ON DRIFT AND ENTROPY GROWTH FOR RANDOM WALKS ON GROUPS

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In this paper we consider drift and entropy growth for symmetric finitary random walks on finitely generated groups. We construct examples of various intermediate asymptotics of the drift for such random walks. We establish general inequalities which connect drift, entropy and exponential growth rate of groups. Then we apply these inequalities to get estimates for entropy in particular examples.

1. Introduction. Let $G$ be a group and $\mu$ a symmetric measure on it. We consider random walks on $G$ induced by this measure. In this paper we assume that the support of $\mu$ is finite and generates the group $G$. We consider two functions

$$H(n) = H^{G,\mu}(n) = -\sum_{g \in G} (\mu^{*n}(g)) \ln(\mu^{*n}(g))$$

and

$$L(n) = L^{G,\mu}(n) = E_{\mu^{*n}} l(g).$$

Here $l$ denotes the word length, corresponding to the fixed set of generators, and $\mu^{*n}$ is the $n$th convolution of $\mu$. The function $H(n)$ is called the entropy and $L(n)$ is called the drift (or escape) of the random walk. $L(n)$ shows how fast (on average) the random walk is moving away from the origin.

It is known that $H(n)$ is asymptotically linear if and only if $L(n)$ is asymptotically linear [14] and if and only if the Poisson boundary of the random walk is nontrivial [11]. In particular, it is so for any nonamenable group. On the other hand, for many examples (e.g., for any Abelian group) $L(n)$ is asymptotically $\sqrt{n}$.

Until recently it was unknown whether there exist simple random walks on groups with intermediate growth rate of $L(n)$. The question of finding such random walks appears in [9]. I learned about this question from A. M. Vershik in 1999. The problem of finding asymptotics for $H(n)$ is discussed in [1].
First examples of random walks on groups with intermediate growth rate of $L(n)$ were found by the author in [3, 6]. In these examples $L(n) \asymp n^{1-1/2^k}$ (for any positive integer $k$) and $L(n) \asymp n / \ln(n)$. Related questions about the drift (escape) for these groups were also considered in [12].

In this paper we find new possibilities for the rate of $L(n)$. In particular, we show that $L(n)$ can be asymptotically equal to

$$\frac{n}{\ln(\ln(\cdots \ln(n) \cdots))}.$$ 

We also estimate the growth of the entropy for random walks on these groups.

The structure of the paper is as follows. In Section 2 we state that certain functions are concave. We use this auxiliary lemma in the next section.

In Section 3 we consider a two-dimensional simple random walk. We find asymptotics of some class of functionals depending on local times of this two-dimensional random walk.

In Section 4 we construct examples of groups and we apply the results of the previous section to find asymptotics of the drift in these groups.

In Section 5 we give some general estimates for the entropy $H(n)$. We apply these estimates to the examples considered in Section 4. These examples show that there are infinitely many possibilities for asymptotics of the entropy. They also show that the growth of the entropy (as well as of the drift) can be very close to linear and yet sublinear.

The main results of this paper were announced in [7].

2. An auxiliary lemma.

**Lemma 1.** Take $0 < \alpha \leq 1$.

(i) Let

$$\tilde{L}_{k,\alpha}(x) = \frac{x^{\alpha}}{(\ln(\ln(\cdots \ln(x) \cdots))^{\alpha})} .$$

Then $\tilde{L}_{k,\alpha}(x)$ is concave on the segment $[T_k, \infty)$, where

$$T_k = \exp(\exp(\cdots \exp(4k) \cdots)) \cdots .$$

(ii) There exist a continuous increasing function $L_{k,\alpha} : [0, \infty) \to [0, \infty)$ and $X_{k,\alpha} > 0$ such that $L_{k,\alpha}(x)$ is concave, $L_{k,\alpha}(0) = 0$ and $L_{k,\alpha}(x) = \tilde{L}_{k,\alpha}(x)$ for any $x > X_{k,\alpha}$.

We prove this lemma in the Appendix.
3. Some functionals of two-dimensional random walk. We say that the random walk is simple if $\mu$ is equidistributed. In this section we consider a simple random walk on $\mathbb{Z}^2$. Let $b_z^{(n)}$ be the number of times the random walk has visited the element $z$, $z \in \mathbb{Z}^2$, from the moment 0 up to the moment $n$. Let $R^{(n)}$ be the number of different elements visited until the moment $n$ ($R^{(n)}$ is called range of the random walk). First we formulate a simple property of the range.

**Lemma 2.** There exist $q_1, q_2 > 0$ such that, for any $n > 1$,

$$\Pr\left[ R^{(n)} \geq q_1 \frac{n}{\ln(n)} \right] \geq q_2.$$

**Proof.** In fact, $R(n)/(n/\ln(n)) \to \pi$ almost surely [2]. □

**Lemma 3.** Let $f$ be a concave strictly increasing function such that $f(0) = 0$. Consider a simple random walk on $\mathbb{Z}^2$. There exists $K > 0$ such that

$$E\left( \sum_{z \in \mathbb{Z}^2} f(b_z^{(n)}) \right) \leq K f(\ln(n)) \frac{n}{\ln(n)}.$$

**Proof.** Note that

$$\sum_{z \in \mathbb{Z}^2} f(b_z^{(n)}) \leq f(n/R^{(n)}) R^{(n)},$$

since $f$ is concave. Hence

$$E\left[ \sum_{z \in \mathbb{Z}^2} f(b_z^{(n)}) \right] \leq E\left[ f(n/R^{(n)}) R^{(n)} \right].$$

Note that $xf(n/x)$ is concave on $(0, \infty)$, since $f(x)$ is concave on $(0, \infty)$. For a smooth $f$ we know that $(xf(n/x))'' = n^2 f''(n/x) / x^3$. To prove this in the general case it is sufficient to consider an approximation of $f$ by twice differentiable concave functions.

Consequently,

$$E\left[ f\left( \frac{n}{R^{(n)}} \right) R^{(n)} \right] \leq E[R^{(n)}] f\left( \frac{n}{E[R^{(n)]}} \right) \propto f(\ln(n)) \frac{n}{\ln(n)}.$$

Then there exists $K > 0$ such that

$$E\left( \sum_{z \in \mathbb{Z}^2} f(b_z^{(n)}) \right) \leq K f(\ln(n)) \frac{n}{\ln(n)}.$$

This completes the proof of the lemma. □

The following lemma gives an estimate from the other side.
Lemma 4. Let $f$ be a strictly increasing function on $[0, \infty)$ such that $f(0) = 0$ and $f(Cx) \leq Cf(x)$ for any $C > 1$. Then for $n$ large enough and for some positive $\varepsilon$ we have

$$E \left[ \sum_{z \in \mathbb{Z}^2} f(b_z^{(n)}) \right] \geq \varepsilon f(\ln(n)) \frac{n}{\ln(n)}.$$

Proof. Note that for any $\varepsilon_1 > 0$ there exists $K > 0$ such that for $n$ large enough

$$\Pr[b_0^{(n)} \geq K \ln(n)] \geq 1 - \varepsilon_1.$$

(This follows from [5], Theorem 1.) Let $n \geq 4$ and let $m = \lceil n/2 \rceil$. Since $m > 1$, Lemma 2 implies that

$$\Pr \left[ R(m) \geq q_1 \frac{m}{\ln(m)} \right] \geq q_2.$$

Let $x_1^{(n)}, \ldots, x_s^{(n)}$ be different points visited by the random walk up to the moment $n$, enumerated in the order of visiting.

Let $\beta_i^{(n)} = b_{x_i^{(n)}}^{(n)}$.

Take $\varepsilon_1$ such that $\varepsilon_1 \leq 1/2$.

Note that, for any $0 \leq i \leq q_1 \frac{m}{\ln(m)}$,

$$\Pr[\beta_i^{(n)} \geq K \ln(m)] \geq \Pr[\beta_i^{(n)} \geq K \ln(m), R(m) \geq i]$$

$$= \Pr[\beta_i^{(n)} \geq K \ln(m) | R(m) \geq i] \Pr[R(m) \geq i]$$

$$\geq \Pr[\beta_i^{(n)} \geq K \ln(m) | R(m) \geq i] q_2$$

$$\geq \Pr[\beta_0^{(n)} \geq K \ln(m)] q_2 \geq (1 - \varepsilon_1) q_2 \geq \frac{q_2}{2}.$$

Hence for any $n$ large enough,

$$E \left[ \sum_{z \in \mathbb{Z}^2} f(b_z^{(n)}) \right] \geq \frac{q_2}{2} f \left( K \ln \left( \left\lceil \frac{n}{2} \right\rceil \right) \right) q_1 \frac{n/2}{\ln(n/2)} \asymp f(\ln(n)) \frac{n}{\ln(n)}. \quad \square$$

Corollary 1. (i) Let $L_{k,\alpha}(x)$ be the function defined in Lemma 1 ($0 < \alpha \leq 1$). Then

$$E \left[ \sum_{z \in \mathbb{Z}^2} L_{k,\alpha}(b_z^{(n)}) \right] \asymp L_{k+1,\alpha}(n).$$
(ii) Let \( f(x) = x^\alpha \) (\( 0 < \alpha \leq 1 \)). Then
\[
\mathbb{E}\left[ \sum_{z \in \mathbb{Z}^2} f(b_z^{(n)}) \right] \asymp \frac{n}{\ln(n)}(1-\alpha).
\]

**Proof.** This corollary follows from Lemmas 1, 3 and 4 since, for \( n > N \),
\[
\frac{n}{\ln(n)} L_{k,\alpha}(\ln(n)) \asymp \frac{n}{\ln(n)} \frac{\ln(n)}{\ln(\ln(n) \ldots)} \asymp L_{k+1,\alpha}(n)
\]
and
\[
\frac{n}{\ln(n)} \ln(n)^{1-\alpha} = \frac{n}{\ln(n)^\alpha}.
\]

4. **Main result.** First we recall the definition of the wreath product.

**Definition 1.** Let \( C, D \) be groups and denote by \( \sum C \times D \) the space of functions \( f: C \to D \) with finite support. The wreath product of \( C \) and \( D \) is a semidirect product of \( C \) and \( \sum C \times D \), where \( C \) acts on \( \sum C \times D \) by shifts: if \( c \in C \), \( f \in \sum C \times D \), then \( c \cdot f(x) = f(xc^{-1}) \), \( x \in C \). Let \( C \wr D \) denote the wreath product.

Note that if \( C \) is infinite, then the group of finite \( D \)-valued configurations \( \sum C \times D \) is infinitely generated. However, the wreath product \( C \wr D \) is finitely generated whenever \( C \) and \( D \) are finitely generated.

**Lemma 5.** Let \( a_1, a_2, \ldots, a_k \) generate \( A \) and let \( \mu \) be the measure equidistributed on these generators and their inverses.

Then for some finite symmetric measure \( v \) on \( B = \mathbb{Z}^2 \wr A \) the drift of the corresponding simple random walk satisfies
\[
L_{B,v}(n) \asymp \mathbb{E} \sum_{z \in \mathbb{Z}^2} L_{A,\mu}(b_z^{(n)}).
\]

**Proof.** The proof of this lemma is similar to that of Lemma 3 in [6].

For any \( a \in A \), \( \tilde{a}^e \) denotes the function from \( \mathbb{Z}^2 \) to \( A \) such that \( \tilde{a}^e(0) = a \) and \( \tilde{a}^e(x) = e \) for any \( x \neq 0 \). Let \( a^e = (0, a^e) \). Let \( w'_1, w'_2 \) be the standard generators of \( \mathbb{Z}^2 \) and \( w_1 = (w'_1, e), w_2 = (w'_2, e) \).

Consider the following set of generators of \( B \):
\[
(a_j^e)^p w_x(a_n^e)^q,
\]
where \( p, q = 0, 1 \) or \(-1, s = 1 \) or \( 2 \) and \( 1 \leq j, n \leq k \).

Consider the simple random walk on \( B \), corresponding to this set of generators.

Let \( \mu_2 \) be the measure on \( A \) such that \( \mu_2(e) = 1/2 \) and \( \mu_2(a_j) = \mu_2(a_j^{-1}) = 1/(4k) \) for any \( 1 \leq j \leq k \). The random walk on \( B = \mathbb{Z}^2 \wr A \) is realized as
follows: Let $X_n$, $n \geq 0$, be a simple symmetric random walk on $\mathbb{Z}^2$ and denote by $\{b^{(n)}_z : z \in \mathbb{Z}^2\}$ and $R^{(n)}$ its occupation process and its range, as introduced in the previous section.

Let $\xi^{(z)}_n$ be independent and identically distributed symmetric random walks on the group $A$, with steps distributed with respect to $\mu_2$. The random walk on $B = \mathbb{Z}^2 \wr A$ is

$$U_n = (X_n, \phi_n)$$

with

$$\phi_n(z) = \xi^{(z)}_n 2b^{(n)}_z - \delta(z,0) - \delta(z,X_n).$$

Let $c^{(n)}_z = l_A(\phi_n(z)).$

Then

$$\frac{1}{2} \sum_{z \in \mathbb{Z}^2} c^{(n)}_z \leq l_B(U_n) \leq 2 \left( \sum_{z \in \mathbb{Z}^2} c^{(n)}_z + R \right).$$

Hence

$$\frac{1}{2} \mathbb{E} \left[ \sum_{z \in \mathbb{Z}^2} c^{(n)}_z \right] = \frac{1}{2} \sum_{z \in \mathbb{Z}^2} \mathbb{E}[c^{(n)}_z] \leq \mathbb{E}[l_B(U_n)] \leq 2 \left( \sum_{z \in \mathbb{Z}^2} \mathbb{E}[c^{(n)}_z] + \mathbb{E}[R] \right).$$

It is clear that

$$L^{A,\mu_2}(n) \asymp L^{A,\mu}(n).$$

Note that

$$\mathbb{E}[c^{(n)}_z | X_k, k = 0, 1, 2, \ldots, n] = L^{A,\mu_2}(2b^{(n)}_z - \delta(z,0) - \delta(z,X_n)).$$

Hence

$$\mathbb{E}[\min(L^{A,\mu_2}(2b), L^{A,\mu_2}(2b-1), L^{A,\mu_2}(2b-2))] \leq \mathbb{E}[c^{(n)}_z | b^{(n)}_z = b] \leq \mathbb{E}[\max(L^{A,\mu_2}(2b), L^{A,\mu_2}(2b-1), L^{A,\mu_2}(2b-2))].$$

There exist $C_2, C_3 > 0$ such that

$$C_2 L^{A,\mu}(n) \leq L^{A,\mu_2}(2n-2), L^{A,\mu_2}(2n-1), L^{A,\mu_2}(2n) \leq C_3 L^{A,\mu}(n).$$

Hence

$$C_2 \mathbb{E} \left[ \sum_{z \in \mathbb{Z}^2} L^{A,\mu}(b^{(n)}_z) \right] \leq \sum_{z \in \mathbb{Z}^2} \mathbb{E}[c^{(n)}_z] \leq C_3 \mathbb{E} \left[ \sum_{z \in \mathbb{Z}^2} L^{A,\mu}(b^{(n)}_z) \right].$$

This completes the proof of the lemma. \qed
THEOREM 1. (i) Let $F$ be a finite group. Consider the following groups that are defined recurrently:

$$G_1 = \mathbb{Z}^2 \cdot F; \quad G_{i+1} = \mathbb{Z}^2 \cdot G_i.$$  

Then for some simple random walk on $G_i$ and for any $n$ large enough, we have

$$L^{G_i}(n) \asymp \frac{n}{\ln(\ln \cdots \ln n \cdots)}.$$

(ii) Consider the following groups that are defined recurrently:

$$F_1 = \mathbb{Z}; \quad F_{i+1} = \mathbb{Z} \cdot F_i;$$

let

$$H_{1,i} = \mathbb{Z}^2 \cdot F_i, \quad H_{j+1,i} = \mathbb{Z}^2 \cdot H_{j,i}.$$  

Then for some simple random walk on $H_{j,i}$ and for any $n$ large enough, we have

$$L^{H_{j,i}}(n) \asymp \frac{n}{\sqrt[3]{\ln(\ln \cdots \ln(n) \cdots)}}.$$

PROOF. (i) We prove the theorem by induction on $i$. Base $i = 1$. In this case $G_i = \mathbb{Z}^2 \cdot F$ and $L(n)$ is asymptotically equal to $n/\ln(n)$ [3]. The induction step follows from the previous lemma and Corollary 1.

(ii) We prove the statement by induction on $j$. For $H_{1,i} = F_i$ the asymptotics of the drift is found in [6]. It is proven there that, for some random walk on $F_i$,

$$L_{F_i}(n) \asymp n^{1-1/2^k}.$$  

The induction step follows from the previous lemma and Corollary 1. □

5. Estimates of the entropy. In this section we give estimates for the entropy of a random walk. It is known (see [1]) that for a wide class of measures on nilpotent groups $H(n) \asymp \ln(n)$. As mentioned before, $H(n)$ is asymptotically linear for any nonamenable group. In this section we study intermediate examples.

Let $v(n)$ be the growth function of the group [i.e., $v(n) = \#\{g \in G : l(g) \leq n\}$]. Let

$$l = \lim_{n \to \infty} L(n)/n,$$

$$h = \lim_{n \to \infty} H(n)/n,$$

$$v = \lim_{n \to \infty} \ln(v(n))/n.$$
It is a well-known fact that these limits exist, since $L(n), H(n)$ and $v(n)$ are subadditive. See, e.g., [15]. It is known (see [9, 15]) that

$$h \leq v l.$$ 

The following lemma generalizes this fact.

**Lemma 6.** For any $\varepsilon > 0$ there exists $C > 0$ such that

$$H(n) \leq (v + \varepsilon)L(n) + \ln(n) + C.$$ 

**Proof.** Let $a_i^{(n)} = \Pr_{\mu^n}[l(g) = i]$. Then by definition

$$L(n) = \sum_{i=0}^{n} i a_i^{(n)}.$$ 

Comparing $\mu^n$ with the measure which is equidistributed on every sphere in the group we get

$$H(n) \leq \sum_{i=1}^{n} a_i^{(n)} \ln(v(i)/a_i^{(n)})$$

$$= \sum_{i=1}^{n} a_i^{(n)} \ln(v(i)) + \sum_{i=1}^{n} a_i^{(n)} (-\ln(a_i^{(n)})) \leq \sum_{i=1}^{n} a_i^{(n)} \ln(v(i)) + \ln(n).$$ 

For any $\varepsilon > 0$ there exists $K > 0$ such that $v(i) \leq K (v + \varepsilon)^i$. Hence

$$H(n) \leq \sum_{i=1}^{n} a_i^{(n)} (i(v + \varepsilon) + \ln(K)) + \ln(n)$$

$$= (v + \varepsilon) \sum_{i=1}^{n} a_i^{(n)} i + \ln(K) + \ln(n)$$

$$= (v + \varepsilon)L(n) + \ln(K) + \ln(n).$$ 

□

Another lemma estimates the entropy from the other side:

**Lemma 7.** (i) There exists $C > 0$ such that

$$H(n) \geq CE_{\mu^n} l^2(g)/n - \ln(n) \geq CL^2(n)/n - \ln(n).$$

(ii) There exists $K > 0$ such that

$$L(n) \leq K \sqrt{n(\ln(v(n)) + \ln(n))}.$$
PROOF. (i) Let \( p_n(x) \) be the probability of hitting \( x \) after \( n \) steps. In [14] it is shown that there exist \( K_1, K_2 > 0 \) such that, for any \( x \) and \( n \),
\[
p_n(x) \leq K_1 n^{3/4} \exp(-K_2 l(x)^2/n) \leq K_1 n \exp(-K_2 l(x)^2/n).
\]
Then note that
\[
- \sum_{g \in G : l(g) = i} \mu^{*n}(g) \ln(\mu^{*n}(g)) \geq (-\ln(K_1) - \ln(n) + K_2 i^2/n)a_i^{(n)}.
\]
Hence
\[
H(n) \geq -\ln(K_1) - \ln(n) + K_2 \sum_{i=0}^{n} i^2/na_i^{(n)}
\]
\[
= -\ln(K_1) - \ln(n) + K_2/nE_{\mu}^{*n}l^2(g) \geq C/nE_{\mu}^{*n}l^2(g) - \ln(n).
\]
The last inequality follows from the fact that, for some \( C_2 > 0 \), \( H(n) \geq C_2 \).

(ii) This follows from the first part of the lemma, since \( H(n) \leq \ln(v(n)) \). \( \square \)

As a corollary from the two previous lemmas we get the following theorem.

**THEOREM 2.** Let \( G_i \) be the groups defined in Theorem 1. Then for some random walk on \( G_i \), we have
\[
K_1 n/\ln(\ln(\cdots \ln(n) \cdots)) \leq H_{G_i}(n) \leq K_2 n/\ln(\ln(\cdots \ln(n) \cdots))
\]
for some positive constants \( K_1 \) and \( K_2 \). In particular, all \( G_i \) have different asymptotics of the entropy.

Moreover, using arguments from [8] one can show that \( H_{G_i}(n) \) in Theorem 2 is asymptotically equivalent to the right-hand term of the second inequality, that is, for some \( K_3 > 0 \),
\[
H_{G_i}(n) \geq K_3 n/\ln(\ln(\cdots \ln(n) \cdots))
\]

**APPENDIX**

**PROOF OF THE AUXILIARY LEMMA.** In this Appendix we give the proof of Lemma 1.

(i) First we consider the case \( \alpha = 1 \). Let \( m_k(x) = (\ln(\ln(\cdots \ln(x) \cdots))) \). We want to prove that \( x/m_k(x) \) is concave on \([T_k, \infty)\). Note that
\[
\left( \frac{x}{m_k} \right)''(x) = \frac{-xm_k''(x)m_k^2(x) - 2m_k'(x)m_k^2(x) + 2xm_k(x)(m_k'(x))^2}{m_k^4(x)}.
\]
Since $m_k(x) > 0$ on $[T_k, \infty)$, it suffices to prove that
\[ 2x(m'_k(x))^2 - x m_k(x)m''_k(x) \leq 2m'_k(x)m_k(x). \]
To prove this we will show that
\[ 2x(m'_k(x))^2 < 1/2m_k(x)m'_k(x) \]
and that
\[ m_k(x)(-m''_k(x))x \leq \frac{3}{2}m'_k(x)m_k(x). \]
Note that
\[ m'_k(x) = \frac{1}{x \ln(x) \ln(\ln(x)) \cdots \ln(\ln(\cdots \ln(x) \cdots ))}. \]
We see that $0 < m'_k(x) < 1/x$ and we know that $m_k(x) > 4$, $x \in [T_k, \infty)$. This proves the first inequality. Now let
\[ r_k(x) = \frac{1}{m'_k(x)} = x \ln(x) \ln(\ln(x)) \cdots \ln(\ln(\cdots \ln(x) \cdots )) \cdot \ln(\ln(\cdots \ln(x) \cdots )) \cdot \ln(\ln(\cdots \ln(x) \cdots )). \]
Note that
\[ r'_k(x) = m_1(x)m_2(x) \cdots m_{k-1}(x) + \sum_{i=1}^{k-1} r_k(x) \frac{m'_i(x)}{m_i(x)}. \]
Note that for $1 \leq i \leq k$ it holds that $m'_i(x) \leq 1/x$. Since $m_k(x) \geq 2k$ and, for $1 \leq i \leq k-1$, $m_i(x) \geq 2k$, we get
\[ r'_k(x)x \leq 1.5r_k(x). \]
This implies that
\[ (-m''_k(x))x = \frac{xr'_k(x)}{r^2_k(x)} \leq \frac{3}{2r_k(x)} = \frac{3}{2}m'_k. \]
So we have proven the second inequality.

Now consider $0 < \alpha \leq 1$. We have already proven that
\[ h_k(x) = \frac{x}{(\ln(\cdots \ln(x) \cdots ))} \]
is concave on $[T_k, \infty)$.
Let
\[ g_k(x) = x h_k\left(\frac{1}{x}\right). \]
Since $g_k''(x) = x^{-3}h_k''(1/x)$ we get that $g_k(x)$ is concave on $(0, 1/T_k]$.

Note that if $f$ is an increasing function and both $f$ and $g$ are concave, then $f(g)$ is concave. In fact, in this case, for any $0 \leq t \leq 1$, it holds that

$$f(g(tx + (1-t)y)) \leq f(tg(x) + (1-t)g(y)) \leq tf(g(x)) + (1-t)f(g(y)).$$

Note that $f(x) = x^\alpha$ is concave and increasing on $[0, \infty)$ for $0 < \alpha \leq 1$ and that $g_k(x)$ is positive on $(0, 1/T_k]$. Therefore $f(g_k(x))$ is concave on $(0, 1/T_k]$. But this implies that

$$\frac{x}{(\ln(\ln\cdots\ln(x))\cdots)} = xf\left(g_k\left(\frac{1}{x}\right)\right)$$

is concave on $[T_k, \infty)$.

(ii) Let $\beta_{k,\alpha} = \tilde{L}_{k,\alpha}(T_k)/(2T_k)$. Consider the function $y_{k,\alpha}(x) = \beta_{k,\alpha}x$. Note that $y_{k,\alpha}(T_k) < \tilde{L}_{k,\alpha}(T_k)$ and that there exists $N_{k,\alpha} > 0$ such that $y_{k,\alpha}(x) > \tilde{L}_{k,\alpha}(x)$ for any $x > N_{k,\alpha}$. Take maximal $z$ such that $y_{k,\alpha}(z) = \tilde{L}_{k,\alpha}(z)$. Let $L_{k,\alpha}(x) = y_{k,\alpha}(x)$ if $0 \leq x \leq z$ and $L_{k,\alpha}(x) = \tilde{L}_{k,\alpha}(x)$ if $x \geq z$. Note that $y_{k,\alpha}(x)$ is concave for $x \in [0, z]$, that $\tilde{L}_{k,\alpha}(x)$ is concave if $x \geq z$ and that $\tilde{L}_{k,\alpha}''(z) \leq y_{k,\alpha}''(z)$. This implies that $L_{k,\alpha}$ is concave. □

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REFERENCES

[1] DERRIENNIC, Y. (1986). Entropie, theoremes limite et marches aleatoires. Probability Measures on Groups VIII, Lecture Notes in Math. 1210 241–284. Springer, Berlin.

[2] DVORETZKY, A. and ERDÖS, P. (1951). Some problems on random walk in space. Proc. Second Berkeley Symp. Math. Statist. Probab. 353–367. Univ. California Press, Berkeley.

[3] DYUBINA, A. (1999). An example of growth rate for random walk on group. Russian Math. Surveys 54 159–160.

[4] DYUBINA, A. (1999). Characteristics of random walks on wreath product of groups. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov (POMI) 256 31–37.

[5] ERDÖS, P. and TAYLOR, S. J. (1960). Some problems concerning the structure of random walk paths. Acta Math. 11 37–63.

[6] ERSCHLER (DYUBINA), A. (2001). On the asymptotics of drift. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov (POMI) 283 251–257.

[7] ERSCHLER (DYUBINA), A. (2001). Drift and entropy growth for random walk on groups. Russian Math. Surveys 56 179–180.

[8] ERSCHLER (DYUBINA), A. (2002). Liouville property for groups and manifolds. Preprint.

[9] GUIVARCH, Y. (1980). Sur la loi des grands nombres et le rayon spectral d’une marche aleatoire. Astérisque 74 47–98.

[10] GUIVARCH, Y. (2000). Marches aléatoires sur les groupes. In Development of Mathematics 1950–2000 (J. P. Pier, ed.). Birkhäuser, Basel.
[11] Kaimanovich, V. A. and Vershik, A. M. (1983). Random walks on discrete groups: Boundary and entropy. *Ann. Probab.* **11** 457–490.

[12] Revelle, D. (2001). Rate of escape of random walk on wreath products and related groups. Preprint.

[13] Spitzer, F. (1964). *Principles of Random Walk*. Van Nostrand, Princeton.

[14] Varopoulos, N. (1985). Long range estimates for Markov chains. *Bull. Sci. Math.* **109** 225–252.

[15] Vershik, A. M. (1999). Characteristics of groups and their relations. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov (POMI)* **256** 1–11.