ON THE EXISTENCE OF ACCESSIBILITY
IN A TREE-INDEXED PERCOLATION MODEL

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Abstract. The study of tree-indexed stochastic processes is of special interest in understanding evolutionary biological questions. An instance of such a process is obtained by considering a phylogenetic or evolutionary tree, with vertices representing species and edges representing evolutionary relationships among them, and associating a (random) fitness value to each individual.

In this work we consider the accessibility percolation model on spherically symmetric trees. The model is defined by associating an absolute continuous random variable $X_v$ to each vertex $v$ of the tree. The main question to be considered is the existence or not of an infinite path of nearest neighbours $v_1, v_2, v_3, \ldots$ such that $X_{v_1} < X_{v_2} < X_{v_3} < \cdots$ and which span the entire graph. The event defined by the existence of such path is called percolation.

We obtain sufficient conditions under which either there is percolation or not with positive probability. In particular, we show that if the tree has a linear growing function given by $f(i) = \lceil (i + 1)\alpha \rceil$ then there is a percolation threshold at $\alpha_c = 1$ such that there is percolation if $\alpha > 1$ and there is no percolation if $\alpha \leq 1$.

Our model may be seen as a simple stochastic model for evolutionary trees. Different levels of the spherically symmetric tree represent different generations of species and the degree of each vertex represents the mean number of offspring species which appear in fixed intervals of time. In this sense, the varying environment of the tree is translated as a varying mutation rate.

1. Introduction

Many percolation models were inspired by physics and developed in order to answer questions of physical and mathematical interest. This summarizes, roughly speaking, the beginning of a modern theory with interesting rigorous results and a wide range of applications. The standard model of percolation appears as a mathematical model for the first time in the work of Broadbent and Hammersley in 1957 — see [4]. The main purpose of that work was to study how random properties of a porous medium influence the transport of a fluid through it. In order to accomplish it, a lattice is used to represent the medium, where vertices are associated to pores and bonds to channels, and a family of Bernoulli independent and identically distributed random variables with parameter $p$ is used to represent when a given channel is open or not to the spread of a fluid. The first question to be considered is the existence or not of an infinite component of open channels and how this event depends on $p$. The answer gives rise to a result known as phase transition, which guarantees that there is a critical value of the parameter $p$ at which infinite components appear. The reader can find more details about

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mathematical formulation and main results for the standard percolation model and related spatial processes in [3, 8, 22].

The resulting mathematical theory was quickly developed and became one of the main branches of contemporary probability. On the other hand, new percolation models appear in the literature as an alternative to understand issues of biological and physical interest. Some instances are the AB percolation, invasion percolation, oriented percolation and continuum percolation models, among many others — for more references see [8, 18]. As a theoretical tool, combined with coupling techniques, percolation theory is also useful in the construction and analysis of stochastic processes [5].

We consider the accessibility percolation model introduced recently by [20] which is inspired by evolutionary biological questions. In that work an n-tree is considered and a continuous random variable $X_v$ is associated to each vertex $v$. A question of interest is to study the existence of paths of nearest neighbors $v_1, v_2, v_3 \ldots$ such that

$$X_{v_1} < X_{v_2} < X_{v_3} < \cdots$$

This type of path is called accessibility path. In [20], the authors considered the case in which independent and identically distributed (iid) random variables are attached to the vertices of the tree. They derived an asymptotic result for the probability of having at least one accessibility path connecting the root with the $h$ level of the tree. Additionally, they proved the existence of phase transition whenever $n = n(h) = \alpha h$, which is completed later by [21]. Recently, a related problem was analyzed in [2]. In this work we define the accessibility percolation model on spherically symmetric trees and establish sufficient conditions under which either there is an infinite accessible path with positive probability or not.

It is a well known fact that the study of tree-indexed stochastic processes is of interest in understanding evolutionary biological questions. An instance of such a process is obtained by considering a phylogenetic or evolutionary tree, with vertices representing species and edges representing evolutionary relationships among them, and associating a (random) fitness value to each individual. Our model may be seen as a simple stochastic model for evolutionary trees. Different levels of the spherically symmetric tree represent different generations of species and the degree of each vertex represents the mean number of offspring species which appear in fixed intervals of time. In this sense, the varying environment of the tree is translated as a varying mutation rate. Related stochastic models for phylogenetic trees are proposed in [9, 15]. In that work the authors consider a model which considers a birth and death component and a fitness component. Then, depending on the value of the (constant) mutation rate, the authors obtain a phylogenetic tree consistent with an influenza tree and also with an HIV tree.

Our methods are constructive and based mainly on a comparison between our model and a suitable branching processes in varying environment. Our approach constitutes an alternative to the methods previously used in the literature to analyze the accessibility percolation model.

The paper is organized as follows. Section [2] gathers the formal notations and definitions of the model and states the main results of the work. The proofs are included in Section [3] and last section is devoted to a discussion of possible generalizations.
2. The model and main results

2.1. Trees. We consider an infinite, locally finite, rooted tree $T = (\mathcal{V}, \mathcal{E})$. We denote the root of $T$ by $0$. Here $\mathcal{V}$ stands for the set of vertices and $\mathcal{E} \subset \{\{u, v\} : u, v \in \mathcal{V}, u \neq v\}$ stands for the set of edges. If $\{u, v\} \in \mathcal{E}$, we say that $u$ and $v$ are neighbors, which is denoted by $u \sim v$. The degree of a vertex $v$, denoted by $d(v)$, is the number of its neighbors. A path in $T$ is a finite sequence $v_0, v_1, \ldots, v_n$ of distinct vertices such that $v_i \sim v_{i+1}$ for each $i$. Since $T$ is a tree, there is a unique path connecting any pair of distinct vertices $u$ and $v$. Therefore we may define the distance between them, which is denoted by $\text{dist}(u, v)$, as the number of edges in such path. For each $v \in \mathcal{V}$ define $|v| := \text{dist}(0, v)$.

For $u, v \in \mathcal{V}$, we say that $u \leq v$ if $u$ is one of the vertices of the path connecting $0$ and $v$; $u < v$ if $u \leq v$ and $u \neq v$. We call $v$ a descendant of $u$ if $u \leq v$ and denote by $T^u = \{v \in \mathcal{V} : u \leq v\}$ the set of descendants of $u$. On the other hand, $v$ is said to be a successor of $u$ if $u \leq v$ and $u \sim v$. For $n \geq 1$, we denote by $\partial T_n$, the set of vertices at distance $n$ from the root. That is, $\partial T_n = \{v \in \mathcal{V} : |v| = n\}$.

In this work we deal with spherically symmetric trees (SST.) which are trees where the degree of any vertex depends only on its distance from the root. In other words, for any $v \in \mathcal{V}$ we have $d(v) = f(|v|)$ where $f := (f(i))_{i \geq 0}$ is a given sequence of natural numbers. The function $f$ is called the growth function of the tree. Such trees will be denoted by $T_f$. We point out that there is no more information in $T_f$ than that contained in the sequence $(|\partial T_n|)_{n \geq 1}$.

2.2. The Accessibility percolation Model. Let $T = (\mathcal{V}, \mathcal{E})$ be a SST. To each $v \in \mathcal{V}$ we associate a random fitness value. More precisely, let $\mathcal{X} = \{X_v : v \in \mathcal{V}\}$ be a family of absolute continuous, independent and identically distributed (iid) and non-negative random variables assuming their values in a common set $S$. Denote by $\mathcal{Q}$ its common law defined in some $\sigma$-algebra $\mathcal{H}$. Now we formally describe the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where all our analysis is developed. Let $\Omega = S^\mathcal{V}$ be the space of ordered sequences of real numbers $\omega = (\omega_v)_{v \in \mathcal{V}}$ where $\omega_v \in S$ for each $v$. Give $\Omega$ the corresponding product $\sigma$-algebra and denote it by $\mathcal{F}$. The probability measure $\mathbb{P}$ is the product measure in $(\Omega, \mathcal{F})$ with one-dimensional marginals given by $\mathcal{Q}$.

In order to formalize what “percolation” means in our context and state the results, we need some extra definitions. A path $v_0, v_1, \ldots, v_n$ in $T$ is said to be an accessible path if $X_{v_0} < X_{v_1} < X_{v_2} < \cdots < X_{v_n}$. We denote by $v_0 \xrightarrow{\mathcal{P}} v_n$ the event that $v_n$ is connected to $v_0$ through an accessible path (see Fig. 2.2).

For each $n \in \mathbb{N}$, let $\Lambda_n$ be the event that $\partial T_n$ is accessible from the root. In other words,

$$\Lambda_n := \Lambda_n(T) = \{0 \xrightarrow{\mathcal{P}} v, \text{ for some } v \in \partial T_n\}.$$

Let $\mathcal{AP}(T, \mathcal{X}) = \{v \in \mathcal{V} : 0 \xrightarrow{\mathcal{P}} v\}$. We call this model the accessibility percolation model on $T$.

Definition 2.1. We say that there is percolation in $\mathcal{AP}(T, \mathcal{X})$ if the event $\cap_{n \in \mathbb{N}} \Lambda_n$ occurs with positive probability.

The main object of study is the percolation probability $\theta(T, \mathcal{X})$ given by

$$\theta(T, \mathcal{X}) = \mathbb{P}[\cap_{n \in \mathbb{N}} \Lambda_n].$$
Figure 1. A realization of the accessibility percolation model on a tree. In this example $u$ and $v$ are the only vertices, at distance 3 from the root, which are connected to 0 through an accessible path.

Since $\mathcal{X}$ is a family of iid random variables, the value of $\theta(T, \mathcal{X})$ does not depend on the distribution of $\mathcal{X}$. Thus, we can safely remove $\mathcal{X}$ from $\theta$. Indeed, the probability of having an accessibility path of length $n$ equals $1/(n+1)!$

**Remark 2.1.** Since $(\Lambda_n)_{n \in \mathbb{N}}$ forms a decreasing sequence of events, then $\theta(T) := \mathbb{P}(\cap_n \Lambda_n) = \lim_{n \to \infty} \mathbb{P}(\Lambda_n)$.

It will be useful to compare, in some way, accessibility percolation models corresponding to different trees. Let $T_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $T_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two trees with common root 0. We say that $T_1$ is dominated by $T_2$ and write $T_1 \prec T_2$ if $\mathcal{V}_1 \subset \mathcal{V}_2$ and $\mathcal{E}_1 \subset \mathcal{E}_2$. If $T_1 \prec T_2$, then a simple coupling argument shows that every accessible path connecting 0 to $\partial T_1_n$ is also an accessible path connecting 0 to $\partial T_2_n$. This, in turns, implies that $\theta(T)$ is a non-decreasing function in $T$. More precisely, the following result holds.

**Lemma 2.2.** If $T_1 \prec T_2$, then $\theta(T_1) \leq \theta(T_2)$.

### 2.3. Phase transition.

The aim of this work is to analyze the behavior of the accessibility percolation model on spherically symmetric trees. One instance of such type of tree is the factorial tree, denoted by $T_!$, whose growth function is given by $f(i) = i + 1$, $i \geq 1$. Note that

$$\mathbb{P}(\Lambda_n) \leq \frac{|\partial T_{1,n}|}{(n+1)!} = \frac{1}{n+1}.$$  

By letting $n \to \infty$ (see Remark 2.1), we have $\theta(T_i) = 0$. Lemma 2.2 implies that it is necessary to consider trees growing much faster than the factorial tree in order to be able to have percolation with positive probability. This behavior is not surprising since the accessibility percolation model penalizes long accessibility paths. Thus, we turn our attention to SST $T_\alpha$ with growth function given by $f(i) = \lceil (i+1)\alpha \rceil$, where $\alpha > 0$ is a constant. By Lemma 2.2, $\theta(\alpha) := \theta(T_\alpha)$ is non-decreasing in $\alpha$.
and the critical parameter $\alpha_c := \inf\{\alpha : \theta(\alpha) > 0\}$ is well defined. Our main result localizes the exact point where phase transition occurs.

**Theorem 2.3.** Consider the $\text{AP}(T_\alpha, \mathcal{X})$ model. Then, the critical parameter is given by $\alpha_c = 1$. More precisely, $\theta(\alpha) = 0$ if $\alpha \leq 1$ and $\theta(\alpha) > 0$ if $\alpha > 1$.

The method of proof of Theorem 2.3 above carries over to SST where the constant $\alpha$ determining the growth function is allowed to vary according to generation. Indeed, we obtain a sufficient condition for having percolation, with positive probability, for trees with growth function given by $f(i) = (i + 1)\alpha_i$, where $\alpha_i := (\alpha_i)_{i \geq 0}$ is a sequence of positive integers. In this case we denote the percolation function by $\theta(\alpha)$.

**Theorem 2.4.** Consider the $\text{AP}(T_\alpha, \mathcal{X})$ model, with $T_\alpha$ a SST with growth function given by $f(i) = (i + 1)\alpha_i$, for all $i \geq 0$. If

$$\liminf_n \left( \prod_{i=0}^{n-1} \alpha_i \right)^{1/n} > 1,$$

then $\theta(\alpha) > 0$.

**Remark 2.2.** Note that $\prod_{i=0}^{n-1} \alpha_i = o(n)$ is a sufficient condition for $\theta(\alpha) = 0$. In particular, this is an instance of a sequence $\alpha$ such that $\liminf_n \left( \prod_{i=0}^{n-1} \alpha_i \right)^{1/n} = 1$.

**Martingales** The existence of a supercritical phase in the accessibility percolation model is related to the behavior of branching processes in varying environment. The use of martingales in the analysis of branching process is well known. Then, one may wonder if it is possible to pursue a martingale approach in order to prove the main result of this work. In what follows we argue that in our case this is not possible.

Without lost of generality assume that the fitness assigned to the root 0 is $X_0 = 0$. With this assumption the probability of having an accessibility path of length $n$ equals $1/n!$.

Let $N_n$ be the number of accessible paths connected to the root at distance $n$. Then

$$N_n = \sum_{i=1}^{|\partial T_n|} 1_i,$$

where $\{1_i\}_i$ is a sequence of Bernoulli r.v.’s which are not independent and have parameter $1/n!$! Furthermore,

$$\mathbb{E}(N_n) = \frac{|\partial T_n|}{n!} = \frac{\prod_{i=0}^{n-1} (i + 1)\alpha_i}{n!}.$$

For $n \geq 1$, let $\mathcal{F}_n = \sigma(X_v, v \in \partial T_n)$ be a sequence of $\sigma-$algebras. Note that $\{\mathcal{F}_n\}_{n \geq 1}$ is a filtration and that $N_n$ is adapted to $\mathcal{F}_n$. Set $M_0 = 1$ and define $M_n = N_n/\prod_{i=0}^{n-1} \alpha_i$ for $n \geq 1$. We claim that $\{M_n\}_{n \geq 0}$ is a (nonnegative) martingale with respect to the filtration $\{\mathcal{F}_n\}_{n \geq 1}$ defined above. Suppose that a path $0, v_1, \ldots, v_n$ is accessible, where $v_n \in \partial T_n$. The vertex $v_n$ has $[(n + 1)\alpha]$ successors in $\partial T_{n+1}$. Let $v_{n+1}$ be a vertex in $\partial T_{n+1}$. Given that the path $0, v_1, \ldots, v_n$ is accessible, the
probability that the path 0, v₁, ..., vₙ, vₙ₊₁ is accessible is
\[ P(X_{v_{n+1}} > X_{v_n} | X_{v_n} > X_{v_{n-1}} > ... > X_{v_2} > 0) = \frac{1}{n+1}. \]  
(2)

Then for each accessible path 0, v₁, ..., vₙ you expect to have exactly \( \alpha_n \) accessible paths at distance \( n + 1 \). Then
\[ E(M_{n+1} | \mathcal{F}_n) = \alpha_n \frac{N_n}{\prod_{i=0}^{n} \alpha_i} = M_n. \]

Hence \( \{M_n\}_{n \geq 0} \) is a martingale and \( M_n \to M_\infty \) almost surely, where \( M_\infty \) is a non-negative r.v. satisfying \( E(M_\infty) \leq 1 \). Since \( E(M_\infty^2) \) is not uniformly bounded, we can not use a martingale approach to give a shorter proof of Theorem 2.4 and state that \( P(M_\infty > 0) > 0 \). For further reading on martingale theory see [10], for instance.

3. Proofs

This section is entirely devoted to the proof of the results. The approach pursued in this part is constructive. The main idea is to compare the accessibility percolation model with a suitable branching process. More precisely, we define a Galton-Watson branching process in a varying environment whose survival implies percolation in our model. This process is a generalization of the standard branching process in which the offspring distribution of any individual depends on its generation. The embedding of branching processes is a powerful technique in the analysis of stochastic growth models — see for instance [11, 12, 13, 14].

3.1. Comparison with a branching process. To begin with the construction consider a SST \( T \) with growth function \( f \). For any \( u \in \mathcal{V} \) and \( n \geq 1 \), define
\[ T_u^n = \{ v \in T^u : |v| \leq |u| + n \} \]
and
\[ \partial T_u^n = \{ v \in T^u : |v| = |u| + n \}. \]

Now fix an integer \( n \geq 1 \) and take \( Z_0^n = \{0\} \). For \( j = 1, 2, \ldots \) define
\[ Z_j^n = \bigcup_{u \in Z_{j-1}^n} \{ v \in \partial T_u^n : u \overset{\text{ap}}{\longrightarrow} v \} \]
and, for \( j = 0, 1, \ldots \), let \( Z_j^n = |Z_j^n| \) denotes its cardinality.

Lemma 3.1. For every fixed \( n \geq 1 \), \( (Z_j^n)_{j \geq 0} \) is a Galton-Watson branching process in a varying environment whose survival implies percolation in the accessibility percolation model. In addition, for \( j \geq 0 \), the mean offspring number of an individual of the \( j \)-th generation satisfies
\[ \mu_j^n = \frac{|\partial T_u^n|}{(n + 1)!}, \]  
(3)
where \( v \) is a vertex of \( T \) and \( |v| = jn \).
Proof. The proof follows from the very construction. Since we assign independent fitnesses to each site, it is not difficult to see that individuals from the \( j \)-th generation have an iid offspring number with mean given by (3), for \( j = 1, 2, \ldots \). Moreover, there is independence between different generations. This implies that the sequence \((Z_j^n)_{j \geq 0}\) is a Galton-Watson branching process in a varying environment.

\[ \Box \]

3.2. Proof of Theorem 2.3 Fix \( \alpha > 0 \) and consider a SST \( T_\alpha \) with growth function given by \( f(i) = [(i + 1)\alpha]^i \). If \( \alpha \leq 1 \), \( T_\alpha \prec T_1 \), where \( T_1 \) is the factorial tree. Since \( \theta(T) = 0 \), we may conclude from Lemma 2.2 that \( \theta(T_\alpha) = 0 \). Now we use comparison with a branching process in order to prove that \( \alpha > 1 \) implies \( \theta(\alpha) > 0 \). According to Lemma 3.1 it is enough to find sufficient conditions under which the process \((Z_j^n)_{j \geq 0}\) survives with positive probability. Indeed,

\[ \liminf_{j \to \infty} \mu_j^n > 1, \tag{4} \]

is a sufficient condition for the survival of the branching process. This is a direct consequence of Theorem 1 of \cite{6}. This theorem states that a Galton-Watson process in a varying environment survives with positive probability if it is uniformly supercritical and the quotient between the offspring number of an individual of the \( j \)-th generation and its mean is dominated by an integrable random variable. In our case the constant random variable \((n + 1)!\) does the work. In this case, the mean offspring \( \mu_j^n \) of an individual of the \( j \)-th generation satisfies the following inequalities

\[ \mu_j^n > \prod_{i=0}^{n-1} [(i + 1)\alpha]^i \frac{(n + 1)!}{(n + 1)} \]

Thus, if we assume \( \alpha > 1 \), it follows that for \( n \) large enough

\[ \liminf_{j \to \infty} \mu_j^n \geq \frac{\alpha^n}{(n + 1)} > 1. \]

This completes the proof.

3.3. Proof of Theorem 2.4 As in Theorem 2.3 we prove the result by comparing the accessibility percolation model with a branching process. It follows from Lemma 3.1 that there exists a branching process in varying environment whose survival implies percolation. Also, if \( \nu_j \) denotes the mean offspring number of an individual of the \( j \)-th generation, then

\[ \nu_j = \frac{\prod_{i=j}^{j+n-1} (i + 1)\alpha_i}{(n + 1)!}. \]

Since \( \liminf_n \left( \prod_{i=0}^{n-1} \alpha_i \right)^{1/n} > 1 \) implies

\[ \ell := \lim_{n \to \infty} \min_{j} \frac{1}{n} \log \left( \prod_{i=j}^{j+n-1} \alpha_i \right) > 0, \]

assessability percolation on trees
we may take a constant \( c, 0 < c < \ell \), for which there exists an integer \( N = N(c) \) such that for any \( n \geq N \),

\[
\min_j \frac{1}{n} \log \left( \prod_{i=j}^{j+n-1} \alpha_i \right) > c.
\]

Therefore

\[
\left( \prod_{i=jn}^{jn+n-1} \alpha_i \right) \geq e^{cn}
\]

for all \( j \) and \( n \geq N \). From this, we get that, for \( n \) large enough,

\[
\liminf_{j \to \infty} \nu^n_j \geq \frac{e^{cn}}{(n + 1)} > 1.
\]

The proof is complete.

4. Discussion

In this section we discuss some connections and generalizations that may be explored in further applications.

Records. The accessibility percolation model was motivated by biological issues. However, this model is also related to record theory. In order to compare both models we assume, without loss of generality, that accessibility paths are defined by considering random variables in decreasing order. Note that this does not change our results. Now consider a sequence of competitions such that on the \( n \)-th edition there are \( \alpha(n) \) participants where \( \alpha(1) = 1 \). Assume that the scores obtained in the different editions are given by a sequence \( Y_n = (Y^n_1, \ldots, Y^n_{\alpha(n)}) \), \( n = 1, 2, \ldots \), of iid random vectors composed by absolutely continuous, iid random variables, each having continuous cumulative distribution function (cdf) \( F \). Then, the score of the winner of the \( n \)-th edition is given by \( X_n = \max \{Y^n_1, \ldots, Y^n_{\alpha(n)}\} \) and its cdf by \( F_n(x) = F(x)^{\alpha(n)} \). So defined, this is the well known \( F^\alpha \) record model, also referred as the \( F^\alpha \) scheme in [19]. See [1, Chapter 6] for a review of results of \( F^\alpha \) record models.

Now we define record time sequence \((S_n)_{n \geq 1}\) as follows. Let \( S_0 = 1 \) and, for \( n \geq 1 \),

\[
S_n = \inf \{ j : X_j > X_{S_{n-1}} \}.
\]

The (asymptotic) distribution of these random variables, as well as the one of the inter-records \( S_n - S_{n-1} \), are well studied in record theory. A different question that arises is if it is possible to have a record at each edition of the competition with positive probability. In other words, is it possible to have

\[
\mathbb{P} \left( \bigcap_{n=1}^{\infty} \{ S_n = n \} \right) > 0 \quad (5)
\]

Theorem [2,3] is related to the answer of this question provided \( \alpha(n) = \lceil (n+1)\alpha \rceil \). Indeed, a simple coupling argument shows that to have a record at each edition of the competition implies accessibility percolation in our model, which occurs with positive probability if, and only if, \( \alpha > 1 \).
**Homogeneous trees and varying environment.** Let \( T_d = T \) be the homogenous rooted tree in which every vertex has \( d + 1 \) neighbors, where \( d \geq 2 \), except the root which has \( d \) neighbors. If we consider the accessibility percolation model on \( T \), then we obtain \( \theta(T) = 0 \). Thus, in order to define a non-trivial accessibility percolation model on homogeneous trees we must assume some change on the distribution of the random fitness. In this section we consider the accessibility percolation model on \( T \) with varying environment indexed by generation. The environment is completely determined by a sequence of iid random variables \((X_n)_{n \geq 0}\) where \( X_n \sim U(a_n, 1) \) with \( a_n < a_{n+1} \) for all \( n \).

For each \( n \), consider a sequence \( \xi^n = (\xi^n_1, \ldots, \xi^n_d) \) of iid random variables where each \( \xi^n_i \) is a copy of \( X_n \). Then, the sequence \( \xi = (\xi^n)_{n \geq 0} \) determines the accessibility percolation model on \( T \). We observe that in the case of varying environment the probability of having an accessible path of length \( n \) is

\[
\mathbb{P}[v_0 \xrightarrow{a.p.} v_n] = \prod_{k=0}^{n-1} \frac{a_{k+1} - a_k}{1 - a_k}.
\]

(6)

It follows from Lemma 3.1 that

\[
\liminf_{j \to \infty} d^n \prod_{i=jn}^{jn+n-1} \frac{a_{i+1} - a_i}{1 - a_i} > 1
\]

(7)

is a sufficient condition for having (accessibility) percolation on \( T \) with positive probability.

Now, we exhibit two sequences showing the non-triviality of the problem above. First, consider the sequence \((a_i)_{i \geq 0}\) given by \( a_i = 1 - \beta^{-i} \), for \( i = 0, 1, 2, \ldots \), where \( \beta > 1 \). A direct computation gives \((a_{i+1} - a_i)/(1 - a_i) = 1 - \beta^{-1}\). Then,

\[
\liminf_{j \to \infty} d^n \prod_{i=jn}^{jn+n-1} \frac{a_{i+1} - a_i}{1 - a_i} > 1
\]

if, and only if, \( d^{-1} + \beta^{-1} < 1 \). Thus, condition (7) is verified by the sequence \((a_i)_{i \geq 0}\). On the other hand, consider the sequence \((b_i)_{i \geq 0}\) given by \( b_i = 1 - i^{-1} \), for \( i = 0, 1, 2, \ldots \). Then, \((b_{i+1} - b_i)/(1 - b_i) = (i + 1)^{-1} \). Therefore,

\[
\liminf_{j \to \infty} d^n \prod_{i=jn}^{jn+n-1} \frac{b_{i+1} - b_i}{1 - b_i} = 0
\]

which in turns implies that

\[
\liminf_{j \to \infty} d^n \prod_{i=jn}^{jn+n-1} \frac{b_{i+1} - b_i}{1 - b_i} = 0.
\]

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