Research Article

The Sub-Riemannian Limit of Curvatures for Curves and Surfaces and a Gauss–Bonnet Theorem in the Rototranslation Group

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The rototranslation group \( \mathcal{RT} \) is the group comprising rotations and translations of the Euclidean plane which is a 3-dimensional Lie group. In this paper, we use the Riemannian approximation scheme to compute sub-Riemannian limits of the Gaussian curvature for a Euclidean \( C^2 \)-smooth surface in the rototranslation group away from characteristic points and signed geodesic curvature for Euclidean \( C^2 \)-smooth curves on surfaces. Based on these results, we obtain a Gauss–Bonnet theorem in the rototranslation group.

1. Introduction

The Gauss–Bonnet theorem connects the intrinsic differential geometry of a surface with its topology and has many applications in physics and mathematics. For example, Petters used the Gauss–Bonnet theorem to study the global geometry of caustics for multiple lens planes in the impulse approximation [1]. Gibbons and Werner showed that it is possible to calculate the deflection angle in weak field limits using the Gauss–Bonnet theorem and the optical geometry [2]. In this method, they found that the focusing of the light rays emerges as a topological effect. In 2018, Övgün et al. used the Gauss–Bonnet theorem to obtain the deflection angle by the photons coupled to the Weyl tensor in a Schwarzschild black hole and Schwarzschild-like black hole in bumblebee gravity in the weak limit approximation [3]. Their computations about the weak gravitational lensing of the Kerr-MOG black hole utilized the method of Gauss–Bonnet first prescribed by Gibbons and Werner [2], which reveals the ignored role of topology in gravitational lensing. In 2019, they studied the weak gravitational lensing by the Kerr-MOG black hole and showed that the MOG effect could be taken into account in the gravitational lensing experiment [4]. In 2020, they applied the RVB method, which considers the topological fractions together with the Gauss–Bonnet theorem and different spacetimes including the nonasymptotically flat ones. This approach showed that Hawking radiation possesses a topological effect coming from the Euler characteristic of the spacetime. The Ricci scalar of the spacetime encodes all the information about the spacetime, which means that it can determine the temperature of a black hole with the Euler characteristic of the metric [5]. They also employed the Gauss–Bonnet theorem to compute the deflection angle by a NAT black hole in the weak limit approximation [6]. In 2021, Chen et al. investigated the photon sphere, shadow, and QNMs of the four-dimensional charged Einstein–Gauss–Bonnet black hole [7]. In this paper, we focus on the Gauss–Bonnet theorem in the rototranslation group.

The rototranslation group, \( \mathcal{RT} \), is the group of Euclidean rotations and translations of the plane equipped with a particular sub-Riemannian metric. More precisely, \( \mathcal{RT} \) is a three-dimensional topological manifold diffeomorphic to \( \mathbb{R}^2 \times S^1 \) with coordinates \( (x, y, \theta) \). The sub-Riemannian geometry of the rototranslation group is in contrast to the well-known case of the Heisenberg group in mathematics, and it provides geometrical models in mechanics and robotics [8, 9]. In [10], the rototranslation group \( \mathcal{RT} \) and its universal cover \( \mathcal{G} \) were introduced. The main theorem states that a straight ruled surface in \( \mathcal{G} \) is horizontally minimal.
Among more recent works, data representations in orientation scores as a function on the rototranslation group have been used for template matching with cross-correlation [11]. Bekkers et al. recognized a curved geometry on the position-orientation domain, which they identified with the rototranslation group. Templates were then optimized in a B-spline basis, and smoothness was defined with respect to the curved geometry. In [12], illusory patterns were identified by a suitable modulation of the geometry of the rototranslation group and computed as the associated geodesics via the fast marching algorithm. In [13,14], Balogh et al. used a Riemannian approximation scheme to define a notion of the intrinsic Gaussian curvature for a Euclidean C^2-smooth surface in the Heisenberg group H^1 away from characteristic points and signed geodesic curvature for Euclidean C^2-smooth curves on surfaces. We also obtain a Gauss–Bonnet theorem in the rototranslation group.

In Section 2, we provide a short introduction to the rototranslation group and the notion which we will use throughout the paper, such as the Levi-Civita connection and curvature in the Riemannian approximants of the rototranslational group. In Section 3, we compute the sub-Riemannian limit of the curvature of curves in the rototranslation group. In Sections 4 and 5, we compute sub-Riemannian limits of the geodesic curvature of curves on surfaces and the Riemannian Gaussian curvature of surfaces in the rototranslation group. In Section 6, we obtain the Gauss–Bonnet theorem in the rototranslation group. In Section 7, we summarize this paper as conclusions.

2. Levi-Civita Connection and Curvature in the Riemannian Approximants of the Rototranslation Group

In this section, some basic notions in the rototranslation group will be introduced. The rototranslation group \(R\mathcal{T}\) is the group comprising rotations and translations of the Euclidean plane. It is a 3-dimensional Lie group, isomorphic to \(\mathbb{R}^2 \times S^1\) with multiplication given by

\[
(x, y, \theta) * (x', y', \theta') = (x + x' \cos \theta - y' \sin \theta, y + x' \sin \theta + y' \cos \theta, \theta + \theta'),
\]

for all \((x, y, \theta), (x', y', \theta') \in \mathbb{R}^2 \times S^1\). In this model, the natural element of \(R\mathcal{T}\) is \((0, 0, 0)\), and the inverse element of \((x, y, \theta)\) is \((-x \cos \theta - y \sin \theta, x \sin \theta - y \cos \theta, -\theta)\).

Let

\[
X_1 = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y},
\]

\[
X_2 = \frac{\partial}{\partial \theta},
\]

\[
X_3 = \sin \theta \frac{\partial}{\partial x} - \cos \theta \frac{\partial}{\partial y},
\]

with brackets

\[
[X_1, X_2] = X_2,
\]

\[
[X_2, X_3] = X_1,
\]

\[
[X_1, X_3] = 0.
\]

Then, we compute sub-Riemannian limits of the Gaussian curvature for a Euclidean C^2-smooth surface in the rototranslation group away from characteristic points and signed geodesic curvature for Euclidean C^2-smooth curves on surfaces. We also obtain a Gauss–Bonnet theorem in the rototranslation group.

\[
\frac{\partial}{\partial x} = \cos \theta X_1 + \sin \theta X_3,
\]

\[
\frac{\partial}{\partial y} = \sin \theta X_1 - \cos \theta X_3,
\]

and span\(\{X_1, X_2, X_3\} = T(R\mathcal{T})\). Let \(H = \text{span}\{X_1, X_3\}\) be the horizontal distribution on \(R\mathcal{T}\). Let \(\omega_1 = \cos \theta dx + \sin \theta dy, \omega_2 = d\theta, \) and \(\omega = \sin \theta dx - \cos \theta dy\). Then, \(H = \ker \omega\). To describe the Riemannian approximants to \(R\mathcal{T}\), for the constant \(L > 0\), we define a metric \(g_L = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + L \omega \otimes \omega\) so that \(X_1, X_2, X_3 = L^{1/2} X_i\) are orthonormal bases on \(T(R\mathcal{T})\) with respect to \(g_L\). It is easy to check that \(g = g_1\) is the Riemannian metric on \(R\mathcal{T}\).

To compute the sectional, Ricci, and scalar curvatures of the rototranslation group with respect to \(g_1\), we use the Levi-Civita connection \(\nabla^L\) on \((R\mathcal{T}, g_1)\). A straightforward calculation shows the following proposition.
Proposition 1. Let $\mathcal{RT}$ be the rototranslation group, relative to the coordinate frame $X_1, X_2, X_3$; then, the Levi-Civita connection on $\mathcal{RT}$ is given by

$$
\begin{align*}
\nabla^L_{X_j} X_k &= 0, \quad 1 \leq j \leq 3, \\
\nabla^L_{X_i} X_2 &= \frac{L-1}{2L} X_3, \\
\nabla^L_{X_i} X_1 &= \frac{1-2}{4L} X_2, \\
\nabla^L_{X_i} X_3 &= \frac{1-L}{2L} X_2, \\
\nabla^L_{X_i} X_1 &= \frac{(L+1)}{2L} X_2,
\end{align*}
$$

Proof. It follows from a direct application of the Koszul identity, which here simplifies

$$
2\langle \nabla^L_{X_j} X_k, X_k \rangle_L = \langle [X_j, X_k], X_k \rangle_L - \langle [X_j, X_k], X_j \rangle_L + \langle [X_k, X_j], X_j \rangle_L.
$$

(6)

When $j = 1$, we compute $\langle \nabla^L_{X_j} X_k, X_k \rangle_L = \langle [X_k, X_j], X_k \rangle_L$. It follows that $\langle \nabla^L_{X_j} X_i, X_i \rangle_L = 0, \langle \nabla^L_{X_j} X_i, X_j \rangle_L = \langle [X_i, X_j], X_j \rangle_L = 0$, and $\langle \nabla^L_{X_j} X_j, X_k \rangle_L = 0$. Hence, $\nabla^L_{X_1} X_1 = 0$. Similarly, $\nabla^L_{X_2} X_2 = 0$ and $\nabla^L_{X_3} X_3 = 0$. By the following equation,

$$
\begin{align*}
\nabla^L_{X_j} X_k &= \langle [X_j, X_k], X_k \rangle_L - \langle [X_j, X_k], X_j \rangle_L + \langle [X_k, X_j], X_j \rangle_L \\
&= \langle [X_j, X_k], X_k \rangle_L + \langle [X_k, X_j], X_j \rangle_L,
\end{align*}
$$

we get $\nabla^L_{X_1} X_2 = (L - 1/2L)X_3$. Other cases follow the similar way.

Define the curvature of the connection $\nabla^L$ by

$$
R^L(X, Y)Z = \nabla^L X \nabla^L_Y Z - \nabla^L Y \nabla^L_X Z - \nabla^L_{[X,Y]} Z.
$$

We get the following proposition.

Proposition 2. Let $\mathcal{RT}$ be the rototranslation group; then,

$$
\begin{align*}
R^L(X_1, X_2) X_1 &= \frac{3L^2 - 2L - 1}{4L} X_2, \\
R^L(X_1, X_2) X_2 &= \frac{-3L^2 + 2L + 1}{4L} X_1, \\
R^L(X_1, X_2) X_3 &= 0, \\
R^L(X_1, X_3) X_1 &= \frac{-L^2 + 2L - 1}{4L} X_3, \\
R^L(X_1, X_3) X_2 &= 0, \\
R^L(X_1, X_3) X_3 &= \frac{L^2 - 2L + 1}{4L} X_1,
\end{align*}
$$

Proof. It is a direct computation using

$$
R^L(X, Y)Z = \nabla^L_X \nabla^L_Y Z - \nabla^L_Y \nabla^L_X Z - \nabla^L_{[X,Y]} Z.
$$

(11)

Taking

$$
R^L(X_1, X_2) X_1 = \left( \nabla^L_{X_1} \nabla^L_{X_2} - \nabla^L_{X_2} \nabla^L_{X_1} - \nabla^L_{[X_1,X_2]} \right) X_1,
$$

(12)

for example, we compute

$$
\begin{align*}
\nabla^L_{X_1} \left( \nabla^L_{X_1} X_1 \right) &= \frac{-(L+1)}{2L} \nabla^L_{X_1} X_3 = \frac{L^2 - 1}{4L} X_2, \\
\nabla^L_{X_1} \left( \nabla^L_{X_1} X_1 \right) &= 0, \\
\nabla^L_{[X_1,X_2]} X_1 &= \frac{(L+1)}{2} X_2.
\end{align*}
$$

Hence,

$$
R^L(X_1, X_2) X_1 = \frac{L^2 - 1}{4L} X_2 - \frac{(L+1)}{2} X_2 = \frac{3L^2 - 2L - 1}{4L} X_2.
$$

(14)

We compute the sectional curvatures of the two planes spanned by the basis vectors $\bar{X}_i$ and $\bar{X}_j$:...
\( \mathcal{K}_{ij} = \langle R^L(\bar{X}_i, \bar{X}_j)\bar{X}_k, \bar{X}_l \rangle_L \), where \( \bar{X}_i = X_i \) for \( i = 1, 2 \) and \( \bar{X}_3 = L^{-1/2}X_3 \).

In fact, the full Riemannian curvature tensor

\[
R^L_{ijkl} = \langle R^L(\bar{X}_i, \bar{X}_j)\bar{X}_k, \bar{X}_l \rangle_L \text{ is}
\]

\[
K_{12} = \langle R^L(\bar{X}_1, \bar{X}_2)\bar{X}_1, \bar{X}_2 \rangle_L = \frac{3L^2 - 2L - 1}{4L},
\]

\[
K_{13} = \langle R^L(\bar{X}_1, \bar{X}_3)\bar{X}_1, \bar{X}_3 \rangle_L = -\frac{L^2 + 2L - 1}{4L},
\]

\[
K_{23} = \langle R^L(\bar{X}_2, \bar{X}_3)\bar{X}_2, \bar{X}_3 \rangle_L = -\frac{L^2 - 2L + 3}{4L},
\]

(15)

\[
R^L_{ijkl} = \begin{cases} 
\frac{3L^2 - 2L - 1}{4L} & \text{if } (ijkl) = (1212) \text{ or } (2121), \\
-\frac{3L^2 + 2L + 1}{4L} & \text{if } (ijkl) = (1221) \text{ or } (2112), \\
-\frac{L^2 + 2L - 1}{4L} & \text{if } (ijkl) = (1313), (3131), (2323) \text{ or } (3232), \\
\frac{L^2 - 2L + 1}{4L} & \text{if } (ijkl) = (1331), (3113), (2332) \text{ or } (3223), \\
0 & \text{otherwise}.
\end{cases}
\]

(16)

In order to compute the Kretschmann scalar, from

\[
R^L_{ijkl} = g^{ij} g^{kl} g^{ji} g^{lk} R^L_{ijkl},
\]

it follows that we can write

\[
R^L_{ijkl} = \begin{cases} 
\frac{3L^2 - 2L - 1}{4L} & \text{if } (ijkl) = (1212) \text{ or } (2121), \\
-\frac{3L^2 + 2L + 1}{4L} & \text{if } (ijkl) = (1221) \text{ or } (2112), \\
-\frac{L^2 + 2L - 1}{4L} & \text{if } (ijkl) = (1313), (3131), (2323) \text{ or } (3232), \\
\frac{L^2 - 2L + 1}{4L} & \text{if } (ijkl) = (1331), (3113), (2332) \text{ or } (3223), \\
0 & \text{otherwise}.
\end{cases}
\]

(17)
Recall that the Kretschmann scalar is defined by $\mathcal{K} = R^L_{ijkl}R^L_{ijkl}$. We calculate that
\begin{equation}
\mathcal{K} = 4\left((R^L_{1212})^2 + (R^L_{1313})^2 + (R^L_{2323})^2\right) = \frac{11L^4 - 20L^3 + 10L^2 - 4L + 3}{4L^2}.
\end{equation}

Next, the Ricci curvature $\text{Ric}_i = \mathcal{K}_{ii} + \mathcal{K}_{ij} + \mathcal{K}_{j1}$ is
\begin{equation}
\begin{aligned}
\text{Ric}_1 &= \frac{L}{2} - \frac{1}{2L}, \\
\text{Ric}_2 &= -L + \frac{1}{2L} - 1, \\
\text{Ric}_3 &= \frac{L}{2} + \frac{1}{2L},
\end{aligned}
\end{equation}
while the scalar curvature
\begin{equation}
\sigma = \text{Ric}_1 + \text{Ric}_2 + \text{Ric}_3 = -L + \frac{1}{2L} - 1.
\end{equation}

It can be observed that the Kretschmann scalar and the sectional, Ricci, and scalar curvatures all diverge as $L \to \infty$.

3. The Sub-Riemannian Limit of the Curvature of Curves in the Rototranslation Group

In this section, we will compute the sub-Riemannian limit of the curvature of curves in the rototranslation group.

**Definition 1.** Let $\gamma: [a,b] \to (\mathcal{M}, g_L)$ be a Euclidean $C^1$-smooth curve; we say that $\gamma$ is regular if $\gamma \neq 0$ for every $t \in [a,b]$. Moreover, we say that $\gamma(t)$ is a horizontal point of $\gamma$ if
\begin{equation}
\omega(\gamma(t)) = (\sin \theta \, dx - \cos \theta \, dy)\left(\ddot{\gamma}_1(t) \frac{\partial}{\partial x} + \ddot{\gamma}_2(t) \frac{\partial}{\partial y} + \dot{\theta}(t) \frac{\partial}{\partial \theta}\right) = 0,
\end{equation}
where $\gamma(t) = (\gamma_1(t), \gamma_2(t), \dot{\theta}(t))$.

**Definition 2.** Let $\gamma: [a,b] \to (\mathcal{M}, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(\mathcal{M}, g_L)$. The curvature $\kappa^L_{\gamma}$ of $\gamma$ at $\gamma(t)$ is defined as
\begin{equation}
\kappa^L_{\gamma} = \sqrt{\frac{\langle \nabla^2_{\gamma} \gamma, \gamma \rangle_{L}}{\|\nabla \|_{L}^6}}.
\end{equation}

**Proposition 3.** Let $\gamma: [a,b] \to (\mathcal{M}, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(\mathcal{M}, g_L)$. Then,
\begin{align}
\kappa^L_{\gamma} &= \sqrt{\frac{\langle \nabla^2_{\gamma} \gamma, \gamma \rangle_{L}}{\|\nabla \|_{L}^6}} \\
&= \sqrt{\frac{1}{L^2} \left(\langle \dot{\gamma}_1(t) \cos \theta + \dot{\gamma}_2(t) \sin \theta \rangle_{L} - \langle \dot{\gamma}_1(t) \sin \theta + \dot{\gamma}_2(t) \cos \theta + L(\omega(\gamma(t)))) \rangle_{L} - \langle \dot{\gamma}_1(t) \cos \theta + \dot{\gamma}_2(t) \sin \theta \rangle_{L} - \langle \dot{\gamma}_1(t) \sin \theta + \dot{\gamma}_2(t) \cos \theta + L(\omega(\gamma(t)))) \rangle_{L}}{\|\nabla \|_{L}^6}}.
\end{align}
In particular, if \( y(t) \) is a horizontal point of \( y \),

\[
k^L_\gamma = \left\{ \begin{array}{l}
\left[ \dot{y}_1(t) \cos \theta + \dot{y}_2(t) \sin \theta + \ddot{\theta}(t) \left( -\dot{y}_1 \sin \theta + \dot{y}_2 \cos \theta \right) \right]^2 + (\ddot{\theta}(t))^2 \\
+ L \left[ \dot{y}_1(t) \cos \theta + \dot{y}_2(t) \sin \theta \right] \frac{1}{L} q + h \frac{d}{dt} \left( \omega(\dot{y}(t)) \right) \end{array} \right\}^{1/2}
\]

(24)

\[
\dot{y}(t) = \dot{y}(t) \frac{\partial}{\partial x} + \dot{y}_2(t) \frac{\partial}{\partial y} + \ddot{\theta}(t) \frac{\partial}{\partial \theta}
\]

\[
= \dot{y}_1(t) \left( \cos \theta X_1 + \sin \theta X_3 \right) + \dot{y}_2(t) \left( \sin \theta X_1 - \cos \theta X_3 \right) + \ddot{\theta}(t) X_2
\]

(25)

By Proposition 1 and (25), we have

\[
\nabla_{L}^L X_1 = \left( \dot{y}_1(t) \cos \theta + \dot{y}_2(t) \sin \theta \right) \nabla_{L}^L X_1 + \ddot{\theta}(t) \nabla_{L}^L X_1 + \omega(\dot{y}(t)) \nabla_{L}^L X_1
\]

\[
= \ddot{\theta}(t) \frac{L+1}{2L} X_3 + \omega(\dot{y}(t)) \frac{L+1}{2} X_2,
\]

\[
\nabla_{L}^L X_2 = \left( \dot{y}_1(t) \cos \theta + \dot{y}_2(t) \sin \theta \right) \nabla_{L}^L X_2 + \ddot{\theta}(t) \nabla_{L}^L X_2 + \omega(\dot{y}(t)) \nabla_{L}^L X_2
\]

\[
= \ddot{\theta}(t) \frac{L-1}{2L} X_3 + \omega(\dot{y}(t)) \frac{-1}{2} X_1,
\]

(26)

\[
\nabla_{L}^L X_3 = \left( \dot{y}_1(t) \cos \theta + \dot{y}_2(t) \sin \theta \right) \nabla_{L}^L X_3 + \ddot{\theta}(t) \nabla_{L}^L X_3 + \omega(\dot{y}(t)) \nabla_{L}^L X_3
\]

\[
= \ddot{\theta}(t) \frac{1-L}{2} X_2 + \omega(\dot{y}(t)) \frac{1+L}{2} X_1.
\]
By (25) and (26), we have

\[
\begin{align*}
\nabla^L_y \gamma &= \nabla^L_y (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta) X_1 + \dot{\gamma}(t) X_2 + \omega(\dot{\gamma}(t)) X_3 \\
&= \gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta) X_1 + (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta) \nabla^L_y X_1 + \gamma(t) X_2 \\
&+ \dot{\gamma}(t) \nabla^L_y X_2 + \omega(\dot{\gamma}(t)) X_3 + \omega(\dot{\gamma}(t)) \nabla^L_y X_3 \\
&= \left[ \gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta - \dot{\gamma}_1(t) \sin \theta + \dot{\gamma}_2(t) \cos \theta + L \omega(\dot{\gamma}(t)) \right] X_1 \\
&+ \left[ (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta) \omega(\dot{\gamma}(t))(1 - L) + \dot{\theta}(t) \right] X_2 + \\
&\left[ (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta) \dot{\theta}(t) - \frac{1}{L} \frac{d}{dt}(L \omega(\dot{\gamma}(t))) \right] X_3.
\end{align*}
\]

(27)

By (22) and (25), we get

\[
\begin{align*}
\| \nabla^L_y \gamma \|^2 &= \left[ (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta + \dot{\gamma}_1(t) \sin \theta + \dot{\gamma}_2(t) \cos \theta + L \omega(\dot{\gamma}(t))) \right]^2 \\
&+ \left[ (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta) \omega(\dot{\gamma}(t))(1 - L) + \dot{\theta}(t) \right]^2 \\
&+ L \left[ (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta) \dot{\theta}(t) \left( -\frac{1}{L} \frac{d}{dt}(L \omega(\dot{\gamma}(t))) \right)^2, \\
\| \dot{\gamma} \|^2 &= \left[ (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta)^2 + (\dot{\theta}(t))^2 + L (\omega(\dot{\gamma}(t)))^2 \right]^2 \\
\langle \nabla^L_y \gamma, \dot{\gamma} \rangle &= \left[ (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta) \\
&= \left[ \gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta + \dot{\gamma}_1(t) \sin \theta + \dot{\gamma}_2(t) \cos \theta + L \omega(\dot{\gamma}(t))) \right] \\
&= + \dot{\theta}(t) \left[ (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta) \omega(\dot{\gamma}(t))(1 - L) + \dot{\theta}(t) \right] \\
&= + L \omega(\dot{\gamma}(t)) \left[ (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta) \dot{\theta}(t) \left( -\frac{1}{L} \frac{d}{dt}(L \omega(\dot{\gamma}(t))) \right)^2, \\
\| \dot{\gamma} \|^2 &= \left[ (\gamma_1(t) \cos \theta + \gamma_2(t) \sin \theta)^2 + (\dot{\theta}(t))^2 + L (\omega(\dot{\gamma}(t)))^2 \right]^3.
\end{align*}
\]

(28)

By the definition of \( \kappa^L_y \), we get Proposition 3.

\[ \square \]

**Definition 3.** Let \( \gamma: [a, b] \longrightarrow (\mathcal{R}, g_L) \) be a Euclidean \( C^2 \)-smooth regular curve in the Riemannian manifold \( (\mathcal{R}, g_L) \); we define the intrinsic curvature \( \kappa^L_y \) of \( \gamma \) at \( y(t) \) to be

\[
\kappa^L_y = \lim_{L \to \infty} \kappa^L_y,
\]

(29)

if the limit exists.

We introduce the following notation: for continuous functions \( f, g: (0, +\infty) \longrightarrow \mathbb{R} \),

\[
f_1(L) = f_2(L), \text{ as } L \longrightarrow +\infty \iff \lim_{L \to +\infty} \frac{f_1(L)}{f_2(L)} = 1.
\]

(30)

**Proposition 4.** Let \( \gamma: [a, b] \longrightarrow (\mathcal{R}, g_L) \) be a Euclidean \( C^2 \)-smooth regular curve in the Riemannian manifold \( (\mathcal{R}, g_L) \). Then,
\[ \kappa_\gamma^\infty = \left( \frac{\dot{\theta}^2 + (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta)^2}{|\omega(\dot{y}(t))|} \right) \text{ if } \omega(\dot{y}(t)) \neq 0. \]  
\[ \kappa_\gamma^\infty = \left\{ \left[ \left( \dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta + \dot{\theta}(\dot{y}_1 \sin \theta + \dot{y}_2 \cos \theta) \right)^2 + \dot{\theta}^2 \right] \cdot \left[ (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta)^2 + \dot{\theta}^2 \right]^{-2} - \left[ \left( \dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta \right) \left( \dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta + \dot{\theta}(\dot{y}_1 \sin \theta + \dot{y}_2 \cos \theta) \right) + \dot{\theta} \right]^2 \right\} \left( \frac{1}{2} \right) \text{ if } \omega(\dot{y}(t)) = 0 \text{ and } \frac{d}{dt} \left( \omega(\dot{y}(t)) \right) = 0, \]  
\[ \lim_{L \to -\infty} \kappa_\gamma^L = \frac{|(d/dt)(\omega(\dot{y}(t))|}{(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta)^2 + \dot{\theta}^2} \text{ if } \omega(\dot{y}(t)) = 0 \text{ and } \frac{d}{dt} \left( \omega(\dot{y}(t)) \right) \neq 0. \]

**Proof.** When \( \omega(\dot{y}(t)) \neq 0 \), we have

\[ \|\nabla^L_{\dot{y}} \|_{L^2}^2 \sim \omega(\dot{y}(t))^2 \left( \dot{\theta}^2 + (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta)^2 \right) \left( \dot{\theta}^2 + (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta)^2 \right) \left( \dot{\theta}^2 + (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta)^2 \right) \text{ as } L \to +\infty, \]
\[ \|\dot{y}\|_{L^2}^2 \sim L \omega(\dot{y}(t))^2, \]
\[ \langle \nabla^L_{\dot{y}} \dot{y}, \dot{y} \rangle_{L^2} \sim O(1) \text{ as } L \to +\infty. \]

Therefore,
\[ \frac{\|\nabla^L_{\dot{y}} \dot{y}\|_{L^2}^2}{\|\dot{y}\|_{L^2}^2} \to \frac{\dot{\theta}^2 + (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta)^2}{\omega(\dot{y}(t))^2} \text{ as } L \to +\infty, \]
\[ \frac{\langle \nabla^L_{\dot{y}} \dot{y}, \dot{y} \rangle_{L^2}^2}{\|\dot{y}\|_{L^2}^2} \to 0 \text{ as } L \to +\infty. \]

If \( \omega(\dot{y}(t)) \neq 0 \), by (22), we have
\[ \kappa_\gamma^\infty = \left( \frac{\dot{\theta}^2 + (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta)^2}{|\omega(\dot{y}(t))|} \right) \]  
By (24) and \( (d/dt)(\omega(\dot{y}(t)) = 0, \) we have
\[ \lim_{L \to -\infty} \frac{\kappa_\gamma^L}{\sqrt{L}} = \frac{|(d/dt)(\omega(\dot{y}(t))|}{(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta)^2 + \dot{\theta}^2}. \]

4. The Sub-Riemannian Limit of the Geodesic Curvature of Curves on Surfaces in the Rototranslation Group

In this section, we will compute the sub-Riemannian limit of the geodesic curvature of curves on surfaces in the
rototranslation group. We will say that a surface \( \Sigma \subset (\mathcal{RT}, g_L) \) is regular if \( \Sigma \) is a Euclidean \( C^2 \)-smooth compact and oriented surface. In particular, we will assume that there exists a Euclidean \( C^2 \)-smooth function \( u: \mathcal{RT} \rightarrow \mathbb{R} \) such that

\[
\Sigma = \{(x_1, x_2, \theta) \in \mathcal{RT}: u(x_1, x_2, \theta) = 0\}, \tag{39}
\]

and

\[
u_hu = X_1(u)X_1 + X_2(u)X_2. \quad \text{A point } x \in \Sigma \text{ is called a characteristic if } \nabla H u(x) = 0. \]

We define the characteristic set

\[
C(\Sigma) = \{(x_1, x_2, \theta) \in \Sigma | \nabla H u(x_1, x_2, \theta) = 0\}. \tag{40}
\]

Our computations will be local and away from characteristic points of \( \Sigma \). Let us first define \( p = X_1 u, q = X_2 u, \) and \( r = \nabla \Sigma \). We then define

\[
l = \sqrt{p^2 + q^2}, \quad l_L = \sqrt{p^2 + q^2 + r^2}, \quad \bar{p} = \frac{p}{l}, \quad \bar{q} = \frac{q}{l}, \quad \bar{r}_L = \frac{r}{l_L}, \tag{41}
\]

In particular, \( \bar{p}^2 + \bar{q}^2 = 1 \). These functions are well defined at every noncharacteristic point. Let

\[
v_1 = \bar{p}X_1 + \bar{q}X_2 + \bar{r}_L \nabla \Sigma \quad e_1 = \bar{q}X_1 - \bar{p}X_2, \tag{42}
\]

\[
e_2 = \bar{r}_L p X_1 + \bar{r}_L q X_2 - \frac{l}{l_L} \nabla \Sigma.
\]

Then, \( v_1 \) is the Riemannian unit normal vector to \( \Sigma \), and \( e_1 \) and \( e_2 \) are the orthonormal basis of \( \Sigma \). On \( T\Sigma \), we define a linear transformation \( J_L: T\Sigma \rightarrow T\Sigma \) such that

\[
J_L(e_1) = e_2, J_L(e_2) = -e_1. \tag{43}
\]

For every \( U, V \in T\Sigma \), we define \( V^{\Sigma L}_U V = \pi^{\Sigma L}_U V \), where \( \pi: T\Sigma \rightarrow T\Sigma \) is the projection. Then, \( V^{\Sigma L}_U \) is the Levi-Civita connection on \( \Sigma \) with respect to the metric \( g_L \). By (27), (42), and

\[
\nabla_{\bar{v}_1} \bar{y} = \langle \bar{v}_1, e_1 \rangle \bar{e}_1 + \langle \bar{v}_1, e_2 \rangle \bar{e}_2, \tag{44}
\]

we have

\[
\nabla^{\Sigma L}_{\bar{v}_1} \bar{y} = \left[ \bar{q} \left( \bar{y}_1 \cos \theta + \bar{y}_2 \sin \theta + \bar{q} \left( -\bar{y}_1 \sin \theta + \bar{y}_2 \cos \theta + L \omega(\bar{y}(t)) \right) \right) - \bar{p} \left( \bar{y}_1 \cos \theta + \bar{y}_2 \sin \theta + \bar{q} \left( -\bar{y}_1 \sin \theta + \bar{y}_2 \cos \theta + L \omega(\bar{y}(t)) \right) \right) \right] e_1
\]

\[
+ \left[ \bar{r}_L \bar{q} \left( \bar{y}_1 \cos \theta + \bar{y}_2 \sin \theta + \bar{q} \left( -\bar{y}_1 \sin \theta + \bar{y}_2 \cos \theta + L \omega(\bar{y}(t)) \right) \right) - \bar{r}_L \bar{p} \left( \bar{y}_1 \cos \theta + \bar{y}_2 \sin \theta + \bar{q} \left( -\bar{y}_1 \sin \theta + \bar{y}_2 \cos \theta + L \omega(\bar{y}(t)) \right) \right) \right] e_2, \tag{45}
\]

Moreover, if \( \omega(\bar{y}(t)) = 0 \), then

\[
\nabla^{\Sigma L}_{\bar{v}_1} \bar{y} = \left[ \bar{q} \left( \bar{y}_1 \cos \theta + \bar{y}_2 \sin \theta + \bar{q} \left( -\bar{y}_1 \sin \theta + \bar{y}_2 \cos \theta \right) \right) - \bar{p} \left( \bar{y}_1 \cos \theta + \bar{y}_2 \sin \theta + \bar{q} \left( -\bar{y}_1 \sin \theta + \bar{y}_2 \cos \theta \right) \right) \right] e_1
\]

\[
+ \left[ \bar{r}_L \bar{q} \left( \bar{y}_1 \cos \theta + \bar{y}_2 \sin \theta + \bar{q} \left( -\bar{y}_1 \sin \theta + \bar{y}_2 \cos \theta \right) \right) - \bar{r}_L \bar{p} \left( \bar{y}_1 \cos \theta + \bar{y}_2 \sin \theta + \bar{q} \left( -\bar{y}_1 \sin \theta + \bar{y}_2 \cos \theta \right) \right) \right] e_2. \tag{46}
\]
Definition 4. Let $\Sigma \subset (\mathcal{R}, g_L)$ be a regular surface and $\gamma: [a, b] \rightarrow \Sigma$ be a Euclidean $C^2$-smooth regular curve. The geodesic curvature $\kappa^L_{\gamma, \Sigma}$ of $\gamma$ at $t(t)$ is defined as

$$\kappa^L_{\gamma, \Sigma} = \left\| \nabla_{\dot{\gamma}} \right\|_{\Sigma_L}$$

(47)

Definition 5. Let $\Sigma \subset (\mathcal{R}, g_L)$ be a regular surface and $\gamma: [a, b] \rightarrow \Sigma$ be a Euclidean $C^2$-smooth regular curve. We define the intrinsic geodesic curvature $\kappa^\infty_{\gamma, \Sigma}$ of $\gamma$ at $t(t)$ to be

$$\kappa^\infty_{\gamma, \Sigma} = \lim_{L \rightarrow \infty} \kappa^L_{\gamma, \Sigma}$$

(48)

if the limit exists.

Proposition 5. Let $\Sigma \subset (\mathcal{R}, g_L)$ be a regular surface and $\gamma: [a, b] \rightarrow \Sigma$ be a Euclidean $C^2$-smooth regular curve. Then,

$$\kappa^\infty_{\gamma, \Sigma} = \left\| \frac{\theta \dot{p} + \overline{p}(\gamma_1 \cos \theta + \gamma_2 \sin \theta)}{\omega(\dot{\gamma}(t))} \right\|$$

if $\omega(\dot{\gamma}(t)) \neq 0,$

$$\kappa^\infty_{\gamma, \Sigma} = 0$$

if $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0,$

(49)

$$\lim_{L \rightarrow \infty} \frac{\kappa^L_{\gamma, \Sigma}}{\sqrt{L}} = \left\| \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right\|$$

(50)

if $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0.$

Proof. By (25) and $\dot{\gamma} \in T_\Sigma,$ we have

$$\dot{\gamma}(t) = (\gamma_1 \cos \theta + \gamma_2 \sin \theta)X_1 + \theta(t)X_2 + \omega(\dot{\gamma}(t))X_3.$$  

(51)

By (46), we have

$$\dot{\gamma}(t) = a\dot{\gamma}_1 + b\dot{\gamma}_2 = a(\overline{\dot{q}}X_1 - \overline{\dot{p}}X_2) + b(\overline{\dot{r}}_1\overline{\dot{p}}X_1 + \overline{\dot{r}}_2\overline{\dot{q}}X_2 - \frac{l}{L^2}X_3).$$

(52)

By the aforementioned first equation and the second equation, we get

$$\begin{cases}
    a\overline{q} + b\overline{r}_1\overline{p} = \gamma_1 \cos \theta + \gamma_2 \sin \theta,
    
    -a\overline{p} + b\overline{r}_2\overline{q} = \theta(t),
    
    -\frac{bl}{L^2} = \omega(\dot{\gamma}(t)).
\end{cases}$$

(53)

Solving the above equations, we get

$$\begin{cases}
    a = (\gamma_1 \cos \theta + \gamma_2 \sin \theta)\overline{q} - \overline{p}(\theta),
    
    b = -\frac{1}{L} \omega(\dot{\gamma}(t)).
\end{cases}$$

(54)

Then, we get

$$\dot{\gamma} = \left[ -\overline{p}(\theta) + \overline{q}(\gamma_1 \cos \theta + \gamma_2 \sin \theta) \right] e_1 - \frac{l}{L}(1/2) \omega(\dot{\gamma}(t))e_2.$$  

(55)

By (26), we have

$$\dot{\gamma}(t) = \gamma_1 \cos \theta + \gamma_2 \sin \theta \gamma(t) + \gamma_2 \cos \theta + \omega(\dot{\gamma}(t))X_3.$$  

(56)

Similarly, we have that when $\omega(\dot{\gamma}(t)) \neq 0,$
\[
\|\dot{y}\|_{\Sigma,L} = \sqrt{\left[ -\overline{p}\dot{\theta} + \overline{q}(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \right]^2 + \left( \frac{1}{L^2} \right)^2 L^{1/2} (\dot{\omega}(\dot{y}(t))) L^{1/2} |\omega(\dot{y}(t))|} \text{ as } L \rightarrow +\infty. \tag{57}
\]

By (45) and (55), we have
\[
\langle \mathcal{V}_y^\Sigma, \dot{y} \rangle_{\Sigma,L} = \left[ -\overline{p}\dot{\theta} + \overline{q}(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \right]
\begin{align*}
&= \left\{ -\overline{p}\left[ (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \omega \left( (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) (1 - L) + \ddot{\theta}(t) \right) \right] \right. \\
&\quad+ \overline{q}\left[ (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \omega \left( (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) (-\dot{y}_1 \sin \theta + \dot{y}_2 \cos \theta + L \omega(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta)) \right) \right] \right. \\
&\quad- \frac{L}{L^2} \left[ \left( \dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta \right) \omega \left( (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) (1 - L) + \ddot{\theta}(t) \right) \right] \\
&\left. \left. + \frac{L}{L^2} \left[ (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \omega \left( (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) (1 - L) + \ddot{\theta}(t) \right) \right] \right\} \sim M_0 L, \tag{58}
\end{align*}
\]

where \( M_0 \) does not depend on \( L \). By (47), we have
\[
k_{y,\Sigma}^\infty = \lim_{L \rightarrow \infty} k_{y,\Sigma}^L = \lim_{L \rightarrow \infty} \sqrt{\left[ \overline{p}\dot{\theta} + \overline{q}(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \right]^2 + \frac{1}{L^2} \left[ \overline{p}\dot{\theta} + \overline{q}(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \right]^2 \omega^2 (\dot{y}(t))} \left| \omega(\dot{y}(t)) \right| \sim M_0 L, \tag{59}
\]

if \( \omega(\dot{y}(t)) \neq 0 \). When \( \omega(\dot{y}(t)) = 0 \) and \( (d/dt)(\omega(\dot{y}(t)) = 0) \), we have
\[
\left\| \nabla_{\Sigma}^{\Sigma_L} \varphi \right\|_{\Sigma_L}^2 = \left\{ \begin{align*}
\& \left[ \tilde{\varphi} \left( \gamma_1 \cos \theta + \gamma_2 \sin \theta \right) - \tilde{\varphi} \right]^2 \\
&+ \left[ \frac{\tilde{\varphi L}}{L} \left[ \gamma_1 \cos \theta + \gamma_2 \sin \theta \right] - \tilde{\varphi} \right]^2 \\
&+ \frac{\tilde{\varphi L}}{L} \left( \frac{(\gamma_1 \cos \theta + \gamma_2 \sin \theta) - \tilde{\varphi}}{L} \right)^2 \\
&\sim \left[ \tilde{\varphi} \left( \gamma_1 \cos \theta + \gamma_2 \sin \theta \right) - \tilde{\varphi} \right]^2 \\
&\text{as } L \to +\infty,
\end{align*} \right.
\]

By (60)–(62) and (47), we get

\[
k_{\gamma,\Sigma}^{\cos} = \sqrt{\frac{A^2 - A^2 B^2}{B^2}} = 0.
\]

If \( \omega'(t) \) is defined as

\[
k_{\gamma,\Sigma}^{\cos} = \sqrt{\frac{(\omega'(t))^2}{\Lambda}} - \frac{\Lambda}{\omega'(t)} \text{,}
\]

where \( \Lambda \) is defined by (43).

\[
k_{\gamma,\Sigma}^{\cos} = \lim_{L \to +\infty} k_{\gamma,\Sigma}^{\cos}.
\]

\section*{Definition 6.}
Let \( \Sigma \subset (\mathcal{R}, g_\Sigma) \) be a regular surface. Let \( \gamma: [a, b] \to \Sigma \) be a Euclidean C2-smooth regular curve. The signed geodesic curvature \( k_{\gamma,\Sigma}^{\cos} \) of \( \gamma \) at \( t \) is defined as

\[
k_{\gamma,\Sigma}^{\cos} := \left. \frac{\langle \nabla_{\gamma}^{\Sigma_L} \gamma, J_L \gamma \rangle_{\Sigma_L}}{\left\| \gamma \right\|_{\Sigma_L}^2} \right|_{\gamma}',
\]

where \( J_L \) is defined by (43).

\section*{Definition 7.}
Let \( \Sigma \subset (\mathcal{R}, g_\Sigma) \) be a regular surface. Let \( \gamma: [a, b] \to \Sigma \) be a Euclidean C2-smooth regular curve. We define the intrinsic geodesic curvature \( k_{\gamma,\Sigma} \) of \( \gamma \) at the noncharacteristic point \( \gamma'(t) \) to be

\[
k_{\gamma,\Sigma} = \lim_{L \to +\infty} k_{\gamma,\Sigma}^{\cos}.
\]

When \( \omega'(t) = 0 \) and \( \frac{d}{dt}(\omega'(t)) \neq 0 \), we have

\[
k_{\gamma,\Sigma}^{\cos} = \frac{\tilde{\varphi} - \tilde{\varphi} L}{\omega'(t)} \quad \text{if } \omega'(t) \neq 0,
\]

\[
k_{\gamma,\Sigma}^{\cos} = 0 \quad \text{if } \omega'(t) = 0 \text{ and } \frac{d}{dt}(\omega'(t)) = 0,
\]

if the limit exists.

\section*{Proposition 6.}
Let \( \Sigma \subset (\mathcal{R}, g_\Sigma) \) be a regular surface. Let \( \gamma: [a, b] \to \Sigma \) be a Euclidean C2-smooth regular curve. Then

\[
\lim_{L \to +\infty} \sqrt{\frac{L}{\left\| \gamma \right\|_{\Sigma_L}^2}} \frac{L \left[ \frac{d}{dt} \left( \omega'(t) \right) \right]^2}{\sqrt{\left[ \frac{\Lambda}{\omega'(t)} - \frac{\Lambda}{\omega'(t)} \right]^2}} = \frac{\left\| \frac{d}{dt} \left( \omega'(t) \right) \right\|}{\sqrt{\left[ \frac{\Lambda}{\omega'(t)} - \frac{\Lambda}{\omega'(t)} \right]^2}}
\]

if \( \omega'(t) = 0 \) and \( \frac{d}{dt}(\omega'(t)) \neq 0 \).
Proof. By (43) and (55), we get

\[
J_L(y) = \left[ -\overline{\rho} \dot{\theta} + \overline{\varphi}(\gamma_1 \cos \theta + \gamma_2 \sin \theta) \right] J_L(e_1) - \frac{L}{I} L^{(1/2)} \omega(\dot{y}(t)) J_L(e_2)
\]

(68)

By (45) and the above equation, we have

\[
\langle \nabla \Sigma^{\gamma L} J_L(y) \rangle = \frac{I}{L} L^{(1/2)} \omega(\dot{y}(t)) \left[ \overline{\rho} \left( \gamma_1 \cos \theta + \gamma_2 \sin \theta \right) \right] \left[ \dot{\theta} + \overline{\varphi}(\gamma_1 \cos \theta + \gamma_2 \sin \theta) \right]
\]

(69)

So, we get

\[
\kappa_{\gamma L}^{\Sigma} = \frac{\langle \nabla \Sigma^{\gamma L} J_L(y) \rangle_{\Sigma L}}{\| \dot{y} \|^3_{\Sigma L}} = \frac{L^{(3/2)} \left[ \overline{\rho} \dot{\theta} + \overline{\varphi}(\gamma_1 \cos \theta + \gamma_2 \sin \theta) \right] \omega^2(\dot{y}(t))}{L^{(3/2)} |\omega(\dot{y}(t))|^3}
\]

(70)

Furthermore, When \( \omega(\dot{y}(t)) = 0 \) and \( (d/dt)(\omega(\dot{y}(t))) = 0 \), we get

\[
\kappa_{\gamma L}^{\Sigma} = \lim_{L \to +\infty} \kappa_{\gamma L}^{\Sigma, L} = \frac{\overline{\rho} \dot{\theta} + \overline{\varphi}(\gamma_1 \cos \theta + \gamma_2 \sin \theta)}{|\omega(\dot{y}(t))|} - M_0 L^{-(1/2)} \text{ as } L \to +\infty.
\]

(71)
So, \( k_{\gamma;\Sigma}^{\text{CO},q} = 0 \). When \( \omega(\dot{y}(t)) = 0 \) and 
\( (d/dt)(\omega(\dot{y}(t))) \neq 0 \), we have

\[
\langle \nabla^E_{\dot{y}} \dot{y}, I_L(\dot{y}) \rangle_{L^2} = \left[ -\overline{\nabla} \theta + q(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \right] \frac{1}{L} - \frac{L}{L^{(1/2)}} (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \frac{1}{L} \left( \frac{1}{L} \right) \frac{d}{dt} \omega(\dot{y}(t)) \right) \right]
\]

\[
\sim \left[ -\overline{\nabla} \theta + q(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \right] \frac{1}{L} \frac{d}{dt} \omega(\dot{y}(t))L^{(1/2)} \text{ as } L \to + \infty,
\]

\[
\sim - \left[ -\overline{\nabla} \theta + q(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \right] \frac{d}{dt} \omega(\dot{y}(t))L^{(1/2)} \text{ as } L \to + \infty.
\]

We get

\[
k_{\gamma;\Sigma}^{\text{CO},q} = \lim_{L \to \infty} \frac{k_{\gamma;\Sigma}^{\text{CO},q}}{\sqrt{L}} = \lim_{L \to \infty} \frac{-\overline{\nabla} \theta + q(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \left( \frac{d}{dt} \omega(\dot{y}(t)) \right) L^{(1/2)}}{-\overline{\nabla} \theta + q(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \sqrt{L}}
\]

\[
= \frac{\left[ \overline{\nabla} \theta - q(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \left( \frac{d}{dt} \omega(\dot{y}(t)) \right) \right]}{\left[ -\overline{\nabla} \theta + q(\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \right]^{3/2}}.
\]

5. The Sub-Riemannian Limit of the Riemannian Gaussian Curvature of Surfaces in the Rototranslation Group

In this section, we will compute the sub-Riemannian limit of the Riemannian Gaussian curvature of surfaces in the \( \mathcal{RT} \) group. We define the second fundamental form \( II^L \) of the embedding of \( \Sigma \) into \( (\mathcal{RT}, g_L) \):

\[
II^L = \begin{pmatrix}
\langle \nabla^L_{e_1} v_{1L}, e_1 \rangle_L & \langle \nabla^L_{e_1} v_{1L}, e_2 \rangle_L \\
\langle \nabla^L_{e_2} v_{1L}, e_1 \rangle_L & \langle \nabla^L_{e_2} v_{1L}, e_2 \rangle_L 
\end{pmatrix}.
\]

We have the following theorem.

**Theorem 1.** The second fundamental form \( II^L \) of the embedding of \( \Sigma \) into \( (\mathcal{RT}, g_L) \) is given by

\[
II^L = \begin{pmatrix}
h_{11} & h_{12} \\
h_{21} & h_{22}
\end{pmatrix},
\]

where

\[
h_{11} = \frac{1}{L} \left( X_1 (\overline{\nabla}) + X_2 (\overline{q}) \right) - \frac{\overline{p}_1 q}{\sqrt{L}},
\]

\[
h_{12} = h_{21} = -\frac{1}{L} \langle e_1, \nabla H \rangle_L - \frac{1}{2 \sqrt{L}} \left( \overline{p}_1^2 - \overline{q}_1^2 \right) - \frac{\sqrt{L}}{2} - \frac{\overline{r}_1^2}{2 \sqrt{L}} \left( \overline{p}_1^2 - \overline{q}_1^2 \right),
\]

\[
h_{22} = \frac{p_1^2}{L} \left( e_2, \nabla_H (\overline{r}) \right)_L + \overline{X}_3 (\overline{r}_1) + \frac{1}{\sqrt{L}} \overline{p}_1 q \overline{r}_1 + \frac{1}{\sqrt{L}} \overline{p} \overline{q} \overline{r}_1.
\]
Proof. Since \( \langle e_1, v_L \rangle_L = 0 \) and \( \langle e_2, v_L \rangle_L = 0 \), we have
\[
\langle \nabla^L_{e_1} v_L, e_1 \rangle_L = -\langle \nabla^L_{e_1} v_L, e_2 \rangle_L = -\langle \nabla^L_{e_2} v_L, v_L \rangle_L.
\]
(78)

Using the definition of the connection, identities in (5), and grouping terms, we have
\[
\nabla^L_{e_1} e_1 = \nabla^L_{\overline{q}_X X_L} \overline{q}_X X_1 - \overline{p}_X X_2 = \overline{q} (X_1 (\overline{q}) X + \overline{q} \nabla^L_{X_L} X_1 - X_1 (\overline{p}) X_2 - \nabla^L_{X_L} X_2)
\]
\[
- \overline{p} (X_2 (\overline{q}) X + \overline{q} \nabla^L_{X_L} X_1 - X_2 (\overline{p}) X_2 - \nabla^L_{X_L} X_2)
\]
\[
= \overline{q} (X_1 (\overline{q}) X_1 - X_1 (\overline{p}) X_2 + \frac{L}{2} X_1) - \overline{p} (X_2 (\overline{q}) X_1 + \frac{L}{2} X_2)
\]
\[
= \overline{q} X_1 (\overline{q}) X_1 - X_1 (\overline{p}) X_2 + \frac{L}{2} X_1 - \overline{p} X_2 (\overline{q}) X_1 - \overline{p} X_2 (\overline{p}) X_2 - \frac{L}{2} X_2
\]
(79)

Since \( \overline{p}^2 + \overline{q}^2 = 1 \), we have \( \overline{p} X \overline{p} + \overline{q} X \overline{q} = 0 \), \( i = 1, 2, 3 \). Thus, \( \overline{q} X \overline{q} = -\overline{p} X \overline{p} \) and \( \overline{q} X \overline{q} = -\overline{p} X \overline{p} \), and we have
\[
\nabla^L_{e_1} e_1 = -\overline{p} (X_1 (\overline{p}) + X_2 (\overline{q})) X_1 - \overline{q} (X_1 (\overline{p}) + X_2 (\overline{q})) X_2 + \frac{\overline{p} \overline{q}}{L} X_3.
\]
(80)

Next, we compute the inner product of this with \( v_L \), we obtain
\[
h_{11} = -\langle \nabla^L_{e_1} e_1, v_L \rangle_L = \overline{p} \overline{p} (X_1 (\overline{p}) + X_2 (\overline{q})) + \overline{q} \overline{q} (X_1 (\overline{p}) + X_2 (\overline{q})) - \frac{\overline{p} \overline{q}}{L} \sqrt{L}
\]
\[
= \frac{\overline{p}}{L} X_1 (\overline{p}) + X_2 (\overline{q}) + \frac{\overline{q}}{L} X_1 (\overline{p}) + X_2 (\overline{q}) - \frac{\overline{p} \overline{q}}{L} \sqrt{L}
\]
\[
= \frac{1}{L} (\overline{p}^2 + \overline{q}^2) X_1 (\overline{p}) + \frac{1}{L} (\overline{p}^2 + \overline{q}^2) X_2 (\overline{q}) - \frac{\overline{p} \overline{q}}{L} \sqrt{L} = \frac{1}{L} X_1 (\overline{p}) + X_2 (\overline{q}) - \frac{\overline{p} \overline{q}}{L} \sqrt{L}
\]
(81)

To compute \( h_{12} \) and \( h_{21} \), using the definition of the connection, we obtain
\[
\nabla^L_{e_1} e_2 = \nabla^L_{\overline{p} X_1 - \overline{q} X_2} \overline{p} X_1 + \overline{q} X_2 - \frac{1 + L}{2} L X_3 = \overline{q} \nabla^L_{\overline{q} X_1} \overline{p} X_1 + \overline{q} X_2 - \frac{1}{L} L X_3 - \frac{1 + L}{2} L X_3
\]
\[
- \overline{p} \nabla^L_{\overline{q} X_1} \overline{p} X_1 + \overline{q} X_2 - \frac{1}{L} 2 X_3
\]
\[
= \overline{q} (X_1 (\overline{q} X_1 X_1 + (\overline{q} X_1) X_2 + \overline{q} \overline{p} L X_1 X_2 - \overline{X}_1 (\overline{p} L X_1 X_2 - \overline{X}_1) L X_3 - \frac{1}{L} L X_3 - \overline{X}_1) L X_3)
\]
\[
- \overline{p} (X_2 (\overline{p} X_1 + X_2 (\overline{p} X_1 + X_2 (\overline{p} X_1 + X_2 (\overline{p} X_1) X_2 - X_2 X_1 (\overline{p} L X_1 X_2 - \overline{X}_1) L X_3 - \frac{1}{L} L X_3 - \overline{X}_1) L X_3)
\]
\[
= \overline{q} X_1 (\overline{q} X_1 - X_1 (\overline{p} X_2 + \overline{q} X_2) X_2 - X_2 \overline{X}_2 X_2 - \frac{1}{L} X_3 - \overline{X}_1) L X_3)
\]
\[
+ \left( \frac{q}{\sqrt{L}} X_1 \left( \frac{1}{L} \right) + \frac{\overline{p}}{\sqrt{L}} X_2 \left( \frac{1}{L} \right) + \overline{q} X_2 \left[ 1 + \frac{1}{L} (\overline{p}^2 - \overline{q}^2) \right] \right) X_3,
\]
(82)
Next, we compute the inner product of this with $v_L$. Using the product rule and the identity $\overline{q}_L \overline{P} = \overline{P}_L \overline{q}$, we obtain

$$
\langle \nabla e_2, v_L \rangle_L = \left( \overline{P}_L \overline{q}_L \overline{P} + \overline{q}_L \overline{P} \right) \overline{X}_r \overline{r}_L - \left( \overline{P}_L \overline{q}_L \overline{P} + \overline{q}_L \overline{P} \right) \overline{X}_r \overline{r}_L + \overline{P}_L \overline{r}_L \overline{q}_L \overline{X}_r \overline{P} + \overline{P}_L \overline{r}_L \overline{q}_L \overline{X}_r \overline{q}_L \overline{X}_r \overline{q}_L
$$

To simplify this, we find $\overline{P}_L \overline{q}_L \overline{P} + \overline{q}_L \overline{P}^2 = (\overline{q}_L \overline{P}^2 + \overline{q}_L \overline{P}^2) = \overline{q}_L (\overline{P}^2 + \overline{q}^2) = \overline{q}_L \overline{P}_L \overline{P} + \overline{q}_L \overline{P}_L \overline{P} = \overline{P}_L (\overline{P}^2 + \overline{q}^2)

Finally, we use the identity $\overline{X}_r \overline{r}_L - \overline{P}_L \overline{X}_r \overline{r}_L = 0$. Under these simplifications, we get

$$
\langle \nabla e_2, v_L \rangle_L = \overline{P}_L \overline{q}_L \overline{X}_r \overline{P} + \overline{P}_L \overline{r}_L \overline{q}_L \overline{X}_r \overline{q}_L \overline{X}_r \overline{q}_L = \overline{P}_L \overline{r}_L \overline{q}_L \overline{X}_r \overline{P} + \overline{P}_L \overline{r}_L \overline{q}_L \overline{X}_r \overline{q}_L
$$

Finally, we use the identity $(11L) - (11L)c_H \overline{r}_L = t_{12} c_H \overline{r}_L$ in the above equation:

$$
\langle \nabla e_2, v_L \rangle_L = \frac{1}{L} \langle e_1, c_H \overline{r}_L \rangle_L + \frac{1}{2 \sqrt{L}} (\overline{P}_L \overline{q}_L \overline{P} + \overline{q}_L \overline{P}^2) + \frac{1}{2 \sqrt{L}} (\overline{P}_L \overline{q}_L \overline{P} + \overline{q}_L \overline{P}^2)
$$

Therefore,

$$
\nabla_L e_2 = \nabla_L (\overline{P}_L \overline{r}_L \overline{q}_L \overline{X}_r \overline{r}_L - (11L) c_H \overline{r}_L) \left( \overline{P}_L \overline{r}_L \overline{q}_L \overline{X}_r \overline{r}_L - \frac{1}{L} \overline{P}_L \overline{q}_L \overline{P} \right) \overline{X}_r \overline{r}_L
$$

Taking the inner product with $v_L$ yields

$$
\langle \nabla_L e_2, v_L \rangle_L = \frac{1}{L} \langle e_1, c_H \overline{r}_L \rangle_L + \frac{1}{2 \sqrt{L}} (\overline{P}_L \overline{q}_L \overline{P} + \overline{q}_L \overline{P}^2)
$$
\[ \langle \nabla^L_{e_1} e_2, v \rangle_L = \frac{1}{L} \left( X_1(\bar{p}) + X_2(\bar{q}) - \frac{\bar{p} \bar{q}}{L} - \frac{l}{L} \langle e_2, \nabla^L_H \nabla^L_H \rangle_L \right) \]

Under some similar simplifications to Theorem 4.3 in [8], we get

\[ h_{22} = \langle \nabla^L_{e_1} e_2, v \rangle_L = \frac{l^2}{L} \langle e_2, \nabla^L_H \nabla^L_H \rangle_L + X_3(\bar{r}_L) + \frac{1}{\sqrt{L}} \frac{l}{L} \langle e_2, \nabla^L_H \nabla^L_H \rangle_L. \]  

The Riemannian mean curvature \( H^L \) of \( \Sigma \) is defined by

\[ H^L := \text{tr}(11^L) = \frac{1}{L} \left( X_1(\bar{p}) + X_2(\bar{q}) - \frac{\bar{p} \bar{q}}{L} - \frac{l^2}{L} \langle e_2, \nabla^L_H \nabla^L_H \rangle_L \right) \]

Similar to Proposition 3.8 in [15], away from the characteristic point, the horizontal mean curvature \( H^L_\infty \) of \( \Sigma \in (\mathcal{T}, g_L) \) is given by

\[ H^L_\infty = \lim_{L \to \infty} H^L = X_1(\bar{p}) + X_2(\bar{q}). \]  

Let

\[ R^{\Sigma,L}(e_1, e_2) = \langle - R^{\Sigma,L}(e_1, e_2) e_1, e_2 \rangle_{\Sigma,L}, \]

\[ R^L(e_1, e_2) = \langle - R^L(e_1, e_2) e_1, e_2 \rangle_L. \]  

By the Gauss equation, we have

\[ R^{\Sigma,L}(e_1, e_2) = R^L(e_1, e_2) + \text{det}(11^L). \]  

**Proof.** We compute

**Proposition 7.** Away from characteristic points, we have

\[ R^{\Sigma,\infty}(e_1, e_2) = -\langle e_1, \nabla_H \left( \frac{X_3 u}{|H|} \right) \rangle - \frac{(X_3 u)^2}{L^2} \text{ as } L \to +\infty. \]  

\[ \boxed{\text{Proposition 7. Away from characteristic points, we have}} \]

\[ R^{\Sigma,\infty}(e_1, e_2) = -\langle e_1, \nabla_H \left( \frac{X_3 u}{|H|} \right) \rangle - \frac{(X_3 u)^2}{L^2} \text{ as } L \to +\infty. \]
\[ R^l(e_1, e_2) = R^l(\bar{\varphi} X_1 - \bar{\varphi} X_2, \bar{\varphi} L \bar{\varphi} X_1 + \bar{\varphi} q X_2 - \frac{L}{l_2} \sqrt{l_2} X_3)(\bar{\varphi} X_1 - \bar{\varphi} X_2) \]

\[ = \frac{lq}{l_2} R^l(X_1, X_1) X_1 + \frac{iq}{l_2} R^l(X_1, X_2) X_2 \]

\[ - \frac{lq}{l_2} R^l(X_2, X_1) X_1 - \frac{iq}{l_2} R^l(X_2, X_2) X_1 \]

\[ - \frac{lq}{l_2} R^l(X_1, X_1) X_2 - \frac{iq}{l_2} R^l(X_1, X_2) X_2 + \frac{iq}{l_2} R^l(X_2, X_3) X_2 \]

\[ + \frac{iq}{l_2} R^l(X_2, X_1) X_2 + \frac{iq}{l_2} R^l(X_3, X_3) X_2 \]

\[ = \frac{iq}{l_2} R^l(X_1, X_2) X_1 - \frac{lq}{l_2} R^l(X_1, X_2) X_2 + \frac{iq}{l_2} R^l(X_2, X_3) X_2 \]

\[ - \frac{iq}{l_2} R^l(X_2, X_1) X_2 - \frac{iq}{l_2} R^l(X_3, X_3) X_2 \]

\[ = - \frac{3L^2 + 2L + 1}{4L} \frac{lq}{l_2} R^l(X_1, X_2) X_2 + \frac{3L^2 - 2L - 1}{4L} X_2 + \left[ \frac{\bar{\varphi}^2}{l_2} \frac{L - 2L - 1}{4L} + \frac{iq}{l_2} R^l(X_2, X_3) X_2 \right] X_3, \]

\[ \mathcal{X}^l(e_1, e_2) = - (R^l(e_1, e_2) e_1, e_2) = - \left( \frac{3}{4} \frac{L + 1}{4} \right) R^l \frac{lq}{l_2} - \frac\bar{\psi}\frac{lq}{l_2} \left( \frac{L - 1}{4L} \right) \]

\[ - \frac{\bar{\psi}^2}{l_2} \left( \frac{L}{4} + \frac{3}{4} \right) \frac{lq}{l_2} - \frac{\bar{\psi}^2}{l_2} \left( \frac{L}{4} + \frac{3}{4} \right) \frac{lq}{l_2} \frac{L - 1}{4L}. \]

To simplify this, we find \( -(3/4)LR_1^2 \bar{\varphi}^2 - (3/4)LR_1^2 \bar{\varphi}^2 = -(3/4)LR_1^2 \bar{\varphi}^2 \sim -(3/4)(X_3u)^2 \) as \( L \to \infty \).

\[ \bar{\psi}^2 (l/l_1)^2 \left( L/4 + \bar{\varphi}^2 (l/l_1)^2 \right) L/4 = (l/l_1)^2 \left( L/4 \right) \]

\[ \bar{\psi}^2 \left( l/l_1 \right)^2 \left( 1/2 \right) - \bar{\psi}^2 \left( l/l_1 \right)^2 \left( 1/2 \right) \]

\[ \sim \left( (l/l_1)^2 \right), \left( l/l_1 \right)^2 \sim 1 \text{ as } L \to \infty. \]

Finally, we get

\[ \text{det}(IL^l) = L_{11}^2 - h_{12}^2 = \frac{L}{4} \left( e_1, \nabla_H \left( \frac{X_3u}{\nabla_H u} \right) \right) + \frac{1}{2} \left( \bar{\psi}^2 - \bar{\psi}^2 \right) + O(L^{-2}) \]

\[ \text{ds}_L = ||\gamma||_L \, dt. \]

\[ \text{Lemma 1. Let } \gamma: [a, b] \to (\mathcal{S}, g_L) \text{ be a Euclidean } C^2 \text{-smooth and regular curve. Let} \]

\[ \text{ds} = |\omega(\gamma(t))| \, dt, \]

\[ \text{d}s = \frac{1}{2} \left( \frac{1}{|\omega(\gamma(t))|} \right) \left[ (\dot{\gamma}_1(t) \cos \theta + \dot{\gamma}_2(t) \sin \theta)^2 + (\dot{\theta}(t))^2 \right] \, dt. \]

Then,
\[
\lim_{L \to \infty} \frac{1}{\sqrt{L}}\int_Y ds_L = \int_a^b \, ds.
\]  \quad (100)

\[
\frac{1}{\sqrt{L}}\int_Y ds_L = \frac{1}{\sqrt{L}} \int \sqrt{\left(\dot{\gamma}_1(t)\cos \theta + \dot{\gamma}_2(t)\sin \theta\right)^2 + \left(\dot{\theta}(t)\right)^2} \, dt.
\]  \quad (102)

When \(\omega(\dot{y}(t)) \neq 0\), we have
\[
\frac{1}{\sqrt{L}}\int_Y ds_L = \frac{1}{\sqrt{L}} \left[ \left(\dot{\gamma}_1(t)\cos \theta + \dot{\gamma}_2(t)\sin \theta\right)^2 + \left(\dot{\theta}(t)\right)^2 + L(\omega(\dot{y}(t))^2) \right] \, dt.
\]  \quad (101)

\[
\|\dot{y}(t)\|_L = \sqrt{\left(\dot{\gamma}_1(t)\cos \theta + \dot{\gamma}_2(t)\sin \theta\right)^2 + \left(\dot{\theta}(t)\right)^2 + L(\omega(\dot{y}(t))^2)},
\]  \quad (103)

similar to the proof of Lemma 6.1 in [5], we can prove
\[
\lim_{L \to \infty} \frac{1}{\sqrt{L}} \int_Y \|\dot{y}(t)\|_L \, dt = \lim_{L \to \infty} \frac{1}{\sqrt{L}} \int_a^b \|\dot{y}(t)\|_L \, dt
\]
\[
= \left[ \int_a^b \lim_{L \to \infty} \frac{1}{\sqrt{L}} \sqrt{\left(\dot{\gamma}_1(t)\cos \theta + \dot{\gamma}_2(t)\sin \theta\right)^2 + \left(\dot{\theta}(t)\right)^2 + L(\omega(\dot{y}(t))^2) \right] \, dt
\]
\[
= \int_a^b |\omega(\dot{y}(t))| \, dt = \int_a^b ds.
\]  \quad (104)

When \(\omega(\dot{y}(t)) = 0\), we have
\[
\frac{1}{\sqrt{L}}\int_Y ds_L = \sqrt{L^{-1} \left( (\dot{\gamma}_1(t)\cos \theta + \dot{\gamma}_2(t)\sin \theta)^2 + (\dot{\theta}(t))^2 \right) + \omega(\dot{y}(t))^2} \, dt.
\]  \quad (105)

Using the Taylor expansion, we can prove
\[
\frac{1}{\sqrt{L}}\int_Y ds_L = ds + d\xi L^{-1} + O(L^{-2}) \quad \text{as} \quad L \to +\infty.
\]  \quad (106)

From the definition of \(ds_L\) and \(\omega(\dot{y}(t)) = 0\), we get
\[
\frac{1}{\sqrt{L}}\int_Y ds_L = \frac{1}{\sqrt{L}} \sqrt{\left(\dot{\gamma}_1(t)\cos \theta + \dot{\gamma}_2(t)\sin \theta\right)^2 + (\dot{\theta}(t))^2} \, dt.
\]  \quad (107)

Then,
\[
\frac{1}{\sqrt{L}}\int_Y ds_L = \frac{1}{\sqrt{L}} \sqrt{\left(\dot{\gamma}_1(t)\cos \theta + \dot{\gamma}_2(t)\sin \theta\right)^2 + (\dot{\theta}(t))^2} \, dt.
\]  \quad (108)

\[
\frac{1}{\sqrt{L}}\int_Y ds_L = d\sigma_L + d\sigma_{L^{-1}} + O(L^{-2}), \quad \text{as} \quad L \to +\infty.
\]  \quad (109)

Proposition 8. Let \(\Sigma \subset (\mathcal{R}, g_L)\) be a Euclidean \(C^2\)-smooth surface, \(\Sigma = \{u = 0\}\), and \(d\sigma_{L^{-1}}\) denote the surface measure on \(\Sigma\) with respect to the Riemannian metric \(g_L\). Let
\[
\lim_{L \to \infty} \frac{1}{\sqrt{L}}\int_{\Sigma} ds_L = \int_{\Sigma} \left\{ \left[ \left( (f_1)_u (f_3)_u - (f_1)_u (f_3)_u \right) \sin \theta + \left( (f_2)_u (f_3)_u - (f_2)_u (f_3)_u \right) \cos \theta \right]^2 + \left( (f_1)_u (f_2)_u - (f_2)_u (f_1)_u \right)^2 \right\}^{1/2} \, du_1 du_2.
\]  \quad (110)
Proof. It is well known that
\[
\begin{align*}
g_L(X_{1},\cdot) &= \omega_1, \\
g_L(X_{2},\cdot) &= \omega_2, \\
g_L(X_{3},\cdot) &= \omega. 
\end{align*}
\tag{111}
\]
We define \(e_1^* = g_L(e_1,\cdot)\) and \(e_2^* = g_L(e_2,\cdot)\); then,
\[
\begin{align*}
e_1^* &= g_L(e_1,\cdot), \\
e_2^* &= g_L(e_2,\cdot).
\end{align*}
\tag{112}
\]
Therefore,
\[
\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = \frac{1}{\sqrt{L}} e_1^* \wedge e_2^* = \frac{1}{L} (\omega_2 - \omega_1) \wedge \omega + \frac{1}{\sqrt{L}} \omega_1 \wedge \omega_2.
\tag{113}
\]
Recalling
\[
\tilde{r}_L = \frac{(X_3 u) L^{-\frac{1}{2}}}{\sqrt{p^2 + q^2 + L^{-1} (X_3 u)^2}}
\tag{114}
\]
and the Taylor expansion
\[
\frac{1}{L} - \frac{1}{2L} (X_3 u)^2 L^{-1} + O(L^{-2}) \quad \text{as} \quad L \longrightarrow + \infty,
\tag{115}
\]
we get (109). By (5), we have
\[
f_{u_1} = (f_1)_{u_1} \partial x_1 + (f_2)_{u_1} \partial x_2 + (f_3)_{u_1} \partial \theta \\
= (f_1)_{u_1} (\cos X_1 + \sin X_2) + (f_2)_{u_1} (\cos X_1 - \sin X_2) + (f_3)_{u_1} X_2
\tag{116}
\]
\[
f_{u_2} = (f_1)_{u_2} \partial x_1 + (f_2)_{u_2} \partial x_2 + (f_3)_{u_2} \partial \theta \\
= (f_1)_{u_2} (\cos X_1 + \sin X_2) + (f_2)_{u_2} (\cos X_1 - \sin X_2) + (f_3)_{u_2} X_2
\]
Let
\[
\begin{align*}
\tau_L &= \begin{vmatrix}
X_1 & X_2 & X_3 \\
(f_1)_{u_1} \cos \theta + (f_2)_{u_1} \sin \theta & (f_3)_{u_1} \cos \theta & \sqrt{L} (f_1)_{u_1} \sin \theta - (f_2)_{u_1} \cos \theta \\
(f_1)_{u_2} \cos \theta + (f_2)_{u_2} \sin \theta & (f_3)_{u_2} \cos \theta & \sqrt{L} (f_1)_{u_2} \sin \theta - (f_2)_{u_2} \cos \theta, \\
\end{vmatrix}
\end{align*}
\tag{117}
\]
We know that \(d\sigma_{\Sigma,L} = \sqrt{\det(g_{ij})} du_1 du_2\), \(g_{ij} = g_L(f_{u_i}, f_{u_j})\), and
\[
\det(g_{ij}) = \|\tau_L\|^2 = L \left[ (f_1)_{u_1} (f_3)_{u_1} - (f_1)_{u_2} (f_3)_{u_2} \right]^2 \sin \theta + \left[ (f_1)_{u_1} (f_3)_{u_1} - (f_2)_{u_1} (f_3)_{u_1} \right] \cos \theta \tag{118}
\]
so by the dominated convergence theorem, we get
Theorem 2. Let $\Sigma \subset (\mathcal{R}, g_L)$ be a regular surface with finitely many boundary components $(\partial \Sigma)_i$, $i \in \{1, \ldots, n\}$, given by Euclidean $C^2$-smooth regular and closed curves $\gamma_i: [0, 2\pi] \rightarrow (\partial \Sigma)_i$. Suppose that the characteristic set $C(\Sigma)$ satisfies $\mathcal{H}^1(C(\Sigma)) = 0$ where $\mathcal{H}^1(C(\Sigma))$ denotes the Euclidean 1-dimensional Hausdorff measure of $C(\Sigma)$ and that $\|\nabla_H u\|_{L^1}^2$ is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure near the characteristic set $C(\Sigma)$; then,

$$\int_{\Sigma} \mathcal{H}^{	ext{L}}_{\Sigma, L} \, d\Sigma + \sum_{i=1}^{n} \int_{\gamma_i} \mathcal{H}^{	ext{co}, z}_{\gamma_i, L} \, ds = 0. \quad (120)$$

Proof. Using the similar discussions in [13, 14], we assume that all points satisfy $\omega(\gamma_i(t)) \neq 0$ on the curve $\gamma_i$. Recalling the result in Proposition 4 indicates

$$\kappa^L_{\gamma_i} = \kappa^\text{co}, z_{\gamma_i, L} + O(L^{-1/2}). \quad (121)$$

According to the Gauss–Bonnet theorem, we get

$$\int_{\Sigma} \mathcal{H}^{	ext{L}}_{\Sigma, L} \frac{1}{\sqrt{L}} \, d\Sigma + \sum_{i=1}^{n} \int_{\gamma_i} \kappa^L_{\gamma_i} \frac{1}{\sqrt{L}} \, ds = 2\pi \chi(\Sigma) \quad (122)$$

Let $L$ go to the infinity, and using the dominated convergence theorem, (121), (122), (109), Proposition 6, and Lemma 1, we get the desired result. \qed

7. Conclusion

This paper dealt with an interesting question of the Gauss–Bonnet theorem in the rotation group from the Riemannian approximation scheme. The main result of this paper is Theorem 2, which is the Gauss–Bonnet-type theorem in the rotation group. To prove Theorem 2, we obtained the sub-Riemannian limit of the curvature of curves, sub-Riemannian limits of the geodesic curvature of curves on surfaces, and the Riemannian Gaussian curvature of surfaces in the rotation group.

As a future work, we plan to proceed to study Gauss–Bonnet theorems in the rotation group with the general left-invariant metric and other three-dimensional Riemannian Lie groups which were classified in [18]. In these conditions, Gauss–Bonnet theorems can be obtained through the Riemannian approximation scheme took by Balogh et al. [13, 14]. The Gauss–Bonnet theorem connects the intrinsic differential geometry of a surface with its topology and has many applications. Therefore, it will be interesting to extend the Gauss–Bonnet theorem to other different Lie groups. We believe that the results to be obtained will have some geometric applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest in this work.

Authors’ Contributions

All the authors contributed equally to the writing of this paper and read and approved the final manuscript.

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