Degenerate Bose Gas: A New Tool for Accurate Frequency Measurement

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We propose a new way of detecting frequencies using superradiant Rayleigh scattering from degenerate Bose gases. A measurement of the time evolution of population at the initial momentum state could determine an unknown frequency with respect to a known one at which the pump laser’s frequency modulates. A range of frequencies from kHz to \(~\text{MHz}\) could be determined with a fractional uncertainty $10^{-6}$.

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Bose Einstein condensate (BEC) atoms have long-range spatial coherence \cite{1} which offers the possibility to study quantum optics in a new domain where the atom-photon interactions could be altered \cite{2}. The interest in superradiant Rayleigh scattering and matter-wave amplification originates due to long coherence time of the condensate \cite{1, 2}. The first photon scattered from the condensate leaves a perturbed BEC, which induce the next photons to scatter along the same direction. Superradiant radiation occurs due to spontaneous Rayleigh scattering of photons from an elongated BEC \cite{3}. Due to momentum conservation atoms end up in a different momentum state after each scattering \cite{4}. These scatterings are equally probable in all directions so atoms could go to various momentum states. However the elongated shape of the condensate introduces self amplification of a particular mode and forms a matter wave grating \cite{4, 7}. This redistribute atoms in different momentum states \cite{4, 5}. Reverse avalanche occurs in the scattered photon mode which is the well known Dicke superradiance \cite{8}. This mechanism is also known as collective atomic recoil lasing (CARL). That was first described for a free electron laser system \cite{9, 10}, however it was first observed in an atomic vapor cell \cite{11}. Damping of photons limit intensity of the superradiance \cite{12, 13}. This can be improved by storing the BEC in an optical cavity which will enhance the atom-photon interactions. The influence of atomic motion on the superradiant light scattering from a moving BEC was investigated theoretically and experimentally \cite{14, 15}. In this letter we propose a new way of detecting frequencies using the technique of population transfer to a different momentum state in superradiance. An analytic model is described for detection of an unknown frequency with respect to a reference one. Also we give an experimental scheme for realization of our proposal.

We consider the superradiant radiation occurs when a off-resonant pump laser beam impinges along the axial direction of a cigar-shaped BEC. The atom-photon interaction initiates after the condensate is released from its confining potential in an optical dipole trap (ODT), as shown in Fig. 1 (a). An ensemble of degenerate atoms maintain its elongated shape even after 1-10 ms time of flight (TOF). Atom-photon interactions are fast processes than the time at which superradiant dynamics develop. This is limited by the decay of pump photons from the condensate and is faster than the expansion rate of the BEC. We consider BEC is very dilute after expansion and hence intra-atomic interactions are ignored. In an anisotropic condensate, correlation between scattered photons is enhanced when scattering occurs along the elongated direction (z-axis) of the condensate. This is known as end-fire mode in the regime of Dicke superradiance \cite{10}. Here an atom scatters photons from the pump laser beam of wave-vector $k_p$ and propagates along the z-direction. The recoil photon along the opposite direction results in a net momentum change $\Delta p = 2\hbar k_p$ of the atom as shown in Fig. 1 (b). Internal state of atoms do not change but they spread between two momentum states separated by $\Delta p$. Hence atomic center of mass motion changes but different momentum states get coupled.

The Hamiltonian describing the superradiant Rayleigh scattering is

$$\hat{H} = \hat{H}_a + \hat{H}_p + \hat{H}_{a-p},$$

which consists of energy associated with free atoms $\hat{H}_a$, pump laser field $\hat{H}_p$ and interaction of atoms with the photons in the pump laser light $\hat{H}_{a-p}$. We will explore dynamics along the z-axis of the elongated condensate. For a two level atom of mass $m$ the atomic Hamiltonian is

$$\hat{H}_a = \int \! dz \left[ \hat{\psi}_g^\dagger(z) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \right) \hat{\psi}_g(z) \right. + \hat{\psi}_e^\dagger(z) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \hbar \omega_a \right) \hat{\psi}_e(z) \right],$$

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locking technique [17]. The moving mirror of the cavity will shift the lock point of
after releasing from ODT. the expanded BEC will fall through a large cross section pump light after a delay of 1-2
2-rotated linear polarization after passing twice through the
Modulation of $\omega_p$ can be obtained by coupling the pump light to an unstable Febry-Pero t cavity. One mirror of
where $c_s$ is speed of phonons in the BEC.
where $\hat{\psi}_g (\hat{\psi}_e)$ and $\hat{\psi}_g^\dagger (\hat{\psi}_e^\dagger)$ are annihilation and creation operators of the atom in its ground (excited) states which are separated by a frequency $\omega_a$ and $\hbar$ is the Planck constant. The bosonic field operators satisfy the equal time commutation relations $[\hat{\psi}_j(z), \hat{\psi}_{j'}^\dagger(z')] = \delta_{jj'}\delta(z,z')$ also $[\hat{\psi}_j(z), \hat{\psi}_{j'}^\dagger(z') \neq 0$, where subscripts $j, j'$ refers to both $|g\rangle$ and $|e\rangle$ states respectively. In this case one can neglect the excited state population in Eq. (2) and spontaneous emission since frequency of the pump laser $\omega_p \ll \omega_a$. Hamiltonian of the optical pump field of strength $\eta$ is
\begin{equation}
\hat{H}_p = -i\hbar \eta (\hat{c}_p - \hat{c}_p^\dagger),
\end{equation}
where $\hat{c}_p$ ($\hat{c}_p^\dagger$) are the annihilation (creation) operator of photons, which satisfy the commutation relation $[\hat{c}_p, \hat{c}_p^\dagger] = 1$. The interaction Hamiltonian of atoms with the photons in the pump laser light is
\begin{equation}
\hat{H}_{a-p} = -\hbar g e_p \int dz \hat{\psi}_e^\dagger (z) e^{ikp z} e^{i\phi(t)} \hat{\psi}_g (z) + \int dz \hat{\psi}_g^\dagger (z) e^{-ikp z} e^{-i\phi(t)} \hat{\psi}_e (z) + h. c.,
\end{equation}
where $h. c. \text{ is the Hermitian conjugate.}$ The coupling strength $g = d\sqrt{\omega_p/(2\hbar e_o V)}$ between atoms and photons in the pump light [15] depends on induced electric dipole moment $d$ of the atom, volume $V$ of the condensate interacting with pump beam and permittivity $e_o$. The phase difference between the pump and the scattered photons is
\begin{equation}
\phi(t) = \delta (1 - e \sin \Omega t)t,
\end{equation}
where $\delta = \omega_p - \omega_a$ is their frequency difference. Frequency of the pump laser modulates at $e \sin \Omega t$ which evolves $\phi(t)$ during the atom-photon interaction time. The perturbing modulation amplitude is small, $e \delta < \delta$.

Figure 1: In superradiant scattering: (a) schematic of the experiment, (b) population transfer to the higher momentum state and (c) interaction of a two level atom with a far off-resonant photon, where $c_s$ is speed of phonons in the BEC.
negligible. Using $\psi$ motion arises under the assumption that the atoms in the BEC are delocalized and their momentum uncertainty is $n$ function can be written in the basis of eigenfunctions homogeneous density distribution for simplifying the calculation. In that case evolution of the ground state wave-

obtained. Equation of motion of atoms in the $n$ which is the total number of atoms in the condensate. This yields an equation of motion for the atoms

\[ \frac{d}{dt} \psi_{|g\rangle}(z) = \frac{i}{2m} \nabla^2 \psi_{|g\rangle}(z) - i \frac{2g^2}{\Delta} \tilde{c}_p \psi_{|g\rangle}(z) \]

where $\Delta = \omega_p - \omega_n$ is the detuning of the pump laser light and $\tilde{c}_p = \tilde{c}_p e^{i\omega p t}$. Photons in the pump laser beam acquire an equation of motion

\[ \frac{d}{dt} \tilde{c}_p = -i \frac{2g^2}{\Delta} \tilde{c}_p \int dz \psi_{|g\rangle}^\dagger(z) \cos (2k_p z + \phi(t)) \psi_{|g\rangle}(z) - \kappa \tilde{c}_p + \eta, \]

where $\kappa \leq c/2L$ is the damping rate of photons in the BEC of length $L$ and $c$ is the velocity of light. Now in the following, the bosonic operators $\hat{\psi}_{|g\rangle}$ and $\tilde{c}_p$ are substituted by the coherent condensate wavefunction $\langle \hat{\psi}_{|g\rangle} \rangle = \psi_{|g\rangle}$ and the classical light field amplitude $\tilde{c}_p$ respectively.

Since $L$ is much larger than the radiation wavelength, one can apply periodic boundary condition. We also assume homogeneous density distribution for simplifying the calculation. In that case evolution of the ground state wavefunction can be written in the basis of eigenfunctions

\[ \psi_{|g\rangle}(z, t) = \sum_{n=0}^{\infty} U_n(t) e^{2i k_p z} e^{i\phi(t)}, \]

where eigenvalues are $n \cdot \Delta p$ for $n = 0, 1, 2 \ldots$ and population at the $n$th eigenstate is $\rho_n(t) = U_n(t)^* U_n(t)$. This atomic motion arises under the assumption that the atoms in the BEC are delocalized and their momentum uncertainty is negligible. Using $\psi_{|g\rangle}(z, t)$ from Eq. (9) in the Eqs. (7) and (8), three coupled ordinary differential equations are obtained. Equation of motion of atoms in the $n$th level with momentum $p_0$ is

\[ \frac{d}{dt} U_n = -i 4 n^2 \omega_r U_n - i \dot{\phi}(t) U_n - i \frac{g^2 N}{\Delta} \alpha [2U_n + U_{n+1}]. \]

Similarly equation of motion of atoms shifted to the $(n + 1)$th level with momentum $p_0 + \Delta p$ is

\[ \frac{d}{dt} U_{n+1} = -i 4 (n + 1)^2 \omega_r U_{n+1} - i \dot{\phi}(t) U_{n+1} - i \frac{g^2 N}{\Delta} \alpha [2U_{n+1} + U_n]. \]

Equation of motion of photons in the pump laser beam is

\[ \frac{d}{dt} \tilde{c}_p = -i \frac{2g^2 N}{\Delta} \tilde{c}_p \rho_{n,n+1} - \kappa \tilde{c}_p + \eta, \]

where $\omega_r = \hbar k_p^2/2m$ is the single photon recoil frequency, $\alpha = \tilde{c}_p^2 \tilde{c}_p$ is the intensity of the light field and $\rho_{i,j} = U_i^* U_j$ is the coherence between $i$th and $j$th eigenstates. The quantity $\rho_{i,j}$ reduces to the density of a state when $i = j$. We have introduced rescaled light amplitude $\tilde{c}_p \to \sqrt{N} \tilde{c}_p$ in the Eq. (12).
Figure 2: (Color online) The solid lines show (a) population at the initial momentum state, (b) \( \Phi(t) \) at modulation frequencies \( \Omega = 4\omega_r \) (green), 3.45\( \omega_r \) (red) and 1.2\( \omega_r \) (orange). A strong modulation of the pump laser at \( \epsilon = 0.8 \) is considered for all spectra. The dashed lines show \( \tau \), where the first minima in the \( \rho_{n, \text{min}} \) and the first maxima \( \Phi_{\text{max}} \) appears, which are distinguished by their respective colors. (c) Shift of \( \tau(\Phi_{\text{max}}) \) as \( \tau(\rho_{n, \text{min}}) \) shifts with change of \( \Omega \) from blue (blue) to red (red) detuning relative to \( 4\omega_r \) (green) and the solid line is a linear fit to it.

For simplicity we consider the mass flow occurs from \( n \)th to a \((n + 1)\)th level. That gives a population difference between two momentum states

\[
\Delta \rho(t) = \rho_n(t) - \rho_{n+1}(t),
\]

and \( \rho_n + \rho_{n+1} = N/V \) remains conserved. A new set of equations can be obtained for the rate of change of coherence function

\[
\frac{d}{dt} \rho_{n,n+1} = i[4(1 + 2n)\omega_r + \dot{\phi}(t)]\rho_{n,n+1} - \mathcal{D} \rho_{n,n+1} + i\alpha g^2 N \Delta \Delta \rho,
\]

and for the population difference

\[
\frac{d}{dt} \Delta \rho = -4\alpha g^2 N \Delta \Im \left[ \rho_{n,n+1} \right],
\]

where, \( \Im \) indicates imaginary part of the correlation function. The decoherence rate of atoms \( \mathcal{D} \) arises due to Doppler broadening, inhomogeneity in the condensate and phase diffusion. The phase diffusion mechanism depends on \( \delta \) and \( p_0 \). Equations (14)-(15), are then equivalent to the well known Maxwell-Bloch equations for a two-level system [6,18].
In the quantum regime, \( \Delta p/\Delta \text{photon} \) at small \( \alpha \) is 

\( \Delta \text{frequency} \). The CARL phenomena starts when the build up laser power 

\( \sim D^2 \text{-transition of} \) atoms by their colors.

Solution of the Eqns. (14) and (15) identify quantum and semiclassical regimes of the superradiance scattering [15].

Assuming an experiment can resolve \( \sim 1 \% \) atom number fluctuation, a range of frequencies from 2 \( \times 10^3 \) to about a 

\( 3 \times 10^4 \) MHz could be detected by this technique.

Figure 3: At \( \Omega_r = 4 \omega_r \) (green) the relation between \( \Omega_u \) and shift of \( \tau(\rho_{n, \min}) \). Example blue and red detuned \( \Omega_u \) are indicated by their colors.
have limited run time. The proposed techniques could be incorporated for frequency stabilization of the flywheel oscillators.

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