T-systems, Y-systems, and cluster algebras: Tamely laced case

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Dedicated to Professor Tetsuji Miwa on his 60th birthday

The T-systems and Y-systems are classes of algebraic relations originally associated with quantum affine algebras and Yangians. Recently they were generalized to quantum affinizations of quantum Kac-Moody algebras associated with a wide class of generalized Cartan matrices which we say tamely laced. Furthermore, in the simply laced case, and also in the nonsimply laced case of finite type, they were identified with relations arising from cluster algebras. In this note we generalize such an identification to any tamely laced Cartan matrices, especially to the nonsimply laced ones of nonfinite type.

Keywords: T-systems; Y-systems; quantum groups; cluster algebras

1. Introduction
The T-systems and Y-systems appear in various aspects for integrable systems. Originally, the T-systems are systems of relations among the Kirillov-Reshetikhin modules in the Grothendieck rings of modules over quantum affine algebras and Yangians. The T and Y-systems are related to each other by certain changes of variables. See, for example, Ref. 1 and references therein for more information and background.

Let $I = \{1, \ldots, r\}$ and let $C = (C_{ij})_{i,j \in I}$ be a (generalized) Cartan matrix in Ref. 2; namely, it satisfies $C_{ij} \in \mathbb{Z}$, $C_{ii} = 2$, $C_{ij} \leq 0$ for any $i \neq j$, and $C_{ij} = 0$ if and only if $C_{ji} = 0$. We assume that $C$ is symmetrizable, i.e., there is a diagonal matrix $D = \text{diag}(d_1, \ldots, d_r)$ with $d_i \in \mathbb{N} := \mathbb{Z}_{>0}$ such that $B = DC$ is symmetric. We always assume that there is no common divisor for $d_1, \ldots, d_r$ except for 1. Following Ref. 4, we say that a Cartan matrix $C$ is tamely laced if it is symmetrizable and satisfies the following
condition due to Hernandez:\(^3\)

\[
\text{If } C_{ij} < -1, \text{ then } d_i = -C_{ji} = 1. \quad (1)
\]

Recently, the T-systems were generalized by Hernandez\(^3\) to the quantum affinizations of the quantum Kac-Moody algebras associated with *tamely laced* Cartan matrices. Subsequently, the corresponding Y-systems were also introduced by Kuniba, Suzuki, and the author.\(^3\)

Remarkably, these T and Y-systems are identified with (a part of) relations among the variables for *cluster algebras*,\(^5,6\) which are a class of commutative algebras closely related to the representation theory of quivers. For the T and Y-systems associated with *simply laced* Cartan matrices of *finite type*, this identification is a topic intensively studied by various authors recently with several reasons (periodicity, categorification, positivity, dilogarithm identities, etc.)\(^1,6–16\) In Ref. 4, such an identification was generalized to the *simply laced* Cartan matrices. In Refs. 15 and 16, it was also extended to the *nonsimply laced* Cartan matrices of *finite type*.

In this note we present a generalization of the above identification to *any tamely laced* Cartan matrices, especially to the *nonsimply laced ones of nonfinite type*, thereby justifying Sec. 6.5 of Ref. 4 which announced that such a generalization is possible. Basically it is a straightforward extension of the simply laced ones\(^4\) and the nonsimply laced ones of finite type,\(^15,16\) but it is necessarily more complicated. At this time we do not have any immediate application of such a generalization. However, we believe that this is a necessary step toward further study of the intriguing interplay of two worlds — the representation theories of quantum groups and quivers — through cluster algebras.

2. T and Y-systems

In this section we recall the definitions of (restricted) T and Y-systems. See Ref. 4 for more detail.

With a tamely laced Cartan matrix \(C\), we associate a *Dynkin diagram* \(X(C)\) in the standard way: For any pair \(i \neq j \in I\) with \(C_{ij} < 0\), the vertices \(i\) and \(j\) are connected by max\(\{|C_{ij}|, |C_{ji}|\}\) lines, and the lines are equipped with an arrow from \(j\) to \(i\) if \(C_{ij} < -1\). Note that the condition (1) means

(i) the vertices \(i\) and \(j\) are not connected, if \(d_i, d_j > 1\) and \(d_i \neq d_j\),
(ii) the vertices \(i\) and \(j\) are connected by \(d_i\) lines with an arrow from \(i\) to \(j\) or not connected, if \(d_i > 1\) and \(d_j = 1\),
(iii) the vertices \(i\) and \(j\) are connected by a single line or not connected, if \(d_i = d_j\).
As usual, we say that a Cartan matrix $C$ is simply laced if $C_{ij} = 0$ or $-1$ for any $i \neq j$. If $C$ is simply laced, then it is tamely laced.

For a tamely laced Cartan matrix $C$, we set integers $t$ and $t_a$ $(a \in I)$ by

$$t = \text{lcm}(d_1, \ldots, d_r), \quad t_a = \frac{t}{d_a},$$

(2)

For an integer $\ell \geq 2$, we set

$$I_\ell := \{(a, m, u) \mid a \in I; m = 1, \ldots, t_a \ell - 1; u \in \frac{1}{\ell} \mathbb{Z}\}.$$  

(3)

For $a, b \in I$, we write $a \sim b$ if $C_{ab} < 0$, i.e., $a$ and $b$ are adjacent in $X(C)$.

First, we introduce the $T$-systems and the associated rings.

**Definition 2.1.** Fix an integer $\ell \geq 2$. For a tamely laced Cartan matrix $C$, the level $\ell$ restricted $T$-system $T_\ell(C)$ associated with $C$ (with the unit boundary condition) is the following system of relations for a family of variables $T_\ell = \{T^{(a)}_m(u) \mid (a, m, u) \in I_\ell\}$,

$$T^{(a)}_m(u - \frac{d_a}{t}) T^{(a)}_m(u + \frac{d_a}{t}) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + \prod_{b \sim a} T^{(b)}_{m,m}(u) \quad \text{if } d_a > 1,$$

(4)

$$T^{(a)}_m(u - \frac{d_a}{t}) T^{(a)}_m(u + \frac{d_a}{t}) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + \prod_{b \sim a} S^{(b)}_{m}(u) \quad \text{if } d_a = 1,$$

(5)

where $T^{(a)}_0(u) = 1$, and furthermore, $T^{(a)}_{t_a \ell}(u) = 1$ (the unit boundary condition) if they occur in the right hand sides in the relations. The symbol $S^{(b)}_m(u)$ is defined as follows. For $m = 0, 1, 2, \ldots$ and $0 \leq j < d_b$,

$$S^{(b)}_{d_m+j}(u) = \left\{ \prod_{k=1}^{j} T^{(b)}_{m+1}\left(u + \frac{1}{\ell}(j + 1 - 2k)\right) \right\} \times \left\{ \prod_{k=1}^{d_m-j} T^{(b)}_{m+1}\left(u + \frac{1}{\ell}(d_b - j + 1 - 2k)\right) \right\}.$$  

(6)

For the later use, let us formally write (4) and (5) in a unified manner

$$T^{(a)}_m(u - \frac{d_a}{t}) T^{(a)}_m(u + \frac{d_a}{t}) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + \prod_{(b,k,v) \in \mathcal{I}_\ell} T^{(b)}_{k}(v) G^{(b,k,v;a,m,u)}.$$  

(7)

**Definition 2.2.** Let $\mathcal{T}_\ell(C)$ be the commutative ring over $\mathbb{Z}$ with identity element, with generators $T^{(a)}_m(u)^{\pm 1}$ $(a, m, u) \in I_\ell$ and relations $T_\ell(C)$
together with $T_m^{(a)}(u)T_m^{(a)}(u)^{-1} = 1$. Let $\mathcal{T}_l^1(C)$ be the subring of $\mathcal{T}_l(C)$ generated by $T_m^{(a)}(u)$ ($\ell, m, u \in \mathcal{I}_l$).

Similarly, we introduce the Y-systems and the associated groups.

**Definition 2.3.** Fix an integer $\ell \geq 2$. For a tamely laced Cartan matrix $C$, the **level $\ell$ restricted Y-system $\mathcal{Y}_\ell(C)$ associated with $C$** is the following system of relations for a family of variables $Y = \{Y_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}_\ell \}$,

$$Y_m^{(a)}(u - \frac{d_a}{t}) Y_m^{(a)}(u + \frac{d_a}{t}) = \frac{\prod_{b,b=\ell-a} Z_{m,n}^{(b)}(u)}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})} \text{ if } d_a > 1,$$

$$Y_m^{(a)}(u - \frac{d_a}{t}) Y_m^{(a)}(u + \frac{d_a}{t}) = \frac{\prod_{b,b=\ell-a} (1 + Y_{m}^{(b)}(u))}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})} \text{ if } d_a = 1,$$

where $Y_0^{(a)}(u)^{-1} = Y_{1,\ell}^{(a)}(u)^{-1} = 0$ if they occur in the right hand sides in the relations. Besides, $Y_{m/d_b}^{(b)}(u) = 0$ in (9) if $m/d_b \notin \mathbb{N}$. The symbol $Z_{m,n}^{(b)}(u)$ ($p \in \mathbb{N}$) is defined as follows.

$$Z_{m,n}^{(b)}(u) = \prod_{j=-p+1}^{p-1} \left\{ \prod_{k=1}^{p-j} \left( 1 + Y_{p+m+j}^{(b)}(u + \frac{1}{4}(p - |j| + 1 - 2k)) \right) \right\}. \tag{10}$$

One can write (8) and (9) in a unified manner as

$$Y_m^{(a)}(u - \frac{d_a}{t}) Y_m^{(a)}(u + \frac{d_a}{t}) = \prod_{(b,k,v) \in \mathcal{I}_\ell} \frac{(1 + Y_k^{(b)}(u))^{G(b,k,v;a,m,u)}}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}, \tag{11}$$

where $^tG(b,k,v;a,m,u) := G(a,m,u;b,k,v)$.

A **semifield** $(\mathbb{P}, \oplus)$ is an abelian multiplicative group $\mathbb{P}$ endowed with a binary operation of addition $\oplus$ which is commutative, associative, and distributive with respect to the multiplication in $\mathbb{P}$.

**Definition 2.4.** Let $\mathcal{Y}_\ell(C)$ be the semifield with generators $Y_m^{(a)}(u)$ ($\ell, m, u \in \mathcal{I}_\ell$) and relations $\mathcal{Y}_\ell(C)$. Let $\mathcal{Y}_\ell^1(C)$ be the multiplicative subgroup of $\mathcal{Y}_\ell(C)$ generated by $Y_m^{(a)}(u)$, $1 + Y_m^{(a)}(u)$ ($\ell, m, u \in \mathcal{I}_\ell$). (Here we use the symbol $+ \oplus$ instead of $\oplus$ for simplicity.)
3. Cluster algebra with coefficients

In this section we recall the definition of cluster algebras with coefficients following Ref. 6. The description here is minimal to fix convention and notion. See Ref. 6 for more detail and information.

Let \( I \) be a finite set, and let \( B = (B_{ij})_{i,j \in I} \) be a skew symmetric (integer) matrix. Let \( x = (x_i)_{i \in I} \) and \( y = (y_i)_{i \in I} \) be \( I \)-tuples of formal variables. Let \( \mathbb{P} = \mathbb{Q}(y) \) be the universal semifield of \( y = (y_i)_{i \in I} \), namely, the semifield consisting of the subtraction-free rational functions of \( y \) with usual multiplication and addition (but no subtraction) in the rational function field \( \mathbb{Q}(y) \). Let \( \mathbb{Q}\mathbb{P} \) denote the quotient field of the group ring \( \mathbb{ZP} \) of \( \mathbb{P} \).

For the above triplet \((B, x, y)\), called the initial seed, the cluster algebra \( \mathcal{A}(B, x, y) \) with coefficients in \( \mathbb{P} \) is defined as follows.

Let \((B', x', y')\) be a triplet consisting of skew symmetric matrix \( B' \), an \( I \)-tuple \( x' = (x'_i)_{i \in I} \) with \( x'_i \in \mathbb{Q}\mathbb{P}(x) \), and an \( I \)-tuple \( y' = (y'_i)_{i \in I} \) with \( y'_i \in \mathbb{P} \). For each \( k \in I \), we define another triplet \((B'', x'', y'') = \mu_k(B', x', y')\), called the mutation of \((B', x', y')\) at \( k \), as follows.

(i) Mutations of matrix.

\[
B''_{ij} = \begin{cases} 
-B'_{ij} & i = k \text{ or } j = k, \\
B'_{ij} + \frac{1}{2}(B'_{ik}B'_{kj} + B'_{ik}B'_{kj}) & \text{otherwise}.
\end{cases}
\]  

(ii) Exchange relation of coefficient tuple.

\[
y''_i = \begin{cases} 
y'_i^{-1} & i = k, \\
y'_i \left( \frac{y'_k}{1 + y'_k} \right)^{B'_{ki}} & i \neq k, \; B'_{ki} \geq 0, \\
y'_i(1 + y'_k)^{-B'_{ki}} & i \neq k, \; B'_{ki} \leq 0.
\end{cases}
\]

(iii) Exchange relation of cluster.

\[
x''_i = \begin{cases} 
y'_k \prod_{j:B'_{ji} > 0} x'_j^{B'_{jk}} + \prod_{j:B'_{ji} < 0} x'_j^{-B'_{jk}} & i = k, \\
(1 + y'_k)x'_k & i \neq k.
\end{cases}
\]

It is easy to see that \( \mu_k \) is an involution, namely, \( \mu_k(B'', x'', y'') = (B', x', y') \). Now, starting from the initial seed \((B, x, y)\), iterate mutations and collect all the resulted triplets \((B', x', y')\). We call \((B', x', y')\) the seeds, \(y'\) and \(y'_k\) a coefficient tuple and a coefficient, \(x'\) and \(x'_k\), a cluster and a cluster variable, respectively. The cluster algebra \( \mathcal{A}(B, x, y) \) with coefficients in \( \mathbb{P} \) is the \( \mathbb{ZP} \)-subalgebra of the rational function field \( \mathbb{Q}\mathbb{P}(x) \) generated by all the cluster variables. Similarly, the coefficient group \( G(B, y) \) associated
with $A(B, x, y)$ is the multiplicative subgroup of the semifield $\mathbb{P}$ generated by all the coefficients $y'_i$ together with $1 \oplus y'_i$.

It is standard to identify a skew-symmetric (integer) matrix $B = (B_{ij})_{i,j \in I}$ with a quiver $Q$ without loops or 2-cycles. The set of the vertices of $Q$ is given by $I$, and we put $B_{ij}$ arrows from $i$ to $j$ if $B_{ij} > 0$. The mutation $Q'' = \mu_k(Q')$ of a quiver $Q'$ is given by the following rule: For each pair of an incoming arrow $i \rightarrow k$ and an outgoing arrow $k \rightarrow j$ in $Q'$, add a new arrow $i \rightarrow j$. Then, remove a maximal set of pairwise disjoint 2-cycles. Finally, reverse all arrows incident with $k$.

4. Cluster algebraic formulation: The case $|I| = 2$; $t$ is odd

4.1. Cartan matrix $M_t$

We are going to identify $T_\ell(C)$ and $Y_\ell(C)$ as relations for cluster variables and coefficients of the cluster algebra associated with a certain quiver $Q_\ell(C)$.

To begin with, we consider the case $I = \{1, 2\}$, which will be used as building blocks of the general case. Without loss of generality we may assume that a Cartan matrix $C$ is indecomposable, i.e., $X(C)$ is connected. Thus, we assume that our tamely laced Cartan matrix $C$ has the form $(t = 1, 2, \ldots)$

$$C = M_t := \begin{pmatrix} 2 & -1 \\ -t & 2 \end{pmatrix}, \quad D = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}. \quad (15)$$

We have the data $d_1 = t$, $d_2 = 1$, $t = \text{lcm}(d_1, d_2)$, $t_1 = 1$, $t_2 = t$, and the corresponding Dynkin diagram looks as follows (2 lines in the middle and there is no arrow for $t = 1$):

```
  1  2
```

We ask the reader to refer to Refs. 4, 15, and 16, where the cases $t = 1$ (type $A_2$), 2 (type $B_2$), and 3 (type $G_2$), respectively, are treated in detail.

It turns out that we should separate the problem depending on the parity of $t$. In this section we consider the case when $t$ is odd.

4.2. Parity decompositions of $T$ and $Y$-systems

For a triplet $(a, m, u) \in \mathcal{I}_t$, we set the parity conditions $P_+$ and $P_-$ by

$$P_+ : m + tu \text{ is odd for } a = 1; \quad m + tu \text{ is even for } a = 2,$$

$$P_- : m + tu \text{ is even for } a = 1; \quad m + tu \text{ is odd for } a = 2. \quad (16)$$
We write, for example, \((a, m, u) : \mathbf{P}_+\) if \((a, m, u)\) satisfies \(\mathbf{P}_+\). We have \(I_\ell = I_{\ell+} \sqcup I_{\ell-}\), where \(I_{\ell\pm}\) is the set of all \((a, m, u) : \mathbf{P}_\pm\). Define \(\mathcal{T}_\ell^\circ (M_t)\) \((\varepsilon = \pm)\) to be the subring of \(\mathcal{T}_\ell^\circ (M_t)\) generated by \(T_m^\circ (u)\) \(\((a, m, u) \in I_{\ell\pm}\)\). Then, we have \(\mathcal{T}_\ell^\circ (M_t)_{+} \simeq \mathcal{T}_\ell^\circ (M_t)_{-}\) by \(T_m^\circ (u) \mapsto T_m^\circ (u + \frac{1}{t})\) and

\[
\mathcal{T}_\ell^\circ (M_t) \simeq \mathcal{T}_\ell^\circ (M_t)_{+} \otimes_{\mathbb{Z}} \mathcal{T}_\ell^\circ (M_t)_{-}. \tag{17}
\]

For a triplet \((a, m, u) \in I_\ell\), we introduce another parity conditions \(\mathbf{P}'_+\) and \(\mathbf{P}'_-\) by

\[
\mathbf{P}'_+ : m + tu \text{ is even for } a = 1; m + tu \text{ is odd for } a = 2,
\]
\[
\mathbf{P}'_- : m + tu \text{ is odd for } a = 1; m + tu \text{ is even for } a = 2. \tag{18}
\]

Since \(\mathbf{P}'_\pm = \mathbf{P}_\mp\), it may seem redundant, but we use this notation to make the description unified for both odd and even \(t\). We have

\[
(a, m, u) : \mathbf{P}'_+ \iff (a, m, u \pm \frac{d}{t}) : \mathbf{P}_+ . \tag{19}
\]

Let \(I_{\ell\pm}\) be the set of all \((a, m, u) : \mathbf{P}'_\pm\). Define \(Y^\circ_\ell (M_t)\) \((\varepsilon = \pm)\) to be the subring of \(Y^\circ_\ell (M_t)\) generated by \(Y_m^\circ (u), 1 + Y_m^\circ (u)\) \(\((a, m, u) \in I_{\ell\pm}\)\). Then, we have \(Y^\circ_\ell (M_t)_{+} \simeq Y^\circ_\ell (M_t)_{-}\) by \(Y_m^\circ (u) \mapsto Y_m^\circ (u + \frac{1}{t}), 1 + Y_m^\circ (u) \mapsto 1 + Y_m^\circ (u + \frac{1}{t})\), and

\[
Y^\circ_\ell (M_t) \simeq Y^\circ_\ell (M_t)_{+} \times Y^\circ_\ell (M_t)_{-}. \tag{20}
\]

### 4.3. **Quiver** \(Q_\ell (M_t)\)**

With the Cartan matrix \(M_t\) and \(\ell \geq 2\) we associate a quiver \(Q_\ell (M_t)\) as below. First, as a rather general example, the case \(t = 5\) is given in Fig. 1, where the right columns in the five quivers \(Q_1, \ldots, Q_5\) are identified. Also we assign the empty or filled circle \(\circ/\bullet\) and the sign \(+/−\) to each vertex as shown. For a general odd \(t\), the quiver \(Q_\ell (M_t)\) is defined by naturally extending the case \(t = 5\). Namely, we consider \(t\) quivers \(Q_1, \ldots, Q_t\). In each quiver \(Q_i\) there are \(\ell - 1\) vertices (with \(\circ\)) in the left column and \(t \ell - 1\) vertices (with \(\bullet\)) in the right column. The arrows are put as clearly indicated by the example in Fig. 1. The right columns in all the quivers \(Q_1, \ldots, Q_t\) are identified.

Let us choose the index set \(I\) of the vertices of \(Q_\ell (M_t)\) so that \(i = (i, i') \in I\) represents the vertex at the \(i\)'th row (from the bottom) of the left column in \(Q_i\) for \(i = 1, \ldots, t\), and the one of the right column in any quiver for \(i = t + 1\). Thus, \(i = 1, \ldots, t + 1\), and \(i' = 1, \ldots, \ell - 1\) if \(i \neq t + 1\) and \(i' = 1, \ldots, t \ell - 1\) if \(i = t + 1\).
we identify the right columns in all the quivers $Q_{\ell}(M_t)$ with $t=5$ for even \( \ell \) (upper) and for odd \( \ell \) (lower), where we identify the right columns in all the quivers $Q_1, \ldots, Q_5$. 
For $k \in \{1, \ldots, t\}$, let $\mathbf{I}_{+,k}$ (resp. $\mathbf{I}_{-,k}$) denote the set of the vertices $i$ in $Q_k$ with property $\circ$ and $+$ (resp. $\circ$ and $-$). Similarly, let $\mathbf{I}_{+}$ (resp. $\mathbf{I}_{-}$) denote the set of the vertices $i$ with property $\bullet$ and $+$ (resp. $\bullet$ and $-$). We define composite mutations,

$$
\mu_{+,k}^\circ = \prod_{i \in \mathbf{I}_{+,k}} \mu_i, \quad \mu_{-,k}^\circ = \prod_{i \in \mathbf{I}_{-,k}} \mu_i, \quad \mu_{+}^\circ = \prod_{i \in \mathbf{I}_{+}} \mu_i, \quad \mu_{-}^\circ = \prod_{i \in \mathbf{I}_{-}} \mu_i.
$$

(21)

Note that they do not depend on the order of the product.

For a permutation $w$ of $\{1, \ldots, t\}$, let $\tilde{w}$ be the permutation of $\mathbf{I}$ such that $\tilde{w}(i, i') = (w(i), w(i'))$ for $i \neq t+1$ and $(t+1, i')$ for $i = t+1$. Let $\tilde{w}(Q_t(M_t))$ denote the quiver induced from $Q_t(M_t)$ by $\tilde{w}$. Namely, if there is an arrow $i \to j$ in $Q_t(M_t)$, then, there is an arrow $\tilde{w}(i) \to \tilde{w}(j)$ in $\tilde{w}(Q_t(M_t))$. For a quiver $Q$, let $Q^{\text{op}}$ denote the opposite quiver.

**Lemma 4.1.** Let $Q(0) := Q_t(M_t)$. We have the following periodic sequence of mutations of quivers:

$$
Q(0) \xleftarrow{\mu_{+,1}^\circ, \mu_{-,1}^\circ} Q_{(1)} \xleftrightarrow{\mu_{+,2}^\circ, \mu_{-,2}^\circ} Q(3) \xleftrightarrow{\mu_{+,3}^\circ, \mu_{-,3}^\circ} Q(4) \xleftrightarrow{\mu_{+,4}^\circ, \mu_{-,4}^\circ} Q(2) = Q(0).
$$

(22)

Here, the quiver $Q(p/t)$ $(p = 1, \ldots, 2t)$ is defined by

$$
Q(p/t) := \begin{cases} 
\tilde{w}_p(Q(0))^{\text{op}} & \text{p: odd} \\
\tilde{w}_p(Q(0)) & \text{p: even},
\end{cases}
$$

(23)

and $w_p$ is a permutation of $\{1, \ldots, t\}$ defined by

$$
w_p = \begin{cases} 
(1 + r_1, \ldots, 1 + r_p) & \text{p: odd} \\
(r_1, \ldots, r_p) & \text{p: even},
\end{cases}
$$

(24)

where $(ij)$ is the transposition of $i$ and $j$.

**Proof.** Let $Q_1, \ldots, Q_t$ be the subquivers in the definition of $Q_t(M_t)$ as in Fig. 1. By the sequence of mutations (22), one can easily check that $Q_1$ mutates as

$$
Q_1 \leftrightarrow Q_1^{\text{op}} \leftrightarrow Q_2 \leftrightarrow Q_3^{\text{op}} \leftrightarrow \cdots Q_t^{\text{op}} \leftrightarrow Q_1 \leftrightarrow \cdots Q_3 \leftrightarrow Q_2^{\text{op}} \leftrightarrow Q_1.
$$

(26)
$Q_2$ mutates as

$$Q_2 \leftrightarrow Q_3^{op} \leftrightarrow \cdots \leftrightarrow Q_t^{op} \leftrightarrow Q_t \leftrightarrow \cdots \leftrightarrow Q_2^{op} \leftrightarrow Q_2,$$  \hfill (27)

$Q_3$ mutates as

$$Q_3 \leftrightarrow Q_2^{op} \leftrightarrow Q_1 \leftrightarrow Q_1^{op} \leftrightarrow \cdots \leftrightarrow Q_t^{op} \leftrightarrow Q_t \leftrightarrow \cdots \leftrightarrow Q_3^{op} \leftrightarrow Q_3,$$  \hfill (28)

and so on. The result is summarized as (23).

**Example 4.2.** The mutation sequence (22) for $t = 5$ is explicitly given in Figs. 2 and 3, where only a part of each quiver is presented. (Caution: the mutations of the top and bottom arrows may look erroneous but they are correct because of the effect from the omitted part.) The encircled vertices are the mutation points of (22) in the forward direction.

### 4.4. Embedding maps

Let $B = B_t(M_t)$ be the skew-symmetric matrix corresponding to the quiver $Q_t(M_t)$. Let $A(B, x, y)$ be the cluster algebra with coefficients in the universal semifield $\mathbb{Q}_{sf}(y)$, and let $G(B, y)$ be the coefficient group associated with $A(B, x, y)$ as in Section 3.

In view of Lemma 4.1 we set $x(0) = x$, $y(0) = y$ and define clusters $x(u) = (x_i(u))_{i \in \mathcal{I}}$ ($u \in \mathbb{Z}/t\mathbb{Z}$) and coefficient tuples $y(u) = (y_i(u))_{i \in \mathcal{I}}$ ($u \in \mathbb{Z}/t\mathbb{Z}$) by the sequence of mutations

$$
\cdots \xrightarrow{\mu^i_{t+1}, \mu^i_{t}} (B(0), x(0), y(0)) \xrightarrow{\mu^i_{1}, \mu^i_{0}} (B(1), x(1), y(1)) \xrightarrow{\mu^i_{1}, \mu^i_{0}, t-1} \cdots \xrightarrow{\mu^i_{1}, \mu^i_{0}} (B(2), x(2), y(2)) \xrightarrow{\mu^i_{1}, \mu^i_{0}} \cdots,
\hfill (29)
$$

where $B(u)$ is the skew-symmetric matrix corresponding to $Q(u)$.

For $(i, u) \in \mathcal{I} \times \mathbb{Z}/t\mathbb{Z}$, we set the parity condition $p_+$ by

$$p_+ : \begin{cases} 
  i \in \mathcal{I}_+, \quad u \equiv \frac{p}{t}, \quad 0 \leq p \leq t - 1, \quad p: \text{even} \\
  i \in \mathcal{I}_+, \quad u \equiv \frac{p}{t}, \quad 0 \leq p \leq t - 1, \quad p: \text{odd} \\
  i \in \mathcal{I}_+, \quad u \equiv \frac{t}{t}, \quad t \leq p \leq 2t - 1, \quad p: \text{even} \\
  i \in \mathcal{I}_+, \quad u \equiv \frac{p}{t}, \quad t \leq p \leq 2t - 1, \quad p: \text{odd}
\end{cases}
\hfill (30)
$$

where $\equiv$ is modulo $2\mathbb{Z}$. We define the condition $p_-$ by $(i, u) : p_- \iff (i, u - 1/t) : p_+$. Plainly speaking, each $(i, u) : p_+$ (resp. $p_-$) is a mutation point of (29) in the forward (resp. backward) direction of $u$.

There is a correspondence between the parity condition $p_+$ here and $P_+, P'_+$ in (16).
Fig. 2. (Continues to Fig. 3.) The mutation sequence of the quiver $Q_t(M_t)$ in (22) for $t = 5$. Only a part of each quiver is presented. The encircle vertices correspond to the mutation points in the forward direction.

**Lemma 4.3.** Below $\equiv$ means the equivalence modulo $2\mathbb{Z}$.

(i) The map $g : \mathcal{I}_+ \rightarrow \{(1, u) : p_+\}$

$(a, m, u - \frac{d}{2}) \mapsto \begin{cases} ((2j + 1, m), u) & a = 1; m + u \equiv \frac{2j}{t} \\ (j = 0, 1, \ldots, (t-1)/2) \end{cases}$

$(2t - 2j, m), u) & a = 1; m + u \equiv \frac{2j}{t} \\ (j = (t+1)/2, \ldots, t-1) \end{cases}$

$((t + 1, m), u) & a = 2$
is a bijection.

(ii) The map \( g' : \mathcal{I}'_+ \to \{(i, u) : p_+\} \)

\[
(a, m, u) \mapsto \begin{cases} 
((2j + 1, m), u) & a = 1; m + u \equiv \frac{2j}{t} \\
 & (j = 0, 1, \ldots, (t - 1)/2) \\
((2t - 2j, m), u) & a = 1; m + u \equiv \frac{2j}{t} \\
 & (j = (t + 1)/2, \ldots, t - 1) \\
((t + 1, m), u) & a = 2 
\end{cases}
\]  

(32)

is a bijection.
Proof. The fact (i) is equivalent to (ii) due to (19). So, it is enough to prove (ii). Let us examine the meaning of the map (32) in the case \( t = 5 \) with Fig. 2. Each encircled vertex therein corresponds to \((i, u) : p_+\), and some \((a, m, u)\) is attached to it by \(g'\). For example, in \( Q(0)\), \((1, m, 0)\) \((m: \text{even})\) are attached to the vertices with \((o, +)\) in the first quiver (from the left), and \((2, m, 0)\) \((m: \text{odd})\) are attached to the vertices with \((\bullet, +)\). Similarly, in \( Q(1/5)\), \((1, m, 1/5)\) \((m: \text{odd})\) are attached to the vertices with \((o, +)\) in the fourth quiver (from the left), and \((2, m, 1/5)\) \((m: \text{even})\) are attached to the vertices with \((\bullet, -)\). Then, one can easily confirm that \(g'\) is indeed a bijection. A general case is verified similarly. □

We introduce alternative labels \(x_1(u) = x_m^{(a)}(u-d_a/t)\), \((a, m, u-d_a/t) \in I_{t+}\) for \((i, u) = g((a, m, u-d_a/t))\) and \(y_1(u) = y_m^{(a)}(u)\), \((a, m, u) \in I_{t+}\) for \((i, u) = g'((a, m, u))\), respectively.

Remark 4.4. In the case \( t = 1 \), i.e., the simply laced case, the map \(g\) in Lemma 4.3 reads \((a, m, u) \mapsto ((a, m), u + 1)\), thus, differs from the simpler one \((a, m, u) \mapsto ((a, m), u)\) used in Refs. 1 and 4. Either will serve as a natural parametrization and the transferring from one to the other is easy.

4.5. T-system and cluster algebra

We show that the T-system \(T_{t}(M_t)\) naturally appears as a system of relations among the cluster variables \(x_i(u)\) in the trivial evaluation of coefficients. (The quiver \(Q_t(M_t)\) is designed to do so.) Let \(A(B, x)\) be the cluster algebra with trivial coefficients, where \((B, x)\) is the initial seed. Let \(1 = \{1\}\) be the trivial semifield and \(\pi_1 : \mathbb{Q}_{sf}(y) \to 1, y_1 \mapsto 1\) be the projection. Let \([x_1(u)]_1\) denote the image of \(x_1(u)\) by the algebra homomorphism \(A(B, x, y) \to A(B, x)\) induced from \(\pi_1\). It is called the trivial evaluation.

Lemma 4.5. Let \(G(b, k, v; a, m, u)\) be the one in (7). The family \(\{x_m^{(a)}(u) \mid (a, m, u) \in I_{t+}\}\) satisfies a system of relations

\[
x_m^{(a)}(u - \frac{d_a}{t}) x_m^{(a)}(u + \frac{d_a}{t}) = \frac{y_m^{(a)}(u)}{1 + y_m^{(a)}(u)} \prod_{(b, k, v) \in I_{t+}} x_k^{(b)}(v)^{G(b, k, v; a, m, u)} + \frac{1}{1 + y_m^{(a)}(u)} x_m^{(a)}(u) x_{m-1}^{(a)}(u) x_{m+1}^{(a)}(u),
\]

where \((a, m, u) \in I_{t+}\). In particular, the family \(\{[x_m^{(a)}(u)]_1 \mid (a, m, u) \in I_{t+}\}\) satisfies the T-system \(T_{t}(M_t)\) in \(A(B, x)\) by replacing \(T_m^{(a)}(u)\) with
Then, the attached variable \( x \) which should equal to \( x \)\( a, m, u \).

Let us demonstrate how to obtain these relations in the case \( t = 5 \) using Figs. 2 and 3. For example, consider the mutation at \((1, 2), 0 \). Then, the attached variable \( x_2^{(1)}(-1) \) is mutated to

\[
\frac{1}{x_2^{(1)}(-1)} \left\{ \frac{y_2^{(1)}(0)}{1 + y_2^{(1)}(0)} x_2^{(2)}(0) + \frac{1}{1 + y_2^{(1)}(0)} x_1^{(1)}(0) x_3^{(1)}(0) \right\},
\]

which should equal to \( x_2^{(1)}(1) \). Also, consider the mutation at, say, \((2, 9), 0 \). Then, the attached variable \( x_9^{(2)}(-1/5) \) is mutated to

\[
\frac{1}{x_9^{(2)}(-\frac{1}{5})} \left\{ \frac{y_9^{(2)}(0)}{1 + y_9^{(2)}(0)} x_1^{(1)}(0) x_2^{(1)}(0) - \frac{3}{5} x_1^{(1)}(-\frac{1}{5}) x_2^{(1)}(\frac{1}{5}) x_2^{(1)}(\frac{3}{5}) + \frac{1}{1 + y_9^{(2)}(0)} x_8^{(2)}(0) x_1^{(2)}(0) \right\},
\]

which should equal to \( x_9^{(2)}(1/5) \). They certainly agree with (4) and (5). (The quiver \( Q_\ell(M_i) \) is designed to do so.)

**Definition 4.6.** The \( T \)-subalgebra \( A_T(B, x) \) of \( A(B, x) \) associated with the sequence (29) is the subring of \( A(B, x) \) generated by \([x_i(u)]\) \((i, u) \in I \times \frac{1}{\tau} \mathbb{Z})

By Lemma 4.5, we have the following embedding.

**Theorem 4.7.** The ring \( T_{\ell}^\tau(M_i) \) is isomorphic to \( A_T(B, x) \) by the correspondence \( T_{\ell}^{(a)}(u) \rightarrow [x_m^{(a)}(u)]_1 \).

**Proof.** The map \( \rho : T_{\ell}^{(a)}(u) \rightarrow [x_m^{(a)}(u)]_1 \) is a ring homomorphism due to Lemma 4.5. We can construct the inverse of \( \rho \) as follows. For each \( i \in I \), let \( u_i \in \frac{1}{\tau} \mathbb{Z} \) be the smallest nonnegative \( u_i \) such that \((i, u) : p_+). Then, thanks to Lemma 4.3 (i) there is a unique \((a, m, u_i - d_a/t) \in \mathcal{I}_\ell^+ \) such that \( g((a, m, u_i - d_a/t)) = (i, u_i) \). We define a ring homomorphism \( \tilde{\varphi} : Z[x_1^{\pm 1}]_{i \in I} \rightarrow \mathcal{T}_\ell(M_i) \) by \( x_i^{\pm 1} \rightarrow T_{\ell}^{(a)}(u_i - d_a/t)^{\pm 1} \). Thus, we have \( \tilde{\varphi} : [x_m^{(a)}(u_i - d_a/t)]_1 \rightarrow T_{\ell}^{(a)}(u_i - d_a/t) \). Furthermore, one can prove that \( \tilde{\varphi} : [x_m^{(a)}(u)]_1 \rightarrow T_{\ell}^{(a)}(u) \) for any \((a, m, u) \in \mathcal{I}_\ell^+ \) by induction on the forward and backward mutations, applying the same T-systems for the both sides. By the the restriction of \( \tilde{\varphi} \) to \( A_T(B, x) \), we obtain a ring homomorphism \( \varphi : A_T(B, x) \rightarrow T_{\ell}^\tau(M_i)_+ \), which is the inverse of \( \rho \).
4.6. Y-system and cluster algebra

The Y-system $\mathcal{Y}_t(M_t)$ also naturally appears as a system of relations among the coefficients $y_t(u)$.

The following lemma follows from the exchange relation of coefficients and the property of the sequence (22).

**Lemma 4.8.** The family $\{y^{(a)}_m(u) \mid (a, m, u) \in \mathcal{I}_t^+\}$ satisfies the Y-system $\mathcal{Y}_t(M_t)$ by replacing $Y^{(a)}_m(u)$ with $y^{(a)}_m(u)$.

**Proof.** Again, let us demonstrate how to obtain these relations in the case $t = 5$ using Figs. 2 and 3. For example, consider the mutation at $((1, 2), 0)$. Then, the attached variable $y^{(1)}_2(0)$ is mutated to $y^{(1)}_2(0)^{-1}$. Then, the following factors are multiplied to $y^{(1)}_2(0)^{-1}$ during $u = \frac{1}{5}, \ldots, \frac{9}{5}$:

\[
(1 + y^{(2)}_1(\frac{5}{7})), \\
(1 + y^{(2)}_1(\frac{5}{7}))(1 + y^{(2)}_1(\frac{6}{7})), \\
(1 + y^{(2)}_2(\frac{3}{7}))(1 + y^{(2)}_2(\frac{5}{7}))(1 + y^{(2)}_2(\frac{7}{7})), \\
(1 + y^{(2)}_1(\frac{5}{7}))(1 + y^{(2)}_1(\frac{6}{7}))(1 + y^{(2)}_1(\frac{7}{7}))(1 + y^{(2)}_2(\frac{9}{7})), \\
(1 + y^{(2)}_2(\frac{5}{7}))(1 + y^{(2)}_2(\frac{7}{7}))(1 + y^{(2)}_2(\frac{9}{7})), \\
(1 + y^{(2)}_2(\frac{5}{7}))(1 + y^{(2)}_2(\frac{7}{7}))(1 + y^{(2)}_2(\frac{9}{7})), \\
(1 + y^{(2)}_1(\frac{5}{7}))(1 + y^{(2)}_1(\frac{6}{7}))(1 + y^{(2)}_1(\frac{7}{7}))(1 + y^{(2)}_2(\frac{9}{7})), \\
(1 + y^{(2)}_1(\frac{5}{7}))(1 + y^{(2)}_1(\frac{6}{7}))(1 + y^{(2)}_1(\frac{7}{7}))(1 + y^{(2)}_2(\frac{9}{7})).
\]

The result should equal to $y^{(1)}_2(2)$. Also, consider the mutation at, say, $((2, 9), 0)$. Then, the attached variable $y^{(2)}_9(0)$ is mutated to $y^{(2)}_9(0)^{-1}$. Then, the following factors are multiplied to $y^{(2)}_9(0)^{-1}$ at $u = \frac{1}{5}$:

\[
(1 + y^{(2)}_9(\frac{5}{7}))(1 + y^{(2)}_9(\frac{6}{7}))(1 + y^{(2)}_9(\frac{7}{7}))(1 + y^{(2)}_9(\frac{9}{7})).
\]

The result should equal to $y^{(2)}_9(\frac{5}{7})$. They certainly agree with (8) and (9)\[.\]

**Definition 4.9.** The Y-subgroup $\mathcal{G}_Y(B, y)$ of $\mathcal{G}(B, y)$ associated with the sequence (29) is the subgroup of $\mathcal{G}(B, y)$ generated by $y_t(u) \ (i, u) \in I \times \frac{1}{2} \mathbb{Z}$ and $1 + y_t(u) \ ((i, u) \in \mathcal{I}_t^+ \ or \ \mathcal{I}_t^-)$. 
By Lemma 4.8, we have the following embedding.

**Theorem 4.10.** The group $\mathcal{Y}_Y^0(M_t)_+$ is isomorphic to $\mathcal{G}_Y(B, y)$ by the correspondence $Y^0_m(u) \mapsto y^0_m(u)$ and $1 + Y^0_m(u) \mapsto 1 + y^0_m(u)$.

**Proof.** The map $\rho : Y^0_m(u) \mapsto y^0_m(u)$, $1 + Y^0_m(u) \mapsto 1 + y^0_m(u)$ is a group homomorphism due to Lemma 4.5. We can construct the inverse of $\rho$ as follows. For each $i \in I$, let $u_i \in \mathbb{Z}$ be the largest nonpositive $u_i$ such that $(i, u_i) : p_+$. Then, thanks to Lemma 4.3 (ii) there is a unique $(a, m, u_i) \in I^*_t$ such that $g'((a, m, u_i)) = (i, u_i)$. We define a semifield homomorphism $\tilde{\varphi} : Q_{sf}(y_i)_{i \in I} \rightarrow Y_f(M_t)$ as follows. If $u_i = 0$, then $y_i \mapsto Y^0_m(0)$. If $u_i < 0$, we define

$$
\tilde{\varphi}(y_i) = Y^0_m(u_i)^{-1} \prod_{(b, k, v)} \frac{1 + Y^0_k(v)}{1 + Y^0_k(v)^{-1}}.
$$

(37)

where the product in the numerator is taken for $(b, k, v) : I^*_t$ such that $u_i < v < 0$ and $B_H(v) < 0$ for $(j, v) = g'((b, k, v))$, and the product in the denominator is taken for $(b, k, v) : I^*_t$ such that $u_i < v < 0$ and $B_H(v) > 0$ for $(j, v) = g'((b, k, v))$. Then, we have $\tilde{\varphi} : y^0_m(u_i) \mapsto Y^0_m(u_i)$. Furthermore, one can prove that $\tilde{\varphi} : Y^0_m(u) \mapsto Y^0_m(u)$ for any $(a, m, u) \in I^*_t$ by induction on the forward and backward mutations, applying the same Y-systems for both sides. By the restriction of $\tilde{\varphi}$ to $\mathcal{G}_Y(B, x)$, we obtain a group homomorphism $\varphi : \mathcal{G}_Y(B, x) \rightarrow \mathcal{Y}_Y^0(M_t)_+$, which is the inverse of $\rho$. \hfill \Box

5. Cluster algebraic formulation: The case $|I| = 2; t$ is even

In this section we consider the case $|I| = 2$ when $t$ is even. Basically it is parallel to the former case and we omit proofs.

5.1. Parity decompositions of $T$ and $Y$-systems

For a triplet $(a, m, u) \in \mathcal{I}_t$, we reset the ‘parity conditions’ $P_+$ and $P_-$ by

$$
P_+ : tu \text{ is even if } a = 1; \ m + tu \text{ is even if } a = 2,
$$

$$
P_- : tu \text{ is odd if } a = 1; \ m + tu \text{ is odd if } a = 2.
$$

(38)

Let $\mathcal{I}_t$ be the set of all $(a, m, u) : P_\varepsilon$. Define $\mathcal{Y}_\varepsilon^0(M_t)$ ($\varepsilon = \pm$) to be the subring of $\mathcal{Y}_t^0(M_t)$ generated by $T_m^0(u)$ ($(a, m, u) \in \mathcal{I}_t$). Then, we have
\[ T^\circ_\ell(M_t) = T^\circ_\ell(M_t) \] holds.

For a triplet \((a, m, u) \in \mathcal{I}_\ell\), we set another ‘parity conditions’ \(P'_+\) and \(P'_-\) by

\[
P'_+: \quad tu \text{ is even if } a = 1; \quad m + tu \text{ is odd if } a = 2,
P'_-: \quad tu \text{ is odd if } a = 1; \quad m + tu \text{ is even if } a = 2.
\]

We have

\[ (a, m, u) : P'_+ \iff (a, m, u \pm \frac{a}{\ell}) : P_. \] (40)

Let \(\mathcal{I}'_\ell\) be the set of all \((a, m, u) : P'_\). Define \(Y^\circ_\ell(M_t) \) (\(\varepsilon = \pm\)) to be the subgroup of \(Y^\circ_\ell(M_t)\) generated by \(Y^\circ_t(a)(u), 1 + Y^\circ_t(a)(u) ((a, m, u) \in \mathcal{I}'_\ell)\). Then, we have \(Y^\circ_\ell(M_t)_+ \simeq Y^\circ_\ell(M_t)_-\) by \(Y^\circ_t(a)(u) \mapsto Y^\circ_t(a)(u + \frac{a}{\ell}), 1 + Y^\circ_t(a)(u) \mapsto 1 + Y^\circ_t(a)(u + \frac{a}{\ell})\), and the decomposition (20) holds.

5.2. Quiver \(Q_\ell(M_t)\)

With the Cartan matrix \(M_t\) and \(\ell \geq 2\) we associate the quiver \(Q_\ell(M_t)\).

As a rather general example, the case \(t = 4\) is given by Fig. 4, where the right columns in the four quivers \(Q_1, \ldots, Q_4\) are identified. Also we assign the empty or filled circle \(o/\bullet\) and the sign \(+/-\) to each vertex as shown. For a general even \(t\), the quiver \(Q_\ell(M_t)\) is defined by naturally extending the case \(t = 4\). Even though it looks quite similar to the odd \(t\) case in Fig. 1, there is one important difference due to the parity of \(t\); that is, when \(t\) is even, any vertex in the right column of \(Q_i\) has the sign ‘−’ whenever it is connected to a vertex in the left column by a horizontal arrow. This is not so when \(t\) is odd.

Let us choose the index set \(I\) of the vertices of \(Q_\ell(M_t)\) as before so that \(i = (i, i') \in I\) represents the vertex at the \(i'\)th row (from the bottom) of the left column in \(Q_i\) for \(i = 1, \ldots, t\), and the one of the right column in any quiver for \(i = t + 1\). We use the same notations \(I^\circ_{\pm,k} I^\bullet_{\pm}\) as before. For \(k, k' \in \{1, \ldots, t\}, k \neq k'\), let \(I^\circ_{\pm,k,k'} = I^\circ_{\pm,k} \sqcup I^\circ_{\pm,k'}\). We define composite mutations,

\[
\mu^\circ_{\pm,k,k'} = \prod_{i \in I^\circ_{\pm,k,k'}} \mu_i, \quad \mu^\circ_{-,k,k'} = \prod_{i \in I^\circ_{-,k,k'}} \mu_i, \quad \mu^\bullet = \prod_{i \in I^\bullet} \mu_i, \quad \mu^\circ = \prod_{i \in I^\circ} \mu_i.
\]

(41)

Lemma 5.1. Let \(Q(0) := Q_\ell(M_t)\). We have the following periodic sequence
we identify the right columns in all the quivers $Q_1, \ldots, Q_4$.

of mutations of quivers:

$$Q(0) \xrightarrow{\mu^*} Q\left(\frac{t+1}{t}\right) \xrightarrow{\mu^*} Q\left(\frac{t+2}{t}\right) \xrightarrow{\mu^*} Q\left(\frac{t+3}{t}\right) \xrightarrow{\mu^*} Q\left(\frac{t+4}{t}\right),$$

$$Q(1) \xrightarrow{\mu^*} Q\left(\frac{t+1}{t}\right) \xrightarrow{\mu^*} Q\left(\frac{t+2}{t}\right) \xrightarrow{\mu^*} Q\left(\frac{t+3}{t}\right) \xrightarrow{\mu^*} Q\left(\frac{t+4}{t}\right)$$

$$Q\left(\frac{t+1}{t}\right) \xrightarrow{\mu^*} Q\left(\frac{t+2}{t}\right) \xrightarrow{\mu^*} Q\left(\frac{t+3}{t}\right) \xrightarrow{\mu^*} Q\left(\frac{t+4}{t}\right)$$

$$Q(2) = Q(0).$$

(42)
Here, the quiver $Q(p/t)$ ($p = 1, \ldots, 2t$) is defined by

$$Q(p/t) := \begin{cases} \tilde{w}_p(Q) & p: \text{odd} \\ \check{w}_p(Q) & p: \text{even}, \end{cases}$$

(43)

and $w_p$ is a permutation of $\{1, \ldots, t\}$ defined by

$$w_p = \begin{cases} r_+r_- \cdots r_+ & (p \text{ terms}) \quad p: \text{odd} \\ r_+r_- \cdots r_- & (p \text{ terms}) \quad p: \text{even}, \end{cases}$$

(44)

$$r_+ = (23)(45) \cdots (r-2, r-1), \quad r_- = (12)(34) \cdots (r-1, r),$$

(45)

where $(ij)$ is the transposition of $i$ and $j$.

**Example 5.2.** The mutation sequence (42) for $t = 4$ is explicitly given in Fig. 5, where only a part of each quiver is presented as before.

### 5.3. Embedding maps

Let $B = B_t(M_t)$ be the corresponding skew-symmetric matrix to the quiver $Q_t(M_t)$ for even $t$. Let $A(B, x, y)$ be the cluster algebra with coefficients in the universal semifield, and let $G(B, y)$ be the coefficient group associated with $A(B, x, y)$ as before.

In view of Lemma 5.1 we set $x(0) = x$, $y(0) = y$ and define clusters $x(u) = (x_i(u))_{i \in \mathbb{Z}/t}$ and coefficient tuples $y(u) = (y_i(u))_{i \in \mathbb{Z}/t}$ by the sequence of mutations

$$\cdots \xrightarrow{\mu^+} (B(0), x(0), y(0)) \xrightarrow{\mu^+ \mu^+} (B(1), x(\frac{1}{t}), y(\frac{1}{t})) \xrightarrow{\mu^+ \mu^+} \cdots,$$

(46)

where $B(u)$ is the skew-symmetric matrix corresponding to $Q(u)$.

For $(i, u) \in \mathbb{I} \times \frac{1}{t}\mathbb{Z}$, we set the parity condition $p_+$ by

$$p_+: \begin{cases} i \in \mathbb{I}_p^+ \cup \mathbb{I}_{p+1,t-p}^+ & u \equiv \frac{p}{t}, 0 \leq p \leq t-1, p: \text{even} \\ i \in \mathbb{I}_p^- \cup \mathbb{I}_{p+1,t-p}^- & u \equiv \frac{p}{t}, t \leq p \leq 2t-1, p: \text{even} \\ i \in \mathbb{I}_p^- & u \equiv \frac{p}{t}, 0 \leq p \leq 2t-1, p: \text{odd}, \end{cases}$$

(47)

where $\equiv$ is modulo $2\mathbb{Z}$. Again, each $(i, u) : p_+$ is a mutation point of (42) in the forward direction of $u$.

**Lemma 5.3.** Below $\equiv$ means the equivalence modulo $2\mathbb{Z}$.
Fig. 5. The mutation sequence of the quiver $Q_t(M_4)$ in (42) for $t = 4$. 
(i) The map \( g : \mathcal{I}_t^+ \to \{ (i, u) : p_+ \} \)
\[
(a, m, u) \mapsto \begin{cases}
((2j + 1, m), u) & a = 1; m + u \equiv \frac{2j}{t} \\
((2t - 2j, m), u) & a = 1; m + u \equiv \frac{2j}{t} \\
((t + 1, m), u) & a = 2
\end{cases}
\]
\[
(j = 0, 1, \ldots, t/2 - 1)
\]
\[
(j = t/2, \ldots, t - 1)
\]
\[
(48)
\]

is a bijection.

(ii) The map \( g' : \mathcal{I}_t' \to \{ (i, u) : p_+ \} \)
\[
(a, m, u) \mapsto \begin{cases}
((2j + 1, m), u) & a = 1; m + u \equiv \frac{2j}{t} \\
((2t - 2j, m), u) & a = 1; m + u \equiv \frac{2j}{t} \\
((t + 1, m), u) & a = 2
\end{cases}
\]
\[
(j = 0, 1, \ldots, t/2 - 1)
\]
\[
(j = t/2, \ldots, t - 1)
\]
\[
(49)
\]

is a bijection.

5.4. T-system, Y-system, and cluster algebra

All the properties depending on the parity of \( t \) are now absorbed in the quiver \( Q_t(M_t) \), the mutation sequence \( (46) \), and the embedding maps \( g \) and \( g' \) in Lemma 4.3. Lemmas 4.5, 4.8, and Theorems 4.7, 4.10 are true for even \( t \).

6. Cluster algebraic formulation: Tree case

In this section we extend Theorems 4.7 and 4.10 to any tamely laced Cartan matrix \( C \) whose Dynkin diagram is a tree, by patching parity conditions and quivers introduced in Secs. 4 and 5. This is an intermediate step for treating the most general case in Sec. 7.

6.1. Parity decompositions of T and Y-systems

Throughout this section we assume that \( C \) is a tamely laced and indecomposable Cartan matrix whose Dynkin diagram \( X(C) \) is a tree, i.e., without cycles.

We decompose the index set \( I \) of \( X(C) \) into two parts \( I = I_+ \cup I_- \) such that the following two rules are satisfied:
(I) If \(a\) and \(b\) are adjacent in \(X(C)\) and both \(d_a\) and \(d_b\) are odd, then either \(a \in I_+\), \(b \in I_-\) or \(a \in I_+, b \in I_-\) holds.

(II) If \(a\) and \(b\) are adjacent in \(X(C)\) and at least one of \(d_a\) and \(d_b\) is even, then either \(a, b \in I_+\) or \(a, b \in I_-\) holds.

To each \(a \in I\) with \(d_a\) even, we also attach the ‘color’ \(c_a = \alpha\) or \(\beta\) satisfying the following condition:

(III) If \(a\) and \(b\) are adjacent in \(X(C)\), then \(c_a \neq c_b\).

See Fig. 6 for an example. (The coloring is not used in this subsection.)

For a triplet \((a, m, u) \in I_\ell\), we set the parity conditions \(Q_+\) as follows.

\[
Q_+ = \begin{cases} 
(o+) m + tu \text{ is even} & d_a \text{ is odd, } a \in I_+ \\
(o-) m + tu \text{ is odd} & d_a \text{ is odd, } a \in I_- \\
(e+) tu \text{ is odd} & d_a \text{ is even, } a \in I_+ \\
(e-) tu \text{ is even} & d_a \text{ is even, } a \in I_- 
\end{cases}
\]  

(50)

Let \(Q_-\) be the negation of \(Q_+\). Suppose that \(a\) and \(b\) in \(I\) with \(d_a \geq d_b\) are adjacent in \(X(C)\). Due to the condition (1), we have four possibilities: (i) \(d_a\) is odd and \(d_b = 1\), (ii) \(d_a = d_b\), and \(d_a\) is odd and not 1, (iii) \(d_a\) is even and \(d_b = 1\), (iv) \(d_a = d_b\), and \(d_a\) is even. For (i), the condition \(Q_+\) is compatible with \(P_\pm\) in (16) with \(t = d_a\) therein. For (ii), the condition \(Q_+\) is compatible with \(P_\pm\) in (38) with \(t = d_a\) therein. For (ii) and (iv), one can directly check that the condition \(Q_+\) is compatible with (4). Therefore, we have the parity decomposition

\[
\mathcal{T}_\ell^\varepsilon(C) \simeq \mathcal{T}_\ell^\varepsilon(C)_+ \otimes_{\mathbb{Z}} \mathcal{T}_\ell^\varepsilon(C)_-, 
\]  

(51)

where \(\mathcal{T}_\ell^\varepsilon(C)\) \((\varepsilon = \pm)\) is the subring of \(\mathcal{T}_\ell^\varepsilon(C)\) generated by \(T_m^{(a)}(u)\) \(\langle (a, m, u) : Q_\varepsilon \rangle\).

Similarly, for a triplet \((a, m, u) \in I_\ell\), we set the parity conditions \(Q_+\)
as follows.

\[ Q'_+ = \begin{cases} 
\text{(o\,+) } m + tu \text{ is odd} & d_a \text{ is odd, } a \in I_+ \\
\text{(o\,-) } m + tu \text{ is even} & d_a \text{ is odd, } a \in I_- \\
\text{(e\,+) } tu \text{ is odd} & d_a \text{ is even, } a \in I_+ \\
\text{(e\,-) } tu \text{ is even} & d_a \text{ is even, } a \in I_- 
\end{cases} \]  \quad (52)

We have

\[ (a, m, u) : Q'_+ \iff (a, m, u \pm \frac{d_a}{2}) : Q'_+. \]  \quad (53)

Let \( Q'_- \) be the negation of \( Q'_+ \). Then, we have the parity decomposition

\[ Y^\varepsilon(C) \simeq Y^\varepsilon(C)_+ \times Y^\varepsilon(C)_-, \]  \quad (54)

where \( Y^\varepsilon(C)_\varepsilon \) (\( \varepsilon = \pm \)) is the subring of \( Y^\varepsilon(C) \) generated by \( Y^\varepsilon_m(u) \), \( 1 + Y^\varepsilon_m(u) \) \((a, m, u) : Q'_\varepsilon\).

6.2. Construction of quiver \( Q_\varepsilon(C) \)

Let us construct a quiver \( Q_\varepsilon(C) \) for \( C \) and \( \ell \). We do it in two steps. In Step 1, to each adjacent pair \((a, b)\) of the Dynkin diagram \( X(C) \) we attach a certain quiver \( Q(a, b) \). In Step 2, these quivers are ‘patched’ at each vertex.

**Step 1.** \( Q(a, b) \).

Recall that \( t \) is the one in (2). Below suppose that \( a \) and \( b \) are adjacent in \( X(C) \) and \( d_a \geq d_b \).

**Case (i).** \( d_a \text{ is odd and } d_b = 1 \). (a) The case \( a \in I_- \). We set the quiver \( Q(a, b) \) by the quiver \( Q'(\ell': M_{t'}) \) in Sec. 4.3 with \( t' = d_a \) and \( \ell' = t\ell/d_a \). We assign +/− as in Sec. 4.3. (We do not need to assign •/◦ here.)

(b) The case \( a \in I_+ \). We set the quiver \( Q(a, b) \) by the quiver \( Q(1) \) obtained from \( Q(0) = Q'_\ell(M_t) \) in Sec. 4.3 with \( t' = d_a \) and \( \ell' = t\ell/d_a \). We assign +/− in the opposite way to Sec. 4.3.

**Case (ii).** \( d_a = d_b \), and \( d_a \text{ is odd and not } 1 \). We can assume that \( a \in I_- \) and \( b \in I_+ \). We set the quiver \( Q(a, b) \) as a disjoint union of quivers \( Q_1, \ldots, Q_{d_a} \) specified as follows. The quivers \( Q_1, Q_3, \ldots, Q_{d_a} \) are the quiver \( Q'_\ell(M_{t'}) \) in Sec. 4.3 with \( t' = 1 \) and \( \ell' = t\ell/d_a \). We assign +/− as in Sec. 4.3. The quivers \( Q_2, Q_4, \ldots, Q_{d_a-1} \) are the opposite quiver of \( Q'_\ell(M_{t'}) \) in Sec. 4.3 with \( t' = 1 \) and \( \ell' = t\ell/d_a \). We assign +/− in the opposite way to Sec. 4.3.

**Case (iii).** \( d_a \text{ is even and } d_b = 1 \). (a) The case \( a \in I_+ \) and \( c_a = \alpha \). We set the quiver \( Q(a, b) \) by the quiver \( Q'_\ell(M_{t'}) \) in Sec. 5.2 with \( t' = d_a \) and \( \ell' = t\ell/d_a \). We assign +/− as in Sec. 5.2.
(b) The case \( a \in I_+ \) and \( c_a = \beta \). We set the quiver \( Q(a, b) \) by the quiver \( Q(1) \) obtained from \( Q(0) = Q_{\ell'}(M_{t'}) \) in Sec. 5.2 with \( t' = d_a \) and \( \ell' = t\ell/d_a \). For \( \bullet/\circ \) in Sec. 5.2, we assign \(+/-\) to vertices with \( \bullet \) as in Sec. 5.2, while we assign \(+/-\) to vertices with \( \circ \) in the opposite way to Sec. 5.2.

(c) The case \( a \in I_- \) and \( c_a = \alpha \). We set the quiver \( Q(a, b) \) by the quiver \( Q(-1/t') \) obtained from \( Q(0) = Q_{\ell'}(M_{t'}) \) in Sec. 5.2 with \( t' = d_a \) and \( \ell' = t\ell/d_a \). For \( \bullet/\circ \) in Sec. 5.2, we assign \(+/-\) to vertices with \( \circ \) as in Sec. 5.2, while we assign \(+/-\) to vertices with \( \bullet \) in the opposite way to Sec. 5.2.

(d) The case \( a \in I_- \) and \( c_a = \beta \). We set the quiver \( Q(a, b) \) by the quiver \( Q((t' - 1)/t') \) obtained from \( Q(0) = Q_{\ell'}(M_{t'}) \) in Sec. 5.2 with \( t' = d_a \) and \( \ell' = t\ell/d_a \). We assign \(+/-\) in the opposite way to Sec. 5.2.

Case (iv). \( d_a = d_b \), and \( d_a \) is even. We can assume that \( c_a = \alpha \) and \( c_b = \beta \). We set the quiver \( Q(a, b) \) as a disjoint union of quivers \( Q_1, \ldots, Q_{d_a} \) specified as follows. The quivers \( Q_1, Q_3, \ldots, Q_{d_a - 1} \) are the quiver \( Q_{\ell'}(M_{t'}) \) in Sec. 4.3 with \( t' = 1 \) and \( \ell' = t\ell/d_a \). We assign \(+/-\) as in Sec. 4.3. The quivers \( Q_2, Q_4, \ldots, Q_{d_a} \) are the opposite quiver of \( Q_{\ell'}(M_{t'}) \) in Sec. 4.3 with \( t' = 1 \) and \( \ell' = t\ell/d_a \). We assign \(+/-\) in the opposite way to Sec. 4.3.

Throughout Step 1, we regard the left column(s) of \( Q(a, b) \) (with length \( t\ell/d_a - 1 \)) as attached to \( a \) and the right column(s) of \( Q(a, b) \) (with length \( t\ell/d_b - 1 \)) as attached to \( b \).

Step 2. \( Q_{\ell'}(C) \).

The quiver \( Q_{\ell'}(C) \) is defined by patching the above quivers \( Q(a, b) \) at each vertex. Namely, fix \( a \in I_+ \), and take all \( b \)'s which are adjacent to \( a \). If \( d_a = 1 \), we identify the columns attached to \( a \) in \( Q(b, a) \) for all \( b \). If \( d_a > 1 \), for each \( i = 1, \ldots, d_a \), we identify the columns attached to \( a \) in the \( i \)th quivers \( Q_i \) of \( Q(a, b) \) or \( Q(b, a) \) (depending on the sign and color of \( a \)) for all \( b \). (For Cases (i) and (ii), \( Q_i \) appears in the construction of \( Q_{\ell'}(M_{t'}) \).)

Some basic examples are given below.

**Example 6.1.** The two examples below mostly clarify the situation involving Cases (i) and (ii).

1. Let \( C \) be the Cartan matrix with the following Dynkin diagram.

   ![Dynkin diagram](image)

   + - + - + -
The corresponding quiver $Q_{\ell}(C)$ for $\ell = 3$ is given as follows.

Here, the long columns at the same horizontal positions in three quivers are identified with each other. The encircled vertices are the mutation points at $Q(0) = Q_{\ell}(C)$, which will be described in the next subsection. The same remark applies below.

(2) Let $C$ be the Cartan matrix with the following Dynkin diagram.

The corresponding quiver $Q_{\ell}(C)$ for $\ell = 3$ is given as follows.

Example 6.2. The four examples below mostly clarify the situation involving Cases (iii) and (iv).

(1) Let $C$ be the Cartan matrix with the following Dynkin diagram.

The corresponding quiver $Q_{\ell}(C)$ for $\ell = 3$ is given as follows.
(2) Let \( C \) be the Cartan matrix with the following Dynkin diagram.

\[
\begin{array}{ccccc}
  & + & +, \alpha & +, \beta & +, \alpha & +, \beta & + \\
\end{array}
\]

The corresponding quiver \( Q_\ell(C) \) for \( \ell = 3 \) is given as follows.

\[
\begin{array}{c}
Q_1 \\
Q_2
\end{array}
\]

(3) Let \( C \) be the Cartan matrix with the following Dynkin diagram.

\[
\begin{array}{ccccc}
  & - & -, \alpha & -, \beta & -, \alpha & - \\
\end{array}
\]

The corresponding quiver \( Q_\ell(C) \) for \( \ell = 3 \) is given as follows.

\[
\begin{array}{c}
Q_1 \\
Q_2
\end{array}
\]

(4) Let \( C \) be the Cartan matrix with the following Dynkin diagram.

\[
\begin{array}{ccccc}
  & - & -, \alpha & -, \beta & -, \alpha & -, \beta & - \\
\end{array}
\]

The corresponding quiver \( Q_\ell(C) \) for \( \ell = 3 \) is given as follows.

\[
\begin{array}{c}
Q_1 \\
Q_2
\end{array}
\]

6.3. Mutation sequence

We set \( Q(0) = Q_\ell(C) \) and define a periodic sequence of mutations of quivers

\[
Q(0) \xrightarrow{\mu(0)} Q\left(\frac{1}{2}\right) \xrightarrow{\mu\left(\frac{1}{2}\right)} Q\left(\frac{3}{2}\right) \xrightarrow{\mu\left(\frac{3}{2}\right)} \cdots \xrightarrow{\mu\left(\frac{2t-1}{2}\right)} Q(2) = Q(0) \quad (55)
\]

by patching the ones in (22) and (42). Let \( M_a(k/t) \) \((a \in I, k = 0, \ldots, 2t-1)\) be the set of the mutation points of \( \mu(k/t) \) in the columns attached to \( a \).
It is defined as follows. (Below we use the assignment of $+/−$ specified in Sec. 6.2. Also we use the similar notations in the ones in (22) and (42), e.g., $\Gamma^a_{+\cdot \cdot \cdot i}$ denotes the set of vertices in the column attached to $a$ of $i$th quiver $Q_i$ with property $+$.)

(i) $d_a$: odd. (cf. (22))

\[ M_a(0) = \Gamma^a_{+\cdot \cdot \cdot 1}, \quad M_a(\frac{1}{t}) = \Gamma^a_{+\cdot \cdot \cdot d_a-1}, \quad M_a(\frac{2}{t}) = \Gamma^a_{+\cdot \cdot \cdot 3}, \ldots, \]
\[ M_a\left(\frac{2d_a-2}{t}\right) = \Gamma^a_{-\cdot \cdot \cdot 2}, \quad M_a\left(\frac{2d_a-1}{t}\right) = \Gamma^a_{-\cdot \cdot \cdot d_a-1}, \quad M_a\left(\frac{2d_a}{t}\right) = M_a(0), \ldots \quad (56) \]

In particular, for $d_a = 1$,

\[ M_a(0) = \Gamma^a_{+\cdot \cdot \cdot 1}, \quad M_a(\frac{1}{t}) = \Gamma^a_{-\cdot \cdot \cdot 1}, \quad M_a(\frac{2}{t}) = M_0(0), \ldots \quad (57) \]

(ii) $d_a$: even, $a \in I_+$ (cf. (42))

\[ M_a(0) = \Gamma^a_{+\cdot \cdot \cdot d_a}, \quad M_a(u) = 0, \quad M_a(\frac{2}{t}) = \Gamma^a_{+\cdot \cdot \cdot 3, d_a-2}, \ldots, \]
\[ M_a\left(\frac{2d_a-2}{t}\right) = \Gamma^a_{-\cdot \cdot \cdot d_a-1, 2}, \quad M_a\left(\frac{2d_a-1}{t}\right) = \Gamma^a_{-\cdot \cdot \cdot d_a-1}, \quad M_a\left(\frac{2d_a}{t}\right) = M_a(0), \ldots \quad (58) \]

(iii) $d_a$: even, $a \in I_-$ (cf. (42))

\[ M_a(0) = 0, \quad M_a(u) = \Gamma^a_{+\cdot \cdot \cdot d_a}, \quad M_a(\frac{2}{t}) = 0, \quad M_a(\frac{2}{t}) = \Gamma^a_{+\cdot \cdot \cdot 3, d_a-2}, \ldots, \]
\[ M_a\left(\frac{2d_a-2}{t}\right) = 0, \quad M_a\left(\frac{2d_a-1}{t}\right) = \Gamma^a_{-\cdot \cdot \cdot d_a-1, 2}, \quad M_a\left(\frac{2d_a}{t}\right) = M_a(0), \ldots \quad (59) \]

6.4. $T$-system, $Y$-system, and cluster algebra

Now it is straightforward to repeat the formulation in Secs. 4 and 5. The compatibility of mutations is the only issue, but it has been already taken care of in the construction of $Q_\ell(C)$ as self-explained in Examples 6.1 and 6.2.

Let $\textbf{I}$ be the index set of the quiver $Q_\ell(C)$. Let $B$ the skew-symmetric matrix corresponding to $Q_\ell(C)$. Using the sequence (55), we define cluster variables $x_i(u)$ and coefficients $y_i(u)$ ($i \in \textbf{I}, u \in \frac{1}{t}\mathbb{Z}$) as before. Define the T-subalgebra $A_T(B, x)$ and Y-subgroup $A_T(B, y)$ as parallel to Definitions 4.6 and 4.9.

Repeating the same argument as before, we obtain the conclusion of this section.

**Theorem 6.3.** Let $C$ be any tamely laced and indecomposable Cartan matrix whose Dynkin diagram is a tree. Then, the ring $\mathcal{T}_T^+(C)$ is isomorphic to $A_T(B, x)$. The group $\mathcal{Y}_T^+(C)$ is isomorphic to $G_Y(B, y)$. 
7. Cluster algebraic formulation: General case

It is easy to extend Theorem 6.3 to any tamely laced Cartan matrix $C$ with suitable modification. Due to the lack of the space, we concentrate on describing the construction of the quiver $Q_\ell(C)$. Throughout the section we assume that $C$ is a tamely laced and indecomposable Cartan matrix.

Before starting, we introduce some preliminary definitions. We call a subdiagram $Y$ of $X(C)$ an even block, if $Y$ is a maximal indecomposable subdiagram of $X(C)$ such that $d_a$ of each vertex $a$ of $Y$ is even. Due to the condition (1), $d_a$ is constant for any vertex $a$ of $Y$. Below we suppose that $X(C)$ has $n$ even blocks $Y_1, \ldots , Y_n$. ($n$ may be zero.) Let $X'(C)$ be the diagram obtained from $X(C)$ by shrinking each even block $Y_i$ into a vertex ‘⊗’ while keeping any line from $Y_i$ to its outside. For example, for the following $X(C)$

\[ \begin{array}{c}
\bullet - \bullet - \bullet - \bullet - \bullet - \bullet
\end{array} \]  

$X'(C)$ is given by

\[ \begin{array}{c}
\bullet
\end{array} \]

7.1. The case $X'(C)$ is bipartite

Let us assume that $X'(C)$ is bipartite, i.e., it contains no odd cycle.

First, consider the case when all the even blocks $Y_1, \ldots , Y_n$ are also bipartite. Then, $X(C)$ admits a decomposition and a coloring of $I$ satisfying Conditions (I)–(III) in Sec. 6.1, and one can construct $Q_\ell(C)$ as in Sec. 6.2.

Next, consider the case when some of the even blocks, say, $Y_1, \ldots , Y_k$ are nonbipartite. Then, $X(C)$ does not admit a coloring of $I$ satisfying Condition (III) in Sec. 6.1. Following Ref. 4, we define the bipartite double $Y^\#$ of any tamely laced Dynkin diagram $Y$ as follows. Let $J$ be the vertex set of $Y$. The vertex set $J^\#$ of $Y^\#$ is the disjoint union $J^\# = J_+ \sqcup J_-$, where $J_+ = \{ j_+ \mid j \in J \}$ and $J_- = \{ j_- \mid j \in J \}$; furthermore, we write a line (or multiple line with arrow) in $Y^\#$ from $i_+$ to $j_-$ and also from $i_-$ to $j_+$ if and only if there is a line (or multiple line with arrow) from $i$ to $j$ in $Y$. Let $\tilde{X}(C)$ be the diagram obtained from $X(C)$ by replacing each nonbipartite even block $Y_i$ ($i = 1, \ldots , k$) with its bipartite double $Y_i^\#$, while connecting $i_\pm$ in $Y_i$ to any vertex $j$ outside $Y_i$ by a line (or multiple line with arrow) if and only if $i$ and $j$ are connected in $X(C)$ by a line (or multiple line with arrow). The diagram $\tilde{X}(C)$ now admits a decomposition and coloring satisfying Conditions (I)–(III) in Sec. 6.1. See Fig. 7 for an example. Then, we repeat the construction of the quiver $Q_\ell(C)$ in Sec. 6.2 for the diagram
Theorem 7.1. Let $C$ be any tamely laced and indecomposable Cartan matrix such that $X'(C)$ is bipartite. Let $B$ the skew-symmetric matrix corresponding to the quiver $Q(\ell)(C)$ defined above. Then, the ring $T_\ell(C)_{+}$ is isomorphic to $A_T(B,x)$. The group $\mathcal{G}(\ell)(C)$ is isomorphic to $\mathcal{I}_T(B,y)$. 

7.2. The case $X'(C)$ is nonbipartite

Let us assume that $X'(C)$ is nonbipartite. Then, $X(C)$ does not admit a decomposition of $I$ satisfying Conditions (I) and (II) in Sec. 6.1; consequently, neither $T(\ell)(C)$ nor $\mathcal{G}(\ell)(C)$ admits the parity decomposition.

First, consider the case when all the even blocks $Y_1, \ldots, Y_n$ of $X(C)$ are bipartite. We take the bipartite double $X'(C)^\# \cdot X'(C)$. Then, in $X'(C)^\#$, restore each even block of $X(C)$, which appears twice in $X'(C)^\#$, in place of $\otimes$. The resulted diagram $\tilde{X}(C)$ now admits a decomposition and a coloring satisfying Conditions (I)—(III) in Sec. 6.1. See Fig. 8 for an example. Now we repeat the construction of the quiver $Q(\ell)(C)$ in Sec. 6.2 for the diagram $\tilde{X}(C)$. We write the resulted quiver as $Q(\ell)(C)$. 

\[ \tilde{X}(C) \] with the following modification: In Step 1 of Sec. 6.2, in Cases (iii) and (iv), we only take the $d_a/2$ subquivers $Q_1, Q_3, \ldots, Q_{d_a-1}$ for those $Q(a,b)$ involving the vertices of $Y_1^\# \cdot Y_2^\# \cdot \ldots \cdot Y_k^\#$. We write the resulted quiver as $Q(\ell)(C)$. Accordingly, we also replace $I_{a,1}, I_{a,3}, \ldots$ in (58) and (59) with $I_{a,1}, I_{a,3}, \ldots$. The rest are defined in the same way as in Sect. 6.4.

Now we have the first main result of the paper.
Next, consider the case when some of the even blocks of $X(C)$, say, $Y_1, \ldots, Y_k$ are nonbipartite. Then, in the above construction of $\tilde{X}(C)$, we further replace each nonbipartite even block $Y_i (i = 1, \ldots, k)$ with its bipartite double $Y_i^\#$ as in Sec. 7.1. We write the resulted diagram as $\tilde{X}(C)$. Then, repeat the construction of the quiver $Q_\ell(C)$ in Sec. 7.1 for the diagram $\tilde{X}(C)$. We write the resulted quiver as $Q_\ell(C)$.

The rest are defined in the same way as before. Then, as in the simply laced case,\(^4\) we have the counterpart of Theorem 7.1, which is the second main result of the paper.

**Theorem 7.2.** Let $C$ be any tamely laced and indecomposable Cartan matrix such that $X'(C)$ is nonbipartite. Let $B$ the skew-symmetric matrix corresponding to the quiver $Q_\ell(C)$ defined above. Then, the ring $T_\ell(C)$ is isomorphic to $A_T(B, x)$. The group $Y_\ell(C)$ is isomorphic to $G_Y(B, y)$.

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