PARISIAN RUIN PROBABILITY OF AN INTEGRATED GAUSSIAN RISK MODEL

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Abstract: In this paper we investigate the Parisian ruin probability for an integrated Gaussian process. Under certain assumptions, we find the Parisian ruin probability and the classical ruin probability are on the log-scale asymptotically the same. Moreover, for any small interval required by the risk process staying below level zero, the Parisian ruin probability and the classical one are the same also in the premise asymptotic behavior. Furthermore, we derive an approximation of the conditional ruin time.

Key Words: integrated Gaussian process; Parisian ruin; method of moments; exact asymptotics

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

Gaussian risk processes have been investigated in numerous research paper. With motivation from [8] the risk reserve process of an insurance company can be modelled by a stochastic process \( \{R_u(t), t \geq 0\} \) given as

\[
R_u(t) = u + ct - \int_0^t Z(s) \, ds, \quad t \geq 0,
\]

where \( u \geq 0 \) is the initial reserve, \( c > 0 \) is the rate of premium received by the insurance company and \( \{Z(t), t \geq 0\} \) is a centered Gaussian process with almost surely continuous sample paths. Commonly the process \( \{Z(t), t \geq 0\} \) is referred to as the loss rate of the insurance company. In order to take into account the time-value of money, in this contribution for a given real-valued measurable function \( \delta(\cdot) \) we shall consider the more general risk process

\[
R_u(t) = u + c \int_0^t e^{-\delta(s)} \, ds - \int_0^t e^{-\delta(s)} Z(s) \, ds, \quad t \geq 0.
\]  

An important quantity of interest for such a risk process is the calculation of the ruin probability over the finite time-horizon \([0, S]\)

\[
\psi_{\delta \upsilon}^{\upsilon}(u) := \mathbb{P} \left\{ \inf_{t \in [0, S]} R_u(t) < 0 \right\}
\]

\[
= \mathbb{P} \left\{ \sup_{t \in [0, S]} \left( \int_0^t e^{-\delta(s)} Z(s) \, ds - c \int_0^t e^{-\delta(s)} \, ds \right) > u \right\}.
\]

Since it is not possible to calculate \( \psi_{\delta \upsilon}^{\upsilon}(u) \) for any fixed \( u \) explicitly, one resorts to asymptotic theory analysing the ruin probability as the initial reserve \( u \) becomes large. The recent contribution \([2]\) derived the exact tail asymptotics of \( \psi_{\delta \upsilon}^{\upsilon}(u) \) as \( u \to \infty \) under some restrictions on \( Z \). Moreover, therein an approximation of the ruin time as \( u \to \infty \) is derived.
A more general concept than the classical ruin probability is the Parisian ruin probability, which in our context is defined for some given \( T_u > 0 \) as

\[
\mathcal{P}_S(u, T_u) := \mathbb{P} \left\{ \inf_{t \in [0,S]} \sup_{s \in [t,t+T_u]} R_u(s) < 0 \right\}.
\]

Clearly, in the particular case that \( T_u = 0 \) we have that the Parisian ruin probability equals the classical ruin probability. Parisian ruin and Parisian ruin time have been recently discussed for self-similar Gaussian processes in [3, 4], whereas the recent publication [1] discusses the classical Brownian motion risk model. With motivation from these recent contributions we shall analyse the asymptotic behavior of \( \mathcal{P}_S(u, T_u) \) as \( u \to \infty \). Our objectives are two-fold. First we are interested in a large deviation type result for the Parisian ruin probability. Specifically, under two weak restrictions on \( Z \) and the inflation/deflation rate function \( \delta(\cdot) \) we shall show in our first result that

\[
\lim_{u \to \infty} \frac{\log(\mathcal{P}_S(u, T_u))}{u^2} = -\frac{1}{2\sigma^2(S)},
\]

where \( \sigma^2(t) \) is given by

\[
\sigma^2(t) := \text{Var}(R_u(t)) = 2 \int_0^t \int_0^t e^{-(w-v)\delta(w)-\delta(v)} \text{Cov}(Z(w), Z(v)) dw dv.
\]

The above result is shown for quite general \( T_u \), in particular it holds for \( T_u = 0 \). Hence, the Parisian ruin probability and the classical ruin probability are on the log-scale asymptotically the same, i.e.,

\[
\log(\mathcal{P}_S(u, T_u)) \sim \log \psi_S(u), \quad u \to \infty,
\]

where \( \sim \) stands for asymptotic equivalence when \( u \to \infty \). Then we show in our main result, for any small interval that required by the risk reserve process staying below level zero, it is possible to derive the exact asymptotic of the Parisian ruin probability as \( u \to \infty \). Such a result reveals, that the asymptotic of Parisian ruin probability and the classical one are the same also in the precise asymptotic behavior.

Brief organization of the rest of the paper: Section 2 presents our main results where additionally to the large deviation and the precise asymptotic of the Parisian ruin probability. Furthermore, we obtain an approximation of the Parisian ruin time. All the proofs are relegated to Section 3 which concludes this contribution.

2. Results

In this section we shall present two results. The first one gives the large deviation asymptotic and the precise asymptotic of the Parisian ruin probability. The second result is concerned with the Parisian ruin time. In theoretical investigations, the analysis of ruin time is of interests since it gives more information on how and when the ruin occurs. Our results are derived under the following conditions on the risk reserve process \( R_u(t) \).

\textbf{A1.} The claim rate process \( Z(t) \) is a centered, non-degenerate Gaussian process with continuous sample path and nonnegative covariance, i.e., \( \text{Cov}(Z(s), Z(t)) \geq 0 \) for any \( s, t \geq 0 \).

\textbf{A2.} The inflation/deflation rate function \( \delta(\cdot) \) is locally bounded.
Clearly, condition A2 is always met in practical applications. Condition A1 is a weak one, it is satisfied by many Gaussian processes, for instance, Ornstein–Ohlenbeck process, Slepian process and the fractional Brownian motion with Hurst index $H \in (0,1)$. Note that $H = 1/2$ corresponds to the Brownian motion. 

Next, we give two asymptotic results which are based on the characteristics of pre-specified time $T_u$.

**Theorem 2.1.** Let $\{R_u(t), t \geq 0\}$ be the reserve process evolving as (1.1) and satisfy assumptions A1-A2. Define $\sigma^2(\cdot)$ by (1.3) and set $\delta(t) := \int_0^t e^{-\delta(s)}ds$. For any bounded delayed time $T_u \geq 0$ we have

$$\lim_{u \to \infty} \frac{\log(\mathcal{P}_S(u, T_u))}{u^2} = \lim_{u \to \infty} \frac{\log(\psi_S(u))}{u^2} = -\frac{1}{2\sigma^2(S)}.$$  

Furthermore, for any small interval $T_u \to 0$ as $u \to \infty$, then

$$\mathcal{P}_S(u, T_u) \sim \psi_S(u) \sim \mathbb{P}\left\{N > (u + c\delta(S))/\sigma(S)\right\}$$  

holds as $u \to \infty$, with $N$ a $N(0,1)$ random variable.

Another quantity of interest is the conditional distribution of the ruin time for the surplus process $R_u(s)$. We define the ruin time as

$$\tau(u) := \inf\{t \geq T_u : t - \kappa_{t,u} \geq T_u, R_u(t) < 0\}, \quad \text{with} \quad \kappa_{t,u} = \sup\{s \in [0, t] : R_u(s) \geq 0\}.$$  

**Theorem 2.2.** Under the conditions of Theorem 2.1, for small interval $T_u$, we have

$$\lim_{u \to \infty} \mathbb{P}\left\{u^2(S + T_u - \tau(u)) \leq x \mid \tau(u) < S + T_u\right\} = 1 - \exp\left(-\frac{\sigma'(S)}{\sigma^3(S)}x\right), \quad x \geq 0.$$  

Below we shall present two illustrating examples. It is worth noting that all the examples given in [2] are adapted to our model.

**Example 2.3.** Let $\{Z(t), t \geq 0\}$ be a standard fractional Brownian motion with Hurst index $H \in (0,1)$, i.e., it is a centered Gaussian process with a.s. continuous sample paths and covariance function $\text{Cov}(Z(t), Z(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H})$. If $\delta(t) = t, t \geq 0$, then

$$\bar{\delta}(t) = 1 - e^{-t},$$  

$$\sigma^2(t) = 2 \int_0^t \int_0^x e^{-x-y}(x^{2H} + y^{2H} - (x-y)^{2H})dydx$$  

$$= \Gamma(2H + 1, t)(1 - 2e^{-t}) + e^{-2t}\Gamma^*(2H + 1, t),$$  

where $\Gamma(a, t) = \int_0^t x^{a-1}e^{-x}dx$ and $\Gamma^*(a, t) = \int_0^t x^{a-1}e^xdx$. It is easy to see conditions A1 and A2 are naturally satisfied. Consequently, Theorem 2.1 implies, for $T_u \to 0$ as $u \to \infty$,

$$\mathcal{P}_S(u, T_u) = \frac{1}{u} \sqrt{\frac{\Gamma(2H + 1, S)(1 - 2e^{-S}) + e^{-2S}\Gamma^*(2H + 1, S)}{2\pi}} \times \exp\left(-\frac{(u + c(1 - e^{-S}))^2}{2\Gamma(2H + 1, S)(1 - 2e^{-S}) + e^{-2S}\Gamma^*(2H + 1, S)}\right)(1 + o(1)).$$  

Furthermore, according to Theorem 2.2 the convergence in (2.2) holds with

$$\frac{\sigma'(S)}{\sigma^3(S)} = \frac{e^{-S}(S^{2H} + \Gamma(2H + 1, S)) - e^{-2S}(S^{2H} + \Gamma^*(2H + 1, S))}{(\Gamma(2H + 1, S)(1 - 2e^{-S}) + e^{-2S}\Gamma^*(2H + 1, S))^2}.$$
Example 2.4. Let \( \{Z(t) = \frac{B(t)}{\sqrt{t}}, t \geq 0\} \) be a scaling Brownian motion with \( B \) a standard Brownian motion. If further \( \delta(t) = t, t \geq 0 \), then by Taylor formula

\[
\sigma^2(t) = 2 \int_0^t \int_0^x e^{-x-y} \sqrt{y/x} \ dy \ dx
\]

\[
= 2 \int_0^t \frac{\sqrt{z}(1 - e^{-t(1+z)})}{(1 + z)^2} \ dz - 2 \int_0^t \frac{t\sqrt{z}e^{-t(1+z)}}{1 + z} \ dz
\]

\[
= \frac{2}{3} t^2 + \sum_{k=3}^{\infty} (-1)^k \frac{2(k-1)}{k!} t^k \int_0^1 (1 + z)^{k-2} \sqrt{z} \ dz.
\]

Applying Theorem 2.1 once again, we obtain

\[
\mathcal{P}_S(u, T_u) = \frac{1}{u} \sqrt{\frac{2}{3} S^2 + \sum_{k=3}^{\infty} (-1)^k \frac{2(k-1)}{k!} S^k \int_0^1 (1 + z)^{k-2} \sqrt{z} \ dz}
\]

\[
\times \exp \left( -\frac{(u + c(1 - e^{-S}))^2}{\frac{2}{3} S^2 + \sum_{k=3}^{\infty} (-1)^k \frac{2(k-1)}{k!} S^k \int_0^1 (1 + z)^{k-2} \sqrt{z} \ dz} \right) (1 + o(1)),
\]

for \( T_u \to 0 \) as \( u \to \infty \). Finally, by Theorem 2.2 the convergence in (2.2) holds with

\[
\frac{\sigma'(S)}{\sigma^3(S)} = \frac{S - \frac{1}{2} e^{-S} \Gamma \left( \frac{3}{2}, S \right)}{\left( \frac{2}{3} S^2 + \sum_{k=3}^{\infty} (-1)^k \frac{2(k-1)}{k!} S^k \int_0^1 (1 + z)^{k-2} \sqrt{z} \ dz \right)^2}.
\]

3. Proofs

Before the demonstration, for notational simplicity, we define

\[
Y(t) := \int_0^t e^{-\delta(s)} Z(s) \ ds, \quad R(s, t) := \text{Cov}(Z(s), Z(t))
\]

\[
g_u(t) := \frac{u + c \delta(t)}{\sigma(t)}, \quad X_u(t) := \frac{Y(t) g_u(S)}{\sigma(t) g_u(t)}
\]

\[
\sigma^2_{X_u}(t) := \text{Var}(X_u(t)), \quad r_{X_u}(s, t) := \text{Corr}(X_u(s), X_u(t)).
\]

Then, we can reformulate (1.2) as

\[
(3.1) \quad \mathcal{P}_S(u, T_u) = \mathbb{P} \left\{ \sup_{t \in [0, S]} \inf_{s \in [t, t + T_u]} X_u(s) > g_u(S) \right\}.
\]

**Proof of Theorem 2.1** From assumption A1, we know in fact \( Z(t) \) is continuous in the mean squared sense. This means that the covariance function \( R(s, t) \) is a bivariate continuous function and is strictly positive for \( |t - s| \) sufficiently small due to the non-degeneracy. Therefore,

\[
(3.2) \quad \frac{\partial \sigma^2(t)}{\partial t} = 2 \int_0^t e^{-\delta(s) - \delta(t)} R(s, t) \ ds > 0
\]

and for \( 0 < s \leq t \)

\[
\text{Cov}(Y(t), Y(s)) \geq \sigma^2(s).
\]

Then by the Slepian Lemma (cf. [7])

\[
\mathcal{P}_S(u, T_u) \leq \mathbb{P} \left\{ \sup_{t \in [0, S]} Y(t) > u \right\} \leq \mathbb{P} \left\{ \sup_{t \in [0, S]} B(\sigma^2(t)) > u \right\} = 2\Psi \left( \frac{u}{\sigma(S)} \right),
\]

where \( \Psi \) is the distribution function of a standard normal distribution.
where $B(\cdot)$ is a standard Brownian motion and $\Psi(\cdot)$ denotes the tail distribution of a standard normal random variable. Moreover, appealing to Theorem 2.1 in [2], we get a sharper upper bound

(3.3) \[ \mathcal{P}_S(u, T_u) \leq \psi_S(u) = \Psi(g_u(S))(1 + o(1)), \quad \text{as } u \to \infty. \]

For the lower bound, put $T = \sup_{u > 0} T_u$ and $m = \sup_{t \in [0,S+T]} e^{-\delta(t)}$. Due to our assumptions, $T$ and $m$ are both finite. Then drawing on similar arguments as used in the proof of Theorem 3.1 in [4] (only replacing therein $c$ by $cm$ and $\rho_S$ by $(\inf_{t \in [S,S+T]} \partial \sigma^2(t)/\partial t)^{-1}$, note that the latter is finite and the concavity in the aforementioned paper is not necessary) we get

$$ \mathcal{P}_S(u, T_u) \geq \mathbb{P} \left\{ \inf_{t \in [S,S+T]} Y(t) - cmT > u \right\} \geq C \frac{\sigma^2(S)}{u} \Psi \left( \frac{u + cmS}{\sigma(S)} \right) (1 + o(1)) $$

for some positive constant $C$, as $u \to \infty$.

The first claim follows straightforwardly from combination of the above inequalities concerning $\mathcal{P}_S(u, T_u)$. Next, we derive a lower bound of $\mathcal{P}_S(u, T_u)$ for $T_u \to 0$ as $u \to \infty$ by considering three separate cases. **Case I: $T_u = o\left(\frac{1}{u}\right)$**

First, differentiating $\sigma_{X_u}(s)$ yields

$$ \sigma'_{X_u}(s) = \frac{\sigma'(s) u + c\tilde{\delta}(S)}{\sigma(S)} u + c\tilde{\delta}(s) - \frac{ce^{-\delta(s)} \sigma(s)(u + c\tilde{\delta}(S))}{(u + c\tilde{\delta}(s)) \sigma(S)}, \tag{3.4} $$

which together with (3.2) implies $\sigma'_{X_u} > 0$ for sufficiently large $u$. Consequently, $\sigma_{X_u}(s) > 1$ for all $s > S$. Secondly, given arbitrary $\theta > S$, then for any $s, t \in [S, \theta]$

$$ 1 - r_{X_u}(s,t) \leq \frac{\text{Var}(Y(t) - Y(s))}{2\sigma(s) \sigma(t)} = \frac{\int_s^t \int_s^t R(w,v)e^{-\delta(w) - \delta(v)} dw dv}{2\sigma(s) \sigma(t)} $$

$$ \leq C(t - s)^2, \tag{3.5} $$

where $C = \max_{w,v \in [S,\theta]} R(w,v) e^{-\delta(w) - \delta(v)}/(2\sigma^2(S))$. Next, for any $\varepsilon > 0$, put $C_{\varepsilon} = C(1 + \varepsilon)$ and define a centered Gaussian process $\{\xi_{\varepsilon}(t), t \geq 0\}$ with covariance function $\text{Cov}(\xi_{\varepsilon}(t), \xi_{\varepsilon}(s)) = e^{-C_{\varepsilon}(t-s)^2}$. Lastly, in view of Slepian Lemma, for $T_u = o(u^{-1})$ and any sufficiently small $\varepsilon_1 \in (0,1)$

$$ \mathcal{P}_S(u, T_u) \geq \mathbb{P} \left\{ \inf_{s \in [S,S+T_u]} X_u(s) > g_u(S) \right\} $$

$$ \geq \mathbb{P} \left\{ \inf_{s \in [S,S+\varepsilon_1 u^{-1}]} \frac{X_u(s)}{\sigma_{X_u}(s)} > g_u(S) \right\} $$

$$ \geq \mathbb{P} \left\{ \inf_{s \in [0,\varepsilon_1]} \xi_{\varepsilon}(su^{-1}) > g_u(S) \right\} $$

$$ = \tilde{\mathcal{H}}_2(\tilde{\alpha}, \Psi(g_u(S))(1 + o(1))) \tag{3.6} $$

as $u$ sufficiently large, where the last equality follows from Lemma 5.1 in [4], $\tilde{\alpha} = \sqrt{C_{\varepsilon}/\sigma(S)}$ and $\tilde{\mathcal{H}}_2(\cdot)$ is described by the following generalized Picands constant

$$ \tilde{\mathcal{H}}_\alpha(T) = \mathbb{E} \left\{ \exp \left( \inf_{s \in [0,T]} \left( \sqrt{2}B_\alpha(s) - s^\alpha \right) \right) \right\} \in (0, \infty), \quad T \geq 0, $$
with $B_\alpha(\cdot)$ a standard fractional Brownian motion with Hurst index $\alpha/2 \in (0,1]$. Therefore, letting $\varepsilon_1 \to 0$ in (3.6) yields the lower bound
\begin{equation}
\mathcal{P}_S(u, T_u) \geq \Psi(g_u(S)) (1 + o(1)), \quad u \to \infty.
\end{equation}

**Case II:** $T_u = O(\frac{1}{u})$
Without loss of generality, suppose $T_u = T/u$ for some positive constant $T$. Then, similar to the case I, for any constant $Q > 0$ and sufficiently large $u$ ($> 2Q\sigma'(S)/\sigma(S)$)
\begin{equation}
\mathcal{P}_S(u, T_u) \geq \mathbb{P} \left\{ \inf_{s \in [S, S+T_u]} X_u(s) > g_u(S) \right\}
\geq \mathbb{P} \left\{ \inf_{s \in [S, S+T-u^{-1}]} \frac{X_u(s)}{\sigma_X(u)} \left( 1 + Q \frac{u}{s} \right) > g_u(S) \right\}
\geq \mathbb{P} \left\{ \inf_{s \in [0, T]} \xi(s) \left( 1 + Q \frac{u}{s} \right) > g_u(S) \right\}
\end{equation}
\begin{equation}
= \tilde{\mathcal{H}}_2^Q(T) \Psi(g_u(S)) (1 + o(1)),
\end{equation}
where $\tilde{\mathcal{H}}_2^Q(\cdot)$ is described by the following generalized Piterbarg constant
\begin{equation}
\tilde{\mathcal{H}}_2^Q(T) = \mathbb{E} \left\{ \exp \left( \inf_{s \in [0, T]} \left( \sqrt{2} B(s) - s^\alpha + Qs \right) \right) \right\} \in (0, \infty), \quad T \geq 0, \quad \alpha \in (0, 2].
\end{equation}

Letting $Q \to \infty$ in (3.8) gives the same lower bound as (3.7).

**Case III:** $\frac{1}{u} = o(T_u)$ and $T_u \to 0$ as $u \to \infty$
From the proof above we see that the small constant $\varepsilon$ plays an insignificant role. Hence, for the sake of notational convenience, we use $C$ and $\xi(\cdot)$ instead of $C_\varepsilon$ and $\xi(\cdot)$, and put $\hat{\xi}(s) = \xi(s)\sigma_X(S + S)$ for $s \geq 0$. Then, using Slepian Lemma again yields
\begin{equation}
\mathcal{P}_S(u, T_u) \geq \mathbb{P} \left\{ \inf_{s \in [S, S+T_u]} X_u(s) > g_u(S) \right\}
\geq \mathbb{P} \left\{ \inf_{s \in [0, T]} \hat{\xi}(s) > g_u(S) \right\}
\end{equation}
\begin{equation}
= \mathbb{P} \left\{ \hat{\xi}(0) > g_u(S), \quad \hat{\xi}(T_u) > g_u(S) \right\}
- \mathbb{P} \left\{ \hat{\xi}(0) > g_u(S), \quad \exists s \in (0, T_u) \text{ s.t. } \hat{\xi}(s) \leq g_u(S), \quad \hat{\xi}(T_u) > g_u(S) \right\}.
\end{equation}

In the following, we first show (3.9) is asymptotically equivalent to $\Psi(g_u(S))$ as $u \to \infty$, and then appeal to the method of moments (see [7]) to show that (3.10) is negligible with respect to the former probability. Note, for a bivariate normal variable,
\begin{equation}
\mathbb{P} \left\{ \hat{\xi}(0) > g_u(S), \quad \hat{\xi}(T_u) > g_u(S) \right\}
= \mathbb{P} \left\{ \xi(0) > g_u(S), \quad \xi(T_u) > g_u(S + T_u) \right\}
= \frac{1}{\sqrt{2\pi}} \int_{g_u(S)}^\infty \mathbb{P} \left\{ N > \frac{g_u(S + T_u) - x\xi(T_u)}{\sqrt{1 - r^2(T_u)}} \right\} e^{-\frac{x^2}{2}} dx
\end{equation}
\begin{equation}
= \frac{1}{\sqrt{2\pi}g_u(S)} e^{-\frac{g_u^2(S)}{2} \frac{1}{g_u(S)}} \int_0^\infty \mathbb{P} \left\{ N > \frac{g_u(S + T_u) - (g_u(S) + x/g_u(S))\xi(T_u)}{\sqrt{1 - r^2(T_u)}} \right\} e^{-\frac{x^2}{2g_u^2(S)} - x} dx.
\end{equation}
where \( r_\xi(t) = e^{-Ct^2} \) is the correlation function of stationary Gaussian process \( \xi(t) \). Furthermore, the fraction part within the probability in (3.11) can be rewritten as

\[
(3.12) \quad \frac{g_u(S)(\sigma(S) - \sigma(S + T_u)r_\xi(T_u)) + c(\tilde{\delta}(S + T_u) - \tilde{\delta}(S)) - \sigma(S + T_u)r_\xi(T_u)x/g_u(S)}{\sigma(S + T_u)\sqrt{1 - r_\xi^2(T_u)}}.
\]

With Taylor formula, simple calculations indicate (3.12) tends to \(-\infty\) as \( u \to \infty \). Consequently, (3.11) is asymptotically equivalent to \( \Psi(g_u(S)) \).

Next, thanks to the Bulinskaya’s theorem, see Theorem E.4 in [7], the probability of contingency of any level \( u \) by the process \( \hat{\xi}(\cdot) \) is equal to zero. Therefore, the event in (3.10) implies that the number of crossings of the level \( g_u(S) \) by the process \( \hat{\xi}(\cdot) \) is greater than one. Denote by \( N_{g_u(S)}[0, T_u] \) the number of crossings, by \( p_{st}(\cdot, \cdot, \cdot) \) the distribution density of the vector \( (\hat{\xi}(s), \hat{\xi}(t), \hat{\xi}'(s), \hat{\xi}'(t)) \) and by \( p_{st}(\cdot, \cdot, \cdot) \) the distribution density of the vector \( (\xi(s), \xi(t), \xi'(s), \xi'(t)) \). Then, appealing to the Theorem E.2 of [7] and using the symmetry of a normal density, we get a series of upper bounds for (3.10),

\[
(3.10) \quad \leq \quad \mathbb{P}\{N_{g_u(S)}[0, T_u] \geq 2\} \\
\quad \leq \quad \mathbb{E}[N_{g_u(S)}[0, T_u] \{N_{g_u(S)}[0, T_u] - 1\}] \\
\quad = \quad \int_0^{T_u} \int_0^{T_u} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |xy| p_{st}(g_u(S), g_u(S), x, y)dx dy ds dt \\
\quad \leq \quad \int_0^{T_u} \int_0^{T_u} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{x^2 + y^2}{2} p_{st}(g_u(S), g_u(S), x, y)dx dy ds dt \\
\quad \leq \quad \int_0^{T_u} \int_0^{T_u} \int_{-\infty}^{\infty} y^2 p_{st}(g_u(S), g_u(S), y)dy ds dt \\
\quad = \quad \int_0^{T_u} \int_0^{T_u} \varphi_{st}(g_u(S), g_u(S)) \int_{-\infty}^{\infty} y^2 \varphi_{st}(y|g_u(S), g_u(S)) dy ds dt \\
(3.13) \quad = \quad \int_0^{T_u} \int_0^{T_u} \varphi_{st}(g_u(S), g_u(S)) \left( m_u^2(s, t) + \sigma_u^2(s, t) \right) ds dt,
\]

where \( \varphi_{st}(g_u(S), g_u(S)) \) represents the distribution density of \((\hat{\xi}(s), \hat{\xi}(t))\). Specifically,

\[
(3.14) \quad \varphi_{st}(g_u(S), g_u(S)) = \frac{\sigma_{X_u}^{-1}(S + s)\sigma_{X_u}^{-1}(S + t)}{2\pi \sqrt{1 - r_\xi^2(t - s)}} \exp \left( -\frac{g_u^2(S)}{2} \left( \frac{(\sigma_{X_u}^{-1}(S + t) - r_\xi(t - s)\sigma_{X_u}^{-1}(S + s))^2}{1 - r_\xi^2(t - s)} + \sigma_{X_u}^{-2}(S + s) \right) \right).
\]

The expected value \( m_u(s, t) \) and the variance \( \sigma_u^2(s, t) \) of the conditional density \( \varphi_{st}(y|g_u(S), g_u(S)) \) of random variable \( \hat{\xi}'(t) \) given \( \hat{\xi}(s) = \hat{\xi}(t) = g_u(S) \), according to Lemma 3.1, are equal to

\[
m_u(s, t) = g_u(S) \left( \frac{\sigma_{X_u}'(S + t)}{\sigma_{X_u}(S + t)} + \frac{r_\xi(t - s)(\sigma_{X_u}(S + t) - r_\xi(t - s)\sigma_{X_u}(S + s))}{\sigma_{X_u}(S + s)(1 - r_\xi^2(t - s))} \right)
\]

and

\[
\sigma_u^2(s, t) = \sigma_{X_u}^2(S + t) \left( 2C - \frac{r_\xi^2(t - s)^2}{1 - r_\xi^2(t - s)} \right).
\]
Applying the Taylor formula again, after some technical calculations we have for \(s, t \in [0, T_u]\)

\[
g_u^{-1}(S) \frac{m_u(s, t)}{|t-s|} \to -C \quad \text{and} \quad \frac{\sigma_u^2(s, t)}{|t-s|^2} \to 2C^2
\]

uniformly as \(u \to \infty\). Similar calculations show that the quantity in the square brackets of (3.14) converges to \((2C)^{-1}\sigma'(S)^2/\sigma^2(S) + 1\) uniformly as \(u \to \infty\). Substituting these asymptotic results back into (3.13) yields an upper bound in the form

\[
\text{Constant} \times T_u^2 g_u^2(S) \exp \left(-\frac{g_u^2(S)}{2} \left(1 + \frac{\sigma'(S)^2}{2C\sigma^2(S)}\right)\right),
\]

which is negligible with respect to \(\Psi(g_u(S))\) as \(u \to \infty\).

In summary, for all three different cases of \(T_u\), we have the lower bound (3.7). This together with upper bound (3.3) completes the proof. \(\square\)

**Remark 3.1.** The attentive reader may have found that the method of moments in Case III can be also applied to Case I and Case II. However, the method used in Case I and Case II, as the authors have tried before, failed to solve Case III. The method of moments, also known as Rice method, has long been used to estimate the distribution of the maximum of a random process (See the monograph [6] and recent paper [5]).

**Proof of Theorem 2.2** From the definition of ruin time \(\tau(u)\) in (2.1), we know

\[
P\{\tau(u) \leq S + T_u\} = P_{S_u}(u, T_u).
\]

Then, by (3.1), for any \(x > 0\)

\[
P\{u^2(S + T_u - \tau(u)) > x | \tau(u) < S + T_u\} = \frac{P\left\{\sup_{t \in [0, S_x(u)]} \inf_{s \in [t, t + T_u]} \tilde{X}_u(s) > g_u(S_x(u))\right\}}{P\left\{\sup_{t \in [0, S]} \inf_{s \in [t, t + T_u]} X_u(s) > g_u(S)\right\}},
\]

with \(S_x(u) := S - xu^{-2}\) and \(\tilde{X}_u(t) := \frac{Y(t) g_u(S_x(u))}{\sigma(t)}\). We need to find an exact asymptotic for the numerator. First, as in the proof of Theorem 2.1 case III, just replacing \(S\) by \(S_x(u)\), we have

\[
P_{S_x(u)}(u, T_u) \geq P\left\{\inf_{s \in \{S_x(u), S_x(u) + T_u\}} \tilde{X}_u(s) > g_u(S_x(u))\right\} \geq \Psi\left(g_u(S_x(u))\right) (1 + o(1))
\]

as \(u \to \infty\). Next, appealing to the upper bound given in the proof of Theorem 2.4 in [2], we get

\[
P_{S_x(u)}(u, T_u) \lesssim P\left\{\sup_{t \in [0, S_x(u)]} \tilde{X}_u(t) > g_u(S_x(u))\right\} = \Psi\left(g_u(S_x(u))\right) (1 + o(1))
\]

as \(u \to \infty\). Therefore,

\[
P\{u^2(S + T_u - \tau(u)) > x | \tau(u) < S + T_u\} \sim \frac{\Psi\left(g_u(S_x(u))\right)}{\Psi\left(g_u(S)\right)} \sim \exp\left(\frac{g_u^2(S) - g_u^2(S_x(u))}{2}\right), \quad u \to \infty.
\]

Some standard algebra yields

\[
g_u^2(S) - g_u^2(S_x(u)) \sim \frac{-2\sigma'(S)}{\sigma^3(S)} x, \quad u \to \infty.
\]
In other words,
\[
\lim_{u \to \infty} \mathbb{P}\{u^2(S + T_u - \tau(u)) > x | \tau(u) < S + T_u\} = \exp\left(-\frac{\sigma'(S) x}{\sigma^3(S)}\right),
\]
which completes the proof. 

**Lemma 3.1.** Let \((X, Y, Z)\) be a centered Gaussian vector with values in \(\mathbb{R}^3\), the conditional distribution of \(Z\) given \(X = x\) and \(Y = y\) is a Gaussian random variable with expected value
\[
\mathbb{E}[Z|X = x, Y = y] = (x, y)Q^{-1}b
\]
and variance
\[
\mathbb{V}ar(Z|X = x, Y = y) = \mathbb{V}ar(Z) - b^TQ^{-1}b,
\]
where \(Q\) is the covariance matrix of random variables \(X\) and \(Y\), \(b = (\text{Cov}(X, Z), \text{Cov}(Y, Z))^T\).

**Proof of Lemma 3.1** Decomposing \(Z\) as sum like
\[
Z = \alpha X + \beta Y + \Gamma
\]
such that \(\Gamma\) is independent of both \(X\) and \(Y\). This yields \((\alpha, \beta)^T = Q^{-1}b\). 

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