Projective representations of mapping class groups in combinatorial quantization

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Abstract. Let \( \Sigma_{g,n} \) be a compact oriented surface of genus \( g \) with \( n \) open disks removed. The graph algebra \( \mathcal{L}_{g,n}(H) \) was introduced by Alekseev–Grosse–Schomerus and Buffenoir–Roche and is a combinatorial quantization of the moduli space of flat connections on \( \Sigma_{g,n} \). We construct a projective representation of the mapping class group of \( \Sigma_{g,n} \) using \( \mathcal{L}_{g,n}(H) \) and its subalgebra of invariant elements. Here we assume that the gauge Hopf algebra \( H \) is finite-dimensional, factorizable and ribbon, but not necessarily semi-simple. We also give explicit formulas for the representation of the Dehn twists generating the mapping class group; in particular, we show that it is equivalent to a representation constructed by V. Lyubashenko using categorical methods.

1 Introduction

Let \( \Sigma_{g,n} \) be a compact oriented surface of genus \( g \) with \( n \) open disks removed. It is readily seen that \( \Sigma_{g,n} \setminus D \) (where \( D \) is an open disk) is homeomorphic to the tubular neighborhood of the graph \( \Gamma \) whose edges are the generators of the fundamental group of the surface (see Figure 1); we will denote \( \Sigma_{g,n}^o = \Sigma_{g,n} \setminus D \). This particular choice of graph is not a loss of generality.

Let \( G \) be an algebraic Lie group (generally assumed connected and simply-connected, e.g. \( G = \text{SL}_2(\mathbb{C}) \)). A lattice gauge field theory on \( \Gamma \) is a discretization of the moduli space of flat \( G \)-connections on \( \Sigma_{g,n}^o \). It consists of a set of discrete connections \( A = G^{2g+n} \), a gauge group \( \mathcal{G} = G \) and an algebra of functions \( \mathbb{C}[A] = \mathbb{C}[G]^{\otimes 2g+n} \) (see e.g. [Wit91, 2.3], [Lab13 Chap. 2] for the general definitions). There is also a notion of discrete holonomy defined in a natural way. The gauge group acts on \( A \) (and dually on \( \mathbb{C}[A] \) on the right) by conjugation; the invariant functions are called classical observables.

Lattice gauge field theory on \( \Gamma \) is another description of the character variety of \( \Sigma_{g,n}^o \). More precisely, the discrete holonomy is a bijection between the set \( A/\mathcal{G} \) of discrete \( G \)-connections up to gauge equivalence and \( \text{Hom}(\pi_1(\Sigma_{g,n}^o), G)/G \). The space \( A \) is endowed with a Poisson structure defined by Fock and Rosly [FR98]. This Poisson structure descends to \( A/\mathcal{G} \) and moreover, \( \mathbb{C}[A/\mathcal{G}] = \mathbb{C}[\text{Hom}(\pi_1(\Sigma_{g,n}^o), G)]^G \), namely the space of functions on the character variety. Under this isomorphism, the Fock–Rosly Poisson structure corresponds to that given by the Goldman bracket, or equivalently, by the Atiyah–Bott symplectic form.

The previous remarks apply to the original surface \( \Sigma_{g,n} \) if we consider the subset of discrete flat connections \( A_j \) instead of \( A \). These are the discrete connections whose holonomy along the boundary of the unique face of the graph \( \Gamma \) is trivial.

It is worthwhile to describe the algebra of functions \( \mathbb{C}[A] \) in terms of matrix coefficients \( T^I_j \in \mathbb{C}[G] \) (where \( I \) is a finite-dimensional \( G \)-module), since they linearly span \( \mathbb{C}[G] \). For instance, we can construct a function \( \hat{A}(k)^I_j \in \mathbb{C}[A] \) by putting the function \( T^I_j \) over the edge \( a_k \) and the trivial
function 1 on the other edges. In particular, we get a matrix $A(k)$ with coefficients in $\mathbb{C}[\mathcal{A}]$, see Figure 1. The coefficients of such matrices span $\mathbb{C}[\mathcal{A}]$ as an algebra. The action of the gauge group is by conjugation, for instance $A \cdot g = g A g^{-1}$, where $g = T(g)$ is the representation of $g$ on $I$.

In the works of Alekseev [Ale91], Alekseev–Grosse–Schomerus [AGS93, AGS96] and Buffenoir–Roche [BR95, BR96], the Lie group $G$ is replaced by a quantum group $U_q(\mathfrak{g})$, with $\mathfrak{g} = \text{Lie}(G)$. The notions described above can be generalized in this setting. Then the graph algebra $L_{g,n}(U_q(\mathfrak{g}))$ is a quantization of the Fock-Rosly Poisson structure on $\mathcal{A}$. The multiplication in $L_{U_q(\mathfrak{g})}$ is endowed with an action of $U_q(\mathfrak{g})$, analogous to the action of the gauge group $G$ on $\mathbb{C}[\mathcal{A}]$. The multiplication in $L_{g,n}(U_q(\mathfrak{g}))$ is designed so that it is an $U_q(\mathfrak{g})$-module-algebra with respect to this action. In particular, we have a subalgebra of invariant elements $L_{g,n}^{\text{inv}}(U_q(\mathfrak{g}))$, which is a quantized analogue of the algebra of classical observables of the initial lattice gauge field theory.

These quantized algebras of functions and their generalizations appear in various works, e.g. [BFKB98b, BNR02, MW15, BZBJ18, AGPS18].

The definition of the algebras $L_{g,n}(U_q(\mathfrak{g}))$ is purely algebraic and we can replace the quantum group $U_q(\mathfrak{g})$ by any ribbon Hopf algebra $H$. The representation theory of $L_{g,n}(H)$ and of its subalgebra of invariant elements is investigated in [Ale94] when $H$ is the quantum group $U_q(\mathfrak{g})$ for $q$ generic and in [AS96a] when $H$ is finite-dimensional and semi-simple, or a semisimple truncation of quantum group at a root of unity (the latter being defined in the setting of quasi-Hopf algebras).

Moreover, in [AS96a, AS96b], a projective representation of the mapping class group of $\Sigma_{g,n}$ based on $L_{g,n}(H)$ is described. This representation is an analogue in the quantized setting of the obvious representation of the mapping class groups on $\mathbb{C}[\mathcal{A}]$ and $\mathbb{C}[\mathcal{A}_f]$.

In this paper, we consider the algebras $L_{g,n}(H)$ from a purely algebraic viewpoint, under the general assumption that the gauge algebra $H$ is finite-dimensional, factorizable and ribbon, but not necessarily semi-simple. The algebras $L_{0,1}(H)$ and $L_{1,0}(H)$, which are the building blocks of the theory (see Definition 3.3), and the associated projective representation of $\text{SL}_2(\mathbb{Z})$, have already been studied under these assumptions in [Fai18a].

In section 3 we first quickly recall the definition and main properties of $L_{0,1}(H)$ and $L_{1,0}(H)$. Then we recall the definition of $L_{g,n}(H)$, and we show that the Alekseev isomorphism [Ale94], which is a fundamental tool to construct representations of $L_{g,n}(H)$, holds under our assumptions. In particular, when $n = 0$, the Alekseev isomorphism implies that $L_{g,0}(H)$ is isomorphic to a matrix algebra (because the Heisenberg double is a matrix algebra, see subsection 2.2 and (21)) and that the only indecomposable (and simple) representation of $L_{g,0}(H)$ is $(H^*)^{\otimes g}$.

We construct representations of the subalgebras of invariant elements $L_{g,n}^{\text{inv}}(H)$ in section 4 with a generalization of the method used in [Ale94]. More precisely, for each representation $V$ of $L_{g,n}(H)$ we associate a representation $\text{Inv}(V) \subset V$ of $L_{g,n}^{\text{inv}}(H)$, defined by the requirement that the holonomy of a connection along the boundary of the unique face of the graph $\Gamma$ acts trivially on it.

In section 5.3 we construct a projective representation of the mapping class groups $\text{MCG}(\Sigma_{g,0})$...
and $\text{MCG}(\Sigma_{g,0})$ (we discuss the case $n > 0$ in subsection 5.4). The idea of the construction is to associate an automorphism $\tilde{f}$ of $\mathcal{L}_{g,0}(H)$ to each element $f$ of the mapping class group (Proposition 5.1), called the lift of $f$. To define such a lift, we just replace generators of the fundamental group by matrices of generators of $\mathcal{L}_{g,0}(H)$ (up to some normalization), see (27) and (28). Since $\mathcal{L}_{g,0}(H)$ is isomorphic to a matrix algebra, this automorphism is inner and we get an element $\tilde{f} \in \mathcal{L}_{g,0}(H)$, unique up to scalar. Then to $f$ we associate the representation of $\tilde{f}$ on $(H^*)^g$ (in the case of $\Sigma_{g,0}$) and on $\text{Inv}((H^*)^g)$ (in the case of $\Sigma_{g,0}$). This construction was first introduced by Alekseev and Schomerus in [AS96a] and [AS96b] in the semi-simple setting. Here we generalize and complete this approach with detailed proofs in the non-semi-simple setting.

Finally, we give explicit formulas for the representation of the Dehn twists about the curves depicted in Figure 4 (Theorem 5.12), and in particular this allows us to prove that the representation of the mapping class group described above is equivalent (Theorem 6.4) to another one constructed by Lyubashenko using categorical techniques based on the coend of a ribbon category $\mathcal{C}$ satisfying some assumptions [Lyu95a, Lyu95b, Lyu96]. For this equivalence we take $\mathcal{C} = \text{mod}_l(H)$, the category of finite-dimensional left modules. For works based on the Lyubashenko representation, see e.g. [FSS12, FSS14].

Although the two representations are equivalent, the combinatorial quantization provides additional structure and tools. Indeed, it also gives rise to a representation of the quantized version of the classical observables $\mathcal{L}_{g,n}^{\text{inv}}(H)$; this is interesting because these quantum observables are related to skein theory [BFKB98a, BFKB98b]. Moreover, as a deformation of the algebra of functions on the character variety, combinatorial quantization is a natural and explicit setting to derive mapping class group representations.

To sum up, the main results of this paper are:

- The construction of a projective representation of $\text{MCG}(\Sigma_{g,0})$ and $\text{MCG}(\Sigma_{g,0})$ (Theorem 5.10),
- Explicit formulas for the representation of the Dehn twists about the curves of Figure 4 (Theorem 5.12),
- The equivalence with the Lyubashenko representation for $\text{mod}_l(H)$ (Theorem 6.4).

Let us conclude with a few remarks about our results and further work. First, as already said, all our constructions are explicit; this feature of the theory could be helpful to make computations when one studies the representation of the mapping class group for a given $H$ (see for instance the proof of [Fai18b] Theorem 6.4) for computations in the case of the torus with $H = U_q(\mathfrak{sl}(2))$. Second, for $H = U_q(\mathfrak{sl}(2))$, our representations of the mapping class group should be associated to logarithmic conformal field theory in arbitrary genus. For the torus $\Sigma_{1,0}$, this is indeed the case: combining the results of [FGST06] and [Fai18b], the projective representation of $\text{SL}_2(\mathbb{Z})$ obtained via the combinatorial quantization is equivalent to that coming from logarithmic conformal field theory. Hence, a natural problem is to study in depth the representation of the mapping class group obtained for $H = U_q(\mathfrak{sl}(2))$ (basis of the representation space, explicit formulas for the action on this basis and structure of the representation). Another question is to study the relation between $\mathcal{L}_{g,n}^{\text{inv}}(\mathcal{C}_q(\mathfrak{sl}(2)))$ and skein theory (work in progress).

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**Notations.** If $A$ is a $\mathbb{C}$-algebra, $V$ is a finite-dimensional $A$-module and $x \in A$, we denote by $x \in \text{End}_\mathbb{C}(V)$ the representation of $x$ on the module $V$. Similarly, if $X \in A^{\otimes n}$ and if $V_1, \ldots, V_n$ are $A$-modules, we denote by $X_{V_1 \otimes \ldots \otimes V_n}$ the representation of $X$ on $V_1 \otimes \ldots \otimes V_n$. Here we consider only finite-dimensional representations, hence $H$-module implicitly means finite-dimensional $H$-module.
Let \( \text{Mat}_m(A) = \text{Mat}_m(\mathbb{C}) \otimes A \). Every \( M \in \text{Mat}_m(A) \) is written as \( M = \sum_{i,j} E^i_j \otimes M^j_i \), where \( E^i_j \) is the matrix with 1 at the intersection of the \( i \)-th row and the \( j \)-th column and 0 elsewhere. If \( f : A \to A \) is a morphism, then we define \( f(M) \) by \( f(M) = \sum_{i,j} E^i_j \otimes f(M^j_i) \). Let moreover \( N = \sum_{i,j} E^i_j \otimes N^j_j \in \text{Mat}_n(\mathbb{C}) \otimes A \). We embed \( M, N \) in \( \text{Mat}_m(\mathbb{C}) \otimes \text{Mat}_n(\mathbb{C}) \otimes A = \text{Mat}_{mn}(A) \) by

\[
M_1 = \sum_{i,j} E^i_j \otimes I_n \otimes M^j_i, \quad N_2 = \sum_{i,j} I_m \otimes E^i_j \otimes N^j_j
\]

where \( I_k \) is the identity matrix of size \( k \). Then \( M_1 N_2 \) (resp. \( N_2 M_1 \)) contains all the possible products of coefficients of \( M \) (resp. of \( N \)) by coefficients of \( N \) (resp. of \( M \)): \( (M_1 N_2)_{ik} = M^j_j N^j_k \) (resp. \( (N_2 M_1)_{ik} = N^j_j M^j_k \)).

In order to simplify notation we use Sweedler’s notation (see [Kas95, Not. III.1.6]) without summation sign for coproducts, that is we write

\[
\Delta(x) = x' \otimes x'', \quad \Delta^{(2)}(x) = (\Delta \otimes \text{id}) \circ \Delta(x) = x' \otimes x'' \otimes x''' \quad \ldots, \quad \Delta^{(n)}(x) = x^{(1)} \otimes \ldots \otimes x^{(n+1)}.
\]

We write the universal \( R \)-matrix as \( R = a_i \otimes b_i \) with implicit summation on \( i \) and define \( R' = b_i \otimes a_i \). We also denote \( RR' = X_i \otimes Y_i \), \( (RR')^{-1} = X_i \otimes Y_i \).

The symbol “\( ? \)” will mean a variable in functional constructions. For instance if \( H \) is a finite-dimensional Hopf algebra and \( \varphi, \psi \in H^* \), \( a, b \in H \), then for all \( x, y \in H \), \( \varphi(\cdot a) : x \mapsto \varphi(xa) \) and \( \varphi(\cdot a) \psi(\cdot b) : x \mapsto \varphi(xa) \psi(bx) \) (thanks to the dual Hopf algebra structure on \( H^* \), see below).

## 2 Preliminaries

In all this paper, \( H \) is a finite-dimensional, factorizable, ribbon Hopf algebra.

### 2.1 Factorizable ribbon Hopf algebras

We recall basic facts about Hopf algebras. For more details, see [Kas95].

If \( I \) is a (finite-dimensional) \( H \)-module, we denote by \( \overline{I} = \overline{I} \in \text{Mat}_{\dim(I)}(H^*) \) the matrix defined by \( \overline{I}(x) = \overline{x} \). Since \( H \) is finite-dimensional, the coefficients of the matrices \( \overline{I} \) span \( H^* \) when \( I \) runs in the set of \( H \)-modules. We assume that \( H \) is factorizable, which means that the coefficients of the matrices \( (\overline{I} \otimes \text{id})(RR') \) span \( H \) when \( I \) runs in the set of \( H \)-modules. Let \( R^{(+)} = R, \quad R^{(-)} = R^{-1} \), and consider the matrices \( \overline{L}^{(\pm)} = (\overline{I} \otimes \text{id})(R^{(\pm)}) \). Since \( H \) is factorizable, the coefficients of the matrices \( \overline{L}^{(+)} \), \( \overline{L}^{(-)} \) generate \( H \) as an algebra when \( I \) runs in the set of \( H \)-modules. As a consequence of the properties of the universal \( R \)-matrix (see [Kas95, VIII.2]), we have the following relations:

\[
\begin{align*}
\overline{L}^{(\epsilon)}_1 \overline{L}^{(\epsilon)}_2 &= \overline{L}^{(\epsilon)}_{12}, \quad \Delta^{(1)} \overline{L}^{(\epsilon)}_a &= \sum_{i} \overline{L}^{(\epsilon)}_b i \otimes \overline{L}^{(\epsilon)}_a i, \\
\overline{R}^{(\epsilon)}_{12} \overline{L}^{(\sigma)}_1 \overline{L}^{(\sigma)}_2 &= \overline{L}^{(\sigma)}_1 \overline{L}^{(\sigma)}_{12} \overline{R}^{(\epsilon)}_{12} \quad \forall \epsilon, \sigma \in \{\pm\} \\
\overline{R}^{(\epsilon)}_{12} \overline{L}^{(\sigma)}_1 \overline{L}^{(\sigma)}_2 &= \overline{L}^{(\sigma)}_2 \overline{L}^{(\sigma)}_{12} \overline{R}^{(\epsilon)}_{12} \quad \forall \epsilon, \sigma \in \{\pm\}.
\end{align*}
\]

(1)

Recall that the Drinfeld element \( u \) (see [Kas95, VIII.4]) and its inverse are:

\[
u = S(b_i) a_i = b_i S^{-1}(a_i), \quad u^{-1} = S^{-2}(b_i) a_i = S^{-1}(b_i) S(a_i) = b_i S^2(a_i).
\]

(2)
We assume that $H$ contains a ribbon element $v$ (see \cite[XIV.6]{Kas95}); it satisfies
\[ v^2 = uS(u), \quad \Delta(v) = (R'R)^{-1}v \otimes v, \quad S(v) = v. \] (3)

Then $H$ contains a canonical pivotal element $g = uv^{-1}$. It satisfies $\Delta(g) = g \otimes g$ and $S^2(x) = gxg^{-1}$ for all $x \in H$.

We denote by $O(H)$ the vector space $H^*$ endowed with the dual Hopf algebra structure, which in terms of matrix coefficients is:
\[ \frac{I}{J} \frac{I}{J} T_1 T_2 = \frac{I}{J} \frac{I}{J} 1_{H^*} = T, \quad \Delta(T^\mu) = \sum_i T^\mu_i \otimes T^\mu_i, \quad \epsilon(T) = I_{\dim(H)}, \quad S(T) = T^{-1} \] (4)

where $\mathbb{C}$ is the trivial representation, so $\frac{C}{T} = \epsilon$, the counit of $H$. In particular, in $O(H)$ holds the following exchange relation:
\[ \frac{I}{J} \frac{I}{J} R_{12} T_1 T_2 = \frac{I}{J} \frac{I}{J} T_2 T_1 R_{12}. \]

Since $H$ is finite-dimensional, it exists right and left integrals $\mu^r, \mu^l \in O(H)$ defined by
\[ \forall \varphi \in O(H), \quad \mu^r \varphi = \epsilon(\varphi)\mu^r, \quad \varphi \mu^l = \epsilon(\varphi)\mu^l. \]

They are unique up to scalar and we fix $\mu^l = \mu^r \circ S$. Moreover, it holds
\[ \forall h \in H, \quad \mu^r(h^?)\varphi = \mu^r(h^?)\varphi(S^{-1}(h^n)), \]
\[ \mu^l = \mu^r(g^?), \]
\[ \forall x, y \in H, \quad \mu^r(xy) = \mu^r(S^2(y)x), \quad \mu^l(xy) = \mu^l(S^{-2}(y)x), \]
\[ \forall x, y \in H, \quad \mu^r(gxy) = \mu^r(gyx), \quad \mu^l(g^{-1}xy) = \mu^l(g^{-1}gy). \]

These properties are well-known, for proofs see e.g. \cite[Prop. 5.3, Lemma 5.9, Lemma 5.10]{Fai18b} and the references therein; \cite{5} is easy, \cite{8} is an obvious consequence of \cite{7}.

2.2 Heisenberg double of $O(H)$

Let $H$ be a Hopf algebra. We recall the definition of the Heisenberg double $\mathcal{H}(O(H))$ (see e.g. \cite[4.1.10]{Mon93}). As a vector space, $\mathcal{H}(O(H)) = O(H) \otimes H$. We identify $\psi \otimes 1$ with $\psi \in O(H)$ and $1 \otimes h$ with $h \in H$. Then the structure of algebra on $\mathcal{H}(O(H))$ is defined by the following conditions:

- $O(H) \otimes 1$ and $1 \otimes H$ are subalgebras of $\mathcal{H}(O(H))$,
- Under the previous identifications, we have the exchange rule
  \[ h\psi = \psi(\psi h)h'' \] (9)
  where $\psi(?z) \in O(H)$ is defined by $\psi(?z)(x) = \psi(xz)$.

In terms of matrices, the exchange relation is
\[ \frac{I}{J} \frac{I}{J} L_1^{(\pm)} L_2 = \frac{I}{J} \frac{I}{J} L_2 L_1^{(\pm)} R_{12}^{(\pm)}. \] (10)

There is a faithful representation $\triangleright$ of $\mathcal{H}(O(H))$ on $O(H)$ (see \cite[Lem. 9.4.2]{Mon93}) defined by
\[ \psi \triangleright \varphi = \psi \varphi, \quad h \triangleright \varphi = \varphi(\psi h). \] (11)

Hence we have an injective morphism $\rho : \mathcal{H}(O(H)) \to \text{End}_\mathbb{C}(H^*)$; by equality of the dimensions, it follows that
\[ \mathcal{H}(O(H)) \cong \text{End}_\mathbb{C}(H^*). \]
In particular, the elements of $\mathcal{H}(\mathcal{O}(H))$ can be defined by their action on $\mathcal{O}(H)$ under $\triangleright$. In terms of matrices, the representation $\triangleright$ is

$$
\begin{align*}
\begin{array}{ll}
I & T_1 \triangleright T_2 = T_{12}, \\
I & L_1(\pm) \triangleright T_2 = (a_i(\pm))_1 b_i(\pm) \triangleright T_2 = (a_i(\pm))_1 T_2(b_i(\pm))_2 = T_2 R_{12}^{I}\end{array}
\end{align*}
$$

(12)

where $R^{I} = a_i(\pm) \otimes b_i(\pm)$.

For $h \in H$, let $\tilde{h} \in \mathcal{H}(\mathcal{O}(H))$ be defined by

$$
\tilde{h} \triangleright \varphi = \varphi(S^{-1}(h)).
$$

(13)

It is easy to see that

$$
\forall g \in H, \forall \psi \in \mathcal{O}(H), \quad \tilde{g} \tilde{h} = \tilde{g} h, \quad \tilde{g} \psi = \psi(S^{-1}(h')? h'.
$$

(14)

Applying this to the matrices $L^{I}$ of generators of $H$, we define

$$
L^{I} = \tilde{a}_i \tilde{b}_i, \quad L^{(-)} = S^{-1}(\tilde{b}_i) \tilde{a}_i \in \text{Mat}_{\text{dim}(I)}(\mathcal{H}(\mathcal{O}(H)))
$$

or equivalently $L^{I}(\pm) \triangleright T_2 = R_{12}^{I} T_2$. Using the standard properties of the $R$-matrix, it is not difficult to show the following relations:

$$
\begin{align*}
\begin{array}{ll}
\tilde{l}^{(e)} \tilde{l}^{(e)} = \tilde{l}^{(e)} \tilde{l}^{(e)}, \\
\tilde{l}^{(e)} \tilde{l}^{(e)} = \tilde{l}^{(e)} \tilde{l}^{(e)} , \\
R_{12}^{(e)} \tilde{l}^{(e)} \tilde{l}^{(e)} = \tilde{l}^{(e)} \tilde{l}^{(e)} \tilde{l}^{(e)} \tilde{l}^{(e)} \forall e, \sigma \in \{\pm\}, \\
R_{12}^{(e)} \tilde{l}^{(e)} \tilde{l}^{(e)} \tilde{l}^{(e)} \tilde{l}^{(e)} = \tilde{l}^{(e)} \tilde{l}^{(e)} \tilde{l}^{(e)} \tilde{l}^{(e)} \forall e, \sigma \in \{\pm\}.
\end{array}
\end{align*}
$$

(15)

3 Definition of $\mathcal{L}_{g,n}(H)$ and the Alekseev isomorphism

Recall that $H$ is a finite-dimensional factorizable ribbon Hopf algebra. The algebras $\mathcal{L}_{g,n}(H)$ where introduced by Alekseev for $H = U_q(\mathfrak{g})$, which gave a presentation of them by generators and relations close to [19]. Here we will define $\mathcal{L}_{g,n}(H)$ using the braided tensor product, as in [AS95a]. This has the advantage to show immediately that $\mathcal{L}_{g,n}(H)$ is a $H$-module-algebra and to emphasize the role of the two building blocks of the theory, namely $\mathcal{L}_{0,1}(H)$ and $\mathcal{L}_{1,0}(H)$. We quickly recall the main properties of these building blocks, and we refer to [Fal15] for more details about them under our assumptions on $H$.

3.1 Definition and properties of $\mathcal{L}_{0,1}(H)$ and $\mathcal{L}_{1,0}(H)$

Let $\mathcal{T}(H^*)$ be the tensor algebra associated to $H^*$, which by definition is spanned by all the formal products $\psi_1 \cdots \psi_n$ of elements of $H^*$, modulo the obvious multilinear relations. There is a canonical injection $j : H^* \to \mathcal{T}(H^*)$ and we denote $\hat{M} = j(T)$.

**Definition 3.1.** The loop algebra $\mathcal{L}_{0,1}(H)$ is the quotient of $\mathcal{T}(H^*)$ by the following fusion relations:

$$
\begin{align*}
\begin{array}{ll}
\hat{I} \otimes \hat{J} & \hat{M}_{12} = \hat{M}_{1}(\hat{R}')_{12} \hat{M}_{2}(\hat{R}')_{12}^{-1} \\
\end{array}
\end{align*}
$$

for all finite-dimensional $H$-modules $I, J$.

See [11] for an explicit description of the product in $\mathcal{L}_{0,1}(H)$. The matrix coefficients $\hat{M}_{ij}^j$ for all $I, i, j$ linearly span $\mathcal{L}_{0,1}(H)$. The following exchange relation, called reflection equation, holds in $\mathcal{L}_{0,1}(H)$:

$$
\begin{align*}
\begin{array}{ll}
\hat{I} \hat{J} & \hat{R}_{12} \hat{M}_{1}(\hat{R}')_{12} \hat{M}_{2} = \hat{M}_{2} \hat{R}_{12} \hat{M}_{1}(\hat{R}')_{12}.
\end{array}
\end{align*}
$$
An important fact is that $\mathcal{L}_{0,1}(H)$ is endowed with a structure of left $\mathcal{O}(H)$-comodule-algebra $\Omega : \mathcal{L}_{0,1}(H) \to \mathcal{O}(H) \otimes \mathcal{L}_{0,1}(H)$ (i.e. $\Omega$ is a morphism of algebras, see [Kas95, Def. III.7.1]) defined by

$$\Omega(M^*_h) = T^*_iS(T^i_h) \otimes M^j_i.$$  

If we view $\mathcal{O}(H)$ and $\mathcal{L}_{0,1}(H)$ as subalgebras of $\mathcal{O}(H) \otimes \mathcal{L}_{0,1}(H)$ in the canonical way, then $\Omega$ is simply written $\Omega(M) = TMS(T)$. Equivalently, evaluating the coaction $\Omega$ on $H$, $\mathcal{L}_{0,1}(H)$ is endowed with a structure of right $H$-module-algebra (see [Kas95, Def. V.6.1] for this notion) defined by

$$\overset{l}{M} \cdot h = h'MS(h'')$$  \hspace{1cm} (16)

for $h \in H$. Moreover, if we endow $H$ with the right adjoint action defined by $a \cdot h = S(h')ah''$ ($a, h \in H$), then

$$\Psi_{0,1} : \mathcal{L}_{0,1}(H) \to H$$

$$\overset{l}{M} \mapsto (T \otimes \text{id})(RR') = L^l(+)L^{-1}$$  \hspace{1cm} (17)

is an isomorphism of $H$-module-algebras. In particular, $\mathcal{L}^{\text{inv}}_{0,1}(H) \cong \mathcal{Z}(H)$, where $\mathcal{L}^{\text{inv}}_{0,1}(H)$ is the space of coinvariants (that is, the elements such that $x \cdot h = \varepsilon(h)$ for all $h \in H$ or equivalently $\Omega(x) = 1 \otimes x$).

Moreover, the matrices $\overset{l}{M}$ are invertible.

Now consider the free product $\mathcal{L}_{0,1}(H) * \mathcal{L}_{0,1}(H)$. Let $j_1$ (resp. $j_2$) be the canonical algebra embeddings in the first (resp. second) copy of $\mathcal{L}_{0,1}(H)$ in $\mathcal{L}_{0,1}(H) * \mathcal{L}_{0,1}(H)$, and define $\overset{l}{A} = j_1(M)$, $\overset{l}{B} = j_2(M)$.

**Definition 3.2.** The handle algebra $\mathcal{L}_{1,0}(H)$ is the quotient of $\mathcal{L}_{0,1}(H) * \mathcal{L}_{0,1}(H)$ by the following exchange relations:

$$\overset{l}{A_1} R_{12} \overset{l}{B_1} R'_{12} A_2 = A_2 R_{12} B_1 R'_{12}$$

for all finite-dimensional $H$-modules $I, J$.

Similarly to $\mathcal{L}_{0,1}(H)$, $\mathcal{L}_{1,0}(H)$ is endowed with a structure of left $\mathcal{O}(H)$-comodule-algebra structure $\Omega : \mathcal{L}_{1,0}(H) \to \mathcal{O}(H) \otimes \mathcal{L}_{1,0}(H)$ defined by

$$\Omega(A) = T^lAS(T), \hspace{0.5cm} \Omega(B) = T^lBS(T).$$

As previously, it is equivalent to deal with the right action defined by $\overset{l}{A} \cdot h = h'MS(h'')$, $\overset{l}{B} \cdot h = h'BS(h'')$. The map

$$\Psi_{1,0} : \mathcal{L}_{1,0}(H) \to \mathcal{H}(\mathcal{O}(H))$$

$$\overset{l}{A} \mapsto L^l(+)L^{-1}$$

$$\overset{l}{B} \mapsto L^l(+)TL^{-1}$$

is an isomorphism of algebras (see [Fal18b] for a proof). It follows that $\mathcal{L}_{1,0}(H)$ is isomorphic to a matrix algebra, and in particular has trivial center.

### 3.2 Braided tensor product and definition of $\mathcal{L}_{g,n}(H)$

Let $\text{mod}_r(H)$ be the category of finite-dimensional right $H$-modules (or, equivalently, of finite-dimensional left $H$-comodules). The braiding in $\text{mod}_r(H)$ is given by:

$$c_{I,J} : I \otimes J \to J \otimes I$$

$$v \otimes w \mapsto w \cdot a_i \otimes v \cdot b_i$$
with \( R = a_i \otimes b_i \). Let \((A, m_A, 1_A)\) and \((B, m_B, 1_B)\) be two algebras in \( \text{mod}_r(H) \) (that is, \( H \)-module-algebras), and define:

\[
\begin{align*}
m_{A \otimes B} &= (m_A \otimes m_B) \circ (\text{id}_A \otimes c_{B,A} \otimes \text{id}_B) : A \otimes B \to A \otimes B, \\
1_{A \otimes B} &= 1_A \otimes 1_B : C \to A \otimes B.
\end{align*}
\]

This endows \( A \otimes B \) with a structure of algebra in \( \text{mod}_r(H) \), denoted \( A \widehat{\otimes} B \) and called braided tensor product of \( A \) and \( B \) (see [Ma95, Lemma 9.2.12]). Note that \( \widehat{\otimes} \) is associative.

There are two canonical algebra embeddings \( j_A, j_B : A, B \to A \widehat{\otimes} B \) respectively defined by \( j_A(x) = x \otimes 1_B, j_B(y) = 1_A \otimes y \). We identify \( x \in A \) (resp. \( y \in B \)) with \( j_A(x) \in A \widehat{\otimes} B \) (resp. \( j_B(y) \)). Under these identifications, the multiplication rule in \( A \widehat{\otimes} B \) is entirely given by:

\[
\forall x \in A, \forall y \in B, \ yx = (x \cdot a_i)(y \cdot b_i).
\]

Since \( L_{0,1}(H) \) and \( L_{1,0}(H) \) are algebras in \( \text{mod}_r(H) \), we can apply the braided tensor product to them.

**Definition 3.3.** \( L_{g,n}(H) \) is the \( H \)-module-algebra \( L_{1,0}(H) \widehat{\otimes} g \widehat{\otimes} L_{0,1}(H) \widehat{\otimes} n \).

It is useful to keep in mind that the \( H \)-module-algebra \( L_{g,n}(H) \) is associated with the surface \( \Sigma_{g,n} \setminus D \); in order to make this precise we now define the matrices introduced in Figure 8. There are canonical algebra embeddings \( j_i : L_{0,1}(H) \hookrightarrow L_{g,n}(H) \) for \( 1 \leq i \leq g \) and \( j_i : L_{0,1}(H) \hookrightarrow L_{g,n}(H) \) for \( g + 1 \leq i \leq g + n \), given by \( j_i(x) = 1^{\otimes i-1} \otimes x \otimes 1^{\otimes g+n-i} \). Define

\[
\begin{align*}
I(i) &= j_i(A), \\
B(i) &= j_i(B) \quad \text{for } 1 \leq i \leq g \quad \text{and} \\
M(i) &= j_i(M) \quad \text{for } g + 1 \leq i \leq g + n.
\end{align*}
\]

Relation (§) indicates that \( L_{g,n}(H) \) is an exchange algebra. Let us write the exchange relations in a matrix form. Let \( U \) be \( A \) or \( B \) or \( M \), let \( V \) be \( A \) or \( B \) or \( M \) and let \( i < j \). Then, by definition of the right action and by (§):

\[
\begin{align*}
V(j)_{2} U(i)_{1} &= (a'_{k})_{1} U(i)_{1} S(a'_{k})_{1} (b'_{k})_{2} V(j)_{2} S(b'_{k})_{2} = (a_{k})_{1} R_{12} U(i)_{1} R_{12}^{-1} V(j)_{2} R_{12} S(b_{k})_{2},
\end{align*}
\]

where for the second equality we applied properties of the \( R \)-matrix and obvious commutation relations in \( \text{End}_{C}(I) \otimes \text{End}_{C}(J) \otimes L_{g,n}(H) \). Using that \( a_{n} a_{l} \otimes S(b_{l})b_{m} = 1 \otimes 1 \) together with obvious commutation relations, we obtain the desired exchange relation:

\[
V(j)_{2} R_{12} U(i)_{1} R_{12}^{-1} V(j)_{2} = V(j)_{2} R_{12} U(i)_{1} R_{12}^{-1}.
\]

To sum up, the presentation of \( L_{g,n}(H) \) by generators and relations is:

\[
\begin{align*}
I_{12} U(i)_{12} &= U(i)_{12} R_{12}^{-1} U(i)_{2} R_{12}^{-1} \quad \text{for } 1 \leq i \leq g + n \\
I_{12} R_{12} U(i)_{12} &= U(i)_{12} R_{12}^{-1} U(i)_{2} R_{12}^{-1} \quad \text{for } 1 \leq i < j \leq g + n \\
I_{12} B(i)_{12} R_{12}^{-1} A(i)_{2} &= A(i)_{2} B(i)_{12} R_{12}^{-1} \quad \text{for } 1 \leq i \leq g
\end{align*}
\]

where \( U(i) \) (resp. \( V(i) \)) is \( A(i) \) or \( B(i) \) if \( 1 \leq i \leq g \) and is \( M(i) \) if \( g + 1 \leq i \leq g + n \). Such a presentation was first introduced in [Alek94] and [AGS95]. Recall that the first line of relations is the \( L_{0,1}(H) \)-fusion relation on each loop, the second line is the exchange relation of the braided tensor product and the third line is the \( L_{1,0}(H) \)-exchange relation.

**Notation.** Let \( N = \tilde{I}m_{1} N_{1} \cdots \tilde{I}m_{n} N_{n} \in \text{Mat}_{\text{dim}(I)}(L_{g,n}(H)) \), where \( m, n_{i} \in \mathbb{Z} \) and each \( N_{i} \) is one of the \( A(j), B(j), M(k) \) for some \( j \) or \( k \). By definition of the right action on \( L_{g,n}(H) \), we have a morphism of \( H \)-modules \( j_{N} : L_{0,1}(H) \to L_{g,n}(H) \) defined by \( j_{N}(M) = N \). Let \( x \in H \cong L_{0,1}(H) \), then we denote \( x_{N} = j_{N}(x) \). The following lemma is an obvious fact.
Lemma 3.4. If $N$ satisfies the fusion relation of $L_{0,1}(H)$, $I_{i}^{\otimes J} = I_{i}^{J}, \ N_{12} = N_{i}(i_{1}(R))_{12} \ N_{i}(i_{2}(R))_{12}^{-1}$, then $j_{N}$ is a morphism of $H$-module-algebras: $(xy)_{N} = x_{N}y_{N}$.

See e.g. [24] for an application of this lemma.

3.3 The Alekseev isomorphism

Consider the tensor product algebra $L_{1,0}(H)^{\otimes g} \otimes L_{0,1}(H)^{\otimes n}$. We have canonical algebra embeddings $j_{i}: L_{1,0}(H) \hookrightarrow L_{1,0}(H)^{\otimes g} \otimes L_{0,1}(H)^{\otimes n}$ for $1 \leq i \leq g$ and $j_{i}: L_{0,1}(H) \hookrightarrow L_{1,0}(H)^{\otimes g} \otimes L_{0,1}(H)^{\otimes n}$ for $g + 1 \leq i \leq g + n$, defined by $j_{i}(x) = 1^{\otimes i} \otimes x \otimes 1^{\otimes g + n - i}$. Define $A(i) = j_{i}(A), B(i) = j_{i}(B)$ for $1 \leq i \leq g$ and $M(i) = j_{i}(M)$ for $g + 1 \leq i \leq g + n$. We underline these matrices to avoid confusion with prior matrices having coefficients in $L_{g,n}(H)$. By definition, the exchange relation between copies in $L_{1,0}(H)^{\otimes g} \otimes L_{0,1}(H)^{\otimes n}$ is simply

$$U(i)(j) = V(j)(i)$$

where $i \neq j, U(i), V(i)$ is $A(i)$ or $B(i)$ if $1 \leq i \leq g$ and is $M(i)$ if $g + 1 \leq i \leq g + n$.

The next result is due to Alekseev (see [Ale94]). Consider the matrices $M^{(-)} = \Psi_{g,n}^{-1}(L^{(-)})$ and $C^{(-)} = \Psi_{1,0}^{-1}(L^{(-)} \tilde{L}^{(-)})$. Let

$$(A_{1} = I_{\text{dim}(L)}, \quad \Lambda_{i} = \frac{C^{(-)}(1) \cdots C^{(-)}(i - 1)}{C^{(-)}(i)} \quad \text{for } 2 \leq i \leq g + 1, \quad C^{(-)} = \frac{1}{\Gamma_{g+1}^{(-)}(g + 1) \cdots C^{(-)}(i - 1)} \quad \text{for } g + 2 \leq i \leq g + n)$$

be matrices with coefficients in $L_{1,0}(H)^{\otimes g} \otimes L_{0,1}(H)^{\otimes n}$ (with $I_{s}$ the identity matrix of size $s$).

Proposition 3.5. The map

$$\alpha_{g,n}: L_{g,n}(H) = L_{1,0}(H)^{\otimes g} \otimes L_{0,1}(H)^{\otimes n} \rightarrow L_{1,0}(H)^{\otimes g} \otimes L_{0,1}(H)^{\otimes n}$$

is an isomorphism of algebras, which we call the Alekseev isomorphism.

Proof: In order to show that it is a morphism of algebras, one must check using various exchange relations that the defining relations (19) of $L_{g,n}(H)$ are preserved under $\alpha_{g,n}$. This is a straightforward but tedious task and we will not give the details. Let us prove that $\alpha_{g,n}$ is bijective. We first show that $\alpha_{g,0}$ is surjective for all $g$ by induction. For $g = 1$, $\alpha_{1,0}$ is the identity. For $g \geq 2$, we embed $L_{g-1,0}(H)$ in $L_{g,0}(H)$ in an obvious way by $I(A_{i}) \mapsto A_{i}$ and $I(B_{i}) \mapsto B_{i}$ for $1 \leq i \leq g - 1$. Then the restriction of $\alpha_{g,0}$ to $L_{g-1,0}(H)$ is $\alpha_{g-1,0}$, and by induction we assume that $\alpha_{g-1,0}(L_{g-1,0}(H)) = L_{1,0}(H)^{\otimes g - 1}$. Since $A_{i} \in \text{Mat}_{\text{dim}(L)}(L_{1,0}(H)^{\otimes g - 1} \otimes C^{\otimes g + 1 - i})$, there exists matrices $N_{i}(1 \leq i \leq g)$ such that $\alpha_{g,0}(N_{i}) = A_{i}$. Then $\alpha_{g,0}(N_{i}^{-1}U_{i})N_{i} = U_{i}$, with $U = A$ or $B$ and $\alpha_{g,0}$ is surjective. Similarly, for $g$ fixed and $n \geq 1$, we can embed $L_{g,n-1}(H)$ into $L_{g,n}(H)$ and reproduce the same reasoning. Hence $\alpha_{g,n}$ is surjective for all $g, n$. Since the domain and the range of $\alpha_{g,n}$ have the same dimension, it is an isomorphism.

We can now generalize the isomorphisms $\Psi_{0,1}$ and $\Psi_{1,0}$ by

$$\Psi_{g,n} = (\Psi_{1,0} \otimes \Psi_{0,1}) \circ \alpha_{g,n}: L_{g,n}(H) \sim H(O(H))^{\otimes g} \otimes H^{\otimes n}$$.  

(21)
In particular $\mathcal{L}_{g,0}(H)$ is a matrix algebra, since $\mathcal{H}(\mathcal{O}(H))$ is.

Thanks to $\Psi_{g,n}$, the representation theory of $\mathcal{L}_{g,n}(H)$ is entirely determined by the representation theory of $H$. Indeed, the only indecomposable (and simple) representation of $\mathcal{H}(\mathcal{O}(H)) \cong \text{End}_\mathbb{C}(H^*)$ is $H^*$, thus it follows that the indecomposable representations of $\mathcal{L}_{g,n}(H)$ are of the form

$$(H^*)^g \otimes I_1 \otimes \ldots \otimes I_n$$

where $I_1, \ldots, I_n$ are indecomposable representations of $H$. We will denote the action of $\mathcal{L}_{g,n}(H)$ on $(H^*)^g \otimes I_1 \otimes \ldots \otimes I_n$ by $\triangleright$, namely:

$$x \triangleright \varphi_1 \otimes \ldots \otimes \varphi_g \otimes v_1 \otimes \ldots \otimes v_n = \Psi_{g,n}(x) \cdot \varphi_1 \otimes \ldots \otimes \varphi_g \otimes v_1 \otimes \ldots \otimes v_n \quad (22)$$

for $x \in \mathcal{L}_{g,n}(H)$, where $\cdot$ is the action component-by-component of $\Psi_{g,n}(x)$ on $(H^*)^g \otimes I_1 \otimes \ldots \otimes I_n$.

## 4 Representation of $\mathcal{L}_{g,n}^\text{inv}(H)$

Recall that an element $x \in \mathcal{L}_{g,n}(H)$ is invariant if $x \triangleright h = \varepsilon(h)x$ for all $h \in H$, or equivalently, if $\Omega(x) = 1 \otimes x$. In this section we construct representations of the subalgebra of invariants $\mathcal{L}_{g,n}^\text{inv}(H)$.

For this, we use an idea introduced in [Ale94] (the matrices $C$), but adapted to our assumptions on $H$.

### 4.1 The matrices $C_{g,n}$

We first consider the case of $\mathcal{L}_{1,0}(H)$. Let us define matrices

$$C = v^2 BA^{-1}B^{-1}A, \quad C^{(\pm)} = \Psi_{1,0}^{-1}((L^{(\pm)}L^{(\pm)\dagger}))$$

(23)

**Lemma 4.1.** The following equality holds in $\mathcal{L}_{1,0}(H)$:

$$C = C^{(+)}C^{(-)-1}.$$ 

Moreover, the matrices $C$ satisfy the fusion relation of $\mathcal{L}_{0,1}(H)$:

$$C^{(i)\otimes j} = C_{ij} = C_{1}(R_{i})_{12}C_{2}(R_{j})_{12}^{-1}.$$ 

**Proof:** We have

$$\Psi_{1,0}\left(v^2 BA^{-1}B^{-1}A\right) = L^{(+)\dagger}\left(v^2 TL^{(+)\dagger}L^{(-)}S(T)\right)L^{(-)\dagger}.$$ 

Let us simplify the middle term:

$$v^2 TS(a_i)S^{-1}(b_j)b_ia_jS(T) = v^2 TS(a_i)S^{-1}(b_j)S(a_j^i)S(b_j^i)S(T)b_i^ia_j^i$$

$$= v^2 TS(a_ia_k)S^{-1}(b_jb_k)S(a_j^i)S(b_k^i)S(T)b_ia_k^i.$$ 

The first equality is the exchange relation $[3]$ in $\mathcal{H}(\mathcal{O}(H))$ and the second follows from the properties of the $R$-matrix. The third equality is obtained as follows: denoting $m : H \otimes H \to H$ the multiplication, we can write

$$v^2 S(a_ia_k)S^{-1}(b_jb_k)S(a_j^i)S(b_k^i) \otimes b_ia_k = vS(a_ia_k)S^{-1}(b_kb_j)g^{-1} \otimes b_ia_k$$

$$= v \otimes 1(m \circ (S \otimes S^{-1}) \otimes \text{id})(R_{13}R_{12}R_{32})g^{-1} \otimes 1 = v \otimes 1(m \circ (S \otimes S^{-1}) \otimes \text{id})(R_{32}R_{12}R_{13})g^{-1} \otimes 1$$

$$= vS(a_k)a_iS^{-1}(b_kb_j)g^{-1} \otimes a_kb_i = vS(a_i)S(S^{-2}(b_kb_j))S^{-1}(b_kb_j)g^{-1} \otimes a_kb_i = S(a_i)S(b_i) \otimes a_kb_i.$$
We used formula (2) for $u^{-1}$ twice, a Yang-Baxter relation and the standard properties for $g$ and $v$. Now, we have:

\[
T_1 S(b_i a_i)_1 S(T)_1 a_i b_i \triangleright T_2 = T_1 S(b_i a_i)_1 S(T)_1 T_2 (a_i b_i)_2 = S(b_i a_i)_1 (a_i b_i)_2 T_2 = (a_i b_i)_1 \tilde{b}_i \tilde{a}_i \triangleright T_2.
\]

For the second equality, we used that for any $h \in H$:

\[
\langle S(b_i a_i)_1 S(T)_1 T_2 (a_i b_i)_2, h \rangle = S(h b_i a_i)_1 (h'' a b_i)_2 = S(b_i a_i h')_1 (a i b_i h'')_2 = \langle S(T)_1 S(b_i a_i)_1 (a_i b_i)_2 T_2, h \rangle.
\]

Since $\triangleright$ is faithful, we finally get

\[
\text{Hence } \quad \Psi_{1,0}(C) = \tilde{L}^{(+) \tilde{L}^{(+)} \tilde{L}^{(-)} \tilde{L}^{(-)}}^{-1} = \Psi_{1,0}(\tilde{C}^{(+)} \tilde{C}^{(-)})^{-1}
\]

as desired. To prove the fusion relation, it suffices to consider $\Psi_{1,0}(C) = \tilde{L}^{(+) \tilde{L}^{(+)} \tilde{L}^{(-)} \tilde{L}^{(-)}}^{-1}$ and to use the exchange relations in (15). This is a straightforward computation left to the reader. □

We now give the general definition. For $i \leq g$, let $C(i)$ be the embedding of $I$ previously defined on the $i$-th copy of $L_{1,0}(H)$ in $L_{g,n}(H)$.

**Definition 4.2.** $C_{g,n} = I \cdot C(1) \ldots C(g) M(g + 1) \ldots M(g + n)$.

Geometrically (see Figure 1), for each $I$ the matrix $C_{g,n}$ corresponds to the holonomy along the boundary of the unique face of the graph $\Gamma$ defined in the Introduction.

There is a decomposition analogous to Lemma 4.1 which was the case $g = 1, n = 0$. Indeed, let

\[
\tilde{C}_{g,n}^{(±)} = \alpha_{g,n}^{-1} \left( C^{(±)}(1) \ldots C^{(±)}(g) \tilde{M}^{(±)}(g + 1) \ldots \tilde{M}^{(±)}(g + n) \right) \in \text{Mat}_{\dim(I)}(L_{g,n}(H))
\]

**Proposition 4.3.** The following equality holds in $L_{g,n}(H)$:

\[
C_{g,n} = C_{g,n}^{(+)} C_{g,n}^{(-)}^{-1}.
\]

Moreover, the matrices $C_{g,n}$ satisfy the fusion relation of $L_{0,1}(H)$:

\[
I \otimes^J (C_{g,n})_{12} = (C_{g,n})_1 (R')_{12} I J (C_{g,n})_2 (R')_{12}^{-1}.
\]

**Proof:** The first claim is a simple consequence of the definition of $\alpha_{g,n}$ and of Lemma 4.1. The fusion relation is a consequence of a more general fact which is easy to show, namely: if $i_1 < \ldots < i_k$ and if $X^1(i_1), \ldots, X^k(i_k)$ are matrices satisfying the fusion relation of $L_{0,1}(H)$, then their product $X^1(i_1) \ldots X^k(i_k)$ also satisfies the fusion relation of $L_{0,1}(H)$. □

The image of these matrices have simple expressions in $H(\mathcal{O}(H))^\otimes g \otimes H^\otimes n$:

**Lemma 4.4.** It holds

\[
\Psi_{g,n}(C_{g,n}^{(±)}) = \tilde{a}_i b_i^{(2g-1+n)} b_i^{(2g+n)} \otimes \ldots \otimes b_i^{(1+n)} b_i^{(2+n)} \otimes b_i^{(n)} \otimes \ldots \otimes b_i^{(1)}
\]

\[
\Psi_{g,n}(C_{g,n}^{(-)}) = S^{-1}(b_i) a_i^{(2g-1+n)} a_i^{(2g+n)} \otimes \ldots \otimes a_i^{(1+n)} a_i^{(2+n)} \otimes a_i^{(n)} \otimes \ldots \otimes a_i^{(1)}
\]

\[
\Psi_{g,n}(C_{g,n}) = X_i Y_i^{(2g-1+n)} Y_i^{(2g+n)} \otimes \ldots \otimes Y_i^{(1+n)} Y_i^{(2+n)} \otimes Y_i^{(n)} \otimes \ldots \otimes Y_i^{(1)}
\]

where $X_i \otimes Y_i = RR'$ and the superscripts mean iterated coproduct.
Proof: As an immediate consequence of quasitriangularity, we have for all $n \geq 2$

$$(\text{id} \otimes \Delta^{(n-1)})(R) = a_i \otimes b_i^{(1)} \otimes \ldots \otimes b_i^{(n)} = a_{i_1} \ldots a_{i_n} \otimes b_{i_1} \otimes \ldots \otimes b_{i_1}.$$  

with implicit summation on $i_1, \ldots, i_n$. It follows that

$$\Psi_{g,n}(C_{g,n}^{(+)}) = L^{(+)}(1) L^{(+)}(1) \ldots L^{(+)}(g) L^{(+)}(g + 1) \ldots L^{(+)}(g + n)$$

$$= a_{i_1} \ldots a_{i_{2g+n}} b_{i_1}^{(2g-1+n)} b_{i_1}^{(2g+n)} \otimes \ldots \otimes b_{i_1}^{(1+n)} b_{i_1}^{(2+n)} b_{i_1}^{(n)} \otimes \ldots \otimes b_{i_1}^{(1)}$$

as desired. The second is shown similarly since $R^{-1}$ is also an universal $R$-matrix. The third is an immediate consequence.

The matrices $C_{g,n}$ satisfying the fusion relation of $L_{0,1}(H)$, we can apply Lemma 3.4 and define a representation of $H$ on $V = (H^*)^{\otimes g} \otimes I_1 \otimes \ldots \otimes I_n$ by

$$h \cdot v = h_{C_{g,n}} \triangleright v.$$  

(24)

Since $H$ is factorizable, each $h \in H$ is a linear combination of coefficients of the matrices $X_i Y_i$. Hence, $h_{C_{g,n}}$ is a linear combination of coefficients of the matrices $C_{g,n}$. It follows from Lemma 4.3 that this representation is explicitly given by

$$h \cdot \varphi_1 \otimes \ldots \otimes \varphi_g \otimes v_1 \otimes \ldots \otimes v_n$$

$$= \varphi_1(S^{-1}(h^{(2g-1+n)}) h^{(2g+n)}) \otimes \ldots \otimes \varphi_g(S^{-1}(h^{(1+n)}) h^{(2+n)}) \otimes h^{(n)} v_1 \otimes \ldots \otimes h^{(1)} v_n.$$  

(25)

4.2 Determination and representation of $L_{g,n}^{\text{inv}}(H)$

The matrices $C_{g,n}$ introduced above allow one to give a simple characterization of the invariant elements of $L_{g,n}^{\text{inv}}(H)$ and to construct representations of them. We begin with a technical lemma.

Lemma 4.5. It holds

$$\left( C_{g,n}^{(\pm)} \right)_1 \left( C_{g,n}^{(\pm)} \right)^{-1} = R_{12}^{(\pm)} L_{12}^{(\pm)}$$

where $U$ is $A$ or $B$.

Proof: Applying the isomorphisms $\Psi_{0,1}$ and $\Psi_{1,0}$ and using relations (1), (10) and (15), it is easy to show the result for $L_{0,1}(H)$ and $L_{1,0}(H)$. We get similarly:

$$I_{12}^{J} C_{1}^{(\pm)} C_{2}^{(\pm)} = C_{2}^{(\pm)} C_{1}^{(\pm)} I_{12}^{J} M_{1}^{(\mp)} M_{2}^{(\pm)} = M_{2}^{(\mp)} M_{1}^{(\pm)} I_{12}^{J}.$$  

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Using these preliminary facts, we can carry out the general computation. For instance, for $i \leq g$
\[
\alpha_{g,n} \left( \mathcal{C}^{(\pm)}_{g,n} \right)^{-1} U(i)_2 \mathcal{C}^{(\pm)}_{g,n}^{-1}
= \mathcal{R}^{(-)}(1)_1 \cdots \mathcal{R}^{(-)}(i)_1 \mathcal{R}^{(-)}(i-1)_2 \cdots \mathcal{R}^{(-)}(i-1)_2^{-1} \cdots \mathcal{R}^{(\pm)}(i)_1 \cdots \mathcal{R}^{(\pm)}(1)_1^{-1}
= \mathcal{R}^{(-)}(1)_1 \cdots \mathcal{R}^{(-)}(i)_1 \cdots \mathcal{R}^{(-)}(i-1)_2 \cdots \mathcal{R}^{(-)}(i-1)_2^{-1} \cdots \mathcal{R}^{(\pm)}(i)_1 \cdots \mathcal{R}^{(\pm)}(1)_1^{-1}
= \mathcal{R}^{(-)}(1)_2 \cdots \mathcal{R}^{(-)}(i)_1 \cdots \mathcal{R}^{(-)}(i-1)_2 \cdots \mathcal{R}^{(-)}(i-1)_2^{-1} \cdots \mathcal{R}^{(\pm)}(i)_1 \cdots \mathcal{R}^{(\pm)}(1)_1^{-1}
\]

The case $i > g$ is treated in a similar way. □

For $(V,\triangleright)$ a representation of $\mathcal{L}_{g,n}(H)$, let
\[
\text{Inv}(V) = \left\{ v \in V \mid \forall I, \mathcal{C}_{g,n} \triangleright v = \mathbb{I}_{\dim(I)v} \right\} = \{ v \in V \mid \forall h \in H, \ h \cdot v = \varepsilon(h)v \}
\]
where $\mathbb{I}_k$ is the identity matrix of size $k$, and the action $\cdot$ of $H$ on $V$ is defined in [24] and [25].

**Theorem 4.6.** 1) An element $x \in \mathcal{L}_{g,n}(H)$ is invariant under the action of $H$ (or equivalently under the coaction $\Omega$ of $\mathcal{O}(H)$) if, and only if, for every $H$-module $I$, $\mathcal{C}_{g,n,x} = x\mathcal{C}_{g,n}$.

2) Let $V$ be a representation of $\mathcal{L}_{g,n}(H)$. Then $\text{Inv}(V)$ is stable under the action of invariant elements and thus provides a representation of $\mathcal{L}_{g,n}^{\text{inv}}(H)$.

**Proof:** 1) The right action of $H$ on $\mathcal{L}_{g,n}(H)$ is by definition
\[
\mathcal{U}(i) : h = h\mathcal{U}(i)S(h^\circ)
\]
where $\mathcal{U}(i)$ is $A(i)$ or $B(i)$ if $i \leq g$ and is $M(i)$ if $i > g$. Then, denoting $a_j^{(\pm)} \otimes b_j^{(\pm)} = R^{(\pm)}$,
\[
\mathcal{J} \mathcal{U}(i)_2 \cdot S^{-1}(L^{(\pm)}) = \mathcal{U}(i)_2 \cdot S^{-1}(b_j^{(\pm)}) = S^{-1}(b_j^{(\pm)}) \mathcal{U}(i)_2 \mathcal{J} a_k^{(\pm)} = \mathcal{R}^{(\pm)}(i)_1 \mathcal{R}^{(\pm)}(i)_2^{-1}
\]
where the last equality is Lemma [4.3]. Observe that the matrix $\mathcal{R}^{(\pm)}(i)_1$ contains all the elements obtained by acting by the coefficients of $S^{-1}(L^{(\pm)})$ on the coefficients of $\mathcal{U}(i)$. The coefficients of the $S^{-1}(L^{(\pm)}) = \mathcal{L}^{(\pm)}$ generate $H$ as an algebra, hence we deduce that an $x \in \mathcal{L}_{g,n}(H)$ is invariant if, and only if, $\mathcal{C}_{g,n,x} = x\mathcal{C}_{g,n}$. Since $H$ is factorizable, it is generated by the coefficients of the matrices $X_i \otimes Y_i$ and we see by Lemma [4.3] that the subalgebra of $\mathcal{L}_{g,n}(H)$ generated by the coefficients of the matrices $\mathcal{C}_{g,n}^{(\pm)}$ equals the subalgebra generated by the coefficients of the matrices $\mathcal{C}_{g,n}^{(\pm)}$. Thus $x$ commutes with the $\mathcal{C}_{g,n}^{(\pm)}$. If, and only if, $x$ commutes with the $\mathcal{C}_{g,n}$.

2) If $v \in \text{Inv}(V)$ and $x \in \mathcal{L}_{g,n}^{\text{inv}}(H)$, we have
\[
\mathcal{C}_{g,n} \triangleright (x \triangleright v) = \mathcal{C}_{g,n} x \triangleright v = x \mathcal{C}_{g,n} \triangleright v = \mathbb{I}_{\dim(I)}(x \triangleright v)
\]
which shows that \( x \triangleright v \in \text{Inv}(V) \). 

By definition, a \( H \)-connection \( \nabla = (h_e)_{e \in E} \) (with \( E = \{ b_1, a_1, \ldots, b_g, a_g, m_{g+1}, \ldots, m_{g+n} \} \), the set of edges) is flat if its holonomy along the boundary \( c = b_1 a_1^{-1} b_1^{-1} a_1 \ldots b_g a_g^{-1} b_g^{-1} a_g m_{g+1} \ldots m_{g+n} \) of the unique face of the graph \( \Gamma \) is trivial:

\[
\text{Hol}^\nabla(C) = h_{b_1} h_{b_1}^{-1} h_{a_1} \ldots h_{b_g} h_{a_g}^{-1} h_{b_g}^{-1} h_{a_g} h_{m_{g+1}} \ldots h_{m_{g+n}} = 1.
\]

Hence, the subrepresentation \( \text{Inv}(V) \) implements this flatness constraint. This constraint was directly implemented in \( L_{g,n}(H) \) (and not just on representations) in [AGS96, AS96a] by means of characteristic projectors, giving rise to the moduli algebra, a quantum analogue of \( \mathbb{C}[A_{\text{f}}/G] \) (see Introduction). However, the definition of these projectors requires the \( S \)-matrix, which has nice properties in the semi-simple case only. In [AS96a], the representation space of the mapping class group is the moduli algebra. Here we do not consider the moduli algebra; in particular we will not need to generalize these projectors to construct the projective representation of the mapping class group.

5 Projective representations of mapping class groups

Let \( \Sigma_{g,n} \) be the compact orientable surface of genus \( g \) with \( n \) open disks removed. For simplicity we consider the case of \( \Sigma_g \) (\( n = 0 \)). The particular features in this case are that the presentation of the mapping class group is easier and that the associated algebra \( L_{g,0}(H) \cong \mathcal{H}(\mathcal{O}(H))^{\otimes g} \) is isomorphic to a matrix algebra.

We will discuss the case of \( n > 0 \) in subsection 5.4.

5.1 Mapping class group of \( \Sigma_g \)

Let \( D \subset \Sigma_g \) be an embedded open disk and define \( \Sigma_g^o = \Sigma_g \setminus D \). We put a basepoint on the boundary circle \( c = \partial(\Sigma_g^o) \) and we take the curves \( a_i, b_i \; (1 \leq i \leq g) \) represented in Figure 2 as generators for the free group \( \pi_1(\Sigma_g^o) \).

![Figure 2: Surface \( \Sigma_g^o \) and generators of \( \pi_1(\Sigma_g^o) \).](image)

With these generators,

\[
c = b_1 a_1^{-1} b_1^{-1} a_1 \ldots b_g a_g^{-1} b_g^{-1} a_g.
\]

Recall from the Introduction the embedded oriented graph \( \Gamma = (\{ \bullet \}; \{ b_1, a_1, \ldots, b_g, a_g \}) \) with vertex \( \bullet \) and edges the generators of the fundamental group. It is readily seen that \( \Sigma_g^o \) is homeomorphic to the thickening of \( \Gamma \) represented in Figure 3.

Simple closed curves on a surface will simply be called circles. Elements of \( \pi_1(\Sigma_g^o) \) will be called loops. We consider circles up to free homotopy. In particular, if \( \gamma \in \pi_1(\Sigma_g^o) \), we denote by \( [\gamma] \) the free homotopy class of \( \gamma \). For \( \alpha \) a circle, we denote by \( \tau_\alpha \) the Dehn twist about it. If \( \gamma \in \pi_1(\Sigma_g^o) \), then \( \tau_\gamma \) is a shortand for \( \tau_{[\gamma]} \), thus defined as follows: consider a circle \( \gamma' \) freely homotopic to \( \gamma \) and which does not intersect the boundary circle \( c \); then \( \tau_{[\gamma]} = \tau_{[\gamma']} \).

If \( S \) is a compact oriented surface, we denote by \( \text{MCG}(S) \) its mapping class group, that is the group of isotopy classes of orientation preserving homeomorphisms of \( S \) which fix the boundary
The correspondence of notations with \cite{FM12, Figure 5.7} is Sect. 5.2.1. Let \( g \) pointwise.

There exists presentations of \( \text{MCG}(\Sigma_g) \) and \( \text{MCG}(\Sigma_o) \) due to Wajnryb \cite{Waj83} (also see \cite{FM12, Sect. 5.2.1}). Let

\[
\begin{align*}
d_1 &= a_1, \quad d_i = a_{i-1}b_ia_i^{-1}b_i^{-1} \quad \text{for} \ 2 \leq i \leq g, \\
e_1 &= a_1, \quad e_i = b_1a_i^{-1}b_1^{-1}a_1 \cdots b_{i-1}a_{i-1}^{-1}b_{i-1}a_i^{-1}b_i^{-1} \quad \text{for} \ 2 \leq i \leq g.
\end{align*}
\]

The correspondence of notations with \cite[Figure 5.7]{FM12} is \( c_0 = [e_2], \ c_2j = [b_j], \ c_{2j-1} = [d_j] \). The free homotopy class of these loops are depicted in Figure 4 below.

The Dehn twists \( \tau_{e_2}, \tau_{b_i}, \tau_{d_i} \) are called the Humphries generators. Then \( \text{MCG}(\Sigma_g) \) is generated by the Humphries generators together with four families of relations called disjointness relations, braid relations, 3-chain relation and lantern relation, see \cite[Theorem 5.3]{FM12}. The presentation of \( \text{MCG}(\Sigma_g) \) is obtained as the quotient of \( \text{MCG}(\Sigma_o) \) by the hyperelliptic relation:

\[
(\tau_{b_1} \tau_{d_1} \cdots \tau_{b_1} \tau_{d_1} \tau_{d_2} \cdots \tau_{d_g} \tau_{b_g}) \ o = \ o (\tau_{b_1} \tau_{d_1} \cdots \tau_{b_1} \tau_{d_1} \tau_{d_2} \cdots \tau_{d_g} \tau_{b_g})
\]

where \( o \) is any word in the Humphries generators which equals \( \tau_{a_g} \).

The action of the Humphries generators on the fundamental group is easily computed. We just indicate the non-trivial actions:

\[
\begin{align*}
\tau_{e_2}(a_1) &= e_2^{-1}a_1e_2, \quad \tau_{e_2}(b_1) = e_2^{-1}b_1e_2, \quad \tau_{e_2}(b_2) = e_2^{-1}b_2, \\
\tau_{b_1}(a_i) &= b_1^{-1}a_i, \\
\tau_{a_1}(b_1) &= b_1a_1 \quad \text{(recall that} \ a_1 = d_1), \\
\tau_{d_i}(a_{i-1}) &= d_i^{-1}a_{i-1}d_i, \quad \tau_d(b_{i-1}) = b_{i-1}d_i, \quad \tau_d(b_i) = d_i^{-1}b_i \quad \text{(with} \ i \geq 2).
\end{align*}
\]

In the sequel, we will be concerned with positively oriented, non-separating simple loops in \( \pi_1(\Sigma_g) \). We say that a simple loop is positively oriented if its orientation is clockwise, as indicated in Figure 5. Recall that a loop is non-separating if it does not cut the surface into two connected components and that it is simple if it does not contains self-crossings (up to homotopy). It is clear that these properties are preserved by Dehn twists, hence the set of such loops is stable under the action of \( \text{MCG}(\Sigma_g) \). Note that the loops \( a_i, b_i, d_i, e_i \) satisfy these properties.
5.2 Dehn twists as automorphisms of $L_{g,0}(H)$

In $\pi_1(\Sigma_g)$, we have the curves $a_i, b_i$ while in $L_{g,0}(H)$, we have the matrices $\hat{A}(i), \hat{B}(i)$. Using this, we can lift the action of the Humphries generators on $\pi_1(\Sigma_g)$ to $L_{g,0}(H)$ by replacing loops by matrices, up to some normalization, as we shall see now.

First, we lift some of the loops introduced above. We replace the generators of $\pi_1(\Sigma_g)$ by matrices of generators of $L_{g,0}(H)$ (see Figure 4), up to some normalization. More precisely, we define

$$\hat{D}_j = i^2 A(j-1) B(j) A(j)^{-1} B(j)^{-1}, \quad \hat{E}_2 = i^4 B(1) A(1)^{-1} B(1)^{-1} A(1)^{-1} B(2) A(2)^{-1} B(2)^{-1}$$

with $2 \leq j \leq g$. Then $\hat{D}_j$ and $\hat{E}_2$ satisfy the fusion relation of $L_{0,1}(H)$:

$$\hat{D}_j \otimes \hat{D}_j = \hat{D}_j (1) (R')_{12} (\hat{D}_j)_{2 (R')}_{12}, \quad \hat{E}_2 = \hat{E}_2 (1) (R')_{12} (\hat{E}_2)_{2 (R')}_{12}.$$

This is easy to check: we observe that $\hat{D}_j = \hat{A}(1) \hat{C}(2) \hat{A}(2)^{-1}, \hat{E}_2 = \hat{C}(1) \hat{C}(2) \hat{A}(2)^{-1}$ and we use Lemma 4.1 and relations (19) to write the fusion and reorder the matrices. Note that the normalizations by powers of $v$ are necessary to have the $L_{0,1}(H)$-fusion relation on these elements (see the proof of Proposition 5.1 below for an example of computation).

Now, we lift the action of the Humphries generators on the fundamental group (27). More precisely, let us define maps $\tilde{\tau}_{e_2}, \tilde{\tau}_{b_1}, \tilde{\tau}_{d_j} : L_{g,0}(H) \rightarrow L_{g,0}(H)$ by:

$$\tilde{\tau}_{e_2}(A(1)) = E_2^{-1} A(1) E_2, \quad \tilde{\tau}_{e_2}(B(1)) = E_2^{-1} B(1) E_2, \quad \tilde{\tau}_{e_2}(B(2)) = v E_2^{-1} B(2),$$

$$\tilde{\tau}_{b_1}(A(i)) = v^i B(i)^{-1} A(i),$$

$$\tilde{\tau}_{a_1}(B(1)) = v^{-1} B(1) A(1) \quad \text{ (recall that } a_1 = d_1),$$

$$\tilde{\tau}_{d_j}(A(j-1)) = D_j^{-1} A(j-1) D_j, \quad \tilde{\tau}_{d_j}(B(j-1)) = v^{-1} B(j-1) D_j, \quad \tilde{\tau}_{d_j}(B(j)) = v D_j^{-1} B(j),$$

for $j \geq 2$, and the other matrices are fixed.

**Proposition 5.1.** 1) The maps $\tilde{\tau}_{e_2}, \tilde{\tau}_{b_1}, \tilde{\tau}_{d_j}$ are automorphisms of $L_{g,0}(H)$.

2) The assignment \[
\tau_{e_2} \mapsto \tilde{\tau}_{e_2}, \quad \tau_{b_1} \mapsto \tilde{\tau}_{b_1}, \quad \tau_{d_j} \mapsto \tilde{\tau}_{d_j}
\]

extends to a morphism of groups $\text{MCG}(\Sigma_g) \rightarrow \text{Aut}(L_{g,0}(H))$.

**Proof:** 1) We have to check that these maps are compatible with the defining relations (19). This relies on straightforward but tedious computations. For instance, let us show that $\tilde{\tau}_{d_j}(B(j-1))$ satisfies the fusion relation. First, it is easy to establish the following exchange relation:

$$\hat{D}_j \otimes \hat{D}_j = \hat{D}_j (1)_2 \hat{B}_2 (\hat{D}_j)_{1 (R')}_{12} = (\hat{D}_j)_{1 (R')}_{12} \hat{B}_2 (j-1)_2.$$
Hence
\[ I \otimes J B(j - 1)_{12}(D_i)_{12} = I \otimes J B(j - 1)_{12}(D_i)_{12}I \otimes J I \otimes J = B(j - 1)_{12}(D_i)_{12}I \otimes J I \otimes J \]
\[ = I \otimes J B(j - 1)_{12}(D_i)_{12}(R')_{12}B(j - 1)_{12}(D_i)_{12}(R')_{12}^{-1}I \otimes J I \otimes J \]
\[ = I \otimes J B(j - 1)_{12}(D_i)_{12}(R')_{12}I \otimes J I \otimes J I \otimes J = B(j - 1)_{12}(D_i)_{12}(R')_{12}^{-1}I \otimes J I \otimes J \]
\[ = B(j - 1)_{12}(D_i)_{12}(R')_{12}^{-1}I \otimes J I \otimes J I \otimes J = B(j - 1)_{12}(D_i)_{12}(R')_{12}^{-1}I \otimes J I \otimes J \]
\[ = B(j - 1)_{12}(D_i)_{12}(R')_{12}^{-1}I \otimes J I \otimes J I \otimes J = B(j - 1)_{12}(D_i)_{12}(R')_{12}^{-1}I \otimes J I \otimes J \]

For the second equality we applied a trick based on (3). The aim is to replace \( R'_{-1} \) by \( R \) in order to apply the previously established exchange relation. It follows that \( v^{-1}B(j - 1)D_j \) satisfies the fusion relation, as desired. This computation shows how the normalizations by powers of \( v \) arises in order to satisfy the fusion relation. These normalizations have no importance when one checks the compatibility with the other defining relations of \( L_{g,0}(H) \).

2) Straightforward verification using Wajnryb’s relations.

**Definition 5.2.** The lift of an element \( f \in \text{MCG}(\Sigma^g) \), denoted by \( \widehat{f} \), is its image by the morphism of Proposition 5.7.

Let \( u_i \) be one of the generators of \( \pi_1(\Sigma^g) \), let \( f \in \text{MCG}(\Sigma^g) \) and let \( f(u_i) = a_{i_1}^{m_1}b_{j_1}^{n_1} \cdots a_{i_k}^{m_k}b_{j_k}^{n_k} \) with \( m_\ell, n_\ell \in \mathbb{Z} \). Then it follows from the definition of \( \tau_{e_2}, \tau_{b_i}, \tau_{d_i} \) that
\[ f(U(i)) = v^N A(i_1)^{m_1}fA(j_1)^{n_1} \cdots A(i_k)^{m_k}fA(j_k)^{n_k} \] (29)
where \( U(i) = A(i) \) (resp. \( U(i) = B(i) \)) if \( u_i = a_i \) (resp. \( u_i = b_i \)) and for some \( N \in \mathbb{Z} \). In other words, \( f \) and \( \widehat{f} \) are formally identical except for the normalizations by some power of \( v \).

Recall that \( L_{g,0}(H) \cong \text{End}_\mathbb{C}((H^*)^g) \) is a matrix algebra. By the Skolem-Noether theorem, every automorphism of \( L_{g,0}(H) \) is inner. Hence to each \( f \in \text{MCG}(\Sigma^g) \) is associated an element \( \widehat{f} \in L_{g,0}(H) \), unique up to scalar, such that
\[ \forall x \in L_{g,0}(H), \quad \widehat{f}(x) = \widehat{f}x\widehat{f}^{-1}. \] (30)

We now determine the elements \( \widehat{\tau}_\gamma \) associated to Dehn twists.

**Lemma 5.3.** We have \( \widehat{\tau}_{a_1} = v_{A(1)}^{-1} \). In other words:
\[ \forall x \in L_{g,0}(H), \quad \widehat{\tau}_{a_1}(x) = v_{A(1)}^{-1}xv_{A(1)}. \]

**Proof:** We have \( v_{A(1)}^{-1}A(1) = A(1)v_{A(1)}^{-1} = v_{A(1)}^{-1}A(1)v_{A(1)}^{-1} \). Indeed, since \( v^{-1} \) is central in \( H, \) \( v_{A(1)}^{-1} \) is central in the subalgebra generated by the coefficients of the matrices \( A(1) \). Next, let \( j_1 : \mathcal{H}(\mathcal{O}(H)) \rightarrow \mathcal{H}(\mathcal{O}(H))^g \) be the canonical embedding on the first copy. Observe that for all \( x \in H, \Psi_{g,0}(x_{A(1)}) = j_1(x) \). Then:
\[ \Psi_{g,0}(v_{A(1)}^{-1}B(1)) = j_1(v^{-1}L^{(+)}_T L^{(-)}_T)^{-1} = j_1(v^{-1}L^{(+)}_T L^{(-)}_T)^{-1} \]
\[ = j_1(v^{-1}L^{(+)}_T L^{(-)}_T)^{-1}b_{\delta_1}a_{j_1}b_{\nu_1}L^{(-)}_T = j_1(v^{-1}L^{(+)}_T L^{(-)}_T)^{-1}b_{\delta_1}a_{j_1}b_{\nu_1}L^{(-)}_T \]
\[ = \Psi_{g,0}(v_{A(1)}^{-1}B(1)) = \Psi_{g,0}(v_{A(1)}^{-1}B(1)) \]
We used the exchange relation (9) of $\mathcal{H}(\mathcal{O}(H))$ together with (3) and the definition of the matrices $\tilde{L}^{(\pm)}$. Finally, recall the matrices (20) which occur in the definition of the Alekseev isomorphism. The same argument as in the proof of Lemma 4.4 shows that

$$
\Psi_{I,0}^{g}(\Lambda_i) = S^{-1}(b_i) a^{(2i-1)}_\epsilon a^{(2i)}_\epsilon \otimes \ldots \otimes a^{(1)}_\epsilon b^{(2)}_\epsilon.
$$

From this we see that $j_1(v^{-1})$ commutes with $\Psi_{I,0}^{g}(\Lambda_i)$. Eventually it follows that $\Psi_{g,0}(v^{-1}_{A(1)})$ commutes with $\Psi_{g,0}(U(i)) = \Psi_{g,0}(\tau_{ai}(U(i)))$, where $U$ is $A$ or $B$.

If $\gamma_1, \gamma_2$ are non-separating circles on a surface, it is well known that there exists a homeomorphism $f$ such that $f(\gamma_1)$ is freely homotopic to $\gamma_2$ (see e.g. [FM12 Sect. 1.3.1]). Here we need to consider fixed-point homotopies.

**Lemma 5.4.** Let $\gamma_1, \gamma_2$ be positively oriented, non-separating simple loops in $\pi_1(\Sigma^\alpha_g)$, then there exists $f \in \text{MCG}(\Sigma^\alpha_g)$ such that $f(\gamma_1) = \gamma_2$ in $\pi_1(\Sigma^\alpha_g)$.

**Proof:** We know that there exists $\eta \in \text{MCG}(\Sigma^\alpha_g)$ such that $\eta(\gamma_1) = \gamma'_2 = \alpha^\epsilon \gamma^{+\epsilon}_2 \alpha^{-\epsilon}$ in $\pi_1(\Sigma^\alpha_g)$ for some loop $\alpha$ and some $\epsilon \in \{\pm 1\}$. $\gamma'_2$ is positively oriented, non-separating and simple since $\gamma_1$ is, and thus we can assume that $\alpha$ is simple and does not intersect $\gamma_2$ (except at the basepoint). There are six possible configurations for the loops $\alpha$ and $\gamma_2$ in a neighbourhood of the basepoint:

1. \[ \begin{array}{c}
\gamma_2 \\
\bullet \\
\end{array} \]
2. \[ \begin{array}{c}
\gamma_2 \\
\alpha \\
\end{array} \]
3. \[ \begin{array}{c}
\alpha \\
\gamma_2 \\
\end{array} \]
4. \[ \begin{array}{c}
\alpha \\
\gamma_2 \\
\bullet \\
\end{array} \]
5. \[ \begin{array}{c}
\gamma_2 \\
\alpha \\
\bullet \\
\end{array} \]
6. \[ \begin{array}{c}
\gamma_2 \\
\alpha \\
\bullet \\
\end{array} \]

In case 1, $\gamma'_2 = \alpha \gamma_2 \alpha^{-1}$, and then $\tau_\alpha(\gamma'_2) = \alpha^{-1} \gamma'_2 \alpha = \gamma_2$. Case 2 is impossible because none of the four possible loops $\alpha^\epsilon \gamma^{\pm\epsilon}_2 \alpha^{-\epsilon}$ is simple. In case 3, $\gamma_2 = \alpha \gamma_2 \alpha^{-1}$. For $\beta = \alpha \gamma_2$, we have $\tau_\beta(\alpha) = \beta^{-1} \alpha \beta$, $\tau_\beta(\gamma_2) = \beta^{-1} \gamma_2 \beta$, and thus $\tau_\beta(\gamma'_2) = \gamma_2$. In case 4, $\gamma'_2 = \alpha^{-1} \gamma_2 \alpha$. For $\delta = \gamma_2 \alpha$, we get similarly to case 3 that $\tau^{-1}_\delta(\gamma'_2) = \tau \gamma_2 \delta^{-1} = \gamma_2$. In case 5, $\gamma'_2 = \alpha^{-1} \gamma_2 \alpha$. Observe that $\tau_\alpha(\gamma'_2) = \gamma_2 \alpha$, $\tau_\alpha(\gamma_2) = \gamma_2^{-1} \alpha$, and then

$$
\tau^{-1}_\alpha \tau^{-2}_{\gamma_2} \tau^{-1}_{\alpha^{-1}}(\alpha^{-1} \gamma^{-1}_{2} \alpha) = \tau^{-1}_\alpha \tau^{-2}_{\gamma_2} (\gamma^{-1}_2 \alpha) = \tau^{-1}_\alpha (\gamma_2 \alpha) = \gamma_2.
$$

In case 6, $\gamma'_2 = \alpha^{-1} \gamma_2 \alpha^{-1}$, and we get similarly to case 5 that $\tau_\alpha \tau^{-2}_{\gamma_2} \tau^{-1}_\alpha(\gamma'_2) = \gamma_2$.

**Example.** We have

$$
\tau_{a_i} \tau_{a_i}(b_i) = a_i,
\tau^{-1}_{y_i} \tau^{-1}_{\beta_{a_i}}(d_i) = b_{i-1},
\tau^{-1}_{y_i} \tau^{-1}_{\alpha_{a_i}} \tau^{-1}_{y_{a_i}} (a_i) = b_{i-1},
\tau^{-1}_{\gamma_2} \tau^{-1}_{\epsilon_x} \tau_{\gamma_2} (e_2) = b_i,
$$

where $y_i = a_{i-1} b_i$. This allows to transform any of the loops $a_i, b_i, d_i, e_2$ into $a_1$.

**Lemma 5.5.** Let $f, g \in \text{MCG}(\Sigma^\alpha_g)$ such that $f(a_1) = g(a_1)$. Then $\tilde{f}(A(1)) = \tilde{g}(A(1))$.

**Proof:** Let $\eta \in \text{MCG}(\Sigma^\alpha_g)$ be such that $\eta(a_1) = a_1$. A priori, $\tilde{f}(A(1)) = \tilde{b}^{N \cdot f}(A(1))$ (see [29]) and we must show that $N = 0$. Let $\text{MCG}(\Sigma^\alpha_g)[a_1]$ be the stabilizer of the free homotopy class $[a_1]$. There is a surjection $p : \text{MCG}(\Sigma^\alpha_g) \to \text{MCG}(\Sigma^\alpha_g)[a_1]$. $\text{MCG}(\Sigma^\alpha_g)[a_1]$ is generated by $\tau_{a_i}, \tau_{b_i}, \tau_{e_2}, \tau_{\alpha_i}, \tau_{\epsilon_x}, \tau_\beta$ with $i \geq 2$, where $\tau_\beta$ is such that $p(\tau_\beta) = \tau_{\epsilon_x}$ (see [FM12 Figure 4.10]). It
follows that $MCG(\Sigma^o_1) = \langle \tau_d, \tau_b, \tau_{e_2}, \tau_{a_g}, \tau_{e_g} \rangle_{i \geq 2}$. For each of these generators $h$, it is possible to verify directly that

$$h(a_1) = \gamma_h a_1 \gamma_h^{-1} \text{ in } \pi_1(\Sigma^o_1), \text{ with } \gamma_h = a_i^{m_1} b_j^{n_1} \ldots a_k^{m_k} b_k^{n_k} (m_\ell, n_\ell \in \mathbb{Z}),$$

In other words, $h(a_1)$ and $\tilde{h}(A(1))$ are formally identical, without any power of $v$ in the expression of $\tilde{h}(A(1))$ (note that the normalization of $\Gamma_h$ by a power of $v$ vanishes in the conjugation). We deduce that this property is true for any $h \in MCG(\Sigma^o_g)[a_1]$. Since $\eta(a_1) = a_1$ in $\pi_1(\Sigma^o_1)$, then in particular $\eta \in MCG(\Sigma^o_g)[a_1]$, and thus $\tilde{\eta}(A(1)) = A(1)$, as desired.

This lemma justifies the following definition.

**Definition 5.6.** Let $\gamma \in \pi_1(\Sigma^o_g)$ be a positively-oriented, non-separating simple loop, and let $f \in MCG(\Sigma^o_g)$ be such that $f(a_1) = \gamma$. The lift of $\gamma$ is $\tilde{\gamma} = \tilde{f}(A(1)).$

Some comments are in order. First, if $\gamma = a_i^{m_1} b_j^{n_1} \ldots a_k^{m_k} b_k^{n_k}$ with $m_\ell, n_\ell \in \mathbb{Z}$ is a positively oriented, non-separating simple loop, then $\tilde{\gamma} = v^{-N} A(i_1)^{m_1} B(j_1)^{n_1} \ldots A(i_k)^{m_k} B(j_k)^{n_k}$ with $N \in \mathbb{Z}$. In other words, $\gamma$ and $\tilde{\gamma}$ are formally identical except for the normalization by a power of $v$. Note that by definition every lift satisfies the fusion relation of $L_{0,1}(H)$. We mention again that the normalization by a power of $v$ is required to satisfy the fusion relation (see e.g. the proof of Proposition 5.1).

We can now answer the question of what are the elements implementing lifting of Dehn twists by conjugation. We use the notation introduced at the end of subsection 3.2.

**Proposition 5.7.** For any non-separating circle $\gamma$ on $\Sigma^o_g$, we have $\tilde{\tau}_\gamma = v^{-1}_\gamma$. In other words:

$$\forall x \in L_{g,0}(H), \quad \tilde{\tau}_\gamma(x) = v^{-1}_\gamma x v_\gamma.$$

If $\gamma, \delta \in \pi_1(\Sigma^o_g)$ are positively oriented non-separating simple loops such that $[\gamma] = [\delta]$, then $v_\gamma$ is proportional to $v_\delta$.

**Proof:** We represent the circle $[\gamma]$ by a positively-oriented, non-separating simple loop $\gamma \in \pi_1(\Sigma^o_g)$. Let $f \in MCG(\Sigma^o_g)$ be such that $f(a_1) = \gamma$, then

$$\tilde{\tau}_\gamma = \tilde{f}_\gamma(a_1) = \tilde{f}(\tau_{a_1}) \tilde{f}^{-1} = \tilde{f}(\tau_{a_1}) \tilde{f}^{-1}.$$

Hence, by Lemma 5.3

$$\tilde{\tau}_\gamma \left( \tilde{f}(x) \right) = \tilde{f}(\tau_{a_1}(x)) = \tilde{f} \left( v^{-1}_{A(1)} x v_{A(1)} \right) = v^{-1}_\gamma \tilde{f}(x) v_\gamma.$$

Replacing $x$ by $\tilde{f}^{-1}(x)$, we get the result. The second claim follows from a similar reasoning together with the fact that $\tau_\gamma$ depends only of the free homotopy class of $\gamma$.

Note that an analogous result in the semi-simple setting has been stated without proof in [AS96a]. The notation $v^{-1}_\gamma$ does not appear in their work; instead, they express this element in a basis of characters, which is possible in the semi-simple case only.

**Corollary 5.8.** For all $f \in MCG(\Sigma^o_g)$, it holds $\tilde{f} \in L_{g,0}^{inv}(H)$.

**Proof:** Let $\gamma$ be a positively-oriented, non-separating simple loop. Then $\tilde{\gamma}$ satisfies the fusion relation of $L_{0,1}(H)$, and thus $j_\gamma$ is a morphism of $H$-module-algebras (Lemma 5.3). Hence, since $v^{-1} \in \mathcal{Z}(H) = L_{0,1}^{inv}(H)$, we have $v^{-1}_\gamma \in L_{g,0}^{inv}(H)$. In particular, the statement is true for the Humphries generators thanks to Proposition 5.1 and thus for any $f$. 

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5.3 Representation of the mapping class group

The only additional fact needed is the following lemma.

**Lemma 5.9.** It holds: \( v_{A(g)}^{-1} = v_{A(g)}^{-1} \).

**Proof:** Denote as usual \( X_i \otimes Y_i = RR', \ \overline{X}_i \otimes \overline{Y}_i = (RR')^{-1} \) and let \( \mu^l \) be the left integral on \( H \) (unique up to scalar). Using basic facts about integrals and (3), we have (see [Far18b Prop. 5.3]):

\[
\mu^l(vX_i)Y_i = \mu^l(v\overline{X}_i)\overline{Y}_i = \mu^l(v)^{-1}.
\]

Let us write \( \mu^l(v)^{-1}\mu^l(v) = \sum_{i,j,l} c^l_{i,j} f^l_j \) with \( c^l_{i,j} \in \mathbb{C} \). Then, using the identification \( \overline{M} = (X_i)Y_i \) between \( L_{0,1}(H) \) and \( H \), the fact that \( \overline{M}^{-1} = (X_i)Y_i \) and the equalities above, we get

\[
v_{A(g)}^{-1} = j_{A(g)} \left( \sum_{i,j,l} c^l_{i,j} f^l_j \right) = j_{A(g)} \left( \sum_{i,j,l} c^l_{i,j} (\overline{M})^{-1}_j \right) = j_{A(g)}^{-1} \left( \sum_{i,j,l} c^l_{i,j} f^l_j \right) = v_{A(g)}^{-1},
\]

where the morphisms \( j_\bullet \) are defined at the end of subsection 3.2. We used that \( j_{A(g)} \) is a morphism of algebras (see Lemma 3.1).

It is clear that the lemma holds for the lift of any positively oriented, non-separating simple loop, but we do not need this.

Recall that we have a representation of \( L_{g,0}(H) \) on \( (H^*)^g \), let us denote it \( \rho \). We also have the induced representation of \( L_{g,0}^{inv}(H) \) on \( \text{Inv}((H^*)^g) \), let us denote it \( \rho_{inv} \). Also recall that the elements \( \hat{f} \) are defined in (30). We can now state the representation of the mapping class groups \( \text{MCG}(\Sigma_g^0) \) and \( \text{MCG}(\Sigma_g) \). An analogous result in the semi-simple setting has been given without proof in [AS96a].

**Theorem 5.10.** 1) The map

\[
\text{MCG}(\Sigma_g^0) \rightarrow \text{GL}((H^*)^g)
\]

\[
f \mapsto \rho(\hat{f})
\]

is a projective representation.

2) The map

\[
\text{MCG}(\Sigma_g) \rightarrow \text{GL}(\text{Inv}((H^*)^g))
\]

\[
f \mapsto \rho_{inv}(\hat{f})
\]

is a projective representation.

**Proof:** 1) This is an immediate consequence of Proposition 5.1.

2) We must show that the hyperelliptic relation (20) is projectively satisfied. The word \( \omega \) can be constructed as follows: take \( f \in \text{MCG}(\Sigma_g^0) \) such that \( f(a_1) = a_g \) and express it as a word in the Humphries generators \( f = \tau_{\gamma_1} \cdots \tau_{\gamma_n} \). Then \( \tau_{a_g} = f \tau_{a_1} f^{-1} \). The automorphism \( \tau_{a_g} \) is implemented by conjugation by \( \hat{f} v_{A(1)}^{-1} \hat{f}^{-1} \) and also by conjugation by \( v_{A(g)}^{-1} \) (Proposition 5.7). Hence, \( \hat{f} v_{A(1)}^{-1} \hat{f}^{-1} \sim v_{A(g)}^{-1} \) where \( \sim \) means proportional. Now, let \( H = \tau_{a_g} \tau_{a_2} \cdots \tau_{a_1} \tau_{d_1} \tau_{b_1} \cdots \tau_{d_2} \tau_{b_2} \). A computation gives \( \hat{f}(A(g)) = A(g)^{-1} C_{g,0} \). Thus

\[
\hat{f} v_{A(1)}^{-1} \hat{f}^{-1} \sim \hat{f} v_{A(g)}^{-1} \hat{f}^{-1} = \hat{f}(v_{A(g)}^{-1}) = v_{A(g)}^{-1} C_{g,0}.
\]

By definition of \( \text{Inv}((H^*)^g) \) and Lemma 5.9, we have

\[
\rho_{inv}(v_{A(g)}^{-1} C_{g,0}) = \rho_{inv}(v_{A(g)}^{-1}) = \rho_{inv}(v_{A(g)}^{-1}).
\]

It follows that \( \rho_{inv}(\hat{f} v_{A(1)}^{-1} \hat{f}^{-1}) \sim \rho_{inv}(\hat{f} v_{A(g)}^{-1} \hat{f}^{-1}) \). This shows that the map is well-defined since \( \text{MCG}(\Sigma_g) \) is the quotient of \( \text{MCG}(\Sigma_g^0) \) by the hyperelliptic relation and that it is a projective representation.\( \square \)
5.4 Discussion for the case \( n > 0 \)

Let us consider the general case \( n > 0 \), see Figure 1. Denote \( \Sigma_{g,n}^o = \Sigma_{g,n} \setminus D \), where \( D \) is an embedded open disk. Recall that by definition the mapping class group fixes pointwise the boundary.

In general, \( L_{g,n}(H) \) is not a matrix algebra and we cannot claim directly the existence and unicity up to scalar of the elements \( \tilde{f} \). Nevertheless, we now describe an extension of the previous construction which should not be difficult to apply.

- Consider a generating set \( \tau_{c_1}, \ldots, \tau_{c_k} \) of \( \text{MCG}(\Sigma_{g,n}^o) \) (this consists only of Dehn twists when \( g > 1 \), see [PM12, Figure 4.10]) and compute the action on \( \pi_1(\Sigma_{g,n}^o) \). Here the \( c_i \) are loops in \( \pi_1(\Sigma_{g,n}^o) \) written in terms of the generators depicted in Figure 1.

- Determine the lifts \( \tilde{c}_i \) of the loops \( c_i \) (i.e. replace generators of \( \pi_1(\Sigma_{g,n}^o) \) by matrices of generators of \( L_{g,n}(H) \)) and then determine the normalisations by powers of \( v \) needed to satisfy the fusion relation.

- Determine the lifts \( \tilde{\tau}_{c_k} \) of the generators as automorphisms of \( L_{g,n}(H) \) (i.e. replace generators of \( \pi_1(\Sigma_{g,n}^o) \) by matrices of generators of \( L_{g,n}(H) \) in the action of \( \tau_{c_1}, \ldots, \tau_{c_k} \) on \( \pi_1(\Sigma_{g,n}^o) \), and then determine the normalisations by powers of \( v \) needed to satisfy the fusion relation and check that the other relations of (19) are satisfied.

- Show that the assignment \( \tau_{c_k} \mapsto \tilde{\tau}_{c_k} \) extends to a morphism of groups \( \text{MCG}(\Sigma_{g,n}^o) \to \text{Aut}(L_{g,n}(H)) \) (this is a just a tedious verification using a presentation of \( \text{MCG}(\Sigma_{g,n}^o) \)). Thus the lift \( \tilde{f} of f \in MCG(\Sigma_{g,n}^o) \) is still defined.

- It is clear that Lemma 5.4 still holds, so in particular for each \( i \) there exists \( f_i \in \text{MCG}(\Sigma_{g,n}^o) \) such that \( f_i(a_1) = c_i \).

- It is clear that Lemma 5.3 still holds, so that \( \tilde{\tau}_{c_i}(x) = v^{-1}_i x v_i \) (by reproducing the proof of Proposition 5.7 with the \( f_i \)). Since the \( \tau_{c_i} \) are a generating set, it follows that for each \( f \in \text{MCG}(\Sigma_{g,n}^o) \), there exists an element \( \tilde{f} \in L_{g,n}^\text{inv}(H) \), unique up to an invertible central element such that \( \tilde{f}(x) = \tilde{f} x \tilde{f}^{-1} \).

- Since \( Z(\mathcal{H}(O(H))^{\otimes g}) \cong \mathbb{C} \), we have for all \( c \in Z(L_{g,n}(H)) \):

\[
\Psi_{g,n}(c) = 1 \otimes \ldots \otimes 1 \otimes c_1 \otimes \ldots \otimes c_n
\]

with \( c_i \in Z(H) \). Let \( V = (H^*)^{\otimes g} \otimes S_1 \otimes \ldots \otimes S_n \), where \( S_1, \ldots, S_n \) are simple representations of \( H \). Then \( \Psi_{g,n}(c) \) acts by scalar on \( V \) thanks to Schur lemma. Let \( \rho \) (resp. \( \rho_{\text{inv}} \)) be the representation of \( L_{g,n}(H) \) (resp. \( L_{g,n}^\text{inv}(H) \)) on \( V \) (resp. \( \text{Inv}(V) \)). Then the elements \( \rho(\tilde{f}) \) and thus \( \rho_{\text{inv}}(\tilde{f}) \) are unique up to scalar. It should not be difficult to check that the corresponding generalisation of Theorem 5.10 is true.

5.5 Explicit formulas for the representation of some Dehn twists

We will compute explicitly the representation on \( (H^*)^{\otimes g} \) of the Dehn twists \( \tau_\gamma \), where the curves \( \gamma \) are represented in Figure 4. Thanks to Proposition 5.7, this amounts to compute the action of \( v^{-1}_\gamma \) on \( (H^*)^{\otimes g} \).

We recall that the action \( \triangleright \) of \( L_{g,0}(H) \) on \( (H^*)^{\otimes g} \) is defined using \( \Psi_{g,0} \) in (22) and that we denote the associated representation by \( \rho \). Also recall the definition of the elements \( \tilde{h} \) in (13) and the notation \( RR' = X_i \otimes Y_i \). Note that

\[
X_i \otimes Y_i' \otimes Y_i'' = a_i X_i b_k \otimes Y_i \otimes b_j a_k.
\]

Recall from [Fai18b] the elements \( v_A^{-1}, v_B^{-1} \in L_{1,0}(H) \) and their action on \( H^* \):

\[
\begin{align*}
v_A^{-1} \triangleright \varphi &= \varphi^{v^{-1}} = \varphi(v^{-1}), \\
v_B^{-1} \triangleright \varphi &= \mu^1(v)^{-1} (\mu^1(g^{-1}v?) \varphi)^{-1}
\end{align*}
\]
where \( \varphi^h = \varphi(h?) \) for \( h \in H \) and \( \mu^l \) is the left integral on \( H \).

We will need the following generalization of [Fai18b, Lemma 5.7] (in which we restricted to \( \varphi \in \text{SLF}(H) \)).

**Lemma 5.11.** For all \( \varphi \in H^* 
\)

\[
(v_A^{-1}v_B^{-1}v_A^{-1})^2 \triangleright \varphi = \frac{\mu^l(v^{-1})}{\mu^l(v)} \varphi(S^{-1}(a_i)g^{-1}v^{-1}S(?b_i) \\
(v_A^{-1}v_B^{-1}v_A^{-1})^{-2} \triangleright \varphi = \frac{\mu^l(v)}{\mu^l(v^{-1})} \varphi(b_jS^{-1}a_jg^{-1}v)
\]

**Proof:** Write \( \varphi = \sum_{I,i,j} \Phi_{I,j}^l \Phi_{I,j}^l = \sum_I \text{tr} \left( \Phi_I^l \right) \) with \( \Phi_{I,j}^l \in \mathbb{C} \) and let \( z(\varphi) = \sum_I \text{tr} \left( b_i \Phi_1 S^{-1}(a_i) M \right) \in \mathcal{L}_{0,1}(H) \). Then \( z(\varphi) \triangleright \varepsilon = \varphi \) (where \( z(\varphi)_B = j_B(z(\varphi)) \), see notation at the end of section 3.2 and \( \varepsilon \) is the counit of \( H \)). Indeed

\[
z(\varphi)_B \triangleright \varepsilon = \sum_I \text{tr} \left( \frac{b_i}{b_i} \Phi_1 S^{-1}(a_i) L^{(-1)} T L^{(-1)} \triangleright \varepsilon \right) = \sum_I \text{tr} \left( \frac{b_i}{b_i} \Phi_1 S^{-1}(a_i) b_j T b_j \right) = \sum_I \text{tr} \left( \Phi_1 T \right) = \varphi.
\]

We simply used (22), (12), the cyclicity of the trace and the equality \( S^{-1}(a_i)a_j \otimes b_j b_i = 1 \otimes 1 \). Observe that

\[
(\overline{t}_a \overline{t}_b \overline{t}_a)^2 \otimes (B) = t_a^2 A^{-1} B^{-1} A = B^{-1} C
\]

where \( C = C_{1,0} \) is defined in (23). Hence:

\[
\frac{\mu^l(v)}{\mu^l(v)} z(\varphi)_{B^{-1}} \triangleright \varepsilon = \frac{\mu^l(v^{-1})}{\mu^l(v)} z(\varphi)_{B^{-1}} \triangleright \varepsilon
\]

We used Proposition [5.7] the formula of [Fai18b, Lemma 5.7] applied to \( \varepsilon \), and the fact that \( t_B \triangleright \varepsilon = \mathbb{I}_{\dim(H)\varepsilon} \) (which follows from 25). Now we compute

\[
z(\varphi)_{B^{-1}} \triangleright \varepsilon = \sum_I \text{tr} \left( b_i \Phi_1 S^{-1}(a_i) L^{(-1)} T \triangleright \varepsilon \right) = \sum_I \text{tr} \left( b_i \Phi_1 S^{-1}(a_i) S^{-1}(b_j) S^{-1}(a_j) S(T) \right) = \sum_I \text{tr} \left( \Phi_1 S^{-1}(a_i) b_j S(T) \right)
\]

We used (11) and (2). The second formula is easily checked. \( \square \)

**Theorem 5.12.** The following formulas hold:

\[
v_{A(i)}^{-1} \triangleright (\varphi_1 \otimes \ldots \otimes \varphi_g) = \varphi_1 \otimes \ldots \otimes \varphi_i \otimes (v_A^{-1} \triangleright \varphi_i) \otimes \varphi_{i+1} \otimes \ldots \otimes \varphi_g,
\]

\[
v_{B(i)}^{-1} \triangleright (\varphi_1 \otimes \ldots \otimes \varphi_g) = \varphi_1 \otimes \ldots \otimes \varphi_i \otimes (v_B^{-1} \triangleright \varphi_i) \otimes \varphi_{i+1} \otimes \ldots \otimes \varphi_g,
\]

\[
v_{D_i}^{-1} \triangleright (\varphi_1 \otimes \ldots \otimes \varphi_g) = \varphi_1 \otimes \ldots \otimes \varphi_i \otimes \varphi_{i+1} \otimes (S^{-1}(a_j)a_k b_k v_n^{-1} b_j) \otimes \varphi_i (S^{-1}(a_i) S^{-1}(v^{-1})a_m b_m b_i)
\]

\[
\otimes \varphi_{i+1} \otimes \ldots \otimes \varphi_g,
\]

\[
v_{E_i}^{-1} \triangleright (\varphi_1 \otimes \ldots \otimes \varphi_g) = \varphi_1 (S^{-1}(v^{(2i-2)-1}) \otimes \ldots \otimes \varphi_i \otimes (S^{-1}(v^{(2i-1)-1}) \otimes \varphi_{i+1} \otimes \ldots \otimes \varphi_g)
\]

with \( i \geq 2 \) for the two last formulas.
The rest of the section is devoted to the proof of that theorem. First, it is useful to record that
\[
\Psi_{1,0}^g(I) A_i = \Psi_{1,0}^g \left( I^{(-)}(1) \cdots I^{(-)}(i-1) \right) = S^{-1}(b_j) a_j^{(2i-3)} a_j^{(2i-2)} \cdots a_j^{(2)} a_j^{(1)} 1^g i + 1
\]
\[
\Psi_{1,0}^g \left( I^{(+)}(1) \cdots I^{(+)}(i-1) \right) = I_j b_j^{(2i-3)} b_j^{(2i-2)} \cdots b_j^{(2)} 1^g i + 1
\]
where the matrix \( I_k \) is defined in (20). The proof is a simple computation analogous to that of Lemma 4.4. Second, recall from the proof of Lemma 5.9 that
\[
\mu^I(v) = \mu^I(vX_i) Y_i = v^{-1}
\]
We will write \( \mu^I(v) \cdots \mu^I(v) = \sum_I \text{tr} \left( c_I T \right) \). Then \( v^{-1} = \sum_I \text{tr} \left( c_I M \right) \) under the identification \( \mathcal{L}_{0,1}(H) = H \).

- Proof of the formula for the action of \( v_{A(i)}^{-1} \). By definition and by (33), we have
\[
\Psi_{g,0}(v_{A(i)}^{-1}) = \sum_I \text{tr} \left( c_I \Psi_{1,0}^g(I_A(i) \Lambda_i)^{-1} \right)
\]
\[
= \sum_I \text{tr} \left( c_I S^{-1}(b_j) X_k b_j \right) a_j^{(2i-3)} a_j^{(2i-2)} \cdots a_j^{(2)} a_j^{(1)} 1^g i Y_k \otimes 1^g i
\]
\[
= \mu^I(v) \mu^I(v S^{-1}(b_j) X_k b_j) a_j^{(2i-3)} a_j^{(2i-2)} \cdots a_j^{(2)} a_j^{(1)} 1^g i Y_k \otimes 1^g i
\]
\[
= \mu^I(v) \mu^I(v S^{-1}(b_j) X_k b_j) a_j^{(2i-3)} a_j^{(2i-2)} \cdots a_j^{(2)} a_j^{(1)} 1^g i Y_k \otimes 1^g i
\]
and the formula follows. We used (17), the formula \( R^{-1} = a_I \otimes S^{-1}(b_i) \) and (33).

- Proof of the formula for the action of \( v_{B(i)}^{-1} \). This the same proof as for \( v_{A(i)}^{-1} \) (the conjugation by \( \Lambda_i \) vanishes thanks to (17)).

- Proof of the formula for the action of \( v_{A(i)}^{-1} \). 2. We first compute the action of \( A(i-1) A(i) \). We have
\[
\Psi_{g,0}(I_A(i-1) A(i)) = \Psi_{1,0}^g \left( I_{A(i-1)} A(i) \Lambda_i^{-1} \right)
\]
\[
= \Psi_{1,0}^g \left( I_{A(i-1)} A(i) \left(C^{(-)}(i-1) A(i) C^{(-)}(i-1) -1 \Lambda_i^{-1} \right) \right)
\]
Hence:
\[
\Psi_{g,0}(v_{A(i-1) A(i)}) = \sum_I \text{tr} \left( c_I \Psi_{1,0}^g(I_{A(i-1)} A(i) C^{(-)}(i-1) A(i) C^{(-)}(i-1) -1 \Lambda_i^{-1} \right)
\]
\[
= \mu^I(v) \mu^I(v S^{-1}(b_j) X_k b_j) a_j^{(2i-3)} a_j^{(2i-2)} \cdots a_j^{(2)} a_j^{(1)} 1^g i Y_k \otimes 1^g i
\]
\[
= \mu^I(v) \mu^I(v X_k S^{-1}(b_j) X_m b_n) 1^{g-i-2} a_j^{(2)} a_j^{(1)} 1^g i Y_k \otimes 1^g i
\]
\[
= \mu^I(v) \mu^I(v X_k S^{-1}(b_j) X_m b_n) 1^{g-i-2} a_j^{(2)} a_j^{(1)} 1^g i Y_k \otimes 1^g i
\]

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We used the fact that \( X_k S^{-1}(b_l) \otimes a_l' \otimes Y_k a_p' = X_k S^{-1}(b_p) S^{-1}(b_l) \otimes a_l \otimes Y_k a_p = a_k S^{-1}(b_l) \otimes a_l \otimes b_k \). We see that we can assume without loss of generality that \( g = 2 \), \( i = 2 \) since the action is “local”. Moreover, this can be simplified. Let \( F : H^* \to H^* \) be the map defined by

\[
F(\varphi) = \varphi(a_j?b_j), \quad F^{-1}(\varphi) = \varphi(S^{-1}(a_j)?b_j).
\]

We compute:

\[
(F^{-1} \otimes \text{id}) \circ \rho \left( v_{A(1) A(2)}^{-1} \right) \circ (F \otimes \text{id})(\varphi \otimes \psi)
= \mu'(v)^{-1} \mu'(v b_k a_k X_m b_n) \varphi(a_j S^{-1}(a_n') a_l S^{-1}(a_o) b_o b_k a'_n b_j) \otimes \psi(?Y_m)
= \mu'(v)^{-1} \mu'(v a_k b_l X_m b_n) \varphi(S^{-1}(a_n') a_l S^{-1}(a_o) b_o b_k a'_n b_j) \otimes \psi(?Y_m) = (*).
\]

We used the formula \( R \Delta = \Delta^{op} R \). Now, we have a Yang-Baxter identity

\[
a_k b_l \otimes a_j a_l \otimes b_j b_k = R_{13} R_{23} R_{21} = R_{21} R_{23} R_{13} = b_l a_k \otimes a_l a_j \otimes b_j b_k
\]

which allows us to continue the computation:

\[
(*') = \mu'(v)^{-1} \mu'(v b_k a_k X_m b_n) \varphi(S^{-1}(a_n') a_l S^{-1}(a_o) b_o b_k a'_n b_j) \otimes \psi(?Y_m)
= \mu'(v)^{-1} \mu'(v a_k b_l X_m b_n) \varphi(S^{-1}(a_n') a_l b_k a'_n b_j) \otimes \psi(?Y_m)
= \mu'(v)^{-1} \mu'(v a_k X_m b_n) \varphi(?a_o b_k a'_n) \otimes \psi(?Y_m)
= \mu'(v)^{-1} \mu'(v X_m) \varphi(?Y'_m) \otimes \psi(?Y'_m) = \varphi(?v''^{-1}) \otimes \psi(?v''^{-1}).
\]

We used basic properties of the \( R \)-matrix and relations \([31], [34]\). We have thus shown that

\[
v_{A(1)A(2)}^{-1} \varphi \otimes \psi = \varphi(S^{-1}(a_j) a_k a'_n b_k v''^{-1} b_j) \otimes \psi(?v''^{-1}).
\]

Recall that \( \tilde{D}_2 = \tilde{I}^2 \tilde{A}(1) \tilde{B}(2) \tilde{A}(2)^{-1} \tilde{B}(2)^{-1} \). Hence \( (\tilde{a}_2 \tilde{a}_2 \tilde{a}_2 \tilde{a}_2)^{-2} (v_{A(1)A(2)}^{-1}) = \tilde{v}_{D_2}^{-1} \). It follows that \( \tilde{D}_2 \) and by Proposition 5.7 and Lemma 5.11,

\[
v_{D_2}^{-1} \varphi \otimes \psi = \left( v_{A(2)A(2)}^{-1} v_{B(2)A(2)}^{-1} \right)^{-2} v_{A(1)A(2)}^{-1} \left( v_{A(2)B(2)}^{-1} v_{A(2)}^{-1} \right)^{-2} \varphi \otimes \psi
= \varphi(S^{-1}(a_j) a_k a'_n b_k v''^{-1} b_j) \otimes \psi(S^{-1}(a_l) g^{-1} v^{-1} S(v''^{-1}) S(?b_l)
= \varphi(S^{-1}(a_j) a_k a'_n b_k v''^{-1} b_j) \otimes \psi(S^{-1}(a_l) S(v''^{-1}) S(v''^{-1}) S(?b_l))
= \varphi(S^{-1}(a_j) a_k a'_n b_k v''^{-1} b_j) \otimes \psi(S^{-1}(a_l) S(v''^{-1}) S(?b_l))
\]

which is the announced formula.

- **Proof of the formula for the action of \( v_{E_i}^{-1} \), \( i \geq 2 \).** We first compute the action of \( \tilde{C}(1) \ldots \tilde{C}(i - 1) \tilde{A}(i) \). We have

\[
\Psi_{g,0} \left( \tilde{C}(1) \ldots \tilde{C}(i - 1) \tilde{A}(i) \right) = \Psi_{g,0} \left( \tilde{C}(1)^{(+)}(1) \ldots \tilde{C}(i - 1)^{(+)}(i - 1) \left( \tilde{C}(1)^{(-)}(1) \ldots \tilde{C}(i - 1)^{(-)}(i - 1) \right)^{-1} L_i A(i) A_i^{-1} \right)
= \Psi_{g,0} \left( \tilde{C}(1)^{(+)}(1) \ldots \tilde{C}(i - 1)^{(+)}(i - 1) A(i) \right) \left( \tilde{C}(1)^{(-)}(1) \ldots \tilde{C}(i - 1)^{(-)}(i - 1) \right)^{-1}
= \tilde{I}_{a_j a_k b_l b_j (i - 2)} A_i^{-1} a_i (i - 3) a_i (i - 2) \ldots \otimes b_j (i - 2) a_i (i - 2) \otimes Y_k \otimes 1 \otimes g^{-i}
= \tilde{I}_{X_k Y_k (2i - 2)} Y_k (2i - 1) \otimes \ldots \otimes Y_k (2) Y_k (1) \otimes Y_k (1) \otimes 1 \otimes g^{-i}
\]
Let us first quickly recall the Lyubashenko representation in the general framework of a ribbon obtained thanks to morphisms which we recall now. The first is an algebra structure $H_{g,n}$ allowed to apply his construction to modules. Here we will show that these two representations are equivalent. For the case of the torus $\varphi_1 \otimes \ldots \otimes \varphi_g = \varphi_1 (S^{-1} (v^{(2i-2)}-1)?v^{(2i-1)}-1) \otimes \ldots \otimes \varphi_i-1 (S^{-1} (v^{(2i-1)}-1)?v^{(3i-1)}) \otimes \varphi_i(?v^{(1i-1)}) \otimes \varphi_{i+1} \ldots \otimes \varphi_g.$

Recall that $\tilde{E}_i = \frac{I}{2} \tilde{C}(i) \ldots \tilde{C}(i-1) \tilde{A}(i) \frac{I}{2} \tilde{B}(i) \frac{1}{2} \tilde{I}$. Hence $(\tilde{a}_i \tilde{b}_i \tilde{a}_i)^{-1} (\tilde{C}(i) \ldots \tilde{C}(i-1) \tilde{A}(i)) = \tilde{E}_i$. As previously, it follows from Proposition 5.7 and Lemma 5.11 that

$$v_{E_i}^{-1} \triangleright (\varphi_1 \otimes \ldots \otimes \varphi_g) = \left( v_{A(i)}^{-1} v_{B(i)}^{-1} v_{A(i)}^{-1} \right)^{-2} v_{C(1) \ldots C(i-1) A(i)} v_{A(i)}^{-1} v_{B(i)}^{-1} v_{A(i)}^{-1} \triangleright (\varphi_1 \otimes \ldots \otimes \varphi_g)$$

$$= \varphi_1 (S^{-1} (v^{(2i-2)}-1)?v^{(2i-1)}-1) \otimes \ldots \otimes \varphi_i-1 (S^{-1} (v^{(2i-1)}-1)?v^{(3i-1)}) \otimes \varphi_i (S^{-1} (a_i) S^{-1} (v^{(1i-1)}) a_k b_k b_j) \otimes \varphi_{i+1} \ldots \otimes \varphi_g,$$

which is the announced formula. \qed

6 Equivalence with the Lyubashenko representation

In a series of papers [Lyu95a, Lyu95b, Lyu96], V. Lyubashenko has constructed projective representations of $\text{MCG}(\Sigma_{g,n})$ by categorical techniques based on the coend of a ribbon category. Our assumptions on $H$ allow to apply his construction to $\text{mod}_l(H)$, the ribbon category of finite-dimensional left $H$-modules. Here we will show that these two representations are equivalent. For the case of the torus $\text{MCG}(\Sigma_{1,0})$ and $\text{MCG}(\Sigma_{1,0})$, we have already shown in [Fai18b] that the projective representation obtained thanks to $L_{1,0}(H)$ is equivalent to the Lyubashenko-Majid representation [LM94].

6.1 The Lyubashenko representation for $\text{mod}_l(H)$

Let us first quickly recall the Lyubashenko representation in the general framework of a ribbon category $C$ satisfying some assumptions (see [Lyu95b]).

Let $K = \int X^* \otimes X$ be the coend of the functor $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$, $F(X,Y) = X^* \otimes Y$ and let $i_X : X^* \otimes X \to K$ be the associated dinatural transformation (see [ML98, IX.6]). Thanks to the universal property of the coend $K$, Lyubashenko defined several morphisms; we will need some of them which we recall now. The first is an algebra structure $K \otimes K \to K$. Consider the following family of morphisms (for each $X, Y \in C$)

$$d_{X,Y} : X^* \otimes X \otimes Y^* \otimes Y \xrightarrow{id_{X^*} \otimes e_X \otimes id_Y} X^* \otimes Y^* \otimes X \otimes Y \xrightarrow{id_{X^*} \otimes id_Y \otimes \otimes e_X} X^* \otimes Y^* \otimes Y \otimes X \xrightarrow{\sim} (Y \otimes X)^* \otimes Y \otimes X \xrightarrow{\text{inv} \otimes X} K.$$

Since the family $d_{X,Y}$ is dinatural in $X$ and $Y$, it exists a unique $m_K : K \otimes K \to K$ such that $d_{X,Y} = m_K \circ (i_X \otimes i_Y)$, which is in fact an associative product on $K$. Actually, $K$ is endowed with a Hopf algebra structure whose structure morphisms are similarly defined using the universal property, but we do not need this here.
Next, consider the following families of morphisms

\[
\begin{align*}
\alpha_X : X^* \otimes X & \xrightarrow{\theta_X \otimes \text{id}_X} X^* \otimes X \xrightarrow{i_X} K, \\
\beta_{X,Y} : X^* \otimes X \otimes Y^* \otimes Y & \xrightarrow{\text{id}_{X^*} \otimes (\text{cyc}, \text{oc}_{X,Y}, \text{cyc}) \otimes \text{id}_Y} X^* \otimes X \otimes Y^* \otimes Y \xrightarrow{i_X \otimes i_Y} K, \\
\gamma_X^Y : X^* \otimes X \otimes Y & \xrightarrow{\text{id}_{X^*} \otimes (\text{cyc}, \text{oc}_{X,Y})} X^* \otimes X \otimes Y \xrightarrow{i_X \otimes \text{id}_Y} K \otimes Y.
\end{align*}
\]

(36)

The families \(\alpha_X\) and \(\gamma_X^Y\) (with \(Y\) fixed) are dinatural in \(X, Y\). Hence by the universal property of \(K\), there exists unique morphisms \(T : K \to K, O : K \otimes K \to K \otimes K, Q_Y : K \otimes Y \to K \otimes Y\) such that

\[
\alpha_X = T \circ i_X, \quad \beta_{X,Y} = O \circ (i_X \otimes i_Y), \quad \gamma_X^Y = Q_Y \circ (i_X \otimes \text{id}_Y).
\]

(37)

Finally, the morphism \(S : K \to K\) is defined by \(S = (\varepsilon_K \otimes \text{id}_K) \circ O \circ (\text{id}_K \otimes \Lambda_K)\), where \(\Lambda_K\) is the two-sided cointegral on \(K\).

Let \(X\) be any object of \(\mathcal{C}\) and \(V_X = \text{Hom}_\mathcal{C}(X, K^{\otimes g})\). The Lyubashenko representation \(Z_X : \text{MCG}(\Sigma^g) \to \text{PGL}(V_X)\) \([\text{Lyu95b}]\) Section 4.4\] takes the following values:

\[
\begin{align*}
Z_X(\tau_{a_i}) &= \text{Hom}_\mathcal{C}(X, \text{id}_K^{\otimes g-i} \otimes T \otimes \text{id}_K^{i-1}), \\
Z_X(\tau_{b_i}) &= \text{Hom}_\mathcal{C}(X, \text{id}_K^{\otimes g-i} \otimes (\varepsilon_K \circ O \circ (\varepsilon_K \otimes \varepsilon_K) \otimes \text{id}_K^{i-1}), \\
Z_X(\tau_{d_i}) &= \text{Hom}_\mathcal{C}(X, \text{id}_K^{\otimes g-i} \otimes (\varepsilon_K \circ (T \otimes T) \otimes \text{id}_K^{i-2}) \text{ for } i \geq 2, \\
Z_X(\tau_{e_i}) &= \text{Hom}_\mathcal{C}(X, \text{id}_K^{\otimes g-i} \otimes ((T \otimes \theta_{K^\otimes i-1}) \otimes \text{id}_K^{i-2}) \text{ for } i \geq 2.
\end{align*}
\]

(38)

Recall that the curves \(a_i, b_i, d_i, e_i\) are represented in Figure 4. Since these Dehn twists are a generating set, we have an operator \(Z_X(f)\) for all \(f \in \text{MCG}(\Sigma^g)\). If moreover we take \(X = 1\), the unit object of \(\mathcal{C}\), then this defines a representation \(Z_1 : \text{MCG}(\Sigma^g) \to \text{PGL}(V_1)\) of the mapping class group of \(\Sigma_g\).

Now, let us explicit the above formulas to the case of \(\mathcal{C} = \text{mod}_1(H)\). Recall that the category \(\text{mod}_1(H)\) has braiding \(c_{i,j} : X \otimes Y \to Y \otimes X\) and twist \(\theta_X : X \to X\) given by

\[
c_{X,Y}(x \otimes y) = b_i \cdot y \otimes a_i \cdot x, \quad \theta_X(x) = v^{-1} \cdot x
\]

and that the action on the dual module \(V^*\) is \(h \cdot \varphi = \varphi(S(h) \cdot ?)\) for all \(\varphi \in V^*, h \in H\), see [Kas95] Chap. XIII–XIV] for more details.

It is well-known (and not difficult to see) that \(K\) is \(H^*\) endowed with the coadjoint action:

\[
\forall h \in H, \forall \varphi \in K, \quad h \varphi = \varphi(S(h') \cdot h'')
\]

and that the dinatural transformation of \(K\) is

\[
i_X(\psi \otimes x) = \psi(\varphi \cdot x) \in K.
\]

Note that \(\psi(\varphi \cdot x)\) is just a matrix coefficient of the module \(X\). The dinatural family \(d_{X,Y}\) of (35) is

\[
d_{X,Y}(\varphi \otimes x \otimes \psi \otimes y) = \psi(S(b_i) \cdot b_j \cdot y) \varphi(a_j a_i \cdot x)
\]

where in the right of the equality it is the usual product in \(H^*: \langle fg, h \rangle = f(h') g(h'')\). To compute the product \(m_K\) in \(K\) explicitly, observe that \(i_{H_{\text{reg}}} \varphi \otimes 1 = \varphi\), where \(H_{\text{reg}}\) is the regular representation of \(H\). Thus

\[
m_K(\varphi \otimes \psi) = m_K \circ (i_{H_{\text{reg}}} \otimes i_{H_{\text{reg}}}) (\varphi \otimes 1 \otimes \psi \otimes 1) = d_{H_{\text{reg}}, H_{\text{reg}}} (\varphi \otimes 1 \otimes \psi \otimes 1) = \psi(S(b_i) \cdot b_j) \varphi(a_j a_i) = \varphi(a_j a_i) \psi(S(b_i) b_j)
\]

where we used \(R \Delta = \Delta^{op} R\) for the last equality. Moreover, the unit element of \(K\) is \(1_K = \varepsilon\), the counit of \(H\). We record the following lemma, already given in [Lyu95b].
Lemma 6.1. Assume $C = \text{mod}_1(H)$, and let $\mu^r \in H^*$ be the right integral on $H$ (unique up to scalar). Then $\mu^r$ is the two-sided cointegral in $K$ (unique up to scalar):

$$\forall \varphi \in K, \ m_K(\mu^r \otimes \varphi) = m_K(\varphi \otimes \mu^r) = \varepsilon_K(\varphi)\mu^r$$

where $\varepsilon_K(\varphi) = \varphi(1)$.

Proof: Using (1), we get

$$m_K(\mu^r \otimes \varphi) = \mu^r(a_j^? a_i^?) \varphi(S(b_i^?) b_j^?) = \mu^r(S^2(a_j^?) a_i^?) \varphi(S(b_i^?) b_j^?) = \mu^r \varphi = \varphi(1)\mu^r.$$  

We used (7) and the basic properties of $R$ [Kas05 VIII.2]. Similarly:

$$m_K(\varphi \otimes \mu^r) = \mu^r(S(b_i^?) b_j^?) \varphi(\mu^r \varphi(1)\mu^r).$$

The dinatural families of (30) are

$$\alpha_X(\varphi \otimes x) = \varphi(v^{-1} \cdot x), \quad \beta_{X,Y}(\varphi \otimes x \otimes \psi \otimes y) = \varphi(b_j a_i \cdot x) \otimes \psi(S(a_j b_i)^? \cdot y) = \varphi(\mu^r v^{-1} \cdot x) \otimes \psi(S(v''^n) v \cdot y),$$

$$\gamma_X(\varphi \otimes x \otimes y) = \varphi(b_j a_i \cdot x) \otimes a_j b_i \cdot y = \varphi(\mu^r v^{-1} \cdot x) \otimes v''^n v \cdot y.$$  

where we used (3). It follows that the morphisms defined in (37) are

$$T(\varphi) = T \circ i_{H_{\text{reg}}}(\varphi \otimes 1) = \alpha_{H_{\text{reg}}}(\varphi \otimes 1) = \varphi(v^{-1}),$$

$$O(\varphi \otimes \psi) = O \circ (i_{H_{\text{reg}}} \otimes i_{H_{\text{reg}}})(\varphi \otimes 1 \otimes \psi \otimes 1) = \beta_{H_{\text{reg}}, H_{\text{reg}}}(\varphi \otimes 1 \otimes \psi \otimes 1) = \varphi(\mu^r v^{-1}) \otimes \psi(S(v''^n) v),$$

$$Q_Y(\varphi \otimes y) = Q_Y \circ (i_{H_{\text{reg}}} \otimes id_Y)(\varphi \otimes 1 \otimes y) = \gamma_{H_{\text{reg}}, Y}(\varphi \otimes 1 \otimes y) = \varphi(\mu^r v^{-1}) \otimes v''^n v \cdot y.$$  

Note that $(T \otimes \theta_Y) \circ Q_Y(\varphi \otimes y) = \varphi(\mu^r v^{-1}) \otimes v''^n \cdot y$ (see (38)). Finally, thanks to Lemma 6.1, the morphism $S$ is

$$S(\varphi) = \varphi(\mu^r v^{-1}) \mu^r(S(v''^n) v) = \varphi(S^{-1}(v''^n) v) \mu^r(v^{-1})$$

where the second equality is due to the equality $v^{-1} \otimes S(v''^n) = S^{-1}(v''^n) \otimes v^{-1}$ (which follows from $S(v^{-1}) = v^{-1}$). Moreover, we will need the following lemma to prove the equivalence of the representations.

Lemma 6.2. Let $\rho$ be the representation of $L_{1,0}(H)$ on $H^*$, then the following formulas hold:

$$T = \rho(v_A^{-1}) = (v^{-1})_*,$$  
$$S = \mu^l(v^{-1}) g^{-1}_s \circ \rho(v_A^2 v_B) \circ g_s,$$  
$$S^{-1} \circ T \circ S = (g^{-1} v)_s \circ \rho(v_B^{-1}) \circ (g v^{-1})_*,$$

where $h_*(\varphi) = \varphi(\mu^r)$ for all $h \in H$ and $\varphi \in H^*$.

Proof: The formula for $T$ is obvious. Propositions 4.10 and 5.3 of [Fai18b] give $\rho(v_B)$ and then we compute using (3) and (5):

$$\rho(v_B)(\varphi) = v_B \triangleright \varphi = \mu^l(v^{-1})^{-1} \left(\mu^l(g^{-1} v^{-1} \varphi) v^{-1}ight) = \mu^l(v^{-1})^{-1} \left(\mu^r(g v^{-1} \varphi) v^{-1}ight)$$

$$= \mu^l(v^{-1})^{-1} \mu^r(v^{-1} g v^{-1}) \varphi(S^{-1}(v''^n) g v^{-1})$$

$$= \mu^l(v^{-1})^{-1} (g v^{-1}) \left(\mu^r(v v^{-1} \varphi) \right) \left(\mu^l(g^{-1}) \varphi, S^{-1}(v''^n) \right)$$

$$= \mu^l(v^{-1})^{-1} (g v^{-1}) \circ S \circ g^{-1}_s(\varphi) = \mu^l(v^{-1})^{-1} \rho(v_A^2) \circ g_s \circ S \circ g^{-1}_s(\varphi)$$

where $\varphi^h = \varphi(h)$ for $h \in H$. The last claimed formula follows from $S = \mu^l(v^{-1})(g^{-1} v)_s \circ \rho(v_A v_B v_A) \circ (g v^{-1})_s$ and the fact that $v_A, v_B \in L_{1,0}(H)$ satisfy the braid relation $v_A v_B v_A = v_B v_A v_B$ (see [Fai18b, Prop. 5.5]).  

\[\square\]
For the representation space, we take \( X = H_{\text{reg}} \), so that \( V_X = \text{Hom}_H(H_{\text{reg}}, K^\otimes g) \cong K^\otimes g \). Then by the previous formulas, we get the Lyubashenko projective representation of \( \text{MCG}(\Sigma) \) \( \text{applied to } \mod_2(H) \):

\[
\begin{align*}
Z_{H_{\text{reg}}} (\tau_a)(\varphi_1 \otimes \ldots \otimes \varphi_g) &= \varphi_1 \otimes \ldots \otimes \varphi_{g-i+1}(v^{-1}) \otimes \ldots \otimes \varphi_g, \\
Z_{H_{\text{reg}}} (\tau_b)(\varphi_1 \otimes \ldots \otimes \varphi_g) &= \varphi_1 \otimes \ldots \otimes (g^1v)_0 \circ \rho(v^{-1})_0 \circ (g^1v')_1 \otimes \ldots \otimes \varphi_g, \\
Z_{H_{\text{reg}}} (\tau_d)(\varphi_1 \otimes \ldots \otimes \varphi_g) &= \varphi_1 \otimes \ldots \otimes \varphi_{g-i+1}(?v^{-1}) \otimes \varphi_{g-i+2}(v^{-1}) \otimes \ldots \otimes \varphi_g, \\
Z_{H_{\text{reg}}} (\tau_e)(\varphi_1 \otimes \ldots \otimes \varphi_g) &= \varphi_1 \otimes \ldots \otimes \varphi_{g-i} \otimes \varphi_{g-i+1}(v^{-1}) \otimes \varphi_{g-i+2}(v^{-1}) \otimes \ldots \otimes \varphi_g(S(v^{(2i-2)}-1)v^{(2i-1)}-1),
\end{align*}
\]

with \( i \geq 2 \) for the two last formulas. If we take \( X = C \), we get

\[
V_C = \text{Hom}_H(C, K^\otimes g) = (K^\otimes g)^{\text{inv}} = \{ f \in K^\otimes g | \forall h \in H, \ h \cdot f = \varepsilon(h)f \}
\]

where by definition of the action of \( H \) on \( K \), the action of \( H \) on \( K^\otimes g \) is

\[
h \cdot \varphi_1 \otimes \ldots \otimes \varphi_g = \varphi_1(S(h^{(1)})h^{(2)}) \otimes \ldots \otimes \varphi_g(S(h^{(2g-1)})h^{(2g)}).
\]

Then \( Z_C \) is a projective representation of \( \text{MCG}(\Sigma) \) (note that \( Z_C \) is just \( Z_{H_{\text{reg}}} \) restricted to \( (K^\otimes g)^{\text{inv}} \)).

To conclude this section, we explain how to see \( \mathcal{L}_{0,1}(H) \) as a coend. Interpreting slightly differently the fusion relation of Definition 3.1, we can view \( \mathcal{L}_{0,1}(H) \) as \( H^* \) endowed with a new product. Indeed, we know that \( \mathcal{L}_{0,1}(H) \) is generated by matrix coefficients \( \tilde{M}^I_J \) and due to (17), \( \dim(\mathcal{L}_{0,1}(H)) = \dim(H^*) \); hence \( \mathcal{L}_{0,1}(H) \cong H^* \) as vector spaces and we identify them: \( \tilde{M}^I_J = \mathcal{L}_{0,1}(H) \). To avoid confusion, we exceptionally denote by \( \star \) (resp. \( \ast \)) the product of \( O(H) \) (resp. \( \mathcal{L}_{0,1}(H) \)); both are products on \( H^* \) thanks to the identification. Due to (18) and to obvious commutation relations, the \( \mathcal{L}_{0,1} \)-fusion relation on \( H^* \) is given by

\[
I_{T_1 \ast T_2} = S^{-1}(a_i)_{a_j} \otimes b_j b_i = 1 \otimes T_1 \ast T_2 \text{ together with obvious commutation relations, we get}
\]

\[
I_{T_1 \ast T_2} = S^{-1}(a_i)_{a_j} I_{T_1 \ast T_2(b_j)_{a_j}} (R')^{-1}_{12} = I_{T_1 \ast T_2(b_j)_{a_j}} (S^{-1}(a_i)_{a_j})_2.
\]

Since every element of \( H^* \) is a linear combination of matrix elements \( T^I_J \) for certain \( I, i, j \), the product in \( \mathcal{L}_{0,1}(H) \) is

\[
\varphi \ast \psi = \varphi(b_i b_j) \ast \psi(S^{-1}(a_i)_{a_j}).
\]

Moreover, we define a left \( H \)-module structure on \( \mathcal{L}_{0,1}(H) \) by \( h \cdot \varphi = \varphi \cdot S^{-1}(h) = \varphi(S^{-1}(h^{(1)})h^{(2)}) \) (see (19)). Since \( h \cdot (\varphi \ast \psi) = (h^{(1)} \cdot \varphi) \ast (h^{(2)} \cdot \psi), \mathcal{L}_{0,1}(H) \) is an algebra in \( \text{mod}_I(H^{\text{cop}}) \), where \( H^{\text{cop}} \) is \( H \) with opposite coproduct. Moreover, in \( H^{\text{cop}} \), we replace \( \Delta \) by \( \Delta^{\text{op}} \), \( R \) by \( R' \) and \( S \) by \( S^{-1} \) so that the formulas for the product and the \( H \)-action in the coend of \( \text{mod}_I(H^{\text{cop}}) \) are exactly those of \( \mathcal{L}_{0,1}(H) \). We state this as a proposition.

**Proposition 6.3.** It holds:

\[
\mathcal{L}_{0,1}(H) = \int_{X \in \text{mod}_I(H^{\text{cop}})} X^* \otimes X.
\]

### 6.2 Equivalence of the representations

Recall that we denote by \( \rho \) (resp. \( \rho_{\text{inv}} \)) the representation of \( \mathcal{L}_{g,0}(H) \) on \( (H^*)^\otimes g \) (resp. \( \text{Inv}((H^*)^\otimes g) \)). Also recall the map \( F : H^* \rightarrow H^* \)

\[
F(\varphi) = \varphi(a_i b_i), \quad F^{-1}(\varphi) = \varphi(S^{-1}(a_i) b_i)
\]
(already used in the proof of Theorem 5.12) and let \( \sigma : (H^*)^{\otimes g} \rightarrow (H^*)^{\otimes g} \) be the permutation

\[
\sigma(\varphi_1 \otimes \varphi_2 \otimes \ldots \otimes \varphi_{g-1} \otimes \varphi_g) = \varphi_g \otimes \varphi_{g-1} \otimes \ldots \otimes \varphi_2 \otimes \varphi_1.
\]

It satisfies \( \sigma^{-1} = \sigma \).

**Theorem 6.4.** The representation of Theorem 5.10 and the Lyubashenko representation of MCG(\( \Sigma_g \)) and MCG(\( \Sigma_g \)) are equivalent. More precisely:

1) The isomorphism of vector spaces

\[
(F \circ S)^{\otimes g} \circ \sigma : K^{\otimes g} \rightarrow (H^*)^{\otimes g}
\]

\[
\varphi_1 \otimes \ldots \otimes \varphi_g \mapsto \varphi_g(b_iS(?)a_i) \otimes \ldots \otimes \varphi_1(b_iS(?)a_i)
\]

is an intertwiner between the two representations:

\[
[(F \circ S)^{\otimes g} \circ \sigma] \circ Z_{H_{\text{reg}}}(f) = \rho(\hat{f}) \circ [(F \circ S)^{\otimes g} \circ \sigma].
\]

2) The isomorphism of vector spaces

\[
(F \circ S)^{\otimes g} \circ \sigma : (K^{\otimes g})^{\text{inv}} \rightarrow \text{Inv}( (H^*)^{\otimes g})
\]

is an intertwiner between the two representations:

\[
[(F \circ S)^{\otimes g} \circ \sigma] \circ Z_{C}(f) = \rho_{\text{inv}}(\hat{f}) \circ [(F \circ S)^{\otimes g} \circ \sigma].
\]

**Proof:** 1) We show that this isomorphism intertwines the formulas of Theorem 5.12 and of (39). Thanks to the properties of \( v \) (3), it is clear that \( (F \circ S)^{\otimes g} \circ \sigma \circ Z(\tau_a) = \rho(v_{A(\lambda)}^{-1}) \circ (F \circ S)^{\otimes g} \circ \sigma \).

Next, thanks to (32), (3) and (5), we have

\[
v_B^{-1} \triangleright \varphi = \mu^l(v^{-1})^* \mu^r(gv^{-1}v') \varphi(vS^{-1}(gv')).
\]

Hence, for \( \varphi \in H^* \),

\[
\rho(v_B^{-1}) \circ (F \circ S)(\varphi) = \mu^l(v^{-1})^* \mu^r(gv^{-1}v') \varphi(vb_igv'n'a_i) = \mu^l(v^{-1})^* \mu^r(gY_j) \varphi(v^2b_igX_ja_i)
\]

\[
= \mu^l(v^{-1})^* \mu^r(gS (a_j)S^{-1}(b_k)) \varphi(v^2gS^{-2}(b_j)b_aj_a_i) = (\ast)
\]

with \( X_i \otimes Y_i = (RR')^{-1} \). We have a Yang-Baxter relation

\[
S(a_j)S^{-1}(b_k) \otimes S^{-2}(b_i)b_j \otimes a_k a_i = a_jS^{-1}(b_k) \otimes S^{-1}(b_jS^{-1}(b_i)) \otimes a_k a_i = (\text{id} \otimes S^{-1} \otimes \text{id})(R_{12}R_{31}^{-1}R_{32}^{-1})
\]

\[
= (\text{id} \otimes S^{-1} \otimes \text{id})(R_{32}R_{31}^{-1}R_{12}) = S^{-1}(b_k)S(a_j) \otimes b_jS^{-2}(b_i) \otimes a_ia_k
\]

which allows us to continue the computation:

\[
(\ast) = \mu^l(v^{-1})^* \mu^r(gS^{-1}(b_k)S(a_j)) \varphi(v^2gb_jS^{-2}(b_i)a_ia_k) = \mu^l(v^{-1})^* \mu^r(gS^{-1}(b_k)S(a_j)) \varphi(vS^2(b_j)ak)
\]

\[
= \mu^l(v^{-1})^* \mu^r(gS^{-1}(a_jb_k)) \varphi(vb_aj_k) = \mu^l(v^{-1})^* \mu^r(gvS^{-1}(v'^{-1})) \varphi(v^2v'^{-1}).
\]

We used (2) and (3). On the other hand, we compute

\[
(F \circ S) \circ Z(\tau_b)(\varphi) = (F \circ S) \circ (S^{-1} \circ T \circ S)(\varphi) = (F \circ S) \circ (g^{-1}v) \circ \rho(v_B^{-1}) \circ (g^{-1}v) \circ \rho(\hat{f})(\varphi)
\]

\[
= F \circ S(\mu^l(v^{-1})^* \mu^r(v'^{-1}) \varphi(S^{-1}(v'))) = \mu^l(v^{-1})^* \mu^r(v'b_iS(?)a_i) \varphi(S^{-1}(v'))
\]

\[
= \mu^l(v^{-1})^* \mu^r(vS^2(a_i)Y_jb_iS(?)) \varphi(vS^{-1}(X_j))
\]

\[
= \mu^l(v^{-1})^* \mu^r(vS^2(a_i)S(a_j)S^{-1}(b_k)b_iS(?)) \varphi(vS^{-1}(b_aj_k)) = (\ast \ast).
\]

We used Lemma 6.2 (7) and (3). As previously, we have a Yang-Baxter relation

\[
S^2(a_i)S(a_j) \otimes S^{-1}(b_k)b_i \otimes b_ja_k = S(a_j)S^2(a_i) \otimes b_iS^{-1}(b_k) \otimes a_ka_b
\]
which allows us to continue the computation:

\[(**)=\mu^{i}(v)^{-1}\mu^{r}(vS(a_{j})S^{2}(a_{b})b_{S}^{-1}(b_{k})S(\gamma))\varphi(vS^{-1}(a_{k}b_{j}))
=\mu^{i}(v)^{-1}\mu^{r}(S(a_{j})gS^{-1}(b_{k})S(\gamma))\varphi(vS^{-1}(a_{k}b_{j}))
=\mu^{i}(v)^{-1}\mu^{r}(gS^{-1}(b_{k})S(\gamma))\varphi(vS^{-1}(a_{k}b_{j}))
=\mu^{i}(v)^{-1}\mu^{r}(gS^{-1}(b_{k})S(\gamma))\varphi(vS^{-1}(a_{k}))
=\mu^{i}(v)^{-1}\mu^{r}(\gamma)\varphi(vS^{-1}(a_{k})).\]

We used \([2]\) to simplify \(S^{2}(a_{j})b_{i}=S^{-1}(b_{k})S(a_{j})\) and the properties of \(\mu^{i}\) and \(\mu^{r}\) recorded in section \([2]\). Hence, it holds \(\rho(v_{B}^{-1})\circ(F\circ S)=(F\circ S)\circ Z(\gamma),\) which clearly implies that \(\rho(v_{B}^{-1})\circ(F\circ S)^{\otimes g}\circ\sigma=(F\circ S)^{\otimes g}\circ\sigma\circ Z(\gamma),\) where \(\rho(v_{B}^{-1})\circ(F\circ S)^{\otimes g}\circ\sigma=(F\circ S)^{\otimes g}\circ\sigma\circ Z(\gamma).\) Let us now proceed with \(\tau_{d},(i\geq 2):\)

\[(F\circ S)^{\otimes g}\circ\sigma\circ Z(\tau_{d})\circ\sigma\circ(S^{-1}\circ F^{-1})^{\otimes g}(\varphi_{1}\otimes\ldots\varphi_{g})
=F(F\circ S)^{\otimes g}\circ\sigma\circ Z(\tau_{d})\varphi_{g}\varphi_{1}(S^{-1}(a_{j})S^{-1}(\gamma))\varphi_{g}(S^{-1}(a_{j})S^{-1}(\gamma))b_{j}\otimes\ldots\otimes\varphi_{g}(S^{-1}(a_{j})S^{-1}(\gamma))b_{j}\)
=\varphi_{g}\varphi_{g}(S^{-1}(a_{j})S^{-1}(\gamma))b_{j}\otimes\ldots\otimes\varphi_{1}(S^{-1}(a_{j})S^{-1}(\gamma))b_{j}\otimes\varphi_{g}(S^{-1}(a_{j})S^{-1}(\gamma))b_{j}\otimes\ldots\otimes\varphi_{g}
=\rho(v_{E}^{-1})(\varphi_{1}\otimes\ldots\varphi_{g}).\]

Finally, for \(\tau_{e},(i\geq 2):\)

\[(F\circ S)^{\otimes g}\circ\sigma\circ Z(\tau_{e})\circ\sigma\circ(S^{-1}\circ F^{-1})^{\otimes g}(\varphi_{1}\otimes\ldots\varphi_{g})
=F(F\circ S)^{\otimes g}\circ\sigma\circ Z(\tau_{e})\varphi_{g}\varphi_{g}(S^{-1}(a_{j})S^{-1}(\gamma))b_{j}\otimes\ldots\otimes\varphi_{g}(S^{-1}(a_{j})S^{-1}(\gamma))b_{j}\)
=\varphi_{g}(S^{-1}(a_{j})S^{-1}(\gamma))b_{j}\otimes\ldots\otimes\varphi_{g}(S^{-1}(a_{j})S^{-1}(\gamma))b_{j}\otimes\varphi_{g}(S^{-1}(a_{j})S^{-1}(\gamma))b_{j}\otimes\ldots\otimes\varphi_{g}
=\rho(v_{E}^{-1})(\varphi_{1}\otimes\ldots\varphi_{g}).\]

We used \(\Delta^{\otimes g}R=R\Delta\) for the last equality.

2) It is not difficult to see that \((F\circ S)^{\otimes g}\circ\sigma:K^{\otimes g}\rightarrow(H^{*})^{\otimes g}\) is a morphism of \(H\)-modules, where \(K^{\otimes g}\) is endowed with the action \([10]\) and \((H^{*})^{\otimes g}\) is endowed with the action \([27]\) (with \(n=0\)). Hence, the restriction of \((F\circ S)^{\otimes g}\circ\sigma\) to \((K^{\otimes g})^{\text{inv}}\) indeed takes values in \(\text{Inv}((H^{*})^{\otimes g})\). Since \(Z_{C}(f)=(Z_{H^{\text{reg}}}(f))_{|((K^{\otimes g})^{\text{inv}})}\) and \(\rho_{\text{inv}}(f)=\rho(f)_{|\text{Inv}((H^{*})^{\otimes g})}\),
the result follows from the first part of the theorem. \(\square\)

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