NEW SYMMETRIES IN MATHEMATICAL PHYSICS EQUATIONS

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Abstract

An algorithm for studying the symmetrical properties of the partial differential equation of the type 
\(L\phi(x) = 0\) is proposed. By symmetry of this equation we mean the operators \(Q\) satisfying commutational relations of order \(p > 1\) on the solutions \(\phi(x)\): \([L \ldots [L, Q] \ldots] \phi(x) = 0\). It is shown, that within the framework of the proposed method with \(p = 2\) the relativistic D’Alembert and Maxwell equations are the Galilei symmetrical ones. Analogously, with \(p = 2\) the Galilei symmetrical Schrödinger equation is the relativistic symmetrical one. In both cases the standard symmetries are realized with \(p = 1\).

1 Introduction

The symmetry properties of mathematical physics equations contain the important information about objects of research.

Some receptions are offered for symmetry research: the classical and the modified Lie methods [1, 2], the non-Lie ones [2, 3], the renormgroup concept [4], the method for search of the conditional symmetries [5, 6], the theoretical-algebraic approach [7]. The purpose of the present work is the formulation of the new method for obtaining the additional information about symmetrical properties of equations. For research we choose the D’Alembert, Maxwell and Schrödinger equations.

2 Method of Research

We begin with a definition of symmetry, which we shall name as extended one.

Let the equation be given in the space \(R^n(x)\)

\[L\phi(x) = 0,\]  \hspace{1cm} (1)

where \(L\) is a linear operator.

Definition 1 By the symmetry of Eq. (1) we shall mean a set of operators \(\{Q^{(p)}\}\), \(p = 1, 2, \ldots, n, \ldots\), if the result of the successive \((p - 1)\)-fold action of the operator \(L\) on an operator \(Q^{(p)}\) transforms a nonzero solution \(\phi(x)\) into another solution \(\phi'(x) = L^{(p-1)}Q^{(p)}\phi(x) \neq 0\).

From Definition follows, that operators \(Q^{(p)}\) satisfy commutational relations of order \(p\)

\[[L \ldots [L, Q^{(p)}] \ldots]_{(p-\text{fold})}\phi(x) = 0.\]  \hspace{1cm} (2)

The extended Definition:
• includes the understanding of symmetry in the case when Eq. (2) is fulfilled on a set of arbitrary functions, that is equivalent to \([L \ldots [L, Q^{(p)}] \ldots]_{(p_{-fold})} = 0\);

• contains the standard understanding of symmetry, when \([L, Q^{(1)}] \phi(x) = 0\);

• includes the understanding of symmetry in quantum mechanics sense \([L, Q^{(1)}] = 0\);

• differs from the standard one, as in the framework of the latter by the operators of symmetry we should mean not the operators \(\{Q^{(p)}\}\), but the operators \(\{X^{(1)} = L^{(p-1)}Q^{(p)}\}\).

The question is how practically to find them. In the present work it is decided by analogy with the modified Lie algorithm [2]. Below we consider the case, when \(p = 2\).

Let us introduce a set of operators

\[
Q^{(1)} = \xi_1(x) \partial_a + \eta_1(x);
\]

\[
Q^{(2)} = \xi_2(x) \partial_a + \eta_2(x),
\]

which have the following commutation properties

\[
[L, Q^{(1)}] = \zeta_1(x) L;
\]

\[
[L, Q^{(2)}] = \zeta_2(x) L.
\]

The given expressions are operator’s versions of the extended Definition of symmetry. Here \(\partial_a = \partial/\partial x^a; \ a = 0, 1, \ldots n - 1; \ \xi_a(x), \ \eta(x), \ \zeta(x)\) are unknown functions; the summation is carried out over a twice repeating index; the unknown functions may be found by equaling the coefficients at identical derivatives in the left and in the right parts of ratios and by integrating the set of determining differential equations available.

After integrating the general form of the operators \(Q\) may be recorded as a linear combination of the basic elements \(Q^{(1)}_\alpha, Q^{(2)}_\beta\), on which, by analogy with [3] we impose the condition to belong in Lie algebras:

\[
A^1 : [Q^{(1)}_\alpha, Q^{(1)}_\beta] = C_{\alpha\beta\gamma} Q^{(1)}_\gamma;
\]

\[
A^2 : [Q^{(1)}_\alpha, Q^{(2)}_\delta] = C_{\alpha\delta\chi} Q^{(1,2)}_\chi;
\]

\[
Q^{(2)}_\mu, Q^{(2)}_\nu = C_{\mu\nu\sigma} Q^{(1,2)}_\sigma.
\]

Here \(C_{\alpha\beta\gamma}, C_{\mu\nu\sigma}\) are the structural constants; operators \(Q^{(1,2)}_\chi, Q^{(1,2)}_\sigma\) belong to the sets of operators \(\{Q^{(1)}_\alpha, Q^{(2)}_\beta\}\).

With help of the Lie equations we transfer from Lie algebras to the Lie groups

\[
dx^\alpha/\theta = \xi^\alpha(x'),
\]

where \(x^{\alpha}(\theta = 0) = x^a; \ a = 0, 1, \ldots n - 1; \ \theta\) is a group parameter [2].

For the law of field transformation to be found, instead of integrating the Lie equations \(d\phi'(x')/\theta = \eta(x')\phi'(x'), \ \phi'(x')|_{(\theta = 0)} = \phi(x)\) we shall take the reception [4], which we shall illustrate by example of one-component field.
Let us introduce such a weight function $\Phi(x)$ in the field transformation law, that

$$\phi'(x') = \Phi(x)\phi(x).$$  \hfill (10)

We choose the function $\Phi(x)$ so that Eq. (1) should transform into itself in accordance with the generalized understanding of symmetry because of the following additional condition (the set of engaging equations)

$$A\Phi(x)\phi(x) = 0, \quad L\phi(x) = 0.$$  \hfill (11)

The former is obtained by replacing the variables in the initial equation $L'\phi'(x') = 0$. Formula (10) corresponds to a linearization of the transformed unprimed equation at replacing $x' = x'(x)$, $\phi'(x') = \phi'(\Phi(x)\phi(x))$, if $A = L$, we shall call the symmetry the classical symmetry and if $A \neq L$, we shall refer it to as the generalized one. By solving Set (11), the weight function $\Phi(x)$ can be put in conformity to each field function $\phi(x)$ for ensuring the transition $L' \rightarrow L$.

Instead of solving Set (11), the weight function may be found on the base of the symmetry approach. As far as $\phi'(x') = Q^{(1)'}\phi(x')$ is a solution too, and $\phi'(x') = \Phi(x)\phi(x)$ we have

$$\Phi(x) = \frac{\phi'(x' \rightarrow x)}{\phi(x)} \in \left\{ \frac{\phi(x' \rightarrow x)}{\phi(x)}; \frac{1}{\phi(x)}; \frac{Q^{(1)'}\phi(x' \rightarrow x)}{\phi(x)}; \frac{[L', Q^{(2)'}]\phi(x' \rightarrow x)}{\phi(x)}; \ldots \right\}.$$  \hfill (12)

Here the dots correspond to a consecutive action of the operators $Q^{(1)'}_{\alpha}$ and $[L', Q^{(2)'}_{\mu}]$ on a solution $\phi(x')$. Thus, for the function $\Phi(x)$ to be found it is necessary to turn to the unprimed variables in the primed solution $\phi'(x')$, and to divide the result available by the unprimed solution $\phi(x)$. By solving Set (11), the weight function $\Phi(x)$ can be put in conformity to each field function $\phi(x)$ for ensuring the transition $L' \rightarrow L$.

After finding the weight functions $\Phi(x)$ the task about the symmetry of Eq. (1) for one-component field may be thought as completed in the definite sense, namely: the set of the operators of symmetry and the corresponding Lie algebras are indicated for $p = 1$ and $p = 2$; the groups of symmetry are restored by the given algebras; with help of the weight functions the transformational properties of field $\phi(x)$ are determined.

The proposed method allows generalization to the case of multicomponent field and a symmetry of order more high, than $p = 2$.

3 Application of the Method

3.1 The Galilei symmetry of D’Alembert equation, $p=2$

$$L_D\phi(x) = \Box \phi(x) = (\partial_t/c^2 - \triangle)\phi(x) = 0$$  \hfill (13)

$$\phi(x) = \exp(-ik.x) = \omega(t - n.x/c)$$  \hfill (14)

Generator of space-time transformations and its commutational properties:

$$H_1 = x_0 \partial_1; \quad [\Box, H_1] = 0.$$  \hfill (15)
Conditions of transfer of equation (13) into itself:

\[ ((\partial_0 + \beta \partial_1)^2/\lambda^2 - \Delta)\Phi_D(x)\phi(x) = 0; \quad \Box \phi(x) = 0. \tag{16} \]

Weight function:

\[ \Phi_D(x) = \exp\{-i(1 - \lambda)k.x - \beta \omega(n.xt - x/c)\}. \tag{17} \]

Transformational properties of solution (14):

\[ \exp(-ik'.x') = \exp\{-i(1 - \lambda)k.x - \beta \omega(n.xt - x/c)\}\exp(-ik.x). \tag{18} \]

Here \( x_0 = x^0 = ct, \) \( t \) is the time; \( x_k = -x^k, \) \( k = 1, 2, 3, \) \( x^{1,2,3} = x, y, z \) are the space variables; \( \omega \) is the frequency.

### 3.2 The Lorentz symmetry of Schrödinger equation, \( p=2 \)

Let us investigate equation, which we name relativistic Schrödinger equation, and next transfer to the known non-relativistic one. We have:

\[ L_S\psi(x) = (i\hbar \partial_t + \frac{\hbar^2 \sqrt{1 - \beta^2}}{2m_0} \Delta)\phi(x) = (i\hbar \partial_t + \frac{c^2 \hbar^2}{2W} \Delta)\psi(x) = 0; \tag{19} \]

\[ \psi_1(x) = \exp[-\frac{i}{\hbar}(\beta^2/2Wt - P.x)] = \exp[-\frac{mv^2}{2\hbar}(t - \frac{s.x}{v^2})]; \]

\[ \psi_2(x) = \exp[-\frac{i}{\hbar}(Wt - \sqrt{2}\frac{P.x}{\beta})] = \exp[-\frac{mc^2}{\hbar}(t - \frac{s.x}{c/\sqrt{2}})]. \tag{20} \]

Generator of space-time transformations and its commutational properties:

\[ M_{01} = x_0 \partial_1 - x_1 \partial_0, \quad [L_S[L_S, M_{01}]] = 0. \tag{21} \]

Conditions of transfer of equation (19) into itself:

\[ \{i\hbar(\partial_t + V \partial_x) + \frac{c^2 \hbar^2}{2W(1 - \frac{V^2}{c^2})}[(\partial_x + V \partial_t/c^2)^2 - \frac{(\partial_y + \partial_z)^2}{1 - \frac{V^2}{c^2}}] + \partial_{yy} + \partial_{zz}\}\Psi = 0; \quad L_S\psi = 0. \tag{22} \]

Weight functions:

\[ \Psi_{11} = \exp\{-i\frac{W}{2\hbar(1 - \beta^2)}[(\beta_v^2 - 2\beta^2 - \beta_v^2)(1 - \beta^2) - \beta \beta_x(\beta_v^2 - 2)t - (\beta_v^2 - 2)(\beta - \beta^2\beta_x)\frac{x}{c}]; \]

\[ \Psi_{22} = \exp\{-i\frac{W}{2\hbar(1 - \beta^2)}[(1 - \frac{\sqrt{2}}{\beta_v})(\beta^2 - \beta \beta_x)t + ((1 - \frac{\sqrt{2}}{\beta_v})(\beta^2\beta_x - \beta) + \sqrt{2}\beta_x(\frac{1}{\beta_v} - \frac{1}{\beta_v'})\frac{x}{c} + \sqrt{2}(1 - \beta^2)(\frac{1}{\beta_v} - \frac{1}{\beta_v'})(\beta_y y + \beta_z z)]}; \]

\[ \Psi_{12} = \Psi_{11}\psi_1/\psi_2; \quad \Psi_{21} = \Psi_{22}\psi_2/\psi_1. \tag{23} \]
Transformational properties of solutions \([21]\):

\[
\psi'_1(x') = \Psi_{11}(x)\psi_1(x) = \Psi_{12}(x)\psi_2(x); \quad \psi'_2(x') = \Psi_{21}(x)\psi_1(x) = \Psi_{22}(x)\psi_2(x). \tag{24}
\]

In the non-relativistic approximation the equation \([19]\) and his solution \(\psi_1(x)\) and the weight function \(\Psi_{11}(x)\) obtain the known view \(L_s\psi(x) = 0 \rightarrow (i\hbar \partial_t + h^2\Delta/2m_0)\psi(x) = 0; \psi_1(x) = \exp[-i(m_0v^2/2h)(t - s.x/(v/2))] = \exp[-i(Et - x.p/\hbar)]; \Psi_{11}(x) = \exp[-i(-Et + xP)/\hbar]\). where \(E = m_0v^2/2; E = m_0V^2/2; p = m_0v; P = m_0V, W = mc^2, P = mv, s = v/v, n = c/c; V\) is the velocity of inertial reference \(K'\) relative to \(K; \beta = V/c; \lambda = c'/c = (1 - 2\beta n_x + \beta^2)^{1/2}; \beta_x = v_x/c, \beta_y = v_y/c, \beta_z = v_z/c, \beta_v = v/c; v = (v_x, v_y, v_z)\) is the speed of a particle; \(c\) is the speed of light; \(m = m_0/(1 - \beta^2)^{1/2}\) is the relativistic mass, \(m_0\) is the rest mass of a particle; \(\beta_v^2 = [\beta^2(1 - \beta^2) + \beta_0^2 - 2\beta x + \beta^2\beta_x^2]/(1 - \beta x)^2\).

The second solution \([20]\) is \(\psi_2(x) \rightarrow \exp[-i(m_0c^2/\hbar)(t - s.x/(c/\sqrt{2})). \) It is the new solution of the non-relativistic Schrödinger equation \([1]\).

### 3.3 The Galilei symmetry of Maxwell equations, p=2

\[
\nabla \cdot \mathbf{E} = 0; \quad \nabla \times \mathbf{H} - \frac{1}{c}\partial_t \mathbf{E} = 0; \quad \nabla \cdot \mathbf{H} = 0; \quad \nabla \times \mathbf{E} + \frac{1}{c}\partial_t \mathbf{H} = 0; \tag{25}
\]

\[
\mathbf{(E, H)} = (l, m)\exp(-ik.x) = (l, m)\omega(t - n.x/c), \tag{26}
\]

where \(l, m\) are the vectors of polarization. We find the field transformation law as

\[
\begin{align*}
E'_1 &= \Phi_D(x)E_1; \\
H'_1 &= \Phi_D(x)H_1; \\
E'_2 &= \Phi_D(x)k(E_2 + h_{23}H_3); \\
H'_2 &= \Phi_D(x)k(H_2 + e_{23}E_3); \\
E'_3 &= \Phi_D(x)k(E_3 + h_{32}H_2); \\
H'_3 &= \Phi_D(x)k(H_3 + e_{32}E_2). \tag{27}
\end{align*}
\]

Here \(\Phi_D\) is the weight function \([17]\); \(e_{23}, e_{32}, h_{23}, h_{32}\) are parameters of transformations. Bear in mind the expressions \((27)\) and replacing the variables in Eq. \((27)\) we receive the system of engaging equations analogous to Sys. \([10]\) and \([22]\). Insertion the solution \((26)\) and weight function \([17]\) in this system leads to superdefined system of algebraic equations for determination of parameters \(k, e_{23}, \ldots\). The system has solutions:

\[
k = \frac{n_x(\beta - n_x) + \lambda}{1 - n_x^2}; \quad e_{23} = \frac{n_x(\lambda - 1) + \beta}{n_x(\beta - n_x) + \lambda}; \quad h_{23} = -\frac{n_x(\lambda - 1) + \beta}{n_x(\beta - n_x) + \lambda}; \tag{28}
\]

where \(e_{23} = -e_{32}, h_{23} = -h_{32} = e_{32}\). The parameters and weight function have the following transformational properties because of the Galilei addition theorem of velocities \(\beta'' = \beta + \lambda \beta'\) and transformation law of guiding cosines \(n'_x = (n_x - \beta)/\lambda, n'_y = n_y/\lambda, n'_z = n_z/\lambda\)

\[
\Phi'' = \Phi'; \quad d'' = (d' + d)/(1 + d'd); \quad k'' = k'k(1 + d'd), \tag{29}
\]

where \(d = (e_{23}, e_{32}, h_{23}, h_{32}), \beta'' = V''/c, \beta' = V'/c, \beta = V/c, \lambda = c'/c\). For comparison, in relativistic theory with \(\Phi = 1, k = 1/(1 - \beta^2)^{1/2}, e_{23} = \beta, h_{23} = -\beta\) the relations \((29)\) are valid because of the relativistic addition theorem of velocities \(\beta'' = (\beta' + \beta)/(1 + \beta'\beta)\). Transformations of fields \((27)\) hold invariance of the forms \(E'H' = k^2\Phi'^2E'H = 0, \quad E'^2 - H'^2 = k^2\Phi^2(E^2 - H^2) = 0\). When parameter \(\beta \to 0\), the weight function \(\Phi_D(x) \approx 1\), the transformations of the electric and magnetic fields have a limit \(E \approx E + \beta \times H, \quad H \approx H - \beta \times E\).
coincident with the known non-relativistic one. It is the common limit for the relativistic and Galilei transformations of electromagnetic field. This result is similar to the one, known as non-relativistic limit of the space-time transformations \(x' = x - Vt, \ y' = y, \ z' = z, \ t' = t, \ c' = c\) in the relativistic theory. Being neither Lorentz transformations nor Galilei ones, these transformations are the same limit for both Lorentz and Galilei space and time transformations indeed.

4 Conclusion

In summary it is possible to state, that the concept of symmetry is conventional. Dividing equations into the relativistic and the Galilei-invariant equations makes sense only in the case, when \(p = 1\). In more general case, when \(p \geq 1\), equations have cumulative symmetrical properties complying with the principles of relativity in the relativistic, in the Galilei, as well as in the other versions. In particular, Poincaré group \(P_{10}\) is the classical group of symmetry of D’Alembert and Maxwell equations with \(p = 1\) and reflects the property of relativistic invariance of these equations. The Galilei group \(G_{10}\) is the generalized group of symmetry of D’Alembert and Maxwell equations with \(p = 2\) and reflects the property of invariance of these equations relative to the space and time transformations of the classical physics.

Analogous situation takes place in the case of Schrödinger equation with the difference that the Galilei group is not the classical one but the generalized group of symmetry of the equation with \(p = 1\) and the Poincaré group is the generalized group of symmetry with \(p = 2\).

Both groups are the subgroups of the 20-dimentional group of inhomogeneous linear space and time transformations \(IGL(4,R)\) in the space \(R^4(x)\). This group is the maximal linear group symmetry \(\mathbb{1}\) of the discussed equations in view of relations

\[
[\Box, P_a] = [L^r_s, P_a] = 0; \ [\Box[\Box, G_{ab}]] = [L^r_s[L^r_s, G_{ab}]] = 0,
\]

(30)

where \(P_a = \partial_a, \ G_{ab} = x^a \partial_b\). Owing to this relations, the Galilei symmetry of the D’Alembert and of the free Maxwell equations and relativistic symmetry of the free Schrödinger equation (the new symmetries of these equations) are not exotic but their natural properties because the generators of corresponding groups may be composed from the generators of Lie algebra of \(IGL(4,R)\) group.

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