A formula with hypervolumes of six 4-simplices and two discrete curvatures

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Abstract

One of the generalizations of the pentagon equation to higher dimensions is the so-called “six-term equation”. Geometrically, it corresponds to one of the “Alexander moves”, that is elementary rebuildings of simplicial complexes, namely, replacing a “cluster” of three 4-simplices by another “cluster”, also of three 4-simplices and with the same boundary. We present a formula containing the euclidean volumes of the simplices in the first cluster in its l.h.s., and those in the second cluster — in its r.h.s. The formula also involves “discrete curvatures” appearing when we slightly deform the euclidean space.

As an artist and scholar I prefer the specific detail to the generalization, images to ideas, obscure facts to clear symbols, and the discovered wild fruit to the synthetic jam.

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1 Introduction

This short note continues the paper [1] and the short note [2]. We present a formula that naturally corresponds to one of the “Alexander moves” [3], i.e., “elementary rebuildings” of simplicial complexes. Our formula belongs to a four-dimensional space and deals with three “initial” 4-simplices in its l.h.s. and three “final” ones in its r.h.s. Recall that a similar formula for a three-dimensional space was obtained in [2], and this was done on the basis of “duality formulas” (which are valid, themselves, for any-dimensional space) from [1].

2 Derivation of the formula

Consider six points $A, B, C, D, E, F$ in the four-dimensional euclidean space. There exist fifteen distances between them, which we denote, like in papers [1], [2], as

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1The duality in [1] can be said to be dealing with “branched polymers” known in quantum gravity.

2Yet, as will be seen below, we will allow them sometimes to “go out in the fifth dimension”.

There exist six 4-simplices with vertices in our points, and we denote those simplices as $\overrightarrow{AB}$, $\overrightarrow{AC}$, and so on. There exist six 4-simplices with vertices in our points, and we denote those simplices as $\overrightarrow{A}$, ..., $\overrightarrow{F}$, where, say, $\overrightarrow{A}$ is the simplex $BCDEF$ (not containing the vertex $A$). The four-dimensional hypervolume we will denote as $V$, e.g., $V_A$ is the hypervolume of the simplex $\overrightarrow{A}$. We will need also the areas of two-dimensional faces ($S_{ABC}$ being the area of face $ABC$ and so on) and the “defect angles”, or “discrete curvatures” concentrated in those faces (the defect angle $\omega_{ABC}$ corresponds to the face $ABC$, etc.).

A defect angle means the following. With arbitrary distances $l_{AB}$, ..., $l_{EF}$, the points $A$, ..., $F$ may not necessarily be placed in the (“flat”) four-dimensional Euclidean space. Any five of those points, however, can be placed there, thus forming a 4-simplex with vertices in those points, and then one can calculate the “dihedral angles” between its three-dimensional hyperfaces. There are three such “dihedral angles” at the two-dimensional face $ABC$ — they correspond to tetrahedra $D$, $E$, and $F$. In the flat case, the sum of those angles is $2\pi$, and in the general case, it equals, by definition, $2\pi - \omega_{ABC}$.

All our considerations will take place in a small neighborhood of the flat case $\omega_{ABC} = 0$. Arguments perfectly analogous to those in [3], but using [1, formulas (15, 16)] instead of [1, formulas (11, 12)], yield

$$\frac{1}{12} \frac{S_{ABC} l_{AB} dl_{AB}}{V_T V_P V_P} = \frac{d\omega_{ABC}}{V_T V_P V_P},$$

if only $l_{AB}$ of all distances can vary. Similarly, one can write

$$\frac{1}{12} \frac{S_{DEF} l_{DE} dl_{DE}}{V_T V_P V_P} = \frac{d\omega_{DEF}}{V_T V_P V_P},$$

if only $l_{AB}$ can vary.

If both $l_{AB}$ and $l_{DE}$ can change, but the zero curvature is fixed:

$$\omega_{ABC} \equiv 0,$$

which is obviously equivalent to

$$\omega_{DEF} \equiv 0,$$

then $dl_{AB}$ and $dl_{DE}$ are related by

$$\frac{l_{AB} dl_{AB}}{V_T V_P} = \frac{l_{DE} dl_{DE}}{V_T V_P},$$

(cf. [1, (16)]), and similarly one can write out the relation between the differentials of any pair of distances.

Consider $\omega_{ABC}$ as a function of fifteen distances. In a neighborhood of the flat configuration, we have

$$d\omega_{ABC} = c_{AB} dl_{AB} + \cdots + c_{EF} dl_{EF},$$
where all the ratios of coefficients $c_{\ldots}$ are fixed (at least, up to a sign) by the formula (5) and the like formulae for other pairs of distances, e.g.,

\[
\left| \frac{c_{AB}}{c_{DE}} \right| = \left| \frac{l_{AB} V_A V_B}{l_{DE} V_D V_E} \right|, \tag{7}
\]

etc. If now we write, analogously,

\[
d\omega_{DEF} = c'_{AB} dl_{AB} + \cdots + c'_{EF} dl_{EF}, \tag{8}
\]

then the coefficients $c'_{\ldots}$ will, obviously, have the same ratios. Thus, the differentials of curvatures $\omega_{ABC}$ and $\omega_{DEF}$ as functions of all distances are proportional, namely, from (1, 2 and 5) we find:

\[
\left| \frac{d\omega_{ABC}}{S_{ABC} V_A V_B V_C} \right| = \left| \frac{d\omega_{DEF}}{S_{DEF} V_D V_E V_F} \right|. \tag{9}
\]

Let us, finally, write this in the form of the desired “six-term equation”:

\[
\frac{S_{ABC} \delta (\omega_{ABC})}{V_A V_B V_C} = \frac{S_{DEF} \delta (\omega_{DEF})}{V_D V_E V_F}. \tag{10}
\]

Here $\delta$ is the Dirac delta function, and instead of writing out the absolute value signs, we assume that the signs of (oriented) hypervolumes and areas are chosen “properly”. It is implied that both sides of (10) can be integrated in any of $dl_{AB}, \ldots, dl_{EF}$.

3 Remarks

1. Our equation (10) corresponds to a “move of type 3 $\leftrightarrow$ 3”, i.e., three simplices are transformed into three new ones. Nontrivial seems the question of what to do with the other Alexander moves, that is $2 \leftrightarrow 4$ and $1 \leftrightarrow 5$. Similarly, in the paper [2] the moves $2 \leftrightarrow 3$ are analyzed, while $1 \leftrightarrow 4$ requires further investigation.

2. Our formulas are likely to be useful for quantum gravity. Namely, they may help to find the most symmetric integration measure for “functional integrals” in the discrete Regge-type models of space-time.

3. The triangulated manifold where our rebuildings take place is not bound to be flat — see the similar Remark 2 in the end of paper [2].

References

[1] I. G. Korepanov, Multidimensional analogs of geometric s $\leftrightarrow$ t duality, solv-int/9911008, accepted for publication in Theor. Math. Phys.

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[3] J. W. Alexander, The combinatorial theory of complexes, Ann. Math. 31 292 (1930).