Global smooth solution to the simplified Ericksen-Leslie system in dimension three

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Abstract

In this paper, we consider Cauchy problem of simplified Ericksen-Leslie system in dimension three. We establish the unique existence of global smooth solution under some nonlinear conditions on initial data. However, we do not need small conditions on initial data.

Keywords: Global smooth solution, Liquid crystal.

1 Introduction

In this paper we consider the following hydrodynamic system modeling the flow of liquid crystal materials on $\mathbb{R}^3$ (see [1, 2, 3, 4, 8]):

\[
\begin{aligned}
\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla P &= -\lambda \nabla \cdot (\nabla d \otimes \nabla d), \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\
\frac{\partial d}{\partial t} + u \cdot \nabla d &= \gamma (\Delta d + |\nabla d|^2 d), \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\
\nabla \cdot u &= 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\
|d| &= 1, \\
\end{aligned}
\]

with initial data:

\[
(u(0, x), d(0, x)) = (u_0(x), d_0(x)) \quad x \in \mathbb{R}^3.
\]

Where $u(t, x) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ stands for the velocity field of the flow, $d(t, x) : \mathbb{R}^+ \times \mathbb{R}^3 \to S^2$, the unit sphere in $\mathbb{R}^3$, is a unit-vector field that represents the macroscopic molecular orientation of the liquid crystal material and $P(t, x) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$ is the pressure function. The constants $\nu, \lambda, \gamma$ are positive constants that stand for the viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time for the molecular orientation field. Since the value of $\nu, \lambda, \gamma$ do not play role in our work, in this paper, we assume that $\nu = \lambda = \gamma = 1$. $\nabla d \otimes \nabla d$ denotes the $3 \times 3$ matrix whose
the \((i, j)\) entry is given by \(\nabla_i d \cdot \nabla_j d\) for \(1 \leq i, j \leq 3\). It is easy to see that \(\nabla d \circ \nabla d = (\nabla d)^T \nabla d\), where \((\nabla d)^T\) denotes the transpose of matrix \(\nabla d\).

System (1.1) is a simplified version of the Ericksen-Leslie model. General Ericksen-Leslie model reduces the Ossen-Frank model in the static case, for the hydrodynamics of nematic liquid crystals developed during the period from 1958 to 1968 [1, 2, 3]. Since the general Ericksen-Leslie system is very complicated, we only study a simplified model of the Ericksen-Leslie system which can derive without destroying the basic structure. It is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow field \(u(t, x)\), and the macroscopic description of the microscopic orientation configurations \(d(t, x)\) of rod-like liquid crystals. The system (1.1)-(1.2) is a system of the Navier-Stokes equation coupled with the harmonic map flows.

In a series of papers, Lin [5] and Lin-Liu [3, 6] initiated the mathematical analysis of the system (1.1)-(1.2). Since the Erichsen-Leslie system (1.1)-(1.2) with \(|d| = 1\) is complicated, Lin and Liu [3, 6] proposed to consider an approximation model of Ericksen-Leslie system by Ginzburg-Landau functional. More precisely, they replaced the Dirichlet functionals

\[
\frac{1}{2} \int_\Omega |\nabla d|^2 dx
\]

for \(d : \Omega \to S^2\) by the Ginzburg-Landau functionals

\[
\int_\Omega \left(\frac{1}{2} |\nabla d|^2 + \frac{(1 - |d|^2)^2}{4\epsilon}\right)dx
\]

for \(d : \Omega \to \mathbb{R}^3(\epsilon > 0)\). In [3], Lin and Liu proved the global existence of solutions in dimensions two or three. In [6], Lin and Liu proved partial regularity of weak solutions in dimension three. Furthermore, Lin and Liu in [7] proved existence of solutions for the general Ericksen-Leslie system and also analyzed the limits of weak solutions as \(\epsilon \to 0\). Recently, Lin and Liu in [7] studied the system (1.1)-(1.2) in two dimensions. They established the global existence and partial regularity of the global weak solution and performed the blow-up analysis at each singular time. Hong [9] proved the global existence of the system (1.1)-(1.2) in two dimensions independently.

In dimension three, system (1.1) become more complicated. For the initial-boundary value problem, Tan and Yin [10] prove the local existence and global existence for small initial-boundary data in positive index Besov space. For Cauchy Problem, Wang [11] prove the global existence for small initial data in \(BM0^{-1} \times BMO\). Without small condition for initial data, the global existence results of system (1.1)-(1.2) have not been established. In this paper, our aim is to construct global smooth solution to system (1.1)-(1.2) without small initial data. However, we should add some nonlinear conditions on the initial data. Our work is partially inspired by [12]. In that paper, the author consider Navier-Stokes equations. But, compare to Navier-Stokes equations, system (1.1) is more complicated. The main difficulties are due to the higher order nonlinear item \(|\nabla d|^2 d\) and the mutual effect between \(u\) and \(d\). Before stating our main theorem, we introduce some function space. Firstly, we define

\[
H(\mathbb{R}^3) = \{u \in (L^2(\mathbb{R}^3))^3 | \nabla \cdot u = 0\}
\]

2
equipped with $L^2(\mathbb{R}^3)$ norm. Then, we introduce the space $H^\infty(\mathbb{R}^3)$ which is introduced firstly in [12]. Let $H^\infty(\mathbb{R}^3)$ be the completion of $\cap_{m \geq 0} H^m(\mathbb{R}^3)$ endowed with norm

$$\| u \|_{H^\infty} = \max_{m \geq 0} \sum_{j=1}^3 \| \partial^m u \|_{L^2(\mathbb{R}^3)}, \quad \forall u \in H^\infty(\mathbb{R}^3),$$

i.e.

$$H^\infty(\mathbb{R}^3) = \{ u | u \in \cap_{m \geq 0} H^m(\mathbb{R}^3), \| u \|_{H^\infty} < \infty \}.$$

The space $H^\infty(\mathbb{R}^3)$ introduced in [12] will play an important role in the estimates of nonlinear items. For simplicity of presentation, $(L^q(\mathbb{R}^3))^3$ (resp. $(H^m(\mathbb{R}^3))^3$) denotes by $L^q$ (resp. $H^m$), where $q \geq 1$ and $m \geq 0$.

Now, we state our main theorem as following:

**Theorem 1.1.** Assume that $u_0 \in H^\infty \cap H$, $d_0 | = 1$, $\nabla d_0 \in H^\infty$. Moreover, we need that $u_0$, $\nabla u_0$, $u_0 \cdot \nabla d_0$, $| \nabla d_0 |^2$ and $| \nabla d_0 |^2 d_0$ are belong to $H^\infty$. Then for any $T > 0$ there exists an unique solution $(u, d)$ to system (1.1)-(1.2) such that

$$u \in C([0, T]; H^\infty), \quad u \cdot \nabla u, \nabla P \in C([0, T]; H^\infty),$$

$$\nabla d \in C([0, T]; H^\infty), \quad u \cdot \nabla d, | \nabla d |^2, | \nabla d |^2 d \in C([0, T]; H^\infty).$$

The present paper is written as follows. In Section 2, we establish some basic estimates of linear system of (1.1)-(1.2). In Section 3, we prove the existence of local smooth solution of system (1.1)-(1.2). In Section 4, we give an uniform estimate for the local smooth solution obtained in Section 3 and complete the proof of Theorem 1.1.

## 2 Linear estimates

In this section, we establish some estimates to the following linear system of (1.1)-(1.2):

$$\begin{cases}
\frac{\partial u}{\partial t} - \Delta u + \nabla P = f, & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\
\frac{\partial d}{\partial t} - \Delta d = g, & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\
\nabla \cdot u = 0, & \text{in } \mathbb{R}^+ \times \mathbb{R}^3,
\end{cases}$$

with initial data

$$\begin{cases}
u |_{t=0} = u_0 \\
d |_{t=0} = d_0
\end{cases}$$

Taking the divergence of the first equation of (2.1), we have

$$\Delta P = \nabla \cdot f.$$ 

In the rest of the paper, we denote the Leray projector on divergence-free vector field by $P$. The solutions of (2.1)-(2.2) are written

$$u(t) = S(t)u_0 + \int_0^t S(t - \tau)f(\tau)d\tau,$$

$$d(t) = S(t)d_0 + \int_0^t S(t - \tau)g(\tau)d\tau.$$
where \( S(t) \) denote the semigroup operator \( e^{t\Delta} \). The basic estimates of (2.1) are established as following:

**Lemma 2.1** Assume that \( u_0 \in H^\infty \cap H, \; d_0 \in L^\infty, \nabla d_0 \in H^\infty, \; f, g \in L^\infty(0,T;H^\infty) \) for any \( T > 0 \). Moreover, we need that \( u_0 \cdot \nabla u_0, u_0 \cdot \nabla d_0, |\nabla d_0|^2 \) and \( |\nabla d_0|^2 d_0 \) are belong to \( H^\infty \). Then there exits an unique global solution \((u,d)\) to the problem (2.1)-(2.2) such that

\[
u \in C(0,T;H^\infty), d \in C(0,T;L^\infty), \nabla d \in C(0,T;H^\infty), \quad \forall T > 0,
\]

and the following estimates are valid:

\[
\| u(t,\cdot) \|_{H^\infty} \leq \| u_0 \|_{H^\infty} + \int_0^t \| f(\tau,\cdot) \|_{H^\infty} \, d\tau,
\]

\[
\| u \cdot \nabla u(t,\cdot) \|_{H^\infty} \leq \{ \| u_0 \cdot \nabla u_0 \|_{H^\infty}^2 + C \| u_0 \|_{H^t}^2 \tau + t + F(t) \} e^t, \quad \forall t \geq 0,
\]

\[
\| S(t) \nabla u_0 \|_{L^2} \leq C t^{-\frac{1}{2}} \| u_0 \|_{L^2}, \quad \forall t > 0.
\]

\[
\| d(t,\cdot) \|_{L^\infty} \leq \| d_0 \|_{L^\infty} + \int_0^t \| g(\tau,\cdot) \|_{H^\infty} \, d\tau,
\]

\[
\| \nabla d(t,\cdot) \|_{H^\infty} \leq \| \nabla d_0 \|_{H^\infty} + \int_0^t \| g(\tau,\cdot) \|_{H^\infty} \, d\tau,
\]

\[
\| \nabla d(t,\cdot) \|_{H^\infty}^2 \leq \{ \| \nabla d_0 \|_{H^\infty}^2 + \| \nabla d_0 \|_{H^\infty} \| \nabla d \|_{H^\infty} \} e^t,
\]

\[
\| \nabla d \|_{H^\infty}^2 \| d(t,\cdot) \|_{H^\infty}^2 \leq \{ \| \nabla d_0 \|_{H^\infty}^2 d_0 + \int_0^t \| d \|_{L^\infty} \| g \|_{H^\infty} \| \nabla d \|_{H^\infty} \, d\tau
\]

\[
+ \int_0^t \| \nabla d \|_{H^\infty} \| g \|_{H^\infty} \, d\tau + \int_0^t \| \nabla d \|_{H^\infty} \| \nabla d \|_{H^\infty} \, d\tau
\]

\[
+ \int_0^t \| \nabla d \|_{H^\infty} \| \nabla d \|_{H^\infty} \, d\tau \} e^t,
\]

\[
\| u \cdot \nabla d(t,\cdot) \|_{H^\infty} \leq \{ \| u_0 \cdot \nabla d_0 \|_{H^\infty}^2 + \int_0^t \| f \|_{H^\infty} \| \nabla d \|_{H^\infty} \, d\tau
\]

\[
+ \int_0^t \| g \|_{H^\infty} \| u \|_{H^\infty} \| \nabla d \|_{H^\infty} \, d\tau + \int_0^t \| u \|_{H^\infty} \| \nabla d \|_{H^\infty} \, d\tau \} e^t,
\]

\[
\| u \cdot \nabla d(t,\cdot) \|_{H^\infty} \leq \{ \| u_0 \cdot \nabla d_0 \|_{H^\infty}^2 + \int_0^t \| f \|_{H^\infty} \| \nabla d \|_{H^\infty} \, d\tau
\]

\[
+ \int_0^t \| g \|_{H^\infty} \| u \|_{H^\infty} \| \nabla d \|_{H^\infty} \, d\tau + \int_0^t \| u \|_{H^\infty} \| \nabla d \|_{H^\infty} \, d\tau \} e^t,
\]
where

\[
F(t) = \int_0^t \left( \int_0^s \| f(\tau, \cdot) \|_{H^\infty} \, d\tau \right)^4 \, ds
+ \int_0^t \left( \int_0^s \| f(\tau, \cdot) \|_{H^\infty}^2 \| u_0 \|_{H^\infty} + \int_0^\tau \| f(s, \cdot) \|_{H^\infty} \, ds \right)^2 \, d\tau,
\]
\[
G(t) = \int_0^t \left( \int_0^s \| g(\tau, \cdot) \|_{H^\infty} \, d\tau \right)^4 \, ds
+ \int_0^t \| g(\tau, \cdot) \|_{L^\infty}^2 \| \nabla_d u_0 \|_{H^\infty} + \int_0^\tau \| g(s, \cdot) \|_{H^\infty} \, ds \right)^2 \, d\tau.
\]

Proof. Since \( f, g \in L^\infty(0, T; H^\infty) \), it is easy to show \( u \in C(0, T; H^\infty), d \in C(0, T; L^\infty), \nabla d \in C(0, T; H^\infty), \forall T > 0 \). The proof of (2.6)-(2.8) is the same to the lemma 2.1 of [12], we omit it. Using this facts \( \| S(t) d_0 \|_{L^\infty} \leq \| d_0 \|_{L^\infty}, H^\infty \subset L^\infty \) and (2.5), we get (2.9). From equation (2.5) and the estimate \( \| S(t) v \|_{L^2} \leq \| v \|_{L^2} \), we derive that

\[
(2.14) \quad \| \partial^m d(t, \cdot) \|_{L^2} \leq \| \partial^m d_0 \|_{L^2} + \int_0^t \| \partial^m g \|_{L^2} \, d\tau
\leq \| \nabla_d d_0 \|_{H^\infty} + \int_0^t \| g \|_{H^\infty} \, d\tau, \forall m \geq 1.
\]

Taking the maximum with respect to \( m \geq 1 \), we obtain (2.10).

Using the second equation of (2.1), we derive that

\[
\partial_t (\| \nabla d \|^2) - \Delta (\| \nabla d \|^2) = 2(\nabla g \cdot \nabla d + | \nabla^2 d |^2)
\]

Let \( \partial^m \) denote by \( D^m \). By the above equation, we derive:

\[
(2.15) \quad \partial_t (\| D^m | \nabla d |^2 \|_{L^2} \cdot \Delta (\| D^m | \nabla d |^2 \) = 2D^m (\nabla g \cdot \nabla d - | \nabla^2 d |^2).
\]

Taking the scalar product of \( D^m | \nabla d |^2 \) on the equation and then integrating over \( \mathbb{R}^3 \), we have

\[
(2.16) \quad \frac{d}{dt} \| D^m | \nabla d |^2 \|_{L^2}^2 + 2 \| \nabla D^m | \nabla d |^2 \|_{L^2}^2
= (-1)^m 2 \int_{\mathbb{R}^3} (\nabla g \cdot \nabla d - | \nabla^2 d |^2) D^{2m} | \nabla d |^2 \, dx.
\leq \| D^{2m} (\| \nabla d |^2 \) \|_{L^2}^2 + C(\| \nabla g \cdot \nabla d \|_{L^2}^2 + \| \nabla^2 d |^2 \|_{L^2}^2), \leq \| \nabla d |^2 \|_{L^\infty}^2 + C(\| g \|_{H^\infty} \| \nabla d \|_{H^\infty}^2 + \| \nabla d \|_{H^\infty}^2).
\]

Integrating (2.16) from 0 to \( t \), we have

\[
(2.17) \quad \| D^m | \nabla d |^2 \|_{L^2}^2 - \| D^m | \nabla d_0 |^2 \|_{L^2}^2
\leq \int_0^t \| \nabla d |^2 \|_{H^\infty}^2 + C(\| g \|_{H^\infty} \| \nabla d \|_{H^\infty}^2 + \| \nabla d \|_{H^\infty}^2) \, d\tau.
\]
Taking the maximum with respect $m \geq 0$ on the left-hand side of (2.17) and using the definition of $H^\infty$, we deduce that

\[
\begin{align*}
(2.18) \quad \| \nabla d \|^2_{H^\infty} &\leq \| \nabla d_0 \|^2_{H^\infty} + \int_0^t \| \nabla d \|^2_{H^\infty} \, dt \\
&+ C \int_0^t (\| \nabla d \|^2_{H^\infty} \| \nabla d \|^2_{H^\infty} + \| \nabla d \|^2_{H^\infty}) \, dt.
\end{align*}
\]

Using (2.10) and Gronwall inequality, we derive the estimate (2.11).

From the second equation of (2.1), we get

\[
(2.19) \quad \partial_t (|\nabla d|^2) - \Delta (|\nabla d|^2) = 2 (\nabla g \cdot \nabla d) \partial_t g - 2 |\nabla^2 d|^2 \partial_t d - 2 (\partial_t |\nabla d|^2 \partial_t d).
\]

In this paper, the repeated indexes denote summation. Repeating the Progress of (2.15)-(2.16), we get

\[
(2.20) \quad \frac{d}{dt} \| D^m (|\nabla d|^2) \|^2_{L^2} + 2 \| \nabla D^m (|\nabla d|^2) \|^2_{L^2} \\
\leq \| D^{2m} (|\nabla d|^2) \|^2_{L^2} + C (\| \nabla g \cdot \nabla d \|_{L^2}^2 + \| \nabla d \|^2 g \|_{L^2}^2) \\
+ \| \nabla^2 d \|^2 \| \nabla d \|_{L^2}^2 + \| \partial_t |\nabla d|^2 \partial_t d \|_{L^2}^2) \\
\leq \| \nabla d \|^2_{H^\infty} + C (\| \nabla g \|_{L^\infty}^2 + \| \nabla d \|^2_{H^\infty}) \\
+ \| \nabla d \|^2_{H^\infty} + \| \nabla d \|^2_{H^\infty} + \| \nabla d \|^2_{H^\infty}.
\]

Repeating the progress of (2.17)-(2.18), we deduce (2.12).

Now, we prove (2.13). Using system (2.1), we have

\[
(2.21) \quad \partial_t (u \cdot \nabla d) - \Delta (u \cdot \nabla d) = f \cdot \nabla d + u \cdot \nabla g - (\nabla P) \cdot \nabla d - 2 (\partial_t u \cdot \nabla d).
\]

Repeating the progress of (2.15)-(2.16), we get

\[
(2.22) \quad \frac{d}{dt} \| D^m (u \cdot \nabla d) \|^2_{L^2} + 2 \| \nabla D^m (u \cdot \nabla d) \|^2_{L^2} \\
\leq \| D^{2m} (u \cdot \nabla d) \|^2_{L^2} + C (\| f \cdot \nabla d \|_{L^2}^2 + \| u \cdot \nabla g \|_{L^2}^2) \\
+ \| \nabla P \cdot \nabla d \|_{L^2}^2 + \| \partial_t u \cdot \nabla \partial_t d \|_{L^2}^2).
\]

Notice the equation (2.3), we derive that $\| \nabla P \|_{L^2} \leq \| f \|_{L^2}$. Using this fact, we write (2.22) as

\[
(2.23) \quad \frac{d}{dt} \| D^m (u \cdot \nabla d) \|^2_{L^2} + 2 \| \nabla D^m (u \cdot \nabla d) \|^2_{L^2} \\
\leq \| (u \cdot \nabla d) \|^2_{H^\infty} + C (\| f \|^2_{H^\infty} + \| u \|^2_{H^\infty}) \\
+ \| f \|^2_{H^\infty} + \| (u \cdot \nabla d) \|^2_{H^\infty}.
\]

Repeating the progress of (2.17)-(2.18), we obtain (2.13). \qed

### 3 The local well-posedness

In this section, we prove the local existence of system (1.1)-(1.2). Firstly, we write (1.1)-(1.2) as the integral equations:

\[
(3.1) \begin{aligned}
    u(t) &= S(t) u_0 - \int_0^t S(t-s) P(u \cdot \nabla u + \nabla \cdot (\nabla d \otimes \nabla d)) \, ds \\
    d(t) &= S(t) d_0 + \int_0^t S(t-s) (\nabla d \cdot \nabla d - u \cdot \nabla d) \, ds.
\end{aligned}
\]
The main result of this section is the following local existence theorem.

**Theorem 3.1** Assume that \( u_0 \in H^\infty \cap H, \ | d_0 | = 1, \nabla d_0 \in H^\infty \). Moreover, we need that \( u_0 \cdot \nabla u_0, u_0 \cdot \nabla d_0, | \nabla d_0 |^2 \) and \( | \nabla d_0 |^2 d_0 \) are belong to \( H^\infty \). Then there exists \( T > 0 \) only depended on initial data \((u_0, d_0)\) and a unique solution \((u, d)\) of the system (1.1)-(1.2) such that

\[
u \in C(0, T; H^\infty), \ d \in C(0, T; L^\infty), \nabla d \in C(0, T; H^\infty)\]

Moreover, we have that

\[
u \cdot \nabla u, u \cdot \nabla d, | \nabla d |^2, | \nabla d |^2 d \in C(0, T; H^\infty).
\]

**Proof.** Let \( T > 0 \) and define

\[(3.2)\]

\[
E = \{ (v, b) | v \in L^\infty(0, T; H^\infty), b \in L^\infty(0, T; L^\infty), \nabla b \in L^\infty(0, T; H^\infty) \]

\[
\| v \|_{L^\infty(H^\infty)} + \| b \|_{L^\infty(L^\infty)} + \| \nabla b \|_{L^\infty(H^\infty)} + \| \nabla b \|^2 \| L^\infty(H^\infty) \]

\[
+ \| v \cdot \nabla v \|_{L^\infty(H^\infty)} + \| v \cdot \nabla b \|_{L^\infty(H^\infty)} + \| \nabla b \|^2 b \|_{L^\infty(H^\infty)} \]

\[
\leq 2\epsilon (\| u_0 \|_{H^\infty} + \| d_0 \|_{L^\infty} + \| \nabla d_0 \|_{H^\infty} + \| \nabla d_0 \|^2 \| d_0 \|_{H^\infty})
\]

Endowed with the metric

\[
dist((u^1, b^1), (u^2, b^2)) = \| u^1 - u^2 \|_{L^\infty(H^2)} + \| d^1 - d^2 \|_{L^\infty(L^\infty)} \]

\[
+ \| \nabla (d^1 - d^2) \|_{L^\infty(H^2)}
\]

It is clearly that \( E \) is complete metric space. For any \((v, b) \in E\), we define a map \( \Gamma \) as follows:

\[
\Gamma^1(v, b) = S(t)u_0 - \int_0^t S(t - s)P(v \cdot \nabla v + \nabla \cdot (\nabla b \circ \nabla b))ds,
\]

\[
\Gamma^2(v, b) = S(t)d_0 + \int_0^t S(t - s)\{ \nabla b \|^2 b - v \cdot \nabla b \}ds.
\]

Then \((u, d) = \Gamma(v, b)\) is a solution to system (2.1)-(2.2) with \( f = -v \cdot \nabla v - \nabla \cdot (\nabla b \circ \nabla b) \) and \( g = | \nabla b |^2 b - v \cdot \nabla b \). It is easy to show that

\[
\| f \|_{H^\infty} \leq \| v \cdot \nabla v \|_{H^\infty} + \| \nabla b \|^2 \| H^\infty \|
\]

\[
\| g \|_{H^\infty} \leq \| v \cdot \nabla b \|_{H^\infty} + \| \nabla b \|^2 \| H^\infty \|
\]

We should find \( T \) such that the map \( \Gamma : E \rightarrow E \) is a strict contraction. Denote

\[
r = e(\| u_0 \|_{H^\infty} + \| d_0 \|_{L^\infty} + \| \nabla d_0 \|_{H^\infty} + \| \nabla d_0 \|^2 \| H^\infty \|
\]

\[
+ \| u_0 \cdot \nabla u_0 \|_{H^\infty} + \| u_0 \cdot \nabla d_0 \|_{H^\infty} + \| \nabla d_0 \|^2 \| d_0 \|_{H^\infty}).
\]

7
For $0 \leq t \leq T \leq 1$, the estimate (2.6) implies that

\begin{equation}
\| \Gamma^1(v, b)(t, \cdot) \|_{H^\infty} \leq \| u_0 \|_{H^\infty} + \int_0^t (\| v \cdot \nabla v \|_{H^\infty} + \| \nabla d \|_{L^\infty(H^\infty)}) \, ds
\leq \| u_0 \|_{H^\infty} + \mathcal{T}(\| v \cdot \nabla v \|_{L^\infty(H^\infty)} + \| \nabla d \|_{L^\infty(H^\infty)})
\leq r + 2rT.
\end{equation}

Using estimate (2.7) and (3.3), we have that

\begin{equation}
\| \Gamma^1(v, b) \cdot \nabla \Gamma^1(v, b)(t, \cdot) \|_{H^\infty}^2
\leq c(\| u_0 \cdot \nabla u_0 \|_{H^\infty}^2 + C \| u_0 \|_{H^\infty}^2 T
+ C \| u_0 \|_{H^\infty}^2 (\| v \cdot \nabla v \|_{L^\infty(H^\infty)} + \| \nabla d \|_{L^\infty(H^\infty)})^2 T
+ C(\| v \cdot \nabla v \|_{L^\infty(H^\infty)} + \| \nabla b \|^2_{L^\infty(H^\infty)})^2 T
\leq r^2 + C r^4 T.
\end{equation}

By estimate (2.9) and (3.4), we get that

\begin{equation}
\| \Gamma^2(v, b)(t, \cdot) \|_{L^\infty}
\leq \| d_0 \|_{L^\infty} + (\| v \cdot \nabla b \|_{L^\infty(H^\infty)} + \| \nabla b \|^{2}_{L^\infty(H^\infty)}) T
\leq r + 2rT.
\end{equation}

Using (2.10) and (3.4), it is easy to show that

\begin{equation}
\| \nabla \Gamma^2(v, b)(t, \cdot) \|_{H^\infty} \leq r + 2rT.
\end{equation}

Using (2.11) and (3.4), we can get that

\begin{equation}
\| \nabla \Gamma^2(v, b) \|^2 \| \nabla \Gamma^1(v, b)(t, \cdot) \|_{H^\infty}^2
\leq c(\| \nabla d_0 \|^2_{H^\infty} + C \| \nabla d_0 \|_{H^\infty}^2 T
+ C \| \nabla d_0 \|_{H^\infty}^2 (\| v \cdot \nabla b \|_{L^\infty(H^\infty)} + \| \nabla b \|^2_{L^\infty(H^\infty)})^2 T
+ C(\| v \cdot \nabla b \|_{L^\infty(H^\infty)} + \| \nabla b \|^{4}_{L^\infty(H^\infty)})^2 T
\leq r^2 + C r^4 T.
\end{equation}

Together with (2.12), (3.3) and (3.4), the similar computation deduce that

\begin{equation}
\| \nabla \Gamma^2(v, b) \|^2 \| \Gamma^2(v, b) \|_{H^\infty} \leq r^2 + C (r^4 + r^6) T.
\end{equation}

Combining (2.13), (3.3) and (3.4), we obtain that

\begin{equation}
\| \Gamma^1(v, b) \cdot \nabla \Gamma^2(v, b) \|_{H^\infty} \leq r^2 + C r^4 T.
\end{equation}

Using (3.5)-(3.11), we have that

\begin{equation}
\| \Gamma^1(v, b)(t, \cdot) \|_{H^\infty} + \| \Gamma^1(v, b) \cdot \nabla \Gamma^1(v, b)(t, \cdot) \|_{H^\infty}
+ \| \Gamma^2(v, b)(t, \cdot) \|_{L^\infty} + \| \nabla \Gamma^2(v, b)(t, \cdot) \|_{H^\infty}
+ \| \nabla \Gamma^2(v, b)(t, \cdot) \|^2 \| \Gamma^1(v, b) \cdot \nabla \Gamma(v, b)(t, \cdot) \|_{H^\infty}
+ \| \nabla \Gamma^2(v, b) \|^2 \| \Gamma^1(v, b) \cdot \nabla \Gamma(v, b)(t, \cdot) \|_{H^\infty}
\leq r + C T^2 (r^2 + r^3),
\end{equation}

\vspace{1cm}

8
where $C$ is universal positive constant. Taking $T_0 = \min\{1, \frac{1}{c_{(H^4)\ell}}\}$, then

$$\Gamma : E \to E$$

for all $T \leq T_0$.

Let $(u^1, d^1) = \Gamma(v^1, b^1), (u^2, d^2) = \Gamma(v^2, b^2)$ and $0 \leq t \leq T \leq T_0$. By the definition of $\Gamma$, we have

\begin{equation}
\label{eq:3.13}
(u^1(t, x) - u^2(t, x)) = \int_0^T S(t-s)\nabla \cdot \left(\nabla b^2 \circ \nabla b^2 - \nabla b^1 \circ \nabla b^1\right) ds,
\end{equation}

\begin{equation}
\label{eq:3.14}
d^1(t, x) - d^2(t, x) = \int_0^T (S(t-s)\nabla \cdot \left(\nabla b^1 \circ \nabla b^1 - |\nabla b^2|^2 b^2 + v^2 \cdot \nabla b^2 - v^1 \cdot \nabla b^1\right) ds.
\end{equation}

Using estimate (2.8), we have that

\begin{equation}
\label{eq:3.15}
\| (u^1 - u^2)(t, \cdot) \|_{L^2} \leq \int_0^t (t-s)^{-\frac{3}{2}} C\| v^1 - v^2 \|_{L^\infty(L^2)} \left(\| v^1 \|_{L^\infty(H^\infty)} + \| v^2 \|_{L^\infty(H^\infty)}\right)
\end{equation}

\begin{equation}
\nonumber
+ \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} \left(\| \nabla b^1 \|_{L^\infty(H^\infty)} + \| \nabla b^2 \|_{L^\infty(H^\infty)}\right) ds
\end{equation}

\begin{equation}
\label{eq:3.16}
\leq CT^\frac{3}{2}r\left(\| v^1 - v^2 \|_{L^\infty(L^2)} + \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)}\right),
\end{equation}

where we using the fact $\| f \|_{L^\infty} \leq \| f \|_{H^\infty}$. Using Newton-Leibniz formula and estimate (2.8), we get that

\begin{equation}
\label{eq:3.17}
\| \nabla^2 u^1 - \nabla^2 u^2 \|_{L^2} \leq \int_0^t \| S(t-s)\nabla \cdot (\nabla v^2 \circ \nabla v^2 - v^1 \circ v^1 + \nabla b^2 \circ \nabla b^2 - \nabla b^1 \circ \nabla b^1) \|_{L^2} ds
\end{equation}

\begin{equation}
\nonumber
\leq \int_0^t (t-s)^{-\frac{3}{2}} C\| \nabla^2 v^1 - \nabla^2 v^2 \|_{L^\infty(L^2)} \left(\| v^1 \|_{L^\infty(H^\infty)} + \| v^2 \|_{L^\infty(H^\infty)}\right)
\end{equation}

\begin{equation}
\nonumber
+ \| v^1 - v^2 \|_{L^\infty(L^2)} \left(\| v^1 \|_{L^\infty(H^\infty)} + \| v^2 \|_{L^\infty(H^\infty)}\right)
\end{equation}

\begin{equation}
\nonumber
+ \| \nabla^3 b^1 - \nabla^3 b^2 \|_{L^\infty(L^2)} \left(\| \nabla b^1 \|_{L^\infty(H^\infty)} + \| \nabla b^2 \|_{L^\infty(H^\infty)}\right)
\end{equation}

\begin{equation}
\nonumber
+ \| \nabla^2 b^1 - \nabla^2 b^2 \|_{L^\infty(L^2)} \left(\| \nabla b^1 \|_{L^\infty(H^\infty)} + \| \nabla b^2 \|_{L^\infty(H^\infty)}\right)
\end{equation}

\begin{equation}
\nonumber
+ \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} \left(\| \nabla b^1 \|_{L^\infty(H^\infty)} + \| \nabla b^2 \|_{L^\infty(H^\infty)}\right),
\end{equation}

where, we using this fact $\| \nabla^m f \|_{L^\infty(L^2)} \leq \| f \|_{H^\infty}, m \geq 0$. Using interpolation $\| \nabla^m f \|_{L^2} \leq \| f \|_{L^2} + \| \nabla^2 f \|_{L^2}$, we write (3.16) as

\begin{equation}
\label{eq:3.18}
\| \nabla^2 u^1 - \nabla^2 u^2 \|_{L^2} \leq CT^\frac{3}{2}r\left(\| v^1 - v^2 \|_{L^\infty(L^2)} + \| \nabla^2 v^1 - \nabla^2 v^2 \|_{L^\infty(L^2)}
\end{equation}

\begin{equation}
\nonumber
+ \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} + \| \nabla^3 b^1 - \nabla^3 b^2 \|_{L^\infty(L^2)}\right).
\end{equation}
By (3.14), we derive that

\begin{equation}
\nabla d^1 - \nabla d^2 \n\end{equation}

\begin{equation}
= \int_0^t S(t-s)^\frac{4}{3} \nabla (v^2 \cdot \nabla b^2 - v^1 \cdot \nabla b^1 + |\nabla b^1|^2 b^1 - |\nabla b^2|^2 b^2) ds.
\end{equation}

Using estimate (2.8), we get

\begin{equation}
\nabla d^1 - \nabla d^2 \|_{L^2} \leq \int_0^t \langle t-s \rangle^{\frac{4}{3}} C \{ \| v^1 - v^2 \|_{L^\infty(L^2)} (\| \nabla b^1 \|_{L^\infty(H^\infty)} + \| \nabla b^2 \|_{L^\infty(H^\infty)}) + \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} (\| v^1 \|_{L^\infty(H^\infty)} + \| v^2 \|_{L^\infty(H^\infty)}) + \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} (\| b^1 \|_{L^\infty(H^\infty)} + \| b^2 \|_{L^\infty(H^\infty)}) + \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} \} ds.
\end{equation}

Using Newton-Leibniz formula and estimate (2.8), we have

\begin{equation}
\nabla d^1 - \nabla d^2 (t, \cdot) \|_{L^2} \leq \int_0^t \langle t-s \rangle^{\frac{4}{3}} C \{ \| \nabla v^1 - \nabla v^2 \|_{L^\infty(L^2)} (\| \nabla b^1 \|_{L^\infty(H^\infty)} + \| \nabla b^2 \|_{L^\infty(H^\infty)}) + \| \nabla v^1 - \nabla v^2 \|_{L^\infty(L^2)} (\| \nabla b^1 \|_{L^\infty(H^\infty)} + \| \nabla b^2 \|_{L^\infty(H^\infty)}) + \| \nabla v^1 - \nabla v^2 \|_{L^\infty(L^2)} (\| v^1 \|_{L^\infty(H^\infty)} + \| v^2 \|_{L^\infty(H^\infty)}) + \| \nabla v^1 - \nabla v^2 \|_{L^\infty(L^2)} (\| v^1 \|_{L^\infty(H^\infty)} + \| v^2 \|_{L^\infty(H^\infty)}) + \| \nabla v^1 - \nabla v^2 \|_{L^\infty(L^2)} (\| b^1 \|_{L^\infty(H^\infty)} + \| b^2 \|_{L^\infty(H^\infty)}) + \| \nabla v^1 - \nabla v^2 \|_{L^\infty(L^2)} (\| b^1 \|_{L^\infty(H^\infty)} + \| b^2 \|_{L^\infty(H^\infty)}) + \| \nabla v^1 - \nabla v^2 \|_{L^\infty(L^2)} (\| b^1 \|_{L^\infty(H^\infty)} + \| b^2 \|_{L^\infty(H^\infty)}) + \| \nabla v^1 - \nabla v^2 \|_{L^\infty(L^2)} (\| b^1 \|_{L^\infty(H^\infty)} + \| b^2 \|_{L^\infty(H^\infty)}) + \| \nabla v^1 - \nabla v^2 \|_{L^\infty(L^2)} (\| b^1 \|_{L^\infty(H^\infty)} + \| b^2 \|_{L^\infty(H^\infty)}) \} ds.
\end{equation}
Using interpolation \( \| \nabla f \|_{L^2} \leq \| \nabla^2 f \|_{L^2} + \| f \|_{L^2} \), we write (3.21) as
\[
\| (d^1 - d^2)(t, \cdot) \|_{L^\infty} \leq CT\| v^1 - v^2 \|_{L^\infty(L^2)} + \| \nabla^2 v^1 - \nabla^2 v^2 \|_{L^\infty(L^2)} + \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} + \| b^1 - b^2 \|_{L^\infty(L^2)}
\]
\[
\leq CT^2(r + r^2)\| v^1 - v^2 \|_{L^\infty(H^2)} + \| \nabla v^1 - \nabla v^2 \|_{H^\infty} + \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} + \| b^1 - b^2 \|_{L^\infty(L^2)}
\]
\[
+ \| \nabla v^1 - \nabla v^2 \|_{L^\infty(H^2)} + \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} + \| b^1 - b^2 \|_{L^\infty(L^2)}
\]
\[
\leq CT^2(r + r^2)\| v^1 - v^2 \|_{L^\infty(H^2)} + \| \nabla v^1 - \nabla v^2 \|_{H^\infty} + \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} + \| b^1 - b^2 \|_{L^\infty(L^2)}
\]
where, we use this facts \( \| S(t)f \|_{L^\infty} \leq \| f \|_{L^\infty} \) and \( \| \nabla^m f \|_{L^\infty} \leq \| f \|_{H^\infty} \). By interpolation in \( \mathbb{R}^3 \), it holds that \( \| f \|_{L^\infty} \leq \| f \|_{L^2} + \| \nabla^2 f \|_{L^2} \). Using this estimate, we write (3.23) as
\[
\| (d^1 - d^2)(t, \cdot) \|_{L^\infty} \leq CT^2(r + r^2)\| v^1 - v^2 \|_{L^\infty(H^2)} + \| \nabla v^1 - \nabla v^2 \|_{H^\infty} + \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} + \| b^1 - b^2 \|_{L^\infty(L^2)}
\]
\[
\leq CT^2(r + r^2)\| v^1 - v^2 \|_{L^\infty(H^2)} + \| \nabla v^1 - \nabla v^2 \|_{H^\infty} + \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} + \| b^1 - b^2 \|_{L^\infty(L^2)}
\]
\[
\leq CT^2(r + r^2)\| v^1 - v^2 \|_{L^\infty(H^2)} + \| \nabla v^1 - \nabla v^2 \|_{H^\infty} + \| \nabla b^1 - \nabla b^2 \|_{L^\infty(L^2)} + \| b^1 - b^2 \|_{L^\infty(L^2)}
\]
where, we use the fact \( T \leq T_0 \leq 1 \). Taking \( T \leq \min\{ \frac{1}{CT_0}, T_0 \} \). Then \( \Gamma : E \rightarrow E \) is a strict contraction. By the principle of contraction mapping, there exist a function pair \((u, d)\) satisfying system (1.1)-(1.2) except \( d \mid = 1 \). Since \( S(t) \) is a \( C_0 \) semigroup in \( H(\mathbb{R}^3) \), we derive that
\[
u \in C(0, T; H^\infty), d \in C(0, T; L^\infty), \nabla d \in C(0, T; H^\infty)
\]
Applying the maximum principle to the equation for \( | d |^2 \), one can easily see that \( | d | = 1 \). This implies that \((u, d)\) is a unique solution of system (1.1)-(1.2).

4 Global solution

In this section, our aim is to extend the local solution which is established in Section 3 to a global solution. It is enough to prove the global existence by establishing the following uniformly estimate.
Theorem 4.1 Assume that $u_0 \in H^\infty \cap H$, $|d_0| = 1$, $\nabla d_0 \in H^\infty$. Moreover, we need that $u_0 \cdot \nabla u_0$, $u_0 \cdot \nabla u_0$, $|\nabla d_0|^2$ and $|\nabla d_0|^2 d_0$ are belong to $H^\infty$. Then for the solution $(u, d)$ of the system (1.1)-(1.2) established in Theorem 3.1, the following estimate is valid:

$$
\begin{align*}
(4.1) & \quad \| u(t) \|_{H^\infty}^2 + \| \nabla d(t) \|_{H^\infty}^2 + \| \nabla d(t) \|_{L^2}^2 dt(t) \|_{H^\infty}^2 \\
& + \| \nabla d \|_{H^\infty}^2 + \| u \cdot \nabla d \|_{H^\infty}^2 + \| u \cdot \nabla u \|_{H^\infty}^2 \\
& \leq \| u_0 \|_{H^\infty}^2 + \| \nabla d_0 \|_{H^\infty}^2 + \| \nabla d_0 \|_{L^2}^2 d_0 \|_{H^\infty}^2 \\
& + \| \nabla d_0 \|_{H^\infty}^2 + \| u_0 \cdot \nabla d_0 \|_{H^\infty}^2 + \| u_0 \cdot \nabla u_0 \|_{H^\infty}^2 \} e_{C_1+C_2},
\end{align*}
$$

where, the constants $C_1, C_2$ only depend on initial data $(u_0, d_0)$.

Proof. Since $u \in C(0, T; H^\infty)$ and $\nabla d \in C(0, T; H^\infty)$, Using system (1.1)-(1.2), we can get the following basic energy inequality by standard energy estimate.

$$
(4.2) \quad \| u(t) \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + 2 \int_0^T \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla d|^2 d)^2 dx ds \\
\leq \| u_0 \|_{L^2}^2 + \| \nabla d_0 \|_{L^2}^2.
$$

Taking $D^{m}$ on the first equation of (1.1), we get

$$
(4.3) \quad \partial_t D^m u - \Delta D^m u = -D^m (\nabla P + u \cdot \nabla u + \nabla \cdot (\nabla d \nabla d)).
$$

Taking the scalar product of $D^m u$ then integrating over $\mathbb{R}^3$, we get

$$
(4.4) \quad \frac{d}{dt} \| D^m u(t) \|_{L^2}^2 + 2 \| \nabla D^m u \|_{L^2}^2 \\
\leq \| D^{2m} u \|_{L^2}^2 + C(\| \nabla P \|_{L^2}^2 + \| u \cdot \nabla u \|_{L^2}^2 + \| \nabla \cdot (\nabla d \nabla d) \|_{L^2}^2).
$$

Noticing that $-\Delta P = \nabla \cdot (u \cdot \nabla u + \nabla \cdot (\nabla d \nabla d))$, we obtain the estimate

$$
(4.5) \quad \| \nabla ^{m+1} P \|_{L^2} \leq \| \nabla^m (u \cdot \nabla u) \|_{L^2} + \| \nabla^m (\nabla \cdot (\nabla d \nabla d)) \|_{L^2} \\
\leq \| u \cdot \nabla u \|_{H^\infty} + \| \nabla d \|_{H^\infty}^2, \quad \forall m \geq 0.
$$

Together (4.4) and (4.5), we have

$$
(4.6) \quad \frac{d}{dt} \| D^m u(t) \|_{L^2}^2 + 2 \| \nabla D^m u \|_{L^2}^2 \\
\leq \| u \|_{H^\infty}^2 + C(\| u \cdot \nabla u \|_{H^\infty}^2 + \| \nabla d \|_{H^\infty}^2).
$$

Using the first equation of (1.1), we derive that

$$
\begin{align*}
D^m \{ u \cdot \nabla (u \cdot \nabla u) \} + D^m \{(u \cdot \nabla u) \cdot \nabla u \} \\
+ D^m \{ u \cdot \nabla (\nabla P) \} + D^m \{ (\nabla P) \cdot \nabla u \} \\
= -D^m \{ u \cdot \nabla (\nabla d \nabla d) \} - \{ \nabla \cdot (\nabla d \nabla d) \} \cdot \nabla u \}.
\end{align*}
$$

Repeat the similar progress of (4.3)-(4.6), we have

$$
(4.7) \quad \frac{d}{dt} \| D^m \{ u \cdot \nabla (u \cdot \nabla u) \} \|_{L^2}^2 + 2 \| \nabla D^m \{ u \cdot \nabla (u \cdot \nabla u) \} \|_{L^2}^2 \\
\leq \| u \cdot \nabla u \|_{H^\infty} + C(\| u \|_{L^2}^2 \| \nabla^2 u \|_{L^\infty}^2 + \| u \|_{L^2}^2 \| \nabla u \|_{H^\infty}^2 \\
+ \| \nabla u \|_{L^2}^2 \| \nabla u \|_{L^2}^2 + \| u \|_{L^2}^2 \| \nabla^2 P \|_{L^\infty}^2 + \| \nabla u \|_{L^2}^2 \| \nabla P \|_{L^\infty}^2 \\
+ \| u \|_{L^2}^2 \| \nabla (\nabla d \nabla d) \|_{L^\infty}^2 + \| \nabla u \|_{L^2}^2 \| \nabla \cdot (\nabla d \nabla d) \|_{L^\infty}^2).
Replacing $g$ by $|\nabla d|^2 \cdot d - u \cdot \nabla d$ in (2.16), we have that

$$
\frac{d}{dt} \| D^m \nabla d \|^2_{L^2} + 2 \| \nabla D^m \nabla d \|^2_{L^2} \\
\leq \| \nabla d \|^2_{L^\infty} + C(\| u \cdot \nabla d \|^2_{L^\infty} + \| \nabla d \|^2 d \|^2_{L^\infty}).
$$

Replacing $g$ by $|\nabla d|^2 \cdot d - u \cdot \nabla d$ in (2.19), we get

$$
\| \nabla g \cdot \nabla d \|^2_{L^2} \leq \| \nabla d \|^2_{L^2} (\| u \cdot \nabla d \|^2_{L^\infty} + \| \nabla d \|^2 d \|^2_{L^\infty}),
$$

$$
\| \nabla^2 d \|^2_{L^2} = \| \nabla^2 d \|^2_{H^1} \leq \| \nabla d \|^2_{L^2} \| \nabla d \|^2_{L^\infty},
$$

where, we use Gagliardo-Nirenberg’s inequality. By (4.11)-(4.12) and (2.16), we derive that

$$
\frac{d}{dt} \| D^m \nabla d \|^2_{L^2} + 2 \| \nabla D^m \nabla d \|^2_{L^2} \\
\leq \| \nabla d \|^2_{L^\infty} + C(\| u \cdot \nabla d \|^2_{L^\infty} + \| \nabla d \|^2 d \|^2_{L^\infty}).
$$

Replacing $g$ by $|\nabla d|^2 \cdot d - u \cdot \nabla d$ in (4.15), we have

$$
\| \nabla g \cdot \nabla d \|^2_{L^2} \leq \| \nabla d \|^4 d + \| \nabla d \|^3 u \|^2_{L^2} \\
\leq \| \nabla d \|^2_{L^\infty} \| \nabla d \|^2_{L^2} + \| u \|^2_{L^2} \| \nabla d \|^2_{L^{12}} \\
\leq \| \nabla d \|^2_{L^\infty} \| \nabla d \|^2_{H^\infty} + \| u \|_{L^2} \| u \|^2_{H^\infty} \| \nabla d \|^2_{L^2} \| \nabla d \|^2_{H^\infty},
$$

where, we use Gagliardo-Nirenberg’s inequality and the fact $\| f \|^2_{H^m} \leq \| f \|^2_{H^\infty}$. By similar computations, we have

$$
\| \nabla d \|^2 \cdot d \|^2_{L^2} \leq \| d \|^2_{L^\infty} \| \nabla d \|^2_{L^2} \| \nabla d \|^2_{H^\infty}.
$$

By similar computations, we have

$$
\| \nabla d \|^2 \cdot d \|^2_{L^2} \leq \| d \|^2_{L^\infty} \| \nabla d \|^2_{L^2} \| \nabla d \|^2_{H^\infty}.
$$

$$
\| \nabla d \|^2 \cdot d \|^2_{L^2} \leq \| d \|^2_{L^\infty} \| \nabla d \|^2_{L^2} \| \nabla d \|^2_{H^\infty}.
$$

$$
\| \nabla d \|^2 \cdot d \|^2_{L^2} \leq \| d \|^2_{L^\infty} \| \nabla d \|^2_{L^2} \| \nabla d \|^2_{H^\infty}.
$$

$$
\| \nabla d \|^2 \cdot d \|^2_{L^2} \leq \| d \|^2_{L^\infty} \| \nabla d \|^2_{L^2} \| \nabla d \|^2_{H^\infty}.
$$

$$
\| \nabla d \|^2 \cdot d \|^2_{L^2} \leq \| d \|^2_{L^\infty} \| \nabla d \|^2_{L^2} \| \nabla d \|^2_{H^\infty}.
$$
Replacing \( f \) and \( g \) by \(-u \cdot \nabla d - \nabla \cdot (\nabla d \circ \nabla d)\) and \(| \nabla d |^2 d - u \cdot \nabla d \) in (2.22) respectively. It holds that

\[
\begin{align*}
&\| f \cdot \nabla d \|_{L^2_t L^2_x} \leq \| \nabla d \|_{L^2_t L^2_x} (\| u \cdot \nabla u \|_{H^\infty} + \| \nabla d \|_{H^\infty}^2), \\
&\| u \cdot \nabla g \|_{L^2_t L^2_x} \leq \| u \cdot \nabla d \|_{H^\infty} + \| \nabla d \|_{H^\infty}^2 d \|_{H^\infty}^2, \\
&\| \nabla P \cdot \nabla d \|_{L^2_t L^2_x} \leq \| \nabla d \|_{L^2_t L^2_x} (\| u \cdot \nabla u \|_{H^\infty} + \| \nabla d \|_{H^\infty}^2), \\
&\| \partial_t u \cdot \nabla \partial_t d \|_{L^2_t L^2_x} \leq \| \nabla u \|_{L^2_t L^2_x} \| \nabla d \|_{H^\infty}.
\end{align*}
\]

where, we use (4.5) in the estimate (4.21). Using (4.19)-(4.22) and (2.22), we have

\[
\begin{align*}
&\frac{d}{dt} \| D^m (u \cdot \nabla d) \|_{L^2_t L^2_x} + 2 \| \nabla D^m (u \cdot \nabla d) \|_{L^2_t L^2_x} \\
&\leq \| u \cdot \nabla d \|_{H^\infty} + \| u \cdot \nabla u \|_{H^\infty} + \| \nabla d \|_{H^\infty}^2 + \| \nabla d \|_{H^\infty}^2 d \|_{H^\infty}^2 + \| \nabla u \|_{L^2_t L^2_x} \| \nabla d \|_{H^\infty}.
\end{align*}
\]

Let

\[
G(t) = \| u(t) \|_{H^\infty} + \| u \cdot \nabla u(t) \|_{H^\infty} + \| \nabla d(t) \|_{H^\infty}^2 + \| \nabla d(t) \|_{H^\infty}^2 d \|_{H^\infty}^2.
\]

Using (4.6), (4.8), (4.10), (4.13), (4.18), (4.23) and the definition of \( H^\infty \), we get

\[
\begin{align*}
G(t) - G(0) & \leq \int_0^t C (1 + \| u \|_{L^2_x} + \| \nabla d \|_{L^2_x} + \| \nabla u \|_{L^2_x} + \| \nabla d \|_{L^2_x}^5 + \| \nabla d \|_{L^2_x}^6 G(s) ds \end{align*}
\]

where, we use the fact \( | d | = 1 \). Noticing (4.2) and Using Gronwall’s inequality, we have

\[
G(t) \leq G(0) e^{C_1 + C_2 t}
\]

where \( C_1 \) and \( C_2 \) only depend on the initial data \((u_0, d_0)\). \qed

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