BLOWUP TIME AND BLOWUP MECHANISM OF SMALL DATA SOLUTIONS TO GENERAL 2-D QUASILINEAR WAVE EQUATIONS

BINGBING DING
School of Mathematical Sciences, Jiangsu Provincial Key Laboratory for Numerical Simulation of Large Scale Complex Systems, Nanjing Normal University
Nanjing, 210023, China

INGO WITT
Mathematical Institute, University of Göttingen, Bunsenstr. 3-5
Göttingen, D-37073, Germany

HUICHENG YIN*
School of Mathematical Sciences, Jiangsu Provincial Key Laboratory for Numerical Simulation of Large Scale Complex Systems, Nanjing Normal University
Nanjing, 210023, China

Abstract. For the 2-D quasilinear wave equation
\[ \sum_{i,j=0}^{2} g_{ij}(\nabla u) \partial_{ij}^2 u = 0, \]
whose coefficients are independent of the solution \( u \), the blowup result of small data solution has been established in [1, 2] when the null condition does not hold as well as a generic nondegenerate condition of initial data is assumed. In this paper, we are concerned with the more general 2-D quasilinear wave equation
\[ \sum_{i,j=0}^{2} g_{ij}(u, \nabla u) \partial_{ij}^2 u = 0, \]
whose coefficients depend on \( u \) and \( \nabla u \) simultaneously. When the first weak null condition is not fulfilled and a suitable nondegenerate condition of initial data is assumed, we shall show that the small data smooth solution \( u \) blows up in finite time, moreover, an explicit expression of lifespan and blowup mechanism are also established.

1. Introduction and main results. In this paper, we focus on the blowup problem of small data smooth solution to such a general 2-D quasilinear wave equation
\[
\begin{align*}
&\sum_{i,j=0}^{2} g_{ij}(u, \nabla u) \partial_{ij}^2 u = 0, \\
&(u(0, x), \partial_t u(0, x)) = \varepsilon(\varphi_0(x), \varphi_1(x)),
\end{align*}
\]
where \( x_0 = t, \ x = (x_1, x_2), \ \nabla = (\partial_0, \partial_1, \partial_2), \ \varepsilon > 0 \) is small enough, \( \varphi_1(x) \in C_0^\infty(B(0, M)) \ (i = 0, 1) \) with \( B(0, M) \) being a disk of radius \( M \) centered at the...

2000 Mathematics Subject Classification. 35L05, 35L72.
Key words and phrases. Weak null condition, lifespan, blowup, blowup system, Nash-Moser-Hörmander iteration.

The first author and the third author were supported by the NSFC (No. 11571177) and by the Priority Academic Program Development of Jiangsu Higher Education Institutions. The second author was supported by the DFG via the Sino-German project "Analysis of PDEs and application."

* Corresponding author.
origin, and the coefficients \( g_{ij}(u, \nabla u) = g_{ji}(u, \nabla u) \) \((0 \leq i, j \leq 2)\) are \( C^\infty \) smooth on their arguments.

Without loss of generality, one can write \( g_{ij}(u, \nabla u) = c_{ij} + d_{ij}u + \sum_{k=0}^{2} e_{ij}^k \partial_k u + O(|u|^2 + |\nabla u|^2)\), where \( c_{ij}, d_{ij} \) and \( e_{ij}^k \) are constants, \( d_{00} = 0, d_{ij} = d_{ji} \neq 0 \) for at least one \((i, j) \neq (0, 0)\) (if \( d_{ij} = 0 \) holds for all \((i, j)\), then problem (1) has been treated in [1, 2, 3]), and \( \sum_{i,j,k=0}^{2} c_{ij} \partial_j^2 = \partial_i^2 - \Delta \). In addition, it is assumed that \( \sum_{i,j,k=0}^{2} e_{ij}^k \partial_k u \partial_j^2 u \) does not satisfy the null condition, which means \( \sum_{i,j,k=0}^{2} e_{ij}^k \xi_k \xi_j \neq 0 \) for the variables \((\xi_0, \xi_1, \xi_2)\) with \( \xi_0^2 = \xi_1^2 + \xi_2^2 \) and \((\xi_1, \xi_2) \neq 0\) (i.e., the first weak null condition does not hold by using the terminology in [3] and [27]).

Introducing polar coordinates \((r, \theta)\) in \( \mathbb{R}^2 \) as follows

\[
\begin{align*}
x_1 &= r \cos \theta, \\
x_2 &= r \sin \theta,
\end{align*}
\]

where \( r = \sqrt{x_1^2 + x_2^2} \) and \( \theta \in [0, 2\pi] \). Later on we will need the function

\[
F_0(\sigma, \theta) = F_0(\sigma, \omega) = \frac{1}{2\pi \sqrt{2}} \int_0^{+\infty} \frac{R(s, \omega; \varphi_1) - \partial_s R(s, \omega; \varphi_0)}{\sqrt{s - \sigma}} ds, \tag{2}
\]

where \( \sigma \in \mathbb{R}, \omega \equiv (\omega_1, \omega_2) = (\cos \theta, \sin \theta) \), and \( R(s, \omega; v) \) is the Radon transform of the smooth function \( v(x) \), i.e., \( R(s, \omega; v) = \int_{x \cdot \omega = s} v(x) dS \). From Theorem 6.2.2 of [15], one knows that the function \( F_0(\sigma, \theta) \neq 0 \) unless \((\varphi_0(x), \varphi_1(x)) \equiv 0\). Moreover, \( F_0(\sigma, \theta) \equiv 0 \) for \( \sigma \geq M \) and \( \lim_{\sigma \to -\infty} F_0(\sigma, \theta) = 0 \).

Set \( F_1(\sigma, \theta) = \sum_{i,j=0}^{2} d_{ij} \delta_i \delta_j F_0(\sigma, \theta) \) and \( F_2(\sigma, \theta) = \sum_{i,j,k=0}^{2} e_{ij}^k \delta_i \delta_j \delta_k F_0(\sigma, \theta) \) with \((\delta_0, \delta_1, \delta_2) = (-1, 1, 1)\). Define the function

\[
G_0(\sigma, \theta) = \frac{1}{F_1(\sigma, \theta)} \ln \left( 1 + \frac{F_1(\sigma, \theta)}{F_2(\sigma, \theta)} \right) \text{ for } (\sigma, \theta) \in A, \tag{3}
\]

where \( A = \{ (\sigma, \theta) \in (-\infty, M) \times [0, 2\pi] : F_1(\sigma, \theta) \neq 0, F_2(\sigma, \theta) 
eq 0, 1 + \frac{F_1(\sigma, \theta)}{F_2(\sigma, \theta)} > 0 \} \).

Denote by

\[
\tau_0 = \inf_{(\sigma, \theta) \in B} G_0(\sigma, \theta) \tag{4}
\]

with the domain \( B = \{ (\sigma, \theta) \in A : G_0(\sigma, \theta) > 0 \} \). Here we specially emphasize that \( \tau_0 > 0 \) can be shown as long as \((\varphi_0(x), \varphi_1(x)) \neq 0\) (see Lemma 2.1 below). In order to state and show the main conclusion, as in [1] and [2], we require the following non-degeneracy assumption:

\((ND)\) There exists a unique minimum point \((\sigma_0, \theta_0) \in B\) such that \( \tau_0 = G_0(\sigma_0, \theta_0) \) and the Hessian matrix \((\nabla_{\sigma, \theta}^2 G_0)(\sigma_0, \theta_0) > 0 \).

Note that \( \tau_0 \) can be achieved at some interior point of the open set \( B \) in the rather general cases (one can see Remark 10 below), then assumption \((ND)\) can be regarded to be suitable.

**Theorem 1.1.** If \( \varphi_0(x) \neq 0 \) or \( \varphi_1(x) \neq 0 \), under assumption \((ND)\), problem (1) has a \( C^\infty \) solution for \( 0 \leq t < T_\varepsilon \), where \( T_\varepsilon \) stands for the lifespan of smooth solution \( u \) which satisfies

\[
\lim_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} = \tau_0 > 0. \tag{5}
\]
Moreover, there exists a point \( M_x = (T_x, x_x) \) and a positive constant \( C > 1 \) independent of \( \varepsilon \) such that

(i) \( u(t,x) \in C^1([0,T_x] \times \mathbb{R}^2) \) and \( \|u\|_{L^\infty([0,T_x] \times \mathbb{R}^2)} + \|\nabla u\|_{L^\infty([0,T_x] \times \mathbb{R}^2)} \leq C \varepsilon \).

(ii) \( u \in C^2([0,T_x] \times \mathbb{R}^2) \) and satisfies for \( t < T_x \)

\[
\frac{1}{C(T_x - t)} \leq \|\nabla^2 u(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{T_x - t}.
\]  

(6)

**Remark 1.** With respect to the blowup problem of small data solution to \( \partial_t^2 u - \sum_{i=1}^2 \partial_i(c_i^2(u)\partial_i u) = 0, \) in [9] we have shown that the blowup mechanism of smooth solution \( u \) is of ODE-type, which means \( \nabla u \) start to develop singularities from the lifespan time \( T_x \) while \( u \) is still continuous up to \( T_x \). Here Theorem 1.1 illustrates that the blowup mechanism of smooth solution to (1) is of geometric type, which means only \( \nabla^2 u \) develop singularities from \( T_x \) while \( u \) and \( \nabla u \) are still continuous up to \( T_x \). Our Theorem 1.1 is similar to the “lifespan theorems” in [1, 2], where such 2-D nonlinear wave equations \( \partial_t^2 v - \Delta v + \sum_{0 \leq i,j,k \leq 2} \partial_{ij}^k v \partial_{ij}^k v = 0 \) with \( g_{ij}^k \) are studied when the null conditions are not fulfilled.

**Remark 2.** For the 3-D wave equation \( \partial_t^2 u - \sum_{0 \leq i,j \leq 3, (i,j) \neq (0,0)} (\delta_{ij} + d_{ij})u \partial_{ij}^2 u = 0 \) \((d_{ij} \in \mathbb{R} \text{ and } d_{ij} \neq 0 \text{ for some } (i, j) \neq (0, 0))\) with small initial data \((u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x))\), in [4, 25, 26] it was shown that the smooth solution exists globally. On the other hand, for the \( n \)-dimensional nonlinear wave equation \((n = 2, 3)\) with coefficients depending only on the first order derivatives of the solution, \( \partial_t^2 u - c^2(\partial_t u) \Delta u = 0 \) and, more generally, \( \sum_{i,j=0}^n g_{ij}(\nabla u) \partial_{ij}^2 u = 0 \), \( t = x_0, x = (x_1, \ldots, x_n) \), \( g_{ij}(\nabla u) = c_{ij} + \sum_{k=0}^n d_{ij}^k \partial_k u + O(|\nabla u|^2) \), and the linear part \( \sum_{i,j=0}^n c_{ij} \partial_{ij}^2 u \) is strictly hyperbolic with respect to time \( t \), it is known that small data smooth solutions exist globally or almost globally if related null conditions hold (see [3, 5, 20, 27, 29, 30] and the references therein), otherwise small data smooth solutions blow up in finite time (see [1, 2, 6, 8, 11, 14, 15, 16, 17, 18, 19, 24, 28] and so on).

**Remark 3.** In terms of the results in [23], one can only derive that the lifespan \( T_x \) of smooth solution to (1) satisfies \( T_x \geq \frac{C}{\varepsilon} \) for small \( \varepsilon > 0 \). On the other hand, similar to the proof of Theorem 2.3 in [24], where the 3-D quasilinear wave equations \( \Box u = \sum_{i=0}^3 \partial_i G_i(u, \nabla u) + G_4(\nabla u, \nabla^2 u) \) with \( G_i \) \((0 \leq i \leq 4)\) being quadratic forms are treated, one can further obtain \( T_x \geq \frac{C}{\varepsilon^2} \). Here we especially point out that a precise bound of \( \varepsilon^2 T_x \) has been given in our Theorem 1.1.

**Remark 4.** If problem (1) admits a symmetric solution \( u(t, r) \) when \( \varphi_0(x) \) and \( \varphi_1(x) \) are symmetric, then by the similar arguments as in [10] and [17], we can prove (6) still holds even if assumption (ND) is removed.

**Remark 5.** Since \( d_{00} = 0 \) and \( d_{ij} = d_{ji} \neq 0 \) for at least one \((i, j) \neq (0, 0)\), then under the restriction conditions \( \xi_0 = -1 \) and \( \xi_1^2 + \xi_2^2 = 1, \sum_{i,j=0}^3 d_{ij} \xi_i \xi_j \neq 0 \) holds except only finite points \((\xi_1^0, \xi_2^0)\) (this is easily obtained by utilizing the standard form of quadratic polynomials under suitable coordinate rotations). Analogously, this is also correct for \( \sum_{i,j,k=0}^3 c_{ij}^k \xi_k \xi_i \xi_j \). These simple facts will be used in proving \( \tau_n > 0 \) as in Lemma 2.1 of [8].

**Remark 6.** In [6] and [28], for the 3-D quasilinear wave equations \( \sum_{i,j=0}^3 g_{ij}^k (\partial u) \partial_{ij}^2 u = 0 \), the authors utilize Christodoulou’s geometric approach to
Consider the general 3-D quasilinear wave equation
\[ \sum_{i,j=0}^{3} g_{ij}(u, \nabla u) \partial_{ij}^2 u = 0 \] with \( (u(0, x), \partial_t u(0, x)) = \varepsilon (u_0(x), u_1(x)) \). When the corresponding weak null condition is not fulfilled, and an analogous (ND) condition for the 2-D quasilinear wave equation is posed, we have shown in [8] that the lifespan \( T_\varepsilon \) of small data smooth solution \( u \) satisfies
\[ \lim_{\varepsilon \to 0} \varepsilon \ln T_\varepsilon = \tau_0 > 0, \]
where the constant \( \tau_0 \) is completely determined by the coefficients \( g_{ij} \) and the data \( (u_0(x), u_1(x)) \). Otherwise, if the weak null condition holds, we have proved in [7] that the small data solution \( u \) exists globally.

**Remark 7.** Consider the general 3-D quasilinear wave equation
\[ \sum_{i,j=0}^{3} g_{ij}(u, \nabla u) \partial_{ij}^2 u = 0 \] for the 2-D quasilinear wave equation is posed, we have shown in [8] that the lifespan corresponding weak null condition is not fulfilled, and an analogous (ND) condition for the 2-D quasilinear wave equation is posed, we have shown in [8] that the lifespan \( T_\varepsilon \) of small data smooth solution \( u \) satisfies
\[ \lim_{\varepsilon \to 0} \varepsilon \ln T_\varepsilon = \tau_0 > 0, \]
where the constant \( \tau_0 \) is completely determined by the coefficients \( g_{ij} \) and the data \( (u_0(x), u_1(x)) \). Otherwise, if the weak null condition holds, we have proved in [7] that the small data solution \( u \) exists globally.

**Remark 8.** For problem (1), so far the systematic results as in [7] and [8] for the general 3-D wave equations \( \sum_{i,j=0}^{3} g_{ij}(u, \nabla u) \partial_{ij}^2 u = 0 \) have not been established. For examples, even for the cases of \( g^{ij}(u, \nabla u) = c^{ij} + d^{ij} u \) or \( g^{ij}(u, \nabla u) = c^{ij} + d^{ij} u^2 \), the global existence or blowup results on the small data smooth solution \( u \) of (1) are not known yet.

Near the blowup point \( M_\varepsilon \) we can give a more accurate description on the behavior of solution \( u \) as in [1, 2] and [8].

**Theorem 1.2** (Geometric Blowup Theorem). Choose the constants \( \tau_1, A_0, A_1 \) and \( \delta_0 \) such that \( 0 < \tau_1 < \tau_0, A_0 < \sigma_0 < A_1 < M_0, A_0 \) and \( A_1 \) are close to \( \sigma_0 \), and \( \delta_0 > 0 \) is sufficiently small. Denote by \( \mathcal{D} \) the domain
\[ \mathcal{D} = \{(s, \theta, \tau) : A_0 \leq s \leq A_1, \theta_0 - \delta_0 \leq \theta \leq \theta_0 + \delta_0, \tau_1 \leq \tau \leq \tau_\varepsilon \}, \]
where \( \tau_\varepsilon = \varepsilon \sqrt{T_\varepsilon} \). Then there exist a subdomain \( \mathcal{D}_0 \) of \( \mathcal{D} \) containing a point \( m_\varepsilon = (s_\varepsilon, \theta_\varepsilon, \tau_\varepsilon) \) and functions \( \phi(s, \theta, \tau), w(s, \theta, \tau), v(s, \theta, \tau) \in C^3(\mathcal{D}_0) \) with the following properties:

In the domain \( \mathcal{D}_0 \), \( \phi \) satisfies
\[ \left\{ \begin{array}{l}
\partial_s \phi(s, \theta, \tau) \geq 0, \\
\partial_s \phi(s, \theta, \tau) = 0 \leftrightarrow (s, \theta, \tau) = m_\varepsilon,
\end{array} \right. \]
\[ \partial^2_{s\tau} \phi(m_\varepsilon) < 0, \quad \nabla_{s,\theta} \partial_s \phi(m_\varepsilon) = 0, \quad \nabla_{s,\theta}^2 \partial_s \phi(m_\varepsilon) > 0. \] (H)

In addition, \( \partial_s w = v \partial_s \phi \) and
\[ \partial_s v(m_\varepsilon) \neq 0. \] (7)

Let the function \( G(\phi, \theta, \tau) \) be defined by \( G(\Phi) = w(s, \theta, \tau) \) and \( \partial_s G(\Phi) = v(s, \theta, \tau) \) in the domain \( \Phi(\mathcal{D}_0) \), where \( \Phi \) is a map such that \( \Phi(s, \theta, \tau) = (\phi(s, \theta, \tau), \theta, \tau) \). Then the solution of (1) can be written as \( u(t, x) = \frac{\varepsilon}{\sqrt{r}} G(r - t, \theta, \varepsilon \sqrt{r}) \) near the point \( M_\varepsilon = (T_\varepsilon, (\cos \theta_\varepsilon, T_\varepsilon + \phi(s_\varepsilon, \theta_\varepsilon, \tau_\varepsilon)) \) and for \( t < T_\varepsilon \).

**Remark 9.** Theorem 1.2 provides a more accurate description of the solution \( u \) near the blowup point \( M_\varepsilon \) for \( t \leq T_\varepsilon \) than Theorem 1.1 as in [1, 2] and [8]. For instances, \( G, \nabla G \in C(\Phi(\mathcal{D}_0)) \) can be directly derived from \( (w(s, \theta, \tau), v(s, \theta, \tau)) \in C^3(\mathcal{D}_0) \) and (H); in addition, by \( \partial^2_{\phi} G = \frac{\partial u}{\partial s} \phi \) and \( u(t, x) = \frac{\varepsilon}{\sqrt{r}} G(r - t, \theta, \varepsilon \sqrt{r}) \),
it follows from a direct computation and condition \((H)\) that there exists a positive constant \(C\) independent of \(\varepsilon\) such that
\[
\frac{1}{C(T_\varepsilon - t)} \leq \|\nabla^2 u\|_{L^\infty(\Phi(\varepsilon))} \leq \frac{C}{T_\varepsilon - t}.
\]

To prove Theorem 1.1, at first, as in [15, Chapter 6] and [8], by constructing a suitable approximate solution \(u_\alpha\) to (1) and then considering the difference of the exact solution \(u\) and \(u_\alpha\), applying the Klainerman-Sobolev inequality in [21], and further establishing a delicate energy estimate, we finally obtain this lower bound of \(T_\varepsilon\). Next we derive the required upper bound of \(T_\varepsilon\). Motivated by the “geometric blowup” method of [1, 2], as in [8], we introduce the blowup system of (1) to study the blowup point. In the process to treat the resulting blowup system, as in [1, 2], asymptotic expansion of \(T\) as well as a precise description of the behavior of \(u\) close to the blowup point. In the process to treat the resulting blowup system, as in [1, 2], we will use the Nash-Moser-Hörmander iteration method to overcome the difficulties introduced by the free boundary \(t = T_\varepsilon\) and the complicated nonlinear blowup system. To this end, the linearized blowup system should be solved. However, due to the simultaneous appearances of \(u\) and \(\nabla u\) in the coefficients \(g_{ij}(u, \nabla u)\), the resulting blowup system of (1) has some different features from that in [1, 2] and [8] (one can see (52)-(53) in §3 below). For examples, compared with the linearized blowup system of \(\partial_\alpha^2 u - \Delta u + \sum_{0 \leq i,j,k \leq 2} g_{ij} \partial_k u \partial_\alpha^2 u = 0\) in [2], some coefficients \(\alpha_1\) and \(\alpha_2\) in (52) do not admit the smallness property, moreover, there are more terms to be treated in (53) than those in the corresponding (3.1.1b) of [2]. Thanks to the multipliers chosen in [1, 2], through integration by parts we can derive the energy estimates of solutions to the linearized blowup system directly, and subsequently its solvability is shown. Based on this and the standard Nash-Moser-Hörmander iteration, the proof of Theorem 1.2 can be completed. In addition, compared with the treatments on the 3-D problem in [8], here the treatments on the 2-D problem (1) have some distinct features due to the different large time behaviors of solutions to 2-D and 3-D linear wave equations.

The paper is organized as follows: In §2, as in [15], we construct a suitable approximate solution \(u_\alpha(t, x)\) to (1) and establish related estimates, which allows us to obtain the required lower bound on the lifespan \(T_\varepsilon\). In §3, the blowup system of (1) is derived and solved, which enables us to prove Theorem 1.2. The proof of Theorem 1.1 is completed in §4.

Throughout the paper, we will use the following notations:

- \(Z\) denotes one of the Klainerman vector fields in \(\mathbb{R}^+ \times \mathbb{R}^2\), i.e.,
- \(\partial_t, \partial_1, \partial_2, S = t\partial_t + \sum_{j=1}^2 x_j \partial_j, H_i = x_i \partial_t + t \partial_i, i = 1, 2, R = x_1 \partial_2 - x_2 \partial_1.\)
2. The lower bound of the lifespan $T_\varepsilon$. In this section, we will establish the estimate on the lower bound of $T_\varepsilon$ for the smooth solution to problem (1). As in the proof of [15, Theorem 6.5.3], by constructing the approximate solution $u_a$ of (1), and then estimating the difference of $u_a$ and the solution $u$, we can derive the lower bound of $T_\varepsilon$ by continuity induction argument. Our new gradients in this procedure are how to construct the approximate solution and look for the precise blowup time for the nonlinear profile equation of (1) and subsequently treat the solution $u$ itself and $\nabla u$ other than only treat the derivatives of solutions in [15]. Although some related procedures are analogous to those in [8], for reader’s convenience, we shall give some detailed proofs.

Set the slow time variable $\tau = \varepsilon \sqrt{1 + t}$, and assume the solution $u$ of (1) can be approximated by

$$\varepsilon \sqrt{r} V(q, \omega, \tau), \quad r > 0,$$

where $q = r - t$, $\omega = (\omega_1, \omega_2) = \frac{r}{\varepsilon} \in \mathbb{S}^1$, and $V(q, \omega, \tau)$ is solved by the following equation

$$\begin{cases}
\partial_{q\tau}^2 V - \sum_{i,j=0}^2 \left( d_{ij} V + \sum_{k=0}^2 e_{ij}^k \hat{\omega}_k \partial_q V \right) \hat{\omega}_i \hat{\omega}_j \partial_q^2 V = 0, \\
V(q, \omega, 0) = F_0(q, \omega), \\
\text{supp} V \subseteq \{ q \leq M \},
\end{cases}
$$

(8)

where $(\hat{\omega}_0, \hat{\omega}_1, \hat{\omega}_2) = (-1, \omega_1, \omega_2)$, and $F_0(q, \omega)$ has been defined in (2).

Before studying the blowup problem of (8), we require cite the following two lemmas, whose proofs are almost analogous to Lemma 2.1-2.2 in [8] and are then omitted here.

**Lemma 2.1.** Assume that $\tau_0$ is defined in (4), then $\tau_0 > 0$ is a determined constant as long as $(\varphi_0(x), \varphi_1(x)) \not\equiv 0$.

**Lemma 2.2.** Define the function

$$\tilde{G}_0(\sigma, \theta) = \begin{cases}
G_0(\sigma, \theta) & \text{for } (\sigma, \theta) \in A, \\
1 & \text{for } (\sigma, \theta) \in D,
\end{cases}
$$

if $D = \{ (\sigma, \theta) \in (-\infty, M) \times [0, 2\pi] : F_1(\sigma, \theta) = 0, F_2(\sigma, \theta) > 0 \} \neq \emptyset$. Denote by

$$\tilde{\tau}_0 = \min \{ \tau_0, \inf_{(\sigma, \theta) \in D} \tilde{G}_0(\sigma, \theta) \}.$$

Then

$$\tilde{\tau}_0 = \tau_0.$$

**Remark 10.** Note that as in the analysis of [8], one can show that assumption $(ND)$ is generic.

Based on Lemma 2.1-Lemma 2.2, with respect to problem (8), we have

**Lemma 2.3.** (8) has a $C^\infty$ solution for $0 \leq \tau < \tau_0$, where $\tau_0 = \min_{(\sigma, \theta) \in B} G_0(\sigma, \theta)$. 

Proof. Set \( w(q, \omega, \tau) = \partial_\tau V(q, \omega, \tau) \). Then it follows from (8) that
\[
\begin{align*}
\partial_\tau w - \sum_{i,j=0}^2 \left( d_{ij}V + \sum_{k=0}^2 c_{ij}^k \hat{\omega}_k w \right) \hat{\omega}_i \hat{\omega}_j w = 0, \quad (q, \tau) \in (-\infty, M] \times [0, \tau_0), \\
w(q, \omega, 0) = \partial_\tau F_0(q, \omega).
\end{align*}
\]

The characteristics \( q = q(\omega, \tau; s) \) of (9) starting from the point \((\omega, 0)\) is defined as
\[
\begin{align*}
\frac{dq}{d\tau}(\omega, \tau; s) &= - \sum_{i,j=0}^2 \left( d_{ij}V + \sum_{k=0}^2 c_{ij}^k \hat{\omega}_k w \right) \hat{\omega}_i \hat{\omega}_j (q(\omega, \tau; s), \omega, \tau), \\
q(\omega, 0; s) &= s.
\end{align*}
\]

Along this characteristics curve, we have that for \( \tau < \tau_0 \),
\[
w(q(\omega, \tau; s), \omega, \tau) = \partial_\tau F_0(s, \omega) = \partial_\tau V(q(\omega, \tau; s), \omega, \tau).
\]

On the other hand, by (10) and (11), we can obtain
\[
\begin{align*}
\partial_{\tau}^2 q(\omega, \tau; s) &= - \sum_{i,j=0}^2 \left( d_{ij} \partial_\tau F_0(s, \omega) \partial_\tau q(\omega, \tau; s) + \sum_{k=0}^2 c_{ij}^k \hat{\omega}_k \partial_\tau^2 F_0(s, \omega) \right) \hat{\omega}_i \hat{\omega}_j, \\
\partial_\tau q(\omega, 0; s) &= 1.
\end{align*}
\]

This derives \( \partial_\tau q(\omega, \tau; s) = \exp(-F_1(s, \omega)\tau) \left( 1 + \frac{F_2(s, \omega)}{F_1(s, \omega)} \right) - \frac{F_2(s, \omega)}{F_1(s, \omega)} > 0 \) if \( F_1(s, \omega) \neq 0 \) and \( \partial_\tau q(\omega, \tau, s) = 1 - \tau F_2(s, \omega) > 0 \) if \( F_1(s, \omega) = 0 \) when \( 0 \leq \tau < \tau_0 \), respectively. Then
\[
q(\omega, \tau; s) = q(\omega, \tau; M) + \int_M^s \left( \exp(-F_1(\rho, \omega)\tau) \left( 1 + \frac{F_2(\rho, \omega)}{F_1(\rho, \omega)} \right) - \frac{F_2(\rho, \omega)}{F_1(\rho, \omega)} \right) d\rho.
\]

Here we have used the property of \( \lim_{\tau \to 0}(e^{\tau y}(1 - \frac{y}{2}) + \frac{y}{2}) = 1 - \tau y \).

Note that \( q(\omega, \tau; M) = M \) such that \( V(q, \omega, \tau) \) satisfies the boundary condition \( V|_{q=M} = 0 \). Hence
\[
\begin{align*}
V(q(\omega, \tau; s), \omega, \tau) &= \int_M^s \left( \exp(-\tau F_1(\rho, \omega)) \partial_\tau F_0(\rho, \omega) + \left( \exp(-\tau F_1(\rho, \omega)) - 1 \right) \frac{F_2(\rho, \omega) \partial_\tau F_0(\rho, \omega)}{F_1(\rho, \omega)} \right) d\rho.
\end{align*}
\]

By Lemma 2.1-Lemma 2.2 and the implicit function theorem, we obtain the smooth function \( s = s(q, \omega, \tau) \) for \( \tau < \tau_0 \). Hence
\[
V(q, \omega, \tau) = \int_M^{s(q, \omega, \tau)} \left( \exp(-\tau F_1(\rho, \omega)) \partial_\tau F_0(\rho, \omega) \right.
\]
\[
\left. + \left( \exp(-\tau F_1(\rho, \omega)) - 1 \right) \frac{F_2(\rho, \omega) \partial_\tau F_0(\rho, \omega)}{F_1(\rho, \omega)} \right) d\rho
\]
is a smooth solution of (8) for \( 0 \leq \tau < \tau_0 \).

According to the Chapter 6 of [15], we know that \( F_0(q, \omega) \in C^\infty(\mathbb{R} \times S^1) \), \( \text{supp}F_0 \subseteq (-\infty, M] \times S^1 \), and
\[
|\partial_\omega^k \partial_q F_0(q, \omega)| \leq C_{k, \alpha}(1 + |q|)^{-\frac{3}{2} - k}.
\]
In addition, from the expression of \( V(q, \omega, \tau) \) we can conclude that if \( \tau \leq b < \tau_0 \), we have
\[
|\partial^m_{q, \omega, \tau} V(q, \omega, \tau)| \leq C_{a,t,m}^b (1 + |q|)^{- \frac{3}{2} - m}
\]
and
\[
|\partial^m_{q} V(q, \omega, \tau)| \leq C^b_{a,t}.
\]

We now start to construct an approximate solution to (1) for \( 0 \leq \tau = \varepsilon \sqrt{1 + t} < \tau_0 \). Let \( w_0 \) be the solution of the following linear wave equation:
\[
\begin{cases}
\partial^2_t w - \Delta w = 0, \\
w(0, x) = \phi_0(x), \\
\partial_t w(0, x) = \phi_1(x).
\end{cases}
\]
Choosing a \( C^\infty \) function \( \chi(s) \) such that \( \chi(s) = 1 \) for \( s \leq 1 \) and \( \chi(s) = 0 \) for \( s \geq 2 \). We set for \( 0 \leq \tau = \varepsilon \sqrt{1 + t} < \tau_0 \),
\[
u_a(t, x) = \varepsilon \chi(\varepsilon t) w_0(t, x) + \varepsilon \sqrt{\varepsilon} (1 - \chi(\varepsilon t)) \chi(-3\varepsilon q)V(q, \omega, \tau).
\]
By Theorem 6.2.1 of [15] and Lemma 2.3, we then know \( |Z^\alpha u_a| \leq C_{a,b} \varepsilon (1 + t)^{-1/2} \) for \( \tau \leq b < \tau_0 \) and all multi-index \( \alpha \). Set
\[
J_a = \partial^2_t u_a - \Delta u_a + \sum_{i,j=0}^2 (d_{ij} u_a + \sum_{k=0}^2 c_{ij}^k \partial_k u_a) \partial^2_{ij} u_a + O(|u_a|^2 + |\nabla u_a|^2) \sum_{i,j=0}^2 \partial^2_{ij} u_a,
\]
we have

Lemma 2.4.
\[
\int_0^{t^2 / \varepsilon^2 - 1} ||Z^\alpha J_a(t, \cdot)||_{L^2} dt \leq C_{a,b} \varepsilon^\frac{3}{2}.
\]

Proof. We divide this proof into the following three cases.

Case A. \( \frac{2}{\varepsilon} \leq t \leq b^2 / \varepsilon^2 - 1 \).

In this case, \( \chi(\varepsilon t) = 0 \) and \( u_a = \frac{\varepsilon}{\sqrt{\varepsilon}} \chi(-3\varepsilon q)V(q, \omega, \tau) \), then
\[
J_a = -\varepsilon^2 \left( \partial^2_{t\tau} V - \sum_{i,j=0}^2 d_{ij} \tilde{\omega}_i \tilde{\omega}_j \partial^2_{ij} V - \sum_{i,j,k=0}^2 c_{ij}^k \tilde{\omega}_i \tilde{\omega}_j \tilde{\omega}_k \partial_{ij} \tilde{V} \partial^2_{k\tau} V \right)
\]
\[
+ \frac{\varepsilon}{(1 + t)^{3/2}},
\]
where \( \tilde{V}(q, \omega, \tau) = \chi(-3\varepsilon q)V(q, \omega, \tau) \). Thanks for (14)-(15) and the explicit expression of \( V \), then
\[
\partial^2_{\tau\tau} V - \sum_{i,j=0}^2 d_{ij} \tilde{\omega}_i \tilde{\omega}_j \partial^2_{ij} V - \sum_{i,j,k=0}^2 c_{ij}^k \tilde{\omega}_i \tilde{\omega}_j \tilde{\omega}_k \partial_{ij} \tilde{V} \partial^2_{k\tau} V
\]
\[
\chi(1 - \chi) \partial^2_{t\tau} V - 3\varepsilon \chi' \partial_t V - \sum_{i,j,k=0}^2 d_{ij} \tilde{\omega}_i \tilde{\omega}_j V(9\varepsilon^2 \chi'' V - 6\varepsilon \chi' \partial_q V)
\]
\[
- \sum_{i,j,k=0}^2 c_{ij}^k \tilde{\omega}_i \tilde{\omega}_j \tilde{\omega}_k \{ (-3\varepsilon \chi' V + \chi \partial_q V)(9\varepsilon^2 \chi'' V - 6\varepsilon \chi' \partial_q V) - 3\varepsilon \chi' V \chi \partial^2_{q\tau} V \},
\]
which derives that the estimate $|Z^\alpha J_a| \leq C_{\alpha,b} \varepsilon (1 + t)^{-\frac{3}{2}} + C_{\alpha,b} \varepsilon^3 (1 + t)^{-1} \psi(-3\varepsilon q)$ holds, where $\psi(s)$ is a cutoff function satisfying $\psi(s) = 1$ for $s \in [1, 2]$, and otherwise $\psi(s) = 0$.

Case B. $t \leq \frac{1}{2}$.

In this case, $\chi(\varepsilon t) = 1$ and $u_a = \varepsilon u_0$. This derives $J_a = \varepsilon^2 \sum_{k=0}^2 \sum_{i,j=0}^2 \partial_i \partial_j u_a + O(|\varepsilon u_0|^2 + |\nabla u_0|^2) \sum_{i,j=0}^2 \partial_i \partial_j u_a$. It follows from a direct computation that

$$|Z^\alpha J_a| \leq C_{\alpha} \varepsilon^2 (1 + t)^{-1} (1 + |q|)^{-2}.$$ 

Case C. $\frac{1}{2} \leq t \leq \frac{2}{7}$.

A direct computation yields

$$u_a = \varepsilon u_0 + \varepsilon (1 - \chi(\varepsilon t))(r^{-\frac{3}{2}} \chi(\varepsilon q) - V) - w_0$$

and then

$$J_a = J_1 + J_2 + J_3 + J_4$$

with

$$J_1 = \varepsilon (\partial_i \partial_j u_a + \sum_{k=0}^2 \varepsilon^2 \partial_i \partial_j u_a + O(|w_0|^2 + |\nabla w_0|^2) \sum_{i,j=0}^2 \partial_i \partial_j u_a,$$

$$J_2 = \varepsilon (\partial_i \partial_j (V(q) - F_0(q))) = -\varepsilon^2 V \frac{\varepsilon^2}{4(1 + t)^{3/2}} \partial_t \partial_i \partial_j w_0,$$

$$J_3 = \varepsilon (\partial_i \partial_j (\chi(\varepsilon q) - V) - w_0),$$

$$J_4 = \varepsilon (\partial_i \partial_j (\chi(-3\varepsilon q) - 1)w_0).$$

It is easy to see $|Z^\alpha J_1| \leq C_{\alpha,b} \varepsilon^2 (1 + t)^{-1} (1 + |q|)^{-1}$.

Due to $(\partial_i \partial_j (V(q) - F_0(q))) = -\varepsilon^2 V \frac{\varepsilon^2}{4(1 + t)^{3/2}} \partial_t \partial_i \partial_j w_0$, and the fact $(\partial_i \partial_j - \Delta)w = r^{-1/2}((\partial_i + \partial_j)(\partial_t - \partial_r) - r^{-2}(1 + \partial_i \partial_j))(r^{1/2}v)$, then we have $|Z^\alpha J_2| \leq C_{\alpha,b} \varepsilon^3 (1 + |q|)^{-1}$.

Moreover, by the Theorem 6.2.1 of [15], we know for any constant $l > 0$, if $r \leq lt$,

$$|Z^\alpha (w_0 - r^{-\frac{3}{2}} F_0)| \leq C(1 + t)^{-\frac{3}{2}} (1 + |q|)^{\frac{3}{2}}.$$ (17)

On the other hand, according to the fact of $\partial_i \partial_j - \Delta = \frac{1}{r-t} (S + \omega_1 H_1 + \omega_2 H_2)(\partial_t - \partial_r) - \partial_i \partial_j - \frac{1}{2} \Delta \omega$, we conclude $|Z^\alpha J_3| \leq C_{\alpha,b} \varepsilon^2 (1 + t)^{-\frac{3}{2}}$.

Since the support of $J_4$ on the variable $q$ lies in $(-\infty, -\frac{1}{3\varepsilon})$, and apply the fact for any $\phi(t, r) \in C^1$

$$|\partial_\phi| \leq \frac{C}{1 + |t - r|} \sum_{|\beta| = 1} |Z^\beta \phi|,$$ (18)

we can get the estimate $|Z^\alpha J_4| \leq C_{\alpha,b} \varepsilon^3 (1 + t)^{-\frac{3}{2}}$.

All the above analysis yield

$$|Z^\alpha J_a| \leq C_{\alpha,b} \varepsilon^2 (1 + t)^{-\frac{3}{2}} + C_{\alpha,b} \varepsilon^3 (1 + t)^{-1} (1 + |q|)^{-1}.$$

Collecting the estimates above, we arrive at

$$\|Z^\alpha J_a(t, \cdot)\|_{L^2} \leq C_{\alpha,b} \varepsilon^2 (1 + t)^{-\frac{3}{2}} + C_{\alpha,b} \varepsilon^3 (1 + t)^{-1}, \quad 2 \varepsilon \leq t \leq e^{5/\varepsilon} - 1,$$

$$\|Z^\alpha J_a(t, \cdot)\|_{L^2} \leq C_{\alpha,b} \varepsilon^2 (1 + t)^{-\frac{3}{2}}, \quad t \leq \frac{2}{\varepsilon}.$$
Consequently,
\[
\int_0^{b/\varepsilon - 1} \|Z^\alpha u_a(t, \cdot)\|_{L^2} dt \leq C_{\alpha, b}\varepsilon^{\frac{3}{2}},
\]
and Lemma 2.4 is proved.

For the latter requirements, we list a conclusion which has been shown in [24].

**Lemma 2.5.** Suppose \( f(t, x) \in C^1(\mathbb{R}_+ \times \mathbb{R}^2) \), moreover, \( \text{supp} f \subset \{(t, x) : r \leq M + t\} \), then we have
\[
\|(1 + |t - r|)^{-1} f(t, \cdot)\|_{L^2} \leq C\|\partial_r f(t, \cdot)\|_{L^2}.
\]

Based on the preparations above, next we establish

**Proposition 1.** For sufficiently small \( \varepsilon > 0 \) and \( 0 < \tau = \varepsilon \sqrt{1 + t} \leq b < \tau_0 \), then (1) has a \( C^\infty \) solution which admits for all \( |\alpha| \leq 2 \),
\[
\|Z^\alpha \partial(u - u_a)\| \leq C_b\varepsilon^{\frac{3}{2}} (1 + t)^{-\frac{1}{2}} (1 + |t - r|)^{-\frac{1}{2}}.
\] (19)

**Proof.** Set \( v = u - u_a \). Then one has
\[
\begin{aligned}
\partial_t^2 v - \Delta v + \sum_{i,j=0}^2 (d_{ij} u + \sum_{k=0}^2 e_k^{ij} \partial_k u) \partial_{ij}^2 v + O(|u|^2 + |\nabla u|^2) \sum_{i,j=0}^2 \partial_{ij}^2 v \\
= -J_u - \sum_{i,j=0}^2 (d_{ij} v + \sum_{k=0}^2 e_k^{ij} \partial_k v) \partial_{ij}^2 u_a \\
- O(|u|^2 + 2|u_a v| + |\nabla v|^2 + 2|\nabla u_a \cdot \nabla v|) \sum_{i,j=0}^2 \partial_{ij}^2 u_a,
\end{aligned}
\] (20)

We make the induction hypothesis, for some \( T \leq \frac{b^2}{\varepsilon^2} - 1 \),
\[
|Z^\alpha \partial v| \leq \varepsilon (1 + t)^{-\frac{1}{2}} (1 + |t - r|)^{-\frac{1}{2}}, \quad |\alpha| \leq 2, \quad t \leq T,
\] (21)
which further implies for \( |\alpha| \leq 2 \) and \( t \leq T \),
\[
|Z^\alpha v| \leq C\varepsilon (1 + t)^{-\frac{1}{2}} (1 + |t - r|)^{\frac{1}{2}}.
\] (22)
To prove the validity of (21), we will show for sufficiently small \( \varepsilon > 0 \),
\[
|Z^\alpha \partial v| \leq \frac{1}{2} \varepsilon (1 + t)^{-\frac{1}{2}} (1 + |t - r|)^{-\frac{1}{2}}, \quad |\alpha| \leq 2, \quad t \leq T,
\] (23)
and then utilize the continuity method to obtain \( \varepsilon \sqrt{1 + T} = b \).

Applying \( Z^\alpha \) on two hand sides of (20) yields
\[
\partial_t^2 Z^\alpha v - \Delta Z^\alpha v + \sum_{i,j=0}^2 (d_{ij} u + \sum_{k=0}^2 e_k^{ij} \partial_k u) \partial_{ij}^2 Z^\alpha v + O(|u|^2 + |\nabla u|^2) \sum_{i,j=0}^2 \partial_{ij}^2 Z^\alpha v = F, \quad (24)
\]
Next we establish the estimate of \( \| G \| \). Multiplying 
\[
\sum_{i,j} \text{and noting the fact of } |E| = \sum_{i,j} \]

First, we note that it is sufficient to estimate \( (A) \) The treatment on 
\[
\sum_{\alpha_1 + \alpha_2 = \alpha_1 \delta_{\alpha_1} |\alpha_1| \geq 1} Z^{\alpha_1} (d_{ij} u + \sum_{k=0}^2 c_{ij}^k \partial_k u) Z^{\alpha_2} \partial_{ij}^2 v \]

Next we establish the estimate of \( \| \partial Z^\alpha v(t, \cdot) \|_{L^2} \) from the equation (24). Define the energy

\[
E(t) = \frac{1}{2} \sum_{|\alpha| \leq 4} \int_{\mathbb{R}^2} \left( |\partial_t Z^\alpha v|^2 + |\nabla_x Z^\alpha v|^2 - \sum_{i,j=0}^2 (d_{ij} u + \sum_{k=0}^2 c_{ij}^k \partial_k u)(\partial_i Z^\alpha)v(\partial_j Z^\alpha v) \right) dx. 
\]

Multiplying \( \partial_t Z^\alpha v \) \((|\alpha| \leq 4)\) on both sides of (24) and integrating by parts, and noting the fact of \( |\partial^3 u| = |\partial^3 u_a + \partial^3 v| \leq C\varepsilon (1 + t)^{-\frac{1}{2}} \) \((|\beta| = 1, 2)\) from the construction of \( u_a \) and assumption (21), then we arrive at

\[
E'(t) \leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t) + \sum_{|\alpha| \leq 4} \int_{\mathbb{R}^2} |F| |\partial_t Z^\alpha v| dx. 
\]

We now treat each term in the integration \( \sum_{|\alpha| \leq 4} \int_{\mathbb{R}^2} |F| |\partial_t Z^\alpha v| dx. \)

(A) The treatment on \( \sum_{\alpha_1 + \alpha_2 = \beta, |\alpha_1| \geq 1} \int_{\mathbb{R}^2} |Z^{\alpha_1} (d_{ij} u + \sum_{k=0}^2 c_{ij}^k \partial_k u + \sum_{k=0}^2 |u|^2 + \sum_{k=0}^2 |\nabla u|^2) Z^{\alpha_2} \partial_{ij}^2 v| |\partial_t Z^\alpha v| dx \) with \( |\beta| \leq |\alpha| \).

It is sufficient to estimate

\[
\sum_{\alpha_1 + \alpha_2 = \beta, |\alpha_1| \geq 1} \int_{\mathbb{R}^2} |Z^{\alpha_1} (d_{ij} u + \sum_{k=0}^2 c_{ij}^k \partial_k u) Z^{\alpha_2} \partial_{ij}^2 v| |\partial_t Z^\alpha v| dx. 
\]

First, we note that
(i) For $|\delta| \leq 2$, by the assumption (22), we have
\[
|(1 + |t - r|)^{-1}Z^\delta(u_a + v)| \leq \frac{C_b\varepsilon}{\sqrt{1 + t}}.
\] (26)

(ii) For $|\alpha_1| + |\alpha_2| = |\beta| \leq 4$ with $|\alpha_1| \geq 1$, by (26) and (18), we have
\[
\int_{\mathbb{R}^2} |Z^{\alpha_1}(d_{ij} u + e^k_{ij} \partial_k u)(Z^{\alpha_2} \partial_j^2 v)\partial_i Z^\alpha v|dx
\leq C_b \int_{\mathbb{R}^2} (|Z^{\alpha_1} u_a| + |Z^{\alpha_1} \partial_k u_a|) \cdot |(Z^{\alpha_2} \partial_j^2 v)(\partial_t Z^\alpha v)|dx
\]
\[+
C \int_{\mathbb{R}^2} (|Z^{\alpha_1} v| + |Z^{\alpha_1} \partial_k v|) \cdot |(Z^{\alpha_2} \partial_j^2 v)(\partial_t Z^\alpha v)|dx
\]
\[\leq \frac{C_b\varepsilon}{\sqrt{1 + t}} E(t) + C \sum_{|\gamma| \leq |\alpha_2| + 1} \int_{\mathbb{R}^2} (1 + |t - r|)^{-1} |Z^{\alpha_1} v| \cdot |Z^\gamma \partial v| \cdot |\partial_t Z^\alpha v|dx
\]
\[+
C \sum_{|\gamma| \leq |\alpha_2| + 1} \int_{\mathbb{R}^2} |Z^{\alpha_1} \partial_k v| \cdot |Z^\gamma \partial v| \cdot |\partial_t Z^\alpha v|dx.
\] (27)

Note that there is at most one number larger than 2 between $|\alpha_1|$ and $|\gamma|$. If $|\alpha_1| > 2$, then $|\gamma| \leq 2$. Thus by Lemma 2.5 on $(1 + |t - r|)^{-1}Z^{\alpha_1} v$ and assumption (21), we arrive at
\[
\int_{\mathbb{R}^2} (1 + |t - r|)^{-1} |Z^{\alpha_1} v| \cdot |Z^\gamma \partial v| \cdot |\partial_t Z^\alpha v|dx + \int_{\mathbb{R}^2} |Z^{\alpha_1} \partial_k v| \cdot |Z^\gamma \partial v| \cdot |\partial_t Z^\alpha v|dx
\]
\[\leq \frac{C_b\varepsilon}{\sqrt{1 + t}} E(t).
\] (28)

If $|\gamma| > 2$ and then $|\alpha_1| \leq 2$. It follows from (22) that $(1 + |t - r|)^{-1}Z^{\alpha_1} v \leq C\varepsilon(1 + t)^{-\frac{1}{2}}(1 + |t - r|)^{-\frac{1}{2}}$, which leads to
\[
\int_{\mathbb{R}^2} (1 + |t - r|)^{-1} |Z^{\alpha_1} v| \cdot |Z^\gamma \partial v| \cdot |\partial_t Z^\alpha v|dx + \int_{\mathbb{R}^2} |Z^{\alpha_1} \partial_k v| \cdot |Z^\gamma \partial v| \cdot |\partial_t Z^\alpha v|dx
\]
\[\leq \frac{C_b\varepsilon}{\sqrt{1 + t}} E(t).
\] (29)

Substituting (28)-(29) into (27) yields
\[
\sum_{\alpha_1 + \alpha_2 = \beta, |\alpha_1| \geq 1} \int_{\mathbb{R}^2} |Z^{\alpha_1}(d_{ij} u + \sum_{k=0}^{2} e^k_{ij} \partial_k u)Z^{\alpha_2} \partial_j^2 v||\partial_i Z^\alpha v|dx \leq \frac{C_b\varepsilon}{\sqrt{1 + t}} E(t). \quad (30)
\]

(B) The treatment on $\int_{\mathbb{R}^2} |Z^\beta((d_{ij} u + \sum_{k=0}^{2} e^k_{ij} \partial_k u + O(|u|^2 + |\nabla u|^2))\partial_j^2 v) \cdot \partial_i Z^\alpha v|dx$ with $|\beta| < |\alpha|$. 
We only need to treat the term \[ \int_{\mathbb{R}^2} |(d_{ij}u + \sum_{k=0}^2 e_{ij} \partial_k u) Z^\beta \partial_{ij}^2 v \cdot \partial_t Z^\alpha v| \, dx \] since the other terms have been estimated in (A). By (21), we have
\[
\int_{\mathbb{R}^2} |(d_{ij}u + \sum_{k=0}^2 e_{ij} \partial_k u) Z^\beta \partial_{ij}^2 v \cdot \partial_t Z^\alpha v| \, dx \\
\leq C \sum_{|\gamma| \leq |\beta|+1} \int_{\mathbb{R}^2} \left| |1 + |t - r||^{-1} u \cdot |Z^\gamma \partial v| \cdot |\partial_t Z^\alpha v| \right| \, dx \\
+ C \int_{\mathbb{R}^2} |\partial u| \cdot |Z^\beta \partial^2 v| \cdot |\partial_t Z^\alpha v| \, dx \\
\leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t). \tag{31}
\]

(C) The treatment on \( \int_{\mathbb{R}^2} |\partial_t Z^\alpha v| \cdot |Z^\beta J_a| \, dx \) with \( |\beta| \leq |\alpha| \leq 4 \).

For this case, we have
\[
\int_{\mathbb{R}^2} |\partial_t Z^\alpha v| \cdot |Z^\beta J_a| \, dx \leq \| Z^\beta J_a(t, \cdot) \|_{L^2} \sqrt{E(t)}. \tag{32}
\]

(D) The treatment on \( \int_{\mathbb{R}^2} |Z^\beta ((d_{ij} v + \sum_{k=0}^2 e_{ij} \partial_k v + O(|v|^2 + 2|u_a v| + |\nabla v|^2)
+ 2|\nabla u_a \cdot \nabla v|)) \partial_{ij}^2 v_a| \cdot |\partial_t Z^\alpha v| \, dx \) with \( |\beta| \leq |\alpha| \leq 4 \).

A direct computation yields
\[
\int_{\mathbb{R}^2} |Z^\beta ((d_{ij} v + \sum_{k=0}^2 e_{ij} \partial_k v + O(|v|^2 + 2|u_a v| + |\nabla v|^2)
+ 2|\nabla u_a \cdot \nabla v|)) \partial_{ij}^2 v_a| \cdot |\partial_t Z^\alpha v| \, dx \\
\leq C \sum_{|\beta_1| + |\beta_2| \leq |\beta| + 1, |\beta_1| \leq |\beta|} \int_{\mathbb{R}^2} \left| |(1 + |t - r||^{-1} (Z^{\beta_1} v)(Z^{\beta_2} \partial u_a))| \\
+ |(Z^{\beta_1} \partial v)(Z^{\beta_2} \partial u_a))| \right| \partial_t Z^\alpha v \, dx \\
\leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t). \tag{33}
\]

(E) The treatment on \( \int_{\mathbb{R}^2} |(d_{ij} v + \sum_{k=0}^2 e_{ij} \partial_k v + O(|u|^2 + |\nabla u|^2)) [Z^\alpha, \partial_{ij}^2 v] \cdot \partial_t Z^\alpha v| \, dx \).

Since \( [Z^\alpha, \partial_{ij}^2] = \sum_{|\beta| \leq |\alpha|-1} C_{\alpha \beta}^{ij} \partial_\beta^2 Z^\beta \), we have
\[
\int_{\mathbb{R}^2} |(d_{ij} u + \sum_{k=0}^2 e_{ij} \partial_k u + O(|u|^2 + |\nabla u|^2)) [Z^\alpha, \partial_{ij}^2 v] \cdot \partial_t Z^\alpha v| \, dx \\
\leq C \sum_{|\gamma| \leq |\alpha|} \int_{\mathbb{R}^2} |(1 + |t - r||^{-1} |u| + |\partial u|)| \cdot |\partial Z^\gamma v| \cdot |\partial_t Z^\alpha v| \, dx \tag{34}
\]
\[
\leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t)
\]

Substituting (30)-(34) into (25) yields
\[
E'(t) \leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t) + \sum_{|\beta| \leq 4} \| Z^\beta J_a(t, \cdot) \|_{L^2} \sqrt{E(t)}. \tag{35}
\]
Thus, by Lemma 2.2.4 and Gronwall’s inequality we get
\[ \| \partial Z^\alpha v(t, \cdot) \|_{L^2} \leq C \varepsilon^{\alpha/2}, \quad |\alpha| \leq 4, \]
and then
\[ \| Z^\alpha \partial v(t, \cdot) \|_{L^2} \leq C \varepsilon^{\alpha/2}, \quad |\alpha| \leq 4. \quad (36) \]
By (36) and Klainerman-Sobolev’s inequality (see [15] or [21]), we have
\[ |Z^\alpha \partial v| \leq C \varepsilon^{\alpha/2} (1 + t)^{-\frac{\alpha}{2}} (1 + |t - r|)^{-\frac{\alpha}{2}}, \quad |\alpha| \leq 2, \quad t \leq T, \quad (37) \]
which means for small \( \varepsilon > 0, \)
\[ |Z^\alpha \partial v| \leq \frac{1}{2} \varepsilon (1 + t)^{-\frac{\alpha}{2}} (1 + |t - r|)^{-\frac{\alpha}{2}}, \quad |\alpha| \leq 2, \quad t \leq T. \]
Therefore, we complete the proofs of (23) and further (19) together with (37). \( \square \)

Proposition 1 yields \( \lim_{\varepsilon \to 0} \varepsilon (1 + T_x)^{1/2} \geq \tau_0, \) and hence
\[ \lim_{\varepsilon \to 0} \varepsilon \sqrt{T_x} \geq \tau_0. \quad (38) \]

3. Proof of Theorem 1.2. In this section, we will use the coordinates \((t, r, \theta)\) instead of \((t, x)\) to study problem (1), and set
\[ \sigma = r - t, \quad \tau = \varepsilon \sqrt{t}. \]

We write \( u(t, x) = \frac{\varepsilon}{\sqrt{t}} G(\sigma, \theta, \tau) \) for \( r > 0 \) and introduce the following notations:
\[ \bar{\omega} = (0, -\sin \theta, \cos \theta), \quad \bar{\omega} = (0, \cos \theta, \sin \theta), \]
\[ \hat{\omega} = (-1, \cos \theta, \sin \theta), \quad \Xi = \tau^2 + \varepsilon^2 \sigma, \]
\[ a_{ij} = d_{ij} G + \sum_{k=0}^{2} e_{ij}^k \left( \bar{\omega}_k \partial_y G + \varepsilon^2 \left( -\frac{\bar{\omega}_k}{2\Xi} G + \frac{1}{2\Xi} \bar{\omega}_k \partial_y G + \frac{1}{2\tau} \delta_{ij} \partial_y G \right) \right) \]
\[ + O\left( \frac{\varepsilon^2}{\Xi^{1/2}} G^2 \right) + \sum_{k=0}^{2} \frac{\varepsilon^2}{\Xi^{1/2}} \left( \bar{\omega}_k \partial_y G + \bar{\omega}_k \partial_y G + \frac{\varepsilon^2}{2\Xi} \bar{\omega}_k \partial_y G + \frac{\varepsilon^2}{2\tau} \delta_{ij} \partial_y G \right)^2. \]

Then it follows from a direct computation that the equation in (1) has such a form
\[ P(G) = \sum_{i,j=0}^{2} p_{ij}(y, G, \nabla y G) \partial_{ij} G + \varepsilon^2 p_0(y, G, \nabla y G) = 0 \quad (39) \]
with
\[ p_{00} = \sum_{i,j=0}^{2} a_{ij} \bar{\omega}_i \bar{\omega}_j, \quad p_{01} = p_{10} = \varepsilon^2 \sum_{i,j=0}^{2} a_{ij} (\bar{\omega}_i \bar{\omega}_j + \bar{\omega}_j \bar{\omega}_i), \]
\[ p_{02} = p_{20} = \frac{\varepsilon^2}{4\tau} \sum_{i,j=0}^{2} a_{ij} (\bar{\omega}_j \bar{\delta}_0^i + \bar{\omega}_0 \bar{\delta}_i^j) - \frac{\varepsilon^2}{2\tau}, \quad p_{11} = -\frac{\varepsilon^2}{\Xi^{3/2}} + \frac{\varepsilon^4}{\Xi^{2}} \sum_{i,j=0}^{2} a_{ij} \bar{\omega}_i \bar{\omega}_j, \]
\[ p_{12} = p_{21} = \frac{\varepsilon^4}{4\tau^2} \sum_{i,j=0}^{2} a_{ij} (\bar{\omega}_j \bar{\delta}_0^i + \bar{\omega}_0 \bar{\delta}_i^j), \quad p_{22} = \frac{\varepsilon^2}{4\tau^2} + \frac{\varepsilon^4}{4\tau^2} \sum_{i,j=0}^{2} a_{ij} \delta_i^j \delta_0^i, \]
where \( y = (y_0, y_1, y_2) = (\sigma, \theta, \tau), \partial_i = \partial_{y_i}, (0 \leq i \leq 2), \) and \( p_0 \) is a smooth function.

Introduce a transformation \( \Phi \) as follows
\[ \Phi(s, \theta, \tau) = (\sigma = \phi(s, \theta, \tau), \theta, \tau). \quad (40) \]
Set \(w(s, \theta, \tau) = G(\phi(s, \theta, \tau), \theta, \tau)\) and \(v(s, \theta, \tau) = \partial_{s}G(\phi(s, \theta, \tau), \theta, \tau)\). It is obvious that \(\partial_{s}w = v\partial_{s}\phi\). If we can find smooth functions \((\phi, w, v)\) and some point \(m_{\varepsilon} = (s_{\varepsilon}, \theta_{\varepsilon}, \tau_{\varepsilon})\) such that \(\partial_{s}\phi(m_{\varepsilon}) = 0\) and \(\partial_{s}v(m_{\varepsilon}) \neq 0\) hold, then the singularity of the second order derivatives of \(G\) at \(M_{\varepsilon} = (\phi(s_{\varepsilon}, \theta_{\varepsilon}, \tau_{\varepsilon}), \theta_{\varepsilon}, \tau_{\varepsilon})\) is derived by \(\partial^{2}_{s}G = \frac{\partial_{w}}{\partial_{s}w}\). Such a kind of blowup is called the geometric blowup in \([1, 2]\). The following conclusion is established in \([2]\) by a direct computation, which can guide us to construct the blowup system of (1) conveniently.

**Proposition 2.** Set \(\vec{\partial} = (0, \partial_{1}, \partial_{2})\) and \(\vec{\phi} = (\vec{\phi}_{0}, \vec{\phi}_{1}, \vec{\phi}_{2}) = (-1, \partial_{1}\phi, \partial_{2}\phi)\). Then one has

\[
P(G)(\Phi) = \frac{\partial_{s}v}{\partial_{s}\phi}I_{1} + I_{2},
\]

where

\[
I_{1} = \sum p_{ij}(\phi, \theta, \tau, w, \partial w - \hat{\phi}v)\hat{\phi}_{i}\hat{\phi}_{j},
\]

\[
I_{2} = \sum p_{ij}(\phi, \theta, \tau, w, \partial w - \hat{\phi}v)(\partial^{2}_{ij}w - v\partial_{ij}\phi - (\hat{\phi}_{i}\partial_{j}v + \hat{\phi}_{j}\partial_{i}v))
\]

\[
+ \varepsilon^{2}p_{0}(\phi, \theta, \tau, w, \partial w - \hat{\phi}v).
\]

Based on Proposition 2, in order to solve the nonlinear equation \(P(G) = 0\), it suffices to solve the following system on \((\phi, w, v)\)

\[
\begin{cases}
I_{1} = 0, \\
I_{2} = 0, \\
I_{3} = \partial_{s}w - v\partial_{s}v = 0,
\end{cases}
\tag{41}
\]

which is called the blowup system of (1) according to the terminology in \([1, 2]\).

We now give some illustrations on the existence of local solution to (41). According to the analysis in §2, we know that \(P(G) = 0\) can be solved with the corresponding initial data on \(t = \tau_{1}/\varepsilon^{2}\) in the strip

\[
D_{S} = \{(\sigma, \theta, \tau) : -C_{0} \leq \sigma \leq M, \theta_{0} - \delta_{0} \leq \theta \leq \theta_{0} + \delta_{0}, \tau_{1} \leq \tau \leq \tau_{1} + \eta\},
\]

where \(C_{0} > 0\) is a large constant, \(\tau_{1} > 0\) is a fixed constant and \(\eta > 0\) is a small constant satisfying \(\eta < \tau_{0} - \tau_{1}\).

By the expression of \(I_{1} = \sum p_{ij}(\phi, \theta, \tau, G(\phi, \theta, \tau), \nabla G(\phi, \theta, \tau))\hat{\phi}_{i}\hat{\phi}_{j} = 0\), we have

\[
\frac{\partial I_{1}}{\partial (\sigma, \phi, \phi)} = \frac{\varepsilon^{1/2}}{\partial s} + O(\varepsilon^{2}) > 0
\]

for small \(\varepsilon\) and smooth function \(\phi\), and then by implicit function theorem one obtains

\[
\partial_{s}\phi = E(\varepsilon, \theta, \tau, \phi, \partial_{\phi}\phi),
\tag{42}
\]

where \(E\) is a smooth function on its arguments. With the initial data \(\phi(s, \theta, \tau_{1}) = s\), (42) has a unique solution \(\phi\) for a sufficient small \(\eta > 0\). Set

\[
\vec{w} = G(\vec{\phi}, \theta, \tau), \quad \vec{v} = \partial_{\phi}G(\vec{\phi}, \theta, \tau),
\]

Then \((\vec{\phi}, \vec{w}, \vec{v})\) is a local solution to the blowup system (41) since the local existence of \(G\) is known by (38). Moreover, from the uniqueness result on the solution \(u\) of (1) for \(t \in [0, (\tau_{1} + \eta)^{2}/\varepsilon^{2}]\) we know that \(\vec{v}\) and \(\vec{\phi} - s\) are smooth and flat on \(\{s = M\}\).

In order to solve the blowup system (41), as in \([1, 2]\), we will use the Nash-Moser-Hörmander iteration method under the restriction \((H)\). For this end, we will divide this into the following five steps.
3.1. Structure of the linearized blowup system. Denote \((\dot{\phi}, \dot{w}, \dot{v})\) by the corresponding unknown solution to the linearized blowup system of (41). Similar to Theorem 3 in [2], set \(\dot{z} = \dot{w} - v\dot{\phi}\) and note that \(\sum_{i,j,k=0}^2 e_{ij}^k \partial_k u \partial_j^2 u\) does not satisfy the null condition, then it follows from a direct computation that the linearized system of (41) can be changed into such forms

\[
\begin{align*}
\mathcal{L}_1(\dot{\phi}, \dot{z}) &= Z_1 \partial_s \dot{z} - \varepsilon^2 (\partial_s \phi) Q \dot{z} + a_0 \partial_s \dot{z} + \varepsilon^2 a_1 \partial_s \dot{z} + 2 \dot{z} + b_1 Z_1 \dot{\phi} + b_2 \dot{\phi} = \dot{f}_1, \\
\mathcal{L}_2(\dot{\phi}, \dot{z}) &= Z_1^2 \dot{\phi} + a_3 Z_1 \dot{\phi} + a_4 \dot{\phi} + \varepsilon^2 c_0 Q \dot{z} + \varepsilon^2 Z_1 (a_5 \partial_t + a_6 \partial \theta) \dot{z} \\
&+ \varepsilon^2 a_7 \partial \dot{z} + a_8 Z_1 \dot{\dot{z}} + \varepsilon^2 a_9 \dot{\dot{z}} = \dot{f}_2,
\end{align*}
\]

where

\[
\begin{align*}
Z_1 &= \sum_{i,j} p_{ij} (\dot{\phi} \partial_j \dot{\phi} + \dot{\phi} \partial_j \dot{\phi}) = (1 + O(\varepsilon))(\partial_t + O(\varepsilon^2) \partial \theta), \\
Q &= \varepsilon^{-2} \sum_{i,j} p_{ij} \partial^2_{ij} = (-1 + O(\varepsilon^2)) \partial^2_t + O(\varepsilon^2) \partial^2 \theta + \left( \frac{1}{4\tau} + O(\varepsilon) \right) \partial^2 \theta, \\
b_1 &= \partial_s \psi + \sum_{i,j,k} e_{ij}^k \partial_i \omega_j \partial_k v + \sum_{i,j} d_{ij} \partial_i \omega_j v \partial_s \phi + O(\varepsilon^{1/2}), \\
b_2 &= Z_1 \partial_s v + b_1 \sum_{i,j} d_{ij} \partial_i \omega_j v + O(\varepsilon^{1/2}), \\
c_0 &= \sum_{i,j,k} e_{ij}^k \partial_i \omega_j \partial_k v + O(\varepsilon^2),
\end{align*}
\]

and \(a_i (i = 0, 1, \ldots, 9)\) are smooth functions.

On the other hand, in terms of Proposition II.2 in [2], \(\dot{v}\) can be determined by the first equation \(I_1(\dot{\phi}, \dot{w}, \dot{v}) = \dot{f}_3\) in the linearized blowup system of (41).

For the convenience to obtain the weighted energy estimate on (43)-(44) later on, as in [2] we choose a “nearly horizontal” surface \(\Sigma\) through \(\{\tau = \tau_1, s = M\}\) instead of the initial plane \(\{\tau = \tau_1, s = 1\}\), \(\Sigma\) being a characteristic surface of the operator \(Z_1 \partial_s - \varepsilon^2 \partial_s \dot{\phi} Q\) whose corresponding coefficients are computed on \((\dot{\phi}, \dot{w}, \dot{v})\). Note that if we define the characteristic surface \(\Sigma\) by \(\tau = \psi(s, \theta) + \tau_1\), then \(\psi\) satisfies

\[
\left\{ \begin{array}{l}
(1 + O(\varepsilon^2) \partial \psi) \partial_s \psi + \varepsilon^2 \partial_s \dot{\phi} \left( \frac{1}{4\tau} + O(\varepsilon) - O(\varepsilon^2) \partial \psi - (1 + O(\varepsilon^2))(\partial \psi)^2 \right) = 0, \\
\psi(M, \theta) = 0.
\end{array} \right.
\]

(45)

For small \(\varepsilon > 0\) it is easy to know that (45) has a smooth solution \(\psi(s, \theta)\) in the domain \(D_S\). Choosing a truncation function \(\chi \in C^\infty(\mathbb{R})\) with \(\chi(t) = 1\) for \(t \leq \frac{1}{2}\), and \(\chi(t) = 0\) for \(t \geq 1\), and making the following transformation

\[
X = s, \quad Y = \theta, \quad T = \tau - \tau_1 - \psi(s, \theta) \chi(t - \tau_1),
\]

(46)

then we will work later on in the following domain

\[
D_1 = \{(X, Y, T) : C_0 \leq X \leq M, \theta_0 - \delta_0 \leq Y \leq \theta_0 + \delta_0, 0 \leq T \leq \tau_\varepsilon - \tau_1 \},
\]

which is actually an unknown domain since we do not know what the precise \(\tau_\varepsilon\) is so far. By (46), the characteristic surface \(\Sigma\) becomes \(\{T = 0\}\).

3.2. The construction of an approximate solution to (41). As the first step to use Nash-Moser-Hörmander iteration method, it is required to construct an approximate solution \((\phi_a, w_a, v_a)\) of (41) such that \(\phi_a\) satisfies (H) at some point.
For $\varepsilon = 0$, the blowup system (41) becomes

\[
\begin{aligned}
\partial_t \phi + \sum_{i,j} \left( d_{ij} w + \sum_k e_{ij}^k \dot{w}_k v \right) \dot{w}_j = 0, \\
- \partial_t v = 0, \\
\partial_x w - v \partial_x \phi = 0
\end{aligned}
\]

with the initial-boundary value conditions

\[
\phi(X, Y, 0) = X, \quad \phi(M, Y, T) = M, \\
v(X, Y, 0) = \partial_x F_0(s(X, Y, \tau_1), Y), \quad v(M, Y, T) = 0,
\]

where $s(X, Y, \tau_1)$ is determined by $X = M + \int_M^s \left\{ \exp(-F_1(\rho, Y)\tau_1)(1 + \frac{F_2(\rho, Y)}{F_1(\rho, Y)}) - \frac{F_2(\rho, Y)}{F_1(\rho, Y)} \right\} d\rho$.

Hence, an exact solution of (47) is

\[
\begin{aligned}
\bar{\phi}_0 &= M + \int_M^s \left\{ \exp(-(T + \tau_1)F_1(\rho, Y)) \left( 1 + \frac{F_2(\rho, Y)}{F_1(\rho, Y)} \right) - \frac{F_2(\rho, Y)}{F_1(\rho, Y)} \right\} d\rho, \\
\bar{v}_0 &= \partial_x F_0(s(X, Y, \tau_1), Y), \\
\bar{w}_0 &= \int_M^s \partial_x F_0(\rho, Y) \left\{ \exp(-(T + \tau_1)F_1(\rho, Y)) \right\} d\rho.
\end{aligned}
\]

Note that (41) has a local solution $(\bar{\phi}, \bar{w}, \bar{v})$ for $0 \leq T \leq \eta$ whose existence has been known in the previous statements. We now glue together $(\bar{\phi}, \bar{w}, \bar{v})$ with $(\phi_0, \bar{w}_0, \bar{v}_0)$ to yield an approximate solution of (41)

\[
\begin{aligned}
\phi_a(X, Y, T) &= \chi\left( \frac{T}{\eta} \right) \bar{\phi}(X, Y, T) + (1 - \chi\left( \frac{T}{\eta} \right)) \phi_0(X, Y, T), \\
v_a(X, Y, T) &= \chi\left( \frac{T}{\eta} \right) \bar{v}(X, Y, T) + (1 - \chi\left( \frac{T}{\eta} \right)) \bar{v}_0(X, Y, T), \\
w_a(X, Y, T) &= \chi\left( \frac{T}{\eta} \right) \bar{w}(X, Y, T) + (1 - \chi\left( \frac{T}{\eta} \right)) \bar{w}_0(X, Y, T).
\end{aligned}
\]

Substituting the approximate solution $(\phi_a, v_a, w_a)$ into the blowup system (41) yields $I_i = f_i^a$ $(i = 1, 2, 3)$, where $f_i^a$ is smooth, flat on $\{X = M\}$ and zero near $\{T = 0\}$. In addition, under the assumption (ND), we can verify that $\phi_a$ satisfies (H) at the point $(\bar{\sigma}, \bar{\theta}_0, \bar{\tau}_0 - \tau_1)$, which can be shown in next step, here $\bar{\sigma} = M + \int_M^\sigma \left\{ \exp(-F_1(\rho, \theta_0)\tau_1)(1 + \frac{F_2(\rho, \theta_0)}{F_1(\rho, \theta_0)}) - \frac{F_2(\rho, \theta_0)}{F_1(\rho, \theta_0)} \right\} d\rho$.

3.3. The condition (H) for the approximate solution $\phi_a$. We now prove that the approximate solution $\phi_a$ given in (48) satisfies the condition (H) at the point $(\bar{\sigma}, \bar{\theta}_0, \bar{\tau}_0 - \tau_1)$.

Indeed, it follows from a direct computation that

\[
\partial_x \bar{v}_0(X, Y, T) = \left( \frac{\exp(-(T + \tau_1)F_1)}{\exp(-\tau_1 F_1)}(1 + \frac{F_2}{F_1}) - \frac{F_2}{F_1} \right)(\bar{s}(X, \tau_1, Y), Y).
\]
Note that \((\sigma_0, \theta_0)\) is the interior minimum point of \(G_0(\sigma, \theta)\), then we have \(\nabla_{\sigma, \theta} G_0(\sigma_0, \theta_0) = 0\). Thus, by the expression of \(q(\theta, \tau, s)\) in Lemma 2.3,
\[
\nabla_{s, \theta} \partial_s q(\theta_0, \tau_0, \sigma_0) = F_2(\sigma_0, \theta_0) \nabla_{\sigma, \theta} G_0(\sigma_0, \theta_0) = 0
\]
holds. This, together with \(\partial_s q(\theta_0, \tau_0, \sigma_0) = 0\), yields
\[
\nabla_{X,Y} (\partial_X \tilde{\phi}_0)(\bar{\sigma}, \theta_0, \tau_0 - \tau_1) = 0.
\]
In addition, a careful calculation derives that
\[
\nabla_{s, \theta}^2 \partial_s q(\theta_0, \tau_0, \sigma_0) = F_2(\sigma_0, \theta_0) \nabla_{\sigma, \theta}^2 G_0(\sigma_0, \theta_0)
\]
is a positive definite matrix. On the other hand, at the point \(M_0 = (\bar{\sigma}, \theta_0, \tau_0 - \tau_1)\), it follows from direct computation that \(\nabla_{X,Y}^2 \partial_X \tilde{\phi}_0(M_0)\) is a positive definite matrix.

Next we verify \((H)\) condition for the function \(\phi_a\).

i) Due to \(\partial_X \tilde{\phi}(X, Y, 0) = 1\), without loss of generality, we assume \(\partial_X \tilde{\phi}(X, Y, T) > 0\) for \(T \leq \eta\). In addition, it follows from the expression of \(\partial_X \tilde{\phi}_0(X, Y, T)\) that \(\partial_X \tilde{\phi}_0(X, Y, T) \geq 0\) and \(\partial_X \tilde{\phi}_0(X, Y, T) = 0 \iff (X, Y, T) = (\bar{\sigma}, \theta_0, \tau_0 - \tau_1)\). Therefore, \(\partial_X \tilde{\phi}_0(X, Y, T) \geq 0\) holds, moreover, \(\partial_X \phi_a(X, Y, T) = 0\) if and only if \(T \geq \eta\) and \(\partial_X \tilde{\phi}_0(X, Y, T) = 0\), which means
\[
\partial_X \tilde{\phi}_0(X, Y, T) = 0 \iff (X, Y, T) = (\bar{\sigma}, \theta_0, \tau_0 - \tau_1).
\]

ii) Due to \(\eta < \tau_0 - \tau_1\), then \(\phi_a(X, Y, T) = \tilde{\phi}_0(X, Y, T)\) holds in the neighborhood of \((\bar{\sigma}, \theta_0, \tau_0 - \tau_1)\). Thus,
\[
\partial_T \partial_X \phi_a(\sigma, \theta_0, \tau_0 - \tau_1) = \partial_T \partial_X \tilde{\phi}_0(\sigma, \theta_0, \tau_0 - \tau_1) < 0,
\]
\[
\nabla_{X,Y}^2 \phi_a(\bar{\sigma}, \theta_0, \tau_0 - \tau_1) = \nabla_{X,Y}^2 \tilde{\phi}_0(\bar{\sigma}, \theta_0, \tau_0 - \tau_1) \text{ is positive definite.}
\]

3.4. Reduction to a Goursat problem on a fixed domain. To be free to adjust the height of the domain \(D_1\), we perform a change of variables depending on a nonnegative parameter \(\lambda\) close to zero as follows
\[
X = x, \quad Y = y, \quad T = T(\rho, \lambda) = (\tau_0 - \tau_1)(\rho + \lambda \rho(1 - \chi_1(\rho))), \quad (49)
\]
where \(\chi_1\) is 1 near 0 and 0 near 1. From now on we will work on a fixed subdomain of \(D_1\)
\[
D_2 = \{(x, y, \rho) : -C_0 \leq x \leq M, \theta_0 - \delta_0 \leq y \leq \theta_0 + \delta_0, 0 \leq \rho \leq 1\}.
\]
At this time, for \(\lambda = \lambda_0 = 0\), the approximate solution of \((41)\) is
\[
\begin{cases}
\phi_0(x, y, \rho) = \phi_a(x, y, (\tau_0 - \tau_1)\rho), \\
v_0(x, y, \rho) = v_a(x, y, (\tau_0 - \tau_1)\rho), \\
w_0(x, y, \rho) = w_a(x, y, (\tau_0 - \tau_1)\rho),
\end{cases}
\]
moreover, \(\phi_0\) satisfies \((H)\) in \(D_2\) at the point \((\bar{\sigma}, \theta_0, 1)\).

On the characteristic surfaces \(\{x = M\}\) and \(\{\rho = 0\}\) of equation \((41)\), we naturally pose the following boundary value conditions:
\[
\phi \text{ and } \phi - \phi_0 \text{ are flat on } \{x = M\} \text{ and } \{\rho = 0\} \text{ respectively.} \quad (51)
\]

We now turn to the linearized equations \((43)\) and \((44)\). Under the changes of variables \((46)\) and \((49)\), it follows from a direct computation that \((43)\) and \((44)\) have
Lemma 3.1. Then we can obtain the following energy estimate.

\[ ZS\ddot{z} - \varepsilon^2(S\phi)N\ddot{z} + \alpha_1 S\dot{z} + \varepsilon^2l_1(\nabla\dot{z}) + \alpha_2 \dot{z} + \beta_1 Z\dot{\phi} + \beta_2 \dot{\phi} = \tilde{F}_1, \]

\[ Z^2\dot{\phi} + \alpha_3 Z\dot{\phi} + \varepsilon^2\gamma_0 N\ddot{z} + \varepsilon^2 Z(\alpha_5 Z + \alpha_6 \partial_y)\dot{z} + \varepsilon^2l_2(\nabla\dot{z}) + \alpha_7 Z\ddot{z} + \alpha_8 \dot{z} = \tilde{F}_2, \]  

(52) \quad (53)

where

\[ Z = \partial_{\rho} + \varepsilon^2 z_0 \partial_y, \quad S = \partial_x = \partial_{\rho} + \varepsilon^2 \partial_{\rho}, \quad N = N_1 Z^2 + 2\varepsilon^2 N_2 Z \partial_y + N_3 \partial_y^2; \]

\[ \beta_1 = b_1, \beta_2 = (\partial_{\rho} T)b_2 + O(\varepsilon), \quad \gamma_0 = (\partial_{\rho} T)c_0 + O(\varepsilon), \]

\[ N_1 = \frac{1}{4\varepsilon^2 \partial_{\rho} T} + O(\varepsilon) > 0 \quad \text{and} \quad N_3 = -\partial_{\rho} T + O(\varepsilon) < 0. \]

Here we specially point out that although (52) and (53) are somewhat similar to the linearized equations (3.1.1a) and (3.1.1b) in [2], however, the coefficients \( \alpha_1 \) and \( \alpha_2 \) in (52) are not bounded quantities other than the ones of \( O(\varepsilon^2) \) in (3.1.1a) of [2], in addition, there are more terms in (53) than those in (3.1.1a) of [2] due to the simultaneous appearances of the solution \( u \) and its first order derivatives \( \nabla u \) in the coefficients of \( \{1, 2 \} \). Due to the differences among (52)-(53) and (3.1.1a) - (3.1.1b) in [2], we will derive the energy estimates of solutions to (52)-(53) directly by choosing suitable multipliers and integrating by parts other than change the main part of (52) into a third order scalar equation to derive the related estimates by introducing a new unknown function \( \tilde{k} \) with \( \dot{z} = Z\tilde{k} \) as in [1, 2].

In the process of solving (52)-(53), we require to choose a subdomain \( D_3 \) of \( D_2 \) which is a domain of the first order differential operator \( Z \) so that the point \((\tilde{\sigma}, \tilde{\theta}_0, 1)\) and is bounded by the planes \( \{x = -C_0\}, \{x = M\}, \{y = 0\}, \{\rho = 0\}, \{\rho = 1\}, S_+ \) and \( S_- \). Here, \( S_+ \) and \( S_- \) do not intersect in \( D_2 \), and their normal directions are \( (-\eta, \nu, 1) \) and \( (-\eta, -\nu, 1) \), \( \nu \) is some appropriate positive constant. In addition, \( \tilde{\sigma} \) is assumed to be a smooth function \( \phi \) in \( D_3 \) and a constant \( \Lambda \) in (49) close to \( \phi_0 \) and \( \lambda_0 \) respectively, where the function \( \phi \) also satisfies (H) for some point \((\tilde{x}_0, \tilde{y}_0, 1)\) (this can be achieved by the implicit function theorem established in [1] in terms of the properties of \( \phi_0 \) satisfying (H) at the point \((\tilde{\sigma}, \tilde{\theta}_0, 1)\)).

3.5. The tame estimate and solvability of (52)-(53). Note that \( \rho = 0 \) is a characteristic surface of the operator \( ZS - \varepsilon^2(S\phi)N \), then

\[ s_0 - (S\phi)N_1 = 0. \]

(54)

As in [1], set

\[ A = S\phi, \quad \delta = 1 - t, \quad g = \exp h(x - t), \quad p = \delta \sqrt{g}, \quad |\cdot|_0 = \|\cdot\|_{L^2(D_3)}, \]

then we can obtain the following energy estimate.

**Lemma 3.1.** There exist \( C > 0, \varepsilon_0 > 0, \eta_0 > 0 \) and \( h_0 > 0 \) such that for \( \phi \) satisfying (H), (54) and

\[ |\phi - \phi_0|_{C^1(D_3)} + |w - w_0|_{C^1(D_3)} + |v - v_0|_{C^1(D_3)} \leq \eta_0, \]

(55)
where the functions $\dot{\phi}$ and $\dot{z}$ are smooth and flat on $\{t = 0\}$ and $\{x = M\}$.

**Proof.** Set $P = ZS - \varepsilon^2 AN$ and choose the multiplier $M\dot{z} = aS\dot{z} + dZ\dot{z}$ as in [1], where the functions $a$ and $d$ will be determined later on. Then through integrations by parts we have

$$
\int_{D_3} (P\dot{z})(M\dot{z}) dx dy d\rho
= \int_{D_3} K_1(S\dot{z})^2 + \int_{D_3} K_2(\partial_y \dot{z})^2 + \int_{D_3} K_3(Z\dot{z})^2 + \int_{D_3} K_4(S\dot{z})(\partial_y \dot{z})
+ \int_{D_3} K_5(S\dot{z})(Z\dot{z}) + \int_{D_3} K_6(\partial_y \dot{z})(Z\dot{z}) + I_1 - I_0 - J_1 + L_+ + L_-
$$

with

$$K_1 = -\frac{1}{2}(Az) - \frac{1}{2}\varepsilon^2(\partial_y z_0)a,$$

$$K_2 = -\frac{1}{2}\varepsilon^2 S(Aa N_3) - \frac{1}{2}\varepsilon^2 Z(Ad N_3) - \varepsilon^8 AaN_2(\partial_\rho s_0)z_0 - \varepsilon^6 AaN_2(\partial_x z_0) - \varepsilon^{10} z_0^2(\partial_y s_0)AaN_2 - \varepsilon^8 s_0(\partial_\rho z_0)AaN_2 - \frac{1}{2}\varepsilon^4(\partial_\rho s_0)AaN_3$$

$$- \varepsilon^6 AaN_3(\partial_y s_0)z_0 - \frac{1}{2}\varepsilon^4(\partial_y z_0)Ad N_3 + \varepsilon^4 Ad N_3(\partial_y z_0),$$

$$K_3 = -\frac{1}{2}(Sd) - \frac{1}{2}\varepsilon^2(\partial_y s_0)d + \varepsilon d(\partial_\rho s_0) + \varepsilon z_0(\partial_\rho s_0)d - \frac{1}{2}\varepsilon^4(\partial_\rho s_0)AaN_1$$

$$- \frac{1}{2}\varepsilon^2 S(Aa N_1) + \varepsilon^4 AaN_1(\partial_\rho s_0) + \varepsilon^6 AaN_2(\partial_\rho s_0) + \frac{1}{2}\varepsilon^4(\partial_\rho z_0)Ad N_1$$

$$+ \frac{1}{2}\varepsilon^2 Z(Ad N_1) + \varepsilon^4 \partial_\rho (Ad N_2),$$

$$K_4 = \varepsilon^6(\partial_\rho z_0)AaN_2 + \varepsilon^4 Z(Ad N_2) + \varepsilon^6 AaN_2(\partial_\rho z_0) + \varepsilon^2 \partial_\rho (AaN_3),$$

$$K_5 = \varepsilon^4 AaN_1(\partial_\rho z_0) + \varepsilon^2 Z(Aa N_1) + \varepsilon^4 \partial_\rho (AaN_2),$$

$$K_6 = -\varepsilon^4 d(\partial_\rho s_0)z_0 - \varepsilon^2 d(\partial_x z_0) - \varepsilon^6 z_0^2(\partial_y s_0)d - \varepsilon^4 ds_0(\partial_\rho z_0) - \varepsilon^6 AaN_1(\partial_\rho s_0)z_0$$

$$- \varepsilon^6(\partial_\rho s_0)AaN_2 - \varepsilon^4 S(Aa N_2) - \varepsilon^8 AaN_2(\partial_\rho s_0)z_0 + \varepsilon^6 AaN_2(\partial_\rho s_0)$$

$$+ \varepsilon^8 z_0(\partial_\rho s_0)AaN_2 + \varepsilon^6 AaN_3(\partial_\rho s_0) + 2\varepsilon^6 Ad N_2(\partial_\rho z_0) + \varepsilon^2 \partial_\rho (Ad N_3)$$

and

$$I_1 = \int_{\{t = 1\}} \left\{ \frac{1}{2} a(S\dot{z})^2 + \frac{1}{2}\varepsilon^2 s_0(\partial_x z_0) - \varepsilon^2 AaN_1(S\dot{z})(S\dot{z}) - \frac{1}{2}\varepsilon^4 s_0 AaN_1(S\dot{z})^2$$

$$- \varepsilon^4 AaN_2(\partial_\rho \dot{z})(S\dot{z}) + \varepsilon^6 s_0 AaN_2(\partial_\rho \dot{z})(Z\dot{z}) + \frac{1}{2}\varepsilon^4 s_0 AaN_3(\partial_\rho \dot{z})^2 \right\} \frac{\delta g}{A} (1 + \delta h)(S\dot{z})^2 + \varepsilon^2 \int_{D_3} \delta g(A + \delta h)(\partial_\rho \dot{z})^2 + h|pZ\dot{z}|^2 + h|p\dot{\phi}|^2 + h|p\dot{\phi}|^2$$

$$\leq C|p\dot{F}|^2 + C \int_{D_3} \delta g|\dot{F}|^2,$$
\[-\frac{1}{2}\varepsilon^2 \text{Ad} N_1(\dot{Z}^2) + \frac{1}{2}\varepsilon^2 \text{Ad} N_3(\partial_\theta \dot{Z})^2 \right\} dS,\]

\[I_0 = 0,\]

\[J_1 = \int_{\{x=-c_0\}} \left\{ \frac{1}{2} d(\dot{Z}^2) + \frac{1}{2}\varepsilon^2 \text{Ad} N_1(\dot{Z}^2) + \varepsilon^4 \text{Ad} N_2(\partial_\theta \dot{Z})(\dot{Z}) \right.\]

\[\left. + \frac{1}{2}\varepsilon^2 \text{Ad} N_3(\partial_\theta \dot{Z})^2 \right\} dS,\]

\[L_\pm = \int_{S_\pm} \left\{ \frac{1}{2} a(\dot{S}^2) + \frac{1}{2}\varepsilon^2 \nu a_0(\dot{S}^2) - \frac{1}{2}\eta d(\dot{Z}^2) + \frac{1}{2}\varepsilon^2 s_0 d(\dot{Z}^2) \right.\]

\[\left. - \varepsilon^2 \text{Ad} N_1(\dot{Z})(\dot{S}) + \varepsilon^4 \nu a_0(\dot{S})(\dot{Z}) - \frac{1}{2}\varepsilon^2 \eta \text{Ad} N_3(\dot{Z})^2 + \frac{1}{2}\varepsilon^2 s_0 \text{Ad} N_3(\dot{Z})^2 \right\} dS.\]

Choosing \( a = A^{-1} \delta^2 g \) and \( d = -\delta^2 g \), and using (H) condition of \( \dot{\phi} \), (54) and the geometric property of \( D_3 \) as in the proof of Proposition 3.3 in [2], we can obtain that by (57) and a careful computation

\[I_1 = \int_{\{t=1\}} \frac{1}{2} a(\dot{S}^2) dS \geq 0, \quad J_1 \leq 0, \quad L_\pm \geq 0\]

and

\[\int_{D_3} \frac{1}{A} g(1 + \delta h)(\dot{S}^2) + \varepsilon^2 \int_{D_3} g(A + \delta h)(\partial_\theta \dot{Z})^2 + h|pZ\dot{Z}|^2 \leq C|p|p|Z|\right|_0^2. \quad (58)\]

Note that for any \( \theta > 0 \), there exists \( C > 0 \) such that for all \( h \geq 1 \) and smooth \( \psi \) satisfying \( \psi|_{t=0} = 0 \), the following inequality holds

\[\int_{D_3} (\delta^{\theta-1} + h\delta^\theta) g \psi^2 \leq C \int_{D_3} \delta^{\theta+1} g(Z\psi)^2. \quad (59)\]

In addition,

\[P\dot{Z} = \dot{F}_1 - \alpha_1 S\dot{Z} - \varepsilon^2 l_1(\nabla \dot{Z}) - \alpha_2 \dot{Z} - \beta_1 Z\dot{\phi} - \beta_2 \dot{\phi}. \quad (60)\]

Substituting (60) into the right hand side of (58) and using (59) for \( \theta = 2 \) and the function \( \dot{Z} \) instead of \( \psi \), we have for sufficiently large \( h \)

\[\int_{D_3} \frac{1}{A} g(1 + \delta h)(\dot{S}^2) + \varepsilon^2 \int_{D_3} g(A + \delta h)(\partial_\theta \dot{Z})^2 + h|pZ\dot{Z}|^2 \leq C|p(\dot{F}_1 - \beta_1 Z\dot{\phi} - \beta_2 \dot{\phi})|\right|_0^2. \quad (61)\]

here we point out that the “largeness” of \( \alpha_1 \) and \( \alpha_2 \) does not play essential role since the parameter \( h > 0 \) can be chosen larger.
Next we estimate $\beta_1$ and $\beta_2$ in (52) or (61). If we substitute $$(\tilde{\phi}_0, \tilde{w}_0, \tilde{v}_0)$$ into the expressions of $\beta_1$ and $\beta_2$, then a direct computation yields

$$\beta_1(\tilde{\phi}_0, \tilde{w}_0, \tilde{v}_0) = O(\varepsilon^{1/2}), \quad \beta_2(\tilde{\phi}_0, \tilde{w}_0, \tilde{v}_0) = O(\varepsilon^{1/2}).$$

On the other hand, by the estimate (19) and under the coordinate system $(X, Y, T)$ in (46), we have

$$\partial_T \tilde{\phi} + \sum_{i,j} \left( d_{ij} \tilde{w} + \sum_k \varepsilon_{ij} \tilde{\omega}_k \tilde{v} \right) \tilde{\omega}_j + O(\varepsilon) = 0,$$

$$\partial_T \tilde{v} + O(\varepsilon) = 0,$$

$$\partial_X \tilde{w} - \tilde{v} \partial_X \tilde{\phi} = 0,$$

$$\tilde{\phi}(X, Y, 0) = \tilde{\phi}_0(X, Y, 0),$$

$$\tilde{v}(X, Y, 0) = \tilde{v}_0(X, Y, 0) + O(\varepsilon^{1/2}),$$

$$\tilde{\phi}(M, Y, T) = \tilde{\phi}_0(M, Y, T) = M,$$

$$\tilde{v}(M, Y, T) = \tilde{v}_0(M, Y, T) = 0,$$

(62)

subsequently it follows from the expressions of $(\phi_0, w_0, v_0)$, $\beta_1, \beta_2, (62)$ and a direct computation that

$$\beta_1(\phi_0, w_0, v_0) = O(\varepsilon^{1/2}), \quad \beta_2(\phi_0, w_0, v_0) = O(\varepsilon^{1/2}).$$

(63) together with (55) yields that $\beta_1(\phi, w, v)$ and $\beta_2(\phi, w, v)$ are also small when $\eta_0 > 0$ is small. By (52), we have

$$\varepsilon^2 N \dot{z} = \frac{1}{A}(Z S \dot{z} + \alpha_1 S \dot{z} + \varepsilon^2 \dot{l}_1(\nabla \dot{z}) + \alpha_2 \dot{z} + \beta_1 Z \dot{\phi} + \beta_2 \dot{\phi} - \dot{F}_1).$$

(64)

Substituting (64) into (53) and using (59) for $\vartheta = 3$ yield

$$|p(Z \dot{\phi} + \alpha_3 \dot{\phi} + \varepsilon^2 \alpha_5 Z \dot{z} + \varepsilon^2 \alpha_6 \partial_y \dot{z} + \alpha_7 \dot{z} + \gamma_0 A^{-1} S \dot{z})|^2 \leq C \int \delta^2 g \{ |\dot{F}_2|^2 + A^{-2} |\dot{F}_1|^2 + A^{-2} |\dot{\phi}|^2 + A^{-4} |S \dot{z}|^2 + \varepsilon^4 A^{-2} |\nabla \dot{z}|^2 + A^{-2} |\dot{z}|^2 + \beta_1^2 A^{-2} |Z \dot{\phi}|^2 \}.\quad (65)$$

Note $\delta \leq CA$, then it follows from (65) that

$$|pZ \dot{\phi}|^2 \leq C \int \delta^4 g |\dot{F}_2|^2$$

$$+ C \int \delta^2 g \{ |\dot{F}_1|^2 + |\dot{\phi}|^2 + A^{-2} |S \dot{z}|^2 + \varepsilon^4 |\nabla \dot{z}|^2 + |\dot{z}|^2 + \beta_1^2 |Z \dot{\phi}|^2 \}.\quad (66)$$

Then making use of the smallness property of $\beta_1$ and the inequality $h|p \dot{\phi}|^2 \leq C |Z \dot{\phi}|^2$, we have from (66) that

$$|pZ \dot{\phi}|^2 + h|p \dot{\phi}|^2 \leq C \int \delta^4 g |\dot{F}_2|^2$$

$$+ C \int \delta^2 g \{ |\dot{F}_1|^2 + A^{-2} |S \dot{z}|^2 + \varepsilon^4 |\nabla \dot{z}|^2 \} + Ch^{-1} |pZ \dot{z}|^2.\quad (67)$$

Combining (67) with (61), we can complete the proof of (56).

Next we establish the higher order tame estimates for (52)-(53) as in [8].
Lemma 3.2. There exist \( \varepsilon_0 > 0, \eta_0 > 0 \) and an integer \( n_0 \) such that for smooth functions \( (\phi, w, v) \) satisfying (H), (54) and
\[
|\phi - \phi_0|_{C^4(D_3)} + |w - w_0|_{C^4(D_3)} + |v - v_0|_{C^4(D_3)} \leq \eta_0,
\]
for all \( 0 \leq \varepsilon \leq \varepsilon_0 \) and all integer \( s \), we have
\[
|\dot{\phi}|_{H^s(D_3)} + |\dot{z}|_{H^s(D_3)} \leq C_s((|\phi, w, v|_{H^{s+n_0}(D_3)})(|\dot{F}_1|_{H^s(D_3)} + |\dot{F}_2|_{H^s(D_3)}), \quad (68)
\]
where \( \dot{\phi} \) and \( \dot{z} \) are the solutions of (52)-(53), which are smooth and flat on \( \{t = 0\} \) and \( \{x = M\} \).

Proof. Denote by \( T = \{Z, S, \partial_y\} \). For \( k \in \mathbb{N} \) and \( T \in T \), we have by a direct computation that
\[
[T^l, P] = \varepsilon^2 \sum_{a+b+c=l+1, a \geq 1, a+b+c \leq l} C_{abc} Z^a S^b \partial_y^c
\]
and
\[
[T^l, Z^2] = \varepsilon^2 \sum_{a+b+c=l+1, a \geq 2, a+b+c \leq l} D_{abc} Z^a S^b \partial_y^c,
\]
where \( C_{abc} \) and \( D_{abc} \) are smooth functions of \( (\phi, w, v) \).

Taking the derivative up to order \( l \) on two sides of (52)-(53), we arrive at
\[
\begin{align*}
(P + \alpha_1 S + \varepsilon^2 l_1 \nabla + \alpha_2) T^{l+1} \dot{\phi} + \beta_1 T^l \dot{\phi} + \beta_2 T^l \dot{\phi} &+ \varepsilon^2 \sum_{a+b+c=l, a \geq 1, a+b+c \leq l} \bar{C}_{abc} Z^a S^b \partial_y^c \dot{\phi} \\
+ \sum_{a+b+c=l} \bar{C}_{abc} Z^a S^b \partial_y^c \dot{\phi} + \sum_{a+b+c \leq l} C_{abc} Z^a S^b \partial_y^c \dot{\phi} &+ T^l \dot{F}_1, \\
Z^2 T^l \phi + \alpha_3 Z T^l \phi + \alpha_4 T^l \phi + \varepsilon^2 \gamma_0 N T^l \dot{\phi} + \varepsilon^2 Z(\alpha_5 Z + \alpha_6 \partial_y) T^l \dot{\phi} + \varepsilon^2 l_2 (\nabla T^l \dot{\phi}) &+ \sum_{a+b+c \leq l} D_{abc} Z^a S^b \partial_y^c \dot{\phi} \\
+ \alpha_7 Z T^l \dot{\phi} + \alpha_8 T^l \dot{\phi} + \varepsilon^2 \sum_{a+b+c = l+1} \bar{D}_{abc} Z^a S^b \partial_y^c \dot{\phi} + \sum_{a+b+c = l} D_{abc} Z^a S^b \partial_y^c \dot{\phi} &+ \varepsilon^2 \sum_{a+b+c = l+1} \bar{D}_{abc} Z^a S^b \partial_y^c \dot{\phi} + \sum_{a+b+c = l} D_{abc} Z^a S^b \partial_y^c \dot{\phi} = T^l \dot{F}_2,
\end{align*}
\]
where \( \bar{C}_{abc} \) and \( \bar{D}_{abc} \) are smooth functions of \( (\phi, w, v) \).

Applying Lemma 3.1 for \( (T^l \dot{\phi}, T^l \dot{\phi}) \) together with a direct computation yields
\[
\sum_{T \in T} \left( \int_{D_3} \frac{1}{A} \delta g(1 + \delta h)(ST^l \dot{\phi})^2 + \varepsilon^2 \int_{D_3} \delta g(A + \delta h)(\partial_y T^l \dot{\phi})^2 \\
+ h[pZT^l \dot{\phi}]^2_0 + [pZT^l \dot{\phi}]^2_0 + h[pT^l \dot{\phi}]^2_0 \right) \leq C \sum_{T \in T} \left( ||pT^l \dot{F}_1||^2_0 + C \int_{D_3} \delta g(T^l \dot{F}_2)^2 \right).
\]

Noting that the space \( \tilde{H}^s = \{ f \in L^2(D_3) : T^l f \in L^2(D_3), l \leq s \} \) is equivalent to the usual Sobolev space \( H^s(D_3) \) when \( (\phi, w, v) \in H^{s+n_0}(D_3) \) with suitable large \( n_0 \in \mathbb{N} \), then we can get the desired tame estimate (68) from (69) by direct computations.

Based on Lemma 3.1-Lemma 3.2, by the standard Picard iteration and fixed point theorem we can obtain the following conclusion as in Proposition 3.4 of [2]:
Lemma 3.3. Let \((\phi, w, v)\) and \(\varepsilon\) satisfy the assumptions of Lemma 3.2, then for all smooth \(F_1\) and \(F_2\) which are flat on \(\{t = 0\}\) and \(\{x = M\}\), there exists a unique smooth solution \((\dot{\phi}, \dot{z})\) to (52)-(53), flat on \(\{t = 0\}\) and \(\{x = M\}\). Moreover, \((\dot{\phi}, \dot{z})\) satisfies the tame estimate (68).

Therefore, due to Lemma 3.2-Lemma 3.3, by the standard Nash-Moser-Hörmander iteration method (see [1, 2]) and Sobolev imbedding theorem, one can complete the proof of Theorem 1.2 in some domain \(D_0\) (\(D_0\) is just only here \(D_3\)).

4. Proof of Theorem 1.1. Based on Theorem 1.2, as in [8], we now start to prove Theorem 1.1.

(i) \(u \in C^1(\Phi(D_3))\) and \(\|u\|_{C^1(\Phi(D_3))} \leq C\varepsilon^2\) hold.

As in [8], by \(\phi, w \in C^3(D_3)\) and (H) of Theorem 1.2, one can obtain \(G \in C(\Phi(D_3))\) from \(G(\Phi) = w\). Thus, it follows that \(u(t, x) = \frac{\varepsilon}{\sqrt{T}} G(r - t, \theta, \varepsilon \sqrt{T}) \in C^1(\Phi(D_3))\) and \(\|u\|_{C^1(\Phi(D_3))} \leq C\varepsilon^2\).

(ii) \(\frac{1}{C(T_\varepsilon - t)} \leq \|\nabla^2 u(t, \cdot)\|_{L^\infty(\Phi(D_3))} \leq \frac{C}{T_\varepsilon - t}\) holds.

Note that we have obtained the \(C^3\)-solutions \((\phi, w, v)\) to the (41) in the domain \(D_3\) by Theorem 1.2. Therefore, the solution to (1) is obtained in the domain \(\Phi(D_3)\) under the coordinate system \((s, \theta, \tau)\). Next, we go back to the original coordinate system \((r, \theta, t)\).

For \((s, \theta, \tau) \in D_3\) close to the point \(m_\varepsilon\), by Taylor’s formula, there exists \(\bar{\tau} = \lambda \tau + (1 - \lambda)\tau_\varepsilon\) with \(0 < \lambda < 1\) such that
\[
\partial_s \phi(s, \theta, \tau) = \partial_s \phi(s, \theta, \tau_\varepsilon) + \partial^2_{ss} \phi(s, \theta, \bar{\tau})(\tau - \tau_\varepsilon).
\]

Furthermore, another number \(\bar{\eta} \in (0, 1)\) makes the point \((s, \bar{\theta}) = (\bar{s} \bar{\eta}s + (1 - \bar{\eta})s_\varepsilon, \bar{\eta} \theta + (1 - \bar{\eta})\theta_\varepsilon)\) satisfy
\[
\partial_s \phi(s, \theta, \tau) = \frac{1}{2} (s - s_\varepsilon, \theta - \theta_\varepsilon) \nabla_{s, \theta}^2 \phi(s, \bar{\theta}, \tau_\varepsilon)(s - s_\varepsilon, \theta - \theta_\varepsilon)^T + \partial^3_{s\bar{\theta} s} \phi(s, \theta, \bar{\tau})(\tau - \tau_\varepsilon). \tag{70}
\]

In addition, we assume \(-2c_0 \leq \partial^2_{\tau^2} \phi \leq -c_0\) in \(D_3\) due to \(\partial^2_{\tau^2} \phi(m_\varepsilon) < 0\) and \(\phi \in C^3(D_0)\), here \(c_0 > 0\) is a constant. Together with (70) and (H) yields
\[
\partial_s \phi(s, \theta, \tau) \geq c_0 (\tau - \tau_\varepsilon) = c_0 \varepsilon \frac{T_\varepsilon - t}{\sqrt{T_\varepsilon} + \sqrt{t}} \geq \frac{c_0 \varepsilon}{3} \frac{T_\varepsilon - t}{\sqrt{t}}. \tag{72}
\]

On the other hand, using the fact \(\nabla^2_{s, \theta} \partial_s \phi(m_\varepsilon) > 0\), if \(|(s - s_\varepsilon, \theta - \theta_\varepsilon)| < \tau_\varepsilon - \tau\), from (71) it is easy to get
\[
|\partial_s \phi(s, \theta, \tau)| \leq 3c_0(\tau - \tau_\varepsilon) \leq \frac{3c_0 \varepsilon}{2} \frac{T_\varepsilon - t}{\sqrt{t}}. \tag{73}
\]

According to the expression \(u = \frac{\omega}{\sqrt{T}} G\) and \(v = \partial_s G\), we have
\[
\frac{1}{C(T_\varepsilon - t)} \leq \|\nabla^2 u(t, \cdot)\|_{L^\infty(\Phi(D_3))} \leq \frac{C}{T_\varepsilon - t}, \tag{74}
\]

(iii) \(u \in C^1([0, T_\varepsilon] \times \mathbb{R}^2) \cap C^2((0, T_\varepsilon] \times \mathbb{R}^2) \setminus \{M_\varepsilon\}\) and \(\|u\|_{C^1([0, T_\varepsilon] \times \mathbb{R}^2)} \leq C\varepsilon\) hold.

Outside of \(M_\varepsilon\) for \(t \leq T_\varepsilon\), due to the assumption (ND), then the smooth solution of (8) does not blow up in \((\{t \leq T_\varepsilon\} \times \mathbb{R}^2) \setminus \{M_\varepsilon\}\). Therefore, similar to the proof
on Proposition 1, we can get $u \in C^2((0, T_\varepsilon] \times \mathbb{R}^2) \setminus \{M_\varepsilon\})$, furthermore, in the domain $(\{t \leq T_\varepsilon\} \times \mathbb{R}^2) \setminus \{\Phi(D_3)\}$ one has

$|u| \leq C\varepsilon$ and $|\nabla^\alpha u| \leq C\varepsilon(1 + t)^{-1/2}$ for $|\alpha| = 1$ or 2.

This, together with (i), derives $\|u\|_{C^1([0, T_\varepsilon] \times \mathbb{R}^2)} \leq C\varepsilon$.

(iv) $\lim_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} = \tau_0$ holds.

By Theorem 1.2 and the related Nash-Moser-Hörmander iteration process, we can conclude $\lim \tau_\varepsilon = \tau_0$ for the solution $u$ whose corresponding variables $(r - t, \theta, \varepsilon \sqrt{t})$ lie in $\Phi(D_3)$. This implies the lifespan $T_\varepsilon$ satisfies

$$\lim_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} \leq \tau_0. \quad (75)$$

Combining (75) with (38) yields

$$\lim_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} = \tau_0.$$

Thus, collecting (i)-(iv), we complete the proof of Theorem 1.1.

REFERENCES

[1] S. Alinhac, Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions, Ann. of Math., 149 (1999), 97–127.
[2] S. Alinhac, Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions. II, Acta Math., 182 (1999), 1–23.
[3] S. Alinhac, The null condition for quasilinear wave equations in two space dimensions. II, Amer. J. Math., 123 (2001), 1071–1101.
[4] S. Alinhac, An example of blowup at infinity for quasilinear wave equations, Asterisque, 284 (2003), 1–91.
[5] D. Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data, Comm. Pure Appl. Math., 39 (1986), 267–282.
[6] D. Christodoulou and S. Miao, Compressible Flow and Euler’s Equations, Surveys of Modern Mathematics, 9. International Press, Somerville, MA; Higher Education Press, Beijing, 2014.
[7] Ding Bingbing, Liu Yingbo and Yin Huicheng, The small data solutions of general 3-D quasilinear wave equations. I, SIAM Journal on Mathematical Analysis, 47 (2015), 4192–4228.
[8] Ding Bingbing, Witt Ingo and Yin Huicheng, The small data solutions of general 3-D quasilinear wave equations. II, J. Differential Equations, 261 (2016), 1429-1471.
[9] Ding Bingbing, Witt Ingo and Yin Huicheng, Blowup of classical solutions for 2-D quasilinear wave equations with small initial data, Quart. Appl. Math., 73 (2015), 773–796.
[10] Ding Bingbing, Witt Ingo and Yin Huicheng, On the blowup of classical solutions to the 3-D pressure-gradient systems, J. Differential Equations, 252 (2012), 3608–3629.
[11] P. Godin, Lifespan of solutions of semilinear wave equations in two space dimensions, Comm. Partial Differential Equations, 18 (1993), 895–916.
[12] Guo Fei, Wave-breaking phenomena, decay properties and limit behaviour of solutions of the Degasperis-Procesi equation, Proc. Roy. Soc. Edinburgh Sect. A, 142(2012), 805–824.
[13] Guo Fei and Peng Weimei, Blowup solutions for the generalized two-component Camassa-Holm system on the circle, Nonlinear Anal., 105 (2014), 120–133.
[14] L. Hörmander, The Lifespan of Classical Solutions of Nonlinear Hyperbolic Equations, Mittag-Leffler report No.5, 1985.
[15] L. Hörmander, Lectures on Nonlinear Hyperbolic Equations, Mathematiques & Applications 26, Springer Verlag, Heidelberg, 1997.
[16] A. Hoshiga, The asymptotic behaviour of the radially symmetric solutions to quasilinear wave equations in two space dimensions, Hokkaido Math. J., 24 (1995), 575–615.
[17] F. John, Blow-up of radial solutions of $u_{tt} = c^2(u_t)\Delta u$ in three space dimensions, Mat. Apl. Comput., 4 (1985), 3–18.
[18] F. John and S. Klainerman, Almost global existence to nonlinear wave equations in three space dimensions, Comm. Pure Appl. Math., 37 (1984), 443–455.
[19] M. Keel, H. Smith and C. D. Sogge, Almost global existence for quasilinear wave equations in three space dimensions, *J. Amer. Math. Soc.*, 17 (2004), 109–153.

[20] S. Klainerman, *The Null Condition and Global Existence to Nonlinear Wave Equations*, Lectures in Appl. Math., 23, Amer. Math. Soc., Providence, RI, 1986.

[21] S. Klainerman, Remarks on the global Sobolev inequalities in the Minkowski space $\mathbb{R}^{n+1}$, *Comm. Pure Appl. Math.*, 40 (1987), 111–117.

[22] Lei Yutian, Singularity analysis of Ginzburg-Landau energy related to p-wave superconductivity, *Z. Angew. Math. Phys.*, 64 (2013), 1249–1266.

[23] Li Ta-tsien and Chen Yun-mei, Initial value problems for nonlinear wave equations, *Comm. Partial Differential Equations*, 13 (1988), 383–422.

[24] H. Lindblad, On the lifespan of solutions of nonlinear wave equations with small initial data, *Comm. Pure Appl. Math.*, 43 (1990), 445–472.

[25] H. Lindblad, Global solutions of nonlinear wave equations, *Comm. Pure Appl. Math.*, 45 (1992), 1063–1096.

[26] H. Lindblad, Global solutions of quasilinear wave equations, *Amer. J. Math.*, 130 (2008), 115–157.

[27] H. Lindblad, M. Nakamura and C. D. Sogge, Remarks on global solutions for nonlinear wave equations under the standard null conditions, *J. Differential Equations*, 254 (2013), 1396–1436.

[28] J. Speck, Shock formation in small-data solutions to 3D quasilinear wave equations, preprint, arXiv:1407.6320.

[29] Wu Sijue, Global wellposedness of the 3-D full water wave problem, *Invent. Math.*, 184 (2011), 125–220.

[30] Wu Sijue, Almost global wellposedness of the 2-D full water wave problem, *Invent. Math.*, 177 (2009), 45–135.

Received January 2016; revised March 2016.

*E-mail address*: 13851929236@163.com

*E-mail address*: iwitt@uni-math.gwdg.de

*E-mail address*: huicheng@nju.edu.cn, 05407@njnu.edu.cn