On Transmit Beamforming for MISO-OFDM Channels With Finite-Rate Feedback

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Abstract

With finite-rate feedback, we propose two feedback methods for transmit beamforming in a point-to-point MISO-OFDM channel. For the first method, a receiver with perfect channel information, quantizes and feeds back the optimal transmit beamforming vectors of a few selected subcarriers, which are equally spaced. Based on those quantized vectors, the transmitter applies either constant or linear interpolation with the remaining beamforming vectors. With constant interpolation, we derive an approximate capacity upper bound and the optimal cluster size that maximize the sum capacity. For linear interpolation, we derive a closed-form expression for the phase rotation by utilizing the correlation between OFDM subcarriers. For the second proposed method, a channel impulse response is quantized with a uniform scalar quantizer. At the transmitter, the channel frequency response can be reconstructed from the quantized impulse response and the optimal beamforming vectors can then be computed. With channel quantization, we also approximate an upper bound of the sum capacity. We show that switching between the two methods for different feedback-rate requirements can perform better than the existing schemes.

Index Terms

Multiple-input single-output (MISO), OFDM, transmit beamforming, feedback, RVQ, beamforming interpolation, channel quantization.

This work was supported by the 2010 Telecommunications Research and Industrial Development Institute (TRIDI) scholarship and joint funding from the Thailand Commission on Higher Education, Thailand Research Fund, and Kasetsart University under grant MRG5580236.

The material in this paper was presented in part at the Electrical Engineering/Electronics, Computer, Telecommunications and Information Technology Conference (ECTI), Huahin, Thailand, May 2012, and the IEEE International Conference on Communications (ICC), Budapest, Hungary, June 2013.

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I. INTRODUCTION

Equipping a transmitter and/or a receiver with multiple antennas creates a multiantenna wireless channel whose capacity depends on the channel information available at the transmitter and/or receiver. In multiantenna channels, transmit beamforming has been shown to increase channel capacity by directing transmit signal toward the strongest channel mode [1]. With channel information, the receiver can compute the optimal beamforming vector that maximizes channel capacity and feeds the vector back to the transmitter. Due to a finite feedback rate, the beamforming vector needs to be quantized. Several quantization schemes and codebooks have been proposed and analyzed, and the corresponding performance was shown to depend on the codebook design and the number of available feedback bits [2], [3], see references therein.

In this work, we consider transmit beamforming for multiple-input single-output (MISO) orthogonal frequency-division multiplexing (OFDM), which converts a wideband channel into parallel narrowband subchannels. For each subchannel or subcarrier, the optimal beamforming vector is different and needs to be quantized and fed back. The total number of feedback bits required increases with the number of subcarriers, which can be large. References [4]–[8] have proposed to reduce the feedback amount while maintaining performance. In [4], [6], the optimal transmit beamforming vectors of selected subcarriers, which are a few subcarriers apart, are quantized while the remaining ones are approximated to equal the quantized vector of the closest subcarrier. In [5] and [7], the remaining transmit beamforming vectors are proposed to be linearly interpolated and spherically interpolated, respectively. In [8], a channel impulse response is vector quantized and fed back to the transmitter where the frequency response can be reconstructed.

Given a low feedback rate, we propose to quantize the optimal beamforming vector at every few subcarriers with the random vector quantization (RVQ) codebook proposed by [2] and to either use the same quantized vector for the whole subcarrier cluster or linearly interpolate the remaining beamforming vectors in the cluster from the quantized vectors. For the first proposed method termed constant interpolation, we derive an approximate upper bound of the sum capacity over all subcarriers. The analytical approximation can predict the performance trend well and gives the optimal cluster size precisely. The optimal cluster size depends mainly on the available feedback rate, and the frequency selectivity of the channel.

For the linear interpolation method, we propose a closed-form expression for the phase-rotation...
parameter based on a correlation between the transmit beamformers of subcarriers in the cluster. In earlier work by [5], the parameter was exhaustively searched. Our modified linear interpolation requires fewer minimum feedback bits than that in [5].

When the feedback rate is high, we propose to quantize the channel impulse response with a uniform scalar quantizer and derive the approximate capacity upper bound for the MISO channel. The scalar quantization used in the proposed method is less complex than the vector quantization used in [8]. The proposed scalar quantization of the channel impulse response is shown to perform well with a high feedback rate. Similar results were observed by [2] where the optimal beamformer and not the channel response was scalar quantized.

II. System Model

We consider a point-to-point, discrete-time, MISO-OFDM channel with $N$ subcarriers. A transmitter is equipped with $N_t$ antennas while a receiver is equipped with a single antenna. We assume that the transmit antennas are placed sufficiently far apart that they are independent. For each transmit-receive antenna pair, a transmitted signal propagates through a frequency-selective Rayleigh fading channel with order $L$. Applying a discrete Fourier transform, the frequency response for the $n$th subcarrier and the $n_t$th transmit antenna is given by

$$h_{n,n_t} = \sum_{l=0}^{L-1} g_{l,n_t} e^{-j2\pi ln/N}$$

where $g_{l,n_t}$ is a complex channel gain for the $l$th path between the $n_t$th transmit and receive antenna pairs. Assuming a rich scattering, $g_{l,n_t}$ for all $L$ paths and all $N_t$ transmit antennas are independent complex Gaussian distributed with zero mean and variance $\frac{1}{L}$. Let $h_n$ denote an $N_t \times 1$ channel vector of the $n$th subcarrier, whose entry is $h_{n,n_t}$ shown in (1). Thus,

$$h_n = [h_{n,1} \ h_{n,2} \ \cdots \ h_{n,N_t}]^T.$$ 

Assuming a transmit beamforming or a rank-one precoding, the received signal on the $n$th subcarrier is given by

$$r_n = h_n^\dagger v_n x_n + z_n, \quad 1 \leq n \leq N$$

where $v_n$ is an $N_t \times 1$ unit-norm beamforming vector, $x_n$ is a transmitted symbol with zero mean and unit variance, and $z_n$ is an additive white Gaussian noise with zero mean and variance.
Thus, a resulting sum capacity over $N$ subcarriers is given by
\[ C = \sum_{n=1}^{N} E \left[ \log(1 + \rho |h_n^\dagger v_n|^2) \right] \]  
where the expectation is over the distribution of $h_n$. We assume a uniform power allocation for all subcarriers and hence, the background signal-to-noise ratio (SNR) for each subcarrier $\rho = 1/\sigma_z^2$.

From (4), we note that the sum capacity is a function of transmit beamforming vectors $\{v_1, v_2, \ldots, v_N\}$. A receiver with perfect channel information can optimize the sum capacity over the transmit beamforming vectors and send the selected beamforming vectors to the transmitter via a feedback channel. Since the feedback channel between the receiver and the transmitter has a finite rate, quantizing the transmit beamforming vectors is required. In this study we apply a random vector quantization (RVQ) codebook whose entries are independent, isotropically distributed (i.i.d.) vectors to quantize a transmit beamforming vector. RVQ is simple, however has been shown to perform close to the optimum codebook \[2], \[9\].

We assume $B$ total feedback bits per update. For an equal-bit-per-subcarrier allocation, each beamforming vector is quantized with $B/N$ bits. Let us denote the RVQ codebook by $V = \{w_1, w_2, \ldots, w_{2^{B/N}}\}$ with $2^{B/N}$ entries. The receiver selects for the $n$th subcarrier the entry in the codebook that maximizes an instantaneous achievable rate as follows:
\[ \hat{v}_n = \arg \max_{w \in V} \log(1 + \rho |h_n^\dagger w|^2) \]  
and the associated capacity for the $n$th subcarrier is given by
\[ C_n = E \left[ \log(1 + \rho |\hat{h}_n^\dagger \hat{v}_n|^2) \right] \]
\[ = E \left[ \log(1 + \rho \|h_n\|^2 |\hat{h}_n^\dagger \hat{v}_n|^2) \right] \]
\[ \leq \log(1 + \rho E[\|h_n\|^2 |\hat{h}_n^\dagger \hat{v}_n|^2]) \]
\[ = \log(1 + \rho E[|h_n|^2 E[|\hat{h}_n^\dagger \hat{v}_n|^2]]) \]
\[ = \log(1 + \rho N_t E[|\hat{h}_n^\dagger \hat{v}_n|^2]) \]
where $\hat{h}_n = h_n/\|h_n\|$ is a unit-norm channel vector that points in the same direction as $h_n$. Evaluating the capacity is not tractable due to logarithm. However, we are able to derive an
upper bound by applying Jensen’s inequality. Analyzing the upper bound gives us insights into the performance of quantized transmit beamforming. To obtain (11), we refer to [9] that $\|h_n\|^2$ and $|\hat{h}_n^\dagger \hat{v}_n|^2$ are independent and $E[\|h_n\|^2] = N_t$ due to Rayleigh fading.

Thus, the sum capacity is upper bounded by

$$C \leq \sum_{n=1}^{N} \log\left(1 + \rho N_t E|\hat{h}_n^\dagger \hat{v}_n|^2\right). \tag{12}$$

We note that the sum capacity depends on the number of feedback bits per subcarrier which could be small due to a large number of subcarriers in a practical OFDM system. Hence, this may result in a large quantization error, which leads to a substantial performance loss.

## III. Interpolating Transmit Beamforming Vectors

Feeding back transmit beamforming vectors of all subcarriers requires quantizing $NN_t$ complex coefficients and thus, a large number of feedback bits. We note that adjacent subcarriers in OFDM are highly correlated since the number of channel taps is much lower than that of subcarriers ($L \ll N$). The optimal transmit beamformers, which depend on channel matrices, are also highly correlated. In this section, we apply two interpolation methods to reduce the number of feedback bits while maintaining the performance.

First we evaluate a squared correlation between normalized channel vectors of subcarrier $n$ and $n + q$ defined by

$$E \left[ |\hat{h}_n^\dagger \hat{h}_{n+q}|^2 \right] = E \left[ \frac{|\hat{h}_n^\dagger \hat{h}_{n+q}|^2}{\|h_n\|^2 \|h_{n+q}\|^2} \right]. \tag{13}$$

Evaluating (13) is not tractable for a finite-size system. Hence, we approximate the average squared correlation as follows.

**Lemma 1:** A squared correlation between the $n$th and $n + q$th normalized channel vectors is approximated as follows:

$$E \left[ |\hat{h}_n^\dagger \hat{h}_{n+q}|^2 \right] \approx \frac{L^2 + N_t \varphi^2(q)}{L^2 N_t + \varphi^2(q)} \tag{14}$$

$$\triangleq \psi(q, N_t) \tag{15}$$

where $\varphi(x) = \frac{\sin(\frac{\pi x}{2})}{\sin(\frac{\pi}{4})}$.

The proof of Lemma 1 is shown in Appendix A.

As subsequent numerical example in Section V will show that approximation in Lemma 1 closely predicts the result of a finite-size system. The correlation in (14) depends on $L$, $N$, $N_t$, $N_t$. 

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and most importantly, $q$, which indicates how far apart the two channel vectors are. When $q \to 0$, 
$\varphi(q) \to L$ and the squared correlation becomes $\|\hat{h}_n\|^4 \to 1$.

A. Constant Interpolation

In the first method, we group adjacent contiguous subcarriers into a cluster and apply the
same quantized beamforming vector for all subcarriers in the cluster. We denote the number of
contiguous subcarriers in one cluster by $M$. Thus, the number of clusters is given by $K \triangleq \lfloor N/M \rfloor$
with a possible few remaining subcarriers. The number of feedback bits allocated for each cluster
is equal to $B/K$. All $B/K$ bits are used to quantize the beamforming vector of the centered
subcarrier for odd $M$ and one subcarrier off the center for even $M$. Therefore, the beamforming
vector used for the $k$th cluster is given by

$$
\hat{v}_{kM+m} = \begin{cases} 
\arg \max_{w \in V} |\hat{h}^\dagger_{kM+m+1} w|^2 & \text{for odd } M \\
\arg \max_{w \in V} |\hat{h}^\dagger_{kM+\frac{m}{2}} w|^2 & \text{for even } M 
\end{cases}
$$

where $1 \leq m \leq M$ and $0 \leq k \leq K - 1$. If $N/M$ is not an integer, then there exist some remaining
subcarriers, which do not belong in any cluster. We propose to set the transmit beamforming for
these subcarriers to be that of the last cluster as follows:

$$
\hat{v}_{KM+q} = \hat{v}_{KM} \quad \text{for } 1 \leq q \leq N - KM
$$

With constant interpolation, a capacity for each subcarrier can be approximately bounded by
Proposition [1].

**Proposition 1:** The approximate upper bound on ergodic capacity of the $(n+q)$th subcarrier
is given by

$$
C_{n+q} \lesssim C_{n+q} = \log(1 + \rho N_t \gamma(n + q, B/K))
$$

where

$$
\gamma(n + q, B/K) \triangleq \psi(q, N_t) \cdot (1 - 2^{B/K} \beta(2^{B/K}, \frac{N_t}{N_t - 1}))
+ (1 - \psi(q, N_t)) \cdot \frac{(2^{B/K} \beta(2^{B/K}, \frac{N_t}{N_t - 1}))}{N_t - 1}
$$

and the beta function $\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt$.

The proof of Proposition [1] is shown in Appendix [B].
With Proposition 1 we obtain the approximate upper bound for the sum capacity for a single cluster with odd $M$ as follows:

$$C_{\text{cluster}} = \sum_{q=-\frac{M-1}{2}}^{\frac{M-1}{2}} C_{n+q} = \log(1 + \rho N_t \gamma(0, B/K)) + 2 \sum_{q=1}^{\frac{M-1}{2}} \log(1 + \rho \gamma(q, B/K)).$$  \hspace{1cm} (20)

With even $M$, the sum capacity for a single cluster is approximately upper bounded by

$$C_{\text{cluster}} = \log(1 + \rho N_t \gamma(0, B/K)) + 2 \sum_{q=1}^{\frac{M-1}{2}} \log(1 + \rho \gamma(q, B/K))$$

$$+ \log(1 + \rho \gamma(\frac{M}{2}, B/K)).$$  \hspace{1cm} (21)

Given $B$ feedback bits and other system parameters, we would like to determine the number of subcarriers $M^*$, which maximizes the sum capacity of all $N$ subcarriers given by

$$M^* = \arg \max_{1 \leq M \leq N} \left\{ KC_{\text{cluster}} + \sum_{r=1}^{N-KM} \log(1 + \rho \gamma(r + M, B/K)) \right\}$$  \hspace{1cm} (22)

where the first term accounts for the approximate capacity of the $K$ clusters and the second term accounts for the approximate capacity of a few remaining subcarriers. Solving (22) can be accomplished by either integer programming for which there exist many available tools or by exhaustive search.

Although the optimization in (22) is based on the approximate upper bound of the actual capacity, subsequent numerical examples in Section V show that the solution to (22) accurately predicts the optimal cluster size.

Besides sum capacity, another important performance metric is the average received power across subcarriers defined as follows:

$$\eta_{\text{AVE}} \triangleq \frac{1}{N} \sum_{n=1}^{N} \rho E \left[ |h_n^\dag \hat{v}_n|^2 \right]$$  \hspace{1cm} (23)

$$= \frac{\rho N_t}{N} \sum_{n=1}^{N} E \left[ |\hat{h}_n |^2 \right].$$  \hspace{1cm} (24)

where it is shown in the proof of Proposition 1 that

$$E \left[ |\hat{h}_n |^2 \right] \approx \gamma(n, B/K).$$  \hspace{1cm} (25)

Therefore, the average received power can be approximated as follows.
Corollary 1: For odd $M$,
\[
\eta_{\text{AVE}} \approx \frac{p N_t}{N} \left\{ K \gamma(0, B/K) + 2K \sum_{q=1}^{M/2} \gamma(q, B/K) + \sum_{r=1}^{N-KM} \gamma(r + \frac{M}{2}, B/K) \right\},
\]
and for even $M$
\[
\eta_{\text{AVE}} \approx \frac{p N_t}{N} \left\{ K \gamma(0, B/K) + 2K \sum_{q=1}^{M-1} \rho \gamma(q, B/K) + K \gamma(\frac{M}{2}, B/K) + \sum_{r=1}^{N-KM} \gamma(r + \frac{M}{2}, B/K) \right\}.
\]

These analytical expressions give a more accurate result than those in Proposition 1 since there is no Jensen’s inequality involved as demonstrated by later numerical results.

B. Linear Interpolation

To increase the performance, we propose to modify a linear interpolation proposed by [5]. Similar to the constant interpolation, all subcarriers are grouped into $K$ clusters each cluster consists of $M$ contiguous subcarriers and a possible last cluster with a few remaining subcarriers. For each cluster, the optimal beamforming vector of the first subcarrier is selected from an RVQ codebook with either $B/K$ bits or $B/(K+1)$ bits, depending on the total number of clusters.

All other beamforming vectors in a cluster are linear combinations of the quantized beamforming vector of the first subcarrier in the cluster and that in the next cluster as follows [5]:
\[
\hat{v}_{kM+m} \triangleq \frac{(1 - c_m) \hat{v}_{kM} + c_m e^{j \theta_m} \hat{v}_{(k+1)M}}{\|(1 - c_m) \hat{v}_{kM} + c_m e^{j \theta_m} \hat{v}_{(k+1)M}\|} \tag{28}
\]
for $1 \leq m \leq M - 1$ and $0 \leq k \leq K - 1$, where $c_m = \frac{m}{M}$ is a linear weight and $\theta_m$ is a phase-rotation parameter. We note that for the last cluster, we choose to interpolate with $\hat{v}_1$ instead of $\hat{v}_N$ to save some feedback bits. Due to DFT, $\tilde{h}_1$ is similar to $\tilde{h}_N$ and hence, $\hat{v}_1$ is also similar to $\hat{v}_N$.

In [5], $\theta_m$ is chosen to maximize the sum capacity in (4) by performing exhaustive search over the received power in each cluster as follows:
\[
\theta_m = \arg \max_{\theta \in \Theta} \sum_{m=kM+1}^{(k+1)M} |h_{m}^\dagger \hat{v}_m|^2 \tag{29}
\]
where $\Theta$ is a phase-rotation codebook.

To avoid search complexity, here we propose to determine the phase rotation based on a correlation between the optimal beamformers of neighboring subcarriers. We note that the optimal transmit beamforming vector for the $n$th subcarrier is matched to the normalized channel vector $v_{n}^{\text{opt}} = \bar{h}_{n}$.

Evaluating a correlation between the optimal beamformer and the interpolated beamformer that are $m$ subcarriers apart, $E|\langle v_{kM}^{\text{opt}} \rangle v_{kM+m}|^{2}$, follows similar steps to the proof of Lemma 1. This correlation is most likely close to the correlation between the optimal beamformers, which is approximated to be $\psi(m, N_{t})$ in (15). Based on this assumption, we set $E|\langle v_{kM}^{\text{opt}} \rangle v_{kM+m}|^{2}$ to $\psi(m, N_{t})$ and solve for the phase-rotation parameter given by

**Proposition 2:**

\[
\theta_{m} = \arccos \frac{U(m)}{V(m)}
\]

where

\[
U(m) = (1 - c_{m})^{2}(\psi(m, N_{t}) - N_{t} + 1) + c_{m}^{2}(N_{t}\psi(m, N_{t}) - \frac{N_{t}}{L^{2}}\varphi^{2}(M) + 1)
\]

and

\[
V(m) = \frac{2}{L}(1 - c_{m})c_{m}(N_{t} - N_{t}\psi(m, N_{t}) + 1)\varphi(M)\cos\left(\frac{\pi M(L - 1)}{N}\right).
\]

The proof is shown in Appendix C.

Analyzing the sum capacity of this linear interpolation is not tractable, but some numerical results will be shown in Section V.

**IV. QUANTIZING CHANNEL IMPULSE RESPONSE**

When the available feedback rate is sufficiently high, quantizing the channel impulse response directly can perform well [10]. Here we propose to quantize all channel taps of all transmit-receive antenna pairs with a scalar uniform quantizer. A uniform quantizer is simple and performs close to the optimal quantizer when the number of quantization bits is high. Real and imaginary parts of all channel taps are quantized independently with the same number of bits, which is $\frac{B}{2N_{t}L}$. Thus, the quantized $l$th channel tap for the $n_{t}$ antenna pair is given by

\[
\hat{g}_{l,n_{t}} = \hat{g}_{l,n_{t},r} + j\hat{g}_{l,n_{t},i}
\]

\[
= Q(g_{l,n_{t},r}) + jQ(g_{l,n_{t},i})
\]
where $g_{l,n,r}$ and $g_{l,n,i}$ are real and imaginary parts of $g_{l,n}$, respectively, $Q(\cdot)$ is the uniform scalar quantizer with $2^B$ steps, while variables with hats denote outputs of the quantizer. Here we select a step size of the quantizer by the existing rule of thumb for Gaussian input (cf. [11, p. 125])

$$\Delta = \frac{4E[(g_{l,n,r})^2]}{2^B} = \frac{1}{\sqrt{L}} 2^{\frac{3}{2} - \frac{B}{2NtL}},$$

which changes with the variance of the channel tap and the number of quantization bits. Then, the transmitter computes a DFT of the quantized channel impulse response to obtain an approximate frequency response as follows:

$$\hat{h}_{n,n} = \sum_{l=0}^{L-1} \hat{g}_{l,n} e^{-j\frac{2\pi ln}{N}},$$

which is the $n_t$th entry of the quantized $N_t \times 1$ channel vector for the $n$th subcarrier denoted by $\hat{h}_n = [\hat{h}_{n,1} \hat{h}_{n,2} \cdots \hat{h}_{n,N_t}]^T$.

Based on $\hat{h}_n$, the transmitter transmits signal in the direction of the quantized channel vector, namely, $\hat{h}_n/\|\hat{h}_n\|$ and the corresponding sum rate over all subcarriers is given by

$$C = \sum_{n=1}^{N} E \left[ \log(1 + \rho \frac{|\hat{h}_n^\dagger \hat{h}_n|^2}{\|\hat{h}_n\|^2}) \right]$$

$$= NE \left[ \log(1 + \rho \frac{|\hat{h}_n^\dagger \hat{h}_n|^2}{\|\hat{h}_n\|^2}) \right]$$

$$\leq N \log(1 + \rho E \left[ \frac{|\hat{h}_n^\dagger \hat{h}_n|^2}{\|\hat{h}_n\|^2} \right])$$

$$\lesssim N \log(1 + \rho \frac{E[|\hat{h}_n^\dagger \hat{h}_n|^2]}{E[\|\hat{h}_n\|^2]}$$

where in (38), we use the fact that the distribution of subcarriers is identical and in (39) and (40), we apply Jensen’s inequality and approximate an expectation of the quotient by a quotient of the two expectations. Consequently, we obtained the approximate upper bound of the sum capacity.

Since real and imaginary parts of each channel tap are independent and Gaussian distributed with zero mean and variance $\frac{1}{2L}$, we can easily show that

$$E[\|\hat{h}_n\|^2] = N_t(1 - 2L E[(\hat{g}_r - g_r)^2])$$

and

$$E[|h_n \hat{h}_n^\dagger|^2] = N_t(1 + \frac{1}{L} - (2L - 1) E[(\hat{g}_r - g_r)^2]$$

$$+ 2L E[\hat{g}_r^2 g_r^2] + 4L(N_tL - 1) E^2[\hat{g}_r g_r])$$

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where we have dropped indices \( n_t \) and \( l \) from \( g_{l,n_t,r} \) for clarity. The mean squared error is given by

\[
E[(\hat{g}_r - g_r)^2] = \int (Q(x) - x)^2 f_{g_r}(x) \, dx
\]

and the correlation and its second moment are given by

\[
E[\hat{g}_r g_r] = \int x Q(x) f_{g_r}(x) \, dx
\]

\[
E[\hat{g}_r^2 g_r^2] = \int x^2 Q^2(x) f_{g_r}(x) \, dx.
\]

where \( f_{g_r}(\cdot) \) denotes the probability density function (pdf) of \( g_{l,n_t,r} \).

Each term in (43)-(45) can be computed numerically. However, to obtain some insight on how the sum capacity depends on the feedback rate and other channel parameters, we approximate each term in a high feedback-rate regime. It was shown that for large \( B \),

\[
E[(\hat{g}_r - g_r)^2] \approx \frac{\Delta^2}{12} = \frac{2}{3L^2} \frac{\rho}{N_t L}.
\]

Applying the property of the optimum quantizer [13], we obtain

\[
E[\hat{g}_r g_r] \approx \frac{1}{2L} - E[(\hat{g}_r - g_r)^2].
\]

As \( B \to \infty \), \( \hat{g}_r \to g_r \). Hence,

\[
\lim_{B \to \infty} E[\hat{g}_r^2 g_r^2] = \frac{3}{4L^2}.
\]

Substituting (46)-(48) into (41) and (42), we obtain the approximate upper bound for a sum capacity for the MISO channel with large \( B \) as follows

\[
C \lesssim N \log(1 + \rho(1 - \frac{1}{L} + (N_t L - 1)\Omega_B + \frac{3}{4L^2\Omega_B}))
\]

where \( \Omega_B = \frac{1}{L} - \frac{4}{3L^2} 2^{-\frac{\rho}{N_t L}} \).

V. NUMERICAL RESULTS

To illustrate the performance of the proposed interpolations, Monte Carlo simulation is performed with 3000 channel realizations. Fig. 1 shows a correlation between subcarriers \( E[\bar{h}_n^\dagger \bar{h}_{n+q}]^2 \) from simulation results and the analytical approximation in Lemma 1 with \( N_t = 5 \), \( N = 64 \), \( L = 4 \) and \( 8 \), respectively. From this figure, we see that the correlation between subcarriers decreases as expected when the subcarriers are further apart and note that the analytical approximation derived in Lemma 1 predicts the simulation results quite accurately.
Fig. 2 shows a sum capacity of a $2 \times 1$-OFDM channel versus total feedback bits $B$ for various feedback schemes. The performance upper bound is a result of an ideal unquantized optimal beamforming and is shown with a solid line. From Fig.2 as the number of feedback bits $B$ increases, the sum capacity also increases as expected. Given the cluster size $M$ at 8, the proposed linear interpolation outperforms the constant interpolation for all feedback ranges. However, with optimal $M^*$ computed from (22), the constant interpolation performs better than the linear one in a moderate feedback regime. We also compare the results of our proposed method with that from Choi and Heath [5]. We note that the linear interpolation method is better than ours in moderate and high feedback regimes, but ours performs better with a very low number of feedback bits. We also remark that the method by Choi and Heath is more computationally complex due to the exhaustive search it requires to find the phase-rotation parameter $\theta_m$ and also it requires some minimum $B$ to operate.

Furthermore, we note that with large $B$, directly quantizing channel taps performs close to the performance upper bound as predicted by [10], and is less complex due to the uniform quantizer. As expected, channel quantization closes to the optimal at extreme feedback as mentioned in [10]. From this figure, we can conclude that with roughly one feedback bit per subcarrier, direct channel-tap quantization is preferred, and with less than one bit per subcarrier, a combination of quantized beamforming and interpolation is preferred.

Fig. 3 show the sum capacity over all subcarriers with constant interpolation for different numbers of feedback bits $B$, channel taps $L$, and subcarriers $N$. We set the number of transmit antennas $N_t = 3$, cluster size $M = 8$, and SNR at 10 dB. We note that the capacity increases with $B$ as expected and decreases with $L$. As the channel becomes more frequency selective, the cluster size should be reduced to maintain performance. We also note the larger gap between the approximate analytical upper bound derived in Proposition 1 and capacity from the simulation results as the system size increases from 64 to 128. The gap can be generally attributed to Jensen’s inequality.

In Fig. 4 we plot the average received power per subcarrier $\eta_{AVE}$ for constant interpolation with $B$. The solid lines show the analytical approximation given in Corollary 1 while the square and circular markers show the simulation results. We observe that in this example, the analytical results are very close to those from the simulation. Unlike capacity analysis, Jensen’s inequality is not used in deriving $\eta_{AVE}$. From the figure, we see that about one feedback bit per subcarrier

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gives us close to the infinite feedback performance.

Fig. 5 shows the sum capacity with constant interpolation with cluster size $M$ from both the analytical approximation from Proposition 1 and the simulation results when the number of total feedback bits is severely limited at 16 bits. Different plots correspond to different $L$ values. For small $M$, more beamforming vectors are quantized and fed back from the receiver, but with a smaller number of feedback bits per cluster. For large $M$, the opposite is true. Thus, there exists an optimal $M$ that maximizes the capacity. We can observe from this figure, selecting optimal $M = 16$ performs 35% better than that for feeding back every subcarrier ($M = 1$) for $L = 4$. Comparing the analytical approximation and the simulation results, we observe that the gap is quite substantial (due to Jensen’s inequality); however, the analytical result still can accurately predict the optimal $M$. For a flat fading channel ($L = 1$), all subcarrier gains are the same and thus, the optimal $M^* = 1$. For frequency selective fading ($L \geq 1$), subcarriers are less correlated and the optimal $M^*$ decrease with $L$.

Fig. 6 shows the optimal number of subcarriers $M^*$ obtained from the analytical bound approximation with different numbers of channel taps and total feedback bits. In this figure, we observe that $M^*$ decreases when $L$ increases. In other words, when the channel becomes more frequency selective, cluster size should be reduced. Furthermore, with more available feedback bits, cluster size should also be reduced.

In Fig. 7, we compare a capacity per subcarrier of a $3 \times 1$ channel obtained from simulation and the approximate upper bound (49) for a direct quantization of channel taps. A number of channel taps $L$ varies between 4 and 16. From the figure, the approximate upper bound exhibits the same performance trend as the simulation results and the gap between the two is about 12%. Again we can attribute the gap between the two results to Jensen’s inequality. Although the approximation is derived for a large feedback rate, it seems to predict well the simulation result even with relatively small $B$. In addition, we observe from the simulation results that approximately 3 bits per real coefficient are needed to achieve close to the capacity maximum. While the number of fading paths $L$ increases, $B$ also increases to achieve close to the capacity maximum.
VI. CONCLUSIONS

We have proposed two feedback methods for MISO-OFDM channels. Beamforming interpolation with RVQ performs well with limited feedback while direct quantization of the channel impulse response performs well with large feedback. Thus, switching between the two methods for different feedback rates is recommended. We analyzed the sum capacity with constant interpolation and RVQ and showed that the analytical results can predict the performance trend and accurately predict the optimal cluster size.

For linear interpolation, we have derived a closed-form expression for a phase rotation to avoid exhaustive search and the minimum number of feedback bits. Furthermore, we have analyzed the capacity with direct quantization of channel taps, which depends on the feedback rate and the number of antennas and channel taps.

Future work can take different directions. In the problem considered, the MISO channel was investigated. Extending our results to multiple-input multiple-output (MIMO) beamforming is not straightforward and thus, MIMO beamforming could be a good problem to consider. In addition, here we considered channels with a uniform power delay profile. Other practical channel models might be of interest.

APPENDIX

A. Proof of Lemma 1

We approximate

\[ E \left[ |\bar{h}_{n}^\dagger \bar{h}_{n+q}|^2 \right] \approx \frac{E \left[ |\bar{h}_{n}^\dagger \bar{h}_{n+q}|^2 \right]}{E \left[ \|h_{n}\|^2 \|h_{n+q}\|^2 \right]}. \]  

(50)

First, we evaluate \( E \left[ |h_{n}^\dagger h_{n+q}|^2 \right] \) as follows

\[
E \left[ |h_{n}^\dagger h_{n+q}|^2 \right] = E \left[ \left| \sum_{m=1}^{N_t} h_{n,m}^* h_{n+q,m} \right|^2 \right] 
= \sum_{m_1=1}^{N_t} E \left[ |h_{n,m_1}|^2 |h_{n+q,m_1}|^2 \right] + \sum_{m_2=1}^{N_t} \sum_{m_2 \neq m_1} E[h_{n,m_1}^* h_{n+q,m_1}] E[h_{n,m_2}^* h_{n+q,m_2}] 
= N_t E \left[ |h_{n,m_1}|^2 |h_{n+q,m_1}|^2 \right] + N_t (N_t - 1) |E[h_{n,m_1}^* h_{n+q,m_1}]|^2 
\]

(52)

(53)

where we apply the assumption that channel gains across antennas are i.i.d.
Next we evaluate each term in (53) by substituting (1).

\[
E \left[ |h_n|^2 |h_{n+q}|^2 \right] = \sum_{l_1,l_2,l_3,l_4=1}^{L} E \left[ g_{l_1} g_{l_2}^* g_{l_3} g_{l_4}^* e^{-j2\pi \frac{(l_1 - l_2 + l_3 - l_4)n + (l_3 - l_4)q}{N}} \right] \tag{54}
\]

where we omit the antenna subscript \( m_1 \) for brevity. It is straightforward to show that

\[
E[|g_{l_1} g_{l_2}^* g_{l_3} g_{l_4}^* e^{-j2\pi \frac{(l_1 - l_2 + l_3 - l_4)n + (l_3 - l_4)q}{N}}] = \begin{cases} 
\frac{2}{L} & : l_1 = l_2 = l_3 = l_4, \\
1 - \frac{1}{L} & : (l_1 = l_2) \neq (l_3 = l_4), \\
\frac{1}{L}(\varphi^2(q) - 1) & : (l_1 = l_3) \neq (l_2 = l_4), \\
0 & : \text{otherwise}.
\end{cases} \tag{55}
\]

Substituting (55) into (54) gives

\[
E \left[ |h_n|^2 |h_{n+q}|^2 \right] = 1 + \frac{1}{L^2} \varphi^2(q) \tag{56}
\]

Also,

\[
E \left[ |h_n|^2 |h_{n+q}|^2 \right] = \sum_{l_1=1}^{L} E[|g_{l_1}|^2 e^{-j2\pi l_1 q \frac{n}{N}}] + \sum_{l_1=1}^{L} \sum_{l_2=1}^{L} e^{-j2\pi l_1 l_2 \frac{n}{N}} e^{-j2\pi (l_3 - l_4) q \frac{n}{N}} \tag{57}
\]

\[
= \sum_{l_1=1}^{L} \frac{1}{L} e^{-j2\pi l_1 q \frac{n}{N}} \tag{58}
\]

\[
= \frac{1}{L} e^{-j2\pi (l_1 - 1) q \frac{n}{N}} \tag{59}
\]

where the second term in (57) is equal to zero.

Substituting (56) and (59) into (53) gives

\[
E \left[ |h_n|^2 |h_{n+q}|^2 \right] = N_t \left[ 1 + \frac{1}{L^2} \varphi^2(q) \right] \tag{60}
\]

Following similar steps as the above evaluation of \( E \left[ |h_n|^2 |h_{n+q}|^2 \right] \), we can show that

\[
E \left[ ||h_n||^2 ||h_{n+q}||^2 \right] = N_t^2 + \frac{N_t}{L^2} \varphi^2(q). \tag{61}
\]

Substituting (60) and (61) into (50) yields the Lemma.
B. Proof of Proposition I

To derive the upper bound on $C_{n+q}$ in (11), we need to determine $E|\hat{h}_{n+q}^\dagger \hat{v}_{n+q}|^2$. With constant interpolation, $\hat{v}_{n+q}$ is set to equal the representative beamforming of a cluster, which is $q$ subcarriers away. Therefore, we would like to evaluate $E|\hat{h}_{n+q}^\dagger \hat{v}_n|^2$. To accomplish this goal, we project $\hat{h}_{n+q}$ onto $\hat{h}_n$ and its $N_t - 1$-dimensional orthogonal complement denoted by $\hat{h}_n^\perp$.

Let $\{u_1, u_2, \ldots, u_{N_t - 1}\}$ be a basis of $\hat{h}_n^\perp$. Hence, we can write $\hat{h}_{n+q}$ as a linear combination of its projection onto $\hat{h}_n$ and the basis of $\hat{h}_n^\perp$ as follows.

$$\hat{h}_{n+q} = (\hat{h}_n^\perp \hat{h}_{n+q}) \hat{h}_n + \sum_{i=1}^{N_t-1} (u_i^\dagger \hat{h}_{n+q}) u_i. \quad (62)$$

With (62), we have

$$E\left[|\hat{h}_{n+q}^\dagger \hat{v}_n|^2\right] = E\left[|\hat{h}_{n+q}^\dagger (\hat{h}_n^\perp \hat{h}_n) \hat{h}_n + \sum_{i=1}^{N_t-1} (\hat{h}_{n+q}^\dagger u_i)(u_i^\dagger \hat{v}_n)|^2\right] \quad (63)$$

$$= E\left[|\hat{h}_{n+q}^\dagger \hat{h}_n|^2|\hat{h}_n^\dagger \hat{v}_n|^2\right] + \sum_{i=1}^{N_t-1} E\left[|\hat{h}_{n+q}^\dagger u_i|^2|u_i^\dagger \hat{v}_n|^2\right]$$

$$+ 2E \Re \left\{ (\hat{h}_{n+q}^\dagger \hat{h}_n)(\hat{h}_n^\dagger \hat{v}_n) \sum_{i=1}^{N_t-1} (\hat{h}_{n+q}^\dagger u_i)(u_i^\dagger \hat{v}_n) \right\} \quad (64)$$

where $\Re \{x\}$ is the real part of $x$. Similar to [9], it can be shown that $|\hat{h}_{n+q}^\dagger \hat{h}_n|^2$ and $|\hat{h}_n^\dagger \hat{v}_n|^2$ are independent. In [9], $E|\hat{h}_n^\dagger \hat{v}_n|^2$ was also analyzed while $E|\hat{h}_{n+q}^\dagger \hat{h}_n|^2 \approx \psi(q, N_t)$ from Lemma 1. Thus,

$$E[|\hat{h}_{n+q}^\dagger \hat{h}_n|^2|\hat{h}_n^\dagger \hat{v}_n|^2] = E|\hat{h}_{n+q}^\dagger \hat{h}_n|^2 E|\hat{h}_n^\dagger \hat{v}_n|^2$$

$$\approx \psi(q, N_t) \left(1 - 2^{B/K} \beta(2^{B/K}, \frac{N_t}{N_t - 1})\right). \quad (65)$$

For the second term on the right-hand side of (64), we have that similar to the first term,

$$E\left[|\hat{h}_{n+q}^\dagger u_i|^2|u_i^\dagger \hat{v}_n|^2\right] = E|\hat{h}_{n+q}^\dagger u_i|^2 E|u_i^\dagger \hat{v}_n|^2. \quad (66)$$

We can evaluate the second term in (67) as follows. Similar to (62), we can write $\hat{v}_n$ as a linear combination of its projection onto basis $\{\hat{h}_n, u_1, \ldots, u_{N_t-1}\}$ as follows:

$$\hat{v}_n = (\hat{h}_n^\dagger \hat{v}_n) \hat{h}_n + \sum_{i=1}^{N_t-1} (u_i^\dagger \hat{v}_n) u_i. \quad (68)$$
Evaluating \(|\hat{v}_n^\dagger \hat{v}_n|^2\) with (68) and applying the fact that \(\| \hat{v}_n \| = 1\) results in
\[
|\hat{h}_n^\dagger \hat{v}_n|^2 + \sum_{i=1}^{N_t-1} |u_i^\dagger \hat{v}_n|^2 = 1. \tag{69}
\]
We take expectation on both sides and substitute a closed-form expression of \(E|\hat{h}_n^\dagger \hat{v}_n|^2\) from [9]. Also, \(E|u_i^\dagger \hat{v}_n|^2\) is the same for all \(1 \leq i \leq N_t - 1\) due to identical distributions. Thus, from (69), we have
\[
E|u_i^\dagger \hat{v}_n|^2 = \frac{2B/K \beta \left( \frac{2B/K}{N_t/N_t-1} \right)}{N_t - 1}. \tag{70}
\]
Similar to the steps that derive (70), we can show that
\[
E|\bar{h}_n^\dagger \bar{v}_n|^2 \approx 1 - \psi(q, N_t) \frac{2B/K \beta \left( \frac{2B/K}{N_t/N_t-1} \right)}{N_t - 1}. \tag{71}
\]
Applying (70) and (71), we have
\[
N_t - 1 \sum_{i=1}^{N_t-1} E\left[ |\bar{h}_n^\dagger \bar{v}_n|^2 |u_i^\dagger \hat{v}_n|^2 \right] \approx (1 - \psi(q, N_t)) \cdot \frac{2B/K \beta \left( \frac{2B/K}{N_t/N_t-1} \right)}{N_t - 1}. \tag{72}
\]
Evaluating the final term of the right-hand side of (64) is not tractable. However we note that for both small and large feedback, the term is close to zero due to \(\bar{h}_n^\dagger \bar{v}_n\) and \(u_i^\dagger \hat{v}_n\), respectively. Thus, we approximate
\[
E\Re \left\{ (\bar{h}_{n+q}^\dagger \bar{h}_n)(\bar{h}_n^\dagger \hat{v}_n) \sum_{i=1}^{N_t-1} (\bar{h}_{n+q}^\dagger u_i)(u_i^\dagger \hat{v}_n) \right\} \approx 0. \tag{73}
\]
Finally, substituting (66), (72), and (73) in (64) yields Proposition 1.

C. Proof of Proposition 2

Applying the linear interpolation (28) and assuming optimal, unquantized beamforming, we have
\[
E\| \psi_{opt}^\dagger v_{kM+m} \|^2 \approx \frac{E\left| h_{kM}^\dagger \left\{ (1 - c_m) h_{kM} + c_m e^{j\vec{\theta}_m} h_{(k+1)M} \right\} \right|^2}{E\| (1 - c_m) h_{kM} + c_m e^{j\vec{\theta}_m} h_{(k+1)M} \|^2}. \tag{74}
\]
Here we propose to set phase rotation \(\theta_m\) by solving
\[
\frac{E\left| h_{kM}^\dagger \left\{ (1 - c_m) h_{kM} + c_m e^{j\vec{\theta}_m} h_{(k+1)M} \right\} \right|^2}{E\| (1 - c_m) h_{kM} + c_m e^{j\vec{\theta}_m} h_{(k+1)M} \|^2} = \psi(m, N_t). \tag{75}
\]
where \(\psi(m, N_t)\) is defined in Lemma 1.
Similar to steps shown in the proof of Lemma 1, we can show that

\[
E \left| h_{kM} \left\{ (1 - c_m) h_{kM} + c_m e^{j\theta_m} h_{(k+1)M} \right\} \right|^2
= (1 - c_m)^2 (N_t + 1) + c_m^2 \left( \frac{N_t}{L} \varphi(M)^2 + 1 \right)
+ 2(1 - c_m)c_m \cos \theta_m \left( \frac{\pi M(L-1)}{N} \right) \left( \frac{N_t + 1}{L} \right) \varphi(M) \quad (76)
\]

and

\[
E \| (1 - c_m) h_{kM} + c_m e^{j\theta_m} h_{(k+1)M} \| = (1 - c_m)^2 (N_t + 1)
+ c_m^2 N_t + 2 \frac{N_t}{L} (1 - c_m)c_m \cos \theta_m \left( \frac{\pi M(L-1)}{N} \right) \varphi(M). \quad (77)
\]

Substituting (76) and (77) into (74) and solving for \( \theta_m \) gives Proposition 2.

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Fig. 1. Correlation between subcarriers $E[|h_n^* h_{n+q}|^2]$ from both simulation and analytical results with $N_t = 5$, $N = 64$, and $L = 4$ and 8.
Fig. 2. Sum capacity of a $2 \times 1$ OFDM channel with various feedback schemes with total number of feedback bits $B$ and $N = 64$, $L = 12$, and background SNR at 10 dB.
Fig. 3. Sum capacity of the constant interpolation with the number of total feedback bits $B$ for $N = 64$ and 128, $N_t = 3$, $M = 8$, and SNR at 10 dB.
Fig. 4. Average received power per subcarrier $\eta_{\text{AVE}}$ with the number of total feedback bits $B$ for $N = 64$, $N_t = 3$, $M = 8$, and SNR at 10 dB.
Fig. 5. Sum capacity with different cluster size $M$ and different number of channel taps $L$ for $N = 64$, $N_t = 4$, $B = 16$, and SNR at 10 dB.
Fig. 6. Optimal $M^*$ shown with $L$ and $B$ for $N = 64$, $N_t = 4$, and SNR at 10 dB.
Fig. 7. Comparison between capacity obtained from simulation and the analytical upper bound approximation for channel-tap quantization with $N = 64$, $N_t = 3$, and SNR at 10 dB.