DECRY OF MAGNETIC FIELDS IN THE EARLY
UNIVERSE∗

MARK HINDMARSH
Astronomy Centre and Centre for Theoretical Physics
University of Sussex
Brighton BN1 9QH
U.K.
E-mail: m.b.hindmarsh@sussex.ac.uk

M. CHRISTENSSON AND A. BRANDENBURG
NORDITA
Blegdamsvej 17
DK-2100 Copenhagen Ø
Denmark
E-mail: mattias@nordita.dk, brandenb@nordita.dk

We study the evolution of a stochastic helical magnetic field generated in the early
Universe after the electroweak phase transition, using standard magnetohydrody-
namics (MHD). We find how the coherence length ξ, magnetic energy $E_M$ and
magnetic helicity $H$ evolve with time. We show that the self-similarity of the
magnetic power spectrum alone implies that $\xi \sim t^{1/2}$. This in turn implies that
magnetic helicity decays as $H \sim t^{-2s}$, and that the magnetic energy decays as
$E_M \sim t^{-0.5 - 2s}$, where $s$ is inversely proportional to the magnetic Reynolds num-
ber $Re_M$. These laws improve on several previous estimates.

1. Introduction

Magnetic fields are found everywhere in the Universe, from planets to
galaxy clusters1,2. On the galactic scale and above, the strength is of order
a few $\mu$Gauss, maintained by dynamo action, with a characteristic timescale
of roughly a rotation period, $10^8$ yr. A seed is required to start the dynamo,
and a simple calculation1 based on the age of a typical galaxy shows that
the seed field must have been about $10^{-20}$ Gauss.

*Presented at Strong & Electroweak Matter 2002, Heidelberg, Oct 2–5 2002
There are many ideas for the origin of seed field. Most conservatively, a Biermann battery operated at the era of reionisation. More speculatively, a field could have been generated in the early Universe, which would have given rise to stochastic, homogeneous and isotropic magnetic and velocity fields, characterized by their power spectra and initial length scales. In order to know the strength and coherence scale of this field today, we must study freely decaying magnetohydrodynamic (MHD) turbulence. This talk reports earlier work where we have developed a new framework for understanding decaying 3D MHD turbulence, in the case where the fields are close to being maximally helical, as in the mechanisms proposed earlier.

Of particular interest are how the coherence length $\xi$, the magnetic energy $E_M$ and magnetic helicity $H$ evolve with time. Their growth and decay are thought to be power laws, resulting from an inverse cascade, in which power is transferred locally in $k$-space from small to large scales. This is not a new problem: there are models both with and without helicity, and numerical simulations have also been performed. We have now developed what we believe is the definitive answer for the evolution of helical magnetic fields between the electroweak phase transition and the time of $e^+e^-$ annihilation.

2. 3D MHD simulations of decaying turbulence

The MHD equations in an expanding Universe are most conveniently expressed in terms of conformally rescaled fields $\mathbf{B}$ (where $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field in terms of the magnetic vector potential $\mathbf{A}$), $\mathbf{u}$ (the velocity) and dissipation parameters $\nu$ (kinematic viscosity), $\eta$ (magnetic diffusivity). The current density $\mathbf{J} = \nabla \times \mathbf{B}$.

In the ideal limit $\nu = \eta = 0$, there is an important conserved quantity in addition to the total energy, which is the magnetic helicity $H = \int \mathbf{A} \cdot \mathbf{B} \, d^3x$, also known as the Abelian Chern-Simons number. Also important is the magnetic Reynolds number $Re_M = \xi v/\eta$, where $\xi$ and $v$ are the typical length scale and velocity of the system, because it measures the relative size of the non-linear term in the induction equation.

We solve the MHD equations numerically, taking $\mathbf{u}$ and $\mathbf{B}$ to be homogeneous and isotropic Gaussian random fields drawn from a power-law distribution with a high wavenumber cut-off. The initial helicity, ranged from identically zero to maximal, where the inequality $|H(k)| \leq 2k^{-1}E_M(k)$ is saturated. Helicity is not exactly conserved in our simulations, and providing that the magnetic energy spectrum $E_M(k)$ decays faster than $k^{-2}$,
we can define a helicity scale $\xi_H$ such that

$$\dot{H} = -2\eta H/\xi_H^2.$$  \hspace{1cm} (1)

If we assume that the evolution of $\xi_H$ is described by a power law $\xi_H \sim t^r$, it is clear that only if $r = 0.5$ does the magnetic helicity show a power law decay $H = H_0(t/t_0)^{-2s}$ where $s = (\xi_{\text{diff}}/\xi_H)^2$, in terms of the diffusion scale $\xi_{\text{diff}} = 2\pi\sqrt{\eta t}$. Additional length scales we consider are the integral scale $\xi_I = 2\pi \int \frac{dk}{k} E_M(k)/\int E_M(k)$, the relative helicity scale $\xi_R = \pi |H|/E_M$ and the magnetic Taylor microscale $\xi_T = 2B_{r\text{ms}}/J_{r\text{ms}}$, where $B_{\text{rms}}$ and $J_{\text{rms}}$ are the RMS magnetic field and current density respectively.

It is plausible that all these scales are proportionally related and Fig. 1 shows that this is indeed the case. From the bound on the helicity power spectrum one can show $H_{\text{REL}} = (\xi_R/\xi_I) \leq 1$, so if this bound remains approximately saturated, and the helicity scale goes as $\xi_H \sim t^{0.5}$, the decay law for the magnetic energy is

$$E_M \sim t^{-0.5-2s}.$$  \hspace{1cm} (2)

Given $H_{\text{REL}} = \xi_R/\xi_I$, it is seen from Fig. 1 that $H_{\text{REL}}$ is indeed of order unity and does not decay markedly with time.

To characterize the decay laws we define

$$Q(t) = -t\dot{E}_M/E_M, \quad R(t) = -t\dot{H}/2H.$$  \hspace{1cm} (3)

In Fig. 2a we have plotted $R(t)$ versus the quantity $s(t) = (\xi_{\text{diff}}/\xi_H)^2$ for several runs with different initial conditions. This figure tells us that the value of $R$ is approximately independent of time which confirms the
power law decay of $H$. Fig. 2a also indicates that the quantity $s$ is also approximately independent of time, hence reinforcing the relation $\xi_H \sim t^{0.5}$.

Taking the relative helicity to be constant, it follows from the power-law behavior of $H$ that Eq. (2) is the energy decay law. In the limit of exact conservation of magnetic helicity, $s \rightarrow 0$, the magnetic energy must decay according to the power law $E_M \sim t^{-0.5}$.

The parameter $s$ has interesting physical significance. Note that if $\xi_H \simeq vt$, where $v$ is the RMS velocity, (i.e. if the eddy turn-over time is $t$) then $s \simeq (2\pi)^2/Re_M$, where $Re_M$ is the magnetic Reynolds number evaluated using the helicity scale $\xi_H$. We have measured the ratio $f = vt/\xi_H$ and $Re_M^{-1}$ for all runs, and find that they are both approximately constant, and are linearly related. This means there should be a linear relation between $s$ and $Re_M^{-1}$, and hence a quadratic relation between $Q(t)$, the energy decay exponent defined in Eq. (3), and $Re_M^{-1}$. Fig. 2b, showing $Q$ and $Re_M^{-1}$, confirms that this is indeed the case, with asymptote at large $Re_M$ consistent with $Q = 0.5$. Finally, assuming only that the magnetic energy power spectrum is self-similar, and that Ohmic dissipation, if not dominant, always contributes a constant fraction to the energy loss, one can show\(^7\) that the characteristic length scale of the field $\xi$ must scale as $t^{0.5}$, thus justifying the assumption made at the start of this section.

3. Discussion and conclusions

To summarize, we have studied the evolution of decaying 3D MHD turbulence involving maximally helical magnetic fields. For finite magnetic
diffusivity there emerges an important quantity $s = (\xi_{\text{diff}}/\xi_H)^2$, where $\xi_H$ is the helicity scale defined in Eq. (1), and $\xi_{\text{diff}}$ is the diffusion scale. We find $\xi_H \simeq vt$, where $v$ is the RMS velocity, and hence that $s \propto Re_M^{-1}$, the magnetic Reynolds number evaluated using the helicity scale. The magnetic field coherence length (which can be equally well expressed as the integral, helicity or relative helicity scales) goes as $\xi \sim t^{-0.5}$, magnetic helicity $H_M \sim t^{-2s}$ and magnetic energy $E_M \sim t^{-0.5-2s}$. A corollary is that $Re_M$ is constant once the system has reached self-similarity. Furthermore, we can extrapolate to the limit of very large magnetic Reynolds numbers, useful for example in the early Universe, to find $H$ constant and $E_M \sim t^{-0.5}$.

References

1. Ya.B. Zeldovich, A.A. Ruzmaikin and D.D. Sokoloff, Magnetic Fields in Astrophysics (Gordon & Breach, New York, 1983); A.A. Ruzmaikin, A.A. Shukurov and D.D. Sokoloff, Magnetic Fields in Galaxies (Kluwer, Dordrecht, 1988).
2. P.P. Kronberg, Rep. Prog. Phys. 57 (1994) 325.
3. N. Y. Gnedin, A. Ferrara and E. G. Zweibel, Astrophys. J. 539, 505 (2000) [arXiv:astro-ph/0001066].
4. D. Grasso and H. R. Rubinstein, Phys. Rept. 348, 163 (2001) [arXiv:astro-ph/0009061].
5. D. Biskamp and W.C. Müller, Phys. Rev. Lett. 83, 2195 (1999); W.C. Müller and D. Biskamp, Phys. Rev. Lett. 84, 475 (2000).
6. M. Christensson, M. Hindmarsh and A. Brandenburg, Phys. Rev. E 64 (2001) 056405 [arXiv:astro-ph/0011321].
7. M. Christensson, M. Hindmarsh and A. Brandenburg, arXiv:astro-ph/0209119.
8. M. Joyce and M. E. Shaposhnikov, Phys. Rev. Lett. 79, 1193 (1997) [arXiv:astro-ph/9703005]; T. Vachaspati, Phys. Rev. Lett. 87, 251302 (2001) [arXiv:astro-ph/0101261].
9. A. Pouquet, U. Frisch, and J. Leorat, J. Fluid. Mech. 77, 321 (1976).
10. P. Olesen, Phys. Lett. B 398 (1997) 321 [arXiv:astro-ph/9610154].
11. D. T. Son, Phys. Rev. D 59 (1999) 063008 [arXiv:hep-ph/9803412].
12. G. B. Field and S. M. Carroll, Phys. Rev. D 62 (2000) 103008 [arXiv:astro-ph/9811206].
13. T. Shiromizu, Phys. Lett. B 443 (1998) 127 [arXiv:astro-ph/9810339].
14. D. Biskamp, Nonlinear Magnetohydrodynamics (Cambridge Univ. Press, Cambridge, 1993).
15. A. Brandenburg, K. Enqvist and P. Olesen, Phys. Rev. D 54 (1996) 1291 [arXiv:astro-ph/9602031].
16. K. Subramanian and J. D. Barrow, Phys. Rev. D 58 (1998) 083502 [arXiv:astro-ph/9712083].
17. A. Brandenburg, Astrophys. J. 550, 824 (2001).