RECOGNITION OF AFFINE-EQUIVALENT POLYHEDRA BY THEIR NATURAL DEVELOPMENTS

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Abstract: The classical Cauchy rigidity theorem for convex polytopes reads that if two convex polytopes have isometric developments then they are congruent. In other words, we can decide whether two convex polyhedra are isometric or not by only using their developments. We study a similar problem of whether it is possible to understand that two convex polyhedra in Euclidean 3-space are affine-equivalent by only using their developments.

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1. Introduction

The famous Cauchy Rigidity Theorem [1; 2, Chapter III; 3; 4, Subsection 6.4] can be formulated as follows: *If the developments of two convex polyhedra in \( \mathbb{R}^3 \) are isometric, then the polyhedra are congruent.* In other words, this theorem allows us to answer the question of whether “two convex polyhedra are congruent” by only using the developments of the polyhedra. We seem naturally to pose the problem: “Is it possible to understand whether two convex polyhedra are affine-equivalent or not provided that we are only given the developments of the polyhedra?”

The author is unaware that this problem or any relevant results have appeared in the literature. The first steps to solving the problem are made in this article.

Note that the problem seems to be related not only to the Cauchy Rigidity Theorem and synthetic geometry but also to the problems of computer vision, pattern recognition, and image processing. We refer the interested reader, e.g., to the book [5], although in it, as well as in other available articles on computer vision known to us, the problem under study is not addressed directly.

Obviously, if \( P \) and \( P' \) in \( \mathbb{R}^3 \) are affine-equivalent polyhedra then their corresponding faces are also affine-equivalent to each other. However, the converse is not true. Indeed, consider two convex triangular bipyramids \( P \) and \( P' \) such that the length of every edge of \( P \) is equal to 1, and the length of one edge of \( P' \) connecting a vertex of valency 3 with a vertex of valency 4 is not equal to 1, while the length of every other edge of \( P' \) is equal to 1. The affine equivalence of the corresponding faces of \( P \) and \( P' \) follows obviously from the fact that every two triangles are affine-equivalent to each other. However, as is easy to see, \( P \) and \( P' \) themselves are not affine-equivalent.

Therefore, the main problem of this article can be reformulated as follows: “What extra conditions on the developments of two convex polyhedra (in addition to the condition of the affine equivalence of the corresponding faces) guarantee the affine equivalence of the polyhedra themselves?”

The article is organized as follows: In Section 2, we refine the terminology and formulate the results of other authors in a form convenient for our purposes. In Section 3, we prove that if polyhedra are simple (i.e., if exactly three faces are incident to each vertex), then the affine equivalence of the faces suffices to imply the affine equivalence of the polyhedra themselves (i.e., no additional conditions are needed). In Section 4, we study our problem for suspensions, i.e., for the polyhedra combinatorially equivalent...
to regular convex \( n \)-gonal bipyramids. In Section 5, we study the auxiliary local problem of the affine equivalence of two polyhedra, each of which is homeomorphic to a disk and contains only three faces. Finally, in Section 6 we describe some algorithm suitable for polyhedra of any combinatorial structure for recognizing affine-equivalent polyhedra from their natural developments on using ideas of Sections 3–5. For some pairs of polyhedra, the algorithm can certify that they are not affine-equivalent. Note that the algorithm is applicable not only to convex or closed polyhedra.

2. Refinement of the Terminology

In this article, we call a polyhedron a connected two-dimensional polyhedral surface in Euclidean 3-space composed of finitely many convex polygons that are referred to as faces. The faces of a polyhedron are not necessarily triangular; a polyhedron can either have a nonempty boundary or be closed, can either be convex (i.e., coincide with the boundary or a part of the boundary of a convex set) or nonconvex, can have an arbitrary topological structure, and may have self-intersections. All these possibilities are not excluded in the sequel until the corresponding restriction will be explicitly formulated. On the other hand, we always assume that every polyhedron under study is connected, i.e., we can go from each face to another by crossing not vertices but edges.

Take a convex polyhedron \( P \) and cut \( P \) along its edges into flat faces. We obtain finitely many convex polygons on the plane. At the same time, let us remember the “gluing rules,” i.e., firstly, which edge of the polygon should be glued with which edge of another polygon and, secondly, which vertex of one glued edge should be glued with which vertex of the other in order to get the original polyhedron. We call the so-obtained finite set of convex polygons together with the “gluing rules” the natural development of \( P \).

The natural development is uniquely determined by a convex polyhedron.

Assume given two combinatorially equivalent convex polyhedra \( P \) and \( P' \) and their natural developments \( R \) and \( R' \). We call \( R \) and \( R' \) isometric if their “gluing rules” are compatible with the combinatorial equivalence of \( P \) and \( P' \) and each polygon in \( R \) is congruent to the corresponding polygon in \( R' \).

Using the notion of the natural development of a polyhedron, we can formulate the classical Cauchy Rigidity Theorem for convex polyhedra as follows:

**Theorem 1.** Let \( P \) and \( P' \) be closed convex polyhedra in Euclidean 3-space. If the natural developments of \( P \) and \( P' \) are isometric, then \( P \) and \( P' \) are congruent.

Theorem 1 was first proved by Augustin-Louis Cauchy in 1813 in [6] and considered to be one of the most glorious achievements of geometry. Theorem 1 significantly influenced the development of synthetic geometry and is well presented in scientific [2, Chapter III; 4, Section 6.4; 7; 8, Section 23.1; 9], educational [10, Addition K; 11, Chapter 24], and popular science [1, Chapter 14; 3, Chapter III, § 14; 12; 13, Theorem 24.1] literature. Of recent articles using or generalizing Theorem 1, we mention [14] where the local rigidity of zonohedra is proved in \( \mathbb{R}^3 \); the articles [15–17] where the rigidity of a polyhedron is provided in \( \mathbb{R}^3 \) provided that the polyhedron is homeomorphic to the sphere, torus or pretzel and its every face is a unit square; the article [18] in which Theorem 1 is translated to the case of circular polytopes in the 2-sphere \( \mathbb{S}^2 \); and the article [19] containing a rather unexpected application of Theorem 1 to finding a sufficient condition for a convex polyhedron \( P \) to realize by using the isometries of the ambient space \( \mathbb{R}^3 \) all “combinatorial symmetries” of \( P \), i.e., all edge length preserving maps of the natural development of \( P \) onto itself.

A convex polyhedron can be regarded as a metric space if we define that the distance between every two points is as the infimum of the lengths of the curves connecting these points and lying entirely on the polyhedron. Each point of this metric space other than a vertex of the polyhedron has a neighborhood isometric to a disk in \( \mathbb{R}^2 \). We can say that, after removing finitely many points, this space is locally Euclidean. It is convenient to represent such metric space as a disjoint union of finitely many convex polygons in \( \mathbb{R}^2 \) whose edges are identified so that each edge belongs to exactly two polygons, and the lengths of every edge segment calculated in each of these two polygons are the same. We call this disjoint union of finitely many convex polygons in \( \mathbb{R}^2 \) together with the edge identification rule connected.
if, starting from each polygon, it is possible to pass to any other, successively crossing the sides (not vertices) of the polygons of this disjoint union. Finally, we call a connected disjoint union of finitely many convex polygons in $\mathbb{R}^2$ together with the edge identification rule an abstract development.

Obviously, an abstract development can be constructed for each polyhedron, not just for a convex one (moreover, to construct an abstract development, there is no need to start with a polyhedron; we can start with a set of convex polygons on the plane). It is also obvious that the natural development of a polyhedron $P$ is an abstract development of $P$. Fig. 1 shows that the converse is not true. Indeed, in Fig. 1(a) a regular tetrahedron $P$ with vertices $x_0, x_1, x_2, x_3$ is shown. In Fig. 1(b) and Fig. 1(c), two developments of $P$ with vertices $\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3$ are shown. Moreover, for $j = 0, 1, 2, 3$, the vertex $x_j$ of $P$ corresponds to the vertex $\bar{x}_j$ of its development. The development shown in Fig. 1(b) is natural. In contrast, the abstract development of $P$ shown in Fig. 1(c) is not natural. The latter follows, e.g., from the fact that triangle $\bar{x}_1\bar{x}_2\bar{x}_0$ on the abstract development shaded in Fig. 1(c) is located in two faces $x_1x_2x_3$ and $x_2x_3x_0$ of $P$ simultaneously. In this case, it is said that the face $\bar{x}_1\bar{x}_2\bar{x}_0$ and the edge $\bar{x}_1\bar{x}_0$ of the abstract development “break” and are not a genuine face and a genuine edge of $P$, respectively.

![Diagram of a regular tetrahedron and its developments](image)

Fig. 1. (a) A regular tetrahedron $P$, (b) the natural development of $P$, and (c) an abstract but not natural development of $P$.

A.D. Alexandrov proved the generalization of Theorem 1 in [2, Page 100, the uniqueness claim of Theorem 1∗]:

**Theorem 2.** Let $P$ and $P'$ be closed convex polyhedra in Euclidean 3-space, and let $R$ and $R'$ be their abstract developments. If $R$ and $R'$ are isometric, then $P$ and $P'$ are congruent.

In other words, Theorem 2 shows that in Theorem 1 there is no need to check that the developments of $P$ and $P'$ are natural.

Theorem 2 is of interest not only in its own right, but also plays an essential role in the demonstration of the following theorem on the existence of a convex polyhedron which was also proved by A.D. Alexandrov [2, Page 100, the existence claim of Theorem 1∗]:

**Theorem 3.** Given an abstract development homeomorphic to the sphere and having the sums of the angles at the vertices $\leq 2\pi$, we can glue a closed convex polyhedron in $\mathbb{R}^3$.

The concept is intuitively obvious of the “sum of the angles at the vertex $\bar{x}$ of an abstract development” which is of use in Theorem 3. To find the sum, denoted by $\delta_{\bar{x}}$, you need to select the flat angles with the vertex $\bar{x}$ in all polygons of the abstract development and calculate the sum of the values of these angles. Note that $2\pi - \delta_{\bar{x}}$ is called the curvature of the vertex $\bar{x}$. Theorem 3 requires actually that the curvature of each vertex be nonnegative.

Note also that the example of an abstract development in Fig. 1(c) shows that if in the statement of Theorem 3 we replace the words “abstract development” with the words “natural development,” then we get a false statement. That is, we can say that Theorem 3 is true precisely because the faces and edges of the development are allowed to break in the isometric realization as a polyhedron.

For completeness, we mention that at present there are several substantially different proofs of Theorem 3.
Initially Theorem 3 was published by A.D. Alexandrov in 1941 in the short note [20], then in 1942 in the detailed article [21], and later was included in his book [2] first published in 1950. A.D. Alexandrov deduced Theorem 3 from Brouwer’s Domain Invariance Theorem and Theorem 2. He showed also that, by approximating every metric of positive curvature by polyhedral metrics of positive curvature and passing to the limit, we obtain the positive solution of the generalized Weyl problem; i.e., that we can assert that every two-dimensional sphere-homeomorphic manifold with an intrinsic metric of positive curvature admits an isometric embedding into $\mathbb{R}^3$ by means of a closed convex surface (which is not necessarily smooth and may even degenerate into a doubly covered convex domain in the plane). Various aspects of the Weyl problem (such as the existence, stability and smoothness of a surface depending on the smoothness of the metric) are studied in numerous books and articles; see, e.g., [22–25] and the references therein.

In 1943, at the Steklov Mathematical Institute of the Academy of Sciences of the USSR, Lyusternik gave a talk (see [26]) in which he showed that Theorem 3 follows from the already known facts (see [27]) about the Weyl problem for analytic metrics. Namely, he proposed to approximate each polyhedral metric of positive curvature by analytic metrics of positive curvature and realize the latter by means of analytic convex surfaces, and finally, passing to the limit, obtain the desired convex polyhedron that realizes the original polyhedral metric. Therefore, the Lyusternik method is usually called analytical; it is presented in detail in [28, §12].

Another proof of Theorem 3 was proposed by Volkov in his PhD Thesis defended at Leningrad State University in 1955 under the direction of A.D. Alexandrov. The proof by Volkov is usually called variational, since it is based on the fact that the desired convex polyhedron delivers the minimum to some function defined on a space of “3-dimensional developments” (the space was specially invented by Volkov for this proof). The proof was presented in [29,30] and in details, in [31,32]. Among current publications, we mention the articles [33,34] with a few proofs of Theorem 3 that also base on solving some variational problem and are close to the Volkov proof.

The above arguments involving Theorem 3 and the Weyl problem, show the importance of the concept of abstract development of a polyhedron. Nevertheless, in this article we only deal with the natural developments of polyhedra. The fact is that the problem of recognizing whether a given abstract development is natural or not is in itself very difficult and we do not expect to contribute here to its solution. To justify this fact, let us quote A.D. Alexandrov: “To determine the structure of a polyhedron from a development, i.e., to indicate its genuine edges in the development, is a problem whose general solution seems hopeless” [2, Page 100, Subsection 2.3.3]. As far as we know, even now there is very little progress in solving this problem with “hopeless” general solution.

We are aware of the only particular case described in [35] in which it is possible to directly indicate the genuine edges of a polyhedron on its abstract development. There is proved in [35] that if an abstract development $R$ satisfies the conditions of Theorem 3 and there exists a simple edge cycle $\gamma \subset R$ consisting of the shortest paths connecting in series all vertices of $R$ with positive curvature; and for every two of vertices with positive curvature which are not connected by an edge of $\gamma$ there are exactly two shortest paths connecting them; then the convex polyhedron $P$ existing by Theorem 3 is degenerate in the sense that $P$ is located in a plane, has only two faces, $\gamma$ is the common boundary of these faces, and $\gamma$ consists of all genuine edges of $P$.

At the same time, researchers in various branches of exact sciences proposed quite a few computer algorithms for reconstructing the spatial shape of a convex surface from its intrinsic metric; see, e.g., [33,36–39]. Some of these algorithms work for polyhedra as well. Having computed the spatial shape of a polyhedron, we can claim in a sense that the genuine edges are found as well. However, all of these algorithms are numerical. Among them, the one in [37] is distinguished by a particularly thorough theoretical study. The algorithm bases on solving some differential equation that is derived in [33] and allows one to compute the spatial shape of a polyhedron $P$ with arbitrary precision from an abstract development of $P$. Moreover, [37] yields a pseudopolynomial bound on the running time of the algorithm.

Summarizing what was said above about the relationship between the concepts of abstract and natu-
ral developments, we can say that our desire to deal with natural developments is by no means accidental.

Let us continue to clarify the terminology of this article. We say that two combinatorially equivalent polyhedra $P$ and $P'$ in $\mathbb{R}^3$ are combinatorially-affine equivalent (or co-affine, for short), if there is a non-degenerate affine transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ mapping $P$ onto $P'$ and sending the vertices, edges, and faces of $P$ onto the vertices, edges, and faces of $P'$ corresponding to the former due to the combination equivalence of $P$ and $P'$. In this case, $A$ is also called combinatorial-affine or co-affine, for brevity.

Combinatorially-affine equivalent polygons are defined similarly. For short, we call them co-affine polygons.

We can now clarify the main problem studied in this article:

**Problem 1.** Given two combinatorially equivalent polyhedra $P$ and $P'$ in $\mathbb{R}^3$ whose corresponding faces are co-affine, what additional conditions on natural developments of $P$ and $P'$ guarantee that $P$ and $P'$ are co-affine or, conversely, are not co-affine?

In Sections 3–6, we obtain the partial solutions to Problem 1 for some classes of polyhedra in Euclidean 3-space. Wherein, we will use the so-called Cayley–Menger determinants and their properties. Let us recall the former in a formulation and notation convenient for us.

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Let $k \geq 2$ and let $x_0, x_1, \ldots, x_k$ be arbitrary points in $\mathbb{R}^k$. Denote by $d_{ij} = d(x_i, x_j)$ the Euclidean distance between $x_i$ and $x_j$, $i, j = 0, 1, \ldots, k$. The determinant

$$
\text{cm}(x_0, x_1, \ldots, x_k) \overset{\text{def}}{=} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{01}^2 & \cdots & d_{0k}^2 \\ 1 & d_{10}^2 & 0 & \cdots & d_{1k}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & d_{k0}^2 & d_{k1}^2 & \cdots & 0 \end{vmatrix}
$$

is called the Cayley–Menger determinant of $x_0, x_1, \ldots, x_k$; see, e.g., [40, formula 9.7.3.2]. However, sometimes we will say that (1) defines the Cayley–Menger determinant of the simplex $T$ with vertices $x_0, x_1, \ldots, x_k$, and we will use denotation $\text{cm}(T)$.

We need the two properties of the Cayley–Menger determinant:

(a) The $k$-dimensional volume $\text{vol}(x_0, x_1, \ldots, x_k)$ of the simplex with vertices $x_0, x_1, \ldots, x_k$ is related to the Cayley–Menger determinant of $x_0, x_1, \ldots, x_k$ by the formula [40, Lemma 9.7.3.3]:

$$
[\text{vol}(x_0, x_1, \ldots, x_k)]^2 = \frac{(-1)^{k+1}}{2^k k!} \text{cm}(x_0, x_1, \ldots, x_k).
$$

(b) Suppose given $k(k + 1)/2$ arbitrary positive reals $d_{ij}$ ($i, j = 0, 1, \ldots, k$) such that $d_{ij} = d_{ji}$ and $d_{ii} = 0$. The inequality

$$
(-1)^{k+1} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{11}^2 & \cdots & d_{1k}^2 \\ 1 & d_{20}^2 & 0 & \cdots & d_{2k}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & d_{k0}^2 & d_{k1}^2 & \cdots & 0 \end{vmatrix} > 0
$$

is a necessary and sufficient condition for the existence of a nondegenerate simplex in $\mathbb{R}^k$ (i.e., the simplex not lying in any hyperplane) with vertices $x_0, x_1, \ldots, x_k$ such that $d_{ij}$ is equal to the Euclidean distance between $x_i$ and $x_j$; see [40, Theorem 9.7.3.4, Remark 9.7.3.5, and Exercise 9.14.23].

Finally, let us agree on some notation of use in Sections 3–6 without further explanation. Given a polyhedron $P$, a development $R$ of $P$ (it does not matter whether $R$ is natural or abstract), and a vertex $x$ of $P$, we denote by $\bar{x}$ the vertex of $R$, corresponding to $x$. If $y$ is another vertex of $P$, then we let $d(x, y)$ stand for the Euclidean distance between the points $x$ and $y$ in $\mathbb{R}^3$, and let $\rho(\bar{x}, \bar{y})$ signify the distance between $\bar{x}$ and $\bar{y}$ in $R$. If, along with $P$, we are given a polyhedron $P'$ combinatorially equivalent to $P$, and $R'$ is a development of $P'$; then we denote by $x'$ the vertex of $P'$ corresponding, by combinatorial equivalence to the vertex $x$ of $P$, and by $\bar{x}'$ the vertex of $R'$, corresponding (by the construction of development) to the vertex $x'$ of $P'$. If $y'$ is another vertex of $P'$, then by $d(x', y')$ we denote the Euclidean distance between $x', y' \in \mathbb{R}^3$, and by $\rho'(\bar{x}', \bar{y}')$ the distance between $\bar{x}'$ and $\bar{y}'$ in $R'$. 

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3. Simple Polyhedra

A closed convex polyhedron is strictly convex if none of its dihedral angles is equal to \( \pi \). A closed strictly convex polyhedron is called simple if each of its vertices is incident to exactly three edges (or, which is the same, to exactly three faces). The examples of simple closed polyhedrons are well known: three of five Platonic solids (namely: the tetrahedron, the cube, and the dodecahedron), six of thirteen Archimedean solids (namely: the truncated tetrahedron, the truncated cube, the truncated octahedron, the truncated cuboctahedron, the truncated icosahedron, and the truncated icosidodecahedron (see [41, 42])), and each strictly convex \( n \)-gonal prism for every \( n \geq 3 \).

**Theorem 4.** Let \( P \) and \( P' \) be simple polyhedra in \( \mathbb{R}^3 \) combinatorially equivalent to each other and let the corresponding faces of the natural developments of \( P \) and \( P' \) be co-affine. Then \( P \) and \( P' \) are co-affine.

**Proof.** Given a vertex \( x \) of \( P \), construct an affine transformation \( A_x : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) as follows: Since \( P \) is assumed simple, there are exactly three of the vertices of \( P \) incident to \( x \). Let us denote them by \( x_1, x_2, \) and \( x_3 \). Since \( P \) is assumed strictly convex, the linear span of three vectors \( x_1 - x, x_2 - x, \) and \( x_3 - x \) coincides with \( \mathbb{R}^3 \); i.e., these vectors are linearly independent. Similarly, since \( P' \) is assumed strictly convex, the vectors \( x'_1 - x', x'_2 - x', \) and \( x'_3 - x' \) are also linearly independent. Therefore, there is a unique affine transformation \( A_x : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) such that \( A_x(x_j) = x'_j \) for each \( j = 1, 2, 3 \), and \( A_x(x) = x' \).

Let \( F \) denote an arbitrary face of \( P \) incident to \( x \). For definiteness, let \( F' \) be incident to \( x, x_1, \) and \( x_2 \). Let \( F' \) denote the face of \( P' \) corresponding to \( F \) by combinatorial equivalence. Then \( F' \) is incident to \( x', x'_1, \) and \( x'_2 \). Let us verify that

\[
A_x(F) = F'.
\]  

Indeed, on the natural development of \( P \) there is a polygon corresponding to \( F \). Let us denote it by \( \tilde{F} \). Owing to the construction of the natural development, \( F \) and \( \tilde{F} \) are congruent. Similarly, on the natural development of \( P' \), there is a polygon \( \tilde{F}' \) corresponding to \( F' \). Note that \( \tilde{F}' \) is congruent to \( F' \). By the hypothesis of Theorem 4, \( \tilde{F} \) and \( \tilde{F}' \) are co-affine. Hence the faces \( F \) and \( F' \) are co-affine. This means that there is an affine mapping \( B_F : \text{aff}(F) \rightarrow \text{aff}(F') \) mapping the affine hull \( \text{aff}(F) \) of \( F \) into the affine hull \( \text{aff}(F') \) of \( F' \) and is such that \( B_F(F) = F', B_F(x) = x', B_F(x_1) = x'_1, \) and \( B_F(x_2) = x'_2 \). Since \( A_x(x) = x' = B_F(x), A_x(x_1) = x'_1 = B_F(x_1) \) and \( A_x(x_2) = x'_2 = B_F(x_2), \) the restriction of \( A_x \) to \( \text{aff}(F) \) coincides with \( B_F \). Thus, \( A_x(F) = B_F(F) = F', \) i.e., (4) is proved.

Now let us check that if the vertices \( x \) and \( y \) of the polyhedron \( P \) are connected by an edge then

\[
A_x = A_y.
\]  

To this end, as before we denote by \( x_1, x_2, \) and \( x_3 \) the three vertices of \( P \) incident to \( x \). For definiteness, we assume that \( y = x_3 \). Similarly, let us denote by \( y_1, y_2, \) and \( y_3 \) the three vertices of \( P \) incident to \( y \). Also, we assume that \( x = y_3 \) and the vertices \( x_1, x = y_3, y = x_3, \) and \( y_2 \) are incident to a face of \( P \), while the vertices \( x_2, x = y_3, y = x_3, \) and \( y_1 \) are incident to another face of \( P \). Denote by \( F_1 \) the face of \( P \) incident to \( x_1, x = y_3, y = x_3, \) and \( y_2 \). Let \( \text{aff}(F_1) \) stand for the affine hull of \( F_1 \). Denote by \( F'_1 \) the face of the polyhedron \( P' \) corresponding to \( F_1 \) by virtue of the combinatorial equivalence of \( P \) and \( P' \). Denote by \( \text{aff}(F'_1) \) the affine hull of \( F'_1 \). Arguing as in the previous paragraph, we see that there is an affine mapping \( B_{F_1} : \text{aff}(F_1) \rightarrow \text{aff}(F'_1) \) coinciding with both \( A_x \) and \( A_y \) on \( F_1 \). This implies that \( A_x(x_1) = x'_1 = A_y(x_1) \). In a similar way, using the face \( F_2 \), incident to \( x_2, x = y_3, y = x_3, \) we find that \( A_x(x_2) = x'_2 = A_y(x_2) \). Thus, we showed that the affine transformations \( A_x \) and \( A_y \) coincide with each other at \( x, x_1, x_2, \) and \( x_3 \). Considering that the vectors \( x_1 - x, x_2 - x, \) and \( x_3 - x \) are linearly independent, we obtain (5).

Finally, let us check that the transformation \( A_x \) is the same for all vertices \( x \) of \( P \). Indeed, since \( P \) is connected, every two of its vertices can be connected by an edge path. If the vertex \( y \) follows the vertex \( x \) in this edge path; then, as proved in the previous paragraph, \( A_y = A_x \). Hence, \( A_x \) is the same at the initial and final vertices of the edge path, i.e., \( A_x \) independent of \( x \). Let us denote \( A_x \) by \( A \). Then \( A \) is an affine transformation sending \( P \) to \( P' \) so that the elements of \( P \) arrive to the corresponding elements of \( P' \) by combinatorial equivalence. Thus, \( P \) and \( P' \) are co-affine. \( \square \)
Theorem 4 shows that, for simple polyhedra, the co-affinity of faces already implies the co-affinity of polyhedra. From the above proof of Theorem 4, it is clear that, with a suitable refinement of terminology, similar statements can be formulated and proved both for some nonconvex polyhedra and for projectively equivalent polyhedra. Of all the possible generalizations of Theorem 4, we explicitly state only one applicable not only to simple polytopes.

**Theorem 5.** Let closed strictly convex polyhedra $P$ and $P'$ in $\mathbb{R}^3$ have the same combinatorial structure, and their corresponding faces are co-affine. And let there is an edge path $\gamma$ on $P$ such that
(i) each vertex of $\gamma$ is incident to exactly three edges of $P$, and
(ii) for every face of $P$, there exists a vertex of $\gamma$ incident to this face.
Then $P$ and $P'$ are co-affine.

The proof of Theorem 5 can be carried out similarly to the above proof of Theorem 4. We leave it to the reader as an easy exercise. For every $n \geq 4$, each $n$-gonal trapezohedron may serve as an example of a not simple polyhedron satisfying the conditions of Theorem 5. Recall that an $n$-gonal trapezohedron is the convex polyhedron dual to an $n$-gonal convex antiprism; see [43, 44]. Fig. 2(a) shows a 4-gonal antiprism; Fig. 2(b) shows a 4-gonal trapezohedron; and Fig. 2(c) shows a 5-sided nonclosed edge path $\gamma$ on a 4-gonal trapezohedron satisfying the conditions of Theorem 5.

![Fig. 2.](image)

Fig. 2. (a) A 4-gonal antiprism, (b) a 4-gonal trapezohedron, and (c) the solid broken line represents a 5-sided nonclosed edge path $\gamma$ on a 4-gonal trapezohedron, satisfying the conditions of Theorem 5.

Theorem 4 allows us to prove the following:

**Theorem 6.** Given $n \geq 4$, let $M_n$ be the set of all closed strictly convex polyhedra with $n$ faces in $\mathbb{R}^3$ which is endowed with the Hausdorff metric. There is a dense open subset $\Omega_n \subset M_n$ such that, for all $P, P' \in \Omega_n$, if $P$ and $P'$ are combinatorially equivalent and every two corresponding faces of $P$ and $P'$ are co-affine then so are $P$ and $P'$.

**Proof.** Denote by $\Omega_n$ the set of all simple closed strictly convex polyhedra in $\mathbb{R}^3$ with $n$ faces. Note that $\Omega_n$ is an everywhere dense open subset of $M_n$. Indeed, it suffice to slightly move the planes of the faces parallel to themselves so that only three faces are incident to each vertex. For $P, P' \in \Omega_n$, the conclusion of Theorem 6 follows from Theorem 4. □

In a sense, Theorem 6 is similar to the statement that, in the space of all closed simplicial polyhedra of a given combinatorial structure in $\mathbb{R}^3$, there is a dense open set whose each point corresponds to a rigid polyhedron. This was firstly demonstrated by Gluck in [45]. Recently, Gluck’s proof was adapted to prove the rigidity of almost all triangulated circle polyhedra; see [46].

4. Suspensions

As stated in Section 1, our study of Problem 1 is mainly motivated by the Cauchy Rigidity Theorem. The history of this theorem began in 1794, when in the first edition of the famous textbook by Legendre [47, Note XII, pp. 321–334] it was formulated as a conjecture, while its proofs were given only for
some classes of polyhedra which include octahedra. In [48, §1], Sabitov writes that in the editions from 2nd to 9th the part of Note XII related to the Rigidity Theorem was removed by Legendre, and restored only in the 10th and subsequent editions after Cauchy proved the Rigidity Theorem in the general case [6] in 1813 on significantly using Legendre’s ideas. Therefore, we consider it appropriate to study the various classes of polyhedra for which we can make some progress in solving Problem 1. In doing so, we would first of all like to find the arguments suitable for solving Problem 1 for octahedra.

![Diagram of suspensions](image)

**Fig. 3.** (a) An $n$-gonal suspension for $n = 5$, (b) the tetrahedron $T_j$ with the vertices $x_0, x_j, x_{j+1},$ and $x_{n+1}$ of the suspension is grayed out; the segment $x_0x_{n+1}$, shown by a thin line, is an edge of $T_j$, but not an edge of the suspension.

In this section we study suspensions, i.e., polyhedra combinatorially equivalent to a regular convex $n$-gonal bipyramid for $n \geq 3$. Note that suspensions are not necessarily convex or self-intersection free. From the edges of a suspension $P$ we can form a closed cycle $L$ that passes without repetition through all but two vertices of $P$. Denote the vertices of $L$ by $x_j$, $j = 1, \ldots, n$, labeled in the cyclic order prescribed by $L$. The cycle $L$ is the equator of $P$. Two vertices of $P$ that do not lie on the equator are denoted by $x_0$ and $x_{n+1}$ and called the south and north poles of the suspension. From the above, it is clear that each vertex of the equator is incident to both the south and north poles, while the south and north poles are not incident to each other. For $n = 4$, a suspension is combinatorially equivalent to an octahedron. A 5-gonal suspension is shown in Fig. 3(a). Suspensions play an important role in the theory of flexible polyhedra; for instance, they were studied in the articles [49–55].

Let $P$ be a suspension with $n$ vertices on the equator. Let $T_j$ stand for a tetrahedron with vertices $x_0, x_j, x_{j+1},$ and $x_{n+1}$, if $j = 1, 2, \ldots, n-1$, and with vertices $x_0, x_n, x_1,$ and $x_{n+1}$, if $j = n$; see Fig. 3(b). Note that although the segment $x_0x_{n+1}$ is an edge of $T_j$, it is nevertheless not an edge of $P$. In other words, $x_0x_{n+1}$ is the only edge of $T_j$ whose length cannot be found directly from the natural development of $P$. Define the polynomials $q_j(t)$, $j = 1, 2, \ldots, n$, by the formulas

\[
\begin{vmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & [\rho(\bar{x}_0, \bar{x}_j)]^2 & [\rho(\bar{x}_0, \bar{x}_{j+1})]^2 \\
1 & [\rho(\bar{x}_{j+1}, \bar{x}_0)]^2 & 0 & [\rho(\bar{x}_{j+1}, \bar{x}_{j+1})]^2 \\
1 & t^2 & [\rho(\bar{x}_{n+1}, \bar{x}_j)]^2 & [\rho(\bar{x}_{n+1}, \bar{x}_{j+1})]^2 \\
\end{vmatrix}
\]  

\[
\begin{vmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & [\rho(\bar{x}_n, \bar{x}_0)]^2 & [\rho(\bar{x}_n, \bar{x}_1)]^2 \\
1 & [\rho(\bar{x}_1, \bar{x}_0)]^2 & 0 & [\rho(\bar{x}_1, \bar{x}_1)]^2 \\
1 & t^2 & [\rho(\bar{x}_{n+1}, \bar{x}_0)]^2 & [\rho(\bar{x}_{n+1}, \bar{x}_1)]^2 \\
\end{vmatrix}
\]

(6)
Since $\rho$ denotes the distance in the natural development of $P$; therefore, $\rho(\vec{x}, \vec{y}) = d(x, y)$ for every pair $\{x, y\}$ of vertices of $T_j$ other than the pair $\{x_0, x_{n+1}\}$. Hence, putting $t_* = d(x_0, x_{n+1})$ and comparing formulas (6) and (1), we get

$$q_j(t_*) = \left\{ \begin{array}{ll} \text{cm}(x_0, x_j, x_{j+1}, x_{n+1}), & \text{if } j = 1, 2, \ldots, n - 1, \\ \text{cm}(x_0, x_n, x_1, x_{n+1}), & \text{if } j = n. \end{array} \right. \quad (7)$$

Now comparing (7) and (2), for all $j = 1, 2, \ldots, n$ we get the following expression for the volume of $T_j$:

$$[\text{vol}(T_j)]^2 = \frac{1}{2^33!}q_j(t_*). \quad (8)$$

Let $P'$ be another suspension with $n$ vertices on the equator, for which there is fixed a combinatorial equivalence with the suspension $P$, i.e., there is given a combinatorial equivalence between the vertices (i.e., vertices, edges and faces) of $P'$ onto the set of elements of $P$. As usual, we denote by $x_j'$ the vertex of $P'$ which corresponds to the vertex $x_j$ of $P$ by this combinatorial equivalence. Let $T_j'$ denote the tetrahedron with vertices $x_0', x_j', x_{j+1}'$, and $x_{n+1}'$, if $j = 1, 2, \ldots, n - 1$; and with vertices $x_0', x_n', x_1'$, and $x_{n+1}'$, if $j = n$. Note that segment $x_0'x_{n+1}'$ is the only edge of $T_j'$ whose length cannot be found directly from the natural development of $P'$. We define the polynomials $q_j'(t')$, $j = 1, 2, \ldots, n$, by the formulas

$$q_j'(t') = \left\{ \begin{array}{ll} \text{cm}(x_0', x_j', x_{j+1}', x_{n+1}'), & \text{if } j = 1, 2, \ldots, n - 1, \\ \text{cm}(x_0', x_n', x_1', x_{n+1}'), & \text{if } j = n. \end{array} \right. \quad (9)$$

Since $\rho'$ denotes the distance in the natural development of $P'$; therefore, $\rho'(\vec{x}', \vec{y}') = d(x', y')$ for every pair $\{x', y'\}$ of vertices of $T_j'$ other than the pair $\{x_0', x_{n+1}'\}$. Hence, putting $t_*' = d(x_0', x_{n+1}')$ and comparing formulas (9) and (1), we get

$$q_j'(t_*') = \left\{ \begin{array}{ll} \text{cm}(x_0', x_j', x_{j+1}', x_{n+1}'), & \text{if } j = 1, 2, \ldots, n - 1, \\ \text{cm}(x_0', x_n', x_1', x_{n+1}'), & \text{if } j = n. \end{array} \right. \quad (10)$$

Now comparing (10) and (2), for all $j = 1, 2, \ldots, n$ we get the expression for the volume of $T_j'$:

$$[\text{vol}(T_j')]^2 = \frac{1}{2^33!}q_j'(t_*'). \quad (11)$$

Note that the polynomials $q_j$ and $q_j'$, defined by (6) and (9), are uniquely determined by the natural developments of $P$ and $P'$. Using these polynomials we can formulate the partial answer to Problem 1 for suspensions:

**Theorem 7.** Let $P$ and $P'$ be suspensions in $\mathbb{R}^3$. Suppose that $P$ and $P'$ have the same number of vertices $n \geq 3$ on the equator and, for each $j = 1, 2, \ldots, n$, the polynomials $q_j$ and $q_j'$ are defined by (6) and (9) via the natural developments of $P$ and $P'$. Suppose also that the system of $n \geq 3$ algebraic equations

$$q_j(t') = \delta q_j(t), \quad j = 1, 2, \ldots, n,$$

$$277$$
for the three variables $\delta$, $t$, and $t'$ has no positive real solution (i.e., a solution such that $\delta > 0$, $t > 0$, and $t' > 0$). Then $P$ and $P'$ are not co-affine.

**Proof.** We argue by contradiction. Suppose that $P$ and $P'$ are co-affine and denote by $A : \mathbb{R}^3 \to \mathbb{R}^3$ the corresponding co-affine transformation (in particular, the latter means that $P' = A(P)$). By definition, put $\delta_* = (\det A)^2$, $t_* = d(x_0, x_{n+1})$ and $t'_* = d(x_0', x'_{n+1})$. According to (11) and (8), for every $j = 1, 2, \ldots, n$, we have

$$q_j'(t'_*) = 2^3 \cdot 3! \cdot [\text{vol}(T'_j)]^2 = 2^3 \cdot 3! \cdot (\det A)^2 [\text{vol}(T_j)]^2 = \delta_* q_j(t_*)$$

Therefore, the triplet of the positive reals $\delta_*$, $t_*$, and $t'_*$ is a solution to (12). Thus, we have come to a contradiction with the conditions of Theorem 7, which completes the proof. \qed

5. The Local Realizability Problem for a Natural Development

Some algorithm is proposed in [50] for determining if a given suspension is flexible. This fact enticed us to look for an algorithmic solution to Problem 1. In Section 6, we describe the algorithm that receives two combinatorially equivalent natural developments as input, and as output either guarantees that no polyhedra with the natural developments are co-affine, or responds that it cannot give such guarantee co-affine. Section 5 is devoted to the necessary preparatory work.

Recall that, in Section 2, we denoted by $\tilde{x}$ the vertex of the natural development of a polyhedron corresponding to the vertex $x$ of the polyhedron. In Sections 5 and 6, we use the same notation even if only the development is given and the polyhedron has yet to be found (or it has to be proved that the required polyhedron does not exist).

A set $Z$ of three faces of an abstract development $R$ is a *patch* of $R$ incident to a vertex $\tilde{x}_0$ of $R$ if each of the faces of $Z$ is incident to $\tilde{x}_0$ and $Z$ is connected; i.e., we can go from each face of $Z$ to any other face of $Z$ by crossing not vertices but edges of $Z$. The proofs of the following properties of patches, being straightforward, are left to the reader: Every patch is a development; every patch is homeomorphic to a disk; $\tilde{x}_0$ can be either an inner or a boundary point of a patch incident to $\tilde{x}_0$.

Our approach to some algorithmic solution of Problem 1 bases on reducing the latter to the following:

**Problem 2.** Let abstract developments $R$ and $R'$ be combinatorially equivalent to each other, let $\tilde{x}_0$ be a vertex of $R$, let $Z$ be a patch of $R$ incident to $\tilde{x}_0$, and let $Z'$ be the patch of $R'$ corresponding to $Z$ by the combinatorial equivalence of $R$ and $R'$. Suppose also that each face of $Z$ is co-affine to the corresponding face of $Z'$. Are there polyhedra $P$ and $P'$ in $\mathbb{R}^3$ such that

(i) $P$ and $P'$ are homeomorphic to the disk;
(ii) $Z$ is the natural development of $P$;
(iii) $Z'$ is the natural development of $P'$;
(iv) $P$ and $P'$ are co-affine?

For the convenience of speech, property (ii) is expressed by the words “a natural development $Z$ is isometrically realized as a polyhedron $P$.” Note that Problem 2 deals with the possibility of realizing only patches $Z$ and $Z'$ as co-affine polyhedra, regardless of whether or not it is possible to realize the “ambient” abstract developments $R$ and $R'$. Therefore, Problem 2 can be called local.

Let $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$ be sequentially numbered vertices of a patch $Z$ incident to a vertex $\tilde{x}_0$ of an abstract development $R$, and let $n$ be the total number of vertices of $R$ incident to $\tilde{x}_0$. Let us study Problem 2 for each of the following cases separately: $n = 3$ (Fig. 4(a)); $n = 4$ (Fig. 4(b)); and $n \geq 5$ (Fig. 4(c)).
Therefore, the Cayley–Menger determinant (1) of \( x \) by (13) is positive. Moreover, in the case two adjacent faces of any polyhedron lie in the same plane. Therefore, according to (3), the real parameters \( r, s, \) and \( t \). (a) The case \( n = 3 \), (b) the case \( n = 4 \), and (c) the case \( n \geq 5 \).

\[ \text{Case } n = 3. \text{ Suppose that there is a polyhedron } P \text{ in } \mathbb{R}^3 \text{ for which } Z \text{ is the natural development of } P. \text{ Then } P \text{ includes the star of a vertex } x_0 \text{ of a nondegenerate tetrahedron } T \text{ with vertices } x_0, x_1, x_2, \text{ and } x_3. \text{ As usual, let us denote by } \rho(\bar{x}, \bar{y}) \text{ the distance in } R \text{ between points } \bar{x} \text{ and } \bar{y}. \text{ Since we assumed that } Z \text{ is the natural development of } P, \rho(\bar{x}, \bar{y}) = d(x, y) \text{ for every pair } \{x, y\} \text{ of vertices of } T, \text{ where } d(x, y) \text{ denotes the Euclidean distance between } x, y \in \mathbb{R}^3. \text{ (There is no need to assume that } x \text{ and } y \text{ are connected by an edge of } P; \text{ it suffices that they are incident to one and the same face of } P.) \text{ Therefore, the Cayley–Menger determinant (1) of } T \text{ can be expressed in terms of the distances on } R:\]

\[ \text{cm}(x_0, x_1, x_2, x_3) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & [d(x_0, x_1)]^2 & [d(x_0, x_3)]^2 \\ 1 & [d(x_1, x_0)]^2 & 0 & [d(x_1, x_3)]^2 \\ 1 & [d(x_2, x_0)]^2 & [d(x_2, x_1)]^2 & 0 \\ 1 & [d(x_3, x_0)]^2 & [d(x_3, x_1)]^2 & [d(x_3, x_2)]^2 \end{vmatrix} \text{ def } 3q. \tag{13} \]

Note that \( T \) is nondegenerate because we assumed that \( Z \) is the natural development of \( P \) and because no two adjacent faces of any polyhedron lie in the same plane. Therefore, according to (3), the real \( 3q \) defined by (13) is positive. Moreover, in the case \( n = 3 \), the inequality \( 3q > 0 \) is necessary and sufficient for \( Z \) to be isometrically realizable as the natural development of some polyhedron \( P \); i.e., for property (ii) from the statement of Problem 2 to be fulfilled.

Similarly, putting by definition

\[ \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & [\rho'(\bar{x}_0, \bar{x}_1')]^2 & [\rho'(\bar{x}_0, \bar{x}_3')]^2 \\ 1 & [\rho'(\bar{x}_1, \bar{x}_0')]^2 & 0 & [\rho'(\bar{x}_1, \bar{x}_3')]^2 \\ 1 & [\rho'(\bar{x}_2, \bar{x}_0')]^2 & [\rho'(\bar{x}_2, \bar{x}_1')]^2 & 0 \\ 1 & [\rho'(\bar{x}_3, \bar{x}_0')]^2 & [\rho'(\bar{x}_3, \bar{x}_1')]^2 & [\rho'(\bar{x}_3, \bar{x}_2')]^2 \end{vmatrix} = 3q', \tag{14} \]

we make sure that the inequality \( 3q' > 0 \) is necessary and sufficient for \( Z' \) to be isometrically realizable as the natural development of some polyhedron \( P' \); i.e., for property (iii) from the statement of Problem 2 to be fulfilled.

![Fig. 4. Schematic representation of the star (grayed out) of a vertex \( \bar{x}_0 \) of valency \( n \) in an abstract development \( R \) (part of which is shown by solid lines) and a patch \( Z \) incident to \( \bar{x}_0 \) (depicted in bold lines). The dotted lines show the segments that are not edges of either \( R \) or \( Z \); their lengths are taken as "free parameters" \( r, s, \) and \( t \). (a) The case \( n = 3 \), (b) the case \( n = 4 \), and (c) the case \( n \geq 5 \).](image-url)
It was already noted that the tetrahedron $T$ with vertices $x_0$, $x_1$, $x_2$, and $x_3$ is nondegenerate, i.e., $T$ does not lie in any plane in $\mathbb{R}^3$. Similarly, we can assert that the tetrahedron $T'$ with vertices $x'_0$, $x'_1$, $x'_2$, and $x'_3$ is nondegenerate too. Hence, there is a uniquely determined affine transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $A(x_j) = x'_j$ for all $j = 0, \ldots, 3$. The fact that $A$ maps $P$ onto $P'$ was already established in the proof of Theorem 4. Moreover, we can find the absolute value of the Jacobian of $A$ by the formula

$$| \det A| = \text{vol}(T'_j)/\text{vol}(T_j) = \sqrt{3}q'/\sqrt{3}q.$$  

This formula shows that, for $n = 3$, the assignment of the patches $Z$ and $Z'$ uniquely determines the value of $| \det A|$. Thus, we proved the following

**Lemma 1.** In the case $n = 3$, the answer to Problem 2 is positive if and only if both reals $3q$ and $3q'$ given by (13) and (14) are positive.

**Proof.** The “if” statement was proven above. To check the converse, we need to argue likewise in reverse order. □

**Case $n = 4$.** The corresponding patch $Z$ is shown schematically in Fig. 4(b). Suppose that there is a polyhedron $P$ in $\mathbb{R}^3$ such that $Z$ is the natural development of $P$. Then $P$ contains the star of a vertex $x_0$ of an octahedron $O$ for which $x_0$ is the south pole, and the points $x_1$, $x_2$, $x_3$, and $x_4$ lie on the equator. Since we assumed that $Z$ is the natural development of $P$; therefore, $\rho(\bar{x}, y) = d(x, y)$ for every two-point set $\{x, y\} \subset \{x_0, x_1, x_2, x_3, x_4\}$, except for the pairs $\{x_1, x_3\}$ and $\{x_2, x_4\}$. (There is no need to assume that $x$ and $y$ are connected by an edge of $P$; it suffices that they are incident to one and the same face of $P$.) The segments $x_1x_3$ and $x_2x_4$ are usually called *small diagonals* of $P$. Their lengths in $\mathbb{R}^3$ will be denoted by $t$ and $s$, respectively, and will be called *free parameters*, since they are not directly expressed in terms of $R$.

Given $j = 0, 1, \ldots, 4$, we denote by $T_j$ the tetrahedron that is the convex hull of the set $\{x_0, x_1, x_2, x_3, x_4\}\{x_j\}$. In the Cayley–Menger determinant (1) for $T_j$, replace the Euclidean distances $d_{ij} = d(x_i, x_j)$ by the following:

$$d(x_i, x_j) = \begin{cases} \rho(\bar{x}_i, \bar{x}_j), & \text{if } (i, j) \notin \{(1, 3), (2, 4), (3, 1), (4, 2)\}, \\ t, & \text{if } (i, j) \in \{(1, 3), (3, 1)\}, \\ s, & \text{if } (i, j) \in \{(2, 4), (4, 2)\}. \end{cases}$$

The resulting polynomial depends on either $t$, or $s$, or both free parameters $t$ and $s$ at once. As a result, we get:

$$4q_0(t, s) \overset{\text{def}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \rho(\bar{x}_1, \bar{x}_2) & \rho(\bar{x}_1, \bar{x}_4) \\ 1 & \rho(\bar{x}_2, \bar{x}_1)^2 & 0 & \rho(\bar{x}_2, \bar{x}_3)^2 \\ 1 & \rho(\bar{x}_4, \bar{x}_1)^2 & s^2 & \rho(\bar{x}_4, \bar{x}_3)^2 \end{vmatrix},$$

$$4q_1(s) \overset{\text{def}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \rho(\bar{x}_0, \bar{x}_2)^2 & \rho(\bar{x}_0, \bar{x}_4)^2 \\ 1 & \rho(\bar{x}_2, \bar{x}_0)^2 & 0 & \rho(\bar{x}_2, \bar{x}_3)^2 \\ 1 & \rho(\bar{x}_4, \bar{x}_0)^2 & s^2 & \rho(\bar{x}_4, \bar{x}_3)^2 \end{vmatrix},$$

$$4q_2(t) \overset{\text{def}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \rho(\bar{x}_0, \bar{x}_1)^2 & \rho(\bar{x}_0, \bar{x}_4)^2 \\ 1 & \rho(\bar{x}_1, \bar{x}_0)^2 & 0 & \rho(\bar{x}_1, \bar{x}_3)^2 \\ 1 & \rho(\bar{x}_4, \bar{x}_0)^2 & t^2 & \rho(\bar{x}_4, \bar{x}_3)^2 \end{vmatrix},$$

$$4q_3(s) \overset{\text{def}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \rho(\bar{x}_0, \bar{x}_1)^2 & \rho(\bar{x}_0, \bar{x}_4)^2 \\ 1 & \rho(\bar{x}_1, \bar{x}_0)^2 & 0 & \rho(\bar{x}_1, \bar{x}_3)^2 \\ 1 & \rho(\bar{x}_4, \bar{x}_0)^2 & t^2 & \rho(\bar{x}_4, \bar{x}_3)^2 \end{vmatrix},$$

$$4q_4(t) \overset{\text{def}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \rho(\bar{x}_0, \bar{x}_1)^2 & \rho(\bar{x}_0, \bar{x}_4)^2 \\ 1 & \rho(\bar{x}_1, \bar{x}_0)^2 & 0 & \rho(\bar{x}_1, \bar{x}_3)^2 \\ 1 & \rho(\bar{x}_4, \bar{x}_0)^2 & t^2 & \rho(\bar{x}_4, \bar{x}_3)^2 \end{vmatrix},$$

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A: plane. Therefore, we have expect that in a "generic situation" it has only finitely many solutions. By definition, put polynomials distances letters has a solution the natural development of Note that system (22) contains 5 equations for 5 unknowns Put Suppose The above results can be summarized as follows: Given \( j = 0, 1, \ldots, 4 \), denote by \( T'_j \) the tetrahedron that is the convex hull of \( \{ x'_0, x'_1, x'_2, x'_3, x'_4 \} \backslash \{ x'_j \} \). By definition, put \( d'_{ij} = d(x'_i, x'_j) \). In the Cayley–Menger determinant (1) for \( T'_j \), replace the Euclidean distances \( d_{ij} = d(x_i, x_j) \) by the following:

\[
d(x'_i, x'_j) = \begin{cases} 
\rho'(x'_i, x'_j), & \text{if } (i, j) \notin \{(1, 3), (2, 4), (3, 1), (4, 2)\}, \\
t', & \text{if } (i, j) \in \{(1, 3), (3, 1)\}, \\
s', & \text{if } (i, j) \in \{(2, 4), (4, 2)\}.
\end{cases}
\]

Here \( \rho' \) denotes the distance in \( R' \), while \( t' \) and \( s' \) are new free parameters. As a result, we get the polynomials \( 4q_0(t', s') \), \( 4q_1(s') \), \( 4q_2(t') \), \( 4q_3(s') \), and \( 4q_4(t') \) given by formulas (16)–(20), in which the letters \( \rho, \bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, t, \) and \( s \) are replaced by the same letters endowed with the prime.

If the answer to Problem 2 is positive, then there exists a uniquely determined affine transformation \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) such that \( A(x_j) = x'_j \) for all \( j = 0, \ldots, 4 \). This follows from our assumptions: Patch \( Z \) is the natural development of \( P \); the faces of \( P \) are nondegenerate; and no adjacent faces lie in the same plane. Therefore, we have

\[
\begin{align*}
|\det A|^2 &= \frac{(\text{vol} T'_0)^2}{(\text{vol} T_0)^2} = \frac{4q_0(d'_{13}, d'_{24})}{4q_0(d_{13}, d_{24})}, \\
|\det A|^2 &= \frac{(\text{vol} T'_1)^2}{(\text{vol} T_1)^2} = \frac{4q_1(d'_{24})}{4q_1(d_{24})}, \\
|\det A|^2 &= \frac{(\text{vol} T'_2)^2}{(\text{vol} T_2)^2} = \frac{4q_2(d'_{13})}{4q_2(d_{13})}, \\
|\det A|^2 &= \frac{(\text{vol} T'_3)^2}{(\text{vol} T_3)^2} = \frac{4q_3(d'_{24})}{4q_3(d_{24})}, \\
|\det A|^2 &= \frac{(\text{vol} T'_4)^2}{(\text{vol} T_4)^2} = \frac{4q_4(d'_{13})}{4q_4(d_{13})}.
\end{align*}
\]

The above results can be summarized as follows:

**Lemma 2.** Suppose \( n = 4 \) and the answer to Problem 2 is positive. Then the system of algebraic equations

\[
\begin{align*}
4q_0(t', s') &= 4q_0(t, s)\alpha, \\
4q_1(s') &= 4q_1(s)\alpha, \\
4q_2(t') &= 4q_2(t)\alpha, \\
4q_3(s') &= 4q_3(s)\alpha, \\
4q_4(t') &= 4q_4(t)\alpha
\end{align*}
\]

has a solution \( \alpha, t, s, t', s' \) such that \( \alpha > 0, t > 0, s > 0, t' > 0, \) and \( s' > 0 \).

**Proof.** Put \( \alpha = |\det A|^2, t = d_{13}, s = d_{24}, t' = d'_{13}, \) and \( s' = d'_{24} \). Then use formulas (21). \( \square \)

Note that system (22) contains 5 equations for 5 unknowns \( \alpha, t, s, t', \) and \( s' \). Therefore, we can be expect that in a "generic situation" it has only finitely many solutions.

**Case** \( n \geq 5 \). The corresponding patch \( Z \) is shown schematically in Fig. 4(c). Suppose that there is a polyhedron \( P \) in \( \mathbb{R}^3 \) for which \( Z \) is the natural development of \( P \). Then the star of the vertex \( x_0 \)
of \( P \) contains the vertices \( x_j, j = 1, \ldots, 5 \), and each segment \( x_0x_j \) is an edge of \( P \). Since we assumed that \( Z \) is the natural development of \( P \); therefore, \( \rho(\tilde{x}, \tilde{y}) = d(x, y) \) for every two-point set \( \{x, y\} \subset \{x_0, x_1, x_2, x_3, x_4\} \), except for the pairs \( \{x_1, x_3\}, \{x_2, x_4\}, \text{ and } \{x_1, x_4\} \). (There is no need to assume that \( x \) and \( y \) are connected by an edge of \( P \); it suffices that they are incident to one and the same face of \( P \).) As usual, the segments \( x_1x_3, x_2x_4, \) and \( x_1x_4 \) are called \emph{small diagonals} of \( P \). Their lengths in \( \mathbb{R}^3 \) will be denoted by \( t, s, \) and \( r \) respectively, and will be called \emph{free parameters}, since they are not directly expressed in terms of distances in \( R \).

Given \( j = 0, 1, \ldots, 4 \), we denote by \( T_j \) the tetrahedron that is the convex hull of \( \{x_0, x_1, x_2, x_3, x_4\} \setminus \{x_j\} \). In the Cayley–Menger determinant (1) for \( T_j \), replace the Euclidean distances \( d_{ij} = d(x_i, x_j) \) by the following expressions:

\[
d(x_i, x_j) = \begin{cases} 
\rho(\tilde{x}_i, \tilde{x}_j), & \text{if } (i, j) \notin \{(1, 3), (1, 4), (2, 4), (3, 1), (4, 1), (4, 2)\}, \\
t, & \text{if } (i, j) \in \{(1, 3), (3, 1)\}, \\
s, & \text{if } (i, j) \in \{(2, 4), (4, 2)\}, \\
r, & \text{if } (i, j) \in \{(1, 4), (4, 1)\}.
\end{cases}
\]

The resulting polynomial depends on one or few free parameters \( t, s, \) and \( r \).

\[
5q_0(t, s, r) \overset{\text{def}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & [\rho(\tilde{x}_1, \tilde{x}_2)]^2 & t^2 \\ 1 & t^2 & [\rho(\tilde{x}_3, \tilde{x}_2)]^2 & s^2 \\ 1 & r^2 & [\rho(\tilde{x}_4, \tilde{x}_3)]^2 & 0 \end{vmatrix},
\]

(23)

\[
5q_1(s) \overset{\text{def}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & [\rho(\tilde{x}_0, \tilde{x}_2)]^2 & t^2 \\ 1 & [\rho(\tilde{x}_2, \tilde{x}_0)]^2 & [\rho(\tilde{x}_0, \tilde{x}_3)]^2 & s^2 \\ 1 & [\rho(\tilde{x}_3, \tilde{x}_0)]^2 & [\rho(\tilde{x}_2, \tilde{x}_3)]^2 & 0 \end{vmatrix},
\]

(24)

\[
5q_2(t, r) \overset{\text{def}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & [\rho(\tilde{x}_0, \tilde{x}_1)]^2 & r^2 \\ 1 & t^2 & [\rho(\tilde{x}_1, \tilde{x}_0)]^2 & s^2 \\ 1 & [\rho(\tilde{x}_4, \tilde{x}_0)]^2 & [\rho(\tilde{x}_1, \tilde{x}_3)]^2 & 0 \end{vmatrix},
\]

(25)

\[
5q_3(s, r) \overset{\text{def}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & [\rho(\tilde{x}_0, \tilde{x}_1)]^2 & r^2 \\ 1 & [\rho(\tilde{x}_1, \tilde{x}_0)]^2 & [\rho(\tilde{x}_2, \tilde{x}_1)]^2 & s^2 \\ 1 & [\rho(\tilde{x}_4, \tilde{x}_0)]^2 & [\rho(\tilde{x}_2, \tilde{x}_3)]^2 & 0 \end{vmatrix},
\]

(26)

\[
5q_4(t) \overset{\text{def}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & [\rho(\tilde{x}_0, \tilde{x}_1)]^2 & t^2 \\ 1 & r^2 & [\rho(\tilde{x}_1, \tilde{x}_0)]^2 & s^2 \\ 1 & [\rho(\tilde{x}_3, \tilde{x}_0)]^2 & [\rho(\tilde{x}_1, \tilde{x}_2)]^2 & 0 \end{vmatrix}.
\]

(27)

The five points \( x_0, x_1, x_2, x_3, \) and \( x_4 \) lying in \( \mathbb{R}^3 \) can be considered as the vertices of a degenerate 4-dimensional simplex \( T \). Its 4-dimensional volume is equal to zero. Therefore, replacing the edge lengths in the Cayley–Menger determinant (1) for \( T \) by either the corresponding distances in the patch \( Z \) or the
Given $j = 0, 1, \ldots, 4$, denote by $T'_j$ the tetrahedron that is the convex hull of $\{x'_0, x'_1, x'_2, x'_3, x'_4\} \setminus \{x_j'\}$. In the Cayley–Menger determinant (1) for $T'_j$ and $T'$, replace the Euclidean distances $d'_i = d(x'_i, x'_j)$ by the following:

$$d(x'_i, x'_j) = \begin{cases} 
\rho'(\widetilde{x}'_i, \widetilde{x}'_j), & \text{if } (i, j) \notin \{(1, 3), (1, 4), (2, 4), (3, 1), (4, 1), (4, 2)\}, \\
t', & \text{if } (i, j) \in \{(1, 3), (3, 1)\}, \\
s', & \text{if } (i, j) \in \{(2, 4), (4, 2)\}, \\
r', & \text{if } (i, j) \in \{(1, 4), (4, 1)\}.
\end{cases}$$

As a result, we get the polynomials $5q_0(t', s', r')$, $5q_1(s')$, $5q_2(t', r')$, $5q_3(s', r')$, $5q_4(t')$, and $5q'(t', s', r')$, given by formulas (23)–(28), in which the letters $\rho$, $\widetilde{x}_0$, $\widetilde{x}_1$, $\widetilde{x}_2$, $\widetilde{x}_3$, $\widetilde{x}_4$, $t$, $s$, and $r$ are replaced by the same letters endowed with the prime.

If the answer to Problem 2 for a patch $Z$ is positive, then there exists a uniquely determined affine transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $A(x_j) = x'_j$ for all $j = 0, \ldots, 4$. This follows from our assumptions: Patch $Z$ is the natural development of $P$; the faces of $P$ are nondegenerate; and no adjacent faces lie in the same plane. Therefore,

$$\begin{align*}
|\det A|^2 &= \frac{(\text{vol } T'_0)^2}{(\text{vol } T_0)^2} = \frac{5q_0(d'_1, d'_2, d'_3, d'_4)}{q_0(d_{13}, d_{24}, d_{14})}, \\
|\det A|^2 &= \frac{(\text{vol } T'_1)^2}{(\text{vol } T_1)^2} = \frac{5q_1(d'_2)}{q_1(d_{24})}, \\
|\det A|^2 &= \frac{(\text{vol } T'_2)^2}{(\text{vol } T_2)^2} = \frac{5q_2(d'_3, d'_4)}{q_2(d_{13}, d_{14})}, \\
|\det A|^2 &= \frac{(\text{vol } T'_3)^2}{(\text{vol } T_3)^2} = \frac{5q_3(d'_4)}{q_3(d_{24}, d_{14})}, \\
|\det A|^2 &= \frac{(\text{vol } T'_4)^2}{(\text{vol } T_4)^2} = \frac{5q_4(d'_1)}{q_4(d_{13})}.
\end{align*}$$

The above results for the case $n \geq 5$ can be summarized as follows:

**Lemma 3.** Suppose $n \geq 5$ and the answer to Problem 2 is positive. Then the system of algebraic equations

$$\begin{align*}
5q_0(t', s', r') &= 5q_0(t, s, r)\alpha, \\
5q_1(s') &= 5q_1(s)\alpha, \\
5q_2(t', r') &= 5q_2(t, r)\alpha, \\
5q_3(s', r') &= 5q_3(s, r)\alpha, \\
5q_4(t') &= 5q_4(t)\alpha, \\
5q(t, s, r) &= 0, \\
5q'(t', s', r') &= 0
\end{align*}$$

has a solution $\alpha, t, s, r, t', s'$, and $r'$ such that $\alpha > 0$, $t > 0$, $s > 0$, $r > 0$, $t' > 0$, $s' > 0$, and $r' > 0$.

**Proof.** Put $\alpha = |\det A|^2$, $t = d_{13}$, $s = d_{24}$, $r = d_{14}$, $t' = d'_{13}$, $s' = d'_{24}$, and $r' = d'_{14}$. Then use formulas (29). □
Note that system (30) contains 7 equations for the 7 unknowns $\alpha$, $t$, $s$, $r$, $t'$, $s'$, and $r'$. Therefore, we can expect that in a “generic situation” (30) has only finitely many solutions.

The results on Problem 2 in Section 5 can be summarized as follows:

**Theorem 8.** Let $R$ and $R'$ be the natural developments of combinatorially equivalent polyhedra $P$ and $P'$ in $\mathbb{R}^3$. Suppose that $P$ and $P'$ are co-affine and $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the corresponding co-affine map. Then there exists a positive real $\alpha_*$ such that

(i) For every patch $Z$ incident to an arbitrary vertex $\bar{x}_0$ of valency 3 of $R$, the equality $\alpha_* = \sqrt{3q/3q'}$ holds true, where the expressions $3q$ and $3q'$ are given by the formulas (13) and (14) respectively.

(ii) For every patch $Z$ incident to an arbitrary vertex $\bar{x}_0$ of valency 4 of $R$, the system of algebraic equations (22) has a solution $\alpha$, $t$, $s$, $r$, $t'$, $s'$, and $r'$ such that $\alpha = \alpha_*$ and $t > 0$, $s > 0$, $t' > 0$, $s' > 0$.

(iii) For every patch $Z$ incident to an arbitrary vertex $\bar{x}_0$ of valency $\geq 5$ of $R$, the system of algebraic equations (30) has a solution $\alpha$, $t$, $s$, $r$, $t'$, $s'$, and $r'$ such that $\alpha = \alpha_*$ and $t > 0$, $s > 0$, $r > 0$, $t' > 0$, $s' > 0$, $r' > 0$.

**Proof.** By definition, put $\alpha_* = |\det A|$. Claim (i) follows from (15). Then, according to Lemma 1, both numbers $3q$ and $3q'$ are strictly positive. Claims (ii) and (iii) follow from Lemmas 2 and 3, respectively. □

Note that, in Theorem 8, polyhedra $P$ and $P'$ are not assumed to be convex or closed.

6. Algorithmic Solution to the Problem of Recognition of Co-Affine Polyhedra from Their Natural Developments

Let $P$ and $P'$ be combinatorially equivalent polyhedra in $\mathbb{R}^3$ which are not assumed convex, closed, or co-affine. And let $R$ and $R'$ be the natural developments of $P$ and $P'$. Let us organize an item-by-item examination of all vertices $\bar{x}_0$ of $R$ and all patches $Z$ of $R$ incident to $\bar{x}_0$.

- If $\bar{x}_0$ is a vertex of valency 3, then we put $\alpha(\bar{x}_0, Z) = \sqrt{3q/3q'}$, where expressions $3q$ and $3q'$ are defined by (13) and (14), respectively. (Note that numerical values of $3q$ and $3q'$ are strictly positive. According to Lemma 1, this follows from our assumption that $P$ and $P'$ do actually exist, and $R$ and $R'$ are their natural developments.)

- If $\bar{x}_0$ is a vertex of valency 4, then we distinguish the two cases: If the system of algebraic equations (22) has a solution $\alpha$, $t$, $s$, $r$, $t'$, and $s'$ such that $\alpha > 0$, $t > 0$, $s > 0$, $t' > 0$, and $s' > 0$, then we put $\alpha(\bar{x}_0, Z) = \alpha$; and if (22) has no solution with the indicated properties, then we put $\{\alpha(\bar{x}_0, Z)\} = \emptyset$.

- If $\bar{x}_0$ is a vertex of valency $n \geq 5$, then we again distinguish the two cases: If (30) has a solution $\alpha$, $t$, $s$, $r$, $t'$, $s'$, and $r'$ such that $\alpha > 0$, $t > 0$, $s > 0$, $r > 0$, $t' > 0$, $s' > 0$, then we put $\alpha(\bar{x}_0, Z) = \alpha$; and if (30) has no solution with the indicated properties, then we put $\{\alpha(\bar{x}_0, Z)\} = \emptyset$.

Finally, we find the set $\bigcap\{\alpha(\bar{x}_0, Z)\}$, i.e., the intersection of the sets $\{\alpha(\bar{x}_0, Z)\}$, when $\bar{x}_0$ runs through the set of all vertices of $R$ and $Z$ runs through the set of all patches of $R$ incident to $\bar{x}_0$.

Basing on Theorem 8, we assert that if $\bigcap\{\alpha(\bar{x}_0, Z)\} = \emptyset$, then $P$ and $P'$ are not co-affine.

Note that the algorithm of Section 6 includes, as subproblems, the solving of the systems of polynomial equations (22) and (30) in the set of positive reals $\mathbb{R}_+$. It follows from the Tarski–Seidenberg Theorem [56, 57] that we can eliminate the variables $\alpha$, $t$, $s$, $r$, $t'$, $s'$, and $r'$ from (22) and (30) so that, as a result, the solvability of (22) and (30) in $\mathbb{R}_+$ is equivalent to the validity of some quantifier-free Boolean formula whose atomic formulas are polynomial equations or inequalities containing only the quantities directly accessible from the natural developments $R$ and $R'$, i.e., containing only the distances $\rho(\bar{x}_i, \bar{x}_j)$ and $\rho'(\bar{x}'_i, \bar{x}'_j)$ between the points lying on the same faces. There are even general algorithms that implement this process of eliminating variables. So we can say that subproblems (22) and (30) are algorithmically solvable. However, the computational complexity of those algorithms is so high that they cannot be used in modern computers. Therefore, we expect that in the practical application of our results, the solvability of systems (22) and (30) in $\mathbb{R}_+$ will be studied numerically.
7. Concluding Remarks

In 2022 there was published article [58] in which Problem 1 was considered only for convex octahedra in $\mathbb{R}^3$. Initially, [58] was conceived by the author as a continuation of the present article, but was published earlier. In [58], basically the same ideas are used as in the present article. The main result of [58] is that for convex octahedra the necessary conditions for co-affinity, expressed in terms of natural developments, are also sufficient. At the same time, the theorems of the present article and [58] are much inferior to the Cauchy Rigidity Theorem. In order to obtain a more satisfactory solution to Problem 1, some fundamentally new ideas seem to be required.

At the suggestion of the author, A.V. Sherstobitov, who at that time planned to enroll in graduate school in geometry, proved in 2016 several statements similar to Theorems 4 and 5. However, his plan remained unrealized and the statements remained unpublished.

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