Linear functional equations and their solutions in Lorentz spaces

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Abstract
Assume that $\Omega \subset \mathbb{R}^k$ is an open set, $V$ is a separable Banach space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $f_1, \ldots, f_N: \Omega \to \Omega$, $g_1, \ldots, g_N: \Omega \to \mathbb{K}$, $h_0: \Omega \to V$ are given functions. We are interested in the existence and uniqueness of solutions $\varphi: \Omega \to V$ of the linear functional equation
$$\varphi = \sum_{k=1}^{N} g_k \cdot (\varphi \circ f_k) + h_0$$
in Lorentz spaces. The investigation of the existence of the solutions in the case $L^1([0, 1], \mathbb{R})$, that is in the Lorentz space $L^{1,1}([0, 1], \mathbb{R})$, was motivated by Nikodem’s paper On $\epsilon$-invariant measures and a functional equation published in Czechoslovak Math. J., 41(116)(4):565–569, 1991. We have studied it in Some classes of linear operators involved in functional equations published in Ann. Funct. Anal., 10(3):381–394, 2019. Inspired by Matkowski’s Dissertationes Math. paper Integrable solutions of functional equations from 1975, we studied the Orlicz case in our paper Linear functional equations and their solutions in generalized Orlicz spaces published in Aequationes Mathematicae in 2021 in volume 95, number 6. Here, we build on the ideas from that paper, using the connection between Lorentz and Orlicz spaces as detailed in the paper Fine behavior of functions whose gradients are in an Orlicz space authored by Malý, Swanson, and Ziemer and published in Studia Math. 190(1), 33–71 in 2009. However, we do not only look at real-valued functions, but allow separable Banach spaces as target spaces. As in the paper about Orlicz spaces, an important tool is a result by Hajłasz, Change of variables formula under minimal assumptions as published in Colloq. Math. 64(1), 93–101 (1993), concerning an optimal change of variable formula. It is used for proving that we can apply Banach’s fixed point theorem.

Keywords Linear operators · Approximate differentiability · Luzin’s condition N · Functional equations · Lorentz spaces

Mathematics Subject Classification Primary 47A50; Secondary 26A24 · 39B12 · 47B38

In memoriam: Jan Malý (1955–2021).

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Introduction

Throughout this paper we fix $k, N \in \mathbb{N}$, an open set $\Omega \subset \mathbb{R}^k$, a separable Banach space $(V, \|\cdot\|_V)$ over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and functions $f_1, \ldots, f_N : \Omega \to \Omega, g_1, \ldots, g_N : \Omega \to \mathbb{K}$ and $h_0 : \Omega \to V$. We are interested in the existence and uniqueness of a special solution $\varphi : \Omega \to V$ of the following linear equation

$$\varphi(x) = \sum_{n=1}^N g_n(x)\varphi(f_n(x)) + h_0(x).$$  \hfill (1)

Different solutions of equation (1) have been studied by many authors (see e.g. [12, Chapter XIII], [13, Chapter 6], [3, Chapter 5], [2, Sect. 4] and the references therein). In order to specify what special solution we are talking about, we need to introduce some notations. But before, let us note that this paper is a continuation of the authors’ paper [19] and a reader who is already familiar with the content of that paper may want to jump to Sect. 3.

Denote by $\mathcal{F}$ the linear space of all functions $\psi : \Omega \to V$ and fix a subspace $\mathcal{F}_0$ of $\mathcal{F}$. Then define the operator $P : \mathcal{F}_0 \to \mathcal{F}$ by

$$P\psi = \sum_{n=1}^N g_n \cdot (\psi \circ f_n),$$  \hfill (2)

and observe that it is linear and equation (1) can be written in the form

$$\varphi = P\varphi + h_0.$$  \hfill (3)

If equation (1) has a solution $\varphi \in \mathcal{F}_0$ such that $P\varphi \in \mathcal{F}_0$, then $h_0 \in \mathcal{F}_0$. Conversely, if $h_0 \in \mathcal{F}_0$, then for every solution $\varphi \in \mathcal{F}_0$ of equation (1) we have $P\varphi \in \mathcal{F}_0$. Therefore, if we want to look for solutions of equation (1) in $\mathcal{F}_0$, then it is quite natural to assume that $h_0 \in \mathcal{F}_0$ and

$$P(\mathcal{F}_0) \subset \mathcal{F}_0.$$  \hfill (4)

Remark 1.1 See [19, Remark 1.1], cf. [18, Remark 1.2]. Assume that $\mathcal{F}_0$ is equipped with a norm, $h_0 \in \mathcal{F}_0$, and the operator $P$ given by (2) satisfies (4) and is continuous. If the series

$$\sum_{n=0}^{\infty} P^n h_0$$  \hfill (5)

converges, in the norm, to a function $\varphi \in \mathcal{F}_0$, then (3) holds.

From now on, the series (5) will be called the elementary solution of equation (1) in $\mathcal{F}_0$, provided that it is a well-defined solution of equation (1) belonging to $\mathcal{F}_0$. Let us note that it can happen that equation (1) has a solution in $\mathcal{F}_0$, however its elementary solution in $\mathcal{F}_0$ can fail to exist (see [18, Example 1.4]).

The investigation of the existence of the elementary solution of equation (1) in the case $\mathcal{F}_0 = L^1([0, 1], \mathbb{R})$ was motivated by [21] and studied in [18]. Next, inspired by [17], the existence of the elementary solution of equation (1) in the case where $\mathcal{F}_0$ is a generalized Orlicz space was examined in [19].

The basic result on the existence and uniqueness of the elementary solution of equation (1) in $\mathcal{F}_0$ reads as follows.
Theorem 1.2 (see [19, Theorem 1.2], cf. [18, Theorem 3.2]) Assume that \( \| \cdot \| \) is a complete norm in \( \mathcal{F}_0 \) and let \( h_0 \in \mathcal{F}_0 \). If the operator \( P \) given by (2) satisfies (4) and is a contraction with contraction factor \( \alpha \), then the elementary solution of equation (1) in \( \mathcal{F}_0 \) exists, it is the unique solution of equation (1) in \( \mathcal{F}_0 \) and \( \| \sum_{k=m}^{\infty} P^k h_0 \| \leq \frac{\alpha^m}{1-\alpha} \| h_0 \| \).

2 Preliminaries

Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be measure spaces. We say that \(G: X \to Y\) satisfies Luzin’s condition \( N \) if for every set \( N \subset Y \) of measure zero the set \( G(N) \) is also of measure zero. When we will integrate a function \( \Phi \) has a finite range and \( \Phi \) is measurable, then as usual we denote by \( F \) the determinant of the Jacobi matrix of \( \Phi \).

Lemma 2.1 (see [19, Lemma 2.1]) Assume that \((X, \mathcal{M}, \mu)\) is a complete \( \sigma \)-finite measure space. Let \( F: X \to X \), \( H: X \to V \) and \( G: X \to \mathbb{R} \) be measurable functions. If for all sets \( N \subset X \) of measure zero the set \( F^{-1}(N) \) is also of measure zero, then the functions \( H \circ F \) and \( G \cdot (H \circ F) \) are measurable.

The next result we want to apply is a change of variable formula from [6]. To formulate this theorem, we need to introduce some definitions and notions.

Let \( F: \Omega \to \mathbb{R}^k \) be measurable. We say that a linear mapping \( L: \mathbb{R}^k \to \mathbb{R}^k \) is an approximate differential of \( F \) at \( x_0 \in \Omega \) if for every \( \varepsilon > 0 \) the set

\[
\left\{ x \in \Omega \setminus \{x_0\} : \frac{\| F(x) - F(x_0) - L(x - x_0) \|}{\| x - x_0 \|} < \varepsilon \right\}
\]

has \( x_0 \) as a density point (see [24, Sect. 2], cf. [22, Chapter IX.12]). We say that \( F \) is approximately differentiable at \( x_0 \) if the approximate differential of \( F \) at \( x_0 \) exists. To simplify notation, we will denote the approximate differential of a function \( F: \Omega \to \mathbb{R}^k \) at \( x_0 \) by \( F'(x_0) \). Moreover, if a function \( F: \Omega \to \mathbb{R}^k \) is almost everywhere approximately differentiable, then as usual we denote by \( F' \) the function \( \Omega \ni x \mapsto F'(x) \), adopting the convention that \( F'(x) = 0 \) for every point \( x \in \Omega \) at which \( F \) is not approximately differentiable. If \( E \subset \Omega \), then the function \( N_F(\cdot, E): \mathbb{R}^k \to \mathbb{N} \cup \{\infty\} \) defined by

\[
N_F(y, E) = \text{card}(F^{-1}(y) \cap E)
\]

is called the Banach indicatrix of \( F \).

As we are working with functions equal almost everywhere, we need the following observation.

Lemma 2.2 (see [18, Lemma 2.1], cf. [19, Lemma 2.1]) Let \( F_1, F_2: \Omega \to \mathbb{R}^k \) be functions such that \( F_1 = F_2 \) almost everywhere. If \( F_1 \) is approximately differentiable almost everywhere, then \( F_2 \) is as well. Moreover, whenever \( F_1 \) or \( F_2 \) is approximately differentiable at a point where the functions agree, the other function is as well, and the approximate derivatives agree at this point.

Now we are in a position to formulate the change of variable formula in which \( J_F \) denotes the determinant of the Jacobi matrix of \( F \).
Theorem 2.3 (see [6, Theorem 2]) Assume that $F : \Omega \to \mathbb{R}^k$ is a measurable function satisfying Luzin’s condition N and being almost everywhere approximately differentiable. If $H : \mathbb{R}^k \to \mathbb{R}$ is a measurable function, then for every measurable set $E \subset \Omega$ the following statements are true:

(i) The functions $(H \circ F)|J_F|$ and $HN_F(\cdot, E)$ are measurable;

(ii) If $H \geq 0$, then
\[
\int_E (H \circ F)(x)|J_F(x)| \, dx = \int_{\mathbb{R}^k} H(y)NF(y, E) \, dy;
\]

(iii) If one of the functions $(H \circ F)|J_F|$ and $HN_F(\cdot, E)$ is integrable (for $(H \circ F)|J_F|$ integrability is considered with respect to $E$), then so is the other and (6) holds.

Now we are ready to formulate the main assumption about the functions that were fixed at the beginning of this paper. The assumption reads as follows.

(H) The functions $f_1, \ldots, f_N$ are measurable and almost everywhere approximately differentiable and satisfy Luzin’s condition N. For all $n \in \{1, \ldots, N\}$ and sets $M \subset \mathbb{R}^k$ of measure zero the set $f_n^{-1}(M)$ is of measure zero. There exists $K \in \mathbb{N}$ such that for every $n \in \{1, \ldots, N\}$ the set $\{x \in \Omega : \text{card } f_n^{-1}(x) > K\}$ is of measure zero. The functions $g_1, \ldots, g_N$ and $h_0$ are measurable. Moreover,

\[
L = \max\{l \in \{1, \ldots, N\} : l_{n_1, \ldots, n_l} > 0 \text{ for some } n_1 < n_2 < \cdots < n_l\},
\]

where the number $l_{n_1, \ldots, n_l}$ is the $k$-dimensional Lebesgue measure of the intersection $\bigcap_{l=1}^l f_{n_l}(\Omega)$ for all $l \in \{1, \ldots, N\}$ and distinct $n_1, \ldots, n_l \in \{1, \ldots, N\}$.

3 Lorentz spaces of complex valued functions

In this paper we are interested in the existence of the elementary solution of equation (1) in $\mathcal{F}_0$ being a Lorentz space. In fact, we are interested in assumptions guaranteeing that the elementary solution of equation (1) in a given Lorentz space exists, and moreover, that equation (1) has no other solutions in this Lorentz space.

Lorentz spaces were introduced originally in [14]. They play an important role when studying generalizations of Sobolev maps and are used to obtain sharp conditions for Sobolev maps to be differentiable almost everywhere or to send sets of measure zero to sets of measure zero (see [10] for more details). Lorentz spaces are also important in interpolation theory (see e.g. [5, Theorem 3.5.15] or [4, Theorem 4.4.13]).

We begin our investigations with Lorentz spaces that consist of complex or real valued functions, i.e. with the case where $V = \mathbb{K}$. For the convenience of the reader, following [16] we recall some basic definitions and facts for our need. More details on Lorentz spaces can be found e.g. in [4, 5, 23].

Definition 3.1 (see [16, Definition 2.2]) A non-decreasing left-continuous and convex function $\Psi : [0, \infty) \to [0, \infty]$ is said to be a Young function, if

\[
\lim_{t \to 0^+} \Psi(t) = \Psi(0) = 0 \quad \text{and} \quad \lim_{t \to \infty} \Psi(t) = \infty.
\]

From now on the symbol $\Psi$ is reserved for Young functions only.
Definition 3.2 (see [16, Definition 2.4]) A Young function \( \psi : [0, \infty) \to [0, \infty) \) is said to satisfy condition \( \Delta_2 \) globally, if there exists a real number \( d \in (1, \infty) \) such that
\[
\psi(2t) \leq d \psi(t) \quad \text{for every } t \in (0, \infty).
\]

To the end of this paper we fix a strictly increasing Young function \( \psi \) satisfying the condition \( \Delta_2 \) globally and such that
\[
\lim_{t \to 0^+} \frac{\psi(t)}{t} = \lim_{t \to \infty} \frac{t}{\psi(t)} = 0.
\] (7)

According to [16, page 35] the fixed function \( \psi \) has left-sided and right-sided derivatives, which coincide except on a possibly countable set. When we write \( \psi'_r \), we refer henceforth to the right-sided derivative. Note that \( \psi \) satisfies (7) if and only if \( 0 < \psi'_r(t) < \infty \) for every \( t \in (0, \infty) \), \( \psi'_r(0) = 0 \) and \( \lim_{t \to \infty} \psi'_r(t) = \infty \).

We now define a function \( \tau : [0, \infty) \to [0, \infty) \) by putting
\[
\tau(t) = \begin{cases} 
0, & \text{if } t = 0, \\
\frac{1}{\psi(\frac{1}{t})}, & \text{if } t \in (0, \infty).
\end{cases}
\]

Note that the just defined function \( \tau \) is a strictly increasing Young function satisfying condition \( \Delta_2 \) globally and
\[
\lim_{t \to 0^+} \frac{\tau(t)}{t} = \lim_{t \to \infty} \frac{t}{\tau(t)} = 0
\]
(see [16, page 68]).

Denote by \( \mu \) the \( k \)-dimensional Lebesgue measure on \( \mathbb{R}^k \) and by \( \mathcal{M}_K \) the space of all Lebesgue measurable functions from \( \Omega \) to \( K \).

Definition 3.3 (see [16, Definition 9.1]) If \( f \) is in \( \mathcal{M}_K \), then the function \( \mu_f : [0, \infty] \to [0, \infty] \) defined by
\[
\mu_f(s) = \mu(\{x \in \Omega : |f(x)| > s\})
\]
is said to be the distribution function of \( f \).

The linear space \( L^{\psi,1}(\Omega, \mathbb{K}) \) consisting of all functions \( f \in L^1_{\text{loc}}(\Omega, \mathbb{K}) \) (i.e. \( f \in L^1(K, \mathbb{K}) \) for every compact set \( K \subset \Omega \)) satisfying the following condition
\[
\int_0^\infty \tau^{-1} (\mu_f(s)) \, ds < \infty
\]
is called the Lorentz space (see Definition 9.2 in [16]). The Lorentz space equipped with the norm
\[
\|f\|_{L^{\psi,1}(\Omega, \mathbb{K})} = \int_0^\infty \tau^{-1} (\mu_f(s)) \, ds
\] (8)
is a Banach space (see [16, page 68]; we will give a sketch of a proof of this fact at the beginning of Sect. 4).

The next observation relates Lorentz and Orlicz spaces. But before, let us recall the definition of Orlicz spaces.
Definition 3.4 (see [16, Definition 2.7]) Let $\Omega \subset \mathbb{R}^k$ and let $\Psi : [0, \infty) \to [0, \infty]$ be a Young function. The Orlicz spaces $L^\Psi(\Omega, \mathbb{K})$ is the set of all measurable functions $u : \Omega \to \mathbb{K}$ such that

$$\int_\Omega \Psi \left( \frac{|u(x)|}{t} \right) \, dx < \infty$$

for some $t > 0$.

Equipping $L^\Psi(\Omega, \mathbb{K})$ with the Luxemburg norm

$$\|u\|_{L^\Psi(\Omega, \mathbb{K})} = \inf \left\{ t > 0 : \int_\Omega \Psi \left( \frac{|u(x)|}{t} \right) \, dx \leq 1 \right\}$$

it becomes a Banach space (see e.g. [11, Theorem 2.4]).

Lemma 3.5 (see [16, Lemmas 9.3 and 9.4]) Assume that $h \in M_{\mathbb{K}}$.

(i) If $h \in L^{\Psi,1}(\Omega, \mathbb{K})$ with $\|h\|_{L^{\Psi,1}(\Omega, \mathbb{K})} = 1$, then there exists a Young function $\Psi$ such that

$$\int_\Omega \Psi(|h(x)|) \, dx \leq 1 = \int_0^\infty (\tau_\Psi(t))^{-1} \left( \frac{1}{\Psi(t)} \right) \, dt. \quad (9)$$

(ii) If $\Psi$ is a Young function satisfying (9), then $h \in L^{\Psi,1}(\Omega, \mathbb{K})$ and $\|h\|_{L^{\Psi,1}(\Omega, \mathbb{K})} \leq 2 \|h\|_{L^\Psi(\Omega, \mathbb{K})}$.

The next result is a counterpart of [19, Theorem 4.6] for Lorentz spaces.

Theorem 3.6 Assume that (H) holds and let $h_0 \in L^{\Psi,1}(\Omega, \mathbb{K})$. If there exists a real constant $\alpha \in \left[ 0, \frac{1}{2} \right]$ such that

$$|g_n(x)| \leq \alpha \min \left\{ \frac{|J_{\tau_n}(x)|}{KL}, \frac{1}{N} \right\} \text{ for all } n \in \{1, \ldots, N\} \text{ and }$$

almost all $x \in \Omega$,

then the elementary solution of equation (1) in $L^{\Psi,1}(\Omega, \mathbb{K})$ exists, it is the unique solution of equation (1) in $L^{\Psi,1}(\Omega, \mathbb{K})$ and

$$\left\| \sum_{k=m}^\infty P^k h_0 \right\|_{L^{\Psi,1}(\Omega, \mathbb{K})} \leq \frac{(2\alpha)^m}{1 - 2\alpha} \|h_0\|_{L^{\Psi,1}(\Omega, \mathbb{K})}.$$

Proof We want to apply Theorem 1.2. For this purpose it suffices to check that $Ph \in L^{\Psi,1}(\Omega, \mathbb{K})$ and $\|Ph\|_{L^{\Psi,1}(\Omega, \mathbb{K})} < 2\alpha$ for every $h \in L^{\Psi,1}(\Omega, \mathbb{K})$ with $\|h\|_{L^{\Psi,1}(\Omega, \mathbb{K})} = 1$.

Fix $h \in L^{\Psi,1}(\Omega, \mathbb{K})$ with $\|h\|_{L^{\Psi,1}(\Omega, \mathbb{K})} = 1$. By assertion (i) of Lemma 3.5 there exists a Young function $\Psi$ such that (9) holds. Thus $h \in L^\Psi(\Omega, \mathbb{K})$ and $\|h\|_{L^\Psi(\Omega, \mathbb{K})} \leq 1$. Applying [19, Remark 4.5 and Lemma 4.4] (note that we may apply these results as we are actually dealing with the norms of $h$ and $Ph$, respectively), we conclude that $Ph \in L^\Psi(\Omega, \mathbb{K})$ and $\|Ph\|_{L^\Psi(\Omega, \mathbb{K})} \leq \alpha$. This jointly with assertion (ii) of Lemma 3.5 implies that $Ph \in L^{\Psi,1}(\Omega, \mathbb{K})$ and $\|Ph\|_{L^{\Psi,1}(\Omega, \mathbb{K})} \leq 2 \|Ph\|_{L^\Psi(\Omega, \mathbb{K})} \leq 2\alpha$. □

Now, we fix an integer $m$ with $m > 1$ and consider the Lorentz space $L^{\Psi_m,1}(\Omega, \mathbb{K})$ that is generated by the Young function of the form $\psi_m(t) = mt^m$ for every $t \in [0, \infty)$ (see [5, Sect. 3.4.1]; cf. [10] where mappings with derivatives in those Lorentz spaces are considered). The following result follows from Theorem 3.6.
Corollary 3.7 Assume that \((H)\) holds. Let \(m > 1\) and let \(h \in L^{\psi_m,1}(\Omega, \mathbb{K})\). If (10) holds with a real constant \(\alpha \in [0, \frac{1}{2})\), then the elementary solution of equation (1) in \(L^{\psi_m,1}(\Omega, \mathbb{K})\) exists, it is the unique solution of equation (1) in \(L^{\psi_m,1}(\Omega, \mathbb{K})\) and

\[
\left\| \sum_{k=m}^{\infty} p_k h_0 \right\|_{L^{\psi_m,1}(\Omega, \mathbb{K})} \leq \frac{(2\alpha)^m}{1-2\alpha} \left\| h_0 \right\|_{L^{\psi_m,1}(\Omega, \mathbb{K})}.
\]

4 Lorentz spaces of vector valued functions

We begin with a definition, which collects the nice properties of Lebesgue spaces. To formulate it, we denote by \(M^+_K\) the subclass of \(M_K\) of functions that are almost everywhere nonnegative and accept that the symbol \(a_k \uparrow a\) means that \((a_k)_{k \in \mathbb{N}} \in [0, \infty]^\mathbb{N}\) is increasing and converges to \(a \in [0, \infty]\).

Definition 4.1 (see [5, Definition 3.1.1]) A Banach function norm on \((\Omega, \mu)\) is a map \(\rho : M^+_K \to [0, \infty]\) such that for all functions \(f, g \in M^+_K\), sequences \((f_k)_{k \in \mathbb{N}}\) of functions from \(M^+_K\), scalars \(\lambda \geq 0\) and \(\mu\)-measurable sets \(E \subset \Omega\), the following conditions hold:

(P1) \(\rho(f) = 0 \iff f = 0\) \(\mu\)-a.e., \(\rho(\lambda f) = \lambda \rho(f)\) and \(\rho(f + g) \leq \rho(f) + \rho(g)\);
(P2) \(g \leq f\) \(\mu\)-a.e. \(\implies \rho(g) \leq \rho(f)\);
(P3) \(f_k \uparrow f\) \(\mu\)-a.e. \(\implies \rho(f_k) \uparrow \rho(f)\);
(P4) \(\mu(E) < \infty \implies \rho(\chi_E) < \infty\);
(P5) \(\mu(E) < \infty \implies \int_E f \ d\mu \leq C(E) \rho(f)\) with some \(C(E) < \infty\).

If \(\rho\) is a Banach function norm on \((\Omega, \mu)\), then the set \(X^K = \{ f \in M_K : \rho(|f|) < \infty\}\) is called a Banach function space; as usual, we identify functions that are equal a.e.

The basic result on Banach functions spaces reads as follows.

Theorem 4.2 (see [5, Theorem 3.1.3]) Any Banach function space \(X^K\) equipped with the norm \(\| f \|_{X^K} = \rho(|f|)\) is a Banach space.

Now, we will give a brief sketch of the fact that the Lorentz space \(L^{\psi,1}(\Omega, \mathbb{K})\) equipped with the norm defined by (8) is complete. According to Theorem 4.2 it suffices to show that the map \(\rho : M^+_K \to [0, \infty]\) given by

\[
\rho(f) = \int_0^\infty \tau^{-1}(\mu_f(s)) \ ds
\]

is Banach function norm on \((\Omega, \mu)\).

We only look at the more involved conditions of Definition 4.1. Let us focus first on the triangle inequality in (P1). For this purpose we need the following definition in which we use the convention that \(\inf \emptyset = \infty\).

Definition 4.3 (see [5, Definition 3.2.3]) The nonincreasing rearrangement of an almost everywhere finite function \(f \in M_K\) is the function \(f^* : [0, \infty] \to [0, \infty]\) defined by

\[
f^*(t) = \inf \{ \lambda \in [0, \infty] : \mu_f(\lambda) \leq t \}.
\]
Switching from the distribution function $\mu_f$ to the nonincreasing rearrangement $f^*$ we have

$$\rho(f) = \int_0^\infty f^*(\tau(s)) \, ds.$$  

(11)

We know that $\tau$ is convex, but following the proof of the theorem in [1], we can prove that it is even bi-Lipschitz on compact intervals. This allows us to use Theorem 2.3 and obtain

$$\rho(f) = \int_0^\infty f^*(\tau^{-1}(s))' \, ds.$$  

(12)

Since $\tau$ is convex, it has a nondecreasing right derivative (see e.g. [20, Theorem 1.3.3] or [9, Theorems 3.7.3 and 3.7.4]). Finally, replacing this derivative with a nondecreasing function that is right continuous and agrees with the original function almost everywhere, we apply the following result.

**Proposition 4.4** (see [15, Proposition 2.7]) If $\varphi : (0, \infty) \to \mathbb{R}$ is right continuous, nonnegative and nonincreasing, then the operator

$$f \mapsto \int_0^\infty \varphi(t) f^*(t) \, dt$$

is subadditive.

Condition (P3) can be proven by using the fact that $\tau^{-1}$ is continuous on compact intervals, whereas condition (P5) can be seen by applying [5, Proposition 3.2.5] and the fact that $t \mapsto \tau^{-1}(t)/t$ is nonincreasing, which follows from [1, Lemma].

Banach function spaces are originally defined as spaces of $\mathbb{K}$-valued functions. We will show that the corresponding spaces for functions with Banach space targets are Banach spaces as well. We begin with a formal definition of these spaces.

Denote by $\mathcal{M}_V$ the space of all Bochner–measurable functions from $\Omega$ to $V$.

**Definition 4.5** The set

$$XV = \{ f \in \mathcal{M}_V : \| f \|_V \in X\mathbb{K} \}$$

is called a Banach function space (of vector valued functions); as in the $\mathbb{K}$-valued case we identify functions that are equal a.e.

It is easy to check that the formula

$$\| f \|_{XV} = \| \| f \|_V \|_{X\mathbb{K}}$$

(13)

defines a norm on the Banach function space $XV$. We want to prove that the just defined norm is complete. The next lemma, inspired by [5, Lemma 3.1.2], is the basic step in the proof.

**Lemma 4.6** Assume that $(f_n)_{n \in \mathbb{N}}$ ia a sequence of functions from the Banach function space $XV$ such that $\lim_{n \to \infty} f_n(x)$ exists for almost all $x \in \Omega$. If $\lim \inf_{n \to \infty} \| f_n \|_{XV} < \infty$, then $\lim_{n \to \infty} f_n \in XV$ and

$$\| \lim_{n \to \infty} f_n \|_{XV} \leq \lim \inf_{n \to \infty} \| f_n \|_{XV}.$$
Proof} It is clear that the formula

\[
 f(x) = \begin{cases} 
 \lim_{n \to \infty} f_n(x), & \text{if } \lim_{n \to \infty} f_n(x) \text{ exists} \\
 0, & \text{otherwise}
\end{cases}
\]

defines a function belonging to \( \mathcal{M}_\mathbb{K} \).

For every \( n \in \mathbb{N} \) we define the function \( g_n : \Omega \to [0, \infty] \) by putting

\[
g_n(x) = \inf_{m \geq n} \| f_m(x) \|_\mathbb{K}.
\]

Then for almost all \( x \in \Omega \) we have

\[
\lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} \inf_{m \geq n} \| f_m(x) \|_\mathbb{K} = \lim_{n \to \infty} \inf_{m \geq n} \| f_m(x) \|_\mathbb{K} = \| f(x) \|_\mathbb{K}.
\]

Hence

\[
\| f \|_{\mathbb{K}} = \| f \|_{\mathbb{K}} \| f \|_{\mathbb{K}} = \lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} \inf_{m \geq n} \| f_m \|_{\mathbb{K}}.
\]

Therefore,

\[
\inf_{m \geq n} \| f_m \|_{\mathbb{K}} \leq \inf_{l \geq n} \inf_{m \geq n} \| f_m \|_{\mathbb{K}} \leq \inf_{l \geq n} \| f_l \|_{\mathbb{K}},
\]

which jointly with (14) gives

\[
\| f \|_{\mathbb{K}} = \lim_{n \to \infty} \inf_{l \geq n} \| f_l \|_{\mathbb{K}} = \lim_{n \to \infty} \inf_{n \to \infty} \| f_n \|_{\mathbb{K}} = \lim_{n \to \infty} \inf_{n \to \infty} \| f_n \|_{\mathbb{K}} < \infty.
\]

In consequence, \( f \in \mathbb{K} \).

\[
\text{Theorem 4.7} \quad \text{The Banach function space } \mathbb{K} \text{ equipped with the norm defined by (13) is a Banach space.}
\]

Proof} We will follow the proof of [5, Theorem 3.1.3].

Assume that \( (f_n)_{n \in \mathbb{N}} \) is a Cauchy sequence of functions from \( \mathbb{K} \) and let \( (g_n)_{n \in \mathbb{N}} \) be one of its subsequences such that \( \| g_{n+1} - g_n \|_{\mathbb{K}} \leq \frac{1}{n+1} \) for every \( n \in \mathbb{N} \). Setting \( g_0 = 0 \), for every \( n \in \mathbb{N} \) we put \( h_n = g_n - g_{n-1} \) and note that \( h_n \in \mathcal{M}_\mathbb{K} \). Next for all \( x \in \Omega \) and \( N \in \mathbb{N} \) we put

\[
G_N(x) = \sum_{n=1}^{N} \| h_n(x) \|_{\mathbb{K}} \quad \text{and} \quad G(x) = \sum_{n=1}^{\infty} \| h_n(x) \|_{\mathbb{K}}.
\]

Then \( G_N, G \in \mathcal{M}_\mathbb{K} \) and

\[
\| G_N \|_{\mathbb{K}} \leq \sum_{n=1}^{\infty} \| h_n \|_{\mathbb{K}} \|_{\mathbb{K}} \leq \sum_{n=1}^{\infty} \| h_n \|_{\mathbb{K}} \leq \| h_1 \|_{\mathbb{K}} + 1 < \infty \tag{15}
\]

for every \( N \in \mathbb{N} \). Since \( G_N \uparrow G \), it follows by (P3) that \( G \in \mathbb{K} \).

Fix \( \varepsilon > 0 \) and \( E \subset \Omega \) such that \( \mu(E) < \infty \). By (P5), we see that

\[
\lim_{N \to \infty} \mu(\{x \in E : |G(x) - G_N(x)| > \varepsilon\}) \leq \lim_{N \to \infty} \frac{1}{\varepsilon} \int_{E} |G(x) - G_N(x)| \, d\mu(x)
\]
Thus the sequence \((G_N)_{N \in \mathbb{N}}\) converges to \(G\) in Lebesgue measure on \(E\). Applying the Riesz theorem (see e.g. [8, Theorem 11.26]) there exists a subsequence of \((G_N)_{N \in \mathbb{N}}\) which converges to \(G\) a.e. on \(E\). Since the \(k\)-dimensional Lebesgue measure is \(\sigma\)-finite, we can apply the diagonal argument to obtain that there exists a subsequence of \((G_N)_{N \in \mathbb{N}}\) which converges to \(G\) a.e. on \(\Omega\). As we know that \(G \in X^K\), it follows that the series \(\sum_{n=1}^{\infty} \|h_n(x)\|_V\) is finite for almost all \(x \in \Omega\). Thus

\[
\sum_{n=1}^{\infty} h_n(x) \in V
\]

for almost all \(x \in \Omega\).

Now, we define the function \(g \in XV\) by putting

\[
g(x) = \begin{cases} 
\lim_{n \to \infty} g_n(x), & \text{if } \sum_{n=1}^{\infty} h_n(x) \in V \\
0, & \text{otherwise}.
\end{cases}
\]

Our goal is to show that \(g\) is the limit of \((f_n)_{n \in \mathbb{N}}\); note that to achieve this, it is enough to show that it is the limit of \((g_n)_{n \in \mathbb{N}}\).

Fix \(m \in \mathbb{N}\). Then

\[
\liminf_{n \to \infty} \|g_m - g_n\|_{X^V} = \liminf_{n \to \infty} \left\| \sum_{k=m+1}^{n} h_k \right\|_{X^V} \leq \liminf_{n \to \infty} \sum_{k=m+1}^{\infty} \|h_k\|_{X^V} \leq \frac{1}{2^m}.
\]

(16)

Applying now Lemma 4.6 we conclude that \(g_m - g \in XV\) and \(\|g_m - g\|_{X^V} \leq \liminf_{n \to \infty} \|g_m - g_n\|_{X^V}\). Hence \(g = g_m - (g_m - g) \in XV\) and by (16) we have

\[
\lim_{m \to \infty} \|g_m - g\|_{X^V} \leq \lim_{m \to \infty} \liminf_{n \to \infty} \|g_m - g_n\|_{X^V} \leq \lim_{m \to \infty} \frac{1}{2^m} = 0.
\]

Hence \((g_n)_{n \in \mathbb{N}}\) converges to some \(g\). Since \((f_n)_{n \in \mathbb{N}}\) is a Cauchy sequence and \((g_n)_{n \in \mathbb{N}}\) one of its subsequences, it follows that \((f_n)_{n \in \mathbb{N}}\) converges to \(g\), which completes the proof. 

We end this paper with a counterpart of Theorem 3.6 for vector valued functions. We omit its proof as the boundedness of the considered operator can be proven by looking at the norm of the function instead of at the function itself.

**Theorem 4.8** Assume that \((H)\) holds and let \(h_0 \in L^{\psi,1}(\Omega, V)\). If there exists a real constant \(\alpha \in [0, \frac{1}{2})\) such that

\[
|g_n(x)| \leq \alpha \min \left\{ \frac{|J f_n(x)|}{KL}, \frac{1}{N} \right\} \quad \text{for all } n \in \{1, \ldots, N\} \quad \text{and}
\]

almost all \(x \in \Omega\),
then the elementary solution of equation (1) in $L^{\psi,1}(\Omega, V)$ exists, it is the unique solution of equation (1) in $L^{\psi,1}(\Omega, V)$ and

$$
\sum_{k=m}^{\infty} P^k h_0 \leq \frac{(2\alpha)^m}{1 - 2\alpha} \|h_0\|_{L^{\psi,1}(\Omega, V)}.
$$

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Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

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