DOMINATION NUMBER IN THE ANNIHILATING-SUBMODULE GRAPH OF MODULES OVER COMMUTATIVE RINGS

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Abstract. Let $M$ be a module over a commutative ring $R$. The annihilating-submodule graph of $M$, denoted by $AG(M)$, is a simple graph in which a non-zero submodule $N$ of $M$ is a vertex if and only if there exists a non-zero proper submodule $K$ of $M$ such that $NK = (0)$, where $NK$, the product of $N$ and $K$, is denoted by $(N : M)(K : M)M$ and two distinct vertices $N$ and $K$ are adjacent if and only if $NK = (0)$. This graph is a submodule version of the annihilating-ideal graph and under some conditions, is isomorphic with an induced subgraph of the Zariski topology-graph $G(\tau_T)$ which was introduced in (The Zariski topology-graph of modules over commutative rings, Comm. Algebra., 42 (2014), 3283–3296). In this paper, we study the domination number of $AG(M)$ and some connections between the graph-theoretic properties of $AG(M)$ and algebraic properties of module $M$.

1. Introduction

Throughout this paper $R$ is a commutative ring with a non-zero identity and $M$ is a unital $R$-module. By $N \leq M$ (resp. $N < M$) we mean that $N$ is a submodule (resp. proper submodule) of $M$.

Define $(N :_R M)$ or simply $(N : M) = \{ r \in R \mid rM \subseteq N \}$ for any $N \leq M$. We denote $(0 : M)$ by $Ann_R(M)$ or simply $Ann(M)$. $M$ is said to be faithful if $Ann(M) = (0)$. Let $N, K \leq M$. Then the product of $N$ and $K$, denoted by $NK$, is defined by $(N : M)(K : M)M$ (see \cite{6}). Define $ann(N)$ or simply $annN = \{ m \in M \mid m(K : M) = 0 \}$.

The prime spectrum of $M$ is the set of all prime submodules of $M$ and denoted by $Spec(M)$, $Max(M)$ is the set of all maximal submodules of $M$, and $J(M)$, the jacobson radical of $M$, is the intersection of all elements of $Max(M)$, respectively.

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There are many papers on assigning graphs to rings or modules (see, for example, [4, 7, 12, 13]). The annihilating-ideal graph $AG(R)$ was introduced and studied in [13]. $AG(R)$ is a graph whose vertices are ideals of $R$ with nonzero annihilators and in which two vertices $I$ and $J$ are adjacent if and only if $IJ = (0)$. Later, it was modified and further studied by many authors (see [11, 2, 3, 15, 20]).

In [7], the present authors introduced and studied the graph $G(\tau_T)$ (resp. $AG(M)$), called the Zariski topology-graph (resp. the annihilating-submodule graph), where $T$ is a non-empty subset of $\text{Spec}(M)$.

$AG(M)$ is an undirected graph with vertices $V(AG(M)) = \{N \leq M | \text{there exists } (0) \neq K < M \text{ with } NK = (0)\}$. In this graph, distinct vertices $N, L \in V(AG(M))$ are adjacent if and only if $NL = (0)$ (see [9, 10]). Let $AG(M)^*$ be the subgraph of $AG(M)$ with vertices $V(AG(M)^*) = \{N < M \text{ with } (N : M) \neq \text{Ann}(M)\}$ there exists a submodule $K < M$ with $(K : M) \neq \text{Ann}(M)$ and $NK = (0)$. By [7, Theorem 3.4], one concludes that $AG(M)^*$ is a connected subgraph. Note that $M$ is a vertex of $AG(M)$ if and only if there exists a nonzero proper submodule $N$ of $M$ with $(N : M) = \text{Ann}(M)$ if and only if every nonzero submodule of $M$ is a vertex of $AG(M)$. Clearly, if $M$ is not a vertex of $AG(M)$, then $AG(M) = AG(M)^*$. In [8, Lemma 2.8], we showed that under some conditions, $AG(M)$ is isomorphic with an induced subgraph of the Zariski topology-graph $G(\tau_T)$.

In this paper, we study the domination number of $AG(M)$ and some connections between the graph-theoretic properties of $AG(M)$ and algebraic properties of module $M$.

A prime submodule of $M$ is a submodule $P \neq M$ such that whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in (P : M)$ or $e \in P$ [17].

The notations $Z(R)$ and $\text{Nil}(R)$ will denote the set of all zero-divisors, the set of all nilpotent elements of $R$, respectively. Also, $Z_R(M)$ or simply $Z(M)$, the set of zero divisors on $M$, is the set $\{r \in R | rm = 0 \text{ for some } 0 \neq m \in M\}$. If $Z(M) = 0$, then we say that $M$ is a domain. An ideal $I \leq R$ is said to be nil if $I$ consist of nilpotent elements.

Let us introduce some graphical notions and denotations that are used in what follows: A graph $G$ is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function $\psi_G$ that associates an unordered pair of distinct vertices with each edge. The edge $e$ joins $x$ and $y$ if $\psi_G(e) = \{x, y\}$, and we say $x$ and $y$ are adjacent. The number of edges incident at $x$ in $G$ is called the degree of the vertex $x$ in $G$ and is denoted by $d_G(x)$ or simply $d(v)$. A path in graph $G$ is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where $x_{i-1}$ and $x_i$ are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between $x_{i-1}$ and $x_i$. The distance between two vertices $x$ and $y$, denoted $d(x, y)$, is the length of the shortest path from $x$ to $y$. The diameter of a connected graph $G$ is the maximum distance between two distinct vertices of $G$. For any vertex $x$ of a connected graph $G$, the eccentricity of $x$, denoted $e(x)$, is the maximum of the distances from $x$ to the other vertices of $G$. The set of vertices with minimum eccentricity is called the center of the graph $G$, and this minimum eccentricity value is the radius of $G$. For some $U \subseteq V(G)$, we denote by $N(U)$, the set of all vertices of $G \setminus U$ adjacent to at least one vertex of $U$ and $N[U] = N(U) \cup \{U\}$.

A graph $H$ is a subgraph of $G$, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $\psi_H$ is the restriction of $\psi_G$ to $E(H)$. A subgraph $H$ of $G$ is a spanning subgraph of $G$ if
A clique of a graph is a complete subgraph and the supremum of the sizes of cliques in $G$, denoted by $\gamma(G)$, is called the clique number of $G$. Let $\chi(G)$ denote the chromatic number of the graph $G$, that is, the minimal number of colors needed to color the vertices of $G$ so that no two adjacent vertices have the same color. Obviously $\chi(G) \geq cl(G)$.

A subset $D$ of $V(G)$ is called a dominating set if every vertex of $G$ is either in $D$ or adjacent to at least one vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the number of vertices in a smallest dominating set of $G$. A total dominating set of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex is adjacent to a vertex in $S$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. A dominating set of cardinality $\gamma(G)$ ($\gamma_t(G)$) is called a $\gamma$-set ($\gamma_t$-set). A dominating set $D$ is a connected dominating set if the subgraph $\langle D \rangle$ induced by $D$ is connected. The connected domination number of $G$, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of $G$. A dominating set $D$ is a clique dominating set if the subgraph $\langle D \rangle$ induced by $D$ is complete in $G$. The clique domination number $\gamma_{cl}(G)$ of $G$ equals the minimum cardinality of a clique dominating set of $G$. A dominating set $D$ is a paired-dominating set if the subgraph $\langle D \rangle$ induced by $D$ has a perfect matching. The paired-domination number $\gamma_{pr}(G)$ of $G$ equals the minimum cardinality of a paired-domination set of $G$.

A vertex $u$ of $v$ in $G$, if $uv$ is an edge of $G$, and $u \neq v$. The set of all neighbors of $v$ is the open neighborhood of $v$ or the neighbor set of $v$, and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of $v$ in $G$.

Let $S$ be a dominating set of a graph $G$, and $u \in S$. The private neighborhood of $u$ relative to $S$ in $G$ is the set of vertices which are in the closed neighborhood of $u$, but not in the closed neighborhood of any vertex in $S \setminus \{u\}$.

A bipartite graph is a graph whose vertices can be divided into two disjoint sets $U$ and $V$ such that every edge connects a vertex in $U$ to one in $V$; that is, $U$ and $V$ are each independent sets and complete bipartite graph on $n$ and $m$ vertices, denoted by $K_{n,m}$, where $V$ and $U$ are of size $n$ and $m$, respectively, and $E(G)$ connects every vertex in $V$ with all vertices in $U$. Note that a graph $K_{1,m}$ is called a star graph and the vertex in the singleton partition is called the center of the graph. We denote by $P_n$ a path of order $n$ (see [14]).

In section 2, a dominating set of $AG(M)$ is constructed using elements of the center when $M$ is an Artinian module. Also we prove that the domination number of $AG(M)$ is equal to the number of factors in the Artinian decomposition of $M$ and we also find several domination parameters of $AG(M)$. In section 3, we study the domination number of the annihilating-submodule graphs for reduced rings.
with finitely many minimal primes and faithful modules. Also, some relations
between the domination numbers and the total domination numbers of annihilating-
submodule graphs are studied.

The following results are useful for further reference in this paper.

**Proposition 1.1.** Suppose that $e$ is an idempotent element of $R$. We have the
following statements.

(a) $R = R_1 \times R_2$, where $R_1 = eR$ and $R_2 = (1 - e)R$.
(b) $M = M_1 \times M_2$, where $M_1 = eM$ and $M_2 = (1 - e)M$.
(c) For every submodule $N$ of $M$, $N = N_1 \times N_2$ such that $N_1$ is an $R_1$-
submodule $M_1$, $N_2$ is an $R_2$-submodule $M_2$, and $(N :_R M) = (N_1 :_{R_1}
M_1) \times (N_2 :_{R_2} M_2)$.
(d) For submodules $N$ and $K$ of $M$, $NK = N_1 K_1 \times N_2 K_2$ such that $N =
n_1 \times N_2$ and $K = K_1 \times K_2$.
(e) Prime submodules of $M$ are $P \times M_2$ and $M_1 \times Q$, where $P$ and $Q$ are prime
submodules of $M_1$ and $M_2$, respectively.

**Proof.** This is clear. \qed

We need the following results.

**Lemma 1.2.** (See [5] Proposition 7.6.1) Let $R_1, R_2, \ldots, R_n$ be non-zero ideals of
$R$. Then the following statements are equivalent:

(a) $R = R_1 \times \ldots \times R_n$;
(b) As an abelian group $R$ is the direct sum of $R_1, \ldots, R_n$;
(c) There exist pairwise orthogonal idempotents $e_1, \ldots, e_n$ with $1 = e_1 + \ldots +
e_n$, and $R_i = Re_i$, $i = 1, \ldots, n$.

**Lemma 1.3.** (See [16] Theorem 21.28.) Let $I$ be a nil ideal in $R$ and $u \in R$ be
such that $u + I$ is an idempotent in $R/I$. Then there exists an idempotent $e$ in $uR$
such that $e - u \in I$.

**Lemma 1.4.** (See [9] Lemma 2.4.) Let $N$ be a minimal submodule of $M$ and let
$Ann(M)$ be a nil ideal. Then we have $N^2 = (0)$ or $N = eM$ for some idempotent $e \in R$.

**Proposition 1.5.** Let $R/Ann(M)$ be an Artinian ring and let $M$ be a finitely
generated module. Then every nonzero proper submodule $N$ of $M$ is a vertex in
$AG(M)$.

**Theorem 1.6.** (See [9] Theorem 2.5.) Let $Ann(M)$ be a nil ideal. There exists
a vertex of $AG(M)$ which is adjacent to every other vertex if and only if $M =
eM \oplus (1 - e)M$, where $eM$ is a simple module and $(1 - e)M$ is a prime module
for some idempotent $e \in R$, or $Z(M) = Ann((N : M)M)$, where $N$ is a nonzero
proper submodule of $M$ or $M$ is a vertex of $AG(M)$.

**Theorem 1.7.** (See [9] Theorem 3.3.) Let $M$ be a faithful module. Then the
following statements are equivalent.

(a) $|AG(M)^*| = 2$.
(b) $AG(M)^*$ is a bipartite graph with two nonempty parts.
(c) $AG(M)^*$ is a complete bipartite graph with two nonempty parts.
(d) Either $R$ is a reduced ring with exactly two minimal prime ideals, or $AG(M)^*$
is a star graph with more than one vertex.
Corollary 1.8. (See [11 Corollary 3.5].) Let $R$ be a reduced ring and assume that $M$ is a faithful module. Then the following statements are equivalent.

(a) $\chi(AG(M)^*) = 2$.

(b) $AG(M)^*$ is a bipartite graph with two nonempty parts.

(c) $AG(M)^*$ is a complete bipartite graph with two nonempty parts.

(d) $R$ has exactly two minimal prime ideals.

Proposition 1.9. (See [15, Proposition 3.9].) Every minimal dominating set in a graph $G$ is a maximal irredundant set of $G$.

2. Domination Number in the Annihilating-Submodule Graph for Artinian Modules

The main goal in this section, is to obtain the value certain domination parameters of the annihilating-submodule graph for Artinian modules.

Recall that $M$ is a vertex of $AG(M)$ if and only if there exists a nonzero proper submodule $N$ of $M$ with $(N : M) = Ann(M)$ if and only if every nonzero submodule of $M$ is a vertex of $AG(M)$. In this case, the vertex $N$ is adjacent to every other vertex. Hence $\gamma(AG(M)) = 1 = \gamma((AG(M)))$. So we assume that throughout this paper $M$ is not a vertex of $AG(M)$, then $AG(M) = AG(M)^*$.

We start with the following remark which completely characterizes all modules for which $\gamma((AG(M))) = 1$.

Remark 2.1. Let $Ann(M)$ be a nil ideal. By Theorem 1.6, there exists a vertex of $AG(M)$ which is adjacent to every other vertex if and only if $M = eM \oplus (1 - e)M$, where $eM$ is a simple module and $(1 - e)M$ is a prime module for some idempotent $e \in R$, or $Z(M) = Ann((N : M)M)$, where $N$ is a nonzero proper submodule of $M$ or $M$ is a vertex of $AG(M)$. Now, let $Ann(M)$ be a nil ideal and $M$ be a domain module. Then $\gamma((AG(M))) = 1$ if and only if $M = eM \oplus (1 - e)M$, where $eM$ is a simple module and $(1 - e)M$ is a prime module for some idempotent $e \in R$.

Theorem 2.2. Let $M$ be a f.g Artinian local module. Assume that $N$ is the unique maximal submodule of $M$. Then the radius of $AG(M)$ is 0 or 1 and the center of $AG(M)$ is $\{K \subseteq \text{ann}(N) | K \not= \{0\} \text{ is a submodule in } M\}$.

Proof. If $N$ is the only non-zero proper submodule of $M$, then $AG(M) \cong K_1$, $e(N) = 0$ and the radius of $AG(M)$ is 0. Assume that the number of non-zero proper submodules of $M$ is greater than 1. Since $M$ is f.g Artinian module, there exists $m \in \mathbb{N}$, $m > 1$ such that $N^m = (0)$ and $N^{m-1} \neq (0)$. For any non-zero submodule $K$ of $M$, $KN^{m-1} \subseteq NN^{m-1} = (0)$ and so $d(N^{m-1}, K) = 1$. Hence $e(N^{m-1}) = 1$ and so the radius of $AG(M)$ is 1. Suppose $K$ and $L$ are arbitrary non-zero submodules of $M$ and $K \subseteq \text{ann}(N)$. Then $KL \subseteq KN = (0)$ and hence $e(K) = 1$. Suppose $(0) \neq K' \not\subseteq \text{ann}(N)$. Then $K'N \neq (0)$ and so $e(K') > 1$. Hence the center of $AG(M)$ is $\{K \subseteq \text{ann}(N) | K \not= \{0\} \text{ is a submodule in } M\}$.

Corollary 2.3. Let $M$ be a f.g Artinian local module and $N$ is the unique maximal submodule of $M$. Then the following hold good.

(a) $\gamma(AG(M)) = 1$.

(b) $D$ is a $\gamma$-set of $AG(M)$ if and only if $D \subseteq \text{ann}(N)$.
Proof. (a) Trivial from Theorem 2.6
(b) Let $D = \{K\}$ be a $\gamma$-set of $AG(M)$. Suppose $K \not\subseteq \text{ann}(N)$. Then $KN \neq (0)$ and so $N$ is not dominated by $K$, a contradiction. Conversely, suppose $D \subseteq \text{ann}(N)$. Let $K$ be an arbitrary vertex in $AG(M)$. Then $KL \subseteq NL = (0)$ for every $L \in D$. i.e., every vertex $K$ is adjacent to every $L \in D$. If $|D| > 1$, then $D \setminus \{L'\}$ is also a dominating set of $AG(M)$ for some $L' \in D$ and so $D$ is not minimal. Thus $|D| = 1$ and so $D$ is a $\gamma$-set by (a).

Theorem 2.4. Let $M = \bigoplus_{i=1}^{n} M_i$, where $M_i$ is a f.g Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Then the radius of $AG(M)$ is 2 and the center of $AG(M)$ is $\{K \subseteq J(M) | K \neq (0)\}$ is a submodule in $M$.

Proof. Let $M = \bigoplus_{i=1}^{n} M_i$, where $M_i$ is a f.g Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Let $J_i$ be the unique maximal submodule in $M_i$ with nilpotency $n_i$. Note that $\text{Max}(M) = \{N_1, \ldots, N_n | N_i = M_1 \oplus \cdots \oplus M_{i-1} \oplus J_i \oplus M_{i+1} \oplus \cdots \oplus M_n, 1 \leq i \leq n\}$ is the set of all maximal submodules in $M$. Consider $D_i = (0) \oplus \cdots \oplus (0) \oplus J_n^{n-1} \oplus (0) \oplus \cdots \oplus (0)$ for $1 \leq i \leq n$. Note that $J(M) = J_1 \oplus \cdots \oplus J_n$ is the Jacobson radical of $M$ and any non-zero submodule in $M$ is adjacent to $D_i$ for some $i$. Let $K$ be any non-zero submodule of $M$. Then $K = \bigoplus_{i=1}^{n} K_i$, where $K_i$ is a submodule of $M_i$. 

Case 1. If $K = N_i$ for some $i$, then $KD_j \neq (0)$ and $KN_j \neq (0)$ for all $j \neq i$. Note that $N(K) = \{(0) \oplus \cdots \oplus (0) \oplus L_i \oplus (0) \oplus \cdots \oplus (0) | J_i L_i = (0), L_i$ is a nonzero submodule in $M_i\}$. Clearly $N(K) \cap N(N_j) = (0)$, $d(K, N_j) \neq 2$ for all $j \neq i$, and so $K - D_i - D_j - N_j$ is a path in $AG(M)$. Therefore $e(K) = 3$ and so $e(N) = 3$ for all $N \in \text{Max}(M)$.

Case 2. If $K \neq D_i$ and $K_i \subseteq J_i$ for all $i$. Then $KD_i = (0)$ for all $i$. Let $L$ be any non-zero submodule of $M$ with $KL \neq (0)$. Then $LD_j = (0)$ for some $j$, $K - D_j - L$ is a path in $AG(M)$ and so $e(K) = 2$.

Case 3. If $K_i = M_i$ for some $i$, then $KD_i \neq (0)$, $KN_i \neq (0)$ and $KD_j = (0)$ for some $j \neq i$. Thus $K - D_j - D_i - N_i$ is a path in $AG(M)$, $d(K, N_i) = 3$ and so $e(K) = 3$. Thus $e(K) = 2$ for all $K \subseteq J(M)$. Further note that in all the cases center of $AG(M)$ is $\{K \subseteq J(M) | K \neq (0)\}$ is a submodule in $M$.

In view of Theorems 2.2 and 2.4 we have the following corollary.

Corollary 2.5. Let $M = \bigoplus_{i=1}^{n} M_i$, where $M_i$ is a simple module for all $1 \leq i \leq n$ and $n \geq 2$. Then the radius of $AG(M)$ is 1 or 2 and the center of $AG(M)$ is $\bigcup_{i=1}^{n} D_i$, where $D_i = (0) \oplus \cdots \oplus (0) \oplus M_i \oplus (0) \oplus \cdots \oplus (0)$ for $1 \leq i \leq n$.

Theorem 2.6. Let $M = \bigoplus_{i=1}^{n} M_i$, where $M_i$ is a f.g Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Then $\gamma(AG(M)) = n$.

Proof. Let $N_i$ be the unique maximal submodule in $M_i$ with nilpotency $n_i$. Let $\Omega = \{D_1, D_2, \ldots, D_n\}$, where $D_i = (0) \oplus \cdots \oplus (0) \oplus J_n^{n-1} \oplus (0) \oplus \cdots \oplus (0)$ for $1 \leq i \leq n$. Note that any non-zero submodule in $M$ is adjacent to $D_i$ for some $i$. Therefore $N[\Omega] = V(AG(M))$, $\Omega$ is a dominating set of $AG(M)$ and so $\gamma(AG(M)) \leq n$. Suppose $S$ is a dominating set of $AG(M)$ with $|S| < n$. Then there exists $N \in \text{Max}(M)$ such that $NK \neq (0)$ for all $K \in S$, a contradiction. Hence $\gamma(AG(M)) = n$.

In view of Theorem 2.6 we have the following corollary.
Corollary 2.7. Let $M = \bigoplus_{i=1}^{n} M_i$, where $M_i$ is a f.g Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Then

(a) $ir(AG(M)) = n$.
(b) $\gamma_{c}(AG(M)) = n$.
(c) $\gamma_1(AG(M)) = n$.
(d) $\gamma_{cl}(AG(M)) = n$.
(e) $\gamma_{pr}(AG(M)) = n$, if $n$ is even and $\gamma_{pr}(AG(M)) = n + 1$, if $n$ is odd.

Proof. Consider the $\gamma$-set of $AG(M)$ identified in the proof of Theorem 2.7. By Proposition 1.9, $\Omega$ is a maximal irredundant set with minimum cardinality and so $ir(AG(M)) = n$. Clearly $\langle \Omega \rangle$ is a complete subgraph of $AG(M)$. Hence $\gamma_{c}(AG(M)) = \gamma_{t}(AG(M)) = \gamma_{cl}(AG(M)) = n$. If $n$ is even, then $\langle \Omega \rangle$ has a perfect matching and so $\Omega$ is a paired dominating set of $AG(M)$. Thus $pr(AG(M)) = n$. If $n$ is odd, then $\langle \Omega \cup K \rangle$ has a perfect matching for some $K \in V(AG(M)) \setminus \Omega$, and so $\Omega \cup K$ is a paired dominating set of $AG(M)$. Thus $\gamma_{pr}(AG(M)) = n$ if $n$ even and $\gamma_{pr}(AG(M)) = n + 1$ if $n$ is odd.

Let $M = \bigoplus_{i=1}^{n} M_i$, where $M_i$ is a f.g Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Then by Theorem 2.4 radius of $AG(M)$ is 2. Further, by Theorem 2.6, the domination number of $AG(M)$ is equal to $n$, where $n$ is the number of distinct maximal submodules of $M$. However, this need not be true if the radius of $AG(M)$ is 1. For, consider the ring $M = M_1 \oplus M_2$, where $M_1$ and $M_2$ are simple modules. Then $AG(M)$ is a star graph and so has radius 1, whereas $M$ has two distinct maximal submodules. The following corollary shows that a more precise relationship between the domination number of $AG(M)$ and the number of maximal submodules in $M$, when $M$ is finite.

Corollary 2.8. Let $M$ be a finite module and $\gamma((AG(M))) = n$. Then either $M = M_1 \oplus M_2$, where $M_1, M_2$ are simple modules or $M$ has $n$ maximal submodules.

Proof. When $\gamma((AG(M))) = 1$, proof follows from [9] Corollary 2.12. When $\gamma((AG(M))) = n$, then $M$ cannot be $M = M_1 \oplus M_2$, where $M_1, M_2$ are simple modules. Hence $M = \bigoplus_{i=1}^{m} M_i$, where $M_i$ is a f.g Artinian local module for all $1 \leq i \leq m$ and $m \geq 2$. By Theorem 2.6, $\gamma((AG(M))) = m$. Hence by assumption $m = n$, i.e., $M = \bigoplus_{i=1}^{n} M_i$, where $M_i$ is a f.g Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. One can see now that $M$ has $n$ maximal submodules.

3. The relationship between $\gamma_{t}(((AG(M))))$ and $\gamma((AG(M)))$

The main goal in this section is to study the relation between $\gamma_{t}((AG(M)))$ and $\gamma((AG(M)))$.

Theorem 3.1. Let $M$ be a module. Then

$\gamma_{t}(AG(M)) = \gamma((AG(M)))$ or $\gamma_{t}((AG(M))) = \gamma((AG(M))) + 1$.

Proof. Let $\gamma_{t}((AG(M))) \neq \gamma((AG(M)))$ and $D$ be a $\gamma$-set of $AG(M)$. If $\gamma((AG(M))) = 1$, then it is clear that $\gamma_{t}((AG(M))) = 2$. So let $\gamma((AG(M))) > 1$ and put $k = Max\{n | \text{there exist } L_1, \ldots, L_n \in D : \cap_{i=1}^{n} L_i \neq \emptyset\}$. Since $\gamma_{t}((AG(M))) \neq \gamma((AG(M)))$, we have $k \geq 2$. Let $L_1, \ldots, L_k \in D$ be such that $\cap_{i=1}^{k} L_i \neq \emptyset$. Then $S = \{\cap_{i=1}^{k} L_i, \text{ann}L_1, \ldots, \text{ann}L_k\} \cup D \setminus \{L_1, \ldots, L_k\}$ is a $\gamma_{t}$-set. Hence $\gamma_{t}((AG(M))) = \gamma((AG(M))) + 1$. 

□
In the following result we find the total domination number of $AG(M)$.

**Theorem 3.2.** Let $S$ be the set of all maximal elements of the set $V(AG(M))$. If $|S| > 1$, then $\gamma_t((AG(M))) = |S|$.

**Proof.** Let $S$ be the set of all maximal elements of the set $V(AG(M))$, $K \in S$ and $|S| > 1$. First we show that $K = \text{ann}(\text{ann}K)$ and there exists $m \in M$ such that $K = \text{ann}(m)$. Let $K \in S$. Then $\text{ann}K \neq 0$ and so there exists $0 \neq m \in \text{ann}K$. Hence $K \subseteq \text{ann}(\text{ann}K) \subseteq \text{ann}(m)$. Thus by the maximality of $K$, we have $K = \text{ann}(\text{ann}K) = \text{ann}(m)$. By Zorn’ Lemma it is clear that if $V(AG(M)) \neq \emptyset$, then $S \neq \emptyset$. For any $K \in S$ choose $M \in M$ such that $K = \text{ann}(m)$. We assert that $D = \{Rm_K|K \in S\}$ is a total dominating set of $AG(M)$. Since for every $L \in V(AG(M))$ there exists $K \in S$ such that $L \subseteq K = \text{ann}(m)$, $L$ and $Rm_K$ are adjacent. Also for each pair $K, K' \in S$, we have $(Rm_K)(Rm_{K'}) = 0$. Namely, if there exists $m \in (Rm_K)(Rm_{K'})$, then $K = K' = \text{ann}(m)$. Thus $\gamma_t((AG(M))) \leq |S|$. To complete the proof, we show that each element of an arbitrary $\gamma_t$-set of $AG(M)$ is adjacent to exactly one element of $S$. Assume to the contrary, that a vertex $L'$ of a $\gamma_t$-set of $AG(M)$ is adjacent to $K$ and $K'$, for $K, K' \in S$. Thus $K = K' = \text{ann}L'$, which is impossible. Therefore $\gamma_t((AG(M))) = |S|$.

**Theorem 3.3.** Let $R$ be a reduced ring, $M$ is a faithful module, and $|\text{Min}(R)| < \infty$. If $\gamma((AG(M))) > 1$, then $\gamma_t((AG(M))) = \gamma((AG(M))) = |\text{Min}(R)|$.

**Proof.** Since $R$ is reduced, $M$ is a faithful module, and $\gamma((AG(M))) > 1$, we have $|\text{Min}(R)| > 1$. Suppose that $\text{Min}(R) = \{p_1, \ldots, p_n\}$. If $n = 2$, the result follows from Corollary 1.3. Therefore, suppose that $n \geq 3$. Define $p_iM = p_1 \ldots p_{i-1}p_{i+1} \ldots p_nM$, for every $i = 1, \ldots, n$. Clearly, $p_iM \neq 0$, for every $i = 1, \ldots, n$. Since $R$ is reduced, we deduce that $p_iM p_j M = 0$. Therefore, every $p_iM$ is a vertex of $AG(M)$. If $K$ is a vertex of $AG(M)$, then by [11 Corollary 3.5], $(K : M) \subseteq Z(R) = \cup_{i=1}^n p_i$. It follows from the Prime Avoidance Theorem that $(K : M) \subseteq p_i$, for some $i$, $1 \leq i \leq n$. Thus $p_iM$ is a maximal element of $V(AG(M))$, for every $i = 1, \ldots, n$. From Theorem 3.2 $\gamma_t((AG(M))) = |\text{Min}(R)|$. Now, we show that $\gamma((AG(M))) = n$. Assume to the contrary, that $B = \{J_1, \ldots, J_{n-1}\}$ is a dominating set for $AG(M)$. Since $n \geq 3$, the submodules $p_iM$ and $p_jM$, for $i \neq j$ are not adjacent (from $p_i p_j = 0 \subseteq p_k$ it would follow that $p_i \subseteq p_k$, or $p_j \subseteq p_k$ which is not true). Because of that, we may assume that for some $k < n - 1$, $J_i = p_iM$ for $i = 1, \ldots, k$, but none of the other of submodules from $B$ are equal to some $p_iM$ (if $B = \{p_1M, \ldots, p_{n-1}M\}$, then $p_nM$ would be adjacent to some $p_iM$, for $i \neq n$). So, every submodule in $\{p_{k+1}M, \ldots, p_nM\}$ is adjacent to a submodule in $\{J_{k+1}, \ldots, J_{n-1}\}$. It follows that for some $s \neq t$, there is an $l$ such that $(p_sM)J_l = 0 = (p_tM)J_l$. Since $p_s \not\subseteq p_t$, it follows that $J_l \subseteq p_tM$, so $J_l^2 = 0$, which is impossible, since the ring $R$ is reduced. So $\gamma_t((AG(M))) = \gamma((AG(M))) = |\text{Min}(R)|$.

**Theorem 3.4.** Let $R$ be a reduced ring, $M$ is a faithful module, and $|\text{Min}(R)| < \infty$, then the following are equivalent.

(a) $\gamma((AG(M))) = 2$.
(b) $AG(M)$ is a bipartite graph with two nonempty parts.
(c) $AG(M)$ is a complete bipartite graph with two nonempty parts.
(d) $R$ has exactly two minimal primes.

Proof. Follows from Theorem 3.3 and Corollary 1.8.

In the following theorem the domination number of bipartite annihilating-submodule graphs is given.

**Theorem 3.5.** Let $M$ be a faithful module. If $AG(M)$ is a bipartite graph, then $\gamma(\langle AG(M) \rangle) \leq 2$.

Proof. Let $M$ be a faithful module. If $AG(M)$ is a bipartite graph, then from Theorem 1.7, either $R$ is a reduced ring with exactly two minimal prime ideals, or $AG(M)$ is a star graph with more than one vertex. If $R$ is a reduced ring with exactly two minimal prime ideals, then the result follows by Corollary 3.4. If $AG(M)$ is a star graph with more than one vertex, then we are done.

The next theorem is on the total domination number of the annihilating-submodule graphs of Artinian modules.

**Theorem 3.6.** Let $M = \bigoplus_{i=1}^{n} M_i$, where $M_i$ is a f.g Artinian local module for all $1 \leq i \leq n$, $n \geq 2$, and $M \neq M_1 \oplus M_2$, where $M_1$, $M_2$ are simple modules. Then $\gamma_t(\langle AG(M) \rangle) = \gamma_r(\langle AG(M) \rangle) = |\text{Min}(R)|$.

Proof. By Proposition 1.8, every nonzero proper submodule of $M$ is a vertex in $AG(M)$. So, the set of maximal elements of $V(AG(M))$ and $\text{Max}(M)$ are equal. Let $M = \bigoplus_{i=1}^{n} M_i$, where $(M_i, J_i)$ is a f.g Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Let $\text{Max}(M) = \{N_i = M_1 \oplus \ldots \oplus M_{i-1} \oplus J_i \oplus M_{i+1} \oplus \ldots \oplus M_n | 1 \leq i \leq n\}$. By Theorem 3.2, $\gamma_t(\langle AG(M) \rangle) = \gamma_r(\langle AG(M) \rangle) = |\text{Max}(M)|$. In the sequel, we prove that $\gamma_t(\langle AG(M) \rangle) = n$. Assume to the contrary, the set $\{K_1, \ldots, K_{n-1}\}$ is a dominating set for $AG(M)$. Since $M \neq M_1 \oplus M_2$, where $M_1$, $M_2$ are simple modules, we find that $K_i N_s = K_i N_t = 0$, for some $i, t, s$, where $1 \leq i \leq n - 1$ and $1 \leq t, s \leq n$. This means that $K_i = 0$, a contradiction.

The following theorem provides an upper bound for the domination number of the annihilating-submodule graph of a Noetherian module.

**Theorem 3.7.** If $R$ is a Noetherian ring and $M$ a f.g module, then $\gamma(\langle AG(M) \rangle) \leq \lambda(\text{Ass}(M)) < \infty$.

Proof. By [19], since $R$ is a Noetherian ring and $M$ a f.g module, $\lambda(\text{Ass}(M)) < \infty$. Let $\text{Ass}(M) = \{p_1, \ldots, p_n\}$ where $p_i = \text{ann}(m_i)$ for some $m_i \in M$ for every $i = 1, \ldots, n$. Set $A = \{Rm_i | 1 \leq i \leq n\}$. We show that $A$ is a dominating set of $AG(M)$. Clearly, every $Rm_i$ is a vertex of $AG(M)$, for $i = 1, \ldots, n$ $(p_i M)(m_i R) = 0$. If $K$ is a vertex of $AG(M)$, then [19] Corollary 9.36 implies that $(K : M) \subseteq Z(M) = \bigcup_{i=1}^{n} p_i$. It follows from the Prime Avoidance Theorem that $(K : M) \subseteq p_i$, for some $i, 1 \leq i \leq n$. Thus $K(Rm_i) = 0$, as desired.

The remaining result of this paper provides the domination number of the annihilating-submodule graph of a finite direct product of modules.

**Theorem 3.8.** For a module $M$, which is a product of two (nonzero) modules, one of the following holds:

(a) If $M \cong F \times D$, where $F$ is a simple module and $D$ is a prime module, then $\gamma(AG(M)) = 1$.
(b) If \( M \cong D_1 \times D_2 \), where \( D_1 \) and \( D_2 \) are prime modules which are not simple, then \( \gamma(AG(M)) = 2 \).

(c) If \( M \cong M_1 \times D \), where \( M_1 \) is a module which is not prime and \( D \) is a prime module, then \( \gamma(AG(M)) = \gamma(AG(M_1)) + 1 \).

(d) If \( M \cong M_1 \times M_2 \), where \( M_1 \) and \( M_2 \) are two modules which are not prime, then \( \gamma(AG(M)) = \gamma(AG(M_1)) + \gamma(AG(M_2)) \).

Proof. Parts (a) and (b) are trivial.

(c) With no loss of generality, one can assume that \( \gamma(AG(M_1)) < \infty \). Suppose that \( \gamma(AG(M_1)) = n \) and \( \{K_1, \ldots, K_n\} \) is a minimal dominating set of \( AG(M_1) \). It is not hard to see that \( \{K_1 \times 0, \ldots, K_n \times 0 \} \) is the smallest dominating set of \( AG(M) \).

(d) We may assume that \( \gamma(AG(M_1)) = m \) and \( \gamma(AG(M_2)) = n \), for some positive integers \( m \) and \( n \). Let \( \{K_1, \ldots, K_m\} \) and \( \{L_1, \ldots, L_n\} \) be two minimal dominating sets in \( AG(M_1) \) and \( AG(M_2) \), respectively. It is easy to see that \( \{K_1 \times 0, \ldots, K_m \times 0, L_1 \times 0 \ldots 0 \times L_n\} \) is the smallest dominating set in \( AG(M) \).

\[ \square \]

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