UNIFORM ATTRACTORS OF 3D NAVIER-STOKES-VOIGT EQUATIONS WITH MEMORY AND SINGULARLY OSCILLATING EXTERNAL FORCES

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Dedicated to the memory of Professor Geneviève Raugel
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Abstract. We consider a three-dimensional Navier-Stokes-Voigt equations with memory in lacking instantaneous kinematic viscosity, in presence of Ekman type damping and singularly oscillating external forces depending on a positive parameter $\varepsilon$. Under suitable assumptions on the memory term and on the external forces, we prove the existence and the uniform (w.r.t. $\varepsilon$) boundedness as well as the convergence as $\varepsilon$ tends to 0 of uniform attractors $A^\varepsilon$ of a family of processes associated to the model.

1. Introduction. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and let $\alpha > 0$ and $\beta, \vartheta \geq 0$ be given constants. We consider the Navier-Stokes-Voigt (NSV for short) equations with memory and a singularly oscillating external force, and Ekman type damping in the unknown velocity $u = u(x,t)$ and the pressure $p = p(x,t)$

$$\begin{align*}
\partial_t (u - \alpha \Delta u) - \int_0^\infty \kappa(s) \Delta u(t-s) ds + \beta(-\Delta)^{-\vartheta} u + (u \cdot \nabla) u + \nabla p &= f^\varepsilon(x,t), \\
\text{div } u &= 0, \\
u(x,t) &= 0, \\
u(x,\tau) &= u^\tau(x), \\
u(x,\tau+s) &= q^\tau(x,s),
\end{align*}$$

(1)

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where
\[ f^*(x, t) = f_0(x, t) + \varepsilon^{-\rho} f_1(x, t/\varepsilon), \quad \varepsilon \in (0, 1], \rho > 0. \]

Along with (1), we consider the limit equations
\[
\begin{aligned}
\partial_t (u - \alpha \Delta u) - \int_0^\infty \kappa(s) \Delta u(t-s) ds + \beta(-\Delta)^{-\vartheta} u + (u \cdot \nabla) u + \nabla p &= f_0(x, t), \\
\text{div } u &= 0, \\
u(x, t) &= 0, \quad x \in \partial \Omega, t > \tau, \\
u(x, \tau) &= u_\tau(x), \quad x \in \Omega, \\
u(x, \tau - s) &= q_\tau(x, s), \quad x \in \Omega, s > 0,
\end{aligned}
\]

without rapid and singular oscillations, which formally corresponds to the limiting case \( \varepsilon = 0 \) in (1). Here the energy dissipation is related not to instantaneous stress but to the past history of the stress and this makes dissipation weaker.

The NSV equations was first introduced by Oskolkov in [20] as a model of motion of certain linear viscoelastic fluids. This system was also proposed by Cao, Lunasin and Titi in [7] as a regularization, for small values of \( \alpha \), of the 3D Navier-Stokes equations for the sake of direct numerical simulations. In the last years, the existence and long-time behavior of the 3D NSV equations has attracted the attention of many mathematicians in both autonomous and non-autonomous cases (cf. [2, 14, 16, 17, 20, 24, 26, 27]). We also refer the interested reader to [3, 19, 28] for recent results on the time decay rates of solutions to the NSV equations in the whole space \( \mathbb{R}^3 \).

However, NSV equations with memory have been less studied, except two recent works [15, 12] concerning the existence and long-time behavior of solutions in terms of the existence of a global attractor when the external force is time-independent, i.e. the autonomous case. An interesting open issue suggested in [12, Sect. 9] is to analyze the non-autonomous NSV equations with memory in presence of singularly oscillating external forces. The present paper is an attempt to answer this question. We will study the existence, the uniform boundedness (w.r.t. \( \varepsilon \)) and the upper semicontinuity of uniform attractors for a family of processes associated to model (1) under a very general assumption on the memory kernel \( \kappa \) (as in [13, 21]) and with singularly oscillating non-autonomous external forces.

To do this, we assume that the memory kernel and the external force satisfy the following conditions:

**\textbf{(H1)}** The convolution (or memory) kernel \( \kappa \) is a nonnegative summable function of total mass \( \int_0^\infty \kappa(s) ds = 1 \) having the explicit form
\[ \kappa(s) = \int_s^\infty \mu(r) dr, \]
where \( \mu \in L^1(\mathbb{R}^+) \) is a decreasing (hence nonnegative) piecewise absolutely continuous function. In particular, \( \mu \) is allowed to exhibit (infinitely many) jumps. Moreover, we require that
\[ \kappa(s) \leq \theta \mu(s) \]
for some \( \theta > 0 \) and every \( s > 0 \). As shown in [13], this is completely equivalent to the requirement that
\[ \mu(r + s) \leq Ne^{-\delta r} \mu(s), \]
for some \( N \geq 1, \delta > 0 \), every \( r \geq 0 \) and almost every \( s > 0 \).
It is noticed that the condition in (H1) of the memory term is weaker than the usual condition in [8, 12] in the sense that μ can be weakly singular at the origin. For instance, we can take μ(s) = \( \frac{c e^{-a s}}{s^{1-b}} \) with \( c \geq 0 \) and \( a, b > 0 \).

(H2) The external forces \( f_0, f_1 \in L^2_\text{loc}(\mathbb{R}; V^\varepsilon) \), the space of translation bounded functions in \( L^2_\text{loc}(\mathbb{R}; V^\varepsilon) \), for some \( \varrho > \phi(\vartheta) := \vartheta - \frac{1}{2} \) if \( \vartheta > \frac{1}{2} \) and \( \varrho = 0 \) if \( 0 \leq \vartheta \leq \frac{1}{2} \), that is,

\[
\|f_i\|_{L^2_\varepsilon} := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f_i(y)\|^2 dy = M_i < \infty. \tag{3}
\]

We highlight the fact that in the physically relevant case of classical Ekman damping \( \beta u \) which corresponds to the case \( \vartheta = 0 \) (the strongest damping within this class), we only need the usual assumption \( f_0, f_1 \in L^2(\mathbb{R}; H) \).

A straightforward consequence of (3) is

\[
\int_t^{t+1} \|f_1(y/\varepsilon)\|^2 dy = \varepsilon \int_{t/\varepsilon}^{(t+1)/\varepsilon} \|f_1(y)\|^2 dy \leq \varepsilon (1 + 1/\varepsilon) M_1 \leq 2M_1,
\]

so that

\[
\|f_1(\cdot/\varepsilon)\|^2_{L^2_\varepsilon} \leq 2M_1, \forall \varepsilon \in (0, 1].
\]

Hence,

\[
\|f^\varepsilon\|^2_{L^2_\varepsilon} \leq 2\|f_0\|^2_{L^2_\varepsilon} + 2\varepsilon^{-2\varrho}\|f_1(\cdot/\varepsilon)\|^2_{L^2_\varepsilon} \leq 2M_0 + 4M_1 \varepsilon^{-2\varrho}.
\]

For \( f^\varepsilon \) being translation bounded in \( L^2_\text{loc}(\mathbb{R}; V^\varepsilon) \), we denote by \( \mathcal{H}_w(f^\varepsilon) \) the closure of the set \( \{f_\varepsilon(\cdot + \tau) | \tau \in \Omega \} \) in \( L^2_\text{loc}(\mathbb{R}; V^\varepsilon) \) with the weak topology. Noting that, as in [9, Chapter 5, Proposition 4.2], we have: For all \( \sigma \in \mathcal{H}_w(f^\varepsilon) \) and any fixed positive number \( \varepsilon \), then

\[
\|\sigma\|^2_{L^2_\varepsilon} \leq \|f^\varepsilon\|^2_{L^2_\varepsilon}. \tag{4}
\]

As emphasized in [12], due to the fact that the instantaneous viscosity term has been replaced by a memory term incorporating hereditary effects, the mathematical treatment of the model (1) becomes much harder than the model in [15], where the instantaneous and hereditary kinematic viscosity coexist. In the present paper, following the general lines of the approach in [12] and with the framework in [8], we overcome the intrinsic difficulty of reconstructing the necessary control on the high modes by nontrivially combining suitable energy functionals with a recent Grönwall-type lemma with parameter (see [22]). As a result, we get the existence of uniform attractors \( \mathcal{A}^\varepsilon \) for a family of processes generated by (1). Then, exploiting techniques used in the recent paper [1], we are able to prove the uniform boundedness (w.r.t. \( \varepsilon \)) and convergence of the uniform attractors \( \mathcal{A}^\varepsilon \) as \( \varepsilon \) converges to 0. In particular, we have solved the open question posed in [12] and the obtained result is a counterpart of the result in [24] for the case with memory. It is also worth noticing that our assumption on the memory kernel is very general and weaker than the classical Dafermos condition, so even in the case of time-independent external force, our result on the existence of an attractor is an improvement of the corresponding one in [12].

The paper is organized as follows. In Section 2, for convenience of the reader, we recall the functional setting and prove some auxiliary results. In Section 3, we prove the existence of uniform attractors for the family of processes generated by
the model. The uniform boundedness and the convergence of the uniform attractors are respectively shown in Section 4 and Section 5.

For brevity, throughout the paper, $C \geq 0$ and $Q(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ will stand for a generic constant and a generic increasing function, respectively.

2. Preliminaries and auxiliary results.

2.1. Functional setting and formulation of the problem. We will use the Hilbert space $(H, (\cdot, \cdot), \| \cdot \|)$ given by

$$H = \{ u \in [L^2(\Omega)]^3 : \text{div} \ u = 0, u \cdot n|_{\partial \Omega} = 0 \},$$

where $n$ is the outward normal to $\partial \Omega$, and the Hilbert space

$$V = \{ u \in [H^1_0(\Omega)]^3 : \text{div} \ u = 0 \},$$

with the inner product and norm

$$(u, v)_1 = (\nabla u, \nabla v) \text{ and } \|u\|_1 = \|\nabla u\|.$$

We denote by $V'$ its dual space. We will also use the space

$$W = V \cap [H^2(\Omega)]^3.$$

Let $P : [L^2(\Omega)]^3 \to H$ be the Leray orthogonal projection onto $H$, we consider the Stokes operator on $H$

$$A = -P\Delta \text{ with domain } D(A) = W.$$ 

It is well known that $A$ is a positive self-adjoint operator with compact inverse (see e.g. [25]). This allows us to define the scale of compactly nested Hilbert spaces

$$V^r = D(A^\frac{r}{2}), \ r \in \mathbb{R},$$

endowed with the inner products and norms

$$(u, v)_r = (A^\frac{r}{2} u, A^\frac{r}{2} v) \text{ and } \|u\|_r = \|A^\frac{r}{2} u\|.$$ 

In particular,

$$V^{-1} = V', \ V^0 = H, \ V^1 = V, \ V^2 = W.$$

More generally (see e.g. [6]),

$$V^r = V \cap [H^r(\Omega)]^3, \ 1 \leq r \leq 2,$$

$$V^r \subset V \cap [H^r(\Omega)]^3, \ r > 2.$$ 

We also recall the Poincaré inequality

$$\sqrt{\lambda_1} \|u\| \leq \|u\|_1, \ \forall u \in V,$$

where $\lambda_1 > 0$ is the first eigenvalue of $A$.

We now recall a basic interpolation result (see e.g. [18]).

Lemma 2.1. Let $a < b < c$. Then for all $u \in V^c$, we have

$$\|u\|_b \leq \|u\|_c^{\frac{b-a}{c-a}} \|u\|_a^{1-\frac{b-a}{c-a}}, \ \text{where } y = \frac{b-a}{c-a}.$$
Let $b$ be the trilinear form on $V \times V \times V$ given by

$$b(u, v, w) = \int_{\Omega} (u \cdot \nabla)v.wdx = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

which satisfies the relation

$$b(u, v, v) = 0.$$

We define the associated bilinear form $B : V \times V \rightarrow V'$ by

$$\langle B(u, v), w \rangle = b(u, v, w).$$

We also introduce the weighted $L^2$-spaces $\mathcal{M} = L^2_\mu(\mathbb{R}^+; V)$ and $\mathcal{M}_1 = L^2_\mu(\mathbb{R}^+; W)$ with norms

$$\|\eta^t\|_{\mathcal{M}} = \int_0^\infty \mu(s)\|\eta^t(s)\|_V^2 ds \quad \text{and} \quad \|\eta^t\|_{\mathcal{M}_1} = \int_0^\infty \mu(s)\|\eta^t(s)\|_{W}^2 ds.$$

Define the Hilbert spaces

$$\mathcal{H} = V \times \mathcal{M} \quad \text{and} \quad \mathcal{H}_1 = W \times \mathcal{M}_1$$

with the Euclidean product norms

$$\|(u, \eta^t)\|_\mathcal{H}^2 = \alpha\|u\|_V^2 + \|u\|_W^2 + \|\eta^t\|_{\mathcal{M}}^2 \quad \text{and} \quad \|(u, \eta^t)\|_\mathcal{H}_1^2 = \alpha\|u\|_W^2 + \|u\|_V^2 + \|\eta^t\|_{\mathcal{M}_1}^2.$$

We begin with rephrasing (1) as a non-autonomous dynamical system on a suitable phase space. To this aim, as in the pioneering work of Dafermos [11] we consider for $t \geq 0$, the auxiliary past history variable

$$\eta^t(s) = \int_0^s u(t-r)dr.$$

Notice that $\eta^t$ satisfies the boundary condition $\eta^t(0) := \lim_{s \to 0} \eta^t(s) = 0$ and formally fulfills the equation

$$\partial_t \eta^t(x, s) = -\partial_x \eta^t(x, s) + u(x, t). \quad (5)$$

Setting $\mu(s) = -k'(s)$, problem (1) can be transformed into the following system

$$\partial_t(u + \alpha A u) + \int_0^\infty \mu(s) A \eta^t(s) ds + \beta A^{-\vartheta} u + B(u, u) = f^\vartheta(t) \quad (6)$$

in the unknown $(u(t), \eta^t)$, endowed with initial conditions $(u(\tau), \eta^\tau) = (u_\tau, \eta^\tau) \in \mathcal{H}$.

We give the definition of weak solutions.

**Definition 2.2.** Given $z_\tau = (u_\tau, \eta^\tau) \in \mathcal{H}$, a function $z = (u, \eta) \in C([0, \infty); \mathcal{H})$, with $\partial_t u \in L^2_{loc}(\tau, \infty; V)$, is a weak solution to (6) with initial datum $z(\tau) = (u(\tau), \eta^\tau) = (u_\tau, \eta^\tau)$ if for every test function $v \in V$ and almost every $t \geq \tau$

$$\langle \partial_t u, v \rangle + \alpha \langle \partial_x u, v \rangle + \int_0^{\infty} \mu(s) \langle \eta(s), v \rangle ds + \beta A^{-\vartheta} u + b(u, u, v) = \langle f^\vartheta, v \rangle,$$

where $\eta$ fulfills the explicit representation

$$\eta^t(s) = \begin{cases} \int_s^t u(t-r)dr, & 0 < s \leq t, \\ \eta^0(s-t) + \int_0^t u(t-r)dr, & s > t. \end{cases}$$

By using the compactness method, we can prove the following result on the existence and uniqueness of solutions to the model (6) (see also [12]).
Theorem 2.3. Assume that hypotheses (H1)-(H2) hold. Then for any \((u_\tau, \eta_\tau) \in \mathcal{H}\) and \(\sigma \in \mathcal{H}_w(f^\tau)\) given, problem (\(6\)) (with \(\sigma\) in place of \(f^\tau\)) has a unique global weak solution \(z = (u, \eta^\tau)\). Moreover, the weak solution depends continuously on the initial data on \(\mathcal{H}\).

2.2. Some auxiliary results. We denote by \(\mathcal{T}\) the space of all non-negative locally summable functions \(g\) on \(\mathbb{R}^+\) such that
\[
\|g\|_\mathcal{T} = \sup_{t \geq 0} \int_t^{t+1} g(s)ds < \infty.
\]
If \(g \in \mathcal{T}\) then the following inequality holds for every \(\varepsilon \in (0,1]\) and \(t \geq \tau \geq 0\),
\[
\int_\tau^t e^{-\nu(t-s)} g(s)ds \leq C \|g\|_\mathcal{T}.
\] (7)
The following Gronwall-type lemma is the main tool in the proof.

Lemma 2.4. [22] Let \(\Lambda_\varepsilon\) be a family of absolutely continuous nonnegative functions on \([\tau, \infty)\) satisfying for some \(K > 0\) and for any \(\varepsilon \in (0, \varepsilon_0)\), for some small \(\varepsilon_0 > 0\), the differential inequality
\[
\frac{d}{dt} \Lambda_\varepsilon(t) + K\varepsilon \Lambda_\varepsilon(t) \leq c\varepsilon^p [\Lambda_\varepsilon(t)]^q + g(t)\varepsilon^r,
\]
where \(g \in \mathcal{T}\) satisfies (7) and the nonnegative parameters \(p, q, r\) fulfill
\[
p - 1 > (q - 1)(1 + r) \geq 0.
\]
Moreover, let \(E\) be a continuous non-negative function on \([\tau, \infty)\) such that
\[
\frac{1}{m}E(t) \leq \Lambda_\varepsilon(t) \leq mE(t)
\]
for every \(\varepsilon > 0\) small and some \(m \geq 1\). Then, there exist \(\nu > 0\) and an increasing positive function \(Q(\cdot)\) such that
\[
E(t) \leq Q(E(\tau))e^{-\nu(t-\tau)} + C\|g\|_\mathcal{T}.
\]
Exploiting some ideas of [13], we introduce the probability measure \(\hat{\mu}\) on \(\mathbb{R}^+\) as
\[
\hat{\mu}(P) = \frac{1}{k} \int_P \mu(s)ds,
\]
for any (measurable) set \(P \subset \mathbb{R}^+\), where \(k := k(0)\). For any \(\delta > 0\), we consider the sets
\[
P_\delta = \{s \in \mathbb{R}^+: \mu'(s) + \delta \mu(s) > 0\}
\]
and
\[
N_\delta = \{s \in \mathbb{R}^+: \mu'(s) + \delta \mu(s) \leq 0\}.
\]
Clearly, \(P_\delta \cup N_\delta = \mathbb{R}^+\), except possibly a nullset. Besides, we have
\[
\lim_{\delta \to 0^+} \hat{\mu}(P_\delta) = 0.
\]
Then, for \(\eta^\delta \in \mathcal{M}\), we denote
\[
\mathcal{P}_\delta[\eta] = \int_{P_\delta} \mu(s)\|\eta^\delta(s)\|^2 ds
\]
and
\[
\mathcal{N}_\delta[\eta] = \int_{N_\delta} \mu(s)\|\eta^\delta(s)\|^2 ds.
\]
Observe that $\mathcal{P}_d[\eta] + \mathcal{N}_d[\eta] = \|\eta^f\|_{\mathcal{M}}^2$. Now, in order to deal with the (possible) singularity of $\mu(s)$ at zero, we fix $\tau_0 > 0$ such that

$$\int_0^{\tau_0} \mu(s) ds \leq \frac{k}{2},$$

and we define $\mu_* : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as

$$\mu_*(s) = \mu(\tau_0) \chi(0,\tau_0](s) + \mu(s) \chi(\tau_0,\infty](s),$$

where $\chi$ denotes the characteristic function.

Let $z(t) = (u(t), \eta^f(s))$ be the unique solution to (6) with initial datum $z_\tau \in \mathcal{H}$. We now introduce the following auxiliary functionals

$$\Phi_1(t) = -\frac{4}{k} \int_0^\infty \mu_*(s)(u(t), \eta^f(s)) ds,$$

$$\Phi_2(t) = 2\beta \|u(t)\|_{\varphi}^2,$$

$$\Psi(t) = \int_0^\infty \left( \int_s^\infty \mu(y) \chi_{\mathbb{R}^+}(y) dy \right) \|\eta^f - u\|_{L^2}^2 ds.$$

**Lemma 2.5.** Assume that hypotheses (H1)-(H2) hold and put $\kappa(0) = k$. Then, for every $\alpha > 0$ the following inequality holds

$$\frac{d}{dt} \Phi_1(t) + \|u\|^2 \leq \frac{8\mu(\tau_0)}{k^2} \int_0^\infty -\mu'(s)\|\eta^f(s)\|_{L^2}^2 ds + \left( \alpha \beta + 4\mu(P_\beta) \right) \|u_1\|^2$$

$$+ \frac{4}{k\alpha} \mathcal{P}_d[\eta] + \frac{1}{\kappa} \mathcal{N}_d[\eta].$$

(8)

**Proof.** The time-derivative of $\Phi_1$ is given by

$$\frac{d}{dt} \Phi_1 = -\frac{4}{k} \int_0^\infty \mu_*(s)(u_1 + \eta^f) ds - \frac{4}{k} \int_0^\infty \mu_*(s)(u, \partial_\tau \eta^f) ds.$$

According to (5), we get

$$\frac{d}{dt} \Phi_1 + \frac{4}{k} \int_0^\infty \mu_*(s) \|u\|^2 ds$$

$$\leq -\frac{4}{k} \int_0^\infty \mu_*(s)(u_1 + \eta^f) ds + \frac{4}{k} \int_0^\infty \mu_*(s)(u, \partial_\tau \eta^f) ds.$$

Integrating by parts, we have

$$\frac{4}{k} \int_0^\infty \mu_*(s)(u, \partial_\tau \eta^f) ds = -\frac{4}{k} \int_0^\infty \mu'(s)(u, \eta^f) ds$$

$$\leq \frac{4\sqrt{2\mu(\tau_0)}}{k} \|u\|_1 \left( \int_0^\infty -\mu'(s)\|\eta^f(s)\|_{L^2}^2 ds \right)^{1/2}$$

$$\leq \|u\|_1^2 + \frac{8\mu(\tau_0)}{k^2} \int_0^\infty -\mu'(s)\|\eta^f(s)\|_{L^2}^2 ds,$$
and
\[- \frac{4}{k} \int_0^\infty \mu_*(s)(u_t, \eta_t(s))_1 ds \leq 4 \int_0^\infty \mu_*(s) \|u_t\|_1 \|\eta_t\|_1 ds \]
\[= \frac{4}{k} \left( \int_{P_{\delta}} \mu(s) \|u_t\|_1 \|\eta_t\|_1 ds + \int_{N_{\delta}} \mu(s) \|u_t\|_1 \|\eta_t\|_1 ds \right) \]
\[\leq \frac{4}{k} \left( \int_{P_{\delta}} \mu(s) ds \|u_t\|_1^2 + \frac{1}{4} \int_{P_{\delta}} \mu(s) \|\eta_t\|_1^2 ds \right) \]
\[+ \frac{\alpha a}{4} \int_{N_{\delta}} \mu(s)ds \|u_t\|_1^2 + \frac{1}{\alpha a} \int_{N_{\delta}} \mu(s) \|\eta_t\|_1^2 ds \]
\[\leq (\alpha a + 4\hat{\mu}(P_{\delta})) \|u_t\|_1^2 + \frac{4}{kaa} N_{\delta}[\eta] + \frac{1}{k} P_{\delta}[\eta], \text{ where } \frac{1}{k} \int_{N_{\delta}} \mu(s) ds \leq 1.\]

On the other hand, from the assumption on \(\mu\) and the definition of \(\mu_*\), we get
\[\int_0^\infty \mu_*(s) \|u_t\|_1^2 ds \geq \frac{4}{k} \left( \int_0^\infty \mu(s) ds - \int_0^\infty \mu(s) ds \right) \|u\|_1^2 \geq 2\|u\|_1^2.\]

Collecting the above inequalities, we complete the proof of (8).

In order to prove (9), we multiply the first equation in (6) by \(u_t\) to obtain
\[\frac{d}{dt} \Phi_2 + 4\|u_t\|_1^2 + 4\alpha \|u_t\|_1^2 \]
\[= 4(f^t, u_t) - 4 \left( \int_{P_{\delta}} \mu(s)(\eta_t(s), u_t)_1 ds + \int_{N_{\delta}} \mu(s)(\eta_t(s), u_t)_1 ds \right) - 4b(u, u, u_t).\]

Since
\[4(f^t, u_t) - 4 \left( \int_{P_{\delta}} \mu(s)(\eta_t(s), u_t)_1 ds + \int_{N_{\delta}} \mu(s)(\eta_t(s), u_t)_1 ds \right) \]
\[\leq 4\|f^t\|_{-1} \|u_t\|_1 + 4 \left( \int_{P_{\delta}} \mu(s)\|\eta_t\|_1 \|u_t\|_1 ds + \int_{N_{\delta}} \mu(s)\|\eta_t\|_1 \|u_t\|_1 ds \right) \]
\[\leq \frac{1}{2\alpha} \|f^t\|_{-1}^2 + \frac{\alpha}{2} \|u_t\|_1^2 \]
\[+ 4 \left( k\hat{\mu}(P_{\delta}) \|u_t\|_1^2 + \frac{1}{4} P_{\delta}[\eta] + \frac{\alpha}{4k} \int_{N_{\delta}} \mu(s) ds \|u_t\|_1^2 \right) \]
\[\leq \left( \frac{3\alpha}{2} + 4k\hat{\mu}(P_{\delta}) \right) \|u_t\|_1^2 + \frac{4k}{\alpha} N_{\delta}[\eta] + P_{\delta}[\eta] + C\|f^t\|_1^2,\]
while, by interpolation and the Young inequality,
\[-4b(u, u, u_t) \leq C\|u\|_1^2 \|u_t\|_1^2 \|u_t\|_1 \leq \frac{\alpha}{2} \|u_t\|_1^2 + C\|u\|_1^2,\]
we obtain the desired differential inequality.

\[\square\]

**Lemma 2.6.** The following inequality holds
\[\frac{d}{dt} \Psi \leq -\frac{1}{2} P_{\delta}[\eta] + 2\hat{\mu}(P_{\delta})\|u(t)\|_1^2.\]
An integration by parts then yields

\[
\frac{d}{dt} \Psi = 2 \int_0^\infty \left( \int_s^\infty \mu(y) \chi_{P_s}(y) dy \right) ((\eta^t(s) - u), \partial_s \eta^t) ds
\]

in place of \( \sigma \) of continuous processes.

3. Existence of uniform attractors. Theorem 2.3 allows us to define a family of continuous processes \( \{U_\sigma(t, \tau)\}_{\sigma \in \mathcal{H}(f^c)} \) as follows

\[ U_\sigma(t, \tau) : \mathcal{H} \to \mathcal{H}, \]

where \( U_\sigma(t, \tau)z_\sigma \) is the unique weak solution of (6) (with \( \sigma \) in place of \( f^c \)), at the time \( t \) with the initial datum \( z_\sigma \) at \( \tau \).

The aim of this section is to prove the following result.

**Theorem 3.1.** Assume that hypotheses (H1)-(H2) hold. Then for any fixed positive number \( \varepsilon \), the family of processes \( \{U_\sigma(t, \tau)\}_{\sigma \in \mathcal{H}(f^c)} \) associated to problem (1) possesses a uniform attractor \( \mathcal{A}^c \) in the space \( \mathcal{H} \). Moreover,

\[ \mathcal{A}^c = \bigcup_{\sigma \in \mathcal{K}(f^c)} K_\sigma(s), \quad \forall s \in \mathbb{R}, \quad (10) \]

where \( K_\sigma(s) \) is the kernel section at time \( s \) of the process \( U_\sigma(t, \tau) \).

To prove this theorem, we need to show that \( U_\sigma(t, \tau) \) has a uniform absorbing set \( B_0 \) in \( \mathcal{H} \) and \( U_\sigma(t, \tau) \) is uniform asymptotically compact in \( \mathcal{H} \).

### 3.1. Existence of a uniform absorbing set.

**Lemma 3.2.** Let the hypotheses (H1)-(H2) hold. Then the family of processes \( \{U_\sigma(t, \tau)\}_{\sigma \in \mathcal{H}(f^c)} \) associated to problem (6) has a uniform absorbing set in \( \mathcal{H} \), i.e., there exist \( R_0 \geq 0 \) and \( \nu_0 > 0 \) such that

\[ \|(u(t, \eta^t))\|_{\mathcal{H}}^2 \leq Q(\|(u_\tau, \eta^\tau)\|_{\mathcal{H}}) e^{-\nu_0(t-\tau)} + R_0, \quad (11) \]

for every \( (u_\tau, \eta^\tau) \in \mathcal{H} \). Moreover,

\[ \int_\tau^t \|u_s(y)\|^2 dy \leq (t - \tau + 1)Q(\|(u_\tau, \eta^\tau)\|_{\mathcal{H}}). \quad (12) \]

**Proof.** Putting \( E(t) := \frac{1}{2} (\|u\|^2 + \alpha \|u\|^2 + \|\eta^t\|^2_{L^2}) \). Multiplying the first equation in (1) (with \( \sigma \) in place of \( f^c \)) by \( u \) in \( H \), we obtain

\[ \frac{d}{dt} E(t) + \beta \|u\|^2 - \int_0^\infty \mu'(s) \|\eta^t(s)\|^2 ds = (\sigma(t), u). \]

Now, we consider the functional

\[ \Lambda(t) := NE(t) + \nu a \Phi_1(t) + a^2 \Phi_2(t) + 2\Psi(t), \]

Proof. Since \( \partial_s \eta^t = -\partial_s \eta^t + u \), we have

\[ \frac{d}{dt} \Psi = 2 \int_0^\infty \left( \int_s^\infty \mu(y) \chi_{P_s}(y) dy \right) ((\eta^t(s) - u), \partial_s \eta^t) ds \]

An integration by parts then yields

\[ \frac{d}{dt} \Psi = -\mathcal{P}_\delta[\eta] + 2 \int_{P_s} \mu(s)(\eta^t, u)_{\mathcal{H}} ds. \]

Since \( \int_s^\infty \mu(y) \chi_{P_s}(y) dy = -\mu(s) \chi_{P_s}(s) \), by using the Cauchy inequality, the conclusion follows. \( \square \)
then
\[ E(t) \leq \Lambda(t) \leq 2NE(t). \] (13)

From Lemma 2.5, Lemma 2.6 and selecting \( \delta > 0 \) small enough such that \( \hat{\mu}(P_{\delta}) \leq a^2 \), we deduce that
\[
\frac{d}{dt} \Lambda + N\beta\|u\|_{\varnothing}^2 + a(\nu - 4\varrho)\|u\|_1^2 + a^2(2\alpha - \nu\alpha - 4\nu\varrho - 4\varrho a^2)\|u_t\|_1^2
\leq \left( N - \frac{8\mu(\tau_*)\nu a}{k^2} \right) \int_0^t \mu'(s)\|\eta(s)\|_{L^2}^2 ds + \frac{4(\nu + k^2a^2)}{\alpha k}N_{\delta}[\eta]
- \left( 1 - \frac{\nu a}{k} - \varrho^2 \right) \rho_{\delta}[\eta] + Ca^2\|\sigma\|^2 + N(\sigma(t), u) + Ca^2\|u\|\|u\|_1^2.
\]
Observe that
\[
\left( N - \frac{8\mu(\tau_*)\nu a}{k^2} \right) \int_0^\infty \mu'(s)\|\eta(s)\|_{L^2}^2 ds + \frac{4(\nu + k^2a^2)}{\alpha k}N_{\delta}[\eta]
\leq \frac{1}{\delta} \int_0^\infty \mu'(s)\|\eta(s)\|_{L^2}^2 ds + \frac{4(\nu + k^2a^2)}{\alpha k}N_{\delta}[\eta]
\leq - \left( 1 - \frac{4(\nu + k^2a^2)}{\alpha k} \right) N_{\delta}[\eta],
\]
where \( N \) is large enough. Collecting all the above inequalities and choosing \( \nu < \min\{\frac{\varrho k}{\alpha a}, \frac{1}{4}\} \), a small enough, we finally obtain
\[
\frac{d}{dt} \Lambda + 2a\nu\Lambda + \alpha a^2\|u_t\|_1^2 \leq G_a,
\] (14)
where \( \nu_0 > 0 \) is independent of \( a \), and
\[ G_a = -N\beta\|u\|_{\varnothing}^2 + N(\sigma(t), u) + Ca^2\|\sigma\|^2 + Ca^2\|u\|\|u\|_1^2. \]
Putting
\[ p = \frac{4(1 + \varrho)}{1 + 2\varrho}, \quad q = \frac{3 + 4\varrho}{1 + 2\varrho}, \quad r = \frac{\varrho - \vartheta}{1 + \vartheta} < 1. \]
We can see that \( p, q, r \) comply with the hypotheses of Lemma 2.4. Then, applying Lemma 2.1 with \( a = -\varrho, b = -\vartheta, c = 1 \) and \( y = r \), we get
\[
N(\sigma(t), u) \leq N\|\sigma\|_e\|u\|_1 \|u\|_{\varnothing}^{-r}
\leq \frac{N\beta}{2} \|u\|_{\varnothing}^2 + C\|\sigma\|_e^2 \|u\|_1^\varrho \|u\|_{\varnothing}^{2\varrho}
\leq \frac{N\beta}{2} \|u\|_{\varnothing}^2 + a\nu_0\Lambda + \frac{C\|\sigma(t)\|_e^2}{\alpha^2}. \] (15)

Using Lemma 2.1 once again with \( a = -\varrho, b = 0, c = 1 \) and \( y = \frac{\varrho}{1 + \varrho} \), and then using the Young inequality with exponents \( 2(1 + \varrho) \) and \( p/2 \), we get
\[
Ca^2\|u\|\|u\|_1 \leq Ca^2\|u\|_1^{\frac{1}{1+\varrho}} \|u\|_1^{\frac{2}{1+\varrho}} \leq Ca^2\|u\|_1^{\frac{1}{1+\varrho}} \|u\|_1^{\frac{2}{1+\varrho}} \Lambda^{\frac{2\varrho}{p}} \leq \frac{N\beta}{2} \|u\|_{\varnothing}^2 + Ca^2\Lambda^{\frac{2\varrho}{p}}. \] (16)

Combining (15), (16) and since \( V^e \mapsto H \), we get
\[
G_a \leq a\nu_0\Lambda + Ca^2\Lambda^{\frac{2\varrho}{p}} + \frac{C\|\sigma(t)\|_e^2}{\alpha^2}. \] (17)

From (17) and (14), we obtain
\[
\frac{d}{dt} \Lambda + a\nu_0\Lambda + \alpha a^2\|u_t\|_1^2 \leq Ca^2\Lambda^{\frac{2\varrho}{p}} + \frac{C\|\sigma(t)\|_e^2}{\alpha^2}. \] (18)
On the other hand, from (4), we have
\[
\int_{\tau}^{t} e^{-\nu_1(t-s)} \|\sigma(s)\|_{\sigma}^2 ds \\
\leq \int_{\tau}^{t} e^{-\nu_1(t-s)} \|\sigma(s)\|_{\sigma}^2 ds + \int_{t-1}^{t-1} e^{-\nu_1(t-s)} \|\sigma(s)\|_{\sigma}^2 ds + \ldots
\]

(19)
\[
\leq (1 + e^{-\nu_1} + e^{-2\nu_1} + \ldots) \|\sigma\|_{L^2_t}^2 \leq \frac{1}{1 - e^{-\nu_1}} \|\sigma\|_{L^2_t}^2.
\]

Hence, an application of the Gronwall lemma and (13) yields (11). Integrating (18) from \(\tau\) to \(t\) and using (11), we obtain (12). \(\square\)

3.2. Uniform asymptotic compactness. To make the asymptotic regular estimates, we decompose the solution \(U_\sigma(t,\tau)z_\tau = z(t) = (u(t), \eta^t), z_\tau = (u_\tau, \eta^\tau),\) of problem (6) into the sum
\[
U_\sigma(t,\tau)z_\tau = D(t,\tau)z_\tau + K_\sigma(t,\tau)z_\tau,
\]
where \(D(t,\tau)z_\tau = z_1(t)\) and \(K_\sigma(t,\tau)z_\tau = z_2(t),\) that is, \(z = (u, \eta^t) = z_1 + z_2,\) the decomposition is as follows
\[
u = v + w, \quad \eta^t = \xi^t + \zeta^t,
\]
\[
z_1 = (v, \xi^t), \quad z_2 = (w, \zeta^t),
\]
where \(z_1(t)\) solves the following equation
\[
\begin{aligned}
\partial_t (v + \alpha Av) + B(u, v) + \int_0^\infty \mu(s)A\xi^t(s) ds &= 0, \\
\partial_t \xi^t &= \partial_s \xi^t + v, \\
(v(\tau), \xi^\tau) &= z_\tau,
\end{aligned}
\]
(20)
and
\[
\begin{aligned}
\partial_t (w + \alpha Aw) + B(u, w) + \int_0^\infty \mu(s)A\zeta(s) ds + \beta A^{-\theta} w &= \sigma_1(t), \\
\zeta_t &= \partial_s \zeta + w, \\
(w(\tau), \zeta^\tau) &= (0, 0),
\end{aligned}
\]
(21)
with \(\sigma_1(t) = \sigma(t) - \beta A^{-\theta} v(t).\)

By the standard Galerkin method, for each \(\sigma \in \mathcal{H}_w(f),\) one can prove the existence and uniqueness of solutions to equations (20) and (21). We now prove some estimates for these solutions.

**Lemma 3.3.** Assume that hypotheses (H1)-(H2) hold. Then the solutions of equation (20) satisfy the following estimate: there is a constant \(\nu_2 > 0\) and there exist \(T > \tau\) large enough, which depends on \(\|\sigma\|_{L^2_t}, \|(u_\tau, \eta^\tau)\|_{\mathcal{H}},\) and \(Q(\cdot)\) independent of \(z,\) such that
\[
\|D(t,\tau)z_\tau\|_{\mathcal{H}}^2 \leq Q(\|z_\tau\|_{\mathcal{H}}) e^{-\nu_2(t-\tau)}, \text{ for all } t \geq T,
\]
where \(Q(\cdot)\) is an increasing function on \([0, \infty)\).

**Proof.** For a small and \(N > 1\) (to be chosen), we consider the functional
\[
\Lambda(t) := NE(t) + \nu a\Phi_1(t) + a^2\Phi_2(t) + 2\Psi(t),
\]
as in the proof of Lemma 3.2. Repeating the arguments as in the proof of that lemma (for proving (11)), we obtain the following differential inequality
\[
\frac{d}{dt} \Lambda + \nu_2 \Lambda \leq 0,
\]
where \( \nu_2 > 0 \) and
\[
Ca^2 b(u, v, \nu_t) \leq Ca^2 \|u\|_{L^2} \|\nabla v\|_{L^2} \leq CR_0 a^2 \|v\|_1 \|v_t\|_1
\leq a^2 \left(C(\alpha, R_0)\|v\|_1^2 + \frac{\alpha}{2}\|v_t\|_1^2\right).
\]
An application of the Gronwall lemma completes the proof.

**Lemma 3.4.** Let the assumptions (H1)-(H2) hold. Then, there exist \( T > \tau \) large enough, which depends on \( \|\sigma\|_{L_2^\infty}, \|u_\tau, \eta^\tau\|_H, \) and \( N_0 > 0, \) such that
\[
\|K_\sigma(t, \tau)z_\tau\|_{L^2} \leq N_0, \text{ for all } t \geq T.
\] (22)

**Proof.** From (21) and as in the proof of Lemma 3.2, we see that
\[
\|w, \eta^t\|_H \leq C,
\] (23)
and
\[
\|\sigma_1(t)\| \leq \|\sigma(t)\| + \beta\|A^{-\beta}v\| \leq \|\sigma(t)\| + C.
\]
For \( a \) small and \( N > 1 \) (to be chosen), as in the proof of Lemma 3.2, we define the functional
\[
\Lambda(t) := NE(t) + \nu a\Phi_1(t) + a^2\Phi_2(t) + 2\Psi(t).
\]
For \( \zeta \in M_1, \) we denote
\[
\mathcal{P}_a[\zeta] = \int_{P_a} \mu(s)\|\zeta(s)\|_{L^2}^2 \, ds \quad \text{and} \quad \mathcal{N}_a[\zeta] = \int_{N_a} \mu(s)\|\zeta(s)\|_{L^2}^2 \, ds.
\]
By multiplying the first equation of (21) by \( Aw \) in \( H, \) we obtain
\[
\frac{d}{dt} E(t) + b(u, w, Aw) + \beta\|w\|_{1-\beta}^2 - \int_0^\infty \mu'(s)\|\zeta^t(s)\|_{L^2}^2 \, ds = \langle \sigma_1(t), Aw \rangle. \tag{24}
\]
Besides, reasoning exactly as in the proof of Lemma 2.5, we obtain the following differential inequality
\[
\frac{d}{dt} \Phi_1(t) + \|w\|_2^2 \leq \frac{8\mu(\tau_1)}{k^2} \int_0^\infty -\mu'(s)\|\zeta^t(s)\|_{L^2}^2 \, ds + (\alpha a + 4k\hat{\mu}(P_0))\|w_t\|_2^2
\]
\[
+ \frac{4}{k\alpha a}\mathcal{N}_a[\zeta] + \frac{1}{k}\mathcal{P}_a[\zeta]. \tag{25}
\]
Thus,
\[
\frac{d}{dt} \Phi_2(t) + 4\|w_t\|_2^2 + 4\alpha\|w_t\|_2^2
\]
\[
\leq 4(\sigma_1, Aw_t) - 4\int_0^\infty \mu_*(s)(\zeta^t(s), w_t)_2 \, ds - 4b(u, w, Aw_t).
\]
Using (11) we have
\[
-4b(u, w, Aw_t) \leq CR_0\|w\|_2 \|w_t\|_2 \leq C\|w\|_2^2 + \alpha\|w_t\|_2^2,
\]
\[
4(\sigma_1, Aw_t) \leq C\|\sigma\|^2 + \alpha\|w_t\|_2^2 + C,
\]
\[
-4\int_0^\infty \mu_*(s)(\zeta^t(s), w_t)_2 \, ds \leq (\alpha + 4\hat{\mu}(P_0))\|w_t\|_2^2 + \frac{4}{\alpha}\mathcal{N}_a[\zeta] + \mathcal{P}_a[\zeta].
\]
Thus,
\[
\frac{d}{dt} \Phi_2(t) + 2\alpha \|w_t\|^2_2 \leq \frac{4}{\alpha} N_\delta[\zeta] + \mathcal{P}_\delta[\zeta] + C \|\sigma\|^2 + C. \tag{26}
\]
Collecting (24)-(26) and Lemma 2.6, we get
\[
\frac{d}{dt} \Lambda + \frac{\alpha (\nu - 4a)}{k} \|w\|^2_2 + a^2 \|w_t\|^2_2
\]
\[
+ \left(1 - \frac{\nu a}{k} - a^2\right) \mathcal{P}_\delta[\eta] + \left(1 - \frac{4\nu + 4ka^2}{\alpha k}\right) N_\delta[\zeta]
\]
\[
\leq Ca^2 \|\sigma\|^2 + N(\sigma_1(t), w) - Nb(u, w, Aw) + C,
\]
where
\[
\left(N - \frac{8\mu(\tau_\star)\nu a}{k^2}\right) \int_0^\infty \mu'(s) \|\zeta(s)\|^2_2 ds + \frac{4\nu + 4ka^2}{\alpha k} N_\delta[\zeta]
\]
\[
\leq - \left(1 - \frac{4\nu + 4ka^2}{\alpha k}\right) N_\delta[\zeta].
\]
On the other hand, from (11) and (23), we obtain
\[
N(\sigma_1(t), w) - Nb(u, w, Aw) \leq N \|\sigma_1\| \|w\|_2 + N \|u\|_{L^3} \|\nabla w\|_{L^3} \|Aw\|
\]
\[
\leq \frac{\nu a}{2} \|w\|^2_2 + C \|\sigma\|^2 + C.
\]
Therefore, we get
\[
\frac{d}{dt} \Lambda + \nu_2 \Lambda \leq C \|\sigma\|^2 + C,
\]
for some $\nu_2 > 0$. An application of the Gronwall lemma, (3) and (4) gives that
\[
\|w\|^2_2 + \|\zeta\|^2_{\mathcal{M}_1} \leq C,
\]
which completes the proof. \qed

In addition, for any $\xi_\tau \in \mathcal{M}$, the Cauchy problem (see e.g. [5, 23])
\[
\begin{cases}
\partial_t \xi^t = -\partial_s \xi^t + w_t, & t > \tau, \\
\xi^\tau = \xi_\tau,
\end{cases}
\]
has a unique solution $\xi^t \in C((\tau, \infty); \mathcal{M})$, and
\[
\xi^t(s) = \begin{cases}
\int_0^s w(t - r) dr, & \tau < s \leq t, \\
\xi_\tau(s - t) - \xi_\tau + \int_\tau^t w(t - r) dr, & s > t.
\end{cases}
\tag{27}
\]
So, for the equation (27), thanks to $\xi^0(x, s) = 0$, we have
\[
\xi^t(s) = \begin{cases}
\int_0^s w(t - r) dr, & \tau < s \leq t, \\
\int_0^s w(t - r) dr, & s > t.
\end{cases}
\tag{28}
\]
Let $B_0$ be the bounded absorbing set obtained in Lemma 3.2, we now prove the following result.

**Lemma 3.5.** Assume (H1)-(H2) hold. Setting
\[
K_T = PK_\sigma(T, \tau)B_0,
\]
for $T > 0$ large enough, where $\{K_\sigma(t, \tau)\}_{t \geq \tau}$ is the solution process of (21), $P : \mathcal{H} \to \mathcal{M}$ is the projection operator. Then there is a positive constant $N_1 = N_1(\|B_0\|_\mathcal{H})$ such that
Proof of Theorem 3.1.\( \text{(i) } \mathcal{K}_T \text{ is bounded in } \mathcal{M}_1 \cap H^1_\mu(\mathbb{R}^+; H^2_0(\Omega)^3), \\text{(ii) } \sup_{\xi \in \mathcal{K}_T} \|\xi(s)\|^2_{H^2_0(\Omega)} \leq N_1. \) Moreover, \( \mathcal{K}_T \) is relatively compact in \( \mathcal{M}. \)

Proof. From (28) we have

\[ \partial_s \xi^\ell(s) = \begin{cases} w(t-s), & 0 < s \leq t, \\ 0, & s > t, \end{cases} \]

which, combining with Lemma 3.4, implies (i).

After that, using (28) once again, we can easily deduce that

\[ \|\xi^T(\xi)\|^2_{H^2_0(\Omega)} = \left\{ \begin{array}{ll} \int_0^s \|w^\varepsilon(T-r)\|^2_{H^2_0(\Omega)} \, dr, & 0 < s \leq T, \\ \int_s^T \|w^\varepsilon(T-r)\|^2_{H^2_0(\Omega)} \, dr, & s > T. \end{array} \right. \]

By (22), we know that (ii) holds. Because \( H^2_0(\Omega) \hookrightarrow H^2_0(\Omega) \) compactly, we deduce that \( \mathcal{K}_T \) is relatively compact in \( \mathcal{M} \) thanks to the following lemma in [23].

Lemma 3.6. Assume that \( \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+) \) is a nonnegative function satisfying if there exists \( s_0 \in \mathbb{R}^+ \) such that \( \mu(s_0) = 0 \), then \( \mu(s) = 0 \) for all \( s \geq s_0 \). Moreover, let \( B_0, B_1, B_2 \) be Banach spaces, here \( B_0, B_2 \) are reflexive and satisfy

\[ B_0 \hookrightarrow B_1 \hookrightarrow B_2, \]

where the embedding \( B_0 \hookrightarrow B_1 \) is compact. Let \( \mathcal{C} \subset L^2_\mu(\mathbb{R}^+, B_1) \) satisfy

(i) \( \mathcal{C} \) in \( L^2_\mu(B_0) \cap H^1_\mu(B_2) \),

(ii) \( \sup_{\eta \in \mathcal{C}} \|\eta(s)\|^2_{B_1} \leq h(s), \forall s \in \mathbb{R}^+, h(s) \in L^1_\mu(\mathbb{R}^+) \).

Then \( \mathcal{C} \) is relatively compact in \( L^2_\mu(\mathbb{R}^+; B_1) \). \( \square \)

3.3. Proof of Theorem 3.1. By Lemma 3.5, the family of processes \( U_\varepsilon(t, \tau) \) has a bounded absorbing \( B_0 \) in \( \mathcal{H} \). Moreover, \( U_\varepsilon(t, \tau) \) is uniform asymptotically compact in \( \mathcal{H} \) due to Lemmas 3.3 and 3.5. Therefore, the family of process \( U_\varepsilon(t, \tau) \) has the uniform attractor \( \mathcal{A} \) in \( \mathcal{H} \).

4. Uniform boundedness and convergence of the uniform attractors. In this section, we will prove the following facts concerning the family of uniform attractors \( \{\mathcal{A}^\varepsilon\}_{\varepsilon \in [0,1]} \) of the processes generated by (1) and (2):

(i) The family \( \mathcal{A}^\varepsilon \) is uniformly (w.r.t. \( \varepsilon \)) bounded in \( \mathcal{H} \):

\[ \sup_{\varepsilon \in [0,1]} \|\mathcal{A}^\varepsilon\|_{\mathcal{H}} < \infty; \]

(ii) The uniform attractor \( \mathcal{A}^\varepsilon \) converges to \( \mathcal{A}^0 \) as \( \varepsilon \to 0^+ \) in the standard Hausdorff semi-distance in \( \mathcal{H} \):

\[ \lim_{\varepsilon \to 0^+} \{\text{dist}_{\mathcal{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0)\} = 0. \]

4.1. Uniform boundedness of the uniform attractors. To this end, setting

\[ F(t, \tau) = \int_0^t f_1(s) \, ds, t \geq \tau, \]

we assume that

\[ \sup_{t \geq \tau, \tau \in \mathbb{R}} \left( \|F(t, \tau)\|^2_{L^1} + \int_t^{t+1} \|F(s, \tau)\|^2_{L^1} \, ds \right) \leq \ell^2. \]  

(29)
Proposition 1. Assume that \( f_1 \in L^2_0(\mathbb{R}; L^2(\Omega)) \) and satisfies (29). Then, the solution \( v(t) \) to the problem
\[
\begin{aligned}
\begin{cases}
\partial_t (v + \alpha Av) + \int_0^\infty \mu(s) A\eta_1^s(s) \, ds = f_1(t/\varepsilon), \\
\partial_t \eta_1^s = -\partial_s \eta_1^s + v, \\
(v(0), \eta_1^0) = (0, 0),
\end{cases}
\end{aligned}
\]
(30)
with \( \varepsilon \in (0, 1] \), satisfies the inequality
\[
\| (v(t), \eta_1^s) \|_H^2 + \int_t^{t+1} \| (v(r), \eta_1^r) \|_H^2 \, dr \leq C\ell^2 \varepsilon^2, \quad \forall t \geq \tau,
\]
(31)
where \( C \) is a constant independent of \( f_1 \).

Proof. Without loss of generality, we may assume \( \tau = 0 \). Denoting
\[
V(t) = \int_0^t v(y) \, dy, \quad \text{and} \quad \eta_1^t = \int_0^t \eta_1^y(s) \, dy.
\]
Integrating (30) in time from 0 to \( t \), we see that the function \( V(t) \) solves the problem
\[
V_t + \alpha AV_t + \int_0^\infty \mu(s) A\eta_1^s(t) \, ds = F_\varepsilon(t), \quad V|_{\partial \Omega} = 0, \quad V|_{t=0} = 0,
\]
(32)
where
\[
F_\varepsilon(t) = \int_0^t f_1(s/\varepsilon) \, ds = \varepsilon \int_0^{t/\varepsilon} f_1(s) \, ds = \varepsilon F(t/\varepsilon, 0).
\]
It follows from (30) that
\[
\sup_{t \geq 0} \| F_\varepsilon(t) \|_{-1} \leq \ell \varepsilon,
\]
(33)
and
\[
\sup_{t \geq 0} \int_t^{t+1} \| F_\varepsilon(s) \|_{-1}^2 \, ds \leq 2\ell^2 \varepsilon^2.
\]
Indeed, (33) is straightforward, whereas
\[
\int_t^{t+1} \| F_\varepsilon(s) \|_{-1}^2 \, ds = \varepsilon^3 \int_{t/\varepsilon}^{(t+1)/\varepsilon} \| F(s, 0) \|_{-1}^2 \, ds \leq \varepsilon^3 (1 + \frac{1}{\varepsilon}) \sup_{t \geq 0} \left( \int_t^{t+1} \| F(s, 0) \|_{-1}^2 \, ds \right) \leq 2\ell^2 \varepsilon^2.
\]
Putting \( E(t) := \frac{1}{2} \left( \| V \|^2 + \alpha \| V \|_H^2 + \| \eta_1^t \|_{-1}^2 + \| \eta_1^t \|_{-1}^2 \right) \). By multiplying the first equation of system (32) by \( V \), we obtain
\[
\frac{d}{dt} E(t) - \int_0^\infty \mu(s) \| \eta_1^s \|_H^2 \, ds = (F_\varepsilon(t), V) \leq C \| F_\varepsilon(t) \|_{-1} + \frac{\alpha^2}{N} \| V \|_1^2.
\]
Now, for \( a \) small and \( N > 1 \) (to be chosen), we define the functional
\[
\Lambda_a(t) := NE(t) + \nu a \Phi_1(t) + a^2 \Phi_2(t) + 2\Psi(t),
\]
as in the proof of Lemma 3.2. Reasoning exactly as in the proof of Lemma 3.2, we obtain the following differential inequality

\[
\frac{d}{dt} \Lambda_\alpha + N_\beta \|V\|_{-\infty}^2 + a(\nu - 4a)\|V\|_1^2 + a^2(4\alpha - \nu \alpha - 4a \nu - 4ka^2)\|\dot{V}\|_1^2 \\
\leq \left( N - \frac{8\mu(\tau_\nu)\nu a}{k^2} \right) \int_0^\infty \mu'(s)\|\Gamma(s)\|^2_1 ds + \frac{4(\nu + k^2a^2)}{ak} \mathcal{N}_\delta[\Gamma]\right)
\]

On the other hand, multiplying (32) by \( V \) we have

\[
\left| \int_0^\infty \mu(s)\langle \nabla \Gamma, \nabla V\rangle ds \right|
\]

where \( \mu \) and \( \tau_\nu \) are positive constants. Letting \( \nu_0 > 0 \) be small enough such that \( \hat{\mu}(P_\delta) \leq a^2 \). Observe that

\[
\left( N - \frac{8\mu(\tau_\nu)\nu a}{k^2} \right) \int_0^\infty \mu'(s)\|\Gamma(s)\|^2_1 ds + \frac{4(\nu + k^2a^2)}{ak} \mathcal{N}_\delta[\Gamma]
\]

so

\[
\|V\|^2 + \alpha\|V\|_1^2 + \|\Gamma\|^2_\Lambda \leq C \int_0^t e^{-\alpha_1(t-s)}\|F_\varepsilon(s)\|^2_1 ds,
\]

where we have used the facts that \( V(0) = 0 \) and \( \Gamma(0) = 0 \). Since

\[
\int_0^t e^{-\alpha_1(t-s)}\|F_\varepsilon(s)\|^2_1 ds \\
= \int_0^t e^{-\alpha_1(t-s)}\|F_\varepsilon(s)\|^2_1 ds + \int_{t-1}^{t-1} e^{-\alpha_1(t-s)}\|F_\varepsilon(s)\|^2_1 ds + \cdots \\
\leq \int_0^t \|F_\varepsilon(s)\|^2_1 ds + e^{-\alpha_1} \int_{t-1}^{t-1} \|F_\varepsilon(s)\|^2_1 ds + \cdots \\
\leq \frac{1}{1 - e^{-\alpha_1}} \sup_{t \geq 0} \int_t^{t+1} \|F_\varepsilon(s)\|^2_1 ds \leq C\varepsilon^2,
\]

so

\[
\|V\|^2 + \alpha\|V\|_1^2 + \|\Gamma\|^2_\Lambda \leq C\varepsilon^2. \tag{34}
\]

On the other hand, multiplying (32) by \( V_t \), we obtain

\[
\|V_t\|^2 + \|V_t\|_1^2 \leq |(F_\varepsilon(t), V_t)_{V,V}| + \left| \int_0^\infty \mu(s)\langle \nabla \Gamma, \nabla V_t\rangle ds \right|
\]

Applying the Hölder and Cauchy inequalities, we have

\[
\|V_t\|^2 + \|V_t\|_1^2 \leq C(a_0)\|F_\varepsilon(t)\|^2_1 + a_0\|V_t\|^2_1 + C(a_0)\|\Gamma\|^2_\Lambda + a_0 \int_0^\infty \mu(s)ds\|V_t\|_1^2.
\]
Choosing $a_0$ small enough, then using (32), (34), and noting that $\mu \in L^1(\mathbb{R}^+)$, we deduce that

$$\|V_i\|^2 + \|V_i\|^2 \leq C\ell^2\varepsilon^2,$$

i.e.

$$\|v\|^2 + \|v\|^2 \leq C\ell^2\varepsilon^2.$$  \hfill (35)

Finally, multiplying the second equation of the system (30) by $\Delta \eta_1^i$ in $H$, we get

$$\frac{1}{2} \frac{d}{dt} \|\eta_1^i\|^2_M \leq \int_0^\infty \mu'(s)\|\eta_1^i\|^2ds + a_1N_\delta[\eta_1^i] + a_1P_\delta[\eta_1^i] + C\|V\|^2_1$$

$$\leq -(1 - a_1)N_\delta[\eta_1^i] + a_1P_\delta[\eta_1^i] + C\|V\|^2_1.$$  

By Lemma 2.6, we obtain

$$\frac{d}{dt} \left(\|\eta_1^i\|^2_M + \frac{1}{2}\Psi(t)\right) + 2(1 - a_1)N_\delta[\eta_1^i] + \left(\frac{1}{4} - 2a_1\right)P_\delta[\eta_1^i] \leq C\|V\|^2.$$  

From (34) and choosing $a_1 < 1/8$, we obtain

$$\frac{d}{dt} \left(\|\eta_1^i\|^2_M + \frac{1}{2}\Psi(t)\right) + \delta_0 \left(\|\eta_1^i\|^2_M + \frac{1}{2}\Psi(t)\right) \leq C\ell^2\varepsilon^2.$$  

Hence using the Gronwall inequality we arrive at

$$\|\eta_1^i\|^2_M + \frac{1}{2}\Psi(t) \leq C\ell^2\varepsilon^2.$$  

Thus,

$$\|\eta_1^i\|^2_M \leq C\ell^2\varepsilon^2,$$  \hfill (36)

where

$$\Psi(t) \geq -P_\delta[\eta_1^i] - \mu(P_\delta)||V||_1^2 \geq -P_\delta[\eta_1^i] - C\ell^2\varepsilon^2.$$  

Combining (35) and (36), we get the estimate (31) as desired. \hfill $\square$

**Theorem 4.1.** Let hypotheses (H1)-(H2) and (29) hold. Then the uniform attractors $\mathcal{A}_\varepsilon$ are uniformly (w.r.t. $\varepsilon$) bounded in $H$, that is,

$$\sup_{\varepsilon \in (0,1]} \|\mathcal{A}_\varepsilon\|_H < \infty.$$

**Proof.** Let $z(t) = (u, \eta^i)$ be the solution to (1) with the initial datum $z_\tau \in \mathcal{H}$. For $\varepsilon > 0$, we consider the problem

$$\begin{aligned}
&v_t + \alpha A v_t + \int_0^\infty \mu(s) A\eta^i_t(s)ds = \varepsilon^{-\rho}f_1(t/\varepsilon), \\
&\partial_t \eta^i_t = -\partial_s \eta^i_t + v, \\
&(v(\tau), \eta^i_t) = (0, 0).
\end{aligned}$$

Proposition 1 provides the estimate

$$\|v(t)\|^2_1 + \|\eta^i_t\|^2_M \leq C\ell^2\varepsilon^{2(1-\rho)}, \quad \forall t \geq \tau.$$  \hfill (37)

Then, we introduce the function $(w(t), \eta^2_t) = z(t) - (v(t), \eta^i_t)$ which satisfies the problem

$$\begin{aligned}
w_t + \alpha A w_t + \beta A^{-\theta} w + \int_0^\infty \mu(s) A\eta^2_t(s)ds + B(u, u) = f_0(t) - \beta A^{-\theta} v, \\
\partial_t \eta^2_t = -\partial_s \eta^2 + w, \\
w|_{t=\tau} = u_\tau, \quad \eta^2_t = \eta_\tau.
\end{aligned}$$

For $a \in (0, 1/2)$ and $N > 1$ (to be chosen), we define the functional

$$\Lambda_w(t) := NE(t) + nu_1(t) + a^2\Phi_2(t) + 2\Psi(t),$$
where

\[ E(t) = \frac{1}{2} \left( \|w(t)\|^2 + \|\dot{w}(t)\|^2 + \|\eta(t)\|_{H^1}^2 \right), \]

\[ \Phi_1(t) = -\frac{4}{k} \int_0^\infty \mu_\ast(s)(w, \eta_2(s)) \, ds, \]

\[ \Phi_2(t) = 2\beta\|w(t)\|_{-\vartheta}^2, \]

and \( \Psi(t) = \int_0^\infty \left( \int_s^\infty \mu(y) \chi_{\mathcal{P}_0}(y) \, dy \right) \|\eta_2(s) - w\|_{H^1}^2 \, ds. \)

It can be verified that

\[ E(t) \leq \Lambda_w(t) \leq N E(t) \quad \text{for all } t \geq \tau, N \text{ large enough.} \quad (38) \]

Reasoning exactly as in the proof of Lemma 3.2, we obtain the following differential inequality

\[ \frac{d}{dt} \Lambda_w + N \beta \|w\|_{-\vartheta}^2 + a\left( \frac{\nu}{2} - 4a \right) \|w\|_1^2 \]

\[ + a^2(2\alpha - \nu\alpha - 4a
\]  
\[ = \left( 1 - \frac{\nu a}{k} - a^2 \right) \mathcal{P}_\delta[\eta_2] + C\|f_0(t)\|^2 + N(f_0(t) - \beta A^{-\vartheta} v, w) \]

\[ + C\|v\|_{-\vartheta}^2 + Ca^2\|w\|\|v\|_1^2 + C\|v\|\|v\|_1^2, \]

where

\[ 4a^2(f_0(t) - \beta A^{-\vartheta} v, w) \leq C\|f_0(t)\|^2 + a^4\|w_1\|^2 + a^4\|w_1\|_{-\vartheta}^2 + C\|v\|_{-\vartheta}^2, \]

and

\[ -a^2(B(u, u), w) \]

\[ = -a^2(b(u, u, w_1) \]

\[ = -a^2(b(w, w, w_1) + b(w, v, w_1) + b(v, w, w_1) + b(v, v, w_1)) \]

\[ \leq Ca^2\|w\|_1\|w_1\|_1 + Ca^2\|v\|_1\|w_1\|_1 + Ca^2\|v\|_1\|w_1\|_1 + Ca^2\|v\|_1\|w_1\|_1 \]

\[ \leq \frac{\alpha}{2}\|w_1\|^2 + C\|v\|\|w_1\|^3 + C\|v\|\|v\|_1^3 + \frac{Ca^2\ell^2\xi^{(1-\rho)}}{\nu} \|w_1\|_1^2 + \frac{a\nu}{2}\|w_1\|^2. \]

As in estimates (15) and (16), we have

\[ N(f_0(t) - \beta A^{-\vartheta} v, w) \leq N\beta(\|A^{-\vartheta} v, w\| + N(\|f_0(t), w\|) \]

\[ \leq \frac{N\beta}{2} \|w\|_{-\vartheta}^2 + C\|v\|_{-\vartheta}^2 + a\nu\Lambda_w + \frac{C\|f_0(t)\|^2}{a^2}, \]

\[ Ca^2\|w\|\|w_1\|^3 \leq \frac{N\beta}{2} \|w\|_{-\vartheta}^2 + Ca^2\Lambda_w, \]

and

\[ \left( 1 - \frac{8\mu(\tau_\ast\nu a)}{k^2} \int_0^\infty \mu'(s)\|\eta'(s)\|^2 ds + \frac{4\nu + a^2 k^2}{\nu} \mathcal{P}_\delta[\eta] \right) \]

\[ \leq \frac{1}{2}\int_0^\infty \mu'(s)\|\eta'(s)\|^2 ds + \frac{4\nu + a^2 k^2}{\nu} \mathcal{P}_\delta[\eta] \]
\[
\leq - \left( 1 - \frac{2\nu}{\alpha R} \right) \mathcal{N}_{\delta}^{\nu},
\]
where \(N\) is large enough. Collecting all the above inequalities and choosing \(\nu < \min\left\{ \frac{\alpha R}{2}, \frac{1}{2} \right\}\) and \(a\) a small enough, we finally obtain
\[
\frac{d}{dt} \Lambda_w + a\nu_0 \Lambda_w \leq c_0 \left[ \Lambda_w + \frac{C_2}{\nu} + C_{21} \varepsilon^2 (1-\rho) + C_{22} \varepsilon^4 (1-\rho) \right],
\]
where \(\nu_0 > 0\) is independent of \(\varepsilon\). Hence, using (38) and Lemma 2.4, we obtain
\[
\| (w(t), \eta^2_2) \|_{A^0}^2 \leq Q(\| (w(\tau), \eta^2_2) \|_{A^0}^2) e^{-\nu_1 (t-\tau)} + C (1 + \varepsilon^2 (1-\rho) + \varepsilon^4 (1-\rho)),
\]
where we have used the fact that (see (19) for a similar proof)
\[
\int_{t-\tau}^t e^{-\nu_1 (t-s)} \| f_0(s) \|_{A^0}^2 ds \leq \frac{1}{1 - e^{-\nu_1}} \| f_0 \|_{A^0}^2.
\]
Noting \(z(t) = (v, \eta^1_1) + (w, \eta^2_2)\) and using (37) once again, we have for all \(t \geq \tau\),
\[
\| z(t) \|_{A^0}^2 \leq Q(\| z(\tau) \|_{A^0}^2) e^{-\nu_1 (t-\tau)} + C (1 + \varepsilon^2 (1-\rho) + \varepsilon^4 (1-\rho)), \quad \forall t \geq \tau.
\]
Hence, the family of processes \( \{ \Omega_\varepsilon \} \) has a bounded absorbing set \( B^* \), which is independent of \(\varepsilon\) (because \(\rho < 1\)). Since \( A^0 \subset B^* \), the proof is complete. \(\square\)

4.2. Convergence of the uniform attractors. The main result of this subsection is to establish the upper semicontinuity of the uniform attractors \( A^\varepsilon \) as \( \varepsilon \to 0 \).

**Theorem 4.2.** Let (H1)-(H2) and (29) hold. Then, for every \( \rho \in [0,1] \), the uniform attractor \( A^\varepsilon \) converges to \( A^0 \) with respect to the Hausdorff semidistance in \( \mathcal{H} \) as \( \varepsilon \to 0^+ \), i.e.,
\[
\lim_{\varepsilon \to 0^+} \{ \text{dist}_{\mathcal{H}} (A^\varepsilon, A^0) \} = 0.
\]

In order to prove this theorem, we make a comparison between some particular solutions to (6) corresponding to \( \varepsilon > 0 \) and \( \varepsilon = 0 \), respectively, starting from the same initial data. We denote
\[
u^\varepsilon(t) = U_\varepsilon(t, \tau) u^\varepsilon,
\]
with \( u^\varepsilon \) belonging to the absorbing set \( B^* \) found in the previous subsection. In particular, for \( \varepsilon = 0 \), since \( u^\varepsilon \in B^* \), we get
\[
\| u^\varepsilon(t) \|_{A^0}^2 \leq R_0^2,
\]
for some \( R_0 = R_0(\rho) \), as the size of \( B^* \) depends on \( \rho \).

On the other hand, to prove the convergence of the uniform attractors, we actually need consider whole family of equations
\[
\partial_t (\hat{u} + \alpha \mu_0 \hat{u}) + \int_0^{\infty} \mu(s) A^\varepsilon \hat{u} + \beta A^\varepsilon \hat{u} + B(\hat{u}, \hat{u}) = \hat{f}^\varepsilon(t),
\]
with the external force \( \hat{f} = \hat{f}^\varepsilon \in \mathcal{H}_w(\varepsilon) \). To this end, we observe that every function \( \hat{f}_1 \in \mathcal{H}_w(\varepsilon) \) fulfills the inequality (29).

For any \( \varepsilon \in [0, 1] \), we denote
\[
u^\varepsilon(t) = U_{\hat{f}_1}(t, \tau) u^\varepsilon,
\]
where \( u^\varepsilon \) belongs to the absorbing set \( B^* \). Then
\[
\hat{u}^\varepsilon(t) = U_{\hat{f}_1}(t, \tau) \hat{u}^\varepsilon
\]
Proof. Since the deviation is the solution to (40) with the external force $\widehat{f}^\varepsilon = \widehat{f}_0 + \varepsilon^{-\rho} \widehat{f}_1(\cdot, \varepsilon) \in \mathcal{H}_w(f^\varepsilon)$. By Theorem 4.1, we have the uniform bound
$$\sup_{t \in [0, 1]} \|z^\varepsilon(t)\| \leq C, \quad \forall t \geq \tau.$$ Next, we define the deviation
$$z(t) = z^\varepsilon(t) - z^0(t) = (r(t), \zeta^\varepsilon).$$

**Lemma 4.3.** For every $\varepsilon \in (0, 1]$, we have the estimate
$$\|z(t)\| \leq C(2, \varepsilon) e^{\varepsilon^2 (1-\rho) (R_0^2 + \ell^2)} e^{C(t-\tau)}, \quad \forall t \geq \tau,$$
for some positive constant $C$ independent of $\varepsilon, \tau, \widehat{f}^\varepsilon$.

Proof. Since the deviation $q(t)$ solves
$$\partial_t (r - \alpha Ar) - \int_0^\infty \mu(s) A\zeta^\varepsilon(s) ds + \beta A^{-\theta} r + B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0) = \varepsilon^{-\rho} f_1(t, \varepsilon),$$
$$\partial_t \zeta^\varepsilon = -\partial_t r + \varepsilon, \quad (r^\varepsilon, \zeta^\varepsilon) = (0, 0),$$
the difference $(w(t), \eta^\varepsilon_2) = z(t) - (v(t), \eta^\varepsilon_1)$ clearly satisfies the equations
$$\partial_t (w - \alpha Aw) - \int_0^\infty \mu(s) A\zeta^\varepsilon_2(s) ds + \beta A^{-\theta} w + B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0) = 0,$$
$$\partial_t \eta^\varepsilon_2 = -\partial_t \eta^\varepsilon_2 + w,$$
with initial condition $(w(\tau), \eta^\varepsilon_2) = (0, 0)$. Taking the scalar product the first equation by $w$, we obtain
$$\frac{d}{dt}(\|w\|^2 + \alpha \|w\|_1^2 + \|\eta^\varepsilon_2\|_{L^2}) - 2 \int_0^\infty \mu'(s) \|\eta^\varepsilon_2\|_1^2 ds$$
$$+ 2 \beta \|w\|_{L^2}^2 + 2 \langle B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0), w \rangle = 0.$$ (41)

From the equality
$$B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0) = B(u^0, w + v) + B(w + v, u^0) + B(w + v, w + v),$$
we have
$$\langle B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0), w \rangle$$
$$= b(w, v, w) + b(w, u^0, w) + b(v, u^0, w) + b(w, v, w) + b(v, v, w).$$ (42)

We now proceed to estimate each term in the right-hand side as follows
$$|b(w, v, w)| \leq C \|v\|_1 \|w\|_1^2,$$
$$|b(v, v, w)| \leq C \|v\|_1^2 \|w\|_1 \leq C \|v\|_1^2, $$ (43)
$$|b(w, u^0, w)| \leq C \|w\|_1 \|u^0\|_1 \|w\|_1 \leq C \|u^0\|_1 \|w\|_1^2,$$
$$|b(u^0, v, w)| + |b(v, u^0, w)| \leq C \|u^0\|_1 \|v\|_1 \|w\|_1 \leq C \|u^0\|_1 \|v\|_1^2 \|w\|_1^2.$$ Therefore,
$$\langle B(u^\varepsilon, u^\varepsilon) - B(u^0, u^0), w \rangle$$
$$\leq C (1 + \|v\|_1 + \|u^0\|_1) \|w\|_1^2 + C \|w\|_1^4 + C \|u^0\|_1^2 \|v\|_1^2$$ (44)
$$\leq C (1 + \|w\|_1 + \|u^0\|_1) \|w\|_1^2 + \alpha \|w\|_1^2 + \|\eta^\varepsilon_2\|_{L^2} + C \|v\|_1^2 + C \|u^0\|_1^2 \|v\|_1^2.$$ Exploiting (41)-(44), we readily obtain
$$\frac{d}{dt} y(t) \leq hy(t) + G,$$
exists a complete bounded trajectory

\begin{proof}[Proof of Theorem 4.2]

where

\begin{align*}
y(t) &= \|w\|^2 + \alpha \|w\|^2 + \|\eta_2\|^2_M, \\
b(t) &= C(1 + \|v\|^2 + \|u_0\|^1), \\
G(t) &= C\|v\|^2 + C\|u_0\|^2\|v\|^2, \\
\end{align*}

Since \((w(\tau), \eta^\tau) = (0, 0)\), the Gronwall lemma implies that

\begin{equation}
y(t) \leq \int_\tau^t G(s)e^{\int_s^t h(r) dr} ds \leq e^{\int_\tau^t h(s) ds} \int_\tau^t G(s) ds. \tag{45}
\end{equation}

On the other hand, from (37) and (39), we learn that

\begin{equation}
\int_\tau^t h(s) ds \leq \int_\tau^t C(1 + \|v(s)\|^1 + \|u_0(s)\|^1) ds \leq C(R_0 + \ell)(t - \tau + 1), \tag{46}
\end{equation}

and

\begin{equation}
\int_\tau^t G(s) ds \leq \int_\tau^t C\|v\|^2 + C\|u_0\|^2\|v\|^2 ds \leq C\ell^2 \varepsilon^2(1 - \rho) + R_0^2 \ell^2 \varepsilon^2(1 - \rho)(t - \tau + 1) \tag{47}
\end{equation}

Using the triangle inequality, we get

\begin{align*}
y(t) &\leq C\ell^2 \varepsilon^2(1 - \rho)(R_0^2 + \ell^2)(t - \tau + 1) e^{C(R_0 + \ell)(t - \tau)} \\
&\leq C\ell^2 \varepsilon^2(1 - \rho)(R_0^2 + \ell^2) e^{R_1(t - \tau)}, \quad \text{where } R_1 = R_3(\rho, \ell).
\end{align*}

The desired conclusion follows then by comparison. \(\square\)

**Proof of Theorem 4.2.** For \(\varepsilon > 0\), let \(z^\varepsilon \in \mathcal{A}^\varepsilon\). Thus, in view of (10), there exists a complete bounded trajectory \(\tilde{z}^\varepsilon(t)\) of (40), with the external force

\begin{equation}
\tilde{z}^\varepsilon = \tilde{f}_0 + \varepsilon^{-\rho} \tilde{f}_1(. / \varepsilon) \in \mathcal{H}_w(f^\varepsilon), \quad \text{where } \tilde{f}_0 \in \mathcal{H}_w(f_0), \quad \tilde{f}_1 \in \mathcal{H}_w(f_1),
\end{equation}

such that \(\tilde{z}^\varepsilon(0) = z^\varepsilon\). By Lemma 4.3 with \(t = 0\),

\begin{equation}
\|z^\varepsilon - U_{\tilde{f}_0}(0, \tau)\tilde{z}^\varepsilon(\tau)\|_{\mathcal{H}} \leq C\ell \varepsilon^{1 - \rho}(R_0 + \ell)e^{C\tau}, \quad \forall \tau \leq 0.
\end{equation}

On the other hand, it is known (see e.g., [9]) that the set \(\mathcal{A}^0\) attracts \(U_{\tilde{f}_0}(t, \tau)B^\varepsilon\), uniformly not only with respect to \(\tau \in \mathbb{R}\), but also with respect to \(\tilde{f}_0 \in \mathcal{H}_w(f^0)\). Then, for every \(\delta > 0\), there is \(\tau = \tau(\delta) \leq 0\) independent of \(\varepsilon\) such that

\begin{equation}
\text{dist}_{\mathcal{H}}(U_{\tilde{f}_0}(0, \tau)\tilde{z}^\varepsilon(\tau), \mathcal{A}^0) \leq \delta.
\end{equation}

Using the triangle inequality, we get

\begin{equation}
\text{dist}_{\mathcal{H}}(z^\varepsilon, \mathcal{A}^0) \leq C\ell \varepsilon^{1 - \rho}(R_0 + \ell)e^{C\tau} + \delta.
\end{equation}

Because \(z^\varepsilon \in \mathcal{A}^\varepsilon\) is arbitrary, we deduce that

\begin{equation}
\limsup_{\varepsilon \to 0^+} \{\text{dist}_{\mathcal{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0)\} \leq \delta.
\end{equation}

Letting \(\delta \to 0\) completes the proof. \(\square\)
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