Whitney categories and the Tangle Hypothesis

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(joint with Conor Smyth)

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Small categories ‘are’ presheaves on $\Delta$ — finite ordinals and order-preserving maps — with a sheaf-like property.
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The nerve of a category

The nerve $N: \text{Cat} \to PSh(\Delta)$ is fully faithful, with essential image those simplicial sets $S$ satisfying the Segal condition

$$S_k \xrightarrow{\sim} S_1 \times_{S_0} \cdots \times_{S_0} S_1 \quad \forall k \in \mathbb{N}.$$  

(We think of $[k]$ as a concatenation of directed intervals rather than as a geometrical simplex.)
Small dagger categories ‘are’ presheaves on $\mathbf{D\Delta} —$ finite ordinals and order-preserving or reversing maps — with a sheaf-like property.
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The dagger nerve of a dagger category [Joy10]

The dagger nerve $DN: DCat \to PSh(\Delta)$ is fully faithful, with essential image those dagger simplicial sets $S$ satisfying the Segal condition

$$S_k \sim S_1 \times S_0 \cdots \times S_0 S_1 \quad \forall k \in \mathbb{N}.$$  

(We think of $[k]$ as a concatenation of undirected intervals

$$\bullet \cdots \bullet$$

rather than as a sequence of arrows.)
Remark

The above realisation of $[k]$ is a typical 1d stratified space. The idea of [SW11] is to define higher categories with duals by

\[ D\Delta \sim \text{category of higher dim stratified spaces} \]

Segal condition \sim sheaf-like condition for presheaves on above
Remark

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$$D\Delta \sim \text{category of higher dim stratified spaces}$$

Segal condition $\sim \text{sheaf-like condition for presheaves on above}$

Definition (Higher category with duals — preliminary version)

Presheaf on a (suitable) category of stratified spaces satisfying a (suitable) sheaf-like property.
Definition (Whitney stratified manifold)
Manifold with a locally-finite partition into disjoint locally-closed submanifolds \( \{ S_i \} \) (the strata) satisfying Whitney’s condition \( B \).
Stratified spaces

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**Examples**

A real or complex projective analytic variety admits a Whitney stratification by subvarieties. A compact manifold can be stratified by the flow of Morse–Smale vector field.
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Compact union of cellular strata in a Whitney stratified manifold.
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**Examples**

Geometric simplex \( \Delta_n \subset \mathbb{R}^{n+1}, S^n, \mathbb{R}\mathbb{P}^n, \mathbb{C}\mathbb{P}^n, \) Grassmannians...
Stratified and prestratified maps

Definition (Stratified map)

Smooth $f: X \to Y$, where $X$, $Y$ stratified spaces, such that

- $f^{-1}T$ is a union of strata for each stratum $T \subset Y$
- $f|_S: S \to T$ is a submersion for each $S \subset f^{-1}T$.

Definition (Prestratified map)

Smooth $f: X \to Y$ which becomes stratified after refining the stratification of $X$. 

Example

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![Diagram](image-url)
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### Definition (Categories of stratified spaces)

Objects are compact cellular stratified spaces of ambient dim $n$ and respective morphisms are

- $\text{Str}_n$: germs of stratified maps

### Definition (Whitney $n$-category)

Presheaf on $\text{hPStr}_n$ whose pullback along $\text{Str}_n/\text{uni}$ $\rightarrow \text{hPStr}_n$ is a sheaf. In particular

$$W(X) = \lim_{i \in S(X)} W(S_i)$$

where $S(X)$ is the poset of strata of $X$. Let $\text{Whit}_n$ be the full subcategory of such presheaves.
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Example (Sheaf condition $\implies$ Segal condition)

$$W(\cdots) = W(\cdot) \times_{W(\cdot)} \cdots \times_{W(\cdot)} W(\cdot)$$
Example (Sheaf condition $\Rightarrow$ Segal condition)

$$W(\bigcirc_\ldots\bigcirc) = W(\bigcirc) \times_{W(\bigcirc)} \ldots \times_{W(\bigcirc)} W(\bigcirc)$$

Objects / morphisms

$$X \leftrightarrow \text{template for a pasting diagram, }$$

$$W(X) \leftrightarrow \text{set of pasting diagrams, or } X\text{-morphisms.}$$
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Boundary:
Source/Target
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Structure

- Boundary
- Source/Target
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- Map to point
- Identities
- Reflection
- Dual

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Example (Low-dimensional cases)

$0\text{Whit} \simeq \text{Set}$ and $D\Delta \to \text{hPStr}_1$ induces $1\text{Whit} \simeq D\text{Cat}$. 
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Example (Representable Whitney categories)

Let \( \text{Rep}(X) = hPStr_n (-, X) \in n\text{Whit} \). By Yoneda \( \text{Rep}(X) \) is free on one \( X \)-morphism, i.e. \( n\text{Whit}(\text{Rep}(X), W) \cong W(X) \).
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Example (Framed tangles)
Define $n\text{Tang}^{fr}_k \in (n + k)\text{Whit}$ by

$$n\text{Tang}^{fr}_k(X) = \{ \text{codim } k \text{ framed sbmflds } \upharpoonright \text{ to strata} \}/\text{isotopy}.$$
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Formal properties

- $n$Whit is complete and cocomplete.
Properties of Whitney categories

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- \( n\text{Whit} \) is complete and cocomplete.
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Formal properties

- $nWhit$ is complete and cocomplete.
- The inclusion $nWhit \hookrightarrow PSh(hPStr_n)$ has a left adjoint.
- There is a ‘dagger nerve’ $nWhit \rightarrow PSh(D\theta_n)$ induced by $D\theta_n \rightarrow hPStr_n$ where $D\theta_1 = D\Delta$ and $D\theta_n = D\Delta \vee D\theta_{n-1}$. 
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Formal properties

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Definition (Category of morphisms)

For objects $w_0, w_1 \in W(pt)$ there is a Whitney $(n-1)$-category

$$W(w_0, w_1)(X) = \{\omega \in W(X \times [0,1]) : \omega|_{X \times i} = p^* w_i, i = 0, 1\}$$

of morphisms between $w_0$ and $w_1$, where $p: X \rightarrow pt$. 
Tangles and prestratified maps to $S^k$

The Pontrjagin–Thom construction

Choosing a generic framed point $p \in S^k$ yields a correspondence

- isotopy classes of framed tangles in $X$ ↔ homotopy classes of prestratified maps $X \to S^k$
The Pontrjagin–Thom construction

Choosing a generic framed point \( p \in S^k \) yields a correspondence

\[
\text{isotopy classes of} \leftrightarrow \text{homotopy classes of} \\
\text{framed tangles in } X \leftrightarrow \text{prestratified maps } X \to S^k
\]

Consider \([p] \in n\text{Tang}^\text{fr}_k(S^k)\). Since Rep\((S^k)\) free we obtain

\[
 PT: \text{Rep}(S^k) \to n\text{Tang}^\text{fr}_k: \left[ X \xrightarrow{f} S^k \right] \mapsto [f^{-1}(p) \subset X]
\]

in \((n + k)\text{Whit.} \) Pontrjagin–Thom \( \Rightarrow \) \( PT \) is an isomorphism.
### Definition (k-tuply monoidal Whitney n-category)

A Whitney \((n + k)\)-category \(W\) with \(W(X) = 1\) for \(\text{dim } X < k\).
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A morphism in \((n + k)\)Whit between \(k\)-tuply monoidal Whitney \(n\)-categories.
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### Example
\(PT: \text{Rep}(S^k) \to n\text{Tang}^\text{fr}_k\) is a \(k\)-tuply monoidal functor.
The Whitney Tangle Hypothesis

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A Whitney \((n + k)\)-category \(W\) with \(W(X) = 1\) for \(\dim X < k\).

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A morphism in \((n + k)\)Whit between \(k\)-tuply monoidal Whitney \(n\)-categories.

Example
\[PT: \text{Rep}(\mathbb{S}^k) \to n\text{Tang}^fr_k\] is a \(k\)-tuply monoidal functor.

Theorem (Whitney Tangle Hypothesis, c.f. [BD95])
\(n\text{Tang}^fr_k\) is the free \(k\)-tuply monoidal Whitney \(n\)-category on one \(\mathbb{S}^k\)-morphism.
John. Baez and James. Dolan.  
Higher-dimensional algebra and topological quantum field theory.  
*J. Math. Phys.*, 36(11):6073–6105, 1995.

André Joyal.  
Dagger not evil.  
Posted on Category Theory mailing list, January 2010.

Scott Morrison and Kevin Walker.  
Blob homology.  
*Geom. Topol.*, 16(3):1481–1607, 2012.

Charles Rezk.  
A Cartesian presentation of weak $n$-categories.  
*Geom. Topol.*, 14(1):521–571, 2010.
Conor Smyth and Jon Woolf. Whitney categories and the Tangle Hypothesis. arXiv:1108.3724 (major revision in progress), 2011.