Detection of transverse entanglement in phase space

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Transverse entanglement between pairs of photons can be detected through intensity correlation measurements in the near and far fields. We show theoretically and experimentally that at intermediate zones, it is also possible to detect transverse entanglement performing only intensity correlation measurements. Our results are applicable to a number of physical systems.

Detection and quantification of entanglement is essential for the development of many applications in the field of quantum information. Several tasks proposed to take advantage of the entanglement properties of quantum systems can be experimentally tested with photons produced from spontaneous parametric down-conversion (SPDC). This is a versatile system, since SPDC photons can be prepared in entangled states of many different degrees of freedom, such as polarization, time-bins, orbital angular momentum, as well as transverse spatial variables. The latter concerns correlations between the transverse components of the wave vectors of the signal and idler photons, which have been extensively studied and utilized in the last decade. They arise due to the localization of the emission of photon pairs and the phase matching conditions for the non-linear interaction between the pump, signal and idler fields. Even though the quantum nature of spatial correlations was already evident, the formal relationship with entanglement has only been demonstrated a few years ago. Transverse entanglement was detected through the violation of a non-separability criteria, based on intensity correlation measurements performed in the near and far fields. Entanglement in continuous variables (CV) is a rich research subject, because several quantum information tasks can be optimized using high dimensional Hilbert spaces. SPDC is a natural option for the experimental investigation of transverse spatial entanglement, which can be present in other quantum systems.

So far, CV transverse entanglement detection has been based on intensity correlation measurements performed in the near and in the far field. An interesting entanglement “migration” effect was shown recently by Chan et. al, in which entanglement moves from the real to the imaginary part of the two-photon wave function during propagation. In order to be able to detect entanglement in this case, it would be necessary to perform phase-sensitive measurements.

In this Letter, we show theoretically and experimentally that it is always possible to detect entanglement by performing intensity correlation measurements, even outside near and far field zones. We demonstrate the connection between the variances of two observables and the variances of these same observables rotated in phase space. We encounter the conditions for which entanglement detection is possible with intensity measurements, and others for which it is impossible. This connection allows one to circumvent problems like the migration of entanglement by performing proper phase space rotations on the observables. Though we consider the particular case of propagation of transverse correlations of photon pairs, our results can be used to improve detection of entanglement in other CV systems.

Our approach is based on the propagation of the signal and idler fields using the formalism of the Fractional Fourier Transform (FRFT), which is parameterized by the angle $\alpha$. The FRFT appears naturally in a number of physical systems and describes rotation in phase space. In particular, it is possible to completely describe the propagation of a light field through the order $\alpha$ of the FRFT. For instance, the field at the source is given by a FRFT of order $\alpha = 0$ and the usual Fourier transform, associated to Fraunhofer diffraction in the far field, is given by an FRFT of order $\alpha = \pi/2$. Free propagation can always be described in terms of an FRFT operation up to a quadratic phase term, which can be considered essentially unity in the near and far field. In Ref. [3], as is customary, detection of entanglement was actually performed using lenses to obtain the intensity correlations in the near field image ($\alpha = \pi$) and far field ($\alpha = \pi/2$). Likewise, any FRFT of order $\alpha$ can be implemented perfectly with lenses.

We consider the experimental arrangement sketched in Fig. 1(a), where signal and idler photons from SPDC are sent through FRFT systems of order $\alpha_s$ and $\alpha_i$, respectively.

Following Ref. [17], we write the two-photon wave-
and we obtain

\[ \hat{q} \equiv \hat{q}_s, i \equiv \hat{q}_s, j, k \]

as the dimensionless operators. For example, defining the dimensionless operators \( \hat{\rho}_\pm \equiv \hat{\rho}_s \pm \hat{\rho}_i \) and \( \hat{q}_\pm \equiv \hat{q}_s \pm \hat{q}_i \), the separability criteria of Duan, Giedke, Cirac and Zoller (DGCZ) \( [11] \) establishes that if one of the two inequalities

\( \langle \Delta \hat{\rho}_- \rangle_\Psi^2 + \langle \Delta \hat{q}_- \rangle_\Psi^2 \geq 2 \) or \( \langle \Delta \hat{\rho}_+ \rangle_\Psi^2 + \langle \Delta \hat{q}_+ \rangle_\Psi^2 \geq 2 \)

is violated, then the state \( \Psi \) is non separable and therefore is entangled.

Using (11) and (2), we have

\[ \langle \Delta \hat{\rho}_\pm \rangle_\Psi^2 = \sigma_\pm^2, \]

\[ \langle \Delta \hat{q}_\pm \rangle_\Psi^2 = 1/\sigma_\pm^2, \]

\[ \langle \Delta \hat{q}_\pm \rangle_\Psi^2 = 1/\sigma_\pm^2, \]

and we obtain

\[ \langle \Delta \hat{\rho}_- \rangle_\Psi^2 + \langle \Delta \hat{q}_- \rangle_\Psi^2 \geq \sigma_\pm^2 + 1/\sigma_\pm^2. \]

The right hand side (RHS) of Eq. (4) can be smaller than 2 for small \( \sigma_- \) and large \( \sigma_+ \). In the case of SPDC, this is readily achievable, as these two parameters are independent and experimentally accessible.

The variances in inequality (3) refer to position and momentum variables of the signal and idler fields in the source plane, which are related to the intensity distributions in the near and far field. It is well known that the propagation of a light field characterized by a FRFT is equivalent to a rotation of the transverse variables in phase space \( [10] \), given that these variables are properly adimensionalized \( [21] \). The dimensionless operators transform as

\[ \hat{\rho}_j \rightarrow \hat{\rho}_{\alpha_j} = \cos \alpha_j \hat{\rho}_j + \sin \alpha_j \hat{q}_j \]

\[ \hat{q}_j \rightarrow \hat{q}_{\alpha_j} = -\sin \alpha_j \hat{\rho}_j + \cos \alpha_j \hat{q}_j, \]

where \( j = s, i \). Therefore, it is possible to write the DGCZ inequality for rotated transverse variables of the fields, \( \hat{\rho}_\pm' \equiv \hat{\rho}_s - \hat{\rho}_i \) and \( \hat{q}_\pm' \equiv \hat{q}_s + \hat{q}_i \), in terms of the variables \( \hat{\rho}_- \) and \( \hat{q}_- \) at the source \( [22] \):

\[ \langle \Delta \hat{\rho}_- \rangle_{\Psi}^2 + \langle \Delta \hat{q}_- \rangle_{\Psi}^2 = \frac{1 + \cos(\alpha_i + \alpha_s)}{2} \left( \langle \Delta \hat{\rho}_- \rangle_{\Psi}^2 + \langle \Delta \hat{q}_- \rangle_{\Psi}^2 \right) \]

\[ + \frac{1 - \sin(\alpha_i + \alpha_s)}{2} \left( \langle \Delta \hat{\rho}_+ \rangle_{\Psi}^2 + \langle \Delta \hat{q}_+ \rangle_{\Psi}^2 \right) \]

\[ - \frac{\sin(\alpha_i + \alpha_s)}{2} \left( \langle \hat{\rho}_- \hat{q}_- \rangle_{\Psi} - \langle \hat{\rho}_+ \hat{q}_+ \rangle_{\Psi} \right) \]

\[ + \frac{\sin(\alpha_i + \alpha_s)}{2} \left( \langle \hat{\rho}_- \hat{q}_+ \rangle_{\Psi} - \langle \hat{\rho}_+ \hat{q}_- \rangle_{\Psi} \right). \]

Eq. (4) shows that whenever \( \alpha_i + \alpha_s \) is zero, the sum of variances for the rotated variables coincide with the sum of variances for the variables at the source. This shows that, for any propagation of the signal field, characterized by \( \alpha_s \), it is possible to find a propagation of the idler field \( \alpha_i \), so that intensity correlation measurement will violate the DGCZ inequality, in or out of the near and far field. We also note that Eq. (4) does not depend on the state (Eq. 11) and is applicable to any bipartite continuous variable systems.

For states of the form (11), the last two lines of the RHS of Eq. (6) are zero. Then considering an entangled state satisfying

\[ \langle \Delta \hat{\rho}_- \rangle_{\Psi}^2 + \langle \Delta \hat{q}_- \rangle_{\Psi}^2 = \sigma_\pm^2 + 1/\sigma_\pm^2 \leq 2, \]

a necessary condition to detect entanglement is

\[ \cos(\alpha_i + \alpha_s) > \frac{S_1 + S_2 - 4}{S_1 - S_2} \geq 0, \]

where we define \( S_1 \equiv \sigma_\pm^2 + 1/\sigma_\pm^2 \) and \( S_2 \equiv \sigma_\pm^2 + 1/\sigma_\pm^2 \). We note that for \( \cos(\alpha_i + \alpha_s) = 0 \), intensity correlation measurements never evidence entanglement, regardless of the state.

We have experimentally tested these conditions, using pairs of twin photons generated by SPDC in a 5nm long lithium iodate crystal (LiIO3) with a c.w. diode laser oscillating at 405nm, as shown in FIG. 1a). Optical FRFT systems, such as the one shown in FIG. 1b) were used in each of the down-converted fields. This system, with \( z_\sigma = 2f \sin^2(\alpha/2) \), is able to implement a FRFT in the range \( 0 \leq \alpha \leq \pi \). For \( \alpha > \pi \) we use a series of FRFT systems, respecting the additivity condition of maintaining

\[ f' = f \sin \alpha \]

the same for each. To describe all FRFTs as rotations in the same phase space, we use dimensionless coordinates \( \rho = \sqrt{k/f} \hat{\rho} \) and \( q = \sqrt{f/k} \hat{q} \).
In our experimental setup, \( f' = 25/\sqrt{2} \text{cm} \). Signal and idler photons were detected with single photon counting modules and 10nm FWHM bandwidth interference filters centered at 810nm. Horizontal slits (100\( \mu \)m) were mounted on translation stages and scanned vertically in steps of 50\( \mu \)m to register the detection position. In all measurements, the \((-+\) (+-)) correlations were measured in all cases by scanning the detectors with equal steps in the same (opposite) directions.

First, we measured the \( \rho_- \) and \( q_+ \) distributions at the source, using imaging \( (\alpha_s = \alpha_i = \pi) \) and Fourier transform \( (\alpha_s = \alpha_i = \pi/2) \) lens configurations. The dimensionless variances were \( \Delta^2(\rho_-) = 0.93 \pm 0.01 \) and \( \Delta^2(q_+) = 0.073 \pm 0.004 \). Applying the DGCZ inequality we obtain

\[
\Delta(\rho_-)^2 + \Delta(q_+)^2 = 1.00 \pm 0.01 \leq 2, \tag{8}
\]

indicating that the state is entangled.

Next, we measured the intensity correlations for the signal and idler fields at intermediate zones. We chose FRFT orders \( \alpha_s = \alpha_i = 3\pi/4 \), so that \( \cos(\alpha_s + \alpha_i) = 0 \), which does not satisfy the condition of Eq. (7). The coincidence counts \( C(\rho^i_{\omega s} - \rho^s_{\omega s}) \) and \( C(\rho^i_{\omega s} + \rho^s_{\omega s}) \) are plotted in FIG. 2(a) and 2(b), respectively. We obtain \( \Delta^2(\rho^i_{\omega s} - \rho^s_{\omega s}) = 13.6443 > 2 \) and \( \Delta^2(\rho^i_{\omega s} + \rho^s_{\omega s}) = 39.1473 > 2 \), which clearly indicates that these intensity correlations cannot be used to violate the DGCZ inequality. We also tested an intermediate zone configuration following the condition given by Eq. (7). We used three additive FRFT lens systems to perform a 3rd order FRFT on the signal field, while maintaining the \( \alpha_i = 3\pi/4 \) order FRFT on the idler field, so that \( \alpha_s + \alpha_i = 2\pi \). Coincidence counts \( C(\rho^i_{\omega s} - \rho^s_{\omega s}) \) are plotted in FIG. 2(c), and the dimensionless variance is \( \Delta^2(\rho^i_{\omega s} - \rho^s_{\omega s}) = 0.038 \pm 0.005 \). Coincidence counts \( C(\rho^i_{\omega s} + \rho^s_{\omega s}) \), plotted in FIG. 2(d), were measured performing an inverse Fourier transform of the signal and idler fields at the planes of FRFT of order \( 3\pi/4 \) and \( 3\pi/2 \), corresponding to FRFT of orders \( 3\pi/2 \) and \( \pi \), respectively. The dimensionless variance is \( \Delta^2(\rho^i_{\omega s} + \rho^s_{\omega s}) = 0.069 \pm 0.003 \). With our experimental data, we are now able to verify entanglement at intermediate zones:

\[
\Delta^2(\rho^i_{\omega s} - \rho^s_{\omega s}) + \Delta^2(\rho^i_{\omega s} + \rho^s_{\omega s}) = 0.107 \pm 0.006 < 2. \tag{9}
\]

The experimental values obtained in Eqs. (8) and (9) are not equal as expected from Eq. (6). This discrepancy can be explained by the experimental imperfections. It is difficult to characterize every source of experimental error and their precise effect on the measurement results. However, we notice that these imperfections contribute by broadening the coincidence distributions. In this respect, our measurement results are upper limits to the actual variances. To evaluate the expected variances, we characterized the initial state \( | \rangle \) at the source by measuring the width \( w \) of the intensity distribution of the pump beam. The dimensionless variance at the source is \( \sigma_{\text{source}}^2 = (4w^2)(k f') = 47 \pm 2 \). The dimensionless variance \( \sigma_{\text{source}}^2 \) is given by \( \sigma_{\text{source}}^2 = (k f')0.455D/K = 0.006 \), where \( D \) is the length of the nonlinear crystal and \( K \) is the pump beam wavenumber. With these values, we predict a violation of the DGCZ inequality: \( \langle (\Delta \rho_-)^2 \rangle_{\text{th}} + \langle (\Delta q_+)^2 \rangle_{\text{th}} = 0.027 \pm 0.001 \leq 2 \), which is smaller than both experimental values \( \text{Eq. (8) and (9)} \). Therefore it is clear that without the experimental imperfections, we should have observed even stronger violations for both cases. A possible imperfection is the error in lens positioning. The effect of this type of imperfection can be estimated by calculating the propagation through the different lens systems in each field \( \text{Eq. (9)} \) and including a 1% error in \( z \) for all lens systems.

Taking the worst case scenario, we obtain the following predictions for each variance: \( \Delta^2(\rho^i_{\omega s} - \rho^s_{\omega s})_{\text{th}} = 0.09 \) and \( \Delta^2(\rho^i_{\omega s} + \rho^s_{\omega s})_{\text{th}} = 0.04 \), \( \Delta^2(\rho_+)^2_{\text{th}} = 0.84 \) and \( \Delta^2(q_+)_{\text{th}} = 0.03 \). These variances are much closer to the experimental values. Thus, considering small experimental imperfections, the theoretical prediction agrees with both Eqs. (8) and (9), as expected from Eq. (6).

Let us now discuss the application of these results to a situation similar to that of Ref. [17], in which it is shown that the transverse intensity correlations decrease as the field propagates, and then are recovered again in the far-field. For a certain propagation distance, the DGCZ or similar inequality will be satisfied, because the real part of the wavefunction becomes separable and the entanglement is present only in the imaginary part. An analysis similar to that of Ref. [17] in terms of FRFT’s yields a sep-
results demonstrate that, given a signal field propagating through spatial entanglement of photons, our results are applicable to a number of physical systems.

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FIG. 3: (Color online) Both signal and idler fields propagate through lens systems that implement a FRFT of order \(\alpha_{sep} = \tan^{-1}(\sigma_+ \sigma_-)\). The dashed detectors will observe no intensity correlation. Additional FRFT systems are directly applicable to spatial entanglement in other configurations. Though our experiment was conducted using spatial entanglement of photons, this is the order of the FRFT implemented on both fields. Substituting this condition in Eq. (7) shows that the DGCZ inequality will be violated in the same way as it would be for the field in the source.

In conclusion, we have demonstrated theoretically and experimentally, that it is possible to detect transverse entanglement performing intensity correlation measurements, not only in the near and far fields, but also at intermediate propagation planes. This is achieved using optical systems that implement Fractional Fourier Transforms (FRFTs) according to the condition \(\alpha_s + \alpha_i = 2\pi\), where \(\alpha_s\) and \(\alpha_i\) are the orders of the transforms on the signal and idler fields. We also show that entanglement is never registered when \(\alpha_i + \alpha_s (\text{mod } 2\pi) = \pi/2\). These results demonstrate that, given a field propagating characterized by \(\alpha_s\), one can always find an FRFT \(\alpha_i\) which can be used to detect entanglement with intensity correlations alone. Though our experiment was conducted using spatial entanglement of photons, our results are directly applicable to spatial entanglement in other systems. Since the Fractional Fourier Transform

\[\tan s \alpha = \tan \alpha (\text{mod } 2\pi)\]