)^4 \sigma^2,$$

$$h_{tx} = -\frac{1}{4}(1 + x)^{-3}(2 + x)^4 am \sigma,$$

$$h_{xx} = \frac{1}{4}(1 + x)^{-4}(2 + x)^4 (am)^2$$

and

$$h_{\theta\phi} = h_{\phi\theta} \csc^2 \theta = \frac{1}{4}(1 + x)^{-2}(2 + x)^4 (am)^2.$$

(112)

It is clear from (112) that in the limit as $t \to \infty$, when $\sigma \to \sigma_\infty$ and $am \to (am)_\infty$ in the gMcV space-time, the components of $h_{ab}$ in the gMcV and the corresponding Kottler space-times do coincide.

Similarly, the second fundamental form $K_{ab} = h^c_a h^d_b D_c(n_d)$ of the surface has non-zero components

$$K_{tt} = \frac{1}{4}(am)^{-1} x^{-1}(1 + x)^{-2}(2 + x)^4 \sigma^2 [x \sigma + (2 + x) \xi]$$

(113)

$$K_{tx} = -\frac{1}{4} x^{-1}(1 + x)^{-3}(2 + x)^4 \sigma [x \sigma + (2 + x) \xi]$$

(114)

$$K_{xx} = \frac{1}{4}(am)x^{-1}(1 + x)^{-4}(2 + x)^4 [x \sigma + (2 + x) \xi]$$

(115)

and

$$K_{\theta\phi} = K_{\phi\theta} \csc^2 \theta = \frac{1}{4}(am)x^{-1}(1 + x)^{-2}(2 + x)^4 [x \sigma + (2 + x) \xi],$$

(116)

Here, $H$ is shorthand for the Hubble parameter $\dot{a}/a$. 

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and it is again evident that the components of $K_{ab}$ in the gMcV and Kottler space-times coincide in the limit $t \to \infty$.

What is questionable about the claims made by Kaloper, Kleban and Martin [8] is not that a McVittie space-time with $H \to H_0 > 0$ can be matched onto a Kottler solution, but that the Kottler solution can be interpreted as describing a “black hole”. Although the gMcV line element covers only the coordinate patch $x > 0$ and the Kottler solution is regular at $x = 0$, the static form (4) of the Kottler line element has a curvature singularity at $R = 0$, and it is known that the region between $R = 0$ and the inner apparent horizon is everywhere anti-trapped.

Furthermore, the maximal analytic extension of the Kottler solution consists of an infinite chain of isometric copies of the original Kottler patch (4), as has been demonstrated by Bažański and Ferrari [16], on the basis of a more qualitative discussion of the problem by Gibbons and Hawking [17]. There is no cosmological singularity in the Kottler solution, and ingoing radial null geodesics are past-incomplete as well as future-incomplete, so each link in the chain of Kottler patches is formed by joining the future endpoints of the geodesics in one patch to the past endpoints of the geodesics in the next patch.

As a result, there is no prospect that the geodesic completion of any of the incomplete expanding gMcV space-times will contain a horizon surrounded by a trapped region, rather than an anti-trapped one. The search for a realistic line element describing a non-rotating black hole embedded in an expanding universe therefore needs to proceed in other directions.

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