A SIMPLE PROOF OF THE KARUSH-KUHN-TUCKER THEOREM WITH FINITE NUMBER OF EQUALITY AND INEQUALITY CONSTRAINTS

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Abstract. We provide a simple and short proof of the Karush-Kuhn-Tucker theorem with finite number of equality and inequality constraints. The proof relies on an elementary linear algebra lemma and the local inverse theorem.

1. Introduction

Let $X$ be a normed real linear space. We denote by $X'$ the space of linear mapping from $X$ to $\mathbb{R}$ and by $X^*$ the dual space of $X$ i.e. the space of linear and continuous mapping from $X$ to $\mathbb{R}$. The infinite dimensional version of the famous Karush-Kuhn-Tucker theorem with finite number of equality and inequality constraints reads as follows.

Theorem 1.1. Let $\Omega$ be an open space of $X$ and $\{f_i: 0 \leq i \leq n + m\}$ a family of continuously differentiable functions from $\Omega$ to $\mathbb{R}$ where $n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}$. Let $x^*$ be a solution of the constraint minimization problem

$$\begin{cases}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) = 0, 1 \leq i \leq n, \\
& f_i(x) \leq 0, n + 1 \leq i \leq n + m,
\end{cases}$$

such that the family $\{f'_i(x^*) : i \in J(x^*)\}$ is linearly independent in $X^*$, where $J(x^*) = \{i: 1 \leq i \leq n + m \text{ and } f_i(x^*) = 0\}$.

Then there exist $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ and $(\mu_1, \cdots, \mu_m) \in ([0, +\infty[^m$ such that

$$f'_0(x^*) + \sum_{i=1}^n \lambda_i f'_i(x^*) + \sum_{j=1}^m \mu_j f'_{j+n}(x^*) = 0$$

and

$$\mu_j f_{j+n}(x^*) = 0, \forall 1 \leq j \leq m.$$
This famous theorem is a natural extension of the classical Lagrange multipliers theorem to the case of the minimization problem with finite number of equality and inequality constraints. Its finite dimensional version has been originally derived independently by Karush [7] and Kuhn and Tucker [8]. Since there, different proofs of the generalization of the Karush, Kuhn and Tucker theorem (KKT) to the infinite dimensional setting have been provided in many works (see, for instance, [1,5,6,9] and references therein). In three recent papers [2–4], Brezhneva, Tretyakov, and Wright have given some elementary and different proofs of the KKT Theorem respectively with equality constraints, inequality constraints and linear equality, and nonlinear inequality constraints. In this short note, inspired essentially by the paper [4], we give a new, detailed and simple proof of the KKT theorem with finite number of mixed equality and inequality constraints. Our proof relies essentially on a very simple but powerful lemma from linear algebra and the classical local inverse theorem in the finite dimensional setting.

2. Proof of the Karush, Kuhn and Tucker Theorem

Before starting the proof of Theorem 1.1, we introduce the following simple notations.

**Notation 1.** Let $N \in \mathbb{N}$.

1. The canonical basis of $\mathbb{R}^{N}$ is the vector family $\{e_1, \cdots, e_N\}$ defined be: $e_1 = (1, 0, \cdots, 0), e_2 = (0, 1, 0, \cdots, 0), \cdots, e_N = (0, \cdots, 0, 1)$.
2. $B_{\mathbb{R}^N}(0, r)$ is the open ball of $\mathbb{R}^{N}$ with center $0$ and radius $r > 0$.
3. $I_N$ is the unity matrix of size $(N, N)$.
4. $Id_{\mathbb{R}^N}$ is the identity mapping from $\mathbb{R}^{N}$ into itself.

Next, we prove the following elementary linear algebra lemma.

**Lemma 2.1.** Let $\{T_i : 1 \leq i \leq n\}$ be a finite family of a linear independent elements of $X'$. Then there exists a family $\{v_i : 1 \leq i \leq n\}$ of elements of $X$ such that

\begin{equation}
T_i(v_j) = \delta_{ij}, \quad \forall 1 \leq i, j \leq n
\end{equation}

where $\delta_{ij}$ is the Kronecker’s symbol.

**Remark 2.1.** The family $\{v_i : 1 \leq i \leq n\}$ will be called a quasi primal basis of $X$ associated to the family $\{T_i : 1 \leq i \leq n\}$.

**Proof.** Define the linear mapping $T : X \to \mathbb{R}^{n}$, $T(v) = (T_1(v), \cdots, T_n(v))$. Let us prove that $T$ is not surjective. Suppose that this is not true; then there exists a vector $\alpha = \cdots$
Let us first notice that up to replace $\Omega$ by the open subset $\{ \nu \in \Omega : f_i(x) < 0, \forall n \leq i \leq n + m \text{ and } i \notin J(x^*) \}$ and to set $\mu_{j-n} = 0$ for every $j \notin J(x^*)$, we can assume without loss of generality that $J(x^*) = \{ i : 1 \leq i \leq n + m \}$. Now we will first prove that the family $\{ f'_i(x^*) : 0 \leq i \leq n + m \}$ is linearly dependant in $X^*$. We argue by contradiction. According to Lemma 2.1 there exits a quasi primal basis $\{ v_i : 0 \leq i \leq n + m \}$ of $X$ associated to the family $\{ f'_i(x^*) : 0 \leq i \leq n + m \}$. Since $x^*$ belongs to the open subset $\Omega$, there exists a real number $r_0 > 0$ such that the mapping defined for every $t = (t_i)_{0 \leq i \leq n+m}$ in $B_{R^{m+n+1}}(0, r_0)$ by

$$\Phi(t) = (f_0(\sigma(t)), \cdots, f_{m+n}(\sigma(t)))$$

where

$$\sigma(t) = x^* + \sum_{i=0}^{m+n} t_i v_i,$$

is continuously differentiable and its Jacobian matrix at $t = 0$ is

$$J_{\Phi}(0) = [ f'_i(x^*)(v_j) ]_{0 \leq i,j \leq m+n} = I_{m+n+1}.$$

Therefore, $\Phi'(0) = Id_{R^{m+n+1}}$; hence by applying the local inverse theorem, we deduce the existence of a real number $r_1 \in [0, r_0]$ such that $\Phi$ is a $C^1$ diffeomorphism from $U_1 \equiv B_{R^{m+n+1}}(0, r_1)$ to an open neighbourhood $V_1$ of $\Phi(0) = (f_0(x^*), 0, \cdots, 0)$ in $R^{m+n+1}$. For $\nu > 0$ small enough, the vector $y_\nu \equiv (f_0(x^*) - \nu, 0, \cdots, 0)$ belongs to $V_1$; let $t_\nu = \Phi^{-1}(y_\nu)$. It is clear that the vector $x_\nu = \sigma(t_\nu)$ belongs to $\Omega$ and satisfies

$$f_0(x_\nu) = f_0(x^*) - \nu,$$

$$f_i(x_\nu) = 0, \forall 1 \leq i \leq n + m,$$
which contradicts the definition of \( x^* \). Thus, the family \( \{f'_i(x^*) : 0 \leq i \leq n + m\} \) is linearly dependant in \( X^* \). On the other hand, since \( \{f'_i(x^*) : 1 \leq i \leq n + m\} \) is linearly independent in \( X^* \), we infer the existence of \( (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m) \in \mathbb{R}^{m+n} \) such that

\[
f'_0(x^*) + \sum_{i=1}^{n} \lambda_i f'_i(x^*) + \sum_{j=1}^{m} \mu_j f'_{j+n}(x^*) = 0.
\]

It remains to prove that \( \mu_j \geq 0 \) for every \( 1 \leq j \leq m \). According to Lemma 2.1, there exists \( \{w_1, \ldots, w_{m+n}\} \) a quasi primal basis of \( X \) associated to the family \( \{f'_i(x^*) : 1 \leq i \leq n + m\} \). Proceeding as previously, we deduce that there exists \( r > 0 \) and a neighbourhood \( V \) of 0 in \( \mathbb{R}^{m+n} \) such that the mapping \( \varphi : B_{\mathbb{R}^{m+n}}(0, r) \rightarrow V \) defined

\[
\varphi(t) = (f_1(s(t)), \ldots, f_{m+n}(s(t)),
\]

where

\[
s(t = (t_i)_{1 \leq i \leq m+n}) = x^* + \sum_{i=1}^{m+n} t_i w_i,
\]

is a \( C^1 \) diffeomorphism. Let \( 1 \leq j_0 \leq m \) be a fixed integer. Since \( V \) is an open neighbourhood of 0 in \( \mathbb{R}^{m+n} \), there exists \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in [\varepsilon_0, \varepsilon_0[ \), \( -\varepsilon e_{j_0+n} \in V \), where \( (e_1, \ldots, e_{m+n}) \) is the canonical basis of \( \mathbb{R}^{m+n} \). Hence, for every \( \varepsilon \in ]\varepsilon_0, \varepsilon_0[, \) the vector

\[
\tilde{x}(\varepsilon) = s(\varphi^{-1}(-\varepsilon e_{j_0+n}))
\]

belongs to \( \Omega \) and satisfies \( f_{j_0+n}(\tilde{x}(\varepsilon)) = -\varepsilon \) and \( f_i(\tilde{x}(\varepsilon)) = 0 \) for every \( i \in \{1, \ldots, m + n\} \setminus j_0 + n \). Hence, for every \( \varepsilon \in ]0, \varepsilon_0[, \)

\[
\frac{f_0(\tilde{x}(\varepsilon)) - f_0(\tilde{x}(0))}{\varepsilon} = \frac{f_0(\tilde{x}(\varepsilon)) - f_0(x^*)}{\varepsilon} \geq 0.
\]

Letting \( \varepsilon \to 0 \), we obtain

\[
f'_0(x^*) (\frac{d\tilde{x}}{d\varepsilon}(0)) \geq 0.
\]

For every \( \varepsilon \in ]-\varepsilon_0, \varepsilon_0[, \) define

\[
\tilde{t}(\varepsilon) = (\tilde{t}_1(\varepsilon), \ldots, \tilde{t}_{m+n}(\varepsilon)) = \varphi^{-1}(-\varepsilon e_{j_0+n}).
\]

First, since \( \varphi(\tilde{t}(\varepsilon)) = -\varepsilon e_{j_0+n} \) and \( \varphi'(0) = Id_{\mathbb{R}^{m+n}} \), we have \( \frac{d\tilde{t}}{d\varepsilon}(0) = -e_{j_0+n} \). Using now the fact that

\[
\tilde{x}(\varepsilon) = s(\tilde{t}(\varepsilon)) = x^* + \sum_{i=1}^{m+n} \tilde{t}_i(\varepsilon) w_i,
\]

we deduce that

\[
\frac{d\tilde{x}}{d\varepsilon}(0) = -w_{j_0+n}.
\]
Finally, combining (2.2), (2.3), and (2.4) yields
\[ \mu_{j_0} = -f'_0(x^*)(w_{j_0+n}) \geq 0, \]
which completes the proof of the theorem. \(\square\)

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