TOPOLOGICAL RIGIDITY OF QUASITORIC MANIFOLDS

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Abstract. Quasitoric manifolds are manifolds that admit an action of the torus that is locally as the standard action of $T^n$ on $\mathbb{C}^n$. It is known that the quotients of such actions are nice manifolds with corners. We prove that such manifolds are equivariantly rigid i.e., that any other manifold that is $T^n$-homotopy equivalent to a quasitoric manifold, is $T^n$-homeomorphic to it.

1. Introduction

Toric varieties are studied extensively in algebraic geometry and combinatorics ([5], [10]). The main tool in their study is the simplicial complex that is determined by the fan of the toric variety. This simplicial complex is actually the quotient of the toric variety by the torus action. The combinatorial properties of the simplicial complex reflect the algebraic and geometric properties of the variety and vice versa. A topological analogue of toric varieties was by Davis–Januszkiewicz ([3]), called quasitoric manifolds. Quasitoric manifolds are manifolds that admit an action of the torus $T^n$ which is locally standard such that the quotient space is a simple polytope. Locally standard actions are those where, locally, $T^n$ acts by the standard coordinate wise multiplication on $\mathbb{C}^n$. As in the toric variety case, the combinatorial properties of the polytope provide information about the topological structure of the manifold. Furthermore, the manifolds can be reconstructed from the polytope and an appropriate assignment of subgroups of $T^n$ to the faces of the polytope.

In this paper, we consider a further generalization considering locally standard $T^n$-actions on manifolds, and we call them pseudotoric manifolds. In this case, the quotient space is a nice manifold with corners. As before, we show that the combinatorial properties of the manifold with corners are reflected to the topology of the pseudotoric manifold. Also, the pseudotoric manifold can be reconstructed by an appropriate assignment of subgroups of $T^n$ to its faces. The main theorem of the paper is the following.

Theorem (Main Theorem). Let $M^{2n}$ be a quasitoric manifold and $N^{2n}$ a locally linear $T^n$-manifold. Let $f : N^{2n} \rightarrow M^{2n}$ be an equivariant homotopy equivalence. Then $f$ is equivariantly homotopic to an equivariant homeomorphism.

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The idea of the proof is the same as the one used in the Coxeter group case \([9], \[14], \[15]\). After all, the reconstruction of the quasitoric and pseudotoric manifolds, from their quotient spaces, is similar to the construction of the Coxeter complex of a Coxeter group, a similarity that was made precise in \([3]\). First it is proved that \(N^{2n}\) is a quasitoric manifold. Let \(X = M^{2n}/T^n\) and \(Y = N^{2n}/T^n\). Then \(X\) is a simple polytope and \(f\) induces a map \(\phi : Y \to X\) that is a face-preserving homotopy equivalence. As in the references for the Coxeter group case, we show inductively that there is a face-preserving homotopy from \(\phi\) to a face-preserving homeomorphism \(\chi\). The homeomorphism \(\chi\) lifts to a \(T^n\)-homeomorphism between \(N^{2n}\) and \(M^{2n}\) that is homotopic to \(f\).

The main theorem, loosely, can be considered as a version of a stratified Borel Conjecture. Let \(\pi : M \to X\). Over the interior \(\sigma\) of faces of \(X\), the map \(\pi\) is a fiber bundle with fiber \(T_\sigma\), where \(T_\sigma\) is the isotropy group of \(\sigma\). So, \(M^{2n}\) admits a stratification by open aspherical manifolds.

For non-singular toric varieties better rigidity theorems are known. Let \(M\) and \(N\) be two toric manifolds. In \([6]\) and \([8]\) it was shown that if \(H^*(ET \times_T M)\) and \(H^*(ET \times_T N)\) are isomorphic as \(H^*(BT)\)-algebras, then \(M\) and \(N\) are algebraically isomorphic. Actually a slightly stronger result was proved in the above references.

In \([18]\), a generalization of the locally standard actions is given, called local torus actions. Our methods do not directly generalize to this case. In \([17]\), the generalization of the quotient map \(\pi : M^{2n} \to X\) is given. It is called local standard torus fibration. Again, our methods can not be applied directly to the stratified rigidity problem for such \(M^{2n}\).

2. Preliminaries and Notation

We consider \(S^1\) as the standard subgroup of \(\mathbb{C}^\ast\), the multiplicative group of non-zero complex numbers. Furthermore \(T^n < (\mathbb{C}^\ast)^n\). We refer to the standard representation of \(T^n\) by diagonal matrices in \(U(n)\) as the standard action of \(T^n\) on \(\mathbb{C}^n\). The orbit of the action is the positive cone:

\[
\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) : x_1 \geq 0\}.
\]

Most of the \(T^n\)-actions considered in this paper satisfy the following property.

**Definition 2.1.** Let \(M^{2n}\) be a \(2n\)-dimensional manifold with an action of \(T^n\). The action is called **locally standard** if for every \(x \in M^{2n}\) there is a \(T^n\) invariant neighbourhood \(U\) of \(x\) and a homeomorphism \(f : U \to W\) where \(W\) is an open set in \(\mathbb{C}^n\) invariant under the standard action of \(T^n\), and an automorphism \(\phi : T^n \to T^n\) such that \(f(ty) = \phi(t)f(y)\) for all \(y \in U\).

**Definition 2.2.** An action \(G \times X \to X\) is effective if there is no non-trivial element of \(G\) that stabilizes \(X\) pointwise. In other words the intersection of all isotropy subgroups is trivial.

**Remark 2.3.**

1. If the action of \(T^n\) is effective and it does not have any finite isotropy groups, then the action is locally standard by the slice theorem (\([17]\)).
(2) If $M^{2n}$ is smooth and $H^{\text{odd}}(M) = 0$, then the action is locally standard \((7)\).

The next definition formalizes the local properties of the quotient space of a locally standard $T^n$-action.

**Definition 2.4.** A space $X$ is an $n$-manifold with corners if it is a Hausdorff space equipped with an atlas of open sets each one homeomorphic to an open subset of $\mathbb{R}^n_+$ such that the overlap maps are local homeomorphisms that preserve the natural stratification of $\mathbb{R}^n_+$.

The quotient of a locally standard action is a manifold with corners \((2), (7)\).

**Remark 2.5.** For any $n$-manifold with corners $X$ we have the following.

(1) For each $x \in X$ and a chart $\sigma$, define $c(x)$ to be the number of coordinates of $\sigma(x)$ that are 0. The number $c(x)$ is independent of the choice of the chart and so $c$ defines a map $c: X \to \mathbb{N}$. For $0 \leq k \leq n$, a connected component of $c^{-1}(k)$ is a stratum of codimension $k$. The closure of a stratum is called a closed stratum.

(2) Let $x \in X$. Define

$$Y(x) = \{ C : C \text{ closed codimension-one stratum that contains } x \}.$$  

The manifold with corners $X$ is called nice if $|Y(x)| = 2$, whenever $c(x) = 2$.

(3) The slice theorem implies that the quotient space of a locally standard $T^n$-action is a nice manifold with corners \((7)\).

(4) A facet in an $n$-manifold with corners is the closure of a connected component of the codimension 1 stratum. A non-empty intersection of $k$ facets is called a codimension-$k$ preface \((k = 1, \ldots, n)\). In general, prefaces of codimension $> 1$ may be disconnected. A connected component of a preface is called a face. The manifold $X$ itself is considered to be a codimension-0 face. The $k$-skeleton of a manifold with corners $X$ is the set of all faces of codimension greater than or equal to $k$ and it is denoted by $X^{(k)}$.

(5) An $n$-manifold with corners is called simple if the codimension $n$ faces are contained in exactly $n$ facets. This definition is in analogy with the definition of simple polyhedra.

For the sequel of the paper we assume that $M^{2n}$ is an $n$-dimensional manifold with a locally standard action of $T^n$.

The following definition is a special case of torus actions. It generalizes the definition of quasitoric manifolds given in \((11), (3)\).

**Definition 2.6.** Given a nice manifold with corners $X^n$, a manifold $M^{2n}$ with a locally standard $T^n$-action is called a pseudotoric manifold over $X^n$ if there is a projection $\pi: M^{2n} \to X^n$ whose fibers are the orbits of the action.

**Remark 2.7.** (1) Quasitoric manifolds are pseudotoric manifolds so that the quotient space is not just a manifold with corners but it is a simple polytope.
(2) Remember that a torus manifold $M^{2n}$ is a smooth $T^n$-manifold with an effective $T^n$-action and $M^T \neq \emptyset$ ([7]).

(3) Definition [2.6] implies that, under the projection $\pi$, points with the same isotropy groups are mapped to the relative interior of a face of $X^n$. Thus the action of $T^n$ is free over the open stratum of $X^n$ and the vertices of $X^n$ i.e. the 0-dimensional faces, correspond to the fixed points of the action.

Let $\pi : M^{2n} \to X^n$ be the projection defined above. A codimension-1 connected component of a fixed point set of a circle in $T^n$ is called a characteristic submanifold of $M$. The images of the characteristic submanifolds are the facets of $X$.

**Definition 2.8.** A manifold with corners $X$ is called face acyclic if all the faces, including $X$ itself, are acyclic, that is, its reduced integral homology $\tilde{H}_*X$ is trivial. We call $X$ a homology polytope if all its prefaces are acyclic (in particular they are connected). Thus $X$ is a homology polytope if and only if it is face acyclic and every non-empty intersection of characteristic submanifolds is connected.

**Remark 2.9.** A simple convex polytope is an example of a manifold with corners that is a homology polytope. A quasitoric manifold can be defined as a locally standard manifold whose orbit space is a simple convex polytope with the standard face structure.

### 3. The canonical model

We will show how to reconstruct the pseudotoric manifold from a manifold with corners $X$ and some linear data on the set of facets of $X$. We use the construction in [7] that generalizes the construction of quasitoric manifolds in [1] and [3]. We write $T = T^n$.

First, we will see some of the properties of characteristic submanifolds of a pseudotoric manifold. Let $M_i = \pi^{-1}(X_i)$ be the characteristic submanifolds, where $X_i$ are the facets of $X$ ($i = 1, \ldots, k$). Let

$$\Lambda : \{X_1, \ldots, X_k\} \to \text{Hom}(S^1, T) \cong \mathbb{Z}^n$$

such that $\Lambda(X_i)$ is a primitive vector that determines the circle subgroup of $T$ that fixes $M_i$. The main property of these data is that if $X_{i_1} \cap \ldots \cap X_{i_m} \neq \emptyset$ then $\Lambda(X_{i_1}), \ldots, \Lambda(X_{i_m})$ is a part of $\mathbb{Z}$-basis of the integral lattice $\text{Hom}(S^1, T)$.

Now, we give the inverse of the construction. We start with a simple manifold with corners $X$ and a map $\Lambda$ that satisfies the above condition about the non-empty intersections of the facets. Given a point $x \in X$, the smallest face which contains $x$ is the intersection $X_{i_1} \cap \ldots \cap X_{i_m}$ of all facets with $x \in X_{i_j}$. We define $T(x)$ to be the subtorus of $T$ generated by the circle subgroups corresponding to $\Lambda(X_{i_1}), \ldots, \Lambda(X_{i_m})$. We define:

$$M_X(\Lambda) = T \times X/\sim, \quad (t, x) \sim (t', x') \iff x = x', \text{ and } t^{-1}t' \in T(x).$$
The space $M_X(\Lambda)$ is a closed manifold and the torus $T$ acts on it by acting on the first coordinate. This follows as in the case of quasitoric manifolds ([3], Proposition 1.8, [7], Proposition 4.5, and [18] Lemma 5.2 and Theorem 5.5).

**Lemma 3.1.** Let $M$ be a pseudotoric manifold with orbit space $X$, and the map $\Lambda$ defined as above. If $X$ is contractible then there is a $T$-equivariant homeomorphism $M_X(\Lambda) \to M$ covering the identity on $X$.

**Proof.** The idea is to construct a continuous map $f : T \times X \to M$ so that $f(T \times \{x\}) = \pi^{-1}(x)$. This is done by subsequent "blowing up the singular strata". The condition on the contractibility guarantees that the resulting principal $T$-bundle over $X$ is trivial, i.e. there is an equivariant homeomorphism $\hat{f} : T \times X \to \hat{M}$ inducing the identity on $X$. Then the map $f$ descends to the required equivariant homeomorphism. The details are in [3] (also in [7] and [18]). □

**Remark 3.2.** In [3] and [7] the result is proved under the condition that $M$ is a smooth manifold. In [18] it is proved for topological manifolds that admit a local torus action, generalizing the concept of locally standard torus manifolds.

We call $(X, \Lambda)$ a characteristic pair. Hence, under the hypothesis $X$ is contractible, a characteristic pair $(X, \Lambda)$ completely determines $M_X(\Lambda)$.

Now we investigate the natural properties of the construction. Let $\phi : X \to Y$ a map between manifolds with corners. It is called skeletal if it preserves skeleta i.e. $\phi(X^{(k)}) \subset Y^{(k)}$. Similarly, a homotopy is called skeletal if the maps at each level are skeletal.

**Proposition 3.3.** Let $(X^n, \Lambda)$ and $(Y^n, \Lambda')$ be two characteristic pairs and $\sigma : T \to T$ a continuous automorphism. Let $\phi : X \to Y$ a skeletal map that satisfies $\sigma(\Lambda(X_i)) < \Lambda'(\phi(X_i))$ for each facet $X_i$ of $X$. Then $\phi$ induces a $\sigma$-equivariant map $\phi_* : M_X(\Lambda) \to M_Y(\Lambda')$.

**Proof.** Define that map $\phi_*$, the obvious way:

$$\phi_* : M_X(\Lambda) \to M_Y(\Lambda'), \quad \phi_*(t, x) = (\sigma(t), \phi(x)).$$

We need to show that the map is well-defined. Let $(t, x) = (t', x)$ in $M_X(\Lambda)$. Then $t^{-1}t' \in T(x)$ where $x$ belongs to the relative interior of the face that is determined by the intersection of facets $X_{i_1}, \ldots, X_{i_m}$ and $T(x)$ is the subgroup generated by $\Lambda(X_{i_1}), \ldots, \Lambda(X_{i_m})$. Then

$$\phi(x) \in \phi(X_{i_1}) \cap \ldots \cap \phi(X_{i_m}).$$

It should be noticed that the faces $\phi(X_{i_j})$ are not necessarily facets. Therefore there are facets $Y_{j_1}, \ldots, Y_{j_s}$ of $Y$ so that $\phi(x)$ belongs to the relative interior of their intersection. Since the map $\phi$ is skeletal,

$$Y_{j_1} \cap \ldots \cap Y_{j_s} = \phi(X_{i_1}) \cap \ldots \cap \phi(X_{i_m}).$$
Proposition 3.5. Let \( \sigma(\Lambda(X_i)) < \Lambda'(\phi(X_i)) \) and therefore, the subgroup generated by \( \Lambda'(\phi(X_i)) \), \( r = 1, \ldots, m \), contains the subgroup generated by \( \sigma(\Lambda(X_i)) \), \( r = 1, \ldots, m \), which implies that \( \langle \sigma(\Lambda(X_i)), r = 1, \ldots, m \rangle \) is contained in the subgroup generated by \( \Lambda'(Y_j) \), \( p = 1, \ldots, s \). But \( \langle \Lambda'(Y_j), p = 1, \ldots, s \rangle \) is actually the subgroup \( T(\phi(x)) \). Thus \( \sigma(t^{-1}t') \) belongs to the subgroup \( T(\phi(x)) \). That implies, \( (\sigma(t), \phi(x)) = (\sigma(t'), \phi(x)) \) in \( M^0_2(\Lambda') \).

By the construction, the map is obviously \( \sigma \)-equivariant. \( \square \)

**Corollary 3.4.** If \( \phi_s : X \to Y, s \in [0, 1], \) is a skeletal homotopy so that \( \sigma(\Lambda(F)) < \Lambda'(\phi_s(F)) \), for each \( s \) and each facet \( F \) of \( X \). Then \( \phi_0, s \simeq_\sigma \phi_{1,*} \).

We now investigate the reverse construction.

**Proposition 3.5.** Let \( f : M_X(\Lambda) \to M_Y(\Lambda') \) a \( \sigma \)-equivariant map, with \( \sigma \) as before. Let \( \phi : X \to Y \) be the map induced on the quotients. Then

1. The map \( \phi \) is skeletal.
2. \( \sigma(\Lambda(F)) < \Lambda'(\phi(F)) \), for each facet \( F \) of \( X \).
3. There is a \( \sigma \) equivariant homotopy such that \( f \simeq_\sigma \phi_s \).

**Proof.** The equivariance implies that \( \phi \) is skeletal. Let \( x \in X \) and \( g \in \Lambda(X_i) \) for some facet \( X_i \). Then one can easily show that \( \phi(gx) = \sigma(g)\phi(x) \) and so \( \sigma(\Lambda(X_i)) \leq \Lambda'(\phi(X_i)) \) since \( \phi \) is skeletal map. Now, for each face \( F \) of \( X \) we denote \( T_F \) the isotropy subgroup of \( F \) under the action of \( T \). Let \( C_F = \{ t_{i,F} : i_F \in I_F \} \) be a complete set of coset representatives of \( T_F \) in \( T \) such that if \( F' \subseteq F \) then \( T_F \leq T_{F'} \) and \( C_{F'} \subseteq C_F \). We require that \( 1 \in C_F \) for each \( F \). Now let \( f(1, x) = (t_x, \phi(x)) \), where \( t_x \) is in \( C_F \) and \( x \) belongs to the relative interior of \( F \). Since \( t_x \in T \),

\[
t_x = (\exp(2\pi \lambda_{1,x} i), \ldots, \exp(2\pi \lambda_{n,x} i)), \quad (\lambda_{1,x}, \ldots, \lambda_{n,x}) \in \mathbb{R}^n.
\]

Define the line-segment path from 0 to \( \lambda_{i,x} \) in \( \mathbb{R} \),

\[
\alpha_{i,x} : [0, 1] \to \mathbb{R}, \quad \alpha_{i,x}(s) = \lambda_{i,x} s.
\]

Define the path

\[
\alpha_x = (\exp(\alpha_{1,x}), \ldots, \exp(\alpha_{n,x})) : [0, 1] \to T.
\]

Define a homotopy

\[
\Phi : M_X(\Lambda) \times [0, 1] \to M_Y(\Lambda'), \Phi((t, x), s) = (\sigma(t)\alpha_x(s), \phi(x)).
\]

It is obvious that \( \Phi \) is well-defined, \( \sigma \)-equivariant and starts from \( \phi_s \) and ends to \( f \). We will show that \( \Phi \) is continuous. Let \( F \) be a face of \( X \). Then the map \( \{1\} \times F \to M_X(\Lambda) \) is an embedding. Thus \( f|\{1\} \times F \) is continuous. So the map

\[
\{1\} \times F \to T \times Y, \quad (1, x) \mapsto (t_x, \phi(x)), \quad \text{where } f(1, x) = (t_x, \phi(x))
\]
is continuous. From the definition of $\alpha_x$ the map
\[(\{1\} \times F) \times [0,1] \to T \times Y, \quad ((1, x), s) \mapsto (\alpha_x(s)t_x, \phi(x))\]
is continuous. Composing with the quotient map, we see that $\Phi|\{1\} \times F$ is continuous. Also, the map $\Phi$ agrees on the intersection of two faces $\{1\} \times F$ and $\{1\} \times F'$ because of the choices we made for the coset representatives. Thus the map $\Phi$ is on the image of $\{1\} \times X$. By equivariance, $\Phi$ is continuous on $M_X(\Lambda)$.

\[\square\]

4. Rigidity

Let $(X^n, \Lambda)$ be a characteristic pair with $X^n$ a simple nice manifold with corners and $M^{2n} = M_X(\Lambda)$ the corresponding pseudotoric manifold. We assume that all the faces of $X^n$ (and $X^n$ itself) are homeomorphic to contractible manifolds with boundary. That means that $X$ is a homology polytope. That is the situation when $M$ is a quasitoric manifold. In that case the co-dimension 0 faces have trivial isotropy subgroups. Let $N^{2n}$ be a 2n-dimensional $T = T^n$-manifold and $f : N^{2n} \to M^{2n}$ a $T$-equivariant homotopy equivalence.

Lemma 4.1. The action of $T$ on $N^{2n}$ is effective.

Proof. We assume that that is not the case. So there is $t \in T$ that fixes $N^{2n}$ pointwise. Let $G = \langle t \rangle$. Then $N^G = N^{2n} \simeq M^G$ since $f$ is an equivariant homotopy equivalence. But $M^G$ is a closed proper submanifold of $M^{2n}$, because the action on $M^{2n}$ is effective. Thus $\dim(N^G) = \dim(M^G) < \dim(M^{2n}) = \dim(N^{2n})$, a contradiction. \[\square\]

Proposition 4.2. The action of $T$ on $N^{2n}$ is locally standard. Thus, $N^{2n}/T = Y$, that is, $Y$ is a manifold with corners.

Proof. Since $T$ acts effectively on $N^{2n}$, it is enough to show that $N^{2n}$ has no finite subgroups. Suppose that $F$ is a finite isotropy group of $N^{2n}$. Then $N^F \neq \emptyset$. But $N^F \simeq M^F = \emptyset$, a contradiction. Thus, $N^{2n}$ has no finite isotropy subgroups and thus the action of $T$ is locally standard ([17], Example 2.1). \[\square\]

Corollary 4.3. Let $N^{2n}$ and $Y$ be as above. Then there is a characteristic map $\Lambda'$ so that $N^{2n} \simeq_T M_Y(\Lambda')$.

Proof. For each facet, $F$ of $Y$, define $\Lambda'(F) = T_F$, the isotropy subgroup of $F$. Define $M_Y(\Lambda')$ to be the quotient space $T \times Y/ \sim$ with $(t, x) \sim (t', x')$ if and only if $x = x'$ and $t^{-1}t' \in T(x)$. The $T$-homotopy equivalence $f$ induces a skeletal homotopy equivalence $\phi : Y \to X$. Thus $Y$ is contractible. The result follows from lemma 3.1. \[\square\]

We start with some reductions to the rigidity problem. Let $f : N^{2n} \to M^{2n}$ a $T$-homotopy equivalence as before. Let $\phi : Y \to X$ be the quotient map induced by $f$. By Proposition 3.3 the map $f$ is $T$-homotopic to $\phi_*$. 

We need a version of the Poincaré Conjecture. For an \( n \)-dimensional manifold with boundary \((M, \partial M)\) the relative structure set \( S(M, \partial M) \) is the set of equivalence classes of pairs \((N, f)\) with \( N \) an \( n \)-dimensional manifold with boundary and \( f : N \to M \) a homotopy equivalence such that \( \partial f : \partial N \to \partial M \) is a homeomorphism.

**Lemma 4.4.** Let \((M, \partial M)\) be a compact contractible manifold with boundary. Then the relative structure set \( S(M, \partial M) = \ast \).

**Proof.** In [15], the calculation is reduced to the structure set of the sphere. For \( n \geq 5 \) the result is due to Smale ([16]), for \( n = 4 \) to Freedman ([4]), and for \( n = 3 \) to Perelman ([11], [12], [13]). For \( n = 1, 2 \) it is a classic result. \( \square \)

**Proposition 4.5.** With the above notation, the map \( \phi_* \) is \( T \)-homotopic to a \( T \)-homeomorphism.

**Proof.** We will use the method that was used in [9], [14], and [15]. We will show that \( \phi \) is skeletal homotopic to a skeletal homeomorphism. Then, he result follows from Corollary 3.4. We will construct a skeletal homotopy by induction on faces. Notice that because \( f \) is a \( T \) equivariant homotopy, the skeletal map preserves (closed) faces. That means that if \( F_1 \) is a face of \( Y \) of codimension-\( k \), then \( \phi \) maps \( F_1 \) to \( F_2 \) and \( \phi|_{F_1} : F_1 \to F_2 \) where \( F_2 \) is a face of \( X \) of codimension-\( k \). Also, each closed face is homeomorphic to a contractible manifold with boundary.

We start the induction. The zero faces correspond to the \( T \)-fixed point sets. Thus, we have the same number of zero faces. The restriction of \( \phi \) to zero faces is a homeomorphism. Now, let a face \( F_1 \) be a face of \( Y \) and \( \partial F_1 \) its boundary. We assume that there is skeletal homeomorphism \( h_{\partial F_1} \) skeletal homotopic to \( \phi|_{\partial F_1} \). Using the homotopy extension property, there is a map \( \phi' : F_1 \to F_2 \) that is homotopic to \( \phi|_{F_1} \) and it extends the map \( h_{\partial F_1} \). Because all the maps and homotopies are skeletal at the boundary, they are skeletal in the closed face \( F_1 \). By Lemma 4.4, \( \phi' \) is homotopic to a homeomorphism relative to the boundary. As before, all homotopies are skeletal. Continuing this way, we get a skeletal homeomorphism \( h : Y \to X \) that is skeletal homotopic to \( \phi \). Also, \( \Lambda'(F) < \Lambda(\phi(F)) = \Lambda(h(F)) \). Thus \( \phi_* \simeq_T h_* \), which is a \( T \)-homeomorphism. \( \square \)

**Theorem 4.6** (Rigidity for Pseudotoric Manifolds). Let \( M \) be a pseudotoric manifold over a nice, simple manifold with corners \( X \). We assume that all the faces of \( X \) (and \( X \) itself) are homeomorphic to contractible manifolds with corners. Let \( N \) a locally linear \( T \)-manifold and \( f : N \to M \) a \( T \)-equivariant homotopy equivalence. Then \( f \) is \( T \)-homotopic to a \( T \)-homeomorphism.

**Proof.** For Proposition 4.2, the action of \( T \) on \( N \) is locally standard. For Corollary 4.3, \( N \cong_T M_Y(\Lambda') \) for some manifold with corners \( Y \) and a characteristic function \( \Lambda' \). Then the map \( f \) induces a skeletal map \( \phi : Y \to X \). By Proposition 5.5 \( \phi \) is skeletal, homotopic to a skeletal homeomorphism \( h \). Thus

\[
f \simeq_T \phi_* \simeq_T h_*
\]

and the last map is a \( T \)-homeomorphism. \( \square \)
The following is an immediate consequence of our Theorem.

**Corollary 4.7.** Let $M$ be a quasitoric manifold. Let $N$ a locally linear $T^n$-manifold and $f : N \to M$ a $T^n$-homotopy equivalence. Then $f$ is $T^n$-homotopic to a $T^n$-homeomorphism.

Also, a slightly more general result holds.

**Corollary 4.8.** Let $M$ be a pseudotoric manifold over a nice, simple manifold with corners $X$. We assume that all the faces of $X$ (and $X$ itself) are homeomorphic to contractible manifolds with corners. Let $\sigma : T \to T$ be a continuous automorphism. Let $N$ a locally linear $T$-manifold and $f : N \to M$ a $\sigma$-equivariant homotopy equivalence. Then $f$ is $\sigma$-homotopic to a $\sigma$-homeomorphism.

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