WEIGHTED $\ell_q$ APPROXIMATION PROBLEMS ON THE BALL
AND ON THE SPHERE

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Abstract. Let $L_{q,\mu}$, $1 \leq q < \infty$, $\mu \geq 0$, denote
the weighted $L_q$ space with the classical Jacobi weight $w_\mu$ on
the ball $\mathbb{B}^d$. We consider the weighted least $\ell_q$ approxima-
tion problem for a given $L_{q,\mu}$-Marcinkiewicz-Zygmund family
on $\mathbb{B}^d$. We obtain the weighted least $\ell_q$ approximation
errors for the weighted Sobolev space $W_{r,\mu}^q$, $r > (d + 2\mu)/q$,
which are order optimal. We also discuss the least squares quadrature
induced by an $L_{2,\mu}$-Marcinkiewicz-Zygmund family, and get the
quadrature errors for $W_{r,\mu}^2$, $r > (d + 2\mu)/2$, which are also
order optimal. Meanwhile, we give the corresponding the weighted
least $\ell_q$ approximation theorem and the least squares quadrature
errors on the sphere.

1. Introduction

Let $\mathbb{B}^d = \{x \in \mathbb{R}^d : |x| \leq 1\}$ denote the unit ball of $\mathbb{R}^d$, where $x \cdot y$ is the usual
inner product and $|x| = (x \cdot x)^{1/2}$ is the usual Euclidean norm. Let $\Pi_n^d$ be the space
of all polynomials in $d$ variables of total degree at most $n$. For the classical Jacobi
weight on $\mathbb{B}^d$

$$w_\mu(x) = b_\mu^d(1 - |x|^2)^{-\mu/2}, \quad \mu \geq 0, \quad b_\mu^d = \left( \int_{\mathbb{B}^d} (1 - |x|^2)^{-\mu/2} dx \right)^{-1},$$

we denote by $L_{p,\mu} \equiv L_p(\mathbb{B}^d, w_\mu(x)dx)$, $0 < p < \infty$, the space of all Lebesgue
measurable functions $f$ on $\mathbb{B}^d$ with finite quasi-norm

$$\|f\|_{p,\mu} := \left( \int_{\mathbb{B}^d} |f(x)|^p w_\mu(x) dx \right)^{1/p}.$$

When $p = \infty$ we consider the space of continuous functions $C(\mathbb{B}^d)$ with the uniform
norm. In particular, $L_{2,\mu}$ is a Hilbert space with inner product

$$\langle f, g \rangle_\mu := \int_{\mathbb{B}^d} f(x)g(x)w_\mu(x)dx,$$ for $f, g \in L_{2,\mu}$.

For $n \in \mathbb{N}$ we take $l_n$ points in $\mathbb{B}^d$ and $l_n$ positive numbers

$$X_n := \{x_{n,k} : k = 1, 2, \ldots, l_n\} \text{ and } \tau_n := \{\tau_{n,k} : k = 1, 2, \ldots, l_n\},$$

and assume that $l_n \to \infty$ as $n \to \infty$. This yields a family $(X, \tau) := (X_n, \tau_n)_{n \geq 1}$. We
assume that the family $(X, \tau)$ constitutes a Marcinkiewicz-Zygmund (MZ) family
on $\mathbb{B}^d$ defined as follows.

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weight; Weighted least $\ell_q$ approximation; Ball; Sphere.
Definition 1.1. Suppose that $\mathcal{X} = \{\mathcal{X}_n\} = \{x_{n,k} : k = 1, 2,\ldots, l_n, n = 1, 2,\ldots\}$ is a doubly-indexed set of points in $\mathbb{B}^d$, and $\tau = \{\tau_n\} = \{\tau_{n,k} : k = 1, 2,\ldots, l_n, n = 1, 2,\ldots\}$ is a doubly-indexed set of positive numbers. Then for $0 < q < \infty$, the family $(\mathcal{X}, \tau)$ is called an $L_{q,\mu}$-Marcinkiewicz-Zygmund family, denoted by $L_{q,\mu}$-MZ, if there exist constants $A, B > 0$ independent of $n$ such that

$$A\|P\|_{q,\mu}^q \leq \sum_{k=1}^{l_n}|P(x_{n,k})|^q\tau_{n,k} \leq B\|P\|_{q,\mu}^q,$$  \hspace{1cm} (1.1)

for all $P \in \Pi^d_n$.

The ratio $\kappa = B/A$ is the global condition number of $L_{q,\mu}$-MZ family $(\mathcal{X}, \tau)$, and $\mathcal{X}_n = \{x_{n,k} : k = 1, 2,\ldots, l_n\}$ is the $n$-th layer of $\mathcal{X}$.

Remark 1.2. For $q = \infty$, we say that the family $(\mathcal{X}, \tau)$ is $L_{\infty}$-MZ if there exist constants $A > 0$ independent of $n$ such that

$$A\|P\|_{\infty} \leq \max_{1 \leq k \leq l_n} |P(x_{n,k})| \leq \|P\|_{\infty}$$

for all $P \in \Pi^d_n$.

The global condition number of $L_{\infty}$-MZ family is $1/A$.

It follows from [5,19] that such $L_{q,\mu}$-MZ families on $\mathbb{B}^d$ exist. Necessary density conditions for $L_{2,\mu}$-MZ families on $\mathbb{B}^d$ were obtained in [1,4]. There are many papers devoted to studying MZ families on the sphere and compact manifold (see [7,14,16,18,20]).

Remark 1.3. It follows from [14] that if $P \in \Pi^d_n$ and $P(x_{n,k}) = 0$, $k = 1,\ldots, l_n$, then $P = 0$. This means that usually $\mathcal{X}_n$ contains more than $\dim \Pi^d_n$ points, so that it is not an interpolating set for $\Pi^d_n$. We set

$$\mu_n := \sum_{k=1}^{l_n}\tau_{n,k}\delta_{x_{n,k}},$$  \hspace{1cm} (1.2)

where $\delta_z(g) = g(z)$ for a function $g$ is the evaluation operator. For any $f \in C(\mathbb{B}^d)$, we define for $0 < q < \infty$,

$$\|f\|_{(q)} := \left(\int_{\mathbb{B}^d}|f(x)|^q d\mu_n(x)\right)^{1/q} = \left(\sum_{k=1}^{l_n}|f(x_{n,k})|^q\tau_{n,k}\right)^{1/q},$$

and for $q = \infty$,

$$\|f\|_{(\infty)} := \max_{1 \leq k \leq l_n} |f(x_{n,k})|.$$

Hence, for $0 < q \leq \infty$, $(\Pi^d_n, \| \cdot \|_{(q)})$ is a Frechet space. It follows from [14] that the $L_{q,\mu}$-norm of a polynomial of degree at most $n$ on $\mathbb{B}^d$ is comparable to the discrete version given by the weighted $l_q$-norm of its restriction to $\mathcal{X}_n$.

This paper is concerned with constructive polynomial approximation on $\mathbb{B}^d$ which uses function values (the samples) at the points in $\mathcal{X}_n$ (sometimes called standard information). For an $L_{q,\mu}$-MZ family $(\mathcal{X}, \tau)$, we usually sample a continuous function $f$ on the $n$-th layer $\mathcal{X}_n$ and apply the samples to construct an approximation to $f$. We use the weighted least $l_q$ algorithms. This means that our problem is to solve a sequence of weighted least $l_q$ approximation problems with samples taken from the samples $\mathcal{X}_n$. We recall the following definition.
Definition 1.4. Let \( 0 < q \leq \infty \), and let \((X, \tau)\) be an \( L_{q, \mu} \)-MZ family. For \( f \in C(\mathbb{B}^d) \), we define the weighted least \( \ell_q \) approximation by

\[
L_{n,q}(f) := \arg \min_{P \in \Pi_n} \left( \sum_{k=1}^{l_n} |f(x_{n,k}) - P(x_{n,k})|^{q} \right)^{1/q}.
\] (1.3)

That is, \( L_{n,q}(f) \) is any function in \( \Pi_n \) satisfying

\[
\| f - L_{n,q}(f) \|_q = \min_{P \in \Pi_n} \| f - P \|_q.
\]

Remark 1.5. Clearly, for \( f \in C(\mathbb{B}^d) \) and \( 0 < q < \infty \), the minimizer \( L_{n,q}(f) \) exists. Hence, this definition is well defined. For \( 1 < q < \infty \), \( L_{n,q}(f) \) is unique. However, if \( 0 < q \leq 1 \) or \( q = \infty \), then \( L_{n,q}(f) \) may be not unique.

Remark 1.6. If \( q \neq 2 \), then \( L_{n,q} \) is not linear. That is, there exist \( f_1, f_2 \in C(\mathbb{B}^d) \) such that

\[
L_{n,q}(f_1 + f_2) \neq L_{n,q}(f_1) + L_{n,q}(f_2).
\]

However, for \( f \in C(\mathbb{B}^d) \) and \( P_n \in \Pi_n \), by the definition of \( L_{n,q}(f) \) we have

\[
\| f - L_{n,q}(f) \|_q = \min_{P \in \Pi_n} \| f - P \|_q
\]

\[
= \| (f + P_n) - L_{n,q}(f + P_n) \|_q
\]

\[
\leq \| f + P_n \|_q.
\] (1.4)

The most interesting case is \( q = 2 \) in which \( L_{n,2} \) has many good properties. Indeed, \( L_{n,2} \) is a bounded linear operator on \( C(\mathbb{B}^d) \) satisfying that \( L_{n,2}^2 = L_{n,2} \), and the range of \( L_{n,2} \) is \( \Pi_n \). If we define the discretized inner product on \( C(\mathbb{B}^d) \) by

\[
\langle f, g \rangle_{(2)} := \sum_{k=1}^{l_n} \tau_{n,k} f(x_{n,k}) g(x_{n,k}),
\]

then \( L_{n,2} \) is just the orthogonal projection onto \( \Pi_n \) with respect to the discretized inner product \( \langle \cdot, \cdot \rangle_{(2)} \). Hence, we obtain for \( f \in C(\mathbb{B}^d) \),

\[
L_{n,2}(f)(x) = \langle f, D_n(x, \cdot) \rangle_{(2)} = \sum_{k=1}^{l_n} \tau_{n,k} f(x_{n,k}) D_n(x, x_{n,k}),
\]

where \( D_n(x, y) \) is the reproducing kernel of \( \Pi_n \) with respect to the discretized inner product \( \langle \cdot, \cdot \rangle_{(2)} \). We call \( L_{n,2}(f) \) the weighted least squares polynomial, and \( L_{n,2} \) the weighted least squares operator.

Following Gröchenig in [8], for \( L_{2, \mu} \)-MZ family we can use the frame theory to construct the quadrature on \( \mathbb{B}^d \)

\[
I_n(f) = \sum_{k=1}^{l_n} w_{n,k} f(x_{n,k}).
\]

It was shown in [13] that

\[
w_{n,k} = \tau_{n,k} \int_{\mathbb{B}^d} D_n(x, x_{n,k}) w_\mu(x) dx,
\]

and

\[
I_n(f) = \int_{\mathbb{B}^d} L_{n,2}(f)(x) w_\mu(x) dx.
\] (1.5)
Such quadrature $I_n$ is called the least squares quadrature.

Given an $L_2$-MZ family $(\mathcal{X}, \tau)$ on a usual compact space $\mathcal{M}$ with some structure, Gröchenig in [8] studied the weighted least squares approximation $L_n^{\mathcal{M}}(f)$ and the least squares quadrature $I_n^{\mathcal{M}}$ from the samples $\mathcal{X}_n$ of a function $f$, and obtained the following approximation theorems and quadrature errors as follows. For $r > d/2$, we have

$$\|f - L_n^{\mathcal{M}}(f)\|_2 \leq c(1 + \kappa^2)^{1/2}n^{-r-d/2}\|f\|_{H^r(\mathcal{M})},$$  

(1.6) and

$$\left| \int_{\mathcal{M}} f(x)dm(x) - I_n^{\mathcal{M}}(f) \right| \leq c(1 + \kappa^2)^{1/2}n^{-r-d/2}\|f\|_{H^r(\mathcal{M})},$$  

(1.7) where $c$ depends on $d$, $r$, but not on $f$, $\kappa$ or $(\mathcal{X}, \tau)$. $\mu$ is a probability measure on $\mathcal{M}$, and $H^r(\mathcal{M})$ is the Sobolev space on $\mathcal{M}$ (see [8]). However, the obtained error estimates are not optimal due to the generality of $\mathcal{M}$. Lu and Wang in [18] investigated the weighted least squares approximation $L_n^{\mathcal{M}}$ and the least squares quadrature $I_n^{\mathcal{M}}$ on the sphere $\mathbb{S}^d$, and obtained the following optimal error estimates. For $r > d/2$, we have

$$\|f - L_n^{\mathcal{M}}(f)\|_2 \leq c(1 + \kappa^2)^{1/2}n^{-r}\|f\|_{H^r(\mathbb{S}^d)},$$  

(1.8) and

$$\left| \int_{\mathbb{S}^d} f(x)d\sigma(x) - I_n^{\mathcal{M}}(f) \right| \leq c(1 + \kappa^2)^{1/2}n^{-r}\|f\|_{H^r(\mathbb{S}^d)},$$  

(1.9) where $c$ depends only on $d$ and $r$, and $H^r(\mathbb{S}^d)$ is the Sobolev space on $\mathbb{S}^d$.

Gröchenig commented in [8] that Marcinkiewicz-Zygmund families with respect to general $q$-norms seemed to require different techniques. In this paper we consider $L_{q,\nu}$-MZ families on $\mathbb{B}^d$ for $1 \leq q < \infty$. We use the weighted least $\ell_q$ approximation to obtain the optimal approximation errors of the weighted Sobolev classes $BW^r_{q,\mu}$ on $\mathbb{B}^d$ (see Section 2 for definition of $BW^r_{q,\mu}$). The techniques we used are different from the ones in [8] even in the case $q = 2$. We remark that $W^r_{q,\mu}$ is just the Sobolev space $H^r(\mathbb{B}^d)$ given in [8], and if $r > (d + 2\mu)/q$, then the weighted Sobolev space $W^r_{q,\mu}$ can be compactly embedded into $C(\mathbb{B}^d)$. Our main results can be formulated as follows.

**Theorem 1.7.** Let $1 \leq p \leq \infty$, $1 \leq q < \infty$, and $\mu \geq 0$. Suppose that $(\mathcal{X}, \tau)$ is an $L_{q,\mu}$-MZ family with global condition number $\kappa = B/A$, $L_n^{\mathcal{X}}$ is the weighted least $\ell_q$ approximation defined by (1.3). If $f \in W^r_{p,\mu}$, $r > (d + 2\mu)\max\{1/p, 1/q\}$, then we have

$$\|f - L_n^{\mathcal{X}}(f)\|_{q,\mu} \leq C(1 + \kappa^{1/q})n^{-r+(d+2\mu)(1/p-1/q)+}\|f\|_{W^r_{p,\mu}},$$  

(1.10) where $C > 0$ are independent of $f$, $n$, $\kappa$, and $(\mathcal{X}, \tau)$, and $a_+ = \max\{a, 0\}$.

The following theorem follows from Theorem 1.7 and (1.5) immediately.

**Theorem 1.8.** Let $\mu \geq 0$. Suppose that $(\mathcal{X}, \tau)$ is an $L_{2,\mu}$-MZ family with global condition number $\kappa = B/A$, $L_n^{\mathcal{X}}$ and $I_n$ are the weighted least squares approx-imation and the least squares quadrature, respectively. If $f \in H^r(\mathbb{B}^d) \equiv W^r_{2,\mu}$, $r > (d + 2\mu)/2$, then we have

$$\|f - L_n^{\mathcal{X}}(f)\|_{2,\mu} \leq C(1 + \kappa^{1/2})n^{-r}\|f\|_{H^r(\mathbb{B}^d)},$$  

(1.11) and

$$\left| \int_{\mathbb{B}^d} f(x)w(x)dx - I_n(f) \right| \leq C(1 + \kappa^{1/2})n^{-r}\|f\|_{H^r(\mathbb{B}^d)},$$  

(1.12)
where $C > 0$ are independent of $f$, $n$, $\kappa$, and $(X, \tau)$.

We also give the corresponding weighted least $\ell_q$ approximation and least squares quadrature results on $S^d$.

The contribution of this paper contains three aspects. First, we obtain the corresponding results for Marcinkiewicz-Zygmund families with respect to general $q$-norms by a different method from [8]. Second, the obtained error estimates (1.10) for $1 \leq q = p < \infty$ and (1.12) are asymptotically optimal in a variety of Sobolev space settings (as explained in Remark 1.9 below). Third, we reduce dependence on the global condition number in (1.11) by replacing the constant $(1 + \kappa^2)^{1/2}$ in (1.0) with the constant $1 + \kappa^{1/2}$.

Throughout the paper, the notation $a_n \asymp b_n$ means $a_n \lesssim b_n$ and $a_n \gtrsim b_n$. Here, $a_n \lesssim b_n (a_n \gtrsim b_n)$ means that there exists a constant $c > 0$ independent of $n$ such that $a_n \leq cb_n (b_n \leq ca_n)$.

Remark 1.9. Let $F$ be a class of continuous functions on $D$, and $(X, \| \cdot \|_X)$ be a normed linear space of functions on $D$, where $D$ is a subset of $\mathbb{R}^d$, $\nu$ is a probability measure on $D$. For $N \in \mathbb{N}$, the sampling numbers (or the optimal recovery) of $F$ in $X$ are defined by

$$g_N(F, X) := \inf_{\xi_1, \ldots, \xi_N \in D} \sup_{f \in F} \| f - \varphi(f(\xi_1), \ldots, f(\xi_N)) \|_X,$$

where the infimum is taken over all $N$ points $\xi_1, \ldots, \xi_N$ in $D$ and all mappings $\varphi$ from $\mathbb{R}^N$ to $X$. And the optimal quadrature errors of $F$ are defined by

$$e_N(F; \text{INT}) := \inf_{\lambda_1, \ldots, \lambda_N \in \mathbb{R}} \sup_{\xi_1, \ldots, \xi_N \in D} \left| \int_D f(x) d\nu(x) - \sum_{j=1}^{N} \lambda_j f(\xi_j) \right|,$$

where the infimum is taken over all $N$ points $\xi_1, \ldots, \xi_N$ in $D$ and all $N$ numbers $\lambda_1, \ldots, \lambda_N$.

It follows from [19] that there exist $L_{q,\mu}$-MZ families on $B^d$ with $l_n \asymp N \asymp n^d$. Combining with [12] Theorem 3.5], for such $L_{q,\mu}$-MZ family, and $1 \leq q \leq p < \infty$, $r > (d + 2\mu)/q$, we obtain,

$$\sup_{f \in BW_{p,\mu}^r} \| f - L_{n,q}(f) \|_{q,\mu} \asymp N^{\mu/d} \asymp g_N(BW_{p,\mu}^r, L_{q,\mu}),$$

which implies that the weighted least $\ell_q$ approximation operators $L_{n,q}$ are asymptotically optimal algorithms in the sense of optimal recovery.

Also, for the least squares quadrature rules $I_n$, it follows from (1.12) and [12] Theorem 1.1] that for $r > (d + 2\mu)/2$,

$$\sup_{f \in BH^r(B^d)} \left| \int_{B^d} f(x)w_\mu(x) dx - I_n(f) \right| \asymp N^{-\mu/d} \asymp e_N(BH^r(B^d); \text{INT}),$$

which means that the least squares quadrature rules $I_n$ are the asymptotically optimal quadrature formulas for $BH^r(B^d)$.

The outline of this paper is as follows. In the next section, we recall some basic results about harmonic analysis on the ball, introduce the filtered approximation and prove some auxiliary lemmas. In Section 3 we give the proof of Theorem 1.1. Finally, in Section 4 we give the corresponding weighted least $\ell_q$ approximation and least squares quadrature results on $S^d$. 


2. Preliminaries

2.1. Harmonic analysis on the ball.

Let $\mathbb{B}^d$ denote the unit ball of $\mathbb{R}^d$, and $L_{p,\mu}$, $1 \leq p < \infty$, the weighted $L_p$ space on $\mathbb{B}^d$. We denote by $\mathcal{V}^d_n(w_\mu)$ the space of all polynomials of degree $n$ which are orthogonal to lower degree polynomials in $L_{2,\mu}$, and

$$a_n^d := \dim \mathcal{V}^d_n(w_\mu) = \binom{n + d - 1}{n} \approx n^{d-1}.$$  

It is well known (see [6, p.38 or p.229]) that

$$D_\mu P = -(n + 2\mu + d - 1)P, \quad \text{for all } P \in \mathcal{V}^d_n(w_\mu),$$

where the second-order differential operator

$$D_\mu := \triangle - (x \cdot \nabla)^2 - (2\mu + d - 1)x \cdot \nabla,$$

and $\triangle$, $\nabla$ are the Laplace operator, the gradient operator, respectively.

The standard Hilbert theory states that the spaces $\mathcal{V}^d_n(w_\mu)$ are mutually orthogonal in $L_{2,\mu}$. Let $\{\phi_{nk} \equiv \phi_{nk}^d : k = 1, 2, \ldots, a_n^d\}$ be a fixed orthonormal basis for $\mathcal{V}^d_n(w_\mu)$. Then

$$\{\phi_{nk} : k = 1, 2, \ldots, a_n^d, \ n = 0, 1, 2, \ldots\}$$

is an orthonormal basis for $L_{2,\mu}$.

The orthogonal projector $\text{Proj}_n : L_{2,\mu} \rightarrow \mathcal{V}^d_n(w_\mu)$ can be written as

$$(\text{Proj}_n f)(x) = \sum_{k=1}^{a_n^d} \langle f, \phi_{nk} \rangle \phi_{nk}(x) = \langle f, P_n(w_\mu; x, \cdot) \rangle, \quad x \in \mathbb{B}^d,$$

where $P_n(w_\mu; x, y) = \sum_{k=1}^{a_n^d} \phi_{nk}(x)\phi_{nk}(y)$ is the reproducing kernel of $\mathcal{V}^d_n(w_\mu)$. See [6] for more details about $P_n(w_\mu; x, y)$.

Given $r > 0$, define the fractional power $(-D_\mu)^{r/2}$ of the operator $-D_\mu$ on $f$ by

$$(-D_\mu)^{r/2}(f) := \sum_{k=0}^{\infty} (k(k + 2\mu + d - 1))^{r/2} \text{Proj}_k f,$$

in the sense of distribution. Using this operator we define the weighted Sobolev space as follows: for $r > 0$ and $1 \leq p \leq \infty$,

$$W_{p,\mu}^r := \{f \in L_{p,\mu} : \|f\|_{W_{p,\mu}^r} := \|f\|_{p,\mu} + \|(-D_\mu)^{r/2}(f)\|_{p,\mu} < \infty\},$$

while the weighted Sobolev class $BW_{p,\mu}^r$ is defined to be the unit ball of the weighted Sobolev space $W_{p,\mu}^r$. Note that if $r > (d + 2\mu)/p$, then $W_{p,\mu}^r$ is compactly embedded into $C(\mathbb{B}^d)$.

We introduce a metric $d$ on $\mathbb{B}^d$ by

$$d(x, y) := \arccos \left( |x| \sqrt{1 - |x|^2} + \sqrt{1 - |y|^2} \right), \quad x, y \in \mathbb{B}^d.$$

For $r > 0$, $x \in \mathbb{B}^d$ and a positive integer $n$, we set

$$B(x, r) := \{y \in \mathbb{B}^d \mid d(x, y) \leq r\} \quad \text{and} \quad W_{\mu}(n; x) := (\sqrt{1 - |x|^2} + n^{-1})^{2\mu}.$$

It follows from [19, Equation (4.23)] that

$$\frac{1}{2^\mu(1 + nd(x, y))^{2\mu}} \leq \frac{W_{\mu}(n; x)}{W_{\mu}(n; y)} \leq 2^\mu(1 + nd(x, y))^{2\mu}, \quad x, y \in \mathbb{B}^d. \quad (2.1)$$
For $\epsilon > 0$, we say that a finite subset $\Lambda \subset B^d$ is maximal $\epsilon$-separated if

$$B^d \subset \bigcup_{y \in \Lambda} B(y, \epsilon) \quad \text{and} \quad \min_{y \neq y'} d(y, y') \geq \epsilon.$$ 

Such a maximal $\epsilon$-separated set $\Lambda$ exists and satisfies (see [19, Lemma 5.2])

$$1 \leq \sum_{\xi \in \Lambda} \chi_{B(\xi, \epsilon)}(x) \leq C_d, \quad \text{for any } x \in B^d. \quad (2.2)$$

### 2.2. The filtered approximation.

Now we introduce the filtered approximation on the ball as in [19]. In the filtered approximation, the terms in the Fourier series are to be modified by multiplication by $\eta(k/n)$, where $\eta \in C^\infty([0, \infty))$ is a "$C^\infty$-filter" satisfying

$$\chi_{[0,1]} \leq \eta \leq \chi_{[0,2]}.$$ 

Here, $\chi_A$ denotes the characteristic function of $A$ for $A \subset \mathbb{R}$. The filtered approximation of $f$ is defined by

$$V_n(f)(x) := \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \text{Proj}_k(f)(x) = \langle f, K_{n,\eta}(x, \cdot) \rangle_\mu, \quad (2.3)$$

where

$$K_{n,\eta}(x, y) = \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) P_k(w_\mu; x, y) = \sum_{k=0}^{2n-1} \eta\left(\frac{k}{n}\right) P_k(w_\mu; x, y).$$

Evidently, $V_n(f) \in \Pi_d^{2n-1}$, and for any $f \in \Pi_n^d$ we have

$$f(x) = \int_{\mathbb{B}^d} f(y) K_{n,\eta}(w_\mu; x, y) w_\mu(y) dy, \quad x \in B^d. \quad (2.4)$$

It follows from [19, Theorem 4.2] that for $x, y \in B^d$,

$$|K_{n,\eta}(x, y)| \lesssim \frac{n^d}{\sqrt{W_\mu(n; x)} \sqrt{W_\mu(n; y)}(1 + nd(x, y))^{d+\mu+1}} := g_n(x, y). \quad (2.5)$$

By (2.5) and [19, Lemma 4.6] we have

$$\max_{y \in B^d} \int_{B^d} |K_{n,\eta}(x, y)| w_\mu(x) dx \lesssim \max_{y \in B^d} \int_{B^d} g_n(x, y) w_\mu(x) dx \lesssim 1. \quad (2.6)$$

For $n \in \mathbb{N}$ we define

$$E_n(f)_{p,\mu} := \inf\{\|f - P\|_{p,\mu} : P \in \Pi_n^d\}.$$ 

It is well known (see [19, 23]) that for $1 \leq p \leq \infty$, $r > 0$, and $f \in W^r_{p,\mu}$,

$$\|f - V_n(f)\|_{p,\mu} \lesssim E_n(f)_{p,\mu} \lesssim n^{-r} \|f\|_{W^r_{p,\mu}}. \quad (2.7)$$

### 2.3. Some auxiliary lemmas.

In this subsection, we give some lemmas which will be needed in the next section.

Let $n$ be a positive integer. Suppose that $\Omega_n$ is a finite set of $\mathbb{B}^d$, and $\Gamma_n := \{\mu_\omega : \omega \in \Omega_n\}$ is a set of positive numbers. The induced measure $\lambda_n$ by $\Gamma_n$ is defined by

$$\lambda_n := \sum_{\omega \in \Omega_n} \mu_\omega \delta_\omega, \quad (2.8)$$
where $\delta_z(f) = f(z)$ for a function $f$ is the evaluation operator. Hence, for any continuous function $f$ defined on $\mathbb{B}^d$, we have

$$
\int_{\mathbb{B}^d} f(x)d\lambda_n(x) = \sum_{\omega \in \Omega_n} \mu_\omega f(\omega).
$$

We call the induced measure $\lambda_n$ satisfies the regularity condition with a constant $N$ if the inequality

$$
\int_{\mathbb{B}(y, 1/n)} d\lambda_n(x) \leq N \int_{\mathbb{B}(y, 1/n)} w_\mu(x)dx
$$

holds. That is,

$$
\sum_{\omega \in \Omega_n \cap \mathbb{B}(y, 1/n)} \mu_\omega \leq N w_\mu(\mathbb{B}(y, 1/n)), \quad \text{for any } y \in \mathbb{B}^d, \quad (2.9)
$$

holds, where $w_\mu(E) := \int_E w_\mu(x)dx$ for any measurable $E \subset \mathbb{B}^d$.

**Lemma 2.1.** ([21, Theorem 3.1]) Suppose that $n \in \mathbb{N}$, $\Omega_n$ is a finite subset of $\mathbb{B}^d$, and $\Gamma_n := \{\mu_\omega : \omega \in \Omega_n\}$ is a set of positive numbers. If there exist $p_0 \in (0, \infty)$ and $M > 0$ such that for any $f \in \Pi_d^m$,

$$
\sum_{\omega \in \Omega_n} \mu_\omega |f(\omega)|^{p_0} \leq M \int_{\mathbb{B}^d} |f(x)|^{p_0} w_\mu(x)dx, \quad (2.10)
$$

then the following regularity condition

$$
\sum_{\omega \in \Omega_n \cap \mathbb{B}(y, 1/n)} \mu_\omega \leq C M w_\mu(\mathbb{B}(y, 1/n)), \quad \text{for any } y \in \mathbb{B}^d, \quad (2.11)
$$

holds, where $C > 0$ depends only on $d, \mu,$ and $p_0$.

**Lemma 2.2.** Suppose that $n \in \mathbb{N}$, $\Omega_n$ is a finite subset of $\mathbb{B}^d$, and $\Gamma_n := \{\mu_\omega : \omega \in \Omega_n\}$ is a set of positive numbers. If the induced measures $\lambda_n$ by $\Gamma_n$ satisfies the regularity condition (2.9) with a constant $N$, then for any $1 \leq p < \infty$, $m \in \mathbb{N}$, $m \geq n$, and $f \in \Pi_d^m$, we have

$$
\sum_{\omega \in \Omega_n} \mu_\omega |f(\omega)|^p \leq CN\left(\frac{m}{n}\right)^{d+2\mu} \int_{\mathbb{B}^d} |f(x)|^pw_\mu(x)dx, \quad (2.12)
$$

where $C > 0$ depends only on $d, \mu,$ and $p$.

**Lemma 2.2** is an improvement of [21, Corollary 3.3] with the exponent $d + 2\mu + 1$ in [21, Corollary 3.3] replaced by $d + 2\mu$ in (2.12). Its proof is based on the following lemma.

**Lemma 2.3.** Suppose that $n \in \mathbb{N}$, $\Omega_n$ is a finite subset of $\mathbb{B}^d$, and $\Gamma_n := \{\mu_\omega : \omega \in \Omega_n\}$ is a set of positive numbers. If the induced measure $\lambda_n$ by $\Gamma_n$ satisfies the regularity condition (2.9) with a constant $N$, then for any $m \in \mathbb{N}$, $m \geq n$, we have

$$
\max_{y \in \mathbb{B}^d} \left(\sum_{\omega \in \Omega_n} \mu_\omega |K_{m,0}(\omega, y)|\right) \leq C N\left(\frac{m}{n}\right)^{d+2\mu}, \quad (2.13)
$$

where $C > 0$ is independent of $m, n$ and $N$. 

Proof. For $x, y \in \mathbb{B}^d$, and $m \geq n$, set

$$g_m(x, y) = \frac{m^d}{\sqrt{W_{\mu}(m; x)} \sqrt{W_{\mu}(m; y)} (1 + md(x, y))^{d+\mu+1}}.$$ 

Note that

$$\frac{W_{\mu}(n; x)}{W_{\mu}(m; x)} = \left(\frac{1/n + \sqrt{1 - |x|^2}}{1/m + \sqrt{1 - |x|^2}}\right)^{2\mu} = \left(\frac{1 + n \sqrt{1 - |x|^2}}{1 + m \sqrt{1 - |x|^2}}\right)^{2\mu} \left(\frac{m}{n}\right)^{2\mu} \leq \left(\frac{m}{n}\right)^{2\mu}.$$ 

It follows that

$$g_m(x, y) \leq \left(\frac{m}{n}\right)^{2\mu} \frac{m^d}{\sqrt{W_{\mu}(n; x)} \sqrt{W_{\mu}(n; y)} (1 + nd(x, y))^{d+\mu+1}}.$$ 

According to (2.15), we have

$$|K_{m, n}(x, y)| \lesssim g_m(x, y) \leq \left(\frac{m}{n}\right)^{d+2\mu} g_n(x, y). \tag{2.14}$$

Let $\Lambda_n$ be a maximal $\frac{1}{n}$-separated set on $\mathbb{B}^d$. For $\xi \in \Lambda_n$ and $x, x' \in B(\xi, \frac{1}{n})$, by (2.1) we have

$$g_n(x', y) \asymp g_n(\xi, y) \asymp g_n(x, y),$$

which leads to

$$\max_{x \in B(\xi, \frac{1}{n})} g_n(x, y) \lesssim \min_{x \in B(\xi, \frac{1}{n})} g_n(x, y). \tag{2.15}$$

It follows from (2.14) that for any $y \in \mathbb{B}^d$,

$$\sum_{\omega \in \Omega_n^*} \mu_\omega |K_{m, n}(\omega, y)| = \int_{\mathbb{B}^d} |K_{m, n}(x, y)| d\lambda_n(x)$$

$$\lesssim \left(\frac{m}{n}\right)^{d+2\mu} \int_{\mathbb{B}^d} g_n(x, y) d\lambda_n(x)$$

$$\leq \left(\frac{m}{n}\right)^{d+2\mu} \int_{\mathbb{B}^d} g_n(x, y) \left(\sum_{\xi \in \Lambda_n} \chi_{B(\xi, \frac{1}{n})}(x)\right) d\lambda_n(x)$$

$$\leq \left(\frac{m}{n}\right)^{d+2\mu} \max_{\xi \in \Lambda_n} \int_{\mathbb{B}(\xi, \frac{1}{n})} g_n(x, y) d\lambda_n(x)$$

$$\lesssim N \left(\frac{m}{n}\right)^{d+2\mu} \sum_{\xi \in \Lambda_n} \min_{x \in B(\xi, \frac{1}{n})} g_n(x, y) \int_{B(\xi, \frac{1}{n})} w_\mu(x) dx$$

$$\leq N \left(\frac{m}{n}\right)^{d+2\mu} \sum_{\xi \in \Lambda_n} \int_{\mathbb{B}(\xi, \frac{1}{n})} g_n(x, y) \chi_{B(\xi, \frac{1}{n})}(x) w_\mu(x) dx$$

$$= N \left(\frac{m}{n}\right)^{d+2\mu} \int_{\mathbb{B}^d} g_n(x, y) \sum_{\xi \in \Lambda_n} \chi_{B(\xi, \frac{1}{n})}(x) w_\mu(x) dx$$

$$\lesssim N \left(\frac{m}{n}\right)^{d+2\mu} \int_{\mathbb{B}^d} g_n(x, y) w_\mu(x) dx \lesssim N \left(\frac{m}{n}\right)^{d+2\mu},$$
where in the second and the second last inequalities we used (2.2); in the fourth inequality we used the regularity condition and (2.15); and in the last inequality we used (2.6). This completes the proof of Lemma 2.3. □

Now we turn to prove Lemma 2.2.

Proof of Lemma 2.2

The proof is standard (see [21, Corollary 3.3]). For the convenience of the readers we give the proof.

Applying (2.8) and the Hölder inequality, we have for \( f \in \Pi^d_m \) with \( m \geq n \), \( 1 \leq p < \infty \), and \( x \in B^d \),

\[
|f(x)| = \left| \int_{B^d} f(y) K_{m,\eta}(x, y) w_\mu(y) dy \right| \\
\leq \int_{B^d} |f(y)||K_{m,\eta}(x, y)|^\frac{1}{p} |K_{m,\eta}(x, y)|^\frac{1}{p'} w_\mu(y) dy \\
\leq \left( \int_{B^d} |f(y)|^p |K_{m,\eta}(x, y)| w_\mu(y) dy \right)^\frac{1}{p} \left( \int_{B^d} |K_{m,\eta}(x, y)| w_\mu(y) dy \right)^{\frac{1}{p'}} \\
\lesssim \left( \int_{B^d} |f(y)|^p |K_{m,\eta}(x, y)| w_\mu(y) dy \right)^\frac{1}{p},
\]

(2.16)

where in the last inequality we used (2.6). It follows from (2.16) and Lemma 2.3 that

\[
\sum_{\omega \in \Omega_n} \mu_\omega |f(\omega)|^p \lesssim \sum_{\omega \in \Omega_n} \mu_\omega \int_{B^d} |f(y)|^p |K_{m,\eta}(\omega, y)| w_\mu(y) dy \\
= \int_{B^d} |f(y)|^p \left( \sum_{\omega \in \Omega_n} \mu_\omega |K_{m,\eta}(\omega, y)| \right) w_\mu(y) dy \\
\leq \int_{B^d} |f(y)|^p w_\mu(y) dy \max_{y \in B^d} \left( \sum_{\omega \in \Omega_n} \mu_\omega |K_{m,\eta}(\omega, y)| \right) \\
\lesssim N \left( \frac{m}{n} \right)^{d+2\mu} \int_{B^d} |f(x)|^p w_\mu(x) dx.
\]

Lemma 2.2 is proved. □

Finally we give the Nikolskii inequalities on \( B^d \).

Lemma 2.4. ([1] Proposition 2.4]) Let \( 1 \leq p, q \leq \infty \) and \( \mu \geq 0 \). Then for any \( P \in \Pi^d_n \) we have,

\[
\|P\|_{q,\mu} \lesssim n^{(d+2\mu)(1/p-1/q)} \|P\|_{p,\mu}.
\]

(2.17)

3. Proof of Theorem 1.7

In this section, we give the following lemma from which Theorem 1.7 follows immediately.

Lemma 3.1. Let \( 1 \leq p \leq \infty \), \( 1 \leq t, q < \infty \), and \( \mu \geq 0 \). Suppose that \((X, \tau)\) is an \( L_{t,\mu}\)-MZ family with global condition number \( \kappa = B/A \), \( L_{n,t} \) is the weighted least \( \ell_t \)
approximation defined by (1.3). If \( f \in W_{p,\mu}^r \), \( r > (d+2\mu)\max\{1/p, 1/t\} \), then we have
\[
\|f - L_{n,t}(f)\|_{q,\mu} \leq C(1 + \kappa^{1/t})n^{-r+(d+2\mu)(\frac{1}{p} - \frac{1}{q})+\frac{1}{t}}\|f\|_{W_{p,\mu}^r},
\]
where \( C > 0 \) is independent of \( f, n, \kappa, \) and \( (X, \tau) \).

**Proof.** For \( n \in \mathbb{N} \), we choose a nonnegative integer \( s \) such that
\[
2^s \leq n < 2^{s+1},
\]
and for \( f \in W_{p,\mu}^r \), we define
\[
\sigma_1(f) = V_1(f), \quad \sigma_j(f) = V_{2j-1}(f) - V_{2j-2}(f), \quad \text{for } j \geq 2.
\]
Note that
\[
\sigma_j(f) \in \Pi_{2^s}^d, \quad \text{and } V_{2^{s-1}}(f) = \sum_{j=1}^{2^s} \sigma_j(f) \in \Pi_{2^s}^d.
\]
By (2.17) we get
\[
\|\sigma_j(f)\|_{p,\mu} \leq \|f - V_{2^{s-1}} (f)\|_{p,\mu} +\|f - V_{2^{s-2}}(f)\|_{p,\mu} \leq E_{2^{s-1}}(f)_{p,\mu} + E_{2^{s-2}}(f)_{p,\mu} \lesssim 2^{-j}r\|f\|_{W_{p,\mu}^r}. \quad (3.2)
\]
For \( f \in W_{p,\mu}^r, r > (d+2\mu)/p \), we have
\[
\|f - L_{n,t}(f)\|_{q,\mu} \leq \|f - V_{2^{s-1}}(f)\|_{q,\mu} + \|L_{n,t}(f) - V_{2^{s-1}}(f)\|_{q,\mu}. \quad (3.3)
\]
First we estimate \( \|f - V_{2^{s-1}}(f)\|_{q,\mu} \). Note that the series \( \sum_{j=s+1}^{\infty} \sigma_j(f) \) converges to \( f - V_{2^{s-1}}(f) \) in \( L_{q,\mu} \) norm. Thus, by the Nikolskii inequality (2.17) and (3.2), we obtain
\[
\|f - V_{2^{s-1}}(f)\|_{q,\mu} \lesssim \sum_{j=s+1}^{\infty} \|\sigma_j(f)\|_{q,\mu} \lesssim \sum_{j=s+1}^{\infty} (2^j)^{(d+2\mu)(\frac{1}{t} - \frac{1}{q})+\|\sigma_j(f)\|_{p,\mu}} \leq \sum_{j=s+1}^{\infty} (2^j)^{-r+(d+2\mu)(\frac{1}{p} - \frac{1}{q})+\|f\|_{W_{p,\mu}^r}} \lesssim (2^{s+1})^{-r+(d+2\mu)(\frac{1}{p} - \frac{1}{q})+\|f\|_{W_{p,\mu}^r}} \lesssim n^{-r+(d+2\mu)(\frac{1}{p} - \frac{1}{q})+\|f\|_{W_{p,\mu}^r}}. \quad (3.4)
\]
Next we estimate \( \|L_{n,t}(f) - V_{2^{s-1}}(f)\|_{q,\mu} \). We note that \( (L_{n,t}(f) - V_{2^{s-1}}(f)) \in \Pi_{n}^d \). It follows from (2.17), (1.1), and (1.2) that
\[
\|L_{n,t}(f) - V_{2^{s-1}}(f)\|_{q,\mu} \lesssim n^{(d+2\mu)(\frac{1}{t} - \frac{1}{q})+(\frac{1}{p} - \frac{1}{q})+\|L_{n,t}(f) - V_{2^{s-1}}(f)\|_{(t)}} \leq A^{-1/t}n^{(d+2\mu)(\frac{1}{t} - \frac{1}{q})+(\frac{1}{p} - \frac{1}{q})+\|f\|_{(t)}} \leq 2A^{-1/t}n^{(d+2\mu)(\frac{1}{t} - \frac{1}{q})+\|f - V_{2^{s-1}}(f)\|_{(t)}} \lesssim A^{-1/t}n^{(d+2\mu)(\frac{1}{p} - \frac{1}{q})+\|f\|_{(t)}} \lesssim A^{-1/t}n^{(d+2\mu)(\frac{1}{p} - \frac{1}{q})+\sum_{j=s+1}^{\infty} \|\sigma_j(f)\|_{(t)}}. \quad (3.5)
\]
Hence, for \( j \leq 2.2 \), that is, according to (1.1), it is easy to see that (2.10) is true for 
\[ \{ \tau_{n,k} \}_{k=1}^{n} \] with \( p_{0} = t \) and \( N = B \). Note that \( \sigma_{j}(f) \in \Pi_{2j}^{d} \), and \( 2^{j} \geq n \) for \( j \geq s + 1 \). It follows from Lemma 2.2 that for \( j \leq s + 1 \),
\[
\| \sigma_{j}(f) \|_{(t)} = \sum_{k=1}^{n} | \sigma_{j}(x_{n,k}) | \tau_{n,k}
\]
\[
\lesssim B \left( \frac{2j}{n} \right)^{d+2\mu} \| \sigma_{j}(f) \|_{t}^{e}
\]
\[
\lesssim B \left( \frac{2j}{n} \right)^{d+2\mu} (2j)^{(d+2\mu)(\frac{1}{2} - \frac{1}{q})+} \| \sigma_{j}(f) \|_{p,\mu}
\]
\[
\lesssim B \left( \frac{2j}{n} \right)^{d+2\mu} (2j)^{-r+t(d+2\mu)(\frac{1}{2} - \frac{1}{q})+} \| f \|_{W_{r,\mu}^{p}}
\]
\[
= Bn^{-d-2\mu} (2j)^{-r+1} \| f \|_{W_{r,\mu}^{p}},
\]
that is,
\[
\| \sigma_{j}(f) \|_{(t)} \lesssim B^{1/t} n^{-((d+2\mu)/t)} (2j)^{-r+(d+2\mu)(\frac{1}{2} - \frac{1}{q})+} \| f \|_{W_{r,\mu}^{p}}. \tag{3.6}
\]
Hence, for \( r > (d + 2\mu) \) \( \max \{ 1/p, 1/t \} \), by (3.5) and (3.3) we obtain
\[
\| L_{n,t}(f) - V_{2^{j-1}}(f) \|_{q,\mu} \lesssim A^{-1/n} n^{(d+2\mu)(\frac{1}{d} - \frac{1}{q})+} \sum_{j=s+1}^{\infty} \| \sigma_{j}(f) \|_{(t)}
\]
\[
\lesssim \kappa^{1/t} n^{(d+2\mu)(\frac{1}{d} - \frac{1}{q})+} \sum_{j=s+1}^{\infty} (2j)^{-r+(d+2\mu)(\frac{1}{2} - \frac{1}{q})+} \| f \|_{W_{r,\mu}^{p}}
\]
\[
\lesssim \kappa^{1/t} n^{-r+(d+2\mu)(\frac{1}{d} - \frac{1}{q})+} \| f \|_{W_{r,\mu}^{p}}. \tag{3.7}
\]
By (3.3), (3.4), and (3.7), we get (3.1), which completes the proof of Lemma 3.1.

\[ \square \]

Remark 3.2. Let \( 1 \leq p \leq \infty \), \( \mu \geq 0 \), and \( r > (d + 2\mu)/p \). Suppose that \( (\mathcal{X}, \tau) \) is an \( \ell_{p} \)-MZ family with global condition number \( \kappa = 1/A \), \( L_{n,\infty} \) is the weighted least \( \ell_{q} \) approximation defined by (1.3). For \( f \in W_{r,\mu}^{p} \), by (3.4) we have
\[
\| f - V_{2^{j-1}}(f) \|_{(\infty)} \lesssim \| f - V_{2^{j-1}}(f) \|_{(\infty)} \lesssim n^{-r+(d+2\mu)/p} \| f \|_{W_{r,\mu}^{p}}.
\]

It follows that
\[
\| f - L_{n,\infty}(f) \|_{(\infty)} \leq C(1 + \kappa) n^{-r+(d+2\mu)/p} \| f \|_{W_{r,\mu}^{p}},
\]
where \( C > 0 \) is independent of \( f, n, \kappa, \) and \( (\mathcal{X}, \tau) \).

4. Weighted least \( \ell_{q} \) approximation on the sphere

In this section, we discuss the weighted least \( \ell_{q} \) approximation problem on the unit sphere \( S^{d} \) in \( \mathbb{R}^{d+1} \). Let \( L_{p}(S^{d}) \), \( 0 < p < \infty \), denote the space of all Lebesgue measurable functions \( f \) on \( S^{d} \) with the finite quasi-norm
\[
\| f \|_{p} := \left( \int_{S^{d}} | f(x) |^{p} d\sigma(x) \right)^{1/p},
\]
where $d\sigma(x)$ is the rotationally invariant measure on $S^d$ normalized by $\int_{S^d} d\sigma(x) = 1$. When $p = \infty$ we consider the space of continuous functions $C(S^d)$ with the uniform norm. In particular, $L_2(S^d)$ is a Hilbert space with inner product

$$(f, g) := \int_{S^d} f(x)g(x)d\sigma(x), \quad \text{for } f, g \in L_2(S^d).$$

We denote by $H_n(S^d)$ the space of all spherical harmonics of degree $n$, i.e., the space of the restrictions to $S^d$ of all homogeneous harmonic polynomials of exact degree $n$ on $\mathbb{R}^{d+1}$, and by $\Pi_n(S^d)$ the space of all spherical polynomials of degree not exceeding $n$.

It is well known that the spaces $H_n(S^d)$, $n = 0, 1, 2, \ldots$, are mutually orthogonal in $L_2(S^d)$, and

$$\triangle_0 P = -n(n + d - 1)P, \quad \text{for all } P \in H_n(S^d),$$

where $\triangle_0$ is the Laplace-Beltrami operator on the sphere $S^d$. Let

$$\{Y_{nk} \equiv Y_{nk}^d : k = 1, 2, \ldots, b_n^d\}$$

be a fixed orthonormal basis for $H_n(S^d)$, where $b_n^d = \dim H_n(S^d)$. Then

$$\{Y_{nk} : k = 1, 2, \ldots, b_n^d, \quad n = 0, 1, 2, \ldots\}$$

is an orthonormal basis for $L_2(S^d)$.

The orthogonal projector $H_n : L_2(S^d) \to H_n(S^d)$ can be written as

$$H_n f(x) = \sum_{k=1}^{b_n^d} (f, Y_{nk})Y_{nk}(x) = (f, E_n(x, \cdot)),\$$

where $E_n(x, y) = \sum_{k=1}^{b_n^d} Y_{nk}(x)Y_{nk}(y)$ is the reproducing kernel of $H_n(S^d)$. See [5] for more details.

Given $r > 0$, define the fractional power $(-\triangle_0)^{r/2}$ of the operator $-\triangle_0$ on $f$ by

$$(-\triangle_0)^{r/2}(f) := \sum_{k=0}^{\infty} (k(k + d - 1))^{r/2} H_k f,$$

in the sense of distribution. Using this operator we define the Sobolev space on $S^d$ as follows: for $r > 0$ and $1 \leq p \leq \infty$,

$$W_p^r(S^d) := \{ f \in L_p(S^d) : \| f \|_{W_p^r} := \| f \|_p + \| (-\triangle_0)^{r/2}(f) \|_p < \infty\},$$

while the Sobolev class $BW_p^r(S^d)$ is defined to be the unit ball of the Sobolev space $W_p^r(S^d)$. We remark that $W_2^r(S^d)$ is just the Sobolev space $H^r(S^d)$ given in [3], and if $r > d/p$, then $W_p^r(S^d)$ is compactly embedded into $C(S^d)$.

Similar to the case on $\mathbb{B}^d$, we give the definitions of $L_q$-Marcinkiewicz-Zygmund family and the weighted least $\ell_q$ approximation on $S^d$ as follows.

**Definition 4.1.** Suppose that $\mathcal{X} = \{X_n\} = \{x_{n,k} : k = 1, 2, \ldots, l_n, n = 1, 2, \ldots\}$ is a doubly-indexed set of points in $S^d$, and $\tau = \{\tau_n\} = \{\tau_{n,k} : k = 1, 2, \ldots, l_n, n = 1, 2, \ldots\}$ is a doubly-indexed set of positive numbers. Then for $0 < q < \infty$, the
family \((X, \tau)\) is called an \(L_q\)-Marcinkiewicz-Zygmund family on \(S^d\), denoted by \(L_q\)-MZ, if there exist constants \(A, B > 0\) independent of \(n\) such that

\[
A \|P\|_q^q \leq \sum_{k=1}^{l_n} |P(x_{n,k})|^q \tau_{n,k} \leq B \|P\|_q^q, \quad \text{for all } P \in \Pi_n(S^d). \tag{4.1}
\]

The ratio \(\kappa = B/A\) is the global condition number of \(L_q\)-MZ family \((X, \tau)\), and \(X_n = \{x_{n,k} : k = 1, 2, \ldots, l_n\}\) is the \(n\)-th layer of \(X\). Similarly, we can define \(L_\infty\)-MZ family.

**Remark 4.2.** Similar to the case on \(\mathbb{B}^d\), we set

\[
\mu_n := \sum_{k=1}^{l_n} \tau_{n,k} d_{x_{n,k}}.
\]

For any \(f \in C(S^d)\), we define for \(0 < q < \infty\)

\[
\|f\|_q := \left( \int_{S^d} |f(x)|^q d\mu_n(x) \right)^{1/q} = \left( \sum_{k=1}^{l_n} |f(x_{n,k})|^q \tau_{n,k} \right)^{1/q}.
\]

and for \(q = \infty\),

\[
\|f\|_\infty := \max_{1 \leq k \leq l_n} |f(x_{n,k})|.
\]

It follows from (4.1) that the \(L_q\)-norm of a polynomial of degree at most \(n\) on \(S^d\) is comparable to the discrete version given by the weighted \(\ell_q\)-norm of its restriction to \(X_n\). It follows from [3, 5, 15, 17] that such MZ families exist if the families are dense enough.

**Definition 4.3.** Let \(0 < q \leq \infty\), and let \((X, \tau)\) be an \(L_q\)-MZ family on \(S^d\). For \(f \in C(S^d)\), we define the weighted least \(\ell_q\) approximation on \(S^d\) by

\[
L_{n,q}^S(f) := \arg \min_{P \in \Pi_n(S^d)} \left( \sum_{k=1}^{l_n} |f(x_{n,k}) - P(x_{n,k})|^q \tau_{n,k} \right)^{1/q}. \tag{4.2}
\]

That is, \(L_{n,q}^S(f)\) is any function in \(\Pi_n(S^d)\) satisfying

\[
\|f - L_{n,q}^S(f)\|_q = \min_{P \in \Pi_n(S^d)} \|f - P\|_q.
\]

**Remark 4.4.** Similar to the case on \(\mathbb{B}^d\), for \(f \in C(S^d)\) and \(0 < q \leq \infty\), the minimizer \(L_{n,q}^S(f)\) exists. Hence, this definition is well defined. If \(1 < q < \infty\), \(L_{n,q}^S(f)\) is unique. If \(q = 2\), then \(L_{n,2}^S(f)\) is linear. However, if \(0 < q \leq 1\) or \(q = \infty\), then \(L_{n,q}^S(f)\) may be not unique, and if \(q \neq 2\), then the operator \(L_{n,q}^S\) is not linear.

For \(L_\infty\)-MZ family on \(S^d\) with \(q = 2\), \(L_{n,2}^S\) is a bounded linear operator on \(C(S^d)\) satisfying that \((L_{n,2}^S)^2 = L_{n,2}^S\), and the range of \(L_{n,2}^S\) is \(\Pi_n(S^d)\). If we define the discretized inner product on \(C(S^d)\) by

\[
\langle f, g \rangle_{[2]} := \sum_{k=1}^{l_n} \tau_{n,k} f(x_{n,k}) g(x_{n,k}),
\]
then $L_{n,2}^S$ is just the orthogonal projection onto $\Pi_n(\mathbb{S}^d)$ with respect to the discretized inner product $\langle \cdot, \cdot \rangle_2$. Hence, we get for $f \in C(\mathbb{S}^d)$,

$$L_{n,2}^S(f)(x) = \langle f, D_{n}^S(x, \cdot) \rangle_2 = \sum_{k=1}^{l_n} \tau_{n,k} f(x_{n,k}) D_{n}^S(x, x_{n,k}),$$

where $D_{n}^S(x, y)$ is the reproducing kernel of $\Pi_n(\mathbb{S}^d)$ with respect to the discretized inner product $\langle \cdot, \cdot \rangle_2$. We call $L_{n,2}^S(f)$ the weighted least squares polynomial on $\mathbb{S}^d$, and $L_{n,2}^S$ the weighted least squares operator on $\mathbb{S}^d$.

Following Gröchenig in [3], for $L_2$-MZ family on $\mathbb{S}^d$ we can also use the frame theory to construct the quadrature formula

$$I_{n}^S(f) = \sum_{k=1}^{l_n} W_{n,k} f(x_{n,k}).$$

It was shown in [13] that

$$W_{n,k} = \tau_{n,k} \int_{\mathbb{S}^d} D_{n}^S(x, x_{n,k})d\sigma(x),$$

and

$$I_{n}^S(f) = \int_{\mathbb{S}^d} I_{n,2}^S(f)(x)d\sigma(x). \quad (4.3)$$

Such quadrature $I_{n}^S$ is called the least squares quadrature on $\mathbb{S}^d$.

Analogous to the case on $\mathbb{B}^d$, we obtain the following two theorems. The proofs are similar to the ones of Theorems 1.7 and 1.8.

**Theorem 4.5.** Let $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Suppose that $(\mathcal{X}, \tau)$ is an $L_q$-MZ family on $\mathbb{S}^d$ with global condition number $\kappa$, $L_{n,q}^S$ is the weighted least $\ell_q$ approximation defined by (4.2). For $f \in W_r^p(\mathbb{S}^d)$, $r > d \max\{1/p, 1/q\}$, we have for $1 \leq q < \infty$

$$\|f - L_{n,q}^S(f)\|_q \leq C(1 + \kappa^{1/q}) n^{-r + \frac{d}{p} - \frac{1}{q}} \|f\|_{W_r^p}, \quad (4.4)$$

and for $q = \infty$,

$$\|f - L_{n,q}^S(f)\|_\infty \leq C(1 + \kappa) n^{-r + \frac{d}{p}} \|f\|_{W_r^p},$$

where $C > 0$ depends on $r, d, p, q$, but not on $f, n, \kappa$ or $(\mathcal{X}, \tau)$.

**Theorem 4.6.** Suppose that $(\mathcal{X}, \tau)$ is an $L_2$-MZ family on $\mathbb{S}^d$ with global condition number $\kappa$, $L_{n,2}^S$ and $I_{n}^S$ are the weighted least squares approximation and the least squares quadrature, respectively. If $f \in H^r(\mathbb{S}^d) \equiv W_2^r(\mathbb{S}^d)$, $r > d/2$, then we have

$$\|f - L_{n,2}^S(f)\|_2 \leq C(1 + \kappa^{1/2}) n^{-r} \|f\|_{H^r(\mathbb{S}^d)}, \quad (4.5)$$

and

$$\left| \int_{\mathbb{S}^d} f(x)d\sigma(x) - I_{n}^S(f) \right| \leq C(1 + \kappa^{1/2}) n^{-r} \|f\|_{H^r(\mathbb{S}^d)}, \quad (4.6)$$

where $C > 0$ depends on $r, d$, but not on $f, n, \kappa$ or $(\mathcal{X}, \tau)$.

Theorem 4.5 is new, and Theorem 4.6 is a slight improvement of [13] Theorem 1.2. Indeed, we only reduce dependence on the global condition number in (4.5) and (4.6) by replacing the constant $(1 + \kappa^2)^{1/2}$ in (1.8) and (1.9) with the constant $1 + \kappa^{1/2}$.
Remark 4.7. It follows from \cite{14,15,16} that there exist $L_q$-MZ families on $\mathbb{S}^d$ with $l_q N \asymp n^d$. For such $L_q$-MZ family, combining (4.4) with \cite[Theorem 1.2]{22}, we obtain for $1 \leq p, q \leq \infty$, $r > d \max \{1/p, 1/q\}$,
\[
\sup_{f \in BW^r_p(\mathbb{S}^d)} \|f - L^S_{n,q}(f)\|_q \asymp N^{-\frac{1}{d}} + (\frac{1}{p} - \frac{1}{q})^+ \asymp \gamma N(BW^r_p(\mathbb{S}^d), L_q(\mathbb{S}^d)),
\]
which implies that the weighted least $\ell_q$ approximation operators $L^S_{n,q}$ are asymptotically optimal algorithms in the sense of optimal recovery for $1 \leq p, q \leq \infty$.

For the least squares quadrature rules $I^S_n$, it follows from (4.6) and \cite{2,9,22} that for $r > d/2$,
\[
\sup_{f \in BH^r(\mathbb{S}^d)} \left| \int_{\mathbb{S}^d} f(x) d\sigma(x) - I^S_n(f) \right| \asymp N^{-\frac{1}{d}} \asymp \epsilon N(BH^r(\mathbb{S}^d); \text{INT}),
\]
which means that the least squares quadrature rules $I^S_n$ are the asymptotically optimal quadrature formulas for $BH^r(\mathbb{S}^d)$.

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