On the role of the atom–cavity detuning in bimodal cavity experiments

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Abstract
The coherent evolution of the atom–cavity state in bimodal (cavity) experiments has been analysed for a realistic time dependence in detuning the atomic transition frequency. Apart from a ‘smooth switch’ of the atomic resonance from one to the second mode of a bimodal cavity, we considered also an additional (effective) interaction between the field modes of the cavity, known as a ‘communication channel’. Comparison of our model computations has been made especially with the measurements by Rauschenbeutel et al (2001 Phys. Rev. A 64 050301) who demonstrated for the first time the entanglement of the field modes in a bimodal cavity. It is shown that the agreement between the (theoretically) predicted and experimental phase shifts can be improved by allowing a ‘communication’ between the two field modes during a short but finite switch of the atomic transition frequency from one mode to the other. We therefore suggest that the details of the atom–cavity detuning should be taken into account for the future interpretation of bimodal cavity experiments.

1. Introduction
During the last few decades, entanglement has been recognized as a key feature of quantum mechanics that describes not only the correlation in composite quantum systems but is useful also for applications. Since the famous Bohr–Einstein debate and the seminal work of Einstein, Podolsky and Rosen [1] in 1935, indeed, a large number of entanglement studies has helped improve our present understanding of nonlocal and nonclassical phenomena as they occur in the microscopic world. In quantum engineering and quantum information theory [2], moreover, entanglement has been found crucial for implementing new (quantum) information protocols, such as super-dense coding [3], quantum cryptography [4], or even simple quantum algorithms [5]. However, despite all the progress in the design and description of entangled quantum systems, their manipulation and controlled interaction with the environment still remain a great challenge for experiment owing to the fragile nature of most quantum states. Among various other implementations, an excellent control over light fields and atoms has been achieved especially by using neutral atoms that are coupled to a high-finesse optical cavity [6, 7].

In a recent experiment by Rauschenbeutel and co-workers [8], for example, the two modes of a superconducting (bimodal) cavity were prepared in a maximally entangled state by using circular Rydberg atoms. In this experiment, an entanglement of the two field modes was achieved by properly adjusting the detuning of the atomic (transition) frequency of the Rydberg atom while it passes through the bimodal cavity. The entangled state produced in the cavity is probed later by a second atom which, after being detected, reveals the coherent evolution of the superposition of the cavity states. The coherence of the (two-qubit) cavity state was demonstrated by varying the delay time after which the probe atom interacts with the cavity. Different delay times then give rise to an oscillation in the final-state probability for the second atom to be found in either the ground or excited state of the two-level configuration, as supposed for the Rydberg atom. For this final-state probability, an analytical expression was derived by assuming an idealized time evolution of the quantum state of both the photon field and the atoms, throughout the periods of atom–cavity interactions. Despite the high quality of the cavities today, however, such an ideal time evolution neglects a number of relevant effects, including the relaxation of the cavity field, the influence of external and internal stray fields, or imperfections due to the
cavity mirrors and cavity geometry. In particular the proper matching of the atomic transition frequency to (the frequency of) one or the other cavity mode (the so-called atom–cavity detuning) plays an essential role on the superposition of the cavity states, because the atom has to mediate the interactions between the (cavity) modes while passing through the cavity.

The experiments by Rauschenbeutel et al [8] nicely demonstrated how the field modes of a bimodal cavity become entangled or disentangled in a well-controlled way by manipulating the (de-) tuning of the atom–cavity interaction. In contrast to the single-mode cavities, the use of bimodal cavities has been found an important step towards the manipulation of complex quantum states and for performing various fundamental tests in quantum theory [9–17]. For these bimodal cavities, it is important to know how (sensitively) the produced cavity state depends on the details and the particular shape of the detuning process. Since the two nearly degenerate frequencies of the cavity modes are fixed by the geometry of the cavity, a detuning of the atom (transition) frequency with regard to one of the field modes automatically affects also the detuning with regard to the other mode. In our discussions below, we shall often refer to this detuning of the atomic transition frequency (of the Rydberg atoms passing through the cavity) with regard to the field modes briefly as the atom–cavity detuning.

In the present work, we examine how the coherent evolution of the cavity state of bimodal cavities, i.e. the superposition of the two-cavity modes, is affected by a realistic time dependence of the atom–cavity detuning and how the atoms, when passing through the cavity, may interact with both cavity modes simultaneously. To exhibit these effects, we make use of two models for the atom–cavity detuning in which the (idealized) step-wise detuning of the atomic transition frequency from one to the other cavity mode is replaced by (i) a smooth detuning that happens in a short but finite (‘switching’) time interval and (ii) a simultaneous interaction of the atom with both cavity modes leading to a wave mixing in the cavity. We shall refer to these models as the (separate) single-mode interaction and communication channel model. In the latter model, the wave-mixing effects mainly arise due to imperfections of the cavity mirrors that cause the cavity modes to interact (communicate) with each other, cf section 2. To follow the time evolution of the atom–cavity interaction, we combine analytical solutions for the Jaynes–Cummings Hamiltonian with numerical simulations, if the atom–cavity interaction is not resonant. For both models, the single-mode interaction and communication channel, the final-state probability for the ‘probe’ atom to be found in either the ground or excited state is then compared with the experiments in [8]. Using the cavity parameters from these experiments, we show that more realistic assumptions about the shape (and model) of the atom–cavity interaction improves the agreement between the theoretical predictions and experiment. A detailed account of the atom–cavity detuning will therefore play an crucial role also for manipulating and analysing (more) complex cavity experiments in the future.

In section 2, we start from the Jaynes–Cummings Hamiltonian to describe the interaction of a two-level atom with the two field modes of a bimodal cavity. For this, we first (re-) derive the expression for the final-state probability in subsection 2.1 that the probe atom is detected in the excited state by using the matrix formalism and by assuming an idealized step-wise change in the atom–cavity interaction. This derivation sets the framework for introducing our more realistic models concerning the shape and explicit form of the atom–cavity interaction in sections 2.2 (single-mode interaction model) and 2.3 (communication channel). In the latter model, we modify the Jaynes–Cummings Hamiltonian as to allow the atom to interact with both cavity modes simultaneously. In section 3 then, the predictions from these models for the final-state probability are compared with the data of [8]. In particular, here we display and discuss how the ‘switching time’ affects the outcome of the experiments. Finally, a few conclusions are given in section 4.

2. Entanglement of two field modes using a bimodal cavity

The use of the resonant atom–cavity interaction regime is the simplest way to generate entangled states between atoms and/or the field modes of a cavity. For a sufficiently high quality factor of the cavity mirrors, namely, this regime implies a ‘strong’ atom–field coupling for which the dissipation of field energy in course of the atom–cavity evolution becomes negligible. Such a small dissipation of the photon field plays a crucial role for engineering of coherent states in the framework of cavity QED. Apart from the quality of the mirrors, the correct matching of the atomic transition frequency to the frequency of cavity field modes (the so-called detuning) is also an important ingredient in order to achieve the resonant atom–cavity interaction regime.

In the following, let us first recall the basic components and notions of the cavity QED experiments by Haroche and co-workers [6]. In these experiments, circular Rydberg atoms in quantum states with principal quantum numbers 50 and 51 are treated as ‘two-level’ systems, being in the ground state $|g\rangle$ or the exited state $|e\rangle$, respectively. Using the Stark shift technique, the frequency of atomic $e \leftrightarrow g$ transition can be tuned in a well-controlled way to the nearly degenerate frequencies of the two cavity field modes. If, for the moment, we consider only a single-mode cavity, then the evolution of the atom–cavity state is described (for both a resonant and a non-resonant interaction of the atom with the cavity) by the Jaynes–Cummings Hamiltonian [18]

$$H = \hbar \omega_0 S_z + \hbar \Omega (S_x a_1 + a_1^* S_-) + \hbar \omega_1 (a_1^* a_1 + \frac{1}{2}),$$  \hspace{1cm} (1)

where $\omega_0$ is the atomic $e \leftrightarrow g$ transition frequency, $\omega_1$ is the frequency of the cavity field, and $\Omega$ is the atom-field coupling frequency. In the Hamiltonian (1), moreover, $a_1$ and $a_1^*$ denote the annihilation and creation operators for a photon in the cavity, acting upon the Fock states $|n\rangle$, while $S_-$ and $S_z$ are the spin lowering and raising operators that act upon the atomic states $|e\rangle$ and $|g\rangle$, which are the ‘eigenstates’ of the spin operator $S_z$ with eigenvalues $+1/2$ and $-1/2$, respectively. Furthermore, if there is not more than one photon
In the cavity, the overall atom-field state for a resonant atom–cavity interaction, i.e. for a zero detuning \((0 = \omega_0 - \omega_1)\), evolves according to [19]

\[
\begin{align*}
|e, 0\rangle &\rightarrow \cos(\Omega t/2)|e, 0\rangle - i\sin(\Omega t/2)|g, 1\rangle, \\
|g, 1\rangle &\rightarrow \cos(\Omega t/2)|g, 1\rangle - i\sin(\Omega t/2)|e, 0\rangle.
\end{align*}
\]  

(2a)

(2b)

In the literature, this time evolution of the atom–cavity states is known also as Rabi rotation where \(t\) designates the effective atom–cavity interaction time in the laboratory and \((\Omega \cdot t)\) the angle of rotation. Note that neither the state \(|e, 1\rangle\) nor \(|g, 0\rangle\) appears in the time evolution \((2a)-(2b)\), in line with our physical perception that the ‘photon’ of the \(e \leftrightarrow g\) transition is ’stored’ either by the atom or the cavity, but cannot occur twice in the system.

In contrast to single-mode cavities, a bimodal cavity possesses the feature of two nearly degenerate light modes of (usually) orthogonal polarization. Since the frequency of the cavity modes are fixed geometrically by the design of the cavity, it is only the atomic frequency that can be tuned by means of external fields in order that the Rydberg atom interacts resonantly with either the first or the second cavity field mode. In practice, this detuning is done by applying a well-adjusted time-varying electric field across the gap between the cavity mirrors [20] so that the desired (Stark) shift of the atomic transition frequency \(\omega_0(t)\) is achieved. By proper adjustment of \(\omega_0(t)\), one can induce the desired Rabi rotation of the atomic state interacting either with regard to the first or the second cavity mode [8]. In fact, the development of bimodal cavities has been found important not only for the manipulation of complex quantum states but provides one also with an additional ‘photonic qubit’ in the framework of quantum information that may interact independently with the Rydberg atoms (‘atomic qubits’).

Below, we shall denote the two cavity modes by \(M_1\) and \(M_2\) and suppose that they are associated with the frequencies \(\omega_1\) and \(\omega_2\) such that \(\omega_1 - \omega_2 \approx \delta > 0\). Since, moreover, the frequencies of the two field modes are fixed, only the atomic \(e \leftrightarrow g\) transition frequency can be changed and, as mentioned before, we shall briefly refer to the (de-)tuning of the atomic frequency with regard to the cavity modes as atom–cavity detuning.

For a resonant interaction of the Rydberg atoms with both cavity modes, one needs to be able to switch the atomic frequency from one to the other mode. Let \(\Delta(t)\) denote the (time-dependent) atom–cavity detuning and let us assume that the atom is in resonance with the cavity mode \(M_1\) for \(\Delta(t) = \omega_0(t) = 0\). It is then in resonance with \(M_2\) for \(\Delta(t) = -\delta\). For a sufficiently large frequency shift \(\delta\) of the two cavity modes, moreover, the atom can be in resonance only with one of the modes and, hence, the time evolution of the (interacting) atom–cavity system \(A - M_1 - M_2\) can be ‘divided’ into separate time evolutions owing to the \(A - M_1\) or \(A - M_2\) resonant interactions [8]. In practice, however, neither mode of the cavity can be frozen out completely because of the rather small shift \(\delta \approx 2\Omega\) in the experiments in [6, 8, 9], the decoherence in the photon field [10] as well as imperfections of the cavity mirrors, i.e. the local roughness and deviations from the spherical geometry. For any theoretical treatment of the decoherence and, especially, the imperfections of the cavity, we must proceed beyond the Jaynes–Cummings model (1) by allowing the atom to couple with both cavity modes at the same time [11, 12]. Such a coupling then supports also the coupling between the cavity modes and is therefore referred to as a communication channel in the literature [10], cf section 2.3. It is the goal of this work to demonstrate how a realistic atom–cavity interaction (coupling) can be analysed quantitatively and can improve the theoretical predictions on the various probabilities, when compared with experiment [8].

2.1. The step-wise model for detuning the atom–cavity interaction

With this short reminder on the development of bimodal cavities and the Jaynes–Cummings model for the atom–cavity interaction, we are now prepared to discuss all necessary steps for obtaining the coherent superposition of the cavity mode state as reported recently in the experiments of Rauschenbeutel et al [8]. In this reference, an idealized time evolution of the atom–cavity system was considered, and it was shown how the two cavity modes can be brought into the maximally entangled Bell state

\[
|\Psi\rangle = \frac{1}{\sqrt{2}}(e^{i\theta}|0\rangle \bar{1} + |1\rangle \bar{0}),
\]

(3)

where \(|0\rangle, |1\rangle\) denote the Fock states of the cavity mode \(M_1\), and \(\bar{0}, \bar{1}\) the Fock states of mode \(M_2\), respectively.

In a step-wise model of the atom–cavity interaction, the coherent superposition (3) can be produced as follows. In the derivations below, we shall often apply the matrix representation of the time evolution operators in order to discuss the individual steps. The use of the matrix representation later facilitates also the application of our two models concerning a more realistic atom–cavity interaction. Suppose the cavity is initially ‘empty’, i.e. in the state \(|0\rangle \equiv |0\rangle \times |0\rangle\), and is crossed by the (so-called) source atom \(A_s\) prepared in the excited state \(|e\rangle\). Being within the cavity, this atom is first tuned in resonance with the cavity mode \(M_1\) (\(\Delta = 0\)) for a Rabi rotation \(\Omega t_1 = \pi/2\), and followed by a second rotation \(\Omega t_2 = \pi\), now being in resonance with the second mode \(M_2(\Delta = -\delta)\). In the step-wise model of the atom–cavity interaction, it is essential that the detuning from modes \(M_1\) to \(M_2\) can be achieved instantaneously, i.e. within a period that is completely negligible to the time of interaction with the individual modes. This idealized sequence of atom–cavity interactions is shown in figure 1(a) where the Rabi rotations are denoted by the black diamonds, containing the angle of rotation, and where the spatio-temporal evolution can be seen for both, the cavity modes as well as the atoms.

Figure 1(b) displays the step-wise change in the atom–cavity detuning from the modes \(M_1\) to \(M_2\). After the time \(\frac{\pi}{\Omega}\), the source atom has passed the cavity (and turns out to be decoupled from the state of the cavity modes).

To understand the time evolution of the atom–cavity system as a whole, we can consider successively the resonant interaction of the atom with the two cavity modes \(M_1\) and \(M_2\). From figure 1(b), we see that the (time-dependent)
Hamiltonian can be written in the form

\[ H_s(t) = \theta_1(t)H_1 + \theta_2(t)H_2 \]

with the two 'step functions'

\[ \theta_1(t) = 1 - \theta \left( t - \frac{\pi}{2\Omega} \right), \quad \theta_2(t) = \theta \left( t - \frac{\pi}{2\Omega} \right), \]

and where \( \theta \equiv 1; \mu = 1, 2 \)

\[ H_\mu = \omega_0 S_\mu + \frac{\Omega}{2} (S_\mu a_\mu + a_\mu S_\mu) + \omega_\mu \left( a_\mu^* a_\mu + \frac{1}{2} \right) \]

(5)

refers to the Jaynes–Cummings Hamiltonian as discussed above. Following the text by Puri [19], here we introduce the operators

\[ N_\mu = a_\mu^* a_\mu + S_\mu + \frac{1}{2} \]

(6)

such that \([H_\mu, N_\mu] = 0\). In particular, the last relation enables us in the case that the atom is in resonance with the field mode \((\omega_0 = \omega_\mu)\) to cast the Hamiltonian (5) into the equivalent but more convenient form

\[ H_\mu = \omega_\mu N_\mu + \frac{\Omega}{2} (a_\mu^* S_- + S_+ a_\mu) \]

(7)

Of course, the Hilbert space associated with the Hamiltonian (4) is given by the product space of the three (sub)systems: \(A_1([|g\rangle, |e\rangle]), M_1([|0\rangle, |1\rangle]), \text{and } M_2([|0\rangle, |1\rangle])\). For the sake of convenience, let us introduce an explicit notation for the 2\(^3\) basis states

\[ |V_1\rangle = |e, 0, 0\rangle, \quad |V_2\rangle = |g, 1, 1\rangle, \quad |V_3\rangle = |e, 1, 1\rangle, \quad |V_4\rangle = |g, 0, 0\rangle, \quad |V_5\rangle = |e, 1, 0\rangle, \quad |V_6\rangle = |g, 0, 1\rangle, \]

\[ |V_7\rangle = |e, 0, 1\rangle, \quad |V_8\rangle = |g, 1, 0\rangle, \]

which naturally fulfill the relations \((i, j = 1, \ldots, 8)\)

\[ \langle V_i | V_j \rangle = \delta_{ij} \quad \text{and} \quad \sum_j \langle V_j | V_i \rangle = I. \]

Using these basis states, the wavefunction (of the atom–cavity system) can be always written in the form

\[ |\Psi(t)\rangle = \sum_i c_i(t)|V_i\rangle \]

with \( c_i(t) = \sum_j U_{ij}(t)c_j(0) \)

(10)

and where \( U_{ij}(t) = (V_i|U(t)|V_j) \) refers to the matrix representation of the time evolution operator \( U(t) = \exp \left( \frac{i}{\hbar} \int_0^t H(z) \, dz \right) \). By this definition, the time-evolution matrix \( U_{ij}(t) \) for the atom–cavity state at \( t = \frac{3\pi}{2\Omega} \) as given by the time-dependent Hamiltonian (4) is factorized:

\[ U_{ij}^{(t)} \left( \frac{3\pi}{2\Omega} \right) = \sum_k U_{ik}^{(2)} \left( \frac{\pi}{2\Omega} \right) U_{kj}^{(1)} \left( \frac{\pi}{2\Omega} \right), \]

(11)

where the operator \( U^{(1)}(t) \) refers to the evolution due to the \( A_1 \) – \( M_1 \) interaction and \( U^{(2)}(t) \) to that due to \( A_1 \) – \( M_2 \). Since, as mentioned above, the initial atom–cavity state is \( |V_1\rangle = |e, 0, 0\rangle \), we find \( c_i(0) = \delta_{i1} \), and the factorization in (11) is equivalent to the requirement

\[ \left[ \int_0^t \theta_1(z) H_1 \, dz, \int_0^t \theta_1(z) H_2 \, dz \right] = 0, \]

(12)

in accordance with the Hamiltonian (4).

For the separate steps of the resonant atom–cavity interaction described by the Hamiltonian (7), the Schrödinger equation is given by solutions (2a)–(2b). Omitting the details of the derivation, for which we refer the reader to the literature [19], the matrix representation of the operators \( U^{(1)}(t) \) and \( U^{(2)}(t) \) in the basis (8a)–(8b) is given by

\[ U_{ij}^{(1)}(t) = \begin{pmatrix} x(t) & y(t) \\ \bar{x}(t) & \bar{y}(t) \end{pmatrix} \]

(13)
and

\[ U_{ij}^{(2)}(t) = \begin{pmatrix} x(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & x'(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & x'(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & x(t) \end{pmatrix}, \]

(14)

and where

\[ x(t) = \bar{x}(t) = \cos \left( \frac{\Omega t}{2} \right), \quad y(t) = \bar{y}(t) = -i \sin \left( \frac{\Omega t}{2} \right). \]

(15)

In this notation, the prime refers to the additional phase factor \( \square(t) = e^{i\Omega t} \cdot \square(t) \) that arises from the energy difference \( \hbar \delta \) of the two cavity modes and is accumulated during the \((A - M_2)\) Rabi rotation \([8]\). Substituting the expressions \((11), (13)\) and \((14)\) into the wavefunction \((10)\) and by making use of the initial condition \( c_i(0) = \delta_{ij} \), we then obtain at the time \( t = \frac{3\pi}{2\Omega} \) two non-zero coefficients

\[ c_6 \left( \frac{3\pi}{2\Omega} \right) = -i \frac{e^{i\pi}}{\sqrt{2}}, \quad c_9 \left( \frac{3\pi}{2\Omega} \right) = -i \frac{e^{i\pi}}{\sqrt{2}}. \]

(16)

and, hence, the total wavefunction (up to an irrelevant phase factor) is

\[ \Psi \left( \frac{3\pi}{2\Omega} \right) = \frac{1}{\sqrt{2}} (e^{i\pi} [0, \bar{1}] + [1, \bar{0}]) |g\rangle. \]

(17)

This wavefunction is equivalent to those derived in \([8]\) and is the same as displayed above in \((3)\) where \( \psi = \delta \pi/2\Omega \).

Unfortunately, the cavity state is inaccessible for direct measurements. In order to ‘prove’ the coherent superposition \((17)\) of the two cavity modes, another (probe) atom \( A_p \) has to be sent through the cavity after a time delay \((T - \frac{3\pi}{2\Omega})\). In the experiments in \([8]\), this time delay was chosen between about 0, . . . , 710 \(\mu s\). The purpose of the probe atom is to ‘read off’ the state of the cavity modes and to copy this information upon its own state. Similar to the source atom, the probe atom \( A_p \) interacts with the cavity during the time interval \((T + \frac{3\pi}{2\Omega})\) being in resonance with the cavity modes \( M_1 \) and \( M_2 \) as shown in figure 1(c). Using the arguments from above, the whole time evolution of the atom–cavity system from zero up to the time \((T + \frac{3\pi}{2\Omega})\) is given by applying the evolution matrix

\[ U_{ij} \left( T + \frac{3\pi}{2\Omega} \right) = \sum_{k,m,l} U_{ik}^{(3)} \left( \frac{T + 3\pi}{2\Omega} \right) U_{lm}^{(3)} \left( \frac{3\pi}{2\Omega} \right) \times U_{mj}^{(3)} \left( T - \frac{3\pi}{2\Omega} \right) U_{ij}^{(3)} \left( \frac{3\pi}{2\Omega} \right). \]

(18)

In this matrix, \( U_{ij}^{(3)} \) is given by \((11)\) and refers to the interaction of the cavity with the source atom, while the other matrices \( U_{ml}^{(3)}, U_{kj}^{(3)} \) and \( U_{ik}^{(3)} \) describe the three subsequent steps: the free time evolution during the time interval \([\frac{3\pi}{2\Omega}, T]\)

\[ U_{ij}^{(3)}(t) = \text{diag}[1, 1, 1, 1, 1, 1, 1], \]

(19)

which is diagonal in the basis \((8a)–(8b)\) and where a relative phase shift \( e^{i\phi} \) is accumulated due to the energy difference \( \hbar \delta \) of the two cavity modes. Afterwards, during the interval \([T, T + \frac{3\pi}{2\Omega}]\), the further evolution of the atom–cavity system is driven by the Hamiltonian (cf figure 1(c))

\[ H_p(t) = \begin{cases} H_1, & T \leq t \leq T + \frac{\pi}{\Omega} \\ H_2, & T + \frac{\pi}{\Omega} < t \leq T + \frac{3\pi}{2\Omega} \end{cases} \]

(20)

Since the atom–cavity interaction is well described by the Jaynes–Cummings Hamiltonian \((7)\), the matrices \( U_{ij}^{(4)}(t) \) and \( U_{ij}^{(5)}(t) \) coincide with the matrices \((13)\) and \((14)\), except that in \( U_{ij}^{(4)}(t) \) the relative phase \( e^{i\phi} \) occurs. Including this phase factor, \( U_{ij}^{(5)}(t) \) then becomes

\[ U_{ij}^{(5)}(t) = \begin{pmatrix} x(t) & 0 & 0 & 0 & 0 & 0 & y(t) \\ 0 & \bar{x}(t) & 0 & 0 & 0 & \bar{y}(t) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\phi} & 0 & 0 \\ 0 & y(t) & 0 & 0 & 0 & x(t) & 0 \\ 0 & \bar{y}(t) & 0 & 0 & 0 & \bar{x}(t) & 0 \end{pmatrix}. \]

(21)

Having the complete time evolution \((18)\), we can easily determine the probability \( P(T) \) of finding the probe atom in the excited state \( |e\rangle \), after it has crossed the cavity and hit the detector \( D \). This is the (final-state) probability that has been measured during the experiment \([8]\) and that provides us with the information about the superposition of the cavity mode states \((17)\). In practice, of course, the atom–cavity interaction is not ideal and has to be replaced by some realistic model for the atom–cavity interaction. For the idealized sequence of interaction, however, the (complete) time evolution owing to the matrix \((18)\) at the time \( t = \frac{3\pi}{2\Omega} + T \) implies only the two non-zero coefficients

\[ c_1 \left( \frac{3\pi}{2\Omega} + T \right) = -\frac{1}{2} e^{i\phi} (1 + e^{i\pi} e^{i\phi T}), \]

\[ c_6 \left( \frac{3\pi}{2\Omega} + T \right) = \frac{i}{2} e^{i\phi} (1 - e^{i\pi} e^{i\phi T}); \]

and, hence, the final-state probability to find \( A_p \) in the excited state is simply

\[ P(T) = \left| c_1 \left( \frac{3\pi}{2\Omega} + T \right) \right|^2 = \frac{1 + \cos(\omega T + \phi)}{2}. \]

(22)

Obviously, this probability oscillates between zero and one with the frequency \( \omega \equiv \delta \) and with the phase \( \phi \equiv \pi \delta/2\Omega \). In the experiments by Rauschenbeutel et al \([8]\) and Raimond et al \([6]\), the shift between the cavity mode frequencies takes the value \( \delta/2\pi = 128.3 \, \text{kHz} \), while the atom–cavity coupling constant is \( \Omega/2\pi = 47 \, \text{kHz} \).
In the previous subsection, it was shown how the (two) cavity modes become entangled with each other by detuning the atom–cavity interaction due to the non-resonant part of the atom–cavity interaction described mathematically by the Hamiltonian (7) from above, and where $H_{\text{res}}(t)$ and $H_{\text{non-res}}(t)$ are given by

$$ H_{\text{res}}(t) = S_i \Delta(t) + \frac{\Omega}{2} \left( a_i^\dagger S_- + S_a a_i \right). \quad (24) $$

Here, the superscript ‘$\text{res}$’ in (23) is introduced in order to distinguish the Hamiltonian (23) from that of the step-wise detuning (4). Note that, for the present smooth change of the atom–cavity detuning, $H_{\text{res}}$ and $H_{\text{non-res}}$ together describe the evolution of the cavity states is affected by a more realistic time-dependence of the detuning. In this section, therefore, we shall consider a smooth ‘switch’ from the $A_{ij} = M_i$ to the $A_{ij} = M_j$ resonant interaction within a finite time period, which we will denote by $\tau/\Omega$. For instance, according to the experimental setup [8] this period is of the length $\tau/\Omega \approx 1 \mu s$, which corresponds to $7\%$ of the overall $\pi$ interaction time of the atom with the cavity, or about an angle $\pi/10$ in units of Rabi rotations, which is no longer negligible (as we considered in the previous model). Below, we shall assume a smooth behaviour for this finite switch as displayed in figures 2(a) and (c), where the cavity is non-resonant to the source atom during the interval $\left[ \frac{\pi}{2\Omega}, \frac{3\pi}{2\Omega} \right]$ and with the probe atom during $\left[ T + \frac{2\pi}{2\Omega}, T + \frac{2\pi}{2\Omega} \right]$, respectively. This smooth switch of the atomic resonance from one to the other cavity mode can be described mathematically by the Hamiltonian

$$ H^\prime_{ij}(t) = \begin{cases} H_1, & 0 \leq t < \frac{\pi - \tau}{2\Omega} \\ H_{\text{res}}(t), & \frac{\pi - \tau}{2\Omega} \leq t < \frac{\pi}{2\Omega} \\ H_{\text{non-res}}(t), & \frac{\pi}{2\Omega} \leq t < \frac{\pi + \tau}{2\Omega} \\ H_2, & \frac{\pi + \tau}{2\Omega} \leq t \leq \frac{3\pi}{2\Omega} \end{cases} \quad (23) $$

where $H_1$ and $H_2$ are the resonant Jaynes–Cummings Hamiltonians (7) from above, and where $H_{\text{res}}(t)$ and $H_{\text{non-res}}(t)$ are given by

$$ H_{\text{res}}(t) = S_i \Delta(t) + \frac{\Omega}{2} \left( a_i^\dagger S_- + S_a a_i \right). \quad (24) $$

$$ H_{\text{non-res}}(t) = S_i \left[ \Delta(t) + \delta \right] + \frac{\Omega}{2} \left( a_i^\dagger S_- + S_a a_i \right). \quad (25) $$

Here, the superscript ‘$\text{res}$’ in (23) is introduced in order to distinguish the Hamiltonian (23) from that of the step-wise detuning (4). Note that, for the present smooth change of the atom–cavity detuning, $H_{\text{res}}$ and $H_{\text{non-res}}$ together describe the evolution of the resonator and cavity interaction up to the time $\frac{3\pi}{2\Omega}$. That is the resonant part of the atom–cavity interaction due to $H_1$ is shortened by the time interval $\frac{\pi}{2\Omega}$ while the remaining time up to $\frac{3\pi}{2\Omega}$ follows the non-resonant Jaynes–Cummings Hamiltonian $H_{\text{non-res}}$ in line with figures 2(a)–(b). Similarly, the $A_i - M_j$ interaction is first non-resonant and modelled by the Hamiltonian $H_{\text{non-res}}$ for the interval $\left[ \frac{\pi}{2\Omega}, \frac{3\pi}{2\Omega} \right]$, and becomes resonant with the cavity mode $M_i$, as seen in figures 2(c)–(d). Obviously, the Hamiltonian (23) can be understood also as an extension of the ‘step-wise’ model from section 2.1 by introducing an additional short period for the non-resonant atom–cavity interaction. For the Hamiltonian (23), therefore, the time evolution matrix still factorizes and can be expressed at $t = \frac{3\pi}{2\Omega}$ as

$$ U_{ij}^{(3)}(t) = \sum_{k,l,m} U_{ij}^{(2)} \left( \frac{\pi - \tau}{2\Omega} \right) U^{(1)}_{ij} \left( \frac{\pi - \tau}{2\Omega} \right) \times U_{\text{non-res}} \left( \frac{\pi - \tau}{2\Omega} \right) U_{ij}^{(1)} \left( \frac{\pi - \tau}{2\Omega} \right), \quad (26) $$

where the matrices $U_{ij}^{(1)}(t)$ and $U_{ij}^{(2)}(t)$ are the same as in (13) and (14), and where the matrix $U_{\text{non-res}}(t)$ has still the form (13)
but with $x(t)$, $\tilde{x}(t)$, $y(t)$, and $\tilde{y}(t)$ now being solutions of the differential equations [19]:

$$i \frac{dx}{dt} = \frac{\Omega}{2} \tilde{x}(t) + \frac{1}{2} \Delta(t)y(t),$$

$$i \frac{dy}{dt} = \frac{\Omega}{2} \tilde{y}(t) - \frac{1}{2} \Delta(t)x(t),$$

$$i \frac{dx}{dt} = \frac{\Omega}{2} \tilde{x}(t) + \frac{1}{2} \Delta(t)y(t),$$

$$i \frac{dy}{dt} = \frac{\Omega}{2} \tilde{y}(t) - \frac{1}{2} \Delta(t)x(t).$$

(27)

(28)

Analogously, the matrix $U(\tau)$ interaction is described by the matrix in model is again (shift) behave as a function of the 'switching' parameter section 3, we display and discuss how the frequency and phase time for which the cavity is non-resonant to the atom.

$\omega \tau$

(22) from the step-wise model, this probability still oscillates numerically in the various matrices. Like the cosine function of the need to solve the functions $x(t)$, $\tilde{x}(t)$, $y(t)$, and $\tilde{y}(t)$ that are obtained from (27) and (28) by replacing $\Delta(t) \to \Delta(t) + \delta$.

No exact analytic solutions are known for the relations (26)–(28). Therefore, in order to derive the time evolution matrix (26), we need to solve these equations for $x(t)$, $\tilde{x}(t)$, $y(t)$, and $\tilde{y}(t)$ numerically up to $t = \frac{\tau}{\Omega}$, and then to substitute their values into the matrices $U(\tau)$ (t). As discussed above, the final-state probability to find the probe atom in the state $|e\rangle$ is obtained by performing a similar procedure for the $A_p - M_1 - M_2$ interaction during the time interval $[T, T + \frac{3\tau}{2\Omega}]$. During the time $\frac{3\tau}{2\Omega}$, of course, the 'free' evolution of the cavity state still follows the trivial matrix (19). Putting these pieces together, hence, the complete time evolution of the wavefunction for such a smooth change in the atom–cavity interaction is described by the matrix

$$U^{(1)}_{ij}(T) = \sum_{i,j} U^{(1)}_{ik} \left( \frac{\pi - \tau}{2\Omega} \right) U^{(1)}_{kj} \left( \frac{\pi}{2\Omega} \right) \times U^{(1)}_{ml} \left( \frac{\tau}{2\Omega} \right) \times U^{(1)}_{np} \left( T - \frac{3\tau}{2\Omega} \right) U^{(1)}_{pj} \left( \frac{3\pi}{2\Omega} \right),$$

(29)

where the matrices $U^{(1)}_{ij}(t)$, $U^{(1)}_{ij}(t)$, $U^{(1)}_{ij}(t)$ and $U^{(1)}_{ij}(t)$ coincide with matrices (19), (21), (14), and (26), respectively. For the initial condition $c_i(0) = \delta_i$, the final-state probability in model is again

$$P_(T) = \left| \langle N_1 | \Psi_c(T) \rangle \right|^2 = c_1 \left( T + \frac{3\pi}{2\Omega} \right)$$

(30)

which, however, cannot be given in a closed form because of the need to solve the functions $x(t)$, $\tilde{x}(t)$, $y(t)$, and $\tilde{y}(t)$ numerically in the various matrices. Like the cosine function (22) from the step-wise model, this probability still oscillates between zero and one with the frequency $\omega_0$, and phase $\phi_0$. In section 3, we display and discuss how the frequency and phase (shift) behave as a function of the 'switching' parameter $\tau$, i.e. the time for which the cavity is non-resonant to the atom.

2.3. Communication channel model: allowing the mutual interaction of the cavity modes

Up to the present, we have always assumed that, at a given time $t$, the atom interacts either with the cavity mode $M_1$ or the mode $M_2$, implying a sudden change in the interaction from one cavity mode to the other. In our notation above, this means that the overall atom–cavity interaction is understood as a set of pure $A_p - M_1$ or $A_p - M_2$ (non-resonant interactions. However notice that at the time $t = \frac{\pi}{2\Omega}$ in figure 2(b), for instance, the atomic transition frequency is equally far from the (nearly degenerate) frequencies of both cavity modes and, hence, we expect the atom to ‘feel’ the (non-resonant) contribution of $M_1$ and $M_2$ simultaneously. As additional sources for such a simultaneous interaction also serve the imperfections of the mirrors, stray fields, decoherence, etc. Finally, this simultaneous interaction also implies a mutual interaction among the cavity modes to which one therefore refers as a communication channel [10, 19].

To describe this ‘communication’ between the two cavity modes, we need to proceed beyond the Jaynes–Cummings model in order to allow the atom to couple with both cavity modes at the same time. In addition, in order to preserve the ‘correspondence’ with the previous two models, this simultaneous interaction must happen only inside the time intervals $[T, T + \frac{3\tau}{2\Omega}]$ and $[T + \frac{\tau}{2\Omega}, T + \frac{3\tau}{2\Omega}]$, given by the same ‘switching’ parameter $\tau$ as in section 2.2. Below, we shall model the total (source) atom–cavity interaction within the interval $[0, \frac{3\tau}{2\Omega}]$ by means of the time-dependent Hamiltonian

$$\hat{H}_c(t) = \begin{cases} H_1, & 0 \leq t \leq \frac{\pi - \tau}{2\Omega} \\ H_2, & \frac{\pi - \tau}{2\Omega} < t \leq \frac{\pi + \tau}{2\Omega} \end{cases}$$

(31)

with

$$H_2(t) = f(t) \hat{H}_c(t) + f(t) \hat{H}_c(t) + f(t) f(t) \hat{H}_c(t)$$

(32)

$$H_1 = \lambda (a_1a_2 + a_2a_1)$$

(33)

where $\hat{H}_c$ and $\hat{H}_c$ denote the Hamiltonians (7) and (24)–(25), respectively. The temporal behaviour of the detuning $\Delta(t)$ used in $\hat{H}_c(t)$ is the same as we considered in the previous section and which is displayed in figure 2. The two (dimensionless) functions $f(t)$ and $f(t)$ displayed in figure 3(a), thus determine the contribution of the non-resonant Jaynes–Cummings Hamiltonians $\hat{H}_c(t)$ and the communication channel Hamiltonian $\hat{H}_c$ to the overall time-dependent Hamiltonian (31). In this model, again, the resonant $A_p - M_1$ and $A_p - M_2$ interactions are shortened by the time interval $\tau / \Omega$ and the evolution of the (atom–cavity) state is governed by the Hamiltonian $\hat{H}_c(t)$ with the non-resonant part $\hat{H}_c(t)$ and the communication channel $\hat{H}_c$. The Hamiltonian $\hat{H}_c$ is the simplest expression that fulfills the above requirements and preserves the cavity field energy. The contribution of $\hat{H}_c$ to the Hamiltonian $\hat{H}_c(t)$ is given by the time-dependent dimensionless product $f(t) f(t)$. The shape of this coupling term is displayed in figure 3(b) by a dashed line, which has its maximal value $1 / 4$ at $t_{max} = \pi / 2\Omega$.

Making use of the functions $f(t)$ in the Hamiltonian (31), we expect to describe the atom–cavity interaction during the switching period $\tau / \Omega$ in a realistic fashion. Moreover, below we shall consider also the coupling strength $\lambda$ of the cavity.
modes such that the ‘effective’ coupling $\lambda f_1(t) f_2(t)$ satisfies the relation

$$\lambda f_1(t_{\text{max}}) f_2(t_{\text{max}}) = \Omega. \quad (34)$$

The meaning of the last relations can be understood as follows: the strongest $M_1 - M_2$ effective coupling should be (or, at least, should not exceed) the composition of the $A_r - M_1$ and $A_r - M_2$ couplings given by the values of $\Omega/2$, respectively. A simple calculation implies the value of $\lambda$ to be of the order of $\sim 1$ MHz. For this value, indeed, a good agreement with the experimental data of [8] has been found within the expected time ($\tau/\Omega \leq 1 \mu s$), i.e. for that time which is needed to ‘switch’ the atomic frequency between the two cavity modes (see section 3). Therefore, the Hamiltonian (31) exposes the right ‘correspondence’ property with regard to the models we have introduced in the previous two sections. This can be readily seen if we assume $f_1(t) = \theta_1(t)$ and $f_2(t) = \theta_2(t)$; then, expression (31) reduces to the Hamiltonian $H_f(t)$ (23) and, if we additionally impose the limit $\tau \rightarrow 0$, the Hamiltonian (31) simplifies to $H_f(t)$ in expression (4).

The time evolution matrix at $t = \frac{3\pi}{2\Omega}$ as described by the Hamiltonian (31) is given by

$$\tilde{U}_{ij}^{(5)} \left( \frac{3\pi}{2\Omega} \right) = \sum_{k,l} U_{ik}^{(2)} \left( \frac{\pi - \tau}{2\Omega} \right) U_{jl}^{(3)} \left( \frac{\tau}{2\Omega} \right) U_{ij}^{(1)} \left( \frac{\pi - \tau}{2\Omega} \right), \quad (35)$$

where $U_{ij}^{(1)}(t)$ and $U_{ij}^{(2)}(t)$ refer to the expressions (13) and (14), and where the matrix elements of $U_{ij}^{(3)}(t)$ must fulfill the Schrödinger equation

$$i \frac{dU_{ij}^{(3)}(t)}{dt} = \sum_k [V_i | H_{\tau}(t) | V_k] U_{kj}^{(3)}(t). \quad (36)$$

This set of differential equations cannot be solved analytically and need to be integrated numerically during the interval $t = \frac{\pi}{2\Omega}$. Indeed, a similar procedure has to be performed later also for the $A_p - M_1 - M_2$ interaction during the time $[T, T + \frac{3\pi}{2\Omega}]$ as seen from figures 3(c), (d) and figures 2(c), (d). Combining the pieces together, the complete time evolution of the atom–cavity interaction therefore becomes

$$\tilde{U}_i^{(1)} \left( T + \frac{3\pi}{2\Omega} \right) = \sum_{k,m} U_{ik}^{(5)} \left( \frac{\pi - \tau}{2\Omega} \right) U_{jm}^{(3)} \left( \frac{\tau}{2\Omega} \right) U_{ij}^{(1)} \left( \frac{\pi - \tau}{2\Omega} \right) \times$$

$$\times U_{ml}^{(4)} \left( \frac{\pi}{2\Omega} \right) U_{ln}^{(5)} \left( T + \frac{3\pi}{2\Omega} \right) \tilde{U}_{nl}^{(3)} \left( \frac{3\pi}{2\Omega} \right), \quad (37)$$

where the matrices $U_{ij}^{(3)}(t)$, $U_{ij}^{(5)}(t)$, $U_{ij}^{(1)}(t)$, and $\tilde{U}_{ij}^{(1)}(t)$ coincide with the matrices (19), (21), (14), and (35) respectively. Moreover, the final-state probability is given for the initial conditions $c_0 = 0$ by

$$\tilde{P}_i(T) = |\langle V_i | \tilde{U}_i(T) \rangle|^2 = |c_1 \left( T + \frac{3\pi}{2\Omega} \right)|^2. \quad (38)$$

Although no closed expression can be given for this probability (due to the numerical parts in the time evolution), it still oscillates between zero and one with the frequency $\tilde{\omega}_\tau$ and the phase $\phi_\tau$ that is different from the two other models in sections 2.1 and 2.2. In the next section, we shall discuss how these frequency and phase (shifts) behave as functions of the ‘switching’ parameter $\tau$.

### 3. Comparison with experiment and discussion

The expression (22) of the final-state probability describes the time evolution of an idealized experiment [8] with a step-wise change in the resonant interaction between the two cavity modes. This expression neglects of course a number of additional effects, such as the cavity field relaxation, the influence of external stray fields, or imperfections due to the cavity mirrors and cavity geometry. The most significant
distortion of the idealized probability in (22) is given by the cavity field relaxation during the interval $\left[\frac{\pi}{\Omega}, T\right]$, i.e. during the ‘free’ evolution of the cavity field, since during the $A_j - M_j - M_i$ and $A_p - M_1 - M_2$ interactions, the system evolves in the strong coupling (resonant) regime, and hence, the dissipation of the cavity energy is negligible. This field relaxation mainly arises from the interaction of the cavity with the environment and its effect on the final-state probability (22) has been prognosticated in [10]. Therefore, our investigations enable us to further improve the agreement with experiment.

We now compare the predictions for the frequency and phase of the final-state probability, obtained with the models from section 2, with those from experiment [8]. In these measurements, the probability for detecting the probe atom $A_p$ in the excited state $|e\rangle$ was recorded and displayed for the four time intervals $I_i = [48; 57] \mu s$, $I_2 = [200; 207] \mu s$, $I_3 = [400, 408] \mu s$ and $I_4 = [699, 706] \mu s$, respectively. From the observed probability (cf figures 2(a)–(d) in [8]), the four ‘experimental’ frequencies and phases have been extracted by a fit to the analytical function (40), where the cosine arguments have been considered as unknown values, separately for each time interval $I_i$, ($v = 1, \ldots, 4$), and are displayed by the dots in figure 4. For the sake of convenience, moreover, these experimental data were normalized to the reference frequency $\omega = 2\pi \times 128.3$ kHz and phase $\phi \equiv \pi \delta/2\Omega = 4.29$ of the ‘idealized’ probability (22). As seen from figure 4, clear deviations of the extracted experimental phase $\phi_e$ occur with regard to the reference phase $\phi$.

To understand how a smooth change in the atom–cavity detuning $\Delta(t)$ affects the frequency and phase of the predicted final-state probability, a number of steps have to be carried out for both models of sections 2.2 and 2.3: (i) to evaluate numerically the final-state probability $P_\nu(T)$ or $\tilde{P}_\nu(T)$ for the time delay intervals $I_i$ from the experiment; (ii) to fit the numerical results from step (i) to the analytical function (40) in order to extract the cosine arguments for every time interval; and (iii) to repeat the previous two steps for different values of the ‘switching’ parameter $\tau$. The cosine arguments obtained from this procedure can then be compared to the experimental dots from figure 4 in order to determine the parameter $\tau$ that agrees better with the experiment.
Figure 5. Comparison of the relative phase $\phi_{T}/\phi$ for a smooth change of the atom–cavity detuning (cf section 2.2) with experiment. (a) Results are shown for the ‘switching’ parameter $\tau/\Omega^{-1} = 3.6 \mu s$, for which the best agreement with the experimental data (dots) is found. The theoretical values (crosses) are joined by dashed lines to guide the eye of the reader. Figure (b) displays the relative phase $\phi_{T}/\phi$ as a function of $\tau$ (curves $C_{\nu}$). Every curve corresponds to a phase $\phi_{\nu}$ as obtained for the time intervals $I_{\nu}$. The vertical dashed line indicates the $\tau$ value from plot (a).

Figure 6. The same as figure 5 but for the communication-channel model in section 2.3, $\tilde{\phi}_{T}/\phi$. (a) Results are shown again for the ‘switching’ parameter $\tau/\Omega^{-1} = 0.33 \mu s$ with the best agreement with the experimental data (dots). The theoretical values (crosses) are joined by dashed lines to guide the eye of the reader. Figure (b) displays the relative phase $\tilde{\phi}_{T}/\phi$ as a function of $\tau$ (curves $C_{\nu}$). Every curve corresponds to a phase $\tilde{\phi}_{\nu}$ as obtained for the time intervals $I_{\nu}$. The vertical dashed line indicates the $\tau$ value from plot (a).

Figure 5 compares the relative phase $\phi_{T}/\phi$ for a smooth change of the atom–cavity detuning as modelled in section 2.2. In figure 5(a), results are shown for the ‘switching’ parameter $\tau/\Omega = 3.6 \mu s$, for which the best agreement with the experimental data (dots) is found. In addition, figure 5(b) displays the relative phase $\phi_{T}/\phi$ as a function of $\tau$ for each time intervals $I_{\nu}$. As seen from this figure, our mixed analytic–numerical treatment of the atom–cavity state gives (as expected) the same final-state probability (22) in the limit $\tau \to 0$. Note that the difference between $\omega_{T}$ and $\omega$ is about two orders of magnitude smaller than the differences in the phases (and are therefore not displayed here). This confirms that the frequency of the final-state probability is less sensitive to ‘undesired’ effects in the atom–cavity interaction than the phase shift. The best agreement between our predictions and experiment is found for a ‘switching time’ $\tau/\Omega = \tau/\Omega = 3.6 \mu s$ that is far outside of what is expected for the switch of the atomic transition frequency from one to the other cavity mode in the experiment [8]. Therefore, despite its rather large effect on the phase-shift of the final-state probability, the finite switching time cannot explain the observations from the experiment as taken alone.

In the communication channel model from section 2.3, in contrast, the atom interacts for a short period with both cavity modes simultaneously, leading to an effective ‘communication’ between the cavity modes. Figure 6(a) displays again the relative phase $\phi_{T}/\phi$ for that switching parameter, $\tau/\Omega = 0.33 \mu s$, for which the best agreement with the experimental data (dots) is found. This value nicely fits to the expected time that is needed to ‘switch’ the atomic frequency between the two cavity modes. Moreover, the relative phase $\tilde{\phi}_{T}/\phi$ as a function of $\tau$ is shown in figure 6(b) for each time intervals $I_{\nu}$ separately. We therefore conclude that a finite switch together with an effective communication of the two cavity modes, i.e. the model of the communication channel, is appropriate for describing the atom–cavity interaction when the atom passes through the cavity.

Let us mention once more that numerical simulations were performed only ‘while’ the atomic frequency is switched between the cavity modes ($\tau/\Omega^{-1} \leq 1 \mu s$), both for the single-mode (section 2.2) and the communication-channel model (section 2.3). For all other periods, the time evolution of the atom–cavity state was followed analytically (15) owing to a purely resonant atom–cavity interaction. The slight deviation between the predicted and observed phases (figures 6(a) and 5(a)) clearly indicates that there is no perfect agreement with the experimental data. However, no error bar has been given on the final-state probability in [8], and hence this slight disagreement should be later on refined in accordance with error bars which occur in cavity QED experiments.
4. Summary and outlook

The coherent evolution of the atom–cavity state in bimodal (cavity) experiments has been analysed for a realistic time dependence of the detuning of the atomic transition frequency. Apart from a ‘smooth switch’ of the atomic resonance from one cavity mode to the other during a short but finite time interval (section 2.2), we considered also an additional (effective) interaction, a so-called communication channel, between the two cavity modes (section 2.3). While, in the former case, the atom interacts at any time with just one mode of the photon field, either resonantly or non-resonantly, a simultaneous interaction with both field modes is modelled by means of the communication channel. The outcome of the experiment is the final-state probability for that the probe atom is observed in the (excited) Rydberg state; this probability is analysed by combining the analytical solution of the Jaynes–Cummings Hamiltonian with numerical simulations for all those time intervals during which the atom–cavity interaction is tuned from one cavity mode to the other.

Comparison of our model computations has been made especially with the recent measurements by Rauschenbeutel and co-workers [8]. To this end, a series of computations have been carried out for the final-state probability as a function of the delay time $T$ for different (realistic) choices of the ‘switching time’ $\tau$, the model parameter, during which the atom is non-resonant to the cavity modes. Making use of typical cavity parameters as reported in [8], it is demonstrated that the agreement between the predicted and experimental phase can be improved by allowing a ‘communication’ between the two cavity modes, cf figures 5 and 6. Together with the recent study by Bosco de Magalhaes and Nemes [10], who investigated the damping of the probability amplitude due to the coupling of the cavity to the environment, the use of the communication channel from above brings the theoretical predictions in good agreement with experiment. For the interpretation of future bimodal experiments, therefore, it seems appropriate to take into account both, the decoherence and the details of the atom–cavity detuning dynamics.

In fact, several proposals have been reported for performing bimodal cavity experiments [11–17]. In the proposal by Zubairy et al [13], for instance, a bimodal cavity is utilized to realize a quantum state gate in which the two qubits are given by the cavity modes. Based on this gate, the authors suggested a scheme that enables one to implement Grover’s search algorithm. Another fruitful branch of the bimodal cavity applications represent proposals [14–17], where the schemes for engineering of various entangled states between the atomic and/or photonic qubits have been reported. All these papers ignore, however, the effects of the ‘non-instantaneous’ detuning of the atom–cavity interaction from one cavity mode to the other, as well as the effective interaction among the cavity modes. Further investigation is therefore required to better understand how well these schemes may work in practice. Finally, we mention the papers [11, 12] where it has been proven that the coupling of both cavity modes to a common reservoir, as in line with $H_{\text{coupled}}$ (39), induces the tunnelling of a field state from one cavity mode to another mode of the same cavity, and thus, opens a way to implement the environment-assisted (short-distance) teleporting inside a bimodal cavity. We note that in order to follow the time evolution of such quantum systems embedded into a reservoir or under the external noise and to analyse different (entanglement or separability) measures, a ‘quantum simulator’ has been developed recently in our group [21] that can be utilized for such studies in the future.

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References

[1] Einstein A, Podolsky B and Rosen N 1935 Phys. Rev. 47 777
[2] Nielsen M A and Chuang I L 2000 Quantum Computation and quantum Information (Cambridge: Cambridge University Press)
[3] Bennett C H and Wiesner S J 1992 Phys. Rev. Lett. 69 2881
[4] Ekert A K 1991 Phys. Rev. Lett. 67 661
[5] Grover L K 1997 Phys. Rev. Lett. 79 325
[6] Raimond J M, Brune M and Haroche S 2001 Rev. Mod. Phys. 73 565
[7] Kuhr S et al 2007 Appl. Phys. Lett. 90 164101
[8] Rauschenbeutel A, Bertet P, Osanghi S, Nogues G, Brune M, Raimond J M and Haroche S 2001 Phys. Rev. A 64 050301
[9] Haroche S and Raimond J M 2006 Exploring the Quantum: Atoms, Cavities, and Photons (Oxford: Oxford University Press)
[10] Bosco de Magalhaes A R and Nemes M C 2004 Phys. Rev. A 70 053825
[11] Iara de Queir S, Souza S, Cardoso W B and de Almeida N G 2007 Phys. Rev. A 76 034101
[12] Bosco de Magalhaes A R and Nemes M C 2005 Phys. Lett. A 339 294
[13] Zubairy M S, Kim M and Scully M O 2003 Phys. Rev. A 68 033820
[14] Biswas A and Agarwal G S 2004 J. Mod. Phys. Opt. 51 1627
[15] Zou X and Mathis W 2005 J. Mod. Opt. 52 2001
[16] Ikram M and Saif F 2002 Phys. Rev. A 66 043404
[17] Wildfeuer C and Schiller D H 2003 Phys. Rev. A 67 053801
[18] Jaynes E T and Cummings F W 1963 Proc. IEEE 51 89
[19] Puri R R 2001 Mathematical Methods of Quantum Optics (Berlin: Springer)
[20] Maitre X, Hagley E, Nogues G, Wunderlich C, Goy P, Brune M, Raimond J M and Haroche S 1997 Phys. Rev. Lett. 79 769
[21] Radtke T and Fritzscbe S 2006 Comput. Phys. Commun. 175 145
[22] Radtke T and Fritzscbe S 2006 Comput. Phys. Commun. 176 617