The purpose of this short note is to give a proof of the following identity between (logarithmic) Mahler measures

\begin{equation}
m(y^2 + 2xy + y - x^3 - 2x^2 - x) = \frac{5}{7} m(y^2 + 4xy + y - x^3 + x^2),
\end{equation}

which is one of many examples that arise from the comparison of Mahler measures and special values of L-functions [Bo], [De], [RV]. Let us recall that the logarithmic Mahler measure of a Laurent polynomial \( P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) is defined as

\begin{equation}
m(P) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| d\theta_1 \cdots d\theta_n.
\end{equation}

The conjecture of Bloch–Beilinson [Be], [BG] for elliptic curves predicts that both sides of (1) are rationally related to \( L'(E, 0) \) (and hence to each other), where \( E \) is the elliptic curve of conductor 37

\begin{equation}
E : \quad y^2 + y = x^3 - x,
\end{equation}

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and \( L(E, s) \) is its \( L \)-function. More precisely, we expect that the two numbers \( a \) and \( b \) defined by

\[
m(y^2 + 2xy + y - x^3 - 2x^2 - x) = a \, L'(E, 0), \\
m(y^2 + 4xy + y - x^3 + x^2) = b \, L'(E, 0)
\]

are rational. A proof of this fact is not without reach but will not be attempted here, we will prove instead that \( a/b = 5/7 \).

1. Computing in \( K_2(E) \)

We first recall the definition of the group \( K_2(A) \) of an elliptic curve \( A \). Given a field \( F \) the group \( K_2(F) \) can be defined as \( F^* \otimes F^* \) modulo the Steinberg relations \( x \otimes (1 - x) \) for \( x \neq 0, 1 \) in \( F \).

Given a discrete valuation \( v \) on \( F \) with maximal ideal \( \mathcal{M} \) and residue field \( k \) we have the tame symbol at \( v \) defined by

\[
(x, y)_v \equiv (-1)^{v(x)v(y)} \frac{x^{\nu(y)}}{y^{\nu(x)}} \mod \mathcal{M},
\]

which determines a homomorphism

\[
\lambda_v : K_2(F) \rightarrow k^*
\]

For an elliptic curve \( A \) defined over \( \mathbb{Q} \) we let \( K_2(A) \) be the elements of \( K_2(\mathbb{Q}(A)) \) annihilated by all \( \lambda_v \) with \( v \) the valuations associated to \( \mathbb{Q} \) points of \( A \).

Our \( E \) appears as a fiber in several of Boyd’s families of elliptic curves (see [Bo], [RV] for a discussion of these families). For example, in its original form \( y^2 + y = x^3 - x \), but also as the two Weierstrass equations

\[
(4) \quad E_1 : \quad y_1^2 + 4x_1y_1 + y_1 = x_1^3 - x_1^2
\]

and

\[
(5) \quad E_2 : \quad y_2^2 + 2x_2y_2 + y_2 = x_2^3 + 2x_2^2 + x_2.
\]
It is easy to check that

\[(6)\]
\[
x_1 = x - 1
\]
\[
y_1 = y - 2x + 2
\]

and

\[(7)\]
\[
x_2 = x - 1
\]
\[
y_2 = -x + y + 1
\]

give isomorphisms
\[E \simeq E_1, \quad E \simeq E_2.\]

It follows from [RV] therefore, that some integer multiple of each of

\[\xi = \{x, y\}, \quad \xi_1 = \{x_1, y_1\}, \quad \xi_2 = \{x_2, y_2\}\]

is in \(K_2(E)\).

The divisors of the six functions \(x, y, x_1, y_1, x_2, y_2\) are supported on \(E(\mathbb{Q})\), which is generated by the point \(P\) with \(x = 0, y = 0\). More precisely, we have

\[(x) = [P] + [-P] - 2[O]
\]
\[(y) = [P] + [2P] + [-3P] - 3[O]
\]

\[(8)\]
\[
(x_1) = [2P] + [-2P] - 2[O]
\]
\[
(y_1) = 2[2P] + [-4P] - 3[O]
\]
\[
(x_2) = [-2P] + [2P] - 2[O]
\]
\[
(y_2) = [2P] + 2[-P] - 3[O]
\]

where \([O]\) denotes the point at infinity on \(E\).

Given a pair of functions \(f\) and \(g\) on \(E\) with divisors supported on \(E(\mathbb{Q})\)

\[(f) = \sum_{n \in \mathbb{Z}} a_n [nP], \quad (g) = \sum_{n \in \mathbb{Z}} b_n [nP]\]
we define

\[(f) \triangle (g) = \sum_{m,n} a_n b_m [(n - m)P],\]

which we will view as an element of

\[\mathbb{Z}[E(\mathbb{Q})]^– = \mathbb{Z}[E(\mathbb{Q})]/\sim,\]

where \(\sim\) is the equivalence relation determined by

\[-nP \sim [nP], \quad n \in \mathbb{Z}.\]

We may and will represent elements of \(\mathbb{Z}[E(\mathbb{Q})]^–\) as vectors \([a_1, a_2, \ldots]\) with \(a_i \in \mathbb{Z}\) almost all zero where

\([a_1, a_2, \ldots] \leftrightarrow \sum_{n=1}^{\infty} a_n [nP]\)

In fact, we will only consider elements where \(a_n = 0\) for \(n > 6\) and hence simply write \([a_1, \ldots, a_6]\).

We now compute

\[(x) \triangle (y) = [1, 2, -3, 1, 0, 0]\]

\[(x_1) \triangle (y_1) = [0, 5, 0, -4, 0, 1]\]

\[(x_2) \triangle (y_2) = [-6, 2, 2, -1, 0, 0].\]

On the other hand, we also find

\[(-y) \triangle (1 + y) = [-8, -7, 8, 1, 0, -1]\]

\[(x - y) \triangle (1 - x + y) = [-9, 5, -5, 5, 0, -1]\]

and verify easily that

\[7(x) \triangle (y) + (x_1) \triangle (y_1) = -2(-y) \triangle (1 + y) + (x - y) \triangle (1 - x + y)\]

\[5(x) \triangle (y) + (x_2) \triangle (y_2) = -(-y) \triangle (1 + y) + (x - y) \triangle (1 - x + y).\]

2. The regulator
Let
\begin{equation}
    r : \quad K_2(E) \longrightarrow \mathbb{R}
\end{equation}
be the regulator map. It can be defined as follows. If \( f, g \) are two non-constant functions on \( E \) with \( \{ f, g \} \in K_2(E) \) then
\begin{equation}
    r(\{ f, g \}) = \int_\gamma \eta(f, g),
\end{equation}
where
\begin{equation}
    \eta(f, g) = \log |f| \, d \arg g - \log |g| \, d \arg f
\end{equation}
and \( \gamma \) is a closed path not going through poles or zeroes of \( f \) or \( g \) which generates the subgroup \( H_1(E, \mathbb{Z})^- \) of \( H_1(E, \mathbb{Z}) \) where complex conjugation acts by \(-1\), properly oriented. The fact that the integral only depends on the homology class of \( \gamma \) is a consequence of \( \{ f, g \} \in K_2(E) \), see [RV] for details. (However, note that in [RV] we inaccurately said \( \gamma \) should generate the cycles fixed by complex conjugation; we take the opportunity to correct this.)

The regulator may also be expressed in terms of the elliptic dilogarithm [BG], [Za]
\begin{equation}
    \mathcal{L} : \quad E(\mathbb{C}) \longrightarrow \mathbb{R}.
\end{equation}
In our context, this works as follows. We extend it by linearity to \( \mathbb{Z}[E(\mathbb{Q})]^- \) and since \( \mathcal{L} \) is odd it actually gives a map
\begin{equation}
    \mathcal{L} : \quad \mathbb{Z}[E(\mathbb{Q})]^- \longrightarrow \mathbb{R}.
\end{equation}
If \( f, g \) are two non-constant functions on \( E \) with divisors supported on \( E(\mathbb{Q}) \) and such that \( \{ f, g \} \in K_2(E) \) then
\begin{equation}
    r(\{ f, g \}) = c \, \mathcal{L} \left( (f) \circ (g) \right),
\end{equation}
for some explicit non-zero constant \( c \), which is not relevant for our purposes. In particular, in the case that \( g = 1 - f \)
\begin{equation}
    \mathcal{L} \left( (f) \circ (1 - f) \right) = 0.
\end{equation}
The above discussion extends naturally to $K_2(E) \otimes \mathbb{Q}$, which contains $\xi, \xi_1$ and $\xi_2$.

It follows from (12) therefore, that

$$
r(\xi_1) = -7r(\xi) \quad r(\xi_2) = -5r(\xi).
$$

3. The regulator and Mahler’s measure

In [RV] we showed that if $P_k(x, y) = 0$ is one of Boyd’s families of elliptic curves and $k$ is such that $P_k$ does not vanish on the torus $|x| = |y| = 1$ then

$$
r(\{x, y\}) = c_k \pi m(P_k)
$$

for some nonzero integer $c_k$. We will now make this precise for

$$P_k(x, y) = y^2 - kxy + y - x^3 + x^2.$$

We consider the region $\mathcal{K}$ of $k \in \mathbb{C}$ such that $P_k$ vanishes somewhere on the torus. It is the image of the torus under the rational map

$$R : \quad (x, y) \mapsto \frac{y^2 + y - x^3 + x^2}{xy}.$$  

We can get a pretty good idea of what $\mathcal{K}$ looks like by graphing the image of a grid under $(\theta_1, \theta_2) \mapsto R(e^{2\pi i \theta_1}, e^{2\pi i \theta_2})$. Dividing the square $0 \leq \theta_1 < 1, 0 \leq \theta_2 < 1$ in 40 equal parts we obtain

**Figure 1.** Region $\mathcal{K}$

It is not hard to verify directly that the boundary of $\mathcal{K}$ meets the real axis at $k = -4$ and $k = 2$.

If $k \notin \mathcal{K}$ then as $x$ moves counterclockwise on the circle $|x| = 1$ one root $y_1(x)$ of $P_k(x, y) = 0$ satisfies $|y_1(x)| < 1$ and the other $y_2(x)$ satisfies $|y_2(x)| > 1$ and in particular $y_1(x)$ and $y_2(x)$ do not meet. To see this, note
that when \( x = 1 \) the roots are 0 and \( k - 1 \). Hence, for \(|k|\) large these roots are one inside and the other outside the unit circle. The claim follows since the roots depends continuously on \( k \). We let \( \sigma_k \) be the resulting smooth closed path \((x, y_1(x))\) on the elliptic curve \( E_k \) determined by \( P_k(x, y) = 0 \).

Using Jensen’s formula we find that

\[
m(P_k) = \frac{1}{2\pi i} \int_{\sigma_k} \log |y| \frac{dx}{x}
\]

and note that since \(|x| = 1\) on \( \sigma_k \) we can write this identity as

\[
(22) \quad m(P_k) = \frac{1}{2\pi} \int_{\sigma_k} \eta(x, y).
\]

We now show that for real and \( k \notin K \) the homology class of \( \sigma_k \) generates \( H_1(E_k, \mathbb{Z}) \). We complete the square and write \( P_k = (2y - kx + 1)^2 - f(x) \), where \( f(x) = 4x^3 + (k^2 - 4)x^2 - 2kx + 1 \). The discriminant \( \Delta(k) = k^4 - k^3 - 8k^2 + 36k - 1 \) of \( f \) has two real roots \( \alpha = -3.7996\ldots \) and \( \beta = .3305\ldots \). Hence, for \( k < \alpha \) or \( k > \beta \), \( \Delta(k) > 0 \) and \( f \) has three real roots \( e_1 < e_2 < e_3 \). As \(|k|\) increases the roots of \( f \) tend to \( e_1 = -\infty \) and \( e_2 = e_3 = 0 \) and by continuity the circle \(|x| = 1\) encircles \( e_2 \) and \( e_3 \) once. Since \( f \) is negative in the interval \( e_2 < x < e_3 \) the period

\[
\int_{\sigma_k} \frac{dx}{2y - kx + 1}
\]

is purely imaginary and our claim follows.

Combined with (14) and (22) this proves that in fact

\[
(23) \quad r(\{x, y\}) = \pm 2\pi m(P_k), \quad k \in \mathbb{R}, \quad k \notin K.
\]

By continuity (23) also holds for \( k = -4 \) and \( k = 2 \), which are on the boundary of \( K \). In particular, in the notation of \( \S 2 \), we obtain the identity

\[
(24) \quad r(\xi_1) = \pm 2\pi m(y^2 + 4xy + y - x^3 + x^2).
\]

A completely analogous analysis yields

\[
(25) \quad r(\xi_2) = \pm 2\pi m(y^2 + 2xy + y - x^3 - 2x^2 - x)
\]
(and again $k = -2$ is on the boundary of the corresponding set $K$). Putting together (19), (24) and (25) (and a simple check for the right sign) we obtain (1).

**Remarks**

1. We should point out that we do not expect $m(y^2 + y - x^3 + x)$ to be rationally related to either side of (1) (and numerically it indeed does not appear to be). The reason is that $y^2 + y - x^3 + x$ vanishes on the torus and in fact $k = 0$ is in the interior of the region $K$ corresponding to the Boyd family $y^2 - kxy + y - x^3 + x$. Hence the analogue of (22) gives the integral of $\eta(x, y)$ on a non-closed cycle.

2. One can prove in a similar way an identity relating either side of (1) with $m(y^2 + 2xy + y - x^3 + x^2)$.

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