Data Collection versus Data Estimation: A Fundamental Trade-off in Dynamic Networks with Known and Unknown Models

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Abstract—An important question that often arises in networked systems is whether to collect the real-time data or to estimate them based on the previously collected data, where various factors should be taken into account such as how informative the data are at each time instant for state estimation, how costly and credible the collected data are, and how rapidly the data vary with time. The above question can be formulated as a dynamic decision-making problem with imperfect information structure, where a decision maker wishes to find an efficient way to switch between data collection and data estimation and the quality of the estimation depends on the collected data (i.e., duality effect). In this paper, the evolution of the state of each node is modeled by an exchangeable Markov process for discrete features and an equivariant linear system for continuous features, where the data of interest are defined in the former case as the empirical distribution of the states, and in the latter case as the weighted average of the states. When the data are collected, they may or may not be credible, according to a Bernoulli distribution. Based on a novel planning space, a Bellman equation is proposed to identify a near-optimal strategy whose computational complexity is logarithmic with respect to the inverse of the desired maximum distance from the optimal solution, and polynomial with respect to the number of nodes. A reinforcement learning algorithm is developed for the case when the model is not known exactly, and its convergence to the near-optimal solution is shown subsequently. In addition, a certainty threshold is introduced that determines when data estimation is more desirable than data collection, as the number of nodes increases. For the special case of linear dynamics, a separation principle is constructed wherein the optimal estimate is computed by a Kalman-like filter, irrespective of the probability distribution of random variables (i.e., not necessarily Gaussian). It is shown that the complexity of finding the proposed sampling strategy, in this special case, is independent of the size of the state space and the number of nodes. Examples of a sensor network, a communication network and a social network are provided.

Index Terms—Partially observable Markov decision process, reinforcement learning, separation principle, networked systems.

I. INTRODUCTION

The trade-off between the cost and value of information emerges in different types of networks, where it is important to monitor the status of the network by analyzing its real-time data; however, there is often a cost associated with collecting the data. Hence, the question arises as when an estimate of the network data would be more desirable than monitoring the operation of the network.

A partially observable Markov decision process (POMDP) model is presented in this paper to address the above trade-off in invariant/equivariant networks. The data are defined as the empirical distribution and weighted average of the states of the nodes for finite and infinite state spaces, respectively, and the per-step cost function incorporates the two competing concepts of the above trade-off: the cost of information and the value of information. It is to be noted that the empirical distribution and weighted average (i.e., linear regression) of states are the data of interest in various applications, specially the ones modelled by deep neural networks that have recently received much attention [1], [2]. For example, the empirical distribution appears in Markov-chain deep teams wherein the nodes are partitioned into several sub-populations such that the system is invariant to the permutation of nodes in each sub-population [3]. The weighted average, on the other hand, emerges in linear quadratic deep teams wherein the nodes in each sub-population are not necessarily exchangeable and can have different weights [4]. See other examples in social networks [5], epidemics [6], cyber-security networks [7] and smart grids [8]; to name only a few.

In general, finding the optimal solution of an infinite-horizon discounted cost POMDP is an undecidable problem [9] and even finding a near-optimal solution is NP-hard [11]. On the other hand, it is not always possible to have sufficiently accurate knowledge of the model of nodes. Thus, it is important to learn the solution when the exact model is not available, and this clearly adds to the complexity of the problem. Due to the complexity of finding a near-optimal solution, most of the existing work is mainly focused on developing various approximation methods for POMDPs, some of which are briefly reviewed in the next paragraph.

In grid-based approaches, an approximate value function is computed at a fixed number of points (called grids), and then interpolated over the entire belief space [12]. The advantage of such approaches is that their computational complexity remains the same at every iteration (i.e., does not increase with time); however, their drawback is that the fixed points may not be reachable. In point-based methods, the reachability problem is addressed by considering the reachable set only, where an approximate value function is calculated iteratively in terms of $\alpha$-vectors over a finite number of points in the

1The computational complexity of finding the optimal solution of the finite-horizon POMDP is PSpace-complete whereas that of MDP is P-complete [10].
2The notion of $\alpha$-vectors was introduced in [13] to solve the finite-horizon POMDP, and was later enhanced by pruning dominated vectors [14], [15].
reducible set \[16\]. Note that unlike grid-based approaches, here the points are not fixed and may change as the value function changes. In policy-search methods such as finite-state controllers, attention is restricted to a certain class (structure) of strategies, and the objective is to find the best strategy in the class using policy iteration and gradient-based techniques \[17\].

The above methods often use the belief space as the planning space, whose size grows exponentially with the number of nodes and time horizon; thus, they suffer from the curse of dimensionality. In addition, the dynamics of the belief state depends on the transition probability matrix (i.e., it is a model-based state). Hence, it is not clear how the above POMDP solvers may be used in the case where the model is not exactly known \[18\]. For more details on POMDPs, the interested reader is referred to \[18\], \[19\] and references therein.

In this paper, it is desired to find a near-optimal strategy that determines, at each time instant, whether to collect the data or to estimate them. In contrast to POMDP solvers mentioned in the previous paragraph, we exploit the structure of the problem to efficiently solve the resultant POMDP by using a different planning space, whose size grows exponentially with the number of nodes and time horizon; thus, they suffer from the curse of dimensionality. In addition, the dynamics of the belief state depends on the transition probability matrix (i.e., it is a model-based state). Hence, it is not clear how the above POMDP solvers may be used in the case where the model is not exactly known \[18\]. For more details on POMDPs, the interested reader is referred to \[18\], \[19\] and references therein.

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computing a near-optimal scheduling strategy, where the proposed planning space requires no knowledge about the underlying probability distributions and corresponding matrices (Theorem 8). It is to be noted that our proposed strategy is nonlinear even in the case of the linear dynamics with quadratic cost function.

4) The computational complexity of the proposed strategy is linear with respect to the approximation index, to be defined later, and polynomial with respect to the number of nodes. For the special case of linear quadratic cost functions, the complexity of computing the strategy is independent of the number of nodes as well as the dimension of the space.

5) We generalize the obtained results to the following cases: (a) multiple decision makers and multiple estimators; (b) multiple reset actions; (c) partially exchangeable and partially equivariant networks, and (d) Markovian noises and credibility (see Section VII for details).

II. Problem Formulation

Consider a population of \( n \in \mathbb{N} \) nodes. Let \( s^i_t \in S \) denote the state of node \( i \in \mathbb{N}_n \) at time \( t \in \mathbb{N} \), where \( S \) is a known finite set. Denote by \( m_t \in M(n) \) the empirical distribution of states at time \( t \), i.e.,

\[
m_t(s) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(s^i_t = s), \quad s \in S,
\]

where \( M(n) = \{ (\alpha_1, \ldots, \alpha_{|S|}) | \alpha_i \in \{0, \frac{1}{n}, \ldots, 1\}, i \in \mathbb{N}_{|S|}, \sum_{i=1}^{|S|} \alpha_i = 1 \} \) is the set of empirical distributions over state space \( S \) with \( n \) samples. In the sequel, the empirical distribution of states is interchangeably referred to as data. It is shown in [20] that any set of exchangeable Markov processes \( s_t = \text{vec}(s^1_t, \ldots, s^n_t) \) can be equivalently expressed as a set of Markov processes coupled through the empirical distribution of states. Therefore, let the state of the \( i \)-th node at time \( t \) evolve according to the following dynamics:

\[
s^i_{t+1} = \mathcal{F}(s^i_t, m_t, w^i_t), \quad i \in \mathbb{N}_n, t \in \mathbb{N},
\]

where \( w^i_t \in \mathcal{W} \) is the local noise of node \( i \) at time \( t \). In addition, consider a decision maker that wishes to find an affordable way to sample the data over time horizon such that the estimate of the data, constructed based on the previously sampled data, is reliable. Let \( a_t \in A := \{0, 1\} \) denote the action of the decision maker at time \( t \in \mathbb{N} \), where \( a_t = 1 \) means that the decision maker collects the data and \( a_t = 0 \) means that it does not collect them.

In real-world applications, it is possible that the collected data are not credible due to, for instance, misinformation induced by fake news in social networks, packet drop in communication networks, and faulty encoders and decoders in sensor networks. Denote by \( q \in [0, 1] \) the probability that the collected data are credible (correct). When the data are not credible, they are discarded. Note that the evaluation of the credibility of the data may be viewed as an exogenous process that can have arbitrary Markov-chain dynamics. Since this extension does not add much complexity to our analysis, it is not considered here for simplicity of notation.

Denote by \( o_t \in O := M(n) \cup \{\text{blank}\} \) the observation of the decision maker at time \( t \), where \( \text{blank} \) implies that the decision maker receives either no observation, when \( a_t = 0 \), or potentially incorrect (unreliable) data, when \( a_t = 1 \). Subsequently, for any \( t \in \mathbb{N} \) and \( m \in M(n) \),

\[
\mathbb{P}(o_{t+1} = \text{blank}|m_{t+1} = m, a_t = 0) = 1,
\]

and

\[
\mathbb{P}(o_{t+1} = m|m_{t+1} = m, a_t = 1) = q,
\]

\[
\mathbb{P}(o_{t+1} = \text{blank}|m_{t+1} = m, a_t = 1) = 1 - q,
\]

where initially \( o_1 = m_1 \). The decision maker determines its action at time \( t \) based on its information by that time, i.e.,

\[
a_t = g_t(o_{1:t}, a_{1:t-1}),
\]

where \( g_t : O^t \times A^{t-1} \rightarrow A \) is called the control law of the decision maker. Denote by \( g := \{ g_1, g_2, \ldots \} \) the strategy of the decision maker. Note that the information structure of the decision maker is imperfect because the decision maker has access only to the collected data.

Let \( \mathbf{w}_t = \text{vec}(w^1_t, \ldots, w^n_t) \in \mathcal{W}^n, t \in \mathbb{N} \), with a probability distribution function \( \mathbb{P}_\mathbf{w} \). Let also \( \mathbf{1} \eta_t \in \{0, 1\} \) denote the underlying random variable of the credibility process at time \( t \) such that \( \mathbb{P}(\eta_t = 1) = q \). It is assumed that the primitive random variables \( \{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \eta_1, \eta_2, \ldots \} \) are defined on a common probability space, are mutually independent, and have finite variances. At any time \( t \in \mathbb{N} \), the decision maker constructs an estimate of the data \( m_t \), denoted by \( \hat{m}_t \in M(n) \), according to an estimator function \( h \) as follows:

\[
h_t = h(\mathbb{P}(m_t|\eta_{1:t}, a_{1:t-1})).
\]

The objective of the decision maker is to design a strategy that not only keeps the estimation error small but also incurs minimal collection cost. To this end, we define the following per-step cost \( c : M(n)^2 \times A \rightarrow \mathbb{R}_{\geq 0} \), i.e.,

\[
c(m_t, \hat{m}_t, a_t),
\]

where \( c(m_t, \hat{m}_t, 0) \) is the cost when the decision maker chooses not to collect the data at time \( t \), and \( c(m_t, \hat{m}_t, 1) \) is the cost when it is decided to collect the data.

A. Linear dynamics

Since the role of topology is implicitly described in [2], we present an equivariant linear network in this subsection whose topology can be described explicitly in terms of the eigenvalues and eigenvectors of its adjacency matrix. In particular, consider an undirected weighted graph with a real-valued symmetric adjacency matrix \( \mathbf{A} \). One standard way to vectorize the graph is to use the spectral decomposition [23], i.e.,

\[
\mathbf{A} \approx \sum_{l=0}^{\infty} \alpha(l) \mathbf{1}_l(\mathbf{A})^l | \sum_{l=0}^{\infty} \alpha(l) (\sum_{j=1}^{n} \lambda_j \mathbf{V}(:,j) \mathbf{V}(:,j)^T)^l,
\]

where \( \alpha(l) \in \mathbb{R}, \mathbf{V} \) is an orthogonal matrix consisting of the eigenvectors of \( \mathbf{A} \), \( \Lambda \) is a diagonal matrix of the corresponding eigenvalues, and \( \mathbf{A}' = \mathbf{V} \Lambda \mathbf{V}^T = \sum_{j=1}^{n} \lambda_j^l \mathbf{V}(:,j) \mathbf{V}(:,j)^T, \)
$l \in \mathbb{N}_+$. Let $D \ll n$ be the number of dominant eigenvalues; then, the dynamics of the network can be expressed as follows:

$$s_{t+1} = As_t + \sum_{l=0}^{L} \alpha(l) \left( \sum_{d=1}^{D} \lambda_d V(:, d) V(:, d)^T \right) s_t$$

$$= \sum_{d=1}^{D} A_d \mathbf{V}(;, d) m^d_t,$$

where $m^d_t := \mathbf{V}(;, d)^T s_t = \sum_{i=1}^{n} \mathbf{V}(i, d) s^i_t$ and $A_d := \sum_{l=0}^{L} \alpha(l) \lambda_d^l$, $d \in \mathbb{N}_D$. Hence, for each mode $d \in \mathbb{N}_D$:

$$m^d_{t+1} = \mathbf{V}(;, d)^T s_{t+1} = A_d m^d_t,$$

where $\mathbf{V}(;, d)^T \mathbf{V}(;, j)$ is equal to Kronecker delta function $\delta_{d,j}$ for any $j, d \in \mathbb{N}_n$. Based on the above vectorized representation, consider now a network of $n \in \mathbb{N}$ nodes wherein the state of node $i \in \mathbb{N}_n$ at time $t \in \mathbb{N}$ is a vector denoted by $s^i_t \in \mathbb{R}^d$. Let $m^d_t$ be the weighted average of the states at time $t$ associated with the $d$-th dominant mode, i.e.

$$m^d_t = \frac{1}{n} \sum_{i=1}^{n} v^i_t \cdot m^d_t,$$

where $\frac{1}{n} \sum_{i=1}^{n} (v^i_t)^2 = 1$, and $m^1_{t:}\infty$ is a random process with zero mean and finite covariance matrix $\Sigma_{t:}\infty = \Sigma_{\text{max}} \in \mathbb{R}^{d \times d}$. Let the dynamics of the augmented weighted average be described by the following linear dynamics:

$$m_{t+1} = \sum_{s=0}^{n} \sum_{m} m_t(s) \mathbb{I}(f(s, m, w) = s') \frac{1}{n} \sum_{j=1}^{n} (w^j_t = w),$$

where $m_{t+1}(s') := \sum_{s=0}^{n} \sum_{m} m_t(s) \mathbb{I}(f(s, m, w) = s') \frac{1}{n} \sum_{j=1}^{n} (w^j_t = w)$.

Proof. The proof follows directly from the fact that the empirical distribution of states and also the primitive random variables are invariant to the permutation of nodes.

Assumption 1. The primitive random variables $w^1, \ldots, w^n$ are exchangeable at any time $t \in \mathbb{N}$.

Proposition 1. Let Assumption 1 hold. The empirical distribution $m_t$, $t \in \mathbb{N}$, is a Markov process and evolves almost surely at any state $s \in \mathcal{S}$ as follows:

$$m_{t+1}(s') = \sum_{s=0}^{n} \sum_{m} m_t(s) \mathbb{I}(f(s, m, w) = s') \frac{1}{n} \sum_{j=1}^{n} (w^j_t = w).$$

Under Assumption 2, the dynamic equation (2) can also be described in terms of transition probability matrix at any $s', s \in \mathcal{S}$ and $m \in \mathcal{M}(n)$ as follows:

$$T(s', s, m) = \mathbb{P}(s_{t+1} = s'| s_t = s, m_t = m) = \sum_{w \in \mathcal{W}} P_W(w^t = w) \mathbb{I}(s' = f(s, m, w)), $$

where the probability of transitioning to state $s'$ from state $s$, given empirical distribution $m$, is equal to the probability of realizations $w$ resulting in this transition. For any $s', s \in \mathcal{S}$ and $m \in \mathcal{M}(n)$, define vector-valued function $\phi_m(s) : \mathbb{S}^2 \times \mathcal{M}(n) \rightarrow \mathcal{P}([0, 1, \ldots, n \cdot m(s)])$ as follows:

$$\phi_m(s')(s', s, m) = \delta_0(n \cdot m(s)) + \mathbb{I}(m(s) > 0) \binom{n \cdot m(s)}{T(s', s, m)},$$

where $\delta_0(n \cdot m(s))$ is a Dirac measure with the domain set $\{0, 1, \ldots, n \cdot m(s)\}$ and a unit mass concentrated at zero. In addition, let $\phi : \mathcal{S} \times \mathcal{M}(n) \rightarrow \mathcal{P}([0, 1, \ldots, n])$ be the convolution of $\phi_m(s')(s', s, m)$ over all states $s \in \mathcal{S} = \{s_1, \ldots, s_n\}$, i.e.,

$$\bar{\phi}(s', m) = \phi_m(s_1)(s', s_1, m) \ast \ldots \ast \phi_m(s_n)(s', s_n, m),$$

where $\bar{\phi}(s', m)$ is a vector of size $n + 1$. When the primitive random variables are independently and identically distributed, the evolution of data has a special structure as described in the next theorem.

### III. DYNAMICS OF DATA

Prior to addressing Problems 1 and 2 it is necessary to analyze the evolution of data in time. Many natural systems obey some form of the invariance principle. For example, the outcome of an election is independent of voters’ identity (index of the voters); or, the spectrum of the adjacency matrix of an undirected graph does not depend on the specific labeling of the nodes, or the power demand of a user in a smart grid is often independent of other users’ demands. Hence, it is reasonable to assume that the local noises are exchangeable (index-invariant) and i.i.d. (independently and identically distributed). It is to be noted that these assumptions are standard assumptions in statistical models and data science [24].

Assumption 1. The primitive random variables $w^1, \ldots, w^n$ are exchangeable at any time $t \in \mathbb{N}$.

Proposition 1. Let Assumption 1 hold. The empirical distribution $m_t$, $t \in \mathbb{N}$, is a Markov process and evolves almost surely at any state $s \in \mathcal{S}$ as follows:

$$m_{t+1}(s') = \sum_{s=0}^{n} \sum_{m} m_t(s) \mathbb{I}(f(s, m, w) = s') \frac{1}{n} \sum_{j=1}^{n} (w^j_t = w).$$

Proof. The proof follows directly from the fact that the empirical distribution of states and also the primitive random variables are invariant to the permutation of nodes.

Assumption 2. The primitive random variables $w^1, \ldots, w^n$ are i.i.d. at any time $t \in \mathbb{N}$ with probability function $P_W$.

Under Assumption 2, the dynamic equation (2) can also be described in terms of transition probability matrix at any $s', s \in \mathcal{S}$ and $m \in \mathcal{M}(n)$ as follows:

$$T(s', s, m) = \mathbb{P}(s_{t+1} = s'| s_t = s, m_t = m) = \sum_{w \in \mathcal{W}} P_W(w^t = w) \mathbb{I}(s' = f(s, m, w)),$$

where the probability of transitioning to state $s'$ from state $s$, given empirical distribution $m$, is equal to the probability of realizations $w$ resulting in this transition. For any $s', s \in \mathcal{S}$ and $m \in \mathcal{M}(n)$, define vector-valued function $\phi_m(s) : \mathbb{S}^2 \times \mathcal{M}(n) \rightarrow \mathcal{P}([0, 1, \ldots, n \cdot m(s)])$ as follows:

$$\phi_m(s')(s', s, m) = \delta_0(n \cdot m(s)) + \mathbb{I}(m(s) > 0) \binom{n \cdot m(s)}{T(s', s, m)},$$

where $\delta_0(n \cdot m(s))$ is a Dirac measure with the domain set $\{0, 1, \ldots, n \cdot m(s)\}$ and a unit mass concentrated at zero. In addition, let $\phi : \mathcal{S} \times \mathcal{M}(n) \rightarrow \mathcal{P}([0, 1, \ldots, n])$ be the convolution of $\phi_m(s')(s', s, m)$ over all states $s \in \mathcal{S} = \{s_1, \ldots, s_n\}$, i.e.,

$$\bar{\phi}(s', m) = \phi_m(s_1)(s', s_1, m) \ast \ldots \ast \phi_m(s_n)(s', s_n, m),$$

where $\bar{\phi}(s', m)$ is a vector of size $n + 1$. When the primitive random variables are independently and identically distributed, the evolution of data has a special structure as described in the next theorem.
Theorem 1. Let Assumption 1 hold. Given $m_t \in M(n)$ at time $t \in \mathbb{N}$, the transition probability matrix of the empirical distribution can be computed as follows:

$$P(m_{t+1}(s') = \frac{y}{n} | m_t) = \hat{\phi}(s', m_t)(y + 1), s' \in S, y \in \mathbb{N}_n.$$  

Proof. The proof is presented in Appendix A.

To simplify the notation, denote by $T_m(m', m)$ the transition probability matrix of the empirical distribution, given by Proposition 1 and Theorem 1, i.e.,

$$T_m(m', m) := P(m_{t+1} = m' | m_t = m), m', m \in M(n).$$

In general, the complexity of computing $T_m$ in time is exponential with respect to the number of nodes $n$. However, when the noises are exchangeable, this complexity reduces to polynomial time according to Proposition 1 (because the space of empirical distributions grows polynomially with respect to $n$). The above complexity can be further alleviated in time when the noises are i.i.d., according to Theorem 1.

Theorem 2. Suppose Assumption 2 holds and the dynamics of the states in $\mathbb{N}$ are decoupled, i.e., $s_{i+1} = f(s_i, w'_i), i \in \mathbb{N}_n, t \in \mathbb{N}$. It follows from Theorem 1 that the transition probability matrix $T_m(m', m), m \in M(n)$, whose size $|M(n)|^n$ increases with $n$, can be identified by the transition probability matrix $T(s', s)$, whose size $|S|^2$ is independent of $n$.

Proof. The proof follows directly from (13), on noting that $T(s', s, m)$ reduces to $T(s', s)$ for decoupled dynamics. In this case, knowing function $T(s', s), s', s \in S$, is enough to compute the function $\phi_m(s), m \in M(n)$, and subsequently function $\hat{\phi}$ in (13). Therefore, one can determine the global interactions $T_m(m', m), m', m \in M(n)$ by identifying the local interactions $T(s', s), s', s \in S$.

Remark 1. A consequence of Theorem 2 is that the transition probability matrix of data $T_m$ can be identified by $|S|^2$ scalars, which is a considerable reduction in the parameter space.

IV. A NEAR-OPTIMAL STRATEGY FOR PROBLEMS 1 AND 2

In this section, we propose near-optimal strategies for Problems 1 and 2. At any time $t \in \mathbb{N}$, let $x_t \in M(n)$ denote the last credible data (i.e., the last observation of the decision maker that is not blank). Let also $y_t \in \mathbb{N}^+$ be the number of blanks up to time $t$, after the last credible data $x_t$. The initial value of $(x_t, y_t)$ is $(m_1, 0)$ because $o_1 = m_1$. When data are credible upon request (i.e., $q = 1$), the number of blanks $y_t$ has an inverse relationship with the frequency of collecting data that is $1/(y + 1)$.

In the following lemma, we identify the dynamics of $(x_{t+1}, y_{t+1})$ at time $t \in \mathbb{N}$, given $(x_t, y_t, o_{t+1})$.

Lemma 1. There exists a function $\hat{f} : M(n) \times \mathbb{N}^+ \times O \rightarrow M(n) \times \mathbb{N}^+$ such that $(x_{t+1}, y_{t+1}) = \hat{f}(x_t, y_t, o_{t+1}), t \in \mathbb{N}$:

$$\hat{f}(x_t, y_t, o_{t+1}) := \begin{cases} (x_t, y_t + 1), & o_{t+1} = \text{blank}, \\ (o_{t+1}, 0), & o_{t+1} \neq \text{blank}. \end{cases}$$

Proof. The proof follows from the definition of $x_t$ and $y_t$.

Now, let the observation $o_t \in O, t \in \mathbb{N}$, be rewritten as:

$$o_t = x_t, \text{if } y_t = 0, \text{ and } o_t = \text{blank}, \text{if } y_t \neq 0. \quad (17)$$

According to (16) and (17), one can conclude that sets $\{o_{t+1}, o_{t+1-1}\}$ and $\{x_{t+1}, x_{t+1}, o_{t+1-1}\}$ have equivalent information as each set can be fully described the other one, i.e.,

$$P(m_t | x_{t+1}, y_{t+1}, o_{t+1-1}) = T_m^{|m_t, x_t\rangle}, \quad (18)$$

where $T_m^{|m_t, x_t\rangle}$ is the transition matrix (15) to the power of $y_t$.

In the next lemma, we demonstrate that $(x_{t+1}, y_{t+1})$ has Markovian dynamics by showing that the posterior probability of $o_{t+1}$ given the history $(x_{t+1}, y_{t+1}, o_{t+1})$ depends only on the information at time $t$, i.e., $(x_t, y_t, o_t)$.

Lemma 2. Let Assumption 1 hold. Given any realization $x_{t+1}, y_{t+1}$ and $o_{t+1}, t \in \mathbb{N}$, the following equality holds irrespective of the strategy $g$:

$$P(o_{t+1} | x_{t+1}, y_{t+1}, o_{t+1}) = (1 - a_t q) \hat{g}(o_{t+1} = \text{blank}) + a_t q T_m^{|m_t, x_t\rangle} (y_{t+1}, o_{t+1} = \text{blank}). \quad (19)$$

Proof. The proof is presented in Appendix B.

We prove that the conditional expectation of the per-step cost given the history by time $t$ can be described by the information at time $t$.

Lemma 3. Given any realization $x_{t+1}, y_{t+1}$ and $o_{t+1}, t \in \mathbb{N}$, there exists a function $\hat{c} : M(n) \times \mathbb{N}^+ \times A \rightarrow \mathbb{R}_{\geq 0}$ such that:

$$E[c(m_t, m_{t+1}) | x_{t+1}, y_{t+1}, o_{t+1}] = \hat{c}(x_t, y_t, o_t) := \sum_{m \in M(n)} c(m, h(T_m^{|m_t, x_t\rangle} (m, x_t)), o_t) T_m^{|m_t, x_t\rangle} (m, x_t).$$

where the above equality holds irrespective of strategy $g$.

Proof. The proof is presented in Appendix C.

From the results of Lemmas 1 and 3, an optimal strategy for Problem 1 is identified by the following theorem.

Theorem 3. Let Assumption 1 hold. The optimal solution of Problem 1 is given by the following Bellman equation such that for any $x \in M(n)$ and $y \in \mathbb{N}^+$,

$$V(x, y) = \min_{a \in A} \left( \hat{c}(x, y, a) + \gamma E[V(\hat{f}(x, y, o)) \right], \quad (21)$$

where the above expectation is taken over all observations $o \in O$ with probability distribution (19).

Proof. The proof follows from the fact that $(x_t, y_t)$ are sufficient statistic to identify the optimal solution of Problem 1. More precisely, given observations $o_{t+1}, t \in \mathbb{N}$, state $(x_t, y_t)$ is observable at any time $t$ and has Markovian dynamics according to Lemma 1. In addition, given any history $(x_{t+1}, y_{t+1}, o_{t+1}),$ the conditional probability of $o_{t+1}$ and the conditional expectation of the per-step cost $c(m_t, m_{t+1})$ do not depend on the strategy $g$ according to Lemmas 2 and 3, respectively. Consequently, the optimal solution of Problem 1 can be identified by dynamic programming decomposition, and
Bellman equation (21) is obtained from well-known results in Markov decision theory \cite[Proposition 5.4.1, Volume 1]{markov}. Since the planning space in Theorem 3 is countably infinite, the solution of the Bellman equation (21) is intractable, in general. As a result, we propose an \( \varepsilon \)-optimal solution based on a truncation technique whose performance is within an arbitrary neighbourhood (determined by \( \varepsilon \)) of the optimal performance associated with the solution of Problem 1.

Remark 2. Under some practical constraints such as limited energy resources for data collection or saturation of states, the feasible set of the dynamic program (21) may reduce to a finite set, yielding a tractable optimization problem.

A. An \( \varepsilon \)-optimal solution for Problem 7

Denote by \( c_{\text{max}} \) an upper bound on the per-step cost (7). For any \( k \in \mathbb{N} \), define the following Bellman equation at any \( x, m^* \in \mathcal{M}(n) \) and \( y \in \{0,1\}^k \),

\[
\tilde{V}_k(x, y) = \min(\tilde{V}_k^0(x, y), \tilde{V}_k^1(x, y)),
\]

where

\[
\begin{aligned}
\tilde{V}_k^0(x, y) &:= \sum_{m \in \mathcal{M}(n)} c(m, h(T_m^k(m, x), 0)) T_m^k(m, x), \\
&+ (1 + \gamma^\varepsilon \tilde{V}_k(x, y + 1) + \gamma (y = k) \tilde{V}_k(m^*, 0)), \\
\tilde{V}_k^1(x, y) &:= \sum_{m \in \mathcal{M}(n)} c(m, h(T_m^k(m, x), 0)) T_m^k(m, x) \\
&+ (1 - \gamma^\varepsilon \tilde{V}_k(x, y + 1) + \gamma (y = k) \tilde{V}_k(m^*, 0)) \\
&+ \gamma (\sum_{m' \in \mathcal{M}(n)} T_m^k(m', x) \tilde{V}_k(m', 0)).
\end{aligned}
\]

Theorem 4. Suppose Assumption 7 holds. Given any \( \varepsilon \in \mathbb{R}_{>0} \), let \( k(\varepsilon) \in \mathbb{N} \) be sufficiently large such that

\[
k(\varepsilon) \geq \log \left( \frac{(1 - \gamma)\varepsilon}{2c_{\text{max}}} \right) / \log(\gamma).
\]

Then, any solution to the Bellman equation (22) is an \( \varepsilon \)-optimal solution for Problem 7, i.e.,

\[
g^*_k(x, y) := \begin{cases} 0, & \tilde{V}_k^0(x, y) \leq \tilde{V}_k^1(x, y), \\ 1, & \tilde{V}_k^0(x, y) > \tilde{V}_k^1(x, y). \end{cases}
\]

Proof. The proof is presented in Appendix D.

To numerically compute the solution of the Bellman equation (22), one can use value iteration, policy iteration, or any other existing approximate method \cite{markov, dpection}. Note that the space of Bellman equation (22) (i.e., \( \mathcal{M}(n) \times \mathbb{N}_k^2 \)) grows polynomially with the number of nodes \( n \) and linearly with the approximation index \( k \). Therefore, the following result holds.

Corollary 1. The computational complexity of the proposed solution in Theorem 7 is linear with respect to the approximation index \( k(\varepsilon) \), logarithmic with respect to the inverse of the size of the desired neighborhood \( \varepsilon \), and polynomial with respect to the number of nodes \( n \).

Corollary 2. For the special case of single node, i.e., \( n = 1 \), there is no loss of optimality in replacing space \( \mathcal{M}(n) \) by space \( \mathcal{S} \) in Theorems 3 and 4.

Proof. The proof follows on noting that when \( n = 1 \), spaces \( \mathcal{M}(n) \) and \( \mathcal{S} \) have equivalent information, i.e., \( m_t = \delta_n \) at any time \( t \in \mathbb{N} \).

It is to be noted that the application domain of the present work is different from applications such as sensor selection, where the objective is to dynamically choose a subset of sensors in order to monitor a time-varying phenomenon \cite{sensor1, sensor2}. For example, in sleep sensor scheduling control \cite{sensor3}, the dynamics of the target (phenomenon) is decoupled from the scheduling (sampling) strategy, the value of data at each time instant is binary (i.e., it is zero if data is observed and it is one otherwise), and the planning space consists of the belief state of the target as well as the residual sleep times of sensors \cite[Theorem 3.1]{sensor4}. In contrast, the phenomenon considered in this paper is an estimate of data, generated by the estimator, that is influenced by the sampling strategy and leads to a dual effect \cite{detection}. The value of data depends on various parameters such as the cost and estimator functions and is not necessarily binary variables. Furthermore, the proposed dynamic program is based on a planning space, defined as the last credible data and the elapsed time since then (which is not a belief space). In addition, the action set here does not depend on the number of nodes and the state space and the dynamics of the phenomenon (data) do, and the solution methodology is amenable to the incompleteness of the model structure.

B. An \( \varepsilon \)-optimal solution for Problem 2

Two different approaches are considered here to find a near-optimal strategy for Problem 2. The first one is an indirect (model-based) approach which involves two steps: supervised learning and planning. Given a large number of training samples, one can utilize supervised-learning (parametrization) techniques such as linear regression and logistic regression to identify the model, and then find the (planning) solution of the Bellman equation (22) by using methods such as value iteration and policy iteration \cite{markov, detection}. In general, the total number of unknown parameters that should be learned to solve equation (22) using this approach is equal to \( |\mathcal{M}(n)|^2 + 2(k + 1) |\mathcal{M}(n)| + 1 \), with \( |\mathcal{M}(n)|^2 \) scalars for transition probability matrix \( T_m \), \( |\mathcal{M}(n)| \) scalars for the per-step cost \( c \), and 1 scalar for the credibility of data \( q \). In practice, the first approach is feasible when the number of unknown parameters is relatively small. For example, the deep Chapman-Kolmogorov equation in Theorem 1 can be parametrized by a small number of variables according to Theorem 2 for the case when the dynamics of nodes are decoupled and the random variables are i.i.d., which is a non-trivial way of parametrization.

Proposition 2. Let Assumption 2 hold and the per-step cost function as well as the estimator function be given. When the dynamics of the states are decoupled and \( n \) is large, it is more efficient to first learn the model and then solve the Bellman equation (22) to obtain an \( \varepsilon \)-optimal solution for Problem 2.

Proof. The proof follows on noting that the total number of unknown parameters in (22) is \(|\mathcal{S}|^2 + 1\), which is independent of the number of nodes \( n \) (i.e., \(|\mathcal{S}|^2 \) parameters correspond to the transition probability matrix \( T_m \) from Theorem 2 and one parameter corresponds to the credibility probability \( q \)).

The second approach, on the other hand, is a direct (model-free) approach that finds an \( \varepsilon \)-optimal solution of the Bellman
Algorithm 1 Proposed Q-Learning Algorithm

1: Given any $\varepsilon \in \mathbb{R}_{>0}$, let $k(\varepsilon) \in \mathbb{N}$ satisfy inequality (23).
2: Let $x_1 = m_1$, $y_1 = 0$, $Q_1(x, y, a) = 0$ and $\alpha_1(x, y, a) = 1$, $\forall x \in \mathcal{M}(n)$, $y \in \mathbb{N}_n^*$, $a \in \mathcal{A}$.
3: At iteration $\tau \in \mathbb{N}$, give any state $(x, y) \in \mathcal{M}(n) \times \mathbb{N}_n^*$ and any action $a \in \mathcal{A}$, update the corresponding Q-function and learning rate as follows:

$$
\begin{align*}
Q_{\tau+1}(x, y, a) &= (1 - \alpha_\tau(x, y, a))Q_\tau(x, y, a) \\
&+ \alpha_\tau(x, y, a)(c' + \gamma \min_{a' \in \mathcal{A}} Q_{\tau}(x', y', a')), \\
\end{align*}
$$

where $c'$ is the immediate cost, $(x', y') = f(x, y, a)$ is the next state with observation $o$ and $\lambda$ determines proper learning rates $\alpha_{\tau} \in [0, 1]$, $\tau \in \mathbb{N}$, such that $\sum_{\tau=1}^\infty \alpha_\tau(x, y, a) = \infty$ and $\sum_{\tau=1}^{\infty} (\alpha_\tau(x, y, a))^2 < \infty$.
4: Let $\tau = \tau + 1$, and go to step 3 until termination.

Let also the infinite-population estimator $h$ be defined as:

$$
\hat{m}_t = \hat{T}(\ldots \hat{T}(\hat{m}_1)), \quad t \in \mathbb{N}\backslash\{1\}. \quad (25)
$$

Remark 3. When the dynamics are decoupled and $n = \infty$, nonlinear dynamics (24) reduces to a linear one, i.e. $\hat{T}p$.

Lemma 4. Let Assumption 3 hold. There exists a positive scalar $K_p$, such that

$$
\|\hat{T}(p) - \hat{T}(\hat{p})\| \leq K_p\|p - \hat{p}\|, \quad p, \hat{p} \in \mathcal{P}(\mathcal{S}).
$$

Proof. The proof follows from equation (25), Assumption 3 and the fact that the Lipschitz property is preserved by summation and multiplication.

Lemma 5. Let Assumption 2 hold. Given any $m_i \in \mathcal{M}(n)$ and $\hat{m}_i \in \mathcal{P}(\mathcal{S})$, $t \in \mathbb{N}$, the following inequality is satisfied:

$$
\mathbb{E}[\|m_{t+1} - \hat{m}_{t+1}\| | m_t, \hat{m}_t] \leq K_p\|m_t - \hat{m}_t\| + O(\frac{1}{\sqrt{n}}).
$$

Proof. The proof follows directly from the triangle inequality, Lemma 4 and [29, Lemma 2].

Assumption 4. Assume that $\gamma K_p < 1$. Note that this inequality always holds when the dynamics of states in (2) are decoupled because in this case $K_p = 1$ satisfies Lemma 7.

Theorem 6. Let Assumptions 2, 3 and 4 hold. The total expected discounted cost associated with using the estimator (25) is bounded at all times and converges to zero at the rate $1/\sqrt{n}$:

$$
\mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} c(m_t, \hat{m}_t, 0)\right] \leq \frac{\gamma K_c}{(1 - \gamma)(1 - \gamma K_p)} O\left(\frac{1}{\sqrt{n}}\right), \quad (26)
$$

where $O\left(\frac{1}{\sqrt{n}}\right)$ depends on the variance of local noises.

Proof. The proof is presented in Appendix [10].

V. NETWORKS WITH A LARGE NUMBER OF NODES: ASYMPTOTIC ANALYSIS

In this section, we show that estimating data tends to be more efficient than collecting it when the number of nodes is sufficiently large. To this end, the following assumption is made on the model.

Assumption 3. There exist positive scalars $K_T$ and $K_c$ such that for any $s', s \in \mathcal{S}$, $a \in \mathcal{A}$, and $m, m' \in \mathcal{P}(\mathcal{S})$,

1. $|T(s', m) - T(s', m')| \leq K_T|m - m'|$;
2. $c(m, \hat{m}, a) \leq K_c|m - \hat{m}|$.

It is to be noted that Assumption 3 is mild because any polynomial function in $m$ is Lipschitz with respect to $m$ due to the fact that $m$ is confined to a bounded interval. Let function $\hat{T} : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ denote a set of ordinary difference equations, i.e., discrete-time (finite-state) Fokker-Planck-Kolmogorov equation such that:

$$
\hat{T}(p) := \sum_{s \in \mathcal{S}} p(s)T(\cdot, s, p), \quad p \in \mathcal{P}(\mathcal{S}). \quad (24)
$$

Definition 1 (Certainty Threshold). The right-hand side of (26) is defined as the certainty threshold. This threshold depends on the number of nodes as well as the Lipschitz constants of the transition probabilities and cost function introduced in Assumption 3.

Corollary 3. Let Assumptions 2, 3 and 4 hold. If the collection cost is greater than the certainty threshold, then the optimal solution to Problem 1 is to use estimator (25) at all times.

Proof. Consider two scenarios, where in the first one the strategy is to always use the estimator function (25), and in the second one the strategy is to collect data at least once. The costs of both scenarios until the first data collection are the same. The proof now follows directly from Theorem 6.

Remark 4 (De Finetti’s theorem). It is worth mentioning that when $n = \infty$, exchangeable random variables behave as i.i.d. variables, according to de Finetti’s theorem [30].
VI. LINEAR DYNAMICS: A SPECIAL CASE

A. Optimal estimator

The model presented in Subsection II-A has various applications, e.g., in remote-state estimation wherein an encoder sends a Markovian process to a decoder over an unreliable link under Transmission Control Protocol (TCP), where an acknowledgement is sent to the encoder upon the reception of the data at the decoder [31], [32]. The objective of the encoder and decoder is to collaborate in such a way that the cost function (10) is minimized, where \( g \) is the transmission strategy of the encoder and \( h \) is the estimation strategy of the decoder. By using majorization theory and imposing some conditions such as symmetric and unimodal probability distribution of random variables, it is shown in [31], [32] that the optimal estimator is Kalman-like. In what follows, we extend the above findings by a simple proof technique using the proposed planning space, and subsequently establish a separation theorem that holds for the general case of multi-dimensional dynamic systems with arbitrary probability distribution, without resorting to majorization theory or any other conditions on the random variables. To this end, define \( x_t \in \mathbb{R}^{d_r} \) as the last (credible) observed data and \( y_t \in \mathbb{N}^* \) as the elapsed time since then.

**Theorem 7 (Separation principle).** The problem of finding the optimal estimator in (10) is separated from that of the optimal scheduling strategy. In particular, the optimal estimator has a structure similar to the minimum mean-square estimator with the following Kamlan-like update rule:

\[
\hat{m}_{t+1} = A\hat{m}_t + L(y_{t+1})(x_{t+1} - A\hat{m}_t),
\]

where \( L(y_{t+1}) = 0 \) if \( y_{t+1} = 0 \) and \( L(y_{t+1}) = 0 \) if \( y_{t+1} \neq 0 \). This result holds regardless of the sampling strategy, the order of dynamics and the probability distribution of the initial state, credibility of data and additive noises.

**Proof.** The proof is presented in Appendix [C].

**Remark 5.** The result of Theorem 7 also holds for the time-varying finite-horizon case since the proof technique presented above does not depend on the time-homogeneity of the model.

B. Sampling (scheduling) strategy

Consider a centralized networked control system wherein the joint state is perfectly observed and the optimal joint action is a state-feedback strategy. In such a case, every node must broadcast its state at each time instant so that everyone can observe the joint state in order to compute its control action accordingly. In practice, however, sharing information is costly, the quality of data transmission is sometimes compromised, and data packets may be lost in the communication channels (e.g., in erasure channels). Thus, it is important to be able to implement the centralized solution in a distributed manner. To this end, one can solve a (local) scheduling problem at every node in order to decide when to broadcast its state [33]. Once the information is broadcast, other nodes can update their estimates about the state of the node, which is used in their strategies. For the case where every node uses a minimum mean-square estimator and attention is restricted to a threshold-type scheduling, a dynamic-program-based solution can be developed whose information state is the estimation error (that is a continuous variable in \( \mathbb{R}^{d_r} \) with a model-dependent dynamics) [33]. It is shown in [31], [32] that such threshold-type policies are optimal under some technical assumptions and symmetrical model structures.

In this work, however, we use a dynamic program with an information state different from [31]-[33], which is a discrete variable between 0 and \( k \), independent of the state space dimension \( d_r \), with a model-free dynamics. Therefore, it is computationally easy to find a near-optimal strategy by solving our proposed dynamic program.

**Assumption 5.** Suppose that the per-step cost (10) takes the following structure: \( c_t(m_t - \hat{m}_t, a_t) \leq c_{\text{max}}, \) where \( c_{\text{max}} \) is a positive constant and the estimator is the minimum mean-square estimator, i.e., \( \hat{m}_t = \mathbb{E}[m_t | o_{1:t}, a_{1:t-1}], t \in \mathbb{N}. \)

From the dynamics of the mean-square estimator given by (27), it follows that for any \( t \in \mathbb{N}^* \):

\[
m_t - \hat{m}_t = 1(y_t > 0) \sum_{\tau=1}^{y_t} A^{-1} \tilde{w}_{t-\tau}.
\]

Since \( w_{1:1} \) is an i.i.d. random process, one has:

\[
c_t(y_t, a_t) = \mathbb{E}[c_t(m_t - \hat{m}_t, a_t) | x_{1:t}, y_{1:t}, a_{1:t}],
\]

where the above per-step cost does not depend on \( x_t \); hence, it can be denoted by \( c(y_t, a_t), t \in \mathbb{N}. \) It is also possible to consider an alternative assumption described below.

**Assumption 6.** For any \( t \in \mathbb{N}, a_t \in A \) and \( m_t \in M(n) \), let functions \( z(a_t) \) and \( \ell(m_t, a_t) \) in (10) be equal to 1 and \( \ell_0 \), respectively, where \( \ell \in \mathbb{R}_{\geq 0}. \) In addition, matrix \( A \) is symmetric and all its eigenvalues are within the unit circle.

Under Assumption 5, (28) can be computed as follows:

\[
c_t(y_t, a_t) = \mathbb{E}[(m_t - \hat{m}_t)^T(m_t - \hat{m}_t) + \ell_0 | x_{1:t}, y_{1:t}, a_{1:t}]
\]

\[
= 1(y_t > 0) \text{Tr}(A^{-1} \Sigma_w) + \ell_0,
\]

\[
= \text{Tr}((I - A^-1 A^\top) \Sigma_w) + \ell_0 = \text{Tr}((I - A^-1 A^\top) \Sigma_w) + \ell_0.
\]

**Theorem 8 (Sampling strategy).** Let either Assumption 5 or Assumption 6 hold. There is no loss of optimality in restricting attention to the space of elapsed time after the last credible data (i.e., there is no need to know the data). More precisely, select a sufficiently large \( k \in \mathbb{N} \) such that inequality (23) is satisfied, and simplify the dynamic program (22) as follows:

\[
\tilde{V}_k(y) = \min(\bar{V}_k^0(y), \tilde{V}_k^1(y)), \quad y \in \mathbb{N}_k^d,
\]

where \( \bar{V}_k^0(y) := c(y, 0) + \gamma(1(y < k)\bar{V}_k(y + 1) + 1(y = k)\bar{V}_k(0)) \) and \( \bar{V}_k^1(y) := c(y, 1) + (1 - q)\gamma(1(y < k)\tilde{V}_k(y + 1) + 1(y = k)\tilde{V}_k(0)) + q \tilde{V}_k(0). \) Then, a near-optimal sampling strategy is to collect data when \( \bar{V}_k^0(y_t) < \tilde{V}_k^1(y_t). \) A similar

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\( ^4 \) For finite-horizon analysis, matrix \( A \) can be any arbitrary matrix.
Remark 7 (Mean-field approximation). In the case where observations are received with a fixed time delay, the mean-field approximation in the previous paper, including Theorems 7 and 8, extend naturally to the case of non-linear dynamics, where the measurement noise destroys the tractability provided by the proposed Kalman filter. However, we demonstrate that the resultant optimization problem can be solved numerically by various computational tools.

Corollary 4. Let Assumptions 2 and 3 hold. The certainty equivalence threshold in the case of linear dynamics with quadratic cost function converges to zero at the rate $1/n$, which is faster than the generic rate of $1/\sqrt{n}$. In this case, $\Sigma w = \frac{1}{n} \sum_{i=1}^{n} (v_i^d, v_i^d) \Sigma_i w \leq \frac{1}{n} \Sigma_{\max}$.

Remark 6 (Time delay). All the results presented in this paper, including Theorems 7 and 8, extend naturally to the case where observations are received with a fixed time delay $\tau \in \mathbb{N}^*$, by simply replacing $y_t$ with $y_t + \tau$.

Remark 7 (Mean-field approximation). When $n = \infty$ and the dynamics are decoupled, the infinite-population (linear) model presented in Remark 3 may be used in Theorem 8 to provide a scale-free approximation.

Remark 8 (Age of information). Note that the cost function (29) is exponential with respect to $y_t$, reflecting the fact that the quality of the minimum mean-square estimator deteriorates exponentially in the absence of credible data. Nonetheless, it is possible to consider a simpler cost function, e.g., an affine or quadratic cost function in $y_t$, which can be obtained more efficiently. This case is then related to real-time status updating, where $y_t$ is the age of information, representing the freshness of data, and the objective is to monitor a phenomenon of interest in a timely manner.

C. Noisy observations with Gaussian random variables

In this subsection, we show that the presence of measurement noise destroys the tractability provided by the proposed planning space and finding an $\varepsilon$-optimal solution becomes NP-hard. However, we demonstrate that the resultant optimization problem is a deterministic non-linear dynamic optimization problem, that may be solved numerically by various computational tools.

Suppose Assumption 4 holds and local noises are Gaussian. Let $o_t^i \in \mathbb{R}^{d_o}$, $d_o \in \mathbb{N}$, be the noisy observation of node $i$ at time $t$ such that $o_t^i = C o_t^i + \xi_t^i$, where $\xi_t^i \in \mathbb{R}^{d_o}$ is an i.i.d. Gaussian random process with zero-mean and finite covariance matrix $\Sigma_t^i \in \mathbb{R}^{d_o \times d_o}$. In addition, it is assumed that the measurement noises $\{\xi_t^i \}_{t \in \mathbb{N}}$ are mutually independent across nodes, and are also independent from the previously defined primitive random variables. Then, $o_t^i := \frac{1}{n} \sum_{i=1}^{n} o_t^i = C \tilde{m}_t + \xi_t$, where $\xi_t := \frac{1}{n} \sum_{i=1}^{n} o_t^i$ with the covariance matrix $\Sigma_t := \frac{1}{n} \sum_{i=1}^{n} (o_t^i, o_t^j) \Sigma_i.\xi_t$.

For simplicity of presentation, assume that $q = 1$ and that the horizon is finite. Therefore, $a_{t+1} = C(a_t) m_{t+1} + E(a_t) \xi_{t+1}$, where $a_t = \text{vec}(o_t^1, \ldots, o_t^n)$, $C(a_t) := \text{diag}(C_1, \ldots, C_D)$, and $E(a_t) = a_t$. In general, blank observation does not carry the same information that zero observation does. However, when attention is restricted to Gaussian random variables, the conditional expectation of the state, given zero observation, is equal to that given the blank observation, because the innovation processes associated with both cases are zero. Subsequently, from (37), one can use the celebrated Kalman filter to compute the optimal state-estimate based on the noisy observations. In particular, given any realization $a_{t+1}$, the minimization in (10) reduces to a mean-square optimization problem, where the best nonlinear estimator is known to be $\hat{m}_t = E[m_t|a_{t+1}, a_{t+1}]$. In a way similar to (37), define the following covariance matrix:

$$P_{t+1} = AP_t A^T + \Sigma w - AP_t C_t^T (a_t) (C(a_t) P_t C_t^T (a_t) + E(a_t) \Sigma^d E_t(a_t))^{-1} C(a_t) P_t A^T, \quad t \in \mathbb{N}, \quad (31)$$

where $P_t = 0_{d \times d}$. Then, the optimal estimator is given by the following Kalman filter: $\hat{m}_{t+1} = A \hat{m}_t + L(a_t) (a_{t+1} - C(a_t) \hat{m}_t)$, where $\hat{m}_t = 0_{d \times 1}$ and the observer gain is described by:

$$L(a_t) = (AP_t C_t^T (a_t)) (C(a_t) P_t C_t^T (a_t) + E(a_t) \Sigma^d E_t(a_t))^{-1}.$$

Subsequently, the optimization problem associated with the optimal scheduling strategy for any finite horizon $H$ reduces to a deterministic non-convex nonlinear optimization problem as follows: $\min_{a_{t+1}} \sum_{t=1}^{H} \gamma_t^{-1} (\operatorname{Tr}(P_{t+1}) + \ell_t)$. To find a solution to the above optimization problem, one can write a dynamic program based on the history space $\{a_{t+1} \mid H\}$, whose cardinality grows exponentially with respect to the horizon (i.e., $2^H$). Alternatively, one can write a dynamic program based on the information state $P_t$ (which is a continuous variable in $\mathbb{R}^{d \times d}$ with nonlinear model-dependent dynamics). For a reasonably large horizon $H$, both dynamic programs can be very difficult to solve analytically.

VII. GENERALIZATION TO COMPLEX NETWORKS

The main focus of the previous sections was to study the trade-off between data collection and data estimation, and for this reason, the simplest model structure was considered in order for the excessive number of parameters not to obscure the main results. In this section, we show how our results can naturally be extended to more complex applications.

A. Multiple decision makers and estimators

Let $n \in \mathbb{N}$ denote the number of decision makers and $n(k) \in \mathbb{N}$ denote the number of estimators whose accessibility to data is decided by the decision maker $k \in \mathbb{N}_n$. In such a setup, for any $j \in \mathbb{N}(k)$, estimator $j$ provides a different estimate $\hat{m}_t(k)$ of the states of $n$ nodes. Thus, the objective is to minimize the following social cost function:

$$\mathbb{E} \left[ \sum_{t=1}^{\infty} \gamma_t^{-1} \sum_{k=1}^{n} C^d \left( m_t, \hat{m}_t, \ldots, \hat{m}_t(k), a_t^k \right) \right].$$

Since
the state dynamics is not influenced by the actions of decision makers, and on the other hand, the above cost function is additive, the optimization problem of each decision maker (i.e., Problems 1 and 2) can be solved separately, with possibly different parameters. Therefore, the proposed concept of planning space is applicable here.

B. Multiple reset actions

So far, the trade-off between data collection and data estimation has been formulated as a binary decision, where 0 means that the data are not collected and 1 means that the data are collected (which can be credible or not credible). It is possible to generalize the above decision to multiple decision options where 0 means that the data are not collected and i ∈ N means that the data are collected through the i-th option (which can be different routes, channels, sensors, receivers, etc.). In such a case, one has to define the last credible data xt rendered and elapsed time yt rendered for each option i so that when the credible data xt are received at a particular time t via the i-th option, its elapsed time resets to zero (i.e., yt = 0) [21, Remark 2]. This extension is similar to a bandit problem, where each option represents a bandit. Consequently, an immediate application of the data collection/estimation trade-off is the trade-off between exploration and exploitation that arises in numerous learning problems, where a decision maker wishes to sequentially choose when to explore (collect the data of interest) and when to exploit the learned model (which is data estimation based on the previously collected data).

For multiple reset actions, see an example of machine maintenance problem with three actions in [38].

C. Partially exchangeable and partially equivariant networks

Consider partially exchangeable networks for the Markov-chain model and partially equivariant networks for the linear model, where the population of nodes can be partitioned into a few distinct sub-populations in which the nodes are exchangeable and equivariant, respectively. In this case, the data dynamics becomes more complex, but the proposed approach still works because the dynamics does not depend on the action of the decision maker. For an example of partially exchangeable network, see a leader-follower network in [39] with n exchangeable followers and one non-exchangeable leader.

D. Markovian noises and credibility

Depending on the data (state of the system), Assumptions 1 and 2 can be generalized to the case in which local noises have their own Markov-chain dynamics. In such a case, dynamic programming decomposition is still valid, with the only difference that the state of the Markov chain is added to the system state. Note that Assumptions 1 and 2 are not required for the linear case (i.e., Theorems 7 and 9). Similarly, the credibility can be a Markovian process, e.g., see [40] for the spread of fake news in a homogeneous network.

VIII. Applications

A. A sensor network with data of different importance level

We consider a sensor (decision maker) that measures a Markovian source st at time t ∈ N, such as the temperature of a room or the battery of a smart house, and wishes to report it to a data center. Let S := Zd+1, d ∈ N, be the state space, and st evolve in time as follows:

\[ s_{t+1} = \begin{cases} s_{\text{max}}, & s_t > d_s, \\ s_t + w_t, & |s_t| \leq d_s, \\ s_{\text{min}}, & s_t < -d_s, \end{cases} \]

where \( s_{\text{max}}, s_{\text{min}} \in S \) are the saturation levels, and for any t ∈ N, \( w_t \in W := Z_{d_w} \) is an i.i.d. process with probability distribution function \( P_W \). The state \( s_{t+1} \) is successfully received by the data center upon transmission (i.e., \( a_t = 1 \)) with probability \( q \in [0,1] \) at time t ∈ N, i.e., for any s ∈ S, \( P(o_{t+1} = s|s_{t+1} = s, a_t = 1) = q \). In practice, it is not efficient for the sensor to measure \( s_t \) and transmit it to the data center at each time instant t ∈ N because there is often a cost associated with sensing and transmitting. Let \( \hat{s}_t \in S \) be the last state received by time t ∈ N at the data center. The objective of the sensor is to find an efficient transmission law that not only keeps the estimation error small at the data center, but also incurs minimal measurement and transmission cost at the sensor. The following performance index is defined:

\[ J = \mathbb{E} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} |s_t - \hat{s}_t| + \ell a_t \right], \]

where \( \gamma \in (0,1) \) is the discount factor and \( \ell \in \mathbb{R}_{>0} \) is the transmission cost. Note that the estimation error \( |s_t - \hat{s}_t| \) in the above cost function is penalized by \( |s_t| \); the rationale for using such a penalty term lies in that fact that in some applications the saturation levels represent warning zones, in the sense the estimation error around such zones carries more weight than that around normal operating zones: hence, classical threshold-based strategies that treat all states equally are not practical.

Example 1. Suppose that \( s_t \) is the energy level of a battery. The battery is charged by some renewable generation sources and discharged as serving demands. Initially the nominal value of the battery is \( s_1 = 0 \). At each time t ∈ N, one unit energy is added to \( s_t \) with probability \( p_g \) and one unit energy is depleted from \( s_t \) with probability \( p_d \); assume that the probability of the generation is independent of that of the consumption. Let \( w_t \in \{-1,0,1\} \) denote the change in the energy level of the battery at time t, i.e., \( P(w_t = 1) = p_g(1 - p_d), P(w_t = -1) = p_d(1 - p_g) \), and \( P(w_t = 0) = p_g \times p_d + (1 - p_g)(1 - p_d) \). The objective is to find the optimal frequency for transmitting the state of the battery under the transmission cost \( \ell \). Let \( p_g = p_d = 0.8, d_w = 1, d_s = s_{\text{max}} = -s_{\text{min}} = 99, q = 0.95, \gamma = 0.9 \), and \( \ell = 100 \). Due to the incompleteness of the information structure at the decision making level, the conventional belief space is large, i.e. \( \mathbb{P}(s_t | o_{1:t}) \in \mathbb{R}^{200} \). In addition, reinforcement learning in the belief space is conceptually difficult because the dynamics of the belief state depends on the model (i.e., it is a model-dependent planning space). To this end, we use a new information state based on which the proposed strategies in both planning and reinforcement
learning cases are tractable, and their performances are very close to the optimal performance. Simulations provided in this example confirm the theoretical results, where the number of states in the new planning space is $10^5 = 200 \times 50$.

In particular, let $x_t \in S$ denote the last credible observation of the data center by time $t \in \mathbb{N}$ (i.e., $x_t = \tilde{s}_t$) and $y_t \in \mathbb{N}^*$ denote the elapsed time after receiving $x_t$. A near-optimal transmission law can be obtained by solving the Bellman equation (22) in Theorem 4 for a sufficiently large approximation index $k \in \mathbb{N}$, where space $\mathcal{M}(n)$ and transition probability matrix $T_m$ are replaced by space $S$ and transition probability matrix $T$, respectively, according to Corollary 2. After exhaustive numerical simulations, it is observed that the optimal strategy is obtained for any approximation index $k \geq 70$. According to Figure 1, the frequency of transmitting the energy level to the data center increases as the energy level approaches the warning (saturation) thresholds. When the generation probability $p_G$, consumption probability $p_d$, and successful delivery probability $q$ are all unknown, one can use Q-learning algorithm (Algorithm 1) to obtain the optimal solution. For the purpose of display, the convergence of the algorithm is depicted at state $(x, y) = (0, 50)$ in Figure 2. It is shown that $\min_{a \in A} Q(0, 50, a)$ converges to the optimal value function $V(0, 50) = 160.83$. In this example, the Q-learning algorithm (Algorithm 1) is trained offline, where at each training sample a batch update is performed over the entire state-action pairs (Q-functions), also known as synchronized parallel Q-learning, with step sizes inversely proportional to the number of visits (updates) to each pair of state and action. A Mac Pro laptop with 2.7 GHz Intel Core i5 runs the known model case in 250 seconds and unknown model case in 6 hours.

B. A communication network with packet drop

Inspired by recent developments in networked control systems [31–33] and deep teams [3], [4], we consider a networked remote estimation problem for a deep linear quadratic control system, where $n$ networked controllers use a deep structured optimal state-feedback strategy. In this case, the dynamics of the deep state (weighted average of the states of the controllers) is in the form of (9). Now, consider an authority (e.g., an independent service operator in a smart grid) wishing to collect the deep state, and transmit it over an unreliable channel, and eventually receive it at a decoder, located far away from the control site. The objective is to find an efficient way to construct a reliable estimate at the decoder with minimum possible collection cost, while taking into account the topology of the network and unreliability of the transmission (formulated as packet drop). A block diagram of the above system is displayed in Figure 3.

Example 2. Consider two topologies, a complete graph and a star graph, whose dominant eigenvalues of their Laplacians are $(n - 1)$ and $n \sqrt{1 - \frac{1}{n}}$, respectively. Suppose $L = 1$, $\alpha(0) = 0$, and $\alpha(1) = \frac{A}{\sqrt{n-1}}$, $A \in \mathbb{R}$, in the vectorized dynamics (8), where $v^C = \text{vec}(1, \ldots, 1)$ and $v^S = \sqrt{n-1} \text{vec}(\sqrt{1 - \frac{1}{n}})$. The dominant eigenvalues of the complete and star graphs, respectively. Hence, the dynamics of the weighted average of the dominant mode of the complete graph is described by:

$$m_{t+1}^C = Am^C_t + \bar{w}_t^C,$$

where $m_t^C = \frac{1}{n} \sum_{i=1}^n s_t^i$ and $w_t^C = \frac{1}{n} \sum_{i=1}^n w_t^i$. Similarly, the dynamics of the dominant mode of the star graph is given by:

$$m_{t+1}^S = \frac{1}{n} \sum_{i=1}^n v^S(i)s_t^i + \bar{w}_t^S,$$

where $m_t^S = \frac{1}{n} \sum_{i=1}^n v^S(i)s_t^i = \frac{1}{\sqrt{2n}} s_t^1 + \frac{1}{\sqrt{2n}} \sum_{i=2}^n s_t^i$ and $w_t^S = \frac{1}{\sqrt{2n}} w_t^1 + \frac{1}{\sqrt{2n}} \sum_{i=2}^n w_t^i$ with $s_t^i$ and $w_t^i$ denoting the state and local noise of the central node. The per-step cost function is given by (14) under Assumption 6. In addition, suppose that the probability that the data are dropped is $1 - q = 0.1$ and the discount factor is $\gamma = 0.85$. The local noises are i.i.d. random variables with zero mean (that are not necessarily Gaussian or symmetric with unimodal distribution).

From Theorem 8 one can find a near-optimal strategy for a sufficiently large approximation index $k \in \mathbb{N}$. The optimal sampling strategy is shown in Figure 4 with respect to the number of nodes $n$ and the variance of local noises $\Sigma^n$. In addition, the optimal estimate is constructed at the decoder based on the result of Theorem 7. Since the data in a complete graph spread faster than a star graph, it is observed in Figure 4 that the data must be sampled more frequently in the complete graph, as the variance of noise increases. Also, according to Figure 4, the certainty threshold of the complete
maximum a posteriori probability (MAP) estimator to estimate the percentage of opinions from its previously conducted surveys, i.e.,
\[
\hat{m}_t = \arg\max_{m \in \mathcal{M}(n)} (P(m_t = m | o_{1:t}, a_{1:t})).
\] (32)

Given a discount factor \( \gamma \in (0, 1) \), the objective is to minimize the following expected total discounted cost:
\[
J = \mathbb{E}\left[ \sum_{t=1}^{\infty} \gamma^{t-1} (D_{KL}(m_t || \hat{m}_t) + \ell a_t) \right],
\]
where \( D_{KL}(m_t || \hat{m}_t) = \sum_{s \in \mathcal{S}} m_t(s) \log \frac{m_t(s)}{\hat{m}_t(s)} \) denotes the Kullback–Leibler divergence. At any time \( t \in \mathbb{N} \), let \( x_t \in \mathcal{M}(n) \) be the empirical distribution of the opinions collected at the last survey and \( y_t \in \mathbb{N}^n \) be the elapsed time after the last survey. From (18) and (32), \( \hat{m}_t = \arg\max_{m \in \mathcal{M}(n)} T_m(m, x_t) \), where the transition probability matrix \( T_m \) is given by Theorem 1. A near-optimal strategy can be obtained by Theorem 4 for a sufficiently large \( k \).

**Example 3.** Consider an election between two candidates \( A \) and \( B \), i.e., \( \mathcal{S} = \{A, B\} \). An agency is interested to conduct polls among \( n \in \mathbb{N} \) voters. With a slightly abuse of notation, let \( m_t := m_t(A) \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\} \) represent the empirical distribution of the voters who prefer candidate \( A \) at time \( t \in \mathbb{N} \). Since the state space \( \mathcal{S} \) is binary, \( m_t \) is sufficient for identifying the empirical distribution of the voters favoring candidate \( B \), i.e., 1 - \( m_t \). Note that the empirical distribution \( m_t \) takes \( n+1 \) different values, and its belief state is \( \mathbb{P}(m_t(A) | x_t) \in \mathbb{R}^{n+1} \). In contrast, our proposed planning space is a discrete space with \((n+1) \times k\) values, which is a considerable reduction with respect to \( n \). Let the number of voters be \( n = 50 \). In addition, let \( \mathbb{P}(s_{t+1} = A | s_t = A) = 0.95 \) be the probability that voter \( i \in \mathbb{N}_n \) chooses candidate \( A \) at the next time instant if this is currently the voter’s favorite candidate. Similarly, \( \mathbb{P}(s_{t+1} = B | s_t = B) = 0.98 \) is the probability that voter \( i \) chooses candidate \( B \) at the next time instant if \( B \) is the voter’s current choice. Suppose that the cost of running a poll is \( \ell = 0.02 \), the discount factor is \( \gamma = 0.8 \), and the probability of the credibility of a poll is \( q = 0.95 \). Exhaustive numerical simulations demonstrate that the optimal solution is attained for any approximation index \( k \geq 50 \). The optimal strategy is displayed in Figure 5 which demonstrates that when the number of voters in favor of candidate \( A \) is \( 45 \), the agency should run a poll after the elapsed time since the latest credible observation exceeds 10. When, on the other hand, the number of voters in favor of candidate \( B \) is \( 45 \), the agency should wait slightly longer (11 time instants) before conducting a new poll. This difference is due to the fact that voters are more likely to change their opinion if candidate \( A \) is their current choice. The simulation time is 116 seconds for a computer with the specification described in Example 1.

**IX. CONCLUSIONS**

In this paper, the trade-off between data collection and estimation in networks with both known and unknown models.

\[\text{For } \epsilon = 10^{-5}, \text{inequality } (23) \text{ holds for any } k \geq 50.\]
Fig. 5. Optimal strategy for conducting polls in Example 3. The unsmooth surface of the solution is due to the fact that the MAP estimator is not smooth.

was investigated. Some important properties of the dynamics of data were studied first, and an ε-optimal solution was subsequently provided using the Bellman equation. The proposed solution was then extended to the case where the model is not completely known using two different learning-based approaches. It was also shown at what point estimating data tends to be more desirable than collecting data as the number of nodes increases. The special case of linear dynamics was studied in more detail and a separation principle was presented accordingly. Three numerical examples were provided to demonstrate the effectiveness of the proposed strategies.

For future research directions, it would be interesting to study the computational complexity of the proposed approach under various approximation methods such as stochastic approximation, linearization, particle filtering, quantization, randomization, and Monte-Carlo simulation as well as different simplifying, yet realistic, assumptions such as ergodicity and the myopicity of the decision process. In particular, the reader is referred to [26] and references therein for various stochastic numerical methods that can be used to obtain a more efficient way of computation, albeit at the cost of losing the performance guarantee.

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APPENDIX A

PROOF OF THEOREM 1

From (1) and (2) it follows that for any \( s' \in S, t \in N, \)
\[
n \cdot m_{t+1}(s') = \sum_{i=1}^{n} \mathbb{1}(s'_{i+1} = s') = \sum_{i=1}^{n} \mathbb{1}(f(s'_t, m_t, w'_t) = s').
\]
\[
= \sum_{s' \in S} \sum_{i=1}^{n} \mathbb{1}(s'_i = s) \mathbb{1}(f(s, m_t, w'_t) = s').
\]
(33)

For every state \( s \in S \), the inner summation in equation (33) has \( n(1 - m_s(s)) \) zero terms because there are only \( nm_s(s) \) terms with state \( s \) according to the definition of empirical distribution (1). These \( nm_s(s) \) possibly non-zero terms are independent binary random variables according to Assumption 2 with the following success probability:
\[
\mathbb{P}(f(s, m_t, w'_t) = s') = 1 \sum_{w' \in W} P_{W}(w'_t = w) \mathbb{1}(f(s, m_t, w) = s') = T(s', s, m_t).
\]
Let \( \phi_{m}(s', s, m_t) \) denote the probability distribution function of the sum of these \( nm_s(s) \) Bernoulli random variables, which is a binomial distribution with \( nm_s(s) \) trials and success probability \( T(s', s, m_t) \). Now, the probability distribution function of \( nm_{t+1}(s') \) is the probability distribution function of the outer summation in equation (33) consisting of \( |S| \) independent random variables (due to Assumption 2), each of which has the probability distribution function \( \phi_{m}(s', s, m_t) \). Therefore, the probability distribution of \( nm_{t+1}(s') \) can be expressed as the convolution of the probability distribution functions \( \phi_{m}(s', s, m_t) \) over space \( S \), denoted by \( \phi(s', m_t) \).

APPENDIX B

PROOF OF LEMMA 2

One can write:
\[
\mathbb{P}(\text{a}_{t+1} \mid x_{1:t}, y_{1:t}, a_{1:t}) = \sum_{m'} \mathbb{P}(m_{t+1} = m' \mid x_{1:t}, y_{1:t}, a_{1:t}) \times \mathbb{P}(\text{a}_{t+1} \mid m_{t+1} = m', x_{1:t}, y_{1:t}, a_{1:t}).
\]
(34)

From (3) and (4), the second term in the right-hand side of equation (34) is given by:
\[
\mathbb{P}(\text{a}_{t+1} \mid m_{t+1} = m', x_{1:t}, y_{1:t}, a_{1:t}) = a_t q \mathbb{1}(\text{a}_{t+1} = m') + (1 - a_t) q \mathbb{1}(\text{a}_{t+1} = \text{blank}).
\]
(35)

The first term in the right-hand side of equation (34) is:
\[
\mathbb{P}(m_{t+1} = m' \mid x_{1:t}, y_{1:t}, a_{1:t}) = \sum_{m \in M(n)} \mathbb{P}(m_{t+1} = m' \mid m_t = m, x_{1:t}, y_{1:t}, a_{1:t}) \times \mathbb{P}(m_t = m \mid x_{1:t}, y_{1:t}, a_{1:t}).
\]
(36)

where (a) follows from Proposition 1 the fact that \( w_t \) is independent of the information up to time \( t \), i.e.
The proof consists of two parts. In the first part, we define a "virtual" finite-state Markov decision process (MDP), and in the second part, we show that the optimal solution of this MDP is an ε-optimal solution of Problem 1.

Part 1: For any $t \in \mathbb{N}$ and finite $k \in \mathbb{N}$, define a so-called "virtual" finite-state MDP with state $(\tilde{x}_t, \tilde{y}_t) \in \mathcal{M}(n) \times \mathbb{N}_k^*$ and action $\tilde{a}_t \in \mathcal{A}$ as well as the initial state $(\tilde{x}_1, \tilde{y}_1) = (m_1, 0)$. At time $t \in \mathbb{N}$, state $(\tilde{x}_t, \tilde{y}_t)$ evolves according to function $\tilde{f} : \mathcal{M}(n) \times \mathbb{N}_k^* \times \mathcal{O} \to \mathcal{M}(n) \times \mathbb{N}_k$ as follows: $(\tilde{x}_{t+1}, \tilde{y}_{t+1}) = \tilde{f}(\tilde{x}_t, \tilde{y}_t, \tilde{a}_t)$ such that

$$\tilde{f}(\tilde{x}_t, \tilde{y}_t, \tilde{a}_t) := \begin{cases} (\tilde{x}_t, \tilde{y}_t + 1), & \tilde{a}_t = \text{blank}, \tilde{y}_t < k, \\ (m^*, 0), & \tilde{a}_t = \text{blank}, \tilde{y}_t = k, \\ (\tilde{a}_t, 0), & \tilde{a}_t \neq \text{blank}, \end{cases}$$

(37)

where $\tilde{a}_t \in \mathcal{O}$ is a noise process and $m^* \in \mathcal{M}(n)$ is an arbitrary empirical distribution. The probability distribution of the noise $\tilde{a}_t$ is identical to that of the observation in the original model (19), i.e., $P(\tilde{a}_{t+1} \mid \tilde{x}_{t+1}, \tilde{y}_{t+1}, \tilde{a}_t) = (1 - a_t q)\sum(\tilde{a}_{t+1} = \text{blank}) + a_t q T_{m+1}^{\tilde{a}_t}(\tilde{x}_t, \tilde{y}_t)\sum(\tilde{a}_{t+1} \neq \text{blank})$. The per-step cost of the virtual model introduced above is the restriction function $\hat{c}$ given by (20), over space $\mathcal{M}(n) \times \mathbb{N}_k^* \times \mathcal{A}$, i.e., at time $t \in \mathbb{N}$, $c(\tilde{x}_t, \tilde{y}_t, \tilde{a}_t) := \sum_{m \in \mathcal{M}(n)} c(m, h(T_{m}^{\tilde{a}_t}(m, \tilde{x}_t)), a_t)T_{m}^{\tilde{a}_t}(m, \tilde{x}_t)$. The strategy of the virtual MDP is given by $\tilde{\pi}(\tilde{x}_t, \tilde{y}_t, \tilde{a}_t) = (\tilde{x}_t, \tilde{y}_t, \tilde{a}_t)$, and its performance is described by $J(\tilde{f}) = E[\sum_{t=1}^\infty \gamma^{t-1} \hat{c}(\tilde{x}_t, \tilde{y}_t, \tilde{a}_t)]$. From the standard results in Markov decision theory (25), the optimal solution of the virtual model is obtained by solving the following Bellman equation for any $\tilde{x} \in \mathcal{M}(n) , \tilde{y} \in \mathbb{N}_k^*$:

$$\tilde{V}_k(\tilde{x}, \tilde{y}) = \min_{\tilde{a} \in \mathcal{A}}(\tilde{c}(\tilde{x}, \tilde{y}, \tilde{a}) + \gamma E[\tilde{V}_k(\tilde{f}(\tilde{x}, \tilde{y}, \tilde{a}))]) ,$$

(38)

where the above expectation is taken over all noises $\tilde{o} \in \mathcal{O}$.

Part 2: Let $J^*$ be the performance under the optimal solution of Bellman equation (21) and $\tilde{J}^*$ be the performance under the optimal solution of Bellman equation (38). We compute an upper bound on the relative distance between $J^*$ and $\tilde{J}^*$, i.e., $|J^* - \tilde{J}^*|$, as follows:

$$\min_g \mathbb{E}[\sum_{t=1}^\infty \gamma^{t-1} \hat{c}(x_t, y_t, a_t) - \min_g \mathbb{E}[\sum_{t=1}^\infty \gamma^{t-1} \hat{c}(\tilde{x}_t, \tilde{y}_t, \tilde{a}_t)]]$$

$$= \min_g \mathbb{E} \left[ \sum_{t=1}^k \gamma^{t-1} \hat{c}(x_t, y_t, a_t) + \sum_{t=k+1}^\infty \gamma^{t-1} \hat{c}(x_t, y_t, a_t) \right]$$

$$- \min_g \mathbb{E} \left[ \sum_{t=1}^k \gamma^{t-1} \hat{c}(\tilde{x}_t, \tilde{y}_t, \tilde{a}_t) - \sum_{t=k+1}^\infty \gamma^{t-1} \hat{c}(\tilde{x}_t, \tilde{y}_t, \tilde{a}_t) \right]$$

(39)

(40)

where $\gamma$ follows from the triangle inequality, the fact that $c_{\max}$ is an upper-bound for the per-step cost and the minimization of the expected cost of the original model and that of the virtual model up to time $k$ are essentially the same as both models start from identical initial state $(m_1, 0)$, follow the same dynamics, and incur the same cost up to time $k$. Finally, it is concluded from Parts 1 and 2 that when $k$ is sufficiently large such that $2\gamma^k c_{\max} \leq \epsilon$, the optimal solution of the Bellman equation (38) is an $\epsilon$-optimal solution of Problem 1. The proof is completed by incorporating equations (37) in the Bellman equation (38).
function \( \tilde{f} \) is independent of the model; (ii) the expectation and variance of \( o(x, y, a) \) are respectively zero and finite, and (iii) equation (40) is a contraction mapping in the infinity norm due to the discount factor \( \gamma < 1 \), i.e., for any \( Q \) and \( Q' \):
\[
\| F(Q) - F(Q') \| \leq \gamma \max_{x', y', a'} |Q((x', y'), (a', a')) - Q'((x', y'), (a', a'))|,
\]
where \( F \) denotes the function form of equation (40) such that \( Q^* = F(Q^*) \). Therefore, the following stochastic approximation iteration converges to \( Q^* \) under standard assumptions in [41] Theorem 4], i.e., for \( \tau \in \mathbb{N} \),
\[
Q_{\tau+1}(x, y, a) = Q_\tau(x, y, a) + \alpha_\tau(x, y, a)(c' + \gamma \min_{a' \in A} Q_\tau(f(x, y, a), a')) - Q_\tau(x, y, a).
\]
To avoid repetition, the reader is referred to [41] Theorem 4 for the convergence proof of the Q-learning algorithm proposed in Algorithm 1. The proof is completed by noting that the obtained greedy strategy \( g_\varepsilon \) is an \( \varepsilon \)-optimal solution, according to Theorem 4.

**APPENDIX F**

**PROOF OF THEOREM 5**

The total expected discounted cost for any finite horizon \( H \in \mathbb{N} \), under no collection action is given by:
\[
\mathbb{E}\left[ \sum_{t=1}^{H} \gamma^{t-1}c(m_t, \hat{m}_t, 0) \right] \leq \mathbb{E}\left[ \sum_{t=1}^{H} K_c \gamma^{t-1} \| m_t - \hat{m}_t \| \right] + K_c \sum_{t=2}^{H} \gamma^{t-1} \mathbb{E}\left[ K_p \gamma^{t-1} \| m_1 - \hat{m}_1 \| \right] + \gamma \sum_{t=1}^{H-1} K_p \gamma^{t-1} \mathbb{E}\left[ K_p \gamma^{t-1} \| m_1 - \hat{m}_1 \| \right] + O\left( \frac{1}{\sqrt{n}} \right).
\]

\[
\mathbb{E}\left[ \sum_{t=1}^{H} \gamma^{t-1}c(m_t, \hat{m}_t, 0) \right] \leq K_c \sum_{t=2}^{H} \gamma^{t-1} \mathbb{E}\left[ K_p \gamma^{t-1} \| m_1 - \hat{m}_1 \| \right] + K_c \sum_{t=2}^{H} \gamma^{t-1} \mathbb{E}\left[ K_p \gamma^{t-1} \| m_1 - \hat{m}_1 \| \right] + \gamma \sum_{t=1}^{H-1} K_p \gamma^{t-1} \mathbb{E}\left[ K_p \gamma^{t-1} \| m_1 - \hat{m}_1 \| \right] + O\left( \frac{1}{\sqrt{n}} \right).
\]

\[
= K_c \sum_{t=1}^{H} \gamma^{t-1} \mathbb{E}\left[ K_p \gamma^{t-1} \| m_1 - \hat{m}_1 \| \right] + \gamma \sum_{t=1}^{H-1} K_p \gamma^{t-1} \mathbb{E}\left[ K_p \gamma^{t-1} \| m_1 - \hat{m}_1 \| \right] + \gamma \sum_{t=1}^{H-1} K_p \gamma^{t-1} \mathbb{E}\left[ K_p \gamma^{t-1} \| m_1 - \hat{m}_1 \| \right] + O\left( \frac{1}{\sqrt{n}} \right).
\]

where (a) follows from Assumption 3 and the monotonicity of the expectation operator; (b) follows from Lemma 5 by applying it recursively; (c) follows from the fact that \( m_1 = \hat{m}_1 \); (d) rearranges the terms; (e) follows from the fact that \( \gamma K_p \) is non-negative along with the inequality \( t \leq H \), and (f) follows from Assumption 4. The proof is completed by tending horizon \( H \) to \( \infty \).

**APPENDIX G**

**PROOF OF THEOREM 7**

Given any strategy \( g \), the information set \( \{ o_{t+1}, a_{t+1} \} \) can be equivalently expressed by the set \( \{ x_{1:t}, y_{1:t} \} \), according to (5) and (17). Hence, the generic estimator \( h \) can be represented by a strategy-dependent estimator \( \hat{m}_t = h^g(x_{1:t}, y_{1:t}) \). From (9) and the definition of the set \( \{ x_{1:t}, y_{1:t} \} \), one has:
\[
m_t = A^{y_t} x_t + \mathbb{1}(y_t > 0) \sum_{\tau=1}^{t} A^{\tau-1} \hat{w}_{t-\tau}.
\]

Since local noises from time \( t - y_t \) to time \( t \) have zero-mean and are mutually independent over time (and so is \( \hat{w}_{t-\tau} \)), the per-step cost function can be described as follows:
\[
\mathbb{E}[((m_t - \hat{m}_t)^T(m_t - \hat{m}_t)z(a_t) + \ell(m_t, a_t) | x_{1:t}, y_{1:t}, g_{1:t}) = \mathbb{E}[(A^{y_t} x_t - h^g(x_{1:t}, y_{1:t}))^T(A^{y_t} x_t - h^g(x_{1:t}, y_{1:t})) \times z(g_t(x_{1:t}, y_{1:t})) + \mathbb{1}(y_t > 0)z(g_t(x_{1:t}, y_{1:t})) \sum_{\tau=1}^{t} \mathbb{E}[\hat{w}_{t-\tau})^T(A^{\tau-1})^T A^{\tau-1} \hat{w}_{t-\tau}]
\]

\[
+ \mathbb{E}[\ell(A^{y_t} x_t + \mathbb{1}(y_t > 0) \sum_{\tau=1}^{t} A^{\tau-1} \hat{w}_{t-\tau}, g_t(x_{1:t}, y_{1:t}))].
\]

Therefore, for any control horizon \( H \), any sample path \( \{ x_{1:H}, y_{1:H} \} \) and any strategy \( g_{1:H} \), the first term in the right-hand side of the above equation is the only term that is affected by the choice of \( h^g \), which yields the unique minimizer \( h^g(x_{1:t}, y_{1:t}) = A^{y_t} x_t \). Note that the structure of this estimator is independent of the probability distribution of the underlying random variables, strategy \( g \), and the order of dynamics, and has the following update rule: \( \hat{m}_{t+1} = A^{y_{t+1}} x_{t+1} \), i.e., equation (27). This estimator has the same structure as the minimum mean-square estimator \( \mathbb{E}[m_t | o_{1:t}, a_{1:t-1}] \). Hence,
\[
\hat{m}_{t+1} = \begin{cases} x_{t+1} = o_{t+1}, & o_{t+1} \neq \text{blank} \text{ (i.e., } y_{t+1} = 0), \\ A^{1+y_{t+1}} x_t = A \hat{m}_t, & o_{t+1} = \text{blank} \text{ (i.e., } y_{t+1} > 0). \end{cases}
\]