Anomalous dynamics of magnetic field-driven propagating magnetic domain walls

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Exact solutions of magnetic field-driven propagating topological solitons are found in the easy-plane phase of ferromagnetic spin-1 Bose-Einstein condensates, manifesting themselves as dark-soliton-like magnetic domain walls. Such solitons have two types: a low energy solution with positive inertial mass and a higher energy solution with negative inertial mass. Both types become identical at the maximum speed, a new speed bound that is different from speed limits set by the elementary excitations. The physical mass, which accounts for the number density dip, is negative for both types. In a finite one-dimensional system subject to a linear potential, the soliton undergoes oscillations, and transitions between the two types occurring at the maximum speed.

Introduction— Inertial mass quantifies the resistance of matter to forces imposed on it, and is a fundamental characterization of elementary particles [1] and of emergent quasi-particles in quantum many body systems [2].

In the context of quantum fluids, inertial mass of non-linear excitations, determined by kinetic energy and interaction energy, is a key quantity governing their dynamics. In a one-dimensional (1D) harmonically trapped scalar Bose-Einstein condensate (BEC), due to the density dip, a grey soliton experiences an effective force from the surrounding liquid (similar to buoyant force). Since the inertial mass of a grey soliton is negative, it moves as an ordinary particle, i.e., oscillates around the trap center [3, 4]. The sign of inertial mass also conveys the stability. Typically, long wavelength transverse deformations of solitons with negative inertial mass will be enhanced and eventually lead to a breakdown of a soliton at high dimensions [5], known as the snake instability [6, 7]. Most of the relevant solitons have negative inertial mass, including dark solitons in scalar BECs, phase domain walls in coherently coupled BECs for strong coherent coupling strengths [8–11], magnetic solitons in both binary [12] and anti-ferromagnetic spin-1 BECs [13, 14]. A dark-soliton-like magnetic domain wall (MDW) found recently in ferromagnetic spin-1 BECs shows the stability against snake fluctuations as well as white noise in two dimensions (2D) [15], suggesting a positive inertial mass. To explore novel dynamics of these domain walls/solitons in 1D and 2D requires finding propagating solutions. To date, however, only stationary solutions have been uncovered [15].

In this Letter we report exact solutions of magnetic field-driven propagating MDWs in the easy-plane phase of a spin-1 ferromagnetic BEC, obtained for a large spin-dependent interaction strength $g_s$. The transverse magnetization (the magnetic field is along the $z$-axis) has the typical profile of a dark soliton and the number density has a dip at the core. Two types of such MDWs are found. One type has positive inertial mass and the other type has negative inertial mass with higher excitation energy. Unlike most traveling solitons, the moving speed is not limited by group velocities of elementary excitations but has a new speed bound, at which the two types of solitons become identical. We study dynamics of the soliton in a hard-wall trapped quasi-1D system with a superimposed linear potential and find transitions between the two types when it reaches the maximum speed, leading to an oscillatory motion.

Spin-1 BECs— The Hamiltonian density of a spin-1 condensate reads

$$\mathcal{H} = \hbar^2 |\nabla \psi|^2 / 2M + g_s |\psi|^4 / 2 + g_s |\psi|^2 S^3 \psi|^2 + q |\nabla \psi|^2 \psi^g S^2 \psi,$$

where the three-component wavefunction $\psi = (\psi_s, \psi_0, \psi_{-1})^T$ describes the atomic hyperfine state $|F = 1, m = \pm 1, 0, -1\rangle$. $M$ is the atomic mass, $g_s > 0$ is the density interaction strength, $g_s$ is the spin-dependent interaction strength, $S = (S_x, S_y, S_z)$ with $S_{jx,y,z}$ being the spin-1 matrices [16], and $q$ denotes the quadratic Zeeman energy. The spin-dependent interaction term allows for spin-mixing collisions between $m = 0$ and $m = \pm 1$ atoms. At the mean-field level, the dynamics of the field $\psi$ is governed by the Gross-Pitaevskii equations (GPEs)

$$i\hbar \frac{\partial \psi_{\pm 1}}{\partial t} = [H_0 + g_s (n_0 - n_{\pm 1} - n_{\mp 1} + q)] \psi_{\pm 1} + g_s \psi_0^* \psi_{\pm 1},$$

$$i\hbar \frac{\partial \psi_0}{\partial t} = [H_0 + g_s (n_{+ 1} + n_{- 1})] \psi_0 + 2g_s \psi_0^* \psi_{+ 1} \psi_{- 1},$$

where $H_0 = -\hbar^2 \nabla^2 / 2M + g_s n_0 n_m = |\psi_m|^2$ and $n = \sum n_m$. Spin-1 BECs support magnetic order [17–21], quantified by the order parameter magnetization $\mathbf{F} \equiv \psi^* \mathbf{S} \psi$. This identifies ferromagnetic order $|\mathbf{F}| > 0$ for $g_s \lesssim 0$ (\textsuperscript{87}Rb, \textsuperscript{7}Li) and anti-ferromagnetic order $\mathbf{F} = 0$ for $g_s > 0$ (\textsuperscript{23}Na).

Magnetic field-driven propagating MDWs— We consider a uniform ferromagnetic ($g_s < 0$) spin-1 BEC with total number density $n_0$. In the presence of a magnetic field along the $z$-axis ($0 < q < -2g_s n_0$), the uniform ground state is transversally magnetized (easy-plane phase) [20, 21], characterized by the transverse magnetization $F_\perp \equiv F_x + i F_y = \sqrt{8n_{-1}^0 n_0^0} e^{i\tau}$ and $F_z = n_0^0 - n_{-1}^0 = 0$, where $n_{-1}^0 = (1 - \tilde{q}) n_0 / 4$ and $n_0^0 = n_0 (1 + \tilde{q}) / 2$ are the component densities, and $\tilde{q} \equiv \mp -q / (2g_s n_0)$. The SO(3) symmetry is broken by the magnetic field and the system processes the remnant SO(2) symmetry, parameterized by the rotational angle about the $z$-axis $\tau$. In the easy-plane phase, a dark-soliton-like MDW of Ising-type that connects regions magnetized in opposite directions (a topological excitation associated with $\mathbb{Z}_2$ symmetry breaking), has been
found to be stable against the snake instability in 2D [15, 22]. Such MDWs cannot propagate in the \( q \to 0 \) limit due to the conservation of magnetization. Moving MDWs, if there are any, must be driven by magnetic fields. However, motion is not automatically induced by magnetic fields, since stationary MDWs were found for \( q > 0 \) [15].

In the following we consider a 1D system and focus on exactly solvable cases occurring at \( g_g = -g_n/2 \) [23]. We find two types of propagating MDWs, their transverse magnetizations and total number densities read

\[
P_{\perp,II}^{(1)}(x, t) = -e^{i\tau}\sqrt{n_b^2 - \frac{q^2}{g_n^2}}\tanh\left(\frac{x - Vt}{\ell_{\perp,II}}\right),
\]

\[
n_{\perp,II}^{(1)}(x, t) = n_b - \frac{g_n q - M V^2 \mp Q}{2g_n} \sech^2\left(\frac{x - Vt}{\ell_{\perp,II}}\right),
\]

where \( V \) is the moving velocity,

\[
\ell_{\perp,II} = \sqrt{\frac{2\hbar^2}{M (g_n q - M V^2 \mp Q)}},
\]

and

\[
Q = \sqrt{M^2 V^4 + q^2 - 2g_n M n_b V^2}.
\]

The minus (plus) sign in front of \( Q \) specifies type-I (II) MDW. Hereafter, unless specified, we choose \( \tau = 0 \) for convenience. The inequality \( Q^2 > 0 \) gives rise to the upper bound of the traveling speed [26]

\[
V \leq \sqrt{\frac{g_n n_b}{M}} \sqrt{1 - \left(\frac{q}{g_n n_b}\right)^2} \equiv c_{\text{MDW}}.
\]

When \( q \to 0 \), \( c_{\text{MDW}} \to 0 \), in consistent with MDWs being stationary in the absence of magnetic fields [15]. The speed bound Eq. (7) is markedly different from the group velocities of low-lying elementary excitations which normally set the speed limits [27]. In the easy-plane phase, the gap-less branches of the elementary excitations involve spin waves of magnetization \( \mathbf{F} \) (dominantly) and mixed waves of \( F_z \) and \( n \), with group velocities at long wavelengths \( c_m = \sqrt{q/(2M)} \) and \( c_{mp} = \sqrt{n_b(g_n + g_i)/M} \), respectively [28]. Strikingly, for \( 1 > q/g_n n_b > \sqrt{3}/2 \), \( c_{\text{MDW}} > c_{mp} > c_m \), implying that the MDWs can propagate with speed greater than \( c_m \) and \( c_{mp} \). This can happen because a propagating MDW does not involve spin currents. Another conspicuous feature is that the soliton profile does not vanish at \( V = c_{\text{MDW}} \) (see Fig. 1). The velocity of grey solitons in scalar BECs is bounded by the speed of sound, and at this velocity the soliton disappears [27]. At the transition point \( q = g_n n_b \), the easy-plane phase becomes unstable, signalled by the divergence of \( \ell_{\perp} \). The corresponding wavefunctions at the exactly soluble point are shown in Table I, having profiles of vector solitons.

Similar to scalar gray solitons, the density dip of the type-II MDW becomes shallower for greater velocities [Fig. 1(d)]. In contrast, for the type-I MDW the density dip behaves anomalously and deepens with increasing velocity [Fig. 1(b)]. Crucially, at the maximum velocity \( V = c_{\text{MDW}} \), \( Q = 0 \) and the two types of MDWs become identical upon a U(1) gauge transformation, namely \( \psi(x, t) = i\psi(x, t) \) (see Table I).

**Currents**— Moving MDWs involve nematic degrees of freedom and internal spin currents. Since the magnetization is zero at the core of moving MDWs, there is no spin current, i.e., \( J_i^F = h/(2M)\langle \psi^\dagger S_i \psi \rangle \) is zero. According to the continuity equation \( \partial F_i/\partial t + \nabla \cdot J_i^F = K_c \), the time evolution of magnetic domains enclosed by the MDWs is governed by the source term \( K_{ic} = 2(2q/\hbar)\sum_i N_i \delta_{i0} N_{iz} \) [29, 30], where \( N_i = \psi^\dagger \tilde{N}_i \psi \) is the nematic tensor, \( \tilde{N}_i = (S_i S_j + S_j S_i)/2 \) and \( i, j \in \{x, y, z\} \). For \( q = 0 \), \( K_c = 0 \), and MDWs must stay still.

The continuity equations of particle number in each spin state read \( \partial n_{z1}/\partial t + \nabla \cdot J_{z1} = J_{z1\to0} = 0 \), and \( \partial n_{0}/\partial t + \nabla \cdot J_0 + \sum_{i=-1,0} J_{0\to-i} = 0 \), where \( J_{z1\to0} = h/(2M)\langle \psi^\dagger_{z1} S_i \psi_{z1,0} \rangle \) are the component number current densities [31], and

\[
J_{z1\to0} = -J_{0\to-z1} = \frac{g_r}{\hbar} \left[ (\psi^\dagger_i)^2 \psi_{-z1} - h.c. \right]\]

are the internal spin currents, reflecting the internal coherent spin exchange dynamics: \( |00\rangle \leftrightarrow |+1\rangle \leftrightarrow |10\rangle \) [17–19]. Rewriting Eq. (8) in terms of wavefunction phases \( \theta_{z1,0} \) and densities, we obtain \( J_{z1\to0} = (2\eta_{0z1}(g_r/\hbar)\sin[2(\theta_{z1} - \theta_0)]) \) which suggests an analogy to Josephson currents [32, 33]. It is important to note that these built-in currents are invariant under SO(2) rotations (\( e^{-i\pi S_z} \)). Table I shows the expressions of currents at the exactly solvable point.

**Excitation energy and inertial mass**— The excitation energy of MDWs can be obtained by evaluating the difference of grand canonical energies \( \delta K = K_{\text{MDW}} - K_g \), where
\[
\psi
\]

\[
\psi_{\pm}(x, t) = \sqrt{n_b^{\pm}} \left[ a^\dagger \tan\left( \frac{\pi V}{\ell} x \right) + i \delta^\dagger \right]
\]

\[
\alpha^\dagger = -\frac{\sqrt{\frac{M V}{2q} + Q}}{2q}, \quad \delta^\dagger = -\sqrt{\frac{g^{\dagger} Q}{q}(g^{\dagger} - M n_b V^2 - Q)}
\]

\[
\beta^\dagger = \sqrt{\frac{g^{\dagger} Q}{2q}}, \quad \kappa^\dagger = -\frac{g^{\dagger} Q}{q(g^{\dagger} + M n_b V^2 - Q)}
\]

\[
\mathcal{K} = \frac{2}{\sqrt{q}} \sqrt{n_b^{\pm} \delta \hbar^2 \beta^\dagger \left( x^{\dagger} V t \right)} \text{ sech}^2 \left( x^{\dagger} V t \right)
\]

\[
J_{\pm} = \frac{4 \sqrt{q}}{\hbar} \delta \hbar^2 \left( x^{\dagger} V t \right) \tan \left( x^{\dagger} V t \right) \text{ sech}^2 \left( x^{\dagger} V t \right)
\]

\[
K_{\text{MDW}} = \int dx \left( \mathcal{H}[\psi] - \mu n_b \right), \quad K_{\text{SC}} = \int dx \left( \mathcal{H}[\psi_n] - \mu n_b \right), \quad \psi_n \quad \text{is the ground state wavefunction and } \mu = (g_n + g_\alpha) n_b + q/2 \text{ is the chemical potential. For Type-I MDWs, we obtain}
\]

\[
\delta K^\dagger(q, V^2) = \frac{\hbar(g_n n_b + q) [3M V^2 f + 2q^2 (q - Q)]}{3 \sqrt{2} g_n q (M V^2 + q - Q)} \left( \frac{V}{\sqrt{g_n - M V^2 - Q}} \right), \quad (9)
\]

where

\[
f = \sqrt{2(g_n n_b - q)(g_n n_b - M V^2 + Q)(q^2 - q Q - g_n M n_b V^2)}
\]

\[
+ (g_n n_b - q)(q - M V^2) + q(3 - g_n n_b). \quad (10)
\]

Expanding Eq. (9) around \( V = 0 \), we have \( \delta K^\dagger(q, V^2) = \delta K^\dagger(q, 0) + M^\dagger M^\dagger V^2/2 + \alpha(V^2) \) where \( \delta K^\dagger(q, 0) = \sqrt{2} \hbar(g_n n_b - q)^{3/2} \left( \frac{3 g_n}{M} \nabla \right) \) and the inertial mass is

\[
M^\dagger \equiv 2 \frac{\delta \delta K^\dagger}{\delta V^2} \bigg|_{V=0} = \frac{n_b \hbar}{q} \sqrt{\frac{n_b q \hbar}{2}} \left( 1 - \frac{q}{g_n n_b} \right)^{3/2} > 0. \quad (11)
\]

As \( q \to 0 \), \( M^\dagger \to +\infty \) and the MDW becomes infinitely heavy, consistent with the absence of propagation at zero magnetic field due to the conservation of magnetization [15]. In contrast to the normal behavior of grey solitons, the excitation energy (\( \delta K^\dagger \)) of the type-I MDW increases monotonically with increasing \( V^2 \) [Fig. 1(e)], in accordance with the anomalous behavior of the density [Fig. 1(b)]. Following conventional arguments [5] the positive inertial mass explains the stability of MDWs against transverse snake perturbations in 2D [15].

The physical mass is defined as \( M_{\text{phys}} = M \Delta N \), where \( \Delta N = \int dx [n(x) - n_b] \). For type-I MDWs, the density has a dip [Fig. 1(b)] and we obtain \( M_{\text{phys}} = -2 \hbar^2/(g_n \ell^3) < 0 \). In the presence of an external potential \( U \), solitons with negative physical mass experience an effective force pointing in the opposite direction to \(-\nabla U\). For a scalar grey soliton the inertial and the physical masses are both negative and it exhibits normal particle-like behavior, e.g., oscillations in a harmonic potential [3, 4]. Whereas a Type-I MDW in a harmonic potential would be expelled, i.e. moves away from the potential minimum.

The excitation energy of the Type-II MDW is

\[
\delta K^\dagger(q, V^2) = \frac{\sqrt{2} \hbar(g_n n_b - M V^2 + Q)^{3/2}}{3 g_n \sqrt{M}} \quad (12)
\]

with \( \delta K^\dagger \partial V^2 < 0 \) [Fig. 1(e)]. Expansion of Eq. (12) leads to \( \delta K^\dagger(q, V^2) = \delta K^\dagger(q, 0) + M^\dagger M^\dagger V^2/2 + \alpha(V^2) \), where \( \delta K^\dagger(q, 0) = \sqrt{2} \hbar(g_n n_b - q)^{3/2} \left( \frac{3 g_n}{M} \nabla \right) \) and the inertial mass

\[
M^\dagger \equiv 2 \frac{\partial \delta K^\dagger}{\partial V} \bigg|_{V=0} = -\frac{\sqrt{2} \hbar g_n n_b + q)^{3/2}}{\sqrt{2} g_n q} < 0 \quad (13)
\]

Consistently, \( M^\dagger \) diverges as \( q \to 0 \). The physical mass \( M_{\text{phys}} = -2 \hbar^2/(g_n \ell^3) < 0 \). Thus, the inertial and physical mass of the type-II MDW is similar to those of ordinary grey/dark solitons. Excitation energies of type-I and type-II MDWs coincide smoothly at the maximum speed [Fig. 1(e)], making transitions between the two types of MDWs possible under certain circumstances.

**Oscillations between type-I and type-II MDWs** — As discussed earlier the MDW does not vanish as \( V \to c_{\text{MDW}} \), so a natural question is what will happen if it is further accelerated? Let us consider a hard-wall trapped quasi-1D spin-1 BEC subjected to a linear potential whose gradient is along the positive x-axis. A \( V = 0 \) type-I MDW is initially placed near the left end of the system, and the later dynamic shows, surprisingly, a periodic motion. The MDW accelerates until it reaches the maximum speed (the local value of \( c_{\text{MDW}} \) [34])
at which point it smoothly transforms into a type-II MDW. Due to the sign change of the inertial mass (or more generally ∂δK/∂V² > 0 → ∂δK/∂V² < 0), it starts to accelerate in the opposite direction. After reaching the turning point, the MDW starts to move to the left. It converts back to the type-I MDW and experiences positive acceleration again when gaining the maximum speed. Later it returns to the initial configuration. Note that there is no sign change of the physical mass. Numerical simulations show that this process continues without decay (Fig. 2).

![Graph](image_url)

**FIG. 2.** Oscillations of a MDW in a hard-wall trapped spin-1 BEC with a superimposed linear potential [35]. The system size is 200\(\xi_0\), \(g_0/g_\text{m} = -1/2\) and \(\bar{q} = q/(2\bar{g}_0\bar{n}_0) = 0.3\). Here \(\bar{n}_0\) is the average density, \(\bar{n}_0 = h/\sqrt{\hbar^0\bar{n}_0}\) and \(\xi_0 = h/\sqrt{\hbar^0\bar{n}_0}\) is the density healing length. Upper and middle panels show spin and density dynamics of a MWD, respectively. The transverse magnetization is always zero at the core (see also Fig. S3 [28]) and the topological characteristic, i.e., the sign change of \(F_x\) is kept. Bottom panel shows the velocity of the MDW as a function of time, obtained by taking the derivative of its position with respect to time. The slope refers to the acceleration of the MDW and indicates the sign of the inertial mass (positive; blue; negative; red). Transition between type-I and type-II MDWs occur when the slope changes the sign, happening at the maximum speed. Here \(c_{\text{MDW}}\) is the local speed limit for the (background) density at the position where \(dV_x/dt\) changes the sign.

During the motion the total number density profile of the soliton has only minor changes with respect to the local background density (see Fig. 2 and Fig. S3 [28]). However internal oscillations (driven by the external potential) between \(m = \pm 1\) and \(m = 0\) spin states near the core take place though the internal currents \(J_{x1\to0}\) (Fig. 3 and Fig. S4 [28]), inducing transformations between type-I and type-II MDWs. Accounting for the varying density and the potential energy, we map the MDW energy \(\delta K\) extracted from the simulation to its corresponding values for a uniform system [28], and find that it oscillates between lower branch (type-I) and higher branch (type-II) (Fig. 3(c)), as predicted.

It should be noted that here we adopt linear potentials to have a transparent illustration and the transitions between the two types MDWs could occur in other situations.

**Conclusion—** We find a propagating magnetic domain wall corresponding to a soliton with negative physical mass and positive inertial mass in the easy-plane phase of a ferromagnetic spin-1 BEC. It can convert to its higher energy counterpart with negative physical and inertial mass at a novel maximum speed that is different from group velocities of elementary excitations, inducing oscillations in a linear potential [37]. Here we focused on the exactly solvable case, but it exists over the whole easy-plane phase with no qualitative differences. Advances in engineering optical potential [38–40] and nondestructive spin-sensitive imaging methods [41] open the possibility of experimental investigations of the magnetic domain wall dynamics. Recently a \(^7\)Li spin-1 BEC has been prepared in the strong spin interacting regime close to the exactly solvable point [42].

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In coherently coupled BECs, the Son-Stephanov phase domain [43] has positive inertial mass for weak coherent coupling strengths [8–10]. However, a long wall fragments into smaller wall [43] has positive inertial mass for weak coherent coupling.

Then, the components can be expressed as

\[ K_{\text{II}} = \sqrt{1 - q/g_{\text{II}}} \]

\[ K_{\text{III}} = -\sqrt{1 - q/g_{\text{III}}} \]

\[ K_{\text{IV}} = -\sqrt{1 - q/g_{\text{IV}}} \]

The other branch is

\[ y = V/\sqrt{g_{\text{III}}/M} \quad \text{and} \quad y = V/\sqrt{g_{\text{IV}}/M} \]

where the condition \( 1 - y = \sqrt{y^2 + (q/g_{\text{III}})^2} = 2y > 0 \) cannot be satisfied if \( y > 1 \). Hence this branch is not a solution. Other constraints

\[ q - Q > 0 \quad \text{and} \quad qV^2 > 0 \]

\[ (Q - q) + g_{\text{III}}V^2 < 0 \]

\[ (Q - q) + g_{\text{IV}}V^2 < 0 \]

which is always satisfied as long as \( q/g_{\text{III}} < 1 \) and \( MV^2/g_{\text{IV}} < q/g_{\text{IV}} \). Since

\[ 1 - \sqrt{1 - (q/g_{\text{III}})^2} < q/g_{\text{III}} \]

these constraints are automatically satisfied.

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[30] Explicitly, there are \( K_{\text{I}} = -2qN_{12}/\hbar \quad K_{\text{II}} = 2qN_{31}/\hbar \quad K_{\text{III}} = 0 \).

[31] The total number current density is

\[ J_y = \sum_{y=1}^{L} J_{\text{y}} \]

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[33] For \( q = 0 \), the Sine-Gordon representation of the magnetic domain wall allows non-trivial internal spin currents, however they are not invariant under SO(3) spin rotations [15].

[34] Since the density varies, the value of \( C_{\text{MDW}} \) also changes in space.

[35] The linear potential should be chosen such that everywhere in the bulk is in the easy-plane phase, namely \( q < -2g_{\text{I}} \sin[n(x)] \).

[36] The total (grand canonical) energy is \( E_{\text{MDW}} + E_{\text{U}} \), where \( E_{\text{U}} = \int dx U(x) \) is the potential energy and \( U \) is the external potential.

[37] In a binary BEC, a bright-dark soliton with constant total density experiences oscillations when a constant force is imposed on the bright soliton component [46], where the sign change of the inertial mass is due to interplay between the bright and dark soliton components. The mechanism is vastly different from what we discussed in this paper.

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Supplemental Material for “Anomalous dynamics of magnetic field-driven propagating magnetic domain walls”

ELEMENTARY EXCITATIONS IN THE EASY-PLANE PHASE

Let us denote $\psi_s$ as the ground state wavefunction in the easy-plane phase $(0 < q < -2g_n n_b)$. Substituting the perturbed wavefunction $\psi = \psi_s + \delta \psi$ with $\delta \psi = e^{i(ux)e^{-i\theta} + v(x)e^{i\theta}}$ into 1D Gross-Pitaevskii equations (Eq. (2) in the main text) and keeping the leading order terms, we obtain the bosonic Bogoliubov-de Gennes (BdG) equations

\[ E \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{GP} + X - \mu \\ -\Delta^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (S1) \]

where $\epsilon$ is a dimensionless small number, $E = \hbar \omega$, $\mathcal{L}_{GP} = -\hbar^2 \nabla^2 / 2M + g_n \psi_s \psi_T + g_s \sum_{i=1}^3 \psi^\dagger_i S_i \psi_i + q \mu^2$, $X = g_s \psi_s \psi_T + g_s \sum_{i=1}^3 (S_i \psi_s)(S_i \psi_T)^T$ and $\mu = (g_n + g_s) n_b + q/2$ with $n_b = |\psi_s|^2$. Note that $\mathcal{L}_{GP}\psi_s = \mu \psi_s$. Since the system has translational symmetry, it is natural to parameterize the perturbations according to the wave-vector $k_z$: $u(x) = u e^{ik_z x}$ and $v(x) = v e^{ik_z x}$. Solving Eq. (S1), we obtain

\[ E_m(k) = \pm \frac{\hbar \sqrt{(k_z^2 \hbar^2 + 2Mq)k_z^2}}{2M}, \]

\[ E_{mp}(k) = \pm \frac{\sqrt{g_s \left(g_s (2g_n k_z^2 M_n b^2 + k_z^2 h^4 - 2M^2 q^2) + 8g_s^3 \hbar^2 M^2 n_b^2 - 2g_s^2 k_z^2 M_n b h^2 + 2MT k \right) - 2g_s M}}{2M}, \]

\[ E_{gap}(k) = \pm \frac{\sqrt{g_s \left(g_s (2g_n k_z^2 M_n b^2 + k_z^2 h^4 - 2M^2 q^2) + 8g_s^3 \hbar^2 M^2 n_b^2 - 2g_s^2 k_z^2 M_n b h^2 - 2MT k \right) - 2g_s M}}{2M}. \]
where \( \Gamma = \sqrt{g_s \left( g_n (n_s + 3g_s)^2 - q^2 (g_n + 2g_s) \right) \hbar^2 k^2 - 2g_s M n_b (g_n + 3g_s) \left( 4g_s^2 n_s^2 - q^2 \right) \hbar^2 k^2 + g_s M^2 \left( q^2 - 4g_s^2 n_s^2 \right)^2} \). For small \( k_s \), \( \Gamma_k \approx g_s n_b (g_n + 3g_s) \hbar^2 k^2 - g_s M (4g_s^2 n_s^2 - q^2) \) and the spectrum of the two gap-less modes read

\[
E_m(k_x) = \pm c_m \hbar k_x \quad \text{and} \quad E_{mp}(k_x) = \mp c_{mp} \hbar k_x,
\]

where

\[
c_m = \sqrt{\frac{q}{2M}} \quad \text{and} \quad c_{mp} = \sqrt{\frac{n_b (g_n + g_s)}{M}}.
\]

The spectrum, and the fluctuations and spin currents associated with the gap-less excitations are shown in Fig. S1.

**WAVEFUNCTIONS OF PROPAGATING MDWS**

Fig. S2 shows examples of moving MDW wavefunctions presented in Table I in the main text. The fact that \( \text{Im}(\psi^I) \) and \( \text{Re}(\psi^I) \) are constants admit exact solutions at \( g_s = -g_n/2 \). Similarly, for type-II MDWs, \( \text{Re}(\psi^II) \) and \( \text{Im}(\psi^II) \) are constants. Away from the exact solvable point, the constant components develop humps or dips near the domain wall core depending on the value of \( g_s \).

**OSCILLATIONS IN A LINEAR POTENTIAL**

**Mapping to a uniform system**

It is possible to extract the MDW energy \( \delta K \) from the simulated dynamics to compare with analytical predictions which are valid for a uniform system. We construct a mapping \( \tilde{\psi}(t) \rightarrow \tilde{\psi}^m(t) = \tilde{\psi}^m(t)/\tilde{\psi}_g^m \) for each spin state \( (m = -1, 0, +1) \), where \( \tilde{\psi}_g \) is the ground state in the presence of the potentials (linear+hard-wall), \( \tilde{\psi}_b \) is the uniform ground state with density \( \tilde{n}_b \). The mapped wavefunction \( \tilde{\psi}^m(t) \) describes a MDW in a uniform system with density \( \tilde{n}_b \) and the corresponding excitation energy reads \( \delta K[\tilde{\psi}^m(t)] = K[\tilde{\psi}^m(t)] - K[\tilde{\psi}_g^m] \), where \( K[\tilde{\psi}^m(t)] = \int dx \{ \mathcal{H}[\tilde{\psi}^m(t)] - (\mu \tilde{\psi}^* \tilde{\psi}) \} \), \( K[\tilde{\psi}_g^m] = \int dx \{ \mathcal{H}[\tilde{\psi}_g^m] - (\mu \tilde{\psi}^*_g \tilde{\psi}_g) \} \),

\[
\mathcal{H}[\psi] = \frac{\hbar^2 |\nabla \psi|^2}{2M} + \frac{g_n}{2} |\psi|^2 + \frac{g_s}{2} |\psi|^2 S^z \psi^2 + q |\psi|^2 S^z \psi
\]

and \( \mu = (g_n + g_s)\tilde{n}_b + q/2 \) is the chemical potential.
Dynamics in components

Here we present further details of oscillations presented in the main text. Fig. S3 shows number densities and the magnetization density at different stages of the oscillation and Fig. S4 shows the internal dynamics of component densities.

**FIG. S3.** A complete circle of the oscillation of a MDW in a linear potential. The parameters are the same as in Fig. 3 in the main text. The black arrows specify the evolution direction. From left to right: densities of initial state [type-I MDW with $V = 0$] (blue), at the maximum velocity (black), at the turning point (red), at the negative maximum speed (back), and of the final state [returning the initial state] (blue).

**FIG. S4.** Internal oscillations between $m = \pm 1$ and $m = 0$ spin states during the motion described in Fig. 2 in the main text. Top and middle panels show component number densities (subtracting their background values) in $m = \pm 1$ and $m = 0$ spin states, respectively. Bottom panel shows oscillations of number of missing particles (due to the density dip) in $m = \pm 1$ states $r_c \delta N_{\pm 1}$ (solid line) and $\delta N_0$ (dashed line). At the maximum speed, $r_c \delta N_{\pm 1} = \delta N_0$ and transitions between type-I (blue) and type-II (red) MDWs occur. Here $r_c \equiv (\delta N_0/\delta N_{\pm 1})_{V=\text{MDW}} = 2 \sqrt{|g_n n(x_c) + q|/|g_n n(x_c) - q|}$, $n(x_c)$ is the background density at position $x_c$ where the transition occurs.