1 Introduction.

Let $X$ be an algebraic variety, let $\text{coh}(X)$ be the category of coherent sheaves on $X$. One of ways to describe the derived category $D^b(\text{coh}(X))$ is to construct an exceptional collection in it.

An object $E$ of a triangulated $\mathbb{C}$-linear category $\mathcal{C}$ is called \textit{exceptional} if $\text{Hom}(E, E) = \mathbb{C}$ and $\text{Hom}^i(E, E) = 0$ for $i \neq 0$. An ordered collection $(E_1, \ldots, E_n)$ is called \textit{exceptional} if all $E_k$ are exceptional and $\text{Hom}^i(E_k, E_l) = 0$ for $k > l$ and all $i \in \mathbb{Z}$. A collection in $\mathcal{C}$ is said to be \textit{full} if objects of the collection generate the category $\mathcal{C}$.

Of course, not all categories possess a full exceptional collection, but it is proved that bounded derived categories of sheaves on some special varieties do. We mention two results of this kind. Exceptional collections of sheaves on smooth Del Pezzo surfaces were constructed and investigated by D. Orlov and S. Kuleshov [Or, KO]. Y. Kawamata proved that full exceptional collections exist on toric varieties with quotient singularities.

In this paper we construct full exceptional collections of sheaves on some special hypersurfaces in weighted projective spaces. Namely, we consider hypersurfaces of degree $4n - 2$ in the weighted projective spaces $\mathbb{P}(1, 2n - 1, 4n - 3)$. These surfaces are singular non-toric Del Pezzo surfaces, they have a unique singular point. We consider these singular surfaces as smooth stacks in the following sense. For any commutative $\mathbb{N}$-graded algebra $S$ finitely generated over the field $\mathbb{C} = S_0$ we can define a stack structure on the projective variety $\text{Proj} S$ by the action of $\mathbb{C}^*$ on $\text{Spec} S \setminus \{0\}$ associated with the grading. A sheaf on such stack is by definition a $\mathbb{C}^*$-equivariant sheaf on $\text{Spec} S \setminus \{0\}$. According to [AKO], the category $\text{coh}^{\mathbb{C}^*}(\text{Spec} S \setminus \{0\})$ of $\mathbb{C}^*$-equivariant coherent sheaves on $\text{Spec} S \setminus \{0\}$ is equivalent to the quotient category $\text{qgr}(S) = \text{gr}(S)/\text{tors} S$, where $\text{gr}(S)$ is the category of finitely generated graded $S$-modules and $\text{tors} S$ is its subcategory of torsion modules. By Serre theorem, if $S$ is generated by its homogeneous component $S_1$, then the category $\text{qgr}(S)$ is equivalent to the category $\text{coh}(\text{Proj} S)$ of coherent sheaves on $\text{Proj} S$.

Since the surfaces we consider are embedded into the weighted projective space, they are of the form $\text{Proj} A$, where $A = \mathbb{C}[x_0, \ldots, x_3]/(f)$ is the quotient of the algebra of weighted polynomials. In our case $A_1$ does not generate $A$ and in fact the categories $\text{coh}^{\mathbb{C}^*}(\text{Spec} A \setminus \{0\}) \cong \text{gr}(A)$ and $\text{coh}(\text{Proj} A)$ are not equivalent. We construct exceptional collections in the first category, all the work is done in terms of modules over the algebra $A$.

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Any hypersurface $X = \text{Proj} S$ of degree $d$ in the weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ possesses the exceptional collection $(S, S(1), \ldots, S(\kappa - 1))$ in the category $\mathbb{D}^b(\text{qgr}(S))$, where $S = \mathbb{C}[x_0, \ldots, x_n]/(f(x))$ is the quotient algebra and $\kappa = d - \sum a_i$. In our case this gives the collection $(A, A(1), \ldots, A(2n))$, which is not full. To construct a full collection we consider modules $\chi_j$ with support in the singular point $P \in X$. They correspond to the same skyscraper sheaf $\mathcal{O}_P$ but carry different equivariant structures. Using modules $\chi_j$ we can form the collection $(A, \ldots, A(2n), \chi_{2n}, \chi_{2n-1}, \ldots, \chi_4, \chi_3)$. This collection is full but it is not exceptional. To obtain an exceptional collection we make mutations with modules $\chi_4$ and $\chi_3$ in the above collection.

Recall that left and right mutations $L_X(Y)$ and $R_Y(X)$ of objects $X$ and $Y$ in a triangulated category are defined by the triangles

$$L_X(Y) \to \text{Hom}(X,Y) \otimes X \to Y \quad \text{and} \quad X \to \text{Hom}(X,Y) \ast \otimes Y \to R_Y(X).$$

A left mutation of an object over an exceptional collection is defined by induction: $L_{(E_i)}(X) = L_{E_1}(X)$ and $L_{(E_1, \ldots, E_i)}(X) = L_{E_1}(L_{(E_2, \ldots, E_i)}(X))$; a right mutation is defined similarly. For properties of mutations see [Bo].

One of full exceptional collections in $\mathbb{D}^b(\text{qgr}(A))$ is the following:

$$(A, A(1), \ldots, A(2n - 2), G_{2n - 1}, A(2n - 1), G_{2n}, A(2n), \chi_{2n}, \chi_{2n - 1}, \ldots, \chi_6, \chi_5).$$

Here $G_{2n} = L_{A(2n)} \otimes \chi_{2n} \otimes \sum \chi_{2n - 2} \otimes \chi_8 \chi_6). \chi_4\{n - 2\}$ and $G_{2n - 1} = L_{A(2n - 1)} \otimes \chi_{2n - 1} \otimes \chi_{2n - 3} \otimes \chi_7 \chi_5\{n - 2\}$ are twisted ideals in the algebra $A$. We prove this in section 3, theorems 3.1 and 3.3.

Given an exceptional collection $(E_1, \ldots, E_m)$ we can define a graded associative algebra $A = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{1 \leq k \leq l \leq m} \text{Hom}^i(E_k, E_l)$, where multiplication is given by the composition law. If this algebra is concentrated in degree 0 (i.e., $\text{Hom}^i(E_k, E_l) = 0$ for all $k, l$ and $i \neq 0$), the triangulated category generated by the collection can be recovered from the algebra $A$. Explicitly, this category is equivalent to $\mathbb{D}^b(\text{mod} - A)$ (see [Bo]). In general case the algebra $A$ also contains a lot of information on the category generated by the collection. We compute the algebra of morphisms for one of exceptional collections in $\mathbb{D}^b(\text{qgr}(A))$ in section 4.

2 Definition and properties of basic objects.

Fix an integer $n$, $n \geq 2$.

Let $B$ be the graded algebra of polynomials $\mathbb{C}[x_0, x_1, x_2, x_3]$ generated by homogeneous variables $x_0, x_1, x_2, x_3$ of degrees $1, 2, 2n - 1, 4n - 3$ respectively.

Then $\text{Proj} B$ is the weighted projective space $\mathbb{P}(1, 2, 2n - 1, 4n - 3)$. It is a singular variety with three singular points, these points are cyclic quotient singularities of types $\frac{1}{2}(1, 1, 1)$, $\frac{1}{2n - 1}(1, 2, -1)$, $\frac{1}{4n - 3}(1, 2, 2n - 1)$.

2.1 Definition and properties of surface $X$.

Consider a hypersurface $X$ in $\mathbb{P}(1, 2, 2n - 1, 4n - 3)$ defined by a general homogeneous polynomial of degree $d = 4n - 2$. One can easily check that such surface is unique up to an automorphism of the weighted projective space. To be more precise, the following lemma holds:

Lemma 2.1. Suppose $f(x) \in B$ is a homogeneous polynomial of degree $4n - 2$ and coefficients of monomials $x_0x_3, x_2^2$ and $x_1^{2n - 1}$ in $f(x)$ are not equal to zero. Then there exists a homogeneous change of coordinates in $\mathbb{P}(1, 2, 2n - 1, 4n - 3)$ that takes $f(x)$ to $f(x') = x_0'x_3' + (x_1')^{2n - 1} + (x_2')^2$. 

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Further on we assume that
\[
f(x) = x_0 x_3 + x_1^{2n-1} + x_2^2.
\] (1)

Below we list some geometric properties of $X$.

**Proposition 2.2.**

(a) The surface $X$ is rational.
(b) The surface $X$ has unique singular point $P = (0:0:0:1)$.
(c) The point $P$ is a cyclic quotient singularity of type $\frac{1}{4n-3}(1, 4)$. The configuration of exceptional curves on the minimal resolution of $P \in X$ looks as follows:

\[-n \quad -2 \quad -2 \quad -2\]

(d) $X$ is a Del Pezzo surface with Pic $X = \mathbb{Z}$.

**Proof.** (a,b,c) The proof can be done by direct calculation in coordinates. The configuration of exceptional curves on the minimal resolution is determined by the expansion of $\frac{4n-3}{4}$ into a continuous fraction (see [Fu, sect. 2.6]):

\[
\frac{4n-3}{4} = n - \frac{1}{2 - \frac{1}{2}}.
\]

(d) Consider the intersection of $X$ with the hyperplane $x_0 = 0$. Since this intersection is a curve in $\mathbb{P}(2, 2n-1, 4n-3)$ given by the irreducible polynomial $x_1^{2n-1} + x_2^3 = 0$, it is an irreducible divisor. The complement $X \cap \{x_0 \neq 0\}$ is isomorphic to an affine plane so that Cl$(X \cap \{x_0 \neq 0\}) = 0$. We conclude that the group Cl $X$ is generated by the hyperplane section of $X$ (see [Ha, prop. II.6.5]). Therefore Pic $X = \mathbb{Z}$.

From the adjunction formula for $\mathbb{P}(1, 2, 2n - 1, 4n - 3)$ and $X$ it follows that the anticanonical divisor $-K_X$ is effective. Since Cl $X = \mathbb{Z}$, the divisor $-K_X$ is ample. \qed

Below we give an explicit construction of $X$ in terms of birational transformations.

Suppose $F_n$ is the Hirzebruch surface, $F_n = \mathbb{P}_1(\mathcal{O} \oplus \mathcal{O}(n))$. Blowing up three points as shown on the picture below, we obtain the surface $\tilde{X}$. 
Proposition 2.3. The surface $\tilde{X}$ is a minimal resolution of singularities of $X$; the exceptional divisor consists of the proper transforms of $E_0, E_1, E_2, E_3$.

Proof. The proof follows from the classification of Del Pezzo surfaces over $\mathbb{C}$ of Picard rank one with unique cyclic quotient singularity, made in [Ko, theorem 4.1]. We skip the details.

2.2 Definition and properties of basic modules.

Let $A$ be the graded algebra

$$A = B/(f) = \mathbb{C}[x_0, x_1, x_2, x_3]/(x_0x_3 + x_1^{2n-1} + x_2^3),$$

it is the homogeneous coordinate algebra of $X$. For any graded $A$-module $M$ let $M(k)$ be the same module with grading shifted by $k$: $M(k)_i = M_{k+i}$. Now we define $A$-modules that will be useful for a construction of exceptional collections.

Consider $A$-modules

$$\chi = \chi_0 = A/(x_0, x_1, x_2) \cong \mathbb{C}[x_3] \quad \text{and} \quad \chi_j = \chi(j).$$

Obviously, the support of $\chi_j$ is the singular point $P \in X$. The homogeneous components $(\chi_j)_k$ of $\chi_j$ are one-dimensional for $k \geq -j$, $k \equiv -j \pmod{4n-3}$ and zero in other cases.

There exists a monomorphism $\chi_k \twoheadrightarrow \chi_{k+4n-3}$, it is given by multiplication by $x_3$. Note that the cokernel of this monomorphism is a torsion module, hence the modules $\chi_k$ and $\chi_{k+4n-3}$ correspond to isomorphic objects in $\text{qgr}(A)$. We see that in $\text{qgr}(A)$ there exist $4n-3$ nonisomorphic objects $\chi_j$.

Define $A$-modules $Q_{j+2r,j}$ by

$$Q_{j+2r,j} = (A/(x_0, x_1^{r+1}, x_2)) (j + 2r) \cong (\mathbb{C}[x_1, x_3]/(x_1^{r+1})) (j + 2r)$$

for any integers $j$ and $r$ such that $0 \leq r < 2n-1$. As above, their support is the singular point $P \in X$. Homogeneous components of $Q_{j+2r,j}$ of degree $k$ are one-dimensional for $k \geq -(j + 2r)$, $k \equiv -(j + 2r), \ldots, -(j + 2), -j \pmod{4n-3}$ and zero in other cases. It follows from the definitions that $\chi_j$ and $Q_{j,j}$ are isomorphic.

Consider the sequences ($r \geq 0, s > 0, r + s < 2n-1$)

$$0 \rightarrow Q_{j+2r,j} \rightarrow Q_{j+2r+2s,j} \rightarrow Q_{j+2r+2s,j+2r+2} \rightarrow 0$$

where the first map is multiplication by $x_1^s$ and the second one is a factorization. It can be checked that these sequences are exact. As an important special case of (2), we obtain the following exact sequences:

$$0 \rightarrow Q_{j+2r-2,j} \rightarrow Q_{j+2r,j} \rightarrow \chi_{j+2r} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \chi_j \rightarrow Q_{j+2r,j} \rightarrow Q_{j+2r,j+2} \rightarrow 0.$$

From (3) it follows that the module $Q_{j+2k,j}$ has a filtration

$$0 \subset Q_{j,j} \subset Q_{j+2,j} \subset Q_{j+4,j} \subset \ldots \subset Q_{j+2k-2,j} \subset Q_{j+2k,j}$$

with quotients isomorphic to $\chi_j, \chi_{j+2}, \chi_{j+4}, \ldots, \chi_{j+2k-2}, \chi_{j+2k}$. 

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Consider the sequence
\[ 0 \to \chi_j \xrightarrow{x_2} (A/(x_0, x_1)) (j+2n-1) \to \chi_{j+2n-1} \to 0, \] where the first map is multiplication by \( x_2 \) and the second one is a factorization. Since \( A/(x_0, x_1) \cong \mathbb{C}[x_2, x_3]/(x_2^2) \), this sequence is exact.

**Lemma 2.4.** There exist exact sequences
\[ 0 \to A(-3) \to A(-1) \oplus A(-2) \to A \to A/(x_0, x_1) \to 0 \quad \text{and} \quad 0 \to A(-2n) \to A(-1) \oplus A(-2n+1) \to A \to Q_{0,-4n+4} \to 0. \]

**Proof.** We can take the Kozhul complexes for \((x_0, x_1)\) and \((x_0, x_2)\) as required sequences. Indeed, the pairs \((x_0, x_1)\) and \((x_0, x_2)\) are regular; therefore the Kozhul complexes are resolutions of \(A/(x_0, x_1)\) and \(A/(x_0, x_2)\); \( A/(x_0, x_1^2, x_2) \cong Q_{0,-4n+4} \). The latter isomorphisms follow from the definitions of \(A\) and \(Q_{0,-4n+4}\). \(\square\)

### 2.3 Ext groups for basic modules.

In this section we calculate Ext groups for objects \(A(i)\) and \(\chi_j\) in the category \(\text{qgr}(A)\).

Recall that \(\text{qgr}(A)\) is the quotient category \(\text{gr}(A)/\text{tors}A\), where \(\text{gr}(A)\) is the category of finitely generated graded \(A\)-modules and \(\text{tors}A \subset \text{gr}(A)\) is its dense subcategory of torsion modules. Similarly, let \(\text{Gr}(A)\) be the category of all graded \(A\)-modules, \(\text{Tors}A \subset \text{Gr}(A)\) its subcategory of torsion modules, and \(\text{QGr}(A)\) the quotient category \(\text{Gr}(A)/\text{Tors}A\). Since the category \(\text{QGr}(A)\) has enough injective objects, one can define the functors \(\text{Ext}^i(M, \cdot)\) on \(\text{QGr}(A)\) as derived functors of \(\text{Hom}(M, \cdot)\). This also defines functors \(\text{Ext}^i\) on \(\text{qgr}(A)\), see \([AZ]\) for details.

The following well-known (see, e.g., \([Ba]\)) result about Ext groups on weighted projective spaces is needed for the sequel.

**Proposition 2.5.** Suppose \(B = \mathbb{C}[x_0, x_1, \ldots, x_m]\) is the graded algebra of polynomials in variables \(x_i\) of degrees \(a_i\) respectively; then the following equalities hold in \(\text{qgr}(B)\):

\begin{align*}
\text{Hom}(B, B(k)) &= B_k \text{ for all } k \geq 0, \text{ Hom}(B, B(k)) = 0 \text{ otherwise}; \\
\text{Ext}^i(B, B(k)) &= 0 \text{ for all } k \text{ and } i \text{ such that } 1 \leq i \leq m - 1; \\
\text{Ext}^m(B, B(k)) &= B^*_{-s-k} \text{ for all } k \leq -s, \text{ where } s = \sum a_i; \text{ Ext}^m(B, B(k)) = 0 \text{ in other cases.}
\end{align*}

Moreover, there exist natural isomorphisms for all \(i = 0, \ldots, m\):

\[ \text{Ext}^i(B, N) \cong \text{Ext}^{m-i}(N, B(-s))^*, \text{ where } s = \sum a_i. \]

The next proposition is the main result of the subsection.

**Proposition 2.6.** Let \(A\) be the algebra defined in subsection \([2.2]\). Then the following equalities hold in \(\text{qgr}(A)\):

\begin{enumerate}
\item[(a)] \(\text{Hom}(A(k), A(l)) = A_{l-k} \text{ for all } k \leq l,\)
\item[(b)] \(\text{Ext}^2(A(k), A(l)) = A^*_{k-l-(2n+1)} \text{ if } k-l \geq 2n+1,\)
\item[(c)] \(\text{Ext}^i(A(k), A(l)) = 0 \text{ otherwise.}\)
\end{enumerate}
(b) \( \text{Hom}(A(k), \chi_j) = \mathbb{C} \) if \( k \equiv j \pmod{4n - 3} \),
\[ \text{Ext}^i(A(k), \chi_j) = 0 \] in other cases.

(c) \( \text{Ext}^2(\chi_k, A(j)) = \mathbb{C} \) if \( j \equiv k - (2n + 1) \pmod{4n - 3} \),
\[ \text{Ext}^i(\chi_k, A(j)) = 0 \] in other cases.

(d) \( \text{Hom}(\chi_k, \chi_j) = \mathbb{C} \) if \( k \equiv j \pmod{4n - 3} \),
\[ \text{Ext}^1(\chi_k, \chi_j) = \mathbb{C} \] if \( j \equiv k - 2 \pmod{4n - 3} \) or \( j \equiv (2n - 1) \pmod{4n - 3} \),
\[ \text{Ext}^2(\chi_k, \chi_j) = \mathbb{C} \] if \( j \equiv k - (2n + 1) \pmod{4n - 3} \),
\[ \text{Ext}^i(\chi_k, \chi_j) = 0 \] otherwise.

Proof. Obviously, \( \text{Ext}^i(M, N) \cong \text{Ext}^i(M(j), N(j)) \), therefore it suffices to consider the case \( k = 0 \). First we prove the following lemma.

**Lemma 2.7.** Suppose \( B = \mathbb{C}[x_0, \ldots, x_m] \) is the polinomial algebra with grading \( \deg(x_i) = a_i, f \in B \) is a homogeneous polinomial of degree \( d \), and \( A = B/(f) \) is the quotient algebra. Then for any \( A \)-module \( M \) there exists a natural isomorphism \( \text{Ext}^i_{\text{QGr}(A)}(A, M) \cong \text{Ext}^i_{\text{QGr}(B)}(B, M) \).

Proof. Define the functors \( \sigma^*: \text{Gr}(B) \to \text{Gr}(A) \) by \( \sigma^*(N) = N \otimes_B A \) and \( \sigma'_*: \text{Gr}(A) \to \text{Gr}(B) \) by \( \sigma'_*(M) = M \). Then \( \sigma^* \) is the left adjoint functor to \( \sigma'_* \). Functors \( \sigma^* \) and \( \sigma'_* \) can be extended to adjoint functors \( \sigma^* \) and \( \sigma_* \), defined on categories \( \text{QGr}(B) \) and \( \text{QGr}(A) \). Since \( \text{QGr}(B) \) has enough \( \sigma^* \)-acyclic objects (for any \( M \in \text{QGr}(B) \) there exists an epimorphism \( \oplus \alpha B(k_{\alpha}) \to M \) ), the derived functor \( L\sigma^* \) is well-defined; since \( \sigma_* \) is exact, the derived functor \( R\sigma_* = L\sigma_* \) is well-defined. According to [Ko lemma 15.6], the functors \( L\sigma^* \) and \( R\sigma_* \) are adjoint, i.e., for any objects \( X \in \text{D}^+(\text{QGr}(A)) \) and \( Y \in \text{D}^-(\text{QGr}(B)) \) we have
\[ \text{Hom}_{\text{D}(\text{QGr}(A))}(L\sigma^*(Y), X) \cong \text{Hom}_{\text{D}(\text{QGr}(B))}(Y, R\sigma_*(X)). \] (7)

The required isomorphism is obtained by applying the above formula to the \((-j)\)-complex \( X = M[j] \) and the 0-complex \( Y = B[0] \).

Now we can prove proposition 2.6
(a) From the above lemma it follows that \( \text{Ext}^i_{\text{QGr}(A)}(A, A(l)) \cong \text{Ext}^i_{\text{QGr}(B)}(B, A(l)) \). We compute groups \( \text{Ext}^i_{\text{QGr}(B)}(B, A(l)) \) by applying the functor \( \text{Hom}(B, \cdot) \) to the exact sequence
\[ 0 \to B(l-(4n-2)) \xrightarrow{f(x)} B(l) \to A(l) \to 0. \]

Groups \( \text{Ext}^i(B, B(j)) \) are calculated in proposition 2.5, we obtain the result by straightforward calculations.

(b, d) Let \( D_{x_3} = \{ x_3 \neq 0 \} \) be the neighbourhood of the singular point \( P \in X \), let \( s: D_{x_3} \to X \) be its embedding into \( X \); \( s \) corresponds to the localization morphism \( A \to A[x_3^{-1}] \).

Consider the following functors associated with the embedding \( s \). The direct image functor \( s_* : \text{Gr}(A[x_3^{-1}]) \to \text{QGr}(A) \) is defined by \( s_*(M) = M \) and the inverse image functor \( s^*: \text{QGr}(A) \to \text{Gr}(A[x_3^{-1}]) \) is the localization. Note that the localization of a torsion module is zero; this shows that \( s^* \) is well-defined. Since functors \( s_* \) and \( s^* \) are exact and \( s_* \) is the right adjoint functor to \( s^* \), we have natural isomorphisms
\[ R^i \text{Hom}_{\text{QGr}(A)}(M, s_*(N)) \cong R^i \text{Hom}_{A[x_3^{-1}]}(s^*(M), N) \] (8)
for any $A$-module $M$ and any $A[x_3^{-1}]$-module $N$.

Now consider the module $X = A[x_3^{-1}]/(x_1, x_2) \cong \mathbb{C}[x_3, x_3^{-1}]$ over the algebra $A[x_3^{-1}]$. Note that the submodule of $s_*(X)$ generated by its components of nonnegative degree, is isomorphic to the module $\chi$. Since the quotient module $s_*(X)/\chi$ is a torsion module, we see that $\chi$ and $s_*(X)$ are isomorphic in the category $\text{QGr}(A)$. For the same reason, $\chi(j) \cong s_*(X(j))$.

(b) From formula (8) for $M = A$ and $N = X(j)$ it follows that $\text{Ext}^i_{\text{QGr}(A)}(A, \chi j) \cong \text{Ext}^i_{\text{Gr}(A[x_3^{-1}])}(A[x_3^{-1}], X(j))$. But the functor $\text{Hom}(A[x_3^{-1}], \cdot)$ is exact on the category of graded $A[x_3^{-1}]$-modules, in fact, we have $\text{Hom}(A[x_3^{-1}], N) = N_0$. This makes further calculation trivial.

(d) Now we compute the groups $\text{Ext}(\cdot, X(j))$. Since $s^*s_*X \cong X$, applying formula (8) to $M = s_*(X)$, $N = X(j)$ we obtain $\text{Ext}^i_{\text{QGr}(A)}(\chi, \chi j) \cong \text{Ext}^i_{\text{Gr}(A[x_3^{-1}])}(X, X(j))$.

Note that the algebra $A[x_3^{-1}]$ is isomorphic to $\mathbb{C}[x_1, x_2, x_3, x_3^{-1}]$ and the following Kozhul complex is a free resolution of the module $X = A[x_3^{-1}]/(x_1, x_2)$

$$0 \to A[x_3^{-1}]/(-2n - 1) \to A[x_3^{-1}]/(-2n + 1) \oplus A[x_3^{-1}]/(-2) \to A[x_3^{-1}] \to X \to 0.$$ 

According to the definition of $\text{Ext}(X, X(j))$, we apply the functor $\text{Hom}(\cdot, X(j))$ to the above resolution and take homologies. Using the results of part (b), we get:

$\text{Hom}(X, X) = \mathbb{C}$, $\text{Ext}^1(X, X(-2)) = \mathbb{C}$, $\text{Ext}^1(X, X(-2n + 1)) = \mathbb{C}$, $\text{Ext}^2(X, X(-2n - 1)) = \mathbb{C}$, other $\text{Ext}$ groups are zero. Part (d) of the proposition is proved.

The required equalities follow from part (b) and the following lemma.

**Lemma 2.8.** Suppose $B = \mathbb{C}[x_0, \ldots, x_m]$ is the polynomial algebra with grading $\text{deg}(x_i) = \alpha_i$, $f \in B$ is a homogeneous polynomial of degree $d$, and $A = B/(f)$ is the quotient algebra. Then for any $A$-module $M$ there exists a natural isomorphism

$$\text{Ext}^i_{\text{QGr}(A)}(A, M) \cong \text{Ext}^i_{\text{QGr}(B)}(M, A(d - s)),$$

where $s = \sum \alpha_i$.

**Proof.** The proof is by reduction to duality for weighted projective space. We have

$$\text{Ext}^i_{\text{QGr}(A)}(A, N) \cong \text{Ext}^i_{\text{QGr}(B)}(B, N) \cong \text{Ext}^i_{\text{QGr}(B)}(N, B(-s))^*.$$  \hfill (9)

The first isomorphism above follows from lemma 2.8 and the second one follows from proposition 2.5.

Define the functor $\sigma^d$: $\text{Gr}(B) \to \text{Gr}(A)$ by $\sigma^d(N) = \oplus_{k \in \mathbb{Z}} \text{Hom}_{\text{Gr}(B)}(A, N(k))[-k] = \text{Hom}^B(A, N)$. Then $\sigma^d$ is the right adjoint to $\sigma'_d$. The functors $\sigma'_s$ and $\sigma^d$ can be extended to adjoint functors $\sigma_*$ and $\sigma^!$, defined on the categories $\text{QGr}(A)$ and $\text{QGr}(B)$. Since $\text{QGr}(B)$ has enough injective objects, the derived functor $R\sigma^!: \text{D}^+(\text{QGr}(B)) \to \text{D}^+(\text{QGr}(A))$ is well-defined. From [Ku lemma 15.6] it follows that the derived functors $L\sigma_*$ and $R\sigma^!$ are adjoint, i.e., for any objects $X \in \text{D}^-(\text{QGr}(A))$ and $Y \in \text{D}^+(\text{QGr}(B))$ we have

$$\text{Hom}_{\text{D}(\text{QGr}(B))}(L\sigma_*(X), Y) \cong \text{Hom}_{\text{D}(\text{QGr}(A))}(X, R\sigma^!(Y))$$  \hfill (10)

If we put $X = N[0]$ and $Y = B(-s)[j]$ in formula (10), we obtain

$$\text{Ext}^j_{\text{QGr}(B)}(N, B(-s)) \cong \text{Hom}_{\text{D}(\text{QGr}(A))}(N[0], R\sigma^!(B(-s)[j])).$$  \hfill (11)

We need to compute $R^i\sigma^!(B(-s)[j]) = R^{i+j}(\sigma^!B(-s)) = R^{i+j}\text{Hom}(A, B(-s)).$
First we calculate the groups $R^i \text{Hom}(B(r), B(-s))$ for any $r$. From definitions it follows that $R^i \text{Hom}(B(r), B(-s)) = \oplus_{k \in \mathbb{Z}} \text{Ext}^i_{\text{QGr}(B)}(B(r), B(-s+k))[-k]$. Suppose $i > 0$. Then $\text{Ext}^i(B(r), B(-s+k)) = 0$ for $k \gg 0$ by proposition 2.5. Hence the object $R^i \text{Hom}(B(r), B(-s))$ equals zero in the quotient category $\text{QGr}(B)$. For $i = 0$ we have $\text{Hom}(B(r), B(-s)) = B(-s-r)$ by proposition 2.5.

Now consider the exact sequence $0 \to B(-d) \xrightarrow{j} B \to A \to 0$. Applying the functor $\text{Hom}(\cdot, B(-s))$ to this sequence, we obtain a long exact sequence of derived functors. We already know terms $R^i \text{Hom}(B, B(-s))$ and $R^i \text{Hom}(B(-d), B(-s))$; finally we get

$$\text{Hom}(A, B(-s)) = 0, \quad R^1 \text{Hom}(A, B(-s)) = A(d-s), \quad R^i \text{Hom}(A, B(-s)) = 0 \quad \text{for } i > 1.$$ 

Therefore, the object $R\sigma^i(B(-s)) \in D^+(\text{QGr}(A))$ is the 1-complex $A(d-s)[-1]$.

Combining (9) and (11), we have

$$\text{Ext}^i_{\text{QGr}(A)}(A, N) \cong \text{Hom}_{D(\text{QGr}(A))}(N[0], A(d-s)[m-i-1]) \cong \text{Ext}^m_{\text{QGr}(A)}(N, A(d-s))^*.$$ 

This completes the proof of the lemma. \hfill \Box

Proposition 2.6 is completely proved. \hfill \Box

**Lemma 2.9.** The following maps, given by Yoneda multiplication:

$$\text{Hom}(A(j), \chi_j) \otimes \text{Ext}^2(\chi_j+2n+1, A(j)) \to \text{Ext}^2(\chi_j+2n+1, \chi_j) \quad \text{and}$$

$$\text{Ext}^2(\chi_j+2n+1, A(j)) \otimes \text{Hom}(A(j + 2n + 1), \chi_j+2n+1) \to \text{Ext}^2(A(j + 2n + 1), A(j))$$

are nontrivial.

**Proof.** Let us prove that the first map is nontrivial. For the second map the proof is analogous. Consider exact sequences from subsection 2.2 twisted by $j$:

$$0 \to A(j - 3) \to A(j - 1) \oplus A(j - 2) \to A(j) \to (A/(x_0, x_1))(j) \to 0 \quad \text{and}$$

$$0 \to \chi_j+2n-2 \to (A/(x_0, x_1))(j) \to \chi_j \to 0.$$

We claim that the quotient epimorphisms $A(j) \to (A/(x_0, x_1))(j)$ and $(A/(x_0, x_1))(j) \to \chi_j$ induce isomorphisms between groups $\text{Ext}^2(\chi_j+2n+1, M)$, where $M = A(j)$, $(A/(x_0, x_1))(j)$ and $\chi_j$. To check this, it suffices to apply the functor $\text{Hom}(\chi_j+2n+1, \cdot)$ to the above sequences and to use proposition 2.6. \hfill \Box

### 3 Construction of exceptional collections.

In this section we construct full exceptional collections in the bounded derived category of $\text{qgr}(A)$, where $A$ is the algebra, defined in subsection 2.2.

Let $G_j = (x_0, x_1^{j-1}, x_2)(j)$ and $H_j = (x_0, x_1, x_2)(j)$ be the twisted ideals in $A$.

**Theorem 3.1.** The collection of objects

$$(A, A(1), A(2), \ldots, A(2n - 1), G_{2n}, A(2n), H_{2n+1}, \chi_{2n}, \chi_{2n-1}, \ldots, \chi_6, \chi_5)$$

in $\text{qgr}(A)$ is exceptional for $n > 2$. 

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Lemma 3.2. For any $0 \leq k \leq n - 2$ the following statements hold:

(a) $Q_{j+2k,j}[-k] = L_{(\chi_{j+2k},\chi_{j+2k-2},\ldots,\chi_{j+2})}(\chi_j)$.

(b) $\text{Ext}^i(Q_{j+2k,j}, \chi_{j+2l}) = 0$ for all $i$ and $k, l = 0, \ldots, n - 2$ except for two cases:
   - $\text{Hom}(Q_{j+2k,j}, \chi_{j+2k}) = \mathbb{C}$, this space is generated by the epimorphism $Q_{j+2k,j} \to \chi_{j+2k}$;
   - $\text{Ext}^2(Q_{j+2k,j}, \chi_{j+2n-4}) = \mathbb{C}$ if $0 \leq k \leq n - 2$.

(c) $\text{Ext}^i(\chi_{j+2k+2}, Q_{j+2k,j}) = 0$ if $i \neq 1$;
   - $\text{Ext}^1(\chi_{j+2k+2}, Q_{j+2k,j})$ is represented by an extension of type (5);
   - the composition $\text{Hom}(Q_{j+2k,j}, \chi_{j+2k}) \otimes \text{Ext}^1(\chi_{j+2k+2}, Q_{j+2k,j}) \to \text{Ext}^1(\chi_{j+2k+2}, \chi_{j+2k})$ is nontrivial.

(d) For $k < n - 2$ the collection of objects $(\chi_{j+2k}, \chi_{j+2k-2}, \ldots, \chi_{j+2}, \chi_j)$ in $\text{qgr}(A)$ is exceptional. The object $Q_{j+2k,j} = L_{(\chi_{j+2k},\chi_{j+2k-2},\ldots,\chi_{j+2})}(\chi_j)[k]$ is exceptional.

Proof. The proof is by induction on $k$. Case $k = 0$ is trivial. Assume that the lemma is proved for $k - 1$.

(a) By the induction hypothesis we have $Q_{j+2k-2,j}[1-k] = L_{(\chi_{j+2k-2},\ldots,\chi_{j+2})}(\chi_j)$. We need to check that $L_{\chi_{j+2k}}(Q_{j+2k-2,j}) = Q_{j+2k,j}[-1]$. According to part (c) of the inductive assumption, the mutation $L_{\chi_{j+2k}}(Q_{j+2k-2,j})$ is given by the triangle $Q_{j+2k,j}[-1] \to \chi_{j+2k}[-1] \to Q_{j+2k-2,j}$, which is defined by the exact sequence

$$0 \to Q_{j+2k-2,j} \to Q_{j+2k,j} \to \chi_{j+2k} \to 0. \quad (12)$$

(b) To prove this, we apply the functor $\text{Hom}(_, \chi_{j+2l})$ to the exact sequence (12), taking into account proposition 2.6 and part (b) of the inductive assumption. In the case $l = k - 1$, we use part (c) of the inductive assumption to check that the arising morphism is nonzero.

(c) Since the object $Q_{j+2k+2,j}$ has a filtration (5), it follows from proposition 2.6 that $\text{Ext}^1(\chi_{j+2k+2}, Q_{j+2k-2,j}) = 0$. Now it remains to apply the functor $\text{Hom}(\chi_{j+2k+2}, _)\to \text{Ext}^i(\chi_{j+2k+2}, \chi_{j+2k})$ to the exact sequence (12).

Part (d) follows from proposition 2.6 and general properties of mutations in exceptional collections, see [15]. Lemma 3.2 is proved.

By definitions of the modules $G_{2n}, Q_{2n,4},$ and $H_{2n+1}$, we have the following exact sequences:

$$0 \to G_{2n} \to A(2n) \to Q_{2n,4} \to 0 \quad \text{and} \quad (13)$$

$$0 \to H_{2n+1} \to A(2n + 1) \to \chi_{2n+1} \to 0. \quad (14)$$

It can be easily checked (see lemma 3.2) that sequences (13) and (14) determine mutations $H_{2n+1} = L_{A(2n+1)}(\chi_{2n+1})$ and $G_{2n} = L_{A(2n)}(Q_{2n,4}) = L_{(A(2n)\cdot \chi_{2n},\chi_{2n-2},\ldots,\chi_6)}(\chi_4)[n - 2]$.

Lemma 3.3. Objects $G_{2n}$ and (for $n > 2$) $H_{2n+1}$ are exceptional in the category $\text{qgr}(A)$.  

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Proof. Since $H_{2n+1}$ is the mutation $L_{A(2n+1)}(\chi_{2n+1})$ in the exceptional (for $n > 2$) collection $(A(2n + 1), \chi_{2n+1})$, it is exceptional. Thus, we must show that $\text{Hom}(G_{2n}, G_{2n}) = \mathbb{C}$ and $\text{Ext}^i(G_{2n}, G_{2n}) = 0$ for $i \neq 0$. We do this in 6 steps.

1. $\text{Ext}^i(A(2n), G_{2n}) = 0$.
   This follows from $G_{2n} = L_{A(2n)}(Q_{2n,4})$.

2. $\text{Ext}^i(Q_{2n,4}, Q_{2n-2,4}) = 0$.
   To prove this, consider the long exact sequence obtained from the sequence of the type (3).

   $$0 \to Q_{2n-2,4} \to Q_{2n,4} \to \chi_{2n} \to 0$$  \hspace{1cm} (15)

   by applying the functor $\text{Hom}(\cdot, Q_{2n-2,4})$. All required $\text{Ext}$ groups are computed in lemma 3.2; d.

3. $\text{Hom}(Q_{2n,4}, Q_{2n,4}) = \mathbb{C}$, $\text{Ext}^2(Q_{2n,4}, Q_{2n,4}) = \mathbb{C}$, $\text{Ext}^i(Q_{2n,4}, Q_{2n,4}) = 0$ for $i \neq 0, 2$.
   This can be proved by applying the functor $\text{Hom}(Q_{2n,4}, \cdot)$ to the exact sequence (15). Take into account lemma 3.2, and results of step 2.

4. $\text{Ext}^2(Q_{2n,4}, A(2n)) = \mathbb{C}$, $\text{Ext}^i(Q_{2n,4}, A(2n)) = 0$ for $i \neq 2$,
   moreover the map $\text{Ext}^2(Q_{2n,4}, A(2n)) \otimes \text{Hom}(\chi_4, Q_{2n,4}) \to \text{Ext}^2(\chi_4, A(2n))$ is nontrivial.
   Note that $Q_{2n,6,4} \in \{x_6, x_8, \ldots, \chi_{2n}\}$ and $A(2n) \in \{x_6, x_8, \ldots, \chi_{2n}\}^\perp$. It follows that $\text{Ext}^i(Q_{2n,6,4}, A(2n)) = 0$ for all $i$. Now apply the functor $\text{Hom}(\cdot, A(2n))$ to the exact sequence $0 \to \chi_4 \to Q_{2n,4} \to Q_{2n,6} \to 0$.

5. $\text{Ext}^i(Q_{2n,4}, G_{2n}) = \mathbb{C}$, $\text{Ext}^i(Q_{2n,4}, G_{2n}) = 0$ for $i \neq 1$.

   Apply the functor $\text{Hom}(Q_{2n,4}, \cdot)$ to the exact sequence (13). To obtain the result, we must check that the composition $Q_{2n,4} \to A(2n)[2] \to Q_{2n,4}[2]$ is nonzero. A fortiori, it is sufficient to check that the following composition of nontrivial morphisms $\chi_4 \to Q_{2n,4} \to A(2n)[2] \to Q_{2n,4}[2] \to \chi_{2n}$ is nonzero. The composition $\chi_4 \to Q_{2n,4} \to A(2n)[2]$ is nonzero by step 4 and the composition $A(2n) \to Q_{2n,4} \to \chi_{2n}$ is nonzero by definition. Finally, the composition $\chi_4 \to A(2n)[2] \to \chi_{2n}[2]$ is nonzero by lemma 2.9.

6. $G_{2n}$ is an exceptional object.
   Apply the functor $\text{Hom}(\cdot, G_{2n})$ to the exact sequence (13). Using results of steps 1 and 5, we get $\text{Hom}(G_{2n}, G_{2n}) = \mathbb{C}$, $\text{Ext}^i(G_{2n}, G_{2n}) = 0$ for $i \neq 0$.
   Lemma 3.3 is proved.

Proof of theorem 3.7. It follows from proposition 2.6 that the collection $(A, A(1), \ldots, A(2n - 1), A(2n), \chi_{2n}, \chi_{2n-1}, \ldots, \chi_6, \chi_5)$ is exceptional. By lemma 3.3 objects $G_{2n}$ and $H_{2n+1}$ are exceptional. It remains to check that the following groups vanish:

1. $\text{Ext}^i(G_{2n}, A(k)) = 0$ for $k = 0, \ldots, 2n - 1$.
   This is obvious. Actually, by construction $G_{2n}$ is an object of the subcategory

   $$\langle A(2n), \chi_{2n}, \chi_{2n-2}, \ldots, \chi_6, \chi_4 \rangle$$ \hspace{1cm} (16)

   on the other hand, the modules $A, A(1), \ldots, A(2n - 1)$ are right orthogonal to this subcategory.

2. $\text{Ext}^i(\chi_j, G_{2n}) = 0$ for $j = 5, 6, \ldots, 2n - 1, 2n$.
   For even $j$ this follows from $G_{2n} = L_{A(2n)}(A\chi_{2n}, \chi_{2n-2}, \ldots, \chi_6)(\chi_4)[n - 2]$ and general properties of mutations, see [3.6]. For odd $j$ we note that by proposition 2.6, $\chi_j$ is left orthogonal to the subcategory (13) and $G_{2n}$ belongs to this subcategory.

3. $\text{Ext}^i(A(2n), G_{2n}) = 0$.
   This is proved in step 1 of lemma 3.3.

4. $\text{Ext}^i(H_{2n+1}, A(k)) = 0$ for $k = 1, \ldots, 2n$, $\text{Ext}^i(\chi_j, H_{2n+1}) = 0$ for $j = 6, \ldots, 2n$. 


This follows from the exact sequence \(14\) and proposition 2.6.

5. \(\text{Ext}^i(H_{2n+1},A) = 0\) and \(\text{Ext}^i(\chi_5,H_{2n+1}) = 0\).

Apply the functors \(\text{Hom}(\cdot,A)\) and \(\text{Hom}(\chi_5,\cdot)\) to the exact sequence \(13\). To obtain the result, we should show that the compositions of nontrivial maps \(A(2n+1) \to \chi_{2n+1} \to A[2]\) and \(\chi_5 \to A(2n+1)[2] \to \chi_5[2]\) are nontrivial. This is proved in lemma 2.9.

6. \(\text{Ext}^i(H_{2n+1},G_{2n}) = 0\).

Indeed, we have \(G_{2n} \in \langle A(2n),\chi_{2n},\ldots,\chi_6,\chi_4 \rangle \subset \langle A(2n+1),\chi_{2n+1} \rangle^\perp \ni H_{2n+1}\).

This finishes the proof of theorem 3.1. \(\square\)

**Remark.** It is not hard to check that for all \(n \geq 2\) the collection

\[
(A, A(1), A(2), \ldots, A(2n-2), G_{2n-1}, A(2n-1), G_{2n}, A(2n), \chi_{2n}, \chi_{2n-1}, \ldots, \chi_6, \chi_5)
\]

is exceptional. \(\square\)

**Theorem 3.4.** The exceptional collections

\[
(A, A(1), A(2), \ldots, A(2n-1), G_{2n}, A(2n), H_{2n+1}, \chi_{2n}, \chi_{2n-1}, \ldots, \chi_6, \chi_5) \quad \text{and} \quad (A, A(1), A(2), \ldots, A(2n-2), G_{2n-1}, A(2n-1), G_{2n}, A(2n), \chi_{2n}, \chi_{2n-1}, \ldots, \chi_6, \chi_5)
\]

in the category \(D^b(qgr(A))\) are full.

**Proof.** First we prove that the collection \(18\) is full.

Let \(D_0\) be the subcategory of \(D^b(qgr(A))\) generated by objects of the collection \(18\). We claim that the modules \(A(k)\) for all \(k \in \mathbb{Z}\) belong to \(D_0\).

As it was pointed before lemma 3.3, \(G_{2n} = \mathbb{L}_{(A(2n),\chi_{2n},\chi_{2n-2},\ldots,\chi_6)}(\chi_4)[n-2]\), hence \(\chi_4 \in D_0\). Similarly, \(G_{2n-1} = \mathbb{L}_{(A(2n-1),\chi_{2n-1},\chi_{2n-3},\ldots,\chi_5)}(\chi_3)[n-2]\) and \(\chi_3 \in D_0\).

Consider the first exact sequence from lemma 2.4. Twisting it by \(k\), we obtain

\[
0 \to A(k-3) \to A(k-1) \oplus A(k-2) \to A(k) \to (A/(x_0,x_1))(k) \to 0.
\]

For \(k = 3, \ldots, 2n\) the terms \(A(k-3), A(k-1) \oplus A(k-2)\), and \(A(k)\) of the complex \(19\) belong to \(D_0\), therefore we get \((A/(x_0,x_1))(k) \in D_0\). Further, consider the exact sequence \(0 \to \chi_{k-2n+1} \to (A/(x_0,x_1))(k) \to \chi_k \to 0\) of type \(16\). For \(k = 3, \ldots, 2n\) its terms \(\chi_{k-2n+1}\) and \((A/(x_0,x_1))(k)\) belong to \(D_0\), it follows that \(\chi_{k-2n+1} \cong \chi_{k+2n-2} \in D_0\). Thus the objects \(\chi_{2n+1}, \chi_{2n+2}, \ldots, \chi_{4n-4}, \chi_{4n-3} = \chi_0, \chi_1\) belong to the subcategory \(D_0\). Let’s prove that \(D_0\) contains \(\chi_2\).

Consider the second exact sequence from lemma 2.4. Twisting it by \(2n\), we get \(0 \to A \to A(2n-1) \oplus A(1) \to A(2n) \to Q_{2n-2n+4} \to 0\). Since three left terms in this sequence belong to the subcategory \(D_0\), the right term \(Q_{2n-2n+4}\) also belongs to \(D_0\). We know that \(Q_{2n-2n+4}\) has a filtration \(15\) with quotients isomorphic to the modules \(\chi_{2n+4}, \chi_{2n+6}, \ldots, \chi_2, \chi_0, \chi_2, \ldots, \chi_{2n}\). All the above modules, except \(\chi_2\), belong to \(D_0\). Therefore \(\chi_2 \in D_0\).

We see that the subcategory \(D_0\) contains the objects \(\chi_j\) for all \(j\). It follows that \(D_0\) contains the objects \((A/(x_0,x_1))(k)\) for all \(k\). Now consider the complex \(19\). We proved that for any integer \(k\) the right term of \(19\) belongs to \(D_0\). Taking \(k = 2n+1, 2n+2, \ldots\) and arguing as above we deduce that objects \(A(2n+1), A(2n+2), \ldots\) belong to the subcategory \(D_0\). Similarly, putting \(k = 2, 1, 0, -1, \ldots\) we show that \(D_0\) contains \(A(-1), A(-2), \ldots\).
We want to prove that the subcategory $D_0$ is equivalent to the whole category $D^b(qgr(A))$. Since a category generated by an exceptional collection is admissible (see [Bo]), it suffices to show that the right orthogonal $D_0^\perp = \{ Y \in D^b(qgr(A)) \mid \text{Hom}(D_0, Y) = 0 \}$ to $D_0$ is zero.

Assume the converse. Let $Y$ be an object in $D_0^\perp$ such that $Y \neq 0$. Choose an integer $j_0$ such that $H^{j_0}(Y) \neq 0$. From [AZ, prop. 7.4(1), 3.11(3)] it follows that for any finitely generated graded $A$-module $M$ for $k \gg 0$ we have $\text{Hom}_{qgr(A)}(A(-k), M) = M_k$ and $\text{Ext}^i_{qgr(A)}(A(-k), M) = 0$ if $i > 0$. Choose an integer $k$ such that $\text{Ext}^i(\langle -k \rangle, H^j(Y)) = 0$ if $i > 0$, $j \in \mathbb{Z}$ and $\text{Hom}(A(-k), H^{j_0}(Y)) \neq 0$. Use the spectral sequence with $E^{pq}_2 = \text{Ext}^q(A(-k), H^p(Y))$ to compute $\text{Hom}^q(A(-k), Y)$. By the choice of $k$, $E^{pq}_2 = 0$ for $q > 0$. Thus the spectral sequence degenerates in the second term, $E^{pq}_2 = E^{pq}_\infty$, and we get $\text{Hom}^{j_0}(A(-k), Y) = E^{j_00}_2 = \text{Hom}_{qgr(A)}(A(-k), H^{j_0}(Y)) \neq 0$. This contradiction concludes the proof.

Now we prove that the collection (17) is full. Let us show that the mutation $G' = L_{A(2n-1), G_{2n}, A(2n)}(H_{2n+1})$ is isomorphic to $G_{2n-1}[k]$ for some $k$; this implies that the collection (17) is full. Indeed, $G'$ belongs to the intersection of two orthogonals $S = \langle A, \ldots, A(2n-2) \rangle \cap \langle A(2n-1), G_{2n}, \ldots, \chi_5 \rangle$. Since the collection (18) is full, the subcategory $S$ is generated by the exceptional object $G_{2n-1}$ and therefore it is equivalent to the derived category of vector spaces $D^b(C-\text{vect})$. Any exceptional object in $D^b(C-\text{vect})$ is of the form $C[k]$; it follows that $G' \cong G_{2n-1}[k]$ for some $k$.

Theorem 3.4 is completely proved. 

4 Computation of the morphism algebra, associated with exceptional collection.

Starting with the collection $(\chi_{2n}, \chi_{2n-2}, \ldots, \chi_8, \chi_6)$ we can obtain the collection $(\chi_6, R_{\chi_6} (\chi_8), R_{\chi_8, \chi_6} (\chi_{10}), \ldots, R_{\chi_{2n-2}, \ldots, \chi_6} (\chi_{2n}))$ by a series of mutations. The latter collection is equivalent up to shifts of objects to the collection $(Q_{6.6}, Q_{8.6}, \ldots, Q_{2n-2.6}, Q_{2n.6})$, which has a nice property — higher Ext groups between its objects are zero.

Mutating as above the collection (17), we obtain the collection

$$(A, A(1), \ldots, A(2n-1), G_{2n}, A(2n), H_{2n+1}, Q_{6.6}, Q_{8.6}, \ldots, Q_{2n.6}, Q_{5.5}, Q_{7.5}, \ldots, Q_{2n-1.5}).$$ (20)

The number of nonzero higher Ext groups for this collection is much less than for the collections (18) and (17). In this section we explicitly describe the morphism algebra for the collection (20). That is, we compute groups $\text{Hom}$ and $\text{Ext}$ between objects and describe the composition law.

Let $\overline{A_k}$ be the subspace of $A_k$ defined as follows. If $m \leq n - 2$, then $\overline{A_{2m}}$ is the subspace of $A_{2m}$ generated by all monomials except $x_1^m$; in other cases $\overline{A_k}$ is the whole space $A_k$. By definition, $\overline{A_k}$ is the homogeneous component $(G_0)_k$ of degree $k$ of the ideal $G_0 = \ker(A \to Q_{0.4-2n})$.

**Proposition 4.1.** All nontrivial Ext groups between objects of the collection (20) are listed below.

(a) $\text{Hom}(A(k), A(l)) = A_{l-k}$, polynomials from $A_{l-k}$ act on $A(k)$ by multiplication;

(b) $\text{Hom}(A(k), Q_{j+2r,j}) = \mathbb{C} = \langle x_1^{r+(j-k)/2} \rangle$ if $j \leq k \leq j + 2r, k \equiv j \pmod{2}$, the generator sends 1 into $[x_1^{r+(j-k)/2}]$ (further on, $[\ ]$ denotes a coset in a quotient module).
(c) $\text{Hom}(Q_{j+2r,j}, Q_{j+2s,j}) = \mathbb{C} = \langle x_1^{s-r} \rangle$ if $j = 5, 6, r \leq s$, the generator sends $[1]$ into $[x_1^{s-r}]$.

(d) $\text{Hom}(A(k), G_{2n}) = A_{2n-k}$, $\text{Hom}(G_{2n}, A(2n)) = \mathbb{C}$, the generator is the embedding of the ideal, $\text{Ext}^1(G_{2n}, A(2n)) = \mathbb{C}$, $\text{Ext}^1(G_{2n}, Q_{2n,0}) = \mathbb{C}$;

(e) $\text{Hom}(A(k), H_{2n+1}) = A_{2n+1-k}$ if $0 \leq k \leq 2n$, polynomials from $A_{2n+1-k}$ act by multiplication, $\text{Hom}(G_{2n}, H_{2n+1}) = \mathbb{C} = \langle x_0 \rangle$, the generator acts by multiplication by $x_0$, $\text{Hom}(H_{2n+1}, Q_{2n-1,5}) = \mathbb{C}$, the generator is given by $f(x) \mapsto \frac{1}{x_1}[f(x)]$.

Proof. (a) See proposition 2.6.
(b) To check this, note that $Q_{j+2r,j}$ has the filtration [5].
(c) To compute $\text{Ext}^i(Q_{j+2r,j}, Q_{j+2s,j})$, apply the functor $\text{Hom}(Q_{j+2r,j}, \cdot)$ to the exact sequence $0 \rightarrow Q_{j+2r,j} \rightarrow Q_{j+2s,j} \rightarrow Q_{j+2s,j} \rightarrow 0$. Notice that $\text{Ext}^i(Q_{j+2r,j}, Q_{j+2s,j}) = 0$ to get the result. Use the filtrations [3] to check the equalities $\text{Ext}^i(Q_{0+2r,0}, Q_{5+2s,5}) = 0$.
(d) To calculate $\text{Ext}^i(A(k), G_{2n})$ apply the functor $\text{Hom}(-, A(2n))$ to the sequence [13]; the required groups $\text{Ext}(Q_{2n,4}, A(2n))$ were computed in step 4 of lemma [3.3]. It is not hard to check that $\text{Ext}^i(G_{2n, \chi_{2n}}) = \mathbb{C}$, other groups $\text{Ext}(G_{2n, \chi_{2n}'}), \mathbb{Z} \leq j \leq 2n$, are zero. Now, passing from $\chi$ to $Q$ we can compute the groups $\text{Ext}^i(G_{2n}, Q_{j+2r,j})$.
(e) The groups $\text{Ext}^i(A(k), H_{2n+1})$ can be easily computed by the definition of $H_{2n+1}$. Note that there exists a unique morphism $\varphi_0: A(2n) \rightarrow A(2n + 1)$ from objects $A(2n), \chi_{2n}, \ldots, \chi_6, \chi_4$ to objects $A(2n + 1), \chi_{2n+1}$, this yields the statements concerning $\text{Ext}^i(A_{2n}, H_{2n+1})$. To calculate $\text{Ext}^i(H_{2n+1}, Q)$ apply the functor $\text{Hom}(\cdot, Q_{j+2r,j})$ to the sequence [14]. It remains to check that the formula $f(x) \mapsto \frac{1}{x_1}[f(x)]$ gives a well-defined nonzero morphism $H_{2n+1} \rightarrow Q_{2n-1,5}$.

In the next proposition we give an explicit description of composition maps $\text{Ext}^i(E_l, E_m) \otimes \text{Ext}^j(E_l, E_i) \rightarrow \text{Ext}^{i+j}(E_l, E_m)$ for objects $E_i$ of the collection [20]. We omit trivial cases where one of the groups $\text{Ext}^i(E_l, E_m)$, $\text{Ext}^j(E_k, E_l)$ or $\text{Ext}^{i+j}(E_k, E_m)$ is zero.

**Proposition 4.2.** The composition law in the morphism algebra for the collection [20] is the following:

(a) $A(k) \rightarrow A(l) \rightarrow A(m)$. The composition $A_{m-l} \otimes A_{l-k} \rightarrow A_{m-k}$ is the multiplication of polynomials.

(b) $A(j + 2k) \rightarrow A(j + 2l) \rightarrow Q_{j+2r,j}$. The composition $\mathbb{C}\langle x_1^{r-l} \rangle \otimes A_{2l-2k} \rightarrow \mathbb{C}\langle x_1^{r-k} \rangle$ is given by $x_1^{r-l} \otimes f \mapsto c_{l-k} x_1^{r-l}$, where $c_{l-k}$ is the coefficient of $x_1^{r-k}$ in $f \in A_{2l-2k}$.

(c) $A(j+2k) \rightarrow Q_{j+2r,j} \rightarrow Q_{j+2s,j}$. The composition $\mathbb{C}\langle x_1^{s-r} \rangle \otimes \mathbb{C}\langle x_1^{r-k} \rangle \rightarrow \mathbb{C}\langle x_1^{s-k} \rangle$ is the multiplication.

(d) $Q_{j+2q,j} \rightarrow Q_{j+2r,j} \rightarrow Q_{j+2s,j}$. The composition $\mathbb{C}\langle x_1^{s-r} \rangle \otimes \mathbb{C}\langle x_1^{r-q} \rangle \rightarrow \mathbb{C}\langle x_1^{s-q} \rangle$ is the multiplication.

(e) $A(k) \rightarrow A(l) \rightarrow G_{2n}$. The composition $A_{2n-l} \otimes A_{l-k} \rightarrow A_{2n-k}$ is a restriction of the multiplication $\overline{A_{2n-l}} \otimes \overline{A_{l-k}} \rightarrow A_{2n-k}$.

(f) $A(k) \rightarrow G_{2n} \rightarrow A(2n)$. The composition $\overline{A_{2n-k}} \otimes \overline{A_{2n-k}} \rightarrow \overline{A_{2n-k}}$ is the multiplication.
(g) $G_{2n} \to A(2n)[1] \to Q_{2n,6}[1]$. The composition $\text{Hom}(A(2n), Q_{2n,6}) \otimes \text{Ext}^1(G_{2n}, A(2n)) \to \text{Ext}^1(G_{2n}, Q_{2n,6})$ is nonzero.

(h) $A(k) \to G_{2n} \to H_{2n+1}$. The composition $\mathbb{C}(x_0) \otimes A_{2n-k} \to A_{2n-1-k}$ is the multiplication.

(i) $G_{2n} \to A(2n) \to H_{2n+1}$. The composition $\mathbb{C}(x_0) \otimes \mathbb{C} \to \mathbb{C}(x_0)$ is the multiplication.

(j) $A(2k+1) \to H_{2n+1} \to Q_{2n-1,5} (k \geq 2)$. The composition $\mathbb{C} \otimes A_{2n-2k} \to \mathbb{C}(x_1^{n-k-1})$ is given by $1 \otimes f \mapsto c_{n-k} x_1^{-k-1}$, where $c_{n-k}$ is the coefficient of $x_1^{n-k}$ in $f$.

Proof. These statements, except for probably part (g), easily follow from the description of Hom groups given in proposition 4.1. Let us check that the composition of nonzero morphisms $G_{2n} \to A(2n)[1] \to Q_{2n,6}[1]$ is nonzero. In fact, we show that the composition $\chi_4[-1] \to Q_{2n,4}[-1] \to G_{2n} \to A(2n)[1] \to Q_{2n,6}[1] \to \chi_2[1]$ is nontrivial, where the morphism $Q_{2n,4}[-1] \to G_{2n}$ is given by the extension (13) and the left and right morphisms are defined in subsection 2.2. Indeed, by the proof of proposition 4.1d this composition is a composition of nontrivial morphisms $\chi_4[-1] \to A(2n)[1] \to \chi_2[1]$. It is nonzero by lemma 2.9.

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