A theoretical study on the triaxial superdeformed and normal deformed wells in Lu isotopes

S. Sayyah and A. Kardan

School of Physics, Damghan University, Damghan, Iran

Received June 6, 2017; Revised September 21, 2017; Accepted October 5, 2017; Published November 23, 2017

We have theoretically studied the potential barrier between triaxial superdeformed (TSD) and normal deformed (ND) wells in $^{161,163,167,168}_{\text{Lu}}$ and $^{168}_{\text{Hf}}$ nuclei within the framework of the cranked Nilsson–Strutinsky model. We show that the theoretical TSD–ND barrier has disappeared or is narrow or shallow where the experimental TSD–ND electromagnetic transitions are observed, while a theoretical high barrier is seen in nuclei excluding the experimental TSD–ND decay. Also, our calculations show that the barrier between TSD and ND wells decreases with increasing the neutron number.

Subject Index D11

1. Introduction

Shape coexistence is a very peculiar nuclear phenomenon which is observed, for example, in the mass region with $Z \sim 72$ and $N \sim 94$ [1,2]. In this region, normal deformed (ND), terminating, and triaxial superdeformed (TSD) states are confirmed not only in experiments but also in theoretical studies [3–5]. The lutetium nucleus with $Z = 71$ is in the center of this region [3,5]. The most TSD bands, as well as their decay to ND rotational bands, are observed in the Lu isotopes with $A = 161–168$ [4,6–9].

Knowledge of the transitions between TSD and ND states is crucial to determine the spin, parity, and excitation energy of the TSD states. One can determine the spin and parity of TSD bands using their transition to a band with known parity and spin. For example, the observed TSD bands in $^{161}_{\text{Lu}}$ have not been linked to an ND-level scheme, thus their spin and the bandhead energy can only be estimated [10], while in $^{163}_{\text{Lu}}$, transitions between the TSD1 band and the ND band are detected, and have successfully been used to determine the spin and parity of TSD1 and subsequently to define these quantities for the bands TSD2, TSD3, and TSD4 as well [7]. The electromagnetic transitions from the TSD to ND bands are observed even at high spins in $^{167,168}_{\text{Lu}}$, whereas the TSD–ND barrier is expected to disappear at low spin [11,12].

Contrary to the Lu isotopes, no connections are detected between the TSD and ND bands in $^{168}_{\text{Hf}}$, which is in the Lu neighborhood. For example, the spin and parity of the TSD1 band are not certainly specified because there is no decay towards other bands such as ND bands [13]. Referring to experimental observations and conducted studies, these decays have been observed only in some nuclei such as Lu isotopes. Now this question arises: why is the decay between the TSD and ND bands seen only in some of the Lu isotopes and not in a nucleus such as $^{168}_{\text{Hf}}$? The essence of these decays points to the tunneling phenomenon in quantum mechanics. Regarding this phenomenon, the possibility of barrier surmounting is a function of the width and height of the potential barrier.
There are many experimental studies on the transitions between the TSD and ND states, while, so far, few theoretical studies have been performed to investigate the TSD–ND potential barrier.

As an example, the decay out of superdeformed bands to the normal bands are studied quantum mechanically with an exact calculation of the matrix elements for tunneling in Ref. [14]. It is indicated that the rapidity and universality of the decay-out profiles can be explained straightforwardly within our two-state dynamical model by the decrease of the centrifugal barrier between the superdeformed and normal-deformed energy wells with decreasing spin [14]. Also, in Ref. [15], decay-out properties of the thermally excited superdeformed bands are studied theoretically in $^{152}$Dy, $^{143}$Eu, and $^{192}$Hg. The tunneling path in the two-dimensional deformation energy surface, which is calculated with the cranked Nilsson–Strutinsky (CNS) model [16,17], shows that the decay out brings about a characteristic decrease in the effective number of excited superdeformed rotational bands [15].

Our motivation in pursuing the present investigation is to study the TSD and ND wells in $^{161,163,167,168}$Lu and also $^{168}$Hf nuclei. We also study the TSD–ND barrier variations with increasing spin and neutron numbers.

We have performed the calculations within the framework of the CNS model.

2. Summary of the theory

Here, we briefly review a microscopic–macroscopic model for the calculation of the total energy a rotating nucleus. In this model, both collective and single-particle behavior of the nucleus are considered simultaneously, as the nucleons move independently of each other in a deformed and rotating mean field generated by the nucleons themselves. The mean-field Hamiltonian used to describe a nucleon in the rotating nucleus is the cranked modified oscillator Hamiltonian [16]

$$h = h_{HO}(\varepsilon_2, \gamma) - \kappa \hbar \omega_0 \left[ 2 \ell \cdot s + \mu (\ell^2 - (\ell^2)_N) \right] + V_4(\varepsilon_2, \gamma) - \omega_j x,$$

(1)

where $h_{HO}(\varepsilon_2, \gamma)$ is an anisotropic harmonic-oscillator Hamiltonian:

$$h_{HO}(\varepsilon_2, \gamma) = \frac{p^2}{2m} + \frac{1}{2} m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2).$$

(2)

The frequencies $\omega_x$, $\omega_y$, and $\omega_z$ are defined by the following general equation [3]:

$$\omega_j = \omega_j(\varepsilon_2, \gamma) \left[ 1 - \frac{2}{3} \varepsilon_2 \cos \left( \gamma + \frac{2\pi v_j}{3} \right) \right], \quad j \in \{x, y, z\},$$

(3)

where $v_x = 1$, $v_y = -1$, and $v_z = 0$. The hexadecapole potential $V_4(\varepsilon_4, \gamma)$ is an expansion in terms of the spherical harmonics—see Ref. [16]. In Eq. (1), $\omega_j x$ is the cranking term and refers to the nucleus rotation around the $x$ axis. The cranked Nilsson model includes a three-dimensional space of deformation parameters: axial quadrupole deformation, $\varepsilon_2$; non-axial quadrupole deformation (triaxial), $\gamma$; and hexadecapole deformation, $\varepsilon_4$. Diagonalization of the Hamiltonian in Eq. (1) gives the eigenvalues as a function of the deformation parameters $\epsilon_i^0(\varepsilon_2, \gamma, \varepsilon_4)$ which are referred to as the single-particle energies in the rotating system or, more properly, Routhians. The total spin and energy are given by summing over the occupied states [18]:

$$I \approx I_x = \sum_{\text{occ}} \langle j_x \rangle_i,$$

(4)

$$E_{sp}(\varepsilon_2, \gamma, \varepsilon_4, I) = \sum_{\text{occ}} \epsilon_i^0(\varepsilon_2, \gamma, \varepsilon_4).$$

(5)
The renormalized total energy is calculated using the shell correction method. This shows the total energy as a summation of two terms: a macroscopic part, which is the rotating liquid-drop energy, $E_{\text{rlid}}$, and a microscopic energy, $E_{\text{sh}}$, which is obtained from quantal shell corrections.

$$E_{\text{tot}}(I) = E_{\text{sh}}(I) + E_{\text{rlid}}(I). \quad (6)$$

The shell energy is calculated using the Strutinsky procedure \[19,20\], and the Lublin–Strasbourg drop (LSD) model is applied to evaluate the liquid-drop energy \[17,21\]. In this model, in addition to the parity, one other symmetry survives which is associated with $180^\circ$ rotation around the cranking axis, so that the single-particle orbitals are labelled by it (signature) as $\alpha = +1/2$ and $\alpha = -1/2$, as well as the parity.

### 3. Calculational procedure

In Eq. (1), to evaluate the single-particle energies, the single-particle parameters $\kappa$ and $\mu$, the deformation parameters, and the rotational frequency should be specified. In these calculations, the $A = 150$ \[22–24\] and standard \[16\] parameters are used for Lu and Hf isotopes, respectively; see also Ref. \[3\].

The calculations are carried out at the following grid points:

$$x = 0.0[0.02]0.46, \quad y = 0.0[0.02]0.46, \quad \epsilon_4 = -0.06[0.01]0.06, \quad \omega/\omega_0 = 0.0[0.015]0.24,$$

where $(x, y)$ are Cartesian coordinates in the $(\epsilon_2, \gamma)$ plane. The $(x, y)$ coordinates are connected with $(\epsilon_2, \gamma)$ by the expressions

$$x = \epsilon_2 \cos(\gamma + 30^\circ), \quad y = \epsilon_2 \sin(\gamma + 30^\circ).$$

Compared to our previous studies \[3,5\], we have chosen a dense mesh for more accuracy.

First we obtain the eigenvalues of the Hamiltonian introduced in Eq. (1) in all the points of the deformation space. They are referred to as the single-particle energies in the rotating system. Then, for all the points of the deformation space and spin, we calculate the macroscopic energy of the nucleus and subsequently the shell energy. At the end, the total energies from Eq. 6 are minimized with respect to the triaxial and hexadecapole deformation parameters for different $\epsilon_2$ and spin values and for four combinations of parity and signature: $(\pi, \alpha) = (+, 0)$, $(+, 1)$, $(-, 0)$, and $(-, 1)$ for even-$A$ nuclei, and $(\pi, \alpha) = (+, +1/2)$, $(+, -1/2)$, $(-, +1/2)$, and $(-, -1/2)$ for odd-$A$ nuclei.

The total energy is evaluated as a smoothed function of the $\epsilon_2$ parameter by some interpolations. For better understanding, see Fig. 1, where the energies were minimized only with respect to the hexadecapole parameters and plotted in the $(\epsilon_2, \gamma)$ plane for $^{163}$Lu.

### 4. Results and discussion

In Figs. 2 to 4, the total energies are plotted as a function of the $\epsilon_2$ parameter for $^{161,163,167,168}$Lu and $^{168}$Hf nuclei. In these diagrams, the solid lines refer to the spin at which the existence of a decay is confirmed experimentally.

In $^{161}$Lu, the TSD1 band is observed from spin $I = 21^+ / 2h$ to $I = 89^+ / 2h$ with $(\pi, \alpha) = (+, +1/2)$. The experiment demonstrates that there is no transition to the observed ND bands for...
Fig. 1. Calculated potential-energy surfaces versus quadrupole deformation $\varepsilon_2$ and the triaxiality parameter $\gamma$ of $^{163}$Lu with $(\pi, \alpha) = (+, +1/2)$ for spins from $I = 10.5\hbar$ and $I = 20.5\hbar$. The contour lines are separated by 0.25 MeV and the $\gamma$ plane is marked at 15° intervals. Dark regions represent low energy with absolute minima labeled with dots.

Fig. 2. Energy as a function of axial deformation parameter $\varepsilon_2$ for (a) $^{161}$Lu and (b) $^{163}$Lu. Note that the minimization is performed with respect to $\gamma$ as well as the hexadecapole deformation parameter $\varepsilon_4$. The solid lines represent the spins where the transition has been observed experimentally.

the TSD1 band [10]. Therefore, one can expect that a high barrier between TSD and ND minima prevents such a transition. In order to study this barrier, the total energy is plotted as a function of $\varepsilon_2$ for the configuration $(\pi, \alpha) = (+, +1/2)$ for $^{161}$Lu in Fig. 2(a). As pointed out, the minimization of total energy is performed with respect to $\gamma$ as well as the hexadecapole deformation parameter $\varepsilon_4$. Therefore, note that the triaxial parameter $\gamma$ is different for each point of the diagram. Figure 2(a) shows four minima in the potential energy surfaces: an oblate ND minimum with $\varepsilon_2 \sim 0.2$, two prolate ND minima which are very close together in energy with $\varepsilon_2 \sim 0.18, 0.23$, a TSD minimum with $(\varepsilon_2, \gamma) \sim (0.43, 20^\circ)$, and also a TSD minimum with $(\varepsilon_2, \gamma) \sim (0.3, 40^\circ)$ at spins $I < 16.5\hbar$. Among these minima, the prolate shapes are the lowest in energy (yrast) at the plotted region of spin. As illustrated in Fig. 2(a), with decreasing the spin from $I = 24.5\hbar$ to $I = 10.5\hbar$, the height of the potential barrier between the TSD well with $\varepsilon_2 \sim 0.43$ and the prolate ND minima does not disappear and is constant at a value of approximately 1.5 MeV. Therefore, our calculations show that
because of a considerable potential barrier between two minima of TSD and ND, the decay of the TSD band to the ND band is expected to happen with a low probability, which is consistent with observations.

The isotope $^{163}\text{Lu}$ is one of the best candidates for observing the TSD bands experimentally [6,7]. In this isotope, four TSD bands are observed that, in addition to the internal decay, decay to the ND bands $[411]1/2^+$ and $[523]7/2^-$ by TSD1 [7]. According to the experimental data, the TSD1 band decays toward the ND bands at the spins $I = 13/2\hbar–25/2\hbar$ [7]. Considering Weisskopf estimates and the defined spin and parity for these bands, the type of decay can be determined which are most likely to be from the multi-polar types of M1 and E2. The calculations are illustrated in Fig. 2(b), where there is a TSD minimum in $(\varepsilon_2, \gamma) \sim (0.42, 20^\circ)$ and a prolate ND minimum in $\varepsilon_2 \sim 0.23$. 

Fig. 3. As Fig. 2, but for (a) $^{167}\text{Lu}$ and (b) $^{168}\text{Lu}$.

Fig. 4. As Fig. 2, but for $^{168}\text{Hf}$. 

As the spin decreases, the potential barrier between TSD and ND minima decreases until it reaches the lowest value at spin \( I = 8.5\hbar \). Therefore, it is expected that at spin \( I = 8.5\hbar \), the decay occurs between these two minima with the most possibility. Decays are observed around this spin in the experiment well.

It is expected that with increasing spin, the TSD well goes down in energy, thus the TSD–ND barrier increases, and consequently the transition probability decreases. Transitions from TSD to ND bands are observed at higher spins in \(^{167}\text{Lu}\) and \(^{168}\text{Lu}\).

In \(^{167}\text{Lu}\), the TSD1 band is directly connected to the ND Bands 8 and 9. At low spins (\( I < 17\hbar \)), it decays to Band 9 via three transitions and to Band 8 via another three transitions \([11]\). At higher spins, \( I = 61/2\hbar \) and \( 65/2\hbar \), it interacts once again with Band 9 via three further transitions and is linked via the 880 keV transition to the \( 61/2^+ \) state in Band 5 \([11]\). These transitions should be because of a well-defined barrier between the TSD–ND wells. The calculated total energies are plotted for \(^{167}\text{Lu}\) at spins \( I = 26.5\hbar – 40.5\hbar \) for the positive parity yrast state in Fig. 3(a). The TSD and ND minima are observed in \( (\varepsilon_2, \gamma) \sim (0.43, 20^\circ) \) and \( (\varepsilon_2, \gamma) \sim (0.24, 0^\circ) \), respectively. With decreasing spin from \( I = 40.5\hbar \), the potential barrier magnitude between the TSD and ND wells decreases, reaches its minimum value at spin \( 32.5\hbar \), and fades away at spin \( 30.5\hbar \). Thus it leads to the increase in the decay possibility at the spins \( I = 32.5\hbar \) and \( 30.5\hbar \) where the transitions are seen, experimentally.

In Fig. 3(b), with decreasing spin, \( \varepsilon_2 \) of the ND minimum varies from 0.22 to 0.28, while that of the TSD minimum is almost constant \( (\varepsilon_2, \gamma) \sim (0.48, 20^\circ) \) for the \(^{168}\text{Lu}\). With decreasing spin from \( 36\hbar \) to \( 22\hbar \), the barrier between the two considered minima decreases specifically. The potential barrier reaches its minimum value at spin \( 26\hbar \) and fades away and disappears completely at spin \( 24\hbar \). Therefore, it is expected that decay occurs at spin \( 24\hbar \) with the most possibility. Regarding the experimental observations, the E2 transition from the TSD band at spin \( 24\hbar \) to the ND band is confirmed \([12]\).

The trend of the potential barrier in Figs. 2 and 3 shows that with increasing neutron number, the TSD–ND barrier decreases and completely disappears in \(^{168}\text{Lu}\). Therefore, it is predicted that TSD–ND transitions are observed with more intensity and higher probability in isotopes heavier than \(^{168}\text{Lu}\), while with a very low probability in those lighter than \(^{161}\text{Lu}\).

We have also studied the potential barrier in \(^{168}\text{Hf}\), which is in the Lu neighborhood. The observed TSD bands in \(^{168}\text{Hf}\) have not been linked to the normal deformed bands, therefore their spin and excitation energy can only be estimated \([13]\). The TSD1 band in this nucleus is observed in the spin limit of \( 33\hbar \) to \( 61\hbar \) \([13]\). In Fig. 4, the calculated energy is plotted with respect to the parameter \( \varepsilon_2 \) for \(^{168}\text{Hf}\) and spins \( I = 29\hbar – 41\hbar \). As can be seen, the TSD minimum, which is seen only at spins \( I > 35\hbar \), has the deformation \( (\varepsilon_2, \gamma) \sim (0.45, 20^\circ) \).

5. Summary

In summary, the TSD–ND barriers in \(^{161,163,167,168}\text{Lu}\) and \(^{168}\text{Hf}\) nuclei have been studied within the cranked Nilsson–Strutinsky framework. Within this interpretation, the observed transitions in the TSD bands to the ND bands at specific spins can be explained as a shallow or narrow barrier, while no decay is seen as a high barrier. The results show significant consistency with experiment, and thus an illustrative physical understanding is obtained about the electromagnetic transitions of the TSD to ND bands. With increasing neutron number from \( N = 90 \) in \(^{161}\text{Lu}\) to \( N = 97 \) in \(^{168}\text{Lu}\), the TSD–ND barrier decreases until it disappears completely. These observations show that the barrier between TSD and ND wells vanishes with increasing \( N \).
References

[1] K. Heyde and J. L. Wood, Rev. Mod. Phys. 83, 1467 (2011).
[2] I. Ragnarsson, B. G. Carlsson, A. Kardan, and H.-L. Ma, Acta. Phys. Pol. B 46, 477 (2015).
[3] A. Kardan, I. Ragnarsson, H. Miri-Hakimabad, and L. Rafat-Motevali, Phys. Rev. C 86, 014309 (2012).
[4] W. Korten and S. Lunardi, Achievements with the EUROBALL Spectrometer, LNL-INFN (REP) 201/2004, 1997–2003, eds 2003.
[5] A. Kardan and S. Sayyah, Int. J. Mod. Phys. E 25, 1650044 (2016).
[6] S. Ødegård et al., Phys. Rev. Lett. 86, 5866 (2001).
[7] D. R. Jensen et al., Eur. Phys. J. A 19, 173 (2004).
[8] S. Törmänen et al., Phys. Lett. B 454, 8 (1999).
[9] P. Bringel et al., Phys. Rev. C 75, 044306 (2007).
[10] P. Bringel et al., Phys. Rev. C 73, 054314 (2006).
[11] D. G. Roux et al., Phys. Rev. C 92, 064313 (2015).
[12] Y. Li, Triaxial Strongly Deformed Band and High Spin States in $^{168}$Lu, Master’s thesis, Mississippi State University, 2003.
[13] R. B. Yadav et al., Phys. Rev. C 78, 044316 (2008).
[14] B. R. Barrett, J. Bürki, D. M. Cardamone, C. A. Stafford, and D. L. Stein, Phys. Lett. B 688, 110 (2010).
[15] K. Yoshida, M. Matsuo, and Y. R. Shimizu, Nucl. Phys. A 696, 85 (2001).
[16] T. Bengtsson and I. Ragnarsson, Nucl. Phys. A 436, 14 (1985).
[17] B. G. Carlsson and I. Ragnarsson, Phys. Rev. C 74, 011302(R) (2006).
[18] I. Ragnarsson and S. G. Nilsson Shapes and Shells in Nuclear Structure (Cambridge University Press, Cambridge, 1995).
[19] G. Andersson et al., Nucl. Phys. A 268, 205 (1976).
[20] V. M. Strutinsky, Nucl. Phys. A 122, 1 (1968).
[21] K. Pomorski and J. Dudek, Phys. Rev. C 67, 044316 (2003).
[22] T. Bengtsson, Nucl. Phys. A 512, 124 (1990).
[23] T. Bengtsson, Nucl. Phys. A 496, 56 (1989).
[24] R. B. Yadav et al., Phys. Rev. C 80, 064306 (2009).