Non-conservative Evolution of Cataclysmic Variables

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Abstract—We suggest a new mechanism to account for the loss of angular momentum in binaries with non-conservative mass exchange. It is shown that in some cases the loss of matter can result in increase of the orbital angular momentum of a binary. If included into consideration in evolutionary calculations, this mechanism appreciably extends the range of mass ratios of components for which mass exchange in binaries is stable. It becomes possible to explain the existence of some observed cataclysmic binaries with high donor/accretor mass ratio, which was prohibited in conservative evolution models.

1 Introduction

Cataclysmic variables (CVs) are close binary systems consisting of a low-mass main-sequence (MS) star that fills its Roche lobe and a white dwarf. Main-sequence star (mass-donating star, usually referred to as star 2 or secondary) loses matter through the vicinity of the inner Lagrangian point \( L_1 \). White dwarf (accretor star, referred to as star 1 or primary) accretes at least a fraction of this matter via accretion disk or via accretion columns in the polar zones in the case when white dwarf has a strong magnetic field.

Physics and evolution of CVs as well as evolution of similar to them low-mass X-ray binaries were investigated starting from the late sixties (see, e.g., [1–10] and references therein). These studies were motivated by the recognition of the
fact that evolution of CVs is determined by losses of orbital angular momentum due to radiation of gravitational waves [11,12] and magnetically coupled stellar wind [13–15]. A number of studies investigated the influence of the loss of angular momentum associated with the material outflow from the system on the CV evolution (see, e.g., [17–19]). However, in the absence of gas dynamical simulations of the mass transfer in the binaries, these processes were considered under the parametric approach.

Recent three-dimensional (3D) gas dynamical simulations of the structure of gaseous flows in semi-detached binaries proved an important role of a circumbinary envelope (see, e.g. [20,21]). These calculations also show that during mass exchange a significant fraction of matter leaves the system. The present work is mainly devoted to the numerical investigation of the evolution of CVs using the data of 3D gas dynamical calculations on the losses of mass and angular momentum from close binaries. The main attention is paid to the stability of mass exchange against runaway mass loss and its dependence on the donor to accretor mass ratio $q = M_2/M_1$.

It is known that the loss of matter by the donor star results in the violation of its hydrostatic and thermal equilibrium. Hydrostatic equilibrium is restored adiabatically, i.e. in dynamical time scale, while thermal equilibrium is restored in thermal time scale. This transition to a new equilibrium state leads to a change of donor radius. The sign of radius variation depends on the convective and radiative stability of outer envelope of the star. For stars with masses $M < \sim M_\odot$ with deep convective envelope as well as for white dwarfs mass loss results in increase of the radius, while for stars with radiative envelopes it leads to the shrinkage of the star. Mass exchange in close binary is unstable when in the course of evolution mass-losing star tends to overfill the Roche lobe. It can occur when radius of the donor $R_2$ increases faster (or decreases slower) than the effective radius of Roche lobe $R_{RL}$.

The case when the radius of the donor increases while the radius of the Roche lobe decreases is also possible. Thus, the question of stability of mass exchange against runaway is determined by the balance of derivatives $\partial R_2/\partial M_2$ and $\partial R_{RL}/\partial M_2$. It means that unstable mass exchange is possible even for stars which contract when losing matter.

The study of conditions of stable mass exchange is motivated by two related problems: i) it is necessary to explain why $\sim 10\%$ of CVs with well-determined masses of components have combination of donor mass and mass ratio of components which is “forbidden” in the evolutionary models based on conventional assumptions on the conservation of angular momentum of a binary; ii) for population synthesis studies of CVs it is necessary to distinguish progenitors of CVs among all binaries containing a white dwarf and a low-mass companion (i.e., specify in which binaries of this type stable mass exchange is possible). A problem

\footnote{The effective radius of the Roche lobe $R_{RL}$ is determined as the radius of a sphere with a volume equal to the volume of the Roche lobe.}
similar to the last one appears in the studies of low-mass X-ray binaries.

The paper is organized as follows. In Section 2 we describe major factors affecting the evolution of CVs and conditions of stable mass exchange in binaries. The reasons which motivated us to rule out the conservative approximation for mass exchange are considered in Section 3. Sections 4 and 5 describe some results of 3D gas dynamical simulations of the flow structure in semi-detached binaries and introduce the model for description of angular momentum losses due to non-conservative mass exchange. Results of evolutionary calculations for conservative and “non-conservative” models are compared in Section 6. Criteria of stable mass exchange for “non-conservative” model with mass and angular momentum loss are determined in Section 7. Section 8 summarizes the main results of the work.

2 Major factors affecting the evolution of CVs

In the context of the present work the term “stable mass exchange” means dynamical stability. In other words, in the course of mass exchange the donor star can be out of thermal equilibrium and mass transfer rate may exceed the rate corresponding to the Kelvin time scale \( \dot{M} \simeq 3 \times 10^{-7} R L / M \), where radius \( R \), luminosity \( L \), and mass of the star \( M \) are in solar units and mass transfer rate \( \dot{M} \) is in \( M_\odot \ yr^{-1} \). Let us define unstable mass exchange as the situation when the radius of the donor changes faster than the effective radius of Roche lobe: \( \dot{R}_2 > \dot{R}_{RL} \). Using the derivatives of the radii w.r.t. mass of the donor we can rewrite the condition of the stability of mass transfer as:

\[
\zeta \equiv \frac{\partial \ln R_2}{\partial \ln M_2} > \frac{\partial \ln R_{RL}}{\partial \ln M_2} \equiv \zeta_{RL}.
\]

The effective radius of Roche lobe can be estimated, for instance, using interpolation formula given by Eggleton [22]:

\[
R_{RL} \approx 0.49 A \frac{q^{2/3}}{0.6q^{2/3} + \ln(1 + q^{1/3})},
\]

where \( A \) is the semimajor axis of the orbit. For \( q \ll 1 \) the approximation suggested by Paczyński [23] is more convenient:

\[
R_{RL} \approx \frac{2}{3^{1/3}} A \left( \frac{q}{1 + q} \right)^{1/3}.
\]

Derivative of the Roche lobe radius can be written as

\[
\frac{\partial \ln R_{RL}}{\partial \ln M_2} = \frac{\partial \ln A}{\partial \ln M_2} + \frac{\partial \ln (R_{RL}/A)}{\partial \ln M_2}.
\]
It is seen from (4) and (3) that the variation of effective radius of Roche lobe is defined by the change of donor and accretor masses $M_2$ and $M_1$ as well as by the change in the separation $A$, and, hence, it is influenced by the loss of mass and angular momentum from the system. As a rule, only systemic angular momentum is considered as the orbital momentum, i.e. the sum of angular momenta of two stars, which are considered as point masses. Momenta of the spin of the components, momentum of the accretion disc (if it’s present), as well as momentum of gaseous flows inside the system as a rule are not included into the total momentum. Deviation of the momentum of the donor from that of the point mass object (which can be significant, see [24]) usually is not taken into account as well. Such a simplified approach is compelled, since it is very difficult to include all the abovementioned factors into calculations. Having in mind all these simplifications, one gets for the orbital momentum of a binary system with the circular orbit

$$J = \sqrt{\frac{GM_1^2 M_2^2 A}{M_1 + M_2}}, \quad (4)$$

where $G$ is the gravitational constant.

According to the widely accepted models, the evolution of cataclysmic binaries is determined by the loss of angular momentum from the system via gravitational waves radiation (GWR) and/or magnetic stellar wind (MSW) of the donor as well as the mass exchange between components. In the standard models of evolution it’s accepted that the mass exchange among components does not change the systemic angular momentum, and influences it indirectly – through a possible loss of mass from the system and angular momentum ablation by leaving matter (term with subscript LOSS in Eq. (5) below). Therefore, within conventional models, the equation for the change of orbital momentum can be written as:

$$\frac{dJ}{dt} = \left( \frac{\partial J}{\partial t} \right)_{GWR} + \left( \frac{\partial J}{\partial t} \right)_{MSW} + \left( \frac{\partial J}{\partial t} \right)_{LOSS} \quad (5)$$

Let us consider the terms of expression (5):

1) Loss of the systemic angular momentum due to GWR

The change of the systemic orbital momentum as a result of GWR is given by a formula (see, e.g., [25]):

$$\left( \frac{\partial \ln J}{\partial t} \right)_{GWR} = -\frac{32G^3}{5c^3} \frac{M_1 M_2 (M_1 + M_2)}{A^4}, \quad (6)$$

where $c$ is the speed of light. Intensity of GWR is characterized by a very strong dependence on the separation of components and respectively on the orbital pe-
period. This process is essential for short-period systems with $P_{\text{orb}} \lesssim 10^4$, since only then the time scale of angular momentum loss becomes shorter than the Hubble time.

2) Loss of the systemic angular momentum due to MSW of the donor

A mechanism for the loss of the angular momentum of the system by means of the MSW was suggested by Schatzman [13] and by Mestel [14]. If the star has a convective envelope and, subsequently, a surface magnetic field, its own axial rotation is braked by the magnetic stellar wind, and the angular momentum loss rate can be essential even for small mass loss rate. The subsequent synchronization of the spin of the donor and orbital rotation caused by tidal interaction between components results in the loss of the orbital momentum of the system and reduction of the separation of components $A$. To take this effect into account, Verbunt and Zwaan [16] extrapolated the data on rotation of F-type stars [26] to the K and M type components of CVs and suggested a widely used semi-empirical formula which allows to determine the temporal behavior of the orbital momentum:

$$\left( \frac{\partial \ln J}{\partial t} \right)_{\text{MSW}} = -0.5 \times 10^{-28} \cdot Ck^2G\frac{(M_1 + M_2)^2 R_2^4}{M_1 A^5},$$  \hspace{1cm} (7)

were $k$ is the radius of gyration of the donor star; $C$ is a numerical factor determined from the comparison of theoretical calculations and observational data. Well-defined reduction of magnetic field for stars with masses less than $\sim 0.3 M_\odot$ permits to suggest that vanishing of a radiative core of a star when its mass decreases to this threshold leads to an abrupt “switch-off” of the so-called “$\alpha \Omega$”-dynamo mechanism that is responsible for the generation of stellar magnetic fields [27]. At the instant of cessation (or sharp reduction) of MSW the time scale of angular momentum loss [defined by Eq. (7)] is shorter than the thermal time scale of a donor with $M \sim 0.3 M_\odot$. Thus, the latter is out of thermal equilibrium and its radius is larger than the radius of thermally equilibrium star of the same mass. When the action of MSW and corresponding loss of angular momentum stop, the rate of reduction of the semimajor axis declines, the donor shrinks down to the equilibrium radius and loses the contact with Roche lobe [27]. Meanwhile, the system continues to lose angular momentum due to GWR, components continue to approach each other and after some time the optical star fills its Roche lobe again. Further evolution of the system is determined by the loss of angular momentum via GWR.

Cessation of MSW after the star becomes completely convective is widely adopted hypothesis explaining satisfactorily the so-called orbital period gap in

\[\text{For discussion on the response of star on mass loss in different time scales see, e.g., [39].}\]
CVs. The value of coefficient $C$ in Eq. (7) can be determined by comparison of theoretical width of period gap and observational one. In the present study we adopted $C = 3.0$ in accordance to [7].

3) Loss of the systemic angular momentum due to mass loss

Mass loss from the system is mainly considered as a parameter fine tuning of which permits to get an agreement with observational minimal period of CVs (e.g. [1]). More definite assumptions are made only for specific cases when examining evolution of the systems, in which: i) the rate of accretion on white dwarf is limited by the hydrogen burning rate ($\sim 10^{-7} \div 10^{-6} M_\odot \text{yr}^{-1}$), and the mass loss rate by the donor is much higher than this limit, but does not exceed Eddington limit for the dwarf (that is close to $1.5 \times 10^{-5} M_\odot \text{yr}^{-1}$, see, e.g., [28, 29]), the mass excess being lost by means of stellar wind [30]; ii) mass is lost due to outbursts occurring on white dwarf (see, e.g., [17, 31–33]). In both cases it is usually assumed that the specific angular momentum of the matter leaving the system is equal to the specific momentum of the accretor.

To describe the loss of mass and momentum out of the system, it is convenient to introduce parameters describing the degree of non-conservativity of the evolution respective to the mass [34]:

$$
\beta = -\frac{\dot{M}_1}{\dot{M}_2} = -\frac{\partial M_1}{\partial M_2}
$$

and to the angular momentum:

$$
\psi = \frac{\dot{J}}{\dot{M}} = \frac{\partial \ln J}{\partial \ln M},
$$

(8)

where $M = M_1 + M_2$.

With these parameters, the expression for the loss of momentum by the matter leaving the system gets the following form:

$$
\left(\frac{\partial J}{\partial t}\right)_{\text{LOSS}} = (1 - \beta) \dot{M}_2 \psi \frac{J}{M}.
$$

1The latter two studies considered also the loss of momentum due to interaction of the donor with the envelope of a Nova.

2Sometimes specific angular momentum lost from the system is measured in units of $\Omega A^2$ and used instead of $\psi$:

$$
\alpha = \frac{\dot{J}}{\dot{M}} / \Omega A^2 = \frac{J}{\dot{M} \mu} = \frac{q}{(1 + q)^2} \psi, \quad \mu = \frac{M_1 M_2}{M}.
$$
In the calculations of the cataclysmic binary evolution an equation for the variation of the semiaxis of the orbit is used directly, instead of the equation for the variation of momentum over time. Let us differentiate the expression \( (4) \), substitute the result into equation \( (5) \), and use the relation \( \dot{M}_1 = -\beta \dot{M}_2 \). Then the equation for the variation of orbital semiaxis can be written as:

\[
\frac{dA}{dt} = \left( \frac{\partial A}{\partial t} \right)_{\text{EXCH}} + \left( \frac{\partial A}{\partial t} \right)_{\text{LOSS}} + \left( \frac{\partial A}{\partial t} \right)_{\text{GWR}} + \left( \frac{\partial A}{\partial t} \right)_{\text{MSW}}. 
\] 

Here, index ‘EXCH’ corresponds to the change of \( A \) resulting from the mass transfer between components. Note that while the term \( (\partial J/\partial t)_{\text{EXCH}} \) is absent in the equation \((5)\), the appropriate term \( (\partial A/\partial t)_{\text{EXCH}} \) is present in the equation \((9)\), being a natural consequence of the dependence of the orbital momentum both on semimajor axis of the orbit and masses of components. When mass exchange is conservative, the evolution of the orbital separation is determined by the assumption that the mass transfer does not change the orbital momentum of the system. In the course of mass exchange the matter that initially had specific angular momentum of the donor is transferred onto accretor and finally gets the specific momentum of the accretor. The assumption of conservation of the orbital momentum implies that if the mass of the donor is lower than the mass of accretor, the excess momentum of accreted gas should transform into the orbital one, and the mass exchange acts in the direction of increasing the orbital semiaxis. If the donor is more massive than the accretor, the lacking momentum of the accreted gas should be taken from the orbital momentum, therefore, the mass exchange acts in the direction of shrinking of the binary system.

The numerical investigation of the evolution of cataclysmic binaries consists of simultaneous calculations of the evolution of the donor and of the variation of the orbital separation in time. Let us consider the processes that determine the evolution of the orbital semiaxis. Using parameters \( \beta \) and \( \psi \), we can write formulae for the terms of equation \((9)\) (here masses and distances are given in solar units and time is given in years):

\[
\left( \frac{\partial A}{\partial t} \right)_{\text{EXCH}} = 2A \frac{M_2 - M_1}{M_1 M_2} \beta \dot{M}_2, 
\] 

\[
\left( \frac{\partial A}{\partial t} \right)_{\text{LOSS}} = -(1 - \beta)A \frac{2M_1(1 - q\psi) + M_2}{M_2(M_1 + M_2)} M_2, 
\] 

\[
\left( \frac{\partial A}{\partial t} \right)_{\text{GWR}} = -1.65 \times 10^{-9} \frac{M_1 M_2 (M_1 + M_2)}{A^3}, 
\] 

\[
\left( \frac{\partial A}{\partial t} \right)_{\text{MSW}} = -6 \times 10^{-7} \frac{C(M_1 + M_2)^2}{M_1} \left( \frac{R_2}{A} \right)^4. 
\]
It is seen from Eqs (10)–(13) that changing of $R_{RL}$ is mainly determined by the value of $q$. On the other hand, changing of $R_2$ depends on $M_2$ and $\dot{M}_2$. Thus the most convenient way for studying of limits of stable mass exchange is analysis of “ratio of donor-star mass to accretor mass $q$ – donor-star mass $M_2$” diagram.

In a number of cases one may simplify expression (3) and obtain analytical formulae for the derivative of the Roche lobe radius. In particular, when the angular momentum changes due to GWR and MSW can be neglected, one may deduce from (10) and (11) the expression for the change of separation $A$ [the first term in the formula (3)] as a function of mass ratio $q$ and the parameters of the loss of matter and momentum $\beta$ and $\psi$:

$$\frac{\partial \ln A}{\partial \ln M_2} = \frac{2\psi(1 - \beta)q + \beta q + 2\beta q^2 - q - 2}{1 + q}.$$  

(14)

Using the expression (2) for $R_{RL}$ we can express the second term of (3) as:

$$\frac{\partial \ln (R_{RL}/A)}{\partial \ln M_2} = \frac{1}{3} - \frac{1}{3} \frac{(1 - \beta)q}{1 + q}.$$  

(15)

Using (14) and (15), for the case of totally conservative mass exchange ($\beta = 1$), we obtain:

$$\frac{\partial \ln R_{RL}}{\partial \ln M_2} = 2q - \frac{5}{3}.$$  

(16)

From the same equations, for the so-called ‘Jeans mode’ of mass loss, when the mass exchange in the system does not occur, but the donor loses its mass due to stellar wind and matter leaves the system carrying away specific angular momentum of the donor ($\beta = 0$, $\psi = q$), we obtain:

$$\frac{\partial \ln R_{RL}}{\partial \ln M_2} = \frac{1 - 3q}{1 + q}.$$  

(17)

For the case of stellar wind from the accretor when all matter lost by the donor is transferred onto accretor but then leaves the system carrying away specific angular moment of the accretor (formally, one should take here $\beta = 0$ and $\psi = q^{-1}$) we obtain:

$$\frac{\partial \ln R_{RL}}{\partial \ln M_2} = \frac{2q^2 - q - \frac{5}{3}}{1 + q}.$$  

(18)

The comparison of the rates of variation of the radius of the donor and effective radius of the Roche lobe (3) determines the state of stability/instability of mass exchange in any phase of evolution. It is possible to determine the boundaries of stable mass exchange region by two different methods:
• It is possible to calculate derivatives of the radius of the donor w.r.t. its mass directly for different $M_2$ and to compare it with derivatives of the effective radius of Roche lobe depending on $M_1$, $M_2$, and $A$. This method was used, for instance, in [35] and it permits to restrict a region of $q - M_2$ diagram where mass exchange is dynamically stable and also to detect the regions where mass exchange should occur in different time scales: thermal, nuclear, or in the time scale of angular momentum loss.

• Alternatively, a requirement that the rate of accretion should not exceed the Eddington limit for the dwarf can be accepted as a criterion of stable mass exchange. This assumption seems to be more reasonable since evolutionary calculations often suggest very high rate of mass loss during very initial moments of time after overfilling the Roche lobe, while after that $\dot{M}$ falls off rapidly. In fact, this approach involves a restriction on dynamical stability of mass transfer but permits loss of matter in thermal time scale.

3 Stable mass exchange region in conservative on mass model of the CV evolution

Figure 1 depicts the boundary of the stable mass exchange region in conventional conservative model of the evolution of CV. Boundary 1 was found in [35], and boundary 2 was found in [36]. These boundaries coincide well for of donors of low mass but differ a little for large ones. Unfortunately, in [36] the procedure of the calculation of the boundary isn’t described in detail. We may assume that this discrepancy is due to the differences in the codes, in a different treatment of possible weakening of donor MSW at masses greater than $\sim 1M_\odot$, and different techniques for computation of stellar radii derivatives in this region.

Figure 1 also shows CVs with known masses of components from the catalog [37]. The total number of binaries in the diagram is 80, and 11 of them are in the region of unstable mass exchange. For 8 of these 11 systems uncertainties of parameters are known, they are shown in Fig. 1 as error bars. It is seen that the location of stars in a region of unstable mass exchange can not be explained by uncertainties in masses of components. In Fig. 1a we plot CVs with the best determined masses [38], parameters for a number of systems being different compared to the data from [37]. Nevertheless, even for well-determined systems the problem of “improper” location of CVs remains – 3 out of 22 systems lie in the “forbidden” region.

To explain this disagreement we suggest a model of CVs evolution which involves loss of mass and angular momentum from the system. This model is based on 3D gas dynamical simulations of the flow structure in close binary systems.
Fig. 1: “Mass ratio of donor to accretor $q$ – donor mass $M_2$” diagram. Asterisks $\ast$ and circles $\circ$ show observed stars from the catalogue of Ritter & Kolb [37] (marker $\circ$ indicates that data are not reliable). For some CVs error bars are also shown. Dashed lines – boundary of stable mass exchange region in the conservative model according to [35] (dashed line 1) and according to [36] (dashed line 2). Dot-dashed lines show the upper ($M_1 \leq 1.4M_\odot$) and lower ($M_1 \geq 0.15M_\odot$) cutoffs of white dwarf mass.
Fig. 1a: The same as in Fig. 1 but with observed stars from [38].
4 Calculations of the rate of an angular moment loss in 3D gas dynamical models

The above-stated analysis proves that the rate of change of the distance between components $A$ in evolutionary calculations depends heavily on two parameters: the degree of non-conservativity of mass exchange w.r.t. the matter $\beta$ and w.r.t. to the angular momentum $\psi$. These parameters can be determined only by means of 3D gas dynamical simulations of mass transfer in binary systems. These investigations were made in [20] for a binary system with mass ratio of components $q = 1/5$, and in [21] for a binary with $q = 1$. Besides that, we have specially made a numerical simulation of a binary system with $q = 5$ to study the influence of parameter $q$ on the flow structure.

Analysis of the obtained results proves that in the binary systems considered by us for different values of parameter $q$ mass transfer is non-conservative and the non conservativity degree w.r.t. the matter is $\beta \sim 0.4 \div 0.6$. Simulations of flows in the 3D numerical models prove also that the matter leaves the system with a significant angular momentum. The non conservativity degree of mass exchange w.r.t. the matter can be easily determined directly from gas dynamical model, while the determination of non conservativity degree w.r.t. the angular momentum is a sophisticated problem. Let us consider the equations which determines the transport of angular momentum in the system. We should consider Navier-Stokes equations for a correct analysis because viscosity plays a significant role in the redistribution of angular momentum. From the steady-state gas dynamical equations in a rotating frame we can deduce the equation for the angular momentum transfer

\[
u \frac{\partial \lambda}{\partial x} + v \frac{\partial \lambda}{\partial y} + w \frac{\partial \lambda}{\partial z} + \frac{1}{\rho} \left( (r - r_{CM}) \times \text{div} \mathcal{P} \right)_z = - \left( (r - r_{CM}) \times \text{grad} \Phi \right)_z ,
\]

where, as usually, $\mathbf{v} = (u, v, w)$ is the velocity vector, $\rho$ is the density, $\Phi$ is the Roche potential, $\Omega$ is the orbital angular velocity, $\mathcal{P}$ is the stress tensor, $r_{CM}$ is the radius-vector of the center of masses of the system, and angular momentum $\lambda$ (in a laboratory, i.e. inertial or observer’s frame) is defined by expression:

\[
\lambda = (x - x_{CM})v - yu + \Omega \left( (x - x_{CM})^2 + y^2 \right) \\
= (x - x_{CM}) \left( v + \Omega(x - x_{CM}) \right) - y(u - \Omega y) .
\]

Let us write down the equation for angular momentum transfer in a divergent form:

\[
\text{div}(\rho \lambda \mathbf{u}) + \left( (r - r_{CM}) \times \text{div} \mathcal{P} \right)_z = -\rho \left( (r - r_{CM}) \times \text{grad} \Phi \right)_z .
\]
From this equation, we can obtain the integral law of change of angular momentum as follows:

\[
\int_{\Sigma} \rho \lambda \mathbf{u} \cdot d\mathbf{n} + \int_{\Sigma} \left( (\mathbf{r} - \mathbf{r}_{CM}) \times \mathbf{P} \right) \cdot d\mathbf{n} = \int_{V} \rho \left( (\mathbf{r} - \mathbf{r}_{CM}) \times \nabla \Phi \right) \cdot dV \equiv \Pi.
\]

The term \( \Pi \) determines the change of the angular momentum due to non-central nature of the force field, defined by the Roche potential. By definition, the value

\[
\mathbf{F}_\lambda = \rho \lambda \mathbf{u} + \mathbf{P} \mathbf{r}', \quad \text{where} \quad \mathbf{r}' = (-y, x - x_{CM}, 0)
\]

(19)
is the specific flux of angular momentum \( \lambda \). As a result, for the steady-state case we obtain:

\[
- \int_{\Sigma_1} \mathbf{F}_\lambda \cdot d\mathbf{n} = \int_{\Sigma_2} \mathbf{F}_\lambda \cdot d\mathbf{n} + \int_{\Sigma_3} \mathbf{F}_\lambda \cdot d\mathbf{n} - \Pi,
\]

where \( \Sigma_1 \) and \( \Sigma_2 \) are the boundaries of the donor and the accretor, respectively, and \( \Sigma_3 \) is the outer boundary of the calculation domain. By analogy to the flux of matter

\[
\dot{M} = - \int_{\Sigma_3} \mathbf{F}_m \cdot d\mathbf{n} = - \int_{\Sigma_3} \rho \mathbf{u} \cdot d\mathbf{n},
\]

one may estimate the flux of angular momentum from the system as

\[
\dot{J} = - \int_{\Sigma_3} \mathbf{F}_\lambda \cdot d\mathbf{n},
\]

(20)

and to use it in the expression (8), determining the non-conservativity parameter w.r.t. the angular momentum.

Note that numerical simulations were conducted for Eulerian equations for inviscid gas and not for Navier-Stokes equations. Respectively, to calculate integral (20) we use isotropic term describing gas dynamical pressure in the expression for the specific flux of angular momentum: \( \mathbf{P}_{\alpha\beta} = P \cdot \delta_{\alpha\beta} \). This substitution is correct because on the outer border where the integral in (20) is estimated, viscosity does not play a significant role.

Using expression (20) in the gas dynamical calculations we obtained \( \psi \sim 6 \) corresponding to \( \alpha = 0.83 \) (see definition of \( \alpha \) in the footnote on page 9) for the binary system with \( q = 1/5 \) [20], and \( \psi \sim 5 \) that corresponds to \( \alpha = 1.25 \). [21]

\[1\] Here, as usual, \( \delta_{\alpha\beta} \) is Kronecker’s symbol:

\[
\delta_{\alpha\beta} = \begin{cases} 
1, & \alpha = \beta, \\
0, & \alpha \neq \beta.
\end{cases}
\]
for a binary with $q = 1$. The last value is in good agreement with the results of two-dimensional calculations of binary system with equal masses of components \[39\] where $\alpha = 1.65$ was obtained.

However, the use of these estimates for $\psi$ in evolutionary calculations proves that this rate of an angular moment loss is so great that in the majority of binary systems the mass exchange rapidly becomes a runaway process and the rate of mass loss by the donor increases without any limit. Apparently, the application of formula \[20\] to the estimation of the angular momentum loss in the evolutionary calculations is not quite correct. since conjunction of gas dynamical and evolutionary models is not completely consistent. The considered gas dynamical model does not take into account the change in the position of stars in time and, hence, the change in the angular momentum of the gas due to the non-central nature of the force field is not compensated by respective change in the orbital angular momentum.

In a more general gas dynamical model, in which the change in the position of stars is taken into account, one may correctly estimate the change of the systemic angular momentum in the form of angular momentum flux through the external boundary for the “donor + accretor + gas” system. However, to take into account such gas dynamical calculations appears rather complicated. Such estimates are possible only for a specific stage in the life of a binary system. Moreover, thus obtained gas dynamical results cannot be used directly in the standard evolutionary models that ignore the presence of circumbinary envelope in the system. Therefore, a simplified model for estimation of the angular momentum loss developed on the basis of gas dynamical calculations will be a promising approach in the nearest future.

5 A simplified model for estimation of angular momentum loss rate in semi-detached binaries

Gas dynamical simulations of mass transfer in semi-detached binaries confirm that outflow of matter from the donor occurs through a small vicinity of the inner Lagrangian point $L_1$. In this case specific angular momentum of lost matter (in corotating frame) can be estimated as $\lambda_{L_1} = \Omega \Delta^2$, where $\Delta = |x_c - x_{L_1}|$. Respectively, the flow of angular momentum from the donor is equal to

$$F^{(2)}_\lambda = \lambda_{L_1} \dot{M}_2 = \Omega \Delta^2 \dot{M}_2.$$ 

Due to non-conservativity of mass exchange, only a fraction $\beta$ of the matter outflowing through $L_1$ is accreted. Specific angular momentum of accreted matter is equal to the angular momentum of accretor (here we neglect the finite radius of accretor and/or residual angular momentum of accreted matter, and
this approximation appears to be reasonable for CVs). It means that accreted matter has zero angular momentum in the rotating frame, and, hence, does not accelerate the rotation of accretor. Thus the question on the efficiency of the angular momentum transfer from the spin rotation of accretor to the orbital rotation of the binary can be ignored. To summarize, we can describe the flow of accreted angular momentum as

\[ F^{(1)}_{\lambda} = \lambda_{\text{accr}} \beta \dot{M}_2 = \Omega \left( \frac{M_2}{M} \right)^2 A^2 \beta \dot{M}_2. \]

These expressions for the angular momentum flows do not contradict the general formula for specific flow of angular momentum [see (19)]. Indeed, viscosity does not play a significant role neither in the vicinity of the surface of the donor nor in the vicinity of the accretor where the matter has already lost its angular momentum and the flow has radial direction in the rotating frame. It means, that in these cases the stress tensor \( \mathcal{P} \) is reduced to the isotropic pressure as well, and does not deposit into the integral of specific flow of angular momentum. Our assumptions permit to write the expression for the flow of angular momentum lost from the system in the following form:

\[ F^{\text{loss}}_{\lambda} = \eta \left( F^{(2)}_{\lambda} - F^{(1)}_{\lambda} \right) = \eta \left( \Omega \Delta^2 \dot{M}_2 - \Omega \left( \frac{M_2}{M} \right)^2 A^2 \beta \dot{M}_2 \right), \]

where \( \eta \) is a parameter defining the fraction of the circumbinary envelope angular momentum that is lost with the matter leaving the system. Then \( 1 - \eta \) is the fraction of the circumbinary envelope angular momentum that returns back into the system via tidal interaction. Henceforth we adopt for the parameter \( \eta \) a value of 1 that corresponds to the case when the angular momentum of the circumbinary envelope is totally lost with the matter leaving the system. In this case the specific angular momentum of the matter leaving the system is equal to

\[ \lambda_{\text{loss}} = \frac{F^{\text{loss}}_{\lambda}}{\dot{M}_{\text{loss}}} = \Omega \Delta^2 \dot{M}_2 - \beta \left( \frac{M_2}{M} \right)^2 \Omega A^2 \dot{M}_2 \]

or in the units of systemic specific angular momentum (\( \lambda_{\text{syst}} = \Omega A^2 M_1 M_2 / M^2 \)):

\[ \psi = \frac{\lambda_{\text{loss}}}{\lambda_{\text{syst}}} = \frac{\left( f(q) - \frac{1}{1+q} \right)^2 - \beta \left( \frac{q}{1+q} \right)^2}{1 - \beta \cdot \frac{(1+q)^2}{q} \cdot (1+q)^2}, \tag{21} \]

where \( f(q) = x_{L_1}/A \) is dimensionless distance from the center of the donor to \( L_1 \).

We would like to stress that formula (21) was derived under the assumption of aligned synchronous rotation of components of the binary system. It is known that the loss of angular momentum due to outflow of the matter from the vicinity of \( L_1 \) can result in the development of a misalignment of the vector of spin rotation
of the donor and the vector of the orbital rotation of the binary [10]. The flow structure in the binaries with misaligned asynchronous rotation was studied by the authors in [11]. Having in mind the simplified character of evolutionary models we don’t consider the effect of the misalignment in the present work.

The graph of $\psi(q)$ is shown in Fig. 2 for various values of non-conservativity degree w.r.t. the matter $\beta$. The values $\psi = 0$ and $\psi = 1$ are also shown in the Figure by dashed lines. In the region $\psi > 1$ each unit of the matter leaving the system carries away an angular moment that is higher than the mean specific angular momentum of the system. This results in diminishing of systemic specific angular momentum:

$$\frac{d}{dt} \left( \frac{J}{M} \right) < 0.$$

In the region $0 < \psi < 1$ each unit of the matter leaving the system carries away an angular moment that is lower than the mean specific angular momentum of the system, and, hence,

$$\frac{d}{dt} \left( \frac{J}{M} \right) > 0.$$

Figure 2 shows that the value of $\beta$ becomes negative in a certain range of $q$ for the majority of cases of non-conservative mass exchange (except $\beta \sim 0$). This means that the escape of the matter to infinity causes redistribution of the angular momentum in the “donor + accretor + gas” system. This results in the growth of the total angular momentum of the binary: $\dot{J} > 0$. It is evident that such a “pumping” of angular momentum into the system can stabilize the mass exchange in CV. Indeed, consideration of orbit semiaxis change due to conservative mass exchange shows that the binary system widens for $q < 1$ only. For the “non-conservative” model the range of $q$ corresponding to the widening of binary due to mass and angular momentum losses is larger – for $\beta = 0.5$ it happens for $0 < q < 2.8$.

Note that $\psi(q, \beta)$ dependence has a curios feature [see Fig. 2 and formula (21)] – all curves $\psi(q)$ for different values of $\beta$ pass through the same point. This point corresponds to the case when the center of mass of the binary system is located exactly equidistantly between $L_1$ and accretor, and in this case $\psi$ does not depend on $\beta$. Fortunately, this specific feature does not influence the analysis of solution.

Three-dimensional gas dynamical simulation of mass transfer in CVs confirms conclusion that the outflow of matter from the system can lead both to decrease and to increase of the total angular momentum of a binary. The typical flow structures [see Fig. 3a,b] for binaries with $M_2/M_1 = 1:5$ and $M_2/M_1 = 5:1$ demonstrate that in an inertial frame gas can have different directions of rotation (in both figures orbital rotation corresponds to the counter-clockwise direction). Respectively, the change of angular momentum has opposite signs in the obtained
Fig. 2: Non-conservativity degree of the evolution respective to angular momentum $\psi$ as a function of mass ratio $q$ for various values of non-conservativity degree of the evolution respective to mass $\beta$ [see Eq. (21)]. Thin dashed lines correspond to the values of $\psi = 0$ and $\psi = 1$. Bold dashed line shows a dependence $\psi = (1 + q)^2 / q$ corresponding to the value $\alpha = 1$ (see definition of $\alpha$ in the footnote on p. 13).
Fig. 3: Velocity vectors in equatorial plane for 3D gas dynamical simulation of a binary system with $M_2/M_1 = 1:5$ ($q = 1/5$, top panel) and with $M_2/M_1 = 5:1$ ($q = 5$, lower panel). Position of the accretor is marked by an asterisk $\ast$. Shaded region marks the donor. Dashed lines show Roche equipotentials. A vector in the top right corner corresponds to the velocity equal to $3A\Omega$. 

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solutions – for $q = \frac{1}{5}$ the systemic angular momentum decreased, and for $q = 5$ it increased.

It seems reasonable to explain the different types of gas dynamical solutions using the following qualitative speculation:

- The flow of the gas leaving the system consists of the matter of the circumbinary envelope that has an angular momentum large enough to overcome the gravitational attraction of the accretor. Initial velocity of this matter (in rotational frame) has the same direction as the orbital movement of the accretor.

- Velocity of the gas leaving the system changes under the action of the gravitational attraction of both components of the binary system, centrifugal force, Coriolis force, and pressure gradient. The change of azimuthal velocity is determined mainly by the Coriolis force and gravitation forces, since centrifugal force is axially-symmetrical and deviation of pressure gradient from the axial symmetry is small as well. Action of Coriolis force results in deflecting of the stream in the direction opposite to that of the orbital movement. Consequently, this permits to subdivide the whole flow leaving the system through the vicinity of $L_2$ into two parts: the first part retains the direction of original movement, and the direction of movement of the second one is changed.

- It is evident that in an axially-symmetrical gravitational field the action of Coriolis force can not produce a solution when the direction of gas movement in the inertial frame is opposite to the orbital one. The situation is changed drastically for the case of non-axially-symmetrical gravitational field of a binary system. In this case gravitational attraction of the donor can accelerate the gas of the stream and result in the solution with opposite direction of movement in the inertial frame.

- It follows from Eq. (21) and Fig. 3 that there is a range of values of $q$ where gas moves in the direction opposite to the orbital one in the inertial frame. These values of $q$ correspond to the cases when attraction of the donor is large enough to accelerate the gas properly (left end of the range), but not too large to produce accretion of gas on the surface of the donor (right end of the range). The values of $q$ out of this range correspond to the case when the flow directed along the orbital movement prevails.

Appearance of an additional range of values of $q$ where the “non-conservative” mass exchange is accompanied by increase of systemic angular momentum results in expansion of the stable mass exchange region. To illustrate this fact let us consider a graph that is often used in the qualitative studies of the stability of mass transfer in CV. This is a $q$ – $\zeta$ graph with curves corresponding to the
analytical expressions for derivatives of effective radius of the Roche lobe w.r.t. the mass of the donor $\zeta_{RL}$, and straight lines corresponding to the values of derivatives of the radius of the star w.r.t. to its mass $\zeta_*$, the latter being known from the investigations of the internal structure of star (see, e.g., [33]). Mass exchange is stable for the range of $q$ where curve $\zeta_{RL}$ passes below the straight line $\zeta_*$ for the donor of corresponding type. This graph is presented in Fig. 4. It shows the derivative of effective radius of Roche lobe $\zeta_{RL} = \partial \ln R_{RL}/\partial \ln M_2$ as a function of $q$ for a scenario in which the loss of angular momentum is defined by formula (21) for various values of the parameter $\beta$. Dependencies $\zeta_{RL}(q)$ for the scenario of the completely conservative mass exchange and for the case when the outflowing matter carries away specific angular momentum of the donor or specific angular momentum of accretor are shown as well. The same Fig. 4 also shows the values of derivative $\zeta_* = \partial \ln R_2/\partial \ln M_2$ for completely convective and degenerate stars ($\zeta_* = -1/3$), for thermally equilibrium stars of MS with $M \sim M_\odot$ ($\zeta_* = 0.6$), and for subgiants with degenerate low-mass helium cores ($\zeta_* = 0$).

The analysis of Fig. 4 proves that the outflow of matter and loss of angular momentum from the system in accordance to law (21) stabilize mass exchange in a wider range of $q$ compared to the conventional models of conservative mass exchange. It is also worth noting that for conservative scenario of mass exchange its time scale for stars with masses $\sim M_\odot$ is determined by the time scale of the loss of angular momentum or the nuclear time scale for $q < \sim 1.2$ [32], while for above-considered “non-conservative” model this limit on $q$ is sufficiently higher.

We would like to stress also that in the case under consideration there exists a region of unstable mass exchange where value of $q$ is very small. The reason of instability [i.e. the violation of criteria (1)] is fast increase of the ratio of the lost angular momentum to the mean specific momentum of binary system [see Eq. (14) and (21)]. This circumstance can lead to the destruction of a donor of very low mass. Thus, CVs that begin their evolution in a stable mode finish it by a catastrophic disruption of the donor when $q$ becomes lower than a certain threshold.

### 6 Evolution of CVs with losses of mass and angular momentum

To take into account results of 3D gas dynamical simulations of the mass transfer in CVs and to incorporate proper estimates of the loss of angular momentum we investigated the evolution of CVs on the basis of various assumptions on its conservativity.

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1 Another scenario of development of mass exchange instability for low $q$ is also known. It may be caused by low efficiency of interaction between accretion disk and orbital movement (see, e.g., [42][43]).
Fig. 4: Dependence of logarithmic derivative of effective radius of Roche lobe w.r.t. the mass of the donor $\zeta_{RL} = \partial \ln R_{RL} / \partial \ln M_2$ versus mass ratio $q$ for various values of non-conservativity degree of the evolution by mass $\beta$ [bold lines; see Eqs (3) and (21)]. Thin lines show dependence $\zeta_{RL}(q)$ for the case of conservative mass exchange [‘cons’, formula (16)], the case of Jeans mode of mass loss [‘Jeans’, formula (17)], and the case of stellar wind from accretor [‘accr-wind’, formula (18)]. Dashed lines show the values of the logarithmic derivative of the radius of the donor w.r.t. its mass $\zeta_* = \partial \ln R_2 / \partial \ln M_2$ for completely convective and degenerate stars ($\zeta_* = -1/3$), for thermally equilibrium stars of MS with $M \sim M_\odot$ ($\zeta_* = 0.6$), and for subgiants with degenerate low-mass helium cores ($\zeta_* = 0$).
Evolution of the donor was calculated using a modified version of the code for numerical modeling of low-mass stars which was used in the previous works of the authors (see, e.g., [7,10,44–48]). The code uses opacity tables of Huebner et al. [49] with addition of data of Alexander et al. [50] for low-temperature region. Equation of state was adopted from Fontaine et al. [51] with modifications suggested by Denisenkov [52]. Rates of nuclear reaction were adopted from [53,54].

All runs were carried out under assumption that the donor fills its Roche lobe immediately after arriving on ZAMS. Masses of the donors were adopted in the range from $0.1M_{\odot}$ to $1.2M_{\odot}$ and for various values of $q$. Chemical composition was adopted as follows: $X = 0.70$, $Y = 0.28$, $Z = 0.02$. For the mixing length parameter a value of $l/H_p = 1.8$ was adopted. We used the method of calculation of the donor mass loss rate for known stellar radius and effective radius of Roche lobe suggested by Kolb & Ritter [55]. Based on the results of gas dynamical calculations, we assumed that the fraction of matter accreted by white dwarf is $\beta = 0.5$, and that angular momentum carried away by the matter leaving the system is defined by expression (21). We didn’t include into the model mass and angular momentum loss due to outbursts.

Let us consider evolution of CV with parameters corresponding to the stable mass exchange in conservative model. In Fig. 5 ("orbital period – logarithm of donor-star mass loss rate" diagram) two evolutionary tracks are shown. One of the tracks is calculated in the conservative approximation (track 1), and another one – in the "non-conservative" approximation (track 2). Initial masses of both donor and accretor were equal to $1.0M_{\odot}$. It is seen that these two tracks are almost identical. During the early phase evolution is driven by angular momentum loss due to the MSW of the donor. This results in decrease of the orbital semiaxis and orbital period in the course of evolution. Decrease of the mass of the donor as well as its radius lead to decrease of $(\partial A/\partial t)_{\text{MSW}}$ [see Eq. (13)], and consequently to decrease of mass loss rate by the donor $\dot{M}_2$. For the "non-conservative" track, the value of $\dot{M}_2$ is remarkably lower than for the "conservative" one, because in this phase the value of $\psi$ is negative (see Fig. 2) and net loss of orbital angular momentum in the "non-conservative" model is lower than in the conservative one. The situation is inverted in later phases of evolution as $q$ reaches the values corresponding to large angular momentum loss.

After the star becomes completely convective, MSW from the donor stops. Mass of the star at this instant is determined by the deviation from the thermal equilibrium. Single MS stars become completely convective at $M \sim 0.36M_{\odot}$, while mass-losing components of binaries do at lower masses – $M \sim 0.25M_{\odot} \div 0.30M_{\odot}$ (under adopted assumption on chemical composition and opacity of stellar matter).

In the "conservative" track mixing occurs at $M_2 = 0.265M_{\odot}$ (when the orbital period is equal to $P_{\text{orb}} = 3^{1/2}27$), and in the "non-conservative" track – at $M_2 =$
Fig. 5: Evolutionary tracks in the “orbital period – logarithm of the donor mass loss rate” plane for binary systems with initial masses of components $M_1 = 1.0 M_\odot$, $M_2 = 1.0 M_\odot$ (top panel) and $M_1 = 0.28 M_\odot$, $M_2 = 1.0 M_\odot$ (lower panel). Tracks 1 and 3 (dashed lines) were calculated in the conservative approximation, and tracks 2 and 4 (solid lines) were calculated in the “non-conservative” one.
Thus, values of donor mass and orbital period differ only a little for the “conservative” track and for the “non-conservative” one.

The rate of angular momentum loss decreases abruptly and contraction of binary systems becomes slower after the donor MSW is stopped. As a result donor contracts ceases to lose mass. Since at the moment of cessation of mass loss the radius of the donor is larger than the radius of thermally equilibrium MS star, the star contracts to the equilibrium radius. As a result the ratio of stellar radius to the effective radius of Roche lobe decreases even more. In this phase the evolution of binary system is determined by GWR only. During this detached phase the binary can not be identified as CV, and certain range of the orbital periods becomes devoid of stars. This explains the origin of so called period gap of CVs.

Observed period gap of CVs lies between $2^{h}1$ and $3^{h}1$ [37]. The boundaries of the “theoretical” period gap depend on adopted rate of angular momentum loss due to MSW and on input physics parameters for evolutionary code that determine theoretical radii of stars. We used a code that gives somewhat over-estimated value for the lower boundary. Gap boundaries for the “conservative” track are equal to $2^{h}5 \div 3^{h}3$, and for the “non-conservative” one they are equal to $2^{h}4 \div 3^{h}5$. This discrepancy is related to the different rate of angular momentum loss just prior to gap, and with different masses of accretors after termination of semi-detached phase of evolution as well.

Note that the period gap is characterized not by total absence of CVs but rather by deficiency of CVs. The presence of CVs in the period gap can be explained as follows: i) for CVs with masses of donors equal to $0.25M_{\odot} \div 0.4M_{\odot}$ mass exchange begins immediately in gap; ii) in the case when the donor fills its Roche lobe not on ZAMS but later, when hydrogen is almost burn out, then as mass decreases down to $\sim 0.3M_{\odot}$ complete mixing does not occur due to presence a helium core in the star (see, e.g., [46,47]). Though, it must be admitted that the number of such systems is very small.

The phase of evolution after period gap is characterized by lower rate of mass loss by the donor, because evolution now is driven by GWR. In this phase the rate of angular momentum loss from the system is sufficiently lower compared to the donor MSW driven phase of evolution. Another feature of this phase of evolution is existence of a minimum period of binary. Short-period cutoff is related to the onset of significant degeneracy of the material of the donor and consequently to the change of mass-radius dependence law [56,57]: for small masses of donors $M_{2} \sim 0.05M_{\odot}$ their radii increase as mass decreases and the exponent of the mass-

1Since we were interested in peculiarities of CV evolution that are related to its possible non conservativity, we formally continued our calculations beyond accretor mass of $M_{Ch} = 1.4M_{\odot}$ while in reality in this case the evolution of binary has to be terminated by a thermonuclear explosion of accretor. The continuation of calculation for $M_{1} > M_{Ch} \approx 1.4M_{\odot}$ can be advocated by the spin-up of accretor due to accretion because in this case the critical mass can remarkable exceed $M_{Ch}$. 

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radius law tends to $-\frac{1}{3}$. Minimal period corresponds to the value of exponent equal to $+\frac{1}{3}$\footnote{This follows from approximation (3) for the effective radius of Roche lobe $R_{RL} \propto A(M_2/M)^{1/3}$, condition $R_2 \approx R_{RL}$, third Kepler’s law $P_{\text{orb}} \propto A^{3/2} M^{-1/2}$, and definition of $\zeta^*$ (\ref{eq:zeta}). As a result we have: \[ \frac{\dot{P}}{P} = \frac{3}{2} \frac{A}{A} - \frac{1}{2} \frac{\dot{M}}{\dot{M}} / M, \] \[ \frac{\dot{R}_2}{R_2} = \frac{1}{3} \frac{\dot{M}_2}{M_2} - \frac{1}{3} \frac{\dot{M}}{M} + \frac{\dot{A}}{A}, \] \[ \frac{R_2}{R_2} = \zeta^* \frac{\dot{M}_2}{M_2}, \] and, finally \[ \frac{\dot{P}}{P} \propto (\zeta^* - \frac{1}{3}) \frac{\dot{M}_2}{M_2}. \] This conclusion is independent of the assumption of conservativity or non-conservativity of mass exchange.}. For “conservative” track the minimum period is equal to 75 minutes and for “non-conservative” one it is equal to 81 minutes. The difference in the values of $P_{\text{min}}$ is related to the difference in the total mass of the system and to the difference in the deviations of the radii of donor stars from thermal equilibrium values. The latter deviations are caused by different rates of mass loss. After passing the minimum the orbital period begins to increase, and in this phase the rate of mass loss by the donor quickly decreases. It is seen from Fig. \ref{fig:track} that difference between “conservative” and “non-conservative” tracks becomes larger in this stage. This can be naturally explained by increase of the rate of angular momentum loss with decrease of $q$ in the “non-conservative” case.

“Conservative” and “non-conservative” tracks have appreciably different values of mass transfer rates after passing the minimum of orbital period. This fact is of importance since the probability of discovery of a CV in a given range of log $P$ is proportional to

$$p(\log P) \propto \left( \frac{-\dot{M}_2}{\dot{P} / P} \right)^{\gamma},$$

\label{eq:prob}

where $\gamma$ is a positive constant $\simeq 1$ for visual magnitude limited sample \cite{visual}.

After passing the minimum of the orbital period (i.e. in period range $2^\text{h} \div 2^\text{h}5$) the mean value of the probability of discovery of CV for the “non-conservative” track is 1.2 times larger than for the “conservative” one. Thus, despite a substantial difference in the rate of mass exchange, the probability of discovery of CV differs only a little in two cases under consideration. This is due to faster crossing of the corresponding period range in the “non-conservative” case. Nevertheless, predicted difference in $\dot{M}_2$ for “conservative” and “non-conservative” cases could be reflected in the distribution of CVs over types of variability since the character of variability of a CV depends on the rate of mass exchange.
Figure 5 also shows the evolutionary tracks for stars that would be dynamically unstable within the conservative model. Initial masses of the components were equal to 1.0\,M_\odot and 0.28\,M_\odot. Of course, this combination of masses is rather extreme but we adopted these parameters for the sake of illustration.

Track 3 was calculated under the conventional assumptions on the loss of angular momentum due to GWR and MSW only. This is a typical case of unstable mass exchange with unlimited growth of mass exchange rate and fast convergence of the time scale of mass loss to the dynamical one, i.e. runaway. Strictly speaking this run should be considered as purely illustrative because in this case a common envelope has to form that increases the rate of angular momentum loss even more (it was not taken into account in calculations).

Track 4 was calculated within the model in which the binary loses matter and angular momentum in accordance with Eq. (21). It resembles the tracks for conventional case of stable evolution of CV and differs slightly from track 2 by a higher rate of mass loss in the initial phase of evolution. The latter is caused by some increase of (\partial A/\partial t)_\text{MSW} at large \( q \) [see Eq. (13)]. It results in more strong deviations of the donor from thermal equilibrium and corresponding change of the upper border of period gap: it is equal to 4\,h4 for track 4, and 3\,h5 for track 2. The minimum of period for track 4 is equal to 76 minutes, i.e. it is nearly the same as for track 1 but slightly less than for track 2.

7 Boundary of stable mass exchange region

Boundaries of stable mass exchange region were determined as follows: several sets of evolutionary tracks were calculated for systems with predefined masses of donors and values of \( q \) that were varied with step 0.1. The value of \( q \) was considered as critical if for a given evolutionary track the mass accretion rate didn’t exceed Eddington’s limit and if for the next evolutionary track (with \( q \) higher by 0.1) it did. Note that consequent increasing of \( q \) leads to unlimited growth of mass loss rate (at values of \( q \) greater by 0.2 – 0.3 than critical). Note also, that results of such procedure are in good agreement with technique based on comparison of derivatives of stellar radius and effective Roche lobe radius.

The new boundary of the stable mass exchange region is shown in Fig. 6 (under assumption of “non-conservative” evolution). It is well seen that the new region of stable mass exchange contains all observed CVs with estimated masses of components. Thus, within the “non-conservative” model the problem of location of observed systems outside the region of stable mass exchange does not arise.

On the other hand, in the \( q - M_2 \) diagram all observed CVs are concentrated near \( q \lesssim 1 \), while “non-conservative” model supposes sufficiently higher values of \( q \). To investigate this phenomenon, we have calculated a set of “non-conservative” tracks with initial donor masses from \( M_2^{\text{init}} = 0.1\,M_\odot \) up to \( M_2^{\text{init}} = 1.2\,M_\odot \) with
Fig. 6: The same as in Figs 1 and 1a but with boundary of stable mass exchange region found out in the present work (bold line 3).
Fig. 7: Evolutionary tracks in the “donor to accretor mass ratio $q$ – donor mass $M_2$” plane for CVs with different initial donor masses for the “non-conservative” model (dotted lines, a number near the beginning of each track is initial mass of star in $M_\odot$). Bold lines are isochrones with times in years. Thin dashed line – boundary of the region of stable mass exchange in the conservative model according to [35]; solid line – the same boundary for “non-conservative” model proposed in present work. The short-dashed lines show the upper ($M_1 \leq 1.4M_\odot$) and lower ($M_1 \geq 0.15M_\odot$) cutoffs of white dwarf mass. Asterisks $\ast$ and circles $\circ$ show observed stars from the catalog of Ritter & Kolb [37]. Triangles $\triangle$ show observed stars from [38].
step $0.1M_\odot$ and with such values of $q$ that tracks start in the vicinity of the new boundary of stable mass exchange region. These tracks are shown in Fig. 7 ($q - M_2$ diagram). Based on calculations of these tracks, it is possible to get isochrones that are also shown in Fig. 7 for the moments of time equal to $10^6$, $10^7$, $10^8$, $10^9$, $10^{10}$ years (time is measured from the onset of mass exchange between components). It is this “extreme” set of tracks that was chosen since these systems spent the majority of time between an old boundary of the stable mass exchange region and the new one. It is clear, that systems with smaller values of $q$ (tracks of which begin to the left from new boundary) spent there less time. Thus, CVs with parameters that are close to the “extreme” ones have most chances to be found out in this region.

With respect to the location of isochrones in Fig. 7 all tracks can be subdivided into two distinct groups: i) tracks for systems with massive donors (from $0.4M_\odot$ up to $1.2M_\odot$), for which the evolution prior to the period gap is defined mainly by the loss of systemic angular momentum due to the donor MSW; ii) tracks for systems of low-mass donors (from $0.1M_\odot$ up to $0.3M_\odot$), for which this stage is absent, because their donors are initially completely convective. For the tracks of the first group a high ($\sim 10^{-6} \div 10^{-9} M_\odot$ yr$^{-1}$) rate of mass loss by the donor prior the period gap is typical, and, hence, donor mass and mass ratio decrease rapidly. This results in a fast evolution in the $q - M_2$ diagram. As a result, the part of the diagram near the new boundary is crossed very quickly. This explains the absence of observed CVs near new boundary. For the tracks of the second group (dropping a short initial phase with large rate of mass exchange) the evolution of binary system is characterized by low and nearly constant mass loss rate ($\sim 10^{-10} M_\odot$ yr$^{-1}$) for a long time until period minimum. Consequently, the mass of the donor and mass ratio of components change slowly. Hence, if beginning the evolution near the new boundary of stable mass exchange region, the system should remain there long enough. The absence of CVs in this region can be explained as follows: masses of dwarfs in zero-age CVs can not be higher than $\sim 0.15M_\odot$ that is minimum mass of helium core for a star leaving MS.

8 Conclusions

Results of three-dimensional gas dynamical simulations of the flow structure in semi-detached binaries show that during mass exchange a significant fraction of the matter leaves the system. On the other hand, a number of observed CVs have a combination of donor mass and mass ratio of components forbidden in the evolutionary models based on conventional assumptions on the loss of angular momentum of binaries, since in these cases mass exchange should be unstable. This contradiction can be resolved in the framework of the model suggested here.

\footnote{Note, that the isochrones for $10^9$ years and $10^8$ years for tracks with initial donor mass $M_2^{\text{init}} \geq 0.4M_\odot$ merge, since this interval of time binaries spend in the period gap.}
This model takes into account losses of mass and angular momentum from the system according to the results of gas dynamical simulations. It is shown that approximation for outflowing angular momentum as the difference of specific angular momentum of the donor matter in \( L_1 \) and specific angular momentum of accreted matter (being multiplied by appropriate mass flows) permits to explain observations satisfactorily. Note, that the new “non-conservative” model practically does not change such parameters of evolutionary tracks as limits of period gap and minimal period.

Thus, the main conclusion of our study is: if we use a “non-conservative” approximation for CV evolution, we obtain a new boundary of the region of stable mass exchange in \( q - M_2 \) plane which explains the distribution of observed CVs better than conventional models.

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