PC-Spaces

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Abstract. The present study concentrates on the new generalizations of the Jordan curve theorem. In order to achieve our goal, new spaces namely PC-space and strong PC-space are defined and studied their properties. One of the main concepts that use to define the related classes of spaces is paracompact space. In addition, the property of being PC-space and strong PC-space is preserved by defining a new type of function so called para-perfect function.

1. Introduction

One of the standard results in algebraic topology is the theorem of Jordan curve “a simple closed curve C in complex plane separates the plan into two regions which are bounded and unbounded components with C the boundary of each”. This theorem was first proposed in 1887 by French mathematician Camille Jordan [1]. Then, many generalizations of the Jordan curve theorem are discussed by a group of researchers. In 1990, Khalimsky et al. [2] stated a generalization in Z^2 equipped with the khalimsky topology. In 1999, Micael [3] introduced and studied J-space s and strong J-spaces which are considered to be generalizations of properties of the Jordan curve theorem. In 2007, Gao [4] introduced the concept of LJ-spaces exploited the common generalization of Lindelöf spaces and J-spaces. In 2016, Dawood and Gasim [5] used the concept of countably compact to get many generalizations of this theorem.

In this paper, new generalizations of the Jordan curve theorem are introduced. To get these generalizations, the concept of paracompact space is used to define new spaces which are called PC-space and strong PC-space. We study the relations between the two related classes of spaces and provide some main results.

The article is formed as follows: Some basic notations are given in Section 2. The concept of PC-space and strong PC-space is introduced and studied their properties in Section 3. The paper is ended with a conclusion in Section 4.

2. Fundamental Concepts

Some fundamental concepts are reviewed to be used throughout the present study. In the sequel, all spaces are supposed to be Hausdorff.

Definition 2.1.[5] A cover \( \mathcal{G} = \{ \mathcal{G}_i \}_{i \in I} \) of \( \mathcal{A} \) is called locally finite if, for all point \( a \in \mathcal{A} \), there exists a neighbourhood intersecting only finitely many elements in \( \mathcal{G} \).

Definition 2.2. [5] Let \( \mathcal{G} = \{ \mathcal{G}_i \}_{i \in I} \) be a cover of \( \mathcal{A} \). If \( \mathcal{Y} = \{ \mathcal{Y}_j \}_{j \in J} \) is the second cover of \( \mathcal{A} \), then \( \mathcal{Y} \) is called a refinement of \( \mathcal{G} \) if for each \( \mathcal{Y}_j \in \mathcal{Y} \) there exists \( \mathcal{G}_i \in \mathcal{G} \) with \( \mathcal{Y}_j \subseteq \mathcal{G}_i \).
Definition 2.3. [5] A topological space \( \mathcal{A} \) is called paracompact if any open cover of \( \mathcal{A} \) has a locally finite open refinement.

Proposition 2.4. [6] Any closed subset of a paracompact space be para-compact.

Proposition 2.5. [7] Every compact space is paracompact.

It is clear that the real number with the usual topology is paracompact space but it is not a compact space.

Proposition 2.6. [7] Every metric space is paracompact.

Proposition 2.7. [8] The image of paracompact space under a closed and continuous function is paracompact.

Definition 2.8. [7] A space \( \mathcal{A} \) is called a locally connected if for all \( a \in \mathcal{A} \), and each neighbourhood \( \mathcal{U} \) of \( a \) there is a connected neighbourhood \( \mathcal{H} \) of \( a \) where \( \mathcal{H} \subset \mathcal{U} \).

3. Main Results

Now, we define \( PC \)-space and strong \( PC \)-space and study their properties.

Definition 3.1. A \( PC \)-space is a topological space \( \mathcal{A} \) whenever \( \{ p, q \} \) is a closed cover of \( \mathcal{A} \) such that \( p \cap q \) paracompact, then \( p \) or \( q \) is paracompact.

Definition 3.2. A space \( \mathcal{A} \) is said to be strong \( PC \)-space if every paracompact \( J \subset \mathcal{A} \) is contained in a paracompact \( D \subset \mathcal{A} \) with \( \mathcal{A} \setminus D \) connected.

Proposition 3.3. Any strong \( PC \)-space be a \( PC \)-space.

Proof. Assume \( \mathcal{A} \) is a strong \( PC \)-space and \( p, q \) are closed subsets of \( \mathcal{A} \) with \( p \cap q \) is paracompact and \( \mathcal{A} = p \cup q \). So, there is a paracompact \( D \subset \mathcal{A} \) where \( p \cap q \subset D \) and \( \mathcal{A} \setminus D \) is connected. Therefore, \( \{ p \setminus (\mathcal{A} \setminus D), q \setminus (\mathcal{A} \setminus D) \} \) is a disjoint closed cover of \( \mathcal{A} \setminus D \). However, \( \mathcal{A} \setminus D \) is connected, so \( \mathcal{A} \setminus D \) must be in \( p \setminus (\mathcal{A} \setminus D) \) or in \( q \setminus (\mathcal{A} \setminus D) \). Hence, \( \mathcal{A} \setminus D \subset p \) or \( \mathcal{A} \setminus D \subset q \). By complementation, we have \( p^c \subset D \) or \( q^c \subset D \), and since \( p \cap q \subset D \), so \( p \subset D \) or \( q \subset D \). Then \( p \) or \( q \) is paracompact by Proposition 2.1. Hence, \( \mathcal{A} \) is \( PC \)-space.

Proposition 3.4. Every paracompact space be a strong \( PC \).

Proof. Suppose the space \( \mathcal{A} \) is a paracompact and \( J \subset \mathcal{A} \) be also paracompact, then \( \mathcal{A} \) is a paracompact with \( J \subset \mathcal{A} \) and \( \mathcal{A} \setminus \mathcal{A} = \emptyset \) is connected.

Corollary 3.5. Any paracompact space is \( PC \)-space.

Proof. The proof follows through Proposition 3.4 and Proposition 3.3.

Examples 3.6.

1. Any compact space is a strong \( PC \).
2. Any metric space is a strong \( PC \).
3. Any usual topology with \( \mathbb{R} \) is a strong \( PC \).
4. The indiscrete topology on any set is strong \( PC \)-space and \( PC \)-space.
5. The cofinite topology on the set of natural numbers is strong \( PC \)-space and \( PC \)-space.

Lemma 3.7. Let \( \mathcal{W} \) be a closed non-paracompact set in a space \( \mathcal{A} \) and \( p \subset \mathcal{W} \) is paracompact. Then, there is a non-paracompact closed \( q \subset \mathcal{W} \) such that \( p \cap q = \emptyset \).
Proof. Assume the open cover $\mathcal{G}$ of $\mathcal{W}$ with no a locally finite open refinement, and $p \subset \mathcal{W}$ is paracompact. Then, $p$ has an open cover $\mathcal{G}$ and has a locally finite open refinement $\gamma$ covering $p$. Put $q = \mathcal{W} \setminus \cup \gamma$, so $q$ is closed in $\mathcal{W}$ with $p \cap q = \emptyset$.

Now, suppose $q$ is paracompact. So, there exists a locally finite open refinement $\gamma^*$ covering $q$. Thus, $q \subset \bigcup \gamma^*$ and so $\mathcal{W} \setminus \cup \gamma \subset \bigcup \gamma^*$. That implies $\mathcal{W} \subset (\bigcup \gamma) \cup (\bigcup \gamma^*)$. However, $(\bigcup \gamma) \cup (\bigcup \gamma^*)$ is a locally finite open refinement for $\mathcal{G}$ which is contradiction. Hence, $q$ is a non-paracompact.

The following theorem gives some criteria which are equivalent to be $PC$-spaces.

Theorem 3.8. The following statements are equivalent:

$\mathcal{A}$ is a $PC$-space,

For any $p \subset \mathcal{A}$ with paracompact boundary, $\overline{p}$ or $(\overline{\mathcal{A} \setminus p})$ is para-compact.

If the closed subsets $p$, $q$ in $\mathcal{A}$ with $p \cap q = \emptyset$ and $bd(p)$ or $bd(q)$ is paracompact, then $p$ or $q$ is paracompact.

Proof. Statement (1) implies statement (2). Suppose $p \subset \mathcal{A}$ such that $bd(p)$ is paracompact. However, we have $bd(p) = \overline{p} \cap (\mathcal{A} \setminus p)$. Thus, $\{\overline{p} \cap (\mathcal{A} \setminus p)\}$ is a closed cover of $\mathcal{A}$ with $\overline{p} \cap (\mathcal{A} \setminus p)$ is paracompact. Since $\mathcal{A}$ is $PC$-space so $\overline{p}$ or $(\mathcal{A} \setminus p)$ is paracompact.

Statement (2) implies statement (3). Let $p$ and $q$ be disjoint closed subsets of $\mathcal{A}$ with $bd(p)$ or $bd(q)$ is paracompact. Assume that $bd(p)$ is aracompact, so $\overline{p}$ or $(\mathcal{A} \setminus p)$ is paracompact by (2). However, $p = \overline{p}$ and $q \subset (\mathcal{A} \setminus p) \subset (\mathcal{A} \setminus p)$: since $p$ or $q$ is paracompact.

Statement (3) implies statement (1). We want to prove $\mathcal{A}$ is a $PC$-space. Assume $\{p, q\}$ be a closed cover of $\mathcal{A}$ where $p \cap q$ paracompact. Suppose that $q$ is non-paracompact. Since $p \cap q \subset q$ is a paracompact, so by lemma 3.7 there is a non-paracompact closed $D \subset q$ where $D \cap (p \cap q) = \emptyset$. So, $D \cap p = \emptyset$. However, $bd(p)$ is a paracompact because it is a closed subset of $p \cap q$. By (3) $p$ or $D$ is paracompact, but $D$ is non-paracompact. Hence, $p$ must be paracompact.

Theorem 3.9. A topological space $\mathcal{A}$ is a $PC$-space if and only if $J \subset \mathcal{A}$ is paracompact where $S = \{p, q\}$ is an open cover of $\mathcal{A} \setminus J$ with $p \cap q = \emptyset$, then $\mathcal{A} \setminus S_1$ or $\mathcal{A} \setminus S_2$ is paracompact.

Proof. The "if" part. Assume $\mathcal{A}$ is a $CP$-space and a set $J$ is paracompact in $\mathcal{A}$ where $S = \{p, q\}$ is an open cover of $\mathcal{A} \setminus J$ with $p \cap q = \emptyset$. Thus, $\{A \cup J \mid \mathcal{A} \setminus J\}$ is a closed cover of $\mathcal{A}$ with $\mathcal{A} \cup p \cap \mathcal{A} \cup p = \mathcal{A} \cup p \cup q$ which is a closed set in $J$ because of $\mathcal{A} \setminus J \subset \mathcal{A} \cup p$. Then, $\mathcal{A} \cup p \cap \mathcal{A} \cup q$ is paracompact by Proposition 2.4. However, $\mathcal{A}$ is $PC$-space, so $\mathcal{A} \cup p$ or $\mathcal{A} \cup q$ is paracompact.

The "only if" part. Let $\{p, q\}$ be a closed cover of $\mathcal{A}$ with $p \cap q$ paracompact. So, $\{A \cup J \mid \mathcal{A} \setminus J\}$ is an open cover of $\mathcal{A} \setminus p \cap q$ with $\mathcal{A} \cup p \cap \mathcal{A} \cup q = \emptyset$. By hypothesis, $\mathcal{A} \setminus (\mathcal{A} \setminus p)$ or $\mathcal{A} \setminus (\mathcal{A} \setminus q)$ is paracompact, that is $p$ or $q$ is paracompact. Hence, $\mathcal{A}$ is $PC$-space.

Theorem 3.10. If the closed cover $\{A_1, A_2\}$ of a space $\mathcal{A}$ with $A_1 \cap A_2$ is paracompact, then $\mathcal{A}$ is a $PC$-space if and only if $A_1$ and $A_2$ are $PC$-spaces and $A_1$ or $A_2$ is paracompact.

Proof. The "if" part. Suppose that $\mathcal{A}$ is a $PC$-space, then $A_1$ or $A_2$ is paracompact by definition of $PC$-space. Suppose that $\mathcal{A}_1$ is paracompact, it follows that $\mathcal{A}_1$ is $PC$-space by Corollary 3.5, now to show that $A_2$ is $PC$-space. Let $\{p, q\}$ be a closed cover of $A_2$ with $p \cap q$ is paracompact. Hence, $\mathcal{A}$ has a closed cover $\{p, q\} \cup A_1$ where $p \cap (q \cup A_1)$ is paracompact. So $p$ or $q \cup A_1$ is paracompact because of $\mathcal{A}$ is $PC$-space. However, $q$ is a closed subset of $q \cup A_1$, so $p$ or $q$ is paracompact.

The "only if" part. Assume that $A_1$ and $A_2$ are $PC$-spaces and suppose that $A_2$ is paracompact, we have to show that $\mathcal{A}$ is $PC$-space. Let $\{p, q\}$ be a closed cover of $\mathcal{A}$ with $p \cap q$ is paracompact. Now, let $p_i = p \cap A_i$ and $q_i = q \cap A_i$ where ($i = 1, 2$). Consequently, $\{p_1, q_2\}$ is a closed cover of $A_1$, which is $PC$-space, with $p_1 \cap q_1 = (p \cap A_1) \cap (q \cap A_1) = (p \cap q) \cap A_1$ which is a closed subset.
of $p \cap q$. Hence, $p_1 \cap q_1$ is paracompact and $p_2$ or $q_2$ is paracompact. If $p_1$ is paracompact, then $p = p_1 \cup p_2$ is paracompact since $p_2$ is a closed subset of paracompact $A_2$. In the same way, if $q_1$ is paracompact, then so is $q$.

Theorem 3.11. If the closed cover $\{A_1, A_2\}$ of a topological space $A$ with $A_1 \cap A_2$ is paracompact. Then $A$ is a strong $PC$-space if and only if $A_1$ and $A_2$ are strong $PC$-spaces and $A_1$ or $A_2$ is paracompact.

Proof. The "if" part. Assume that $A$ is a strong $PC$-space, it follows by Proposition 3.3 that $A$ is $PC$-space, and thus $A_1$ or $A_2$ is paracompact. Suppose that $A_1$ is paracompact, so by Proposition 3.4 that $A_1$ is strong $PC$-space, so it remains to show that $A_2$ is strong $PC$-space. Let $J_2 \subseteq A_2$ be paracompact. Define $J = J_2 \cup A_1$, then $J$ is a paracompact subset of $A$ which is a strong $PC$-space, so there is a paracompact closed set $D$ in $A$ where $J \subset D$ and $A \setminus D$ is connected. Let $D_2 = D \cap A_2$, then $D_2 \subseteq A_2$ is paracompact since $D_2$ is a closed subset of paracompact set $D$. Also $J_2 \subseteq D_2$ since $J_2 \subseteq J \subset D$ implies that $J_2 \cap A_2 \subset D \cap A_2$. Also, we note $A_1 \subset J \subset D$ and that implies $A_2 \setminus D_2 = A \setminus D$, and hence $A_2 \setminus D_2$ is connected.

The "only if" part. Assume that $A_1$ and $A_2$ are strong $PC$-space and suppose that $A_1$ is a paracompact and let $J \subset A$ be paracompact. Define $J_2 = (J \cup A_1) \cap A_2$, so $J_2$ is paracompact since it is a closed subset of the paracompact set $J \cup A_1$, so $J_2$ is a paracompact subset of the strong $PC$-space $A_2$, then there is a closed paracompact subset $D_2$ of $A_2$ where $D_2 \subseteq D_2$ and $A_2 \setminus D_2$ is connected. Now, let $D = D_2 \cup A_1$, then $D$ is a closed paracompact subset of $A$ and $J \subset D$ and $A \setminus D = A_2 \setminus D_2$ is connected. Hence, $A$ is a strong $PC$-space.

Next, we introduce new types of functions that help study the preservation of strong $PC$-space and $PC$-space.

Definition 3.12. A continuous function $f : A \rightarrow B$ is called para-perfect if it is closed and $f^{-1}(b)$ is a paracompact space for every paracompact subset $b$ of $B$.

Definition 3.13. A closed function $f : A \rightarrow B$ is called boundary para-perfect if the boundary of $f^{-1}(b)$ is paracompact subset of $A$ for all $b \in B$.

Definition 3.14. A continuous closed function $f : A \rightarrow B$ is called quasi-para-perfect if $f^{-1}(b)$ is paracompact subset of $A$ for all $b \in B$.

Remark 3.15. It is clear that every para-perfect function is a quasi-para-perfect because it is known that the only topology that can be defined on a singleton set is the indiscrete topology. Therefore, it is paracompact and its reverse image due to the para-perfect function is paracompact.

The following proposition gives the condition of function which guarantees that the image of a $PC$-space is $PC$-space.

Proposition 3.16. If $f$ is a closed surjective para-perfect function from a $PC$-space $A$ to a space $B$. Then $B$ is $PC$-space.

Proof. Let $\{p, q\}$ be a closed cover of $B$ with $p \cap q$ paracompact, then $\{f^{-1}(p), f^{-1}(q)\}$ is a cover of closed sets in $A$. However, $f^{-1}(p \cap q) = f^{-1}(p) \cap f^{-1}(q)$ is paracompact since $f$ is para-perfect. But $A$ is $PC$-space, so $f^{-1}(p)$ or $f^{-1}(q)$ is paracompact. So, $f(f^{-1}(p))$ or $f(f^{-1}(q))$ is paracompact since $f$ is closed and surjection function. Hence, $p$ or $q$ is paracompact and then $B$ is $PC$-space.

Proposition 3.17. If $f$ is a closed bijective para-perfect function from a space $A$ to a $PC$-space $B$, then $A$ is also $PC$-space.
Proof. Let \( \{p, q\} \) be a closed cover of \( \mathcal{A} \) with \( p \cap q \) paracompact, then \( \{f(p), f(q)\} \) is a closed cover of \( \mathcal{B} \) with \( f(p \cap q) \) is paracompact since \( f \) is closed bijective function. However, \( f(p) \cap f(q) = f(p \cap q) \), so \( f(p) \) or \( f(q) \) is paracompact because of \( \mathcal{B} \) is \( PC \)-space. So, \( f^{-1}(f(p)) \) or \( f^{-1}(f(q)) \) is paracompact because of \( f \) is paraperfect. Hence, \( p \) or \( q \) is paracompact. Therefore, \( \mathcal{A} \) is \( PC \)-space.

Proposition 3.18. If \( f \) is a homeomorphism function and \( \mathcal{A} \) be \( PC \)-space. Then \( \mathcal{B} \) is also \( PC \)-space.

Proof. The proof is the same as Proposition 3.16.

Proposition 3.19. Let \( f : \mathcal{A} \rightarrow \mathcal{B} \) be an open surjection para- perfect function. If \( \mathcal{A} \) is a strong \( PC \)-space , then \( \mathcal{B} \) is a strong \( PC \)-space.

Proof. To prove \( \mathcal{B} \) is strong \( PC \) -space. let \( J \subset \mathcal{B} \) be a paracompact, then a set \( f^{-1}(J) \) is a paracompact in \( \mathcal{A} \) because of \( f \) is para- perfect function. However, \( \mathcal{A} \) is a strong \( PC \)-space so there exist a closed paracompact \( \mathcal{D} \subset \mathcal{A} \) where \( f^{-1}(J) \subset \mathcal{D} \) and \( \mathcal{A} \setminus \mathcal{D} \) are connected. So, \( f(\mathcal{D}) \) is a paracompact closed set in \( \mathcal{B} \) where \( f(f^{-1}(J)) \subset f(\mathcal{D}) \) and \( \mathcal{B} \setminus f(\mathcal{D}) = f(\mathcal{A} \setminus \mathcal{D}) \) is connected since \( f \) is continuous. However, \( f(f^{-1}(J)) = J \), so \( J \subset f(\mathcal{D}) \) and a set \( f(\mathcal{D}) \) is paracompact and closed in \( \mathcal{B} \) with \( \mathcal{B} \setminus f(\mathcal{D}) \) is connected. Hence, \( \mathcal{B} \) is a strong \( PC \)-space.

Proposition 3.20. Every boundary para-perfect continuous function from a \( PC \)-space to a non-paracompact space is a quasi-paraperfect.

Proof. Assume \( f : \mathcal{A} \rightarrow \mathcal{B} \) is a boundary para-perfect continuous function such that \( \mathcal{A} \) is a \( PC \)-space and \( \mathcal{B} \) is a non- paracompact space. To prove that \( f \) is a quasi- perfect, let \( b \in \mathcal{B} \). Then, \( f^{-1}(b) \) is a subset of \( \mathcal{A} \) with paracompact boundary. From Theorem 3.8 that \( f^{-1}(b) \) or \( (\mathcal{A} \setminus f^{-1}(b)) \) is paracompact. However, \( (\mathcal{A} \setminus f^{-1}(b)) \) is not paracompact because of if \( (\mathcal{A} \setminus f^{-1}(b)) \) is paracompact, then \( \mathcal{B} = \{b\} \cup f(\mathcal{A} \setminus f^{-1}(b)) \) is paracompact, which is a contradiction. So \( f^{-1}(b) \) is paracompact, but \( f^{-1}(b) = f^{-1}(b) \). Thus \( f \) is quasi- perfect because of \( f^{-1}(b) \) is paracompact.

Conclusion
In this study, new generalizations of the Jordan curve theorem are considered. New topological spaces namely \( PC \)-space and strong \( PC \)-space is defined to get these generalizations. Many propositions concerning these two related classes are provided. We give some criteria which are equivalent to be \( PC \)-spaces and prove that every strong \( PC \)-space is \( PC \)-space. Additionally, new functions have been defined such as the para-perfect function which is preserved the property \( PC \)-space and the property strong \( PC \)-space.

References
[1] Jordan C 1983 Cours d’analyse de l’École polytechnique Gauthier-Villars et fils
[2] Khalimsky E, Kopperman R and Meyer P R 1990 Computer graphics and connected topologies on finite ordered sets Topology and its Applications 1 36(1) 1-7
[3] Michael E 2000 \( J \)-spaces Topology and its Applications 102(3) 315-39
[4] Gao Y Z 2007 \( LJ \)-spaces Czechoslovak Mathematical Journal 57(4) 1223-37
[5] Dawood N A and Gasim S G 2016 On \( CJ \)-Topological Spaces Journal of Advances in Mathematics 12(2) 5949-5952
[6] Dieudonné J A 1994 Une généralisation des espaces compacts J. Math. Pures. Appl. 23 65-76
[7] Nagata J I 1985 Modern general topology Elsevier
[8] Michael E 1957 Another note on paracompact spaces Proceedings of the American Mathematical Society 8(4) 822-8