ON THE HAMMING WEIGHT OF REPEATED ROOT CYCLIC AND NEGACYCLIC CODES OVER GALOIS RINGS

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Abstract. Repeated root Cyclic and Negacyclic codes over Galois rings have been studied much less than their simple root counterparts. This situation is beginning to change. For example, repeated root codes of length \( p^s \), where \( p \) is the characteristic of the alphabet ring, have been studied under some additional hypotheses. In each one of those cases, the ambient space for the codes has turned out to be a chain ring. In this paper, all remaining cases of cyclic and negacyclic codes of length \( p^s \) over a Galois ring alphabet are considered. In these cases the ambient space is a local ring with simple socle but not a chain ring. Nonetheless, by reducing the problem to one dealing with uniserial subambients, a method for computing the Hamming distance of these codes is provided.

1. Introduction

Cyclic and negacyclic codes have been studied extensively in many contexts, beginning with their linear versions over finite fields and continuing on to the study of such codes over a finite ring alphabet \( A \). A common element in the study of these codes is that they are precisely the submodules of the free module \( A^n \) that correspond to the ideals of a suitable ring \( R_n \) which is isomorphic to \( A^n \) as an \( A \)-module. The ring \( R_n \) is either the quotient \( \frac{A[x]}{(x^n-1)} \) (for the cyclic case) or the quotient ring \( \frac{A[x]}{(x^{ps}+1)} \) (for the negacyclic case). In either case, we refer to the ring \( R_n \) as the ambient space or ambient ring for the codes. While the literature on cyclic and negacyclic codes over chain rings (such as Galois rings) has grown in leaps and bounds (see [4, 10, 11, 18, 19, 21]), in most instances the studies have been focused only on the cases where the characteristic of the alphabet ring is coprime to the code length, the so-called simple root codes. A few of the contributions to the study of the cases where the characteristic of the alphabet ring is not coprime to the code length (repeated root codes) are [1, 2, 5, 9, 17, 12]. In this paper we focus on the repeated root case where the code length is in fact a power of a prime.

Let \( p \) be a prime and consider cyclic and negacyclic codes of length \( p^s \) over \( GR(p^s, m) \). The study of such codes for the negacyclic case when \( p = 2 \) and \( m = 1 \) was undertaken in [8], where it was shown that the ambient \( \frac{Z_2[x]}{(x^{ps}+1)} \) is a chain ring. This result was extended to the case when \( m \) is arbitrary in [6]. The distances for

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most of these codes was calculated there. The chain ring structure of the ambient
was heavily used to accomplish this goal.

When \( a = 1 \), the Galois ring \( GR(p^a, m) \) is just the Galois field \( F_{p^m} \). Codes over
\( F_{p^m} \) were consider in [7]. There it was shown that for arbitrary \( p \), the ambient
space \( \frac{F_{p^m}[x]}{(x^m - 1)} \) is a chain ring. Once again the chain structure of the ambient space
was used to compute all the code distances. Then it was shown also in [7] that
\( \frac{F_{p^m}[x]}{(x^m - 1)} \oplus \frac{F_{p^m}[x]}{(x^m - 1)} \) when \( p \) is odd, which allows all the negacyclic results to be carried
over to the cyclic code case. It should be noted that over a field of characteristic 2,
there is no distinction between cyclic and negacyclic codes since \( \frac{F_{2^m}[x]}{(x^m - 1)} = \frac{F_{2^m}[x]}{(x^m - 1)} \).

In all cases mentioned so far, the codes correspond to principal ideals. This is a
consequence of the fact that the code ambients are chain rings. In the remaining
cases, which comprise the primary subject of this paper, the code ambients are no
longer chain rings and in fact, not even PIRs. There are three remaining cases:
negacyclic codes over \( GR(p^a, m) \) for odd prime \( p \) and \( a > 1 \) of length \( p^a \); cyclic
codes of the same type; cyclic codes over \( GR(2^a, m) \) for \( a > 1 \) of length \( p^a \). In
this paper these remaining cases are considered and a method for computing the
Hamming distance of any code is provided.

Now, simple root cyclic codes over \( Z_{p^m} \) were studied in [4] where a generating set
for such codes was formulated and it was also proved that these codes are principal
ideals of the ambient ring. An alternative generating set was given for codes over
\( Z_4 \) in [18]. This result was extended to \( Z_{p^m} \) in [10] where they also showed the
connection between the two formulations. These results were in turn extended to
simple root cyclic codes over Galois rings in [20].

In a series of papers ([19],[15],[16],[14]), the idea of Gröbner basis was extended
to principal ideal rings and was used to prove the existence of generating sets with
certain desirable properties for cyclic and negacyclic codes over chain rings. Specifically,
they showed that given this generating set, the code distance can be determined from one particular element in the generating set. In Section 3, we will use
this theory to determine all minimum code distances.

In Section 2, the necessary background on Galois rings is given together with
other results that are needed throughout the paper. Section 3 considers the class
of codes in \( \frac{GR(p^a, m)[x]}{(x^m - 1)} \) where \( p \) is an odd prime and \( a > 1 \). In that section it is
shown that \( \frac{GR(p^a, m)[x]}{(x^m - 1)} \) is a local ring with simple socle that is not a chain ring.
Then a method for computing Hamming distances is shown. Section 4 examines
cyclic codes. When \( p \) is odd, there is a one-one correspondence between cyclic and
negacyclic codes over \( GR(p^a, m) \) of length \( p^a \) for odd prime \( p \) which is shown. The
remainder of the section is devoted to cyclic codes over \( GR(2^a, m) \) for \( a > 1 \) of
length \( 2^a \). It is shown that \( \frac{GR(2^a, m)[x]}{(x^m - 1)} \) has a very similar structure to \( \frac{GR(p^a, m)[x]}{(x^m - 1)} \)
from Section 3. Again a method for computing Hamming distances is shown.

2. Preliminaries

In this paper, the word ring means finite commutative ring with identity. The
only exception is when we talk about the (infinite) ring \( R[x] \) of polynomials with
coefficients in the ring \( R \). A local ring is a ring with a unique maximal ideal. Given a
commutative ring \( R \), the Jacobson radical of \( R \), denoted by \( J(R) \), is the intersection
of all maximal ideals of \( R \) and the socle of \( R \), denoted by \( soc(R) \), is the sum of all
minimal ideals of \( R \). A polynomial \( f(x) \in R[x] \) is regular if it is not a zero divisor.
The following is a characterization of regular polynomials in polynomial rings over local rings.

**Lemma 2.1** (Theorem XIII.2, [13]). Let $R$ be a finite local commutative ring and $f(x) \in R[x]$ where $f(x) = a_0 + \cdots + a_n x^n$ for $a_i \in R$. The following are equivalent:

1. $f$ is a regular polynomial.
2. $(a_0, \ldots, a_n) = R$.
3. $a_i$ is a unit for some $0 \leq i \leq n$.
4. $f(x) \pmod{p} \neq 0$.

Polynomial rings over local rings admit a division algorithm for certain polynomials.

**Lemma 2.2** (Proposition 3.4.4, [3]). Let $R$ be a finite local commutative ring and $f(x), g(x) \in R[x]$ where $g(x)$ is regular. Then there exist $q(x), r(x) \in R[x]$ such that

$$f(x) = g(x)q(x) + r(x)$$

with $\deg(r) < \deg(g)$ or $r(x) = 0$.

A chain ring is a ring whose ideals are linearly ordered by inclusion. The following characterization of chain rings is well-known:

**Lemma 2.3.** Let $R$ be a finite commutative ring. The following are equivalent:

1. $R$ is a chain ring.
2. $R$ is a local principal ideal ring.
3. $R$ is a local ring with maximal ideal that is principal.

Galois rings constitute a very important family of finite chain rings. They can be defined as follows: Let $f(x) \in Z_{p^n}[x]$ be a basic irreducible polynomial (a basic irreducible polynomial in $Z_{p^n}[x]$ is an irreducible polynomial in $Z_{p^n}[x]$ whose reduction modulo $p$ is irreducible in $Z_p[x]$) and $m = \deg(f)$. Then the Galois ring $GR(p^n, m) = Z_{p^n}[x]/(f(x))$. It is well-known that different choices of $m$ and $a$ yield non-isomorphic Galois rings while, on the other hand, distinct choices of $f(x)$ with the same degree $m$ yield the same Galois ring up to isomorphism. We now list a few pertinent details about these rings. For a more detailed account of the theory of Galois rings including proofs of the results we mention here, see [13] or [20].

Every Galois ring $R = GR(p^n, m)$ contains a $(p^m - 1)^{th}$ primitive root of unity $\zeta$. Every $r \in R$ has a $p$-adic expansion $r = \zeta_0 + \zeta_1 p + \cdots + \zeta_{m-1} p^{m-1}$ where $\zeta_i \in \{0, 1, \zeta, \zeta^2, \ldots, \zeta^{p^m-2}\}$, the Teichmüller set $T_m$ of $R$.

Given a polynomial $f(x)$ in any polynomial ring $R[x]$, $f$ can be viewed in the form $f(x) = \sum_{i=0}^{k} a_i (x + 1)^i$ where $a_i \in R$. So, for $f \in GR(p^n, m)[x]$, $f(x) = \sum_{i=0}^{k} \sum_{j=0}^{a} \zeta_j p^j (x + 1)^i$ where $\zeta_j \in \{0, 1, \zeta, \zeta^2, \ldots, \zeta^{p^m-2}\}$.

The next two Lemmas are results on negacyclic code ambients over Galois rings which will be needed in the proceeding sections. Defining multiplication of $r \in \frac{GR(p^n, m)[x]}{(x^p + 1)}$ by $M \in \frac{GR(p^n, m)[x]}{(x^p + 1)}$ as multiplication in $\frac{GR(p^n, m)[x]}{(x^p + 1)}$ mod $p$, $\frac{GR(p^n, m)[x]}{(x^p + 1)}$ can be made into an $\frac{GR(p^n, m)[x]}{(x^p + 1)}$-module. In light of this, the following lemma is easy to see.

**Lemma 2.4.** For any prime $p$, the $\frac{GR(p^n, m)[x]}{(x^p + 1)}$-modules $p^{n-1} \frac{GR(p^n, m)[x]}{(x^p + 1)}$ and $\frac{GR(p^n, m)[x]}{(x^p + 1)}$ are isomorphic.
Lemma 2.5 ([7], Proposition 3.2). For any prime $p$, the ambient ring $\frac{GR(p,m)[x]}{(xp^n+1)}$ is a chain ring with exactly the following deals,

$$GR(p,m)[x] = ((x+1)^0) \supseteq \cdots \supseteq ((x+1)^p) = 0.$$ 

In [15] an algorithm was given to find a Gröbner basis for ideals of a polynomial ring over a PIR. Later in [19], it is shown that any ideal of a residue ring of a polynomial ring over a chain ring has a Gröbner basis with certain additional properties. Since for any prime $p$, $GR(p^a, m)$ is a chain ring, ideals of $\frac{GR(p^a, m)[x]}{(xp^n+1)}$ will have such a Gröbner basis. The following Lemma is a restatement of that result.

Lemma 2.6 (adapted from Theorem 4.1 in [19]). For any prime $p$, given an ideal $I \triangleleft \frac{GR(p^a, m)[x]}{(xp^n+1)}$, for $i \in \{0, \ldots, r\}$ there exist $j_i \in \mathbb{Z}$ and $f_i \in \frac{GR(p^a, m)[x]}{(xp^n+1)}$ where $0 \leq r \leq a - 1$ such that

$$I = \langle p^{j_0} f_0, \ldots, p^{j_r} f_r \rangle$$

and

1. $0 \leq j_0 < \cdots < j_r \leq a - 1$,
2. $f_i$ monic for $i = 0, \ldots, r$,
3. $p^i > \deg(f_0) > \cdots > \deg(f_r)$,
4. $p^{j_{i+1}} f_i \in (p^{j_{i+1}} f_{i+1}, \ldots, p^{j_r} f_r)$,
5. $p^{j_0}(xp^n + 1) \in (p^{j_0} f_0, \ldots, p^{j_r} f_r)$ in $GR(p^a, m)[x]$.

One can show further that the set of generators in Lemma 2.6 is a strong Gröbner basis in the sense of [19]. While interesting, this fact will not be used here.

The following Lemma is a special case of Kummer’s Theorem. We include the proof for completeness.

Lemma 2.7. Let $p$ be a prime. Let $k \leq \frac{p^n}{2}$ and $l$ be the largest integer such that $p^l \mid k$. Then $p^{n-l} \mid \binom{p^n}{k}$.

Proof. For $k \leq p$, the result holds. Now we proceed in 3 cases. First assume there is an $l > 0$ such that $p^l \mid k - 1$ and it is the largest such integer. Then $p^{n-l} \mid \binom{p^n}{k-1}$. Since $p^l \mid k - 1$, $p \mid k$ and $p^l \mid p^n - k + 1$. So, $p^l \mid \frac{p^n - k + 1}{k}$. Hence, $p^{n-l+1} \mid \binom{p^n}{k-1} = \binom{p^n}{k}$. Now, assume $p \mid k - 1$ and $p \nmid k$. Then $p \nmid p^n - k + 1$. So, for any $l$ such that $p^l \mid \binom{p^n}{k-1}$, $p^l \mid \binom{p^n}{k}$. Noting the previous case, $p^n \mid \binom{p^n}{k}$.

Now, assume there is an $l > 0$ such that $p^l \mid k$ and it is the largest such integer. Then $p \mid k - 1$ and so $p \nmid p^n - k + 1$. Again noting the previous cases, $p^{n-l} \mid \binom{p^n}{k}$.

3. Negacyclic codes in $\frac{GR(p^a, m)[x]}{(xp^n+1)}$ for odd prime $p$

As mentioned earlier, all ambient rings previously studied in the literature are chain rings. They are $\frac{GR(p^a, m)[x]}{(xp^n+1)}$, $\frac{GR(p^a, m)[x]}{(xp^{n-1}+1)}$ and $\frac{GR(p^a, m)[x]}{(xp^{n-1}+1)}$ for $a,m,p,s \in \mathbb{Z}$ where $a \geq 1$, $m \geq 1$, $p$ is prime and $s \geq 0$. In the following sections the remaining cases will be studied. In these remaining cases, the ambient spaces are not chain rings. We will show, however, that they are local rings with simple socle.

In this section, the structure of $\frac{GR(p^a, m)[x]}{(xp^n+1)}$ where $p$ is an odd prime and $a > 1$ is studied so that in the following section the structure details can be used to find Hamming distance of all codes. Since $s = 0$ is the trivial case also assume $s > 0$.

We start by showing that $x+1$ is nilpotent. The calculation of its exact nilpotency is saved for Corollary 3.7.
Proposition 3.1. In $\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}$, $(x + 1)$ is nilpotent.

Proof. 

$$(x + 1)^{p^r} = x^{p^r} + (p^r - 1)x^{p^r - 1} + \cdots + (p^r - 1)x + 1 = x^r + p\alpha(x) = p\alpha(x)$$

where $\alpha(x) \in \frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}$. Then $(x + 1)^{p^r} = p^a(\alpha(x))^a = 0$. \hfill \Box

Proposition 3.2. The ambient ring $\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}$ is local with radical $J\left(\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}\right) = \langle p, x + 1 \rangle$.

Proof. Let $I$ be the set of non-invertible elements. Let $f \in \langle p, x + 1 \rangle$. By Proposition 3.1 $(x + 1)$ is nilpotent. Since $p$ is also, $f$ is nilpotent and hence not invertible. So, $\langle p, x + 1 \rangle \subset I$. Now, let $f \in I$. We can write $f(x) = \sum_{i=0}^{k} a_i(x + 1)^i$ where $a_i \in GR(p^a, m)$. So, $f$ is invertible if and only if $a_0$ is invertible. The $p$-adic expansion $a_0 = \sum_{i=0}^{a-1} b_i p^i$ where $b_i \in \{0, 1, \zeta, \zeta^2, \ldots, \zeta^{p^{a-2}}\}$ assures that $a_0$ is invertible if and only if $b_0 \neq 0$. The assumption that $f$ not invertible implies therefore, that $p \mid a_0$ and this shows that $f \in \langle p, x + 1 \rangle$. So, $I \subset \langle p, x + 1 \rangle$. Since $I$ contains all invertible elements, it is the unique maximal ideal and therefore, $\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}$ is local. \hfill \Box

Proposition 3.3. The socle $soc\left(\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}\right)$ of $\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}$ is the simple module $\langle p^{a-1}(x + 1)(p^r - 1) \rangle$.

Proof. Using Lemma 2.5, it can be shown that $\langle p^{a-1}(x + 1)(p^r - 1) \rangle \subset soc\left(\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}\right)$. Let $a(x) \in soc\left(\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}\right)$. Since $J(\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}) = \langle p, x + 1 \rangle$, $pa(x) = 0$ and $(x + 1)a(x) = 0$ so $a(x) \in \langle p^{a-1}(x + 1)(p^r - 1) \rangle$. Hence, $soc\left(\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}\right) = \langle p^{a-1}(x + 1)(p^r - 1) \rangle$.

By Lemmas 2.4 and 2.5, it is clear that $soc\left(\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}\right)$ is simple. \hfill \Box

Proposition 3.4. In the ambient ring $\frac{GR(p^a, m)[x]}{(x^{p^r} + 1)}$

1. $p \not\in \langle x + 1 \rangle$,
2. $x + 1 \not\in \langle p \rangle$,
3. $GR(p^a, m)[x]_{(x^{p^r} + 1)}$ is not a chain ring,
4. $(p, x + 1)$ is not a principal ideal.

Proof. Assume $p \in \langle x + 1 \rangle$. So, $p = (x + 1)f(x) + (x^{p^r} + 1)g(x)$ in $GR(p^a, m)[x]$. When $x = -1$, $p = 0$ which is a contradiction in this case since $a \neq 1$. Hence, $p \not\in \langle x + 1 \rangle$.

Now assume $x + 1 \in \langle p \rangle$. Then $x + 1 = pf(x) + (x^{p^r} + 1)g(x)$. Now, comparing coefficients, $1 = pf_0 + g_0$ which implies $1 \equiv g_0$ modulo $p$. Also, $0 = pf_{p^r} + g_{p^r}$ which implies $g_0 \equiv -g_{p^r}$ modulo $p$. In general, $0 = pf_{kp^r} + g_{(k-1)p^r} + g_{kp^r}$ for $k \geq 1$. So, $g_{kp^r} \neq 0$ for $k \geq 0$ which is a contradiction since $g$ is a polynomial. Hence, $x + 1 \not\in \langle p \rangle$.

Finally, since $0 \not\in \langle p \rangle \not\subseteq \langle x + 1 \rangle$ and $0 \not\in \langle x + 1 \rangle \not\subseteq \langle p \rangle$, $GR(p^a, m)[x]_{(x^{p^r} + 1)}$ is not a chain ring. Since any local ring with principal maximal ideal is a chain ring by
Lemma 2.3. \( \frac{GR(p^a,m)}{x^{p^a+1}} \) cannot have principal maximal ideal. Hence, \( \langle p, x + 1 \rangle \) is 2-generated.

\[ \text{Theorem 3.5. The ambient ring } \frac{GR(p^a,m)}{x^{p^a+1}} \text{ is a finite local ring with simple socle but not a chain ring.} \]

\[ \text{Proof. Result of Propositions 3.2, 3.3 and 3.4.} \]

\[ \text{Example 1. To illustrate Theorem 3.5, we provide the following figure. It shows the ideal lattice of } \frac{Z_{32}[x]}{(x^2+1)}. \text{ Notice that the radical is } \langle 3, x + 1 \rangle \text{ and the socle is } \langle 3(x + 1)^2 \rangle. \text{ More importantly, we see that the ring is not a chain ring.} \]

\[ \text{Now we are ready to develop our main structural Lemma.} \]

\[ \text{Lemma 3.6. In } \frac{GR(p^a,m)}{x^{p^a+1}} \text{ for } t \geq 0, \]

\[ (x + 1)^{p^t+t(p-1)p^{t-1}} = p^{t+1}b_t(x)(x + 1)^{p^{t-1}} + a_t(x) \]

\[ \text{where } b_t(x) \text{ is invertible and } p^{t+2}|a_t(x). \]

\[ \text{Proof. We proceed by induction on } t. \text{ For } t = 0, \]

\[ 0 = x^{p^0} + 1 = (x + 1 - 1)^{p^0} + 1 \]

\[ = (x + 1)^{p^0} - \left( \frac{p^0}{p^0 - 1} \right)(x + 1)^{p^{0-1}} + \left( \frac{p^0}{p^0 - 2} \right)(x + 1)^{p^{0-2}} - \cdots + \left( \frac{p^0}{1} \right)(x + 1) \]
By Lemma 2.7
\[(x + 1)^{p^s} = (p^s(x + 1))^{p^s-1} - (p^{s-2})(x + 1)^{p^s-2} + \cdots - (p^s(x + 1)) = (p^s(x + 1))^{p^s-1} + \alpha_0(x) \]
for some \(\alpha_0(x)\) such that \(p^2|\alpha_0(x)\) and \(\beta_0(x)\) invertible.

Now assume the result holds for \(t - 1\). So there exist some \(\alpha_{t-1}(x)\) such that \(p^{t+1}|\alpha_{t-1}(x)\) and \(\beta_{t-1}(x)\) invertible where \((x + 1)^{p^{t+1}} = p^{t+1}\beta_{t-1}(x)(x + 1)^{p^{t-1}} + \alpha_{t-1}(x)\). So
\[(x + 1)^{p^{t+1}(p-1)p^{t-1}} = (x + 1)^{p^{t+1}(p-1)p^{t-1}}(x + 1)^{(p-1)p^{t-1}} \]
\[= \left[p^{t+1}\beta_{t-1}(x)(x + 1)^{p^{t-1}} + \alpha_{t-1}(x)\right] \left(x + 1\right)^{(p-1)p^{t-1}} \]
\[= p^{t+1}\beta_{t-1}(x)(x + 1)^{p^{t-1}} + p^{t+1}\beta_{t-1}(x)\alpha_{t-1}(x) + \alpha_{t-1}(x)(x + 1)^{(p-1)p^{t-1}} \]
\[= p^{t+1}\left[\beta_{t-1}(x)b(x) + \frac{\alpha_{t-1}(x)}{p^{t+1}}(x + 1)^{(p-2)p^{t-1}}\right] + p^{t+1}\beta_{t-1}(x)\alpha_{t-1}(x) \]
\[= p^{t+1}\beta(x)(x + 1)^{p^{t-1}} + \alpha(x) \]

\[\square\]

**Corollary 3.7.** In \(\frac{GR(p^s, m)}{(x^p + 1)}\), the nilpotency of \(x + 1\) is \(p^s a - p^{s-1}(a - 1)\).

**Proof.** By Lemma 3.6,
\[(x + 1)^{p^s + (a-2)(p-1)p^{s-1}} = p^{a-1}b(x)(x + 1)^{p^{s-1}} + a(x) \]
for some \(b(x)\) is invertible and \(a(x)\) such that \(p^a|a(x)\). So, \(a(x) = 0\) and
\[(x + 1)^{p^s + (a-2)(p-1)p^{s-1}} = p^{a-1}b(x)(x + 1)^{p^{s-1}} \]
So,
\[(x + 1)^{p^s + (a-1)(p-1)p^{s-1}} = p^{a-1}b(x)(x + 1)^{p^{s-1}} \]
meaning
\[(x + 1)^{p^s + (a-1)(p-1)p^{s-1}} = p^{a-1}b(x)(x + 1)^{p^{s-1}} \neq 0. \]
Finally,
\[(x + 1)^{p^s + (a-1)(p-1)p^{s-1}} = p^{a-1}b(x)(x + 1)^{p^s} = 0. \]
Hence the nilpotency of \(x + 1\) is \(p^s + (a - 1)(p - 1)p^{s-1} = p^s a - p^{s-1}(a - 1). \) \[\square\]
Given Remark 1. It should be clear that the isomorphism in Lemma 2.4 is an isometry when the Hamming weight in $\frac{GR(p^s,m)}{\langle x^p+1 \rangle}$ is defined similarly to the weight in $\frac{GR(p^s,m)}{\langle x^p+1 \rangle}$.

Theorem 4.11 of [7] provides the distances for any code in $\frac{GR(p^s,m)}{\langle x^p+1 \rangle}$, which we include here.

**Lemma 3.8 (Theorems 4.11, [7]).** In $\frac{GR(p^s,m)}{\langle x^p+1 \rangle}$, for $0 \leq i \leq p^s$

$$d((x+1)^i)) = \begin{cases} 1 & \text{if } i = 0, \\ \beta + 2 & \text{if } \beta p^{s-1} + 1 \leq i \leq (\beta + 1)p^{s-1} \text{ where } 0 \leq \beta \leq p - 2, \\ (t+1)p^k & \text{if } p^s - p^{s-k} + (t-1)p^{s-k-1} + 1 \leq i \leq p^s - p^{s-k} + tp^{s-k-1}, \\ & \text{where } 1 \leq t \leq p - 1 \text{ and } 1 \leq k \leq s - 1, \\ 0 & \text{if } i = p^s, \end{cases}$$

Using one additional result from [19] the distances of all codes in $\frac{GR(p^s,m)}{\langle x^p+1 \rangle}$ can be found. Moreover, Lemma 2.6 is algorithm based, so using all these results, an algorithm exists for finding the distances of these codes.

**Lemma 3.9 (adapted from Theorem 6.1 in [19]).** Given a set of generators for a code $I \triangleleft \frac{GR(p^s,m)}{\langle x^p+1 \rangle}$ as in Lemma 2.6, $d(I) = d(\langle y^{a-1}f_r \rangle)$.

**Remark 2.** Given Remark 1 and Lemmas 3.8 and 3.9, the distance of any code in $\frac{GR(p^s,m)}{\langle x^p+1 \rangle}$ can be determined. Let $I \triangleleft \frac{GR(p^s,m)}{\langle x^p+1 \rangle}$. We can find $f_1, \ldots, f_r \in \frac{GR(p^s,m)}{\langle x^p+1 \rangle}$ such that $I = \langle p^{a_0}f_0, \ldots, p^{a_r}f_r \rangle$ where this set satisfies the properties of Lemma 2.6. Then Lemma 3.9 implies that $d(I) = d(\langle y^{a-1}f_r \rangle)$. Next we view $f_r$ in canonical form. Write

$$f(x) = \beta_0(x + 1)^{a_0} + \beta_1(x + 1)^{a_1} + \cdots + \beta_{a-1}(x + 1)^{a_{a-1}} \alpha_{a-1}(x)$$

where $\beta_k \in \mathbb{F}_m$ and $\alpha_k(x) \in \frac{GR(p^s,m)}{\langle x^p+1 \rangle}$ is invertible. Note since $f_r$ is monic, $\beta_0 \neq 0$. So, $p^{a-1}f_r = p^{a-1}\beta_0(x + 1)^{a_0} \alpha_{a_0}(x)$. Since $\beta_0$ and $\alpha_0(x)$ are units, $d(I) = d(\langle y^{a-1}f_r \rangle) = d(\langle y^{a-1}(x + 1)^{a_0} \rangle)$. In light of Remark 1, the distance $d(\langle y^{a-1}(x + 1)^{a_0} \rangle)$ can be found using Lemma 3.8.

4. **Cyclic codes in $\frac{GR(p^s,m)}{\langle x^p-1 \rangle}$ for arbitrary prime $p$**

Let us first consider the case when $p$ is an odd prime. It is easy to see, arguing as in Proposition 5.1 in [8], that $\frac{GR(p^s,m)}{\langle x^p+1 \rangle} \cong \frac{GR(p^s,m)}{\langle x^p-1 \rangle}$ by sending $x$ to $-x$. Hence, all the results can in Section 3 translate easily into results about cyclic codes. Let us therefore focus solely on the case when $p = 2$ for the remainder of this section.

Remember that in [6], it was shown that $\frac{GR(2^s,m)}{\langle x^p+1 \rangle}$ is a chain ring. They also computed the Hamming distance for most of the codes. Let us now consider the cyclic case i.e. the code ambient $\frac{GR(2^s,m)}{\langle x^p-1 \rangle}$. It turns out that when $a > 1$, this is not a chain ring but the structure is very similar to the ring considered in Section
3. In this section assume \( a > 1 \) and as before \( s > 0 \) since \( s = 0 \) produces the trivial case. Most of the proofs here are very similar to their analogs in Section 3. We include only the proofs that need fundamental modification.

**Proposition 4.1.** In \( \frac{GR(2^a, m)[x]}{(x^2 - 1)} \), \((x + 1)\) is nilpotent.

**Proof.**

\[
(x + 1)^{2^t} = x^{2^t} + \binom{2^t}{2^s - 1} x^{2^t - 1} + \cdots + \binom{2^t}{1} x + 1 - 1 + 1
= x^{2^t} - 1 + 2\alpha(x)
= 2\alpha(x)
\]

where \( \alpha(x) \in \frac{GR(2^a, m)[x]}{(x^2 - 1)} \). Then \((x + 1)^{2^a} = 2^a (\alpha(x))^2 = 0.\)

The following propositions can be obtained from the parallel results in Section 3 by replacing \( p \) with 2 and \( x^p + 1 \) with \( x^{2^t} - 1 \) and using Proposition 4.1 in lieu of Proposition 3.1 when needed.

**Proposition 4.2.** The ambient ring \( \frac{GR(2^a, m)[x]}{(x^2 - 1)} \) is local with radical \( J \left( \frac{GR(2^a, m)[x]}{(x^2 - 1)} \right) = (2, x + 1) \).

**Proposition 4.3.** The socle \( \text{soc} \left( \frac{GR(2^a, m)[x]}{(x^2 - 1)} \right) \) of \( \frac{GR(2^a, m)[x]}{(x^2 - 1)} \) is the simple module \( \langle 2^{a-1}(x + 1)(2^{s-1}) \rangle \).

**Proposition 4.4.** In \( \frac{GR(2^a, m)[x]}{(x^2 - 1)} \)

1. \( 2 \notin \langle x + 1 \rangle \),
2. \( x + 1 \notin (2) \),
3. \( GR(2^a, m)[x] / (x^2 - 1) \) is not a chain ring,
4. \((2, x + 1)\) is not a principal ideal.

**Theorem 4.5.** The ambient ring \( \frac{GR(2^a, m)[x]}{(x^2 - 1)} \) is a finite local ring with simple socle but not a chain ring.

**Proof.** Result of Propositions 4.2, 4.3 and 4.4.

The following Lemma is similar to Lemma 3.6 with a subtle difference in the divisor of \( a_t(x) \) which is used in the last line of the proof.

**Lemma 4.6.** Assume \( s > 1 \). In \( \frac{GR(2^a, m)[x]}{(x^2 - 1)} \) for \( t \geq 0 \),

\[
(x + 1)^{2^t + t2^{s-1}} = 2^{t+1} b_t(x)(x + 1)^{2^{s-1}} + a_t(x)
\]

where \( b_t(x) \) is invertible and \( 2^{t+2}(x + 1)|a_t(x) \).

**Proof.** We proceed by induction on \( t \). For \( t = 0 \),

\[
0 = x^{2^t} - 1 = ((x + 1) - 1)^{2^t} - 1
= (x + 1)^{2^s} - \binom{2^s}{2^s - 1}(x + 1)^{2^{s-1}} + \binom{2^s}{2^s - 2}(x + 1)^{2^{s-2}}
- \cdots - \binom{2^s}{1}(x + 1)
\]
By Lemma 2.7 and the fact that $s > 1$

$$(x + 1)^{2^s} = \left(\frac{2^s}{2} - 1\right) (x + 1)^{2^s - 1} - \left(\frac{2^s}{2} - 2\right) (x + 1)^{2^s - 2} + \cdots + \left(\frac{2^s}{2} - \frac{s}{1}\right) (x + 1)$$

$$= \left(\frac{2^s}{2} - 1\right) (x + 1)^{2^s - 1} + a_0(x)$$

$$= 2b_0(x)(x+1)^{2^s-1} + a_0(x)$$

for some $a_0(x)$ such that $2^s(x + 1)|a_0(x)$ and $b_0(x) = \frac{2^s}{2} - 1$ which is invertible.

Now assume the result holds for $t - 1$. So there exists some $a_{t-1}(x)$ such that $2^t(x + 1)|a_{t-1}(x)$ and $b_{t-1}(x)$ invertible where $(x + 1)^{2^t+(t-1)2^{s-1}} = 2^t b_{t-1}(x)(x + 1)^{2^{s-1}} + a_{t-1}(x)$. So

$$(x + 1)^{2^s+t2^{s-1}} = (x + 1)^{2^s+(t-1)2^{s-1}}(x + 1)^{2^{s-1}}$$

$$= \left[2^t b_{t-1}(x)(x + 1)^{2^{s-1}} + a_{t-1}(x)\right] (x + 1)^{2^{s-1}}$$

$$= 2^t b_{t-1}(x)(x + 1)^{2^{s-1}} + a_{t-1}(x)(x + 1)^{2^{s-1}}$$

$$= 2^t b_{t-1}(x) \left[2b_0(x)(x+1)^{2^{s-1}} + a_0(x)\right] + a_{t-1}(x)(x + 1)^{2^{s-1}}$$

$$= 2^{t+1} b_{t-1}(x) b_0(x)(x + 1)^{2^{s-1}} + 2^t b_{t-1}(x) a_0(x) + a_{t-1}(x)(x + 1)^{2^{s-1}}$$

$$= 2^{t+1} b_{t-1}(x)(x + 1)^{2^{s-1}} + a_t(x)$$

where $2^{t+2}(x + 1)|a_t(x)$ and $b_t(x)$ invertible.

\[ \Box \]

**Corollary 4.7.** In $\displaystyle \frac{GR(2^n,m)}{(x^2-1)}$, the nilpotency of $x + 1$ is $(a + 1)2^{s-1}$.

**Proof.** We first conclude the case when $s = 1$. From the fact that

$$0 = x^2 - 1 = (x + 1) - 1 = (x + 1)^2 - 2(x + 1) + 1 - 1 = (x + 1)^2 - 2(x + 1)$$

we see $(x + 1)^2 = 2(x + 1)$. So, $(x + 1)^{i+1} = 2^i (x + 1)$ for $i > 0$. Since $(x + 1)^a = 2^a (x + 1) \neq 0$ and $(x + 1)^{a+1} = 2^a (x + 1) = 0$, the nilpotency of $x + 1$ is $a + 1$.

Now, assume $s > 1$. By Lemma 4.6,

$$(x + 1)^{2^s+(a-2)2^{s-1}} = 2^{a-1} b(x)(x + 1)^{2^{s-1}} + a(x)$$

for some $b(x)$ is invertible and $a(x)$ such that $2^a | a(x)$. So, $a(x) = 0$ and

$$(x + 1)^{2^s+(a-2)2^{s-1}} = 2^{a-1} b(x)(x + 1)^{2^{s-1}}.$$

So,

$$(x + 1)^{2^s+(a-2)2^{s-1}} (x + 1)^{2^{s-1}-1} = 2^{a-1} b(x)(x + 1)^{2^{s-1}}$$

meaning

$$(x + 1)^{2^s+(a-1)2^{s-1}-1} = 2^{a-1} b(x)(x + 1)^{2^{s-1}} \neq 0.$$ 

Finally,

$$(x + 1)^{2^s+(a-1)2^{s-1}} = 2^{a-1} b(x)(x + 1)^{2^{s-1}} = 0.$$ 

Hence the nilpotency of $x + 1$ is $2^a + (a - 1)2^{s-1} = (a + 1)2^{s-1}$. \[ \Box \]
As was the case with most of the structure results, the Hamming distance results in Section 3 can easily be adapted to this setting. The main results needed are the Hamming Distances for the codes in $\frac{GR(2,m)[x]}{(x^2-1)}$. These distances again were obtained in [7].

Lemma 4.8 (Corollary 4.12, [7]). In $\frac{GR(2,m)[x]}{(x^2-1)}$, for $0 \leq i \leq 2^s$

$$d(((x+1)^i)) = \begin{cases} 1 & \text{if } i = 0, \\ 2 & \text{if } 1 \leq i \leq 2^{s-1}, \\ 2^{k+1} & \text{if } 2^s - 2^{s-k} + 1 \leq i \leq 2^s - 2^{s-k} + 2^{s-k-1}, \\ 0 & \text{if } i = 2^s, \end{cases}$$

Remark 3. Given Lemma 4.8, the same method as in Section 3 of Remark 2 can be applied here to compute the Hamming distances for codes in $\frac{GR(2,m)[x]}{(x^2-1)}$.

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