Higgs and Fermions in $D_4 - D_5 - E_6$ Model based on $Cl(0,8)$ Clifford Algebra

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Abstract

In the $D_4 - D_5 - E_6$ model of a series of papers (hep-ph/9301210, hep-th/9302030, hep-th/9306011, and hep-th/9402003) an 8-dimensional spacetime with Lagrangian action

$$\int_{V_8} F_8 \wedge \star F_8 + \partial^2 \Phi_8 \wedge \star \partial^2 \Phi_8 + \bar{\Phi}_8 S_{8\pm} + GF + GH$$

is reduced to a 4-dimensional Lagrangian.

In [11], the gauge boson terms were seen to give $SU(3) \times SU(2) \times U(1)$ for the color, weak, and electromagnetic forces and gravity of the MacDowell-Mansouri type [4], which has recently been shown by Nieto, Obregón, and Socorro [6] in gr-qc/9402029 to be equivalent, up to a Pontrjagin topological term, to the Ashtekar formulation.

This paper discusses the Higgs and spinor fermion terms.
1 Introduction

The $D_4-D_5-E_6$ model of physics starts out with an 8-dimensional spacetime that is reduced to a 4-dimensional spacetime.

The 8-dimensional Lagrangian (up to gauge-fixing and ghost terms) is:

$$\int_{V_8} F_8 \wedge \star F_8 + \partial_8^2 \Phi_8 \wedge \star \partial_8^2 \Phi_8 + \partial_8 S_{8\pm} + GF + GH$$

where $F_8$ is the 28-dimensional $Spin(8)$ curvature, $\star$ is the Hodge dual, $\partial_8$ is the 8-dimensional covariant derivative, $\Phi_8$ is the 8-dimensional scalar field, $\partial_8$ is the 8-dimensional Dirac operator, $V_8$ is 8-dimensional spacetime, $S_{8\pm}$ are the + and − 8-dimensional half-spinor fermion spaces, and $GF$ and $GH$ are gauge-fixing and ghost terms.

(hep-th/9402003 [11] had a typo error of $S_{8+}$ or $S_{8-}$ instead of $S_{8\pm}$.)

This paper describes the Higgs mechanism and the spinor fermions of the 4-dimensional Lagrangian. Results of the preceding papers in this series [8, 9, 10, 11] are assumed. They are hep-ph/9301210, hep-th/9302030, hep-th/9306011, and hep-th/9402003).

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Summary of Some Material from Earlier Papers:

\[ S_{8\pm} = \mathbb{R}P^1 \times S^7 \, [4]; \]

\[ S_{8\pm} \oplus S_{8\pm} = (\mathbb{R}P^1 \times S^7) \oplus (\mathbb{R}P^1 \times S^7) \]

is the full fermion space of first generation particles and antiparticles, and is the Silov boundary of the 32(real)-dimensional bounded complex domain corresponding to the \textit{TypeV} HJTS \( E_6/(Spin(10) \times U(1)) \, [4] \)

\[ (\text{hep-th/9302030} \, [2]) \]

had erroneously used \( \times \) instead of \( \oplus \).)

after dimensional reduction, the weak force gauge group is \( SU(2) \, [11]; \)

with respect to \( SU(2) \) of the Higgs and weak force, the 4-dimensional spacetime manifold has global type \( M = S^2 \times S^2 \, [7]; \)

the Higgs and weak force \( SU(2) \) acts effectively on a submanifold of the half-spinor fermion space \( S_{8\pm} = \mathbb{R}P^1 \times S^7 \), that is, \( Q_3 = \mathbb{R}P^1 \times S^2 \), which is Silov boundary of the 6(real)-dimensional bounded complex domain corresponding to the \textit{TypeIV} \( 3 \) HJTS \( D_3 = Spin(5)/(SU(2) \times U(1)) \, [8] \)

\[ (\text{hep-th/9302030} \, [2]) \]

had erroneously listed \( SU(3)/SU(2) \times U(1) \) instead of \( Spin(5)/SU(2) \times U(1) \times U(1) \).

after dimensional reduction, the Higgs scalar is the 4th component of the column minimal ideal of \( Cle(0,6) \) that contains \( W_+, W_- \), and \( W_0 \) of the \( SU(2) \) weak force, and so is an \( SU(2) \) scalar \([11]; \)

the \( SU(2) \) gauge group of the vector bosons \( W_+, W_- \), and \( W_0 \) and the \( SU(2) \) gauge group of the scalar Higgs, insofar as they are independent, can be considered as one \( Spin(4) = SU(2) \times SU(2) \) gauge group \([4, 11]\); and

the electromagnetic \( U(1) \) of the \( D_4 - D_5 - E_6 \) model comes from the \( U(3) \) containing the color force \( SU(3) \), and so is ”unified” with the \( SU(3) \) color force rather than with the \( SU(2) \) weak force \([11]\).

The last statement is different from most formulations of the standard model, but is similar to the formulation of O’Raifeartaigh (section 9.4 of [7]) of the standard model as \( S(U(3) \times U(2)) \) rather than \( SU(3) \times U(2) \) or \( SU(3) \times SU(2) \times U(1) \). O’Raifeartaigh states that the unbroken gauge symmetry is actually \( U(3) \) rather than \( SU(3) \times U(1) \).
2 Scalar part of the Lagrangian

The scalar part of the 8-dimensional Lagrangian is

$$\int_{V_8} \partial_s^2 \Phi_8 \wedge \ast \partial_s^2 \Phi_8$$

As shown in chapter 4 of Göckeler and Schücker [2], $\partial_s^2 \Phi_8$ can be represented as an 8-dimensional curvature $F_{H8}$, giving

$$\int_{V_8} F_{H8} \wedge \ast F_{H8}$$

When spacetime is reduced to 4 dimensions, denote the surviving 4 dimensions by 4 and the reduced 4 dimensions by $\perp 4$.

Then, $F_{H8} = F_{H44} + F_{H444} + F_{H\perp 4\perp 4}$, where
- $F_{H44}$ is the part of $F_{H8}$ entirely in the surviving spacetime;
- $F_{H444}$ is the part of $F_{H8}$ partly in the surviving spacetime and partly in the reduced spacetime; and
- $F_{H\perp 4\perp 4}$ is the part of $F_{H8}$ entirely in the reduced spacetime;

The 4-dimensional Higgs Lagrangian is then:

$$\int (F_{H44} + F_{H444} + F_{H\perp 4\perp 4}) \wedge \ast (F_{H44} + F_{H444} + F_{H\perp 4\perp 4}) = \int (F_{H44} \wedge \ast F_{H44} + F_{H444} \wedge \ast F_{H444} + F_{H\perp 4\perp 4} \wedge \ast F_{H\perp 4\perp 4}).$$

As all possible paths should be taken into account in the sum over histories path integral picture of quantum field theory, the terms involving the reduced 4 dimensions, $\perp 4$, should be integrated over the reduced 4 dimensions.

Integrating over the reduced 4 dimensions, $\perp 4$, gives

$$\int (F_{H44} \wedge \ast F_{H44} + \int_{\perp 4} F_{H444} \wedge \ast F_{H444} + \int_{\perp 4} F_{H\perp 4\perp 4} \wedge \ast F_{H\perp 4\perp 4}).$$

2.1 First term $F_{H44} \wedge \ast F_{H44}$

The first term is just $\int F_{H44} \wedge \ast F_{H44}$.

Since they are both $SU(2)$ gauge boson terms, this term, in 4-dimensional spacetime, just merges into the $SU(2)$ weak force term $\int F_w \wedge \ast F_w$. 
2.2 Third term $\int_{\perp 4} F_{H \perp 4 \perp 4} \wedge \star F_{H \perp 4 \perp 4}$

The third term, $\int_{\perp 4} F_{H \perp 4 \perp 4} \wedge \star F_{H \perp 4 \perp 4}$, after integration over $\perp 4$, produces terms of the form

$$\lambda (\overline{\Phi} \Phi)^2 - \mu^2 \Phi \Phi$$

by a process similar to the Mayer mechanism.

The Mayer mechanism is based on Proposition 11.4 of chapter 11 of volume I of Kobayashi and Nomizu [3], stating that:

$$2F_{H \perp 4 \perp 4}(X, Y) = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]),$$

where $\Lambda$ takes values in the $SU(2)$ Lie algebra.

If the action of the Hodge dual $\star$ on $\Lambda$ is such that

$$\star \Lambda = -\Lambda$$

and $\star [\Lambda, \Lambda] = [\Lambda, \Lambda]$,

then

$$F_{H \perp 4 \perp 4}(X, Y) \wedge \star F_{H \perp 4 \perp 4}(X, Y) = (1/4)([\Lambda(X), \Lambda(Y)]^2 - \Lambda([X, Y])^2).$$

If integration of $\Lambda$ over $\perp 4$ is $\int_{\perp 4} \Lambda \propto \Phi = (\Phi^+, \Phi^0)$, then

$$\int_{\perp 4} F_{H \perp 4 \perp 4} \wedge \star F_{H \perp 4 \perp 4} = (1/4) \int_{\perp 4} [\Lambda(X), \Lambda(Y)]^2 - \Lambda([X, Y])^2 =$$

$$= (1/4)[\lambda (\overline{\Phi} \Phi)^2 - \mu^2 \Phi \Phi],$$

where $\lambda$ is the strength of the scalar field self-interaction, $\mu^2$ is the other constant in the Higgs potential, and where $\Phi$ is a 0-form taking values in the $SU(2)$ Lie algebra.

The $SU(2)$ values of $\Phi$ are represented by complex

$SU(2) = Spin(3)$ doublets $\Phi = (\Phi^+, \Phi^0)$.

In real terms, $\Phi^+ = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and $\Phi^0 = (\Phi_3 + i\Phi_4)/\sqrt{2}$, so $\Phi$ has 4 real degrees of freedom.

In terms of real components, $\overline{\Phi} \Phi = (\Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \Phi_4^2)/2$.

The nonzero vacuum expectation value of the

$$\lambda (\overline{\Phi} \Phi)^2 - \mu^2 \Phi \Phi$$

term is $v = \mu/\sqrt{\lambda}$, and

$$< \Phi^0 > = < \Phi_3 > = v/\sqrt{2}.$$

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In the unitary gauge, $\Phi_1 = \Phi_2 = \Phi_4 = 0$, and 
$\Phi = (\Phi^+, \Phi^0) = (1/\sqrt{2})(\Phi_1 + i\Phi_2, \Phi_3 + i\Phi_4) = (1/\sqrt{2})(0, v + H)$,
where $\Phi_3 = (v + H)/\sqrt{2}$, 
v is the Higgs potential vacuum expectation value, and 
$H$ is the real surviving Higgs scalar field.

Since $\lambda = \mu^2/v^2$ and $\Phi = (v + H)/\sqrt{2}$, 
$(1/4)[\lambda(\Phi^\dagger\Phi)^2 - \mu^2\Phi^\dagger\Phi] = 
(1/16)(\mu^2/v^2)(v + H)^4 - (1/8)\mu^2(v + H)^2 = 
(1/16)[\mu^2v^2 + 4\mu^2vH + 6\mu^2H^2 + 4\mu^2H^3/v + \mu^2H^4/v^2 - 2\mu^2v^2 - 
-4\mu^2vH - 2\mu^2H^2] = 
(1/4)\mu^2H^2 - (1/16)\mu^2v^2[1 - 4H^3/v^3 - H^4/v^4].$

2.3 Second term $F_{H^4\perp^4} \wedge \ast F_{H^4\perp^4}$

The second term, 
$\int_{\perp^4} F_{H^4\perp^4} \wedge \ast F_{H^4\perp^4},$
gives $\int \partial \Phi \partial \Phi$, by a process similar to the Mayer mechanism.

From Proposition 11.4 of chapter 11 of volume I of Kobayashi and Nomizu [3]:

$2F_{H^4\perp^4}(X,Y) = [\Lambda(X), \Lambda(Y)] - \Lambda([X,Y]),$
where $\Lambda$ takes values in the $SU(2)$ Lie algebra.

For example, if the $X$ component of $F_{H^4\perp^4}(X,Y)$ is in the surviving 4 spacetime and the $Y$ component of $F_{H^4\perp^4}(X,Y)$ is in $\perp^4$, then

the Lie bracket product $[X,Y] = 0$ so that $\Lambda([X,Y]) = 0$ and therefore 
$F_{H^4\perp^4}(X,Y) = (1/2)[\Lambda(X), \Lambda(Y)] = (1/2)\partial_X\Lambda(Y).$

The total value of $F_{H^4\perp^4}(X,Y)$ is then $F_{H^4\perp^4}(X,Y) = \partial_X\Lambda(Y).$
Integration of $\Lambda$ over $\perp 4$ gives
\[
\int_{Y \perp 4} \partial_X \Lambda(Y) = \partial_X \Phi,
\]
where, as above, $\Phi$ is a 0-form taking values in the $SU(2)$ Lie algebra.

As above, the $SU(2)$ values of $\Phi$ are represented by complex $SU(2) = Spin(3)$ doublets $\Phi = (\Phi^+, \Phi^0)$.

In real terms, $\Phi^+ = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and $\Phi^0 = (\Phi_3 + i\Phi_4)/\sqrt{2}$, so $\Phi$ has 4 real degrees of freedom.

As discussed above, in the unitary gauge, $\Phi_1 = \Phi_2 = \Phi_4 = 0$, and $\Phi = (\Phi^+, \Phi^0) = (1/\sqrt{2})(\Phi_1 + i\Phi_2, \Phi_3 + i\Phi_4) = (1/\sqrt{2})(0, v + H)$, where $\Phi_3 = (v + H)/\sqrt{2}$.

$v$ is the Higgs potential vacuum expectation value, and $H$ is the real surviving Higgs scalar field.

The second term is then:
\[
\int (\int_{\perp 4} - F_{H4\perp 4} \wedge \star F_{H4\perp 4}) = \int (\int_{\perp 4} (-1/2)[\Lambda(X), \Lambda(Y)] \wedge \star[\Lambda(X), \Lambda(Y)]) = \int \partial \Phi \wedge \star \partial \Phi
\]
where the $SU(2)$ covariant derivative $\partial$ is
\[
\partial = \partial + \sqrt{\alpha_{\text{w}}} (W_+ + W_-) + \sqrt{\alpha_{\text{w}}} \cos \theta_w^2 W_0, \text{ and } \theta_w \text{ is the Weinberg angle.}
\]

Then $\partial \Phi = \partial(v + H)/\sqrt{2} = [\partial H + \sqrt{\alpha_{\text{w}}} W_+(v + H) + \sqrt{\alpha_{\text{w}}} W_-(v + H) + \sqrt{\alpha_{\text{w}}} W_0(v + H)]/\sqrt{2}$.

In the $D_4 - D_5 - E_6$ model the $W_+, W_-, W_0$, and $H$ terms are considered to be linearly independent.

$v = v_+ + v_- + v_0$ has linearly independent components $v_+, v_-$, and $v_0$ for $W_+, W_-$, and $W_0$.

$H$ is the Higgs component.

$\partial \Phi \wedge \star \partial \Phi$ is the sum of the squares of the individual terms.

Integration over $\perp 4$ involving two derivatives $\partial_X \partial_X$ is taken to change the sign by $i^2 = -1$. 

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Then:
\[ \partial \Phi \wedge \star \partial \Phi = (1/2)(\partial H)^2 + 
+ (1/2)[\alpha_w v^2 W^+ W^+ + \alpha_w v^2 W^- W^- + \alpha_w v^2 W^+ W^- + \alpha_w v^2 W^+ W^- + \alpha_w W^+ W^- + \alpha_w W^+ W^- [H^2 + 2vH]. \]

Then the full curvature term of the weak-Higgs Lagrangian,
\[ \int F_w \wedge \star F_w + \partial \Phi \wedge \star \partial \Phi + \lambda (\Phi \Phi)^2 - \mu^2 \Phi \Phi, \]
is, by the Higgs mechanism:
\[ \int [F_w \wedge \star F_w + 
+ (1/2)[\alpha_w v^2 W^+ W^+ + \alpha_w v^2 W^- W^- + \alpha_w v^2 W^+ W^- + \alpha_w W^+ W^- + \alpha_w W^+ W^- [H^2 + 2vH] + 
+ (1/2)(\partial H)^2 + (1/4)\mu^2 H^2 - 
- (1/16)\mu^2 v^2[1 - 4H^3/v^3 - H^4/v^4]]. \]

The weak boson Higgs mechanism masses, in terms of \( v = v_+ + v_- + v_0 \), are:
\[(\alpha_w/2)v_+^2 = m^2_{W^+} ; \]
\[(\alpha_w/2)v_-^2 = m^2_{W^-} ; \text{ and} \]
\[(\alpha_w/2)v_0^2 = m^2_{W^+} , \]

with \( v = v_+ + v_- + v_0 = ((\sqrt{2})/\sqrt{\alpha_w})(m_{W^+} + m_{W^-} + m_{W_0}). \)

Then:
\[ \int [F_w \wedge \star F_w + 
+ m^2_{W^+} W^+ W^+ + m^2_{W^-} W^- W^- + m^2_{W_0} W_0 W_0 + 
+ (1/2)[\alpha_w W^+ W^+ + \alpha_w W^- W^- + \alpha_w W^+ W^- + \alpha_w W^+ W^- [H^2 + 2vH] + 
+ (1/2)(\partial H)^2 + (1/2)(\mu^2/2)H^2 - 
- (1/16)\mu^2 v^2[1 - 4H^3/v^3 - H^4/v^4]]. \]
2.4 Higgs Mass

The Higgs vacuum expectation value $v = (v_+ + v_- + v_0)$ is the only particle mass free parameter.

In the $D_4 - D_5 - E_6$ model, $v$ is set so that the electron mass $m_e = 0.5110 \text{MeV}$.

Therefore, $(\sqrt{\alpha_w}/\sqrt{2})v = m_{W^+} + m_{W^-} + m_{W^0} = 260.774 \text{GeV}$,

the value chosen so that the electron mass (which is to be determined from it) will be 0.5110 MeV.

In the $D_4 - D_5 - E_6$ model, $\alpha_w$ is calculated to be $\alpha_w = 0.2534577$, so $\sqrt{\alpha_w} = 0.5034458$ and $v = 732.53 \text{GeV}$.

The Higgs mass $m_H$ is given by the term

$(1/2)(\partial H)^2 - (1/2)(\mu^2/2)H^2 = (1/2)[(\partial H)^2 - (\mu^2/2)H^2]

$ to be $m_H^2 = \mu^2/2 = \lambda v^2/2$, so that $m_H = \sqrt{(\mu^2/2) = \sqrt{\lambda v^2}/2}$.

$\lambda$ is the scalar self-interaction strength. It should be the product of the "weak charges" of two scalars coming from the reduced 4 dimensions in $Spin(4)$, which should be the same as the weak charge of the surviving weak force $SU(2)$ and therefore just the square of the $SU(2)$ weak charge, $\sqrt{(\alpha_w^2)} = \alpha_w$, where $\alpha_w$ is the $SU(2)$ geometric force strength.

Therefore $\lambda = \alpha_w = 0.2534576$, $\sqrt{\lambda} = 0.5034458$, and $v = 732.53 \text{GeV}$, so that the mass of the Higgs scalar is $m_H = v\sqrt{(\lambda/2)} = 260.774 \text{GeV}$. 
3 Spinor Fermion part of the Lagrangian

Consider the spinor fermion term \( \int S_{8^\pm} \bar{\psi}_8 S_{8^\pm} \).

For each of the surviving 4-dimensional 4 and reduced 4-dimensional \( \bot 4 \) of 8-dimensional spacetime, the part of \( S_{8^\pm} \) on which the Higgs \( SU(2) \) acts locally is \( Q_3 = \mathbb{R}P^1 \times S^2 \).

It is the Silov boundary of the bounded domain \( D_3 \) that is isomorphic to the symmetric space \( \overline{D_3} = Spin(5)/SU(2) \times U(1) \).

The Dirac operator \( \bar{\psi}_8 \) decomposes as \( \bar{\psi} = \bar{\psi}_4 + \bar{\psi}_{\bot 4} \), where \( \bar{\psi}_4 \) is the Dirac operator corresponding to the surviving spacetime 4 and \( \bar{\psi}_{\bot 4} \) is the Dirac operator corresponding to the reduced \( 4 \bot 4 \).

Then the spinor term is \( \int S_{8^\pm} \bar{\psi}_4 S_{8^\pm} \) + \( \int S_{8^\pm} \bar{\psi}_{\bot 4} S_{8^\pm} \).

The Dirac operator term \( \bar{\psi}_{\bot 4} \) in the reduced \( \bot 4 \) has dimension of mass.

After integration \( \int S_{8^\pm} \bar{\psi}_{\bot 4} S_{8^\pm} \) over the reduced \( \bot 4 \), \( \bar{\psi}_{\bot 4} \) becomes the real scalar Higgs scalar field \( Y = (v + H) \) that comes from the complex \( SU(2) \) doublet \( \Phi \) after action of the Higgs mechanism.

If integration over the reduced \( \bot 4 \) involving two fermion terms \( S_{8^\pm} \) and \( S_{8^\pm} \) is taken to change the sign by \( i^2 = -1 \), then, by the Higgs mechanism,

\[
\int S_{8^\pm} \bar{\psi}_{\bot 4} S_{8^\pm} \rightarrow \int (\int S_{8^\pm} \bar{\psi}_{\bot 4} S_{8^\pm}) \rightarrow \rightarrow \int S_{8^\pm} Y Y S_{8^\pm} = - \int S_{8^\pm} Y (v + H) S_{8^\pm},
\]

where:

- \( H \) is the real physical Higgs scalar, \( m_H = v\sqrt{\lambda/2} = 261 \text{ GeV} \), and \( v \) is the vacuum expectation value of the scalar field \( Y \), the free parameter in the theory that sets the mass scale.

Denote the sum of the three weak boson masses by \( \Sigma_3 \).

\[
v = \Sigma_3 (\sqrt{2})/\sqrt{\alpha_w},
\]

where

\[
v = 260.774 \times 0.5034458 = 732.53 \text{ GeV},
\]

a value chosen so that the electron mass will be 0.5110 MeV.
3.1 Yukawa Coupling and Fermion Masses

$Y$ is the Yukawa coupling between fermions and the Higgs field.

$Y$ acts on all 28 elements

(2 helicity states for each of the 7 Dirac particles and 7 Dirac antiparticles)

of the Dirac fermions in a given generation, because all of them are in the

same Spin(8) spinor representation.

Denote the sum of the first generation Dirac fermion masses by $\Sigma f_1$.

Then $Y = (\sqrt{2})\Sigma f_1 / v$, just as $\sqrt{(\alpha w)} = (\sqrt{2})\Sigma m_w / v$.

$Y$ should be the product of two factors:

$e^2$, the square of the electromagnetic charge $e = \sqrt{\alpha_E}$, because in the

term $\int (\int_{D4} S_{8+} \, \phi_{D5} S_{8+}) \to - \int S_{8+} Y (v + H) S_{8+}$ each of the Dirac fermions $S_{8\pm}$ carries electromagnetic charge proportional to $e$; and

$1/g_w$, the reciprocal of the weak charge $g_w = \sqrt{\alpha_w}$, because an SU(2)

force, the Higgs SU(2), couples the scalar field to the fermions.

Therefore $\Sigma f_1 = Y v / \sqrt{2} = (e^2 / g_w) v / \sqrt{2} = 7.508$ GeV and

$\Sigma f_1 / \Sigma m_w = (e^2 / g_w) v / g_w v = e^2 / g_w = \alpha_E / \alpha_w$.

The Higgs term $- \int S_{8+} Y (v + H) S_{8+} = - \int S_{8+} Y \, v S_{8+} - \int S_{8+} Y \, H S_{8+} = - int S_{8\pm} (\sqrt{2} \Sigma f_1 / v) S_{8\pm}$.

The resulting spinor term is of the form $\int [S_{8\pm}(\theta - Y v)S_{8\pm} - \overline{S_{8\pm} Y H S_{8\pm}}]$

where $(\theta - Y v)$ is a massive Dirac operator.

How much of the total mass $\Sigma f_1 = Y v / \sqrt{2} = 7.5 GeV$ is allocated to

each of the first generation Dirac fermions is determined by calculating the

individual fermion masses in the $D_4 - D_5 - E_6$ model, and

those calculations also give the values of
$\Sigma_{f_2} = 32.9\text{GeV}$, $\Sigma_{f_3} = 1,629\text{GeV}$, and individual second and third generation fermion masses.

The individual tree-level lepton masses and quark constituent masses are:

- $m_e = 0.5110\text{ MeV}$ (assumed);
- $m_{\nu_e} = m_{\nu_{\mu}} = m_{\nu_\tau} = 0$;
- $m_d = m_u = 312.8\text{ MeV}$ (constituent quark mass);
- $m_\mu = 104.8\text{ MeV}$;
- $m_s = 625\text{ MeV}$ (constituent quark mass);
- $m_c = 2.09\text{ GeV}$ (constituent quark mass);
- $m_t = 130\text{ GeV}$ (constituent quark mass).

The following formulas use the above masses to calculate Kobayashi-Maskawa parameters:

- phase angle $\epsilon = \pi/2$
- $\sin \alpha = [m_e + 3m_d + 3m_u]/\sqrt{[m_e^2 + 3m_d^2 + 3m_u^2] + [m_\mu^2 + 3m_s^2 + 3m_c^2]}$
- $\sin \beta = [m_e + 3m_d + 3m_u]/\sqrt{[m_e^2 + 3m_d^2 + 3m_u^2] + [m_\tau^2 + 3m_b^2 + 3m_t^2]}$
- $\sin \tilde{\gamma} = [m_\mu + 3m_s + 3m_c]/\sqrt{[m_\tau^2 + 3m_b^2 + 3m_t^2] + [m_\mu^2 + 3m_s^2 + 3m_c^2]}$
- $\sin \gamma = \sin \tilde{\gamma} \sqrt{\Sigma_{f_2}/\Sigma_{f_3}}$

The resulting Kobayashi-Maskawa parameters are:

|     | $d$          | $s$   | $b$     |
|-----|--------------|-------|---------|
| $u$ | 0.975        | 0.222 | $-0.00461i$ |
| $c$ | $-0.222 - 0.000191i$ | $0.974 - 0.0000434i$ | 0.0423 |
| $t$ | $0.00941 - 0.0049i$ | $-0.0413 - 0.00102i$ | 0.999  |
4 Parity Violation, W-Boson Masses, and $\theta_\text{w}$

In the $D_4 - D_5 - E_6$ model prior to dimensional reduction, the fermion particles are all massless at tree level.

The neutrinos obey the Weyl equation and must remain massless and left-handed at tree level after dimensional reduction.

The electrons and quarks obey the Dirac equation and acquire mass after dimensional reduction.

After dimensional reduction, the charged $W^\pm$ of the $SU(2)$ weak force can interchange Weyl fermion neutrinos with Dirac fermion electrons.

4.1 Massless Neutrinos and Parity Violation

It is required (as an ansatz or part of the $D_4 - D_5 - E_6$ model) that the charged $W^\pm$ neutrino-electron interchange must be symmetric with the electron-neutrino interchange, so that the absence of right-handed neutrino particles requires that the charged $W^\pm$ $SU(2)$ weak bosons act only on left-handed electrons.

It is also required (as an ansatz or part of the $D_4 - D_5 - E_6$ model) that each gauge boson must act consistently on the entire Dirac fermion particle sector, so that the charged $W^\pm$ $SU(2)$ weak bosons act only on left-handed fermions of all types.

Therefore, for the charged $W^\pm$ $SU(2)$ weak bosons, the 4-dimensional spinor fields $S_{8\pm}$ contain only left-handed particles and right-handed antiparticles.

So, for the charged $W^\pm$ $SU(2)$ weak bosons, $S_{8\pm}$ can be denoted $S_{8\pm L}$.

4.2 $W_0$, $Z$, and $\theta_\text{w}$

The neutral $W_0$ weak bosons do not interchange Weyl neutrinos with Dirac fermions, and so may not entirely be restricted to left-handed spinor particle
fields $S_{8\pm L}$, but may have a component that acts on the full right-handed and left-handed spinor particle fields $S_{8\pm} = S_{8\pm L} + S_{8\pm R}$.

However, the neutral $W_0$ weak bosons are related to the charged $W_{\pm}$ weak bosons by custodial $SU(2)$ symmetry, so that the left-handed component of the neutral $W_0$ must be equal to the left-handed (entire) component of the charged $W_{\pm}$.

Since the mass of the $W_0$ is greater than the mass of the $W_{\pm}$, there remains for the $W_0$ a component acting on the full $S_{8\pm} = S_{8\pm L} + S_{8\pm R}$ spinor particle fields.

Therefore the full $W_0$ neutral weak boson interaction is proportional to $(m_{W_{\pm}}^2/m_{W_0}^2)$ acting on $S_{8\pm L}$ and $(1 - (m_{W_{\pm}}^2/m_{W_0}^2))$ acting on $S_{8\pm} = S_{8\pm L} + S_{8\pm R}$.

If $(1 - (m_{W_{\pm}}^2/m_{W_0}^2))$ is defined to be sin $\theta_w$ and denoted by $\xi$, and

if the strength of the $W_{\pm}$ charged weak force (and of the custodial $SU(2)$ symmetry) is denoted by $T$,

then the $W_0$ neutral weak interaction can be written as: 
$W_{0L} \sim T + \xi$ and $W_{0R} \sim \xi$.

The $D_4 - D_5 - E_6$ model allows calculation of the Weinberg angle $\theta_w$, by $m_{W_+} = m_{W_-} = m_{W_0} \cos \theta_w$.

The Hopf fibration of $S^3$ as $S^1 \to S^3 \to S^2$

gives a decomposition of the $W$ bosons into the neutral $W_0$ corresponding to $S^1$ and the charged pair $W_+$ and $W_-$ corresponding to $S^2$.

The mass ratio of the sum of the masses of $W_+$ and $W_-$ to the mass of $W_0$ should be the volume ratio of the $S^2$ in $S^3$ to the $S^1$ in $S3$.

The unit sphere $S^3 \subset R^4$ is normalized by $1/2$.

The unit sphere $S^2 \subset R^3$ is normalized by $1/\sqrt{3}$. 

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The unit sphere \( S^1 \subset R^2 \) is normalized by \( 1/\sqrt{2} \).

The ratio of the sum of the \( W_+ \) and \( W_- \) masses to the \( W_0 \) mass should then be \( (2/\sqrt{3})V(S^2)/(2/\sqrt{2})V(S^1) = 1.632993 \).

The sum \( \sum m_W = m_{W_+} + m_{W_-} + m_{W_0} \) has been calculated to be \( \nu \sqrt{\alpha_w} = 517.798\sqrt{0.2534577} = 260.774GeV \).

Therefore, \( \cos \theta_w^2 = m_{W_+}^2/m_{W_0}^2 = (1.632993/2)^2 = 0.667 \), and \( \sin \theta_w^2 = 0.333 \), so \( m_{W_+} = m_{W_-} = 80.9GeV \) and \( m_{W_0} = 98.9GeV \).

### 4.3 Corrections for \( m_Z \) and \( \theta_w \)

The above values must be corrected for the fact that only part of the \( w_0 \) acts through the parity violating \( SU(2) \) weak force and the rest acts through a parity conserving \( U(1) \) electromagnetic type force.

In the \( D_4 - D_5 - E_6 \) model, the weak parity conserving \( U(1) \) electromagnetic type force acts through the \( U(1) \) subgroup of \( SU(2) \), which is not exactly like the \( D_4 - D_5 - E_6 \) electromagnetic \( U(1) \) with force strength \( \alpha_E = 1/137.03608 = \epsilon^2 \).

The \( W_0 \) mass \( m_{W_0} \) has two parts:
- the parity violating \( SU(2) \) part \( m_{W_0e} \) that is equal to \( m_{W_\pm} \); and
- the parity conserving part \( m_{W_00} \) that acts like a heavy photon.

As \( m_{W_0} = 98.9 \text{ GeV} = m_{W_0e} + m_{W_00} \), and as \( m_{W_0\pm} = m_{W_\pm} = 80.9\text{GeV} \), we have \( m_{W_00} = 18\text{GeV} \).

Denote by \( \tilde{\alpha}_E = \epsilon^2 \) the force strength of the weak parity conserving \( U(1) \) electromagnetic type force that acts through the \( U(1) \) subgroup of \( SU(2) \).

The \( D_4 - D_5 - E_6 \) electromagnetic force strength \( \alpha_E = \epsilon^2 = 1/137.03608 \) was calculated using the volume \( V(S^1) \) of an \( S^1 \subset R^2 \), normalized by \( 1/\sqrt{2} \).

The \( \tilde{\alpha}_E \) force is part of the \( SU(2) \) weak force whose strength \( \alpha_w = \epsilon^2 \)
was calculated using the volume $V(S^2) \text{of an } S^2 \subset R^3$, normalized by $1/\sqrt{3}$.

Also, the $D_4 - D_5 - E_6$ electromagnetic force strength $\alpha_E = e^2$ was calculated using a 4-dimensional spacetime with global structure of the 4-torus $T^4$ made up of four $S^1$ 1-spheres,

while the $SU(2)$ weak force strength $\alpha_w = w^2$ was calculated using two 2-spheres $S^2 \times S^2$, each of which contains one 1-sphere of the $\tilde{\alpha}_E$ force.

Therefore $\tilde{\alpha}_E = \alpha_E (\sqrt{2}/\sqrt{3})(2/4) = \alpha_E / \sqrt{6}$, $\tilde{E} = e / 4\sqrt{6} = e / 1.565$, and the mass $m_{W_0}$ must be reduced to an effective value

$m_{W_0}^{\text{eff}} = m_{W_0} / 1.565 = 18 / 1.565 = 11.5 \text{ GeV}$

for the $\tilde{\alpha}_E$ force to act like an electromagnetic force in the 4-dimensional spacetime of the $D_4 - D_5 - E_6$ model:

$\tilde{E}m_{W_0} = e (1/5.65) m_{W_0} = em_{Z_0}$,

where the physical effective neutral weak boson is denoted by $Z$ rather than $W_0$.

Therefore, the correct $D_4 - D_5 - E_6$ values for weak boson masses and the Weinberg angle are:

$m_{W_+} = m_{W_-} = 80.9 \text{ GeV};$

$m_Z = 80.9 + 11.5 = 92.4 \text{ GeV};$ and

$\sin \theta_w^2 = 1 - (m_{W_\pm} / m_Z)^2 = 1 - 6544.81 / 8537.76 = 0.233$.

Radiative corrections are not taken into account here, and may change the $D_4 - D_5 - E_6$ value somewhat.
5 Some Errata for Previous Papers

hep-th/9402003 \[11] had a typographical error of only $S_{8^+}$ or $S_{8^-}$ instead of $S_{8^\pm}$. The correct 8-dim Lagrangian is:

$$\int_{V_8} F_8 \wedge \ast F_8 + \partial_8^2 \Phi_8 \ast \partial_8^2 \Phi_8 + \overline{S_{8^\pm}} \partial_8 S_{8^\pm} + GF + GH$$

hep-th/9302030 \[9] had erroneously used $\times$ instead of $\oplus$ for the fermion spinor space. The correct full fermion space of first generation particles and antiparticles is

$S_{8^+} \oplus S_{8^\pm} = (RP^1 \times S^7) \oplus (RP^1 \times S^7)$.

It is the Silov boundary of the 32(real)-dimensional bounded complex domain corresponding to the TypeV HJTS $E_6/(Spin(10) \times U(1))$.

hep-th/9302030 \[9] had erroneously listed $SU(3)/SU(2) \times U(1)$, instead of $Spin(5)/SU(2) \times U(1) = Spin(5)/Spin(3) \times U(1)$, as the TypeIV3 HJTS corresponding to the 6(real)-dimensional bounded complex domain on whose Silov boundary the gauge group $SU(2)$ naturally acts.

The corrected table is:

The $Q$ and $D$ manifolds for the gauge groups of the four forces are:

| Gauge Group | Hermitian Symmetric Space | Type of $D$ | $m$ | $Q$ |
|-------------|--------------------------|-------------|-----|-----|
| Spin(5)     | $\frac{Spin(7)}{Spin(5) \times U(1)}$ | $IV_5$      | 4   | $RP^1 \times S^4$ |
| SU(3)       | $\frac{SU(4)}{SU(3) \times U(1)}$ | $B^6$ (ball) | 4   | $S^5$ |
| SU(2)       | $\frac{Spin(5)}{SU(2) \times U(1)}$ | $IV_3$      | 2   | $RP^1 \times S^2$ |
| U(1)        | –                        | –           | 1   | –   |
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