Superradiant instability in slowly rotating Kerr-Newman black holes

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Abstract

We study the superradiant instability of slowly rotating Kerr-Newman (sKN) black holes under a charged massive scalar perturbation. These black holes resemble closer the Reissner-Nordström black holes than the Kerr-Newman black holes. From the scalar potential analysis, we find that the superradiant instability is not allowed in the sKN black holes because the condition for a trapping well is not compatible with the superradiance condition. However, the rate of energy extraction might grow exponentially if the sKN black hole is placed inside a reflecting cavity. Finally, we obtain two conditions for the trapping well to possess quasibound states in the sKN black holes by analyzing asymptotic scalar potential and far-region wave functions.

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1 Introduction

Superradiance in the black hole physics is a radiation enhancement process that allows for energy extraction from the black hole at the classical level [1]. It has been suggested the ‘rotating black hole-mirror bomb’ idea [2] which states that if the superradiance emerging from a perturbed black hole were reflected back onto the black hole by a mirror, an initial perturbation could be made to grow without bound [3]. This instability is caused by a reflecting mirror, but the superradiant instability could occur naturally if a perturbed scalar has a rest mass [4]. On the other hand, if the asymptotic spacetime is de Sitter, it has been shown that the addition of a mass to the perturbed scalar acts in exactly the opposite way [5] [6] [7] [8]. That is, it depletes superradiant instabilities that are present when the perturbed scalar is massless.

For a Kerr black hole, if the scalar has a mass $\mu$, its mass would act as a reflecting mirror. The superradiant instability depends on two parameters: $a = J/M$ and $M\mu$ ($J$ angular momentum and $M$ mass of the black hole). The instability gets stronger as $a$ and $\mu$ increase [3] but as $a$ decreases with a fixed $\mu$, the unstable modes disappear below a critical value of $a$ because the superradiance condition of $\omega < \omega_c = m\Omega_H$ ($m$ azimuthal number and $\Omega_H$ angular velocity at horizon) violates for small $a$. One has found a superradiant instability of the Kerr black hole for $M\mu \gg 1$ [9], $M\mu \ll 1$ [10], and $M\mu \leq 0.5$ [11] for the first corotating mode ($\ell = m = 1$). A scalar potential including a shape of barrier-well-mirror is responsible for generating quasibound states (resonance spectra [4] or quasistationary levels [11]) for superradiant instability when satisfying $\omega < \omega_c$ and $\omega < \mu$ [12] [13]. The later denotes the bound state condition for getting asymptotic bound states. It is worth noting that the presence of a trapping well is essential to achieve the superradiant instability because scalar modes could be localized in the trapping well and amplified by superradiance to form quasibound states, thus triggering an instability.

If there is no trapping well under a massive scalar wave propagation, the black hole seems to be superradiantly stable. We note here that a shortened form of the potential were employed to sketch the superradiant instability for rotating black branes and strings [14] because $\Psi_{\ell m} = rR_{\ell m}$ and a modified tortoise coordinate $z_s$ defined by $dz_s = r^2dr/\Delta$ are used by following Ref. [15].

On the other hand, quasibound states have complex frequencies ($\omega = \omega_R + i\omega_I$) as the flux passes one way through the outer horizon when solving the Teukolsky equation.
directly. \( \omega_R \) represents oscillations and the imaginary part of the frequency determines the rate at which the perturbation decays (\( \omega_I < 0 \)) or grows (\( \omega_I > 0 \)) with time. For \( \omega_R = \omega_c \) and \( \omega_R < \mu \) (stationary resonances), \( \omega_I \) vanishes and scalar clouds for extremal Kerr black holes \([16]\) and nearly extremal Kerr black holes \([17]\) have been found. Making use of this threshold for superradiant instability has led to Kerr black holes with scalar hair \([18]\). This implies that the hairy Kerr black holes could be found from the growth of dominant superradiant modes.

Also, the superradiance could represent an amplification of charged massive scalar waves impinging on a static Reissner-Nordström (RN) black hole, provided the frequency \( \omega \) and the charge \( q \) of the scalar wave obey the superradiance condition \( \omega < q \Phi_H \) (\( \Phi_H \) electric potential at horizon) \([19]\). Some aspects of superradiance \([20]\) and the absorption cross section of a charged massive scalar \([21]\) have been studied in the RN black hole background. We wish to point out that the studies in the literature \([15, 22, 23]\) have used a shortened potential to show that contrary to the Kerr case, in the RN case the gravitational attraction between RN black hole and charged massive scalar cannot provide a confinement mechanism which may trigger the superradiant instability. At this stage, it is important to note that the superradiant instability does not arise naturally from a charged massive scalar propagation around the RN black holes. However, the superradiant instability of a charged massive scalar could be obtained if a cavity is introduced to surround the RN black hole \([24, 25, 26, 27]\). This is considered as the ‘charged black hole-mirror bomb’. In this case, numerical techniques have used to show the lower bound \( q > \mu \) which considered as a necessary condition for the superradiant instability \([24]\).

In the Kerr-Newman (KN) black hole background, the superradiant instability condition for a charged massive scalar with \( M\mu \leq 1 \) was firstly obtained as \( qQ < \mu M \) which is regarded as a condition for having a trapping well \([15]\). However, this condition is not satisfied when imposing a superradiance condition (\( \omega < \omega_c \) with \( \omega_c = m \Omega_H + q \Phi_H \)) and thus, it may be regarded as a condition for bound states \([25]\). Also, we note that their effective potential \( V_{\text{eff}}(r) \) is not a correct form. Scalar clouds with \( \omega = \omega_c \) and \( \omega < \mu \) were obtained in \([28, 29]\) and the absorption cross section of a charged massive scalar was recently computed to give a negative cross section for corotating spherical waves \([30]\). Recently, it was reported that the condition for no trapping well is given by two of \( qQ > \mu M \) and \( r_-/r_+ \leq 1/3 \) \([31]\). However, their potential based on the analysis is incorrect.
The superradiant stability of a charged massive scalar based on the correct potential was discussed in the KN black hole background [32].

We would like to stress that most black holes are born very slowly rotating [33]. For example, black holes born from single stars rotate very slowly for \( a = 0.01 \) with \( M = 1 \) and fairly slow rotating black holes born from single stars are regarded as those having \( a \leq 0.1 \). But, it holds only at the very first instances of the black hole formation and other studies [34, 35, 36] have demonstrated that accretion can spin up black holes near extremity. Now, it is curious to introduce a slowly rotating Kerr-Newman (sKN) black hole. This black hole could be obtained from the KN black hole by confining the first order in \( a \) (that is, by taking slow rotation approximation). We note that the sKN black holes take after more the RN black holes than the KN black holes. So, it is interesting to investigate superradiant instability of a charged massive scalar propagating around the sKN black holes. Here, the rotation parameter \( a \) is not considered as an important parameter, in comparison with \( q \) and \( M\mu \) because we confine it to be \( a \leq 0.1 \) for keeping the sKN black holes. In this work, we wish to study the superradiant instability of the sKN black holes under a charged massive scalar perturbation mainly by analyzing its asymptotic potential and far-region wave function. We obtain two conditions of a trapping well (39) for getting quasibound states, whereas the conditions of no trapping well for obtaining bound states are given by (40).

2 Scalar propagation on the sKN black holes

Firstly, we introduce the KN black hole expressed in terms of Boyer-Lindquist coordinates

\[
\text{ds}_{KN}^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} \left( (r^2 + a^2) d\phi - adt \right)^2
\]  

(1)

with

\[
\Delta = r^2 - 2Mr + a^2 + Q^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \text{and} \quad a = \frac{J}{M}.
\]  

(2)

Here, \( M, Q, \) and \( J \) represent the mass, charge, and angular momentum of the KN black hole. In addition, the electromagnetic potential is given by

\[
A_{\mu} dx^\mu = \frac{Qr}{\rho^2} \left( -dt, 0, 0, a \sin^2 \theta d\phi \right).
\]  

(3)
The outer and inner horizons are found by demanding $\Delta = (r - \tilde{r}_+)(r - \tilde{r}_-) = 0$ as

$$
\tilde{r}_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}.
$$

(4)

Taking the slow rotation approximation, we find the slowly rotating Kerr-Newman (sKN) black hole (so-called Lense-Thirring solution) by keeping up to $O(a)$-order [37, 38, 39, 40]

$$
ds^2_{sKN} = \bar{g}_{\mu\nu}dx^\mu dx^\nu
- f_{RN}(r)dt^2 + \frac{dr^2}{f_{RN}(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + 2a(f_{RN}(r) - 1)\sin^2\theta dt d\phi
$$

(5)

with the RN metric function and its electromagnetic potential

$$
f_{RN}(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad \bar{A}_\mu dx^\mu = \frac{Q}{r}(dt, 0, 0, a \sin^2\theta d\phi).
$$

(6)

Clearly, the line element (5) is stationary but non-static because $dt \to -dt$ changes the signature of the metric and it is also axially symmetric (invariance under $d\theta \to -d\theta$). We note that the sKN spacetime [5] inherits the hidden Killing symmetries of the full solution [1] to $O(a)$-order [41]. In this case, the outer and inner horizons are given by those of the RN black holes as

$$
r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}.
$$

(7)

A charged massive scalar perturbation $\Phi$ on the background of sKN black holes is described by

$$
(\nabla^\mu - iq\bar{A}^\mu)(\nabla_\mu - iq\bar{A}_\mu)\Phi - \mu^2\Phi = 0.
$$

(8)

Considering the axis-symmetric background [5], it is convenient to separate the scalar perturbation into modes

$$
\Phi(t, r, \theta, \phi) = \sum_{\ell m} e^{-i\omega t + im\phi} P^m_\ell(\theta) R_{\ell m}(r),
$$

(9)

where $P^m_\ell(\theta)$ is an associate Legendre polynomial with $-m \leq \ell \leq m$ ($Y_{\ell m}(\theta, \phi) \sim P^m_\ell(\theta)e^{im\phi}$) and $R_{\ell m}(r)$ satisfies a radial part of the wave equation. Substituting (9) into (8), we have an associate Legendre equation for $P^m_\ell(\theta)$ and a radial equation for $R_{\ell m}(r)$ with $\tilde{\Delta} = r^2f_{RN}(r)$ [28]

$$
\frac{1}{\sin\theta}\partial_\theta \left(\sin\theta\partial_\theta P^m_\ell(\theta)\right) + \left[\ell(\ell + 1) - \frac{m^2}{\sin^2\theta}\right] P^m_\ell(\theta) = 0,
$$

(10)
\[ \hat{\Delta} \partial_r \left( \hat{\Delta} \partial_r R_{\ell m}(r) \right) + U(r) R_{\ell m}(r) = 0 \]  
(11)

with

\[ U(r) = (\omega^2 - q Q r)^2 - 2 a m (2 M \omega r - q Q r - \omega Q^2) - \hat{\Delta} [\mu^2 r^2 + \ell (\ell + 1)]. \]  
(12)

It is worth noting that Eq. (11) is usually used to obtain exact solutions.

Now, we introduce the tortoise coordinate \( r_* \) defined by

\[ r_* = \int \frac{dr}{f_{RN}(r)} = r + \frac{r_+^2}{r_+ - r_-} \ln(r - r_+) - \frac{r_-^2}{r_+ - r_-} \ln(r - r_-) \]  
(13)

to derive the Schrödinger-type equation. Then, the radial equation (11) takes a form of the Schrödinger-type equation when setting \( \Psi_{\ell m} = r R_{\ell m} \)

\[ \frac{d^2 \Psi_{\ell m}(r_*)}{dr_*^2} + V(r) \Psi_{\ell m}(r_*) = 0, \]  
(14)

where \( V(r) \) is found to be \(^{[29]}\)

\[ V(r) = \left( \omega - \frac{q Q}{r} \right)^2 - \frac{2 a m (2 M \omega r - q Q r - \omega Q^2)}{r_+^4} - f_{RN}(r) \left[ \mu^2 + \frac{\ell (\ell + 1)}{r^2} + \frac{2 (M r - Q^2)}{r_+^4} \right]. \]  
(15)

In the non-rotating limit of \( a \to 0 \), we could recover the scalar potential from Eq. (15) for studying superradiance in the RN black hole \(^{[20, 21, 24, 25]}\). In case of \( Q = 0 \) and \( \mu = 0 \), one finds the scalar potential from Eq. (15) for investigating the Lense-Thirring black hole \(^{[42]}\). In the asymptotic limit, one has \( V(r \to \infty) = \omega^2 - \mu^2 \), while one gets \( V(r \to r_+) = (\omega - \omega_c) (\omega - \tilde{\omega}_c) > 0 \) with \( f_{RN}(r_+) = 0 \) in the near-horizon limit. Here, we have two critical frequencies of \( \omega_c = q Q / r_+ \) and \( \tilde{\omega}_c = \omega_c + 2 a m / r_+^2 > \omega_c \). Taking the asymptotic limit of Eq. (14) and its near-horizon limit, one has the solutions

\[ \Psi \sim e^{-i \sqrt{\omega^2 - \mu^2} r_* (\leftarrow)} + R e^{i \sqrt{\omega^2 - \mu^2} r_* (\rightarrow)}, \quad r_* \to +\infty (r \to \infty), \]  
(16)

\[ \Psi \sim T e^{-i \sqrt{(\omega - \omega_c) (\omega - \tilde{\omega}_c)} (\leftarrow)}, \quad r_* \to -\infty (r \to r_+), \]  
(17)

where \( T(R) \) are the transmission (reflection) amplitudes.

Imposing the flux conservation, we obtain the relation between reflection and transmission coefficients as

\[ |R|^2 = 1 - \frac{\sqrt{(\omega - \omega_c) (\omega - \tilde{\omega}_c)}}{\sqrt{\omega^2 - \mu^2}} |T|^2, \]  
(18)
which means that only waves with $\omega > \mu$ propagate to infinity and the superradiant scattering may occur ($\rightarrow$, $|R|^2 > |I|^2$) whenever $\omega < \omega_c$ (superradiance condition) is satisfied because outgoing waves at the outer horizon reinforce the outgoing waves at infinity. The absorption cross section is given by

$$\sigma = \sum_{\ell=0}^{\infty} \sigma_\ell,$$  \hspace{1cm} (19)

where the partial absorption cross section $\sigma_\ell$ takes the form

$$\sigma_\ell = \frac{\pi}{\omega^2 - \mu^2} (2\ell + 1)(1 - |R|^2) = \frac{\pi \sqrt{(\omega - \omega_c) \sqrt{(\omega - \tilde{\omega}_c)}}}{(\omega^2 - \mu^2)^{3/2}} (2\ell + 1) |T|^2.$$  \hspace{1cm} (20)

On the other hand, one may choose the scalar modes to have an exponentially decay as it tends to zero at infinity

$$R_{\ell m} \sim e^{-\sqrt{\mu^2 - \omega^2} r} \rightarrow 0$$  \hspace{1cm} (21)

with the bound state condition of $\omega < \mu$.

When solving Eq. (11) directly, the frequency $\omega$ is permitted to be complex (small complex modification) as [11]

$$\omega = \omega_R + i\omega_I.$$  \hspace{1cm} (22)

In this case, the sign of $\omega_I$ determines the solution which is decaying ($\omega_I < 0$) or growing ($\omega_I > 0$) in time. Considering the asymptotic solution form in Eq. (21), the quasibound state condition can be obtained if $\text{Re}[\sqrt{\mu^2 - \omega^2}] > 0$, tending to zero at infinity, in addition to ingoing at the outer horizon. We note that the boundary condition for the well-known quasinormal modes is ingoing at the outer horizon and purely outgoing (and divergent for $\text{Re}[\sqrt{\mu^2 - \omega^2}] < 0$) at infinity. In both quasibound and quasinormal cases [13], imposing a pair of boundary conditions leads to a discrete spectrum of complex frequencies.

Finally, we wish to mention four cases for a charged massive scalar propagating around the sKN black holes based on the potential analysis:

Case (i) superradiant scattering: $\omega < \omega_c$ and $\omega > \mu$.
Case (ii) superradiant stability: $\omega < \omega_c$ and $\omega < \mu$ without a positive trapping well.
Case (iii) stationary resonances (marginally stable): $\omega = \omega_c$ and $\omega < \mu$.
Case (iv) superradiant instability: $\omega < \omega_c$ and $\omega < \mu$ with a positive trapping well.

Solving the radial equation (11) leads to the real part of frequency ($\omega_R$) and the imaginary
part \((\omega_1)\). In this case, one describes again the last three cases:

Case (ii): \(\omega_1 < 0\) and \(\omega_R < \omega_c\). The solution is stable (decaying in time).

Case (iii): \(\omega_1 = 0\) and \(\omega_R = \omega_c\).

Case (iv): \(\omega_1 > 0\) and \(\omega_R < \omega_c\). The solution is unstable (growing in time).

3 Potential analysis for superradiant instability

We wish to rewrite Eq. (14) as

\[
\frac{d^2 \Psi_{\ell m}(r_s)}{dr_s^2} + \left[ \omega^2 - V_{sKN}(r) \right] \Psi_{\ell m}(r_s) = 0, \tag{23}
\]

where the potential \(V_{sKN}(r)\) is given by

\[
V_{sKN}(r) = \mu^2 - \frac{2(M \mu^2 - qQ \omega)}{r} + \frac{Q^2(\mu^2 - q^2) + \ell(\ell + 1)}{r^2} + \frac{2M - 2M \ell(\ell + 1) + 2am(2M \omega - qQ)}{r^3} - \frac{4M^2 + 2Q^2 - Q^2 \ell(\ell + 1) + 2amQ^2 \omega}{r^4} + \frac{6MQ^2}{r^5} - \frac{2Q^4}{r^6}. \tag{24}
\]

An asymptotic form of the potential is given by

\[
V_a(r) = \mu^2 - \frac{2(M \mu^2 - qQ \omega)}{r} \tag{25}
\]

which appears in the asymptotic region. This potential could be used to find the condition for a trapping well as

\[
V_a'(r) > 0 \quad \rightarrow \quad M \mu^2 > qQ \omega, \tag{26}
\]

while the condition for no trapping well is realized as

\[
V_a'(r) < 0 \quad \rightarrow \quad M \mu^2 < qQ \omega. \tag{27}
\]

However, Eq. (26) [Eq. (27)] is not a sufficient condition for a trapping well [no trapping well]. We have to find the other conditions. For this purpose, we introduce the far-region potential appeared in the large \(r\) region

\[
V_{fr}(r) = \mu^2 - \frac{2(M \mu^2 - qQ \omega)}{r} + \frac{Q^2(\mu^2 - q^2) + \ell(\ell + 1)}{r^2}, \tag{28}
\]

where the last term plays a crucial role of making a trapping well.
Figure 1: (Left) Superradiant scattering potential $V_{sKN}(r)$ as function of $r \in [r_+ = 1.8, 100]$ with $M = 1, Q = 0.6, \omega = 0.05, a = 0.1, m = 1, q = 0.1, \ell = 1, \mu = 0.02$. The height of potential barrier is $0.114(\gg \omega^2)$ at $r = 2.64$. (Right) Superradiant stable potential $V_{sKN}(r)$ as function of $r \in [r_+ = 1.7, 100]$ with $M = 1, Q = 0.7, \omega = 0.04, a = 0.1, m = 10, q = 2, \ell = 10, \mu = 0.05$. The height of barrier is $4.6(\gg \omega^2)$ at $r = 2.72$. We check the conditions of $\omega < \mu$ and $\omega < \omega_c = 0.8$ to have a superradiant stability with $V'_a(r) < 0$.

In the non-rotating limit of $a \to 0$, one finds the potential for a charged massive scalar propagating around the RN black holes \cite{20, 21, 24, 25} as

$$V_{RN}(r) = \mu^2 - \frac{2(M\mu^2 - qQ\omega)}{r} + \frac{Q^2(\mu^2 - q^2)}{r^2} + \frac{\ell(\ell + 1)}{r^3}$$
$$+ \frac{2M - 2M\ell(\ell + 1)}{r^3} - \frac{4M^2 + 2Q^2 - Q^2\ell(\ell + 1)}{r^4} + \frac{6MQ^2}{r^5} - \frac{2Q^4}{r^6}(29)$$

whose asymptotic and far-region potentials are still given by Eqs. \cite{25} and \cite{28}.

At this stage, it is worth mentioning that $V_{sKN}(r)$ and $V_{RN}(r)$ are $\omega$-dependent potentials, compared to the standard potentials appeared in Schrödinger-like problems. However, some information on the superradiant instability could be extracted by investigating these potentials.

Hereafter, we choose $M = 1$ such that $M\mu$ becomes $\mu$ for a simple analysis. First of all, we consider the superradiant scattering [Case (i)]. We display the corresponding potential in (Left) Fig. 1, indicating that $\mu, \omega, q\Phi_H \ll 1$ and $\mu < \omega_c < \omega$ with $a = 0.1$ to give the negative absorption cross section. In addition, the superradiant stability [Case (ii)] could be achieved for $\omega < \mu, \omega < \omega_c$, and $q > \mu$ as is shown in (Right) Fig. 1. It includes no trapping well, being consistent with $V'_a(r) < 0$.

It is curious to note that (Left) Fig. 2 corresponds to a superradiantly stable potential
Figure 2: (Left) Superradiant stable potential $V_{s\text{KN}}(r)$ as function of $r \in [r_+ = 1.8, 100]$ with $M = 1, Q = 0.6, \omega = 0.02, a = 0.1, m = 1, q = 0.1, \ell = 1, \mu = 0.05$. The height of barrier is $0.11(\gg \omega^2)$ at $r = 2.66$. We check the conditions of $\omega < \mu$ and $\omega < \omega_c = 0.03$ to have a superradiant stability, but $V_a''(r) > 0$ implies a trapping well. (Right) Asymptotic forms of $V_{s\text{KN}}(r) \simeq V_b(r)$ indicate a tiny well located at $r = 1535$. $V_a(r)$ approaches them for $r > 1535$.

Figure 3: Stationary resonances potential $V_{s\text{KN}}(r)$ as function of $r \in [r_+ = 1.999, 100]$ with $M = 1, Q = 0.01, \omega = 0.001, a = 0.1, m = 10, q = 0.2, \ell = 10, \mu = 0.4$. We have $V_{s\text{KN}}(r_+) = 0 \simeq \omega^2$ and the height of potential barrier is $4.14(\gg \omega^2)$ at $r = 3.01$. We note $\omega = \omega_c = 0.001$ and $\omega < \mu$ to meet the condition for obtaining scalar clouds.

because we could not find a trapping well for $\omega < \omega_c = 0.03$ and $\omega < \mu$. In this case, however, we observe $V_a''(r) > 0$ which may imply the superradiant instability. So, $V_a''(r) > 0$ contradicts to our expectation of no trapping well. We wish to resolve it. We find from
Figure 4: (Left) Superradiant unstable potential $V_{sKN}(r)$ as function of $r \in [r_+ = 1.999, 100]$ with $M = 1, Q = 0.01, \omega = 2.95, a = 0.1, m = 13, q = 20, \ell = 13, \mu = 3$. The potential at $r = r_+$ is 2.43 and a peak of potential is 10.93($\omega^2$) at $r = 3.41$. A trapping well (local minimum) is located at $r = 18.24$. We note $\omega < \mu$, but $\omega > \omega_c = 0.1$ fails to satisfy the superradiance condition ($\omega < \omega_c$). (Right) Superradiant unstable potential $V_{RN}(r)$ with $a = 0$. One has $V_{RN}(r_+) = 0.58$ and a peak of potential is 10.59($\omega^2$) at $r = 3.59$. A trapping well is located at $r = 18.06$. With $\omega < \mu$, we note that $\omega > \omega_c = q\Phi_H = 0.1$ fails to satisfy the superradiance condition.

(Right) Fig. 2 that a tiny well is located at a very large distance of $r = 1535$ in $V_{sKN}(r) \simeq V_a(r)$, but it does not affect the superadiant stability. It indicates that $V'_a(r) > 0$ implies either a trapping well or a tiny well. Hence, one has to find the other condition for a trapping well in the next section.

To visualize stationary resonances [Case (iii)], we observe the corresponding potential $V_{sKN}(r)$ with $a = 0.1$ and $\mu = 0.4$ in Fig. 3. This is similar to (Left) Fig. 2, showing the superradiant stability except $\omega = \omega_c$. But, the effect of imposing $\omega = \omega_c$ appears in the near-horizon region.

As a specific example for Case (iv), we wish to introduce a sKN potential [(Left) Fig. 4) with $a = 0.1$ and $q > \mu$. It seems that quasibound states of a charged massive scalar do not contain superradiant states. That is, the condition for a trapping well and the superradiance condition ($\omega < \omega_c$) cannot be satisfied simultaneously because of $\omega > \omega_c$. This implies that the superradiant instability is not found from a charged massive scalar propagating around the sKN black holes. In this case, the scalar mass $\mu$ is no longer a reflecting mirror. The same feature is found from the RN potential [(Right) Fig. 4] [24].

However, there might be a way to obtain the superradiant instability in the sKN black
hole background. If a mirror (cavity) is placed at some radial coordinate $r_m$ outside the outer horizon, the asymptotic boundary condition is modified such that the scalar mode vanishes exactly at $r = r_m$ [$R_{\ell m}(r_m) = 0$] instead of Eq. (21) and its proper frequency may be determined by imposing the position of the mirror. The scalar mode might have frequencies that are in the superradiant regime ($\omega_R < \omega_c$) because one can place the mirror at arbitrarily close to the black hole horizon ($r_m \to r_+$ in the $qQ \to \infty$ limit). In the non-rotating limit of $a \to 0$ (the RN black hole) \cite{24, 25}, one has found the real and imaginary part of the resonance frequency as \cite{26}

$$\omega_R = \omega_c (1 - x_m), \quad \omega_I = \omega_c \sqrt{\frac{x_m^3}{r}}, \quad (30)$$

where

$$\omega_c = \frac{qQ}{r_+}, \quad x_m = \frac{r_m - r_+}{r_+}, \quad \tau = \frac{r_+ - r_-}{r_+} \quad (31)$$

in the asymptotic regime of $qQ \gg \tau/x_m \gg 1$. We note here that $\omega_R < \omega_c$ and $\omega_I > 0$ implies a superradiant instability of the charged black hole-mirror system.

4 Far-region and asymptotic wave functions

It is crucial to find the scalar wave forms in the far-region to distinguish between quasibound states (trapping well) and bound states (no trapping well). This is because the condition of $V'_{a}(r) > 0$ in Eq. (26) is not a sufficient condition for getting a trapping well.

In the far-region where we may take $r_* \simeq r$, we obtain an equation from (23) together with (28) as

$$\left[ \frac{d^2}{dr^2} + \omega^2 - V_{a}(r) \right] \Psi_{\ell m}(r) = 0 \quad (32)$$

whose solution is given exactly by the confluent Hypergeometric function $U(a, b; cr)$ as

$$\Psi_{\ell m}(r) = c_1 e^{-\sqrt{\mu^2 - \omega^2} r} \left( 2\sqrt{\mu^2 - \omega^2} r \right)^{\frac{1}{2} + k} \times U\left( \frac{1 + 2k}{2} - \frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}}, 1 + 2k; 2\sqrt{\mu^2 - \omega^2} r \right) \quad (33)$$

with

$$k = \frac{1}{2} \sqrt{1 + 4[\ell(\ell + 1) + Q^2(\mu^2 - q^2)]}. \quad (34)$$

Here, we observe a bound state of $e^{-\sqrt{\mu^2 - \omega^2} r}$ with $\omega < \mu$ appeared in (21). Furthermore,
Figure 5: (Left) Bound state function $\Psi_{1010}(r)$ and its confluent hypergeometric function $U(12.7, 22; 0.06r)$ as $r \in [10, 100]$ without trapping well. (Right) Its asymptotic wave function $\Psi_{1010}^A(r)$ represents an asymptotic bound state.

Figure 6: Bound state function $\Psi_{11}(r)$, its confluent hypergeometric function $U(1.97, 4; 0.09r)$, and its asymptotic wave function $\Psi_{11}^A(r)$ as $r \in [10, 100]$ without trapping well.

Figure 7: (Left) Quasi-bound state of $\Psi_{1313}(r)$ as function of $r \in [22, 100]$ with trapping well. Here, we start with $r = 18$ because a local minimum of $V_{KN}(r)$ in Eq. (24) is located at $r = 18.2$. (Right) Confluent hypergeometric function $U(-1.42, 28; 1.1r)$ represents an increasing function of $r$ approximately.

we find some information from the large $r$-form of $U(a, b; cr)$ as

$$U(a, b; cr \to \infty) \to \frac{1}{(cr)^a} \left[ 1 - \frac{a(1 + a - b)}{cr} + O\left(\frac{1}{cr}\right)^2 \right]$$

(35)
which implies approximately that one finds a decreasing function $U(a, b; cr)$ for a positive $a$, whereas one has an increasing function for a negative $a$. Substituting Eq. (35) into Eq. (33) leads to the asymptotic wave function as

$$
\Psi_{\ell m}^A(r) \simeq e^{-\sqrt{\mu^2 - \omega^2} r} \left(2\sqrt{\mu^2 - \omega^2} r\right)^{\frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}}}.
$$

(36)

We consider three cases with $q > \mu$ only. Considering the potential without trapping well whose asymptotic derivative is negative ($V'_a(r) < 0$) [(Right) Fig. 1], $\Psi_{1010}(r)$ in Fig. 5 shows an exponentially decaying mode (bound state). Also, we have a rapidly decreasing function $U(12.7, 22; 0.06r)$ and the asymptotic wave function $\Psi_{1010}^A(r)$ is an exponentially decreasing function. This case represents no trapping well clearly.

We introduce an interesting potential without apparently trapping well whose asymptotic derivative is positive ($V'_a(r) > 0$) [(Left) Fig. 2]. Its wave function $\Psi_{11}(r)$ and its confluent hypergeometric function $U(1.97, 4; 0.1r)$ indicate monotonically decreasing modes [Fig. 6]. Also, its asymptotic wave function $\Psi_{11}^A(r)$ is a slowly decreasing function. Although this potential includes a tiny well located at $r = 1535$ [see (Right) Fig. 2], it could be neglected effectively. After analyzing far-region wave function, we conclude that it does not include an apparently trapping well.

Let us observe a radial mode $\Psi_{\ell m}(r)$ for a trapping well [see (Left) Fig. 4]. As is shown in (Left) Fig. 7, Eqs. (33) and (36) show quasi-bound states. In this case, one has an increasing function $U(-1.42, 28; 1.1r)$ appeared in (Right) Fig. 7.

Therefore, the quasibound state could be achieved when the first argument of $U(a, b; cr)$ is negative as

$$
\begin{align*}
a < 0 & \rightarrow \frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}} > k + \frac{1}{2}, \\
& \text{which is considered as the other condition for trapping well. On the other hand, the bound state could be found when the first argument of } U(a, b; x) \text{ is positive as} \\
a > 0 & \rightarrow \frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}} < k + \frac{1}{2},
\end{align*}
$$

(37) and (38)

which is regarded as the other condition for no trapping well.

At this stage, we have the same condition (37) for having a trapping well under a charged massive scalar propagating around the RN black holes [see (Right) Fig. 5] because its asymptotic and far-region potentials is given exactly by and (25) and (28), respectively. Also, no trapping well is allowed under the condition of (38).
Finally, we obtain two conditions for getting a trapping well as

\[ M\mu^2 > qQ\omega \quad [V'_a(r) > 0] \quad \text{and} \quad \frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}} > k + \frac{1}{2} \]  \quad (39)

On the other hand, two conditions for no trapping well are given by

\[ M\mu^2 < qQ\omega \quad [V'_a(r) < 0] \quad \text{and} \quad \frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}} < k + \frac{1}{2} \]  \quad (40)

where the former condition could be written as

\[ \frac{M\mu}{qQ} < \frac{\omega}{\mu} < 1, \]  \quad (41)

which may imply one condition for the superradiant stability as \( qQ > M\mu \) appeared in Ref.\[31\]. The other case of \( M\mu^2 > qQ\omega \quad [V'_a(r) > 0] \quad \text{and} \quad \frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}} < k + \frac{1}{2} \) represents a tiny well located at a very large distance (\( r = 1535 \)), which is considered as no trapping well apparently. This explains why one needs to have two conditions to specify a trapping well.

5 Discussions

We have studied the superradiant instability of slowly rotating Kerr-Newman (sKN) black holes under a charged massive scalar perturbation. It is worth noting that these black holes take after more the RN black holes than the KN black holes because the rotation parameter restricting \( a \leq 0.1 \) is not an important one, in comparison with \( q \) and \( \mu \).

For \( q > \mu \), we have obtained a potential with a trapping well. However, the superradiance condition of \( \omega < \omega_c \) fails to be satisfied, being similar to the RN black holes. If a mirror (cavity) is introduced at some radius \( r_m \) outside the outer horizon, the superradiance condition could be managed to be satisfied.

Importantly, we obtain two conditions of a trapping well for getting quasibound states. On the other hand, two conditions of no trapping well for obtaining bound states are given by (40). Furthermore, we could apply these conditions to a charged massive scalar propagating around the RN black holes, too.

Finally, it is worth noting that the superradiant instability could be achieved if all of two conditions for a trapping well, superradiance condition (\( \omega < \omega_c \)), and bound state condition (\( \omega < \mu \)) are satisfied. Also, the superradiant stability is found when all of two
conditions for no trapping-well \([43]\), superradiance condition \(\omega < \omega_c\), and bound state condition \((\omega < \mu)\) are satisfied.

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