The classical \( n \)-queens problem asks in how many different ways \( n \) mutually nonattacking queens can be placed on an \( n \times n \) chessboard. In [2], Christopher Hanusa and Arvind Manhankali studied a generalization of this problem, and they introduced a dynamical system that they describe as “reminiscent of billiards” to aid their study. We will call this dynamical system the chess billiard, and in this article we will study it from a purely dynamical point of view.

The chess billiard is defined as follows. Consider a convex planar domain \( P \) and fix a direction \( \theta_1 \). Foliate the plane by lines in this direction. Each line intersects the boundary \( \partial P \) in either no point, one point, two points, or a segment. When the intersection is nonempty, this defines an involution \( T_1 \) of \( \partial P \) by defining respectively the map \( T_1 \) to be the identity, to exchange the two points, or to be the central symmetry of the segment about its center. We consider a second involution \( T_2 \) corresponding to a direction \( \theta_2 \), and then the chess billiard map is the composition of the two involutions: \( T_2 \circ T_1 \) (see Figure 1). The chess billiard map turns out to be a circle homeomorphism.

The chess billiard map was independently introduced in various other articles (without reference to chess). Fritz John used it to study the Dirichlet problem for hyperbolic equations [3]. Vladimir Arnold mentioned this map as his motivation for the study of KAM theory [1]. It was discussed in [9] in the context of the Sobolev equation, approximately describing fluid oscillations in a rapidly rotating tank. It also arose as certain special cases of the pseudo-Riemannian billiards studied by Boris Khesin and Serge Tabachnikov [5]. It was also studied in a special case by Dmitri Khmelev [6].

Our article has two aims. We first collect various well-known results on circle homeomorphisms and apply them to the chess billiard map to deduce certain interesting results, in particular answering some questions posted in [2]. The second goal is to prove new results about the chess billiard. We prove some general results on periodic points; then we turn to the study of the chess billiard in a polygon. In particular, we prove results about the chess billiard map in triangles, in the square, and in other centrally symmetric domains. Our results on the square are undoubtedly the most interesting of the article.

The ultimate goal of the study of chess billiards is to decide for which convex domains and directions the rotation number is rational.

Structure and Main Results
We first define the rotation number of a circle homeomorphism and summarize some of its properties. We then give the formal definition of the chess billiard map \( S \) and show that it is a circle homeomorphism and apply the theory of circle homeomorphisms, due to Poincaré and Denjoy, to them.

In the next section, we give a necessary and sufficient condition for \( S \) to have a fixed point. This allows us to understand fixed points in strictly convex domains and to show that the chess billiard map is purely periodic in every triangle. In this section, using monotonicity and continuity of the rotation number, we show that in a strictly convex domain, the rotation number of the chess billiard map achieves all values in \([0, 1)\). We also show that there exist directions for which the chess billiard map in the square has no periodic orbit.

We next analyze the periodic directions of the chess billiard map in the square, giving a partial answer to questions 7.5 and 7.6 of [2]. We show that for an open dense set of directions \((\theta_1, \theta_2)\), the chess billiard has a periodic orbit. However, the set of directions that have a neutral cylinder is small. We also give a necessary and sufficient condition for a corner to be a periodic point, and we give a sufficient condition for the existence of neutral cylinders.

Finally, in the short final section, we study centrally symmetric domains. Our results give complete or partial answers to several questions raised in [2].

The Rotation Number
The main tool of our article is the rotation number of an orientation-preserving circle homeomorphism \( f : \mathbb{S}^1 \to \mathbb{S}^1 \), which was introduced by Poincaré to study the precession of the perihelion of the planetary orbits. In this section, we give its definition and main properties.

Suppose that a circle has unit circumference, and consider the natural projection \( \pi : \mathbb{R} \to \mathbb{S}^1 \). A lift of \( f \) is a map \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) satisfying \( \pi \circ \tilde{f} = \pi \circ f \) and \( \tilde{f}(x + 1) = \tilde{f}(x) + 1 \). The limit
The rotation number takes its value in the circle $[0, 1)$. Furthermore, whenever $\rho(f) = \frac{p}{q} \in \mathbb{Q}$, the fraction $\frac{p}{q}$ is in reduced form; i.e., $p$ and $q$ are positive and relatively prime.

2. If $f$ is a homeomorphism, then its rotation number satisfies $\rho(f^{-1}) = 1 - \rho(f)$.

3. If $f$, $g$, $F$ are orientation-preserving circle homeomorphisms such that $F \circ f = g \circ F$, then $\rho(f) = \rho(g)$.

The Chess Billiard Is a Circle Homeomorphism in Disguise

We begin by describing the chess billiard slightly more formally. Consider a strictly convex planar domain $P$ and a foliation of $P$ by a family of unoriented parallel lines (see Figure 2). The choice of families of lines is parameterized by $\theta \in [0, \pi)$, i.e., the projective line. In some of our arguments it will be convenient to parameterize $\theta$ by $[0, 2\pi)$, and when we do so, it will be clear from the context. Since $P$ is convex, we can parameterize its boundary $Q := \partial P$ by normalized arc length with respect to a fixed orientation. Consider a direction $(\theta_1, \theta_2)$ on the torus $[0, \pi)^2$. Throughout, we will refer to either $(\theta_1, \theta_2)$ or one of its components as a direction; which one will be clear from the context. Since $P$ is strictly convex, the foliation in direction $\theta_i$ induces a bijection $T_i : \partial P \to \partial P$ ($i = 1, 2$), which is an orientation-reversing homeomorphism with two fixed points.

We define the chess billiard map $S = S_{\theta_1, \theta_2} : Q \to Q$ by $S := T_2 \circ T_1$. Since $S$ is the composition of two orientation-reversing homeomorphisms, we immediately conclude that $S$ is an orientation-preserving circle homeomorphism. A degenerate case occurs when these two families coincide.

If $P$ is convex but not strictly convex, then the exact same definition works in the case that the direction of each interval in $\partial P$ is transverse to the two foliations, for example if $P$ is a convex polygon and neither of the two foliations is parallel to a side of $P$. We call the direction $(\theta_1, \theta_2)$ exceptional if either $\theta_1$ or $\theta_2$ is parallel to a side of $P$.

We begin by describing the chess billiard slightly more formally. Consider a direction $T_i$ on every line segment in $Q$ to be the central symmetry with respect to the center of that segment (see Figure 2). Again, the map $T$ is an orientation-reversing homeomorphism.

Our chess billiard map is defined on all of $Q$. The map from [2] is defined only on the subset of $Q$ consisting of points of $Q$ that are not corners of $P$. The two definitions agree on this smaller set. Furthermore, in [2], the authors consider only points in $Q$ for which the direction $\theta_i$ points toward the interior of $P$, while we allow foliations tangent to an interval in $Q$.

If $P$ is not convex, then for certain directions, we have orbits that graze the boundary, and any possible definition of the dynamical system will lead to a discontinuous map (see Figure 4). Nonetheless, we can define a piecewise continuous chess map, for example by defining the map by one-sided continuity at these points (answering part of Question 7.8 of [2]). Such a definition departs from our nice framework, and thus in this article we will not consider such domains, although their study is certainly interesting. In the case of a nonconvex polygon, the resulting map is an affine interval exchange transformation; these have been studied since Gilbert Levitt’s article in the 1980s [8].

The study of the dynamics of orientation-preserving circle homeomorphisms was initiated by Poincaré and developed by Arnaud Denjoy, Michael Herman, and others. To summarize the application of their results to our setting, we need some definitions from dynamical systems theory.

Suppose that $f : X \to X$ is a continuous map of a compact metric space $X$. A point $x \in X$ is called periodic if there is an $n \in \mathbb{N}$ such that $f^n(x) = x$, and the least such $n$ is called the period of $x$. If $n = 1$, we refer to $x$ as a fixed point. A map $f$ is called minimal if every orbit $\{f^n(x) : n \geq 0\}$ is dense in $X$. The map $f$ is called uniquely ergodic if for each continuous map $\phi : X \to \mathbb{R}$, there exists a constant $c(\phi)$ such that

$$\frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) \to c(\phi)$$

uniformly for $x \in X$ as $n \to \infty$. The $\omega$-limit set of $x \in X$ is defined as

![Figure 2](image1.png)

**Figure 2.** The circle foliated by a family of parallel lines and the induced bijection.

![Figure 3](image2.png)

**Figure 3.** The square foliated by a family of parallel lines that are parallel to a side, and the induced bijection.
Assume that the two foliations do not coincide, i.e., if \( f \) is a homeomorphism and \( x_-, x_+ \in X \) are fixed points of \( f \), then \( y \in X \) is said to be heteroclinic to \( x_- \) and \( x_+ \) if \( f^{-n}(y) \to x_- \) and \( f^{n}(y) \to x_+ \) as \( n \to \infty \).

Now we can summarize the main results of the above authors applied to our setting. If the rotation number of \( f \) is a supporting line to \( x \) at \( x \), then \( x \) is a fixed point. As we saw above.

If the rotation number of \( S \) is irrational, then:

1. The \( \omega \)-limit set \( \omega(x) \) is independent of \( x \in Q \), and either \( \omega(x) = Q \) or \( \omega(x) \) is a perfect and nowhere dense subset of \( Q \).
2. The map \( S \) is uniquely ergodic.

These results immediately give answers to Questions 7.1, 7.2, 7.3, 7.4, and 7.13 posed in [2].

### Periodic Orbits

#### Fixed Points of \( S \)

If the two foliations coincide, then \( S \) is the identity map. Assume that the two foliations do not coincide, i.e., \( \theta_1 \neq \theta_2 \).

If \( P \) is strictly convex, then the map \( T_i \) fixes the point \( x \) if and only if a line of the foliation is a supporting line\(^1\) to \( Q \) at \( x \). If \( P \) is convex but not strictly convex, then the necessary and sufficient condition is more complicated. The point \( x \) is a fixed point if and only if either (i) a line of the foliation is a supporting line to \( Q \) at \( x \), and \( x \) is isolated in this intersection, or (ii) a line of the foliation is tangent to \( Q \) in an interval, and \( x \) is the midpoint of this interval.

If a point \( x \in \partial P \) is fixed by both \( T_1 \) and \( T_2 \), then it is a fixed point of \( S \), and thus the rotation number of \( S \) is zero. On the other hand, if \( x \) is not fixed by at least one of the \( T_i \), then \( S(x) \neq x \), since by assumption, the two foliations are not parallel.

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\(^1\)For the definitions and proof of this result, see, for example, [4, Propositions 11.2.2 and 11.2.5, Theorem 11.2.9].

\(^2\)A supporting line \( L \) of a planar curve \( C \) is a line that contains at least one point of \( C \), and \( C \) lies completely in one of the two closed half-planes defined by \( L \).

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We have thus shown that if \( \theta_1 \neq \theta_2 \), then the map \( S_{\theta_1, \theta_2} \) has a fixed point if and only if there is a point \( x \in \partial P \) such that the lines through \( x \) in these two directions are supporting lines at \( x \). Moreover, each fixed point is semistable, i.e., repelling from one side and attracting from the opposite side.

Since support lines are tangent lines when the domain is smooth, the uniqueness of tangent lines implies that a \( C^1 \) strictly convex domain cannot have any fixed point (unless \( \theta_1 = \theta_2 \)).

Now we will prove another nice application of the above result. If \( P \) is a triangle, then the map \( S_{\theta_1, \theta_2} \) has a periodic point for each direction \( (\theta_1, \theta_2) \). If \( (\theta_1, \theta_2) \) is not exceptional, then this is a fixed point, while in the exceptional case, the period of this point is either 1, 2, or 3.

When \( \theta_1 = \theta_2 \), the map \( S \) is the identity map, so to prove this result, we need to consider the case in which they are not equal. First consider the case that \( (\theta_1, \theta_2) \) is not exceptional. Consider the lines of the foliation in the direction \( \theta_1 \) that intersect \( P \); the extremal ones are supporting lines that pass through two distinct vertices of \( P \). The same holds for \( \theta_2 \), and since \( P \) has only three vertices, there is a corner for which both directions must have supporting lines, and thus a fixed point, as we saw above.

Turning to the case that \( (\theta_1, \theta_2) \) is exceptional, we begin by treating the case in which only one direction is parallel to a side, say \( \theta_1 \). It is also a supporting line of the corner opposite this side. If \( \theta_2 \) is a supporting line of this corner, then again, we conclude that this corner is fixed (see Figure 6). Otherwise, \( \theta_2 \) is a supporting line of the two endpoints of the side parallel to \( \theta_1 \); these endpoints are fixed by \( T_2 \) and exchanged by \( T_1 \), and thus they are exchanged by \( S \).

Finally, when both directions are parallel to different sides, the map \( S \) cyclically exchanges the vertices of the triangle, and thus has period 3 (Figure 5). This finishes the proof.

If \( P \) is a square and both foliation directions are in the same quadrant, then \( S \) has two fixed points, both semistable (this is essentially contained in [2]; however, the authors did not define the dynamics at the two fixed points). More generally, any convex polygon or even any convex table with a corner has an open set of pairs of directions for which \( S \) has a fixed point. Examples of strictly convex domains and of convex polygons with exactly one fixed point exist (see Figure 6). This point is repelling on one side and attracting on the other.

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Figure 4. Any possible definition of the dynamics at the marked points leads to a discontinuous map.

\[ \omega(x) := \bigcap_{n \in \mathbb{N}} \{ f^k(x) : k > n \} \]

Figure 5. A period-3 orbit in a triangle.
Connections
Let \( F := \{ x : T_1(x) = x \text{ or } T_2(x) = x \} \). If \( x \in F \) and there is an \( n \geq 1 \) such that \( S^n(x) \in F \), then we call the orbit segment \( \{ x, S(x), \ldots, S^n(x) \} \) a connection. When an \( S \)-orbit passes through a point of \( F \), it reverses direction and retraces the orbit in the opposite direction. If this phenomenon happens twice, then the orbit must be periodic, bouncing back and forth between two points of \( F \). Thus we have shown that if there is a connection, then the map \( S \) has a periodic point. This result generalizes one direction of the characterization of fixed points from the previous subsection.

The Rotation Number Achieves All Values
The map \( (\theta_1, x) \mapsto T_1(x) \) is a monotone continuous function of \( \theta_1 \) as long as the direction \( \theta_1 \) is not parallel to a segment in \( \partial P \), and thus the chess billiard map \( S_{\theta_1, \theta_2} \) is a continuous function of \( \theta_1 \). Since the rotation number depends continuously on the map, we conclude that for each fixed \( \theta_2 \), if the direction \( \theta_1 \) is not parallel to a segment in \( \partial P \), then:

1. the point \( \theta_1 \) is a point of continuity of the rotation number map;
2. the map \( \theta_1 \mapsto \rho(S_{\theta_1, \theta_2}) \) is a monotone function of \( \theta_1 \).

These results also hold with \( \theta_1 \) and \( \theta_2 \) interchanged.

Next, we will show that for a strictly convex table and a fixed direction \( \theta_1 \), the rotation number of \( S \) achieves all values in the circle \( [0, 1) \) as we vary \( \theta_1 \in [0, \pi) \).

To see this, we think of \( \theta_1 \) and \( \theta_2 \) as oriented directions. For \( \theta_2 = \theta_1 \) and \( \theta_2 = \theta_1 + \pi \), we have \( S = \text{id} \). In both cases, the rotation number is 0. As we saw in the fixed-point subsection, for fixed \( \theta_1 \), the rotation number is nonzero for all \( \theta_2 \in (\theta_1, \theta_1 + \pi) \). Furthermore, the rotation number is monotonic and continuous in \( \theta_2 \). Recall that the rotation number takes values in the circle \( [0, 1) \). Combining these facts, we conclude that the rotation number varies from 0 to 1 as \( \theta_2 \) varies from \( \theta_1 \) to \( \theta_1 + \pi \), in the sense that \( \lim_{\theta_2 \to \theta_1 + \pi} \rho(S_{\theta_1, \theta_2}) = 1 \). This finishes the proof.

Polygons, of course, are not strictly convex, and the above argument yields only conditional results. The sides of \( P \) divide the set of directions into open sectors \( C_i \). Arguing as above, we conclude that for each fixed pair of sectors \( C_i, C_j \) either the map \( (\theta_1, \theta_2) \mapsto \rho(S_{\theta_1, \theta_2}) \) is constant or its image is a nondegenerate interval; thus either the collection of rotation numbers of a polygon is a finite set of rational numbers or it contains an irrational number (and thus has an aperiodic orbit).

As we have noted, the first possibility occurs in every triangle. The second possibility can occur as well. There exists a direction \( (\theta_1, \theta_2) \) such that the chess billiard map \( S_{\theta_1, \theta_2} \) in the square has an irrational rotation number.

To see this, notice that if \( \theta_1^{(0)} = \pi/4 \) and \( \theta_2^{(0)} = 3\pi/4 \), then every \( S \)-orbit in the square has period 2. On the other hand, if \( \tan \theta_1^{(1)} = 1/3 \) and \( \tan \theta_2^{(1)} = -2/3 \), then a simple geometric exercise shows that every point has period 3 (see Figure 7). The continuity of the map \( (\theta_1, \theta_2) \mapsto \rho(S_{\theta_1, \theta_2}) \) in the sectors containing these directions implies that its image contains an interval. This finishes the proof.

The proof used explicit periodic directions. We do not know of a general existence result for \( n \)-gons with \( n \geq 4 \).

The Circle
We illustrate the previous results with the simplest case, the circle. Recalling that \( \theta_i \in [0, \pi) \), we denote the normalized parameters by \( \psi_i = \theta_i / \pi \) for \( i = 1, 2 \). We will show that:

1. the map \( S \) is the rotation of the circle through the angle \( 2\alpha \), where \( \alpha := \psi_2 - \psi_1 \) \((\mod 1)\);
2. \( \alpha \in \mathbb{Q} \) if and only if there is a connection.

Thus if \( \alpha = p/q \in \mathbb{Q} \) is in reduced form, then all orbits are periodic with period \( q \). Otherwise, the map \( S \) is minimal and uniquely ergodic with respect to the length measure.

Statement 1 above follows from the elementary property of circles summarized in Figure 8: the angle \( xOy \) is twice the angle between \( \theta_1 \) and \( \theta_2 \).

For statement 2, the “only if” part of the statement was proved in the subsection “Connections” for general domains. For the other implication, note that the fixed points of \( T_1 \) are \( \pm 1 \), and the fixed points of \( T_2 \) are \( e^{\pm 2\pi i \psi} \). By statement 1, all four points are periodic and thus belong to connections, concluding the proof.

If the rotation number is \( p/q \) in reduced form and \( q \) is even, it is easy to check that \( S^{q/2}(1) = -1 \) and \( S^{q/2}(-1) = 1 \). However, when \( q \) is odd, both 1 and \(-1 \) have period \( q \) (see Figure 9).

![Figure 6](image6.png)

![Figure 7](image7.png)

![Figure 8](image8.png)
Figure 9. Examples of period 2 and 3 are drawn in black, while the corresponding connections are drawn in blue and red.

**Khmelev’s Result**

Dmitri Khmelev [6] considered the map $S$ in a strictly convex domain that is sufficiently smooth everywhere except at one point whose first derivative has a jump discontinuity. He showed that the rotation number $\rho(S_{\theta_1, \theta_2})$ takes a rational value for almost all values of $\theta_1, \theta_2$ (see his article for the precise smoothness assumptions).

**Periodic Orbits in Polygons**

Fix $P$, and suppose that the rotation number associated to a pair of directions $\theta_1, \theta_2$ is rational and equal to $p/q$ in reduced form. Let $I$ be the maximal interval contained in a side of $P$, perhaps degenerated to a point, such that $S^q(x) = x$ for all $x \in I$. In analogy to terminology from interval exchanges and translation surfaces, the set $C(I) := \bigcup_{n=0}^{\infty} S^n I$ is called a periodic cylinder. In the case that $I$ degenerates to a point, a periodic cylinder is simply a periodic orbit of period $q$, and if it is a nondegenerate interval, we will call it a neutral cylinder. In [2], neutral cylinders play an important role and are called “treachery” (see [2, Theorem 5.7]). Notice that a cylinder is a forward and backward invariant set, i.e., $S(C(I)) = S^{-1}(C(I)) = C(I)$. By continuity, a periodic cylinder $C(I)$ is always a closed set.

We will now show that if $P$ is a convex polygon with $k$ sides, and $(\theta_1, \theta_2)$ is such that the rotation number $\rho(S)$ is equal to $p/q$, then there are at most $3k - 4$ periodic cylinders for $S$. To see this, foliate $P$ by lines in the direction $\theta_1$. There are two supporting lines in this foliation, and each supporting line passes through one or two corners of $P$; thus there are at most $k - 2$ lines that pass through other corners of $P$. Consider the intersection of these lines with $\partial P$. This intersection consists of the $k$ corners plus (at most) $k - 2$ other points, so (at most) $2k - 2$ points. These points define a partition $I_1$ of $Q$ into at most $2k - 2$ intervals on which $T_1$ is affine. This partition is preserved by the involution $T_1$; i.e., $T_1(I_1) = I_1$.

The same construction yields at most $2k - 2$ points in $Q$ for the direction $\theta_2$ and defines a partition $I_2$ into at most $2k - 2$ intervals on which $T_2$ is affine.

The partition $T_1(I_1) \setminus I_2 = I_2 \setminus I_1$ consists of maximal intervals on which $S$ is affine. Note that the $k$ vertices are common endpoints of all the intervals in these two partitions. Thus the cardinality of this set is at most $(2k - 2) + (2k - 2) - k = 3k - 4$. Therefore, the map $S^q$ has at most $q(3k - 4)$ intervals of affinity. Consider the graph of $S^q$. Each piece of affinity contains at most one periodic interval $I$, and there are $q$ such intervals in a periodic cylinder. Thus the number of periodic cylinders is at most $q(3k - 4)/q = 3k - 4$.

**Periodic Orbits in the Square**

Consider the square $[0, 1]^2$. In order to answer Question 7.6 from [2], we will show that the chess billiard map $S_{\theta_1, \theta_2}$ in the square has a periodic point for an open dense set of directions $(\theta_1, \theta_2) \in [0, \pi)^2$.

Our proof uses another cross section to the chess billiard flow, which relies on the symmetries of the square. Throughout the proof, we suppose that the directions $\theta_1$ and $\theta_2$ are not exceptional, and we suppose furthermore that they are in different quadrants, since if they are in the same quadrant, the map has a fixed point. It suffices to treat the case $\theta_1 \in (0, \pi/2)$ and $\theta_2 \in (\pi/2, \pi)$.

Let $D$ denote the diagonal $x + y = 1$ of the square. We define a map $F : D \to D$. We give two different descriptions of this map. We start at a point in $D$, flow in the direction $\theta_1$ (toward the right) until we reach the boundary of the square, then flow in the direction $\theta_2$ until we return to the boundary of the square, and finally again flow in the direction $\theta_2$ until we return to $D$. The point to which we have returned is in $D$, but we can be flowing either to the right or to the left depending on whether or not the flow in the direction $\theta_2$ had crossed the diagonal; if we are flowing to the right, call this point $F(x)$, while if we are flowing to the left, we apply a central symmetry to obtain $F(x)$.

Another way to define $F$ is via unfolding. This is shown in Figures 10 and 11. The direction in the bottom left-hand square and in every other square is $\theta_2$. The direction in the other squares is unfolded, and thus the angle is $\pi - \theta_2$. For convenience, we use the notation $\phi_1 = \theta_1, \phi_2 = \pi - \theta_2$. We remark that the points in the interval $A_1$ have crossed the diagonal $D$ during the flow in the direction $\theta_2$, and they arrive at $D$ with the same orientation, while the points in $A_2 \cup A_3$ do not cross $D$.

In the original chess billiard flow, when they return to $D$, we need to apply the central symmetry to define $F$, but this is not needed in the unfolded picture.

The graph of the map $F : D \to D$ has two possible forms, which are shown in Figure 12. The set $D$ decomposes
into three segments $A_1, A_2, A_3$ such that the derivative $F'|_{A_i}$ is constant for each $i$; we call their images $A'_i = F(A_i)$. Let $a_i := |A_i|$, where $| \cdot |$ denotes the length of a segment. In the case $\frac{\pi}{4} < \phi_2 < \frac{\pi}{2}$, the central symmetry of Figure 11 about the point $(1/2, 3/2)$ yields $|A'_1| = |A_1|$, while the central symmetry of the figure about the point $(1, 1)$ yields $|A'_1| = |A'_2|, |A'_3| = |A_3|$, and thus

$$\frac{|A'_1|}{|A_1|} = \frac{|A'_2|}{|A_2|} = \frac{|A'_3|}{|A_3|}.$$ 

Notice that these symmetries imply that the point $(a_1 + a_2, F(a_1 + a_2))$ of the graph of $F$ lies on the antidiagonal marked in dots in Figure 12, i.e.,

$$a_1 + a_2 + F(a_1 + a_2) = 1.$$ (Similar symmetries arise in the case $\phi_3 \in (0, \pi/4).$

The length of $D$ is $\sqrt{2}$. We parameterize $D$ with normalized arc length and note that $F(0) = F(1)$. Thus we can think of $D$ as a circle. Elementary plane geometry (see Figures 10 and 11) yields

$$a_1 = \begin{cases} 
1 - \tan(\phi_2) & \frac{\sin(\pi/2 - \phi_1)}{\sin(\pi/4 + \phi_1)} \quad \text{if } 0 < \phi_2 < \frac{\pi}{4}, \\
1 - \cot(\phi_2) & \frac{\sin(\phi_1)}{\sin(3\pi/4 - \phi_1)} \quad \text{if } \frac{\pi}{4} < \phi_2 < \frac{\pi}{2}.
\end{cases}$$

There are two special cases. If $\phi_2 = \pi/4$, the interval $A_1$ disappears and there are only two intervals; on the other hand, if $\phi_1 = \phi_2$, then $F$ is a circle rotation by the length $a_1$.

It is not hard to check that if we increase $\phi_2$ (i.e., decrease $\theta_2$), then the graphs of the resulting maps $F = F_{\phi_1, \phi_2}$ and $F = F_{\phi_1, \phi_2 + \delta}$ do not intersect (see Figure 13). Here $\delta$ varies in the interval $(\phi_2, \pi - \phi_2)$.

Note that this nonintersection of graphs does not hold for the graphs of the original maps $S$, which is the reason for the introduction of the cross section $D$. It enables us to apply standard techniques for families of circle homeomorphisms. Proposition 11.1.9 of [4] implies that the rotation number map $h \mapsto \rho(F_h)$ is strictly increasing at $h_0$ whenever $\rho(F_{h_0})$ is irrational, while Proposition 11.1.10 of [4] implies that if $\rho(F_{h_0})$ is rational, then there is a nondegenerate interval containing $h_0$ on which $\rho(F_h)$ is constant and thus rational.

Now consider a chess billiard orbit in the square in a nonexceptional direction passing through a corner. We will show that such an orbit is periodic if and only if it is a connection.

To see this, we first note that the “only if” assertion holds because in a polygon, a connection in a nonexceptional direction must connect vertices, and as we saw before, connections imply the existence of a periodic point.

Vertices that are fixed points are connections. Now consider the case in which the periodic orbit is not a fixed point and passes through a corner $a$; thus the directions must be in different quadrants, and so one of the directions is a supporting line at $a$. Thus $a$ acts as a reflector in the sense that after hitting this corner, the orbit retraces itself backward. The orbit going through the corner $a$ is periodic (and is not a fixed point), and thus it must make its way back to $a$. Since this is the only mechanism for retracing an orbit, the only way in which this can happen is by retracing the orbit once again: the orbit must hit a different corner, i.e., it is a connection. This finishes the proof.

Our next result is that if there is a neutral cylinder in the square chess billiard in a nonexceptional direction, then there is a periodic connection in this direction. These last two results give a partial answer to Question 7.6 from [2]. An example of this was given in Figure 7, which immediately generalizes to slopes $1/(3n)$ and $-2/(3n)$ for all $n > 1$; see Figure 14.

**Figure 10.** The map $F : D \to D$, case $\phi_2 \in (0, \pi/4).$
To show this, we need to consider several cases. If \( \theta_1 \) and \( \theta_2 \) are parallel, then all points are fixed by \( S \). Thus each side of \( P \) is a periodic cylinder, hence a neutral cylinder, and each corner of \( P \) is a connection. Suppose now that \( \theta_1 \) and \( \theta_2 \) are not parallel and \( (\theta_1, \theta_2) \) is nonexceptional. Consider the cylinder \( C(I) \) of periodic orbits of the neutral cylinder, and let \( q \) denote its period. By definition, \( C(I) = C(S(I)) = \ldots = C(S^{q-1}(I)) \), and thus we can choose \( 0 \leq j < q \) such that \( S(I) = [a, b] \), where \( a \) is a corner of the polygon. Otherwise, we could extend \( I \) to a larger interval. Since \( C(I) \) is closed, the orbit of the corner \( a \) is periodic and thus a connection. This finishes the proof.

Now we will show that for the square, the set

\[ \{(\theta_1, \theta_2) : S \text{ has a neutral cylinder}\} \]

is a union of at most countably many Jordan arcs. As we just saw, to justify this, it suffices to prove that the set

\[ \{(\theta_1, \theta_2) : S \text{ has a connection in this direction}\} \]

is a union of at most countably many Jordan arcs.

To see this, we will use the following implication of a strengthening of the implicit function theorem due to Kumagai [7]: Let \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a continuous mapping, and suppose that \( f(\theta_1^0, \theta_2^0) = 0 \). If there exist open neighborhoods \( C \subset \mathbb{R} \) and \( E \subset \mathbb{R} \) of \( \theta_1^0 \) and \( \theta_2^0 \), respectively, such that for all \( \theta_1 \in C \), \( f(\theta_1, \cdot) : E \to \mathbb{R} \) is locally one-to-one, then there exist open neighborhoods \( C_0 \subset \mathbb{R} \) and \( E_0 \subset \mathbb{R} \) of \( \theta_1^0 \) and \( \theta_2^0 \) such that for every \( \theta_1 \in C_0 \), the equation

\[ f(\theta_1, \theta_2) = 0 \]

has a unique solution \( \theta_2 = g(\theta_1) \in E_0 \), where \( g \) is a continuous function from \( C_0 \) into \( E_0 \).

Consider a neutral cylinder in the direction \((\theta_1^0, \theta_2^0)\) and one of the associated connections. Suppose that this connection starts at a corner \( a \). In the representation \( F : D \to D \), by a slight abuse of notation we will denote the associated point in \( D \) on this connection by \( a \), so that

\[ a = F^n_{\theta_1^0, \theta_2^0}(a) \]

Consider a lift \( \tilde{F} : \mathbb{R} \to \mathbb{R} \) of \( F \). Then there is an \( m \in \mathbb{Z} \) such that \( a + m = F^n_{\theta_1^0, \theta_2^0}(a) \). If we can apply Kumagai’s implicit function theorem to the function

\[ f(\theta_1, \theta_2) := F^n_{\theta_1, \theta_2}(a) - (a + m) \]

then a one-dimensional set of the form \( \theta_2 = g(\theta_1) \) is the set of solutions of the equation \( f(\theta_1, \theta_2) = 0 \) for some continuous function \( g \), i.e., a Jordan arc.

We saw earlier in this section that there is an interval \( E \) such that for each \( \theta_1 \in E \), the function \( F^n(\theta_1, \cdot) \) is a strictly monotonic map on \( E \). This immediately implies that \( F^n(\theta_1, \cdot) \)

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**Figure 11.** The map \( F : D \to D \), case \( \phi_2 \in (\pi/4, \pi/2) \).

**Figure 12.** The map \( F \) for \( \phi_2 \in (0, \pi/4) \) and \((\pi/4, \pi/2) \).
is a strictly monotonic map on $E$. Thus $f(\theta_1, \cdot)|_P$ is locally one-to-one, so we can apply Kumagai’s theorem, which completes the proof.

### Centrally Symmetric Domains

Our final result is about a centrally symmetric domain $Q$, for example a circular or square table. We will show that if $\theta_1, \theta_2$ are such that the rotation number of $S$ is irrational, then for all $x \in Q$, the $\omega$-limit set $\omega(x) \subset Q$ is centrally symmetric.

To see this, consider two points $x^-, x^+ \in \partial P$ such that the segment $x^-x^+$ passes through the center of symmetry of $P$ and is in the direction $\theta_1$. For each $n \geq 0$, the points $S^n(x^-)$ and $S^n(x^+)$ are centrally symmetric to each other, and thus the $\omega$-limit sets of these two points are centrally symmetric to each other. Thus $\omega(x^-) = \omega(x^+)$, since as we saw earlier, $\omega(x)$ does not depend on $x$. This finishes the proof.

### Acknowledgments

The project leading to this publication has received funding from Excellence Initiative of Aix-Marseille University - A*MIDEX and Excellence Laboratory Archimedes LabEx (ANR-11-LABX-0033), French “Investissements d’Avenir” programs. The first author, A. N., thanks the program CEFIPRA project No. 5801-1/2017 for their support. We thank the referees for useful suggestions that have improved our article.

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