A new minimal chordal completion

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Abstract

In this paper, we present a minimal chordal completion $G^*$ of a graph $G$ satisfying
the inequality $\omega(G^*) - \omega(G) \leq i(G)$ for the non-chordality index $i(G)$ of $G$. In terms
of our chordal completions, we partially settle the Hadwiger conjecture and the
Erdős-Faber-Lovász Conjecture, and extend the known $\chi$-bounded class by adding
it the family of graphs with bounded non-chordality indices.

Keywords. hole cover, local chordalization, NC property, non-chordality index, chordal
graph, graph coloring

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1 Introduction

Throughout this paper, we only consider simple graphs. We mean by a clique of a graph
both the vertex set of a complete subgraph and the complete subgraph itself. The clique
number of a graph $G$ is defined to be the number of vertices in a maximum clique and
denoted by $\omega(G)$. A hole of a graph $G$ is an induced cycle of length at least 4 in $G$. A
chordal graph is a graph without hole.

A graph $H$ is called a chordal completion of a graph $G$, if $H$ is a chordal spanning
supergraph of $G$. See [4] for a survey on chordal completion. In this paper, we present a
chordal completion of a graph $G$ which is efficient in the following sense: Only edges joining
two vertices on holes of $G$ are added to obtain our chordal completion (Theorem 1.8).

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Furthermore, we show that any minimal chordal completion of a graph can be obtained by joining two vertices on holes of \( G \) by edges (Proposition 4.12). As a matter of fact, for a nonnegative integer \( k \), we give a sufficient condition for a graph \( G \) which has a chordal completion \( G^* \) satisfying the inequality \( \omega(G^*) - \omega(G) \leq k \) (Theorem 4.8). This is a strong point of our chordal completion which differentiate it from other chordal completions. For example, it is shown that a graph \( G \) has treewidth at most \( k \) if and only if it has a chordal completion \( G^* \) satisfying \( \omega(G^*) \leq k + 1 \). Yet, this characterization gives no information on \( \omega(G) \), accordingly no significant information on \( \omega(G^*) - \omega(G) \).

The notion “NC property” plays a key role throughout this paper. We say that a hole \( H \) contains a vertex \( v \) (resp. an edge \( e \)) if \( v \) (resp. \( e \)) is a vertex (resp. an edge) on \( H \). We denote the set of holes in a graph \( G \) by \( \mathcal{H}(G) \) and the set of holes in \( G \) containing \( u \) by \( \mathcal{H}(G, u) \).

A nonempty subset \( X \) of \( V(G) \) is called a hole cover of \( G \) provided that every hole in \( G \) contains at least one vertex of \( X \). Note that, if \( G \) has no hole, that is, \( G \) is a chordal graph, then any nonempty vertex set is a hole cover of \( G \).

For a vertex \( u \) of a graph \( G \), we say that \( u \) satisfies the non-consecutive property (NC property for short) if any hole in \( \mathcal{H}(G, u) \) and any hole not in \( \mathcal{H}(G, u) \) do not share consecutive edges. A vertex subset \( C \) of \( G \) is said to satisfy the NC property in \( G \) if every vertex in \( C \) satisfies the NC property and every hole in \( G \) contains at most one vertex in \( C \). We say that a graph satisfies the NC property if it has a hole cover satisfying the NC property. It is easy to see that

\[(z) \text{ If a hole cover } C \text{ of } G \text{ satisfies the NC property in } G, \text{ then a nonempty set } C \setminus A \text{ is a hole cover satisfying the NC property in } G - A \text{ for any proper subset } A \text{ of } V(G) \text{ not including } C.\]

Then it is immediately true that any induced subgraph of a graph satisfying the NC property satisfies the NC property. See Figure 1 for a graph not satisfying the NC property. To see why, suppose to the contrary that there exists a hole cover \( C \) of \( G \) with the NC property. To cover the hole \( H_2 \), \( C \) must contain a vertex on \( H_2 \). Suppose that a vertex in \( V(H_1) \cap V(H_2) \) is contained in \( C \). Since \( C \) is a hole cover satisfying the NC property, a vertex in \( V(H_3) \setminus V(H_2) \) must be contained in \( C \) to cover \( H_3 \). Then, however, those two vertices are on the hole of length 8 surrounding \( H_2 \) and \( H_3 \), which contradicts the assumption that \( C \) satisfies the NC property. Even if a vertex in \( V(H_2) \cap V(H_3) \) is contained in \( C \), we may reach a contradiction by applying a similar argument to the holes \( H_1 \) and \( H_2 \). Therefore we may conclude that there is no hole cover of \( G \) satisfying the NC property. If we delete the vertex \( x \), we obtain a graph satisfying the NC property as \( \{u, v, w\} \) is a hole cover of the new graph with the NC property.

Let \( G \) be a graph with a hole \( H \). For a vertex \( u \) on \( H \), we locally chordalize the hole \( H \) by \( u \) in the following manner: we join \( u \) and each vertex on \( H \) nonadjacent to \( u \).

For a graph \( G \) and a vertex \( u \) satisfying the NC property, locally chordalizing all the holes in \( \mathcal{H}(G, u) \) does not create any new hole (Theorem 2.6). Based on this observation,
we found that, a hole cover $C$ of a graph $G$ can be partitioned into $C_1, \ldots, C_k$ for some positive integer $k$ so that

(i) $C_i$ is a hole cover of the graph $G_i$ satisfying the NC property,

(ii) $G_i^*$ is chordal,

where $G_0 = G_0^* = G - C$, $G_i$ is the graph defined by $V(G_i) = V(G_{i-1}^*) \cup C_i$ and

$$E(G_i) = E(G_{i-1}^*) \cup E \left( G - \bigcup_{j=i+1}^k C_j \right),$$

and $G_i^*$ is a chordal completion of $G_i$ obtained by applying local chordalizations recursively by the vertices in $C_i$ for each $i = 1, \ldots, k$ (Theorem 4.1). Our chordal completion is $G_k^*$ obtained for a hole cover with the smallest number $k$ of partitions in Theorem 4.1. The smallest number $k$ is called the non-chordality index of $G$ and denoted by $i(G)$ (see Definition 4.3).

For a given graph $G$, a function $f : V(G) \to \{1, 2, \ldots, k\}$ is called a proper $k$-coloring of $G$ if $f(u) \neq f(v)$ for any adjacent vertices $u$ and $v$. The chromatic number $\chi(G)$ of a graph $G$ is defined to be the least positive integer $k$ such that there exists a proper $k$-coloring of $G$. It is well-known that, for a graph $G$,

$$\chi(G) \leq \chi_l(G) \leq \chi_{DP}(G)$$

where $\chi_l(G)$ and $\chi_{DP}(G)$ are the list-chromatic number and the DP-chromatic number of $G$, respectively (see [5], [2], [1] for the definitions).

For a positive integer $k$, a graph $G$ is $k$-degenerate if any subgraph of $G$ contains a vertex having at most $k$ neighbors in it. Dvořák and Postle [1] observed that if a graph $G$ is $k$-degenerate, then $\chi_{DP}(G) \leq k + 1$. It is easy to check that every chordal graph $G$ is $(\omega(G) - 1)$-degenerate and so

$$\omega(G) \leq \chi(G) \leq \chi_l(G) \leq \chi_{DP}(G) \leq \omega(G).$$
Therefore, 

\[(\S)\] for a chordal graph \(G\), \(\chi(G) = \chi_l(G) = \chi_{DP}(G) = \omega(G)\).

The observation \((\S)\) directed our attention to the idea that, for a chordal completion \(G^*\) of a graph \(G\), the chromatic number of \(G\) is bounded above by the clique number of \(G^*\).

By \(\tag{1}\), an upper bound of \(\chi_{DP}(G)\) (resp. \(\chi_l(G)\)) is an upper bound of \(\chi_l(G)\) (resp. \(\chi(G)\)). In this vein, it is interesting to check whether or not \(\chi_{DP}(G) \leq k\) (resp. \(\chi_l(G) \leq k\)) when \(\chi_l(G) \leq k\) (resp. \(\chi(G) \leq k\)) for a positive integer \(k\).

We obtain sharp upper bounds for the chromatic number, the list chromatic number, and the DP chromatic number of a graph in terms of non-chordality index and partially settle the Hadwiger conjecture (Theorems \ref{thm:nc_property_chromatic_number} \ref{thm:nc_property_list_chromatic_number} \ref{thm:nc_property_dp_chromatic_number}). Other than obtaining sharp upper bounds for chromatic numbers, we partially settle the Erdős-Faber-Lovász Conjecture (Theorem \ref{thm:efl_conjecture}), and extend the known \(\chi\)-bounded class by adding to it the family of graphs with bounded non-chordality indices. (Theorem \ref{thm:bounded_class_extension}).

## 2 Graphs with the NC property

In this section, we devote ourselves to proving the following theorem.

**Theorem 2.1.** Let \(G\) be a graph with the NC property. Then \(\chi_{DP}(G) \leq \omega(G) + 1\). If \(G\) is \(K_n\)-minor-free, then \(\chi_{DP}(G) \leq n - 1\).

As a corollary of Theorem \ref{thm:nc_property_chromatic_number} we can prove a special case of Four Color Theorem.

**Corollary 2.2.** For a planar graph \(G\) with the NC property, \(\chi_{DP}(G) \leq 4\).

**Lemma 2.3.** Given a graph \(G\) and a cycle \(C\) of \(G\) with length at least four, suppose that a section \(Q\) of \(C\) forms an induced path of \(G\) and contains a path \(P\) with length at least two none of whose internal vertices is incident to a chord of \(C\) in \(G\). Then \(P\) can be extended to a hole \(H\) in \(G\) so that \(V(P) \subsetneq V(H) \subset V(C)\) and \(H\) contains a vertex on \(C\) not on \(Q\).

**Proof.** Let \(v_i\) and \(v_j\) be the origin and the terminal of \(P\). Since \(P\) is an induced path of length at least two, \(v_i\) and \(v_j\) are nonadjacent. Now we take a shortest \((v_j, v_i)\)-path \(P'\) with some vertices on the \((v_i, v_j)\)-section of \(C\) other than \(P\). Since \(v_i\) and \(v_j\) are nonadjacent, \(P'\) has length at least two. Therefore \(PP'\) is a cycle of length at least four. By the hypothesis, none of the internal vertices of \(P\) is incident to a chord of \(C\). In addition, \(P\) and \(P'\) are induced paths, so \(H := PP'\) is actually a hole in \(G\). Note that \(V(H) \subset V(C)\). If every vertex on \(H\) were on \(Q\), then \(Q\) would have a chord as \(V(H) \subset V(Q)\), which is impossible. Therefore \(H\) contains a vertex on \(C\) not on \(Q\). \(\square\)
Given a graph $G$ and nonempty vertex sets $S_1$ and $S_2$, we denote the set of edges joining vertices of $S_1$ and vertices of $S_2$ by $[S_1, S_2]$. For simplicity, we use $[v, S]$ instead of $\{v\}, S\}$ for a vertex $v$ and a nonempty vertex set $S$ of a graph $G$.

**Lemma 2.4.** Given a graph $G$, suppose that there exist a hole $H$, an induced path $P$, and two nonadjacent vertices $u$ and $v$ on $H$ not on $P$ satisfying the properties that

(i) $v$ is nonadjacent to any vertex on $P$ in $G$;

(ii) there exist an internal vertex on a $(u, v)$-section of $H$ and an internal vertex on the other $(u, v)$-section of $H$ such that each of them is adjacent to a vertex on $P$.

Then there is a hole not containing $u$ but containing two consecutive edges on $H$ incident to $v$ and containing a vertex on $P$ but not on $H$.

**Proof.** Let $P = z_1z_2 \cdots z_r$ ($r \geq 1$). By the hypothesis that $u$ and $v$ are not on $P$, $z_i \neq u, v$ for each $i = 1, \ldots, r$. Since $u$ and $v$ are nonconsecutive vertices on $H$, we may give a sequence of $H$ as follows:

$$H = vx_1x_2 \cdots x_py_qy_{q-1} \cdots y_1v \ (p, q \geq 1).$$

For notational convenience, we let $S_x = \{x_1, \ldots, x_p\}$ and $S_y = \{y_1, \ldots, y_q\}$. Let $\alpha = \min \{i \in \{1, \ldots, p\} \mid [x_i, V(P)] \neq \emptyset\}$ and $\beta = \min \{j \in \{1, \ldots, q\} \mid [y_j, V(P)] \neq \emptyset\}$. By the property (ii), $[S_x, V(P)] \neq \emptyset$ and $[S_y, V(P)] \neq \emptyset$ and so $\alpha$ and $\beta$ exist. Among the vertices on $P$ which are adjacent to $x_\alpha$ and among the vertices on $P$ which are adjacent to $y_\beta$, we take $z_\gamma$ and $z_\delta$ from them, respectively, with the smallest distance on $P$. Let $P^*$ be the $(z_\gamma, z_\delta)$-section of $P$. Then $C := vx_1x_2 \cdots x_\alpha P^* y_\beta y_{\beta-1} \cdots y_1v$ is a cycle not containing $u$. We also note that $C$ contains $x_1v$ and $y_1v$, which are consecutive edges on $H$ incident to $v$. It is easy to check that $C$ has length at least four. No two vertices in $V(C) \setminus V(P^*)$ or in $V(P^*)$ can form a chord of $C$ since the vertices in $V(C) \setminus V(P^*)$ are on the hole $H$ and $P^*$ is an induced path. Moreover, a vertex in $V(C) \setminus V(P^*)$ and a vertex in $V(P^*)$ cannot form a chord of $C$ by the choice of $\alpha, \beta, z_\gamma,$ and $z_\delta$. Therefore we can conclude that $C$ is a hole in $G$. Since $u$ is not on $C$, $C$ is distinct from $H$. We note that $C$ and $H$ both are holes and the vertices on $C$ other than the ones on $P^*$ lie on $H$. Therefore there must be a vertex on $P^*$ not on $H$. Since $P^*$ is a section of $P$, $C$ contains a vertex on $P$ but not on $H$. $\square$

Let $G$ be a graph with a hole $H$. For a vertex $u$ on $H$, we recall that locally chordalizing the hole $H$ by $u$ means the following procedure: we join $u$ and each vertex on $H$ nonadjacent to $u$. We call an edge added in the process of a local chordalization of a hole a **newly added edge**.

**Remark 2.5.** Note that, for a graph $G$, locally chordalizing the holes in $\mathcal{H}(G, u)$ by a vertex $u$ will destroy all the holes in $\mathcal{H}(G, u)$. That is, if $H \in \mathcal{H}(G, u)$, then $H \notin \mathcal{H}(G^*, u)$ where $G^*$ is the graph resulting from the local chordalization by $u$. 

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Theorem 2.6. Let $G$ be a graph and $u$ be a vertex of $G$ satisfying the NC property. Then locally chordalizing all the holes in $\mathcal{H}(G, u)$ by $u$ does not create any new hole.

Proof. Let $G^*$ be the graph obtained by locally chordalizing all the holes in $\mathcal{H}(G, u)$ by $u$. Suppose to the contrary that $G^*$ has a hole, say $H^*$, not in $G$. Obviously $H^*$ contains $u$ and at least one newly added edge. Then, since $u$ is adjacent to exactly two vertices on $H^*$, $H^*$ contains one newly added edge or two newly added edges.

Let $uv$ be a newly added edge and

$$H^* = uu_1u_2 \cdots u_pvu \quad (p \geq 2).$$

Next, we define a cycle $C$ by considering two cases.

Case 1. $H^*$ contains $uv$ as the only newly added edge. By the definition of local chordalization, there exists a hole $H_1$ in $\mathcal{H}(G, u)$ containing $v$ on which $u$ and $v$ are not consecutive. Then $u$ is adjacent to all the vertices on $H_1$ in $G^*$. However, $u$ is not adjacent to $u_k (k = 2, \ldots, p)$ in $G^*$, so we can conclude that $u_k (k = 2, \ldots, p)$ is not on $H_1$. If $u_1$ is on $H_1$, then $u_1$ is adjacent to $u$ in $H_1$. Thus, if $u_1$ is on $H_1$, then $uu_1u_2 \cdots u_pP$ is a cycle in $G$ for the $(v, u)$-section, denoted by $P$, of $H_1$ not containing $u_1$.

If $u_1$ is not on $H_1$ and $u_1$ is not adjacent to any vertex on one of the $(v, u)$-sections of $H_1$ except $u$, then we denote such a section by $P'$.

Now we define the cycle $C$ as follows:

$$C = \begin{cases} 
  uu_1u_2 \cdots u_pP & \text{if } u_1 \text{ is on } H_1; \\
  uu_1u_2 \cdots u_pP' & \text{if } u_1 \text{ is not on } H_1 \text{ and } u_1 \text{ is not adjacent to any vertex on one of the } (v, u)-
  \text{sections of } H_1 \text{ except } u.
\end{cases}$$

See (a) and (b) of Figure 2 for an illustration.

Case 2. $H^*$ contains another newly added edge $uw$. Then $u_1 = w$. Assume that there is a hole $H_2$ in $G$ which contains $u, v, w$. Then no two of $u, v, w$ are consecutive on $H_2$. Let $Q$ be the $(v, w)$-section of $H_2$ containing $u$. Since $u$ is adjacent to all the vertices on $H_2$ but is not adjacent to $u_i$ in $G^*$, we may conclude that $u_i$ is not on $H_2$ for each $i = 2, \ldots, p$. Therefore $wu_2u_3 \cdots u_pQ$ is a cycle in $G$. Now we let

$$C = wu_2u_3 \cdots u_pQ.$$ 

See Figure 2(c) for an illustration.

It is obvious that the cycle $C$ defined in each case has length at least four and $Pu_1$, $P'u_1$, and $Q$ are induced paths of $G$ including $u$ and the two vertices right next to $u$ on $C$. Moreover, $u$ is not adjacent to any vertex on $C$ except the two vertices right next to $u$, and the two vertices right next to $u$ on $C$ are not adjacent in $G$. Thus, by Lemma 2.3, the path $U$ composed of $u$ and the two vertices right next to it can be extended to a hole $H$ in
We let $u$ conclude that $w, S$ is adjacent to all the vertices on $S$ so that $H$ is not adjacent to $w, S$ in $G;$. Therefore the case (i) cannot happen.

Now we assume the case (ii). Since $v$ and $w$ are not consecutive vertices on $H^*$, $w$ is not adjacent to $v$ in $G$. Since $uw$ and $uw$ are newly added edges, there exist a hole $H_3$ containing $u$ and $w$ in $G$. By the case (ii) assumption, $w$ is not on $H_3$ and $v$ is not on $H_4$. Let $H_3 = vx_1x_2 \cdots x_qy_{r-1} \cdots y_1v$ and $H_4 = wz_1z_2 \cdots z_quw_1u_{l-1} \cdots u_1w (q, r, s, t \geq 1)$. See Figure 2(d) for an illustration. Since $u$ is adjacent to all the vertices on $H_3$ (resp. $H_4$) and is not adjacent to $v$, this contradicts the hypothesis that $u$ satisfies the NC property. Therefore the case (i) cannot happen.

Now we consider the sequence $Q := vx_1x_2 \cdots x_quz_sz_{s-1} \cdots z_1w$. As being sections of $H_3$ and $H_4$, respectively, the two subsequences $vx_1x_2 \cdots x_qu$ and $uz_sz_{s-1} \cdots z_1w$ of $Q$ are induced paths in $G$. In addition, since $[S_x \cup \{v\}, S_y \cup \{v\}] = \emptyset$, $Q$ is an induced path in $G$. Consider the cycle $C := Qvuwv$; Since $u$ is on $H_3$, $H_4$, and $H^*$, $u$ is not incident to any chord of $C$ in $G$. Then we apply Lemma 2.4 with $C$, $Q$, and $x_uz_s$ for $P$ to reach a contradiction as before.

**Corollary 2.7.** Suppose that a graph $G$ has a hole cover $C = \{u_1, \ldots, u_k\}$ satisfying the NC property and that $G_0 = G$ and, for $i = 1, \ldots, k$, $G_i$ is the graph obtained by locally chordalizing the holes in $\mathcal{H}(G_{i-1}, u_i)$ by $u_i$. Then $G_k$ is chordal. Moreover, the resulting chordal graph is independent of the order of $u_1, \ldots, u_k$ by which the local chordalizations are performed.
Figure 2: The cycle $C$ defined in the proof of Theorem 2.6. The gray colored edges represent the newly edges on the hole $H^*$ in $G^*$ and $w$ in (c) and (d) turns out to be $u_1$.

Proof. By induction on the size $k$ of a hole cover satisfying the NC property. If $k = 1$, then $G_k$ is chordal by Theorem 2.6. Suppose that the statement is true for any graph with a hole cover with size $k - 1$ satisfying the NC property. Now we locally chordalize the holes in $H(G, u_1)$ by $u_1$ to obtain $G_1$. By Theorem 2.6, $C \setminus \{u_1\}$ is a hole cover of $G_1$. By (2), $C \setminus \{u_1\}$ still satisfies the NC property in $G_1$. Therefore, by the induction hypothesis, $G_k$ is chordal.

It is sufficient to show the uniqueness for the case $k = 2$. Let $G'$ and $G''$ be the graphs obtained by locally chordalizing the holes in $H(G, u_2)$ by $u_2$ and the holes in $H(G', u_1)$ by $u_1$, respectively.

Since $C$ satisfies the NC property, no hole in $G$ contains two vertices in $C$. Therefore, by Theorem 2.6, $H(G, u_1) = H(G', u_1)$ and $H(G_1, u_2) = H(G, u_2)$, which implies $G_2 = G''$.

Let $G$ be a graph with a hole cover $C$ satisfying the NC property. Corollary 2.7 says that a chordal graph can be obtained by applying local chordalizations recursively by the vertices in $C$ and the resulting chordal graph is the same no matter which order of the vertices is taken. The uniqueness of the resulting chordal graph allows us to denote it by a notation, say $\hat{G}(C)$. In the rest of this paper, we derive some noteworthy theorems by
utilizing $\hat{G}(C)$ for graphs $G$ having hole covers $C$ satisfying the NC property.

**Lemma 2.8.** Let $G$ be a graph with a hole cover $C$ satisfying the NC property. Suppose that vertices $u$ and $w$ in $C$ are adjacent in $G$. Then any newly added edge incident to $u$ and any newly added edge incident to $w$ are not adjacent in $G(C)$.

**Proof.** Suppose to the contrary that there exist a newly added edge incident to $u$ and a newly added edge incident to $w$ which are adjacent in $G(C)$. Let $uv$ and $wv$ be such edges for some $v \in V(G)$. Then, by the definition of local chordalization, neither $uv$ nor $wv$ is an edge in $G$ and there exist $H_u \in \mathcal{H}(G, u)$ and $H_w \in \mathcal{H}(G, w)$ sharing the vertex $v$.

To reach a contradiction, suppose that there exist an internal vertex on a $(u, v)$-section of $H_u$ and an internal vertex on the other $(u, v)$-section of $H_u$ each of which is adjacent to $w$. Then, by Lemma 2.4 with $P = w$, there is a hole in $G$ not containing $u$ but containing two consecutive edges on $H_u$ incident to $v$, which contradicts the hypothesis that $C$ satisfies the NC property. Therefore there exists one of the $(u, v)$-sections of $H_u$ such that $w$ is not adjacent to any internal vertex on it. Let $Q$ be such a section. By symmetry, we may conclude that there exists one of the $(v, w)$-sections of $H_w$ such that $u$ is not adjacent to any internal vertex on it. Let $R$ be such a section.

Let $W$ be the concatenation of $Q$ and $R$ at $v$. Then $W$ is a $(u, w)$-walk in $G - uv$. Now $W$ contains a $(u, w)$-path $S$ as an induced subgraph in $G - uv$. By the previous argument, the vertex immediately following $u$ on $S$ cannot be on $R$ while the vertex immediately preceding $w$ on $S$ cannot be on $Q$. Therefore we may conclude that the length of $S$ is at least three. Thus $S$ and the edge $uv$ form a hole in $G$. However, this hole contains both $u$ and $w$, which is impossible as $C$ satisfies the NC property.

**Theorem 2.9.** Let $G$ be a graph with a hole cover $C$ satisfying the NC property. Suppose that a vertex set $K$ forms a clique in $\hat{G}(C)$ but not in $G$. Then there exists a vertex $u \in K \cap C$ such that $K \setminus \{u\}$ is a clique in $G$.

**Proof.** Since $K$ is a clique in $\hat{G}(C)$ but is not a clique in $G$, $K \cap C \neq \emptyset$. Suppose that $K \cap C$ is not a clique in $G$. Then there exist two vertices $x$ and $y$ in $K \cap C$ such that $xy \notin E(G)$. This implies that there exists a hole in $G$ containing both $x$ and $y$, which is impossible by the hypothesis that $C$ satisfies the NC property. Therefore $K \cap C$ is a clique in $G$. However, $K$ is not a clique in $G$, so there exist vertices $u \in K \cap C$ and $v \in K \setminus C$ such that $uw$ is a newly added edge. We claim that every newly added edge whose end vertices belong to $K$ is incident with $u$ by contradiction. Suppose that there exists a newly added edge $zw$ such that $\{z, w\} \subset K \setminus \{u\}$. By the definition of $\hat{G}(C)$, we may assume $z \in C$ and $w \notin C$. Since $K \cap C$ is a clique in $G$, $zu \in E(G)$. Then Lemma 2.8 implies that $v \neq w$, and $uw$ and $zw$ are edges in $G$. If $vw$ is a newly added edge, then either $v$ or $w$ belongs to $C$, which is not the case. Therefore $vw \notin E(G)$. Then the cycle $uzwvwu$ is obviously a hole in $G$ containing $u$ and $z$, which contradicts the hypothesis that $C$ satisfies the NC property. Thus we have shown that every newly added edge in $K$ is incident with $u$. Hence $K \setminus \{u\}$ is a clique in $G$. 

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Corollary 2.10. Let $G$ be a graph with a hole cover $C$ satisfying the NC property. Then
$\omega(\hat{G}(C)) \leq \omega(G) + 1$. Furthermore the equality holds if and only if

\[(\dagger) \text{ There exists a vertex } u \in C \text{ such that the set } \left( \bigcup_{H \in \mathcal{H}(G,u)} V(H) \right) \cup N_G(u) \setminus \{u\} \text{ contains a maximum clique } K \text{ of } G.\]

Proof. By Theorem 2.9 $\omega(\hat{G}(C)) \leq \omega(G) + 1$. Furthermore, by the same theorem, $\omega(\hat{G}(C)) = \omega(G) + 1$ if and only if there is a clique $K$ in $\hat{G}(C)$ of size $\omega(G) + 1$ and there is a vertex $u \in K \cap C$ such that $K \setminus \{u\}$ forms a clique in $G$, which is equivalent to $(\dagger)$. \qed

Theorem 2.11. Let $G$ be a graph with a hole cover $C$ satisfying the NC property. Then every clique of $\hat{G}(C)$ is a minor of $G$.

Proof. Let $K$ be a clique in $\hat{G}(C)$ of size $n$. If $K$ is a clique in $G$, then we are done. Suppose that $K$ is not a clique in $G$. By Theorem 2.9 there exists a vertex $u \in K \cap C$ such that $K \setminus \{u\}$ is a clique in $G$. Therefore $|V(H) \cap (K \setminus \{u\})| \leq 2$ for every $H \in \mathcal{H}(G,u)$. Furthermore, every newly added edge whose end vertices are in $K$ is incident with $u$.

Let $uv_1, \ldots, uv_l$ be the newly added edges whose end vertices are in $K$ and $X = \{v_1, \ldots, v_l\}$. Take a vertex $v_i \in X$. Then there exists $H \in \mathcal{H}(G,u)$ containing $v_i$. Since $uv_i$ is a newly added edge, $u$ and $v_i$ are not consecutive on $H$. Then each of the $(u, v_i)$-sections of $H$ contains at least one internal vertex. In addition, $V(H) \cap X \subset V(H) \cap (K \setminus \{u\})$. Since we have shown that $|V(H) \cap (K \setminus \{u\})| \leq 2$, $|V(H) \cap X| \leq 2$. Since $v_i \in V(H) \cap X$, $V(H)$ contains at least one vertex in $X$ other than $v_i$. Thus one of the $(u, v_i)$-sections of $H$ does not contain any vertex in $K$ as an internal vertex. Let $P_i$ be such a section. In $G$, we contract the edges on $P_1$ except the edge incident to $v_1$ to obtain the edge $e_1$ joining $u$ and $v_1$. Then $P_2$ is transformed to a $(u, v_2)$-walk $W_2$ in the graph $G_1$ resulting from the contractions and still does not contain any vertex in $K$ other than $u$ and $v_2$ by the way of contractions and by the choice of $P_i$. In $G_i$, we contract the edges on $W_2$ except the edge incident to $v_2$ to obtain the graph $G_2$ and the edge $e_2$ joining $u$ and $v_2$ in $G_2$. We may repeat this process until we obtain the graph $G_l$ from $G_{l-1}$ and the edge $e_l$ joining $u$ and $v_l$ in $G_l$. Now, $G_l$ contains the vertices of $K$ and the edges $uv_1, \ldots, uv_l$ so that $K$ is clique of size $n$ in $G_l$. \qed

Now we have the following corollary.

Corollary 2.12. Let $G$ be a graph with a hole cover $C$ satisfying the NC property. If $G$ is $K_n$-minor-free, then $\hat{G}(C)$ is $K_n$-free.

Now we are ready to give a proof of Theorem 2.7.

A proof of Theorem 2.7 Since $\hat{G}(C)$ is a chordal completion,

$$\chi_{DP}(G) \leq \chi_{DP}(\hat{G}(C)) = \omega(\hat{G}(C)).$$

By Corollary 2.10, $\omega(\hat{G}(C)) \leq \omega(G) + 1$, so $\chi_{DP}(G) \leq \omega(G) + 1$. Moreover, by Corollary 2.12 if $G$ is $K_n$-minor-free, then $\omega(\hat{G}(C)) \leq n - 1$ and so $\chi_{DP}(G) \leq n - 1$. \qed
3 A partial result on the Erdős-Faber-Lovász Conjecture

The following is one of the versions equivalent to the conjecture given by Erdős, Faber, and Lovász in 1972.

Conjecture 3.1. If $G$ is the union of $k$ edge-disjoint copies of $K_k$ for a positive integer $k$, then $\chi(G) = k$.

In this section, we show that the above conjecture is true for the graphs satisfying the NC property by deriving the following theorem.

Theorem 3.2. If a graph $G$ satisfying the NC property is the union of $k$ edge-disjoint copies of $K_k$ for a positive integer $k$, then $\chi_{DP}(G) = k$.

We start by showing the following lemmas. A vertex is said to be a simplicial vertex if its neighbors form a clique.

Lemma 3.3. Let $G$ be a graph and $L$ be a maximal clique of $G$. Suppose that every vertex in $G – L$ is a simplicial vertex in $G$. Then $G$ is chordal.

Proof. It suffices to prove the lemma when $G$ is connected. Suppose to the contrary that $G$ has a hole $H$. Since $L$ is complete and $H$ is a hole in $G$, $|V(H) \cap L| \leq 2$. Then $V(H) \setminus L$ forms an induced path in $G$ and, by the hypothesis that any vertex in $G – L$ is a simplicial vertex in $G$, $|V(H) \setminus L| \leq 2$. Since $H$ is a hole, $4 \leq |V(H)| = |V(H) \cap L| + |V(H) \setminus L| \leq 4$ and so $|V(H) \cap L| = 2$ and $|V(H) \setminus L| = 2$. Since $V(H) \setminus L := \{u,v\}$ and $V(H) \cap L := \{x,y\}$ are cliques in $G$, $uv$ and $xy$ are edges in $G$. Since $H$ is a hole, $u$ cannot be a simplicial vertex in $G$ and we reach a contradiction.

In a graph, we say that a clique $K$ covers an edge $e$ if $e$ is an edge of $K$.

Lemma 3.4. Let $G$ be a union of $k$ edge-disjoint copies of $K_k$ and $\mathcal{L}$ be the set of those $k$ copies of $K_k$ for a positive integer $k$. Then $\omega(G) = k$. Furthermore, if a maximal clique of $G$ with size $k$ does not belong to $\mathcal{L}$, then $G$ is chordal.

Proof. Since $G$ contains $K_k$, $\omega(G) \geq k$. We prove that any maximal clique of $G$ not belonging to $\mathcal{L}$ has size at most $k$ to show $\omega(G) \leq k$. Let $L$ be a maximal clique of $G$ with size $l$ which does not belong to $\mathcal{L}$. For each vertex $u$ in $L$, let $n_u$ be the minimum number of cliques in $\mathcal{L}$ needed to cover the edges in the edge cut $[u, L \setminus \{u\}]$. Since each edge of $G$ is covered by a unique maximal clique in $\mathcal{L}$, $n_u$ is the number of cliques in $\mathcal{L}$ which share an edge with $L$. Since $L$ is a maximal clique of $G$ and does not belong to $\mathcal{L}$, the edges on $L$ are covered by at least two cliques in $\mathcal{L}$ and so $n_u \geq 2$ for each $u \in L$. Now let $u^*$ be a vertex in $L$ with the minimum $p := n_{u^*}$. By the observation that $n_u \geq 2$ for each $u \in L$, $p \geq 2$. Let $L_1, \ldots, L_p$ be the cliques in $\mathcal{L}$ which cover the edges in $[u^*, L \setminus \{u^*\}]$. Let $l_i = |L \cap L_i| – 1$ for each $i = 1, \ldots, p$. Without loss of generality, we may assume

$$l_1 \geq l_2 \geq \cdots \geq l_p \geq 1.$$  

(2)
Suppose that there exist distinct vertices $u_1$ and $u_2$ in $L \cap L_i$ for some $i \in \{1, \ldots, p\}$ such that an edge in $[u_1, L \setminus L_i]$ and an edge in $[u_2, L \setminus L_i]$ are covered by the same clique $K$ in $\mathcal{L}$. Then $K \neq L_i$. However, since $K$ is a clique, $u_1u_2$ is covered by $K$, a contradiction to the hypothesis. Therefore

(⋆) two edges in $[L \cap L_i, L \setminus L_i]$ are covered by distinct cliques in $\mathcal{L}$ for $i = 1, \ldots, p$ unless they have a common end in $L \cap L_i$.

Since $L_1, \ldots, L_p$ are mutually edge-disjoint,

$$l = \left| L \cap \bigcup_{i=1}^{p} L_i \right| = \left| \left( \bigcup_{i=1}^{p} (L \cap L_i) \setminus \{u^*\} \right) \cup \{u^*\} \right|$$

$$= \sum_{i=1}^{p} \left| (L \cap L_i) \setminus \{u^*\} \right| + 1 = \sum_{i=1}^{p} l_i + 1. \quad (3)$$

Since $p \geq 2$, $L_1$ and $L_2$ exist. Each edge in $[(L \cap L_1) \setminus \{u^*\}, (L \cap L_2) \setminus \{u^*\}]$ is covered by exactly one clique in $\mathcal{L}$ by the hypothesis. Since any edge in $[(L \cap L_1) \setminus \{u^*\}, (L \cap L_2) \setminus \{u^*\}]$ is not incident to $u^*$, any clique in $\mathcal{L}$ covering an edge in $[(L \cap L_1) \setminus \{u^*\}, (L \cap L_2) \setminus \{u^*\}]$ cannot be $L_i$ for any $i = 1, \ldots, p$. Therefore we need at least $p + l_1l_2$ cliques in $\mathcal{L}$ to cover the edges in $[u^*, L \setminus \{u^*\}] \cup [(L \cap L_1) \setminus \{u^*\}, (L \cap L_2) \setminus \{u^*\}]$ and so

$$p + l_1l_2 \leq |\mathcal{L}| = k. \quad (4)$$

For each vertex $u$ in $L \cap L_1$, $n_u \geq p$ and so there are at least $p$ cliques in $\mathcal{L}$ needed to cover the edges in $[u, L \setminus \{u\}]$. By (⋆), we need at least $p + l_1(p - 1)$ distinct cliques in $\mathcal{L}$ to cover the edges in $[u^*, L \setminus \{u^*\}] \cup [(L \cap L_1) \setminus \{u^*\}, L \setminus L_1]$ and so

$$p + l_1(p - 1) \leq k. \quad (5)$$

If $l_2 \geq p$, then

$$l = \sum_{i=1}^{p} l_i + 1 \quad \text{(by (3))}$$

$$\leq l_1p + 1 \quad \text{(by (2))}$$

$$< l_1l_2 + p \quad \text{(by the case assumption and the fact that } p \geq 2)$$

$$\leq k. \quad \text{(by (4))}$$

Therefore we have shown that $l < k$ if $l_2 \geq p$ and so the “furthermore” part is vacuously true.
Now assume \( l_2 \leq p - 1 \). Then
\[
\begin{align*}
  l &= \sum_{i=1}^{p} l_i + 1 & \text{(by (5))} \\
  &\leq (p - 1)l_1 + l_2 + 1 & \text{(by (2))} \\
  &\leq (p - 1)l_1 + p & \text{(by the assumption that } l_2 \leq p - 1) \\
  &\leq k & \text{(by (5))}
\end{align*}
\]

To show the “furthermore” part, suppose \( l = k \). Then each of the three inequalities above becomes the equality. Now, if \( p = 2 \), then \( l_2 = 1 \) and \( l = l_1 + l_2 + 1 = l_1 + 2 = k \), which implies \( l_1 = k - 2 \). If \( p \geq 3 \), then, by (2), \( l_1 = \cdots = l_p = p - 1 \) and \( k = p^2 - p + 1 \).

**Case 1.** \( p = 2 \). Let \( L \cap L_1 = \{u^*, u_1, u_2, \ldots, u_{k-2}\} \) and \( L \cap L_2 = \{u^*, v\} \). Since \( u_i \) and \( v \) belong to \( L \), \( u,v \) is an edge in \( G \) for each \( i = 1, \ldots, k - 2 \). Since \( \mathcal{L} \) is an edge clique cover of \( G \), there is a clique in \( \mathcal{L} \) covering the edge \( u_i v \) for each \( i = 1, \ldots, k - 2 \). By (*), no clique in \( \mathcal{L} \) contains \( u_i, u_j, v \) for \( 1 \leq i < j \leq k - 2 \). Therefore, by relabelling the cliques in \( \mathcal{L} \) if necessary, we may assume \( L_{i+2} \) is a clique covering \( u_i v \) for each \( i = 1, \ldots, k - 2 \). Then \( (L_1 \cap L_2) \cap L = \{u^*\}, (L_1 \cap L_i) \cap L = \{u_{i-2}\} \) for \( i = 3, \ldots, k \), and \( (L_i \cap L_j) \cap L = \{v\} \) for \( 2 \leq i < j \leq k \). Therefore \( L_i \) and \( L_j \) share exactly one vertex in \( L \) for distinct \( i, j \) in \( \{1, \ldots, k\} \).

**Case 2.** \( p \geq 3 \). Then \( l_1 = \cdots = l_p = p - 1 \) and \( k = p^2 - p + 1 \). Let \( L \cap L_1 = \{u^*, u_1, \ldots, u_{p-1}\} \) and \( L \cap L_2 = \{u^*, w_1, \ldots, w_{p-1}\} \). Since \( L \) is a clique in \( G \), \( v_i \) and \( w_j \) are adjacent in \( G \) and the edge \( v_i w_j \) must be covered by a clique in the edge clique cover \( \mathcal{L} \) for any \( i, j \in \{1, \ldots, p - 1\} \). Let \( K_{i,j} \in \mathcal{L} \) be a clique which covers the edge \( v_i w_j \) for \( i, j \in \{1, \ldots, p - 1\} \) and let \( \mathcal{K} = \{K_{i,j} \mid i, j \in \{1, \ldots, p - 1\}\} \). Suppose \( K_{i,j} = L_t \) for some \( i, j \in \{1, \ldots, p - 1\} \) and \( t \in \{1, \ldots, p\} \). Then the edges \( u^* w_j \in [L \cap L_1, L \setminus L_1] \) and \( v_i w_j \in [L \cap L_1, L \setminus L_1] \) are covered by \( K_{i,j} \), which is impossible by (*). Therefore \( K_{i,j} \) cannot be any of \( L_1, \ldots, L_p \). By (*), \( K_{i,j} \neq K_{i',j'} \) if \( (i, j) \neq (i', j') \). Therefore \( |\mathcal{K}| = (p - 1)^2 \) and
\[
|\{L_1, \ldots, L_p\} \cup \mathcal{K}| = p + (p - 1)^2 = p^2 - p + 1.
\]

Since \( |\mathcal{L}| = k = p^2 - p + 1 \), \( \mathcal{L} = \{L_1, \ldots, L_p\} \cup \mathcal{K} \).

To apply Lemma 3.3, we first claim that \( M \cap N \subset L \) for any distinct cliques \( M, N \in \mathcal{L} \). Take two distinct cliques \( M \) and \( N \) in \( \mathcal{L} \). If \( M \) and \( N \) belong to \( \{L_1, \ldots, L_p\} \), then \( M \cap N = \{u^*\} \subset L \). Suppose that one of \( M \) and \( N \) is in \( \{L_1, \ldots, L_p\} \) and the other is in \( \mathcal{K} \). Without loss of generality, we may assume \( M = L_t := \{u^*, x_1, \ldots, x_{p-1}\} \) and \( N = K_{i,j} \) for some \( t \in \{1, \ldots, p\} \) and \( i, j \in \{1, \ldots, p - 1\} \). By the hypothesis that the cliques in \( \mathcal{L} \) are mutually edge-disjoint,
\[
L_t \cap K_{i,j} = \begin{cases} 
\{v_i\} & \text{if } t = 1 \\
\{w_j\} & \text{if } t = 2.
\end{cases}
\]
Therefore $M \cap N = L_t \cap K_{i,j} \subset L$ for $t = 1, 2$. Assume $3 \leq t \leq p$. Note that
\[
E_{1t} := \{(v_1, \ldots, v_{p-1}), (x_1, \ldots, x_{p-1})\} \subset [L \cap L_1, L \setminus L_1] \cap [L \cap L_t, L \setminus L_t].
\] (6)

Suppose that an edge $v_r x_s$ is covered by $L_a$ for some $a \in \{1, \ldots, p\}$. Then the edges $u^* x_s$ and $u^* v_r$ are covered by $L_a$. However, $u^*$ and $v_r$ belong to $L \cap L_1$, $\{u^* x_s, v_r x_s\} \in [L \cap L_1, L \setminus L_1]$, and we reach a contradiction to $(\ast)$. Therefore each edge in $E_{1t}$ should be covered by a clique in $K$. Since $K \subset L$, it follows from $(\ast)$ that each clique in $K$ covers at most one edge in $E_{1t} \subset [L \cap L_1, L \setminus L_1] \cap [L \cap L_t, L \setminus L_t]$. Since $|K| = (p - 1)^2 = |E_{1t}|$, each clique in $K$ covers exactly one edge in $E_{1t}$. Therefore $K_{i,j}$ covers $v_r x_s$ for some $r, s \in \{1, \ldots, p - 1\}$. Thus $L_t \cap K_{i,j}$ contains the vertex $x_s$. By the hypothesis that the cliques in $L$ are mutually edge-disjoint, $L_t \cap K_{i,j} = \{x_s\} \subset L$. Hence $M \cap N \subset L$ for $M = L_t$ and $N = K_{i,j}$. Finally we suppose that $M$ and $N$ belong to $K$. Then $M = K_{i,j}$ and $N = K_{i',j'}$ for some $i, i', j, j' \in \{1, \ldots, p - 1\}$ with $(i, j) \neq (i', j')$. If $i = i'$, then $M \cap N = \{v_i\} \subset L$ by the hypothesis. Suppose $i \neq i'$. Take a vertex $y \in L \setminus L_1$. Since $L$ is a clique and $\{v_i, v_{i'}, y\} \subset L$, $v_i y$ and $v_{i'} y$ are edges of $G$ and should be covered by cliques in $K$. We note that $L_b$ covers $u^* y$ if $L_b$ covers $v_i y$ or $v_{i'} y$ for any $b \in \{1, \ldots, p\}$. Therefore, by the hypothesis that the cliques in $L$ are mutually edge-disjoint, $v_i y$ and $v_{i'} y$ are covered by cliques in $K$. Let $K_{c,d}$ be a clique in $K$ covering $v_i y$. Then $v_c, v_i, y$ belong to $K_{c,d}$. Since $K_{c,d}$ is a clique, $v_c$ and $y$ are adjacent. Then $v_{i'} y$ and $v_i y$ belong to $[L \cap L_1, L \setminus L_1]$ and are covered by $K_{c,d}$. Thus, by $(\ast)$, $v_i = v_c$ and so $i = c$. Similarly, $v_{i'} y$ is covered by $K_{c',d'}$ for some $d' \in \{1, \ldots, p - 1\}$. By the hypothesis on $L$, $K_{i,d}$ and $K_{i',d'}$ are the unique cliques in $L$ covering $v_i y$ and $v_{i'} y$, respectively. As $K_{i,d}$ and $K_{i',d'}$ are uniquely determined by $y$, we may denote $K_{i,d}$ and $K_{i',d'}$ by $A(y)$ and $B(y)$, respectively. Now we define a function $F : L \setminus L_1 \to \{(K_{i,q}, K_{i',q'}) \mid 1 \leq q, q' \leq p - 1\}$ by $F(y) = (A(y), B(y))$ for $y \in L \setminus L_1$. Then $F$ is well-defined. By the hypothesis on $L$ again, $A(y) \cap B(y) = \{y\}$ for each $y \in L \setminus L_1$ and so $F$ is injective. Since the domain and the codomain of $F$ have the same cardinality $(p - 1)^2$, $F$ is bijective. Since $M$ and $N$ belong to $K$, $(M, N)$ is contained in the codomain of $F$ and so there exists a vertex $z \in L \setminus L_1$ such that $F(z) = (M, N)$. Then $M = A(z)$ and $N = B(z)$, so $M \cap N = A(z) \cap B(z) = \{z\} \subset L$. Hence we have shown that $M \cap N \subset L$ for any distinct cliques $M$ and $N$ in $L$.

In both cases, we have shown that $M \cap N \subset L$ for any distinct cliques $M$ and $N$ in $L$. Now we will show that every vertex in $G - L$ is simplicial in $G$. Take a vertex $v$ in $G - L$. Suppose to the contrary that $v$ is not a simplicial vertex in $G$. Then $v$ has two neighbors $z_1$ and $z_2$ which are nonadjacent in $G$. Since $L$ is an edge clique cover of $G$, $L$ contains a clique covering $v z_1$ and a clique covering $v z_2$. Since $z_1$ and $z_2$ are nonadjacent, these two cliques are distinct. However, they share a vertex $v$ which is not in $L$. This contradicts our claim that the intersection of any two cliques in $L$ is a subset of $L$. Therefore every vertex in $G - L$ is a simplicial vertex in $G$. Thus, by Lemma 3.3, $G$ is chordal. \[\Box\]

A proof of Theorem 3.2. Let $G$ be a graph satisfying the NC property which is the union of $k$ edge-disjoint copies $L_1, \ldots, L_k$ of $K_k$. Obviously $\chi_{DP}(G) \geq k$. By Lemma 3.4
obtain the chordal graph $G$. Let $\mathcal{L} = \{L_1, \ldots, L_k\}$. Then $\mathcal{L}$ is an edge clique cover consisting of cliques of size $k$.

Fix $i \in \{1, \ldots, k\}$. Then $|L_i \cap L_j| \leq 1$ for any $j \in \{1, \ldots, k\} \setminus \{i\}$. Since $L_i$ has $k$ vertices, $L_i$ has a vertex $v$ not contained in $L_j$ for any $j \in \{1, \ldots, k\} \setminus \{i\}$. Then $v$ is a simplicial vertex of $G$. Since $i$ is arbitrarily chosen, $L_i$ has a simplicial vertex for any $i = 1, \ldots, k$.

If $G$ is chordal, then $\chi_{DP}(G) = \omega(G) = k$ by (§). Now we suppose that $G$ is non-chordal. Then, by the “furthermore part” of Lemma 3.4, any clique not belonging to $\mathcal{L}$ has size less than $k$. Since $L_i$ has a simplicial vertex of $G$, we may take a simplicial vertex from $L_i$ and denote it by $v_i$ for each $i = 1, \ldots, k$. Let $G' = G - \{v_1, \ldots, v_k\}$. Then $G'$ still satisfies the NC property. Since any clique not belonging to $\mathcal{L}$ has size less than $k$, $\omega(G') = k - 1$. Let $\mathcal{C}$ be a hole cover of $G'$ satisfying the NC property. Then $\hat{G}'(\mathcal{C})$ is chordal by definition and, by Corollary 2.10, $\omega(\hat{G}'(\mathcal{C})) \leq \omega(G') + 1 = k$. Let $G^*$ be the graph obtained from $\hat{G}'(\mathcal{C})$ by adding the vertices $v_1, \ldots, v_k$ and the edges which were incident to $v_1, \ldots, v_k$ in $G$. Then $G$ is a spanning subgraph of $G^*$. Since $v_1, \ldots, v_k$ are simplicial vertices of $G$, they are still simplicial vertices of $G^*$. Therefore, the fact that $\hat{G}'(\mathcal{C})$ is chordal implies that $G^*$ is chordal. Moreover, we note that exactly $k - 1$ edges are added for $v_i$ for each $i = 1, \ldots, k$ to obtain $G^*$ from $\hat{G}'(\mathcal{C})$. Then, since $\omega(\hat{G}'(\mathcal{C})) \leq k$,

$$k \leq \chi_{DP}(G) \leq \chi_{DP}(G^*) = \omega(G^*) \leq k$$

and so $\chi_{DP}(G) = k$. \hfill \qed

4 A minimal chordal completion of a graph

4.1 Non-chordality indices of graphs

Given a graph $G$, we apply a sequence of local chordalizations to obtain a chordal completion $G^*$ of $G$ as follows: Let $\mathcal{C} = \{v_1, \ldots, v_l\}$ be a hole cover of $G$ and $G_0 = G^*_0 = G - \mathcal{C}$. By the definition of hole cover, $G^*_0$ is chordal. Let $G_1$ be the graph with

$$V(G_1) = V(G^*_0) \cup \{v_1\} \quad \text{and} \quad E(G_1) = E(G^*_0) \cup E\left(G - \bigcup_{j=2}^{l} \{v_j\}\right).$$

Obviously $\{v_1\}$ is a hole cover of $G_1$ satisfying the NC property. By Corollary 2.7 we obtain the chordal graph $G_1^* = \hat{G}_1(\{v_1\})$. Let $G_2$ be the graph with

$$V(G_2) = V(G_1^*) \cup \{v_2\} \quad \text{and} \quad E(G_2) = E(G_1^*) \cup E\left(G - \bigcup_{j=3}^{l} \{v_j\}\right).$$

Again, $\{v_2\}$ is a hole cover of $G_2$ satisfying the NC property. Let $G_2^* = \hat{G}_2(\{v_2\})$ and we repeat this process until we obtain the chordal graph $G_i^* = \hat{G}_i(\{v_1\})$ as a desired graph.
Now we have shown the following theorem.

**Theorem 4.1.** Let $G$ be a graph with a hole cover $\mathcal{C}$. Then $\mathcal{C}$ can be partitioned into $\mathcal{C}_1, \ldots, \mathcal{C}_k$ for some positive integer $k$ so that

(i) $\mathcal{C}_i$ is a hole cover of the graph $G_i$ satisfying the NC property,

(ii) $G^*_i$ is chordal,

where $G_0 = G^*_0 = G - \mathcal{C}$; $G_i$ is the graph defined by $V(G_i) = V(G^*_{i-1}) \cup \mathcal{C}_i$.

$$E(G_i) = E(G^*_{i-1}) \cup E \left( G - \bigcup_{j=i+1}^k \mathcal{C}_j \right),$$

and $G^*_i = \widehat{G}_i(\mathcal{C}_i)$ for each $i = 1, \ldots, k$.

Let $G$ be a graph with a hole cover $\mathcal{C}$. We call an ordered partition $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ of a hole cover $\mathcal{C}$ satisfying the conditions (i) and (ii) in Theorem 4.1 a local chordalization partition of $\mathcal{C}$. Then the graphs $G_i, G^*_i$ are uniquely determined by the given local chordalization partition $\hat{\mathcal{C}} := (\mathcal{C}_1, \ldots, \mathcal{C}_k)$ of $\mathcal{C}$. We call the process of obtaining $G_i$ and $G^*_i$ the chordalization chain corresponding to $\hat{\mathcal{C}}$. Especially, we write the process of obtaining $G_i$ from $G^*_{i-1}$ as $G^*_{i-1} \prec \mathcal{C}_i G_i$ (in the context that $G^*_{i-1}$ is a proper subgraph of $G_i$, we use “strictly less” notation) for $i = 1, \ldots, k$. Then the chordalization chain corresponding to $\hat{\mathcal{C}}$ may be represented as

$$G_0 = G^*_0 \prec \mathcal{C}_1 G_1 \leq G^*_1 \prec \mathcal{C}_2 G_2 \leq G^*_2 \prec \cdots \prec \mathcal{C}_k G_k \leq G^*_k.$$

We note that $G^*_k$ is a chordal completion of $G$. By the way, the last chordal completion in the chordalization chain corresponding to $\hat{\mathcal{C}}$ is a minimal chordal spanning supergraph of $G$.

**Proposition 4.2.** Let $G$ be a graph, $\hat{\mathcal{C}} = (\mathcal{C}_1, \ldots, \mathcal{C}_k)$ be a local chordalization partition of a hole cover $\mathcal{C}$ of $G$, and $G^*$ be the last graph in the chordalization chain corresponding to $\hat{\mathcal{C}}$. Then $G^*$ is a minimal chordal completion of $G$.

**Proof.** Let $H$ be a graph that is a spanning supergraph of $G$ and a proper subgraph of $G^*$. Then $E(G^*) \setminus E(H) \neq \emptyset$. By definition, each edge of $E(G^*) \setminus E(H)$ is incident to one of vertices in $\mathcal{C}$. Let $s$ be the smallest index such that some vertices in $\mathcal{C}_s$ are incident to edges in $E(G^*) \setminus E(H)$. Now let $B$ be the set of edges in $E(G^*) \setminus E(H)$ which are incident to vertices in $\mathcal{C}_s$. By the definition of local chordalization, $G^*_s - B$ is not chordal. Thus there exists a hole $\hat{C}$ in $G^*_s - B$. By the choice of $s$, the edges in $E(G^*) \setminus E(G^*_s)$ are incident to
vertices in $\bigcup_{j=s+1}^\ell C_j$. By definition, $(\bigcup_{j=s+1}^\ell C_j) \cap V(G^*_s) = \emptyset$. Since $V(G^*_s) = V(G^*_s - B)$, the edges in $E(G^*_s) \setminus E(G^*_s)$ cannot be chords of $C$. Since $E(H) \subset E(G^*_s)$, the edges in $E(H) \setminus E(G^*_s)$ cannot be chords of $C$. Therefore $C$ is a hole in $H$ and so $H$ is not chordal. Hence we have shown that $G^*$ is a minimal chordal completion of $G$. \hfill \Box

Now we are ready to introduce a parameter of a graph which measures the number of steps of adding new edges to reach one of its chordal completion.

**Definition 4.3.** The non-chordality index of a graph $G$, denoted by $i(G)$, is defined as follows: If $G$ is chordal, $i(G) = 0$. If $G$ is not chordal, then $i(G)$ is defined to be the smallest $k$ over all the hole covers of $G$ in Theorem 4.1.

**Remark 4.4.** A graph $G$ satisfies the NC property if and only if $G$ satisfies $i(G) \leq 1$.

**Example 4.5.** We consider the graph $G$ given in Figure 1. Since $G$ does not satisfy the NC property, $i(G) \geq 2$ by Remark 4.4. It is easy to check that $C = \{u, v, w, x\}$ is a hole cover of $G$. See Figure 3 for an illustration. Since $G^*_2$ is a chordal completion of $G$, $i(G) \leq 2$. Thus $i(G) = 2$.

In this section, we prove the following statement.

**Theorem 4.6.** For any graph $G$, $\chi_{DP}(G) \leq \omega(G) + i(G)$. Especially, if $G$ is non-chordal and $K_n$-minor-free, then $\chi_{DP}(G) \leq n - 2 + i(G)$.

In order to do that, we show the following theorem first.

**Theorem 4.7.** Let $G$ be a graph, $\tilde{C} = (C_1, \ldots, C_\ell)$ be a local chordalization partition of a hole cover $C$ of $G$, and $G^*$ be the last graph in the chordalization chain corresponding to $\tilde{C}$. If a vertex set $K$ of $G$ forms a clique in $G^*$, then there exists a subset $C^*$ of $K \cap C$ such that $K \setminus C^*$ is a clique in $G$ and $|C^* \cap C_i| \leq 1$ for each $i = 1, \ldots, \ell$.

**Proof.** Let $G_0 = G^*_0 < C_1 < G_1 < C_2 < G_2 < \cdots < C_\ell < G_\ell = G^*$ be the chordalization chain corresponding to $\tilde{C}$ for graphs $G_i$ and chordal graphs $G^*_i$. Then $C_i$ is a hole cover of the graph $G_i$ satisfying the NC property for each $i = 1, \ldots, \ell$. For each $i = 0, 1, \ldots, \ell$, we add the vertices in $\bigcup_{j=i+1}^\ell C_j$ to $G^*_i$ and then restore the edges in $G$ to obtain $H_i$, that is, $H_i$ is the spanning supergraph of $G$ with the edge set $E(G) \cup E(G^*_i)$. Then, by the definitions of $G_i$ and $G^*_i$, $H_\ell = G^*_\ell$ and, for each $i = 0, \ldots, \ell - 1$, $H_i = \bigcup_{j=i+1}^\ell C_j = G^*_i$, $H_i = \bigcup_{j=i+2}^\ell C_j = G_{i+1}$,
Figure 3: A chordalization chain $G_0 = G_0^* \prec_{\{u,v,w\}} G_1 \leq G_1^* \prec_{\{x\}} G_2 \leq G_2^*$ for a local chordalization partition $\tilde{C} = (\{u,v,w\}, \{x\})$ of $G$
and, since $G_{i+1}^* = \overline{G_{i+1}}(C_{i+1})$, 

$$H_{i+1} - \bigcup_{j=i+2}^{\ell} C_j = \left( H_i - \bigcup_{j=i+2}^{\ell} C_j \right) (C_{i+1}). \quad \text{(7)}$$

We claim that if $L$ is a clique in $H_{i+1}$ but is not a clique in $H_i$, then $L \setminus \{u\}$ is a clique in $H_i$ for some vertex $u \in L \cap C_{i+1}$ for each $i = 0, 1, \ldots, \ell - 1$. Suppose $L$ is a clique in $H_{i+1}$ but not a clique in $H_i$ for some $i \in \{0, 1, \ldots, \ell - 1\}$. Then $L^* := L \setminus \bigcup_{j=i+2}^{\ell} C_j$ is a clique in $H_{i+1} - \bigcup_{j=i+2}^{\ell} C_j$. Since two vertices joined by an edge in $H_{i+1}$ but not in $H_i$ belongs to $V(G_{i+1}^*) = V(G) \setminus \bigcup_{j=i+2}^{\ell} C_j$, $L^*$ is not a clique in $H_i - \bigcup_{j=i+2}^{\ell} C_j$. We note that (7) holds and $C_{i+1}$ is a hole cover of $G_{i+1} = H_i - \bigcup_{j=i+2}^{\ell} C_j$ satisfying the NC property. Thus, by Theorem 2.9 there exists a vertex $u \in L^* \cap C_{i+1}$ such that $L^* \setminus \{u\}$ is a clique in $H_i - \bigcup_{j=i+2}^{\ell} C_j$. For the same reason why $L^*$ is not a clique in $H_i - \bigcup_{j=i+2}^{\ell} C_j$, $L \setminus \{u\}$ is still a clique in $H_i$.

Now we take a clique $L_0 := K$ in $H_\ell$. For $i = 0, 1, \ldots, \ell - 1$, we sequentially obtain a clique $L_{i+1}$ in $H_{\ell-i-1}$ in the following way. If $L_i$ is a clique in $H_{\ell-i-1}$, then we let $L_{i+1} = L_i$. If $L_i$ is not a clique in $H_{\ell-i-1}$, then, by the claim which has been proven above, there exists a vertex $u \in L_i \cap C_{\ell-i-1}$ such that $L_i \setminus \{u\}$ is a clique in $H_{\ell-i-1}$ and we let $L_{i+1} = L_i \setminus \{u\}$. Let $C^* = K \setminus L_\ell$. Then $K \setminus C^*$ equals $L_\ell$ and so is a clique as $L_\ell$ is a clique in $H_0 = G$. Moreover, since at most one vertex in $C_{\ell-i}$ was deleted to obtain $L_{i+1}$ from $L_i$, we have $C^* \subset C$ and $|C^* \cap C_i| \leq 1$ for each $i = 1, \ldots, \ell$, which completes the proof.

**Theorem 4.8.** Let $G$ be a graph, $\hat{C} = (C_1, \ldots, C_{\ell(G)})$ be a local chordalization partition of a hole cover $C$ of $G$, and $G^*$ be the last graph in the chordalization chain corresponding to $\hat{C}$. Then, for an induced subgraph $H$ of $G$, $\omega(H^*) \leq \omega(H) + i(G)$ where $H^*$ is the subgraph of $G^*$ induced by $V(H)$. Especially, if $G$ is non-chordal and $K_n$-minor-free, then $\omega(G^*) \leq n - 2 + i(G)$.

**Proof.** If $G$ is chordal, then the first part of the statement is immediately true as we may take $G$ as $G^*$ and the second statement is vacuously true. Thus we may assume $G$ is non-chordal. Then $\ell := i(G) \geq 1$. Let 

$$G_0 = G_0^* < c_1, G_1 \leq G_1^* < c_2, G_2 \leq G_2^* < \cdots < c_\ell, G_\ell \leq G_\ell^* = G^*. \quad \text{(4)}$$

be the chordalization chain corresponding to $\hat{C}$. Clearly $H^*$ is a chordal completion of $H$. Let $K$ be a maximum clique of $H^*$. Then $K$ is a clique in $G^*$. By Theorem 4.7 there exists a subset $C^*$ of $K \cap C$ such that $K \setminus C^*$ is a clique in $G$ and $|C^* \cap C_i| \leq 1$ for each $i = 1, \ldots, \ell$. Then 

$$|C^*| = \left| C^* \cap \bigcup_{j=1}^{\ell} C_j \right| \leq \sum_{j=1}^{\ell} |C^* \cap C_j| \leq \ell.$$
Now we note that $K \setminus C^*$ is a clique in $G$, $K \subset V(H)$, and $H$ is an induced subgraph of $G$. Thus $K \setminus C^*$ is a clique in $H$ and so $|K \setminus C^*| \leq \omega(H)$. Therefore $\omega(H^*) = |K| \leq |K \setminus C^*| + |C^*| \leq \omega(H) + \ell$ and so the first statement is true.

To show the “especially” part, assume that $G$ is $K_n$-minor-free. Let $Z$ be the graph with the vertex set $V(G)$ and the edge set $E(G) \cup E(G_1^*)$. Then $\bigcup_{j=2}^\ell C_j$ is a hole cover of $Z$ and $(C_2, \ldots, C_\ell)$ is a local chordalization partition of $\bigcup_{j=2}^\ell C_j$. By the definition of non-chordality index, $i(Z) \leq \ell - 1$. Let

$$Z_0 = Z_0^* < C_2 Z_1 \leq Z_1^* < C_3 Z_2 \leq Z_2^* < \cdots < C_\ell Z_{\ell-1} \leq Z_{\ell-1}^*$$

be the chordalization chain corresponding to $(C_2, \ldots, C_\ell)$. By the way, $Z_0 = Z_0^* = G_1^*$, $Z_i = G_{i+1}$ and $Z_i^* = G_{i+1}^*$ for $i = 1, \ldots, \ell - 1$. To reach a contradiction, suppose that $Z$ has a clique $L$ of size $n$. Then $L^* := L \setminus \bigcup_{j=2}^\ell C_j$ is a clique in $G_1^*$. By the definition of $Z$, the edges in $L$ but not in $L^*$ belong to $G$. Since $C_1$ is a hole cover of $G_1$ satisfying the NC property, by Theorem 2.11, $L^*$ is a minor of $G_1$ as $G_1^* = \overline{G_1}(C_1)$. As $G_1$ is a subgraph of $G$ and the edges in $L$ but not in $L^*$ belong to $G$, we may conclude that $L$ is a minor of $G$ with size $n$, which is a contradiction. Therefore $Z$ is $K_n$-free and so $\omega(Z) \leq n - 1$. Take a maximum clique $K$ of $G^*$. If $K$ is a clique of $Z$, then $\omega(G^*) = |K| \leq \omega(Z) \leq n - 1 \leq n - 2 + i(G)$ and so the inequality holds. Suppose that $K$ is not a clique of $Z$. By Theorem 4.7, there exists a subset $C^{**}$ of $K \cap \bigcup_{j=2}^\ell C_j$ such that $K \setminus C^{**}$ is a clique in $Z$ and $|C^{**} \cap C_i| \leq 1$ for each $i = 2, \ldots, \ell$. Then

$$|C^{**}| = \left| C^{**} \cap \bigcup_{j=2}^\ell C_j \right| \leq \sum_{j=2}^\ell |C^{**} \cap C_j| \leq \ell - 1.$$ 

Thus

$$n - 1 \geq \omega(Z) \geq |K \setminus C^{**}| \geq |K| - |C^{**}| \geq \omega(G^*) - (\ell - 1)$$

and the “especially” part is true. $lacksquare$

A proof of Theorem 4.6. Take a graph $G$ and let $G^*$ be a chordal completion of $G$ given in Theorem 4.8. Then, since $G^*$ is chordal, $\chi_{DP}(G^*) = \omega(G^*)$ by (8). Thus, by Theorem 4.8

$$\chi_{DP}(G) \leq \chi_{DP}(G^*) = \omega(G^*) \leq \omega(G) + i(G)$$

and, if $G$ is non-chordal and $K_n$-minor-free, then the right hand side of the second inequality may be replaced with $n - 2 + i(G)$. $lacksquare$

By (1), Theorem 4.6 gives $\chi_1(G) \leq \omega(G) + i(G)$ for a graph $G$ and $\chi_1(G) \leq n - 2 + i(G)$ if $G$ is non-chordal and $K_n$-minor-free. Actually, the inequality $\chi_1(G) \leq \omega(G) + i(G)$ is sharp and accordingly so is the first inequality given in Theorem 4.6. To show it, we need the following proposition.

Given a graph $G$, we denote the independence number and the vertex cover number of $G$ by $\alpha(G)$ and $\beta(G)$, respectively. It is well known that $\alpha(G) + \beta(G) = |V(G)|$. 

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Proposition 4.9. Every graph $G$ is $\beta(G)$-degenerate.

Proof. Take a graph $G$. Let $I$ be an independent set of $G$ with size $\alpha(G)$. Take a subgraph $H$ of $G$. Suppose $V(H) \cap I \neq \emptyset$. Then, as $I$ is an independent set of $G$, $V(H) \cap I$ is an independent set of $H$. Thus any vertex in $V(H) \cap I$ has degree at most $|V(H) \setminus I| \leq |V(G) \setminus I| = \beta(G)$. If $V(H) \cap I = \emptyset$, then $|V(H)| \leq |V(G) \setminus I| = \beta(G)$, and so any vertex of $H$ has degree at most $\beta(G) - 1$. Hence $G$ is $\beta(G)$-degenerate. 

We recall that if a graph $G$ is $k$-degenerate, then $\chi_{DP}(G) \leq k + 1$, from which the corollary below is immediately true. As a matter of fact, the corollary enhances the known inequality $\chi(G) \leq \beta(G) + 1$ for a graph $G$.

Corollary 4.10. For a graph $G$, $\chi_{DP}(G) \leq \beta(G) + 1$.

Consider a complete graph $K_n$ with $n \geq 2$. Then $\alpha(K_n) = 1$, $\beta(K_n) = |V(K_n)| - 1$, and $\chi(K_n) = \chi_i(K_n) = \chi_{DP}(K_n) = |V(K_n)| = \beta(K_n) + 1$. Hence the upper bound for $\chi_{DP}(K_n)$ in Corollary [4.11] is sharp.

For a complete graph $K_n$ with $n \geq 2$, the inequality given in Corollary [4.11] is sharp even for $\chi(K_n)$ and $\chi_i(K_n)$ as we have seen above. Yet, it is not necessarily in that way as it is known that $\beta(C_4) = 2$, $\chi(C_4) = \chi_i(C_4) = 2 < \beta(C_4) + 1$, and $\chi_{DP}(C_4) = 3 = \beta(C_4) + 1$. Now we are ready to present the following theorem, which implies that the inequality $\chi_i(G) \leq \omega(G) + i(G)$ is sharp (and so $\chi_{DP}(G) \leq \omega(G) + i(G)$ is sharp).

Theorem 4.11. For a positive integer $s$ and a nonnegative integer $t$, there is a graph $G$ with $\chi(G) = \omega(G) = s + 1$, $i(G) = t$, and $\chi_i(G) = s + t + 1$.

Proof. If $t = 0$, then we let $G = K_{s+1}$. Suppose $t \geq 1$. We may represent $t$ as the sum of $s$ nonnegative integers, that is, $t = \sum_{i=1}^{s} m_i$ for nonnegative integers $m_1, m_2, \ldots, m_s$. Let $G$ be a graph isomorphic to $K_{1+m_1,1+m_2,\ldots,1+m_s}$ where $m = (s + t)^{s+t}$. Let $V_1, V_2, \ldots, V_s$, and $V_{s+1}$ be the partite sets of $G$ with $|V_i| = m_i + 1$ for $i = 1, \ldots, s$ and $|V_{s+1}| = m$. Now we take a vertex $v_i$ from $V_i$ for $i = 1, \ldots, s$. Then $\mathcal{C} := \bigcup_{i=1}^{s} (V_i \setminus \{v_i\})$ is a hole cover of $G$ with size $\sum_{i=1}^{s} m_i = t$. Let $C_1, C_2, \ldots, C_t$ be all the singleton subsets of $\mathcal{C}$. Then it is easy to check that $(C_1, C_2, \ldots, C_t)$ is a local chordalization partition of $\mathcal{C}$. Thus $i(G) \leq t$.

On the other hand, it is obvious that $\omega(G) = s + 1$. Then, as it is easy to check that a complete multipartite graph is perfect,

$$\chi(G) = \omega(G) = s + 1.$$  

Since $|V_{s+1}| = m$ and $|V(G) \setminus V_{s+1}| = \sum_{i=1}^{s}(1 + m_i) = s + t$,

$$\chi_{DP}(G) \leq s + t + 1 \quad (8)$$

by Corollary [4.10]. In addition, $\bigcup_{i=1}^{s} V_i$ and $V_{s+1}$ form two disjoint vertex sets of $G$ with sizes $s + t$ and $(s + t)^{s+t}$, respectively, so $G$ contains $K_{s+t(s+t)}^{s+t}$ as a subgraph. Then,
from the observation made by Gravier \cite{3} that $\chi_l(K_{k,k,k}) > k$ for any positive integer $k$, we obtain

$$\chi_l(G) \geq s + t + 1. \quad (9)$$

Thus, by (1), (8), and (9), $s + t + 1 \leq \chi_l(G) = \chi_{DP}(G) \leq s + t + 1$ and so $\chi_l(G) = s + t + 1$.

Since $\omega(G) = s + 1$, $i(G) \geq t$ by Theorem 4.6. As we have shown that $i(G) \leq t$, we complete the proof.

It is worthy of attention that Theorem 4.11 guarantees the existence of a graph $G$ with $i(G) = t$ for any nonnegative integer $t$.

We recall that $\omega(G) \leq \chi(G) \leq \chi_l(G) \leq \chi_{DP}(G)$ for a graph $G$ and that the gaps between $\omega(G)$ and $\chi(G)$, between $\chi(G)$ and $\chi_l(G)$, and between $\chi_l(G)$ and $\chi_{DP}(G)$ can be arbitrarily large. Yet, Theorem 4.6 tells us that the sum of those gaps cannot exceed $i(G)$. Especially, if $G$ satisfies the NC property, then those gaps cannot exceed one and at most one of them can be one.

4.2 Making a local chordalization really local

In this section, we devote ourselves to convincing readers that the “local” in our terminology “local chordalization” makes a sense.

Let $G$ be a non-chordal graph and $\Omega(G) = \bigcup_{H \in \mathcal{H}(G)} V(H)$. We define a relation $\sim_G$ on $\Omega(G)$ so that, for $u, v \in \Omega(G)$,

$$u \sim_G v \iff \text{either } u \text{ and } v \text{ are on the same hole or there exists a sequence } H_1, \ldots, H_t \text{ of distinct holes in } \mathcal{H}(G) \text{ such that } u \in H_1, v \in H_t, \text{ and } H_i \text{ and } H_{i+1} \text{ share a vertex for each } i = 1, \ldots, t - 1.$$ 

It is easy to see that $\sim_G$ is an equivalence relation and that, for each hole in $G$, the vertices on the hole belong to the same equivalence class.

**Proposition 4.12.** Let $G$ be a non-chordal graph, $H$ be a hole in $G$, and $S$ be the equivalence class under $\sim_G$ containing $V(H)$. If adding a chord of $H$ to $G$ yields a new hole $H^*$, then $V(H^*) \subset S$.

**Proof.** Since $H$ is a hole, there are two nonadjacent vertices $u$ and $v$ on $H$. Suppose that adding the edge joining $u$ and $v$ to $G$ creates a new hole $H^*$. Obviously $uv$ is a chord of $H$ in $G + uv$. Let $x$ be a vertex in $H^*$ other than $u$ and $v$. It suffices to show $x \in S$ to complete the proof. If $x$ is on $H$, then we are done. Thus we may assume that $x$ is not on $H$.

**Case 1.** $x$ is adjacent to an internal vertex of each of the two $(u, v)$-sections of $H$. Since $u, v, x$ are on the hole $H^*$ with $u$ and $v$ consecutive on $H^*$, $x$ is nonadjacent to one of $u$ and $v$ in $G + uv$. Without loss of generality, we may assume that $x$ is nonadjacent to
v in $G + uv$. Obviously $x$ is nonadjacent to $v$ in $G$. By applying Lemma 2.4 for $P = \{x\}$, there exists a hole in $G$ containing $x$ and $v$. Therefore $x \sim_G v$. Since $v \in S$, $x \in S$.

**Case 2.** One of the two $(u, v)$-sections of $H$ has no internal vertex that is adjacent to $x$. Let $R$ be such a $(u, v)$-section. Then none of $x$ and its neighbors on $H^*$ is an internal vertex on $R$. While traversing along the $(x, v)$-section (resp. $(x, u)$-section) of $H^*$ not containing $u$ (resp. $v$), let $y$ (resp. $z$) be the first vertex at which we meet $R$. Let $Q_1$ be the $(y, z)$-section of $H^*$ containing $x$, $Q_2$ be the $(y, z)$-section of $R$, and $Q = Q_1Q_2$. By the choices of $y$ and $z$, $Q$ is an induced cycle of $G$ containing $x$ and a vertex on $H$. Since two neighbors of $x$ on $H^*$ are nonadjacent in $G$, $Q$ is a hole in $G$. Since $Q$ contains $x$ and a vertex on $H$, $x \in S$. □

**Remark 4.13.** Let $G$ be a non-chordal graph and $\Omega(G)/\sim_G$ be the set of equivalence classes under $\sim_G$. Take an equivalence class $S \in \Omega(G)/\sim_G$, a hole $H$ with $V(H) \subset S$, and vertices $u$ and $v$ on $H$ which are not consecutive. Proposition 4.12 implies that the equivalence classes in $\Omega(G)/\sim_G$ except $S$ are still equivalence classes under $\sim_{G+uv}$, and if there are other equivalence classes under $\sim_{G+uv}$, they are disjoint subsets of $S$. Therefore $\Omega(G + uv) \subset \Omega(G)$.

**Remark 4.14.** Let $G$ be a non-chordal graph and $\ell = i(G)$. By the definition of $i(G)$, there exist a hole cover $\mathcal{C}$ of $G$ and a local chordalization partition $\tilde{\mathcal{C}} = (C_1, \ldots, C_\ell)$ of $\mathcal{C}$. Let

$$G_0 = G_0^* < c_1 G_1 \leq G_1^* < c_2 G_2 \leq G_2^* < \cdots < c_\ell G_\ell \leq G_\ell^* =: G^*$$

be the chordalization chain corresponding to $\tilde{\mathcal{C}}$. Let $H$ be the subgraph of $G$ induced by $\Omega(G)$. Then, by the definition of induced subgraph, all the holes in $H$ are contained in $G$. By the definition of $\Omega(G)$, all the holes in $G$ are contained in $H$. Therefore $\mathcal{H}(G) = \mathcal{H}(H)$, $\Omega(G) = \Omega(H)$, and $\mathcal{C}$ is a hole cover of $H$. Thus the equivalence classes under $\sim_G$ are the equivalence classes under $\sim_H$. We recall that

$$G_0 = G_0^* = G - \mathcal{C};$$

$$V(G_i) = V(G_{i-1}^*) \cup C_i, \quad E(G_i) = E(G_{i-1}^*) \cup E \left( G - \bigcup_{j=i+1}^{\ell} C_j \right);$$

$$G_i^* = \tilde{G}_i(\mathcal{C}_i).$$

for each $i = 1, \ldots, \ell$. Let $H_0 = H_0^* = H - \mathcal{C}$. Since $H$ is an induced subgraph of $G$, $H_0$ is an induced subgraph of $G_0$ by (14). Furthermore, $G_0$, $G_0^*$, $H_0$, and $H_0^*$ are chordal and so $\mathcal{H}(G_0^*) = \mathcal{H}(G_0) = \mathcal{H}(H_0) = \mathcal{H}(H_0^*) = \emptyset$ and $\Omega(G_0^*) = \Omega(G_0) = \Omega(H_0) = \Omega(H_0^*) = \emptyset$. Let $H_1$ be the graph defined by $V(H_1) = V(H_0^*) \cup C_1$ and

$$E(H_1) = E(H_0^*) \cup E \left( H - \bigcup_{j=2}^{\ell} C_j \right).$$
Since $H$ and $H_0^*$ are induced subgraphs of $G$ and $G^*_0$, respectively, $H_1$ is an induced subgraph of $G_1$ and $\mathcal{H}(H_1) \subset \mathcal{H}(G_1)$ by (12). Take a hole $\Omega_1$ in $G_1$. Since $G_1$ is an induced subgraph of $G$, $V(\Omega_1) \subset \Omega(G) \setminus \bigcup_{i=2}^\ell C_i$. Since $\Omega(G) = V(H) \cup V(H) \setminus \bigcup_{i=2}^\ell C_i = V(H_1)$, $V(\Omega_1) \subset V(H_1)$. Since $H_1$ is an induced subgraph of $G$, $\Omega_1$ is a hole in $H_1$. Thus we have shown that $\mathcal{H}(H_1) = \mathcal{H}(G_1)$. Hence, since $C_1$ is a hole cover of $G_1$ satisfying the NC property, it is a hole cover of $H_1$ satisfying the NC property and so we obtain $\hat{H}_1(C_1) =: H_1^*$. Since $\mathcal{H}(H_1) = \mathcal{H}(G_1)$ and $H_1$ is an induced subgraph of $G_1$, $H_1^*$ is an induced subgraph of $G_1^*$. Let $H_2$ be the graph defined by $V(H_2) = V(H_1^*) \cup C_2$ and

$$E(H_2) = E(H_1^*) \cup E\left( H - \bigcup_{j=3}^\ell C_j \right).$$

Then $\Omega(G) \setminus \bigcup_{i=3}^\ell C_i = V(H_2)$. Since $H$ and $H_1^*$ are induced subgraphs of $G$ and $G_1^*$, respectively, $H_2$ is an induced subgraph of $G_2$ and $\mathcal{H}(H_2) \subset \mathcal{H}(G_2)$ by (12). Take a hole $\Omega_2$ in $G_2$. Since $G_1^*$ is chordal, $\Omega_2$ must contain a vertex $v$ in $C_2$. By the way, since $C_2$ is a hole cover of $G_2$ satisfying the NC property, $\Omega_2$ contains exactly one vertex in $C_2$ and so $v$ is the only vertex on $\Omega_2$ that is contained in $C_2$.

Since $G$ is non-chordal, there exist a hole in $G$. The chain given in (10) is the shortest, one of the holes in $G$ must be in $G_1$. Thus there exists an edge in $E(G_1^*) \setminus E(G_1)$. Take an edge $e$ in $E(G_1^*) \setminus E(G_1)$. Then there is a hole in $G$ such that $e$ is its chord in $G + e$. By Proposition 4.12, $\Omega(G + e) \subset \Omega(G)$.

If $E(G_1^*) \setminus E(G_1) = \{e\}$, then, by Proposition 4.12, $V(\Omega_2) \subset \Omega(\Omega_2) \subset \Omega(G + e) \subset \Omega(G)$ and so $V(\Omega_2) \subset \Omega(G)$. Suppose that $E(G_1^*) \setminus (E(G_1) \cup \{e\}) \neq \emptyset$ and take an edge $e'$ in $E(G_1^*) \setminus (E(G_1) \cup \{e\})$. Then there is a hole $C$ in $G$ such that $e'$ is its chord in $G + e'$. Now there is a hole in $G + e$ such that $e'$ is its chord in $G \cup \{e, e'\}$. For, if $C$ is a hole in $G + e$, then it is such a hole. Otherwise, by the definition of local chordalization, $e$ is a chord of $C$ and $e'$ is a chord of a hole from $C + e$.

By applying Proposition 4.12 for $G + e$ and an edge $e'$, $\Omega(G \cup \{e, e'\}) \subset \Omega(G)$. We may repeat this argument to conclude that $\Omega(G \cup (E(G_1^*) \setminus E(G_1))) \subset \Omega(G)$. Since $G_2$ is an induced subgraph of $G \cup (E(G_1^*) \setminus E(G_1))$ and $\Omega_2$ is a hole in $G_2$,

$$V(\Omega_2) \subset \Omega(\Omega_2) \subset \Omega(G \cup (E(G_1^*) \setminus E(G_1))) \subset \Omega(G),$$

and so $V(\Omega_2) \subset \Omega(G)$. Therefore we have shown that $V(\Omega_2) \subset \Omega(G)$ whether or not $E(G_1^*) \setminus (E(G_1) \cup \{e\}) \neq \emptyset$. Thus the vertices on $\Omega_2$ belong to $\Omega(G) \setminus \bigcup_{i=3}^\ell C_i$. Since $\Omega(G) \setminus \bigcup_{i=3}^\ell C_i = V(H_2)$ and $H_2$ is an induced subgraph of $G_2$, $\Omega_2$ is a hole in $H_2$ and so $\mathcal{H}(G_2) \subset \mathcal{H}(H_2)$. Thus $\mathcal{H}(G_2) = \mathcal{H}(H_2)$. Hence, since $C_2$ is a hole cover of $G_2$ satisfying the NC property, it is a hole cover of $H_2$ satisfying the NC property and so we obtain
Figure 4: \( \Omega(G_1) = \Omega(G_2) \), so \( i(G_1) = i(G_2) \) by the argument given in Remark 4.14.

We may repeat this process to obtain \( H_3, H_3^*, \ldots, H_\ell, H_\ell^* \) such that

\[
V(H_i) = V(H_{i-1}) \cup C_i, \quad E(H_i) = E(H_{i-1}) \cup E\left(H - \bigcup_{j=i+1}^\ell C_j\right),
\]

and \( \mathcal{H}(G_i) = \mathcal{H}(H_i) \) for \( i = 3, \ldots, \ell \). Noting that \( \mathcal{H}(G) = \mathcal{H}(H) \) and \( G_i^* \) (resp. \( H_i^* \)) is a chordal completion of \( G \) (resp. \( H \)), we may conclude that \( i(H) \leq \ell = i(G) \).

To show that \( i(G) \leq i(H) \), we need to introduce the chordalization chain corresponding to a local chordalization partition \( \tilde{\mathcal{C}}' \) of a hole cover \( \mathcal{C}' \) of \( H \) terminating at \( H_i^* \). By mimicking the previous argument constructing the chordalization chain corresponding to \( \tilde{\mathcal{C}} \) for \( H \), we may construct the chordalization chain corresponding to \( \tilde{\mathcal{C}}' \) for \( G \) to conclude \( i(G) \leq i(H) \). Thus \( i(G) = i(H) \) and it is sufficient to apply local chordalization process to the induced subgraph \( H \) of \( G \), which is a local structure, to obtain a desired chordal completion of \( G \). In this vein, we may claim that the “local” in our terminology “local chordalization” is meaningful in another respect.

**Example 4.15.** The graph \( G_2 \) in Figure 4 is obtained from \( G_1 \) by replacing the vertex \( v \) of \( G_1 \) by the complete graph \( K_n \). Then \( \Omega(G_1) = \Omega(G_2) \). By the argument given in Remark 4.14 \( i(G_1) = i(G_2) \). Yet, the treewidths of \( G_1 \) and \( G_2 \) are 2 and \( n-1 \), respectively.

By the argument given in Remark 4.14 the following proposition is true.

**Proposition 4.16.** For a non-chordal graph \( G \), \( i(G) = \max\{i(G[S_1]), \ldots, i(G[S_r])\} \) where \( S_1, \ldots, S_r \) are the equivalence classes under \( \sim_G \).

**Proof.** Let \( G \) be a graph and \( \tilde{\mathcal{C}} = (C_1, \ldots, C_{i(G)}) \) be a local chordalization partition of a hole cover \( \mathcal{C} \) of \( G \). Then, for each \( j = 1, \ldots, i(G) \), \( C \cap S_j \) is a hole cover of \( G[S_j] \). In
addition, by the argument given in Remark 4.13, a subset of \(\{C_1 \cap J_1, \ldots, C_i(G) \cap J_i\}\) forms a local chordalization partition of \(C \cap S_i\). Thus \(i(G[S_j]) \leq i(G)\) for each \(j = 1, \ldots, i(G)\) and so \(i(G) \geq \max\{i(G[S_1]), \ldots, i(G[S_r])\}\).

Now let \(\tilde{C}^j = (C_1^j, \ldots, C_i(G[S_j]))^j\) be a local chordalization partition of a hole cover \(C^j\) of \(G[S_j]\) for each \(j = 1, \ldots, r\). Clearly \(\bigcup_{j=1}^r C^j\) is a hole cover of \(G\). In addition, by the argument given in Remark 4.13, \(\bigcup_{j=1}^r C^j\) where \(C_p^j = \emptyset\) for any \(j = 1, \ldots, r\) and any \(p\), \(i(G[S_j]) < p \leq \max\{i(G[S_1]), \ldots, i(G[S_r])\}\). Hence \(\max\{i(G[S_1]), \ldots, i(G[S_r])\} \geq i(G)\).

The join, denoted by \(G_1 \lor G_2\), of two graphs \(G_1\) and \(G_2\) is the graph with the vertex set \(V(G_1) \cup V(G_2)\) and the edge set \(E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1)\text{ and } v \in V(G_2)\}\). We denote by \(I_m\) an empty graph with \(m\) vertices.

**Theorem 4.17.** Suppose that a non-chordal graph \(G\) does not contain \(I_m \lor K_n\) for positive integers \(m \geq n\) as a subgraph and \(\omega(G[\Omega(G)]) + i(G) \leq m\). Then there is a chordal completion \(G^*\) of \(G\) with \(\omega(G^*) < m + n\).

**Proof.** Since \(G\) is non-chordal, \(i(G) \geq 1\). Let \(H\) be the subgraph of \(G\) induced by \(\Omega(G)\). By the argument in Remark 4.14, \(i(G) = i(H)\). Let \(H^*\) be the subgraph of \(G^*\) induced by \(\Omega(G)\) where \(G^*\) is a chordal completion of \(G\) obtained in Remark 4.14. Then \(H^*\) is a chordal completion of \(H\).

Suppose to the contrary that \(\omega(G^*) \geq m + n\). Then there is a clique \(K\) of size \(m + n\) in \(G^*\). Clearly \(K \cap \Omega(G)\) forms a clique in \(G^*\). Since \(H^*\) is an induced subgraph of \(G^*\), \(K \cap \Omega(G)\) forms a clique in \(H^*\). By Theorem 4.18, \(|K \cap \Omega(G)| \leq \omega(H) + i(G)\). By the hypothesis, \(|K| = m + n\), \(|K \setminus \Omega(G)| \geq n\). By the definition of local chordalization and Remark 4.13, the end vertices of each of the edges newly added to obtain \(G^*\) belong to \(\Omega(G)\), so \(K \setminus \Omega(G)\) still forms a clique in \(G\) and each vertex in \(K \cap \Omega(G)\) is adjacent to each vertex in \(K \setminus \Omega(G)\) in \(G\). By moving \(m - |K \cap \Omega(G)|\) vertices in \(K \setminus \Omega(G)\) into \(K \cap \Omega(G)\) if \(|K \cap \Omega(G)| < m\), we may claim that \(G\) contains \(I_m \lor K_n\) as a subgraph. This contradicts the hypothesis, so we conclude that \(\omega(G^*) < m + n\).

The following corollary is an immediate consequence of Theorem 4.17.

**Corollary 4.18.** Suppose a graph \(G\) does not contain \(I_m \lor K_n\) for positive integers \(m \geq n\) as a subgraph and \(\omega(G[\Omega(G)]) + i(G) \leq m\). Then \(\chi_{DP}(G) < m + n\).

**Remark 4.19.** Since \(K_{2,4}\) is non-chordal and has a hole cover which is a singleton, \(i(K_{2,4}) = 1\). Then, by Theorem 4.18, \(\chi_{DP}(K_{2,4}) \leq 3\). Yet, \(\chi_{DP}(K_{2,4}) \leq 5\) by Corollary 4.18. Thus, for \(\chi_{DP}(K_{2,4})\), Theorem 4.16 gives a better upper bound than Corollary 4.18.

On the other hand, for a certain graph \(G\), Corollary 4.18 gives a better upper bound of \(\chi_{DP}(G)\) than Theorem 4.16. To see why, consider the graph \(G\) given in Figure 6. If \(G\) contains a subgraph isomorphic to \(I_8 \lor K_4\), then \(G\) would have at least four vertices.
with degree at least 11, which does not happen in $G$ as the two vertices common to $K_6$ and $K_{11}$ are the only vertices with degree at least 11. Hence $G$ does not contain $I_8 \lor K_4$ as a subgraph.

It is easy to check that $\omega(G) = 11$ and $\omega(G[\Omega(G)]) = 5$. The graph $G[\Omega(G)]$ is represented by using bold edges in Figure 5 and happens to be the graph given in Figure 3. Therefore $i(G) = 2$. Then Theorem 4.6 gives rise to $\chi_{DP}(G) \leq 11 + 2$ while Corollary 4.18 gives rise to $\chi_{DP}(G) \leq 11$. Furthermore, since $\omega(G) = 11$, $\chi_{DP}(G)$ is actually equal to 11.

![Figure 5: A graph $G$ which shows that Theorem 4.17 may be regarded as an improvement of Theorem 4.8. The vertices enclosed by a dotted ellipse form a clique.](image)

## 5 New $\chi$-bounded classes

A class $\mathcal{F}$ of graphs is said to be $\chi$-bounded if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that for every graph $G \in \mathcal{F}$ and every induced subgraph $H$ of $G$, $\chi(H) \leq f(\omega(H))$.

We may extend the notion of $\chi$-boundedness as follows. A class $\mathcal{F}$ of graphs is said to be $\chi_l$-bounded (resp. $\chi_{DP}$-bounded) if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that for every graph $G \in \mathcal{F}$ and every induced subgraph $H$ of $G$, $\chi_l(H) \leq f(\omega(H))$ (resp. $\chi_{DP}(H) \leq f(\omega(H))$).

A graph $G$ is called perfect graph if $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$.

We may also extend the notion of perfect graph as follows. We say that a graph $G$ is list-perfect (resp. DP-perfect) if $\chi_l(H) = \omega(H)$ (resp. $\chi_{DP}(H) = \omega(H)$) for every induced subgraph $H$ of $G$.
We denote the class of perfect graphs, the class of list-perfect graphs, and the class of DP-perfect graphs by $\mathcal{P}$, $\mathcal{P}_l$, and $\mathcal{P}_{DP}$, respectively.

By (1), a $\chi_{DP}$-bounded graph class is $\chi_l$-bounded and a $\chi_l$-bounded graph class is $\chi$-bounded. In the proof of Theorem 4.11, we have shown that for any positive integer $s$ and any nonnegative integer $t$, there exist a complete multipartite graph $G$ with $\omega(G) = s + 1$ and $\chi_l(G) = s + t + 1$, which implies that the class of complete multipartite graphs is not $\chi_l$-bounded. Any complete multipartite graph is, however, perfect, which implies that the class of complete multipartite graphs is $\chi$-bounded. Accordingly, a $\chi$-bounded class is not necessarily $\chi_l$-bounded. Furthermore, $\mathcal{P}_{DP} \subset \mathcal{P}_l \subset \mathcal{P}$ by (1). Yet, $\mathcal{P}_{DP} \subsetneq \mathcal{P}_l \subsetneq \mathcal{P}$ as $K_{2,4}$ is perfect but not list-perfect and $C_4$ is list-perfect but not DP-perfect.

Note that $\omega(C_n) = 2$ and $\chi_{DP}(C_n) = 3$ for even integer $n \geq 4$. Thus no graph in $\mathcal{P}_{DP}$ contains a hole of even length. Since a graph containing a hole of odd length is not perfect, no graph in $\mathcal{P}_{DP}$ contains a hole of odd length. Therefore $\mathcal{P}_{DP}$ is included in the class of chordal graphs. Thus, by (§), $\mathcal{P}_{DP}$ is the class of chordal graphs.

Now we present new $\chi$-bounded classes.

**Theorem 5.1.** A family of graphs the non-chordality index of each of which does not exceed $k$ for some nonnegative integer $k$ is $\chi_{DP}$-bounded.

**Proof.** Take a family $\mathcal{F}$ of graphs the non-chordality index of each of which does not exceed $k$ for a nonnegative integer $k$. Let $f : \mathbb{N} \to \mathbb{R}$ be a function defined by $f(x) = x + k$. Take a graph $G$ in $\mathcal{F}$. Then $i(G) \leq k$. Let $H$ be an induced subgraph of $G$. By the first part of (§), there exists a chordal completion $H^*$ of $H$ such that $\omega(H^*) \leq \omega(H) + i(G)$. Thus $\chi_{DP}(H) \leq \chi_{DP}(H^*) = \omega(H^*) \leq \omega(H) + i(G) \leq f(\omega(H))$. Hence the theorem is true.

By Remark 4.4, the following corollary is immediately true.

**Corollary 5.2.** The class of graphs with the NC property is $\chi_{DP}$-bounded.

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