FOURIER RESTRICTION TO SMOOTH ENOUGH CURVES

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Abstract. We prove Fourier restriction estimates to arbitrary compact $C^N$ curves for any $N > d$ in the (sharp) Drury range, using a power of the affine arclength measure as a mitigating factor. In particular, we make no nondegeneracy assumption on the curve.

1. Introduction

The boundedness of restriction operators $\mathcal{R}: L^p(\mathbb{R}^d) \to L^q(\gamma; d\sigma)$ associated with curves $\gamma: \mathbb{R} \to \mathbb{R}^d$ has been studied for decades. The first natural choice of measure $d\sigma$ is the Euclidean arclength measure. In this case the maximal range of $p$ and $q$ for which a restriction operator exists depends on the order of vanishing of the torsion.

Definition 1.1. For a compact interval $I \subset \mathbb{R}$ and a curve $\gamma \in C^d(I; \mathbb{R}^d)$, define the torsion $\tau(t) = |\det[y'(t), \ldots, y^{(d)}(t)]|$. Furthermore, for $\epsilon \geq 0$ define the weight

$$w_\epsilon(t) = \tau(t)^{\frac{\epsilon}{d+1}}.$$  

In particular, $\omega_0(t)dt$ is the well-studied affine arclength measure, which is also interesting due to the affine invariance of the problem. See [23] for background on affine geometry. Using the affine arclength measure, Drury [20] discovered a proof for the optimal range of $p$ and $q$ in the least-degenerate case: when $\gamma$ is the moment curve $y(t) = (t, t^2, \ldots, t^d)$. For various classes of curves, many authors (see below for the history) have shown that the affine arclength measure compensates for vanishing of the torsion, so restriction operators exist for nearly the same range of $p$ and $q$ as the moment curve. Several authors have also used the overdamped affine arclength measure, $\epsilon > 0$ in (1), to attain the exact range of $p$ and $q$ for the moment curve. The case of a general curve $y \in C^\infty(I)$ has long been expected to behave similarly. Building on techniques in [12, 16, 20, 35], our main result establishes the boundedness of restriction operators for arbitrary compact curves that are smooth enough. In the theorem, $C^N := C^{[N], N-[N]}$.

Theorem 1.2. Let $d \geq 2$, $I \subset \mathbb{R}$ be a compact interval, $N \in \mathbb{R}$ such that $N > d$, and $y \in C^N(I; \mathbb{R}^d)$. For $\epsilon \geq 0$ let $w_\epsilon$ be the weight defined in (1). Let

$$1 \leq p < \frac{d^2 + d + 2}{d^2 + d}$$

and

$$1 \leq q < \frac{2^+}{d^2+d} p' \quad \text{if } \epsilon > \sum_{j=1}^d \frac{1}{N-j},$$
$$1 \leq q < \frac{\sum_{j=1}^d \frac{d}{N-j}}{1+\sum_{j=1}^d \frac{d}{N-j}} p' \quad \text{if } 0 \leq \epsilon \leq \sum_{j=1}^d \frac{1}{N-j}.$$
Then there is $C = C(I, d, \gamma, N, p, q, \epsilon) > 0$ such that for any $f \in L^p(\mathbb{R}^d)$,

$$
\left( \int_I |\hat{f}(\gamma(t))|^q w(t) \, dt \right)^{\frac{1}{q}} \leq C \|f\|_{L^p(\mathbb{R}^d)}.
$$

Remark 1.3. When $\gamma \in C^\infty(I)$, Theorem 1.2 gives a restriction bound for $q$ on the scaling line $q = \frac{2}{d^2 + d} p'$ in the overdamped case with any $\epsilon > 0$ and for all $q < \frac{2}{d^2 + d} p'$ with the affine arclength measure.

Remark 1.4. In the case $\epsilon \leq \sum_{j=1}^d \frac{1}{N-j}$, the range of $q$ in (2) is empty whenever

$$
1 < \frac{2}{d^2 + d} + \epsilon \frac{1}{1 + \frac{d^2 + d}{2} \sum_{j=1}^d \frac{1}{N-j} p'}.
$$

Thus, to obtain restriction estimates for all $p$ in the range $1 \leq p < \frac{d^2 + d + 2}{d^2 + d}$, we need

$$
\sum_{j=1}^d \frac{1}{N-j} \leq \frac{4}{(d^2 + d)^2} + \frac{\epsilon(d^2 + d + 2)}{d^2 + d}.
$$

If $N$ is much larger than $d$, then (3) is true even in the undamped case $\epsilon = 0$. When $N$ is closer to $d$, there is always some $0 < \epsilon_0 < \sum_{j=1}^d \frac{1}{N-j}$ such that (3) holds for all $\epsilon \geq \epsilon_0$. See Figure 1 below, which uses the extension operator instead of the restriction operator for visual clarity.

![Figure 1](image)

**Figure 1.** Range of $q'$ and $p'$ for which the extension operator associated with an arbitrary $\gamma \in C^{d+1}$ is bounded from $L^q$ to $L^p$, with shading indicating the amount of damping required.

In the case of the moment curve, where the torsion is constant, the offspring curve method is available to prove optimal restriction estimates. That method includes an analysis of the function

$$
\Phi_\gamma(t_1, \ldots, t_d) = \gamma(t_1) + \cdots + \gamma(t_d).
$$
When \( \gamma \) is the moment curve, the Jacobian \( J_{\Phi} \) is (up to a multiplicative constant) equal to the Vandermonde determinant
\[
v(t_1, \ldots, t_d) = C_d \prod_{1 \leq i < j \leq d} (t_j - t_i).
\]

To prove Theorem 1.2, we apply the Drury method to intervals on which the torsion of \( \gamma \) is comparable to some dyadic value. We also need to ensure that \( \Phi_{\gamma} \) is well-behaved on each interval. The main difficulty is controlling the number of intervals we study. This is the content of the following theorem:

**Theorem 1.5.** Let \( I \subset \mathbb{R} \) be a compact interval. If \( N \in \mathbb{N} \) with \( N > d \) and \( \gamma \in C^N(I; \mathbb{R}^d) \), then there is a family of intervals \( I_{k,l} \) such that
\[
I = \{ t \in I : \tau(t) = 0 \} \cup \bigcup_{k \geq C_d} \bigcup_{l=1}^{N_k} I_{k,l},
\]
\[
2^{-k-2} \leq \tau(t) \leq 2^{-k+1} \text{ for } t \in I_{k,l},
\]
\[
(t_1, \ldots, t_d) \mapsto \Phi_{\gamma}(t_1, \ldots, t_d) \text{ is } 1\to1 \text{ for } t_1 < \cdots < t_d \in I_{k,l},
\]
\[
|J_{\Phi_{\gamma}}(t_1, \ldots, t_d)| \geq C_d 2^{-k} |v(t_1, \ldots, t_d)| \text{ for } (t_1, \ldots, t_d) \in (I_{k,l})^d,
\]
\[
N_k \leq C_{d,N} 2^k \sum_{j=1}^d \frac{1}{j}, \text{ and }
\]
\[
\sum_{l=1}^{N_k} |I_{k,l}| \leq C_{d,N,N'} (k + C_d \cdot \gamma)^d |I|.
\]

**History.** In [33], Stein traces the roots of Fourier restriction theory to observations about the continuity of Fourier transforms of radial functions. Laurent Schwartz and many others independently observed that if \( 1 < p < \frac{d+1}{d} \) and \( f \in L^p(\mathbb{R}^d) \) is a radial function, then \( \hat{f}(r) \) is continuous for all \( 0 < r < \infty \). Thus, \( \hat{f} \) can be thought of as a function on \( S^{d-1} \), even though that set has measure 0. Stein wondered if one could give a similar statement about Fourier restrictions of nonradial functions. He successfully proved such a result in 1967, but did not publish because it was unclear what purpose such a lemma would have.

In 1970, Fefferman [21], in collaboration with Stein, improved on Stein’s lemma in 2 dimensions and showed that the \( n \)-dimensional lemma could be used to make progress on the multiplier problem for the ball. Interest in the restriction problem picked up, and by 1974 the case of curves had largely been solved by Zygmund [37], Hörmander [25], and Sjölin [31]. While Zygmund and Hörmander dealt with nondegenerate curves (\( \tau \neq 0 \)), Sjölin brought in the measures \( w(t) \cdot dt \) to understand curves with vanishing curvature. The case of \( d = 2 \) and \( \gamma \in C^\infty \) in Theorem 1.2 is due to Sjölin [31]. There have been several more papers [8, 9, 22, 26, 27, 30, 32] that have answered some remaining questions in 2 dimensions. In the most recent of these (in 2021), Fraccaroli [22] proved a restriction theorem in the optimal range for all continuous convex curves, which is the first result that did not require \( C^2 \).
Unfortunately, the techniques that work well in two dimensions do not carry over to higher dimensions. Thus, it was a few more years before any results were known. Prestini broke into the high-dimensional setting by proving a restriction theorem for curves with nonvanishing torsion in 1978 \[28\] for \(d = 3\) and 1979 \[29\] for \(d \geq 4\). However, her range of \(p\) was not sharp: it was \(1 \leq p < \frac{d^2+2d}{d^2+2d-2}\) (compare with \[2\]). She also did not attain bounds on the scaling line \(q = \frac{2}{d^2+d} p'\), which is seen to be the largest possible value of \(q\) by inspecting Knapp examples. In 1982, Christ \[14\] extended Prestini’s theorem to include the scaling line, and then in 1985 \[13\] he provided restriction estimates for curves of finite type with the same range of \(p\) and for \(q\) up to the scaling line. Furthermore, those bounds included \(q\) on the scaling line in a restricted range of \(p\). The range \(1 \leq p < \frac{d^2+2d}{d^2+2d-2}\) is called the Christ-Prestini range. See also \[30\] for the first result for a curve with vanishing torsion.

At a similar time that Christ was beginning the study of curves with finite type, Drury \[20\] was concluding the study of curves with nonvanishing torsion. He proved a restriction theorem for nondegenerate \(C^d\) curves in the optimal range \(1 \leq p < \frac{d^2+2d+2}{d^2+2d}\) and \(1 \leq q \leq \frac{2}{d^2+d} p'\). Optimality of the range of \(p\) is due to Arkhipov, Karacuba, and Čubarikov \[1\] (see also \[2\]). Further results for nondegenerate curves appear in \[3, 7\].

Shortly thereafter, Drury and Marshall \[17, 18\] improved the known estimates for curves of finite type, and then in 1990 Drury \[19\] further improved the results for curves \(\gamma(t) = (t, t^2, \ldots, t^k)\), \(k \geq 4\).

Little further progress was made for curves with vanishing torsion in higher dimensions until 2008, when Bak, Oberlin, and Seeger \[4\] solved the monomial curve case. In that and a subsequent paper \[6\], they also obtained endpoint results. Shortly thereafter, Dendrinos and Müller \[15\] obtained results for perturbed monomial curves. General polynomial curves were covered for a restricted range of \(p\) by Dendrinos and Wright \[16\], and then for the full range of \(p\) by Stovall \[35\]. See also \[24\] for results with general measures.

In addition to monomial curves, Bak, Oberlin, and Seeger \[5\] also proved restriction theorems for simple curves

\[ \gamma(t) = (t, t^2, \ldots, t^d, \phi(t)) \]

such that \(\phi \in C^d\) with \(\phi^{(d)}\) satisfying a certain inequality. Chen, Fan, and Wang \[12\] were able to dispense with the inequality, but at the cost of enforcing \(\phi \in C^N\) for some \(N > d\). By a change of variables, the case of \(d = 2\) and \(\gamma \in C^N\) in Theorem 1.2 is due to Chen, Fan, and Wang \[12\]. Another result of this nature appears in \[36\].

With the polynomial case solved, it is likely that an argument based on \(\varepsilon\)-removal and polynomial approximation could be used to solve the general \(C^\infty\) case off the scaling line. Thus, the most interesting new consequences of Theorem 1.2 are (in dimension \(d \geq 3\)) the scaling line estimates for \(C^\infty(\mathbb{R}^d)\) curves with \(\varepsilon > 0\) and the nontrivial range of \(p\) and \(q\) for \(C^N(\mathbb{R}^d)\) curves with \(\varepsilon = 0\).
Outline of proof. Section 2 uses Theorem 2.4, which is a stronger version of Theorem 1.5, as a black box to prove Theorem 1.2. It begins with a restriction result on each interval in the decomposition given by Theorem 2.4 and then combines these estimates into a restriction inequality on the whole interval. Sections 3 and 4 are devoted to a proof of Theorem 2.4. Section 3 constructs a decomposition for Theorem 2.4 in two steps. The first step decomposes \( I \) into intervals on which \( \gamma \) is well-behaved, and the secondary decomposition creates intervals where certain auxiliary curves are similarly well-behaved. Finally, Section 4 finishes the proof of the geometric inequality (8) and the condition (7) on each interval in the decomposition.

Notation. Let \( \tilde{I} \) be a compact interval, \( d \in \mathbb{N}, N \in \mathbb{R} \) with \( N > d \), and \( \gamma \in C^d(\tilde{I}; \mathbb{R}^d) \). These will remain fixed throughout this paper, and we will prove Theorem 1.2 and Theorem 1.5 with these fixed values. \( C \) denotes an arbitrary constant that may change line by line and is always allowed to depend on the dimension \( d \) and the interval \( \tilde{I} \). Any subscripts indicate additional dependence: for instance, \( C_\gamma \) is a constant that depends only on \( \gamma \), the dimension, and the original interval. For two numbers \( A \) and \( B \), write \( A \asymp B \) if there exist constants \( C \) and \( C' \) such that

\[
CB \leq A \leq C'B.
\]

Once again, subscripts indicate additional dependence. Logarithms are taken in base 2 purely for the convenience of calculations in Section 3.

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2. Proof of Theorem 1.2

In this section, we use a strengthening of Theorem 1.5 to prove Theorem 1.2. The first step is to prove a restriction estimate on each interval of the decomposition (6). Define the family of offspring curves

\[
\Upsilon = \left\{ \gamma_h(t) = \frac{1}{m} \sum_{j=1}^{m} \gamma(t + h_j) : m \in \mathbb{N}, h \in \mathbb{R}^m, 0 \leq h_1 \leq \cdots \leq h_m \right\}.
\]

For an interval \( I = [a, b] \), set \( I_h = [a - h_1, b - h_m] \). We will use induction to show that a restriction bound holds uniformly for \( \gamma_h \in \Upsilon \).

Proposition 2.1. Let \( I \subseteq \tilde{I} \) be a compact interval and \( k \in \mathbb{Z} \). Suppose that for every \( \gamma_h \in \Upsilon \),

\[
(t_1, \ldots, t_d) \mapsto \Phi_{\gamma_h}(t_1, \ldots, t_d) \text{ is } 1\text{-to-}1 \text{ for } t_1 < \cdots < t_d \in I_h \text{ and}
\]

\[
|J_{\Phi_{\gamma_h}}(t_1, \ldots, t_d)| \geq C 2^{-k} |\nu(t_1, \ldots, t_d)| \text{ for } (t_1, \ldots, t_d) \in I_h^d.
\]

Then for

\[
1 \leq p < \frac{d^2 + d + 2}{d^2 + d} \quad \text{and} \quad q = \frac{2p'}{d^2 + d'},
\]

...
we have the restriction inequality
\begin{equation}
\left( \int_{I_0} |f(Y_h(t))|^q dt \right)^{\frac{1}{q}} \leq 2^{k} C_p \|f\|_{L^p(\mathbb{R}^d)}
\end{equation}
for all $f \in L^p(\mathbb{R}^d)$ and all $Y_h \in \mathcal{Y}$.

**Proof.** We adapt Drury’s argument from [20]. By duality, it suffices to study the extension operator
$$
\mathcal{E}_h g(x) = \int_{I_0} e^{iy_h(t) \cdot x} g(t) dt.
$$
We will show that
$$
\|\mathcal{E}_h g\|_{L^{p'}(\mathbb{R}^d)} \leq 2^{k} C_p \|g\|_{L^{p'}(\mathcal{I})},
$$
for
$$
1 \leq q' < \frac{d^2 + d + 2}{2}, \quad \frac{d^2 + d}{2p'} + \frac{1}{q'} = 1.
$$
The proof is by induction on $q'$. Hausdorff-Young shows that the base case $q' = 1$ and $p' = \infty$ is true. The induction hypothesis is that for some $1 \leq q'_0 < \frac{d^2 + d + 2}{d^2 + d}$ and $p'_0$ defined by
$$
\frac{d^2 + d}{2p'_0} + \frac{1}{q'_0} = 1,
$$
the following inequality holds uniformly for $Y_h \in \mathcal{Y}$:
\begin{equation}
\|\mathcal{E}_h g\|_{L^{p'_0}(\mathbb{R}^d)} \leq 2^{k} C_{p'_0} \|g\|_{L^{p'_0}(\mathcal{I})}.
\end{equation}
Fix $Y_h \in \mathcal{Y}$. For ease of notation, set $\zeta = Y_h$, $I = I_h$, and $\mathcal{E} = \mathcal{E}_h$. To improve the bound (12), we first write
$$
\left( \mathcal{E}_h g \left( \frac{x}{d} \right) \right)^d = \left( \int_I e^{i \zeta(t) \cdot \frac{x}{d}} g(t) dt \right)^d = \int_A e^{ix \cdot \frac{1}{d} \Sigma_{j=1}^{d} \zeta(t_j)} \prod_{j=1}^{d} g(t_j) dt_1 \ldots dt_d.
$$
Set
$$
A = \{ (t_1, \ldots, t_d) \in I^d : t_1 < \cdots < t_d \}.
$$
By symmetry in $t_1, \ldots, t_d$,
$$
\left( \mathcal{E}_h g \left( \frac{x}{d} \right) \right)^d = d! \int_A e^{ix \cdot \frac{1}{d} \Sigma_{j=1}^{d} \zeta(t_j)} \prod_{j=1}^{d} g(t_j) dt_1 \ldots dt_d.
$$
With the change of variables
$$
t = t_1, \quad h_j = t_j - t_1 \text{ for } 1 \leq j \leq d,
$$
and with $B$ the image of $A$ under this change of variables, we observe that
$$
\left( \mathcal{E}_h g \left( \frac{x}{d} \right) \right)^d = d! \int_B e^{ix \cdot \frac{1}{d} \Sigma_{j=1}^{d} \zeta(t + h_j)} \prod_{j=1}^{d} g(t + h_j) dt dh_2 \ldots dh_d.
$$
For fixed $h_2, \ldots, h_d$, each curve
\[ t \mapsto \frac{1}{d} \sum_{j=1}^{d} \zeta(t + h_j) \]
is an offspring curve in the family $\Upsilon$. Let $\vartheta$ be the Vandermonde determinant (5) and define
\[ TG(x) = \int e^{ix \cdot \frac{1}{d} \sum_{j=1}^{d} \zeta(t + h_j)} G(t, h)\vartheta(h)\,dh. \]

**Lemma 2.2.** We have the bound
\begin{equation}
\|TG\|_{L^p_0} \leq 2^k C_p \|G\|_{L^1_{\rho_0}(L^2_{\Lambda'}; [\vartheta(h)])}.
\end{equation}

**Proof.** An application of Minkowski’s inequality for integrals shows that
\[ \|TG\|_{L^p_0} \leq \int \left\| \int e^{ix \cdot \frac{1}{d} \sum_{j=1}^{d} \zeta(t + h_j)} G(t, h)\,dt \right\|_{L^p_0(\mathbb{R}^d)} |\vartheta(h)|\,dh. \]
Employing the induction hypothesis (12), we obtain
\[ \|TG\|_{L^p_0} \leq 2^k C_p \int \|G(\cdot, h)\|_{L^2_{\rho_0}(\Lambda_0)} |\vartheta(h)|\,dh. \]
The lemma now follows from the inequality
\[ 2^k C_p \int \|G(\cdot, h)\|_{L^2_{\rho_0}(\Lambda_0)} |\vartheta(h)|\,dh \leq 2^k C_p \|G\|_{L^1_{\rho_0}(L^2_{\Lambda'}; [\vartheta(h)])}. \]

**Lemma 2.3.** We have the bound
\begin{equation}
\|TG\|_{L^2(\mathbb{R}^d)} \leq 2^k C \|G\|_{L^2(\rho_0[L^2_{\Lambda'}; [\vartheta(h)])}. \end{equation}

**Proof.** Set
\[ y = \frac{1}{d} \sum_{j=1}^{d} \zeta(t + h_j). \]
This change of variables is injective because of (9). The Jacobian is
\[ J(t, h) = \frac{1}{d^d} J_\rho_0(t, t + h_2, \ldots, t + h_d). \]
The geometric inequality (10) guarantees that
\begin{equation}
J(t, h) \geq C 2^{-k} \vartheta(h).
\end{equation}
With these variables, set
\[ F(y) = \mathbb{1}_B(t, h)G(t, h) \frac{\vartheta(h)}{J(t, h)}, \]
Applying the change of variables to $T$, we see that
\[ TG(x) = \int e^{iy \cdot x} F(y)\,dy = \tilde{F}(x). \]
Plancherel gives $\| F \|_2 = \| F \|_{L^2}$, so $\| T G \|_{L^2(\mathbb{R}^d)} = \| F \|_{L^2(dg)}$. Changing variables back and unwinding the definition of $F$ yields

$$
\| T G \|_{L^2(\mathbb{R}^d)} = \left( \int \frac{v(h)}{f(h) + \epsilon} |G(t, h)|^2 v(h) dt dh \right)^{\frac{1}{2}}.
$$

Lines 15 and 16 combine to demonstrate that

$$
\| T G \|_{L^2(\mathbb{R}^d)} \leq 2^k C \left( \int |G(t, h)|^2 v(h) dt dh \right)^{\frac{1}{2}}.
$$

Since

$$
\left( \int |G(t, h)|^2 v(h) dt dh \right)^{\frac{1}{2}} \leq \| G \|_{L^p(d \nu)}
$$

the inequality 14 is true. \(\square\)

Interpolation of 13, 14, and the trivial $L^1(L^1) \to L^\infty$ estimate establishes

$$
\| T G \|_{L^p} \leq 2^\frac{k}{p} C_{a, b} \| G \|_{L^p, (L^2; \nu(h))}
$$

for all $(a^{-1}, b^{-1})$ in the triangle with vertices $(1, 1)$, $(1, \frac{1}{d})$, and $\left(\frac{1}{2}, \frac{1}{2}\right)$, with $c$ satisfying

$$
\frac{(d + 2)(d - 1)}{2} a^{-1} + b^{-1} + \frac{d^2 + d}{2} c^{-1} = \frac{d^2 + d}{2}.
$$

In particular, the choice of

$$
G(t, h) = |v(h)|^{-1} \int_{j=1}^d g(t + h_j)
$$

has

$$
\| G \|_{L^p, (L^2; \nu(h))} = \left( \int |v(h)|^{-(a-1)} \left( \int_{\mathbb{R}^d} |g(t + h_1) \cdots g(t + h_d)|^b dt \right)^{\frac{2}{b}} dh \right)^{\frac{2}{b}}.
$$

As noted in 20, $v(0, h')^{-1} \in L^d_{\nu', \infty}$, so we can apply Hölder’s inequality to obtain

$$
\| G \|_{L^p, (L^2; \nu(h))} \leq \| g \|_{L^q(\mathbb{R}^d)}^d
$$

for

\[
\begin{align*}
1 < a < \frac{d+2}{2}, \\
a \leq b < \frac{2a}{d+2-da}, \text{ and} \\
\frac{d}{q} = \frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} - \frac{d(d-1)}{2}.
\end{align*}
\]

On the other hand, by the definition of $G$ (18),

$$
TG(x) = \frac{1}{d!} \left( \mathcal{E} g \left( \frac{x}{d} \right) \right)^d.
$$

Combining 17, 19 and 20, we see that

$$
\| \mathcal{E} g \|_{L^{p'}} \leq C_{p} \frac{2^k}{s} \| g \|_{L^q(\mathbb{R}^d)},
$$

for

\[
\begin{align*}
1 < a < \frac{d+2}{2}, \\
a \leq b < \frac{2a}{d+2-da}, \text{ and} \\
\frac{d}{q} = \frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} - \frac{d(d-1)}{2}.
\end{align*}
\]
for
\[(22)\quad \frac{d}{q'} = \frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} - \frac{d(d-1)}{2},\]

where \(p' = \frac{d^2+d}{2}q\), and \(a\) and \(b\) satisfy (Figure 2):
\[
\begin{align*}
\frac{d}{a^2} &< a^{-1} < 1, \\
b^{-1} &< a^{-1}, \\
(d+2)a^{-1} - 2b^{-1} &< d, \text{ and} \\
(q_0' - 2)a^{-1} + q_0' b^{-1} &\geq q_0' - 1.
\end{align*}
\]

The point \((a^{-1}, b^{-1}) = \left(\frac{d}{a^2}, \frac{2}{a^2} + \frac{d-2}{(d+2)q_0}\right)\) lies on the boundary of this region and satisfies
\[
\frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} - \frac{d(d-1)}{2} < \frac{d}{q_0'}.
\]

Taking \((a^{-1}, b^{-1})\) slightly inside of the region and using real interpolation, we obtain
\[
\|\mathcal{K}g\|_{L^{p'}} \leq C_p 2^{\frac{k}{p'}} \|g\|_{L^{p'}}, \quad p' = \frac{d^2+d}{2}q, \quad \frac{d}{q'} > \frac{2}{(d+2) + (d+2)q_0'}.\]

This closes the induction and proves Proposition 2.1.

Now we turn to the deduction of Theorem 1.2 from the following stronger version of Theorem 1.5. Recall that the symbol “≈” depends only on the dimension \(d\) and the original interval \(I\), unless otherwise specified by subscripts. In particular, there is no dependence on \(m\) or \(h\) in what follows.
Theorem 2.4. Let $\tilde{I} \subset \mathbb{R}$ be a compact interval. If $N \in \mathbb{R}$ with $N > d$ and $\gamma \in C^N(\tilde{I}; \mathbb{R}^d)$, then there is a family of intervals $I_{k,l}$ such that for every $m \in \mathbb{N}$ and $h \in \mathbb{R}^m$,

\begin{equation}
\tilde{I} = \{ t \in \tilde{I} : \tau(t) = 0 \} \cup \bigcup_{k \geq C_y} \bigcup_{l=1}^{N_k} I_{k,l},
\end{equation}

\begin{equation}
\tau_h(t) \approx 2^{-k} \text{ for } t \in (I_{k,l})_h,
\end{equation}

$(t_1, \ldots, t_d) \mapsto \Phi_y(t_1, \ldots, t_d)$ is 1-to-1 for $t_1 < \cdots < t_d \in (I_{k,l})_h$.

$|J_{\Phi_y}(t_1, \ldots, t_d)| \geq C2^{-k}|\sigma(t_1, \ldots, t_d)|$ for $(t_1, \ldots, t_d) \in (I_{k,l})_h^d$,

\begin{equation}
N_k \leq C_{N, \gamma}(k + C_\gamma)^d 2^{k \sum_{j=1}^{d} \frac{1}{q_j}}, \text{ and }
\end{equation}

\begin{equation}
\sum_{k \geq C_y} \sum_{l=1}^{N_k} |I_{k,l}| \leq C_{N, \gamma}(k + C_\gamma)^d.
\end{equation}

Deduction of Theorem 1.2. Let $\{I_{k,l}\}$ be the intervals given in Theorem 2.4. By (23),

$$
\int_{\tilde{I}} |\hat{f}(\gamma(t))|^q \omega_x(t) \, dt \leq \sum_{k \geq C_y} \sum_{l=1}^{N_k} \int_{I_{k,l}} |\hat{f}(\gamma(t))|^q \omega_x(t) \, dt.
$$

Utilizing the bound (24) on the size of the torsion on each interval,

$$
\sum_{k \geq C_y} \sum_{l=1}^{N_k} \int_{I_{k,l}} |\hat{f}(\gamma(t))|^q \omega_x(t) \, dt \leq C \sum_{k \geq C_y} \sum_{l=1}^{N_k} 2^{-\frac{2k}{d+dp}} \int_{I_{k,l}} |\hat{f}(\gamma(t))|^q \, dt.
$$

An application of Hölder’s inequality shows that for any $q \leq \frac{2}{d+dp} p'$,

$$
\sum_{k \geq C_y} \sum_{l=1}^{N_k} 2^{-\frac{2k}{d+dp}} \int_{I_{k,l}} |\hat{f}(\gamma(t))|^q \, dt \leq \sum_{k \geq C_y} \sum_{l=1}^{N_k} 2^{-\frac{2k}{d+dp}} |I_{k,l}| 1^{-\frac{(d+dp)q}{2p'}} \left( \int_{I_{k,l}} |\hat{f}(\gamma(t))|^{\frac{2p'}{d+dp}} \, dt \right)^{\frac{(d+dp)q}{2p'}},
$$

where $|I_{k,l}|$ is the length of the interval $I_{k,l}$. Each interval $I_{k,l}$ satisfies the hypotheses of Proposition 2.1. so by (11),

$$
\sum_{k \geq C_y} \sum_{l=1}^{N_k} 2^{-\frac{2k}{d+dp}} |I_{k,l}| 1^{-\frac{(d+dp)q}{2p'}} \left( \int_{I_{k,l}} |\hat{f}(\gamma(t))|^{\frac{2p'}{d+dp}} \, dt \right)^{\frac{(d+dp)q}{2p'}} \leq \sum_{k \geq C_y} \sum_{l=1}^{N_k} 2^{-\frac{2k}{d+dp}} |I_{k,l}| 1^{-\frac{(d+dp)q}{2p'}} 2^q C_p^d \|f\|_{L^p(\mathbb{R}^d)}^q.
$$
Another application of Hölder’s inequality yields
\[
\sum_{k \geq C_N} \sum_{l=1}^{N_k} 2^{-d/2} \cdot k^\frac{d}{2p'} \cdot N_k \cdot \epsilon_j \cdot |I_{k,l}| \leq C_p^q \|f\|_{L_P(\mathbb{R}^d)} \sum_{k \geq C_N} 2^{-d/2} \cdot k^\frac{d}{2p'} \cdot N_k \cdot \epsilon_j |I_{k,l}|.
\]
With the bounds on \(N_k\) and \(\epsilon_j\) on the total lengths of the intervals \(I_{k,l}\),
\[
C_p^q \|f\|_{L_P(\mathbb{R}^d)} \sum_{k \geq C_N} 2^{-d/2} \cdot k^\frac{d}{2p'} \cdot N_k \cdot \epsilon_j \cdot |I_{k,l}| \leq C_{p,q,N,\epsilon} \|f\|_{L_P} \sum_{k \geq 1} k^d 2^{-d/2} \cdot k^\frac{d}{2p'} \cdot \epsilon_j ^{d/2} \cdot |I_{j-1,1}|^{1/2}.
\]
The sum converges whenever
\[
\frac{(d^2 + d)q}{2p'} \sum_{j=1}^{d} \frac{1}{N-j} - \frac{2}{d^2 + d} - \epsilon + \frac{q}{p'} < 0,
\]
which occurs in either of the cases:
\[
\begin{cases}
  1 \leq q < \frac{2}{d^2 + d} p' & \text{if } \epsilon > \sum_{j=1}^{d} \frac{1}{N-j} \\
  1 \leq q < \frac{2}{d^2 + d} \frac{\epsilon + p'}{\Sigma_{j=1}^{d} \frac{1}{N-j}} & \text{if } 0 \leq \epsilon \leq \sum_{j=1}^{d} \frac{1}{N-j}.
\end{cases}
\]

3. The Decomposition

This section contains the decomposition for Theorem 2.4, of which Theorem 1.5 is essentially a special case. First, we will create an initial decomposition using Lemma 8 from [12] to find intervals on which we can prove Theorem 2.4 for the original curve \(\gamma\). Then, we will use polynomial approximation and Lemma 2.3 from [35] to decompose further into intervals on which offspring curves are well-behaved.

More concretely, the methods in [16] that we need to prove Theorem 2.4 on each interval in our final decomposition require an examination of minors of the torsion matrix for all offspring curves. With that in mind, for a curve \(\zeta\), a permutation \(\sigma \in S_d\) (the symmetric group on \(d\) elements), and \(1 \leq j \leq d\), define
\[
L_{\sigma,j}^\zeta(t) = \det \begin{pmatrix} 
\zeta'_{\sigma(1)}(t) & \cdots & \zeta'_{\sigma(j)}(t) \\
\vdots & \ddots & \vdots \\
\zeta'_{\sigma(j)}(t) & \cdots & \zeta'_{\sigma(d)}(t) 
\end{pmatrix}.
\]
Whenever \(j = d\), we will omit \(\sigma\) since \(|L_{\sigma,d}^\zeta|\) does not depend on \(\sigma\). We also omit \(\sigma\) when \(\sigma\) is the identity. Recall that \(\gamma \in C_N\) for some \(d < N \in \mathbb{R}\). The main result of this section is the following proposition.
Proposition 3.1. For every $k_d \in \mathbb{Z}$, there is a family of intervals $\{I_l\}$ and permutations $\sigma_l$ such that for every $m \in \mathbb{N}$ and $h \in \mathbb{R}^m$,

$$\{t \in I : 2^{-k_d - 1} \leq |y_d^h(t)| \leq 2^{-k_d}\} \subseteq \bigcup_l I_l,$$

$$|L_{j,l}^h (t)| \approx 2^{-k_j}, \quad \forall t \in (I_l)_h, \; 1 \leq j \leq d,$$ and

(27) $$\# \{I_l\} \leq C_{N,Y} (k_d + C_Y)^d 2^{k_d \sum_{j=1}^d \frac{1}{j^2}},$$

$$\sum_l |I_{k,l}| \leq C(k_d + C_Y)^d.$$

The initial decomposition. We first prove Proposition 3.1 in the special case $h = 0$.

Proposition 3.2. For every $k_d \in \mathbb{Z}$, there is a family of intervals $\{I_l\}$ with

(28) $$\sum_{I \in \mathcal{I}_{k_d}} |I| \leq C(k_d + C_Y)^d \text{ and } \# \{I_l\} \leq C_{N,Y} 2^{k_d \sum_{j=1}^d \frac{1}{j^2}},$$

such that

$$\{t \in I : 2^{-k_d - 1} \leq |y_d^h(t)| \leq 2^{-k_d}\} \subseteq \bigcup_{I \in \mathcal{I}_{k_d}} I.$$

Furthermore, there are constants $A_j$ depending only on $Y, j$, and $d$ such that on each interval $I \in \mathcal{I}_{k_d}$, there is a permutation $\sigma \in S_d$ and $k_j \in \mathbb{Z}$ with $A_j \leq k_j \leq k_d + A_j + \log(d ||y||_{Cd})$ such that

(29) $$2^{-k_j - 2} \leq |L_{\sigma,j}^h (t)| \leq 2^{-k_j + 1}, \quad t \in I, \; 1 \leq j \leq d.$$

The first step in proving Proposition 3.2 is to show that for each $t \in I$, there is a permutation $\sigma$ such that the $L_{\sigma,j}^h$’s are generally decreasing in $j$.

Lemma 3.3. There are constants $A_j$ such that for every $t \in I$, there is a permutation $\sigma$ such that if

$$2^{-k_d - 1} \leq |y_d^h(t)| \leq 2^{-k_d},$$

then there is $k_j \in \mathbb{Z}$ with $A_j \leq k_j \leq k_d + A_j + \log(d ||y||_{Cd}^d)$ such that

$$2^{-k_j - 1} \leq |y_{\sigma,j}^h (t)| \leq 2^{-k_j}.$$

Proof. Fix $t \in I$. For each $j$, let $k_j$ be the unique integer that satisfies

$$2^{-k_j - 1} \leq |y_{\sigma,j}^h (t)| < 2^{-k_j}.$$

For any permutation $\sigma$,

$$y_{\sigma,j} = j ||y||_{Cd}^j.$$

Hence,

(30) $$k_j \geq - \log(j ||y||_{Cd}^j).$$
To get an upper bound on \( k_j \), we'll show by induction that there is a permutation \( \sigma \) such that

\[
|L^\gamma_{\sigma,j}(t)| \geq \frac{j!}{d! \|y\|_C^d} |L^\gamma_d(t)| \quad \text{for all } 1 \leq j \leq d.
\]

The base case of \( j = d \) is true for every permutation. In the induction step, suppose that \( \sigma(d), \ldots, \sigma(j + 2) \) have been specified (with nothing specified if \( j = d - 1 \)). Let \( M \) be the torsion matrix with the last \( d - j - 1 \) columns deleted and rows \( \sigma(d), \ldots, \sigma(j + 2) \) deleted. Then \( |\det M| = |L^\sigma_{j+1}| \). For \( 1 \leq i, l \leq j \), let \( M_{i,l} \) be the minor of \( M \) obtained by deleting the \( i \)th row and \( l \)th column. Using the cofactor expansion, we find that

\[
|L^\sigma_{j+1}(t)| = \frac{1}{(j + 1)! \|y\|_C^d} \sum_{i=1}^{j+1} (-1)^{i+j+1} y_i^{(d)}(t) \det M_{i,l} \leq \|y\|_C^d \sum_{i=1}^{j+1} |\det M_{i,l+1}|.
\]

Hence there is some \( i \) such that

\[
|\det M_{i,l+1}| \geq \frac{1}{(j + 1)! \|y\|_C^d} |L^\sigma_{j+1}(t)|.
\]

Thus, for every permutation \( \sigma \) that sends \( i \) to \( j + 1 \),

\[
|L^\gamma_{\sigma,j}(t)| \geq \frac{1}{(j + 1)! \|y\|_C^d} |L^\sigma_{j+1}(t)|.
\]

By the induction hypothesis,

\[
|L^\gamma_{\sigma,j}(t)| \geq \frac{1}{(j + 1)! \|y\|_C^d} \frac{(j + 1)!}{d! \|y\|_C^{d-j-1}} |L^\gamma_d(t)| = \frac{j!}{d! \|y\|_C^{d-j}} |L^\gamma_d(t)|.
\]

This completes the induction, so \([31]\) holds. Putting together \([30]\) and \([31]\),

\[
- \log(j! \|y\|_C^d) \leq k_j \leq - \log \left( \frac{j!}{d! \|y\|_C^{d-j}} \right) + k_d.
\]

The lemma follows by setting \( A_j = - \log(j! \|y\|_C^d) \).

We can combine Lemma 3.3 and the following lemma from [12] \( d \) times to prove Proposition 3.2.

**Lemma 3.4** ([12] Lemma 8). Let \( \varphi \in C^{1/\alpha}(I) \) with \( \alpha > 0 \). For every \( k \in \mathbb{Z} \), there exist disjoint intervals \( \{I_{k,j} \subseteq I\} \) such that \( 2^{-k-2} \leq |\varphi(t)| \leq 2^{-k+1} \) for all \( t \in I_{k,j} \) and

\[
\{ t \in I : 2^{-k-1} \leq |\varphi(t)| \leq 2^{-k} \} \subseteq \bigcup_{j=1}^{N_k} I_{k,j};
\]

moreover, there is a constant \( B_\alpha \) such that \( N_k \leq B_\alpha 2^{\alpha k} \) for every \( k \).

**Remark 3.5.** The proof of the above lemma in [12] shows that we can take

\[
B_\alpha = \|\varphi\|_C^{\frac{1}{\alpha}} 2^{\alpha + 4}. \quad \frac{1}{4} 2^{\alpha + 4}.
\]
Proof of Proposition 3.2. For an interval $J$, a permutation $\sigma$, $1 \leq j \leq d$, and $k \in \mathbb{Z}$, let $\mathcal{J}_j^\sigma(J,k)$ be the set of intervals from Lemma 3.4 with $\varphi = L_j$ and $\frac{1}{\alpha} = N - j$. To simplify the notation, set

$$B_j = \|y\| \frac{1}{CN} 4^\frac{j}{j_+} 4^N - j + 1.$$  

Then for any interval $J$, the number of intervals of $\mathcal{J}_J^\sigma(J,k)$ is at most $B_j 2^N$. Combining Lemma 3.3 with $d$-many applications of Lemma 3.4, we see that

$$\{t \in I : 2^{-k_d-1} \leq |L_j(t)| \leq 2^{-k_d} \}$$

is bounded by

$$\sum_{k_j = A_j}^{k_d + A_j + \log(dj)} B_j 2^N \leq B_j \frac{\sum_{k_d + A_j}^{k_d + A_j} \log(dj)}{2^N - 1}.$$  

Recalling that all logarithms are taken in base 2 and using the fact that $N - j > 1$,

$$B_j \frac{\sum_{k_d + A_j}^{k_d + A_j} \log(dj)}{2^N - 1} \leq C\|y\| \frac{1}{CN} 4^N \left( \frac{\log(dj)}{j_+} \right)^\frac{1}{j_+} \leq C\|y\| \frac{1}{CN} 4^N \|y\| \sum_{j=1}^{d-1} \frac{kd}{2^N} - \frac{kd^2}{2^N}.$$  

Hence, the set $\{t \in I : 2^{-k_d-1} \leq |L_j(t)| \leq 2^{-k_d} \}$ is covered by at most

$$C 4^N 4^N \|y\| \sum_{j=1}^{d-1} \left( \frac{kd}{2^N} \right)^{d-j} 2^kd \sum_{j=1}^{d-1} \frac{kd}{2^N} = C_N 4^N \|y\| \sum_{j=1}^{d-1} \frac{kd}{2^N}.$$  

many intervals that satisfy (29). Moreover, the sum of the lengths of the intervals in each pair of unions (32) is bounded by

$$(k_d + \log(dj)) |I_{j+1}|,$$

since the intervals in $\mathcal{J}_J^\sigma(I_{j+1}, k_j)$ are disjoint for every $k_j$. Therefore, the total length of all the intervals in the initial decomposition at scale $k_d$ is at most

$$d! (k_d + \log(dj)) |I_k| = C(k_d + C_d)^d.$$  

\[\square\]
The secondary decomposition. We now proceed to the general $h \in \mathbb{R}^m$ in Proposition 3.1. Our initial decomposition gave a family of intervals where $|L_{j,a}^Y| \approx 2^{-kj}.$ We finish the proof of Proposition 3.1 by applying the following proposition to each $\zeta_j = (\gamma_1, \ldots, \gamma_j)$ in turn. We need to ensure the intervals in the initial decomposition are small for this proposition. By the upper bounds (27) on the total length and the number of intervals, we can freely shrink the intervals to be of size at most $C_{N,j} 2^{-k_d} \sum_{j=1}^{\frac{1}{2}}$ while retaining the necessary upper bound (28) on the total number of intervals.

**Proposition 3.6.** Let $I = [a, b]$ with $b - a \leq 1$ and let $\zeta \in C^N(I; \mathbb{R}^j).$ There is a constant $A$ depending only on $N$ and $j$ such that if $|L_{j,a}^\zeta| \approx 2^{-k}$ on $I$ and

$$b - a \leq A N^{-j} \|\zeta\|_{C^N} 2^{-k}$$

there is a decomposition $I = \bigcup_{i=1}^{C_N} I_i$ into disjoint intervals such that for every $m \in \mathbb{N}$ and $h \in \mathbb{R}^m,$

$$|L_{j,a}^{\zeta_h}(t)| \approx 2^{-k} \quad \forall t \in (I_i)_h.$$

**Proof.** By Taylor’s theorem, for any $t \in I,$

$$\zeta(t) = \sum_{i=0}^{[N]-1} \frac{\zeta^{(i)}(a)}{i!}(t - a)^i + \frac{\zeta^{([N])}(z_t)}{[N]!} (t - a)^{[N]}$$

$$= \sum_{i=0}^{[N]} \frac{\zeta^{(i)}(a)}{i!}(t - a)^i + \frac{\zeta^{([N])}(z_t) - \zeta^{([N])}(a)}{[N]!} (t - a)^{[N]}.$$

Set

$$P(t) = \sum_{i=0}^{[N]} \frac{\zeta^{(i)}(a)}{i!}(t - a)^i \quad \text{and} \quad R(t) = \frac{\zeta^{([N])}(z_t) - \zeta^{([N])}(a)}{[N]!} (t - a)^{[N]}.$$

Then $R \in C^N(I),$ $R^{(i)}(a) = 0$ for $0 \leq i \leq [N],$ and

$$|R^{(i)}(t)| \leq |t - a|^{N-i} \|\zeta\|_{C^N} \quad \forall t \in I.$$

For any $h \in \mathbb{R}^m$ and $t \in I_h,$

$$L_{j,a}^{\zeta_h}(t) = \det[P'_h(t) + R'_h(t), \ldots, p_{(j)}^h(t) + R_{(j)}^h(t)]$$

$$= \det[P'_h(t), \ldots, p_{(j)}^h(t)]$$

$$+ \det[R'_h(t), P''_h(t), \ldots, p_{(j)}^h(t)]$$

$$+ \det[P'_h(t) + R'_h(t), R''_h(t), P'''_h(t), \ldots, p_{(j)}^h(t)]$$

$$\ldots$$

$$+ \det[P'_h(t) + R'_h(t), \ldots, P_{(j-1)}^h(t) + R_{(j-1)}^h(t), R_{(j)}^h(t)]$$

$$=: L_{j,a}^{p_h}(t) + L_{P,R,j,h}(t).$$

We will show that $|L_{P,R,j,h}| \leq 2^{-k}$ with an implicit constant depending on $A$ in (33) (which we can make as small as we need), and then we will divide $I$ into intervals where
\(|L_j^{P_h}| \approx 2^{-k}\). Combined, these imply that \(|L_j^{\xi h}| \approx 2^{-k}\). We start with \(|L_{P,R,j,h}|\). Applying \(34\), we see that
\[|L_{P,R,j,h}(t)| \leq C|I|^{N-j}\|\xi\|_{CN}^j \quad \forall t \in I_h.\]
Let \(A\) be a small constant to be chosen shortly. Assuming the bound \(33\) on the size of each interval holds, we conclude that
\[|L_{P,R,j,h}(t)| \leq AC2^{-k} \quad \forall t \in I_h.\]
The next lemma will give a decomposition to deal with \(|L_j^{P_h}|\).

**Lemma 3.7** \((35), \text{Lemma } 2.3\). Fix \(j \geq 2\) and \(l \in \mathbb{N}\). There exists a decomposition of \([-1,1]\) into disjoint intervals \(I_i, 1 \leq i \leq C_l\), such that for every \(I_i\) and every degree \(l\) polynomial \(Q: \mathbb{R} \to \mathbb{R}\) satisfying
\[|L_j^Q(t)| \approx 1, \quad t \in [-1,1],\]
every offspring curve \(Q_h(t)\) satisfies
\[|L_j^{Q_h}(t)| \approx_{L,j} 1 \quad \forall t \in (I_i)_h.\]
For \(t \in [-1,1]\), set
\[Q(t) = 2^k\left(2 \frac{2^{\frac{1}{2} l}}{b-a}\right)^{\frac{l+1}{2}} \left(P_1\left(\frac{b-a}{2}(t+1)+a\right), \ldots, P_j\left(\frac{b-a}{2}(t+1)+a\right)\right)\]
Before we can apply Lemma \(3.7\), we calculate
\[|L_j^Q(t)| = 2^k\left(2 \frac{2^{\frac{1}{2} l}}{b-a}\right)^{\frac{l+1}{2}} L_j^P\left(\frac{b-a}{2}(t+1)+a\right) \quad \forall t \in [-1,1].\]
By the calculation in \(35\) with \(h = 0\), we have
\[L_j^{\xi}(t) = L_j^P(t) + L_{P,R,j,0}(t).\]
Since \(|L_j^{\xi}| \approx 2^{-k}\) and \(|L_{P,R,j,0}(t)| \leq AC2^{-k}\) by \(36\), we can choose \(A\) small enough that \(|L_j^P| \approx 2^{-k}\). Therefore,
\[|L_j^Q(t)| \approx 1 \quad \forall t \in [-1,1].\]
Applying Lemma \(3.7\), we obtain a decomposition \([-1,1] = \cup_{i=1}^C J_i\) into disjoint intervals such that
\[|L_j^{Q_h}(t)| \approx 1 \quad \forall t \in J_i.\]
Setting
\[I_i = \left\{ t \in I : \frac{b-a}{2}(t-1) + a \in J_i \right\},\]
we see that
\[|L_j^{P_h}(t)| \approx 2^{-k} \quad \forall t \in I_i.\]
As before, we can choose \(A\) small enough that we can combine \(35\), \(36\), and \(37\) to conclude \(|L_j^{P_h}(t)| \approx 2^{-k}\). \(\square\)
4. The Geometric Inequality

To finish the proof of Theorem 2.4, we need to show that $\Phi \gamma$ given in (4) is 1-to-1 for $t_1 < \cdots < t_d$ and that the geometric inequality (8) holds on each interval given in Lemma 3.2. The following proposition shows that the geometric inequality (8) holds on each interval in our decomposition for $\gamma$, all offspring curves $\gamma h$, and all truncations of these curves.

**Proposition 4.1.** Let $I \subset \mathbb{R}$, $n \in \mathbb{N}$, $\xi : I \to \mathbb{R}^n$, and

$$\Phi \xi (t_1 \cdots t_n) = \xi(t_1) + \cdots + \xi(t_n).$$

Assume that there are $k_j \in \mathbb{Z}$, $1 \leq j \leq n$, such that (38)

$$|L^j_\xi| \approx 2^{-k_j} \text{ on } I.$$

Then the Jacobian

$$J_{\Phi \xi}(t_1, \ldots, t_n) = \det[\xi'(t_1) \cdots \xi'(t_n)]$$

satisfies

$$|J_{\Phi \xi}(t_1, \ldots, t_n)| \approx_n 2^{-k_n} |\nu(t_1, \ldots, t_n)|$$

for all $(t_1, \ldots, t_n) \in I^n$.

The above proposition shows that if $I$ is an interval in the decomposition, $1 \leq n \leq d$, and $\xi = ((\gamma_h)_1, \ldots, (\gamma_h)_n)$, then the Jacobian $J_{\Phi \gamma_h}$ is single-signed and nonzero in the region $A = \{(t_1, \ldots, t_n) \in I^d : t_1 < \cdots < t_n\}$. With that, an argument of Steinig [34] (see also [11, 16]) shows that $\Phi \gamma h$ is 1-to-1 on $A$.

**Proposition 4.2** (Steinig). $\Phi \gamma h$ is 1-to-1 on $A = \{(t_1, \ldots, t_n) \in I^d : t_1 < \cdots < t_d\}$.

For the convenience of the reader, we recall Steinig’s argument.

**Proof.** Assume for contradiction that there are $\bar{\gamma} \neq \gamma_1 \in A$ such that

(39) \quad $\gamma_h(s_1) + \cdots + \gamma_h(s_d) = \gamma_h(t_1) + \cdots + \gamma_h(t_d)$.

We can rewrite (39) as

$$\sum_{j=1}^{m} \varepsilon_j \gamma_h(u_j) = 0$$

for some even integer $m \in [2, 2d]$, $u_1 < \cdots < u_m \in I$, $\varepsilon_j \in \{-1, 1\}$, and $\sum_{j=1}^{m} \varepsilon_j = 0$. Let

$$\alpha_l = \sum_{j=1}^{l} \varepsilon_j, \quad 1 \leq l \leq m.$$

Then the sequence of $\alpha_l$’s has at most $d - 1$ changes of sign. Define the step function $\phi(u)$ to be $\alpha_j$ when $u \in (u_j, u_{j+1})$. We have

(40) \quad $0 = \sum_{j=1}^{m} \varepsilon_j \gamma_h(u_j) = \sum_{j=1}^{m-1} \alpha_j [\gamma_h(u_j) - \gamma_h(u_{j+1})] = -\int_{u_1}^{u_m} \phi(u) \gamma_h'(u) du$. 

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Let $I_i$, $1 \leq i \leq n$ be the ordered, maximal intervals where $\phi$ is constant and nonzero. Since the sequence of $\alpha_i$’s has at most $d - 1$ changes of sign, $n \leq d$. Let $M$ be the $n \times n$ matrix whose $(i, j)$’th entry is given by

$$\int_{I_i} |\phi(u)| (\gamma_u)'(u) du.$$  

Setting $\zeta = ((\gamma_1), \ldots, (\gamma_n))$,

$$\det M = \int_{I_1} \cdots \int_{I_n} |\phi(u_1)| \cdots |\phi(u_n)| \det [\zeta'(u_1) \cdots \zeta'(u_n)] du_1 \cdots du_n.$$  

By (40), the rows of $M$ are linearly dependent, so $\det M = 0$. On the other hand, $J_{\phi_\zeta}$ is single-signed and nonzero. Thus

$$0 = \det M = \int_{I_1} \cdots \int_{I_n} |\phi(u_1)| \cdots |\phi(u_n)| \det [\zeta'(u_1) \cdots \zeta'(u_n)] du_1 \cdots du_n > 0,$$

so we have reached a contradiction. □

**Proof of Proposition 4.1.** The proof comes in two steps. Both parts of this proof are adaptations of methods in [16]. Some minor differences arise because we are not dealing with polynomials.

First, we will define a sequence of iterated integrals $\mathfrak{I}_1, \ldots, \mathfrak{I}_n$ such that

$$J_{\phi_\zeta} (t_1, \ldots, t_n) = \mathfrak{I}_n(t_1, \ldots, t_n).$$  

The equality (41) will be shown in Lemma 4.3. Then, using the inductive definition of the iterated integrals, we will show in Lemma 4.5 that

$$|\mathfrak{I}_n(t_1, \ldots, t_n)| \approx n 2^{-k_n} v(t_1, \ldots, t_n).$$

To that end, let

$$\mathfrak{I}_1(t_1) = \frac{L_{n-2}(t_1)L_n(t_1)}{[L_{n-1}(t_1)]^2}.$$  

For $2 \leq m \leq n$ and $t_1, \ldots, t_m \in \Gamma^m$, define

$$\mathfrak{I}_m(t_1, \ldots, t_m) = \left( \prod_{j=1}^m \frac{L_{n-m-1}(t_j)L_{n-m+1}(t_j)}{[L_{n-m}(t_j)]^2} \right) \int_{t_1}^{t_2} \cdots \int_{t_{m-1}}^{t_m} \mathfrak{I}_{m-1}(\bar{s}) ds_1 \cdots ds_{m-1},$$

with the convention that $L_0 = L_{-1} \equiv 1$. As mentioned, the proof of Proposition 4.1 will be complete following the proofs of Lemmata 4.3 and 4.5.

**Lemma 4.3.** $\mathfrak{I}_n$ defined in (42) satisfies (41).

**Proof.** Define $f_{i,0} = \xi_i$ for $1 \leq i \leq n$, and for $1 \leq j \leq i - 1$ define

$$f_{i,j} = \frac{f_{i,j-1}'}{f_{j,j-1}'}.$$  

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Assume for now that the denominator is always nonzero; this follows from (44) and the condition (38). We will show that

\[ \mathfrak{I}_{n-j+1}(t_1, \ldots, t_{n-j+1}) = \det \begin{pmatrix} f'_{j-1}(t_1) & \cdots & f'_{j-1}(t_{n-j+1}) \\ \vdots & \ddots & \vdots \\ f'_{n-j-1}(t_1) & \cdots & f'_{n-j-1}(t_{n-j+1}) \end{pmatrix} \]

for all \( 1 \leq j \leq n \). In particular, when \( j = 1 \) we see that

\[ \mathfrak{I}_n(t_1, \ldots, t_n) = \det \begin{pmatrix} f'_{1,0}(t_1) & \cdots & f'_{1,0}(t_{n}) \\ \vdots & \ddots & \vdots \\ f'_{n,0}(t_1) & \cdots & f'_{n,0}(t_{n}) \end{pmatrix} = \det \begin{pmatrix} \zeta'_1(t_1) & \cdots & \zeta'_1(t_{n}) \\ \vdots & \ddots & \vdots \\ \zeta'_n(t_1) & \cdots & \zeta'_n(t_{n}) \end{pmatrix} = J_{\Phi \zeta}(t_1, \ldots, t_n). \]

The proof of (43) requires two ingredients. First, we need to write down the exact relationship between each \( f'_{ij} \) and various derivatives of \( \zeta \). Second, we need an iterative way of writing the left-hand side of (43).

For the first ingredient, we will need to define auxiliary matrices

\[ L_{\zeta_{i_1}, \ldots, \zeta_{i_l}}(t) = \det \begin{pmatrix} \zeta'_{i_1}(t) & \cdots & \zeta^{(l)}_{i_1}(t) \\ \vdots & \ddots & \vdots \\ \zeta'_{i_l}(t) & \cdots & \zeta^{(l)}_{i_l}(t) \end{pmatrix}. \]

If \( A \) is the \( (j + 1) \times (j + 1) \) matrix defining \( L_{\zeta_{i_1}, \ldots, \zeta_{i_l}} \), and if \([r_1, \ldots, r_j; c_1, \ldots, c_j] \) denotes the determinant of the matrix obtained from \( A \) by deleting rows \( r_1, \ldots, r_j \) and columns \( c_1, \ldots, c_j \), then an application of Sylvester’s Determinant Identity (see [10]) gives

\[ [j; j + 1] \cdot \det A = [j + 1; j + 1] \cdot [j; j] - [j + 1; j] \cdot [j; j + 1]. \]

Unwinding all the definitions, we see that

\[
\begin{align*}
[j, j + 1; j, j + 1] &= L_{\zeta_{i_1}, \ldots, \zeta_{i_j}}, \\
[j + 1; j + 1] &= L_{\zeta_{i_1}, \ldots, \zeta_{i_j}}, \\
[j; j] &= (L_{\zeta_{i_1}, \ldots, \zeta_{i_{j-1}}, \zeta})', \\
[j + 1; j] &= (L_{\zeta_{i_1}, \ldots, \zeta_j})', \quad \text{and} \\
[j; j + 1] &= L_{\zeta_{i_1}, \ldots, \zeta_{i_{j-1}}, \zeta}. 
\end{align*}
\]

Thus, we have

\[ L_{\zeta_{i_1}, \ldots, \zeta_{i_{j-1}}, \zeta} \cdot L_{\zeta_{i_1}, \ldots, \zeta_j} = L_{\zeta_{i_1}, \ldots, \zeta_j} \cdot (L_{\zeta_{i_1}, \ldots, \zeta_{j-1}}, \zeta) - (L_{\zeta_{i_1}, \ldots, \zeta_j})' \cdot L_{\zeta_{i_1}, \ldots, \zeta_{j-1}}, \zeta'. \]

Since \( L_{\zeta_{i_1}, \ldots, \zeta_j} = L_{j} \) is bounded away from 0 by (38), the above shows that

\[
\left( \frac{L_{\zeta_{i_1}, \ldots, \zeta_{j-1}}, \zeta}{L_{\zeta_{i_1}, \ldots, \zeta_j}} \right)' = \frac{L_{\zeta_{i_1}, \ldots, \zeta_{j-1}}, \zeta \cdot L_{\zeta_{i_1}, \ldots, \zeta_j}}{(L_{\zeta_{i_1}, \ldots, \zeta_j})^2}.
\]

Induction in \( i \) and \( j \) then gives

\[ f'_{i,j} = \frac{f'_{i,j-1}}{f'_{j-1}} = \left( \frac{L_{\zeta_{i_1}, \ldots, \zeta_{j-1}}, \zeta}{L_{\zeta_{i_1}, \ldots, \zeta_j}} \right)' = \frac{L_{\zeta_{i_1}, \ldots, \zeta_{j-1}}, \zeta \cdot L_{\zeta_{i_1}, \ldots, \zeta_j}}{(L_{\zeta_{i_1}, \ldots, \zeta_j})^2}. \]
The second ingredient is covered by the following calculus lemma in \[16\]:

**Lemma 4.4** (\[16\] Lemma 5.1). Let \( \{g_j\}_{j=1}^l \) be smooth functions on an open interval \( J \subset \mathbb{R} \) such that \( g_1 \) never vanishes on \( J \). If \( f_i = \frac{g_i}{g_j}, 2 \leq i \leq l \), then for \( (t_1, \ldots, t_l) \in J^l \),

\[
\begin{align*}
\det \begin{pmatrix} g_1(t_1) & \ldots & g_1(t_l) \\ \vdots & & \vdots \\ g_n(t_1) & \ldots & g_n(t_l) \end{pmatrix} &= \prod_{i=1}^l g_1(t_i) \int_{t_i}^{t_{i+1}} \cdots \int_{t_{l-1}}^{t_l} \det \begin{pmatrix} f_2'(s_1) & \ldots & f_2'(s_{l-1}) \\ \vdots & & \vdots \\ f_n'(s_1) & \ldots & f_n'(s_{l-1}) \end{pmatrix} ds_1 \ldots ds_{l-1}.
\end{align*}
\]

Using this lemma (noting that \( f_j' \neq 0 \)), we have

\[
\begin{align*}
\det \begin{pmatrix} f_{j,j-1}'(t_1) & \ldots & f_{j,j-1}'(t_{n-j+1}) \\ \vdots & & \vdots \\ f_{n,j-1}'(t_1) & \ldots & f_{n,j-1}'(t_{n-j+1}) \end{pmatrix} \\
&= \prod_{i=1}^{n-j+1} f_{j,j-1}'(t_i) \int_{t_i}^{t_{i+1}} \cdots \int_{t_{n-j}}^{t_{n-j-1}} \det \begin{pmatrix} f_{j+1,j}'(s_1) & \ldots & f_{j+1,j}'(s_{n-j}) \\ \vdots & & \vdots \\ f_n'(s_1) & \ldots & f_n'(s_{n-j}) \end{pmatrix} ds_1 \ldots ds_{n-j}.
\end{align*}
\]

As in \[16\], combining (44) and (45) iteratively gives us the equality (43), thus proving the lemma.

**Lemma 4.5.** Under the assumption (38), for \( 1 \leq m \leq n \) we have

\[
(46) \quad \mathfrak{I}_m(t_1, \ldots, t_m) \approx_m 2^{(m+1)k_m - mk_{m-1} - k_n} \mathcal{V}(t_1, \ldots, t_m),
\]

where \( k_0 = k_{-1} = 0 \). In particular,

\[
|\mathfrak{I}_n(t_1, \ldots, t_n)| \approx_n |2^{-k_n} \mathcal{V}(t_1, \ldots, t_n)|.
\]

**Proof.** We proceed by induction. In the base case \( m = 1 \), the Vandermonde determinant \( \mathcal{V}(t) \) is simply the constant function 1. Furthermore,

\[
\mathfrak{I}_1(t_1) = \frac{L_n^\xi(t_1)L_n^\xi(t_2)}{[L_n^\xi(t_1)]^2}.
\]

For every \( t_1 \), the assumption (38) shows

\[
|\mathfrak{I}_1(t_1)| \approx 2^{-k_n - 2k_n + 2k_{n-1}}
\]

so the base case is complete. In the inductive step, assume that (46) holds for some \( 1 \leq m - 1 \leq n \). From the definition of \( \mathfrak{I}_m \) in (42) and the condition (38),

\[
\mathfrak{I}_m(t_1, \ldots, t_m) \approx_m 2^{mk_{n-m} - mk_{n-m} - k_{n-m+1}} \int_{t_1}^{t_2} \cdots \int_{t_{m-1}}^{t_m} \mathfrak{I}_{m-1}(s_1, \ldots, s_{m-1}) ds_1 \ldots ds_{m-1}
\]

so the inductive step is complete.
By the induction hypothesis,
\[ \mathcal{I}_m(t_1, \ldots, t_m) \approx_m 2^{\frac{(m+1)k_n-mk_{n-m-1}-k_n}{2}} \int_{t_1}^{t_2} \ldots \int_{t_{m-1}}^{t_m} v(s_1, \ldots, s_{m-1}) \, ds_1 \ldots ds_{m-1}. \]
The integrand is a homogeneous polynomial of degree \( \frac{(m-1)(m-2)}{2} \). Thus, the integral is a homogeneous polynomial of degree \( \frac{(m-1)(m-2)}{2} + m - 1 = \frac{m(m-1)}{2} \). Hence, there is some polynomial \( P \) such that
\[ \int_{t_1}^{t_2} \ldots \int_{t_{m-1}}^{t_m} v(s_1, \ldots, s_{m-1}) \, ds_1 \ldots ds_{m-1} = P(t_1, \ldots, t_m) \prod_{1 \leq i < j \leq m} (t_j - t_i) \]
\[ = P(t_1, \ldots, t_m) v(t_1, \ldots, t_m). \]
Moreover, for any \( 1 \leq i < j \leq m \), the integral is 0 whenever \( t_j = t_i \). Since \( v(t_1, \ldots, t_m) \) already has degree \( \frac{m(m-1)}{2} \), \( P \) must be a constant, so
\[ \mathcal{I}_m(t_1, \ldots, t_m) \approx_m 2^{\frac{(m+1)k_n-mk_{n-m-1}-k_n}{2}} v(t_1, \ldots, t_m). \]
This closes the induction and finishes the proof of the lemma.

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