GRADING THE TRANSLATION FUNCTORS IN TYPE A

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ABSTRACT. We deal with the representation theory of quantum groups and Hecke algebras at roots of unity. We relate the philosophy of Andersen, Jantzen and Soergel on graded translated functors to the Lascoux, Leclerc and Thibon-algorithm. This goes via the Murphy standard basis theory and the idempotents coming from the Murphy-Jucys operators. Our results lead to a guess on a tilting algorithm outside the lowest $p^2$ alcove, which at least in the $SL_2$-case coincides with Erdmann’s results.

1. Introduction

This work is concerned with the modular representation theory of the symmetric group and of the Hecke algebra. The main sources of inspiration are the work of Lascoux, Leclerc and Thibon [LLT] on the crystal basis of the Fock module and the paper of Andersen, Jantzen and Soergel [AJS] on the representation theory of Frobenius kernels.

The philosophy of [AJS] is to provide the representation theory – including the Jantzen translation functors – with a grading, that explains the fact that the Kazhdan-Lusztig algorithm constructs polynomials rather than numbers. Under the Schur functor, (certain of) the translation functors correspond to the classical $r$–inducing and $r$–restricting operators from the modular representation theory of the symmetric group. On the other hand, the action of the quantum group $U_q(\widehat{sl}_l)$ on the Fock space can be viewed as $q$-deformations of these operators on the Grothendieck group level. So according to the above philosophy, it is natural to look for a representation theoretical meaning of these deformed operators.

We here propose to connect them to the invariant form on the Specht modules. This is quite a different approach than that of [AJS]. Our calculations are considerably simpler than those of [AJS],
on the other hand we have to pass to the characteristic zero situation
on the way and therefore we are not able to construct a grading on
the representation category itself.

We still believe that our results and methods provides new insight
to modular representation theory. For example, we show how our
results lead to a guess on a tilting algorithm outside the lowest $p^2$
alcove, which at least in the $SL_2$-case coincides with Erdmann’s
results.

The paper is organized as follows. In the next section we review
briefly the representation theory of the symmetric group, following
James’s book [J]. In the third section we perform the calculations for
the restriction functors. In the fourth section we treat the induction
functors. These are the hardest calculations of the paper and, as a
matter of fact, we were forced to go via the representation theory of
the Hecke algebra where we can rely on Murphy’s standard basis. In
the fifth section we treat the duality. In the last section we present
the $SL_2$-algorithm.

2. Preliminaries

We will use the terminology of James book [J]. So let $k$ be an
arbitrary field and let $S_n$ denote the symmetric group on $n$ letters
acting on $\{1, \ldots, n\}$ on the right. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a
partition of $n$ into $r$ parts and let $S_\lambda$ be the corresponding Young
subgroup of $S_n$ which is defined as the row stabilizer of the $\lambda$-tableau
$\lambda$ in which the numbers $1, 2, \ldots, n$ are placed in the diagram along
the rows. The permutation module $M^\lambda$ is obtained by inducing the
trivial representation $k$ from $S_\lambda$ to $S_n$. It has a basis consisting
of so called $\lambda$-tabloids; these are equivalence classes of $\lambda$-tableaux,
where two tableaux $t_1$ and $t_2$ are said to be equivalent if there is an
element $\pi$ of the row stabilizer $R_{t_1}$ of $t_1$, such that $t_1\pi = t_2$. We
denote the tabloid class of the tableau $t$ by $\{t\}$.
(Although we shall usually assume that $\lambda$ is a partition, the above constructions and statements would also make sense for compositions, i.e. for unordered partitions. For instance, if $V$ is an $r$-dimensional vector space over $k$, we can make $V^\otimes n$ into an $S_n$-module by place permutation and as such we then have that

$$V^\otimes n = \bigoplus_\lambda M^\lambda$$

with $\lambda$ running over all compositions $\lambda$ of $n$ in less than $r$ parts. In other words, there is a tabloid basis of $V^\otimes r$ as well.)

Let $C_t$ be the column stabilizer of the $\lambda$-tableau $t$. We associate then to $t$ the element $e_t \in M(\lambda)$ as follows:

$$e_t = \sum_{\pi \in C_t} \text{sign} \pi \{t\pi\}$$

The Specht module $S(\lambda)$ labeled by $\lambda$ is defined as the subspace of $M(\lambda)$ generated by the $e_t$'s with $t$ running over the set of all $\lambda$-tableaux.

### 3. Restriction Functors

In this section we take a careful look at the proof of the branching rule given in James’s book [J]. According to this the restricted module $\text{Res}_{S_{n-1}}^{S_n} S(\lambda)$ has a Specht filtration, in which $S(\mu)$ occurs as a subquotient if and only if $\mu$ is a subdiagram of $\lambda$ such that the difference $\lambda \setminus \mu$ consists of exactly one node.

This filtration can be constructed in terms of the standard basis $\{e_t \mid t \text{ standard tableau}\}$ of $S(\lambda)$ in the following way: Let $r_1 < r_2 < \ldots < r_m$ be the row numbers of the rows from which a node can be removed to leave a diagram. Denote by $V_i$ the subspace of $S(\lambda)$ generated by the $e_t$'s where $n$ is in the $r_1$’th, $r_2$’th, $\ldots$ or $r_i$’th row of $t$. Then we have a filtration of $S_{n-1}$-modules

$$V_1 \subseteq V_2 \subseteq \ldots \subseteq V_m, \quad V_i/V_{i-1} \cong S(\lambda^i)$$
with \( \lambda^i \) denoting the partition that is obtained from \( \lambda \) by removing the last node from the \( r_i \)'th row. This construction is independent of the characteristic of \( k \).

Recall the \( S_n \)-invariant form \( \langle \cdot, \cdot \rangle_\lambda \) on the permutation module \( M(\lambda) \) which is defined by the formula

\[
\langle \{t_1\}, \{t_2\} \rangle_\lambda = \delta_{\{t_1\}, \{t_2\}}
\]

Its restriction to the Specht module \( S(\lambda) \subset M(\lambda) \) is also denoted \( \langle \cdot, \cdot \rangle_\lambda \).

The following theme is central to the paper: Assume that \( k = \mathbb{Q} \). Then \( \langle \cdot, \cdot \rangle_\lambda \) is nondegenerate. Let \( U_i \) be the complement with respect to \( \langle \cdot, \cdot \rangle_\lambda \) of \( V_{i-1} \) in \( V_i \). We then have that

\[
U_i \cong V_i / V_{i-1} \cong S(\lambda^i)
\]

Here \( \iota \) is induced from the injection of \( U_i \subseteq V_i \), while \( \pi \) is the map that takes \( e_t \in S(\lambda) \) to \( e_{t'} \in S(\lambda^i) \), where \( t' \) is obtained from \( t \) by removing the node containing \( n \). We can now define a new scalar product \( \langle \cdot, \cdot \rangle'_\lambda \) on \( U_i \) by pulling back the standard product \( \langle \cdot, \cdot \rangle_\lambda \) on \( S(\lambda^i) \) to \( U_i \). Since \( U_i \cong S(\lambda^i) \) is simple when \( \text{char} k = 0 \), we have that \( \langle \cdot, \cdot \rangle_\lambda \) and \( \langle \cdot, \cdot \rangle'_\lambda \) only differ by a scalar in \( k \) which we denote by \( m_i^2 \):

\[
\langle \cdot, \cdot \rangle_\lambda = m_i^2 \langle \cdot, \cdot \rangle'_\lambda
\]

James and Murphy [JM] calculated the scalars \( m_i^2 \). The expression provides the inductive step in their determination of the Gram matrix and so they call it the branching rule for the Gram matrix. It goes as follows:

\[
m_i^2 = \prod_{a=1}^{r_i-1} \frac{H_{ac}}{H_{ac} - 1}
\]  

(1)

where \((r_i, c)\) are the coordinates of the last node of the \( r_i \)'th row we are looking at and \( H_{ac} \) denotes the hook length of the hook centered at the node \((a, c)\).

**Example:**
Here we find $m_i^2 = \frac{8}{7} \cdot \frac{7}{6} \cdot \frac{5}{4} \cdot \frac{3}{2} = \frac{5}{2}$.

Let us fix $l \in \mathbb{N}$. In the LLT setup which we consider next, $l$ is the order of the root of unity. We are especially interested in the “quantum $l$-adic valuation” of the numbers $m_i^2$ which we define as follows

$$\nu_q(m_i^2) = \nu_p \left( \prod_{a=1}^{r_i-1} \frac{[H_{ac}]_v}{[H_{ac} - 1]_v} \right)$$

where $[n]_v = \frac{v^{n+1} - v^{-n-1}}{v - v^{-1}}$ is the usual Gaussian integer, while $\nu_p : \mathbb{Z}[v, v^{-1}] \rightarrow \mathbb{Z}$ is the valuation with respect to the $l$’th cyclotomic polynomial. Notice that if all the hook lengths $H_{ij}$ are less than $l$, this “quantum valuation” coincides with the usual $l$-adic valuation of the number $m_i^2$.

Let us consider the $q$-analogue of the Fock space $\mathcal{F}_q$ as introduced by Hayashi. It is defined as

$$\mathcal{F}_q = \bigoplus_{\lambda} \mathbb{Q}(q) |\lambda\rangle$$

with basis parameterized by the set of all partitions $\lambda$. There is an action of the quantum group $U_q(\hat{\mathfrak{sl}}_l)$ on $\mathcal{F}_q$ ($q$ is here transcendent), making $\mathcal{F}_q$ into an integrable module. Using this, Lascoux, Leclerc and Thibon defined a kind of Kazhdan-Lusztig algorithm, which first of all calculates the global crystal basis of $\mathcal{F}_q$, but also, by a theorem of Ariki, the decomposition numbers of the representation theory of the Hecke algebra at an $l$’th root of unity. We are aiming at a connection between this algorithm and the numbers $\nu_p(m_i^2)$.
coming from the branching rule. Let us therefore give the precise formulas for the action of $U_q(\widehat{sl}_l)$ on $\mathcal{F}_q$.

In order to do this we need the concept of an $i$-node of the Young diagram $Y(\lambda)$ of $\lambda$. We start by filling in the nodes of $Y(\lambda)$ with integers, increasingly along the rows, decreasingly along the columns and starting with 0 in the $(1,1)$’th position. We then reduce these numbers mod $l$, the resulting diagram is called the $l$-diagram of the partition. We call a node an $i$-node $i \in \mathbb{Z}/l\mathbb{Z}$ if is has been filled in with $i$. A node of the diagram is called removable if it can be removed to leave another diagram, and a (virtual) node that can be added to the diagram to yield another Young diagram, is called an indent node. Finally, we define for a subdiagram $\mu$ of $\lambda$ such that $\gamma = \lambda \setminus \mu$ consist of just one node, $I^r_i(\lambda, \mu)$ (resp. $R^r_i(\lambda, \mu)$) as the number of indent (removable) $i$-nodes situated to the right of $\gamma$. Let now $f_i, k_i$ and $e_i$ be the standard generators of the quantum group $U_q(\widehat{sl}_l)$. Define

$$N^r_i(\lambda, \mu) = I^r_i(\lambda, \mu) - R^r_i(\lambda, \mu), \quad N^l_i(\lambda, \mu) = I^l_i(\lambda, \mu) - R^l_i(\lambda, \mu)$$

(3)

Then $f_i$ and $e_i$ operate on $\mathcal{F}_q$ by the following formulas

$$f_i \lambda = \sum_{\mu} q^{N^r_i(\lambda, \mu)} |\mu\rangle \quad e_i \mu = \sum_{\lambda} q^{-N^l_i(\lambda, \mu)} |\lambda\rangle$$

(4)

where the first sum is over all partitions $\mu$ such that $\lambda \setminus \mu$ consists of one $i$-node and analogously in the second expression. There are similar formulas for the action of the other generators $k_i$ and $D$ of $U_q(\widehat{sl}_l)$, see eg. [LLT].

We can now formulate the following Theorem.

**Theorem 1.** The numbers $N^r_i(\lambda, \mu)$ and $\nu_q(m_i^2)$ coincide.

**Proof.** Let us consider the $(i,j)$ hook of the $l$-diagram.
Here \( a \) denotes the arm length of the hook, while \( b \) denotes the foot length of the hook. We now have that

\[
H_{ij} = a + b - 1 \equiv (\text{hand} - (j - i) + 1) + (j - i - \text{foot} + 1) \\
\equiv \text{hand} - \text{foot} + 1 \mod l
\]

with \( \text{hand} \) (resp. \( \text{foot} \)) denoting the residue of the hand node (foot node) of the hook.

Let us now calculate the constant \( \nu_q(m_i^2) \) at the partition \( \lambda \). We can wlog. assume that the \( (i, l) \)’th node is a 0-node. The \( l \)-diagram will then have the following form:

Let \( \lambda' > \lambda_i \) be some row length of the diagram and let \( s, t \) be such that

\[
\{s, s + 1, \ldots, t\} = \{j \mid \lambda_j = \lambda'\}.
\]
Then the contribution to $\nu_q(m_i^2)$ coming from $\{s, s+1, \ldots, t\}$ equals
\[ \nu_p \left( \frac{[H_{tl}]_v}{[H_{st} - 1]_v} \right) \]

Now, according to the above calculation $l$ divides the hook length $H_{il}$ if and only if the last node of the $i$th row is a $l - 1$ node. On the other hand, this means that there is an indent 0 node in the position to the right of this row. Analogously, we have that $l \mid H_{il} - 1$ if and only if the last node of the $i$th row is a 0-node: thus a removable 0-node. We conclude that the $t'$th row will contribute (with 1) to $\nu_q(m_i^2)$ if and only if it gives rise to an indent node, while the $t'$th row will contribute (with $-1$) to $\nu_q(m_i^2)$, if and only its end node is a removable one.

It just remains to consider the rows of length $\lambda_i$. So let $i'$ be given by
\[ \{i, i + 1, \ldots, i'\} = \{j \mid \lambda_j = \lambda_i\} \]
But then the corresponding rows will contribute with $H_{i'i}$ to $m_i^2$. And once again $l$ divides the hook length if and only if there is an indent 0-node in the position beyond the $i'$th row, so also here things match up. We have proved the Theorem. 

**Remark:** The action of $e_i$ is a deformation of Robinson’s $r$-restriction functors, so we would have expected $\nu_q(m_i^2) = -N_i^l(\lambda, \mu)$ in the above Theorem to get the complete match with (4). Let us therefore consider the action $e_i^l$ (resp. $f_i^l$) on $F_q$ which is equal to that of $e_i$ (resp. $f_i$) but with $q^{-N_i^l(\lambda, \mu)}$ (resp. $q^{N_i^r(\lambda, \mu)}$) replaced by $q^{N_i^r(\lambda, \mu)}$ (resp. $q^{-N_i^l(\lambda, \mu)}$). Define furthermore
\[ \varphi : F_q \to F_q \quad \lambda \mapsto q^{[\lambda]_l} \lambda \]
where $[\lambda]_l$ is the $l$-weight of the partition $\lambda$, i.e. the number of $l$-skewhooks one should remove to arrive at the $l$-core. Then we have that
\[ \varphi \circ e_i^l = e_i \circ \varphi, \quad \varphi \circ f_i^l = f_i \circ \varphi \]
which comes from the formula, [LLT, equation 13]
\[ -[\lambda]_l = -[\mu]_l + N_i^l(\lambda, \mu) + N_i^r(\lambda, \mu) \]
It is in other words basically the same problem to determine the lower global basis $G^l(\lambda)$ with respect to $f_i^l, e_i^l$ as with respect to $f_i, e_i$. The relation is

$$G^l(\lambda) = q^{[\lambda]} G(\lambda) \quad (5)$$

4. **Induction functors.**

We wish in this section to prove that the $f_i$-operators on the Fock space can be realized in a similar way as the $e_i$-operators. This is of course a conclusion one might expect. Actually the LLT algorithm only uses the $f_i$-operators so from our point of view, this conclusion is more important than the one for the $e_i$-operators.

A first approach towards this result might be to use a Frobenius reciprocity argument. We were however unable to find any arguments along that line. A reason why this approach apparently does not work may be that not all functors admit adjoints.

A major difference between the induction functors and the restriction functors is that, unlike the restricted module, the induced module does not come with a natural basis which is compatible with the Specht filtration, so to obtain results for the induction functors one cannot just copy the calculations done for the restriction functors. Now, it is possible to extract a basis of the induced Specht module from the chapter on the Littlewood Richardson rule in James book [J], which indeed is compatible with the Specht filtration – this basis is however not well suited for the diagonalization process – the calculations explode very quickly.

We choose to work with the Hecke algebra setting. Although this seems like a further complication of the problem, it provides us with a more natural setting for the simultaneous induction and diagonalization. The reason for this is first of all Murphy’s standard basis which turns out to behave well with respect to the diagonalization process. The results on the symmetric group case can then be obtained by specializing the Hecke algebra parameter $q$ to 1.
Let us now review the basic definitions of the representation theory of the Hecke algebra $H_n$ of type A; contrary to the symmetric group case the Specht modules are here constructed as ideals in $H_n$ itself.

Let the standard generators of the Hecke algebra $H_n$ of type A be $T_w$, $w \in S_n$; they satisfy the relations

$$T_w T_v = \begin{cases} T_{wv} & \text{if } l(wv) = l(w) + 1 \\ q T_{wv} + (q - 1) T_w & \text{otherwise} \end{cases} \tag{6}$$

$$T_v T_w = \begin{cases} T_{vw} & \text{if } l(vw) = l(w) + 1 \\ q T_{vw} + (q - 1) T_w & \text{otherwise} \end{cases} \tag{7}$$

with $l(\cdot)$ denoting the standard length function on $S_n$. One checks that these relations make $H_n$ into an associative algebra over $\mathbb{Z}[q, q^{-1}]$ with unit element $T_1$ and with $\{T_w \mid w \in S_n\}$ as a basis.

Now for $X \subseteq S_n$ we define the following elements of $H_n$

$$\iota(X) = \sum_{w \in X} T_w, \text{ and } \epsilon(X) = \sum_{w \in X} (-q)^{-l(w)} T_w$$

Let $h^*$ denote the image of $h \in H_n$ under the antiautomorphism of $H_n$ induced by the map $T_w \mapsto T_{w^{-1}}$, $w \in S_n$. For any row standard $\lambda$-tableau $t$ we define $d(t) \in S_n$ as the element of $d \in S_n$ satisfying $t = t^\lambda d$. We denote the row stabilizer of $t^\lambda$ by $S_\lambda$ and can introduce the following elements of $H_n$

$$x_{st} = T_{d(s)}^* \iota(S_\lambda) T_{d(t)} \text{ and } y_{st} = T_{d(s)}^* \epsilon(S_\lambda) T_{d(t)}$$

With $\lambda$ running through all partitions of $n$ and $s, t$ through all row standard $\lambda$-tableaux, the $x_{st}$ (as well as the $y_{st}$) form a basis of the Hecke algebra: the Murphy standard basis of $H_n$. It is a cellular basis in the sense of Graham and Lehrer [GL], indeed it is in many senses the prototype of a cellular basis.

Let $s$ and $t'$ be row standard $\lambda$ and $\lambda'$ tableaux, respectively; then we define

$$z_{st} = T_{d(s)}^* x_{\lambda\lambda'} T_{w_{\lambda'}} y_{\lambda'\lambda} T_{d(t')}$$
where the subscript $\lambda$ stands for the tableau $t^\lambda$ and where $w_\lambda \in S_n$ is defined by the property that the elements of $t^\lambda w_\lambda$ are entered by columns. The Specht module $S(\lambda)$ is now the right ideal of $H_n$ generated by $z_{\lambda\lambda}$, it has a basis consisting of

$$\{ z_{\lambda t} \mid t \text{ is a standard } \lambda \text{ tableau} \}.$$

Our first task will be to describe a basis of $\text{Ind} S(\lambda)$: the $H_n$-Specht module induced up to $H_{n+1}$. Let then $\lambda_e$ be the partition of $n+1$ which is equal to $\lambda$ except that $(\lambda_e)_1 = \lambda_1 + 1$. Dually we introduce the partition $\lambda_e$ of $n+1$ by $\lambda_e = ((\lambda')^e)'$. For a pair of partitions $(\lambda, \mu)$ we write $(\lambda, \mu)' := (\lambda', \mu')$ and similarly for pairs of tableaux.

We were unable to find the following Lemma in the literature:

**Lemma 1.** $\text{Ind} S(\lambda) = \text{span}\{ x_{\lambda\lambda} T_{w_\lambda} y_{(\lambda^e,s)'} \mid s \ \lambda^e\text{-tableau} \}$

**Proof.** By the definition we have that

$$S(\lambda) = \text{span}\{ x_{\lambda\lambda} T_{w_\lambda} y_{\lambda's'} \mid s \ \lambda\text{-tableau} \}$$

Now the transpositions $\{(1, n + 1), (2, n + 1), \ldots, (n, n + 1)\}$ form a set of coset representatives of $S_n \setminus S_{n+1}$ and we have

$$\text{Ind} S(\lambda) = S(\lambda) \otimes_{H_n} H_{n+1} = \bigoplus_{j=1}^{n} S(\lambda) T_{(j,n+1)}$$

as $\mathbb{Z}[q, q^{-1}]$-modules. But then the fact that $T_{(j,n+1)}$ is invertible implies that $\text{Ind} S(\lambda)$ is free over $\mathbb{Z}[q, q^{-1}]$. We can also work out the rank by passing to the quotient field $\mathbb{Q}(q)$ and using Frobenius reciprocity: we know everything about the restriction functors. The rank is given by the branching rule.

Now it is clear from the definition that $y_{\lambda'\lambda'} = y_{(\lambda')_e (\lambda')_e} = y_{(\lambda^e, \lambda^e)'}$ and then

$$x_{\lambda\lambda} T_{w_\lambda} y_{\lambda'\lambda'} = x_{\lambda\lambda} T_{w_\lambda} y_{(\lambda^e, \lambda^e)'}$$

Combining these equations we see that the right hand side of the Lemma is a quotient of the left hand side. But by the next Lemma the $H_{n+1}$-module

$$\text{span}\{ x_{\lambda\lambda} T_{w_\lambda} y_{(\lambda^e,s)'} \mid s \ \lambda^e\text{-tableau} \}$$

is free over $\mathbb{Z}[q, q^{-1}]$. We can also work out the rank by passing to the quotient field $\mathbb{Q}(q)$ and using Frobenius reciprocity: we know everything about the restriction functors. The rank is given by the branching rule.
has a Specht filtration and is thus free over $\mathbb{Z}[q, q^{-1}]$ as well. Since also this rank is given by the branching rule we are done. □

Recall the total order $<$ on partitions introduced eg. in [M1]. See also loc. cit. for the extension of this ordering to tableaux, of arbitrary shapes, which we shall also denote by $<$. For a partition $\lambda$, the tableau $t^\lambda$ dominates all tableaux of shape $\lambda$. We then have the next Lemma:

**Lemma 2.** The $H_{n+1}$-module $I = \text{span}\{ x_{\lambda\lambda} T_{w\lambda} y_{(\lambda^e, s)} \mid s \ \lambda^e$-tableau $\}$ has a Specht filtration

$$0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_m = I$$

where the $\lambda^i$s are the partitions that can be obtained from $\lambda$ by adding exactly one node to $\lambda$ ordered such that $\lambda^1 \leq \lambda^2 \leq \ldots \leq \lambda^m$.

**Proof.** Using Lemma 3.5 in [M1] (the Garnir relations) one can express $y_{(\lambda^e, s)}$ for all $\lambda^e$-tableaux $s$ as a linear combination of the elements

$$\{ y_{u'v'} \mid u, v \text{ standard } \mu$-tableau such that $\mu \leq \lambda^e \} \quad (8)$$

On the other hand it is a general fact (see the remarks before Lemma 3.6 of [M1]) that for any $w \in S_n$ and any partitions $\tau, \nu$ of $n$ we have

$$x_{\tau\tau} T_w y_{\nu\nu} \neq 0 \Rightarrow \tau \leq \nu'$$

(9)

We have that $x_{\lambda\lambda} = x_{\lambda_e \lambda_e}$ and $y_{u'v'} = T_{d(u')}^* y_{\mu'\mu'} T_{d(v')}$ so we can use this general fact on the partitions $\lambda_e$ and $\mu'$ of $n + 1$. We find that only $\mu$ with $\lambda_e \leq \mu$, can make $x_{\lambda\lambda} T_{w\lambda} y_{u'v'}$, nonzero, when $y_{u'v'}$ is from the set (8). But the only $\mu$ that satisfy the simultaneous inequalities

$$\lambda_e \leq \mu \leq \lambda^e$$

are those that can be obtained from $\lambda$ by adding exactly one node.

We need a more precise analysis of the $\mu$-tableaux $u$ that can occur as first index in $y_{u'v'}$ from (8), in such a way that $x_{\lambda\lambda} T_{w\lambda} y_{u'v'}$ is nonzero. These must first of all satisfy $u' \geq t^{(\lambda^e)'}$; this is once
again according to Lemma 3.5 in [M1]. Let \( u'\{ \hat{n} + 1 \} \) denote the tableau that is obtained from \( u' \) by removing the \( n + 1 \)-node. Then by definition of the order on tableaux we have \( u'\{ \hat{n} + 1 \} \geq t^\lambda \) or equivalently \( u\{ \hat{n} + 1 \} \leq t_\lambda \). But then also the underlying partition \( P\left(u\{ \hat{n} + 1 \}\right) \) of the tableau \( u\{ \hat{n} + 1 \} \) satisfies that

\[
P\left(u\{ \hat{n} + 1 \}\right) \leq \lambda.
\]

On the other hand, since \( x_{\lambda\lambda} T_{w_{\lambda}} y_{w'w'} = x_{\lambda\lambda} T_{w_{\lambda}} y_{w'w'} \neq 0 \) we have (as before) that

\[
\lambda_e \leq P(u)
\]

Since \( \mu \) is obtained from \( \lambda \) by adding one node, this is only possible if

\[
P\left(u\{ \hat{n} + 1 \}\right) = \lambda
\]

and since \( t_\lambda \) is the minimal tableau of shape \( \lambda \) we conclude from \( u\{ \hat{n} + 1 \} \leq t_\lambda \) that in fact

\[
u\{ \hat{n} + 1 \} = t_\lambda.
\]

So all in all we have proved that the only tableaux \( u \) that can occur as the first index of \( y_{\mu'\nu'} \) are those that are obtained from \( t_\lambda \) by adding an extra node which must contain the number \( n + 1 \).

We must now show that all these partitions are actually needed. Let \( \mu \) be a partition of \( n - 1 \) and let \( t \) be the tableau obtained from \( t^\mu \) by attaching \( n \) to the \( i \)th row. Define

\[
\alpha(\lambda) = \sum_i i\lambda_i
\]

Then one has the formula (see the proof of Lemma 4.4 in [M1]).

\[
q^\alpha(t) - l(d(i)) y_{tt} = q^i \left( 1 - \sum_{c=j}^{n-1} q^{c-m} T_{(c,n)} \right) q^\alpha(\mu) y_{\mu\mu} + q^{i-1} \sum_{c=n}^{n-1} q^{c-n} T_{(c,n)} q^\alpha(\mu) y_{\mu\mu}
\]

(10)

where \( \{j, \ldots, m\} \) are the nodes of the \( i \)th row of \( P(t) \). Notice that there seems to be a, for our purposes irrelevant, sign error in loc.cit.
We can apply the antiautomorphism \( \ast \) to get a similar formula with the \( T_{(c,m)} \) operating on the right.

Let \( \lambda^i \) be the partition of \( n + 1 \) obtained from \( \lambda \) by adjoining a node to the \( i \)’th row (if possible). Let \( t^i \) be the \( \lambda^i \)-partition obtained from \( t_\lambda \) by filling in the extra node with \( n + 1 \).

Then one gets from the above formula (10), or rather its \( \ast \)-version, and multiplication with an appropriate \( T_w \) that the

\[
x_{\lambda\lambda} T_{w_\lambda} y_{(t^i,s)'}
\]

will all occur in the induced module for all \( \lambda^i \)-tableaux \( s \).

We finally claim that

\[
\{ x_{\lambda\lambda} T_{w_\lambda} y_{(t^i,s)'} \mid s \text{ standard } \lambda^i \text{-tableaux, } i = 1, \ldots \}
\]

(11)
is a linearly independent set. To do so we may specialize \( q = 1 \); then we have that

\[
x_{\lambda\lambda} T_{w_\lambda} y_{(t^i,t^i)'} = x_{\lambda\lambda} T_{w_\lambda} y_{\lambda',\lambda'} \left( 1 - T_{(j,n)} - \cdots - T_{(m-1,n)} \right)
\]

and since the \((i, n)\) are coset representatives of \( S_{n-1} \) in \( S_n \) we deduce that at least the

\[
x_{\lambda\lambda} T_{w_\lambda} y_{(t^i,t^i)'}
\]

are linearly independent with \( i \) varying. Now letting \( s \) vary over standard \( \lambda \)-tableaux the

\[
x_{\lambda\lambda} T_{w_\lambda} y_{\lambda',s'}
\]

are linearly independent (a basis of the Specht module), so the argument used before can be generalized.

We have now shown that the elements from (11) form a basis of the module from the Lemma. Defining

\[
V_i := \{ x_{\lambda\lambda} T_{w_\lambda} y_{(t^i,s)'} \mid s \text{ standard } \lambda^k \text{-tableaux, } k = 1, \ldots , i \}
\]
it follows from Lemma 3.5 of [M1] that the \( V_i \)’s are \( H_{n+1} \) modules, and since clearly \( V_i/V_{i-1} \cong S(\lambda^i) \), the proof of the Lemma is finished. \( \square \)
We now come to the diagonalizing procedure; in the rest of this section we shall be working over \( \mathbb{Q}(q) \). The idea is to reuse Murphy’s construction of Young’s seminormal form, based on the idempotents \( E_t \). So let us recall the definition and basic properties of these.

Let \( \lambda \vdash n \) and let \( t \) be a \( \lambda \)-tableau. The (generalized) residue of the \((i, j)\) node of \( t \) is defined to be \( [j - i]_q \) where

\[
[k]_q := 1 + q + \ldots + q^{k-1}
\]

The residue of the node occupied by \( m \) in \( t \) is denoted \( r_t(m) \) and the set of possible residues for standard tableaux by \( \mathcal{R}(m) \).

For any tableau \( t \) we define

\[
E_t := \prod_{m=1}^{n} \prod_{c \in \mathcal{R}(n) \setminus \{r_t(m)\}} \frac{L_m - c}{r_t(m) - c}
\]

where \( L_m \) is the \( q \)-analogue of the Murphy-Jucys operator introduced in [M1]:

\[
L_m = q^{-1}T_{(m-1,m)} + q^{-2}T_{(m-2,m)} + \ldots + q^{-m}T_{(1,m)}
\]

The set

\[
\{ E_t \mid t \text{ standard} \}
\]

is a complete set of orthogonal idempotents of the Hecke algebra \( H_n \) (defined over \( \mathbb{Q}(q) \)) while \( E_t = 0 \) for \( t \) nonstandard.

Furthermore we have the following key formula

\[
y_{st} E_u = \delta_{tu} y_{st} + \sum_{\sigma, \tau \mid \sigma \tau \text{ standard } (\sigma, \tau) > (s,t)} a_{\sigma \tau} y_{\sigma \tau} \tag{12}
\]

where \( a_{\sigma \tau} \in \mathbb{Q}(q) \). (This version of the formula is obtained by combining Theorem 4.5 and the \( \eta \)-version of (5.1) of [M1]). It implies that if we set \( f_t := z_{\lambda t} E_t \) we get a new basis (Youngs seminormal form) of the Specht module

\[
\{ f_t \mid t \text{ standard } \lambda-\text{tableau} \}
\]

The orthogonality of the idempotents implies that the \( f_t \) are orthogonal with respect to the bilinear form on \( S(\lambda) \). Moreover one
can calculate the length of the \( f_t \); here once again one of the main ingredients is the above formula. All this theory is developed in the papers \([M1],[M2]\) and \([M3]\).

We now focus on the subquotient \( S(\lambda^i) \) of \( \text{Ind}\ S(\lambda) \) which arises from adjoining a node to the \( i \)'th row of \( \lambda \). Recall the \( \lambda^i \)-tableau \( t^i \) which is obtained from \( t^\lambda \) by letting the new node be an \( n + 1 \) node. Let \( d^i := d(t^i)^{-1} \). The lowest vector of the subquotient \( S(\lambda^i) \) in \( \text{Ind}\ S(\lambda) \) is now the coset of

\[
x_{\lambda\lambda} T_{w_\lambda} T_{d^i}^* y_{(\lambda^i,\lambda^i)^{\prime}} = x_{\lambda\lambda} T_{w_\lambda} T_{d^i} y_{(\lambda^i,\lambda^i)^{\prime}}
\]

(13)

Let \( t_i \) be the lowest \( \lambda^i \)-tableau, in which the \( \{1, \ldots, n + 1\} \) are filled in increasingly along the columns. (Note that \( t_i \) is not equal to the conjugate of \( t^i \)). Then

\[
* x_{\lambda\lambda} T_{w_\lambda} T_{d^i} y_{(\lambda^i,\lambda^i)^{\prime}} E_{t_i}
\]

(14)

is equal to \( x_{\lambda\lambda} T_{w_\lambda} T_{d^i} y_{(\lambda^i,\lambda^i)^{\prime}} \mod \) the submodule \( V_{i-1} \) of \( \text{Ind}\ S(\lambda) \) corresponding to the \( \lambda^j \)'s with \( j < i \). Furthermore, \( * \) is orthogonal to \( V_{i-1} \); here the bilinear form on \( \text{Ind}\ S(\lambda) \) is the restriction of the one on the Hecke algebra \( H_{n+1} \) – it is given by

\[
\langle a, b \rangle := \text{coefficient of } T_1 \text{ in } ab^*
\]

Let \( U_i \) be the orthogonal complement of \( V_{i-1} \) in \( V_i \). Then \( U_i \cong S(\lambda^i) \) canonically and – like in the restriction functor case – we get two forms on \( U_i \), which we wish to compare.

We can view \( * \) as the lowest vector of \( U_i \); it is mapped to

\[
** x_{\lambda_i\lambda^i} T_{w_{\lambda^i}} y_{(\lambda^i,\lambda^i)^{\prime}}
\]

(15)

under the above isomorphism \( U_i \cong S(\lambda^i) \). So our task is to calculate the lengths of \( * \) and \( ** \) and compare. The quotient of the two lengths is the number we are looking for.

We need a further piece of notation. Let \( \lambda \vdash n \). Denote by \( h_{ij} \) the hook length of the hook centered at \( (i, k) \) and define the hook
product \( h_\lambda \) by
\[
h_\lambda = \prod_{(i,j) \in \lambda} [h_{ij}]_q
\]

Let \( t \) be a \( \lambda \)-tableau; if \( n \) is in the \( i \)th row of \( t \) we define the hook-quotient \( \gamma_{tn} \) to be
\[
\gamma_{tn} = \frac{\lambda_i}{\prod_{j=1}^{\lambda_i} [h_{ij}]_q [h_{ij} - 1]_q}
\]
For \( m < n \) we define \( \gamma_{tm} \) similarly, except that we this time remove all the nodes \( m + 1, \ldots, n \) from \( t \). Finally we let
\[
\gamma_t = \prod_{m=2}^{n} \gamma_{tm}
\]
In the case of \( t^\lambda \), we then get
\[
\gamma_\lambda = \prod_{i>0} \prod_{j=1}^{\lambda_i} [j]_q
\]
We abbreviate \( \lambda_i' \) for \( (\lambda^i)' \). We now have the following Lemma:

**Lemma 3.** Let \( * \) be as above. Then
\[
\langle *, * \rangle = \gamma_{\lambda_i'} \gamma_\lambda \prod_{j=1}^{\lambda_i} \frac{[h_{ij} - 1]_q}{[h_{ij}]_q}
\]

**Proof.** First of all we see from Theorem 4.5, including the remarks after the Theorem, and Lemma 6.1 in \([M1]\) that
\[
* = q^{n-\alpha(\lambda)} \gamma_{\lambda_i'} x_{\lambda \lambda} T_{w_\lambda} T_{d^i} E_{t_i}
\]
where \( \alpha \) is the function on partitions (and tableaux) introduced in the proof of the preceding Lemma. However, since we are really only interested in a certain valuation of \( \langle *, * \rangle \), we shall from now on omit the \( q \)-power of the expression. Hence we get
\[
\langle *, * \rangle = \gamma_{\lambda_i'}^2 \langle x_{\lambda \lambda} T_{w_\lambda} T_{d^i} E_{t_i}, x_{\lambda \lambda} T_{w_\lambda} T_{d^i} E_{t_i} \rangle
\]
\[
\gamma_\lambda \gamma_{\lambda_i'}^2 \langle x_{\lambda \lambda}, T_{w_\lambda} T_{d^i} E_{t_i}, T_{d^i}^* T_{w_\lambda}^* \rangle
where we used that \( x_{\lambda \lambda}^2 = \gamma_{\lambda} x_{\lambda \lambda} \) which follows from the definitions (or from Theorem 4.5 and Lemma 6.1 in [M1]).

In order to calculate \( T_{w_\lambda} T_{d_i} E_{t_i} T_{d_i}^* T_{w_\lambda}^* \) we shall use Theorem 6.4 of [M1], which gives formulas for the product \( \zeta_{us} T_v \) where \( v = (i-1, i) \) and \( \zeta_{us} := E_u x_{us} E_s \) (actually [M1] uses the basis \( \xi_{us} \) in the definition of \( \zeta_{us} \); however this basis is related to the standard basis \( x_{us} \) by an upper triangular matrix with ones on the diagonal and thus gives rise to the same \( \zeta_{us} \)). To get the action on the \( E_t \)'s we then use the formula

\[
\zeta_{tt} = \gamma_t E_t
\]

which is also proved in [M1].

Let us quote the mentioned Theorem 6.4.

**Theorem 2.** Let \( s, u \) be standard tableaux of the same shape, \( v = (i-1, i) \), \( t = sv \). Let \((a, b)\) and \((a', b')\) be the nodes occupied by \( i - 1 \) and \( i \) respectively in \( s \) and let \( h = b - b' - a + a' \); then

\[
\zeta_{us} T_v = \begin{cases} 
\frac{1}{[h]_q} \zeta_{us} & \text{if } |h| = 1, \\
\frac{1}{[h]_q} \zeta_{us} + \zeta_{ut} & \text{if } h > 1, \\
\frac{1}{[h]_q} \zeta_{us} + \frac{q [h+1]_q [h-1]_q}{[h]_q^2} \zeta_{ut} & \text{if } h < -1, 
\end{cases}
\]

By applying * we get formulas for the left multiplication with \( T_v \) as well.

The plan is now to write \( d_i \) and \( w_\lambda \) as a product of basic transpositions and use the above formulas. Let \( s^i \) be the \( \lambda^i \)-tableau obtained from \( t^\lambda \) by letting the new node be an \( n + 1 \)-node. Recalling that \( t_i \) is the smallest \( \lambda^i \)-tableau, the result of applying this to \( T_{w_\lambda} T_{d_i} E_{t_i} T_{d_i}^* T_{w_\lambda}^* \) will be a sum of \( \zeta_{us} \) with \( u \) and \( s \) less than \( s^i \).

However, since

\[
x_{\lambda \lambda} E_{\tau} = E_{\tau} x_{\lambda \lambda} = 0 \quad \text{for } \tau < \lambda \tag{16}
\]
(by Theorem 4.5, Lemma 6.1 and (5.5) of [M1]) and since \( \zeta_{us} := E_u x_{us} E_s \) we actually only need a small part of the sum to determine \( \langle *, * \rangle \).

Notice first of all that
\[
d_i = (k, k + 1, \ldots, n + 1) = (n + 1, n) \cdot \ldots \cdot (k + 1, k)
\]
where \( k \) is the number in the position \((i, \lambda_i + 1)\) of \( t^\lambda \). This is a reduced presentation of \( d_i \) and so \( T_{d_i} = T_{(n+1,n)} \cdot \ldots \cdot T_{(k,k+1)} \). The product \( T_{d_i} E_{t_i} T_{d_i}^* \) will therefore only involve \( \zeta_{us} \)'s in which \( n + 1 \) occurs in \( u \) and \( s \) in positions higher than in \( s^i \); removing \( n + 1 \) from these will lead to partitions smaller than or equal to \( \lambda \). But by (16) we can neglect those \( u \) and \( s \) where \( n + 1 \) is in a strictly higher position.

We can repeat this argument on the remaining numbers \( 1, 2, \ldots, n \) and find that only for \( u = s = s^i \) there will be a contribution to \( \langle *, * \rangle \) from \( T_{w^\lambda} T_{d_i} E_{t_i} T_{d_i}^* T_{w^\lambda} \).

Let \( s \) and \( t \) be standard tableaux with \( t = s v \), \( s < t \) for \( v = (i, i - 1) \). Then we are in the situation \( h < -1 \) of the Theorem, so we get (once again ignoring the \( q \)-power):
\[
\frac{1}{\gamma_s} T_v E_s T_v = \frac{1}{\gamma_s} T_v \zeta_{ss} T_v = \frac{1}{\gamma_s} \left( \frac{|h+1|_q}{|h|_q} \right)^2 \zeta_{tt} + \text{lower terms}
\]
\[
= \frac{1}{\gamma_i} \zeta_{tt} + \text{lower terms}
\]
\[
= \frac{1}{\gamma_i} E_t + \text{lower terms}
\]

We are now in position to calculate the length \( \langle *, * \rangle \). Using the above formula we get
\[
\langle *, * \rangle = \gamma_{\lambda} \gamma_{\lambda'}^2 \langle x_{\lambda\lambda}, T_{w^\lambda} T_{d_i} E_{t_i} T_{d_i}^* T_{w^\lambda}^* \rangle
\]
\[
= \frac{\gamma_{t_i}}{\gamma_{s_i}} \gamma_{\lambda} \gamma_{\lambda'}^2 \langle x_{\lambda\lambda}, E_{s^i} \rangle
\]
\[
= \frac{\gamma_{t_i}}{\gamma_{s_i}} \gamma_{\lambda} \gamma_{\lambda'}^2 \langle E_{s^i}, E_{s^i} \rangle
\]
For the last equality we factored \( E_{\lambda} \) out of \( E_{s^i} \), applied the formula \( x_{\lambda\lambda} E_{\lambda} = \gamma_{\lambda} x_{\lambda\lambda} \) and plugged in the extra factor once again.
But the length of the idempotents is known, see the last page of [M1], it is
\[ \langle E_t, E_t \rangle = \frac{1}{h_\lambda} = \frac{1}{\gamma_t \gamma_t'} \]
This holds at least for \( t = t^\lambda \) by loc.cit. However one checks that the argument of Theorem 6.6 actually is valid for any \( \lambda \)-tableau \( t \) and thus the length \( \langle E_t, E_t \rangle \) is independent of \( t \).

All in all we have
\[ \langle *, * \rangle = \frac{\gamma_i}{\gamma_s} \gamma^{2} \gamma_{\lambda'}^{2} \frac{1}{\gamma_{\lambda'} \gamma_i} \]
\[ = \gamma_{\lambda'} \gamma \prod_{j=1}^{\lambda_i} \frac{[h_{ij}-1]_q}{[h_{ij}]_q} \]
and the proof of the Lemma is finished. \( \square \)

Let us return to \( \text{Ind}_S(\lambda) \) and its filtration
\[ 0 \subseteq V_1 \subseteq V_2 \subseteq \ldots V_m = I \text{ with } V_i / V_{i-1} \cong S(\lambda^i) \]
The projection map goes as follows:
\[ \pi_i : V_i \to S(\lambda^i) \]
\[ x_{\lambda \lambda} T_{w_{\lambda}} y_{(t \lambda)t'} \mapsto \begin{cases} x_{\lambda' \lambda'} T_{w_{\lambda'}} y_{(\lambda' \lambda')'} & \text{if } j = i \\ 0 & \text{if } j > i \end{cases} \]
We already saw that \( x_{\lambda \lambda}^2 = \gamma_{\lambda} x_{\lambda \lambda} \) and \( y_{\lambda \lambda}^2 = \gamma_{\lambda} y_{\lambda \lambda} \). Since furthermore \( x_{\lambda \lambda} y_{t \lambda t} = 1 - \text{ignoring } q\text{-powers} \) – we find
\[ \langle x_{\lambda' \lambda'} T_{w_{\lambda'}} y_{(\lambda' \lambda')'}, x_{\lambda' \lambda'} T_{w_{\lambda'}} y_{(\lambda' \lambda')'} \rangle = \gamma_{\lambda'} \gamma_{(\lambda')'} \]
A similar argument shows, using (13), that
\[ \langle x_{\lambda \lambda} T_{w_{\lambda}} y_{t' t'}, x_{\lambda \lambda} T_{w_{\lambda}} y_{t' t'} \rangle = \gamma_{\lambda} \gamma_{(\lambda')'} \]
So we conclude that the isomorphism \( V_i / V_{i-1} \cong S(\lambda^i) \) stretches all squared lengths by the factor \([\lambda_i + 1]_q\).

Recall \( U_i \) the orthogonal complement of \( V_{i-1} \) in \( V_i \), we have \( U_i \cong S(\lambda^i) \). We consider then two \( S_{n+1} \)-invariant forms on \( U_i \), namely the usual one \( \langle *, * \rangle \) coming from the embedding \( U_i \subseteq \text{Ind} S(\lambda) \) and \( \langle *, * \rangle_1 \) which is the pullback of the form on \( S(\lambda^i) \) along \( \pi_i \). The two forms differ by a constant \( m \):
\[ \langle *, * \rangle = m \langle *, * \rangle_1 \]
The next theorem gives the promised statements about $m$:

**Theorem 3.** Let $l > 1$ and let $\nu_l$ be the $[l]_{q}$-adic valuation $\mathbb{Z}[q, q^{-1}]$ defined in (2). Consider the $l$-diagram of $\lambda^i$ and assume that $\tau = \lambda^i \setminus \lambda$ is a $k$-node. Then

$$\nu_q(m) = N^l_i(\lambda^i, \lambda)$$

where $N^l_i(\lambda^i, \lambda)$ is the number of removable $k$-nodes situated to the left of $\tau$ minus the number of indent $k$-nodes situated to the left of $\tau$.

**Proof.** This follows by combining the previous Theorem with (14), (15), (17), (18), along with an argument similar to the one of Theorem 1. □

5. Duality

We now return to the symmetric group case. We wish to investigate the above logic on the duality of the symmetric group.

Recall that for any right module $M$ of a finite group $G$ the dual module is defined by

$$M^* := \text{Hom}_k(M, k); \quad (fg)(m) = f(mg), \quad f \in M^*, \quad m \in M, \quad g \in G$$

There is then the following classical result:

$$S(\lambda)^* \cong S(\lambda') \otimes k_{alt}$$

The way James [J] proves this result is as follows: $S(\lambda)$ is by construction a submodule of the permutation module $M(\lambda)$, which comes with an invariant form $\langle \cdot, \cdot \rangle_\lambda$, which is nondegenerate independently of the field $k$. One can construct a surjection

$$\pi : \quad M(\lambda) \twoheadrightarrow S(\lambda') \otimes k_{alt}$$

and can prove that $\ker \pi = S(\lambda)^\perp$. Since $M(\lambda)/S(\lambda)^\perp \cong S(\lambda)^*$, the isomorphism $S(\lambda)^* \cong S(\lambda') \otimes k_{alt}$ now follows.

Working over $\mathbb{Q}$ the restriction of $\pi$ to $S(\lambda)$ is an isomorphism, i.e. $S(\lambda)^* \cong S(\lambda)$. So in that case there are two invariant forms on $S(\lambda') \otimes k_{alt}$: the first one is $\langle \cdot, \cdot \rangle_{\lambda'} \otimes 1$ which comes from the
embedding \( S(\lambda') \subseteq M(\lambda') \), the second one is obtained by carrying over the form on \( S(\lambda) \) to \( S(\lambda') \otimes k_{alt} \). As usual the two forms differ by a scalar \( m \in \mathbb{Q} \); we wish to calculate the \( \nu_q \)-adic valuation of it.

Let us therefore take a closer look at the way \( \pi \) is defined. For a \( \lambda \)-tableau \( t \) we denote by \( \kappa_t \) the alternating column sum \( \kappa_t = \sum_{C(t)} \text{sgn} \sigma \sigma \) and by \( \rho_t \) the row sum \( \rho_t = \sum_{R(t)} \sigma \) of \( t \). So our generators \( e_t \) of the Specht module \( S(\lambda) \) have the form

\[
e_t = \{ t \} \kappa_t \quad - \text{here } \{ t \} \text{ was the tabloid class of } t.
\]

With this notation, \( \pi \) is the map given by

\[
\pi : S(\lambda) \longrightarrow S(\lambda') \otimes k_{alt}
\]

\[
e_t = \{ t \} \mapsto \{ t' \} \kappa'_t \rho'_t \otimes 1
\]

Using \( \rho'^2_t = |R(t)| \rho_t \), we find that

\[
\langle \pi(e_t), \pi(e_t) \rangle = \langle \{ t' \} \kappa'_t \rho'_t, \{ t' \} \kappa'_t \rho'_t \rangle
\]

\[
= \langle t', t' \kappa'_t \rho'^2_t \kappa'_t \rangle
\]

\[
= |R(t)| \langle t', t' \kappa'_t \rho'_t \kappa'_t \rangle
\]

\[
= |R(t)| h_{\lambda'} \langle t', t' \kappa'_t \rangle
\]

\[
= |R(t)| h_{\lambda'}
\]

One again \( h_{\lambda} \) denotes the product of all hook lengths in \( \lambda \); for the second last equality we used Lemma 23.2 in James’s book [J].

On the other hand we have that:

\[
\langle e_t, e_t \rangle = \langle \{ t \} \kappa_t, \{ t \} \kappa_t \rangle = |C_t|
\]

Since

\[
|C_t| = |R'_t|
\]

we deduce that our constant \( m \) is equal to \( h_{\lambda} \). But by 2.7.40 the number of \( p \)-divisible hooks equals the \( p \)-weight of \( \lambda \), i.e.

\[
\nu_q(m) = |\lambda|_p
\]

But this is exactly the \( q \)-power that appears in [LLT]'s duality, see Theorem 7.2 of loc. cit.
6. The $SL_2$-situation.

In this section we shall comment on the problem of determining the tilting modules outside the lowest $p^2$-region. This is probably the most difficult problem in the modular representation theory, there is so far not even a conjecture around. In type A this problem is known to be equivalent to determining the decomposition matrix of the symmetric group.

In fact, if $T(\lambda) \in \text{mod-}GL_m$ is the tilting module labeled by the partition $\lambda$ of $n$, $m \geq n$, we have, see [E]:

$$[T(\lambda), \Delta(\mu)] = [S(\mu), D(\lambda)] \quad \lambda \in \text{Par}_p, \mu \in \text{Par}$$

where Par denotes the partitions of $n$ while Par$_p$ denotes the $p$-regular partitions of $n$. Based on this formula, or rather a $q$-analogue of it relating the relevant $q$-Schur and Hecke-algebras, one can show that the LLT-algorithm is nothing but an extension of Soergel’s tilting algorithm to singular weights.

Now the calculations done in the previous chapters lead naturally to the following guess on how to obtain the general tilting modules: perform the LLT-algorithm as usual but replace everywhere the quantum valuations by ordinary $p$-adic valuations.

In this section we shall see that this idea seems to work at least in the $SL_2$-situation. On the other hand we point out right from the beginning that already in the $SL_3$-case the direct generalization of the resulting algorithm does not work. See however the work of A. Cox, [AC].

Let us recall the work of K. Erdmann [E] on the tilting modules for $SL_2$. We start out with some notation: any natural number $a$ has a $p$-adic decomposition

$$a = a_0 + a_1p + a_2p^2 + \ldots a_kp^k$$

For any other natural number $b$, we shall say that $a$ contains $b$ if in the $p$-adic decomposition of $b$

$$b = b_0 + b_1p + b_2p^2 + \ldots b_lp^l$$
$l < k$ and for all $i$: $b_i = a_i$ or $b_i = 0$.

The representation theory of $SL_2$ is parameterized by $\mathbb{N}$, i.e. for every natural number $n$ there is a tilting module $T(n)$ with highest weight $n$ and so on. K. Erdmann has now obtained the following result, [E]:

**Theorem 4.** Let $p \neq 2$. Then the multiplicity of the Weyl module $\Delta(s)$ in $T(m)$ is given by the formula:

\[
[T(m), \Delta(s)] = \begin{cases} 
1 & \text{if } m + 1 \text{ contains } \frac{1}{2}(m - s) \\
0 & \text{otherwise}
\end{cases}
\]

**Example:** Let us take $p = 3$ and $m = (p^3 + p^2 + 2) - 1$. The Theorem then gives rise to the following alcove pattern, where the dots indicate Weyl composition factors in $T(m) -$ all multiplicities are one.

\[
\begin{array}{c|c|c|c}
\hline
& & & m \\
\hline
-1 & \cdots & \cdots & p^3 - 1 \\
\hline
\end{array}
\]

The rule for obtaining the picture is the following: reflect first $m$ in the last $p$-wall before $m$, then in the last $p^2$-wall before $m$ and so on. Finally “symmetrize” the picture. The number of $\Delta$-factors in $T(m)$ hence equals $2^l$, where $l = \#\{i|a_i \neq 0\}$ for $m + 1 = a_0 + a_1 p + \ldots + a_k p^k$.

We shall present the effect of the LLT-algorithm in a series of examples. The representation theory of $Gl_2$ is parameterized by Young diagrams with at most two lines. The diagram $(\lambda_1, \lambda_2)$ passes under the restriction to the $Sl_2$-weight $m = \lambda_1 - \lambda_2$. Let us therefore consider the LLT algorithm on Young-diagrams with at most two lines. It adds nodes in different ways and we simply neglect all diagrams with more than two lines appearing. It only involves the $f_i$-operators, which when adding a node to the second line may produce a $q$-power (namely $q^{N_i}$, where $N_i^r$ is 1 if the first line has an addable $i$-node, $-1$ if it has a removable $i$-node).
Example 1: Take \( l = 5 \), \((\lambda_1, \lambda_2) = (15, 9)\). The corresponding \( Sl_2 \) weight is \( m = 6 \).

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
\end{array}
\]

The last two nodes of the 5-diagram are different, corresponding to \( m \) being a regular \( Sl_2 \)-weight. We can add a 0-node and a 3-node, i.e. only \( f_0 \) and \( f_3 \) operate non trivially on the diagram. The action is as follows:

\[
\begin{align*}
  f_0 (\lambda_1, \lambda_2) &= (\lambda_1 + 1, \lambda_2), \text{ i.e. } f_0 (m) = (m + 1) \\
  f_3 (\lambda_1, \lambda_2) &= (\lambda_1, \lambda_2 + 1), \text{ i.e. } f_3 (m) = (m - 1)
\end{align*}
\]

Example 2: Let \((\lambda_1, \lambda_2)\) satisfy \( \lambda_1 - \lambda_2 \equiv -1 \mod l \), i.e. the corresponding \( Sl_2 \)-weight is a Steinberg weight. Then the \( l \)-diagram has the form

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & \ldots & \ldots & r \\
0 & 1 & 2 & \ldots & \ldots & r
\end{array}
\]

where \( r \equiv \lambda_1 - 1 \equiv \lambda_2 - 2 \mod p \). We can thus only operate non trivially with \( f_{r+1} \) and obtain the following formula:

\[
\begin{align*}
  f_{r+1} (\lambda_1, \lambda_2) &= (\lambda_1 + 1, \lambda_2) + q (\lambda_1, \lambda_2 + 1) \text{ i.e.} \\
  f_{r+1} (m) &= (m + 1) + q (m - 1)
\end{align*}
\]

Example 3: Let \((\lambda_1, \lambda_2)\) satisfy \( \lambda_1 - \lambda_2 \equiv -2 \mod p \). Then \( m + 1 \) is a Steinberg weight. We can operate non trivially only with \( f_{\lambda_1} \) and find that

\[
\begin{align*}
  f_{\lambda_1} (\lambda_1, \lambda_2) &= (\lambda_1 + 1, \lambda_2) \text{ i.e. } f_{\lambda_1} (m) = (m + 1)
\end{align*}
\]
Example 4: Let \((\lambda_1, \lambda_2)\) satisfy \(\lambda_1 - \lambda_2 \equiv 0 \mod p\). Then \(m - 1\) is a Steinberg weight and we now get

\[ f_{\lambda_2}(\lambda_1, \lambda_2) = q^{-1}(\lambda_1, \lambda_2 + 1) \text{ i.e } f_{\lambda_2}(m) = q^{-1}(m - 1) \]

Example 5: Let \((\lambda_1, \lambda_2)\) be as in example 3. We can then compose the operators of example 3 and 2 to obtain a kind of translation through the wall. The result is

\[ f_{\lambda_1+1} f_{\lambda_1}(\lambda_1, \lambda_2) = (\lambda_1 + 2, \lambda_2) + q(\lambda_1 + 1, \lambda_2 + 1) \text{ i.e } f_{\lambda_1+1} f_{\lambda_1}(m) = (m + 2) + q(m), \]

Example 6: Let \((\lambda_1, \lambda_2)\) be as in example 4. We can then compose with example 2 and this time translate backwards through the wall. The result is

\[ f_{\lambda_1} f_{\lambda_2}(\lambda_1, \lambda_2) = q^{-1}(\lambda_1 + 1, \lambda_2 + 1) + (\lambda_1, \lambda_2 + 2) \text{ i.e } (m) \mapsto q^{-1}(m) + (m - 2) \]

(The combination of these examples proves that the LLT algorithm is equivalent to Soergel’s tilting algorithm in the \(Sl_2\)-case.)

Let us now consider the modification of the algorithm mentioned in the beginning of the section. Let \((\lambda_1, \lambda_2)\) be a two line partition with Young diagram as follows:

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Let \(h\) be the length of the hook indicated in the diagram. When adding a node to the second line, we thus multiply by the \(q\)-power with exponent \(\nu\left(\frac{h}{h-1}\right)\) where \(\nu\) is ordinary \(p\)-adic valuation.

One now goes through the various examples. Example 5, which was translation through the wall, takes the following form:
while example 6 takes the form

\[
\frac{1}{p^a - 1} \rightarrow \frac{q^a}{p^a - 1}
\]

We can now start the modified algorithm. There is however one more detail to point out. When running the modified algorithm, there will soon be negative \(q\)-powers occurring. This is not the case in Soergel’s algorithm, but certainly the case in the LLT-algorithm, generally. Hence, we shall proceed as LLT do, not only subtracting inductively known tilting characters when a 1 is appearing, but also when a negative \(q\)-power is appearing. To be precise: if some negative \(q\)-power appears, it is possible to subtract an expression on the form \(\gamma T\) where \(\gamma \in \mathbb{Z}[q, q^{-1}]\) with \(\gamma(q^{-1}) = \gamma(q)\) and where \(T\) is a known tilting character to finally arrive at an expression involving only positive \(q\)-powers.

Let us work out the case \(p = 3\) up to the weight \(m = (p^3 + p^2 + 2) - 1\); this was also our first example. In the lowest \(p^2\)-alcove the two algorithms agree, so we jump directly to the largest weight in that alcove.
GRADING THE TRANSLATION FUNCTORs IN TYPE A

\[ q^2 q \quad q \quad 1 \]

\[ p^3 - 1 \]

\[ q^2 \quad q \quad q + q^3 \quad q \quad 1 \]

\[ q + q^3 \]

\[ p^3 - 1 \]

\[ q^2 \quad q \quad q \quad 1 \]

\[ p^3 - 1 \]

\[ q^2 \quad q^2 \quad q \quad 1 \]

\[ p^3 - 1 \]

\[ q^3 \quad 1 \]

\[ p^3 - 1 \]

\[ q^3 \quad q^2 \quad 1 \]

\[ q^5 + q \]

\[ q^3 q^2 \quad q^2 + q^{-2} \quad q \quad 1 \]

\[ q^5 + q \]
So for all weights smaller than our chosen $m$ we get results that are compatible with Erdmann’s Theorem.

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