Reverse generalized Bessel matrix differential equation, polynomial solutions, and their properties

M. Abul-Dahab\textsuperscript{a*†}, M. Abul-Ezb\textsuperscript{b}, Z. Kishkab\textsuperscript{b} and D. Constales\textsuperscript{c}

Communicated by W. Sprößig

This paper is devoted to the study of reverse generalized Bessel matrix polynomials (RGBMPs) within complex analysis. This study is assumed to be a generalization and improvement of the scalar case into the matrix setting. We give a definition of the reverse generalized Bessel matrix polynomials \( \theta_n(A; B; z), z \in \mathbb{C} \), for parameter (square) matrices \( A \) and \( B \), and provide a second-order matrix differential equations satisfied by these polynomials. Subsequently, a Rodrigues-type formula, a matrix recurrence relationship, and a pseudo-generating function are then developed for RGBMPs. © 2013 The Authors Mathematical Methods in the Applied Sciences Published by John Wiley & Sons, Ltd.

Keywords: generalized Bessel matrix polynomials; Bessel matrix differential equation; matrix functional calculus

1. Introduction

Generalized Bessel polynomials (GBP) are defined explicitly by

\[ y_n(z, a) = \sum_{k=0}^{n} \binom{n}{k} (n + a - 1)_k \left( \frac{z}{2} \right)^k, \]

where \( a \) is a real number, \( (\alpha)_0 = 1 \) and \( (\alpha)_k = \alpha (\alpha + 1) \ldots (\alpha + k - 1) \), for \( k \geq 1 \). In hypergeometric notation, we have

\[ y_n(z, a) = {}_2F_0 \left( -n, a + n - 1; \frac{-z}{2} \right). \]

These polynomials satisfy the second-order linear differential equation

\[ z^2y'' + (az + 2)y' - n(a + n - 1)y = 0. \tag{1.1} \]

An early appearance of the generalized Bessel polynomials was in the (1929) papers by Bochner [1] and Romanovsky [2], and they have appeared after that in papers by many other authors [3–10].

The importance of GBP is realized in (1949) when Krall and Frink [11] found their connection with the wave equation in spherical coordinates. At about the same time, Thompson [12] independently discovered these polynomials in his study of electrical networks. For a historical survey and discussion of many interesting properties, we refer to the definitive book by Grosswald [5]. It has been long recognized that the GBP are closely related to the so-called reverse generalized Bessel polynomials (RGBP), which are defined by

\[ \theta_n(z; a) = z^n y_n \left( \frac{1}{z}; a \right). \]

\textsuperscript{a}Department of Mathematics, Faculty of Science, South Valley University, Qena 83523, Egypt
\textsuperscript{b}Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt
\textsuperscript{c}Department of Mathematics Analysis, Ghent University, Galglaan2, B-9000 Ghent, Belgium
\textsuperscript{*Correspondence to: M. Abul-Dahab, Department of Mathematics, Faculty of Science, South Valley University, Qena 83523, Egypt.
\textsuperscript{†}E-mail: mamabuldahbab@yahoo.com

This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

© 2013 The Authors Mathematical Methods in the Applied Sciences Published by John Wiley & Sons, Ltd.
These RGBP $\theta_n(z; a)$ satisfy the following second-order linear differential equation:

$$z \theta'' - (2n - 2 + a + 2z) \theta' + 2n \theta = 0.$$  

(1.2)

The phase reverse can be justified because, if

$$y_n(z; a) = b_0 z^n + b_1 z^{n-1} + \ldots + b_{n-1} z + b_n,$$

then

$$\theta_n(z; a) = z^n \left[ b_0 \left( \frac{1}{z} \right)^n + b_1 \left( \frac{1}{z} \right)^{n-1} + \ldots + b_{n-1} \left( \frac{1}{z} \right) + b_n \right]$$

$$= b_0 + b_1 z + \ldots + b_{n-1} z^{n-1} + b_n z^n.$$

Clearly, it is seen that the RGBP is a polynomial with the same coefficients but in reverse order. The differential equation for the generalized and RGBP have a basic difference. In fact in (1.1), $z = 0$ is an irregular singular point, and $z = \infty$ is a regular singular point. However, in (1.2), the point at origin $z = 0$ is a regular singular point, while the point at infinity represents an irregular singularity, which is preferable. The polynomial solutions of (1.1) and (1.2) are called Bessel polynomials (see [9]) and form a set of orthogonal polynomials on the unit circle in the complex plane.

Now, owing to the significance of the earlier mentioned work related to Bessel polynomials, we should record that many authors became interested to develop the scalar cases of the classical sets of orthogonal polynomials into orthogonal matrix polynomials. Of those authors, we mention L. Jodar et al. [13–16] and the references there in [17–28].

Orthogonal matrix polynomials comprise an emerging field of study, with important results in both theory and applications continuing to appear in the literature. Some results in this field can be found in [17, 20, 29]; applications to matrix integration may be found in [14, 30]. Important connections between orthogonal matrix polynomials and matrix differential equations appear in [26, 31].

Generalization of the concept of orthogonality from the scalar case has been considered in different ways [13, 18–21] and development of other extensions, such as a Rodrigues-type formula [25], a second-order Sturm–Liouville differential equation [25], or a three-term recurrence [16], for example, continue to emerge as they have been discovered. Applications of matrix polynomials also grow, and active areas in recent literature include statistics, group representation theory [27], scattering theory [22], differential equations [15, 24, 26, 32], Fourier series expansions [23], interpolation and quadrature [13, 30], splines [33], and medical imaging [34].

In the scalar case, the aforementioned GBP $y_n(z; a)$ have already been developed into the matrix setting $Y_n(A; B; z), z \in \mathbb{C}$, for parameter matrices $A$ and $B$ in a recent work [28]. It should be observed that the matrices $A, B$ (and further down $C$) are commuting if and only if they are simultaneously diagonalizable (see [35]). In this paper, we establish a structure for the RGBMPs. It is well recognized in the field, however, that the non-commutativity of matrix multiplication usually results in the development of matrix analogs that does not have the relative simplicity found in the scalar situation.

This paper, then, is concerned with matrix polynomials

$$P_n(z) = A_0 z^n + A_{n-1} z^{n-1} + A_{n-2} z^{n-2} + \ldots + A_0$$

in which the coefficients $A_i$ are members of $\mathbb{C}^{N \times N}$, the space of complex matrices of order $N$, and $z$ is a complex number. $P_n(z)$ is of degree $n$ if $A_n$ is not the zero matrix; for orthogonal matrix polynomials, the leading coefficient, $A_n$, being nonsingular is important [13, 16] and [24]. In Section 2, we summarize basic facts and properties to be used in the following sections. Section 3 provides the definition of the RGBMP $\Theta_n(A; B; z)$, for parameter matrices $A$ and $B$. The section also includes development of second-order matrix differential equations, which are satisfied by the $\Theta_n(A; B; z)$. A Rodrigues-type formula and recurrence relations for the RGBMP $\Theta_n(A; B; z)$ are obtained in Section 4. A pseudo-generating function for $\Theta_n(A; B; z)$ is given in Section 5.

Throughout this paper, for a matrix $A \in \mathbb{C}^{N \times N}$, its spectrum is denoted by $\sigma(A)$. The two-norm of $A$, which will be denoted by $\|A\|$, is defined by

$$||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||},$$

where for a vector $y$ in $\mathbb{C}^N$, $||y||_2 = (y^H y)^{1/2}$ is Euclidean norm of $y$. I and $0$ stand for the identity matrix and the null matrix in $\mathbb{C}^{N \times N}$, respectively.

### 2. Preliminaries

There are some basic facts and notations used throughout the development in Sections 3–5. They are listed here for easy referral in the sequel as ‘facts’ or ‘notations’, respectively, and references are given where appropriate.

**Notation 2.1**

For $A \in \mathbb{C}^{N \times N}$, the matrix version of the symbol is

$$(A)^{(k)} = (A) (A - I) \ldots (A - (k - 1) I), \quad k \geq 1,$$
and the Pochhammer symbol (the shifted factorial) is
\[(A)_k = A(A + 1) \cdots (A + (n - 1))I, \quad n \geq 1; (A)_0 = I.\]

Note that if \(A = -jl\), where \(j\) is a positive integer, then \((A)_k = 0\) whenever \(k > j\) (cf. [36]).

**Notation 2.2**

Relying to [28], one can easily obtain
\[
\frac{(-1)^k}{(n-k)!}l = \frac{(-n)_k}{n!}l = \frac{(-nl)_k}{n!}; \quad 0 \leq k \leq n.
\]

**Notation 2.3**

If \(P(n)\) is a polynomial with the zeros \(u_1, u_2, \ldots, u_m\), then referring to [5] one has
\[
P(\delta)z^k = (k - u_1)(k - u_2) \cdots (k - u_m)z^k = P(k)z^k; \quad \delta = z \frac{d}{dz}.
\]

and for any function \(f(z), z \in \mathbb{C}\), we shall have
\[
(\delta - k_1)(\delta - k_2) \cdots (\delta - k_r)f(z) = z(\delta - k_1 + 1) \cdots (\delta - k_r + 1)f(z).
\]

**Fact 2.1**

For an arbitrary matrix \(A \in \mathbb{C}^{N \times N}\),
\[
\frac{d^k}{dz^k} \left[t^{A+nl}\right] = (A + l)_m \left[(A + l)_{m-k}\right]^{-1} t^{A+(m-k)}l, \quad k = 0, 1, 2, 3, \ldots
\]

**Fact 2.2** (see [36].)

If \(f(z)\) and \(g(z)\) are holomorphic functions of the complex variable \(z\), which are defined in an open set \(\Omega\) of the complex plane, and \(A\) is a matrix in \(\mathbb{C}^{N \times N}\) such that \(\sigma(A) \subset \Omega\), then
\[
f(A)g(A) = g(A)f(A).
\]

Hence, if \(B\) in \(\mathbb{C}^{N \times N}\) is a matrix for which \(\sigma(B) \subset \Omega\), and if \(AB = BA\), then
\[
f(A)g(B) = g(B)f(A).
\]

**Fact 2.3** (see [28].)

If \(A \in \mathbb{C}^{N \times N}\), and \(z\) is any complex number, then the matrix exponential \(e^{Az}\) is defined to be
\[
e^{Az} = I + Az + \ldots + \frac{A^N}{N!}z^n + \ldots,
\]

\[
\frac{d^k}{dz^k} \left[e^{Az}\right] = A^k e^{Az} = e^{Az} A^k, \quad k = 0, 1, 2, 3, \ldots.
\]

Let \(\alpha(A) = \max \{|Re(z); z \in \sigma(A)\}\). Then
\[
\|e^{tA}\| \leq e^{\alpha(A) t} \sum_{s=0}^{N-1} \left(\frac{\|A\| N^s t^s}{s!}\right); \quad t \geq 0,
\]

and
\[
\|A^n\| \leq n^{\alpha(A)} \sum_{s=0}^{N-1} \left(\frac{\|A\| N^s \ln n}{s!}\right); \quad n \geq 1.
\]

**Fact 2.4**

The reciprocal scalar Gamma function denoted by \(\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}\) is an entire function of the complex variable \(z\). Thus, for any \(A \in \mathbb{C}^{N \times N}\), Riesz–Dunford functional calculus [36] shows that \(\Gamma^{-1}(A)\) is well defined and is, indeed, the inverse of \(\Gamma(A)\). Furthermore, if
\[
A + nl \quad \text{is invertible for all integer } n \geq 0,
\]

then
\[
(A)_n = \Gamma(A + nl)\Gamma^{-1}(A).
\]
Fact 2.5 (see [28].) Let $A$ and $B$ be parameter commuting matrices in $\mathbb{C}^{N \times N}$ satisfying the spectral condition (2.1). For any natural number $n \geq 0$, the $n$-th generalized Bessel matrix polynomial $Y_n(A, B; z)$ is defined by

$$Y_n(A, B; z) = \sum_{k=0}^{n} \binom{n}{k} (A + (n + k - 2)I)^{(k)} (zB^{-1})^k,$$

(2.3)

where $\binom{n}{k}$ is a binomial coefficient. This matrix polynomial is a solution of the following matrix differential equation:

$$z^2 Y_n''(A, B; z) + (Az + B) Y_n'(A, B; z) = nI(A + (n - 1)I)Y_n(A, B; z).$$

(2.4)

By means of the notation of the hypergeometric matrix series, the generalized Bessel matrix polynomials are given by

$$Y_n(A, B; z) = \left(-nl, A + (n - 1)I; -; -zB^{-1}\right).$$

Also, we see that the generalized Bessel matrix polynomials are essentially Laguerre's matrix polynomials and Whittaker matrix functions. In fact, we have

$$Y_n(A, B; z) = n! \left(-zB^{-1}\right)^n L_n^{-2nl-A+1} \left(\frac{B}{z}\right),$$

and

$$Y_n(A, B; z) = e^{B/2z} \left(zB^{-1}\right)^{-A/2} W_{-A/2,(A-I)/2+nl} \left(\frac{B}{z}\right).$$

An immediate consequence of the definition (2.3) is the integral representation:

$$Y_n(A, B; z) = \Gamma^{-1}(A + (n - 1)I) \int_0^\infty e^{(A+(n-2)I)t} F_0(-nl; -; -t z B^{-1}) \, e^{-t} \, dt$$

$$= \Gamma^{-1}(A + (n - 1)I) \int_0^\infty e^{(A+(n-2)I)t} \left(l + t z B^{-1}\right)^n \, e^{-t} \, dt.$$

(2.5)

The generalized Bessel matrix polynomials are orthogonal on the unit circle with respect to the matrix weight function (cf. [28])

$$\rho(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \Gamma(A) \Gamma^{-1}(A + (n - 1)I) \left(-\frac{B}{z}\right)^n,$$

(2.6)

which satisfies the related matrix nonhomogeneous equation

$$\left(z^2 \rho(z)\right)' = (Az + B) \rho(z) - \frac{1}{2\pi i} [(A-I)(A-2I)]z.$$

(2.7)

For $n \neq m$, we have

$$\int_Y \rho(z) Y_m(A, B; z) Y_n(A, B; z) \, dz = \mathbf{0}.$$

3. Definition and matrix differential equations

The RGBMPs are defined in (3.1) in the succeeding text, then the second-order differential equations they satisfy are derived, as stated in Theorems 3.1. and 3.2 of this section.

Definition 3.1

Let $A$ and $B$ be commuting matrices in $\mathbb{C}^{N \times N}$ satisfying the spectral condition (2.1). For any natural number $n \geq 0$, the $n$-th RGBMP $\Theta_n(A, B; z)$ is defined by

$$\Theta_n(A, B; z) = z^n Y_n \left(A, B; \frac{1}{z}\right) = \sum_{k=0}^{n} \binom{n}{k} (B^{-1})^k (A + (n + k - 2)I)^{(k)} (z)^{n-k}.$$

(3.1)

The first four terms of the RGBMPs $\Theta_n(z) = \Theta_n(A, B; z)$ are

$$\Theta_0(z) = 1,$$

$$\Theta_1(z) = zI + A B^{-1},$$

$$\Theta_2(z) = z^2 I + 2(A + I)(z B^{-1}) + (A + I)(A + 2I)(B^{-1})^2.$$
\[
\Theta_3(z) = z^2 + 3(A + 2I) \left( z^2 B^{-1} \right) + 3z(A + 2I)(A + 3I) \left( B^{-1} \right)^2 + (A + 2I)(A + 3I)(A + 4I) \left( B^{-1} \right)^3.
\]

**Remark 3.1**
If we extend the definition (3.1) of \( \Theta_n(z) \) formally to negative subscripts, we obtain \( \Theta_{-n}(z) = z^{-2n+1} \Theta_{n-1}(z) \), and replacing \( n \) by \( n + 1 \), it follows that \( \Theta_{-(n+1)}(z) = z^{-(2n+1)} \Theta_{n}(z) \), which will be useful in the sequel.

**Theorem 3.1**
For each natural number \( n \geq 0 \), the RGBMs \( \Theta_n(A; B; z) \) will satisfy the following matrix differential equation:
\[
z \Theta''_n(z) - (zB + A + 2(n - 1)I) \Theta'_n(z) + nB \Theta_n(z) = 0,
\]
which is equivalent to
\[
dl(\delta l - A - (2n - 1)I) \Theta_n(z) = Bz(\delta l - nI) \Theta_n(z); \quad \delta = \frac{dz}{dz}.
\]

**Proof**
Consider the generalized Bessel matrix differential equation
\[
z^2 Y''_n(A; B; z) + (zA + B) Y'_n(A; B; z) = n(A + (n - 1)I) Y_n(A; B; z),
\]
The corresponding generalization of \( \Theta_n(A; B; z) \) is obtained most conveniently by setting
\[
Y_n(A; B; z) = z^n \Theta_n \left( A, B; \frac{1}{z} \right).
\]
Applying the chain rule in (3.5), we have
\[
Y''_n(A; B; z) = n^2 z^{n-1} \Theta_n \left( A, B; \frac{1}{z} \right) - n z^n \Theta'_n \left( A, B; \frac{1}{z} \right) \quad \text{and}
\]
\[
Y'_n(A; B; z) = n(n - 1) z^{n-2} \Theta_n \left( A, B; \frac{1}{z} \right) - n z^{n-3} \Theta'_n \left( A, B; \frac{1}{z} \right) - (n - 2) z^{n-3} \Theta''_n \left( A, B; \frac{1}{z} \right) + z^{n-4} \Theta'_n \left( A, B; \frac{1}{z} \right)
\]
Substituting (3.4) and (3.5) in (3.6) and using routine computations, we obtain the matrix differential equation satisfied by \( \Theta_n(z) \) in the form
\[
z \Theta''_n(z) - (zB + A + 2(n - 1)I) \Theta'_n(z) + nB \Theta_n(z) = 0.
\]
Equation (3.2) is easily put in the form (3.3) by employing the relations \( \delta \Theta_n(z) = z \Theta'_n(z) \) and \( \delta l(\delta l - I) \Theta_n(z) = z^2 \Theta''_n(z) \).

**Theorem 3.2**
Let \( A, B, \) and \( C \) be commuting matrices in \( \mathbb{C}^{N \times N} \). Then for each natural number \( n \geq 0 \), we have that \( W(z) = e^{-Cz} \Theta_n(A; B; z) \) is a solution of the matrix differential equation
\[
zW''(z) - [2(n - 1)I + A + (B - 2C)z]W'(z) + [C(C - B)z + (B - 2C)n + C(2I - A)]W(z) = 0.
\]

**Proof**
Differentiate the equation \( W(z) = e^{-Cz} \Theta_n(A; B; z) \) twice and substitute results in (3.2), we obtain the matrix differential equation satisfied by \( W(z) \).

Theorem 3.2 leads to the following corollaries.

**Corollary 3.1**
For each natural number \( n \geq 0 \), if \( C = B \) in (3.7), then \( W(z) = e^{-Bz} \Theta_n(A; B; z) \) will be a solution of the matrix differential equation
\[
zW''(z) - [2(n - 1)I + A - zB]W'(z) + [B(2I - nI - A)]W(z) = 0,
\]
or
\[
dl(\delta l - A - (2n - 1)I) W(z) = -Bz(\delta l - A - (n - 2)I) W(z).
\]

**Corollary 3.2**
If \( 2C = B = 2I \) in (3.7), then \( W(z) = e^{-zl} \Theta_n(A; 2I; z) \) is a solution of the matrix differential equation
\[
zW''(z) - [2(n - 1)I + A]W'(z) + [(2I - A - zl)]W(z) = 0.
\]
Corollary 3.3
If \( A = 2i \) in (3.9), then \( W(z) = e^{-Bz} \Theta_n(2i, B; z) \) will be a solution of the matrix differential equation
\[
\delta l(\delta l - (2n + 1)i) W(z) = -Bz(\delta l - nl) W(z). \tag{3.11}
\]

Remark 3.2
It is worthy to mention that in (3.3) and (3.9), we shall obtain the two solutions in the form
\[
e^{-Bz} \Theta_n(2i, B; z), \quad \Theta_n(2i, -B; z).
\]

4. The analogue of Rodrigues’ formula and recurrence relations

Two more basic properties of the RGBMPs \( \Theta_n(A, B; z) \) are developed in this section and that they enjoy a Rodrigues’ formula, which is obtained from Theorem 4.1 and with the help of Definition (3.1). Also, some recurrence relations for the RGBMPs are given.

4.1. Rodrigues’ formula

The following lemma will be useful in the sequel:

Lemma 4.1
For the matrix \( D \) in \( \mathbb{C}^{N \times N} \), we have
\[
(\delta l - (n + m - 1)i)(\delta l - (n + m - 2)i) \ldots (\delta l - nl) e^{-Dz} = z^{n+m} \frac{d^m}{dz^m} \left( z^{-n} e^{-Dz} \right); \quad m = 1, 2, 3, \ldots \tag{4.1}
\]

Proof
From Fact 2.3, we can write
\[
(\delta l - nl) e^{-Dz} = -(zD + nl) e^{-Dz} = z^{n+1} \frac{d}{dz} \left( z^{-n} e^{-Dz} \right),
\]
and
\[
\delta l (\delta l - nl) e^{-Dz} = (n + 1)^{1+n} \frac{d}{dz} \left( z^{-n} e^{-Dz} \right) + z^{n+2} \frac{d^2}{dz^2} \left( z^{-n} e^{-Dz} \right),
\]
so that
\[
(\delta l - (n + 1)i)(\delta l - nl) e^{-Dz} = z^{n+2} \frac{d^2}{dz^2} \left( z^{-n} e^{-Dz} \right).
\]
An induction on \( m \) can complete the proof of (4.1).

Now, we show that (3.9) has the solution
\[
\chi(z) = (\delta l - A - (n - 1)i)(\delta l - A - nl) \ldots (\delta l - A - 2(n - 1)i) e^{-Bz}. \tag{4.2}
\]
We obtain successively,
\[
\delta l(\delta l - (2n - 1)i) \chi(z) = (\delta l - A - (n - 1)i) \ldots (\delta l - A - 2(n - 1)i) (\delta l - A - (2n - 1)i) \delta l e^{-Bz}
\]
\[
= (\delta l - A - (n - 1)i) \ldots (\delta l - A - 2(n - 1)i) (-Bz) e^{-Bz}
\]
\[
= -Bz(\delta l - A - (n - 2)i) \ldots (\delta l - A - 2(n - 1)i) e^{-Bz}
\]
\[
= -Bz(\delta l - A - (n - 2)i) \left[ (\delta l - A - (n - 1)i) \ldots (\delta l - A - 2(n - 1)i) e^{-Bz} \right]
\]
\[
= -Bz(\delta l - A - (n - 2)i) \chi(z),
\]
as claimed. Recalling that the coefficient of \( z^n \) in \( \Theta_n(A, B; z) \) is unity, we have
\[
\Theta_n(A, B; z) = (-B)^{-n} e^{Bz} \left[ (\delta l - A - (n - 1)i) \ldots (\delta l - A - 2(n - 1)i) e^{-Bz} \right].
\]
Applying (4.1), we see that
\[
\Theta_n(A, B; z) = (-B)^{-n} e^{Bz} z^{A+(n-1)i} \frac{d^n}{dz^n} \left( e^{-z(A+(n-1)i)} e^{-Bz} \right).
\]
This result can be expressed in the form.
Theorem 4.1
Let \( A \) and \( B \in \mathbb{C}^{N \times N} \) satisfy (2.1). Then the RGBMPs \( \Theta_n(A, B; z) \) defined in (3.1) may be expressed as

\[
\Theta_n(A, B; z) = (-B)^{-n} e^{Bz} \frac{d^n}{dz^n} \left\{ z^{-((A+(n-1))l)} e^{-Bz} \right\},
\]

for \( n = 0, 1, 2, 3, \ldots \).

Remark 4.1
Using relation 4.3, the RGBMPs \( \Theta_n(A, B; z) \) have the following matrix hypergeometric series representation:

\[
\Theta_n(A, B; z) = (-1)^n \Gamma^{-1} (-A - (2n - 2)L) \Gamma (-A + (n - 2)L) \Gamma (-nl; -A - (2n - 2)L; Bz).
\]

From this representation, we have connection with Laguerre’s matrix polynomials in the form

\[
\Theta_n(A, B; z) = (-1)^n B^{n} I_n^{-2nA} I_n(I_n+A)(Bz).
\]

4.2. Recurrence relations
Among the infinitely many recurrence relations for the RGBMPs, we list the following two as being the most useful or interesting ones.

\[
(A + (n - 1)L)(A + 2(n - 1)L) \Theta_{n+1}(A, B; z) = \left[(A + 2nL)(A + 2(n - 1)L)B^{-1} + (A - 2L)z\right] (A + (2n - 1)L) \Theta_n(A, B; z)
+ n(A + 2nL)z^2 \Theta_{n-1}(A, B; z),
\]

\[
((A - 2L)B^{-1} + nl) \Theta'_n(A, B; z) = n \Theta_n(A, B; z) - nz \Theta_{n-1}(A, B; z).
\]

It can easily verify these relations through a Rodrigues’ formula for the RGBMPs.

Other recurrence relations for the RGBMPs \( \Theta_n(A, B; z) \) may be derived from the relations in (4.4), (4.6), and (4.7). It should be noted that these relations continue to hold for negative values of \( n \), as may be verified by using \( \Theta_{-n}(A, B; z) = \Theta_{n-1}(A, B; z) \), see Remark 3.1.

5. A pseudo-generating function for \( \Theta_n(A, B; z) \)

The last major property developed here is a pseudo-generating function for the RGBMPs \( \Theta_n(A, B; z) \) derived from Rodrigues’ formula, then using Theorem 4.1, we then have

Theorem 5.1
Let \( A \) and \( B \) be matrices in \( \mathbb{C}^{N \times N} \) satisfying the spectral condition (2.1), and let \( z \) be a complex number. Then a pseudo-generating function for \( \Theta_n(A, B; z) \) is given by

\[
\frac{(1 - w)^{2l-A} e^{Bzw}}{1 - 2w} = \sum_{n=0}^{\infty} \frac{(w(1 - w))^n}{n!} \Theta_n(A, B; z),
\]

for sufficiently small values of \( w \).

Proof
The following auxiliary formula, derived in [3], will be useful in the sequel.

\[
\sum_{n=0}^{\infty} \frac{(n-k)[w(1 - w)]^n}{n!} = (1 - 2w)^{-1} (1 - w)^{k+1}.
\]

Relations (4.3), (5.1), and (5.2) lead to

\[
\sum_{n=0}^{\infty} \frac{(w(1 - w))^n}{n!} \Theta_n(A, B; z) = e^{Bz} \sum_{n=0}^{\infty} \frac{(-1)^n(w(1 - w))^n}{n!} \sum_{m=0}^{\infty} \frac{((m-2n+2)L-A)n(-Bz)^m}{m!}
= e^{Bz} \sum_{m=0}^{\infty} \frac{(-Bz)^m}{m!} \sum_{n=0}^{\infty} \frac{(A + (n - m - 1)L)n[w(1 - w)]^n}{n!}
= e^{Bz} (1 - 2w)^{-1} \sum_{m=0}^{\infty} \frac{(-Bz)^m(1 - w)^{m+2l-A}}{m!}
= (1 - 2w)^{-1} (1 - w)^{2l-A} e^{Bzw}.
\]

As required.
Remark 5.1
The operational methods of the present work enable us to repair one omission in [28], namely the failure to supply a generating function for the matrix polynomials $Y_n(A, B; z)$. Indeed, replacing $z$ by $\frac{1}{t}$ and $w$ by $zw$ in (5.1), we shall have

\[
(1 - zw)^{2(1 - A)t} \exp \left( -\left( \frac{B}{2} \left( 1 - \frac{1}{2} - 2zt \right) \right) \right) = \sum_{n=0}^{\infty} \frac{(B/2)^n}{n!} Y_n(A, B; z) t^n,
\]  
and setting $2w(1 - zw) = t$ or $2zw = 1 - (1 - 2zt)\frac{1}{t}$ in (5.4), we find

\[
\left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - 2zt} \right]^{2(1-A)} (1 - 2zt)^{-\frac{1}{2}} \exp \left[ B \left( 1 - \frac{1}{2} - 2zt \right) \right] = \sum_{n=0}^{\infty} \frac{(B/2)^n}{n!} Y_n(A, B; z) t^n.
\]  
which may serve as a generating function for the matrix polynomials $Y_n(A, B; z)$.

Invoking to $Y_n(A, B; z)$ by $z^n \Theta_n (A, B; \frac{1}{t})$ in (5.5), we obtain

\[
\left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - 2zt} \right]^{2(1-A)} (1 - 2zt)^{-\frac{1}{2}} \exp \left[ B \left( 1 - \frac{1}{2} - 2zt \right) \right] = \sum_{n=0}^{\infty} \frac{(B/2)^n}{n!} z^n \Theta_n (A, B; \frac{1}{t}) t^n.
\]  
Replacing $z$ by $\frac{1}{t}$, one gets

\[
\sqrt{\frac{z}{z - 2t}} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{\frac{z}{z - 2t}} \right]^{2(1-A)} \exp \left[ \frac{Bz}{2} \left( 1 - \frac{1}{\sqrt{\frac{z}{z - 2t}}} \right) \right] = \sum_{n=0}^{\infty} \frac{(B/2)^n}{n!} \Theta_n (A, B; z) \left( \frac{t}{z} \right)^n.
\]  
Therefore the following result can be obtained.

Theorem 5.2
Let $A$ and $B$ be two matrices in $\mathbb{C}^{N \times N}$ and satisfy the spectral condition (2.1), and let $z$ and $t$ be complex numbers. Then the generating function for the RGBMPs is given by

\[
\sqrt{\frac{z}{z - 2t}} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{\frac{z}{z - 2t}} \right]^{2(1-A)} \exp \left[ \frac{Bz}{2} \left( 1 - \frac{1}{\sqrt{\frac{z}{z - 2t}}} \right) \right] = \sum_{n=0}^{\infty} \frac{(B/2)^n}{n!} \Theta_n (A, B; z) \left( \frac{t}{z} \right)^n.
\]  

6. Concluding comments

The material developed in Sections 3–5 provides several important properties of the reverse generalized Bessel matrix polynomials $\Theta_n(A, B; z)$, $z \in \mathbb{C}$ introduced in (3.1), under the spectral conditions (2.1) on the parameter matrices $A$ and $B$. The $\Theta_n(A, B; z)$ are first shown to satisfy more than one second-order differential equation as indicated in (3.2), (3.7), (3.8), (3.10), and (3.11). Then, with the added condition of commutativity, $AB = BA$, a Rodrigues’ type formula and recurrence relations for the $\Theta_n(A, B; z)$ are derived.

As for the orthogonality property of $\Theta_n(A, B; z)$, it should be observed that the expansion of the $\chi(z)$ defined by (4.2) lacks all terms with indices between $(A + (n - 1))l$ and $(A + 2(n - 1))l$ inclusive. The expansion may in fact be divided by this gap into two parts furnishing, respectively, a polynomial and a series solution. An important consequence of this gap in the expansion of $\chi(z)$ is the following. From (4.2), we have

\[
e^{-Bz} \Theta_n (A, B; z) = \sum_r c_r z^r + \sum_{r=-1}^{\infty} c_{A+(2n+1)r} z^{A+(2n+1)r},
\]  
where $c_0 = B^{-n} (A + (n - 1))$, $c_{A+(2n+1)r} = n! (-B)^{A+(n-1)r} \Gamma^{-1}(A + 2n!l)$. Of course, the splitting formula (6.1) is natural in the non-matrix case, that is, if $A$ is an integer. For non-integral $A$, the formula (6.1) will not be true. The exception is where $A$ would be an integer multiple a $l$ of the unit matrix (then it splits as in the scalar case), or when all eigenvalues of $A$ are integers, in which case the same subset of the range of indices might vanish (but this case is not elegant). Therefore, the generalization to the matrix setting of this question of orthogonality appears to be complicated and not very fruitful. We hope that it can be solved.

References
1. Bohner S. Über Sturm–Liouvillische polynomsysteme. Mathematische Zeitschrift 1929; 29:730–736.
2. Romanovsky V. Sur quelques classes nouvelles des polynômes orthogonaux. Comptes Rendus de l’Académie des Sciences de Paris - Series I - Mathematics 1929; 188:1023–1025.
3. Burchnell JL. The Bessel polynomials. *Canadian Journal of Mathematics* 1951; **3**:62–67.

4. Martinez JR. Transfer functions of generalized Bessel polynomials. *IEEE Transactions on Circuits and Systems* 1977; **24**:325–328.

5. Grosswald E. *Bessel Polynomials*, Lecture Notes in Math, Vol. 698. Springer: New York, 1978.

6. Abul-Ez M. Bessel polynomial expansions in spaces of holomorphic functions. *Journal of Mathematical Analysis and Applications* 1998; **221**:177–190.

7. Dunster TM. Uniform asymptotic expansions for the reverse generalized Bessel polynomials, and related functions. *SIAM Journal on Mathematical Analysis* 2001; **32**:987–1013.

8. Doha EH, Ahmed HM. Recurrences and explicit formulae for the expansion and connection coefficients in series of Bessel polynomials. *Journal of Physics A: Mathematical and General* 2004; **37**:8045–8063.

9. Polat ZS. Studies on the generalized and reverse generalized Bessel polynomials. MSc Thesis, The Graduate School of Natural and Applied Sciences of the Middle East Technical University, April 2004. http://etd.lib.metu.edu.tr/upload/12604961/index.pdf.

10. López JL, Temmeb NM. Large degree asymptotics of generalized Bessel polynomials. *Journal of Mathematical Analysis and Applications* 2011; **377**:30–42.

11. Krawtchouk M, Frink O. A new class of orthogonal polynomials: the Bessel polynomials. *Transactions of the American Mathematical Society* 1949; **65**:100–115.

12. Thompson WE. Delay network having maximally flat frequency characteristics. *Proceedings of the Institute Electrical Engineers* 1949; **96**:487.

13. Jódar L, Defez E, Ponsoda E. Orthogonal matrix polynomials with respect to linear matrix moment functions: theory and applications. *Approximation Theory and its Applications* 1996; **12**:96–115.

14. Jódar L, Defez E, Ponsoda E. Orthogonal matrix polynomials and systems of second order differential equations. *Differential Equations and Dynamical Systems* 1995; **3**:269–288.

15. Defez E, Jódar L, Company R, Ponsoda E. Orthogonal matrix polynomials and systems of second order differential equations. *Computers & Mathematics with Applications* 2000; **48**:789–803.

16. Defez E, Jódar L, Law A, Ponsoda E. Three-term recurrences and matrix orthogonal polynomials. *Utilitas Mathematica* 2000; **57**:129–146.

17. Duran AJ, López-Rodriguez P. Density questions for the truncated matrix moment problem. *Canadian Journal of Mathematics* 1997; **49**:708–721.

18. Duran AJ, López-Rodriguez P. Orthogonal matrix polynomials: Zeros and Blumenthal’s theorem. *Journal of Approximation Theory* 1996; **84**:96–118.

19. Duran AJ. On orthogonal polynomials with respect to a positive definite matrix of measures. *Canadian Journal of Mathematics* 1995; **47**:88–112.

20. Duran AJ, Van Assche W. Orthogonal matrix polynomials and higher order recurrence relations. *Linear Algebra and its Applications* 1995; **219**:261–318.

21. Daux A, Jokung-Nguena O. Orthogonal polynomials in a non-commutative algebra. Non-normal case. *IMACS Annals on Computing and Applied Mathematics* 1991; **9**:237–242.

22. Geronimo JS. Scattering theory and matrix orthogonal polynomials on the real line, applications. *Approximation Theory and its Applications* 1996; **12**:96–115.

23. Defez E, Jódar L. Some applications of the Hermite matrix polynomials series expansions. *Journal of Computational and Applied Mathematics* 1998; **99**:105–117.

24. Defez E, Jódar L, Law A. Jacobi matrix differential equation, polynomial solutions, and their properties. *Computers & Mathematics with Applications* 2004; **48**:789–803.

25. Defez E, Jódar L, Chebyshev matrix polynomials and second order matrix differential equations. *Utilitas Mathematica* 2002; **61**:107–123.

26. Jódar L, Company R, Navarro E. Laguerre matrix polynomials and systems of second order differential equations. *Applied Numerical Mathematics* 1994; **15**:53–63.

27. James AT. Special functions of matrix and single argument in statistics. In *Theory and Application of Special Functions*, Askey RA (ed.). Academic Press: New York, 1975; 497–520.

28. Kishka ZMG, Shehata A, Abul-Dahab M. The generalized Bessel matrix polynomials. *Journal of Mathematical and Computational Science* 2012; **2**:305–316.

29. Sastre Martinez J. Laguerre matrix polynomials: series expansion and applications. *Ph.D. Thesis*, Universidad Politecnica de Valencia, September 2003.

30. Sinap A, Van Assche W. Polynomial interpolation and Gaussian quadrature for matrix valued functions. *Linear Algebra and its Applications* 1994; **207**:71–114.

31. Jódar L, Company R. Hermite matrix, polynomials and second order differential equations. *Approximation Theory and its Applications* 1996; **12**:20–30.

32. Jódar L, Cortés JC. Closed form general solution of the hypergeometric matrix differential equation. *Mathematical and Computer Modelling* 2000; **32**:1017–1028.

33. Defez E, Villanueva-Oller J, Villanueva RJ, Law A. Matrix cubic splines for progressive 3D imaging. *Journal of Mathematical Imaging and Vision* 2002; **17**:41–53.

34. Defez E, Hervás A, Law A, Villanueva-Oller J, Villanueva RJ. Progressive transmission of images: PC-based computations, using orthogonal matrix polynomials. *Mathematical and Computer Modelling* 2000; **32**:1125–1140.

35. Brualdi R, Cvetković D. *A Combinatorial Approach to Matrix Theory and Its Applications*. CRC Press: Taylor and Francis Group, LLC. New York, 2009.

36. Jódar L, Cortés JC. On the hypergeometric matrix function. *Journal of Computational and Applied Mathematics* 1998; **99**:205–217.