A NEW CLASS OF EXAMPLES OF GROUP-VALUED MOMENT MAPS

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Abstract. The purpose of this paper is to construct new examples of group-valued moment maps. As the main tool for construction of such examples we use quasi-symplectic implosion which was introduced in [HJS06]. More precisely we show that there are certain strata of \(D\text{Sp}(n)_{\text{impl}}\), the universal implored space, where it is singular but whose closure is a smooth quasi-Hamiltonian \(\text{Sp}(n) \times T\) space.

1. Introduction

The notion of group-valued moment map, which was introduced by Alekseev, Malkin and Meinrenken [AMM98], is a natural generalization of classical Hamiltonian spaces. In contrast to their classical counterpart, the moment map takes values in a Lie group instead of the dual of the Lie algebra. Quasi-Hamiltonian manifolds and their moment maps share many of the features of the Hamiltonian ones, such as reduction, cross-section and implosion.

The motivation of [AMM98] for developing the theory of group-valued moment map came from one particularly important result. They show that the moduli space \(M(\Sigma)\) of flat connection on a closed Riemann surface \(\Sigma\) of genus \(k\) is a quasi-Hamiltonian quotient of \(G^{2k}\), which possesses natural quasi-Hamiltonian \(G\)-structure. Therefore it is a symplectic manifold, result earlier obtained by Atiyah and Bott. They go further generalizing it to the case \(M(\Sigma, C)\), the moduli space of flat connection on punctured Riemann surface with fixed conjugacy classes representing homotopy classes of loops around punctures.

By imitating symplectic implosion [GJS02], J. Hurtubise, L. Jeffrey, and R. Sjamaar introduced the notion of group-valued implosion [HJS06]. It is somewhat similar to quasi-symplectic reduction, but instead of quotienting by the whole stabilizer subgroup it reduces it by its certain subgroup. While an imploded cross-section is almost always singular, the quasi-symplectic quotients are not. For example, using result of [GJS02], one can show the imploded cross-section of \(D(G)\) is singular unless the commutator subgroup of \(G\) is a product of copies of \(\text{SU}(2)\). Usually they are not even orbifold unless their universal cover of \([G, G]\) is a product of copies of \(\text{SU}(2)\). However, like in case of singular quotients [SL91], using imploded cross-section theorem one can show that imploded spaces partition into symplectic manifolds.

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It was observed in [HJS06] that there are certain strata of $D(G)_{\text{impl}}$ where it is singular, but whose closure is smooth. This observation lead them to construct new class of examples of quasi-Hamiltonian manifolds. In particular when $G$ is $A$-type i.e. $G = \text{SU}(n)$, there is a one dimensional face of the alcove whose corresponding stratum has a smooth closure diffeomorphic to $S^{2n}$. Motivated by this example, we study the implosion for type $C$ groups, i.e. $G = \text{Sp}(n)$ unitary quaternionic group. We show that there is a certain stratum of $D(G)_{\text{impl}}$ which has a smooth closure diffeomorphic to $\mathbb{HP}^n$. On the other hand, it also gives new examples of multiplicity-free quasi-Hamiltonian spaces with non-effective $G \times T$ action.

The organization of this paper is as follows. In section 2 we recall the definition, basic properties and related examples of group-valued moment map. In the section 3 we review the definition and basic properties of group-valued implosion. In this section we also give an example of “Spinning Sphere” constructed in [HJS06], as motivating example of our own construction. In the section 4, we give a construction of quasi-Hamiltonian structure on $\mathbb{HP}^n$ using an implosion.

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2. Quasi-Hamiltonian Manifolds

Let $G$ be compact, connected Lie group with Lie algebra $\mathfrak{g}$. Given $G$-manifold $M$, there is induced infinitesimal Lie algebra action:

\begin{equation}
\xi_M(x) = \frac{d}{dt}|_{t=0} \exp(-t\xi).x \quad \text{for} \quad \xi \in \mathfrak{g}.
\end{equation}

Recall that a Hamiltonian $G$-manifold is a symplectic $G$-manifold $(M, \omega)$ with an equivariant map, called moment map, $\Phi : M \to \mathfrak{g}^*$ satisfying relation

\begin{equation}
\iota(\xi_M)\omega = d\langle \Phi, \xi \rangle.
\end{equation}

Imitating the Hamiltonian case, [AMM98] introduced the notion of so called quasi-Hamiltonian $G$-manifolds. Recall that the Maurer-Cartan forms $\theta_L, \theta_R \in \Omega^1(G, \mathfrak{g})$ are defined by $\theta_{L,g}(L(g) \cdot \xi) = \xi$ and $\theta_{R,g}(R(g) \cdot \xi) = \xi$ for $\xi \in \mathfrak{g}$, where $L(g)$ denotes left multiplication and $R(g)$ right multiplication by $g$. Let $(\cdot, \cdot)$ be some choice of inner product on $\mathfrak{g}$. Then there is a closed bi-invariant three-form on $G$

\begin{equation}
\chi = \frac{1}{12}\left(\theta_L, [\theta_L, \theta_L]\right) - \frac{1}{12}\left(\theta_R, [\theta_R, \theta_R]\right).
\end{equation}

**Definition 1.** A quasi-Hamiltonian $G$-manifold is a smooth $G$-manifold $M$ equipped with with $G$-invariant two-form $\omega$ and $G$-equivariant map $\Phi : M \to G$, called group-valued moment map, such that the following properties hold:

(i) $d \omega = -\Phi^* \chi$

(ii) $\ker \omega_x = \{\xi_M | \xi \in \text{Ker}(Ad_{\Phi(x)} + 1)\}$ for all $x \in M$
(iii) \( \iota(\xi_M)\omega = \frac{1}{2}\Phi^*(\theta_L + \theta_R, \xi) \)

Basic examples of quasi-Hamiltonian manifolds are provided by conjugacy classes and double \( D(G) \). One can think of them as analogs of coadjoint orbits and cotangent bundle respectively.

**Conjugacy classes.** Let \( C \) be a conjugacy class in \( G \). Define an invariant two-form

\[
\omega_g(v_\xi, v_\eta) = \frac{1}{2}((\eta, Ad_g \xi) - (\xi, Ad_g \eta)) \quad \text{for} \quad g \in C,
\]

where \( v_\xi \) and \( v_\eta \) are fundamental vector fields induced by conjugation action on \( C \). Then \( (C, \omega) \) with the moment map \( \Phi : C \hookrightarrow G \) is a quasi-Hamiltonian \( G \)-space. Moreover \( \omega \) is uniquely determined by \( \Phi \).

**Double \( D(G) \).** It has been remarked in [AMM98], the \( D(G) \) plays the same role in the category of quasi-Hamiltonian spaces, as \( T^*G \) does in Hamiltonian one. As the space \( D(G) \) is defined:

\[
D(G) := G \times G.
\]

The \( G \times G \) action on \( D(G) \) is given by:

\[
(g_1, g_2)(u, v) = (g_1ug_2^{-1}, Ad_{g_2}v).
\]

Define a moment map \( \Phi : D(G) \longrightarrow G \times G \) by \( \Phi = \Phi_1 \times \Phi_2 \) where

\[
\Phi_1(u, v) = Ad_uv^{-1}, \quad \Phi_2(u, v) = v
\]

and two-form

\[
\omega = -\frac{1}{2}(Ad_u \Phi_1^*\theta_L, \Phi_2^*\theta_L) - \frac{1}{2}(u^*\theta_L, v^*(\theta_L + \theta_R)).
\]

The following statement was shown in [AMM98, Proposition 3.2].

**Proposition 1.** The \( (D(G), \Phi, \omega) \) is a quasi-Hamiltonian \( G \times G \)-manifold.

Large class of examples of quasi-Hamiltonian manifolds are constructed in [AMM98] by two operations called “Fusion” and “Exponentiation”.

**Fusion.** Unlike in Hamiltonian case, group-valued moment maps does not behave well under restriction to subgroups or taking products. But under slight modification one can still define these notions in the category of quasi-Hamiltonian spaces.

**Theorem 1** (Internal Fusion). Let \( (M, \omega, \Phi) \) be a quasi-Hamiltonian \( G \times G \times H \)-manifold, with moment map \( \Phi = \Phi_1 \times \Phi_2 \times \Phi_3 : M \rightarrow G \times G \times H \). Let \( G \times H \) act via embedding \( (g, h) \rightarrow (g, g, h) \) Then \( M \) equipped with the two-form \( \omega + \frac{1}{2}(\Phi_1^*\theta_L, \Phi_2^*\theta_R) \) and the moment map \( \Phi_1, \Phi_2 \times \Phi_3 : M \rightarrow G \times H \) is a quasi-Hamiltonian \( G \times H \)-manifold.

The most important class of examples produced by this operation is fusion product \( M = M_1 \circledast M_2 \), where \( M_1 \) is quasi-Hamiltonian \( G \)-manifold and \( M_2 \) quasi-Hamiltonian \( G \times H \)-manifold. The underlying space for \( M \) is defined as a Cartesian product \( M_1 \times M_2 \) while the quasi-Hamiltonian structure is obtained by fusing the two copies of \( G \) in \( G \times G \times H \).
Exponentiation and linearization. Let \((M, \omega_0, \Phi_0)\) be a Hamiltonian \(G\)-manifold. In this section we will see how one can get from Hamiltonian manifold a quasi-Hamiltonian one and vice-versa. First using the inner product on \(g\) one can regard \(\Phi_0\) as a map into \(g\). Then composing \(\Phi_0\) with \(\exp : g \to G\) one gets a map \(\Phi : M \to G\). Then according to [AMM98, Proposition 3.4] by slightly changing 2-form \(\omega = \omega_0 + \Phi_0^* \varpi\), the triple \((M, \omega, \Phi)\) defines a quasi-Hamiltonian \(G\)-manifold.

The “inverse functor” called linearization, is constructed in following way. Let \((M, \omega, \Phi)\) be a q-Hamiltonian \(G\)-manifold. Moreover, assume that there exists an \(Ad\)-invariant open \(U\) in \(g\) such that \(\exp : U \to G\) is a diffeomorphism onto an open subset containing \(\Phi(M)\), and let \(\log : \exp U \to U\) be its inverse. Then the linearization is the Hamiltonian \(G\)-manifold \((M, \omega_0, \Phi_0)\), where \(\Phi_0 = \log \circ \Phi\) and \(\omega_0 = \omega - \Phi_0^* \varpi\).

Quasi-symplectic reduction. One other important feature of quasi-Hamiltonian spaces is reduction.

Let \((M, \omega, \Phi)\) be a quasi-Hamiltonian \(G\)-manifold such that \(G\) is the product \(G_1 \times G_2\) where \(G_2\) torus. Let \(\Phi = (\Phi_1, \Phi_2)\) be corresponding components of moment map \(\Phi\). We want to reduce the space with respect to the first factor. Suppose that \(g \in G_1\) be regular value so that \(\Phi_1^{-1}(g)\) is a smooth manifold. The centralizer \((G_1)_g\) acts locally freely on the submanifold \(\Phi_1^{-1}(g)\). Then the quasi-symplectic quotient at \(g\) is defined as topological space:

\[
M/\!/_{G_1} = \Phi_1^{-1}(g)/(G_1)_g.
\]

In good cases this quotient is a symplectic orbifold. Under above assumptions:

**Theorem 2.** ([AMM98, Theorem 5.1]) The restriction of \(\omega\) to \(\Phi_1^{-1}(g)\) is closed and \((G_1)_g\)-basic. The form \(\omega_g\) on the orbifold \(M/\!/_{G_1}\) induced by \(\omega\) is nondegenerate. The map \(M/\!/_{G_1} \to G_2\) induced by \(\Phi_2\) is a moment map for the induced \(G_2\)-action on \(M/\!/_{G_1}\).

In case when \(G_2\) is nonabelian \(M/\!/_{G_1}\) is not symplectic, but a quasi-Hamiltonian \(G_2\)-orbifold.

In singular case the quotient stratifies into symplectic manifolds according to orbit type. Let \(g\) be a arbitrary element of \(G_1\). For each subgroup \(H\) define a \((G_1)_g\)-invariant submanifold \(M_{(H)}\) consisting of all points such that the stabilizer \((G_1)_g \cap (G_1)_x\) is conjugate to \(H\). Put \(Z = \Phi^{-1}(g)\) and \(Z_{(H)} = Z \cap M_{(H)}\). Let \(Z_i\) be the collection of connected components of \(Z_{(H)}\), where \(H\) ranges over conjugacy classes of \((G_1)_g\). Then we have decomposition:

\[
M/\!/_{G_1} = \bigsqcup_{i \in I} Z_i/(G_1)_g.
\]

**Theorem 3.** ([HJS06]) The decomposition \((2.9)\) is a locally normally trivial stratification of \(M/\!/_{G_1}\) into symplectic submanifolds. Moreover, the stratification is \(G_2\)-invariant and the continuous map \(\Phi_2 : M/\!/_{G_1} \to G_2\) induced by \(\Phi_2\) restricts to a moment map for the \(G_2\) action on each stratum.
3. Group-valued imploded cross-section.

Let $G$ be a simply connected compact Lie group with maximal torus $T$. Recall that the symplectic implosion is an “abelianization functor”, which transforms a Hamiltonian $G$-manifold into a Hamiltonian $T$-space preserving some of properties of the manifold, but in the expense of producing singularities \cite{GJS02}. However, the singularities are not arbitrary in the sense that it “stratifies” into symplectic submanifolds in such a way that $T$ action preserves the stratification.

Now let $(M, \omega, \Phi)$ be a quasi-Hamiltonian $G$-space. In \cite{AMM98}, it was shown that like in Hamiltonian case one can prove a convexity theorem. But the moment map image instead of a Weyl chamber, one have to consider in an alcove. Here the assumption of simply connectedness of the group is crucial, since otherwise the description of a space of conjugacy classes is quite complicated.

Let $C^\vee$ be the chamber in $\mathfrak{t}$ dual to $C$ and let $A$ be the unique (open) alcove contained in $C^\vee$ such that $0 \in \mathfrak{A}$. Using the exponential map one can identify $\overline{A}$ with space of conjugacy classes $T/W \cong G/AdG$, where $W$ is the corresponding Weyl group. Let denote by $G_\sigma$ the centralizer of $g$ in $G$. For points $m_1, m_2 \in \Phi^{-1}(\exp \mathfrak{A})$ define $m_1 \sim m_2$ if $m_2 = gm_1$ for some $g \in [G_{\Phi(m_1)}, G_{\Phi(m_1)}]$. One can check that $\sim$ is indeed equivalence relation.

**Definition 2.** The imploded cross-section of $M$ is the quotient space $M_{\text{impl}} = \Phi^{-1}(\exp \mathfrak{A})/\sim$, equipped with the quotient topology. The imploded moment map $\Phi_{\text{impl}}$ is the continuous map $M_{\text{impl}} \to T$ induced by $\Phi$.

$M_{\text{impl}}$ has many nice properties that smooth manifolds posses. It is Hausdorff, locally compact and second countable. The action of $T$ preserves $\Phi^{-1}(\exp \mathfrak{A})$ and descends to a continuous action on $M_{\text{impl}}$.

We have decomposition of $M_{\text{impl}}$ into orbit spaces:

\begin{equation}
M_{\text{impl}} = \prod_{\sigma \leq A} \Phi^{-1}(\exp \sigma)/[G_\sigma, G_\sigma],
\end{equation}

where $\sigma$ ranges over the faces of alcove $A$ and $K_\sigma$ is the centralizer of $\exp \sigma$. Let us denote piece $\Phi^{-1}(\exp \sigma)/[G_\sigma, G_\sigma]$ by $X_\sigma$. Using quasi-Hamiltonian cross-section \cite[Theorem 3.4]{HJS06} each $X_\sigma$ stratifies into symplectic manifolds. Now let $\{X_i | i \in I\}$ be the collection of the all strata of the all pieces $X_\sigma$ where $\sigma$ ranges over the faces of alcove. Then imploded cross-section $M_{\text{impl}}$ the disjoint union:

\begin{equation}
M_{\text{impl}} = \bigsqcup_{i \in I} X_i
\end{equation}

such that each piece $X_i$ is symplectic manifold:

**Theorem 4.** \cite{HJS06} The decomposition \eqref{eq:impl_cross_section_decomposition} of the imploded cross-section is a locally finite partition into locally closed subspaces, each of which is a symplectic manifold. There is a unique open stratum, which is dense, in $M_{\text{impl}}$ and symplectomorphic to the principal
cross section of $M$. The action of the maximal torus $T$ on $M_{\text{impl}}$ preserves the decomposition and the imploded moment map $\Phi_{\text{impl}} : M_{\text{impl}} \to T$ restricts to a moment map for the $T$-action on each stratum.

Therefore we call $M_{\text{impl}}$ a stratified quasi-Hamiltonian $T$-space.

**Imploded cross-section of the double.** In the example of $q$-Hamiltonian manifolds we have seen that $D(G) := G \times G$ possesses quasi-Hamiltonian $G \times G$-structure.

Let $M$ be an arbitrary quasi-Hamiltonian $G$-space. Fusing it with $D(G)$ one obtains a $q$-Hamiltonian $G \times G$-manifold $M \oplus D(G)$. Now define $j : M \to M \oplus D(G)$ by $j(m) = (m, 1, \Phi(m))$. Then one of the main results of [HJS06] states:

**Theorem 5** (universality of imploded double). Let $M$ be a quasi-Hamiltonian $G$-manifold. The map $j$ induces a homeomorphism

\begin{equation}
(3.3) \quad j_{\text{impl}} : M_{\text{impl}} \xrightarrow{\approx} (M \oplus D(G)_{\text{impl}})//G
\end{equation}

which maps strata to strata and whose restriction to each stratum is an isomorphism of quasi-Hamiltonian $T$-manifolds.

**A smoothness criterion and quasi-Hamiltonian structure on $S^{2n}$.** We have seen that in previous section that in order to construct implosion of a given manifold, it suffices to know the implosion of double of corresponding Lie group. The implosion of double is singular space, however the singularities on certain strata are removable. In order to show that one has to use the explicit correspondence between $D(G)$ and $T^*G$.

Identify $g$ with $g^*$ using bi-invariant inner product on $g$. Trivializing $T^*G$ in a left-invariant manner, define $G \times G$-equivariant map $H = \text{id} \times \exp : T^*G \to D(G)$. Let $(T^*G, \omega_0, \Psi_0)$ be a Hamiltonian $G \times G$ manifold, where $\omega_0$ is the canonical symplectic form on cotangent bundle and a moment map $\Psi_0(g, \lambda) = (-Ad_g \lambda, \lambda)$. Let $O$ be the set of all $\xi \in t$ with $(2\pi i)^{-1} \alpha(\xi) < 1$ for all positive roots $\alpha$ and $U = (AdG)O$.

**Lemma 1.** ([HJS06]) The triple $(T^*G, H^*\omega, H^*\Psi)$ is the exponentiation of $(T^*G, \omega_0, \Psi_0)$. In particular, $G \times U$ is a quasi-Hamiltonian $G \times G$-manifold.

Now using local diffeomorphism given by $H$ and that of result of [GJS02] we have:

**Theorem 6** (Smoothness criterion). Let $\sigma$ be a face of $\mathcal{A}$ satisfying $[G_\sigma, G_\sigma] \cong \text{SU}(2)^k$ (resp. $[g_\sigma, g_\sigma] \cong \text{su}(2)^k$) for some $k \geq 0$ and possessing a vertex $\xi$ such that $\exp \xi$ is central. Then $D(G)_{\text{impl}}$ is a smooth quasi-Hamiltonian $G \times T$-manifold (resp. orbifold) in a neighborhood of the stratum corresponding to $\sigma$.

The partial converse of this result is also true. Suppose that $\sigma$ contains a vertex $\xi$ such that $\exp \xi$ is central and $D(G)_{\text{impl}}$ is smooth in a neighborhood of the corresponding stratum. Then $[G_\sigma, G_\sigma] \cong \text{SU}(2)^k$. On the other hand, there are certain strata where $D(G)_{\text{impl}}$ is singular, but their closure is a smooth quasi-Hamiltonian manifold.

Let $G$ be $\text{SU}(n)$. Consider an edge $\sigma_{01}$ of an alcove with centralizer $G_{01} = \text{S(U(1) \times U(n - 1))}$. By the argument above we know that for $n > 3$ the corresponding stratum
$X_{01}$ in $X$ consists of genuine singularities. Nevertheless the following result asserts that it is smooth quasi-Hamiltonian manifold and in fact $S^{2n}$.

**Theorem 7.** ([HJS06]) The closure of the stratum $X_{01}$ of $X = DSU(n)_{\text{impl}}$ is a smooth quasi-Hamiltonian $U(n)$-manifold diffeomorphic to $S^{2n}$. Furthermore antipodal map of $S^{2n}$ corresponds to involution of $X_{01}$ obtained by lifting symmetry of the alcove $\mathcal{A}$ that reverses the edge $\sigma_{01}$.

4. Imploded cross-section of $\text{Sp}(n)$

In the last section, we saw a construction using an imploded cross-section yielding an example of sphere with quasi-Hamiltonian structure. In this section using construction of somewhat similar nature, we will show that $\text{HP}^n$ have quasi-Hamiltonian structure as well.

Let $G = \text{Sp}(n)$, the group of unitary $n \times n$ matrices over the quaternions, with maximal torus $T = \{ \text{diag}(e^{2\pi i x_1}, ..., e^{2\pi i x_n}) \}$. Identify $t$ with $\mathbb{R}^n$ via the map $x \mapsto 2\pi i \text{diag}(x_1, ..., x_n)$. Then the simple roots will have form:

$$\begin{align*}
(2\pi i)^{-1}\alpha_k(x) &= x_k - x_{k+1} \quad \text{for} \quad k = 1, ..., n - 1 \quad \text{and} \quad (2\pi i)^{-1}\alpha_n(x) = -2x_1
\end{align*}$$

with minimal root $(2\pi i)^{-1}\alpha_n(x) = 2x_n$. The corresponding alcove is the $n$-simplex $0 < x_n < ... < x_1 < 1/2$. We will abuse our notation and denote $\sigma_{01}$ the edge of the simplex with vertices $\sigma_0$, $\sigma_1$ corresponding to $I$ and $\text{diag}(-1,1,...,1)$. Under exponential map this edge corresponds to torus elements of the form $\text{diag}(t,1,...,1)$ with centralizer $G_{01} = U(1) \times \text{Sp}(n-1)$. The centralizers of the vertices are $G_0 = \text{Sp}(n)$ and $G_1 = \text{Sp}(1) \times \text{Sp}(n-1)$ respectively. By (3.1), the corresponding strata are given by:

$$X_\sigma = G/[G_{0,\sigma}, G_{\sigma}] \times \exp \sigma.$$

Therefore

$$X_0 = \{7\} \times \{I\} \cong \{pt\};$$

$$X_{01} = \text{Sp}(n)/\text{Sp}(n-1) \times \{ \text{diag}(t,1,...,1) | t \in (0,1) \} \cong S^{4n-1} \times (0,1),$$

$$X_1 = \text{Sp}(n)/(\text{Sp}(n-1) \times \text{Sp}(1)) \times \text{diag}(-1,1,...,1) \cong \text{HP}^{n-1}.$$  

where overline denotes a coset corresponding to an element. Consider the closure of the stratum corresponding to $\sigma_{01}$. $\bar{X}_{01} = X_0 \cup X_{01} \cup X_1$. Notice that $X_0 \cup X_{01} \cong \mathbb{H}^n$ and $X_1 \cong \text{HP}^{n-1}$, and one would expect that $\bar{X}_{01} \cong \text{HP}^n$. We will prove this by directly constructing a homeomorphism from $\bar{X}_{01}$ to $\text{HP}^n$. Define a map:

$$\mathcal{G} : \bar{X}_{01} \to \text{HP}^n, \quad (\bar{g}, x) \mapsto \left[ \sqrt{\frac{2x_1}{1-2x_1}}, gv \right],$$

where $v = (1,0,...,0) \in \mathbb{H}^n$. One can easily check that it is well-defined, i.e., does not depend on the equivalence class of $g$ in $\text{Sp}(n)/\text{Sp}(n-1)$. $\mathcal{G}$ is a continuous, bijective
on $X_{01}$ (or $0 < x_1 < 1$) and continuously extends to $X_{01}$. Indeed on $X_0$ (or $x_1 = 0$) $\mathcal{G}(\bar{g}, x) = [1, 0, \ldots, 0]$ and on $X_1$ (or $x_1 = 1$) $\mathcal{G}(\bar{g}, x) = [0, g.v]$.

In [HJS06] it was shown that imploded space is Hausdorff, locally compact and second countable. Therefore $\mathcal{G}$ is a homeomorphism. Define a smooth structure on $\bar{X}_{01}$ by pulling back a smooth structure on $\text{HP}^n$ via homeomorphism $\mathcal{G}$. Then the inverse map $\mathcal{F}: \text{HP}^n \to \bar{X}_{01}$ is smooth and defined as:

$$\mathcal{F}(z) \mapsto \left( f_{p, n-q}(z) \right) \times \text{diag}(e^{2\lambda \pi i}, 1, \ldots, 1),$$

where $\lambda$ and $f_{p, n-q}(z)$ are given by:

$$\lambda = \frac{\sum_{l=2}^{n+1} |Z_l|^2}{\sum_{l=1}^{n+1} |Z_l|^2},$$

$$f_{p, n-q} = \begin{cases} 
\frac{Z_{p+1}Z_1}{|Z_1|\sqrt{\sum_{l=2}^{n+1} |Z_l|^2}} & \text{if } q = n - 1 \\
\frac{|Z_{p+1}|Z_{p+1}}{\sqrt{\sum_{l=2}^{n+1} |Z_l|^2}} & \text{if } p - q < 2 \text{ and } q \leq n - 2 \\
\frac{\sqrt{\sum_{l=2}^{p-1} |Z_l|^2}Z_{p+1}}{\sqrt{\sum_{l=2}^{n+1} |Z_l|^2}Z_{p+1}} & \text{if } p - q = 2 \text{ and } q \leq n - 2 \\
0 & \text{if otherwise}
\end{cases}$$

There is an action of $\text{Sp}(n) \times T$ on $\text{HP}^n$:

$$(g, t)[Z_1, \ldots, Z_{n+1}] = [Z_1t_1, k.(Z_2, \ldots, Z_{n+1})],$$

where we regard $\text{H}^{n+1}$ as a right $\text{H}$-module. Then one can easily show

**Lemma 2.** The map $\mathcal{G}$ is a $\text{Sp}(n) \times T$-equivariant.

First, let us recall the quasi-Hamiltonian structure on stratum $X_\sigma$. Since the moment map $\Phi_2$ defined as in [2.6] is transversal to all faces of the alcove, using quasi-symplectic reduction with quasi-Hamiltonian cross-section theorem one can show:

**Lemma 3.** (HJS06) For every $\sigma \leq \mathcal{A}$ the subspace $X_\sigma = G/[G_\sigma, G_\sigma] \times \exp \sigma$ of $D(G)_{\text{impl}}$ is a quasi-Hamiltonian $G \times T$-manifold. The moment map $X_\sigma \to G \times T$ is the restriction to $X_\sigma$ of the continuous map $\Phi_{\text{impl}} \to G \times T$ induced by $\Phi: D(G) \to G \times G$.

Next we compute the corresponding 2-form $\omega_\sigma$ on $X_\sigma$. Let $(g, x)$ be an arbitrary point on $X_\sigma$. A tangent vector at $(g, x)$ is of the form $((L_g)_* \xi, (L_x)_* \eta)$ where $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{z} + \mathfrak{z}(\mathfrak{g}_\sigma)^+$ for some $\zeta \in \mathfrak{z}(\mathfrak{g}_\sigma)^+ = [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]$ (Lemma A.3, HJS06). Then a simple calculation yields:

$$\omega_\sigma((L_g)_* \xi_1, (L_x)_* \eta_1), (L_g)_* \xi_2, (L_x)_* \eta_2)) = -\frac{1}{2}((\text{Ad}_x - \text{Ad}_{x^{-1}})\xi_1, \xi_2) - (\xi_1, \eta_2) + (\xi_2, \eta_1).$$

One can check that it does not depend on equivalence class of $\xi_i$ in $\mathfrak{g}/[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]$. Consider 2-form $\omega_{01}$ on an open stratum $X_{01}$. In what follows we compute the pull back of this
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2-form on via $\mathcal{F}$ and show that it extends smoothly to all of $\mathbf{HP}^n$. Since $\omega_{01}$ is $\mathbf{Sp}(n) \times T$-invariant, it suffices to consider vectors of the form $z_0 = [t, 1, 0, ..., 0]$, where

$$t = \frac{|Z_1|}{\sqrt{\sum_{l=2}^{n+1} |Z_l|^2}}.$$ 

The tangent space at $z_0$ will be:

$$T_{z_0} \mathbf{HP}^n = \{(w_1, ..., w_{n+1}) \in \mathbf{H}^{n+1} | tw_1 + w_2 = 0\},$$

where $w_i = w_{1i} + w_{2i} + w_{3i}j + w_{4i}k$. Let us first find corresponding tangent vectors at $\mathcal{F}(z_0)$, or more precisely corresponding pull-backs $\xi_i, \eta_i$ to elements of Lie algebra as in (4.11). Let $v$ and $w$ be tangent vectors of form (4.12). Note, since $\mathcal{F}(z_0) = (\bar{I}, \text{diag}(\exp(\lambda \pi i), 1, ..., 1)) =: (g, x)$, the first component of the image is already an element of Lie algebra, while the second one has to be translated by an appropriate element of Lie group (that is $x$). Denote by $(A^v_{p,q})$ and $(B^w_{p,q})$ ($(A^w_{p,q})$ and $(B^w_{p,q})$) the matrix representation of $\xi_1$ and $\eta_1$ (correspondingly $\xi_2$ and $\eta_2$). Then substituting these to the first term of (4.11) and using above expression for $\mathcal{F}(z_0)$ we have:

$$((Ad_x - Ad_{x^{-1}})\xi_1, \xi_2) = \text{Re}\left(\left[\exp(\lambda \pi i)A^v_{11} \exp(-\lambda \pi i) - \exp(-\lambda \pi i)A^v_{11} \exp(\lambda \pi i)\right] - \sum_{p=2}^{n} A^v_{1p} A^w_{p1} + \sum_{p=2}^{n} A^v_{1p} \left[\exp(-\lambda \pi i) - \exp(\lambda \pi i)\right] A^w_{p1}\right),$$

where the inner product is given by $(A, B) = \text{Re}(tr(AB^t))$ and

$$A^v_{11} = -(t + t^{-1})(v_{12}i + v_{13}j + v_{14}k),$$

and by skew-symmetry:

$$A^v_{p1} = -A^v_{1p} = v_{(p+1)1} + v_{(p+1)2}i + v_{(p+1)3}j + v_{(p+1)4}k.$$ 

There are similar relations to (4.14) and (4.15) if we replace $v$ by $w$. Thus we can rewrite (4.13) in the following form

$$((Ad_x - Ad_{x^{-1}})\xi_1, \xi_2) = 2\sin(2\pi \lambda)(t + t^{-1})(v_{13}w_{14} - w_{13}v_{14}) - 4\sin(\pi \lambda) \sum_{p=3}^{n+1} (v_{p1}w_{p2} - w_{p1}v_{p2} - v_{p3}w_{p4} + w_{p3}v_{p4}).$$
Hence, corresponding two-form will be:

\begin{equation}
2 \sin(2\pi\lambda) (t + t^{-1}) dx_{13} dx_{14} - 4 \sin(\pi\lambda) \sum_{p=3}^{n+1} (dx_{p1} dx_{p2} - dx_{p3} dx_{p4}),
\end{equation}

where \(x\)'s are just real coordinates for \(Z\)'s, such that \(Z_l = x_{i1} + x_{i2} + x_{i3} + x_{i4}\). For the remaining part of (4.11) we have:

\begin{equation}
- (\xi_1, \eta_2) + (\xi_2, \eta_1) = \text{Re}(-A_{11}^v \bar{B}^{w}_{11} + A_{11}^w \bar{B}^{v}_{11}),
\end{equation}

where

\begin{equation}
B^{v}_{11} = \frac{-2\pi i t}{t^2 + 1} v_{11},
\end{equation}

and corresponding two-form will be:

\begin{equation}
2\pi dx_{11} dx_{12}.
\end{equation}

Combining (4.17) with (4.20) yields:

\begin{equation}
\mathcal{F}^*\omega_{01} = 2\pi dx_{11} dx_{12} - 2 \sin(2\pi\lambda) (t + t^{-1}) dx_{13} dx_{14} +
4 \sin(\pi\lambda) \sum_{p=3}^{n+1} (dx_{p1} dx_{p2} - dx_{p3} dx_{p4}).
\end{equation}

It is a smooth two-form defined on open dense subset of \(\mathbb{HP}^n\). Moreover we can show

**Lemma 4.** The two-form \(\mathcal{F}^*\omega_{01}\) extends smoothly on all of \(\mathbb{HP}^n\).

**Proof.** It suffices to check two critical cases \(Z_1 = 0\), a line at infinity, and \([1,0,...,0]\), a point at infinity. As \(|Z_1|\) approaches to 0, \(\lambda\) tends to 1 and therefore the third expression on the right hand side of (4.21) vanishes. Now since \(\lambda = 1 - t^2\), we have \(t \to 0\) and hence

\[2 \sin(2\pi\lambda) (t + t^{-1}) \to -4\pi.
\]

So in the neighborhood of \(Z_1 = 0\) the two-form \(\mathcal{F}^*\omega_{01}\) can be written as

\[2\pi dx_{11} dx_{12} + 4\pi dx_{13} dx_{14}.
\]

In the similar fashion one can show that in the neighborhood of \([1,0,...,0]\) it is given by:

\[2\pi dx_{11} dx_{12} - 4\pi dx_{13} dx_{14}.
\]

This finishes the proof of this lemma. \(\square\)

Notice that the obtained 2-form is given in dehomogenized coordinates:

\begin{equation}
\begin{bmatrix}
\frac{|Z_1|}{\sqrt{\sum_{l=2}^{n+1} |Z_l|^2}}, & \frac{Z_2 \bar{Z}_1}{|Z_1| \sqrt{\sum_{l=2}^{n+1} |Z_l|^2}}, & \ldots, & \frac{Z_{n+1} \bar{Z}_1}{|Z_1| \sqrt{\sum_{l=2}^{n+1} |Z_l|^2}}
\end{bmatrix}
\end{equation}

For \(Z = x_1 + x_2 i + x_3 j + x_4 k\) define \(\text{Im}_i(Z) = x_2\), then we have:

\begin{equation}
\text{Im}_i(d\bar{Z}_pdZ_p) = 2(dx_{p1} dx_{p2} - dx_{p3} dx_{p4}).
\end{equation}
Now using (4.22) and (4.23) in homogenous coordinates the first two terms vanishes, our 2-form will take form:

\[
4\sin(\lambda \pi) \left( \left| Z_1 \right|^2 \sum_{l=2}^{n+1} \left| Z_l \right|^2 \right)^{-1} \left[ \sum_{p=3}^{n+1} \left| Z_p \right|^2 \text{Im}_i(dZ_1d\bar{Z}_1) - \right.
\]

\[
\text{Im}_i(Z_1d\bar{Z}_p dZ_p \bar{Z}_1) + \left( \sum_{p=3}^{n+1} \left| Z_p \right|^2 \text{Im}_i((Z_1d\bar{Z}_1)) - \text{Im}_i(Z_1\bar{Z}_p dZ_p \bar{Z}_1) \right) \times
\]

\[
\left( \frac{Z_1d\bar{Z}_1 + dZ_1 \bar{Z}_1}{\left| Z_1 \right|^2} + \frac{\sum_{l=2}^{n+1} (Z_l d\bar{Z}_l + dZ_l \bar{Z}_l)}{\sum_{l=2}^{n+1} \left| Z_l \right|^2} \right).
\]

(4.24)

Last thing we need to show that there is well-defined smooth moment map. Define a map \( \Phi : \mathbb{HP}^n \to \text{Sp}(n) \times T \) such that the following diagram commutes:

(4.25)

\[
\begin{array}{ccc}
\mathcal{G}_{|X_\sigma} & \xrightarrow{\Phi_\sigma} & \mathbb{HP}^n \\
\Phi \downarrow & & \downarrow \\
\text{Sp}(n) \times T
\end{array}
\]

for each face \( \sigma \) in the closure. Then it has to be of the form:

(4.26)

\[
[Z_1, \ldots, Z_{n+1}] \mapsto (AB^{-1}A^{-1}, B)
\]

where \( A = (A_{pq}) \) and \( B = (B_{pq}) \) are matrices:

\[
A_{p1} = \frac{Z_{p+1}Z_1}{\left| Z_1 \right| \sqrt{\sum_{l=2}^{n+1} \left| Z_l \right|^2}}
\]

(4.27)

\[
B = \text{diag}(\exp(\lambda \pi i), 1, \ldots, 1)
\]

Notice that we only defined the first column of the \( A_{pq} \) since it is uniquely determined by its first column. Evidently, \( \Phi \) is uniquely determined and \( \text{Sp}(n) \times T \)-equivariant. We have to show that it is smooth. From the construction one can see that \( B_{pq} \) are smooth. As for the first component of \( \Phi \), using the fact \( A \in \text{Sp}(n) \) we have:

\[
AB^{-1}A^{-1} = \text{Id}_n + C,
\]

where \( C = (C_{pq}) \):

\[
C_{pq} = A_{p1}B_{1q}A_{q1} - A_{p1}\bar{A}_{q1},
\]

or to be more precisely:

(4.28)

\[
C_{pq} = \left( \left| Z_1 \right|^2 \sum_{l=2}^{n+1} \left| Z_l \right|^2 \right)^{-1} Z_{p+1} \left[ \bar{Z}_1 \exp(\pi i \lambda) Z_1 - \left| Z_1 \right|^2 \right] \bar{Z}_{q+1}.
\]
We can easily see that it is smooth for $Z_1 \neq 0$ and $\sum_{i=2}^{n+1} |Z_i|^2 \neq 0$. Otherwise using almost the same argument as in Lemma 4 we can show it is smooth in these two cases as well. Now summarizing these facts we have:

**Theorem 8.** The closure of the stratum $X_{01}$ of $X = D Sp(n)_{\text{impl}}$ is a smooth quasi-Hamiltonian $Sp(n) \times T$-manifold diffeomorphic to $n$-dimensional quaternionic projective space with 2-form and moment map determined by (4.24) and (4.26) correspondingly.

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