A Theorem on Light-Front Quantum Models*

WAYNE N. POLYZOU

Department of Physics and Astronomy
The University of Iowa
Iowa City, Iowa 52242

ABSTRACT

I give a sufficient condition for a relativistic front-form quantum mechanical model to be scattering equivalent to a relativistic front-form quantum model with an interaction-independent front-form spin.

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1. Introduction

In this paper I present a sufficient condition for a relativistic front-form quantum model with an interaction dependent spin operator to be scattering equivalent to a relativistic front-form quantum model with an interaction independent spin.

Two quantum mechanical models are scattering equivalent if they are unitarily equivalent and have the same scattering matrix. Thus, scattering equivalent models represent equivalent representations of a given physical system. In formulating models it is desirable to work in a representation that has simplifying features. An undesirable feature of relativistic light-front quantum mechanics is that the front-form Hamiltonian and two components of the total angular momentum necessarily involve interactions. This complication is unavoidable. However, if the angular momentum generators are replaced by the spin operator it does not follow that the spin operator is interaction dependent. In some applications the spin operator naturally involves interactions, but it is also possible to construct models with an interaction independent spin operator.

The advantage of working in a representation with a non-interacting spin operator is that it is straightforward to make approximations that preserve the relativistic invariance. For models with an interacting spin the problem of making approximations that preserve the relativistic invariance is a non-linear problem. This non-linearity is the problem that makes it difficult to find relativistically invariant truncations of front-form field theories.

The motivation for seeking scattering equivalences with models having an non-interacting spin operator is to eliminate the difficulty in making relativistically invariant approximations to more general front-form models. In addition, the use of phenomenological models with a non-interacting front-form spin is justified when the existence of such a scattering equivalence can be established.

In this paper I prove a theorem that gives sufficient conditions for the existence of a scattering equivalence that relates front-form models with an interaction dependent spin to front-form models with an interaction independent spin. The theorem proved in this paper applies to systems where particle number is conserved. The proof uses standard methods based on two-Hilbert
space scattering theory\textsuperscript{[1]} with simple estimates in the strong topology. Thus, to the extent that the abstract structure is preserved, the proof can be extended to settings where particle number is not conserved. In models with a production vertex there are other fundamental issues that need to be addressed such as the definition of an elementary particle and the definition of the kinematic representation of the Poincaré group. These issues are not considered in this paper.

A brief description of front-form quantum mechanics is given below. Section two introduces the notation used in the formulation of the multichannel scattering theory. The theorem is stated and proved in section 3.

A relativistic quantum mechanical model is a model of a system of particles formulated on a model Hilbert space with the dynamics given by a unitary representation, $U(\Lambda, x)$, of the Poincaré group. In the representation $U(\Lambda, x)$ $\Lambda$ denotes a Lorentz transformation and $x$ denotes the displacement of a space-time translation. This representation defines the dynamics of the model since the Poincaré group contains the time-evolution subgroup. In the absence of interactions the dynamics of this system is given by another representation of the Poincaré group, $U_0(\Lambda, x)$, called the kinematic (non-interacting) representation. Subgroups on which the interacting and kinematic representations are identical are called kinematic subgroups. Dirac\textsuperscript{[2]} investigated the three largest kinematic subgroups. The largest kinematic subgroup is the 7 parameter subgroup of the Poincaré group that maps the light-front, $x^+ = x^0 + x^3 = 0$, into itself. Relativistic models with this kinematic subgroup are called front-form models.

Front-form models have three independent one-parameter subgroups that necessarily involve interactions. They can be taken as the one-parameter subgroups that generate translations normal to the light front and rotations about two independent space axes tangent to the light front. Although the total angular momentum operator is necessarily interaction dependent in front-form models, it is possible to satisfy the commutation relations of the Poincaré Lie algebra with a kinematic total front-form spin operator.\textsuperscript{[3][4]} Phenomenological relativistic models with a kinematic front-form spin have been employed to model many systems\textsuperscript{[5][6][7][8][9][10][11]} where Poincaré invariance is an important symmetry.

There are two relevant examples of representations of the Poincaré group with an interaction dependent spin operator. The first example is light-front quantum field theory. The second
example \cite{3,4} is any representation generated by taking tensor products of interacting representations that have kinematic spin operators. Tensor products of interacting representations with kinematic spins have an interacting spin and satisfy the conditions of the theorem. The field theory case is interesting because, due to the infinite number of degree of freedom, the interacting and kinematic representation of the Poincaré group may act on different Hilbert spaces \cite{12}. Thus any application to the field theoretic case must be coupled with a reduction to a finite number of degrees of freedom. Because of this, a complete treatment of the field theoretic case is beyond the scope of this paper.

2. Multi Channel Scattering Theory

The Hilbert space $\mathcal{H}$ for a system of $N$ particles is the $N$-fold tensor product of one-particle Hilbert spaces. The transformation properties of this system are defined by a representation, $U(\Lambda, x)$, of the Poincaré group on $\mathcal{H}$. The dynamics is given by the time-evolution subgroup, which I denote by

$$T(t) = U[I, (t, 0, 0, 0)] \quad (2.1)$$

If the $N$-particles are partitioned into a $k$-cluster partition $a$ and the interactions between the particles in different clusters are set to 0, then the representation $U(\Lambda, x)$ becomes a new representation $U_a(\Lambda, x)$. This representation physically corresponds to a set of subsystems that do not interact with each other and, for a $k$-cluster partition $a$, it can be represented as the diagonal of the tensor product of $k < N$ subsystem representations:

$$U_a(\Lambda, x) := U_{a_1}(\Lambda, x) \otimes \cdots \otimes U_{a_k}(\Lambda, x). \quad (2.2)$$

The $N$-cluster partition corresponds to turning off all of the interactions. The representation of the Poincaré group associated with the $N$-cluster partition is the diagonal of the tensor product of $N$ one-particle representations:

$$U_0(\Lambda, x) := U_1(\Lambda, x) \otimes \cdots \otimes U_1(\Lambda, x). \quad (2.3)$$

N-times

The representations of $U(\Lambda, x)$ and $U_0(\Lambda, x)$ are identical when $(\Lambda, x)$ is an element of a
kinematic subgroup. It follows that for each partition \( a \),

\[
U_a(\Lambda, x) = U_0(\Lambda, x)
\]

(2.4)

for \((\Lambda, x)\) in the kinematic subgroup.

The infinitesimal generators of the Poincaré group are the four momentum \( P_\mu \) and the antisymmetric angular momentum tensor \( M_{\mu\nu} \). The spin is related to the Pauli-Lubanski vector which is defined by

\[
W^\mu = \frac{1}{2} \epsilon^{\mu\alpha\beta\gamma} P_\alpha M_{\beta\gamma}.
\]

(2.5)

The Poincaré group has two independent invariant polynomials in the infinitesimal generators,

\[
M^2 = -P^\mu P_\mu, \quad W^2 = W^\mu W_\mu,
\]

(2.6)

where the spectrum of \( M^2 \) has a positive lower bound and the spin \( j^2 \) is defined by

\[
j^2 = W^2/M^2.
\]

(2.7)

There are many spin vectors, \( \vec{j} \), which are functions of the generators satisfying \( SU(2) \) commutation relations and the relation \( \vec{j} \cdot \vec{j} = W^2/M^2 \). In front-form models the spin vector is taken as the front-form spin [3].

A single notation is used to treat bound states and scattering states. Bound states are treated as one fragment scattering channels.

\( N \)-particle bound states are simultaneous eigenstates of \( M^2 \) and \( W^2 \) with eigenvalues in the point spectrum of both of these operators. These eigenstates can be expressed as linear superpositions of simultaneous eigenstates of the mass, the light-front components \((\vec{p} := p^+, p^1, p^2)\) of the four momentum, the spin, and the z-component of the front-form spin:

\[
|f_\alpha\rangle = \int d\vec{p}^+ d^2p_\perp \sum_{\mu = s}^s |m, s, d, \vec{p}, \mu\rangle f(\vec{p}, \mu)
\]

(2.8)

where \( f(\vec{p}, \mu) \) is square integrable, and \( d \) is a degeneracy quantum number. The degeneracy quantum number represents relativistically invariant internal quantum numbers, such as isospin,
that distinguish different types of bound states with the same mass and spin. The notation $\alpha$ is used to denote the collection of quantum numbers $\{m, s, d\}$ and each distinct $\alpha$ is called a one-body channel. There is a one-body channel associated with each type of $N$-body bound state. The square integrable functions $f(p, \mu)$ span a channel Hilbert space $\mathcal{H}_\alpha$.

The generalized eigenstates $|m, s, d, p, \mu\rangle$ define an isometric mapping from the channel space $\mathcal{H}_\alpha$ to $\mathcal{H}$ by:

$$\Phi_\alpha |f\rangle := |f_\alpha\rangle$$ (2.9)

where the normalization

$$\langle m, s, d, p', \mu' |m, s, d, p, \mu\rangle = \delta(p-p')\delta_{\mu\mu'}$$ (2.10)

is assumed. Because $\Phi_\alpha$ is an isometry with range all of $\mathcal{H}_\alpha$ it follows that

$$U_\alpha(\Lambda, x) := \Phi_\alpha^\dagger U(\Lambda, x)\Phi_\alpha$$ (2.11)

defines a unitary representation of the Poincaré group on $\mathcal{H}_\alpha$.

Channels with more than one fragment are associated with scattering states. Scattering states $|\Psi_{\alpha}^{\pm}(t)\rangle$ are solutions of the time-dependent Schrödinger equation that satisfy either the outgoing ($+$) or incoming ($-$) wave asymptotic condition

$$\lim_{t \to \pm \infty} \| |\Psi_{\alpha}^{\pm}(t)\rangle - |\phi_{\alpha}(t)\rangle \| = 0,$$ (2.12)

where $|\phi_{\alpha}(t)\rangle$ represents a system of non-interacting particles and/or bound fragments and $\alpha$ distinguishes different scattering channels. In order to precisely define a channel, note that the distinct bound fragments define a partition $a$ of the $N$ particles into $n_a$ clusters, where two particles are in the same cluster of partition $a$ if they are in the same asymptotic fragment. For the partition $a$ to be associated with a scattering channel each fragment should either have only one particle or be in a bound state of the subsystem of particles in the fragment. The scattering state can be labelled by the quantum numbers of each bound particle
or fragment, \(\{m_1, s_1, d_1, p_1, \mu_1, \cdots m_{n_a}, s_{n_a}, d_{n_a}, \mu_{n_a}\}\). The subset of quantum numbers
\(\{m_1, s_1, d_1, m_2, s_2, d_2, \cdots m_{n_a}, s_{n_a}, d_{n_a}\}\), which do not include the momenta and magnetic quantum numbers of the particles, label the \(n_a\) fragment scattering channels which are denoted by \(\alpha\).

With this definition of channel, if some of the particles are identical, there is a distinct channel for each permutation of particles that changes the partition \(a\) to a partition \(a' \neq a\). Channels that differ by the exchange of identical particles are not physically distinguishable. In this paper these channels are treated as being distinguishable with the understanding the cross section are computed by averaging over equivalent initial channels and summing over equivalent final channels. The individual bound states and bound clusters are assumed to have the required symmetry under the exchange of identical particles.

With this definition, particles are formally treated as distinguishable. To each channel \(\alpha\) there is a unique partition \(a = a(\alpha)\) of the system into asymptotic fragments. It can happen that different channels \(\alpha\) and \(\beta\) are associated with same partition; i.e. \(a(\alpha) = a(\beta)\). It can also happen that there are no channels associated with some partitions, such as partitions containing two-neutron or two-proton clusters.

Given a scattering channel, the generalized eigenstates of the four momentum and 3-component of the front-form spin for each bound fragment define a mapping from the tensor product of the \(k\)-bound state channel spaces

\[ \mathcal{H}_\alpha := \mathcal{H}_{\alpha_1} \otimes \cdots \otimes \mathcal{H}_{\alpha_2} \]  

(2.13)

to \(\mathcal{H}\) where each \(\mathcal{H}_{\alpha_i}\) is the space of square integrable functions of the front-form momenta and magnetic quantum number of the \(i^{th}\) bound fragment. The mapping, \(\Phi_\alpha\), is defined by

\[ \Phi_\alpha |f_1 \cdots f_k\rangle := \sum_{\mu_1, \cdots, \mu_k} \int dp_1 \cdots dp_k |m_1, s_1, d_1, p_1, \mu_1\rangle \otimes \cdots \otimes |m_k, s_k, d_k, p_k, \mu_k\rangle \times f_1(p_1, \mu_1) \cdots f_k(p_k, \mu_k). \]  

(2.14)
There is a natural representation of the Poincaré group on the channel space $\mathcal{H}_\alpha$ given by:

$$U_\alpha(\Lambda, x) := U_{\alpha_1}(\Lambda, x) \otimes \cdots \otimes U_{\alpha_k}(\Lambda, x)$$  \hspace{1cm} (2.15)$$

where $U_{\alpha_i}(\Lambda, x)$ is the irreducible representation of the Poincaré group associated with the bound system of particles in the $i^{th}$ cluster of $a$ as defined by (2.11).

It is a consequence of (2.14), (2.8), and (2.11) that

$$U_{a(\alpha)}(\Lambda, x) \Phi_\alpha = \Phi_\alpha U_{a}(\Lambda, x).$$  \hspace{1cm} (2.16)$$

The scattering asymptote corresponding to the channel $\alpha$ in (2.12) is given by

$$|\phi_\alpha(t)\rangle := T_{a(\alpha)}(t)|f_1 \cdots f_k\rangle = \Phi_\alpha T_{a}(t)|f_1 \cdots f_k\rangle.$$  \hspace{1cm} (2.17)$$

where $T_{a(\alpha)}(t)$ and $T_{a}(t)$ are the time evolution subgroups of $U_{a(\alpha)}(\Lambda, x)$ and $U_{a}(\Lambda, x)$ respectively. The asymptotic condition can be reformulated for all channels simultaneously (trivially including all $N$-particle bound states) in a two Hilbert space language. The asymptotic Hilbert space, $\mathcal{H}_f$, is the direct sum of all channel Hilbert spaces:

$$\mathcal{H}_f := \bigoplus_{\alpha} \mathcal{H}_\alpha.$$  \hspace{1cm} (2.18)$$

The sum of the channel injection operators, $\Phi_\alpha$, defines a mapping from $\mathcal{H}_f$ to $\mathcal{H}$ by:

$$\Phi := \sum_{\alpha} \Phi_\alpha$$  \hspace{1cm} (2.19)$$

where each $\Phi_\alpha$ in the sum is understood to act on the subspace $\mathcal{H}_\alpha$ of $\mathcal{H}_f$. The asymptotic representation of the Poincaré group on $\mathcal{H}_f$ is defined by

$$U_f(\Lambda, x) := \sum_{\alpha} U_{\alpha}(\Lambda, x)$$  \hspace{1cm} (2.20)$$

where each $U_{\alpha}(\Lambda, x)$ acts on the subspace $\mathcal{H}_\alpha$. Scattering solutions, $|\Psi^\pm(0)\rangle$, are defined in terms
of two-Hilbert space wave operators which are mappings from $\mathcal{H}_f$ to $\mathcal{H}$ defined by:

$$|\Psi^\pm(0)\rangle := \Omega^\pm(T, \Phi, T_f)|\phi_f\rangle$$

(2.21)

$$\Omega^\pm(T, \Phi, T_f) := s - \lim_{t \to \pm\infty} T(t)\Phi T_f(-t)$$

(2.22)

where $T(t)$ and $T_f(t)$ are the time evolution subgroups of $U(\Lambda, x)$ and $U_f(\Lambda, x)$ respectively, and $|\phi_f\rangle$ is a vector in $\mathcal{H}_f$. A similar notation is used for the individual channel scattering state vectors and channel wave operators

$$|\Psi^\pm_\alpha(0)\rangle := \Omega^\pm_\alpha(T, \Phi, T_f)|\phi_\alpha\rangle = \Omega^\pm_\alpha|\phi_\alpha\rangle$$

(2.23)

$$\Omega^\pm_\alpha := \Omega^\pm_\alpha(T, \Phi_\alpha, T_\alpha) := s - \lim_{t \to \pm\infty} T(t)\Phi_\alpha T_\alpha(-t)$$

(2.24)

where $|\phi_\alpha\rangle$ is a vector in the channel Hilbert space $\mathcal{H}_\alpha$.

The scattering operator is the following mapping from $\mathcal{H}_f$ to itself:

$$S := \Omega^\dagger(T, \Phi, T_f)|\phi_f\rangle = \Omega^\dagger(T, \Phi, T_f)\Omega^\pm(T, \Phi, T_f).$$

(2.25)

The scattering operator will be unitary if

$$\Omega^\pm(T, \Phi, T_f)\Omega^\dagger(T, \Phi, T_f) = \Omega^\pm(T, \Phi, T_f)\Omega^\dagger(T, \Phi, T_f),$$

(2.26)

which is the property that the wave operators are asymptotically complete.

The wave operators are Poincaré invariant provided they satisfy the intertwining relations:

$$U(\Lambda, x)\Omega^\pm(T, \Phi, T_f) = \Omega^\pm(T, \Phi, T_f)U_f(\Lambda, x)$$

(2.27)

for all Poincaré transformations. This implies that the scattering operator is Poincaré invariant

$$[U_f(\Lambda, x), S]_\pm = 0.$$  

(2.28)

If a representation of the Poincaré group is transformed by a unitary transformation, it does not follow that the original theory and the transformed theory have the same scattering operator.
Unitary transformations that preserve the scattering operator are called scattering equivalences. In the two Hilbert space setting a sufficient condition for $U'(\Lambda, x) = A^\dagger U(\Lambda, x) A$ to be scattering equivalent to $U(\Lambda, x)$ is \[^3\]

$$\lim_{t \to \pm \infty} \|(A - 1)\Phi T_f(-t)|\xi\| = 0$$

(2.29)

for both time limits. An operator $A$ satisfying (2.29) is said to be asymptotically equivalent to the identity with respect to $\Phi$. When (2.29) holds it follows that $\Omega_{\pm}(T, A \Phi, T_f) = \Omega_{\pm}(T, \Phi, T_f)$ which implies:

$$\Omega_{\pm}' = \Omega_{\pm}(T', \Phi, T_f) = \Omega_{\pm}(A^\dagger T A, \Phi, T_f) = A^\dagger \Omega_{\pm}(T, A \Phi, T_f) = A^\dagger \Omega_{\pm}(T, \Phi, T_f) = A^\dagger \Omega_{\pm}$$

(2.30)

and

$$S' = \Omega_{\pm}' \Omega_{\mp}' = \Omega_{\pm}^1 A A^\dagger \Omega_{\mp} = \Omega_{\pm}^1 \Omega_{\mp} = S.$$  

(2.31)

### 3. Statement and Proof of the Theorem

In this section sufficient conditions for the existence of a scattering equivalence between a front form-model with an interaction dependent spin and a front-form model with an interaction independent spin are established by proving the following theorem:

**Theorem:** Let $U(\Lambda, x)$ be the representation of the Poincaré group for a model of a system of $N$ interacting particles and let $U_a(\Lambda, x)$ be the representation obtained from $U(\Lambda, x)$ by turning off the interactions between particles in different clusters of the partition $a$. Assume that $U(\Lambda, x)$ (and consequently $U_a(\Lambda, x)$) has the kinematic subgroup of the light-front and that the model satisfies:

1. The wave operators exist and are asymptotically complete .
2. The wave operators are Poincaré invariant .
3. There exits unitary operators $A_\alpha$ on $H$ that are kinematically invariant and satisfy

   a.) $\lim_{t \to \pm \infty} \|(A_{\alpha(x)} - 1)\Phi_\alpha T_{\alpha}(t)|\xi_\alpha\| = 0$

   (3.29)
and

b.) The operator \( \sum_s P(s)A_a P_a(s) \) has a bounded inverse for each partition \( a \) of \( N \)-particles, where \( P(s) \) and \( P_a(s) \) are the orthogonal projectors on the invariant spin \( s \) subspaces associated with the representations \( U(\Lambda, x) \) and \( U_a(\Lambda, x) \) respectively.

Under these conditions the representation \( U(\Lambda, x) \) is scattering equivalent to a representation \( \bar{U}(\Lambda, x) \) with the kinematic subgroup of the light-front and with a kinematic front-form spin operator.

The first two assumptions of the theorem are sufficient to ensure the model has a reasonable physical interpretation. The first assumption implies the unitarity of the scattering matrix while the second implies the Poincaré invariance of the scattering matrix.

The third assumption is the nontrivial assumption. It is a mild condition because of the freedom available to choosing the operators \( A_a \). The simplest choice is to choose \( A_a = 1 \). This choice is appropriate for models that already have a non-interacting \( j^2 \). It is also appropriate for models with interacting spins provided that \( \sum_s P(s)P_a(s) \) remains invertible. If the sum is not invertible, then the theorem permits modifications of the condition by the insertion of an operator \( A_a \) between the projectors. For the case of tensor products of models with non-interacting spins, it is know that if \( A_a \) is taken as \( A_a = B^\dagger B_a \) where \( B \) and \( B_a \) are the Sokolov or packing operators for the models \(^{[3][4]}\), then the third condition of the theorem is satisfied, although in this example it is not known whether condition 3b.) is satisfied for \( A_a = 1 \).

The proof of the theorem is based on a number of lemmas.

By assumption the wave operators, \( \Omega_{\pm}(T, \Phi, T_f) \), exist as unitary mappings from \( \mathcal{H}_f \) to \( \mathcal{H} \), are asymptotically complete,

\[
\Omega_+(T, \Phi, T_f)\Omega_+^\dagger(T, \Phi, T_f) = \Omega_-(T, \Phi, T_f)\Omega_-^\dagger(T, \Phi, T_f),
\]

and Poincaré invariant,

\[
U(\Lambda, x)\Omega_{\pm}(T, \Phi, T_f) = \Omega_{\pm}(T, \Phi, T_f)U_f(\Lambda, x).
\]

In addition \( U(\Lambda, x) \) has the kinematic subgroup of the light front, which means that \( U(\Lambda, x) = U_0(\Lambda, x) \) for Poincaré transformations \( (\Lambda, x) \) that leave the null plane \( x^+ = 0 \) invariant.
The technical assumption is that the sum

\[ X_a := \sum_s P(s)A_a P_a(s) \]  

(3.3)

has a bounded inverse for each partition \( a \) where \( P_a(s) \) is the projection on the invariant subspace of \( U_a(\Lambda, x) \) on which \( j^2_a := W^2_a/M^2_a \) has eigenvalue \( s(s + 1) \), and \( P(s) \) is the corresponding projector associated with \( j^2 \). The unitary operator \( A_a \) is any kinematically invariant operator that is asymptotically equivalent to the identity with respect to \( \Phi_\alpha \) for all \( \alpha \) with \( a = a(\alpha) \). The freedom to choose \( A_a \) can be used to make the \( X_a \) have a bounded inverse.

The proof of the theorem follows as a consequence of the lemmas that follow.

**Lemma 1:** With \( P(s), P_a(s), \) and \( A_a \) as defined in the statement of the theorem

\[ X_a := \sum_s P(s)A_a P_a(s) \]  

(4.4)

satisfies

\[ \|X_a\| \leq 1. \]  

(3.5)

**Proof:** Let \( |\xi(s)\rangle \) denote a unit normalized vector in the range of \( P_a(s) \). Then

\[ \|X_a|\xi(s)\rangle\| = \|\sum_r P(r)A_a P_a(r)|\xi(s)\rangle\| = \|P(s)A_a|\xi(s)\rangle\| \leq \|P(s)A_a\| \|\xi(s)\| = 1. \]  

(3.6)

Next observe that any normalizable state can be expanded in the form

\[ |\xi\rangle = \sum_s c_s|\xi(s)\rangle \quad \text{with} \quad \sum_s |c_s|^2 < \infty \]  

(3.7)

where \( |\xi(s)\rangle \) are unit normalized vectors in the range of \( P_a(s) \). The orthogonality of the \( P(s) \)'s imply

\[ \|X_a|\xi\rangle\| = \sum_s |c_s|^2 \|P(s)A_a|\xi(s)\rangle\|^2 \leq \sum_s |c_s|^2 = \|\xi\|. \]  

(3.8)

The lemma follows by dividing the left side of the equation by the right.
The next lemma is the key result that is needed to establish scattering equivalence with a system with non-interacting $j^2$.

**Lemma 2:** For $|\xi\rangle \in H_\alpha$ the assumptions of the theorem imply

$$\lim_{t \to \pm \infty} \| [X^a - 1] \Phi_\alpha T_\alpha (-t) |\xi\rangle \| = 0 \quad (3.9)$$

and

$$\lim_{t \to \pm \infty} \| [X^a_\dagger - 1] \Phi_\alpha T_\alpha (-t) |\xi\rangle \| = 0. \quad (3.10)$$

**Proof:** By the Poincaré invariance of the wave operators the wave operators intertwine the asymptotic and interacting spin operators:

$$j^2 \Omega_{\alpha \pm} = \Omega_{\alpha \pm} j^2. \quad (3.11)$$

This implies

$$P(s) \Omega_{\alpha \pm} = \Omega_{\alpha \pm} P_\alpha (s) \quad (3.12)$$

for each value of $s$, where $P_\alpha (s)$ is the projector on the invariant spin $s$ subspace associated with the representation $U_\alpha (\Lambda, x)$ of $H_\alpha$. For $Q(s) := [1 - P(s)]$ the orthogonality of $P(s)$ and $Q(s)$ implies

$$0 = Q(s) P(s) \Omega_{\alpha \pm} = Q(s) \Omega_{\alpha \pm} P_\alpha (s). \quad (3.13)$$

Using the definition of the channel wave operator (2.24) in (3.13) and letting it act on a spin $s$ channel state, $|\xi\rangle = P_\alpha (s) |\xi\rangle$, gives

$$0 = \lim_{t \to \pm \infty} \| Q(s) T(t) \Phi_\alpha T_\alpha (-t) |\xi\rangle \| =$$

$$\lim_{t \to \pm \infty} \| Q(s) T(t) A_{a(\alpha)} \Phi_\alpha T_\alpha (-t) |\xi\rangle \| =$$

$$\lim_{t \to \pm \infty} \| [1 - P(s)] T(t) A_{a(\alpha)} P_\alpha (s) \Phi_\alpha T_\alpha (-t) |\xi\rangle \| =$$
\[
\lim_{t \to \pm \infty} \|[1 - P(s)]A_{a(\alpha)}P_{a(\alpha)}(s) \Phi_\alpha T_\alpha(-t)\| = \quad (3.14)
\]

\[
\lim_{t \to \pm \infty} \|[1 - \sum_r P(r)A_{a(\alpha)}P_{a(\alpha)}(r)] \Phi_\alpha T_\alpha(-t)\| =
\]

\[
\lim_{t \to \pm \infty} \|[1 - A_{a(\alpha)}(s)A_{a(\alpha)}(s)] \Phi_\alpha T_\alpha(-t)\| = \quad (3.15)
\]

where the relation \([P(s), T(t)]_\alpha = 0\), equation (2.16) and the asymptotic equivalence

\[
\lim_{t \to \pm \infty} \|[1 - X_{a(\alpha)}] \Phi_\alpha T_\alpha(-t)\| = 0 
\]

were used to obtain (3.15). This proves the first part of lemma 2. To prove the second part note

\[
\lim_{t \to \pm \infty} \|[1 - X^\dagger_{a(\alpha)}] \Phi_\alpha T_\alpha(-t)\| =
\]

\[
\lim_{t \to \pm \infty} \|[1 - \sum_r P_{a(\alpha)}(r)A_{a(\alpha)}^\dagger P(r)] \Phi_\alpha T_\alpha(-t)\| =
\]

\[
\lim_{t \to \pm \infty} \|[P_{a(\alpha)}(s)A_{a(\alpha)}^\dagger A_{a(\alpha)} - \sum_r P_{a(\alpha)}(r)A_{a(\alpha)}^\dagger P(r)P_{a(\alpha)}(s)] \Phi_\alpha T_\alpha(-t)\| \leq
\]

\[
\lim_{t \to \pm \infty} \|[\sum_{r \neq s} P_{a(\alpha)}(r)A_{a(\alpha)}^\dagger P(r)Q(s)P_{a(\alpha)}(s)] \Phi_\alpha T_\alpha(-t)\| +
\]

\[
\|[P_{a(\alpha)}(s)A_{a(\alpha)}^\dagger (P(s) - A_{a(\alpha)})P_{a(\alpha)}(s) \Phi_\alpha T_\alpha(-t)\| \leq
\]

\[
\lim_{t \to \pm \infty} \|[\sum_{r \neq s} P_{a(\alpha)}(r)A_{a(\alpha)}^\dagger P(r)Q(s)] + \|[P_{a(\alpha)}(s)A_{a(\alpha)}^\dagger (P(s) - 1)P_{a(\alpha)}(s) \Phi_\alpha T_\alpha(-t)\| \| (3.17)
\]

which vanishes by (3.14) (the coefficient is bounded by lemma 1). Equation (3.16) was used again in (3.17). The completes the proof of lemma 2.

The operators \(X_{a(\alpha)} \Phi_\alpha\) and \(X^\dagger_{a(\alpha)} \Phi_\alpha\) are suitable injection operators for a scattering theory but neither one is unitary. The next lemma provides a unitary injection operator constructed out of \(X_a\) and \(X^\dagger_a\).
Lemma 3: For $|\xi\rangle \in H_\alpha$: the assumptions of the theorem imply:

$$\lim_{t \to \pm \infty} \| [Y_{a(\alpha)} - 1] \Phi_\alpha T_\alpha(-t)|\xi\rangle \| = 0$$  \hfill (3.18)

where

$$Y_{a(\alpha)} := (X_{a(\alpha)}X_{a(\alpha)}^\dagger)^{-1/2}X_{a(\alpha)}.$$

Proof: For the proof of this theorem the notation $a$ is used as an abbreviation for $a(\alpha)$. Since $X_a$ is bounded by lemma 1 and has bounded inverse by assumption it follows that $(X_a^\dagger)^{-1} = (X_a^{-1})^\dagger$ exists and that $X_aX_a^\dagger$ is a bounded positive operator with bounded inverse. Consequently, $(X_aX_a^\dagger)^{-1/2}$ exists and is bounded with bounded inverse. To prove the theorem I consider the inequalities:

$$\| [Y_a - 1] \Phi_\alpha T_\alpha(-t)|\xi\rangle \| =$$

$$\| [(X_aX_a^\dagger)^{-1/2}X_a - 1] \Phi_\alpha T_\alpha(-t)|\xi\rangle \| \leq$$

$$\| [(X_aX_a^\dagger)^{-1/2}] \| [X_a - 1] \Phi_\alpha T_\alpha(-t)|\xi\rangle \| + \| [(X_aX_a^\dagger)^{-1/2} - 1] \Phi_\alpha T_\alpha(-t)|\xi\rangle \| \leq$$

$$\| [(X_aX_a^\dagger)^{-1/2}] \| [X_a - 1] \Phi_\alpha T_\alpha(-t)|\xi\rangle \| + \| [(X_aX_a^\dagger)^{-1/2}] \| [X_aX_a^\dagger - 1] \Phi_\alpha T_\alpha(-t)|\xi\rangle \|. \hfill (3.20)$$

As $t \to \pm \infty$ the first term vanishes by lemma 2 and the boundedness of $(X_aX_a^\dagger)^{-1/2}$. The second term is bounded by:

$$\| [(X_aX_a^\dagger)^{-1/2}] \| [X_a - 1] \Phi_\alpha T_\alpha(-t)|\xi\rangle \| \leq$$

$$\| [(X_aX_a^\dagger)^{-1/2}] \| [X_a - 1] \Phi_\alpha T_\alpha(-t)|\xi\rangle \| \leq$$

$$\| [(X_aX_a^\dagger)^{-1/2}] \| [X_a - 1] \Phi_\alpha T_\alpha(-t)|\xi\rangle \| + \| [X_a - 1] \Phi_\alpha T_\alpha(-t)|\xi\rangle \|. \hfill (3.21)$$

which also vanishes as $t \to \pm \infty$ by lemma 2. Note that the operator norms in the above expression are all finite. This completes the proof of lemma 3.
The next lemma establishes that $Y_a$ is unitary, commutes with the kinematic subgroup of the light front, and intertwines $j^2$ and $j_a^2$:

**Lemma 4:** Under the assumptions of the theorem the operator $Y_a$ defined in the previous lemma is unitary, commutes with the kinematic subgroup of the light front, and satisfies $j^2 Y_a = Y_a j_a^2$.

**Proof:** Unitarity follows by computation:

$$Y_a^{-1} = [(X_a X_a^\dagger)^{-1/2} X_a]^{-1} = X_a^{-1} (X_a X_a^\dagger)^{1/2} = X_a^\dagger (X_a^\dagger)^{-1} X_a^{-1} (X_a X_a^\dagger)^{1/2} =$$

$$X_a^\dagger (X_a X_a^\dagger)^{-1/2} (X_a X_a^\dagger)^{1/2} = X_a^\dagger X_a X_a^\dagger = Y_a^\dagger.$$

(3.22)

The intertwining property also follows by computation:

$$j^2 X_a = j^2 \sum_s P(s) A_a P_a(s) = \sum_s s(s + 1) P(s) A_a P_a(s) = \sum_s P(s) A_a P_a(s) s(s + 1) =$$

$$\sum_s P(s) A_a(\alpha) P_a(s) j_a^2 = X_a j_a^2.$$

(3.23)

Similarly it can be shown that $j_a^2 X_a^\dagger = X_a^\dagger j_a^2$. It follows that

$$[(X_a X_a^\dagger), j^2]_- = 0 \Rightarrow [(X_a X_a^\dagger)^{-1/2}, j^2]_- = 0.$$

(3.24)

Combining (3.24) with (3.23) gives

$$j^2 Y_a = j^2 (X_a X_a^\dagger)^{-1/2} X_a = (X_a X_a^\dagger)^{-1/2} j^2 X_a = (X_a X_a^\dagger)^{-1/2} X_a j_a^2 = Y_a j_a^2.$$

(3.25)

That $Y_a$ commutes with the kinematic subgroup follows because each $P(s), P_a(s)$, and $A_a$ commute with the kinematic subgroup. This completes the proof of lemma 4.

**Proof of the Theorem:** The first step in the proof of the theorem is to construct a scattering equivalence that eliminates the interaction dependence in $j^2$. The second step is to construct another scattering equivalence that removes the interaction dependence in $\hat{z} \times \hat{j}$. I follow closely the discussion in [3].
To prove the theorem I define the transformed representation of the Poincaré group:

\[ U_y(\Lambda, x) := Y_0^\dagger U(\Lambda, x) Y_0 \]  

(3.26)

and the transformed channel injection operators

\[ \Phi_{y\alpha} = Y_0^\dagger Y_{a(\alpha)} \Phi_\alpha \]  

(3.27)

where \( Y_0 \) is the \( Y_{a(\alpha)} \) corresponding to the \( N \)-cluster partition \( a = 0 \). The full-two Hilbert space injection operator for this representation is defined by

\[ \Phi_y := \sum_{\alpha} \Phi_{y\alpha} \]  

(3.28)

where the sum runs over all channels including bound (one-cluster channels).

Lemma three implies that for

\[ \bar{\Phi} := \sum_{\alpha} Y_\alpha \Phi_\alpha \]  

(3.29)

that \( \Omega_{\pm}(T, \Phi, T_f) = \Omega_{\pm}(T, \Phi, T_f) \) which gives the relation

\[ \Omega_{y\pm} := \Omega_{\pm}(T_y, \Phi_y, T_f) = \Omega_{\pm}(Y_0^\dagger T Y_0, Y_0^\dagger \Phi, T_f) = Y_0^\dagger \Omega_{\pm}(T, \Phi, T_f) = Y_0^\dagger \Omega_{\pm}. \]  

(3.30)

This equation establishes both the existence of the transformed wave operators and the scattering equivalence with the original theory.

The transformed injection operators are constructed to satisfy the intertwining properties similar to (2.16):

\[ \Phi_y U_f(\Lambda, x) |\xi_\alpha\rangle = \bar{U}_{a(\alpha)}(\Lambda, x) \Phi_y |\xi_\alpha\rangle \]  

(3.31)

where

\[ \bar{U}_a(\Lambda, x) := Y_0^\dagger Y_a U_a(\Lambda, x) Y_a^\dagger Y_0 \]  

(3.32)

and where for each partition \( a, \bar{U}_a(\Lambda, x) \) is a unitary representation of the Poincaré group with (1) the kinematic subgroup of the light front and (2) the same spin operator as the noninteracting
system. The first claim follows because $U_a(\Lambda, x)$ has the kinematic subgroup of the light front for each $a$ and $Y_a^\dagger Y_0$ commutes with the kinematic subgroup by lemma four. The second claim follows by lemma four because

$$Y_a^\dagger Y_0 j^2 \equiv j^2 Y_0 = j^2_a Y_a^\dagger Y_0.$$ (3.33)

The asymptotic completeness and Poincaré invariance of the wave operators $\Omega_{y\pm}$ follow from the corresponding properties of the untransformed wave operators and the relation (3.30):

$$\Omega_{y\pm}^\dagger = Y_0^\dagger \Omega_{\pm}.$$ (3.34)

It follows that $U_y(\Lambda, x)$ and $U(\Lambda, x)$ are scattering equivalent. This does not complete the proof of the theorem because although $U_y(\Lambda, x)$ has the kinematic subgroup of the light-front and satisfies $j_y^2 = j_0^2$, it does not follow that $\vec{j}_y = \vec{j}_0$. Specifically the raising and lowering operators for the front-form spin may be interaction dependent in this representation. Given that this is the case, the Lorentz invariance of the scattering operator implies

$$[j_{f\pm}, \Omega_{y\pm}^\dagger, \Omega_{y-}] = 0$$ (3.35)

where $j_{f\pm}$ are the asymptotic raising and lowering operators. Since the scattering matrix also commutes with $j_f^2$ and $\vec{z} \cdot \vec{j}_f$, the matrix elements have the form:

$$f(\beta_s \mu | S | \beta' s' \mu') = \delta_{ss'} \delta_{\mu \mu'} f(\beta | \hat{S} (s) | \beta')$$ (3.36)

where $\beta$ denotes the remaining quantum numbers. The matrix elements of the wave operators between the non-interacting and asymptotic basis vectors satisfy

$$0(\beta_s \mu | \Omega_{y\pm} | \beta' s' \mu') = \delta_{ss'} \delta_{\mu \mu'} 0(\beta | \hat{\Omega}_{\pm} (s, \mu) | \beta')$$ (3.37)

because the wave operators intertwine to the $y$ representation, where $j^2$ and $\vec{z} \cdot \vec{j}$ are kinematic but the raising and lowering operators may be non-trivial. The non-triviality of the raising and
lowering operators results in the $\mu$ dependence in matrix elements of the reduced wave operator $\hat{\Omega}_\pm (s, \mu)$. Equation (3.36) implies that

$$f \langle \beta | \hat{\Omega}_\mp (s, \mu) \hat{\Omega}_\mp -(s, \mu) | \beta' \rangle_f = f \langle \beta | \hat{\Omega}_\mp (s, s) \hat{\Omega}_\mp -(s, s) | \beta' \rangle_f$$

(3.38)

independent of $\mu$. Asymptotic completeness then implies

$$0 \langle \beta s \mu | Z | \beta' s' \mu' \rangle_0 := \delta_{ss'} \delta_{\mu\mu'} 0 \langle \beta | \hat{\Omega}_\mp -(s, s) \hat{\Omega}_\mp +(s, s) | \beta' \rangle_0 = \delta_{ss'} \delta_{\mu\mu'} 0 \langle \beta | \hat{\Omega}_\mp -(s, s) \hat{\Omega}_\mp +(s, \mu) | \beta' \rangle_0$$

(3.39)

is unitary and independent of the choice of asymptotic condition. It is useful to introduce the notation

$$0 \langle \beta s \mu | \bar{\Omega} \mp | \beta' s' \mu' \rangle_f := \delta_{ss'} \delta_{\mu\mu'} 0 \langle \beta | \bar{\Omega} \mp -(s, s) | \beta' \rangle_f .$$

(3.40)

In this notation (3.38) becomes

$$Z = \bar{\Omega}_\mp \Omega_\mp^\dagger = \hat{\Omega}_\mp - \Omega_\mp^\dagger$$

(3.41)

from which it follows that

$$\bar{\Omega}_\mp = Z \Omega_\mp .$$

(3.42)

The structure of $\bar{\Omega}_\mp$ implies that the spin raising and lowering operators satisfy

$$j_{0\pm} \bar{\Omega}_\mp = \bar{\Omega}_\mp j_{f\pm}$$

(3.43)

since by construction there is no $\mu$ dependence in the kernel of the matrix elements of $\bar{\Omega}_\mp$ in the free-asymptotic representation.

The result is that the desired representation is given by

$$\tilde{U}_y(\Lambda, x) := Z \gamma_0^\dagger U(\Lambda, x) Y_0 Z^\dagger$$

(3.44)

with channel injection operators given by

$$\Phi_\alpha = Z \gamma_0^\dagger Y_\alpha \Phi_\alpha$$

(3.45)

To show that this system leads to a scattering theory with the desired properties note the
wave operators exist since:

\[ \Omega_{y\pm}(\bar{T}_y, \Phi_y, T_f) = \Omega_{y\pm}(ZT_yZ^\dagger, Z\Phi_y, T_f) = Z\Omega_{y\pm}(T_y, \Phi_y, T_f) = ZY_0\Omega_{y\pm}(T, \Phi, T_f) \quad (3.46) \]

exists. The scattering equivalence to the original wave operators is a consequence of (3.46).

These operators are asymptotically complete since

\[ \bar{\Omega}_{y+} \bar{\Omega}_{y+}^\dagger = Z\Omega_{y+} \Omega_{y+}^\dagger = Z\Omega_{y+} \Omega_{y+}^\dagger = \bar{\Omega}_{y-} \bar{\Omega}_{y-} \quad (3.47) \]

and Poincaré invariant since

\[ \bar{U}(\Lambda, x)\Omega_{y\pm} = ZU_y(\Lambda, x)\Omega_{y\pm} = Z\Omega_{y\pm}U_f(\Lambda, x) = \bar{\Omega}_{y\pm}U_f(\Lambda, x). \quad (3.48) \]

The kinematic invariance follows because \( Z \) and \( Y_0 \) commute with the kinematic subgroup. Finally the intertwining properties of the wave operators, \( Y_0 \) and \( Z \) ensure that the spin vector is kinematic. This completes the proof of the theorem.

4. Conclusion:

In this paper I have given a sufficient condition for a front-form quantum model with an interacting spin to be scattering equivalent to a front-form model with a kinematic front-form spin. The essential property is that there exist a set of kinematically invariant unitary operators \( A_a \) such that \( A_{a(\alpha)} \) is asymptotically equivalent to the identity with respect to the channel injection operator \( \Phi_a \) and has the property that the operators \( X_a = \sum_s P(s)A_aP_a(s) \), have bounded inverses.

The simplest case is when \( \sum_s P(s)P_a(s) \) has a bounded inverse. In this case all of the operators \( A_a \) can be taken to be the identity. When this fails the freedom to choose the operators \( A_a \) can be utilized, although choice of \( A_a \) is limited by the asymptotic conditions. The theorem is also applicable to the case where the model satisfies spacelike cluster properties. In that case if the operators \( A_a \) are taken to be \( A_a = B^\dagger B_a \) where \( B \) and \( B_a \) are the Sokolov or packing operators that map representations that cluster properly to representation with a kinematic spin, all of the conditions of the theorem are satisfied.
In general models need to be considered on a case to case basis. The ability to choose the operators $A_a$ leaves a fair amount of flexibility in establishing the scattering equivalence. As it stands, the theorem does not directly apply to the interesting case of field theories. The basic elements of the proof should extend to the field theoretic case although it is not known how to control the usual difficulties that arise in models with an infinite numbers of degrees of freedom.

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