Unified Optimal Analysis of the (Stochastic) Gradient Method

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Abstract

In this note we give a simple proof for the convergence of stochastic gradient (SGD) methods on \( \mu \)-convex functions under a (milder than standard) \( L \)-smoothness assumption. We show that for carefully chosen stepsizes SGD converges after \( T \) iterations as
\[
O \left( L \| x_0 - x^* \|^2 \exp \left[ - \frac{\mu}{4L} T \right] + \frac{\sigma^2}{T} \right)
\]
where \( \sigma^2 \) measures the variance in the stochastic noise. For deterministic gradient descent (GD) and SGD in the interpolation setting we have \( \sigma^2 = 0 \) and we recover the exponential convergence rate. The bound matches with the best known iteration complexity of GD and SGD, up to constants.

1 Introduction

We consider the unconstrained optimization problem
\[
f^* := \min_{x \in \mathbb{R}^n} f(x),
\]
where \( f : \mathbb{R}^n \to \mathbb{R} \) is a convex continuously differentiable function. We consider a stochastic approximation scenario—comprising the classic deterministic setting—where only unbiased estimates of the gradient of \( f \) are available and study the convergence rate of stochastic gradient descent (SGD).

Formally, we assume that we have an increasing sequence of \( \sigma \)-fields \( \{F_t\}_{t \geq 0} \), such that \( x_0 \in \mathbb{R}^n \) is \( F_0 \) measurable and such that for all \( t \geq 0 \) the iterates of SGD are given as:
\[
x_{t+1} = x_t - \gamma_t g_t,
\]
where \( \{\gamma_t\}_{t \geq 0} \) denotes a sequence of (positive) stepsizes and \( g_t \in \mathbb{R}^n \) is a stochastic gradient of \( f \), satisfying the following three assumptions.

Assumption 1 (Unbiased gradient oracles). Almost surely,
\[
E [ g_t | F_t ] = \nabla f(x_t), \quad \forall t \geq 0,
\]
where here \( \nabla f(x_t) \) denotes the gradient of \( f \) at \( x_t \).

Assumption 2 ((\( L, \sigma \))-smoothness). There exists two constants \( L, \sigma^2 \geq 0 \), s.t.
\[
E \left[ \| g_t \|^2 | F_t \right] \leq 2L(f(x_t) - f^*) + \sigma^2, \quad \forall t \geq 0.
\]

This assumption generalizes the standard smoothness assumption as we will explain in Section 2 below. We further assume that \( f \) is \( \mu \)-convex (with respect to the optimum \( x^* \)—a slight relaxation of the standard assumption, see e.g. (Necoara et al., 2019)) and denote by \( x^* \) a minimizer of \( f \) in \( \mathbb{R}^n \).

Assumption 3 (\( \mu \)-convexity). There exists \( x^* \in \arg \min_{x \in \mathbb{R}^n} f(x) \) and a constant \( \mu \geq 0 \) with
\[
\frac{\mu}{2} \| x - x^* \|^2 + f(x) - f^* \leq \langle \nabla f(x), x - x^* \rangle, \quad \forall x \in \mathbb{R}^n.
\]

Note that this assumption is weaker than the standard strong convexity (for \( \mu > 0 \)) or convexity (for \( \mu = 0 \)) assumptions that require such an inequality to hold for arbitrary pairs \( x, y \in \mathbb{R}^d \) and not only for \( y = x^* \).

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1.1 Contribution

Let \( \{x_t\}_{t \geq 0} \) denote the iterates of (1). We show that for appropriate step sizes \( \gamma_t \) and an appropriately defined average iterate after \( T \) iterations, \( \bar{x}_T := \frac{1}{T} \sum_{t=1}^{T} w_t x_t \) for weights \( w_t \geq 0 \) and \( W_T := \sum_{t=0}^{T} w_t \), it holds for \( R = \|x_0 - x^*\|:\)

\[
E f(\bar{x}_T) - f^* + \mu E \|x_{T+1} - x^*\|^2 = O \left( \min \left\{ LR^2 \exp \left( - \frac{\mu T}{4L} \right) + \frac{\sigma^4}{\mu^2 L^2}, \sigma R \sqrt{T} \right\} \right).
\]

We further also give a simpler proof that shows, up to polylogarithmic factors\(^1\),

\[
E f(\bar{x}_T) - f^* + \mu E \|x_{T+1} - x^*\|^2 = \tilde{O} \left( LR^2 \exp \left( - \frac{\mu T}{2L} \right) + \sigma^2 \right),
\]

and that only relies on constant step sizes in (1).

This analysis unifies the analyses of gradient descent and SGD for smooth functions. In the deterministic case and in the interpolation setting (where \( \sigma^2 = 0 \)), we recover the exponential convergence rates of these algorithms (up to a factor 4 in the exponent) when \( \mu > 0 \). Furthermore, the result for convergence in function values is tight up to absolute (non-problem specific) constants (Nesterov, 2004). Similarly, in the stochastic setting we recover the best known rates not only for the function values but also for the squared distance of the last iterate to the optimum (Nemirovski et al., 2009; Shamir and Zhang, 2013).

1.2 Related Work

Whilst the first analyses of stochastic gradient descent (SGD) (Robbins and Monro, 1951) focused on asymptotic results (Chung, 1954), the focus shifted to non-asymptotic results in recent years.

For \( \mu > 0 \), Bach and Moulines (2011) give a bound \( E \|x_T - x^*\|^2 = \tilde{O}(\ell^2 R^2 \exp[-\frac{\mu T}{4}] + \sigma^2 \ell^2 \mu) \), this was later improved by Needell et al. (2016) to \( E \|x_T - x^*\|^2 = \tilde{O}(\ell^2 R^2 \exp[-\frac{\mu T}{4}] + \sigma^2 \ell^2 \mu) \). Up to polylogarithmic factors this is the same rate as we show here in a slightly more general setting—however, their result only covers the distance \( \|x_T - x^*\|^2 \) of the iterates. Deducing from this result a rate for the function values via the smoothness inequality \( f(x_T) - f^* \leq \frac{\ell^2}{\mu} \|x_T - x^*\|^2 \) introduces a superficial condition number factor \( \frac{\ell}{\mu} \).

In the quest of deriving optimal rates—up to constant factors—in function suboptimality, different averaging schemes have been studied (Ruppert, 1988; Polyak, 1990; Rakhlin et al., 2012; Shamir and Zhang, 2013). Lacoste-Julien et al. (2012) give a simple proof for \( f(x_T) - f^* = O(\frac{G^2}{\mu T}) \), where here \( G^2 \geq \sigma^2 \) is an upper bound on the gradient norms, \( E \left[ \|g_t\|^2 \right] \leq G^2 \). Analyses under this assumption are not optimal in the deterministic setting where \( \sigma^2 = 0 \), but \( G^2 > 0 \) in general.

The \((L, \sigma^2)\)-smoothness assumption appeared in this form recently in e.g. (Grimmer, 2019), though very similar conditions have been studied in the literature (Bertsekas and Tsitsiklis, 1996; Schmidt and Roux, 2013; Needell et al., 2016; Bottou et al., 2018; Gower et al., 2019). We will discuss a few of these in Section 2 below. In contrast to the bounded gradient assumption, these assumption allow to choose \( \sigma^2 = 0 \) in non-trivial situations and thus allow to recover faster rates in general. However, adapting the proof technique from (Lacoste-Julien et al., 2012) to the relaxed assumptions considered here (cf. Lemma 7 below, or (Stich et al., 2018; Grimmer, 2019)) gives \( f(x_T) - f^* = O \left( \frac{L^2 R^2}{\mu T^2} + \frac{\sigma^4}{\mu^2 L^2} \right) \), where the dependence on the initial distance \( R \) is not optimal, i.e. not exponentially decreasing as in (Bach and Moulines, 2011). Gower et al. (2019) generalize the results of (Needell et al., 2016) for the convergence of the distance \( \|x_T - x^*\|^2 \) to the setting considered here and obtain the same rate as stated earlier in this subsection.

To keep our focus, we do not discuss obvious generalizations of our bounds to other settings here. For instance convergence under average smoothness or importance sampling (Bach and Moulines, 2011; Needell et al., 2016) or expected smoothness conditions (Gower et al., 2018).
2 Motivating Examples

In this section we give a few examples that motivate Assumption 2.

Example 1 (Gradient Descent). In the non-stochastic setting, we have $g_t = \nabla f(x_t), \forall t \geq 0$. If $f$ is $L$-smooth, then $f$ is also $(L, 0)$-smooth, as seen by the choice $\mathbf{y} = x^*$ in the following inequality that holds for convex $L$-smooth functions [Nesterov, 2004, Theorem 2.1.5]:

$$\frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|^2 \leq f(x) - f(y) - \langle \nabla f(y), x - y \rangle, \quad \forall x, y \in \mathbb{R}^n. \tag{5}$$

Hence, we recover the $O(L \|x_0 - x^*\|^2 \exp\left(-\frac{\sigma^2 t}{L}\right))$ convergence rate which coincides with the best known rate [Nesterov, 2004, Theorem 2.1.15] for the function value convergence in this setting.

Example 2 (Stochastic Gradient Descent). In the stochastic setting, we have $g_t = \nabla f(x_t) + \xi_t$, where $\{\xi_t\}_{t \geq 0}$ are independent, zero-mean noise terms, with uniformly bounded second moment $E[\xi_t^2] \leq \sigma^2$ for a constant $\sigma^2 \geq 0$. Again, by relying on (5), we see that Assumption 2 is satisfied for $L$-smooth functions:

$$E \left[\|g_t\|^2 \mid F_t\right] = \|\nabla f(x_t)\|^2 + E \left[\|\xi_t\|^2 \mid F_t\right] \leq (2L(f(x_t) - f^*) + \sigma^2).$$

Hence, we recover the $O\left(\frac{\sigma^2}{\sigma^2 T}\right)$ convergence rate of SGD for the function values and the $O\left(\frac{\sigma^2}{\sigma^2 T}\right)$ rate for the last iterate—which are the best known rates [Rakhlin et al., 2012, Shamir and Zhang, 2013].

Example 3 (Empirical Risk Minimization). In machine learning applications the objective function often has a known sum structure, $f(x) := \frac{1}{m} \sum_{i=1}^m f_i(x)$ for $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ convex and $L$-smooth. By picking one index $i \sim \text{a.a.r.} \ [m]$, uniformly at random, $g_t := \nabla f_i(x_t)$ is an unbiased, $(2L, \frac{\sigma^2}{m} \sum_{i=1}^m \|\nabla f_i(x^*)\|^2)$-smooth gradient oracle, as can be seen from (cf. Needell et al., 2016):

$$E \left[\|g_t\|^2 \mid F_t\right] = E_t \|\nabla f_i(x_t) - \nabla f_i(x^*) + \nabla f_i(x^*)\|^2 \leq 2E_t \|\nabla f_i(x_t) - \nabla f_i(x^*)\|^2 + 2E_t \|\nabla f_i(x^*)\|^2 \leq 4L(f(x_t) - f^*) + 2E_t \|\nabla f_i(x^*)\|^2. \tag{6}$$

When the loss at the optimum vanishes, i.e. $\nabla f_i(x^*) = 0, \forall i \in [m]$—the so called interpolation setting—we have $\sigma^2 = 0$ and we recover linear convergence of SGD, as e.g. in [Schmidt and Roux, 2013, Needell et al., 2016, Ma et al., 2018].

The above observation also holds in more general settings, such as e.g. under expected smoothness or weak growth conditions (cf. Gower et al., 2018, 2019).

The constant $L$ is tight here [Nesterov, 2004]. However—as a side remark—we like to point out that our proof reveals the improved bound $E(f(S_T) - f^* + \mu E[\|x_{T+1} - x^*\|^2] = O\left(\frac{\mu \|x_0 - x^*\|^2 \exp\left(-\frac{\sigma^2 T}{L}\right)}{T^2} \right)$ if $T = \Omega\left(\frac{\sigma^2}{\mu L}\right)$ is sufficiently large (an assumption that appears sometimes in the literature—though does not improve the worst-case complexity for arbitrary $T$).
3 Convergence Analysis Part I—Deriving a Recursion

Following standard techniques, we prove the following lemma:

Lemma 1. For $x_0 \in \mathbb{R}^d$, let $\{x_t\}_{t \geq 0}$ denote the iterates of SGD (1) generated on a function $f$ under Assumptions 3 for stepsizes $\gamma_t \leq \frac{1}{2L}$, $\forall t \geq 0$.

$$E\|x_{t+1} - x^*\|^2 \leq (1 - \mu \gamma_t)E\|x_t - x^*\|^2 - \gamma_t (E f(x_t) - f^*) + \gamma_t^2 \sigma^2.$$  \hspace{1cm} (6)

Proof. By definition,

$$E\left[\|x_{t+1} - x^*\|^2 | F_t\right] = E\left[\|x_t - x^*\|^2 - 2\gamma_t \langle g_t, x_t - x^* \rangle + \gamma_t^2 \|g_t\|^2 | F_t\right]$$

$$\overset{3}{=} \|x_t - x^*\|^2 - 2\gamma_t (\nabla f(x_t), x_t - x^*) + \gamma_t^2 E\left[\|g_t\|^2 | F_t\right]$$

$$\overset{3}{\leq} \|x_t - x^*\|^2 - 2\gamma_t \left(\frac{\mu}{2} \|x_t - x^*\|^2 + f(x_t) - f^*\right) + \gamma_t^2 (2L(f(x_t) - f^*) + \sigma^2),$$

where we also used $\mu$-convexity in the last inequality. By re-arranging and taking expectation on both sides, we get:

$$E\|x_{t+1} - x^*\|^2 \leq (1 - \mu \gamma_t)E\|x_t - x^*\|^2 - 2\gamma_t (1 - L \gamma_t)(Ef(x_t) - f^*) + \gamma_t^2 \sigma^2,$$

and the claim follows by observing $(1 - L \gamma_t) \geq \frac{1}{2}$ for $\gamma_t \leq \frac{1}{2L}$. \hfill \Box

A classic convergence result (cf. Section 1.2). Lemma 3 is a key tool that allows to derive convergence results for SGD. To exemplify, we show there a first result for $\mu$-convex functions with $\mu > 0$. By choosing constant stepsizes $\gamma_t \equiv \gamma \leq \frac{1}{2L}$ (to be specified below) and relaxing (6) to $E\|x_{t+1} - x^*\|^2 \leq (1 - \mu \gamma)E\|x_t - x^*\|^2 + \gamma^2 \sigma^2$ we obtain after unrolling the recurrence,

$$E\|x_{T+1} - x^*\|^2 \leq (1 - \mu \gamma)^T \|x_0 - x^*\|^2 + \gamma^2 \frac{\sigma^2}{\mu}.$$  \hspace{1cm} (7)

This intermediate results shows that SGD with constant stepsizes reduces the initial error term $\|x_0 - x^*\|^2$ linearly, but only converges towards a $\frac{\sigma^2}{\mu}$-neighborhood of $x^*$ (cf. discussions in e.g. Bach and Moulines, 2011; Bottou et al., 2018). To obtain a convergence guarantee that holds for arbitrary accuracy, we need to choose the stepsize $\gamma$ carefully:

- If $\frac{1}{2L} \geq \frac{\ln(\max(2, \mu \|x_0 - x^*\|^2 / T \sigma^2))}{\mu T}$ then we choose $\gamma = \frac{\ln(\max(2, \mu \|x_0 - x^*\|^2 / T \sigma^2))}{\mu T}$.

- Otherwise $\frac{1}{2L} < \frac{\ln(\max(2, \mu \|x_0 - x^*\|^2 / T \sigma^2))}{\mu T}$ then we pick $\gamma = \frac{1}{2L}$.

With these choices of $\gamma$, we can show\footnote{We refer the readers to the proof of Lemma 2 below for detailed computations in a very similar setting.}

$$E\|x_{T+1} - x^*\|^2 = \hat{O}\left(\|x_0 - x^*\|^2 \exp \left[-\frac{\mu T}{2L} + \frac{\sigma^2}{\mu^2 T}\right]\right).$$

This result does not show convergence of the function values $f(x_t) - f^*$ and the $\hat{O}(\cdot)$ notation hides logarithmic factors. In the next section, we show how we can address both these issues.
4 Convergence Analysis Part II—Solving the Recursion

In this section, we consider two non-negative sequences \(\{r_t\}_{t \geq 0}, \{s_t\}_{t \geq 0}\), that satisfy the relation

\[ r_{t+1} \leq (1 - a\gamma_t)r_t - b\gamma_ts_t + c\gamma_t^2, \quad (8) \]

for all \(t \geq 0\) and for parameters \(b > 0, a, c \geq 0\) and non-negative stepsizes \(\{\gamma_t\}_{t \geq 0}\) with \(\gamma_t \leq \frac{1}{d}, \forall t \geq 0\), for a parameter \(d \geq a, d > 0\).

By considering the special case \(r_t = \|x_t - x^*\|^2, s_t = (\mathbb{E}f(x_t) - f^*), a = \mu, b = 1, c = \sigma^2\) and \(d = 2L\), we see that \((8)\) comprises the setting of Lemma 2, and thus the three lemmas that follow below will prove the claims from Section 1.1.

**Constant Stepsizes (with Log Terms).** First, we derive a suboptimal (up to polylogarithmic factors) solution of \((8)\).

**Lemma 2.** Let \(\{r_t\}_{t \geq 0}, \{s_t\}_{t \geq 0}\) as in \((8)\) and \(a > 0\). Then there exists a constant stepsize \(\gamma_t \equiv \gamma \leq \frac{1}{d}\) such that for weights \(w_t := (1 - a\gamma)^{-(t+1)}\) and \(W_T := \sum_{t=0}^{T} w_t\) it holds:

\[
\frac{b}{W_T} \sum_{t=0}^{T} s_tw_t + aT_{t+1} = \tilde{O}(dr_0\exp\left(-\frac{aT}{d}\right) + \frac{c}{aT}).
\]

**Decreasing Stepsizes (Avoiding Log Terms).** In Lemma 2 above we collected suboptimal logarithmic terms. The averaging scheme with exponentially decreasing weights has a too short effective window to reduce the variance at the optimal \(\mathcal{O}\left(\frac{1}{T}\right)\) rate. In contrast, averaging schemes with polynomial weights can in general achieve the optimal \(\mathcal{O}\left(\frac{1}{T}\right)\) decrease of the statistical term, but do not decrease the optimization term exponentially fast (see e.g. (Lacoste-Julien et al., 2012), (Shamir and Zhang, 2013)). This suggests that a combination of these averaging strategies might yield the best results. We analyze a simple two-phase scheme, that first performs \(\frac{T}{a}\) iterations without averaging and then switches to suffix averaging scheme for the remaining iterations (this analysis could be generalized to \(a\)-suffix averaging as in (Rakhlin et al., 2012)).

**Lemma 3.** Let \(\{r_t\}_{t \geq 0}, \{s_t\}_{t \geq 0}\) as in \((8)\), \(a > 0\). Then there exists stepsizes \(\gamma_t \leq \frac{1}{d}\) and weights \(w_t \geq 0, W_T := \sum_{t=0}^{T} w_t\), such that:

\[
\frac{b}{W_T} \sum_{t=0}^{T} s_tw_t + aT_{t+1} \leq 32dr_0\exp\left(-\frac{aT}{2d}\right) + \frac{36c}{aT}.
\]

**Sublinear Rate (a = 0).** The previous lemma allows to derive the main result presented in this note. For completeness, we also recite a lemma that solves the recursion in the special case \(a = 0\).

**Lemma 4 (Karimireddy et al., 2019).** Let \(\{r_t\}_{t \geq 0}, \{s_t\}_{t \geq 0}\) as in \((8)\) for \(a \geq 0\). Then there exists a constant stepsize \(\gamma_t \equiv \gamma \leq \frac{1}{d}\) such that:

\[
\frac{b}{T+1} \sum_{t=0}^{T} s_t \leq \frac{dr_0}{T+1} + \frac{2\sqrt{c\gamma_0}}{\sqrt{T+1}}.
\]

**SGD convergence rates.** To conclude this section, we now briefly summarize our main result that follows by replacing the variables in Lemmas 2–4 by the values stated at the beginning of this section, and observing \(f(\bar{x}_T) \leq \frac{1}{W_T} \sum_{t=0}^{T} w_tf(x_t)\) for convex \(f\).
Theorem 5. For $x_0 \in \mathbb{R}^d$, let $\{x_t\}_{t \geq 0}$ denote the iterates of SGD generated on a function $f$ under Assumptions 1–3 for stepsizes $\gamma_t \leq \frac{1}{2L}, \forall t \geq 0$. Then there exists stepsizes $\gamma_t \leq \frac{1}{2L}$ and weights $w_t \geq 0$ such that it holds for all $T \geq 0$:

$$
\mathbb{E} f(\bar{x}_T) - f^* + \mu \mathbb{E} \|x_{T+1} - x^*\|^2 \leq \min \left\{ \frac{64LR^2}{4L} \exp \left[ \frac{\mu T^2}{4L} \right] + \frac{36\sigma^2}{\mu T} + \frac{2LR^2}{T} + \frac{2\sigma R}{\sqrt{T}} \right\},
$$

where here $R := \|x_0 - x^*\|$, and again $W_T := \sum_{t=0}^T w_t$ and $\bar{x}_T := \sum_{t=0}^T w_t x_t$.

Proof. The theorem follows from the decrease Lemma 6 and Lemmas 3 and 4 with $a = \mu$, $b = 1$, $c = \sigma^2$ and $d = 2L$. For this we observe that all stepsizes $\gamma_t \leq \frac{1}{2L}, \forall t \geq 0$.

$$\square$$

We here did not explicitly state the (suboptimal) convergence result for tuned constant stepsizes that follows directly from Lemma 2.

5 Discussion

We study the iteration complexity of the (stochastic) gradient descent method and recover—simultaneously—the best known rates for the function value suboptimality for an average iterate of SGD and the distance to the optimal solution of the last iterate of SGD. Our analysis focuses on the general stochastic setting, but—as a special case—we also recover the exponential convergence rates in the deterministic setting. This unified analysis address several shortcomings of previous works.

Whilst we only consider (strongly) convex functions here, further extension of the framework to larger function classes would obviously be an interesting future direction. We would like to remark that Assumption 2 potentially also covers a much larger class of functions than the few examples discussed in Section 2. For instance, the approximate gradient oracles introduced in (Devolder et al., 2014) satisfy this assumption as well (cf. (Devolder et al., 2014, Theorem 1)) and, interestingly, Hölder continuous functions (which are in general not continuously differentiable) still admit approximate gradient oracles. However, as these oracles are not unbiased in general, Assumption 4 prevents the immediate application our framework in this extended setting (though, extension of our results under mild relaxations of the unbiasedness Assumption 1 similar as e.g. in (Bottou et al., 2018), are immediately possible).

A small drawback our results is that one needs knowledge of $T$ and the problem parameters $\mu, L, \sigma$ to implement the schemes presented here (to decide on the stepsize, and for switching to the suffix averaging). In practice, some of these limitations can be remedied by the doubling trick. Further, just knowing $T$ up to some constant factor approximation is sufficient to recover the optimal rate up to constant factors.

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A Technical Lemmas

A.1 Constant Stepsizes (with Log Terms)

Proof of Lemma 2. We start by re-arranging (8) and multiplying both sides with \( w_t \):

\[
bs_t w_t \leq \frac{w_t (1 - a\gamma)t}{\gamma} - \frac{w_t r_{t+1}}{\gamma} + c\gamma w_t = \frac{w_{t-1} r_t}{\gamma} - \frac{w_t r_{t+1}}{\gamma} + c\gamma w_t.
\]

By summing from \( t = 0 \) to \( t = T \), we obtain a telescoping sum:

\[
\frac{b}{W_T} \sum_{t=0}^{T} s_t w_t \leq \frac{1}{\gamma W_T} (w_0 (1 - a\gamma)r_0 - w_T r_{T+1}) + c\gamma,
\]

and hence

\[
\frac{b}{W_T} \sum_{t=0}^{T} s_t w_t + \frac{w_T r_{T+1}}{\gamma W_T} \leq \frac{r_0}{\gamma W_T} + c\gamma.
\]

With the estimates

- \( W_T = (1 - a\gamma)^{-T} \sum_{t=0}^{T} (1 - a\gamma)^t \leq \frac{w_0}{\delta} \) (here we leverage \( a\gamma \leq \frac{\delta}{2} \leq 1 \)),
- and \( W_T \geq w_T = (1 - a\gamma)^{-T} \),

we can further simplify the left and right hand sides:

\[
\frac{b}{W_T} \sum_{t=0}^{T} s_t w_t + a r_{T+1} \leq (1 - a\gamma)^{(T+1)} \frac{r_0}{\gamma} + c\gamma \leq \frac{r_0}{\gamma} \exp \left\{-a\gamma(T+1)\right\} + c\gamma. \tag{9}
\]

Now the lemma follows by carefully tuning \( \gamma \). Consider the two cases:

- If \( \frac{1}{d} \geq \frac{\ln(\max\{2, a^2 r_0 T^2/c\})}{aT} \) then we choose \( \gamma = \frac{\ln(\max\{2, a^2 r_0 T^2/c\})}{aT} \) and get that Equation (9) is

  \[
  \tilde{O} \left( a r_0 T \exp\left\{-\ln(\max\{2, a^2 r_0 T^2/c\})\right\} \right) + \tilde{O} \left( \frac{c}{aT} \right) = \tilde{O} \left( \frac{c}{aT} \right),
  \]

  as in case 2 \( \geq a^2 r_0 T^2/c \) it holds \( a r_0 T \leq \frac{2c}{aT} \).

- If otherwise \( \frac{1}{d} < \frac{\ln(\max\{2, a^2 r_0 T^2/c\})}{aT} \) then we pick \( \gamma = \frac{1}{d} \) and get that Equation (9) is

  \[
  dr_0 \exp \left\{-\frac{aT}{d} \right\} + \frac{c}{d} \leq dr_0 \exp \left\{-\frac{aT}{d} \right\} + \frac{c\ln(\max\{2, a^2 r_0 T^2/c\})}{aT} = \tilde{O} \left( dr_0 \exp \left\{-\frac{aT}{d} \right\} + \frac{c}{aT} \right).
  \]

Collecting these two cases concludes the proof.

A.2 Decreasing Stepsizes (Avoiding Log Terms)

For the proof of Lemma 3 we need auxiliary results, for both of which we do not claim much novelty here.

Lemma 6. Let \( \{r_t\}_{t \geq 0}, \{s_t\}_{t \geq 0} \) be as in for \( a > 0 \) and for constant stepsizes \( \gamma_t \equiv \gamma := \frac{1}{d}, \forall t \geq 0 \). Then it holds for all \( T \geq 0 \):

\[
r_T \leq r_0 \exp \left\{-\frac{aT}{d} \right\} + \frac{c}{ad}.
\]
Lemma 7. Let \( \{r_t\}_{t \geq 0}, \{s_t\}_{t \geq 0} \) as in\(^\square\) for \( a > 0 \) and for decreasing stepsizes \( \gamma_t := \frac{2}{a(k+t)} \), \( \forall t \geq 0 \), with parameter \( \kappa := \frac{2d}{a} \), and weights \( w_t := (\kappa + t) \). Then
\[
\frac{b}{W_T} \sum_{t=0}^{T} s_t w_t + a r_{T+1} \leq \frac{2a\kappa^2 r_0}{T^2} + \frac{2c}{aT},
\]
where here again \( W_T := \sum_{t=0}^{T} w_t \).

Proof. We start as in the proof of Lemma\(^\square\)
\[
bs_t w_t \leq \frac{w_t(1 - a\gamma_t)r_t}{\gamma_t} - \frac{w_t r_{t+1}}{\gamma_t} + c\gamma_t w_t = a(\kappa + t)(\kappa + t - 2)r_t - a(\kappa + t)^2 r_{t+1} + \frac{c}{a}
\]
\[
\leq a(\kappa + t - 1)^2 r_t - a(\kappa + t)^2 r_{t+1} + \frac{c}{a},
\]
where the equality follows from the definition of \( \gamma_t \) and \( w_t \) and the inequality from \( (\kappa + t)(\kappa + t - 2) = (\kappa + t - 1)^2 - 1 \leq (\kappa + t - 1)^2 \). Again we have a telescoping sum:
\[
\frac{b}{W_T} \sum_{t=0}^{T} s_t w_t + \frac{a(\kappa + T)^2 r_{T+1}}{W_T} \leq \frac{2a\kappa^2 r_0}{W_T} + \frac{c(T+1)}{aW_T},
\]
with
- \( W_T = \sum_{t=0}^{T} w_t = \sum_{t=0}^{T} (\kappa + t) = \frac{(2\kappa + T)(T+1)}{2} \geq \frac{T(T+1)}{2} \geq \frac{T^2}{2} \),
- and \( W_T = \frac{(2\kappa + T)(T+1)}{2} \leq \frac{2(\kappa + T)(T+1)}{2} \leq (\kappa + T)^2 \) for \( \kappa = \frac{2d}{a} \geq 1 \).

By applying these two estimates we conclude the proof. \( \square \)

We can now combine the findings of these two lemmas.

Proof of Lemma\(^\square\) For integer \( T \geq 0 \), we choose stepsizes and weights as follows:
- If \( T \leq \frac{d}{a} \), \( \gamma_t = \frac{1}{d} \), \( w_t = (1 - a\gamma_t)^{-(t+1)} = \left(1 - \frac{a}{d}\right)^{-(t+1)} \),
- If \( T > \frac{d}{a} \) and \( t < t_0 \), \( \gamma_t = \frac{1}{d} \), \( w_t = 0 \),
- If \( T > \frac{d}{a} \) and \( t \geq t_0 \), \( \gamma_t = \frac{2}{a(\kappa + t - t_0)} \), \( w_t = (\kappa + t - t_0)^2 \),

for \( \kappa = \frac{2d}{a} \) and \( t_0 = \left\lfloor \frac{T}{2} \right\rfloor \). We will now show that these choices imply the claimed result.
We start with the case $T \leq \frac{d}{a}$. This case is similar to the proof of Lemma 2 and it suffices to consider Equation 9 for the choice $\gamma = \frac{1}{d}$. We observe that Equation 9 simplifies to
\[
dr_0 \exp \left[ -\frac{aT}{d} \right] + \frac{c}{d} \leq dr_0 \exp \left[ -\frac{aT}{d} \right] + \frac{c}{aT}.
\]
If $T > \frac{d}{a}$, then we obtain from Lemma 6 that
\[
r_{t_0} \leq r_0 \exp \left[ -\frac{aT}{2d} \right] + \frac{c}{ad}.
\]
From Lemma 7 we have for the second half of the iterates:
\[
\frac{b}{W_T} \sum_{t=0}^{T} s_tw_t + ar_{T+1} = \frac{b}{W_T} \sum_{t=t_0}^{T} s_tw_t + ar_{T+1} \leq \frac{8\kappa^2 r_{t_0}}{T^2} + \frac{4c}{aT}.
\]
Now we observe that the restart condition $r_{t_0}$ satisfies:
\[
\frac{a\kappa^2 r_{t_0}}{T^2} = \frac{a\kappa^2 r_0 \exp \left( -\frac{aT}{2d} \right)}{T^2} + \frac{\kappa^2 c}{dT^2} \leq 4ar_0 \exp \left[ -\frac{aT}{2d} \right] + \frac{4c}{aT},
\]
because $T \geq \frac{d}{a}$. These inequalities show the claim.

A.3 Sub-linear Convergence rate

Proof of Lemma 4. We start by re-arranging (8) and summing from $t = 0$ to $t = T$:
\[
\frac{b}{T+1} \sum_{t=0}^{T} s_t \leq \frac{1}{T+1} \sum_{t=0}^{T} \left( \frac{r_t}{\gamma} - \frac{r_{t+1}}{\gamma} \right) + c\gamma \leq \frac{r_0}{\gamma(T+1)} + c\gamma.
\]
Now we carefully select $\gamma$.
- If $\frac{1}{\gamma} \leq \frac{r_0}{c(T+1)}$, then we pick $\gamma = \frac{1}{\gamma}$ with $\gamma \leq \frac{\sqrt{T}}{\sqrt{c(T+1)}}$ and verify
  \[
  \frac{r_0}{\gamma(T+1)} + c\gamma \leq \frac{dr_0}{T+1} + \frac{\sqrt{c r_0}}{\sqrt{T+1}}.
  \]
- If otherwise $\frac{1}{\gamma} > \frac{r_0}{c(T+1)}$ then we pick $\gamma = \frac{\sqrt{T}}{\sqrt{c(T+1)}}$ to obtain
  \[
  \frac{r_0}{\gamma(T+1)} + c\gamma \leq \frac{2\sqrt{c r_0}}{\sqrt{T+1}}.
  \]