1. Introduction

In this paper we introduce a new global integral which represents the standard $L$ function attached to a cuspidal representation of the exceptional group $G_2(\mathbb{A})$. Here $\mathbb{A}$ is the adele ring of a global field $F$. In [PS-R] the authors introduced a global integral which represents...
the standard $L$ function for a classical group $H$. To describe their construction, for $i = 1, 2$, let $\sigma_i$ denote two cuspidal representations of $H(\mathbb{A})$. Their global integral is given by

$$\int_{H(F) \times H(F) \backslash H(\mathbb{A}) \times H(\mathbb{A})} \varphi_{\sigma_1}(h_1) \varphi_{\sigma_2}(h_2) E((h_1, h_2), s) dh_1 dh_2$$

Here $E(\cdot, s)$ is a certain Eisenstein series. Unfolding the integral, one obtains the bilinear form

$$<\pi_1(h)\varphi_{\sigma_1}, \varphi_{\sigma_2}> = \int_{H(F) \backslash H(\mathbb{A})} \varphi_{\sigma_1}(h_1)\varphi_{\sigma_2}(h_1) dh_1$$

as inner integration. Using that, the authors established the fact that the global integral is Eulerian. Moreover, assuming that $\sigma_1$ and $\sigma_2$ are contragredient, then this integral is nonzero for some choice of data. Hence, the above global integral is nonzero for all cuspidal representation $\sigma_1$ of $H(\mathbb{A})$. Integrals of this type are now known as “doubling integrals.”

In this paper we construct a doubling integral which represents the standard $L$ function for $G_2(\mathbb{A})$. The global integral we construct uses a certain Eisenstein series defined on the exceptional group $E_8(\mathbb{A})$. It is introduced in section 2 integral (3). Thus, as in the classical groups, we obtain an integral construction which represents the standard $L$ function, and which is not zero for all cuspidal representations. By attaching a character $\chi$ to the Eisenstein series, our construction actually represents the twisted $L$ function.

A global construction for this $L$ function is given in [G1], which is valid only for generic cuspidal representations. Another construction for this $L$ function, which is valid also for cuspidal representations which are not generic, is given in [S]. In contrast to our construction and to [G1], this global integral unfolds to a non-unique model.

In this paper we prove the very basic properties of our construction. In section 2, after some preparation we introduce the global integral, unfold it, and show that it is Eulerian. In the third section, using MacDonald’s formula, we carry out the unramified computations. In sections 4 and 5 we carry out some computations which we use in the proof of the unramified calculations. For some of the calculations, we used the software LiE [L] and another software program [E], written by the second named author. This is mainly due to the sheer size of the calculations, owing to the large number of roots in $E_8$, the size of the Weyl group, etc., as well as to the large number of such calculations which must be performed. It is worth noting that any individual calculation which was performed with software can also be checked by hand.

We also mention, that as in [PS-R] pages 50-51, the local unramified computation gives us a definition of a generating function for the standard $L$ function. Thus, it follows from section 3 equation (16), that if we define $N(s)$ to be the normalizing factor of the Eisenstein
series (cf. section 4) and we define

$$\Delta_s(g) = N(s) \int_{U_0(F)} f(w_0 z u(1, g), s \psi_U(u) du$$

then

$$\int_{G_2(F)} \omega_{\pi}(g) \Delta_s(g) dg = L(\pi, s)$$

We expect that our global construction will have certain applications in the study of this specific $L$ function, and its twists by a character $\chi$. The first, is the study of the poles of this $L$ function. The construction in [G1] implies that for generic cuspidal representations, this $L$ function can have at most a simple pole. As follows from section 4, one expects that the Eisenstein series we use would have at most double poles. In fact one expects that there are cuspidal representations, which are CAP with respect to the Borel subgroup, whose $L$ functions will have a double pole. Such CAP representations were constructed in [G-G-J].

In the near future we hope to prove the following

**Conjecture 1.** The twisted partial standard $L$ function $L^S(\pi \otimes \chi, s)$ can have at most a double pole.

2. **The Global Integral**

In this section we introduce the global integral, and carry out the unfolding process. We work with the unique split $F$-group of type $E_8$, which we assume to be equipped with a choice of maximal torus $T$ and Borel subgroup $B = TU_{\text{max}}$. Here $U_{\text{max}}$ is a maximal unipotent subgroup of $E_8$. For $H \subset E_8$ a $T$-stable subgroup, $\Phi(H, T)$ is the set of roots of $T$ in $H$. Also for any reductive group $H$ with maximal torus $S$, $W(H, S)$ is the Weyl group of $H$ relative to $S$. For $\Phi(E_8, T)$ and $W(E_8, T)$, we may write $\Phi$ and $W$ respectively. In this paper we shall label the roots of $E_8$ by $\alpha_i$, $(1 \leq i \leq 8)$. The labeling we use is as in [G-S]. Let $w_i$ denote the simple reflection corresponding to the root $\alpha_i$. We shall denote the product $w_{i_1} \ldots w_{i_n}$ by $w[i_1 \ldots i_n]$. Write $U_\alpha$ for the one-dimensional unipotent subgroup attached to the root $\alpha$, and equip $G$ with a realization $\{x_\alpha : \mathbb{G}_a \to U_\alpha \mid \alpha \in \Phi(E_8, T)\}$, consisting of an isomorphism $\mathbb{G}_a \to U_\alpha$ for each root $\alpha$. We assume that the structure constants are determined as in [G-S]. For $1 \leq i \leq 8$, the product $x_{\alpha_i}(1) x_{-\alpha_i}(-1) x_{\alpha_i}(1)$ is a representative for the simple reflection $w_i$. This, in turn, determines a standard representative in $G(F)$ for any word in the simple reflections. We shall often abuse notation by conflating this representative with the Weyl word it represents.
2.1. **Eisenstein Series.** For $1 \leq i \leq 8$ we let $P_i$ denote the standard maximal parabolic subgroup of $E_8$ such that $\alpha_i$ is a root of the unipotent radical and the remaining simple roots of $E_8$ are roots of the standard Levi factor. We consider the group $P_2$. It’s Levi factor, $M_2$, is isomorphic to $\{g \in GL_8 : \det g \text{ is a square}\}$. The group $M_2$ has a rational character whose square is $\det$. (Indeed, $M_2$ acts on the highest weight vectors in the second fundamental representation of $E_8$ by such a representation.) Denote this rational character by $\det^{1/2}$. The modular quasicharacter $\delta_{P_2}$ of $P_2$ is the seventeenth power of $\det^{1/2}$.

Let $\chi$ be a character of $\mathbb{A}^\times$ trivial on $F^\times$. Regard $\chi$ as a representation of $M_2(\mathbb{A})$ by composing with $\det^{1/2}$. Consider the induced representation $Ind_{P_2(\mathbb{A})}^{E_8(\mathbb{A})} \delta_{P_2} \chi$. This representation has an automorphic realization as a space of Eisenstein series. Specifically, for $f_{s,\chi} \in Ind_{P_2(\mathbb{A})}^{E_8(\mathbb{A})} \delta_{P_2} \chi$ the corresponding Eisenstein series is defined by $E(h, f_{s,\chi}) := \sum_{\gamma \in P_2(F) \setminus E_8(F)} f_{s,\chi}(\gamma g)$ for $\Re(s)$ large and by meromorphic continuation elsewhere.

2.2. **Fourier Coefficient.** Next we describe a Fourier coefficient which plays an important role in this construction. As explained in [G2] one can associate with every unipotent orbit of a given group which defined over an algebraic closed field, a set of Fourier coefficients. In [G2] it is explained how to do it in the classical groups, but it is similar in the exceptional groups. The Fourier coefficient we will define is attached to the unipotent orbit $2A_2$ of $E_8$.

In the notation fixed above, $P_1$ denotes the standard maximal parabolic subgroup of $E_8$ whose Levi factor $M_1$ is the product of a derived subgroup isomorphic to $Spin_{14}$ and a one-dimensional torus. Let $U$ denote its unipotent radical of $P_1$. Then $U$ is a two step unipotent group whose dimension is 78. The center $Z(U)$ is 14 dimensional and can be identified with the standard (“vector”) representation of $Spin_{14}$. The quotient $U/Z(U)$ is 64 dimensional and can be identified with a half-spin representation of $Spin_{14}$.

Let $\psi$ denote a nontrivial additive character of the group $F \setminus \mathbb{A}$. The choice of $\psi$ identifies $F$ with the Pontrjagin dual of $F \setminus \mathbb{A}$. Characters of $U(\mathbb{A})$ trivial on $U(F)$ are then identified with the $F$-points of the rational representation of $M_1$ which is dual to $U/Z(U)$. This would correspond to the other half-spin representation of $Spin_{14}$.

Over an algebraically closed field, this representation of $M_1$ has an open orbit. As recorded in the tables on p. 405 of [C] and p.200 of [Ka], the identity component is isomorphic to $G_2 \times G_2$. Moreover, using the discussion on p. 267 of [Ki], it’s not hard to show that the number of components in the stabilizer is two. We now choose a character of $U(F) \setminus U(\mathbb{A})$ which corresponds to a point in general position.

Define the character $\psi_U$ of $U(F) \setminus U(\mathbb{A})$ as follows. Using the fixed realization $\{x_\alpha\}$, write $u \in U(\mathbb{A})$ as $u = x_{11221111}(r_1)x_{11122111}(r_2)x_{12232210}(r_3)x_{11233210}(r_4)u'$, where $u'$ is a product of elements $x_\alpha(u_\alpha)$ corresponding to roots $\alpha$ which are not among the four listed above. Then
we define $\psi_U(u) = \psi(r_1 + r_2 + r_3 + r_4)$. It’s not difficult to calculate the identity component of the stabilizer of this character, which turns out to be isomorphic to $G_2 \times G_2$, proving that the character is indeed in general position.

To be precise, the images of
\[
\begin{align*}
&x_{\pm00001000}(r); \ x_{\pm01000000}(r)x_{\pm00100000}(r)x_{\pm00010000}(r); \ x_{\pm01010000}(r)x_{\pm00110000}(r)x_{\pm00011000}(r) \\
&x_{\pm01011000}(r)x_{\pm01110000}(r)x_{\pm00111000}(r); \ x_{\pm01111000}(r); \ x_{\pm01121000}(r)
\end{align*}
\]
generate a subgroup of $E_8$ which is isomorphic to $G_2 \times G_2$. One may check that an element of this group which normalizes/centralizes its maximal torus also normalizes/centralizes the full maximal torus of $E_8$. This induces an embedding of the Weyl group of this copy of $G_2$ into that of $E_8$. The images of the two simple reflections are $w_4$ and $w[235]$.

A second copy of $G_2$ is generated by the images of the following homomorphisms:
\[
\begin{align*}
&x_{\pm00000010}(r); \ x_{\pm00000001}(r)x_{\pm01011100}(r)x_{\pm00111000}(r); \ x_{\pm01000011}(r)x_{\pm01011110}(r)x_{\pm00111100}(r) \\
&x_{\pm01122210}(r)x_{\pm01111110}(r)x_{\pm00111111}(r); \ x_{\pm01122211}(r); \ x_{\pm01122221}(r).
\end{align*}
\]
In this case, the simple reflections in the Weyl group map to $w_7$ and $w[657423456]$ in the Weyl group of $E_8$.

Finally, suitable representatives for the Weyl element $w[657486576]$ stabilize $\psi_U$ and act on the two copies of $G_2$ by reversing them.

We shall denote the stabilizer of $\psi_U$ by $H^\pm$ and its identity component by $H$. Note that $T \cap H$ and $U_{\text{max}} \cap H$ are a maximal torus and maximal unipotent subgroup of $H$, respectively. The one parameter subgroups listed above equip each copy of $G_2$ with a realization and the structure constants of the two realizations. Thus, by equipping $G_2$ (regarded as an abstract group) with a realization having the same structure constants, we pin down a specific identification of $H$ with $G_2 \times G_2$.

One may now form the Fourier coefficient mapping
\[
\varphi \mapsto \varphi^{(U,\psi_U)}(h) := \int_{U(F)\backslash H(F)} \varphi(uh)\psi_U(u)du, \quad (\varphi \in C^\infty(HU(F)\backslash E_8(A)))
\]
with image in the space $C^\infty(H(F)U(A)\psi_U \backslash E_8(A))$, of smooth, left $H(F)$-invariant, $(U(A),\psi_U)$-equivariant functions $E_8(A) \to \mathbb{C}$.

2.3. **Global Integral.** In order to define a global integral, we apply the Fourier coefficient mapping defined in section 2.2 to the Eisenstein series of section 2.1. The result is a smooth function of uniformly moderate growth $H(F) \backslash H(A) \to \mathbb{C}$. It may therefore be integrated
over $H(F) \backslash H(\mathbb{A})$ against a cusp form. Thus, let $\pi = \pi_1 \otimes \pi_2$ denote an irreducible cuspidal representation of $H(\mathbb{A})$.

The global integral we consider is

$$\int_{G_2(F) \backslash G_2(\mathbb{A})} \int_{G_2(F) \backslash G_2(\mathbb{A})} \varphi_1(g_1)\varphi_2(g_2) \int_{U(F) \backslash U(\mathbb{A})} E(u(g_1, g_2), f_{s, \chi}) \psi_U(u) \, du \, dg_1 \, dg_2$$

Here $G_2 \times G_2$ is embedded in $E_8$ as the stabilizer of the character $\psi_U$ as described above. Also, $\varphi_{\pi_1}$ and $\varphi_{\pi_2}$ are vectors in the space of $\pi_1$ and $\pi_2$ respectively.

### 2.4. Main Global Result

The main result of this section is the following

Theorem 1. Let $w_{\text{ng}}$ denote the shortest element of $W(M_2, T)w_TW(M_1, T)$, where $w_T$ is the longest element of $W$. Let $\nu_0 = w[345678243546576]$, and let $w_0 = w_{\text{ng}}\nu_0$. Let $U_0 = U \cap w_0^{-1}U_{\text{max}}w_0 = \prod_{\alpha \in \Phi(U, T); \nu_0 \alpha < 0} U_{\alpha}$. Denote

$$< \pi_1(g)\varphi_{\pi_1}, \varphi_{\pi_2} > = \int_{G_2(F) \backslash G_2(\mathbb{A})} \varphi_1(g_1)\varphi_2(g_1) \, dg_1,$$

and let $z = 1_2x_{000011100001110}x_{000000111}(-1)$. Then the integral (3) is equal to

$$\int_{G_2(\mathbb{A}) \backslash U_0(\mathbb{A})} \int \langle \pi_1(g)\varphi_{\pi_1}, \varphi_{\pi_2} \rangle f_{s, \chi}(w_0 u(g, 1)) \psi_U(u) \, du \, dg$$

for all $\text{Re}(s)$ large.

Proof. Throughout this proof, we assume $s$ lies in the domain of absolute convergence for $E(f_{s, \chi}, g)$. For $\gamma_0 \in E_8(F)$ define

$$E_{\gamma_0}(g, f_{s, \chi}) = \sum_{\gamma \in (\gamma_0^{-1}P_2(\gamma_0)UH(F)) \backslash UH(F)} f_{s, \chi} \gamma g.$$  

Clearly, the sum is absolutely convergent. Moreover $E_{\gamma_0}(g, f_{s, \chi})$ is left $UH(F)$-invariant for each $\gamma_0$ and

$$E(g, f_{s, \chi}) = \sum_{\gamma_0 \in P_2(F) \backslash E_8(F)/UH(F)} E_{\gamma_0}(g, f_{s, \chi}).$$

Since $E_{\gamma_0}$ is $UH(F)$-invariant, we can consider its Fourier coefficient $E_{\gamma_0}^{(U, \psi_U)}$.

Rather than look for an exact set of representatives for the double cosets $P_2(F) \backslash E_8(F)/UH(F)$ it is convenient to work with a larger subset of $E_8(F)$ which clearly contains a set of representatives. Write $L_{4,7}$ for the subgroup of $E_8$ generated by $U_{\pm \alpha_4}, U_{\pm \alpha_7}$. Clearly, $L_{4,7} \subset H$. Let $U_{\text{max}} = \prod_{\alpha < 0} U_{\alpha}$ denote the maximal unipotent subgroup of $E_8$ opposite $U_{\text{max}}$ and for $w \in W$ let $N_w = U_{\text{max}} \cap w^{-1}U_{\text{max}} w = \prod_{\alpha > 0, \nu_0 \alpha < 0} U_{\alpha}$. Let

$$\dot{W}(M_2, L_{4,7}) = \{ \sigma \in W(E_8, T) : \sigma \text{ is of minimal length in } W(M_2, T) \cdot \nu \cdot [w[4], w[7]] \sigma \}.$$
Here \( w[4] \) and \( w[7] \) denote the subgroup of \( W(E_8, T) \) generated by the \( w[4] \) and \( w[7] \). Then it follows from the Bruhat decomposition that the set
\[
\{ \sigma \delta \mid \sigma \in \tilde{W}(M_2, L_{4,7}) \delta \in M_1 \cap N_{\sigma}(F) \}.
\]
is a set of double coset representatives for \( P_2(F) \backslash E_8(F) / U(F) L_{4,7}(F) \), and hence surjects onto \( P_2(F) \backslash E_8(F) / U(F) H(F) \).

We say that \( w \in W \) is left \( M_2 \) reduced if it is the shortest element of \( W(M_2, T) \cdot w \). Assume that \( \gamma_0 = w \mu \), with \( w \in W \) left \( M_2 \) reduced and \( \mu \in M_1(F) \). This certainly includes the case \( \gamma_0 = \sigma \delta \) as in (5). Then it can be shown that \( \gamma_0 \mu \gamma_0^{-1} \in P_2 \) if and only if \( \gamma_0 \mu \gamma_0^{-1} \in P_2 \) and \( \gamma_0 \mu \gamma_0^{-1} \in P_2 \). It follows that
\[
E_{\gamma_0}(U, \psi_U)(h, f, \chi) = \int_{U \cap w^{-1}P_2w} \sum_{\gamma \in (H \cap \gamma_0^{-1}P_2 \gamma_0)(F) \backslash H(F)} f_{s, \chi}(\gamma_0 \gamma \mu \gamma_0^{-1}) \psi_U(u) \, du
\]
(6)
\[
= \int_{U \cap w^{-1}P_2w} \sum_{\gamma \in (H \cap \gamma_0^{-1}P_2 \gamma_0)(F) \backslash H(F)} f_{s, \chi}(w \mu \gamma_0 \mu \gamma_0^{-1}) [\mu \cdot \psi_U](u) \, du.
\]
where, \( U^w := U \cap w^{-1}P_2w \), and \( [\mu \cdot \psi_U](u) := \psi_U(\mu^{-1}u \mu) \) for \( \mu \in M_1(F), u \in U(A) \).

**Lemma 1.** Take \( w \in W \) left \( M_2 \) reduced, and \( \mu \in M_1(F) \).

1. If the character \( \mu \cdot \psi_U \) is nontrivial on the group \( U^w(A) \), then \( E_{w \mu}(U, \psi_U) = 0 \).
2. Set \( \gamma_0 = w \mu \). If the group \( (H \cap \gamma_0^{-1}P_2 \gamma_0) \) contains the unipotent radical of a parabolic subgroup of \( H \), then \( E_{\gamma_0}(U, \psi_U) \) is orthogonal to cuspforms \( \Gamma(F) \backslash H(A) \rightarrow \mathbb{C} \).
3. For any \( \gamma_0 \in E_8(F) \), if the function \( E_{\gamma_0}(U, \psi_U) \) is zero (resp. orthogonal to cuspforms), then \( E_{\gamma_0}(U, \psi_U) \) is zero (resp. orthogonal to cuspforms) for all \( \gamma_0 \in P_2(F) \cdot \gamma_0 \cdot U(F) H^\pm(F) \).

**Remark 1.** “Orthogonal to cuspforms” just means that the integral
\[
J_{\gamma_0}(f_{s, \chi}, \varphi) := \int_{H(F) \backslash H(A)} E_{\gamma_0}(U, \psi_U)(h, f_{s, \chi}) \varphi(h) \, dh,
\]
is zero for any \( f_{s, \chi} \in \text{Ind}_{P_1(A)}^{E_8(A)} \delta_P \chi \) and any cuspform \( \varphi : H(F) \backslash H(A) \rightarrow \mathbb{C} \). Note that (7) is precisely the contribution of \( E_{\gamma_0} \) to our global integral. The function \( E_{\gamma_0}(U, \psi_U)(f_{s, \chi}) \) need not be \( L^2 \), so strictly speaking there is no inner product space here. However, it is of uniformly moderate growth, so it follows from the decay properties of cuspforms that (7) is absolutely convergent.

**Proof.** (1) The function \( g \mapsto f_{s, \chi}(wg) \) is left-invariant by \( U^w(A) \). So, (6) can be written as a double integral with the inner integration being
\[
\int_{U \cap w^{-1}P_2w} [\mu \cdot \psi_U](u) \, du.
\]
(2) The group \( H \cap \gamma_0^{-1}P_2\gamma_0 \) normalizes \( U \) and also the subgroup \( U \cap \gamma_0^{-1}P_2\gamma_0 \). Moreover, \( H(\mathbb{A}) \) stabilizes \( \psi_U \). The function \( f_{s,\chi} \) is left-invariant, by the \( \mathbb{A} \)-points of any unipotent subgroup of \( P_2 \). Consequently, the function
\[
h \mapsto \int_{U^{(F)} \setminus U(\mathbb{A})} f_{s,\chi}(\gamma_0 u h) \psi_U(u) \, du
\]
is left-invariant by the \( \mathbb{A} \)-points of any unipotent subgroup of \( H \cap \gamma_0^{-1}P_2\gamma_0 \). The second part follows.

(3) It follows from the definition of \( E_{\gamma_0} \) that \( E_{p\gamma_0 h} = E_{\gamma_0} \) for \( p \in P_2(F) \) and \( h \in H(F) \).

Preuve. If \( \alpha \in \text{Supp}_\Phi(\psi_U) \) the \( \psi_U \) is nontrivial on \( U_\alpha \). The condition \( \sigma \alpha > 0 \) implies that \( U_\alpha \subset U^\sigma \), and ensures that \( \psi_U \) is nontrivial on \( U^\sigma \). What must be shown is that \( [\delta \cdot \psi_U] \) remains nontrivial on \( U_\alpha(\mathbb{A}) \), for all \( \delta \in N_\nu(F) \). This follows from the fact that \( N_\nu \) is contained in the standard maximal unipotent and no two elements of \( \text{Supp}_\Phi(\psi_U) \) differ by a positive root.

\[\text{Lemma 2. Let} \]
\[\text{Supp}_\Phi(\psi_U) = \{11221111, 11122111, 12232210, 11233210\}. \]

Take \( \gamma_0 = \sigma \delta \) as in (5) If \( \sigma \alpha > 0 \) for some \( \alpha \in \text{Supp}_\Phi(\psi_U) \), then \( E_{\gamma_0}(f_{s,\chi}, \varphi) = 0 \).

\[\text{Proof. If } \alpha \in \text{Supp}_\Phi(\psi) \text{ the } \psi_U \text{ is nontrivial on } U_\alpha \text{. The condition } \sigma \alpha > 0 \text{ implies that } U_\alpha \subset U^\sigma, \text{ and ensures that } \psi_U \text{ is nontrivial on } U^\sigma. \text{ What must be shown is that } [\delta \cdot \psi_U] \text{ remains nontrivial on } U_\alpha(\mathbb{A}), \text{ for all } \delta \in N_\nu(F). \text{ This follows from the fact that } N_\nu \text{ is contained in the standard maximal unipotent and no two elements of } \text{Supp}_\Phi(\psi_U) \text{ differ by a positive root.} \]

\[\text{Proposition 1. The set } \hat{W}(M_2, L_{4,7}) \text{ as in (5) has 6576 elements, of which all but 25 map at least one of the four roots listed in lemma 2 to a positive root.} \]

\[\text{Proof. One can check this using the computer package LiE [L].} \]

Let \( S = \{\sigma \in \hat{W}(M_2, L_{4,7}) : \sigma \alpha > 0 \quad \forall \alpha \in \text{Supp}_\Phi(\psi)\}. \) According to proposition [L] \( S \) has 25 elements. Now, for \( \sigma \in \hat{W}(M_2, L_{4,7}) \) and \( \delta \in (N_\nu \cap M)(F) \), one has \( P_2(F)\sigma \delta L_{4,7}(F) \subset P_2(F)\sigma \delta U H(F) \subset P_2(F)\sigma \delta P_1(F). \) So, it makes sense to sort these 25 elements according to the image in \( P_2(F)\sigma \delta P_1(F). \) Another straightforward computer check using [L] yields the next lemma.

\[\text{Lemma 3. If } \sigma \in S \text{ then } P_2 \sigma P_1 \text{ contains either} \]
\[w_{\text{sh}} := w[243154234564234576542314354287654231435426543765428765431]\]
\[w_{\text{ln}} := w[2431542345642314354276542314354265437654287654231435426543765428765431]. \]
If \( S_* = \{ \sigma \in S : w_* \in P_2\sigma P_1 \} \), \( * = \text{sht}, \text{lng} \), then \( S_{\text{sht}} \) has 9 elements and \( S_{\text{lng}} \) has 16. The 9 elements of \( S_{\text{sht}} \) are all in the same \( P_2(F), H(F) \) double coset, and the shortest of them is \( w_{\text{sht}}w[56] \).

**Proposition 2.** For each \( \sigma \in S_{\text{sht}} \), the restriction of \( \delta \cdot \psi_U \) to \( U^\sigma \) is trivial if and only if \( \delta \in H(F) \).

**Proof.** One can check this on a case-by-case basis using the program “DCA3” from the egut package \([E]\).

**Corollary 1.** If \( \sigma \in S_{\text{sht}} \) then \( E_{\sigma,\delta}^{(U,\psi_U)} \) is orthogonal to cuspforms.

**Proof.** We use all three parts of lemma \([4]\). It is clear that \( E_{\text{sht},w[56]}^{(U,\psi_U)} \) is orthogonal to cuspforms by part (2). Hence the same holds for any element of \( P_2(F)w_{\text{sht}}H(F) \) by part (3), and the proposition shows that \( E_{\sigma,\delta}^{(U,\psi_U)} \) is identically zero the rest of the time by part (1).

**Lemma 4.** Suppose that \( \sigma \in S_{\text{lng}} \) and \( w_{\text{lng}}^{-1}\sigma \) can be written as a word in the simple reflections without using \( w[4] \). Then \( E_{\sigma,\delta}^{(U,\psi_U)} \) is orthogonal to cuspforms for all \( \delta \in N_\sigma \cap M_1(F) \). The same is true with “4” replaced by “7”.

**Proof.** Set \( \nu = w_{\text{lng}}^{-1}\sigma \). Since \( w_{\text{lng}} \) is left \( M_1 \)-reduced, it follows that \( \sigma \alpha < 0 \iff \nu \alpha < 0 \) for \( \alpha \in \Phi(M_1, T) \). From this it follows that \( N_\sigma \cap M_1 = N_\nu \).

Let \( P_4^{M_1} := M_1 \cap P_4 \), i.e., the maximal standard parabolic subgroup of \( M_1 \) obtained by intersecting \( M_1 \) with the standard maximal parabolic subgroup \( P_4 \) of \( G \). Thus, the only simple root \( \alpha \) of \( M_1 \) such that \( U_\alpha \) is contained in the unipotent radical of \( P_4^{M_1} \) is \( \alpha_4 \). Suppose that \( \nu \) can be expressed as a word in the simple reflections without using 4. Then \( \nu \) is actually in the Levi \( M_4^{M_1} \) of \( P_4^{M_1} \), and, consequently, so is \( N_\nu \). Clearly, the unipotent radical \( U_4^{M_1} \) of \( P_4^{M_1} \) is contained in \( Q_\nu \), and hence in \( \delta^{-1}\nu^{-1}Q_\nu \nu \delta \) for any \( \delta \in N_\nu \). Since the intersection of \( U_4^{M_1} \) with \( H \) is the unipotent radical of a parabolic subgroup of \( H \). The lemma follows from lemma \([4]\) part (2). The previous argument also works if “4” is replaced by “7” throughout, which completes the proof.

Inspecting the 16 elements of \( S_{\text{lng}} \) yields a reduction.

**Lemma 5.** Let \( S'_{\text{lng}} = \{ \sigma \in S_{\text{lng}} \mid w_{\text{lng}}^{-1}\sigma \) contains a 4 and a 7 \} then \( S'_{\text{lng}} \) has 8 elements.

**Proposition 3.** Let \( w_0 = w_{\text{lng}}\nu_0 \) where \( \nu_0 = w[345678243546576] \). For three of the eight elements \( \sigma \in S'_{\text{lng}} \), the function \( E_{\sigma,\delta}^{(U,\psi_U)} \) is orthogonal to cuspforms for all \( \delta \in N_\sigma \cap M_1 \). The remaining five have the property that \( P_2\sigma(N_\sigma \cap M) \) intersects \( P_2w_0N_{w_0}H^\pm \), and \( \delta \cdot \psi_U \) is nontrivial on \( U^\sigma \) unless \( \sigma \delta \in P_2(F)w_0N_{w_0}(F)H^\pm(F) \).
Proof. This can be checked on a case by case basis using the egut program DCA3. □

The results of the previous section imply that $E(U, \psi)_{\gamma_0}$ is orthogonal to cuspforms whenever $P_2(F)\gamma_0 H^\pm \cap w_0(N_{w_0} \cap M_1)(F)$ is empty. In this section we must study $E(U, \psi)_{\gamma_0}$ for $\gamma_0 \in w_0 N_{w_0}(F)$. Recall that $w_0 = w_{\text{lng}} \nu_0$ where $\nu_0 = w[345678243546576]$. And that $N_{w_0} \cap M_1$ can be described more simply as $N_{\nu_0}$.

We have

$$\Phi(N_{\nu_0}, T) = \{00000100, 00000110, 00001111, 00001100, 00011111, 00011110, 00011100, 00111111, 00111110, 00111100, 01122221, 01122220, 01122221, 01122211, 01122220, 01122210, 01122211, 01122221\}.$$  

Write $\delta \in N_{\nu_0}$ as $\prod_{\alpha \in \Phi(N_{\nu_0}, T)} \alpha(x_\alpha)$ with the product taken in the order the roots are listed above. Since $H$ contains every element of $E_8$ of the form $x_00000001(r)x_00111100(r), x_00000011(r)x_00111110(r), x_0101111(r)x_01122210(r)x_00111111(-r), x_01122211(r), \text{ or } x_01122221(r)$,

It follows that every element of $\sigma N_{\nu_0}(F)$ lies in the same $P_2(F), H(F)$-double coset as $\nu_0 \delta$ for some $\delta$ such that

$$\delta_00011111 = \delta_00001111 = \delta_00000110 = \delta_00000100 = \delta_00001111 = 0.$$ (11)

For such $\delta$, the condition $\delta \cdot \psi_U |_{U^\sigma} \equiv 1$ implies

$$\delta_00001111 = \delta_00000111 = \delta_00000101 = \delta_00000010 = \delta_00011111 = 0.$$ (12)

Lemma 6. The set

$$\{x_00001100(\delta_00001100)x_00011100(\delta_00011100)x_00001110(\delta_00001110)x_00000011(\delta_00000111)x_00001111(\delta_00011110)\}$$

is a four-dimensional abelian subgroup $D_0 \subset N_{\nu}$, which is normalized by $M_0 := (T \cap H) \cdot SL_2^7 \cdot SL_2^5 \subset H \cap \nu^{-1}Q_w \nu$.

The subset consisting of elements which satisfy (12) is a union of three orbits for this action, represented by the identity, $x_00011100(1)$ and $x_00011100(1)x_00001111(1)x_00000011(-1)$.

Proof. First, $\nu \cdot \alpha_4 = \alpha_7$, while $\nu \cdot \alpha_7 = \alpha_4$. Both lie in $\Phi(L_w, T)$, and this proves that $M_0 \subset H \cap \nu^{-1}Q_w \nu$. Verifying that the set $D_0$ is indeed a subgroup is as simple as checking that no two of the roots

$$00001100, 00011100, 00001110, 00000111, 00011110.$$ (14)
does as well is as simple as verifying that when either $\alpha_4$ or $\alpha_7$ is subtracted from a root listed in (14), the result is either not a root, or one of the other roots listed in (14).

The action of $M_0$ on $D_0$ can be identified with the action of $GL^2_2$ on $G_\alpha \times \text{Mat}_2$ via
\[
(g_1, g_2) \cdot (x, X) = (\det g_1 \det g_2^{-1} x, g_1 X g_2^{-1}),
\]
in such a way that the subvariety defined by (12) corresponds to that defined by $x = -\det X$. It’s clear that

- this subset is preserved by the action (15),
- it is the union of three orbits, namely, the trivial orbit, $\{(0, 0)\}$, $\{(0, X) : X \neq 0, \det X = 0\}$ and $\{(-\det X, X) : \det X \neq 0\}$.
- the elements given above do indeed represent these orbits.

\[\square\]

**Corollary 2.** If $\gamma_0 \notin P_2(F)w_0zH^\pm(F)$, then $E_{\gamma_0}^{(U,w_u)}$ is orthogonal to cuspforms.

**Proof.** Indeed, one can check using LiE that $w_0\alpha_i > 0$ for $i = 2, 3, 4, 5$. Hence $w_0^{-1}Hw_0$ contains the full unipotent radical of the copy of $G_2$ generated by (1).

Let $G_1$ denote the subgroup of $E_8$, isomorphic to $\text{Spin}_8$, generated by $U_{\pm\alpha_i}$ for $i = 2, 3, 4, 5$. Consider the maximal parabolic subgroup of $G_1$ whose unipotent radical contains $U_{\alpha_i}$. Let $V_4$ denote this unipotent radical. Then $x_{000011110}(1)V_4x_{000011110}(-1) \subset V_4U_{01121110U}01122110$, and $w_0\alpha > 0$ for each $\alpha \in \{01121110, 01122110\} \cup \Phi(V_4, T)$. Hence $w_0x_{000011110}(1)$ conjugates $(V_4 \cap H)$ into $P_2$. The result follows from lemma 1, part (2).

This completes the proof that the global integral (3) is equal to $J_{\gamma_0}(f_{s,\chi}, \varphi_{\pi_1}, \varphi_{\pi_2})$, defined by (7), for $\gamma_0 = w_0z$.

**Remark 2.** For each $\gamma_0 \in E_8(F)$, $J_{\gamma_0}$ is a bilinear form between $\text{Ind}_{P_2(F)}^{E_8(F)} \delta_{F,X}$ and the space of cuspforms on $H(F) \backslash H(\mathbb{A})$. Define $\text{Supp}(J)$ to be the set of $\gamma_0 \in E_8(F)$ such that $J_{\gamma_0}$ is nonzero. Then we have shown that $\text{Supp}(J)$ is a union of $P_2(F), UH^\pm(F)$ double cosets, and that it vanishes off of a single $P_2(F), UH$-double coset. It is therefore worth asking whether $P_2(F)w_0zUH^\pm(F) = P_2(F)w_0zUH(F)$, for if this is not so, then we have proved that $J_{\gamma_0}$ vanishes identically. Thankfully, the answer is yes. Indeed, if we define $z = x_{-00011100}(1)x_{-00011110}(1)x_{00000111}(-1)$, then $z^{-1}z^{-1}$ is a representative for the Weyl element $w[657486576]$. Recall that certain representatives for this Weyl element lie in $H^\pm(F) \backslash H(F)$. Moreover one can choose $t_0 \in T(F)$ such that $z^{-1}z^{-1}t_0 \in H^\pm(F) \backslash H(F)$ and $t_0^{-1}z^{-1}t_0 = z$. As $w_0zt_0w_0 \in P_2(F)$, it follows that $z \cdot (z^{-1}z^{-1}t_0)$ and $z$ are in the same $P_2(F), H(F)$ double coset.
Let \( J_{\gamma_0} \) be defined as in (7). Then
\[
J_{\gamma_0}(f_{s,\chi}, \varphi) = \int_{(H \cap \gamma_0^{-1} P_2 \gamma_0)(F) \backslash H(A)} \varphi(h) \int_{U \gamma_0(F) \backslash U(A)} f_{s,\chi}(\gamma_0 u h) \psi_U(u) \, du \, dh.
\]
We have shown that the global integral \( \mathbf{3} \) is equal to \( J_{w_0 z}(f_{s,\chi}, \varphi_{\pi_1} \varphi_{\pi_2}) \).

**Lemma 7.** Define \( w_0, z \) and \( U_0 \) as in theorem \( \mathbf{4} \). Then
\[
\int_{U \gamma_0 z(F) \backslash U(A)} f_{s,\chi}(w_0 z u h) \psi_U(u) \, du = \int_{U_0(A)} f_{s,\chi}(\gamma_0 u h) \psi_U(u) \, du.
\]

**Proof.** Note that \( U^{\gamma_0} = \prod_{\alpha \in \Phi(U, T) : w_0 \alpha > 0} U_\alpha \) while \( U_0 = \prod_{\alpha \in \Phi(U, T) : w_0 \alpha < 0} U_\alpha \). It follows that \( U = U^{\gamma_0} U_0 \). One calculates that \( \{ \alpha \in \Phi(U, T) : w_0 \alpha > 0 \} \) equals
\[
\{(11110000); (11111000); (11121000); (11221000); (12232100); (12232110); (12232111)\}.
\]
For each value of \( \alpha \), one easily checks that none of \( \alpha - 00011100, \alpha - 00001110, \alpha - 00000111 \)
is a root. It follows that \( z \) normalizes \( U_0 \), and hence \( U = U^{\gamma_0} U_0 \).

Since \( f_{s,\chi}(w_0 z u h) \) is left-invariant by \( U^{\gamma_0 z}(A) \), and since \( \psi_U \) is trivial on \( U^{\gamma_0 z}(A) \), the result follows.

**Lemma 8.** The group \( H \cap (w_0 z)^{-1} P_2 w_0 z \) is the diagonally embedded copy of \( G_2 \), i.e., the subgroup of \( H \) generated by \( x_{\pm00010000}(r)x_{\pm00000010}(r) \), and
\[
x_{\pm01000000}(r)x_{\pm00100000}(r)x_{\pm00001000}(-r)x_{\pm00000001}(r)x_{00111100}(r)x_{01011100}(-r).
\]

**Proof.** The group \( M_1 \cap w_{\text{reg}}^{-1} P_2 w_{\text{reg}} \) is the maximal parabolic subgroup of \( M_1 \) whose unipotent radical contains \( U_{\alpha_3} \). Denote this group by \( P_{3}^{M_1} \). Then \( H \cap (w_0 z)^{-1} P_2 w_0 z = H \cap (\nu_0 z)^{-1} P_{3}^{M_1} \nu_0 z = \{ h \in H \mid \nu_0 z h z^{-1} \nu_0^{-1} \in P_{3}^{M_1} \} \). Now, \( M_1 \) is isogenous to \( SO_{14} \). The kernel of this isogeny is contained in the maximal torus, and hence in every parabolic subgroup, and this means \( \nu_0 z h z^{-1} \nu_0^{-1} \) lies in \( P_{3}^{M_1} \) if and only if its image in \( SO_{14} \) lies in the corresponding parabolic subgroup of \( SO_{14} \). This reduces the lemma to a straightforward matrix calculation.

This completes the proof of theorem \( \mathbf{1} \).

### 3. The Unramified Computations

3.1. **Notation.** In this section the unramified local zeta integral corresponding to integral \( \mathbf{4} \) will be computed. Therefore, let \( F \) now denote a nonarchimedean local field, and \( \pi \) an unramified representation of \( G_2(F) \). In this section we write \( T_{E_8} \) for our fixed maximal torus of \( E_8 \) and use \( T \) for a fixed maximal torus of \( G_2 \). Let \( f \) denote a section of the induced representation \( \text{Ind}_{P_2(F)}^{E_8(F)} \delta_{P_2} \), where \( s \in \mathbb{C} \) with \( \text{Re}(s) \) large. (We no longer need to consider
characters of the type $\delta^\nu$; if $\chi$ is unramified, then it is equal to $\delta^\nu$ for some $\nu \in \mathbb{C}$ and we may absorb $\nu$ into $s$.) Let $\omega_\pi$ denote the normalized spherical function for $\pi$. Let $w_0 = w_{\text{un}}\nu_0$ where

$$w_{\text{un}} = w_{[24315423456423143542654234354265437654287654231435426543765428765431]}$$

and $\nu_0 = w_{[345678245673456]}$. Let $U$ denote the unipotent radical of the standard maximal parabolic subgroup of $E_8$ with Levi isomorphic to the product of $\text{Spin}_{14}$ and a one-dimensional torus, and let $U_0$ be the $T_{E_8}$-stable subgroup of $U$ such that $\Phi(U, T_{E_8}) \setminus \Phi(U_0, T_{E_8})$ equals

$$\{(11110000); (11111000); (11121000); (12232100); (12232110); (12232111)\}.$$ 

This group can also be described as $U \cap w_0^{-1}T_{\text{max}}w_0$, or $\prod_{\alpha > 0; w_0\alpha < 0} U_\alpha$. Define a character $\psi_U$ of $U$ (which may then be restricted to $U_0$) by the formula

$$\psi_U(u) = \psi_U(x_{11221111}(r_1) x_{11122111}(r_2) x_{12232110}(r_3) x_{11233210}(r_4) u') = \psi(r_1 + r_2 + r_3 + r_4),$$

for any $u' \in U$ which lies in the product of the groups $U_\alpha$ with $\alpha \in \Phi(U, T_{E_8})$ and $\alpha \notin \{11221111, 11122111, 12232210, 11233210\}$. Embed $G_2 \times G_2$ into $E_8$ as the identity component of the stabilizer of $\psi_U$ in $M_1$, as in section 2.2. Finally, let $z = x_{00011100}(1)x_{00011110}(1)x_{00000111}(-1)$.

Then the local zeta integral is given by

$$I(s, \pi) := \int_{G_2(F)} \int_{U_0(F)} \omega_s(g) f(w_0zu(1, g), s) \psi_U(u) \, du \, dg$$

Throughout this section, we shall often abuse notation and denote the $F$-points of an algebraic group $H$ by "$H$," suppressing the ubiquitous "$(F)$" 's. We denote the long and short simple roots of $G_2$ by $\beta_{\text{un}}$ and $\beta_{\text{sht}}$ respectively.

We embed $G_2$ into $SO_{8}$ in such a way that

$$T = \{t = \text{diag}(t_1 t_2^2, t_1 t_2, t_2, 1, 1, t_2^{-1}, t_1^{-1} t_2^{-1}, t_1^{-1} t_2^{-2})\}.$$ 

This puts coordinates $t_1, t_2$ on $T$. Another way to define the same coordinates is that $t_1 = \beta_{\text{un}}(t)$ and $t_2 = \beta_{\text{sht}}(t)$.

We also write $h : T \hookrightarrow E_8$ for the embedding $h(\beta_{\text{sht}}(a_1) \beta_{\text{un}}(a_2)) = \alpha_2 \alpha_3 \alpha_5(a_1) \alpha_4(a_2)$ determined by our chosen embedding $G_2 \hookrightarrow E_8$. Composing with the projection $M_1 \to SO_{14}$ gives the embedding

$$T \hookrightarrow SO_{14}, \quad t \mapsto \begin{pmatrix} I_3 & t \\ t & I_3 \end{pmatrix},$$

which we also denote $h$. (Here $T$ is identified with a subgroup of $SO_8$.)

We also write $K$ for $G_2(\mathfrak{o})$. 

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Remark 3. We saw in lemma 8 that the $H \cap (w_0z)^{-1}P_2w_0z$ is the diagonally embedded copy of $G_2$ inside of $H = G_2 \times G_2$. Hence $\{(1,g) : g \in G_2\} \subset H$ maps isomorphically onto $(H \cap (w_0z)^{-1}P_2w_0z) \setminus H$. In fact we could take the integral in (16) over any embedded copy of $G_2$ with this property.

3.2. Main Local Result.

Theorem 1. For $\Re(s)$ sufficiently large,

$$I(s, \pi) = \frac{L(s, \pi, \mathrm{St})}{N(s)},$$

where $L(s, \pi, \mathrm{St})$ is the local $L$ function attached to $\pi$ and the 7 dimensional “standard” representation of $G_2(\mathbb{C})$, and $N(s)$ is the normalizing factor of the Eisenstein series, given explicitly (see section 3) by

$$N(s) = \zeta(17s) \prod_{i=2}^{6} \zeta(17s - i) \prod_{i=5}^{8} \zeta(34s - 2i) \zeta(51s - 21).$$

Proof. The proof occupies the rest of section 3 and comes in several steps. Write $T$ for the maximal torus of $G_2$, and write $T^+$ for the subset $\{t \in T : |\beta_{\mathrm{ing}}(t)|, |\beta_{\mathrm{sh}}(t)| \leq 1\}$. As a subset of $SO_8$, this is

$$\{\text{diag}(t_1t_2^2, t_1t_2, t_2, 1, 1, t_2^{-1}, t_1^{-1}t_2^{-1}, t_1^{-1}t_2^{-2}) \in T : |t_1|, |t_2| \leq 1\}.$$

For the rest of the section, we define $t_1 := \beta_{\mathrm{ing}}(t)$ and $t_2 := \beta_{\mathrm{sh}}(t)$ as coordinates on the torus $T$. Define

$$\psi_{U,t}(x_{10111111}(r_1)x_{12232111}(r_2)x_{11232111}(r_3)x_{11232111}(r_4)x_{11232111}(r_5)x_{11232111}(r_6)u') =$$

$$\psi\left(\sum_{i=1}^{4} r_i + t_2r_5 + t_1t_2r_6\right),$$

$$U' := \nu_0U_0\nu_0^{-1} = U \cap w_0^{-1}U_{\max}w_0,$$ and

$$I(s, t) := \int_{U'_0} f(w_0u, s) \psi_{U,t}(u) du. \quad (17)$$

Then we prove in section 3.3 that

$$I(s, \pi) = \int_{T^+} \omega_\pi(t) \delta(t)|t_1t_2^2|^{17s-5}I(s, t) dt, \quad (18)$$

where $\delta(t)$ is the measure of the double coset $KtK$.

It’s value is given by $\delta_B(t)^{-1}Q/Q_t$, where

$$Q = \frac{(1 - q^{-2})(1 - q^{-6})}{(1 - q^{-1})^2} = (1 + 2q^{-1} + 2q^{-2} + 2q^{-3} + 2q^{-4} + 2q^{-5} + q^{-6}).$$
Then theorem 3 states that
\[ Q_t := \begin{cases} 
Q, & |t_1| = |t_2| = 1, \\
1 + q^{-1}, & 0 = \max(|t_1|, |t_2|) > \min(|t_1|, |t_2|), \\
1 & 0 > \max(|t_1|, |t_2|).
\end{cases} \]

This is proved in section 3.2 of [M]. See also [Cas].

Every term in the integrand of (18) depends only on the $p$-adic valuations of $t_1$ and $t_2$ of $T$. To be precise, suppose that $v(t_2) = m$ and $v(t_1) = n$. Here $v$ is the $p$-adic valuation. Then $\delta_B(t) = q^{-6n-10m}$. Define $Q_{(m,n)} = Q_t$ for $t \in T$ with $t_1 = p^m$ and $t_2 = p^n$.

Now, set $x = q^{-17s}$, and define
\[ Z(x, q) = (1 - x)(1 - xq^2)(1 - xq^3)(1 - xq^4)(1 - x^2q^{10})(1 - x^2q^{12}) \]
and
\[ I_0(n, m; x, q) := (1 - xq^6)(1 - x^3q^{21}) - (1 - xq^5)(1 - xq^6)(xq^8)^{m+1} - (1 - xq^8)(1 - xq^5)(xq^8)^m(xq^7)^{n+1}. \]
Then theorem 3 states that
\[ I(s, t) = \frac{Z(x, q)I_0(n, m; x, q)}{(1 - xq^7)(1 - xq^8)}. \]
Moreover, $|t_1t_2|^{17s-5} = (xq^3)^{2m+n}$. This leaves $\omega_\pi(t)$. To write a formula for $\omega_\pi(t)$, it is convenient to identify the pair $(m, n)$ with the weight $m\omega_1 + n\omega_2$ in the weight lattice $\Lambda$ of $L\mathcal{G}_2^0 = G_2(\mathbb{C})$. This is compatible with the natural identification of $\Lambda$ with $T/T(\mathfrak{o})$ which is built into the definition of the $L$-group. Let $\tau$ be an element of the $L$-group associated to the representation $\pi$ (this determines $\tau$ up to the action of $W$ which is enough). We write $\lambda$ additively, and therefore denote the value of $\lambda \in \Lambda$ at $\tau$ by $\tau^\lambda$ as opposed to $\lambda(\tau)$. Define $S_0$ to be the finite set of weights of $L\mathcal{G}_2^0$ which have at least one expression as a sum of distinct positive roots, and define polynomials $P_\nu$, $(\nu \in S_0)$ by the condition
\[ \left( \prod_{\alpha > 0} 1 - q^{-1}\tau^{-\alpha} \right) = \sum_{\nu \in S_0} P_\nu(q^{-1})\tau^{-\nu}. \]
Then
\[ \omega_\pi(t) = \frac{1}{q^{5m-3n}} \sum_{\nu \in S_0} P_\nu(q^{-1}) \frac{A_{\nu + \rho - \nu}(\tau)}{A_{\rho}(\tau)}, \quad \text{where} \quad A_{\lambda}(\tau) = \sum_{w \in W} (-1)^{\ell(w)}\tau^{w\lambda}, \quad (\lambda \in \Lambda). \]
Here $\ell$ denotes the length function on $W$ and $Q$ is defined as in the formula for the measure of $KtK$ given above. It is not difficult to derive this expression for $\omega_\pi(t)$ from the one given in [Ca].

Plugging all of this into (18) yields
\[ I(s, \pi) = \frac{Z(x, q)}{(1 - xq^7)(1 - xq^8)} \sum_{m,n=0}^\infty x^{2m+n}q^{15m+8n}I_0(m, n; x, q) \sum_{\nu \in S_0} P_\nu(q^{-1}) \frac{A_{(m,n)-\nu + \rho}(\tau)}{A_{\rho}(\tau)}. \]
The inner summation must be calculated. This is accomplished in theorem \[2\] which states that
\[
(19) \quad \sum_{m,n=0}^{\infty} \frac{x^{2m+n}q^{15m+8n}}{Q_{(m,n)}} \sum_{\nu \in S_0} P_{\nu}(q^{-1}) \frac{A_{(m,n)-\nu+\rho}(\tau)}{A_{\rho}(\tau)} = (1 - xq^5)(1 - xq^6)(1 - xq^7)(1 - xq^8)(1 - x^2q^{14})(1 - x^2q^{21}) \sum_{r=0}^{\infty} \chi_{(r,0)}(\tau)x^r q^{8r},
\]
where \(\chi_\lambda\) is the character of the irreducible finite dimensional representation of \(G_2(\mathbb{C})\) having highest weight \(\lambda\). The decomposition of the symmetric algebra of \(S^t\) is described by the theorem of Brion [Br], and this description implies that
\[
L(s, \pi, S^t) = (1 - x^2q^{16})^{-1} \sum_{r=0}^{\infty} \chi_{(r,0)}(\tau)x^r q^{8r}.
\]
Plugging this in, and noting that \((1 - xq^5)(1 - xq^6)(1 - x^2q^{14})(1 - x^2q^{16})(1 - x^3q^{21})Z(x, q)\) is precisely \(N(s)\) yields theorem \[1\] \(\square\)

### 3.3. Transformation of the integral.

**Proposition 4.** Let \(I(s, \pi)\) be defined by \([16]\), and \(I(s, t)\) by \([17]\). Then
\[
I(s, \pi) = \int_{T^+} I_{\pi}(t) |t_1 t_2|^{17s-5} I(s, t) \, dt.
\]

**Proof.** Using the bi-K invariant property of \(\omega_\pi\), integral \([16]\) is equal to
\[
(20) \quad \int_{T^+ \psi_0} \omega_\pi(t)f(w_0 zuh(t), s)\psi_U(u)\delta(t) \, du \, dt,
\]
where \(\delta(t)\) denotes the measure of the double coset \(KtK\).

Conjugate \(h(t)\) across \(u\). It normalizes \(U_0\) preserving \(\psi_U\), but the change of variables \(u \mapsto h(t)uh(t)^{-1}\) produces a factor of
\[
|t_1 t_2|^{-5} = \left| t_1 t_2^2 \right|^{-5}.
\]
from the measure. (Here \(t_1 = \beta_{\text{ling}}(t)\) and \(t_2 = \beta_{\text{hlt}}(t)\).)

Next define \(z(t) = h(t)^{-1}zh(t)\). Then \(z(t) = x_{00011100}(t_2)x_{000001111}(t_1 t_2)x_{000000111}(t_1 t_2^2)\). This calculation can be done in LiE or with matrices, since the projection \(M_1 \rightarrow SO_{14}\) restricts to an isomorphism on \(G_2 \cdot U\). The image of \(t\) was described above, and the image of \(z\) is the unipotent matrix \(I_{14} + e_{1,4}' + e_{2,5}' + e_{3,6}'\) in \(SO_{14}\). Now conjugate \(z\) across \(u_0\) to obtain that integral \([20]\) is equal to
\[
(21) \quad \int_{T^+ \psi_0} \omega_\pi(t)f(w_0 uh(t)z(t), s)[z(t) \cdot \psi_U](u)\delta(t) \, du \, dt
\]
Notice that \(z(t)\) is in the maximal compact and hence can be ignored.
Write \( w_0 = w_{\text{sg}} \nu_0 \) as in Theorem\textsuperscript{1} We conjugate \( h(t) \) to the left and then \( \nu_0 \) to the right, and we obtain

\[
(22) \quad \int_{T+U_0^{'}} \omega_\pi(t) f(w_{\text{sg}}u, s)[\nu_0 z(t) \cdot \psi_U](u) \delta(t)|t_1 t_2^2|^{-5} \delta_{P_2}(w_0 h(t) w_0^{-1}) \, du \, dt.
\]

The character \( \nu_0 z(t) \cdot \psi_U(u) \) is precisely \( \psi_U, t \), and \( \delta_{P_2}(w_0 h(t) w_0^{-1}) = |t_1 t_2^2|^{17s} \). This gives the result. \( \square \)

We remark that \( U_0' \) may be defined as \( \nu_0 U_0 \nu_0^{-1} \), as \( U \cap w_0^{-1} U_{\text{max}} \), or as the \( T_{E_6} \)-stable subgroup of \( U \) such that \( \Phi(U, T_{E_6}) \setminus \Phi(U_0', T_{E_6}) \) is the set

\[
\{(12343210); (12343211); (12343221); (12344321); (12354321); (13354321)\}
\]
Thus \( \dim U = 71. \)

3.4. **Proof of Main Unramified Identity.** In this section we prove (19). Set

\[
I_0(n, m; x, q) = 1 - x q^6 - x^3 q^{21} + x^4 q^{27} - x^{m+1} q^{8(m+1)} + x^{m+2} q^{8m+14} - x^{n+m+1} q^{7(n+1)+8m} + x^{n+m+2} q^{7n+8m+12} + x^{n+m+2} q^{7n+8m+15} - x^{n+m+3} q^{7n+8m+20} + x^{m+2} q^{8m+13} - x^{m+3} q^{8m+19}
\]

\[
= (1 - x q^6)(1 - x^3 q^{21}) - (x q^8)^{m+1} (1 - x q^5)(1 - x q^6) - (x q^7)^{n+1} (x q^8)^n (1 - x q^5)(1 - x q^8)
\]
Identify \( \mathbb{Z}^2 \) with the weight lattice of \( G_2(\mathbb{C}) \) via the mapping \( (n, m) \mapsto n \varpi_1 + m \varpi_2 \), and regard \( I_0 \) as a function defined on the weight lattice as well.

Write \( (n, m) \) for the trace of the semisimple conjugacy class in \( G_2(\mathbb{C}) \) attached to \( \pi \), acting on the irreducible finite dimensional representation of \( G_2(\mathbb{C}) \) with highest weight \( n \varpi_1 + m \varpi_2 \). Here \( \varpi_1 \) and \( \varpi_2 \) are the fundamental weights.

Define \( S_0 \) and \( P_\nu \) as in the previous section and set

\[
Q_{\varpi} = \begin{cases} 
Q, & \varpi = 0 \\
1 + q^{-1}, & \varpi = n \varpi_1 \text{ or } n \varpi_2, \\
1, & \text{otherwise},
\end{cases}
\]
for \( \varpi \) a dominant weight of \( G_2(\mathbb{C}) \).

Define

\[
p(\varpi) = \frac{1}{Q_{\varpi}} \sum_{\nu \in S_0} P_\nu(q^{-1}) \frac{\varpi + \rho - \nu}{|\varpi + \rho - \nu|} \chi_{|\varpi + \rho - \nu|}(\tau),
\]
and \( p(n, m) = p(n \varpi_1 + m \varpi_2) \).

Define

\[
z_0(x, q) = (1 - x q^5)(1 - x q^6)(1 - x q^7)(1 - x q^8)(1 - x^2 q^{14})(1 - x^3 q^{21}).
\]
Theorem 2. We have

\begin{equation}
\sum_{n,m=0}^{\infty} p(n, m) I_0(n, m; x, q)x^{n+2m}q^{8n+15m} = z_0(x, q) \sum_{r=0}^{\infty} (r, 0)x^r q^{8r}
\end{equation}

Proof. For \( \varpi \) a dominant weight of \( G_2(\mathbb{C}) \), define \( J(\varpi; x, q) = I_0(\varpi; x, q)\tau_0^\varpi / Q_\varpi \), where \( \tau_0 \) is an element of the standard maximal torus \( L^T \) of \( G_2(\mathbb{C}) \), chosen so that \( \tau_0^\varpi = xq^8 \) and \( \tau_0^\varpi = x^2q^{15} \). (Here, we employ the exponential notation for the weights. That is \( \varpi \) is a function from \( L^T \) to \( \mathbb{C}^\times \), and its value at \( \tau \in L^T \) is denoted \( \tau^\varpi \)). Then

\[ J(\varpi; x, q) = \frac{1}{Q_\varpi} ((1 - xq^6)(1 - x^3q^{21})\tau_0^\varpi - (xq^8)(1 - xq^5)(1 - xq^6)\tau_1^\varpi - (xq^7)(1 - xq^5)(1 - xq^8)\tau_2^\varpi), \]

where \( \tau_1, \tau_2 \in L^T \) satisfy

\[ \tau_1^{\varpi} = xq^8, \quad \tau_2^{\varpi} = x^3q^{23}, \quad \tau_1^{\varpi} = x^2q^{15}, \quad \tau_2^{\varpi} = x^3q^{23}. \]

Set

\[ J_0(x, q) = (1 - xq^6)(1 - x^3q^{21}), \quad J_1(x, q) = (xq^8)(1 - xq^5)(1 - xq^6), \quad J_2(x, q) = (xq^7)(1 - xq^5)(1 - xq^8), \]

so that

\[ J(\varpi; x, q) = J_0(x, q)\tau_0^\varpi - J_1(x, q)\tau_1^\varpi - J_2(x, q)\tau_2^\varpi. \]

Write \( \Lambda^{++} \) for the semigroup of dominant weights of \( G_2(\mathbb{C}) \). For \( \lambda \in \Lambda^{++} \), write \( \chi(\lambda) \) for the character of the irreducible finite dimensional representation of \( G_2(\mathbb{C}) \) with highest weight \( \lambda \).

Then \( \chi(\lambda) \) appears in the expression for \( p(\varpi) \) given in the previous section if and only if \( \{(w, \nu) \mid w \in W, \nu \in S_0, \varpi + \nu + \rho = w(\lambda + \rho)\} \) is nonempty. If this set is nonempty, then the coefficient of \( \chi(\lambda) \) in this expression for \( p(\varpi) \) is governed by

\[ D(\lambda, \varpi) := \{(w, S) \mid w \in W, \nu \in S_0, S \subset \Phi^+, \varpi + \Sigma(S) + \rho = w(\lambda + \rho)\}, \]

where \( \Sigma(S) := \sum_{\alpha \in S} \alpha \). To be precise

\[ p(\varpi) = \sum_{\lambda \in \Lambda^{++}} \sum_{(w, S) \in D(\lambda, \varpi)} (-1)^{\ell(w)}(-q^{-1}|S|) \]

Thus, it must be shown that for \( \lambda \in \Lambda^{++} \)

\[ \sum_{\varpi \in \Lambda^{++}} \sum_{(w, S) \in D(\lambda, \varpi)} J(\varpi; x, q)(-1)^{\ell(w)}(-q^{-1}|S|) = \begin{cases} z_0(x, q)(xq^8)^r, & \varpi = r\varpi_1, \\ 0, & \text{otherwise.} \end{cases} \]

This will be reduced to a finite number of cases.

Lemma 9. Given \( \lambda, \varpi \in \Lambda^{++} \), \( D(\lambda, \varpi) \) is nonempty if and only if \( \varpi \in \lambda + S_0 \).
Proof. The set \( \rho - S_0 = \{ \frac{1}{2} \Sigma(S) - \frac{1}{2} \Sigma(S^c) \mid S \subset \Phi \} \) is clearly \( W \)-stable. (Here \( S^c \) denotes the complement of \( S \) in \( \Phi \).) So

\[
\exists S \subset \Phi \text{ s.t. } w(\varphi - \Sigma(S) + \rho) = \lambda + \rho \iff \exists S \subset \Phi \text{ s.t. } w\varphi - \Sigma(S) + \rho = \lambda + \rho \iff w\varphi \in \lambda + S_0.
\]

Now, the convex hull of \( \lambda + S_0 \) is a convex dodecagon \( D \) with edges parallel to the roots. The center of mass of the dodecagon \( D \) is \( \lambda + \rho \) and lies in the positive Weyl chamber. Let \( \ell \) be any line which is parallel to a root and does not pass through \( \lambda + \rho \). Then \( \ell \) partitions the plane into two half-planes. Let \( H^+ \) be the half-plane containing \( \lambda + \rho \) and \( H^- \) the other. Then the reflection of \( (D \cap H^+) \) over \( \ell \) contains \( H^- \). Applying this geometric observation to the reflections in the Weyl group, it is clear that \( \{ w \in W : w\varphi \in \lambda + S_0 \} \) must contain the identity if it is nonempty. (In fact, one can say more. If \( w\varphi \in \lambda + S_0 \) and \( w = w[i_1 i_2 \ldots i_k] \) is a reduced expression for \( w \), then \( w[i_j \ldots i_k]w \in \lambda + S_0 \) for \( 1 \leq j \leq k \).)

Thus we need to show that

(24)

\[
\sum_{\nu \in S_0} J(\lambda + \nu; x, q) \sum_{w \in W} \frac{(-1)^{\ell(w)}}{w(\lambda + \rho) \cap \lambda + \nu + \rho - S_0} \sum_{S \subset \Phi} (-q^{-1})^{\left| S \right|} = \begin{cases} z_0(x, q)(xq^8)^r, & \lambda = r\varphi_1, \\ 0, & \text{otherwise.} \end{cases}
\]

Now, \( \{ w \mid w(\lambda + \rho) \in \varphi + \rho - S_0 \} \) for some \( \varphi \in \Lambda^+ \) is given by

\[
\begin{cases} \{ e \}, & \lambda = n\varphi_1 + m\varphi_2, \ n \geq 5, m \geq 3, \\ \{ e, w[1] \}, & \lambda = n\varphi_1 + m\varphi_2, \ n \leq 4, m \geq 3, \\ \{ e, w[2] \}, & \lambda = n\varphi_1 + m\varphi_2, \ n \geq 5, m \leq 2. \end{cases}
\]

Here \( e \) is the identity element of the Weyl group.

Lemma 10. Fix \( n \) with \( 0 \leq n \leq 4 \) and \( \nu \in S_0 \). For \( \lambda = n\varphi_1 + m\varphi_2 \), the expression

\[
P(n, \nu; q^{-1}) := \sum_{w \in W} \frac{(-1)^{\ell(w)}}{w(\lambda + \rho) \cap \lambda + \nu + \rho - S_0} \sum_{S \subset \Phi} (-q^{-1})^{\left| S \right|}
\]

is independent of \( m \geq 3 \). Likewise, for fixed \( m \leq 2 \) it is independent of \( n \geq 5 \).

Proof. We prove the first statement. The proof of the second statement is symmetric. Indeed, for \( m \geq 3 \) and \( n \leq 4 \), the given expression is

\[
\sum_{S \subset \Phi} (-q^{-1})^{\left| S \right|} + \sum_{S \subset \Phi} (-q^{-1})^{\left| S \right|}.
\]

and \( w[1] \lambda - \lambda = n\alpha_1 \), independent of \( m \).
Observe that for $n \leq 4$ and $m \geq 3$, $\{\nu \in S_0 \mid \lambda + \nu \in \Lambda^+\}$ is also independent of $m$. 

Thus, for all $m \geq 3$, the left hand side of equation (24) is equal to

$$\sum_{\nu \in S_0} \sum_{\nu = a + b_2, a \geq -n} J(\lambda + \nu; x, q) P(n, \nu; q^{-1})$$

$$= \sum_{\nu \in S_0} (J_0(x, q) \tau_0^{\lambda + \nu} - J_1(x, q) \tau_1^{\lambda + \nu} - J_2(x, q) \tau_2^{\lambda + \nu}) P(n, \nu; q^{-1})$$

$$= (x^2 q^{15})^m \left( (xq)^n J_0(x, q) \sum_\nu P(n, \nu; q^{-1}) \tau_0^{\nu} \right)$$

$$- (x^3 q^{23})^m \left( (xq)^n J_1(x, q) \sum_\nu P(n, \nu; q^{-1}) \tau_1^{\nu} + (x^2 q^{15})^n J_2(x, q) \sum_\nu \tau_2^{\nu} P(n, \nu; q^{-1}) \right).$$

Now $m \mapsto (x^2 q^{15})^m$ and $m \mapsto (x^3 q^{23})^m$ are two linearly independent functions of $m$. Therefore, after checking equation (24) for two distinct values of $m \geq 3$, one may deduce that

$$\sum_{\nu} P(n, \nu; q^{-1}) \tau_0^{\nu} = \left( (xq)^n J_1(x, q) \sum_\nu P(n, \nu; q^{-1}) \tau_1^{\nu} + (x^2 q^{15})^n J_2(x, q) \sum_\nu \tau_2^{\nu} P(n, \nu; q^{-1}) \right) = 0,$$

and thence that (24) holds for all $m \geq 3$. The same method allows one to reduce the case when $m \leq 2$ is fixed and $n \geq 5$ is arbitrary to checking two cases, and to reduce the case $m \geq 3, n \geq 5$ to checking three cases. Overall, it suffices to check all pairs $(n, m)$ with $n \leq 6$ and $m \leq 4$. This is easily accomplished using LiE \[\text{[L]}\].

**Remark 4.** Equation (24) has another simple proof in the case $n \geq 5, m \geq 3$. In that case, equation (24) reduces to

$$\sum_{\nu \in S_0} J(\lambda + \nu; x, q) \sum_{S \subset \Phi^+} (-q^{-1})^{|S|} = 0.$$

The left hand side is equal to

$$\sum_{S \subset \Phi^+} (-q^{-1})^{|S|} J(\lambda + \Sigma(S); x, q)$$

$$= J_0(x, q) \prod_{\alpha \in \Phi^+} (1 - q^{-1} \tau_0^{\alpha}) \tau_0^{\lambda} - J_1(x, q) \prod_{\alpha \in \Phi^+} (1 - q^{-1} \tau_1^{\alpha}) \tau_1^{\lambda} - J_2(x, q) \prod_{\alpha \in \Phi^+} (1 - q^{-1} \tau_2^{\alpha}) \tau_2^{\lambda}.$$

But one can fairly easily check that for each $i = 0, 1, 2$, there is a positive root $\alpha$ such that $\tau_i^{\alpha} = q$.

4. **The normalizing factor of the Eisenstein series**

In this section we compute the normalizing factor of the Eisenstein series. The Eisenstein series we use, denoted by $E(h, s)$, is attached to the induced representation $\text{Ind}_{P_2(\mathbb{A})}^G S_2 \chi$. Here, $\chi = \prod_v \chi_v$ is a character of $\mathbb{A}^\times / \mathbb{F}^\times$ which has been identified with a character of
Theorem 3. Let $I(s, t)$ be defined by (17), let

$$Z(x, q) = (1 - x)(1 - xq^2)(1 - xq^3)(1 - xq^4)(1 - x^2q^{10})(1 - x^2q^{12})$$

and

$I_0(n, m; x, q) := (1 - xq^6)(1 - x^3q^{21}) - (1 - xq^5)(1 - x^5q^8)(xq)(xq^8)^{m + 1} - (1 - xq^8)(1 - xq^5)(xq^8)^m(xq^7)^{n + 1}.$

Then

$$I(s, t) = \frac{Z(x, q)I_0(n, m; x, q)}{(1 - xq^7)(1 - xq^8)},$$

where $x = q^{17s}$, $n$ is the $p$-adic valuation of $t_1$, and $m$ is that of $t_2$. 

Conjecture 1. For $Re(s) > 1/2$, the possible poles of the Eisenstein series are as follows.

- If $\chi$ is trivial, then the Eisenstein series $E(h, s)$ can have a double pole at the points $\frac{10}{117}$, $\frac{11}{117}$, and $\frac{12}{117}$. At the points $\frac{9}{17}$, $\frac{13}{17}$, $\frac{14}{17}$, $\frac{15}{17}$, and 1, it can have a simple pole.
- If $\chi$ is nontrivial quadratic, then the Eisenstein series can have simple poles at $\frac{10}{117}$, $\frac{11}{117}$, $\frac{12}{117}$, $\frac{13}{117}$, $\frac{14}{117}$, $\frac{15}{117}$, and 1.
- If $\chi$ is nontrivial cubic, then the Eisenstein series can have a simple pole at $\frac{10}{117}$.
- If the order of $\chi$ exceeds 3 then the Eisenstein series is holomorphic in $Re(s) > \frac{1}{2}$.

5. Calculation of $I(s, t)$
5.1. **First reduction.** The purpose of this section is to compute the period \( I(s, t) \) appearing in [18]. The first step is to reduce the study of \( I(s, t) \) to the study of a simpler period \( J(a, b, c) \), which we now define.

Throughout section 5 we denote the maximal torus of \( E_8 \) by \( T \). If \( w \) is an element of the Weyl group, \( W \), of \( E_8 \), then we define \( U_w = U_{\max} \cap w^{-1}U_{\max}w = \prod_{\alpha > 0 : w\alpha < 0} U_\alpha \). In this notation, the group \( U_0 \) appearing in the definition of \( I(s, t) \) can also be described as \( U_{w_{\text{Iag}}} \).

**Definition 1.** For \( a, b, c \in F^\times \), define

\[
J(a, b, c) = \int_{U_w} f_s(wu)\psi(u_{01000000} + au_{01000000} + u_{10000000} + bu_{00001110} + cu_{00000111}) \, du,
\]

\[
\quad w = w[243154234565423145765423187].
\]

**Theorem 4.**

\[
I(s, t) = \begin{cases} (1 + q^{-51s+18})J(1, 1, 1), & t_1, t_2 \in \mathfrak{o}^\times, \\ J(1, 1, t_1) - q^{-68s+26}J(1, 1, p^{-2}t_1), & t_2 \in \mathfrak{o}^\times, t_1 \notin \mathfrak{o}^\times, \\ J(1, t_2, t_1 t_2) - q^{-34s+14}J(p, p^{-1}t_2, p^{-1}t_1, t_2) + q^{-81s+35}J(1, p^{-2}t_2, p^{-2}t_1 t_2), & t_2 \notin \mathfrak{o}^\times. \end{cases}
\]

5.1.1. **Tools.** Before proceeding to the details of the proof we review a few standard techniques which are used in the calculations. We consider integrals of the following type:

\[
\int_V f_s(wu)\psi_V(u) \, du,
\]

where \( w \in W \),

\[
V \subset \{ u = x_{\beta_1}(u_1) \ldots , x_{\beta_N}(u_N) : u_i \in F, \ (1 \leq i \leq N) \},
\]

is defined by a finite set of conditions of one of the following three types:

\[
|u_i| \leq 1, \quad |u_i| > 1, \quad u_i = c, \ c \in F, \ \text{constant}.
\]

Also \( \beta_1, \ldots , \beta_N \) are distinct roots such that \( w\beta_i \notin \Phi(P, T), \ (1 \leq i \leq N), \) and

\[
\psi_V(x_{\beta_1}(u_1) \ldots , x_{\beta_N}(u_N)) = \psi \left( \sum_{i=1}^{N} c_i u_i \right), \quad \text{for some} \ c_1, \ldots , c_N \in F.
\]

The variable \( u_i \) is said to be free if it does not appear in any of the conditions which define \( V \). Of course, \( I(s, t) \) is of this type.

The basic technique is to split an integral of this type up according to whether \( |u_N| \leq 1 \) of \( |u_N| > 1 \) and plug in the Iwasawa decomposition for each to obtain two integrals of the same type with a smaller value of \( N \). Some care must be taken with regard to the order of the terms in the product to avoid venturing outside of this relatively simple class of integrals.

In addition to this basic technique, there are a few additional tricks that can be used.
Indeed, it’s clear that the terms may be reordered arbitrarily if we assume that
\[
\beta_i + \beta_j \text{ is a root } \implies \begin{cases} 
\beta_k = \beta_i + \beta_j \text{ for some } 1 \leq k \leq N, \\
c_k = 0, \\
u_k \text{ is free.}
\end{cases}
\]

Lemma 11. Suppose that \(u_N\) is not constant, and \(c_N \notin \mathfrak{o}\). Then the section integral (26) is zero.

Proof. Introduce \(x_{\beta_N}(z)\) at right with \(z \in \mathfrak{o}\) such that \(\psi(c_N z) \neq 1\), and make a change of variables in \(u_N\). □

Lemma 12 (Killing). Let
\[
(27)\quad V' = \{x_{\beta_1}(u_1) \ldots, x_{\beta_{N-1}}(u_{N-1})\}, \quad \psi_{V'}(x_{\beta_1}(u_1) \ldots x_{\beta_{N-1}}(u_{N-1})) = \psi\left(\sum_{i=1}^{N-1} c_i u_i\right).
\]
Assume that
\[
\int_{V'} f(wv'x_{\beta_N}(u_N)x_{\alpha}(z))\psi_{V'}(v') dv = \psi(az + \varepsilon u_N z) \int_{V'} f(wv'x_{\beta_N}(u_N)) \cdot \psi_{V'}(v') dv,
\]
where \(a \in \mathfrak{o}\) and \(\varepsilon \in \mathfrak{o}^\times\). Then one may restrict \(u_N\) to \(\mathfrak{o}\) without affecting the value of the integral. In this situation we say we can kill the root \(\beta_N\).

Proof. Simply integrate \(z\) over \(\mathfrak{o}\). □

Lemma 13. Suppose that \(c_N \in \mathfrak{o}^\times\), and that there is a cocharacter \(h_0 : GL_1 \to T\) with the property that \(\langle h_0, \beta_N \rangle = 1\), and \(\langle h_0, \beta_i \rangle = 0\) for all \(i \neq N\) with \(c_i \neq 0\), and that the variable \(u_N\) in (26) is subject to the bound \(|u_N| > 1\). Then the section integral (26) is equal to the integral
\[
-\int_{V'} f_s(wux_{\beta_N}(p^{-1}))\psi_{V'}(u) du.
\]
Here, \(V'\) and \(\psi_{V'}\) are defined as in 12.

Proof. Introduce \(h_0(\varepsilon)\) at right and integrate \(\varepsilon\) over \(\mathfrak{o}^\times\). Conjugating \(h_0(\varepsilon)\) across and making suitable changes of variable, one obtains an inner integration of
\[
q^k \int_{|\varepsilon|=1} \psi(c_N \varepsilon p^{-k}) d\varepsilon = \begin{cases} 
-1, & k = 1 \\
0, & \text{otherwise},
\end{cases}
\]
which gives the result. □

Remark 5. A sufficient condition for the existence of a cocharacter with the given properties is that \(#\{i : c_i \neq 0\} \leq 8\), and that there is an element of \(SL(8, \mathbb{Z})\) with the property that each root \(\beta_i\) with \(c_i \neq 0\) is one of the rows.
Lemma 14. Keep the notation and assumptions of lemma 12, but now assume that
\[ \int_{V'} f(wv'x_{\beta}(u_N)x_\alpha(z))\psi_{V'}(v') \, dv = \psi((a + bu_k + \varepsilon u_N)z) \int_{V'} f(wv'x_{\beta}(u_N) \cdot \psi_{V'}(v') \, dv, \]
for some \( a, b \in o, \varepsilon \in o^\times \), and \( k \in \{1, \ldots, N - 1\} \). Then the section integral \((26)\) is equal to
\[ \int_{V'} f_s(wu)\psi \left( \sum_{i=1}^{N-1} c_i u_i + c_N bu_k \right) \, du, \]
where \( V' \) is the subset of \( V \) defined by the condition \( u_N = -\varepsilon^{-1}bu_k \).

Proof. The proof is the same as that of lemma 12. \qed

Lemma 15. Suppose that \( w = w_1 w_2 \). Then one may conjugate \( w_2 \) from right to left without changing the value of the integral.

Remark 6. In general \( w_2 x_{\beta}(u_i) w_2^{-1} = x_{w_2 \beta}(\pm u_i) \). Normally, we make changes of variables at the same time to remove this signs. This may introduce signs into \( \psi_{V'} \).

If \( S \) is any subset of \( \Phi(G, T) \) which is closed under addition, then the product of the groups \( U_\alpha, \alpha \in S \), is a group, denoted \( U_S \). Note that \( S = \Phi(U_S, T) \). It will frequently be convenient to describe a subgroup \( U_S \) of \( U_w \) by specifying the complement of \( \Phi(U_S, T) \) in \( \Phi(U_w, T) \).

Proof of first reduction theorem. Using lemma 12, one kills \( \int_{U''} f_s(w_{1n}ux_{1111110}(m_2)x_{1121100}(m_3)x_{10111111}(m_1)x_{10000000}(n))\psi_{U',t}(u) \, du \), deducing that \( I(s, t) \), is equal to an integral of the same type, a 61-dimensional subgroup \( U' \subset U_0 \), let \( S'' \) be the complement of \( \{11111110, 11121100, 10111111, 10000000\} \) in \( S' \). Number the elements of \( S'' \), as \( \beta_1, \ldots, \beta_{57} \), and let
\[ U'' = \{ x_{\beta_1}(u_{\beta_1}) \ldots, x_{\beta_{57}}(u_{\beta_{57}}) : u_i \in F, (1 \leq i \leq 57) \}. \]

Define
\[ II(s, t; m_1, m_2, m_3, n) = \int_{U''} f_s(w_{1n}ux_{1111110}(m_2)x_{1121100}(m_3)x_{10111111}(m_1)x_{10000000}(n))\psi_{U',t}(u) \, du. \]
Thus
\[ I'(s, t) = \int_F \int_F \int_F \int_{U''} II(s, t; m_1, m_2, m_3, n)\psi(m_1) \, dm_2 \, dm_3 \, dm_1 \, dn. \]

We then consider various cases, based on the absolute values of \( n, m_1, t_1, \) and \( t_2 \). Write \( I_{o, o}(s, t) \) for the integral over \( n \in o \) and \( m_1 \in o \), \( I_{o, F \setminus o}(s, t) \) for the integral over \( n \in o \) and \( m_1 \in F \setminus o \), and so on, and \( I_{F \setminus o} \) for the integral over \( n \in F \setminus o \).

Using lemma 12 then lemma 15 yields \( I_{o, o} = J(1, t_2, t_1 t_2) \).
Proposition 5. One has
\[ I_{F \setminus \alpha}(s, t) = \begin{cases} q^{-51s+18}J(1, 1, 1), & t_1, t_2 \in \alpha^x, \\ 0, & \text{otherwise.} \end{cases} \]

Proof. The four terms in
\[ x_{11111110}(m_2)x_{11211100}(m_3)x_{10111111}(m_1)x_{10000000}(n) \]
all commute with one another. We rearrange them as
\[ x_{11111110}(m_2)x_{10000000}(n)x_{11211100}(m_3)x_{10111111}(m_1). \]
And write \( I'_{F \setminus \alpha}(s, t) = I'_{F \setminus \alpha, F \setminus \alpha}(s, t) + I'_{F \setminus \alpha, F \setminus \alpha}(s, t) \), where
\[
I'_{F \setminus \alpha, F \setminus \alpha}(s, t) := \int \int \int \int I(s, t; m_1, m_2, m_3, n) \psi(m_1) \, dm_2 \, dm_3 \, dm_1 \, dn
\]
\[
= \int \int \int \int I(s, t; 0, m_2, m_3, n) \, dm_2 \, dm_3 \, dn
\]
\[
I'_{F \setminus \alpha, F \setminus \alpha}(s, t) := \int \int \int \int I(s, t; m_1, m_2, m_3, n) \psi(m_1) \, dm_2 \, dm_3 \, dm_1 \, dn.
\]
In \( I'_{F \setminus \alpha, \alpha}(s, t) \), the root 01111110 kills 11121100, and then, plugging in the Iwasawa decomposition of \( x_{\alpha}(n) \) and simplifying yields
\[
I'_{F \setminus \alpha, \alpha}(s, t) = \int \int I(s, t; 0, m_2, 0, 0) \, dm_3 \int |n|^{-51s+17} \, dn.
\]
The first integral on the right hand side may also be obtained by killing one term in \( I'_{\alpha, \alpha} \). So it is equal to \( I'_{\alpha, \alpha} \) and hence to \( J(1, t_2, t_1t_2) \).

Next, lemma 13 implies that
\[
I_{F \setminus \alpha, F \setminus \alpha}(s, t) = -\int \int \int I(s, t; p^{-1}, m_2, m_3, n) \, dm_2 \, dm_3 \, dn.
\]
and then lemma 14 with \( \alpha = 01111110 \) yields
\[
I_{F \setminus \alpha, F \setminus \alpha}(s, t) = \int \int I(s, t; p^{-1}, m_2, t_1t_2p^{-1}, n) \, dm_2 \, dn.
\]
Plugging in the Iwasawa decomposition for \( x_{\alpha}(n) \) and simplifying gives
\[
I_{F \setminus \alpha, F \setminus \alpha}(s, t) = -\int \int f_s(u_m u x_{11111110}(m_2)) \psi(t_1)(u) \, du \, dm_2 \int \psi(n^{-1}p^{-2}t_1t_2^2) \, dn.
\]
As before, the integral on the left equals \( J(1, t_2, t_1t_2) \). Now,
\[
\int |n|^{-51s+17} \psi(n^{-1}p^{-2}t_1t_2^2) \, dn = \int |n|^{-51s+17} \, dn,
\]

unless \( t_1 t_2^2 \in \mathfrak{o} \), in which case
\[
\int_{|n|>q} |n|^{-51s+17} \psi(n^{-1}p^{-2}t_1 t_2^2) \, dn = \int_{|n|>q} |n|^{-51s+17} \, dn,
\]
while
\[
\int_{|n|=q} |n|^{-51s+17} \psi(n^{-1}p^{-2}t_1 t_2^2) \, dn = -q^{-51s+17}.
\]
Combined with the fact that
\[
\int_{|n|=q} |n|^{-51s+17} \, dn = q^{-51s+18} - q^{-51s+17},
\]
this yields the result. \( \square \)

**Proposition 6.**

(29)
\[
I_{\alpha,F,F-o} = \begin{cases} 
0, & |t_1| = |t_2| = 1, \\
-J(p, p^{-1}t_2, p^{-1}t_1 t_2)q^{-34s+14} + q^{-85s+35}J(1, p^{-2}t_2, p^{-2}t_1 t_2), & |t_2| < 1, \\
-q^{-68s+26}J(1, 1, p^{-2}t_1), & |t_2| = 1, |t_1| < 1.
\end{cases}
\]

**Proof.** This follows from the same type of arguments using the lemmas from section 5.1.1. We omit the details. \( \square \)

Assembling the pieces we obtain the theorem. \( \square \)

### 5.2. Second Reduction.

**Proposition 7.** Let \( f \) be the normalized spherical vector in the induced representation attached to the character
\[
\prod_{i=1}^{8} \alpha_i^\lambda(t_i) \mapsto \prod_{i=1}^{8} |t_i|^{t_i}, \quad \text{with} \\
[s_1, \ldots, s_8] = [17s - 6, 17s - 6, 17s - 6, -34s + 14, 17s - 6, -17s + 7, 17s - 6, 17s - 5],
\]
and let
\[
J_0(b, c) := \int_{U_w[57687]} f(w[57687]) u \psi(bu_0001110 + cu_00000111) \, du.
\]
Then
\[
J(a, b, c) = \frac{\zeta(17s - 6)\zeta(17s - 7)}{\zeta(17s - 4)\zeta(17s - 3)\zeta(17s - 2)\zeta(17s)\zeta(34s - 10)} (1 - |a|^{17s-7} q^{-17s+7}) J_0(b, c).
\]

**Proof.** First, write \( w[243154234565423145765423187] = w''w' \) with \( w'' = w[243154234654237654], w' = w[131257687] \). We have

(30)
\[
J(a, b, c) = \int_{U_w'} (M(s, w'')f_s)(w'u) \psi(u_{00100000} + au_{01000000} + u_{10000000} + bu_{0001110} + cu_{00000111}) \, du,
\]
26
where
\[
M(s, w'') f_s(g) = \int_{U_{w''}} f_s(w'' u g) \, du
\]
is the standard intertwining operator. By the Gindikin-Karpelevich formula we have
\[
M(s, w'') f_s = \frac{\zeta(17s - 6)^4 \zeta(34s - 13)}{\zeta(17s - 4) \zeta(17s - 3) \zeta(17s - 2) \zeta(17s) \zeta(34s - 10)} f.s
\]
Next, observe that \( w' \) and \( U_{w'} \) are contained in the standard Levi subgroup of \( E_8 \) which has derived group isomorphic to \( SL_2 \times SL_3 \times SL_5 \). One may factor \( w' \) as \( w' = w[131]w[2]w[57687] \), and also factor the integral in (30) into a product of 3 simpler integrals corresponding to the three components of the Levi and the three factors of \( w' \). The integrals corresponding to \( w[131] \) and \( w[2] \) are Jacquet integrals, equalling \( \zeta(17s - 6)^{-2} \zeta(34s - 13)^{-1} \) and \( \zeta(17s - 7)^{-1} \zeta(17s - 8)^{-1} \), respectively. The proposition follows.

5.3. An \( SL_5 \) period. From Theorem A and Proposition B, the computation of \( I(s, t) \), is reduced to the computation of an integral \( J_0(b, c) \) over a unipotent subgroup of the copy of \( SL_5 \) in \( E_8 \) which is generated by \( x_{\pm \alpha_5}, x_{\pm \alpha_6}, x_{\pm \alpha_7} \) and \( x_{\pm \alpha_8} \). In this section we compute \( J_0(b, c) \).

**Proposition 8.** The function \( J_0(b, c) \) depends only on the \( p \)-adic valuations of \( b \) and \( c \). Take integers \( B \) and \( C \) with \( B \leq C \), and take \( E \) equal to either an integer or \( \infty \). Define \( J_2(B, C, E) \) to be
\[
\frac{(1 - xq^6)(1 - (xq^7)^{B+1})}{(1 - xq^7)^2(1 - x^2q^{13})}
\begin{cases}
(1 - x^2q^{12})(1 - xq^6) - (1 - xq^6)(1 - x^2q^{13})(xq^7)^{C+1} & \text{min}(B, C, E) \geq 0, \quad E \geq C, \\
+ (1 - q^{-1})(xq^6)(1 - xq^7)(x^2q^{13})^{C+1}, & \text{min}(B, C, E) \geq 0, \quad E < C, \\
(1 - x^2q^{12})(1 - xq^6)(1 - x^2q^{13})^{E+1} & \quad E \geq C, \\
- (xq^7)^{C+1}(1 - xq^5)(1 - x^2q^{13})(1 - (xq^6)^{E+1}), & \quad E < C, \\
0 & \quad \text{min}(B, C, E) < 0.
\end{cases}
\]
where \( x = q^{-s} \). Then for all \( b, c \in F \) with \( |b| = q^{-B} \) and \( |c| = q^{-C} \),
\[
J_0(b, c) = J_2(B, C, \infty) + (1 - q^{-1}) \sum_{\ell=1}^{B} J_2(B - \ell, C - \ell, \infty)(xq^\ell)^{B-\ell} + (1 - q^{-1}) \sum_{k=1}^{B} (x^2q^{13})^k J_2(B - k, C, \infty)
\]
\[
+ (1 - q^{-1})^2 \sum_{k=1}^{B} \left[ \sum_{\ell=0}^{k-1} J_2(B - k, C - k + \ell)q^{-\ell} + \sum_{\ell=1}^{B-k} J_2(B - k - \ell, C - \ell, C - k - \ell)(xq^\ell) \right].
\]

**Proof.** Define
\[
n^-(x_1, x_2, x_3, x_4, x_5) := x_{\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_4} x_{\alpha_5 + \alpha_6 + \alpha_7} x_{\alpha_5 + \alpha_6 + \alpha_7} x_{\alpha_5 + \alpha_6 + \alpha_7} x_{\alpha_5 + \alpha_6 + \alpha_7} \in E_8.
\]
This gives an explicit parametrization of the group \( U_{w[57687]} \). Next, define
\[
\tilde{J}_1(b, c, e) = \int_{F^4} f_\chi(n^-(0, x_2, x_3, x_4, x_5)\psi(bx_5 + cx_4 + ex_2x_3) \, dx
\]
\[ \hat{J}_2(b, c, e) = \int_{F^4} f_\chi(n^-(0,0,x_3,x_5))\psi(bx_5 + cx_4 + ex_3) \, dx \]
\[ \hat{J}_4(c, e) = \int_{F^2} f_\chi(n^-(0,0,x_3,0))\psi(cx_4 + ex_3) \, dx. \]

Then by plugging in the Iwasawa decompositions of \(x_{\alpha_7}(x_1)\) and then \(x_{\alpha_6+\alpha_7}(x_2)\), one finds that
\[ J_0(b, c) = \hat{J}_1(b, c, 0) + \int_{F \setminus \mathcal{O}} \hat{J}_1(bx_1, b, bx_1)|x_1|^{-34s+13} \, dx_1 \]
\[ \hat{J}_1(b, c, e) = \int_{\mathcal{O}} \hat{J}_2(b, c, x_2) \, dx_2 + \int_{F \setminus \mathcal{O}} \hat{J}_2(bx_2, cx_2, ex_2)|x_2|^{-17s+7} \, dx_2. \]

Moreover, if \(1_\mathcal{O}\) is the characteristic function of \(\mathcal{O}\), then
\[ \hat{J}_2(b, c, e) = \frac{\zeta(17s-7)}{\zeta(17s-6)}(1 - |b|^{17s-7}q^{-17s+7})\hat{J}_4(c, e) \cdot 1_\mathcal{O}(b). \]

(The integration in \(x_5\) amounts to an \(SL_2\) Jacquet integral.) Likewise
\[ \hat{J}_4(c, e) = \frac{\zeta(17s-7)}{\zeta(17s-6)} \left( 1_\mathcal{O}(c) \int_{\mathcal{O}} (1 - |c|^{17s-7}q^{-17s+7})\psi(ex_3) \, dx_3 \right. \]
\[ + \int_{F \setminus \mathcal{O}} 1_\mathcal{O}(cx_3)(1 - |cx_3|^{17s-7}q^{-17s+7})\psi(ex_3)|x_3|^{-34s+12} \, dx_3 \left. \right). \]

At this point it is clear that \(\hat{J}_4\), and hence all of the other integrals, depend only on the absolute values, or, equivalently, \(p\)-adic valuations, of their arguments. Introducing the notation \(x := q^{-17s}\), we have
\[ \hat{J}_4(p^C, p^E) = 1_\mathcal{O}(c) 1_\mathcal{O}(e) \frac{1 - xq^6}{1 - qx^7} \left( 1 - (xq^7)^{C+1} \right) + \sum_{m=1}^{C} \left( 1 - (xq^7)^{C-m+1} \right) (x^2q^{12})^m \int_{|x_3|=q^m} \psi(ex_3) \, dx_3 \right). \]

Denote this quantity by \(J_4(C, E)\). Recall that for any \(e \in F\) with \(|e| = q^{-E}\),
\[ \int_{|x_3|=q^m} \psi(ex_3) \, dx_3 = \begin{cases} (1 - q^{-1})q^m, & m \leq E, \\ -q^E, & m = E + 1, \\ 0, & m > E + 1. \end{cases} \]

It follows that \(J_4(C, E) = 0\) if either \(C\) or \(E\) is negative, and that otherwise
\[ J_4(C, E) = \frac{1}{(1-xq^7)(1-x^2q^{13})} \left\{ \begin{array}{ll} (1 - x^2q^{12})(1 - xq^6) - (1 - xq^5)(1 - x^2q^{13})(xq^7)^{C+1} \\ + (1 - q^{-1})(xq^6)(1 - xq^7)(x^2q^{13})^{C+1}, & E \geq C, \\ (1 - x^2q^{12})(1 - xq^6)(1 - (x^2q^{13})^{E+1}) \\ - (xq^7)^{C+1}(1 - x^2q^{13})(1 - (xq^6)^{E+1}), & E < C. \end{array} \right. \]

It follows from (33) that
\[ \hat{J}_2(b, c, e) = \frac{1 - xq^6}{1 - xq^7}(1 - (xq^7)^{B+1})J_4(C, E)1_\mathcal{O}(b) = J_2(B, C, E), \]
whenever \( b, c \in F^\times, e \in F \) have valuations \( B, C, E \) respectively (with the convention that the valuation of 0 is \( \infty \)). Plugging in to (32), and using the fact that \( B \leq C \) and that the volume of \( \{ y \in F : |y| = q^k \} \) is \( q^k(1 - q^{-1}) \) for each \( k \in \mathbb{Z} \), gives the result. \hfill \Box

Now, take \( X = (X_1, \ldots, X_6) \) to be a sextuple of indeterminates, and consider the ring \( R_1 := \mathbb{C}(x, q)[X] \) of polynomials in \( X \) with coefficients in the field \( \mathbb{C}(x, q) \) of rational functions. We define two elements of this ring by

\[
J_2^1(X) = (1 - X_1X_2^3) \left((1 - x^2q^{12})(1 - xq^6) - (1 - xq^5)(1 - x^2q^{13})X_3X_4^7\right)
+ (1 - q^{-1})(xq^6)(1 - x^2q^7)X_3^2X_4^{13}
\]

\[
J_2^2(X) = (1 - X_1X_2^7) \left((1 - x^2q^{12})(1 - xq^6)(1 - X_5^2X_6^{13}) - X_3X_4^7(1 - xq^5)(1 - x^2q^{13})(1 - X_5X_6^6)\right),
\]

then we have

\[
J_2(B, C, E) = \frac{(1 - xq^6)}{(1 - xq^7)^2(1 - x^2q^{13})} \begin{cases} J_2^1(x^{C+1}, q, q^{C+1}, q^{E+1}), & E \geq C, \\ J_2^2(x^{B+1}, q^{B+1}, x^{C+1}, q^{C+1}, q^{E+1}), & E < C. \end{cases}
\]

Let \( R_2 \) be the ring of Laurent polynomials \( \mathbb{C}(x, q)[X_1, X_2, X_3, X_4, X_5^{-1}, X_6^{-1}] \). Then \( J_2^1 \in R_2 \), and each of the four summations above corresponds to a linear operator \( R_1 \to R_2 \). For example,

\[
\sum_{\ell=1}^{B} (x^{n_1}q^{n_2})^{B+1-\ell}(x^{n_3}q^{n_4})^{C+1-\ell}(xq^\ell)
= \frac{\sum_{\ell=1}^{B} (x^{n_1}q^{n_2})^{B+1}(x^{n_3}q^{n_4})^{C+1}(x^{1-n_1-n_3}q^{8-n_2-n_4} - (x^{1-n_1-n_3}q^{8-n_2-n_4})^{B+1})}{1 - x^{1-n_1-n_3}q^{8-n_2-n_4}},
\]

for all \( n_1, \ldots, n_4 \in \mathbb{Z} \) with \((n_1 + n_3, n_2 + n_4) \neq (1, 8)\). It follows that

\[
\sum_{\ell=1}^{B} J_2(B - \ell, C - \ell, \infty) = [T_1, J_2^1](x^{B+1}, q^{B+1}, x^{C+1}, q^{C+1}),
\]

where \( T_1 \) is the \( \mathbb{C}(x, q) \)-linear map \( R_1 \to R_2 \) defined on monomials by

\[
T_1 \left( \prod_{i=1}^{6} X_i^{n_i} \right) = \prod_{i=1}^{4} X_i^{n_i}(x^{1-n_1-n_3}q^{8-n_2-n_4} - X_1^{1-n_1-n_3}q^{8-n_2-n_4}).
\]

In similar fashion, we can define operators corresponding to the other three summations in (31). Specifically, if

\[
T_2 \left( \prod_{i=1}^{6} X_i^{n_i} \right) = \prod_{i=1}^{4} X_i^{n_i} \frac{x^{2-n_1}q^{13-n_2} - X_1^{2-n_1}X_2^{13-n_2}}{1 - x^{2-n_1}q^{13-n_2}},
\]

\[
T_3 \left( \prod_{i=1}^{6} X_i^{n_i} \right) = \prod_{i=1}^{4} X_i^{n_i} \cdot X_3^{n_3} X_4^{n_4} \left[ \frac{x^{2-n_1-n_5}q^{14-n_2-n_6}}{(1 - x^{2-n_1-n_5}q^{14-n_2-n_6})(1 - x^{2-n_1}q^{13-n_2})} + \frac{X_1^{2-n_1}X_2^{13-n_2}}{1 - x^{2-n_1}q^{13-n_2}} \right]
+ \frac{1}{1 - x^{n_5}q^{n_6-1}} \times \left( \frac{-X_1^{2-n_1-n_5}X_2^{14-n_2-n_6}}{1 - x^{2-n_1-n_5}q^{14-n_2-n_6}} + \frac{X_1^{2-n_1}X_2^{13-n_2}}{1 - x^{2-n_1}q^{13-n_2}} \right)
\]


Proposition 9. The value of \( J_0(p^B, q^C) = J_0(x^{B+1}, q^{B+1}, x^{C+1}, q^{C+1}) \), where

\[
J_0 = \frac{1 - xq^6}{(1 - xq^7)^2(1 - x^2q^{13})}[J_1^2 + (1 - q^{-1})(T_1 + T_2)]J_2^1 + (1 - q^{-1})^2(T_3 + T_4)]J_2^1.
\]

Then \( J_0(p^B, q^C) = J_0(x^{B+1}, q^{B+1}, x^{C+1}, q^{C+1}) \), where

\[
J_0 = \frac{1 - xq^6}{(1 - xq^7)^2(1 - x^2q^{13})}[J_2^3 + (1 - q^{-1})(T_1 + T_2)]J_2^1 + (1 - q^{-1})^2(T_3 + T_4)]J_2^1.
\]

**Proof.** Straightforward calculations give each of the components of the sum. We record the results:

\[
[T_1, J_2^1] = (1 - xq^6)(1 - x^2q^{12}) \left( \frac{xq^8}{1 - xq^7} - \frac{q}{1 - q} X_1 X_2^7 + \frac{q(1 - xq^7)}{(1 - xq^8)(1 - q)} X_1 X_2^7 \right)
- (1 - xq^5)(1 - x^2q^{13}) \left( \frac{q}{1 - q} + \frac{1}{1 - xq^6} X_1 X_2^7 - \frac{1 - xq^7}{(1 - q)(1 - xq^6)} X_2 \right) X_3 X_4^7
+ (1 - q^{-1})(1 - xq^7)(xq^6) \left( \frac{-1}{1 - xq^6} + \frac{1}{1 - x^2q^{12}} X_1 X_2^7 + \frac{xq^5(1 - xq^7)}{(1 - xq^6)(1 - x^2q^{13})} X_2 \right) X_3 X_4^{13}
\]

\[
[T_2, J_2^1] = [T_2, (1 - X_1 X_2^7)]J_2^1 = \left( \frac{x^2q^{13}}{1 - x^2q^{13}} - \frac{xq^6 X_1 X_2^7}{1 - xq^6} + \frac{xq^6 X_2 X_3 X_4^1(1 - xq^7)}{(1 - xq^6)(1 - x^2q^{13})} \right) J_2^1
\]

\[
[T_3, J_2^1] = ((1 - xq^6)(1 - x^2q^{12}) - (1 - xq^5)(1 - x^2q^{13}) X_3 X_4^1) \cdot [T_3, (1 - X_1 X_2^7)] + c_2(X_1, X_2) X_3 X_4^{13},
\]

where

\[
T_3, (1 - X_1 X_2^7) = \frac{x^2q^{14}}{(1 - x^2q^{14})(1 - x^2q^{13})} - \frac{X_3 X_4^7 X_2}{(1 - xq^7)(1 - xq^6)}
- \frac{X_2 X_3 X_4^1 xq^6(1 - xq^7)}{(1 - q^{-1})(1 - x^2q^{13})} + \frac{X_2 X_3 X_4^1 xq^7}{(1 - q^{-1})(1 - x^2q^{14})}
\]
\[
\begin{align*}
c_2 &= (1 - xq^5)(1 - x^2q^{13}) \left[ \frac{xq^8}{(1 - xq^8)(1 - x^2q^{13})} - \frac{X_1X_2^7q}{(1 - xq^6)(1 - q)} \right] \\
&\quad + \frac{X_1X_2^3q}{(1 - xq^5)(1 - xq^8)} - \frac{X_1X_2^3xq^6(1 - xq^7)}{(1 - xq^6)(1 - x^2q^{13})(1 - x^2q^5)} \\
&\quad - (1 - xq^6)(1 - x^2q^{12}) \left[ \frac{q}{(1 - q)(1 - x^2q^{13})} - \frac{X_1X_2^3x^{-1}q^{-6}}{(1 - xq^6)(1 - x^{-1}q^{-6})} \right] \\
&\quad + \frac{X_2x^{-1}q^{-6}}{(1 - x^2q^{12})(1 - q)} - \frac{X_3^2X_2^3xq^6(1 - xq^7)}{(1 - xq^6)(1 - x^2q^{13})(1 - x^2q^{12})} 
\end{align*}
\]

Finally

\[
[T_4J_2^2] = (1 - xq^6)(1 - x^2q^{12})Q(0, 0, 0, 0) - (1 - xq^5)(1 - x^2q^{13})Q(1, 7, 0, 0)X_3X_4^7 \\
\quad - [(1 - xq^6)(1 - x^2q^{12})Q(2, 13, 0, 0) - (1 - xq^5)(1 - x^2q^{13})Q(1, 7, 1, 6)]X_3^2X_4^{13}.
\]

where

\[
Q(n_3, n_4, n_5, n_6) = \frac{x^{3-n_3-2n_5}q^{22-n_4-2n_6}}{(1 - x^{2-n_5}q^{14-n_6})(1 - x^{1-n_3-n_5}q^{8-n_4-n_6})} \\
\quad - \frac{X_1X_2^5x^{1-n_3-2n_5}q^{8-n_4-2n_6}}{(1 - x^{1-n_5}q^{7-n_6})(1 - x^{1-n_3-n_5}q^{1-n_4-n_6})} \\
\quad - \frac{x^{1-n_5}q^{7-n_6}X_1^2-n_5X_2^{14-n_6}}{(1 - x^{n_3+1}q^{n_4+6})(1 - x^{1-n_5}q^{7-n_6})(1 - x^{2-n_5}q^{14-n_6})} \\
\quad + \frac{x^{1-n_5}q^{7-n_6}X_1^{1-n_3-n_5}X_2^{8-n_4-n_6}}{(1 - x^{n_3+1}q^{n_4+6})(1 - x^{1-n_3-n_5}q^{1-n_4-n_6})(1 - x^{1-n_3-n_5}q^{8-n_4-n_6})}.
\]

Totalling up the contributions, multiplying by \( \frac{(1-xq^6)}{(1-xq^5)(1-xq^{13})} \), and simplifying gives the result.

\[\square\]

5.4. Final calculation of \( I(s, t) \). In this section we complete the proof of theorem. First, by proposition 7, \( J(a, b, c) = P_0(x, q)(1 - (xq^7)^{A+1})J_0(b, c) \), where

\[
P_0(x, q) = \frac{(1 - x)(1 - xq^2)(1 - xq^3)(1 - xq^4)(1 - x^2q^{10})}{(1 - xq^6)(1 - xq^7)},
\]

and by theorem 4

\[
I(s, t) = \begin{cases} 
J(1, 1, 1)(1 + x^3q^{18}), & t_1, t_2 \in o^x \\
J(1, 1, t_1) - x^4q^{26}J(1, 1, \frac{t_1}{p}), & t_2 \in o^x, t_1 \notin o^x \\
J(1, t_2, t_1t_2) - x^2q^{14}J(p, \frac{t_2}{p}, \frac{t_1t_2}{p}) + x^5q^{35}J(1, \frac{t_2}{p}, \frac{t_1t_2}{p}), & t_2 \notin o^x.
\end{cases}
\]

[10]
5.4.2. **Case 2:** \( q \). Multiplying by \( (1 \) empty sums. Hence these terms will specialize to zero in that case. Moreover \( X \) has only to check that \( n \) where \( R \) on the ring \( m \)

\[
5.4.1. \quad J \in \mathbb{Z}
\]

so we obtain

\[
J_0(t_2,t_1t_2) - x^2q^{14}J(p, t_2, t_1t_2) + x^5q^{35}J(1, t_2, t_1t_2)
\]

\[
= (1 - xq^7) \left( J_0(t_2,t_1t_2) - x^2q^{14}(1 + xq^7)J_0(t_2, t_1t_2) + x^5q^{35}J_0(t_2, t_1t_2) \right)
\]

\[
= (1 - xq^7)[T_0J_0](x^{m+1}, q^{m+1}, x^{n+m+1}, q^{n+m+1}),
\]

where \( n \) and \( m \) are the \( p \)-adic valuations of \( t_1 \) and \( t_2 \), respectively, and \( T_0 \) is a linear operator on the ring \( R \) defined by

\[
T_0 \left( \prod_{i=1}^{4} X_i^{n_i} \right) = \prod_{i=1}^{4} X_i^{n_i}(1 - x^{2-n_1-n_3}q^{4-n_2-n_4})(1 - x^{3-n_1-n_3}q^{21-n_2-n_4}).
\]

Applying the operator \( T_0 \) to \( J_0 \) as computed in the previous section gives

\[
[T_0J_0] = \frac{(1 - xq^6)(1 - x^2q^{12})}{(1 - xq^7)(1 - xq^8)} \left[ (1 - xq^6)(1 - x^3q^{21}) - (1 - xq^6)(1 - x^5q^7)X_3X_4^7 \right] - q^{-1}X_2X_3X_4^7(1 - x^5q^4)(1 - xq^8).
\]

Multiplying by \( (1 - xq^7)P_0(x, q) = (1 - xq^6)(1 - x^2q^{12})Z(x, q) \), and plugging in \( X_2 = q^{m+1}, X_3 = x^{n+m+1} \) and \( X_4 = q^{n+m+1} \) gives the proof in this case.

5.4.2. **Case 2:** \( t_2 \in \mathfrak{o}^\times \). In this case the summations corresponding to \( T_1, T_3 \) and \( T_4 \) are empty sums. Hence these terms will specialize to zero in that case. Moreover

\[
(J_2^1 + (1 - q^{-1})T_2J_2^1) = \frac{1}{(1 - xq^6)(1 - xq^8)}J_4^1(x, q, X_1, X_2)J_4^1(x, q, X_3, X_4).
\]

The condition \( t_2 \in \mathfrak{o}^\times \) translates to \( X_1 = x, X_2 = q \), and

\[
J_4^1(x, q, X_1, X_2) = (1 - xq^6)(1 - xq^7)(1 - x^2q^{13}),
\]

so we obtain \( J_0 = \frac{1 - xq^6}{(1 - xq^6)(1 - xq^8)}J_4^1(x, q, X_3, X_4) \) in this case. Now,

\[
I = P_0(x, q)(1 - xq^7) \left\{ \begin{array}{ll}
(1 + x^3q^{18})J_0(x, q, x, q), & t_1 \in \mathfrak{o}^\times, \\
J_0(x, q, X_3, X_4) - x^4q^{26}J_0(x, q, X_3x^{-2}, X_4q^{-2}), & t_2 \notin \mathfrak{o}^\times.
\end{array} \right.
\]

Simplifying

\[
J_0(x, q, X_3, X_4) - x^4q^{26}J_0(x, q, X_3x^{-2}, X_4q^{-2})
\]

\[
= \frac{1 - xq^6}{(1 - xq^7)(1 - x^2q^{13})} \left( (1 - xq^6)(1 - x^2q^{12})(1 - x^4q^{26}) - (1 - xq^5)(1 - x^2q^{13})(1 - x^2q^{12})X_3X_4^7 \right).
\]

If \( X_3 = x \), and \( x = q \), this coincides with \((1 + x^3q^{18})J_0(x, q, x, q) \). It follows that the case \( t_1 \in \mathfrak{o}^\times \) does not need to be handled separately. After incorporating \((1 - xq^7)P_0(x, q) \), one has only to check that

\[
I_0(n, 0; x, q) = (1 - xq^8) \left( (1 - xq^6)(1 + x^2q^{13}) - (1 - xq^8)x^{n+1}q^{7n+7} \right),
\]

and this is straightforward.
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