THE LOCAL POLYNOMIAL HULL NEAR A DEGENERATE CR SINGULARITY – BISHOP DISCS REVISITED

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Abstract. Let \( S \) be a smooth real surface in \( \mathbb{C}^2 \) and let \( p \in S \) be a point at which the tangent plane is a complex line. How does one determine whether or not \( S \) is locally polynomially convex at such a \( p \) — i.e. at a CR singularity? Even when the order of contact of \( T_p(S) \) with \( S \) at \( p \) equals 2, no clean characterisation exists; difficulties are posed by parabolic points. Hence, we study non-parabolic CR singularities. We show that the presence or absence of Bishop discs around certain non-parabolic CR singularities is completely determined by a Maslov-type index. This result subsumes all known facts about Bishop discs around order-two, non-parabolic CR singularities. Sufficient conditions for Bishop discs have earlier been investigated at CR singularities having high order of contact with \( T_p(S) \). These results relied upon a subharmonicity condition, which fails in many simple cases. Hence, we look beyond potential theory and refine certain ideas going back to Bishop.

1. Introduction and statement of results

Given a real surface \( S \subset \mathbb{C}^2 \) and a point \( p \in S \) at which \( T_p(S) \) is a complex line, it would be interesting to characterise when \( S \) is locally polynomially convex at \( p \). A number of questions in function theory — ranging from the existence of Stein neighbourhood bases for imbedded real discs, to studying removable singularities for CR functions — would be aided by such a characterisation. Moreover, very little is currently known about when a germ of a surface is locally polynomially convex at a point of complex tangency of high order. For the record: a compact \( K \subset \mathbb{C}^n \) is said to be locally polynomially convex at a point \( p \in K \) if there exists a closed ball \( \overline{B}(p) \) centered at \( p \) such that \( K \cap \overline{B}(p) \) is polynomially convex.

We will abbreviate the phrase “point of complex tangency” to CR singularity. Consider a CR singularity \( p \in S \subset \mathbb{C}^2 \) where the order of contact of \( T_p(S) \) with \( S \) equals 2 — i.e. a non-degenerate CR singularity. Bishop showed [3] that there exist holomorphic coordinates \((z, w)\) centered at \( p \) such that \( S \) is locally given (barring one manifestly locally polynomially convex case) by an equation of the form \( w = |z|^2 + \gamma(z^2 + \overline{z}^2) + G(z) \), where \( \gamma \geq 0 \), \( G(z) = O(|z|^3) \), and three distinct situations arise. In Bishop’s terminology, the CR singularity \( p = (0, 0) \) is said to be elliptic if \( 0 \leq \gamma < 1/2 \), parabolic if \( \gamma = 1/2 \), and hyperbolic if \( \gamma > 1/2 \). Bishop [3] showed that when \( p \in S \) is elliptic, the polynomially convex hull of \( S \) near \( p \) contains a one-parameter family of non-constant analytic discs attached to \( S \) that shrink to \( p \). On the other hand, Forstnerič & Stout [7] (also refer to [13] by Stout) showed that when \( p \) is hyperbolic, \( S \) is locally polynomially convex at \( p \). Very little is known beyond this about polynomial convexity.

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at a CR singularity. What can we say about \((S, p)\) if, for instance, \(p\) is a degenerate CR singularity? Some conditions for polynomial convexity are given in [1] and in [2].

However, even if \(p\) is a CR singularity in \(S\) where the order of contact of \(T_p(S)\) with \(S\) equals 2, Jörıcke’s results in [3] show that the situation is far from orderly when \(p\) is a parabolic point. One would thus expect some assumptions on the pair \((S, p)\), when \(p\) is a degenerate CR singularity, for a reasonably coherent theory to emerge. This motivates the following:

**Definition 1.1.** Let \(S\) be a smooth real surface in \(\mathbb{C}^2\), and let \(p \in S\) be an isolated CR singularity. We say that \(p\) is non-parabolic if:

a) \(T_p(S)\) has finite order of contact with \(S\) at \(p\); and

b) given holomorphic coordinates \((z, w)\) centered at \(p\) such that \(S\) is locally given by an equation of the the form

\[
S \cap U_p : \quad w = \mathcal{F}_m(z) + O(|z|^{m+1}),
\]

where \(U_p\) is a small neighbourhood of \(p\) and \(\mathcal{F}_m\) is a homogeneous polynomial of degree \(m\), the graph \(\Gamma(\mathcal{F}_m)\) has an isolated CR singularity at \((0, 0)\).

Note that when \(p \in S\) is either elliptic or hyperbolic, it is non-parabolic in the sense of Definition 1.1. We wish to extend what we know about local polynomial convexity at a non-parabolic, non-degenerate CR singularity to the degenerate case. When \((S, p)\) is presented in the Bishop normal form near a non-parabolic, non-degenerate \(p\), we have holomorphic coordinates \((z, w)\) in which — using the the notation of (1.1) — \(\mathcal{F}_2\) is real-valued. This last fact is of importance to Bishop’s proofs in [3, Section 3].

We need to reformulate this for non-degenerate CR singularities in order to extend the Bishop/Forstnerič-Stout dichotomy to the degenerate setting. Hence, we need the following:

**Definition 1.2.** Let \(S\) be a smooth real surface in \(\mathbb{C}^2\), and let \(p \in S\) be an isolated CR singularity. Suppose \(T_p(S)\) has finite order of contact \(2 \leq m < \infty\) with \(S\) at \(p\). We say that \(S\) is thin at \(p\) if there exist holomorphic coordinates \((z, w)\) centered at \(p\) such that \(S\) is locally the graph

\[
S \cap U_p : \quad w = \mathcal{F}_m(z) + O(|z|^{m+1}),
\]

and such that the leading homogeneous component of the graphing function, denoted by \(\mathcal{F}_m\), is real-valued.

When, for the pair \((S, p)\), \(p\) is a non-parabolic, non-degenerate CR singularity (in which case \(S\) is always thin at \(p\)) the works [3] and [7], when read together, imply that the local polynomial convexity of \(S\) at \(p\) is determined precisely by the sign of a certain Maslov-type index, denoted by \(\text{Ind}_M(S, p)\). Specifically:

\((*)\) When \(p \in S\) is a non-parabolic, non-degenerate (hence thin) CR singularity, \(S\) is locally polynomially convex at \(p\) if and only in \(\text{Ind}_M(S, p) \leq 0\).

The reader is directed to Section 2 for the set-up needed to define the index \(\text{Ind}_M(S, p)\) precisely. The goal of this paper is to attempt to extend \((*)\) to non-parabolic CR singularities in general. The following theorem says, among other things, that \(\text{Ind}_M(S, p) > 0 \implies S\) is not locally polynomially convex at \(p\). We conjecture that the converse is true too. The grounds for this conjecture is a partial converse that we present in Theorem 1.4. We begin, however, with the following theorem.
Theorem 1.3. Let \( \mathcal{S} \) be a smooth real surface in \( \mathbb{C}^2 \) and let \( p \in \mathcal{S} \) be a CR singularity. Assume that \( p \) is non-parabolic and that \( \mathcal{S} \) is thin at \( p \). If \( \text{Ind}_\mathcal{F}(\mathcal{S}, p) > 0 \), then \( \mathcal{S} \) is not locally polynomially convex at \( p \).

In fact, there exists a \( C^1 \)-smooth family of analytic discs whose boundaries are contained in \( \mathcal{S} \). More precisely: there exist a neighbourhood \( U_p \ni p \), an open interval \( (0, R_0) \), and a function \( g : (0, R_0) \to A^\alpha(\mathbb{D}; \mathbb{C}^2) \) that is of class \( C^1 \) on \( (0, R_0) \) (for an arbitrary but fixed \( \alpha \in (0, 1) \)), where each \( g(r) \) is a non-constant analytic disc satisfying

\[ \begin{align*}
&i) \ g(r)[\partial \mathcal{D}] \subset (\mathcal{S} \setminus \{p\}) \cap U_p \ \forall r \in (0, R_0) \ ; \text{ and} \\
&ii) \ g(r)(\zeta) \to \{p\} \ \text{for each} \ \zeta \in \mathcal{D} \ \text{as} \ r \to 0^+.
\end{align*} \]

Theorem 1.3 shows the existence of smoothly-varying Bishop discs when \( \text{Ind}_\mathcal{F}(\mathcal{S}, p) > 0 \). Here, \( A^\alpha(\mathbb{D}; \mathbb{C}^2) \) denotes the class of all \( C^2 \)-valued holomorphic maps on \( \mathbb{D} \) whose restrictions to \( \partial \mathbb{D} \) are in the Hölder class \( C^\alpha(\partial \mathbb{D}) \). Such a relationship between \( \text{Ind}_\mathcal{F}(\mathcal{S}, p) \) and Bishop discs is reminiscent of Wiegerinck’s results in [14]. The CR singularities studied by Wiegerinck are, for the most part, also non-parabolic, degenerate CR singularities, but they are required to satisfy an additional analytical hypothesis. The key point of departure of this article from [14] is summarised by these two observations:

- Although the surfaces \( (\mathcal{S}, p) \) studied in [14] are not necessarily thin at \( p \), Wiegerinck’s hypotheses do not hold true in general when \( \mathcal{S} \) is thin at \( p \).

- We provide some evidence for the conjecture that if \( \mathcal{S} \) is thin at \( p \), local polynomial convexity at \( p \) depends solely on whether or not \( \text{Ind}_\mathcal{F}(\mathcal{S}, p) \le 0 \). In contrast, there does not seem to be a clear-cut discriminant for local polynomial convexity if thinness is replaced by the hypotheses in [14].

We refer the reader to [14] Theorems 3.3, 3.4 for a precise statement of Wiegerinck’s hypotheses, but we translate to the requirement that \( \mathcal{F}_m \) — in the notation of (1.1) — must be subharmonic and non-harmonic. One of the motivations of this paper is to develop tools to show the existence of Bishop discs in the absence of such subharmonicity conditions. This is a meaningful motivation because of the following:

Fact (see Example 4.2). There exist polynomials \( \mathcal{F}_m : \mathbb{C} \to \mathbb{R} \), homogeneous of degree \( m \), such that

- 0 is an isolated CR singularity of the graph \( \Gamma(\mathcal{F}_m) \) satisfying \( \text{Ind}_\mathcal{F}(\Gamma(\mathcal{F}_m), 0) > 0 \); and

- \( \mathcal{F}_m \) is not subharmonic.

Example 4.2 rules out the possibility of simply applying the results of [14] to deduce Theorem 1.3. Let us now turn to a partial converse of Theorem 1.3.

Theorem 1.4. Let \( \mathcal{S} \) be a smooth real surface in \( \mathbb{C}^2 \) and let \( p \in \mathcal{S} \) be a CR singularity. Assume that \( p \) is non-parabolic and that \( \mathcal{S} \) is thin at \( p \). Suppose \( \text{Ind}_\mathcal{F}(\mathcal{S}, p) \le 0 \).

1) Let \( (z, w) \) be holomorphic coordinates centered at \( p \) such that (by hypothesis) \( \mathcal{S} \) is locally defined by

\[ \mathcal{S} \cap U_p : \ w = \mathcal{F}_m(z) + \mathcal{R}(z) \quad (\mathcal{R}(z) = O(|z|^{m+1}) \ \text{and} \ |z| \ \text{small}), \]

and such that \( \mathcal{F}_m \) is a real-valued polynomial that is homogeneous of degree \( m \). If \( \mathcal{R} \) is real-valued, then \( \mathcal{S} \) is locally polynomially convex at \( p \).

2) In general, given any \( \alpha \in (0, 1) \), it is impossible to find a continuous one-parameter family \( g : (0, 1) \to A^\alpha(\mathbb{D}; \mathbb{C}^2) \) of immersed, non-constant analytic discs having the following properties:
• \( g(t)(\partial \mathbb{D}) \subset (\mathcal{S} \setminus \{p\}) \cap U_p \forall t \in (0, 1); \)
• \( g(t)(e^t) \) is a simple closed curve in \( \mathcal{S} \forall t \in (0, 1); \) and
• \( g(t)(\zeta) \longrightarrow \{p\} \) for each \( \zeta \in \partial \mathcal{D} \) as \( t \longrightarrow 0^+. \)

The point of Part (2) of Theorem 1.4 is to observe that, although we do not know whether \( \text{Ind}_{\mathcal{M}}(\mathcal{S}, p) \leq 0 \) implies that \( \mathcal{S} \) is locally polynomially convex at \( p, \) the local polynomially convex hull of \( (\mathcal{S}, p) \) does not contain non-constant analytic discs (with boundaries in \( \mathcal{S} \)) that shrink to \( p. \) Note also that each part of Theorem 1.4 can be viewed as a partial converse to Theorem 1.3. These lead us to suggest the following conjecture:

**Conjecture 1.5.** Let \( \mathcal{S} \) be a smooth real surface in \( \mathbb{C}^2 \) and let \( p \in \mathcal{S} \) be a CR singularity. Assume that \( p \) is non-parabolic and that \( \mathcal{S} \) is thin at \( p. \) \( \mathcal{S} \) is locally polynomially convex at \( p \) if and only if \( \text{Ind}_{\mathcal{M}}(\mathcal{S}, p) \leq 0. \)

Before proceeding to the proofs, we would like to point out a couple of new inputs required in the proof of Theorem 1.3 and to sketch the main ingredients of our approach. Our proof consists of the following parts:

- **Part I.** We work in the coordinate system \((z, w)\) centered at \( p \) in which \((\mathcal{S}, p)\) is presented locally as shown in (1.2). We prove a general result:

\[
\text{Ind}_{\mathcal{M}}(\Gamma(\mathcal{F}_m), 0) = -\frac{\#(\mathcal{F}_m(e^{r+i\tau})^{-1}\{0\} \cap [0, 2\pi])}{2} + 1
\]

(the notation \(\#[\mathcal{S}]\) stands for the cardinality of the set \(\mathcal{S}\)). This tells us, since \(\text{Ind}_{\mathcal{M}}(\mathcal{S}, p) > 0,\) that we may assume (after making a holomorphic change of coordinate if necessary) that \(\mathcal{F}_m(z) > 0 \forall z \in \mathbb{C} \setminus \{0\}.\)

- **Part II.** We see that \(\mathcal{F}_m^{-1}\{1\}\) is a simple closed real-analytic curve. Let \(g\) denote the boundary-value of the normalised Riemann mapping of \(\mathbb{D}\) onto the region enclosed by \(\mathcal{F}_m^{-1}\{1\}.\) Then, the curves \(\varphi_r : \partial \mathbb{D} \longrightarrow \mathbb{C}^2, \ r > 0,\) given by \(\zeta \longmapsto (rg(\zeta), r^m)\) are closed curves in \(\Gamma(\mathcal{F}_m)\) that bound analytic discs. We view \(\mathcal{S},\) equivalently the graph \(\Gamma(\mathcal{F}_m + \mathcal{R}),\) as a small perturbation of \(\Gamma(\mathcal{F}_m),\) and attempt to obtain small corrections, say \(\psi_r, \) of \(\varphi_r \forall r \in (0, R_0),\) for \(R_0 > 0\) sufficiently small, such that \((\varphi_r + \psi_r)\) are curves in \(\Gamma(\mathcal{F}_m + \mathcal{R})\) that bound analytic discs. This gives us a family of functional equations, involving the harmonic-conjugate operator, parametrised by the interval \((0, R_0).\) The desired \(\psi_r, \) \(r \in (0, R_0),\) are derived from the fixed points of these equations.

- **Part III.** One way to obtain fixed points is to show that the functions involved in the aforementioned equations are contractions. This is the approach of Kenig & Webster in [9]. In making the required estimates, Kenig and Webster are aided by the following remarkable fact:

\[\text{(*) If, in addition to the hypotheses in Theorem 1.3, the polynomial } \mathcal{F}_m \text{ is quadratic, then given any } l \in \mathbb{N}, l \geq 3, \text{ there exists a holomorphic coordinate system } (z, w) \text{ such that } (\mathcal{S}, p) \text{ has a local representation of the form } (1.2) \text{ and such that } \text{Im}(\mathcal{R})(z) = O(|z|^{l+1}).\]

This fact is good enough to show that the Bishop discs foliate a \(C^\infty-\text{smooth 3-manifold with boundary. Unfortunately, the conclusion of (*) is not true in general if } m > 2, \) which takes a toll on our estimates. Hence, the conditions under which Kenig and Webster are able to use a type of Reflection Principle do not seem to be achievable. However, we do get estimates that are good
enough to show that a certain auxiliary functional equation — abstracted from
the proof of the Implicit Function Theorem — admits a fixed point (in [9], the
result (▲) completely eliminates the need to study such an equation). A careful
look at the proof of the Implicit Function Theorem reveals that the last fact
automatically implies $C^1$-smooth variation of the Bishop discs.

One final expository remark is in order here: one could naively set up a functional
equation of the kind we allude to in Part II above, and hope to show that
$r \mapsto -r \psi(r)$ is of class $C^1$ via the Implicit Function Theorem. The problem is that, owing to the
presence of the CR singularity, the relevant Fréchet (partial-)derivative of the non-
linear functional involved is non-surjective at all the obvious zeros of this functional!
The reader’s attention is drawn to the note in Step 2 of Section [3]. It is this fact that
leads to the (unavoidable) technicalities of the approach outlined above.

All this still leaves open certain questions that may be tractable. For instance, does
the local polynomial hull of $(S, p)$ contain any analytic discs with boundaries in $S$ other
than the discs $g(r)$, $r \in (0, R_0)$? The reader is directed to Section [6] for a rigorous
discussion of some open questions.

Since an important part of both Theorems [1.3] and [1.4] is based on a good understand-
ing of $\text{Ind}_{M}(S, p)$, we shall begin with a discussion on this index in the next section.
The proofs of Theorems [1.3] and [1.4] will be presented in Sections [3] and [5] respectively.
A discussion on the non-subharmonicity of the local graphing functions of the $(S, p)$
that we consider in this paper will be presented in Section [4].

2. SOME FACTS ABOUT THE MASLOV-TYPE INDEX

Given a smooth real surface $S \subset \mathbb{C}^2$, the term “Maslov-type index” might refer to
three inter-related numbers that apply to slightly different contexts. They are:

a) The index $\text{Ind}_{M, \gamma}(S)$ of a closed path: This applies to a closed path $\gamma : S^1 \to S$, where $S$ is a totally-real submanifold of a region $\Omega \subseteq \mathbb{C}^2$.

b) The index $\text{Ind}_{M}(S, p)$ of a CR singularity $p$: This applies to a pair $(S, p)$, where $S$ is an orientable real 2-submanifold of some region $\Omega \subseteq \mathbb{C}^2$ having an isolated CR singularity at $p \in S$.

c) The index $\text{Ind}_{M, \psi}(S)$ of an analytic disc $\psi$: This applies to an analytic disc $\psi \in \mathcal{O}(D; \mathbb{C}^2) \cap \mathcal{C}(\overline{D}; \mathbb{C}^2)$ with $\psi(\partial D) \subset S$, where $S$ is a totally-real submanifold of some region $\Omega \subseteq \mathbb{C}^2$.

In this paper, it is the first two indices that will be relevant to our discussions. Before
making the proper definitions, we will need one piece of notation. We set

$$G_{\text{tot, R}}(\mathbb{C}^2) := \text{the manifold of oriented totally-real planes in } \mathbb{C}^2,$$

where the differentiable structure on $G_{\text{tot, R}}(\mathbb{C}^2)$ is the one that makes it a submanifold of
the Grassmanian $G(2, \mathbb{R}^4)$ of oriented 2-subspaces of $\mathbb{R}^4$. We are now in a position
to make our definitions. In doing so, we follow the constructions by Forstnerič in [6].
Here, we make one remark: we wish to define the concepts (a) and (b) above with the
least amount of technicality possible, and to draw upon some computations in [9] that
pertain to graphs in $\mathbb{C}^2$. Hence, in the definitions below we will assume that the bundle
$\gamma^* TS|_{\gamma(S^1)}$ is a trivial bundle (where $\gamma : S^1 \to S$ is as in (a)), although the notion of
$\text{Ind}_{M, \gamma}(S)$ is not restricted to the trivial-bundle case.
Definition 2.1. Let $S$ be a totally-real submanifold of a region $\Omega \subseteq \mathbb{C}^2$. Let $\gamma : S^1 \rightarrow S$ be a smooth, closed path such that the pullback $\gamma^* TS|_{\gamma(S^1)}$ is a trivial bundle (equivalently, $S$ is orientable along $\gamma$). Let $\Theta_{\gamma} : S^1 \rightarrow G_{\text{tot.}}(\mathbb{C}^2)$ denote the tangent map, i.e., $\Theta_{\gamma}(\zeta) := T_{\gamma(\zeta)}(S)$. There is a well-defined Gauss map $\mathfrak{G} : G_{\text{tot.}}(\mathbb{C}^2) \rightarrow \mathbb{C}\setminus\{0\}$ given by

$$\mathfrak{G}(P) := \det [X^P_1, X^P_2], \quad (X^P_1, X^P_2) \text{ a positively oriented orthonormal basis of } P;$$

this being well-defined because, given two positively oriented orthonormal bases $(X^P_1, X^P_2)$ and $(Y^P_1, Y^P_2)$, $Y^P_j = A(X^P_j)$, $j = 1, 2$, for some $A \in \text{SL}(2, \mathbb{R})$. The composition $\Theta_{\gamma} \circ \mathfrak{G}$ induces a homomorphism in homology $H_1(\Theta_{\gamma}) : H_1(S^1; \mathbb{Z}) \rightarrow H_1(\mathbb{C}\setminus\{0\}; \mathbb{Z})$. The degree of this homomorphism is called the Maslov-type index of the path $\gamma$, denoted by $\text{Ind}_{M,\gamma}(S)$.

Definition 2.2. Let $S$ be a real orientable 2-submanifold of some region $\Omega \subseteq \mathbb{C}^2$ that has an isolated CR singularity at $p \in S$. Then there is an $S$-open neighbourhood of $p$, say $W_p$, that is contractible to $p$ and such that $p$ is the only CR singularity in $W_p$. Let $W_p$ have the orientation induced by the complex line $T_p(S)$. Let $\gamma : S^1 \rightarrow W_p \setminus \{p\}$ be a smooth, simple closed curve that has positive orientation with respect to the orientation of $W_p$. Then, we define the Maslov-type index of the CR singularity $p$, written as $\text{Ind}_{M}(S, p)$, by $\text{Ind}_{M}(S, p) := \text{Ind}_{M,\gamma}(W_p \setminus \{p\})$.

We note that $\text{Ind}_{M}(S, p)$ is well-defined because $\text{Ind}_{M,\gamma}(W_p \setminus \{p\})$ depends only on the homology class of $\gamma$ in $W_p \setminus \{p\}$. When $S$ is the graph $\Gamma(F)$ of some function $F$ that is $C^1$-smooth near $0 \in \mathbb{C}$, with $\Gamma(F)$ having an isolated CR singularity at the origin, then $\gamma^* TS|_{\gamma(S^1)}$ is trivial for any $\gamma : S^1 \rightarrow \Gamma(F) \setminus \{(0,0)\}$ as in Definition 2.2. Using an explicit frame for $\gamma^* TF|_{\gamma(S^1)}$, Forstnerič has shown that:

Lemma 2.3 ([6], Lemma 8). Let $\Omega$ be a domain in $\mathbb{C}$ containing $0$ and let $F \in C^1(\Omega; \mathbb{C})$. Suppose that the graph $\Gamma(F)$ has an isolated CR singularity at $0$. Let $\gamma : S^1 \rightarrow \Omega \setminus \{0\}$ be a smooth, positively-oriented, simple closed curve that encloses $0$ and encloses no other points belonging to $(\partial F/\partial \bar{z})^{-1}\{0\}$. Then

$$\text{Ind}_{M}(\Gamma(F), 0) = \text{Wind} \left( \frac{\partial F}{\partial \bar{z}} \circ \gamma, 0 \right),$$

where the expression on the right-hand side denotes the winding number around $0$.

We are now in a position to prove a key lemma. This was informally stated in Part I of our outline, in Section 1 of the proof of Theorem 1.3.

Lemma 2.4. Let $\mathcal{F}_m : \mathbb{C} \rightarrow \mathbb{R}$ be a polynomial that is homogeneous of degree $m$ and such that $(\partial \mathcal{F}_m / \partial \bar{z})^{-1}\{0\} = \{0\}$. Then

$$\text{Ind}_{M}(\Gamma(\mathcal{F}_m), 0) = -\frac{\#(\mathcal{F}_m(e^{i\theta})^{-1}\{0\} \cap [0, 2\pi])}{2} + 1$$

(the notation $\#(S)$ denotes the cardinality of the set $S$).

Proof. Let us define the real-analytic, $2\pi$-periodic function $f$ by the relation $\mathcal{F}_m(z) = |z|^m f(\theta)$, where we write $z = |z|e^{i\theta}$. Then, we compute

$$\frac{\partial \mathcal{F}_m}{\partial \bar{z}}(e^{i\theta}) = \frac{e^{i\theta}}{2} \left( mf(\theta) + if'(\theta) \right).$$

We record two facts:
a) Since $f \in C^\infty(\mathbb{R})$ and $2\pi$-periodic, $\# \{ \theta \in [0, 2\pi) : f(\theta) = 0 \}$ is an even number.
b) Since $(\partial \mathcal{F}_m/\partial z)^{-1}\{0\} = \{0\}$, $f(\theta)$ and $f'(\theta)$ cannot simultaneously vanish for any $\theta \in [0, 2\pi)$.

Thus, we have two closed paths $\gamma_1, \gamma_2 : [0, 2\pi] \to \mathbb{C} \setminus \{0\}$, defined by:

$$\gamma_1(\theta) := mf(\theta) + if'(\theta),$$

$$\gamma_2(\theta) := e^{i\theta}.$$

Recalling that the winding number is additive across factors, we get:

$$\text{Wind}\left( \frac{\partial \mathcal{F}_m}{\partial z}(e^{i\cdot}), 0 \right) = \text{Wind}(\gamma_1, 0) + \text{Wind}(\gamma_2, 0).$$

Hence, in view of the above and Lemma 2.3, it suffices for us to show that

$$(\ref{2.4}) \quad \text{Wind}(\gamma_1, 0) = -\frac{\# \{ \theta \in [0, 2\pi) : f(\theta) = 0 \}}{2}.$$  

Let us first consider the case when $f^{-1}\{0\} \neq \emptyset$. Without loss of generality, we may assume that $f(0) = 0$. Let

$$0 = \theta_1 < \theta_1 < \ldots < \theta_N < 2\pi$$

denote the distinct zeros of $f|_{[0,2\pi]}$. Let $\phi : [0,2\pi] \to \mathbb{R}$ be a function having the following properties (recall that by (b) above $f$ has only simple zeros):

- $\phi(\theta_j) = 0$, $j = 1, \ldots, N$;
- $\phi'(\theta_j)f'(\theta_j) > 0$, $j = 1, \ldots, N$;
- $|\phi'(\theta_j)| > |f'(\theta_j)|$, $j = 1, \ldots, N$;
- $\left(\phi|_{(\theta_{j-1}, \theta_j)}\right)'$ has precisely one simple zero in $(\theta_{j-1}, \theta_j)$, $j = 1, \ldots, N$; and
- $\phi$ has a $C^\infty$-smooth periodic extension to $\mathbb{R}$.

In view of the third property of $\phi$, there exists a constant $K > 0$ such that

$$(\ref{2.5}) \quad |\phi'(\theta_j)| > |f'(\theta_j)| > K, \quad j = 1, \ldots, N.$$  

Define the homotopy $H : [0, 2\pi] \times [0, 1] \to \mathbb{C}$ by

$$H(\theta, t) := m[(1-t)f(\theta) + t\phi(\theta)] + i[(1-t)f'(\theta) + t\phi'(\theta)].$$

Note that, by construction

$$\text{Re}(H)(\theta, t) = 0 \iff f(\theta) = 0,$$

$$f(\theta) = 0 \implies |(1-t)f'(\theta) + t\phi'(\theta)| > K.$$  

Hence, in fact, $H([0,2\pi] \times [0, 1]) \subset \mathbb{C} \setminus \{0\}$. Thus $\gamma_1$ is homotopic in $\mathbb{C} \setminus \{0\}$ to the path $\Gamma_1 : [0, 2\pi] \to \mathbb{C} \setminus \{0\}$ given by

$$\Gamma_1(\theta) = m\phi(\theta) + i\phi'(\theta), \quad \theta \in [0, 2\pi].$$

By construction, the number of times that $\Gamma_1$ winds around the origin is half the number of times that $\Gamma_1$ intersects the real axis. But, since, by construction, $\Gamma_1$ is oriented clockwise, we get, by homotopy invariance of the winding number:

$$(\ref{2.6}) \quad \text{Wind}(\gamma_1, 0) = \text{Wind}(\Gamma_1, 0) = -\frac{\# \{ \theta \in [0, 2\pi) : f(\theta) = 0 \}}{2}.$$
In the case when \( f^{-1}\{0\} = \emptyset \), \( \gamma_1 \) never crosses the real axis. Hence
\[
(2.7) \quad f^{-1}\{0\} = \emptyset \iff \text{Wind}(\gamma_1, 0) = 0.
\]
From \((2.6)\) and \((2.7)\) we see that \((2.4)\) has been established. This establishes our result. \(\square\)

The last result in this section provides a Maslov-index calculation for the graph of a homogeneous polynomial \( F_m \) that is, in contrast to Lemma \(2.4\), complex-valued. It will find no application later in this paper, but we present it as it might be of independent interest.

**Lemma 2.5.** Let \( F_m \) be a non-analytic, complex-valued polynomial that is homogeneous of degree \( m \) and such that \((\partial F_m/\partial \overline{z})^{-1}\{0\} = \{0\}\). Define the polynomial \( Q_m \in \mathbb{C}[z, w] \) by the relation
\[
Q_m(z, \overline{z}) = \frac{\partial F_m}{\partial \overline{z}}(z, \overline{z})
\]
by making explicit the dependence of \( \partial F_m/\partial \overline{z} \) on \( z \) and \( \overline{z} \). Let \( p_m \) be the polynomial defined as \( p_m(z) := Q_m(z, 1) \). Then,
\[
\text{Ind}_{\mathcal{M}}(\Gamma(F_m), 0) = 2 \left( \sum \left\{ \mu(\zeta) : \zeta \in p_m^{-1}\{0\} \cap \mathbb{D} \right\} \right) - (m - 1),
\]
where \( \mu(\zeta) \) denotes the multiplicity of \( \zeta \) as a zero of the polynomial \( p_m \).

**Proof.** Note that, by hypothesis, the path \((\partial F_m/\partial \overline{z})(e^{i\theta})\) does not pass through the origin. Hence, in view of \((2.1)\), we can explicitly compute the desired winding number to get:
\[
(2.8) \quad \text{Ind}_{\mathcal{M}}(\Gamma(F_m), 0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\partial^2 F_m(e^{i\theta})i e^{i\theta} - \partial^2 F_m(e^{i\theta})i e^{-i\theta}}{\partial \overline{z} F_m(e^{i\theta})} d\theta.
\]
We now compute that
\[
(2.9) \quad \partial^2_{\overline{z} z} F_m(e^{i\theta})i e^{i\theta} - \partial^2_{\overline{z} z} F_m(e^{i\theta})i e^{-i\theta} = i e^{i\theta} \left[ \frac{1}{z^{m-1}} Q_m(z^2, 1) \right]_{z=e^{i\theta}},
\]
\[
(2.10) \quad \partial_{\overline{z}} F_m(e^{i\theta}) = \frac{1}{z^{m-1}} Q_m(z^2, 1) \bigg|_{z=e^{i\theta}}.
\]
From \((2.8)\), \((2.9)\) and \((2.10)\), we get
\[
\text{Ind}_{\mathcal{M}}(\Gamma(F_m), 0) = \frac{1}{2\pi i} \oint_{S^1} \frac{\left[ \frac{1}{z^{m-1}} p_m(z^2) \right]'}{\frac{1}{z^{m-1}} p_m(z^2)} \bigg|_{z=\zeta} d\zeta.
\]
Since, by hypothesis, the denominator in the above integral never vanishes, the Argument Principle gives us
\[
\text{Ind}_{\mathcal{M}}(\Gamma(F_m), 0) = 2 \left( \sum \left\{ \mu(\zeta) : \zeta \in p_m^{-1}\{0\} \cap \mathbb{D} \right\} \right) - (m - 1).
\]
\(\square\)
3. The proof of Theorem 1.3

We introduce some notations that will be needed in the proof of Theorem 1.3. First, we define the Banach space $C^\alpha(\partial \mathbb{D}; F)$, $\alpha \in (0, 1)$, where $F$ will stand for either $\mathbb{R}$ or $\mathbb{C}$ in the following proof, as

$$C^\alpha(\partial \mathbb{D}; F) := \left\{ f : \partial \mathbb{D} \to F : \sup_{\theta \in \mathbb{R}} |f(e^{i\theta})| + \sup_{\theta \neq \phi \in \mathbb{R}} \frac{|f(e^{i\theta}) - f(e^{i\phi})|}{|\theta - \phi|^\alpha} < \infty \right\},$$

where the norm on this Banach space is:

$$\|f\|_{C^\alpha} := \sup_{\theta \in \mathbb{R}} |f(e^{i\theta})| + \sup_{\theta \neq \phi \in \mathbb{R}} \frac{|f(e^{i\theta}) - f(e^{i\phi})|}{|\theta - \phi|^\alpha}.$$

We will also have occasion to use the following abbreviation

$$[f]_\alpha := \sup_{\theta \neq \phi \in \mathbb{R}} \frac{|f(e^{i\theta}) - f(e^{i\phi})|}{|\theta - \phi|^\alpha}.$$

In what follows, $A(\partial \mathbb{D})$ will denote the class of restrictions to the unit circle of functions that are holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. For any $f \in C(\partial \mathbb{D}; F)$ we will denote the Fourier series of $f$ as follows:

$$f \sim \sum_{n \in \mathbb{Z}} \hat{f}(n)e^{in\theta}.$$  

It is well known that if $f \in C^\alpha(\partial \mathbb{D}; F)$ with $\alpha \in (0, 1)$, then any harmonic conjugate on $\mathbb{D}$ of the Poisson integral of $f$ extends to a function on $\overline{\mathbb{D}}$ and its restriction to $\partial \mathbb{D}$, say $h_f$, is of class $C^\alpha(\partial \mathbb{D}; F)$. In this paper, $\mathcal{H}[f]$ will denote that $h_f$ which satisfies (in our Fourier-series notation) $\mathcal{H}_f(0) = 0$. In terms of Fourier series:

$$\mathcal{H}[f] \sim \sum_{n \in \mathbb{Z}} -i \text{sgn}(n)\hat{f}(n)e^{in\theta}.$$  

We call $\mathcal{H}[f]$ the conjugate of $f$. Recall that the operator $\mathcal{H} : C^\alpha(\partial \mathbb{D}; F) \to C^\alpha(\partial \mathbb{D}; F)$ is a certain singular-integral operator that is bounded on $C^\alpha(\partial \mathbb{D}; F)$. We shall use this fact (which we assume the reader is familiar with) in Step 1 of our proof below.

**The proof of Theorem 1.3** Let $(\mathcal{S}, p)$ be as stated in the hypothesis of the theorem. By definition, there is a neighbourhood $U_p$ of $p$ and holomorphic coordinates $(z, w)$ centered at $p$ such that $\mathcal{S}$ is locally defined by

$$(3.1) \quad \mathcal{S} \cap U_p : \quad w = f_m(z) + \mathcal{R}(z) \quad (\text{for } |z| \text{ small}),$$

where $f_m$ is a real-valued polynomial that is homogeneous of degree $m$, and $\mathcal{R}(z) = O(|z|^{m+1})$. From this last fact, and from (2.1) in Lemma 2.3 we see that

$$\text{Ind}_M(\Gamma(f_m + \mathcal{R}), 0) = \text{Ind}_M(\Gamma(f_m), 0).$$

This is seen by considering the relevant winding numbers of small circles centered at $z = 0$. Now note that the index $\text{Ind}_M(\mathcal{S}, p)$ is, by construction, invariant under holomorphic changes of coordinate. Hence

$$\text{Ind}_M(\Gamma(f_m), 0) = \text{Ind}_M(\Gamma(f_m + \mathcal{R}), 0) = \text{Ind}_M(\mathcal{S}, p) > 0.$$  

Applying (2.2) to the above statement, we may conclude, without loss of generality, that

$$(3.2) \quad f_m(z) > 0 \quad \forall z \in \mathbb{C} \setminus \{0\}.$$
Note that as Φ(ψ, α can define the region in
D : the region in C enclosed by F−1{1},
G : the unique Riemann mapping of D onto D such that G(0) = 0, G′(0) > 0.

Step 1. Constructing the relevant Bishop’s Equation
Let us fix an α ∈ (0, 1). Define the mapping A : Cα(∂D; R) → Cα(∂D; C) ∩ A(∂D) by
A[ψ] := ψ + iδ[ψ].

Recall that δ[ψ] denotes the conjugate of ψ. It is well-known that for each α ∈ (0, 1),
there exists a γα > 0 such that
||δ[ψ]|| < γα ||ψ||_{Cα} ∀ψ ∈ Cα(∂D; R).

Define the open set Ωα ⊂ Cα(∂D; R) by
Ωα := {ψ ∈ Cα(∂D; R) : \sqrt{1 + γα^2} ||ψ||_{Cα} < 3ρ/8}.

Finally, define the function Φ : Ωα × (0, 3ρ/4) → Cα(∂D; R) by
Φ(ψ, r) := −(kr)^m + \mathcal{F}_m \circ (g_r + e^{iA[ψ]}) + (\text{Re} \mathcal{R}) \circ (g_r + e^{iA[ψ]})
+ δ[\text{Im} \mathcal{R}] \circ (g_r + e^{iA[ψ]})
= ∂z\mathcal{F}_m(g_r)e^{iA[ψ]} + ∂z\mathcal{F}_m(g_r)e^{iA[ψ]} + Q(g_r, e^{iA[ψ]})
+ (\text{Re} \mathcal{R}) \circ (g_r + e^{iA[ψ]}) + δ[\text{Im} \mathcal{R}] \circ (g_r + e^{iA[ψ]})

where we define
Q(X, Y) := \sum_{j=2}^m \sum_{\mu+\nu=j} \frac{1}{\mu!\nu!} \partial_x^\mu \partial_y^\nu \mathcal{F}_m(X)Y^\mu Y^\nu.

We are now in a position to assert the following:

Fact A. If, for some (ψ_0, r_0) ∈ Ωα × (0, 3ρ/4), Φ(ψ_0, r_0) = 0, then there is an
analytic disc F ∈ O(D; C^2) ∩ Cα(D), which is a small perturbation of the analytic disc
(g_r, (kr)^m), such that F(∂D) ⊂ S.

To justify the above assertion, note that as Φ(ψ_0, r_0) is identically zero,
Φ(ψ_0, r_0) + iA[\text{Im} \mathcal{R}] \circ (g_{r_0} + e^{iA[ψ_0]})
is the boundary value of a holomorphic function. However

\[(3.5) \quad \Phi(\psi_0, r^0) + iA[(\text{Im} R) \circ (g_{r^0} + e^{i\cdot A}[\psi_0])] = -((kr^0)^n + m)^n + F_m \circ (g_{r^0} + e^{i\cdot A}[\psi_0]) + R \circ (g_{r^0} + e^{i\cdot A}[\psi_0]).\]

Clearly, the Poisson integral of the function

\[(g_{r^0}, (kr^0)^n) + (e^{i\cdot A}[\psi_0], iA[(\text{Im} R) \circ (g_{r^0} + e^{i\cdot A}[\psi_0])])\]

is an analytic disc \(F := (F_1, F_2), \) and by (3.5)

\[F_2(\zeta) = F_m \circ F_1(\zeta) + R \circ F_1(\zeta) \quad \forall \zeta \in \partial D,\]

which is precisely the fact asserted above.

To show that the analytic discs described in Theorem \(1.3\) vary smoothly with respect to the parameter \(r,\) we have to establish that each of these discs exists. To this end, the above discussion helps in setting the following

**Intermediate Goal.** To solve the equation \(\Phi(\psi, r) = 0\) for all sufficiently small values of the parameter \(r.\)

**Step 2.** Setting up an equivalent equation to the functional equation \(\Phi(\psi, r) = 0\)

Consider the linear operator (which is bounded from \(C^\alpha(\partial D; \mathbb{R})\) to \(C^\alpha(\partial D; \mathbb{R})\) owing to (3.3) above)

\[\Lambda_r : \psi \mapsto \partial_z F_m(g_r)e^{i\cdot A}[\psi] + \partial_{\bar{z}} F_m(g_r)e^{i\cdot A}[\psi] = 2\text{Re}\left\{ \partial_z F_m(g_r)e^{i\cdot A}[\psi] \right\}.\]

**Note.** Before we engage in technicalities, we ought to point out the difficulties inherent in this problem. First note that:

*The Fréchet derivative \(\partial_\psi \Phi|_{(\psi,0)}\) is not invertible for any \(\psi \in \Omega_\alpha.\)*

Suppose that could show that the Fréchet derivative \(\partial_\psi \Phi|_{(\psi^0, r^0)}\) is invertible for some \((\psi^0, r^0) \in \Omega_\alpha \times (0, 3\rho/4) = \text{Dom}(\Phi).\) With this, we would still be unable to invoke the Implicit Function Theorem to either assert the existence of analytic discs attached to \(S\) or to infer their smooth dependence on \(r\) via the smooth dependence of \(\psi\) on \(r\) in a neighborhood of \(r_0.\) This is because it must first be established that \(\Phi(\psi^0, r^0) = 0!\)

This is precisely our Intermediate Goal above. To achieve this, we prefer to analyse the operators \(\Lambda_r\) — rather than the Fréchet derivatives \(\partial_\psi \Phi|_{(\cdot, r)}\) — as the \(\Lambda_r\)’s are easier to estimate. We begin our analysis with the following:

**Claim.** \(\Lambda_r\) is an isomorphism.

To show that \(\Lambda_r\) is surjective, note that it suffices to show that given any \(f \in C^\alpha(\partial D; \mathbb{R}),\) there exists a function \(a_f \in A^0_0(\partial D),\) where

\[A^0_0(\partial D) := \{ h \in C^\alpha(\partial D; \mathbb{C}) : \hat{h}(0) \in \mathbb{R}, \text{ and } \hat{h}(j) = 0 \ \forall j \leq -1\},\]

such that \(2\text{Re}\left\{ \partial_z F_m(g_r)e^{i\cdot A}a_f \right\} = f.\) Note that, from the discussion preceding Step 2, it can be inferred that

\[F_m \circ (r \kappa G) = (kr)^n \text{ vanishes on } \partial D.\]

Thus, recalling that \(\kappa(G)(\zeta) \neq 0 \ \forall \zeta \in \partial D,\) there exists a \(\delta > 0\) and a function \(R \in C^\alpha(\text{Ann}(0; 1 - \delta, 1 + \delta))\) such that

- \(R(z) > 0 \ \forall z \in \text{Ann}(0; 1 - \delta, 1 + \delta);\) and
\begin{itemize}
  \item \(T_m \circ (r\kappa \tilde{G}) - (kr)^m = r^m R(z)(|z|^2 - 1) \forall z \in \text{Ann}(0; 1 - \delta, 1 + \delta)\).
\end{itemize}

By the chain rule (recall that \(g_r\) is the restriction of a holomorphic function):

\begin{equation}
\tag{3.6}
e^{ir}(\partial_z T_m) \circ (g_r) = \frac{\partial(T_m \circ (kr \tilde{G}) - (kr)^m)}{\partial z} \bigg|_{\partial D} \times \frac{e^{ir}}{kr \tilde{G}'}
\end{equation}

\begin{equation}
\tag{3.7}
= r^m (\frac{\partial R(\frac{z}{|z|^2 - 1})}{\partial D}) \frac{e^{ir}}{kr \tilde{G}'}.
\end{equation}

The second equality follows from the fact that \((|z|^2 - 1)\partial_z R(z)\) vanishes on \(\partial \mathbb{D}\). Hence, the desired \(a_f\) is a solution to the equation

\begin{equation}
\tag{3.8}
2r^{m-1} R(e^{ir}) \text{Re} \left( \frac{a_f}{G'} \right) = f \quad \text{with } a_f \in A_0^0(\partial \mathbb{D}).
\end{equation}

It was shown by Privalov that – owing to the normalisation condition that \(a_f\) belong to \(A_0^0(\partial \mathbb{D})\) – the equation (3.8) has a unique solution in \(A_0^0(\partial \mathbb{D})\) given by

\[
a_f(\zeta) = \frac{\kappa \tilde{G}'(\zeta)}{2r^{m-1}} \mathcal{A} \left[ \frac{f}{R(e^{ir})} \right] \quad \forall \zeta \in \partial \mathbb{D}.
\]

This establishes that \(\Lambda_r\) is surjective, and the uniqueness of \(a_f\) establishes that it is injective. Hence the claim.

To complete the discussion on the invertibility of \(\Lambda_r\) we note that \(\Lambda_r^{-1} = \Lambda_r\), where

\[
\Lambda_r[f] = \text{Re} \left\{ \frac{\kappa \tilde{G}'(e^{ir})}{2r^{m-1}} \mathcal{A} \left[ \frac{f}{R(e^{ir})} \right] \right\}.
\]

Furthermore, from the fact that \(\mathcal{A} = \mathbb{I}_{C^\alpha} + i \mathcal{J}\), and from the estimate (5.3), we get the following important estimate: there exists a \(K_\alpha > 0\) such that

\begin{equation}
\tag{3.9}
\|\Lambda_r[f]\|_{C^\alpha} \leq K_\alpha r^{1-m} \|f\|_{C^\alpha} \quad \forall f \in C^\alpha(\partial \mathbb{D}; \mathbb{R}).
\end{equation}

Finally, by applying \(\Lambda_r\) to the equation (5.3), we see that solving the equation

\[
\Phi(\psi, r) = 0, \quad (\psi, r) \in \Omega_\alpha \times (0, 3\rho/4)
\]

is equivalent to solving

\[
\psi + \Lambda_r \left[ Q(g_r, e^{ir} \mathcal{A}[\psi]) + (\text{Re} R) \circ (g_r + e^{ir} \mathcal{A}[\psi]) + \mathcal{J}((\text{Im} R) \circ (g_r + e^{ir} \mathcal{A}[\psi])) \right]
= \psi - H(\psi; r)
\]

In view of this, the goal presented at the end of Step 1 is modified as follows:

**Modified Intermediate Goal.** To find a fixed point of the map \(H(\cdot; r) : \Omega_\alpha \to C^\alpha(\partial \mathbb{D}; \mathbb{R})\) for each sufficiently small value of the parameter \(r\).

**Step 3. Some estimates**

We shall use the contraction mapping principle to establish the modified goal above. For this purpose, we will (for a fixed \(r > 0\)) determine the image under \(H(\cdot; r)\) of a small closed ball in \(C^\alpha(\partial \mathbb{D}; \mathbb{R})\) centered at 0. We will also show that \(H(\cdot; r)\) is a contraction on this ball. This requires some estimates.
Since $R(z) = O(|z|^{m+1})$, it follows that there is a large positive constant $L > 0$ that is independent of $r > 0$ such that

$$\|(ReR) \circ g_r\|_\infty \leq L \left(\frac{r}{\rho}\right)^{m+1},$$

and

$$\|(ReR) \circ g_r\|_{\alpha} \leq L \left(\frac{r}{\rho}\right)^{m+1} \|ReR\|_{e^1} \kappa \rho \sup_{\theta \neq \phi \in \mathbb{R}} \frac{|g(e^{i\theta}) - g(e^{i\phi})|}{|\theta - \phi|^\alpha}. \tag{3.11}$$

We now set the stage for showing that for each $r > 0$ sufficiently small, $H(\cdot, r)$ is a contraction on the closed ball $\overline{B_C(0; r^{1+\delta})}$, where we pick and fix $\delta \in (1/2, 1)$. Furthermore, we shall work with $r \in (0, r_1)$, where $r_1 > 0$ is so small that $r/(100\sqrt{1 + \gamma_0^2})$.

Next, we estimate, using the fundamental theorem of calculus:

$$\left|Q(g_r, e^{i\cdot}A[\psi_1]) - Q(g_r, e^{i\cdot}A[\psi_2])\right|_{\alpha} \leq \sum_{j=2}^{m} \sum_{\mu + \nu = j} \frac{r^{m-j}}{\mu!\nu!} \|\partial^\mu \partial^\nu \mathcal{J}_m(\kappa g) (A(\mu, \nu, \psi_1) - A(\mu, \nu, \psi_2))\|_{\alpha}$$

$$\leq \sum_{j=2}^{m} \sum_{\mu + \nu = j} \frac{r^{m-j}}{\mu!\nu!} \sup_{\zeta \in \mathbb{B}} |\partial^\mu \partial^\nu \mathcal{J}_m(\zeta)|$$

$$\times \sqrt{1 + \gamma_\alpha^2} \left|A[\psi_1 - \psi_2]\right|_{\alpha} \left(\mu r^{(1+\delta)(j-1)} + \nu r^{(1+\delta)(j-1)}\right) \tag{3.13}$$

It is the bound $\|\psi_j\|_{C^\alpha} \leq r^{(1+\delta)}$, $j = 1, 2$, that leads to the second inequality above.

Next, we estimate, using the fundamental theorem of calculus:

$$\left|(ReR) \circ (g_r + e^{i\cdot}A[\psi_1]) - (ReR) \circ (g_r + e^{i\cdot}A[\psi_2])\right|_{\alpha} \leq 2 \sup_{\theta \in \mathbb{R}} \left|\int_0^1 \partial_z (ReR)(g_r(e^{i\theta}) + e^{i\theta}(tA[\psi_1] + (1-t)A[\psi_2])(e^{i\theta}))dt\right| \left|A[\psi_1 - \psi_2]\right|_{\alpha}.$$

Since $R(z) = O(|z|^{m+1})$, and since the constraint (3.12) ensures that

$$\text{range}(g_r + te^{i\cdot}A[\psi_1] + (1-t)e^{i\cdot}A[\psi_2]) \in \text{dil}_r[\text{domain}(R)] \forall t \in [0, 1]$$

(where $\text{dil}_r$ denotes the dilation on $\mathbb{C}$ by a factor of $r$), there is a large positive constant $L > 0$ that is independent of $r > 0$ such that

$$\|(ReR) \circ (g_r + e^{i\cdot}A[\psi_1]) - (ReR) \circ (g_r + e^{i\cdot}A[\psi_2])\|_{\alpha} \leq L \sqrt{1 + \gamma_\alpha^2} \|\psi_1 - \psi_2\|_{C^\alpha} \left(\frac{r}{\rho}\right)^m. \tag{3.14}$$
To simplify the presentation of our next estimate, let us set
\[ C(j, \alpha) := \|e^{ij}\|_{C_\alpha}, \quad j \in \mathbb{Z}, \quad M_\alpha := \max (\|A[\psi_1]\|_{C_\alpha}, \|A[\psi_2]\|_{C_\alpha}), \]
and write \( j = \mu + \nu, 2 \leq j \leq m \). Note that:
\[
\begin{align*}
\partial_\nu \partial_\mu \mathcal{F}_m(g_r)A(\mu, \nu, \psi_1) &- \partial_\nu \partial_\mu \mathcal{F}_m(g_r)A(\mu, \nu, \psi_2) \\
n &= r^{m-j} \partial_\nu \partial_\mu \mathcal{F}_m(\kappa g)e^{i(\mu-\nu)} \\
&\times \left( \frac{\partial \mathcal{F}_m}{\partial \nu_1} A(\mu, \nu - t - 1, \psi_2) + A(\psi_1 - \psi_2) \sum_{s=0}^{\mu-1} A(s, \nu, \psi_1)A[\psi_2]^{\mu-s-1} \right).
\end{align*}
\]
Then, it is easy to see that
\[
\begin{align*}
&\left[ \partial_\nu \partial_\mu \mathcal{F}_m(g_r)A(\mu, \nu, \psi_1) - \partial_\nu \partial_\mu \mathcal{F}_m(g_r)A(\mu, \nu, \psi_2) \right]_\alpha \\
&\leq C_\alpha r^{m-j} \|F_m\|_{C^{j+1}(\mathbb{R})} \sup_{\theta \neq \phi \in \mathbb{R}} \left| \frac{g(e^{i\theta}) - g(e^{i\phi})}{|\theta - \phi|^{\alpha}} \right| j M_\alpha^{-1} \|A[\psi_1 - \psi_2]\|_{C^{j+1}} \\
&+ C_\alpha r^{m-j} \|F_m\|_{C^{j+1}(\mathbb{R})} j M_\alpha^{-1} \left( \|A[\psi_1 - \psi_2]\|_{C^{j+1}} (C(\mu - \nu, \alpha) + (j - 1)) + \|A[\psi_1 - \psi_2]\|_{C_\alpha} \right)
\end{align*}
\]
From this estimate, and the fact that \( 0 \leq M_\alpha \leq \sqrt{1 + \gamma_\alpha^2 r^{1+\delta}} \), we conclude that there exists a constant \( C_\alpha > 0 \), depending only on \( \alpha \), such that
\[
(3.15) \quad [Q(g_r, e^{i\cdot}A[\psi_1]) - Q(g_r, e^{i\cdot}A[\psi_2])]_\alpha \leq C_\alpha \sum_{j=2}^{m} r^{(m-1)+\delta(j-1)} \|\psi_1 - \psi_2\|_{C_\alpha}.
\]
Finally, using exactly the same technique that led to the estimate (3.14), we compute:
\[
\begin{align*}
&\left[ (\text{Re} \mathcal{R}) \circ (g_r + e^{i\cdot}A[\psi_1]) - (\text{Re} \mathcal{R}) \circ (g_r + e^{i\cdot}A[\psi_2]) \right]_\alpha \\
&\leq 2 \sup_{\theta \in \mathbb{R}} \left| \text{Re} \left\{ \int_0^1 \partial_\nu (\text{Re} \mathcal{R})(g_r(e^{i\theta})) + \partial_\nu (\text{Re} \mathcal{R})(g_r(e^{i\theta}))(e^{i\theta})dt \right\} \right| \|A[\psi_1 - \psi_2]\|_{C_\alpha} \\
&+ L \left( \frac{r}{\rho} \right)^m \|A[\psi_1 - \psi_2]\|_{C^{j+1}} \int_0^1 \rho \left[ \kappa g + \frac{e^{i\cdot}(tA[\psi_1] + (1-t)A[\psi_2])}{r} \right] dt.
\end{align*}
\]
Arguing in an analogous manner as above, we conclude that there exists a large uniform constant \( L > 0 \) and a \( C_\alpha > 0 \), depending only on \( \alpha \), such that:
\[
(3.16) \quad [ (\text{Re} \mathcal{R}) \circ (g_r + e^{i\cdot}A[\psi_1]) - (\text{Re} \mathcal{R}) \circ (g_r + e^{i\cdot}A[\psi_2]) ]_\alpha \\
\leq 2L \left( \frac{r}{\rho} \right)^m \left( \sqrt{1 + \gamma_\alpha^2 + C_\alpha r^\delta} \right) \|\psi_1 - \psi_2\|_{C_\alpha}.
\]
We are now in a position to write down three key estimates that we need. In each of the three estimates, there exists a constant \( C_\alpha > 0 \) that depends only on \( \alpha \) such that the following inequalities hold. Firstly, from (3.13) and (3.15) we get
\[
(3.17) \quad \|Q(g_r, e^{i\cdot}A[\psi_1]) - Q(g_r, e^{i\cdot}A[\psi_2])\|_{C_\alpha} \leq C_\alpha \sum_{j=2}^{m} r^{(m-1)+\delta(j-1)} \|\psi_1 - \psi_2\|_{C_\alpha}.
\]
Next, from (3.14) and (3.16), we get
\[
(3.18) \quad \| (\text{Re} \mathcal{R}) \circ (g_r + e^{i\cdot}A[\psi_1]) - (\text{Re} \mathcal{R}) \circ (g_r + e^{i\cdot}A[\psi_2]) \|_{C_\alpha} \leq C_\alpha (1 + \rho^\delta) r^m \|\psi_1 - \psi_2\|_{C_\alpha}.
\]
Finally, note that the same arguments that lead to (3.14) and (3.16) also yield exactly analogous estimates for \((\text{Im} \mathcal{R}) \circ (g_r + e^{iA}[\psi])\). This observation, coupled with the bound (3.3) for the operator \(\mathcal{S}\) gives us

\[(3.19) \quad \| \mathcal{S} \left( (\text{Im} \mathcal{R}) \circ (g_r + e^{iA}[\psi_1]) - (\text{Im} \mathcal{R}) \circ (g_r + e^{iA}[\psi_2]) \right) \|_{C^\alpha} \leq C_{\alpha} (1 + r^\delta) r^m \| \psi_1 - \psi_2 \|_{C^\alpha}.\]

All these estimates hold for \(\psi_1, \psi_2 \in \overline{B}_{C^\alpha}(0; r^{1+\delta})\).

**Step 4. Completing the proof**

Applying the bounds (3.9) for the operator \(\mathcal{A}_r\) to the estimates (3.17), (3.18) and (3.19), we see that there exists a constant \(L > 0\) such that

\[
\|H(\psi_1; r) - H(\psi_2; r)\|_{C^\alpha} \leq \frac{L}{r^{m-1}} \left( r^m (1 + r^\delta) + \sum_{j=2}^m r^{(m-1)+\delta(j-1)} \right) \| \psi_1 - \psi_2 \|_{C^\alpha}
\]

\[
\leq 2L r^\delta \| \psi_1 - \psi_2 \|_{C^\alpha} \quad \forall \psi_1, \psi_2 \in \overline{B}_{C^\alpha}(0; r^{1+\delta}) \text{ and } \forall r \in (0, r_2),
\]

where \(r_2 \in (0, r_1)\) is so small that the second inequality is valid for all \(r \in (0, r_2)\). Furthermore, we deduce from the estimates (3.10) and (3.11) (and by the same argument that leads to the estimate (3.19)) that there exists a constant \(K > 0\) such that

\[(3.21) \quad \|H(0; r)\|_{C^\alpha} = \|\mathcal{A}_r ([\text{Re} \mathcal{R}] \circ g_r + \mathcal{S}[(\text{Im} \mathcal{R}) \circ g_r])\|_{C^\alpha} \leq K r^{2} \quad \forall r \in (0, 3\rho/4).\]

Let \(r_3 > 0\) be so small that:

\[
2L r^\delta \leq 1/2 \quad \text{and} \quad K r^2 \leq r^{1+\delta}/2 \quad \forall r \in (0, r_3).
\]

Set \(R_0 := \min(3\rho/4, r_2, r_3)\). Then, for any \(r \in (0, R_0)\),

\[(3.22) \quad \psi_1, \psi_2 \in \overline{B}_{C^\alpha}(0; r^{1+\delta}) \implies \|H(\psi_1; r) - H(\psi_2; r)\|_{C^\alpha} \leq (1/2) \| \psi_1 - \psi_2 \|_{C^\alpha} \quad [\text{due to (3.20)}],
\]

and, furthermore

\[
\psi \in \overline{B}_{C^\alpha}(0; r^{1+\delta}) \implies \|H(\psi; r)\|_{C^\alpha} \leq (1/2) \| \psi \|_{C^\alpha} + \|H(0; r)\|_{C^\alpha} \quad [\text{due to (3.22)}]
\]

\[
\leq r^{1+\delta}. \quad [\text{due to (3.21)}]
\]

This last fact and the estimate (3.22) enable us to apply the contraction mapping principle to \(H(\cdot; r) : \overline{B}_{C^\alpha}(0; r^{1+\delta}) \to \overline{B}_{C^\alpha}(0; r^{1+\delta})\) for each \(r \in (0, R_0)\) — owing to which we get:

**Fact B.** For each \(r \in (0, R_0)\), there exists a unique \(\psi_r \in \overline{B}_{C^\alpha}(0; r^{1+\delta})\) such that \(H(\psi_r; r) = \psi_r\).

Before proceeding any further, we remark that, shrinking \(R_0 \geq 0\) further if necessary, \(\| \psi_r \|_{C^\alpha} \) is not comparable to \(\|g_r\|_{C^\alpha} \approx r \) \(\forall r \in (0, R_0)\), which ensures that the desired
analytic discs will be non-constant. Now consider the two maps $\Theta : \Omega_\alpha \times (0, 3\rho/4) \rightarrow C^\alpha(\partial \mathbb{D}; \mathbb{R}) \times (0, 3\rho/4)$ and $G : (0, R_0) \rightarrow \mathbb{B}_{C^\alpha}(0; R_0^{1+\delta})$ with the definitions

$$
\Theta(\psi, r) := (\Phi(\psi, r), r), \quad G(r) := \psi_r,
$$

where $\psi_r$ is as described in Fact B above. The total derivative of $\Theta$ at the point $(\psi, r)$ has the matrix representation

$$
D\Theta(\psi, r) = \begin{bmatrix} \Lambda_r & \partial_r \Phi(\psi, r) \\ 0 & 1 \end{bmatrix} : C^\alpha(\partial \mathbb{D}; \mathbb{R}) \oplus \mathbb{R} \rightarrow C^\alpha(\partial \mathbb{D}; \mathbb{R}) \oplus \mathbb{R},
$$

where $\Lambda_r$ is as defined in Step 2, and $\partial_r \Phi$ denotes the partial Fréchet derivative with respect to $r$. It is easy to show that the latter exists, and that $D\Theta$ varies continuously with $(\psi, r) \in \Omega_\alpha \times (0, 3\rho/4)$. Owing to Claim in Step 2, $D\Theta$ is an isomorphism (see computation of the inverse below). We now appeal to the strategy of the proof of the Inverse/Implicit Function Theorem to show that $G$ is of class $C^1$. Specifically: if, for a fixed $(\psi_o, r^0)$, we show that $(\psi, r)$ is a fixed point of the auxiliary function

$$
\Psi_\rho : \begin{pmatrix} \psi \\ r \end{pmatrix} \mapsto \begin{pmatrix} \psi \\ r \end{pmatrix} + \begin{bmatrix} \Lambda_r & \partial_r \Phi(\psi, r) \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\Phi(\psi, r) \\ \rho - r \end{bmatrix}
$$

for each $\rho \in (0, R_0)$, then a close study of the proof of the Inverse/Implicit Function Theorem reveals that we can conclude that there is a small interval $I(r^0)$ containing $r^0$ — with $I(r^0) \subseteq (0, R_0)$ — such that $G|_I(r^0)$ is of class $C^1$. To this end, we compute

$$
\Psi_\rho(\psi, r) = \begin{pmatrix} \psi \\ r \end{pmatrix} + \begin{bmatrix} \Lambda_r & -\Lambda_r \partial_r \Phi(\psi, r^0) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\Phi(\psi, r) \\ \rho - r \end{bmatrix}
$$

$$
= \begin{pmatrix} \psi - \Lambda_r \partial_r \Phi(\psi, r) + O(|r - \rho|) \\ \rho \end{pmatrix}
$$

(3.23)

$$
= \begin{pmatrix} (1 - (r/r^0)^{m-1}) \psi + (r/r^0)^{m-1} H(\psi, r) + O(|r - \rho|) \\ \rho \end{pmatrix}.
$$

Now, as $H(\psi, \rho) = (\psi, r) \forall \rho \in (0, R_0)$, the above calculation reveals that $\Psi_\rho(\psi, \rho) = (\psi, r) \forall \rho \in (0, R_0)$. By our preceding remarks, there exists an interval $I(r^0)$ containing $r^0$ — with $I(r^0) \subseteq (0, R_0)$ — such that $G|_I(r^0)$ is of class $C^1$. Since smoothness is a local property, we have just concluded that $G$ is smooth on $(0, R_0)$. Recalling the discussions at the end of Step 1 and Step 2 of this proof, we see that the desired analytic discs $g(r)$ are the analytic discs defined by:

$$
g(r)|_{\partial \mathbb{D}} = (g_r, (kr)^m) + (\epsilon^i A[G(r)] + iA(\ln \mathcal{R}) \circ (g_r + \epsilon^i A[G(r)]))
$$

The analytic discs per se are the Poisson integrals of the functions on the right-hand side above. Standard facts about the Poisson integral imply that $g : (0, R_0) \rightarrow A^\alpha(\mathbb{D}; \mathbb{C}^2)$ is of class $C^1$, given that $G$ is smooth on $(0, R_0)$. The italicized remark after Fact B reminds us that $g(r)$ is non-constant for each $r \in (0; R_0)$. Also, by Fact B, the Hölder norms and hence the sup-norms of $g(r)$ shrink to zero as $r \rightarrow 0^+$. □
4. A comparison of Theorem 1.3 with previous results

Theorem 1.3 is reminiscent of some of results in [14] about analytic discs in the polynomially-convex hull around a degenerate CR singularity. We paraphrase Wiegerinck’s results to the context that we have been studying.

Result 4.1 (paraphrasing parts of Theorem 3.3 and 3.5, [14]). Let \( \varphi \) be \( C^{m+1} \)-smooth function defined in a neighbourhood of \( 0 \in \mathbb{C} \) that vanishes to order \( m \) at 0. Write

\[
\varphi(z) = F_m(z) + R(z) \quad (\text{with } |z| \text{ sufficiently small}),
\]

where \( F_m \) is polynomial that is homogeneous of degree \( m \), and \( R(z) = O(|z|^{m+1}) \). Suppose \( (0,0) \) is an isolated CR singularity of \( \Gamma(\varphi) \) and that \( F_m \) is real-valued. If \( \text{Ind}_M(\Gamma(\varphi), 0) > 0 \) and \( F_m \) is a subharmonic, non-harmonic function, then \( \Gamma(\varphi) \) is not locally polynomially convex at \( (0,0) \).

Results like the above rely strongly on the results of Chirka & Shcherbina [4] (also refer to [11] by Shcherbina), which can be used to analyse the structure of the polynomially-convex hulls of graphs of functions defined on certain classes of sets in \( \mathbb{C}^2 \) that are homeomorphic to the 2-sphere. The potential-theoretic ideas used in [11] and [4] shape the hypotheses of the results therein. Those hypotheses lead to certain subharmonicity conditions being imposed in the results of [14]. For example, in the setting of Result 4.1, they translate into the requirement that \( F_m \) be a subharmonic, non-harmonic function. This raises the following question: with the hypotheses imposed on \( F_m \) in Theorem 1.3, is it possible that \( F_m \) is automatically subharmonic? If this were the case, then Theorem 1.3 would be a special case of the results in [14].

We demonstrate in this section that the answer to the above question is negative. There are pairs \( (\mathcal{S}, p) \), where \( p \) is an isolated degenerate CR singularity, to which Wiegerinck’s hypotheses do not apply but which admit Bishop discs. The point of Theorem 1.3 was to demonstrate some techniques for examining the local polynomially-convex hull near an isolated CR singularity that do not require any subharmonicity-type conditions. We now present a one-parameter family of relevant counterexamples.

Example 4.2. For each \( C \in (1/3, 2/3) \), there exists an \( \varepsilon_C > 2/3 \) such that the real-valued, homogeneous polynomial

\[
F_C(z) := \frac{C}{2}(z^4 + \overline{z}^4) + \varepsilon_C(z^3 \overline{z} + z \overline{z}^3) + |z|^4
\]

has the following properties:

a) 0 is an isolated CR singularity of \( \Gamma(F_C) \) satisfying \( \text{Ind}_M(\Gamma(F_C), 0) > 0 \); and
b) \( F_m \) is not subharmonic.

To arrive at a polynomial with the above properties, let us first examine

\[
F(z; \varepsilon, C) := \frac{C}{2}(z^4 + \overline{z}^4) + \varepsilon(z^3 \overline{z} + z \overline{z}^3) + |z|^4.
\]

Then

\[
\partial^2_{z \overline{z}} F(z; \varepsilon, C) = 3\varepsilon(z^2 + \overline{z}^2) + 4|z|^2 = (6\varepsilon \cos 2\theta + 4)|z|^2,
\]

where, as usual, we write \( z = |z|e^{i\theta} \). Then, clearly

\[
F(\cdot; \varepsilon, C) \text{ fails to be subharmonic } \iff \varepsilon > 2/3.
\]
From Lemma 2.4, we realise that
\[ \text{Ind}_M(\Gamma(F), 0) > 0 \iff \mathcal{F}_C(e^{i\theta}) \neq 0 \quad \forall \theta \in \mathbb{R}. \]
Hence, to begin with, we shall examine whether there are any values of the parameter \( C \) such that
\[ \mathcal{F}(e^{i\theta}; 2/3, C) = 2C(\cos 2\theta)^2 + (4/3)\cos 2\theta + (1 - C) > 0 \quad \forall \theta \in \mathbb{R}. \]
We will then perturb the parameter \( \varepsilon \) away from \( \varepsilon = 2/3 \) so as to ensure that positivity is preserved, but subharmonicity fails. To this end, we set \( X := \cos 2\theta \) in the above inequality to get
\[ 2CX^2 + (4/3)X + (1 - C) > 0. \]
Note that:
\[ \begin{align*}
(4.3) \iff & \left\{ \begin{array}{l}
C > 0, \\
(4/3)^2 - 8C(1 - C) < 0.
\end{array} \right. \\
& \iff C \in (1/3, 2/3).
\end{align*} \]
This shows that for each \( C \in (1/3, 2/3) \), \( \mathcal{F}(\cdot; 2/3, C) > 0 \) on \( \mathbb{C} \setminus \{0\} \).

Finally, note that
\[ \begin{align*}
& \bullet \ S^1 \times \{2/3\} \times \{C\} \text{ is a compact subset of } S^1 \times (\mathbb{R}_+) \times (1/3, 2/3); \text{ and} \\
& \bullet \ \mathcal{F}|_{S^1 \times (2/3) \times \{C\}} > 0 \text{ for each } C \in (1/3, 2/3).
\end{align*} \]
Since \( \mathcal{F} \) is continuous on \( S^1 \times (\mathbb{R}_+) \times (1/3, 2/3) \), there exists a \( \delta(C) > 0 \) such that
\[ (\varepsilon, C) \in B^2((2/3, C); \delta(C)) \implies \mathcal{F}(e^{i\varepsilon}; \varepsilon, C) > 0. \]

We now pick an \( \varepsilon_C \in (2/3, 2/3 + \delta(C)) \) and define \( \mathcal{F}_C := \mathcal{F}(\cdot; \varepsilon_C; C) \). From (4.2), (4.3) and (4.4), we conclude that \( \mathcal{F}_C \) satisfies property (a). We have chosen \( \varepsilon_C > 2/3 \); hence, by (4.1), \( \mathcal{F}_C \) fails to be subharmonic.

5. The proof of Theorem 1.4

A non-trivial result that we will require is the following theorem by Forstnerič, which we shall paraphrase:

**Result 5.1** (paraphrasing Theorem 2, [6]). Let \( M \) be a maximally totally-real \( C^4 \)-smooth submanifold of \( \mathbb{C}^2 \) and let \( g \in A^\alpha(\mathbb{D}; \mathbb{C}^2) \) be an immersed analytic disc with boundary in \( M \), such that the tangent bundle \( TM \) is trivial over an \( M \)-open neighbourhood of \( g(\partial\mathbb{D}) \). If \( \text{Ind}_{M,g(e^{i\cdot})} \leq 0 \), then there is an open neighbourhood \( \Omega \subset A^\alpha(\mathbb{D}; \mathbb{C}^2) \) of \( g \) such that the only analytic discs \( F \in \Omega \) with boundary in \( M \) are of the form \( g \circ \varphi \), where \( \varphi \in \text{Aut}(\mathbb{D}) \).

**Remark 5.2.** Theorem 2 in [6] has been stated — in the notation of Result 5.1 — only for \( g \in A^{1/2}(\mathbb{D}; \mathbb{C}^2) \). However, the observations made in [6, Remark 1] about Theorem 1 apply as well to Theorem 2 in [6]. In other words, we can allow \( g \in A^\alpha(\mathbb{D}; \mathbb{C}^2) \) in the hypothesis of the latter theorem — as paraphrased above.

We are now ready to provide

**The proof of Theorem 1.4.** We first consider Part (1). Let \( (S, p) \) be as described in the hypothesis of the theorem. As before, we may work with the graph \( \Gamma(\mathcal{F}_m + \mathcal{R}) \), where \( \mathcal{F}_m \) and \( \mathcal{R} \) have the same meanings as in (3.1). Since \( \text{Ind}_M(S, p) \) is invariant
under a holomorphic change of coordinate, arguing exactly as in the proof of Theorem 1.2.16, \( \text{Ind}_M(\Gamma(F_m),0) \leq 0 \). By hypothesis, and the formula (2.2) in Lemma 2.1, we conclude that \( F_m \) changes sign. To see this, we rely on the fact that \((0,0)\) is an isolated CR singularity of \( \Gamma(F_m) \). We have discussed that in this case — see equation (2.3) — if \( F_m \) has zeros, then it has only simple zeros. This fact — combined with the fact that, by the formula (2.2), \( F_m(e^{\gamma})^{-1}\{0\} \neq \emptyset \) — implies that \( F_m \) must change sign. Then, each level set \( F_m^{-1}\{c\}, \ c \in \mathbb{R} \), is a finite union of disjoint arcs in \( \mathbb{C} \).

Since \( \mathcal{R}(z) = O(|z|^{m+1}) \), there exists a \( \delta > 0 \) which is sufficiently small that the level sets of \((F_m + \mathcal{R})|_{D(0,\delta)}\), i.e. the sets

\[
\{z \in D(0;\delta) : (F_m + \mathcal{R})(z) = c\}
\]

do not separate \( \mathbb{C} \) for each \( c > 0 \). We now appeal to the following:

**Result 5.3** (Theorem 1.2.16, [12]). If \( X \subset \mathbb{C}^n \) is compact and if \( \mathcal{P}(X) \) contains a real-valued function \( f \), then \( X \) is polynomially convex if and only if each fiber \( f^{-1}\{c\}, \ c \in \mathbb{R} \), is polynomially convex.

We clarify that, for \( X \subset \mathbb{C}^n \) compact,

\[
\mathcal{P}(X;\mathbb{C}^n) := \text{the uniform algebra on } X \text{ generated by }
\]

the class \( \{P|_X : P \in \mathbb{C}[z_1, \ldots, z_n] \} \).

Taking \( X := \Gamma(F_m + \mathcal{R}; D(0;\delta)) \) and \( f(z, w) := w \), and observing that each of the sets in (5.1) is polynomially convex, we conclude from Result 5.3 that \( \Gamma(F_m + \mathcal{R}) \) is locally polynomially convex at \((0,0)\) — or, equivalently, that \( \mathcal{S} \) is locally polynomially convex at \( p \).

We now consider Part (2). As before, we shall work in the coordinate system \((z, w)\) with respect to which \((\mathcal{S}, p)\) has the representation (3.1). Suppose, for some \( \alpha \in (0,1) \), there exists a continuous one-parameter family \( g : (0,1) \to A^\alpha(\mathbb{D};\mathbb{C}^2) \) of immersed, non-constant analytic discs with the following three properties:

a) \( g(t)(\partial \mathbb{D}) \subset (\mathcal{S}\setminus\{p\}) \cap U_p \forall t \in (0,1) \).

b) \( g(t)(e^{\gamma}) \) is a simple closed curve in \( \mathcal{S} \forall t \in (0,1) \).

c) \( g(t)(\zeta) \to \{p\} \) for each \( \zeta \in \mathbb{D} \) as \( t \to 0^+ \).

Let \( \delta_0 > 0 \) be so small that for every smooth, positively-oriented, simple closed path \( \gamma : S^1 \to D(0;\delta_0) \setminus \{(0,0)\} \),

\[
\text{Wind} \left( \frac{\partial(F_m + \mathcal{R})}{\partial z} \circ \gamma, 0 \right) = \text{Wind} \left( \frac{\partial F_m}{\partial z} \circ \gamma, 0 \right).
\]

Such a \( \delta_0 > 0 \) exists because \( \mathcal{R}(z) = O(|z|^{m+1}) \). Then, in view of Lemma 2.3 and the properties (a)–(c), (5.2) allows us to infer that there exists a \( t_0 \in (0,1) \) such that

\[
\text{Ind}_{M,g(t)(e^{\gamma})}(\Gamma(F_m + \mathcal{R})) = \text{Ind}_{M}(\Gamma(F_m + \mathcal{R}),0) = \text{Ind}_{M}(\Gamma(F_m),0) \leq 0, \ \forall t \in (0,t_0).
\]

We pick a \( t^* \in (0,t_0) \). By Result 5.1 \( \exists \varepsilon > 0 \) such that for any analytic disc \( F \in A^\alpha(\mathbb{D};\mathbb{C}^2) \) with boundary in \( \Gamma(F_m + \mathcal{R}) \setminus \{(0,0)\} \) such that \( 0 < \|F - g(t^*)\|_{C^0} < \varepsilon \), \( F = g(t^*) \circ \varphi \), where \( \varphi \in \text{Aut}(\mathbb{D}) \). However, this leads to a contradiction because, owing
to the continuity of \( g \) and to (c) above, there must exist a \( t' \in (0, t_0) \), \( t' \neq t^* \), such that

\[
0 < \| g(t^*) - g(t') \|_{C_\alpha} < \varepsilon, \text{ and } \quad \text{Image}(g(t^*)) \neq \text{Image}(g(t')).
\]

Hence, our assumption about the existence of \( g : (0, 1) \to A^\alpha(\mathbb{D}; \mathbb{C}^2) \) must be wrong, which establishes Part (2).

\[\text{\(\square\)}\]

6. SOME OPEN PROBLEMS ON THE LOCAL HULL OF \((S, p)\)

We conclude this article with a couple of open problems. The first of these has already been discussed in the Introduction.

**Problem 1.** Prove or disprove the following

**Conjecture.** Let \( S \) be a smooth real surface in \( \mathbb{C}^2 \) and let \( p \in S \) be a CR singularity. Assume that \( p \) is non-parabolic and that \( S \) is thin at \( p \). \( S \) is locally polynomially convex at \( p \) if and only if \( \text{Ind}_M(S, p) \leq 0 \).

The “only if” part has already been established above. We have been able to provide some evidence that the “if” might also hold, or, at any rate, that the complex structure of the local polynomial hull around the exceptional point \( p \) would have unexpected features. A possible disproof of the “if” part would involve looking for analytic discs with boundaries in \( S \) where the boundaries pass through \( p \). Alternatively, general tools for investigating the above problem might be found in [5] by Duval & Sibony.

Questions on the fine structure of the local polynomially-convex hull of the pair \((S, p)\) satisfying the hypotheses of Theorem 1.3 remain open. For instance:

**Problem 2.** Let \((S, p)\) satisfy the hypotheses of Theorem 1.3. Is there a neighbourhood \( \tilde{U}_p \subset U_p \) containing \( p \) (where \( U_p \) is as in Theorem 1.3) such that if \( H : \mathbb{D} \to \mathbb{C}^2 \) is an analytic disc with \( H(\partial \mathbb{D}) \subset (S \cap \tilde{U}_p) \), then \( H \) is just a reparametrisation of \( \tilde{g}_s \) for some \( r^0 \in (0, R_0) \)?

The answer to the above question is “Yes,” when \( p \) is a non-degenerate CR singularity. The reason for this is that (in the notation of Section 1) \( \mathcal{F}_2 + \text{Re}(\mathcal{R}) \) is subharmonic in the non-degenerate case. This allows the use of some easy arguments from potential theory — see Proposition 4.3 in [9]. In general, we can — in view of Example 1.2 — no longer rely on \( \mathcal{F}_m + \text{Re}(\mathcal{R}) \) being subharmonic.

We make one final remark. Once the analytic disc \( g(r) \) with boundary in \((S \setminus \{p\}) \) (which is a totally real submanifold of \( U_p \setminus \{p\} \)) is obtained, the reader may ask why we do not simply invoke Forstnerič’s results in [9] to deduce the local regularity of the family \( \{g(r) : r \in (0, R_0)\} \). Problem 2 hints at the obstacle to this approach. Unless one establishes that the question in Problem 2 has an affirmative answer, one cannot rule out that, for some \( r^0 \in (0, R_0) \), the smooth family \( \{H(t) : t \in (-\varepsilon, \varepsilon)\} \subset A^\alpha(\mathbb{D}; \mathbb{C}^2) \) — given by Forstnerič’s result — passing through \( g(r^0) \) and satisfying \( H(t)(\partial \mathbb{D}) \subset (S \setminus \{p\}) \forall t \in (-\varepsilon, \varepsilon) \) does not coincide (taking reparametrisations into account) with \( \{g(r) : r \in I(r^0)\} \).

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References

[1] G. Bharali, *Surfaces with degenerate CR singularities that are locally polynomially convex*, Michigan Math. J. 53 (2005), 429-445.
[2] G. Bharali, *Polynomial approximation, local polynomial convexity, and degenerate CR singularities*, J. Funct. Anal. 236 (2006), 351-368.
[3] E. Bishop, *Differentiable manifolds in complex Euclidean space*, Duke Math J. 32 (1965), 1-21.
[4] E.M. Chirka and N.V. Shcherbina, *Pseudoconvexity of rigid domains and foliations of hulls of graphs*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., Serie IV, 22 (1995), 707-735.
[5] J. Duval and N. Sibony, *Polynomial convexity, rational convexity, and currents*, Duke Math. J. 79 (1995), 487-513.
[6] F. Forstnerič, *Analytic disks with boundaries in a maximal real submanifold of \( C^2 \)*, Ann. Inst. Fourier 37 (1987), 1-44.
[7] F. Forstnerič and E.L. Stout, *A new class of polynomially convex sets*, Arkiv Mat. 29 (1991), 51-62.
[8] B. Jöricke, *Local polynomial hulls of discs near isolated parabolic points*, Indiana Univ. Math. J. 46 (1997), 789-826.
[9] C.E. Kenig and S. Webster, *The local hull of holomorphy of a surface in the space of two complex variables*, Invent. Math. 67 (1982), 1-21.
[10] S.N. Mergelyan, *Uniform approximations of functions of a complex variable* (Russian), Uspekhi Mat. Nauk 7 (48) (1952), 31-122.
[11] N.V. Shcherbina, *On the polynomial hull of a graph*, Indiana Univ. Math. J. 42 (1993), 477-503.
[12] E.L. Stout, *Polynomial Convexity*, Progress in Mathematics 261, Birkhäuser, 2007.
[13] E.L. Stout, *Polynomially convex neighborhoods of hyperbolic points*, Abstracts AMS 7 (1986), 174.
[14] J. Wiegerinck, *Locally polynomially convex hulls at degenerated CR singularities of surfaces in \( C^2 \)*, Indiana Univ. Math. J. 44 (1995), 897-915.

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