SOME APPLICATIONS OF RECOLLEMENTS TO GORENSTEIN
PROJECTIVE MODULES

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Abstract. We apply the technique of recollement to study the Gorenstein projective modules. First, we construct a recollement of defect Gorenstein categories for the upper triangular matrix algebras. Then we use the defect Gorenstein category to give a categorical interpretation of the Gorenstein properties of the upper triangular matrix algebra obtained by X-W Chen, B. Xiong and P. Zhang respectively. Finally, as a generalization of the structure of the cluster-tilted algebra of type $A$, we define the gluing Nakayama algebra, and use recollement to describe its singularity category clearly.

1. Introduction

In the study of B-branes on Landau-Ginzburg models in the framework of Homological Mirror Symmetry Conjecture, D. Orlov rediscovered the notion of singularity categories [25, 26, 27]. The singularity category of an algebra $A$ is defined to be the Verdier quotient of the bounded derived category with respect to the thick subcategory formed by complexes isomorphic to those consisting of finitely generated projective modules. It measures the homological singularity of an algebra in the sense that an algebra $A$ has finite global dimension if and only if its singularity category vanishes.

In the meantime, the singularity category captures the stable homological features of an algebra [8]. A fundamental result of R. Buchweitz [8] and D. Happel [17] states that for a Gorenstein algebra $A$, the singularity category is triangle equivalent to the stable category of Gorenstein projective(also called Cohen-Macaulay) $A$-modules. Buchweitz’s Theorem says that there is an exact embedding $\Phi : \text{Gproj} A \to D_{sg}(A)$ given by $\Phi(M) = M$, where the second $M$ is the corresponding stalk complex at degree 0, and $\Phi$ is an equivalence if and only if $A$ is Gorenstein. In general, to provide a categorical characterization of Gorenstein algebras, P. A. Bergh, D. A. Jørgensen and S. Oppermann defined the Gorenstein defect category $D_{def}(A) := D_{sg}(A)/\text{Im} \Phi$ and proved that $A$ is Gorenstein if and only if $D_{def}(A) = 0$ [6].

Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with bimodule $AM_B$. X-W. Chen proved that if $A, B$ are Gorenstein, then the upper triangular matrix algebra $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is Gorenstein if and only if $AM$ and $MB$ are finitely generated, $\text{proj.dim}_A M < \infty$ and $\text{proj.dim}_B M < \infty$ [11, Theorem 3.3]. Furthermore, the upper triangular matrix rings play very well to recollement. A recollement of triangulated categories is a

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of triangulated categories and functors satisfying various conditions, which describes the middle term as being “glued together” from a triangulated subcategory and another one. The recollement setup was first introduced by A. Beilinson, J. Bernstein and P. Deligne in [4], which plays an important role in algebraic geometry and representation theory, see for instance [24, 13, 14, 18, 21, 31]. We know that $\mathcal{D}^b(\Lambda)$ admits a recollement relative to $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ if $\text{proj.dim} M_B < \infty$ [14]. P. Zhang proved that if $A, B$ are Gorenstein algebras and $A M, M_B$ are projective, then $\text{Gproj}\Lambda$ admits a recollement relative to $\text{Gproj}A$ and $\text{Gproj}B$. The author together with P. Liu generalized this, proved that $\text{Gproj}(\Lambda)$ admits a recollement relative to $\text{Gproj}(A)$ and $\text{Gproj}(B)$ if $\text{proj.dim}_A M < \infty$ and $\text{proj.dim}_B M_B < \infty$ for any Artin rings $A$ and $B$ [23]. In [24], the machinery of localization to the three categories in a recollement is introduced, and we also find conditions for the quotient categories to form a new recollement. We refer the reader to [22] for localization theory of triangulated categories.

The aim of the present article is threefold. First is to construct a recollement of Gorenstein defect categories for upper triangular matrix algebra $\Lambda$.

**Theorem 1.1.** Let $A$ and $B$ be two finite-dimensional algebras, $A M_B$ an $A$-$B$-bimodule such that $A M$ and $M_B$ are finitely generated projective modules, and $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. If $\text{proj.dim}_B \text{Hom}_A(M, A) < \infty$, then $\text{D}_{\text{def}}(\Lambda)$ admits a recollement relative to $\text{D}_{\text{def}}(A)$ and $\text{D}_{\text{def}}(B)$.

Second is to give a categorical interpretation of the Gorenstein property of upper triangular matrix algebra $\Lambda$.

**Theorem 1.2.** Let $A$ and $B$ be two finite-dimensional algebras, $A M_B$ an $A$-$B$-bimodule, and $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. We also assume that $\text{proj.dim}_A M < \infty$ and $\text{proj.dim}_B M_B < \infty$. Then

(a) If $B$ is a Gorenstein algebra, then $\text{D}_{\text{def}}(\Lambda) \simeq \text{D}_{\text{def}}(A)$.

(b) If $A$ is a Gorenstein algebra and $\text{inj.dim}_B M_B < \infty$, then $\text{D}_{\text{def}}(\Lambda) \simeq \text{D}_{\text{def}}(B)$.

It is difficult to describe the exact subcategory of Gorenstein projective modules. In recent years, C. M. Ringel describe the Gorenstein projective modules over Nakayama algebra clearly [28]. Finally, as a generalization of the structure of the cluster-tilted algebra of type $A$, we give the definition of gluing Nakayama algebra, see Definition 4.2. We study their Gorenstein property, and get the following statement.

**Theorem 1.3.** For any gluing Nakayama algebra $\Gamma$, if the gluing components $A_1 = kQ_1/I_1$, $A_2 = kQ_2/I_2$, $\cdots$, $A_m = kQ_m/I_m$ of $\Gamma$ are Gorenstein, then

$$\text{Gproj}(\Gamma) \simeq \bigsqcup_{i=1}^m \text{Gproj}(A_i).$$

As a corollary, we get the following. Let $\Lambda = kQ/I$ be a cluster-tilted algebra of type $A$. Then

$$\text{Gproj}(\Lambda) \simeq \bigsqcup_{t(Q)} \text{mod} S_3,$$
where \( t(Q) \) is the number of the triangles of \( Q \), and \( S_3 \) is the selfinjective cluster-tilted algebra of type \( A_3 \). This result is gotten by M. Kalck in [19], see also [10].

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2. Preliminary

In this paper, we always assume that \( k \) is a field and all algebras are finite-dimensional algebra over \( k \) and modules are finitely generated, and all the categories we consider are \( k \)-linear with finite-dimensional Hom spaces.

Let us first recall the definition of recollement for triangulated categories from [4].

**Definition 2.1.** [4] A recollement of triangulated categories is a diagram of triangulated categories and triangulated functors

\[
\begin{array}{ccccccc}
\mathcal{D}' & \xrightarrow{i^*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{D}'' \\
\xleftarrow{i_*} & & \xleftarrow{j_*} & & \xleftarrow{j_*} \\
\end{array}
\]

satisfying

(R1) \((i^*, i_*)\), \((i_*, i^*)\), \((j_*, j^*)\) and \((j^*, j_*)\) are adjoint pairs;

(R2) \(i_*, j_*, j^*\) are full embeddings;

(R3) \(j^*i_* = 0\) (and hence, by adjoint properties, \(i^*j_* = 0\) and \(i^*j_* = 0\));

(R4) Each object \(X\) in \(\mathcal{D}\) determines distinguished triangles

\[
i_! \! i^! X \to X \to j_* j^* X \to \cdot \quad \text{and} \quad j^! j_* X \to X \to i_* i^* X \to \cdot
\]

in \(\mathcal{D}\), where the arrows to and from \(X\) are counit and unit morphisms.

Parallel to the recollement of triangulated categories, Franjou and Pirashvili defined recollement of abelian categories: Let \(\mathcal{D}, \mathcal{D}', \mathcal{D}''\) be abelian categories. The diagram \((\dagger)\) of additive functors is an abelian category recollement of \(\mathcal{D}\) relative to \(\mathcal{D}'\) and \(\mathcal{D}''\), if \((R1), (R2)\) and \((R5)\) are satisfied, where

\(R5\) \(\text{Im } i_* = \ker j^*\).

The following theorem is useful to construct recollements of quotient categories.

**Theorem 2.2.** [23] Let

\[
\begin{array}{ccccccc}
\mathcal{D}' & \xrightarrow{i^*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{D}'' \\
\xleftarrow{i_*} & & \xleftarrow{j_*} & & \xleftarrow{j_*} \\
\end{array}
\]

be a recollement of triangulated categories and \(\mathcal{T}\) be an épaisse subcategory of \(\mathcal{D}\). Let \(\mathcal{T}_1 = i^* \mathcal{T}\) and \(\mathcal{T}_2 = j^* \mathcal{T}\). If \(i_* i^* \mathcal{T} \subseteq \mathcal{T}\) and \(j_! j^* \mathcal{T} \subseteq \mathcal{T}\), then there exists a recollement of localizations

\[
\begin{array}{ccccccc}
\mathcal{D}' / \mathcal{T}_1 & \xrightarrow{i^*} & \mathcal{D} / \mathcal{T} & \xrightarrow{j^*} & \mathcal{D}' / \mathcal{T}_2 \\
\xleftarrow{i_*} & & \xleftarrow{j_*} & & \xleftarrow{j_*} \\
\end{array}
\]

2.1. Gorenstein projective modules and Gorenstein algebras. Let \(A\) be an finite-dimensional \(k\)-algebra. Let \(\text{mod } A\) be the category of finitely generated left \(A\)-modules. With \(D = \text{Hom}_k(\cdot, k)\) we denote the standard duality with respect to the ground field. Then \(A D(A A)\) is an injective cogenerator for \(\text{mod } A\). For an arbitrary \(A\)-module \(A X\) we denote by \(\text{proj}. \text{dim}_A X\) (resp. \(\text{inj}. \text{dim}_A X\)) the projective dimension (resp. the injective dimension) of the module \(A X\).
A complex

\[ P^* : \ldots \to P^{-1} \to P^0 \to P^1 \to \ldots \]

of finitely generated projective \( A \)-modules is said to be \textit{totally acyclic} provided it is acyclic and the Hom complex \( \text{Hom}_A(P^*, A) \) is also acyclic \[15\]. An \( A \)-module \( M \) is said to be (finitely generated) \textit{Gorenstein projective} provided that there is a totally acyclic complex \( P^* \) of projective \( A \)-modules such that \( M \cong \text{Ker} d^0 \) \[15\]. Dually, one can define \textit{Gorenstein injective} modules. We denote by \( \text{Gproj}_A \) (\( \text{Ginj}_A \)) the full subcategory of \( \text{mod}_A \) consisting of Gorenstein projective modules (resp. Gorenstein injective modules).

Let \( \mathcal{X} \) be a subcategory of \( \text{mod}_A \). Then \( {}^\perp \mathcal{X} \) := \( \{ M | \text{Ext}_i^A(M, X) = 0, \text{ for all } X \in \mathcal{X}, i \geq 1 \} \). Dually, we can define \( \mathcal{X}^\perp \). In particular, we define \( {}^\perp A := {}^\perp (\text{proj}_A) \).

\textbf{Lemma 2.3.} (a) \[5\]

\[ \text{Gproj}(A) = \{ M \in \text{mod}_A | \exists \text{ an exact sequence } 0 \to M \to T^0 \to T^1 \to \ldots, \text{ with } T^i \in \text{proj}_A, \text{Ker} d^i \in {}^\perp \text{proj}_A, \forall i \geq 0 \} \).

(b) \[31\] If \( M \) is Gorenstein projective, then \( \text{Ext}_i^A(M, L) = 0 \), \( \forall i > 0 \), for all \( L \) of finite projective dimension.

(c) \[31\] If \( P^* \) is a totally acyclic complex, then all \( \text{Im} d^i \) are Gorenstein projective; and any truncations

\[ \ldots \to P^i \to \text{Im} d^i \to 0, \quad 0 \to \text{Im} d^i \to P^{i+1} \to \ldots \]

and

\[ 0 \to \text{Im} d^i \to P^{i+1} \to \ldots \to P^j \to \text{Im} d^j \to 0, i < j \]

are \( \text{Hom}_A(\_, \text{proj}_A) \)-exact.

\textbf{Definition 2.4.} \[11\] \[2\] \[17\] A finite-dimensional algebra \( A \) is called a \textit{Gorenstein algebra} if \( A \) satisfies \( \text{proj.dim}_A D(A_A) < \infty \) and \( \text{inj.dim}_A A < \infty \). Given an \( A \)-module \( X \). If \( \text{Ext}_i^A(X, A) = 0 \) for all \( i > 0 \), then \( X \) is called a \textit{Cohen-Macaulay module} of \( A \).

Observe that for a Gorenstein algebra \( A \), we have \( \text{inj.dim}_A A = \text{inj.dim}_A A_A \). \[14\] Lemma 6.9]; the common value is denoted by \( \text{Gd}_A \). If \( \text{Gd}_A \leq d \), we say that \( A \) is \( d \)-Gorenstein.

\textbf{Theorem 2.5.} \[15\] Let \( A \) be a Gorenstein algebra. Then

(1) \( P^* \) is an exact sequence of projective left \( A \)-modules, then \( \text{Hom}_A(P^*, A) \) is again an exact sequence of projective right \( A \)-modules.

(2) A module \( G \) is Gorenstein projective if and only if there is an exact sequence \( 0 \to G \to P^0 \to P^1 \to \ldots \) with each \( P^i \) projective.

(3) \( \text{Gproj}_A = {}^\perp A \).

For a module \( M \), take a short exact sequence

\[ 0 \to \Omega M \to P \to M \to 0 \]

with \( P \) projective. The module \( \Omega M \) is called a \textit{syzygy module} of \( M \).

\textbf{Theorem 2.6.} \[3\] Let \( A \) be an finite-dimensional algebra and let \( d \geq 0 \). Then the following statements are equivalent:

(1) the algebra \( A \) is \( d \)-Gorenstein;

(2) \( \text{Gproj}_A = \Omega^d(\text{mod}_A) \);

(2') \( \text{Ginj}_A = \Omega^{-d}(\text{mod}_A) \).

In this case, we have \( \text{Gproj}_A = {}^\perp A \).
So for a Gorenstein algebra, the definition of Cohen-Macaulay module coincides with the one of Gorenstein projective.

For an algebra $A$, we define the singularity category of $A$ to be the quotient category $\text{D}^b(\text{proj}) := \text{D}^b(A)/\text{K}^b(\text{proj})_A$.\footnote{\cite{17,23}}

**Theorem 2.7.** \cite{17} Let $A$ be a Gorenstein algebra. Then $\text{Gproj}(A)$ is a Frobenius category with the projective modules as the projective-injective objects. $\text{Gproj}(A)$ is triangulated equivalent to the singularity category $D_{sg}(A)$ of $A$.

3. Gorenstein defect category

Let $A$ be an algebra over $k$. Buchweitz’s Theorem \cite[Theorem 4.4.1]{8} says that there is an exact embedding $\Phi : \text{Gproj}(A) \hookrightarrow D_{sg}(A)$ given by $\Phi(M) = M$. To measure the difference of $\text{Gproj}(A)$ and $D_{sg}(A)$, Bergh, Jørgensen and Oppermann define the Gorenstein defect category $D_{def}(A) := D_{sg}(A)/\text{Im}(\Phi).$

Let $A$ and $B$ be finite-dimensional algebras, $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. A left $A$-module is identified with a triple $\left( X, Y, \phi \right)$, where $X \in \text{mod } A$, $Y \in \text{mod } A$, and $\phi : M \otimes_B Y \rightarrow X$ is a morphism of $A$-modules. A morphism $\left( X, Y, \phi_0 \right) \rightarrow \left( X', Y', \phi_0' \right)$ is a pair $\left( f, g \right)$, where $f \in \text{Hom}_A(X, X')$, $g \in \text{Hom}_B(Y, Y')$, such that $\phi'(id_M \otimes g) = f \phi_0$. Note that $\left( \begin{pmatrix} P \\ 0 \end{pmatrix} \right)$ and $\begin{pmatrix} G(Q) \\ Q \end{pmatrix}_{id}$ are precisely the indecomposable projective modules, where $P$ and $Q$ are indecomposable as $A$ and $B$ modules respectively.

Dually, we have the description of right $A$-module via row vectors. Precisely, a right $A$-module is identified with $(X, Y)_\phi$, where $X_A$ is a right $A$-module and $Y_B$ is a right $B$-module, and $\psi : X \otimes_A M \rightarrow Y$ is a right $B$-module morphism.

We have the following recollement of abelian categories.

**Theorem 3.1.** \cite{31} Let $A$ and $B$ be finite-dimensional algebras, $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Denote by $G = M \otimes_B \Lambda$. We have the following recollement of abelian categories:

$$
\begin{array}{ccc}
\text{mod } A & \rightarrow & \text{mod } \Lambda \\
\downarrow i^* & & \downarrow j^* \\
\text{mod } B & \rightarrow & \text{mod } \Lambda
\end{array}
$$

where $i^*$ is given by $\left( \begin{pmatrix} X \\ Y \end{pmatrix} \right)_\phi \mapsto \text{Coker}(\phi)$; $i_*$ is given by $X \mapsto \left( \begin{pmatrix} X \\ 0 \end{pmatrix} \right)$; $i^!$ is given by $\left( \begin{pmatrix} X \\ Y \end{pmatrix} \right)_\phi \mapsto X$; $j_*$ is given by $Y \mapsto \left( \begin{pmatrix} G(Y) \\ Q \end{pmatrix} \right)_{id}$; $j^*$ is given by $\left( \begin{pmatrix} X \\ Y \end{pmatrix} \right)_\phi \mapsto Y$; $j^!$ is given by $Y \mapsto \left( \begin{pmatrix} 0 \\ Y \end{pmatrix} \right)$.

**Lemma 3.2.** \cite{12} Let $F_1 : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor between abelian categories which has an exact right adjoint $F_2$. Then the pair $(D^b(F_1), D^b(F_2))$ is adjoint, where $D^b(F_1)$ is the induced functor from $D^b(\mathcal{A})$ to $D^b(\mathcal{B})$ ($D^b(F_2)$ is defined similarly). Moreover, if $F_1$ is fully faithful, then so is $D^b(F_1)$.

The following well-known result is very helpful.
Theorem 3.3. [14] Let $A$ and $B$ be finite-dimensional algebras, $\Lambda M_B$ an $A$-$B$-bimodule such that $\Lambda M$ and $M_B$ are projective, and $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Then we have the following recollement:

$$
\begin{array}{c}
\Lambda(i^*) \\
D^b(A) \\
\Lambda(j^*) \\
\end{array}
\begin{array}{c}
\Lambda(j_*) \\
D^b(\Lambda) \\
\Lambda(i_*) \\
\end{array}
\begin{array}{c}
D^b(i^*) \\
D^b(j^*) \\
D^b(j_*) \\
\end{array}
$$

where $\Lambda(i^*), D^b(i_*), D^b(i^*), D^b(j_*,i^*), D^b(j_*,), D^b(j_*)$ are the derived functors of these in Theorem 3.3.

Now we prove the main results in this section.

Theorem 3.4. Let $A$ and $B$ be finite-dimensional algebras, $\Lambda M_B$ a finitely generated $A$-$B$-bimodule such that $\Lambda M$ and $M_B$ are projective, and $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$.

If $\text{proj.dim}_B \text{Hom}_A(M,A) < \infty$,

then $D_{\text{def}}(A)$ admits a recollement relative to $D_{\text{def}}(A)$ and $D_{\text{def}}(B)$.

Proof. By Buchweitz’s theorem, we identify the stable category of Gorenstein projective objects with a thick subcategory of the singularity category.

By [23, Theorem 2.5], we get the following recollement:

$$
\begin{array}{c}
\Lambda(i^*) \\
D_{\text{sg}}(A) \\
\Lambda(j^*) \\
\end{array}
\begin{array}{c}
\Lambda(j_*) \\
D_{\text{sg}}(\Lambda) \\
\Lambda(i_*) \\
\end{array}
\begin{array}{c}
\Lambda(i^*) \\
\Lambda(j^*) \\
\Lambda(j_*) \\
\end{array}
$$

where $i^*, i_*, j^*, j_*$ and $j^*$ are induced by the six structure functors in Theorem 3.3.

Denote by $G$ the functor $M \otimes_B -$, and $F$ the functor $\text{Hom}_A(M,-)$. Then $G$ and $F$ are exact functors since $\Lambda M$ and $M_B$ are projective.

For any $\begin{pmatrix} X \\ Y \end{pmatrix} \in G_{\text{proj}} A$, we have the following exact sequence:

$$
\begin{array}{c}
0 \rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} P_0 \oplus G(Q_0) \\ Q_0 \end{pmatrix} \rightarrow \begin{pmatrix} P_1 \oplus G(Q_1) \\ Q_1 \end{pmatrix} \rightarrow \ldots,
\end{array}
$$

with $\begin{pmatrix} P_i \oplus G(Q_i) \\ Q_i \end{pmatrix} \in \text{proj}(\Lambda), \ker d^i \in \text{proj}(\Lambda), \forall i \geq 0$.

where $P_i \in \text{proj}(A), Q_i \in \text{proj}(B)$. It is easy to see that $\phi : G(Y) \rightarrow X$ is monic since $\phi_0$ is monic. Apply $j^*$ to the exact sequence, since

$$
\begin{array}{c}
\begin{pmatrix} P_i \\ Q_i \end{pmatrix} \rightarrow Q_i \in \text{proj}(B)
\end{array}
$$

for all $i \geq 0$, we get an exact sequence

$$
0 \rightarrow Y \rightarrow Q_0 \xrightarrow{j^*d^i} Q_1 \xrightarrow{j^*d^i} \ldots.
$$

In fact, $\ker(j^*d^i) = j^*(\ker d^i)$ for any $i \geq 0$. Note that $j^*$ is exact and preserves projective, we get that

$$
\text{Ext}^k_B(j^* \ker d^i, Q) \cong \text{Ext}^k_A(\ker d^i, j_* Q) = \text{Ext}^k_A(\ker d^i, \begin{pmatrix} 0 \\ Q \end{pmatrix}) = 0, \forall k > 0
$$
since \( \text{proj.dim} \left( \begin{array}{c} 0 \\ Q \end{array} \right) \leq 1 \) and \( \ker d^i \) is Gorenstein projective. So \( \tilde{\iota}^* \left( \begin{array}{c} X \\ Y \end{array} \right) \in \text{Gproj}\Lambda \).

Denote \( \ker d^i \) in the exact sequence [1] by \( \left( \begin{array}{c} X_i \\ Y_i \end{array} \right) \). Then \( \ker d^i \in \text{Gproj}\Lambda \), we also get that \( \psi_i \) is monic. Applying \( \iota^* \) to the exact sequence [1], we get an exact sequence since \( \phi \) and \( \psi_i \) are monic:

\[
0 \to \text{Coker} \phi \to P_0 \xrightarrow{\iota^* d^i} P_1 \xrightarrow{\iota^* d^i} \cdots.
\]

Note that \( \ker(\iota^* d^i) = \iota^* (\ker d^i) \), for \( i \geq 0 \). Let

\[
T^i : \cdots \to T_1^i \to T_0^i \to \ker d^i \to 0
\]

be a projective resolution of \( \ker d_i^i \). Applying \( \iota^* \) to it, we get a projective resolution \( \iota^*(T^i) \) of \( \iota^* \ker d^i \). So

\[
\text{Ext}_A^k(\ker d^i, i_* P) \cong H^k \text{Hom}_A(T^i, i_* P) \cong H^k \text{Hom}_A(i_* T^i, P) \cong \text{Ext}_A^k(\iota^* \ker d^i, P)
\]

for any \( k \geq 0 \) and \( P \in \text{proj}. A \). Since \( i_* P = \left( \begin{array}{c} P \\ 0 \end{array} \right) \) is projective, we get that

\[
\text{Ext}_A^k(\ker d^i, i_* P) \cong 0 \text{ for } k \geq 1, \text{ and then } \text{Ext}_A^k(\iota^* \ker d^i, P) = 0. \text{ So } \tilde{\iota}^*(\left( \begin{array}{c} X \\ Y \end{array} \right)) \in \text{Gproj}\Lambda.
\]

Applying \( i^! \) to the exact sequence [1], we get an exact sequence:

\[
0 \to X \to P_0 \oplus G(Q_0) \xrightarrow{i^! d_{0}} P_1 \oplus G(Q_1) \xrightarrow{i^! d_1} \cdots.
\]

In fact \( i^! \) admits a right exact functor \( i^* : \text{mod} A \to \text{mod} \Lambda \) given by \( X \mapsto \left( \begin{array}{c} X \\ F(X) \end{array} \right) \), where \( \psi_X = \alpha_{X,F(X)}^{-1}(\text{id}_{F(X)}) \). For any projective \( A \)-module \( P \), there are two exact sequences:

\[
0 \to \left( \begin{array}{c} P \\ 0 \end{array} \right)_0 \to \left( \begin{array}{c} P \\ F(P) \end{array} \right)_{\psi_P} \to \left( \begin{array}{c} 0 \\ F(P) \end{array} \right)_0 \to 0,
\]

\[
0 \to \left( \begin{array}{c} GF(P) \\ 0 \end{array} \right)_0 \to \left( \begin{array}{c} GF(P) \\ F(P) \end{array} \right)_{\text{id}} \to \left( \begin{array}{c} 0 \\ F(P) \end{array} \right)_0 \to 0.
\]

It is easy to see that \( \text{proj. dim}\left( \begin{array}{c} GF(P) \\ 0 \end{array} \right)_0 = \text{proj. dim}_A GF(P) \leq \text{proj. dim}_B F(P) \) since \( G \) is exact and preserves projective, and \( \text{proj. dim}\left( \begin{array}{c} GF(P) \\ F(P) \end{array} \right)_{\text{id}} = \text{proj. dim}_B F(P) \).

By the assumption

\[
\text{proj. dim}_B \text{Hom}_A(M, A) < \infty,
\]

we get that

\[
\text{proj. dim}\left( \begin{array}{c} 0 \\ F(P) \end{array} \right)_0 < \infty
\]

and then

\[
\text{proj. dim}\left( \begin{array}{c} P \\ F(P) \end{array} \right)_{\psi_P} < \infty.
\]

Since \( i^! \) is exact and preserves projectives, similar to \( i^* \), we get that

\[
\text{Ext}_B^k(i^! \ker d^i, P) \cong \text{Ext}_A^k(\ker d^i, i_* P) = \text{Ext}_A^k(\ker d^i, \left( \begin{array}{c} P \\ F(P) \end{array} \right)_{\psi_P}) = 0, k \geq 1.
\]

Thus \( \tilde{\iota}^*(\left( \begin{array}{c} X \\ Y \end{array} \right)) \in \text{Gproj}\Lambda. \)
Similarly, we can check that \( \tilde{i}_* (\text{Gproj}_A) \subseteq \text{Gproj}_A \). Then
\[
(2) \quad \tilde{i}^*(\text{Gproj}(A)) = \text{Gproj}(A)
\]
by \( \tilde{i}_* \tilde{i} \simeq \text{id} \). From it, we get that
\[
(3) \quad \tilde{i}_* \tilde{i}^*(\text{Gproj}(A)) \subseteq \text{Gproj}(A)
\]
and
\[
(4) \quad \tilde{i}_* \tilde{i}^*(\text{Gproj}(A)) \subseteq \text{Gproj}(A).
\]
By \( \text{R4} \) and \( \text{Gproj}_A \) is an épaisse subcategory of \( D_{\text{def}}(A) \), we get that
\[
(5) \quad \tilde{j}_* \tilde{j}^*(\text{Gproj}(A)) \subseteq \text{Gproj}(A).
\]
On the other hand, for any \( Y \in \text{Gproj}_B \), we have the following exact sequence:
\[
(6) \quad 0 \to Y \to Q^0 \xrightarrow{d^0} Q^1 \xrightarrow{d^1} \cdots , \text{with } Q^i \in \text{proj} \cdot B, \ker d^i \in \bot \text{proj} \cdot B, \forall i \geq 0.
\]
Apply \( j \) to it, we get the following exact sequence:
\[
0 \to \left( \begin{array}{c}
G(Y) \\
Y
\end{array} \right) \xrightarrow{\text{id}} \left( \begin{array}{c}
G(Q^0) \\
Q^0
\end{array} \right) \xrightarrow{j \cdot d^0} \left( \begin{array}{c}
G(Q^1) \\
Q^1
\end{array} \right) \xrightarrow{j \cdot d^1} \cdots .
\]
For any projective object \( \left( \begin{array}{c}
P \\
Q
\end{array} \right) \in \text{mod } A \), we have
\[
\text{Ext}_A^k(j \cdot \ker d^i, \left( \begin{array}{c}
P \\
Q
\end{array} \right) \phi ) \equiv \text{Ext}_B^k(\ker d^i, j^* \left( \begin{array}{c}
P \\
Q
\end{array} \right) ) = \text{Ext}_B^k(\ker d^i, Q) = 0, \forall k > 0.
\]
So \( \tilde{j}_*(Y) \in \text{Gproj}_A \). In fact, we have proved that
\[
(7) \quad \tilde{j}^*(\text{Gproj}_A) = \text{Gproj}_B
\]
since \( \tilde{j}_* \tilde{j} \simeq \text{id} \).

From the Equalities (2), (3), (5), (7) and Theorem 2.2, we get that \( D_{\text{def}}(A) \) admits a recollement relative to \( D_{\text{def}}(A) \) and \( D_{\text{def}}(B) \).

**Corollary 3.5.** Let \( A \) be an finite-dimensional algebras. Then \( D_{\text{def}}(\left( \begin{array}{c}
A \\
0
\end{array} \right)) \) admits a recollement relative to \( D_{\text{def}}(A) \) and \( D_{\text{def}}(A) \).

For any finite-dimensional algebra \( A \), we have that \( D_{\text{def}}(A) \) is zero if and only if \( A \) is Gorenstein [6]. X-W. Chen proved the following: if \( A, B \) are Gorenstein, then the upper triangular matrix algebra \( \Lambda = \left( \begin{array}{cc}
A & M \\
0 & B
\end{array} \right) \) is Gorenstein if and only if \( \text{proj} \cdot A M < \infty \) and \( \text{proj} \cdot \text{dim } M_B < \infty \) [11, Theorem 3.3]. Following Chen, B. Xiong and P. Zhang proved that if \( \text{proj} \cdot \text{dim } M < \infty \) and \( \text{proj} \cdot \text{dim } M_B < \infty \), then \( \Lambda \) is Gorenstein if and only if \( A \) and \( B \) are Gorenstein [29, Theorem 2.2]. In the following, we will try to interpret these results by the Gorenstein defect categories.

**Lemma 3.6.** [15] Let \( R \) be a Gorenstein ring, \( N \) a left \( R \)-module. Then \( N \) has finite projective dimension if and only if \( N \) has finite injective dimension.

**Theorem 3.7.** Let \( A \) and \( B \) be finite-dimensional algebras, \( AM_B \) an \( A-B \)-bimodule, and \( \Lambda = \left( \begin{array}{cc}
A & M \\
0 & B
\end{array} \right) \). We also assume that \( \text{proj} \cdot \text{dim } A M < \infty \) and \( \text{proj} \cdot \text{dim } M_B < \infty \).

(a) If \( B \) is a Gorenstein algebra, then \( D_{\text{def}}(\Lambda) \simeq D_{\text{def}}(A) \).

(b) If \( A \) is a Gorenstein algebra and \( \text{inj} \cdot \text{dim } M_B < \infty \), then \( D_{\text{def}}(\Lambda) \simeq D_{\text{def}}(B) \).
Proof. (a) By [23], we get that \( D_{sg}(\Lambda) \) admits a recollement relative to \( D_{sg}(A) \) and \( D_{sg}(B) \) as the diagram (*) in the proof of Theorem 3.1 shows, where the six structure functors are induced by the derived functors of the ones in Theorem 3.1. In fact, \( i_*, \iota^*, j^*, j_* \) are exact.

For any \( X \in \text{Gproj}(A) \), there is an exact sequence:

\[
0 \to X \to P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} \cdots ,
\]

with \( P_i \in \text{proj}. \Lambda, \ker \; d_i \in \mathcal{I}. \text{proj}. \Lambda, \forall i \geq 0 \).

Applying \( \iota_* \) to the exact sequence (5), we get the following exact sequence:

\[
0 \to \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} \xrightarrow{i_*,d_0} \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} \xrightarrow{i_*,d_1} \cdots .
\]

Note that \( \begin{pmatrix} P_i \\ 0 \end{pmatrix} \) are projective for \( i \geq 0 \). For any projective \( \Lambda \)-module \( U, \; i^! U \cong P \oplus M \otimes_B Q \) for some projective \( \Lambda \)-module \( P \), and projective \( B \)-module \( Q \). Since \( \text{proj} \dim \; M_B < \infty \), we get that \( \text{proj} \dim \; M \otimes_B Q < \infty \), so \( \text{Ext}^k_A(\ker \; d_i, U) \cong \text{Ext}^k_A(\ker \; d_i, i^! U) \cong \text{Ext}^k_A(\ker \; d_i, P \oplus M \otimes_B Q) = 0, \forall k > 0 \).

Thus \( \tilde{i}_*(\text{Gproj}(\Lambda)) \subseteq \text{Gproj}(A) \). So \( \tilde{i}_* \) induces a full functor from \( D_{def}(\Lambda) \) to \( D_{def}(\Lambda) \), which is also denoted by \( \tilde{i}_* \). In the following, we will prove that \( \tilde{i}_*: D_{def}(\Lambda) \to D_{def}(\Lambda) \) is full faithful and dense.

First, we prove that \( \tilde{i}^*(\text{Gproj}(\Lambda)) \subseteq \text{Gproj}(A) \). For any \( \begin{pmatrix} X \\ Y \end{pmatrix} \in \text{Gproj}(A) \), we have the following exact sequence:

\[
0 \to \begin{pmatrix} X \\ Y \end{pmatrix} \xrightarrow{i_*} \begin{pmatrix} P_0 \oplus M \otimes_B Q_0 \\ Q_0 \end{pmatrix} \xrightarrow{d'_0} \begin{pmatrix} P_1 \oplus M \otimes_B Q_1 \\ Q_1 \end{pmatrix} \xrightarrow{d'_1} \cdots ,
\]

with \( \begin{pmatrix} P_i \oplus M \otimes_B Q_i \\ Q_i \end{pmatrix} \in \text{proj}. \Lambda, \ker \; d_i \in \mathcal{I}. \text{proj}. \Lambda, \forall i \geq 0 \).

In fact, \( \phi_i = \begin{pmatrix} 0 \\ \text{id} \end{pmatrix} \). So \( \phi : M \otimes_B Y \to X \) is monic. Applying \( \iota^* \) to the exact sequence (10), we get an exact sequence:

\[
0 \to \text{Coker} \phi \to P_0 \xrightarrow{\iota^*d'_0} P_1 \xrightarrow{\iota^*d'_1} \cdots .
\]

Note that \( \ker(\iota^*d_i) = \iota^*(\ker \; d_i) \), for \( i \geq 0 \). So \( \text{Ext}^k_A(\ker(\iota^*d_i), P) \cong \text{Ext}^k_A(\ker(\iota^*d_i), P) \cong \text{Ext}^k_A(\ker \; d_i, \iota_*P) = 0, \forall k > 0 \)

follows from that \( \iota_*P = \begin{pmatrix} P \\ 0 \end{pmatrix} \) is projective. So \( \tilde{i}^*(\begin{pmatrix} X \\ Y \end{pmatrix} \phi) \in \text{Gproj}(A) \).

For any \( X_1, X_2 \in D_{sg}(A) \), and \( f : X_1 \to X_2 \), if \( \tilde{i}_*(f) = \begin{pmatrix} f \\ 0 \end{pmatrix} : \begin{pmatrix} X_1 \\ 0 \end{pmatrix} \to \begin{pmatrix} X_2 \\ 0 \end{pmatrix} \)

is zero in \( D_{def}(\Lambda) \), then it factors through some object \( \begin{pmatrix} X \\ Y \end{pmatrix} \phi \in \text{Gproj}(A) \).

Applying \( \tilde{i}^* \), we get that \( f : X_1 \to X_2 \) factors through \( \text{Coker} \phi = \iota^*(\begin{pmatrix} X \\ Y \end{pmatrix} \phi) \in \text{Gproj}(A) \). It follows that \( \tilde{i}_*: D_{def}(\Lambda) \to D_{def}(\Lambda) \) is full faithful.

For any \( Y \in \text{Gproj}(B) \), we have the following exact sequence:

\[
0 \to Y \to Q^0 \xrightarrow{d^0} Q^1 \xrightarrow{d^1} \cdots , \text{with} \; Q^i \in \text{proj}. \; B, \ker \; d^i \in \mathcal{I}. \text{proj}. \; B, \forall i \geq 0.
\]
Since \( \text{proj. dim } M_B < \infty \), we get that \( \text{inj. dim}_B DM < \infty \). Then \( \text{proj. dim } B DM < \infty \) by Lemma \ref{lem:proj_dim}. We get that \( \text{DTor}_0^B(M, \ker d) \cong \text{Ext}_B^1(\ker d, DM) \cong 0 \). So

\[
0 \to M \otimes_B Y \to M \otimes_B Q^0 \to M \otimes_B Q^1 \to \cdots
\]

is also exact. Therefore, applying \( \Lambda = \bigoplus_{i=0}^\infty Q_i \) we have \( \text{Ext}^n_B(M, \Lambda) \cong 0 \). By Theorem \ref{thm:main}, (a) is trivial.

**Proof.**

Let \( \Lambda \) be a minimal injective resolution of \((0, M)\). From these, we can prove that \( \tilde{j} \): \( \text{Gproj}(B) \to \text{Gproj}(\Lambda) \) for \( B \) is a Gorenstein algebra.

For any object \( T \in \text{Gproj}(\Lambda) \), we have the following distinguished triangle in \( \text{Gproj}(\Lambda) \):

\[
\tilde{j}_* j^* T \to T \to \tilde{i}_* i^* T \to T.
\]

But \( \tilde{j} j^* T \in \text{Gproj}(\Lambda) \) and then \( T \cong \tilde{i} i^* T \) in \( F_{\text{def}}^k(\Lambda) \), so \( \tilde{i}_* : D_{\text{def}}(A) \to D_{\text{def}}(\Lambda) \) is dense. Therefore, \( D_{\text{def}}(A) \cong D_{\text{def}}(\Lambda) \).

For (b), by the proof of (a), we get that

\[
\tilde{i}_*(\text{Gproj}(A)) = \tilde{i}_*(\text{Gproj}(A)) \subseteq \text{Gproj}(\Lambda).
\]

Because \( \text{inj. dim}_B M_B < \infty \), from the proof of (a), we also get that \( \tilde{j} \): \( \text{Gproj}(B) \subseteq \text{Gproj}(\Lambda) \). So \( \tilde{j} \) induces a functor from \( D_{\text{def}}(B) \) to \( D_{\text{def}}(\Lambda) \), which is also denoted by \( \tilde{j} \).

Similar to the proof of Theorem \ref{thm:main}, we get that \( \tilde{j}^*(\text{Gproj}(\Lambda)) \subseteq \text{Gproj}(B) \).

From these, we can prove that \( \tilde{j} : D_{\text{def}}(B) \to D_{\text{def}}(\Lambda) \) is full faithful and dense similar to the proof of (a).

\[\square\]

**Corollary 3.8.** Let \( A \) and \( B \) be finite-dimensional algebras, \( AM_B \) an \( A-B \text{-bimodule} \), and \( \Lambda = \left( \begin{array}{cc} A & M \\ 0 & B \end{array} \right) \). We also assume that \( \text{proj. dim}_A M < \infty \) and \( \text{proj. dim } B < \infty \).

(a) If \( B \) is a Gorenstein algebra, then \( \Lambda \) is Gorenstein if and only if \( A \) is.

(b) If \( A \) is a Gorenstein algebra, \( \Lambda \) is Gorenstein if and only if \( B \) is.

**Proof.** By Theorem \ref{thm:main} (a) is trivial.

For (b), if \( B \) is Gorenstein, then \( \text{inj. dim } M_B < \infty \) since \( \text{proj. dim } M_B < \infty \), so \( D_{\text{def}}(\Lambda) \cong 0 \), then \( \Lambda \) is Gorenstein. On the other hand, because \( \text{proj. dim } M_B < \infty \), we get that \( \text{proj. dim } D_{\text{def}}(M) \) is \( \text{proj. dim } M_B < \infty \). Note that the injective \( \Lambda \text{-modules} \) are precisely of form \( (J, 0) \oplus (\text{Hom}_B(M, I), I) \), where \( J, I \) are injective \( A \text{-module} \) and \( B \text{-module} \) respectively. Let

\[
0 \to (0, M) \to (J_0 \oplus \text{Hom}_B(M, I_0), I_0)_{\psi_0} \to \cdots \to (J_r \oplus \text{Hom}_B(M, I_r), I_r)_{\psi_r} \to 0
\]

be a minimal injective resolution of \((0, M)\). Then \( I_i, J_i \) are injective as \( B \text{-module} \) and \( A \text{-module} \) respectively. Furthermore, it is easy to see that

\[
0 \to M \to I_0 \to \cdots \to I_r \to 0
\]

is also exact. So \( \text{inj. dim } M_B < \infty \). By Theorem \ref{thm:main}, we get that \( D_{\text{def}}(B) \cong D_{\text{def}}(\Lambda) \cong 0 \). So \( B \) is also Gorenstein.

\[\square\]
4. Gluing of Nakayama algebras

Let $Q$ be a finite quiver. For any arrow $\alpha$ in $Q$, we denote by $s(\alpha), t(\alpha)$ the source and the target of $\alpha$ respectively. The path algebra $kQ$ of $Q$ is an associative algebra with an identity. If $I$ is an admissible ideal of $kQ$, the pair $(Q, I)$ is said to be a bound quiver. The quotient algebra $kQ/I$ is said to be the algebra of the bound quiver $(Q, I)$, or simply, a bound quiver algebra.

In this section, we shall freely identify the representations of $(Q, I)$ with left modules over $kQ/I$.

An algebra $A$ is called a Nakayama algebra if it is both right and left serial. That is, $A$ is a Nakayama algebra if and only if every indecomposable projective $A$-module and every indecomposable injective $A$-module are uniserial.

Lemma 4.1. A basic and connected algebra $A$ is a Nakayama algebra if and only if its ordinary quiver $Q_A$ is one of the following two quivers:

(a) Nakayama quiver of Type I:

\[ \cdots \rightarrow v_2 \leftarrow v_1 \rightarrow v_3 \leftarrow \cdots \leftarrow v_{n-1} \leftarrow v_n. \]

(b) Nakayama quiver of Type II:

An oriented cyclic quiver with $n$ vertices.

For any Nakayama algebra $A = kQ_A/I$, we always assume that any path of length greater than $n$ is zero in $A$. Furthermore, we assume that the Nakayama algebra $A$ is Gorenstein.

As a generalization of the structure of the cluster-tilted algebra of type $A$, we give the definition of gluing Nakayama algebra as follows.

Definition 4.2. Let $A_1 = kQ_1/I_1$, $A_2 = kQ_2/I_2$, $\ldots$, $A_n = kQ_m/I_n$ be $m$ Nakayama algebras. We define a gluing quiver of $Q_1, Q_2, \ldots, Q_m$ inductively.

Given two vertices $v_1 \in Q_1$ and $v_2 \in Q_2$, we define a connected quiver $Q_{1,2}$ from $Q_1, Q_2$ by identifying $v_1$ to $v_2$. This is called a gluing quiver of $Q_1$ and $Q_2$.

Let $v_{1,2} \in Q_{1,2}$ and $v_3 \in Q_3$ be two vertices. We define a connected quiver $Q_{1,2,3}$ from $Q_{1,2}$ and $Q_3$ by identifying $v_{1,2}$ to $v_3$. Inductively, we can define a quiver $Q = Q_{1,\ldots,m}$ from $Q_{1,\ldots,m-1}$ and $Q_m$.

In this way, $Q$ is called the gluing quiver of $Q_1, Q_2, \ldots, Q_m$ and $A = kQ/I$ is called the gluing Nakayama algebra, where the ideal $I$ is generated by $\{I_1, I_2, \ldots, I_n\}$. $Q_1, Q_2, \ldots, Q_m$ are called to be the gluing components of $Q$; $A_1, A_2, \ldots, A_n$ are called to be the gluing components of $A$.

Example 4.3. (1) Let $Q_1, Q_2, \ldots, Q_m$ be $m$ oriented cyclic quivers where $Q_i$ has $n_i$ vertices, $1 \leq i \leq m$. Given any vertex $v_i \in Q_i$ for all $1 \leq i \leq m$. The gluing quiver $Q$ is defined by identifying all the vertices $v_i$ to a vertex. This is called the
simple gluing Nakayama quiver of type II, denoted by \((Q, v)\), where the vertex \(v\) of \(Q\) is glued by \(v_i, 1 \leq i \leq m\), and called the glued vertex. The special case of \(m = 2\) is showed in the following Figure 2.

(2) Let \((Q_1, v_1), (Q_2, v_2)\) be two simple gluing Nakayama quiver of type II, and \(Q_0\) be a Nakayama quiver of type I as Lemma 4.1 (a) shows. The gluing quiver \(Q\) from \(Q_0, Q_1, Q_2\), is defined to be the quiver by identifying the vertices \(v_1 \in Q_1\) and \(1 \in Q_0\), and the vertices \(v_2 \in Q_2\) and \(n \in Q_0\). The special case of \(Q_0\) is of type \(A_2\) is showed in the following Figure 3.

\[Q_2 \quad v_2 \quad v_1 \quad Q_1\]

Figure 3. The gluing quiver from two simple gluing Nakayama quiver of type II and one of type I.

**Lemma 4.4.** \([11]\) Let \(A\) and \(B\) be two Gorenstein algebras, \(_AM_B\) an \(A\)-\(B\)-bimodule such that \(_AM\) and \(_MB\) are projective, and \(\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}\). Then \(\Lambda\) is Gorenstein. In particular, if \(\text{Gd} A \neq \text{Gd} B\), then \(\text{Gd}(\Lambda) = \max\{\text{Gd} A, \text{Gd} B\}\); Otherwise, \(\text{Gd}(\Lambda) \leq \text{Gd} A + 1\).

**Proof.** The proof of this lemma is similar to \([11]\) Theorem 3.3]

Let \(G = M \otimes_B - , F = \text{Hom}_A(M, -)\). They are exact functors. For any indecomposable injective \(\Lambda\)-module, it is of form \(\begin{pmatrix} 0 \\ I \end{pmatrix}\) or \(\begin{pmatrix} J \\ F(J) \end{pmatrix}\) where \(I\) and \(J\) are injective \(B\)-module and \(A\)-module respectively. For \(\begin{pmatrix} 0 \\ I \end{pmatrix}\), we know that \(\text{proj} \text{dim}_B I \leq \text{Gd} B\). There is an exact sequence:

\[0 \to \begin{pmatrix} G(I) \\ 0 \end{pmatrix} \to \begin{pmatrix} G(I) \\ I \end{pmatrix}_{\text{id}} \to \begin{pmatrix} 0 \\ I \end{pmatrix} \to 0.\]

It is easy to see that

\[\text{proj} \text{dim}_A \begin{pmatrix} G(I) \\ I \end{pmatrix}_{\text{id}} = \text{proj} \text{dim}_B I \leq \text{Gd} B\]

and

\[\text{proj} \text{dim}_A \begin{pmatrix} G(I) \\ 0 \end{pmatrix} = \text{proj} \text{dim}_A G(I) \leq \text{proj} \text{dim}_B I \leq \text{Gd} B\]

since \(G\) is exact and preserves projectives. So \(\text{proj} \text{dim}_A \begin{pmatrix} 0 \\ I \end{pmatrix} \leq \text{Gd} B + 1\).
For \( \left( \begin{array}{c} J \\ F(J) \end{array} \right) \), there is an exact sequence:

\[
0 \rightarrow \left( \begin{array}{c} J \\ 0 \end{array} \right) \rightarrow \left( \begin{array}{c} J \\ F(J) \end{array} \right) \rightarrow \left( \begin{array}{c} 0 \\ F(J) \end{array} \right) \rightarrow 0.
\]

we get that \( \text{proj.dim}_\Lambda \left( \begin{array}{c} J \\ 0 \end{array} \right) = \text{proj.dim}_A J \leq \text{Gd} A \) and

\[
\text{proj.dim}_\Lambda \left( \begin{array}{c} 0 \\ F(J) \end{array} \right) \leq \text{Gd} B + 1
\]

since \( F(J) \) is injective as \( B \)-module. So

\[
\text{proj.dim}_\Lambda \left( \begin{array}{c} J \\ F(J) \end{array} \right) \leq \max \{ \text{Gd} A, \text{Gd} B + 1 \}.
\]

Therefore, the projective dimension of any injective \( \Lambda \)-module is less than \( \max \{ \text{Gd} A, \text{Gd} B + 1 \} \).

Dually, we can prove that for any indecomposable projective \( \Lambda \)-module, its injective dimension is less than \( \max \{ \text{Gd} A + 1, \text{Gd} B \} \). Then \( \Lambda \) is Gorenstein. If \( \text{Gd} A \neq \text{Gd} B \), then \( \text{Gd} \Lambda \leq \max \{ \text{Gd} A, \text{Gd} B \} \). Conversely, it is easy to see that \( \max \{ \text{Gd} A, \text{Gd} B \} \leq \text{Gd} \Lambda \). So \( \text{Gd} \Lambda = \max \{ \text{Gd} A, \text{Gd} B \} \).

If \( \text{Gd} A = \text{Gd} B \), then it is easy to see that \( \text{Gd} \Lambda \leq \text{Gd} A + 1 \). \( \square \)

Let \( A = kQ'/I' \) and \( B = kQ''/I'' \). Let \( Q \) be the quiver glued by \( Q', Q'' \) as the following picture shows:

![Figure 4. The gluing quiver Q from Q' and Q'']()
to $j$; with $\psi_\beta = \phi_\beta$ if $\beta$ is an arrow of $Q_B$, with $\psi_\beta = \text{id}$ if $\beta$ is an arrow of $Q_A$ and $N_\alpha(\beta) = k^j = N_\ell(\beta)$, otherwise zero. $(N_\gamma, \psi_\gamma)$ corresponds to a module $N \in \text{mod } \Lambda$. We define $j_\lambda(Y) := N$, it acts on morphisms naturally. In fact, $j_\lambda$ is an exact fully faithful functor and $(j_\lambda, j)$ is an adjoint pair.

We define another functor $j_\rho : \text{mod } B \to \text{mod } \Gamma$ as follows. Given any $Y \in \text{mod } B$, it corresponds to a representation $(Y, \phi_\alpha)$ of $(Q_B, I_2)$. If the linear space $Y_\gamma$ has dimension $l$, define a representation $(L_j, \psi_\beta)$ of $(Q_\lambda, I_\lambda)$, with $L_j = Y_j$ if $j$ is a vertex of $Q_B$, with $L_j = k^j$ if $j$ is a vertex of $Q_A$ and there is a nonzero path from $j$ to $v$; with $\psi_\beta = \phi_\beta$ if $\beta$ is an arrow of $Q_B$, with $\psi_\beta = \text{id}$ if $\beta$ is an arrow of $Q_A$ and $N_\alpha(\beta) = k^j = N_\ell(\beta)$, otherwise zero. $(L_j, \psi_\beta)$ corresponds to a module $L \in \text{mod } \Lambda$. We define $j_\rho(Y) := L$, it acts on morphisms naturally. In fact, $j_\rho$ is an exact fully faithful functor and $(j_\rho, j_\lambda)$ is an adjoint pair.

It is easy to see that $j_\lambda, j$ preserve projective objects, and $j_\rho$ preserves injective objects.

Dually, we can define $i : \text{mod } \Gamma \to \text{mod } A$ as $j$; $i_\lambda, i_\rho : \text{mod } A \to \text{mod } \Gamma$ as $j_\lambda, j_\rho$.

Similar to the upper triangular matrix algebra, we have the following lemma.

**Lemma 4.6.** Keep the notations as above. If $A$ and $B$ are Gorenstein algebras, then $\Gamma$ is also a Gorenstein algebra. In particular, $\text{Gd}(\Gamma) \leq \max\{1, \text{Gd } A, \text{Gd } B\}$.

**Proof.** Let $\Gamma I = \{I_j, \psi_\gamma\}$ be an indecomposable injective module of $\Gamma$. It corresponds to a vertex $w \in Q_\Gamma$. Without losing generality, we assume that $w \in Q_A$. Let $\lambda A$ be the indecomposable injective module of $A$ corresponding to $w$. Since $A$ is Gorenstein, $\text{proj. dim}_A J < \infty$. Let

$$0 \to P^r \to \cdots \to P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} A \to 0$$

be a minimal projective resolution of $A I$. If $A J_v = 0$, then $i_\lambda(A J) \cong I$. Applying $i_\lambda$ to (12), we get a projective resolution of $\Gamma I$:

$$0 \to i_\lambda(P^r) \to \cdots \to i_\lambda(P^1) \to i_\lambda(P^0) \to \Gamma I \to 0$$

since $i_\lambda$ is an exact functor which also preserves projective objects. So $\text{proj. dim}_\Gamma I \leq \text{Gd } A$.

If $A J_v \neq 0$, let $B R$ be the indecomposable injective module of $B$ corresponding to the vertex $v_\lambda$ (also vertex $v$ in $Q_\Gamma$). Since $B$ is also Gorenstein, let

$$0 \to Q^r \to \cdots \to Q^1 \xrightarrow{e^1} Q^0 \xrightarrow{e^0} B \to 0$$

be the minimal projective resolution of $R$. Then there is an exact sequence

$$0 \to i_\lambda(\ker d^0) \oplus j_\lambda(\ker e^0) \to \Gamma I \to i_\lambda(P^0) \oplus j_\lambda(Q^0) \to \Gamma I \to 0,$$

where $\Gamma T$ is the indecomposable projective module of $\Gamma$ corresponding to the vertex $v$. Since $\text{proj. dim } i_\lambda(\ker d^0) \leq r - 1$, $\text{proj. dim } j_\lambda(\ker e^0) \leq s - 1$, we get $\text{proj. dim}_\Gamma I \leq \max\{1, \text{Gd } A, \text{Gd } B\}$.

Dually, we can prove that every projective $\Gamma$-module has finite injective dimension. So $\Gamma$ is Gorenstein, and from the proof, we also get that $\text{Gd}(\Gamma) \leq \max\{1, \text{Gd } A, \text{Gd } B\}$. \qed

**Lemma 4.7.** Keep the notations as above. If $A$ and $B$ are Gorenstein algebras, then $\text{Gproj}(\Gamma) \cong \text{Gproj}(A) \boxplus \text{Gproj}(B)$.

**Proof.** By Lemma 4.6 we know that $\Gamma$ is also Gorenstein. From Theorem 2.7, we also get that $\text{Gproj}(\Gamma) \cong \text{Dmod}(\Gamma)$, $\text{Gproj}(A) \cong \text{Dmod}(A)$, $\text{Gproj}(B) \cong \text{Dmod}(B)$.

By the construction above, we get that the functors $j_\lambda, j_\rho, j_\lambda, j_\rho$ induce functors on the bounded derived categories, also denoted by the same notations. It is easy to see that $j_\lambda$ and $j$ preserve projective objects, they induces functors on the singularity categories, denoted by $j_\lambda, j$. 


Given any projective object \( Q \in \text{mod } B \). Since \( B \) is Gorenstein, it has a finite injective resolution: \( 0 \to Q \to I^0 \to I^1 \to \cdots \to I^t \to 0 \). Since \( j_\mu \) is exact and preserves injective objects, we get that \( j_\mu(Q) \) has finite injective dimension as \( \Gamma \)-module, but \( \Gamma \) is also Gorenstein, it follows that \( j_\mu(Q) \) has finite projective dimension. So \( j_\mu \) maps \( K^b(\text{proj } B) \) to \( K^b(\text{proj } \Gamma) \), and then it induces a functor on the singularity categories, denoted by \( \tilde{j}_\mu \).

Similarly, \( i_\lambda, i \) induces functors on the singularity categories, denoted by \( \tilde{i}_\lambda, \tilde{i} \).

In particular, it is easy to see that \( \tilde{j}_\lambda, \tilde{j}_\mu \) and \( \tilde{i}_\lambda \) are fully faithful, since \( \tilde{i}_\lambda \cong \text{id} \), etc.. In the following, we prove that \( \text{Im}(\tilde{i}_\lambda)_0 = \text{ker } \tilde{j}_\mu \). For any module \( X \in \text{mod } A \), it corresponds to a representation \( (X_\lambda, \phi_\sigma) \). Then \( j \circ \iota_\lambda(X) = B R^e_r \), where \( B R \) is the indecomposable projective module corresponding to the vertex \( v_2, r = \dim X>V \). So \( \tilde{j} \circ \iota_\lambda = 0 \).

On the other hand, for any module \( Y \in \text{mod } \Gamma \), if \( j(Y) \cong 0 \) in \( K^b(\text{proj } B) \), then \( j(Y) \) has finite projective dimension as \( B \)-module. Let \( Q^0 \xrightarrow{\partial} j(Y) \) be the projective cover of \( j(Y) \). Similarly, let \( P^0 \xrightarrow{\partial'} i(Y) \) be the projective cover of \( i(Y) \). Then there exists the following exact sequence:

\[
0 \to j_\lambda(\ker e^0) \oplus i_\lambda(\ker d^0) \oplus T_{\Sigma\sigma} \to j_\lambda(Q^0) \oplus i_\lambda(P^0) \to Y \to 0,
\]

where \( T \) is the indecomposable projective module corresponding to \( v \) and \( s = \dim Y \). Then \( Y \cong \Sigma(i_\lambda(\ker e^0)) \) in \( D^b_{sg}(\Gamma) \) since \( j_\lambda(\ker e^0) \) has finite projective dimension, where \( \Sigma \) is the suspension functor.

By [14, Theorem 1.1], we get that there is a recollement:

\[
D_{sg}(A) \xrightarrow{i\lambda} D_{sg}(\Gamma) \xrightarrow{j} D_{sg}(B),
\]

where \( \tilde{i}_\mu \) is the left adjoint functor of \( i_\lambda \).

So for any \( X \in D_{sg}(\Gamma) \), there exist a distinguished triangle

\[
\tilde{i}_\lambda \tilde{\iota}_X \to X \to \tilde{j}_\mu \tilde{\iota}_X \xrightarrow{h} \Sigma \tilde{i}_\lambda \tilde{\iota}_X.
\]

We claim that \( h = 0 \) for any \( X \). Since \( D_{sg}(\Gamma) \cong \text{Gproj}(\Gamma) \), we assume that \( X \in \text{Gproj } \Gamma \subseteq \text{mod } \Gamma \). Let \( \dim X_v = r \) and \( I \) the indecomposable projective \( A \)-module corresponding to \( v_1 \). Then it is easy to see that \( h \) factors through \( \Gamma\tilde{i}_\lambda(\Gamma)^{\oplus r} \). By \( A \) is Gorenstein, we get that \( \text{proj } \text{dim } I < \infty \), and then \( \text{proj } \text{dim } \Gamma\tilde{i}_\lambda(\Gamma) < \infty \) since \( \tilde{i}_\lambda \) is exact and preserves projective modules. So \( h = 0 \). It implies the triangle \( \text{L} \) splits, so \( X \cong X_1 \oplus X_2 \), where \( X_1 = \tilde{i}_\lambda \tilde{\iota}_X \in \text{Im}(\tilde{i}_\lambda), X_2 = \tilde{j}_\mu \tilde{\iota}_X \in \text{Im}(\tilde{j}_\mu) \).

In fact, \( \text{Hom}_{D_{sg}(\Gamma)}(\tilde{i}_\lambda(X), \tilde{i}_\lambda(X)) = 0 \). Similar to the above, we can get that \( \text{Hom}_{D_{sg}(\Gamma)}(\tilde{j}_\mu(X), \tilde{j}_\mu(X)) = 0 \). Thus \( D_{sg}(\Gamma) \cong D_{sg}(A) \amalg D_{sg}(B) \). It follows that \( \text{Gproj } \Gamma \cong \text{Gproj } (A) \amalg \text{Gproj } (B) \).

**Proposition 4.8.** For any gluing Nakayama algebra \( \Gamma \), if the gluing components \( A_1 = kQ_1/I_1, A_2 = kQ_2/I_2, \cdots, A_m = kQ_m/I_m \) of \( \Gamma \) are Gorenstein, then \( \Gamma \) is Gorenstein.

**Proof.** We prove it by inducting on \( m \). By Lemma 4.4, we assume that all the gluing components are of type II.

If \( m = 1 \), it is obvious. We assume that for any gluing Nakayama algebra \( A' \) with \( m - 1 \) gluing components is Gorenstein. For \( n \), let \( A \) be the gluing Nakayama algebra with the gluing components \( A_1 = kQ_1/I_1, A_2 = kQ_2/I_2, \cdots, A_{m-1} = kQ_{m-1}/I_{m-1} \). Let \( B = kQ_m/I_m \). With the same proof as Lemma 4.6, we can get \( \Gamma \) is Gorenstein. □
Corollary 4.9. For any gluing Nakayama algebra $\Gamma$, if the gluing components $A_1 = kQ_1/I_1$, $A_2 = kQ_2/I_2$, \ldots, $A_m = kQ_m/I_m$ of $\Gamma$ are selfinjective, then $\Gamma$ is Gorenstein with the Gorenstein dimension at most 1.

\textbf{Proof.} It follows from Lemma 4.4 and Lemma 4.6 inductively. \hfill $\Box$

Theorem 4.10. For any gluing Nakayama algebra $\Gamma$, if the gluing components $A_1 = kQ_1/I_1$, $A_2 = kQ_2/I_2$, \ldots, $A_m = kQ_m/I_m$ of $\Gamma$ are Gorenstein, then

$$\text{Gproj}(\Gamma) \simeq \prod_{i=1}^{m} \text{Gproj}(A_i).$$

\textbf{Proof.} We prove it by induction on $m$. If $m = 1$, it is obvious. For $m$, if there exist $A_i$ such that it is of type I, then by [10, Lemma 3.5] and the inductive hypothesis, we get the conclusion immediately. If all of $A_i$ are of type II, by Lemma 4.7 we also get the conclusion immediately. \hfill $\Box$

Corollary 4.11. For any gluing Nakayama algebra $\Gamma$, if the gluing components $A_1 = kQ_1/I_1$, $A_2 = kQ_2/I_2$, \ldots, $A_m = kQ_m/I_m$ of $\Gamma$ are selfinjective, then

$$\text{Gproj}(\Gamma) \simeq \prod_{i=1}^{m} \text{mod}(A_i).$$

Corollary 4.12. [19, 10] Let $kQ/I$ be a cluster-tilted algebra of type $A$. Then

$$\text{Gproj}(kQ/I) \simeq \bigoplus_{t(Q)} \text{mod}S_3,$$

where $t(Q)$ is the number of the triangles of $Q$, and $S_3$ is the selfinjective cluster-tilted algebra of type $A_3$.\hfill $\Box$

\textbf{Proof.} From [9,7], we know that cluster-tilted algebras of type $A$ are gluing Nakayama algebras, so it follows from Theorem 4.10 immediately. \hfill $\Box$

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