We investigate the geometrical phase associated to the Schrödinger equation in a three level system in Stimulated Raman Adiabatic Passage (STIRAP). We solve explicitly a dual model, in which the pulses are applied in the counterintuitive and intuitive order. We show that when the pulse areas are finite, a pair of magnetic monopoles with opposite charges are created resulting the oscillations of the populations on the final states. The applications of the phase shift include, for example, phase gates in quantum computing, phase manipulation in quantum cryptography and phase interactions in quantum interference.
1 Introduction

Time-dependent two-state models are widely used in quantum mechanics. A lot of effort has been made to investigate simple models describing population transfer at level crossings. The first prototype model was presented by Laudau [1] and Zener [2] where the population was transferred adiabatically. A review of nonadiabatic corrections and solvable two-level systems shows that only in a few cases an exact solution can be found [3].

Another class of time-dependent problems arises when we consider three-level systems. As was pointed out in [4] and [5] three-level systems are related to two-level problems by the SU(2) representation of the rotation group. Three-level systems are used, for example, to describe the Stimulated Raman Adiabatic Passage (STIRAP) in quantum optics as well as neutrino propagation in the medium in high energy physics.

In this article we study an adiabatic three-level system (STIRAP). We calculate explicitly a dual model in which the pulses are applied in the counterintuitive and intuitive order. We show with the examples how the adiabatic phase gets a nontrivial contribution. The phase manipulation has many applications, for example, in quantum computing and quantum cryptography. We also show that when the pulse areas are finite, this corresponds to a case where magnetic monopoles locate at the origin of the parametric space.

2 Adiabatic system

We consider a general time-dependent two-level Schrödinger system written with the generators of SU(2)

$$H_2(t) = \frac{1}{2} \vec{\sigma} \cdot \vec{R}_2(t).$$ (1)

The vector $\vec{R}_2(t)$ is defined as $\vec{R}_2(t) = \Omega_1(t)\hat{i} + \Omega_2(t)\hat{k}$ and $\vec{\sigma}$ contains the Pauli sigma matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$ (2)

according to the rule $\sigma_i \sigma_j = i \epsilon_{ijk} \sigma_k + \delta_{ij}$ ($i, j, k = 1, 2, 3$). The state vector of the system Eq. (1) is $|\Psi\rangle$. By making an unitary transformation $U$ to the state $|\Psi\rangle$, $|\tilde{\Psi}\rangle = U|\Psi\rangle = U(|a_1\rangle + |a_2\rangle)$, we diagonalize the Hamiltonian $U^\dagger H_2 U = D$ with the eigenvalues $\pm R_2/2$. In the adiabatic space Eq. (1) transforms as

$$H_2^{ad}(t) = (D - iU^\dagger \partial_t U)(t) = \frac{1}{2} \vec{\sigma} \cdot \vec{R}_2^{ad}(t),$$ (3)

where $\vec{R}_2^{ad}(t) = -\dot{\phi}(t)\hat{j} + R_2(t)\hat{k}$. Here we have used the notations $R_2^2(t) = \Omega_1^2(t) + \Omega_2^2(t)$ and $\tan \phi(t) = \Omega_1(t)/\Omega_2(t)$. 


We apply the result to the STIRAP problem. It consists of three-levels which are coupled in a sequence \(1 \rightarrow 2 \rightarrow 3\), and the states 1 and 3 are assumed to be in resonance. The Hamiltonian which describes the system is
\[
H_3(t) = \vec{J} \cdot \vec{R}_3(t).
\]
The vector \(\vec{R}_3(t)\) is defined as \(\vec{R}_3(t) = \lambda_1(t)\hat{i} + \lambda_2(t)\hat{k}\) and \(\vec{J}\) contains the modified generators of the rotation group \(SO(3)\)
\[
J_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},
\]
according the rule \([J_i, J_j] = i\epsilon_{ijk} J_k\) \((i, j, k = 1, 2, 3; i \neq j)\). The state vector of Eq.(4) is
\[
|\psi\rangle = b_1 |1\rangle + b_2 |2\rangle + b_3 |3\rangle.
\]
The question is to find \(|b_3(\infty)|^2\) with the initial condition \(|b_1(-\infty)|^2 = 1\). The couplings have to satisfy \(\lim_{t \rightarrow -\infty} \lambda_1(t) = 0\) and \(\lim_{t \rightarrow \infty} \lambda_2(t) = 0\) when the pulses are applied in the counterintuitive case. The \(SU(2)\) representation of the rotation group \(SO(3)\) allows us to present the current three-level problem as a two-level system with the Hamiltonian Eq.(3) and the relations \(\vec{R}_2(t) = \vec{R}_3(t)\). The result is
\[
\begin{align*}
b_1 &= -\sin \phi (a_1 a_2^* + a_1^* a_2) + \cos \phi (|a_1|^2 - |a_2|^2), \\
b_2 &= -(a_1 a_2^* - a_1^* a_2), \\
b_3 &= -\cos \phi (a_1 a_2^* + a_1^* a_2) - \sin \phi (|a_1|^2 - |a_2|^2).
\end{align*}
\]
In the counterintuitive case the corresponding initial condition in the two-level system is \(|a_1(-\infty)|^2 = 1\) which also determines the final population \(|b_3(\infty)|^2 = |2|a_1(\infty)|^2 - 1|^2\).

### 3 Exact adiabatic three-level solution

The couplings \(\lambda_1\) and \(\lambda_2\) are defined so that
\[
\begin{align*}
\lambda_1(t) &= f_1(t)g(t), \\
\lambda_2(t) &= f_2(t)g(t),
\end{align*}
\]
and the corresponding values in the eigenspace become
\[
\begin{align*}
R_2^2 &= (f_1^2 + f_2^2)g^2, \\
\dot{\phi} &= \frac{\dot{f}_1 f_2 - f_1 \dot{f}_2}{f_1^2 + f_2^2}.
\end{align*}
\]
Especially when choosing the functions \(f_1\) and \(f_2\) in such a way that
\[ f_1^2 + f_2^2 = 1, \]  

(13)

the parameters \( R_2 \) and \( \dot{\phi} \) separate from each other in the adiabatic space, i.e. \( R_2 \) is a function of \( g \) only and \( \dot{\phi} \) depends on \( f_1 \) and \( f_2 \) only. By choosing suitable functions for \( g \) we can tune the behavior of the system in the adiabatic space.

### 3.1 Counterintuitive solution

As an example of a separated counterintuitive three-level solution we consider a system whose couplings are

\[
\lambda_1(t) = \Omega_1(t) = A \sqrt{\frac{1 + \tanh(t/T)}{2}} \text{sech}(t/T), \tag{14}
\]

\[
\lambda_2(t) = \Omega_2(t) = A \sqrt{\frac{1 - \tanh(t/T)}{2}} \text{sech}(t/T). \tag{15}
\]

\( A \) and \( T \) are scaling parameters. In the adiabatic space these couplings transform to the parameters

\[
R_2(t) = A \text{sech}(t/T), \tag{16}
\]

\[
\dot{\phi}(t) = \frac{1}{2T} \text{sech}(t/T). \tag{17}
\]

Now the Hamiltonian system of Eq.\((4)\) is trivially solved. A full exact solution is

\[
b_1(t) = \tanh \eta \sin \phi \sin(2I) + \cos \phi [\text{sech}^2 \eta + \tanh^2 \eta \cos(2I)], \tag{18}
\]

\[
b_2(t) = -i2 \tanh \eta \text{sech} \eta \sin^2(I), \tag{19}
\]

\[
b_3(t) = \tanh \eta \cos \phi \sin(2I) - \sin \phi [\text{sech}^2 \eta + \tanh^2 \eta \cos(2I)], \tag{20}
\]

where \( I(t) = AT \cosh \eta \text{artan}[\exp(t/T)] \) and \( \sinh \eta = 1/(2AT) \). The final populations become

\[
b_1(\infty) = \tanh \eta \sin(\pi AT \cosh \eta), \tag{21}
\]

\[
b_2(\infty) = -i2 \tanh \eta \sin^2\left[\frac{\pi AT}{2} \cosh \eta\right], \tag{22}
\]

\[
b_3(\infty) = -1 + \frac{\sinh^2 \eta [1 - \cos(\pi AT \cosh \eta)]}{\cosh^2 \eta}. \tag{23}
\]

We notice that \(|b_3(\infty)|^2\) depends on \( AT \) in a polynomial way. In the corresponding 2-level system, these oscillations relate to the Rabi cycle. When
\[(4n)^2 - (2AT)^2 = 1, \quad (24)\]

and \(n > 0\) is an integer, then \(|b_3(\infty)|^2 = -1\). Oscillations continue to the infinity. Similar numerical results were provided in [5].

We make two remarks here:

- If the \(b_1(-\infty)\) is a real positive number, then the intermediate state \(b_2(t)\) is populated by the phase only, and the final state, \(b_3(t)\) will be a negative number with no phase. During the transition the state \(b_3(t)\) obtains an additional geometrical phase, \(\exp(i\pi)\).

- Generally, the state is a complex number with some phase. However, in the counterintuitive process with the initial condition of no phase, the \(\operatorname{Im}(b_1(t))\), \(\operatorname{Re}(b_2(t))\), and \(\operatorname{Im}(b_3(t))\) are never populated. From this it follows that if we have an initial condition where both real and imaginary parts are populated, the solution can be broken into two separate equations and the solution can be calculated separately.

### 3.2 Exponential pulses

As a comparison we state the result of the exponential pulses in which the pulse areas not finite [5]. The couplings are

\[\lambda_1(t) = A(1 + e^{-t/T})^{-1/2}, \quad (25)\]
\[\lambda_2(t) = A(1 + e^{t/T})^{-1/2}. \quad (26)\]

With the initial condition \(b_1(-\infty) = 1\), the population of the final state is

\[b_3(\infty) = -1 + \operatorname{sech}^2(\pi AT). \quad (27)\]

There are two notes:

- The population on the level 3 does not oscillate. The Hamiltonian degenerates and the zero eigenvalue dominates.

- The population on the level 3 has the same phase shift \(\exp(i\pi)\) in both cases, i.e. when the pulse areas are finite and infinite.

### 3.3 Intuitive order solution

Using the same technique presented before, it is also possible to calculate the case when the pulses are applied in the intuitive order, i.e.
\[
\lambda_1(t) = A \sqrt{\frac{1 - \tanh(t/T)}{2}} \text{sech}(t/T), \quad (28)
\]
\[
\lambda_2(t) = A \sqrt{\frac{1 + \tanh(t/T)}{2}} \text{sech}(t/T), \quad (29)
\]

The calculation is similar as earlier and we just state the result which is

\[
b_1(t) = \sin \phi \cos(2I) + \cos \phi \tanh \eta \sin(2I), \quad (30)
\]
\[
b_2(t) = -i \text{sech} \eta \sin(2I), \quad (31)
\]
\[
b_3(t) = \cos \phi \cos(2I) - \sin \phi \tanh \eta \sin(2I). \quad (32)
\]

When the time is infinite this becomes

\[
b_1(\infty) = \tanh \eta \sin(\pi AT \cosh \eta), \quad (33)
\]
\[
b_2(\infty) = -i \text{sech} \eta \sin(\pi AT \cosh \eta), \quad (34)
\]
\[
b_3(\infty) = \cos(\pi AT \cosh \eta). \quad (35)
\]

We note that \(|b_3(\infty)|^2\) is now pure oscillations. There are two special cases

\[
|b_3(\infty)|^2 = 1, \quad (2n)^2 - (2AT)^2 = 1, \quad (36)
\]
\[
|b_3(\infty)|^2 = 0, \quad \left[2(n - \frac{1}{2})\right]^2 - (2AT)^2 = 1, \quad (37)
\]

and \(n > 0\) is an integer.

We conclude that we have been able to solve both of the cases, counterintuitive and
intuitive order, by using the same structure of the pulses. We call this as a dual model.

### 4 Three dimensional system

We consider a time- and space-dependent Schrödinger equation

\[
i\hbar \frac{\partial}{\partial t} \Psi(R,t) = (T + V)\Psi(R,t), \quad (38)
\]

where \(T\) is a kinetic energy operator and \(V\) is a potential. A well-known numerical
solution approach is a Split operator -method, in which the solution is written in an ex-
ponential form. The operators generally do not commute, but using the Baker-Hausdorff
formula, one can show that within a small time interval, \(\Delta T\), the error will be of order
\(dt^2\). When the time interval is small enough, the numerical result will be accurate. The
Split operator -method solution of Eq. (38) can be written as
\[ \Psi(R, t + \Delta t) = e^{-iV\Delta t/(2\hbar)} e^{-iT\Delta t/\hbar} e^{-iV\Delta t/(2\hbar)} \Psi(R, t) \]
\[ = \hat{O}(T, V) \Psi(R, t). \] (39)

The operator \( T \) is solved in the Fourier space. Let’s assume we have a normalized minimum wave function
\[ \Psi(x) = [2\pi(\Delta x)^2]^{-1/4} \exp\left[ -\frac{(x - \langle x \rangle)^2}{4(\Delta x)^2} + \frac{i\langle p \rangle x}{\hbar} \right], \] (41)
which propagates in an harmonic oscillator potential
\[ V \sim \frac{1}{2} m \omega^2 (x - x_0)^2. \] (42)

According to the Eq.(39) the population on the level is interchanged between the real and imaginary parts of the state and the wave probability remains the same.

Now we consider the Hamiltonian which includes the couplings \( \lambda_1 \) and \( \lambda_2 \)
\[ H_{3D} = \begin{bmatrix} T_1(p) + V_1(x) & \lambda_1(t) & 0 \\ \lambda_1(t) & T_2(p) + V_2(x) & \lambda_2(t) \\ 0 & \lambda_2(t) & T_3(p) + V_3(x) \end{bmatrix}, \] (43)
and the Schrödinger equation is
\[ i\hbar \frac{\partial}{\partial t} \Psi(R, t) = (T + V + \lambda) \Psi(R, t). \] (44)

The Split operator -solution is then
\[ \Psi(R, t + \Delta t) = e^{-i(V+\lambda)\Delta t/(2\hbar)} e^{-iT\Delta t/\hbar} e^{-i(V+\lambda)\Delta t/(2\hbar)} \Psi(R, t) \]
\[ = \hat{F}(\lambda) \hat{O}(T, V) \hat{F}(\lambda) \Psi(R, t), \] (45)

where \( \hat{F}(\lambda) = \exp(-i\lambda\Delta t/(2\hbar)) \). What this equation states, is that in a small time interval, we first make a small STIRAP process step, \( \hat{F}(\lambda) \), from level 1 to 3, then interchange the population between the real and imaginary parts on all 3 states, \( \hat{O}(T, V) \), and then make another small STIRAP process step, \( \hat{F}(\lambda) \), from level 1 to 3. In this solution method these two processes are not connected; one transfers the population between the states and another moves the population within the state. Also, as it was stated in the previous section, in the STIRAP process the full complex wave equation can be separated into two different equations which can be solved separately. Altogether this means that under the STIRAP process, the full wave function which has some phase, is transferred to the final state in a same manner as in our simple exact solution, as a result that the wave function has obtained an additional geometric phase \( \exp(i\pi) \). Depending on the potentials of the states, the wave functions can change their shapes during the transition.
4.1 Phase manipulation

We consider two consecutive STIRAP processes. The first counterintuitive STIRAP process transfers the population from level 1 to 3, and the level 3 obtains the phase shift. Then another STIRAP is applied in the reverse order (intuitive order) and the population is transferred back from state 3 to 1. If the delay between the pulses is $\Delta T$, the geometric phase on state 1 is shifted by $\Delta T$. By changing the value of $\Delta T$, one can rigorously control the geometric phase on the state 1.

The phase manipulation has applications, for example, in the following areas:

- One-qubit phase gate in quantum computing.
- Phase manipulation in quantum cryptography.
- Phase Conjugated Mirror (PCM) type of applications.
- Magnetic charge ($\pm$) annihilation (analogy to electron-positron annihilation).
- In the study of quantum interference.

All these phenomena make use of the phase shift and a complete two-way population transfer between the states.

4.2 Hadamard gate

Another popular phase gate in quantum computing is the Hadamard gate. It has the Hamiltonian

$$H(t) = \begin{bmatrix}
0 & 0 & \lambda_{10}(t) & 0 \\
0 & 0 & \lambda_{11}(t) & 0 \\
\lambda_{10}(t) & \lambda_{11}(t) & 0 & \lambda_2(t) \\
0 & 0 & \lambda_2(t) & 0
\end{bmatrix},$$

(47)

and the state vector is $|\psi\rangle = b_{10}|10\rangle + b_{11}|11\rangle + b_2|2\rangle + b_3|3\rangle$. We use the notations

$$\lambda_{10}(t) = c_{10}\lambda_1(t),$$

(48)

$$\lambda_{11}(t) = c_{11}\lambda_1(t),$$

(49)

$$|1\rangle = c_{10}|10\rangle + c_{11}|11\rangle,$$

(50)

where $c_{10}$ and $c_{11}$ are constants. When defining the constants to have a relation

$$c_{10}^2 + c_{11}^2 = 1,$$

(51)

the Hamiltonian Eq. (47) reduces to the 3-dimensional Hamiltonian Eq. (4). Now the same exact solution of Eqs. (18)-(20) can be applied to find exact solutions to this system also.

One finding is that when the pulse areas are finite, the Rabi cycle is also at present in the Hadamard gate and the final populations on the levels oscillate. The Rabi cycle has crucial importance in quantum computing.
5 Monopoles in adiabatic three-level system

We consider two approaches to show the existence of the monopoles in the STIRAP system when the pulse areas are finite. The first formalism is commonly used in quantum optics while the second approach uses the notation of quantum field theory.

5.1 Approach A

When the pulse areas of the counterintuitive STIRAP system are finite, a soliton with a constant flux can be found. We show this by using the Berry’s adiabatic phase and the group SU(2). The definition of the Berry’s phase is

\[ \gamma_m(C) = \oint_C d\vec{R} \cdot \vec{A}(\vec{R}) = \int_S d\vec{S} \cdot \vec{V}, \]

where \( \vec{R} \) describes the slowly varying parameters in time and \( \vec{A}(\vec{R}) \) is a vector potential of the magnetic field. In the second equality we have used Stoke’s Law and defined \( \vec{V} = \nabla \times \vec{A} \) which is the flux associated to the magnetic field. Writing the Berry’s phase in the form which is manifestly independent of the phase of state \(|m, \vec{R}\rangle\), we get

\[ \gamma_m(C) = -\int_S d\vec{S} \cdot \vec{V}_m, \]

where

\[ \vec{V}_m(\vec{R}) = \text{Im} \sum_{n \neq m} \frac{\langle m, \vec{R} | \nabla R H | n, \vec{R} \rangle \times \langle n, \vec{R} | \nabla R H | m, \vec{R} \rangle}{(E_m(\vec{R}) - E_n(\vec{R}))^2}. \]

When the eigenvalues cross, a field source is at present. Evaluating Eq. (54) for a Hamiltonian Eq. (1) we get

\[ \vec{V}_+(R_2) = \frac{1}{2} \frac{\dot{R}_2}{R_2}, \quad \vec{V}_-(R_2) = -\frac{1}{2} \frac{\dot{R}_2}{R_2}, \]

and the degeneracy exists when \( R_2(-\infty) = R_2(\infty) = 0 \). The Berry’s phase becomes

\[ \gamma_\pm = \pm \frac{1}{2} \Delta \Omega, \]

where \( \Delta \Omega \) is the solid angle subtended by the closed path as seen from the place of degeneracy, \( R_2 = 0 \). The \( \pm \) refers to the direction in which the line integration is traversed. Equations (55) equal with the Wu-Yang magnetic monopole of strength \( \pm 1/2 \) located at the origin in the parameter space [8]. The total flux of the monopole is \( \Phi = \pm 2\pi \).
5.2 Approach B

The transformation from the two-level system Eq. (1) to the adiabatic Hamiltonian Eq. (3) can be seen as a special case of a local gauge transformation

$$D_t \psi^a \equiv \partial_t \psi^a + \epsilon^{abc} A^b_t \psi^c = 0. \quad (57)$$

$D_t$ is a covariant derivative, and $A^a_t$ is a gauge field with the non-Abelian group SU(2), $\epsilon^{abc} A^b_t \psi^c = i(H_2(t)\psi)^a$. One special configuration of $\Psi$ defines a supervacuum and by using the SU(2) rotations we can go to another vacuum, which we call a normal vacuum. We look for soliton solutions which are topologically stable. The energy must be finite and this is achieved by choosing the proper boundary conditions for $A^a_t$.

We show the existence of a string like soliton in the Schrödinger equation, Eq. (4)

$$D_t \Psi = (\partial_t + iA_t)\Psi = 0, \quad (58)$$

where $A_t = H_3(t)$. Here $t$ is some arbitrary parameter which parametrizes the space. First we look the symmetries of Eq. (58). Clearly it is not any more locally gauge invariant. After an infinitesimal rotation we get diagonal matrix elements, which do not belong to the group SO(3). The vacuum consists of three scalar fields, and it has the SO(3) symmetry because the norm is conserved, $(\psi_1^2 + \psi_2^2 + \psi_3^2) = 1$. The supervacuum is determined by fixing the initial condition, $\Psi(\infty)^2 = 1$. The analogy in the field theory is the Higgs field, which specifies the vacuum state. The symmetry of the supervacuum is thus U(1), $M_0 = S^1$, which is needed for a soliton carrying a magnetic charge. The unbroken vacuum has the symmetry SO(3)/U(1).

Again we are looking for a finite energy solution. The eigenvalues of $H_3$ define the energy of the system $\omega_0(t) = 0$, $\omega_\pm(t) = (\lambda_1(t) + \lambda_2(t))^{1/2}$. For a finite energy solution the eigenvalues must vanish $\omega_\pm \to 0$, when $t \to \pm \infty$. It follows that the gauge field also vanishes, $A_t \to 0$ when $t \to \pm \infty$. The manifold of the points at infinity is thus $M_\infty = SO(3)/U(1)$. In order to have a nontrivial solution we need a well defined map from $M_\infty$ into $M_0$. The first homotopy class of the group SO(3), $\pi_1(SO(3)) = Z_2$, assures that a path which forms a circle belongs to the $M_\infty$. The map is then $S^1 \to S^1$. All such maps are cylindrically symmetric and characterized by integers, i.e. $\Psi(r \to \infty) \to \exp^{-in\theta}$. We look for a solution in the form $\Psi = f(\rho)\exp^{-in\theta}$, where $f(\rho) \to 1$ when $r \to \infty$ and $f(\rho)$ vanishes at the origin. From Eq. (58) we get $A_t \to n\partial_\theta$, when $r \to \infty$. We have a string like soliton, whose magnetic field is

$$\int B \cdot dS = \oint_{r \to \infty} A_t dt = n\Delta \theta, \quad (59)$$

with the flux $\Psi = 2\pi n$.

5.3 Monopole confinement

In the case of intuitive order pulse system, the supervacuum does not respect any symmetries and a constant flux tube does not exist. For a counterintuitive pulse system, the supervacuum is $S^1$ invariant under the coupling $\lambda_2(t)$. The system admits then a
constant magnetic flux tube. When the coupling $\lambda_1(t)$ is switched on, the symmetry of the supervacuum is destroyed. Physically this looks very odd, since one would expect magnetic flux to be conserved. One explanation of the breaking up of double tubes is that a pair of magnetic monopoles with opposite magnetic charges are created. The supervacuum does not exist when the pulse areas are infinite.

6 Conclusions

In conclusion, we have shown an approach to obtain exact solutions to adiabatic three-level systems. In the adiabatic space the Hamiltonian becomes solvable. In particular, we calculated explicitly a dual model in a three-level system (STIRAP) where the pulses were applied in the counterintuitive and intuitive order. We showed that when the eigenvalues crosses at the infinity, a pair of magnetic monopoles are created. Interesting is that the monopoles and Rabi cycle are not at present when the pulse areas not finite. Additional phase is added in both cases, but the population on level 3 gets oscillations when the pulse areas are finite.

When two STIRAP processes are applied in a row; first the counterintuitive process and then the intuitive order process, the population is transferred back to the initial state. The phase of the new state can be well controlled. There are many application areas where the phase shift can have a significant impact. Some of these are, for example, phase gates in quantum computing, phase manipulation in quantum cryptography and the phase interactions in quantum interference.
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