On $\tau$-function of conjugate nets

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Abstract

We study a potential introduced by Darboux to describe conjugate nets, which within the modern theory of integrable systems can be interpreted as a $\tau$-function. We investigate the potential using the non-local $\bar{\partial}$ dressing method of Manakov and Zakharov, and we show that it can be interpreted as the Fredholm determinant of an integral equation which naturally appears within that approach. Finally, we give some arguments extending that interpretation to multicomponent Kadomtsev–Petviashvili hierarchy.

Dedicated to F. Calogero for his 70-th birthday.

1 Introduction

Conjugate nets are certain parametrized submanifolds whose theory was the object of investigations of the XIX-th century differential geometry [5, 2, 9]. The basic system of equations of the theory

$$\partial_i \partial_j h_k(u) = (\partial_j \log h_i(u)) \partial_i h_k(u) + (\partial_i \log h_j(u)) \partial_j h_k(u), \quad i, j, k \text{ distinct,}$$

(1.1)

constitutes one-half of the system of the Lamé equations [12] describing the orthogonal coordinate systems. Here the functions $h_i, i = 1, \ldots, N, \ N > 2, \text{ called the Lamé coefficients, depend on}$

parameters $u = (u_1, \ldots, u_N)$ of conjugate nets, and $\partial_i = \frac{\partial}{\partial u_i}$ denote partial derivatives. The

Darboux system (1.1) written in terms of the rotation coefficients $\beta_{ij}$, defined by the compatible system

$$\partial_j \beta_{ik}(u) = \beta_{ij}(u) \beta_{jk}(u), \quad i \neq j,$$

(1.4)

The symmetry $i \leftrightarrow j$ of the system (1.4) allows, in turn, for existence yet another potential field (this fact was known already to Darboux, see [5] p. 363)

$$\partial_i \log \tau(u) = -\beta_{ii}(u),$$

(1.5)
which was identified with the \( \tau \)–function of the multicomponent KP hierarchy.

The \( \tau \)–functions play the central role in establishing the connections between integrable systems and quantum field theory, statistical mechanics or the theory of random matrices. They are often represented in a determinant form or can be identified with the Fredholm determinant of the integral Gel’fand–Levitan–Marchenko equation used to solve the model under consideration. In particular, the determinant formula for \( \tau \)–function of the KP hierarchy follows from the free fermions (or \( gl_{\infty} \)) realization of the hierarchy. Within the context of the Zakharov and Shabat dressing method the \( \tau \)–function of the KP hierarchy was interpreted as the Fredholm determinant in [19].

Manakov and Zakharov introduced in [19] the \( \bar{\partial} \)-dressing method, based on the nonlocal \( \bar{\partial} \) problem (see also [1]) as a generalized version of the inverse scattering method [21]. They rediscovered the Darboux system in the generalized matrix form, as the ”basic set of equations” solvable by the \( \bar{\partial} \)-dressing method. For example, the KP equation was shown in [4] to be a limiting case of the Darboux system.

The multicomponent KP hierarchy and conjugate nets were studied within the \( \bar{\partial} \)-dressing method in a number of papers [3, 7, 13]. However, in that approach the determinant interpretation of the \( \tau \)–function of conjugate nets was somehow missing. In this paper we show that indeed the \( \tau \)–function of conjugate nets can be identified with the Fredholm determinant of the integral equation inverting the nonlocal \( \bar{\partial} \)-problem.

The paper is constructed as follows. In Section 2 we collect the basic elements of the \( \bar{\partial} \)-dressing method and we present the way of solving the Darboux equations within this method. Section 3 is devoted to presentation of the Fredholm determinant interpretation of the \( \tau \)–function of conjugate nets. Finally, in Section 4 we briefly discuss generalization of the above result to the full multicomponent KP hierarchy. In the appendix we recall some standard facts concerning the Fredholm integral equations.

2 The \( \bar{\partial} \)-dressing method and the Darboux equations

In this Section we recall [19], [4] the method of solution of the Darboux equations [10] or [13] within the \( \bar{\partial} \)-dressing approach.

The basis of the \( \bar{\partial} \)-dressing method is the following integro-differential equation in the complex plane \( \mathbb{C} \)

\[
\bar{\partial}(\chi(\lambda) - \eta(\lambda)) + (\hat{S}\chi)(\lambda) = 0.
\]

(2.1)

Here \( \hat{S} \) is an integral operator

\[
(\hat{S}\phi)(\lambda) = \int_{\mathbb{C}} S(\lambda, \lambda')\phi(\lambda') d\lambda' \wedge d\bar{\lambda}',
\]

and the given rational function \( \eta(\lambda) \) is called the normalization; it is assumed that

\[
\chi(\lambda) - \eta(\lambda) \to 0, \quad \text{for} \quad |\lambda| \to \infty.
\]

Due to the generalized Cauchy formula the solution of the non-local \( \bar{\partial} \) problem [21] can be expressed in terms of the solution of the equation

\[
\chi(\lambda) + \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{(\hat{S}\chi)(\lambda')}{\lambda' - \lambda} d\lambda' \wedge d\bar{\lambda}' = \eta(\lambda),
\]

(2.2)

which can be put in the form of the Fredholm integral equation of the second kind

\[
\chi(\lambda) = \eta(\lambda) - \int_{\mathbb{C}} K(\lambda, \lambda'')\chi(\lambda'')d\lambda'' \wedge d\bar{\lambda}'',
\]

(2.3)
with the kernel
\[ K(\lambda, \lambda'') = \frac{1}{2\pi i} \int_{C} \frac{S(\lambda', \lambda'')}{\lambda' - \lambda} d\lambda' \wedge d\lambda. \]  \hspace{1cm} (2.4)

**Remark.** In the paper we always assume that the kernel \( S \) in the non-local \( \bar{\partial} \) problem is such that the Fredholm equation (2.2) is uniquely solvable. Then, by the Fredholm alternative, the homogenous equation with \( \eta = 0 \) has only the trivial solution.

Let us assume that the kernel \( S \) in the non-local \( \bar{\partial} \) problem depends on additional parameters. To get the \( \bar{\partial} \)-dressing method of construction of solutions for the Darboux equations let us introduce the following simple dependence of the kernel \( S \) on the variables \( u_i, i = 1, \ldots, N \)
\[ S(\lambda', \lambda'', u) = g(\lambda', u)^{-1} S_0(\lambda', \lambda'') g(\lambda'', u), \]  \hspace{1cm} (2.5)
where
\[ g(\lambda', u) = \exp \sum_{i=1}^{N} u_i A_i(\lambda'), \quad A_i(\lambda') = \frac{c_i}{\lambda' - \lambda_i}, \]  \hspace{1cm} (2.6)
c_i are non-zero constants, and \( \lambda_i \in \mathbb{C} \) are points of the complex plane. Moreover we assume that \( \lambda_i \neq \lambda_j \) for different \( i \) and \( j \).

Directly one can verify the following result.

**Lemma 1.** The evolution (2.5) of the kernel \( S \) implies that the kernel \( K \) of the integral equation (2.3) is subjected to the equation
\[ \partial_i K(\lambda, \lambda'', u) = A_i(\lambda) K(\lambda, \lambda'', u) + A_i(\lambda'') K(\lambda, \lambda', u) - A_i(\lambda) K(\lambda, \lambda'', u). \]  \hspace{1cm} (2.7)

In consequence of the above formula (2.7) we obtain the following useful and crucial for the \( \bar{\partial} \)-dressing method result.

**Lemma 2.** When \( \chi(\lambda, u) \) is the unique, by assumption, solution of the \( \bar{\partial} \) problem (2.1) with the kernel \( S \) evolving according to (2.5), and with normalization given by \( \eta(\lambda, u) \), then the function
\[ \partial_i \chi(\lambda, u) + A_i(\lambda) \chi(\lambda, u) \] is the solution of the same \( \bar{\partial} \) problem but with the new normalization
\[ \partial_i \eta(\lambda, u) + A_i(\lambda) \eta(\lambda, u) + A_i(\lambda)(\chi(\lambda_i, u) - \eta(\lambda_i, u)). \]

The above Lemma leads to the following Theorem, which gives the system of linear problems for the Darboux equation.

**Theorem 1.** Given solution \( \chi(\lambda, u) \) of the the \( \bar{\partial} \) problem (2.1), (2.5) with the normalization \( \eta(\lambda, u) = 1 \), then the function
\[ \psi(\lambda, u) = \chi(\lambda, u) g(\lambda, u) \]
satisfies with respect to variables \( u_i \) the following system of Laplace equations
\[ \partial_i \partial_j \psi(\lambda, u) = a_{ij}(u) \partial_i \psi(\lambda, u) + a_{ji}(u) \partial_j \psi(\lambda, u), \quad i \neq j, \]  \hspace{1cm} (2.8)
with the coefficients
\[ a_{ij}(u) = \frac{\partial_j \chi(\lambda_i, u)}{\chi(\lambda_i, u)} + A_j(\lambda_i). \]  \hspace{1cm} (2.9)

**Proof.** The idea of the proof is standard within the \( \bar{\partial} \)-dressing method approach. One collects solutions of the \( \bar{\partial} \) problem (2.1) (or, equivalently, the integral equation (2.3)) to obtain new solution with the vanishing normalization. Then, by the Fredholm alternative, such solution must be identically zero. \( \square \)
Define the Lamé coefficients $h_i(u)$, by

$$h_i(u) = \chi(\lambda_i, u) g_i(\lambda_i, u),$$

where

$$g_i(\lambda, u) = \exp \sum_{j=1, j \neq i}^{N} u_j A_j(\lambda);$$

equivalently, they are the "non-singular" parts of the function $\psi(\lambda, u)$ in the points $\lambda_i$

$$h_i(u) = \lim_{\lambda \to \lambda_i} [\psi(\lambda, u) \exp(-u_i A_i(\lambda))]. \quad (2.10)$$

Then the coefficients $a_{ij}(u)$ of the Laplace equations (2.8) read

$$a_{ij}(u) = \partial_j \log h_i(u), \quad i \neq j,$$

and the compatibility condition of (2.8) is the Darboux system (1.1). Alternatively, the Darboux system can be obtained evaluating (removing first the singularity like in equation (2.10)) equation (2.8) in the points $\lambda_k, k \neq i, j$.

**Remark.** To obtain real solutions one needs special symmetry properties of the kernel $S$ with respect to the complex conjugation

$$S(\bar{\lambda}, \lambda, \bar{\lambda'}, \lambda') = S(\lambda, \bar{\lambda}, \lambda', \bar{\lambda'}),$$

and reality of the points $\lambda_i = \bar{\lambda}_i \in \mathbb{R}$, and the parameters $c_i \in \mathbb{R}, i = 1, \ldots, N$, which imply that

$$\psi(\bar{\lambda}, \lambda, u) = \psi(\lambda, \bar{\lambda}, u), \quad \text{and} \quad h_i(u) \in \mathbb{R}.$$

The following result allows to give the $\bar{\partial}$-method of construction of solutions of the Darboux system (1.3) written in terms of the rotation coefficients. Its proof is analogous to the proof of Theorem 1.

**Theorem 2.** Let $\chi_i(\lambda, u), i = 1, \ldots, N$ of the be solutions of the $\bar{\partial}$ problem (2.1) with the normalizations

$$\eta_i(\lambda, u) = A_i(\lambda) g_i(\lambda_i, u)^{-1}, \quad (2.11)$$

then the functions

$$\psi_i(\lambda, u) = \chi_i(\lambda, u) g(\lambda, u)$$

satisfy equations

$$\partial_i \psi(\lambda, u) = h_i(u) \psi_i(\lambda, u). \quad (2.12)$$

Moreover, the functions $\psi_i(\lambda, u)$ satisfy the linear system

$$\partial_j \psi_i(\lambda, u) = \beta_{ij}(u) \psi_j(\lambda, u), \quad j \neq i \quad (2.13)$$

with the rotation coefficients

$$\beta_{ij}(u) = \chi_i(\lambda_j, u) g_j(\lambda_j, u) = \lim_{\lambda \to \lambda_j} [\psi_i(\lambda, u) \exp(-u_j A_j(\lambda))]. \quad (2.14)$$

Evaluating equations (2.12) in the points $\lambda_j, j \neq i$, (removing first the singularity) we obtain equations (1.2). The compatibility of the linear system (2.13) gives the Darboux equations (1.3) in terms of the rotation coefficients. Alternatively, equations (1.3) can be obtained evaluating equations (2.14) in the points $\lambda_k, k \neq i, j$. 

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3 The first potentials and the $\tau$–function

To give the meaning of the $\tau$–function within the $\bar{\partial}$-dressing method we first present the meaning of the potentials $\beta_{ii}$ defined by equations (3.1).

Proposition 1. Within the $\bar{\partial}$-dressing method the potential $\beta_{ii}(u)$ can be identified with the non-singular part of the function $\psi_i$ at the point $\lambda_i$

$$\beta_{ii}(u) = \lim_{\lambda \to \lambda_i} \left( \psi_i(\lambda, u) \exp(-u_i A_i(\lambda)) - A_i(\lambda) \right). \quad (3.1)$$

Proof. Multiplication of both sides of equation (2.13) by $g_i(\lambda, u)^{-1}$ and evaluation of the result in the limit $\lambda \to \lambda_i$ gives

$$\lim_{\lambda \to \lambda_i} \partial_j (\psi_i(\lambda, u) \exp(-u_i A_i(\lambda))) = \beta_{ij}(u) \beta_{ji}(u).$$

The expression in brackets in the LHS of the above equation is singular at $\lambda_i$. Up to a constant term, which vanishes during differentiation, it agrees with

$$\psi_i(\lambda, u) \exp(-u_i A_i(\lambda)) - A_i(\lambda),$$

which is finite at $\lambda_i$, due to the normalization condition (2.11). After this replacement we can exchange differentiation with taking the limit.

Finally, we give the meaning to the next potential, the $\tau$–function, which is related with potentials $\beta_{ii}(u)$ by equations (1.4).

Theorem 3. Within the $\bar{\partial}$-dressing method, the $\tau$–function of conjugate nets can be identified with the Fredholm determinant of the integral equation (2.3) inverting the non-local $\bar{\partial}$-problem (2.4) with kernel evolving according to equation (2.5).

Before proving this theorem we first show the following Lemma; we use here the notation of Appendix A.

Lemma 3.

$$\partial_i K \left( \begin{array}{c} z_1 \cdots z_m \\ z_1 \cdots z_m \\ \end{array} \bigg| u \right) = \sum_{\ell=1}^{m} A_i(z_\ell) K \left( \begin{array}{c} \lambda_i \ z_1 \cdots \ z_\ell \cdots z_m \\ z_1 \cdots z_\ell \cdots z_m \\ \end{array} \bigg| u \right),$$

where the symbol $\bar{z}_{\ell}$ means that $z_\ell$ should be removed from the sequence.

Proof. Denote by $\pi_m$ the set of all permutations of $\{1, \ldots, m\}$. Differentiation of the expression

$$K \left( \begin{array}{c} z_1 \cdots z_m \\ z_1 \cdots z_m \\ \end{array} \bigg| u \right) = \sum_{\sigma \in \pi_m} \text{sgn} \sigma K(z_1, z_{\sigma(1)}, u) \cdots K(z_m, z_{\sigma(m)}, u),$$

with respect to $u_i$ gives, by application of equation (2.13),

$$\partial_i K \left( \begin{array}{c} z_1 \cdots z_m \\ z_1 \cdots z_m \\ \end{array} \bigg| u \right) = \sum_{\ell=1}^{m} \sum_{\sigma \in \pi_m} \text{sgn} \sigma K(z_1, z_{\sigma(1)}, u) \cdots A_i(z_\ell) K(\lambda_i, z_{\sigma(\ell)}, u) \cdots K(z_m, z_{\sigma(m)}, u),$$

where we have used also the following elementary formula valid for any permutation $\sigma \in \pi_m$

$$\sum_{\ell=1}^{m} A_i(z_\ell) = \sum_{\ell=1}^{m} A_i(z_{\sigma(\ell)}).$$

After application of an even number of transpositions in any of the $m$ determinants we obtain the statement of the Lemma.
Proof of Theorem 3. Using Lemma 3 we can derive the following formula for \( i \)-th derivative of the Fredholm determinant \( D_F(u) \)

\[
\partial_i D_F(u) = \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \int_{\mathbb{C}^m} A_i(\lambda') K \left( \begin{array}{ccc} \lambda_i & z_1 & \ldots & z_{m-1} \\ \lambda' & z_1 & \ldots & z_{m-1} \end{array} | u \right) d\lambda' \wedge d\bar{\lambda}' \ldots dz_{m-1} \wedge d\bar{z}_{m-1}.
\]

Comparing with

\[
D_F(\lambda_i, \lambda', u) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{C}^m} K \left( \begin{array}{ccc} \lambda_i & z_1 & \ldots & z_m \\ \lambda & z_1 & \ldots & z_m \end{array} | u \right) dz_1 \wedge d\bar{z}_1 \ldots dz_m \wedge d\bar{z}_m
\]

we obtain that

\[
\partial_i D_F(u) = \int_{\mathbb{C}} D_F(\lambda_i, \lambda', u) A_i(\lambda') d\lambda' \wedge d\bar{\lambda}'.
\]

Notice that the solution of the Fredholm equation (2.3) with normalization \( A_i(\lambda) \) is given, due to (2.11) and (A.4), by

\[
\chi_i(\lambda, u) g_i(\lambda_i, u) = A_i(\lambda) - \int_{\mathbb{C}} \frac{D_F(\lambda', \lambda_i, u)}{D_F(u)} A_i(\lambda') d\lambda' \wedge d\bar{\lambda}'.
\]

In the limit \( \lambda \to \lambda_i \) (compare with (3.1)) we obtain, therefore,

\[
\partial_i \log D_F(u) = -\beta_{ii}(u)
\]

which allows, via equation (1.5), for identification of the \( \tau \)-function with the Fredholm determinant \( D_F(u) \).

4 Conclusion and remarks

We have shown that within the \( \bar{\partial} \)-dresing method the \( \tau \)-function of conjugate nets can be identified with the Fredholm determinant inverting the non-local \( \bar{\partial} \) problem. In fact, this result can be extended to the full \( N \)-component KP hierarchy. To justify this opinion we should:

1. incorporate higher times of the hierarchy into the scheme,
2. give the meaning to the full set of \( \tau \)-functions labelled by the root lattice vectors of the \( A_{N-1} \) root system.

The first task can be done by using the idea of [4], where the KP equation was obtained from the Darboux system (the scalar ”basic set of equations”) for \( N = 3 \). In the same way the higher times can be included into evolution of the kernel. The second problem can be solved by combination of the result of [8] on relation of the Schlesinger transformations on the \( A_{N-1} \) root lattice with the Laplace transformations of conjugate nets, with the construction of such Laplace transformations within the \( \bar{\partial} \)-dresing method given in [7]. The details will be presented elsewhere.

A Elements of the Fredholm theory

We recall in this Appendix some standard facts (see, for example [18]) from the theory of Fredholm integral equations which we use in the paper.

Consider the Fredholm equation of the second kind

\[
f(x) = g(x) - \int_{\Omega} K(x, y) f(y) dy,
\]

(A.1)
where $K(x, y)$ is the kernel of the integral operator, and $g(x)$ is a given function. The Fredholm determinant $D_F$ is defined by the series

$$D_F = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\Omega^m} K(x_1, x_2, \ldots, x_m) \, dx_1 \ldots dx_m,$$

(A.2)

where

$$K(x_1, x_2, \ldots, x_m) = \det \begin{pmatrix} K(x_1, y_1) & K(x_1, y_2) & \cdots & K(x_1, y_m) \\ K(x_2, y_1) & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ K(x_m, y_1) & \cdots & K(x_m, y_2) & K(x_m, y_m) \end{pmatrix},$$

while the Fredholm minor is defined by the series

$$D_F(x, y) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Omega^m} K(x, x_1, x_2, \ldots, x_m) \, dx_1 \ldots dx_m.$$

(A.3)

For non-vanishing Fredholm determinant $D_F$ the solution of the integral equation (A.1) is unique and can be written in the form

$$f(x) = g(x) - \int_{\Omega} \frac{D_F(x, y)}{D_F} g(y) \, dy.$$

(A.4)

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