Vertex-Coloring Graphs with 4-Edge-Weightings

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1 Introduction

Let $G = (V, E)$ be a simple graph. A $k$-edge-weighting is a function $\omega : E \to \{1, \ldots, k\}$. Given an edge-weighting $\omega$, for a vertex $v \in V$, we denote by $s_\omega(v) = \sum_{w \in N(v)} \omega(\{v, w\})$ its weighted degree. We say that $\omega$ is vertex-coloring if for each edge $e = \{u, v\} \in E$, it holds $s_\omega(u) \neq s_\omega(v)$. Obviously, if $G$ contains an isolated edge, no edge-weighting is vertex-coloring. Otherwise, we aim to find a vertex-coloring $k$-edge-weighting with the smallest possible integer $k$. In 2004, Karoński, Łuczak, and Thomason conjectured that for each graph without connected component isomorphic to $K_2$, a vertex-coloring 3-edge-weighting exists [1]. This statement is also known as 1-2-3-conjecture. For instance, if $G$ is a cycle of length not divisible by 4, no 2-edge-weighting is vertex-coloring, thus $k = 3$ is best possible in general.

Addario-Berry, Dalal, McDiarmid, Reed, and Thomason proved the first general upper bound of $k = 30$ [2]. The bound was improved to $k = 16$ by Addario-Berry, Dalal, and Reed [3], to $k = 13$ by Wang and Yu [4], and then, in a significant improvement, to $k = 5$ by Kalkowski, Karoński, and Pfender [5] in 2010. For a random
graph $G(n, p)$, asymptotically almost surely there exists a vertex-coloring 2-edge-weighting [3]. For $d$-regular graphs, an upper bound of $k = 4$ holds in general [6, 7] and the conjecture (i.e., an upper bound of $k = 3$) is confirmed for $d > 10^8$ [6]. Recently, Przybyło verified the conjecture for graphs where the minimum degree is sufficiently large compared to the maximum degree [8]. Furthermore, the conjecture was confirmed for 3-colorable graphs [1] and for dense graphs [9]. Moreover, Dudek and Wajc proved that it is NP-complete to decide whether a given graph $G$ admits a vertex-coloring 2-edge-weighting [10]. For an early overview on the conjecture and on related problems, we refer to the survey of Seamone [11].

The contribution of this note is an improved general upper bound of $k = 4$ with the following result.

**Theorem 1** Let $G = (V, E)$ be a graph without connected component isomorphic to $K_2$. Then there exists an edge-weighting $\omega : E \rightarrow \{1, 2, 3, 4\}$ such that for any two neighbors $u$ and $v$,

$$\sum_{w \in N(u)} \omega(\{u, w\}) \neq \sum_{w \in N(v)} \omega(\{v, w\}).$$

**2 Proof**

We start by introducing notation, giving a high-level overview of the proof, and collecting two auxiliary results. Let $G = (V, E)$ be a graph and let $C = (S, T)$ be a cut. We denote by $E(S)$ the edge set of the induced subgraph $G[S]$ and by $E(S, T)$ the subset of edges having an endpoint in both $S$ and $T$ (the cut edges of $C$). For a vertex $v \in V$, we denote by $N(v)$ its neighborhood.

We will start the proof by identifying a vertex $v_0$ that is handled separately. Next, we will take a maximum cut $C = (S, T)$ of $G[V \setminus \{v_0\}]$ and construct an edge-weighting such that the weighted degree of a vertex $v \neq v_0$ is even if $v \in S$ and odd if $v \in T$. There may still be conflicting edges, that is, neighboring vertices with the same weighted degree. The main idea to solve these conflicts is to modify the edge-weighting along sufficiently many edge-disjoint paths.

To actually find a suitable collection of edge-disjoint paths, we carefully design a flow problem that depends on the set of conflicting edges. For the flow problem, we use the edges of the cut $C$. Because $C$ is chosen as a maximum cut, it turns out that the auxiliary network behind the flow problem is sufficiently dense to guarantee that the resulting maximum flow yields the desired number of edge-disjoint paths. The subsequent lemma gives the formal statement.

**Lemma 2** Let $G = (V, E)$ be a graph and let $C = (S, T)$ be a maximum cut of $G$. Furthermore, let $F \subseteq E(S) \cup E(T)$ and let $\sigma$ be an orientation of the edge set $F$. Let $G_{C,F,\sigma}$ be the auxiliary directed multigraph network constructed as follows.

(i) As vertex set, take $V$, and add a source node $s$ and a sink node $t$.
(ii) For each edge $\{u, v\} \in E(S, T)$, insert the two arcs $(u, v)$ and $(v, u)$, both with capacity 1.
(iii) For each edge \( \{u, v\} \in F \) with corresponding orientation \((u, v) \in \sigma\), insert arcs \((s, u)\) and \((v, t)\), both with capacity 1, potentially creating multi-arcs. Do not insert \((u, v)\).

Then in the network \( G_{C, F, \sigma} \), there exists an \( s-t \)-flow of size \(|F|\).

The cardinalities of \( S \) and \( T \) will determine the parity of the weighted degree of the remaining vertex \( v_0 \). It remains to ensure that the edge-weighting also properly colors \( v_0 \), which will be done by applying the following technical lemma to the induced subgraph \( G[N(v_0) \cup \{v_0\}] \) and using especially property (iv) of its statement.

**Lemma 3** Let \( G = (V, E) \) be a graph with \(|V| \geq 3\), let \( v_0 \in V \) such that \( \deg(v_0) = |V| - 1 \), and let \( g : V \to \mathbb{N}_0 \) be such that for each edge \( \{u, v\} \in E(N(v_0)) \), it holds \( g(u) \neq g(v) \). Then there exists a function \( h : E \to [0, 1, 2] \) such that

(i) \( h([u, v]) \in \{0, 1\} \), whenever \( v_0 \not\in [u, v] \) and \( g(u) + g(v) \) is even,

(ii) \( h([u, v]) = 0 \), whenever \( v_0 \not\in [u, v] \) and \( g(u) + g(v) \) is odd,

(iii) \( s_h(v) := \sum_{w \in N(v)} h([v, w]) \in \{0, 2\} \) for all \( v \in N(v_0) \), and

(iv) \( g(u) + s_h(u) \neq g(v) + s_h(v) \) for each edge \( \{u, v\} \in E \).

We now start proving the theorem. The proofs of Lemma 2 and 3 are deferred to the end of the paper.

**Proof of Theorem 1** Assume w.l.o.g. that \( G \) is connected and has at least three vertices. We give each edge \( e \) the provisiorial weight \( \mu(e) = 2 \), which will be modified later on. Denote by \( s_\mu(v) := \sum_{w \in N(v)} \mu([v, w]) \) the weighted degree of a vertex \( v \in V \) under \( \mu \). Let \( v_0 \) be a vertex which is not an articulation node (for instance, take a leaf of a spanning tree), so that the reduced graph \( H := G[V \setminus \{v_0\}] \) is still connected. Next, take a maximum cut \((S, T)\) of \( H \). Let \( G(S, T) \) be the bipartite subgraph with vertex set \( V(H) = V \setminus \{v_0\} \) and edge set \( E(S, T) \). Observe that \( G(S, T) \) is connected due to the maximality of the cut.

Let \( r \in N(v_0) \) and take a spanning tree \( T' \) of \( G(S, T) \) rooted at \( r \). For each node \( v \not\in r \) in the tree, denote by \( \text{par}(v) \) its parent in \( T' \). We are going to modify \( \mu \) on the edges of \( T' \) so that \( s_\mu(v) \) is even if \( v \in S \) and odd if \( v \in T \). We start with the leafs. For each leaf \( \ell \), put \( \mu([\ell, \text{par}(\ell)]) := 3 \) if \( \ell \in S \). If \( \ell \in S \), we keep \( \mu([\ell, \text{par}(\ell)]) = 2 \). Then indeed, \( s_\mu(\ell) \) is even if and only if \( \ell \in S \).

Afterwards, we iterate the idea level by level towards root \( r \), processing each internal node only after all its child nodes have been handled: We assign to each tree-edge \( [v, \text{par}(v)] \) weight either 2 or 3 such that the parity modulo 2 of \( s_\mu(v) \) becomes correct. Finally, we assign to the edge \( e_0 = \{v_0, r\} \) weight either 1 or 2 such that the parity of \( s_\mu(r) \) becomes correct as well.

So far, neighboring vertices on different sides of the cut \( C \) receive different weighted degrees. We need to ensure that the same happens for neighbors on the same side of \( C \) as well, and we should not forget \( v_0 \). The plan is to give each vertex \( v \in V \) a designated “color” \( f(v) \) such that neighboring vertices always receive different colors. Afterwards, from \( \mu \) we construct a new edge-weighting \( \omega \) such that under \( \omega \), indeed each vertex \( v \) obtains weighted degree \( f(v) \). We start with \( v_0 \) and its neighborhood. Let \( N(v_0) = \{v_1, \ldots, v_m\} \subseteq V(H) \), with arbitrary order. We assign to each \( v_i \in N(v_0) \)
values \( k(v_i) \) and \( g(v_i) \) as follows. We start with \( k(v_1) := 0 \) and \( g(v_1) := s_\mu(v_1) \). For \( i > 1 \), choose \( k(v_i) \in \mathbb{N}_0 \) minimal such that \( g(v_i) := s_\mu(v_i) + 2k(v_i) \) is different from \( g(v_j) \) for all \( j < i \) with \( \{ v_i, v_j \} \in E \). If \( v_i \) has no such neighbors, use \( k(v_i) := 0 \) and \( g(v_i) := s_\mu(v_i) \). A vertex \( v_i \in S \) thus has at least \( k_i \) neighbors in \( S \) with smaller index (and the same is obviously true for \( T \)). For the sake of completeness, set \( g(v_0) := s_\mu(v_0) \).

Assume first that \( \text{deg}(v_0) > 1 \). We apply Lemma 3 to the induced subgraph \( G[N(v_0) \cup \{ v_0 \}] \), which is possible since \( g \) as defined above indeed satisfies the precondition. The lemma yields a function \( h : E(N(v_0) \cup \{ v_0 \}) \rightarrow \{ 0, 1, 2 \} \), which we use as follows.

First, we consider the edge weighting. So far, edges incident to \( v_0 \) have weight either 1 or 2. All other edges have weight 2, except some cut edges \( e \in E(S, T) \) with \( \mu(e) = 3 \). For each edge \( e \in E(N(v_0) \cup \{ v_0 \}) \), we set \( \omega(e) := \mu(e) + h(e) \). For all other edges, we put \( \omega(e) := \mu(e) \). Then, edges \( e \) incident to \( v_0 \) satisfy \( \omega(e) \leq 2 + h(e) \leq 4 \). Regarding the cut edges, recall that for \( u \in S \) and \( v \in T \), \( \mu(u) + s_\mu(v) \) is odd. Hence, by property (ii) of Lemma 3, \( h \) vanishes on cut edges, implying that we still have \( \omega(e) = \mu(e) \in \{ 2, 3 \} \) if \( e \) is a cut edge of \( C \). Finally, if \( e \in E(S) \cup E(T) \), then \( h(e) \in \{ 0, 1 \} \) by (i), implying \( \omega(e) \in \{ 2, 3 \} \) as well. Thus, on edges not incident to \( v_0 \), we can further increase or decrease the weighting \( \omega \) by 1 later on.

Second, we assign to each node \( v \in N(v_0) \cup \{ v_0 \} \) the designated color \( f(v) := g(v) + s_h(v) \). By property (iii) of Lemma 3, we preserve parities, i.e., \( f \) is even-valued on \( S \) and odd-valued on \( T \). By (iv), indeed neighboring nodes receive different designated colors. Furthermore, \( f(v_0) \) already coincides with \( s_\omega(v_0) \), and for all \( v \in N(v_0) \), we have

\[
f(v) - s_\omega(v) = g(v) + s_h(v) - s_\omega(v) = 2k(v).
\]

In the special case \( \text{deg}(v_0) = 1 \), \( r = v_1 \) is the only neighbor of \( v_0 \). Here, we directly put \( \omega \equiv \mu \), and then set \( f(v_0) := s_\omega(v_0) = \mu(e_0) \) and \( f(v_1) := s_\omega(v_1) \). Since \( |V| \geq 3 \) and \( G \) is connected, \( v_1 \) has at least one other incident edge in addition to \( e_0 \). Therefore, \( s_\omega(v_1) > s_\omega(v_0) \) and \( f(v_1) > f(v_0) \). Clearly, for each edge \( e \neq e_0 \) we again have \( \omega(e) \in \{ 2, 3 \} \).

We now turn to the remaining set of vertices \( V \setminus (N(v_0) \cup \{ v_0 \}) := \{ v_{m+1}, \ldots, v_{n-1} \} \), which didn’t yet receive a designated color. Similarly as before, for each \( v_i \), put \( f(v_i) = s_\omega(v_i) + 2k(v_i) \), where \( k(v_i) \geq 0 \) is the minimal integer such that \( f(v_i) \) differs from all \( f(v_j) \) for all its neighbors \( v_j \) with \( 1 \leq j < i \). Hence, for each vertex \( v_i \neq v_0 \), we ensured \( f(v_i) - s_\omega(v_i) = 2k(v_i) \). Moreover, any two neighbors of the graph already have different designated colors. For later reference, denote by \( t(v_i) := f(v_i) - 2k(v_i) \) the current weighted degree of \( v_i \) under \( \omega \).

To actually achieve the desired colors, the weighted degree \( s_\omega(v_i) \) of each node \( v_i \) should further increase by exactly \( 2k(v_i) \). In order to solve this task, we construct a subset \( F \subseteq E(S) \cup E(T) \) and an orientation \( \sigma \) of \( F \) as follows. For each vertex \( v_i \in S \), choose \( k(v_i) \) neighbors \( v_j \in S \) with smaller index (i.e., \( 1 \leq j < i \)), add \( \{ v_i, v_j \} \) to \( F \), and add the orientation \( (v_i, v_j) \) to \( \sigma \). For each vertex \( v_j \in T \), choose \( k(v_i) \) neighbors \( v_j \in T \) with \( 1 \leq j < i \), add \( \{ v_i, v_j \} \) to \( F \), but add orientation \( (v_j, v_i) \) to \( \sigma \) (mind the asymmetry compared to side \( S \)).
Then by applying Lemma 2 to the reduced graph $H$, there is an $s$-$t$-flow of size $|F|$ in the auxiliary multigraph $H_{C,F,\sigma}$. As all edges have capacity 1, there are $|F|$ edge-disjoint $s$-$t$-paths in $H_{C,F,\sigma}$. Consider such a path $p = (s, u_1, \ldots, u_m, t)$, and let $p' = \{u_1, \ldots, u_m\}$ be its induced, undirected subpath in the bipartite graph $G(S, T)$. Unless $u_1 = u_m$ (which happens when $p'$ is an empty path), we modify the weighting $\omega$ of each edge $\{u_i, u_{i+1}\} \in p'$ as follows: increase its weight by 1 if $u_i \in S$, and decrease the weight by 1 if $u_i \in T$. In other words, we alternately increase or decrease the edge weights along the path. The weighted degrees of the internal nodes $u_2, \ldots, u_{m-1}$ thereby do not change, in contrast to those of the endpoints $u_1$ and $u_m$. The weighted degree of $u_1$ increases by 1, if $u_1 \in S$, and decreases by 1, if $u_1 \in T$. Regarding $u_m$, its weighted degree increases by 1, if $u_m \in T$, and decreases by 1, if $u_m \in S$. When $u_1 = u_m$, there is no change on the weighted degree of this node.

By construction of $H_{C,F,\sigma}$, each edge of $F$ led to exactly one arc incident to $s$ and one arc incident to $t$. Thus, for each path $p = (s, u_1, \ldots, u_m, t)$ of the provided edge-disjoint $s$-$t$-paths, there are two uniquely identified $F$-incidences: an edge $f^+ = \{u_1, w_1\} \in F$ with $(u_1, w_1) \in \sigma$, leading to the arc $(s, u_1) \in p$, and an edge $f^- = \{w_m, u_m\} \in F$ with $(w_m, u_m) \in \sigma$, leading to the arc $(u_m, t) \in p$. Note that $f^+ = f^- = \{u_1, u_m\}$ is possible. Vice versa, as we found $|F|$ edge-disjoint $s$-$t$-paths in the auxiliary network $H_{C,F,\sigma}$, for each edge $(u^+, u^-) \in F$ with $(u^+, u^-) \in \sigma$, there are uniquely identified paths starting with $(s, u^+)$ and ending with $(u^-, t)$.

We repeat the described modification on $\omega$ for all $|F|$ paths provided by Lemma 2. Summing up the changes $\omega$ caused by each path $p'$, for $v \in S$, the freshly updated edge-weighting $\omega$ satisfies

$$s_\omega(v) - t(v) = |\{w : (v, w) \in \sigma\}| - |\{w : (w, v) \in \sigma\}|,$$

whereas for $v \in T$,

$$s_\omega(v) - t(v) = |\{w : (w, v) \in \sigma\}| - |\{w : (v, w) \in \sigma\}|.$$

Finally, as a last modification step, we increase the weighting $\omega$ on each edge in $F$ by 1, obtaining

$$s_\omega(v) - t(v) = 2|\{w : (v, w) \in \sigma\}| = 2k(v_i)$$

for all $v \in S$, and

$$s_\omega(v) - t(v) = 2|\{w : (w, v) \in \sigma\}| = 2k(v_i)$$

for all $v \in T$. We conclude that each vertex $v_i$ obtained weighted degree $t(v_i) + 2k(v_i) = f(v_i)$, thus indeed, the constructed edge-weighting $\omega$ gives rise to a proper vertex-coloring of $G$. $\square$

**Proof of Lemma 2** Let $k := |F|$ and consider the network $H := G_{C,F,\sigma}$ with auxiliary vertices $s$ and $t$. Assume by contradiction that there exists no $s$-$t$-flow of value $k$ in $G_{C,F,\sigma}$. Then, by the standard max-flow min-cut theorem [12, e.g.], there exists
a cut $C' = (A_H, B_H)$ of size $\ell < k$, where $s \in A_H$ and $t \in B_H$. Observe that $A_G := A_H \setminus \{s\}$ and $B_G := B_H \setminus \{t\}$ are both subsets of the vertex set $V$ of the original, undirected graph $G = (V, E)$.

By step (iii) of the lemma statement, for each edge $f \in F$, in $H$ there are two uniquely defined arcs outgoing at $s$ and incoming at $t$. Let $F' := \{(u, v) \in F : u \in A_G, v \in B_G\}$ and remark that for each $f = (u, v) \in F \setminus F'$ with $(u, v) \in \sigma$, one of its two identified arcs $(s, u)$ and $(v, t)$ is in the cut $C'$. Next, let $E_1 := E(S \cap A_G, T \cap B_G) \cup E(S \cap B_G, T \cap A_G)$ and notice that for each edge $e = \{(u, v) \in E_1$, either $(u, v)$ or $(v, u)$ is contained in $C'$ as well. It follows

$$|E_1| \leq \ell - |F \setminus F'| = \ell - (k - |F'|) < |F'|.$$  

Finally, consider the cut

$$C'' := ((S \cap A_G) \cup (T \cap B_G), (S \cap B_G) \cup (T \cap A_G))$$

of the original graph $G$. Let $E_2 := E(S \cap A_G, S \cap B_G) \cup E(T \cap A_G, T \cap B_G)$, and observe that $F' \subseteq E_2$. Putting everything together, we deduce

$$|C''| = |C| - |E_1| + |E_2| > |C| - |F'| + |F'| = |C|.$$  

We see that in $G$, $C''$ would be a larger cut than $C$, contradicting the maximality of cut $C$. So, in $H$ there exists an $s$-$t$-flow of value $k$. \hfill $\Box$

**Proof of Lemma 3** In order to prove the statement, we do a case analysis to find a suitable function $h : E \to \{0, 1, 2\}$ that satisfies (i)-(iv). Whenever such a function $h$ is constructed, for each $v \in V$, define $f_h(v) := g(v) + s_h(v)$. Then (iv) is fulfilled if and only if $f_h$ is a proper vertex coloring of $G$.

We start with the case $g(v_0) \neq g(v)$ for all vertices $v \neq v_0$, where we can set $h \equiv 0$. Then (i)-(iv) are all met. Otherwise, let

$$V' := \{v \neq v_0 : g(v) - g(v_0) \equiv 0 \mod 2\} = \{v_1, \ldots, v_{m'}\},$$

and assume w.l.o.g. that $g(v_1) \leq \ldots \leq g(v_{m'})$. Suppose first that $m' = |V'| = 1$. Because we already handled the case $g(v_0) \neq g(v_1)$, we can assume $g(v_0) = g(v_1)$. Take $u \in N(v_0) \setminus V'$ such that $g(u)$ is maximal. Put $h(\{v_0, u\}) := 2$ and $h(e) := 0$ for any other edge $e \neq \{v_0, u\}$. We obtain $f_h(v_1) < f_h(v_0)$. Since for any $w \in V \setminus \{u, v_0, v_1\}$, it holds that

$$f_h(u) = g(u) + 2 \geq g(w) + 2 > g(w) = f_h(w),$$

(iv) is achieved. The other properties clearly hold as well.

For the remaining proof, we can assume $m' \geq 2$. For all $j \in \{0, \ldots, m'\}$, define the function

$$h_j : E \to \mathbb{N}_0, \quad e \mapsto \begin{cases} 2, & \text{if } e = \{v_0, v_i\} \text{ for some } j < i \leq m', \\ 0, & \text{otherwise.} \end{cases}$$
The plan is now to find a suitable function \( h_j \) for as many cases as possible. Clearly, each \( h_j \) directly fulfills (i)-(iii). Regarding (iv), we claim that it is sufficient to verify the property only for edges incident to \( v_0 \). Indeed, by construction, on \( V' \) \( g(u) > g(v) \) directly implies \( f_{h_j}(u) > f_{h_j}(v) \). Hence, \( f_{h_j} \) would already properly color \( V' \cup \{v_0\} \). But \( f_{h_j} \) also preserves the parities mod 2 of \( g \), and \( h_j \) vanishes on \( E \setminus E(V' \cup \{v_0\}) \), so the proper coloring of the remaining vertices is inherited from \( g \).

Let \( x \geq 1 \) be the smallest integer such that \( g(v_0) + 2x \) differs from all \( g(v_1), \ldots, g(v_{m'}) \), and let \( i' \leq m' \) be maximal such that \( g(v_{i'}) < g(v_0) + 2x \). Because \( g(v_0) = g(v_i) \) for some \( v_i, i' \) is well-defined. First consider the case \( i' \leq m' - x \). Here, we set \( h \equiv h_{m' - x} \). Then, for \( i > m' - x \geq i' \), we have \( f_h(v_i) > g(v_i) \geq g(v_0) + 2x \), whereas for \( i \leq m' - x \), it holds \( f_h(v_i) = g(v_i) \neq g(v_0) + 2x \). Hence indeed, \( f_h(v_0) = g(v_0) + 2x \) is different from \( f_h(v_1), \ldots, f_h(v_m) \).

Next, consider the case \( i' > m' - x \) and \( x < m' \), where we put \( h \equiv h_{m' - x - 1} \). For \( i \geq m' - x \) we then have \( f_h(v_i) = g(v_i) + 2 \neq g(v_0) + 2(x + 1) \), whereas for all \( i < m' - x \) it holds that

\[
f_h(v_i) = g(v_i) \leq g(v_{i'}) < g(v_0) + 2x. 
\]

Thus, for all \( v_i \in V' \), it holds \( f_h(v_0) = g(v_0) + 2(x + 1) \neq f_h(v_i) \), and (iv) is again fulfilled.

It remains the case \( x = m' \). Here, for each \( 0 \leq y < m' \), the value \( g(v_0) + 2y \) is attained by one \( g(v_i) \). So we have \( g(v_i) = g(v_0) + 2(i - 1) \) for all \( v_i \in V' \). We distinguish two subcases. If \( m' \) is even, we can use \( h \equiv h_{m'/2} \). For \( i \leq m'/2 \), it then holds \( f_h(v_i) = g(v_i) \leq g(v_0) + m' - 2 \), whereas, for \( i > m'/2 \), it holds \( f_h(v_i) = g(v_i) + 2 \geq g(v_0) + m' + 2 \). Since \( f_h(v_0) = g(v_0) + m' \), (iv) is again achieved.

On the other hand, if \( m' \) is odd, then \( m' \geq 3 \). This situation is a bit inconvenient, because none of the functions \( h_j \) can be used. Instead, let \( z := \frac{m'+3}{2} \leq m' \). Put \( h(\{v_0, v_i\}) := 2 \) for all \( i > z \), and \( h(\{v_0, v_i\}) := 0 \) for all \( i < z - 1 \). Moreover, let \( h(e) := 0 \) whenever \( e \notin E(V' \cup \{v_0\}) \), so (ii) is already satisfied. We want to achieve \( s_h(v_0) = m' - 1 \), but need to be careful to satisfy (iii) and (iv) at the same time. If \( v_z \) and \( v_{z-1} \) do not share an edge, put \( h(\{v_0, v_{z-1}\}) := 2 \), \( h(\{v_0, v_z\}) := 0 \), and \( h(e) := 0 \) for all \( e \in E(N(v_0)) \). Then

\[
f_h(v_z) = f_h(v_{z-1}) = g(v_0) + 2(z - 1) = g(v_0) + m' + 1,
\]

which is fine regarding (iv), as the two nodes are not neighbors.

Vice versa, if the edge \( e' := \{v_z, v_{z-1}\} \) is present in \( E \), for each edge \( e \) of the triangle \( \{v_0, v_{z-1}, v_z\} \), put \( h(e) := 1 \). For all edges \( e \in E(V') \setminus \{e'\} \), set \( h(e) := 0 \), yielding

\[
f_h(v_z - 1) = g(v_0) + 2(z - 1) = g(v_0) + m' + 1.
\]

and

\[
f_h(v_{z-1}) = g(v_0) + 2(z - 1) = g(v_0) + m' + 1.
\]
In both subcases, \( f_h(v_0) = g(v_0) + m' - 1 \) by construction. Moreover, for each \( v_i \) with \( i \geq z + 1 \), it holds that

\[
f_h(v_i) = g(v_i) + 2 = g(v_0) + 2i \geq g(v_0) + m' + 5,
\]

whereas for \( i < z - 1 \), it holds that

\[
f_h(v_i) = g(v_i) = g(v_0) + 2(i - 1) \leq g(v_0) + 2(z - 3) = g(v_0) + m' - 3.
\]

We conclude that \( f_h \) properly colors \( V' \cup \{v_0\} \). Properties (i)-(iii) are clearly achieved with \( h \) in both subcases. By the same argument as above for the functions \( h_j, f_h \) then properly colors the entire set \( V \).

\[\square\]

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