HÖLDER REGULARITY FOR SOLUTIONS TO COMPLEX MONGE-AMPÈRE EQUATIONS

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Abstract. We consider the Dirichlet problem for the complex Monge-Ampère equation in a bounded strongly hyperconvex Lipschitz domain in \( \mathbb{C}^n \). We first give a sharp estimate on the modulus of continuity of the solution when the boundary data is continuous and the right hand side has a continuous density. Then we consider the case when the boundary value function is \( C^{1,1} \) and the right hand side has a density in \( L^p(\Omega) \) for some \( p > 1 \) and prove the Hölder continuity of the solution.

1. Introduction

Let \( \Omega \) be a bounded pseudoconvex domain in \( \mathbb{C}^n \). Given \( \varphi \in C(\partial \Omega) \) and \( 0 \leq f \in L^1(\Omega) \).

We consider the Dirichlet problem:

\[
\text{Dir}(\Omega, \varphi, f) : \begin{cases}
    u \in PSH(\Omega) \cap C(\bar{\Omega}) \\
    (dd^c u)^n = f \beta^n & \text{in } \Omega \\
    u = \varphi & \text{on } \partial \Omega
\end{cases}
\]

where \( PSH(\Omega) \) is the set of plurisubharmonic (psh) functions in \( \Omega \). Here we denote \( d = \partial + \overline{\partial} \) and \( d^c = i/4(\overline{\partial} - \partial) \) then \( dd^c = i/2\partial\overline{\partial} \) and \( (dd^c)^n \) stands for the complex Monge-Ampère operator.

If \( u \in C^2(\Omega) \) and is plurisubharmonic function, the complex Monge-Ampère operator is given by

\[
(dd^c u)^n = \det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \right) \beta^n
\]

where \( \beta = i/2 \sum_{j=1}^n dz_j \wedge d\overline{z}_j \) be the standard Kähler form in \( \mathbb{C}^n \).

In their seminal work, Bedford and Taylor proved that the complex Monge-Ampère operator can be extended to the set of bounded plurisubharmonic functions (see [BT76], [BT82]). Moreover, it is invariant under holomorphic change of coordinates. We refer the reader to [BT76], [De89], [Kl91], [Ko05] for more details on its properties.

This problem has been studied extensively in last decades by many authors. When \( \Omega \) is a bounded strongly pseudoconvex domain with smooth boundary, Bedford and Taylor had showed that \( \text{Dir}(\Omega, \varphi, f) \) has a unique continuous solution \( U := U(\Omega, \varphi, f) \). Furthermore, it was proved in [BT76] that \( U \in Lip_\alpha(\Omega) \) when \( \varphi \in Lip_{2\alpha}(\partial \Omega) \) and \( f^{1/n} \in Lip_\alpha(\Omega) \) (\( 0 < \alpha \leq 1 \)). In the non degenerate case i.e. \( 0 < f \in C^\infty(\Omega) \) and \( \varphi \in C^\infty(\partial \Omega) \), Caffarelli,
Kohn, Nirenberg and Spruck proved in [CKNS85] that \( U \in C^\infty(\overline{\Omega}) \). However a simple example of Gamelin and Sibony shows that the solution is not, in general, better than \( C^{1,1} \)-smooth when \( f \geq 0 \) and smooth ([GS80]). Krylov proved that if \( \varphi \in C^{3,1}(\partial\Omega) \) and \( f^{1/n} \in C^{1,1}(\overline{\Omega}) \), \( f \geq 0 \) then \( U \in C^{1,1}(\overline{\Omega}) \) (see [Kr89]).

For \( B \)-regular domains, Blocki [Bl96] proved the existence of a continuous solution to the Dirichlet problem \( \text{Dir}(\Omega, \varphi, f) \) when \( f \in C(\overline{\Omega}) \).

For a strongly pseudoconvex domain with smooth boundary, Kołodziej demonstrated in [Ko98] that \( \text{Dir}(\Omega, \varphi, f) \) still admit a unique continuous solution under the milder assumption \( f \in L^p(\Omega) \), for \( p > 1 \). Recently Guedj, Kołodziej and Zeriahi studied the H"older continuity of the solution when \( 0 \leq f \in L^p(\Omega) \), for some \( p > 1 \), is bounded near the boundary (see [GKZ08]).

For the complex Monge-Ampère equation on a compact Kähler manifold, H"older continuity of the solution was proved earlier by Kołodziej [Ko08] (see also [DDGHKZ12]).

A viscosity approach to the complex Monge-Ampère equation has been developed in [EGZ11] and [Wan12].

In this paper, we consider the more general case where \( \Omega \) be a bounded strongly hyperconvex Lipschitz domain (the boundary does not need to be smooth) and \( f \in L^p(\Omega) \).

We will generalize the approach of Bedford and Taylor [BT76] by showing an estimate for the modulus of continuity to the solution in terms of the modulus of continuity of the data.

**Theorem A.** Let \( \Omega \subseteq \mathbb{C}^n \) be a bounded strongly hyperconvex Lipschitz domain, \( \varphi \in C(\partial\Omega) \) and \( 0 \leq f \in C(\overline{\Omega}) \). Assume that \( \omega_\varphi \) is the modulus of continuity of \( \varphi \) and \( \omega_{f^{1/n}} \) is the modulus of continuity of \( f^{1/n} \). Then the modulus of continuity of \( U \) has the following estimate

\[
\omega_0(t) \leq \eta (1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\omega_\varphi(t^{1/2}), \omega_{f^{1/n}}(t), t^{1/2}\}
\]

where \( \eta \) is a positive constant depending on \( \Omega \).

Here we will use a new description of the solution given by Proposition 3.3 to get an optimal control for the modulus of continuity of this solution in a strongly hyperconvex Lipschitz domain.

For more general density \( f \in L^p(\Omega) \) for some \( p > 1 \), it was shown in [GKZ08] that the unique solution to \( \text{Dir}(\Omega, \varphi, f) \) belongs to \( C^{0,\alpha}(\Omega) \) for all \( \alpha < 2/(nq+1) \) when \( \varphi \in C^{1,1}(\partial\Omega) \) and \( f \in L^p(\Omega) \) be a bounded function near the boundary. Here we will improve this result and show the following theorem

**Theorem B.** Let \( \Omega \subseteq \mathbb{C}^n \) be a bounded strongly hyperconvex Lipschitz domain. Assume that \( \varphi \in C^{1,1}(\partial\Omega) \) and \( f \in L^p(\Omega) \) for some \( p > 1 \). Then the unique solution \( U \) to \( \text{Dir}(\Omega, \varphi, f) \) is \( \alpha \)-Hölder continuous on \( \Omega \) for any \( 0 < \alpha < 1/(nq + 1) \) where \( 1/p + 1/q = 1 \). Moreover, if \( p \geq 2 \), then the solution \( U \) is \( \alpha \)-Hölder continuous on \( \Omega \) for any \( 0 < \alpha < \min\{1/2, 2/(nq + 1)\} \).

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2. Preliminaries

We recall that a hyperconvex domain is to be a domain in \( \mathbb{C}^n \) admitting a bounded exhaustion function.

Let us define the class of hyperconvex domains which will be considered in this paper.

**Definition 2.1.** A bounded domain \( \Omega \subset \mathbb{C}^n \) is called strongly hyperconvex Lipschitz (shortly SHL) domain if there exists a neighbourhood \( \Omega' \) of \( \bar{\Omega} \) and a Lipschitz plurisubharmonic function \( \rho : \Omega' \to \mathbb{R} \) such that

1. \( \rho < 0 \) in \( \Omega \) and \( \partial \Omega = \{ \rho = 0 \} \),
2. there exists a constant \( c > 0 \) such that \( \dd c \rho \geq c \beta \) in \( \Omega \) in the weak sense of currents.

**Example 2.2.**

1. Let \( \Omega \) be a strictly convex domain that is there exists a Lipschitz defining function \( \rho \) such that \( \rho - c|z|^2 \) is convex for some \( c > 0 \). It is clear that \( \Omega \) is strongly hyperconvex Lipschitz domain.
2. A smooth strictly pseudoconvex bounded domain is a SHL domain (see [HL84]).
3. The nonempty finite intersection of strictly pseudoconvex bounded domains with smooth boundary in \( \mathbb{C}^n \) is a bounded SHL domain. In fact, it is sufficient to put \( \rho = \max \{ \rho_i \} \). More generally a finite intersection of SHL domains is an SHL domain.
4. The domain \( \Omega = \{ z = (z_1, \cdots, z_n) \in \mathbb{C}^n; |z_1| + \cdots + |z_n| < 1 \} (n \geq 2) \) is a bounded strongly hyperconvex Lipschitz domain in \( \mathbb{C}^n \) with non smooth boundary.
5. The unit polydisc in \( \mathbb{C}^n (n \geq 2) \) is hyperconvex with Lipschitz boundary but it is not a strongly hyperconvex Lipschitz.

**Remark 2.3.** Kerezman and Rosay [KR81] proved that in a hyperconvex domain there exists there exists an exhaustion function which is smooth and strictly plurisubharmonic. Furthermore, they proved that any bounded pseudoconvex domain with \( C^1 \)-boundary is hyperconvex domain. This result was extended by Demailly [De87] to bounded pseudoconvex domains with Lipschitz boundary.

Let \( \Omega \subset \mathbb{C}^n \) be a bounded domain. If \( u \in PSH(\Omega) \) then \( \dd c u \geq 0 \) in the sense of currents. We define

\[
\Delta_H u := \sum_{j,k=1}^{n} h_{jk} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_j}
\]

for every positive definite Hermitian matrix \( H = (h_{jk}) \). We can see \( \Delta_H u \) as a positive Radon measure in \( \Omega \).

The following lemma is elementary and important for the sequel (see [Gav77]).

**Lemma 2.4.** ([Gav77]). Let \( Q \) be a \( n \times n \) nonnegative hermitian matrix. Then
where $H^+_n$ denotes the set of all positive hermitian $n \times n$ matrices.

**Example 2.5.** We calculate $\Delta_H(|z|^2)$ for every matrix $H \in H^+_n$ and $\det H = n^{-n}$.

$$\Delta_H(|z|^2) = \sum_{j,k=1}^{n} h_{jk} \delta_{jk} = \text{tr}(H)$$

using the inequality of arithmetic and geometric means, we have:

$$1 = (\det I)^{1/n} \leq \text{tr}(H),$$

hence $\Delta_H(|z|^2) \geq 1$ for every matrix $H \in H^+_n$ and $\det H = n^{-n}$.

Using ideas from the theory of viscosity due to Eyssidieux, Guedj and Zeriahi [EGZ11], we can prove the following result.

**Proposition 2.6.** Let $u \in PSH \cap L^\infty(\Omega)$ and $0 \leq f \in C(\bar{\Omega})$. Then the following conditions are equivalent:

1) $\Delta_H u \geq f^{1/n}$ in the weak sense of distributions, for any $H \in H^+_n$ and $\det H = n^{-n}$.

2) $(dd^c u)^n \geq f \beta^n$ in the weak sense of currents in $\Omega$.

This result is implicitly contained in [EGZ11], but we will give a complete proof for the convenience of the reader.

**Proof.** First, suppose that $u \in C^2(\Omega)$, then by Lemma 2.4 the following

$$\Delta_H u = \sum_{j,k=1}^{n} h_{jk} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \geq f^{1/n}, \forall H \in H^+_n, \det(H) = n^{-n}$$

is equivalent to

$$\left(\det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)\right)^{1/n} \geq f^{1/n}.$$

The last inequality means that

$$(dd^c u)^n \geq f \beta^n.$$

(1 $\Rightarrow$ 2) Let $(\rho_\epsilon)$ be a family of regularizing kernels with supp $\rho_\epsilon \subset B(0, \epsilon)$ and $\int_{B(0, \epsilon)} \rho_\epsilon = 1$, hence the sequence $u_\epsilon = u * \rho_\epsilon$ decreases to $u$, then we see that (1) implies $\Delta_H u_\epsilon \geq (f^{1/n})_\epsilon$. Since $u_\epsilon$ is smooth, we use the first case and get $(dd^c u_\epsilon)^n \geq \left((f^{1/n})_\epsilon\right)^n \beta^n$, hence by applying the convergence theorem of Bedford and Taylor (Theorem 7.4 in [BT82]) we obtain $(dd^c u)^n \geq \beta^n$.

(2 $\Rightarrow$ 1) Fix $x_0 \in \Omega$, and $q$ is $C^2$-function in a neighborhood $B$ of $x_0$ such that $u \leq q$ in this neighborhood and $u(x_0) = q(x_0)$.

First step: we will show that $dd^c q_{x_0} \geq 0$. Indeed, for every small enough ball $B' \subset B$ centered at $x_0$, we have

$$u(x_0) - q(x_0) \geq \frac{1}{V(B')} \int_{B'} (u - q) dV,$$
then we get
\[ \frac{1}{V(B')} \int_{B'} q dV - q(x_0) \geq \frac{1}{V(B')} \int_{B'} u dV - u(x_0) \geq 0. \]

Since \( q \) is \( C^2 \)-smooth and the radius of \( B' \) tends to 0, it follows, form Proposition 3.2.10 in [H94], that \( \Delta q_{x_0} \geq 0 \). For every positive definite Hermitian matrix \( H \) with \( \det H = n^{-n} \), we make linear change of complex coordinates \( T \) such that \( H \) will be \( I \) (the identity matrix) in the new coordinates and \( \bar{Q} = (\partial^2 \bar{q} / \partial w_j \partial \bar{w}_k) \) where \( \bar{q} = q \circ T^{-1} \) then

\[ \Delta_{H} q(x_0) = \text{tr}(H \bar{Q}) = \text{tr}(I \bar{Q}) = \Delta \bar{q}(y_0) \]

Hence \( \Delta_{H} q(x_0) \geq 0 \) for every \( H \in H^+ \) then \( dd^c q_{x_0} \geq 0 \).

Second step: we claim that \( (dd^c q)^n_{x_0} \geq f(x_0) \beta^n \). Suppose that there exists a point \( x_0 \in \Omega \) and a \( C^2 \)-function \( q \) which satisfies \( u \leq q \) in a neighborhood of \( x_0 \) and \( u(x_0) = q(x_0) \) such that \( (dd^c q)^n_{x_0} < f(x_0) \beta^n \). We put

\[ q'(x) = q(x) + \epsilon \left( \| x - x_0 \|^2 - \frac{r^2}{2} \right) \]

for \( 0 < \epsilon \ll 1 \) small enough, we see that

\[ 0 < (dd^c q')^n_{x_0} < f(x_0) \beta^n. \]

Since \( f \) is lower semi-continuous on \( \bar{\Omega} \), there exists \( r > 0 \) such that

\[ (dd^c q')^n \leq f(x) \beta^n \quad \forall x \in B(x_0, r). \]

Then \( (dd^c q')^n \leq f(x) \beta^n \leq (dd^c u)^n \) in \( B(x_0, r) \) and \( q' = q + \epsilon \frac{r^2}{2} \geq q \geq u \) on \( \partial B(x_0, r) \), hence \( q' \geq u \) on \( B(x_0, r) \) by the comparison principle. But \( q'(x_0) = q(x_0) - \epsilon \frac{r^2}{2} = u(x_0) - \epsilon \frac{r^2}{2} < u(x_0) \) contradiction.

Hence, form the first part of the proof, we get \( \Delta_{H} q(x_0) \geq f^{1/n}(x_0) \) for every point \( x_0 \in \Omega \) and every \( C^2 \)-function \( q \) in a neighborhood of \( x_0 \) such that \( u \leq q \) in this neighborhood and \( u(x_0) = q(x_0) \).

Assume that \( f > 0 \) and \( f \in C^\infty(\bar{\Omega}) \), then there exists \( g \in C^\infty(\bar{\Omega}) \) such that \( \Delta_{H} g = f^{1/n} \). Hence \( \varphi = u - g \) is \( H \)-subharmonic (by Proposition 3.2.10', [H94]), from which it follows \( \Delta_{H} \varphi \geq 0 \) and \( \Delta_{H} u \geq f^{1/n} \).

In case \( f > 0 \) is merely continuous, we observe that

\[ f = \sup \{ w; w \in C^\infty, f \geq w > 0 \}, \]

then \( (dd^c u)^n \geq f \beta^n \geq w \beta^n \). Since \( w > 0 \) is smooth, we have \( \Delta_{H} u \geq w^{1/n} \). Therefore, we get \( \Delta_{H} u \geq f^{1/n} \).

In the general case \( 0 \leq f \in C(\bar{\Omega}) \), we observe that \( u^\epsilon(z) = u(z) + \epsilon |z|^2 \) satisfies

\[ (dd^c u^\epsilon)^n \geq (f + \epsilon^n) \beta^n, \]

then

\[ \Delta_{H} u^\epsilon \geq (f + \epsilon^n)^{1/n}. \]

Letting \( \epsilon \) converges to 0, we get

\[ \Delta_{H} u \geq f^{1/n} \quad \text{for all } H \in H^+ \text{ and } \det H = n^{-n}. \]
As a consequence of Proposition 2.6, we give a new description of the classical Perron-Bremermann family of subsolutions to the Dirichlet problem Dir(Ω, φ, f).

**Definition 2.7.** We denote \( V(Ω, φ, f) \) the family of subsolutions of Dir(Ω, φ, f), that is
\[
V(Ω, φ, f) = \{ v \in PSH(Ω) \cap C(Ω), v|_{∂Ω} ≤ φ \text{ and } ∆Hv ≥ f^{1/n} ∀ H ∈ H^n_+, detH = n^{-n} \}.
\]

**Remark 2.8.** We observe that \( V(Ω, φ, f) ≠ ∅ \). Indeed, let \( ρ \) as in Definition 2.1 and \( A, B > 0 \) big enough, then \( Aρ − B \in V(Ω, φ, f) \).

Furthermore, the family \( V(Ω, φ, f) \) is stable under finite maximum, that is if \( u, v \in V(Ω, φ, f) \) then \( max(u, v) ∈ V(Ω, φ, f) \).

### 3. The Perron-Bremermann envelope

Bedford and Taylor proved in [BT76] that the unique solution to Dir(Ω, φ, f) in a bounded strongly pseudoconvex domain with smooth boundary, is given as the envelope of Perron-Bremermann

\[
u = \sup\{v; v ∈ B(Ω, φ, f)\}
\]

where \( B(Ω, φ, f) = \{ v ∈ PSH(Ω) ∩ C(Ω), v|_{∂Ω} ≤ φ \text{ and } (dd^cv)^n ≥ fβ_n \} \).

Thanks to Proposition 2.6, we get the following corollary

**Corollary 3.1.** The two families \( V(Ω, φ, f) \) and \( B(Ω, φ, f) \) coincide, that is \( V(Ω, φ, f) = B(Ω, φ, f) \).

Here we will first give an alternative description of the Perron-Bremermann envelope in a bounded SHL domain.

More precisely, we consider the upper envelope

\[
U(z) = \sup\{v(z); v ∈ V(Ω, φ, f)\}.
\]

3.1. **Continuity of the upper envelope.** Following the same argument in [Wal69, Bl96], we prove the continuity of the upper envelope.

**Theorem 3.2.** Let \( Ω ⊂ \mathbb{C}^n \) be a bounded SHL domain, \( 0 ≤ f ∈ C(Ω) \) and \( φ ∈ C(∂Ω) \). Then the upper envelope

\[
U = \sup\{v; v ∈ V(Ω, φ, f)\}
\]

belongs to \( V(Ω, φ, f) \) and \( U = φ \) on \( ∂Ω \).

**Proof.** Let \( g ∈ C^2(Ω) \) be an approximation of \( φ \) such that \( |g − φ| < ϵ \) on \( ∂Ω \), for \( ϵ > 0 \). Let also \( ρ \) the defining function as in Definition 2.1 and \( A > 0 \) large enough such that \( v_0 := Ap + g − ϵ \) belongs to \( V(Ω, φ, f) \).

Thus \( v_0 ≤ U ≤ h \), where \( h \) be the harmonic extension of \( φ \) to \( Ω \). Then it follows that \( φ − 2ϵ ≤ g − ϵ ≤ U ≤ φ \) on \( ∂Ω \), as \( ϵ \) tends to 0, we see that \( U = φ \) on \( ∂Ω \).

We will prove that \( U \) is continuous on \( Ω \). Fix \( ϵ > 0 \) and \( z_0 \) in a compact set \( K ⊂ Ω \). Thanks to the continuity of \( h \) and \( v_0 \) on \( Ω \), one can find \( δ > 0 \) such that for any \( z_1, z_2 ∈ Ω \) we have

\[
|U(z_1) − U(z_2)| ≤ ϵ.
\]
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\[ |h(z_1) - h(z_2)| < \epsilon, \ |v_0(z_1) - v_0(z_2)| < \epsilon \text{ if } |z_1 - z_2| < \delta. \]

Let \( a \in \mathbb{C}^n \) such that \( |a| < \min(\delta, dist(K, \partial \Omega)) \). Since \( U \) is the upper envelope, we can find \( v \in \mathcal{V}(\Omega, \varphi, f) \) such that \( v(z_0 + a) > U(z_0 + a) - \epsilon \) and we can assume that \( v_0 \leq v \). Hence for all \( z \in \Omega \) and \( w \in \partial \Omega \) we get

\[ -3\epsilon < v_0(z) - \varphi(w) < v(z) - \varphi(w) < h(z) - \varphi(w) < \epsilon, \]

this implies that

(3.1) \[ |v(z) - \varphi(w)| < 3\epsilon \text{ if } |z - w| < \delta. \]

Then for \( z \in \Omega \) and \( z + a \in \partial \Omega \), we have

\[ v(z + a) \leq \varphi(z + a) < v(z) + 3\epsilon. \]

We define the following function

\[ v_1(z) = \begin{cases} v(z) & ; z + a \notin \Omega \\ \max(v(z), v(z + a) - 3\epsilon) & ; z + a \in \Omega \end{cases} \]

which is well defined, plurisubharmonic on \( \Omega \), continuous on \( \overline{\Omega} \) and \( v_1 \leq \varphi \) on \( \partial \Omega \). Indeed, if \( z \in \partial \Omega \), \( z + a \notin \overline{\Omega} \) then \( v_1(z) = v(z) \leq \varphi(z) \). On the other hand, if \( z \in \partial \Omega \) and \( z + a \in \overline{\Omega} \) then we have, from (3.1) that

\[ v(z + a) - 3\epsilon < \varphi(z), \]

so \( v_1(z) = \max(v(z), v(z + a) - 3\epsilon) \leq \varphi(z) \).

Moreover, we note that \( \Delta_H v_1(z + a)) \geq f^{1/n}(z + a) \), hence it follows that

\[ \Delta_H v_1 \geq \min(f^{1/n}, f^{1/n}(z + a)). \]

Let \( \omega \) be the modulus of continuity of \( f^{1/n} \) and define

\[ v_2 = v_1 + \omega(|a|)(v_0 - \|v_0\|_{L^\infty(\Omega)}). \]

We claim that \( v_2 \in \mathcal{V}(\Omega, \varphi, f) \). It is clear that \( v_2 \in PSH(\Omega) \cap C(\overline{\Omega}) \) and \( v_2 \leq \varphi \) on \( \partial \Omega \).

Moreover, one can point out that

\[ \Delta_H v_2 = \Delta_H v_1 + \omega(|a|)\Delta_H v_0 \geq f^{1/n}. \]

In fact, if \( \Delta_H v_1 = f^{1/n}(z + a) \), by suitable choice of \( A \) we get

\[ \Delta_H v_2 = f^{1/n}(z + a) + \omega(|a|)\Delta_H v_0 \geq -\omega(|a|) + \omega(|a|)\Delta_H v_0 + f^{1/n} \geq f^{1/n}. \]

Hence we obtain that

\[ U(z_0) \geq v_1(z_0) + \omega(|a|)v_0(z_0) - \omega(|a|)\|v_0\| \]

\[ \geq v(z_0 + a) - 5\epsilon \quad (\text{where } \omega(|a|) < \frac{\epsilon}{\|v_0\|}) \]

\[ > U(z_0 + a) - 6\epsilon. \]

Since \( |a| \) is small and the last inequality is true for every \( z_0 \in K \), then \( U \) is continuous on \( \Omega \).

As the family \( \mathcal{V}(\Omega, \varphi, f) \) is stable under the operation maximum, we can find a sequence \( u_j \in \mathcal{V}(\Omega, \varphi, f) \) such that \( u_j \) increases almost everywhere to \( U \), then \( u_j \to U \) in \( L^1(\Omega) \). Hence \( \Delta_H U = \lim \Delta_H u_j \geq f^{1/n} \) for all \( H \in H_n^2 \), \( detH = n^{-n} \), this implies \( U \in \mathcal{V}(\Omega, \varphi, f) \). \( \square \)
Proposition 3.3. Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly hyperconvex Lipschitz domain, $0 \leq f \in C(\bar{\Omega})$ and $\varphi \in C(\partial \Omega)$. Then the Dirichlet problem $\text{Dir}(\Omega, \varphi, f)$ has a unique solution $U$. Moreover the solution is given by

$$U = \sup\{v; v \in V(\Omega, \varphi, f)\}$$

where

$$V = \{v \in PSH(\Omega) \cap C(\bar{\Omega}), v|_{\partial \Omega} \leq \varphi \text{ and } \Delta_H v \geq f^{1/n} \forall H \in H_+^n, \det H = n^{-n}\}$$

and $\Delta_H$ be the laplacian associated to a positive definite Hermitian matrix $H$ as in (2.7).

Proof. The uniqueness follows from the comparison principle ([BT76]). On the other hand, Theorem 3.2 implies that our domain $\Omega$ is $B$-regular in the sense of Sibony ([Sib87]). Therefore existence and uniqueness of the solution follows from Theorem 4.1 in [Bl96]. The description of the solution given in the proposition follows from Corollary 3.1 and Theorem 3.2.

Remark 3.4. Let $\varphi_1, \varphi_2 \in C(\partial \Omega)$ and $f_1, f_2 \in C(\bar{\Omega})$, then the solutions $U_1 = U(\Omega, \varphi_1, f_1)$, $U_2 = U(\Omega, \varphi_2, f_2)$ satisfy the following stability estimate

$$\|U_1 - U_2\|_{L^\infty(\Omega)} \leq d^2 \|f_1 - f_2\|_{L^{1/n}(\bar{\Omega})} + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial \Omega)}$$

where $d := \text{diam}(\Omega)$. Indeed, fix $z_0 \in \Omega$ and define

$$v_1(z) = \|f_1 - f_2\|_{L^{1/n}(\bar{\Omega})}(|z - z_0|^2 - d^2) + U_2(z)$$

and

$$v_2(z) = U_1(z) + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial \Omega)}.$$ 

It is clear that $v_1, v_2 \in PSH(\Omega) \cap C(\bar{\Omega})$. Hence, by the comparison principle, we get $v_1 \leq v_2$ on $\Omega$. Then we conclude that

$$U_2 - U_1 \leq d^2 \|f_1 - f_2\|_{L^{1/n}(\bar{\Omega})} + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial \Omega)}$$

Reversing the roles of $U_1$ and $U_2$, we get the inequality (3.2).

We will need in Section 5 an estimate, proved by Blocki in [Bl93], for the $L^n - L^1$ stability of solutions to the Dirichlet problem $\text{Dir}(\Omega, \varphi, f)$

$$\|U_1 - U_2\|_{L^n(\Omega)} \leq \lambda(\Omega)\|\varphi_1 - \varphi_2\|_{L^\infty(\partial \Omega)} + \frac{r^2}{4} \|f_1 - f_2\|_{L^1(\Omega)}^{1/n}$$

where $r = \min\{r' > 0 : \Omega \subset B(z_0, r') \text{ for some } z_0 \in \mathbb{C}^n\}$.

4. The modulus of continuity of Perron-Bremermann envelope

Recall that a real function $\omega$ on $[0, \ell]$, $0 < \ell < \infty$, is called a modulus of continuity if $\omega$ is continuous, subadditive, nondecreasing and $\omega(0) = 0$.

In general, $\omega$ fails to be concave, we denote $\bar{\omega}$ to be the minimal concave majorant of $\omega$. The following property of the minimal concave majorant $\bar{\omega}$ is well known (see [Kor82] and [Ch14]).

Lemma 4.1. Let $\omega$ be a modulus of continuity on $[0, \ell]$ and $\bar{\omega}$ be the minimal concave majorant of $\omega$. Then $\omega(\eta t) < \bar{\omega}(\eta t) < (1 + \eta)\omega(t)$ for any $t > 0$ and $\eta > 0$.
4.1. Modulus of continuity of the solution. Now, we will start the first step to establish an estimate for the modulus of continuity of the solution to \( \text{Dir}(\Omega, \varphi, f) \). For this reason, it is natural to require the relation between the modulus of continuity of \( \Delta U \) and the modulus of continuity of sub-barrier and super-barrier. Thus, we present the following proposition

**Proposition 4.2.** Let \( \Omega \subset \mathbb{C}^n \) be a bounded SHL domain, \( \varphi \in C(\partial \Omega) \) and \( 0 \leq f \in C(\overline{\Omega}) \). Suppose that there exist \( v \in \mathcal{V}(\Omega, \varphi, f) \) and \( w \in SH(\Omega) \cap C(\overline{\Omega}) \) such that \( v = \varphi = -w \) on \( \partial \Omega \), then there is a constant \( C > 0 \) depends on \( \text{diam}(\Omega) \) such that the modulus of continuity of \( \Delta U \) satisfies

\[
\omega_0(t) \leq C \max\{\omega_v(t), \omega_w(t), \omega_{f^{1/n}}(t)\}.
\]

**Proof.** Let us put \( g(t) := \max(\omega_v(t), \omega_w(t), \omega_{f^{1/n}}(t)) \) and \( d := \text{diam}(\Omega) \). As \( v = \varphi = -w \) on \( \partial \Omega \), we have for all \( z \in \Omega \) and \( \xi \in \partial \Omega \)

\[
-g(|z - \xi|) \leq v(z) - \varphi(\xi) \leq \text{diam}(\Omega) - w(z) - \varphi(\xi) \leq g(|z - \xi|).
\]

Hence we get

\[
|u(z) - u(\xi)| \leq g(|z - \xi|); \forall z \in \Omega, \forall \xi \in \partial \Omega.
\]

Fix a point \( z_0 \in \Omega \), for any small vector \( \tau \in \mathbb{C}^n \), we set \( \Omega_{-\tau} := \{z - \tau; z \in \Omega\} \) and define in \( \Omega \cap \Omega_{-\tau} \) the function

\[
v_1(z) = u(z + \tau) + g(|\tau|)|z - z_0|^2 - d^2g(|\tau|) - g(|\tau|)
\]

which is well defined psh function in \( \Omega \cap \Omega_{-\tau} \) and continuous on \( \overline{\Omega_{-\tau}} \). By (4.1), if \( z \in \Omega \cap \partial \Omega_{-\tau} \) we can see that

\[
v_1(z) - u(z) \leq g(|\tau|) + g(|\tau|)|z - z_0|^2 - d^2g(|\tau|) - g(|\tau|) \leq 0.
\]

Moreover, we assert that \( \Delta_H v_1 \geq f^{1/n} \) in \( \Omega \cap \Omega_{-\tau} \) for all \( H \in H^+_{n}, \det H = n^{-n} \). Indeed, we have

\[
\Delta_H v_1(z) \geq f^{1/n}(z + \tau) + g(|\tau|)\Delta_H(|z - z_0|^2)
\]

\[
\geq f^{1/n}(z + \tau) + g(|\tau|)
\]

\[
\geq f^{1/n}(z + \tau) + |f^{1/n}(z + \tau) - f^{1/n}(z)|
\]

\[
\geq f^{1/n}(z)
\]

for all \( H \in H^+_{n} \) and \( \det H = n^{-n} \).

Hence, by the last properties of \( v_1 \), we find that

\[
V_{\tau}(z) = \begin{cases} 
\U(z) & ; z \in \Omega \setminus \Omega_{-\tau} \\
\max(\U(z), v_{1}(z)) & ; z \in \Omega \cap \Omega_{-\tau}
\end{cases}
\]

is well defined function and belongs to \( \text{PSH}(\Omega) \cap C(\overline{\Omega}) \). It is clear that \( \Delta_H V_{\tau} \geq f^{1/n} \) for all \( H \in H^+_{n}, \det H = n^{-n} \). We claim that \( V_{\tau} = \varphi \) on \( \partial \Omega \). If \( z \in \partial \Omega \setminus \Omega_{-\tau} \) then \( V_{\tau}(z) = \U(z) = \varphi(z) \). On the other hand \( z \in \partial \Omega \cap \Omega_{-\tau} \), by (4.2) we get \( V_{\tau}(z) = \max(\U(z), v_{1}(z)) = \U(z) = \varphi(z) \). Consequently \( V_{\tau} \in \mathcal{V}(\Omega, \varphi, f) \) and this implies that

\[
V_{\tau}(z) \leq U(z); \forall z \in \Omega.
\]
Then we have for all \( z \in \bar{\Omega} \cap \Omega_{-\tau} \)
\[
U(z + \tau) + g(|\tau|)|z - z_0|^2 - d^2g(|\tau|) - g(|\tau|) \leq U(z).
\]
Hence,
\[
U(z + \tau) - U(z) \leq (d^2 + 1)g(|\tau|) - g(|\tau|)|z - z_0|^2 \leq Cg(|\tau|).
\]
Reversing the roles of \( z + \tau \) and \( z \), we get
\[
|U(z + \tau) - U(z)| \leq Cg(|\tau|); \forall z, z + \tau \in \bar{\Omega}.
\]
Thus, we finally get
\[
\omega_\varphi(|\tau|) \leq C \max(\omega_v(|\tau|), \omega_\mu(|\tau|), \omega_{f_{1/n}}(|\tau|)).
\]
\[\square\]

**Remark 4.3.** Let \( H_\varphi \) be the harmonic extension of \( \varphi \) in a bounded SHL domain \( \Omega \), we can replace \( w \) in the last proposition by \( H_\varphi \). It is known in the classical harmonic analysis (see [Ai10]) that the harmonic extension \( H_\varphi \) has not, in general, the same modulus of continuity of \( \varphi \).

Let us define, for small positive \( t \), the modulus of continuity
\[
\psi_{\alpha,\beta}(t) = (-\log(t))^{-\alpha}t^\beta
\]
with \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \). It is clear that \( \psi_{\alpha,0} \) is weaker than the Hölder continuity and \( \psi_{0,\beta} \) is the Hölder continuity. It was shown in [Ai02] that \( \omega_{H_\varphi}(t) \leq c\psi_{0,\beta}(t) \) for some \( c > 0 \) if \( \omega_\varphi(t) \leq c_1\psi_{0,\beta}(t) \) for \( \beta < \beta_0 \) where \( \beta_0 < 1 \) depending only on \( n \) and the Lipschitz constant of the defining function \( \rho \). Moreover, a similar result was proved in [Ai10] for the modulus of continuity \( \psi_{\alpha,0}(t) \). However, the same argument of Aikawa gives that \( \omega_{H_\varphi}(t) \leq c\psi_{\alpha,\beta}(t) \) for some \( c > 0 \) if \( \omega_\varphi(t) \leq c_1\psi_{\alpha,\beta}(t) \) for \( \alpha \geq 0 \) and \( 0 \leq \beta < \beta_0 < 1 \).

Hence, this leads us to the conclusion that if there exists a barrier \( v \) to the Dirichlet problem such that \( v = \varphi \) on \( \partial \Omega \) and \( \omega_v(t) \leq \lambda\psi_{\alpha,\beta}(t) \) with \( \alpha, \beta \) as above, then the last proposition gives
\[
\omega_v(t) \leq \lambda_1 \max\{\psi_{\alpha,\beta}(t), \omega_{f_{1/n}}(t)\},
\]
where \( \lambda_1 > 0 \) depending on \( \lambda \) and \( \text{diam}(\Omega) \).

**4.2. Construction of barriers.** In this subsection, we will construct a subsolution to Dirichlet problem with the boundary value \( \varphi \) and estimate its modulus of continuity.

**Proposition 4.4.** Let \( \Omega \subset \mathbb{C}^n \) be a bounded SHL domain, assume that \( \varphi \in C(\partial \Omega) \) and \( 0 \leq f \in C(\overline{\Omega}) \). Then there exists a subsolution \( v \in \mathcal{V}(\Omega, \varphi, f) \) such that \( v = \varphi \) on \( \partial \Omega \) and the modulus of continuity of \( v \) satisfies the following inequality
\[
\omega_v(t) \leq \lambda(1 + \|f\|_{L^{1/n}(\Omega)}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}
\]
where \( \lambda > 0 \) depends on \( \Omega \).

Observe that we do not assume any smoothness on \( \partial \Omega \).
Proof. First of all, let us fix $\xi \in \partial \Omega$, we claim that there exists $v_\xi \in V(\Omega, \varphi, f)$ such that $v_\xi(\xi) = \varphi(\xi)$. It is sufficient to prove that there exists a constant $C > 0$ depending on $\Omega$ such that for every point $\xi \in \partial \Omega$ and $\varphi \in C(\partial \Omega)$, there is a function $h_\xi \in PSH(\Omega) \cap C(\bar{\Omega})$ such that

1) $h_\xi(z) \leq \varphi(z), \forall z \in \partial \Omega$

2) $h_\xi(\xi) = \varphi(\xi)$

3) $\omega_{h_\xi}(t) \leq C\omega_\varphi(t^{1/2})$.

Assume this is true, we fix $z_0 \in \Omega$ and choose $K_1 := \sup_\Omega f^{1/n} \geq 0$, hence

$$\Delta_H(K_1|z - z_0|^2) = K_1\Delta_H|z - z_0|^2 \geq f^{1/n}, \forall H \in H^+_n, det H = n^{-n},$$

we also put $K_2 = K_1|\xi - z_0|^2$. Then for the continuous function

$$\tilde{\varphi}(z) := \varphi(z) - K_1|z - z_0|^2 + K_2,$$

we have $h_\xi$ such that 1),2) and 3) hold.

Then the desired function $v_\xi \in V(\Omega, \varphi, f)$ is given by

$$v_\xi(z) = h_\xi(z) + K_1|z - z_0|^2 - K_2$$

Because, $h_\xi(z) \leq \tilde{\varphi}(z) = \varphi(z) - K_1|z - z_0|^2 + K_2$ on $\partial \Omega$, so $v_\xi(z) \leq \varphi$ on $\partial \Omega$ and $v_\xi(\xi) = \varphi(\xi)$.

Moreover, it is clear that

$$\Delta_H v_\xi = \Delta_H h_\xi + K_1\Delta_H(|z - z_0|^2) \geq f^{1/n}, \forall H \in H^+_n, det H = n^{-n}.$$ 

Furthermore, using the hypothesis of $h_\xi$, we can control the modulus of continuity of $v_\xi$

$$\omega_{v_\xi}(t) = \sup_{|z - y| \leq t} |v_\xi(z) - v_\xi(y)| \leq \omega_{h_\xi}(t) + K_1 \omega_{|z - z_0|^2}(t)$$

$$\leq C\omega_{\tilde{\varphi}}(t^{1/2}) + 4d^{3/2}K_1 t^{1/2}$$

$$\leq C\omega_{\varphi}(t^{1/2}) + 2dK_1(C + 2d^{1/2})t^{1/2}$$

$$\leq (C + 2d^{1/2})(1 + 2dK_1) \max\{\omega_{\varphi}(t^{1/2}), t^{1/2}\}.$$ 

Hence, we conclude that

$$\omega_{v_\xi}(t) \leq \lambda(1 + K_1) \max\{\omega_{\varphi}(t^{1/2}), t^{1/2}\}$$

where $\lambda := (C + 2d^{1/2})(1 + 2d)$ is a positive constant depending on $\Omega$.

Now we will construct $h_\xi \in PSH(\Omega) \cap C(\bar{\Omega})$ which satisfies the three conditions above. Let $B \geq 0$ large enough such that the function

$$g(z) = B\rho(z) - |z - \xi|^2$$

is psh in $\Omega$. Let $\bar{\varphi}$ be the minimal concave majorant of $\omega_{\varphi}$ and define

$$\chi(x) = -\bar{\varphi}((-x)^{1/2})$$

which is convex nondecreasing function on $[-d^2, 0]$. Now fix $r > 0$ so small that $|g(z)| \leq d^2$ in $B(\xi, r) \cap \Omega$ and define for $z \in B(\xi, r) \cap \bar{\Omega}$ the function

$$h(z) = \chi \circ g(z) + \varphi(\xi).$$
It is clear that $h$ is continuous psh function on $B(\xi, r) \cap \Omega$ and we see that $h(z) \leq \varphi(z)$ if $z \in B(\xi, r) \cap \partial \Omega$ and $h(\xi) = \varphi(\xi)$. Moreover by the subadditivity of $\tilde{\omega}_\varphi$ and Lemma 4.1 we have

$$
\omega_h(t) = \sup_{|z-y| \leq t} |h(z) - h(y)| \\
\leq \sup_{|z-y| \leq t} \tilde{\omega}_\varphi \left[ (|z - \xi|^2 - |y - \xi|^2 - B(\rho(z) - \rho(y)))^{1/2} \right] \\
\leq \sup_{|z-y| \leq t} \tilde{\omega}_\varphi \left[ (|z - y|/(2d + B_1))^{1/2} \right] \\
\leq C \omega_h(t^{1/2})
$$

where $C := 1 + (2d + B_1)^{1/2}$ depends on $\Omega$.

Recall that $\xi \in \partial \Omega$ and fix $0 < r_1 < r$ and $\gamma_1 \geq d/r_1$ such that

$$
-\gamma_1 \tilde{\omega}_\varphi \left[ (|z - \xi|^2 - B\rho(z))^{1/2} \right] \leq \inf_{\partial \Omega} \varphi - \sup_{\partial \Omega} \varphi,
$$

for $z \in \partial \Omega \cap \partial B(\xi, r_1)$. Set $\gamma_2 = \inf_{\partial \Omega} \varphi$, then it follows that

$$
\gamma_1 (h(z) - \varphi(\xi)) + \varphi(\xi) \leq \gamma_2 \text{ for } z \in \partial B(\xi, r_1) \cap \bar{\Omega}.
$$

Now let us put

$$
h_\xi(z) = \begin{cases} 
max\{\gamma_1 (h(z) - \varphi(\xi)) + \varphi(\xi), \gamma_2\} ; & z \in \bar{\Omega} \cap B(\xi, r_1) \\
\gamma_2 ; & z \in \bar{\Omega} \setminus B(\xi, r_1)
\end{cases}
$$

which is well defined plurisubharmonic function on $\Omega$, continuous on $\bar{\Omega}$ and satisfies that $h_\xi(z) \leq \varphi(z)$ for all $z \in \partial \Omega$. Indeed, on $\partial \Omega \cap B(\xi, r_1)$ we have

$$
\gamma_1 (h(z) - \varphi(\xi)) + \varphi(\xi) = -\gamma_1 \tilde{\omega}_\varphi(|z - \xi|) + \varphi(\xi) \leq -\tilde{\omega}_\varphi(|z - \xi|) + \varphi(\xi) \leq \varphi(z).
$$

Hence it is clear that $h_\xi$ satisfies the three conditions above.

We have just proved that for each $\xi \in \partial \Omega$, there is a function

$$
v_\xi \in \mathcal{V}(\Omega, \varphi, f), \ v_\xi(\xi) = \varphi(\xi) \text{ and } \omega_{v_\xi}(t) \leq \lambda(1 + K_1) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}.
$$

Let us set

$$
v(z) = \sup \{v_\xi(z) ; \xi \in \partial \Omega\}.
$$

We can note $0 \leq \omega_v(t) \leq \lambda(1 + K_1) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$, then $\omega_v(t)$ converges to zero when $t$ converges to zero. Consequently, we get $v \in C(\bar{\Omega})$ and $v = v^* \in PSH(\Omega)$. Thanks to Choquet lemma, we can choose a nondecreasing sequence $(v_j)$, where $v_j \in \mathcal{V}(\Omega, \varphi, f)$, converging to $v$ almost everywhere. This implies that

$$
\Delta_H v = \lim_{j \to \infty} \Delta_H v_j \geq f^{1/n}, \forall H \in H_n^+, \det H = n^{-n}.
$$

It is clear that $v(\xi) = \varphi(\xi)$ for any $\xi \in \partial \Omega$. Finally, we get $v \in \mathcal{V}(\Omega, \varphi, f)$, $v = \varphi$ on $\partial \Omega$ and $\omega_{v}(t) \leq \lambda(1 + K_1) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. \hfill \Box

**Remark 4.5.** If we assume that $\Omega$ has a smooth boundary and $\varphi$ is $C^{1,1}$-smooth, then it is possible to construct a Lipschitz barrier $v$ to the Dirichlet problem $\text{Dir}(\Omega, \varphi)$ (see Theorem 6.2 in [BT76]).
Corollary 4.6. Under the same assumption of Proposition 4.4. There exists a plurisuperharmonic function \( \tilde{v} \in C(\Omega) \) such that \( \tilde{v} = \varphi \) on \( \partial \Omega \) and
\[
\omega_\tilde{v}(t) \leq \lambda (1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\},
\]
where \( \lambda > 0 \) depends on \( \Omega \).

Proof. We can do the same construction as in the proof of Proposition 4.4 for the function \( \varphi_1 = -\varphi \in C(\partial \Omega) \), then we get \( v_1 \in V(\Omega, \varphi_1, f) \) such that \( v_1 = \varphi_1 \) on \( \partial \Omega \) and
\[
\omega_{v_1}(t) \leq (1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}.
\]
Hence, we set \( \tilde{v} = -v_1 \) which is a plurisuperharmonic function on \( \Omega \), continuous on \( \bar{\Omega} \) and satisfies \( \tilde{v} = \varphi \) on \( \partial \Omega \) and
\[
\omega_{\tilde{v}}(t) \leq \lambda (1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}.
\]

4.3. Proof of Theorem A. Thanks to Proposition 4.4 we obtain a subsolution \( v \in V(\Omega, \varphi, f) \), \( v = \varphi \) on \( \partial \Omega \) and
\[
\omega_v(t) \leq \lambda (1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}.
\]
Observing Corollary 4.6 we get \( w \in PSH(\Omega) \cap C(\bar{\Omega}) \) such that \( w = -\varphi \) on \( \partial \Omega \) and
\[
\omega_w(t) \leq \lambda (1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}
\]
where \( \lambda > 0 \) constant. Applying the Proposition 4.2 we get the wanted result, that is
\[
\omega_u(t) \leq \eta (1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\omega_\varphi(t^{1/2}), \omega_f^{1/n}(t), t^{1/2}\}
\]
where \( \eta > 0 \) depends on \( \Omega \).

Corollary 4.7. Let \( \Omega \) be a bounded SHL domain in \( \mathbb{C}^n \). Let \( \varphi \in C^0,\alpha(\partial \Omega) \) and \( 0 \leq f^{1/n} \in C^{0,\beta}(\bar{\Omega}) \), \( 0 < \alpha, \beta \leq 1 \). Then the solution \( U \) to the Dirichlet problem \( \text{Dir}(\Omega, \varphi, f) \) belongs to \( C^{0,\gamma}(\Omega) \) for \( \gamma = \min(\beta, \alpha/2) \).

The following example illustrates that the estimate of \( \omega_\varphi \) in Theorem A is optimal.

Example 4.8. Let \( \psi \) be a concave modulus of continuity on \([0, 1]\) and
\[
\varphi(z) = -\psi[\sqrt{(1 + \text{Re}z_1)/2}], \text{ for } z = (z_1, z_2, ..., z_n) \in \partial \mathbb{B} \subset \mathbb{C}^n.
\]
It is easy to show that \( \varphi \in C(\partial \mathbb{B}) \) with modulus of continuity
\[
\omega_\varphi(t) \leq C \psi(t)
\]
for some \( C > 0 \).

Let \( v(z) = -(1 + \text{Re}z_1)/2 \in PSH(\mathbb{B}) \cap C(\bar{\mathbb{B}}) \) and \( \chi(\lambda) = -\psi(\sqrt{-\lambda}) \) is convex increasing function on \([-1, 0]\). Hence we get that
\[
u(z) = \chi \circ v(z) \in PSH(\mathbb{B}) \cap C(\bar{\mathbb{B}})
\]
and satisfies \((dd^c u)^n = 0\) in \( \mathbb{B} \) and \( u \equiv \varphi \) on \( \partial \mathbb{B} \). The modulus of continuity of \( U \), \( \omega_0(t) \), has the estimate
\[
C_1 \psi(t^{1/2}) \leq \omega_0(t) \leq C_2 \psi(t^{1/2})
\]
for $C_1, C_2 > 0$. Indeed, let $z_0 = (-1,0,...,0)$ and $z = (z_1,0,...,0) \in \mathbb{B}$ where $z_1 = -1 + 2t$ and $0 \leq t \leq 1$. Hence, by Lemma 4.1, we see that

$$\psi(t^{1/2}) = \psi(\sqrt{|z - z_0|/2}) = \psi(\sqrt{(1 + \text{Re}z_1)/2}) = |U(z) - U(z_0)| \leq \omega_0(2t) \leq 3\omega_0(t).$$

**Definition 4.9.** Let $\psi$ be a modulus of continuity, $E \subset \mathbb{C}^n$ be a bounded set and $g \in \mathcal{C} \cap L^\infty(E)$. We define the norm of $g$ with respect to $\psi$ (ψ-norm) as follows

$$\|g\|_\psi := \sup_{z \in E} |g(z)| + \sup_{z \neq y \in E} \frac{|g(z) - g(y)|}{\psi(|z - y|)}$$

**Proposition 4.10.** Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $\varphi \in \mathcal{C}(\partial \Omega)$ with modulus of continuity $\psi_1$ and $f^{1/n} \in \mathcal{C}(\overline{\Omega})$ with modulus of continuity $\psi_2$. Then there exists a constant $C > 0$ depending on $\Omega$ such that

$$\|U\|_\psi \leq C(1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}$$

where $\psi(t) = \max\{\psi_1(t^{1/2}), \psi_2(t)\}$.

**Proof.** By hypothesis, we see that $\|\varphi\|_{\psi_1} < \infty$ and $\|f^{1/n}\|_{\psi_2} < \infty$. Let $z \neq y \in \Omega$, by Theorem A, we get

$$|U(z) - U(y)| \leq \eta(1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\omega_\varphi(|z - y|^{1/2}), \omega_f^{1/n}(|z - y|)\}$$

$$\leq \eta(1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\|\varphi\|_{\psi_1} \psi_1(|z - y|^{1/2}), \|f^{1/n}\|_{\psi_2} \psi_2(|z - y|)\}$$

$$\leq \eta(1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\} \psi(|z - y|)$$

where $\psi(|z - y|) = \max\{\psi_1(|z - y|^{1/2}), \psi_2(|z - y|)\}$. Hence we have

$$\sup_{z \neq y \in \Omega} \frac{|U(z) - U(y)|}{\psi(|z - y|)} \leq \eta(1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}$$

where $\eta \geq d^2 + 1$ and $d = \text{diam}(\Omega)$ (see Proposition 4.2). From Remark 3.2, we note that

$$\|U\|_{L^\infty(\Omega)} \leq d^2 \|f\|_{L^\infty(\Omega)}^{1/n} + \|\varphi\|_{L^\infty(\partial \Omega)} \leq \eta \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}$$

Then we can conclude that

$$\|U\|_\psi \leq 2\eta(1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}.$$

Finally, it is natural to try to relate the modulus of continuity of $U := U(\Omega, \varphi, f)$ to the modulus of continuity of $U_0 := U(\Omega, \varphi, 0)$ the solution to Bremermann problem in a bounded SHL domain.

**Proposition 4.11.** Let $\Omega$ be a bounded SHL domain in $\mathbb{C}^n$, $f \in C(\overline{\Omega})$ and $\varphi \in C(\partial \Omega)$. Then there exists a positive constant $C = C(\Omega)$ such that

$$\omega_0(t) \leq C(1 + \|f\|_{L^\infty(\Omega)}^{1/n}) \max\{\omega_0(t), \omega_f^{1/n}(t)\}.$$
Proof. First, we search a subsolution \( v \in V(\Omega, \varphi, f) \) such that \( v|_{\partial \Omega} = \varphi \) and estimate its modulus of continuity. Since \( \Omega \) is bounded SHL domain, there exists a Lipschitz defining function \( \rho \) on \( \overline{\Omega} \). Let us define the function

\[
v(z) = U_0(z) + A\rho(z)
\]

where \( A := \|f\|^{1/n}_{L^\infty(\overline{\Omega})} \) and \( c > 0 \) as in the Definition 2.1. It is clear that \( v \in V(\Omega, \varphi, f) \), \( v = \varphi \) on \( \partial \Omega \) and \( \omega_U(t) \leq \tilde{C}\omega_{U_0}(t) \) where \( \tilde{C} := \gamma(1 + \|f\|^{1/n}_{L^\infty(\overline{\Omega})}) \) and \( \gamma \geq 1 \) depends on \( \Omega \).

On the other hand, by the comparison principle we get that \( U \leq U_0 \). Hence \( v \leq U \leq U_0 \) in \( \Omega \) and \( v = U = U_0 = \varphi \) on \( \partial \Omega \).

Thanks to Proposition 4.2, there exists \( \lambda > 0 \) depending on \( \Omega \) such that

\[
\omega_U(t) \leq \lambda \max\{\omega_v(t), \omega_{U_0}(t), \omega_{f^{1/n}}(t)\}.
\]

Hence, the following inequality holds for some \( C > 0 \) depending on \( \Omega \)

\[
\omega_U(t) \leq C(1 + \|f\|^{1/n}_{L^\infty(\overline{\Omega})}) \max\{\omega_{U_0}(t), \omega_{f^{1/n}}(t)\}.
\]

\[ \square \]

5. Hölder continuous solutions for the Dirichlet problem with \( L^p \) density

In this section we will prove the existence and the Hölder continuity of the solution to Dirichlet problem \( \text{Dir}(\Omega, \varphi, f) \) when \( f \in L^p(\Omega), p > 1 \) in a bounded SHL domain.

It is well known in [Ko98] that there exists a weak continuous solution to this problem when \( \Omega \) is a bounded strongly pseudoconvex domain with smooth boundary.

The Hölder continuity of this solution was studied in [GKZ08] under some additional conditions on the density and on the boundary data, that is when \( f \) is bounded near the boundary and \( \varphi \in C^{1,1}(\partial \Omega) \).

An essential method in this study is played by an a priori weak stability estimate of the solution which is still true when \( \Omega \) is a bounded SHL domain. More precisely, we have the following theorem

**Theorem 5.1.** ([GKZ08]). Fix \( 0 \leq f \in L^p(\Omega), p > 1 \). Let \( u, v \) be two bounded plurisubharmonic functions in \( \Omega \) such that \( (dd^c u)^n = f \beta^n \) in \( \Omega \) and \( u \geq v \) on \( \partial \Omega \). Fix \( r \geq 1 \) and \( 0 \leq \gamma < r/(nq+r), 1/p+1/q = 1 \). Then there exists a uniform constant \( C = C(\gamma, n, q) > 0 \) such that

\[
\sup_{\Omega}(v - u) \leq C(1 + \|f\|_{L^p(\Omega)})\|v - u\|_r + \|v - u\|_{L^r(\Omega)}
\]

where \( \tau := \frac{1}{n} + \frac{\gamma q}{r - \gamma(r + nq)} \) and \( (v - u)_+ := \max(v - u, 0) \).

It was constructed in [GKZ08] a Lipschitz continuous barrier to the Dirichlet problem when \( \varphi \in C^{1,1}(\partial \Omega) \) and \( f \) is bounded near the boundary. Moreover, it was shown in this case that the total mass of \( \Delta U \) is finite in \( \Omega \). Finally, they conclude that \( U \in C^{0,\alpha}(\Omega) \) for any \( \alpha < 2/(nq+1) \). However, the following theorem summarizes the work introduced in [GKZ08]
Theorem 5.2. (GKZ08). Let 0 ≤ f ∈ L^p(Ω), for some p > 1 and ϕ ∈ C(∂Ω). Suppose that there exists v, w ∈ PSH(Ω) ∩ C^0,α(Ω) such that v ≤ ϕ ≤ −w on Ω and v = ϕ = −w on ∂Ω. If the total mass of ΔU is finite in Ω, then U ∈ C^0,α(Ω) for α < min{α, 2/(nq + 1)}.

Here let Ω ⊂ C^n be a bounded SHL domain. Using the stability theorem 5.1 we will ensure the existence of the solution to the Dirichlet problem Dir(Ω, φ, f).

Proposition 5.3. Let Ω ⊂ C^n be a bounded SHL domain, φ ∈ C(∂Ω) and f ∈ L^p(Ω) for some p > 1. Then there exists a unique solution U to the Dirichlet problem Dir(Ω, φ, f).

Proof. Let (f_j) be a sequence of smooth functions on ̄Ω which converges to f in L^p(Ω). Thanks to Proposition 5.3 there exists a unique solution U_j to Dir(Ω, φ, f_j) that is U_j ∈ PSH(Ω) ∩ C(Ω), U_j = φ on ∂Ω and (dd^cU_j)^n = f_jβ^n in Ω. We claim that
\begin{equation}
\|U_k - U_j\|_{L^\infty(Ω)} \leq A(1 + \|f_k\|_{L^p(Ω)})^{1/(1 + \|f_j\|_{L^p(Ω)})}\|f_k - f_j\|^\gamma/n_{L^1(Ω)}
\end{equation}
where 0 ≤ γ < 1/(q + 1) fixed, τ := 1 + 2 −n+γqq and A = A(γ, n, q, diam(Ω)).

Indeed, by the stability theorem 5.1 and for r = n, we get that
\[\sup_{Ω} (U_k - U_j) ≤ C(1 + \|f_j\|_{L^p(Ω)})\|U_k - U_j\| + \|\gamma/n_{L^\infty(Ω)} \leq C(1 + \|f_j\|_{L^p(Ω)})\|U_k - U_j\|\]
where 0 ≤ γ < 1/(q + 1) fixed and C = C(γ, n, q) > 0.

Hence by the L^n - L^1 stability theorem in [B93] (see here Remark 3.2), we get
\[\|U_k - U_j\|_{L^\infty(Ω)} \leq C\|f_k - f_j\|^{1/n}_{L^1(Ω)}\]
where C depends on diam(Ω).

Then, by combining the last two inequalities, we get
\[\sup_{Ω} (U_k - U_j) ≤ C\|f_k - f_j\|^{\gamma/n}_{L^1(Ω)}\]
Reversing the roles of U_j and U_k we see that
\[\sup_{Ω} (U_j - U_k) ≤ C\|f_k - f_j\|^{\gamma/n}_{L^1(Ω)}\]
Hence we conclude that
\[\|U_k - U_j\|_{L^\infty(Ω)} ≤ C\|f_k - f_j\|^\gamma/n_{L^1(Ω)}\]
Since U_k = U_j = φ on ∂Ω, we get the inequality (5.1).

Since f_j converges to f in L^p(Ω), there is a uniform constant B > 0 such that
\[\|U_k - U_j\|_{L^\infty(Ω)} ≤ B\]
This implies that the sequence U_j converges uniformly in ̄Ω. Let us put U = lim_j U_j, it is clear that U ∈ PSH(Ω) ∩ C(Ω), U = φ on ∂Ω. Moreover, (dd^cU_j)^n converges to (dd^cU)^n in the sense of currents, then (dd^cU)^n = fβ^n in Ω. The uniqueness of the solution comes from the comparison principle (see [BT76]).

Our next step is to construct H"older continuous sub-barrier and super-barrier to the Dirichlet problem when f ∈ L^p(Ω) for some p > 1 and φ ∈ C^{0,1}(∂Ω).
Proposition 5.4. Let \( \varphi \in C^{0,1}(\partial \Omega) \) and \( 0 \leq f \in L^p(\Omega) \), for some \( p > 1 \). Then there exist \( v, w \in PSH(\Omega) \cap C^{0,\alpha}(\overline{\Omega}) \) where \( \alpha < 1/(nq + 1) \) such that \( v = \varphi = -w \) on \( \partial \Omega \) and \( v \leq u \leq -w \) on \( \Omega \).

Proof. Fix a large ball \( B \subset \mathbb{C}^n \) so that \( \Omega \subset B \subset \mathbb{C}^n \). Let \( \tilde{f} \) be a trivial extension of \( f \) to \( B \). Since \( \tilde{f} \in L^p(\Omega) \) is bounded near \( \partial B \), the solution \( h_1 \) to \( \text{Dir}(B, 0, \tilde{f}) \) is Hölder continuous on \( B \) with exponent \( \alpha_1 < 2/(nq + 1) \) (see [GKZ08]). Now let \( h_2 \) denote the solution to the Dirichlet problem in \( \Omega \) with boundary values \( \varphi - h_1 \) and the zero density. Thanks to Theorem A, we see that \( h_2 \in C^{0,\alpha_2}(\overline{\Omega}) \) where \( \alpha_2 = \alpha_1/2 \). Therefore, the required barrier will be \( v = h_1 + h_2 \). It is clear that \( v \in PSH(\Omega) \cap C(\overline{\Omega}) \), \( v|_{\partial \Omega} = \varphi \) and \( (dd^c v)^n \geq f\beta^n \) in the weak sense in \( \Omega \). Hence, by the comparison principle we get that \( v \leq u \) in \( \Omega \) and \( v = u = \varphi \) on \( \partial \Omega \). Moreover we have \( v \in C^{0,\alpha}(\Omega) \) for any \( \alpha < 1/(nq + 1) \).

Finally, it is enough to set \( w = u(\Omega, -\varphi, 0) \) to obtain a super-barrier to the Dirichlet problem \( \text{Dir}(\Omega, \varphi, f) \). We note that \( w \in PSH(\Omega) \cap C(\overline{\Omega}) \), \( -w = \varphi \) on \( \partial \Omega \) and \( \Omega \leq -w \) on \( \Omega \). Furthermore, by Theorem A, \( w \in C^{0,1/2}(\Omega) \) and then \( w \in C^{0,\alpha}(\Omega) \) for any \( \alpha < 1/(nq + 1) \). \( \Box \)

When \( f \in L^p(\Omega) \) for \( p \geq 2 \), we are able to find a Hölder continuous barrier to the Dirichlet problem with more better Hölder exponent. The following theorem was proved in [Ch14] for the complex Hessian equation and it is enough here to put \( m = n \) for the complex Monge-Ampère equation.

Theorem 5.5. ([Ch14]). Let \( \varphi \in C^{0,1}(\partial \Omega) \) and \( 0 \leq f \in L^p(\Omega) \), \( p \geq 2 \). Then there exist \( v, w \in PSH(\Omega) \cap C^{0,1/2}(\overline{\Omega}) \) such that \( v = \varphi = -w \) on \( \partial \Omega \) and \( v \leq u \leq -w \) in \( \Omega \).

Now we recall the comparison principle for the total mass of laplacian of plurisubharmonic functions.

Lemma 5.6. Let \( u, v \in PSH(\Omega) \cap C(\overline{\Omega}) \) such that \( v \leq u \) on \( \Omega \) and \( u = v \) on \( \partial \Omega \). Then

\[
\int_{\Omega} dd^c u \wedge \beta^{n-1} \leq \int_{\Omega} dd^c v \wedge \beta^{n-1}.
\]

Proof. First assume that there exists an open set \( V \Subset \Omega \) such that \( u = v \) on \( \Omega \setminus V \). Let \( h \in PSH(\Omega) \cap C(\overline{\Omega}) \) such that \( h = 0 \) on \( \partial \Omega \). Then integration by parts yields

\[
\int_{\Omega} h dd^c (v - u) \wedge \beta^{n-1} = \int_{\Omega} (v - u) dd^c h \wedge \beta^{n-1}.
\]

Let \( V_1 \) be an open set such that \( V \Subset V_1 \Subset \Omega \) and define the function \( h = \max(-1, \rho/m) \) where \( \rho \) be the defining function of \( \Omega \) and \( m = |\sup_{\partial V_1} \rho| \). It is clear that \( h \in PSH(\Omega) \cap C(\overline{\Omega}) \), \( h = 0 \) on \( \partial \Omega \) and \( h = -1 \) on \( \partial V_1 \). Since \( u = v \) on \( \Omega \setminus V \), we get

\[
\int_{\Omega} dd^c (v - u) \wedge \beta^{n-1} = \int_{V_1} dd^c (v - u) \wedge \beta^{n-1}.
\]

We note that
Remark 5.7. It is shown in \cite{GKZ08} that we cannot expect a better Hölder exponent than \(2/nq\) (see also \cite{P105}).

\[
\begin{align*}
&\int_{\Omega} dd^c(v - u) \wedge \beta^{n-1} = - \int_{\Omega} hdd^c(v - u) \wedge \beta^{n-1} \\
&\quad = - \int_{\Omega} hdd^c(v - u) \wedge \beta^{n-1} \\
&\quad = - \int_{\Omega}(v - u)dd^c h \wedge \beta^{n-1} \geq 0.
\end{align*}
\]

Hence we obtain
\[
\int_{\Omega} dd^c u \wedge \beta^{n-1} \leq \int_{\Omega} dd^c v \wedge \beta^{n-1}.
\]

Now if we have only \(u = v\) on \(\partial \Omega\), then we define for small \(\epsilon > 0\), the function \(u_{\epsilon} := \max(u - \epsilon, v)\). Then we see that \(v \leq u_{\epsilon}\) on \(\Omega\) and \(u_{\epsilon} = v\) near the boundary of \(\Omega\).

Therefore, we have
\[
\int_{\Omega} dd^c u_{\epsilon} \wedge \beta^{n-1} \leq \int_{\Omega} dd^c v \wedge \beta^{n-1}.
\]

We know by the convergence’s theorem of Bedford and Taylor that \(dd^c u_{\epsilon} \beta^{n-1} \rightharpoonup dd^c u \wedge \beta^{n-1}\) when \(\epsilon \searrow 0\). Thus we have
\[
\int_{\Omega} dd^c u \wedge \beta^{n-1} \leq \int_{\Omega} dd^c v \wedge \beta^{n-1},
\]

which proves the required inequality.

\[
\begin{align*}
&\int_{\Omega} dd^c u_{\epsilon} \wedge \beta^{n-1} \leq \int_{\Omega} dd^c v \wedge \beta^{n-1}.
\end{align*}
\]

5.1. **Proof Theorem B.** Let \(U_0\) the solution to the Dirichlet problem \(Dir(\Omega, 0, f)\). We first claim that the total mass of \(\Delta U_0\) is finite in \(\Omega\). Indeed, let \(\rho\) be the defining function of \(\Omega\), then by Corollary 5.6 in \cite{Ce04} we get that
\[
(5.2) \quad \int_{\Omega} dd^c U_0 \wedge (dd^c \rho)^{n-1} \leq \left( \frac{\int_{\Omega}(dd^c U_0)^{n}}{c^{1/n}} \cdot \left( \frac{\int_{\Omega}(dd^c \rho)^{n}}{1/n} \right)^{(n-1)/n} \right)^{n}.
\]

Since \(\Omega\) is a bounded SHL domain, there exists a constant \(c > 0\) such that \(dd^c \rho \geq c\beta\) in \(\Omega\). Hence the inequality \(5.2\) yields
\[
\begin{align*}
&\int_{\Omega} dd^c u_{\epsilon} \wedge \beta^{n-1} \leq \int_{\Omega} dd^c v \wedge \beta^{n-1}.
\end{align*}
\]

Now we note that the total mass of complex Monge-Ampère measure of \(\rho\) is finite in \(\Omega\) by Chern-Levine-Nirenberg inequality since \(\rho\) is psh and bounded in a neighborhood of \(\Omega\) (see \cite{BT76}). Therefore, the total mass of \(\Delta U_0\) is finite in \(\Omega\).

Let \(\tilde{\varphi}\) be a \(C^{1,1}\)-extension of \(\varphi\) to \(\Omega\) such that \(\|\tilde{\varphi}\|_{C^{1,1}(\Omega)} \leq C\|\varphi\|_{C^{1,1}(\partial \Omega)}\) for some \(C > 0\).

Now, let \(v = A\rho + \tilde{\varphi} + U_0\) where \(A \gg 1\) such that \(A\rho + \tilde{\varphi} \in PSH(\Omega)\). By the comparison principle we see that \(v \leq U\) in \(\Omega\) and \(v = U = \varphi\) on \(\partial \Omega\). Since \(\rho\) is psh in a neighborhood of \(\Omega\) and \(\|\Delta U_0\|_{\Omega} < +\infty\), we get that \(\|\Delta v\|_{\Omega} < +\infty\). Then by Lemma 5.6 we have \(\|\Delta U\| < +\infty\).

The Proposition 5.4 gives the existence of Hölder continuous barriers to the Dirichlet problem. Then using Theorem 5.2 we obtain the final result that is when \(f \in L^p(\Omega)\) for some \(p > 1\), we get \(U \in PSH(\Omega) \cap C^{0,\alpha}(\Omega)\) where \(\alpha < 1/(nq + 1)\).

Moreover, if \(f \in L^p(\Omega)\) for some \(p \geq 2\), we can get better result. By Theorem 5.5 and Theorem 5.2 we see that \(U \in PSH(\Omega) \cap C^{0,\alpha}(\Omega)\) where \(\alpha < \min\{1/2, 2/(nq + 1)\}\).
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