Optimizing Performance of Continuous-Time Stochastic Systems using Timeout Synthesis *

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1 Abstract

We consider parametric version of fixed-delay continuous-time Markov chains (or equivalently deterministic and stochastic Petri nets, DSPN) where fixed-delay transitions are specified by parameters, rather than concrete values. Our goal is to synthesize values of these parameters that, for a given cost function, minimise expected total cost incurred before reaching a given set of target states. We show that under mild assumptions, optimal values of parameters can be effectively approximated using translation to a Markov decision process (MDP) whose actions correspond to discretized values of these parameters.

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1 Introduction

Continuous-time Markov chains (CTMC) are a fundamental model of stochastic systems with a discrete state-space that evolve in continuous-time. Several higher level modelling formalisms, such as stochastic Petri nets and stochastic process algebras, use CTMC as their semantics. As such, CTMC have been applied in performance and dependability analysis in various contexts ranging from aircraft communication protocols (see, e.g. [32]) to models of biochemical systems (see, e.g. [20]).

There are several equivalent definitions of CTMC (see, e.g. [14, 28]). We may conveniently define a (uniformized, finite-state) CTMC to consist of a finite set of states \( S \) coupled with a common rate \( \lambda \) and a stochastic matrix \( P \in \mathbb{R}^{S \times S}_{\geq 0} \) specifying probabilities of transitions between pairs of states. An execution starts in some state. In every step, the CTMC waits for a duration that is chosen at random according to the exponential distribution with the rate \( \lambda \), and then moves to a state \( s' \) randomly chosen with probability \( P(s, s') \).

The practical interpretation of the above semantics is that in every state the system waits for an event to occur and then reacts by changing its state. A typical example is a model of a simple queue to which new customers come in random intervals and are also served in random intervals. However, in practice, events are usually not exponentially distributed, and, in fact, their distributions may be quite far from being exponential. To deal with such events, phase-type approximation technique [27] is usually applied.

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¹ Equivalently, one may assign rates to transitions between states, and then in every step assign duration to each transition and fire the earliest one. Assuming that the rates of transitions leaving each state sum up to \( \lambda \), the two definitions are fully equivalent. Note that this assumption is w.l.o.g. as uniformization does not change the expected total cost we study here.
Unfortunately, as already noted in [27], some distributions cannot be efficiently fit with phase-type approximation. In particular, degenerate distributions of events with fixed delays, i.e., events that occur after a fixed amount of time with probability 1, form a distinguished example of this phenomenon (for more details see [22]). However, as events with fixed delays play a crucial role in many systems, especially in communication protocols [29], time-driven real-time scheduling [31], etc., they should be handled faithfully in modelling and analysis.

Inspired by deterministic and stochastic Petri nets [25] and delayed CTMC [15], we study fixed-delay CTMC (fdCTMC), the CTMC extended with fixed-delay transitions. More concretely, we specify a set of states $S_{fd} \subseteq S$ where fixed-delay transitions are enabled and add a stochastic matrix $F \in \mathbb{R}_{\geq 0}^{S_{fd} \times S_{fd}}$ specifying probabilities of fixed-delay transitions between states. In addition, we consider a delay function $d : S_{fd} \to \mathbb{R}_{>0}$. The semantics can be intuitively described as follows. Imagine a CTMC extended with an alarm clock. At the beginning of an execution, the alarm clock is turned off and the process behaves as the original CTMC. Whenever a state $s$ of $S_{fd}$ is visited and the alarm clock is off at the time, it is turned on and set to ring after $d(s)$ time units. Subsequently, the process keeps behaving as the original CTMC until either a state of $S \setminus S_{fd}$ is visited (in which case the alarm clock is turned off), or the alarm clock rings in a state $s'$ of $S_{fd}$. In the latter case, a fixed-delay transition takes place, which means that the process changes the state randomly according to the distribution $F(s', \cdot)$, and the alarm clock is either turned off or newly set (when entering a state of $S_{fd}$).

In most practical applications mentioned above, fixed-delay events are determined by the design of the system and often strongly influence performance of the system. Indeed, both timeouts in network protocols as well as scheduling intervals in real-time systems directly influence performance of the respective systems and their manual setting usually requires considerable effort and expertise. This motivates us to consider the fixed-time delays $d(s)$ as free parameters of the model, and develop techniques for their optimization with respect to a given performance measure.

**Example** We demonstrate the concept on two different models of sending one segment of data in the *alternating bit protocol*. In the protocol, each segment of data is retransmitted until an acknowledgement is received. The delay between retransmissions has impact on throughput of the protocol as well as on network congestion. In the simpler model below on the left, the data is sent in state `init`. The exp-delay transitions, drawn as solid arrows, model message loss (with probability 0.2) and delivery (with probability 0.8). The fixed-delay transitions, drawn as dashed arrows, cause the data to be retransmitted.

Note that whenever the data is retransmitted, we assume in the model on the left, that the previous message with the data is lost. The more faithful model on the right models up to two messages with the data segment being delivered concurrently. The states $m_0, m_1, m_2$ denote how many messages are being delivered at the moment. For choosing an optimal delay between retransmissions, we need to formalize how to express performance of the protocol.

To express performance properties, we use standard cost (or reward) structures (see, e.g. [30]) that assign numerical rewards to states and transitions. More precisely, we consider the following three cost functions: $R : S \to \mathbb{R}_{\geq 0}$, which assigns a cost rate $R(s)$ to every state $s$ so that the cost $R(s)$ is paid for every unit of time spend in the state $s$, and functions $I_p, I_F : S \times S \to \mathbb{R}_{\geq 0}$ that assign to each exp-delay and fixed-delay transition, respectively, the cost that is immediately paid when the transition is taken. Note that $R$ is usually used to express time spent in individual states, while the other two cost functions...
are used to quantify difficulty of dealing with events corresponding to transitions. The performance measure itself is the expected total cost incurred before reaching a given set of states \( G \) starting in a given initial state \( s_0 \). For this moment, let us denote this measure by \( E_d \), stressing the fact that it depends on the delay function \( d \) which is the only variable quantity in our optimization task:

\[
\inf_{d \in D} E_d - E_d < \varepsilon.
\]

Problem 1 (Cost optimization). For a subspace \( D \subseteq (\mathbb{R}_{\geq 0})^{S_d} \) and a given approximation error \( \varepsilon > 0 \), compute a delay function \( d \in D \) that is \( \varepsilon \)-optimal within \( D \), i.e.

Example (cont.) We can model the expected cost of sending one data segment in our examples as follows. To take into account the expected time of data delivery, we set the cost rate of each state to, e.g., 1. To take into account the expected number of retransmissions, we set the cost of each fixed-delay transition, e.g., to 3. The cost of each exp-delay transition is set to 0. Now the goal for the left model is to find a delay \( d(init) \) optimizing the expected total cost incurred before reaching the state \( OK \). The goal for the model on the right is the same. However, note that it makes no sense to synthesize different delays \( d(m_2) \) and \( d(m_1) \) as the states \( m_2 \) and \( m_1 \) are indistinguishable in the implementation of the protocol.

In this paper we provide algorithms for solving the following two special cases of the cost optimization problem under the assumption that the reward rate \( R(s) \) is positive in every state \( s \):

1. **Unconstrained optimization** where we demand \( D = (\mathbb{R}_{\geq 0})^{S_d} \), i.e. the set of all delay functions. In this paper, we show that the unconstrained optimization can be solved in exponential time.

2. **Bounded optimization under partial observation** where we introduce bounds \( d, d > 0 \) together with an equivalence relation \( \equiv \) on \( S_d \) and demand \( D \) to be the set of all delay functions \( d \) satisfying the following conditions:

   \[
   \begin{align*}
   & d \leq d(s) \leq d \quad \text{for all } s \in S_d, \\
   & d(s) = d(s') \quad \text{whenever } s \equiv s'.
   \end{align*}
   \]

Like in the example above, the equivalence \( \equiv \) can be used to hide information about detailed internal structure of states which is often needed in practical applications. In this paper, we show that the bounded optimization under partial observation can be solved in exponential time.

Also, we consider the corresponding approximate threshold variant: For a given \( x \) decide whether \( \inf_{d \in D} E_d > x + \varepsilon \) or \( \inf_{d \in D} E_d < x - \varepsilon \) (for \( \inf_{d \in D} E_d \in [x - \varepsilon, x + \varepsilon] \) an arbitrary answer may be given). We show that this bounded optimization problem is NP-complete, thus a polynomial time solution of the bounded optimization under partial observation is unlikely.

The assumption that all delays are between fixed thresholds \( d \) and \( d \) is crucial for our methods to work. As we discuss in Section 4 without this assumption the optimization under partial observation becomes much trickier and we leave its solution for future work.

Related work. Various forms of continuous-time stochastic processes with fixed-delay events have already been studied, see e.g. [25][13][1][4]. In particular, as noted above, our definition of fdCTMC is closely related to the original definition of deterministic and stochastic Petri nets [25]. Papers on verification of continuous-time systems with timed automata (TA) specifications [12][4][5] are also related to our work as the constraints in timed automata resemble fixed-delay transitions. None of these works, however, considers synthesis of fixed-delays (or other parameters).

Parameter synthesis techniques have been developed for several models, such as parametric timed automata [2], parametric one-counter automata [16], parametric Markov models [17], etc. In continuous-time stochastic systems, [13] presents algorithms for synthesis of rates in CTMC with respect to
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time-bounded reachability properties and studies synthesis of rates in CTMC in the context of biochemical models. None of these papers considers synthesis of fixed-delays.

Optimal control of continuous-time (Semi)-Markov decision processes (MDP) by time-dependent schedulers [26] [11] [9] [7] can also be viewed as synthesis of discrete parameters in continuous-time systems. However, the schedulers are only allowed to choose actions from finite domains while parameter synthesis calls for continuous domains.

The problem of synthesizing time-outs has been studied in variety of engineering contexts such as vehicle communication systems [21] and avionic subsystems [3] [32]. To the best of our knowledge no generic framework for synthesis of time-outs in continuous-time systems exhibiting random behavior has been developed so far.

Paper outline. Section 2 introduces fixed-delay CTMC and cost structures. Section 3 and Section 4 are devoted to unconstrained optimization and bounded optimization under partial observation, respectively. Due to space constraints, full proofs are in [8].

2 Preliminaries

We use \( \mathbb{N}_0, \mathbb{R}_{\geq 0}, \) and \( \mathbb{R}_{>0} \) to denote the set of all non-negative integers, non-negative real numbers, and positive real numbers, respectively. Furthermore, for a countable set \( A \), we denote by \( \mathcal{D}(A) \) the set of discrete probability distributions over \( A \), i.e. functions \( \mu: A \to \mathbb{R}_{\geq 0} \) such that \( \sum_{a \in A} \mu(a) = 1 \).

\textbf{Definition 1.} A fixed-delay CTMC structure (fdCTMC structure) \( C \) is a tuple \( (S, \lambda, P, S_{fd}, F) \) where

\begin{itemize}
  \item \( S \) is a finite set of states,
  \item \( \lambda \in \mathbb{R}_{>0} \) is a rate of exp-delay transitions,
  \item \( P: S \times S \to \mathbb{R}_{\geq 0} \) is a stochastic matrix specifying probabilities of exp-delay transitions,
  \item \( S_{fd} \subseteq S \) is a set of states where fixed-delay transitions are enabled,
  \item \( F: S_{fd} \times S \to \mathbb{R}_{\geq 0} \) is a stochastic matrix specifying probabilities of fixed-delay transitions.
\end{itemize}

A fixed-delay CTMC (fdCTMC) is a pair \( C(d) = (C, d) \) where \( C \) is a fdCTMC structure and \( d: S_{fd} \to \mathbb{R}_{\geq 0} \) is a delay function which to every state where fixed-delay transitions are enabled assigns a positive delay.

A configuration of a fdCTMC is a pair \((s, d)\) where \( s \in S \) is the current state and \( d \in \mathbb{R}_{\geq 0} \cup \{\infty\} \) is the remaining time before a fixed-delay transition takes place (here we assume that \( d = \infty \) iff \( s \notin S_{fd} \)).

Intuitively, an execution of \( C(d) \) starts in some configuration \((s_0, d_0)\) where \( d_0 \) is equal either to \( d(s_0) \), or to \( \infty \), depending on whether \( s_0 \in S_{fd} \), or not. In every step, assuming that the current configuration is \((s, d)\), the fdCTMC waits for a transition to occur and then moves on to a new configuration \((s', d')\) as follows. If \( s \notin S_{fd} \), then only exp-delay transitions are enabled, which means that the fdCTMC waits for some duration which is exponentially distributed with the rate \( \lambda \), the state \( s' \) is chosen randomly with probability \( P(s, s') \) and \( d' \) is an appropriately updated information about fixed-delay transitions (see below). If \( s \in S_{fd} \), then fixed-delay transitions are enabled and compete with the exp-delay transitions. Then either an exp-delay transition occurs earlier, in which case \( s' \) is chosen as above with the probability \( P(s, s') \), or a fixed-delay transition occurs earlier, in which case \( s' \) is chosen with the probability \( F(s, s') \).

We define a run initiated in \( s_0 \in S \) to be an alternating sequence of configurations and times \((s_0, d_0)t_0(s_1, d_1)t_1 \cdots \) where, as indicated above, \( d_0 = d(s_0) \) if \( s_0 \in S_{fd} \), and \( d_0 = \infty \) otherwise; and for each \( i \in \mathbb{N}_0 \) we have that \( t_i \leq d_i \) and \( t_{i+1} = \text{next}(s_i, d_i; t_i) \) is defined as follows:

\begin{itemize}
  \item If \( t_i < d_i \) (an exp-delay transition occurs),
    \[ \text{next}(s_i, d_i; t_i) = \begin{cases}
        d_i - t_i & \text{if } s_{i+1} \in S_{fd} \text{ and } s_i \in S_{fd} \text{ (fixed-delay transitions remain enabled)}, \\
        d(s_{i+1}) & \text{if } s_{i+1} \in S_{fd} \text{ and } s_i \notin S_{fd} \text{ (fixed-delay transitions become enabled)}, \\
        \infty & \text{if } s_{i+1} \notin S_{fd} \text{ (fixed-delay transitions get disabled)},
    \end{cases} \]
  \item If \( t_i = d_i \) (a fixed-delay transition occurs),
    \[ \text{next}(s_i, d_i; t_i) = \begin{cases}
        s_{i+1} & \text{if } s_{i+1} \in S_{fd} \text{ and } s_i \in S_{fd}, \\
        s_{i+1} & \text{if } s_{i+1} \in S_{fd} \text{ and } s_i \notin S_{fd}, \\
        s_{i+1} & \text{if } s_{i+1} \notin S_{fd},
    \end{cases} \]
\end{itemize}
If $t_i = d_i$ (a fixed-delay transition occurs),

$$\text{next}(s_i, d_i; t_i) = \begin{cases} d(s_{i+1}) & \text{if } s_{i+1} \in S_d \text{ (fixed-delay transitions become enabled again)}, \\ \infty & \text{if } s_{i+1} \notin S_d \text{ (fixed-delay transitions get disabled)}. \end{cases}$$

Next, for every state $s_n$, we define the probability measure $Pr_{fd}^{s_n}$ over the measurable space $(\Omega, \mathcal{F})$ where $\Omega$ is the set of all runs initiated in $s_n$ and $\mathcal{F}$ is a $\sigma$-field over $\Omega$ generated by the set all of cylinder sets of the form $R_{t_0 \cdots t_{n-1}} = \{ (s', d_0) \cdots \in \Omega \mid \forall i \leq n : s'_i = s_i, \forall i < n : t_i \in I_i \}$, here $s_0, \ldots, s_n \in S$ and $I_0, \ldots, I_{n-1} \subseteq \mathbb{R}_{\geq 0}$ are intervals. Given such a cylinder set $R_{t_0 \cdots t_{n-1}}$, we define the probability that a run of $C(d)$ belongs to $R_{t_0 \cdots t_{n-1}}$ by

$$Pr_{fd}^{s_n}(R_{t_0 \cdots t_{n-1}}) := \prod_{i=1}^{n} \Pr_{\omega}(s_i, d_i; t_i)$$

where $d_0 = d(s_0)$ if $s_0 \in S_d$, and $d_0 = \infty$ otherwise; $d_{i+1} = \text{next}(s_i, d_i, t_i)$ for each $i \in \mathbb{N}$; and the probability measure $Pr_{\omega}(s, d, s'; \cdot)$ stands for the probability of moving to $s'$ in a given time interval when staying in $(s, d)$, i.e. for each $s, s' \in S, d \in \mathbb{R}_{\geq 0} \cup \{ \infty \}$, and an interval $[a, b] \subseteq \mathbb{R}_{\geq 0}$,

$$\Pr_{\omega}(s, d, s'; [a, b]) := \begin{cases} 1_{[a,b]} \cdot F(s, s') \cdot (1 - e^{-\lambda(b - a)}) & \text{exp-delay transition in the interval } [a, b] \\ 1_{d \in [a,b]} \cdot F(s, s') \cdot e^{-\lambda a} & \text{fixed-delay transition after time } d \in (a, b) \end{cases}$$

The probability measure $Pr_{fd}^{s_n}$ then extends uniquely to all sets of $\mathcal{F}$. Given a random variable $V$ on runs of $C(d)$, we denote by $\mathbb{E}_{C(d)}^{s_n}[V]$ the expected value of $V$ with respect to the probability measure $Pr_{C(d)}^{s_n}$.

**Total cost before reaching a goal** To allow formalization of optimization properties, we enrich the model in a standard way (see, e.g. [3]) with costs (or rewards). A cost structure over a fdCTMC structure $C$ with state space $S$ is a triple $\text{Cost} = (R, I_P, I_F)$ where $R : S \rightarrow \mathbb{R}_{\geq 0}$ assigns a cost rate to every state, and $I_P, I_F : S \times S \rightarrow \mathbb{R}_{\geq 0}$ assign an impulse cost to every exp-delay and fixed-delay transition, respectively. For a given set of goal states $G \subseteq S$, we denote by $W_{\text{Cost}}^{G}$ the random variable assigning to each run $\omega = (s_0, d_0)0 \cdots$ the total cost before reaching $G$, given by

$$W_{\text{Cost}}^{G}(\omega) = \begin{cases} \sum_{i=0}^{n-1} (t_i \cdot R(s_i) + I_P(s_i, s_{i+1})) & \text{for minimal } n \text{ such that } s_n \in G, \\ \infty & \text{if there is no such } n, \end{cases}$$

where $I_P(s_i, s_{i+1})$ for an exp-delay transition, i.e. when $t_i < d_i$, and equals $I_F(s_i, s_{i+1})$ for a fixed-delay transition, i.e. when $t_i = d_i$.

By a straightforward adaptation of existing methods, the expected total cost of a given fdCTMC can be efficiently approximated (for a proof, see [8]).

**Proposition 2.** There is a polynomial-time algorithm that for a given fdCTMC $C(d)$, initial state $s_0$, cost structure $\text{Cost}$, goal states $G \subseteq S$, and an approximation error $\varepsilon > 0$ computes $x \in \mathbb{R}_{\geq 0}$ such that

$$\left| \mathbb{E}_{C(d)}^{s_0} \left[ W_{\text{Cost}}^{G} \right] - x \right| < \varepsilon,$$

(Recall that $\mathbb{E}_{C(d)}^{s_0}$ denotes the expected value with respect to the probability measure $Pr_{C(d)}^{s_0}$.)

**3 Unconstrained optimization**

**Theorem 3.** There is an algorithm that given a fdCTMC structure $C$, a cost structure $\text{Cost}$ with $R(s) > 0$ for all $s \in S$, an initial state $s_0$, a set of goal states $G$, and $\varepsilon > 0$ computes in exponential time a delay function $d$ such that

$$\left| \inf_d \mathbb{E}_{C(d)}^{s_0} \left[ W_{\text{Cost}}^{G} \right] - \mathbb{E}_{C(d)}^{s_0} \left[ W_{\text{Cost}}^{G} \right] \right| < \varepsilon.$$
We say that an action \( a \) is enabled in a vertex \( v \) if \( T(v, a) \neq \perp \), we denote by \( \text{Act}(v) \) the set of all actions enabled in \( v \), and we assume that \( \text{Act}(v) \) is non-empty for every vertex \( v \). We resolve the non-determinism of actions by a memoryless deterministic (MD) strategy, or simply a strategy\( ^{\mathcal{M}} \). A strategy is a function \( \sigma : V \rightarrow \text{Act} \) which assigns to every vertex \( v \) an action enabled in \( v \). We denote by \( \Sigma(M) \) the set of all MD strategies in \( M \). The behaviour of \( M \) with a fixed strategy \( \sigma \) can be intuitively described as follows: A run starts in a vertex. In every step, assuming that the current vertex is \( v \), the process moves to a new vertex \( v' \) with probability \( T(v, \sigma(v))(v') \). Every strategy \( \sigma \) together with an initial vertex \( v \) uniquely determine a probability measure \( \Pr_{M(\sigma)}^{v} \) on the set of runs, i.e. infinite alternating sequences of vertices and actions \( v_0a_1v_1a_2v_2\cdots \in (V \cdot \text{Act})^\omega \); see \( [8] \) for details. As for fdCTMC, we denote by \( \mathbb{E}_{M(\sigma)}^{v} \) the expectation w.r.t. \( \Pr_{M(\sigma)}^{v} \).

We define a random variable \( W \) assigning to each run the total cost incurred before reaching \( V' \) by

\[
W(v_0a_1v_1a_2\ldots) = \begin{cases} 
\sum_{i=0}^{n-1} A(v_i, a_{i+1}) & \text{if there is a minimal } n \text{ such that } v_n \in V', \\
\infty & \text{otherwise.}
\end{cases}
\]

Given \( v \in V \) and a strategy \( \sigma \), we define \( E_{M(\sigma)}^{v} = \mathbb{E}_{M(\sigma)}^{v}[W] \) the expected total cost incurred before reaching \( V' \) starting in \( v \). Given \( \varepsilon \geq 0 \), we say that a strategy \( \sigma \) is \( \varepsilon \)-optimal in a vertex \( v \) if \( E_{M(\sigma)}^{v} \leq \inf_{\sigma'} E_{M(\sigma')}^{v} + \varepsilon \). A \( \varepsilon \)-optimal strategy in \( v \) is optimal in \( v \). Finally, we define \( \text{Val}[M](v) = \inf_{\sigma} E_{M(\sigma)}^{v} \).

**The reduction:** Recall that we consider a fdCTMC structure \( C = (S, P, S_g, F) \), a cost structure \( \text{Cost} = (\mathcal{R}, I_p, I_f) \), an initial state \( s_0 \), and goal states \( G \). We define a DTMDP \( M = (S, \text{Act}, T, G, \text{Cost}) \):

\[ \text{Act} := \mathbb{R}_{>0} \cup \{\infty\}; \text{where actions } \mathbb{R}_{>0} \text{ are enabled in } s \in S_{\text{fd}} \text{ and action } \infty \text{ is enabled in } s \in S \setminus S_{\text{fd}}. \]

Let \( s \in S_{\text{fd}} \) and let \( d \in \mathbb{R}_{>0} \) be an action of \( M \) enabled in \( s \). Denote by \( d \) the constant delay function returning \( d \) for every state of \( S_{\text{fd}} \). Intuitively, we define \( T(s, d) \) and \( \text{Cost}'(s, d) \) to summarize the behavior of \( C(d) \) starting in the state \( s \) (i.e., starting in the configuration \( (s, d) \)) until the first moment when either a fixed-delay transition is taken or the fixed-delay transitions get deactivated. We capture this moment by a stopping time \( \tau \) defined for all runs \( \omega = (s_0, d_0) t_0 \cdots \) of \( C(d) \) by \( \tau(\omega) := \inf\{n \in \mathbb{N} \mid d_{n-1} = t_{n-1} \text{ or } s_n \notin S_{\text{fd}}\} \). Furthermore, we define

\[ \text{exit}(\omega) := s_{\tau(\omega)}, \quad \text{dcost}(\omega) := \sum_{i=0}^{\tau(\omega)-1} (t_i \cdot \mathcal{R}(s_i) + I_i(\omega)) \]

\[ \text{exit}(\omega) := s_{\tau(\omega)}, \quad \text{dcost}(\omega) := \sum_{i=0}^{\tau(\omega)-1} (t_i \cdot \mathcal{R}(s_i) + I_i(\omega)) \]

\[ \text{exit}(\omega) := s_{\tau(\omega)}, \quad \text{dcost}(\omega) := \sum_{i=0}^{\tau(\omega)-1} (t_i \cdot \mathcal{R}(s_i) + I_i(\omega)) \]
Intuitively, \textit{exit}(\omega) is the state occupied by \(C(d)\) after \(\tau(\omega)\) steps, and \(d\text{cost}(\omega)\) is the cost accumulated up to \(\tau(\omega)\)-th step. Now let
\[
T(s,d)(s') := \Pr^s_{C(d)}(\text{exit} = s') \quad \text{for every } s' \in S,
\]
\[
\text{Cost}(s,d) := \mathbb{E}^s_{C(d)}[d\text{cost}].
\]

For every state \(s \in S \setminus S_{\text{id}}\) with no fixed-delay transitions we model in \(M\) just one step of \(C\) by
\[
T(s,\infty)(s') := \Pr(s, s') \quad \text{for every } s' \in S,
\]
\[
\text{Cost}(s,\infty) := \mathcal{R}(s)/\lambda + \sum_{s' \in S} \Pr(s, s') \cdot I(s, s').
\]

Here, note that \(\text{Cost}(s,\infty)\) is the expected cost incurred in one step from \(s\).

Note that there is a one-to-one correspondence between the delay functions in \(C\) and strategies in \(M\): the only difference is that strategies are also defined in states \(s \in S \setminus S_{\text{id}}\) where they choose the only available action \(\infty\). Thus we use \(d, d', \ldots\) to denote strategies in \(M\). To further simplify notation, we write \(E_{M(d)}\) and \(\text{Val}[M(d)]\) instead of \(E_{M(d)}^{\infty}\) and \(\text{Val}[M(d)](s_{\text{id}})\), respectively. Finally, let us state correctness of the reduction.

\textbf{Proposition 5.} For any delay function \(d\) it holds \(E_{C(d)} = E_{M(d)}\).

In particular, in order to solve the optimization problem for \(C\) it suffices to find an \(\varepsilon\)-optimal strategy (i.e., a delay function) \(d\) in \(M\). For constructing a finite approximation, the following property will be useful.

\textbf{Lemma 6.} Let \(B\) denote \(\left(\frac{|S|}{\min \lambda}\right)^{|S|} \cdot e \cdot \max C \cdot (1 + 1/\lambda)\). We have \(\text{Val}[M] < B\).

\textbf{Approximation algorithm using discretization of \(M\).} So far we have reduced the continuous-time optimization problem in \(\text{idCTMC} \ C\) to a discrete-time optimization problem in \(\text{DTMDP} \ M\). Still, \(M\) has a continuous and unbounded space of actions. In order to obtain a MDP with finitely many actions, we fix a discretization step \(\delta > 0\) and an upper bound \(u \in \mathbb{R}_{>0}\), and define a new DTMDP \(M_{\delta, u} = (S, \text{Act}_{\delta, u}, T_{\delta, u}, G, \text{Cost}_{\delta, u})\) where \(\text{Act}_{\delta, u} = \{k\delta \mid k \in \mathbb{N}, \delta \leq k\delta \leq u\} \cup \{\infty\}\), \(T_{\delta, u}(s, d)(s') = T(s, d)(s')\) approximated up to an absolute error \(|S| \cdot \delta \lambda\) and \(\text{Cost}_{\delta, u}(s, d) = \text{Cost}(s, d)\) approximated up to an absolute error \(\min C \cdot \delta^{1-\varepsilon/2}\). We further require that each transition probability that is positive in \(T(s, d)\) has in \(T_{\delta, u}(s, d)\) value at least \(\delta \lambda\). Thus, the discretized model \(M_{\varepsilon}\) preserves qualitative properties of \(M\):

\textbf{Lemma 7.} We have \(E_{M_{\delta, u}} = \infty\) for some \(d\) if and only if \(\text{Val}[M] = \text{Val}[M_{\varepsilon}] = \infty\).

Thanks to this, we can detect infinite expected total cost in \(M_{\varepsilon}\). Let us address finite expected total costs.

\textbf{Proposition 8.} Let \(\varepsilon > 0\) and \(\text{Val}[M] < \infty\). Then for \(\delta = \varepsilon^2 \cdot B^{-5}\) and \(u = \frac{B}{\min \lambda}\) we have
\[
\text{Val}[M] - \text{Val}[M_{\delta, u}] < \varepsilon.
\]

Before proving Proposition\textsuperscript{5} we let us finish the proof of Theorem\textsuperscript{3} by presenting a simple algorithm for the unconstrained optimization. Let us denote by \(M_{\varepsilon}\) the DTMDP \(M_{\delta, u}\) where \(\delta\) and \(u\) are as specified in Proposition\textsuperscript{5}. Note that given \(M_{\varepsilon}\), the unconstrained optimization becomes almost trivial: Simply compute an optimal (MD) strategy \(d\) for \(M_{\varepsilon}\) in polynomial time using standard techniques for solving MDP with expected total reward objectives (see e.g. [30]). Proposition\textsuperscript{5} and Proposition\textsuperscript{6} then make sure that the delay function \(d\) satisfies \(|E_{C(d)} - \inf_d E_{C(d)}| < \varepsilon\). However, as the definition of \(M_{\varepsilon}\) is based on an uncountable DTMDP \(M\), it does not straightforwardly yield an algorithm for computing

\textsuperscript{3} Note that the definitions are correct since we have \(\mathbb{E}^s_{C(d)}[d\text{cost}] < \infty\) and \(\sum_{s' \in S} \Pr^s_{C(d)}(\text{exit} = s') = 1\) for every \(s \in S\).
M_e. So in [8] we propose a simple procedure which computes M_e directly from C, Cost, and ε in exponential time (it is based on techniques for transient analysis of CTMC).

To summarize, the algorithm first computes M_e in exponential time, and then computes an optimal MD strategy d for M_e (satisfying \(|E_{C(d)} - \inf_{d'} E_{C(d')}| < \epsilon\) in time polynomial in the size of M_e. Observe that this algorithm is more efficient than a naive exploration of the whole discretized space of delay functions (Act_e)^N. Furthermore, we discuss heuristics to improve the computation time in Section 5.

**Proof of Proposition 8** Let us fix ε > 0. As before, to simplify notation, we write E_{M_e(d)} and Val[M_e] instead of E_{M_e(d)} and Val[M_e(s_{opt})], respectively. We need to show that

\[
\begin{align*}
\text{Val}[M_e] &\leq \text{Val}[M] + \epsilon & \text{(in } M_e \text{ we do not miss good strategies)}; \\
\text{Val}[M_e] &\geq \text{Val}[M] - \epsilon & \text{(solutions in } M_e \text{ are not much better than in } M). 
\end{align*}
\]

As regards intuition for (1), note that for every d, its closest strategy d' ∈ Σ(M_e) yields in M_e(d') “similar” one-step transition probabilities and “not much higher” one-step costs, compared to M(d). The crucial condition for bounding the difference E_{M_e(d)} - E_{M(d)} of expected total cost is that in M(d) no state is visited too many times before reaching the target (otherwise, small differences of individual steps accumulate). Formally, denoting by \#s the random variable assigning to each run of M the number of visits to s before reaching G; or ∞ if G is never reached, we have:

- **Lemma 9.** Let d be ε/2-optimal in every state of M and satisfy for s, s' ∈ S: \(E_{\lambda M(d)}[\#s] \leq \frac{\log 1}{\epsilon^2} \cdot 3^3\). Let d' ∈ Σ(M_e) be such that d'(s) = δ · min(μ, [d(s)/δ]) for each s ∈ S. We have for every s ∈ S

\[
E_{M_e(d')} \leq E_{M(d)} + \epsilon/2.
\]

Apart from bounding one-step transition probabilities and costs, the proof of Lemma 9 is standard. The fundamental step in finishing the proof of (1) is to show that there is such an ε/2-optimal d:

- **Lemma 10.** There is d, ε/2-optimal in every state of M, such that for s, s' ∈ S: \(E_{\lambda M(d)}[\#s] \leq \frac{10}{\epsilon^2} \cdot 3^3\).

**Proof sketch.** This proof forms the core of the proof of Theorem 3 and contains several non-trivial observations, so we sketch its main idea here. Note that in general \(E_{\lambda M(d)}[\#s]\) may be arbitrarily high even for nearly optimal d.

Consider the example with two components, a and b, with rate costs 1 and 2, respectively, being switched by fixed-delay transitions with zero cost. The expected time to reach the goal stays the same no matter how often the components are switched. Setting d(a) = 1000 · d(b) guarantees that the expected cost to reach to goal is close to the optimal cost 1 for any value of d(b). Both \(E_{\lambda M(d)}[\#s]\) and \(E_{\lambda M(d)}[\#s]\) get arbitrarily high by decreasing both delays proportionally.

Hence, we construct the desired delay function d from an arbitrary ε/4-optimal delay function d' as follows. We consider a graph where vertices are states with delays below a specific value d_{opt} (which is much smaller than ε but much bigger than δ) and edges are fixed-delay transitions with zero cost. For each bottom strongly connected component of this graph, we need to find an optimal exit state s (which is a in the example above) and we alter d' by increasing the delay of s to d_{opt} and decreasing the delays of all other states in the component to a very small number, compared to d_{opt}. Then we show that each such change does not increase the expected cost much and that the resulting delay function d indeed meets the specified bound.

As regards (2), note that it suffices to show the inequality for d optimal in M_e as follows.

- **Lemma 11.** Let d ∈ Σ(M_e) be any delay function that is optimal in M_e in every state. Then

\[
E_{M_e(d)} \geq E_{M(d)} - \epsilon.
\]


## 4 Bounded optimization under partial observation

In this section, we address the cost optimization problem under partial observation. For an equivalence relation \( \equiv \) on \( S_0 \) and \( d, \overline{d} > 0 \), we set \( D(d, \overline{d}, \equiv) = \{ d \mid \forall s, s' : d \leq d(s) \leq \overline{d}, s \equiv s' \Rightarrow d(s) = d(s') \} \).

> **Theorem 12.** There is an algorithm that for a fdCTMC structure \( C \), a cost structure \( \text{Cost} \) with \( \mathcal{R}(s) > 0 \) for all \( s \in S \), an initial state \( s_{in} \), a set of goal states \( G \), an equivalence relation \( \equiv \) on \( S_{in} \), \( d, \overline{d} > 0 \), and \( \epsilon > 0 \) computes in exponential time a delay function \( d \) such that

\[
\inf_{d \in D(d, \overline{d}, \equiv)} \mathbb{E}_{C(d)}^{s_{in}} \left[ W_{\text{Cost}}^G \right] - \mathbb{E}_{C(d)}^{s_{in}} \left[ W_{\text{Cost}}^G \right] < \epsilon.
\]

One cannot hope for better theoretical complexity as the corresponding threshold problem is NP-complete.

> **Theorem 13.** For a fdCTMC structure \( C \), a cost structure \( \text{Cost} \) with \( \mathcal{R}(s) > 0 \) for all \( s \in S \), an initial state \( s_{in} \), a set of goal states \( G \), an equivalence relation \( \equiv \) on \( S_{in} \), \( d, \overline{d} > 0 \), and \( x \in \mathbb{R}_{\geq 0} \), it is NP-complete to decide

\[
\text{whether } \inf_{d \in D(d, \overline{d}, \equiv)} \mathbb{E}_{C(d)}^{s_{in}} \left[ W_{\text{Cost}}^G \right] > x + \epsilon \text{ or } \inf_{d \in D(d, \overline{d}, \equiv)} \mathbb{E}_{C(d)}^{s_{in}} \left[ W_{\text{Cost}}^G \right] < x - \epsilon
\]

(if the optimal cost lies in the interval \( [x - \epsilon, x + \epsilon] \), an arbitrary answer may be given).

For the rest of this section we fix a fdCTMC structure \( C = (S, \lambda, P, S_{in}, F) \), a cost structure \( \text{Cost} = (\mathcal{R}, I_p, I_f) \), an initial state \( s_{in} \), a set of goal states \( G \), an equivalence relation \( \equiv \) on \( S_{in} \), and \( d, \overline{d} > 0 \). We again write \( E_{C(d)} \) instead of \( \mathbb{E}_{C(d)}^{s_{in}} \left[ W_{\text{Cost}}^G \right] \), and denote by \( \min C \) and \( \max C \) the minimal and the maximal numbers (i.e. costs and probabilities) in \( C \), respectively. To further simplify notation, we write \( D \) instead of \( D(d, \overline{d}, \equiv) \) and denote by \( \text{Val}[C, D] \) the optimal expected total cost \( \inf_{d \in D} E_{C(d)} \) restricted to \( D \); similarly \( \text{Val}[M, D] \) denotes the optimal expected total cost \( \inf_{d \in D} E_{M(d)} \) restricted to strategies of \( M \) in \( D \).

We firstly show correctness of a slightly adapted approximation by a DTMDP. Then we present the EXPTIME optimization algorithm. Lastly, we address the NP-completeness.

### Approximation using discretization of \( M \)

First observe, that the MDP \( M \) introduced in Section 3 can be due to Proposition 5 also applied in the bounded partial-observation setting. Indeed, \( E_{C(d)} = E_{M(d)} \) for each \( d \in D \) and thus, \( \text{Val}[C, D] = \text{Val}[M, D] \). Analogously, we need to bound the value.

> **Lemma 14.** Let \( B_D \) denote \( \left( \frac{|S| e^2}{\delta^2} \right)^{3/2} \cdot 2 \cdot \max C \cdot (\lambda \cdot d + 1/\lambda) \). We have \( \text{Val}[M, D] < B_D \).

Analogously to Section 3, we define for any \( \delta > 0 \), \( \ell, u \in \mathbb{N} \) a discretized MDP \( M_{\ell, u} = (S, \text{Act}_{\ell, u}, T_{\ell, u}, G, \text{Cost}_{\ell, u}^G) \) where \( \text{Act}_{\ell, u} = \{ k \delta \mid k \in \mathbb{N}, \ell \leq k \delta \leq u \} \cup \{ \infty \} \), \( T_{\ell, u}(s, d)(s') = T(s, d)(s') \) approximated up to an absolute error \( |S| \cdot \delta \lambda \), and \( \text{Cost}_{\ell, u}^G(s, d) = \text{Cost}^G(s, d) \) approximated up to an absolute error \( \delta \cdot \max C \). Like in Section 3, we require that each transition probability that is positive in \( T(s, d) \) has in \( T_{\ell, u}(s, d) \) value at least \( \delta \lambda \). We can again rule out the case that \( \text{Val}[M, D] = \infty \) by Lemma 7. Let us prove the error bound.

> **Proposition 15.** Let \( d, \overline{d} > 0 \) and \( \text{Val}[M, D] < \infty \). For \( \delta = \frac{\epsilon}{e \lambda} \cdot (1 - e^{-1/\lambda}) \cdot B_D^{-3} \) we have

\[
\left| \text{Val}[M, D] - \text{Val}[M_{d, \overline{d}}, D] \right| < \epsilon.
\]

Proof sketch. Thanks to the bounds \( d \) and \( \overline{d} \), the set of delay functions \( D \) in \( M \) is compact and by standard arguments there is an optimal delay function. It suffices to show that for any delay function
$d \in D$ optimal in $M$, for $d, e \in \Sigma(M_{\delta(d)})$ obtained from $d$ by rounding it to multiples of $\delta$, and for any strategy $d'$ optimal in $M'_\varepsilon$ such that $d, d' \in D$ we have

$$E_{M(d)} \leq E_{M(d)} + \varepsilon, \quad \text{and} \quad E_{M(d')} \geq E_{M(d')} - \varepsilon.$$  

Similarly to the proof of Proposition \[8\] we get the $\varepsilon$-bound by bounding the derivative of expected total cost w.r.t. $d$. Again, we need to bound in $M(d)$ and in $M'_\varepsilon(d')$ the expected number of visits to any state $s$ before reaching the goal. Proof of this crucial step is based on the bounds $\delta$ and $\varepsilon$ as they guarantee that every transition has its probability and every state its cost bounded from below. 

Note that the proof technique for unconstrained optimization cannot be easily adapted to optimization under partial observation without requiring the bounds $\delta$ and $\varepsilon$. The reason is that local adaptation of the delay function (heavily applied in the proof of Lemma \[10\]) is not possible as the delays are not independent. An example demonstrating this is in \[8\].

To finish the proofs of Theorems \[12\] and \[13\] we now present two algorithms. Let us fix $\varepsilon > 0$ and denote for the rest of this section by $M'_\varepsilon$ the POMDP $M_{\delta(d)}$ where $\delta$ is as specified in Proposition \[15\].

**Optimization algorithm** First, we present the algorithm of Theorem \[12\].

As explained in more detail in \[8\], the algorithm first constructs in exponential time the MDP $M'_\varepsilon$.

Then it finds an optimal strategy $d$ (which also satisfies $|E_{C(d)} - \inf_{d'} E_{C(d')}| < \varepsilon$) by computing $E_{M(d),d}$ for every (MD) strategy $d$ of $M'_\varepsilon$ in the set $D$.

The algorithm runs in EXPTIME because there are $|Act(d)|$ strategies which is exponential in $|C|$ as $|Act(d)|$ is exponential in $|C|$. The correctness follows from Propositions \[5\] \[15\] proving Theorem \[12\].

**The threshold problem is in NP** The NP algorithm for the approximate threshold $x > 0$ consists of

1. **first** guessing the strategy $d$ of $M'_\varepsilon$ that is in the set $D$ such that $E_{M(d)} < x$,
2. **then** constructing just the fragment $M_d$ of $M'_\varepsilon$ used by the guessed strategy $d$. Here $M_d = (S, \{\infty\}, T_d, G, Cost_d')$ where the transition probabilities and costs coincide with $M$ in states from $S \setminus S_{id}$ and in any state $s \in S_{id}$ are defined by $T_d(s, \infty) = T(s, d(s))$ and $Cost'_d(s, \infty) = Cost'(s, d(s))$,
3. **last**, for $\sigma : s \mapsto \infty$, the algorithm computes $y = E_{M_{\delta(d)}(\sigma)}$ by standard methods and accepts iff $y < x$.

Note that the algorithm runs in non-deterministic polynomial time. Indeed, both $d$ and $M_d$ can be constructed in polynomial time (although the whole $M'_\varepsilon$ is exponential). The expected total cost $x$ in $M_{\delta(d)}(\sigma)$ that has polynomial size can be also computed in polynomial time.

Let us show that the algorithm is correct for the problem in Theorem \[15\]. First, assume that $Val(C, D) < x - \varepsilon$. The optimal (MD) strategy $d$ in $M'_\varepsilon$ from the set $D(d, \varepsilon)$ hence satisfies $E_{M(d)} < x$, due to Proposition \[15\]. The algorithm accepts as it non-deterministically guesses $d$. Similarly if $Val(C, D) > x + \varepsilon$, no strategy from the set $D$ guarantees in $M'_\varepsilon$ expected total cost $\leq x$, hence the algorithm does not accept. This concludes the proof that the problem from Theorem \[15\] is in NP.

**The threshold problem is NP-hard** We show the hardness by reduction from SAT. Let $\varphi = \varphi_1 \land \cdots \land \varphi_n$ be a propositional formula in conjunctive normal form (CNF) with $\varphi_i = (l_{i,1} \lor \cdots \lor l_{i,k_i})$ for each $1 \leq i \leq n$ and with the total number of literals $k = \sum_{i=1}^n k_i$. As depicted in the following figure, the fdCTMC structure $C_\varphi$ is composed of $n$ components (one per clause), depicted by rectangles. The component of each clause is formed by a cycle of sub-components (one per literal) connected by fixed-delay transitions. Positive literals are modelled differently from negative literals.
The cost structure $\text{Cost}_\nu$ assigns rate cost 1 to every state, and impulse cost 0 to every transition. The goal states $G_\varphi \subseteq S_\varphi$ are depicted by double circles. We require $s^0_{i,j} \equiv s^0_{t,c,f}$ iff the literals $l_{i,j}$ and $l_{t,c,f}$ have the same variable. Furthermore, we set $d$ to 0.01 and $\bar{d}$ to 16$k$. We obtain the following:

\begin{itemize}
  \item \textbf{Proposition 16.} Let $\varphi$ be a formula in CNF with $k$ literals and let $x := \inf_{d \in [d_\text{min},2\varepsilon]} \mathbb{E}^{\varphi}_{C_\nu}[\text{Cost}_\nu]$. $C_\varphi$ is constructed in time polynomial in $k$ and $x < 17k^2$ if $\varphi$ is satisfiable and $x > 17k^2 + 1$, otherwise.
\end{itemize}

\textbf{Sketch.} If $\varphi$ is satisfiable, let $\nu$ denote the satisfying truth assignment. We set $d(s_m) := d$ and $d(s^0_{i,j}) := \bar{d}$ if $\nu(X) = 1$ and $d(s^0_{i,j}) := d$ if $\nu(X) = 1$, where $X$ is the variable of the literal $l_{i,j}$ and arbitrarily in other states. In $C_\varphi(d)$ in the gadget of any TRUE literal $l_{i,j}$, i.e. with $\nu(l_{i,j}) = 1$, the goal is reached from $s^0_{i,j}$ with probability $> 0.99$ before leaving the gadget. Indeed, the probability to take no exponential transition within time $0.01$ is $> 0.99$ and the probability to take at least $8k$ exponential transitions within time $16k$ is $> 0.99$ for any $k \in \mathbb{N}$. As each clause $\varphi_i$ has at most $k$ literals and at least one TRUE literal, the expected cost incurred in the gadget $\varphi_i$ is at most $(16k \cdot k)/0.99 < 17k^2$.

The other implication is based on the observation that there is no delay length guaranteeing high probability of going directly into the target from both gadgets of a positive and a negative literal.

The reduction proves NP-hardness from Theorem 13 as it remains to set $x := 2k^2 + \frac{1}{2}$ and $\varepsilon := \frac{1}{2}$.

## 5 Conclusions and Future Work

In this paper, we introduced the problem of synthesising time-outs for fixed-delay CTMC. First, for \textit{unconstrained optimization}, we present an approximation algorithm based on a reduction to a discrete-time Markov decision process and a standard optimization algorithm for this model. Second, we approximate the case of \textit{bounded optimization under partial observation} also by a MDP. However, a restriction of the class of strategies twists it basically into a partial-observation MDP (where only memoryless deterministic strategies are considered). We give an exponential approximation algorithm and show that the corresponding decision problem is NP-complete. Even though the theoretical complexity is settled, it still leaves space for heuristic solutions for POMDP with MD strategies.

Observe that the number of actions in the discretized MDPs is only polynomial in bounds on $\text{Val}[M]$ and $\text{Val}[M,D]$. For theoretical complexity, we give the explicit (exponential) bounds $B$ and $B_D$. In practice, one can obtain better bounds by computing $E_{C(d)}$ for an arbitrary $d$ as $E_{C(d)} \geq \inf_d E_{C(d)} = \text{Val}[M]$ (similarly for $\text{Val}[M,D]$). One can set $d$ by some heuristics (e.g. to the constant function $1/\lambda$) or randomly. One can even use the minimum from a series of such computations. In most cases, this yields a significant improvement. For instance, for the 3-state model from Section 1 we get a bound $E_{C(1/\lambda)} \approx 4.9$ instead of the theoretical bound $B \approx 55000$.

Note that the question whether explicit bounds $\underline{d}$ and $\bar{d}$ need to be assumed for optimization under partial observation remains open. In future work, we would also like to address synthesis of \textit{dynamic} time-outs being chosen by history-dependent and randomizing controllers. We believe that such extended setting fits well to an interesting class of communication protocols.

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A DTMDPs

Definition of probability space Every strategy $\sigma$ together with an initial vertex $v$ uniquely determines a probability space $(\text{Run}_{A_M}, \mathcal{F}_{A_M}, \Pr^\nu_{A_M(r)})$, where $\text{Run}_{A_M}$ is the set of all runs, i.e. infinite alternating sequences of vertices and actions $v_0a_1v_1a_2v_2\cdots \in (V \cdot \text{Act})^\omega$; $\mathcal{F}_{A_M}$ is the sigma-field generated by all sets of runs of the form $\{p \mid p$ has prefix $w\}$ for all finite paths $w$, i.e. prefixes of runs ending with a vertex; and $\Pr^\nu_{A_M(r)}$ is the unique probability measure such that for every finite path $w = v_0a_1v_1 \ldots a_nv_n$ it holds:

$$\Pr^\nu_{A_M(r)}(p \mid p$ has prefix $w) = \begin{cases} 0 & \text{if } v_0 \neq v \text{ or } \sigma(v_i-1) \neq a_i \text{ for some } 1 \leq i \leq n, \\ \prod_{j=1}^n T(v_{i-1}, a_i)(v_i) & \text{otherwise.} \end{cases}$$

The probability measure is also denoted by $\Pr_{\sigma}$ where the MDP and the initial state are clear from context. Similarly, the corresponding expectation operator is denoted by $E^\nu_{A_M(r)}$ or $E_{\sigma}$.

Proposition 5. For any delay function $d$ it holds $E_{C(d)}[W_{Cost}^G] = E_{M(d)}[W_{Cost}^G]$.

Proof. Let us fix $d$ and let $R$ and $R'$ denote the set od runs in $C(d)$ and $M(d)$, respectively, that reach $G$. Note that both $R$ and $R'$ can be partitioned into countable sets $\mathcal{R}$ and $\mathcal{R}'$ such that

- all sets of $\mathcal{R}$ are of the form $R_{b_0\cdots b_n}^{s_0\cdots s_n} := \{(s'_0, d_0)0 \cdots n \leq i : n = s_i, (b_i = 1) \Rightarrow (d_i = t_i), s_n \in G\}$
- all sets of $\mathcal{R}'$ are of the form $R_{b_0\cdots b_n}^{s_0\cdots s_n} := \{s'_0s'_1 \cdots n \leq i : s_i' = s_i, s_n \in G\}$

Next, we define a mapping $\rho$ that maps each set $R_{b_0\cdots b_n}^{s_0\cdots s_n}$ to a set $R_{b_0\cdots b_n}^{s_0\cdots s_n}$ such that $s_0'\cdots s_m'$ is obtained from $s_0\cdots s_n$ as follows. Each maximal sequence $s_k \cdots s_l$ such that either $b_l = 1$ or $s_{l+1} \notin S_{id}$, and at the same time for every $k \leq i < l$, $b_i = 0$ and $s_i \in S_{id}$. We replace just by the state $s_k$. Note in all runs from $R_{b_0\cdots b_n}^{s_0\cdots s_n}$, fixed-delay transitions become enabled in the state $s_k$, stay enabled until leaving state $s_i$ which happens either by a fixed-delay transition or by an exp-delay transition into a state from $S \setminus S_{id}$. Furthermore, note that $\rho$ is surjective on $\mathcal{R}'$.

By a straightforward induction in number of such replacement, it is easy to show for each set $R_{b_0\cdots b_n}^{s_0\cdots s_n} \in \mathcal{R}'$ that

$$E^\nu_{C(d)}[W_{Cost}^G \mid R_{b_0\cdots b_n}^{s_0\cdots s_n}] = E^\nu_{M(d)}[W_{Cost}^G \mid \rho^{-1}(R_{b_0\cdots b_n}^{s_0\cdots s_n})].$$

This implies that, denoting by $N$ and $N'$ the set of all runs of $C$ and $M$ that do not reach the target,

$$\Pr^\nu_{C(d)}[N] > 0 \iff \Pr^\nu_{M(d)}[N'] > 0.$$

Furthermore, we get the result as

$$E_{M(d)}[W_{Cost}^G] = \sum_{R \in \mathcal{R}} \Pr^\nu_{M(d)}[R] \cdot E^\nu_{M(d)}[W_{Cost}^G \mid R]$$

$$= \sum_{R \in \mathcal{R}} \Pr^\nu_{C(d)}[R] \cdot E^\nu_{C(d)}[W_{Cost}^G \mid R]$$

$$= E_{C(d)}[W_{Cost}^G].$$

B Algorithms

In this section, we present pseudo-codes for all algorithms.
The algorithm for unconstrained optimization is presented as Algorithm 1. Let us discuss in closer detail, how the transition probabilities and costs are approximated.

**Algorithm 1**: Unconstrained optimization

```plaintext
input : fdCTMC structure C, cost structure Cost with R(s) > 0 for all s ∈ S, initial state s₀, set of goal states G, and ε > 0
output : The delay function d that is ε-optimal in C.

1 Compute δ = ε² · B⁻³ and u = \( \frac{b}{\min} \).
2 Construct \( M_ε \) and Cost' by the procedure Discretize(C, Cost, δ, 0, u)
3 \( d \) := optimal (MD) strategy in \( M_ε \) // standard alg. 30 in PTIME in \( |M_ε| \)
```

### B.1 Unconstrained optimization

The algorithm for unconstrained optimization is presented as Algorithm 1. Let us discuss in closer detail, how the transition probabilities and costs are approximated.

**Approximating transition probabilities and costs** Note that for \( s ∈ S \setminus S_{fd} \) and \( a = ∞ \), the approximation is easy as the probabilities and costs in \( M \) are already given as rational numbers.

Hence, let us fix a state \( s ∈ S_{fd} \) and \( 1 ≤ k ≤ N \). We need to approximate the probabilities up to an absolute error \( |S| · δ · λ \) such that any positive probability is approximated by at least \( δ · λ \). We do it as follows. First, by a simple qualitative analysis, we find states that have positive transient probability at time \( τ \). Then we approximate all probabilities up to an absolute error of \( δ · λ \) as described below. We round all positive probabilities up \( δ · λ \) and reduce the probabilities of all remaining states uniformly so that the sum is 1.

Now, we explain how we approximate in \( C \) up to a fixed absolute error the distribution over states at time \( τ \) and the expected cost accumulated up to time \( τ \) where \( τ \) is the first moment when either the fixed-delay event occurs or the set \( S_{fd} \) is left. This is equivalent to the distribution over states at time \( kδ \) and the expected cost accumulated up to time \( kδ \) when we change the model so that all states outside \( S_{fd} \) are (1) absorbing w.r.t. both exponential and fixed-delay transitions and (2) incur neither rate nor impulse costs. We denote the modified transition matrices by \( \overline{P} \) and \( \overline{F} \) and the modified cost structure by \( (\overline{R}, \overline{I}_p, \overline{F}) \). After this transformation, we can assume that the fixed-delay event occurs at time \( kδ \) in any state. We obtain a behavior of a continuous-time Markov chain up to time \( kδ \) followed by one fixed-delay transition at time \( kδ \). Similarly to the method of uniformization 19, it is easy to show that...
Lemma 18. For any state \( s \in S_{\text{id}} \) and \( 1 \leq k \leq N \),

\[
T(s, k\delta) = \sum_{i=0}^{\infty} \psi_{i\delta,(i)} \cdot \left( 1_1 \cdot \overline{F} \right) \cdot F
\]

\[
\text{Cost}(s, k\delta) = \sum_{i=0}^{\infty} \psi_{i\delta,(i)} \left( \sum_{j=0}^{i-1} \left( 1_1 \cdot \overline{F} \right) \cdot \left( \frac{k\delta \cdot \overline{R}}{i+1} + \overline{F}_Q \right) + \left( 1_1 \cdot \overline{F} \right) \cdot \left( \frac{k\delta \cdot \overline{R}}{i+1} + \overline{F}_R \right) \right)
\]

where \( 1_1 \) denotes the unit vector of state \( s \), \( \psi_{i\delta,(i)} \) denotes the probability mass function of the Poisson distribution with parameter \( k\delta \), and \( \overline{F}_Q, \overline{F}_R : S \rightarrow \mathbb{R}_{\geq 0} \) assign to each state the expected impulse reward of the next exponential or fixed-delay transition, i.e., \( \overline{F}_Q(s) = \sum_{s'} \overline{F}(s, s') \cdot \overline{F}(s', s) \) and \( \overline{F}_R(s) = \sum_{s'} \overline{F}(s, s') \cdot \overline{F}(s', s) \).

Indeed, the distribution over states after \( j \) exponential transitions is \( 1_1 \cdot \overline{P}^j \); the probability that exactly \( i \) exponential transitions occur before time \( k\delta \) is \( \psi_{i\delta,(i)} \); and under the condition that \( i \) transitions occur, the expected time spent in each of the \( i + 1 \) visited states is \( k\delta/(i+1) \).

Finally, note that for any fixed error \( \gamma > 0 \), a bound \( I \) can be easily computed such that \( \sum_{i=0}^{\infty} \psi_{i\delta,(i)} < \gamma/2 \), hence the quantities above can be approximated up to \( \gamma/2 \) by truncating the infinite sums at \( I \).

Furthermore, for each \( 0 \leq i < I \), we can compute an admissible error \( b_i > 0 \) of approximating \( \psi_{i\delta,(i)} \) such that \( \sum_{i=0}^{i-1} b_i \cdot i \cdot r < \gamma/2 \) where \( r \) is the maximal entry of \( \overline{F}_Q \) and \( \overline{F}_R \).

B.2 Bounded optimization under partial observation

The exponential-time algorithm for bounded optimization under partial observation is described as Algorithm 2.

Algorithm 2: Exponential algorithm for bounded optimization under partial observation

input : fdCTMC structure \( C \), cost structure \( \text{Cost} \) with \( R(s) > 0 \) for all \( s \in S \), initial state \( s_{\text{in}} \), set of goal states \( G \), equivalence relation \( \equiv \) on \( S_{\text{id}}, d, \overline{d} > 0 \), and \( \varepsilon > 0 \)

output : delay function \( d \) that is \( \varepsilon \)-optimal in \( C \)

1. Construct \( M_e \) and \( \text{Cost}_e \) by the procedure Discretize\((C, \text{Cost}, \delta, d, \overline{d})\) for \( \delta = \frac{\varepsilon}{6\varepsilon^2} \cdot (1 - e^{-\frac{1}{2d}}) \cdot R_0^{-3} \)

2. for every (MD) strategy \( d \) in \( M_e \) such that \( d(s) = d(s') \) iff \( s \equiv s' \) do

3. \( x_d := \mathbb{E}_{M_e(d)} [W^G_{\text{Cost}}] \) \hspace{1cm} // by standard methods in time polynomial in \( |S| \)

4. return some function \( d \) with minimal \( x_d \)

Let us sketch why the proof technique for unconstrained optimization cannot be easily adapted to optimization under partial observation without requiring the bounds \( d \) and \( \overline{d} \). The reason is that local adaptation of the delay function (heavily applied in the proof of Lemma 10) is not possible as the delays are not independent. Consider on the right a variant of the example from the proof of Lemma 10, with components \( a \) and \( b \) being switched by fixed-delay transitions. All states have cost rate 1 and all transitions have cost 0; furthermore, all states are in one class of equivalence of \( \equiv \). If in state \( a \) or \( b \) more than one exp-delay transition is taken before a fixed-delay transition, a long detour via state 1 is taken. In order to avoid it and to optimize the cost, one needs to set the one common delay as close as possible to 0. Contrarily, in order to decrease the expected number of visits from \( a \) to \( b \) from \( a \) before reaching \( t \) which is crucial for the error bound, one needs to increase the delay.
We need to show that $\mathbb{E}^{\phi}_{\bot} \geq k$ literal, respectively, the goal is reached before leaving the component. It is easy to show that for all $p$ states, in $\mathbb{E}^\tau_{\bot} \geq k$, the overall expected cost is still unrealistic worst case that the expected cost in components of other clauses is $0$, the overall expected cost is $\mathbb{E}^\tau_{\bot} \geq k$, that is not satisfied w.r.t. $\mathcal{C}$. In $\mathbb{E}^\tau_{\bot} \geq k$, the expected cost incurred in the gadget $\nu$ is approximated up to an absolute error $\mathbb{E}^\tau_{\bot} \geq k$, by standard methods in polynomial time.

The algorithm for the threshold problem running in non-deterministic polynomial time is formalized as Algorithm 3.

### C Proof of Proposition 16 (NP hardness)

**Proposition 16.** Let $\varphi$ be a formula in CNF with $k$ literals and let $x := \inf_{\mathbb{E}^\nu_{\bot} \geq k} \mathbb{E}^\tau_{\bot} \geq k \mathbb{E}^\tau_{\bot} \geq k$. $C_v$ is constructed in time polynomial in $k$ and $x < 17k^2$ if $\varphi$ is satisfiable and $x > 17k^2 + 1$, otherwise.

**Proof.** If $\varphi$ is satisfiable, let $\nu$ denote the satisfying truth assignment. We set $d(s_{\bot}) := d$ and $d(s_{\bot}^0) := d$ if $\nu(X) = 1$ and $d(s_{\bot}^1) := d$ if $\nu(X) = 1$, where $X$ is the variable of the literal $l_{ij}$ and arbitrarily in other states. In $\mathcal{C}_v(d)$ in the gadget of any TRUE literal $l_{ij}$, i.e. with $\nu(l_{ij}) = 1$, the goal is reached from $s_{\bot}^0$ with probability $0.99$ before leaving the gadget. Indeed, the probability to take no exponential transition within time $0.01$ is $0.99$ and the probability to take at least $8k$ exponential transitions within time $16k > 0.99$ for any $k \in \mathbb{N}$. As each clause $\varphi_i$ has at most $k$ literals and at least one TRUE literal, the expected cost incurred in the gadget $\nu_i$ is at most $(16k \cdot k)/0.99 < 17k^2$.

As regards the other implication, let $\varphi$ be not satisfiable such that $k \geq 7$ (note that we can assume it as when fixing the number of literals, the SAT problem becomes polynomial) and let $d \in D(d, d, \equiv)$. We need to show that $\mathbb{E}^\nu_{\bot} \geq k \mathbb{E}^\tau_{\bot} \geq k > 17k^2 + 1$. Based on $d$, we construct a truth assignment $\nu$ such that $\nu(X) = 0$ iff for all literals $l_{ij}$ with variable $X$, we have $d(s_{ij}) \leq k$. Since $\varphi$ is unsatisfiable, there must be at least one clause $i$ that is not satisfied w.r.t. $\nu$. It suffices to show that for the initial state of the component of the $i$-th clause we have $E_i := \mathbb{E}^\nu_{\bot} \geq k \mathbb{E}^\tau_{\bot} \geq k > 17k^2 + 1$. Indeed, even for the unrealistic worst case that the expected cost in components of other clauses is $0$, the overall expected cost is still $17k^2 + 1$.

We know that all negative literals have the delay above $k \geq 7$ and all the positive literals have the delay below $k$. Let us assume the worst case that all the delays are $k$. With delay equal $k$, let us by $p_{pos}$ and $p_{neg}$ denote the probabilities that from the initial state of a component for a positive and negative literal, respectively, the goal is reached before leaving the component. It is easy show that for all $k \geq 7$:

\[
p_{pos} := 1 - e^{-k} \cdot \sum_{n=0}^{k} \frac{k^n}{n!} \leq \frac{1}{17k^2 + 1}, \quad \text{and} \quad p_{neg} := e^{-k} \leq \frac{1}{17k^2 + 1}.
\]
Based on this observation, we can under-approximate $E_i$ by

$$E_i \geq \frac{k}{17k^2 + 1} = k \cdot (17k^2 + 1).$$

\section{Proof of quantitative properties of MDPs}

\textbf{Lemma 7} We have $E_{M(d)} = \infty$ for some $d$ if and only if $\text{Val}(M) = \text{Val}(M_e) = \infty$.

\textbf{Proof.} The only if part is obvious. We prove the if part. For strategy $d$ there exists a set of runs that with positive measure with runs that never hit any target state and generate infinite cost. Thus there is at least one bottom strongly connected component (BSCC) in $M(d)$ that is reachable from the initial state and does not contain any of the target states. Because of the fact that we cannot set any fixed delay to zero or $\infty$ from definition of $T$ it follows that for all $d'$ the $M(d)$ and $M(d')$ share the same set of transitions with the only difference in their positive probabilities. Thus the same BSCC is also in $M(d')$ for each $d'$. Similarly this holds for $M_e$ with the only difference that when doing approximation one must ensure that he does not give transition with small probability a zero probability. But this is from definition of approximation ruled out. Thus we know that BSCC is also included $M_e(d_e)$ for each $d_e$.

Now we show that runs that reach this BSCC generate infinite expected cost. In $M$ it is obvious that the expected time taken by each transition is positive. Moreover because the positive cost rate of each state, we have that for each strategy $d$ and $s$ in BSCC it holds that $\mathbb{E}_{M(d)}[s] = \infty$. Because we reach this BSCC with positive probability we have that $\text{Val}(M) = \infty$.

It remains to show that BSCC generates infinite cost also in $M_e$. The only complication can arise from approximation of $Cost'_{t,d}$, because it can happen that we the cost is very small and due to approximation we set it to zero. But because minimal time set to fixed-delay is $\delta$. We can bound the minimal expected time of any transition using Lemma 21. Because we are allowed to make at most $\min C \cdot \delta \cdot (1 - e^{-\delta t})$ of absolute error when computing $Cost'$ we know that we do not set it to zero. We have that for each strategy $d_e'$ and $s$ in BSCC it holds that $\mathbb{E}_{M(d_e')}[s] = \infty$. Because we reach this BSCC with positive probability we have that $\text{Val}(M_e) = \infty$.

\textbf{Lemma 21.} For all $M$, each its state $s \in S$ and $d \in \mathbb{R}_{\geq 0}$ it holds that $Cost'(s, d) \in [\min C \cdot d \cdot (1 - e^{-\delta t}), \max C \cdot d + \max C \cdot \lambda \cdot d]$.

\textbf{Proof.} We first provide bounds on expected time spent in each state $s$ of $M$. We know that we can do several non fixed-delay transitions and either we end up in state not in $S_{\text{fd}}$ or the fixed-delay transition occurs. Thus the maximal expected time spent in $s$ is $d$. However if some series of exponential transition disables the fixed-delay it can shorten the time $d$. Obviously the series of length one can shorten $d$ the most. Thus the minimum expected time spent in $s$ can be boud from below by

$$\int_0^d x \cdot PDF_{\text{Exp} \cdot d}(x) dx + d \cdot (1 - CDF_{\text{Exp} \cdot d}(d)) = \left( \int_0^d x \lambda \cdot e^{-\lambda x} dx + d \cdot e^{-\lambda d} \right) = \left( \frac{1}{\lambda} - e^{-\lambda d} (d + 1/\lambda) + d \cdot e^{-\lambda d} \right) = \frac{1}{\lambda} \cdot (1 - e^{-\lambda d}).$$

Since every state has positive cost rate we have minimal expected cost paid in $s$ is

$$\min C \cdot \frac{1}{\lambda} \cdot (1 - e^{-\lambda d}).$$
As mentioned before the expected time spent in $s$ is $d$. Thus the maximal expected rate cost is $\max C \cdot d$.

However we must also include over-approximation of the impulse costs. For this purpose we need to bound the expected number of steps taken by exponential transitions in time $d$. This is simply the mean of Poisson distributed random variable with parameter $\lambda \cdot d$ what has closed-form formula equal to $\lambda \cdot d$. Thus the maximal expected cost accumulated via impulse rewards is $\max C \cdot \lambda \cdot d$. Altogether the over-approximation of expected cost in $s$ is

$$\max C \cdot d + \max C \cdot \lambda \cdot d$$

Lemma 22. For all $M$, each its state $s \in S$ and $d \in \mathbb{R}_{\geq 0}$ it holds that $\text{Cost}'(s, d) \in [\min C \cdot d \cdot (1 - e^{-\lambda \cdot d}), \max C \cdot \max(d + \lambda \cdot d, 1/\lambda + 1)]$.

Proof. If state $s$ is not in $S_d$ then the expected waiting time is $1/\lambda$. Then the minimal and maximal rate cost is $\min C \cdot 1/\lambda$ and $\max C \cdot 1/\lambda$ respectively. The minimal and maximal impulse costs are 0 and $\max C$. Combining with Lemma 21 the minimal bound is $\min(min C \cdot d \cdot (1 - e^{-\lambda \cdot d}), \min C \cdot 1/\lambda) = \min C \cdot d \cdot (1 - e^{-\lambda \cdot d})$ and the maximal bound is $\max(max C \cdot d + \max C \cdot \lambda \cdot d, \max C \cdot 1/\lambda + \max C) = \max C \cdot \max(d + \lambda \cdot d, 1/\lambda + 1)$.

E Computation of cost in fdCTMC

Proposition 2. There is a polynomial-time algorithm that for a given fdCTMC $C(d)$, initial state $s_m$, cost structure $\text{Cost}$, goal states $G \subseteq S$, and an approximation error $\varepsilon > 0$ computes $x \in \mathbb{R}_{>0}$ such that

$$\left| \mathbb{E}_{s_m \in C(d)}[\mathbf{W}_G^{C}] - x \right| < \varepsilon,$$

(Recall that $\mathbb{E}_{s_m \in C(d)}$ denotes the expected value with respect to the probability measure $\text{Pr}_{s_m \in C(d)}$.)

Proof. This proposition follows directly from results from results presented in [23].

F Discretization for unconstrained optimization

Lemma 10. There is $d, \varepsilon/2$-optimal in every state of $M$, such that for $s, s' \in S : \mathbb{E}_{s \in M(d)}[\mathbf{W}_G^{C}] \leq \frac{2|G|}{\varepsilon \lambda^3} \cdot B^3$.

Proof. The proof of Lemma 10 is based several lemmas defined and proven bellow. The whole process can be seen in three steps:

1. Using Lemma 25 we know that from each state we have an upper bound $B$ on expected cost paid by runs from this state, thus for each $\varepsilon/2$-optimal strategy in every state it holds that the value bound in this state is $B + \varepsilon/2$. 


Using Lemma 28 we know that for all \( \varepsilon > 0 \) there exist functions \( r(\varepsilon) \) and \( D(\varepsilon) \) and \( \varepsilon \)-optimal strategy that hits a target state or generates at least \( r(\varepsilon) \) of expected cost in at most \( D(\varepsilon) \) steps on paths from any state.

We combine the first two items. We know that exists \( \varepsilon/2 \)-optimal strategy \( d \) such that for all \( s, s' \in S \) the following holds

\[
\mathbb{E}_{M(\varepsilon)}[W_{G'}^F] \leq \left( B + \varepsilon \right) \cdot D(\varepsilon) / \varepsilon \cdot D(\varepsilon/2)
\]

\[
= \frac{2 \cdot |S| \cdot B^2}{\varepsilon \cdot \lambda}.
\]

\[\blacksquare\]

**Lemma 25.** Let \( B \) denote \( \left( \frac{|S|}{\min C} \right) \cdot e \cdot \max C \cdot (1 + 1/\lambda) \). For all \( s \in S \) we have \( V(s) < B \).

**Proof.** We use Lemma 26 and directly evaluate \( \max C \) as \( \max C/\lambda + \max C \) and \( \min C \) as \( \min C |S| \). \( e^{-1}/|S|^{S} \) thanks to employing strategy \((1/\lambda)^{|S|}\).

\[\blacksquare\]

**Lemma 26.** Let us fix \( \varepsilon > 0 \). For every vertex \( s \) of an DTMDP \( M \) it holds that

\[
\inf_d \mathbb{E}_{M(\varepsilon)}[W_{G'}^F] \leq \frac{\max C \cdot |V|}{\min Prob}
\]

where \( V \) is the set of all vertices of \( M \), \( \max C \) is an upper bound on the cost function, and \( \min Prob \) be a positive lower bound on non-zero probabilities of the probability transition function. Formally, for all \( v, v' \in V \) such that \( T(v, a')(v') > 0 \) for some \( a' \in \text{Act}(v) \), there is an action \( a \in \text{Act}(v) \) such that \( \min Prob \leq T(v, a)(v') \) and \( \max C \geq \text{Cost}^*(v, a) \).

**Proof.** Intuitively, the bounds defined in this lemma says: if we can reach a vertex \( v' \) in one transition from a vertex \( v \) then we can do it with a probability higher than \( \min Prob \) and pay less than \( \max C \).

Note that for every \( d \) and \( s \in G' \)

\[
\mathbb{E}_{M(\varepsilon)}[W_{G'}^F] = \text{Cost}^*(s, t_s) + \sum_{s'} T(s, t_s)(s') \cdot \mathbb{E}_{M(\varepsilon)}[W_{G'}^F]
\]

where each \( f \)-component of \( d \) corresponding to \( s \) is \( t_s \).

\[\blacksquare\]

**Lemma 6.** Let \( B \) denote \( \left( \frac{|S|}{\min C} \right) \cdot e \cdot \max C \cdot (1 + 1/\lambda) \). We have \( V(M) < B \).

**Proof.** The lemma is corollary of Lemma 25.

\[\blacksquare\]

**Lemma 28.** For all MDP \( M \) and \( \varepsilon > 0 \) exists \( \varepsilon \)-optimal strategy \( d \), \( D(M, \varepsilon) = |S|/T_{\min} \cdot e^{-1}d_{opt}(\varepsilon)/B^{|S|} \) and \( r(M, \varepsilon) = \min_x \text{R}(s) \cdot (1 - e^{-1}d_{opt}(\varepsilon))/\lambda \) such that for all \( s \in S \) it holds that

\[
\mathbb{E}_{M(\varepsilon)}[X] \leq D(M, \varepsilon),
\]

where \( T_{\min} = \min(s, s') \text{Pr}(s, s') \cdot \min_{s, s'} \text{F}(s, s) \), \( d_{opt}(\varepsilon) = \frac{e}{\min(\min C |S|, \max C |S|, 1 / \lambda, B)} \), and \( X \) denotes random variable that assigns each run \( \omega \) number of step in which \( \omega \) reaches target state or state from which the first transition has expected cost at least \( r(M, \varepsilon) \).

For the proof of the Lemma 28 we need the following definitions.
Definition 29. Let us fix MDP $M$.

A state $s \in S_M$ is good iff there is a positive probability that using only fixed delay transitions a target state is reached. We denote the set of good states $S_{\text{good}}$.

Sink of MDP $M$ is a maximal subset $K$ of $S$ that is strongly connected on fixed-delay transitions and every fixed-delay transition from state in $K$ leads only to states in $K$ (i.e. sink is bottom strongly connected component on fixed-delay transitions).

For all $s \in K$ we denote $\pi_K(s)$ the probability that $K$ is left by transition from $s$ (i.e. $\pi_K(s) = \frac{1}{1 - \sum_{s' \in K} T(s, s')(d(s'))}$).

Value of state $s$ in sink for strategy $d$ is $V^*_M(d) = \sum_{s \in S} \pi_K(s) \cdot E^*_M(d) + \text{Cost}(s, d(s))$.

For sink $K$ and each state $s \in K$ we denote $p_K(s)$ the probability of runs for which all the following holds

- run starts in state $s$,
- second state of the run does not belong to $K$,
- a target is reached and the on the way $K$ was not visited again.

We denote $p_K = \sum_{s \in K} \pi_K(s) \cdot p(s)$.

Proof. We choose arbitrary $\varepsilon/2 \cdot |S|$-optimal strategy in each state $d$ and we show that we can change it to strategy $\Pi(d)$ that satisfies the Lemma 28 and causes increase of expected cost by at most $\varepsilon/2$ for each state.

We construct the strategy as follows: In each $K$ not containing any target state we find its state $s \in K$ with optimal value. If $s$ has larger value than $d_{opt}(\varepsilon)$ its fixed-delays is kept with no change in $\Pi(d)$. Otherwise it is inflated to $d_{opt}(\varepsilon)$. If any other state has fixed-delay larger than $d_{opt}(\varepsilon)/B$ then it is set to $d_{opt}(\varepsilon)/B$.

By Lemma 31 we have that the increase in expected cost is at most $\varepsilon/2$ for each state. We show that each state $s$ has small expected number of steps until reaching target or making transition with expected cost at least $r(M, \varepsilon)$.

Lemma 30. For all MDP $M$, $\varepsilon > 0$, $\varepsilon$-optimal strategy in each state $d$, for each sink $K$ it holds that

$$p \geq 1/(B + \varepsilon).$$

Proof. First we choose $s \in K$ that is optimal in $K$ according to $V^*_M(d)$. Then we evaluate the following basic property:

$$V^*_M(d) \geq p(s) \cdot (\text{expected cost of runs that do not visit } K \text{ again})$$

$$\quad + (1 - p) \cdot (\text{expected cost of runs that visit } K \text{ again}) + \text{Cost}(s, d(s))$$

$$\geq (1 - p) \cdot (\text{expected cost of runs that visit } K \text{ again}) + \text{Cost}(s, d(s))$$

$$\geq (1 - p) \cdot (c + \sum_{s \in K} \pi_K(s') \cdot V^*_{M(d/s)} + \text{Cost}(s, d(s)))$$

where $c$ is expected cost accumulated until $K$ is left again by runs starting in $s$ and making the first transition to state not belonging to $K$. The last inequality follows from optimality of $V^*_M(d)$. Observe that $c$ is actually a cost accumulated until stopping time concerning the second leave of $K$. We can leave sink only by exponential transitions. If we consider stopping time concerning the first execution of exponential transition in $K$ after returning to it this stopping time is smaller or equal than the first one. Moreover if we assume that exponential transition is enabled in each state of $K$ the stopping time can be
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the same or decrease. We know that exponential transition is memoryless, thus the expected time until exponential transition occurs is $1/\lambda$. Thus it holds that $c \geq \min, R(s)/\lambda \geq \min C/\lambda$.

Further evaluation of the last inequality gives:

$$V_{M(d)}^s \geq c + V_{M(d)}^s - p_s \cdot (c + \sum_{s' \in K} \pi_K(s') \cdot V_{M(d)}^{s'})$$

$$p_s \cdot (c + \sum_{s' \in K} \pi_K(s') \cdot V_{M(d)}^{s'}) \geq c$$

Further evaluation of the last inequality gives:

$$p_s \geq \frac{c}{(c + \sum_{s' \in K} \pi_K(s') \cdot V_{M(d)}^{s'})} \geq \frac{c}{(c + \lambda + \epsilon)} \geq \frac{\min C/\lambda}{(\min C/\lambda + \lambda + \epsilon)}.$$ 

Thus $p_K = \sum_{s' \in K} \pi_K(s') \cdot p_s \geq \frac{\min C/\lambda}{(\min C/\lambda + \lambda + \epsilon)}$.

\begin{lemma}
For each $\epsilon > 0$ After changing the strategy $d$ to "inflated" strategy $\oplus K(d) = d'$ on sink $K$ in MDP $M$, for each state $s$ it holds that $E_{M(d\oplus)}^s - E_{M(d)}^s \leq \frac{\epsilon}{|S|}$.
\end{lemma}

\begin{proof}
We analyse the change of expected cost of strategies state by state in $K$. If the state was the one where we decreased the fixed-delay then we have obviously decreased the waiting costs. Actually the only way we can increase the expected cost gathered in $K$ is due to increase of fixed-delay in "inflated" state $s$ and by larger amount of entries of $s$ before leaving from $K$. The change of expected cost in one leave of $s$ can be over-approximated by:

$$d_{opt} \cdot \max C + d_{opt} \cdot |S| \cdot B \cdot \lambda + d_{opt} \cdot \max C \cdot |S| \cdot \lambda.$$ 

The probability of leaving $K$ can be underestimated by

$$|S| \cdot \min C |S| \cdot e^{-\lambda d_{opt}} \cdot (\lambda \cdot d_{opt})^{|S|}/|S|^{|S|}.$$ 

Finally the expected impulse cost gathered until state $s$ is reached is overestimated by

$$D(M,e) \cdot \max C \cdot \lambda \cdot d.$$ 

Moreover the expected number of visits of $K$ is at most $1/p_K(s)$. Altogether we get:

$$\frac{|S| \cdot (d_{opt} \cdot \max C + d_{opt} \cdot |S| \cdot B \cdot \lambda + d_{opt} \cdot \max C \cdot |S| \cdot \lambda + D(M,e) \cdot \max C \cdot \lambda \cdot d)}{|S| \cdot \min C |S| \cdot e^{-\lambda d_{opt}} \cdot (\lambda \cdot d_{opt})^{|S|} \cdot p_K(s)}$$

$$\leq \frac{2 \cdot |S|^2 \cdot d_{opt} \cdot B \cdot \lambda + |S| \cdot D(M,e) \cdot \max C \cdot \lambda \cdot d}{|S| \cdot \min C |S| \cdot e^{-\lambda d_{opt}} \cdot (\lambda \cdot d_{opt})^{|S|} \cdot p_K(s)}$$ 

\begin{lemma}
Let $d$ be $\epsilon/2$-optimal in every state of $M$ and satisfy for $s, s' \in S$: $E_{M(d)}^{s_r} \leq \frac{2 |S|}{\epsilon^2} \cdot B^{1}$. Let $d' \in \Sigma(M_e)$ be such that $d'(s) = \delta \cdot \min(u, [d(s)/\delta])$ for each $s \in S$. We have for every $s \in S$

$$E_{M(d')}^s \leq E_{M(d)}^s + \epsilon/2.$$ 

\end{lemma}
The exponential transitions have minimal probability

\[ P_d \cdot c_d + w_d = c_d. \]

Assume we have an arbitrary strategy \( d' \in [d, \bar{d}]^S \) such that it is \( \varepsilon \)-optimal. Then the following holds:

\[
\begin{align*}
(P_{d_{ij'}}, \Delta P_{d_{ij'}}) & \cdot (c_{d_{ij'}} + \Delta c_{d_{ij'}}) + (w_{d_{ij'}} + \Delta w_{d_{ij'}}) = (c_{d_{ij'}} + \Delta c_{d_{ij'}}) \vspace{0.5cm} \\
(P_{d_{ij'}}, \Delta c_{d_{ij'}}) & \cdot (c_{d_{ij'}} + \Delta c_{d_{ij'}}) + (w_{d_{ij'}} + \Delta w_{d_{ij'}}) = (c_{d_{ij'}} + \Delta c_{d_{ij'}}) \\
(P_{d_{ij'}}, \Delta P_{d_{ij'}}) & \cdot (c_{d_{ij'}} + \Delta c_{d_{ij'}}) + (w_{d_{ij'}} + \Delta w_{d_{ij'}}) = (c_{d_{ij'}} + \Delta c_{d_{ij'}}) \\
\Delta P_{d_{ij'}} & \cdot (c_{d_{ij'}} + \Delta w_{d_{ij'}}) = \Delta c_{d_{ij'}} - P_{d_{ij'}} \cdot \Delta c_{d_{ij'}} \\
\Delta P_{d_{ij'}} & \cdot (c_{d_{ij'}} + \Delta w_{d_{ij'}}) = (I - P_{d_{ij'}}) - \Delta P_{d_{ij'}} \cdot \Delta c_{d_{ij'}} \\
(I - P_{d_{ij'}} - \Delta P_{d_{ij'}})^{-1} & \cdot (\Delta P_{d_{ij'}} \cdot (c_{d_{ij'}} + \Delta w_{d_{ij'}})) = \Delta c_{d_{ij'}}
\end{align*}
\]

From lemma 25 we have the bound on expected cost from every state. Using the bound on expected number we plug everything to formula above and show that we generate sufficiently small error.

\[ E_{M_d(t)} \geq E_{M(t)} - \varepsilon. \]

**G Discretization for bounded optimization under partial observation**

**Lemma 14** Let \( D \) denote \( \left\lfloor \frac{|S| \cdot e^2}{\lambda \cdot d \cdot \min C^i} \right\rfloor \cdot 2 \cdot \max C \cdot (\lambda \cdot d + 1/\lambda) \). We have \( \text{Val} [M, D] < B_D. \)

**Proof.** We first find the minimal probability to reach one state from other for any pair of states and then use this value to bound the \( B_D \) from above using Bernoulli trial.

We will fix some strategy \( d = (d_1, \ldots, d) \) in bounds \( d \) and \( \bar{d} \). For any pair of states \( s, s' \) we have a path of length at most \( |S| - 1 \) from \( s \) to \( s' \) and vice versa. Thus path can have transitions corresponding to fixed-delay or exponential transitions of \( C \) from which \( M \) was constructed. Any series of \( i \) transitions in the fixed-delay section has to occur within time bound \( d \) and the transition probability can be under-approximated using Poisson distribution:

\[ \min C^i \cdot e^{-\lambda d} \cdot \frac{\lambda \cdot d^i}{i!}. \]

The exponential transitions have minimal probability \( \min \text{Prob} \). We have to multiply the probabilities of all sections in order to get the minimal probability of the whole path. Obviously the least probability we get when we have all transitions within one fixed-delay section, thus the minimal probability of path of length \( |S| \) (let denote it \( p \) ) is

\[ p = \min C^{|S| - 1} \cdot e^{-\lambda d} \cdot \frac{(\lambda \cdot d)^{|S| - 1}}{(|S| - 1)!} \geq \min C^{S - 1} \cdot e^{-\lambda d} \cdot \frac{(\lambda \cdot d)^{|S| - 1}}{|S|!.} \]
Now to get the maximal expected number of steps we evaluate following

$$|S| \cdot p \cdot \sum_{i=0}^{\inf} (1 - p)^i \cdot (i + 1) = |S|/p.$$  

Finally we have to multiply the maximal expected number of steps by the maximal cost paid by each transition what we evaluate using Lemma\[16\]

$$\max C \cdot (1 + \lambda \cdot d + \max(1/\lambda, d)).$$

Altogether we get the maximal expected cost is

$$\left(\frac{|S| \cdot e^{\lambda d}}{\min C \cdot \lambda \cdot d}\right)^{|S|} \cdot \max C (1 + \lambda \cdot d + \max(1/\lambda, d)) \leq \left(\frac{|S| \cdot e^{\lambda d}}{\min C \cdot \lambda \cdot d}\right)^{|S|} \cdot \max C (1 + \lambda \cdot d + (1/\lambda, d)) \leq \left(\frac{|S| \cdot e^{\lambda d}}{\min C \cdot \lambda \cdot d}\right)^{|S|} \cdot 2 \cdot \max C (\lambda \cdot d + (1/\lambda)).$$

Proposition \[15\] Let $\varepsilon, d, \overline{d} > 0$ and $\text{Val}[M, D] < \infty$.

For $\delta = \frac{\varepsilon}{\overline{d}} \cdot (1 - e^{-\lambda \overline{d}}) \cdot B \overline{d}^{-3}$ we have

$$\left|\text{Val}[M, D] - \text{Val} \left[M_{d, \overline{d} - \delta}, D\right] \right| < \varepsilon.$$

Proof. Let us denote the optimal strategy as $d_{\text{opt}}$ (i.e. it holds $\mathbb{E}_{M(d_{\text{opt}})}[W] = \inf_{d \in \mathbb{R}_0^{|S|}} \mathbb{E}_{M(d)}[W]$). We first show that for each strategy $d \in [d_{\text{opt}} - \delta, d_{\text{opt}} + \delta]

$$E_{M(d)} = E_{M(d_{\text{opt}})} < \varepsilon/2.$$

Assume we have some strategy $d$. Let vector $c_d \in \mathbb{R}_0^{|S|}$ denote the expected accumulated cost before absorption in $M$, $w_d \in \mathbb{R}_0^{|S|}$ be vector of expected costs when leaving a corresponding state (i.e. for all $s$ it holds that $w_d(s) = \text{Cost}'(s, d(s))$) and $P_d$ transition probability matrix of $M(d)$. Then obviously for all strategies $d$ holds

$$P_d \cdot c_d + w_d = c_d.$$

Assume we have an arbitrary strategy $d' \in [d, \overline{d}]^{|S|}$ such that it is $\varepsilon$-optimal. Then the following holds:

$$P_{d'} \cdot c_{d'} + w_{d'} = c_{d'}$$

$$(P_{d_{\text{opt}}} + \Delta P_{d_{\text{opt}}}) \cdot (c_{d_{\text{opt}}} + \Delta c_{d_{\text{opt}}}) + (w_{d_{\text{opt}}} + \Delta w_{d_{\text{opt}}}) = (c_{d_{\text{opt}}} + \Delta c_{d_{\text{opt}}})$$

$$P_{d_{\text{opt}}} \cdot c_{d_{\text{opt}}} + P_{d_{\text{opt}}} \cdot \Delta c_{d_{\text{opt}}} + \Delta P_{d_{\text{opt}}} \cdot (c_{d_{\text{opt}}} + \Delta c_{d_{\text{opt}}}) + w_{d_{\text{opt}}} + \Delta w_{d_{\text{opt}}} = c_{d_{\text{opt}}} + \Delta c_{d_{\text{opt}}}$$

$$P_{d_{\text{opt}}} \cdot \Delta c_{d_{\text{opt}}} + \Delta P_{d_{\text{opt}}} \cdot (c_{d_{\text{opt}}} + \Delta c_{d_{\text{opt}}}) + \Delta w_{d_{\text{opt}}} = \Delta c_{d_{\text{opt}}}$$

$$\Delta P_{d_{\text{opt}}} \cdot c_{d_{\text{opt}}} + \Delta w_{d_{\text{opt}}} = \Delta c_{d_{\text{opt}}} - P_{d_{\text{opt}}} \cdot \Delta c_{d_{\text{opt}}}$$

$$- \Delta P_{d_{\text{opt}}} \cdot \Delta c_{d_{\text{opt}}}$$

$$\Delta P_{d_{\text{opt}}} \cdot c_{d_{\text{opt}}} + \Delta w_{d_{\text{opt}}} = (I - P_{d_{\text{opt}}} - \Delta P_{d_{\text{opt}}}) \cdot \Delta c_{d_{\text{opt}}}$$

$$(I - P_{d_{\text{opt}}} - \Delta P_{d_{\text{opt}}})^{-1} \cdot (\Delta P_{d_{\text{opt}}} \cdot c_{d_{\text{opt}}} + \Delta w_{d_{\text{opt}}}) = \Delta c_{d_{\text{opt}}}$$
We want $\Delta c_{d_{opt}}(s_{in})$ to be less than or equal $\varepsilon$. From rules for matrix multiplication we want to ensure:

$$\sum_{s \in S} (I - P_{d_{opt}} - \Delta P_{d_{opt}})^{-1}(s_{in}, s) \cdot (\Delta P_{d_{opt}} \cdot c_{d_{opt}} + \Delta w_{d_{opt}})(s) \leq \varepsilon/2. \tag{3}$$

From now on we will gradually over-approximate the left hand side term by term. From over-approximation of term $\Delta P_{d_{opt}}$ we will get a term $\delta$. Thus in the end we will bound the final over-approximation of the left hand side by $\varepsilon$ and solve the inequality. From there we will get formula for $\delta$.

Now we bound from above $\Delta w_{d_{opt}}$.

$$\Delta w_{d_{opt}}[s] \leq \delta \cdot |S| \cdot \max_{s', s''}(I_{T}(s', s''), I_{T}(s', s'')) \cdot \lambda + \delta \cdot \max_{s'} R(s')$$

Now we will bound $\Delta P_{d_{opt}}$. From Lemma 18 and from [28] it holds that forall $s, s' \in S$ it holds that

$$\Delta P_{d_{opt}}(s, s') = |T(s, d_{opt}(s)) - T(s, d_{opt}(s)) - \delta(s')|$$

$$\leq \delta \cdot \partial P_{d_{opt}}(s) \cdot \delta \cdot \max_{s'}(P_{s}(d_{opt}(s))) \cdot \lambda + \delta \cdot |S| \cdot \lambda,$$

where $\overline{P}_{s}$ is the transition matrix for $s$ constructed same as in Section B and $Q_{s}$ is a Q-matrix constructed for $\overline{P}_{s}$ with maximal exit rate $\lambda$.

Now we will bound $c_{d_{opt}}$. From properties of expected cost we know that for all $s \in S$ and $d$ it holds that

$$c_{d}[s] \cdot \Pr_{\text{Med}}^{d}(\omega \text{ reaches eventually state } s) \leq c_{d}(s_{in})$$

$$c_{d}[s] \leq \frac{c_{d}(s_{in})}{\Pr_{\text{Med}}^{d}(\omega \text{ reaches eventually state } s)}$$

Using minimal transition probability $\min_{s', s''} T(s', d(s'))(s'')$ in $M(d)$ we can bound the above formula to

$$c_{d}[s] \leq \frac{c_{d}(s_{in})}{(\min_{s', s''} T(s', d(s'))(s''))^{\beta}}$$
From Lemma 18 we know that

\[
\min_{s', s''} T(s', d[s']) (s'') \geq \min_{s', s''} \sum_{i=0}^{\infty} \psi_{d[s']} (i) \cdot \left( 1 \cdot P^t \cdot F \right) (s'') \\
\geq \min_{s', s''} \sum_{i=0}^{\infty} \psi_{d[s']} (i) \cdot \left( 1 \cdot P^t \cdot F \right) (s'') \\
\geq \min_{s'} \sum_{i=0}^{\infty} \psi_{d[s']} (i) = p_{\text{min}}^{\text{SS}} \cdot F_{\text{min}} \cdot \sum_{i=0}^{\infty} \psi_{d[i]} (i) \\
= p_{\text{min}}^{\text{SS}} \cdot F_{\text{min}} \cdot \min_{s'} \sum_{i=0}^{\infty} \psi_{d[i]} (i) \\
\geq p_{\text{min}}^{\text{SS}} \cdot F_{\text{min}} \cdot \min_{i \in \mathbb{N}} \sum_{i=0}^{\infty} \psi_{d[i]} (i) = p_{\text{min}}^{\text{SS}} \cdot F_{\text{min}} \cdot \sum_{i=0}^{\infty} (\lambda t^i) / i! \cdot e^{-\lambda t} \\
\geq p_{\text{min}}^{\text{SS}} \cdot F_{\text{min}} \cdot (\lambda t)^{\text{SS}} \cdot e^{-\lambda t}
\]

where \( P_{\text{min}}, F_{\text{min}} \) are the minimum positive branching probability in \( P, F \) of the \( M \), respectively. Altogether we have that for all \( s \in S \) and strategies \( d \in [d_0, d_1] \) it holds that

\[
\mathbf{c}_d (s) \leq p_{\text{min}}^{\text{SS}} \cdot F_{\text{min}} \cdot (\lambda t)^{\text{SS}} \cdot e^{-\lambda t}
\]

Now we will bound \( \sum_{s \in S} (I - P_{d_0} - \Delta P_{d_0})^{-1} (s_{in}, s) = \sum_{s \in S} (I - P_d)^{-1} (s_{in}, s) \). We know that

\[
\sum_{s \in S} (I - P_d)^{-1} (s_{in}, s) \cdot w_d (s) = \mathbf{c}_d (s_{in}) \\
\sum_{s \in S} (I - P_d)^{-1} (s_{in}, s) \cdot \min_{s'} w_d (s') \leq \mathbf{c}_d (s_{in}) \\
\sum_{s \in S} (I - P_d)^{-1} (s_{in}, s) \leq \mathbf{c}_d (s_{in}) / \min_{s'} w_d (s')
\]

Recall that \( w_d [s'] \) is the expected cost paid by single transition in \( M \). We can use the knowledge of \( d \). Obviously we can either in each state, where a fixed-delay is active, stay \( d \) or some sequence of exponential transitions will disable the fixed-delay transition before \( d \). If no fixed delay is enabled, the expected time spent in state is \( 1/\lambda \). Thus for the minimal cost paid in each step the following holds

\[
w_d (s') \geq \min_{d} \mathcal{R} (s) \cdot \left( \int_{0}^{\infty} x \lambda \cdot e^{-\lambda x} dx + d \cdot e^{-\lambda x} \right) \\
= \min_{d} \mathcal{R} (s) \cdot (1/\lambda - e^{-\lambda x} (d + 1/\lambda) + d \cdot e^{-\lambda x}) \\
= \min_{d} \mathcal{R} (s) \cdot (1/\lambda - e^{-\lambda x} d)
\]

We use the above equation to bound \( \sum_{s \in S} (I - P_{d_0} - \Delta P_{d_0})^{-1} [s_{in}, s] \), i.e.
\[
\sum_{s \in S} (I - P_{d_{op}} - \Delta P_{d_{op}})^{-1}(s_{in}, s) = \sum_{s \in S} (I - P_d)^{-1}(s_{in}, s) \\
\leq \frac{c_d(s_{in})}{\min_{s'} w_d(s')} \\
\leq \min_{s', s''} R(s) \cdot 1/\lambda \cdot (1 - e^{-\lambda d}) \\
\leq c_{d_{opt}}(s_{in}) + \epsilon \\
\min_{s', s''} R(s) \cdot 1/\lambda \cdot (1 - e^{-\lambda d}).
\]

We use all the bounds to bound the left hand side of the inequality \[3\].

\[
\sum_{s \in S} (I - P_{d_{op}} - \Delta P_{d_{op}})^{-1}(s_{in}, s) \cdot (\Delta P_{d_{op}} \cdot c_{d_{op}} + \Delta w_{d_{op}})(s) \leq \\
\leq \sum_{s \in S} (I - P_{d_{op}} - \Delta P_{d_{op}})^{-1}(s_{in}, s) \cdot (\delta \cdot |S|^2 \cdot \lambda \cdot c_{d_{op}} + \Delta w_{d_{op}})(s) \\
\leq \sum_{s \in S} (I - P_{d_{op}} - \Delta P_{d_{op}})^{-1}(s_{in}, s) \cdot (\delta \cdot |S|^2 \cdot (\lambda \cdot c_{d_{op}}(s_{in}) + w)) \\
\leq \min_{s', s''} R(s) \cdot 1/\lambda \cdot (1 - e^{-\lambda d}) \cdot \frac{c_{d_{opt}}(s_{in}) + \epsilon}{\min_{s', s''} R(s)} \\
\leq \min_{s', s''} R(s) \cdot 1/\lambda \cdot (1 - e^{-\lambda d}) \cdot \frac{\delta \cdot |S|^2 \cdot (\lambda \cdot c_{d_{op}}(s_{in}) + w)}{\min_{s', s''} R(s)} \\
\leq \min_{s', s''} R(s) \cdot 1/\lambda \cdot (1 - e^{-\lambda d}) \cdot \frac{\delta \cdot |S|^2 \cdot (\lambda \cdot c_{d_{opt}}(s_{in}) + w)}{\min_{s', s''} R(s)} \\
\leq \min_{s', s''} R(s) \cdot 1/\lambda \cdot (1 - e^{-\lambda d}) \cdot \frac{c_{d_{opt}}(s_{in}) + \epsilon}{\min_{s', s''} R(s)}.
\]

where \( w = \delta \cdot (\max_{s', s''}(I_P(s', s''), I_P(s', s'')) \cdot \lambda + \max_{s'} R(s')) \)

The last inequality follows from the fact that for any strategy \( d \in [d_L, d] \) it holds that \( c_d(s_{in}) \leq c_d(s_{in}) \).

We require that

\[
\frac{c_{d_{opt}}(s_{in}) + \epsilon}{\min_{s', s''} R(s)} \cdot \frac{\delta \cdot |S|^2 \cdot (\lambda \cdot c_{d_{opt}}(s_{in}) + w)}{\min_{s', s''} R(s)} \leq \epsilon/2
\]

thus we get the formula for \( \delta \)

\[
\delta \leq \frac{\epsilon \cdot \min_{s', s''} R(s) \cdot 1/\lambda \cdot (1 - e^{-\lambda d}) \cdot \frac{c_{d_{opt}}(s_{in}) + \epsilon}{\min_{s', s''} R(s)} \cdot \frac{\delta \cdot |S|^2 \cdot (\lambda \cdot c_{d_{opt}}(s_{in}) + w)}{\min_{s', s''} R(s)}}{2 \cdot \frac{c_{d_{opt}}(s_{in}) + \epsilon}{\min_{s', s''} R(s)} \cdot \frac{\delta \cdot |S|^2 \cdot (\lambda \cdot c_{d_{opt}}(s_{in}) + w)}{\min_{s', s''} R(s)}}
\]

\[
\delta \leq \frac{\epsilon}{6 \cdot \lambda} \cdot (1 - e^{-\lambda d}) \cdot B_d^{-3}.
\]

Now we will prove that for every strategy \( d' \) optimal in \( M_{d'} \) obtained by approximating costs and transition probabilities of \( d_c \) of \( M_c \) we have that

\[
E_{M_{d'}} \geq E_{M_{d}} - \epsilon.
\]

We again show that the following holds.

\[
\sum_{s \in S} (I - P_{d'} - \Delta P_{d'})^{-1}(s_{in}, s) \cdot (\Delta P_{d'} \cdot c_{d'} + \Delta w_{d'})(s) \leq \epsilon
\]

Please observe that we can directly use the bounds on \( \sum_{s \in S} (I - P_{d'} - \Delta P_{d'})^{-1}(s_{in}, s) \) and \( c_{d'} \), because we are dealing at worst with \( \epsilon \)-optimal strategy \( d' \).
\[
\sum_{s \in S} (I - P_{d'} - \Delta P_{d'})^{-1}(s_{in}, s) \cdot (\Delta P_{d'} \cdot c_{d'} + \Delta w_{d'})(s) \leq \frac{c_{d_{\min}}(s_{in}) + \varepsilon}{\min_{i} \mathcal{R}(s) \cdot 1/\lambda \cdot (1 - e^{-\lambda t/2})} \cdot \delta \cdot |S| \cdot (maxC + \lambda \cdot |S|^2 \cdot (c_{d_{\min}}(s_{in}) + \varepsilon)) \leq \varepsilon.
\]