Cyclic Uniform 2-Factorizations of the Complete Multipartite Graph

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Abstract
A generalization of the Oberwolfach problem, proposed by Liu (J Comb Des 8:42–49, 2000), asks for a uniform 2-factorization of the complete multipartite graph $K_{m \times n}$. Here we focus our attention on cyclic 2-factorizations, whose 2-factors are disjoint union of cycles all of even length $\ell$. In particular, we present a complete solution for the extremal cases $\ell = 4$ and $\ell = mn$.

Keywords Cycle · 2-factorization · Complete multipartite graph

1 Introduction
Throughout this paper, $K_{m \times n}$ will denote the complete multipartite graph with $m$ parts of same cardinality $n$. If $n = 1$, we may identify $K_{m \times 1}$ with the complete graph on $m$ vertices $K_m$, while $K_{1 \times n}$ is a union of $n$ disjoint vertices. So from now we consider $K_{m \times n}$ with $m, n > 1$. Also, note that $K_{m \times 2}$ is nothing but the cocktail party graph $K_{2m} - I$, namely the graph obtained from $K_{2m}$ by removing a 1-factor $I$, that is, a set of $m$ pairwise disjoint edges.

For any graph $\Gamma$ we write $V(\Gamma)$ for the set of its vertices and $E(\Gamma)$ for the set of its edges. We denote by $(c_0, c_1, \ldots, c_{\ell-1})$ the cycle of length $\ell$ whose edges are $[c_0, c_1], [c_1, c_2], \ldots, [c_{\ell-1}, c_0]$. An $\ell$-cycle system of a graph $\Gamma$ is a set $\mathcal{B}$ of cycles of length $\ell$ whose edges partition $E(\Gamma)$; clearly a graph may admit a cycle system only if the degree of each its vertex is even.

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A 2-factor of a graph $\Gamma$ is a set of cycles whose vertices partition $V(\Gamma)$. A 2-factorization of $\Gamma$ is a set $\mathcal{F}$ of 2-factors such that any edge of $\Gamma$ appears in exactly one member of $\mathcal{F}$. Hence the cycles appearing in the 2-factors of $\mathcal{F}$ form a cycle system of $\Gamma$, called the underlying cycle system of $\mathcal{F}$. A 2-factorization whose 2-factors are all isomorphic to a given 2-factor is said to be uniform. When the underlying cycle system of $\mathcal{F}$ consists of $\ell$-cycles, the uniform 2-factorization is called a $C_\ell$-factorization. In this paper we focus our attention on the case $\ell$ even. In particular we deal with the extremal cases: $\ell = 4$ and $\ell = mn$. In the second case one speaks of a hamiltonian 2-factorization, since each 2-factor is a hamiltonian cycle of $K_{mn}$.

The problem of finding 2-factorizations of $K_{m \times n}$ is a natural generalization of the analogous problem for the complete graph, i.e. of the well-known Oberwolfach problem (OP) proposed by G. Ringel in 1967. Even if OP has been devoted of an extensive research activity, the complete solution has not yet been achieved. However, many partial existence results are known, see for example [1,20] and the references therein. On the other hand very few results have been obtained about 2-factorizations of $K_{m \times n}$. The generalization of OP to the multipartite case was proposed by Liu [11]:

“At a gathering there are $m$ delegations each having $n$ people. Is it possible to arrange a seating of $mn$ people present at $s$ round tables $T_1, T_2, \ldots, T_s$ (where each $T_i$ can accommodate $\ell_i \geq 3$ people and $\sum \ell_i = mn$) for a suitable number of different meals so that each person has every other person not in the same delegation for a neighbor exactly once?”

Liu gave a complete solution for the existence of $C_\ell$-factorizations of $K_{m \times n}$ in [11,12]. Bryant et al. [2] completely solved the case of uniform 2-factorizations of $K_{m \times n}$ whose 2-factors are disjoint union of cycles of even length. Note that other generalizations of OP have been considered, for example, see [7–9,13,16,18]. Here we consider cyclic $C_\ell$-factorizations of $K_{m \times n}$.

In general, given an additive group $G$ of order $v$ and a graph $\Gamma$ such that $V(\Gamma) = G$ one can consider the regular (right) action of $G$ on $V(\Gamma)$ defined by $x \mapsto x + g$, for any $x \in V(\Gamma)$ and any $g \in G$. Clearly, this induces a natural action of $G$ on the subgraphs of $\Gamma$. Following the standard terminology (see, for instance, [6]) a 2-factorization $\mathcal{F}$ of $\Gamma$ is said to be regular under the action of $G$ if for any $F \in \mathcal{F}$ and any $g \in G$, we have also $F + g \in \mathcal{F}$. Note that in this case $\mathcal{F}$ admits $G$ as an automorphism group acting sharply transitively on the vertices of $\Gamma$. If $G$ is a cyclic group, a 2-factorization regular under $G$ is simply said cyclic. A natural reason to looking for a regular 2-factorization $\mathcal{F}$ is that this additional property allows one to economize, sometimes considerably, the description of $\mathcal{F}$; it suffices in fact to give a complete system of representatives for the $G$-orbits on $\mathcal{F}$ rather than to list all the 2-factors of $\mathcal{F}$.

We point out that many authors have investigated the existence of regular solution of OP, even if a (non-regular) solution was already known, for instance see [5,6,14]. Analogously, in this paper, we want to examine when the problem of factorizing $K_{m \times n}$ into cycles of uniform length $\ell$ has a regular (in particular cyclic) solution, even if the problem (with a non-regular solution) has been completely solved in [2,11,12]. The main results of the paper are the following theorems, that completely solve the cases $\ell = 4$ and $\ell = mn$ even.
Theorem 1 A cyclic $C_4$-factorization of $K_{m \times n}$ exists if and only if one of the following cases occurs:

(a) $m$, $n$ are both even;
(b) $m$ is odd, $n \equiv 0 \pmod{4}$ and $(m, n) \neq (p^\alpha, 4)$, where $p$ is a prime.

Theorem 2 Let $mn$ be even. A cyclic hamiltonian $2$-factorization of $K_{m \times n}$ exists if and only if all these three conditions are satisfied:

(a) $n$ is even;
(b) if $n \equiv 2 \pmod{4}$, then $m \equiv 1, 2 \pmod{4}$;
(c) if $n = 2$, then $m \neq p^\alpha$ where $p$ is an odd prime.

The paper is organized as follows. In Sect. 2, we give some preliminary results and we talk about the concept of a 2-starter which plays a fundamental role in all our constructions. Then, in Sect. 3, we provide necessary conditions for the existence of cyclic $C_\ell$-factorizations and in Sect. 4 we introduce some auxiliary sets and notation which allow us to illustrate the constructions in a simpler and more elegant way. Sections 5 and 6 contain direct constructions of cyclic $C_4$-factorizations of $K_{m \times n}$. In Sect. 7 direct constructions of cyclic hamiltonian 2-factorizations of $K_{m \times n}$ are given. Finally, in Sect. 8 we prove Theorems 1 and 2, using the results of the previous sections.

2 Preliminaries

As remarked in the Introduction, a necessary condition for the existence of a 2-factorization of $K_m$ is that $m - 1$ is even. Hence, a $C_4$-factorization of the complete graph cannot exist. On the other hand, the following results about cyclic hamiltonian 2-factorizations are known.

Theorem 3 [5] There exists a cyclic hamiltonian 2-factorization of $K_m$ if and only if $m$ is odd with $m \neq 15$ and $m \neq p^\alpha$ where $p$ is an odd prime and $\alpha > 1$.

Theorem 4 [10] There exists a cyclic hamiltonian 2-factorization of $K_{m \times 2}$ if and only if $m \equiv 1, 2 \pmod{4}$ and $m \neq p^\alpha$ with $p$ an odd prime and $\alpha \geq 1$.

Theorem 5 [15] Let $m$ be even; a cyclic hamiltonian 2-factorization of $K_{m \times n}$ exists if and only if

(a) $n$ is even, and
(b) if $n \equiv 2 \pmod{4}$, then $m \equiv 2 \pmod{4}$.

In the rest of the paper when speaking of a group $G$ we always mean that $G$ is written in the additive notation. Also, if $H$ is a subgroup of $G$, then a complete system of distinct representatives for the left cosets of $H$ in $G$ will be called a left transversal for $H$ in $G$.

Now, we recall the definition of a Cayley graph on a finite group $G$ with connection set $\Omega$, denoted by $Cay[G : \Omega]$. Let $\Omega \subseteq G \setminus \{0\}$ such that for every $\omega \in \Omega$ we also have $-\omega \in \Omega$. The Cayley graph $Cay[G : \Omega]$ is the graph whose vertices
are the elements of $G$ and in which two vertices are adjacent if and only if their difference is an element of $\Omega$. Note that $K_{m \times n}$ can be interpreted as the Cayley graph $\text{Cay}(\mathbb{Z}_{mn} : \mathbb{Z}_{mn} \setminus m\mathbb{Z}_{mn})$, where by $m\mathbb{Z}_{mn}$ we mean the subgroup of index $m$ of $\mathbb{Z}_{mn}$. The vertices of $K_{m \times n}$ will be always understood as elements of $\mathbb{Z}_{mn}$ and the parts of $K_{m \times n}$ are the cosets of $m\mathbb{Z}_{mn}$ in $\mathbb{Z}_{mn}$.

Given a graph $\Gamma$ with vertices in $G$, the list of differences from $\Gamma$ is the multiset so defined:

$$\Delta \Gamma = \{ \pm(x - y) \mid [x, y] \in E(\Gamma) \}.$$ 

More generally, given a collection $\gamma$ of graphs with vertices in $G$, by $\Delta \gamma$ we denote the union (counting multiplicities) of all multisets $\Delta \Gamma$ with $\Gamma \in \gamma$. If $\Gamma$ is a 2-regular graph, i.e. a disjoint union of cycles, we can introduce also the list of partial differences. For this purpose the following notation will be useful.

**Notation 6** Let $c_0, c_1, \ldots, c_{r-1}, x$ be elements of $G$, with $x$ of order $d$. The closed trail

$$C = [c_0, c_1, c_2, \ldots, c_{r-1},$$

$$c_0 + x, c_1 + x, c_2 + x, \ldots, c_{r-1} + x, \ldots,$$

$$c_0 + (d - 1)x, c_1 + (d - 1)x, c_2 + (d - 1)x, \ldots, c_{r-1} + (d - 1)x]$$

will be denoted by

$$[c_0, c_1, \ldots, c_{r-1}]_x.$$ 

Also, we set

$$\phi(C) = \{c_0, c_1, c_2, \ldots, c_{r-1}\},$$

$$\partial C = \{\pm(c_{h+1} - c_h) \mid 0 \leq h \leq r - 2\} \cup \{\pm(c_0 + x - c_{r-1})\}.$$ 

The multiset $\partial C$ is called the list of partial differences from $C$.

We underline that many important results have been obtained with the method of partial differences, e.g., see [4–6].

More generally, given a collection $\mathcal{C}$ of cycles with vertices in $G$, by $\phi(\mathcal{C})$ and $\partial \mathcal{C}$ one means the union (counting multiplicities) of all multisets $\phi(C)$ and $\partial C$ respectively, where $C \in \mathcal{C}$. Notice that the closed trail $[c_0, c_1, \ldots, c_{r-1}]_x$ is a cycle if and only if the elements $c_i$, for $i = 0, \ldots, r - 1$, belong to pairwise distinct cosets of the subgroup $\langle x \rangle$ of $G$.

Now, let $C = (c_0, c_1, \ldots, c_{\ell - 1})$ be an $\ell$-cycle with vertices in $G$. The stabilizer of $C$ under the action of $G$ is the subgroup $\text{Stab}_G(C)$ of $G$ defined by $\text{Stab}_G(C) = \{ g \in G : C + g = C \}$, where $C + g$ is the cycle $(c_0 + g, c_1 + g, \ldots, c_{\ell - 1} + g)$. The $G$-orbit of $C$ is the set $\text{Orb}_G(C)$ of all distinct cycles in the collection $\{C + g : g \in G\}$. Then $\phi(C) = \{c_0, c_1, \ldots, c_{\ell - 1}\}$ and $\partial C = \{\pm(c_{h+1} - c_h) \mid 0 \leq h < \ell / d\}$, where
\[ d = |Stab_G(C)|. \] Note that if \( d = 1 \) then \( \phi(C) = V(C) \) and \( \partial C = \Delta C \). The notation \( (c_0, c_1, \ldots, c_{\ell-1}) \) will be maintained only for a cycle with trivial stabilizer.

As well as many results about regular 1-factorizations of the complete graph have been obtained using the notion of a starter, see for instance [3,17,19], in this paper we obtain cyclic 2-factorizations of \( K_{m\times n} \) using a generalization of the concept of a 2-starter introduced in [6, Definition 2.3].

**Definition 7** Let \( H \) be a subgroup of \( G \). A 2-starter in \( G \) relative to \( H \) is a collection \( \Sigma = \{S_1, \ldots, S_t\} \) of 2-regular graphs with vertices in \( G \) satisfying the following conditions:

(a) \( \partial S_1 \cup \cdots \cup \partial S_t = G \setminus H \);

(b) \( \phi(S_i) \) is a left transversal of some subgroup \( H_i \) of \( G \) containing the stabilizers of all cycles of \( S_i \), for \( i = 1, \ldots, t \).

Note that if \( H = \{0\} \) we find again the definition given in [6]. The importance of investigating the existence of 2-starters is explained in the following theorem.

**Theorem 8** Let \( H \) be a subgroup of \( G \). The existence of a \( G \)-regular 2-factorization of \( K_{|G:H|\times|H|} \) is equivalent to the existence of a 2-starter in \( G \) relative to \( H \).

**Proof** Suppose \( \Sigma = \{S_1, \ldots, S_t\} \) is a 2-starter in \( G \) relative to \( H \). By definition, for \( i = 1, \ldots, t \), there is a suitable subgroup \( H_i \) of \( G \) such that \( \phi(S_i) \) is a left transversal for \( H_i \) in \( G \). Set \( F_i = \bigcup_{A \in S_i} Orb_{H_i}(A) \). Reasoning as in the proof of [6, Theorem 2.4] one can easily see that

\[ \mathcal{F} = Orb_G(F_1) \cup Orb_G(F_2) \cup \cdots \cup Orb_G(F_t) \]

is a 2-factorization of \( K_{|G:H|\times|H|} \).

Suppose now \( \mathcal{F} \) to be a \( G \)-regular 2-factorization of \( K_{|G:H|\times|H|} \). Let \( \{F_1, \ldots, F_t\} \) be a complete system of representatives for the \( G \)-orbits of the factors of \( \mathcal{F} \). For each \( i \), denote by \( H_i \) the stabilizer in \( G \) of \( F_i \) and let \( S_i \) be a complete system of representatives for the \( H_i \)-orbits of the cycles that are contained in \( F_i \). Again reasoning as in the proof of [6, Theorem 2.4], one can prove that \( \Sigma = \{S_1, \ldots, S_t\} \) is a 2-starter in \( G \) relative to \( H \).

In the following, we will apply Theorem 8 for \( G = \mathbb{Z}_{mn} \) and \( H = m\mathbb{Z}_{mn} \).

### 3 Necessary Conditions for the Existence of Cyclic \( C_\ell \)-Factorizations

First of all, we recall that a trivial necessary condition for the existence of a 2-factorization of \( K_{m\times n} \) is that the degree \( (m - 1)n \) of each vertex is even. Also, since we have to organize the \( \ell \)-cycles into 2-factors, \( \ell \) must divide \( |V(K_{m\times n})| = mn \).

**Remark 9** From our hypothesis \( \ell \) even, by the above trivial conditions it follows that \( n \) must be even.
On the other hand, since we are looking for cyclic $C$-factorizations we need also the necessary conditions for the existence of a cyclic $C$-cycle system of $K_{m \times n}$, given in [15].

**Theorem 10** [15, Theorem 3.3] Let $n$ be an even integer. A cyclic $\ell$-cycle system of $K_{m \times n}$ cannot exist in each of the following cases:

(a) $m \equiv 0 \pmod{4}$ and $|\ell|_2 = |m|_2 + 2|n|_2 - 1$;
(b) $m \equiv 1 \pmod{4}$ and $|\ell|_2 = |m - 1|_2 + 2|n|_2 - 1$;
(c) $m \equiv 2, 3 \pmod{4}$, $n \equiv 2 \pmod{4}$ and $\ell \not\equiv 0 \pmod{4}$;
(d) $m \equiv 2, 3 \pmod{4}$, $n \equiv 0 \pmod{4}$ and $|\ell|_2 = 2|n|_2$;

where, given a positive integer $x$, $|x|_2$ denotes the largest $e$ for which $2^e$ divides $x$.

If the cycles of the system are all hamiltonian, that is $\ell = mn$, we obtain the following corollary.

**Corollary 11** Let $n$ be an even integer. A cyclic hamiltonian 2-factorization of $K_{m \times n}$ cannot exist if both $m \equiv 0, 3 \pmod{4}$ and $n \equiv 2 \pmod{4}$.

We remark that if $\ell = 4$ from Theorem 10 we do not obtain non-existence results. On the other hand we prove the following.

**Proposition 12** For any odd prime $p$ and any integer $\alpha \geq 1$, a cyclic $C_4$-factorization of $K_{p^\alpha \times 4}$ cannot exist.

**Proof** In view of Theorem 8 we prove that a 2-starter in $\mathbb{Z}_{4p^\alpha}$ relative to $p^\alpha \mathbb{Z}_{4p^\alpha}$ does not exist. By way of contradiction, suppose the existence of such a 2-starter $\Sigma$. Consider $d = 4p^{\alpha - 1}$. By item (a) of Definition 7, there exists $S \in \Sigma$ such that $d \in \partial S$. In particular, $d \in \partial C$ for a suitable 4-cycle $C$ of $S$. Also, by item (b) of the same definition, $\phi(S)$ is a transversal for some subgroup $H$ of $\mathbb{Z}_{4p^\alpha}$ containing the stabilizers of all cycles of $S$.

Note that a 4-cycle of type $[0]$, cannot exist, otherwise $t = p^\alpha$, in contradiction with item (a) of Definition 7. We point out that this implies that $|\phi(S)|$ has to be even. Since $d \in \partial C$, we may suppose that there are two vertices $x, y \in \phi(S)$ such that $x - y = d$. Recalling that $\phi(S)$ is a transversal of $H$, we obtain that $d \notin H$.

Now, $H$ is a cyclic group, so suppose $H = \langle h \rangle$. Hence $h \nmid d = 4p^{\alpha - 1}$ and so we have three possibilities: $h = p^\alpha$ and $|H| = 4$; $h = 2p^\alpha$ and $|H| = 2$; $h = 4p^\alpha$ and $|H| = 1$. If $|H| \leq 2$, then $|\phi(S)| \geq 2p^\alpha$ which implies the absurd $|\partial S| \geq 4p^\alpha$. So $|H| = 4$ and $|\phi(S)| = p^\alpha$, which is impossible because, as previously remarked, $|\phi(S)|$ must be even.

**Corollary 13** If there exists a cyclic $C_4$-factorization of $K_{m \times n}$ then the following conditions hold:

(a) $n$ is even;
(b) $n \equiv 0 \pmod{4}$ if $m$ is odd;
(c) $n \not\equiv 4$ if $m = p^\alpha$ for some odd prime $p$. 

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4 Auxiliary Sets and Notation

In order to give a uniform construction of cyclic $C_4$-factorizations of $K_{m \times n}$ for any choice of admissible values of $m$ and $n$, it is convenient to fix some notations and define auxiliary sets we will use in the present paper.

Given $k \in \mathbb{Z}$ we denote by $\|k\|$ the set $\|k\| = \{0, 1, 2, \ldots, k\} \subseteq \mathbb{N}$ if $k \geq 0$, $\emptyset$ if $k < 0$.

Furthermore, if $a, b \in \mathbb{Z}$ and $I \subseteq \mathbb{Z}$, then $a + bI$ denotes the subset \{a + bi : i \in I\}. According to this notation, when $I = \emptyset$, then $a + bI = \emptyset$. For instance, $2 + 5 \|3\|$ is the arithmetic progression \{2, 7, 12, 17\}.

Our method for constructing cyclic $C_4$-factorizations of $K_{m \times n}$ is the following. We first construct sets $L = \{(a_i, b_i) : i \in I_L\}$ of edges with $a_i, b_i \in \|mn\| - 1\| \in \mathbb{Z}$ in such a way that the largest possible number of even integers in $\|mn\| \setminus m\mathbb{Z}$ appears in exactly one set $\Delta L = \{\pm(a_i - b_i) : i \in I_L\}$. Starting from the sets of edges previously described, we construct 2-regular graphs $\square L$, whose connected components are 4-cycles, as follows. Let

$$\square L = \bigcup_{i \in I_L} [a_i, b_i]_{\frac{mn}{2}}$$

and define $\phi(L) = \phi(\square L)$. Notice that $\partial(\square L)$ can be written as the following disjoint union:

$$\partial(\square L) = (\Delta L) \cup \left(\frac{mn}{2} - \Delta L\right).$$

Then, we deal with the remaining even integers (if they exist). Next, we consider the odd integers in $\|mn\| - 1\| \setminus m\mathbb{Z}$, not appearing in the sets of partial differences of the 2-regular graphs, previously constructed. More precisely, given a subset $D \subseteq \|mn\| - 1\| \setminus m\mathbb{Z}$ consisting of odd integers, we define for all $d \in D$, the graph $S_d = [0, d]_{\frac{mn}{2}}$. Observe that $\partial S_d = \pm \{d, \frac{mn}{2} - d\}$ and that $\phi(S_d)$ is a transversal of the subgroup of index 2 in $\mathbb{Z}_{mn}$. Finally, we set

$$D_* = \{S_d : d \in D\}.$$

We will show that the set $\Sigma$ of the constructed 2-regular graphs is a 2-starter in $\mathbb{Z}_{mn}$ relative to $m\mathbb{Z}_{mn}$, leaving to the reader the easy verification that, given a 2-regular graph $S \in \Sigma$, the stabilizer of each cycle of $S$ is contained in the subgroup associated to $\phi(S)$, according to Definition 7(b). The existence of a cyclic $C_4$-factorization of $K_{m \times n}$ follows from Theorem 8.
5 Cyclic $C_4$-Factorizations of $K_{m \times n}$, $m$ Even

In this section, we construct cyclic $C_4$-factorizations of $K_{m \times n}$ for $m$ even. We recall that also $n$ must be even.

It is convenient to fix some notation we will use in the next propositions. For any $k, v \in \mathbb{N}$ let

$$Y_k(2, I) = \emptyset,$$

$$Y_k(2v + 2, I) = \{[v - 1 - i, v + 1 + i + km] : i \in \|v - 1\| \setminus I\} \text{ if } v \geq 1,$$

$$Y_k(2v + 1, I) = \{[v - i, v + 1 + i + km] : i \in \|v\| \setminus I\}.$$ 

It is easy to see that

$$\Delta Y_k(2v + 2, I) = \{2 + 2i + km : i \in \|v - 1\| \setminus I\},$$

$$\Delta Y_k(2v + 1, I) = \{1 + 2i + km : i \in \|v\| \setminus I\},$$

$$\phi(Y_0(2v + 2, \emptyset)) = \|2v\| \setminus \{v\},$$

$$\phi(Y_0(2v + 1, \emptyset)) = \|2v + 1\|.$$ 

Also, for fixed $k \geq 0$ and $v \geq 1$, let

$$Y_k(4v) = Y_k(4v, \{0\}) \cup \{[2v - 1, 2v + km], [2v - 2, 4v - 1 + km]\};$$

$$Y_k(4v + 2) = Y_k(4v + 2, \emptyset) \cup \{[2v, 4v + 1 + km]\}.$$ 

Hence, if $m \equiv 0 \pmod{4}$ then

$$\Delta Y_k(m) = \pm \left(4 + km + 2 \left\| \frac{m}{2} - 3 \right\| \right) \cup \pm \left\{1 + km, 1 + \frac{m}{2} + km\right\}$$

and if $m \equiv 2 \pmod{4}$, $m > 2$, then

$$\Delta Y_k(m) = \pm \left(2 + km + 2 \left\| \frac{m}{2} - 2 \right\| \right) \cup \pm \left\{\frac{m}{2} + km\right\}.$$ 

In both cases, $\phi(Y_k(m))$ is a transversal of the subgroup of index $m$ of $\mathbb{Z}_{mn}$. Finally, define also the 4-cycle

$$Z_k = \left(0, 2 + km, -1, \frac{mn}{2} - 3 - km\right).$$

Notice that $\partial Z_k = \Delta Z_k = \pm \left\{2 + km, 3 + km, \frac{mn}{2} - 2 - km, \frac{mn}{2} - 3 - km\right\}$ and that, when $m \equiv 0 \pmod{4}$, $\phi(Z_k)$ is a transversal of the subgroup of index 4 of $\mathbb{Z}_{mn}$.

Lemma 14 For all even integers $n$ there exist a cyclic $C_4$-factorization of $K_{2 \times n}$ and one of $K_{4 \times n}$. 

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Proof Consider $K_{2\times n}$. If $n \equiv 0 \pmod{4}$, take $D = 1 + 2 \| \frac{n-4}{4} \|$ and $\Sigma = D^\star$. If $n \equiv 2 \pmod{4}$, take $D = 1 + 2 \| \frac{n-2}{4} \|$ and $\Sigma = \left\{ [0]_n \right\} \cup D^\star$. It is easy to see that in both cases $\Sigma$ is a 2-starter in $\mathbb{Z}_{2n}$ relative to $2\mathbb{Z}_{2n}$.

Consider $K_{4\times n}$. If $n \equiv 0 \pmod{4}$, let $D = 1 + 4 \| \frac{n-4}{4} \|$ and $\Sigma = \left\{ \mathbb{Z}_k : k \in \| \frac{n-4}{4} \| \right\} \cup D^\star$. If $n \equiv 2 \pmod{4}$, let $D = 1 + 4 \| \frac{n-2}{4} \|$ and $\Sigma = \left\{ [0]_n \right\} \cup \left\{ \mathbb{Z}_k : k \in \| \frac{n-2}{4} \| \right\} \cup D^\star$. In both cases, $\Sigma$ is a 2-starter in $\mathbb{Z}_{4n}$ relative to $4\mathbb{Z}_{4n}$.

Proposition 15 For all even integers $m, n$, there exists a cyclic $C_4$-factorization of $K_{m\times n}$.

Proof We split the proof into five subcases according to the congruence class of $m$ modulo 8 and $n$ modulo 4. In view of Lemma 14 we may assume $m > 4$.

Case 1. Suppose $m, n \equiv 0 \pmod{4}$. Let $m \geq 8$ and $n = 4t + 4$. Let $D$ be the set of the odd integers $d \in \| \frac{mn}{4} - 1 \| \setminus (1 + km, 3 + km, 1 + \frac{m}{2} + km : k \in \| t \|)$ and define

$$\Sigma = \left\{ \mathbb{Z}_k, \Box \mathcal{Y}_k(m) : k \in \| t \| \right\} \cup D^\star.$$ 

Case 2. Suppose $m \equiv 2 \pmod{4}$ and $n \equiv 0 \pmod{4}$. Let $m \geq 6$ and $n = 4t + 4$. Let $D$ be the set of the odd integers $d \in \| \frac{mn}{4} - 1 \| \setminus \left( \frac{m}{2} + m \| t \| \right)$ and define

$$\Sigma = \left\{ \Box \mathcal{Y}_k(m) : k \in \| t \| \right\} \cup D^\star.$$ 

Case 3. Suppose $m, n \equiv 2 \pmod{4}$. Let $m \geq 6$ and $n = 4t + 2$. Let $S_0 = \Box \mathcal{Y}_l(1 + \frac{m}{2}, \emptyset) \cup \left\{ \frac{m-2}{4} \right\}_{mn}$. Clearly, $\partial S_0 = \pm(2 + tm + 2 \| \frac{m-6}{4} \|) \cup \pm \left\{ \frac{mn}{4} \right\}$ and $\phi(S_0)$ is a transversal of the subgroup of index $\frac{m}{2}$ in $\mathbb{Z}_{mn}$. Let $D$ be the set of the odd integers $d \in \| \frac{mn}{4} - 1 \| \setminus \left( \frac{m}{2} + m \| t \| \right)$. Define

$$\Sigma = \left\{ S_0 \right\} \cup \left\{ \Box \mathcal{Y}_k(m) : k \in \| t - 1 \| \right\} \cup D^\star.$$ 

Case 4. Suppose $m \equiv 0 \pmod{8}$ and $n \equiv 2 \pmod{4}$. Let $n = 4t + 2$. Suppose firstly $m = 8$. In this case, let $D = (7 + 8 \| t - 1 \|) \cup \{ 8t + 1 \}$ and define

$$\Sigma = \left\{ [0]_{2n} \right\} \cup \left\{ \mathbb{Z}_k : k \in \| t \| \right\} \cup \left\{ \Box \mathcal{Y}_k(8) : k \in \| t - 1 \| \right\} \cup D^\star.$$ 

Now, suppose $m \geq 16$. Let $S_0 = [0]_{mn}$ and $S_2 = \Box \mathcal{Y}_l \left( \frac{m}{4} \right)$. For $k \in \| t \|$ take $S_4(k) = \mathbb{Z}_k$ and for $k \in \| t - 1 \|$ take $S_6(k) = \Box \mathcal{Y}_k(m)$. Observe that $\phi(S_2)$ is a transversal of the subgroup of index $\frac{m}{2}$ in $\mathbb{Z}_{mn}$ and that $\Delta \mathcal{Y}_l \left( \frac{m}{4} \right) \equiv \pm(4 + mt + 2 \left( \frac{m}{4} - 3 \right)) \cup \pm(1 + mt, 1 + \frac{m}{2} + mt)$. Let $D$ be the set of the odd integers in $\| \frac{mn}{4} - 1 \|$ not belonging to $\partial S_1$. Define

$$\Sigma = \left\{ S_0, S_2 \right\} \cup \left\{ S_4(k) : k \in \| t \| \right\} \cup \left\{ S_6(k) : k \in \| t - 1 \| \right\} \cup D^\star.$$ 

Case 5. Suppose $m \equiv 4 \pmod{8}$ and $n \equiv 2 \pmod{4}$. Let $m \geq 12$ and $n = 4t + 2$. Take $S_0 = [0]_{mn}$ and $S_2 = \Box \mathcal{Y}_l \left( \frac{m}{4} \right)$. Also, for any $k \in \| t - 1 \|$ let $S_4(k) = \mathbb{Z}_k$ and $S_6(k) = \Box \mathcal{Y}_k(m)$. Observe that $\phi(S_2)$ is a transversal of the subgroup of index $\frac{m}{2}$ of
Let $Z_{mn}$ and that $\Delta \mathcal{Y}(m, n) = \pm(2 + mt + 2 \| \frac{m}{4} - \frac{1}{4} \|) \cup \pm(\frac{m}{4} + mt)$. Let $\mathcal{D}$ be the set of the odd integers in $\| \frac{mn}{4} - 1 \|$ not belonging to $\partial S_i$. Define

$$\Sigma = \{S_0, S_2 \cup \{S_4(k), S_6(k) : k \in \|t - 1\| \} \cup \mathcal{D}^\star.$$ 

In all these five cases, $\bigcup_{S \in \Sigma} \partial S = Z_{mn} - mZ_{mn}$ and hence $\Sigma$ is a 2-starter in $Z_{mn}$ relative to $mZ_{mn}$.

**Example 16** Here, for each case of Proposition 15, following its proof we construct the sets necessary to obtain the 2-starter for a choice of $m$ and $n$.

**Case 1.** Let $m = 16$ and $n = 8$, hence $t = 1$. We obtain $\mathcal{D} = (1 + 2 \|15\|) \setminus \{1, 3, 9, 17, 19, 25\}$ and

$$Z_0 = (0, 2, -1, 61), \quad Z_1 = (0, 18, -1, 45),$$

$$\mathcal{Y}_0(16) = \{5, 9, [4, 10], [3, 11], [2, 12], [1, 13], [0, 14]\} \cup \{[7, 8], [6, 15]\},$$

$$\mathcal{Y}_1(16) = \{5, 25, [4, 26], [3, 27], [2, 28], [1, 29], [0, 30]\} \cup \{[7, 24], [6, 31]\}.$$

**Case 2.** Let $m = 6$ and $n = 12$, hence $t = 2$. We obtain $\mathcal{D} = (1 + 2 \|8\|) \setminus \{3, 9, 15\}$ and

$$\mathcal{Y}_0(6) = \{1, 3], [0, 4]\} \cup \{[2, 5]\}, \quad \mathcal{Y}_1(6) = \{1, 9], [0, 10]\} \cup \{[2, 11]\},$$

$$\mathcal{Y}_2(6) = \{1, 15], [0, 16]\} \cup \{[2, 17]\}.$$

**Case 3.** Let $m = 10$ and $n = 6$, hence $t = 1$. We obtain

$$S_0 = \{1, 13\}_{30}, [0, 14]_{30}, [2, 15]\}, \quad \mathcal{Y}_0(10) = \{3, 5], [2, 6], [1, 7], [0, 8]\} \cup \{4, 9\}$$

and $\mathcal{D} = (1 + 2 \|6\|) \setminus \{5\}$.

**Case 4.** Let $m = 32$ and $n = 6$, hence $t = 1$. We obtain $S_0 = [0]_{48}$,

$$\mathcal{Y}_1(16) = \{5, 41], [4, 42], [3, 43], [2, 44], [1, 45], [0, 46]\} \cup \{[7, 40], [6, 47]\},$$

$$\mathcal{Z}_1 = (0, 2, -1, 93), \quad \mathcal{Z}_2 = (0, 34, -1, 61),$$

$$\mathcal{Y}_0(32) = \{13, 17], [12, 18], [11, 19], [10, 20], [9, 21], [8, 22], [7, 23], [6, 24],$$

$$[5, 25], [4, 26], [3, 27], [2, 28], [1, 29], [0, 30]\} \cup \{[15, 16], [14, 31]\}.$$

Hence $\mathcal{D} = (1 + 2 \|23\|) \setminus \{1, 3, 17, 33, 35, 41\}$.

**Case 5.** Let $m = 12$ and $n = 6$, hence $t = 1$. We obtain $S_0 = [0]_{18}$,

$$\mathcal{Y}_1(6) = \{1, 15], [0, 16]\} \cup \{[2, 17]\}, \quad \mathcal{Z}_0 = (0, 2, -1, 33),$$

$$\mathcal{Y}_0(12) = \{3, 7], [2, 8], [1, 9], [0, 10]\} \cup \{[5, 6], [4, 11]\}.$$ 

Hence $\mathcal{D} = (1 + 2 \|8\|) \setminus \{1, 3, 7, 15\}$. 

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6 Cyclic $C_4$-Factorizations of $K_{m \times n}$, $m$ Odd

In this section, we construct cyclic $C_4$-factorizations of $K_{m \times n}$ for $m$ odd. By Corollary 13 we have that $n \equiv 0 \pmod{4}$.

In order to explain our constructions, we need to define some other auxiliary sets. For any $v, k \in \mathbb{N}$ and any subset $I \subset \mathbb{N}$ we define

\[
X_k^c(4v + 2, I) = \{ [2v - 2 - 2i, 2v + 2i + km] : i \in \|v - 1\| \setminus I\},
\]

\[
X_k^o(4v + 2, I) = \{ [2v - 1 - 2i, 2v + 3 + 2i + km] : i \in \|v - 1\| \setminus I\},
\]

\[
X_k^c(4v + 4, I) = \{ [2v - 2 - 2i, 2v + 2 + 2i + km] : i \in \|v - 1\| \setminus I\},
\]

\[
X_k^o(4v + 4, I) = \{ [2v + 1 - 2i, 2v + 3 + 2i + km] : i \in \|v\| \setminus I\}.
\]

Notice that

\[
\Delta X_k^c(4v + 2, I) = \pm \{ 2 + 4i + km : i \in \|v - 1\| \setminus I\},
\]

\[
\Delta X_k^o(4v + 2, I) = \pm \{ 4 + 4i + km : i \in \|v - 1\| \setminus I\},
\]

\[
\Delta X_k^c(4v + 4, I) = \pm \{ 4 + 4i + km : i \in \|v - 1\| \setminus I\},
\]

\[
\Delta X_k^o(4v + 4, I) = \pm \{ 2 + 4i + km : i \in \|v\| \setminus I\},
\]

and that, for $k = 0$ and $I = \emptyset$,

\[
\phi(X_0^c(4v + 2, \emptyset) \cup X_0^o(4v + 2, \emptyset)) = \| 4v + 1 \| \setminus \{ 2v + 1, 4v \},
\]

\[
\phi(X_0^c(4v + 4, \emptyset) \cup X_0^o(4v + 4, \emptyset)) = \| 4v + 3 \| \setminus \{ 2v, 4v + 2 \}.
\]

**Proposition 17** For any $m$ odd and any $n \equiv 0 \pmod{8}$ there exists a cyclic $C_4$-factorization of $K_{m \times n}$.

**Proof** We split the proof into two subcases according to the congruence class of $n$ modulo 16.

Case 1. Suppose $n \equiv 0 \pmod{16}$. Let $n = 16t + 16$, whence $\frac{mn}{4} = 4m(t + 1) = 4v + 4$, i.e. $v = m(t + 1) - 1$. We start removing from the sets $X_0^o\left(\frac{mn}{4}, \emptyset\right)$ and $X_0^c\left(\frac{mn}{4}, \emptyset\right)$ the edges $[a, b]$ such that $b - a$ is a multiple of $m$. First, take $X_0^c\left(\frac{mn}{4}, \emptyset\right)$: we have to remove the edges giving the difference $4 + 4i = \alpha m$ for some $\alpha \in \mathbb{N}$. Clearly $\alpha \equiv 0 \pmod{4}$ and so, setting $\alpha = 4j + 4$, from $4 + 4i = (4j + 4)m$ we get $i = m - 1 + jm$. Since $i \in \|v - 1\|$, it turns out that we have to remove the edges $[2v - 2 - 2i, 2v + 2 + 2i]$ with $i \in I = m - 1 + m \| \frac{v - m}{m} \| = m - 1 + m \| t - 1 \|$. So, the vertices we removed are the elements of the set $(2m - 2 + 2m \| t - 1 \|) \cup (2mt + 4m - 2 + 2m \| t - 1 \|)$ which coincides with the set $(2m - 2 + 2m \| 2t \|) \setminus \{ 2mt + 2m - 2 \}$.

Now, take $X_0^o\left(\frac{mn}{4}, \emptyset\right)$: we remove the edges giving the difference $2 + 4i = \alpha m$ for some $\alpha \in \mathbb{N}$. Clearly $\alpha \equiv 2 \pmod{4}$. Setting $\alpha = 4j + 2$, from $2 + 4i = (4j + 2)m$ we get $i = \frac{m - 1}{2} + jm$. Since $i \in \|v\|$, it turns out that we have to remove the edges $[2v + 1 - 2i, 2v + 3 + 2i]$ with $i \in I = \frac{m - 1}{2} + m \| \frac{2v + 1 - 2m}{2m} \| = \frac{m - 1}{2} + m \| t \|$. So, the vertices we removed are the elements of the set $(m + 2m \| t \|) \cup (2mt + 3m + 2m \| t \|) = m + 2m \| 2t + 1 \|$.
Let
\[ \mathcal{X}(4v + 4) = \mathcal{X}^e(4v + 4, m - 1 + m \parallel t - 1\|) \cup \mathcal{X}^o_0\left(4v + 4, \frac{m - 1}{2} + m \parallel t\|\right). \]

Then
\[ \partial \mathcal{X}(4v + 4) = \pm 2 \left(\left\lfloor \frac{mn}{8} \right\rfloor - 1\right) \backslash \left\lfloor \frac{n}{8} - 1\right\rfloor, \]

and, recalling (2),
\[ \phi(\mathcal{X}(4v + 4)) = \left\lfloor \frac{mn}{4} \right\rfloor - 1 \backslash ((2m - 2 + 2m \parallel 2t + 1\|) \cup (m + 2m \parallel 2t + 1\|)). \]

Taking
\[ A = \{(1 + 2t - 2j)m, 2m(t + 2 + j) - 2\} : j \in \parallel t\|\}
\[ \cup \{(2m(t + 1 - j) - 2, m(2t + 3 + 2j)) : j \in \parallel t\|\}\]

we obtain
\[ \partial A = \pm (m + 2 + 4m \parallel t\|) \cup \pm (3m - 2 + 4m \parallel t\|) \]

and
\[ \phi(A) = (2m - 2 + 2m \parallel 2t + 1\|) \cup (m + 2m \parallel 2t + 1\|). \]

Let now \( B = \mathcal{X}(4v + 4) \cup A, \) whence \( \phi(B) = \left\lfloor \frac{mn}{4} \right\rfloor - 1 \backslash \). Let \( D \) be the set of the odd integers \( d \in \left\lfloor \frac{mn}{4} \right\rfloor - 1 \backslash (m\mathbb{Z} \cup \partial A). \) Define \( S_0 = \square B \) and \( \Sigma = \{S_0\} \cup D^\star. \) It is easy to see that \( \Sigma \) is a 2-starter in \( \mathbb{Z}_{mn} \) relative to \( m\mathbb{Z}_{mn}. \)

Indeed \( \partial S_0 \cup (\cup_{d \in D} \partial S_d) = \mathbb{Z}_{mn} \backslash m\mathbb{Z}_{mn} \) and \( \phi(S_0) \) is a transversal of the subgroup of index \( \frac{mn}{4} \) of \( \mathbb{Z}_{mn}. \)

Case 2. Suppose \( n \equiv 8 \) (mod 16). Let \( n = 16t + 8, \) whence \( \frac{mn}{4} = 2(2t + 1)m = 4v + 2, \) i.e. \( v = tm + \frac{m - 1}{2}. \) As before we start removing from the subsets \( \mathcal{X}^e\left(\frac{mn}{4}, 0\right) \) and \( \mathcal{X}^o\left(\frac{mn}{4}, 0\right) \) the edges \([a, b]\) such that \( b - a \) is a multiple of \( m. \) First, take \( \mathcal{X}^o\left(\frac{mn}{4}, 0\right): \)
we have to remove the edges giving the difference \( 2 + 4i = \alpha m \) for some \( \alpha \in \mathbb{N}. \)

Clearly \( \alpha \equiv 2 \) (mod 4) and so, setting \( \alpha = 4j + 2, \) from \( 2 + 4i = (4j + 2)m \) we

get \( i = \frac{m - 1}{2} + jm. \) Since \( i \in \|v - 1\|, \) it turns out that we have to remove the edges
\([2v - 2 - 2i, 2v + 2i]\) with \( i \in I = \frac{m - 1}{2} + m \left\lfloor \frac{2v - m - 1}{2m} \right\rfloor = \frac{m - 1}{2} + m \|t - 1\|. \) So,

the vertices we removed are the elements of the set \( (2m - 2 + 2m \|t - 1\|) \cup (2m - 2 + 2mt + 2m \|t - 1\|) \)

which coincides with the set \( (2m - 2 + 2m \|2t - 1\|). \)

Now take \( \mathcal{X}^o\left(\frac{mn}{4}, 0\right): \) we remove the edges giving the difference \( 4 + 4i = \alpha m \) for

some \( \alpha \in \mathbb{N}. \) Clearly \( \alpha \equiv 0 \) (mod 4). Setting \( \alpha = 4j + 4, \) from \( 4 + 4i = (4j + 4)m \) we

get \( i = m - 1 + jm. \) Since \( i \in \|v - 1\|, \) it turns out that we have to remove the edges
\([2v - 1 - 2i, 2v + 3 + 2i]\) with \( i \in I = m - 1 + m \left\lfloor \frac{v - m}{m} \right\rfloor = m - 1 + m \|t - 1\|. \)
So, the vertices we removed are the elements of the set \((m + 2m \parallel t - 1\parallel) \cup (2mt + 3m + 2m \parallel t - 1\parallel) = (m + 2m \parallel 2t\parallel) \setminus \{m + 2mt\}.

Let
\[
\mathcal{X}(4v + 2) = \mathcal{X}_0^e \left(4v + 2, \frac{m - 1}{2} + m \parallel t - 1\parallel\right) \\
\cup \mathcal{X}_0^o (4v + 2, m - 1 + m \parallel t - 1\parallel).
\]

Then
\[
\partial \mathcal{X}(4v + 2) = \pm \left(2 \left\| \frac{mn}{4} - 1 \right\| \setminus 2m \left\| \frac{n}{8} - 1 \right\| \right),
\]
and, recalling (1),
\[
\phi \left(\mathcal{X}(4v + 2)\right) = \left\| \frac{mn}{4} - 1 \right\| \setminus (2m - 2 + 2m \parallel 2t\parallel) \cup (m + 2m \parallel 2t\parallel)).
\]

Taking
\[
A = \{[m(2t + 1 - 2j), 2m(t + 1 + j) - 2] : j \in \parallel t\parallel\} \\
\cup \{[2m(t - j) - 2, m(2t + 3 + 2j)] : j \in \parallel t - 1\parallel\}
\]
we obtain
\[
\partial A = \pm (m - 2 + 4m \parallel t\parallel) \cup \pm (3m + 2 + 4m \parallel t - 1\parallel)
\]
and
\[
\phi(A) = (2m - 2 + 2m \parallel 2t\parallel) \cup (m + 2m \parallel 2t\parallel).
\]

Let now \(B = \mathcal{X}(4v + 2) \cup A\), whence \(\phi(B) = \left\| \frac{mn}{4} - 1 \right\|\).

Let \(\mathcal{D}\) be the set of the odd integers \(d \in \left\| \frac{mn}{4} - 1 \right\| \setminus (m\mathbb{Z} \cup \partial A)\). Define \(S_0 = \Box B\) and \(\Sigma = \{S_0\} \cup \mathcal{D}^\star\). Arguing as before one can see that \(\Sigma\) is a 2-starter in \(\mathbb{Z}_{mn}\) relative to \(m\mathbb{Z}_{mn}\).

**Example 18** Following the proof of Proposition 17 we construct the sets necessary to obtain the 2-starter for two choices of \(m\) and \(n\).

**Case 1.** Let \(m = 3\) and \(n = 32\), hence \(t = 1\) and \(v = 5\). We obtain
\[
B = \{[8, 12], [6, 14], [2, 18], [0, 20]\} \cup \{[11, 13], [7, 17], [5, 19], [1, 23]\} \\
\cup \{[9, 16], [3, 22]\} \cup \{[10, 15], [4, 21]\},
\]
which implies \(\mathcal{D} = \{1, 11, 13, 23\}\).
Case 2. Let $m = 5$ and $n = 40$, hence $t = 2$ and $v = 12$. We obtain

\[ B = \{[22, 24], [20, 26], [16, 30], [14, 32], [12, 34], [10, 36], [6, 40], [4, 42], [2, 44] \}
\cup \{[0, 46], [23, 27], [21, 29], [19, 31], [17, 33], [13, 37], [11, 39], [9, 41], [7, 43], [3, 47] \} \cup \{[1, 49], [25, 28], [15, 38], [5, 48] \} \cup \{[18, 35], [8, 45] \}, \]

which gives $D = (1 + 2 \parallel 24) \backslash \{3, 5, 15, 17, 23, 25, 35, 37, 43, 45 \}$.

**Lemma 19** There exists a cyclic $C_4$-factorization of $K_{35 \times 4}$.

**Proof** A 2-starter $\Sigma$ in $\mathbb{Z}_{140}$ relative to $35\mathbb{Z}_{140}$ is given by $\Sigma = \{S_0, S_2, S_4, S_6, S_8\} \cup D^\ast$, where $S_0 = \emptyset \{[0, 12], [1, 11], [2, 10], [3, 9], [4, 8], [5, 7], [6, 13] \}$, $S_2 = \emptyset \{[0, 26], [1, 25], [2, 24], [3, 23], [4, 22], [5, 35], [6, 27] \}$. $S_4 = \emptyset \{[0, 28], [1, 15], [2, 13], [4, 27], [6, 39] \}$, $S_6 = \emptyset \{0, 16, 1, 55\} \cup \{[2, 29]_{70} \cup \{3, 8\}_{70} \cup \{4, 17\}_{70} \}$, $S_8 = \emptyset \{[0, 34], [1, 33], [2, 19], [5, 8], [6, 7] \}$, and $D = \{9, 19, 25, 29, 31 \}$.

**Proposition 20** Let $m$ be an odd integer. There exists a cyclic $C_4$-factorization of $K_{m \times 4}$ if and only if $m \neq p^a$ where $p$ is a prime.

**Proof** In Proposition 12 we have already proved that $m \neq p^a$ is a necessary condition for the existence of a cyclic $C_4$-factorization of $K_{m \times 4}$. We now prove the sufficiency.

Firstly, write $m$ as a product $ab$ with $a > 1$ minimal such that $\text{gcd}(a, b) = 1$. In view of Lemma 19 we can assume $m \neq 35$.

Now, for all $i \in \| \frac{a-3}{2} \|$, take the set

\[ W_i = \{[b - 2 - j, b + j + 2bi] : j \in \| b - 2 \| \} \cup \{[b - 1, 2b - 1 + 2bi] \}. \]

Observe that $\Delta W_i = \pm(2 + 2bi + 2 \| b - 2 \|) \cup \pm\{(2i + 1)b\}$ and that $\phi(W_i)$ is a transversal of the subgroup of index $2b$ in $\mathbb{Z}_{4m}$. In particular,

\[ \bigcup_{i=0}^{\frac{a-3}{2}} \Delta W_i = \pm \left(2 \left\| \frac{ab - b}{2} \right\| 2b\mathbb{Z} \right) \cup \pm \left(\frac{b + 2b}{2} \left\| \frac{a - 3}{2} \right\| \right). \]

We are left to consider the two sets of even differences

\[ A = 2b + 2b \left\| \frac{a - 3}{2} \right\| \quad \text{and} \quad B = b(a - 1) + 2 + 2 \left\| \frac{b - 3}{2} \right\|. \]

Observe that $|A| = \frac{a-1}{2}$ and $|B| = \frac{b-1}{2}$. Also, the elements of $A$ are, modulo $2a$, pairwise distinct and different from 0.

To deal with these even differences, it will be useful to construct two particular sets of edges in the following way. Let $C$ be a set of even cardinality $2N < 2a$ consisting of integers $0 < c_k < ab$ which are, modulo $2a$, non-zero and pairwise distinct. Consider the Euclidean division of $c_k$ by $2a$ and let $r_k$ be its the remainder. Take $r_{k_0} > r_{k_1} > \cdots > r_{k_{2N-1}}$ and consider the corresponding elements $c_{k_0}, c_{k_1}, \ldots, c_{k_{2N-1}}$. Suppose first that $C$ contains $N$ even integers and $N$ odd integers. The set $\| 2a - 1 \| \setminus \{r_{k_0}, \ldots, r_{k_{2N-1}} \}$ contains $a - N$ even integers $x_0 < x_1 <

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\[ \cdots < x_{a-N-1} \text{ and } a-N \text{ odd integers } y_{a-N-1} < y_{a-N-2} < \cdots < y_1 < y_0. \]

Define \( J_1(C) = \{ [x_j, y_j] : j \in \| a-N-1 \| \} \). For simplicity, write the previous edges in the form \([z_j, w_j]\) with \( 0 < d_j = w_j - z_j < 2a \). Clearly, each difference \( d_j \) is odd and appears at most twice in \( \Delta J_1(C) \). Next, suppose that \( C \) contains only even integers and that \( 2N < a \). Let \( J_2(C) = \{ [j, j+c_k] : j \in \| 2N-1 \| \} \) and \( J(C) = J_1(\phi(J_2(C))) \cup J_2(C) \). Notice that \( \Delta J_2(C) = \pm C \) and that the elements of \( \phi(J(C)) \) are pairwise distinct modulo \( 2a \). Hence, \( \phi(J(C)) \) is a transversal of the subgroup of index \( 2a \) in \( \mathbb{Z}_{4m} \).

We now come back to the sets \( A \) and \( B \), describing step by step our construction of a 2-starter in \( \mathbb{Z}_{4m} \) relative to \( m\mathbb{Z}_{4m} \).

**Step 1.** We start with the elements of \( B \) which are 0 modulo 2a: they are precisely the even integers \( f_k = ab - (2k+1)a \) with \( k \in \| q \| \), where \( q = \lfloor \frac{b-a-2}{2a} \rfloor \). Write \( b-a-2 = 2aq + 2\rho \), where \( 0 \leq \rho < a \). Let \( W' \) be the set of edges obtained from \( W_{a-3} \) by replacing the edge \([b - 2 - j, b + j + (a-3)b]\) with the edge \([b - 2 - j, b + j + (a-1)b]\), for all \( j \in \| \rho + a \| \). Observe that \( f_k \in \Delta W' \) and that \( \phi(W') = \phi\left(W_{a-3}\right) \) modulo \( 2b \). After this modification, we have to consider the differences \( f_k = f_k - 2b \) (which are all congruent to \(-2b \) modulo \( 2a \)). Hence, for all \( k \in \| q - 1 \| \), let \( Q_k \) be the 2-regular graph

\[ Q_k = (0, f_k, 1, 2ab + 1 - f_k) \cup \square J_1([0, 1, -2b, 2b + 1]). \]

Observe that \( \phi(Q_k) \) is a transversal of the subgroup of index \( 2a \) in \( \mathbb{Z}_{4m} \). Furthermore, let \( G_k = \{ f_{k+1} + 2, f_{k+1} + 4, \ldots, f_k - 2 \} \). Clearly, \( G_k \) contains \( a-1 \) even elements which are, modulo \( 2a \), non-zero and pairwise distinct: construct \( J(G_k) \).

We conclude this step by setting

\[ \Sigma_1 = \left\{ \square W_i : i \in \| a-5 \| / 2 \right\} \cup \{ \square W' \}, \quad \Sigma_2 = \{ Q_k, \square J(G_k) : k \in \| q - 1 \| \} \]

and \( A^* = A \cup \{ f_{q} \} \). Note that the elements of \( A^* \) are, modulo \( 2a \), non-zero and pairwise distinct.

**Step 2.** In this step we consider sets \( C \) of even cardinality in order to construct \( J(C) \). Let \( 0 \leq \kappa \leq a-1 \) be maximal such that \( \kappa + \rho \leq a - 1 \) is even. Define

\[ S = (a(b-1) + 2 + 2 \| \kappa - 1 \| ) \cup (b(a-1) + 2 + 2 \| \rho - 1 \| ). \]

Then \( S \) is a set of even cardinality, whose elements are, modulo \( 2a \), non-zero and pairwise distinct.

Now, let \( L = a(b-1) + 2(\kappa + 1) + 2 \| a-3 \| - \kappa \). Assume that \( a \equiv 1 \pmod{4} \). Then, \( |A^*| \) is odd. If \( \rho \) is even, then \( |L| \) is even. Construct

\[ Q_q = (0, f_q, 1, 2ab + 1 - f_q) \cup \square J_1([0, 1, -2b, 2b + 1]) \]

and observe that \( \phi(Q_q) \) is a transversal of the subgroup of index \( 2a \) in \( \mathbb{Z}_{4m} \). Set \( \Sigma_3 = \{ \square J(S), \square J(A), \square J(L), Q_q \} \).
If \( \rho \) is odd, then \( |L| \) is odd. Write the set \( A^* \cup L \) as a disjoint union \( T_1 \cup T_2 \) such that \( |T_1| \) is even and maximal with respect to the property that the elements of \( T_j \) are, modulo 2, pairwise distinct, for all \( j = 1, 2 \), allowing \( T_2 \) to be the empty set. Then, take \( \Sigma_3 = \{ \Box J(S), \Box J(T_1), \Box J(T_2) \} \).

Assume that \( a \equiv 3 \pmod{4} \). Then, \( |A^*| \) is even. If \( \rho \) is odd, then \( |L| \) is even: take \( \Sigma_3 = \{ \Box J(S), \Box J(A^*), \Box J(L) \} \). If \( \rho \) is even, then \( |L| \) is odd. Define

\[
T_3 = (0, ab - 1, 1, ab + 2) \cup \Box J((0, 1, ab - 1, ab + 2))
\]

and observe that \( \phi(T_3) \) is a transversal of the subgroup of index 2\( a \) in \( \mathbb{Z}_{4m} \). Take \( \Sigma_3 = \{ \Box J(S), \Box J(A^*), \Box J(L) \} \).

Step 3. Notice that all even differences appear exactly once in \( \Delta(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \), but some odd differences can appear more than once and some other can not appear. So, we need to modify the 2-regular graphs constructed in the previous steps in such a way that the elements of the list \( \Delta(\Sigma_2 \cup \Sigma_3) \) are pairwise distinct and none of them is an odd multiple of \( b \). Let \( d \) be an odd difference occurring more than once in \( \Delta(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \).

We repeat the following procedure until \( \Delta(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \) becomes a set. If \( 0, d + 1, 1, 2ab - d \) \( \in \Sigma_3 \), replace this cycle with \( 0, d + 1, 1 - 2ah, 2ab - d - 2ah \); if \( [z, z + d]_{2ab} \in \Sigma_2 \cup \Sigma_3 \), replace this cycle with \( [z, z + d + 2ah]_{2ab} \). In both cases, take \( h \) in such a way that \( d + 2ah \in \{m - 1\} \setminus \Delta(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \) and call again \( \Sigma_2 \) and \( \Sigma_3 \) the sets obtained after this modification. Notice that this operation preserves \( \phi(\Gamma) \) modulo 2\( a \), for every 2-regular graph \( \Gamma \) which has been modified.

Clearly, we need to show that this procedure is legit. Fix an odd integer \( d \in \{2a\} \) and, given \( \sigma \subseteq \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \), let \( \vartheta_d(\sigma) \) be the number of the integers appearing in \( \Delta\sigma \cap \{m - 1\} \) which are congruent to \( d \) modulo 2\( a \). Then, \( \vartheta_d(\Sigma_1) \leq 1 \) and \( \vartheta_d(\Sigma_2) \leq 4q \). Furthermore

\[
\vartheta_d(\Sigma_3) \leq \begin{cases} 
2 + 2 + 2 + 3 = 9 & \text{if } a \equiv 1 \pmod{4} \text{ and } \rho \text{ is even,} \\
2 + 2 + 2 = 6 & \text{if } a \equiv 1 \pmod{4} \text{ and } \rho \text{ is odd,} \\
2 + 2 + 2 = 6 & \text{if } a \equiv 3 \pmod{4} \text{ and } \rho \text{ is even,} \\
2 + 2 + 2 + 3 = 9 & \text{if } a \equiv 3 \pmod{4} \text{ and } \rho \text{ is odd.}
\end{cases}
\]

If \( \rho \) is even, it suffices to prove that \( \vartheta_d(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \leq 4q + 10 < b - \frac{1}{2} \), namely that

\[
b = 2aq + a + 2 + 2\rho > 8q + 21.
\]

If \( \rho \) is odd, it suffices to prove that \( \vartheta_d(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \leq 4q + 7 < b - \frac{1}{2} \), namely that

\[
b = 2aq + a + 2 + 2\rho > 8q + 15.
\]

It is easy to see that the previous inequalities hold for all odd \( a \geq 5 \) and all \( q \geq 0 \) with the following exceptions:

(i) \( a = 5 \) and \( b \in \{7, 9, 11, 13, 17, 19, 23, 27, 29, 31, 37, 41, 47, 61, 67, 77\} \);

(ii) \( a = 7 \) and \( b \in \{9, 11, 13, 17, 23, 27, 37\} \);

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Example 21 Let $m = 143$. Following the proof of Proposition 20 we construct the elements of a 2-starter $\Sigma$ in $\mathbb{Z}_{572}$ relative to $143\mathbb{Z}_{572}$. We factorize $m$ as $m = ab$, where $a = 11$ and $b = 13$.

For any $i \in \|3\|$ let

$$W_i = \{(11 - j, 13 + j + 26i) : j \in \|11\|\} \cup \{(12, 25 + 26i)\}$$

and instead of $W_4$ we consider

$$W' = \{(0, 128), [1, 127], [2, 126], [3, 125], [4, 124], [5, 123], [6, 122], [7, 121], [8, 120], [9, 119], [10, 118], [11, 143], [12, 129]\}.$$

We obtain $A^* = \{26, 52, 78, 104, 106, 130\}$, $S = \{134, 136, 138, 140\}$ and $L = \{142\}$. Construct

$$J(A^*) = \{(0, 130), [1, 107], [2, 106], [3, 81], [4, 56], [5, 31], [6, 21], [7, 16], [8, 17], [10, 13], [11, 14]\},$$

$$J(S) = \{(0, 140), [1, 139], [2, 138], [3, 137], [4, 21], [9, 20], [10, 19], [11, 18], [12, 17], [13, 16], [14, 15]\},$$

$$T_3 = \square\{(2, 21), [3, 20], [4, 19], [5, 18], [6, 17], [7, 16], [8, 15], [9, 14], [11, 12]\} \cup \{(0, 142), 1, 145\}.$$
Because of the odd differences appearing more than once, we modify the previous sets, as
\[
J(A^*) = \{[0, 130], [1, 107], [2, 106], [3, 81], [4, 56], [5, 31], [6, 21], [7, 16], [8, 39], [10, 13], [11, 36]\},
\]
\[
J(S) = \{[0, 140], [1, 139], [2, 138], [3, 137], [4, 21], [9, 20], [10, 63], [11, 18], [12, 17], [13, 60], [14, 15]\},
\]
\[
T_3 = \square\{[2, 21], [3, 64], [4, 41], [5, 40], [6, 39], [7, 82], [8, 37], [9, 36], [11, 34]\} \cup (0, 142, 1, 145).
\]

Here, we wrote in boldface the modified vertices, according to Step 3.

Finally the elements of \( D \) are the odd integers of \( \| 142 \| \setminus \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 27, 29, 31, 33, 35, 37, 39, 47, 53, 61, 65, 75, 91, 117, 141\}.

Hence \( \Sigma = \{\square W_0, \square W_1, \square W_2, \square W_3, \square W', \square J(A^*), \square J(S), T_3\} \cup D^\star \).

**Proposition 22** For any \( m \equiv 1 \pmod 4 \) and any \( n \equiv 4 \pmod 8 \) with \( n > 4 \), there exists a cyclic \( C_4 \)-factorization of \( K_{m \times n} \).

**Proof** Let \( n = 8t + 12 \). Define \( S_0 = \square A \) where
\[
A = \mathcal{Y}_1(m + 1, \emptyset) \cup \left( \left\{ \frac{m - 1}{2} + \mathcal{X}_{2(t+1)}^\circ(m + 1, \emptyset) \right\} \cup \left\{ \left[ \left\lceil m, \frac{3(m - 1)}{2} \right\rceil \right] \right\} \cup \left\{ \left[ 0, \frac{m - 3}{2} \right] \right\} \right).
\]

Observe that \( \phi(S_0) = \|m - 2\| \cup \{(m, \frac{3m - 3}{2}) \cup (\frac{3m + 1}{2}, \frac{m - 3}{2}) \cup \{m - 1 + 2t + 1)m \} \cup (m + 1 + 2(t + 1)m + \frac{m - 7}{2}) \cup \{\frac{3m - 1}{2} + 2(t + 1)m\} \), and so \( \phi(S_0) \) is a transversal of the subgroup of index \( 2m \) of \( \mathbb{Z}_{mn} \). Also, \( \Delta A = \pm(m + 2 + 2 \|\frac{m - 3}{2}\|) \cup \pm(2 + 2t + 1)m + 2 \|\frac{m - 3}{2}\|) \cup \pm(\frac{m - 3}{2}) \). Now, for all \( k \in \|t\| \), let \( S_2(k) = \square B_k \) where
\[
B_k = \mathcal{X}_{2k}^\circ(2m, \emptyset) \cup \mathcal{X}_{2k}^\circ(2m, \emptyset) \cup \{m, 2m - 2 + 2km\}.
\]

It is easy to see \( \phi(S_2(k)) \) is a transversal of the subgroup of index \( 2m \) of \( \mathbb{Z}_{mn} \) and \( \Delta B_k = \pm(2 + 2k + 2 \|m - 2\|) \cup \pm(m - 2 + 2km) \). Let \( D \) be the set of the odd integers in \( \|\frac{mn}{4} - 1\| \cdot m\mathbb{Z} \) not belonging to \( \Delta(A, B_k) \). Define
\[
\Sigma = \{S_0\} \cup \{S_2(k) : k \in \|t\| \} \cup D^\star.
\]

We get \( \cup_{S \in \Sigma} \partial S = \mathbb{Z}_{mn} - m\mathbb{Z}_{mn} \). Hence \( \Sigma \) is a 2-starter in \( \mathbb{Z}_{mn} \) relative to \( m\mathbb{Z}_{mn} \).

**Example 23** Let \( m = 9 \) and \( n = 12 \). Following the proof of previous proposition we obtain \( A = \{[3, 14], [2, 15], [1, 16], [0, 17] \} \cup \{[6, 26], [4, 28] \} \cup \{[7, 29], [5, 31] \} \cup \{[9, 12] \} \) and \( B_0 = \{[6, 8], [4, 10], [2, 12], [0, 14] \} \cup \{[7, 11], [5, 13], [3, 15], [1, 17] \} \cup \{[9, 16] \} \). Hence, it results \( D = \{1, 5, 19, 21, 23, 25\} \).
Lemma 24 For any \( n = 4 \pmod{8} \) with \( n > 4 \), there exist a cyclic \( C_4 \)-factorization of \( K_{3 \times n} \) and one of \( K_{7 \times n} \).

Proof First consider \( K_{3 \times n} \) with \( n = 8t + 12 \). For all \( k \in \|t\| \) take

\[
S_0(k) = [0, 2 + 6k]_{3n/2} \cup [1, 5 + 6k]_{3n/2} \cup [3, 4 + 6k]_{3n/2}.
\]

Take also

\[
S_2 = \left( 0, \frac{3n}{4} - 1, \frac{3n}{4} + 2 \right) \cup [4, 9]_{3n/2}
\]

and \( D = (11 + 6\|t - 1\|) \setminus 3\mathbb{Z} \). Observe that \( \phi(S_0(k)) \) and \( \phi(S_2) \) are transversals of the subgroup of index 6 in \( \mathbb{Z}_{3n} \). Also,

\[
\partial S_0(k) = \pm \{2 + 6k, 4 + 6k, 1 + 6k, \frac{3n}{2} - 2 - 6k, \frac{3n}{2} - 4 - 6k, \frac{3n}{2} - 1 - 6k \}
\]

and \( \partial S_2 = \pm \{\frac{3n}{4} - 1, \frac{3n}{4} - 2, \frac{3n}{4} + 1, \frac{3n}{4} + 2, 2, 3, \frac{3n}{2} - 5\} \).

The set

\[
\Sigma = \{S_0(k) : k \in \|t\|\} \cup \{S_2\} \cup D^*.
\]

is a 2-starter in \( \mathbb{Z}_{3n} \) relative to \( 3\mathbb{Z}_{3n} \).

Now consider \( K_{7 \times n} \) with \( n = 8t + 12 \). For all \( k \in \|t\| \) take

\[
S_0(k) = [0, 10 + 14k]_{7n/2} \cup [1, 13 + 14k]_{7n/2} \cup [2, 8 + 14k]_{7n/2}
\]

\[
\cup [3, 11 + 14k]_{7n/2} \cup [4, 6 + 14k]_{7n/2} \cup [5, 9 + 14k]_{7n/2}
\]

\[
\cup [7, 12 + 14k]_{7n/2}.
\]

Take also

\[
S_2 = \left( 0, \frac{7n}{4} - 5, \frac{7n}{4} + 8 \right) \cup \left( 4, \frac{7n}{4} + 3, 5, \frac{7n}{4} + 6 \right)
\]

\[
\cup \left( 8, \frac{7n}{4} + 5, 11, \frac{7n}{4} + 14 \right) \cup [6, 9]_{7n/2}.
\]

Observe that \( \phi(S_0(k)) \) and \( \phi(S_2) \) are transversals of the subgroup of index 14 in \( \mathbb{Z}_{7n} \). Also,

\[
\partial S_0(k) = \pm \{2 + 2i + 14k, \frac{7n}{2} - 2 - 2i - 14k : i \in \|5\|\} \cup \pm \{5 + 14k, \frac{7n}{2} - 5 - 14k \}
\]

and \( \partial S_2 = \pm \{\frac{7n}{4} + 1, \frac{7n}{4} + 2, \frac{7n}{4} + 3, \frac{7n}{4} + 5, \frac{7n}{4} + 6, \frac{7n}{4} + 8, 3, \frac{7n}{2} - 3\} \).

Let \( D \) be the set of the odd integers in \( \|\frac{7n}{4} - 1\| \setminus 7\mathbb{Z} \) not belonging to the previous sets of differences.

The set

\[
\Sigma = \{S_0(k) : k \in \|t\|\} \cup \{S_2\} \cup D^*.
\]

is a 2-starter in \( \mathbb{Z}_{7n} \) relative to \( 7\mathbb{Z}_{7n} \).

Lemma 25 For any \( m \equiv 3 \pmod{4} \), there exists a cyclic \( C_4 \)-factorization of \( K_{m \times 12} \).
Proof Suppose firstly \( m \equiv 3 \pmod{8} \). If \( m = 3 \) the statement follows from Lemma 24. So, we may assume \( m \geq 11 \). Define \( S_0 = \emptyset A_0 \) and \( S_2 = \emptyset A_2 \), where

\[
A_0 = \mathcal{X}_0^\circ \left( 2m, \left\{ \frac{3m - 9}{8} \right\} \right) \cup \mathcal{X}_0^\circ \left( 2m, \left\{ \frac{3m - 9}{8} \right\} \right) \\
\cup \left\{ \left\lfloor \frac{m - 3}{4}, m + 1 \right\rfloor, \left\lfloor \frac{7m - 13}{4}, \frac{7m - 1}{4} \right\rfloor, [m, 2m - 2] \right\},
\]

\[
A_2 = \mathcal{Y}_2(m + 1, \emptyset) \cup (m + \mathcal{Y}_2(m - 2, \emptyset)) \cup \left\{ \left\lfloor \frac{m - 1}{2}, 2m - 1 \right\rfloor \right\}.
\]

Also, define \( S_4 = (0, \frac{3m - 5}{2}, 1, \frac{9m + 7}{2}) \cup [3, 10]_{6m} \). Observe that \( \phi(S_0) \) and \( \phi(S_2) \) are both transversals of the subgroup of index \( 2m \) in \( \mathbb{Z}_{12m} \) and that \( \phi(S_4) \) is a transversal of the subgroup of index 6 in \( \mathbb{Z}_{12m} \). Furthermore, \( \Delta A_0 = \pm((2 + 2 \| m - 2 \|) \setminus \left\{ \frac{3m - 9}{8}, \frac{3m - 1}{8} \right\}) \cup \pm\{1, 3, m - 2\} \), \( \Delta A_2 = \pm(2m + 1 + \| m - 2 \|) \cup \pm\{\frac{3m - 1}{2}\} \) and \( \partial S_4 = \pm\{\frac{3m - 5}{2}, \frac{3m - 7}{2}, \frac{9m + 5}{2}, \frac{9m + 7}{2}, 7, 6m - 7\} \). Let \( \mathcal{D} \) be the set of the odd integers \( d \in \| 3m - 1 \| \setminus \{1, 3, 7, m - 2, \frac{3m - 7}{2} \} \cup (2m + 1 + 2 \| \frac{m - 3}{2} \|) \cup m \mathbb{Z} \).

Suppose now \( m \equiv 7 \pmod{8} \). If \( m = 7 \) the statement follows from Lemma 24. So, we may assume \( m \geq 15 \). Define \( S_0 = \emptyset A_0 \) and \( S_2 = \emptyset A_2 \), where

\[
A_0 = \mathcal{X}_0^\circ \left( 2m, \left\{ \frac{3m - 5}{8} \right\} \right) \cup \mathcal{X}_0^\circ \left( 2m, \left\{ \frac{3m + 3}{8} \right\} \right) \\
\cup \left\{ \left\lfloor \frac{m - 11}{4}, m - 7 \right\rfloor, \left\lfloor \frac{7m - 9}{4}, \frac{7m + 11}{4} \right\rfloor, [m, 2m - 2] \right\},
\]

\[
A_2 = \mathcal{Y}_2(m + 1, \emptyset) \cup (m + \mathcal{Y}_2(m - 2, \emptyset)) \cup \left\{ \left\lfloor \frac{m - 1}{2}, 2m - 1 \right\rfloor \right\}.
\]

Also, define \( S_4 = (0, \frac{3m + 11}{2}, -1, \frac{9m - 13}{2}) \cup [2, 9]_{6m} \). Observe that \( \phi(S_0) \) and \( \phi(S_2) \) are both transversals of the subgroup of index \( 2m \) in \( \mathbb{Z}_{12m} \) and that \( \phi(S_4) \) is a transversal of the subgroup of index 6 in \( \mathbb{Z}_{12m} \). Furthermore, \( \Delta A_0 = \pm((2 + 2 \| m - 2 \|) \setminus \left\{ \frac{3m - 5}{2}, \frac{3m + 11}{2} \right\}) \cup \pm\{1, 5, m - 2\} \), \( \Delta A_2 = \pm(2m + 1 + \| m - 2 \|) \cup \pm\{\frac{3m - 1}{2}\} \) and \( \partial S_4 = \pm\{\frac{3m + 11}{2}, \frac{3m + 13}{2}, \frac{9m - 11}{2}, \frac{9m - 13}{2}, 7, 6m - 7\} \). Let \( \mathcal{D} \) be the set of the odd integers \( d \in \| 3m - 1 \| \setminus \{(1, 5, 7, m - 2, \frac{3m + 13}{2}) \cup (2m + 1 + 2 \| \frac{m - 3}{2} \|) \cup m \mathbb{Z} \).

In both cases, \( \Sigma = \{S_0, S_2, S_4\} \cup \mathcal{D} \) is a 2-starter in \( \mathbb{Z}_{12m} \) relative to \( m \mathbb{Z}_{12m} \).

Proposition 26 For any \( m \equiv 3 \pmod{4} \) and any \( n \equiv 4 \pmod{8} \) with \( n > 4 \), there exists a cyclic \( C_4 \)-factorization of \( K_{m \times n} \).

Proof We split the proof into two subcases according to the congruence class of \( m \) modulo 8. By Lemmas 24 and 25 we may assume \( m \geq 11 \) and \( n \geq 20 \). Let \( n = 8t + 20 \).

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Case 1. Suppose \( m = 3 \) (mod 8). Firstly, suppose \( m = 11 \). If \( n = 20 \), define \( S_0(k) = □A_k \), for \( k = 0, 1, 2 \), where

\[
A_0 = \{[0, 18], [4, 14], [6, 12], [8, 10]\} \cup \{[1, 21], [5, 17], [7, 15], [9, 13]\}
\cup \{[11, 20], [2, 25], [16, 19]\},
\]
\[
A_1 = \{[0, 40], [2, 38], [4, 36], [6, 34], [8, 32]\} \cup \{[1, 43], [3, 41], [5, 39], [7, 37], [9, 35]\} \cup \{[11, 42]\},
\]
\[
A_2 = \{[0, 54], [1, 53], [2, 52], [3, 51], [4, 50]\} \cup \{[11, 64], [12, 63], [13, 62], [14, 61], [15, 60]\} \cup \{[5, 21]\}.
\]

Also, define

\[
S_2 = (0, 14, 1, 97) \cup [2, 9]_{110} \cup [3, 8]_{110} \cup [5, 6]_{110}.
\]

Notice that \( φ(S_0(k)) \) and \( φ(S_2) \) are a transversal of the subgroup of index 22 and of index 10, respectively, in \( Z_{220} \). It is easy to see that the set \( Σ = \{S_0(0), S_0(1), S_0(2), S_2\} \cup D^∗ \), where \( D \) is the set of the odd integers of \( \|54\| \{11, 33\} \) not appearing in \( ∅S_t \), is a 2-starter in \( Z_{220} \) relative to \( 11Z_{220} \).

So, suppose \( n > 20 \). For \( k ∈ \|t + 2\| \) define \( S_0(k) = □A_k \) where

\[
A_0 = X_0^e(22, [1]) \cup X_0^o(22, [3]) \cup ([3, 28], [12, 41], [11, 20]),
\]
\[
A_k = X_k^e(22, ∅) \cup X_k^o(22, ∅) \cup ([11, 20 + 2k]) \quad \text{for } 1 ≤ k ≤ t + 1,
\]
\[
A_{t+2} = Y_{2t+4}(12, ∅) \cup (11 + Y_{2t+4}(9, ∅)) \cup ([5, 21]).
\]

Now let \( S_2 = □B \cup (0, 6, −1, \frac{11n−14}{2}) \), where

\[
B = Y_0(21, [3, 4, 5, 10]) \cup ([14, 71], [16, 51]).
\]

Notice that \( φ(S_0(k)) \) and \( φ(S_2) \) are both transversals of the subgroup of index 22 in \( Z_{11n} \). One can check that the set \( Σ = \{S_0(k) : k ∈ \|t + 2\| \} \cup \{S_2\} \cup D^∗ \), where \( D \) is the set of the odd integers of \( \|\frac{11n}{4} − 1\| \{11Z\} \) not appearing in \( ∅S_t \), is a 2-starter in \( Z_{11n} \) relative to \( 11Z_{11n} \).

Assume now \( m ≥ 19 \). For \( k ∈ \|t + 2\| \) define \( S_0(k) = □A_k \) where

\[
A_0 = X_0^e(2m, \left\{\frac{m−3}{8}\right\}) \cup X_0^o(2m, \left\{\frac{3m−9}{8}\right\})
\cup \left\{\frac{m+1}{4}, \frac{11m−9}{4}, \frac{5m−7}{4}, \frac{15m−1}{4}, [m, 2m−2]\right\},
\]
\[
A_k = X_k^e(2m, ∅) \cup X_k^o(2m, ∅) \cup ([m, 2m−2 + 2km]) \quad \text{for } 1 ≤ k ≤ t + 1,
\]
\[
A_{t+2} = Y_{2t+4}(m + 1, ∅) \cup (m + Y_{2t+4}(m − 2, ∅)) \cup \left\{\frac{m−1}{2}, 2m−1\right\}.
\]
Also, let $S_2 = \Box B \cup (0, \frac{m+1}{2}, -1, \frac{mn-m-3}{2})$, where

$$B = \mathcal{Y}_0 \left( 2m - 1, \left\{ \frac{m+1}{4}, \frac{m-3}{2}, \frac{m-1}{2}, m-1 \right\} \right) \cup \left\{ \left[ \frac{m-1}{2}, \frac{13m+1}{4} \right], \left[ \frac{3m-1}{2}, \frac{19m-5}{4} \right] \right\}.$$  

Notice that $\phi(S_0(k))$ and $\phi(S_2)$ are both transversals of the subgroup of index $2m$ in $\mathbb{Z}_{mn}$. Furthermore, $\Delta A_0 = \pm \left( (2+2 \| m-2 \|) \setminus \left( \frac{m+1}{2}, \frac{3m-1}{2} \right) \right) \cup \pm \left( \frac{5m-5}{2}, \frac{5m+3}{2}, m-2 \right)$, $\Delta A_k = \pm \left( 2+2km + 2 \| m-2 \| \right) \cup \pm \left( \frac{5m-5}{2}, \frac{5m+3}{2}, m-2 \right)$ when $1 \leq k \leq t + 1$ and $\Delta A_{t+2} = \pm \left( 1+ (2t+4)m + \| m-2 \| \right) \cup \pm \left( \frac{3m-1}{2} \right)$. Finally, denoting

$$B = \left( (1+2 \| m-2 \|) \setminus \left( \frac{m+1}{2}, \frac{m-3}{2}, m-2, m \right) \right) \cup \left( \frac{11m+3}{4}, \frac{13m-3}{4} \right),$$

we have $\Delta B = \pm B$ and $\partial S_2 = \pm \left\{ b, \frac{mn}{2} - b : b \in B \right\} \cup \pm \left\{ \frac{m+1}{2}, \frac{m+3}{2}, \frac{mn-m-3}{2}, \frac{mn-m-1}{2} \right\}$. Since $m \geq 19$, $(\cup \partial S_0(k)) \cup \partial S_2$ is a set.

Define $\Sigma = \{ S_0(k) : k \in \| t+2 \| \} \cup \{ S_2 \} \cup \mathcal{D}^\bullet$, where $\mathcal{D}$ is the set of the odd integers of $\| \frac{mn}{4} - 1 \| \setminus \mathbb{mZ}$ not appearing in $\partial S_1$. It results that $\Sigma$ is a 2-starter in $\mathbb{Z}_{mn}$ relative to $\mathbb{mZ}_{mn}$.

**Case 2.** Suppose $m \equiv 7 \pmod{8}$. For $k \in \| t+2 \|$ define $S_0(k) = \Box A_k$ where

$$A_0 = \mathcal{X}_0^e \left( 2m, \left\{ \frac{3m-5}{8} \right\} \right) \cup \mathcal{X}_0^o \left( 2m, \left\{ \frac{m-7}{8} \right\} \right) \cup \left\{ \left[ \frac{m-7}{4}, \frac{11m-1}{4} \right], \left[ \frac{5m+1}{4}, \frac{15m-9}{4} \right], [m, 2m-2] \right\}.$$  

$$A_k = \mathcal{X}_2^e(2m, \emptyset) \cup \mathcal{X}_2^o(2m, \emptyset) \cup \left\{ [m, 2m-2+2km] \right\} \text{ if } 1 \leq k \leq t + 1,$$

$$A_{t+2} = \mathcal{Y}_{2t+4}(m+1, \emptyset) \cup (m+1, \emptyset) \cup \left\{ \left[ \frac{m-1}{2}, 2m-1 \right] \right\}.$$  

Also, let $S_2 = \Box B \cup (0, \frac{m+1}{2}, -1, \frac{mn-m-3}{2})$, where

$$B = \mathcal{Y}_0 \left( 2m - 1, \left\{ \frac{m+1}{4}, \frac{m-3}{2}, \frac{m-1}{2}, m-1 \right\} \right) \cup \left\{ \left[ \frac{m-1}{2}, \frac{11m-5}{4} \right], \left[ \frac{3m-1}{2}, \frac{21m+1}{4} \right] \right\}.$$  

Notice that $\phi(S_0(k))$ and $\phi(S_2)$ are both transversals of the subgroup of index $2m$ in $\mathbb{Z}_{mn}$. Furthermore, $\Delta A_0 = \pm \left( (2+2 \| m-2 \|) \setminus \left( \frac{m+1}{2}, \frac{3m-1}{2} \right) \right) \cup \pm \left( \frac{5m-5}{2}, \frac{5m+3}{2}, m-2 \right)$, $\Delta A_k = \pm \left( 2+2km + 2 \| m-2 \| \right) \cup \pm \left( \frac{5m-5}{2}, \frac{5m+3}{2}, m-2 \right)$ when $1 \leq k \leq t + 1$ and $\Delta A_{t+2} = \pm \left( 1+ (2t+4)m + \| m-2 \| \right) \cup \pm \left( \frac{3m-1}{2} \right)$. Finally, denoting

$$B = \left( (1+2 \| m-2 \|) \setminus \left( \frac{m+1}{2}, m-2, m \right) \right) \cup \left( \frac{9m-3}{4}, \frac{15m+1}{4} \right),$$

we have $\Delta B = \pm B$ and $\partial S_2 = \pm \left\{ b, \frac{mn}{2} - b : b \in B \right\} \cup \pm \left\{ \frac{m+1}{2}, \frac{m+3}{2}, \frac{mn-m-3}{2}, \frac{mn-m-1}{2} \right\}$. Since $m \geq 15$, $(\cup \partial S_0(k)) \cup \partial S_2$ is a set.

The set $\Sigma = \{ S_0(k) : k \in \| t+2 \| \} \cup \{ S_2 \} \cup \mathcal{D}^\bullet$, where $\mathcal{D}$ is the set of the odd integers of $\| \frac{mn}{4} - 1 \| \setminus \mathbb{mZ}$ not appearing in $\partial S_1$, is a 2-starter in $\mathbb{Z}_{mn}$ relative to $\mathbb{mZ}_{mn}$. 

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Lemma 28 Let $n \equiv 2 \pmod{4}$, we can rely on Theorem 4.

Remark 29 Let $n \equiv 2 \pmod{4}$ and let $U = \{x_0, x_0 + 2^{a_0}, x_1, x_1 + 2^{a_1}, \ldots, x_b, x_b + 2^{a_b}\}$ be a set containing odd integers not divisible by $m$. Then $W_u = \{0, x_u + 2^{a_x} \}_{2a_x}$ is a hamiltonian cycle of $K_{m \times n}$ for any $u \in ||b||$. Obviously, $\partial (\bigcup_{u=0}^{b} W_u) = \pm U$.

The following definition and lemma are instrumental in proving Proposition 32, where we shall settle the case $n \equiv 2 \pmod{4}$.

Example 27 Let $m = 15$ and $n = 20$, hence we are in Case 2 of the proof of Proposition 26. Following such a proof we construct the following sets of edges

$$A_0 = \{\{12, 14\}, \{10, 16\}, \{8, 18\}, \{6, 20\}, \{4, 22\}, \{0, 26\}\}$$
$$\cup \{\{13, 17\}, \{9, 21\}, \{7, 23\}, \{5, 25\}, \{3, 27\}, \{1, 29\}\}$$
$$\cup \{\{2, 41\}, \{19, 54\}, \{15, 28\}\},$$
$$A_1 = \{\{12, 44\}, \{10, 46\}, \{8, 48\}, \{6, 50\}, \{4, 52\}, \{2, 54\}, \{0, 56\}\}$$
$$\cup \{\{13, 47\}, \{11, 49\}, \{9, 51\}, \{7, 53\}, \{5, 55\}, \{3, 57\}, \{1, 59\}\} \cup \{\{15, 58\}\},$$
$$A_2 = \{\{6, 68\}, \{5, 69\}, \{4, 70\}, \{3, 71\}, \{2, 72\}, \{1, 73\}, \{0, 74\}\}$$
$$\cup \{\{21, 82\}, \{20, 83\}, \{19, 84\}, \{18, 85\}, \{17, 86\}, \{16, 87\}, \{15, 88\}\} \cup \{\{7, 29\}\},$$
$$B = \{\{14, 15\}, \{13, 16\}, \{12, 17\}, \{11, 18\}, \{9, 20\}, \{6, 23\}, \{5, 24\}, \{4, 25\}, \{3, 26\},$$
$$\{2, 27\}, \{1, 28\}\} \cup \{\{7, 40\}, \{22, 79\}\}.$$
**Definition 30** For all positive integers \(s, d\) and all odd integers \(w \geq 3\), set

\[
\mathcal{R}(s, d, w) = s + d \left\| \frac{w - 3}{2} \right\|
\]

and

\[
\varphi(s, d, w) = |\{x \in \mathcal{R}(s, d, w) : \gcd(x, w) = 1\}|.
\]

**Lemma 31** [15, Lemma 4.6] Assume \(\gcd(s, d, w) = 1\). If \(3 \nmid s\) when \(w = 3\), then \(\varphi(s, d, w) > 0\).

Now, we can prove the following.

**Proposition 32** Let \(m\) be an odd integer; let \(n \equiv 2 \pmod{4}\) and \(n > 2\). Then there exists a cyclic hamiltonian 2-factorization of \(K_{m \times n}\) if and only if \(m \equiv 1 \pmod{4}\).

**Proof** The non existence immediately follows from Corollary 11. So assume \(m \equiv 1 \pmod{4}\). In order to prove the existence we will construct a 2-starter \(\Sigma\) in \(\mathbb{Z}_{mn}\) relative to \(m\mathbb{Z}_{mn}\), where each \(S \in \Sigma\) is a hamiltonian cycle. We have to distinguish two cases according to the congruence class of \(n\) modulo 8.

**Case 1.** Suppose \(n = 8t + 10\).

For any \(k \in \|t\|\) let

\[
A_k = [0, 4m - 2 + 4km, 2, 4m - 4 + 4km, \ldots, m - 3, 4m - (m - 1) + 4km, m - 1, 4m - m + 4km, m + 2, 4m - (m + 2) + 4km, m + 4, 4m - (m + 4) + 4km, \ldots, 2m - 1, 4m - (2m - 1) + 4km]_{2m}
\]

and

\[
B = \left[0, \frac{mn}{2} - 1, 1, \frac{mn}{2} - 2, \ldots, \frac{m - 3}{2}, \frac{mn}{2} - \frac{m - 1}{2}, \frac{m - 1}{2}\right]_m.
\]

Note that \(\phi(A_k)\) and \(\phi(B)\) are transversals of the subgroup of index \(2m\) and of index \(m\), respectively, of \(\mathbb{Z}_{mn}\). Also, we have

\[
\partial \left( \bigcup_{k=0}^{t} A_k \cup B \right) = \pm \left(\left(\left(2 + 2 \|2t + 2\|m - 2\|\right) \setminus \left(2m + 2m \|2t\|\right)\right) \cup (1 + 2m \|2t + 1\|) \cup \left\{\frac{m + 1}{2}\right\} \cup (4t + 4)m + 1 \|m - 2\|\right).
\]

It is easy to see that the number of odd integers of the set \(\Theta = \|mn\|_2 \setminus (m \|\frac{n}{2}\| \cup \partial(\bigcup_{k=0}^{t} A_k \cup B))\) is odd.

**Case 1.a.** Assume that 3 does not divide \(m\). Consider the set \(\mathcal{R}(2m - 3, 2m, \frac{n}{2})\) whose elements are all coprime with \(m\). Since \(\gcd(2m - 3, 2m, \frac{n}{2}) = 1\) and \(\gcd(3, 2m - 3) = 1\).
We can apply Lemma 31. Hence there exists an element $\kappa$ in $\mathcal{R}(2m - 1, 2m, \frac{n}{2})$ coprime with $\frac{n}{2}$. So $\kappa$ is coprime with $mn$ and $C = [0]_\kappa$ is a hamiltonian cycle of $K_{mn}$. If $m \geq 13$, then the set $\Theta \setminus \{\kappa\}$ satisfies the condition of Remark 29, so we construct the associated cycles $W_\kappa$.

The case $m = 5$ (and hence $2m - 3 = 7$) requires a special analysis. If $n = 10$ take the cycles $D_0 = [0, 9]_2$, $D_1 = [0, 19]_2$ and, instead of $C = [0]_\kappa$, take $C = [0]_{13}$. It is easy to see that $\Sigma = \{A_0, B, C, D_0, D_1\}$ is a 2-starter in $\mathbb{Z}_{50}$ relative to $5\mathbb{Z}_{50}$. So we can suppose $n \geq 18$. If $\kappa = 7$ then $\Theta_-$ is empty. The sets $\Theta_+$ and $\Theta_-$ satisfy the hypothesis of Remark 29, and hence we construct the cycles $W_\kappa$. Case 1.b. Assume that 3 divides $n$ (which implies $m \geq 9$). Consider the set $\mathcal{R}(2m - 1, 2m, \frac{n}{2})$ whose elements are all coprime with $m$. Since $\gcd(2m - 1, 2m, \frac{n}{2}) = 1$ and $\gcd(3, 2m - 1) = 1$ we can apply Lemma 31. Hence there exists an element $\kappa$ in $\mathcal{R}(2m - 1, 2m, \frac{n}{2})$ coprime with $\frac{n}{2}$. So $\kappa$ is coprime with $mn$ and $C = [0]_\kappa$ is a hamiltonian cycle of $K_{mn}$. If $\kappa = 4m - 1 + jm$ with $j \in \|t-1\|$ the elements of $\Theta \setminus \{\kappa\}$ can be partitioned, as above, into 2 sets $\Theta_+$ and $\Theta_-$ of even size. Hence we construct the cycles $W_\kappa$ as in Remark 29.

Finally, if $\kappa = 10(2j + 1) + 7$ with $j = t$ take $D_0 = [0, \kappa + 10]_2, D_1 = [0, \kappa + 12]_2, D_2 = [0, \kappa + 20]_2$ and $D_3 = [0, \kappa + 22]_2$, which are hamiltonian cycles by Lemma 28. The elements of $\Theta \setminus \partial (C \cup (\cup_i D_i))$ can be partitioned into 2 sets $\Theta_+$ and $\Theta_-$ of even size containing respectively the elements of $\Theta$ greater than $\kappa$ and the elements of $\Theta$ less than $\kappa$. Reasoning as above, we can construct the cycles $W_\kappa$, Case 1.b. Assume that 3 divides $m$ (which implies $m \geq 9$). Consider the set $\mathcal{R}(2m - 1, 2m, \frac{n}{2})$ whose elements are all coprime with $m$. Since $\gcd(2m - 1, 2m, \frac{n}{2}) = 1$ and $\gcd(3, 2m - 1) = 1$ we can apply Lemma 31. Hence there exists an element $\kappa$ in $\mathcal{R}(2m - 1, 2m, \frac{n}{2})$ coprime with $\frac{n}{2}$. So $\kappa$ is coprime with $mn$ and $C = [0]_\kappa$ is a hamiltonian cycle of $K_{mn}$. If $\kappa = 4m - 1 + jm$ with $j \in \|t\|$ the elements of $\Theta \setminus \{\kappa\}$ can be partitioned, as above, into 2 sets $\Theta_+$ and $\Theta_-$ of even size. Hence we construct the cycles $W_\kappa$ as in Remark 29.

If $\kappa = 2m - 1 + jm$ with $j \in \|t\|$ the set $\Theta \setminus \partial (C \cup D)$ can be divided into sets of even size which satisfy the conditions of Remark 29. So, again, we construct the cycles $W_\kappa$.

In both cases 1.a and 1.b, $\Sigma = \{A_i, B, C, D, W_\kappa\}$ is a 2-starter in $\mathbb{Z}_{mn}$ relative to $m\mathbb{Z}_{mn}$.

**Case 2.** Suppose $n = 8t + 6$.

Take

$$A = [0, 2m - 2, 2, 2m - 4, 4, \ldots, m + 1, m - 1]_m,$$

(5)

$B$ as in (4) and for any $k \in \|t-1\|$ take

$$C_k = [0, 6m - 2 + 4km, 2, 6m - 4 + 4km, \ldots, m - 3, 5m + 1 + 4km, m - 1, 5m + 4km, m + 2, 5m - 2 + 4km, m + 4, 5m - 4 + 4km, \ldots, 2m - 1, 4m + 1 + 4km]_{2m}.$$

(6)
Note that

\[
\Lambda = \partial (A \cup B \cup \cup_k C_k) = \pm \left( (2 + 2 \|m(2t + 1) - 2\|) \setminus (2 + 2m \|2t - 1\|) \right) \\
\cup (1 + 2m \|2t\|) \cup \left\{ \frac{m + 1}{2} \right\} \cup (1 + 2m(2t + 1) + \|m - 2\|) \right). 
\]

One can check that \( \| \frac{mn}{2} \| \setminus (m \mathbb{Z} \cup \Lambda) \) is a set satisfying the hypothesis of Remark 29. So we construct the cycles \( W_u \). It is not hard to see that \( \Sigma = \{A, B, C_k, W_u\} \) is a 2-starter in \( \mathbb{Z}_{mn} \) relative to \( m \mathbb{Z}_{mn} \).

**Example 33** Following the proof of Proposition 32, we construct a 2-starter \( \Sigma \) in \( \mathbb{Z}_{mn} \) relative to \( m \mathbb{Z}_{mn} \) for some choice of \( m \) and \( n \).

**Case 1.a.** Let \( m = 5 \) and \( n = 18 \). We obtain

\[
A_0 = [0, 18, 2, 16, 4, 15, 7, 13, 9, 11]_{10}, \quad A_1 = [0, 38, 2, 36, 4, 35, 7, 33, 9, 31]_{10}, \quad B = [0, 44, 1, 43, 2]_5. 
\]

It results \( \partial (A_0 \cup A_1 \cup B) = \pm (((2 + 2 \|18\|) \setminus \{10, 20, 30\}) \cup \{1, 11, 21, 31\} \cup \{3\} \cup \{41, 42, 43, 44\}) \) hence \( \Theta = \{7, 9, 13, 17, 19, 23, 27, 29, 33, 37, 39\} \). Note that \( \Theta \) has odd size. Since 3 does not divide \( m \) we consider \( R(7, 10, 9) = \{7, 17, 27, 37\} \). We can choose for instance \( \kappa = 17 \) which is coprime with \( \frac{5}{2} = 9 \), so we take \( C = [0]_{17} \). Now we have to take the cycles \( D_0 = [0, 27]_8, \quad D_1 = [0, 29]_{16}, \quad D_2 = [0, 37]_4 \) and \( D_3 = [0, 39]_{16} \). It results \( \Theta \setminus \partial (C \cup (\cup_i D_i)) = \{7, 9\} \). So in this case \( \Theta_+ = \emptyset \). It remains to consider only the cycle \( W_0 = [0, 9]_2 \). Hence take \( \Sigma = \{A_i, B, C, D_i, W_0\} \).

**Case 1.b.** Let \( m = 9 \) and \( n = 26 \). We obtain

\[
A_0 = [0, 34, 2, 32, 4, 30, 6, 28, 8, 27, 11, 25, 13, 23, 15, 21, 17, 19]_{18}, \\
A_1 = [0, 70, 2, 68, 4, 66, 6, 64, 8, 63, 11, 61, 13, 59, 15, 57, 17, 55]_{18}, \\
A_2 = [0, 106, 2, 104, 4, 102, 6, 100, 8, 99, 11, 97, 13, 95, 15, 93, 17, 91]_{18}, \\
B = [0, 116, 1, 115, 2, 114, 3, 113, 4]_9. 
\]

We have \( \partial (\cup_i A_i \cup B) = \pm (((2 + 2 \|52\|) \setminus \{18, 36, 54, 72, 90\}) \cup \{1, 19, 37, 55, 73, 91\} \cup \{5\} \cup \{109 + \|7\|\}) \). Hence \( \Theta = \{1 + 2 \|53\|\} \setminus \{1, 5, 9, 19, 27, 37, 45, 55, 63, 73, 81, 91, 99\} \). Since 3 divides \( m \) we consider \( R(17, 18, 13) = \{17, 35, 53, 71, 89, 107\} \). All the elements of \( R(17, 18, 13) \) are coprime with \( \frac{9}{2} \). The more convenient choices are \( 35, 71 \) or \( 107 \) since in each of these cases we have not to construct the cycle \( D_0 \). Choosing \( \kappa = 107 \), we have to take \( C = [0]_{107} \) and following Remark 29, we construct \( W_0 = [0, 7]_4, \quad W_1 = [0, 13]_2, \quad W_2 = [0, 17]_2, \quad W_3 = [0, 23]_2, \quad W_4 = [0, 29]_4, \quad W_5 = [0, 33]_2, \quad W_6 = [0, 39]_4, \quad W_7 = [0, 43]_2, \quad W_8 = [0, 49]_2, \quad W_9 = [0, 53]_2, \quad W_{10} = [0, 59]_2, \quad W_{11} = [0, 65]_4, \quad W_{12} = [0, 69]_2, \quad W_{13} = [0, 75]_4, \quad W_{14} = [0, 79]_2, \quad W_{15} = [0, 85]_2, \quad W_{16} = [0, 89]_2, \quad W_{17} = [0, 95]_2, \quad W_{18} = [0, 101]_4, \quad W_{19} = [0, 105]_2 \). Hence take \( \Sigma = \{A_i, B, C, W_u\} \).
Case 2. Let $m = 13$ and $n = 14$. We obtain

$$A = [0, 24, 2, 22, 4, 20, 6, 18, 8, 16, 10, 14, 12]_{13},$$

$$B = [0, 90, 1, 89, 2, 88, 3, 87, 4, 86, 5, 85, 6]_{13},$$

$$C_0 = [0, 76, 2, 74, 4, 72, 6, 70, 8, 68, 10, 66, 12, 65, 15, 63, 17, 61, 19, 59, 21, 57, 23, 55, 25, 53]_{26}.$$  

It results $\Lambda = \partial(A \cup B \cup C_0) = \{((-2 + 2 \parallel 37\parallel) \setminus \{26, 52\}) \cup \{1, 7, 27, 53\} \cup (79 + \parallel 11\parallel)\}$. Hence we are left to consider $\|91\| \setminus (13Z \cup A) = (3 + 2 \parallel 37\parallel) \setminus \{7, 13, 27, 39, 53, 65\}$. Now, following Remark 29, we construct the cycles $W_0 = [0, 5]_2$, $W_1 = [0, 11]_2$, $W_2 = [0, 17]_2$, $W_3 = [0, 21]_2$, $W_4 = [0, 25]_2$, $W_5 = [0, 31]_2$, $W_6 = [0, 35]_2$, $W_7 = [0, 41]_4$, $W_8 = [0, 45]_2$, $W_9 = [0, 49]_2$, $W_{10} = [0, 55]_4$, $W_{11} = [0, 59]_2$, $W_{12} = [0, 63]_2$, $W_{13} = [0, 69]_2$, $W_{14} = [0, 73]_2$, $W_{15} = [0, 77]_2$. Hence take $\Sigma = \{A, B, C_0, W_n\}$.

Now we consider the case $n \equiv 0 \pmod{4}$.

**Proposition 34** Let $m$ be an odd integer; for any $n \equiv 0 \pmod{4}$ there exists a cyclic hamiltonian 2-factorization of $K_{m \times n}$.

**Proof** Let us first consider the case $n = 4$. Take $A$ as in (5) so that $\partial A = \pm (\{1\} \cup (2 + 2 \parallel m - 2\parallel))$.

If $m \equiv 1 \pmod{4}$, take $F_j = [0, 4j + 5]_2$ for $j \in \| m - 3 \parallel \setminus \{ \frac{m - 5}{4} \}$ and $C = [0]_{m - 2}$ (note that we have $\gcd(m - 2, 4m) = 1$).

If $m \equiv 3 \pmod{4}$, take

$$F_j = \left\{ \begin{array}{ll} [0, 4j + 5]_2 & \text{for } j \in \parallel \frac{m - 3}{2} \parallel \setminus \left\{ \frac{m - 5}{4} \right\}, \\
[0, 4j + 3]_2 & \text{for } j \in \frac{m + 1}{4} + \parallel \frac{m - 7}{4} \parallel, \\
\end{array} \right.$$  

and $C = [0]_{2m - 1}$. Note that $\gcd(2m - 1, 4m) = 1$. In both cases $\Sigma = \{A, C, F_j\}$ is a 2-starter in $\mathbb{Z}_{4m}$ relative to $m\mathbb{Z}_{4m}$.

Suppose now $n \geq 8$. We split the proof into two similar cases according to the congruence class of $m$ modulo 4.

**Case 1.** Suppose $m \equiv 1 \pmod{4}$.

If $n \equiv 4 \pmod{8}$, take $A$ as in (5) and for $k \in \parallel \frac{n - 12}{8} \parallel$ take $C_k$ as in (6). It is easy to see that

$$\partial(A \cup (\bigcup C_k)) = \pm \left( \left( 2 \parallel \frac{mn}{4} - 1 \parallel \setminus 2m \nmid \frac{n - 4}{4} \parallel \right) \cup \left( 1 + 2m \nmid \frac{n - 4}{4} \parallel \right) \right).$$

Now, for any $i \in \parallel \frac{n - 4}{4} \parallel$ and any $j \in \parallel \frac{m - 3}{2} \parallel \setminus \left\{ \frac{m - 5}{4} \right\}$ we take

$$D_{i, j} = [0, 2mi + 4j + 5]_2.$$
The differences arising from the cycles $D_{i,j}$ are the odd integers
\[
\pm \left(1 + 2\left\lceil \frac{mn}{4} - 1 \right\rceil \right) \setminus \left\{1 + 2mi, m - 2 + 2mi, m + 2mi : i \in \left\lceil \frac{n - 4}{4} \right\rceil \right\}.
\]

The only differences left to consider are the odd integers in the set $\pm(m - 2 + 2m \left\lceil \frac{n-4}{4} \right\rceil)$. To obtain them, take $G = [0]_{mn/4-2}$ and finally for $s \in \left\lceil \frac{n-12}{8} \right\rceil$ take
\[
F_s = [0, (2s+1)m - 2]_{mn/2-4};
\]
the $F_s$'s and $G$ are hamiltonian cycles, since $\gcd\left(\frac{mn}{4} - 2, mn\right) = 1$. One can check that set $\Sigma = \{A, C_k, D_{i,j}, F_s, G\}$ is a 2-starter in $\mathbb{Z}_{mn}$ relative to $m\mathbb{Z}_{mn}$.

If $n \equiv 0 \pmod{8}$, for $k \in \left\lceil \frac{n-8}{8} \right\rceil$ take the cycles
\[
\widetilde{C}_k = [0, 4m - 2 + 4km, 2, 4m - 4 + 4km, \ldots, m - 3, 3m + 1 + 4km, \\
m - 1, 4m + 1 + 4km, 2m + 3, 4m - 1 + 4km, 2m + 5, \\
4m - 3 + 4km, \ldots, 3m, 5m + 2 + 4km]_{2m}.
\]  \hspace{1cm} (7)

Also, for any $i \in \left\lceil \frac{n-4}{4} \right\rceil$ and for any $j \in \left\lceil \frac{m-3}{2} \right\rceil \setminus \left\{\frac{m-1}{4}\right\}$ let $D_{i,j} = [0, 2mi + 4j + 3]_2$. Finally, take
\[
G_0 = [0]_{mn/4-1}, \quad G_1 = [0]_{mn/2-1},
\]
and for all $s \in \left\lceil \frac{n-16}{8} \right\rceil$ consider
\[
F_s = [0, 2(s+1)m - 1]_{mn/2-2}.
\]

It is not hard to see that $\Sigma = \{\widetilde{C}_k, D_{i,j}, F_s, G_0, G_1\}$ is a 2-starter in $\mathbb{Z}_{mn}$ relative to $m\mathbb{Z}_{mn}$.

Case 2. Suppose $m \equiv 3 \pmod{4}$.

If $n \equiv 4 \pmod{8}$, take the cycle $A$ as in (5) and for any $k \in \left\lceil \frac{n-12}{8} \right\rceil$ take $C_k$ as in (6).

Also, for any $i \in \left\lceil \frac{n-4}{4} \right\rceil$ and for any $j \in \left\lceil \frac{m-3}{2} \right\rceil \setminus \left\{\frac{m-1}{4}\right\}$ let $D_{i,j} = [0, 2mi + 4j + 5]_2$. Finally, take $G = [0]_{mn/4+2}$ and for $s \in \left\lceil \frac{n-12}{8} \right\rceil$ take also the cycles
\[
F_s = [0, (2s+1)m + 2]_{mn/2+4}.
\]

One can check that $\Sigma = \{A, C_k, D_{i,j}, F_s, G\}$ is a 2-starter in $\mathbb{Z}_{mn}$ relative to $m\mathbb{Z}_{mn}$.

If $n \equiv 0 \pmod{8}$, for $k \in \left\lceil \frac{n-8}{8} \right\rceil$ take the cycles $A_k$ as in (3). Also, for any $i \in \left\lceil \frac{n-4}{4} \right\rceil$ take
\[
D_{i,j} = \begin{cases} [0, 2mi + 4j + 5]_2 & \text{for } j \in \left\lceil \frac{m-7}{4} \right\rceil, \\
[0, 2mi + 4j + 3]_2 & \text{for } j \in \left\lceil \frac{m+1}{4} \right\rceil + \left\lceil \frac{m-7}{4} \right\rceil.
\end{cases}
\]

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Finally, take

\[ G_0 = [0]_{mn/4-1}, \quad G_1 = [0]_{mn/2-1}, \]

and for all \( s \in \left\lVert \frac{n-16}{8} \right\rVert \) take

\[ F_s = [0, 2(s+1)m-1]_{mn/2-2}. \]

It is not hard to see that \( \Sigma = \{A_k, D_{i,j}, F_s, G_0, G_1\} \) is a 2-starter in \( \mathbb{Z}_{mn} \) relative to \( m\mathbb{Z}_{mn} \).

**Example 35** Following the proof of Proposition 34, we construct a 2-starter \( \Sigma \) in \( \mathbb{Z}_{mn} \) relative to \( m\mathbb{Z}_{mn} \) for some choice of \( m \) and \( n \).

**Case 1.** Let \( m = 9 \) and \( n = 20 \). The elements of \( \Sigma \) are

\[
\begin{align*}
A &= [0, 16, 2, 14, 4, 12, 6, 10, 8]_9, \\
C_0 &= [0, 52, 2, 50, 4, 48, 6, 46, 8, 45, 11, 43, 13, 41, 15, 39, 17, 37]_{18}, \\
C_1 &= [0, 88, 2, 86, 4, 84, 6, 82, 8, 81, 11, 79, 13, 77, 15, 75, 17, 73]_{18}, \\
D_{0,0} &= [0, 5]_2, \quad D_{0,2} = [0, 13]_2, \quad D_{0,3} = [0, 17]_2, \quad D_{1,0} = [0, 23]_2, \\
D_{1,2} &= [0, 31]_2, \quad D_{1,3} = [0, 35]_2, \quad D_{2,0} = [0, 41]_2, \quad D_{2,2} = [0, 49]_2, \\
D_{2,3} &= [0, 53]_2, \quad D_{3,0} = [0, 59]_2, \quad D_{3,2} = [0, 67]_2, \quad D_{3,3} = [0, 71]_2, \\
D_{4,0} &= [0, 77]_2, \quad D_{4,2} = [0, 85]_2, \quad D_{4,3} = [0, 89]_2, \quad G = [0]_{43}, \\
F_0 &= [0, 7]_{86}, \quad F_1 = [0, 25]_{86}.
\end{align*}
\]

**Case 2.** Let \( m = 15 \) and \( n = 12 \). The elements of \( \Sigma \) are

\[
\begin{align*}
A &= [0, 28, 2, 26, 4, 24, 6, 22, 8, 20, 10, 18, 12, 16, 14]_{15}, \\
C_0 &= [0, 88, 2, 86, 4, 84, 6, 82, 8, 80, 10, 78, 12, 76, 14, 75, 17, 73, 19, 71, 21, 69, \\
&\quad 23, 67, 25, 65, 27, 63, 29, 61]_{30}, \\
D_{0,0} &= [0, 5]_2, \quad D_{0,1} = [0, 9]_2, \quad D_{0,2} = [0, 13]_2, \quad D_{0,4} = [0, 21]_2, \\
D_{0,5} &= [0, 25]_2, \quad D_{0,6} = [0, 29]_2, \quad D_{1,0} = [0, 35]_2, \quad D_{1,1} = [0, 39]_2, \\
D_{1,2} &= [0, 43]_2, \quad D_{1,4} = [0, 51]_2, \quad D_{1,5} = [0, 55]_2, \quad D_{1,6} = [0, 59]_2, \\
D_{2,0} &= [0, 65]_2, \quad D_{2,1} = [0, 69]_2, \quad D_{2,2} = [0, 73]_2, \quad D_{2,4} = [0, 81]_2, \\
D_{2,5} &= [0, 85]_2, \quad D_{2,6} = [0, 89]_2, \quad G = [0]_{47}, \quad F_0 = [0, 17]_{94}.
\end{align*}
\]

**8 Proof of Theorems 1 and 2**

**Proof of Theorem 1** “⇒” It is Corollary 13. 

“⇐” Case (a) is settled in Proposition 15. Case (b) is considered in Propositions 17, 20, 22 and 26. \( \square \)
Proof of Theorem 2  “⇒” It is sufficient to apply Theorem 4, Remark 9 and Corollary 11.
“⇐” It follows from Theorem 5 and Propositions 32 and 34.

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