CRITICAL METRICS OF THE VOLUME FUNCTIONAL
ON MANIFOLDS WITH BOUNDARY

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Abstract. The goal of this article is to study the space of smooth Riemann-
ian structures on compact manifolds with boundary that satisfies a critical
point equation associated with a boundary value problem. We provide an in-
tegral formula which enables us to show that if a critical metric of the volume
functional on a connected \( n \)-dimensional manifold \( M^n \) with boundary \( \partial M \)
has parallel Ricci tensor, then \( M^n \) is isometric to a geodesic ball in a simply
connected space form \( \mathbb{R}^n \), \( \mathbb{H}^n \) or \( \mathbb{S}^n \).

1. Introduction

An outstanding problem in differential geometry is to find Riemannian
metrics on a given manifold \( M^n \) that provides constant curvature. In this sense, it is crucial
to understand the critical metrics of the Riemannian functionals, as for instance, the
total scalar curvature functional and the volume functional. Einstein and Hilbert
have proven that the critical points of the total scalar curvature functional restricted
to the set of smooth Riemannian structures on \( M^n \) of unitary volume are Einstein
(cf. Theorem 4.21 in [4]). Moreover, the total scalar curvature functional restricted
to a given conformal class is just the Yamabe functional, whose critical points
are constant scalar curvature metrics in that class. Hilbert [12] proved that the
equations of general relativity can be recovered from the total scalar curvature
functional. From this, we have a natural way to prove the existence of Einstein
metrics.

Inspired by a result obtained in [11] as well as in the characterization of the
critical points of the total scalar curvature functional, Miao and Tam (cf. [15]
and [16]) studied variational properties of the volume functional constrained to the
space of metrics of constant scalar curvature on a given compact manifold with
boundary. While Corvino, Eichmair and Miao [7] studied the modified problem of
finding stationary points for the volume functional on the space of metrics whose
scalar curvature is equal to a given constant.

Following the terminology used in [2,3] we recall the definition of Miao-Tam
critical metrics.
Definition 1. A Miao-Tam critical metric is a 3-tuple \((M^n, g, f)\), where \((M^n, g)\) is a compact Riemannian manifold of dimension at least three with a smooth boundary \(\partial M\) and \(f : M^n \to \mathbb{R}\) is a smooth function such that \(f^{-1}(0) = \partial M\) satisfying the overdetermined-elliptic system

\[
\mathcal{L}_g^*(f) = g.
\]

Here, \(\mathcal{L}_g^*\) is the formal \(L^2\)-adjoint of the linearization of the scalar curvature operator \(\mathcal{L}_g\). Such a function \(f\) is called a potential function.

We also recall that

\[
\mathcal{L}_g^*(f) = -(\Delta f)g + \text{Hess} f - f \text{Ric},
\]

where \(\text{Ric}\), \(\Delta\) and \(\text{Hess}\) stand, respectively, for the Ricci tensor, the Laplacian operator and the Hessian form on \(M^n\); see for instance [4]. Therefore, the fundamental equation of Miao-Tam critical metrics (1.1) can be rewritten as

\[
-(\Delta f)g + \text{Hess} f - f \text{Ric} = g.
\]

In [15], it was remarked that Miao-Tam critical metrics arise as critical points of the volume functional on \(M^n\) when restricted to the class of metrics \(g\) with prescribed constant scalar curvature such that \(g|_\partial M = h\) for a prescribed Riemannian metric \(h\) on the boundary. In addition, they showed that such metrics have constant scalar curvature \(R\). For more background see e.g. Proposition 2.1 and Theorem 2.3 in [7]. Some explicit examples of Miao-Tam critical metrics can be found in [15][16]. They include the spatial Schwarzschild metrics and AdS-Schwarzschild metrics restricted to certain domains containing their horizon and bounded by two spherically symmetric spheres. We also remember that the standard metrics on geodesic balls in space forms are Miao-Tam critical metrics. For more details see Theorem 6 in [15].

Miao and Tam [16] posed the question of whether there exist non-constant sectional curvature Miao-Tam critical metrics on a compact manifold whose boundary is isometric to a standard round sphere. In this sense, inspired by ideas outlined by Kobayashi [13], Kobayashi and Obata [14], they proved that a locally conformally flat simply connected, compact Miao-Tam critical metric \((M^n, g, f)\) with boundary isometric to a standard sphere \(\mathbb{S}^{n-1}\) must be isometric to a geodesic ball in a simply connected space form \(\mathbb{R}^n\), \(\mathbb{H}^n\) or \(\mathbb{S}^n\).

In order to proceed, we recall three special tensors in the study of curvature for a Riemannian manifold \((M^n, g)\), \(n \geq 3\). The first one is the Weyl tensor \(W\) which is defined by the following decomposition formula:

\[
R_{ijkl} = W_{ijkl} + \frac{1}{n-2} \left( R_{ik} g_{jl} + R_{jl} g_{ik} - R_{il} g_{jk} - R_{jk} g_{il} \right)
\]

\[
- \frac{R}{(n-1)(n-2)} \left( g_{jl} g_{ik} - g_{il} g_{jk} \right),
\]

where \(R_{ijkl}\) stands for the Riemann curvature operator \(Rm\), whereas the second one is the Cotton tensor \(C\) given by

\[
C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} \left( \nabla_i R g_{jk} - \nabla_j R g_{ik} \right).
\]
Note that $C_{ijk}$ is skew-symmetric in the first two indices and trace-free in any two indices. These two above tensors are related as follows:

\begin{equation}
C_{ijk} = \frac{(n-2)}{(n-3)} \nabla_l W_{ijkl},
\end{equation}

provided $n \geq 4$. We also recall that the Bach tensor \[1\] on a Riemannian manifold $(M^n, g)$, $n \geq 4$, is defined in terms of the components of the Weyl tensor $W_{ikjl}$ as follows:

\begin{equation}
B_{ij} = \frac{1}{n-3} \nabla^k \nabla_l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{ij}^k j^l,
\end{equation}

while for $n = 3$ it is given by

\begin{equation}
B_{ij} = \nabla^k C_{kij}.
\end{equation}

We say that $(M^n, g)$ is Bach-flat when $B_{ij} = 0$. It is easy to check that locally conformally flat metrics as well as Einstein metrics are Bach-flat.

Recently, Barros, Diógenes and Ribeiro [2], based on the techniques outlined in a work of Cao and Chen [5], proved that a Bach-flat simply connected, compact Miao-Tam critical metric with boundary isometric to a standard sphere $S^3$ must be isometric to a geodesic ball in a simply connected space form $R^4$, $H^4$ or $S^4$. Further, they showed that in dimension three the result even is true replacing the Bach-flat condition by the weaker assumption that $M^3$ has divergence-free Bach tensor. For more details, we refer the reader to [2].

At the same time, Miao and Tam [16] studied these critical metrics under Einstein condition. In that case, they were able to remove the condition of boundary isometric to a standard sphere. More precisely, they obtained the following result.

**Theorem 1** (Miao-Tam, [16]). Let $(M^n, g, f)$ be a connected, compact Einstein Miao-Tam critical metric with smooth boundary $\partial M$. Then $(M^n, g)$ is isometric to a geodesic ball in a simply connected space form $R^n$, $H^n$ or $S^n$.

For what follows, we point out that every Riemannian manifold with parallel Ricci tensor has harmonic curvature. But, the converse statement is not true; see [9] and [10] for more details. Indeed, there are examples of compact and non-compact Riemannian manifolds with harmonic curvature but non-parallel Ricci tensor. Here, motivated by the historical development on the study of critical metrics of the volume functional, we shall replace the assumption of Einstein in the Miao-Tam result (cf. Theorem 1) by the parallel Ricci tensor condition, which is weaker than the former. In order to do so, we have established the following result.

**Theorem 2.** Let $(M^n, g, f)$ be a compact, oriented, connected Miao-Tam critical metric with smooth boundary $\partial M$. Then we have:

\[
\int_M f |\text{div} Rm|^2 dM_g + \frac{n}{n-1} \int_M |\text{Ric} - \frac{R}{n} g|^2 dM_g + \int_M f (\Delta |\text{Ric}|^2 - |\nabla \text{Ric}|^2) dM_g = 0.
\]

In the sequel, as an application of Theorem 2 we get the following rigidity result.

**Corollary 1.** Let $(M^n, g, f)$ be a compact, oriented, connected Miao-Tam critical metric with parallel Ricci tensor and smooth boundary $\partial M$. Then $(M^n, g)$ is isometric to a geodesic ball in a simply connected space form $R^n$, $H^n$, or $S^n$. 


It is easy to check that Einstein manifolds $M^n$, $n \geq 3$, have parallel Ricci tensor. Therefore, Corollary 1 clearly improves Theorem 1. Moreover, it is worth pointing out that our arguments designed for the proof of Theorem 2 differ significantly from [16].

Let us also highlight that every Einstein manifold has harmonic Weyl tensor (cf. [9]). Furthermore, we already know that there is no relationship between the Bach-flat condition, considered in [2], and the condition that $M^n$ has harmonic Weyl tensor. Therefore, it is natural to ask which geometric implications has the assumption of the harmonicity of the Weyl tensor on a Miao-Tam critical metric. By using once more Theorem 2 jointly with Theorem 1, it is straightforward to check that if a compact, oriented, connected Miao-Tam critical metric with harmonic Weyl tensor and smooth boundary satisfies

\begin{equation}
\int_M f \Delta |\text{Ric}|^2 dM_g \geq \int_M f |\nabla \text{Ric}|^2 dM_g,
\end{equation}

then $(M^n, g)$ is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$, or $S^n$. Nonetheless, it is very interesting to prove that condition (1.8) can be removed.

2. Preliminaries

In this section we shall present some preliminaries which will be useful for the establishment of the desired results. First, we recall that the fundamental equation of a Miao-Tam critical metric (1.1) becomes

\begin{equation}
-(\Delta f)g + H_{\text{ess}}f - f \text{Ric} = g.
\end{equation}

In particular, tracing (2.1) we have

\begin{equation}
(n - 1)\Delta f + Rf = -n.
\end{equation}

For the sake of simplicity, we now rewrite equation (2.1) in the tensorial language as follows:

\begin{equation}
-(\Delta f)g_{ij} + \nabla_i \nabla_j f - f R_{ij} = g_{ij}.
\end{equation}

Moreover, by using (2.2) it is not difficult to check that

\begin{equation}
f \bar{\text{Ric}} = H_{\text{ess}}f,
\end{equation}

where $\bar{T}$ stands for the traceless of $T$. We also remember that as a consequence of Bianchi identity we have

\begin{equation}
(\text{div} Rm)_{ijkl} = \nabla_k R_{jl} - \nabla_l R_{jk}.
\end{equation}

Under this notation we get the following lemma obtained previously in [2].

**Lemma 1** ([2]). Let $(M^n, g, f)$ be a Miao-Tam critical metric. Then we have:

$$f(\nabla_i R_{jk} - \nabla_j R_{ik}) = R_{ijkl} \nabla_l f + \frac{R}{n - 1}(\nabla_i f \nabla_j g_{lk} - \nabla_j f \nabla_i g_{lk}) - (\nabla_i f R_{jk} - \nabla_j f R_{ik}).$$

**Proof.** Since the proof of this lemma is very short, we include it here for the sake of completeness. First, we use (2.3) to infer

\begin{equation}
(\nabla_i f) R_{jk} + f \nabla_i R_{jk} = \nabla_i \nabla_j \nabla_k f - (\nabla_i \Delta f) g_{jk}.
\end{equation}
Next, since $M^n$ has constant scalar curvature we have from (2.2) that

$$\nabla_i \Delta f = -\frac{R}{n-1} \nabla_i f,$$

which substituted into (2.6) gives

$$(2.7) \quad f \nabla_i R_{jk} = -(\nabla_i f) R_{jk} + \nabla_i \nabla_j \nabla_k f + \frac{R}{n-1} \nabla_i f g_{jk}.$$

Finally, it suffices to apply the Ricci identity to arrive at

$$f(\nabla_i R_{jk} - \nabla_j R_{ik}) = R_{ijkl} \nabla_l f + \frac{R}{n-1} (\nabla_i f g_{jk} - \nabla_j f g_{ik}) - (\nabla_i f R_{jk} - \nabla_j f R_{ik}),$$

as we wanted to prove. $\square$

To conclude this section we recall that, from commutation formulas (Ricci identities), for any Riemannian manifold $M^n$ we have

$$(2.8) \quad \nabla_i \nabla_j R_{ik} - \nabla_j \nabla_i R_{ik} = R_{ijls} R_{sk} + R_{ijks} R_{ls};$$

for more details see [8].

3. Proof of the main result

In this section we shall prove Theorem 2 mentioned in Section 1. First of all, we shall present a fundamental integral formula.

**Lemma 2.** Let $(M^n, g, f)$ be a Miao-Tam critical metric. Then we have:

$$\int_M f |\text{div} R^m|^2 dM_g = 2 \int_M f |\nabla \text{Ric}|^2 dM_g + \int_M (\nabla f, \nabla |\text{Ric}|^2) dM_g$$

$$+ 2 \int_M f (R_{ij} R_{ik} R_{jk} - R_{ijkl} R_{jl} R_{ik}) dM_g$$

$$+ 2 \int_M C_{klj} \nabla_l f R_{jk} dM_g.$$

**Proof.** First, we use (2.5) to infer

$$\int_M f |\text{div} R^m|^2 dM_g = \int_M f (\text{div} R^m)_{jkl} (\text{div} R^m)_{jkl} dM_g$$

$$= \int_M f (\nabla_k R_{lj} - \nabla_l R_{kj}) (\nabla_k R_{lj} - \nabla_l R_{kj}) dM_g,$$

which can be rewritten as

$$(3.1) \quad \int_M f |\text{div} R^m|^2 dM_g = 2 \int_M f |\nabla \text{Ric}|^2 dM_g - 2 \int_M f \nabla_k R_{lj} \nabla_l R_{kj} dM_g.$$

Next, notice that

$$\int_M f \nabla_k R_{lj} \nabla_l R_{kj} dM_g = -\int_M \nabla_l f \nabla_k R_{lj} R_{kj} dM_g - \int_M f \nabla_l \nabla_k R_{lj} R_{kj} dM_g$$

$$+ \int_M \nabla_l (f \nabla_k R_{lj} R_{kj}) dM_g.$$
Hence we use this data in (3.1) to deduce
\[\int_M f|\text{div}R_m|^2 dM_g = 2 \int_M f|\nabla \text{Ric}|^2 dM_g + 2 \int_M \nabla_i f \nabla_k R_{ij} R_{kj} dM_g \]
(3.2)
\[+ 2 \int_M f \nabla_i \nabla_k R_{ij} R_{kj} dM_g - 2 \int_M \nabla_i (f \nabla_k R_{ij} R_{kj}) dM_g.\]

Since $M^n$ has constant scalar curvature we may use (1.4) to infer
\[C_{klj} = \nabla_k R_{lj} - \nabla_l R_{kj},\]
which substituted jointly with (2.8) into (3.2) gives
\[\int_M f|\text{div}R_m|^2 dM_g = 2 \int_M f|\nabla \text{Ric}|^2 dM_g + 2 \int_M \nabla_i \nabla_k R_{ij} R_{kj} dM_g \]
\[- 2 \int_M \nabla_i (f \nabla_k R_{ij} R_{kj}) dM_g.\]

Now, since $f$ vanishes on the boundary, we apply Stokes’s formula and the twice contracted second Bianchi identity to arrive at
\[\int_M f|\text{div}R_m|^2 dM_g = 2 \int_M f|\nabla \text{Ric}|^2 dM_g + 2 \int_M \langle \nabla f, \nabla |\text{Ric}|^2 \rangle dM_g \]
\[+ 2 \int_M \nabla_i f C_{klj} R_{kj} dM_g \]
\[+ 2 \int_M f (R_{ks} R_{sj} R_{kj} - R_{kljs} R_{klj} R_{kj}) dM_g.\]

Finally, it suffices to change the indices of the last integral of the right hand side to obtain
\[\int_M f|\text{div}R_m|^2 dM_g = 2 \int_M f|\nabla \text{Ric}|^2 dM_g + 2 \int_M \langle \nabla f, \nabla |\text{Ric}|^2 \rangle dM_g \]
\[+ 2 \int_M C_{klj} \nabla_i f R_{kj} dM_g \]
\[+ 2 \int_M f (R_{ij} R_{ik} R_{jk} - R_{ijkl} R_{ij} R_{ik}) dM_g.\]

So, the proof is completed. \(\square\)

Before presenting the proof of Theorem 2, we provide another expression for $\int_M f|\text{div}R_m|^2 dM_g$. More precisely, we have established the following lemma.

**Lemma 3.** Let $(M^n, g, f)$ be a Miao-Tam critical metric. Then we have:
\[\int_M f|\text{div}R_m|^2 dM_g = - \int_M \langle \nabla f, \nabla |\text{Ric}|^2 \rangle dM_g - 2 \int_M C_{klj} \nabla_i f R_{kj} dM_g \]
\[- 2 \int_M f (R_{ij} R_{ik} R_{jk} - R_{ijkl} R_{ij} R_{ik}) dM_g \]
\[+ 2 \int_M \nabla_i (\nabla_j R_{ik} R_{jk}) dM_g - 2 \int_M \nabla_i (R_{ijkl} R_{ij} \nabla_k f) dM_g.\]
Proof. To begin with, from (2.3) we achieve
\[ \int_M f R_{ijkl} R_{jl} R_{ik} dM_g = \int_M R_{ijkl} R_{jl} (- (\Delta f + 1) g_{ik} + \nabla_i \nabla_k f) dM_g \]
\[ = - \int_M (\Delta f + 1) |Ric|^2 dM_g + \int_M R_{ijkl} R_{jl} \nabla_i \nabla_k f dM_g. \]

From here it follows that
\[ \int_M f R_{ijkl} R_{jl} R_{ik} dM_g = - \int_M (\Delta f + 1) |Ric|^2 dM_g - \int_M \nabla_i R_{ijkl} R_{jl} \nabla_k f dM_g \]
\[ - \int_M R_{ijkl} \nabla_i R_{jl} \nabla_k f dM_g + \int_M \nabla_i (R_{ijkl} R_{jl} \nabla_k f) dM_g. \]

By using (2.5) we have
\[ \int_M f R_{ijkl} R_{jl} R_{ik} dM_g = - \int_M (\Delta f + 1) |Ric|^2 dM_g - \int_M C_{klj} R_{jl} \nabla_k f dM_g \]
\[ (3.3) - \int_M R_{ijkl} \nabla_i R_{jl} \nabla_k f dM_g + \int_M \nabla_i (R_{ijkl} R_{jl} \nabla_k f) dM_g. \]

On the other hand, it is not difficult to check that
\[ \int_M R_{ijkl} \nabla_i R_{jl} \nabla_k f dM_g = \frac{1}{2} \left[ \int_M R_{ijkl} \nabla_i R_{jl} \nabla_k f dM_g + \int_M R_{ijkl} \nabla_i R_{jl} \nabla_k f dM_g \right] \]
\[ = \frac{1}{2} \left[ \int_M R_{ijkl} \nabla_i R_{jl} \nabla_k f dM_g + \int_M R_{ijkl} \nabla_j R_{il} \nabla_k f dM_g \right] \]
\[ = \frac{1}{2} \left[ \int_M R_{ijkl} \nabla_i R_{jl} \nabla_k f dM_g - \int_M R_{ijkl} \nabla_j R_{il} \nabla_k f dM_g \right] \]
\[ = \frac{1}{2} \int_M R_{ijkl} (\nabla_i R_{jl} - \nabla_j R_{il}) \nabla_k f dM_g. \]

Then, we change the indices $k$ by $l$ in the right hand side to infer
\[ \int_M R_{ijkl} \nabla_i R_{jl} \nabla_k f dM_g = - \frac{1}{2} \int_M R_{ijkl} (\nabla_i R_{jk} - \nabla_j R_{ik}) \nabla_l f dM_g. \]

This data substituted in (3.3) yields
\[ \int_M f R_{ijkl} R_{jl} R_{ik} dM_g = - \int_M (\Delta f + 1) |Ric|^2 dM_g - \int_M C_{klj} R_{jl} \nabla_k f dM_g \]
\[ + \frac{1}{2} \int_M R_{ijkl} (\nabla_i R_{jk} - \nabla_j R_{ik}) \nabla_l f dM_g \]
\[ (3.4) + \int_M \nabla_i (R_{ijkl} R_{jl} \nabla_k f) dM_g. \]

Next, it follows from Lemma 1 that
\[ R_{ijkl} \nabla_i f (\nabla_i R_{jk} - \nabla_j R_{ik}) = f |\nabla_i R_{jk} - \nabla_j R_{ik}|^2 + (\nabla_i R_{jk} - \nabla_j R_{ik})(\nabla_i f R_{jk} - \nabla_j f R_{ik}) \]
\[ - \frac{R}{n-1} (\nabla_i f g_{jk} - \nabla_j f g_{ik})(\nabla_i R_{jk} - \nabla_j R_{ik}). \]
Then, since $M^n$ has constant scalar curvature we can use the twice contracted second Bianchi identity to obtain

$$R_{ijkl} \nabla_i f(\nabla_{jk} R_{ik} - \nabla_{jk} R_{ik}) = f|\nabla_i R_{jk} - \nabla_{jk} R_{ik}|^2 + (\nabla_i R_{jk} - \nabla_{jk} R_{ik})(\nabla_i f R_{jk} - \nabla_{jk} f R_{ik}).$$

From this it follows that

$$R_{ijkl} \nabla_i f(\nabla_{jk} R_{ik} - \nabla_{jk} R_{ik}) = f|\text{div} R^m|^2 + \langle \nabla f, \nabla |\text{Ric}|^2 \rangle - \nabla_i f R_{jk} \nabla_{jk} R_{ik}$$

$$\quad = f|\text{div} R^m|^2 + \langle \nabla f, \nabla |\text{Ric}|^2 \rangle - 2\nabla_i f R_{jk} \nabla_{jk} R_{ik}. \quad (3.5)$$

By using once more the twice contracted second Bianchi identity we immediately have

$$\nabla_i (\nabla_j f R_{ik} R_{jk}) = \nabla_i \nabla_j f R_{ik} R_{jk} + \nabla_i f R_{jk} \nabla_{jk} R_{ik}.$$ 

This combined with $(3.5)$ yields

$$R_{ijkl} \nabla_i f(\nabla_{jk} R_{ik} - \nabla_{jk} R_{ik})$$

$$\quad = f|\text{div} R^m|^2 + \langle \nabla f, \nabla |\text{Ric}|^2 \rangle + 2\nabla_i \nabla_j f R_{ik} R_{jk}$$

$$\quad - 2\nabla_i(\nabla_j f R_{ik} R_{jk})$$

$$\quad = f|\text{div} R^m|^2 + \langle \nabla f, \nabla |\text{Ric}|^2 \rangle + 2(f R_{ij} + (\Delta f + 1)g_{ij}) R_{ik} R_{jk}$$

$$\quad - 2\nabla_i(\nabla_j f R_{ik} R_{jk}).$$

From this it follows that

$$R_{ijkl} \nabla_i f(\nabla_{jk} R_{ik} - \nabla_{jk} R_{ik}) = f|\text{div} R^m|^2 + \langle \nabla f, \nabla |\text{Ric}|^2 \rangle + 2f R_{ij} R_{ik} R_{jk}$$

$$\quad + (\Delta f + 1)|\text{Ric}|^2 - 2\nabla_i (\nabla_j f R_{ik} R_{jk}). \quad (3.4)$$

Whence, on integrating this last expression over $M^n$ and then substituting into $(3.4)$ we arrive at

$$\int_M f R_{ijkl} R_{jl} R_{ik} dM_g = -\int_M C_{klj} R_{jl} \nabla_k f dM_g + \int_M \nabla_i (R_{ijkl} R_{jl} \nabla_k f) dM_g$$

$$\quad + \frac{1}{2} \int_M f|\text{div} R^m|^2 dM_g + \frac{1}{2} \int_M \langle \nabla f, \nabla |\text{Ric}|^2 \rangle dM_g$$

$$\quad + \int_M f R_{ij} R_{ik} R_{jk} dM_g - \int_M \nabla_i (\nabla_j f R_{ik} R_{jk}) dM_g.$$ 

Since the Cotton tensor $C$ is skew-symmetric in the first two indices we get

$$\int_M f|\text{div} R^m|^2 dM_g = -\int_M \langle \nabla f, \nabla |\text{Ric}|^2 \rangle dM_g - 2\int_M C_{klj} \nabla_i f R_{jk} dM_g$$

$$\quad - 2\int_M f (R_{ij} R_{ik} R_{jk} - R_{ijkl} R_{jl} R_{ik}) dM_g$$

$$\quad + 2\int_M \nabla_i (\nabla_j f R_{ik} R_{jk}) dM_g - 2\int_M \nabla_i (R_{ijkl} R_{jl} \nabla_k f) dM_g.$$ 

This gives the requested result. \hfill \Box

We are now in the position to prove Theorem 2.
3.1. **Proof of Theorem**

**Proof.** First of all, we sum up the expressions obtained in Lemma 2 and Lemma 3 to infer

\[
\int_M f |\text{div} Rm|^2 dM_g = \int_M f |\nabla \text{Ric}|^2 dM_g + \int_M \nabla_i (\nabla_j f R_{ik} R_{jk}) dM_g
- \int_M \nabla_i (R_{ijkl} R_{jk} \nabla_l f) dM_g.
\]

Now, we change the indices \(k\) by \(l\) in the third integral of the right hand side to get

\[
\int_M f |\text{div} Rm|^2 dM_g = \int_M f |\nabla \text{Ric}|^2 dM_g + \int_M \nabla_i (\nabla_j f R_{ik} R_{jk}) dM_g
+ \int_M \nabla_i (R_{ijkl} R_{jk} \nabla_l f) dM_g.
\]

(3.6)

Then, substituting Lemma 1 into (3.6) we deduce

\[
\int_M f |\text{div} Rm|^2 dM_g = \int_M f |\nabla \text{Ric}|^2 dM_g + \int_M \nabla_i (\nabla_j f R_{ik} R_{jk}) dM_g
+ \int_M \nabla_i \left[ f C_{ikj} R_{jk} - \frac{R}{n-1} (\nabla_i f R - \nabla_j f R_{ji}) + (\nabla_i f |\text{Ric}|^2 - \nabla_j f R_{ik} R_{jk}) \right] dM_g.
\]

Upon rearranging the terms we use Stokes's formula to arrive at

\[
\int_M f |\text{div} Rm|^2 dM_g
= \int_M f |\nabla \text{Ric}|^2 dM_g
+ \int_M \nabla_i \left[ -\frac{R^2}{n-1} \nabla_i f \frac{R}{n-1} R_{ji} \nabla_j f + \nabla_i f |\text{Ric}|^2 \right] dM_g
\]

\[
= \int_M f |\nabla \text{Ric}|^2 dM_g
+ \int_M \left[ -\frac{R^2}{n-1} \Delta f + \frac{R}{n-1} \nabla_i \nabla_j f R_{ji} + \Delta f |\text{Ric}|^2 + \langle \nabla f, \nabla |\text{Ric}|^2 \rangle \right] dM_g.
\]

Therefore, by using (2.3) and (2.2) we achieve

\[
\int_M f |\text{div} Rm|^2 dM_g
= \int_M \left[ -\frac{R^2}{n-1} \Delta f + \frac{R}{n-1} (f R_{ij} + (\Delta f + 1) g_{ij}) R_{ji} + \Delta f |\text{Ric}|^2 \right] dM_g
+ \int_M f |\nabla \text{Ric}|^2 dM_g + \int_M \langle \nabla f, \nabla |\text{Ric}|^2 \rangle dM_g
\]

\[
= \int_M \left[ \frac{R}{n-1} f |\text{Ric}|^2 + \frac{R^2}{n-1} f + \frac{R f - n}{n-1} |\text{Ric}|^2 \right] dM_g
+ \int_M f |\nabla \text{Ric}|^2 dM_g + \int_M \langle \nabla f, \nabla |\text{Ric}|^2 \rangle dM_g.
\]
This implies that
\[
\int_M f |\text{div} R_m|^2 dM_g = \int_M f |\nabla \text{Ric}|^2 dM_g - \frac{n}{n-1} \int_M (|\text{Ric}|^2 - \frac{R^2}{n}) dM_g \\
+ \int_M \langle \nabla f, \nabla |\text{Ric}|^2 \rangle dM_g \\
= \int_M f |\nabla \text{Ric}|^2 dM_g - \frac{n}{n-1} \int_M |\text{Ric} - \frac{R}{n} g|^2 dM_g \\
+ \int_M \langle \nabla f, \nabla |\text{Ric}|^2 \rangle dM_g \\
= \int_M f |\nabla \text{Ric}|^2 dM_g - \frac{n}{n-1} \int_M |\text{Ric} - \frac{R}{n} g|^2 dM_g \\
- \int_M f \Delta |\text{Ric}|^2 dM_g.
\]

From this we can deduce
\[
\int_M f |\text{div} R_m|^2 dM_g + \frac{n}{n-1} \int_M |\text{Ric} - \frac{R}{n} g|^2 dM_g + \int_M f (\Delta |\text{Ric}|^2 - |\nabla \text{Ric}|^2) dM_g = 0.
\]
This is what we wanted to prove. \(\square\)

3.2. Proof of Corollary 1

Proof. We already know that every Riemannian manifold with parallel Ricci tensor has harmonic curvature (cf. equation (2.5)). Moreover, from Kato’s inequality we have
\[
|\nabla |\text{Ric}| | \leq |\nabla \text{Ric}|.
\]
From this, since \(M^n\) has parallel Ricci tensor, we conclude that \(|\text{Ric}|\) is constant on \(M^n\). Now, it suffices to invoke Theorem 2 to deduce
\[
\frac{n}{n-1} \int_M |\text{Ric} - \frac{R}{n} g|^2 dM_g = 0
\]
and this forces \(M^n\) to be Einstein. Then, we are in a position to use Theorem 1 (see also Theorem 1.1 of [16]) to conclude that \((M^n, g)\) is isometric to a geodesic ball in a simply connected space form \(\mathbb{R}^n\), \(\mathbb{H}^n\) or \(S^n\). So, the proof is completed. \(\square\)

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References

[1] Rudolf Bach, Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs (German), Math. Z. 9 (1921), no. 1-2, 110–135, DOI 10.1007/BF01378338. MR1544454

[2] A. Barros, R. Diógenes, and E. Ribeiro Jr., Bach-flat critical metrics of the volume functional on 4-dimensional manifolds with boundary, J. Geom. Anal. 25 (2015), no. 4, 2698–2715, DOI 10.1007/s12220-014-9532-z. MR3427144

[3] R. Batista, R. Diógenes, M. Ranieri, and E. Ribeiro, Critical Metrics of the Volume Functional on Compact Three-Manifolds with Smooth Boundary, J. Geom. Anal. 27 (2017), no. 2, 1530–1547, DOI 10.1007/s12220-016-9730-y. MR3625163
[4] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987. MR867684

[5] Huai-Dong Cao and Qiang Chen, *On Bach-flat gradient shrinking Ricci solitons*, Duke Math. J. 162 (2013), no. 6, 1149–1169, DOI 10.1215/00127094-2147649. MR3053567

[6] Justin Corvino, *Scalar curvature deformation and a gluing construction for the Einstein constraint equations*, Comm. Math. Phys. 214 (2000), no. 1, 137–189, DOI 10.1007/PL00005533. MR1794269

[7] Justin Corvino, Michael Eichmair, and Pengzi Miao, *Deformation of scalar curvature and volume*, Math. Ann. 357 (2013), no. 2, 551–584, DOI 10.1007/s00208-013-0903-8. MR3096517

[8] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, James Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, and Lei Ni, *The Ricci Flow: techniques and applications. Part I*, Mathematical Surveys and Monographs, vol. 135, American Mathematical Society, Providence, RI, 2007. Geometric aspects. MR2302600

[9] Andrzej Derdziński, *On compact Riemannian manifolds with harmonic curvature*, Math. Ann. 259 (1982), no. 2, 145–152, DOI 10.1007/BF01457307. MR656660

[10] Andrzej Derdziński, *Classification of certain compact Riemannian manifolds with harmonic curvature and nonparallel Ricci tensor*, Math. Z. 172 (1980), no. 3, 273–280, DOI 10.1007/BF01215090. MR581444

[11] Xu-Qian Fan, Yuguang Shi, and Luen-Fai Tam, *Large-sphere and small-sphere limits of the Brown-York mass*, Comm. Anal. Geom. 17 (2009), no. 1, 37–72, DOI 10.4310/CAG.2009.v17.n1.a3. MR2495833

[12] D. Hilbert, *Die grundlagen der physik*, Nach. Ges. Wiss. Göttingen, (1915) 461–472.

[13] Osamu Kobayashi, *A differential equation arising from scalar curvature function*, J. Math. Soc. Japan 34 (1982), no. 4, 665–675, DOI 10.2969/jmsj/03440665. MR669275

[14] Osamu Kobayashi and Morio Obata, *Conformally-flatness and static space-time*, Manifolds and Lie groups (Notre Dame, Ind., 1980), Progr. Math., vol. 14, Birkhäuser, Boston, Mass., 1981, pp. 197–206. MR642858

[15] Pengzi Miao and Luen-Fai Tam, *On the volume functional of compact manifolds with boundary with constant scalar curvature*, Calc. Var. Partial Differential Equations 36 (2009), no. 2, 141–171, DOI 10.1007/s00526-008-0221-2. MR2546025

[16] Pengzi Miao and Luen-Fai Tam, *Einstein and conformally flat critical metrics of the volume functional*, Trans. Amer. Math. Soc. 363 (2011), no. 6, 2907–2937, DOI 10.1090/S0002-9947-2011-05195-0. MR2775792

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