Field Tensor Network States

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We define a class of tensor network states for spin systems where the individual tensors are functionals of fields. The construction is based on the path integral representation of correlators of operators in quantum field theory. These tensor network states are infinite dimensional versions of matrix product states and projected entangled pair states. We find the field-tensor that generates the Haldane-Shastry wave function and extend it to two dimensions. We give evidence that the latter underlies the topological chiral state described by the Kalmeyer-Laughlin wave function.

Tensor networks (TN) are becoming a key tool to describe many-body quantum systems [1]. On the one hand, they can efficiently approximate quantum states of local Hamiltonians in thermal equilibrium, which has led to powerful numerical algorithms with applications in condensed matter and, to some extent, in high-energy physics [2]. On the other hand, they provide us with paradigmatic examples of strongly correlated states and thus allow us to investigate intriguing many-body quantum phenomena. For instance, they offer us a guide to classify symmetry protected topological phases [3, 4], or to understand a large variety of topologically ordered behavior. In fact, states (or models) like the AKLT [5], string-net states [6], or resonating valence-bond states have a very simple description in terms of TN. By simple we mean with a small bond dimension, $D$, which limits the number of coefficients describing the tensors generating the many-body states. The description of such states in terms of TN automatically opens up the possibility of using powerful tools in order to describe their physical properties by just inspecting a simple tensor. In 1D, one can easily describe symmetries and string order parameters [7], or even gauge symmetries [8]. In 2D, apart from obtaining the physical symmetries, one can directly identify the topological properties or type of anyon excitations of the parent Hamiltonian [9].

There exist, however, some classes of states for which no exact expressions in terms of tensor network states of finite bond dimensions exist. Two prominent examples are critical states [10], and chiral topological states of gapped Hamiltonians in one and two dimensional spin lattices, respectively [11, 12]. The reason behind the lack of description as TN for the first stems from the fact that critical states violate the area law [13, 14]. Specifically, the entanglement entropy of a connected region containing $L$ spins scales as $\propto \ln(L)$ [15, 16], whereas for a matrix product state (MPS), the one-dimensional version of tensor network states, it is bounded by $2\ln(D)$; therefore, in the thermodynamic limit for any finite $D$, there always exists some $L$ for which an MPS cannot cope with the amount of entanglement and thus it is impossible that it describes a critical state. The reason for the second class is more subtle and not fully understood; however, there are good reasons to believe that there exist obstructions due to the non-existence of local Wannier states [17] (see, however [18]). In fact, for Gaussian fermionic states, it is not possible to describe gapped chiral topological insulators [19]. We emphasize that here we mean an exact description; in fact, both classes of states may well be approximated efficiently with an error that decreases as $D$ increases [20–22].

The arguments above do not prevent the existence of exact descriptions of critical or chiral topological states with TN of infinite bond dimensions. In [23], it was noted that the conformal field theory (CFT) formulation [24] of the Haldane-Shastry state has similarities with MPS, and in [20–22], the CFT formulation was used to obtain MPS with a discrete, infinite bond dimension describing chiral topological states in 2D. From the tensor network perspective it is, however, desirable to use projected entangled pair states (PEPS) to deal with 2D systems. Furthermore, although the approach of [20–22] can, in principle, be used to describe critical states in 1D with open boundary conditions, it is more appropriate to use periodic boundary conditions for translationally invariant systems.

In this letter we define Field Tensor Networks (FTN) for spin lattices in any dimension, where the bonds in the tensors are functions, the corresponding contractions are accomplished by a path integration, and the tensors themselves are functionals. The virtual space is hence continuous. We show how this approach can be used to describe translationally invariant critical systems, as well as the analogs of PEPS for two dimensional systems. Our construction is reminiscent of recent proposals for constructing tensor networks for quantum fields, where path integration is also employed [25, 26]. In our case, however, we deal with discrete spin lattices and the con-

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1 Even though those do not possess topological order, but rather belong to a symmetry protected topological phase, they constitute a strong evidence for the impossibility of describing chiral topological phases corresponding to gapped local Hamiltonians.
struction is quite different. We give a procedure to compute the FTN for states whose coefficients in the spin basis can be written as vacuum correlators of a quantum field theory with a local action. In particular, we give an explicit construction for free boson CFTs and vertex operators. We also take advantage of the fact that the Haldane-Shastry state \([27, 28]\), a prominent critical state, can be expressed in that form \([23, 29]\) to compute a FTN generating that state. The description allows both periodic and open boundary conditions. We also propose a FTN in 2D and give strong evidence that it represents a Kalmeyer-Laughlin state \([11]\), a prototypical representative of chiral topological order.

**FTN in 1D:** We consider a spin chain of \(N\) spins of dimension \(d\), and a translationally invariant state

\[
|\Psi\rangle = \sum_{s_1, \ldots, s_N=1}^{d} c_{s_1, \ldots, s_N}|s_1, \ldots, s_N\rangle. \tag{1}
\]

Let us start recalling MPS, where

\[
c_{s_1, \ldots, s_N} = \sum_{n_1, \ldots, n_N=1}^{D} A_{n_1, n_2}^{s_1} \cdots A_{n_N, n_1}^{s_N}, \tag{2}
\]

\(D\) is the bond dimension, and for each value of \(s = 1, \ldots, d\), \(A_{n_1, n_2}^{s}\) is a \(D \times D\) matrix. In an analogous way we define (translationally invariant) Field Tensor Network States (FTNS) as

\[
c_{s_1, \ldots, s_N} = \int D[\alpha_1] \cdots D[\alpha_N] A_{1,n_2}^{s_1} \cdots A_{n_N,n_1}^{s_N}. \tag{3}
\]

Here, \(\alpha_n : \mathbb{R} \to \mathbb{R}\) belong to the set of square-integrable functions and also include a constant function, and for each value of \(s = 1, \ldots, d\), \(A_{n_1, n_2}^{s}\) are functionals of \(\alpha_1, \alpha_2\). Note that \(\Psi\) has the same structure as a MPS, where the indices of the matrices are replaced by the functions \(\alpha_n\), and the sum over repeated indices is replaced by a path integral.

We also define the functionals

\[
A_{\alpha_1, \alpha_2}^{s_1, \ldots, s_n} = \int D[\alpha_2] \cdots D[\alpha_n] A_{\alpha_2, \alpha_3}^{s_1} \cdots A_{\alpha_n, \alpha_1}^{s_n}. \tag{4}
\]

fulfilling

\[
A_{\alpha_1, \alpha_2}^{s_1, \ldots, s_{n+1}} = \int D[\alpha_2] \cdots D[\alpha_n] A_{\alpha_2, \alpha_3}^{s_1} \cdots A_{\alpha_n, \alpha_1}^{s_{n+1}}. \tag{5a}
\]

\[
c_{s_1, \ldots, s_N} = \int D[\alpha_1] \cdots D[\alpha_N] A_{\alpha_1, \alpha_2}^{s_1} \cdots A_{\alpha_N, \alpha_1}^{s_N}, \tag{5b}
\]

which we will call “sewing” and “closing” conditions.

**Example:** Particularly interesting examples of FTNS are those for which the coefficient can be written in terms of correlators of a simple CFT in \(1+1\) dimensions and a local action. Specifically, we shall here study a family of critical states in 1D with

\[
c_{s_1, \ldots, s_N} \propto \langle \chi_{s_1} \cdot e^{iqs_1\varphi(r_1)} \cdots \chi_{s_N} \cdot e^{iqs_N\varphi(r_N)} \rangle_0. \tag{7}
\]

where \(\varphi\) is a real scalar field defined on a cylinder of circumference \(\pi N\), \(r_n = (x_n, 0)\) are points in cylindrical coordinates (see Fig. 1), and the expectation value is taken in the vacuum. In this case, using the path integral representation we have

\[
c_{s_1, \ldots, s_N} \propto \int D[\varphi] e^{-S} e^{iq \sum_{n=1}^{N} s_n \varphi(r_n)} \prod_n \chi_{s_n}, \tag{8}
\]

where

\[
S = \frac{1}{8\pi} \int_0^{\pi N} dx \int_{-\infty}^{\infty} dt \{[\partial_x \varphi(x, t)]^2 + [\partial_t \varphi(x, t)]^2\} \tag{9}
\]

is the Euclidean action of the boson field. Notice that (9) vanishes if \(\varphi\) is a constant \(\varphi_0\) which upon integration generates the constraint \(\sum_n s_n = 0\) appearing in (6).

In order to find the FTNS representation of (7), we...
where

\[ P \in \text{interval } (\varphi) \]

chiral states, and as a test case, we next consider a chiral \( \alpha \) with elements

\[ \hat{\alpha}, \hat{S} \]

\[ \text{normalizability. Here } U_L, U_R, \]

\[ n \]

\[ \text{coth } \left( \frac{L}{t} \right) \]

\[ \| \alpha \| \]

\[ \| \alpha \| \]

\[ \hat{\alpha}(t) \]

\[ \hat{v}_{\alpha,\alpha'} \]

\[ \text{chiral version:} \]

\[ \text{We aim at also being able to describe} \]

\[ \text{chiral vertex operators coincides with (6) except that} \]

\[ \sqrt{2q} \]

\[ \text{is replaced by } q. \]

\[ \text{This correlator can be written as in} \]

\[ (8) \]

\[ \text{with a chiral action} [32] \text{employed to study the edge} \]

\[ \text{excitations in the Quantum Hall effect [33]. However,} \]

\[ \text{the slicing of the path integral into the intervals} \]

\[ \text{introduces boundaries that mix the} \]

\[ \text{left and right moving modes of the bosonic field, which} \]

\[ \text{in turn complicates the approach.} \]

\[ \text{We notice, however, that in (16) there are two parts} \]

\[ \text{related by complex conjugation. Moreover, (14) comes} \]

\[ \text{from a Green function with four terms where only one of} \]

\[ \text{them is analytic in the location of the vertex operators.} \]

\[ \text{It is therefore natural to expect that one obtains the} \]

\[ \text{chiral state by selecting only one of those parts. We shall} \]

\[ \text{use this property to define the new tensors} \]

\[ \hat{A}_{\alpha,\alpha'}^{x_1,\ldots,x_L} = e^{\frac{iq}{2} \varphi(x_0)} \sum_j s_j e^{S_L^{(0)}(x_j) + S_L^{(1)} + S_L^{(2)}} \prod_{n=1}^{L} \chi_{\alpha}(z_n), \]

\[ \text{where} \]

\[ S_L^{(0)} = 2q^2 \sum_{L \geq n > m \geq 1} s_n s_m \ln \frac{\sin((x_n - x_m)/(2L))}{\sin((x_n + x_m)/(2L))} \]

\[ S_L^{(1)} = \frac{1}{64\pi^2} \int_\mathbb{R} dt dt' \hat{\alpha}(t) T U_L(t - t') \hat{\alpha}(t') \]

\[ S_L^{(2)} = \frac{q}{4\pi} \int_\mathbb{R} \sum_{n=1}^{L} s_n \hat{\alpha}(t) \left( T \hat{v}_{L,n}(t), \hat{v}_{L,n}(t) \right) \]

and \( \varphi_0 \) is the zero mode of the boson field that has been subtracted from the functions \( \alpha, \alpha' \) to guarantee their normalizability. Here \( \alpha = (\alpha, \alpha') \), \( U \) is a 2 \times 2 matrix with elements

\[ U_{L,11} \text{ or } 22(t) = \frac{2}{L^2} \left[ \frac{1}{\sinh^2 \left( \frac{t}{2L} \right)} - \left( \frac{2L}{t} \right)^2 \right] - 8P' \left( \frac{1}{7} \right) \]

\[ U_{L,12} \text{ or } 21(t) = \frac{2}{L^2 \cosh^2 \left( \frac{t}{2L} \right)} \]

and

\[ \hat{v}_{L,n}(t) = \frac{1}{L} \left( \coth \left( \frac{t - ix_n}{2L} \right), - \tanh \left( \frac{t - ix_n}{2L} \right) \right) \]

where \( P' \left( \frac{1}{7} \right) = -\frac{1}{2} \left( \frac{1}{1 + 2m^2} + \frac{1}{1 - 2m^2} \right) \) is the derivative of the principal value distribution \( P \left( \frac{1}{7} \right) \).

Chiral version: We aim at also being able to describe chiral states, and as a test case, we next consider a chiral formulation of the critical states (6). In this formulation the states are defined in terms of a chiral free boson field \( \varphi(z) \), which depends on \( z \), but not on its conjugate \( \bar{z} \). The states are again given by (7), except that the vertex operators now take the form: \( e^{iq_n \varphi(z_n)} \); where \( q \in \mathbb{R} \) and \( z_n = t + ix_n \) (the wave function obtained with these chiral vertex operators coincides with (6) except that \( \sqrt{2q} \) is replaced by \( q \)). This correlator can be written as in (8) with a chiral action [32] employed to study the edge excitations in the Quantum Hall effect [33]. However, the slicing of the path integral into the intervals \( (x, t) \in [x_n - \delta, x_n + \delta] \times \mathbb{R} \) introduces boundaries that mix the left and right moving modes of the bosonic field, which in turn complicates the approach.

We notice, however, that in (16) there are two parts related by complex conjugation. Moreover, (14) comes from a Green function with four terms where only one of them is analytic in the location of the vertex operators. It is therefore natural to expect that one obtains the chiral state by selecting only one of those parts. We shall use this property to define the new tensors

\[ \bar{A}_{\alpha,\alpha'}^{x_1,\ldots,x_L} = e^{\frac{iq}{2} \varphi(x_0)} \sum_j s_j e^{S_L^{(0)} + S_L^{(1)} + S_L^{(2)}} \prod_{n=1}^{L} \chi_{\alpha}(z_n), \]

where

\[ \bar{S}_L^{(0)} = q^2 \sum_{L \geq n > m \geq 1} s_n s_m \ln \left[ 2 \sin \left( \frac{x_n - x_m}{2L} \right) \right], \]

\[ \bar{S}_L^{(2)} = \frac{q}{4\pi} \int_\mathbb{R} \sum_{n=1}^{L} s_n \hat{\alpha}(t) \hat{v}_{L,n}(t) \]
transforms the rectangle \([x, x_n] \times \gamma_{n,m}(x)\) into the complex upper-half plane in terms of Jacobi elliptic functions. We then carry out the path integral in (22). The main results are (the complete matrix is given in the supplemental material [31]).

\[
A_{\alpha, \beta, \gamma, \delta}^{s} = e^{S^{(1)} + S^{(2)}} \tag{24}
\]

where

\[
S^{(1)} = \frac{1}{64\pi^2} \int_{\mathcal{I}} d\xi d\xi' \tilde{\alpha}(\xi)U(\xi, \xi')\tilde{\alpha}(\xi')^T, \tag{25}
\]

\[
S^{(2)} = \frac{qs}{4\pi} \int_{\mathcal{I}} d\xi \tilde{\beta}(\xi)\tilde{\beta}^T - \tilde{\beta}(\xi)^T, \tag{26}
\]

with \(\tilde{\alpha} = (\alpha, \beta, \gamma, \delta)\), \(\xi_1 = \xi_2 = t\) and \(\xi_3 = \xi_4 = x\). The integration domain \(\mathcal{I}\) is adapted to the type of variables involved. \(U\) is a 4 \times 4 matrix some of whose elements are (the complete matrix is given in the supplemental material [31])

\[
U_{11}(t, t') = 8 \left( \frac{\text{cn}(t) \text{dn}(t) \text{cn}(t') \text{dn}(t')}{\text{sn}(t) - \text{sn}(t')^2} - \frac{1}{(1 - t^2)} - P' \right),
\]

\[
U_{33}(t, t') = 8 \left( \frac{\text{cn}(t) \text{dn}(t) \text{cn}(t') \text{dn}(t')}{(1 - t^2)^2} - \frac{1}{(1 - t^2)} - P' \right),
\]

\[
U_{12}(t, t') = U_{21}(t, t') = 8k4 \frac{\text{cn}(t) \text{dn}(t) \text{cn}(t') \text{dn}(t')}{(1 - k^2 \text{sn}(t) \text{sn}(t')^2)},
\]

\[
U_{34}(t, t') = U_{43}(t, t') = 8k4 \frac{\text{cn}(t) \text{dn}(t) \text{cn}(t') \text{dn}(t')}{(1 - t^2)^2},
\]

\[
U_{22}(t, t') = U_{11}(t, t'), \quad U_{44}(t, t') = U_{33}(t, t'). \tag{27}
\]

Here, \(\text{sn}(t), \text{cn}(t),\) and \(\text{dn}(t)\) are Jacobi elliptic functions of modulus \(k\) and \(\text{sn}(t), \text{cn}(t),\) and \(\text{dn}(t)\) of modulus \(k' = \sqrt{1 - k^2}\), where \(k\) is determined from the aspect ratio \(\delta/\delta'\) of the considered rectangle [31]. In the limit \(k \to 1\), the rectangle degenerates into a strip and we recover the matrix elements (17).

As in the previous example we can truncate this functional to a chiral one

\[
\hat{A}_{\alpha, \beta, \gamma, \delta} = \chi_s e^{S^{(1)} + \tilde{S}^{(2)}} \tag{28}
\]

where the phase factor \(\chi_s\) can be chosen at will and

\[
\tilde{S}^{(2)} = \frac{qs}{4\pi} \int_{\mathcal{I}} d\xi \tilde{\beta}(\xi)\tilde{\beta}(\xi)^T. \tag{29}
\]

We here consider the system on a cylinder, and to remove the virtual degrees of freedom on the boundaries, we take the rectangular regions for the boundary spins to go all the way to infinity. These boundary tensors can be obtained following the same approach as for the tensors in the bulk. We conjecture that sewing these amplitudes we get the wave function

\[
c_{s_1, \ldots, s_N} \propto \delta \sum_n \chi_{s_n} \prod_{n>m} (z_n - z_m)^{qa_{s_n, s_m}} \tag{30}
\]

with \(z_n = t_n + ix_n\). When \(q = 1/\sqrt{2}\), Eq. (30) is a 2D topological state in the same universality class as the bosonic Laughlin state at filling fraction 1/2, and the Kalmeyer-Laughlin wave function is obtained for \(N \to \infty\) [34].

Conclusions: We introduced a new class of TN constructed using functionals of fields that are contracted by means of the path integral of the functions defined on the links of the network. These tensors satisfy sewing and closing conditions that are similar to those employed in the construction of the scattering amplitudes in string theory [35, 36].

We illustrate our approach using a massless boson in 2D that allows us to derive the Haldane-Shastry wave function that describes a critical state in the unification class given by the WZW model \(SU(2)_1\). We also conjecture the field-tensor that generates the Kalmeyer-Laughlin state, which suggests that the chiral PEPS underlying topological chiral states in 2D require infinite bond dimension. The latter suggestion could be further
studied by truncating the field variables to a finite number of modes in which case the field-tensor provides a PEPS with finite bond dimension. We have here focused on lattice states, but utilizing the techniques in [37] to approach the continuum limit of the states, one could similarly describe continuum states. The definition of field tensor network states applies equally well to other types of lattices than those considered here. Our approach also allows a way to study topological chiral states based on the symmetry properties of the field-tensors.

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GENERAL FORMALISM

Let $\phi(x_1,x_2)$ be a real massless scalar field in a simply connected region $M$ of the two dimensional spacetime $\mathbb{R}^2$. The Euclidean free action of this field is given by

$$S_M^{(0)}(\phi) = \frac{1}{8\pi} \int_M d^2x \partial_\mu \phi \partial^\mu \phi.$$  \hfill (S1)

Let $f(x)$ be the values that $\phi(x)$ takes at the boundary $\partial M$ of the region $M$, and $\{q_j\}_{j=1}^N$ a set of $N$ charges located at the positions $x_j \in M$ that corresponds to the charge density

$$\rho = 4\pi i \sum_{j=1}^N q_j \delta^2(x-x_j).$$  \hfill (S2)

We shall associate to these data the path integral

$$Z_M(f,\rho) = \int_{\phi|_{\partial M}=f} |d\phi| e^{-S_M(\phi)},$$  \hfill (S3)

with

$$S_M(\phi) = S_M^{(0)}(\phi) - \frac{1}{4\pi} \int_M d^2x \phi \rho.$$  \hfill (S4)

To compute (S3) we write the scalar field as $\phi(x) = \phi_0 + \tilde{\phi}(x)$, where $\phi_0$ is a constant and $\tilde{\phi}$ is an orthogonal integrable function. Similarly, we write $f(x) = f_0 + \tilde{f}(x)$ so that (S3) becomes

$$Z_M(f,\rho) = \int_{\phi|_{\partial M}=f} d\phi_0 [d\tilde{\phi}] \left( \delta(\phi_0 - f_0) e^{-S_M(\tilde{\phi}) + i\phi_0 \sum_{j=1}^N q_j} = e^{i\tilde{f}_0 \sum_{j=1}^N q_j} \int_{\phi|_{\partial M}=f} [d\tilde{\phi}] e^{-S_M(\tilde{\phi})}, \right.$$  \hfill (S5)

where $\delta(\phi_0 - f_0)$ implements the constraint $\phi(x)|_{\partial M} = f(x)$ between the zero modes. The second factor in (S5) can be computed explicitly obtaining

$$Z_M(f,\rho) \propto e^{i\tilde{f}_0 \sum_{j=1}^N q_j} e^{-S_M(\phi_{cl})},$$  \hfill (S6)

where $\phi_{cl}$ satisfies the Poisson equation,

$$\partial^2_{x_i} \phi_{cl}(x) = -\rho(x), \quad \phi_{cl}(x)|_{x \in M} = \tilde{f}(x),$$  \hfill (S7)

that is solved by

$$\phi_{cl}(x) = -\int_M d^2x' G_M(x,x') \rho(x') - \int_{\partial M} dx_i' \tilde{f}(x') \epsilon_{ij} \partial_j G_M(x, x'),$$  \hfill (S8)

where $dx_i'$ is the line element along the curve $\partial M$ oriented anti-clockwise, $\epsilon_{12} = -\epsilon_{21} = 1$, and $G_M$ is the Green’s function with Dirichlet BCs

$$\partial^2_{x_1} G_M(x,x') = \delta^2(x-x'), \quad G_M(x,x') = 0, \quad \text{if } x \text{ or } x' \in \partial M.$$  \hfill (S9)

From (S4) we get

$$S_M(\phi_{cl}) = S_M(\tilde{f},\rho) = -\frac{1}{8\pi} \int_{\partial M} dx_i \epsilon_{ij} \phi_{cl} \partial_{x_j} \phi_{cl} - \frac{1}{8\pi} \int_M d^2x \phi_{cl} \rho,$$  \hfill (S10)
where we have performed a partial integration using the Gauss theorem,
\[ \int_M d^2x \partial_i V_i = -\int_{\partial M} dx_i \epsilon_{ij} V_j, \quad V_i = \phi_0 \partial_{x_i} \phi_A. \]  
(S11)
Inserting eq.(S8) into (S10) gives
\[
S_M(\tilde{f}, \rho) = \frac{1}{8\pi} \int_M d^2x \int_M d^2x' G_M(x, x') \rho(x) \rho(x') \\
+ \frac{1}{8\pi} \int_{\partial M} dx_k \epsilon_{ij} \epsilon_{kl} \tilde{f}(x) \tilde{f}(x') \partial_{x_i} \partial_{x_j} G_M(x, x') \\
+ \frac{1}{4\pi} \int_{\partial M} dx_i \int_M d^2x' \epsilon_{ij} \tilde{f}(x) \rho(x') \partial_{x_i} G_M(x, x'),
\]
and then
\[
Z_M(\tilde{f}, \rho) \propto e^{i f_0 \sum_{j=1}^N \rho_j} e^{-S_M(\tilde{f}, \rho)}. \]  
(S13)
This expression can also be applied to the case when \( M \) is the sphere \( S^2 \). Since \( S^2 \) has no boundary the last two terms of (S12) are absent. The Green’s function on \( S^2 \) is
\[
G(z, \bar{z}; z', \bar{z}') = \frac{1}{4\pi} \log |z - z'|^2,
\]
(S14)
where \( z = x_1 + ix_2 \) and \( z' = x_1' + ix_2' \). In the rest of the SM we shall use the variable \( z = x + iy \), that corresponds to \( t + ix \) in the main text. We also have to integrate over \( f_0 \). The final result is
\[
Z_{S^2}(\rho) \propto \delta(\sum_{j=1}^N q_j) \prod_{i>j} |z_i - z_j|^{2q_i q_j},
\]
(S15)
that does not vanish under the neutrality condition
\[
\sum_{j=1}^N q_j = 0.
\]
(S16)
In the case discussed in the main text the charges are given by
\[ q_j = qs_j, \quad s_j = \pm 1, \]
(S17)
and hence eq.(S16) becomes \( \sum_{j=1}^N s_j = 0 \).

**THE MPS FUNCTIONAL**

We shall use below the results obtained above to construct the MPS functional. First of all, we shall find the Green’s function that solves eq.(S9). This can be done using the Riemann’s mapping theorem that asserts the existence of a conformal map \( g \) from \( M \) to the upper-half plane \( \mathbb{H} \), when \( M \) is a simply connected region of the complex plane \( \mathbb{C} \)
\[
g : M \to \mathbb{H}, \quad \mathbb{H} = \{ z \in \mathbb{C}; \text{Im} \ z \geq 0 \},
\]
(S18)
such that the boundary of \( M \), is mapped into the real axis, that is \( g : \partial M \to \mathbb{R} \). This map allows us to construct \( G_M \) from the Green’s function \( G_{\mathbb{H}} \) in \( \mathbb{H} \), that is given by
\[
G_{\mathbb{H}}(\zeta, \bar{\zeta}; \zeta', \bar{\zeta}') = \frac{1}{4\pi} \log \frac{(\zeta - \zeta')(\bar{\zeta} - \bar{\zeta}')}{(\zeta - \bar{\zeta})(\zeta' - \bar{\zeta}')}, \quad \zeta, \bar{\zeta} \in \mathbb{H}.
\]
(S19)
Notice that \( G_{\mathbb{H}} \) vanishes if \( \zeta \) or \( \zeta' \) are real satisfying the Dirichlet BCs (S9). The Green’s function \( G_M \) can be obtained replacing \( \zeta \) and \( \zeta' \) by \( g(z) \) and \( g(z') \) respectively,
\[
G_M(z, \bar{z}; z', \bar{z}') = \frac{1}{4\pi} \log \frac{(g(z) - g(z'))(g(\bar{z}) - g(\bar{z}'))}{(g(z) - g(\bar{z}))(g(z') - g(\bar{z}'))}, \quad z, z' \in M.
\]
(S20)
To construct the MPS functional we take the strip $M = \mathbb{R} \times [0, \pi]$. The corresponding conformal map (S18) is given by

$$\zeta = e^z,$$

that replaced into (S20) gives

$$G_M(z, \bar{z}; z', \bar{z}') = \frac{1}{4\pi} \log \frac{\sinh(\frac{z-z'}{2}) \sinh(\frac{\bar{z}-\bar{z}'}{2})}{\sinh(\frac{z-\bar{z}}{2}) \sinh(\frac{\bar{z}-z'}{2})}, \quad z, z' \in M.$$  \hspace{1cm} (S22)

In order to prove the sewing and closing conditions, given in eqs.(5a) and (5b) of the main text, we shall consider a generic strip $M = \mathbb{R} \times \pi[a, b]$ ($a < b$), that can be mapped into $\mathbb{H}$ by the conformal map

$$g(z) = e^{(z-i\pi a)/\Delta}, \quad \Delta = b-a.$$  \hspace{1cm} (S23)

The associated Green’s function is

$$G_M(z, \bar{z}; z', \bar{z}') = \frac{1}{4\pi} \log \frac{\sinh(\frac{z-z'}{2\Delta}) \sinh(\frac{\bar{z}-\bar{z}'}{2\Delta})}{\sinh(\frac{z-\bar{z}}{2\Delta}) \sinh(\frac{\bar{z}-z'}{2\Delta})}, \quad z, z' \in M.$$  \hspace{1cm} (S24)

Choosing $z = x + iy, z' = x' + iy'$ we get

$$G_M(x, y; x', y') = \frac{1}{4\pi} \log \frac{\sinh(x-x' + iy) \sinh(z-\bar{z}')}{\sinh(x-x' - iy) \sinh(\bar{z}-\bar{z}')} \frac{2\Delta}{\Delta},$$

$$\partial_y G_M(x, y; x', y') = \frac{i}{8\pi \Delta} \left[ \coth \frac{x-x' + iy + y'}{2\Delta} - \coth \frac{x-x' - iy + y'}{2\Delta} - \coth \frac{x-x' + iy + y'}{2\Delta} \right],$$

$$\partial_y \partial_y G_M(x, y; x', y') = -\frac{1}{16\pi \Delta^2} \left[ \frac{1}{\sinh^2 \frac{x-x' + iy + y'}{2\Delta}} + \frac{1}{\sinh^2 \frac{x-x' - iy + y'}{2\Delta}} \right] + \frac{1}{2\Delta} \left[ \frac{1}{\sinh^2 \frac{x-x' + iy + y'}{2\Delta}} + \frac{1}{\sinh^2 \frac{x-x' - iy + y'}{2\Delta}} \right].$$

The non chiral MPS functional

The boundary $\partial M$ consists of the straight lines in the plane with $y = \pi a$ and $y = \pi b$. We define the real functions (see fig. S1)

$$f_+(x) = \tilde{f}(x, \pi a), \quad f_-(x) = \tilde{f}(x, \pi b), \quad x \in \mathbb{R}.$$  \hspace{1cm} (S26)

The constant mode $f_0$ is common to both lines and will be treated separately. Its contribution to the functional is simply the phase factor in eq.(S13). The functions $f_\pm(x)$ correspond to $\alpha(t)$ and $\beta(t)$ in the main text. The definitions (S26) lead us to write eq. (S12) as

$$S_M[f_+, f_-, \rho] = \frac{1}{8\pi} \int_M d^2x \int_M d^2x' \left( G_M(x, x') \rho(x) \rho(x') \right) \hspace{1cm} (S27)$$

$$+ \frac{1}{8\pi} \int_R dx \int_R dx' \left( f_+(x), f_-(x) \right) \left( \partial_y \partial_y G_M(x, y; x', y') \right) \left( f_+(x'), f_-(x') \right)$$

$$+ \frac{1}{4\pi} \int_R dx \int_M d^2x' \rho(x') \left( f_+(x), f_-(x) \right) \left( \partial_y G_M(x, y; x', y') \right) \left( f_+(x), f_-(x) \right) + \text{c.c.},$$

where

$$y_+ = \pi(a+\varepsilon), \quad y'_+ = \pi(a+\varepsilon'), \quad y_- = \pi(b-\varepsilon), \quad y'_- = \pi(b-\varepsilon'), \quad 0 < \varepsilon, \varepsilon' \ll 1, \quad \varepsilon \neq \varepsilon'.$$  \hspace{1cm} (S28)
are a regularization of \( y = \pi a, \pi b \). Eqs.(S25) lead to

\[
S_M[f_+, f_-, \rho] = \frac{1}{8\pi} \int_M d^2x \int_M d^2x' G_M(x, x') \rho(x) \rho(x')
\]

\[
- \frac{1}{64\pi^2} \int_R dx \int_R dx' (f_+(x), f_-(x)) \left( \begin{array}{cc} u_+\Delta(x - x') & u_-\Delta(x - x') \\ u_-\Delta(x - x') & u_+\Delta(x - x') \end{array} \right) (f_+(x'))
\]

\[
+ \frac{i}{16\pi^2} \int_R dx \int_M d^2x' \rho(x')(f_+(x), f_-(x)) \left( \begin{array}{cc} v_+\Delta, a(x, x') & v_+, \Delta, a(x, x') \\ -v_-\Delta, a(x, x') & v_-, \Delta, a(x, x') \end{array} \right),
\]

with

\[
u_+\Delta(x - x') = \frac{1}{\Delta^2} \left( \frac{1}{\sinh^2 \frac{x-x'+i\varepsilon}{2\Delta}} + \frac{1}{\sinh^2 \frac{x-x'-i\varepsilon}{2\Delta}} \right),
\]

\[
u_-\Delta(x - x') = \frac{2}{\Delta^2 \cosh^2 \frac{x-x'}{2\Delta}},
\]

\[
u_+\Delta, a(x, x') = \frac{1}{\Delta} \coth \frac{x-x'-iy'+i\pi a}{2\Delta}, \quad v_-\Delta, a(x, x') = \frac{1}{\Delta} \tanh \frac{x-x'+iy'+i\pi a}{2\Delta}.
\]

In equations (S31) and (S32) we have taken the limit \( \varepsilon, \varepsilon' \to 0 \) that gives ordinary functions, while in (S30) we have replaced \( \varepsilon \pm \varepsilon' \) by \( \epsilon \) that in the limit \( \epsilon \to 0 \) becomes a generalized function, namely a distribution. To discuss this issue in more detail we define the function

\[
f_\epsilon(x) = \frac{1}{\sinh^2(x + i\epsilon)} + \frac{1}{\sinh^2(x - i\epsilon)},
\]

that is obtained from (S30) after rescaling the variables. Let us also define the function

\[
g_\epsilon(x) = 2 \left( \frac{1}{\sinh^2 x} - \frac{1}{x^2} \right) + \frac{1}{(x + i\epsilon)^2} + \frac{1}{(x - i\epsilon)^2}
\]

whose first term is regular at \( x = 0 \). We aim at showing that \( f_\epsilon(x) \to g_\epsilon(x) \) in the limit \( \epsilon \to 0 \). Expanding the difference between (S33) and (S34) around \( \epsilon = 0 \) yields

\[
f_\epsilon(x) - g_\epsilon(x) = \left( \frac{6}{x^4} - \frac{2(2 + \cosh(2x))}{\sinh^2 x} \right) \epsilon^2 + O(\epsilon^4).
\]

The term proportional to \( \epsilon^2 \), has the series expansion \(-\frac{2}{15} + O(x^2)\) around \( x = 0 \), so that one can take safely the limit \( \epsilon \to 0 \) obtaining

\[
\lim_{\epsilon \to 0} (f_\epsilon(x) - g_\epsilon(x)) = 0, \quad \forall x \in \mathbb{R}.
\]

The terms depending of \( \epsilon \) in (S34) can be expressed using the principal value distribution,

\[
P \left( \frac{1}{x} \right) = \lim_{\epsilon \to 0} \frac{1}{2} \left( \frac{1}{x + i\epsilon} + \frac{1}{x - i\epsilon} \right)
\]

whose derivative with respect to \( x \) is

\[
P' \left( \frac{1}{x} \right) = -\lim_{\epsilon \to 0} \frac{1}{2} \left( \frac{1}{(x + i\epsilon)^2} + \frac{1}{(x - i\epsilon)^2} \right)
\]

that together with (S34) and (S36) yields

\[
\lim_{\epsilon \to 0} f_\epsilon(x) = 2 \left( \frac{1}{\sinh^2 x} - \frac{1}{x^2} - P' \left( \frac{1}{x} \right) \right).
\]

Rescaling the variables leads finally to

\[
\lim_{\epsilon \to 0} u_+\Delta(x - x') = \frac{2}{\Delta^2} \left( \frac{1}{\sinh^2 \frac{x-x'}{2\Delta}} - \left( \frac{2\Delta}{x-x'} \right)^2 \right) - 8P' \left( \frac{1}{x} \right),
\]

that coincides with the Eq.(14) given in the main text with \( \Delta = L \) and \( x - x' \rightarrow t \).
The chiral MPS functional

Eq. (S29) is the basis of our proposal for a chiral version of the MPS functional. It is obtained by keeping the terms that depend exclusively on $z$ or $z'$ in the Green’s function and in the piece proportional to $\rho(x')$. The terms quadratic in $f_{\pm}$ do not possess a chiral/antichiral factorization and stay the same. The chiral functional is defined by truncating $S_M$ to

$$R_M[f_+ , f_- , \rho] = \frac{1}{32\pi^2} \int_M d^2 x \int_M d^2 x' \rho(x)\rho(x') \log(\frac{x-x'+iy-y'}{2\Delta})$$

We have introduced a constant $\mu$ whose value will be fixed later on. Replacing the charge density (S2) in (S41) gives

$$R_M[f_+ , f_- , \{q_j , z_j \}^N_{j=1}] = - \sum_{N \geq j > k \geq 1} q_j q_k \log(\frac{z_j - z_k}{2\Delta})$$

FIG. S1. Top: graphical representation of the MPS functional (S44) for $M$ = $\mathbb{R} \times \pi[a, b]$, that is

$$\pi a < \text{Im} \ z_j < \pi b, \quad j = 1, \ldots, N.$$  (S43)

We shall define the functional

$$A_M[f_+ , f_- , \{g , z_j \}^N_{j=1}] = \exp \left( -R_M[f_+ , f_- , \{g , z_j \}^N_{j=1}] \right)$$

that is represented in fig. S1. Observe that for $N = 1$ the first term in eq. (S42) does not appear. $A_M[f_+ , g , \{q , z \}]$ is the basic building block to construct the functionals with $N > 1$. This is a consequence of the sewing condition
illustrated in fig. S1,
\[ \int [dg] A_{M_1} [f_+, g, \{ q_0, z_0 \}] A_{M_2} [g, f_-, \{ q_j, z_j \}]_{j=1}^L \propto A_{M_1 \cup M_2} [f_+, f_-, \{ q_j, z_j \}]_{j=0}^L, \]  
(S45)

where \( M_1 = \mathbb{R} \times [-\pi, 0] \) and \( M_2 = \mathbb{R} \times [0, \pi L] \) are two strips with a common boundary \( M_1 \cap M_2 = \mathbb{R} \times \{ 0 \} \). The coordinates of the charges belong to the corresponding intervals, that is \( z_0 \in M_1, z_j=1,\ldots,L \in M_2 \). Equation (S45) can be interpreted geometrically as the sewing of the strips \( M_1 \) and \( M_2 \) along \( M_1 \cap M_2 \) to produce the strip \( M_{L+1} \equiv M_1 \cup M_2 = \mathbb{R} \times [-\pi, \pi L] \).

The functional for \( M_L \) is given by (S42) and (S44) with \( N = L \) and \( \Delta = L \), that is

\[
A_{M_L} [f_+, f_-, \{ q_j, z_j \}]_{j=1}^L = \exp (-R_{M_L} [f_+, f_-, \{ q_j, z_j \}]_{j=1}^L),
\]
(S46)

\[
R_{M_L} [f_+, f_-, \{ q_j, z_j \}]_{j=1}^L = - \sum_{L \geq j > k \geq 1} q_j q_k \log(\mu \sinh \frac{z_j - z_k}{2L})
- \frac{1}{64\pi^2} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dx' (f_+(x), f_-(x)) \left( u_{+,L}(x-x') u_{-,L}(x-x') \right) \left( f_+(x') - f_-(x') \right)
- \frac{1}{4\pi} \sum_{j=1}^L \int_{\mathbb{R}} dx q_j (f_+(x), f_-(x)) \left( v_{+,L}(x, z_j) - v_{-,L}(x, z_j) \right).
\]

If \( L = 1 \) the log term does not appear. Similarly, the functional for \( M_1 \) is given by

\[
A_{M_1} [f_+, f_-, \{ q_0, z_0 \}] = \exp (-R_{M_1} [f_+, f_-, \{ q_0, z_0 \}]),
\]
(S47)

\[
R_{M_1} [f_+, f_-, \{ q_0, z_0 \}] = - \frac{1}{64\pi^2} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dx' (f_+(x), f_-(x)) \left( u_{+,1}(x-x') u_{-,1}(x-x') \right) \left( f_+(x') - f_-(x') \right)
- \frac{1}{4\pi} \int_{\mathbb{R}} dx q_0 (f_+(x), f_-(x)) \left( v_{-,1}(x, z_0) - v_{+,1}(x, z_0) \right),
\]

where from (S30) one has

\[
u_{+,L}(x-x') = \frac{1}{L^2} \left( \frac{1}{\sinh^2 \frac{x-x'+i\varepsilon}{2L}} + \frac{1}{\sinh^2 \frac{x-x'-i\varepsilon}{2L}} \right),
\]
(S48)

\[
u_{-,L}(x-x') = \frac{2}{L^2 \cosh^2 \frac{x-x'}{2L}};
\]

\[
v_{+,L}(x, z') \equiv v_{+,L,0}(x, z') = \frac{1}{L} \coth \frac{x - z'}{2L};
\]

\[
v_{-,L}(x, z') \equiv v_{-,L,0}(x, z') = \frac{1}{L} \tanh \frac{x - z'}{2L}.
\]

In the last term of eq. (S47) we used that

\[
v_{+,1,0}(x, z') = v_{+,1,0}(x, z') = v_{-,1,0}(x, z') = v_{+,1,0}(x, z') = v_{+,1,0}(x, z') = v_{-,1,0}(x, z').
\]
(S49)

**The chiral MPS functional in momentum space**

To perform the path integral (S45), we exploit the translation invariance of \( u_{\pm,L}(x-x') \) working in momentum space. First of all, we define the Fourier transform \( \hat{f}_{\pm}(k) \) of the integrable functions \( f_{\pm}(x) \),

\[
f_{\pm}(x) = \int_{\mathbb{R}} dk \ e^{ikx} \hat{f}_{\pm}(k),
\]
(S50)
The reality of \( f_\pm(x) \) implies that \( \hat{f}_\pm(-k) = \hat{f}_\pm^*(k) \). The Fourier transform of the functions (S48) is

\[
\begin{align*}
\int_R dx \, e^{ikx} u_{+,L}(x) &= -8\pi k \coth(\pi kL), \\
\int_R dx \, e^{ikx} u_{-,L}(x) &= 8\pi k / \sinh(\pi kL), \\
\int_R dx \, e^{ikx} v_{+,L}(x, z') &= 2\pi i e^{ikz'} e^{\pi kL} / \sinh(\pi kL), \quad 0 < \text{Im } z' < \pi L, \\
\int_R dx \, e^{ikx} v_{+,L}(x, z') &= 2\pi i e^{ikz'} e^{-\pi kL} / \sinh(\pi kL), \quad -\pi < \text{Im } z' < 0, \\
\int_R e^{ikx} v_{-,L}(x, z') &= 2\pi i e^{ikz'} / \sinh(\pi kL), \quad -\pi L < \text{Im } z' < \pi L.
\end{align*}
\]

The functionals (S46) and (S47) become in momentum space

\[
R_{M_L}[f_+, f_-, \{q_j, z_j \}_{j=1}^L] = -\sum_{L \geq j > k \geq 1} q_j q_k \log(\mu \sinh \frac{z_j - z_k}{2L})
\]

\[
+ \frac{1}{2} \int_0^\infty dk \left( \hat{f}_+(k) \hat{f}_-(k) \right) \begin{pmatrix} \omega_{+,L}(k) & \omega_{-,L}(k) \\ \omega_{-,L}(k) & \omega_{+,L}(k) \end{pmatrix} \begin{pmatrix} \hat{f}_+^*(k) \\ \hat{f}_-^*(k) \end{pmatrix}
\]

\[
- \frac{i}{2} \sum_{j=1}^L q_j \int_R dk e^{ikz_j} \frac{e^{\pi kL} \hat{f}_+(k) - \hat{f}_-(k)}{\sinh(\pi kL)}
\]

and

\[
R_{M_L}[f_+, f_-, \{q_0, z_0 \}] = \frac{1}{2} \int_0^\infty dk \left( \hat{f}_+(k) \hat{f}_-(k) \right) \begin{pmatrix} \omega_{+,1}(k) & \omega_{-,1}(k) \\ \omega_{-,1}(k) & \omega_{+,1}(k) \end{pmatrix} \begin{pmatrix} \hat{f}_+^*(k) \\ \hat{f}_-^*(k) \end{pmatrix}
\]

\[
- \frac{i q_0}{2} \int_R dk e^{ikz_0} \frac{\hat{f}_+(k) - e^{-\pi k} \hat{f}_-(k)}{\sinh(\pi k)}
\]

where

\[
\omega_{+,L}(k) = k \coth(\pi kL), \quad \omega_{-,L}(k) = -k / \sinh(\pi kL).
\]

The sewing condition

The LHS of (S45) is given by

\[
\int [dg] A_{M_1}[f_+, g, \{q_0, z_0 \}] A_{M_L}[g, f_-, \{q_j, z_j \}_{j=1}^L] = \int [dg] \exp \left( -R_{M_1}[f_+, g, \{q_0, z_0 \}] - R_{M_L}[g, f_-, \{q_j, z_j \}_{j=1}^L] \right).
\]

The exponent of the integrand is the sum

\[
R_{M_1} + R_{M_L} = (1) + (2) + (3) + (4) + (5)
\]

where

\[
(1) = -\sum_{L \geq j > k \geq 1} q_j q_k \log(\mu \sinh \frac{z_j - z_k}{2L}),
\]

\[
(2) = \frac{1}{2} \int_0^\infty dk \left( \hat{g}(k) \hat{f}_-(k) \right) \begin{pmatrix} \omega_{+,L}(k) & \omega_{-,L}(k) \\ \omega_{-,L}(k) & \omega_{+,L}(k) \end{pmatrix} \begin{pmatrix} \hat{g}^*(k) \\ \hat{f}_-^*(k) \end{pmatrix},
\]

\[
(3) = -\frac{i}{2} \sum_{j=1}^L q_j \int_R dk \frac{e^{ikz_j} (e^{\pi kL} \hat{g}(k) - \hat{f}_-(k))}{\sinh(\pi kL)}
\]

\[
(4) = \frac{1}{2} \int_0^\infty dk \left( \hat{f}_+(k) \hat{g}(k) \right) \begin{pmatrix} \omega_{+,1}(k) & \omega_{-,1}(k) \\ \omega_{-,1}(k) & \omega_{+,1}(k) \end{pmatrix} \begin{pmatrix} \hat{f}_+^*(k) \\ \hat{g}^*(k) \end{pmatrix},
\]

\[
(5) = \frac{i q_0}{2} \int_R dk \frac{e^{ikz_0} (\hat{f}_+(k) - e^{-\pi k} \hat{g}(k))}{\sinh(\pi k)}.
\]
Consider the partial sums
\[ 2 + 4 = 7 + 8 \]
\[ 3 + 5 = 9 + 10 \]
where
\[
7 = \frac{1}{2} \int_0^\infty dk \left[ \hat{g}(k)\hat{g}^*(k)(\omega_\pm, L(k) + \omega_\pm, 1(k)) + \hat{g}(k)(\hat{f}_+^*(k)\omega_\pm, L(k) + \hat{f}_+^*(k)\omega_\pm, 1(k)) \right],
\]
\[
8 = \frac{1}{2} \int_0^\infty dk \left[ \hat{f}_+(k)\hat{f}_+^*(k)\omega_\pm, 1(k) + \hat{f}_-(k)\hat{f}_-^*(k)\omega_\pm, L(k) \right],
\]
\[
9 = -\frac{i}{2} \int_0^\infty dk \left[ \hat{g}(k) \left( \sum_{j=1}^L q_j \frac{e^{ikz_j e^{\pi kL}}}{\sinh(\pi k)} - q_0 \frac{e^{ikz_0 e^{-\pi k}}}{\sinh(\pi k)} \right) - \hat{g}^*(k) \left( \sum_{j=1}^L q_j \frac{e^{-ikz_j e^{-\pi k}}}{\sinh(\pi k)} - q_0 \frac{e^{-ikz_0 e^{-\pi k}}}{\sinh(\pi k)} \right) \right],
\]
\[
10 = -\frac{i}{2} \int_0^\infty dk \left[ \hat{g}(k) \left( \sum_{j=1}^L q_j \frac{e^{ikz_j e^{\pi kL}}}{\sinh(\pi kL)} - q_0 \frac{e^{ikz_0 e^{-\pi k}}}{\sinh(\pi k)} \right) - \hat{g}^*(k) \left( \sum_{j=1}^L q_j \frac{e^{-ikz_j e^{-\pi k}}}{\sinh(\pi kL)} - q_0 \frac{e^{-ikz_0 e^{-\pi k}}}{\sinh(\pi k)} \right) \right],
\]
such that
\[
R_{M_1} + R_{M_L} = 1 + 7 + 8 + 9 + 10
\]
The terms depending on \( g(k) \) are
\[
7 + 10 = \int_0^\infty dk \left[ \hat{g}(k)\hat{g}^*(k)\Omega(k) + \hat{g}(k)\alpha(k) + \hat{g}^*(k)\beta(k) \right],
\]
where
\[
\Omega(k) = \frac{1}{2}(\omega_\pm, L(k) + \omega_\pm, 1(k)) = \frac{k}{2} \frac{\sinh(\pi kL + 1)}{\sinh(\pi kL)}
\]
\[
\alpha(k) = \frac{1}{2}(\hat{f}_+^*(k)\omega_\pm, L(k) + \hat{f}_+^*(k)\omega_\pm, 1(k)) - \frac{i}{2} \left( \sum_{j=1}^L q_j \frac{e^{ikz_j e^{\pi kL}}}{\sinh(\pi kL)} - q_0 \frac{e^{ikz_0 e^{-\pi k}}}{\sinh(\pi k)} \right),
\]
\[
\beta(k) = \frac{1}{2}(\hat{f}_-(k)\omega_\pm, L(k) + \hat{f}_+(k)\omega_\pm, 1(k)) + \frac{i}{2} \left( \sum_{j=1}^L q_j \frac{e^{-ikz_j e^{-\pi k}}}{\sinh(\pi kL)} - q_0 \frac{e^{-ikz_0 e^{-\pi k}}}{\sinh(\pi k)} \right).
\]

Now, we perform the functional integral
\[
\int \prod_{k>0} d\hat{g}(k)d\hat{g}^*(k) \exp \left[ -\int_0^\infty dk \left( \hat{g}(k)\hat{g}^*(k)\Omega(k) + \hat{g}(k)\alpha(k) + \hat{g}^*(k)\beta(k) \right) \right] \exp \left( \int_0^\infty dk \frac{\alpha(k)\beta(k)}{\Omega(k)} \right),
\]
where we used
\[
\int dz d\bar{z} e^{-\omega |z|^2 + az + b\bar{z}} = \frac{\pi}{\omega} e^{ab/\omega}.
\]
Upon integration, the LHS of (S55) can be written as \( e^{-\hat{R}_{M_1 \cup M_L}} \) with
\[
\hat{R}_{M_1 \cup M_L} = 1 + 8 + 9 - \int_0^\infty dk \frac{\alpha(k)\beta(k)}{\Omega(k)}.
\]
Our goal is to show that \( e^{-\hat{R}_{M_1 \cup M_L}} \) coincides with \( e^{-\hat{R}_{M_1 \cup M_L}} \). The last integral in (S66) splits as
\[
-\int_0^\infty dk \frac{\alpha(k)\beta(k)}{\Omega(k)} = 11 + 12 + 13
\]
where

\begin{equation}
(11) = - \int_0^\infty \frac{dk}{4\Omega(k)} \left( \sum_{j=1}^L q_j \frac{e^{ikz_j} e^{\pi k L} - q_0 e^{-ikz_j} e^{-\pi k}}{\sinh(\pi k L) \sinh(\pi k)} \right) \left( \sum_{l=1}^L q_l \frac{e^{-ikz_l} e^{-\pi k L} - q_0 e^{ikz_l} e^{\pi k}}{\sinh(\pi k L) \sinh(\pi k)} \right),
\end{equation}

\begin{equation}
(12) = - \int_0^\infty \frac{dk}{4\Omega(k)} \left( \hat{f}_-(k) \omega_{-L}(k) + \hat{f}_+(k) \omega_{-1}(k) \right) \left( \hat{f}_+(k) \omega_{-L}(k) + \hat{f}_-(k) \omega_{-1}(k) \right),
\end{equation}

\begin{equation}
(13) = i \int_0^\infty \frac{dk}{4\Omega(k)} \left[ \left( \hat{f}_+(k) \omega_{-1}(k) + \hat{f}_-(k) \omega_{-L}(k) \right) \left( \sum_{j=1}^L q_j \frac{e^{ikz_j} e^{\pi k L} - q_0 e^{-ikz_j} e^{-\pi k}}{\sinh(\pi k L) \sinh(\pi k)} \right) 
- \left( \hat{f}_+(k) \omega_{-1}(k) + \hat{f}_-(k) \omega_{-L}(k) \right) \left( \sum_{j=1}^L q_j \frac{e^{-ikz_j} e^{-\pi k L} - q_0 e^{ikz_j} e^{\pi k}}{\sinh(\pi k L) \sinh(\pi k)} \right) \right].
\end{equation}

The integrand of (11) behaves as $1/k^2$ for $k \sim 0$. To obtain a finite value we use the following regularization method. We first write (11) as

\begin{equation}
(11) = - \left[ \sum_{L \geq j > l \geq 1} q_j q_l I_{1, L}(z_j - z_l) - \sum_{j=1}^L q_j q_0 I_{2, L}(z_j - z_0) + \sum_{j=1}^L q_j^2 I_{3, L} + q_0^2 I_{4, L} \right],
\end{equation}

with

\begin{equation}
I_{1, L}(z) = \int_0^\infty \frac{dk}{4\Omega(k)} \frac{e^{ikz} + e^{-ikz}}{\sinh^2(\pi k L)},
\end{equation}

\begin{equation}
I_{2, L}(z) = \int_0^\infty \frac{dk}{4\Omega(k)} \frac{e^{ikz + \pi k L} + e^{-ikz - \pi k L}}{\sinh(\pi k L) \sinh(\pi k)},
\end{equation}

\begin{equation}
I_{3, L}(z) = \int_0^\infty \frac{dk}{4\Omega(k)} \frac{1}{\sinh^2(\pi k L)},
\end{equation}

\begin{equation}
I_{4, L}(z) = \int_0^\infty \frac{dk}{4\Omega(k)} \frac{1}{\sinh(\pi k L) \sinh(\pi k L + 1)}.
\end{equation}

The integrands of these functions is even, so one can replace the integration interval from $(0, \infty)$ to $\mathbb{R}$. Next, we replace $\mathbb{R}$ by a contour $\mathcal{R}_\epsilon$ in the complex plane that runs along the negative real axis until the point $(-\epsilon, 0)$, encircles the origin clockwise until the point $(0, \epsilon)$ and then continues along the positive real axis,

\begin{equation}
\mathcal{R}_\epsilon = (-\infty, \epsilon) \cup \{e^{i\theta} | \theta \in (\pi, 0) \} \cup (\epsilon, \infty), \quad 0 < \epsilon \ll 1.
\end{equation}

The regularized integrals (S71) are given by

\begin{equation}
I_{1, L}^\epsilon(z) = \int_{\mathcal{R}_\epsilon} \frac{dk}{2k} \frac{\sinh(\pi k) \cos(k z)}{\sinh(\pi k L) \sinh(\pi k (L + 1))}, \quad 0 < \left|\text{Im}(z)\right| < \pi L
\end{equation}

\begin{equation}
I_{2, L}^\epsilon(z) = \int_{\mathcal{R}_\epsilon} \frac{dk}{2k} \frac{\cosh(ik z + \pi k (L + 1))}{\sinh(\pi k L) \sinh(\pi k (L + 1))}, \quad 0 < \text{Im}(z) < \pi (L + 1)
\end{equation}

\begin{equation}
I_{3, L}^\epsilon(z) = \int_{\mathcal{R}_\epsilon} \frac{dk}{2k} \frac{\sinh(\pi k L)}{2k \sinh(\pi (k L + 1))},
\end{equation}

\begin{equation}
I_{4, L}^\epsilon(z) = \int_{\mathcal{R}_\epsilon} \frac{dk}{4k} \frac{\sinh(\pi k L)}{\sinh(\pi k L) \sinh(\pi (k L + 1))},
\end{equation}

where we included the ranges of the variable $z$ that come from their relation to $z_j$ and $z_0$.

The simplest integral is

\begin{equation}
I_{5,1}^\epsilon = \int_{\mathcal{R}_\epsilon} \frac{dk}{4k} \frac{1}{\sinh(2\pi k)} = -\frac{1}{2} \log 2,
\end{equation}
that can be computed using residue calculus. Similarly

\[ I_{3,L}^e = -\frac{1}{2} \log \frac{L + 1}{L}, \quad I_{4,L}^e = -\frac{1}{2} \log (L + 1). \]  

\( \text{(S78)} \)

To compute \( I_{2,L}^e(z) \) by residue calculus we close the contour \( \mathcal{R}_e \) on a half circle of radius \( R \) in the upper half plane, and take the limit \( R \to \infty \). Next, we find the conditions under which the integration along the half circle vanishes. The modulus square of the integrand of (S74) is given by (not including \( 1/\kappa \) that does not contribute to the integral along the half-circle)

\[ \left| \frac{\cosh(ikz + \pi k(L + 1))}{\sinh(\pi k(L + 1))} \right|^2 = \frac{\text{num}}{\text{den}}, \quad k = Re^{i\theta} \]  

\( \text{(S79)} \)

\[ \text{num} = [\cosh(R(\cos(\theta)\lambda - \sin(\theta)\hat{x})) \cos(R(\sin(\theta)\lambda - \sin(\theta)\hat{x}))]^2 + [\sinh(R(\cos(\theta)\lambda - \sin(\theta)\hat{x})) \sin(R(\sin(\theta)\lambda - \sin(\theta)\hat{x}))]^2, \]

\[ \text{den} = [\sinh(R \cos(\theta)) \cos(R \sin(\theta))]^2 + [\cosh(R \cos(\theta)) \sin(R \sin(\theta))]^2, \]

where \( R \) has been replaced by \( R/(\pi(L + 1)) \) and

\[ \lambda = 1 - \frac{y}{\pi(L + 1)}, \quad y = \text{Im } z, \quad 0 < \lambda < 1, \]

\[ \hat{x} = \frac{x}{\pi(L + 1)}, \quad x = \text{Re } z. \]

In the limit \( R \to \infty \) and \( \theta \neq \pi/2 \), one gets

\[ \frac{\text{num}}{\text{den}} \to \exp(2R(|\cos(\theta)| - |\sin(\theta)\hat{x}| - |\cos \theta|)), \]  

\( \text{(S81)} \)

which vanishes exponentially for \( \hat{x} = 0 \) because \( \theta < \lambda < 1 \), while for \( \hat{x} \neq 0 \) there is a region in \( \theta \) where it diverges. We shall then impose the condition that \( x = 0 \), i.e. \( z = iy \), in which case the integral (S74) can be computed by residue calculus obtaining

\[ I_{2,L}^e(iy) = \sum_{n=1}^{\infty} \frac{\cos(ny/(L + 1))}{n} = -\log \left( 2 \sin \frac{y}{2(L + 1)} \right), \quad 0 < y < \pi(L + 1). \]  

\( \text{(S82)} \)

Similarly, the integral (S73) is given by (with \( z = iy \))

\[ I_{1,L}^e(iy) = \log \frac{\sin \frac{y}{2(L + 1)}}{\sin \frac{y}{2L}}, \quad 0 < |y| < \pi L. \]  

\( \text{(S83)} \)

Gathering the results obtained we have

\[ I_{1,L}^e(z) = \log \left( \frac{\sinh \frac{z}{2(L + 1)}}{\sinh \frac{z}{2L}} \right), \quad 0 < |\text{Im}(z)| < \pi L, \]  

\( \text{(S84)} \)

\[ I_{2,L}^e(z) = -\log \left( -2i \sinh \frac{z}{2(L + 1)} \right), \quad 0 < \text{Im}(z) < \pi(L + 1), \]

\[ I_{3,L}^e = -\frac{1}{2} \log \frac{L + 1}{L}, \quad I_{4,L}^e = -\frac{1}{2} \log (L + 1), \]

that allow us to write eq.(S70) as

\[ (11) = - \left[ \sum_{L \geq j > l \geq 1} q_j q_l \log \frac{\sinh \frac{z_j - z_l}{2(L + 1)}}{\sinh \frac{z_j - z_l}{2L}} + \sum_{j=1}^{L} q_j q_0 \log \left( -2i \sinh \frac{z_j - z_0}{2(L + 1)} \right) - \sum_{j=1}^{L} \frac{q_j^2}{2} \log \frac{L + 1}{L} - \frac{q_0^2}{2} \log (L + 1) \right] \]  

\( \text{(S85)} \)

and adding (S57)

\[ (14) \equiv (11) + (11) = - \left[ \sum_{L \geq j > l \geq 0} q_j q_l \log \sinh(\frac{\mu z_j - z_l}{2(L + 1)}) - \sum_{j=1}^{L} \frac{q_j^2}{2} \log \frac{L + 1}{L} - \frac{q_0^2}{2} \log (L + 1) \right], \]  

\( \text{(S86)} \)
where we have chosen
\[ \mu = -2i , \]  
(S87)
to simplify the expression. Collecting the terms in (S66), (S67) and (S85) we get
\[ \tilde{R}_{M_1 \cup M_L} = 1 + 8 + 9 + 11 + 12 + 13 \]
\[ = 8 + 9 + 12 + 13 + 14 \]  
(S88)
Now, we consider
\[ 15 = 8 + 12 = \frac{1}{2} \int_0^\infty dk \left[ \hat{f}_+(k) \hat{f}_+(k) \left( \omega_{+,1}(k) - \frac{\omega^2_{+,1}(k)}{2\Omega(k)} \right) + \hat{f}_-(k) \hat{f}_+(k) \left( \omega_{+,L}(k) - \frac{\omega^2_{+,L}(k)}{2\Omega(k)} \right) \right] \]
\[ - (\hat{f}_+(k) \hat{f}_+(k) + \hat{f}_-(k) \hat{f}_+(k)) \frac{\omega_{-,L}(k) \omega_{-,L}(k)}{2\Omega(k)} \]
\[ = \frac{1}{2} \int_0^\infty dk \left( \hat{f}_+(k) \hat{f}_+(k) \hat{f}_+(k) \left( \omega_{+,L+1}(k) \omega_{-,L+1}(k) \right) + \hat{f}_-(k) \hat{f}_+(k) \right) , \]
where we used the identities
\[ \omega_{+,1}(k) - \frac{\omega^2_{+,1}(k)}{2\Omega(k)} = \omega_{+,L}(k) - \frac{\omega^2_{+,L}(k)}{2\Omega(k)} = k \cosh(\pi k(L + 1)) = \omega_{+,L+1}(k) , \]
\[ -\frac{\omega_{-,L}(k) \omega_{-,L}(k)}{2\Omega(k)} = -k / \sinh(\pi k(L + 1)) = \omega_{-,L+1}(k) , \]
(S90)
The other summand in (S88) is
\[ 16 = 9 + 13 = \frac{i}{2} \int_R dk \left( \frac{q_i \sinh(\pi k)}{k} \left[ \hat{f}_+(k) \left( 1 + \frac{e^{-\pi k} \omega_{+,1}(k)}{2\Omega(k)} \right) + \hat{f}_-(k) \frac{e^{-\pi k} \omega_{-,1}(k)}{2\Omega(k)} \right] \right) \]
\[ + \frac{i}{2} \int_R \sum_{j=1}^L q_j \sinh(\pi k L) \left[ \hat{f}_+(k) \frac{e^{\pi k L} \omega_{-,L}(k)}{2\Omega(k)} + \hat{f}_-(k) \left( 1 + \frac{e^{\pi k L} \omega_{-,L}(k)}{2\Omega(k)} \right) \right] \]
\[ = \frac{i}{2} \int_R \sum_{j=0}^L q_j \sinh(\pi k(L + 1)) \left( e^{\pi k L} \hat{f}_+(k) - \hat{f}_-(k) e^{-\pi k} \right) , \]
where we used
\[ 1 + \frac{e^{-\pi k} \omega_{+,1}(k)}{2\Omega(k)} = \frac{e^{\pi k L} \sinh(\pi k)}{\sinh(\pi k(L + 1))} , \]
\[ \frac{e^{-\pi k} \omega_{-,1}(k)}{2\Omega(k)} = \frac{e^{\pi k L} \sinh(\pi k L)}{\sinh(\pi k(L + 1))} , \]
\[ \frac{e^{\pi k L} \omega_{-,L}(k)}{2\Omega(k)} = \frac{e^{\pi k L} \sinh(\pi k L)}{\sinh(\pi k(L + 1))} , \]
\[ 1 + \frac{e^{\pi k L} \omega_{-,L}(k)}{2\Omega(k)} = \frac{e^{-\pi k} \sinh(\pi k L)}{\sinh(\pi k(L + 1))} . \]
(S92)
Collecting terms
\[ \tilde{R}_{M_1 \cup M_L} = (1) + (15) + (16) \]
\[ = - \sum_{L \geq j \geq L \geq 0} q_j q_i \log(\mu \sinh \frac{z_j - z_i}{2(L + 1)}) + \sum_{j=1}^L q_j^2 \log \frac{L + 1}{L} + \frac{q_0^2}{2} \log(L + 1) \]
\[ + \frac{1}{2} \int_0^\infty dk \left( \hat{f}_+(k) \hat{f}_+(k) \hat{f}_+(k) \left( \omega_{+,L+1}(k) \omega_{-,L+1}(k) \right) + \hat{f}_-(k) \hat{f}_+(k) \right) \]
\[ - \frac{i}{2} \int_R \sum_{j=0}^L q_j \sinh(\pi k(L + 1)) \left( e^{\pi k L} \hat{f}_+(k) - \hat{f}_-(k) e^{-\pi k} \right) , \]
that using (S52) can be written as
\[ \tilde{R}_{M_1 \cup M_L} [f_+, f_-, \{q_j, z_j\}]_{j=0}^L = R_{M_{L+1}} [f_+, f_-, \{q_j, z_j\}]_{j=0}^L + \sum_{j=0}^L \frac{q_j^2}{2} \log(L + 1) - \sum_{j=1}^L \frac{q_j^2}{2} \log L, \]  

where \( \tilde{z}_j = z_j + \pi \) takes into account that \( M_1 \cup M_L = \mathbb{R} \times [-\pi, \pi L] \) while the expression (S52) is associated to the interval \( M_{L+1} = \mathbb{R} \times [0, \pi(L + 1)] \). The terms proportional to \( q_j^2 \) have the following interpretation. The functional \( A_{M_L} [f_+, f_-, \{z_0, q_0\}] \) corresponds in the CFT formalism to the chiral vertex operator: \( e^{i \eta \phi(z_0)} \), which has conformal dimension \( h = \frac{1}{2} q_0^2 \) [S1]. The MPS amplitude \( A_{M_L} \) should therefore correspond to the operator \( e^{i \sum_{j=1}^L q_j \phi(z_j)} \) with scaling dimension \( \sum_{j=1}^L \frac{1}{2} q_j^2 \). This leads to the definition of the functional 

\[ A_{M_L} [f_0, f_+, f_-, \{q_j, z_j\}] = L^{-\frac{1}{2}} \sum_{j=1}^L q_j e^{i \eta_0 \sum_{j=1}^L q_j} A_{M_L} [f_+, f_-, \{q_j, z_j\}]_{j=1}^L = e^{i \eta_0 \sum_{j=1}^L q_j} \exp \left( -R_{M_L} [f_+, f_-, \{q_j, z_j\}]_{j=1}^L - \sum_{j=1}^L \frac{q_j^2}{2} \log L \right), \]  

where we have included the zero mode \( f_0 \) (recall eq. (S13)). The sewing equation (S45) can finally be written as 

\[ \int \left[ dg \right] A_{M_1} [f_0, f_+, g, \{q_j, z_j\}] A_{M_L} [f_0, g, f_-, \{q_j, z_j\}]_{j=1}^L = A_{M_1 \cup M_L} [f_0, f_+, f_-, \{q_j, z_j\}]_{j=0}^L, \]  

that is equivalent to the equation (5a) in the main text.

**The closing condition**

We first make the identification \( f_+ = f_- \equiv f \) in eq. (S52), 

\[ R_{M_L} [f, f, \{q_j, z_j\}]_{j=1}^L = - \sum_{L \geq j > k \geq 1} q_j q_k \log(\mu \sinh \frac{z_j - z_k}{2L}) \]  

\[ + \frac{1}{2} \int_0^\infty dk \left( \hat{f}(k), \hat{f}(k) \right) \left( \frac{\omega_{+L}(k)}{\omega_{-L}(k)} \frac{\omega_{-L}(k)}{\omega_{+L}(k)} \right) \left( \hat{f}^\ast(k), \hat{f}^\ast(k) \right) \]  

\[ = - \sum_{L \geq j \geq 1} q_j q_k \log(\mu \sinh \frac{z_j - z_k}{2L}) + \int_0^\infty dk \left[ \hat{f}^\ast(k) \hat{f}(k) \tilde{\Omega}(k) + \hat{f}(k) \tilde{\alpha}(k) + \hat{f}^\ast(k) \tilde{\beta}(k) \right], \]  

where 

\[ \tilde{\Omega}(k) = \frac{1}{2} \left( \frac{\omega_{+L}(k)}{\omega_{+L}(k)} + \omega_{-L}(k) \right) = k \tanh(\pi k L/2), \]  

\[ \tilde{\alpha}(k) = \frac{i}{2} \sum_{j=1}^L q_j e^{ik z_j} \left( e^{\pi k L} - 1 \right) = - \frac{i}{2} \sum_{j=1}^L q_j e^{ik z_j} e^{\pi k L / 2}, \]  

\[ \tilde{\beta}(k) = \frac{i}{2} \sum_{j=1}^L q_j e^{-ik z_j} \left( e^{-\pi k L} - 1 \right) = - \frac{i}{2} \sum_{j=1}^L q_j e^{-ik z_j} e^{-\pi k L / 2}, \]  

and integrate over \( f \) (recall eq.(S64)) 

\[ \int \prod_{k > 0} d\hat{f}(k) d\hat{f}^\ast(k) \exp \left( -R_{M_L} [f, f, \{q_j, z_j\}]_{j=1}^L \right) = \exp \left( \sum_{L \geq j \geq 1} q_j q_k \log(\mu \sinh \frac{z_j - z_k}{2L}) + \int_0^\infty dk \frac{\tilde{\alpha}(k) \tilde{\beta}(k)}{\Omega(k)} \right) \]  

(S99)
The integral is given by
\[
\int_0^\infty \frac{dk}{\Omega(k)} \frac{d\tilde{\alpha}(k)\tilde{\beta}(k)}{} = -\int_0^\infty \frac{dk}{k \sinh(\pi k L)} \left( \sum_{L \geq j > k \geq 1} q_j q_k \cos(k(z_j - z_l) + \sum_{j=1}^L \frac{q_j^2}{2}) \right) \tag{S100}
\]
\[
= -\left( \sum_{L \geq j > k \geq 1} q_j q_k \log \left( \frac{\sinh \frac{z_j - z_l}{L}}{\sinh \frac{z_l - z_j}{L}} \right) - \log 2 \sum_{j=1}^L \frac{q_j^2}{2} \right)
\]
where we used the regularized integrals (S73) and (S77). Plugging (S100) into (S99) gives
\[
\int_{k > 0} \prod_{k > 0} \frac{df(k)d\bar{f}^*(k)\exp(-R_{ML} [f, f, \{q_j, z_j\}_{j=1}^L])}{\Omega(k)} = \exp \left( \sum_{L \geq j > k \geq 1} q_j q_k \log(\mu \sinh \frac{z_j - z_k}{L}) + \log 2 \sum_{j=1}^L \frac{q_j^2}{2} \right) \tag{S101}
\]
This equation together with (S95) implies
\[
\int [df]_{AM_L} [f_0, f, \{q_j, z_j\}_{j=1}^L] = e^{i f_0 \sum_{j=1}^L q_j} \exp \left( \sum_{L \geq j > k \geq 1} q_j q_k \log(-2i \sinh \frac{z_j - z_k}{L}) + \sum_{j=1}^L \frac{q_j^2}{2} \log \frac{2}{L} \right) \tag{S102}
\]
\[
= e^{i f_0 \sum_{j=1}^L q_j} \exp \left( \sum_{L \geq j > k \geq 1} q_j q_k \log(-Li \sinh \frac{z_j - z_k}{L}) + \frac{1}{2} (\sum_{j=1}^L q_j)^2 \log \frac{2}{L} \right)
\]
\[
= e^{i f_0 \sum_{j=1}^L q_j} \prod_{L \geq j > k \geq 1} \left( L \sin \frac{y_j - y_k}{L} \right)^{q_j q_k} \left( \frac{L}{2} \right)^{-\frac{1}{2} (\sum_{j=1}^L q_j)^2}.
\]
Finally, integrating over the zero mode \( f_0 \) yields,
\[
\int [df]_{AM_L} [f_0, f, \{q_j, z_j\}_{j=1}^L] = 2\pi \delta(\sum_{j=1}^L q_j) \prod_{L \geq j > k \geq 1} \left( L \sin \frac{y_j - y_k}{L} \right)^{q_j q_k} \tag{S103}
\]
\[
= 2\pi \delta(\sum_{j=1}^L q_j) \prod_{L \geq j > k \geq 1} \langle \prod_{L \geq j > k \geq 1} : e^{iq_j \varphi(z_j)} : e^{iq_k \varphi(z_k)} : \rangle_{cyl}.
\]
where \( \langle .. \rangle_{cyl} \) is the vacuum expectation value of the product of chiral vertex operators, at positions \( z_j = iy_j \) in a cylinder of length \( \pi L \). This result provides a proof of equation (5b) in the main text.

**PEPS FUNCTIONAL**

**The conformal map**

The PEPS functional is obtained when the region \( M \) of the path integral is a rectangle in the complex plane. The conformal map \( g : M \to \mathbb{H} \) can be constructed using the Schwarz-Christoffel formula,
\[
g = F(\sin^{-1}(z), k) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad u \in M, \quad z \in \mathbb{H}, \quad \tag{S104}
\]
where \( F(\phi, k) \) is the incomplete elliptic integral of the first kind defined as
\[
F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \int_0^{\sin \phi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{S105}
\]
and \( k \) is the elliptic modulus that satisfies \( 0 < k^2 < 1 \). The points \( z = \pm 1, \pm 1/k \), on the real axis are mapped into the vertices of the rectangle with coordinates
\[
z = \pm 1 \to u = \pm K(k), \quad \tag{S106}
\]
\[
z = \pm 1/k \to u = \pm K(k) + iK'(k),
\]
where $K(k)$ is the complete elliptic integral of the first kind,

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$  \hspace{1cm} (S107)

and $K'(k) = K(k')$, with $k' = \sqrt{1 - k^2}$ is the complementary modulus. Moving along the real axis in the $z$-plane one goes through the points $-\frac{1}{k}, -1, 1, \frac{1}{k}$, that correspond in the $u$-plane to the points $-K + iK', -K, K, K + iK'$ that form the vertices of a rectangle of width $2K$ and height $K'$. Figure S2 shows these constants that intersect at $k = 0.171573$. In the limit $k \to 1$ one has

$$\lim_{k \to 1} K(k) = \infty, \quad \lim_{k \to 1} K'(k) = \frac{\pi}{2},$$  \hspace{1cm} (S108)

which shows that the rectangle $M = [-K, K] \times [0, K']$ becomes the strip $R \times [0, \frac{\pi}{2}]$.

FIG. S2. Plot of $2K(k)$ (red curve) and $K'(k)$ (blue curve) as a function of the elliptic modulus $k$. The two curves intersect at $k = 0.171573$.

The inverse function of $F$ is the Jacobi amplitude

$$\phi = F^{-1}(u, k) = \text{am}(u, k),$$  \hspace{1cm} (S109)

in terms of which the Jacobi elliptic functions are defined

$$\sin \phi = \sin(\text{am}(u, k)) = \text{sn}(u, k),$$  \hspace{1cm} (S110)
$$\cos \phi = \cos(\text{am}(u, k)) = \text{cn}(u, k),$$

$$\sqrt{1 - k^2 \sin^2 \phi} = \sqrt{1 - k^2 \sin^2(\text{am}(u, k))} = \text{dn}(u, k).$$

For $k = 0$ and $k = 1$ the elliptic function become trigonometric and hyperbolic functions respectively

$$k = 0 : \text{sn}(x) = \sin(x), \quad \text{cn}(x) = \cos(x), \quad \text{dn}(x) = 1,$$
$$k = 1 : \text{sn}(x) = \tanh(x), \quad \text{cn}(x) = \frac{1}{\cosh(x)}, \quad \text{dn}(x) = \frac{1}{\cosh(x)}.$$  \hspace{1cm} (S111)

Using eqs. (S104) and (S110) the conformal map $g : M \to \mathbb{H}$ is given by

$$\zeta = \text{sn}(z, k), \quad z \in M, \quad \zeta \in \mathbb{H}. $$  \hspace{1cm} (S112)

The non chiral functional

The Green’s function $G_M$ on $M = [-K, K] \times [0, K']$ is obtained replacing (S112) into (S19),

$$G_M(z, \bar{z}; z', \bar{z}') = \frac{1}{4\pi} \log \frac{(\text{sn}(z) - \text{sn}(z'))(\text{sn}(z) - \overline{\text{sn}(z')})}{(\text{sn}(z) - \overline{\text{sn}(z')})(\text{sn}(z) - \text{sn}(z'))}, \quad z, z' \in M,$$  \hspace{1cm} (S113)
where \( s(z) \equiv sn(z, k) \). On the boundary of \( M \) we define the functions (see fig.S3)

\[
\begin{align*}
  h_+(x) &= f(x, 0), & h_-(x) &= f(x, K'), & -K \leq x \leq K, \\
  v_+(y) &= f(-K, y), & v_-(y) &= f(K, y), & 0 \leq y \leq K'.
\end{align*}
\]

(S114)

that correspond to the functions \( \alpha_{n,m}, \beta_{n,m}, \gamma_{n,m}, \delta_{n,m} \) defined in eqs. (23) in the main text.

Equation (S12) becomes

\[
S_M = \frac{1}{8\pi} \int_M d^2x \int_M d^2x' G_M(x, x') \rho(x) \rho(x')
\]

(S115)

\[
\begin{align*}
  + \frac{1}{8\pi} \int_{-K}^K dx \int_{-K}^K dx' (h_+(x), h_-(x)) & \left( \frac{\partial_y G_M|_{y_+}}{y_+} - \frac{\partial_y G_M|_{y_-}}{y_-} \right) h_+(x') \\
  + \frac{1}{8\pi} \int_0^{K'} dy \int_0^{K'} dy' (v_+(y), v_-(y)) & \left( \frac{\partial_x G_M|_{x_+}}{x_+} - \frac{\partial_x G_M|_{x_-}}{x_-} \right) v_+(y') \\
  + \frac{1}{8\pi} \int_{-K}^K dx \int_{-K}^K dx' (v_+(y), v_-(y)) & \left( \frac{\partial_y G_M|_{y_+}}{y_+} - \frac{\partial_y G_M|_{y_-}}{y_-} \right) v_+(y') \\
  + \frac{1}{8\pi} \int_0^{K'} dy \int_0^{K'} dy' (v_+(y), v_-(y)) & \left( \frac{\partial_x G_M|_{x_+}}{x_+} - \frac{\partial_x G_M|_{x_-}}{x_-} \right) v_+(y') \\
  + \frac{1}{4\pi} \int_{-K}^K dx \int_M d^2x' \rho(x') (h_+(x), h_-(x)) & \left( \frac{\partial_y G_M(x, y; x', y')}{y_+} - \frac{\partial_y G_M(x, y; x', y')}{y_-} \right) \\
  + \frac{1}{4\pi} \int_0^{K'} dy \int_M d^2y' \rho(x') (v_+(y), v_-(y)) & \left( \frac{\partial_x G_M(x, y; x', y')}{x_+} - \frac{\partial_x G_M(x, y; x', y')}{x_-} \right),
\end{align*}
\]

where

\[
\begin{align*}
x_+ &= -K + \varepsilon, & x_- &= K - \varepsilon, & y_+ &= \varepsilon, & y_- &= K' - \varepsilon & \varepsilon > 0,
\end{align*}
\]

(S116)

and the corresponding primed versions as in eq.(S28) needed to regularize the expressions. Replacing \( z = x + iy \) and \( z' = x' + iy' \) into (S113) we get

\[
G_M(x, y; x', y') = \frac{1}{4\pi} \log \left( \frac{sn(x + iy) - sn(x' + iy')}{sn(x + iy) - sn(x' - iy')}(sn(x - iy) - sn(x' - iy')) \right),
\]

(S117)

which is defined in the rectangle

\[
- K \leq x, x' \leq K, \quad 0 \leq y, y' \leq K'.
\]

(S118)

Since \( k \) is a real parameter, we used that \( \frac{dsn(x + iy)}{dz} = sn(x - iy) \). Taking the derivatives respect to \( x \) and \( y \) and using

\[
\frac{d}{dz} sn(z) = cn(z)sn(z)
\]

(S119)
one obtains

$$\partial_y G_M(x, y; x', y') = \frac{i}{4\pi} \left( \frac{cn(x + iy)dn(x + iy)}{sn(x + iy) - sn(x' + iy')} - \frac{cn(x - iy)dn(x - iy)}{sn(x - iy) - sn(x' - iy')} \right)$$

\(S120\)

$$\partial_x G_M(x, y; x', y') = \frac{1}{4\pi} \left( \frac{cn(x + iy)dn(x + iy)cn(x' - iy')dn(x' - iy')}{(sn(x + iy) - sn(x' + iy'))^2} + \frac{cn(x - iy)dn(x - iy)cn(x' + iy')dn(x' + iy')}{(sn(x - iy) - sn(x' - iy'))^2} \right)$$

and

$$\partial_y \partial_y' G_M(x, y; x', y') = -\frac{1}{4\pi} \left( \frac{cn(x + iy)dn(x + iy)cn(x' + iy')dn(x' + iy')}{(sn(x + iy) - sn(x' + iy'))^2} + \frac{cn(x - iy)dn(x - iy)cn(x' - iy')dn(x' - iy')}{(sn(x - iy) - sn(x' - iy'))^2} \right)$$

$$\partial_x \partial_x' G_M(x, y; x', y') = \frac{1}{4\pi} \left( \frac{cn(x + iy)dn(x + iy)cn(x' + iy')dn(x' + iy')}{(sn(x + iy) - sn(x' + iy'))^2} + \frac{cn(x - iy)dn(x - iy)cn(x' - iy')dn(x' - iy')}{(sn(x - iy) - sn(x' - iy'))^2} \right)$$

Using (S116) together with (see [S2])

\(S121\)

we derive

$$S_M = \frac{1}{8\pi} \int_M dx \int_M dx' G_M(x, x') \rho(x) \rho(x')$$

\(S122\)
where

\[ U_{h_+,h_+}(x,x') = U_{h_-,h_-}(x,x') = 2cn(x)dn(x)cn(x')dn(x') \left( \frac{1}{(sn(x+i\epsilon) - sn(x'+i\epsilon'))^2} + \frac{1}{(sn(x-i\epsilon) - sn(x'-i\epsilon'))^2} + \frac{1}{(sn(x-i\epsilon) - sn(x'+i\epsilon'))^2} \right) \]  

(S123)

\[ U_{h_+,h_+}(x,x') = U_{h_-,h_-}(x,x') = 8k \frac{cn(x)dn(x)cn(x')dn(x')}{(1 - k^2 sn(x)sn(x'))^2} \]

\[ U_{v_+,v_+}(y,y') = U_{v_-,v_-}(y,y') = 2k'^4 \frac{cn(y)sn(y)cn(y')sn(y')}{(dn(y) + dn(y'))^2} \]

\[ U_{v_+,v_-}(y,y') = U_{v_-,v_+}(y,y') = 8k'^4 \frac{cn(y)sn(y)cn(y')sn(y')}{(dn(y) + dn(y'))^2} \]

\[ V_{h_+}(x,x') = \frac{2cn(x)dn(x)}{sn(x) - sn(x'+iy')} \]

\[ V_{v_+}(x,x') = \frac{2k'^2 sn(y)cn(y)}{dn(y)(1 + dn(y)sn(x'+iy'))} \]

and

\[ \tilde{sn}(y) = sn(y|k'), \quad cn(y) = cn(y|k'), \quad \tilde{dn}(y) = dn(y|k'), \quad k'^2 = 1 - k^2. \]  

(S124)

The regularize versions of \( U_{h_+,h_+} \) and \( U_{v_+,v_-} \) is given in eqs.(27) of the main text.

The chiral version of (S122) that we propose is

\[ R_M = - \sum_{N \geq j > k \geq 1} q_j q_k \ln(sn(z_j) - sn(z_k)) \]  

(S125)

\[ - \frac{1}{64\pi^2} \int_{-K}^K dx \int_{-K}^K dx' (h_+(x), h_-(x)) \left( U_{h_+,h_+}(x,x') U_{h_-,h_-}(x,x') \right) \left( h_+(x') \right) \]

\[ - \frac{1}{64\pi^2} \int_{0}^{K'} dy \int_{0}^{K'} dy' (v_+(y), v_-(y)) \left( U_{v_+,v_+}(y,y') U_{v_-,v_-}(y,y') \right) \left( v_+(y') \right) \]

\[ - \frac{1}{64\pi^2} \int_{-K}^{K} dx \int_{-K}^{K} dy' (h_+(x), h_-(x)) \left( U_{h_+,h_+}(x,x') U_{h_-,h_-}(x,x') \right) \left( v_+(y') \right) \]

\[ - \frac{1}{4\pi} \sum_{j=1}^{N} \int_{-K}^{K} dx q_j (h_+(x), h_-(x)) \left( \frac{V_{h_+}(x,z_j)}{-V_{h_-}(x,z_j)} \right) \]

\[ - \frac{1}{4\pi} \sum_{j=1}^{N} \int_{0}^{K'} dy q_j (v_+(y), v_-(y)) \left( \frac{-V_{v_+}(y,z_j)}{V_{v_-}(y,z_j)} \right). \]

where we have used the charge density (S2).

Equations (S108) shows that in the limit \( k \rightarrow 1 \), the rectangle \( M = [-K,K] \times [0,K'] \) becomes the strip \( R \times [0, \frac{\pi}{2}] \). Therefore, the PEPS functional (S125) must be closely related to the MPS functional (S42) with \( a = 0 \) and \( \Delta = 1/2 \). The reason for this fact is the following. In the limit \( k \rightarrow 1 \), the conformal map (S112) becomes
\[ g_1(z) = \tanh(z) = \frac{e^{2z} - 1}{e^{2z} + 1} \]  

(S126)

where we used (S111). On the other hand, the conformal map (S23), with \( a = 0, \Delta = 1/2 \), is \( g(z) = e^{2z} \). Notice that \( g_1(z) \) is a Möbius transformation of \( g(z) \), so we expect the wave functions constructed with both functionals to be the same. This issue will be considered elsewhere in more detail.

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