RIEMANN-ROCH THEOREMS FOR HIGHER ALGEBRAIC $K$-THEORY
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In [1] and [2] Baum, Fulton and MacPherson, generalizing the celebrated Grothendieck-Riemann-Roch theorem, proved that given a category $V$ of quasi-projective schemes there is a natural transformation called the Todd class of functors (covariant for proper morphisms) between $K'_0$, the homology algebraic $K$-theory of coherent sheaves and any of the standard homology theories. Here we announce generalizations of the results of [1] and [2] to Quillen's higher algebraic $K$-theory [8] which may help to illuminate the relationship between algebraic $K$-theory and more ordinary cohomology theories.

The statements of our theorems depend on defining global analogues of Quillen’s construction of Chern classes for the $K$-theory of a ring [3], [9]. We can use any of the standard cohomology theories defined on $V$, such as étale or crystalline cohomology or even the Chow ring. All of these theories can be realized for each $X \in V$ as the hypercohomology of a graded complex or pro-complex $\Gamma^j_\bullet, j \in \mathbb{Z}$, of sheaves on the Zariski site of $X$. All of these theories have Chern classes for representations of sheaves of groups and there exist universal classes

$$C_t \in \mathbb{H}^d(X, GL(\mathcal{O}_X), \Gamma^*_t) \quad (d = 1 \text{ or } 2).$$

Using Brown's generalized cohomology “with supports” of simplicial sheaves [6], and the functor $Z_\infty$ of [5] instead of the “+” construction one can mimic in the category of simplicial sheaves the methods of [3] and [9] to obtain Chern classes for all $p > 0$

$$C^Y_{i,p} ; K^Y_p(X) = K_p(X, X - Y) \to \mathbb{H}^{d_l-p}(X, \Gamma^*_t)$$

whose domains are the relative $K$-groups, defined so as to force a Quillen-style localization sequence. One can show that these classes coincide for $p = 0$ with those of Iversen [7]. For $p > 0$ they are group homomorphisms and are compatible with products in the way described by Bloch [3], hence one can define a Chern character with supports, which is a ring homomorphism

$$ch^Y : \bigoplus_{p \geq 0} K_p(X, X - Y) \to \bigoplus_{i,p \geq 0} \mathbb{H}^{d_l-p}(X, \Gamma^*_t) \otimes \mathbb{Q}.$$
THEOREM 1. For $Y \in \mathcal{V}$ define the $\Gamma$-homology groups $H_i(Y, \Gamma_i)$ to be $H_{Y^{d_n-1}}^i(X, \Gamma_{n-1}^*)$ where $Y \subset X$ and $X$ is smooth of dimension $n$. Then there is a natural transformation of covariant functors on the category of proper morphisms in $\mathcal{V}$, with domain the $K'$-theory of coherent sheaves [8],

$$\tau_* = \bigoplus_{p \geq 0} \tau_p: \bigoplus_{p \geq 0} K'_p(Y) \rightarrow \bigoplus_{p, i \geq 0} H_{d_{i+p}}(Y, \Gamma_i) \otimes \mathbb{Q}.$$

$\tau_*$ satisfies the following conditions:

(i) For all $X \in \mathcal{V}$ and $\alpha \in K_p(X)$, $\beta \in K'_q(Y)$,

$$\tau_{p+q}(\alpha \cap \beta) = ch_p(\alpha) \cap \tau_q(\beta).$$

(ii) If $X, Y \in \mathcal{V}$ and $\alpha \in K'_p(X)$, $\beta \in K'_q(Y)$ then $\tau(\alpha \boxtimes \beta) = \tau(\alpha) \boxtimes \tau(\beta)$

($\boxtimes$ = external product in $K'$-theory or $\Gamma$-homology).

(iii) If $f: U \rightarrow X$ is an open immersion in $\mathcal{V}$ and $\alpha \in K'_p(X)$ then

$$\tau_p(f^*\alpha) = f^*\tau_p(\alpha).$$

(iv) If $X \in ob(\mathcal{V})$ is smooth then the structure sheaf $\mathcal{O}_X$ defines an element $[\mathcal{O}_X]$ of $K'_0(X)$ and $\tau_0([\mathcal{O}_X] = Td(X) \cap \eta_X$, where $Td(X)$ is the classical Todd class and $\eta_X \in H_{dn}(X, \Gamma_n)$ is the homology cycle class of $X$ ($n = \text{dimension of } X$).

The construction of $\tau_*$ and the proof of Theorem 1 are adapted from the methods of [1]. However the extension to higher $K$-theory does involve more detailed "functorial" constructions than are necessary for $K_0$. Similar methods to those of Theorem 1 and the paper [2] can be used to prove

THEOREM 2. On the category of quasi-projective algebraic varieties over $\mathbb{C}$ there is a natural transformation

$$\tau_* = \bigoplus_{p \geq 0} \tau_p: \bigoplus_{p \geq 0} K'_p \rightarrow \bigoplus_{p \geq 0} KU^L_p$$

covariant for proper morphisms, where $KU^L_p$ is the 'L = C' or 'locally compact' homology topological $K$-theory associated to the spectrum $BU$. There is also a natural transformation

$$\tau^*: K^* \rightarrow KU^*.$$

These maps $\tau_*, \tau^*$ satisfy the "module", "product" and presheaf properties analogous to (i), (ii) and (iii) of Theorem 1.

If $X$ is a Noetherian scheme, there is a filtration

$$N^kX = \{X^{(0)} = X \supset X^{(1)} \supset \cdots \supset X^{(n)} \supset \cdots\}$$

of its underlying topological space, called the coniveau filtration, defined by

$$X^{(k)} = \{x \in X | \{\tilde{x}\} \text{ has codimension } \geq k \text{ in } X\}.$$
For any simplicial sheaf $F$ on $X$, there is a natural exact couple associated to $N^*X$ and hence a spectral sequence [8], [3]:

$$E^{p,q}_1(F) = \bigoplus_{x \in X(p) - X(p+1)} H^{p+q}_x(X, F.) \Rightarrow H^{p+q}(X, F.).$$

Given a cohomology theory $\bigoplus_{i \geq 0} \Gamma_i^*$, for each $X \in \mathcal{V}$ one obtains maps of coniveau spectral sequences

$$E^{p,q}_r(C_j): E^{p,q}_r(Z_{\infty} B GL(\mathcal{O}_X)) \rightarrow E^{p,q}_r(K(di, \Gamma_i^*))$$

which on $E_1$ terms may be written

$$\bigoplus C_{i-p-q}^X:
\bigoplus_{x \in X(p) - X(p+1)} K_{i-p-q}^X(x) \rightarrow \bigoplus_{x \in X(p) - X(p+1)} H^{d(i+p+q)}_x(X, \Gamma_i^*)$$

$$\downarrow \int \bigoplus_{x \in X(p) - X(p+1)} K_{i-p-q}(k(x)) \rightarrow \bigoplus_{x \in X(p) - X(p+1)} H^{d((i-p)+p+q)}_x(x, \Gamma_i^*)$$

where the vertical isomorphisms come from duality and Quillen's localization theorem [8].

**Theorem 3.** The map $\gamma_i^X$ in the diagram above is equal to $(-1)^i(i-1)!/(i-p-1)! C_{i-p-q}$. This is a consequence of the following Riemann-Roch theorem without denominators and with supports.

**Theorem 4.** Let $j: Y \rightarrow X$ be a closed immersion of smooth schemes in $\mathcal{V}$. Then we have isomorphisms

$$j_*: K_q(Y) \cong K_q(X, X - Y),$$

$$j^*: H^{d(i-p)-q}(Y, \Gamma_{i-p}^*) \rightarrow H^{d(i-q)}(X, \Gamma_i^*)$$

where $p = \text{codim}_X(Y)$. For each $k \geq 0$ there exist polynomials $P_k(T_0, \ldots, T_{k-p}; U_1, \ldots, U_{k-p})$ with integer coefficients such that if $\alpha \in K_q(Y)$ and $q \leq 2k$ then

$$C_k^Y(j_* \alpha) = j!(P_k(rk, C_1, \ldots, C_{k-p}; C_1(N_{X/Y}), \ldots, C_{k-p}(N_{X/Y}))(\alpha)).$$

Here $rk, C_1, \ldots, C_{k-p}$ are the universal Chern classes in $H^*(Y, GL(\mathcal{O}_Y), \Gamma_i^*)$ and the $C_i(N_{X/Y})$ are the Chern classes of the normal bundle of $Y$. The polynomial $P_k$ in these classes defines a class in $H^{dk}(Y, GL(\mathcal{O}_Y), \Gamma_k^*)$ which we apply to $\alpha$.

Theorem 4 is proved by using the "deformation to the normal bundle" construction of [1] to reduce to the case of the zero section $Y \subset N_{X/Y}$ followed by explicit computations. The same methods may be used to prove Theorems 3
and 4 in the case where $X$ is any connected regular one-dimensional scheme and $\Gamma^*$ is étale cohomology, in which case $j_!$ is defined by Grothendieck’s absolute purity theorem (SGA5, I §5.1). In the case $X = \text{Spec}(\mathcal{O}_S)$, $\mathcal{O}_S$ the ring of $S$-integers in a global field, this extends more ad-hoc results of Soulé in [9].

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