MODULATION SPACES OF SYMBOLS FOR
REPRESENTATIONS OF NILPOTENT LIE GROUPS

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ABSTRACT. We investigate continuity properties of operators obtained as values of the Weyl correspondence constructed by N.V. Pedersen (Invent. Math. 118 (1994), 1–36) for arbitrary irreducible representations of nilpotent Lie groups. To this end we introduce modulation spaces for such representations and establish some of their basic properties. The situation of square integrable representations is particularly important and in the special case of the Schrödinger representation of the Heisenberg group we recover the classical modulation spaces used in the time-frequency analysis.

1. INTRODUCTION

The representation theory of the \((2n+1)\)-dimensional Heisenberg group \(\mathbb{H}_{2n+1}\) provides a natural background for the pseudo-differential calculus on \(\mathbb{R}^n\). It is well known that the representation theoretic approach has led to a deeper understanding of the Weyl calculus, which resulted in simplified proofs and improvements for many basic results. A celebrated example in this connection is the Calderón-Vaillancourt theorem on \(L^2\)-boundedness for pseudo-differential operators ([CV72]). This classical theorem was strengthened in the paper [GH99] by using the modulation spaces, which are function (or distribution) spaces defined in terms of the Schrödinger representations of Heisenberg groups. The modulation spaces were introduced in [Fe83] in the framework of harmonic analysis of locally compact abelian groups.

On the other hand, a remarkable Weyl calculus was set up in [Pe94] for arbitrary unitary irreducible representations of any nilpotent Lie group. We shall call it the Weyl-Pedersen calculus. It is a challenging task to understand this interaction of the ideas of pseudo-differential calculus with the representation theory of nilpotent Lie groups.

In the present paper we address the above problem in the shape of the \(L^2\)-boundedness theorems. Specifically, we are going to investigate continuity properties of the operators constructed by the Weyl-Pedersen calculus. For this purpose we introduce the modulation spaces \(M^{r,s}_φ(\pi)\) defined in terms of an arbitrary irreducible representation \(\pi\) of a nilpotent Lie group \(G\). One key feature of our representation theoretic approach is that if \(O\) stands for the coadjoint orbit corresponding to \(\pi\) ([Kir62]), then the symbols of the operators constructed by the Weyl-Pedersen calculus are functions or distributions on the coadjoint orbit \(O\), while the Hilbert space \(L^2(O)\) carries a natural irreducible representation \(\pi^#\) of the nilpotent Lie group.
Therefore our general notion of modulation spaces for irreducible representations allows us to investigate the modulation spaces of symbols for the operators constructed by the Weyl-Pedersen calculus for the representation \( \pi \). This approach also reveals the representation theoretic background of the \( L^2 \)-boundedness theorem of [GH99].

We find several of the familiar properties of the classical modulation spaces, such as:

- continuity of the operators constructed by the Weyl-Pedersen calculus with symbols in an appropriate modulation space \( M_\infty^{\infty,1}(\pi^\#) \) (Corollary 2.26);
- independence on the choice of a window function, and covariance of the Weyl-Pedersen calculus, in the case of square-integrable representations (Theorems 3.3 and 3.5).

Besides the aforementioned reasons, the present research has also been motivated by the recent interest in the magnetic pseudo-differential Weyl calculus on \( \mathbb{R}^n \) (see for instance [MP04], [IMP07], [MP09], and the references therein), which was partially extended to nilpotent Lie groups in the papers [BB09a] and [BB09b]. Specifically, the results of the present paper apply to the Weyl calculus associated with a polynomial magnetic field on \( \mathbb{R}^n \), in particular complementing the \( L^2 \)-boundedness theorem established in [IMP07] for magnetic fields whose components are bounded and so are also their partial derivatives of arbitrarily high degree.

**Notation.** Throughout the paper we denote by \( \mathcal{S}(V) \) the Schwartz space on a finite-dimensional real vector space \( V \). That is, \( \mathcal{S}(V) \) is the set of all smooth functions that decay faster than any polynomial together with their partial derivatives of arbitrary order. Its topological dual — the space of tempered distributions on \( V \) — is denoted by \( \mathcal{S}'(V) \).

We shall also use the convention that the Lie groups are denoted by upper case Latin letters and the Lie algebras are denoted by the corresponding lower case Gothic letters.

For basic notions on Weyl pseudo-differential calculus, we refer to [Hor07], [Fo89] and [Gr01].

## 2. Modulation Spaces for Unitary Irreducible Representations

### 2.1. Preliminaries on semidirect products.

**Definition 2.1.** Let \( G_1 \) and \( G_2 \) be connected Lie groups and assume that we have a continuous group homomorphism \( \alpha: G_1 \to \text{Aut} G_2, g_1 \mapsto \alpha_{g_1} \). The corresponding **semidirect product of Lie groups** \( G_1 \ltimes \alpha G_2 \) is the connected Lie group whose underlying manifold is the Cartesian product \( G_1 \times G_2 \) and whose group operation is given by

\[
(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, \alpha_{g_1}^{-1}(g_2) h_2)
\]

whenever \( g_j, h_j \in G_j \) for \( j = 1, 2 \).

Let us denote by \( \hat{\alpha}: \mathfrak{g}_1 \to \text{Der} \mathfrak{g}_2 \) the homomorphism of Lie algebras defined as the differential of the Lie group homomorphism \( G_1 \to \text{Aut} \mathfrak{g}_2, g_1 \mapsto L(\alpha_{g_1}) \). Then the **semidirect product of Lie algebras** \( \mathfrak{g}_1 \ltimes \hat{\alpha} \mathfrak{g}_2 \) is the Lie algebra whose underlying linear space is the Cartesian product \( \mathfrak{g}_1 \times \mathfrak{g}_2 \) with the Lie bracket given by

\[
\{ (X_1, X_2), (Y_1, Y_2) \} = ([X_1, Y_1], \hat{\alpha}(X_1)Y_2 - \hat{\alpha}(Y_1)X_2 + [X_2, Y_2])
\]
if $X_j, Y_j \in \mathfrak{g}_j$ for $j = 1, 2$. One can prove that $\mathfrak{g}_1 \ltimes \mathfrak{g}_2$ is the Lie algebra of the Lie group $G_1 \ltimes G_2$ (see for instance Ch. 9 in [Ho65]). □

**Remark 2.2.** Let $G_1$ and $G_2$ be nilpotent Lie groups and $\alpha: G_1 \to \text{Aut} G_2$ be a *unipotent automorphism*, in the sense that for every $X_1 \in \mathfrak{g}_1$ there exists an integer $m \geq 1$ such that $\dot{\alpha}(X_1)^m = 0$. Then an inspection of (2.2) shows that $\mathfrak{g}_1 \ltimes \mathfrak{g}_2$ is a nilpotent Lie algebra, hence $G_1 \ltimes G_2$ is a nilpotent Lie group. □

**Example 2.3.** For an arbitrary Lie group $G$ with the center $Z$, let us specialize Definition [2.4] for $G_1 = G_2 := G$ and $\alpha: G \to \text{Aut} G, g \mapsto \alpha_g$, where $\alpha_g(h) = ghg^{-1}$ whenever $g, h \in G$. Then the corresponding semidirect product will always be denoted simply by $G \ltimes G$ and has the following properties:

1. If $G$ is nilpotent, then so is $G \ltimes G$.
2. The Lie algebra of $G \ltimes G$ is $\mathfrak{g} \ltimes_adg$, which will be denoted simply by $\mathfrak{g} \ltimes \mathfrak{g}$, and the center of $G \ltimes G$ is $Z \times Z$.
3. The exponential map of the Lie group $G \ltimes G$ is given by
   \[\exp_{G \ltimes G}(X,Y) = (\exp_G X, \exp_G (-X) \exp_G (X + Y))\]
   for every $(X,Y) \in \mathfrak{g} \ltimes Ad g$.
4. The mapping
   \[\mu: G \times G \to G \ltimes G, \quad (g,h) \mapsto (gh, g)\]
   is an isomorphism of Lie groups, and the corresponding isomorphism of Lie algebras is $L(\mu): \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \ltimes \mathfrak{g}$, $(X,Y) \mapsto (X + Y, X)$.

In fact, property (1) follows by Remark 2.2. Property (2) is a consequence of the fact that $\dot{\alpha} = ad_g: \mathfrak{g} \to \text{Der} \mathfrak{g}$ along with (2.4).

To prove property (3), note that the mapping $\Pi: G \ltimes G \to G, (g_1, g_2) \mapsto g_1g_2$ is a homomorphism of Lie groups, hence we have the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{g} \ltimes \mathfrak{g} & \xrightarrow{L(\Pi)} & \mathfrak{g} \\
\exp_{G \ltimes G} \downarrow & & \downarrow \exp_G \\
G \ltimes G & \longrightarrow & G
\end{array}
\]

where it is easy to see that the Lie algebra homomorphism $L(\Pi): \mathfrak{g} \ltimes \mathfrak{g} \to \mathfrak{g}$ is given by $(X,Y) \mapsto X + Y$. Now let $(X,Y) \in \mathfrak{g} \ltimes \mathfrak{g}$ arbitrary. It is clear that there exists $g \in G$ such that $\exp_{G \ltimes G}(X,Y) = (\exp_G X, g)$, and then the above commutative diagram shows that $\exp_{G}(X+Y) = \Pi(\exp_{G \ltimes G}(X,Y)) = \Pi(\exp_{G} X, g) = (\exp_G X)g$, whence $g = \exp_{G}(-X)\exp_{G}(X + Y)$.

Finally, property (4) follows by a straightforward computation. □

### 2.2. Weyl-Pedersen calculus for unitary irreducible representations.

**Setting 2.4.** Throughout the present section we shall use the following notation:

1. Let $G$ be a connected, simply connected, nilpotent Lie group with the Lie algebra $\mathfrak{g}$. Then the exponential map $\exp_G: \mathfrak{g} \to G$ is a diffeomorphism with the inverse denoted by $\log_G: G \to \mathfrak{g}$.
2. We denote by $\mathfrak{g}^*$ the linear dual space to $\mathfrak{g}$ and by $\langle \cdot, \cdot \rangle: \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the natural duality pairing.
3. Let $\xi_0 \in \mathfrak{g}^*$ with the corresponding coadjoint orbit $O := \text{Ad}_G^*(G)\xi_0 \subseteq \mathfrak{g}^*$. 

(4) The isotropy group at $\xi_0$ is $G_{\xi_0} := \{ g \in G \mid \text{Ad}_G^*(g)\xi_0 = \xi_0 \}$ with the corresponding isotropy Lie algebra $\mathfrak{g}_{\xi_0} = \{ X \in \mathfrak{g} \mid \xi_0 \circ \text{ad}_X = 0 \}$. The center $\mathfrak{z} := \{ X \in \mathfrak{g} \mid [X, g] = \{0\} \}$ clearly satisfies $\mathfrak{z} \subseteq \mathfrak{g}_{\xi_0}$.

(5) Let $n := \dim \mathfrak{g}$ and fix a sequence of ideals in $\mathfrak{g}$,

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

such that $\dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1$ and $[\mathfrak{g}, \mathfrak{g}_j] \subseteq \mathfrak{g}_{j-1}$ for $j = 1, \ldots, n$.

(6) Pick any $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$ for $j = 1, \ldots, n$, so that the set $\{X_1, \ldots, X_n\}$ will be a Jordan-Hölder basis in $\mathfrak{g}$.

(7) The set of jump indices of the coadjoint orbit $\mathcal{O}$ with respect to the above Jordan-Hölder basis is $e := \{ j \in \{1, \ldots, n\} \mid \mathfrak{g}_j \not\subseteq \mathfrak{g}_{j-1} + \mathfrak{g}_{\xi_0} \}$ and does not depend on the choice of $\xi_0 \in \mathcal{O}$ (see also Prop. 2.4.1 in [Pe84]). The corresponding predual of the coadjoint orbit $\mathcal{O}$ is

$$\mathfrak{g}_e := \text{span}\{X_j \mid j \in J_{\xi_0}\} \subseteq \mathfrak{g}. $$

We shall denote $e = \{ j_1, \ldots, j_d \}$ with $1 \leq j_1 < \cdots < j_d \leq n$.

(8) We shall always consider $\mathcal{O}$ endowed with its canonical Liouville measure (see for instance the remark after the statement of the theorem in § 6, Ch. II, Part 2 in [Pu67]).

(9) Let $\pi: G \rightarrow B(\mathcal{H})$ be a fixed unitary irreducible representation associated with the coadjoint orbit $\mathcal{O}$ by Kirillov’s theorem ([Kir62]). □

Remark 2.5. The space of smooth vectors $\mathcal{H}_\infty := \{ v \in \mathcal{H} \mid \pi(\cdot) v \in C^\infty(G, \mathcal{H}) \}$ is a Fréchet space in a natural way and is a dense linear subspace of $\mathcal{H}$ which is invariant under the unitary operator $\pi(g)$ for every $g \in G$. The derivite representation $d\pi: \mathfrak{g} \rightarrow \text{End}(\mathcal{H}_\infty)$ is a homomorphism of Lie algebras defined by

$$(\forall X \in \mathfrak{g}, v \in \mathcal{H}_\infty) \quad d\pi(X)v = \left. \frac{d}{dt} \right|_{t=0} \pi(\exp_G(tX))v.$$ 

We denote by $\mathcal{H}_-$ the space of all continuous antilinear functionals on $\mathcal{H}_\infty$ and the corresponding pairing will be denoted by $(\cdot | \cdot): \mathcal{H}_- \times \mathcal{H}_\infty \rightarrow \mathbb{C}$. just as the scalar product in $\mathcal{H}$, since they agree on $\mathcal{H}_\infty \times \mathcal{H}_\infty$ if we think of the natural inclusions $\mathcal{H}_\infty \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_-$. (See for instance [Ca70] for more details.) □

Remark 2.6. We now recall a few facts from subsection 1.2 in [Pe94] for later use. Let us denote by $\mathfrak{S}_p(\mathcal{H})$ the Schatten ideals of operators on $\mathcal{H}$ for $1 \leq p \leq \infty$. Consider the unitary representation $\pi^{\otimes 2}: G \times G \rightarrow B(\mathfrak{S}_2(\mathcal{H}))$ defined by

$$ (\forall g_1, g_2 \in G) (\forall T \in \mathfrak{S}_2(\mathcal{H})) \quad \pi^{\otimes 2}(g_1, g_2)T = \pi(g_1)T\pi(g_2)^{-1}. $$

It is not difficult to see that $\pi^{\otimes 2}$ is strongly continuous. The corresponding space of smooth vectors is denoted by $B(\mathcal{H})_\infty$ and is called the space of smooth operators for the representation $\pi$. One can prove that actually $B(\mathcal{H})_\infty \subseteq \mathfrak{S}_1(\mathcal{H})$.

For an alternative description of $B(\mathcal{H})_\infty$ let $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}$ with the corresponding universal associative enveloping algebra $U(\mathfrak{g}_\mathbb{C})$. Then the aforementioned homomorphism of Lie algebras $d\pi$ has a unique extension to a homomorphism of unital associative algebras $d\pi: U(\mathfrak{g}_\mathbb{C}) \rightarrow \text{End}(\mathcal{H}_\infty)$. One can prove that for $T \in B(\mathcal{H})$ we have $T \in B(\mathcal{H})_\infty$ if and only if $T(\mathcal{H}) + T^*(\mathcal{H}) \subseteq \mathcal{H}_\infty$ and $d\pi(u)T, d\pi(u)T^* \in B(\mathcal{H})$ for every $u \in U(\mathfrak{g}_\mathbb{C})$. 

Since \( \{ \cdot | f_1,f_2 \in \mathcal{H}_\infty \} \subseteq \mathcal{B}(\mathcal{H})_\infty \subseteq \mathcal{G}_1(\mathcal{H}) \) and \( \mathcal{H}_\infty \) is dense in \( \mathcal{H} \), we get continuous inclusion maps
\[
\mathcal{B}(\mathcal{H})_\infty \hookrightarrow \mathcal{G}_1(\mathcal{H}) \hookrightarrow \mathcal{B}(\mathcal{H}) \hookrightarrow \mathcal{B}(\mathcal{H})_\infty,
\]
where the latter mapping is constructed by using the well-known isomorphism \( (\mathcal{G}_1(\mathcal{H}))^* \cong \mathcal{B}(\mathcal{H}) \) given by the usual semifinite trace on \( \mathcal{B}(\mathcal{H}) \). \( \square \)

**Definition 2.7.** The Fourier transform \( \mathcal{S}(\mathcal{O}) \to \mathcal{S}(\mathfrak{g}_c) \), \( a \mapsto \hat{a} \), defined by
\[
\hat{a}(X) = \int_{\mathcal{O}} e^{-i\langle \xi,X \rangle} a(\xi) d\xi
\]
is an isomorphism of Fréchet spaces. The Lebesgue measure on \( \mathfrak{g}_c \) can be normalized such that the Fourier transform extends to a unitary operator
\[
L^2(\mathcal{O}) \to L^2(\mathfrak{g}_c), \quad a \mapsto \hat{a},
\]
and its inverse is defined by the usual formula (see Lemma 4.1.1. in [Pe94]). We shall always consider the predual \( \mathfrak{g}_c \), endowed with this normalized measure.

If \( f \in \mathcal{H}_\infty \) and \( \phi \in \mathcal{H}_\infty \), or \( f, \phi \in \mathcal{H} \), then we define the corresponding ambiguity function
\[
\mathcal{A}(f,\phi) = \mathcal{A}_{\phi}f : \mathfrak{g}_c \to \mathbb{C}, \quad (\mathcal{A}_{\phi}f)(X) = (f | \pi(\exp_G X)\phi).
\]
For \( \phi \in \mathcal{H}_\infty \) and \( f \in \mathcal{H}_\infty \) we also define \( (\mathcal{A}_{\phi}f)(X) = (\phi | \pi(\exp_G(-X))f) \) whenever \( X \in \mathfrak{g}_c \).

It follows by Proposition 2.8 below that if \( f,\phi \in \mathcal{H} \), then \( \mathcal{A}_{\phi}f \in L^2(\mathfrak{g}_c) \), so we can use the aforementioned Fourier transform to define the corresponding cross-Wigner distribution \( W(f,\phi) \in L^2(\mathcal{O}) \) such that \( \hat{W}(f,\phi) := \mathcal{A}_{\phi}f \). \( \square \)

The second equality in Proposition 2.8 below could be referred to as the Moyal identity since that classical identity (see for instance [Gr01]) is recovered in the special case when \( G \) is a simply connected Heisenberg group.

**Proposition 2.8.** The following assertions hold:

1. If \( \phi \in \mathcal{H} \), then \( \mathcal{A}_{\phi}f \in L^2(\mathfrak{g}_c) \). We have
   \[
   (\mathcal{A}_{\phi_1}f_1 | \mathcal{A}_{\phi_2}f_2)_{L^2(\mathfrak{g}_c)} = (f_1 | f_2)_H \cdot (\phi_2 | \phi_1)_H
   = (W(f_1,\phi_1) | W(f_2,\phi_2))_{L^2(\mathcal{O})}
   \]
   for arbitrary \( \phi_1,\phi_2,f_1,f_2 \in \mathcal{H} \).

2. If \( \phi_0 \in \mathcal{H} \) with \( \|\phi_0\| = 1 \), then the operator \( \mathcal{A}_{\phi_0} : \mathcal{H} \to L^2(\mathfrak{g}_c) \), \( f \mapsto \mathcal{A}_{\phi_0}f \),
is an isometry and we have
   \[
   \int_{\mathfrak{g}_c} (\mathcal{A}_{\phi_0}f)(X) \cdot \pi(\exp_G X)\phi_0 dX = (\phi | \phi_0)f
   \]
   for every \( \phi \in \mathcal{H}_\infty \) and \( f \in \mathcal{H} \). In particular,
   \[
   \int_{\mathfrak{g}_c} (\mathcal{A}_{\phi_0}f)(X) \cdot \pi(\exp_G X)\phi_0 dX = f
   \]
   for arbitrary \( f \in \mathcal{H} \).
Proof. (1) We first prove that (2.4) holds for \( \phi_1, \phi_2, f_1, f_2 \in \mathcal{H}_\infty \). Since \( \mathcal{B}(\mathcal{H}_\infty) \) is contained in the ideal \( \mathcal{S}_1(\mathcal{H}) \) of trace-class operators, it makes sense to define

\[
(\forall A \in \mathcal{B}(\mathcal{H}_\infty)) \quad f_A^* : G \to \mathbb{C}, \quad f_A^*(x) = \text{Tr}(\pi(x)A).
\]

It follows by Th. 2.2.7 in [Pe94] that for the suitably normalized Lebesgue measure on \( \mathfrak{g}_e \) we have for every \( A, B \in \mathcal{B}(\mathcal{H}_\infty) \),

\[
\int_{\mathfrak{g}_e} f_A^*(\exp X)f_B^*(\exp X)dX = \text{Tr}(AB^*). \tag{2.5}
\]

We now denote

\[
(\forall f, \phi \in \mathcal{H}) \quad A_{f,\phi} = (f | \phi)f \in \mathcal{B}(%0.8cm H)
\]

and recall that for arbitrary \( f_1, f_2, \phi_1, \phi_2 \in \mathcal{H} \) we have

\[
A_{f_1,\phi_1}A_{f_2,\phi_2} = A_{f_1,\phi_1}(f_2 | \phi_2)\phi_2 = A_{f_1}(\phi_1 | f_2)\phi_2 = A_{f_1}(\phi_1 | f_2)\phi_2.
\]

It then easily follows that if \( f, \phi \in \mathcal{H}_\infty \), then \( A_{f,\phi} \in \mathcal{B}(\mathcal{H}_\infty) \) and for arbitrary \( X \in \mathfrak{g}_e \) we have

\[
f_{A_{f,\phi}}(\exp X) = \text{Tr}(\pi(\exp X)A_{f,\phi}) = \text{Tr}(A_{\pi(\exp X)f,\phi} = (\pi(\exp X)f | \phi) = (f | \pi(\exp G(-X)\phi),
\]

whence

\[
(\forall X \in \mathfrak{g}_e) \quad f_{A_{f,\phi}}(\exp X) = (A_{\phi}f)(-X). \tag{2.6}
\]

Now, by using (2.5) for \( A := A_{f_1,\phi_1} \) and \( B := A_{f_2,\phi_2} \), we get

\[
(A_{\phi_1}f_1 | A_{\phi_2}f_2)_{L^2(\mathfrak{g}_e)} = \text{Tr}(A_{f_1,\phi_1}A_{f_2,\phi_2}') = \text{Tr}(A_{f_1,\phi_1}A_{\phi_2,f_2}') = \text{Tr}(A_{f_1}(\phi_1 | \phi_2)f_2') = (f_1 | (\phi_1 | \phi_2)f_2') = (f_1 | f_2)_{\mathcal{H}} \cdot (\phi_2 | \phi_1)_{\mathcal{H}}
\]

The second part of (2.4) then follows since the Fourier transform \( L^2(\mathcal{O}) \to L^2(\mathfrak{g}_e) \) is a unitary operator, as we already mentioned in Definition 2.7.

The extension of (2.4) from \( \mathcal{H}_\infty \) to \( \mathcal{H} \) proceeds by a density argument. First note that by (2.4) for \( \phi_1 = \phi_2 =: \phi \in \mathcal{H}_\infty \) and \( f_1 = f_2 =: f \in \mathcal{H}_\infty \) we get \( ||A_{\phi}f|| = ||\phi|| \cdot ||f|| \). Since \( \mathcal{H}_\infty \) is dense in \( \mathcal{H} \), it then follows that the sesquilinear mapping \( \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{H} \), \( (f, \phi) \mapsto A_{\phi}f \) extends uniquely to a sesquilinear mapping \( \mathcal{H} \times \mathcal{H} \to \mathcal{H} \) satisfying

\[
(\forall f, \phi \in \mathcal{H}) \quad ||A_{\phi}f|| = ||\phi|| \cdot ||f||. \tag{2.7}
\]

Now the first part of (2.4) follows as a polarization of (2.7), and then the second part follows by using the Fourier transform \( L^2(\mathcal{O}) \to L^2(\mathfrak{g}_e) \) as above.

(2) It follows at once by Assertion (1) that the operator \( A_{\phi_0} : \mathcal{H} \to L^2(\mathfrak{g}_e) \) is an isometry if \( ||\phi_0|| = 1 \). The other properties then follow immediately; see for instance Proposition 2.11 in [Fe05].

We now draw some useful consequences of Proposition 2.8. We emphasize that Assertion (3) in the following corollary in the special case of square-integrable representations reduces to a theorem of [Co84] and [CM96]. One thus recovers Th. 2.3 in [GZ01] in the case of the Schrödinger representation of the Heisenberg group.

**Corollary 2.9.** If \( \phi_0 \in \mathcal{H}_\infty \) with \( ||\phi_0|| = 1 \), then the following assertions hold:
For every \( f \in \mathcal{H}^{-\infty} \) we have

\[
\int_{\mathfrak{g}_e} \hat{a}(X) \pi(\exp G X) \phi_0 \, dX = f
\]

where the integral is convergent in the weak*-topology of \( \mathcal{H}^{-\infty} \).

(2) If \( f \in \mathcal{H}_\infty \), then the above integral converges in the Fréchet topology of \( \mathcal{H}_\infty \).

(3) If \( f \in \mathcal{H}^{-\infty} \), then we have \( f \in \mathcal{H}_\infty \) if and only if \( A_{\phi_0} f \in \mathcal{S}(\mathfrak{g}_e) \).

Proof. If \( f \in \mathcal{H}^{-\infty} \), we have to prove that

\[
\int_{\mathfrak{g}_e} \hat{a}(X) \pi(\exp G X) \phi_0 \, dX = (f | \phi),
\]

for every \( \phi \in \mathcal{H}_\infty \), that is,

\[
\int_{\mathfrak{g}_e} (f | \pi(\exp G X) \phi_0) \pi(\exp G X) \phi_0 \, dX = (f | \phi).
\]

Since \((f | \cdot)\) is an antilinear continuous functional, the above equation will follow as soon as we have proved that for \( \phi \in \mathcal{H}_\infty \) we have

\[
\int_{\mathfrak{g}_e} \phi(\pi(\exp G X) \phi_0) \pi(\exp G X) \phi_0 \, dX = \phi
\]

with an integral that converges in the topology of \( \mathcal{H}_\infty \). Note that this is precisely Assertion (2). To prove it, we just have to use Proposition 2.8(2) along with the fact that for \( \phi, \phi_0 \in \mathcal{H}_\infty \) the function \( X \mapsto \phi(\pi(\exp G X) \phi_0) = (A_{\phi_0} \phi)(X) \) belongs to \( \mathcal{S}(\mathfrak{g}_e) \) (see Th. 2.2.6 in [Pe94]) while the function \( X \mapsto \pi(\exp G X) \phi_0 \) and all its partial derivatives have polynomial growth.

For Assertion (3), we have just noted that if \( f \in \mathcal{H}_\infty \) then \( A_{\phi_0} f \in \mathcal{S}(\mathfrak{g}_e) \) as a direct consequence of Th. 2.2.6 in [Pe94]. Conversely, if \( f \in \mathcal{H}^{-\infty} \) has the property \( A_{\phi_0} f \in \mathcal{S}(\mathfrak{g}_e) \), then the fact that all the partial derivatives of \( X \mapsto \pi(\exp G X) \phi_0 \) have polynomial growth implies at once that the integral in (2.8) is convergent in the Fréchet space \( \mathcal{H}_\infty \), hence Assertion (1) shows that actually \( f \in \mathcal{H}_\infty \). \( \square \)

Definition 2.10. The Weyl-Pedersen calculus \( \text{Op}^\pi(\cdot) \) for the unitary representation \( \pi \) is defined for every \( a \in \mathcal{S}(\mathfrak{g}) \) by

\[
\text{Op}^\pi(a) = \int_{\mathfrak{g}_e} \hat{a}(X) \pi(\exp G X) \, dX \in \mathcal{B}(\mathcal{H}).
\]

This definition can be extended to an arbitrary tempered distribution \( a \in \mathcal{S}'(\mathfrak{g}) \) by using Th. 4.1.4 and Th. 2.2.7 in [Pe94] to define an unbounded operator \( \text{Op}^\pi(a) \) such that

\[
(\forall b \in \mathcal{S}(\mathfrak{g})) \quad \text{Tr}(\text{Op}^\pi(a) \text{Op}^\pi(b)) = \langle a, b \rangle,
\]

where we recall that \( \langle \cdot, \cdot \rangle : \mathcal{S}'(\mathfrak{g}) \times \mathcal{S}(\mathfrak{g}) \to \mathbb{C} \) stands for the usual pairing between the tempered distributions and the Schwartz functions. We say that \( a \in \mathcal{S}'(\mathfrak{g}) \) is the symbol of the operator \( \text{Op}^\pi(a) \). \( \square \)

We now record some basic properties of the Weyl-Pedersen calculus constructed in Definition 2.10. These are actually direct consequences of Proposition 2.8(1).

Corollary 2.11. The following assertions hold:
In the space of distributions $S$ mapping $S$ to $W$, it follows that the above extension of the cross-Wigner distribution $W$ can consider the rank-one operator uniquely a distribution $A$.

That is, $T_{f_1,f_2} : B(S(\mathcal{O})) \to \mathbb{C}$, $T_{f_1,f_2}(A) := (f_1 \mid Af_2)$. Then for $f_1,f_2 \in H_\infty$ we can define the continuous antilinear functional

$$T_{f_1,f_2} : B(S(\mathcal{O})) \to \mathbb{C}, \quad T_{f_1,f_2}(A) := (f_1 \mid Af_2).$$

That is, $T_{f_1,f_2} \in B(S(\mathcal{O}))$, and then Th. 4.1.4(5) in [Pe94] shows that there exists a unique distribution $a_{f_1,f_2} \in S(\mathcal{O})$ such that $Op^\pi(a_{f_1,f_2}) = T_{f_1,f_2}$. Now define $W(f_1,f_2) := a_{f_1,f_2}$.

We can consider the rank-one operator $S_{f_1,f_2} := (\cdot \mid f_2)f_1 : H_\infty \to H_\infty$ and for arbitrary $A \in B(H_\infty)$ thought of as a continuous linear map $A : H_\infty \to H_\infty$ as above we have

$$\text{Tr}(S_{f_1,f_2}A) = (f_1 \mid Af_2) = T_{f_1,f_2}(A).$$

Thus the trace duality pairing allows us to identify the functional $T_{f_1,f_2} \in B(S(\mathcal{O}))$ with the rank-one operator $(\cdot \mid f_2)f_1$, and then we can write

$$(\forall f_1,f_2 \in H_\infty) \quad Op^\pi(W(f_1,f_2)) = (\cdot \mid f_2)f_1. \quad (2.11)$$

In particular, it follows that the above extension of the cross-Wigner distribution to a mapping $W(\cdot, \cdot) : H_\infty \times H_\infty \to S'(\mathcal{O})$ allows us to generalize the assertion of Corollary 2.11(2) to arbitrary $\phi_1, \phi_2 \in H_\infty$.

**Definition 2.13.** Recall from Th. 4.1.4(5) in [Pe94] that the Weyl-Pedersen calculus $Op^\pi : S'(\mathcal{O}) \to B(H_\infty)$ is a linear isomorphism and a weak*-homeomorphism. We introduce the linear space

$$S^0(\mathcal{O}) := \{ a \in S'(\mathcal{O}) \mid Op^\pi(a) \in B(H) \}$$

(see [2.3]). Then the mapping $Op^\pi$ induces a linear isomorphism $S^0(\mathcal{O}) \to B(H)$, hence there exists an uniquely defined bilinear associative *Moyal product*

$$S^0(\mathcal{O}) \times S^0(\mathcal{O}) \to S^0(\mathcal{O}), \quad (a,b) \mapsto a \# b$$

such that

$$(\forall a,b \in S^0(\mathcal{O})) \quad Op^\pi(a \# b) = Op^\pi(a)Op^\pi(b).$$

The space of distributions $S^0(\mathcal{O})$ is thus made into a $W^*$-algebra such that the mapping $S^0(\mathcal{O}) \to B(H)$, $a \mapsto Op^\pi(a)$ is a *-isomorphism.
Example 2.14. Here are some examples of distributions in $S^0(\mathcal{O})$ which are already available.

1. It follows at once by (2.9) and (2.10) that
   $$\{a \in S'(\mathcal{O}) \mid \hat{a} \in L^1(\mathfrak{g}_e)\} \subseteq S^0(\mathcal{O}).$$

2. The Schwartz space $S(\mathcal{O})$ is a *-subalgebra of $S^0(\mathcal{O})$ and the mapping $\text{Op}^\pi : S(\mathcal{O}) \to B(\mathcal{H}_\infty)$ is an algebra *-isomorphism by Th. 4.1.4 in [Pe94].

3. The space $L^2(\mathcal{O})$ is a *-subalgebra of $S^0(\mathcal{O})$, and $\text{Op}^\pi : L^2(\mathcal{O}) \to \mathcal{S}_2(\mathcal{H})$ is a unitary operator and an algebra *-isomorphism as an easy consequence of Th. 4.1.4 in [Pe94]; see also [Ma07].

4. For every $Y \in \mathfrak{g}$ we have $\exp iY \in S^0(\mathcal{O})$ since it follows at once by (2.9) and (2.10) that $\text{Op}^\pi(\exp iY) = \pi(\exp G Y)$.

See also Corollary 2.26 for the important example $M^{\infty,1}_\phi (\pi^\#) \hookrightarrow S^0(\mathcal{O})$. □

2.3. Modulation spaces.

Definition 2.15. Let $\phi \in \mathcal{H}_\infty \setminus \{0\}$ be fixed and assume that we have a direct sum decomposition $\mathfrak{g}_e = \mathfrak{g}_e^1 + \mathfrak{g}_e^2$.

Then let $1 \leq r, s \leq \infty$ and for arbitrary $f \in \mathcal{H}_{-\infty}$ define

$$\|f\|_{M^{r,s}_\phi} = \left( \int \left( \int |(\mathcal{A}_\phi f)(X_1, X_2)|^r dX_1 \right)^{s/r} dX_2 \right)^{1/s} \in [0, \infty]$$

with the usual conventions if $r$ or $s$ is infinite. Then we call the space

$$M^{r,s}_\phi(\pi) := \{f \in \mathcal{H}_{-\infty} \mid \|f\|_{M^{r,s}_\phi} < \infty\}$$

a modulation space for the irreducible unitary representation $\pi : G \to B(\mathcal{H})$ with respect to the decomposition $\mathfrak{g}_e \simeq \mathfrak{g}_e^1 \times \mathfrak{g}_e^2$ and the window vector $\phi \in \mathcal{H}_\infty \setminus \{0\}$. □

Remark 2.16. Assume the setting of Definition 2.15 and recall the mixed-norm space $L^{r,s}(\mathfrak{g}_e^1 \times \mathfrak{g}_e^2)$ consisting of the (equivalence classes of) Lebesgue measurable functions $\Theta : \mathfrak{g}_e^1 \times \mathfrak{g}_e^2 \to \mathbb{C}$ such that

$$\|\Theta\|_{L^{r,s}} := \left( \int \left( \int |(\Theta(X_1, X_2)|^r dX_1 \right)^{s/r} dX_2 \right)^{1/s} < \infty$$

(cf. [Gr01]). It is clear that $M^{r,s}_\phi(\pi) = \{f \in \mathcal{H}_{-\infty} \mid \mathcal{A}_\phi f \in L^{r,s}(\mathfrak{g}_e^1 \times \mathfrak{g}_e^2)\}$. □

Example 2.17. For any choice of $\phi \in \mathcal{H}_\infty \setminus \{0\}$ in Definition 2.15 we have

$$M^{2,2}_\phi(\pi) = \mathcal{H}.$$ 

Indeed, this equality holds since $\|\mathcal{A}_\phi f\|_{L^2(\mathfrak{g}_e)} = \|\phi\| \cdot \|f\|$ for every $f \in \mathcal{H}$ (see (2.7) in the proof of Proposition 2.8 above). □
2.4. Continuity of Weyl-Pedersen calculus on modulation spaces. In the following lemma we use notation introduced in Example 2.3[4] and Remark 2.6

**Lemma 2.18.** Let $G$ be any Lie group with a unitary irreducible representation $\pi: G \to \mathcal{B}(\mathcal{H})$ and define

$$\bar{\pi}: G \times G \to \mathcal{B}(\mathfrak{S}_2(\mathcal{H})), \quad \bar{\pi}(g,h)T = \pi(gh)T\pi(g)^{-1}.$$  

Then the following assertions hold:

1. The diagram

$$\begin{array}{ccc}
G \times G & \xrightarrow{\bar{\pi}} & \mathcal{B}(\mathfrak{S}_2(\mathcal{H})) \\
\mu & & \downarrow_{\pi \otimes 2} \\
G \times G
\end{array}$$

is commutative and $\bar{\pi}$ is a unitary irreducible representation of $G \times G$.

2. The space of smooth vectors for the representation $\bar{\pi}$ is $\mathcal{B}(\mathcal{H})$.\infty.

3. Let us denote by $\bar{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{g}$ and define

$$\bar{X}_j = \begin{cases} 
(X_j, 0) & \text{for } j = 1, \ldots, n, \\
(X_{j-n}, X_{j-n}) & \text{for } j = n + 1, \ldots, 2n.
\end{cases}$$

Then $\bar{X}_1, \ldots, \bar{X}_{2n}$ is a Jordan-Hölder basis in $\bar{\mathfrak{g}}$ and the corresponding pre-dual for the coadjoint orbit $\bar{\mathcal{O}} \subseteq \bar{\mathfrak{g}}^*$ associated with the representation $\bar{\pi}$ is

$$\bar{\mathfrak{e}} = \mathfrak{e} \times \mathfrak{e} \subseteq \bar{\mathfrak{g}},$$

where $\bar{\mathfrak{e}}$ is the set of jump indices for $\bar{\mathcal{O}}$.

**Proof.** It is clear that the diagram is commutative, and then the mapping $\bar{\pi}$ is a representation since $\pi \otimes 2$ is a representation and $\mu: G \times G \to G \times G$ is a group isomorphism. It is well-known that the representation $\pi \otimes 2$ is irreducible, hence $\bar{\pi}$ is irreducible as well. For the sake of completeness, we recall the corresponding reasoning. Let arbitrary $A \in \mathcal{B}(\mathfrak{S}_2(\mathcal{H}))$ satisfying

$$(\forall (g, h) \in G \times G) \quad A\bar{\pi}(g, h) = \bar{\pi}(g, h)A. \tag{2.12}$$

We have to show that $A$ is a scalar multiple of the identity operator on $\mathfrak{S}_2(\mathcal{H})$. For that purpose, let us define the operators $L_B, R_B: \mathfrak{S}_2(\mathcal{H}) \to \mathfrak{S}_2(\mathcal{H})$ by $L_B X = B X$ and $R_B X = X B$ for $X, B \in \mathcal{B}(\mathcal{H})$. Note that if $h \in G$, then $\bar{\pi}(1, h) = L_{\pi(h)}$. It then follows by (2.12) that $A L_{\pi(h)} = L_{\pi(h)} A$ for every $h \in G$. On the other hand, the representation $\pi$ is irreducible, the linear space span $\{\pi(h) \mid h \in G\}$ is dense in $\mathcal{B}(\mathcal{H})$ in the strong operator topology, and then it easily follows that $A L_B = L_B A$ for every $B \in \mathcal{B}(\mathcal{H})$. This property implies that there exists $A \in \mathcal{B}(\mathcal{H})$ such that $A = R_A$ (see for instance [Ta03]). Now, by using (2.12) for $h = 1$, we get $\pi(g) X \pi(g)^{-1} A = \pi(g) X A \pi(g)^{-1}$ for every $g \in G$ and $X \in \mathcal{B}(\mathcal{H})$, which implies $A \pi(g) = \pi(g) A$ for arbitrary $g \in G$. Since $\pi$ is an irreducible representation, it follows that $A$ is a scalar multiple of the identity operator on $\mathcal{H}$, hence $A = L_A$ is a scalar multiple of the identity operator on $\mathfrak{S}_2(\mathcal{H})$, as we wished for.

\[ \text{This assertion follows by Remark 2.6} \]

\[ \text{It is easy to see that the sequence} \]

$$(0, X_1), \ldots, (0, X_n), (X_1, 0), \ldots, (X_n, 0)$$
is a Jordan-Hölder basis in the direct product $g \times g$, and the coadjoint orbit corresponding to the representation $\pi \otimes 2: G \times G \to B(S_2(\mathcal{H}))$ is $O \times O$. (This follows for instance by the theorem in § 6, Ch. II, Part 2 in [Pu67]). Then the assertion follows by Example 2.14 along with the above Assertion (1).

In the following definition we use an idea similar to one used in [MP09].

**Definition 2.19.** Let $G$ be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi: G \to B(\mathcal{H})$. Assume that $O \subseteq g^*$ is the coadjoint orbit associated with this representation and define

$$\pi^#: G \ltimes G \to B(L^2(O)), \quad \pi^#(\exp X, \exp Y)f = e^{i\langle \cdot, X \rangle} \# e^{i\langle \cdot, Y \rangle} \# f \# e^{-i\langle \cdot, X \rangle},$$

where $\#$ is the Moyal product associated with $\pi$ (see Definition 2.13). We note the following equivalent expression

$$(\forall X, Y \in g) \quad \pi^#(\exp G \times G (X, Y))f = e^{i\langle \cdot, X+Y \rangle} \# f \# e^{-i\langle \cdot, X \rangle}$$ (2.13)

which follows by Example 2.14. The corresponding *ambiguity function* is given by

$$A_{\phi}^# F: g_e \times g_e \to \mathbb{C}, \quad (A_{\phi}^# F)(X, Y) = (F | \pi^#(\exp G \times G (X, Y))F)$$

for $F, F \in L^2(O)$ or for a function $F \in \mathcal{S}(O)$ and a continuous antilinear functional $F: \mathcal{S}(O) \to \mathbb{C}$ denoted by $\Psi \mapsto (F | \Psi)$. □

**Remark 2.20.** To explain the terminology of Definition 2.19 let us see that we really have to do with the ambiguity function of a unitary representation. To this end, recall the unitary operator $\text{Op}^\pi: L^2(O) \to S_2(\mathcal{H})$ (see e.g., Example 2.14) and the representation $\hat{\pi}: G \ltimes G \to B(S_2(\mathcal{H}))$ from Lemma 2.18. It follows by Definition 2.13 and Example 2.14 that the unitary operator $\text{Op}^\pi$ intertwines $\pi^#$ and $\hat{\pi}$, hence we get by Lemma 2.18 that $\pi^# \equiv (\exp G \times G (X, Y))F$ corresponds to the coadjoint orbit $O^# \subseteq (g \ltimes g)^*$ associated with the representation $\pi^#$.

Let us note that the space of smooth vectors for the representation $\pi^#$ is equal to $\mathcal{S}(O)$, as a consequence of Lemma 2.18, since $\text{Op}^\pi: \mathcal{S}(O) \to B(\mathcal{H}_{\infty})$ is a linear isomorphism by Th. 4.1.4 in [Pe94]. □

The next statement points out the representation theoretic background of the computation carried out in the proof of Lemma 14.5.1 in [Gr01].

**Proposition 2.21.** Let $G$ be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi: G \to B(\mathcal{H})$. Pick any predual $g_e \subseteq g$ for the coadjoint orbit $O \subseteq g^*$ corresponding to the representation $\pi$. If either $\phi_1, \phi_2, f_1, f_2 \in \mathcal{H}$ or $\phi_1, \phi_2 \in \mathcal{H}_{\infty}$ and $f_1, f_2 \in \mathcal{H}_{-\infty}$, then

$$(\forall X, Y \in g_e) \quad A_{\phi}^# (\mathcal{W}(f_1, f_2))(X, Y) = (A_{\phi_1} f_1)(X + Y) \cdot (A_{\phi_2} f_2)(X)$$

where $\Phi := \mathcal{W}(\phi_1, \phi_2) \in L^2(O)$, while $\mathcal{W}(\cdot, \cdot)$ and $A_{\phi j}: g_e \to \mathbb{C}$ for $j = 1, 2$ are cross-Wigner distributions and ambiguity functions for the representation $\pi$, respectively.

**Proof.** If we denote $F = \mathcal{W}(f_1, f_2)$, then for arbitrary $X, Y \in g_e$ we have by Definition 2.19, Example 2.14, and Remark 2.20

$$(A_{\phi}^# F)(X, Y) = (F | \pi^#(\exp G \times G (X, Y))\Phi)_{L^2(O)} = (F | \pi^#(\exp G X, (\exp G X)^{-1} \exp G (X + Y)\Phi)_{L^2(O)}$$
Proof. Let

\[ (\text{Op}^\pi(F) | \exp_{G}X, \exp_{G}X^{-1}) \text{Op}^\pi(\Phi))_{\mathfrak{g}_{r}(\mathcal{H})} \]

and

\[ (\text{Op}^\pi(F) | \pi(\exp_{G}(X+Y))\text{Op}^\pi(\Phi)\pi(\exp_{G}(X+Y)^{-1})\mathfrak{g}_{r}(\mathcal{H}). \]

On the other hand Remark 2.12 (particularly (2.11)) shows that

\[ \text{Op}^\pi(F) = (\cdot | f_{2})f_{1} \]

and \( \text{Op}^\pi(\Phi) = (\cdot | \phi_{2})\phi_{1} \), whence

\[ \pi(\exp_{G}(X+Y))\text{Op}^\pi(\Phi)\pi(\exp_{G}(X+Y)^{-1}) = (\cdot | \pi(\exp_{G}(X)\phi_{2})\pi(\exp_{G}(X+Y))\phi_{1}). \]

Then the above computation leads to the formula

\[ (A_{\mu}^{\Phi}F)(X, Y) = (\pi(\exp_{G}(X)\phi_{2}) | f_{2}) \cdot (f_{1} | \pi(\exp_{G}(X+Y))\phi_{1}), \]

which is equivalent to the equation in the statement.

We now prove a generalization of Th. 4.1 in [1004] to irreducible representations of nilpotent Lie groups.

**Theorem 2.22.** Let \( G \) be a simply connected, nilpotent Lie group with a unitary irreducible representation \( \pi: G \to B(\mathcal{H}) \). Let \( O \) be the corresponding coadjoint orbit, pick \( \phi_{1}, \phi_{2} \in \mathcal{H}_{\infty} \setminus \{0\} \), and denote \( \Phi = W(\phi_{1}, \phi_{2}) \in \mathcal{S}(O) \). Assume that \( \mathfrak{g}_{e} \) is a predual to the coadjoint orbit \( O \), and let \( \mathfrak{g}_{e} = \mathfrak{g}_{e}^{1} + \mathfrak{g}_{e}^{2} \) be any direct sum decomposition.

If \( 1 \leq r \leq s \leq \infty \) and \( r_{1}, r_{2}, s_{1}, s_{2} \in [r, s] \) satisfy

\[ \frac{1}{r_{1}} + \frac{1}{r_{2}} = \frac{1}{s_{1}} + \frac{1}{s_{2}} = \frac{1}{r} + \frac{1}{s}, \]

then the cross-Wigner distribution defines a continuous sesquilinear map

\[ W(\cdot, \cdot): M_{\mu_{1}, s_{1}}^{r_{1}, s_{1}}(\pi) \times M_{\mu_{2}, s_{2}}^{r_{2}, s_{2}}(\pi) \to M_{\mu, s}^{r}(\pi^{\#}). \]

**Proof.** Let \( f_{1}, f_{2} \in \mathcal{H}_{\infty} \) and note that for every \( X \in \mathfrak{g}_{e} \) we have

\[ (A_{f_{2}, f_{2}}(X)) = (f_{2} | \pi(\exp_{G}X)\phi_{2}) = (A_{f_{2}, f_{2}})(-X). \]

Therefore by Proposition 2.21 we get

\[ \|W(f_{1}, f_{2})\|_{M_{\mu, s}^{r}(\pi^{\#})} = \left( \int_{\mathfrak{g}_{e}^{2}} F(Y_{2}) dY_{2} \right)^{1/s}, \]

where

\[ F(Y_{2}) = \int_{\mathfrak{g}_{e}^{1}} \int_{\mathfrak{g}_{e}^{2}} \int_{\mathfrak{g}_{e}^{1}} |(A_{\phi_{1}} f_{1})(X_{1} + Y_{1}, X_{2} + Y_{2})\cdot (A_{\phi_{2}} f_{2})(-X_{1}, -X_{2})||^{r} dX_{1} dX_{2} \]

\[ \int_{\mathfrak{g}_{e}^{2}}^{s/r} dY_{1}. \]

On the other hand, it follows by Minkowski’s inequality that for every measurable function \( \Gamma: \mathfrak{g}_{e}^{1} \times \mathfrak{g}_{e}^{2} \times \mathfrak{g}_{e}^{2} \to \mathbb{C} \) and every real number \( t \geq 1 \) we have

\[ \left( \int_{\mathfrak{g}_{e}^{2}} \int_{\mathfrak{g}_{e}^{2}} |\Gamma(Y_{1}, X_{2}, Y_{2})||^{t} dY_{1} dY_{2} \right)^{1/t} \leq \left( \int_{\mathfrak{g}_{e}^{2}} \int_{\mathfrak{g}_{e}^{2}} |\Gamma(Y_{1}, X_{2}, Y_{2})||^{t} dY_{1} dY_{2} \right)^{1/t} \]

whenever \( Y_{2} \in \mathfrak{g}_{e}^{2} \). By (2.16) and (2.17) with \( t := s/r \) and

\[ \Gamma(Y_{1}, X_{2}, Y_{2}) := \int_{\mathfrak{g}_{e}^{2}} |(A_{\phi_{1}} f_{1})(Y_{1} - X_{1}, Y_{2} - X_{2})\cdot (A_{\phi_{2}} f_{2})(X_{1}, X_{2})||^{r} dX_{1}. \]
we get

\[ F(Y_2) \leq \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \Gamma(Y_1, X_2, Y_2)^{s/r} dY_1 \right)^{r/s} dX_2 \right)^{s/r} \]

\[ = \left( \int_{\mathbb{R}^n} \left\| \Gamma(\cdot, X_2, Y_2) \right\|_{L^r / (s/r)} dX_2 \right)^{s/r}. \tag{2.18} \]

Now note that \( \Gamma(\cdot, X_2, Y_2) \) is equal to the convolution product of the functions \( \left| (A_{\phi_1} f_1)(\cdot, Y_2 - X_2) \right| \) and \( \left| (A_{\phi_2} f_2)(\cdot, X_2) \right| \). It follows by Young’s inequality that

\[ \left\| \Gamma(\cdot, X_2, Y_2) \right\|_{L^r / (s/r)} \leq \left( \int_{\mathbb{R}^n} \left| (A_{\phi_1} f_1)(\cdot, Y_2 - X_2) \right| \left| (A_{\phi_2} f_2)(\cdot, X_2) \right| dY_1 \right)^{1/m_1} \times \left( \int_{\mathbb{R}^n} \left| (A_{\phi_2} f_2)(\cdot, X_2) \right|^2 dX_2 \right)^{1/m_2} \]

provided that \( m_1, m_2 \in [1, \infty) \) and \( \frac{1}{m_1} + \frac{1}{m_2} = 1 + \frac{s}{r} \). For \( m_j = \frac{r}{r_j} \), \( j = 1, 2 \), we get

\[ \left\| \theta \right\|_{L^r / (s/r)} \leq \left( \int_{\mathbb{R}^n} \left| f_1 \right|_{M_{r_{1}^{s/r}}(\pi)}^s \left( \int_{\mathbb{R}^n} \left| f_2 \right|_{M_{r_{2}^{s/r}}(\pi)}^r \right)^{r/s} dX_2 \right)^{1/m_2} \]

which in turn implies that \( \mathcal{W}(f_1, f_2) \in M_{\Phi}(\pi) \) if and only if for \( j = 1, 2 \) we have \( f_j \in M_{\Phi}^{r_j}(\pi) \).

**Remark 2.23.** A particularly sharp version of Theorem 2.22 holds for \( r_1 = s_1 \), \( r_2 = s_2 \), and \( r = s \). That is, let \( r, r_1, r_2 \in [1, \infty) \) such that \( \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} \). It follows at once by Proposition 2.21 that for arbitrary \( f_1, f_2 \in \mathcal{H}_{-\infty} \) we have

\[ \mathcal{W}(f_1, f_2) \in M_{\Phi}^{r}(\pi) \]

which in turn implies that \( \mathcal{W}(f_1, f_2) \in M_{\Phi}^{r_j}(\pi) \) and only if for \( j = 1, 2 \) we have \( f_j \in M_{\Phi}^{r_j}(\pi) \).

**Corollary 2.24.** Let \( G \) be a simply connected, nilpotent Lie group with a unitary irreducible representation \( \pi: G \to \mathcal{B}(\mathcal{H}) \), pick \( \phi_1, \phi_2 \in \mathcal{H}_{\infty} \setminus \{0\} \), and denote \( \Phi = \mathcal{W}(\phi_1, \phi_2) \in \mathcal{S}(\mathcal{O}) \). If \( r, r_1, r_2 \in [1, \infty) \) and \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \), then the cross-Wigner
Proof. One can apply Theorem 2.22 with \( r_1 = s_1, r_2 = s_2, \) and \( s = \infty \). Alternatively, a direct proof proceeds as follows. Let \( f_1, f_2 \in \mathcal{H}_{-\infty} \). It follows by Proposition 2.21 along with Hölder’s inequality that for every irreducible representation \( \pi \) defines a continuous sesquilinear map

\[
\mathcal{W}(\cdot, \cdot) : M^{r_2, r_1}(\pi) \times M^{r_1, r_2}(\pi) \to M^{r_\infty}(\pi^\#).
\]

The next corollary provides a partial generalization of Th. 4.3 in [To04].

**Corollary 2.25.** Let \( G \) be a simply connected, nilpotent Lie group with a unitary irreducible representation \( \pi : G \to \mathcal{B}(\mathcal{H}) \), pick \( \phi_1, \phi_2 \in \mathcal{H}_\infty \setminus \{0\} \), and denote \( \Phi = \mathcal{W}(\phi_1, \phi_2) \in \mathcal{S}(\mathcal{O}) \). Assume that \( \mathfrak{g}_e \) is a predual to the coadjoint orbit \( \mathcal{O} \) associated with the representation \( \pi \), and let \( \mathfrak{g}_e = \mathfrak{g}_e^r + \mathfrak{g}_e^s \) be any direct sum decomposition. If \( r, s, r_1, s_1, r_2, s_2 \in [1, \infty] \) satisfy the conditions

\[
r \leq s, \quad r_2, s_2 \in [r, s], \quad \text{and} \quad \frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{s_1} - \frac{1}{s_2} = 1 - \frac{1}{r} - \frac{1}{s},
\]

then the following assertions hold:

1. For every symbol \( a \in M^{r_s, s}(\pi^\#) \) we have a bounded linear operator

\[
\text{Op}^\pi(a) : M^{r_1, s_1}(\pi) \to M^{r_2, s_2}(\pi).
\]

2. The linear mapping \( \text{Op}^\pi(\cdot) : M^{r_s, s}(\pi^\#) \to \mathcal{B}(M^{r_1, s_1}(\pi), M^{r_2, s_2}(\pi)) \) is continuous.

**Proof.** For every \( t \in [1, \infty] \) we are going to define \( t' \in [1, \infty] \) by the equation

\[
\frac{1}{t} + \frac{1}{t_1} = 1. \quad \text{With this notation, the hypothesis implies} \quad \frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{t_1} + \frac{1}{t_1} = \frac{1}{t} + \frac{1}{t'} \\
\text{and moreover} \quad r_1, s_1, r'_2, s_2 \in [r', s']. \quad \text{Therefore we can apply Theorem 2.22 to obtain}
\]

\[
\|\mathcal{W}(f_2, f_1)\|_{M^{r'_1, s'_1}(\pi^\#)} \leq \|f_1\|_{M^{r_1, s_1}(\pi)} \cdot \|f_2\|_{M^{r_2, s_2}(\pi)} \tag{2.20}
\]

whenever \( f_1, f_2 \in \mathcal{H}_{-\infty} \).

On the other hand, if \( a \in M^{r_*, s_*(\pi^\#)} \), then

\[
(\text{Op}^\pi(a)f_1 | f_2) = \langle a | \mathcal{W}(f_2, f_1) \rangle_{L^2(\mathcal{O})} = \langle A^\#_\Phi a | A^\#_\Phi(\mathcal{W}(f_2, f_1)) \rangle_{L^2(\mathfrak{g}_e \times \mathfrak{g}_e)}
\]

by Corollary 2.11(1) and Proposition 2.20(1). Then Hölder’s inequality for mixed-norm spaces (see for instance Lemma 11.1.2(b) in [Gr01]) shows that

\[
\|(\text{Op}^\pi(a)f_1 | f_2)\| \leq \|A^\#_\Phi a\|_{L^{r_*, s_*(\mathfrak{g}_e \times \mathfrak{g}_e)}} \cdot \|A^\#_\Phi(\mathcal{W}(f_2, f_1))\|_{L^{r'_1, s'_1(\mathfrak{g}_e \times \mathfrak{g}_e)}}
\]

\[
= \|a\|_{M^{r_*, s_*(\pi^\#)}} \cdot \|\mathcal{W}(f_2, f_1)\|_{M^{r'_1, s'_1(\pi^\#)}}
\]

\[
\leq \|a\|_{M^{r_*, s_*(\pi^\#)}} \cdot \|f_1\|_{M^{r_1, s_1}(\pi)} \cdot \|f_2\|_{M^{r_2, s_2}(\pi)},
\]

where the latter inequality follows by (2.20). Now the assertion follows by a straightforward argument that uses the duality of the mixed-norm spaces (see Lemma 11.1.2(d) in [Gr01]). \( \square \)
Corollary 2.26. If $G$ be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi: G \to \mathcal{B}(\mathcal{H})$, then the following assertions hold whenever $\Phi = \mathcal{W}(\phi_1, \phi_2)$ with $\phi_1, \phi_2 \in \mathcal{H}_\infty \setminus \{0\}$:

1. For every $a \in M^{1,1}_\Phi(\pi^\#)$ we have $\text{Op}^\pi(a) \in \mathcal{B}(\mathcal{H})$.
2. The linear mapping $\text{Op}^\pi(\cdot): M^{1,1}_\Phi(\pi^\#) \to \mathcal{B}(\mathcal{H})$ is continuous.

Proof. This is the special case of Corollary 2.25 with $r_1 = 1$, $r_2 = s_1 = s_2 = 2$, $r = 1$, and $s = \infty$, since Example 2.17 shows that $M^{2,2}(\pi) = \mathcal{H}$.

We conclude this section by a sufficient condition for a pseudo-differential operator to belong to the trace class. In the special case of the Schrödinger representation of a Heisenberg group, a proof for this result can be found for instance in [Gr96] or [GH99].

Proposition 2.27. Let $G$ be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi: G \to \mathcal{B}(\mathcal{H})$, pick $\phi_1, \phi_2 \in \mathcal{H}_\infty$ with $\|\phi_1\| = \|\phi_2\| = 1$, and denote $\Phi = \mathcal{W}(\phi_1, \phi_2)$ in $\mathcal{S}(\mathcal{O})$. Then for every symbol $a \in M^{1,1}_\Phi(\pi^\#)$ we have $\text{Op}^\pi(a) \in \mathcal{S}_1(\mathcal{H})$ and $\|\text{Op}^\pi(a)\|_1 \leq \|a\|_{M^{1,1}_\Phi(\pi^\#)}$.

Proof. For arbitrary $a \in \mathcal{S}'(\mathcal{O})$ we have by Corollary 2.17(1) and Remark 2.20

$$a = \int \int (\mathcal{A}_{\Phi}^\# a)(X, Y) \cdot \pi^\#(\exp_{G\times G}(X, Y)) \Phi \, dX \, dY,$$

whence by Corollary 2.11

$$\text{Op}^\pi(a) = \int \int (\mathcal{A}_{\Phi}^\# a)(X, Y) \cdot \text{Op}^\pi(\pi^\#(\exp_{G\times G}(X, Y)) \Phi) \, dX \, dY$$

where the latter integral is weakly convergent in $\mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$. On the other hand, for arbitrary $X, Y \in g_e$ we get by (2.13) and Corollary 2.11(2)

$$\text{Op}^\pi(\pi^\#(\exp_{G\times G}(X, Y)) \Phi) = \pi(\exp_{G}(X + Y)) \circ \text{Op}^\pi(\pi(\exp_{G} X)^{-1}) = (\cdot | \pi(\exp_{G} X)\phi_2) \pi(\exp_{G}(X + Y))\phi_1.$$

In particular, $\text{Op}^\pi(\pi^\#(\exp_{G\times G}(X, Y)) \Phi) \in \mathcal{S}_1(\mathcal{H})$ and

$$\|\text{Op}^\pi(\pi^\#(\exp_{G\times G}(X, Y)) \Phi)\|_1 = ||\pi(\exp_{G}(X + Y))\phi_1|| \cdot ||\pi(\exp_{G} X)\phi_2|| = 1.$$ 

It then follows that the integral in (2.21) is absolutely convergent in $\mathcal{S}_1(\mathcal{H})$ for $a \in M^{1,1}_\Phi(\pi^\#)$ and moreover we have

$$\|\text{Op}^\pi(a)\|_1 \leq \int \int |(\mathcal{A}_{\Phi}^\# a)(X, Y)| \, dX \, dY = \|a\|_{M^{1,1}_\Phi(\pi^\#)}$$

which concludes the proof. \qed

3. The case of square-integrable representations

In this section we focus on square-integrable representations of nilpotent Lie groups. A discussion of the crucial role of these representations along with many examples can be found for instance in the monograph [CG90].
3.1. Independence of the modulation spaces on the window vectors.

**Lemma 3.1.** Let $G_1$ and $G_2$ be unimodular Lie groups and assume that we have a group homomorphism $\alpha: G_1 \rightarrow \text{Aut} G_2$, $g_1 \mapsto \alpha_{g_1}$. Consider the semidirect product $G = G_1 \ltimes_{\alpha} G_2$ and for every $h \in G$ and $\phi: G \rightarrow \mathbb{C}$ define $R_h \phi: G \rightarrow \mathbb{C}$, $(R_h \phi)(g) = \phi(gh)$. Fix $r, s \in [1, \infty]$ and consider the mixed-norm space $L^{r,s}(G)$ consisting of the equivalence classes of functions $\phi: G \rightarrow \mathbb{C}$ such that

$$\|\phi\|_{L^{r,s}(G)} := \left( \int_{G_1} \left( \int_{G_2} |\phi(g_1, g_2)|^r dg_2 \right)^{s/r} dg_1 \right)^{1/s} < \infty,$$

with the usual conventions if $r$ or $s$ is infinite. Then the space $L^{r,s}(G)$ is invariant under the right-translation operator $R_h$ for every $h \in G$, and the mapping

$$\rho: G \rightarrow \mathcal{B}(L^{r,s}(G)), \quad h \mapsto R_h|_{L^{r,s}(G)}$$

is a strongly continuous representation of the Lie group $G$ by isometries on the Banach space $L^{r,s}(G)$.

**Proof.** Let $\phi: G \rightarrow \mathbb{C}$ be any measurable function and $h = (h_1, h_2) \in G$. We have $(R_h \phi)(g_1, g_2) = \phi(g_1 h_1, \alpha_{h_1}^{-1}(g_2) h_2)$. Since the group $G_1$ is unimodular, it then follows that for every $g_2 \in G_2$ we have

$$\int_{G_1} |(R_h \phi)(g_1, g_2)|^r dg_1 = \int_{G_1} |\phi(g_1 h_1, \alpha_{h_1}^{-1}(g_2) h_2)|^r dg_1 = \int_{G_1} |\phi(g_1, \alpha_{h_1}^{-1}(g_2) h_2)|^r dg_1.$$

Now, by integrating on $G_2$ both extreme terms in above equality and taking into account that $G_2$ is unimodular, we get

$$(\forall h \in G) \| R_h \phi \|_{L^{r,s}(G)} = \| \phi \|_{L^{r,s}(G)}.$$

With this equality at hand, it is straightforward to prove all the assertions in the statement just as in the classical case when $r = s$. \hfill $\square$

**Remark 3.2.** In the setting of Lemma 3.1, for every $\psi \in L^1(G)$ we can define the bounded linear operator $\rho(\psi): L^{r,s}(G) \rightarrow L^{r,s}(G)$ by

$$(\forall \chi \in L^{r,s}(G)) \quad \rho(\psi)\chi = \int_G \psi(h) R_h \chi dh.$$

Then for every $\chi \in L^{r,s}(G)$ we have

$$(\rho(\psi)\chi)(g) = \int_G \chi(gh)\psi(h) dh \text{ for a.e. } g \in G$$

and

$$\|\rho(\psi)\phi\|_{L^{r,s}(G)} \leq C \|\phi\|_{L^{r,s}(G)},$$

where $C$ denotes the norm of the operator $\rho(\psi)$, hence $C \leq \|\psi\|_{L^1(G)}$. \hfill $\square$

We are now ready to prove a theorem that covers many cases when the modulation spaces for square-integrable representations do not depend on the choice of a window function. The second stage in the proof is inspired by the methods of the theory of coorbit spaces (see [FG88], [FG89a], [FG89b], and also the proof of Prop. 11.3.2(c) in [Gr01]).
Theorem 3.3. Let $G_1$ and $G_2$ be simply connected, nilpotent Lie groups and a unipotent homomorphism $\alpha: G_1 \to \text{Aut} G_2$. Define $G = G_1 \ltimes_{\alpha} G_2$ and assume that the center $\mathfrak{z}$ of $\mathfrak{g}$ satisfies the condition

$$\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{g}_1) + (\mathfrak{z} \cap \mathfrak{g}_2).$$

(3.1)

Assume the irreducible representation $\pi: G \to B(\mathcal{H})$ is square integrable modulo the center of $G$, and pick any Jordan-Hölder basis in $\mathfrak{g}$ such that for the corresponding predual $\mathfrak{g}_c$ for the coadjoint orbit associated with $\pi$ we have $\mathfrak{g}_c = (\mathfrak{g}_c \cap \mathfrak{g}_1) + (\mathfrak{g}_c \cap \mathfrak{g}_2)$.

Then the modulation spaces for the representation $\pi$ with respect to the decomposition $\mathfrak{g}_c \simeq (\mathfrak{g}_c \cap \mathfrak{g}_1) \times (\mathfrak{g}_c \cap \mathfrak{g}_2)$ are independent on the choice of a window vector $\phi \in \mathcal{H}_\infty \setminus \{0\}$.

Proof. The proof has two stages.

1° For the sake of simplicity let us identify the Lie group $G_j$ to its Lie algebra $\mathfrak{g}_j$ by means of the exponential map $\exp_{G_j}$, so that $G_j$ will be just $\mathfrak{g}_j$ with the group operation $\ast$ defined by the Baker-Campbell-Hausdorff series. Let $Z$ be the center of $G$, whose Lie algebra $\mathfrak{z}$ is the center of $\mathfrak{g}$. Then we have a linear isomorphism $\mathfrak{g}_c \simeq \mathfrak{g}/\mathfrak{z}$; $X \mapsto X + \mathfrak{z}$, and we shall endow $\mathfrak{g}_c$ with the Lie algebra structure which makes this map into an isomorphism of Lie algebras.

If we define $G_c := G/Z$, then $G_c$ is a connected, simply connected nilpotent Lie group, whose Lie algebra is just $\mathfrak{g}_c$. Let $\ast_c$ denote the multiplication in $G_c$, which is just the Baker-Campbell-Hausdorff multiplication in $\mathfrak{g}_c$.

Now use assumption (3.1) to see that if $(Y_1, Y_2) \in \mathfrak{z} \subseteq \mathfrak{g} = \mathfrak{g}_1 \ltimes \mathfrak{g}_2$, then 

$(Y_1,0),(0,Y_2) \in \mathfrak{z}$. Now formula (2.2) shows that for every $(X_1, X_2) \in \mathfrak{g}$ we have 

$$0 = [(X_1, X_2), (Y_1, 0)] = ([X_1, Y_1], -\dot{\alpha}(Y_1)X_2),$$

hence $Y_1$ belongs to the center $\mathfrak{z}_1$ of $\mathfrak{g}_1$ and $\dot{\alpha}(Y_1) = 0$. This shows that the closed subgroup $Z_1 := Z \cap G_1$ is contained in the center of $G_1$ and satisfies 

$$Z_1 \subseteq \text{Ker} \alpha.$$ (3.2)

Also $0 = [(X_1, X_2), (0, Y_2)] = (0, \dot{\alpha}(X_1)Y_2 + [X_2, Y_2])$ for every $(X_1, X_2) \in \mathfrak{g}$, whence we see that $Y_2$ belongs both to the center $\mathfrak{z}_2$ of $G_2$ and to Ker $\dot{\alpha}(X_1))$ for arbitrary $X_1 \in \mathfrak{g}_1$. Therefore the closed subgroup $Z_2 := Z \cap G_2$ is contained in the center of $G_2$ and we have 

$$(\forall g_1 \in G_1) \quad \alpha_{g_1}(Z_2) \subseteq Z_2.$$ (3.3)

It follows by (3.2) and (3.3) that the group homomorphism $\alpha: G_1 \to \text{Aut} G_2$ induces a group homomorphism $\alpha: G_1/Z_1 \to \text{Aut} (G_2/Z_2)$ and we have the isomorphisms of Lie groups 

$$G_c \simeq G/Z \simeq (G_1/Z_1) \ltimes_{\alpha} (G_2/Z_2).$$

Moreover $Z \simeq Z_1 \times Z_2$.

2° We now come back to the proof. Fix $r, s \in [1, \infty]$ and let $\phi_1, \phi_2 \in \mathcal{H}_\infty$ be any window functions with $\|\phi_1\| = \|\phi_2\| = 1$. For $j = 1, 2$ and every $f \in \mathcal{H}_{-\infty}$ we have by Corollary 2.4.

$$(A_{\phi_j} f)(X) = (f \mid \pi(\exp_G X)\phi_2)$$

$$= \int_{\mathfrak{z}} \chi(Y)(\pi(\exp_G Y)\phi_1 \mid \pi(\exp_G X)\phi_2) dY$$

$$= \int_{\mathfrak{z}} \chi(Y)(\phi_1 \mid \pi(\exp_G ((-Y) \ast X))\phi_2) dY$$
3.2. Covariance properties of the Weyl-Pedersen calculus.

Assume that the representation \( \pi \) of the Heisenberg group we recover a classical fact (see e.g., [Gr01]). Now note that by Lemma 3.1 and Remark 3.2 that there exists a constant \( C > 0 \) such that for every \( f \in H \) we have \( \|A_{\phi_2}f\|_{L^{r,s}(g,\mathfrak{g}_{c})} \leq \|A_{\phi_1}f\|_{L^{r,s}(g,\mathfrak{g}_{c})} \). Thus we get the continuous inclusion map \( M_{r,s}^{r,s}(\pi) \hookrightarrow M_{r,s}^{r,s}(\pi) \). Now the conclusion follows by interchanging \( \phi_1 \) and \( \phi_2 \).

The previous theorem allows us to omit the window vector in the notation for modulation spaces associated to square-integrable representations.

**Example 3.4.** Theorem 3.3 applies to a wide variety of situations. Let us mention here just a few of them:

1. In the case of the Schrödinger representation of the Heisenberg group \( H_{2n+1} = \mathbb{R}^n \ltimes \mathbb{R}^{n+1} \) we recover the well-known property that the classical modulation spaces used in the time-frequency analysis are independent on the choice of a window function (see for instance Prop. 11.3.2(c) in [Gr01]).

2. We shall see below (see subsection 3.3) that one can give sufficient conditions for the continuity of the operators constructed by the Weyl-Pedersen calculus for the square-integrable representation \( \pi: G \rightarrow \mathcal{B}(H) \) by using spaces of symbols which are modulation spaces \( M_{r,s}^{r,s}(\pi) \) and \( \mathcal{B}(L^2(\mathcal{O})) \) is in turn a square integrable representation to which Theorem 3.3 applies and ensures that the corresponding modulation spaces do not depend on the choice of a window function.

3.2. Covariance properties of the Weyl-Pedersen calculus. We now record the covariance property for the cross-Wigner distributions and its consequence for the Weyl-Pedersen calculus. In the very special case of the Schrödinger representation for the Heisenberg group we recover a classical fact (see e.g., [Gr01]).

**Theorem 3.5.** Assume that the representation \( \pi: G \rightarrow \mathcal{B}(H) \) associated with \( \mathcal{O} \) is square integrable modulo the center of \( G \). Then the following assertions hold:

1. For every \( f, h \in H \) and \( X \in \mathfrak{g} \) we have
   \[
   W(\pi(\exp G X)f, \pi(\exp G X)h)(\xi) = W(f, h)(\xi \circ e^{ad_a X}) \quad \text{for a.e. } \xi \in \mathcal{O}.
   \]
2. For every symbol \( a \in \mathcal{S}'(\mathcal{O}) \) and arbitrary \( g \in G \) we have
   \[
   \text{Op}(a \circ Ad_{\pi(g^{-1})}(g^{-1})) = \pi(g)\text{Op}(a)\pi(g)^{-1}.
   \]

**Proof.** Note that the following assertions hold:

\[
(\forall X \in \mathfrak{g}) \quad \pi(\exp G X) = e^{i\langle \xi_0, X \rangle} \text{id}_H,
\]

(3.4)
\[
(\forall \xi \in \mathcal{O}) \quad |\xi|_3 = \xi|_3, \quad (3.5) \\
|\xi|_{g_e} = 0. \quad (3.6)
\]

Also, it easily follows by Definition 2.7 that for arbitrary \( f, h \in \mathcal{H} \) we have
\[
\mathcal{W}(f, h)(\xi) = \int_{\mathfrak{g}_e} e^{i\langle \xi, X \rangle} (f \mid \pi((\exp_{G})h))dX \\
\text{for a.e. } \xi \in \mathcal{O}.
\]

It then follows that for arbitrary \( X_0 \in \mathfrak{g}_e \) and a.e. \( \xi \in \mathcal{O} \) we have
\[
\mathcal{W}(\pi((\exp_{G})f),\pi((\exp_{G})h))(\xi) \\
= \int_{\mathfrak{g}_e} e^{i\langle \xi, X \rangle} (f \mid \pi((\exp_{G}(\exp_{-X_0}))h))dX \\
= \int_{\mathfrak{g}_e} e^{i\langle \xi, X \rangle} (f \mid \pi(\exp_{G}(e^{\text{ad}_{\xi}},(-X_0)))h))dX.
\]

If we denote by \( \text{pr}_j : \mathfrak{g} \rightarrow \mathfrak{j} \) the natural projection corresponding to the direct sum decomposition \( \mathfrak{g} = \mathfrak{j} + \mathfrak{g}_e \), then we have for every \( \xi \in \mathfrak{g}_e \),
\[
e^{\text{ad}_{\xi}}(X_0) = e^{\text{ad}_{\xi}(-X_0)}X_0 + \text{pr}_j(e^{\text{ad}_{\xi}}(-X_0)X),
\]
where we have endowed \( \mathfrak{g}_e \) with the Lie algebra structure which makes the linear isomorphism \( \mathfrak{g}_e \simeq \mathfrak{j} / \mathfrak{h} \) into an isomorphism of Lie algebras (see also [Ma07]). Therefore, by using (3.5) and (3.6), we get
\[
(\forall \xi \in \mathfrak{g}_e) \quad \pi((\exp_{G}(e^{\text{ad}_{\xi}}(-X_0)))\pi(\exp_{G}(e^{\text{ad}_{\xi}}(-X_0)))
\]

and then the above computation leads to
\[
\mathcal{W}(\pi((\exp_{G})f),\pi((\exp_{G})h))(\xi) \\
= \int_{\mathfrak{g}_e} e^{i\langle \xi, X \rangle} e^{-i\langle \xi, \pi((\exp_{G}(\exp_{-X_0}))h) \rangle}dX \\
= \int_{\mathfrak{g}_e} e^{i\langle \xi, X \rangle} e^{-i\langle (e^{\text{ad}_{\xi}}(-X_0)), \xi_0, Y \rangle}dX \\
= \int_{\mathfrak{g}_e} e^{i\langle \xi, e^{\text{ad}_{\xi}}(-X_0)Y \rangle} e^{-i\langle (e^{\text{ad}_{\xi}}(-X_0)), \xi_0, e^{\text{ad}_{\xi}}(-X_0)Y \rangle}dY
\]

where we used the change of variables \( X \mapsto Y = e^{\text{ad}_{\xi}}(-X_0)X \), which is a measure-preserving diffeomorphism since \( \mathfrak{g}_e \) is a nilpotent Lie algebra. Now note that by using (3.5) we get for a.e. \( \xi \in \mathcal{O} \) and every \( \xi \in \mathfrak{g}_e \),
\[
\langle \xi, e^{\text{ad}_{\xi}}(-X_0)Y \rangle - \langle (e^{\text{ad}_{\xi}}(-X_0)), \xi_0, e^{\text{ad}_{\xi}}(-X_0)Y \rangle
\]
\[
= \langle \xi, e^{\text{ad}_{\xi}}(-X_0)Y \rangle - \langle \xi, \text{pr}_j(e^{\text{ad}_{\xi}}(-X_0)Y) \rangle \\
- \langle \xi_0, e^{\text{ad}_{\xi}}(-X_0)(e^{\text{ad}_{\xi}}(-X_0)Y - \text{pr}_j(e^{\text{ad}_{\xi}}(-X_0)Y)) \rangle
\]
\[
= \langle \xi, e^{\text{ad}_{\xi}}(-X_0)Y \rangle - \langle \xi_0, e^{\text{ad}_{\xi}}X_0Y \rangle \\
- \langle \xi_0, Y \rangle + \langle \xi_0, e^{\text{ad}_{\xi}}X_0Y \rangle
\]
\[
= \langle \xi, e^{\text{ad}_{\xi}}X_0Y \rangle
\]
since \( \langle \xi_0, Y \rangle = 0 \) by (3.6). Thus the conclusion follows by the formula we had obtained above for \( W(\pi(\exp_G X_0)f, \pi(\exp_G X_0)h)(\xi) \), and this completes the proof for \( X \in \mathfrak{g} \). Then the formula extends to arbitrary \( X \in \mathfrak{g} \) by using the fact that \( \mathfrak{g} = \mathfrak{g}_c + \mathfrak{z} \) and taking into account (3.3).

(2) If \( a \in \mathcal{S}(\mathcal{O}) \), then for every \( f, \phi \in \mathcal{H} \) we have

\[
(\text{Op}(a \circ \text{Ad}_G^*(g^{-1}))|_{\mathcal{O}})\phi \mid f)_{\mathcal{H}} = (a \circ \text{Ad}_G^*(g^{-1})|_{\mathcal{O}}) W(f, \phi)_{L^2(\mathcal{O})}
\]

\[
= (a \mid W(f, \phi) \circ \text{Ad}_G^*(g)|_{\mathcal{O}})_{L^2(\mathcal{O})}
\]

\[
= (a \mid W(\pi(g)^{-1} f, \pi(g)^{-1} \phi))_{L^2(\mathcal{O})}
\]

\[
= (\text{Op}(a) \pi(g)^{-1} \phi \mid \pi(g)^{-1} f)_{\mathcal{H}}
\]

\[
= (\pi(g) \text{Op}(a) \pi(g)^{-1} \phi \mid f)_{\mathcal{H}},
\]

where the first and the fourth equalities follow by Corollary 2.11, the second equality is a consequence of the fact that the coadjoint action preserves the Liouville measure on \( \mathcal{O} \), while the third equality follows by Assertion 11 which we already proved.

Thus we obtained the conclusion for \( a \in \mathcal{S}(\mathcal{O}) \), and then it can be easily extended by duality to any \( a \in \mathcal{S}'(\mathcal{O}) \) by using equation (2.10) in Definition 2.10. \( \square \)

3.3. Continuity of Weyl-Pedersen calculus.

**Lemma 3.6.** Let \( G \) be any Lie group with a unitary irreducible representation \( \pi: G \to \mathcal{B}(\mathcal{H}) \) and

\[
\bar{\pi}: G \ltimes G \to \mathcal{B}(G_2(\mathcal{H})), \quad \bar{\pi}(g, h)T = \pi(gh)T \pi(g)^{-1}.
\]

If \( G \) is a unimodular group and \( \pi \) is square integrable modulo the center of \( G \), then \( \bar{\pi} \) is square integrable modulo the center of \( G \ltimes G \).

**Proof.** If \( \pi \) is square integrable modulo the center \( Z \) of \( G \), then there is \( \phi_0 \in \mathcal{H} \setminus \{0\} \) such that the function \( gZ : \{ \pi(g)\phi_0 \mid \phi_0 \} \) is square integrable on \( G/Z \). Let us define the rank-one projection \( T_0 = (\cdot | \phi_0)\phi_0 \). Then we have

\[
\iint_{(G \times G)/(Z \times Z)} |(\bar{\pi}(g, h)T_0 | T_0)^2 | g d\sigma d\tau = \int_{G/Z} \left( \int_{G/Z} |(\pi(h)T \pi(g)^{-1} | T_0)^2 | d\sigma \right) d\tau
\]

\[
= \int_{G/Z} \left( \int_{G/Z} |(\pi(h)T \pi(g)^{-1} | T_0)^2 | d\sigma \right) d\tau.
\]

Since \( T_0 = (\cdot | \phi_0)\phi_0 \), we get \( \pi(h)T \pi(g)^{-1} = (\cdot | \pi(g)\phi_0)\pi(h)\phi_0 \), and then

\[
(\pi(h)T \pi(g)^{-1} | T_0) = (\pi(h)\phi_0 | \phi_0) \cdot (\phi_0 | \pi(g)\phi_0).
\]

Therefore

\[
\iint_{(G \times G)/(Z \times Z)} |(\bar{\pi}(g, h)T_0 | T_0)^2 | g d\sigma d\tau = \left( \int_{G/Z} \left( |(\pi(g)\phi_0 | \phi_0)^2 | d\sigma \right)^2 \right)
\]

hence the function \( (g, h)(Z \times Z) \mapsto |(\bar{\pi}(g, h)T_0 | T_0)| \) is square integrable on the quotient group \( (G \ltimes G)/(Z \times Z) \), and this concludes the proof since \( Z \times Z \) is the center of \( G \ltimes G \) (see Example 2.3). \( \square \)
Remark 3.7. Assume that \( \pi: G \to B(\mathcal{H}) \) is a square-integrable representation of a simply connected, nilpotent Lie group, with the corresponding coadjoint orbit \( \mathcal{O} \subseteq g^* \). Recall the representation \( \pi^\#: G \times G \to B(L^2(\mathcal{O})) \) from Definition 2.19 (see also Remark 2.20). The assumption that \( \pi \) is square integrable modulo the center of \( G \) implies by Theorem 3.3 that \( \pi^\# \) is given by

\[
\pi^\#: G \times G \to B(L^2(\mathcal{O})), \quad \pi^\#(g, \exp Y) f = (\exp^{i(Y)} \# f) \circ \text{Ad}_G^{-1}(g^{-1})|_{\mathcal{O}}.
\]

Since the unitary operator \( \text{Op}^\#: L^2(\mathcal{O}) \to \mathcal{S}_2(\mathcal{H}) \) intertwines \( \pi^\# \) and \( \pi \), we get by Lemma 3.6 that \( \pi^\# \) is also a unitary irreducible representation which is square integrable modulo the center \( Z \times Z \) of \( G \times G \).

Proof. Firstly use Corollary 2.24. Then the conclusion follows since both \( \pi \) and \( \pi^\# \) are square integrable representations (see also Remark 3.7), hence Theorem 3.3 shows that the topologies of the modulation spaces \( M^{r_1, r_2}(\pi) \), \( M^{r_2, r_1}(\pi) \), and \( M^{r_1, \infty}(\pi^\#) \) can be defined by any special choice of window functions.

Corollary 3.8. Let \( G \) be a simply connected, nilpotent Lie group with a unitary irreducible representation \( \pi: G \to B(\mathcal{H}) \) which is square integrable modulo the center of \( G \). If \( r, r_1, r_2 \in [1, \infty] \) and \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \), then the cross-Wigner distribution associated with any predual to the coadjoint of the representation \( \pi \) defines a continuous sesquilinear map

\[
\mathcal{W}(\cdot, \cdot): M^{r_1, r_2}(\pi) \times M^{r_2, r_1}(\pi) \to M^{r_1, \infty}(\pi^\#).
\]

Proof. Just apply Corollary 3.8 with \( r_1 = r_2 = 2 \) and \( r = 1 \); and recall from Example 2.17 that \( M^{2,2}(\pi) = \mathcal{H} \).

In the special case of the Schrödinger representation for the Heisenberg group, the following corollary recovers the assertion of Th. 1.1 in [GH99] concerning the boundedness of pseudo-differential operators defined by the classical Weyl-Hörmander calculus on \( \mathbb{R}^n \).

Corollary 3.9. If \( G \) be a simply connected, nilpotent Lie group with a unitary irreducible representation \( \pi: G \to B(\mathcal{H}) \) which is square integrable modulo the center of \( G \), then the cross-Wigner distribution associated with any predual to the coadjoint of the representation \( \pi \) defines a continuous sesquilinear map

\[
\mathcal{W}(\cdot, \cdot): \mathcal{H} \times \mathcal{H} \to M^{1, \infty}(\pi^\#).
\]

Proof. Firstly use Corollary 3.8 with \( r = 1 \) and \( r_1 = r_2 = 2 \) and \( r_1 = 1 \); and recall from Example 2.17 that \( M^{2,2}(\pi) = \mathcal{H} \).

Corollary 3.10. Let \( G \) be a simply connected, nilpotent Lie group with a unitary irreducible representation \( \pi: G \to B(\mathcal{H}) \) which is square integrable modulo the center of \( G \). If \( r, r', r_1, r_2 \in [1, \infty] \) satisfy the equations \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = 1 - \frac{1}{r'} \), then the following assertions hold:

1. For every symbol \( a \in M^{r', 1}(\pi^\#) \) we have a bounded linear operator

\[
\text{Op}^\#(a): M^{r_1, r_1}(\pi) \to M^{r_2, r_2}(\pi).
\]

2. The linear mapping \( \text{Op}^\#(\cdot): M^{r', 1}(\pi^\#) \to B(M^{r_1, r_1}(\pi), M^{r_2, r_2}(\pi)) \) is continuous.

Proof. Firstly use Corollary 3.8. Then the conclusion follows since Theorem 3.3 shows that the topologies of the modulation spaces involved in the statement can be defined by any special choice of window functions.
Corollary 3.11. If $G$ be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ which is square integrable modulo the center of $G$, then the following assertions hold:

(1) For every $a \in M^{\infty,1}(\pi^\#)$ we have $\text{Op}^\pi(a) \in \mathcal{B}(\mathcal{H})$.

(2) The linear mapping $\text{Op}^\pi(\cdot) : M^{\infty,1}(\pi^\#) \rightarrow \mathcal{B}(\mathcal{H})$ is continuous.

Proof. This is the special case of Corollary 3.10 with $r_1 = r_2 = 2$ and $r = 1$, since Example 2.17 shows that $M^{2,2}(\pi) = \mathcal{H}$. □

4. Schrödinger representations of the Heisenberg groups

We show in the present section that, in the special case of the Heisenberg group, the modulation spaces of symbols defined in our paper are in fact nothing else than the modulation spaces widely used in time-frequency analysis.

Schrödinger representations. Let $\mathcal{V}$ be a finite-dimensional vector space endowed with a nondegenerate bilinear form denoted by $(p, q) \mapsto p \cdot q$. The corresponding Heisenberg algebra $\mathfrak{h}_\mathcal{V} = \mathcal{V} \times \mathcal{V} \times \mathbb{R}$ is the Lie algebra with the bracket

$$[(q, p, t), (q', p', t')] = [(0, 0, p \cdot q' - p' \cdot q)].$$

The Heisenberg group $\mathbb{H}_\mathcal{V}$ is just $\mathfrak{h}_\mathcal{V}$ thought of as a group with the multiplication $*$ defined by

$$X * Y = X + Y + \frac{1}{2} [X, Y].$$

The unit element is $0 \in \mathbb{H}_\mathcal{V}$ and the inversion mapping given by $X^{-1} := -X$. The Schrödinger representation is the unitary representation $\pi_\mathcal{V} : \mathbb{H}_\mathcal{V} \rightarrow \mathcal{B}(L^2(\mathcal{V}))$ defined by

$$(\pi_\mathcal{V}(q, p, t)f)(x) = e^{i(p x + p \cdot q + t)} f(x + q) \quad \text{for a.e. } x \in \mathcal{V} \quad (4.1)$$

for arbitrary $f \in L^2(\mathcal{V})$ and $(q, p, t) \in \mathbb{H}_\mathcal{V}$. This is a square-integrable representation and the corresponding coadjoint orbit of $\mathbb{H}_\mathcal{V}$ is

$$\mathcal{O} = \{ \xi : \mathfrak{h}_\mathcal{V} \rightarrow \mathbb{R} \text{ linear} \mid \xi(0, 0, 1) = 1 \}. \quad (4.2)$$

Let $\xi_0 \in \mathcal{O}$ be the functional satisfying $\xi_0(q, p, 0) = 0$ for every $q, p \in \mathcal{V}$. If we denote $\dim \mathcal{V} = n$, then any basis $\{x_1, \ldots, x_n\}$ in $\mathcal{V}$ naturally gives rise to the Jordan-Hölder basis

$$(x_1, 0, 0), \ldots, (x_n, 0, 0), (0, x_1, 0), \ldots, (0, x_n, 0), (0, 0, 1)$$

in $\mathfrak{h}_\mathcal{V}$ and the corresponding predual of $\mathcal{O}$ is

$$(\mathfrak{h}_\mathcal{V})_c = \mathcal{V} \times \mathcal{V} \times \{0\}.$$

For the sake of an easier comparison with the previously obtained results we shall denote $G = \mathbb{H}_\mathcal{V}$ and $\mathfrak{g} = \mathfrak{h}_\mathcal{V}$ from now on, and in particular we shall denote $\mathfrak{g}_c = (\mathfrak{h}_\mathcal{V})_c$. 
Computing the Moyal product representation. Recall from [Ma07] that for every \( f, h \in \mathcal{S}(\mathcal{O}) \) we have

\[
(f \# h)(\xi) = \int_{\mathfrak{g} \times \mathfrak{g}} e^{i(\xi, X + Y)} e^{i(1/2)\langle \xi_0, [X, Y] \rangle} \hat{f}(X) \hat{h}(Y) dX dY.
\]

It then follows by a duality argument that for every \( f \in \mathcal{S}(\mathcal{O}) \) and \( V \in \mathfrak{g} \) we have

\[
(f \# e^{-i(\cdot, V)})(\xi) = \int_{\mathfrak{g}} e^{i(\xi, X - V)} e^{i(1/2)\langle \xi_0, [X, -V] \rangle} \hat{f}(X) dX,
\]

whence

\[
(\forall V \in \mathfrak{g})(\forall \xi \in \mathcal{O}) \quad (f \# e^{-i(\cdot, V)})(\xi) = e^{-i\langle \xi, V \rangle} f(\xi + (1/2)\xi_0 \circ \text{ad}_{\mathfrak{g}} V).
\] (4.3)

Since \( f \# h = h \# f \), we also get

\[
(\forall V \in \mathfrak{g})(\forall \xi \in \mathcal{O}) \quad (e^{i(\cdot, V)} \# f)(\xi) = e^{i\langle \xi, V \rangle} f(\xi + (1/2)\xi_0 \circ \text{ad}_{\mathfrak{g}} V).
\] (4.4)

Now for arbitrary \( X, Y \in \mathfrak{g}, f \in \mathcal{S}(\mathcal{O}) \), and \( \xi \in \mathcal{O} \) we get

\[
(e^{i(\cdot, X + Y)} \# f)(\xi) = e^{i\langle \xi, X + Y \rangle} (f \# e^{-i(\cdot, X)})(\xi + (1/2)\xi_0 \circ \text{ad}_{\mathfrak{g}} (X + Y))
\]

\[
= e^{i\langle \xi, X + Y \rangle} e^{-i\langle \xi + (1/2)\xi_0 \circ \text{ad}_{\mathfrak{g}} (X + Y), X \rangle}
\]

\[
\times f(\xi + (1/2)\xi_0 \circ \text{ad}_{\mathfrak{g}} (X + Y)) + (1/2)\xi_0 \circ \text{ad}_{\mathfrak{g}} X)
\]

\[
\times f(\xi + (1/2)\xi_0 \circ \text{ad}_{\mathfrak{g}} (X + (2/2)Y)).
\]

By taking into account (2.13), we now see that the unitary irreducible representation \( \pi^\#: G \rtimes G \to \mathcal{B}(L^2(\mathcal{O})) \) is given by

\[
(\pi^\#(\exp_{G \times G}(X, Y)) f)(\xi) = e^{i\langle \xi, Y \rangle + \langle \xi_0, [X, Y] \rangle/2} f(\xi + \xi_0 \circ \text{ad}_{\mathfrak{g}} (X + (2/2)Y))
\] (4.5)

where the latter equation follows by (1.2).

Abstract unitary equivalence. Denote the center of \( G \) by \( Z \), with the corresponding Lie algebra \( \mathfrak{z} \). The above formula yields \( \exp_{G \times G}(\mathfrak{z} \times \{0\}) \subseteq \text{Ker} \pi^\# \), hence we get a unitary irreducible representation \( \pi^\#: (G \times G)/\{Z \times \{1\}\} \to \mathcal{B}(L^2(\mathcal{O})) \). Also note that there exist the natural isomorphisms of Lie groups

\[
(G \times G)/\{Z \times \{1\}\} \simeq (G/Z) \times G \simeq \mathbb{H}_{V \times V}.
\] (4.6)

By specializing (1.5) for \( X, Y \in \mathfrak{z} \) we can see that the representation \( \pi^\# \) has the same central character as the Schrödinger representation of the Heisenberg group \( \mathbb{H}_{V \times V} \), hence they are unitarily equivalent to each other, as a consequence of the Stone-von Neumann theorem.

Specific unitary equivalence. Alternatively, we can exhibit an explicit unitary equivalence as follows. Let us consider the affine isomorphism \( \mathcal{O} \to (V \times V)^*, \xi \mapsto \xi|_{V \times V \times \{0\}} \), and the natural embedding \( V \times V \simeq V \times V \times \{0\} \hookrightarrow \mathfrak{h}_V \). Now for \( X, Y \in V \times V \) and \( t \in \mathbb{R} \) we have \( (X, Y, t) \in \mathbb{H}_{V \times V} \simeq (G \times G)/\{Z \times \{1\}\} \) (see (4.6)) hence

\[
(\pi^\#(\exp_{\mathbb{H}_{V \times V}}(X, Y, t)) f)(\xi) = e^{i\langle \xi, Y \rangle + t + \omega_{\xi_0}(X, Y)/2} f(\xi + \xi_0 \circ \text{ad}_{\mathfrak{g}} (X + (2/2)Y))
\]
where $\omega_0(V, W) := \langle \xi_0, [V, W] \rangle$ whenever $V, W \in \mathcal{V} \times \mathcal{V} \hookrightarrow \mathfrak{g}$. Thence we get

$$\pi_V^\#(\exp_{\mathbb{H} \times \mathbb{V}}(X - (1/2)Y, Y, t)) f(\xi) = e^{i(\langle \xi, Y \rangle + t + \omega_0(X,Y)/2)} f(\xi + \xi_0 \circ \text{ad}_g X).$$

Note that $\psi : \mathfrak{h}_{\mathcal{V} \times \mathcal{V}} \to \mathfrak{h}_{\mathcal{V} \times \mathcal{V}}$, $(X, Y, t) \mapsto (X - (1/2)Y, Y, t)$ is an automorphism of the Heisenberg algebra $\mathfrak{h}_{\mathcal{V} \times \mathcal{V}}$, hence, by denoting by $\Psi : \mathbb{H} \times \mathcal{V} \to \mathbb{H} \times \mathcal{V}$ the corresponding automorphism of the Heisenberg group $\mathbb{H} \times \mathcal{V}$, we get

$$(\pi(\exp_{\mathbb{H} \times \mathbb{V}}(X, Y, t))) f(\xi) = e^{i(\langle \xi, Y \rangle + \omega_0(X,Y)/2 + t)} f(\xi + \omega_0(X,Y)/2 + t)$$

where $\pi := \pi_V^\# \circ \Psi$ is again a representation of the Heisenberg group $\mathbb{H} \times \mathcal{V}$. Then for arbitrary $V \in \mathcal{V} \times \mathcal{V}$ we get

$$(\pi(\exp_{\mathbb{H} \times \mathcal{V}}(X, Y, t))) f(\xi + \omega_0(X,Y)/2 + t) \times f(\xi + \omega_0(V,X)/2 + t)$$

(4.7)

Now let us define the affine isomorphism

$$A : \mathcal{V} \times \mathcal{V} \to \mathcal{O}, \quad V \mapsto \xi_0 + \omega_0 \circ \text{ad}_g V$$

and consider the unitary operator $U : L^2(\mathcal{V} \times \mathcal{V}) \to L^2(\mathcal{O})$, $f \mapsto f \circ A^{-1}$. It follows by the above equation that if we define the Heisenberg group $\mathbb{H} \times \mathcal{V}$ by using the nondegenerate bilinear map

$$(\mathcal{V} \times \mathcal{V}) \times (\mathcal{V} \times \mathcal{V}) \to \mathbb{R}, \quad (V, W) \mapsto -\omega_0(V, W),$$

then the unitary operator $U$ intertwines the representation $\pi : \mathbb{H} \times \mathcal{V} \to \mathcal{B}(L^2(\mathcal{O}))$ and the Schrödinger representation $\pi_{\mathcal{V} \times \mathcal{V}} : \mathbb{H} \times \mathcal{V} \to \mathcal{B}(L^2(\mathcal{V} \times \mathcal{V}))$. In other words, the operator $U$ induces a unitary equivalence of the representation $\pi_V^\#$ with the representation $\pi_{\mathcal{V} \times \mathcal{V}} \circ \Psi^{-1}$.

**Determining the modulation spaces of symbols.** It follows by the above discussion that the operator $U$ induces isomorphisms between the modulation spaces for the representations

$$\pi_V^\# : G \times G \to \mathcal{B}(L^2(\mathcal{O})) \quad \text{and} \quad \pi_{\mathcal{V} \times \mathcal{V}} \circ \Psi^{-1} : \mathbb{H} \times \mathcal{V} \to \mathcal{B}(L^2(\mathcal{V} \times \mathcal{V})).$$

Now note that for arbitrary $r, s \in [1, \infty]$ we have $M^{r,s}(\pi_{\mathcal{V} \times \mathcal{V}} \circ \Psi^{-1}) = M^{r,s}(\pi_{\mathcal{V} \times \mathcal{V}})$ since the norm of any measurable function $f : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ in $L^{r,s}(\mathcal{V} \times \mathcal{V})$ is equal to the norm of the function $(X, Y) \mapsto f((X + (1/2)Y, Y))$ in the same space. Therefore the operator $f \mapsto f \circ A^{-1}$ actually induces an isomorphism from the modulation space $M^{r,s}(\pi_V^\#)$ onto the modulation space $M^{r,s}(\pi_V)$ of the Schrödinger representation $\pi_{\mathcal{V} \times \mathcal{V}} : \mathbb{H} \times \mathcal{V} \to \mathcal{B}(L^2(\mathcal{V} \times \mathcal{V}))$. Finally, recall that $M^{r,s}(\pi_{\mathcal{V} \times \mathcal{V}}) = M^{r,s}(\mathcal{V} \times \mathcal{V})$, where the latter is just the classical modulation space on $\mathcal{V} \times \mathcal{V}$ as used for instance in [Gr01].

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