Polynomials, numerical triangles and sequences associated with the probability distribution of the hyperbolic cosine type for odd values of the parameter

M S Tokmachev
Yaroslav-the-Wise Novgorod State University, Veliky Novgorod, Russian Federation
E-mail: mikhail.tokmachov@novsu.ru

Abstract The article introduces a new class of polynomials that first appeared in the probability distribution density function of the hyperbolic cosine type. With an integer change in one of the parameters of this distribution, polynomials in the form of a product of positive factors are written out with an increasing degree. Earlier, the author found a connection between the distribution of the hyperbolic cosine type and numerical sets, in particular, in the simplest cases with the triangle of coefficients of Bessel polynomials, the triangle of Stirling numbers, sequences of coefficients in the expansion of various functions, etc. Also from the distribution formed numerous numerical sequences, both new and widely known. Consideration of polynomials separately from the density function made it possible to reconstruct numerical sets of coefficients, ordered in the form of numerical triangles and numerical sequences. The connections between the elements of the sets are established. Among the sequences obtained, in the simplest cases, there are those known from others, for example, physical problems. However, the overwhelming majority of the found number sets have not been encountered earlier in the literature. The obvious applications of this research are number theory and algebra. And the interdisciplinarity of the results indicates the possibility of applications and enhances their practical significance in other areas of knowledge.

1. Introduction
For the first time, the three-parameter probability distribution, called by the author the distribution of the hyperbolic cosine type, was published in 1994 [1]. It is one of the characterizations of the distributions by the condition of constancy of the regression of the quadratic statistics on the linear form for resampling. Characterizations of a number of popular distributions (in particular, the normal, gamma distribution) were known earlier [2], [3]. In [4], an attempt was made to systematize all cases of characterization of distributions by the condition of constancy of the regression of quadratic statistics on a linear form, depending on the ratio of the coefficients of this statistics. However, the results obtained turned out to be incomplete and were revised in [5]: this is how new distribution problems of the negative binomial type and three-parameter hyperbolic cosine type appeared, $Ch(m, \beta, \mu)$. The paper also considers the cases of a degenerate distribution and the absence of probability distributions.

The three-parameter distribution of the hyperbolic cosine type is a generalization of the well-known two-parameter Meixner distribution [6], [7] which, in turn, generalizes the one-parameter distribution of the hyperbolic cosine (hyperbolic secant) [8], [9], [10], [11] and, according to the just remark of W. Feller, [10], it is “very curious”. As further research showed, [12], [13], [14], the
distribution is no less curious and interesting both in theoretical terms and in applications. We note that the family of probability distributions obtained in [5] coincides with the Meixner family [15], [16] found using a fundamentally different characterization, and in which the Meixner distribution is a classical two-parameter.

The \( Ch(m, \beta, \mu) \) distribution has a characteristic function of the form

\[
f(t) = \left( \frac{\cosh t - i \frac{\beta}{m} \sinh t}{m \beta} \right)^{-m}, \quad \text{where } \mu, \beta, m \in \mathbb{R}; \quad m > 0, \beta \neq 0. \tag{1}
\]

The probability density \( p_m(x) \) calculated as the inverse Fourier transform of the characteristic function (see, for example, [17], [10]), for type \( f(t) \) (1) for \( f(t) \) has the form

\[
p_m(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} f(t) dt = \frac{2^{m-2}m^{m-1}}{\pi(\beta^2 + \mu^2)^{m/2}} \left( \frac{\beta - i\mu}{\beta + i\mu} \right)^{imx} B\left( \frac{m}{2}, \frac{imx}{2\beta}; \frac{m}{2} + \frac{imx}{2\beta} \right), \tag{2}
\]

where \( i = \sqrt{-1} \), \( B(p; q) \) is the beta function, and the factor with the imaginary unit is converted to the real form:

\[
A \equiv \left( \frac{\beta - i\mu}{\beta + i\mu} \right)^{imx} =
\begin{cases}
  e^{\text{arctg} \frac{2\beta\mu}{\beta^2 - \mu^2}} & \text{for } \beta^2 - \mu^2 > 0, \\
  e^{\text{arctg} \frac{2\beta\mu}{\beta^2 - \mu^2} + \pi \text{ sign } \mu} & \text{for } \beta^2 - \mu^2 < 0, \\
  e^{\frac{\mu}{2} \text{ sign } \mu} & \text{for } \beta^2 - \mu^2 = 0.
\end{cases}
\]

The found moments \( M(X^n) \) of the distribution made it possible to calculate a number of nontrivial integrals of a combination of power, exponential functions, hyperbolic cosine or sine, and a certain class of polynomials [13].

To calculate the moments, moment-generating polynomials were introduced, recurrent differential relations for polynomials of the type \( P_n(m; b) \) presented in [18], [19], as well as algebraic relations for the connection and calculation of the coefficients \( U(n; k, j) \) of these polynomials, were found, see [14], [13]:

\[
P_{n+1}(m; b) = mbP_n(m; b) + (1 + b^2) \frac{\partial P_n(m; b)}{\partial b}
\]

where

\[
P_n(m; b) = \sum_{k=1}^{n} \sum_{j=1}^{n} U(n; k, j)m^{k}b^{j}, \quad b = \frac{\mu}{\beta}, \tag{3}
\]

\[
U(0; 0, 0) = 1; \quad U(0; k, j) = 0 \text{ for } k \neq 0 \text{ or } j \neq 0;
\]

\[
U(n + 1; k, j) = U(n; k - 1, j - 1) + (j - 1)U(n; k, j - 1) + (j + 1)U(n; k, j + 1) \tag{4}
\]

for \( n = 0, 1, 2, \ldots \)....

For various arguments \( n; k, j \), the \( U(n; k, j) \) coefficients are ordered and systematized into a numerical prism (in geometric terms). When fixing one or two parameters of the section, the prisms form numerical triangles or, respectively, integer sequences [14], [13] among which there are widely known (for example, the numerical Stirling triangle, the triangle of the coefficients of Bessel polynomials, sequences of tangential and secant numbers, generalized Euler numbers and others [20]) and new ones. The novelty of these numerical sets was checked against the widely known and fairly complete and updated electronic encyclopedia OEIS (see [21]). Taken together, the results are presented in [13].

In the work under consideration, a different approach to the construction of numerical sets is carried out: the class of polynomials that appear in the distribution density function \( Ch(m, \beta, \mu) \) when the \( m, \) where \( m \in \mathbb{N} \) parameter changes are studied. In this case, the density function, and hence
the polynomials for odd and even values \( m \), are described by close but different relations. In this work, the parameter \( m = 2k - 1 \). The coefficients of the polynomials are considered, when ordering which numerical triangles and integer sequences are formed. Their inclusion in the OEIS encyclopedia was checked [21], and the novelty of the results was assessed.

2. Main points

2.1. Density function. Polynomials

The distribution density function \( p_m(x) \) of the hyperbolic cosine type \( Ch(m, \beta, \mu) \) in general form with arbitrary admissible parameters is represented by relation (2).

When \( m \in \mathbb{N} \) transforming the B-function in (2), it greatly simplifies the expressions, allows them to be represented \( p_m(x) \) in elementary functions.

Let us write down the corresponding formulas for odd values of the parameter \( m \):

\[
p_1(x) = \frac{A^{z \beta}}{z \sqrt{\beta^2 + \mu^2}} \frac{\pi}{2 \beta}; (5)
\]

\[
p_m(x) = \frac{m}{2(\beta^2 + \mu^2)^{m/2} \, \Gamma(m-1)} \text{ch} \left[ \frac{mx}{z \beta} \right] \prod_{s=1}^{m-1} [(2s - 1)^2 \beta^2 + m^2 x^2] \text{ for } m = 3, 5, 7, \ldots (6)
\]

According to [12], [13], the formulas are valid for \( \beta > 0 \). Note that for \( \beta < 0 \), the density formulas \( p_m(x) \) will be somewhat different.

In (5), (6), we separate polynomial factors from the functions \( p_m(x) \), \( m = 1, 3, 5, \ldots \), and write them in the form of independent polynomials:

\[
Q_1 \beta(x) = 1; \quad Q_m \beta(x) = \prod_{s=1}^{m-1} [(2s - 1)^2 \beta^2 + m^2 x^2] \text{ for } m = 3, 5, 7, \ldots (7)
\]

Then, according to (6), (7), the density \( p_m(x) \) is written in the form

\[
p_m(x) = \frac{m}{2(\beta^2 + \mu^2)^{m/2} \, \Gamma(m-1)} \text{ch} \left[ \frac{mx}{z \beta} \right] Q_m \beta(x) \text{ for } m = 1, 3, 5, \ldots (8)
\]

For simplicity, we assume \( \beta = 1 \) and \( Q_m \beta(x) \equiv Q_m(x) \). The polynomials introduced in this way are the subject of further research.

Let us write down concrete expressions for first-order polynomials with odd indices.

\[
Q_1(x) = 1;
\]

\[
Q_3(x) = (1 + 9x^2);
\]

\[
Q_5(x) = (1 + 25x^2)(9 + 25x^2) = 9 + 250x^2 + 625x^4;
\]

\[
Q_7(x) = (1 + 49x^2)(9 + 49x^2)(25 + 49x^2) = 225 + 12691x^2 + 84035x^4 + 117649x^6;
\]

\[
Q_9(x) = (1 + 81x^2)(9 + 81x^2)(25 + 81x^2)(49 + 81x^2) = 11025 + 104619x^2 + 1295144x^4 + 44641044x^6 + 43046721x^8;
\]

\[
Q_{11}(x) = (1 + 121x^2)(9 + 121x^2)(25 + 121x^2)(49 + 121x^2)(81 + 121x^2) = 893025 + 127923741x^2 + 253011210x^4 + 15550762458x^6 + 35369215365x^8 + 25937424601x^{10};
\]

......
Note that for odd $m$, the order of the polynomial $Q_m(x)$ is $m - 1$. All polynomials $Q_m(x)$ are even functions.

2.2. **Polynomial coefficients. Recurrent formulas. Sequences**

In general, we write the polynomial $Q_m(x)$ as

$$Q_m(x) = a_m^{(0)} + a_m^{(2)}x^2 + a_m^{(4)}x^4 + \cdots + a_m^{(m-1)}x^{m-1}, m = 1, 3, 5, \ldots$$  \hspace{1cm} (10)

To study the coefficients of polynomials $Q_m(x)$, we will form them in the form of a table representing a numerical triangle.

**Table 1.** The numerical triangle of the coefficients of the polynomials $Q_m(x)$.

| $x^0$ | $x^2$ | $x^4$ | $x^6$ | $x^8$ | $x^{10}$ | $x^{12}$ |
|-------|-------|-------|-------|-------|----------|----------|
| $Q_1(x)$ | 1     |       |       |       |          |          |
| $Q_3(x)$ | 1     | 9     |       |       |          |          |
| $Q_5(x)$ | 9     | 250   | 625   |       |          |          |
| $Q_7(x)$ | 225   | 12691 | 84035 | 117649 |          |          |
| $Q_9(x)$ | 11025 | 1046196 | 12951414 | 44641044 | 43046721 |          |
| $Q_{11}(x)$ | 893025 | 127923741 | 2530111210 | 15550762458 | 35369215365 | 25937424601 |
| $Q_{13}(x)$ | 108056025 | 21770033454 | 627406084591 | 5960858120932 | 23446548113703 | 39427528668814 | 23298085122481 |
| ... | ... | ... | ... | ... | ... | ... | ...

In table 1, the coefficients of the polynomials $Q_m(x), m = 1, 3, 5, \ldots$, with the same degrees $x$, are ordered by columns and form integer sequences.

**Theorem 1.** For polynomials $Q_m(x), m = 1, 3, 5, \ldots$, the sequence of coefficients $\{a_m^{(0)}\}$ can be restored by the recursive formula

$$a_m^{(0)} = a_m^{(2)} \cdot m^2, \quad (11)$$

i.e.

$$a_m^{(0)} = 1 \cdot 9 \cdot 25 \cdot 49 \cdot \ldots \cdot m^2 = (1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot m)^2. \quad (12)$$

**Proof of Theorem 1.** This statement follows from the structure of polynomials $Q_m(x)$, presented in (12). Each value $a_m^{(0)} = Q_m(0)$ is the product of free terms of the factors of the corresponding polynomial, which is actually stated in the theorem (see (12), also see (7): $Q_{m1}(0) \equiv Q_m(0)$).

**Theorem 2.** For polynomials $Q_m(x), m = 1, 3, 5, \ldots$, the sequence of coefficients can be restored by the recursive formula

$$a_m^{(2k)} = \left(\frac{a_m^{(2k-2)} + a_m^{(2k)}}{m^{2k-2}}\right) \cdot (m + 2)^{2k}, \quad (13)$$

or, which is the same, for $m = 2n - 1$ ($n = 1, 2, 3, \ldots$)

$$a_{2n+1}^{(2k)} = \left(\frac{a_{2n-1}^{(2k-2)} + a_{2n-1}^{(2k)}}{(2n-1)^{2k-2}}\right) \cdot (2n + 1)^{2k}. \quad (14)$$

Let us present special cases of recurrent formulas (13), (14) of the sequences of coefficients.

**Corollary 2.1.** A sequence of coefficients $\{a_m^{(2)}\}$, i.e. coefficients of polynomials at $x^2$ can be restored by the recursive formula
Corollary 2.2. A sequence of coefficients \( \{a_m^{(0)}\} \), i.e. coefficients of polynomials at \( x^4 \) can be restored by the recursive formula

\[
a_{m+2}^{(2)} = \left( a_{m}^{(0)} + a_{m}^{(2)} \right) \cdot (m + 2)^2.
\]

Let us write out the obtained sequences of coefficients of polynomials at various degrees \( x \) and identify them by the known sequences in the OEIS encyclopedia.

The sequence is known in the OEIS as A001818 – squares of double factorials; this sequence is also included, for example, in the encyclopedia [22].

\[
\{a_m^{(0)}\} = 1, 1, 9, 225, 11025, 893025, 108056025, 18261468225, 4108830350625, ...
\]

This sequence does not exist in the OEIS.

\[
\{a_m^{(2)}\} = 9, 250, 12691, 1046196, 127923741, 21770033454, 4922570132775, ...
\]

This sequence does not exist in the OEIS.

\[
\{a_m^{(4)}\} = 625, 84035, 12951414, 2530111210, 627406084591, 194464739503125, ...
\]

This sequence does not exist in the OEIS.

\[
\{a_m^{(6)}\} = 117649, 44641044, 15550762458, 5960858120932, 2627514686671875, ...
\]

This sequence does not exist in the OEIS.

It can be assumed that the following sequences \( \{a_m^{(2k)}\} \) for odd \( m \) and fixed \( k > 3 \) are absent in the OEIS at the moment.

Also in the OEIS there is no sequence of all coefficients of polynomials \( Q_m(x) \) for odd ones \( m \):

\[
(1, 1, 9, 9, 250, 625, 225, 12691, 84035, 117649, 11025, 1046196, 12951414, ...).
\]

Next, we present the sequences \( Q_m(x), m = 1, 3, 5, ... \), for some specific values of the variable \( x \).

\[
\{Q_m(\pm 1)\} = 1, 10, 884, 214600, 101696400, 795163304, 92782304200000, 151115361757776000, ...
\]

\[
\{Q_m(\pm 2)\} = 1, 37, 11009, 8925085, 14088405825, 36650685089525, 142196682940938625, ...
\]

\[
\{Q_m(\pm 3)\} = 1, 82, 52884, 92687400, 316031348880, 1775179004320800, 14867943079616596800, ...
\]

\[
\{Q_m(\pm 4)\} = 1, 145, 164009, 503606545, 3007291935825, 29579663515749425, ...
\]

These sequences are not found in the OEIS. It can be assumed that other sequences of this type are absent in the OEIS, in particular, \( \{Q_m(\pm 5)\} \) etc.

2.3. Study of polynomials

From the structure of polynomials \( Q_m(x), m = 1, 3, 5, ... \), it obviously follows that these are even functions.

The polynomial \( Q_m(x) \) has a degree \( (m - 1) \) and, accordingly, has \( (m - 1) \) roots. All roots are simple, purely imaginary, and due to the parity of the function, they form pairs: if \( \alpha i \) is a root of a polynomial, then \( -\alpha i \) is also a root. Thus, the roots of the polynomials \( Q_m(x), m = 1, 3, 5, ... \), are
located on the imaginary axis in pairs symmetrically to zero, while $x = 0$ is not a root of the polynomial.

Let’s write down these roots.

$$Q_3(x): \left\{ \pm \frac{1}{3}i \right\};$$

$$Q_5(x): \left\{ \pm \frac{1}{5}i; \pm \frac{3}{5}i \right\};$$

$$Q_7(x): \left\{ \pm \frac{1}{7}i; \pm \frac{3}{7}i; \pm \frac{5}{7}i \right\};$$

$$Q_9(x): \left\{ \pm \frac{1}{9}i; \pm \frac{3}{9}i; \pm \frac{5}{9}i; \pm \frac{7}{9}i \right\};$$

$$Q_{11}(x): \left\{ \pm \frac{1}{11}i; \pm \frac{3}{11}i; \pm \frac{5}{11}i; \pm \frac{7}{11}i; \pm \frac{9}{11}i \right\};$$

In general:

$$Q_m(x): \left\{ \pm \frac{1}{m}i; \pm \frac{3}{m}i; \pm \frac{5}{m}i; \ldots; \pm \frac{m-2}{m}i \right\}, m = 3, 5, 7, \ldots$$

Polynomial plots $Q_m(x)$ for different $m$ have the same configuration. A typical polynomial plot $Q_7(x)$ is shown in Figure 1.

![Figure 1. Polynomial plot $Q_7(x)$.](image)

2.4. Polynomials of the form $Q_{m\beta}(x)$ and $Q_{m\beta}(\beta x)$

Earlier in (7) denoted where $\beta > 0$, polynomials of the form

$$\prod_{s=1}^{m-1} [(2s - 1)^2 \beta^2 + m^2 x^2] ~ for ~ m = 3, 5, 7, \ldots.$$  

It is in this generalized form that these polynomials are represented in the density function $p_m(x)$, see the relations (6), (8).

In particular,

$$Q_{1\beta}(x) = 1; ~ Q_{3\beta}(x) = (\beta^2 + 9x^2);$$

$$Q_{5\beta}(x) = (\beta^2 + 25x^2)(9\beta^2 + 25x^2) = 9\beta^4 + 250\beta^2x^2 + 625x^4;$$
Accordingly, the roots of these polynomials will also change in comparison with \( Q_m(x) \):

\[
Q_{3\beta}(x) : \left\{ \pm \frac{\beta}{3} i \right\};
\]
\[
Q_{5\beta}(x) : \left\{ \pm \frac{\beta}{5} i; \pm \frac{3\beta}{5} i \right\};
\]
\[
Q_{7\beta}(x) : \left\{ \pm \frac{\beta}{7} i; \pm \frac{3\beta}{7} i; \pm \frac{5\beta}{7} i \right\};
\]
\[
Q_{9\beta}(x) : \left\{ \pm \frac{\beta}{9} i; \pm \frac{3\beta}{9} i; \pm \frac{5\beta}{9} i; \pm \frac{7\beta}{9} i \right\};
\]
\[
Q_{11\beta}(x) : \left\{ \pm \frac{\beta}{11} i; \pm \frac{3\beta}{11} i; \pm \frac{5\beta}{11} i; \pm \frac{7\beta}{11} i; \pm \frac{9\beta}{11} i \right\};
\]

In general:

\[
Q_{m\beta}(x) : \left\{ \pm \frac{\beta}{m} i; \pm \frac{3\beta}{m} i; \pm \frac{5\beta}{m} i; \ldots; \pm \frac{(m-2)\beta}{m} i \right\}, \quad m = 3, 5, 7, \ldots
\]  \hspace{1cm} (15)

Assuming the coefficients of the polynomials \( Q_{m\beta}(x) \), \( m = 1, 3, 5, \ldots \), at the same degrees \( x \), we obtain new numerical sequences \( \left\{ a_{m\beta}^{(k)} \right\} \).

For example,

\[
\left\{ a_{m\beta}^{(0)} \right\} = 1, \beta^2, 9\beta^4, 225\beta^6, 11025\beta^8, 893025\beta^{10}, 108056025\beta^{12}, 18261468225\beta^{14}, 410883053625\beta^{16}, \ldots.
\]

The resulting sequence is the squares of double factorials multiplied by the corresponding degrees \( \beta^2 \), i.e. is a generalization of the sequence A001818 in the OEIS.

In particular, at \( \beta^2 = 2 \) follows

\[
\left\{ a_{m\beta}^{(0)} \right\} = 1, 2, 36, 1800, 176400, 28576800, 6915585600, 2337467932800, \ldots.
\]

The specified sequence, but with alternating sign, is given in the OEIS as A126934 (for more details on the sequence, see [23], [24], [25]).

Similar sequences \( \left\{ a_{m\beta}^{(2)} \right\} \) and \( \left\{ a_{m\beta}^{(4)} \right\} \) are of the form

\[
\left\{ a_{m\beta}^{(2)} \right\} = 9, 250\beta^2, 12691\beta^4, 1046196\beta^6, 127923741\beta^8, 21770033454\beta^{10}, 4922570132775\beta^{12}, \ldots;
\]
\[
\left\{ a_{m\beta}^{(4)} \right\} = 625, 84035\beta^2, 12951414\beta^4, 2530111210\beta^6, 627406084591\beta^8, 194464739503125\beta^{10}, \ldots;
\]
The indicated sequences and others similar in OEIS are absent even in special cases with specific $\beta$. The considered polynomials $Q_m(\beta x)$ are somewhat simplified for the argument $\beta x$, namely:

$$Q_m(\beta x) = \prod_{s=1}^{m-1} [(2s-1)^2 + m^2 x^2] = \beta^{m-1} Q_m(x), \quad m = 3, 5, 7, \ldots \quad (16)$$

In particular,

$$Q_3(\beta x) = 1; \quad Q_5(\beta x) = \beta^2(1 + 9x^2) = \beta^2 Q_3(x);$$
$$Q_5(\beta x) = \beta^4(1 + 25x^2)(9 + 25x^2) = \beta^4 Q_5(x);$$
$$Q_7(\beta x) = \beta^6(1 + 49x^2)(9 + 49x^2)(25 + 49x^2) = \beta^6 Q_7(x).$$

Obviously, the roots of the polynomials $Q_m(\beta x)$ and $Q_m(x)$ coincide.

Based on (16), a factor $\beta^{m-1}$ will be added to the corresponding sequences of coefficients $\{a_m^{(k)}\}$.

In particular, for the argument $\beta x$, we obtain the sequences

$$\{a_m^{(2)}\} = 9\beta^2, 250\beta^4, 12691\beta^6, 1046196\beta^8, 127923741\beta^{10}, 21770033454\beta^{12}, 4922570132775\beta^{14}, \ldots ;$$

$$\{a_m^{(4)}\} = 625\beta^4, 84035\beta^6, 12951414\beta^8, 2530111210\beta^{10}, 627406084591\beta^{12}, 194464739503125\beta^{14}, \ldots.$$  

Depending on the value, the sequence of coefficients may not be integer.

### 3. Conclusion

The class of polynomials $Q_m(x)$ considered in the work, and also $Q_m(\beta x)$, appeared from an applied problem: the probability distribution density function of the hyperbolic cosine type. Thus, there is automatically a practical application of these polynomials in probability theory.

As already noted, the indicated distribution was previously used by the author as a source of numerical triangles and integer sequences. The coefficients of the polynomials also form numerical triangles and sequences that have not been encountered before in the literature. However, in some of the simplest special cases, sequences of coefficients also arise, known from others, in particular, physical problems. It is possible that the numerous sequences introduced for the first time in this work will find their applications in real scientific research. And the established connections between the elements of the sequences will contribute to the emergence of new results in applied disciplines.

The results obtained can certainly be used in mathematical fields such as number theory, algebra, and information technology.

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