LELONG NUMBERS OF \(m\)-SUBHARMONIC FUNCTIONS

AMEL BENALI AND NOUREDDINE GHILOUFI

Abstract. In this paper we study the existence of Lelong numbers of \(m\)-subharmonic currents of bidimension \((p,p)\) on an open subset of \(\mathbb{C}^n\), when \(m+p \geq n\). In the special case of \(m\)-subharmonic function \(\varphi\), we give a relationship between the Lelong numbers of \(dd^c\varphi\) and the mean values of \(\varphi\) on spheres or balls. As an application we study the integrability exponent of \(\varphi\). We express the integrability exponent of \(\varphi\) in terms of volume of sub-level sets of \(\varphi\) and we give a link between this exponent and its Lelong number.

1. Introduction

In complex analysis and geometry, the notion of Lelong numbers of positive currents has many applications. A famous result due to Siu [7], proves that if \(T\) is a positive closed current of bidimension \((p,p)\) on an open set \(\Omega\) of \(\mathbb{C}^n\), then the level subset \(E_T(c)\) of points \(z\) where the Lelong number \(\nu_T(z)\) of \(T\) at \(z\) is greater than or equal to \(c\) is an analytic set of dimension less than or equal to \(p\) for any real \(c > 0\). In a particular case, if \(u\) is a plurisubharmonic function on \(\Omega\) then the Lelong number \(\nu_u(a)\) of \(u\) at a point \(a \in \Omega\) characterizes the complex singularity exponent \(c_u(a)\) of \(u\) at \(a\). In fact, Skoda [8] stated the following inequalities:

\[
\frac{1}{\nu_u(a)} \leq c_u(a) \leq \frac{n}{\nu_u(a)}.
\]

This result was enhanced by Demailly and Pham [3], who showed a sharp relationship between this exponent and the Lelong numbers of currents \((dd^c\varphi)^j\) at \(a\) for \(1 \leq j \leq n\).

In 2005, Blocki [1] posed a problem about the integrability exponents of \(m\)-subharmonic functions, he conjectured that every \(m\)-subharmonic functions...
function belongs to \( L^q_{loc}(\Omega) \) for any \( q < \frac{nm}{n-m} \) i.e. the integrability exponent (the supremum of \( q \)) is greater than or equal to \( \frac{nm}{n-m} \). This problem may be similar, but different, to the study of the complex singularity exponents of plurisubharmonic functions. To study this similar problem, we need a suitable definition of Lelong numbers of \( m \)-subharmonic functions. In 2016, Wan and Wang [9] give the definition of the Lelong number of an \( m \)-subharmonic function \( \psi \) as the Lelong number of the current \( \ddc \psi \):

\[
\nu_\psi(x) = \lim_{r \to 0^+} \frac{1}{r^{\frac{2n}{m}(m-1)}} \int_{B(x,r)} \ddc \psi \wedge \beta^{n-1}.
\]

Our aim is to find a link between the integrability exponents of \( \psi \) and its Lelong numbers. To reach this aim, we need to know more properties of Lelong numbers.

The paper is organized as follows: in section 2, we introduce the basic concepts which will be employed in the rest of this paper. Indeed, we recall the notions of \( m \)-positive currents, \( m \)-subharmonic functions and their Lelong numbers with some properties.

Section 3 is devoted to the study of the existence of Lelong numbers of \( m \)-positive currents. We start by proving, the main tool in this study, the Lelong-Jensen formula. By this formula we conclude that, for an \( m \)-positive \( m \)-subharmonic current \( T \) on \( \Omega \), the Lelong function \( \nu_T(a,) \) associated to \( T \) at \( a \in \Omega \) is increasing on \( ]0,d(a,\partial \Omega)[ \). Thus, its limit at zero \( \nu_T(a) \) exists. Moreover, we study the case of an \( m \)-negative \( m \)-subharmonic current \( S \), we show that \( \nu_S(a) \) exists with the assumption that \( t \mapsto t^{\frac{2n}{m}+1} \nu_{ddcS}(a,t) \) is integrable in a neighborhood of 0.

In section 4, we give a relationship between the Lelong number of an \( m \)-subharmonic function \( \psi \) at \( a \in \Omega \) and its mean values on spheres and balls. We conclude then that the map \( z \mapsto \nu_\psi(z) \) is upper semi-continuous on \( \Omega \).

Finally, as an application, we study in the last section the integrability exponent \( \iota_K(\psi) \) of an \( m \)-subharmonic function \( \psi \) on a compact subset \( K \) of \( \Omega \). We prove that if \( \psi < 0 \) on a neighborhood of \( K \) then we have

\[
\iota_K(\psi) = \sup \left\{ \alpha > 0; \exists \ C_\alpha > 0, \forall \ t < 0 \ V(\{\psi < t\}) \cap K \leq \frac{C_\alpha}{|t|^{\alpha}} \right\}.
\]
At the end, we show that if \( \nu_{\varphi}(a) > 0 \) then
\[
\frac{n}{n-m} \leq \iota_{n}(\psi) \leq \frac{nm}{n-m}.
\]
In particular if the Blocki conjecture is true then we have \( \iota_{n}(\psi) = \frac{nm}{n-m} \).
We claim that this equality is true for \( m = 1 \) and \( \nu_{\varphi}(a) > 0 \) and this result can be viewed as a partial answer for the Blocki conjecture.

2. Preliminaries

Throughout this paper \( \Omega \) is an open set of \( \mathbb{C}^n \) and \( m \) is an integer such that \( 1 \leq m < n \). We use the operators \( d = \partial + \partial^c \) and \( d^c = \frac{i}{4\pi}(\overline{\partial} - \partial) \) in order to have \( dd^c = \frac{i}{2\pi} \partial \partial \). We set \( \beta = dd^c|x|^2 \) and
\[
\phi_{m}(r) = -\frac{1}{(\frac{n}{m} - 1)r^{2(\frac{n}{m} - 1)}}.
\]
For \( x \in \mathbb{C}^n, r > 0 \) and \( 0 < r_1 < r_2 \) we set
\[
\mathcal{B}(x, r_1, r_2) = \{ z \in \mathbb{C}^n; r_1 < |z - x| < r_2 \}
\]
and
\[
\mathcal{B}(x, r) = \{ z \in \mathbb{C}^n; |z - x| < r \}.
\]
In this part we recall some definitions of \( m \)-positivity cited by Dhouib-Elkhadhra in [4].

Definition 1.

1. A \((1,1)\)-form \( \alpha \) on \( \Omega \) is said to be \( m \)-positive if \( \alpha^j \wedge \beta^{n-j} \geq 0 \) (in sens of currents) for every \( 1 \leq j \leq m \).
2. A \((p,p)\)-form \( \alpha \) on \( \Omega \) is strongly \( m \)-positive if
\[
\alpha = \sum_{k=1}^{N} a_k \alpha_{1,k} \wedge \cdots \wedge \alpha_{p,k}
\]
where \( N = \binom{n}{p} \) and \( \alpha_{1,k}, \ldots, \alpha_{p,k} \) are \( m \)-positive \((1,1)\)-forms and \( a_k \geq 0 \) for every \( k \).
3. A current \( T \) of bidimension \((p,p)\) on \( \Omega \) with \( m + p \geq n \) is said to be \( m \)-positive if \( \langle T \wedge \beta^{n-m}, \alpha \rangle \geq 0 \) for every strongly \( m \)-positive \((m + p - n, m + p - n)\)-test form \( \alpha \) on \( \Omega \).
4. A function \( \varphi : \Omega \to \mathbb{R} \cup \{-\infty\} \) is said to be \( m \)-subharmonic (\( m \)-sh for short) if it is subharmonic and \( dd^c \varphi \) is an \( m \)-positive current on \( \Omega \).

We set \( \mathcal{SH}_m(\Omega) \) the set of \( m \)-subharmonic functions on \( \Omega \).
In general, if $T$ is an $m-$positive current of bidimension $(p,p)$ on $\Omega$ and $a \in \Omega$, the $m-$Lelong function of $T$ at $a$ is defined by:

$$\nu_T(a, r) := \frac{1}{r^m} \int_{B(a,r)} T \wedge \beta_p.$$ 

for $r < d(a, \partial \Omega)$. The Lelong number of $T$ at $a$, when it exists, is

$$\nu_T(a) = \lim_{r \to 0^+} \nu_T(a, r).$$

Here, we give a short list of the most basic properties of $m-$sh functions:

**Proposition 1.** Let $\Omega \subset \mathbb{C}^n$ be a domain.

1. If $\varphi \in \mathcal{C}^2(\Omega)$, then $\varphi$ is $m-$sh if and only if

$$(dd^c \varphi)^k \wedge \beta^{n-k} \geq 0$$

for $k = 1, 2, ..., m$, in the sense of currents.

2. $\mathcal{PSH}(\Omega) = \mathcal{SH}_n(\Omega) \subsetneq \mathcal{SH}_{n-1}(\Omega) \subsetneq ... \subsetneq \mathcal{SH}_1(\Omega) = \mathcal{SH}(\Omega)$.

3. $\mathcal{SH}_m(\Omega)$ is a convex cone.

4. If $\varphi$ is $m-$sh and $\gamma : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^2-$smooth convex, increasing function then $\gamma \circ \varphi$ is also $m-$sh.

5. The standard regularization $\varphi^\star \rho_\varepsilon$ of an $m-$sh function is again $m-$sh.

6. The limit of a uniformly converging or decreasing sequence of $m-$sh functions is either $m-$sh or identically equal to $-\infty$.

Now we recall some classes of $m-$sh functions on $\Omega$, in relation with the definition of the complex Hessian operator called Cegrell classes (see [5] for more details):

- $\mathcal{E}_{0,m}(\Omega)$ is the convex cone of bounded negative $m-$sh function $\varphi$ on $\Omega$ such that

$$\lim_{z \to \partial \Omega} \varphi(z) = 0 \quad \text{and} \quad \int_{\Omega} (dd^c \varphi)^m \wedge \beta^{n-m} < +\infty.$$

- $\mathcal{F}_m(\Omega)$ is the class of negative $m-$sh functions on $\Omega$ such that there exists a sequence $(\varphi_j)_j$ in $\mathcal{E}_{0,m}(\Omega)$ that decreases to $\varphi$ and

$$\sup_j \int_{\Omega} (dd^c \varphi_j)^m \wedge \beta^{n-m} < +\infty.$$

- We denote by $\mathcal{E}_m(\Omega)$ the subclass of negative $m-$sh functions on $\Omega$ that coincides locally with elements of $\mathcal{F}_m$.

In the next, we introduce some properties that will employed in the sequel:
Proposition 2. (See [5])

- \( E_{0,m}(\Omega) \subset F_m(\Omega) \subset E_m(\Omega) \).
- If \( \varphi \in E_{0,m}(\Omega) \) and \( \psi \in \mathcal{SH}_m(\Omega) \), then \( \max(\varphi, \psi) \in E_{0,m}(\Omega) \).
- If \( \varphi \in F_m(\Omega) \) then \( \int_{\Omega} (dd^c \varphi)^p \wedge \beta^{n-p} < +\infty \).

Lemma 1. (See [5]) Suppose that \( \varphi_1, \ldots, \varphi_2 \in F_m(\Omega) \) and \( h \in E_{0,m}(\Omega) \). Then we have

\[
\int_{\Omega} -h dd^c \varphi_1 \wedge \cdots \wedge dd^c \varphi_m \wedge \beta^{n-m} \\
\leq \left( \int_{\Omega} -h (dd^c \varphi_1)^m \wedge \beta^{n-m} \right)^{\frac{1}{m}} \cdots \left( \int_{\Omega} -h (dd^c \varphi_m)^m \wedge \beta^{n-m} \right)^{\frac{1}{m}}.
\]

3. Lelong numbers of \( m \)-subharmonic currents

The aim of this part is to prove the existence of the Lelong number of \( m \)-subharmonic currents. The main tool is the Lelong-Jensen formula.

Proposition 3. (Lelong-Jensen formula) Let \( T \) be a current of bidimension \((p, p)\) on \( \Omega \) such that \( T \) and \( dd^c T \) are of zero order on \( \Omega \). Then for every \( a \in \Omega \) and \( 0 < r_1 < r_2 < d(a, \partial \Omega) \), we have

\[
A(r_1, r_2) := \nu_T(a, r_2) - \nu_T(a, r_1) = \frac{1}{r_2^{m+n+p}} \int_{B(a, r_2)} T \wedge \beta^p - \frac{1}{r_1^{m+n+p}} \int_{B(a, r_1)} T \wedge \beta^p = \frac{1}{r_1^{m+n+p}} \int_{B(a, r_1)} T \wedge \beta^p = \frac{1}{r_2^{m+n+p}} \int_{B(a, r_2)} T \wedge \beta^p
\]

where \( \tilde{\phi}_m(\zeta) = \phi_m(|\zeta|) \).

Proof. Without loss of generality, we can assume that \( a = 0 \). We use \( B(r) \) and \( B(r_1, r_2) \) instead of \( B(0, r) \) and \( B(0, r_1, r_2) \). We set \( S(r) = \partial B(r) \).

Suppose first that \( T \) is of class \( \mathcal{C}^2 \). Then thanks to Stokes formula, we
As have

\[
\int_{r_1}^{r_2} \frac{2tdt}{t^{2n(m+p-n)}} \int_{\mathbb{B}(t)} dd\epsilon T \land \beta^{p-1}
\]

\[
= \int_{r_1}^{r_2} \frac{2tdt}{t^{2n(m+p-n)}} \int_{\mathbb{S}(t)} d\epsilon T \land \beta^{p-1}
\]

\[
= \int_{r_1}^{r_2} 2tdt \int_{\mathbb{S}(t)} d\epsilon T \land (dd\tilde{\phi}_m)^{m+p-n} \land \beta^{n-m-1}
\]

\[
= \int_{\mathbb{B}(r_1, r_2)} d|z|^2 \land d\epsilon T \land (dd\tilde{\phi}_m)^{m+p-n} \land \beta^{n-m-1}
\]

\[
= \int_{\mathbb{B}(r_1, r_2)} d\left(T \land d\epsilon |z|^2 \land (dd\tilde{\phi}_m)^{m+p-n} \land \beta^{n-m-1}\right)
\]

\[
- \int_{\mathbb{E}(r_1, r_2)} T \land (dd\tilde{\phi}_m)^{m+p-n} \land \beta^{n-m}
\]

\[
= \int_{\mathbb{S}(r_2)} T \land d\epsilon |z|^2 \land (dd\tilde{\phi}_m)^{m+p-n} \land \beta^{n-m-1}
\]

\[
- \int_{\mathbb{S}(r_1)} T \land d\epsilon |z|^2 \land (dd\tilde{\phi}_m)^{m+p-n} \land \beta^{n-m-1}
\]

\[
- \int_{\mathbb{E}(r_1, r_2)} T \land (dd\tilde{\phi}_m)^{m+p-n} \land \beta^{n-m}
\]

A simple computation shows that

\[
(3.1)
\]

\[
\int_{\mathbb{S}(r)} T \land d\epsilon |z|^2 \land (dd\tilde{\phi}_m)^{m+p-n} \land \beta^{n-m-1}
\]

\[
= \frac{1}{t^{2n(m+p-n)}} \int_{\mathbb{S}(r)} T \land d\epsilon |z|^2 \land \beta^{p-1}
\]

\[
= \frac{1}{t^{2n(m+p-n)}} \int_{0}^{2tdt} \int_{\mathbb{B}(t)} dd\epsilon T \land \beta^{p-1} + \frac{1}{t^{2n(m+p-n)}} \int_{\mathbb{B}(t)} T \land \beta^p.
\]

The result follows from Equalities (3.1) and (3.2).

If \( T \) is not of class \( C^2 \), we consider the set

\[
E_T := \{r > 0; ||T||(S(r)) \neq 0 \text{ or } ||dd\epsilon T||(S(r)) \neq 0\}.
\]

As \( T \) and \( dd\epsilon T \) are of zero orders, then \( E_T \) is at least countable. Let \( (\rho_\epsilon)_\epsilon \) be a regularizing kernel and \( r \in \mathbb{R} \setminus E_T \), then we have

\[
\lim_{\epsilon \to 0} \int_{\mathbb{B}(r)} T * \rho_\epsilon \land \beta^p = \lim_{\epsilon \to 0} \int_{\mathbb{C}^n} \mathbb{1}_{\mathbb{B}(r)} T * \rho_\epsilon \land \beta^p = \int_{\mathbb{B}(r)} T \land \beta^p.
\]

It follows that if \( 0 < r_1 < r_2 \) are two values outside \( E_T \), then the result is checked by regularization. If \( r_1 \) or \( r_2 \) is in \( E_T \), it suffices to take two
sequences \((r_{1,j})_j\) and \((r_{2,j})_j\) in \(\mathbb{R} \setminus E_T\) which tend respectively to \(r_1\) and \(r_2\) and apply the previous step with \(r_{1,j}\) and \(r_{2,j}\). The result follows by passing to the limit when \(j \to +\infty\).

As a consequence, we have:

**Theorem 1.** Let \(T\) be a current of bidimension \((p, p)\) on \(\Omega\). Assume that \(T\) is \(m\)--positive and \(dd^c T \wedge \beta^{p-1}\) is a positive measure on \(\Omega\) with \(m + p \geq n\). Then the Lelong number of \(T\) exists at every point of \(\Omega\).

This result is due to Wan and Wang for \(T = dd^c \varphi\) where \(\varphi\) is an \(m\)--sh function.

**Proof.** Since \(T\) is an \(m\)--positive current on \(\Omega\) then \(\nu_T(a, \cdot)\) is positive. Moreover, thanks to Lelong-Jensen formula, \(\nu_T(a, \cdot)\) is increasing. It follows that its limit \(\nu_T(a)\) when \(r\) tends to 0 exists.

The case of \(m\)--negative \(m\)--subharmonic currents is so different to the previous case (of \(m\)--positive \(m\)--subharmonic currents). Indeed, let \(T_0 := \tilde{\phi}_m(dd^c \tilde{\phi}_m)^{m-1}\); it is not hard to see that \(T_0\) is \(m\)--negative \(m\)--subharmonic current of bidimension \((n-m+1, n-m+1)\) on \(\mathbb{C}^n\) with \(dd^c T_0 \wedge \beta^{m-n} = \delta_0\) and \(\nu_{T_0}(r) := \nu_{T_0}(0, r) = \frac{c_n}{r^{2(\frac{n}{m}-1)}}\), for some constant \(c_n < 0\). Thus the Lelong number of \(T_0\) at 0 doesn't exist.

It follows that it is legitimate to impose a condition on an \(m\)--negative \(m\)--subharmonic current to ensure the existence of its Lelong number, this will be the aim of Theorem 2. But before giving such a condition, we may study the local behavior of the Lelong function associated to such a current.

**Lemma 2.** Let \(T\) be an \(m\)--negative \(m\)--subharmonic current of bidimension \((p, p)\) on \(\Omega\) with \(m + p - 1 \geq n\). Then for every \(a \in \Omega\) and \(0 < r_0 < d(a, \partial \Omega)\), there exists \(c_0 < 0\) such that for any \(0 < r \leq r_0\) we have:

\[
\nu_T(a, r) \geq \frac{\nu_{dd^c T}(a, r_0)}{1 - \frac{n}{m}} r^{2(1 - \frac{n}{m})} + c_0
\]

**Proof.** Without loss of generality, we can assume that \(a = 0\). For \(r \leq r_0\) we set:

\[
\gamma_T(r) = \nu_T(r) - \frac{\nu_{dd^c T}(r_0)}{1 - \frac{n}{m}} r^{2(1 - \frac{n}{m})}
\]
Thanks to Lelong-Jensen formula, for any $r_1 < r_2 \leq r_0$, one has:

\[
\begin{align*}
\Upsilon_T(r_2) - \Upsilon_T(r_1) &= \nu_T(r_2) - \nu_T(r_1) - \frac{\nu_{dd^c T}(r_0)}{1 - \frac{2}{m}} \left( r_2^{2(1 - \frac{2}{m})} - r_1^{2(1 - \frac{2}{m})} \right) \\
&= 2 \int_{r_1}^{r_2} \left( \frac{1}{t^{2n/(m+p-n)}} - \frac{1}{r_2^{2n/(m+p-n)}} \right) t^{2n/(m+p-1-n)+1} \nu_{dd^c T}(t)dt \\
&\quad + 2 \int_{0}^{r_1} \left( \frac{1}{r_1^{2n/(m+p-n)}} - \frac{1}{r_2^{2n/(m+p-n)}} \right) t^{2n/(m+p-1-n)+1} \nu_{dd^c T}(t)dt \\
&\quad + \int_{\mathbb{B}(r_1,r_2)} T \wedge \beta^{n-m} \wedge (dd^c \tilde{\phi}_m)^{m+p-n} - \frac{\nu_{dd^c T}(r_0)}{1 - \frac{2}{m}} \left( r_2^{2(1 - \frac{2}{m})} - r_1^{2(1 - \frac{2}{m})} \right) \\
&= \int_{\mathbb{B}(r_1,r_2)} T \wedge \beta^{n-m} \wedge (dd^c \tilde{\phi}_m)^{m+p-n} - \frac{\nu_{dd^c T}(r_0)}{1 - \frac{2}{m}} \left( r_2^{2(1 - \frac{2}{m})} - r_1^{2(1 - \frac{2}{m})} \right) \\
&\quad + 2 \int_{r_1}^{r_2} t^{-\frac{2n}{m}} \nu_{dd^c T}(t)dt - 2 \int_{0}^{r_2} \left( \frac{1}{t^{2n/(m+p-1-n)+1}} \nu_{dd^c T}(t)dt - \frac{1}{r_1^{2n/(m+p-1-n)+1}} \nu_{dd^c T}(t)dt \\
&\quad + 2 \int_{0}^{r_1} \frac{1}{r_1^{2n/(m+p-1-n)+1}} \nu_{dd^c T}(t)dt \\
&= \int_{\mathbb{B}(r_1,r_2)} T \wedge \beta^{n-m} \wedge (dd^c \tilde{\phi}_m)^{m+p-n} + 2 \int_{r_1}^{r_2} \left( \nu_{dd^c T}(t) - \nu_{dd^c T}(r_0) \right) t^{-\frac{2n}{m}+1}dt \\
&\quad - 2 \int_{0}^{r_2} \frac{1}{r_1^{2n/(m+p-1-n)+1}} \nu_{dd^c T}(t)dt + 2 \int_{0}^{r_1} \frac{1}{r_2^{2n/(m+p-1-n)+1}} \nu_{dd^c T}(t)dt \leq 0.
\end{align*}
\]

Indeed, since $T$ is $m$-negative then $T \wedge \beta^{n-m} \wedge (dd^c \tilde{\phi}_m)^{m+p-n}$ is a negative measure so

\[
\int_{\mathbb{B}(r_1,r_2)} T \wedge \beta^{n-m} \wedge (dd^c \tilde{\phi}_m)^{m+p-n} \leq 0.
\]

Moreover, as $dd^c T$ is an $m$-positive closed current, then thanks to Theorem 1, $\nu_{dd^c T}$ is an increasing function on $[0,r_0]$. Hence we have

\[
\int_{r_1}^{r_2} \left( \nu_{dd^c T}(t) - \nu_{dd^c T}(r_0) \right) t^{-\frac{2n}{m}+1}dt \leq 0.
\]

Furthermore, if we set

\[
f(r) = -\frac{1}{r^{2n/(m+p-n)}} \int_{0}^{r} t^{2n/(m+p-1-n)+1} \nu_{dd^c T}(t)dt
\]
then $f$ is an absolutely continuous function on $[0, r_0]$ and satisfies:

$$f'(r) = \frac{2n}{m} \frac{(m + p - n)}{m(m + p - n) + 1} \int_0^r t^{\frac{2n}{m}(m+p-1-n) + 1} \nu_{dd^c T}(t) dt - r^{-\frac{2n}{m} + 1} \nu_{dd^c T}(r)$$

for almost every $0 < r < r_0$. As a consequence, $\Upsilon_T$ is a decreasing function on $]0, r_0]$, thus $\Upsilon_T(r) \geq \Upsilon_T(r_0)$ for every $0 < r \leq r_0$. We conclude that we have for every $0 < r \leq r_0$,

$$\nu_T(r) \geq \Upsilon_T(r_0) + \nu_{dd^c T}(r_0) \frac{r^{2(1 - \frac{2n}{m})}}{1 - \frac{2n}{m}}.$$

The result follows by choosing for example $c_0 = \min(0, \Upsilon_T(r_0))$. □

**Theorem 2.** Let $T$ be an $m-$negative $m-$subharmonic current of bidimension $(p, p)$ on $\Omega$. Assume that $t \mapsto t^{-\frac{2n}{m} + 1} \nu_{dd^c T}(z_0, t)$ is integrable in neighborhood of 0 for $z_0 \in \Omega$. Then the Lelong number $\nu_T(z_0)$ of $T$ at $z_0$ exists.

**Proof.** It suffices to prove the result with $z_0 = 0$. For every $0 < r \leq r_0 < d(0, \partial \Omega)$, we set:

$$g(r) = \nu_T(r) + 2 \int_0^r \left( \frac{t^{\frac{2n}{m}(m+p-n)}}{r^{\frac{2n}{m}(m+p-n)} - 1} - 1 \right) t^{-\frac{2n}{m} + 1} \nu_{dd^c T}(t) dt.$$

The assumption implies that the function $g$ is well defined and negative on $]0, r_0[$. Moreover, using the Lelong-Jensen formula, one can prove that for any $0 < r_1 < r_2 \leq r_0$,

$$g(r_2) - g(r_1) = \int_{\mathbb{B}(r_1, r_2)} T \wedge \beta^{n-m} \wedge (dd^c \tilde{\phi}_m)^{m+p-n} \leq 0.$$

It follows that $g$ is a negative decreasing function on $]0, r_0]$, which gives the existence of the limit

$$\lim_{r \to 0^+} g(r) = \lim_{r \to 0^+} \nu_T(r)$$

because $t \mapsto t^{-\frac{2n}{m} + 1} \nu_{dd^c T}(t)$ is integrable in neighborhood of 0 and $((t/r)^{\frac{2n}{m}(m+p-n)} - 1)$ is uniformly bounded. □

**4. Lelong numbers of $m-$subharmonic functions**

In this particular case we give a new expression of Lelong number of $dd^c \varphi$ using the mean values of the $m-$sh function $\varphi$ on spheres and balls
analogous to the case of plurisubharmonic functions; for this reason we set, as usual,
\[ \nu_{\varphi}(a) = \lim_{r \to 0^+} \nu_{\varphi}(a, r) := \lim_{r \to 0^+} \frac{1}{r^{2m/(m-1)}} \int_{B(a, r)} dd^c \varphi \wedge \beta^{n-1} \]
this number and we consider the mean values of \( \varphi \) over the ball and the sphere respectively:
\[ \Lambda(\varphi, a, r) = \frac{n!}{\pi^n r^{2n}} \int_{B(a, r)} \varphi(x) dV(x) \]
\[ \lambda(\varphi, a, r) = \frac{(n-1)!}{2 \pi^n r^{2n-1}} \int_{S(a, r)} \varphi(x) d\sigma(x). \]

The main result of this paper is the following theorem:

**Theorem 3.** Let \( \varphi \) be an \( m \)-sh function on \( \Omega \). Then for any \( a \in \Omega \), the Lelong number of \( \varphi \) at \( a \) is given by the following limits:

\[ \nu_{\varphi}(a) = 2 \lim_{r \to 0^+} \lambda_{\phi_m}(r) = 2 \frac{(n-m+1)}{n} \lim_{r \to 0^+} \frac{\Lambda(\varphi, a, r)}{\phi_m(r)}. \]

In particular, if \( \varphi \) is bounded near \( a \) then \( \nu_{\varphi}(a) = 0 \).

To prove this theorem we need the following lemmas where we prove some more precise results.

**Lemma 3.** Let \( a \in \Omega \), and \( 0 < r_1 < r_2 < d(a, \partial \Omega) \). Then

\[ \lambda(\varphi, a, r_2) - \lambda(\varphi, a, r_1) = \frac{1}{2} \int_{\phi_m(r_1)}^{\phi_m(r_2)} \nu_{\varphi}(a, \phi_m^{-1}(t)) dt. \]

**Proof.** According to Green formula we have
\[
\lambda(\varphi, a, r) - \lambda(\varphi, a, s) = \frac{1}{2n} \left[ \int_{s}^{r} t \Lambda(\Delta \varphi, a, t) dt - \int_{0}^{s} t \Lambda(\Delta \varphi, a, t) dt \right] \\
= \frac{1}{2n} \int_{s}^{r} t \Lambda(\Delta \varphi, a, t) dt \\
= \frac{1}{2n} \int_{s}^{r} t \frac{n!}{\pi^n t^{2n}} \int_{B(a, t)} \Delta \varphi(x) dV(x) dt \\
= \frac{1}{2n} \int_{s}^{r} t \frac{n!}{t^{2n}} \int_{B(a, t)} \frac{2}{(n-1)!} dd^c \varphi \wedge \beta^{n-1} dt \\
= \int_{s}^{r} \frac{1}{t^{2n-1}} \int_{B(a, t)} dd^c \varphi \wedge \beta^{n-1} dt \\
= \int_{s}^{r} \nu_{\varphi}(a, t) dt \\
= \frac{1}{2} \int_{s}^{r} \nu_{\varphi}(a, t) d\phi_m(t). \]
Thus, a changement of variable \( t = \phi_m^{-1}(r) \) gives,

\[
\lambda(\varphi, a, r_2) - \lambda(\varphi, a, r_1) = \frac{1}{2} \int_{\phi_m(r_1)}^{\phi_m(r_2)} \nu_\varphi(a, \phi_m^{-1}(r)) dr.
\]

\[\square\]

**Lemma 4.** Let \( a \in \Omega \). Then \( t \mapsto \lambda(\varphi, a, r) \) is convex increasing of \( t = \phi_m(r) \). Moreover, the following limit exists in \([0, +\infty[\),

\[
\nu_\varphi(a) = 2 \lim_{r \to 0^+} \frac{\lambda(\varphi, r)}{\phi_m(r)} = 2 \frac{\partial^+ \lambda(\varphi, r)}{\partial \phi_m(r)} \bigg|_{r=0}
\]

**Proof.** According to the previous lemma

\[
\lambda(\varphi, a, r_2) - \lambda(\varphi, a, r_1) = \frac{1}{2} \int_{\phi_m(r_1)}^{\phi_m(r_2)} \nu_\varphi(a, \phi_m^{-1}(r)) dr.
\]

It follows that,

\[
2 \frac{\partial^+ \lambda(\varphi, a, r)}{\partial \phi_m(r)} = \nu_\varphi(a, r).
\]

Since the Lelong function \( r \mapsto \nu_\varphi(a, r) \) is increasing on \([0, r_0]\), then the function \( \phi_m(r) \mapsto \lambda(\varphi, a, r) \) is convex increasing.

Furthermore, for \( 0 < r_0 < d(a, \partial \Omega) \) the following limit exists in \([0, +\infty[\)

\[
2 \lim_{r \to 0^+} \frac{\lambda(\varphi, a, r)}{\phi_m(r)} = 2 \lim_{r \to 0^+} \frac{\lambda(\varphi, a, r) - \lambda(\varphi, a, r_0)}{\phi_m(r) - \phi_m(r_0)}
\]

\[= 2 \frac{\partial^+ \lambda(\varphi, a, r)}{\partial \phi_m(r)} \bigg|_{r=0}
\]

\[= \lim_{r \to 0^+} \nu_\varphi(a, r).\]

\[\square\]

Now we can conclude the proof of Theorem 3.

**Proof.** The first equality in (4.1) is proved by Lemma 4. For the second one, using the classical formula:

\[
\Lambda(\varphi, a, r) = 2n \int_0^1 t^{2n-1} \lambda(\varphi, a, rt) dt
\]
and Lemma 4, one can deduce that $\phi_m(r) \rightarrow \Lambda(\varphi, a, r)$ is convex increasing so the second limit in (4.1) exists. Moreover, one has

$$2 \lim_{r \to 0^+} \frac{\Lambda(\varphi, a, r)}{\phi(r)} = 2n \lim_{r \to 0^+} \int_0^1 2 \frac{\lambda(\varphi, a, rt)}{\phi(rt)} t^{2n-\frac{2n}{m}+1} dt$$

$$= 2n \nu_{\varphi}(a) \int_0^1 t^{2n-\frac{2n}{m}+1} dt$$

$$= \frac{n}{n - \frac{n}{m} + 1} \nu_{\varphi}(a).$$

□

**Corollary 1.** Let $\varphi$ be an $m$-sh function on $\Omega$. Then the function $z \mapsto \nu_{\varphi}(z)$ is upper semi-continuous on $\Omega$.

**Proof.** For any $c \in \mathbb{R}$, we set $\Omega_c := \{ z \in \Omega; \nu_{\varphi}(z) < c \}$. To prove that $\Omega_c$ is open we claim that if $c \leq 0$ then $\Omega_c = \emptyset$. So let $c > 0$ and $z \in \Omega_c$. Without loss of generality, we can assume that $\varphi < 0$ on $B(z, r_0)$ for some $0 < r_0 < d(z, \partial \Omega)$. Let $c' \in [\nu_{\varphi}(z), c]$ and $t \in [0, 1]$ such that

$$\frac{c'}{(1 - t)^{2(n+1-\frac{n}{m})}} < c.$$

As

$$\frac{2}{n} \frac{(n + 1 - \frac{n}{m}) \Lambda(\varphi, z, r)}{\phi(r)}$$

decreases to $\nu_{\varphi}(z)$, then there exists $0 < r_1 < r_0$ such that for every $0 < r < r_1$ one has

$$\frac{2}{n} \frac{(n + 1 - \frac{n}{m}) \Lambda(\varphi, z, r)}{\phi(r)} \leq c'.$$

Let $0 < r < r_1$. Then for any $\xi \in B(z, rt)$ one has $B(\xi, r(1 - t)) \subset B(z, r)$. Hence we obtain

$$\Lambda(\varphi, \xi, r(1 - t)) \leq \frac{1}{(1 - t)^{2n}} \Lambda(\varphi, z, r)$$

Thus,

$$\frac{2}{n} \frac{(n + 1 - \frac{n}{m}) \Lambda(\varphi, \xi, r(1 - t))}{\phi(r(1 - t))} \leq \frac{2}{n} \frac{(n + 1 - \frac{n}{m}) \Lambda(\varphi, z, r)}{(1 - t)^{2(n-\frac{2n}{m}+1)}} \leq \frac{c'}{(1 - t)^{2(n-\frac{2n}{m}+1)}}.$$

We conclude that we have

$$\nu_{\varphi}(\xi) \leq \frac{c'}{(1 - t)^{2(n-\frac{2n}{m}+1)}} < c.$$

So $\xi \in \Omega_c$ for every $\xi \in B(z, rt)$.
Since $z \mapsto \nu_\varphi(z)$ is upper semi continuous on $\Omega$ then it is clear that the level sets $E^m_\varphi(c) := \{z \in \Omega; \nu_\varphi(z) \geq c\}$ is closed. Moreover, for any $c > 0$ we have $E^m_\varphi(c) \subset \{\varphi = -\infty\}$ and its Hausdorff dimension
\[
\dim_H(E^m_\varphi(c)) \leq \frac{2n}{m}(m-1).
\]
In particular, For $m = 1$ we have $E^1_\varphi(c)$ is a locally finite set.

Indeed, let $z \in \Omega$ such that $-\infty < \varphi(z) < 0$. Then for any $0 < r < d(z, \partial \Omega)$ one has
\[
\frac{\varphi(z)}{\phi_m(r)} \geq \frac{\lambda(\varphi, z, r)}{\phi_m(r)} \geq 0.
\]
Hence, when we tend $r \to 0^+$ we get $\nu_\varphi(z) = 0$. It follows that $z \not\in E^m_\varphi(c)$ for all $c > 0$.

It is well known that for a plurisubharmonic function $u$, the level set $E^m_u(c)$ is analytic whenever $c > 0$ (this result is due to Siu [7]). Thus, we can ask the following question:

**Problem 1.** Let $\varphi$ be an $m$–sh function on $\Omega$. What can be said about the analyticity of level sets $E^m_\varphi(c)$ for any $c > 0$?

For the maximum of $m$–sh functions on spheres/or balls we have the following proposition:

**Proposition 4.** Let $\varphi$ be an $m$–sh function on $\Omega$ and $a$ be a point of $\Omega$. Then $\phi_m(r) \longmapsto M(\varphi, a, r)$ is a convex increasing function on $[0, d(a, \partial \Omega)]$ where
\[
M(\varphi, a, r) := \sup_{\xi \in B(a, r)} \varphi(\xi).
\]
In particular, the limit
\[
\lim_{r \to 0^+} \frac{M(\varphi, a, r)}{\phi(r)}
\]
exists.

**Proof.** Without loss of generality we can assume that $a = 0 \in \Omega$.
Let $0 < r_1 < r_2 < d(0, \partial \Omega)$. We consider the two following $m$–sh functions on $\Omega$:
\[
u(\zeta) = \frac{\varphi(\zeta) - M(\varphi, 0, r_1)}{M(\varphi, 0, r_2) - M(\varphi, 0, r_1)} \quad \text{and} \quad v(\zeta) = \frac{\phi_m(|\zeta|) - \phi_m(r_1)}{\phi_m(r_2) - \phi_m(r_1)}.
\]
Therefore
\[
\lim_{r \to 0^+} \frac{M(\varphi, a, r)}{\phi(r)}
\]
• for every $z \in \mathbb{B}(r_2)$ one has $u(z) \leq 1$
• and for every $z \in \mathbb{B}(r_1)$ one has $u(z) \leq 0$.

Hence, 

$$\begin{align*}
\{ u(z) &\leq 1 = v(z) \text{ if } |z| = r_2 \\
u(z) &\leq 0 = v(z) \text{ if } |z| = r_1
\end{align*}$$

which gives $u(z) \leq v(z)$ for every $z \in \partial(\mathbb{B}(r_2) \setminus \mathbb{B}(r_1))$.

As $v$ is an $m$--sh function and $dd^c u^m \wedge \beta^{n-m} \geq dd^c v^m \wedge \beta^{n-m} = 0$, then thanks to the comparison principle, we have $u \leq v$ on $\mathbb{B}(r_2) \setminus \mathbb{B}(r_1)$.

Thus, for every $z \in \mathbb{B}(r_2) \setminus \mathbb{B}(r_1)$, 

$$\frac{\varphi(z) - M(\varphi, 0, r_1)}{M(\varphi, 0, r_2) - M(\varphi, 0, r_1)} \leq \frac{\phi_m(|z|)}{\phi_m(r_2) - \phi_m(r_1)}$$

Then, for any $r_1 < r < r_2$ we have 

$$\frac{M(\varphi, 0, r) - M(\varphi, 0, r_1)}{\phi_m(r) - \phi_m(r_1)} \leq \frac{M(\varphi, 0, r_2) - M(\varphi, 0, r_1)}{\phi_m(r_2) - \phi_m(r_1)}.$$ 

It follows that the function 

$$r \mapsto \frac{M(\varphi, 0, r) - M(\varphi, 0, r_1)}{\phi_m(r) - \phi_m(r_1)}$$

is increasing on $]r_1, d(0, \partial \Omega)[$. So we conclude the existence of the limit 

$$\ell_\varphi(0) := 2 \lim_{r \to 0^+} \frac{M(\varphi, 0, r)}{\phi_m(r)}.$$ 

\[\square\]

Claim that we have $\ell_\varphi(a) \leq \nu_\varphi(a)$ for any $a \in \Omega$ and we have equality in some particular cases of $m$--sh functions $\varphi$ on $\Omega$. Hence we can pose the following question:

**Problem 2.** Is it true that for any $m$--sh function $\varphi$ on $\Omega$ and any $a \in \Omega$, we have 

$$\nu_\varphi(a) = 2 \lim_{r \to 0^+} \frac{M(\varphi, a, r)}{\phi_m(r)}.$$ 

In the following, we give an estimate to Lelong number by the mass of $m$--sh function.

**Remark 1.** Let $a$ be a point of $\Omega$ and $\varphi$ be a function in $\mathcal{E}_m(\Omega)$. Then 

$$\nu_\varphi(a) \leq ((dd^c \varphi)^m \wedge \beta^{n-m}([a]))^{\frac{1}{m}}$$
Proof. Without loss of generality we can assume that \( a = 0 \) and \( \varphi \) belongs to \( F_m(\mathbb{B}) \) where \( \mathbb{B} = \mathbb{B}(r_0) \) is a ball. We have

\[
\lim_{s \to 0} \int_{B(0,s)} dd^c \varphi \wedge (dd^c \tilde{\phi}_m)^{m-1} \wedge \beta^{n-m} = \nu_\varphi(0).
\]

Using Lemma 1 it follows that for \( \varrho \geq 1 \),

\[
\nu_\varphi(0) \leq \int_{\mathbb{B}} - \max \left( \frac{\tilde{\phi}_m}{\varrho}, -1 \right) dd^c \varphi \wedge (dd^c \tilde{\phi}_m)^{m-1} \wedge \beta^{n-m} \leq \left[ \int_{\mathbb{B}} - \max \left( \frac{\tilde{\phi}_m}{\varrho}, -1 \right) dd^c \varphi^m \wedge \beta^{n-m} \right] \frac{1}{m} \times \left[ \int_{\mathbb{B}} - \max \left( \frac{\tilde{\phi}_m}{\varrho}, -1 \right) (dd^c \tilde{\phi}_m)^m \wedge \beta^{n-m} \right] \frac{m-1}{m}.
\]

Since \( \tilde{\phi}_m \) is the elementary solution of the complex Hessian equation we infer,

\[
\nu_\varphi(0) \leq \left[ \int_{\mathbb{B}} - \max \left( \frac{\tilde{\phi}_m}{\varrho}, -1 \right) (dd^c \varphi)^m \wedge \beta^{n-m} \right] \frac{1}{m}.
\]

Consequently, when \( \varrho \) goes to \( +\infty \), we obtain

\[
\nu_\varphi(0) \leq (dd^c \varphi)^m \wedge \beta^{n-m}(\{0\}) \frac{1}{m}.
\]

\( \square \)

5. Integrability Exponents of \( m \)--Subharmonic Functions

This part is an application of previous parts where we study the integrability exponents of \( m \)--sh functions. This problem was posed by Blocki [1] in 2005. We express this exponent in terms of volume of sub-level sets of the function, then we find a relationship between it and the Lelong number of the function. In particular we determine this exponent of a 1--sh function when its Lelong number is not equal to zero.

Definition 2. Let \( \varphi \) be an \( m \)--sh function on \( \Omega \) and \( K \) be a compact subset of \( \Omega \). The integrability exponent \( \iota_K(\varphi) \) of \( \varphi \) at \( K \) is defined as

\[
\iota_K(\varphi) = \sup \left\{ c > 0, |\varphi|^c \in L^1(\vartheta(K)) \right\}.
\]

For simplicity, if \( K = \{x\} \) then we denote \( \iota_{\{x\}}(\varphi) \) by \( \iota_x(\varphi) \).
Proposition 5. Let \( \varphi \) be an \( m \)-sh function on \( \Omega \). Then for every compact subset \( K \) of \( \Omega \) we have

\[
i_K(\varphi) = \inf_{x \in K} i_x(\varphi).
\]

Proof. For \( x \in K \), we have

\[
\{ c > 0, |\varphi|^c \in L^1(\partial(K)) \} \subset \{ c > 0, |\varphi|^c \in L^1(\partial(x)) \}.
\]

Therefore, \( i_K(\varphi) \leq i_x(\varphi) \). It follows that

\[
i_K(\varphi) \leq \inf_{x \in K} i_x(\varphi).
\]

Conversely, let \( a < \inf_{x \in K} i_x(\varphi) \). For any \( x \in K \), let \( U_x \) be a neighborhood of \( x \) such that \( |\varphi|^a \in L^1(U_x) \). As \( K \) is compact, then there are \( x_1, \ldots, x_p \in K \) such that \( K \subset \bigcup_{j=1}^p U_{x_j} = U \). According to Borel-Lebesgue Lemma, we have \( |\varphi|^a \in L^1(U) \). Hence \( a \leq i_K(\varphi) \) so we conclude that we have \( \inf_{x \in K} i_x(\varphi) \leq i_K(\varphi) \).

Some quite questions related to the integrability exponents of \( m \)-sh function are still open, among this we can state the following:

Problem 3. Let \( \varphi \) be an \( m \)-sh function on \( \Omega \).

1. Are the maps \( a \mapsto i_a(\varphi) \) and \( \varphi \mapsto i_a(\varphi) \) lower semi-continuous respectively on \( \Omega \) and on the set of locally integrable functions?

2. Let \( I_\varphi := \{ c > 0; |\varphi|^c \in L^1(\partial(z)) \} \). Is \( I_\varphi \) an open set? (openness conjecture).

If the map \( a \mapsto i_a(\varphi) \) is lower semi continuous on \( \Omega \) then for every compact subset \( K \) of \( \Omega \) there exists \( a \in K \) such that \( i_K(\varphi) = i_a(\varphi) \).

Lemma 5. Let \( \varphi, \psi \) be two \( m \)-sh functions on \( \Omega \) and \( K \) be a compact subset of \( \Omega \). If \( \varphi \leq \psi \) in a neighborhood of \( K \) then

\[
i_K(\varphi) \leq i_K(\psi).
\]

Proof. Without loss of generality we can assume that \( \varphi \leq \psi \leq 0 \) on \( \partial(K) \). It follows that,

\[
\{ c > 0, |\varphi|^c \in L^1(\partial(K)) \} \subset \{ c > 0, |\psi|^c \in L^1(\partial(K)) \}
\]

and the result holds.

Lemma 6. Let \( K \) be a compact subset of \( \Omega \) and \( \varphi \) be an \( m \)-sh function on \( \Omega \), negative on a neighborhood of \( K \). For any \( t \in \mathbb{R} \) we set \( A_\varphi(t) = \{ z \in \Omega; \varphi(z) \leq t \} \). Then for every positive number \( 0 < \alpha < i_K(\varphi) \) there exists \( C_\alpha > 0 \) such that for any \( t < 0 \) we have

\[
V(K \cap A_\varphi(t)) \leq \frac{C_\alpha}{|t|^\alpha}.
\]
Proof. Let $0 < \alpha < i_K(\varphi)$ and $t < 0$. If $z \in A_\varphi(t) \cap K$ then $\frac{\varphi(z)}{t} \geq 1$.

It follows that,

$$V(A_\varphi(t) \cap K) \leq \int_{A_\varphi(t) \cap K} \left| \frac{\varphi(z)}{t} \right| \alpha dV(z) \leq \frac{1}{|t|^{\alpha}} \int_K |\varphi(z)|^\alpha dV(z)$$

as $\alpha < i_K(\varphi)$ then

$$C_\alpha := \int_K |\varphi(z)|^\alpha dV(z) < +\infty.$$

□

Theorem 4. Let $K$ be a compact subset of $\Omega$ and $\varphi$ be an $m$--sh function on $\Omega$, negative on a neighborhood of $K$. Then

$$i_K(\varphi) = \sup \left\{ \alpha > 0; \exists C_\alpha > 0, \forall t < 0 \ V(A_\varphi(t) \cap K) \leq \frac{C_\alpha}{|t|^{\alpha}} \right\}$$

A similar result for the complex singularity exponents of plurisubharmonic functions was proved by Kiselman [6].

Proof. In order to simplify the notations, we set

$$\gamma = \sup \left\{ \alpha > 0; \exists C_\alpha > 0, \forall t < 0 \ V(A_\varphi(t) \cap K) \leq \frac{C_\alpha}{|t|^{\alpha}} \right\}.$$

Thanks to Lemma [4] for $\alpha < i_K(\varphi)$ there is $C_\alpha > 0$ such that for any $t < 0$ one has $V(A_\varphi(t) \cap K) \leq \frac{C_\alpha}{|t|^{\alpha}}$, which means that $\gamma \geq i_K(\varphi)$.

In the other hand, let $0 < \alpha_0 < \gamma$, then there exists $C_{\alpha_0} > 0$ such that for any $t < 0$,

$$V(A_\varphi(t) \cap K) \leq \frac{C_{\alpha_0}}{|t|^{\alpha_0}}.$$

It follows that for any $0 < \alpha < \alpha_0$ we have

$$\int_K |\varphi(z)|^\alpha dV(z) = \int_{\mathbb{R}_+} V(A_{|\varphi|^\alpha}(s) \cap K) ds$$

$$= \int_{\mathbb{R}_+} V(A_{\varphi}(-s^{\frac{1}{\alpha}}) \cap K) ds$$

$$\leq \int_0^s V(A_{\varphi}(-s^{\frac{1}{\alpha}}) \cap K) + \int_1^{+\infty} \frac{C_{\alpha_0}}{s^{\alpha_0}} ds$$

$$\leq V(K) + \int_1^{+\infty} \frac{C_{\alpha_0}}{s^{\alpha_0}} ds < +\infty.$$

Thus, $\alpha \leq i_K(\varphi)$ so the result holds. □

The main result of this part is the following theorem:
Theorem 5. Let $\varphi$ be an $m-$sh function on $\Omega$ and $x \in \Omega$. Then

$$\iota_x(\varphi) \geq \frac{n}{n - m}.$$  

Moreover, if $\nu_\varphi(x) > 0$ then

$$\iota_x(\varphi) \leq \frac{nm}{n - m}.$$  

The first part of this result was proved by Blocki [1] where he has conjectured that for any $m-$sh function, the integrability exponent is greater than or equal to $\frac{nm}{n - m}$. If the conjecture is true then we conclude that we have equality in the second statement.

Claim that this result gives that for any $1-$sh function $\varphi$ with non vanishing Lelong number at a point $x$, the integrability exponent of $\varphi$ at $x$ is $\iota_x(\varphi) = \frac{n}{n - 1}$.

Proof. Without loss of generality we can assume that $x = 0 \in \Omega$ and $\varphi \leq 0$ in a neighborhood of 0.

(1) Let $T = dd^c \varphi$ and $\chi$ be a cut-off function with support in a small ball $B(r)$, equal to 1 on $B(\frac{r}{2})$. As $(dd^c \tilde{\phi}_m)^m \wedge \beta^{n-m} = \delta_0$, for $z \in B(\frac{r}{2})$

$$\varphi(z) = \int_{B(r)} \chi(\xi) \varphi(\xi) (dd^c \tilde{\phi}_m(z - \xi))^m \wedge \beta^{n-m}(\xi)$$

$$= \int_{B(r)} \chi(\xi) \varphi(\xi) dd^c \tilde{\phi}_m(z - \xi) \wedge (dd^c \tilde{\phi}_m(z - \xi))^{m-1} \wedge \beta^{n-m}(\xi)$$

$$= \int_{B(r)} dd^c (\chi(\xi) \varphi(\xi) \tilde{\phi}_m(z - \xi) \wedge (dd^c \tilde{\phi}_m(z - \xi))^{m-1} \wedge \beta^{n-m}(\xi)$$

$$= \int_{B(r)} \chi(\xi) dd^c \varphi(\xi) \tilde{\phi}_m(z - \xi) \wedge (dd^c \tilde{\phi}_m(z - \xi))^{m-1} \wedge \beta^{n-m}(\xi)$$

$$\quad + \int_{B(r)} dd^c \chi(\xi) \varphi(\xi) \tilde{\phi}_m(z - \xi) \wedge (dd^c \tilde{\phi}_m(z - \xi))^{m-1} \wedge \beta^{n-m}(\xi)$$

$$\quad + \int_{B(r)} d\chi(\xi) \wedge d^c \varphi(\xi) \tilde{\phi}_m(z - \xi) \wedge (dd^c \tilde{\phi}_m(z - \xi))^{m-1} \wedge \beta^{n-m}(\xi)$$

$$\quad - \int_{B(r)} d^c \chi(\xi) \wedge d\varphi(\xi) \tilde{\phi}_m(z - \xi) \wedge (dd^c \tilde{\phi}_m(z - \xi))^{m-1} \wedge \beta^{n-m}(\xi)$$

$$= \int_{B(r)} \chi(\xi) dd^c \varphi(\xi) \tilde{\phi}_m(z - \xi) \wedge (dd^c \tilde{\phi}_m(z - \xi))^{m-1} \wedge \beta^{n-m}(\xi) + R(z).$$

Where $R$ is a $C^\infty$ function on $B(r)$. Set

$$J(z) = \int_{B(r)} \chi(\xi) T(\xi) \wedge (dd^c \tilde{\phi}_m(z - \xi))^{m-1} \wedge \beta^{n-m}(\xi).$$
As
\[ \nu_T(0, r) = \int_{B(r)} T(\xi) \wedge (dd^c \tilde{\phi}_m(\xi))^{m-1} \wedge \beta^{n-m}(\xi) \]
then for \(0 < \delta < 1\) and \(r\) small enough, one has
\[ \nu_T(0, \frac{r}{2}) \leq J(0) \leq \nu_T(0, r) \leq \nu_T(0) + \delta. \]
By continuity, there exists \(0 < \epsilon < \frac{r}{2}\) such that for every \(z \in B(\epsilon)\) one has
\[ (1 - \delta) \nu_T(0, \frac{r}{2}) < J(z) \leq \nu_T(0) + 2\delta. \]

Fix \(z \in B(\epsilon)\) and let \(\mu_z\) be the probability measure defined on \(B(r)\) by
\[ d\mu_z(\xi) = J^{-1}(\xi)\chi(\xi)T(\xi) \wedge (dd^c \tilde{\phi}_m(z - \xi))^{m-1} \wedge \beta^{n-m}(\xi) \]
It follows that
\[ -\varphi(z) = \int_{B(r)} J(z)(-\tilde{\phi}_m(z - \xi))d\mu_z(\xi) - R(z) \]
As \(R\) is \(C^\infty\) on \(B(r)\) then it is bounded on \(B(\epsilon)\).
Hence, for any \(p \geq 1\) we have
\[ (-\varphi(z))^p \leq \left( \int_{B(r)} J(z)(-\tilde{\phi}_m(z - \xi))d\mu_z(\xi) + C \right)^p \leq C \left( \frac{1}{C} \int_{B(r)} J(z)\tilde{\phi}_m(z - \xi)d\mu_z(\xi) + 1 \right)^p. \]
It is easy to show that
\[(5.1) \ (h + 1)^p \leq h^p + \alpha_1 h^{p-1} + \ldots + \alpha_{[p]-1} h^{p-[p]+1} + \alpha(1 + h), \quad \forall h \geq 0 \]
for some positive constantes \(\alpha_j, \alpha\) depending on \(p\) where \([p]\) is the integer part of \(p\). If we set
\[ h(z) = \frac{-1}{C} \int_{B(r)} J(z)\tilde{\phi}_m(z - \xi)d\mu_z(\xi) \]
then
\[ (-\varphi(z))^p \leq C \left( \sum_{j=0}^{[p]-1} \alpha_j h^{p-j}(z) + \alpha(1 + h(z)) \right). \]
Thus, to prove that \((-\varphi)^p \in L^1(\mathbb{B}(\epsilon))\), it suffices to prove that \(h^s \in L^1(\mathbb{B}(\epsilon))\) for every \(s = p, \ldots, p - [p] + 1\).
Now, applying Jensen’s convexity inequality with the probability measure $\mu_z$, we obtain

$$(-h(z))^s = \frac{1}{C^s} \left( \int_{B(r)} J(z)(-\bar{\phi}_m(z - \xi)) d\mu_z(\xi) \right)^s \leq \frac{1}{C^s} \int_{B(r)} J^s(z)(-\bar{\phi}_m(z - \xi))^s d\mu_z(\xi) \leq a_s \int_{B(r)} (-\bar{\phi}_m(z - \xi))^s \chi(\xi) T(\xi) \wedge (dd^{c}\bar{\phi}_m(z - \xi))^{m-1} \wedge \beta^{n-m}(\xi) \leq a_s \int_{B(r)} (-\bar{\phi}_m(z - \xi))^s T(\xi) |z - \xi|^{-2a_m/(m-1)} \wedge \beta^{n-1}(\xi)$$

where we use $a_s := \frac{(\nu_T(0)+2\delta)^{s-1}}{C^s}$ and the fact that $(dd^{c}\bar{\phi}_m(z - \xi))^{m-1} \wedge \beta^{n-m} \leq |z - \xi|^{-2a_m/(m-1)} \beta^{n-1}(\xi)$. We conclude that for every $z \in B(\varepsilon)$, we have

$$(-h(z))^s \leq a_s \int_{B(r)} \frac{1}{|z - \xi|^{\left(\frac{2n}{m} - 2\right)s + \frac{2n}{m}(m-1)}} T(\xi) \wedge \beta^{n-1}(\xi) \leq a_s \int_{B(r)} \frac{1}{|z - \xi|^{\left(\frac{2n}{m} - 2\right)s + \frac{2n}{m}(m-1)}} d\sigma_T(\xi).$$

Hence

$$\int_{B(\varepsilon)} (-h(z))^p dV(z) \leq a_s \int_{B(\varepsilon)} \int_{B(r)} \frac{1}{|z - \xi|^{\left(\frac{2n}{m} - 2\right)s + \frac{2n}{m}(m-1)}} d\sigma_T(\xi) dV(z)$$

The Fubini theorem implies

$$\int_{B(\varepsilon)} (-h(z))^s dV(z) \leq a_s \int_{B(\varepsilon)} \int_{B(r)} \frac{dV(z)}{|z - \xi|^{\left(\frac{2n}{m} - 2\right)s + \frac{2n}{m}(m-1)}} d\sigma_T(\xi)$$

is finite. Indeed, for $\xi \in B(\varepsilon)$ and $\eta < \varepsilon - |\xi|$ one has

$$\int_{B(\varepsilon)} \frac{dV(z)}{|z - \xi|^{\left(\frac{2n}{m} - 2\right)s + \frac{2n}{m}(m-1)}} = \int_{B(\varepsilon) \cap B(\xi, \eta)} \frac{dV(z)}{|z - \xi|^{\left(\frac{2n}{m} - 2\right)s + \frac{2n}{m}(m-1)}} + \int_{B(\varepsilon) \cap B^c(\xi, \eta)} \frac{dV(z)}{|z - \xi|^{\left(\frac{2n}{m} - 2\right)s + \frac{2n}{m}(m-1)}} \leq C \int_0^\eta t^{-\left(\frac{2n}{m} - 2\right)s - \frac{2n}{m}(m-1) + 2n-1} dt + a$$

the last integral is finite for any $s < \frac{n}{n-m}$.

(2) Assume that $\nu_\varphi(0) > 0$. Using the subharmonicity of $\varphi$ and the fact that $\varphi$ is negative on $B(3r) \subset \Omega$, it is not hard to see that for any $z \in B(r) \setminus \{0\}$

$$\varphi(z) \leq \Lambda(\varphi, z, 2|z|) \leq \frac{1}{4^n} \Lambda(\varphi, 0, |z|).$$
Hence, for every $0 < p < v_0(\varphi)$ we have

$$(-\varphi(z))^p \geq \left(-\frac{1}{4^n} \Lambda(\varphi, 0, |z|)\right)^p.$$ 

Thanks to Theorem 3 we have

$$\frac{2}{n} \frac{n(n - m + 1) \Lambda(\varphi, 0, s)}{\phi_m(s)}$$

decreases to $\nu_\varphi(0)$ as $s \searrow 0$.

It follows that

$$\int_{B(r)} (-\varphi(z))^p dV(z) \geq \frac{1}{4^{mp}} \int_{B(r)} (-\Lambda(\varphi, 0, |z|^n))^p dV(z) \geq \left(\frac{n}{2 \times 4^n(n + 1 - \frac{n}{m})} \right)^p \frac{2\pi^n}{(n-1)!} \int_{0}^{r} \left( 2(n + 1 - \frac{n}{m}) \Lambda(\varphi, 0, t) \right)^p \frac{2\pi^n}{n\phi_m(t)} t^{2n-1-2p(\frac{n}{m}-1)} dt \geq \left(\frac{n}{2 \times 4^n(n + 1 - \frac{n}{m})} \right)^p \frac{2\pi^n}{(n-1)!} \int_{0}^{r} t^{2n-1-2p(\frac{n}{m}-1)} dt$$

As $0 < p < v_0(\varphi)$, then the first integral is finite; hence the last one is too which gives $p < \frac{nm}{n-m}$. \hfill \Box

**Remark 2.** If the Lelong number of an $m-$sh function at a point $a \in \Omega$ vanishes then the integrability exponent of this function at this point can be greater than $\frac{nm}{n-m}$. For example one can consider the function

$$\psi(z) = \frac{1}{|z'|^{2(\frac{n-1}{m}-1)}}$$

where $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$ and $2 \leq m \leq n - 2$. It is simple to see that $\psi$ is an $m-$sh function with $\nu_\psi(0) = 0$ and the integrability exponent of $\psi$ at $0$ is equal to $\frac{m(n-1)}{n-1-m}$ which is greater than $\frac{nm}{n-m}$.

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E-mail address: amelbenali3010@gmail.com

Laboratory of Mathematics and Applications, Faculty of Sciences of Gabès, University of Gabès, 6072 Gabès Tunisia.

E-mail address: noureddine.ghiloufi@fsg.rnu.tn

Department of Mathematics, College of Science, P.O. box 400 King Faisal University, Al-Ahsaa, 31982, Kingdom of Saudi Arabia.