REPRESENTATION DIMENSION AND FINITELY GENERATED COHOMOLOGY

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Abstract. We consider selfinjective Artin algebras whose cohomology groups are finitely generated over a central ring of cohomology operators. For such an algebra, we show that the representation dimension is strictly greater than the maximal complexity occurring among its modules. This provides a unified approach to computing lower bounds for the representation dimension of group algebras, exterior algebras and Artin complete intersections. We also obtain new examples of classes of algebras with arbitrarily large representation dimension.

1. Introduction

In his 1971 notes [Au1], Auslander introduced the notion of the representation dimension of an Artin algebra. This invariant measures how far an algebra is from having finite representation type; it was introduced in order to study algebras of infinite representation type. A non-semisimple algebra is of finite type if and only if its representation dimension is exactly two, and of infinite type if and only if the representation dimension is at least three.

For a long time it was unclear whether there could exist algebras of representation dimension strictly greater than three. Moreover, Igusa and Todorov showed in [IgT] that if this was not the case, i.e. if the representation dimension could not exceed three, then the finitistic dimension conjecture would hold. However, in 2006 Rouquier showed in [Ro2] that the representation dimension of the exterior algebra on a $d$-dimensional vector space is $d + 1$, using the notion of the dimension of a triangulated category (cf. [Ro1]). Other examples illustrating this were subsequently given in [AvI], [BeO], [KrK], [Op1] and [Op2].

In this paper we study selfinjective Artin algebras satisfying a certain finite generation hypothesis on its cohomology groups. We show that the representation dimension of such an algebra is strictly greater than the maximal complexity occurring among its modules. This provides a unified approach to computing the known lower bounds for the representation dimension of group algebras, exterior algebras and Artin complete intersections. We also obtain new examples of classes of algebras with arbitrarily large representation dimension.

2. Representation dimension

Throughout this paper, we let $k$ be a commutative Artin ring and $\Lambda$ an Artin $k$-algebra with Jacobson radical $r$. We denote by $\text{mod} \, \Lambda$ the category of finitely generated $\Lambda$-modules. The representation dimension of $\Lambda$, denoted $\text{repdim} \, \Lambda$, is defined as

$$\text{repdim} \, \Lambda \overset{\text{def}}{=} \inf \{ \text{gl. dim} \, \text{End}_\Lambda(M) \mid M \text{ generates and cogenerates } \text{mod} \, \Lambda \}.$$

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where gl.dim denotes the global dimension of an algebra. Auslander showed that
the representation dimension of a selfinjective algebra is at most its Loewy length,
whereas Iyama showed in [Iya] that this invariant is finite for every Artin algebra.

In order to compute the representation dimension of exterior algebras, Rouquier
used the notion of the dimension of a triangulated category, a concept he introduced
in [Ro1]. We recall here the definitions. Let \( T \) be a triangulated category, and
let \( C \) and \( D \) be subcategories of \( T \). We denote by \( \langle C \rangle \) the full subcategory of \( T \)
consisting of all the direct summands of finite direct sums of shifts of objects in \( C \).
Furthermore, we denote by \( C * D \) the full subcategory of \( T \) consisting of objects \( M \)
such that there exists a distinguished triangle
\[
C \to M \to D \to C[1]
\]
in \( T \), with \( C \in C \) and \( D \in D \). Finally, we denote the subcategory \( \langle C * D \rangle \) by
\( C \circ D \). Now define \( \langle C \rangle_1 \) to be \( \langle C \rangle \), and for each \( n \geq 2 \) define inductively \( \langle C \rangle_n \) to be
\( \langle C \rangle_{n-1} \circ \langle C \rangle \). The dimension of \( T \), denoted \( \dim T \), is defined as
\[
\dim T \overset{\text{def}}{=} \inf \{ d \in \mathbb{Z} | \text{there exists an object } M \in T \text{ such that } T = \langle M \rangle_{d+1} \}.
\]
In other words, the dimension of \( T \) is the minimal number of layers needed to obtain
\( T \) from one of its objects.

The key ingredient in the proof of our main result is the following lemma on
compositions of natural transformations. The lemma is analogous to [Ro1] Lemma
4.11.

**Lemma 2.1.** Let \( T \) be a triangulated category, let \( H_1, \ldots, H_{n+1} \) be cohomological
functors on \( T \), and for each \( 1 \leq i \leq n \) let \( H_i \overset{f_i}{\rightarrow} H_{i+1} \) be a natural transformation.
Furthermore, let \( C_1, \ldots, C_n \) be subcategories of \( T \) closed under shifts, and assume
that for every object \( c \in C_i \) the map \( H_i(c[j]) \overset{f_i}{\rightarrow} H_{i+1}(c[j]) \) vanishes for \( j \gg 0 \)
(respectively, for \( j \ll 0 \)). Then for every object \( w \in C_1 \circ \cdots \circ C_n \) the map
\( H_1(w[j]) \overset{f_n \cdots f_1}{\rightarrow} H_{n+1}(w[j]) \) vanishes for \( j \gg 0 \) (respectively, for \( j \ll 0 \)).

**Proof.** We may assume \( n \geq 2 \). Let \( c_1 \rightarrow c \rightarrow c_2 \rightarrow c_1[1] \) be a triangle in \( T \) with
\( c_1 \in C_1 \) and \( c_2 \in C_2 \). Then for every \( j \in \mathbb{Z} \), there is a commutative diagram
\[
\begin{array}{ccc}
H_1(c_1[j]) & \longrightarrow & H_1(c[j]) & \longrightarrow & H_1(c_2[j]) \\
\downarrow f_1 & & \downarrow f_1 & & \downarrow f_1 \\
H_2(c_1[j]) & \longrightarrow & H_2(c[j]) & \longrightarrow & H_2(c_2[j]) \\
\downarrow f_2 & & \downarrow f_2 & & \downarrow f_2 \\
H_3(c_1[j]) & \longrightarrow & H_3(c[j]) & \longrightarrow & H_3(c_2[j])
\end{array}
\]
with exact rows. By assumption, there is an integer \( j_1 \) such that the vertical upper
left map vanishes for \( j \geq j_1 \), and an integer \( j_2 \) such that the vertical lower right
map vanishes for \( j \geq j_2 \). An easy diagram chase shows that the vertical middle
composition vanishes for \( j \geq \max\{j_1, j_2\} \), hence for every object \( w \in C_1 \circ C_2 \) the map
\( H_1(w[j]) \overset{f_n f_{n-1} \cdots f_1}{\rightarrow} H_3(w[j]) \) vanishes for \( j \gg 0 \). An induction argument now
establishes the lemma. \( \square \)

The triangulated category we shall use is the *stable module category* of \( \Lambda \), in the
case when \( \Lambda \) is selfinjective. This category, denoted \( \text{mod} \Lambda \), is defined as follows:
the objects of \( \text{mod} \Lambda \) are the same as in \( \text{mod} \Lambda \), but two morphisms in \( \text{mod} \Lambda \) are equal
in \( \text{mod} \Lambda \) if their difference factors through a projective \( \Lambda \)-module. The cosyzygy
functor \( \Omega^{-1}_{\Lambda} \): \( \text{mod} \Lambda \to \text{mod} \Lambda \) is an equivalence of categories, and a triangulation of
\( \text{mod} \Lambda \) is given by using this functor as a shift and by letting short exact sequences
in mod $\Lambda$ correspond to triangles. Thus mod$\Lambda$ is a triangulated category, and its
dimension is related to the representation dimension of $\Lambda$ by the following result.

**Proposition 2.2.** [Ro2, Proposition 3.7] If $\Lambda$ is selfinjective and not semisimple,
then $\text{repdim } \Lambda \geq \dim(\text{mod } \Lambda) + 2$.

This result was originally formulated for a finite dimensional algebra over a
field, but it works just as well in our setting, i.e. for an Artin algebra which is not
necessarily finite dimensional over a field.

The main result of this paper relates the representation dimension of $\Lambda$ with
the maximal complexity occurring among its finitely generated modules. Recall
therefore that for a module $M \in \text{mod } \Lambda$ with minimal projective resolution
$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$
say, the complexity of $M$ is defined as
$$\text{cx } M \overset{\text{def}}{=} \inf \{ t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } t_k(P_n) \leq an^{t-1} \text{ for } n \gg 0 \}.$$ 
In general, the complexity of a module may be infinite, whereas it is zero if and
only if the module is projective. The complexity of $M$ can be computed as the rate
of growth of the graded $k$-module $\text{Ext}^*_\Lambda(M, \Lambda/\tau)$, and from the definition we also see
that it equals the complexity of $\Omega^s_\Lambda(M)$ for any $i \in \mathbb{N}$. Moreover, given a short
exact sequence
$$0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0$$
in mod $\Lambda$, it is well known that the inequality $\text{cx } X_n \leq \sup \{ \text{cx } X_v, \text{cx } X_w \}$ holds for
$\{u, v, w\} = \{1, 2, 3\}$. In particular, induction on the length of a module shows that
$\text{cx } X \leq \text{cx } \Lambda/\tau$ for every $X \in \text{mod } \Lambda$. We end this section with the following
elementary lemma, which shows that a module generating mod$\Lambda$ must be of maximal
complexity.

**Lemma 2.3.** Let $\Lambda$ be selfinjective, let $M \in \text{mod } \Lambda$ be a module, and suppose there
exists a number $n \in \mathbb{N}$ such that $(\Lambda)_n = \text{mod } \Lambda$. Then $\text{cx } N \leq \text{cx } M$ for every
$N \in \text{mod } \Lambda$, in particular $\text{cx } M = \text{cx } \Lambda/\tau$.

**Proof.** The result follows from the fact that triangles in mod$\Lambda$ correspond to short
exact sequences in mod$\Lambda$. $\square$

## 3. Finitely generated cohomology

We now introduce a certain “finite generation” assumption on the cohomology
groups of $\Lambda$. Recall that for $\Lambda$-modules $X$ and $Y$, the graded $k$-module $\text{Ext}^*_\Lambda(X, Y)$
is an $\text{Ext}^*_\Lambda(Y, Y) - \text{Ext}^*_\Lambda(X, X)$-bimodule via Yoneda products. Also recall that a
graded $k$-module $\bigoplus V_i$ is of finite type if each $V_i$ is a finitely generated $k$-module.

**Assumption (Fg).** There exists a commutative Noetherian graded $k$-algebra $H = \bigoplus_{i=0}^\infty H^i$ of finite type satisfying the following:

(i) For every $M \in \text{mod } \Lambda$ there is a graded ring homomorphism
$$\phi_M : H \rightarrow \text{Ext}^*_\Lambda(M, M).$$

(ii) For each pair $(X, Y)$ of finitely generated $\Lambda$-modules, the scalar actions
from $H$ on $\text{Ext}^*_\Lambda(X, Y)$ via $\phi_X$ and $\phi_Y$ coincide, and $\text{Ext}^*_\Lambda(X, Y)$ is a finitely
generated $H$-module.

In the assumption, why do we require that the left and right scalar multiplications
on $\text{Ext}^*_\Lambda(X, Y)$ coincide? The reason is that this requirement is what makes
the bifunctor $\text{Ext}^*_\Lambda(\cdot, \cdot)$ preserve maps. To see this, let $f : M \rightarrow M'$ be a homomorphism in mod$\Lambda$. For every $N \in \text{mod } \Lambda$, this map induces a homomorphism
\(\hat{f}: \text{Ext}^*_\Lambda(M', N) \rightarrow \text{Ext}^*_\Lambda(M, N)\) of graded groups. The image of a homogeneous element

\[\theta : 0 \rightarrow N \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow M' \rightarrow 0\]

is the extension \(\theta \hat{f}\) given by the commutative diagram

\[
\begin{array}{ccccccc}
\theta : 0 & \rightarrow & N & \rightarrow & X_n & \rightarrow & \cdots & \rightarrow & X_2 & \rightarrow & Y & \rightarrow & M & \rightarrow & 0 \\
\hat{f} \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\theta \hat{f} : 0 & \rightarrow & N & \rightarrow & X_n & \rightarrow & \cdots & \rightarrow & X_2 & \rightarrow & X_1 & \rightarrow & M' & \rightarrow & 0 \\
\end{array}
\]

in which the module \(Y\) is a pullback. For a homogeneous element \(\eta \in H\) we then get

\[
\hat{f}(\theta \cdot \eta) = \hat{f}(\eta \cdot \theta) = \hat{f}(\phi_N(\eta) \circ \theta) = \phi_N(\eta) \circ (\theta \hat{f}) = \eta \cdot \hat{f}(\theta) = \hat{f}(\theta) \cdot \eta,
\]

showing \(\hat{f}\) is a homomorphism of \(H\)-modules. Similarly, \(\text{Ext}^*_\Lambda(-, -)\) preserves maps in the second argument. The fact that \(\text{Ext}^*_\Lambda(-, -)\) preserves maps is absolutely essential. Note that when using this property, induction on length of modules shows that the finite generation part of the assumption \(\mathbf{F}_g\) is equivalent to \(\text{Ext}^*_\Lambda(\Lambda/\tau, \Lambda/\tau)\) being a finitely generated \(H\)-module.

It should also be noted that when \(\mathbf{F}_g\) holds, then every finitely generated \(\Lambda\)-module has finite complexity, i.e. \(\text{cx} \Lambda/\tau < \infty\). Namely, the \(H\)-module \(\text{Ext}^*_\Lambda(\Lambda/\tau, \Lambda/\tau)\) is finitely generated, and so its rate of growth is not more than that of \(H\). The ring \(H\) is commutative Noetherian and of finite type, and hence its rate of growth is finite.

In the following examples we point out three important situations in which the assumption \(\mathbf{F}_g\) holds.

**Examples.** (i) Suppose \(k\) is a field of positive characteristic \(p\), and let \(G\) be a finite group whose order is divisible by \(p\). Then by a theorem of Evens (cf. [Eve]), the graded commutative group cohomology ring \(H^*(G, k) = \text{Ext}^*_kG(k, k)\) is Noetherian. Moreover, if \(X_1\) and \(X_2\) are finitely generated \(kG\)-modules, then \(\text{Ext}^*_kG(X_1, X_2)\) is a finitely generated \(H^*(G, k)\)-module via the maps

\[- \otimes_k X_i : H^*(G, k) \rightarrow \text{Ext}^*_kG(X_i, X_i),\]

and the right and left scalar actions induced by these maps commute up to a graded sign. The even part \(\bigoplus H^{2i}(G, k)\) of \(H^*(G, k)\) is a commutative \(k\)-algebra, over which \(H^*(G, k)\) is finitely generated as a module.

(ii) Let \((A, m, k)\) be a commutative Noetherian local complete intersection. That is, the completion \(\hat{A}\), with respect to the maximal ideal \(m\), is of the form \(R/(x_1, \ldots, x_c)\), where \(R\) is regular local and \(x_1, \ldots, x_c\) is a regular sequence. We may without loss of generality assume that the length \(c\) of the defining regular sequence is the codimension of \(A\), i.e. \(c = \dim_k (m/m^2) - \dim A\). By [Avr] Section 1 there exists a polynomial ring \(\hat{A}[\chi_1, \ldots, \chi_c]\) in commuting Eisenbud operators, such that for every finitely generated \(\hat{A}\)-module \(X\) there is a homomorphism

\[
\phi_X : \hat{A}[\chi_1, \ldots, \chi_c] \rightarrow \text{Ext}^*_\hat{A}(X, X)
\]

of graded rings. Moreover, for every finitely generated \(\hat{A}\)-module \(Y\), the left and right scalar actions on \(\text{Ext}^*_\hat{A}(X, Y)\) coincide, and the latter is a finitely generated
Λ. Our aim is to show $n$ i.e. Λ is the exterior algebra on an $n$-dimensional vector space. Now if $A$ is Artin, then it is a complete ring since $m$ is nilpotent. Thus $\mathbf{Fg}$ holds in this case.

(iii) Suppose Λ is projective as a $k$-module. Denote by $\Lambda^e$ the enveloping algebra $\Lambda \otimes_k \Lambda^{op}$ of Λ, and by $\text{HH}^*(\Lambda)$ its Hochschild cohomology ring. By [Yon] Proposition 3, this is a graded commutative ring, and equal to $\text{Ext}^*_\Lambda(\Lambda, \Lambda)$ since Λ is $k$-projective. If $X_1$ and $X_2$ are finitely generated Λ-modules, then the right and left scalar actions from $\text{HH}^*(\Lambda)$ on $\text{Ext}^*_\Lambda(X_1, X_2)$, via the maps

$$- \otimes \Lambda X_i : \text{HH}^*(\Lambda) \rightarrow \text{Ext}^*_\Lambda(X_1, X_i),$$

are graded commutative.

In [EHSSY] the finite generation assumption imposed was the following: there exists a commutative Noetherian graded subalgebra $S = \bigoplus_{i=0}^{\infty} S^i$ of $\text{HH}^*(\Lambda)$, with $S^0 = \text{HH}^0(\Lambda)$, such that $\text{Ext}^*_\Lambda(\Lambda/r, \Lambda/r)$ is a finitely generated $S$-module. However, having to deal with such an “unknown” subalgebra of the Hochschild cohomology ring is not satisfactory, and in fact it is not difficult to see (cf. [Sol, Proposition 5.7]) that the assumption is equivalent to the following one: the Hochschild cohomology ring $\text{HH}^*(\Lambda)$ is Noetherian and $\text{Ext}^*_\Lambda(\Lambda/r, \Lambda/r)$ is a finitely generated $\text{HH}^*(\Lambda)$-module. Therefore, as in the first example, we see that $\mathbf{Fg}$ holds by choosing $H$ to be the even part of $\bigoplus \text{HH}^{2i}(\Lambda)$ of $\text{HH}^*(\Lambda)$.

In particular, the assumption $\mathbf{Fg}$ holds for exterior algebras. Namely, suppose $k$ is a field, let $n$ be a number, and denote by Λ the algebra

$$k(x_1, \ldots, x_n)/(x_i^2, x_i x_j + x_j x_i),$$

i.e. Λ is the exterior algebra on an $n$-dimensional vector space. Then by [Sol] Theorem 9.2 and Theorem 9.11, the Koszul dual $\text{Ext}^*_\Lambda(k, k)$ of Λ is the polynomial ring $k[x_1, \ldots, x_n]$, and via the map

$$- \otimes \Lambda k : \text{HH}^*(\Lambda) \rightarrow \text{Ext}^*_\Lambda(k, k)$$

this is a finitely generated $\text{HH}^*(\Lambda)$-module. By choosing $H$ to be the even part of the inverse image of $k[x_1, \ldots, x_n]$, we see that $\mathbf{Fg}$ holds.

Having pointed out these three examples where $\mathbf{Fg}$ holds, we now prove the main result: when Λ is selfinjective and $\mathbf{Fg}$ holds, then the dimension of the stable module category of Λ is at least $c x \Lambda / r - 1$.

**Theorem 3.1.** If Λ is selfinjective and $\mathbf{Fg}$ holds, then $\dim(\text{mod} \Lambda) \geq c x \Lambda / r - 1$.

**Proof.** Denote the complexity of $\Lambda/r$ by $c$. Let $M \in \text{mod} \Lambda$ be a module generating $\text{mod} \Lambda$, i.e. there exists a number $n$ such that $\langle M \rangle_n = \text{mod} \Lambda$. Our aim is to show that $n \geq c$. If $c \leq 1$, then there is nothing to prove, so we may assume $c \geq 2$.

By Lemma 2.3 the module $M$ must have maximal complexity, that is, the equality $c x M = c$ holds. Choose, by [Be1] Proposition 2.1, a homogeneous element $\eta_1 \in H^+$ such that the multiplication map

$$\text{Ext}^i_\Lambda(M, M \oplus \Lambda/r) \rightarrow \text{Ext}^{i+|\eta_1|}_\Lambda(M, M \oplus \Lambda/r)$$

is injective for $i \gg 0$. Applying the map $\phi_M$ to $\eta_1$ gives a short exact sequence

$$\phi_M(\eta_1) : 0 \rightarrow M \xrightarrow{f_1} K_1 \rightarrow \Omega^{|\eta_1|-1}_\Lambda(M) \rightarrow 0,$$

and the arguments used in the proof of [Be1] Theorem 3.2 shows that $c x K_1 = c - 1$. Next, if $c \geq 3$, choose a homogeneous element $\eta_2 \in H^+$ such that the multiplication map

$$\text{Ext}^i_\Lambda(K_1, M \oplus \Lambda/r) \oplus \text{Ext}^i_\Lambda(M, K_1) \rightarrow \text{Ext}^{i+|\eta_2|}_\Lambda(K_1, M \oplus \Lambda/r) \oplus \text{Ext}^{i+|\eta_2|}_\Lambda(M, K_1)$$

is injective for $i \gg 0$. Applying the map $\phi_{K_1}$ to $\eta_2$ gives a short exact sequence

$$\phi_{K_1}(\eta_2) : 0 \rightarrow K_1 \xrightarrow{f_2} K_2 \rightarrow \Omega^{\eta_2}(-1)(K_1) \rightarrow 0,$$
in which $cx K_2 = c - 2$. We continue this process until we end up with a module $K_{c-1}$ of complexity 1. Thus we obtain homogeneous elements $\eta_1, \ldots, \eta_{c-1} \in H^+$, and for each $1 \leq j \leq c - 1$ a short exact sequence

$$
\phi_{K_{c-1}}(\eta_j) : 0 \to K_{j-1} \xrightarrow{f_j} K_j \xrightarrow{\Omega_A^{|\eta_j|-1}} (K_{j-1}) \to 0
$$

with $cx K_j = c - j$ (here $K_0 = M$). For each $j$ the element $\eta_j$ is chosen in such a way that it is regular on $\text{Ext}_A^i(K_{j-1}, M \oplus \Lambda/t) \oplus \text{Ext}_A^i(M, K_{j-1})$ for $i \gg 0$.

For each $1 \leq j \leq c - 1$ and $i \gg 0$, the exact sequence $\phi_{K_{c-1}}(\eta_j)$ induces the two exact sequences

$$
\text{Ext}_A^i(K_j, M) \xrightarrow{(f_j)\ast} \text{Ext}_A^i(K_{j-1}, M) \xrightarrow{\eta_j} \text{Ext}_A^{i+|\eta_j|}(K_{j-1}, M)
$$

and

$$
\text{Ext}_A^i(M, K_{j-1}) \xrightarrow{(f_j)\ast} \text{Ext}_A^i(M, K_j) \xrightarrow{\eta_j} \text{Ext}_A^{i+|\eta_j|+1}(M, K_{j-1}) \xrightarrow{\eta_j} \text{Ext}_A^{i+1}(M, K_{j-1}).
$$

From the upper exact sequence, we see that the map $\text{Ext}_A^i(K_j, M) \xrightarrow{(f_j)\ast} \text{Ext}_A^i(K_{j-1}, M)$ vanishes for $i \gg 0$, since the multiplication map involving $\eta_j$ is injective. From the lower exact sequence we see that the map $\text{Ext}_A^i(M, K_{j-1}) \xrightarrow{(f_j)\ast} \text{Ext}_A^i(M, K_j)$ is surjective for $i \gg 0$.

The latter implies that when $i$ is large, the maps $f_1, \ldots, f_{c-1}$ induce a chain

$$
\text{Ext}_A^i(M, M) \xrightarrow{(f_1)\ast} \text{Ext}_A^i(M, K_1) \xrightarrow{(f_2)\ast} \cdots \xrightarrow{(f_{c-1})\ast} \text{Ext}_A^i(M, K_{c-1})
$$

of epimorphisms. Now choose a homogeneous element $\eta \in H^+$ which is regular on $\text{Ext}_A^i(K_{c-1}, \Lambda/t)$ for $i \gg 0$. Applying $\phi_{K_{c-1}}$ to this element gives an element

$$
\phi_{K_{c-1}}(\eta) : 0 \to K_{c-1} \to K \xrightarrow{\Omega_A^{|\eta|-1}} (K_{c-1}) \to 0
$$

in $\text{Ext}_A^i(K_{c-1}, K_{c-1})$, where the module $K$ is projective. Using the arguments in the proof of [Be2, Corollary 3.2], we see that $\phi_{K_{c-1}}(\eta)$ cannot be nilpotent in $\text{Ext}_A^i(K_{c-1}, K_{c-1})$. Consequently, given any $w \in \mathbb{N}$, there is an integer $i \geq w$ such that $\text{Ext}_A^i(K_{c-1}, K_{c-1})$ is nonzero. Using the exact sequences $\phi_M(\eta_1), \phi_M(\eta_2), \ldots, \phi_M(\eta_{c-1})$ we then see that given any $w \in \mathbb{N}$, there is an integer $i \geq w$ such that $\text{Ext}_A^i(M, K_{c-1})$ is nonzero. This shows that the composition

$$
M \xrightarrow{f_1} K_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{c-1}} K_{c-1}
$$

is nonzero in $\text{mod} \Lambda$.

Now consider the functors $\text{Hom}_A(K_j, -)$ on $\text{mod} \Lambda$, together with the natural transformations

$$
\text{Hom}_A(K_{c-1}, -) \xrightarrow{(f_{c-1})\ast} \text{Hom}_A(K_{c-2}, -) \xrightarrow{(f_{c-2})\ast} \cdots \xrightarrow{(f_1)\ast} \text{Hom}_A(M, -).
$$

Since the map $\text{Ext}_A^i(K_j, M) \xrightarrow{(f_j)\ast} \text{Ext}_A^i(K_{j-1}, M)$ vanishes for $i \gg 0$, the map $\text{Hom}_A(K_j, \Omega_A^i(M)) \xrightarrow{(f_j)\ast} \text{Hom}_A(K_{j-1}, \Omega_A^i(M))$ vanishes for $i \ll 0$. From Lemma [2.1] we conclude that for every module $X \in \langle M \rangle_{c-1}$, the map

$$
\text{Hom}_A(K_{c-1}, \Omega_A^i(X)) \xrightarrow{(f_{c-1} \circ \cdots \circ f_1)\ast} \text{Hom}_A(M, \Omega_A^i(X))
$$

vanishes for $i \ll 0$. However, by [Be1, Theorem 2.3] the module $K_{c-1}$ is periodic in $\text{mod} \Lambda$, that is, there is an integer $p \geq 1$ such that $K_{c-1} \simeq \Omega_A^p(K_{c-1})$ in $\text{mod} \Lambda$. Therefore, since the composition $f_{c-1} \circ \cdots \circ f_1$ is nonzero in $\text{mod} \Lambda$, the map

$$
\text{Hom}_A(K_{c-1}, \Omega_A^{ip}(K_{c-1})) \xrightarrow{(f_{c-1} \circ \cdots \circ f_1)\ast} \text{Hom}_A(M, \Omega_A^{ip}(K_{c-1}))
$$

is nonzero in $\text{mod} \Lambda$. Hence, $\text{Ext}_A^i(M, K_{c-1}) = 0$ for $i \ll 0$. Thus, we obtain a chain of epimorphisms

$$
\text{Ext}_A^i(M, M) \xrightarrow{(f_1)\ast} \text{Ext}_A^i(M, K_1) \xrightarrow{(f_2)\ast} \cdots \xrightarrow{(f_{c-1})\ast} \text{Ext}_A^i(M, K_{c-1})
$$

of epimorphisms, and this shows that $\text{Ext}_A^i(M, K_{c-1}) = 0$ for $i \ll 0$, as desired.
does not vanish for any \( i \in \mathbb{Z} \). This shows that the module \( K_{e-1} \) cannot be an element in \( X = \langle M \rangle_{c-1} \), and so \( n \geq c \). The proof is complete. \( \square \)

Using Proposition 2.2 and Auslander’s upper bound, we obtain the promised result on the representation dimension. We denote by \( \ell \ell(\Lambda) \) the Loewy length of our algebra \( \Lambda \).

**Theorem 3.2.** If \( \Lambda \) is a non-semisimple selfinjective algebra and \( \text{Fg} \) holds, then
\[
\text{cx} \Lambda/r + 1 \leq \text{repdim} \Lambda \leq \ell \ell(\Lambda).
\]

Rouquier showed that the representation dimension of the exterior algebra on an \( n \)-dimensional vector space is exactly \( n + 1 \). It therefore seems natural to ask the following:

**Question.** When \( \text{Fg} \) holds, what is the exact value of \( \text{repdim} \Lambda \)?

The following corollaries to Theorem 3.2 provide lower bounds for the representation dimension of the algebras given in the three examples prior to Theorem 3.1. In particular, we obtain [Op1, Corollary 19], half of [Ro2, Theorem 4.1] and the result of Avramov and Iyengar on the representation dimension of Artin complete intersections (cf. [AvI]).

**Corollary 3.3.** Suppose \( k \) is a field of positive characteristic \( p \), and let \( G \) be a finite group whose order is divisible by \( p \). Then \( \text{repdim} kG \geq p - \text{rank} G + 1 \), that is, the representation dimension of \( kG \) is strictly greater than \( \text{Krulldim} H^*(G, k) \).

**Proof.** By a result of Quillen (cf. [Qu1, Qu2]), the complexity of the trivial \( kG \)-module, i.e. the Krull dimension of the cohomology ring \( H^*(G, k) \), equals the \( p \)-rank of \( G \). \( \square \)

**Corollary 3.4.** Let \( A \) be a commutative Noetherian local complete intersection of codimension \( c \). If \( A \) is Artin, then \( \text{repdim} A \geq c + 1 \).

**Proof.** By a classical result of Tate (cf. [Tat, Theorem 6]), the complexity of the simple module over a complete intersection equals the codimension of the ring. \( \square \)

**Remark.** Let \( A \) be a commutative Noetherian local complete intersection of codimension \( c \). If \( A \) is complete, then the proof of Theorem 3.1 also applies to the stable category of finitely generated maximal Cohen-Macaulay \( A \)-modules. Namely, the dimension of this triangulated category is at least \( c - 1 \).

**Corollary 3.5.** Suppose \( \Lambda \) is semisimple and projective as a \( k \)-module. Furthermore, suppose the Hochschild cohomology ring \( HH^*(\Lambda) \) is Noetherian, and that \( \text{Ext}^*_\Lambda(\Lambda/\tau, \Lambda/\tau) \) is a finitely generated \( HH^*(\Lambda) \)-module. Then \( \text{repdim} \Lambda \geq \text{Krulldim} HH^*(\Lambda) + 1 \). In particular, the representation dimension of the exterior algebra on an \( n \)-dimensional vector space is at least \( n + 1 \).

**Proof.** The Krull dimension of \( HH^*(\Lambda) \) is its rate of growth \( \gamma (HH^*(\Lambda)) \) as a graded \( k \)-module. Therefore, since the \( HH^*(\Lambda) \)-module \( \text{Ext}^*_\Lambda(\Lambda/\tau, \Lambda/\tau) \) is finitely generated, we see that
\[
\text{cx} \Lambda/\tau = \gamma (\text{Ext}^*_\Lambda(\Lambda/\tau, \Lambda/\tau)) \leq \gamma (HH^*(\Lambda)) = \text{Krulldim} HH^*(\Lambda).
\]

Denote the radical of \( \Lambda^e \) by \( r^e \), and let \( B \) be a finitely generated bimodule (i.e. \( B \in \text{mod} \Lambda^e \)). If \( B \) is not simple, then choose an exact sequence
\[
0 \rightarrow S \rightarrow B \rightarrow B' \rightarrow 0
\]
in which \( S \) is simple. This sequence induces an exact sequence
\[
\text{Ext}^*_\Lambda(\Lambda, S) \rightarrow \text{Ext}^*_\Lambda(\Lambda, B) \rightarrow \text{Ext}^*_\Lambda(\Lambda, B')
\]
of $\text{HH}^*(\Lambda)$-modules, all of which are finitely generated by \cite[Proposition 2.4]{EHSST}. Consequently the inequality
\[
\gamma(\text{Ext}_A^*(\Lambda, B)) \leq \max\{\gamma(\text{Ext}_A^*(\Lambda, S)), \gamma(\text{Ext}_A^*(\Lambda, B'))\}
\]
holds, and so induction on length gives
\[
\gamma(\text{Ext}_A^*(\Lambda, B)) \leq \gamma(\text{Ext}_A^*(\Lambda, \Lambda/r)) = \text{cx}_A \Lambda.
\]
In particular, the inequality \(\gamma(\text{HH}^*(\Lambda)) \leq \text{cx}_A \Lambda\) holds. But the complexity of $\Lambda$ as a bimodule equals that of the $\Lambda$-module $\Lambda/r$. Namely, applying $- \otimes \Lambda \Lambda/r$ to the minimal projective bimodule resolution of $\Lambda$ gives the minimal $\Lambda$-projective resolution of $\Lambda/r$. Therefore $\text{cx}_A \Lambda = \text{cx} \Lambda/r$, and this shows that the Krull dimension of $\text{HH}^*(\Lambda)$ equals $\text{cx} \Lambda/r$. □

We end with two examples illustrating Corollary 3.5. These provide new examples of classes of algebras with arbitrarily large representation dimension.

**Examples.** (i) Let $k$ be a field, let $n \geq 1$ be an integer, and let $\Lambda$ be the quantum complete intersection
\[
k\langle X_1, \ldots, X_n \rangle/(X_i^2, \{X_iX_j - q_{ij}X_jX_i\}_{i<j}),
\]
where $0 \neq q_{ij} \in k$. This algebra is finite dimensional of dimension $2^n$, and the complexity of $k$ is $n$. Furthermore, this is a Frobenius algebra; the codimension two argument in the beginning of \cite[Section 3]{BeE} carries over. In particular, this algebra is selfinjective, and it was shown in \cite{Ehs} that $\text{Fg}$ holds if and only if all the $q_{ij}$ are roots of unity. Therefore, when this is the case, then the representation dimension of $\Lambda$ is at least $n + 1$.

(ii) Let $k$ be an algebraically closed field, and let $R$ be a Noetherian Artin-Schelter regular Koszul $k$-algebra of dimension $d$. That is, $R$ is graded connected of global dimension $d$, its Gelfand-Kirillov dimension is finite, and $\text{Ext}_R^i(k, R) \simeq \begin{cases} 0 & i \neq d \\ k & i = d \text{ (up to shift)} \end{cases}$. If $R$ is a finitely generated module over its center, then by \cite[Proposition 9.15]{Sol} the Koszul dual $\Lambda$ of $R$ is selfinjective, satisfies $\text{Fg}$, and $\text{cx} \Lambda/r = d$. Thus in this case the representation dimension of $\Lambda$ is at least $d + 1$.

An example of such an algebra is obtained from the Sklyanin algebras (cf. \cite{Smi}, Section 8): let $E$ be an elliptic curve over $k$, and fix a point $P \in E$ such that $nP = 0$ for some $n \geq 1$. Denote by $\sigma_P: E \to E$ the corresponding translation automorphism. Furthermore, let $d \geq 1$ be an integer, and let $A_d(E, \sigma_P)$ be the $d$-dimensional Sklyanin algebra. This is an Artin-Schelter regular algebra of the above type.

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