Generalized Ginzburg-Landau theory for non-uniform FFLO superconductors

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We derive a generalized Ginzburg-Landau (GL) functional near the tricritical point in the \((T, H)\)-phase diagram for the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) superconducting state, in 1, 2, and 3 dimensions. We find that the transition from the normal to the FFLO state is of second order in 1 and 2 dimensions, and the order parameter with one-coordinate sine modulation corresponds to the lowest energy near the transition line. We also compute the jump of the specific heat and describe the one-dimensional case the transformation of the sine modulation into the soliton-lattice state as the magnetic field decreases. In 3 dimensions however, we find that the transition into an FFLO state is of first order, and it is impossible to obtain an analytic expression for the critical temperature. In this case the generalized GL functional proposed here provides a suitable basis for a numerical study of the properties of the FFLO state, and in particular for computing the critical temperature, and for describing the transition into a uniform state.

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1. INTRODUCTION

As it was shown long time ago by Larkin and Ovchinnikov \[4\] and by Fulde and Ferrell \[2\], at low temperatures and when the magnetic field is acting on the spin of electrons only, a transition from normal (N) to modulated superconducting state (FFLO state) must occur. Due to this non-uniform superconducting state formation the paramagnetic limit at \(T = 0\) becomes larger than the usual Chandrasekhar-Clogston limit \[3\], \(H_p(0) = \frac{\Delta_0}{\mu_B\sqrt{2}}\), where \(\Delta_0 = 1.76 T_c\) is the superconducting gap at \(T = 0\). The \((T, H)\) phase diagram for 3D superconductors was obtained by Saint-James and Sarma \[6\] assuming that the transition \(N \rightarrow FFLO\) is of second order. It happens that the FFLO state only appears at \(T < T^* \approx 0.56 T_c\) \[3\] and that the temperature-dependence of the critical field \(H_{FFLO}(T)\) is strongly influenced by the dimensionality of the system. Indeed, for instance in the 3D case \(H_{FFLO}^{3D}(0) = 0.755 \frac{\Delta_0}{\mu_B}\), and \(H_{2D}^{FFLO}(0) = \frac{\Delta_0}{\mu_B}\) for 2D superconductors \[3\], while this field diverges in the one-dimensional case \[3\].

Up to now there is no conclusive experimental evidence for the FFLO state formation, except perhaps for what concerns \(UB_{c13}\) \[3\]. However, for this heavy-fermion superconductor the applicability of the standard theory of superconductivity is not evident. The main reason for the difficulties in observing such state experimentally resides in the fact that the orbital effect is usually more important than the paramagnetic one, and that the actual critical field is mainly determined by the orbital effect. However, for heavy-fermion superconductors and low-dimensional superconductors (when the field is applied parallel to the planes or chains) the orbital effect can be suppressed, and thence we deal with paramagnetically limited critical field.

The problem of exact structure of the FFLO state is not solved yet even in the framework of the model of pure paramagnetic limit, except for the 1D case where the superconducting order parameter in the FFLO state is described by the Jacobi elliptic function \[4\], \[3\], \[6\]. For this reason in this paper we concentrate on the description of the FFLO phase in the vicinity of the tricritical point \((T^*, H_p(T^*))\) where the characteristic wavevectors of the FFLO state are small compared with the inverse superconducting coherence length \(\xi_0^{-1}\). In this case, the appearance of the non-uniform state is related with a change of the sign of the coefficient \(\beta\) at the gradient term \(\beta(\nabla \psi)^2\) in the free energy. In the standard Ginzburg-Landau (GL) theory the coefficient \(\beta\) is positive, but it happens to be a function of the external field acting on the electron spins (paramagnetic effect) and vanishes at the point \((T^*, H_p(T^*))\), and then becomes negative for \(T < T^*\). A negative \(\beta\) means that the modulated state corresponds to lower energy as compared with the uniform one. So, in order to obtain the modulated vector one needs to include a term with second derivative in the GL functional. Moreover, in BCS theory the vanishing of the gradient term at \((T^*, H_p(T^*))\) is simultaneously accompanied with that of the coefficient \(\gamma\) of the fourth-order term \(\gamma \psi^4\) \[3\]. Due to this particular property, one needs to add higher-order terms such as, \(\psi^6\) and \((\nabla \psi)^2\psi^2\). Note that some particular cases of generalized GL functional were considered in \[3\], \[4\].
In this article we derive a generalized GL functional for 1, 2 and 3D superconductors which provides a thorough description of the FFLO state near the tricritical point. Here, we find that in one and two dimensions the transition from the normal to the FFLO superconducting state is of second order while it is of first order in the three-dimensional case. In any case the state with simple exponential modulation of the order parameter, i.e. \( \psi \sim |\psi_0| e^{iq \cdot r} \), considered in ref. [10], does not yield the minimum energy and always corresponds to a second-order phase transition. In the 1D and 2D cases it is most favourable for the FFLO state to appear with a simple sine structure, i.e. \( \psi \sim \psi_0 \sin(q \cdot r) \). For 2D superconductors such structure has lower energy when compared with the "2D lattice", \( \psi \sim \psi_0 \left( \sin(q \cdot x) + \sin(q \cdot y) \right) \), a result that justifies the choice made in ref. [11] of the FFLO structure for layered superconductors with an order parameter depending only on one coordinate. For 3D superconductors, however, the situation turns out to be more complicated. In this case we find that the transition is of first order for the sine state with one, two, and three-dimensional modulation. Our approach yields a good basis for a numerical study of the structure and properties of the FFLO state.

II. GENERALIZED GINZBURG-LANDAU FUNCTIONAL

In the paramagnetic limit the Hamiltonian of the system can be written in the mean-field approximation, in a \( d \)-dimensional space as follows

\[
H = \int d^d r \left\{ \sum_\sigma \left[ \Phi_\sigma^+(r) \sum^{2d}_m \Phi_\sigma(r) + \sigma \mathcal{H} \Phi_\sigma^+(r)\Phi_\sigma(r) \right] \left( \psi(r) \cdot \Phi_1^+(r)\Phi^-_{-1}(r) + h.c. \right) \right\}
\]

where \( \sigma = \pm 1 \) if the electron spin is parallel (anti-parallel) to the magnetic field \( H \) and \( \mathcal{H} = \mu_B H \). The superconducting order parameter is \( \psi(r) = \lambda \langle \Phi_1(r)\Phi_{-1}(r) \rangle \), where \( \lambda \) is the electron-phonon coupling constant.

Near the transition line the order parameter \( \psi \) is small, thus using the Gorkov procedure [12] for deriving the Ginzburg-Landau functional, we write

\[
F = \int d^d r \frac{\psi^2}{|\lambda|} + \frac{1}{\beta} \sum_\nu \int d^d r_{1,2} G^-(\omega_\nu, r_1 - r_2)G^+(\omega_\nu, r_2 - r_1)\psi(r_1)\psi^+(r_2) + \frac{1}{2\beta} \sum_\nu \int d^d r_{1,2,3,4} G^+_\omega(r_1 - r_2)G^-_{-\omega}(r_4 - r_2)G^-_{-\omega}(r_4 - r_1)G^+_{\omega}(r_4 - r_1) \times \psi(r_1)\psi(r_2)\psi^+(r_3)\psi^+(r_4) + \cdots
\]

where the Green functions here are defined by

\[
G^+_{\omega_\nu}(p) = \frac{1}{i\omega_\nu - \xi_p + \mathcal{H}}, \quad G^-_{\omega_\nu}(p) = \frac{1}{i\omega_\nu - \xi_p - \mathcal{H}}
\]

\[
\xi_p = \frac{p^2}{2m} - \varepsilon_F
\]

Being close to the tricritical point of the \((T, H)\) phase diagram, the spatial modulation of the order parameter is small so that we can expand \( \psi(r_1) \) around \( r_1 \) in Taylor series

\[
\psi(r_1) \simeq \psi(r_1) + \left( (\vec{r}_1 - \vec{r}_2) \cdot \vec{\nabla} \right) \psi + \frac{1}{2!} \left( (\vec{r}_1 - \vec{r}_2) \cdot \vec{\nabla} \right)^2 \psi + O(3)
\]

(2)

After long but straightforward calculations, the free energy can be rewritten as

\[
F = \alpha |\psi|^2 + \beta |\partial \psi|^2 + \gamma |\psi|^4 + \delta |\partial^2 \psi|^2 + \mu |\psi|^2 |\partial \psi|^2 + \eta \left[ (\psi^+)^2(\partial \psi)^2 + \psi^2(\partial \psi^+)^2 \right] + \nu |\psi|^6
\]

(3)

with the coefficients

\[\text{We have also added the sixth-order term because the fourth order one turns out to be small near the tricritical point.}\]
\[ \alpha = -\pi N(0) \cdot (K_1 - K_1^0), \quad \gamma = \frac{\pi N(0)K_3}{4}, \quad \nu = \frac{\pi N(0)K_5}{8} \]
in all dimensions, and in 1D
\[ \beta = \frac{\pi N(0)V_F^2K_3}{4}, \quad \delta = -\frac{\pi N(0)V_F^2K_5}{16}, \quad \mu = 8\eta = -\frac{\pi N(0)V_F^2K_5}{2} \]
in 2D
\[ \beta = \frac{\pi N(0)V_F^2K_3}{8}, \quad \delta = -\frac{3\pi N(0)V_F^2K_5}{16}, \quad \mu = 8\eta = -\frac{\pi N(0)V_F^2K_5}{4}, \]
and in 3D
\[ \beta = \frac{\pi N(0)V_F^2K_3}{12}, \quad \delta = -\frac{\pi N(0)V_F^2K_5}{80}, \quad \mu = 8\eta = -\frac{\pi N(0)V_F^2K_5}{6} \]
where \( V_F \) is the Fermi velocity, \( N(0) \) the electron density of states, and
\[ K_n = 2T \cdot \text{Re} \left[ \sum_{\nu=0}^{\infty} \frac{1}{(\omega_\nu - i\mathcal{H})^n} \right] = (2T)^{n-1} \frac{1}{\pi^n} \text{Re} \left[ \sum_{\nu=0}^{\infty} \frac{1}{(\nu + z)^n} \right], \quad n \geq 1 \]
\[ \omega_\nu = \pi(2\nu + 1)T, \quad z = \frac{1}{2} - i\frac{\mathcal{H}}{2\pi T} \]
or in terms of the \( psi \) function \( \Psi(x) = \frac{d}{dx} \ln \Gamma(x) \), \( K_3 \) and \( K_5 \) can be rewritten as
\[ K_3 = \frac{2T}{(2\pi T)^3} \frac{(-1)^3}{2!} \text{Re} \left( \Psi^{(2)}(z) \right), \quad K_5 = \frac{2T}{(2\pi T)^5} \frac{(-1)^5}{4!} \text{Re} \left( \Psi^{(4)}(z) \right) \]
and near the tricritical point, we have
\[ \alpha = -N(0) \cdot \text{Re} \left( \Psi(\frac{1}{2} - i\frac{\mathcal{H}}{2\pi T}) - \Psi(\frac{1}{2} - i\frac{\mathcal{H}_0}{2\pi T}) \right) \]
\[ = -N(0) \frac{(\mathcal{H} - \mathcal{H}_0)}{2\pi T} \cdot \text{Im} \Psi' \left( \frac{1}{2} - i\frac{\mathcal{H}_0}{2\pi T} \right) \]
where \( \mathcal{H}_0 \) is the field corresponding to the second-order transition into the uniform superconducting state, and it is given by \[ \frac{\ln T_c}{T} = \text{Re} \left[ \Psi(\frac{1}{2} - i\frac{\mathcal{H}_0}{2\pi T}) - \Psi(\frac{1}{2}) \right]. \]
Note also that in the expressions for all coefficients, except \( \alpha \), in the functional \[ \text{(3)} \] we may set \( \mathcal{H} = \mathcal{H}_0 \).

III. MINIMIZATION OF THE FREE ENERGY

Now, we proceed to study different solutions for the order parameter in one, two and three dimensions, and see which solution minimizes the free energy. We will compute the free energy when the order parameter is an exponential, and a sine with one, two and three dimensional modulation.

A. One-dimensional case

If we choose the exponential order parameter
\[ \psi(x) = \psi_0 e^{iqx} \]
then the free energy reads
\[ F_{\text{exp}} = (\alpha + \beta q^2 + \delta q^4) \cdot |\psi_0|^2 + (\gamma + (\mu - 2\eta)q^2) \cdot |\psi_0|^4. \]

Analysing the coefficient at \( |\psi_0|^2 \) we see that the field corresponding to the second-order transition depends on the wave vector \( q \), and that the actual field is the maximum one. Accordingly, we get

\[ q_{\text{max}}^2 = -\frac{\beta}{2\delta}, \quad \alpha = \alpha_0 = \frac{\beta^2}{4\delta}. \]  

The free energy of such solution then reads

\[ F_{\text{exp}} = \frac{-1}{2\pi N(0)} \frac{(\alpha - \alpha_0)^2}{-K_3}. \]  

Similarly, for the sine modulation \( \psi(x) = \psi_0 \sin(qx) \), we find the free energy

\[ F_{\text{sin}} = \frac{-1}{\pi N(0)} \frac{(\alpha - \alpha_0)^2}{-K_3}. \]  

Note that since \( K_3 < 0 \) the denominators in these free energies are positive in the region of existence of the FFLO state, which implies that the transition is of second order. On the other hand, we see that the sine modulation corresponds to the state with (twice) lower energy than the exponential one in one dimension, and is therefore more favourable.

It is important to emphasize here that this sine solution for the order parameter is just a limit of a more general solution on the transition line. Indeed, we show below that as we drift away from the transition line, we need to use an exact solution for the order parameter in one dimension \[7\].

Minimizing the free energy (3) leads to following equation for the order parameter (which is assumed to be a real function)

\[ \psi^{(4)} + \tilde{\alpha} \psi - \tilde{\beta} \psi'' - \tilde{\mu} \left( \psi(\psi')^2 + \psi^2 \psi'' \right) + 2 \tilde{\gamma} \psi^3 + 3 \tilde{\nu} \psi^5 = 0 \]  

where we have redefined

\[ \tilde{\alpha} = \frac{\alpha}{\delta}, \quad \tilde{\beta} = \frac{\beta}{\delta}, \quad \tilde{\gamma} = \frac{\gamma}{\delta}, \quad \tilde{\nu} = \frac{\nu}{\delta}, \quad \tilde{\mu} = \frac{\mu + 2\eta}{\delta}. \]

Now using a similar approach to that of ref. [13], we demonstrate that the (Jacobi) elliptic-sine function

\[ \psi(x,k) = \Delta k \text{sn}(\frac{x}{\xi},k) \]  

is an adequate solution of eq.(7), \( \Delta \) being its amplitude, \( k \) the modulus of the elliptic sine and \( \xi \) is some ”effective coherence length” that will be determined later on. Note that \( \text{sn}(x,k) \to \sin(x) \), as \( k \to 0 \), and this just happens on the transition line, and that the amplitude of \( \psi \) vanishes at the transition. Next, owing to the Jacobi elliptic function \[14\], \( \psi(x,k) \) satisfies the following equation

\[ \xi^2(\psi')^2 + (k^2 + 1)\psi^2 - \frac{\psi^4}{\Delta^2} = \Delta^2 k^2 \]  

Differentiating this equation with respect to \( x \), we obtain

\[ \xi^2 \cdot \psi'' + (k^2 + 1) \cdot \psi - \frac{2}{\Delta^2} \cdot \psi^3 = 0 \]  

and differentiating the latter equation in turn twice and making use of it, leads to the fourth-order equation,

\[ \psi^{(4)} + \left( \frac{k^2 + 1}{\xi^2} \right) \cdot \psi'' - \frac{12}{\xi^4 \Delta^2} \left( \psi(\psi')^2 + \psi^2 \psi'' \right) + \frac{12}{\xi^4 \Delta^2} \cdot \psi^5 - \frac{6(k^2 + 1)}{\xi^4 \Delta^2} \cdot \psi^3 = 0 \]  

\[ ^2 \text{The prime here stands for the derivative with respect to the space coordinate } x. \]
Multiplying (9) by $\psi$ and (10) by $\psi^2$, and adding the results leads to
\[
\frac{1}{\xi^2 \Delta^2} \cdot (\psi(\psi')^2 + \psi^2 \psi'') - \frac{3}{\xi^4 \Delta^4} \cdot \psi^5 + \frac{2(k^2 + 1)}{\xi^4 \Delta^2} \cdot \psi^3 - \frac{k^2}{\xi^4} \cdot \psi = 0
\] (12)
and we also rewrite (10) as follows
\[
\frac{k^2 + 1}{\xi^2} \cdot \psi'' + \frac{(k^2 + 1)^2}{\xi^4} \cdot \psi - \frac{2(k^2 + 1)}{\xi^4 \Delta^2} \cdot \psi^3 = 0
\] (13)

Now, the point is to obtain an equation for $\psi$ that could be identified with eq.(7). For this purpose, we add eq.(11) to the results of multiplying respectively equations (13) and (12) by arbitrary coefficients $A$ and $B$. Then by identifying the coefficients in the resulting equation with those in (7), we obtain the following system of 5 equations for the 5 parameters $\Delta, \xi, k, A, B$:
\[
\tilde{\alpha} = A(k^2 + 1)^2 - Bk^2 \xi^4, \quad \tilde{\beta} = -\frac{(1 + A)(k^2 + 1)}{\xi^2}
\]
\[
\tilde{\gamma} = \frac{(B - A - 3)(k^2 + 1)}{\xi^4 \Delta^2},
\]
\[
\tilde{\mu} = \frac{12 - B}{\xi^2 \Delta^2}, \quad \tilde{\nu} = 4 - B \frac{1}{\xi^4 \Delta^4}
\] (14)

On the other hand, in one dimension the coefficients $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}$ and $\tilde{\nu}$ are obtained from eq.(3),
\[
\tilde{\alpha} = \frac{16(K_1 - K_0)}{V_F^2 K_5}, \quad \tilde{\beta} = -\frac{4K_3}{V_F^2 K_5}
\]
\[
\tilde{\gamma} = -\frac{4K_3}{V_F^2 K_5}, \\
\tilde{\mu} = \frac{10}{V_F^2}, \quad \tilde{\nu} = 2 \frac{1}{V_F^2}
\] (15)

In addition, equating the ratio $\frac{\tilde{\mu}}{\tilde{\nu}}$ in (14) with that in (15) we infer that $B = 2$ and $V_F^2 = \xi^2 \Delta^2$.

Moreover, as explained above, on the line of second-order transition the solution must be $\sin(x)$, i.e. $k \to 0$, whereupon the coefficients in (14) become
\[
\tilde{\alpha} = \frac{A}{\xi^4}, \quad \tilde{\beta} = -\frac{1 + A}{\xi^2}, \quad \tilde{\gamma} = \frac{B - A - 3}{\xi^4 \Delta^2},
\]
\[
\tilde{\mu} = \frac{12 - B}{\xi^2 \Delta^2}, \quad \tilde{\nu} = 4 - B \frac{1}{\xi^4 \Delta^4}
\] (16)

Then, using the condition for the second-order transition,
\[
\tilde{\alpha} = \frac{\tilde{\beta}^2}{4}
\] (17)
we find that $A = 1$.

At this point we wish to stress that in two and three dimensions one can readily check that the elliptic sine is no longer a good solution that transforms into sine on the transition line. Indeed, on this line $A = 1$, so that the identity $\tilde{\gamma} = \frac{\tilde{\beta}^2}{4}$ inferred from eq.(10) is compatible with eq.(17) that is valid in one dimension. However, this is not true in other dimensions since in this identity the right hand side depends on the dimension of space whereas the left hand side doesn’t, see (3).

Away from the transition line defined by eq.(17), the parameter $A$ obtained from (14) can be written in terms of the ”external field” parameter $h = \frac{\tilde{\alpha}}{\tilde{\beta}^2}$ as follows
\[
A(k, h) = -1 + \frac{1}{2h} - \frac{1}{2h} \sqrt{1 - 4h \left(1 + \frac{2k^2}{(k^2 + 1)^2}\right)}
\] (18)
Of course, $A(k, h) \to 1$ as $k \to 0$, since then $h \to \frac{1}{2}$ due to eq.(17).

Note that in this case we have in fact a family of solutions parametrized by $k$. Indeed, using eqs.(14), (17) we can express the parameters $\Delta$ and $\xi$ in terms of the modulus $k$, bearing in mind that $B = 2$ and $V_P^2 = \xi^2 \Delta^2$, see above.

Therefore, in one dimension we have found a solution to the system of equations (14) for all parameters entering in the order parameter $\psi(x, k)$ described by the general Ginzburg-Landau functional (3). Now, let us compute the free energy of this non-uniform order parameter and compare it with that of the uniform state. We shall also study the transition between these two states. For this purpose, we insert the solution (8) into the Ginzburg-Landau functional (3) and integrate over a period of the Jacobi elliptic sine, namely $4 \sinh x$. We obtain the following expression

$$F_{NU}(k, h) = \frac{-k^4}{(k^2 + 1)(A + 1)} \left[ 10 I_2 - (k^2 + 1)(11 + A) I_4 + 14k^2 I_6 \right]$$

where we have defined

$$I_n(k) = \frac{1}{K(k)} \int_0^{\frac{\pi}{2}} \frac{dx \sin^n x}{\sqrt{1 - k^2 \sin^2 x}}$$

or in terms of $K(k)$ and the complete elliptic integral of second kind $E(k)$,

$$I_2 = \frac{1}{k^2} \left( 1 - \frac{E(k)}{K(k)} \right), \quad I_4 = \frac{2 + k^2}{3k^2} - \frac{2(1 + k^2)}{3k^4} \frac{E(k)}{K(k)}$$

$$I_6 = \frac{4(k^2 + 1)}{5k^2} I_4 - \frac{3}{5k^4} \left( 1 - \frac{E(k)}{K(k)} \right)$$

The coefficient $C = \delta V_P^2 (-\tilde{\beta})^3$ is positive since $\tilde{\beta} < 0$.

Note that the most advantageous feature of the free energy (19) is that it depends only on the parameter $k$ once the external field parameter $h$ is fixed. Then, upon minimizing this free energy we determine the modulus $k$ and thereby we obtain the parameters $\xi$ and $\Delta$ from eq.(14),

$$\xi^2 = -\frac{(1 + A(k, h))(k^2 + 1)}{\beta}, \quad \Delta = \frac{V_P}{\xi}$$

On the other hand, the free energy for the uniform order parameter reduces to

$$F_U = \delta \cdot (\tilde{\alpha} \psi^2 + \tilde{\gamma} \psi^4 + \tilde{\nu} \psi^6)$$

where the coefficient $\tilde{\gamma}$ is negative, and hence the transition is of first order here.

Using the same notation as above, the free energy $F_U$ becomes

$$\frac{F_U(h)}{C} = \frac{-1}{18} \left[ 1 + \sqrt{1 - 6h}(\frac{1}{3} - 2h) - h \right]$$

which depends on the sole parameter $h$.

It is seen in figure 1 that the free energies (14) and (20) converge when the external field $h$ becomes equal to the critical value $hc \simeq 0.0925925$. The latter marks the transition from the non-uniform to uniform state. Next, we show in figure 2 the change of the form and period of the order parameter from sine to soliton lattice as we lower the external field. When $h \leq hc$, the period of the order parameter is infinite and the superconducting state becomes uniform, see figure 3.

Before considering the two and three dimensional cases, let us first compute the jump in the specific heat at the second-order transition in one dimension. In this case the free energy for the sine solution is given in eq.(14), and the corresponding jump in the specific heat reads

$$\Delta C = 4N(0)T_c \times \frac{\text{Im} \left( \Psi'(\frac{1}{2} - i \frac{\mathcal{H}_0}{2\pi T}) \cdot \frac{d\mathcal{H}_0}{dT} \right)^2}{\text{Re} \left( \Psi(\frac{1}{2} - i \frac{\mathcal{H}_0}{2\pi T}) \right)}$$

The characteristic feature of such behaviour is the divergence of the jump at the tricritical point.
B. Two-dimensional case

1. Exponential

As above, we start by considering the exponential order parameter

$$\psi(r) = \psi_0 e^{iq \cdot r}$$

for which the free energy reads

$$F = (\alpha + \beta q^2 + \delta q^4) \cdot |\psi_0|^2 + (\gamma + (\mu - 2\eta)q^2) \cdot |\psi_0|^4,$$

and upon minimizing over \(q^2\) and \(|\psi_0|^2\), this becomes

$$F_{\text{exp}} = \frac{1}{\pi N(0)} \frac{(\alpha - \alpha_0)^2}{K_3}. \tag{22}$$

For one-component sine

$$\psi(r) = \psi_0 \sin(q \cdot x)$$

we obtain

$$F_{\text{sin}_x} = \frac{6}{\pi N(0)} \frac{(\alpha - \alpha_0)^2}{K_3} \tag{23}$$

and finally for the two-component sine

$$\psi(r) = \psi_0 (\sin(q \cdot x) + \sin(q \cdot y))$$

we get

$$F_{\text{sin}_{xy}} = \frac{4}{\pi N(0)} \frac{(\alpha - \alpha_0)^2}{K_3} \tag{24}$$

Therefore, since \(K_3 < 0\), we infer that

$$F_{\text{sin}_x} < F_{\text{sin}_{xy}} < F_{\text{exp}}$$

This means that the phase with one-component sine modulation is the most stable in two dimensions. Here again we find that the coefficients of the quartic term in the free energy are positive, which implies that the transition into the FFLO state is of second-order. Our result can justify the choice made in ref. [11] of the structure depending only on one coordinate for the numerical analysis of the FFLO phase in two dimensions.

Finally, as in one dimension we obtain the jump in the specific heat

$$\Delta C = 24 N(0) T_c \times \frac{\text{Im} \Psi'(\frac{1}{2} - i \frac{\Delta_0}{2T}) \cdot \frac{d\Psi(0)}{dT}^2}{\text{Re} \Psi(2)(\frac{1}{2} - i \frac{\Delta_0}{2T})} \tag{25}$$

which diverges at the tricritical point.

C. Three-dimensional case

The free energy of the exponential order parameter \(\psi(r) = \psi_0 e^{iq \cdot r}\), reads

$$F = (\alpha + \beta q^2 + \delta q^4) \cdot |\psi_0|^2 + (\gamma + (\mu - 2\eta)q^2) \cdot |\psi_0|^4. \tag{26}$$

Then using the coefficients in eq. (3) we see that the coefficient of the quartic term in (26) is positive at \(q^2 = q_0^2 = -\frac{\beta}{2\delta}\), thus indicating that the transition is of second order.
On the contrary, for the 1, 2, and 3 component sine we find that this coefficient is negative and consequently the transition is of first order. But, in order to compute the critical temperature it is necessary to add the sixth-order term $v\psi^6$. The transition occurs into a structure with well-developed harmonics and to find the corresponding critical temperature and the structure of the non-uniform state numerical calculations are needed.

The functional (3) provides a good basis for such calculations. Apparently the 3D-lattice state is more stable. Really, one could also study the change in the order of the transition as we drift toward the $T = 0$ point. In fact, it was argued in [15] that as $T \to 0$ the transition into the sine state becomes of second order, whilst the results of ref. [1] showed that at this point the transition into the 3D-lattice state remains of first order, and thus corresponds to a higher critical field and lower energy as compared with the sine state with one-dimensional modulation.

We may speculate that the 3D-lattice is also more favourable near the tricritical point and along the whole transition line.

IV. CONCLUSION

We have constructed a generalized Ginzburg-Landau functional for the FFLO superconductors near the tricritical point in the $(T, H)$-phase diagram, in 1, 2, and 3 dimensions. Using this functional we have shown that the state with exponential modulation of the order parameter ($\psi \sim e^{iqx}$) is always unfavourable and that the transition from the normal to the FFLO state is of second order in 1 and 2 dimensions, where the order parameter has a one-component sine modulation near the transition line. In one dimension we have shown that upon lowering the external field, the order-parameter modulation changes from $\sin(x)$ on the transition line into elliptic sine $\sinh x$, and finally transfers to a uniform state. The proposed generalized Ginzburg-Landau functional for 1D superconductors can be directly applied to the description of the spin-Peierls systems in magnetic field near the corresponding tricritical point [10].

In 3 dimensions we have shown that the transition is of first order, and contrary to 1 and 2 dimensions, one has to take into account higher harmonics of the order parameter to calculate the critical temperature.

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Figure captions

• Figure 1: Plot of free energy as a function of the “external field” $h = \tilde{\alpha} \tilde{\beta}^2$, $C = \delta V F^2 (\tilde{\beta})^3$. The solid line represents the free energy of the uniform state. The line in crosses represents the free energy of the non-uniform soliton-lattice state. These two curves meet at $h_c = 0.0925925$.

• Figure 2: Spatial dependence of the superconducting order parameter in the non-uniform 1D phase for different fields $h = \tilde{\alpha} \tilde{\beta}^2$. The dotted line indicates the solution at $h = 0.165$, and the solid line represents the solution when $h = 0.0927$, i.e. close to the critical value $h_c \approx 0.0925925$. The dimensionless space coordinate $\tilde{x}$ is defined by $\tilde{x} = x \cdot \left( \sqrt{-\gamma V_F} \right)$, and $\tilde{\psi}(\tilde{x}, k) = \frac{\psi(x, k)}{\sqrt{1 + A(k, h)(k^2 + 1)}}$.

• Figure 3: Variation of the period $L$ of the non-uniform phase as a function of the external field $h$. The period diverges as the external field approaches the critical value $h_c$. The dimensionless period $\tilde{L}$ plotted here is defined as $\tilde{L} = L \cdot \left( \sqrt{-\gamma V_F} \right)$, and for the soliton-lattice phase this is $\tilde{L} = 4K(k) \cdot \sqrt{(1 + A(k, h))(k^2 + 1)}$, see text.