BULKY HAMILTONIAN ISOTOPIES OF LAGRANGIAN TORI WITH APPLICATIONS

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Abstract. We exhibit an example of a monotone Lagrangian torus inside the standard symplectic four dimensional unit ball which becomes Hamiltonian isotopic to a standard product torus only when considered inside a strictly larger ball (it is not even not symplectomorphic to a standard torus inside the unit ball). These tori are then used to construct new examples of symplectic embeddings of toric domains into the unit ball which are symplectically knotted in the sense of J. Gutt and M. Usher. In contrast to this, we establish a certain condition on the Gromov width of the complement of a Lagrangian torus inside the unit ball which ensures that it is a standard product torus.

1. Introduction and results

Our focus here is on the Liouville manifold
\[ (\mathbb{C}^2, \omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2) \]
equipped with the standard Liouville form
\[ \lambda_0 = \frac{1}{2} \sum_{i=1}^{2} (x_i dy_i - y_i dx_i), \quad d\lambda_0 = \omega_0. \]
Denote by \(\zeta_0\) the corresponding Liouville vector field, which generates the flow
\[ \phi^{t}_{\lambda_0} : (\mathbb{C}^2, d\lambda_0) \to (\mathbb{C}^2, e^{-t}d\lambda_0), \]
\[ \phi^{t}_{\lambda_0}(z) = e^{t/2}z. \]
To set up notation, we will use \(B^n_x(r)\) and \(D^n_x(r)\) to denote the open and closed balls inside \(\mathbb{C}^n\) of radius \(r > 0\) centred at the point \(x \in \mathbb{C}^n\), and \(S^{2n-1}(r) \subset \mathbb{C}^n\) for the sphere of radius \(r > 0\).

Recall that a Lagrangian submanifold in this case is a half-dimensional submanifold to which \(\lambda_0\) pulls back to a closed form. A Lagrangian isotopy is a smooth isotopy through Lagrangian embeddings; recall the standard fact that such an isotopy can be generated by a global Hamiltonian

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isotopy of the ambient symplectic manifold if and only if the pullbacks of \( \lambda_0 \) are constant in cohomology; see e.g. [27] by A. Weinstein. In general, we will call a smooth isotopy of a subset of a symplectic manifold Hamiltonian if it can be realised by an ambient Hamiltonian isotopy.

The symplectic action class of a Lagrangian is the cohomology class \( \sigma_L := [\lambda_0|_{TL}] \in H^1(L; \mathbb{R}) \) pulled back to it, which by Stokes’ theorem is the value of the symplectic area of any two-chain inside \( \mathbb{C}^2 \) with boundary in that class. A torus is monotone if the symplectic area of a two-dimensional chain with boundary on it is proportional to the so-called Maslov class of the same chain; see V. Arnold [1] for the definition of the latter characteristic class. In particular, it follows that the Lagrangian product tori \( S^1(a) \times S^1(b) \subset \mathbb{C}^2 \) are monotone if and only if \( a = b \). Note that the symplectic action class of the standard monotone product torus \( S^1(r) \times S^1(r) \subset (\mathbb{C}^2, \omega_0) \) takes values that are integer multiples of \( r^2\pi > 0 \). These tori are usually called Clifford tori.

R. Vianna [26] has shown that the classes of monotone Lagrangian tori inside \((\mathbb{C}P^2, \omega_{FS})\) exhibit a very rich and interesting structure. In particular, they consist of infinitely many different Hamiltonian isotopy classes. The result [12, Theorem C] by the author together with E. Goodman and A. Ivrii implies that all of Vianna’s tori can be placed inside the open unit ball

\[ (B^4, \omega_0) = (\mathbb{C}P^2 \setminus \ell_{\infty}, \omega_{FS}) \]

and thus, a fortiori, also give rise to infinitely many different Hamiltonian isotopy classes of monotone Lagrangian tori inside \( B^4 \). In contrast to this, the only known Hamiltonian isotopy classes of Lagrangian tori inside \( \mathbb{C}^2 \) are the product tori, together with linear rescalings of the “exotic” monotone torus [4] constructed by Y. Chekanov, which goes under the name of the Chekanov torus. We refer to the work [14] by A. Gadbled for the presentation that we will use here.

We expect that Vianna’s tori all become Hamiltonian isotopic to standard tori inside a ball which is strictly larger than the unit ball. (This can be confirmed by hand for e.g. certain particular Hamiltonian isotopies that takes Vianna’s first exotic torus constructed in [25] into the unit ball.)

Remark 1.1. Even though all Lagrangian tori are Lagrangian isotopic inside the ball by [12], there is still no classification of Lagrangian tori inside the plane up to Hamiltonian isotopy. Under additional assumptions concerning a certain linking behaviour with a conic; the author established a Hamiltonian classification in [11].

We begin by presenting a criterion for when a monotone Lagrangian torus inside the unit ball is Hamiltonian isotopic to a standard torus inside the ball itself. The characterisation can be reformulated in terms of the so-called Gromov width of the complement of the Lagrangian. This is a symplectic capacity that was introduced by M. Gromov in [15], which for a symplectic
manifold \((X^4,\omega)\) is equal to the supremum
\[
\sup \left\{ \pi \cdot r^2; \exists \varphi: (B^4(r),\omega_0) \hookrightarrow (X,\omega) \right\}
\]
taken over all symplectic embeddings.

**Theorem 1.1.** Let \(L \subset (B^4,\omega_0)\) be a Lagrangian torus inside the unit ball whose symplectic action class takes the values \(Zr^2\pi\) on \(H_1(L)\) for some fixed \(r \geq 1/\sqrt{3}\). There exists a Hamiltonian isotopy inside the ball which takes \(L\) to the standard monotone product torus \(S^1(r) \times S^1(r)\) if and only if one can find a symplectic embedding
\[
\varphi: (D^4(\sqrt{2/3}),\omega_0) \hookrightarrow (B^4,\omega_0),
\]
\[
L \subset B^4 \setminus \varphi(B^4(\sqrt{2/3})),
\]
in the complement of \(L\).

**Remark 1.2.** In particular, this implies that the Gromov width of the complement of a Chekanov torus inside \((B^4,\omega_0)\) is strictly less than \(\pi^2/3\) whenever its symplectic action assumes the values \(Zr^2\pi\) for some fixed \(r \geq 1/\sqrt{3}\). For interesting previous results about the Gromov width of the complement of Lagrangian submanifolds we refer to [3] by P. Biran as well as [4] by P. Biran and O. Cornea.

We then show that the above theorem is sharp in the following sense: Even under the stronger assumption that \(L \subset B^4 \setminus B^4(\sqrt{2/3} - \epsilon)\) is a Lagrangian torus that is Hamiltonian isotopic to \(S^1(r) \times S^1(r)\) inside the full plane \((\mathbb{C}^2,\omega_0)\), there are cases when any such Hamiltonian isotopy must intersect \(S^3 = \partial D^4\) at some moment in time. (In other words, the Hamiltonian isotopy cannot be confined to the unit ball that contains the original Lagrangian.) More precisely, we establish that

**Theorem 1.2.** There exists a Lagrangian torus \(L \subset (B^4,\omega_0)\) which is Hamiltonian isotopic inside \((\mathbb{C}^2,\omega_0)\) to the standard product torus
\[
S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3}),
\]
but where every such Hamiltonian isotopy necessarily satisfies \(\phi_{H_t}^L(L) \cap S^3 \neq \emptyset\) for at least one value \(t_0 \in [0,1]\). In addition, we may assume that one of the following holds:

- \(L \subset B^4 \setminus B^4(\sqrt{2/3} - \epsilon)\), whenever \(\epsilon > 0\) is sufficiently small, or
- \(L \subset B^4(\sqrt{1 - r}) \subset B^4\), whenever \(r \in (0,1/6)\).

The torus \(L\) is constructed in Section 3 by using a probe; see Figures 3 and 4.

In view of the above theorem, the following definition is natural. Consider two subsets \(A_0, A_1 \subset (X^{2n},d\lambda)\) of a Liouville domain with smooth boundary and denote by \((\overline{X}^{2n},d\lambda)\) the completion of \((X,d\lambda)\) to a noncompact Liouville manifold with a convex cylindrical end. In particular,
\[
(\overline{X} \setminus (X \setminus \partial X),d\lambda) \cong ([0,\infty) \times \partial X, d(e^s\lambda|_{\partial X}))
\]
where the latter exact symplectic manifold is the symplectisation of the boundary of \((X, d\lambda)\).

**Definition 1.1.** A Hamiltonian isotopy from \(A_0\) to \(A_1\) inside the completion of \((X, d\lambda)\), i.e. a Hamiltonian isotopy

\[
\phi^t_{H_t} : (X, d\lambda) \rightarrow (X, d\lambda)
\]

which satisfies

\[
\phi^0_{H_t} = \text{Id}, \quad \text{and} \quad \phi^1_{H_t}(A_0) = A_1,
\]

is said to be a bulky Hamiltonian isotopy from \(A_0\) to \(A_1\) relative \(X\) if there exists no smooth one-parameter family \(\phi^t_{H,s}\) of Hamiltonian isotopies of the same kind that satisfies \(\phi^t_{H,s} = \phi^t_{H_t}\) as well as \(\phi^t_{H,s}(A_0) \subset X\) for all \(t \in [0,1]\).

In other words, we can rephrase the above theorem as the statement that “any Hamiltonian from \(L\) to a standard torus inside \(\mathbb{C}^2\) is bulky relative the unit ball.”

The torus \(L\) that we construct in order to prove Theorem 1.2 is much more elementary than the tori constructed by Vianna. In fact, the example that we consider can be identified with the monotone Chekanov torus inside \(\mathbb{C}P^2\), but where the embedding of \(B^4 \hookrightarrow \mathbb{C}P^2\) that contains the torus is obtained by removing a line in the complement of the torus which is different from the “standard line at infinity”; see Figures 3 and 4. In order to distinguish \(L\) from a product torus inside the unit ball it suffices to compactify the ball to \(\mathbb{C}P^2\), and then to use the classical result by Y. Chekanov and F. Schlenk [7] that the monotone Chekanov torus is not Hamiltonian isotopic to a product torus inside \(\mathbb{C}P^2\).

**Remark 1.3.** Another way to distinguish \(L\) and the product torus up to Hamiltonian isotopy inside the ball is to consider the superpotential that counts families of pseudoholomorphic Maslov-two discs with boundary on \(L\); see e.g. the work [2] by D. Auroux. Here it is important to not only consider the count of pseudoholomorphic discs inside the ball. (In this case, that count is the same as for a product torus, by invariance of the potential under Hamiltonian isotopy.) More precisely, it is the terms in the superpotential that count the discs that pass through the line at infinity that distinguish \(L\) from the product torus. The important property here is that, for an almost complex structure on \(\mathbb{C}P^2 = B^4\) which makes the line at infinity holomorphic, the class of pseudoholomorphic discs of Maslov index two that pass through the line at infinity are a priori of minimal symplectic area, given the symplectic action properties of \(L\). Hence, the count of these discs is invariant under deformations of the almost complex structure that keeps the line at infinity holomorphic. However, for a ball which is larger than the unit ball, these discs are no longer of minimal area. For that reason one should not expect them to be invariant.

Additionally, in conjunction with Theorem 1.1, we can conclude that \(L\) is exotic also in the following sense which (at least a priori) is stronger:
Corollary 1.3. The Lagrangian torus $L$ in Theorem 1.2 not in the image of $S^1(1/√3) \times S^1(1/√3)$ under any symplectomorphism $(B^4, \omega_0) \to (B^4, \omega_0)$.

1.1. Application to knotted symplectic embeddings. A typical symplectic embedding problem concerns the question whether there exists an embedding $(Y^{2n}, d\lambda_Y) \hookrightarrow (X^{2n}, Cd\lambda_X)$ of a symplectic manifold into e.g. an open Liouville domain $(X, C\lambda_X)$ for some $C > 0$. Here we assume that $X$ is the interior of a compact Liouville domain with smooth boundary, while $Y$ is compact subset of a symplectic domain with a sufficiently well-behaved boundary. Typically one is interested in the case when an obvious, or even canonical, such embedding exists for all $C \gg 0$ sufficiently large. The natural question is then: how small can $C > 0$ be taken for a symplectic embedding to exist?

The first nontrivial result about symplectic embeddings was Gromov’s famous non-squeezing result [15], which showed that there are interesting symplectic obstructions beyond the obvious volume obstruction. Since then symplectic embedding problems have received a great amount of attention. And especially in dimension four, the situation is rather well understood for some particular cases of $Y$ and $X$. Notably, see the seminal work [22] by D. McDuff, which answers the question when an ellipsoid can be embedded into a ball.

Many of the natural examples of domains of the form $Y \subset (\mathbb{C}^n, \omega_0)$ that have been studied in the literature have the feature that $\partial Y$ is foliated by (possibly degenerate) Lagrangian standard product tori. Most attention has been given to domains for which the standard Liouville vector field $\zeta_0$ moreover is transverse to $\partial Y$. Such domains include closed balls $D^n(r)$, closed ellipsoids $E(a, b) := \{ \pi \| z_1 \|^2/a + \pi \| z_2 \|^2/b \leq 1 \}$, as well as polydiscs $D^2(a) \times D^2(b)$ (the latter has a smooth boundary with corner equal to a Lagrangian product torus). Domains of this type are typically depicted by their image under the standard momentum map

$$
\mu: \mathbb{C}^2 \to \mathbb{R}^2, \\
(z_1, z_2) \mapsto \pi(\| z_1 \|^2, \| z_2 \|^2).
$$

In this manner we obtain a direct connection between symplectic embedding problems and embedding problems for families of Lagrangian tori. This direction was taken in the work [18] by R. Hind and S. Lisi, [9] by K. Cieliebak and K. Mohnke, [16] by J. Gutt and M. Hutchings, and [19] by R. Hind and E. Opshtein.

In the case when there exists a symplectic embedding $(Y^{2n}, d\lambda_Y) \hookrightarrow (X^{2n}, d\lambda_X)$ one can further ask the question whether two different such embeddings have images that can be made to coincide after a symplectomorphism of the ambient space $(X, d\lambda_X)$. It was shown by J. Gutt and M. Usher [17] that this is not necessarily the case, even if such a symplectomorphism exists for the completion of $(X, d\lambda_X)$ to a Liouville manifold $(\overline{X}, d\lambda_X)$, in a number of cases. The same authors calls an embedding is
called \textit{symplectically knotted} (relative some other embedding) if there exists an ambient symplectomorphism inside the completion that takes the image of one embedding to the other, but when no such symplectomorphism exists of the original Liouville domain.

We now show that, in view of Corollary \[1.3\], the embedding of a domain can be shown to be symplectically knotted by considering Lagrangian tori contained inside its boundary.

\textbf{Theorem 1.4.} Let \[Y \subset \{ \|z_1\|^2 + 2\|z_2\|^2 \leq 1 \} \subset (D^4, \omega_0)\] be any closed symplectic domain that satisfies \[Y \cap \{ \|z_1\|^2 + 2\|z_2\|^2 = 1 \} = S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3}).\] There exists a symplectic embedding \[\phi: (Y, \omega_0) \hookrightarrow (B^4, \omega_0)\] for which the monotone Lagrangian torus \[S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3}) \in \partial Y\] is mapped to a torus \[L\] as in Theorem \[1.2\], and such that for any \[\epsilon > 0\], one can find a symplectomorphism \[\Phi: (B^4(1 + \epsilon), \omega_0) \rightarrow (B^4(1 + \epsilon), \omega_0)\] that satisfies \[\Phi(\phi(Y)) = Y.\]

In view of Corollary \[1.3\] we conclude that:

\textbf{Corollary 1.5.} The embedding \[\phi(Y) \subset (B^4, \omega_0)\] is symplectically knotted relative the canonical inclusion \[Y \subset (B^4, \omega_0)\].

\textbf{Proof.} Any symplectomorphism \(\Phi\) that takes \(\phi(Y)\) to \(Y\) must take the Lagrangian torus \(\phi(S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3}))\) to \(S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3})\) by mere considerations of symplectic action. \(\square\)

\textbf{Example 1.4.} The above method in particular yields a symplectically knotted embedding of the polydisc \(Y = D^2(1/\sqrt{3}) \times D^2(1/\sqrt{3}) \subset (B^4, \omega_0)\) into the unit ball, which was not covered in \[17\], and seems to be of a rather different nature than the examples therein. It is unclear to the author if this embedding remains symplectically knotted inside \(B^2(1) \times B^2(1)\); if this is the case, this would answer Question 1.9 in the aforementioned paper.

\textbf{Proof of Theorem 1.4.} The key point is that the Hamiltonian isotopy from \(S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3})\) to \(L\) inside \(C^2\) that was constructed in Section \[3\] can be taken to fix the hypersurface \(\{ \|z_1\|^2 + 2\|z_2\|^2 = 1 \} \subset D^4\) setwise; this is the hyperplane that contains the “probe” \(P_2\) as well as the Lagrangian torus. To see this, it is convenient to extend the embedding \(\psi_2\) of the probe constructed in the same section to a symplectic embedding

\[\Psi_2: B^2(\sqrt{(1-\delta^2)/2}) \times (-\delta, \delta) \times S^1 \rightarrow C^* \times C,\]

\[((r, \theta), (s, \varphi)) \mapsto (s + \sqrt{1 - 2r^2}, \varphi), (r, \theta + 2\varphi)),\]

defined using polar coordinate for some small \(\delta > 0\). Note that \(\omega_0\) pulls back to the product symplectic form \(\omega_0 + sds \wedge d\varphi\) on \(B^2(\sqrt{(1-\delta^2)/2}) \times C^* \times C\).
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\((-\delta, \delta) \times S^1\), while the restriction $\Psi_2|_{s=0} = \psi_2$ is the original embedding of the probe $P_2$ from Section 3.

One can then realise the Hamiltonian isotopy of the torus in the probe by a suitable lift of a Hamiltonian isotopy of $(B^2(\sqrt{(1-\delta^2)/2}), \omega_0)$ that is generated by a compactly supported Hamiltonian, to yield a Hamiltonian isotopy of the product

$$(B^2(\sqrt{(1-\delta^2)/2}) \times ((-\delta, \delta) \times S^1), \omega_0 + s ds \wedge d\varphi)$$

of symplectic manifolds. Finally it suffices to make a cut-off of the Hamiltonian by a suitable smooth bump function. □

2. THE PROOF OF THEOREM 1.1

After the application of the positive Liouville flow $\phi^{4\epsilon}_{\lambda_0}$ to both $L$ and $\varphi(B^4(\sqrt{2/3}))$ for some small $\epsilon > 0$ (recall that $\phi^{4\epsilon}_{\lambda_0}$ is a conformal symplectomorphism) we may in addition assume that the new Lagrangian torus has the symplectic actions $\mathbb{Z}\pi e^{4\epsilon r}/\sqrt{3}$ while being disjoint from the rescaled image $e^{2\epsilon}\varphi(B^4(\sqrt{2/3}))$ of a symplectic ball of slightly larger radius $e^{2\epsilon}\sqrt{2/3}$.

(Here we have made use of the assumption in Theorem 1.1 that the closure of the image of $\varphi$ is contained inside the open unit ball.) If we manage to construct the sought Hamiltonian isotopy in this case, the general case will then also follow immediately. Indeed, it suffices to rescale the produced Hamiltonian isotopy by the negative-time Liouville flow $\phi^{-4\epsilon}_{\lambda_0}$.

In view of the above, we will in the following restrict attention to the case when $r > 1/\sqrt{3}$ and $\varphi: (B^4(e^{2\epsilon}\sqrt{2/3}), \omega_0) \hookrightarrow (B^4 \setminus L, \omega_0)$ satisfies

$$\varphi(B^4(e^{\epsilon}\sqrt{2/3}), \omega_0) \subset B^4 \setminus L.$$  

2.1. A NECK-STRETCHING SEQUENCE. Symplectic reduction applied to the boundary $\partial B^4 = S^3 \to \mathbb{C}P^1$ produces a compactification $\overline{B^4} = \mathbb{C}P^2$ where the latter is equipped with the Fubini–Study symplectic form $\omega_{FS}$ in which a line has symplectic areas equal to $\int_{L} \omega_{FS} = \pi$. In particular, using $\ell_{\infty}$ to denote the line at infinity, we have $(B^4, \omega_0) = (\mathbb{C}P^2 \setminus \ell_{\infty}, \omega_{FS})$.

The main technical ingredient that we will need is neck-stretching around a hypersurface of contact type that can be identified with a small unit normal bundle around $L$. Neck-stretching first appeared in work [13] by Y. Eliashberg, A. Givental, and H. Hofer, and later made precise in the SFT compactness theorem [5] by F. Bourgeois, Y. Eliashberg, H. Hofer, C. Wysocki, and E. Zehnder and independently [8] by K. Cieliebak and K. Mohnke. Roughly speaking, neck-stretching is a conformal limit in which the symplectic manifold splits into several pieces, along with its pseudoholomorphic curves. For us it will be crucial to consider the neck-stretching limits of the foliation of pseudoholomorphic lines of $\mathbb{C}P^2$, which persists for arbitrary compatible almost complex structures by Gromov’s classical result [15].
We follow the same strategy and work in the same setting as in [12, Section 3]; we direct the reader to that article for most technical points concerning the method.

A Weinstein neighbourhood of $L$ becomes a concave cylindrical end
$$((-\infty, \log \delta) \times UT^*L, d(e^t(\alpha))) \hookrightarrow \left(B^4 \setminus \varphi(B^4(e^{\sqrt{2/3}})), \omega_0\right)$$
when considered in the symplectic manifold $B^4 \setminus L$. Here $\alpha = pdq|_{T(UT^*L)}$ is chosen to be the contact form on the unit cotangent bundle for a choice of flat metric on $\mathbb{T}^2$, and $\delta > 0$ is taken sufficiently small. The neck stretching is performed along the hypersurface
$$\{\log (\delta/2)\} \times UT^*L \hookrightarrow B^4 \setminus \varphi(B^4(e^{\sqrt{2/3}}))$$
of contact type that can be seen as an embedded spherical normal bundle of $L$. Stretching the neck amounts to choosing a sequence $J_\tau$, $\tau \geq 0$, of compatible almost complex structures on $\mathbb{C}P^2$. More precisely, all $J_\tau$ are fixed outside of the aforementioned concave end near $L$, and are all equal to the standard integrable complex structure $i$ near the divisor $\ell_\infty$. The limit compatible almost complex structure on $\mathbb{C}P^2 \setminus L$ will be denoted by $J_\infty$. In the cylindrical part of the concave end the sequence of complex structures becomes cylindrical on a larger and larger subset of the noncompact end as $\tau \to +\infty$. In the limit the almost complex structure $J_\infty$ is cylindrical on the entire end. We refer to [12, Sections 3 and 4] for more details, and the precise choices of almost complex structures.

For the analysis that we conduct it is crucial that the cylindrical almost complex structure is chosen with respect to the contact form on $UT^*L$ induced by the flat metric on $L$. The reason is that, for instance, the non-existence of contractible geodesics makes the breaking analysis of pseudoholomorphic curves significantly simpler. Recall that SFT compactness theorem implies that a sequence of finite energy $J_\tau$-holomorphic curves has a subsequence that converges to a pseudoholomorphic building which consists of several levels of punctured finite-energy pseudoholomorphic curves. These finite energy curves are asymptotic to Reeb chords on $UT^*L$, i.e. lifted geodesics for the flat metric in the case under consideration.

We will only be interested in the case of a sequence of $J_\tau$-holomorphic degree one curves in $\mathbb{C}P^2$, which are usually called lines. Recall that there exists a unique line through every two points, or through a point with a given complex tangency, by Gromov’s classical result [15]; any lines is moreover automatically an embedding. In the case of a limit of lines the corresponding building will a priori consist of:

- a non-empty top level consisting of punctured $J_\infty$-holomorphic spheres in $\mathbb{C}P^2 \setminus L$;
- a (possibly zero) number of middle levels consisting of punctured pseudoholomorphic spheres in $\mathbb{R} \times UT^*L$ for the cylindrical almost complex structure $J_{cyl}$; and
• a (possibly empty) **bottom level** consisting of punctured pseudo-holomorphic spheres in $T^* L$ for the almost complex structure $J_{\text{std}}$ defined in [12 Section 4],

such that the spheres moreover can be glued along the punctures to form a continuous map from a sphere into $\mathbb{C}P^2$ of degree one. Of course, it is also possible that the limit just consists of a component in the top level; in this case the sphere has no punctures (it is a compact $J_\infty$-holomorphic sphere of degree one in the usual sense). By positivity of intersection (see [20] by D. McDuff) one can deduce that any component arising in the limit is a (possibly trivial) branched cover of an embedded punctured sphere. Note that the almost complex structures $J_{\text{std}}$ and $J_{\text{cyl}}$ used here have the feature that the canonical $\mathbb{T}^2$-action by isometries on the flat torus $L$ lift to an action by biholomorphisms; see [12 Section 4].

Now comes the point when we will use the existence of the embedding of the symplectic ball as stipulated by the assumptions of the theorem; we choose the neck-stretching so that

\[ \text{(†): the almost complex structures } J_\tau \text{ all coincide with the push-forward of the standard almost complex structure } i \text{ under the symplectomorphism } \varphi \text{ in the subset } \varphi(B^4(e^\epsilon \sqrt{2/3})) \subset B^4 \setminus L \]

holds in addition to the above.

### 2.2. Extracting an SFT-limit of lines.

Choose a generic point

\[ \text{pt} \in \varphi(B^4((e^\epsilon - 1) \sqrt{2/3})) \]

and consider the $J_\tau$-holomorphic lines that pass through pt as well as some second fixed point on $L$. (By Gromov’s result [15] there is always a unique such line.) A sequence of such lines for which $\tau \to +\infty$ has a convergent subsequence by the SFT compactness theorem [5]. Due to the point constraint on $L$ the limit is a pseudoholomorphic building in the class of a “broken” line that passes through both $\text{pt} \in B^4 \setminus L$ as well as some point on the torus $L$.

Using the monotonicity property for the symplectic area of a pseudoholomorphic curve (see [24] by J.-C. Sikorav) applied to the ball

\[ \varphi(B^4_{\text{pt}}(\sqrt{2/3})) \subset \varphi(B^4(e^\epsilon \sqrt{2/3})) \]

and while using (†) we deduce that

\[ \int_{A_{\text{pt}}} \omega_{FS} \geq \pi 2/3 \]

for the unique top level component $A_{\text{pt}} \subset \mathbb{C}P^2 \setminus L$ of the limit building that passes through the point pt. (This uniqueness is a consequence of positivity of intersection; again see [20].) From this we are able to conclude that:
Lemma 2.1. For a generic point $pt \in \varphi(B^4((e^t - 1)\sqrt{2/3}))$ and a generic perturbation of the almost complex structure $J_{\infty}$ in a unit normal bundle of $L \cup \ell_{\infty}$, we can assume that the component $A_{pt}$ is

- disjoint from $\ell_{\infty}$,
- of symplectic area $\pi 2r^2$, where $r < 1/\sqrt{2}$, and
- of Maslov index four and embedded (thus in particular it is not a branched cover).

Proof. Every broken pseudoholomorphic line must consist of a plane that is disjoint from $\ell_{\infty}$ by the flatness of the metric on $L$ used in the construction of the neck-stretching sequence; see [12, Section 3]. Since these punctured spheres inside $\mathbb{C}P^2 \setminus (\ell_{\infty} \cup L)$ are of symplectic area equal to $kr^2\pi > k\pi / 3$ for some $k = 1, 2, 3, \ldots$, by our assumptions, and since $A_{pt}$ is of symplectic area at least $\pi 2/3$ by the above argument based upon monotonicity, we conclude that $A_{pt}$ has area $2r^2\pi$ (i.e. $k = 2$) and that $r < 1/\sqrt{2}$. Furthermore, there can be no other punctured spheres in the top level that are disjoint from $\ell_{\infty}$ except $A_{pt}$. (Here we use the property that a sphere of degree one is of total symplectic area equal to $\pi$.)

To compute the Maslov index of $A_{pt}$, we first observe that it can be at most four for a generic almost complex structure (it is sufficient to perturb near $L$) and positivity of the index; again see [12, Section 3]. Finally, since the point $pt$ was chosen to be generic, we can assume that $A_{pt}$ is of Maslov index at least four, and moreover not a branched cover of a plane of Maslov index two. To that end, recall that the moduli space of planes of Maslov index two evaluates to a three-dimensional chain, and thus so does the multiply covered planes of Maslov index two. (The moduli space of simply covered planes of Maslov index four, on the other hand, evaluates to a five-dimensional chain.)

Recall that the plane $A_{pt}$ must be embedded by positivity of intersection [20], since it is not a branched cover.

Lemma 2.2. The $J_{\infty}$-holomorphic plane $A_{pt} \subset \mathbb{C}P^2 \setminus (\ell_{\infty} \cup L)$ of Maslov index four produced by the above lemma has a simply covered asymptotic Reeb orbit.

Proof. Consider a sequence of $J_{\tau}$-holomorphic lines which satisfy a generic tangency condition at a generic point $pt' \in A_{pt}$ as $\tau \to +\infty$. Using the SFT compactness theorem, we can extract a limit holomorphic building from a convergent subsequence.

We first claim that the limit component is smooth at the point where the tangency is taken. Indeed, positivity of intersection implies that in some neighbourhood of the point $pt'$, the underlying simply covered curve must be smooth; see [20]. There is still the possibility that the building contains a branched cover of the component $A_{pt}$ with a branch point precisely at $pt'$ (such a curve satisfies any prescribed tangency condition). This scenario can be excluded by a symplectic area argument as in the proof of the previous
lemma, using the fact that the symplectic area of a line is equal to \( \pi \). (The hypothetical building would otherwise contain a component of symplectic area at least \( 2 \cdot \int_{A_{pt}} \omega_{FS} = 4r^2 \pi > \pi \).)

To conclude, we have shown that we can find a limit that satisfies any generic tangency condition at any generic point in \( A_{pt} \), satisfying the additional property that the underlying point of the curve is smooth. A dimension analysis as in [12] Section 3 then implies that we can find an unbroken \( J_\infty \)-holomorphic line \( \ell \subset CP^2 \setminus L \) (i.e. a pseudoholomorphic curve without punctures) that satisfies the tangency. (Any component in the top level of a broken line comes in a family of dimension strictly less than four.)

Using the existence of the unbroken line that passes through \( A_{pt} \), together with fact that the connecting morphism \( H_2(B^4, L) \xrightarrow{\delta} H_1(L) \) is an isomorphism, positivity of intersection [20] allows us to conclude that \( A_{pt} \bullet \ell \geq 2 \) if the asymptotic is multiply covered. Since two curves of degree one have algebraic intersection number \( [\ell_\infty] \bullet [\ell_\infty] = 1 \), we finally arrive at the sought contradiction by yet another positivity of intersection argument \( \square \)

![Figure 1](https://via.placeholder.com/150)

**Figure 1.** The numbers indicate the dimension of the moduli space of the respective component (without any asymptotic constraint in the Bott manifolds of periodic Reeb orbits). The asymptotic orbits are lifts of the geodesics on \( L \) in the homology classes \( \pm \eta \in H_1(L) \). Without loss of generality we may assume that the bottom component is a cylinder that intersects \( L \) cleanly in the corresponding geodesic.

### 2.3. A condition for Hamiltonian unknottedness

The monotonicity combined with Lemma [2.3] now implies that the broken line produced in the previous subsection consists of precisely two components in its top level: the embedded plane \( A_{pt} \) together with an embedded plane \( A_\infty \) that passes through \( \ell_\infty \); both are simply covered and have simply covered asymptotics. Further, by the classification of pseudoholomorphic cylinders in [12] Section 4 implies that the component in the bottom level is a standard cylinder, which roughly speaking is the complexification of the geodesic in class \( \pm \eta \in H_1(L) \) to which the planes are asymptotic. Even if the original broken line
does not pass through \( L \), we can replace the cylinder in the bottom level with a cylinder that intersects \( L \) cleanly precisely in the corresponding geodesic; such a configuration is shown in Figure 1.

Since the involved asymptotic orbits are simply covered by Lemma 2.2, the smoothing technique from [12, Section 5] can then be used to produce a smoothing of the above building to an embedded symplectic sphere that intersects \( L \) cleanly along the simply covered closed geodesic in class \( \pm \eta \in H_1(L) \) to which the planes \( A_{pt} \) and \( A_\infty \) are asymptotic. In other words, the assumptions of the below proposition is met, from which the existence of the sought Hamiltonian isotopy then follows.

**Proposition 2.3.** Assume that we can find a tame almost complex structure \( J \) on \( \mathbb{CP}^2 \) which is standard near \( \ell_\infty \) and for which there is a \( J \)-holomorphic line \( \ell \) whose intersection with \( L \) is a simple closed curve of Maslov index four (computed using the trivialisation of \( TB^4 \)). Then \( L \) is Hamiltonian isotopic to a product torus inside \( B^4 \) by a Hamiltonian isotopy supported inside the same ball.

2.4. **The proof of Proposition 2.3.** By the “refined” version of the nearby Lagrangian conjecture for the cotangent bundle \( (T^*T^2, d(p_1d\theta_1 + p_2d\theta_2)) \) of a torus established in [11, Theorem B] it suffices to find a Hamiltonian isotopy of \( L \) supported inside \( B^4 \) that places the torus inside the subset

\[
(\mathbb{CP}^2 \setminus (\ell_\infty \cup \{ z_1z_2 = 0 \}), \omega_{FS}) \cong (T^2 \times U, d(p_1d\theta_1 + p_2d\theta_2)),
\]

and so that the torus moreover becomes homologically essential inside the same neighbourhood. Here \( z_i \) denote the standard affine coordinates on \( \mathbb{CP}^2 \setminus \ell_\infty \cong \mathbb{C}^2 \), and

\[
U = \{ p_1 + p_2 < \pi, \ p_1, p_2 > 0 \} \subset \mathbb{R}^2
\]

is an open convex subset.

To do this we will rely on the techniques from [11, Section 4.2], by which it suffices to find two \( J \)-holomorphic lines \( \ell_i \subset \mathbb{CP}^2 \), \( i = 1, 2 \), which intersects \( \ell_\infty \) in two distinct points, and for which \( L \subset \mathbb{CP}^2 \setminus (\ell_\infty \cup \ell_1 \cup \ell_2) \) is homologically essential. Namely, after a deformation near the nodes of \( \ell_\infty \cup \ell_1 \cup \ell_2 \) that can be performed by hand, there then exists a Hamiltonian isotopy that fixes \( \ell_\infty \) setwise and takes the three lines \( \ell_\infty \cup \{ z_1z_2 = 0 \} \) to the three lines in standard position.

In order to construct the \( J \)-holomorphic lines \( \ell_\infty \) we need to again consider a neck-stretching sequence \( J_\tau \) induced by a flat metric on \( L \). It will furthermore be crucial that:

- \( J_\tau = i \) near \( \ell_\infty \), and
- the line \( \ell \) whose existence was assumed remains \( J_\tau \)-holomorphic for all \( \tau \geq 0 \).

In other words, we want \( \ell \) to converge to a building as shown in Figure 1 when taking the limit \( \tau \to +\infty \). We use \( \pm \eta \in H_1(L) \) to denote the homology
class of the (unoriented) closed geodesic on \( L \) to which the planes involved in the limit are asymptotic.

To ensure that \( J_\tau \) can be made to satisfy the latter bullet point above, we argue as follows.

**Lemma 2.4.** After a Hamiltonian isotopy, \( \ell \) can be made to coincide with a "complexified geodesic" (i.e. a \( J_{\text{std}} \)-holomorphic cylinder explicitly described in [12, Section 4]) for the flat metric on \( L \) inside some Weinstein neighbourhood \( D_{\leq \delta} T^* L \hookrightarrow B^4 \) of \( L \).

**Proof.** Recall the standard fact that any smooth isotopy of \( L \) can be generated by an ambient Hamiltonian isotopy of its Weinstein neighbourhood \( D_{\leq \delta} T^* L \). In this manner we can thus deform \( \ell \) in order to make it intersect \( L \) in a closed geodesic that represents \([\ell \cap L] \in H_1(L)\). The normal form for a symplectic neighbourhood can then readily be used to, first, make \( \ell \) tangent to the complexified geodesic along \( L \) and, second, to make it coincide with the complexified geodesic in a neighbourhood. \( \Box \)

The existence of the broken pseudoholomorphic line arising from the limit of \( \ell \) has the following strong, and for us crucial, implications.

**Lemma 2.5.** For any neck-stretching sequence \( J_\tau \) as above for which \( \ell \) remains pseudoholomorphic, a pseudoholomorphic building that arises as a limit of lines can contain only simply covered components in the bottom and middle levels, together with possibly branched covered cylinders asymptotic to the geodesic in the homology class \( \pm \eta \in H_1(L) \). In addition, if one component is a cylinder, then all components are simply covered cylinders with asymptotics to geodesics in the same homology class (not necessarily the class \( \eta \)).

For a generic \( J_\tau \), it follows that the building consists of at most one component in these levels, which moreover is simply covered.

**Proof.** Recall that the bottom and middle levels are foliated by standard cylinders asymptotic to the geodesics in the homology class \( \pm \eta \in H_1(L) \); see [12, Section 4]. As a consequence, positivity of intersection [20] together with \([\ell_\infty] \cdot [\ell_\infty] = 1\) implies that there can be no nontrivial branched cover of a component different from the aforementioned cylinders. (The nature of the SFT-convergence [5] implies that the \( J_\tau \)-holomorphic lines that converge to the broken configuration have an intersection number with \( \ell \) which is strictly greater than one.)

For the same reasons, under the assumption that the limit building contains a cylinder in its bottom or middle level, it follows that all components are cylinders in the same class.

The consequence under the stronger assumption that \( J_\tau \) is generic can then be shown as follows. An index argument readily implies that there can be at most one cylinder asymptotic to a geodesics in the homology class \( \pm \eta \in H_1(L) \) arising in the limit, and that this cylinder moreover is simply covered. Indeed, otherwise one can readily extract a broken plane of Maslov
index $\mu$ satisfying either $\mu < 2$ or $\mu > 4$ that arises as a sub-building of the limit.

Following the construction of K. Cieliebak and K. Mohnke [9], we now consider the limit of lines that satisfy a generic tangency condition at a point $pt \in L$; also c.f. the author’s work [10] which also is based upon Cieliebak and Mohnke’s technique. The limit building then necessarily consists of precisely three components in its top level: the planes $A_1, A_2, A_\infty \subset \mathbb{C}P^2 \setminus L$ of Maslov index two, where $A_\infty$ intersects $\ell_\infty$ precisely once transversely and where $A_1, A_2 \subset B^4 \setminus L$. The remaining component lives in the bottom level and is a three-punctured sphere $C_0 \subset T^*L$ which passes trough $pt \in L$, where it satisfies the prescribed generic tangency condition; see Figure 2 for a schematic picture. There are two possibilities for the three-punctured sphere $C_0$: either it is a twofold branched cover of a cylinder, or it is embedded; see [10] for more details. In this case $C_0$ is necessarily embedded. Here we have made heavy use of Lemma 2.5 (also in order to conclude that the planes $A_i$ themselves are not broken buildings).

![Figure 2](image_url)

**Figure 2.** Applying the technique of Cieliebak–Mohnke produces a building consisting of three planes in the top level of Maslov index three, where the component $C_0$ in the bottom level satisfies a generic tangent condition at a point $pt \in L$. The goal is then to find a nonbroken pseudoholomorphic line that passes through each of the two planes $A_1, A_2 \subset B^4 \setminus \ell_\infty$.

The sought $J$-holomorphic lines are finally constructed by following an idea due to K. Mohnke [23], in which Lemma 2.5 is the crucial ingredient that we need in order to rule out the appearance of branched covers.

**Proposition 2.6.** One can find an unbroken $J_\infty$-holomorphic line $\ell_i \subset \mathbb{C}P^2 \setminus L, i = 1, 2$, that passes through the plane $A_i$ (but which is disjoint from the other plane $A_j$ with $j \neq i$). It follows that $A_1$ and $A_2$ are asymptotic to geodesics in different (primitive) homology classes in $H_1(L)$.

**Remark 2.1.** It can also be shown that the lines exist under the mere assumption that $A_i$ are asymptotic to different (primitive) homology classes, but we do not need this fact here.
Proof. Take a complex tangency transverse to a point \( p_1 \in A_1 \) and consider the sequence of \( J_r \)-holomorphic lines that satisfy this tangency. Since Lemma 2.5 implies that the limit line is smooth at \( p_1 \), it must still be transverse to \( A_1 \). For a generic \( p_1 \) and tangency, the limit line is not broken. Positivity of intersection and \([\ell_\infty] \cdot [\ell_\infty] = 1\) implies that the limit line is disjoint from \( A_2 \). An elementary topological consideration then implies that \( A_1 \) and \( A_2 \) indeed are asymptotic to geodesics in different homology classes. By symmetry we also obtain the sought line \( \ell_2 \).

Since the lines \( \ell_i \) produced by the above proposition can be perturbed inside \( \mathbb{C}^2 \setminus L \) through \( J \)-holomorphic lines, so that their unique intersection point is contained inside in the complement of \( \ell_\infty \), we have thus finally managed to produce the lines in the sought position. (The linking properties established in the above proposition implies that \( L \) is homologically essential in the complement of the lines.)

3. Proof of Theorem 1.2

![Figure 3](image-url)

Figure 3. The monotone Clifford torus \( L_{\text{Cl}} = S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3}) \) can be isotoped to \( L \) inside the probe \( P_2 \subset D^4 \). We can moreover place \( L \) inside \( B^4(\sqrt{1-r}) \) for any \( r \in (0, 1/6) \).

Probes are a useful tool for constructing Hamiltonian isotopies that was invented by McDuff in [21]. For any integer \( m = 1, 2, 3, \ldots \) we consider the probe

\[
P_m := \mu^{-1} \{ u_1 = \pi - mu_2, \; u_1 > 0 \} \subset \mathbb{C}^2
\]

for the standard moment map

\[
\mu : \mathbb{C}^2 \to (\mathbb{R}_{\geq 0})^2,
\]

\[
\mu(z_1, z_2) := (\pi ||z_1||^2, \pi ||z_2||^2),
\]

where \( || \cdot || \) denotes the Euclidean norm.
on \((\mathbb{C}^2, \omega_0)\). We will consider the foliation
\[
\psi_m : B^2(1/\sqrt{m}) \times S^1 \to P_m \subset \mathbb{C}^* \times \mathbb{C},
\]
\[
((r, \theta), \varphi) \mapsto ((\sqrt{1 - mr^2}, \varphi), (r, \theta + m\varphi)),
\]
by symplectic discs \(B^2(1/\sqrt{m}) \times \{\varphi\}\) in local angular coordinates. Note that this map indeed extends smoothly over \(\{0\} \times S^1\). Since the symplectic form \(\omega_0\) is pulled back to the standard symplectic form \(\omega_0\) on \(B^2(1/\sqrt{m})\) under the map \(\psi_m\), the characteristic distribution on \(P_m\) can be seen to be given by
\[
\ker(\omega_0|_{TP_m}) = \mathbb{R}\partial_\varphi \subset TP_m.
\]
In particular, integrating it, we obtain a trivial symplectic monodromy map on the symplectic disc leaves. This means that

**Lemma 3.1.** For any simple closed curve \(\gamma \subset B^2(1/\sqrt{m})\), the image \(\psi_m(\gamma \times S^1) \subset (\mathbb{C}^2, \omega_0)\) is an embedded Lagrangian torus.

For example, the monotone Clifford torus of symplectic action \(\pi/3\) is given by
\[
L_0 := \psi_2(S^1(1/\sqrt{3}) \times S^1) \subset S^3(\sqrt{2/3}).
\]
Considering a suitable smooth family of simple closed curves that all bound the area \(\pi/3\) inside \(B^2(1/\sqrt{3})\) we obtain a Hamiltonian isotopy
\[
L_t := \psi_2(\gamma_t \times S^1) \subset P_2 \subset (\mathbb{C}^2, \omega_0)
\]
of Lagrangian tori. We will take \(L_0\) to be the Clifford torus while \(L_1\) the torus obtained from the curve \(\gamma_1 \subset B^2(1/\sqrt{2})\) shown in Figures 3 and 4.
Lemma 3.2 (Gadbled [14]). The torus $L_1 \subset B^4$ is Hamiltonian isotopic to the Chekanov torus when considered inside the completion 
\[(\mathbb{C}P^2, \omega_{FS}) \supset (\mathbb{C}P^2 \setminus \ell_\infty, \omega_{FS}) \cong (B^4, \omega_0)\]
of the ball.

Proof. The representative of the Chekanov torus described in [14] differs from $L_1$ simple by the linear change of coordinates 
\[\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \mapsto \begin{bmatrix} Z_3 \\ Z_2 \\ Z_1 \end{bmatrix}.\]
In particular, the torus is clearly Hamiltonian isotopic to the Chekanov torus. □

The claim that $L_1$ is not Hamiltonian isotopic to the standard product torus inside $B^4$ then finally follows from the fact that the monotone Clifford and Chekanov tori inside the compactification $(\mathbb{C}P^2, \omega_{FS})$ are not Hamiltonian isotopic as shown by Chekanov and Schlenk [7].

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