STORAGE OPERATORS and ∀-POSITIVE TYPES in TTR TYPE SYSTEM

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Abstract In 1990, J.L. Krivine introduced the notion of storage operator to simulate "call by value" in the "call by name" strategy. J.L. Krivine has shown that, using Gödel translation of classical into intuitionistic logic, we can find a simple type for the storage operators in AF2 type system. This paper studies the ∀-positive types (the universal second order quantifier appears positively in these types), and the Gödel transformations (a generalization of classical Gödel translation) of TTR type system. We generalize, by using syntactical methods, the J.L. Krivine's Theorem about these types and for these transformations. We give a proof of this result in the case of the type of recursive integers.

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1 Introduction

The strategy of left reduction (iteration of head reduction denoted by ➔) has the following advantages:

- It has good mathematical properties stated by the normalisation Theorem : if a λ-term is normalizable, then we obtain the normal form by left reduction.

- It seems more economic since we compute a λ-term only when we need it.

Now, a drawback of the strategy of left reduction (call by name) is the fact that the argument of a function is computed as many times as it is used. The purpose of storage operators is precisely to correct this drawback.

Let F be a λ-term (a function), and N the set of normal Church integers. During the computation, by left reduction, of (F)θₙ (where θₙ ≃ β n), θₙ may be computed several times (as many times as F uses it). We would like to transform (F)θₙ to (F)n. We also want this transformation depends only on θₙ (and not F). In other words we look for some closed λ-terms T with the following properties :

1We thank R. David, J.L. Krivine, and M. Parigot for helpful discussions.
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• For every \( F, n \in \mathbb{N} \), and \( \theta_n \vdash \beta n \), we have \( (T)\theta_n F \succ (F)n \).

• The computation time of the head reduction \( (T)\theta_n F \succ (F)n \) depends only on \( \theta_n \).

Therefore the first definition: A closed \( \lambda \)-term \( T \) is called storage operator for \( \underline{N} \) if and only if for every \( n \in \mathbb{N} \), and for every \( \theta_n \vdash \beta n \), \( (T)\theta_n f \succ (f)n \) (where \( f \) is a new variable).

It is clear that a storage operator satisfies the required properties. Indeed,

• Since we have \( (T)\theta_n f \succ (f)n \), then the variable \( f \) never comes in head position during the reduction, and we may then replace \( f \) by any \( \lambda \)-term.

• The computation time of the head reduction \( (T)\theta_n F \succ (F)n \) depends only on \( \theta_n \).

We showed (see [12]) that it is not possible to get the normal form of \( \theta_n \). We then change the definition: A closed \( \lambda \)-term \( T \) is called storage operator for \( \underline{N} \) if and only if for every \( n \in \mathbb{N} \), there is a closed \( \lambda \)-term \( \tau_n \vdash \beta n \) (for example \( \tau_n = (\lambda x)\theta_n \), where \( \lambda x \) is a \( \lambda \)-term for the successor), such that for every \( \theta_n \vdash \beta n \), \( (T)\theta_n f \succ (f)\tau_n \) (where \( f \) is a new variable).

If we take \( T_1 = \lambda n((\lambda x)\lambda y(x)\lambda z(y)\lambda f(f)n) \) and \( T_2 = \lambda n\lambda f((\lambda x)\lambda y(x)(\lambda z)y)(\lambda f(f)n) \), then it is easy to check that: for every \( \theta_n \vdash \beta n \), \( (T_1)\theta_n f \succ (f)(\lambda z)(\lambda f(f)n) \) and \( (T_2)\theta_n f \succ (f)(\lambda z)(\lambda f(f)n) \). Therefore \( T_1 \) and \( T_2 \) are storage operators for \( \underline{N} \).

The \( AF2 \) type system is a way of interpreting the proof rules for the second order intuitionistic logic plus equational reasoning as construction rules for terms. In this system we have the possibility to define the data types, the representation in \( \lambda \)-calculus being automatically extracted from the logical definition of the data type. At the logical level the data type are defined by second order formulas expressing the usual iterative definition of the corresponding algebras of terms and the data receive the corresponding iterative definition in \( \lambda \)-calculus. For example, the type of integers is the formula: \( N[x] = \forall X \forall y[X(y) \rightarrow X(sy)] \rightarrow [X(0) \rightarrow X(x)] \) \( (X \) is a unary predicate variable, \( 0 \) is a constant symbol for zero, and \( s \) is a unary function symbol for successor).

If we try to type a storage operator \( T \) in \( AF2 \) type system, we naturally find the type \( \forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\} \) (where \( O \) is a particular 0-ary predicate symbol which represents an arbitrary type). Indeed, if \( \vdash_{AF2} \tau_n : N[s^n(0)] \), and \( f \) is of type \( N[s^n(0)] \rightarrow O \), then \( f : N[s^n(0)] \rightarrow O \vdash_{AF2} (f)\tau_n : O \). It is natural to have \( (T)\theta_n f \) of type \( O \). If \( \vdash_{AF2} \theta_n : N[s^n(0)] \), then the type for \( T \) must be \( \forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\} \).

It is easy to check that \( \vdash_{AF2} T_1, T_2 : \forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\} \).

The type \( \forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\} \) does not characterize the storage operators. Indeed, if we take \( T = \lambda n\lambda f(f)n \), we obtain:

• \( n : N[x], f : N[x] \rightarrow O \vdash_{AF2} (f)n : O \), then, \( \vdash_{AF2} T : \forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\} \).

• For every \( \theta_n \vdash \beta n \), \( (T)\theta_n f \succ (f)\theta_n \), therefore \( T \) is not a storage operator for \( \underline{N} \).
This comes from the fact that the type \( \forall x \{ N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O] \} \) does not take into account the independence of \( \tau_n \) with \( \theta_n \). To solve this problem, we must prevent the use of the first \( N[x] \) in \( \forall x \{ N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O] \} \) as well as its subtypes to prove the second. Therefore, we will replace the first \( N[x] \) by a new type \( N^*[x] \) with the following properties:

1. \( \vdash_{AF2} \tau : N^*[s^a(0)] \) (for example, take \( N^*[x] = \forall X \{ \forall y [F(X,y) \rightarrow F(X,sy)] \rightarrow [F(X,0) \rightarrow F(X,x)] \} \);
2. If \( \nu : N^*[x], x_i : \forall y [F(G,y) \rightarrow F(G,sy)], y_j : F(H,a) \vdash_{AF2} t : N[s^a(0)] \), then \( \vdash_{AF2} t' : N[s^a(0)] \) where \( t' \) is the normal form of \( t \);
3. There is a closed \( \lambda \)-term \( T \), such that \( \vdash_{AF2} T : \forall x \{ N^*[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O] \} \).

A simple solution for the second property is to take a formula \( F(X,a) \) ending with a new constant symbol. Indeed, since \( N[x] \) does not contain this symbol, we cannot use the variables \( \nu, x_i, y_j \) in the typing of \( t' \). We suggest the following proposition:

\[
N^*[x] = \forall X \{ \forall y [(X(y) \rightarrow O) \rightarrow (X(sy) \rightarrow O)] \rightarrow [(X(0) \rightarrow O) \rightarrow (X(x) \rightarrow O)] \}.
\]

It is easy to check that \( \vdash_{AF2} T_1, T_2 : \forall x \{ N^*[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O] \} \) (see [6] and [12]). For each formula \( F \) of \( AF2 \), we indicate by \( F^* \) the formula obtained by putting \(-\) in front of each atomic formulas of \( F \) (\( F^* \) is called the Gödel translation of \( F \)).

J.L. Krivine has shown that the type \( \forall x \{ N^*[x] \rightarrow \neg\neg N[x] \} \) characterize the storage operators for \( \sum_n \) (see [6]). But the \( \lambda \)-term \( \tau_n \) obtained may contain variables substituted by \( \lambda \)-terms \( u_1, \ldots, u_m \) depending on \( \theta_n \). Since the \( \lambda \)-term \( \tau_n \) is \( \beta\eta \)-equivalent to \( \tau \), therefore, the left reduction of the \( \tau_n[u_1/x_1, \ldots, u_m/x_m] \) is equivalent to the left reduction of \( \tau_n \) and the \( \lambda \)-terms \( u_1, \ldots, u_m \) will therefore never be evaluated during the reduction.

Taking into account the above remarks, we modify again the definition: A closed \( \lambda \)-term \( T \) is called a storage operator for \( \sum_n \) if and only if for every \( n \in \mathbb{N} \), there is a \( \lambda \)-term \( \tau_n \simeq_{\beta} \tau \), such that for every \( \theta_n \simeq_{\beta} \tau \), there is a substitution \( \sigma \), such that \( (T)[\theta_n] f \simeq (f)\sigma(\tau_n) \) (where \( f \) is a new variable).

The \( AF2 \) type system is satisfactory from an extensional point of view: one can construct programs for all the functions whose termination is provable in the second order Peano arithmetic. But from an intensional point of view the situation is very different: we cannot always obtain the simple (in term of time complexity, for instance) programs we need. For example we cannot find a \( \lambda \)-term of type \( \forall x \forall y \{ N[x], N[y] \rightarrow N[min(x,y)] \} \) \( (min \) is a binary function symbol defined by equations) in \( AF2 \) type system that computes the minimum of two Church integers in time \( O(min^3) \).

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\[3\] R. David gives a \( \lambda \)-term of type \( N, N \rightarrow N \) \( (N = \forall X \{ [X \rightarrow X] \rightarrow [X \rightarrow X] \}) \) in \( F \) type system that computes the minimum of two Church integers in time \( O(min.Log(min)) \). The notion of storage operators plays an important tool in this construction (see [2]).
The TTR type system is an extension of AF2 based on recursive definitions of types, which is intended to solve the basic problems of efficiency mentioned before. In TTR we have a logical operator \( \mu \) of least fixed point. If \( A \) is a formula, \( C \) an \( n \)-ary predicate symbol which appears and occurs positively in \( A \), \( x_1, \ldots, x_n \) first order variables, and \( t_1, \ldots, t_n \) terms, then \( \mu Cx_1 \ldots x_n A < t_1, \ldots, t_n > \) is a formula called the least fixed point of \( A \) in \( C \) calculated over the terms \( t_1, \ldots, t_n \). The interded logical meaning of the formula \( \mu Cx_1 \ldots x_n A < t_1, \ldots, t_n > \) is \( K(t_1, \ldots, t_n) \), where \( K \) is the least \( X \), such that \( X(x_1, \ldots, x_n) \leftrightarrow A \).

In this paper we study the types \( D \) of TTR, and the transformations *, for which we have the following result: if \( \vdash_{TTR} T : D \rightarrow \neg \neg D \), then for every \( \lambda \)-term \( t \) with \( \vdash_{TTR} t : D \), there are \( \lambda \)-terms \( \tau_t \) and \( \tau'_t \) such that \( \tau_t \simeq_\beta \tau'_t \), \( \vdash_{TTR} \tau'_t : D \), and for every \( \theta_t \simeq_\beta t \), there is a substitution \( \sigma \), such that \( (T)\theta_t f \succ (f)\sigma(\tau_t) \) (where \( f \) is a new variable).

We prove \(^4\) that, to obtain this result, it suffices to assume that:

- The universal second order quantifier appears positively in \( D \) (\( \forall \)-positive type) \(^5\).
- The transformation * satisfies the following properties:
  - If \( A = C(t_1, \ldots, t_n) \), then \( A^* = A \);
  - If \( A = X(t_1, \ldots, t_n) \), then \( A^* = F_X[t_1/x_1, \ldots, t_n/x_n] \) where \( F_X \) is a formula ending with \( \bot \) and having \( x_1, \ldots, x_n, X_1, \ldots, X_r \) as free variables:
    - \( (A \rightarrow B)^* = A^* \rightarrow B^* \);
    - \( (\forall x A)^* = \forall x A^* \);
    - \( (\forall X A)^* = \forall X_1 \ldots X_r A^* \);
    - \( (\mu Cx_1 \ldots x_n A < t_1, \ldots, t_n >)^* = \mu Cx_1 \ldots x_n A^* < t_1, \ldots, t_n > \).

We give the proof of this result in the case of the type of recursive integers.

2 Basic notions of pure \( \lambda \)-calculus

Our notation is standard (see [1] and [5]).

We denote by \( \Lambda \) the set of terms of pure \( \lambda \)-calculus, also called \( \lambda \)-terms.

Let \( t, u, u_1, \ldots, u_n \in \Lambda \), the application of \( t \) to \( u \) is denoted by \( (t)u \). In the same way we write

\(^4\)J.L. Krivine and the author proved independently the same result for AF2 type system (see [7] and [12]).

\(^5\)This types were studied by some authors (in particular R. Labib-Sami), and have remarkable properties (see [8]).
The notation $\lambda$.

The set of free variables of a $\lambda$-term $t$ is denoted by $Fv(t)$.

With each normal $\lambda$-term, we associate a set of $\lambda$-terms $STE(t)$ by induction:
if $t = \lambda x_1...\lambda x_n(y)t_1...t_m$, then $STE(t) = \{ t \} \cup \bigcup_{1 \leq i \leq m} STE(t_i)$.

The logical symbols are $\perp$.
The types will be formulas of second order predicate logic over a given language.
The notation $t[u_1/x_1,...,u_n/x_n]$ represents the result of the simultaneous substitution of $\lambda$-terms $u_1,...,u_n$ to the free variables $x_1,...,x_n$ of $t$ (after a suitable renaming of the bounded variables of $t$).

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Let us recall that a $\lambda$-term $t$ either has a head redex [i.e. $t = \lambda x_1...\lambda x_n(\lambda xu)v_1...v_m$, the head redex being $(\lambda xu)v$], or is in head normal form [i.e. $t = \lambda x_1...\lambda x_n(x)v_1...v_m$].
The notation $t \succ t'$ means that $t'$ is obtained from $t$ by some head reductions, and we denote by $n(t,t')$, the number of steps to go from $t$ to $t'$.

A $\lambda$-term $t$ is said to be solvable if and only if the head reduction of $t$ terminates.

We define an equivalence relation $\sim$ on $\Lambda$ by:
if $t = \lambda x_1...\lambda x_n(y)t_1...t_m$, then $STE(t) = \{ t \} \cup \bigcup_{1 \leq i \leq m} STE(t_i)$.

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We define an equivalence relation $\sim$ on $\Lambda$ by:
if $t = \lambda x_1...\lambda x_n(y)t_1...t_m$, then $STE(t) = \{ t \} \cup \bigcup_{1 \leq i \leq m} STE(t_i)$.

Theorem 2.1 shows that to make the head reduction of $(t)u_1...u_n$ (resp. $t[u_1/x_1,...,u_n/x_n]$), it is equivalent (same result, and same number of steps) to make some steps in the head reduction of $t$, and then make the head reduction of $(t')u_1...u_n$ (resp. $t'[u_1/x_1,...,u_n/x_n]$).

3 Basic notions of typed $\lambda$-calculus

3.1 The $AF2$ type system

The types will be formulas of second order predicate logic over a given language.
The logical symbols are $\perp$ (for absurd), $\rightarrow$ and $\forall$ (and no other ones).

There are individual variables : $x, y, ...$ (also called first order variables) and $n$-ary predicate variables ($n = 0, 1, ...$) : $X, Y, ...$ (also called second order variables).
The terms and the formulas are up in the usual way.
The formula $F_1 \rightarrow (F_2 \rightarrow (... \rightarrow (F_n \rightarrow G)...))$ is denoted by $F_1, F_2, ..., F_n \rightarrow G$, and $F \rightarrow \perp$ is denoted by $\neg F$. The formula $\forall v_1...\forall v_n F$ is denoted by $\forall F$, and the sentence "$v$ is not free in $A$" means that for all $1 \leq i \leq n$, $v_i$ is not free in $A$. 
If \( X \) is a unary predicate variable, \( t \) and \( t' \) two terms, then the formula \( \forall X[Xt \to Xt'] \) is denoted by \( t = t' \), and is said to be equation. A particular case of \( t = t' \) is a formula of the form \( t[u_1/x_1, \ldots, u_n/x_n] = t'[u_1/x_1, \ldots, u_n/x_n] \) or \( t'[u_1/x_1, \ldots, u_n/x_n] = t[u_1/x_1, \ldots, u_n/x_n] \), \( u_1, \ldots, u_n \) being terms of the language.

After, we denote by \( E \) a system of function equations.

A context \( \Gamma \) is a set of the form \( x_1: A_1, \ldots, x_n: A_n \) where \( x_1, \ldots, x_n \) are distinct variables and \( A_1, \ldots, A_n \) are formulas.

We are going to describe a system of typed \( \lambda \)-calculus called second order functional arithmetic (shortened in \( AF^2 \) for Arithmétique Fonctionnelle du second ordre). The rules of typing are the following:

1. \( \Gamma, x: A \vdash_{AF^2} x : A \).
2. If \( \Gamma, x: B \vdash_{AF^2} t : C \), then \( \Gamma \vdash_{AF^2} \lambda x: B \to C \).
3. If \( \Gamma \vdash_{AF^2} u : B \to C \), and \( \Gamma \vdash_{AF^2} v : B \), then \( \Gamma \vdash_{AF^2} (u)v : C \).
4. If \( \Gamma \vdash_{AF^2} t : A \), and \( x \) does not appear in \( \Gamma \), then \( \Gamma \vdash_{AF^2} t : \forall xA \).
5. If \( \Gamma \vdash_{AF^2} t : \forall xA \), then, for every term \( u \), \( \Gamma \vdash_{AF^2} t : A[u/x] \).
6. If \( \Gamma \vdash_{AF^2} t : A \), and \( X \) does not appear in \( \Gamma \), then \( \Gamma \vdash_{AF^2} t : \forall XA \).
7. If \( \Gamma \vdash_{AF^2} t : \forall XA \), then, for every formula \( G \), \( \Gamma \vdash_{AF^2} t : A[G/X(x_1, \ldots, x_n)] \) (*)
8. If \( \Gamma \vdash_{AF^2} t : A[u/x] \), then \( \Gamma \vdash_{AF^2} t : A[v/x] \), \( u = v \) being a particular case of an equation of \( E \).

(*) \( A[G/X(x_1, \ldots, x_n)] \) is obtained by replacing in \( A \) each atomic formula \( X(t_1, \ldots, t_n) \) by \( G[t_1/x_1, \ldots, t_n/x_n] \). To simplify, we write sometimes \( A[G/X] \) instead of \( A[G/X(x_1, \ldots, x_n)] \).

Whenever we obtain the typing \( \Gamma \vdash_{AF^2} t : A \) by means of these rules, we say that "the \( \lambda \)-term \( t \) is of type \( A \) in the context \( \Gamma \), with respect to the equation of \( E \) ".

**Theorem 3.1** ([5],[9]).

1. **Conservation Theorem:** If \( \Gamma \vdash_{AF^2} t : A \), and \( t \to_{\beta} t' \), then \( \Gamma \vdash_{AF^2} t' : A \).
2. **Strong normalization:** If \( \Gamma \vdash_{AF^2} t : A \), then \( t \) is strongly normalizable.

### 3.2 The TTR type system

Let \( X \) be a predicate variable or predicate symbol, and \( A \) a type of \( AF^2 \).

We define the notions "\( X \) is positive in \( A \)" and "\( X \) is negative in \( A \)" by induction:

- If \( X \) does not appear in \( A \), then \( X \) is positive and negative in \( A \);
- If $A = X(t_1, \ldots, t_n)$, then $X$ is positive in $A$, and $X$ is not negative in $A$;

- If $A = B \rightarrow C$, then $X$ is positive (resp. negative) in $A$ if and only if $X$ is negative (resp. positive) in $B$, and $X$ is positive (resp. negative) in $C$;

- If $A = \forall v B$, and $v \neq X$, then $X$ is positive (resp. negative) in $A$ if and only if $X$ is positive (resp. negative) in $B$.

We add to the second order predicate calculus a new logic symbol $\mu$, and we allow a new construction for formulas: if $A$ is a formula, $C$ an $n$-ary predicate symbol which appears positively in $A$, $x_1, \ldots, x_n$ first order variables, and $t_1, \ldots, t_n$ terms, then $\mu C x_1 \ldots x_n A < t_1, \ldots, t_n >$ is a formula called the least fixed point of $A$ in $C$ calculated over the terms $t_1, \ldots, t_n$.

We extend the notions "$X$ is positive in a type" and "$X$ is negative in a type" by the following way: $X$ is positive (resp. negative) in $\mu C x_1 \ldots x_n A < t_1, \ldots, t_n >$ if and only if $X$ is positive (resp. negative) in $A$.

We extend the definition of the substitution by assuming that $C, x_1, \ldots, x_n$ are bounded in the formula $\mu C x_1 \ldots x_n A < t_1, \ldots, t_n >$.

We define on these formulas a binary relation $\subseteq$ by: $A \subseteq B$ if and only if it is obtained by using the following rules:

$$(ax) A \subseteq A$$

$$(\forall i_d) A[G/v] \subseteq B$$

$$(\forall i_d) \frac{A \subseteq B}{\forall v A \subseteq B} \ (1)$$

$$(e) \frac{A \subseteq B[v/y]}{A \subseteq B[w/y]} \ (3)$$

$$(\mu_g) D[\mu C x_1 \ldots x_m D < z_1, \ldots, z_m > / C(z_1, \ldots, z_m)] [t_1/x_1, \ldots, t_m/x_m] \subseteq \mu C x_1 \ldots x_m D < t_1, \ldots, t_m >$$

$$(\mu'_g) \mu C x_1 \ldots x_m D < t_1, \ldots, t_m > \subseteq D[\mu C x_1 \ldots x_m D < z_1, \ldots, z_m > / C(z_1, \ldots, z_m)] [t_1/x_1, \ldots, t_m/x_m]$$

$$(\mu_g) \frac{D[E/C(x_1, \ldots, x_m)] \subseteq E}{\mu C x_1 \ldots x_m D < t_1, \ldots, t_m > \subseteq E[t_1/x_1, \ldots, t_m/x_m]}$$

(1) $G$ is a formula if $v$ is a second order variable, and a term if $v$ is a first order variable.

(2) $v$ is not free in $A$.

(3) $v = w$ is a particular case of an equation of $E$.

$(\mu_d)$ and $(\mu'_g)$ are the rules of factorisation and development of a fixed point.

$(\mu_g)$ expresses the fact that $\mu C x_1 \ldots x_m D < t_1, \ldots, t_m >$ is a least fixed point.
We are going to describe a system of typed $\lambda$-calculus called theory of recursive types (shortened in $TTR$ for Théorie des Types Récursifs) where the types are formulas of language. The rules of typing are the following:

- The typing rules (1),..., (8) of $AF_2$ type system.
- $\text{\Gamma} \vdash_{TTR} t : A \quad A \subseteq B$ 
- $(\subseteq) \quad \text{\Gamma} \vdash_{TTR} t : B$
- $(\forall) \quad \text{\Gamma} \vdash_{TTR} t : \forall x_1,...,\forall x_m[\mu C x_1,...,x_m D < x_1,...,x_m \rightarrow E]$

where $\mu C x_1,...,x_m D < x_1,...,x_m$ is a least fixed point.

The rule $(\forall)$ expresses also the fact that $\mu C x_1,...,x_m D < t_1,...,t_m$ is a least fixed point.

**Theorem 3.2** ([12],[18]).

1) Conservation Theorem If $\text{\Gamma} \vdash_{TTR} t : A$, and $t \rightarrow_{\beta} t'$, then $\text{\Gamma} \vdash_{TTR} t' : A$.

2) Strong normalization If $\text{\Gamma} \vdash_{TTR} t : A$ without using the rule $(\forall)$, then $t$ is strongly normalizable.

3) Weak normalization If $\text{\Gamma} \vdash_{TTR} t : A$, and if all least fixed points of $A$ are positives, then $t$ is normalizable.

The $TTR^\circ$ type system is the subsystem of $TTR$ where we only have propositional variables and constants (predicate variables or predicate symbols are of arity 0). So, first order variables, function symbols, and finite sets of equations are useless. With each predicate variable (resp. predicate symbol) $X$, we associate a predicate variable (resp. a predicate symbol) $X^\circ$ of $TTR^\circ$ type system. For every formula $A$ of $TTR$, we define the formula $A^\circ$ of $TTR^\circ$ obtained by forgetting in $A$ the first order part. If $\text{\Gamma} = x_1 : A_1,...,x_n : A_n$ is a context of $TTR$, then we denote by $\text{\Gamma}^\circ$, the context $x_1 : A_1^\circ,...,x_n : A_n^\circ$ of $TTR^\circ$. We write $\text{\Gamma} \vdash_{TTR^\circ} t : A$ if $t$ is tyable in $TTR^\circ$ of type $A$ in the context $\text{\Gamma}$.

**Theorem 3.3** If $\text{\Gamma} \vdash_{TTR} t : A$, then $\text{\Gamma}^\circ \vdash_{TTR^\circ} t : A^\circ$.

**Proof** By induction on the length of the derivation $\text{\Gamma} \vdash_{TTR} t : A$. □

**Theorem 3.4**

1) Conservation Theorem If $\text{\Gamma} \vdash_{TTR^\circ} t : A$, and $t \rightarrow_{\beta} t'$, then $\text{\Gamma} \vdash_{TTR^\circ} t' : A$.

2) Strong normalization If $\text{\Gamma} \vdash_{TTR^\circ} t : A$ without using the rule $(\forall)$, then $t$ is strongly normalizable.

3) Weak normalization If $\text{\Gamma} \vdash_{TTR^\circ} t : A$, and if all least fixed points of $A$ are positives, then $t$ is normalizable.
We use Theorems 3.2 and 3.3. □

Remark We cannot if the reverse of 2)-Theorem 3.2 is true, but the λ-term
\[ t = \lambda x ((\lambda y ((x)(y)) (\lambda x y) (\lambda x y))(\lambda x x))(\lambda y y) (\lambda x y)(\lambda x x) \]
which is strongly normalizable, and untypable in AF² type system (see [3]) is typable in TTR type system. Indeed, if we take \( B = \mu C (\forall X X \rightarrow C) \), we check easily that \( \vdash_{TTR} t : [B \rightarrow (B \rightarrow B)] \rightarrow B \).

4 Properties of TTR type system

4.1 Permutations Lemmas

Lemma 4.1 1) The typing rules (5), (7), and (8) are admissible.
2) In the typing, we may replace the succession of \( n \) times (\( \subseteq \)) and \( m \) times (\( \subseteq \)) (resp. (6)), by the succession of \( m \) times (\( \subseteq \)) (resp. (6)) and \( n \) times (\( \subseteq \)).
3) If \( \Gamma \vdash_{TTR} t : B \) is derived from \( \Gamma \vdash_{TTR} t : A \), then we may assume that we begin by the applications of (4), (6), and next by (\( \subseteq \)).

Proof Easy. □

Lemma 4.2 1) If \( A \subseteq B \), then, for every sequence of terms and/or formulas \( G \), \( A[G/v] \subseteq B[G/v] \), and we use the same proof rules.
2) If \( \Gamma \vdash_{TTR} t : A \), then, for every sequence of terms and/or formulas \( G \), \( \Gamma[G/v] \vdash_{TTR} t : A[G/v] \), and we use the same typing rules.

Proof By induction on the length of the derivation \( A \subseteq B \) (resp. \( \Gamma \vdash_{TTR} t : A \)). □

Corollary 4.1 If \( \Gamma, x : A \vdash_{TTR} (x)u_1...u_n : B \), then :
\( n = 0 \), and there is \( v_0 \) not free in \( A \) and \( \Gamma \), such that \( \forall v_0 A \subseteq B \),
or
\( n \geq 1 \), and there are types \( C_i, B_i \) (\( i = 1, ..., n \)) and \( v_i(i = 1, n) \) not free in \( A \) and \( \Gamma \), such that \( \forall v_0 A \subseteq C_1 \rightarrow B_1, \forall v_1 B_i \subseteq C_{i+1} \rightarrow B_{i+1} \) \( 1 \leq i \leq n - 1 \), \( \forall v_n B_n \subseteq B \), and \( \Gamma, x : A \vdash_{TTR} u_i : C_i \) \( 1 \leq i \leq n \).

Proof By induction on \( n \). □

Lemma 4.3 1) If \( X \) is positive (resp. negative) in \( D \), and \( A \subseteq B \), then \( D[A/X] \subseteq D[B/X] \) (resp. \( D[B/X] \subseteq D[A/X] \)).
2) We may eliminate the rule (\( \mu_d' \)).

Proof 1) By induction on \( D \).
2) By rule (\( \mu_d \)), we have \( A[\mu C x_1...x_n A < y_1, ..., y_n > /C(y_1, ..., y_n)] \subseteq \mu C x_1...x_n A < x_1, ..., x_n >, \)
then, by 1), \( A[\mu C x_1...x_n A < y_1, ..., y_n > /C(y_1, ..., y_n)] /C(x_1, ..., x_n) \) \( \subseteq A[\mu C x_1...x_n A < x_1, ..., x_n > /C(x_1, ..., x_n)] \), and, by using the rule \((\mu g)\), we obtain \( \mu C x_1...x_n A < t_1, ..., t_n > /C(y_1, ..., y_n)])[t_1/x_1, ..., t_n/x_n]. \)

4.2 Without-arrow types and arrow types

**Definitions**
1) A type \( A \) is said to be without-arrow type if and only if \( A \) does not contain any arrow.
2) Each without-arrow type \( A \) contains a unique atomic formula \( X(t_1, ..., t_n) \). We denote \( X \) by \( At(A) \).

We distinguish between two kinds of without-arrow types:
- A without-arrow type \( A \) is said to be of kind 1 if and only if \( At(A) \) is free in \( A \).
- A without-arrow type \( A \) is said to be of kind 2 if and only if \( At(A) \) is bounded in \( A \).

**Lemma 4.4** 1) If \( A \) is a without-arrow type of kind 1, and \( A \subseteq B \), then \( B \) is a without-arrow type of kind 1, and \( At(B) = At(A) \).
2) If \( A \) is a without-arrow type of kind 2, then, for every type \( B \), we have \( A \subseteq B \).

**Proof** 1) By induction on the length of the derivation \( A \subseteq B \).
2) Easy. \( \square \)

**Definition** A type \( A \) is said to be arrow type if and only if \( A \) contains at least an arrow.

**Lemma 4.5** If \( A \) is an arrow type, and \( A \subseteq B \), then \( B \) is an arrow type.

**Proof** By induction on the length of the derivation \( A \subseteq B \). \( \square \)

**Corollary 4.2** Let \( A \) be an atomic formula. If \( \Gamma \vdash_{TTT} t : A \), then \( t \) does not begin by \( \lambda \). Other words, if \( \Gamma \vdash_{TTT} \lambda x u : B \), then \( B \) is an arrow type.

**Proof** If \( t \) begins by \( \lambda \), then there are \( E, F \), and \( v \), such that \( \forall v(E \rightarrow F) \subseteq A \), therefore, by Lemma 4.5, \( A \) is an arrow type. \( \square \)

**Definition** For every arrow type \( A \), we define the type \( Rep(A) \) as follows, by induction on \( A \):
- \( Rep(E \rightarrow F) = E \rightarrow F \);
- \( Rep(\forall v B) = \forall v Rep(B) \);
- \( Rep(\mu C x_1...x_n B < t_1, ..., t_n >) = Rep(B)[\mu C x_1...x_n B < y_1, ..., y_n > /C(y_1, ..., y_n)][t_1/x_1, ..., t_n/x_n] \).

**Lemma 4.6** If \( A \) is an arrow type, then :
1) there are \( G, D \) and \( v \) such that \( Rep(A) = \forall v(G \rightarrow D) \).
2) \( A \subseteq Rep(A) \), and \( Rep(A) \subseteq A \).
Proof By induction on $A$. □

Remark. The Lemma 4.6 means that if $A$ is an arrow type, then $\text{Rep}(A)$ is an "equivalent" type to $A$ of the form $\forall v (G \to D)$. In the rest of the paper, we denoted $G$ by $A_g$ and $D$ by $A_d$.

Lemma 4.7 Let $A, B$ be two types, and $X, X'$ two predicate variables or predicate symbols, such that $X'$ is not free in $A$.
1) If $X$ is positive in $A$, and $X'$ is positive in $B$, then $X'$ is positive in $A[B/X]$.
2) If $X$ is negative in $A$, and $X'$ is negative in $B$, then $X'$ is negative in $A[B/X]$.
3) If $X$ is positive in $A$, and $X'$ is negative in $B$, then $X'$ is negative in $A[B/X]$.
4) If $X$ is negative in $A$, and $X'$ is positive in $B$, then $X'$ is positive in $A[B/X]$.

Proof By induction on $A$. □

Lemma 4.8 Let $A$ be an arrow type.
1) If $X$ is positive (resp. negative) in $A$, then $X$ is positive (resp. negative) in $\text{Rep}(A)$.
2) If $G$ is a sequence of terms and/or formulas, then $\text{Rep}(A[G/v]) = \text{Rep}(A)[G/v]$.

Proof 1) We argue by induction on $A$. The only non-trivial case is the one where $A = \mu C x_1 ... x_n B < t_1, ..., t_n >$. If $X$ is positive (resp. negative) in $A$, then $X$ is positive (resp. negative) in $B$. By the induction hypothesis, we have $X$ is positive (resp. negative) in $\text{Rep}(B)$, therefore, by Lemma 4.7, $X$ is positive (resp. negative) in $\text{Rep}(A)$.
2) By induction on $A$. □

Theorem 4.1 Let $A, B$ be two arrow types, such that $\text{Rep}(A) = \forall v (A_g \to A_d)$ and $\text{Rep}(B) = \forall v'(B_g \to B_d)$. If $A \subseteq B$, then there is a sequence of terms and/or formulas $G$, such that $B_g \subseteq A_g[G/v]$, and $A_d[G/v] \subseteq B_d$.

Proof We argue by induction on the length of the derivation $A \subseteq B$. Let us look at the rule used in the last step. The only non-trivial cases are:

- $(\text{tr})$: then $A \subseteq D$, and $D \subseteq B$. If $\text{Rep}(D) = \forall v'' (D_g \to D_d)$, by the induction hypothesis, there are sequences $G$ and $G''$ such that $D_g \subseteq A_g[G/v]$, $A_d[G/v] \subseteq D_d$, $B_g \subseteq D_g[G''/v'']$, and $D_d[G''/v''] \subseteq B_d$. It is clear that we may assume that $v''$ is not free in $A_g$ and $A_d$, therefore, by Lemma 4.2, we have $B_g \subseteq A_g[G/v][G''/v'']$, and $A_d[G/v][G''/v''] \subseteq B_d$. Let $G' = G[G''/v'']$, then $B_g \subseteq A_g[G'/v]$, and $A_d[G'/v] \subseteq B_d$.

- $(\mu_d)$: then $A = D[\mu C x_1 ... x_k D < y_1, ..., y_k > / C(y_1, ..., y_k)][t_1/x_1, ..., t_k/x_k]$, and $B = \mu C x_1 ... x_k D < t_1, ..., t_k >$. Therefore, by Lemma 4.8, $\text{Rep}(A) = \text{Rep}(B)$, $A_g = B_g$, and $B_d = A_d$, and so $B_g \subseteq A_g$, and $A_d \subseteq B_d$. 

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Corollary 4.3 Let $\vdash A$ Proof and Lemma 4.9 $x \subseteq T \Rightarrow \exists T$, $x - (G, T)$

By the induction hypothesis, we have $E$, and $D \subseteq g = E_d[t][x], t_k/x_k] = B_d$, $E_d[t][x], t_k/x_k] = B_d$.

By the induction hypothesis, there is a sequence $G$, such that $g \subseteq D_g[E/C(x_1, ..., x_k)][G/v]$ , and $D_d[E/C(x_1, ..., x_k)][G/v] \subseteq D_d$. $C$ is positive in $D$, therefore, by Lemma 4.8, $C$ is negative in $D_d$, and $C$ is positive in $D_d$.

Lemma 4.3, $E \subseteq D_g[\mu C x_1 ... x_k D < y_1, ..., y_k > /C(y_1, ..., y_k)][G/v]$ , and $D_d[\mu C x_1 ... x_k D < y_1, ..., y_k > /C(y_1, ..., y_k)][G/v] \subseteq E_d$, and so, by Lemma 4.2,

$E_g[t_1/x_1, ..., t_k/x_k] \subseteq D_g[\mu C x_1 ... x_k D < y_1, ..., y_k > /C(y_1, ..., y_k)][G/v][t_1/x_1, ..., t_k/x_k]$, and $D_d[\mu C x_1 ... x_k D < y_1, ..., y_k > /C(y_1, ..., y_k)][G/v][t_1/x_1, ..., t_k/x_k] \subseteq E_d[t_1/x_1, ..., t_k/x_k]$.

Let $\mathbf{G}' = \mathbf{G}[t_1/x_1, ..., t_k/x_k]$, then $B_g \subseteq A_g[\mathbf{G}' /v]$, and $A_d[\mathbf{G}' /v] \subseteq B_d$. □

**Corollary 4.3** Let $B$ be an atomic formula. If $\Gamma, x : A \rightarrow B \vdash_{TTR} (x) u_1 ... u_n : C$, then $n = 1$, and $\Gamma, x : A \rightarrow B \vdash_{TTR} u_1 : A$.

**Proof** By Corollary 4.1, we have $\forall v(A \rightarrow B) \subseteq F \rightarrow G$, $\Gamma, x : A \rightarrow B \vdash_{TTR} u_1 : F$, and $v$ is not free in $\Gamma$ and $A \rightarrow B$. Therefore, by Theorem 4.1, $F \subseteq A$, and $B \subseteq G$, then $\Gamma, x : A \rightarrow B \vdash_{TTR} u_1 : A$. If $n > 1$, then $\forall v'G \subseteq H \rightarrow J$, and $v'$ is not free in $\Gamma$ and $A \rightarrow B$. Therefore $\forall v' B \subseteq H \rightarrow J$, and $\forall v' B$ is a without-arrow type of kind 1. A contradiction. □

**Lemma 4.9** If $x_1 : A_1, ..., x_n : A_n \vdash_{TTR} t : A, B_i \subseteq A_i 1 \leq i \leq n$, and $A \subseteq B$, then $x_1 : B_1, ..., x_n : B_n \vdash_{TTR} t : B$.

**Proof** We argue by induction on $t$. The only non-trivial cases are :

- If $t = \lambda x u$, then $x_1 : A_1, ..., x_n : A_n, x : E \vdash_{TTR} u : F, \forall v(E \rightarrow F) \subseteq A$, and $v$ is not free in $E$ and $A_j 1 \leq j \leq n$. We may assume that $v$ is not free in $E$ and $B_j 1 \leq j \leq n$. By the induction hypothesis, we have $x_1 : B_1, ..., x_n : B_n, x : E \vdash_{TTR} u : F$, and so $x_1 : B_1, ..., x_n : B_n \vdash_{TTR} t : B$.

- If $t = (Y) u$, then $\forall v_1 \forall y_0 [\mu C y_1 ... y_m E < y_1, ..., y_m > /D)] \subseteq A$, $x_1 : A_1, ..., x_n : A_n \vdash_{TTR} u : \forall y_1 ... \forall y_m [C(y_1, ..., y_m) \rightarrow D] \rightarrow \forall y_1 ... \forall y_m [E \rightarrow D]$, $C$ is positive in $E$, $C$ is not free in $D$, and $v$ is not free in $A_j 1 \leq j \leq n$. We may assume that $v, C$ are not free in $B_j 1 \leq j \leq n$. By the induction hypothesis, we have $x_1 : B_1, ..., x_n : B_n \vdash_{TTR} u : \forall y_1 ... \forall y_m [C(y_1, ..., y_m) \rightarrow D] \rightarrow \forall y_1 ... \forall y_m [E \rightarrow D]$, and so $x_1 : B_1, ..., x_n : B_n \vdash_{TTR} (Y) u : A$. □
5 \ ∀\text{-positive types}

5.1 Properties of \ ∀\text{-positive types}

Definition We define two sets of types, the set \( \Omega^+ \) of \ ∀\text{-positive types}, and the set \( \Omega^- \) of \ ∀\text{-negative types} in the following way:

- If \( A \) is an atomic type, then \( A \in \Omega^+ \), and \( A \in \Omega^- \);
- If \( T^+ \in \Omega^+ \), and \( T^- \in \Omega^- \), then \( T^- \to T^+ \in \Omega^+ \), and \( T^+ \to T^- \in \Omega^- \);
- If \( T^+ \in \Omega^+ \), then \( \forall x T^+ \in \Omega^+ \);
- If \( T^- \in \Omega^- \), then \( \forall x T^- \in \Omega^- \);
- If \( T^+ \in \Omega^+ \), and \( X \) is not free in \( T^- \), then \( \forall X T^- \in \Omega^- \);
- If \( T^+ \in \Omega^+ \), and \( x_1, \ldots, x_n \) first order variables, \( t_1, \ldots, t_n \) terms, \( C \) an \( n \)-ary predicate symbol which appears and is positive in \( T^+ \), then \( \mu C x_1 \ldots x_n T^+ < t_1, \ldots, t_n > \in \Omega^+ \).

Remarks
1) A least fixed point is not a \ ∀\text{-negative type}.
2) If \( T^+ \in \Omega^+ \), then all least fixed points of \( T^+ \) are positives. Therefore, by 3)-Theorem 3.2, if \( \Gamma \vdash_{TTR} t : T^+ \), then \( t \) is normalizable.

Lemma 5.1 Let \( T^-, T'^- \in \Omega^- \), \( T^+, T'^+ \in \Omega^+ \), and \( X \) a predicate variable or predicate symbol.
1) If \( X \) is positive (resp. negative) in \( T^- \), then \( T^-[T'^-/X] \in \Omega^- \) (resp. \( T^-[T'^+/X] \in \Omega^- \)).
2) If \( X \) is positive (resp. negative) in \( T^+ \), then \( T^+[T'^+/X] \in \Omega^+ \) (resp. \( T^+[T'^-/X] \in \Omega^+ \)).
3) If \( T[F/X] \in \Omega^+ \) (resp. \( T[F/X] \in \Omega^- \)), then \( T \in \Omega^+ \) (resp. \( T \in \Omega^- \)).

Proof 1), 2) By induction on \( T^- \) and \( T^+ \).
3) By induction on \( T \). \( \square \)

Definition With each type \( T \) of \( TTR \), we associate the set \( Fv_2(T) \) of free predicate variables and free predicate symbols of \( T \).

Theorem 5.1 Let \( T^- \in \Omega^- \), and \( T^+ \in \Omega^+ \).
1) If \( T^- \subseteq A \), then \( A \in \Omega^- \), and \( Fv_2(A) \subseteq Fv_2(T^-) \).
2) If \( B \subseteq T^+ \), then \( B \in \Omega^+ \), and \( Fv_2(B) \subseteq Fv_2(T^+) \).

Proof We argue by induction on the length of the derivations \( T^- \subseteq A \), and \( B \subseteq T^+ \). Let us look at the rule used in the last step.
1) The only non-trivial case is \((\mu_d)\).

Then \(T^- = T'[\muCx_1\ldots x_nT' < y_1, \ldots, y_n > / C(y_1, \ldots, y_n)][t_1/x_1, \ldots, t_n/x_n]\), and
\(A = \muCx_1\ldots x_nT' < y_1, \ldots, y_n >\). Since \(T^- \in \Omega^-\), then, by Lemma 5.1, \(\muCx_1\ldots x_nT' < y_1, \ldots, y_n > \in \Omega^-\), which is impossible.

2) The only non-trivial cases are : 

- \((\mu_d)\) : then \(B = D[\muCx_1\ldots x_nD < y_1, \ldots, y_n > / C(y_1, \ldots, y_n)][t_1/x_1, \ldots, t_n/x_n]\), and \(T^+ = \muCx_1\ldots x_nD < t_1, \ldots, t_n >\). Since \(T \in \Omega^+\), then \(D \in \Omega^+\), and so, by Lemma 5.1, \(B \in \Omega^+\), and \(Fv_2(B) = Fv_2(D) - \{C\} = Fv_2(T^+)\).

- \((\mu_g)\) : then \(B = \muCx_1\ldots x_nD < t_1, \ldots, t_n >\), \(T^+ = E[t_1/x_1, \ldots, t_n/x_n]\), and \(D[E/C(x_1, \ldots, x_n)] \subseteq E\). Since \(T^+ \in \Omega^+\), then \(E \in \Omega^+\), and, by the induction hypothesis, \(D[E/C(x_1, \ldots, x_n)] \in \Omega^+\), and \(Fv_2(D[E/C(x_1, \ldots, x_n)]) \subseteq Fv_2(E)\). By Lemma 5.1, we have \(D \in \Omega^+\), and \(Fv_2(D) - \{C\} \subseteq Fv_2(D[E/C(x_1, \ldots, x_n)]) \subseteq Fv_2(E)\), and so \(B \in \Omega^+\), and \(Fv_2(B) = Fv_2(D) - \{C\} \subseteq Fv_2(D[E/C(x_1, \ldots, x_n)]) \subseteq Fv_2(E) = Fv_2(T^+)\). \(\square\)

### 5.2 The TTR\(_0\) type system

We define on the types of TTR a binary relation \(\subseteq_0\) by the following way : 
\(A \subseteq_0 B\) if and only if \(A \subseteq B\), and in the proof we use only the weak version of \((\forall_i)\) :

\[ (\forall_i)_{\theta_0} \quad \frac{A[G/v] \subseteq_0 B}{\forall v A \subseteq_0 B} \]

where \(G\) is a term if \(v\) is an individual variable, and \(G\) is a predicate variable or a predicate symbol having the same arity of \(v\) if \(v\) is a predicate variable.

**Lemma 5.2** If \(A \subseteq_0 B\), then, for every sequence of terms and/or formulas \(G\), \(A[G/v] \subseteq_0 B[G/v]\), and we use the same proof rules.

**Proof** Same proof as 1)-Lemma 4.2. \(\square\)

**Lemma 5.3** Let \(A\) be an arrow type, and \(\text{Rep}(A) = \forall v(A_g \rightarrow A_d)\).

1) If \(A \in \Omega^-\) (resp. \(A \in \Omega^+\)), then \(A_g \in \Omega^+\), and \(A_d \in \Omega^-\) (resp. \(A_g \in \Omega^-\), and \(A_d \in \Omega^+\)).
2) \(A \subseteq_0 \text{Rep}(A)\), and \(\text{Rep}(A) \subseteq_0 A\).

**Proof** By induction on \(A\). \(\square\)

**Lemma 5.4** If \(T^- \in \Omega^-\), \(T^+ \in \Omega^+\), and \(T^- \subseteq T^+\), then \(T^- \subseteq_0 T^+\).

**Proof** By induction on the length of the derivation \(T^- \subseteq T^+\). \(\square\)

**Definition** We denote by TTR\(_0\), the \(TTR\) type system whithout the rules (5), (7), (8) and by replacing the rule (\(\subseteq\)) by :
Theorem 5.2  Let $A_1, \ldots , A_n \in \Omega^−$, $\Gamma = x_1 : A_1, \ldots , x_n : A_n, A \in \Omega^+$, and $t$ a normal $\lambda$-term. If $\Gamma \vdash_{TTR} t : A$, then $\Gamma \vdash_{TTR_0} t : A$, and in this typing each variable is assigned of a $\forall$-negative type, and each $u \in STE(t)$ is typable of a $\forall$-positive type.

Proof We argue by induction on $t$.

- If $t = x_i$ 1 $\leq i \leq n$, then $\forall v A_i \subseteq A$, and $v$ is not free in $\Gamma$. Since $A_i \in \Omega^−$, then $\forall v A_i \in \Omega^−$, and, by Lemma 5.4, $\forall v A_i \subseteq_0 A$. Therefore $\Gamma \vdash_{TTR_0} t : A$.

- If $t = \lambda xu$, then $\Gamma, x : B \vdash_{TTR} u : C$, $\forall v (B \rightarrow C) \subseteq A$, and $v$ is not free in $\Gamma$. Since $\forall v (B \rightarrow C)$ is an arrow type, then, by Lemma 4.5, $A$ is an arrow type. If $Rep(A) = \forall v' (A_d \rightarrow A_d)$, then, by 1)-Lemma 5.3, $A_d \in \Omega^−$, and $A_d \in \Omega^+$. By Theorem 4.1, there is a sequence $G$, such that $A_g \subseteq B[G/v]$ , and $C[G/v] \subseteq A_d$. By 2)-Lemma 4.2, we have $\Gamma, x : B[G/v] \vdash_{TTR} u : C[G/c]$ , and, by Lemma 4.9, $\Gamma, x : A_g \vdash_{TTR} u : A_d$. By the induction hypothesis, we have $\Gamma, x : A_g \vdash_{TTR_0} u : A_d$, and so, by 2)-Lemma 5.3, $\Gamma \vdash_{TTR_0} t : A$.

- If $t = (x_i)u_1 \ldots u_k$ 1 $\leq i \leq n$ and $k \neq 0$, then $\forall v_0 A_i \subseteq C_1 \rightarrow B_1, \forall v_j B_i \subseteq C_{j+1} \rightarrow B_{j+1}$ 1 $\leq j \leq k-1, \forall v_k B_k \subseteq A$ where $v_0, \ldots , v_k$ are not free in $\Gamma$, and $\Gamma \vdash_{TTR} u_j : C_j 1 \leq j \leq k$.

By Theorems 4.1, 5.1, and Lemmas 4.4, 5.4, we have

- $A_i = \forall v_0 A_i', A_i' = C_1' \rightarrow \forall v_1 B_1', B_j = C_j' \rightarrow \forall v_j B_j' 1 \leq j \leq k$,
- $C_j \subseteq C_j'[G_0/v_0'][G_{j-1}/v_{j-1}'][G_{j-1}/v_{j-1}][G_{j-1}/v_{j-1}] \subseteq B_j 1 \leq j \leq k$,
- $\forall v_k \forall v_k' B_k[G_0/v_0'][G_{k-1}/v_{k-1}] \subseteq A$.

Since $\Gamma \vdash_{TTR} u_j : C_j 1 \leq j \leq k$, then $\Gamma \vdash_{TTR} u_j : C_j'[G_0/v_0'][G_{j-1}/v_{j-1}]$, and, by the induction hypothesis, $\Gamma \vdash_{TTR_0} u_j : C_j'[G_0/v_0'][G_{j-1}/v_{j-1}]$. It is easy to check that $\Gamma \vdash_{TTR_0} t : B_k'[G_0/v_0'][G_{k-1}/v_{k-1}]$, then $\Gamma \vdash_{TTR_0} t : \forall v_k \forall v_k' B_k[G_0/v_0'][G_{k-1}/v_{k-1}]$, and $\Gamma \vdash_{TTR_0} t : A$. □

6 Gödel transformation

6.1 $\bot$-types of $TTR$

Definition Let $A$ be a type of $TTR$. We say that $A$ is an $\bot$-type if and only if $A$ is obtained by the following rules :

- $\bot$ is an $\bot$-type.
- If $A$ is an $\perp$-type, then $B \rightarrow A$ is an $\perp$-type for every type $B$.

- If $A$ is an $\perp$-type, then $\forall v A$ is an $\perp$-type for every variable $v$.

- If $A$ is an $\perp$-type, $C$ an $n$-ary predicate symbol which appears and is positive in $A$, $x_1, \ldots, x_n$ first order variables, and $t_1, \ldots, t_n$ terms, then $\mu C x_1 \ldots x_n A < t_1, \ldots, t_n >$ is an $\perp$-type.

**Lemma 6.1** If $A$ is an $\perp$-type, and $A \subseteq B$, then $B$ is an $\perp$-type.

**Proof** By induction on the length of the derivation $A \subseteq B$. □

**Lemma 6.2** Let $t$ be a normal $\lambda$-term, $A_1, \ldots, A_n \in \Omega^-$, $A \in \Omega^+$, $\perp$ does not appear in the types $A_1, \ldots, A_n, A$, and $B_1, \ldots, B_m$ are $\perp$-types. If $\Gamma = x_1 : A_1, \ldots, x_n : A_n, y_1 : B_1, \ldots, y_m : B_m \vdash_{\text{TTR}} t : A$, then $x_1 : A_1, \ldots, x_n : A_n \vdash_{\text{TTR}} t : A$.

**Proof** We argue by induction on $t$.

- If $t$ is a variable, then $t = x_i 1 \leq i \leq n$ or $t = y_i 1 \leq i \leq m$.
  
  - The case $t = x_i$ is trivial.
  
  - If $t = y_i$, then $\forall v B_i \subseteq A$ and $v$ is not free in $\Gamma$. Since $B_i$ is an $\perp$-type, then, by Lemma 6.1, $A$ is an $\perp$-type, and $\perp$ appears in $A$. A contradictoire.

- If $t = \lambda x_{n+1} t'$, then $\Gamma, x_{n+1} : A_{n+1} \vdash_{\text{TTR}} t' : D$, $\forall v (A_{n+1} \rightarrow D) \subseteq A$, $v$ is not free in $\Gamma$. Since $A \in \Omega^+$, then, by Theorem 5.1, we have $A_{n+1} \in \Omega^-, D \in \Omega^+$, and $Fv_2(\forall v (A_{n+1} \rightarrow D)) \subseteq Fv_2(A)$. Therefore $\perp$ does not appear in $A_{n+1}$ and $D$. By the induction hypothesis, we have $x_1 : A_1, \ldots, x_n : A_n, x_{n+1} : A_{n+1} \vdash_{\text{TTR}} t : D$, and so $x_1 : A_1, \ldots, x_n : A_n \vdash_{\text{TTR}} t : A$.

- If $t = (x) u_1 \ldots u_k k \geq 1$, then two case can be see :
  
  - If $x = y_i 1 \leq i \leq m$, then, by Corollary 4.1, we have $\forall v_0 B_i \subseteq C_1 \rightarrow D_1, \forall v_j D_j \subseteq C_{j+1} \rightarrow D_{j+1} 1 \leq i \leq k - 1, \forall v_k D_k \subseteq A$, where $v_0, \ldots, v_k$ are not free in $A$ and $\Gamma$, and $\Gamma \vdash_{\text{TTR}} u_j : C_j 1 \leq j \leq k$. Since $B_i$ is an $\perp$-type, then, by Lemma 6.1, $D_j 1 \leq j \leq k$ and $A$ are $\perp$-types, and $\perp$ appears in $A$. A contradictoire.

  - If $x = x_i 1 \leq i \leq n$, then, by Corollary 4.1, we have $\forall v_0 A_i \subseteq C_1 \rightarrow D_1, \forall v_j D_j \subseteq C_{j+1} \rightarrow D_{j+1} 1 \leq j \leq k - 1, \forall v_k D_k \subseteq A$, where $v_0, \ldots, v_k$ are not free in $A$ and $\Gamma$, and $\Gamma \vdash_{\text{TTR}} u_j : C_j 1 \leq j \leq k$. Since $A_i \in \Omega^-$, then, by Theorem 5.1, we have $C_j \in \Omega^+, D_i \in \Omega^- 1 \leq j \leq k$, and $Fv_2(C_j) \cup Fv_2(D_j) \subseteq Fv_2(A_i) 1 \leq j \leq k$. Therefore $\perp$ does not appear in $C_j 1 \leq j \leq k$. By the inductive hypothesis, we have $x_1 : A_1, \ldots, x_n : A_n \vdash_{\text{TTR}} u_j : C_j 1 \leq j \leq k$, and so $x_1 : A_1, \ldots, x_n : A_n \vdash_{\text{TTR}} t : A$. □
6.2 Gödel transformations

**Definition** With each predicate variable $X$, we associate a finite no empty set of predicate variables $V_X = \{X_1, ..., X_r\}$ having the same arity of $X$, such that: if $X \neq Y$, then $V_X \cap V_Y = \emptyset$. With each $n$-ary predicate variable $X$, and with each sequence of individual variables $x_1, ..., x_n$, we associate a formula $F_X$ such that:

- $F_X$ is an $\bot$-type;
- $F_X$ does not contain any predicate symbol;
- the free variables of $F_X$ are among $x_1, ..., x_n$ and the elements of $V_X$.

For each formula $A$, we define the formula $A^*$ by the following induction way:

- If $A = C(t_1, ..., t_n)$, and $C$ is a predicate symbol, then $A^* = A$.
- If $A = X(t_1, ..., t_n)$, and $X$ is a predicate variable, then $A^* = F_X[t_1/x_1, ..., t_n/x_1]$.
- If $A = B \rightarrow C$, then $A^* = B^* \rightarrow C^*$.
- If $A = \forall x B$, then $A^* = \forall x B^*$.
- If $A = \forall X B$, then $A^* = \forall X_1 ... \forall X_r B^*$, where $V_X = \{X_1, ..., X_r\}$.
- If $A = \mu C x_1 ... x_n D < t_1, ..., t_n >$, then $A^* = \mu C x_1 ... x_n D^* < t_1, ..., t_n >$.

$A^*$ is called the Gödel transformation of $A$.

**Remark.** In order to show that the above transformation is well defined, we need to prove the following Lemma:

**Lemma 6.3** Let $C$ be a predicate variable or a predicate symbol, and $A$ a type of TTR. If $C$ is positive in $A$ (resp. negative in $A$), then $C$ is positive in $A^*$ (resp. negative in $A^*$).

**Proof** By induction on $A$. □

**Lemma 6.4** 1) If $A \subseteq_0 B$, then $A^* \subseteq_0 B^*$, and we use the same proof rules.
2) If $\Gamma \vdash_{TTR_0} t : A$, then $\Gamma^* \vdash_{TTR_0} t : A^*$, and we use the same typing rules.

**Proof** By induction on the length of the derivation $A \subseteq_0 B$ (resp. $\Gamma \vdash_{TTR_0} t : A$). □

**Corollary 6.1** Let $D \in \Omega^+$, and $t$ a normal $\lambda$-term. If $\vdash_{TTR} t : D$, then $\vdash_{TTR} t : D^*$.

**Proof** By induction on the length of the derivation $\vdash_{TTR} t : D$, and we use Theorem 5.2 and Lemma 6.4. □
7 Storage operators

7.1 Definition of storage operators

Definitions

1) Let $T$ be a closed $\lambda$-term, and $D, E$ two closed types of $TTR$ (resp. $TTR^\circ$). We say that $T$ is a storage operator for the pair of types $(D, E)$ if and only if for every $\lambda$-term $t$ with $\vdash_{TTR} t : D$ (resp. $\vdash_{TTR^\circ} t : D$), there are $\lambda$-terms $\tau_t$ and $\tau_t'$ such that $\tau_t \simeq_{\beta} \tau_t'$, $\vdash_{TTR} \tau_t' : E$ (resp. $\vdash_{TTR^\circ} \tau_t' : E$), and for every $\theta_t \simeq_{\beta} t$, $(T)\theta_t f \succ (f)\tau_t[t_1/x_1, ..., t_n/x_n]$, where $Fv(\tau_t) = \{f, x_1, ..., x_n\}$ and $t_1, ..., t_n$ are $\lambda$-terms which depend on $\theta_t$.

2) If $D = E$, we say that $T$ is a storage operator for the type $D$.

Examples The type of recursive integers is the formula:

$$N^r[x] = \mu N x. \Phi(N, x) \ldotp x >$$

where

$$\Phi(N, x) = \forall Y \{\forall y(Ny \rightarrow Xsy), X0 \rightarrow Xx\}$$

($s$ is a unary function symbol for successor and 0 is a constant symbol for zero).

For each integer $n$, we define the recursive integer $\pi$ by induction: $\pi = \lambda f. \lambda x. and \pi + 1 = \lambda f. \lambda x. f(\pi x)$. Let $\mathcal{N}$ be the set of recursive integers.

We have $\mathcal{N} = \{t \mid t$ is a closed normal $\lambda$-term / $\vdash_{TTR} t : N^r[s^n(0)], n \geq 0\}$ (see [19]).

Let $\pi = \lambda n. \lambda f. \lambda x. f(n)$. It is easy to check that $\pi$ is a $\lambda$-term for successor, and $\vdash_{TTR} \pi : \forall y(N^r[y] \rightarrow N^r[sy])$.

Define

$$T_1 = (Y)H$$

where $H = \lambda x. \lambda y. (y)\lambda z. (G)(x)(z)\delta$, $G = \lambda x. \lambda y. (x)\lambda z. (y)(\pi)z$, and $\delta = \lambda f. (f)\pi$;

$$T_2 = \lambda \nu. (\nu)\rho \pi\rho$$

where $\tau = \lambda d. \lambda f. (f)\pi$, and $\rho = \lambda y. (G)$.

Then, for every $\theta_n \simeq_{\beta} \pi$, $(T_i)\theta_n f \succ (f)(\pi)\tau\pi (i = 1, 2)$.

Therefore, for every $n \geq 0$, $T_1$ and $T_2$ are storage operators for $N^r[s^n(0)]$.

Typing of $T_1$

We use in the typing the Godel transformation with $V_X = \{X\}$, and $F_X = \neg X(x_1, ..., x_n)$ for every second order variable $X$ of arity $n$.

- We have $\vdash_{TTR} \pi : N^r[0]$, then $\vdash_{TTR} \delta : \neg \neg N^r[0]$.
- We have $\vdash_{TTR} \pi : \forall y(N^r[y] \rightarrow N^r[sy])$, then

  $x : \neg \neg N^r[y], y : \neg \neg N^r[sy], z : N^r[y] \vdash_{TTR} (y)(\pi)z : \bot$ ; hence :

  $x : \neg \neg N^r[y], y : \neg \neg N^r[sy] \vdash_{TTR} (x)\lambda z. (y)(\pi)z : \bot$ ; therefore :

  $\vdash_{TTR} G : \forall y(\neg \neg N^r[y] \rightarrow \neg \neg N^r[sy])$. 

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• We have \( y : \Phi^*(N, x) \vdash_{TTR} y : \forall y(Ny \rightarrow \neg\neg N^r[sy]), \neg\neg N^r[0] \rightarrow \neg\neg N^r[x] \); thus:
\[ x : \forall x(Nx \rightarrow \neg\neg N^r[x]), y : \Phi^*(N, x), z : Ny \vdash_{TTR} (G)(x)z : \neg\neg N^r[sy] \]; therefore:
\[ x : \forall x(Nx \rightarrow \neg\neg N^r[x]), y : \Phi^*(N, x) \vdash_{TTR} \lambda z(G)(x)z : \forall y(Ny \rightarrow \neg\neg N^r[sy]) \]; hence:
\[ x : \forall x(Nx \rightarrow \neg\neg N^r[x]) \vdash_{TTR} \lambda y((y)\lambda z(G)(x)z)\delta : \forall x(\Phi^*(N, x) \rightarrow \neg\neg N^r[x]) \]; therefore:
\[ \vdash_{TTR} H : \forall x(Nx \rightarrow \neg\neg N^r[x]) \rightarrow \forall x(\Phi^*(N, x) \rightarrow \neg\neg N^r[x]) \).

And finally \( \vdash_{TTR} T_1 : \forall x\{N^r*[x] \rightarrow \neg\neg N^r[x]\} \).

**Typing of \( T_2 \)**

We use in the typing the Gödel transformation with \( V_X = \{X, X'\} \), and
\( F_X = X(x_1, ..., x_n), X'(x_1, ..., x_n) \downarrow \) for every second order variable \( X \) of arity \( n \).

Let \( R = \forall X\forall y\{(X, X \rightarrow \neg\neg N^r[0], X \rightarrow \neg\neg N^r[y]), X \rightarrow \neg\neg N^r[xy]\} \), \( D = R \rightarrow \neg\neg N^r[0] \), and
\( F[x] = R, D, R \rightarrow \neg\neg N^r[x] \).

• \( \vdash_{TTR} \lambda f\bar{0} : \neg\neg N^r[0] \); therefore: \( \vdash_{TTR} \tau : X \rightarrow \neg\neg N^r[0] \), and \( \vdash_{TTR} \tau : R \rightarrow \neg\neg N^r[0] \).

• By the previous typing, we have \( \vdash_{TTR} G : \forall y(\neg\neg N^r[y] \rightarrow \neg\neg N^r[xy]) \); hence:
\( y : X, X \rightarrow \neg\neg N^r[0], X \rightarrow \neg\neg N^r[y], z : X \vdash_{TTR} (G)(y)z\tau z : \neg\neg N^r[xy] \); therefore:
\( \vdash_{TTR} \rho : R \).

• Check that \( \Phi^*(\lambda xF[x]/N, x) \subseteq F[x] \).
\[
\Phi^*(\lambda xF[x]/N, x) = \forall X\forall X'\{\forall y(F[y], Xsy, X'sy \rightarrow \bot), (X0, X'0 \rightarrow \bot) \rightarrow (Xx, X'x \rightarrow \bot)\}.
\]

therefore by specifying \( Xx \) by \( R \), and \( X'x \) by \( \neg N^r[x] \); we obtain:
\( \Phi^*(\lambda xF[x]/N, x) \subseteq \forall y(F[y], R, \neg N^r[xy] \rightarrow \bot), (R, \neg N^r[0] \rightarrow \bot) \rightarrow (R, \neg N^r[x] \rightarrow \bot) \). We need to check that \( R \subseteq \forall y(F[y], R, \neg N^r[xy] \rightarrow \bot) \), this is absolutely true.

Therefore \( N^r*[x] \subseteq F[x] \) and \( \nu : N^r*[x] \vdash_{TTR} \nu : R, D, R \rightarrow \neg\neg N^r[x] \); then:
\( \nu : N^r*[x] \vdash_{TTR} (\nu)\rho\tau\rho : \neg\neg N^r[x] \); and finally \( \vdash_{TTR} T_2 : \forall x\{N^r*[x] \rightarrow \neg\neg N^r[x]\} \).

**7.2 General Theorem**

**Theorem 7.1** Let \( D, E \) be two \( \forall \)-positive closed types of \( TTR \), such that \( \bot \) does not appear in \( E \). If \( \vdash_{TTR} T : D^* \rightarrow \neg\neg E \), then \( T \) is a storage operator for the pair \( (D, E) \).

**Proof** It is a consequence from the following Theorem:

**Theorem 7.2** Let \( D, E \) be two \( \forall \)-positive closed types of \( TTR^2 \), such that \( \bot \) does not appear in \( E \). If \( \vdash_{TTR} T : D^* \rightarrow \neg\neg E \), then \( T \) is a storage operator for the pair \( (D, E) \).

Indeed:
Lemma 7.1 1) If \( T \in \Omega^+ \) (resp. \( T \in \Omega^- \)) then \( T^\circ \in \Omega^+ \) (resp. \( T^\circ \in \Omega^- \)).

2) For each Gödel transformation \(*\) of \( \text{TTR} \), there is a Gödel transformation \(*'\) of \( \text{TTR}^\circ \) such that : for every type \( D \) of \( \text{TTR} \), \( D^{\ast'} = D^\circ *' \).

Proof 1) By induction on \( T \).

2) \(*'\) is the restriction of \(*\) on the types of \( \text{TTR}^\circ \). \( \square \)

Let \( t \) be a normal \( \lambda\)-term, such that \( \vdash_{\text{TTR}} t : D \). If \( \vdash_{\text{TTR}} T : D^{\ast} \rightarrow \neg\neg E \), then, by Theorem 3.3, \( \vdash_{\text{TTR}^\circ} T : D^{\ast'} \rightarrow \neg\neg E^\circ \). By 2)-Lemma 7.1, there is a Gödel transformation \(*'\), such that \( \vdash_{\text{TTR}^\circ} T : D^{\ast'} \rightarrow \neg\neg E^\circ \). Therefore, there are \( \lambda\)-terms \( \tau_i \) and \( \tau_i' \), such that \( \tau_i \preceq_{\beta} \tau_i' \), \( \vdash_{\text{TTR}^\circ} \tau_i' : E^\circ \), and \( (T)t_i \succ (f)\tau_i[t_1/x_1,...,t_n/x_n] \). By 2)-Corollary 6.1, we have \( \vdash_{\text{TTR}} t : D^{\ast} \), then \( f : \neg E \vdash_{\text{TTR}} (T)t_i : \bot \), and \( f : \neg E \vdash_{\text{TTR}} (f)\tau_i[t_1/x_1,...,t_n/x_n] : \bot \). Therefore \( f : \neg E \vdash_{\text{TTR}} (f)\tau_i' : \bot \), and, by Corollary 4.1, \( \vdash_{\text{TTR}} \tau_i' : E \). \( \square \)

We give the proof of Theorem 7.2 in a particular case.

Let \( N^r = \mu N[\forall X\{N \rightarrow X, X \rightarrow X\}] \), and \(*\) the Gödel transformation with \( V_X = \{X\} \), and \( F_X = \neg X(x_1,...,x_n) \) for every second order variable \( X \) of arity \( n \).

We will prove that : If \( \vdash_{\text{TTR}^\circ} T : N^r^{\ast'} \rightarrow \neg\neg N^r \), then \( T \) is a storage operator for \( N^r \).

Because of : if \( t \) is a closed normal \( \lambda\)-term with \( \vdash_{\text{TTR}^\circ} t : N^r \), then \( t = \pi \) for a certain integer \( n \), and it is sufffie to prove that : If \( \vdash_{\text{TTR}^\circ} T : N^r^{\ast'} \rightarrow \neg\neg N^r \), then, for every \( n \geq 0 \), there is an \( m \geq 0 \) and \( \tau \preceq_{\beta} \pi \), such that, for every \( \lambda\)-term \( \theta_n \preceq_{\beta} \pi \), there is a substitution \( \sigma \), such that \( (T)\theta_n f \sim (f)\sigma(\tau) \).

Lemma 7.2 If \( \Gamma' = \Gamma, x : N^r^{\ast'} \vdash_{\text{TTR}^\circ} (x)u_1...u_n : \bot \), then \( n = 3 \), and there is a type \( G \), such that \( \Gamma' \vdash_{\text{TTR}^\circ} u_1 : N^r^{\ast'} \rightarrow \neg G \), \( \Gamma' \vdash_{\text{TTR}^\circ} u_2 : \neg G \), and \( \Gamma' \vdash_{\text{TTR}^\circ} u_3 : G \).

Proof By Corollary 4.1, we have \( \forall v_0 N^{r*} \subseteq A_1 \rightarrow B_1 \), \( \forall v_i B_i \subseteq A_{i+1} \rightarrow B_{i+1} \) \( 1 \leq i \leq n \), \( \forall v_i B_n \subseteq \bot \), \( v_0, ..., v_n \) are not free in \( N^{r*} \) and \( \Gamma \), and \( \Gamma' \vdash_{\text{TTR}^\circ} v_i : A_i \) \( 1 \leq i \leq n \). Since \( \forall v_0 N^{r} \subseteq A_1 \rightarrow B_1 \), then, by Theorem 4.1, there is a formula \( F \), such that \( A_1 \subseteq N^{r} \rightarrow \neg F \) and \( \neg F \rightarrow \neg F \subseteq B_1 \). We have also \( \forall v_1 B_1 \subseteq A_2 \rightarrow B_2 \), then \( \forall v_1(\neg F \rightarrow \neg F) \subseteq A_2 \rightarrow B_2 \), and, by Theorem 4.1, there is a sequence of formulas \( F_1 \), such that \( A_2 \subseteq \neg F[F_1/v_1] \) and \( \neg F[F_1/v_1] \subseteq B_2 \). Now, since \( \forall v_2 B_2 \subseteq A_3 \rightarrow B_3 \), we have \( \forall v_2(\neg F[F_1/v_1]) \subseteq A_3 \rightarrow B_3 \), and, by Theorem 4.1, there is a sequence of formulas \( F_2 \), such that \( A_3 \subseteq F[F_1/v_1,F_2/v_2] \) and \( \bot \subseteq B_3 \).

By Corollary 4.1, we have \( n = 3 \). Let \( G = F[F_1/v_1,F_2/v_2] \). Since \( v_1, v_2 \) are not free in \( N^{r*} \) and \( \Gamma \), we deduce \( \Gamma' \vdash_{\text{TTR}^\circ} u_1 : N^{r*} \rightarrow \neg G \), \( \Gamma' \vdash_{\text{TTR}^\circ} u_2 : \neg G \), and \( \Gamma' \vdash_{\text{TTR}^\circ} u_3 : G \). \( \square \)

Let \( n \geq 0 \).
**Definition** An n-special application $\theta$ is a function from $\{0, 1, ..., n\}$ to $\Lambda$ with the following properties: $\theta(0) \triangleright \bar{0}$ and $\theta(m + 1) \triangleright \lambda f_m \lambda x_m (f_m) \theta(m) \ 0 \leq m \leq n - 1$.

**Lemma 7.3** For every $\theta_n \simeq_{\beta} \bar{n}$, there is an n-special application $\theta$, such that $\theta(n) = \theta_n$.

**Proof** Easy. \qed

**Definitions**

1) Let $0 \leq m \leq n$ and $u = u_{m,1}, u_{m,2}, u_{m,3}, ..., u_{n-1,1}, u_{n-1,2}, u_{n-1,3}$ a sequence of $\lambda$-terms. We denoted by $x_{m,u}$ a constant which does not appear in $u$.

2) Let $\theta$ be an n-special application. The n-special substitution $S_{\theta}$ is the function on the set $\Lambda$ defined by induction:

- If $u = x$, then $S_{\theta}(x) = x$;
- If $u = \lambda x v$, then $S_{\theta}(u) = \lambda y S_{\theta}(v[y/x])$ where $y \not\in Fv(\theta(n))$;
- If $u = (v) w$, then $S_{\theta}(u) = (S_{\theta}(v)) S_{\theta}(w)$;
- If $u = x_{m,u}$, then
  
  $S_{\theta}(u) = \theta(m) \{ S_{\theta}(u_{m,1})/f_m, S_{\theta}(u_{m,2})/x_m, ..., S_{\theta}(u_{n-1,1})/f_{n-1}, S_{\theta}(u_{n-1,2})/x_{n-1} \}$.

An n-special substitution is the application $S_{\theta}$ associated to a some n-special application $\theta$.

**Lemma 7.4** Let $\{U_i \triangleright V_i\}_{1 \leq i \leq r}$ be a sequence of head reductions such that:

$V_i = (x_{m,u}) u_1 u_2 u_3 \ 0 \leq m \leq n, \ [U_{i+1} = (u_1) x_{m-1,u_1,u_2,u_3,u} u_3 \text{ if } m \neq 0, \text{ and } U_{i+1} = (u_2) u_3 \text{ if } m = 0], \text{ and } S_{\theta}$ an n-special substitution. For every $1 \leq i \leq r$, $S_{\theta}(U_1) \sim S_{\theta}(V_i)$.

**Proof** We argue by induction on $i$.

The case $i = 0$ is a consequence of Theorem 2.1. Assume that is true for $i$, and prove it for $i + 1$.

If $V_i = (x_{m,u}) u_1 u_2 u_3 \ 0 \leq m \leq n$, then

$S_{\theta}(V_i) = (\theta(m)[S_{\theta}(u_{m,1})/f_m, S_{\theta}(u_{m,2})/x_m, ..., S_{\theta}(u_{n-1,1})/f_{n-1}, S_{\theta}(u_{n-1,2})/x_{n-1}])$

$S_{\theta}(u_1) S_{\theta}(u_2) S_{\theta}(u_3)$.

- If $m \neq 0$, then $\theta(m) \triangleright \lambda f_{m-1} \lambda x_{m-1} (f_{m-1}) \theta(m-1)$, and $S_{\theta}(V_i) \sim (S_{\theta}(u_1)) \theta(m-1) [S_{\theta}(u_{m-1,1})/f_{m-1}, S_{\theta}(u_{m-1,2})/x_{m-1}, ..., S_{\theta}(u_{n-1,1})/f_{n-1}, S_{\theta}(u_{n-1,2})/x_{n-1}] S_{\theta}(u_3) = S_{\theta}(U_{i+1})$.

- If $m = 0$, then $\theta(m) \triangleright \lambda f \lambda xx$, and $S_{\theta}(V_i) \sim (\lambda f \lambda xx) S_{\theta}(u_1) S_{\theta}(u_2) S_{\theta}(u_3) \sim (S_{\theta}(u_2)) S_{\theta}(u_3) = S_{\theta}(U_{i+1})$.  

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By the induction hypothesis we have $S_\theta(U_1) \sim S_\theta(V_1)$, then $S_\theta(U_1) \sim S_\theta(U_{i+1})$, and, by Theorem 2.1, $S_\theta(U_1) \sim S_\theta(V_{i+1})$. □

**Definition** A context $\Gamma = f : \neg N, x_{n,u_0}, N^r_1, x_{m_1,u_1}, N^r_2, \ldots, x_{m_s,u_s}, N^r_s$ where $0 \leq m_j \leq n$, $1 \leq j \leq s$, is called $n$-good.

**Lemma 7.5** There is a sequence of head reductions $\{U_i \succ V_i\}_{1 \leq i \leq r}$ such that:

- $U_1 = (T)x_n.f$ and $V_r = (f)\tau$ where $\tau \simplf \bar{t}$ for some $l \geq 0$;
- $V_i = (x_{m_i,u_i})u_1u_2u_3$ with $0 \leq m_i \leq n$, and $U_{i+1} = (u_1)x_{m-1,u_i,u_2,u_3}u_3$ if $m \neq 0$, and $U_{i+1} = (u_2)u_3$ if $m = 0$;
- For every $1 \leq i \leq r$, there is an $n$-good context $\Gamma_i$ such that $\Gamma_i \vdash_{\text{TR}} V_i : \bot$.

**Proof** Since $\vdash_{\text{TR}} T : N^r \to \neg N^r$, then $x_n : N^r, f : \neg N^r \vdash_{\text{TR}} (T)x_n.f : \bot$, and, by Corollary 4.3 and Lemma 7.2, we have $(T)x_n.f \succ V_1$ where $V_1 = (f)\tau$ or $V_1 = (x_n)u_1u_2u_3$.

Assume that we have the head reduction $U_k \succ V_k$ and $V_k \neq (f)\tau$. Then $V_k = (x_{m_i,u_i})u_1u_2u_3$ with $0 \leq m_i \leq n$, and, by the induction hypothesis, there is a $n$-good context $\Gamma_k$ such that $\Gamma_k \vdash_{\text{TR}} (x_{m_i,u_i})u_1u_2u_3 : \bot$. By Lemma 7.2, there is a type $G$, such that $\Gamma_k \vdash_{\text{TR}} u_1 : N^r \to \neg G$, $\Gamma_k \vdash_{\text{TR}} u_2 : \neg G$, and $\Gamma_k \vdash_{\text{TR}} u_3 : G$.

- If $m = 0$, let $U_{k+1} = (u_2)u_3$. Let $\Gamma_{k+1} = \Gamma_k$. We have $\Gamma_{k+1} \vdash_{\text{TR}} U_k : \bot$.

- If $m \neq 0$, let $U_{k+1} = (u_1)x_{m-1,u_i,u_2,u_3,u}u_3$. The variable $x_{m-1,u_i,u_2,u_3,u}$ is not used before. Indeed, if it is, by Lemma 7.4, the $\lambda$-term $(T)\pi f$ is not solvable. That is impossible because $f : \neg N^r \vdash_{\text{TR}} (T)\pi f : \bot$. Therefore $\Gamma_{k+1} = \Gamma_k, x_{m-1,u_i,u_2,u_3,u}$: $N^r$ is an $n$-good context and $\Gamma_{k+1} \vdash_{\text{TR}} U_{k+1} : \bot$.

By Corollary 4.3 and Lemma 7.2, we have $U_{k+1} \succ V_{k+1}$ where $V_{k+1} = (f)\tau$ or $V_{k+1} = (x_{s,v})v_1v_2v_3$ with $0 \leq s \leq n$.

This construction always terminates. Indeed, if not, by Lemma 7.4, the $\lambda$-term $(T)\pi f$ is not solvable. That is impossible because $f : \neg N^r \vdash_{\text{TR}} (T)\pi f : \bot$.

Therefore there is $r \geq 0$ and an $n$-good context $\Gamma_r$ such that $V_r = (f)\tau$ and $\Gamma_r \vdash_{\text{TR}} V_r : \bot$. By Lemma 6.2, we have $\tau \simplf \bar{t}$ for some $l \geq 0$. □

Let $\theta_n$ be a $\lambda$-term such that $\theta_n \simplf \bar{\pi}$. By Lemma 7.3, let $\theta$ be an $n$-special application such that $\theta(n) = \theta_n$. Let $S_\theta$ the $n$-special substitution associated to $\theta$. By Lemma 7.4, we have for every $1 \leq i \leq r$, $(T)\theta_n f \sim S_\theta(V_i)$. In particular, for $i = n$, $(T)\theta_n f \sim S_\theta((f)\tau) = (f)S_\theta(\tau)$. Then $T$ is a storage operator for $N^r$. □
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