Magnetohydrodynamic Stability at a Separatrix: Part II

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Abstract

In the first part to this paper[1] it was shown how a simple Magnetohydrodynamic model could be used to determine the stability of a Tokamak plasma’s edge to a Peeling (External Kink) mode. Stability was found to be determined by the value of $\Delta'$, a normalised measure of the discontinuity in the radial derivative of the radial perturbation to the magnetic field at the plasma-vacuum interface. Here we calculate $\Delta'$, but in a way that avoids the numerical divergences that can arise near a separatrice’s X-point. This is accomplished by showing how the method of conformal transformations may be generalised to allow their application to systems with a non-zero boundary condition, and using the technique to obtain analytic expressions for both the vacuum energy and $\Delta'$. A conformal transformation is used again to obtain an equilibrium vacuum field surrounding a plasma with a separatrix. This allows the subsequent evaluation of the vacuum energy and $\Delta'$. For a plasma-vacuum boundary that approximates a separatrix, the growth rate $\gamma$ normalised by the Aflven frequency $\gamma_A$ is then found to have $\ln(\gamma/\gamma_A) = -\frac{1}{2} \ln (q'/q)$. Consequences for Peeling mode stability are discussed.

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I. INTRODUCTION

Modern Tokamaks and future designs for power-plant scale Tokamaks, have plasmas with a separatrix at the plasma’s edge. To determine whether these Tokamak plasmas are susceptible to an ideal Magnetohydrodynamic Peeling mode (External kink) instability, a simple model was generalised from a cylindrical to toroidal Tokamak geometry in the first part [1] to this paper. The conclusions from part (I) were: (a.) Peeling-mode stability is determined by the value of $\Delta'$, a poloidally averaged measure of the discontinuity in the radial derivative of the perturbation to the magnetic field at the plasma-vacuum interface (the separatrix), and (b.) that regardless of the sign of $\delta W$ the growth rate could still be vanishingly small. To determine the consequences of part (I) for the stability of the Peeling mode, this paper evaluates $\Delta'$. It is essential that in this calculation any divergences due to the X-point are incorporated and not accidentally removed or accentuated by the discretisation of space that is usually required by numerical modelling, and ideally that $\Delta'$ is calculated exactly. That is the purpose of this paper. This paper and part (I) are summarised in Ref. [2].

By using analytical methods to study Peeling mode stability in a plasma equilibrium with a separatrix boundary (and an X-point), we hope to avoid difficulties that are encountered with numerical studies, and to gain a better understanding of the physical factors that affect the plasma’s stability. The techniques developed here may also find further applications to related problems using different models of the plasma. The results from this and future work are intended to provide understanding and tests, that will assist the development of codes for stability calculations that incorporate more advanced models of plasma physics and Tokamak geometry. It is also hoped that the methods developed in this paper will have applications outside of plasma physics.

Outline: We review conformal transformations in Section II and describe the Karman-Trefftz transformation [3] in Section III. The Karman-Trefftz transformation provides an example of a transformation from a circular boundary to a separatrix boundary with an X-point. Section IV reviews how a complex potential may be defined and used to calculate how the vacuum magnetic field will transform under a mapping from a system with a circular boundary, to one with a separatrix. For a large aspect ratio system, the vacuum energy is calculated for both a circular cross-section and a separatrix cross-section in Section
obtaining the vacuum energy for a separatrix cross-section in terms of a sum of Fourier co-efficients. The Fourier co-efficients are determined by the plasma-vacuum boundary conditions, and Section VI discusses how these boundary conditions are modified by a conformal transformation. This is where we have departed from the conventional textbook applications of conformal transformations, that require the boundary conditions to be zero. Instead the transformed boundary conditions presented in Section VI provide analytic expressions to determine the Fourier co-efficients in terms of the straight field line angle, that is not yet known. The straight field line angle is calculated in terms of an equilibrium vacuum field and the conformal transformation, in Section VII. Section VIII calculates an equilibrium vacuum field for both the circular boundary and the separatrix boundary systems, subsequently allowing analytic expressions for the safety factor $q$ and the straight field line angle to be obtained at the plasma-vacuum boundary. At this point all the analytic expressions needed to calculate the vacuum energy have been obtained. Section IX investigates how other quantities (in addition to the field and the boundary conditions), change under a conformal transformation. These expressions are tested in Section X by re-calculation of the vacuum energy from a surface integral representation given in the first part to this paper. It is noted that this calculation infers the value for $\Delta'$ in terms of the vacuum energy $\delta W_V$.

Reassured with the results from Section X, Section XI calculates $\Delta'$ directly, and confirms the same answer in terms of $\delta W_V$ found in the previous section. At this stage we have analytic expressions for $\delta W_V$ and $\Delta'$ in terms of a sum of Fourier coefficients, and analytic expressions for the Fourier coefficients. Section XII calculates $\delta W_V$ by evaluating the sum of Fourier coefficients in a number of ways, finding the same result as for an equivalent calculation in a circular cross-section system, and independent of the details of the separatrix geometry. Section XIII provides a discussion that compares and extends previous work, considers the mode structure, and summarises the paper.

II. CONFORMAL TRANSFORMATIONS

A conformal transformation $w(z)$ (e.g. see Ref. [4]), is an analytic function between complex planes $z \rightarrow w(z)$. It has the property that the angle (and direction of the angle), between curves in the plane from which they are mapped, is retained between the resulting curves in the plane onto which it is mapped. This angle-preserving property ensures that
the unit normal to a line in one plane will map to a vector normal to the mapped line, with a consequence that a boundary condition of $0 = \vec{n}_z \cdot \vec{B}_z$ will be transformed to a boundary condition of $0 = \vec{n}_w \cdot \vec{B}_w$ for the boundary in the $w(z)$ plane. More generally, if the arc length around a boundary contour in the $z$-plane is parameterised by $\alpha$, then it will be shown in Section [VI] that $\vec{n}_w \cdot \nabla_w V(w) |_{\alpha} = | \frac{dw}{dz} | \vec{n}_z \cdot \nabla_z V(z) |_{\alpha}$.

An important property of conformal maps is that a function that satisfies Laplace’s equation, will continue to do so after a conformal transformation. In other words, if $\nabla^2 V(z) = 0$, then provided $w(z)$ is a conformal transformation, $\nabla^2_w V(z(w)) = 0$ also. Riemann’s mapping theorem indicates the existence of a conformal transformation from a circle to any closed region. So provided a suitable transformation may be found, and provided the boundary conditions map in a simple enough way (as they often do), then it is possible to find solutions in complicated geometries by solving a problem with a simple circular boundary.

III. KARMAN-TREFFTZ TRANSFORMATION

A mapping that may be used to take us from a circle to a shaped cross section with an X-point, is the Karman-Trefftz transformation [3]. The Karman-Trefftz transformation is a generalisation of the Joukowski transformation that is well known for its use in aerodynamics calculations for the lift from an airplane wing. It maps from a domain that surrounds a circle containing the point $z = -l$ and whose edge passes through $z = l$, to $w(z)$, with

$$\left( \frac{w + nl}{w - nl} \right)^n = \left( \frac{z + l}{z - l} \right)^n$$

(1)

For simplicity we will restrict ourselves to domains of $z$ that are symmetric about the real axis, so that $w(z)$ is also symmetric about the real axis with a boundary that has

$$z = -a + (a + l)e^{i\alpha}$$

(2)

with $a > 0$, so that $\alpha$ will parameterise the boundary curve (in both the $z$ and $w(z)$ planes). Note that $\alpha$ is not the argument of either $z$ or $w(z)$ (if we are centred on $z = a$ in the $z$-plane, $\alpha$ is the poloidal angle). If $n = 3/2$, then the cusp-like point at $z = l$ becomes an X-point (with a $\pi/2$ interior angle at the joining surfaces in the $w(z)$ plane) [3], and $n = 2$ produces the Joukowski transformation. By making $l/a$ arbitrarily small we make the X-point region arbitrarily localised, a situation similar to that described by Webster [3]. This may be seen
by rearranging Eq. 1 to give

\[ w(z) = -nl \frac{(z - l)^n + (z + l)^n}{(z - l)^n - (z + l)^n} \]  

and writing it as an asymptotic expansion in \( l/z \), with

\[ w(z) = z + l \left( \frac{l}{z} \right) \frac{5}{12} \left\{ 1 + \frac{7}{60} \left( \frac{l}{z} \right)^2 + \frac{13}{300} \left( \frac{l}{z} \right)^4 + \ldots \right\} \]

So provided \( |l/z| \) is sufficiently small then \( w(z) = z \).

In summary, the Karman-Trefftz transformation provides an explicit representation of a transformation from a circular boundary (with \( z = -a + (a + l)e^{ia} \)), to a shaped boundary with an X-point.

**IV. THE COMPLEX POTENTIAL**

We will need to know how the vacuum magnetic field (the gradient of a potential) is transformed as we move from a circular cross-section to the X-point geometry. This is most easily accomplished by representing the magnetic field as a complex number whose real and imaginary parts are interpreted as its \( x \) and \( y \) components, and by defining a complex potential \( \Omega \) in terms of the magnetic potential \( V \).

The complex representation for the magnetic field is given in terms of the magnetic potential \( V \), with

\[ B_z = \frac{\partial V}{\partial x} + i \frac{\partial V}{\partial y} \]  

The complex potential \( \Omega \) is defined in terms of \( V \) and the conjugate function of \( V \), such that \( \Omega \) is analytic and satisfies the Cauchy-Riemann equations. Specifically,

\[ \Omega(z) = V(z) + i\psi(z) \]

with \( \psi \) the conjugate function of \( V \). Then the Cauchy-Riemann conditions are satisfied, with

\[ \frac{\partial V}{\partial x} = \frac{\partial \psi}{\partial y} \]
\[ \frac{\partial V}{\partial y} = -\frac{\partial \psi}{\partial x} \]

The Cauchy-Riemann conditions may be used to show that

\[ B_z = \frac{d\Omega}{dz} \]
and hence the field in the transformed system \( w(z) \) may now easily be found from

\[
B_w = \frac{d\Omega}{dw} = \frac{d\Omega}{dz} \frac{dz}{dw} = B_z \frac{dz}{dw}
\]

(9)

V. THE VACUUM ENERGY

Working in terms of the complex magnetic field and the complex potential, we have

\[
\delta W_V = \int |B_w|^2 dw_x dw_y
\]

(10)

where the integral extends from the boundary that is parameterised by \( \alpha \) at \( w(z(\alpha)) \), to infinity. To evaluate the integral we use Eq. 9 so that \( |B_w|^2 = \left| \frac{d\Omega}{dz} \frac{dz}{dw} \right|^2 \) and we change coordinates back to the circular cross-section coordinates, with \( dw_x dw_y = \frac{\partial(x,y)}{\partial(w_x,w_y)} dxdy \) where \( \frac{\partial(x,y)}{\partial(w_x,w_y)} = \left| \frac{dw}{dz} \right|^2 \) because \( w(z) \) is an analytic function. So when we change into the \( z \) coordinates (for the purpose of evaluating the integral Eq. 10), the factors of \( \left| \frac{dw}{dz} \right|^2 \) and \( \left| \frac{dz}{dw} \right|^2 \) cancel to give

\[
\delta W_V = \int |B_z|^2 dxdy
\]

(11)

In a similar way it may be shown that \( \oint_{l_w} B_w \cdot dl_w = \oint_{l_z} B_z \cdot dl_z \), reflecting the fact that the same total current is contained within \( l_z \) and \( l_w \).

Hence the vacuum energy may be found in terms of the vacuum energy for a solution with a circular boundary, which is much easier to calculate. The actual values of \( \delta W_V \) and \( B_z \) are determined by the plasma-vacuum boundary conditions, that will be modified by the transformation. The mapping of the boundary condition and the resulting boundary condition are obtained in Section VII, but for the present we will obtain the general solution in terms of the Fourier coefficients that the boundary conditions will determine.

The vacuum field has \( \nabla \land \vec{B}_V = \nabla . \vec{B}_V = 0 \), so we may write \( \vec{B}_V = \nabla V \) with \( \nabla^2 V = 0 \). We will start from a co-ordinate system with a circular cross section toroidal geometry, then subsequently obtain a 2D problem by taking the large aspect ratio limit. In this co-ordinate system

\[
\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \alpha^2} + \frac{1}{R^2} \frac{\partial^2 V}{\partial \phi^2}
\]

(12)

with \( r, \alpha, \phi \) the radial coordinate, poloidal and toroidal angle respectively. Writing

\[
V = \sum_{p=\infty}^{p=\infty} e^{ip\alpha - in\phi} V_p(r)
\]

(13)
and then projecting out the Fourier components, requires
\[ 0 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial V_p}{\partial r} - p^2 V_p(r) - n^2 \left( \frac{r}{R} \right)^2 V_p(r) \]  
which for any given (finite) \( n \) becomes 2-dimensional when \( r/R \to 0 \), leaving
\[ 0 = r^2 \frac{\partial^2 V_p}{\partial r^2} + r \frac{\partial V_p}{\partial r} - p^2 V_p \]  
that has solutions that tend to zero as \( r \to \infty \), of \( V_p = a_p \left( \frac{a}{r} \right)^{|p|} \), where the \( a_p \) will be determined by the boundary conditions, and \( r_a \) denotes the radial position of the plasma-vacuum surface. Note that it is only because the large aspect ratio limit makes the problem 2-dimensional, that we are able to use conformal transformations in the calculation.

The vacuum energy is
\[ \delta W_V = \frac{1}{2} \int \left| B_1^V \right|^2 d\tilde{r} \]  
Into which we now substitute \( V \), with
\[ V = \sum_{p=-\infty}^{p=\infty} e^{i p \alpha - i n \phi} a_p \left( \frac{r_a}{r} \right)^{|p|} \]  
and integrate with respect to \( \phi \) and \( \alpha \), with \( r = r_a \) at the surface, to get
\[ \delta W_V = 2 \pi^2 R \sum_{p \neq 0} |p| |a_p|^2 \]  
Here and in the remainder of this article \( R \) will be taken as a typical measure of the major radius that is approximately constant and independent of the poloidal angle. \( R \) is identical to the \( R_0 \) of the first part to this paper, but because the rest of this paper considers a large aspect ratio limit, we will simply write \( R \) as opposed to \( R_0 \).

VI. BOUNDARY CONDITIONS

The boundary conditions in the \( w \)-plane determine \( \vec{n} \cdot \vec{B} \) in terms of the plasma perturbation. We will write \( \vec{n} \cdot \vec{B} \) in the \( w \)-plane as \( n_w B_w \). However, to obtain the co-efficients \( a_p \) we need to know \( \vec{n} \cdot \vec{B} \) in the \( z \)-plane (that we write as \( n_z B_z \)). Therefore we need to know how \( \vec{n} \cdot \vec{B} \) is transformed as we map between the \( z \)-plane where the boundary is circular with \( z(\alpha) = -a + (a+l)e^{i\alpha} \), and the \( w \)-plane whose boundary is shaped and contains an X-point. This is calculated next.
Recall that the real and imaginary components are considered as orthogonal vector components. Then the tangent vector \( t_w(\alpha) \) of the surface traced by \( w(z(\alpha)) \) is simply given by
\[
t_w(\alpha) = \frac{\partial w(z(\alpha))}{\partial \alpha}.
\]
(19)

However, using the fact that \( w(z) \) is an analytic function, so that \( \frac{\partial w}{\partial \alpha} = \frac{dw}{dz} \frac{\partial z}{\partial \alpha} \), then
\[
t_w(\alpha) = \frac{\frac{dw}{dz}}{\frac{\partial z}{\partial \alpha}} \frac{\partial z}{\partial \alpha} = \frac{dw}{dz} t_z
\]
(20)

where the tangent \( t_z \) of \( z(\alpha) \) in the \( z \)-plane is again simply given by \( t_z(\alpha) = \frac{\partial z}{\partial \alpha} / |\frac{\partial z}{\partial \alpha}| \). To obtain the unit normals we rotate the tangent vector by \(-\pi/2\), by simply multiplying by \(-i\). Hence
\[
n_w = -it_w = \left( \frac{w'(z)}{|w'(z)|} \right) (-it_z) = \frac{w'(z)}{|w'(z)|} n_z.
\]
(21)

We already know that \( B_w = B_z \frac{dz}{dw} \), so to obtain how \( n_w.B_w \) transforms we need to simplify
\[
n_w.B_w = \left( \frac{w'(z)}{|w'(z)|} n_z \right) \cdot \left( B_z \frac{dz}{dw} \right)
\]
(22)

where the dot product refers to the sum of the product of the real parts, plus the product of the imaginary parts (examples may be found in Appendix XV). We will use \( \frac{dz}{dw} = \frac{1}{\frac{dw}{dz}} = \frac{\frac{dz}{dw}}{\left| \frac{dz}{dw} \right|^2} \), and write \( n_z = e^{i\theta}z \), \( B_z = r_B e^{i\theta_B} \), and \( \frac{dw}{dz} = r_w e^{i\theta w'} \), to give
\[
n_w.B_w = \frac{w'(z)}{|w'(z)|} n_z B_z \frac{w'(z)}{|w'(z)|}^2
= \frac{w'}{w'} \left[ e^{i\theta w'} e^{i\theta z} e^{i\theta B} e^{i\theta w'} \right]
= \frac{w'}{w'} \left[ \cos(\theta w' + \theta z) + i \sin(\theta w' + \theta z) \right] \cdot \left[ \cos(\theta w' + \theta B) + i \sin(\theta w' + \theta B) \right]
= \frac{w'}{w'} \left[ \cos(\theta w' + \theta z) \cos(\theta w' + \theta B) + \sin(\theta w' + \theta z) \sin(\theta w' + \theta B) \right]
= \frac{w'}{w'} \cos(\theta w' - \theta B)
= \frac{w'}{w'} e^{i\theta} e^{i\theta B}
= \frac{w'}{w'} n_z B_z
= \frac{n_z B_z}{\left| \frac{dz}{dw} \right|}
\]
(23)

This calculation is repeated by an alternative method in Section IX.

Knowing how \( \vec{n} \vec{B} \) transforms between \( z \) and the \( w(z) \) plane, we now return to the plasma-vacuum boundary conditions. As shown in part (I), the plasma-vacuum boundary condition is
\[
\nabla \psi \vec{B} \bigg|_{\text{edge}} = \nabla \psi \vec{B}^V \bigg|_{\text{edge}}
\]
(24)
where "edge", refers to the equilibrium position of the surface. Because $\vec{B}_0.\nabla \xi = \nabla \psi.\vec{B}_1$, we therefore require that

$$\vec{B}_0.\nabla \xi \bigg|_{\text{edge}} = \nabla \psi.\vec{B}_1 \bigg|_{\text{edge}}$$

(25)

For a single Fourier mode in straight field line co-ordinates $\xi = \xi_m(\psi)e^{im\theta}$, with $\theta = \frac{1}{q} \int \chi d\chi'$, $q = \frac{1}{q} \oint \nu d\chi'$, $\nu = \frac{J_\chi}{R^2}$, $J_\chi$ the Jacobian of the orthogonal $\chi$, $\psi$, $\phi$ co-ordinate system, with $\psi$ the poloidal flux, $\chi$ the poloidal angle, and $\phi$ the toroidal angle, and $I(\psi)$ is the flux function for which $\vec{B} = I\nabla \phi + \nabla \phi \wedge \nabla \psi$. After taking derivatives $\vec{B}.\nabla \xi$ then gives

$$\nabla \psi.\vec{B}_1 = (im - inq) \frac{I}{qR^2} \xi_m e^{im\theta - in\phi}$$

(26)

This may alternately be written as

$$nw.B_w = im\Delta \frac{\xi_m}{RB_p} \frac{I}{qR^2} e^{im\theta - in\phi}$$

(27)

where $\Delta = \frac{m-nq}{m}$.

Now we transform into the $z$ coordinates, transforming both $nw.B_w = n_z.B_z/|w'(z)|$ and $B_p = \frac{B_{pz}}{|w'(z)|}$, to get

$$n_z.B_z = im\Delta \frac{\xi_m}{R} \frac{|w'(z)|^2}{|B_{pz}|} \frac{I}{qR^2} e^{im\theta - in\phi}$$

(28)

Using Eq. 13 along with $V_k = a_k \left( \frac{\xi_m}{R} \right)^{|k|}$, and that $n_z.B_z = \vec{e}_r.\nabla V$, then gives

$$\sum_{k=-\infty}^{\infty} a_k \frac{-|k|}{r_a} e^{ika} = im\Delta \frac{\xi_m}{R} \frac{|w'(z)|^2}{|B_{pz}|} \frac{I}{qR^2} e^{im\theta}$$

(29)

From which the Fourier coefficients are easily obtained by multiplying by $\frac{e^{-ipa}}{2\pi}$ and integrating from $\alpha = -\pi$ to $\pi$, to give

$$a_p = -\left( \frac{\Delta}{|p|} \right) \frac{\xi_m}{R} \frac{1}{2\pi} \int im \frac{|w'(z)|^2}{|B_{pz}|} \frac{I r_a}{qR^2} e^{im\theta - ipa} d\alpha$$

(30)

In the following section we will see that

$$\theta(\alpha) = \frac{Ir_a}{qR^2} \int \alpha \frac{|w'(z)|^2}{|B_{pz}|} d\alpha$$

(31)

which will allow us to integrate by parts once to get

$$a_p = -\left( \frac{ip}{|p|} \right) \frac{\xi_m}{R} \frac{1}{2\pi} \int e^{im\theta - ipa} d\alpha$$

(32)

To evaluate the coefficients $a_p$, we will need an expression for $\theta(\alpha)$, this is addressed in the following 2 Sections.
VII. THE STRAIGHT FIELD-LINE ANGLE

In the absence of equilibrium skin currents, the plasma’s equilibrium field \( \vec{B}_0 \) equals the vacuum’s equilibrium field \( \vec{B}_0^V \) at the surface between the plasma and the vacuum, with \( \vec{B}_{0|\text{edge}} = \vec{B}_{0}^V|_{\text{edge}} \). Therefore provided that we know the equilibrium vacuum field at the surface, then we also know the plasma’s field at the surface. Consequently, if we know the vacuum field at the surface, then it is possible to calculate the straight field-line variable at the surface. Firstly we note that

\[
\theta = \frac{1}{q} \int \nu d\chi = \frac{I}{qR^2} \int \frac{J_X B_p d\chi}{B_p} = \frac{I}{qR^2} \int^l \frac{dl}{B_p} \tag{33}
\]

An element of arc length parallel to the tangent vector, \( dl_w \) has

\[
dl_w = \frac{\partial w}{\partial \alpha} d\alpha = \frac{dw}{d\alpha} \frac{dz}{d\alpha} d\alpha = \frac{dw}{dz} dl_z \tag{34}
\]

Hence an element of arc length \( |dl_w| \) transforms such that \( |dl_w| = |\frac{dw}{dz}| |dl_z| = |\frac{dw}{dz}| r_a d\alpha \), for a circular cross section of radius \( r_a \) in the \( z \)-plane. Using this plus \( |B_w| = |B_z|/|w'(z)| \), we may write Eq. 33 as

\[
\theta(\alpha) = \frac{I}{qR^2} \int^{l_w} \frac{|dl_w|}{|B_{pw}|} = \frac{Ir_a}{qR^2} \int^\alpha \left| \frac{dw}{dz} \right|^2 \frac{d\alpha}{|B_{pz}|} \tag{35}
\]

Hence if we know the equilibrium field, then we can obtain an analytical expression for the straight field-line coordinate as a function of \( \alpha \) in the \( z \)-plane.

VIII. EQUILIBRIUM VACUUM FIELD

The equilibrium vacuum field must (i) have a potential that satisfies Laplace’s equation in the vacuum region, (ii) have \( \vec{n}.\vec{B}_0 = 0 \) at the plasma-vacuum boundary (including at the strongly shaped X-point containing equilibrium), and (iii) have the field \( B_0 = 0 \) at the X-point. The first part is most easily satisfied - we can take a solution that satisfies Laplace’s equation and \( n_z.B_z = 0 \) for a circular cross section, and after a conformal transformation to a shaped cross-section we will still have \( n_w.B_w = 0 \) and a potential that satisfies Laplace’s equation. To obtain a field with \( B_p = 0 \) at the X-point, we follow a procedure that is equivalent to that when applying the Kutta condition to obtain the flow around an airplane wing (using a conformal transformation). Essentially, in the \( z \)-plane we combine a homogeneous
horizontal field and a circulating field, such that the field becomes zero at a single point on the circular boundary. This is physically equivalent to imposing an external horizontal field, and then driving a current through the plasma. Mathematically it corresponds to taking a complex potential in the $z$-plane of

$$\Omega = B_{p0} \left\{ i(z + a) - i \frac{(a + l)^2}{(z + a)} - 2i(a + l) \ln(z + a) \right\}$$

with a boundary at

$$z = -a + (a + l)e^{i\alpha}$$

with $\alpha \in [0, 2\pi]$, $B_{p0}$ a dimensional constant, and the radius of the circular boundary $r_a = (a + l)$. The sign of $B_{p0}$ determines the direction of the circulation of $B_{pz}$, clockwise ($B_{p0} < 0$) or anticlockwise ($B_{p0} > 0$), and the field $B_{pz}$ is obtained from

$$B_{pz} = \frac{d\Omega}{dz} = i\frac{(z - l)^2}{(z + a)^2} B_{p0}$$

which at the separatrix given by Eq. 37 gives $|B_{pz}| = 2B_{p0}(1 - \cos(\alpha))$. To consider an outermost flux surface that is just inside the separatrix, we may instead consider

$$z = -a + (a + l - \epsilon)e^{i\alpha}$$

with $\epsilon \ll l$. Then for $\epsilon \ll l \ll a$ we have

$$|B_{pz}| \simeq 2B_{p0} \left( 1 - \cos(\alpha) + \frac{\epsilon^2}{2a^2} \right)$$

so instead of $B_{pz} = 0$ at the X-point (that is located at $\epsilon = 0$ and $\alpha = 0$), we have $B_{pz} = \epsilon^2/a^2$. Notice that we have retained the singular perturbation in $\epsilon$ (singular in that although formally $\epsilon^2/a^2 \ll \epsilon/a$, it is the term in $\epsilon^2/a^2$ that qualitatively alters $|B_{pz}|$ by preventing it from being zero), further details are given in Appendix XIV. The field in the transformed space is given by $B_{pw} = B_{pz} \frac{dz}{dw}$, although we shall not need this here. Plots of the equilibrium are given in Figure 1.

As we approach the separatrix the behaviour of $\theta(\alpha)$ is dominated by the zeros in $w'(z)$ and $B_{pz}$ that occur near the X-point. Near the X-point it may be shown (in appendix XIV) that for the case of $n = 3/2$ with field lines crossing perpendicularly to each other, that $|w'(z)|^2$ is given by

$$|w'(z)|^2 \simeq \frac{1}{\sqrt{2}} \left( \frac{3}{2} \right)^4 \frac{a}{l} \sqrt{1 - \cos(\alpha) + \frac{\epsilon^2}{2a^2}}$$

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FIG. 1: The figure shows contour plots of the imaginary part of the complex potential Eq. 36 for the magnetic field, with \( a = l = 1 \). Such plots give streamlines of the magnetic field. The plot on the left is equivalent to a combination of a vertical field and that produced by a current through \( z = i \), with a field strength such that \( B_z = 0 \) at the bottom of the plasma vacuum surface. The plot on the right is obtained from \( \Omega(z) \) by a conformal transformation, writing \( \Omega(z(w)) \) and calculating plots of \( \Omega(z(w)) \) in the \( w \)-plane. \( z(w) \) is obtained from Eq. 1 with \( n = 3/2 \) and \( l = 1 \), that gives:

\[
(z(w)) = \left(\frac{w + 3/2}{2} + \frac{(w-3/2)^{2/3}}{2}\right)(w-3/2)^{2/3} - \left(\frac{w + 3/2}{2} - \frac{(w-3/2)^{2/3}}{2}\right).\]

For the plots the domain of \( z \) and \( w \) were rotated by substituting \( iz \) and \( iw \) respectively, so that the X-point is seen at the bottom of the figure.

We may use Eqs. 40 and 41 to calculate \( q \), with

\[
q = \frac{1}{2\pi} \oint \nu \, d\chi = \frac{1}{2\pi} \int \frac{I r_a}{R^2} \int \frac{|w'(\alpha)|^2}{|B_{pz}|} \, d\alpha = \frac{1}{2\pi} \frac{I r_a}{R^2 B_{p\theta}} \frac{1}{\sqrt{c_a}} \int \frac{d\alpha}{\sqrt{1 - \cos \alpha + \epsilon^2/2a}}
\]

and \( \sqrt{c_a} = 2\sqrt{2} \left(\frac{2}{3}\right)^4 \left(\frac{l}{a}\right) \). Similarly for \( \theta(\alpha) \) we have from Eqs. 35, 40 and 41 that

\[
\theta(\alpha) = \frac{1}{q} \frac{I r_a}{R^2 B_{p\theta}} \frac{1}{\sqrt{c_a}} \int_{-\pi}^{\alpha} \frac{d\alpha}{\sqrt{1 - \cos \alpha + \epsilon^2/2a}}
\]

Because the integral is dominated by the divergence at the X-point where \( \alpha = 0 \), \( q \) may be approximated by

\[
q \approx \frac{1}{2\pi} \frac{I r_a}{R^2 B_{p\theta}} \frac{1}{\sqrt{c_a}} \oint \frac{d\alpha}{\sqrt{a^2 + \frac{\epsilon^2}{2a}}} \approx \frac{c_t \sqrt{2}}{2\pi} \ln \left(\frac{2\alpha}{\epsilon}\right)
\]

with \( c_t \equiv \frac{I r_a}{R^2 B_{p\theta}} \frac{1}{\sqrt{c_a}} \), and similarly

\[
\theta(\alpha) \approx \frac{c_t}{q} \int_{-\pi}^{\alpha} \frac{d\alpha}{\sqrt{a^2 + \frac{\epsilon^2}{2a}}} = \frac{c_t \sqrt{2}}{q} \ln \left(\frac{a_\alpha}{\epsilon} + \sqrt{\frac{a_\alpha^2}{\epsilon^2} + 1}\right)
\]

\[
-\frac{a_\alpha}{\epsilon} + \sqrt{\frac{a_\alpha^2}{\epsilon^2} + 1}\right)
\]

(45)
Therefore as we approach the separatrix, with $\epsilon \to 0$, $q$ has a logarithmic divergence with $q \sim -\ln(\epsilon)$ that is typical for a Tokamak plasma near the separatrix. Hence the qualitative features of $\theta(\alpha)$ could have reasonably been postulated without showing that they also arise from a vacuum field whose potential satisfies Laplace’s equation, has $n_w B_{pw} = 0$ on the separatrix, and with $B_{pz} \to 0$ at the X-point; but it is reassuring to know that this is also the case.

It is interesting to note that in this model for the equilibrium field, the angle at which the field lines meet at the X-point determines how strongly the divergence is there. For example, if instead of meeting at $\pi/2$ the lines make a cusp (tending to parallel as they meet), then $q$ is finite (A cusp is obtained by taking $n = 2$ in the Karman-Trefftz transformation, Eq. 1).

IX. MORE TRANSFORMED QUANTITIES

Now we start to return to our problem of calculating $\Delta'$, by firstly calculating $\delta W_V$ from its surface integral representation that is given in part (I), with

$$\delta W_V = \pi \oint J_x d\chi \left( \frac{i}{n} \frac{\nabla \psi B_1^*}{B^2} \left[ R^2 B_p^2 \frac{i}{n} \frac{\partial}{\partial \psi} \left( \nabla \psi \cdot \vec{B}_1 \right) \right] \right)$$

where $\int$ is over the plasma surface. This requires us to know how $\vec{n} \cdot \nabla$ transforms. The calculation is done partly to reassure us that the transformed quantities are correct, but also because it is a simple step to subsequently obtain $\Delta'$.

First we calculate how $\nabla w$ transforms. We have

$$\nabla z f(z) = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f(z)$$

and

$$\nabla w f(z(w)) = \left( \frac{\partial}{\partial w_x} + i \frac{\partial}{\partial w_y} \right) f(z(w))$$

where $z = x + iy$ and $w = w_x + iw_y$. Now we will use the chain rule to expand $\nabla z f(z(w))$, noting that because $w(z)$ is an analytic function it satisfies the Cauchy-Riemann equations,

$$\frac{\partial w_x}{\partial x} = \frac{\partial w_y}{\partial y}$$

$$\frac{\partial w_x}{\partial y} = -\frac{\partial w_y}{\partial x}$$
and in addition, $\frac{\partial w}{\partial x} = \frac{dw}{dz}$. This gives,

$$\nabla_z f(z(w)) = \frac{\partial f}{\partial w_x} \frac{\partial w_x}{\partial x} + \frac{\partial f}{\partial w_y} \frac{\partial w_y}{\partial x} + i \frac{\partial f}{\partial w_x} \frac{\partial w_x}{\partial y} + i \frac{\partial f}{\partial w_y} \frac{\partial w_y}{\partial y}$$

$$= \frac{\partial w_x}{\partial x} \left( \frac{\partial f}{\partial w_x} + i \frac{\partial f}{\partial w_y} \right) + \frac{\partial w_y}{\partial x} \left( -i \frac{\partial f}{\partial w_x} + \frac{\partial f}{\partial w_y} \right)$$

$$= \left( \frac{\partial}{\partial x} (w_x - iw_y) \right) \left( \frac{\partial f}{\partial w_x} + i \frac{\partial f}{\partial w_y} \right)$$

$$= \frac{\partial w_x}{\partial x} \nabla^w f = \frac{dw}{dz} \nabla^w f$$

Hence we have $\nabla_w = \frac{dw}{dz} \nabla_z$.

Now we consider the transformation of $\vec{n} \cdot \nabla$. In calculating how more complicated expressions transform, the author has found the following identities useful, whose derivations are given in Appendix XV.

$$a.b.c = \bar{a} . b . \bar{c}$$

$$ab.cd = (a.c)(b.d) + (ia.c)(b.id)$$

For example, to calculate $n_w . B_w$ we use Eq. 52 along with the results of Section VI to give

$$\vec{n} . B = n_w . B_w = \left( \frac{w'(z)}{|w'(z)|} \right) n_z \left( \frac{dw}{dz} \right) B_z$$

$$= \frac{1}{|w'(z)|} \left( n_z B_z \right) \left( \frac{dw}{dz} \frac{dz}{dw} \right) + \left( i n_z B_z \right) \left( \frac{dw}{dz} \frac{dz}{dw} \right)$$

as before. In the last step we used,

$$\frac{dw}{dz} = \alpha + i\beta$$

$$\frac{dz}{dw} = 1$$

$$\frac{dz}{dw} i \frac{dz}{dw} = 0$$

that may easily be confirmed by writing $\frac{dw}{dz} = \alpha + i\beta$, so that $\frac{dz}{dw} = 1/(\alpha + i\beta)$, and multiplying out. A similar calculation for $\vec{n} \cdot \nabla$ gives

$$\vec{n} \cdot \nabla = n_w \cdot \nabla_w = \frac{w'(z)}{|w'(z)|} n_z \left( \frac{dw}{dz} \right) \nabla_z$$

$$= \frac{1}{|w'(z)|} \left( \frac{dz}{dw} \frac{dw}{dz} \right) (n_z \nabla_z) + \left( i \frac{dz}{dw} \frac{dw}{dz} \right) (n_z i \nabla_z)$$

X. RECALCULATING $\delta W_V$

Firstly we re-express Eq. 46 using our transformed quantities, then we show this gives us the same result, Eq. 18 from Section V, before showing how the calculation easily generalises to give us $\Delta'$ in terms of $\delta W_V$. 

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Using
\[ \nabla \psi \cdot \vec{B}_1 \bigg|_{\text{edge}} = \nabla \psi \cdot \vec{B} \bigg|_{\text{edge}} = \vec{B} \cdot \nabla \xi \bigg|_{\text{edge}} \]
(56)
gives us
\[ \nabla \psi \cdot \vec{B}_1 \bigg|_{\text{edge}} = -\xi^*_{m}(\psi_a) \left( \frac{I}{qR^2} \right) (im) \left( \frac{m-nq}{m} \right) e^{-im\theta + in\phi} \]
(57)
This is substituted into Eq. 46 to give
\[ \delta W_V = \pi \Delta \int J_\chi d\chi \left( \frac{m}{nq} \right) \left( \frac{I}{R^2 B^2} \right) \xi^* m e^{-im\theta + in\phi} \left[ R^2 B^2 qB \frac{i}{\partial \psi} \Delta \int J \chi d\chi \right] \]
(58)
where \( \Delta = \frac{m-nq}{m} \). Then using \( \frac{\partial}{\partial \psi} = \vec{n} \cdot \nabla \), we get
\[ \delta W_V = \pi \Delta \left( \frac{m}{nq} \right) \frac{\xi^* m}{R} \int d\alpha \left( \frac{I}{B^2} \right) e^{-im\theta + in\phi} \vec{n} \cdot \nabla \left( R B \vec{n} \cdot \vec{B}_1 \right) \]
(59)
where we used \( d\alpha = J_\chi B_p d\alpha \). Now we will transform this equation into coordinates in which the plasma has a circular cross-section, using:
\[ dl_w = |w'(z)| dl_z = |w'(z)| r_\alpha d\alpha \]
(60)
\[ |B_{pw}| = |B_{pw}|/|w'(z)| \]
\[ n_w \cdot B_w = n_w B_w/|w'(z)| \]
\[ n_w \cdot \nabla_w = n_w \nabla_w/|w'(z)| \]
After using the chain rule to expand the term in \( n_w \cdot \nabla_w \), we have
\[ \delta W_V = \pi \Delta \left( \frac{m}{nq} \right) \xi^* m \int d\alpha \left( \frac{r_\alpha}{B_\psi} \right) e^{-im\theta(\alpha) + in\phi} \]
\[ \frac{i}{n} \left[ \frac{|B_{pw}|}{|w'(z)|^2} n_z \cdot \nabla_z (n_z \cdot B_z) + n_z B_z n_z \cdot \nabla_z \left( \frac{|B_{pw}|}{|w'(z)|^2} \right) \right] \]
(61)
The \( n_z \cdot \nabla_z \) operator acting on \( n_z \cdot B_z \) will produce a term of order \( n \) larger than \( n_z \cdot B_z \). Thus usually we would neglect the second term. However here we need to be careful that there are no geometrically driven divergences, this is done in Appendix[XVI] where it is confirmed that the term is of order \( \frac{1}{n} \) smaller and may be neglected. Hence if we retain only the leading order term in \( n \), then rearranging the expression slightly we have,
\[ \delta W_V = \pi \Delta \left( \frac{m}{nq} \right) ^2 \xi^* m \int d\alpha e^{-im\theta(\alpha) + in\phi} \frac{r_\alpha}{B_\psi} \frac{i q}{m} \frac{|B_{pw}|}{|w'(z)|^2} n_z \cdot \nabla_z (n_z \cdot B_z) \]
(62)
After comparison with Eq. 35 for \( \theta(\alpha) \), this may be written as
\[ \delta W_V = \pi \Delta \left( \frac{m}{nq} \right) ^2 \xi^* m \left( \frac{r_\alpha^2 I^2}{R^2 B^2} \right) \int d\alpha e^{-im\theta(\alpha) + in\phi} n_z \cdot \nabla_z (n_z \cdot B_z) \]
(63)
Using Eq. 17
\[ n_z \nabla_z (n_z B_z) = \frac{\partial^2 V}{\partial r^2} \bigg|_{r=r_a} = \sum_{p \neq 0} e^{ip\alpha} |p| \left( \frac{|p| + 1}{r_a^2} \right) \] (64)

Giving
\[ \delta W_V = \pi \Delta \left( \frac{m}{n q} \right)^2 \xi^*_m \left( \frac{r_a^2 I^2}{R^2 B^2} \right) \sum_{p \neq 0} a_p |p| \left( \frac{|p| + 1}{r_a^2} \right) \oint d\alpha \frac{-e^{-im\theta(\alpha)+ip\alpha}}{im\theta'(\alpha)} \] (65)

Next we observe that
\[ \oint d\alpha \frac{1}{m\theta'(\alpha)} e^{ip\alpha-im\theta(\alpha)} = \oint d\alpha \frac{1}{ip} e^{ip\alpha-im\theta(\alpha)} + O \left( \frac{(\epsilon)}{n} \right) \] (66)

that we will justify below, and that appears to be the key result linking the high \( n \) and \( q \) calculations at arbitrary cross-section, to the circular cross section result. Integrating by parts we get
\[ \oint d\alpha e^{ip\alpha-im\theta(\alpha)} = \oint d\alpha \frac{\theta''(\alpha)}{ip} e^{ip\alpha-im\theta(\alpha)} + \frac{1}{ip} \oint d\alpha \frac{\theta''(\alpha)}{m\theta'(\alpha)} e^{ip\alpha-im\theta(\alpha)} \] (67)

To estimate the second term we notice that \( |e^{ip\alpha-im\theta(\alpha)}| \leq 1 \). Then taking \( \theta(\alpha) \) as given by Eq. 44 then we find
\[ \left| \oint \frac{\theta''(\alpha)}{m\theta'(\alpha)} e^{ip\alpha-im\theta(\alpha)} \right| \lesssim \left| \oint a \frac{\frac{\dot{\xi}}{\xi}}{\sqrt{\alpha^2 + \frac{\epsilon^2}{\alpha}}} d\alpha \right| = \frac{q}{mc} \left| \alpha \right| \left| \ln \left( \frac{2\pi^2 \alpha}{\epsilon} \right) \right| \] (68)

where the integral is easily obtained by substituting \( \frac{\alpha}{\epsilon} = \text{Sinh}(u) \). Hence for \( \epsilon/a \sim 1 \) the term is of order \( 1/n \) and may be neglected, and as \( \epsilon/a \to 0 \) the term also tends to zero, and hence may be neglected. Thus in the high-\( n \) limit we get
\[ \delta W_V = \pi \Delta \left( \frac{m}{n q} \right)^2 \xi^*_m \left( \frac{r_a^2 I^2}{R^2 B^2} \right) \sum_{p \neq 0} a_p |p| \left( \frac{|p| + 1}{r_a^2} \right) \oint d\alpha e^{-im\theta(\alpha)+ip\alpha} \] (69)

Taking the complex conjugate of Eq. 32 and rearranging, we get
\[ \oint d\alpha e^{-im\theta(\alpha)+ip\alpha} = ia_p |p| \frac{R}{p} \frac{2\pi}{\Delta} \] (70)

which upon substitution into Eq. 69 gives
\[ \delta W_V = 2\pi^2 R \left( \frac{m}{n q} \right)^2 \left( \frac{I^2}{R^2 B^2} \right) \sum_{p \neq 0} (|p| + 1) |a_p|^2 \] (71)
For the high $m,n$ limit considered here, we expect the $|a_p|^2$ coefficients to be largest for $p \sim m \sim nq$, and hence in the high $m,n$ limit we expect

$$\delta W_V = 2\pi^2 R \sum_{p \neq 0} |p| |a_p|^2$$

(72)

where we have also taken $I^2/(RB^2) \simeq 1$. Hence we have re-obtained Eq. [18] from the high-$n$ expression given in part (I). This gives us confidence in the reliability of the calculations.

In addition $\left[ |n_z \cdot \nabla_z (n_z \cdot B_z) | \right]$ may be estimated by approximating the plasma as a vacuum and solving Laplace’s equation to approximate and obtain the perturbed field both inside and outside the plasma respectively, and correctly matching the fields at the plasma-vacuum boundary. Then we find that $\left[ |n_z \cdot \nabla_z (n_z \cdot B_z) | \right] = 2n_z \cdot \nabla_z (n_z \cdot B_z)$. Hence the above calculation may be used to infer that the term $-\pi |\xi_m|^2 \Delta^2 \Delta'$ appearing in Eq. 70 of the first part to this paper, is equal to $2\delta W_V$, where

$$\Delta' = \left[ \frac{1}{2\pi} \int dlRB_p \frac{I^2}{R^2 B^2} \frac{\partial \psi}{\Delta \nabla \psi} \frac{\psi \nabla \cdot \vec{B}_1}{\nabla \psi} \right]$$

(73)

Hence to evaluate $\Delta'$, we need solely evaluate $\delta W_V$.

**XI. DIRECTLY CALCULATING $\Delta'$**

Here we show how to calculate $\Delta'$ directly, using the same assumptions as in Section [X]. For simplicity in all that follows we will take $I^2/R^2 B^2 \simeq 1$, and firstly use Eq. 24, that

$$\nabla \psi \cdot \vec{B}_1 \bigg|_{edge} = \nabla \psi \cdot \vec{B}_1 \bigg|_{edge}$$

to write

$$\Delta' = \left[ \frac{1}{2\pi} \int dlRB_p \frac{I^2}{R^2 B^2} \frac{\partial \psi}{\Delta \nabla \psi} \frac{\psi \nabla \cdot \vec{B}_1}{\nabla \psi} \right] = \frac{1}{2\pi} \int dlRB_p \left[ \frac{\partial \psi}{\Delta \nabla \psi} \frac{\psi \nabla \cdot \vec{B}_1}{\nabla \psi} \right]$$

(74)

Next we use Eqs. 60 to obtain

$$\Delta' = \frac{1}{2\pi} \int r_a da \frac{1}{n_z \cdot B_z} \left[ |n_z \cdot \nabla_z (n_z \cdot B_z) | \right]$$

(75)

Making the usual approximation that treats the perturbed field near the plasma’s edge as behaving the same as in a vacuum, and also using Eq. 17 gives

$$\left[ |n_z \cdot \nabla_z (n_z \cdot B_z) | \right] = 2\left. \frac{\partial^2 V}{\partial r^2} \right|_{r=r_a} = 2 \sum_{p \neq 0} e^{i \alpha - i \omega |a_p| \frac{r^2}{r_a^2} |p| (|p| + 1)$$

(76)

\[\text{17}\]
where we used Eq. 35 that implies \( \theta'(\alpha) = \frac{1}{\xi m \Delta} |\omega'|^2 |B_{ps}| \), Eq. 30, and that \( \theta'(\alpha) e^{im\theta(\alpha)} = \sum_{p \neq 0} e^{ip\alpha} \frac{1}{2\pi} \oint d\beta e^{im\theta(\beta) - ip\beta} d\beta \). Note that the above equation can be obtained more directly from Eq. 28.

Using the above results, after some cancellations we obtain

\[
\Delta' = 2 \frac{2}{2\pi} \oint d\alpha \frac{R}{\xi m \Delta} e^{-im\theta(\alpha)} \sum_{p \neq 0} e^{ip\alpha} a_p |p| (|p| + 1)
= \frac{1}{\pi} \frac{R}{\xi m \Delta} \sum_{p \neq 0} a_p |p|^2 \oint d\alpha e^{-im\theta(\alpha) + ip\alpha} + \frac{1}{\pi} \frac{R}{\xi m \Delta} \sum_{p \neq 0} a_p |p| \oint d\alpha e^{-im\theta(\alpha) + ip\alpha}
\]

Then using the result Eq. 66 of Section X, that

\[
\oint d\alpha e^{-im\theta(\alpha)} = \frac{1}{ip} \oint d\alpha e^{-im\theta + ip\alpha} + O \left( \frac{\ln(\frac{1}{n})}{n} \right)
\]

we obtain the first term as

\[
\frac{R}{\pi \xi m \Delta} \sum_{p \neq 0} a_p |p|^2 \oint d\alpha e^{-im\theta(\alpha) + ip\alpha} = \frac{2}{\xi m^2 \Delta^2} \sum_{p \neq 0} a_p |p| (\frac{-ip}{|p|}) \oint \frac{\xi m}{R} \frac{1}{2\pi} \oint e^{ip\alpha - im\theta}
= \frac{1}{\xi m^2 \Delta^2} (-1) \sum_{p \neq 0} |p| |a_p|^2
\]

Using Eq. 32 for \( a_p \), and the same approximations as above, the second term gives

\[
\frac{R}{\pi \xi m \Delta} \sum_{p \neq 0} |p| a_p \oint d\alpha e^{-im\theta(\alpha) + ip\alpha} = \frac{R}{\xi m \Delta} \sum_{p \neq 0} |p| a_p \oint e^{-im\theta + ip\alpha} + O \left( \frac{1}{n^2} \right)
= -\frac{1}{\pi} \sum_{p \neq 0} \oint e^{im\theta(\beta) - ip\beta} \frac{1}{2\pi} \oint e^{im\theta(\alpha) - ip\alpha} d\alpha
= -\frac{1}{\pi} \oint d\alpha e^{-im\theta(\alpha)} \sum_{p \neq 0} e^{ip\alpha} \frac{1}{2\pi} \oint e^{im\theta(\beta) - ip\beta} d\beta
= -\frac{1}{\pi} \oint e^{-im\theta(\alpha)} e^{im\theta(\beta) - ip\beta} d\alpha = -2
\]

Hence using all of the above, and Eq. 18 for \( \delta W_V \),

\[
\Delta' = -2 \left\{ \frac{R^2}{\xi m^2 \Delta^2} \sum_{p \neq 0} |p||a_p|^2 + 1 \right\} = -2 \left( \frac{\delta W_V}{2\pi^2 \xi m^2 \frac{\Delta^2}{R^2}} \right) \left\{ 1 + O \left( \frac{1}{\delta W_V} \right) \right\}
\]

(Later we will find that \( \sum_{p \neq 0} |p||a_p|^2 = m^2 \xi m^2 \Delta^2 \) and hence that \( \Delta' \simeq -2m \).)

**XII. EVALUATING THE SUM**

We have found that the vacuum energy \( \delta W_V \) and \( \Delta' \) are both determined from \( \sum_{p=-\infty}^{\infty} |p||a_p|^2 \), that we may in principle evaluate using our analytical expression for \( a_p \). We do that here.
Firstly note that if \( |p| \) is replaced with \( p \) in Eq. 18, then we may easily resum the series, because

\[
\sum_{p=-\infty}^{\infty} p|a_p|^2 = \Delta^2 \left| \sum_{p=-\infty}^{\infty} \frac{\xi_m}{p + \frac{1}{2\pi}} \int d\alpha \ e^{i\alpha(p\delta - m\pi)} \right|^2 \sum_{p=-\infty}^{\infty} p e^{-i\alpha(p\delta - m\pi)} \\
= \frac{\Delta^2}{2\pi} \sum_{p=-\infty}^{\infty} \frac{\xi_m}{p + \frac{1}{2\pi}} \int d\alpha \ e^{-i\alpha(p\delta - m\pi)} \sum_{p=-\infty}^{\infty} p e^{i\alpha(p\delta - m\pi)} \\
= \frac{\Delta^2}{2\pi} \sum_{p=-\infty}^{\infty} \frac{\xi_m}{p + \frac{1}{2\pi}} \int d\alpha \ e^{-i\alpha(p\delta - m\pi)} \sum_{p=-\infty}^{\infty} p e^{i\alpha(p\delta - m\pi)} \\
= \frac{\Delta^2}{2\pi} \sum_{p=-\infty}^{\infty} \frac{\xi_m}{p + \frac{1}{2\pi}} \int d\alpha m\theta'(\alpha) e^{-i\alpha(p\delta - m\pi)} \\
= \frac{\Delta^2}{2\pi} \sum_{p=-\infty}^{\infty} \frac{\xi_m}{p + \frac{1}{2\pi}} \int d\alpha (p\delta - m\pi) e^{-i\alpha(p\delta - m\pi)} \\
= \frac{\Delta^2}{2\pi} \sum_{p=-\infty}^{\infty} \frac{\xi_m}{p + \frac{1}{2\pi}} \int d\alpha (p\delta - m\pi) e^{-i\alpha(p\delta - m\pi)} (83)
\]

where in going from lines 3 to 4 we integrated by parts, and in going from lines 4 to 5 we note that \( \theta'(\beta) e^{-i\alpha(p\delta - m\pi)} \sum_{p \neq 0} e^{i\alpha(p\delta - m\pi)} \int d\alpha \theta(\alpha) e^{-i\alpha(p\delta - m\pi)} d\alpha \). We might expect the values of the coefficients to be peaked for values of \( p \sim m \sim nq \gg 1 \), and so it is likely that \( \sum_{p=-\infty}^{\infty} |p||a_p|^2 \simeq \sum_{p=1}^{\infty} p|a_p|^2 \simeq \sum_{p=-\infty}^{\infty} p|a_p|^2 \). This has been confirmed by calculating the sums using a saddle point approximation. (The details of the calculation are too long to be included here)

It is instructive to recalculate the result using a simple model for \( \theta(\alpha) \), that encapsulates the fact that as we approach the separatrix (with \( q \to \infty \) and the local field line pitch \( \nu \) becoming increasingly peaked near the X-point), the function \( \theta(\alpha) \) becomes increasingly similar to a step function. In this simple model we take \( q \sim \frac{1}{\delta} \), \( \delta \ll 1 \), and \( \nu \sim q \sim \frac{1}{\delta} \) when \( \alpha \in (-\delta, \delta) \), this leads to a very simple model for \( \theta(\alpha) \), with

\[
\theta(\alpha) = \begin{cases} 
0 & \alpha \in (-\pi, -\delta) \\
\frac{(\alpha+\delta)}{2\pi} & \alpha \in (-\delta, \delta) \\
2\pi & \alpha \in (\delta, \pi)
\end{cases} \tag{84}
\]

So that

\[
e^{i\alpha(p\delta - m\pi)} = \begin{cases} 
1 & \alpha \in (-\pi, -\delta) \\
e^{i\alpha(p\delta - m\pi)} & \alpha \in (-\delta, \delta) \\
1 & \alpha \in (\delta, \pi)
\end{cases} \tag{85}
\]

Because \( \theta(\alpha) \) is piecewise linear, it is easy to evaluate \( \int d\alpha \ e^{i\alpha(p\delta - m\pi)} \), that gives

\[
ap = -i \frac{p}{|p|} \Delta \frac{\xi_m}{R} m \frac{\sin(p\delta)}{p\delta - m\pi} \tag{86}
\]

Hence

\[
\delta W_V = 2\pi^2 R \sum_{p \neq 0} \frac{|p||a_p|^2}{|p|}\Delta \frac{\xi_m}{R} m \frac{\sin^2(p\delta)}{|p|\sin(p\delta - m\pi)^2} \\
= 2\pi^2 R \sum_{p \neq 0} \frac{|\xi_m|^2}{|p|\sin^2(p\delta)} \Delta^2 m^2 \sum_{p \neq 0} \frac{\sin^2(p\delta)}{|p|\sin^2(p\delta - m\pi)^2} \\
\simeq 2\pi^2 R \sum_{p \neq 0} \frac{|\xi_m|^2}{|p|\sin^2(p\delta)} \Delta^2 m^2 \left( \int_1^\infty \frac{\sin^2(p\delta)}{|p|\sin^2(p\delta - m\pi)^2} dp + \int_1^\infty \frac{\sin^2(p\delta)}{|p|\sin^2(p\delta + m\pi)^2} dp \right) \\
\rightarrow 2\pi^2 R \sum_{p \neq 0} \frac{|\xi_m|^2}{|p|\sin^2(p\delta)} \Delta^2 m^2 \frac{1}{m} \text{ for } m \gg 1 \tag{87}
\]
the same as was obtained previously by the saddle point approximation. To obtain this result we used
\[ \sum_{p=1}^{\infty} \frac{\sin^2(p\delta)}{|p|(p\delta - m\pi)^2} \simeq \int_{1}^{\infty} \frac{\sin^2(p\delta)}{|p|(p\delta - m\pi)^2} dp = \frac{1}{m} + O\left(\frac{\ln(\delta)}{m^2}\right) = \frac{1}{m} + O\left(\frac{\delta \ln(\delta)}{m}\right) \] (88)
and
\[ \sum_{p=-\infty}^{-1} \frac{\sin^2(p\delta)}{|p|(p\delta - m\pi)^2} \simeq \int_{1}^{\infty} \frac{\sin^2(p\delta)}{|p|(p\delta + m\pi)^2} dp = O\left(\frac{\ln(\delta)}{m^2}\right) = O\left(\frac{\delta \ln(\delta)}{m}\right) \] (89)
where we also used \( m \sim nq \sim \frac{\delta}{\epsilon} \). Notice that the integrals do not diverge at \( p\delta - m\pi = 0 \), because \( \sin(p\delta) = \sin(p\delta - m\pi) = 0 \) for \( p\delta - m\pi = 0 \), this would not be the case if \( m \) were not an integer. Hence not only do we find agreement with the calculation using the saddle point approximation, but we again find that
\[ \sum_{p=\infty}^{\infty} |p||a_p|^2 \rightarrow \sum_{p=\infty}^{\infty} p|a_p|^2 \text{ as } m \sim nq \rightarrow \infty \] (90)
Therefore both methods suggest that provided \( m \gg 1 \) and \( n \gg 1 \), then
\[ \sum_{p=\infty}^{\infty} |p||a_p|^2 \rightarrow \sum_{p=\infty}^{\infty} p|a_p|^2 = \Delta^2 |\xi_m|^2 \frac{m}{R^2} \] (91)
In addition notice that for \( m \sim nq \gg 1 \) the result of neither approximation methods involve \( \delta \) (or \( \epsilon \)), suggesting that the result may be generic and independent of the detailed form of \( \theta(\alpha) \).
Returning to the calculation of \( \Delta' \), Sections \[ and \[ showed that at leading order
\[ \Delta' = -2 \left( \frac{\delta W_V}{2\pi^2 |\xi|^2 \frac{m}{R} \Delta^2} \right) \] (92)
and that using the above results gives
\[ \Delta' = -2m \] (93)
In addition, the work described above suggests that the result is generic for perturbations with \( n \gg 1 \), regardless of whether the plasma cross-section is circular, or shaped with a separatrix boundary that contains an X-point.
XIII. DISCUSSION

A. Scope & Purpose of the Calculation

In the first part to this paper we started from the simplest model used to study Peeling modes, that considers Peeling modes in a cylindrical plasma at marginal stability, then generalised it to a toroidal plasma. According to the model, the energy principle’s \( \delta W \) is determined by the value of \( \Delta' \), that is a normalised measure of the jump in the gradient of the normal component of the perturbed magnetic field. In this second part we have restricted ourselves to systems for which the vacuum magnetic field may be treated as being approximately two dimensional, as is the case for a sufficiently large aspect ratio Tokamak. This allows us to use a conformal transformation in our calculations, and at high toroidal mode number we have obtained analytic expressions for the vacuum energy and \( \Delta' \), whenever the plasma is perturbed by a radial displacement consisting of a Fourier mode in straight field line co-ordinates. These expressions remain valid for a plasma cross-section that approximates a separatrix with an X-point, and appear to be generic, independent of the exact form for \( \theta(\alpha) \). Because it is possible to do this analytically, there is the possibility of making similar analytic progress with other linear plasma instabilities whose plasma equilibria have a separatrix with an X-point. Such calculations can provide physical understanding and useful tests during the development of codes to study the stability of more general geometry Tokamak plasmas, either giving confidence in a code or indicating its limitations.

B. (In)Stability of the Ideal MHD Peeling Mode?

According to the model developed in part (I) and our calculation here of \( \Delta' = -2m \), we can now examine the model’s predictions. According to the model developed in part (I), for the trial function used by Laval et al.\[6\], stability is determined by the sign of

\[
\delta W = -2\pi^2 \frac{|\xi_m|^2}{R} \Delta \left[ \Delta' + \hat{J} \right]
\]

with

\[
\Delta = \frac{m - nq}{nq}
\]

\[
\hat{J} = \frac{1}{2\pi} \int dl \frac{I}{RB_p} \frac{\vec{J} \cdot \vec{B}}{B^2}
\]
\[ \hat{\Delta}' = \left[ \frac{1}{2\pi} \oint dl RB_p \frac{I^2}{R^2B^2} \frac{\partial}{\partial \psi} \mathbf{\nabla} \cdot \mathbf{B}_1 \right] \]  

(97)

and as mentioned in Section V, \( R = R_0 \) is constant for the large aspect ratio limit considered here. \( \delta W \) is minimised with respect to \( \Delta \) (or equivalently, a particular choice of toroidal mode number), finding \( \Delta = -\hat{J}/(2\Delta') \). Then using our result of \( \Delta' = -2m \) we get

\[ \delta W = -\left( \frac{\pi}{2} \right)^2 \frac{\xi_m^2}{R} \left( \frac{\hat{J}^2}{m} \right) \]  

(98)

When checking the dimensions of \( \delta W \) it is essential to remember that \( \xi_m = (\nabla \psi \cdot \mathbf{\xi})_m \sim r RB_p \), then because \( B^2 \sim p \) is an energy per unit volume, \( \delta W \sim r^2 RB_p^2 \) and hence has units of energy. For the process of minimisation \( n \) was treated as a continuous variable (that for \( m \sim nq \gg 1 \) is a reasonable approximation). Now we consider two cases in turn, firstly \( \nabla \phi \cdot \mathbf{J} = 0 \), for which

\[ \hat{J} = \frac{1}{2\pi} \oint dl \frac{I}{RB_p} \frac{\hat{J}B_p^2}{B^2} = \frac{1}{2\pi} \oint dl B_p \frac{-\hat{I}B_p^2}{RB^2} \sim 1 \]  

(99)

So that although \( \delta W < 0 \) for all \( m \), because \( m \sim nq \to \infty \) then \( \delta W \sim \frac{1}{m} \to 0 \). For the case of \( \nabla \phi \cdot \mathbf{J} \neq 0 \) however,

\[ \hat{J} = \frac{1}{2\pi} \oint dl \frac{I}{RB_p} \frac{\hat{J}B_p}{B^2} = \frac{1}{2\pi} \oint dl \frac{I}{B_p} \frac{R \hat{J}B_p}{B^2} \sim \left\langle R \frac{\hat{J}B_p}{B^2} \right\rangle = \frac{1}{2\pi} \oint dl \frac{I}{B_p} = q \left\langle \frac{R \hat{J}B_p}{B^2} \right\rangle \]  

(100)

where \( \langle \rangle \) denotes the poloidal average, and \( q = \frac{1}{2\pi} \oint \frac{dl}{RB_p} \) is the safety factor \( q \). The divergence in \( \nabla \phi \cdot \mathbf{J} \) at the X-point will mean that the poloidal location of the X-point will affect the value of \( \hat{J} \) (as a function of \( q \)), this is also the case for Mercier stability and is discussed by Webster \[5\]. Then using Eqs. [98] [100] and \( m \sim nq \) we have

\[ \delta W \sim -\left( \frac{\pi}{2} \right)^2 \frac{\xi_m^2 q}{R} \left\langle R \frac{\hat{J}B_p}{B^2} \right\rangle^2 < 0 \]  

(101)

Therefore if \( \delta W < 0 \) is taken to indicate instability, the result would indicate that the Peeling mode remains unstable near a separatrix. However as observed in the first part of this paper the growth rate \( \gamma \) has \( \gamma^2 = -\delta W / \int d\psi \rho_0 |\xi|^2 \), with \( \int d\psi \rho_0 |\xi|^2 \) diverging at a rate proportional to \( q'(\psi) \). This gives \( \ln(\gamma) = -\frac{1}{2} \ln(q'/q) \) for the limit of a separatrix with \( q \) and \( q' \) tending to infinity, so that although \( \delta W \) is non-zero and negative, the mode will be marginally stable. This is similar to the calculation of the Mercier coefficient by Webster \[5\],
where \( D_M \) is found from the ratio of two diverging quantities, with \( D_M \sim q/q' \to 0 \) as the separatrix is approached. Note that these results for \( \Delta' \) and the growth rate appear to be generic and independent of the detailed forms of \( \theta(\alpha) \) or the radial structure of the mode.

C. Is the Mode Physically Acceptable?

Because we require a poloidal mode number \( m \sim nq \), then near the separatrix where \( q \to \infty \) we also require \( m \to \infty \). This raises the question: Is the mode physically acceptable? To answer this we reconsider the trial function, that has \( \xi \sim e^{im\theta(\alpha)} \), with \( \theta(\alpha) = \frac{1}{q} \int_\alpha^\nu(\alpha) d\alpha \). When \( m \sim nq \) the trial function becomes \( \xi \sim e^{in\int_\alpha^\nu d\alpha} \), from which we can see that because \( \nu \) is of order one and well behaved everywhere except near the x-point (where it diverges to infinity), then so also is the mode’s structure. As discussed in part (I), the divergence in \( m \sim nq \to \infty \) is only manifested in close proximity to the x-point where the divergence in \( \nu \) causes the mode to oscillate increasingly rapidly as the x-point is approached. Elsewhere \( \nu \) is typically of order 1, and the mode structure is like that for a finite mode number, oscillating at a modest rate of order \( n\nu \ll nq \), and only weakly affected by the proximity of the flux surface to the separatrix. Hence the mode has a simple structure everywhere except for a region close to the x-point where it oscillates so rapidly that MHD would no longer be applicable. The resulting mode structure is consistent with observations of ELMs\[7\], that show filamentary structures that follow the magnetic field lines, and whose poloidal structure near the X-point is difficult to determine.

D. Previous Analytical Work

As mentioned at the outset, Laval et al[6] considered a trial function consisting of a single Fourier mode in a straight field line co-ordinate, that is resonant at a rational surface in the vacuum just outside the plasma’s surface. For that trial function, they found that for a positive non-zero current at the plasma edge, \( \delta W < 0 \), and suggested therefore that the Peeling mode would be unstable for a non-zero positive current at the plasma’s edge. On the basis of the sign of \( \delta W \) our study also finds this, but our study also suggests that the growth rate will asymptote to zero as the outermost flux surface approximates a separatrix, so that the mode will be marginally stable.
Lortz [8] also considered Peeling mode stability, in toroidal plasmas with shaped cross-sections, using a systematic calculation with a trial function whose resonant surface is inside the plasma. An advantage of the calculation by Lortz [8], is that the radial structure of the mode is considered. An unfortunate complication for this discussion is that in the ordering scheme of Lortz [8], the vacuum energy can be neglected. This was not the case for our calculation in Part (I) [1], or of Laval et al [6]. Nonetheless, we will consider the predictions of this calculation in the limit where the outermost flux surface approximates a separatrix.

Connor et al [9] review the calculation of Lortz [8], and use it to consider trial functions with resonant surfaces both inside and outside the plasma. They find that stability of the Peeling mode requires

\[ 1 - 4D_M > \left( \frac{2S}{P} - 1 \right)^2 \]  \hspace{1cm} (102)

where the Mercier co-efficient \( D_M \equiv \frac{-Q}{P} \), and \( P, Q, S \) are defined as,

\[
P = 2\pi (q')^2 \left[ \frac{\int J B^2}{R^2 B_p} d\chi \right]^{-1}
\]

\[
Q = \frac{q'}{2\pi} \int \frac{\partial J}{\partial \psi} d\chi - \frac{(q')^2}{2\pi} \int \frac{J}{B_p^2} d\chi + Ip' \int \frac{J}{R^2 B_p^2} d\chi \left[ \frac{\int J B^2}{R^2 B_p^2} d\chi \right]^{-1} \times \left[ \frac{Ip'}{2\pi} \int \frac{J B^2}{R^2 B_p^2} d\chi \right] - q'
\] \hspace{1cm} (103)

\[
S = P + q' \int \frac{J}{R^2 B_p^2} J d\chi \left[ \frac{\int J B^2}{R^2 B_p^2} d\chi \right]^{-1}
\]

Substituting \( D_M \equiv \frac{-Q}{P} \) into Eq. (102) gives the stability requirement

\[
\frac{Q}{P} - \left( \frac{S}{P} \right)^2 + \frac{S}{P} > 0 \] \hspace{1cm} (104)

Substituting for \( P, Q, \) and \( S \), allows Eq. (104) to be simplified to

\[
p' \left[ \frac{\partial}{\partial \psi} \frac{1}{2\pi} \int J \chi d\chi - q' \frac{1}{2\pi} \int \frac{J}{B_p^2} d\chi \right] + Ip' \left[ \frac{\nu R^2}{I} \frac{1}{2\pi} \int \frac{\partial}{\partial \psi} d\chi - \frac{1}{I^2} \frac{1}{2\pi} \int \frac{J B^2}{R^2 B_p^2} d\chi \right] > 0 \] \hspace{1cm} (105)

This may be simplified further by noting that because \( \nu = IJ/\chi R^2 \), and in a large aspect ratio ordering where \( R \) is taken as approximately constant,

\[
\frac{\partial}{\partial \psi} \frac{1}{2\pi} \int J \chi d\chi = \frac{\partial}{\partial \psi} \frac{1}{2\pi} \int \frac{\nu R^2}{I} d\chi = \frac{R^2}{I} \frac{1}{2\pi} \int \frac{\partial}{\partial \psi} d\chi - R^2 I' \frac{1}{2\pi} \int \frac{J B^2}{R^2 B_p^2} d\chi \] \hspace{1cm} (106)

and

\[
q' = \frac{1}{2\pi} \int \frac{\partial}{\partial \psi} d\chi \] \hspace{1cm} (107)

The Grad-Shafranov equation in \( \psi, \chi, \phi \) co-ordinates, has

\[
\frac{\partial \nu}{\partial \psi} = \frac{\nu}{B_p^2} \left\{ - \frac{\partial}{\partial \psi} \left( p + B^2 \right) + \frac{R^2 B^2}{I} \frac{\partial}{\partial \psi} \left( \frac{I}{R^2} \right) \right\} \] \hspace{1cm} (108)
Therefore if $R$ is taken as approximately constant (as would be the case either in a large aspect ratio limit or if we are sufficiently close to the separatrix that the integral is dominated by the divergence at the X-point and $R$ may be approximated by its value $R = R_X$ there), then using Eq. 108, Eq. 105 simplifies to a condition for stability of

$$0 < \frac{1}{2\pi} \int \nu \frac{I}{R^2} d\chi \left\{ (-p' - \frac{1}{R^2}) \frac{\partial}{\partial \psi} (2p + B^2) \right\}$$

$$= \left( \nabla \phi \cdot \vec{J} \right) I \frac{1}{2\pi} \int \nu \frac{\partial}{\partial \psi} (2p + B^2) d\chi$$

This may be simplified further still, to

$$0 < \frac{1}{2\pi} \int \nu \frac{I}{B_p^2 R^2} \left( -\nabla \phi \cdot \vec{J} \right) \left[ 2 \left( \nabla \phi \cdot \vec{J} \right) - \frac{\partial B_p^2}{\partial \psi} \right] d\chi$$

(109)

Because the integrals are dominated by the divergence of $\nu / B^2_p$ at the X-point and $\frac{\partial B_p^2}{\partial \psi} < 0$ at the X-point, then based on the formulation of Lortz [8, 9], then provided $\nabla \phi \cdot \vec{J} > 0$ the negative expression clearly indicates instability to the Peeling mode. However, if we allow $\nabla \phi \cdot \vec{J}$ to be negative, then the formulation of Lortz [8, 9] also suggests that stability is possible provided

$$0 < \frac{1}{2\pi} \int \nu \frac{I}{B_p^2} \left[ -2 \left| \nabla \phi \cdot \vec{J} \right| - \frac{\partial B_p^2}{\partial \psi} \right] d\chi$$

(111)

Therefore in principle there is a range of negative current values at the plasma edge for which the Peeling mode is stable. The appendix calculates $\frac{\partial B_p^2}{\partial \psi}$ near the X-point for a standard and a “snowflake” [12] divertor. Interestingly, whereas a conventional X-point has a range of negative current values for which the Peeling mode is stable, in the limit of an exact “snowflake” X-point (with flux surfaces meeting at an angle of $\pi/3$), the range of values of negative current for which the Peeling mode is stable, tends to zero. Whether this observation will have consequences for the plasma behaviour in a “snowflake” X-point geometry remains to be seen, but it is a qualitative difference between a conventional X-point and that produced with a “snowflake” divertor.

As mentioned previously, the calculation also considered the radial structure of the mode, with $\xi \sim x^{\lambda \pm}$, $x$ a radial co-ordinate, and

$$\lambda \pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{Q}{P}}$$

(112)

As we approach the separatrix, Webster [5] shows that $D_M = -\frac{Q}{P} \to 0$, giving

$$\lambda_+ \simeq \frac{Q}{P} \to 0$$

$$\lambda_- \simeq -1$$

(113)
and mode structures of $\xi_- \sim \frac{1}{x}$ and $\xi_+ \sim x^{Q/P}$. For the perturbations to satisfy the boundary condition of a mode amplitude that tends to zero in the plasma, this requires us to use $\xi_-$ for resonances outside the plasma (the “external” Peeling mode), and $\xi_+$ for resonances inside the plasma (the “internal” Peeling mode). It should be noted that a potential problem with the analysis of Lortz et al. when applied to Peeling modes, that the mode is taken to be sufficiently localised that the equilibrium quantities (that include $q$ and $q'$), are approximately constant. This is almost certainly not the case near a separatrix.

E. Summary

We have started from a simple model for the Peeling mode, at marginal stability in cylindrical geometry, and in Part (I) of this paper generalised it to toroidal Tokamak geometry. A conclusion of Part (I) is that Peeling mode stability is determined by the value of $\Delta'$, a normalised measure of the discontinuity in the gradient of the normal component of the perturbed magnetic field at the plasma-vacuum boundary. Therefore this paper evaluated $\Delta'$ in a large aspect ratio Tokamak geometry with a separatrix and X-point, but in such a way that the effect of the X-point is captured exactly, without encountering the usual discretisation errors present in most numerical methods. This was possible by generalising the method of conformal transformations beyond textbook presentations, that require a boundary condition of either the function or its normal derivative to be zero. Here we observe that even if the field’s normal derivative is non-zero at the boundary, it is still possible to use the conformal transformation method. In this case instead of obtaining an exact analytic solution (as would be the case if its normal derivative were zero on the boundary), the 2-dimensional problem is reduced to a 1-dimensional problem that may subsequently be solved exactly or approximated. The approach avoids the errors that may arise due to the discretisation of space near an X-point, that are necessarily present in most numerical methods. This paper also calculated analytical expressions for physically realistic examples of the equilibrium vacuum magnetic field, and the straight field line angle at the plasma-vacuum boundary. These and other results are likely to find opportunities for application elsewhere.

It is found that a radial plasma perturbation consisting of a single Fourier mode in straight field line co-ordinates with a high toroidal mode number $n$, in a plasma equilibrium
with a separatrix and an x-point, will produce the same change in the vacuum energy as the equivalent perturbation in a cylindrical equilibrium, with \( \delta W_v = 2\pi^2 \frac{\xi_m^2}{R} \Delta^2 m \). It also results in the same value for \( \Delta' \), with \( \Delta' = -2m \), where \( m \) is the poloidal mode number. Despite our trial function requiring \( m \sim nq \to \infty \), we observe that the trial function has \( \xi \sim e^{i \phi} f^\nu \int dx \) that is physically well behaved for all but a highly localised region near the X-point where MHD will fail to apply. Therefore we believe the trial function is physically acceptable even for a separatrix boundary.

Previous work by Lortz\[8\] and Connor et al\[9\] was considered for an outer flux surface that tends to a separatrix with an X-point. Like Laval et al\[6\] their work predicts the Peeling mode to be unstable if there is a positive current at the plasma’s edge, and it also finds a well behaved radial structure for the mode. Interestingly, for a conventional X-point there is predicted to be a range of small but negative edge-current for which the Peeling mode is stable, but in the limit of an exact snowflake divertor this range shrinks to zero size - a qualitative difference between a conventional and a snowflake divertor. A limitation of the Lortz calculation is that it approximates the equilibrium quantities as constant on the length scale of the plasma instability, this is not necessarily the case for \( q \) or \( q' \) near a separatrix, and therefore the results when applied to a separatrix case should be treated with caution. Likewise, as noted in Part (I), there are potential limitations to the high-\( n \) ordering form of \( \delta W \) used here, and this should be investigated in future work.

Thus we have developed a simple model for the Peeling mode, and found that despite \( \delta W < 0 \), the growth rate \( \gamma \) tends to zero as the outermost flux surface tends to a separatrix with an X-point. As the outermost flux surface approaches a separatrix, the growth rate falls with \( \ln(\gamma/\gamma_A) = -\frac{1}{2} \ln(q'/q) \); this has subsequently been confirmed with ELITE (S. Saarelma, private communication), leading us to believe that the effect of a separatrix on the high toroidal mode number ideal MHD model is now understood. The ideal MHD prediction of marginal stability at the separatrix means that other non-ideal terms such as resistivity, non-linear terms, or terms neglected in the high-\( n \) analysis, will play a role in determining the eventual stability. In general it is hoped that the methods and results contained in this paper will provide new tools for studying plasmas in separatrix geometries, and have potential applications in future studies of plasma stability and more generally outside of plasma physics.
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XIV. \( w'(z) \) NEAR THE X-POINT

To obtain \( w'(z) \) we differentiate both sides of Eq. 1 with respect to \( z \), and rearrange the resulting expression to get

\[
w'(z) = \left( w(z) + nl \right)^2 \frac{(z - l)^{n-1}}{(z + l)^{n+1}} \quad (114)
\]

and hence

\[
|w'(z)|^2 = |w(z) + nl|^4 \frac{|z - l|^{2(n-1)}}{|z + l|^{2(n+1)}} \quad (115)
\]

We have deliberately obtained an implicit expression for \( |w'(z)|^2 \), with \( w'(z) \) given in terms of \( w(z) \). This is because one cannot simply expand \( w'(z) \) in powers of \( \epsilon \), because the expansion will give the incorrect answer as \( \epsilon \to 0 \) compared with the exact result for \( \epsilon \) small but non-zero (i.e. \( \epsilon \neq 0 \) is a “singular perturbation”). Instead by obtaining \( |w'(z)|^2 \) implicitly in the form given by Eq. (115) we need solely be careful with the term \( |z - l|^{2(n-1)} \), because \( |w + nl|^4 \) and \( |z + l|^{2(n+1)} \) are well behaved when expanded in \( \epsilon \), as \( \epsilon \to 0 \). Near the X-point,

\[
|w(z) + nl|^4 = (2nl)^4 + O(\epsilon)
\]

\[
|z + l|^{2(n+1)} = (2l)^{2(n+1)} + O(\epsilon)
\]

(116)
We need to be more careful with \(|z - l|^2\), that with \(z = -a + (a + l - \epsilon)e^{i\alpha}\) gives

\[
|z - l|^2 = ((a + l - \epsilon)\cos(\alpha) - a + l)^2 + (a + l + \epsilon)^2 \sin^2(\alpha)
\approx 2a^2 \left[ 1 - \cos(\alpha) + \frac{\epsilon^2}{2a^2} \right]
\]

(117)

where we have retained the term that provides the singular perturbation that prevents \(|z - l|^2\) becoming zero for \(\epsilon \neq 0\), but neglected all the lower order terms that modify the answer by of order \(\epsilon/a\) and \(l/a\). In principle some plasma cross-sections might require the retention of terms of order \(l/a\), but here we neglect them so as to keep algebraic details to a minimum.

Therefore at leading order we have

\[
|w'(z)|^2 \approx \left(\frac{2n!}{(2l)^{2n+1}}(2a^2)^{(n-1)}\right) \left(1 - \cos(\alpha) + \frac{\epsilon^2}{2a^2}\right)^{(n-1)}
\]

(118)

that for the case we are most interested in here with \(n = 3/2\) (corresponding to an X-point with a \(\pi/2\) interior angle), we have

\[
|w'(z)|^2 \approx \left(\frac{3!}{(2)^3}\right) \left(\frac{2a^2}{l}\right)^{1/2} \left(1 - \cos(\alpha) + \frac{\epsilon^2}{2a^2}\right)^{1/2} = \frac{1}{\sqrt{2}} \left(\frac{3}{2}\right)^4 \left(\frac{a}{l}\right) \sqrt{1 - \cos(\alpha) + \frac{\epsilon^2}{2a^2}}
\]

(119)

which is Eq. [11].

**XV. SOME IDENTITIES INVOLVING COMPLEX NUMBERS**

In the following we write \(a = a_x + ia_y\), \(b = b_x + ib_y\), \(c = c_x + ic_y\), and \(d = d_x + id_y\), and remind the reader that the dot product refers to the sum of, the product of the real parts plus the product of the imaginary parts. For example \(a.b = a_xb_x + a_yb_y\). Multiplication is as usual for complex numbers, for example \(ab = a_xb_x - a_yb_y + i(a_xb_y + a_yb_x)\). Then we find

\[
ab.cd = [(a_xb_x - a_yb_y) + i(a_xb_y + a_yb_x)] \cdot [(c_xd_x - c_yd_y) + i(c_xd_y + c_yd_x)]
\]

\[
= (a_xb_x - a_yb_y)(c_xd_x - c_yd_y) + (a_xb_y + a_yb_x)(c_xd_y + c_yd_x)
\]

\[
= a_x [b_x(c_xd_x - c_yd_y) + b_y(c_xd_y + c_yd_x)] + a_y [b_x(c_xd_y + c_yd_x) - b_y(c_xd_x - c_yd_y)]
\]

\[
= a_x [d_x(b_xc_x + b_yc_y) + d_y(b_xc_y - c_yb_x)] + a_y [d_x(b_yc_x + b_yc_y) + d_x(b_xc_y - b_yc_x)]
\]

\[
= (a_xd_x + a_yd_y)(b_xc_x + b_yc_y) + (a_xd_y - a_yd_x)(b_yc_x - c_yb_x)
\]

\[
= (a.d)(b.c) + (i.a.d)(i.c.b)
\]

(120)
and similarly for \( a.b.c \),

\[
a.b.c = a.((b_x c_x - b_y c_y) + i(b_x c_y + b_y c_x))
\]

\[
= a_x b_x c_x - a_x b_y c_y + a_y b_x c_y + a_y b_y c_x
\]

\[
= c_x (a_x b_x + a_y b_y) + c_y (a_x b_x - a_x b_y)
\]

\[
= (c_x + i c_y).((a_x b_x + a_y b_y) + i(a_y b_x - a_x b_y))
\]

\[
= (c_x + i c_y).((a_x (b_x - i b_y) + a_y (b_y + i b_x))
\]

\[
= (c_x + i c_y).((a_x (b_x - i b_y) - i a_y (-b_x + i b_y))
\]

\[
= c.((a_x + i a_y)(b_x - i b_y))
\]

\[
= c.a\bar{b} = \bar{a}b.c
\]

XVI. 2ND TERM IS ORDER \( \frac{1}{m} \) SMALLER THAN \( \delta W_V \)

Here it is shown that the second term in Eq. 61, here written as \( \delta W_G \), is of order \( 1/m \) smaller than \( \delta W_V \). The term we are interested in is

\[
\delta W_G = \pi \Delta \left( \frac{m}{nq} \right) \xi_m^* \int \frac{d\alpha}{2\pi} r_a \frac{I}{B^2} e^{-im\theta(\alpha)+i\phi} \frac{i}{n} \left[ (n_z B_z) n_z \nabla_z \left( \frac{|B_{pz}|}{|w'(z)|^2} \right) \right]
\]

Using Eq. 41 and Eq. 40 we get

\[
n_z \nabla_z \left( \frac{|B_{pz}|}{|w'(z)|^2} \right) = -B_{p0} \frac{\partial}{\partial \epsilon} \left[ 2\sqrt{2} \left( \frac{2}{3} \right)^4 \frac{1}{a} \sqrt{1 - \cos(\alpha) + \epsilon^2/2a^2} \right]
\]

\[
= -B_{p0} \left( \frac{c_x}{a} \right) \left( \frac{2}{3} \right)^4 \left( \frac{1}{a} \right) \frac{1}{\sqrt{1 - \cos(\alpha) + \epsilon^2/2a^2}}
\]

(123)

where \( c_a = \left[ 2\sqrt{2} \left( \frac{2}{3} \right)^4 \left( \frac{1}{a} \right) \right]^2 \) and is a constant. Substituting the above Eq. 123 into Eq. 122 gives

\[
\delta W_G = \pi \Delta \left( \frac{m}{nq} \right) \xi_m^* \frac{-\epsilon}{a^2} c_a \frac{iq}{n} \int \frac{d\alpha}{2\pi} \frac{R^2 B_{p0}^2}{B^2} e^{-im\theta(\alpha)+i\phi} \left[ n_z B_z \frac{\partial \theta}{\partial \alpha} \right]
\]

(124)

Integrating by parts then gives

\[
\delta W_G = \pi \Delta \left( \frac{m}{nq} \right) \xi_m^* \frac{-\epsilon}{a^2} c_a \frac{q}{nm} \int \frac{d\alpha}{2\pi} \frac{R^2 B_{p0}^2}{B^2} e^{-im\theta(\alpha)+i\phi} \left[ \frac{\partial n_z B_z}{\partial \alpha} \right]
\]

(125)

Using \( n_z B_z = \frac{\partial V}{\partial r} \) and Eq. 17 we get

\[
\frac{\partial}{\partial \alpha} (n_z B_z) \bigg|_{r_a} = \sum_p -\frac{-ip|a_p|e^{ip\alpha} - in\phi}{r_a}
\]

(126)
Substituting this into (125) we get

$$\delta W_G = \pi \Delta \left( \frac{m}{nq} \right) \xi_m \left( \frac{-\epsilon}{a^2} \right) c_a \frac{q}{nm} \frac{R^2 B_{\rho_0}^2}{B^2} \sum_{p \neq 0} \frac{-a_p |p|}{r_a} \oint d\alpha e^{-i m \theta(\alpha) + i \phi}$$

(127)

Where the poloidal dependence in $R^2 B_{\rho_0}^2$ has been neglected due to the large aspect ratio Tokamak ordering. Using Eq. 32 we get

$$\oint e^{-i m \theta + ip \alpha} d\alpha = -ia_p |p| \frac{R}{\Delta \xi_m} 2\pi$$

(128)

which may be inserted into (127) and after some cancellations gives

$$\delta W_G = 2\pi^2 R \left( \frac{\epsilon}{a^2} \right) c_a \frac{R^2 B_{\rho_0}^2}{B^2} \frac{1}{n^2} \frac{1}{r_a} \sum_{p \neq 0} |p|^2 |a_p|^2$$

(129)

The sum $\sum_{p \neq 0} |p|^2 |a_p|^2$ may be evaluated arbitrarily accurately, as is indicated next. We use the alternative expression for $a_p$ given by Eq. 30 of

$$a_p = -\frac{\Delta \xi_m}{|p|} \frac{1}{R} \frac{1}{2\pi} \oint im \beta(\alpha)e^{im \theta - ip \alpha} d\alpha$$

(130)

which gives

$$\sum_p |p|^2 |a_p|^2 = \Delta^2 \frac{\xi_m^2}{R^2} \sum_{p \neq 0} \left( \frac{1}{2\pi} \oint d\beta \ m \beta(\beta)e^{-i m \theta(\beta) + ip \beta} \right) \times \left( \frac{1}{2\pi} \oint d\alpha \ m \beta(\alpha)e^{i m \theta(\alpha) - ip \alpha} \right)$$

$$= \Delta^2 \frac{\xi_m^2}{R^2} \frac{m^2}{2\pi} \oint d\beta \ m \beta(\beta)e^{-i m \theta(\beta)} \sum_{p \neq 0} e^{i p \beta} \frac{1}{2\pi} \oint d\alpha \ m \beta(\alpha)e^{i m \theta(\alpha)} e^{-i p \alpha}$$

(131)

$$= \Delta^2 \frac{\xi_m^2}{R^2} \frac{m^2}{2\pi} \oint \theta'(\beta) e^{-i m \theta(\beta)} \theta'(\beta) e^{i m \theta(\beta)} d\beta$$

$$= \Delta^2 \frac{\xi_m^2}{R^2} \frac{m^2}{2\pi} \oint \left( \theta'(\beta) \right)^2 d\beta$$

where in the penultimate line, we resum the Fourier series by noting that $\theta'(\beta)e^{i m \theta(\beta)} = \sum_{p \neq 0} e^{i p \beta} \frac{1}{2\pi} \oint e^{i p \alpha} \theta'(\alpha)e^{i m \theta(\alpha)}$. We approximate $\oint \theta'(\beta)^2 d\beta$, using the analytic expression for $\theta'(\beta)$ obtained from Eqs. 35, 40, and 41 obtaining

$$\oint \theta'(\beta)^2 d\beta = \frac{1}{q^2} \frac{1}{c_a} \frac{R^2 B_{\rho_0}^2}{B^2} \oint \frac{d\beta}{1 - \cos(\beta) + \epsilon^2/2a^2} \approx \frac{1}{q^2} \frac{1}{c_a} \frac{R^2 B_{\rho_0}^2}{B^2} \frac{2\pi a}{\epsilon}$$

(132)

Hence taking $r_a = a + l \approx a$ we get

$$\delta W_G \approx 2\pi^2 R c_a \frac{R^2 B_{\rho_0}^2}{B^2} \left( \frac{1}{n^2} \right) \left( \Delta^2 \frac{\xi_m^2}{R^2} \frac{m^2}{2\pi} \right) \left( \frac{R^2 B_{\rho_0}^2}{B^2} \frac{1}{c_a} \frac{1}{q^2} \frac{2\pi a}{\epsilon} \right)$$

(133)

$$\delta W_G \approx 2\pi^2 \frac{\xi_m^2}{R} \Delta^2 \left( \frac{1}{m} \right) \delta V$$

where the last line uses the expression for $\delta V \sim \Delta^2 m$ obtained from the main text. Hence, we may neglect this term compared with $\delta V$. 31
XVII. MODIFICATIONS TO PEELING MODE STABILITY FOR A “SNOWFLAKE” DIVERTOR

The integrals that determine the stability of Peeling modes, are dominated by the divergence in $\nu/B_p^2$ that occurs near the X-point in the separatrix. This allows the integrals to be estimated by expanding the poloidal flux functions in the vicinity of the X-point, as has been done in Ref. [12] for both a standard X-point and a “snowflake” divertor’s X-point. Near a standard X-point, Eq. 15 of Ref. [12] gives

$$\Phi = \left( \frac{\tilde{B}}{L} \right) \left( \frac{x^2 - z^2}{2} \right)$$  \hspace{1cm} (134)

with $\tilde{B}$ having the poloidal field’s dimensions and $L$ a length scale, and the poloidal magnetic field given by $B_x = -\frac{\partial \Phi}{\partial z}$ and $B_z = \frac{\partial \Phi}{\partial x}$, leading to

$$B_p^2 = \frac{\tilde{B}^2}{L^2} (x^2 + z^2)$$  \hspace{1cm} (135)

Note that near the X-point, $\nabla \psi \rightarrow R_X \nabla \Phi$, with $R_X$ the major radius at the X-point. Therefore near the X-point $\frac{\partial B_p^2}{\partial \psi} = \frac{1}{R_X} \frac{\partial B_p^2}{\partial \Phi}$, and we calculate $\partial B_p^2/\partial \psi$ from

$$\frac{\partial B_p^2}{\partial \Phi} = \frac{\nabla \Phi \cdot \nabla (B_p^2)}{|\nabla \Phi|^2} = \left( \frac{\tilde{B}}{L} \right) 2 \frac{(x^2 - z^2)}{(x^2 + z^2)}$$  \hspace{1cm} (136)

and use Eq. 134 to substitute for $x^2 = 2 \left( \frac{\Phi L}{\tilde{B}} \right) + 2z^2$, giving $\partial B_p^2/\partial \Phi$ at constant $\Phi$ as

$$\frac{\partial B_p^2}{\partial \Phi} = 2 \left( \frac{\tilde{B}}{L} \right) \left( \frac{\Phi L/\tilde{B}}{z^2 + (\Phi L/\tilde{B})} \right)$$  \hspace{1cm} (137)

and similarly $B_p^2 = \left( \frac{\tilde{B}^2}{L^2} \right) 2 \left( \Phi^2 + (\Phi L/\tilde{B}) \right)$. The minimum value of $z$ is at $x = 0$, giving $z_{\text{min}}^2 = -2\Phi L/\tilde{B}$, and

$$\left. \frac{\partial B_p^2}{\partial \Phi} \right|_{z_{\text{min}}} = -2 \left( \frac{\tilde{B}}{L} \right)$$  \hspace{1cm} (138)

For a “snowflake” divertor, Eq. (2) of Ref. [12] gives

$$\Phi = \frac{AI}{c} \left( x^2 z - \frac{z^3}{3} \right)$$  \hspace{1cm} (139)
with $I$ the plasma current, $A$ has dimensions of the inverse cube of the length scale over which the poloidal magnetic field varies near an X-point, and $c$ is the speed of light. In a similar way to the calculation for the standard X-point, this leads to

$$\frac{\partial B_p^2}{\partial \Phi} = \left( \frac{AI}{c} \right) \frac{4z}{(\frac{c\Phi}{AI}) + z^3}$$  \hspace{1cm} (140)

$$B_p^2 = \left( \frac{A^2I^2}{c^2} \right) \left( \frac{4}{3}z^2 + \left( \frac{c\Phi}{AI} \right) \frac{1}{z} \right)^2$$  \hspace{1cm} (141)

with $z_{\text{min}}^3 = -3\Phi c/AI$, giving

$$\left. \frac{\partial B_p^2}{\partial \Phi} \right|_{z_{\text{min}}} = \left( \frac{AI}{c} \right) \frac{4}{3} \left( \frac{c\Phi}{AI} \right)^{1/3}$$  \hspace{1cm} (142)

that tends to zero as we approach a separatrix. Therefore, whereas a conventional X-point leads to a Peeling mode stability boundary for which the Peeling mode can be stable for a range of small but non-zero negative current at the plasma’s edge, this window of stability tends to zero size for a snowflake divertor.

Finally, for a “snowflake plus” divertor\[12\], a similar calculation using the Eqs. of Ref.\[12\] finds $\partial B_p^2/\partial \Phi \sim \sqrt{\frac{I-I_{d0}}{I_{d0}}}$ with $I$ the current in the divertor coils and $I_{d0}$ the divertor coil current required for an exact snowflake divertor. In that case the range of values of negative edge-current for which Peeling modes are stable, tends to zero as $I \to I_{d0}$.

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