Tkachenko modes of vortex lattices in rapidly rotating Bose-Einstein condensates

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We calculate the in-plane modes of the vortex lattice in a rotating Bose condensate from the Thomas-Fermi to the mean-field quantum Hall regimes. The Tkachenko mode frequency goes from linear in the wavevector, \( k \), to lattice rotational velocities, \( \Omega \), much smaller than the lowest sound wave frequency in a finite system, to quadratic in \( k \) in the opposite limit. The system also supports an inertial mode of frequency \( \gg 2\Omega \). The calculated frequencies are in good agreement with recent observations of Tkachenko modes at JILA, and provide evidence for the decrease in the shear modulus of the vortex lattice at rapid rotation.

The collective modes of the vortex lattice in a rotating superfluid have been of considerable interest since the 1960’s. In a classic series of papers Tkachenko [1] showed that the lattice supports an elliptically polarized oscillatory mode, with the semi-major axis of the ellipse orthogonal to the direction of propagation. Such modes were observed in superfluid helium in 1982 [2]. Reference [3] reformulated the hydrodynamics of rotating superfluids to take into account the elasticity of the vortex lattice (including the normal fluid, dissipation, and line bending – Kelvin – oscillations of the vortex lines in three dimensions) and thus describe the Tkachenko modes; effects of the oscillations of the vortex lines at finite temperature on the long ranged phase correlations of the superfluid were discussed in [4]. The focus of these papers was on superfluid helium, where rotational speeds are always much smaller than characteristic phonon frequencies, of order 1K/h.

Atomic Bose condensates, on the other hand, allow one to study superfluids over a large range of rotational speeds [5–9], from the Thomas-Fermi regime where lattice rotational frequencies, \( \Omega \), are small compared with phonon frequencies – of order the trapping frequency – to the quantum Hall limit of rapid rotations [10–13], where a condensate in a harmonic trap flattens to a very weakly interacting effectively two dimensional system, and rotational speeds can well exceed phonon frequencies. One may identify three distinct physical regimes: first, the “stiff” Thomas-Fermi regime, where \( \Omega \) is small compared with the lowest compressional frequencies, \( sk_0 \), where \( s \) is the sound velocity, \( k_0 \sim 1/R \) is the lowest wavenumber in the finite geometry, and \( R \) is the size of the system transverse to the rotation axis. In this regime the system responds to rotation effectively as an incompressible fluid. At faster rotation, when \( \Omega \gg sk_0 \), but \( \Omega \ll ms^2 \), where \( m \) is the atomic mass, the system is in the “soft” Thomas-Fermi regime, where compression of the superfluid becomes important in the response of the lattice. Finally, when \( \Omega \gg ms^2 \), the system enters the “mean field” quantum Hall regime, in which the condensation is only in lowest Landau orbits. Eventually the vortex lattice melts [14,15], and the system enters a strongly correlated regime [10–12].

Observations of Tkachenko modes in Bose-condensed \(^{87}\text{Rb}\) were reported recently by Coddington et al. [16] at rotation speeds up to 0.975 of the transverse trapping frequency, \( \omega_\rho \), in this regime, \( \Omega \) increases past \( sk_0 \), and although not yet in the quantum Hall regime, the experiments show effects on the modes frequencies of very rapid rotation. The modes in rotating atomic condensates have been the subject of several theoretical investigations, including determination of the Tkachenko modes at slow rotation taking full account of the finite geometry [17], determination of the fundamental modes in the absence of effects of the elastic energy of the lattice [18], and studies in the quantum Hall regime [15,19–21]. In this Letter we derive the modes of the vortex lattice at general rotation speeds. The present analysis is restricted to linearized motion in two dimensions transverse to the rotation axis, and neglects the normal fluid [22]. Our starting point is the conservation laws and superfluid acceleration equation governing the system, including the full elasticity of the lattice. Taking into account the decrease of the shear modulus of the vortex lattice with increasing \( \Omega \), one can fully understand the measured Tkachenko frequencies.

We consider a rotating gas of bosons described by a repulsive short range repulsive interaction, \( U(r) = g\delta^3(r) \), with \( g = 4\pi a_s/m \), where \( a_s \) is the s-wave scattering length, in units in which \( \hbar = 1 \). The angular momentum of the superfluid is carried in vortices, which form a triangular lattice rotating as a solid body at angular velocity \( \Omega \). We work in the frame corotating with the vortex lattice. and denote the deviations of the vortices from their home positions by the continuum displacement field, \( \epsilon(r,t) \). In linear order in the vortex displacements, the long wavelength superfluid velocity, \( v(r,t) \), can be written, following [4], in terms of the long wavelength
vortex lattice displacement field, $\epsilon(r, t)$, and the phase $\Phi(r, t)$ of the order parameter, as

$$v + 2\Omega \times \epsilon = \nabla \Phi/m.$$  (1)

This equation follows from the fact that its curl,

$$\nabla \times v = -2\Omega \nabla \cdot \epsilon,$$  (2)

is the conservation law relating the change in vorticity to a compression of the vortex lattice, while its longitudinal part is trivially the gradient of a scalar [23]. Equation (2) constrains the number of degrees of freedom in two dimensions from five ($n, v, \epsilon$) to four. The time derivative of Eq. (1) is the superfluid acceleration equation,

$$m \left( \frac{\partial}{\partial t} + 2\Omega \times \dot{\epsilon} \right) = -\nabla (\mu - V_{\text{eff}}),$$  (3)

where $\mu$ is the chemical potential, and, for a harmonic confining trap of frequency $\omega_p$ in two dimensions,

$$V_{\text{eff}} = \frac{m}{2} (\omega_p^2 - \Omega^2) r^2.$$  (4)

In the frame corotating with the vortex lattice, the chemical potential $\mu$ is related to the phase by

$$\mu(r, t) - V_{\text{eff}} = -\frac{1}{m} \frac{\partial \Phi(r, t)}{\partial t}.$$  (5)

The local elastic energy density of a triangular lattice in two dimensions has the form [3],

$$E(r) = 2C_1 (\nabla \cdot \epsilon)^2 + C_2 \left[ \left( \frac{\partial \epsilon_x}{\partial x} - \frac{\partial \epsilon_y}{\partial y} \right)^2 + \left( \frac{\partial \epsilon_x}{\partial y} + \frac{\partial \epsilon_y}{\partial x} \right)^2 \right],$$  (6)

where $C_1$ is the compressional modulus, and $C_2$ the shear modulus of the vortex lattice. In an incompressible fluid, $C_2 = n\Omega/8 = -C_1$. The shear modulus $C_2$ in fact decreases with increasing $\Omega$, from $2n\Omega/8$ in the incompressible limit $\Omega$, eventually reaching, in the mean field quantum Hall limit, the value [15,21], $C_2 \approx (81/80\pi^4)ms^2n$. In this limit $C_1 = 0$. The falloff for small $\Omega/ms^2$ has the form,

$$C_2 \approx \frac{\Omega n}{8} \left( 1 - \gamma \frac{\Omega}{ms^2} + \ldots \right);$$  (7)

a first estimate [21] is $\gamma \sim 4$.

The dynamics is specified by the superfluid acceleration equation (3); the continuity equation,

$$\frac{\partial n(r, t)}{\partial t} + \nabla \cdot j(r, t) = 0,$$  (8)

where $n$ is the (smoothed) density, and $j = nv$ is the particle current; and conservation of momentum:

$$m \left( \frac{\partial j}{\partial t} + 2\Omega \times j \right) + \nabla P + n\nabla V_{\text{eff}} = -\sigma,$$  (9)

where $P$ is the pressure; at zero temperature, $\nabla P = n\nabla \mu$, while in equilibrium, $\nabla P + n\nabla V_{\text{eff}} = 0$. The elastic stress, $\sigma$, is given in terms of the total elastic energy, $E_{\text{el}} = \int d^2r E(r)$, by

$$\sigma(r, t) = \frac{\delta E_{\text{el}}}{\delta \epsilon} = -4C_1 \nabla (\nabla \cdot \epsilon) - 2C_2 \nabla^2 \epsilon.$$  (10)

Equations (9) and (3), with $\nabla P = n\nabla \mu$, imply $2\Omega \times (\dot{\epsilon} - v) = \sigma/mn$. The curl of this equation becomes

$$\nabla \cdot (\dot{\epsilon} - v) = \nabla \times \sigma = \frac{C_2}{\Omega mn} \nabla^2 (\nabla \cdot \epsilon),$$  (11)

while its divergence, together with (2), yields

$$\nabla \times \dot{\epsilon} + 2\Omega \nabla \cdot \epsilon = -\nabla \cdot \sigma = \frac{C_2 + 2C_1}{\Omega mn} \nabla^2 (\nabla \cdot \epsilon).$$  (12)

The structure of the modes follows directly from the linearized set of equations: (2), (11), (12), and the divergence of (3). The density oscillations are governed by

$$\left( -\frac{\partial^2}{\partial t^2} + s^2 \nabla^2 \right) n = 2n\Omega \nabla \times \epsilon,$$  (13)

where the sound speed $s$ is given by $ms^2 = \partial P/\partial n$ [24]. In the absence of coupling of density oscillations to compressions of the vortex lattice, this equation is simply that of long wavelength phonons in the condensate. Using Eq. (11) to eliminate $\nabla \times \epsilon$, and ignoring the term of order $\nabla^4$, we find

$$\left( -\frac{\partial^2}{\partial t^2} + \frac{2C_2}{mn} \nabla^2 \right) \nabla \cdot \epsilon = \frac{1}{n} \frac{\partial^2}{\partial n^2} n.$$  (14)

In the absence of coupling of density oscillations to compression of the vortex lattice, this equation would describe a free Tkachenko mode of the vortex lattice [1,3,4] of wavevector $k$ and frequency, $\omega_T = (2C_2/mn)^{1/2}k$.

The coupled equations (13) and (14) yield the full spectrum of long wavelength modes. The frequencies, for given $k$, are solutions of the secular equation,

$$D(k, \omega) \equiv \omega^4 - \omega^2 \left[ 4\Omega^2 + \left( s^2 + \frac{4}{nm} (C_1 + C_2) \right) k^2 \right] + \frac{2s^2C_2}{nm} k^4 = (\omega^2 - \omega_T^2)(\omega^2 - \omega^2_T) = 0.$$  (15)

For $2s^2C_2k^4/nm \ll (4\Omega^2 + (s^2 + 4(C_1 + C_2)/nmk^2)^2$, as is generally the case, the mode frequencies are given by

$$\omega_T^2 = 4\Omega^2 + \left( s^2 + 4(C_1 + C_2)/nm \right) k^2,$$  (16)

and
\[ \omega^2 = \frac{2C_2}{nm} \left( 4\Omega^2 + s^2k^4 \right) \]  

(17)

The first mode is the standard inertial mode of a rotating fluid; for \( \Omega \ll s^2k^2 \) it is a sound wave, while for \( \Omega \gg s^2k^2 \), the mode frequencies begin essentially at \( 2\Omega \). The second mode is the observed elliptically polarized Tkachenko mode. See Fig. 1. As Eq. (12) implies, the longitudinally polarized component is \( \pi/2 \) out of phase with the transversely polarized component; of order \( \omega_T/2\Omega \) smaller for the Tkachenko mode and of similar size for the inertial mode [26]. In the stiff limit, the Tkachenko frequency, \( \omega_T = (\Omega/4m)^{1/2}k \), is linear in \( k \).

By contrast, in the very soft limit, the mode frequency, \( \omega_T \) is quadratic in \( k \) at long wavelengths

\[ \omega_T = \left( \frac{C_2}{2nm} \right)^{1/2} \frac{sk^2}{\Omega}; \]  

(18)

in the quantum Hall regime \( \omega_T \simeq (9/4\pi^2\sqrt{\Omega})(s^2k^2/\Omega) \) [15].

FIG. 1. The inertial and Tkachenko mode frequencies vs. wavevector measured in units of \( 2\Omega/s \). The inertial mode (upper curve) is in units of \( 2\Omega \), while the Tkachenko mode (lower curve) is in units of \( (\Omega^2/ms^2)^{1/2} \). The modes are calculated for \( C_1, C_2 \) in the Thomas-Fermi regime. In the quantum Hall regime, the Tkachenko modes are softer by a factor \( (9/\pi^2)(ms^2/10\Omega)^{1/2} \).

The very soft Tkachenko mode in the rapidly rotating regime leads to infrared singular behavior in the vortex transverse displacement-displacement correlations at finite temperature, and in the order parameter phase correlations even at zero temperature [15,21]. In a finite system the single particle density matrix, \( \langle \psi(r)\psi^\dagger(r') \rangle \), falls algebraically as \( (4N_v/\pi^2)^{-\eta} \) for \( |r - r'| \simeq R \), where \( \eta \simeq \pi^2\sqrt{10N_v}/9N \), \( N_v \) is the total particle number, and \( N_t \) is the total number of vortices present [21]. However, dephasing of the condensate only becomes significant as \( N_v \to N \), and not necessarily before the vortex lattice melts.

\[ R^2 = d^2\tau(1 - x)^{-3/5}, \]  

(19)

where \( x = \Omega^2/\omega_p^2 \), and \( d = 1/\sqrt{m\omega_p} \) is the transverse oscillator length, \( \tau = 1/(15N \bar{b}_a/d)(\omega_z/\omega_p)^{1/5} \), and \( \omega_z \) is the axial trapping frequency. The sound velocity, evaluated in the center of the trap (for simplicity), is

\[ ms^2 = gbn(0) = \frac{\omega_p^2}{2}\tau(1 - x)^{2/5}, \]  

so that

\[ \frac{\Omega}{ms^2} = \frac{2}{\tau(1 - x)^{2/5}}. \]  

(21)

Also, \( (\Omega/\omega_k)^2 = 2x/(1 - x)^2 ; \) in the present experiments, \( \Omega/\omega_k \) reaches \( \sim 1.15 \). Ignoring \( b \simeq 1 \) here [24], as well as the small \( C_1 + C_2 \) term in Eq. (17), we have,

\[ \omega_T^2 = \frac{\omega_p^2}{4\tau} \frac{x^{1/2}(1 - x)^{3/5}}{1 - x(1 - 8/\alpha^2)}. \]  

(22)

FIG. 2. Frequency of the lowest Tkachenko mode. The upper curve is for constant shear modulus, \( C_2 \), while the lower curve includes a decreasing \( C_2 \) (Eq. (7)) with \( \gamma = 4 \). The data (triangles), from Ref. [16], are multiplied by a factor \( (N/2.5 \times 10^6)^{-2/5} \) to compare with theory, which is calculated for \( N = 2.5 \times 10^6 \). The inset shows the Tkachenko frequency over the entire range of \( \Omega \); the upper (short dashed) curve is the mode frequency to lowest order in the wavevector, the middle (dashed) curve includes the full frequency dependence at constant \( C_2 \), and the lower curve includes the decreasing \( C_2 \) as well.
The inset in Fig. 2 shows the Tkachenko mode frequencies as a function of $\Omega/\omega_p$, illustrating the initial square root rise and the eventual falloff for $\Omega \lesssim \omega_p$. The curves are evaluated for the parameters of Ref. [16], $(\omega_p, \omega_z) = 2\pi(8.3, 5.2)$ Hz, at the representative condensate number, $N = 2.5 \times 10^6$. Figure 2 proper shows these curves in the region measured in [16]. The upper curve is calculated from Eq. (22) while the lower includes the decrease (7) in the shear modulus of the vortex lattice, with $\gamma = 4$. The experiments of [16] are at particle numbers $\sim (0.7 - 3) \times 10^7$. According to Eq. (22), the frequencies scale as $N^{-2/5}$. Thus to facilitate comparison with theory, shown for $N = 2.5 \times 10^6$, we have scaled the individual data points down by a factor $(N/2.5 \times 10^6)^{2/5}$, which is equivalent to scaling the theory up by the same factor. The JILA experiments provide evidence for the frequency dependence of the shear modulus of the vortex lattice.

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[22] The analysis of the full three dimensional problem will be published separately by S.A. Gifford and G. Baym.
[23] Although the local superfluid velocity in a rotating fluid is irrotational, its long wavelength average in not [3].
[24] In a weakly interacting homogeneous gas, $s^2 = gn/m$; however here one must take into account the correction to the interaction energy arising from the non-uniformity of the density around each vortex. As derived in [25], the energy density is renormalized to $gn^2b/2$, where $n$ is the mean density in a unit cell of the lattice, and $b = \langle n^2 \rangle /n^2$ is a numerical correction $\approx 1$, dependent on the details of the density distribution in the cell. At slow rotation, $b = 1$, while in the quantum Hall regime $b = 1.16$ numerically, [15]. Neglect of the dependence of $b$ on the mean density implies $ms^2 = gn b$, the result we use here.
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[26] In three dimensions, the number of degrees of freedom remains four, since Eq. (1) now implies two constraints on the six degrees of freedom, $n$, $v$, and $\epsilon$, and system still supports only two long wavelength modes. As the wavevector becomes more aligned with the rotation axis, the Tkachenko mode tends to stiffen while the inertial mode tends to soften. In both modes the vortex motion remains elliptically polarized with the longitudinal component $\sim -i\omega/2\Omega$ times the transverse component [22].