WEIGHTED INEQUALITIES AND UNCERTAINTY PRINCIPLES FOR THE \((k,a)\)-GENERALIZED FOURIER TRANSFORM

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Abstract. We obtain several versions of the Hausdorff–Young and Hardy–Littlewood inequalities for the \((k,a)\)-generalized Fourier transform recently investigated at length by Ben Saïd, Kobayashi, and Ørsted. We also obtain a number of weighted inequalities – in particular Pitt’s inequality – that have application to uncertainty principles. Specifically we obtain several analogs of the Heisenberg–Pauli–Weyl principle for \(L^p\)-functions, local Cowling–Price-type inequalities, Donoho–Stark-type inequalities and qualitative extensions. We finally use the Hausdorff–Young inequality as a means to obtain entropic uncertainty inequalities.

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1. Introduction

Uncertainty principles have long been a mainstay of mathematical physics and classical Fourier analysis alike and are statements of the form that a function and its Fourier transform cannot both be small. A well-known example is the Heisenberg–Pauli–Weyl uncertainty principle to the effect that position and momentum of a quantum particle cannot both be sharply localized. In terms of Fourier analysis it can be paraphrased as the statement that if \(f \in L^2(\mathbb{R}^n)\) and \(\alpha > 0\),

\[
\|f\|_2^2 \leq c_\alpha \left( \int_{\mathbb{R}^n} |x|^{2\alpha} |f(x)|^2 \, dx \right) \left( \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\hat{f}(\xi)|^2 \, d\xi \right),
\]

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or in terms of the Laplace operator $\Delta$ as
\[
\|f\|_2^2 \leq c_\alpha \left( \int_{\mathbb{R}^n} |x|^{2\alpha} |f(x)|^2 \, dx \right) \left( \int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} f(x)|^2 \, dx \right).
\]

Many variations and extensions are outlined in the excellent survey \cite{FS97}, as well as \cite{CP84}, where the following qualitative version is also explained. Consider the sets
\[A_f = \{ x \in \mathbb{R}^n : f(x) \neq 0 \} \quad \text{and} \quad A_\hat{f} = \{ \xi \in \mathbb{R}^n : \hat{f}(\xi) \neq 0 \}
\]
or more generally the analogous sets for functions on a locally compact abelian group. It is easy to prove that if $f \in L^2(\mathbb{R}^n) \setminus \{0\}$, then $|A_f| \cdot |A_\hat{f}| \geq 1$. This is originally due to Matolcsi and Szücs \cite{MS73} and was strengthened considerably by Benedicks, cf. \cite{Ben85}:

**Theorem 1.1.** If $f \in L^1(\mathbb{R}^n)$ and $|A_f| \cdot |A_\hat{f}| < \infty$, then $f = 0$ almost everywhere.

A different proof based on operator theory was given in \cite{AB77} and also yield complementary results that we shall discuss in a later section. We recently established analogues of the Matolcsi–Szücs and more generally the Benedicks–Amrein-Berthier theorems in the framework of harmonic analysis in root systems, and we shall presently establish their analogues for the $(k,a)$-generalized transform $F_{k,a}$ that will be described later in this introduction.

A variation of such qualitative statements is obtained by allowing $f$ and $\hat{f}$ to be negligible small on the complements of given sets $A$, $B$. To fix notation, let $G$ be a locally compact abelian group with dual group $\hat{G}$, and let $A \subset G$, $B \subset \hat{G}$ be measurable subsets. Consider the orthogonal projections $P_A, Q_B$ on $L^2(G)$ defined by $P_A f = 1_A f$ and $Q_B \hat{f} = 1_B \hat{f}$ respectively. The operator $P_A Q_B$ – which also intervenes in \cite{AB77} – is a Hilbert–Schmidt operator, and the essence of the Donoho–Stark uncertainty principle is a statement of the following form: If there is a nonzero $f \in L^2(G)$ such that $\|1_{G \setminus A} f\|_2 \leq \epsilon \|f\|_2$ and $\|1_{\hat{G} \setminus B} \hat{f}\|_2 \leq \delta \|\hat{f}\|_2$ for given constants $\epsilon, \delta > 0$, then $1 - \epsilon - \delta \leq \|P_A Q_B\|_{2 \rightarrow 2}$.

The third version of an uncertainty principle is related to the Heisenberg–Pauli–Weyl inequality but is formulated in terms of the Shannon entropy instead and therefore stronger, cf. \cite{FS97}, Section 5). Following Shannon, the *entropy* of a probability density function $\rho$ on $\mathbb{R}^n$ is defined by
\[
\mathbb{E}(\rho) = - \int_{\mathbb{R}^n} \rho(x) \log(\rho(x)) \, dx.
\]

Hirschman defined entropy without the negative sign but we have adopted the definition from \cite{FS97}. Given a function $f \in L^2(\mathbb{R})$ such that $\|f\|_2 = 1$, it was observed by Hirschman \cite{Hir57} that $\mathbb{E}(|f|^2) + \mathbb{E}(|\hat{f}|^2) \geq 0$, and he made the conjecture that
\[
\mathbb{E}(|f|^2) + \mathbb{E}(|\hat{f}|^2) \geq 1 - \log 2 > 0.
\]
The proof by Hirschman was based on an endpoint differentiation technique applied to the Hausdorff–Young inequality
\[
\|\hat{f}\|_{p'} \leq c_p \|f\|_p, \quad 1 \leq p \leq 2, \quad \frac{1}{p} + \frac{1}{p'} = 1,
\]
and his argument carries over to the case of $\mathbb{R}^n$ without change. The analogue of \cite{Hir57} thereby becomes
\[
\mathbb{E}(|f|^2) + \mathbb{E}(|\hat{f}|^2) \geq n(1 - \log 2),
\]
where $f \in L^2(\mathbb{R}^n)$ with $\|f\|_2 = 1$. Hirschman apparently raised the conjecture \cite{Hir57} after having experimented with Gaussian functions in place of $f$, and indeed the conjectures \cite{Hir57}
and (5) are correct, as observed by Beckner in Section IV.3 of [Bec75]. Among other things, Beckner’s paper records the optimal constant $c_p$ in (4) for $p \in [1, 2]$, thereby extending a result by Babenko [Bab61]. At the same time it was established that Gaussians are optimizers for the Hausdorff–Young inequality, so the method by Hirschman – now applied to the sharp Hausdorff–Young inequality – immediately establishes (5). The same conclusion was made in [BBM75]. We must stress, however, that it was not shown until recently (cf. Theorem 1.5 in [ÖP04]) that normalized Gaussians do in fact serve as minimizers in (3), (5) (Hirschman anticipated such a result but due to the endpoint differentiation one cannot simply deduce this fact from a similar statement about the sharp Hausdorff–Young inequality).

We have recently investigated these topics in the case of the Cherednik–Opdam and Heckman–Opdam transforms associated with a root system, cf. [Joh15a], [Joh15c]. The results in the present paper are complimentary, in the sense that while we also work in a framework of generalized harmonic analysis in root systems, the motivation and the resulting transform are different. In order to motivate the construction of the $(k,a)$-generalized Fourier transform $F_{k,a}$ in [BSK012] we shall briefly recall several alternative descriptions of the the Euclidean Fourier transform $F$, which is defined by

$$F f(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(x) e^{-i x \cdot \xi} \, dx, \quad f \in L^1(\mathbb{R}^N).$$

Alternatively

$$F f(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(x) K(x,\xi) \, dx,$$

where $K(x,\xi)$ is the unique solution to the system of partial differential equations $\partial_{x_j} K(x,\xi) = -i \xi_j K(x,\xi)$, $j = 1, \ldots, N$ subject to the initial value condition $K(0,\xi) = 1$ for $\xi \in \mathbb{R}^N$. A third description was discovered by R. Howe [How88a],

$$F = \exp\left(i \frac{\pi N}{4}\right) \exp\left(i \pi \left(\Delta - ||x||^2\right)\right),$$

where $\Delta$ is the Laplace operator on $\mathbb{R}^N$. This ‘spectral’ description connects $F$ to the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ generated by $\Delta$ and $|| \cdot ||^2$, and to the quantum harmonic oscillator $-(\Delta - ||x||^2)/2$. To be more precise,

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H$$

where $E = ||x||^2/2$, $F = -\Delta/2$, and $H = E + N/2$, where $E = \sum_{j=1}^N x_j \partial_{x_j}$ denotes the Euler operator.

Both of the representations (6) and (7) have their uses, and it is explained in the overview paper [DB12] how to construct various extensions such as a fractional Fourier transform and Clifford algebra-valued analogues. We are concerned with a different kind of extension, where the Euclidean Laplace operator $\Delta$ is replaced by the sum of squares $\Delta_k$ of Dunkl operators associated with a given finite reflection group in $\mathbb{R}^N$. The same $\mathfrak{sl}_2$-commutator relations continue to hold, and an analogue of (4) holds as well. It was observed in [BSK012] that one can introduce an additional parameter to the Dunkl-operator construction, in terms of which the Euclidean harmonic oscillator is naturally replaced by an $a$-deformed Dunkl-harmonic oscillator $||x||^{2-a} \Delta_k - ||x||^a$. The resulting spectrally defined family of operators $F_{k,a}(z) = \exp(\Delta_k(||x||^{2-a} \Delta_k - ||x||^a))$, $Rz \geq 0$, may therefore be regarded as a two-parameter generalization of Howe’s description (7), where $k$ refers to a multiplicity function and $a > 0$. The special case $a = 2$ recovers the Dunkl transform in $\mathbb{R}^N$ and it is therefore natural to
ask for analytical properties of $F_{k,a}(\omega)$ such as a Plancherel theorem or an inversion formula. The case $a = 1$ is related to an integral transform appearing in work by Kobayashi and Mano ([KM05], [KM07], [KM11]) on the other hand. In particular, we obtain uncertainty principles for their integral transform at no additional cost.

There questions were addressed at length in [BSKØ12] and placed in a wider context in [DB12], [DBØSS13], [DBØSS12] but many additional questions were left open. Indeed our motivation was to extend classical results beyond the Plancherel theorem for $F$ — such as the Hausdorff–Young, Hardy–Littlewood and Pitt’s inequalities — to $F_{k,a}$ and simultaneously investigating applications to uncertainty principles, given that the connection to quantum mechanics is apparent. Having a Hausdorff–Young inequality for $F_{k,a}$ is of course key to the proof of the entropic inequality. Since the Hausdorff–Young inequality is easily established by means of interpolation, we found it natural to explore further weighted inequalities arising from more intricate interpolation arguments. These include Hardy–Littlewood inequalities of several kinds but the scope of interpolation is wider.

In fact we have adopted the modern point of view of [BH03] that weighted inequalities such as Pitt’s inequality should be obtained by interpolation arguments that do not rely on explicit information on the transform under consideration. We find this approach sensible, since one of the major technical obstacles in the further investigation of $F_{k,a}$ is a lack of explicit formulae for the kernel that appears in the analogue of (9). As already mentioned several classical uncertainty principles were recently [GJ14] established for a general class of integral transforms, and we presently extend these principles and add further to the list of results. The guiding principle has therefore been to use the description of $F_{k,a}$ as an integral transform in combination with interpolation arguments and spectral considerations. The main results may briefly be summarized as follows.

- The $(k,a)$-generalized Fourier transform of a radial function in $\mathbb{R}^N$ is a modified Hankel transform, and existing uncertainty principles for the Hankel transform therefore have analogues for $F_{k,a}$ acting on $L^p_{\text{rad}}(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. Most notably, we obtain a sharp form of Pitt’s inequality, with the help of which a sharp logarithmic uncertainty inequality follows. This application rests on the recent paper [Omr11]. A similar approach was recently used in [AK13] to obtain analogous inequalities for the Riemann–Liouville operator. Further applications are discussed below.
- We obtain an analogue of Hirschman’s entropic inequality and use it to give a new proof of the Heisenberg–Pauli–Weyl uncertainty principle recently obtained by Ben Said, Kobayashi, and Ørsted, cf. [BSKØ12, Theorem 5.29].
- We obtain large classes of weighted inequalities for $F_{k,a}$, the most important one being Pitt’s inequality. These inequalities are based on rearrangement and interpolation techniques from [BH03] so the constants are not optimal. We also establish several Hardy–Littlewood inequalities that are new already for the Dunkl transform $F_{k,2}$.
- We obtain a variation of the Heisenberg inequality involving a combination of $L^1$- and $L^2$-norms; the result was recently obtained for Dunkl transform by [Gho13] and involve additional classical inequalities of Nash- and Clark-type.
- The Heisenberg–Pauli–Weyl, Donoho–Stark, and Benedicks–Amrein-Berthier principles do not rely on having sharp constants and are established in general along the lines of [GJ14]. These results are collected towards the end of the paper as they do not require new proofs. We do provide a proof of the weaker Matolcsi–Szücs principle, though.
The point about [GJ14] is to use the representation of $F_{k,a}$ as an integral operator with a well-behaved kernel and apply known uncertainty principles for such operators. The point of departure, it seems, was the observation by de Jeu (cf. [AJJ94]) that a Donoho–Stark-type inequality established in the framework of Gelfand pairs by J. Wolf in [Wol92], [Wol94] could be generalized to a large class of integral operators satisfying suitable Plancherel-type estimates. A few years ago Ghobber and Jaming revisited the approach by de Jeu in the setting of the Hankel transform and recently ([GJ14]) extended the scope of their results even further to include, among others, the standard Dunkl transform on $\mathbb{R}^N$ (specifically we refer the reader to Theorem 4.3 and Theorem 4.4 in [GJ14]).

While the connection between, say, the entropic inequality and the Heisenberg–Pauli–Weyl principle is well known in Euclidean analysis and nicely laid out in [FS97], it seems that a similar connection has gone unnoticed in more general settings such as Dunkl theory. At the same time we want to raise awareness of the interesting open question regarding sharp inequalities and the immediate applications to mathematical physics.

It must be pointed out that our version of Pitt’s inequality is not strong enough to establish even a weak form of Beckner’s logarithmic uncertainty principle. We hope that our inclusion of the weak form of Pitt’s inequality will inspire further work in this direction.

2. The deformed Fourier transform and interpolation theorems

The present section is a detailed overview of definitions and results for the deformed Dunkl-type harmonic oscillator introduced by Ben Said, Kobayashi, and Ørsted in [BSKO12] that will be needed later on. A subsection has been devoted to a discussion of the important case of radial functions, where the harmonic analysis simplifies significantly. Since interpolation in Lorentz spaces is not usually encountered in literature regarding harmonic analysis in root systems, we have included some technical remarks towards the end of the section for easy reference.

Let $\langle \cdot, \cdot \rangle$ denote the standard Euclidean inner product on $\mathbb{R}^N$ and let $\| \cdot \|$ be the associated norm. The reflection associated with a non-zero vector $\alpha \in \mathbb{R}^N$ is defined by $r_\alpha(x) = x - 2\langle x,\alpha \rangle \alpha$, $x \in \mathbb{R}^N$. Fix a (reduced) root system $\mathcal{R} \subset \mathbb{R}^N \setminus \{0\}$ and let $\mathfrak{c} \subset O(N,\mathbb{R})$ denote the Coxeter (or Weyl) group generated by the root reflections $r_\alpha$, $\alpha \in \mathcal{R}$. Furthermore let $k: \mathcal{R} \to \mathbb{C}$ be a fixed multiplicity function and write $k_\alpha := k(\alpha)$ for $\alpha \in \mathcal{R}$. In the following we shall need the weight function $\vartheta_k(x) = \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, x \rangle|^{2k_\alpha}$ defined on $\mathbb{R}^N$. For $\xi \in \mathbb{C}^n$ and a fixed multiplicity function $k$ define the 1st order Dunkl operators

$$T_{\xi}(k)f(x) = \partial_\xi f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \langle \alpha, x \rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbb{R}^N),$$

where $\partial_\xi$ is the directional derivative in the direction of $\xi$. It follows from the $\mathfrak{c}$-invariance of the multiplicity function that the definition of $T_{\xi}(k)$ is independent of the choice of positive system $\mathcal{R}^+$. These operators are homogeneous of degree $-1$ and have many convenient properties, for instance

- $L(h) \circ T_{\xi}(k) \circ L(h)^{-1} = T_{h\xi}(k)$ for all $h \in \mathfrak{c}$;
- $T_{\xi}(k)T_{\eta}(k) = T_{\eta}(k)T_{\xi}(k)$ for all $\xi, \eta \in \mathbb{R}^N$;
- $T_{\xi}(k)[fg] = gT_{\xi}(k)f + fT_{\xi}(k)g$ if $f, g \in C^1(\mathbb{R}^N)$ and one of the functions is $\mathfrak{c}$-invariant.

Here $(L(h)f)(x) = f(h^{-1}x)$ is the left-regular action of $h \in \mathfrak{c}$ on a function $f$ on $\mathbb{R}^N$. 

It was established in [Dun91] that for every non-negative multiplicity function \(k\) there is a unique linear isomorphism \(V_k\) – the intertwining operator – on the space \(P(\mathbb{R}^N)\) of polynomial functions on \(\mathbb{R}^N\) such that \(V_k(P_m(\mathbb{R}^N)) = P_m(\mathbb{R}^N)\) for all \(m \in \mathbb{N}\), \(V_k|_{\mathcal{P}_0(\mathbb{R}^N)} = \text{id}\), and \(T_\xi(k)V_k = V_k\partial_\xi\) for all \(\xi \in \mathbb{R}^N\). Here \(\mathcal{P}_m(\mathbb{R}^N)\) denotes the space of polynomials that are homogeneous of degree \(m\).

Let \(\langle k \rangle = \sum_{\alpha \in \mathbb{R}^+} k_\alpha = \frac{1}{2} \sum_{\alpha \in \mathbb{R}} k_\alpha\). According to [Rös99] Theorem 1.2] there exists a unique positive Radon probability measure \(\mu_\xi^k\) on \(\mathbb{R}^N\) that represents the Dunkl intertwining operator \(V_k\) in the sense that

\[
V_k f(x) = \int_{\mathbb{R}^N} f(\xi) d\mu_\xi^k(\xi).
\]

Additionally, for a continuous function \(h(t)\) of a single variable set \(h_y(\cdot) = h(\langle \cdot, y \rangle)\) for \(y \in \mathbb{R}^N\) and define

\[
(\tilde{V}_k h)(x, y) = (V_k h)(x) = \int_{\mathbb{R}^N} h(\langle \xi, y \rangle) d\mu_\xi^k(\xi).
\]

Fix an orthonormal basis \(\{\xi_1, \ldots, \xi_n\}\) of \((\mathbb{R}^N, \langle \cdot, \cdot \rangle)\), and write \(T_j(k) = T_{\xi_j}(k)\) for short. The Dunkl–Laplacian \(\Delta_k := \sum_{j=1}^n T_j(k)\) can be written explicitly as

\[
\Delta_k f(x) = \Delta f(x) + \sum_{\alpha \in \mathbb{R}^+} k_\alpha \left( \frac{2 \langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \|\alpha\|^2 \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle^2} \right),
\]

where \(\nabla\) denotes the usual gradient operator.

A \(k\)-harmonic polynomial of degree \(m \in \mathbb{N}\) is a homogeneous polynomial \(p\) on \(\mathbb{R}^N\) of degree \(m\) such that \(\Delta_k p = 0\). Let \(\mathcal{H}_k^p(\mathbb{R}^N)\) denote the space of \(k\)-harmonic polynomials of degree \(m\). Furthermore, let \(d\sigma\) denote the standard measure on the unit \(N\)-sphere \(S^{N-1}\) in \(\mathbb{R}^N\), and let

\[
d_k = \left( \int_{S^{N-1}} \partial_k(w) d\sigma(w) \right)^{-1}.
\]

In the case \(k \equiv 0\) the number \(\frac{1}{d_k}\) is the volume of the unit sphere in \(\mathbb{R}^N\). Then \(L^2(\mathbb{S}^{N-1}, \partial_k(w) d\sigma(w))\) is a Hilbert space with respect to the inner product

\[
\langle f, g \rangle_k = d_k \int_{S^{N-1}} f(w) \overline{g(w)} \partial_k(w) d\sigma(w).
\]

The function spaces \(\mathcal{H}_k^m(\mathbb{R}^N)|_{\mathbb{S}^{N-1}}, m = 0, 1, \ldots\) are mutually orthogonal with respect to \(\langle \cdot, \cdot \rangle_k\), and

\[
L^2(\mathbb{S}^{N-1}, \partial_k(w) d\sigma(w)) = \bigoplus_{m \in \mathbb{N}} \mathcal{H}_k^m(\mathbb{R}^N)|_{\mathbb{S}^{N-1}}.
\]

**Definition 2.1.** Let \(\vartheta_{k,a}(x) := \|x\|^{a-2} \partial_k(x)\). Define \(L_{k,a}^p(\mathbb{R}^N) = L^p(\mathbb{R}^N, \vartheta_{k,a}(x) dx)\) and \(d\mu_{k,a}(x) = \vartheta_{k,a}(x) dx\). The norm of a function \(f \in L_{k,a}^p(\mathbb{R}^N)\) will be written \(\|f\|_p\) if it is clear from the context that the reference measure is the weighted measure \(\mu_{k,a}\), and \(\|f\|_{L_{k,a}^p}\) otherwise.

In standard polar coordinates on \(\mathbb{R}^N\) it holds that

\[
\vartheta_{k,a}(x) dx = r^{2(k)+N+a-3} \vartheta_k(w) dr d\sigma(w),
\]

implying the existence of a unitary isomorphism

\[
L^2(\mathbb{S}^{N-1}, \partial_k(w) d\sigma(w)) \otimes L^2(\mathbb{R}^+, r^{2(k)+N+a-3} dr) \rightarrow L_{k,a}^2(\mathbb{R}^N)
\]
where $dx$ is the usual Lebesgue measure on $\mathbb{R}^N$. Hence we arrive at the very useful orthogonal decomposition

$$
\bigoplus_{m \in \mathbb{N}} (H^m_k(\mathbb{R}^N)|_{\mathbb{S}^{N-1}}) \otimes L^2(\mathbb{R}^+, r^{2(k)+N+a-3} \, dr) \xrightarrow{\sim} \mathcal{L}^2_{k,a}(\mathbb{R}^N).
$$

Let $\lambda_{k,a,m} = \frac{1}{a}(2m + 2 \langle k \rangle + N - 2)$ and let

$$
L^{(\lambda)}_\ell(t) = \frac{(\lambda + 1)_\ell}{\ell!} \sum_{j=0}^{\ell} \frac{(-\ell)_{j+i}}{(\lambda + 1)_j j!} = \sum_{j=0}^{\ell} \frac{(-1)^j \Gamma(\lambda + \ell + 1)}{(\ell - j)! \Gamma(\lambda + \ell + 1) j!}
$$

denote the usual one-dimensional Laguerre polynomial. For $x = rw \in \mathbb{R}^N$ (with $r > 0$ and $w \in \mathbb{S}^{N-1}$), and $p \in H^m_k(\mathbb{R}^N)$ define

$$
\Phi^{(a)}_\ell(p,x) = p(x)L^{(\lambda_{k,a,m})}_\ell \left( \frac{2}{a} \|x\|^2 \right) \exp \left( -\frac{1}{a} \|x\|^a \right) = p(w)r^m L^{(\lambda_{k,a,m})}_\ell \left( \frac{2}{a} r^a \right) \exp \left( -\frac{1}{a} r^a \right).
$$

Furthermore let $W_{k,a}(\mathbb{R}^N) = \text{span}_\mathbb{C}\{\Phi^{(a)}_\ell(p,\cdot) : \ell, m \in \mathbb{N}, p \in H^m_k(\mathbb{R}^N)\}$.

**Proposition 2.2.** Suppose $k$ is a non-negative multiplicity function on the root system $\mathcal{R}$ and that the parameter $a > 0$ has the property that $a + 2 \langle k \rangle + n - 2 > 0$. Let $\ell, m, n \in \mathbb{N}$, $p \in H^m_k(\mathbb{R}^N)$ and $q \in H^m_k(\mathbb{R}^N)$.

1. $\Phi^{(a)}_\ell(p,\cdot) \in C(\mathbb{R}^N) \cap L^2_{k,a}(\mathbb{R}^N)$;
2. $\int_{\mathbb{R}^N} \Phi^{(a)}_\ell(p,x)\Phi^{(a)}_s(q,x) d\theta_{k,a}(x) dx = \delta_{m,n} \delta_{\ell,s} \frac{a^{\lambda_{k,a,m}} \Gamma(\lambda_{k,a,m} + 1)}{2^{\ell+1} \lambda_{k,a,m} \Gamma(\ell + 1)} \int_{\mathbb{S}^{N-1}} p(w)q(w) d\sigma(w)$;
3. $W_{k,a}(\mathbb{R}^N)$ is dense in $L^2_{k,a}(\mathbb{R}^N)$.

**Proof.** See Proposition 3.12 in [BSK012]. \qed

** Proposition 2.3.** Fix $m \in \mathbb{N}$, $a > 0$, and a multiplicity function $k$ satisfying

$$2m + 2 \langle k \rangle + N + a - 2 > 0.
$$

We set

$$
f^{(a)}_{\ell,m}(r) := \left( \frac{2^{\lambda_{k,a,m} + 1} \Gamma(\ell + 1)}{a^{\lambda_{k,a,m}} \Gamma(\lambda_{k,a,m} + 1)} \right)^{1/2} r^m L^{(\lambda_{k,a,m})}_\ell \left( \frac{2}{a} r^a \right) \exp \left( -\frac{1}{a} r^a \right)
$$

for $\ell \in \mathbb{N}$. Then $\{f^{(a)}_{\ell,m}(\cdot) : \ell \in \mathbb{N}\}$ forms an orthonormal basis in $L^2(\mathbb{R}^+, r^{2(k)+N+a-3} \, dr)$.

**Proof.** See Proposition 3.15 in [BSK012]. \qed

For each $m \in \mathbb{N}$ we fix an orthonormal basis $\{h^{(m)}_j\}_{j \in J_m}$ in $H^m_k(\mathbb{R}^N)|_{\mathbb{S}^{N-1}}$.

** Corollary 2.4.** Suppose $a > 0$ and that the non-negative multiplicity function $k$ satisfies (14). For $\ell, m \in \mathbb{N}$ and $j \in J_m$ set

$$
\Phi^{(a)}_{\ell,m,j}(x) := h^{(m)}_j \left( \frac{x}{\|x\|} \right) f^{(a)}_{\ell,m}(\|x\|).
$$

Then $\{\Phi^{(a)}_{\ell,m,j} : \ell, m \in \mathbb{N}, j \in J_m\}$ forms an orthonormal basis in $L^2_{k,a}(\mathbb{R}^N)$.

**Proof.** See Corollary 3.17 in [BSK012]. \qed
Theorem 2.5. Let \( \omega_k,a \) of \( \mathfrak{sl}(2, \mathbb{R}) \). Let us briefly recall the pertinent details. First recall the standard basis for \( \mathfrak{sl}(2, \mathbb{R}) \):
\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}, \quad h = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]
Upon defining
\[
\mathbb{E}^+_k,a = \frac{i}{a} \|x\|^a, \quad \mathbb{E}^-_k,a = \frac{i}{a} \|x\|^{2-a} \Delta_k, \quad \mathbb{H}_k,a = \frac{N + 2 \langle k \rangle + a - 2}{a} + \frac{2}{a} \sum_{i=1}^N x_i \partial_i,
\]
one can define an \( \mathbb{R} \)-linear map \( \omega_{k,a} : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \text{End}(C^\infty(\mathbb{R}^N \setminus \{0\})) \) by \( \omega_{k,a}(h) = \mathbb{H}_{k,a} \), \( \omega_{k,a}(e^+) = \mathbb{E}^+_k,a \), and \( \omega_{k,a}(e^-) = \mathbb{E}^-_k,a \) - it is easily seen to be a Lie algebra homomorphism, and it extends to a \( \mathbb{C} \)-algebra homomorphism of the universal enveloping algebra \( U(\mathfrak{sl}(2, \mathbb{C})) \) of \( \mathfrak{sl}(2, \mathbb{C}) \). Set \( k = \text{Ad}(c) h \), and \( n^\pm = \text{Ad}(e^\pm) \), where \( c = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ i & 1 \end{pmatrix} \) denotes the Cayley transform. Recall that \( \text{Ad}(c) \) induces a Lie algebra isomorphism \( \mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{su}(1,1) \).

Theorem 2.5. Let \( p \in \mathcal{H}_m^m(\mathbb{R}^N) \) be fixed. Then
(i) \( \omega_{k,a}(k)\Phi_{k,a}^\ell(p,x) = (2\ell + \lambda_{k,a,m} + 1)\Phi_{k,a}^\ell(p,x) \); 
(ii) \( \omega_{k,a}(n^+)\Phi_{k,a}^\ell(p,x) = i(\ell + 1)\Phi_{k,a}^{\ell+1}(p,x) \); 
(iii) \( \omega_{k,a}(n^-)\Phi_{k,a}^\ell(p,x) = i(\ell + 1)\Phi_{k,a}^{\ell-1}(p,x) \).
Here \( \lambda_{k,a,m} = \frac{1}{a}(2m + 2 \langle k \rangle + N - 2) \) as previously, and \( \Phi_{-1}^\ell \equiv 0 \) by convention.

Proof. See Theorem 3.19 in [BSKO12]. \( \square \)

For our later purposes the following result is clearly of importance; it appears as Corollary 3.2.2 in [BSKO12].

Corollary 2.6. Let \( a > 0 \) and \( k \) be as above.
(1) The differential-difference operator \( \Delta_{k,a} = \|x\|^{2-a} \Delta_k - \|x\|^a \) is an essentially self-adjoint operator on \( L_{k,a}^2(\mathbb{R}^N) \);
(2) There is no continuous spectrum of \( \Delta_{k,a} \);
(3) The discrete spectrum of \( -\Delta_{k,a} \) is given by
\[
\{ 2a\ell + 2m + 2 \langle k \rangle + N - 2 + a : \ell, m \in \mathbb{N} \} \text{ if } N \geq 2
\]
\[
\{ 2a\ell + 2 \langle k \rangle + a + 1 : \ell \in \mathbb{N} \} \text{ if } N = 1.
\]
This follows at once from the basic identity \( \Delta_{k,a} = -a\omega_{k,a}(k) \); indeed,
\[
-\frac{1}{a} \Delta_{k,a} \Phi_{k,a}^\ell(p,x) = \omega_{k,a}(k) \Phi_{k,a}^\ell(p,x) = (2\ell + \lambda_{k,a,m} + 1)\Phi_{k,a}^\ell(p,x),
\]
that is
\[
\Delta_{k,a} \Phi_{k,a}^\ell(p,x) = -a(2\ell + \lambda_{k,a,m} + 1)\Phi_{k,a}^\ell(p,m).
\]
Additional work furthermore leads to the following important result, cf. Theorem 3.39 in [BSKO12].

Theorem 2.7. Assume \( a > 0 \) and that the nonnegative multiplicity function \( k \) satisfies the condition \( a + 2 \langle k \rangle + N - 2 > 0 \).
(1) The map \( \mathbb{C}^+ \times L^2_{k,a}(\mathbb{R}^N) \rightarrow L^2_{k,a}(\mathbb{R}^N) \), \((z, f) \mapsto e^{-z\Delta_{k,a}}f\) is continuous.

(2) For any \( p \in \mathcal{H}_k^0(\mathbb{R}^N) \) and \( \ell \in \mathbb{N} \) it holds that \( e^{-z\Delta_{k,a}}\Phi_{\ell}^{(a)}(p, \cdot) = e^{-z(\lambda_{k,a,m} + 2\ell + 1)}\Phi_{\ell}^{(a)}(p, \cdot) \).

(3) The operator norm of \( e^{-z\Delta_{k,a}} \) equals \( \exp(-\frac{1}{a}(2(k) + N + a - 2)\Re(z)) \).

(4) If \( \Re(z) > 0 \), then \( e^{-z\Delta_{k,a}} \) is a Hilbert–Schmidt operator.

(5) If \( \Re(z) = 0 \), then \( e^{-z\Delta_{k,a}} \) is a unitary operator.

In particular the operator \( e^{-z\Delta_{k,a}} \) has a distribution kernel, that is, there exists a ‘kernel’ \( \Lambda_{k,a}(x, y; z) \) such that

\[ e^{-z\Delta_{k,a}}f(x) = \int_{\mathbb{R}^N} \Lambda_{k,a}(x, y; z) \partial_{k,a}(y) \, dy \]

for \( f \in L^2_{k,a}(\mathbb{R}^N) \).

In general no closed expression for \( \Lambda_{k,a}(x, y; z) \) is available; the paper \[BSK012\] lists explicit formulae whenever \( N = 1 \) and \( a > 0 \) is arbitrary, or whenever \( N > 2 \) is arbitrary and \( a \in (1, 2] \). We shall recall these below but for some applications it suffices to have a series expansion, which is valid in general. To this end we first introduce the \( I \)-Bessel function \( I_\lambda(w) = e^{-\frac{1}{2}i\lambda}\Lambda_{\lambda}(e^{\frac{1}{2}w}) \), where \( \Lambda_{\lambda} \) is the standard Bessel function. Moreover define

\[ \tilde{I}_\lambda(w) := (w/2)^{-\lambda}I_\lambda(w) = \frac{1}{\sqrt{\pi}\Gamma(\lambda + \frac{1}{2})} \int_{-1}^{1} e^{w t}(1 - t^2)^{\lambda - \frac{1}{2}} \, dt. \]

It holds that \( |\tilde{I}_\lambda(w)| \leq \Gamma(\lambda + 1)(1 - |\Re(w)|)^{\lambda - \frac{1}{2}} \), which can be seen from the integral representation of \( \tilde{I}(\lambda) \) for \( \lambda > -1/2 \) as follows:

\[ |\tilde{I}_\lambda(w)| \leq \frac{1}{\Gamma(\lambda + \frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} e^{-\ell|\Re(w)|} (1 - t^2)^{\lambda - \frac{1}{2}} \, dt \leq \frac{e^{1/|\Re(w)|}}{\Gamma(\lambda + \frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} (1 - t^2)^{\lambda - \frac{1}{2}} \, dt \leq C e^{1/|\Re(w)|} \]

for some constant \( C \) independent of \( w \). It follows from the identity \( I_\lambda(w) = \frac{1}{2}(I_{\lambda-1}(w) + I_{\lambda+1}(w)) \) that

\[ \tilde{I}_\lambda(w) = \frac{d}{dw} \left( \left(\frac{w}{2}\right)^{-\lambda}I_\lambda(w) \right) = -\frac{\lambda}{2}\left(\frac{w}{2}\right)^{-\lambda-1}\tilde{I}_\lambda(w) + \frac{1}{2}\left(\frac{w}{2}\right)^{-\lambda-1}\tilde{I}_{\lambda-1}(w) + \frac{1}{2}\left(\frac{w}{2}\right)^{-\lambda-1}\tilde{I}_{\lambda+1}(w), \]

and similarly for higher derivatives. Therefore

\[ |\tilde{I}_\lambda(w)| \lesssim e^{1/|\Re(w)|} \]

the constant implied in the notation being independent of \( w \). An analogous estimate holds for higher derivatives \( \tilde{I}_\lambda^{(\ell)} \).

Finally define

\[
\Gamma_{k,a}^{(m)}(r, s; z) = \frac{(rs)^{-\frac{(k) + N - 1}{2}}}{\sinh z (l^a + s^a) \coth z} \frac{2}{\sinh z} \frac{2}{a} \frac{e^{-\frac{1}{a}(l^a + s^a) \coth z}}{I_{l}^{(a)}(2(rs)^{a/2})} \\
= \frac{(rs)^m}{a^{l^a + s^a + 1} \sinh z} \tilde{I}_{l}^{(a)}(2(rs)^{a/2}) \sum_{m \in \mathbb{N}} \Lambda_{k,a}^{(m)}(r, s; z) P_{k,m}(\omega, \eta),
\]

cf. formula (4.11) in \[BSK012\]. According to Theorem 4.20. loc. cit the kernel \( \Lambda_{k,a}(x, y; z) \) is then expressed in terms of the \( \Lambda_{k,a}^{(m)} \) according to

\[
\Lambda_{k,a}(x, y; z) = a^{((2(k) + N - 2)/a)} \Gamma\left( \frac{2}{a} l + m + \frac{N - 2}{2} \right) \sum_{m \in \mathbb{N}} \Lambda_{k,a}^{(m)}(r, s; z) P_{k,m}(\omega, \eta),
\]
where \( x = r\omega, y = s\eta \) (polar coordinates), and

\[
P_{k,m}(\omega, \eta) = \left( \frac{k + m + \frac{N-2}{2}}{k + \frac{N-2}{2}} \right) \left( \tilde{V}_k C_{k+m+\frac{N-2}{2}}(\omega, \eta) \right).
\]

Here \( P_{k,m} \) denotes the reproducing kernel of the space \( \mathcal{H}_k^m(\mathbb{R}^N) \) of spherical \( k \)-harmonic polynomials of degree \( m \).

**Definition 2.8.** The \( (k,a) \)-generalized Fourier transform \( \mathcal{F}_{k,a} \) is the unitary operator

\[
\mathcal{F}_{k,a} = \exp \left[ \frac{i\pi}{2} \left( \frac{1}{a} (2\langle k \rangle + n + a - 2) \right) \right] \exp \left[ \frac{i\pi}{2a} (\|x\|^{2-a}\Delta_k - \|x\|^a) \right]
\]

defined on \( L^2_{k,a}(\mathbb{R}^N) \).

Some notable special cases include:

- \( a = 2, k \equiv 0 \). Then \( \mathcal{F}_{k,a} \) is the Euclidean Fourier transform (see [How88b]):
- \( a = 1, k \equiv 0 \). Then \( \mathcal{F}_{k,a} \) is the Hankel transform and appears in [KMI11] as the unitary inversion operator of the Schrödinger model of the minimal representation of the group \( O(N+1,2) \):
- \( a = 2, k > 0 \). Then we recover the Dunkl transform.

In other words \( \mathcal{F}_{k,a} \) ‘interpolates’ between several types of integral transforms and allows a unified study of these. Key to discovering the properties of \( \mathcal{F}_{k,a} \) is the observation that

\[
\mathcal{F}_{k,a} = e^{i\pi((2\langle k \rangle+N+a-2)/a)} \Omega_{k,a}(\gamma_{\pi/2}),
\]

where \( \gamma_{\pi/2} = \exp(\pi k) \) and \( \Omega_{k,a} \) is the unique unitary representation of the double cover \( \text{SL}(2,\mathbb{R}) \) on \( L^2_{k,a}(\mathbb{R}^N) \) with \( \psi_{k,a} \) as infinitesimal generator.

**Theorem 2.9.** Let \( a > 0 \) be given and assume \( k \) satisfies \( a + 2\langle k \rangle + N > 2 \).

1. (Plancherel formula) The operator \( \mathcal{F}_{k,a} \) is a unitary map of \( L^2_{k,a}(\mathbb{R}^N) \) onto itself.
2. \( \mathcal{F}_{k,a}(\Phi^{(a)}_\ell(p,\cdot)) = e^{-in((\ell+m)/a)} \Phi^{(a)}_\ell(p,\cdot) \) for any \( \ell, m \in \mathbb{N} \) and \( p \in \mathcal{H}_k^m(\mathbb{R}^N) \).
3. \( \mathcal{F}_{k,a} \) is of finite order if and only if \( a \in \mathbb{Q} \). If \( a \in \mathbb{Q} \) is of the form \( a = \frac{q}{q'} \), with \( q, q' \) positive, then \( (\mathcal{F}_{k,a})^{2q} = \text{Id} \). In particular \( \mathcal{F}_{k,a}^{-1} = \mathcal{F}_{k,a}^{2q-1} \).

**Proof.** See Theorem 5.1 in [BSK012]. The last assertion follows by observing that \( (\mathcal{F}_{k,a})^{2q} \) acts on \( \Phi^{(a)}_\ell(p,\cdot) \) as scalar multiplication by \( (e^{-in((\ell+m)/a)})^{2q} = 1 \) for every \( m \in \mathbb{N} \) and \( p \in \mathcal{H}_k^m(\mathbb{R}^N) \). \( \Box \)

**Theorem 2.10** (Inversion formula). Let \( k \) be a non-negative multiplicity function.

1. Let \( r \in \mathbb{N} \) and suppose that \( 2\langle k \rangle + N > 2 - \frac{1}{r} \). Then \( (\mathcal{F}_{k,1/r})^{-1} = \mathcal{F}_{k,1/r} \).
2. Let \( r \in \mathbb{N}_0 \) and suppose that \( 2\langle k \rangle + N > 2 - \frac{2}{2r+1} \). Then \( \mathcal{F}_{k,\frac{2}{2r+1}} \) is a unitary operator of order four on \( L^2_{k,\frac{2}{2r+1}}(\mathbb{R}^N) \). The inversion formula is given as

\[
(\mathcal{F}_{k,\frac{2}{2r+1}}^{-1} f)(x) = (\mathcal{F}_{k,\frac{2}{2r+1}} f)(-x).
\]

**Proof.** See Theorem 5.3 in [BSK012]. \( \Box \)
Finally we mention the following useful and easily verified identities (stated as Theorem 5.6 in [BSKØ12]):

\begin{align}
\mathcal{F}_{k,a} \circ E &= -(E + N + 2 \langle k \rangle + a - 2) \circ \mathcal{F}_{k,a} \\
\mathcal{F}_{k,a} \circ \|x\|^a &= -\|x\|^{2-a} \Delta_k \circ \mathcal{F}_{k,a} \\
\mathcal{F}_{k,a} \circ (\|x\|^{2-a} \Delta_k) &= -\|x\|^a \circ \mathcal{F}_{k,a}.
\end{align}

In particular $\mathcal{F}_{k,a} \circ \Delta_{k,a} = \Delta_{k,a} \circ \mathcal{F}_{k,a}$.

By the Schwartz kernel theorem there exists a distribution kernel $B_{k,a}(x, y)$ such that

$$
\mathcal{F}_{k,a}f(x) = c_{k,a} \int_{\mathbb{R}^N} B_{k,a}(x, y) f(y) dy.
$$

**Example 2.11** (The case $N = 1$, $a > 0$). For $N = 1$ there is but a single choice of root system, $\mathcal{R} = \{ \pm 1 \}$ (up to scaling), and $\mathcal{C} = \{ \text{id}, \sigma \} \simeq \mathbb{Z}/2\mathbb{Z}$, as well as $\langle k \rangle = k > \frac{1}{2}(1-a)$. In this case $\vartheta_{k,a}(x) = |x|^{2k+a-2} dx$,

$$
B_{k,a}(x, y) = \Gamma\left( \frac{2k + a - 1}{a} \right) \left[ \tilde{J}_{2k+1}(\frac{2}{a}|xy|^{a/2}) + \frac{xy}{(ia)^{2/a}} \tilde{J}_{2k+1}(\frac{2}{a}|xy|^{a/2}) \right],
$$

where the branch of $i^{2/a}$ is chosen so that $1^{2/a} = 1$, where $\tilde{I}_{\nu}(w) = \tilde{I}_{\nu}(-iw)$, and $\tilde{I}_{\nu}(w)$ is the normalized Bessel function

$$
\tilde{I}_{\nu}(w) = \left( \frac{w}{2} \right)^{-\nu} I_{\nu}(w) = \sum_{\ell=0}^{\infty} \frac{w^{2\ell}}{\sqrt{\pi} \Gamma(\nu + \ell + 1)} = \frac{1}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{-1}^{1} e^{w t}(1 - t^2)^{\nu - \frac{1}{2}} dt.
$$

In addition, it follows from the aforementioned series expression for $\Lambda_{k,a}(x, y; z)$ in terms of the radial components $\Lambda_{k,a}^{(m)}(x, y; z)$, that

\begin{align}
\Lambda_{k,a}(x, y; z) &= \Gamma\left( \frac{2k + a - 1}{a} \right) e^{-\frac{1}{2}(|x|^a + |y|^a) \coth z} \\
&\quad \times \left[ \tilde{I}_{2k+1}(\frac{2}{a}|y|^a \sqrt{\sinh z}) + \frac{1}{a^{2/a}} \frac{xy}{|\sinh z|^{2/a}} \tilde{I}_{2k+1}(\frac{2}{a}|y|^a \sqrt{\sinh z}) \right].
\end{align}

**Example 2.12** (The case $N \geq 2$, $a \in \{1, 2\}$). It is established in [BSKØ12] Section 4.4 that

$$
h_{k,a}(r, s; z, t) = \frac{\exp[-\frac{1}{a}(r^a + s^a) \coth z]}{\sinh(z)^{(2k+N+a-2)/a}} \times \begin{cases} \\
\Gamma(\langle k \rangle + \frac{N-1}{2}) \tilde{I}_{\langle k \rangle} + \frac{\sqrt{2}(r s)^{1/2}}{\sinh z} (1 + t)^{1/2} & \text{ when } a = 1 \\
\exp\left( \frac{r s t}{\sinh z} \right) & \text{ when } a = 2
\end{cases}
$$

where $\tilde{I}_{\nu}$ is the normalized $I$-Bessel function defined in (11).
Lemma 2.13. Assume $N \geq 1$, $k \geq 0$, $a + 2(k) + N > 2$, and that exactly one of the following additional assumptions holds:

$$
\begin{align*}
(i) & \quad N = 1 \text{ and } a > 0; \\
(ii) & \quad a \in \{1, 2\}; \\
(iii) & \quad k \equiv 0 \text{ and } a = \frac{2}{m} \text{ for some } m \in \mathbb{N}.
\end{align*}
$$

(17)

Then $B_{k,a}$ is uniformly bounded, that is, $|B_{k,a}(\xi, x)| \leq C$ for all $x, \xi \in \mathbb{R}^N$, where $C$ is a finite constant that only depends on $N$, $k$, and $a$.

Proof. The case $N = 1$ follows from the explicit formula for $B_{k,a}$ in example 2.11 or by observing that the one-dimensional Dunkl-kernel $B_{k,2}$ is known to be uniformly bounded by 1. The kernel $B_{k,a}$ is a scaled version and is therefore uniformly bounded by a constant that depends on $a$.

The second case is stated as [BSKØ12, Theorem 5.11], and the remaining case was established in [DB13, Theorem 3].

Convention: We shall replace $\mathcal{F}_{k,a}$ by the rescaled version $\mathcal{F}_{k,a}/C$ but continue to use the same symbol $\mathcal{F}_{k,a}$.

It is presently unknown whether the kernel $B_{k,a}$ is uniformly bounded for all admissible parameters $a$, so the following Hausdorff–Young inequality – which was not stated in [BSKØ12] – might not be valid in general. We list it here since it will be used in section 7 where inequalities for Shannon entropy are obtained.

Proposition 2.14. Assume $N$, $k$, and $a$ meet the assumptions in lemma 2.13. For $p \in [1, 2]$ fixed and $p' := \frac{p}{p-1}$,

$$
\| \mathcal{F}_{k,a} f \|_{L^p_{k,a}} \leq \| f \|_{L^{p'}_{k,a}}
$$

for all $f \in L^p_{k,a}(\mathbb{R}^N)$.

Without the aforementioned convention in place one would have to include a constant on the right hand side due to interpolation. As this constant is a nuisance and tends to cloud later applications of the Hausdorff–Young inequality, we decided to rescale $\mathcal{F}_{k,a}$ to get rid of the interpolation constant.

Proof. Since $\mathcal{F}_{k,a}$ is unitary on $L^2_{k,a}(\mathbb{R}^N)$ according to theorem 2.9 it is of strong type $(2, 2)$. Moreover $|\mathcal{F}_{k,a} f(\xi)| \leq \| f \|_1$ for every $\xi \in \mathbb{R}^N$ by convention and $f \in L^1_{k,a}(\mathbb{R}^N)$ by lemma 17 so $\mathcal{F}_{k,a}$ is of strong type $(1, \infty)$. The conclusion now follows from the Riesz–Thorin interpolation theorem.

A more precise formulation is given as follows. Let $f \in (L^1_{k,a} \cap L^2_{k,a})(\mathbb{R}^N)$. If $1 \leq p \leq 2$, $p' = \frac{p}{p-1}$, then $\| \mathcal{F}_{k,a} F f \|_{p'} \leq c_p \| f \|_p$. Since $L^p$ is dense in $(L^1_{k,a} \cap L^2_{k,a})(\mathbb{R}^N)$ for $1 \leq p \leq 2$, the transform $\mathcal{F}_p$ can be defined uniquely for all $f \in L^p_{k,a}(\mathbb{R}^N)$, $1 \leq p \leq 2$, so that $\mathcal{F}_p : L^p_{k,a}(\mathbb{R}^N) \to L^{p'}_{k,a}(\mathbb{R}^N)$ is a linear contraction with $\mathcal{F}_p f = \mathcal{F}_{k,a} f$ for all $f \in (L^1_{k,a} \cap L^2_{k,a})(\mathbb{R}^N)$.

Lemma 2.15. Assume $N$, $k$, and $a$ meet the assumptions in lemma 2.13 and let $p \in (1, 2]$. The map $\mathcal{F}_p : L^p_{k,a}(\mathbb{R}^N) \to L^{p'}_{k,a}(\mathbb{R}^N)$ is surjective if and only if $p = 2$. 


Proof. The ‘if’-part being the Plancherel theorem for $\mathcal{F}_{k,a}$, assume $p \in (1, 2)$. The ‘only if’-part follows by contradiction as in the proof of [Joh15b Corollary 3.10], with the obvious notational modifications. \qed

A weighted extension will be established in theorem 8.4, a special case of which will be an analogue of Pitt’s inequality.

**Lemma 2.16.** Assume $N$, $k$, and $a$ and meet the assumptions in lemma 2.13. If $f$ belongs to $(L^p_{k,a} \cap L^q_{k,a})(\mathbb{R}^N)$ for some $p_1, p_2 \in [1, 2]$, then $\mathcal{F}_{p_1} f = \mathcal{F}_{p_2} f \mu_{k,a}$-almost everywhere on $\mathbb{R}^N$.

Proof. Choose a sequence $\{g_n\}_{n=1}^\infty$ of simple functions on $\mathbb{R}^N$ such that

$$\lim_{n \to \infty} \|f - g_n\|_{p_1} = \lim_{n \to \infty} \|f - g_n\|_{p_2} = 0.$$ 

Each function $\mathcal{F}_{k,a} g_n$ belongs to $(L^p_{k,a} \cap L^q_{k,a})(\mathbb{R}^N)$ by the Hausdorff–Young inequality, and

$$\lim_{n \to \infty} \|\mathcal{F}_{p_1} f - \mathcal{F}_{k,a} g_n\|_{p_1}' = \lim_{n \to \infty} \|\mathcal{F}_{p_2} f - \mathcal{F}_{k,a} g_n\|_{p_2}' = 0.$$ 

One can therefore extract subsequences $\{\mathcal{F}_{k,a} g_{n_k}\}_{k=1}^\infty$ and $\{\mathcal{F}_{k,a} g_{n_k}\}_{k=1}^\infty$ of $\{\mathcal{F} g_n\}_{n=1}^\infty$ such that $\mathcal{F}_{k,a} g_{n_k} \to \mathcal{F}_{p_1} f$ and $\mathcal{F}_{k,a} g_{n_k} \to \mathcal{F}_{p_2} f$, $\mu_{k,a}$-almost everywhere on $\mathbb{R}^N$, from which it follows that $\mathcal{F}_{p_1} f = \mathcal{F}_{p_2} f$, $\mu_{k,a}$-almost everywhere on $\mathbb{R}^N$ as claimed. \qed

**Lemma 2.17.** Assume $N$, $k$, and $a$ satisfy either (i) or (ii) in lemma 2.13. The Euclidean Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is dense in $L^p_{k,a}(\mathbb{R}^N)$ for $p \in [1, \infty)$ and invariant under $\mathcal{F}_{k,a}$.

Proof. Only the invariance under $\mathcal{F}_{k,a}$ needs to be addressed. In the case $a = 2$, the statement is that $\mathcal{S}(\mathbb{R}^N)$ is invariant under the Dunkl transform, a fact that was established in [dJ93 Corollary 4.8]. In the general one-dimensional case one can redo de Jeu’s proof, especially the boundedness of derivatives of the Dunkl kernel in [dJ93 Corollary 3.7] for the ‘deformed’ kernel function $B_{k,a}$; it follows from the explicit formula in 2.11 that it satisfies the same bounds, implying that the transform $\mathcal{F}_{k,a}$, $a > 0$, leaves $\mathcal{S}(\mathbb{R}^N)$ invariant as well.

The case $N \geq 2$, $a = 1$, is also handled by a direct appeal to the explicit formula for $B_{k,1}$, this time in example 2.12. The estimate for the derivatives of $B_{k,a}$, replacing [dJ93 Corollary 3.7] or [Rös99 Corollary 5.4], is obtained from [12] and example 2.12, together with the Bochner-type integral representation of the intertwining operator $\tilde{V}^k$: One utilities that derivatives of $h_{k,a}(r, s, z, t)$ in the case $a = 1$ – both as a function of $r$ and as a function of $s$ – grows exponentially at the same rate as derivatives of the kernel function $\exp(x^2)$ in the Dunkl-case $a = 2$. The argument by de Jeu that leads from [dJ93 Corollary 3.7] to [dJ93 Corollary 4.8] can therefore be repeated.

The remainder of the section is concerned with interpolation results in Lorentz spaces that will be needed in our proof of the Hardy–Littlewood inequality. The interested reader may consult [SW71 Chapter VI] for detailed proofs and historical remarks. Let $(X, \mu)$ be a $\sigma$-finite measure space and let $p \in (1, \infty)$. Define

$$\|f\|_{p,q}^* = \begin{cases} \left( \frac{q}{p} \int_0^\infty t^{q-p-1} f^*(t)^q \, dt \right)^{1/q} & \text{if } q \leq \infty, \\
\sup_{t > 0} t^{q/p} \lambda_f(t)^{1/p} \text{ if } q = \infty \end{cases}$$

where $\lambda_f$ is the distribution function of $f$ and $f^*$ the non-increasing rearrangement of $f$, that is

$$\lambda_f(s) = \mu(\{x \in X : |f(x)| > s\}) \quad \text{and} \quad f^*(t) = \inf\{s : \lambda_f(s) \leq t\}.$$
By definition, the Lorentz space $L^{p,q}(X)$ consists of measurable functions $f$ on $X$ for which $\|f\|_{p,q} < \infty$.

**Definition 2.18.** Let $(X, d\mu)$ and $(Y, d\nu)$ be $\sigma$-finite measure spaces. A linear operator $T : L^p(X, d\mu) \to L^q(Y, d\nu)$ is strong type $(p, q)$ if it is continuous on $L^p(X, d\mu)$. Moreover, $T$ is weak type $(p, q)$ if there exists a positive constant $K$ independent of $f$ such that for all $f \in L^p(X, d\mu)$ and all $t > 0$,

$$\mu(\{y \in Y : |Tf(y)| > t\}) \leq \left(\frac{K}{s}\|f\|_{L^p(X, d\mu)}\right)^q.$$ 

The infimum if such $K$ is the weak type $(p, q)$ norm of $T$.

Although $\| \cdot \|_{p,q}$ is merely a seminorm in general, the spaces $L^{p,q}(X)$ are useful in interpolation arguments. The following interpolation theorem is classical and can be found as Theorem 3.15 in [SW71] Chapter V. It subsumes the interpolation theorem of Marcinkiewicz, for example.

**Theorem 2.19 (Interpolation between Lorentz spaces).** Suppose $T$ is a subadditive operator of (restricted) weak types $(r_j, p_j)$, $j = 0, 1$, with $r_0 < r_1$ and $p_0 \neq p_1$, then there exists a constant $B = B_0$ such that $\|T\|_{p,q} \leq B\|f\|_{r,q}$ for all $f$ belonging to the domain of $T$ and to $L^{p,q}$, where $1 \leq q \leq \infty$,

$$\frac{1}{p} = 1 - \frac{\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{r} = 1 - \frac{\theta}{r_0} + \frac{\theta}{r_1} \quad \text{and} \quad 0 < \theta < 1.$$ 

**Corollary 2.20** (Paley’s extension of the Hausdorff–Young inequality). If $f \in L^p(\mathbb{R}^n)$, $1 < p \leq 2$, then its Fourier transform $\hat{f}$ belongs to $L^{p'}(\mathbb{R}^n)$ and there exists a constant $B = B_p$ independent of $f$ such that $\|\hat{f}\|_{p'} \leq B_p\|f\|_p$, where $\frac{1}{p} + \frac{1}{p'} = 1$. In particular the Fourier transform is a continuous linear mapping from $L^p(\mathbb{R}^n)$ to the Lorentz space $L^{p'}(\mathbb{R}^n)$ for $1 < p < 2$.

**Proof.** Taking $(r_0, p_0) = (1, \infty)$, $(r_1, p_1) = (2, 2)$ in theorem 2.19 the conditions in (18) translate into $\frac{1}{p} = \frac{q}{2}$ and $\frac{1}{r} = 1 - \frac{\theta}{2}$, that is, $r = p'$. Furthermore take $q = r$. Since $\theta \in (0, 1)$ in the hypothesis of theorem 18, the role of $p$ and $p'$ must be exchanged when we consider the setup in the present corollary. (Since $\frac{2}{p} = \theta \in (0, 1)$ if and only if $p > 2$). With this adjustment in mind, the conclusion to theorem 18 becomes $\|\hat{f}\|_{p'} \leq B\|f\|_p$, $\|f\|_{p'} = B\|f\|_p$. 

As in the proof of corollary 2.20 we obtain the following extension immediately from the interpolation theorem 2.19.

**Corollary 2.21.** Assume $N$, $k$, and $a$ and the meeting the assumptions in lemma 2.12. The $(k, a)$-generalized transform $F_{k,a}$ is a continuous mapping from $L^p_{k,a}(\mathbb{R}^N)$ to $L^p_{k,a}(\mathbb{R}^N)$ whenever $1 < p < 2$.

The preceding two corollaries are stronger than their respective standard forms since $L^{p', p}$ is continuously and properly embedded in $L^p$.

The last result on Lorentz spaces that we will need is due to R. O’Neil, [O’N63], and concerns the pointwise product of two functions.

**Theorem 2.22.** Let $q \in (2, \infty)$ and set $r = \frac{q}{q-2}$. For $g \in L^q(X)$ and $h \in L^{r, \infty}(X)$ it holds that $gh$ belongs to $L^{q,q}(X)$ with $\|gh\|_{q,q} \leq \|g\|_q \|h\|_{r,\infty}$. 

2.1. Hankel transforms and radial functions. It is a very useful fact of classical analysis that the Fourier transform of a radial function on \( \mathbb{R}^n \) is radial and given by a suitable Hankel transform of the radial projection. It was observed in Proposition 2.4 in [RV98] that the Dunkl transform of a radial function in \( L^1(\mathbb{R}^n, \omega_k(x)dx) \) is also radial and expressed in terms of an appropriate Hankel transform. Specifically, if \( f \in (L_{k,a}^1 \cap L_{k,a}^2)(\mathbb{R}^n) \) is of the form \( f(x) = p(x)\psi(||x||) \) for some \( p \in \mathcal{H}_k^0(\mathbb{R}^N) \) and some function \( \psi \in \mathbb{R}_+ \), then

\[
\mathcal{F}_{k,a} f(\xi) = a^{-((2m+2(k)+N-2)/\alpha)} e^{-\frac{\alpha}{\pi}a} p(\xi) H_{\nu} \left[ \frac{2^{m+2(k)+N-2}}{\alpha} (\psi)(||\xi||) \right]
\]

where \( H_{a,\nu}(\psi) := \int_0^\infty \psi(r) \tilde{J}_\nu \left( \frac{2}{a} (rs)^{a/2} \right) r^{a(\nu+1)-1} \, dr \). Recall that

\[
\tilde{J}_\nu(\omega) = \left( \frac{\omega}{2} \right)^{-\nu} J_\nu(\omega) = \sum_{\ell=0}^\infty \frac{(-1)^\ell \omega^{2\ell}}{2^{2\ell} \ell! (\nu + \ell + 1)} = \frac{1}{\Gamma(\nu+1)} j_\nu(\omega),
\]

where \( j_\nu \) is the modified Bessel function that usually appears in the definition of the classical Hankel transform \( \mathcal{H}_\nu \).

**Definition 2.23.** Given parameters \( p \in [1, \infty) \), \( a > 0 \), and \( \nu > -1/2 \), the norm \( \|f\|_{p,a,\nu} \) of a measurable function \( f \) on \( \mathbb{R}_+ \) is defined by

\[
\|f\|_{p,a,\nu} = \left( \int_0^\infty |f(r)|^{p,a(\nu+1)-1} \, dr \right)^{1/p}.
\]

The following elementary observation records a useful scaling property of the norms \( \| \cdots \|_{p,a,\nu} \) as \( a \) varies. The proof is carried out by change of variables and will be left to reader.

**Lemma 2.24.** Given a measurable function \( \psi \) defined on \( \mathbb{R}_+ \), define \( \tilde{\psi}(u) = \psi((\frac{u}{2})^{1/a} u^{2/a}) \). Then

\[
\| H_{a,\nu}(\psi) \|_{p,a,\nu} = \left( \frac{2}{a} \right)^{\frac{\nu+1}{p}} \left( \frac{a}{2} \right)^{\frac{\nu+1}{p}(\nu+1)} \| H_{2,\nu}(\tilde{\psi}) \|_{p,\nu} =: M_{a,p} \| H_{2,\nu}(\tilde{\psi}) \|_{p,\nu}.
\]

Since the density \( \vartheta_{k,a}(x) = \| x \|^{-2} \prod_{\alpha \in \mathbb{R}_+} |(\alpha, x)|^{2k_\alpha} = \| x \|^{-2} \omega_k(x) \) is homogeneous of degree \( 2(k+a) - 2 \), it is clear that the \( L^p \)-norm of a radial function \( f \) of the form \( f(x) = \psi(||x||) \) can be expressed in terms of a suitable \( L^p \)-norm of \( \psi \). This is seen by passing to polar coordinates \( x = r\omega, r > 0, \omega \in \mathbb{S}^{N-1} \) and we collect the precise statement for later reference in the following

**Lemma 2.25.** Let \( 1 \leq p \leq 2 \) and let \( f \in L_{k,a}^p(\mathbb{R}^N) \) be radial of the form \( f(x) = \psi(||x||) \) for a suitable measurable function \( \psi \) on \( \mathbb{R}_+ \). Then

\[
\| f \|_{L_{k,a}^p} = K^{1/p} \| \psi \|_{L_p(\mathbb{R}_+, x^{2(k)+N+a-3} \, ds)} = K^{1/p} \| \psi \|_{p,a,\nu_a},
\]

where \( \nu_a := \frac{2(k)+N-2}{a} \) and \( K = K_{k,a,N} := \int_{\mathbb{S}^{N-1}} \vartheta_{k,a}(\omega) \, d\sigma(\omega) \).

The parameter \( \nu_a \) and the norm \( \| \cdot \|_{p,a,\nu} \) are defined in such a way that we recover, in particular, the results of Rösler and Voit (specifically Proposition 2.4 in [RV98]) by choosing \( a = 2 \).
Proof. For \( f \in L^p_\text{rad}(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \) of the form \( f(x) = \psi(||x||) \) and \( p \in (1, 2] \) fixed it follows by change of variables that
\[
\int_{\mathbb{R}^N} |f(x)|^p \vartheta_{k,a}(x) \, dx = \int_0^\infty \int_{S_{N-1}} |f(r\omega)|^p \vartheta_{k,a}(r\omega) \, d\sigma(\omega) \, r^{N-1} \, dr = \int_0^\infty |\psi(r)|^p \int_{S_{N-1}} r^{2(k)+a-2} \vartheta_{k,a}(\omega) \, d\sigma(\omega) \, r^{N-1} \, dr = \left( \int_{S_{N-1}} \vartheta_{k,a}(\omega) \, d\sigma(\omega) \right) \int_0^\infty |\psi(r)|^p r^{2(k)+a+N-3} \, dr,
\]
which proves the assertion. \( \square \)

Two examples will be needed later so we collect them here:

1. The Gaussian \( \gamma_t : y \mapsto e^{-t||y||^2} \). Here \( \psi(y) = e^{-b^2} \), with
\[
||\psi||_{L^p(\mathbb{R}^+, r^{2(k)+N+a-3} \, dr)} = \frac{\Gamma(\frac{2(k)+N+a-2}{2})}{2(p)^{\frac{2(k)+N+a-2}{2}}},
\]
so lemma 2.25 implies that
\[
\|A\|_{L^p_{k,a}} = K^{1/p} \left( \frac{\Gamma(\nu_a + 1)}{2p^{\nu_a + 1}} \right)^{1/p} t^{-\frac{2k}{p}},
\]
where \( \nu_a = \frac{2(k)+N-a-2}{a} \).

2. The function \( g_a : y \mapsto |y|^{-\alpha} 1_{B_r}(y) \), \( 0 < \alpha < \frac{2(k)+N+a-2}{p} \), where \( B_r = \{x \in \mathbb{R}^N : |x| \leq r\} \). Here \( \psi(y) = y^{-\alpha} 1_{[0,r]}(y), r > 0 \), and
\[
||\psi||_{L^p(\mathbb{R}^+, r^{2(k)+N+a-3} \, dr)} = \left( \frac{2k}{p} \right)^{\frac{a(a+1)}{2}} r^{2(k)+N+a-2-\alpha p}.
\]
so lemma 2.25 implies that
\[
\|g_a\|_{L^p_{k,a}} = \frac{K^{1/p} \left( \frac{\Gamma(\alpha+1)}{a(p+1)} \right)^{1/p}}{p^{\alpha+1}} r^{-\frac{a(a+1)}{p} - \alpha}.
\]

It follows from (19) and Lemma 2.25 that
\[
\|F_{k,a}(f)\|_{L^p_{k,a}} = a^{-\nu_a} \|H_{a,\nu_a}(\psi)\|_{p,a,\nu_a},
\]
where \( \nu_a = \frac{2(k)+N-2}{a} \). Therefore
\[
\|F_{k,a}(f)\|_{L^p_{k,a}} = K^{1/q} a^{-\nu_a} \|H_{a,\nu_a}(\psi)\|_{q,a,\nu_a} = K^{1/q} a^{-\nu_a} M_{a,q,\nu_a} \|H_{2,\nu_a}(\psi)\|_{q,a,\nu_a},
\]
so a sharp Hausdorff–Young theorem for the Hankel transform \( H_{2,\nu_a} \) would imply a sharp Hausdorff–Young inequality for the restriction of \( F_{k,a} \) to radial functions. This seems to be an open problem, however.

On the other hand, Omri [Omr11] recently established a sharp Pitt’s inequality for the classical Hankel transform and used it to derive an analogue of Beckner’s sharp logarithmic inequality. Since this logically implies sharp inequalities for the restriction of \( F_{k,a} \) to radial functions we take the opportunity to make this connection precise by explaining the correspondence between the classical Hankel transform (which is considered in [Omr11]) and the aforementioned transform \( H_{a,\nu_a} \).
To this end fix a parameter \( \nu > -1/2 \). The classical Hankel transform \( \mathcal{H}_\nu f \) of a suitable function \( f \) defined on \((0, \infty)\) is defined by

\[
\mathcal{H}_\nu f(\lambda) = \frac{1}{2^\nu \Gamma(\nu + 1)} \int_0^\infty f(x) j_\nu(\lambda x) x^{2\nu+1} \, dx,
\]

where

\[
j_\nu(x) = \frac{2^\nu \Gamma(\nu + 1)}{\lambda^{\nu+1}} \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{2^{2n} n! \Gamma(n + \nu + 1)}.
\]

In particular \( \mathcal{H}_{2,\nu} f = 2^\nu \mathcal{H}_\nu f \). Aside from this slight change in normalization, one should therefore think of \( \mathcal{H}_{a,\nu} \) as a ‘deformation’ of the classical Hankel transform, and [Omr11, Theorem 3.10] therefore implies a sharp logarithmic inequality for \( \mathcal{H}_{a,\nu} \).

3. Further remarks on the Hausdorff–Young inequality

The Hausdorff–Young inequality for \( \mathcal{F}_{k,a} \) easily followed from general mapping properties and interpolation but the argument left out the possibility of a Hausdorff–Young inequality for \( p > 2 \). It is a classical fact that the Euclidean Fourier transform does not allow a Hausdorff–Young inequality for \( L^p \)-functions when \( p > 2 \), and an explicit counterexample for the Fourier transform can be found in [Tit48, Section 4.11]. We have been unable to find any such statement for the Dunkl transform \( \mathcal{F}_{k,2} \), so we have included the following short section to settle the matter, as it fits nicely into the general theme of (weighted) inequalities for \( \mathcal{F}_{k,a} \).

In this section only, the parameter \( a \) is assumed to be chosen in such a way that the Hausdorff–Young inequality and the inversion formula for \( \mathcal{F}_{k,a} \) are both valid. Comparing with theorem 2.10, the parameters \( N, k, \) and \( a \) must therefore satisfy one of the conditions

- (a) \( N = 1, k \geq 0, a > 0, a + 2k + 1 > 2 \);
- (b) \( N \geq 1, k \geq 0, a + 2(k) + N > 2, \) and \( a = \frac{1}{2r+1} \) for some \( r \in \mathbb{N} \);
- (c) \( N \geq 1, k \geq 0, a + 2(k) + N > 2, \) and \( a = \frac{2}{2r+1} \) for some \( r \in \mathbb{N}_0 \).

Let

\[
\sigma_a h(x) = \begin{cases} h(x) & \text{if (a) or (b) holds} \\ h(-x) & \text{if (c) holds} \end{cases}
\]

Recall that the case (c) subsumes the standard Dunkl transform (corresponding to the particular choice \( r = 0 \)). The inversion formula in theorem 2.10 can then be written succinctly as \( \mathcal{F}_{k,a}^{-1} = \sigma_a \circ \mathcal{F}_{k,a} \). Since the Hausdorff–Young inequality is also required to hold, this range of permissible parameters is considerably narrower, however, as we must additionally assume that the assumptions in lemma 2.13 hold. It leaves us with the three cases considered in the following result, the proof of which is adapted from [Cha00], where further historical remarks may be found.

**Proposition 3.1.** Assume \( a + 2(k) + N > 2, \) and that either

- \( N = 1 \) and \( a > 0 \) (no further constraints),
- \( N \geq 1 \) and \( a \in \{1, 2\} \), or
- \( k \equiv 2 \) and \( a = 2/m \) for some \( m \in \mathbb{N} \).

Let \( p > 2 \) be fixed and \( D \) an \( L^p \)-dense subspace of \( (L^1_{k,a} \cap L^p_{k,a})(\mathbb{R}^N) \). Then there exists no finite constant \( D_p \) such that the inequality \( \| \mathcal{F}_{k,a} f \|_{p'} \leq D_p \| f \|_p \) holds for all \( f \in D \).
Proof. Assume the conclusion is false and let $D_p$ be such a finite constant. Then there exists a continuous linear mapping $T : L^p_{k,a}(\mathbb{R}^N) \to L^{p'}_{k,a}(\mathbb{R}^N)$ such that $Tf = F_{k,a}f$ for all $f \in D$. Since $1 < p' < 2$ it follows from the Hausdorff–Young inequality and from $\sigma_a$ being an $L^{p'}$-isometry that $\|\sigma_a \circ F_{k,a}f\|_{p'} = \|F_{k,a}f\|_{p'} \leq c_{p'} \|f\|_{p'}$ for all $f \in D$. Hence there exists a linear contraction $S : L^{p'}_{k,a}(\mathbb{R}^N) \to L^p_{k,a}(\mathbb{R}^N)$ such that $Sf = \sigma_a \circ F_{p'}f$, where $F_{p'}f$ designates the $(k,a)$-generalized transform of the $L^{p'}$-function $f$, whose existence is guaranteed by the Hausdorff–Young inequality.

Note that a function $f \in D$ automatically belongs to $L^2_{k,a}(\mathbb{R}^N)$ (since $p > 2$) and to $L^1_{k,a}(\mathbb{R}^N)$ by assumption. Therefore $Tf$ belongs to $(L^2_{k,a} \cap L^3_{k,a})(\mathbb{R}^N)$ whenever $f \in D$. In particular the inversion and Plancherel formulae hold for $f$, implying that $S(Tf) = S(F_{2}f) = \sigma_a \circ F_{p'} = \sigma_a \circ F_{2}(F_{2}f) = f$ for all $f \in D$. Here it was used that $F_{p'}(F_{2}f) = F_{2}(F_{2}f)$ $\mu_{k,a}$-almost everywhere on $\mathbb{R}^N$ since $F_{2}f \in (L^2_{k,a} \cap L^{p'}_{k,a})(\mathbb{R}^N)$, according to lemma 2.16.

Since $S \circ T$ is continuous and $D$ dense in $L^p_{k,a}(\mathbb{R}^N)$, it follows that $S \circ T = \text{id}$ on $L^p_{k,a}(\mathbb{R}^N)$, in particular that $S$ is a left-inverse to $T$ and therefore surjective. This would imply that $F_{p'}$ were to be surjective on $L^p_{k,a}(\mathbb{R}^N)$, which – according to lemma 2.15 – it is not. We have therefore arrived at a contradiction, proving the claim. $\square$

The Hausdorff–Young inequality facilitates an extension of $F_{k,a}$ to a continuous map $F_p$ from $L^p_{k,a}(\mathbb{R}^N)$ into $L^{p'}_{k,a}(\mathbb{R}^N)$, where $p \in (1,2)$ and $p' = \frac{p}{p-1}$. By the same reasoning, one obtains a Hausdorff–Young inequality for the inverse transform $F^{-1}_{k,a}$, giving rise to a continuous map $F_q : L^q_{k,a}(\mathbb{R}^N) \to L^{q'}_{k,a}(\mathbb{R}^N)$ that coincides with $F^{-1}_{k,a}$ on $L^2_{k,a}(\mathbb{R}^N)$. This raises the

**Question:** Do $F_p$ and $F_{p'}$ coincide?

We recently answered the analogous question for the one-dimensional Cherednik–Opdam transform in the affirmative, cf. [Joh15b, Theorem 3.9], but the proof relied heavily on having a suitable convolution structure. While such a convolution is available in the Dunkl-case $a = 2$, we have not yet investigated these matters in detail. It would be interesting to develop a strategy of proof that would also subsume the case $a = 1$, at least.

It should also be noted that the proof of proposition 3.1 uses in an essential way the special form of the inversion formula for $F_{k,a}$ when $a \in \mathbb{Q}$, namely that $F^{-1}_{k,a} = \sigma_a \circ F_{k,a}$. This rules out an immediate extension to arbitrary deformation parameters $a \in \mathbb{Q}$, since repeated application of the Hausdorff–Young inequality to higher iterates $F^{-2q-1}_{k,a}$ would break down except when $p = 2$. For the proof it was very convenient, yet perhaps not essential, that the underlying measure space $(\mathbb{R}^N, \mu_{k,a})$ remained the same. One would otherwise have to show separately that $S$ cannot be surjective from $L^{p'}$ onto $L^p$ (which again does not rely on special properties of the transform $F_{k,a}$ or $T$, as long as $S$ is injective).

On the other hand the same methodology appears to be applicable in even dimensions to the Clifford–Fourier transform from [BDSS05], [DBX11] although one would first have to establish a suitable Hausdorff–Young inequality. We intend to return to these matters elsewhere.
4. Hardy–Littlewood inequalities

The classical Hausdorff–Young inequality \( \| \hat{f} \|_q \leq c_p \| f \|_p \), \( 1 \leq p \leq 2, \frac{1}{p} + \frac{1}{q} = 1 \), for the Euclidean Fourier transform can be viewed as a partial extension of the Plancherel theorem to \( L^p \)-functions. More generally, the Fourier transform extends to a continuous mapping from \( L^p(\mathbb{R}^N) \) into the Lorentz space \( L^{p,q}(\mathbb{R}^N) \), a result that is due to Paley. A variation on this theme is provided by the Hardy–Littlewood inequality which may be stated as follows: Fix \( q \geq 2 \) and let \( f \) be a measurable function on \( \mathbb{R}^N \) such that \( x \mapsto \| f(x) \| x \| x \|^{(1-2/q)} \) belongs to \( L^q(\mathbb{R}^N) \). Then \( f \) has a well-defined Fourier transform \( \hat{f} \) in \( L^q(\mathbb{R}^N) \) and there exists a positive constant \( A_q \) independent of \( f \) such that

\[
(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^q d\xi)^{1/q} \leq A_q \left( \int_{\mathbb{R}^N} |f(x)|^q \| x \|^{N(1-2/q)} dx \right)^{1/q}.
\]

By duality and general properties of the Fourier transform, one has the following equivalent formulation: For every \( p \in (1, 2) \) there exists a positive constant \( B_p \) independent of \( f \) such that

\[
(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^p |\xi|^{N(p-2)} d\xi)^{1/p} \leq B_p \left( \int_{\mathbb{R}^N} |f(x)|^p dx \right)^{1/p}.
\]

Note that these inequalities do not involve the dual exponent \( p' \). Our first result is a generalization of (24); it generalizes the analogous statement [AASS09, Lemma 4.1] for the Dunkl transform.

**Proposition 4.1.** Assume \( a + 2(k) + N > 2 \). If \( f \in L^p_{k,a}(\mathbb{R}^N) \) for some \( p \in (1, 2) \), then

\[
(\int_{\mathbb{R}^N} \| \xi \|^{2(k)+\frac{N+a+2}{2}(p-2)} |\mathcal{F}_{k,a} f(\xi)|^p d\mu_{k,a}(\xi))^ {1/p} \leq C_p \left( \int_{\mathbb{R}^N} |f(x)|^p d\mu_{k,a}(x) dx \right)^{1/p}.
\]

**Proof.** Consider measure spaces \( (X, d\mu) \) and \( (Y, d\nu) \), where \( X = Y = \mathbb{R}^N, d\nu(x) = \vartheta_{k,a}(x) dx \), and \( d\mu(\xi) = \| \xi \|^{-4(k)+\frac{N+a+2}{2}} \vartheta_{k,a}(\xi) d\xi \), with \( x, \xi \in \mathbb{R}^N \). Moreover define an operator \( T \) on \( L^2_{k,a}(\mathbb{R}^N) \) by

\[
Tf(\xi) = \| \xi \|^{2(k)+\frac{N+a+2}{2}} \mathcal{F}_{k,a} f(\xi), \quad \xi \in \mathbb{R}^N.
\]

Then \( T \) is of strong type \( (2, 2) \) as an operator acting between Lebesgue spaces on \( (X, d\mu) \) and \( (Y, d\nu) \), since

\[
\| Tf \|_{L^2(\nu)} = \int_{\mathbb{R}^N} |Tf(\xi)|^2 d\nu(\xi) = \int_{\mathbb{R}^N} \| \xi \|^{-4(k)+\frac{N+a+2}{2}} \vartheta_{k,a}(\xi) d\xi \cdot \| f \|_{L^2(\mu)}^2.
\]

by the Plancherel theorem [29].

The operator \( T \) is furthermore of weak type \( (1, 1) \), which finishes the proof by an application of the Marcinkiewicz interpolation theorem. To verify this claim, let \( t > 0 \) and \( f \in L^1_{k,a}(\mathbb{R}^N) \) \( \{0\} \) be fixed, and define sets

\[
A_t(f) = \{ \xi \in \mathbb{R}^N : |Tf(\xi)| > t \} \quad \text{and} \quad E_t(f) = \{ \xi \in \mathbb{R}^N : \| \xi \|^{2(k)+\frac{N+a+2}{2}} > t/\| f \|_1 \}
\]
It follows from the basic inequality \( \| \mathcal{F}_{k,a} f \|_\infty \leq \| f \|_1 \) already used to establish the Hausdorff–Young inequality that \( A_t(f) \subset E_t(f) \). Correspondingly, by passing to polar coordinates,

\[
\mathcal{T}(A_t(f)) = \int_{A_t(f)} \frac{d\vartheta_{k,a}(\xi)}{\| \xi \|^{|4(k) + \frac{N+a-2}{2}}/2} \leq \int_{E_t(f)} \frac{d\vartheta_{k,a}(\xi)}{\| \xi \|^{|4(k) + \frac{N+a-2}{2}}/2} \leq \int_1^\infty \frac{2^{|(k) + \frac{N+a-2}{2}|-1}}{r^{4(k) + \frac{N+a-2}{2}}} dr \quad \text{where } a_t = \left( \frac{t}{\| f \|_1} \right)^{2(\frac{k}{2}) + \frac{N+a-2}{2}}.
\]

\[
eq c'' \left( \frac{\int f^2}{t} \right)^{\frac{1}{2}}
\]

\[\square\]

**Remark 4.2.** An advantage in the above interpolation argument is that the possible lack of information on the integral kernel of \( \mathcal{F}_{k,a} \) is not an issue. Instead one has to compensate by adding power weights.

Two types of improvement can be obtained by using the more refined interpolation theorem between Lorentz spaces, theorem [2.19]. Inequalities with weights more general than the norm power \( \| \cdot \|^{-4(\frac{k}{2}) + \frac{N+a-2}{2}} \) can be obtained and the permissible range of exponents \( p \) can be enlarged. An efficient approach to both is to introduce the following terminology.

**Definition 4.3.** If \( \mu \) is any (positive Radon) measure on \( \mathbb{R}^N \), a *Young function (relative to \( \mu \)) is a measurable function \( \psi : \mathbb{R}^N \to \mathbb{R}^+ \) with the property that \( \mu \{ x \in \mathbb{R}^N : |\psi(x)| \leq t \} \leq t \) for all \( t > 0 \). Given such a \( \psi \), we let \( L_\psi^{(p)}(\mathbb{R}^N) \), \( 2 < p < \infty \), denote the Orlicz-type space of measurable functions \( f \) on \( \mathbb{R}^N \) for which

\[
\| f \|_{(p),\psi} := \left( \int_{\mathbb{R}^N} |f(x)|^p |\psi(x)|^{p-2} d\mu(x) \right)^{1/p} < \infty.
\]

In other words \( f \) belongs to \( L_\psi^{(p)}(\mathbb{R}^N) \) if and only if \( f \psi^{\frac{1}{p}} \) belongs to \( L^p(\mathbb{R}^N, \mu) \).

The choice of measure \( \mu \) is determined by the relevant setting and will always be absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^N \), the point of \( \psi \) being that it allows for an easy proof of weak type \((1,1)\) estimates that are needed for the interpolation arguments. This will become clear in due time, firstly we wish to mention examples of Young functions.

**Example 4.4.**

(i) In \( \mathbb{R}^N \), the function \( \psi : x \mapsto |x|^m \) is a Young function with respect to Lebesgue measure if and only if \( m = N \), since \( \{ x \in \mathbb{R}^N : |x|^m < t \} = |B(0, t^{1/m})| = C t^{N/m} \). Since norms on \( \mathbb{R}^N \) are equivalent, the \( N \)th power of any norm on \( \mathbb{R}^N \) gives rise to a Young function.

(ii) More generally, \( \psi : x \mapsto |x|^m \) is a Young function in \( \mathbb{R}^N \) with respect to the weighted measure \( d\mu_{k,a}(x) = \vartheta_{k,a}(x) dx \) for a unique choice of \( m \). Since

\[
\mu_{k,a}(\{ x \in \mathbb{R}^N : \psi(x) \leq t \}) = \mu_{k,a}(B(0, t^{1/m})) = C t^{2(k) + N+a-2}
\]

it follows that \( \psi \) is a Young function if and only if \( m = 2(k) + N + a - 2 \).

(iii) The function \( \vartheta_{k,a} \) itself is a Young function in \( \mathbb{R}^N \) with respect to the weighted measure \( \mu_{k,a} \), since

\[
\mu_{k,a}(\{ x \in \mathbb{R}^N : \vartheta_{k,a}(x) \leq t \}) \leq t \int_{\{ \vartheta_{k,a}(x) \leq t \}} dx
\]
where the integral is finite since the set \( \{ \vartheta_{k,a}(x) \leq t \} \) is compact (the level set is closed, and moreover bounded since \( \vartheta_{k,a}(x) \asymp |x|^{2-a_2} |x|^{2(k)} \), implying that \( \{ \vartheta_{k,a}(x) \leq t \} \) is contained in a ball \( B(0, \frac{ct}{p}) \) for some finite constant \( c \).

**Theorem 4.5.** Let \( q > 2 \) and \( f \in L^1_{\psi}( \mathbb{R}^N) \), where \( \psi \) is a Young function relative to \( \mu_{k,a} \).

There exists a positive constant \( D_q \) independent of \( f \) such that

\[
\int_{\mathbb{R}^N} |F|q_{k,a}(\xi)|^q \, d\mu_{k,a}(\xi) \leq D_q \|f\|_{L^q_{\psi}}^q.
\]

**Proof.** Let \( f \) be a simple function on \( \Lambda \) and let \( T\lambda = \mathcal{F}(\lambda) \) (we do not need to add weights to the operator that enters the interpolation argument). Then \( \|T\lambda\|_{L^q_{\psi}} \leq C \|f\|_{L^q_{\psi}} \), and by the Plancherel theorem it furthermore holds that \( \|T\lambda\|_{L^q_{\psi}} \leq \|T\lambda\|_{L^q_{\psi}} \leq \|f\|_{L^q_{\psi}} \). By interpolation (cf. theorem 2.19) it follows that \( \|T\lambda\|_{L^q_{\psi}} \leq \|f\|_{L^q_{\psi}} \).

Now define \( g(x) = f(x)\psi(x)^{-\frac{1}{2}} \); then \( g \) belongs to \( L^q_{\psi}( \mathbb{R}^N) \) by hypothesis, since

\[
\|g\|_{L^q_{\psi}} = \int_{\mathbb{R}^N} \|f(x)\psi(x)^{\frac{1}{2}} \, d\mu_{k,a}(x) = \|f\|_{L^q_{\psi}},
\]

It follows from the sublevel set estimate implied by \( \psi \) being a Young function that

\[
\mu_{k,a}(\{x \in \mathbb{R}^N : |\psi(x)|^{r_2} > t\}) = \mu_{k,a}(\{x \in \mathbb{R}^N : |\psi(x)|^{r_1} > t\}) \leq Ct^{-\frac{r_1}{r_2}},
\]

whence \( |\psi(x)|^{r_2} \) belongs to \( L^r_{\psi}( \mathbb{R}^N) \), where \( r = \frac{q}{q-2} \). By an application of O’Neil’s theorem 2.22 it is seen that

\[
\int_{\mathbb{R}^N} |F|q_{k,a}(\xi)|^q \, d\mu_{k,a}(\xi) \leq \|f\|_{L^q_{\psi}}^r \|g\|_{L^q_{\psi}}^s \leq C \int_{\mathbb{R}^N} \|f(x)\psi(x)^{q-2} \, d\mu_{k,a}(x) = C \|f\|_{L^q_{\psi}},
\]

which was the desired conclusion for simple functions. The extension to general functions in \( L^q_{\psi}( \mathbb{R}^N) \) follows by standard density arguments. \( \square \)

The Dunkl-version of the following second version of the Hardy–Littlewood inequality was recently established in [Joh15a, Proposition 4.3] where the connection between a ‘flat’ Heckman–Opdam transform and the (symmetrized) Dunkl transform was noted. The motivation behind this improvement involves several several intermediate results for the spherical Fourier transform on a Riemannian symmetric spaces that need not be repeated here.

**Theorem 4.6.** Let \( 1 < q \leq 2 \) be fixed. For \( f \in L^p_{k,a}( \mathbb{R}^N) \) with \( 1 < p < q \) there exists a finite constant \( C_{p,q} \) independent of \( f \) such that

\[
\left( \int_{\mathbb{R}^N} |F|q_{k,a}(\xi)|^r (|\xi|^q \vartheta_{k,a}(\xi))^{r/2^{q-2}} \, d\mu_{k,a}(\xi) \right)^{1/r} \leq C_{p,q} \|f\|_{L^p_{k,a}}
\]

where \( \frac{1}{r} = 1 - \frac{q}{q-1} \).

**Outline of proof.** Consider the measure spaces \( (\mathbb{R}^N, d\mu_{k,a}) \) and \( (\mathbb{R}^N, d\mu_{k,a}) \), where \( d\mu_{k,a}(x) = \|x|^{-N q} \vartheta_{k,a}(x)^{1-N q} dx \). Define \( T\lambda(\xi) = |F|q_{k,a}(\xi)(|\xi|^q \vartheta_{k,a}(\xi))^{q/2} \). Then \( T \) is of strong type \( q, q' \) and of weak type \( (1, 1) \), the latter following from the estimate \( \|\xi|^{q/2} \vartheta_{k,a}(\xi) \asymp C \|\xi\|^{3-a_2 + 2(k)} \).
Remark 4.7. We obtain a more familiar form of the Hardy–Littlewood inequality in theorem [46] by choosing as Young function a power of the Euclidean norm instead of the density \( \vartheta_{k,a} \), that is \( \psi(x) = \|x\|^{2(k)+N+a-2} \), cf. example [12](ii). The space \( L^p_\psi(\mathbb{R}^N) \) now consists of all measurable functions \( f : \mathbb{R}^N \to \mathbb{C} \) for which
\[
\|f\|_p = \left( \int_{\mathbb{R}^N} |f(x)|^p \|x\|^{(p-2)(2(k)+N+a-2)} \, d\mu_{k,a}(x) \right)^{1/p} < \infty.
\]
For \( f \in L^p_\psi(\mathbb{R}^N) \) with \( 2 \leq p < \infty \) it holds that
\[
\left( \int_{\mathbb{R}^N} |F_{k,a} f(\xi)|^p \, d\mu_{k,a}(\xi) \right)^{1/p} \leq C_p \left( \int_{\mathbb{R}^N} |f(x)|^p \|x\|^{(2(k)+N+a-2)(p-2)} \, d\mu_{k,a}(x) \right)^{1/p},
\]
which is the ‘dual’ form of the Hardy–Littlewood inequality for the Dunkl transform obtained in [AASS09, Lemma 4.1].

Remark 4.8. It is briefly indicated in [BH03, Remark 6, p.34] that some versions of the Hardy–Littlewood inequality can be obtained by another interpolation result in [BH03]. Since we have not provided detailed proofs of these interpolation results in the appendix, we found it appropriate to present a more direct, elementary proof with complete details.

5. Around the Heisenberg–Pauli–Weyl Inequality

The following uncertainty principle appeared as Theorem 5.29 in [BSK012].

Theorem 5.1. Assume \( N \geq 1, k \geq 0 \), and that \( a > 0 \) satisfies \( a + 2 \langle k \rangle + N > 2 \). For all \( f \in L^2_{k,a}(\mathbb{R}^N) \), the \( (k,a) \)-generalized Fourier transform \( F_{k,a} \) satisfies the \( L^2 \)-Heisenberg inequality
\[
\|\cdot\| \cdot \|^{a/2} f \|_{L^2_{k,a}} \| \cdot \| \cdot \|^{a/2} F_{k,a} f \|_{L^2_{k,a}} \geq \left( \frac{2(k) + N + a - 2}{2} \right) \|f\|^2_{L^2_{k,a}}.
\]
The inequality is saturated by functions of the form \( f(x) = \lambda \exp(-c\|x\|^a) \) for some \( \lambda \in \mathbb{C} \), \( c > 0 \).

Remark 5.2. The exponent in the power weight \( \|\cdot\|^{a/2} \) comes from simple scaling. An immediate advantage of the weighted interpolation techniques is that weights with different exponents can be used.

The proof is elementary and based on spectral methods. While this rules out an immediate extension to \( L^p \)-functions, their argument does show an important scaling principle that we shall also use below. The first step is to prove an additive inequality: For every \( f \in L^2_{k,a}(\mathbb{R}^N) \),
\[
\|\cdot\| \cdot \|^{a/2} f \|_{L^2_{k,a}}^2 + \|\cdot\| \cdot \|^{a/2} F_{k,a} f \|_{L^2_{k,a}}^2 \geq (2 \langle k \rangle + N + a - 2) \|f\|^2_{L^2_{k,a}},
\]
with equality if and only if \( f \) is proportional to \( \exp\left(\frac{1}{a}\|x\|^2\right) \). Indeed, it follows from general properties of \( F_{k,a} \) and \( \Delta_{k,a} \) that
\[
\|\cdot\| \cdot \|^{a/2} F_{k,a} f \|_{L^2_{k,a}}^2 = \langle \|\cdot\| \cdot \|^{a/2} F_{k,a} f, F_{k,a} f \rangle_{L^2_{k,a}} = -\langle F_{k,a}(\|\cdot\| \cdot \|^{2-a} \Delta_{k} f), F_{k,a} f \rangle_{L^2_{k,a}} = -\langle \|\cdot\| \cdot \|^{2-a} \Delta_{k} f, F_{k,a} f \rangle_{L^2_{k,a}},
\]
and therefore
\[
\|\cdot\| \cdot \|^{a/2} f \|_{L^2_{k,a}}^2 + \|\cdot\| \cdot \|^{a/2} F_{k,a} f \|_{L^2_{k,a}}^2 = \langle (\|\cdot\| \cdot \|^{a/2} f, F_{k,a} f)\rangle_{L^2_{k,a}} = \langle (\|\cdot\| \cdot \|^{a/2} f, \Delta_{k,a} f)\rangle_{L^2_{k,a}} = \langle (\|\cdot\| \cdot \|^{2-a} \Delta_{k} f, f)\rangle_{L^2_{k,a}} = \langle (\|\cdot\| \cdot \|^{2-a} \Delta_{k} f, F_{k,a} f)\rangle_{L^2_{k,a}}.
\]
Since \( -\Delta_{k,a} \) is self-adjoint on \( L^2_{k,a}(\mathbb{R}^N) \) and has a discrete \( L^2 \)-spectrum with \( 2 \langle k \rangle + N - 2 + a \) being the least eigenvalue, the inequality follows. We shall not discuss the optimality statement.

The following standard scaling argument establishes the \( L^2 \)-Heisenberg inequality. For \( c > 0 \) consider \( f_c(x) = f(cx) \). Then
\[
\|\cdot\| \cdot \|^{a/2} f_c \|_{L^2_{k,a}}^2 = c^{-2(k)-N-2a+2} \|\cdot\| \cdot \|^{a/2} f \|_{L^2_{k,a}}^2
\]
\[
\|f_c\|_{L^2_{k,a}}^2 = c^{-(N+2(k)+a-2)} \|f\|_{L^2_{k,a}}^2.
\]
Analogous formulae hold for $F_{k,a}f_c$; for instance, $\|\| \cdot |a/2 F_{k,a}f_c\|_{L^2_{k,a}}^2 = c^{-(2(k)-N+2)} \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2$.
It now follows from \text{(25)} that
\[
(26) \quad c^{-a} \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 + c^a \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 \geq (2 \langle k \rangle + N + a - 2) \|f\|_{L^2_{k,a}}^2.
\]
The derivative of the left hand side of \text{(26)} as a function in $c$ is $-ac^{-(a+1)} \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 + ac^{a-1} \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2$, which is zero when $c^{2a} \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 = \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2$, that is, $c^a = \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2$. The minimum of the left hand side in \text{(26)} is attained for this value of $c$ and computes to exactly $2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2$, which completes the proof of the aforementioned theorem.

A straightforward extension of their Heisenberg inequality is summarized in the following

**Proposition 5.3.** Assume $N \geq 1$, $k \geq 0$, that $a > 0$ satisfies $a + 2 \langle k \rangle + N > 2$, and that $\alpha, \beta \geq 1$. For every $f \in L^2_{k,a}(\mathbb{R}^N)$ it follows that
\[
\|\| \cdot |a/2 f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 \geq \left(2 \langle k \rangle + N + a - 2 \right) \frac{\alpha \beta}{2} \|\| \cdot |a/2 f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2.
\]

Note that the $L^2$-norm of $f$ on the right hand side is not squared; this is due to scaling and homogeneity but can also be explained heuristically by ‘counting’ norm powers in the left hand side of the inequality: Indeed, $\|f\|$ appears raised to the power $\frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} = 1$.

**Proof.** For $\alpha > 1$ fixed and $\alpha'$ such that $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ it is seen that
\[
\|\| \cdot |a/2 f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 = \left(\int_{\mathbb{R}^N} |x|^{\alpha} |f(x)|^2 \, d\mu_{k,a}(x)\right)^{1/2} \left(\int_{\mathbb{R}^N} |f(x)|^2 \, d\mu_{k,a}(x)\right)^{1/2}\alpha',
\]
and furthermore by Hölder’s inequality that
\[
\|\| \cdot |a/2 f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 \leq \left(\int_{\mathbb{R}^N} |x|^{\alpha-a} |f(x)|^{2/\alpha'} \, d\mu_{k,a}(x)\right)^{1/\alpha} \left(\int_{\mathbb{R}^N} |f(x)|^{2/\alpha'} \, d\mu_{k,a}(x)\right)^{1/\alpha'},
\]
from which we obtain the inequality $\|\| \cdot |a/2 f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 \leq \|\| \cdot |a/2 f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2$, that is
\[
(27) \quad \forall \alpha \geq 1 : \|\| \cdot |a/2 f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 \geq \frac{\|\| \cdot |a/2 f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2}{\|f\|_{L^2_{k,a}}^{1\frac{1}{\alpha}}},
\]
The same argument applied to $F_{k,a}f$ leads to the analogous inequality
\[
(28) \quad \forall \beta \geq 1 : \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 \geq \frac{\|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2 \|\| \cdot |a/2 F_{k,a}f\|_{L^2_{k,a}}^2}{\|F_{k,a}f\|_{L^2_{k,a}}^{1\frac{1}{\beta}}},
\]
We conclude from (27), (28), and Theorem 5.1 that
\[
\| \cdot \|_{\alpha}^{-1/\alpha} f^{1/\alpha} \|_{L^2_{k,a}} \| \cdot \|_{\beta}^{-1/\beta} F_{k,a} f \|_{L^2_{k,a}}^{\alpha/\alpha + \beta/\beta} \geq \left( \frac{\| \cdot \|_{\alpha}^{-1/\alpha} f^{1/\alpha} \|_{L^2_{k,a}} \| \cdot \|_{\beta}^{-1/\beta} F_{k,a} f \|_{L^2_{k,a}}^{1/\beta}}{\| f \|_{L^2_{k,a}}^{1-1/\alpha} \| f \|_{L^2_{k,a}}^{1-1/\beta}} \right)^{\alpha/\alpha + \beta/\beta} \geq \left( \frac{2 \langle k \rangle + N + a - 2}{2} \right)^{\alpha/\alpha + \beta/\beta} \| f \|_{L^2_{k,a}}^{(2-2\alpha + \alpha + \beta) \alpha/\alpha + \beta/\beta},
\]
which is exactly the asserted inequality.

\[ \square \]

**Remark 5.4.** Our theorem 5.3 is a slight improvement of [GJ14, Theorem 4.4 (3)] since we obtain a better constant. This is to be expected, however, since the point of [GJ14] is to obtain uncertainty principles for large classes of integral transforms. In concrete situations more detailed information can be brought to bear, as in theorem 5.1 where an optimal constant could be found. It is unlikely that comparably sharp results for general integral transforms can be established.

As our proof relies on Hölder’s inequality, it cannot include the cases \(0 < \alpha, \beta < 1\). As far as we could ascertain from the existing literature, in particular [CRS07] and [Mar10], most proofs of such an improvement involve heat kernel estimates either directly or disguised in spectral estimates of powers of the Laplacian. The heat kernel for the operator \(-\Delta_{k,a}\) is only known at present in the cases (i) \(N = 1, a > 0\) (where one can ‘deform’ the known one-dimensional Dunkl-heat kernel with the parameter \(a\)), and (ii) \(N \geq 2, a \in \{1, 2\}\) (where the explicit formula was obtained in [BSN11, Theorem 5.1]), and even in those cases it is nontrivial to obtain the required bounds. In this regard the techniques employed in [GJ14] are more suitable, since they merely require that the kernel of the integral transform be suitably bounded.

**Theorem 5.5.** Assume \(N \geq 1, k \geq 0\), that \(a > 0\) satisfies \(a + 2 \langle k \rangle + N > 2\), and that \(0 < \alpha, \beta < 1\). If either

(i) \(N = 1\) and \(a > 0\),
(ii) \(N \geq 2\) and \(a \in \{1, 2\}\),

or

(iii) \(N = 2, a = 2/n\) for some \(n \in \mathbb{N}\),

there exists a finite constant \(c = c(\alpha, \beta)\) such that
\[
\| \cdot \|_{\alpha}^{-1/\alpha} f^{1/\alpha} \|_{L^2_{k,a}} \| \cdot \|_{\beta}^{-1/\beta} F_{k,a} f \|_{L^2_{k,a}}^{\alpha/\alpha + \beta/\beta} \geq c(\alpha, \beta) \| f \|_{L^2_{k,a}}.
\]

**Proof.** The statement follows from [GJ14, Theorem C] since the kernel \(B_{k,a}\) for the \((k, a)\)-generalized Fourier transform \(F_{k,a}\) is uniformly bounded in all the cases listed in the statement of the theorem. \[ \square \]

The last variation on the theme of Heisenberg inequalities incorporates \(L^p\)-norms and is based on the following substitute for the heat kernel decay estimates that were used in [CRS07].
Lemma 5.6. Let \( p \in (1,2] \), \( q = p' = \frac{p}{p-1} \), and \( 0 < \alpha < \frac{a(\nu_a + 1)}{q} = \frac{2(k)+N-1}{q} \). For every \( f \in L^p_{k,a}(\mathbb{R}^N) \) and \( t > 0 \),
\[
\|\gamma_t F_{k,a} f\|_{L^q_{k,a}} \leq \left(1 + \frac{K^{2/q}}{(a(\nu_a + 1) - \alpha q)^{1/p}} \left(\frac{\Gamma(\nu_a + 1)}{2q^{\nu_a + 1}}\right)^{1/q} t^{-\alpha/a}\right) \| \cdot \| \cdot \| \cdot \| f \|_{L^p_{k,a}}.
\]

The constant is not optimal; what is important is the exponent \(-\alpha/a\) in the decay rate of \( t \). Also note that we could have used the weight \( \ell \cdot \| \cdot \| \cdot \| f \|_{L^p_{k,a}} \) on the right hand side, as in proposition 5.3. One would then have to impose the restriction \( 0 < \alpha < \frac{2(k)+N-1}{aq} \) which translates into the condition \( 0 < \alpha < \frac{2(k)+N-1}{aq} \) which involves \( a \). This would lead to an \( a \)-independent decay factor \( t^{-\alpha/2} \) in place of \( t^{-\alpha/a} \), so it is a matter of scaling.

Corollary 5.7. Let \( p \in (1,2] \), \( q = p' = \frac{p}{p-1} \), and \( 0 < \alpha < \frac{2(k)+N-1}{aq} \). For every \( f \in L^p_{k,a}(\mathbb{R}^N) \) and \( t > 0 \),
\[
\|\gamma_t F_{k,a} f\|_{L^q_{k,a}} \leq \left(1 + \frac{K^{2/q}}{(a(\nu_a + 1) - \alpha q)^{1/p}} \left(\frac{\Gamma(\nu_a + 1)}{2q^{\nu_a + 1}}\right)^{1/q} t^{-\alpha/2}\right) \| \cdot \| \cdot \| \cdot \| f \|_{L^p_{k,a}}.
\]

Proof of lemma 5.6. Assume without loss of generality that \( \| \cdot \| \cdot \| f \|_{L^q_{k,a}} \) is finite. Since \( \| x \| /r \alpha \geq 1 \) whenever \( x \in \mathbb{R}^N \), where \( B_r = \{ x \in \mathbb{R}^N : \| x \| \leq r \} \), it holds that \( \| f 1_{\mathbb{R}^N} \| \leq \| x \| /r \alpha \| f \| (x) \) for every \( x \in \mathbb{R}^N \), from which it follows that
\[
\|\gamma_t F_{k,a} (f 1_{\mathbb{R}^N})\|_{L^q_{k,a}} \leq \|\gamma_t L^q_{k,a} F_{k,a} (f 1_{\mathbb{R}^N})\|_{L^q_{k,a}} \leq \| f 1_{\mathbb{R}^N} \|_{L^q_{k,a}} \text{ by proposition 2.4.1}
\]
\[
\leq r^{-\alpha} \| \cdot \| \cdot \| \cdot \| f \|_{L^p_{k,a}}.
\]

In addition it holds by the Hölder inequality that
\[
\|\gamma_t F_{k,a} (f 1_{B_r})\|_{L^q_{k,a}} = \|\gamma_t L^q_{k,a} F_{k,a} (f 1_{B_r})\|_{L^q_{k,a}} \leq \|\gamma_t L^q_{k,a} F_{k,a} (f 1_{\mathbb{R}^N})\|_{L^q_{k,a}} \leq \| f 1_{B_r} \|_{L^q_{k,a}} \| f 1_{\mathbb{R}^N} \|_{L^q_{k,a}} \| \cdot \| \cdot \| f \|_{L^p_{k,a}}
\]
The norms \( \|\gamma_t L^q_{k,a} \) and \( \| f 1_{B_r} \|_{L^q_{k,a}} \) having already been computed in (21) and (22), respectively, we conclude that
\[
\|\gamma_t F_{k,a} (f 1_{B_r})\|_{L^q_{k,a}} \leq \frac{K^{2/q}}{(a(\nu_a + 1) - \alpha q)^{1/p}} \left(\frac{\Gamma(\nu_a + 1)}{2q^{\nu_a + 1}}\right)^{1/q} t^{-\alpha/a} \| \cdot \| \cdot \| f \|_{L^p_{k,a}}
\]
and
\[
\|\gamma_t F_{k,a} f\|_{L^q_{k,a}} \leq \|\gamma_t F_{k,a} (f 1_{B_r})\|_{L^q_{k,a}} + \|\gamma_t F_{k,a} (f 1_{\mathbb{R}^N})\|_{L^q_{k,a}} \leq \left(1 + \frac{K^{2/q}}{(a(\nu_a + 1) - \alpha q)^{1/p}} \left(\frac{\Gamma(\nu_a + 1)}{2q^{\nu_a + 1}}\right)^{1/q} t^{-\alpha/a} \right) \| \cdot \| \cdot \| f \|_{L^p_{k,a}}.
\]

This inequality holds, in particular, for \( r = t^{1/a} \), from which the assertion follows. \( \Box \)

Theorem 5.8. Under the same assumptions as in lemma 5.6 and with \( \beta > 0 \), there exists a finite constant \( c(\alpha, \beta) \) such that
\[
\| F_{k,a} f \|_{L^q_{k,a}} \leq c(\alpha, \beta) \| t \cdot \| \cdot \| \cdot \| f \|_{L^p_{k,a}} \| \cdot \| \cdot \| \cdot \| f \|_{L^p_{k,a}} \| \cdot \| \cdot \| f \|_{L^p_{k,a}} \text{ for all } f \in L^p_{k,a}(\mathbb{R}^N).
\]

The proof that follows provides a rough estimate for \( c(\alpha, \beta) \) but will be far from optimal.
Proof. Fix \( p \in (1, 2] \) and assume \( f \in L^p_{k,a} \) satisfy \( \| \cdot \|^{\alpha} f \|_{L^p_{k,a}} + \| \cdot \|^{\beta} F_{k,a} f \|_{L^p_{k,a}} < \infty \). Moreover assume that \( \beta \leq a \). It follows for all \( t > 0 \) from lemma 5.6 that

\[
\| F_{k,a} f \|^q_{L^q_{k,a}} \leq \| \gamma t F_{k,a} f \|^q_{L^q_{k,a}} + \| (1 - \gamma t) F_{k,a} f \|^q_{L^q_{k,a}} \leq (1 + \frac{(\Gamma(\nu_a + 1))^{1/q}}{(2\nu_a + 1)^{1/q}}) t^{-\alpha/a} \| \cdot \|^\alpha f \|^q_{L^p_{k,a}} + \| (1 - \gamma t) F_{k,a} f \|^q_{L^q_{k,a}}.
\]

Moreover \( \| (1 - \gamma t) F_{k,a} f \|^q_{L^q_{k,a}} = t^{\beta/a} \| (t \cdot \|^\alpha f \| \|^\beta F_{k,a} f \|_{L^q_{k,a}} \), where

\[
\| (t \cdot \|^\alpha f \| \|^\beta F_{k,a} f \|_{L^q_{k,a}} \leq \| (t \cdot \|^\alpha f \| \|^\beta F_{k,a} f \|_{L^q_{k,a}} \| (t \cdot \|^\alpha f \| \|^\beta F_{k,a} f \|_{L^q_{k,a}} \| \|^\beta F_{k,a} f \|_{L^q_{k,a}} = c \| \cdot \|^\beta F_{k,a} f \|_{L^q_{k,a}}
\]

whenever \( 0 < \beta \leq a \). It follows that

\[
\| F_{k,a} f \|^q_{L^q_{k,a}} \leq c (t^{-\alpha/a} \| t \cdot \|^\alpha f \|_{L^p_{k,a}} + t^{\beta/a} \| \cdot \|^\beta F_{k,a} f \|_{L^q_{k,a}}) \]

for all \( t > 0 \). The choice \( t = (\frac{2}{\beta}) \| F_{k,a} f \|^p_{L^p_{k,a}} \) in particular, gives rise to the inequality

\[
\| F_{k,a} f \|^q_{L^q_{k,a}} \leq c \left( \frac{\beta}{\beta + \alpha} + \frac{\alpha}{\beta} \right) \| \cdot \|^\alpha f \|_{L^p_{k,a}}^{\frac{\alpha}{\alpha + \beta}} \| \cdot \|^\beta F_{k,a} f \|_{L^q_{k,a}}^{\frac{\beta}{\beta + \alpha}} \]

for \( \beta \leq a \).

The remaining case \( \beta > a \) can be treated by a slight variation of the arguments already given. Since \( u^{\alpha/2} \leq 1 + u^\beta \) for all \( u \geq 0 \), it follows in particular for \( u \) of the form \( y \|/\epsilon \), \( \epsilon \) an arbitrary positive parameter, that \( (\| \cdot \|/\epsilon)^{\alpha/2} \leq 1 + (\| \cdot \|/\epsilon)^\beta \) for all \( \epsilon > 0 \). There is nothing special about \( a/\beta \), any exponent less than \( a \) would work, since we may then apply the first part of the proof.

Therefore

\[
\| \cdot \|^\alpha F_{k,a} f \|_{L^q_{k,a}} \leq c^{\alpha/2} \| F_{k,a} f \|_{L^q_{k,a}} + c^{\alpha/2 - \beta} \| \cdot \|^\beta F_{k,a} f \|_{L^q_{k,a}}
\]

for all \( \epsilon > 0 \). In particular, by choosing \( \epsilon \) such that \( c^{\alpha/2} \| F_{k,a} f \|_{L^q_{k,a}} = c^{\alpha/2 - \beta} \| \cdot \|^\beta F_{k,a} f \|_{L^q_{k,a}} \)

(which amounts to taking \( \epsilon = \| \cdot \|^\beta F_{k,a} f \|_{L^q_{k,a}}^{-1/\beta} \)), it follows that

\[
\| \cdot \|^\alpha F_{k,a} f \|_{L^q_{k,a}} \leq 2 \| F_{k,a} f \|_{L^q_{k,a}}^{\frac{2\beta - a}{2\beta}} \| \cdot \|^\beta F_{k,a} f \|_{L^q_{k,a}}^{\frac{2\beta - a}{2\beta}}
\]

whence

\[
\| F_{k,a} f \|_{L^q_{k,a}} \leq c \| \cdot \|^\alpha F_{k,a} f \|_{L^q_{k,a}}^{\frac{\alpha}{\alpha + \beta}} \| \cdot \|^\beta F_{k,a} f \|_{L^q_{k,a}}^{\frac{\beta}{\alpha + \beta}} \]

by the first part of the proof

\[
\leq c' \| \cdot \|^\alpha F_{k,a} f \|_{L^q_{k,a}}^{\frac{\alpha}{\alpha + \beta}} \| \cdot \|^\beta F_{k,a} f \|_{L^q_{k,a}}^{\frac{\beta}{\alpha + \beta}}
\]

Elementary algebra now leads to the desired conclusion in the case \( \beta > a \) as well: Isolating all factors with \( \| F_{k,a} f \|_{L^q_{k,a}} \) on the left hand side of the inequality yields the exponent \( 1 - \frac{2\beta - a}{2\beta} \frac{\alpha}{\alpha + \beta} = \frac{a(\alpha + \beta)}{2\beta(\alpha + \beta)} \) and \( \frac{\alpha}{\alpha + \beta} \cdot \frac{2\beta(a + 2)}{\alpha(a + \beta)} \cdot \ldots = \frac{\beta}{\alpha + \beta} \), for example.

The following alternative formulation follows by scaling, just as in corollary 5.7. Note that the exponents \( \frac{\alpha}{\alpha + \beta} \) and \( \frac{\beta}{\alpha + \beta} \) are invariant under rescaling \( \alpha \to \alpha \frac{\beta}{2} \), \( \beta \to \beta \frac{2}{2} \). Moreover \( d(\alpha, \beta) = c(\alpha \frac{\beta}{2}, \beta \frac{2}{2}) \).
Corollary 5.9. Under the same assumptions as in corollary [8], and with \( \beta > 0 \), there exists a finite constant \( d(\alpha, \beta) \) such that
\[
\|F_{k,a} f\|_{L^q_{k,a}} \leq d(\alpha, \beta) \|f\|_{L^p_{k,a}}^\alpha \|F_{k,a} f\|_{L^p_{k,a}}^\beta \text{ for all } f \in L^p_{k,a}(\mathbb{R}^N).
\]

Remark 5.10. It is possible to generate an abundance of additional inequalities similar to the aforementioned ones. The interested reader will quickly be able to generalize the results in [CPS4], Section 2, for example, since these inequalities all arise as the result of simple scaling properties.

6. A variation of the HPW inequality with \( L^1 \)-norms

Another variation involves a mixed \( L^1, L^2 \) lower bound and was recently obtained by Gobber [Gho13] for the Dunkl transform. Its Euclidean counterpart seems to go back to [LM99], [Mor01], where the best constant is determined. The proof is elementary and - like in [2, Section 3] - based on the following two inequalities, the contents of which are somewhat obscure, unfortunately (the complicated exponents all arise as a consequence of scaling and homogeneity properties of the underlying measures).

Lemma 6.1 (Nash-type inequality). Let \( s > 0 \) and assume \( a > 0 \) is chosen in such a way that the Plancherel theorem for \( F_{k,a} \) is valid. Then
\[
\|F_{k,a} f\|_{L^2_{k,a}}^2 = \|f\|_{L^2_{k,a}}^2 \leq C \|f\|_{L^1_{k,a}} \|F_{k,a} f\|_{L^2_{k,a}}^a \|F_{k,a} f\|_{L^2_{k,a}}^2
\]
for every \( f \in (L^1_{k,a} \cap L^2_{k,a})(\mathbb{R}^N) \), where
\[
C = C(k, a, s) = \frac{2}{2k + a + N - 2} \left( \frac{2s}{K} \right)^{2(k) + a + N - 2s} + \left( \frac{2s}{K} \right)^{2(k) + a + N - 2s}.
\]

Proof. For \( f \in L^2_{k,a}(\mathbb{R}^N) \) and \( r > 0 \) fixed, consider the function \( \mathbf{1}_r = \mathbf{1}_{B_r(0)} \). It follows from the Plancherel theorem for \( F_{k,a} \) and the fact \( \mathbf{1}_r(1 - \mathbf{1}_r) \equiv 0 \) that
\[
\|F_{k,a} f\|_{L^2_{k,a}}^2 = \|F_{k,a} f\|_{L^2_{k,a}}^2 \leq \|F_{k,a} f\|_{L^2_{k,a}}^2 \leq \|F_{k,a} f\|_{L^2_{k,a}}^2 = \|(F_{k,a} f) \mathbf{1}_r\|_{L^2_{k,a}}^2 + \|(F_{k,a} f)(1 - \mathbf{1}_r)\|_{L^2_{k,a}}^2,
\]
where
\[
\|(F_{k,a} f) \mathbf{1}_r\|_{L^2_{k,a}}^2 = \int_{B_r(0)} |F_{k,a} f(\xi)|^2 d\mu_{k,a}(\xi) \leq \|F_{k,a} f\|_{L^2_{k,a}}^2 \mu_{k,a}(B_r(0))
\]
and
\[
\|(F_{k,a} f)(1 - \mathbf{1}_r)\|_{L^2_{k,a}}^2 = \int_{\mathbb{R}^N \setminus B_r(0)} |F_{k,a} f(\xi)|^2 d\mu_{k,a}(\xi)
\]

and
\[
\leq \frac{K}{2(k) + a + N - 2} \|F_{k,a} f\|_{L^2_{k,a}}^a \mu_{k,a}(B_r(0))
\]
and
\[
\leq r^{-2s} \int_{\mathbb{R}^N \setminus B_r(0)} \|\xi\|^{2s} |F_{k,a} f(\xi)|^2 d\mu_{k,a}(\xi) = r^{-2s} \|\cdot\|_{L^1_{k,a}}^2 \|F_{k,a} f\|_{L^2_{k,a}}^2.
\]

Therefore
\[
\|F_{k,a} f\|_{L^2_{k,a}}^2 \leq \frac{K}{2(k) + a + N - 2} \|f\|_{L^1_{k,a}}^a \|F_{k,a} f\|_{L^2_{k,a}}^2 + \|\cdot\|_{L^2_{k,a}}^2.
\]
the right hand side of which is minimized when \( r^{2(k)+a+N+2s-2} = \frac{2s}{K} \| \cdot \|^{(k)+\frac{2s}{2}+1+2s} \|^{(k)+\frac{2s}{2}+1+2s} \). □

**Lemma 6.2** (Clarkson-type inequality for \( \partial_{k,a}(x)dx \)). Let \( s > 0 \) and assume \( a > 0 \) is chosen in such a way that the Plancherel theorem for \( \mathcal{F}_{k,a} \) is valid. Then

\[
\| f \|_{L_{k,a}^1} \leq D(k, a, s) \| f \|_{L_{k,a}^2}^{(k)+\frac{2s}{2}+1+2s} \|^{2s} f \|_{L_{k,a}^1}^{(k)+\frac{2s}{2}+1+2s}
\]

for every \( f \in (L_{k,a}^1 \cap L_{k,a}^2)(\mathbb{R}^N) \) where the constant \( D \) is computable yet far from optimal.

**Proof.** Let \( f \in (L_{k,a}^1 \cap L_{k,a}^2)(\mathbb{R}^N) \) and consider \( 1_r = 1_{B_r(0)} \), \( r > 0 \). Since \( \| f \|_{L_{k,a}^1} \leq \| f 1_r \|_{L_{k,a}^1} + \| f(1-1_r) \|_{L_{k,a}^1} \leq \| f \|_{L_{k,a}^2}^1 1_r \|_{L_{k,a}^2} + r^{-2s} \| \cdot \|^{2sf} \|_{L_{k,a}^1} \), it follows that

\[
\| f \|_{L_{k,a}^1} \leq \left( \frac{K}{2(k) + a + N - 2} \right)^{1/2} r^{(k)+\frac{2s}{2}+1+2s} \| \cdot \|^{2s} f \|_{L_{k,a}^1} + r^{-2s} \| \cdot \|^{2sf} \|_{L_{k,a}^1}^{-1}.
\]

The right hand side of which is minimized for

\[
r^{(k)+\frac{2s}{2}+1+2s} = \frac{2s}{(\langle k \rangle + \frac{a+N}{2} - 1)(\frac{K}{2(k) + a + N - 2})^{1/2} \| \cdot \|^{2sf} \|_{L_{k,a}^1} \| f \|_{L_{k,a}^1}^{-1} \}.
\]

□

The following uncertainty-type inequality follows at once by combining the aforementioned two lemmata, which at the same time yields an expression for the constant \( C' \).

**Proposition 6.3.** Let \( s > 0 \) and assume \( a > 0 \) is chosen in such a way that the Plancherel theorem for \( \mathcal{F}_{k,a} \) is valid. Then there exists a constant \( C' > 0 \) such that for all \( f \in (L_{k,a}^1 \cap L_{k,a}^2)(\mathbb{R}^N) \)

\[
\| \cdot \|^{2sf} \|_{L_{k,a}^1} \| \cdot \|^{sF_{k,a}f} \|_{L_{k,a}^2}^2 \geq C' \| f \|_{L_{k,a}^1} \| f \|_{L_{k,a}^2}^2.
\]

7. Inequalities for Shannon entropy

It is the purpose of the present section to establish an analogue of Hirschman’s entropic inequality for the \( (k,a) \)-generalized transform \( \mathcal{F}_{k,a} \) and use it to give a new proof of the Heisenberg–Pauli–Weyl inequality.

**Theorem 7.1.** Assume \( N \), \( k \) and \( a \) satisfy either (i) or (ii) in lemma 2.13. For every \( f \in L^2_{k,a}(\mathbb{R}^N) \) with \( \| f \|_{L_{k,a}^2} = 1 \) it holds that

\[
\mathbb{E}(|f|^2) + \mathbb{E}(|\mathcal{F}_{k,a}f|^2) \geq 0,
\]

where

\[
\mathbb{E}(h) = -\int_{\mathbb{R}^N} \ln(|h(x)|) |h(x)| d\mu_{k,a}(x).
\]

In the case of Euclidean Fourier analysis the idea of proof is to differentiate the Hausdorff–Young inequality with respect to \( p \), use various properties of the Fourier transform to establish the statement for \( f \in L^1 \cap L^2 \) and finish the proof with an approximate identity-argument. This was worked out in some detail by Hirschman but might have been used even earlier. It has since become a standard tool in the field of geometric inequalities, be it
Sobolev or Hardy–Littlewood–Sobolev inequalities in various settings. The idea is elementary and based on the following

**Lemma 7.2.** Let $I = [1, 2]$ and $\psi, \phi$ be real-valued differentiable functions on $I$ such that $\phi(t) \leq \psi(t)$ for $t \in I$ and $\phi(2) = \psi(2)$. Then $\phi'(2^-) \geq \psi'(2^-)$ (one-sided derivatives at $p = 2$).

Note, however, that the lack of convolution structure necessitates a different kind of approximation argument. We shall use the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ instead, in which case its invariance under $\mathcal{F}_{k,a}$, cf. lemma 2.17 becomes important.

**Proof of Theorem 7.1.** First assume that either one of the integrals

$$
\int_{\mathbb{R}^N} |f(x)|^2 \log^+ |f(x)|^2 \, d\mu_{k,a}(x), \quad \int_{\mathbb{R}^N} |f(x)|^2 \log^- |f(x)|^2 \, d\mu_{k,a}(x)
$$

is finite. The quantity $E(|f|^2)$ is therefore well-defined except when either

(a) $E(|f|^2)$ or $E(|\mathcal{F}_{k,a} f|^2)$ is not defined

or

(b) $E(|f|^2) = \pm \infty$ and $E(|\mathcal{F}_{k,a} f|^2) = \mp \infty$. (It suffices to exclude the case $E(|f|^2) = +\infty$, $E(|\mathcal{F}_{k,a} f|^2) = -\infty$).

Let $f \in \mathcal{S}(\mathbb{R}^N)$ and $p \in [1, 2]$ be fixed and define $r(p) = \frac{\|\mathcal{F}_{k,a} f\|_{L^p}}{\|f\|_{L^p}}$ together with

$$
C(p) = \log r(p) = \frac{1}{p'} \log \left( \int_{\mathbb{R}^N} |\mathcal{F}_{k,a} f(\xi)|^{p'} \, d\mu_{k,a}(\xi) \right) - \frac{1}{p} \log \left( \int_{\mathbb{R}^N} |f(x)|^p \, d\mu_{k,a}(x) \right).
$$

Then $C(2) = 0$ and $C(p) \leq 0$ for $1 < p < 2$, by the Hausdorff–Young inequality, and the one-sided derivative $C'(2^-)$ – whenever it exists – will be seen to be strictly positive. Let

$$
\psi : \mathbb{R}^N \times [1, 2] \to \mathbb{R}, \quad (x, p) \mapsto \frac{|f(x)|^2 - |f(x)|^p}{2 - p}.
$$

The functions $\psi_p : \mathbb{R}^N \to \mathbb{R}, x \mapsto \psi(x, p), p \in (1, 2]$, are seen to be integrable with respect to $\delta_{k,a}(x) \, dx$, and $x \psi : (1, 2] \to \mathbb{R}, p \mapsto \psi(x, p)$, converges towards $|f(x)|^2 \log |f(x)|$ as $p \nearrow 2$.

Define

$$
A(p) = \int_{\mathbb{R}^N} |f(x)|^p \, d\mu_{k,a}(x) \quad \text{and} \quad B(q) = \int_{\mathbb{R}^N} |\mathcal{F}_{k,a} f(\xi)|^q \, d\mu_{k,a}(\xi).
$$

Then

$$
\frac{A(2-h) - A(2)}{h} = \int_{\mathbb{R}^N} \frac{|f(x)|^{2-h} - |f(x)|^2}{h} \, d\mu_{k,a}(x) \quad \text{as } h \to 0, h > 0,
$$

that is, $A'(2^-) = \int |f(x)|^2 \log |f(x)| \, d\mu_{k,a}(x)$. An analogous consideration shows that $B'(2^-) = \int |\mathcal{F}_{k,a} f(\xi)|^2 \log |\mathcal{F}_{k,a} f(\xi)| \, d\mu_{k,a}(\xi)$, and it follows that $C'(2^-) = -\frac{1}{2} B'(2^-) - \frac{1}{2} A'(2^-)$. Indeed

$$
\frac{C(p) - C(2)}{2 - p} = \frac{\frac{1}{p} \log B(p') - \frac{1}{p} \log A(p)}{2 - p} - \frac{\frac{1}{p} \log B(2) - \frac{1}{p} \log A(2)}{2 - p} \to -\frac{1}{2} B'(2^-) - \frac{1}{2} A'(2^-) \quad \text{as } p \to 2, 1 < p < 2
$$

where it was used that $B(2) = A(2)$ (by the unitarity of $\mathcal{F}_{k,a}$). Since $A(2) = 1$ by assumption, it even follows that $C'(2^-) = -\frac{1}{2} B'(2^-) - \frac{1}{2} A'(2^-)$ as claimed. In other words, $C'(2^-) = \ldots$
\(\mathbb{E}(|f|^2) + \mathbb{E}(|\mathcal{F}_{k,a}f|^2)\), and it remains to establish that \(C'(2^-) > 0\). This follows from the elementary Lemma 7.2. Since \(r(p) \leq 1\) for \(1 \leq p \leq 2\), with equality at \(p = 2\), we apply the lemma to the function \(p \mapsto \log(r(p))\) to conclude that

\[
C'(2^-) = \frac{r'(2^-)}{r(2)} \geq 0
\]

which yields the asserted entropy inequality under the stronger assumption that \(f \in \mathcal{S}(\mathbb{R}^N)\). Since the sharp Hausdorff–Young inequality is not presently known, it is very likely that the lower bound in (29) can be improved considerably. We have tacitly excluded the case where \(\mathbb{E}(|f|^2) = +\infty\) and \(\mathbb{E}(|\mathcal{F}_{k,a}f|^2) = -\infty\). If we drop the requirement that \(\|f\|_{L^2_{k,a}} = 1\), the resulting entropic inequality becomes

\[
\frac{\mathbb{E}(|f|^2)}{\|f\|^2_{L^2_{k,a}}} + \frac{\mathbb{E}(|\mathcal{F}_{k,a}f|^2)}{\|\mathcal{F}_{k,a}f\|^2_{L^2_{k,a}}} \geq \log \|f\|^2_{L^2_{k,a}} + \log \|\mathcal{F}_{k,a}f\|^2_{L^2_{k,a}}.
\]

Now assume that \(f\) is merely in \(L^2_{k,a}(\mathbb{R}^N)\) with \(\|f\|_2 = 1\), and choose a sequence \(\{f_n\}\) in \(\mathcal{S}(\mathbb{R}^N)\) such that \(\lim_n \|f - f_n\|_2 = 0\). It follows from (30) that

\[
\frac{\mathbb{E}(|f_n|^2)}{\|f_n\|^2_{L^2_{k,a}}} + \frac{\mathbb{E}(|\mathcal{F}_{k,a}f_n|^2)}{\|\mathcal{F}_{k,a}f_n\|^2_{L^2_{k,a}}} \geq \log \|f_n\|^2_{L^2_{k,a}} + \log \|\mathcal{F}_{k,a}f_n\|^2_{L^2_{k,a}}
\]

for all \(n \in \mathbb{N}\), and by Lebesgue’s theorem on majorized convergence that \(\lim_n \mathbb{E}(|f_n|^2) = \mathbb{E}(|f|^2)\). Since \(\|f - f_n\|_2 = \|\mathcal{F}_{k,a}(f - f_n)\|_2 = \|\mathcal{F}_{k,a}f - \mathcal{F}_{k,a}f_n\|_2\), where \(\mathcal{F}_{k,a}f_n\) is a Schwartz function according to lemma 2.17, it follows that \(\lim_n \|\mathcal{F}_{k,a}f - \mathcal{F}_{k,a}f_n\|_2 = 0\) and by Lebesgue that \(\lim_n \mathbb{E}(|\mathcal{F}_{k,a}f_n|^2) = \mathbb{E}(|\mathcal{F}_{k,a}f|^2)\). We conclude that \(\mathbb{E}(|f|^2) + \mathbb{E}(|\mathcal{F}_{k,a}f|^2) \geq 0\).}

Although the entropic inequality for \(\mathcal{F}_{k,a}\) on \(\mathbb{R}^N\) is not sharp, it still yields further inequalities. One can establish the Heisenberg–Pauli–Weyl inequality, for example, in a form that improves proposition 5.3 by allowing more freedom in the choice of power weights.

Let \(\alpha, c\) be fixed, positive numbers and define constants

\[
\sigma_\alpha = \int_{\mathbb{R}^N} e^{-|x|^\alpha} \vartheta_{k,a}(x) \, dx, \quad k_{\alpha,c} = \frac{\sigma_\alpha}{c^{2(k)+a-N-2}}.
\]

Then \(d\gamma(x) = k^{-1}_{\alpha,c} \exp(-|cx|^\alpha) \vartheta_{k,a}(x) \, dx\) defines a probability measure on \(\mathbb{R}^N\), since

\[
\int_{\mathbb{R}^N} d\gamma(x) = \frac{1}{k_{\alpha,c}} \int_{\mathbb{R}^N} e^{-|cx|^\alpha} \vartheta_{k,a}(x) \, dx = \frac{1}{k_{\alpha,c}} c^{N-(2(k)+a-2)} \int_{\mathbb{R}^N} e^{-|x|^\alpha} \vartheta_{k,a}(x) \, dx
\]

\[
= \frac{c^{2(k)+a-N-2}}{\sigma_\alpha} c^{N-2(k)-a+2} \sigma_\alpha = 1
\]

Let \(\phi \in L^1_{k,a}(\mathbb{R}^N)\) with \(\|\phi\|_{L^1_{k,a}} = 1\) be fixed and consider the function defined by \(\psi(x) = k_{\alpha,c} \exp(|cx|^\alpha) |\phi(x)|\). Then \(\|\psi\|_{L^1(\gamma)} = 1\), and it follows from Jensen’s inequality applied to
the convex function $g: [0, \infty) \rightarrow \mathbb{R}: t \mapsto t \ln t$ that
\[
0 = g\left(\int_{\mathbb{R}^N} \psi(x) \, d\gamma(x)\right) = \left(\int_{\mathbb{R}^N} \psi(x) \, d\gamma(x)\right) \ln\left(\int_{\mathbb{R}^N} \psi(x) \, d\gamma(x)\right)
\leq \int_{\mathbb{R}^N} \psi(x) \ln(\psi(x)) \, d\gamma(x)
= \int_{\mathbb{R}^N} |\phi(x)|(\ln k_{\alpha,x} + ||cx||^\alpha + \ln(|\phi(x)|)) \, d\mu_{k,a}(x) \, dx
= \ln k_{\alpha,c} + c^\alpha \int_{\mathbb{R}^N} ||x||^\alpha |\phi(x)| \, d\mu_{k,a}(x) - \mathbb{E}(|\phi|)
\]
that is,
\[
(31) \quad \mathbb{E}(|\phi|) \leq \ln k_{\alpha,c} + c^\alpha (M_\alpha(\phi))^\alpha,
\]
where
\[
M_\alpha(\phi) = \int_{\mathbb{R}^N} ||x||^\alpha |\phi(x)| \, d\mu_{k,a}(x)
\]
is a generalized variance of the probability density $\phi$. In particular (31) holds for $\rho = |f|^2$ resp. $\rho = |\mathcal{F}_{k,a} f|^2$, where $f \in L^2_{k,a}(\mathbb{R}^N)$ with $\|f\|_{L^2_{k,a}} = 1$, that is,
\[
\mathbb{E}(|f|^2) \leq \ln k_{\alpha,c} + c^\alpha \int_{\mathbb{R}^N} ||x||^\alpha |f(x)|^2 \, d\mu_{k,a}(x) = \ln k_{\alpha,c} + c^\alpha \|f\|^2 f^2_{L^2_{k,a}}
\]
and $\mathbb{E}(|\mathcal{F}_{k,a} f|^2) \leq \ln k_{\beta,d} + d^\beta \|f\|_{L^2_{k,a}}^{\beta/2} \mathcal{F}_{k,a} f^2_{L^2_{k,a}}$ for further constants $\beta, d > 0$. It follows from theorem [7.1] that
\[
-2 \ln C \leq \mathbb{E}(|f|^2) + \mathbb{E}(|\mathcal{F}_{k,a} f|^2) \leq \ln(k_{\alpha,c,k_{\beta,d}}) + c^\alpha \|f\|_{L^2_{k,a}}^{\alpha/2} + d^\beta \|f\|_{L^2_{k,a}}^{\beta/2} \mathcal{F}_{k,a} f^2_{L^2_{k,a}}.
\]
For more general $f \in L^2_{k,a}(\mathbb{R}^N)$, $f \neq 0$, we replace $f$ by $f/\|f\|_2$ to obtain the inequality
\[
- \ln(C^2 k_{\alpha,c,k_{\beta,d}}) \|f\|_{L^2_{k,a}}^2 \leq c^\alpha \|f\|_{L^2_{k,a}}^{\alpha/2} + d^\beta \|f\|_{L^2_{k,a}}^{\beta/2} \mathcal{F}_{k,a} f^2_{L^2_{k,a}}.
\]

**Corollary 7.3.** There exists a constant $K = K_\alpha > 0$ such that
\[
\|f\|_{L^2_{k,a}} \cdot \|f\|_{L^2_{k,a}}^{\alpha/2} \mathcal{F}_{k,a} f^2_{L^2_{k,a}} \geq K \|f\|_{L^2_{k,a}}^2 \text{ for all } f \in L^2_{k,a}(\mathbb{R}^N).
\]

The constant $K$ can be computed by working through a scaling/dilation argument similar to the one following remark [5.2] above. Specifically, one chooses $\alpha = \beta$ and $c = d$ in the preceding considerations leading up to (32). Then replace $f$ by its dilation $f_t(x) = f(tx)$ and optimize in the variable $t$ to obtain the stated inequality. This was also how we went from the additive inequality (25) to the uncertainty inequality in theorem [5.1] so this provides an alternative proof of the Heisenberg–Pauli–Weyl uncertainty inequality by Ben Saïd, Kobayashi, and Ørsted.

**Remark 7.4.** In recent years several generalizations of the Shannon entropy and its implications for uncertainty of quantum measurements have appeared in the physics literature, most notably the Renyi entropy and related quantities in information theory, such as the Fisher information. It would take us too far afield to discuss these at any length but the interested reader may consult [BB06].
8. Weighted inequalities

The Hausdorff–Young inequality was but an elementary outcome of applying interpolation techniques to the transform $\mathcal{F}_{k,a}$. It is indeed possible to obtain more general weighted inequalities, and the present section addresses these matters. For our purposes would suffice to consider power weights, but it might be of independent interest to work for more general classes of weights. We shall be interesting in a weighted extension of the Hausdorff–Young inequality and an analogue of Pitt’s inequality.

We remind the reader that the classical Pitt’s inequality can be phrased as follows.

**Theorem 8.1** (Pitt’s inequality). Let $1 < p \le q < \infty$, choose $0 < b < 1/p'$, set $\beta = 1 - \frac{1}{p} - \frac{1}{q} - b < 0$, and define $v(x) = |x|^{|b}|$ for $x \in \mathbb{R}$. There exists a constant $C > 0$ such that

$$\left( \int_{\mathbb{R}} |\hat{f}(\xi)|^{q} |\xi|^{\beta q} \, d\xi \right)^{1/q} \le C \left( \int_{\mathbb{R}} |f(x)|^{p} |x|^{bp} \, dx \right)^{1/p}$$

for all $f \in L^p_v(\mathbb{R})$. In particular $\hat{f}$ is well-defined in this case.

Here $L^p_v(\mathbb{R})$ denotes the space of equivalence classes of measurable functions $f$ on $\mathbb{R}$ for which $\int_{\mathbb{R}} |f(x)|^{p} v(x) \, dx < \infty$, and our initial interest in Pitt’s inequality stems from its prominent role in work by Beckner, most notably [Bec95] and later publications. In particular, Beckner determined the optimal constant in the important special case $p = q = 2$, $b + \beta = 0$: For $f \in \mathcal{F}(\mathbb{R}^N)$ and $0 < b < N$,

$$\int_{\mathbb{R}^N} ||x||^{-b} |\hat{f}(\xi)|^{2} \, d\xi \le C(b) \int_{\mathbb{R}^N} ||x||^{b} |f(x)|^{2} \, dx,$$

where $C(b) = \pi^b \left( \frac{\Gamma\left(\frac{N-b}{4}\right)}{\Gamma\left(\frac{b}{4}\right)} \right)^{2}$. In particular $C(b) = 1$, and the inequality is even an equality, according to the Plancherel theorem.

As far as we know, an analogue of Pitt’s inequality for $\mathcal{F}_{k,a}$—even without sharp constants—is unknown for $N \ge 2$, $k \neq 0$. As already mentioned in the introduction the secondary goal of our paper is to fill this gap. The impetus was provided by the intriguing paper [BH03] where Benedetto and Heinig used interpolation techniques and classical inequalities for rearrangements to establish the following very general weighted inequality for the Euclidean Fourier transform (although the constants that appear are not optimal, it will be important to have some control over them). In order to explain the methodology we must introduce some more terminology.

Let $(X, \mu)$ be a measure space, where we assume for simplicity that $X \subset \mathbb{R}^N$, and let $f : X \to \mathbb{C}$ be $\mu$-measurable. The distribution function $D_f : [0, \infty) \to [0, \infty)$ of $f$ is defined by $D_f(s) = \mu(\{ x \in X : |f(x)| > s \})$. Two functions $f$ and $g$ on measure spaces $(X, \mu)$ and $(Y, \nu)$, respectively, are equimeasurable if $D_f$ and $D_g$ coincide as functions on $[0, \infty)$. The decreasing rearrangement of $f$ defined on $(X, \mu)$ is the function $f^* : [0, \infty) \to [0, \infty)$ defined by $f^*(t) = \inf \{ s \ge 0 : D_f(s) \le t \}$. By convention $\inf \emptyset = \infty$, so that $f^*(t) = \infty$ whenever $D_f(s) > t$ for all $s \in [0, \infty)$.

For a given $\mu$-measurable function $f$ on $X$, $f^*$ is non-negative, decreasing and right continuous on $[0, \infty)$. Moreover $f$ and $f^*$ are equimeasurable when $f^*$ is considered as a Lebesgue measurable function on $[0, \infty)$, and for every $p \in (0, \infty)$ it holds that

$$\int_{X} |f(x)|^{p} \, d\mu(x) = \int_{0}^{\infty} s^{p-1} D_f(s) \, ds = \int_{0}^{\infty} (f^*(t))^{p} \, dt,$$

cf. proposition 1.8 on page 43 in [BS88].
They first establish the following result, which can be traced to old results by Jodeit and Torchinsky (we shall supply more detail in the appendix):

**Theorem 8.2** (Theorem B in \[BH03\]). Let \( q \geq 2 \). There is \( K_q > 0 \) such that, for all \( f \in L^1 + L^2 \) and for all \( s \geq 0 \), the inequality
\[
\int_0^s (\hat{f}^*)^q(t) \, dt \leq K_q \int_0^s \left( \int_0^{1/t} f^*(r) \, dr \right)^q \, dt
\]
holds.

**Theorem 8.3.** Let \( u \) and \( v \) be weight functions on \( \mathbb{R}^N \), suppose \( 1 < p, q < \infty \), and let \( K \) be the constant from theorem \[5.2\] associated with the relevant index \( \geq 2 \). There is a positive constant \( C \) such that, for all \( f \in L^p_v(\mathbb{R}^N, dx) \), the inequality
\[
\left( \int_{\mathbb{R}^N} |\hat{f}(\gamma)|^q u(\gamma) \, d\gamma \right)^{1/q} \leq KC \left( \int_{\mathbb{R}^N} |f(x)|^p v(x) \, dx \right)^{1/p}
\]
holds in the following ranges and with the following constraints on \( u \) and \( v \):

(i) \( 1 < p \leq q < \infty \) and
\[
\sup_{s>0} \left( \int_0^{1/s} u^*(t) \, dt \right)^{1/q} \left( \int_0^s \left( \frac{\alpha}{s} \right)^q (t)^{p'-1} \, dt \right)^{1/p'} \equiv B_1 < \infty;
\]

(ii) for \( 1 < q < p < \infty \) and
\[
\left( \int_0^1 \left( \int_0^{1/s} u^* \right)^{r/q} \left( \int_0^s \left( \frac{\alpha}{s} \right)^{r(p'-1)} \left( s \right)^{p'-1} \, ds \right)^{2/r} \right)^{1/2} \equiv B_2 < \infty,
\]
where \( \frac{1}{r} = \frac{1}{q} - \frac{1}{p} \).

The best constant \( C \) in (30) satisfies
\[
C \leq B_1 \begin{cases} (q')^{1/p'} q^{1/q} & \text{if } 1 < p \leq q, q \geq 2 \\ (p^{1/q} (p')^{1/p'}) & \text{if } 1 < p \leq q < 2 \\ \end{cases}
\]
and \( C \leq B_2 q^{1/q} (p')^{1/q} \) if \( 1 < q < p < \infty \).

In the case of the Euclidean Fourier transform on \( \mathbb{R}^N \), the Pitt inequality is obtained by choosing the weights \( u^*(t) = \|\xi\|^{\alpha} \), \( v^*(t) = \|x\|^l \), \( \alpha < 0 \), \( l > 0 \). Here \( u^*(t) = c_a |t|^{\alpha/N} \) and \((1/v)^*(t) = c_t t^{-l/N} \) for all \( t > 0 \), where \( c_a \) and \( c_t \) are suitable constants. The weight conditions in the aforementioned theorem are thereby valid if and only if \(-N < \alpha, l < N(p-1)\), and
\[
\frac{1}{N} \left( \frac{l}{p} + \frac{\alpha}{q} \right) = \frac{1}{p'} - \frac{1}{q} = 1 - \frac{1}{p} - \frac{1}{q}.
\]

The disadvantage of employing such rearrangement and interpolation methods is that one generally picks up sub-optimal constants. In the special case where \( u \equiv 1 \equiv v \), one does not obtain the Plancherel theorem as a limiting case. We shall provide the details for \( \mathcal{F}_{k,a} \) later in this section.

We outline in an appendix the minor modification required to establish the following analogue of theorem \[8.3\] for \( \mathcal{F}_{k,a} \). For all results on \( \mathcal{F}_{k,a} \) that follow it is to be understood that \( k \geq 0 \) and \( a > 0 \) satisfies \( a + 2 \langle k \rangle + N > 2 \), and either

(i) \( N = 1 \) and \( a > 0 \),

(ii) \( N \geq 2 \) and \( a \in \{1, 2\} \)
(iii) \( N = 2 \) and \( a = 2/n \) for some \( n \in \mathbb{N} \).

**Theorem 8.4.** Let \( u \) and \( v \) be weight functions on \( \mathbb{R}^N \), suppose \( 1 < p, q < \infty \), and let \( K \) be the constant from theorem 8.2 associated with the relevant index \( \geq 2 \). There is a positive constant \( C \) such that, for all \( f \in L^p_v(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \), the inequality

\[
(\int_{\mathbb{R}^N} |\mathcal{F}_{k,a}f(\xi)|^q u(\xi) d\mu_{k,a}(\xi))^{1/q} \leq KC (\int_{\mathbb{R}^N} |f(x)|^p v(x) d\mu_{k,a}(x))^{1/p}
\]

holds in the following ranges and with the following constraints on \( u \) and \( v \):

(i) \( 1 < p \leq q < \infty \) and

\[
\sup_{s > 0} \left( \int_0^{1/s} u^s(t) \, dt \right)^{1/q} \left( \int_0^s \left( \frac{1}{s} \right)^{p'/(p' - 1)} ds \right)^{1/p'} \equiv B_1 < \infty;
\]

(ii) for \( 1 < q < p < \infty \) and

\[
\left( \int_0^\infty \left( \int_0^{1/s} u^s(t) \, dt \right) \left( \int_0^s \left( \frac{1}{s} \right)^{p/(p' - 1)} \frac{1}{s} ds \right)^{1/p'} \right)^{2/r} \equiv B_2 < \infty,
\]

where \( \frac{1}{r} = \frac{1}{q} - \frac{1}{p} \).

The best constant \( C \) in (30) satisfies

\[
C \leq B_1 \begin{cases} \left( \frac{q}{q'} \right)^{1/p} q^{1/q} & \text{if } 1 < p \leq q, q \geq 2 \\ \left( \frac{q}{q'} \right)^{1/p} q^{1/q} & \text{if } 1 < p \leq q < 2 \end{cases}
\]

and \( C \leq B_2 q^{1/q}(p')^{1/q} \) if \( 1 < q < p < \infty \).

Although one cannot expect to obtain a sharp inequality by means of interpolation, it is still important to be able to control the optimal constant \( C \) by means of a quantity \( B_1 \) determined by the weights \( u \) and \( v \).

**Corollary 8.5** (Pitt’s inequality for \( \mathcal{F}_{k,a} \)). Assume \( 1 < p \leq q < \infty \) and that the exponents \( \alpha < 0 \) and \( l > 0 \) satisfy the conditions \( -(2 \langle k \rangle + N + a - 2) < \alpha, l < (2 \langle k \rangle + N + a - 2)(p - 1) \) and

\[
\frac{1}{2 \langle k \rangle + N + a - 2} \left( \frac{l}{p} + \frac{\alpha}{q} \right) = \frac{1}{p'} - \frac{1}{q} = 1 - \frac{1}{p} - \frac{1}{q}.
\]

Then the inequality

\[
(\int_{\mathbb{R}^N} |\mathcal{F}_{k,a}f(\xi)|^q \|\xi\|^\alpha q d\mu_{k,a}(\xi))^{1/q} \leq C (\int_{\mathbb{R}^N} |f(x)|^p \|x\|^l d\mu_{k,a}(x))^{1/p}
\]

holds for all \( f \in L^p_v(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \).

**Proof.** This follows from theorem 8.4 by choosing the weights \( u(\xi) = |\xi|^\alpha, v(x) = |x|^l \), but where the rearrangements \( u^* \) and \( v^* \) are now taken with respect to the weighted measure \( d\mu_{k,a}(x) = \vartheta_{k,a}(x)dx \) on \( \mathbb{R}^N \). For \( u \) we compute that

\[
D_u(s) = \mu_{k,a}(\{\xi \in \mathbb{R}^N : \|\xi\|^\alpha > s\}) = \mu_{k,a}(B_{s^{1/\alpha}}(0)) = c(s^{1/\alpha})^{\alpha(\nu_\alpha + 1)} = c s^{(2 \langle k \rangle + N + a - 2)/\alpha},
\]

from which it follows that \( u^*(t) = ct^{\alpha/(2 \langle k \rangle + N + a - 2)} \). Analogously, \( v^*(t) = ct^{-l/(2 \langle k \rangle + N + a - 2)} \). \( \square \)
Becker’s logarithmic inequality followed from (a sharp form of) Pitt’s inequality in the special case \( p = q = 2 \). Since we do not have a sharp constant in our version of Pitt’s inequality, it will be necessary to compute the constants that are produced by the interpolation machinery. Assume \( p = q = 2 \) and consider weights \( u(\xi) = \|\xi\|^\alpha, v(x) = \|x\|^{-\alpha} \), with \( \alpha < 0 \). In this case the constant \( KC \) in Pitt’s inequality for \( F_{k,a} \) for this choice of weights is therefore bounded according to \( C \leq 2B_1 \), where \( B_1 \) is the quantity that incorporates the weights \( u \) and \( v \) and which we must now compute. Here

\[
B_1 = \sup_{s>0} \left( \int_0^{1/s} u^*(t) \, dt \right)^{1/2} \left( \int_0^s ((\frac{t}{s})^*)^2 \, dt \right)^{1/2}
\]

with

\[
u^*(t) = \inf\{s \geq 0 : D_u(s) \leq t\} = c^{-\frac{\alpha}{2(k)+N+a-2}} t^{\frac{\alpha}{2(k)+N+a-2}},
\]

where \( c = \frac{K}{2(k)+N+a-2} \) and \( K = \int_{\mathbb{R}^N-1} \vartheta_{k,a}(\omega) \, d\sigma(\omega) \). Therefore

\[
\int_0^{1/s} u^*(t) \, dt = c^{-\frac{\alpha}{2(k)+N+a-2}} \int_0^{1/s} t^{\frac{\alpha}{2(k)+N+a-2}} \, dt = C(\alpha)s^{-\frac{2(k)+N+a-2}{2(k)+N+a-2}}
\]

with \( C(\alpha) = \frac{2(k)+N+a-2}{2(k)+N+a-2} c^{-\frac{\alpha}{2(k)+N+a-2}}. \) Since \( 1/v = u \), it follows that

\[
\int_0^{\infty} ((\frac{t}{s})^*)^2 (t) \, dt = C(\alpha)s^{\frac{2(k)+N+a-2}{2(k)+N+a-2}}.
\]

But then \( B_1 = C(\alpha) \), and one notes that \( C(0) = 1 \). This is not strong enough to facilitate a differentiation argument, however: Following the strategy in either [Bec95] or the proof of [Omr11], one would consider the function

\[
\phi(\alpha) = \int_{\mathbb{R}^N} \|\xi\|^\alpha |F_{k,a}(\xi)|^2 \, d\mu_{k,a}(\xi) - 2C(\alpha) \int_{\mathbb{R}^N} \|x\|^{-\alpha} |f(x)|^2 \, d\mu_{k,a}(x).
\]

According to Pitt’s inequality for \( F_{k,a} \) it follows that \( \phi(\alpha) \leq 0 \) for \( \alpha < 0 \). Since we cannot conclude that \( \phi(0) = 0 \), however (this is where the Plancherel theorem for \( F_{k,a} \) would be used), we cannot conclude that \( \phi'(0) \leq 0 \), which is the property that would give rise to a logarithmic uncertainty inequality of the form

\[
\int_{\mathbb{R}^N} \ln(\|\xi\||F_{k,a}f(\xi)|^2 \, d\mu_{k,a}(\xi) + \int_{\mathbb{R}^N} \ln(\|x\||f(x)|^2 \, d\mu_{k,a}(x) \geq D \int_{\mathbb{R}^N} |f(x)|^2 \, d\mu_{k,a}(x)
\]

for a suitable positive constant \( D \).

**Remark 8.6.** The scope of the aforementioned paper [BH03] by Benedetto and Heinig is considerably wider than what we have suggested above. Indeed, the nature of the weight conditions enforced is such that one can work with the \( A_p \)-weights of Muckenhoupt. Since the measure space \((\mathbb{R}^N, \vartheta_{k,a})\) is doubling, there is a vast machinery available to produce further \( A_p \)-weighted inequalities for \( F_{k,a} \). We have decided against such applications, since they would seem somewhat tangential to our main applications: classical weighted inequalities and applications to uncertainty principles.
9. Qualitative nonconcentration uncertainty principles

The previous sections have presented several versions of the Heisenberg–Pauli–Weyl uncertainty principle and a strengthening in terms of entropy. The present section collects uncertainty principles that follow directly from [GJ14]. The purpose will not be to repeat their arguments but merely to point out the fact that one obtains uncertainty principle in addition to those already established. In all of the following results, \( N, k, \) and \( a \) are required to satisfy the conditions in lemma 2.13.

**Theorem 9.1** (Benedicks–Amrein–Berther principle). Let \( S, V \) be measurable subsets of \( \mathbb{R}^N \) with \( \mu_{k,a}(S), \mu_{k,a}(V) < \infty \). There exists a constant \( C = C(k,a,S,V) \) such that for all \( f \in L^2_{k,a}(\mathbb{R}^N) \)

\[
\|f\|_{L^2_{k,a}}^2 \leq C \left( \|f\|_{L^2_{k,a}(\mathbb{R}^N \setminus S)}^2 + \|F_{k,a}f\|_{L^2_{k,a}(\mathbb{R}^N \setminus V)}^2 \right).
\]

We include the following analogue of the Matolcsi–Szücs inequality for completeness, although it is morally much weaker than the Benedicks–Amrein–Berthier result. The latter result can be obtained directly from an adaptation of the methods in [GJ14].

**Proposition 9.2.** If \( f \in L^2(\mathbb{R}^N, \partial_{k,a}) \) is nonzero, then \( \mu_{k,a}(A_f) \cdot \mu_{k,a}(A_{F_{k,a}f}) \geq 1 \), where \( A_f = \{ x \in \mathbb{R}^N : f(x) \neq 0 \} \) and \( A_{F_{k,a}f} = \{ \xi \in \mathbb{R}^N : F_{k,a}f(\xi) \neq 0 \} \).

**Proof.** For an arbitrary \( \mu_{k,a} \)-measurable subset \( E \subset \mathbb{R}^N \) it follows from the inequality \( \|F_{k,a}f\|_\infty \leq \|f\|_{L^1_{k,a}} \) that

\[
\int_E |F_{k,a}f(\xi)|^2 \, d\mu_{k,a}(\xi) \leq \mu_{k,a}(E)\|F_{k,a}f\|_{L^1_{k,a}}^2 \leq \mu_{k,a}(E)\|f\|_{L^2_{k,a}}^2 = \int_{\mathbb{R}^N} |f(x)|^2 \, d\mu_{k,a}(x)^{1/2} \left( \int_{\mathbb{R}^N} |f(x)|^2 \, d\mu_{k,a}(x) \right)^{1/2}.
\]

In particular, with \( E = A_{F_{k,a}f} \), we conclude that

\[
\mu_{k,a}(A_{F_{k,a}f}) \cdot \mu_{k,a}(A_f) \|f\|_{L^2_{k,a}}^2 \geq \int_{A_{F_{k,a}f}} |F_{k,a}f(\xi)|^2 \, d\mu_{k,a}(\xi),
\]

that is, \( \mu_{k,a}(A_{F_{k,a}f}) \cdot \mu_{k,a}(A_f) \geq 1 \), by the Plancherel theorem for \( F_{k,a} \). \( \square \)

**Remark 9.3.** A stronger formulation of the nonconcentration property of \( F_{k,a} \) is captured by the Logvinenko–Sereda theorem (cf. [MS13, Section 10.3]), which was recently obtained for the Hankel transform in [GJ13]. By previous remarks, this extends to a result for \( F_{k,a} \) acting on radial functions in \( L^2_{k,a}(\mathbb{R}^N) \). We intend to address the more general case of arbitrary \( L^2 \)-functions in the near future.

Further uncertainty principles include

(a) a local uncertainty principle, which implies the Heisenberg–Pauli–Weyl uncertainty principle;

(b) qualitative uncertainty principles analogous to the Benedicks–Amrein–Berthier principle and the Donoho–Stark principle.

In (a), one obtains the following version of the uncertainty principle which generalizes Theorem 5.29 in [BSKØ12] to include different powers of the norms involved.
Corollary 9.4 (Global uncertainty principle). For $s, \beta > 0$ there exists a constant $c_{s, \beta, k,a}$ such that for all $f \in L^2(\mathbb{R}^N, \partial_{k,a}(x)\,dx)$

$$\| \cdot \|^2 \cdot \frac{1}{\| f \|^2_{L^2(\mathbb{R}^N, \partial_{k,a})}} \cdot \| \mathcal{F}_{k,a}f \|^2_{L^2(\mathbb{R}^N, \partial_{k,a})} \geq c_{s, \beta, k,a} \| f \|^2_{L^2(\mathbb{R}^N, \partial_{k,a})}.$$  

Remark 9.5. The Dunkl-case $a = 2$ was recently obtained by Soltani [Sol13] by a different method.

Having already mentioned the analogue of the Benedicks–Amrein-Berthier result, we conclude by returning to our starting point, the Donoho–Stark uncertainty principle.

Definition 9.6. Let $S$ and $\Sigma$ be measurable subsets of $\mathbb{R}^N$ with $\mu_{k,a}(S), \mu_{k,a}(\Sigma) < \infty$, and let $\varepsilon, \delta \geq 0$ be given. A function $f \in L^p(\mathbb{R}^N, \partial_{k,a})$ is $(L^p, \varepsilon)$-concentrated on $S$ if $\| f - 1_S f \|_p \leq \varepsilon \| f \|_p$. A function $f \in L^p(\mathbb{R}^N, \partial_{k,a})$ is $(L^p, \delta)$-bandlimited to $\Sigma$ if there exists a function $f_\Sigma \in L^p(\mathbb{R}^N, \partial_{k,a})$ with $\text{supp} \, \mathcal{F}_{k,a}(f_\Sigma) \subset \Sigma$ such that $\| f - f_\Sigma \|_p \leq \delta \| f \|_p$.

Theorem 9.7 (Dohono–Stark principle). Let $S$ and $V$ be measurable subsets of $\mathbb{R}^N$, and let $f \in L^2_{k,a}(\mathbb{R}^N)$ be of unit $L^2$-norm, $\varepsilon$-concentrated on $S$ and $\delta$-bandlimited on $V$ for the $(k,a)$-generalized Fourier transform $\mathcal{F}_{k,a}$. Then

$$\mu_{k,a}(S) \mu_{k,a}(V) \geq \frac{(1 - \sqrt{\varepsilon^2 + \delta^2})^2}{c_{k,a}^2}.$$  

In the Dunkl-case $a = 2$ the result is due to Ghobber and Jaming, while a slightly less precise bound from below was obtained by Kawazoe and Mejjaoli in [KM10] (several variants appear in their Section 8, together with some historical remarks). We have recently extended these results to the Heckman–Opdam transform associated to certain higher rank root systems in $\mathbb{R}^N$, cf. [Joh15c].

10. OPEN PROBLEMS

The sharp Hausdorff–Young inequality for the Hankel transform would imply a sharp entropy inequality for $\mathcal{F}_{k,a}$ acting on radial $L^2$-functions in $\mathbb{R}^N$. We are not aware of a reliable source, however, so this remains an interesting open problem. More generally, one would like to have a sharp Hausdorff–Young inequality for $\mathcal{F}_{k,a}$ acting on arbitrary $L^2$-functions but this appears to be out of reach at the moment. In the case $a = 2$, however, it seems likely that such a result can be obtained in the special case where the Weyl group $W$ associated with the underlying root system is isomorphic to $\mathbb{Z}_2^N$, since a tensorization technique already used by Beckner would reduce to problem to the one-dimensional case, which seems doable.

The same remarks are valid when discussing the logarithmic uncertainty inequality, an important point of which is indeed stability under tensorization, as already exploited by L. Gross. An important first step towards a logarithmic uncertainty inequality for $\mathcal{F}_{k,a}$ acting on arbitrary functions in $\mathbb{R}^N$ would therefore be to consider the one-dimensional case $W = \mathbb{Z}_2$, $a > 0$. Already the Dunkl-case $a = 2$ would be interesting, but it seems that the probabilistic approach by Beckner [Bec75] is inadequate. It would be very interesting to clarify these matters. Perhaps it is less difficult to obtain a sharp version of Pitt’s inequality for $\mathcal{F}_{k,a}$ and then follow the strategy in [Bec97], just like Omri [Omr11] did for the Hankel transform.

We finally wish to point out that the general framework of [BSK012] has been extended to include Clifford algebra-valued functions on $\mathbb{R}^N$ (cf. [DBOS13] and [DBOS12]). The liberal use of interpolation techniques in the present paper were scalar-valued in nature but there are
many extensions of classical interpolation theory to operator- or vector-valued functions. It is therefore to be expected that many of the results we have obtained should have immediate extensions to the Clifford-algebra-valued setting. It would be interesting to investigate these matters in detail.

APPENDIX A. PROOF OF THEOREM 8.4

The present appendix establishes theorem 8.4. The proof is largely contained in [BH03, Section 2] where the details were written out in the case of the Euclidean Fourier transform on \( \mathbb{R}^N \). As the authors remark at the beginning of section 2, loc. cit., and expounded upon in their remark 6c and d, the result (that is, the weighted inequality in theorem 8.4) is essentially valid for any bounded linear operator of type \((1, \infty)\) and \((2, 2)\). Benedetto and Heinig clearly had in mind an Euclidean setup, where Lebesgue measure was used, but some of the references they list – most notably [JT71] – indeed involve \( L^p \)-spaces with respect to weighted Lebesgue measure. Of course \( d\mu_{k,a} \) is also a weighted Lebesgue measure, but we found it impractical to incorporate the density \( \vartheta_{k,a} \) in the weights \( u \) and \( v \). Although \( F_{k,a} \) is of type \((2, 2)\) also from \( L^2 u_{k,a}(\mathbb{R}^N, dx) \) to \( L^2 v_{k,a}(\mathbb{R}^N, d\xi) \), where \( u_{k,a}(\xi) = \vartheta_{k,a}(\xi) \) and \( v_{k,a}(x) = \vartheta_{k,a}(x) \), it seems difficult to compute the decreasing rearrangements of the power weights \( u(\xi) = ||\xi||^{\alpha} \vartheta_{k,a}(\xi) \) and \( v(x) = ||x||^{\beta} \vartheta_{k,a}(x) \) that would be used in the original formulation of [BH03, Theorem 1]. With this approach it is clear that a Pitt-type inequality for \( F_{k,a} \) should hold, but it is difficult to determine the exact range of exponents \( \alpha, \beta \) and powers \( p, q \) for which the inequality is valid.

Lemma A.1 (Hardy’s lemma). Let \( \psi \) and \( \chi \) be non-negative Lebesgue measurable functions on \((0, \infty)\), and assume

\[
\int_0^s \psi(t) \, dt \leq \int_0^s \chi(t) \, dt
\]

for all \( s > 0 \). If \( \varphi \) is non-negative and non-decreasing on \((0, \infty)\), then

\[
\int_0^\infty \varphi(t) \psi(t) \, dt \leq \int_0^\infty \varphi(t) \chi(t) \, dt.
\]

Let us agree to let a weight on a measure space \((X, \mu)\) is a non-negative \( \mu \)-locally integrable functions on \( X \).

Theorem (Theorem A in [BH03]). Let \( u \) and \( v \) be weight functions on \((0, \infty)\) and suppose \( 1 < p, q < \infty \). There exists a positive constant \( C \) such that for all non-negative Lebesgue measurable functions \( f \) on \((0, \infty)\) the weighted Hardy inequality

\[
\left( \int_0^\infty \left( \int_0^t f \right)^q u(t) \, dt \right)^{1/q} \leq C \left( \int_0^\infty f^p v(t) \, dt \right)^{1/p}
\]

is satisfied if and only if

(i) for \( 1 < p \leq q < \infty \),

\[
\sup_{s > 0} \left( \int_s^\infty u(t) \, dt \right)^{1/q} \left( \int_0^s v(t)^{1-p'} \, dt \right)^{1/p'} = A_1 < \infty,
\]

and

(ii) for \( 1 < q < p < \infty \),

\[
\left( \int_0^\infty \left( \int_0^\infty u \right)^{r/q} \left( \int_0^s v^{1-p'} \right)^{r/q'} v(s)^{1-p'} \, ds \right)^{1/r} = A_2 < \infty
\]

where \( \frac{1}{r} = \frac{1}{q} - \frac{1}{p} \).
Moreover, if \( C \) is the best constant in the weighted Hardy inequality, then in case (i) we have \( A_1 \leq C \leq A_1(q')^{1/q'} q^{1/q} \), and in case (ii) we have \((p')^{-1/q'} q^{1/q} A_2 \leq C \leq (p')^{1/q'} q^{1/q} A_2\).

The proof of [BH03] Theorem 1] relies of several classical rearrangement inequalities. Since Benedetto and Heinig formulate these for Lebesgue measure and the Fourier transform, two of their results must be modified slightly. The **decreasing rearrangement** of \( f \) defined on \((X, \mu)\) is the function \( f^* : [0, \infty) \to [0, \infty) \) defined by \( f^*(t) = \inf\{s \geq 0 : D_f(s) \leq t\} \). By convention \( \inf \emptyset = \infty \), so that \( f^*(t) = \infty \) whenever \( D_f(s) > t \) for all \( s \in [0, \infty) \).

**Lemma A.2** (The Hardy–Littlewood rearrangement inequality). Let \( f \) and \( g \) be non-negative \( \mu_{k,a} \)-measurable functions on \( \mathbb{R}^N \). Then

\[
\int_{\mathbb{R}^N} f(x)g(x) \, d\mu_{k,a}(x) \leq \int_0^\infty f^*(t)g^*(t) \, dt
\]

and

\[
\int_0^\infty f^*(t) \frac{1}{(1/g)^*(t)} \, dt \leq \int_{\mathbb{R}^N} f(x)g(x) \, d\mu_{k,a}(x).
\]

**Proof.** The first statement can be found as Theorem 2.2 on page 44 in [BS88]. \( \square \)

**Lemma A.3** (Theorem B in [BH03]; the type estimate of Jodeit and Torchinsky). Let \( q \geq 2 \). There is a constant \( K_q > 0 \) such that, for all \( f \in (L^1 + L^2)(\mathbb{R}^N, \partial_{k,a}(x)dx) \) and for all \( s \geq 0 \), the inequality

\[
\int_0^s (|F_{k,a}(t)|)^q \, dt \leq K_q^q \int_0^{f^*(t)/r} \left( \int_0^t f^*(r) \, dr \right)^q \, dt
\]

holds.

**Proof.** The case \( q = 2 \) is [JT71] Theorem 4.6] and the more general statement for \( q \geq 2 \) is [JT71] Theorem 4.7]. \( \square \)

Jodeit and Torchinsky phrased their results more generally in terms of sublinear operators \( T \) acting between Orlicz spaces \( L_A(\mathbb{R}^n, d\mu) \) and \( L_B(\mathbb{R}^N, d\nu) \), where \( A \) and \( B \) are Young functions. The aforementioned result is obtained by considering power weights as Young functions and using that we already know that \( F_{k,a} \) is of type \((1, \infty)\) and \((2, 2)\) when using the weighted measure \( \mu = \nu = \mu_{k,a} \). A close inspection of [JT71] Section 2] establishes that they form the symmetric rearrangements \( f^* \) and \((Tf)^* \) with respect to \( \mu \) and \( \nu \), respectively, so we do not have to redo their proofs.

**Theorem A.4.** Let \( u \) and \( v \) be weight functions on \( \mathbb{R}^n \), suppose \( 1 < p, q < \infty \), and let \( K \) be the constant from theorem 5.2 associated with the relevant index \( \geq 2 \). There is a positive constant \( C \) such that, for all \( f \in L_p^p(\mathbb{R}^N, \nu \partial_{k,a}(x)dx) \), the inequality

\[
\left( \int_{\mathbb{R}^N} |F_{k,a}f(\xi)|^q u(\xi) \, d\mu_{k,a}(\xi) \right)^{1/q} \leq KC \left( \int_{\mathbb{R}^N} |f(x)|^p v(x) \, d\mu_{k,a}(x) \right)^{1/p}
\]

holds in the following ranges and with the following constraints on \( u \) and \( v \):

(i) \( 1 < p \leq q < \infty \) and

\[
\sup_{s > 0} \left( \int_0^{f^*(t)/s} u^*(t) \, dt \right)^{1/q} \left( \int_0^f ((1/\xi)^*)^p \, dt \right)^{1/p'} \equiv B_1 < \infty;
\]
(ii) for $1 < q < p < \infty$ and
\[
\left(\int_0^\infty \left(\int_0^1 u^r \right)^{r/q} \left(\int_0^s \left(\frac{1}{s}\right)^{(p'-1)} \right)^{r/q'} \left(\left(\frac{1}{s}\right)^{(p'-1)}(s)^{p'-1} ds\right)^{2/r} \right)^{1/q'} \equiv B_2 < \infty,
\]
where $q' = \frac{p}{q} - 1$.

The best constant $C$ in (36) satisfies
\[
C \leq B_1 \begin{cases} 
(q')^{1/p'} q^{1/q} & \text{if } 1 < p \leq q, q \geq 2 \\
(p^{1/q}(p')^{1/p'}) & \text{if } 1 < p \leq q < 2
\end{cases}
\]
and $C \leq B_2 q^{1/q}(p')^{1/q'}$ if $1 < q < p < \infty$.

The proof follows exactly as in [BH03, section 2], except that we replace their Theorem B with the above version for $d\mu_{k,a}$, and use the Hardy–Littlewood rearrangement inequality for rearrangements with respect to the weighted measure $d\mu_{k,a}$ rather than Lebesgue measure on $\mathbb{R}^N$.

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