Renormalization Group Flow in One- and Two-Matrix Models

SABURO HIGUCHI, CHIGAK ITOI, SHINSUKE NISHIGAKI and NORISUKE SAKAI

Abstract

Large-\(N\) renormalization group equations for one- and two-matrix models are derived. The exact renormalization group equation involving infinitely many induced interactions can be rewritten in a form that has a finite number of coupling constants by taking account of reparametrization identities. Despite the nonlinearity of the equation, the location of fixed points and the scaling exponents can be extracted from the equation. They agree with the spectrum of relevant operators in the exact solution. A linearized \(\beta\)-function approximates well the global phase structure which includes several nontrivial fixed points. The global renormalization group flow suggests a kind of \(c\)-theorem in two-dimensional quantum gravity.
1 Introduction

Two-dimensional quantum gravity is important to explore the nonperturbative dynamics of string models and statistical models of random surface as well as to provide a theoretical laboratory for quantum gravity in higher dimensions. The conformal field theory approach has successfully described two-dimensional quantum gravity and evaluated its universal quantities [1]. However, the continuum description has encountered a difficulty where quantum gravity is coupled to the conformal matter field with the central charge \( c > 1 \), despite its importance furnishing off-critical string models in an arbitrary dimensional target space.

As a discretized approach, the matrix model [2, 3] provides a more rigorous and powerful formulation of two-dimensional quantum gravity and off-critical string models. Exact solutions in matrix models have been obtained so far only for cases corresponding to matter theories with \( c \leq 1 \) coupled to quantum gravity [4]–[6]. It is straightforward to define matrix model candidates for quantum gravity coupled to conformal matter with \( c > 1 \), such as multi-Ising models [7]. However, it has been difficult to solve these theories exactly. Therefore it is most desirable to find a reliable approximation method which gives correct results for solved models and reasonable results for so far unsolved models.

Brézin and Zinn-Justin have proposed a large-\( N \) renormalization group (RG) equation as such an approximation method [8, 9]. Subsequently, we have explicitly derived exact RG equations for \( O(N) \)-vector models [10, 11] and one-matrix models [12]. We found that it is crucial to take account of the reparametrization identities in order to obtain a meaningful RG equation. Moreover, the RG equation turns out to be a nonlinear equation for the matrix model in contrast to the linear one for vector models.

The purpose of the present paper is to explain our large-\( N \) RG approach for matrix models in depth and to report on results for a two-matrix model. We obtain correct locations of several nontrivial fixed points and scaling exponents. With this result, we can identify the matrix model representation of the gravitationally dressed operators corresponding to those in the continuum theory. Moreover, we find that the large-\( N \) RG is useful to describe the global phase structure of matrix models. The global picture of the RG flow can be drawn practically by a linearized approximation for the nonlinear RG equation. It should be noted that the linearized approximation is meaningful only after the reparametrization identities are taken into account. We observe from this global flow that an RG trajectory connects a pair of fixed points corresponding to two unitary conformal matter theories coupled to gravity, in accordance with the expectation based on conformal field theories over fixed background [13]. This flow can be understood by means of the usual Kadomtsev-Petviashvili (KP) flow [14]–[16]. On the other hand, we also find that a pair of fixed points are connected by an RG trajectory which is out of the KP hierarchical description. From these examples, we conjecture that a kind of the gravitational analogue of Zamolodchikov’s \( c \)-theorem holds in accordance with some previous expectations [17] for several cases.

The partition function \( Z_N(g_j) \) of a one-matrix model with a general potential
$V(\phi)$ is defined by an integral over an $N \times N$ hermitian matrix $\phi$

$$Z_N(g_j) = \int d\phi \exp \left[ -N \text{tr} V(\phi) \right], \quad V(\phi) = \sum_{k \geq 1} \frac{g_k}{k} \phi^k,$$

(1.1)

In particular, we are interested in the subspace $\{ g_1 = 0, g_2 = 1, g_j \equiv 0 \text{ for } j > 2 \}$.

Let us define the free energy by

$$F(N, g_j) = -\frac{1}{N^2} \log \left[ \frac{Z_N(g_j)}{Z_N(g_2 = 1, \text{others} = 0)} \right]. \quad (1.2)$$

The free energy of the matrix model gives a generating function of connected Feynman diagrams constructed in terms of $j$-point vertices with the coupling $g_j$. By considering the dual of the Feynman diagrams, we can interpret that the dual diagram gives a decomposition of a two-dimensional surface in terms of $j$-gons corresponding to the $j$-point vertices $g_j$. Under this interpretation, the free energy of the matrix model is identified with the partition function of the two-dimensional quantum gravity. One finds that contributions from the different genera are distinguished by powers of $1/N^2$.

For brevity, we consider the triangulation $(g_3 = g, g_j \equiv 0 \text{ for } j > 3)$. The free energy $F^h(g)$ for genus $h$ surfaces is given by the coefficient of $N^{-2h}$ in the $1/N$-expansion, and the number $n$ of triangles is counted by the power of $g$

$$F(N, g) = \sum_{h=0}^{\infty} N^{-2h} F^h(g), \quad F^h(g) = \sum_n g^n \nu_h(n). \quad (1.3)$$

Here $\nu_h(n)$ is the number of Feynman diagrams with $n$ vertices and genus $h$ divided by the symmetry factors. The free energy $F^h(g)$ can be regarded as the partition function of the randomly triangulated surface with genus $h$. Its singular part behaves as

$$F^h_{\text{sing}}(g) \sim (g - g_*)^{-(\Gamma_h - 2)} + \text{less singular terms} \quad (1.4)$$

around a critical point $g_*$. The singular behavior (1.4) implies that the number of triangles $\nu_h(n)$ has an asymptotic form

$$\nu_h(n) \sim n^{\Gamma_h - 3} g_*^{-n} \quad (n \to \infty). \quad (1.5)$$

The average number of triangles $\langle n \rangle$ of surfaces with genus $h$ is estimated by (1.4)

$$\langle n \rangle = \frac{\sum_n n \, g^n \, \nu_h(n)}{\sum_n g^n \nu_h(n)} = g \frac{\partial}{\partial g} \log F^h_{\text{sing}}(g) \sim \frac{(\Gamma_h - 2) g_*}{g_* - g} \to \infty \quad (g \to g_*). \quad (1.6)$$

The average number of triangles in triangulations with any genus diverges simultaneously at $g = g_*$ with the same exponent $-1$.

In matrix models (and vector models [18]) solved so far, the susceptibility exponent $\Gamma_h$ depends linearly upon $h$ as

$$\Gamma_h = \gamma_0 + h \gamma_1. \quad (1.7)$$
Therefore we can take the double scaling limit
\[
\frac{1}{N} \to 0, \quad g \to g_*, \quad \text{with} \quad N^{-2/\gamma_1} (g - g_*)^{-1} = \text{fixed}, \quad (1.8)
\]
and retain nontrivial contributions from all genera to the free energy. In this limit the singular part of the free energy takes the form
\[
F_{\text{sing}}(N, g) = (g - g_*)^{2-\gamma_0} f(N^{2/\gamma_1} (g - g_*)), \quad (1.9)
\]
where \( f \) is an unspecified function.

Let us discuss the double scaling limit (1.8) from the viewpoint of the continuum limit in two-dimensional discretized quantum gravity. We consider the following continuum limit for each genus \( h \) by sending the length \( a \) of edges of triangles (spacing of the random lattice) to zero. If the average number of triangles \( \langle n \rangle \) diverges, we can scale the length to zero so as to keep the ‘physical’ area fixed:
\[
a \to 0, \quad \langle n \rangle \to \infty, \quad \text{with} \quad a^2 \cdot \langle n \rangle = \text{fixed}. \quad (1.10)
\]
Since the scaling behavior (1.6) shows that the average number of triangles diverges by letting \( g \) approach a critical value \( g_* \), we can regard the double scaling limit (1.8) as the continuum limit eq.(1.10) for each genus \( h \)
\[
N^{-2/\gamma_1} \cdot \langle n \rangle = \text{fixed} \iff a^2 \cdot \langle n \rangle = \text{fixed}. \quad (1.11)
\]
Therefore the double scaling limit suggests that \( N^{-1/\gamma_1} \) is proportional to the lattice spacing \( a \). A similar point of view has been expressed previously in refs.[5, 19].

Brézin and Zinn-Justin proposed the following approximation scheme for calculating the location and the exponents of critical points [9]. Integrating out parts of \((N + 1) \times (N + 1)\) components of the matrix field \( \phi \) in eq.(1.1) gives us the following recursion relation in the first nontrivial order of perturbation:
\[
Z_{N+1}(g) = \lambda(N, g) N^2 Z_N(g + \delta g(N, g)). \quad (1.12)
\]
One expects that eq.(1.12) is well approximated by a linear differential equation
\[
\left[ N \frac{\partial}{\partial N} - \beta(g) \frac{\partial}{\partial g} + \gamma(g) \right] F(N, g) = r(g). \quad (1.13)
\]
The homogeneous part (i.e. \( r(g) \) set to zero) of eq.(1.13) determines the singular part of \( F \) around a zero \( g = g_* \) (a fixed point) of the \( \beta \)-function as
\[
F_{\text{sing}}(N, g) = (g - g_*)^{(g_*)/\beta'(g_*)} f(N^{\beta'(g_*)} (g - g_*)). \quad (1.14)
\]
This singular behavior is just the same as that of the exact solutions (1.9) of the double scaled matrix model. Namely, the string susceptibility exponents are identified as
\[
\Gamma_h = \gamma_0 + h \gamma_1 = \left( 2 - \frac{\gamma(g_*)}{\beta'(g_*)} \right) + h \left( \frac{2}{\beta'(g_*)} \right). \quad (1.15)
\]
Therefore once one knows the functions $\beta(g)$ and $\gamma(g)$, one can compute positions and exponents of critical points. In fact, the RG equation computed in the one-loop approximation provides us with a fairly good result [3, 20]. However we have confirmed in ref. [11] that no improvement is achieved if one pursues the perturbation to higher-loops by simply truncating induced interactions of higher orders. We now understand that the naive perturbative method fails due to the reparametrization invariance of the partition function, as explained in sect.2.

Eq. (1.13) resembles RG equations in field theories in its form. Since eq. (1.11) shows that we can identify $N^{-1/\gamma_1}$ with the lattice spacing $a$, we can regard an infinitesimal shift of $N$ as a substitute for a shift of the lattice spacing $a$. Thus it may be appropriate to call a differential equation describing the response of the physical quantity under the infinitesimal change of $N$, such as eq. (1.13), an RG equation for matrix models.

This paper is organized as follows. In sect.2, we derive exact RG equations for one- and two-matrix models via coset and/or eigenvalue methods. In sect.3, we develop a scheme to solve the RG equations and show that they reproduce the spectrum of the exact solutions. In sect.4, we analyze the topology of the RG flow using the linear approximation to the RG equations, and discuss the gravitational $c$-theorem. Possible applications of our method, including generalizations to candidates for $c > 1$ quantum gravity, are addressed in sect.5. The reparametrization identities for the two-matrix model are solved in appendix A.

2 Large-\(N\) renormalization group equation

2.1 One-matrix model I: coset method

RG transformation

In this subsection, we restrict ourselves to the simplest case $m = 3$ and set $g \equiv g_3 \geq 0$ without loss of generality.

To derive a relation (1.12), we employ a decomposition of an $(N+1) \times (N+1)$ hermitian matrix variable $\Phi$ into an $N \times N$ hermitian matrix $\phi$, a complex $N$-vector $v$, its transposed conjugate $v^\dagger$ and a real scalar $\alpha$:

$$\Phi = \begin{pmatrix} \phi & v \\ v^\dagger & \alpha \end{pmatrix}. \quad (2.1)$$

In terms of $\phi, v, v^\dagger$ and $\alpha$, the partition function (1.1) reads

$$Z_{N+1}(g) = \int d\phi dv dv^\dagger d\alpha \exp \left[-(N+1)(\operatorname{tr} V(\phi) + V(\alpha) + v^\dagger(1 + g(\phi + \alpha \mathbf{1}))v)\right]. \quad (2.2)$$

First we integrate over $v$ and $v^\dagger$ exactly to obtain

$$Z_{N+1}(g) = \left(\frac{\pi}{N+1}\right)^N \int d\phi \exp[-(N+1)\operatorname{tr} V(\phi)] \cdot \int d\alpha \exp[-(N+1)V(\alpha) - \operatorname{tr} \log(1 + g(\phi + \alpha \mathbf{1}))]. \quad (2.3)$$
Now we are ready to evaluate the $\alpha$-integral by the saddle point method, systematically in each order of the $1/N$-expansion. The saddle point $\alpha_s = \alpha_s(g, \phi)$ is determined by the saddle point equation

$$\alpha_s + g\alpha_s^2 + \frac{g}{N} \text{tr} \frac{1}{1 + g(\phi + \alpha_s)} = 0 \quad (2.4)$$

as a $U(N)$-invariant function in $\phi$ (i.e. a function in $\text{tr} \phi^i$'s).

If we explicitly retain only the leading part of the $1/N$-expansion, we have

$$\frac{Z_{N+1}(g)}{Z_N(g)} = \left( \frac{\pi}{N + 1} \right)^N \langle \exp[-\text{tr} V(\phi) - NV(\alpha_s)] - \text{tr} \log(1 + g(\phi + \alpha_s)) + O(N^0) \rangle \quad (2.5)$$

Here $\langle \cdots \rangle$ denotes the average $\frac{1}{Z_N} \int d\phi \cdots \exp[-N \text{tr} V(\phi)]$. In the following we repeatedly utilize the factorization property of a multi-point function of $U(N)$-invariants $O, O'$ into a product of one-point functions,

$$\langle O O' \rangle = \langle O \rangle \langle O' \rangle + O(N^{-2}) \quad (2.6)$$

in the large-$N$ limit. In view of this factorization property, eq.(2.5) can further be written as

$$\frac{Z_{N+1}(g)}{Z_N(g)} = \left( \frac{\pi}{N + 1} \right)^N \exp[-\langle \text{tr} V(\phi) \rangle - NV(\langle \alpha_s \rangle)] - \langle \text{tr} \log(1 + g(\phi + \langle \alpha_s \rangle)) \rangle + O(N^0) \quad (2.7)$$

Finally we approximate a difference in eq.(2.7) by a differential to obtain

$$\left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g) = -\frac{1}{2} + \frac{1}{N} \langle \text{tr} V(\phi) \rangle + \langle \frac{1}{N} \text{tr} \log(1 + g(\phi + \langle \alpha_s \rangle)) \rangle + O\left( \frac{1}{N} \right) \quad (2.8)$$

Reparametrization invariance

The right hand side of eq.(2.8) is a function of variables $N, g$ and $\langle (1/N) \text{tr} \phi^j \rangle = j(\partial F/\partial g_j)$ ($j = 1, 2, \ldots$). What we will do here is to express the $\partial F/\partial g_j$-dependence ($j \neq 3$) in terms of $\partial F/\partial g_3$. This may be done by taking into account reparametrization invariance of the partition function (1.1). In this way, we can reduce all the induced interactions to those in the original potential.

For a while, we work on a generic potential (1.1). The partition function (1.1) is expressed as an integral and we can freely reparametrize the integration variable $\phi$ without changing the partition function. Obviously only reparametrizations which are analytic at the origin in the field space are allowed. If we perform reparametrizations $\phi' = \phi + c\phi^{n+1}$ ($n \geq -1$) for (1.1), the coupling constants receive shifts. This reparametrization invariance implies identities

$$\sum_{i=0}^{n} \langle \frac{1}{N} \text{tr} \phi' \frac{1}{N} \text{tr} \phi^{n-i} \rangle = \langle \frac{1}{N} \text{tr} \phi^{n+1} V'(\phi) \rangle = \sum_{k=1}^{m} g_k \langle \frac{1}{N} \text{tr} \phi^{n+k} \rangle \quad (n \geq -1). \quad (2.9)$$
We note that eq. (2.9) constitutes the first ($K = 0$) set of full reparametrization identities (discrete Schwinger-Dyson equations) \[16, 21\]

\[
\int d\phi \text{tr} \frac{d}{d\phi} \left\{ \phi^{n+1} \prod_{k=1}^{K} \text{tr} \phi^{n_k} \exp \left[ -N \text{tr} V(\phi) \right] \right\} = 0, \quad (n \geq -1). \tag{2.10}
\]

Only reparametrization identities in the large-$N$ limit will turn out to be necessary in our context. When we truncate the full set of identities (2.10) to those in the large-$N$ limit, eq. (2.9) exhausts all independent relations, due to the factorization property (2.6).

Eq. (2.9) can be neatly expressed by introducing the resolvent operator \[2\]

\[ \hat{W}(z) = \frac{1}{N} \text{tr} \frac{1}{z 1 - \phi}. \tag{2.11} \]

The expectation value \[\langle \hat{W}(z) \rangle\] is the generating function of one-point functions:

\[ \langle \hat{W}(z) \rangle = \sum_{j=0}^{\infty} z^{-1-j} \langle \frac{1}{N} \text{tr} \phi^j \rangle. \tag{2.12} \]

Using the resolvent, eq. (2.9) is summarized into a single equality for \[\langle \hat{W}(z) \rangle \] \[22\],

\[ \langle \hat{W}(z) \rangle^2 - V'(z) \langle \hat{W}(z) \rangle + Q(z) = 0, \tag{2.13} \]

\[ Q(z) \equiv \sum_{k=1}^{m-1} \frac{V^{(k+1)}(z)}{k!} \left\langle \frac{1}{N} \text{tr} (\phi - z)^{k-1} \right\rangle \]

\[ = \sum_{k=1}^{m-1} \frac{V^{(k+1)}(z)}{k!} \sum_{j=0}^{k-1} (-z)^{k-1-j} \frac{k-1}{j} \left\langle \frac{1}{N} \text{tr} \phi^j \right\rangle. \tag{2.14} \]

Eq. (2.13) is usually referred to as the loop equation. In the above, \[Q(z)\] consists of \[\langle \text{tr} \phi^k \rangle\] for \[1 \leq k \leq m - 2\]. For our purpose, one-point functions \[\langle \text{tr} \phi^k \rangle\] for \[3 \leq k \leq m\] are to be regarded as independent variables, corresponding to the coupling constants present in the original potential. Therefore a special treatment must be applied for \[\langle \text{tr} \phi \rangle\] and \[\langle \text{tr} \phi^2 \rangle\]; we should replace them in terms of \[\langle \text{tr} \phi^k \rangle\] for \[3 \leq k \leq m\] with the help of the first two equations of (2.9)

\[ 0 = \left\langle \frac{1}{N} \text{tr} \phi \right\rangle + \sum_{k=3}^{m} g_k \left\langle \frac{1}{N} \text{tr} \phi^{k-1} \right\rangle, \tag{2.15a} \]

\[ 1 = \left\langle \frac{1}{N} \text{tr} \phi^2 \right\rangle + \sum_{k=3}^{m} g_k \left\langle \frac{1}{N} \text{tr} \phi^k \right\rangle. \tag{2.15b} \]

Now let us denote the one-point function of a single trace as \[a_j = (1/j) \left\langle (1/N) \text{tr} \phi^j \right\rangle\], and that of the resolvent with the above replacement as \[W\]. It satisfies

\[ W(z; g_j; a_j)^2 - V'(z) W(z; g_j; a_j) + Q(z; g_j; a_j) = 0, \tag{2.16} \]

with \[Q(z; g_j; a_j)\] given by

\[ Q(z; g_3, \ldots, g_m; a_3, \ldots, a_m) \]
\begin{align}
&= \sum_{k=1}^{m-1} V^{(k+1)}(z)(-z)^{k-1} \frac{1}{k!} \\
&+ \sum_{k=2}^{m-1} V^{(k+1)}(z)(-z)^{k-1-1} \frac{k-1}{k!} (-1)^{m} \left( \sum_{\ell=4}^{m}(\ell-1)ga_{\ell-1} + g_{3} \left(1 - \sum_{\ell=3}^{m} \ell ga_{\ell}\right) \right) \\
&+ \sum_{k=3}^{m-1} V^{(k+1)}(z)(-z)^{k-1-2} \frac{(k-1)(k-2)}{k!} \left(1 - \sum_{\ell=3}^{m} \ell ga_{\ell}\right) \\
&+ \sum_{j=3}^{m-2} \sum_{k=j+1}^{m-1} V^{(k+1)}(z)(-z)^{k-j-1} \frac{1}{k!} \left( k - 1 \right) ja_{j}. \tag{2.17}
\end{align}

The quadratic nature of the loop equation (2.16) is a peculiarity of the one-matrix models, as compared to the multi-matrix models. For the case \( m = 3 \), the explicit form of \( W(z, g, a \equiv a_{3}) \) reads

\begin{equation}
W(z, g, a) = \frac{1}{2} \left( z + gz^{2} - \sqrt{(z + gz^{2})^{2} - 4(1 + gz - g^{2} + 3g^{3}a)} \right). \tag{2.18}
\end{equation}

Note that, by definition (2.11), only a solution with the large-\( z \) asymptotic behavior \( W(z) \sim 1/z \) is allowed. Moreover, the eigenvalue density distribution \( \rho(z) \), related to \( W(z) \) by \( \rho(z) = -1/\pi \text{Im}W(z) \), should satisfy the trivial identity

\begin{equation}
\int \rho(z)dz = 1. \tag{2.19}
\end{equation}

We have observed that these two requirements are enough to determine \( W \) without ambiguity for the present case, as well as in the case of the two-matrix model.

Another simplification occurs for the averaged form of the saddle point equation (2.4)

\begin{equation}
\langle \alpha_{s} \rangle + g \langle \alpha_{s} \rangle^{2} - W \left( -\frac{1}{g} - \langle \alpha_{s} \rangle, g, a \right) = 0. \tag{2.20}
\end{equation}

By combining it with the loop equation (2.16), we explicitly obtain \( \langle \alpha_{s} \rangle \) in the form

\begin{equation}
\langle \alpha_{s} \rangle = \left\langle \frac{1}{N} \text{tr} \phi \right\rangle = -g + 3g^{2} a \equiv \bar{\alpha}(g, a). \tag{2.21}
\end{equation}

This is in fact an expected result, because \( \langle \alpha_{s} \rangle \equiv \langle \Phi_{N+1,N+1} \rangle \) should be equal to \( \left\langle \frac{1}{N} \text{tr} \phi \right\rangle \equiv \langle (1/N) \sum_{N}^{N} \Phi_{N+1,N+1} \rangle \). This fact helps computation in the coset decomposition method considerably, for the case of two-matrix model as well.

Using eqs. (2.8), (2.18) and (2.21), we obtain a differential equation obeyed by the free energy \( F(N, g) \) of the one-matrix model with cubic coupling,

\begin{equation}
\left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g) = G \left( g, \frac{\partial F}{\partial g} \right) + O \left( \frac{1}{N} \right), \tag{2.22}
\end{equation}

\begin{align*}
G(g, a) &= -\frac{g}{2} a + \frac{1}{2} \bar{\alpha}(g, a)^{2} + \frac{g}{3} \bar{\alpha}(g, a)^{3} + \log \left( 1 + g\bar{\alpha}(g, a) \right) \\
&+ \int_{-\infty}^{-1/g-\bar{\alpha}(g,a)} dz \left( W(z, g, a) - \frac{1}{z} \right).
\end{align*}
Since the above equation describes the response under the shift \(N \rightarrow N + 1\) it deserves the name of RG equation, although it is (non-polynomially) nonlinear unlike the form anticipated in eq. (1.13).

**Comment on the nonlinearity**

One might wonder if we can keep the equation linear at the cost of having an infinite dimensional coupling constant space. We could generalize the partition function to include interactions such as \(\text{tr} \phi^{n_1} \text{tr} \phi^{n_2} \cdots\),

\[
Z_N(g_{n_1n_2\ldots}) = \int d\phi \exp \left[ -NS(\phi) \right],
\]

\[
S(\phi) = \sum_{k=1}^{\infty} \sum_{(n_1, n_2, \ldots, n_k)} g_{n_1n_2\ldots n_k} N^{1-k} \prod_{i=1}^{k} n_i^{-1} \text{tr} \phi^{n_i}.
\]

and enlarge the coupling constant space to \(\{ (g_1, g_2, \ldots, g_{i1}, g_{i2}, \ldots, g_{n_1n_2\ldots n_k}, \ldots) \}\). Then we would expand the right hand side of eq. (2.8) in terms of \(\langle \prod_{i=1}^{k} \frac{1}{N} \text{tr} \phi^{n_i} \rangle \mid_{(g_i) = (0,1,g,0,0,\ldots)} = \left( \prod_{i=1}^{k} n_i \right) : \frac{\partial F}{\partial g_{n_1\ldots n_k}} (N,0,1,g,0,0,\ldots) \) (2.24)

to obtain a linear ‘RG equation’ of the form

\[
\left[ N \frac{\partial}{\partial N} + 2 \right] F(N,0,1,g,0,0,\ldots) = \sum_{(n_1,\ldots,n_k)} \beta_{n_1\ldots n_k}(g) \frac{\partial F}{\partial g_{n_1\ldots n_k}} (N,0,1,g,0,0,\ldots).
\]

(2.25)

It is straightforward to check that there exists no simultaneous zero for each function \(\beta_{n_1\ldots n_k}(g)\). This contradicts the exact solution of the one-matrix model, which has a nontrivial fixed point at \(g_1 = 0, g_2 = 1, g_3 = 432^{-1/4}, g_j \equiv 0 \ (j \geq 4)\). The obvious reason for this apparent contradiction is that the \(\left( \frac{\partial F}{\partial g_{n_1\ldots n_k}} \right) \mid_{(g_j) = (0,1,g,0,0,\ldots)}\) are not independent quantities in the large-\(N\) limit. Rather, they are mutually dependent through an infinite number of relations (2.6) and (2.9) and are therefore ‘\(\beta\)-functions’ for \(g_{n_1\ldots n_k}\). Accordingly we are forced to take the reparametrization identities into account to exclude this ambiguity of \(\beta\)-functions.

### 2.2 One-matrix model II: eigenvalue method

The RG transformation by means of the coset decomposition is in principle applicable for an arbitrary potential as well. We can eliminate terms such as \((v \dagger v)^2\), which are present in the quartic potential case, by introducing auxiliary fields. However, for the model which admits an eigenvalue representation via \(U(N)\)-integration we have a more convenient RG transformation: integration over the \((N + 1)\)-th eigenvalue, the extra degree of freedom. This procedure is particularly useful for one-matrix models with a generic potential \((m \geq 4)\), since it can be applied to these models on the same footing.

We can integrate over the \(U(N)\) variables in (1.1) to obtain an integral over the
eigenvalues \{\lambda_j\} \[2\]

\[ Z_N(g_j) = c_N \int \prod_{i=1}^N d\lambda_i \Delta_N(\lambda_i)^2 \exp \left[ -N \sum_{i=1}^N V(\lambda_i) \right], \] (2.26)

where \(\Delta_N\) denotes the Van der Monde determinant \(\Delta_N(\lambda_j) = \prod_{1 \leq i < j \leq N}(\lambda_i - \lambda_j)\) and \(c_N\) the volume of the \(U(N)\) group \(c_N = \pi^{N(N-1)/2} / \prod_{p=1}^N p!\).

In order to relate \(Z_{N+1}\) with \(Z_N\), we shall integrate over the \((N+1)\)-th eigenvalue \(\lambda \equiv \lambda_{N+1}\) in \(Z_{N+1}\)

\[ Z_{N+1}(g_j) = \int d\Phi \exp [-(N+1) \text{tr} V(\Phi)] \] (2.27)

\[ = c_{N+1} \int \prod_{i=1}^N d\lambda_i \Delta_N^2(\lambda_j) \exp(-(N+1)\sum_{i=1}^N V(\lambda_i)) \int d\lambda \prod_{i=1}^N |\lambda - \lambda_i|^2 e^{-(N+1)V(\lambda)} \]

\[ = \frac{c_{N+1}}{c_N} \int d\phi e^{-(N+1)\text{tr} V(\phi)} \int d\lambda \exp [-(N+1)V(\lambda) + 2 \text{tr} \log |\lambda 1 - \phi|]. \]

The \(\lambda\)-integral can be evaluated by the saddle point method as a power series in \(1/N\) around the saddle point, since the effective potential \((N+1)V(\lambda) - 2 \text{tr} \log |\lambda 1 - \phi|\) is of order \(O(N^1)\). The saddle point equation

\[ V'(\lambda_s) = \frac{2}{N} \text{tr} \frac{1}{\lambda_s 1 - \phi} = 2 \sum_{n=0}^\infty \lambda_s^{-n-1} \frac{1}{N} \text{tr} \phi^n \] (2.28)

implicitly determines the saddle point \(\lambda_s = \lambda_s(g_j, \phi)\) as a \(U(N)\)-invariant function in \(\phi\). By substituting \(\lambda_s\) into the partition function, we find

\[ \frac{Z_{N+1}(g_j)}{Z_N(g_j)} = \frac{c_{N+1}}{c_N} \left\langle \exp \left[ -\text{tr} V(\phi) - NV(\lambda_s) + 2 \text{tr} \log |\lambda_s 1 - \phi| + O(N^0) \right] \right\rangle \] (2.29)

\[ = \frac{c_{N+1}}{c_N} \exp \left[ -\left\langle \text{tr} V(\phi) \right\rangle - NV(\langle \lambda_s \rangle) + 2 \left\langle \text{tr} \log |\langle \lambda_s \rangle 1 - \phi| \right\rangle + O(N^0) \right]. \]

Here the factorization property (2.6) is used in the second line, and \(\langle \lambda_s \rangle \equiv \langle \lambda_s(g_j, \phi) \rangle\) is determined again using the resolvent (2.11)

\[ V'(\langle \lambda_s \rangle) = 2 \left\langle \frac{1}{N} \text{tr} \frac{1}{\lambda_s 1 - \phi} \right\rangle = 2 \left\langle \hat{W}(\langle \lambda_s \rangle) \right\rangle. \] (2.30)

By taking the large-\(N\) limit, we get the following differential equation as an RG equation for the free energy,

\[ \left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g_j) = -\frac{3}{2} + \left\langle \frac{1}{N} \text{tr} V(\phi) \right\rangle + V(\langle \lambda_s \rangle) - 2 \left\langle \frac{1}{N} \text{tr} \log |\langle \lambda_s \rangle 1 - \phi| \right\rangle + O\left(\frac{1}{N}\right). \] (2.31)

As in the case of the coset method, we can express the right hand side of eq. (2.31) in terms of \(g_j\) and \(a_j = \partial F/\partial g_j\) for \(j = 3, \cdots, m\) with the help of reparametrization identities. It is straightforward to express \(\langle \lambda_s \rangle\) in terms of these quantities, since the saddle point equation (2.30) readily takes the form \(V'(\langle \lambda_s \rangle) = 2W(\langle \lambda_s \rangle; g_j; a_j)\). By
combining it with the loop equation (2.16), we find an equivalent and more useful expression

\[ V'(⟨Λ⟩)^2 - 4Q(⟨Λ⟩; g_j; a_j) = 0. \]  

(2.32)

Eq. (2.32) shows that ⟨Λ⟩ falls on one of the edges of the eigenvalue distribution ρ(λ), because the left hand side is exactly the discriminant of the loop equation (2.16). This fact can be understood as follows. We adopt the picture that the N + 1 eigenvalues, confined by the potential V(λ), interact with each other via a repulsive Coulomb potential log |λ_i − λ_j|. Consider the effective potential for the (N + 1)-th eigenvalue, which is generated by other N eigenvalues obeying the distribution function ρ(λ). If the (N + 1)-th eigenvalue falls on a point ρ(λ̃) ≠ 0, it costs an energy loss (of order N−1 log N compared to the total energy) from the Coulomb interaction. On the other hand, if the total action is to be minimized when the (N + 1)-th eigenvalue is placed on an isolated point outside the support of ρ(λ), the original distribution is unlikely to minimize the action for N eigenvalues. Consequently the (N + 1)-th eigenvalue should fall just on one of the edges of ρ(λ). Moreover, the true saddle point should converge in the $g \to 0$ limit to 2(Wigner distribution), and among such roots it should minimize the action $S(λ) \equiv V(λ) - ⟨(2/N)tr log |A_1 - φ|⟩$. We can always select the true saddle point out of the $2m - 2$ roots of eq.(2.32) by imposing these two requirements, and denote it as $\bar{λ} = \bar{λ}(g_i; a_i)$.

In this way, we can express the right hand side of eq.(2.31) in terms of $g_i$ and $\partial F/\partial g_i (i = 3, \ldots, m)$,

\[
\left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g_j) = G_{\text{eig}} \left( g_3, \ldots, g_m; \frac{\partial F}{\partial g_3}, \ldots, \frac{\partial F}{\partial g_m}\right) + O \left( \frac{1}{N} \right),
\]

\[
G_{\text{eig}}(g_3, \ldots, g_m; a_3, \ldots, a_m) \equiv -1 - \sum_{k=3}^{m} \frac{k - 2}{2} g_k a_k + V(\bar{λ}) - 2 log \bar{λ}
\]

\[
-2 \int_{\pm \infty} d\lambda \left( W(\lambda; g_3, \ldots, g_m; a_3, \ldots, a_m) - \frac{1}{\lambda} \right).
\]

The complete set of nonlinear RG equations for the one-matrix model consists of the above equations (2.33), (2.16), and (2.32). So far we have much numerical evidence for the equality of the function $G$ in the coset method and the function $G_{\text{eig}}$ in the eigenvalue method, but have not been successful in showing this analytically.

### 2.3 Two-matrix model

We consider a two-matrix model defined by

\[
Z_N(g_+, g_-, c) = \int dφ_+ dφ_- \exp \left[ -N \text{tr} (V(φ_+, g_+) + V(φ_-, g_-) + cφ_+φ_-) \right],
\]

\[
V(x, g) = \frac{1}{2} x^2 + \frac{g}{3} x^3,
\]

(2.34)

where $φ_+$ are $N \times N$ hermitian matrices. The free energy

\[
F(N, g_+, g_-, c) = -\frac{1}{N^2} \log \frac{Z_N(g_+, g_-, c)}{Z_N(0, 0, 0)}
\]

(2.35)
represents the partition function of the Ising model on random triangulated surfaces [23], and the Ising temperature $T$ is proportional to $-\log^{-1}(-c)$. This model is also known to allow an eigenvalue representation [24]

$$Z_N(g_+, g_-, c) = \frac{(-c\pi^2)^{N(N-1)/2}}{\prod_i^{N} p!} \int \prod_{i=1}^{N} d\lambda_+ id\lambda_- i \Delta_N(\lambda_+) \Delta_N(\lambda_-) \exp \left[ -N \sum_{j=1}^{N} (V(\lambda_+, g_+) + V(\lambda_-, g_+) + c\lambda_+ \lambda_-) \right]. \quad (2.36)$$

However, the integrand is an alternating function in $\lambda_{\pm,i}$, since the Van der Monde determinants is not squared but bi-linear in the above expression. Therefore it is inappropriate to evaluate the $\lambda_{\pm,N+1}$-integral by the saddle point method. This argument also applies to multi-matrix models with a chain-like interaction $c \sum_{a=1}^{g-1} tr \phi_a \phi_{a+1}$. For this reason we present only the coset method in the following.

**RG transformation**

We decompose $(N + 1) \times (N + 1)$ hermitian matrices $\Phi_{\pm}$ into

$$\Phi_{\pm} = \begin{pmatrix} \phi_{\pm} & v_{\pm} \\ v_{\pm}^\dagger & \alpha_{\pm} \end{pmatrix} \quad (2.37)$$

as with the case of one-matrix models. In terms of these variables, the partition function [23] reads

$$Z_{N+1}(g_+, g_-, c) = \int d\phi_+ d\phi_- dv_+ dv_+^\dagger dv_- dv_-^\dagger d\alpha_+ d\alpha_- \exp \left[ -(N + 1) tr (V(\phi_+, g_+) + V(\phi_-, g_-) + c\phi_+ \phi_-) \right] \exp \left[ -(N + 1) (V(\alpha_+, g_+) + V(\alpha_-, g_-) + c\alpha_+ \alpha_-) \right] \exp \left[ -(N + 1) \cdot (v_+^\dagger v_-) \left( \begin{array}{cc} 1 + g_+(\phi_+ + \alpha_+ \mathbf{1}) & c \mathbf{1} \\ c \mathbf{1} & 1 + g_-(\phi_- + \alpha_- \mathbf{1}) \end{array} \right) \left( \begin{array}{c} v_+ \\ v_- \end{array} \right) \right]. \quad (2.38)$$

The part of the action quadratic in $(v_+, v_-)$ and $(v_+^\dagger, v_-^\dagger)$ is presented in the form of a $2N \times 2N$ matrix. We perform the integration over $(v_+, v_-)$-field to obtain

$$Z_{N+1}(g_+, g_-, c) = \left( \frac{\pi}{N + 1} \right)^{2N} \int d\phi_+ d\phi_- \exp \left[ -(N + 1) tr (V(\phi_+, g_+) + V(\phi_-, g_-) + c\phi_+ \phi_-) \right] \exp \left[ -(N + 1) (V(\alpha_+, g_+) + V(\alpha_-, g_-) + c\alpha_+ \alpha_-) \right] \exp \left[ -\text{Tr} \log \left( \begin{array}{cc} 1 + g_+(\phi_+ + \alpha_+ \mathbf{1}) & c \mathbf{1} \\ c \mathbf{1} & 1 + g_-(\phi_- + \alpha_- \mathbf{1}) \end{array} \right) \right]. \quad (2.39)$$

The trace $\text{Tr}$ is taken over a $2N \times 2N$ matrix, whereas $\text{tr}$ is over an $N \times N$ matrix. Again we employ the saddle point method for the $\alpha_{\pm}$-integration. The saddle point
\( \alpha_{s,\pm} = \alpha_{s,\pm}(g_+, g_-, \phi_+, \phi_-) \) is determined by the equations

\[
\alpha_{s,\pm} + c \alpha_{s,\pm} + g_+ \alpha_{s,\pm}^2 + g_- \frac{1}{N} \text{Tr}_\pm \left( \begin{array}{cc} 1 + g_+(\phi_+ + \alpha_{s,\pm}) & c1 \\ c1 & 1 + g_-(\phi_- + \alpha_{s,\pm}) \end{array} \right)^{-1} = 0,
\]

(2.40)

where

\[
\text{Tr}_\pm M \equiv \text{Tr}(P_\pm M), \quad P_+ = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \quad P_- = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right).
\]

(2.41)

As in the case of the one-matrix model, we employ the factorization property (2.4) and approximate \( Z_{N+1}/Z_N \) by a differential to obtain

\[
\left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g_+, g_-, c)
\]

\[= -1 + \left( \frac{1}{N} \text{tr} V(\phi_+, g_+) \right) + \left( \frac{1}{N} \text{tr} V(\phi_-, g_-) \right) + c \left( \frac{1}{N} \text{tr} \phi_+ \phi_- \right)
\]

\[+ V(\langle \alpha_{s,\pm} \rangle, g_+) + V(\langle \alpha_{s,\pm} \rangle, g_-) + c \langle \alpha_{s,\pm} \rangle \langle \alpha_{s,\pm} \rangle
\]

\[+ \left( \frac{1}{N} \text{Tr} \log \left( \begin{array}{cc} 1 + g_+ (\phi_+ + \langle \alpha_{s,\pm} \rangle) & c1 \\ c1 & 1 + g_-(\phi_- + \langle \alpha_{s,\pm} \rangle) \end{array} \right) \right) \right) + O \left( \frac{1}{N} \right),
\]

where

\[
\langle \cdots \rangle = Z_N^{-1} \int d\phi_+ d\phi_- (\cdots) \exp[-N \text{tr} (V(\phi_+, g_+) + V(\phi_-, g_-) + c\phi_+ \phi_-)].
\]

(2.43)

Reparametrization invariance

In what follows we concentrate on the case \( g_+ = g_- \equiv g \) which describes the Ising model coupled to two-dimensional quantum gravity in the absence of an external magnetic field. In addition, we assume that

\[
\langle \text{tr} \phi_+^i \rangle = \langle \text{tr} \phi_-^i \rangle
\]

(2.44)

holds for any \( i \). When we trace on a part \( (T \leq T_c) \) of the critical line, this assumption amounts to defining the free energy as the sum of contributions from two spontaneously magnetized states. A direct consequence of this assumption is

\[
\langle \alpha_{s,\pm} \rangle = \langle \alpha_{s,\pm} \rangle \equiv \langle \alpha_s \rangle.
\]

(2.45)

Under these conditions, the saddle point equation for \( \langle \alpha_s \rangle \) reduces to the form

\[
\langle \alpha_s \rangle + c \langle \alpha_s \rangle + g \langle \alpha_s \rangle^2 + \left( \frac{1}{2N} \text{Tr} \frac{1}{(1/g + \langle \alpha_s \rangle)\sigma_0 + (c/g)\sigma_1 + \phi} \right) = 0,
\]

(2.46)

where

\[
\phi = \left( \begin{array}{cc} \phi_+ & 0 \\ 0 & \phi_- \end{array} \right), \quad \sigma_0 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad \sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right).
\]

(2.47)

The differential equation (2.42) simplifies into

\[
\left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g, c)
\]

\[= -1 + \left( \frac{1}{N} \text{tr} V(\phi_+, g) \right) + \left( \frac{1}{N} \text{tr} V(\phi_-, g) \right) + c \left( \frac{1}{N} \text{tr} \phi_+ \phi_- \right)
\]

\[+ 2V(\langle \alpha_s \rangle, g) + c \langle \alpha_s \rangle^2 + \left( \frac{1}{N} \text{Tr} \log \left( \frac{1}{g + \langle \alpha_s \rangle} \sigma_0 + \frac{c}{g} \sigma_1 + \phi \right) \right) \right) + O \left( \frac{1}{N} \right).
\]

(2.48)
The right hand side of eq. (2.48) consists of terms of the form \( \text{tr} (\phi_n^1 \phi_m^1 \phi_n^2 \phi_m^2 \cdots) \). We observe that all these induced interactions can be reduced solely to the interactions present in the original potential, i.e. \( \text{tr} (\phi_3^3 + \phi_3^3) \) and \( \text{tr} \phi_+ \phi_- \), by recursively using the reparametrization identities

\[
\int d\phi_+ d\phi_- \text{ tr} \left\{ \phi_n^1 \phi_m^1 \phi_n^2 \phi_m^2 \cdots \right\} \cdot \exp \left\{ -N \text{ tr} (V(\phi_+, g) + V(\phi_-, g) + c \phi_+ \phi_) \right\} = 0. (2.49)
\]

Again this procedure can be performed at once by introducing the resolvent operator; in fact we can find a closed set of loop equations (A.2) that expresses the one-point function of the resolvent of the required form (see the appendix A)

\[
W_0(z; g, c; a_g, a_c) = \left\langle \frac{1}{N} \text{Tr} \frac{1}{z \sigma_0 + (c/g) \sigma_1 + \phi} \right\rangle \quad (2.50)
\]

in terms of \( a_g = \left\langle (1/3N) \text{ tr} (\phi_3^3 + \phi_3^3) \right\rangle \) and \( a_c = \left\langle (1/N) \text{ tr} \phi_+ \phi_- \right\rangle \).

Furthermore, by combining the saddle point equation (2.46), or

\[
(1 + c) \left\langle \alpha_s \right\rangle + g \left\langle \alpha_s \right\rangle^2 + \frac{1}{2} W_0 \left( \frac{1}{g} + \left\langle \alpha_s \right\rangle ; g, c, a_g, a_c \right) = 0 \quad (2.51)
\]

and the loop equation \( (A.4) \) we can prove

\[
\left\langle \alpha_s \right\rangle = \left\langle \frac{1}{N} \text{ tr} \phi_\pm \right\rangle = \frac{g}{1 + c} \left( -1 + c a_c + \frac{3g}{2} a_g \right) \equiv \bar{\alpha}(g, c, a_g, a_c) \quad (2.52)
\]

as expected. These three equations (2.48), (A.4) and (2.52) constitute the nonlinear RG equation for the two-matrix model

\[
\left[ N \frac{\partial}{\partial N} + 2 \right] F = G \left( g, c; \frac{\partial F}{\partial g}, \frac{\partial F}{\partial c} \right) + O \left( \frac{1}{N} \right), \quad (2.53)
\]

\[
G(g, c; a_g, a_c) \equiv \frac{g}{2} a_g + (1 + c) \bar{\alpha}^2 + \frac{2}{3} g \bar{\alpha}^3 + 2 \log(1 + g \bar{\alpha}) \\
+ \int_{-\infty}^{1/g + \bar{\alpha}} dz \left( W_0(z; g, c; a_g, a_c) - \frac{2}{z} \right).
\]

3 Fixed points and operator contents

3.1 Solution to the nonlinear RG equation

In the previous section we have derived nonlinear differential equations, governing the critical behavior of the free energy \( F(N, g) \), of the form

\[
\left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g) = G \left( g, \frac{\partial F(N, g)}{\partial g} \right) + O \left( \frac{1}{N} \right) \quad (3.1)
\]
(or its multi-variable extension) instead of the linear one \((1.13)\). To complete the program we now need a formula which relates positions and exponents of critical points to \(g\), just as in the linear case eq.\((1.13)\).

**Sphere limit**

First we concentrate on the leading part \(F^0(g)\) of the free energy in the \(1/N\)-expansion. It is easy to see that \(F^0\) satisfies

\[
2F^0(g) = G\left( g, \frac{\partial F^0(g)}{\partial g} \right)
\]

(3.2)

We assume that \(F^0\) consists of analytic and non-analytic parts around a critical point \(g_*\),

\[
F^0(g) = \sum_{k=0}^{\infty} a_k (g - g_*)^k + \sum_{k=0}^{\infty} b_k (g - g_*)^{k+2-\gamma_0} \quad (\gamma_0 \notin \mathbb{Z}).
\]

(3.3)

Quantities \(a_k, b_k, \gamma_0, g_*\) are unknown and are to be determined. In anticipation of the result, we have used the same notation as the susceptibility exponent \(\gamma_0\) in eqs.\((1.4)\) and \((1.7)\). We expand the function \(G(g, a)\) around \((g_*, a_1)\) as follows

\[
G(g, a) = \sum_{n=0}^{\infty} \beta_n(g)(a - a_1)^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \beta_{nk} \cdot (g - g_*)^k(a - a_1)^n.
\]

(3.4)

We substitute these expressions into the nonlinear RG equation \((3.2)\) and compare the coefficients of various powers of \((g - g_*)\) on both sides. Since \(b_0 \neq 0\) by definition, the most singular term \((g - g_*)^{1-\gamma_0}\) determines the fixed point

\[
0 = \beta_{10}(2 - \gamma_0)b_0.
\]

(3.5a)

The next singular term \((g - g_*)^{2-\gamma_0}\) determines the critical exponent \(\gamma_0\)

\[
2b_0 = (2 - \gamma_0)b_0(\beta_{11} + \beta_{20} \cdot 4a_2).
\]

(3.5b)

So far there appear two unknown coefficients \(a_1\) and \(a_2\) from the analytic part. However, by comparing terms \((g - g_*)\) and \((g - g_*)^2\), we can fix these quantities

\[
2a_1 = \beta_{01},
\]

(3.5c)

\[
2a_2 = \beta_{02} + \beta_{11}2a_2 + \beta_{20}(2a_2)^2.
\]

(3.5d)

These four equations can be rewritten in terms of the function \(G(g, a)\):

\[
0 = G_{g,a}(g_*, a_1),
\]

(3.6a)

\[
\frac{2}{2 - \gamma_0} = G_{g,g,a}(g_*, a_1) + 2a_2G_{a,a,a}(g_*, a_1),
\]

(3.6b)

\[
2a_1 = G_{g,a}(g_*, a_1),
\]

(3.6c)

\[
2a_2 = \frac{1}{2}G_{g,g,a}(g_*, a_1) + 2a_2G_{g,a,a}(g_*, a_1) + 2(a_2)^2G_{a,a,a}(g_*, a_1).
\]

(3.6d)
Obviously eqs. (3.6a) and (3.6b) are the nonlinear counterparts of the usual equations determining fixed points and scaling exponents,

\[ 0 = \beta(g_*) \quad \text{and} \quad 2/(2 - \gamma_0) = \beta'(g_*) \]

respectively. We can solve the above four equations simultaneously to fix four unknowns, the critical coupling \( g_* \), critical exponent \( \gamma_0 \), and coefficients \( a_1, a_2 \). Moreover, the other consistency conditions determine all the other coefficients at each fixed point, \( a_k \) \((k \geq 0)\) and \( b_k/b_0 \) \((k \geq 1)\) recursively except for the overall normalization of the singular term \( b_0 \). Although the equation for \( a_2 \) is quadratic, we can choose the correct branch, that which continues to the unique solution at the origin (the Gaussian fixed point). In this way we can obtain the series expansion of the sphere free energy around the fixed point, up to one integration constant.

It is straightforward to generalize our algorithm to the case of multi-coupling constants. We consider a differential equation

\[
\left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g_j) = G \left( g_j; \frac{\partial F}{\partial g_j} \right) + O \left( \frac{1}{N} \right).
\]

(3.8)

We assume that the leading part \( F^0(N, g_j) \) of the \( 1/N \)-expansion of a solution can be expanded in the form

\[
F^0(N, g_j) = \sum_{n=0}^{\infty} a_{j_1 \cdots j_n} \delta g_{j_1} \cdots \delta g_{j_n} + \sum_{i} (V_{ij} \delta g_j)^{2-\gamma_0(i)} \sum_{n=0}^{\infty} b_{(i)j_1 \cdots j_n} \delta g_{j_1} \cdots \delta g_{j_n},
\]

(3.9)

where the repeated indices are summed over. A transformation matrix \( V_{ij} \) is introduced to diagonalize the scaling matrix. Then we have a set of equations to determine \( g_* \), \( \gamma_0(i) \), \( a_i \) and \( a_{ij} \)

\[
0 = \frac{\partial G}{\partial a_i}, \quad (3.10a)
\]

\[
\sum_k (V^{-1})_{ik} \frac{2}{2 - \gamma_0(k)} V_{kj} = \frac{\partial^2 G}{\partial g_j \partial a_i} + 2a_{jk} \frac{\partial^2 G}{\partial a_i \partial a_k} \equiv G_{ij}, \quad (3.10b)
\]

\[
2a_i = \frac{\partial G}{\partial g_i}, \quad (3.10c)
\]

\[
2a_{ij} = \frac{1}{2} \frac{\partial^2 G}{\partial g_i \partial g_j} + a_{ik} \frac{\partial^2 G}{\partial g_j \partial a_k} + a_{jk} \frac{\partial^2 G}{\partial g_i \partial a_k} + 2a_{ik} a_{j\ell} \frac{\partial^2 G}{\partial a_k \partial a_\ell}, \quad (3.10d)
\]

where they stem from terms of order \( O \left( (V_{ij} \delta g_j)^{1-\gamma_0(i)} \right), O \left( (V_{ik} \delta g_k)^{1-\gamma_0(i)} (V_{jk} \delta g_k) \right), O(\delta g_i) \) and \( O(\delta g_i \delta g_j) \), respectively. We refer to the right hand side \( G_{ij} \) of eq. (3.10b) as the ‘scaling exponent matrix’.

**Exceptional cases**

So far we have implicitly assumed that the exponent \( \gamma_0 \) is irrational. In most cases this assumption is incorrect and higher powers of \( F_{\text{sing}} \) appearing in the right hand side of eq. (3.2) contribute to the analytic part. Therefore eqs. (3.6c–d) which serve to determine the coefficients \( a_1 \) and \( a_2 \) may require modification on such occasions.
However, for \( c < 1 \) theories that we examine in this paper, \( \gamma_0 \) is known to be negative. Then we observe that the potentially dangerous terms

\[
(\partial F_{\text{sing}}/\partial g)^k \sim O(\delta g^{(1-\gamma_0)k}) \quad (k \geq 2)
\]

are always of higher order than \( O(\delta g^2) \). Hence the mixing of such terms does not occur in eqs.(3.6c–d) determining \( a_1 \) and \( a_2 \) which are needed to identify fixed points and scaling exponents.

The function \( \beta_n(g) \) in eq.(3.4) may sometimes develop singularity. In such occasions the \( G \)-function can be separated into its regular and singular parts around a specific point according to the power expansion of each \( \beta_n(g) \). Then we can prove that our algorithm, which originally assumes the analyticity of the \( G \)-function, is still valid after a replacement of \( G \) in eqs.(3.6a–d) or (3.10a–d) with its regular part \( G_{\text{reg}} \).

**Higher genus contributions**

Now we show that eq.(3.1) is sufficient to determine the singular behavior of higher genus free energies \( F^h \). Suppose we retain all terms of subleading order in \( 1/N \) in each procedure, i.e. saddle point evaluation of the \( \alpha \)- or \( \lambda \)-integral, application of the reparametrization identities, and replacement of a difference in \( N \) by differentials. Then we can express this complete RG equation in the form

\[
N \frac{\partial}{\partial N} + 2 \ F(N, g) = G \left( g, \frac{\partial F}{\partial g} \right) + \Delta G(N, g; [F]),
\]

where \([F]\) denotes differential polynomials of \( F(N, g) \).

The partial differential equation (3.12) can be separated into a set of ordinary differential equations for each genus contribution. It is important to realize that the additional contribution \( \Delta G \) introduced into the right hand side of the RG equation carry additional powers of \( 1/N \). Therefore they will not appear in the coefficient \( \beta_n(g) \)

\[
(2 - 2h)F^h(g) = r^h(g) + \sum_{n=1}^{\infty} \beta_n(g) \sum_{h_i \geq 0, h_i + \ldots + h_n = h, i=1}^{\infty} \prod_{i=1}^{n} \frac{dF^{h_i}}{dg}.
\]

The inhomogeneous term \( r^h(g) \) does get various higher order contributions including those terms from \( F^{h'} \) for \( h' \leq h - 1 \).

Now we assume that \( F^{h'} \ (h' \leq h - 1) \) is less singular than \( F^h \). Then the leading singular part in the right hand side of eq.(3.13) is dominated by the terms involving \( dF^h/dg \). Once we realize this structure of the RG equation, we can repeat the same argument as in the case of the sphere to obtain the scaling behavior of the higher genus contributions. The singularity of \( F^{h'} \ (h' \leq h - 1) \) does not affect eqs.(3.5a–d). We find that the fixed point condition is the same as eq.(3.5a) except that the coefficient \( b_0 \) is now replaced by \( b_0^h \) of the singular term of the genus \( h \) free energy. The critical exponent condition becomes

\[
2(1 - h)b_0^h = b_0^h(2 - \Gamma_h)(\beta_{11} + \beta_{20} \cdot 4a_2).
\]
energy. Therefore the same conditions as the sphere case (3.6c) and (3.6d) are sufficient to determine them. In this way we find that the fixed point is universal for any genus, and the critical exponent $\Gamma_h$ for genus $h$ is given by

$$2 - \Gamma_h = (1 - h)\gamma_1, \quad \gamma_1 + \gamma_0 = 2.$$  

(3.15)

This result explains the double scaling behavior for the singular part of the free energy

$$F_{\text{sing}}(N, g) = \sum_{h=0}^{\infty} N^{-2h} F^h_{\text{sing}}(g)$$

$$= (g - g_*)^{2-\gamma_0} f \left( N^{2/\gamma_1}(g - g_*) \right) + \text{less singular terms.}$$  

(3.16)

Similarly we observe that the generalization of eq.(3.15) to the multi-coupling case,

$$2 - \Gamma^{(k)}_h = (1 - h)\gamma^{(k)}_1,$$  

(3.17)

holds for each $k$.

### 3.2 Spectrum of continuum theories

In this subsection we recall the spectrum of operators of continuum theories of quantum gravity and present the various scaling exponents, to prepare for the following subsections.

Operators of $(p, q)$-minimal conformal matter coupled to two-dimensional quantum gravity (hereafter referred to as the $(p, q)$ gravity) derived from matrix models can be identified with hamiltonians of $W_p$-constrained, $p$-reduced KP hierarchy with a source insertion at $t_{p+q}$ [15, 16]. Namely the $k$-th order KP hamiltonian $O_k$ for $k < p + q$ can be identified with the gravitationally dressed $(n, m)$-th primary operator with $k = |pm - qn|$, 

$$O_{|pm-qn|} \sim \int \partial^2 z \, \Psi_{n,m} e^{\beta_{n,m} \varphi}$$  

(3.18)

where $\varphi$ denotes the Liouville field. Operators $O_k$ for $k > p + q$ are the gravitational descendants. The $W_p$ constraint implies that the free energy

$$F(\kappa; t_i) = \sum_{h=0}^{\infty} \kappa^h \left\langle \exp \sum_{k \geq 1} t_k O_k \right\rangle_{h, \text{conn.}}$$  

(3.19)

has the following scaling property:

$$\left\langle \exp \left( t_k O_k + \text{const.} O_{p+q} \right) \right\rangle_{h, \text{conn.}} \sim t_k^{2 - \Gamma^{(k)}_h} \quad (k < p + q),$$

$$\Gamma^{(k)}_h = \gamma^{(k)}_0 + h\gamma_1, \quad \gamma^{(k)}_0 = -\frac{2k}{p + q - k}, \quad \gamma^{(k)}_1 = \frac{2(p + q)}{p + q - k}.$$  

(3.20)

The exponent $\gamma^{(k)}_0$ is related to the more familiar ‘gravitational scaling dimension’ $\Delta^{(k)}$ by

$$\Delta^{(k)} = 1 - \frac{2 - \gamma^{(1)}_0}{2 - \gamma^{(k)}_0}.$$  

(3.21)
These exponents can as well be derived from the Liouville theory; they are related with the background charge $Q$ and the Liouville momentum $\beta_{n,m}$ of a vertex operator by

$$\gamma_0^{(pm-qn)} = 2 + \frac{Q}{\beta_{n,m}}, \quad \gamma_1^{(pm-qn)} = -\frac{Q}{\beta_{n,m}}, \quad (3.22)$$

$$Q \equiv \sqrt{\frac{25 - c}{3}} = \sqrt{\frac{2}{pq}} (p + q),$$

$$\beta_{n,m} \equiv -\sqrt{25 - c - \sqrt{1 - c + 24\Delta_0^{(n,m)}}} = \frac{-p + q - |pq - pm|}{\sqrt{2pq}}.$$

The exponent $\gamma_0^{(1)}$ for the dressed lowest dimensional operator $O_1$ is usually referred to as the string susceptibility exponent $\Gamma_{\text{str}}$.

It is instructive to point out that the so-called boundary operators $O_k$ for $k = 0 \mod p$ or $q$, should be excluded from the spectrum in our formalism for the following reason. Consider the $(p,q)$ gravity as the $W_p$-constrained, $p$-reduced KP hierarchy with a source insertion at $t_{p+q}$. Due to the $p$-reduction condition, KP operators $O_k$ for $k = 0 \mod p$ decouple

$$\langle O_{np} \cdots \rangle \equiv 0. \quad (3.23)$$

On the other hand, KP operators $O_k$ for $k = 0 \mod q$ do not decouple automatically. In turn they can always be absorbed into the redefinition of the operators of lower order than them. For instance, in the case of a unitary theory ($q = p + 1$), shifts in the KP time parameters $t_k$ of the form

$$t_1 \rightarrow t'_1 = t_1 + (\text{polynomial in } t_2, t_3, \cdots, t_{p-1}, t_{p+1}),$$

$$t_2 \rightarrow t'_2 = t_2 + (\text{polynomial in } t_3, \cdots, t_{p-1}, t_{p+1}),$$

$$\cdots$$

$$t_{p-1} \rightarrow t'_{p-1} = t_{p-1} + (\text{polynomial in } t_{p+1}),$$

$$t_{p+1} \rightarrow t'_{p+1} = 0 \quad (3.24)$$

are enough to eliminate $t_{p+1}$ [27]. In this sense these operators are redundant. This type of shift of parameters (of the continuum theory) should automatically be dealt with in our procedure to reduce the coupling space, since we have utilized all possible reparametrization invariance at the discrete level. Consequently KP operators $O_k$ for $k = 0 \mod p$ or $q$ would never appear in the spectrum of our RG equation. In the subsequent subsections we will observe that this is indeed the case. The fact that $p$ and $q$ are treated on equal footing in our formalism should be contrasted with the description by the KP flow. This feature will also be discussed in sect.4.2.

### 3.3 Fixed points of the one-matrix model

Here we present a full account of the result for one-matrix models, part of which has been reported in ref.[12]. We investigate the case of two couplings ($g_3$ and $g_4$) as
an example, by employing the eigenvalue method. The exact solution tells us that there exists a pure gravity critical line parameterized by

\[
g_3 = \frac{2(32z^2 - z^4 - 64zw + 4z^3w + 64w^2)}{z^5 + 2z^4w - 4z^3w^2}, \quad g_4 = \frac{16(16z - z^3 - 32w)}{3(z^5 + 2z^4w - 4z^3w^2)},
\]

and the tricritical point at one end of the line. (See Fig.1).

![Figure 1: The pure gravity critical line in the one-matrix model. It has the tricritical point \((0.3066\ldots, 0.02532\ldots)\) at one end. The dashed curves are unphysical critical lines.](image)

The dashed curves in Fig.1 are unphysical critical lines.

We find four solutions to the set of equations (3.10a–d), as summarized in Table 1:

| \((g_{3*}, g_{4*})\)   | \(\{2 - \gamma_0^{(k)}\}\) | \((p, q), \ \text{c}_{\text{eff}}\) | \(\mathcal{O}_k \sim \int \Psi_{n,m} e^{\beta_{n,m}\varphi}\) |
|------------------------|-----------------|-----------------|-------------------------------|
| \((0.3066\ldots, 0.0253\ldots)\) | 7/3             | (2, 5), \ \text{c}_{\text{eff}} = 2/5 | \(\mathcal{O}_1 \sim \int \Psi_{1,2} e^{\beta_{1,2}\varphi}\) |
|                        | 7/2             | \(\mathcal{O}_3 \sim \int 1 e^{\beta_{3,1}\varphi}\) | \(\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1}\varphi}\) |
| \((432^{-1/4}, 0)\)    | 5/2             | (2, 3), \ \text{c}_{\text{eff}} = 0 | \(\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1}\varphi}\) |
|                        | -6              | \(\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1}\varphi}\) | \(\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1}\varphi}\) |
| \((0, -1/12)\)         | 5/2             | (2, 3), \ \text{c}_{\text{eff}} = 0 | \(\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1}\varphi}\) |
|                        | 5/2             | \(\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1}\varphi}\) | \(\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1}\varphi}\) |
| \((0, 0)\)             | -4              | \(\text{Gaussian}\) | \(\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1}\varphi}\) |
|                        | -2              | \(\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1}\varphi}\) | \(\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1}\varphi}\) |

Table 1: Fixed points of the RG equation for the one-matrix model and associated critical exponents. Each fixed point is found to represent a continuum \((p, q)\) gravity through identification of operators as listed above.

* The saddle point equation for the eigenvalue distribution generically has several branches of solutions. If we approach the origin within the sheet on which the dashed curve lies, the distribution does not converge to the Wigner’s semi-circle distribution. Since the identification of the matrix model with gravity is based on the perturbative expansion with respect to \(g\) around the origin, the dashed curves have nothing to do with gravity.
(2,5) fixed point We find a fixed point \((g_{3*}, g_{4*}) = (0.3066, 0.0253)\) on an edge of the pure gravity critical line. Let us describe how this fixed point is identified with the (2,5) gravity (Lee-Yang edge singularity coupled to gravity) [27]. By diagonalizing the scaling exponent matrix computed

\[
G_{ij} = \begin{pmatrix}
1.5929 & -4.5685 \\
0.1645 & -0.1643 \\
\end{pmatrix},
\]

we obtain \(2/(2 - \gamma_0^{(3)}) = 6/7\), \(4/7\) and \(V_{ij} = \begin{pmatrix} 1 & -6.2087 \\ 1 & -4.7222 \end{pmatrix}\). Therefore the singular part of the free energy is found to behave in the neighborhood of the fixed point as

\[
F_{\text{sing}}(N, g_3, g_4) \sim b \cdot [\delta g_3 - 4.4722 \ldots \delta g_4]^{7/3} + b' \cdot [\delta g_3 - 6.2087 \ldots \delta g_4]^{7/2} + \ldots.
\]

Values of \(g_{3*}\) and \(G_{ij}\) can be expressed as roots of algebraic equations, though we present them in numerical forms for brevity here.

On the other hand, the content of relevant operators in the (2,5) gravity consists of \(O_1\), \(O_3\) and \(O_5\). Among them \(O_5\) is a boundary operator which should be suppressed in our formulation. According to eq. (3.20), \(O_1 \sim \int \Psi_{1,2} e^{\beta_1 \varphi}\) and \(O_3 \sim \int 1 e^{\beta_1 \varphi}\) acquire the scaling exponents \(2 - \gamma_0^{(1)} = 7/3\) and \(2 - \gamma_0^{(3)} = 7/2\) respectively, which are identical to those in eq. (3.27). This completes the identification, since we generically expect to detect all the relevant operators (excluding boundary operators) within the coupling constant space which is wide enough to realize the fixed point.

(2,3) fixed point I We find a fixed point at \((g_{3*}, g_{4*}) = (432^{-1/4}, 0)\) on the critical line. By diagonalizing the scaling exponent matrix computed

\[
G_{ij} = \begin{pmatrix}
4/5 & 17(3^{-3/4} - 2 \cdot 3^{3/4})/20 \\
0 & -1/3 \\
\end{pmatrix},
\]

the singular part of the free energy is found to behave as

\[
F_{\text{sing}}(N, g_3, g_4) \sim b \cdot \left[ \delta g_3 + \left( \frac{3^{1/4}}{4} - \frac{3^{7/4}}{2} \right) \delta g_4 \right]^{5/2} + b' \cdot [\delta g_4]^{-6} + \ldots.
\]

This fixed point is identified with the (2,3) gravity (pure gravity) whose sole relevant and non-boundary operator is the dressed identity \(O_1 \sim \int 1 e^{\beta_1 \varphi}\) with the susceptibility exponent \(2 - \gamma_0^{(1)} = 5/2\). Meanwhile the boundary operator \(O_3\) is suppressed as expected.

Here we stress that exponents of irrelevant operators \(O_k\) for \(k > p + q\), for which \(2 - \gamma_0^{(k)} = \gamma_1^{(k)} < 0\), could not be detected correctly within our large-\(N\) RG equation. It is because identification of \(N\) with the cutoff is based upon the possibility of the double scaling limit [18] where we let \(N\) approach infinity in correlation to a combination of \(g_i\)'s coupled to an operator, sent to a critical value. Obviously this is possible only for positive \(\gamma_1^{(k)}\). Therefore the magnitude of negative exponents such as \(-6\) is an artifact of our large-\(N\) RG and untrustable.
(2,3) fixed point II  At \((g_3^*, g_4^*) = (0, -1/12)\) on the critical line, the \(G\)-function turns out to develop singularities of the form 
\[
\beta_0(g_i) \sim O \left( (V_{ij} \delta g_j)^{5/6} \right), \quad \beta_1(g_i) \sim O \left( (V_{ij} \delta g_j)^{-1/6} \right)
\]
and so forth. These singularities originate from the switchover of the dominant solution \(\tilde{\lambda}\) to the saddle point equation (2.30) or (2.32) as we pass over the \(g_3 = 0\) line [19]. By replacing the \(G\)-function in eqs.(3.10a–d) by its regular part \(G_{\text{reg}}\), we find that \((g_3^*, g_4^*) = (0, -1/12)\) is indeed a fixed point with a scaling exponent matrix
\[
G_{ij} = \begin{pmatrix} 4/5 & 0 \\ 0 & 4/5 \end{pmatrix}.
\]
Since eigenvalues of the scaling exponent matrix are degenerate, the transformation matrix \(V_{ij}\) in the singular part of the free energy
\[
F_{\text{sing}}(N, g_3, g_4) \sim b \cdot [V_{ij} \delta g_j]^{5/2} + b' \cdot [V_{ij} \delta g_j]^{-3/2} + \cdots
\]
remains indeterminate. This fixed point can also be identified with the (2,3) gravity, involving doubling of the dressed identity operator due to the \(\mathbb{Z}_2\)-invariance of the potential\[1\].

Gaussian fixed point  We find a trivial fixed point at \((g_3^*, g_4^*) = (0, 0)\). The scaling exponent matrix indicates the naive scaling behavior
\[
F_{\text{sing}}(N, g_3, g_4) \sim b \cdot [\delta g_3]^{-4} + b' \cdot [\delta g_4]^{-2} + \cdots.
\]

3.4 Fixed points of the two-matrix model

The two-matrix model (2.34) has been investigated in ref. [23], and is known to possess the pure gravity critical line parameterized by
\[
g^2 = -2cz^2 + z \left\{ -c^2 + \left( c + 4z - \sqrt{(1 + c)^2 + 8cz + 16z^2} \right)^2 \right\},
\]
and the tricritical point at one end of the line. (See Fig.2).

\[1\] It is well known that the universal part of the free energy derived from matrix models with \(\mathbb{Z}_2\)-symmetric potential is twice as much as that of matrix models without \(\mathbb{Z}_2\)-symmetry.
Figure 2: The critical line in the two-matrix model with the cubic potential. The gravity is critical along this line. At one end of the line, the Ising model becomes critical as well.

We find three solutions to the RG equations in the physical region of the coupling constant space \( \{(g, c) | g \geq 0, c \leq 0\} \), as summarized in Table 2:

| \((g_*, c_*)\) | \(\{2 - \gamma_0^{(k)}\}\) | \((p, q), c_{\text{eff}}\) | \(\mathcal{O}_k \sim \int \Psi_{n,m} e^{\beta_{n,m} \varphi}\) |
|---|---|---|---|
| \((0.2003\ldots, 0.1589\ldots)\) | 7/3 | \((3, 4), c_{\text{eff}} = 1/2\) | \[\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1} \varphi}\] \[\mathcal{O}_5 \sim \int \Psi_{1,3} e^{\beta_{1,3} \varphi}\] |
| \((432^{-1/4}, 0)\) | 5/2 | \((3, 2), c_{\text{eff}} = 0\) | \[\mathcal{O}_1 \sim \int 1 e^{\beta_{1,1} \varphi}\] |
| \((0, \ast)\) | \(-4\) (Gaussian) | 0 | — |

Table 2: Fixed points of the RG equation for the two-matrix model and associated critical exponents.

**\((3, 4)\) fixed point** We find a fixed point on an edge of the pure gravity critical line, \((g_*, c_*) = \left(\left(10(-85 + 62\sqrt{7})/19683\right)^{1/2}, -(2\sqrt{7} - 1)/27\right)\). By diagonalizing the scaling exponent matrix computed

\[
G_{ij} = \begin{pmatrix}
-0.8391\ldots & 0.1148\ldots \\
-0.0866\ldots & 0.3036\ldots 
\end{pmatrix},
\]

we obtain \(2/(2 - \gamma_0^{(k)}) = 6/7, 2/7\) and \(V_{ij} = \begin{pmatrix} 1 & -0.2074\ldots \\ 1 & 6.3388\ldots \end{pmatrix}\). Therefore the singular part of the free energy is found to behave as

\[
F_{\text{sing}}(N, g, c) = b \cdot [\delta g - 0.2074 \cdots \delta c]^{7/3} + b' \cdot [\delta g + 6.3388 \cdots \delta c]^7 \log[\delta g + 6.3388 \cdots \delta c] + \cdots
\]

(3.35)

\[\dagger\] The logarithmic correction should be present in the free energy for the positive integral value of \(2 - \gamma_0\) to be detected in the scaling exponent matrix. We have borrowed from the exact solution the fact that the correction takes the form \(\delta g^7(\log \delta g)^1\).
This fixed point is identified with the (3,4) gravity (critical Ising model coupled to gravity); relevant and non-boundary operators of the (3,4) gravity consist of $O_1 \sim \int e^{3\beta_1 \phi}$, $O_2 \sim \int \Psi_1 \int e^{3\beta_1 \phi}$, and $O_5 \sim \int \Psi_1 \int e^{3\beta_1 \phi}$. According to eq.(3.20), they acquire the scaling exponents $2 - \gamma_0^{(1)} = 7/3$, $2 - \gamma_0^{(2)} = 14/5$, and $2 - \gamma_0^{(5)} = 7$, respectively. Since we have imposed the zero magnetic field condition $g_+ = g_-$ at the discrete level, the dressed spin operator $O_2$ should not be detected. Critical exponents corresponding to the rest of the operators (the dressed identity and energy) are indeed detected in eq.(3.35) and this completes the identification.

(3,2) fixed point At $(g_*, c_*) = (432^{-1/4}, 0)$ on the critical line, the $c$- and $a_c$-derivatives of the $G$-function turn out to develop singularities. By replacing the $G$-function in eqs.(3.10a–d) by its regular part $G_{\text{reg}}$ as in the (2,3) fixed point II, we find this point is indeed a fixed point. Recalling the result of the one-matrix model, this point obviously corresponds to the (twice tensor product of) pure gravity.

Gaussian fixed line We find that all the points on the line $g = 0$ are fixed points. The scaling exponent matrix indicates the naive scaling behavior

$$F_{\text{sing}}(N, g, c) = b \cdot [\delta g]^{-4} + b' \cdot [\delta c]^0 + \cdots.$$  (3.36)

The existence of the fixed line is a direct consequence of the separation of the Gaussian two-matrix model with arbitrary $c$ into two Gaussian one-matrix models.

Finally it is important to remark that the critical behavior of the second term in eq.(3.27), (3.29), or (3.35) is realized at each fixed point when one approaches the fixed point along the critical line. This means that, by fine-tuning the most relevant perturbation by $O_1$ to zero, we can deform a multi-critical continuum theory while keeping the gravity critical. This point will further be discussed in sect.4.2.

4 Renormalization group flow

4.1 Linear approximation to the RG equation

In this section we discuss the structure of the RG flow in the coupling space of matrix models. For the usual linear RG equation of Callan-Symanzik type

$$\left[ N \frac{\partial}{\partial N} + \gamma(g) \right] F(N, g) = r(g) + \beta(g) \frac{\partial F}{\partial g},$$  (4.1)

the concept of running coupling constant $g(N)$ is naturally introduced by

$$\int_{g_0}^{g(N)} \frac{dg'}{\beta(g')} = \log \left( \frac{N}{N_0} \right),$$  (4.2)

where $g_0 = g(N_0)$ and the $\beta$-function plays a role of vector field describing RG flow.
We consider, for instance, the RG flow for the one-matrix model with the cubic coupling described by eq. (2.22) with \( g_2 = 1, \ g_3 = g, \) others = 0. Expanding the right hand side into a power series in \( \partial F/\partial g \) around \( \partial F/\partial g|_{g=0} = 0 \), we obtain

\[
\left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g) = \sum_{n=0}^{\infty} \beta_n(g) \left( \frac{\partial F}{\partial g} \right)^n + O \left( \frac{1}{N} \right), \tag{4.3}
\]

\[
\beta_1(g) = -\frac{g}{2} + 3g^3 \int_0^{\frac{1}{1-g^2}} \frac{dt}{[(1-t)^2 + 4g^2t^3(1-(1-g^2)t)]^{1/2}},
\]

\[
\beta_2(g) = -\frac{9g^4}{2(1-g^2)} + 9g^8 \int_0^{\frac{1}{1-g^2}} \frac{dt}{[(1-t)^2 + 4g^2t^3(1-(1-g^2)t)]^{3/2}},
\]

\[
\beta_3(g) = -\frac{9g^5(3-3g^2+2g^4)}{2(1-g^2)^3} + 54g^{13} \int_0^{\frac{1}{1-g^2}} \frac{dt}{[(1-t)^2 + 4g^2t^3(1-(1-g^2)t)]^{5/2}},
\]

\[
\ldots.
\]

Expansion in powers of \( g \) shows that \( \beta_2(g) \sim -9/2 \ g^6 + O(g^8), \ \beta_3(g) \sim 9/2 \ g^9 + O(g^{11}) \) and so forth. Therefore we expect that the RG flow should be well approximated by \( \beta_1(g)(\sim -g/2 - 3g^3 \log g^2) \) for the region \( |g| \ll 1 \) where most fixed points lie. In fact, we have approximate values of fixed points

\[
\beta_1(g_*) = 0 \text{ for } g_* = 0, \ 0.219345 \ldots; \ \beta'_1(g_*) = -1/2, \ 0.780698 \ldots. \tag{4.4}
\]

By construction \( \beta_1(g) \) possesses a zero at the Gaussian fixed point with the exact value of the slope. On the other hand the deviations of the location of the nontrivial fixed point and of the slope from the exact values are 0.237\% and 2.472\% respectively. The small magnitude of the errors is obviously due to the suppression factors multiplying the nonlinear terms in eq. (4.3) as stated above. We emphasize that the approximate evaluation of \( \beta_n \) is useful only after the reparametrization identities are taken into account, as is explained in sect.2.1.

By the same token, by linearizing the RG equations (2.33) around the Gaussian values, \( (\partial F^0/\partial g_3, \ \partial F^0/\partial g_4)|_{g_3=g_4=0} = (0, 1/2) \), we have also calculated \( \beta_1 \)-functions \((G_{a_3}(g_3, g_4; 0, 1/2), \ G_{a_4}(g_3, g_4; 0, 1/2)) \) for the one-matrix model with cubic and quartic couplings. Here we exhibit the real part of the \( \beta_1 \)-functions in Fig.3.
Similarly the linearization of eq. (2.53) around \( \left( \frac{\partial F^0}{\partial g}, \frac{\partial F^0}{\partial c} \right) |_{g=c=0} = (0, 0) \) gives \( \beta_1 \)-functions \( G_{a_g}(g, c; 0, 0), G_{a_c}(g, c; 0, 0) \) for the two-matrix model (Fig. 4).

We immediately observe that they reproduce all the fixed points found in the previous section within a very small margin of error. Moreover, IR-repulsive and attractive directions of the RG flow around fixed points (which correspond to perturbations by relevant and irrelevant operators, respectively) are approximated quite well.
4.2 Global aspects of the RG flow

The real parts of \( \beta_1 \)-functions plotted in Figs.3 and 4 do not merely reproduce the local properties of the fixed points but also qualitatively represent the global structure of the RG flow.

Topology of the RG flow

We immediately observe from the figures that pure gravity critical lines are characterized as the renormalized trajectories emanating from multi-critical fixed points. By comparing the exponents with the spectrum of exact solutions \((3.20)\), the trajectory of the one-matrix model in Fig.3 is identified with the one corresponding to the perturbation of the \((2, 5)\) gravity by its dressed \( \Psi_{(1,1)} \) (identity) operator, and that of the two-matrix model in Fig.4 to the perturbation of the \((3, 4)\) gravity by its dressed \( \Psi_{(1,3)} \) (energy) operator. Both of them flow into the pure \((2, 3)\) gravity.

These results lead us to propose the following conjectures:

- perturbation of an UV theories of two-dimensional gravity by its least relevant operators leads to the IR theory with the neighboring order of criticality;
- the effective central charge \( c_{\text{eff}}^{(p,q)} \equiv c^{(p,q)} - 24\Delta_{\text{min}} = 1 - 6/pq \) decreases along the RG flow from the UV to the IR. This \( c_{\text{eff}} \) counts the density of states of the matter sector \([28]\).

In particular, the RG trajectory corresponding to the dressed \( \Psi_{(1,3)} \) perturbation interpolates two neighboring unitary models. This is an intriguing example of the gravitational counterpart of the familiar result on two-dimensional renormalizable field theories \([13]\). One may naively anticipate this result from the knowledge about the statistical and field theories over fixed backgrounds. However, we have vanishing total central charge of the system when we couple the matter with quantum gravity. Therefore it is nontrivial to establish the concept of ‘reduction of degrees of freedom’ along the flow from the UV to the IR in presence of gravity.

We remark on the RG trajectory between two \((2,3)\) fixed points in Fig.3. At first sight it is curious that a RG flow apparently exists to interpolate two distinct fixed points (the UV \((2,3)\) fixed point II and the IR \((2,3)\) fixed point I) corresponding to the same continuum theory. We interpret this fact as follows. The one-parameter subspace \((g_3 = 0, g_4)\) is stable under the RG flow, because our RG transformation maintains the \( \mathbb{Z}_2 \)-invariance \( \phi \rightarrow -\phi \). Moreover, the exact solution shows that the singular part of the free energy is discontinuous as we trace it on the critical line from the region \( g_3 > 0 \) to \( g_3 = 0 \). As the \( g_3 \)-derivatives of the free energy is ill-defined, we can attach no meaning to an RG trajectory connecting the \( \mathbb{Z}_2 \)-invariant subspace \( g_3 = 0 \) with other regions; we can make sense only for the eigenvector of the scaling exponent matrix and the RG trajectory which are along the \( g_4 \) axis around the \((2,3)\)

\[ Z(\beta) = \text{Tr} \ e^{-\beta H} \sim \beta^r \exp(\pi c_{\text{eff}}/6\beta) \quad (H = 2\pi(L_0 + T_0 - c/12)) \]

as \( \beta \rightarrow 0 \). Since \( Z(\beta \rightarrow 0) \) is interpreted as the regularized form of the number of states in the theory, \( c_{\text{eff}} \) describes the asymptotic growth of density of states.

\[ \]
Comparison to the KP flow

Let us further contrast these results with the integrable hierarchical description of two-dimensional gravity \([16]\). We again recall the fact that \((p,q)\) gravities are universally described by the \(W_p\)-constrained, \(p\)-reduced KP hierarchy with a source insertion at \(t_{p+q}\) (\(t_i\) denotes the time parameter coupled to the \(i\)-th order Hamiltonian \(\mathcal{O}_i\) of the hierarchy). We list the sets of relevant operators (usually referred to as gravitational primaries) in \((3,4)\) and \((3,2)\) theories in KdV description of the matrix models as well as the corresponding operators in the Liouville field theory description:

\[
\begin{align*}
\text{(3,4)} & \quad \text{KP operators: } \mathcal{O}_1 \cdot \mathcal{O}_2 \cdot (\mathcal{O}_4) \cdot \mathcal{O}_5 \cdot [\mathcal{O}_7] \\
& \quad \text{DDK operators: } \int \Psi_{(1,1)} \int \Psi_{(1,2)} \int \Psi_{(1,3)} \int \Psi_{(1,4)} \int \Psi_{(1,5)} \int \Psi_{(1,6)} \int \Psi_{(1,7)} S^{(3,4)} \\
& \quad \downarrow \\
\text{(3,2)} & \quad \text{KP operators: } \mathcal{O}_1 \cdot (\mathcal{O}_2) \cdot (\mathcal{O}_4) \cdot [\mathcal{O}_5] \\
& \quad \text{DDK operators: } \int \Psi_{(1,1)} S^{(3,2)}
\end{align*}
\]

Table 3: relevant operators of \((3,4)\) and \((3,2)\) theories

In this table supplemented are two types of KP operators: \(\mathcal{O}_{3+q}\) corresponding to the critical action \(S^{(3,q)}\) parenthesized by \([\ ]\), and boundary operators \(\mathcal{O}_k, k = 0 \text{ mod } q\) parenthesized by \((\ )\). It is naturally expected in the KP description that the \((3,4)\) gravity perturbed by \(\mathcal{O}_5\) should flow into the \((3,2)\) gravity; since \(\mathcal{O}_5\) is relevant, \(t_5\) will increase to infinity along the RG trajectory against a fixed source value of \(t_7\). Then by rescaling each \(t_i\) by \((t_5)^{-i/5}\) so as to render \(t_5\) finite and \(t_7\) infinitesimal while keeping the form of the hierarchy, we can describe the IR theory by the same 3-reduced KP hierarchy having a source insertion at \(t_5\). This theory is regarded as the \((3,2)\) gravity. Our result on the two-matrix model is in accord with this picture.

On the other hand, the sets of relevant operators of \((2,5)\) and \((2,3)\) theories are:

\[
\begin{align*}
\text{(2,5)} & \quad \text{KP operators: } \mathcal{O}_1 \cdot \mathcal{O}_3 \cdot (\mathcal{O}_5) \cdot [\mathcal{O}_7] \\
& \quad \text{DDK operators: } \int \Psi_{(1,2)} \int \Psi_{(1,1)} \int \Psi_{(1,3)} \int \Psi_{(1,4)} \int \Psi_{(1,5)} \int \Psi_{(1,6)} \int \Psi_{(1,7)} S^{(2,5)} \\
& \quad \downarrow \\
\text{(2,3)} & \quad \text{KP operators: } \mathcal{O}_1 \cdot (\mathcal{O}_3) \cdot [\mathcal{O}_5] \\
& \quad \text{DDK operators: } \int \Psi_{(1,1)} S^{(2,3)}
\end{align*}
\]

Table 4: relevant operators in \((2,5)\) and \((2,3)\) theories

It is awkward to interpret the RG flow within the framework of the KP hierarchy alone, as we describe below. One naively expects that the \(\mathcal{O}_5\) operator perturbation would give the \((2,3)\) gravity from the \((2,5)\). However, the boundary operators such as \(\mathcal{O}_5\) is physically redundant, since it can be expressed by means of other primary operators. In fact we have found that the RG flow from the \(\mathcal{O}_3 \sim \int \phi^2\) operator perturbation of the \((2,5)\) gravity leads to the \((2,3)\). Our large-\(N\) RG treatment,
which suppresses appearance of such redundant operators automatically by the use of reparametrization identities, is efficient indeed in such cases for identifying the RG trajectories and perturbing operators.

We remind the reader that the ‘universal’ description of \((p, *)\) theories by \(p\)-reduced KP hierarchy is derived by blowing-up one of the critical regions in the bare coupling space \(\{g_j\}\). In other words the KP time parameters \(\{t_i\}\) coupled to continuum operators are renormalized coupling constants defined with reference to a specific critical point. Hence it is a priori not guaranteed that the \(p\)-reduced KP hierarchy can describe an RG flow interpolating between two distant critical points \((p, q)\) and \((p, q')\) by a limiting procedure with respect to \(t_i\).

5 Discussions

In this paper we have constructed the RG equation for matrix models by regarding \(N\) as the cutoff, and investigated its implication. First we have performed the RG transformation \(N + 1 \rightarrow N\) to obtain the RG flow in the enlarged coupling space. We have reduced it into the original finite dimensional coupling space by the use of reparametrization identities, at the cost of allowing the nonlinearity of the RG equation. These reparametrization identities are the discrete Schwinger-Dyson equations for the sphere correlators (2.9) and (2.49). It is crucial for this procedure that these equations close algebraically to determine the required resolvent. The RG equation is found to give the critical points of the exact solution. It also exactly reproduces the spectrum of relevant operators of the Liouville gravity at its fixed points. After confirming the adequacy of linear approximation, we have investigated the RG flow in the coupling space. From these examples, we conjecture that a kind of the gravitational analogue of Zamolodchikov’s \(c\)-theorem holds in accordance with some previous expectations [17].

One of the characteristics of our treatment is that it suppresses the boundary operators which inevitably arise from the usual KP description of quantum gravity derived from matrix models, but are absent in the BRST analysis of Lian and Zuckerman’s [29]. A direct consequence of this is that we do not need them to unambiguously determine the RG flow. Further, if applied to multi-matrix models it can in principle describe the RG flow which changes both \(p\) and \(q\) of the \((p, q)\) theories that is impossible to realize in the KP description. Work in this direction is in progress.

In order to establish the ‘gravitational \(c\)-theorem’, it is necessary to construct a counterpart of the \(c\)-function in the framework of matrix models. The \(c\)-function is a potential function for the \(\beta\)-function, which should monotonically decrease along the RG trajectory from the UV extremum to the IR one. It might exist, since the \(G\)-function itself plays the role of the \(c\)-functions in nonlinear RG equations.

As originally motivated Brézin and Zinn-Justin, another intriguing application of our method is to solve matrix models which do not allow angular integration. Such models include candidates for a system of \(c > 1\) matter coupled to gravity. For instance, the \(n\)-Ising model over randomly triangulated surfaces can be described
by a matrix model with 2^n-plets of \( N \times N \) matrices \( \phi_1, \cdots, \phi_{2^n} \) as:

\[
Z(N, g, \beta) = \int \prod_{\tau} d^{N^2} \phi_\tau \exp \left\{ -N \text{tr} V(\phi_1, \cdots, \phi_{2^n}) \right\},
\]

\[
V(\phi_1, \cdots, \phi_{2^n}) = \frac{1}{2} \sum_{\tau, \tau'} \phi_\tau \Delta_{\tau\tau'} \phi_{\tau'} + \frac{g}{3} \sum_{\tau} \phi_\tau^3
\]

\[
\tau = (\sigma_1, \sigma_2, \cdots, \sigma_n), \quad \sigma_i = \pm 1,
\]

\[
(\Delta^{-1})_{\tau\tau'} = e^{\beta \tau \cdot \tau'}, \quad \phi \equiv \begin{pmatrix} \phi_1 & 0 \\ \vdots & \ddots \\ 0 & \phi_{2^n} \end{pmatrix}.
\]

It is straightforward to write down the RG and saddle point equations,

\[
\left[ N \frac{\partial}{\partial N} + 2 \right] F(N, g, \beta) = \left\langle \frac{1}{N} \text{tr} V(\phi_1, \cdots, \phi_{2^n}) \right\rangle + \frac{1}{\cosh^n \beta} \frac{\langle \alpha_s \rangle^2}{2} + 2^n \frac{g}{3} \langle \alpha_s \rangle^3 + \left\langle \frac{1}{N} \text{Tr} \log \left( 1 \Delta_{\tau\tau'} + g \langle \alpha_s \rangle 1 \delta_{\tau\tau'} + \phi_\tau \delta_{\tau\tau'} \right) \right\rangle - 2^{n-1} + O \left\langle \frac{1}{N} \right\rangle,
\]

\[
\frac{1}{(2 \cosh \beta)^n} \langle \alpha_s \rangle + g \langle \alpha_s \rangle^2 + g \left\langle \frac{1}{N} \text{tr} \left( \frac{1}{1 \Delta + g \langle \alpha_s \rangle 1 \delta + \phi} \right) \right\rangle_{\tau\tau'} = 0,
\]

where the trace over the \((2^n \cdot N) \times (2^n \cdot N)\) matrix is denoted by Tr, and the trace over \( N \times N \) denoted by tr. We have assumed \( \alpha_{s,\tau} \) does not depend on \( \tau \) in the same way as in the analysis of the two-matrix model. Thus the problem reduces to the computation of the resolvent

\[
W(z) = \left\langle \frac{1}{N} \text{Tr} \frac{1}{z 1 \delta + (1 \Delta + g \phi)} \right\rangle
\]

by using the reparametrization identities. So far we have not yet seen whether the resolvent can be determined by a closed set of Schwinger-Dyson equations or not. Even if we do not obtain a finitely closed set, however, we can exploit available approximation schemes, such as the linearization of the RG equation.

(Non-polynomial) nonlinearity is inevitable in constructing an RG equation obeyed by the free energy \( F(N, g) \). Consequently nonlinearity makes the identification of fixed points and exponents complicated, and makes the concept of running coupling constant unclear. While we are writing the manuscript we noticed two recent papers which may be relevant to this point. In ref.\[30\] the free energy of the Penner matrix model is shown to obey a linear RG equation. It is of great interest to examine whether the linearity of the RG equation is the peculiarity of the Penner model or a generic feature in a certain class of matrix models. A more recent paper \[31\] suggests an interesting possibility for constructing a linear RG equation by using a mapping of matrix models to \( O(N^2) \)-vector models. It was also proposed to consider the RG equation for the first derivative of the free energy \( \partial F/\partial g \). Although generalizations of this mapping to the subleading orders in \( N \) are hard to provide, so far we have observed that a certain simplification occurs when we perform an RG
transformation for $\partial F/\partial g$; there appears neither the tr log-term present in eq. (2.8), nor the integration term in eq. (2.22).

We hope to report on these topics in a future publication.

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**A  Loop equation for the two-matrix model**

In this appendix we exhibit the explicit form of the one-point function of the resolvent operator $\langle \hat{W}_0(z) \rangle$ in eq. (2.50) by using the reparametrization invariance of the partition function (2.34). We will consider the following reparametrizations:

\begin{align}
\phi' &= \phi + \epsilon \sum_{a=\pm} P_a \left( z\sigma_0 + \frac{c}{g} \sigma_1 + \phi \right)^{-1} P_a, \\
\phi' &= \phi + \epsilon \sum_{a=\pm} P_a \left\{ \sigma_1, \left( z\sigma_0 + \frac{c}{g} \sigma_1 + \phi \right)^{-1} \right\} P_a, \\
\phi' &= \phi + \epsilon \sum_{a=\pm} P_a \left( \phi \sigma_1 \left( z\sigma_0 + \frac{c}{g} \sigma_1 + \phi \right)^{-1} + \left( z\sigma_0 + \frac{c}{g} \sigma_1 + \phi \right)^{-1} \sigma_1 \phi \right) P_a, \\
\phi' &= \phi + \epsilon \sum_{a=\pm} P_a \left\{ \sigma_1 \phi \sigma_1, \left( z\sigma_0 + \frac{c}{g} \sigma_1 + \phi \right)^{-1} \right\} P_a.
\end{align}

These reparametrizations maintain $\phi_\pm$ hermitian. Invariance under these reparametrizations induces the following reparametrization identities

\begin{align}
0 &= \frac{1}{2} \left\langle \hat{W}_0(z) \right\rangle^2 + \left( gz^2 - z + \frac{c^2}{g} \right) \left\langle \hat{W}_0(z) \right\rangle + \left( 2cz - \frac{c}{g} \right) \left\langle \hat{W}_1(z) \right\rangle \\
&\quad + c \left\langle \hat{W}_{11}(z) \right\rangle + g \left\langle \frac{1}{N} \text{Tr} \phi \right\rangle + 2 - 2gz, \\
0 &= \frac{1}{2} \left\langle \hat{W}_0(z) \right\rangle \left\langle \hat{W}_1(z) \right\rangle + \left( cz - \frac{c}{g} - \frac{c^2}{g} \right) \left\langle \hat{W}_0(z) \right\rangle + (gz^2 - cz - z) \left\langle \hat{W}_1(z) \right\rangle \\
&\quad - c \left\langle \hat{W}_{11}(z) \right\rangle, \\
0 &= -\frac{c}{2g} \left\langle \hat{W}_0(z) \right\rangle^2 - \frac{1}{2} z \left\langle \hat{W}_0(z) \right\rangle \left\langle \hat{W}_1(z) \right\rangle + \left( \frac{cz}{g} - c^2 \right) \left\langle \hat{W}_0(z) \right\rangle \\
&\quad + (-1 + z^2 - gz^3) \left\langle \hat{W}_1(z) \right\rangle + \left( -\frac{c}{g} + cz \right) \left\langle \hat{W}_{11}(z) \right\rangle - c \left\langle \hat{W}_{101}(z) \right\rangle.
\end{align}
where
\[ \hat{W}_{j_1 j_2 \ldots j_k}(z) = \frac{1}{N} \text{Tr} \sigma_{j_1} \phi \sigma_{j_2} \phi \cdots \phi \sigma_{j_k} \left( z \sigma_0 + \frac{c}{g} \sigma_1 + \phi \right)^{-1}. \]  

After some computations, we obtain a quartic equation obeyed by \( \langle \hat{W}_0(z) \rangle \):

\[ 0 = \left( \frac{1}{2} \langle \hat{W}_0(z) \rangle - \frac{c}{g} - z + cz + g z^2 \right) \]
\[ \cdot \left[ -\frac{c}{2g} (1 + 2c - g z) \langle \hat{W}_0(z) \rangle^2 + 2 \cdot \frac{c^2}{g^2} + \frac{c}{g} - \frac{c^2}{g^2} \right] (c + g z)(1 - g z) \langle \hat{W}_0(z) \rangle \]
\[ + c(1 - g z) \left( \frac{1}{N} \text{Tr} \phi \right) + 2gc \left( \frac{1}{N} \text{Tr} \sigma_1 \phi \sigma_1 \phi \right) \]
\[ - \left\{ \frac{1}{4} \langle \hat{W}_0(z) \rangle^2 + \left[ -cz - \left( \frac{c}{2g} + z \right) (1 - g z) \right] \langle \hat{W}_0(z) \rangle - c \right\} \]
\[ + \frac{z}{g} (1 + 2c - g z)(c + g z)(1 - g z) \]
\[ \cdot \left\{ \frac{1}{2} \langle \hat{W}_0(z) \rangle^2 + \left( -\frac{c}{g} - z + cz + g z^2 \right) \langle \hat{W}_0(z) \rangle + g \left( \frac{1}{N} \text{Tr} \phi \right) + 2 - 2g z \right\} \]

There appear one-point functions \( \langle \text{tr} \phi_+ \phi_- \rangle = (1/2) \langle \text{Tr} \sigma_1 \phi \sigma_1 \phi \rangle \) and \( \langle \text{tr} (\phi_+ + \phi_-) \rangle = \langle \text{Tr} \phi \rangle \) in the above expression. We express the latter in terms of \( \langle \text{tr} \phi_+ \phi_- \rangle \) and \( \langle \text{tr} (\phi_3^1 + \phi_3^2) \rangle \) by solving the first two equations in (2.49),

\[ 0 = \left( \frac{1}{N} \text{tr} \phi_+ \right) + g \left( \frac{1}{N} \text{tr} \phi_2^1 \right) + c \left( \frac{1}{N} \text{tr} \phi_3^1 \right), \]  
\[ 1 = \left( \frac{1}{N} \text{tr} \phi_2^1 \right) + g \left( \frac{1}{N} \text{tr} \phi_3^2 \right) + c \left( \frac{1}{N} \text{tr} \phi_3^3 \right). \]

Promoting \( \partial F/\partial g = \langle (1/3N) \text{tr} (\phi_3^1 + \phi_3^2) \rangle \) and \( \partial F/\partial c = \langle (1/N) \text{tr} \phi_+ \phi_- \rangle \) to independent variables, eq. (A.4) with this replacement determines the one-point function of the resolvent

\[ \langle \hat{W}_0(z) \rangle = W_0 \left( z; g, c; \frac{\partial F}{\partial g}, \frac{\partial F}{\partial c} \right). \]  

References

[1] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, *Mod. Phys. Lett.* **A3** (1988) 819;
F. David, *Mod. Phys. Lett.* **A3** (1988) 1651;
J. Distler and H. Kawai, *Nucl. Phys.* **B321** (1989) 509.
[2] E. Brézin, C. Itzykson, G. Parisi, and J.-B. Zuber, *Commun. Math. Phys.* **59** (1978) 35.

[3] D. Bessis, C. Itzykson, and J.-B. Zuber, *Adv. Appl. Math.* **109** (1980) 1.

[4] F. David, *Nucl. Phys.* **B257[FS14]** (1985) 45; J. Ambjörn, B. Durhuus, and J. Fröhlich, *Nucl. Phys.* **B257[FS14]** (1985) 433; V. A. Kazakov, I. K. Kostov, and A. A. Migdal, *Phys. Lett.* **B157** (1985) 295.

[5] D. J. Gross and A. A. Migdal, *Phys. Rev. Lett.* **64** (1990) 127; E. Brézin and V. A. Kazakov, *Phys. Lett.* **B236** (1990) 114; M. R. Douglas and S. H. Shenker, *Nucl. Phys.* **B335** (1990) 635.

[6] D. J. Gross and I. R. Klebanov, *Nucl. Phys.* **B359** (1991) 3; S. R. Das and A. Jevicki, *Mod. Phys. Lett.* **A5** (1990) 1639.

[7] E. Brézin and S. Hikami, *Phys. Lett.* **B283** (1992) 203; S. Hikami, *Phys. Lett.* **B305** (1993) 327; *Physica A204* (1994) 290; M. Wexler, *Phys. Lett.* **B315** (1993) 67.

[8] J. Carlson, *Nucl. Phys.* **B248** (1984) 536; T. Kubota and Y. X. Cheng, *Mod. Phys. Lett.* **A6** (1991) 2289.

[9] E. Brézin and J. Zinn-Justin, *Phys. Lett.* **B288** (1992) 54.

[10] S. Higuchi, C. Itoi, and N. Sakai, *Phys. Lett.* **B312** (1993) 88.

[11] S. Higuchi, C. Itoi, and N. Sakai, *Prog. Theor. Phys. Suppl.* **114** (1993) 53.

[12] S. Higuchi, C. Itoi, S. Nishigaki, and N. Sakai, *Phys. Lett.* **B318** (1993) 63.

[13] A. B. Zamolodchikov, *JETP Lett.* **43** (1986) 730.

[14] M. R. Douglas, *Phys. Lett.* **B238** (1990) 176; D. J. Gross and A. A. Migdal, *Nucl. Phys.* **B340** (1990) 333.

[15] P. Ginsparg, M. Goulian, M. R. Plesser, and J. Zinn-Justin, *Nucl. Phys.* **B342** (1990) 539; A. Jevicki and T. Yoneya, *Mod. Phys. Lett.* **A5** (1990) 1615.

[16] M. Fukuma, H. Kawai, and R. Nakayama, *Int. J. Mod. Phys.* **A6** (1991) 1385; R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Nucl. Phys.* **B348** (1991) 435.

[17] D. Kutasov, *Mod. Phys. Lett.* **A7** (1992) 2943.

[18] S. Nishigaki, *Mod. Phys. Lett.* **A9** (1994) 631, and references therein.

[19] V. Periwal, *Phys. Lett.* **B294** (1992) 49.

[20] J. Alfaro and P. Damgaard, *Phys. Lett.* **B289** (1992) 342; C. Ayala, *Phys. Lett.* **B311** (1993) 55; Y. Itoh, *Mod. Phys. Lett.* **A8** (1993) 3273.
[21] A. Mironov and A. Morozov, *Phys. Lett.* **B252** (1990) 47;  
L. Alvarez-Gaumé, C. Gomez, and J. Lacki, *Phys. Lett.* **B253** (1991) 56;  
H. Itoyama and Y. Matsuo, *Phys. Lett.* **B262** (1991) 233.

[22] V. A. Kazakov, *Mod. Phys. Lett.* **A4** (1989) 2125.

[23] V. A. Kazakov, *Phys. Lett.* **A119** (1986) 140;  
D. V. Boulatov and V. A. Kazakov, *Phys. Lett.* **B186** (1987) 379.

[24] C. Itzykson and J.-B. Zuber, *J. Math. Phys.* **21** (1980) 411;  
M. L. Mehta, *Commun. Math. Phys.* **79** (1981) 327.

[25] E. Martinec, G. Moore, and N. Seiberg, *Phys. Lett.* **B263** (1991) 190.

[26] M. Fukuma, H. Kawai, and R. Nakayama, *Commun. Math. Phys.* **148** (1992) 101.

[27] M. Staudacher, *Nucl. Phys.* **B336** (1990) 349.

[28] C. Itzykson, H. Saleur, and J.-B. Zuber, *Europhys. Lett.* **2** (1986) 91;  
D. Kutasov and N. Seiberg, *Nucl. Phys.* **B358** (1991) 600.

[29] B. H. Lian and G. J. Zuckerman, *Commun. Math. Phys.* **154** (1993) 613.

[30] D. A. Johnston, Orsay preprint LPTHE-Orsay-94-59 (June 1994), hep-th/9406239.

[31] S. Hikami, Univ. of Tokyo preprint UT-KOMABA/94/13 (June 1994), hep-th/9406153.