Supersymmetric AdS$_4$ black holes and attractors

Sergio L. Cacciatori$^{ac}$ and Dietmar Klemm$^{bc}$

$^a$ Dipartimento di Scienze Fisiche e Matematiche, Università dell’Insubria, Via Valleggio 11, I-22100 Como.
$^b$ Dipartimento di Fisica dell’Università di Milano, Via Celoria 16, I-20133 Milano.
$^c$ INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano.

Abstract: Using the general recipe given in arXiv:0804.0009, where all timelike supersymmetric solutions of $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to abelian vector multiplets were classified, we construct the first examples of genuine supersymmetric black holes in AdS$_4$ with nonconstant scalar fields. This is done for various choices of the prepotential, amongst others for the STU model. These solutions permit to study the BPS attractor flow in AdS. We also determine the most general supersymmetric static near-horizon geometry and obtain the attractor equations in gauged supergravity. As a general feature we find the presence of flat directions in the black hole potential, i.e., generically the values of the moduli on the horizon are not completely specified by the charges. For one of the considered prepotentials, the resulting moduli space is determined explicitly. Still, in all cases, we find that the black hole entropy depends only on the charges, in agreement with the attractor mechanism.

Keywords: Black Holes in String Theory, AdS-CFT Correspondence, Superstring Vacua.
1. Introduction

Since their discovery, the physics of black holes has raised several fascinating problems and puzzles, whose resolution is believed to be crucial for the construction of a future quantum theory of gravity. Indeed, much of what we presently know on quantum effects in strong gravitational fields comes from the study of black holes. Of particular interest in this context are black holes preserving a sufficient amount of supersymmetry, which allows (owing to non-renormalization theorems) to extrapolate a computation of the entropy at weak string coupling (when the system is generically described by a configuration of strings and branes) to the strong-coupling regime, where a description in terms of a black hole is valid [1]. These entropy calculations have been essential for our current understanding of black hole microstates.

In this paper, we shall construct the first examples of genuine BPS black holes in four-dimensional anti-de Sitter space (AdS$_4$) with nontrivial scalar fields turned on. The theory under consideration is $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to abelian vector multiplets, whose timelike supersymmetric backgrounds were classified in [2].

1. Actually we consider also models with scalar potentials that have no critical points. This leads to black holes that asymptote to curved domain walls.
The results of [2] provide a systematic method to construct BPS solutions, without the necessity to guess some suitable ansatze. This facilitates much our analysis here.

The motivation for our interest in supersymmetric AdS black holes is twofold: First, since the discovery of $\mathcal{N} = 6$ Chern-Simons-matter theories [3], which (in a certain limit) are dual to type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$, there has been much interest in supersymmetric geometries that asymptote to $\text{AdS}_4$. In principle, the $\text{AdS}_4/\text{CFT}_3$ correspondence should allow to compute the microscopic entropy of $\text{AdS}_4$ black holes and to compare it then with the macroscopic Bekenstein-Hawking result. This would be very tempting to do for the solutions that we shall present below. The second reason is the attractor mechanism [4–8], which states that the scalar fields on the horizon and the entropy are independent of the asymptotic values of the moduli\(^2\). (The scalars are attracted towards their purely charge-dependent horizon values). Given the importance of the attractor mechanism, it would be very interesting to study the BPS attractor flow in AdS. Some work in this direction has been done in [14, 15]\(^3\), but these papers consider non-supersymmetric attractors, since up to now no BPS black holes in $\text{AdS}_4$ with nonconstant scalars were known\(^4\).

Notice that in gauged supergravity, the moduli fields have a potential, and typically approach the critical points of this potential asymptotically, where the solution approaches AdS. Thus, unless there are flat directions in the scalar potential, the values of the moduli at infinity are completely fixed (in terms of the gauge coupling constants), and it would thus be more precise to state the attractor mechanism in AdS in the form: "The entropy is determined entirely by the charges, and is independent of the values of the moduli on the horizon that are not fixed by the charges". Indeed, we shall encounter below several examples where the scalars on the horizon are not completely specified by the charges, i.e., there are flat directions in the black hole potential and hence a nontrivial moduli space. Yet, in all the cases considered here, the entropy depends only on the charges. Unfortunately, we found no way to prove this in general.

The remainder of this paper is organized as follows: In the next section, we briefly review $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to abelian vector multiplets (presence of $U(1)$ Fayet-Iliopoulos terms), give the general recipe to construct supersymmetric solutions found in [2], and simplify the equations of [2] for the case where there is essentially dependence on one coordinate only. This leads to the analogue of the

\(^2\)Note that this is believed to be the explanation [9] of the fact that the Bekenstein-Hawking entropy of many extremal black holes coincides with a weak coupling calculation [10–13], despite the absence of supersymmetry.

\(^3\)For an analysis of the attractor mechanism in $\mathcal{N} = 2$, $D = 4$ supergravity with $SU(2)$ gauging cf. [16].

\(^4\)The solutions found in [17] are naked singularities.
stabilization equations [6, 7] in AdS space. In section 3 these equations are solved for various prepotentials, amongst others for the STU model. This will lead to a variety of new BPS black hole solutions with nonconstant scalars, whose physical properties are analyzed as well. Generically, these black holes are solitonic in the sense that they have no well-defined limit when the gauge coupling constants go to zero (at least not in an obvious way). Finally, in section 4 we determine the most general supersymmetric static near-horizon geometry and obtain the attractor equations in gauged supergravity. These are then solved in a simple example, and the resulting moduli space for the scalars on the horizon is determined explicitly.

2. Supersymmetric black holes in \( \mathcal{N} = 2, \, D = 4 \) gauged supergravity

We consider \( \mathcal{N} = 2, \, D = 4 \) gauged supergravity coupled to \( n_V \) abelian vector multiplets [18].\(^5\) Apart from the vierbein \( e^a_\mu \), the bosonic field content includes the vectors \( A^I_\mu \) enumerated by \( I = 0, \ldots, n_V \), and the complex scalars \( z^\alpha \) where \( \alpha = 1, \ldots, n_V \). These scalars parametrize a special Kähler manifold, i.e., an \( n_V \)-dimensional Hodge-Kähler manifold that is the base of a symplectic bundle, with the covariantly holomorphic sections

\[
\mathcal{V} = \left( \frac{X^I}{F_I} \right), \quad \mathcal{D}_\bar{\alpha} \mathcal{V} = \partial_\bar{\alpha} \mathcal{V} - \frac{1}{2} (\partial_\bar{\alpha} \mathcal{K}) \mathcal{V} = 0, \tag{2.1}
\]

where \( \mathcal{K} \) is the Kähler potential and \( \mathcal{D} \) denotes the Kähler-covariant derivative. \( \mathcal{V} \) obeys the symplectic constraint

\[
\langle \mathcal{V}, \bar{\mathcal{V}} \rangle = X^I \bar{F}_I - F_I \bar{X}^I = i. \tag{2.2}
\]

To solve this condition, one defines

\[
\mathcal{V} = e^{\mathcal{K}(z, \bar{z})/2} v(z), \tag{2.3}
\]

where \( v(z) \) is a holomorphic symplectic vector,

\[
v(z) = \left( \frac{Z^I(z)}{\partial Z^I F(Z)} \right). \tag{2.4}
\]

\( F \) is a homogeneous function of degree two, called the prepotential, whose existence is assumed to obtain the last expression. The Kähler potential is then

\[
e^{-\mathcal{K}(z, \bar{z})} = -i \langle v, \bar{v} \rangle. \tag{2.5}
\]

\(^5\)Throughout this paper, we use the notations and conventions of [19].
The matrix \( N_{IJ} \) determining the coupling between the scalars \( z^\alpha \) and the vectors \( A^I_\mu \) is defined by the relations

\[
F_I = N_{IJ} X^J, \quad D_\alpha \tilde{F}_I = N_{IJ} D_\alpha \tilde{X}^J.
\]  

(2.6)

The bosonic action reads\(^6\)

\[
e^{-1} L_{\text{bos}} = \frac{1}{16 \pi G} R + \frac{1}{4} (\text{Im } N)_{IJ} F^I_{\mu \nu} F^{J \mu \nu} - \frac{1}{8} (\text{Re } N)_{IJ} e^{-1} \epsilon^{\mu \nu \rho \sigma} F^I_{\mu \nu} F^J_{\rho \sigma} - g_{\alpha \beta} \partial_\mu z^\alpha \partial^\mu \bar{z}^\beta - V,
\]

(2.7)

with the scalar potential

\[
V = -2 g^2 \xi_I \xi_J [(\text{Im } N)^{-1}]_{IJ} + 8 X^I X^J,
\]

(2.8)

that results from U(1) Fayet-Iliopoulos gauging. Here, \( g \) denotes the gauge coupling and the \( \xi_I \) are constants. In what follows, we define \( g_I = g \xi_I \).

The most general timelike supersymmetric background of the theory described above was constructed in [2], and is given by

\[
ds^2 = -4 |b|^2 (dt + \sigma)^2 + |b|^{-2} (dz^2 + e^{2\Phi} dw \, d\bar{w}) ,
\]

(2.9)

where the complex function \( b(z, w, \bar{w}) \), the real function \( \Phi(z, w, \bar{w}) \) and the one-form \( \sigma = \sigma_w dw + \sigma_{\bar{w}} d\bar{w} \), together with the symplectic section \( V \) are determined by the equations

\[
\partial_z \Phi = 2 ig_I \left( \frac{\tilde{X}^I}{b} - \frac{X^I}{b} \right),
\]

(2.10)

\[
4 \partial \bar{\partial} \left( \frac{X^I}{b} - \frac{\tilde{X}^I}{b} \right) + \partial_z \left[ e^{2\Phi} \partial_z \left( \frac{X^I}{b} - \frac{\tilde{X}^I}{b} \right) \right] = 0,
\]

(2.11)

\[
-2 ig_J \partial_z \left\{ e^{2\Phi} \left[ |b|^{-2} (\text{Im } N)^{-1} \right]_{IJ} + 2 \left( \frac{X^J}{b} + \frac{\tilde{X}^J}{b} \right) \left( \frac{X^I}{b} + \frac{\tilde{X}^I}{b} \right) \right\} = 0,
\]

\[
4 \partial \bar{\partial} \left( \frac{F_I}{b} - \frac{\tilde{F}_I}{b} \right) + \partial_z \left[ e^{2\Phi} \partial_z \left( \frac{F_I}{b} - \frac{\tilde{F}_I}{b} \right) \right] = 0,
\]

\[
-2 ig_J \partial_z \left\{ e^{2\Phi} \left[ |b|^{-2} \text{Re } N_{IJ} (\text{Im } N)^{-1} \right]_{IJ} + 2 \left( \frac{F_J}{b} + \frac{\tilde{F}_J}{b} \right) \left( \frac{F_I}{b} + \frac{\tilde{F}_I}{b} \right) \right\}
\]

\[
-8 ig_I e^{2\Phi} \left\{ (\mathcal{I}, \partial_z \mathcal{T}) - \frac{g_J}{|b|^2} \left( \frac{X^J}{b} + \frac{\tilde{X}^J}{b} \right) \right\} = 0,
\]

(2.12)

\( ^6 \)We apologize for using the same letter for the fluxes \( F^I = dA^I \) and the lower part \( F_I \) of the symplectic section \( V \), but the meaning should be clear from the index position.

\( ^7 \)Note that also \( \sigma \) and \( V \) are independent of \( t \).
Here \( \star \) is the Hodge star on the three-dimensional base with metric

\[
ds_3^2 = dz^2 + e^{2\Phi} dwd\bar{w} ,
\]

and we defined \( \partial = \partial_w, \bar{\partial} = \partial_{\bar{w}} \), as well as

\[
\mathcal{I} = \text{Im} \left( \mathcal{V}/\bar{b} \right) .
\]

Given \( b, \Phi, \sigma \) and \( \mathcal{V} \), the fluxes read

\[
F^I = 2(dt + \sigma) \wedge \frac{1}{2} [bX^I + \bar{b}\bar{X}^I] + |b|^{-2} dz \wedge d\bar{w} \left[ \bar{X}^I(\partial \bar{b} + iA_w\bar{b}) + (\mathcal{D}_\alpha X_I)\bar{b}\bar{z}^\alpha - X^I(\bar{\partial}b - iA_{\bar{w}}b) - (\mathcal{D}_{\bar{\alpha}}\bar{X}^I)b\partial z^\alpha \right] + |b|^{-2} dw \wedge d\bar{z} \left[ X^I(\partial b + iA_wb) + (\mathcal{D}_{\alpha}X_I)b\partial z^\alpha - X^I(\partial \bar{b} - iA_{\bar{w}}\bar{b}) - (\mathcal{D}_{\bar{\alpha}}\bar{X}^I)b\bar{\partial} z^\bar{\alpha} \right] - \frac{i}{2} |b|^{-2} e^{2\Phi} dw \wedge d\bar{w} \left[ X^I(\partial \bar{z} + iA_z\bar{z}) + (\mathcal{D}_\alpha X_I)b\partial z^\alpha - X^I(\partial z - iA_zb) - (\mathcal{D}_{\bar{\alpha}}\bar{X}^I)b\bar{\partial} z^\bar{\alpha} - 2ig_J(\text{Im} N)^{-1/2} \right].
\]

In (2.17), \( A_\mu \) is the gauge field of the Kähler U(1),

\[
A_\mu = -i/2 (\partial_\alpha \mathcal{K}_{\mu z^\alpha} - \partial_\alpha \mathcal{K}_{\bar{\mu}} z^\bar{\alpha}).
\]

In order to solve the system (2.10)-(2.14) we shall assume that \( b \) and \( \mathcal{V} \) depend on the coordinate \( z \) only, and use the separation ansatz \( \Phi = \psi(z) + \gamma(w, \bar{w}) \). Furthermore, we are looking for static solutions, i.e., \( \sigma = 0 \). Then (2.14) boils down to

\[
\langle \mathcal{I}, \partial z\mathcal{I} \rangle = |b|^{-2} g_I \left( \frac{\bar{X}^I}{b} + \frac{X^I}{b} \right).
\]

Using this, one can integrate (2.12) once, with the result

\[
e^{2\psi} \partial_z \left( \frac{F^I}{b} - \frac{\bar{F}_I}{b} \right) = -2ig_J e^{2\psi} |b|^{-2}\text{Re} N_{IJL}(\text{Im} N)^{-1/2} L
+ 2 \left( \frac{F^J}{b} + \frac{\bar{F}_J}{b} \right) \left( \frac{X^J}{b} + \frac{\bar{X}_J}{b} \right) = -4\pi i q_I,
\]

\footnote{Whereas in the ungauged case, this base space is flat and thus has trivial holonomy, here we have U(1) holonomy with torsion [2].}
while (2.11) yields
\[
e^{2\psi} \partial_z \left( \frac{X^I}{b} - \frac{\bar{X}^I}{b} \right) - 2ig_J e^{2\psi} \left[ |b|^{-2} (\text{Im} \mathcal{N})^{-1} \right]^{I,J} \\
+ 2 \left( \frac{X^I}{b} + \frac{\bar{X}^I}{b} \right) \left( \frac{X^J}{b} + \frac{\bar{X}^J}{b} \right) = -4\pi ip^I. \tag{2.21}
\]

Here, \( q_I \) and \( p^I \) denote integration constants that will be identified below with the electric and magnetic charge densities respectively. Finally, (2.10) and (2.13) reduce to
\[
\partial_z \psi = 2ig_I \left( \frac{X^I}{b} - \frac{\bar{X}^I}{b} \right), \tag{2.22}
\]
\[-4\partial \bar{\partial} \gamma = \kappa e^{2\gamma}, \quad \kappa = -8\pi gp^I, \tag{2.23}
\]
where we used the contraction of (2.21) with \( g_I \). (2.23) is the Liouville equation and implies that the metric \( e^{2\gamma} d\tau d\bar{\tau} \) has constant curvature \( \kappa \), determined by the magnetic charges \( p^I \). In the following section, we shall solve the system (2.19)-(2.23) explicitly for various prepotentials.

3. Explicit examples

3.1 Prepotential \( F = -iX^0X^1 \)

Let us now solve the above equations for the SU(1,1)/U(1) model with prepotential \( F = -iX^0X^1 \), that has \( n_V = 1 \) (one vector multiplet), and thus just one complex scalar \( \tau \). Choosing \( Z^0 = 1 \), \( Z^1 = \tau \), the symplectic vector \( v \) becomes
\[
v = \begin{pmatrix} 1 \\ \tau \\ -i \tau \\ -i \end{pmatrix}. \tag{3.1}
\]

The Kähler potential, metric and kinetic matrix for the vectors are given respectively by
\[
e^{-K} = 2(\tau + \bar{\tau}), \quad g_{\tau\bar{\tau}} = \partial_\tau \partial_{\bar{\tau}} K = (\tau + \bar{\tau})^{-2}, \tag{3.2}
\]
\[
\mathcal{N} = \begin{pmatrix} -i\tau & 0 \\ 0 & -\frac{i}{\tau} \end{pmatrix}. \tag{3.3}
\]

Note that positivity of the kinetic terms in the action requires \( \text{Re} \tau > 0 \). For the scalar potential one obtains
\[
V = -\frac{4}{\tau + \bar{\tau}} \left( g_0^2 + 2g_0g_1 \tau + 2g_0g_1 \bar{\tau} + g_1^2 \tau \bar{\tau} \right), \tag{3.4}
\]
which has an extremum at $\tau = \bar{\tau} = |g_0/g_1|$. In what follows we assume $g_I > 0$. The Kähler $U(1)$ is

$$A_\mu = \frac{i}{2(\tau + \bar{\tau})} \partial_\mu (\tau - \bar{\tau}) \ . \quad (3.5)$$

In order to solve the system (2.19)-(2.23) we shall take $\tau = \bar{\tau}$ (this includes the extremum of the potential, and thus the AdS vacuum). This implies $\text{Re}N = 0$. Furthermore, we assume that $b$ is imaginary,

$$b = iN(z) \ , \quad N \text{ real} \ . \quad (3.6)$$

Then one has

$$\frac{X^I}{b} + \frac{\bar{X}^I}{b} = \frac{F_I}{b} - \frac{\bar{F}_I}{b} = 0 \ , \quad \langle I, dI \rangle = 0 \ , \quad (3.7)$$

hence (2.19) is trivially satisfied and (2.20) is solved for $q_I = 0$. Defining

$$H^0 = \frac{2X^0}{N} = \frac{1}{N\sqrt{\tau}} \ , \quad H^1 = \frac{2X^1}{N} = \frac{\sqrt{\tau}}{N} \ , \quad (3.8)$$

equ. (2.21) leads to

$$e^{2\psi} \left[ \frac{1}{2} \partial_z H^I + g_I (H^I)^2 \right] = -2\pi p^I \ , \quad \text{no summation over } I \ . \quad (3.9)$$

Inspired by the minimal case [20], we make the ansatz

$$\psi = \ln(az^2 + c) \ , \quad H^I = \frac{\alpha^I z + \beta^I}{az^2 + c} \ , \quad (3.10)$$

with $a, c, \alpha^I, \beta^I \in \mathbb{R}$ constants. Then the remaining equations (2.22) and (3.9) are satisfied iff

$$\alpha^I = \frac{a}{2g_I} \ , \quad \frac{ac}{4g_I} + g_I (\beta^I)^2 + 2\pi p^I = 0 \ , \quad g_I \beta^I = 0 \ . \quad (3.11)$$

Note that these relations imply also

$$g_0 p^0 = g_1 p^1 \ . \quad (3.12)$$

The scalar field and lapse function are respectively given by

$$\tau = \frac{H^1}{H^0} = \frac{g_0}{g_1} \frac{az - 2g_0\beta^0}{az + 2g_0\beta^0} \ , \quad N^2 = (H^0 H^1)^{-1} = \frac{4g_0 g_1 (az^2 + c)^2}{a^2 z^2 - 4(g_0\beta^0)^2} \ . \quad (3.13)$$

In what follows, we shall assume $a > 0$, $c < 0$. Then the solution will have an event horizon at $z = z_h = \sqrt{-c/a}$. The scalar $\tau$ is positive as long as $z > 2|g_0\beta^0|/a$. We want
the dangerous point \( z = 2|g_0\beta^0|/a \) to be hidden behind the horizon, i.e., \( z_h > 2|g_0\beta^0|/a \), which, by using the second relation of (3.11), implies \( p^0 > 0 \). By (3.12) we have then also \( p^1 > 0 \) and thus \( \kappa < 0 \), so that the horizon geometry must be hyperbolic. Notice that for \( \beta^0 = 0 \), the scalar field is constant, and the solution reduces to the one of minimal gauged supergravity discovered in [21].

The above black hole geometry has two scaling symmetries, namely

\[
(t, z, w, a, c, \beta^I, \kappa) \mapsto (t/\lambda, \lambda z, \lambda w, a/\lambda^2, c, \beta^I/\lambda, \kappa/\lambda^2),
\]

and

\[
(t, z, w, a, c, \beta^I, \kappa) \mapsto (t/\lambda, \lambda z, w, a/\lambda, \lambda c, \beta^I, \kappa).
\]

One can use the first to set \( \kappa = -1 \) and then the second (that leaves \( \kappa \) invariant) to choose \( a = 1 \). If we define the parameter \( \nu \) by \( \sinh \nu = 2\sqrt{2}g_0\beta^0 \), the line element reads

\[
ds^2 = -4N^2 dt^2 + \frac{dz^2}{N^2} + \frac{z^2 - \frac{1}{2}\sinh^2 \nu}{4g_0g_1} e^{2\gamma} dw d\bar{w},
\]

where

\[
N^2 = \frac{4g_0g_1(z^2 - \frac{1}{2}\cosh^2 \nu)^2}{z^2 - \frac{1}{2}\sinh^2 \nu}.
\]

The scalar and the fluxes become

\[
\tau = \frac{g_0}{g_1} \frac{z - \frac{1}{\sqrt{2}}\sinh \nu}{z + \frac{1}{\sqrt{2}}\sinh \nu}, \quad F^I = 2\pi i p^I e^{2\gamma} dw \wedge d\bar{w}.
\]

This yields for the magnetic charges \( P^I \)

\[
P^I = \frac{1}{4\pi} \int F^I = p^I V, \quad V \equiv \frac{i}{2} \int e^{2\gamma} dw \wedge d\bar{w},
\]

confirming that the \( p^I \) represent the magnetic charge densities. Note that these obey a Dirac quantization condition: From \( \kappa = -8\pi g_1 p^I = -1 \) and (3.12) we get

\[
p^I = \frac{1}{16\pi g_1},
\]

i.e., the \( p^I \) are quantized in terms of the inverse coupling constants. The entropy density of the one-parameter solution \( (3.16)-(3.18) \) is

\[
s = \frac{S}{V} = 8\pi^2 p^0 p^1.
\]
### 3.2 STU model with $F = -2\sqrt{-X^0X^1X^2X^3}$

Next we consider the STU model with prepotential

$$F = -2\sqrt{-X^0X^1X^2X^3}. \quad (3.22)$$

Choosing $Z^0 = 1$, $Z^1 = \tau^2\tau^3$, $Z^2 = \tau^1\tau^3$, $Z^3 = \tau^1\tau^2$, the symplectic vector $v$ becomes

$$v = (1, \tau^2\tau^3, \tau^1\tau^3, \tau^1\tau^2, -i\tau^1\tau^2\tau^3, -i\tau^1, -i\tau^2, -i\tau^3)^T. \quad (3.23)$$

The Kähler potential and metric are given respectively by

$$e^{-K} = 8 \text{Re} \tau^1 \text{Re} \tau^2 \text{Re} \tau^3, \quad (3.24)$$

$$g_{\alpha\bar{\alpha}} = g_{\alpha\alpha} = (\tau^\alpha + \bar{\tau}^\alpha)^{-2}, \quad \alpha = 1, 2, 3, \quad (3.25)$$

and all other components vanishing. In what follows, we assume $\tau^\alpha$ real and positive. Then the kinetic matrix for the vectors is

$$\mathcal{N} = -i \text{diag}(\tau^1\tau^2\tau^3, \tau^1\tau^2\tau^3, \tau^2, \tau^3, \tau^1\tau^2), \quad (3.26)$$

and hence $\text{Re} \mathcal{N} = 0$. Notice also that

$$(\text{Im} \mathcal{N})^{-1} = -8 \text{diag}((X^0)^2, (X^1)^2, (X^2)^2, (X^3)^2). \quad (3.27)$$

For the scalar potential one obtains

$$V = -4 \left( \frac{g_0g_1}{\tau^1} + g_2g_3\tau^1 + \frac{g_0g_2}{\tau^2} + g_1g_3\tau^2 + \frac{g_0g_3}{\tau^3} + g_1g_2\tau^3 \right), \quad (3.28)$$

which has an extremum at

$$\tau^1 = \left( \frac{g_0g_1}{g_2g_3} \right)^{1/2}, \quad \tau^2 = \left( \frac{g_0g_2}{g_1g_3} \right)^{1/2}, \quad \tau^3 = \left( \frac{g_0g_3}{g_1g_2} \right)^{1/2}. \quad (3.29)$$

Note that for all $g_I$ equal, this model can be embedded into $\mathcal{N} = 8$ gauged supergravity as well [22].

In order to solve the equations (2.19)-(2.22), we use once more the assumption $b = iN(z)$ for $N$ real. Since the $F_I$ are imaginary and $X^I$ real, (2.19) is identically satisfied, (2.20) gives $q_I = 0$ and (2.21) simplifies to

$$e^{2\psi} \left[ \partial_z H^I + 8g_I(H^I)^2 \right] = -2\pi p^I, \quad \text{no summation over} \ I, \quad (3.29)$$

where $H^I \equiv X^I/N$. As before, we employ the ansatz (3.10) for the functions $\psi$ and $H^I$. Then (2.22) and (3.29) are fulfilled if the following constraints hold:

$$\alpha^I = \frac{a}{8g_I}, \quad \frac{ac}{8g_I} + 8g_I(\beta^I)^2 + 2\pi p^I = 0, \quad g_I\beta^I = 0. \quad (3.30)$$
The scalars fields and the lapse function read

$$
\tau^\alpha = \frac{1}{8X^\alpha} = \left(\frac{H^0H^1H^2H^3}{H^0}\right)^{1/2},
$$
\(3.31\)

$$
N^2 = \frac{1}{8}(H^0H^1H^2H^3)^{-1/2} = \frac{(az^2 + c)^2}{8\prod_I(\alpha^Iz + \beta^I)^{1/2}}.
$$
\(3.32\)

For the line element and the fluxes one gets respectively

$$
ds^2 = -4N^2dt^2 + \frac{dz^2}{N^2} + 8\prod_{I=0}^3(\alpha^Iz + \beta^I)^{1/2}e^{2\gamma}dwd\bar{w},
$$
\(3.33\)

$$
F^I = 2\pi ip^Ie^{2\gamma}dw \wedge d\bar{w},
$$
\(3.34\)

so that the \(p^I\) represent again the magnetic charge densities. In what follows, we shall assume \(g_I > 0, a > 0\) (and thus \(\alpha^I > 0\) by \(3.30\)), as well as \(c < 0\), so that there is a horizon at \(z = z_h = \sqrt{-c/a}\). The entropy density can be written in the form

$$
\frac{S}{V} = 2\prod_{I=0}^3 \left(\beta^I + \sqrt{\frac{\pi p^I}{4g_I} + (\beta^I)^2}\right)^{1/2}.
$$
\(3.35\)

The solution is again invariant under the scaling symmetries \(3.14\), \(3.15\) that allow to set \(a = 1, \kappa = 0, \pm 1\) without loss of generality.

We must ensure that the moduli \(\tau^\alpha\) be positive in the whole region outside the horizon. This is guaranteed if \(z_h > -\beta^I/\alpha^I \forall I\). A sufficient condition for this is

$$
-c > (\beta^I/\alpha^I)^2,
$$

which, by using the second relation of \(3.30\), yields \(p^I > 0\) and thus \(\kappa < 0\), so that the horizon geometry is hyperbolic in this case. But the condition

$$
-c > (\beta^I/\alpha^I)^2
$$

is not necessary in general and for suitable choices of the parameters, other geometries are allowed. The constraint \(g_I\beta^I = 0\) with \(g_I > 0\) shows that at least one of the \(\beta^I\) must be negative. It is easy to show that if only one of them is negative, then necessarily \(\kappa < 0\). Indeed, let us assume to be \(\beta^0\) the only negative coefficient. Then, we need to impose only one condition, which is equivalent to

$$
-c > (\beta^0)^2/(\alpha^0)^2 = 64(g_0}\beta^0)^2.
$$
\(3.36\)

Then

$$
-\frac{\kappa}{4} = 2\pi g_0p^I = -\frac{c}{2} - 8\sum_I g_0^2(\beta^I)^2 > -\frac{c}{2} - 8g_0^2(\beta^0)^2 - 8(\sum_{i=1}^3 g_i\beta^i)^2 \\
= -\frac{c}{2} - 16g_0^2(\beta^0)^2 > 16g_0^2(\beta^0)^2 > 0.
$$
\(3.37\)
Let us consider the opposite situation, when $\beta^0 > 0$ is positive and assume for simplicity $g_i = g$, $\beta^i = -\beta$, for $i = 1, 2, 3$ and $g$ and $\beta$ positive. Then $g_0\beta^0 = 3g\beta$. The singularities are then hidden by the horizon if and only if $c + 64g^2\beta^2 < 0$. Now

$$\frac{\kappa}{4} = -2\pi g_1 p^I = \frac{c}{2} + 8 \sum_I g^2_I (\beta^I)^2 = \frac{1}{2}(c + 192g^2\beta^2).$$

Then, a spherical topology ($\kappa = 1$) for the horizon is admitted if, in this example, the parameters satisfy

$$-c = 192g^2\beta^2 - \frac{1}{2}.$$  \hspace{1cm} (3.39)

Combining this with $-c > 64g^2\beta^2$ yields $16g\beta > 1$. A flat horizon ($\kappa = 0$) appears for $-c = 192g^2\beta^2$.

In order to say more about the allowed horizon geometries, we shall now solve the equations (3.30) systematically. To this end, note that once we fix $g_I$ and the charge densities $p^I$ (taking into account the condition $8\pi g_1 p^I = -\kappa$), all other parameters are generically determined by the relations (3.30). Indeed, defining $y_I = g_I\beta^I$ (no summation over $I$), and using $\alpha^I = 1/(8g_I)$ together with

$$c = \frac{\kappa}{2} - 16 \sum_I g^2_I (\beta^I)^2,$$

one is left with four equations for the four unknowns $y_I$:

$$3y^2_1 - y^2_2 - y^2_3 - y^2_0 = -\phi_1,$$

$$-y^2_1 + 3y^2_2 - y^2_3 - y^2_0 = -\phi_2,$$

$$-y^2_1 - y^2_2 + 3y^2_3 - y^2_0 = -\phi_3,$$

$$y_0 + y_1 + y_2 + y_3 = 0,$$

where $\phi_I = \frac{\kappa}{32} + \pi g_1 p^I$ (no summation over $I$) satisfying $\sum_I \phi_I = 0$. The solutions are the intersections between three hypersurfaces of degree 2 in $\mathbb{R}^4$ and a hyperplane, so that we expect generically a maximum of eight isolated points. However, it can happen that these four hypersurfaces do not intersect transversally so that there is a higher-dimensional intersection. This happens when the determinant of gradients vanishes,

$$0 = \det \begin{pmatrix} 3y_1 & -y_2 & -y_3 & -y_0 \\ -y_1 & 3y_2 & -y_3 & -y_0 \\ -y_1 & -y_2 & 3y_3 & -y_0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 16(y_1y_2y_3 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3).$$  \hspace{1cm} (3.45)

- 11 -
Using the hyperplane equation, the degeneracy condition takes the form

\[ 0 = (y_0y_3 - y_1y_2)(y_0 + y_3) = (y_0 + y_3)\det \begin{pmatrix} y_0 & y_1 \\ y_2 & y_3 \end{pmatrix}. \]  

(3.46)

Then, either \( y_0 + y_3 = y_1 + y_2 = 0 \), or there exists some \( \lambda \in \mathbb{R} \) such that \( (y_0, y_1) = \lambda(y_2, y_3) \). Inserted into the hyperplane equation this gives

\[ (1 + \lambda)(y_2 + y_3) = 0, \]

(3.47)

which yields \( y_2 + y_3 = y_0 + y_1 = 0 \) or \( \lambda = -1 \). In conclusion, we see that the degeneracy conditions are equivalent to

\[ y_{\sigma(1)} + y_{\sigma(2)} = y_{\sigma(3)} + y_{\sigma(0)} = 0, \]

(3.48)

where \( \sigma \) is a fixed element in the symmetric group \( S_4 \). We see that (3.48) is compatible with the hyperplane equation. However, the degeneracy appears only when it is compatible with the whole system. Substituting into the system we see that this happens when

\[ \phi_{\sigma(1)} = \phi_{\sigma(2)} = -\phi_{\sigma(3)} = -\phi_{\sigma(0)}, \]

(3.49)

which means

\[ g_{\sigma(1)}p_{\sigma(1)} = g_{\sigma(2)}p_{\sigma(2)}, \quad g_{\sigma(0)}p_{\sigma(0)} = g_{\sigma(3)}p_{\sigma(3)}, \quad g_{\sigma(1)}p_{\sigma(1)} = -g_{\sigma(0)}p_{\sigma(0)} + \frac{1}{16\pi}. \]

(3.50)

When these conditions are satisfied, we see that a free parameter is left by (3.30), let us say \( \beta_0 \). Note that the conditions imply \( \kappa = -1 \). The entropy density (3.35) boils down to

\[ \frac{S}{V} = \frac{\pi}{2} \sqrt{\prod_{i=0}^{3} \frac{p_{I}}{g_{I}}}, \]

(3.51)

which is thus independent of the free parameter.

Let us now study the isolated points. We will see that these allow for solutions with \( \kappa \geq 0 \). To this aim let us set \( Z_I = y_I^2, \ I = 0, 1, 2, 3 \). Then the equations (3.41), (3.42) and (3.43) can be solved for \( Z_i, \ i = 1, 2, 3 \) as functions of \( Z_0 \):

\[ Z_i = Z_0 + \frac{1}{4}(\phi_0 - \phi_i). \]

(3.52)

---

There is a possible degeneracy for each choice of \( \sigma \), however only three of them are indeed distinct.
Equation (3.44) becomes

$$\sqrt{Z_0} + \sigma_1 \sqrt{Z_1} = \sigma_2 \sqrt{Z_2} + \sigma_3 \sqrt{Z_3},$$  \hspace{1cm} (3.53)

where $\sigma_i$ are signs. Taking the square of this relation and using (3.52) we get\(^\text{10}\)

$$2\sigma_2\sigma_3 \sqrt{Z_2 Z_3} - 2\sigma_1 \sqrt{Z_0 Z_1} = Z_0 + Z_1 - Z_2 - Z_3 = -\frac{1}{2}(\phi_2 + \phi_3),$$  \hspace{1cm} (3.54)

that is

$$\sigma_1 \sigma_2 \sigma_3 \sqrt{Z_2 Z_3} = \sqrt{Z_0 Z_1} - \frac{1}{4} \sigma_1 (\phi_2 + \phi_3).$$  \hspace{1cm} (3.55)

Squaring this and making use of

$$Z_2 Z_3 - Z_0 Z_1 = \frac{1}{16} (\phi_0 - \phi_2)(\phi_0 - \phi_3) - \frac{1}{2} Z_0 (\phi_2 + \phi_3),$$  \hspace{1cm} (3.56)

one obtains

$$\sqrt{Z_0 Z_1} = -Z_0 + \frac{(\phi_0 - \phi_2)(\phi_0 - \phi_3)}{(\phi_2 + \phi_3)} - \frac{1}{8} (\phi_2 + \phi_3).$$  \hspace{1cm} (3.57)

Here we assumed $\phi_2 + \phi_3 \neq 0$, otherwise we fall in the degenerate case. Taking the square and using the expression for $Z_1$ as a function of $Z_0$ we finally get

$$Z_I = -\frac{(4\phi_I^2 - \sum_{J=0}^3 \phi_J^2)^2}{64 \prod_{J \neq I} (\phi_J + \phi_I)}.$$  \hspace{1cm} (3.58)

The acceptable solutions are the ones satisfying the condition

$$\prod_{J \neq I} (\phi_J + \phi_I) < 0.$$  \hspace{1cm} (3.59)

Note that this condition is the same for all $I$: indeed $\prod_{J \neq I} (\phi_J + \phi_I)$ does not depend on $I$ because of the identity $\sum J \phi_J = 0$. Then

$$y_I = \frac{\sigma_I (4\phi_I^2 - \sum_{J=0}^3 \phi_J^2)}{8 \sqrt{- \prod_{J \neq I} (\phi_J + \phi_I)}},$$  \hspace{1cm} (3.60)

for certain signs $\sigma_I$. Notice that the denominator is the same for all $Z_I$. By inspection we see that all signs must be equal as well. Indeed, if this is the case, we immediately see that (3.44) is satisfied. Moreover, if $\{Y_I\}$ is a solution then $\{-Y_I\}$ is another

\(^{10}\)We often use tacitly the relation $\phi_0 + \ldots + \phi_3 = 0$. 

\[\text{– 13 –}\]
solution and we can consider only the cases when there is just one negative sign or two negative signs. In the first case, without loss of generality, we can assume that only $y_3$ is negative, so that $\sigma_3 = -1$ and $\sigma_i = 1$ for $i \neq 3$. Plugging (3.60) into (3.44) we get $3\phi_2^2 - \phi_0^2 - \phi_3^2 - \phi_2^2 = 0$ which is equivalent to $y_3 = 0$ and then there is not a real different choice of the sign.

Doing the same for the case of two negative signs, for example $\sigma_2 = \sigma_3 = -1$, we find $\phi_0^2 + \phi_1^2 - \phi_2^2 - \phi_3^2 = 0$ that is equivalent to $0 = \phi_2\phi_3 - \phi_1\phi_0$ which is the degeneracy condition. Then we conclude that

$$y_I = \pm \frac{4\phi_I^2 - \sum_{j=0}^{3} \phi_j^2}{8\sqrt{-\prod_{j \neq I} (\phi_I + \phi_j)},}$$

(3.61)

where the sign is the same for all $I$.

Let us suppose to have chosen the charges and the coupling constants such that $\sum_I g_I p^I = -\kappa/8\pi$. Then, the consistency condition (3.59) takes the form

$$\left(\frac{\kappa}{16\pi} + (g_2 p^2)^2 + (g_3 p^3)^2\right) \left(\frac{\kappa}{16\pi} + (g_1 p^1)^2 + (g_3 p^3)^2\right) \left(\frac{\kappa}{16\pi} + (g_1 p^1)^2 + (g_2 p^2)^2\right) > 0.$$

Note that a general choice of the possible parameters does not allow for a horizon. The existence condition requires $y_I > 0$ when $p_I < 0$.

As an example, let us consider the case $p_i > 0$. Then, we can look for solutions with $\kappa \geq 0$. As $c$ must be negative to have a horizon, we see from (3.40) that

$$0 \leq \kappa < 32 \sum_{i=0}^{3} y_i^2.$$

(3.62)

The solutions are given by (3.61). As $p^0 = -p^1 - p^2 - p^3 - \kappa/8\pi$ is the only negative charge, we have to be careful with the sign of $y_0$ only. But $\phi_i > 0$ so that $3(\sum_{i=1}^{3} \phi_i)^2 - \sum_{i=1}^{3} \phi_i^2 > 0$ and then $y_0$ is positive if we chose the plus sign in (3.61). This provides the desired solution.

Notice that, if all $g_I$ are equal, the 4-charge black holes found in this subsection can be uplifted to 11 dimensions along the lines of [23]. That might be interesting to do.

### 3.3 STU model with $F = -X^1X^2X^3/X^0$

Another interesting model is the one with prepotential $F = -X^1X^2X^3/X^0$. In the ungauged case, this is related by a symplectic transformation to the model with $F = -2(-X^0X^1X^2X^3)^{1/2}$ considered above [15]. However, in the presence of gauging, symplectic covariance is broken, so that this prepotential will lead to different physics.
Choosing $Z^0 = 1$, $Z^\alpha = i\tau^\alpha$, $\alpha = 1, 2, 3$, the symplectic vector $v$ becomes

$$v = (1, i\tau^1, i\tau^2, i\tau^3, -i\tau^1\tau^2\tau^3, \tau^2\tau^3, \tau^1\tau^3, \tau^1\tau^2)^T. \quad (3.63)$$

The Kähler potential and metric are again given by (3.24) and (3.25) respectively. In the following, we assume $\tau^\alpha$ real and positive ("vanishing axions" condition). Then the kinetic matrix for the vectors is

$$\mathcal{N} = -i \text{diag}(\tau^1\tau^2\tau^3, \frac{\tau^2\tau^3}{\tau^1}, \frac{\tau^1\tau^3}{\tau^2}, \frac{\tau^1\tau^2}{\tau^3}), \quad (3.64)$$

and thus $\text{Re}\mathcal{N} = 0$. Notice also that

$$(\text{Im}\mathcal{N})^{-1} = 8 \text{diag}(-(X^0)^2, (X^1)^2, (X^2)^2, (X^3)^2). \quad (3.65)$$

For the scalar potential one gets

$$V = -4 \left( \frac{g_1 g_3}{\tau^1} + \frac{g_1 g_3}{\tau^2} + \frac{g_1 g_2}{\tau^3} \right), \quad (3.66)$$

which has no critical point, so that there are no AdS$_4$ vacua with constant moduli.

In what follows we choose $b$ real. Since $X^0$ and $F_\alpha$ ($\alpha = 1, 2, 3$) are real as well, and $F_0, X^\alpha$ are imaginary, one has

$$\frac{X^\alpha}{b} + \bar{X}^\alpha = \frac{X^0}{b} - \bar{X}^0 = 0, \quad \frac{F_\alpha}{b} - \bar{F}_\alpha = \frac{F_0}{b} + \bar{F}_0 = 0, \quad (3.67)$$

and hence $\langle I, dI \rangle = 0$. If we make the choice $g_0 = 0^{11}$, equ. (2.19) holds identically. (2.20) for $I = \alpha$ and (2.21) for $I = 0$ are also satisfied for $q_\alpha = p^0 = 0$. On the other hand, defining $H^\alpha = X^\alpha/b$, $H^0 = 1/(X^0b)$, the remaining equations of (2.20) and (2.21) boil down to

$$e^{2\psi} \partial_2 H^0 = 16\pi q_0, \quad (3.68)$$

$$e^{2\psi} \left[ \partial_2 H^\alpha - 8i g_\alpha (H^\alpha)^2 \right] = -2\pi ip^\alpha, \quad \text{no summation over } \alpha, \quad (3.69)$$

whereas (2.22) gives

$$\partial_2 \psi = -4i g_\alpha H^\alpha. \quad (3.70)$$

Plugging (3.70) into (3.69) yields

$$e^{2\psi} \left[ \psi'' + \frac{2}{3} \psi'^2 \right] = -24\pi gp, \quad (3.71)$$

$^{11}$Note that this does not affect the scalar potential (3.66).
where the prime indicates a derivative w.r.t. $z$, and we made the further assumption $g_\alpha p^\alpha = g p$ and $g_\alpha H^\alpha = g H$ for $\alpha = 1, 2, 3$ (no summation over $\alpha$). This is equivalent to taking $g_\alpha \tau^\alpha = g \tau$, $\alpha = 1, 2, 3$. Setting $y = e^{2\psi/3}$, (3.71) can be rewritten as

$$y'' = -\frac{16\pi gp}{y^2}, \quad (3.72)$$

and thus

$$y' = \pm \sqrt{C + \frac{32\pi gp}{y}}, \quad (3.73)$$

with $C$ an integration constant. We can chose the upper sign, the other one corresponding to the inversion $z \leftrightarrow -z$. From this equation we see that the relation between $z$ and $y$ is monotonic and we can use $y$ in place of $z$ as a new coordinate. Then (3.69) takes the form

$$\sqrt{C + \frac{32\pi gp}{y}} \frac{dH}{dy} - 8igH^2 = -2\pi i \frac{p}{y^3}. \quad (3.74)$$

This is a Riccati equation with particular solution

$$\tilde{H}(y) = \frac{i}{8gy} \sqrt{C + \frac{32\pi gp}{y}}. \quad (3.75)$$

The general solution is then

$$H(y) = \tilde{H}(y) + \frac{iK}{y^2 \left[ 1 - \frac{K}{2\pi p} \sqrt{C + \frac{32\pi gp}{y}} \right]}, \quad (3.76)$$

where $K$ denotes another integration constant. However, using $\psi = \frac{3}{2} \ln y$ and (3.70) we see that $K = 0$ so that

$$H^\alpha(y) = \frac{i}{8g_{\alpha}y} \sqrt{C + \frac{32\pi gp}{y}}. \quad (3.77)$$

In the same way we can solve (3.68), with the result

$$H^0(y) = \frac{q_0}{32\pi g^2 p^2} \sqrt{C + \frac{32\pi gp}{y}} \left( \frac{2}{3} C - \frac{32\pi gp}{3y} \right) + h^0. \quad (3.78)$$

The function $b$ is given by

$$b^4 = -i(8H^0H^1H^2H^3)^{-1} = \frac{64g_1g_2g_3y^3}{H^0(y) \left( C + \frac{32\pi gp}{y} \right)^{3/2}}. \quad (3.79)$$
The scalar fields $\tau^\alpha$ are obtained from $X^0 = 1/(H^0 b) = e^{X/2}$, yielding

$$\tau^\alpha = \sqrt{g_1 g_2 g_3} \left| \frac{y H^0}{2} \right|^\frac{1}{2} \frac{g_\alpha}{g_\alpha + \frac{32 \pi g p}{y}}.$$  

(3.80)

For the metric and the fluxes one gets

$$ds^2 = -4b^2 dt^2 + b^{-2} \left( \frac{dy^2}{C + \frac{32 \pi g p}{y}} + y^3 e^{2\gamma} dw d\bar{w} \right),$$  

(3.81)

$$F^0 = 4 dt \wedge d(H^0)^{-1}, \quad F^\alpha = 2\pi i p^\alpha e^{2\gamma} dw \wedge d\bar{w},$$  

(3.82)

where $\gamma$ satisfies

$$\partial \bar{\partial} \gamma = 6 \pi g p e^{2\gamma}.$$  

(3.83)

The solution carries thus one electric and three magnetic charges, namely

$$Q_0 = q_0 V, \quad P^\alpha = p^\alpha V,$$  

(3.84)

with $V$ given in (3.19). Note that $e^{2\psi} > 0$ corresponds to $y > 0$. Then, there is an event horizon at $y = 0$, if $p > 0$; and thus (cf. (2.23)) $\kappa = -24 \pi g p < 0$, so that the horizon is hyperbolic. If we assume also $C \geq 0$, $q_0 < 0$, and $h^0 > |q_0| C^{3/2}/(48 \pi g^2 p^2)$, the scalar fields are real and positive for $0 \leq y < \infty$. The values of the scalars on the horizon and the entropy are

$$\tau^\alpha_{\text{hor}} = \sqrt{g_1 g_2 g_3} \left| \frac{q_0}{3 gp} \right|^{1/2} \left( \frac{g_1 g_2 g_3}{g_\alpha} \right)^{1/3} \frac{|Q_0/3|^{1/2}}{(P^1 P^2 P^3)^{1/6}},$$  

(3.85)

$$S = \pi V \left( \frac{|q_0| gp}{3 g_1 g_2 g_3} \right)^{1/2} = \frac{\pi |Q_0/3|^{1/2}}{(g_1 g_2 g_3)^{1/3}}.$$

(3.86)

We see that in this case both $\tau^\alpha_{\text{hor}}$ and $S$ depend on the charges only. The solution (3.81) interpolates between AdS$_2 \times \mathbb{H}^2$ near the horizon and a curved domain wall for $y \to \infty$. The worldvolume of the domain wall is the open Einstein static universe $\mathbb{R} \times \mathbb{H}^2$.

### 3.4 Prepotential $F = \frac{i}{4} X^I \eta_{IJ} X^J$

Let us now consider the SU$(1,n)/(U(1) \times \text{SU}(n))$ model with prepotential $F = \frac{i}{4} X^I \eta_{IJ} X^J$, where the scalar fields are $X^I$, $I = 0, 1, \ldots, n$ and $\eta_{IJ} = \text{diag}(-1, 1, \ldots, 1)$. There are

\[ ^{12}\text{We have chosen also } g > 0. \]
\( n_V = n \) vector multiplets and thus \( n \) complex scalars. If we choose \( Z^0 = 1 \) and \( Z^i = \tau^i \), \( i = 1, \ldots, n \), the symplectic vector becomes

\[
v = \left( 1, \tau^1, \ldots, \tau^n, -\frac{i}{2} \tau^1, \ldots, -\frac{i}{2} \tau^n \right)^T .
\]

(3.87)

The Kähler potential and metric are

\[
e^{-K} = 1 - |\vec{\tau}|^2 ,
\]

(3.88)

\[
g_{ij} = \frac{\delta_{ij}}{1 - |\vec{\tau}|^2} + \frac{\bar{\tau}^i \tau^j}{(1 - |\vec{\tau}|^2)^2} ,
\]

(3.89)

where \( |\vec{\tau}|^2 = \sum_{i=1}^{n} \bar{\tau}^i \tau^i \). Positivity is ensured by \( |\vec{\tau}|^2 < 1 \). In particular

\[
X^0 = \frac{1}{\sqrt{1 - |\vec{\tau}|^2}} , \quad X^i = \frac{\tau^i}{\sqrt{1 - |\vec{\tau}|^2}}
\]

(3.90)

satisfy \( \bar{X}^I \eta_{IJ} X^J = -1 \). Defining \( X_I = \eta_{IJ} X^J \), the kinetic matrix for the vectors is

\[
\mathcal{N}_{IJ} = -\frac{i}{2} \eta_{IJ} - i X_I X_J .
\]

(3.91)

Let us now look for a solution with

\[
\tau^i = \tau^i(z) , \quad b = i N(z) ,
\]

(3.92)

with \( N(z) \) and \( \tau^i(z) \) all real and positive. Then \( \eta_{IJ} X^I X^J = -1 \) so that we get

\[
(\text{Im} \mathcal{N})^{-1/2} = -2(\eta^{IJ} + 2 X^I X^J) ,
\]

(3.93)

and the scalar potential reads

\[
V = 8g^2 - 16 \frac{(g_0 + \bar{g} \cdot \bar{\tau})^2}{1 - |\vec{\tau}|^2} ,
\]

(3.94)

where \( g^2 = g^I g_I \), \( g^I = \eta^{IJ} g_J \) and \( \bar{g} \cdot \bar{\tau} = g_i \tau^i \). (3.94) has an extremum for \( \tau^i = -g_i/g_0 \), with \( V|_{\text{extr.}} = 24g^2 \). In order to have a supersymmetric AdS vacuum, this must be negative, so that \( g_I \) is timelike, \( g^2 < 0 \).

Since

\[
\text{Re} \mathcal{N} = 0 , \quad \frac{X^I}{b} + \frac{\bar{X}^I}{b} = 0 , \quad \frac{F_I}{b} - \frac{\bar{F}_I}{b} = 0 ,
\]

(3.95)

(2.20) is satisfied for \( q_I = 0 \). Moreover, \( \langle I, dI \rangle = 0 \), hence (2.19) holds as well. If we define \( H^I \equiv X^I/N \), (2.21) boils down to

\[
e^{2\psi}[\partial_z H^I + 2g_J(-H^2 \eta^{IJ} + 2H^I H^J)] = -2\pi p^I ,
\]

(3.96)
with \(H^2 = \eta_{IJ}H^I H^J = -1/N^2\). Making use of the ansatz

\[
\psi = \ln(az^2 + c), \quad H^I = \frac{\alpha^I z + \beta^I}{az^2 + c} \tag{3.97}
\]

in eqns. (3.90) and (2.22), one obtains the set of relations

\[
a = 2g_I \alpha^I, \quad \alpha^I = \frac{2g^I \alpha^2}{a}, \quad \alpha^I c - 2g^I \beta^2 = -2\pi p^I, \quad g_I \beta^I = 0, \tag{3.98}
\]

where \(\alpha^2 = \eta_{IJ} \alpha^I \alpha^J\) and similar for \(\beta^2\). Thus

\[
N^2 = -\frac{(az^2 + c)^2}{\alpha^2 z^2 + \beta^2}, \tag{3.99}
\]

which to be positive for large \(z\) requires \(\alpha^2 < 0\). This is indeed satisfied, since \(\alpha^I\) is proportional to \(g^I\), and the latter is timelike. The spacetime metric is

\[
ds^2 = -4N^2 dt^2 + \frac{dz^2}{N^2} + (-\alpha^2 z - \beta^2) e^{2\gamma} dw d\bar{w}, \tag{3.100}
\]

while the scalar fields and fluxes read

\[
\tau^i = \frac{\alpha^i z + \beta^i}{\alpha^0 z + \beta^0}, \quad F^I = 2\pi i p^I e^{2\gamma} dw \wedge d\bar{w}, \tag{3.101}
\]

so that the magnetic charges are \(P^I = p^I V\), with \(V\) given in (3.13). Note that asymptotically for \(z \to \infty\), we have \(\tau^i \to -g_i/g_0\), where the scalar potential becomes extremal. The spacetime approaches AdS\(_4\) in this limit. We assume \(a > 0\) and \(c < 0\) so that there is an event horizon at \(z_h = \sqrt{-c/a}\). In what follows, we shall again use the scaling symmetries (3.14), (3.15) to set \(a = 1\) and \(\kappa = 0, \pm 1\) without loss of generality. This implies then \(\alpha^I = g^I/(2g^2)\).

It is easy to show that the positivity condition \(\bar{\tau}^2 < 1\) for the kinetic terms of the scalars is equivalent to \(\alpha^2 z^2 + \beta^2 < 0\). From (3.99) one sees that this coincides with the condition of having a positive lapse function in the region outside the horizon. If \(\beta^2 < 0\), this is always satisfied. Using

\[
\frac{\kappa}{4} = \frac{c}{2} - 2g^2 \beta^2, \tag{3.102}
\]

that follows from the third equation of (3.98) by contracting with \(g_I\), one sees that \(\kappa < 0\) in this case. When \(\beta^2 \geq 0\) we have a singularity at \(z_s = (-\beta^2/\alpha^2)^{1/2}\), where \(N^2\) diverges and \(\bar{\tau}^2 = 1\). Requiring this singularity to be hidden by the horizon \((z_h > z_s)\) leads to \(-c > -4g^2 \beta^2\). Plugging this into (3.102) yields again \(\kappa < 0\). Thus the geometry of the horizon is always hyperbolic. Finally, taking into account that (3.98) imply \(p^I = g^I/(8\pi g^2)\), we get for the entropy density

\[
\frac{S}{V} = -2\pi^2 p^2, \tag{3.103}
\]

which depends on the charges only (and is positive since \(p^2 = \eta_{IJ} p^I p^J < 0\)).
4. General near-horizon analysis

We now want to analyze the near-horizon limit of a general static supersymmetric black hole solution of the theory under consideration. This will be done without specifying a prepotential, with the aim to obtain the analogue of the attractor equations in gauged supergravity.

As we are interested in the near horizon limit, the scalar fields are taken to be constant. In order to get a spacetime geometry of the product form $\text{AdS}_2 \times \Sigma$, where $\Sigma$ denotes a two-dimensional space with constant curvature, i.e., $S^2$, $\mathbb{R}^2$, $H^2$ or a compact quotient thereof, we must have $|b|^{-1}e^{\psi} = c$, with $c$ an arbitrary positive constant. Using this, (2.13) and (2.22) can be easily integrated, with the result

$$b = 4ig_I\bar{X}^I z + b_0,$$

(4.1)

where $b_0$ denotes a complex integration constant. (2.20) and (2.21) lead respectively to

$$q_I = \frac{c^2}{2\pi} \left[ 4\text{Re}(F_{IJ}g_J) + g_J\text{Re}\mathcal{N}_{IJ}(\text{Im}\mathcal{N})^{-1|JL} \right],$$

(4.2)

$$p^I = \frac{c^2}{2\pi} \left[ 4\text{Re}(X^I g_J\bar{X}^J) + g_J(\text{Im}\mathcal{N})^{-1|J} \right].$$

(4.3)

Note that this can be written more compactly as

$$\left(\begin{array}{c} p^I \\ q_I \end{array}\right) = \frac{c^2}{2\pi} \left[ 4\text{Re}(\mathcal{V}g_J\bar{X}^J) + \mathcal{M}\Omega \mathcal{G} \right],$$

(4.4)

where $\mathcal{M}$ is the matrix introduced in [24],

$$\mathcal{M} = \left(\begin{array}{cc} (\text{Im}\mathcal{N})^{-1} & (\text{Im}\mathcal{N})^{-1}\text{Re}\mathcal{N} \\ \text{Re}\mathcal{N}(\text{Im}\mathcal{N})^{-1} & \text{Re}\mathcal{N}(\text{Im}\mathcal{N})^{-1}\text{Re}\mathcal{N} \end{array}\right),$$

(4.5)

$\Omega$ denotes the symplectic metric,

$$\Omega = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right),$$

(4.6)

and we defined $\mathcal{G} = (g_J, 0)^T$.

Then the fluxes are given by

$$F^I = -16\text{Im}(X^I g_J\bar{X}^J) dt \wedge dz + 2\pi i p^I e^{2\gamma} dw \wedge d\bar{w}.$$  

(4.7)

The magnetic and electric charges read respectively

$$P^I = \frac{1}{4\pi} \int_{\Sigma_\infty} F^I = p^I V, \quad Q_I = \frac{1}{4\pi} \int_{\Sigma_\infty} G_I = q_I V,$$

(4.8)
where $G_{IJ} = N_{IJ} F^{+J}$ [19] and $V$ is given in (3.19). By using some relations of special geometry, one obtains for the central charge

$$Z = X^I Q_I - F_I P_I = \frac{V c^2}{2\pi} i g_{ij} X^J.$$  \hfill (4.9)

Before we continue, a small digression on the constant $c$ introduced above is in order. From (2.22) it is clear that $\psi$ is defined only up to an arbitrary constant, so that we are free to shift

$$\psi \rightarrow \psi - \ln \lambda,$$  \hfill (4.10)

which implies $c \rightarrow c/\lambda$, so that one can set $c$ equal to any value. In order for $\Phi$ to be invariant, (4.10) must be compensated by a shift in $\gamma$,

$$\gamma \rightarrow \gamma + \ln \lambda.$$  \hfill (4.11)

From the Liouville equation (2.23) we see that the magnetic charge densities scale as $p^I \rightarrow p^I/\lambda^2$. Using (4.11) in (3.19), one gets $V \rightarrow \lambda^2 V$, so that the product $c^2 V$, and thus the charges $P^I$, $Q_I$, and $Z$, remain invariant, as it must be.

Finally, the entropy of the black hole with this near-horizon geometry is

$$S = \frac{A_{\text{hor}}}{4} = \frac{c^2 V}{4}.$$  \hfill (4.12)

Using the expression (4.9) for the central charge, this can be rewritten as

$$S = \frac{\pi Z}{2 i g_{ij} X^J}.$$  \hfill (4.13)

Multiplying (4.4) with $V$ and eliminating $c^2 V$ by means of (4.3), one gets

$$\left( \begin{array}{c} P_I \\ Q_I \end{array} \right) = \frac{Z}{i g_{ij} X^J} \left[ 4 \text{Re}(\mathcal{V} g_K \bar{X}^K) + \mathcal{M} \Omega \mathcal{G} \right],$$  \hfill (4.14)

which represents the analogue of the attractor equations [6,7] for static supersymmetric black holes in gauged supergravity. If one were able to solve (4.14) in order to obtain the moduli in terms of the charges, one could plug the result into (4.13) to show that the entropy does not depend on the values of the scalars on the horizon. However, the eqns. (4.14) are nonlinear, and in general might not be invertible. In fact, in section 3.1 we encountered an explicit example where invertibility breaks down: The horizon value of the scalar $\tau$ is given in terms of the arbitrary (charge-independent) constant $\nu$, i.e., $\tau$ is not stabilized; in other words the black hole potential has a flat direction. Nevertheless, as can be seen from (3.21), the entropy is completely determined by the
charges\textsuperscript{13}, and is thus still independent of $\nu$, in agreement with the attractor mechanism. Unfortunately we do not know of any way to show this for the case (4.13) of a generic prepotential.

Let us take a closer look at the SU(1,1)/U(1) model of section 3.1, without making the assumption $\tau = \bar{\tau}$ that was adopted there in order to obtain an explicit black hole solution. We wish to determine the moduli space spanned by the flat directions in the black hole potential. To this end, we write down the attractor equations for the parametrization (3.1). It is easy to show that (4.14) are equivalent to

\begin{align}
Q_0 &= Q_1 = 0, \\
g_0P^0 - g_1P^1 &= 0,
\end{align}

and thus the attractor equations imply only some constraints on the charges, but do not involve the complex scalar $\tau$, which remains completely arbitrary. The moduli space of BPS attractors with prepotential $F = -iX^0X^1$ is therefore SU(1,1)/U(1). Notice that in ungauged $\mathcal{N} = 2$, $D = 4$ supergravity coupled to abelian vector multiplets, there are no flat directions in the 1/2-BPS attractor flow [8] (at least as long as the metric of the scalar manifold is strictly positive definite), but there is a nontrivial moduli space for non-BPS flows [25, 26]. The new feature appearing in gauged supergravity is thus the presence of flat directions in the black hole potential also in the BPS case.

Acknowledgments

This work was partially supported by INFN and MIUR-PRIN contract 20075ATT78. We would like to thank Bianca Letizia Cerchiai, Alessio Marrani, Andrea Mauri and Wafic A. Sabra for useful discussions, and Andrea Borghese and Diego S. Mansi for collaboration in the early stages of this project.

References

[1] A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy,” Phys. Lett. B 379 (1996) 99 [arXiv:hep-th/9601029].

[2] S. L. Cacciatori, D. Klemm, D. S. Mansi and E. Zorzan, “All timelike supersymmetric solutions of $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to abelian vector multiplets,” JHEP 0805 (2008) 097 [arXiv:0804.0009 [hep-th]].

[3] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “$\mathcal{N} = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP 0810 (2008) 091 [arXiv:0806.1218 [hep-th]].

\textsuperscript{13}Note that in section 3.1 we have scaled $\kappa$ to be $-1$, which implies that the volume $V$ is independent of the moduli, so that $S$ depends on the product $P^0P^1$ only.
[4] S. Ferrara, R. Kallosh and A. Strominger, “$\mathcal{N} = 2$ extremal black holes,” Phys. Rev. D 52 (1995) 5412 [arXiv:hep-th/9508072].

[5] A. Strominger, “Macroscopic entropy of $\mathcal{N} = 2$ extremal black holes,” Phys. Lett. B 383 (1996) 39 [arXiv:hep-th/9602111].

[6] S. Ferrara and R. Kallosh, “Supersymmetry and attractors,” Phys. Rev. D 54 (1996) 1514 [arXiv:hep-th/9602136].

[7] S. Ferrara and R. Kallosh, “Universality of supersymmetric attractors,” Phys. Rev. D 54 (1996) 1525 [arXiv:hep-th/9603090].

[8] S. Ferrara, G. W. Gibbons and R. Kallosh, “Black holes and critical points in moduli space,” Nucl. Phys. B 500 (1997) 75 [arXiv:hep-th/9702103].

[9] A. Dabholkar, A. Sen and S. P. Trivedi, “Black hole microstates and attractor without supersymmetry,” JHEP 0701 (2007) 096 [arXiv:hep-th/0611143].

[10] D. M. Kaplan, D. A. Lowe, J. M. Maldacena and A. Strominger, “Microscopic entropy of $\mathcal{N} = 2$ extremal black holes,” Phys. Rev. D 55 (1997) 4898 [arXiv:hep-th/9609204].

[11] G. T. Horowitz, D. A. Lowe and J. M. Maldacena, “Statistical entropy of nonextremal four-dimensional black holes and U-duality,” Phys. Rev. Lett. 77 (1996) 430 [arXiv:hep-th/9603195].

[12] A. Dabholkar, “Microstates of non-supersymmetric black holes,” Phys. Lett. B 402 (1997) 53 [arXiv:hep-th/9702050].

[13] R. Emparan and G. T. Horowitz, “Microstates of a neutral black hole in M theory,” Phys. Rev. Lett. 97 (2006) 141601 [arXiv:hep-th/0607023].

[14] J. F. Morales and H. Samtleben, “Entropy function and attractors for AdS black holes,” JHEP 0610 (2006) 074 [arXiv:hep-th/0608044].

[15] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, “$D = 4$ black hole attractors in $\mathcal{N} = 2$ supergravity with Fayet-Iliopoulos terms,” Phys. Rev. D 77 (2008) 085027 [arXiv:0802.0141 [hep-th]].

[16] M. Huebscher, P. Meessen, T. Ortín and S. Vaulà, “Supersymmetric $\mathcal{N} = 2$ Einstein-Yang-Mills monopoles and covariant attractors,” Phys. Rev. D 78 (2008) 065031 [arXiv:0712.1530 [hep-th]].

[17] W. A. Sabra, “Anti-de Sitter BPS black holes in $\mathcal{N} = 2$ gauged supergravity,” Phys. Lett. B 458 (1999) 36 [arXiv:hep-th/9903143].
[18] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fré and
T. Magri, “\(\mathcal{N} = 2\) supergravity and \(\mathcal{N} = 2\) super Yang-Mills theory on general scalar
manifolds: Symplectic covariance, gaugings and the momentum map,” J. Geom. Phys.
23 (1997) 111 [arXiv:hep-th/9605032].

[19] A. Van Proeyen, “\(\mathcal{N} = 2\) supergravity in \(d = 4, 5, 6\) and its matter couplings,” extended
version of lectures given during the semester “Supergravity, superstrings and
M-theory” at Institut Henri Poincaré, Paris, november 2000;
http://itf.fys.kuleuven.ac.be/~toine/home.htm#B

[20] S. L. Cacciatori, M. M. Caldarelli, D. Klemm and D. S. Mansi, “More on BPS solutions
of \(\mathcal{N} = 2, D = 4\) gauged supergravity,” JHEP 0407 (2004) 061 [arXiv:hep-th/0406238].

[21] M. M. Caldarelli and D. Klemm, “Supersymmetry of anti-de Sitter black holes,” Nucl.
Phys. B 545 (1999) 434 [arXiv:hep-th/9808097].

[22] M. J. Duff and J. T. Liu, “Anti-de Sitter black holes in gauged \(\mathcal{N} = 8\) supergravity,”
Nucl. Phys. B 554 (1999) 237 [arXiv:hep-th/9901149].

[23] M. Cvetic et al., “Embedding AdS black holes in ten and eleven dimensions,” Nucl.
Phys. B 558, 96 (1999) [arXiv:hep-th/9903214].

[24] A. Ceresole, R. D’Auria and S. Ferrara, “The symplectic structure of \(\mathcal{N} = 2\)
supergravity and its central extension,” Nucl. Phys. Proc. Suppl. 46 (1996) 67
[arXiv:hep-th/9509160].

[25] S. Ferrara and A. Marrani, “On the moduli space of non-BPS attractors for \(\mathcal{N} = 2\)
symmetric manifolds,” Phys. Lett. B 652 (2007) 111 [arXiv:0706.1667 [hep-th]].

[26] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, “stu black holes unveiled,”
arXiv:0807.3503 [hep-th].