Black Strings in Asymptotically Safe Gravity

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Abstract

In this paper, we study black strings in asymptotic safety gravity (ASG) scenario. The ASG approach is introduced by implementing a gravitational running coupling constant directly in the black string metric. We calculate, analytically, the Hawking temperature, entropy, and heat capacity of the improved black string metric. We show that the ASG corrections provide a remnant mass, whose value depends on the dimension of spacetime, after the complete evaporation of the black string. The corrected entropy is given by in terms of the hypergeometric function. The heat capacity is always positive, as in the usual case. However, differently of the Schwarzschild, in the black string case we do not have phase transitions.

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I. INTRODUCTION

The research in black hole physics has received renewed interest in the last few years. This is mainly due to the present era of high precision measurements which allowed, for example, the direct measurement of gravitational waves by LIGO and the image of supermassive black holes by EHT [1, 2]. However, the black hole interior is riddled with a spacetime singularity. To better understand the properties of these singularities, it is interesting to study their cylindrical counterparts, called black strings.

Black strings/black branes are solutions of the Einstein field equations in D-dimensions [3]. These solutions can be thought of as generalizations of black holes with translational symmetry and can be asymptotically flat if the cosmological constant is null or asymptotically anti-de-Sitter (AAdS) for a negative cosmological constant [4]. In particular the AAdS spacetime had great importance due to the AdS/CFT conjecture [5]. They can be used to test the duality of this theory and gives an alternative to understand the physics of gauge field theories [6–8].

Black string also can be considered as a D1-brane with a horizon, whose boundary is located in a D2-brane satisfying the Dirichlet conditions [9, 10]. In particular, a solution of black string in the usual four dimensional spacetime in cylindrical coordinates was found by Lemos in [11]. It is shown that this black string solution can be used to describe a three dimensional black hole through a dimensional reduction approach, whose properties were determined in [11]. Since then, these black string solutions in 4D cylindrical spacetime was used in several works [12–16]. A generalization of the solution found by Lemos for a D-dimensional spacetime was found in [17]. He found a charged rotating black string as solution of the Einstein-Maxwell field equations with cosmological constant in D-dimensions.

A subject of great importance is the interaction of the quantum nature of particles and geometry. When we take these interactions into account new phenomena can be predicted, such as the radiation emitted by black holes [18]. The process of the emission of radiation brings the possibility of interpreting black holes as thermal systems with temperature and entropy. In this way, a black hole evaporates according as emit radiation. In the final evaporation stage, where the mass of the black hole is of the order of the Planck mass, we can expect that only a quantum theory of gravity can describe the black hole system. However, when we try to describe quantum gravity along the lines of quantum field theory we obtain a non-renormalizable theory.

One way of implementing quantum corrections in general relativity is the asymptotic safety conjecture proposed by Weinberg [19]. This conjecture states that a theory with non-Gaussian fixed points such that all the running coupling constants of the theory tends to them in the ultraviolet (UV) limit, the theory is free of divergences, and therefore, is renormalizable and predictive. The search for fixed points in the case of
gravitational interaction has been done [20–34]. Beyond that, the asymptotic safety conjecture ensures that the theory has no ghosts [35–37]. The asymptotic safety conjecture has been applied in several gravitational systems, such as black holes [38–42] and wormholes [43–47] and can be related to other alternative way to describe quantum gravity [48].

In the asymptotic safety approach we have to compute the gravitational flow $\Gamma_k[g_{\mu\nu}]$, which is the functional of the metric and satisfies the exact renormalization group equation (ERGE) [20]

$$k\partial_k \Gamma_k = \frac{1}{2} \text{Tr}[(\Gamma_k^{(2)} + R_k)^{-1} k\partial_k R_k],$$

where $k$ is the parameter of the renormalization group, $\Gamma_k^{(2)}$ is the Hessian of $\Gamma_k[g_{\mu\nu}]$ and $R_k$ is the cutoff function, responsible for eliminating the small moment modes, that is, the infrared (IR) divergences. Due to the difficulty to solve the ERGE, we have to use the truncation methods to obtain approximate solutions for $\Gamma_k$. The most natural truncation consists in an expansion of $\Gamma_k$ in the basis $\sqrt{g}$ and $\sqrt{g}R$, which is equivalent to associating the gravitational flow with the Einstein-Hilbert action, and therefore this method is called Einstein-Hilbert truncation. However, it is possible to consider other terms beyond the Ricci scalar in the asymptotic safety approach. We can consider higher derivatives terms such as the squared Ricci tensor and the Kretschmann scalar in the ansatz for the gravitational flow [33]. In particular, black holes in the asymptotic safety with higher derivatives terms were studied [33, 34, 49–51]. With this, the fundamental constants of the theory become a function of the renormalization group parameter $k$, and their form is determined in each method cited above. In general, the quantum corrections from these methods can be studied by putting the running coupling constants directly in the classical solutions, and then, the properties of the improved solutions can be done.

In this paper, we study black strings in the asymptotic safety gravity context. The improvement is made by promoting Newton’s constant as a function of the position determined in the asymptotic safety gravity with higher derivatives in [49]. Then, we compute the thermodynamic quantities such as the temperature, entropy, and heat capacity for the improved black string solution. This paper is organized as follows: in the next section, we make the improvement in the black string metric. In section III we compute and analyze the Hawking temperature, entropy, and heat capacity of the improved black string solution. In section IV we conclude the paper.
II. IMPROVED BLACK STRING METRIC IN ASG

Let us consider the static case of a neutral black string solution in $d$-dimensions, which is a particular case of the solution reported in [17]. The metric has the following form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2 + \sum_{i=1}^{d-3} dx_i^2,$$  \hspace{1cm} (2)

where we use coordinates $(t, r, \varphi, x_{d-3}, \ldots, x_1)$ with the ranges $-\infty < t < \infty$, $0 \leq r < \infty$, $0 \leq \varphi < 2\pi$ and $-\infty < x_i < \infty$ for each $i$. Note that $(x_{d-3}, \ldots, x_1)$ is the euclidean submanifold with $d-3$ dimensions. The function $f(r)$ is given by

$$f(r) = \frac{r^2}{l^2} - \frac{m}{r^{d-3}},$$  \hspace{1cm} (3)

where $m$ is the linear density of mass and $l$ is a parameter that is related with the cosmological constant $\Lambda$ through the relation $\Lambda = -\frac{(d-1)(d-2)}{2l^2}$. These are solutions of the Einstein equations in $d$-dimensions and is a generalisation of the particular case of black string solution in a usual 4D spacetime found by Lemos [11]. Although the suitable name for the object described by the metric (2) is black string, this object can be interpreted as a cylindrical black hole or a black brane.

To make the ASG improvement in the black string metric (2), we will consider the quantum gravity corrections in the infrared limit (IR) due to asymptotic safety gravity with higher derivatives. As shown in [49], these corrections are made turning the Newton’s constant $G_d$ into a running coupling constant $G(r)$ with the form

$$G(r) = G_d \left(1 - \frac{\xi}{r^2}\right),$$  \hspace{1cm} (4)

where $G_d$ is the Newton’s constant in $d$-dimensions and $\xi$ is a parameter that measures the ASG effects and has dimensions of length squared. This improvement was used in the Schwarzschild metric in [49], where the authors studied the horizon structure and the thermodynamics of the improved Schwarzschild black hole. Recently, the authors in [52] determined the circular orbits of massive particles and the radiation emitted by the accretion disk of the improved Schwarzschild black hole.

Using (4) in (2), the improved black string metric remains static with cylindrical symmetry, but the function $f(r)$ is changed to

$$f(r) = \frac{r^2}{l^2} - \frac{m_{\text{eff}}(r)}{r^{d-3}},$$  \hspace{1cm} (5)

where $m_{\text{eff}}(r)$ is the effective mass per unit length given by

$$m_{\text{eff}}(r) = m \left(1 - \frac{\xi}{r^2}\right),$$  \hspace{1cm} (6)
in units $c = G_d = 1$. Thus, we can interpret the ASG improvement as introducing a mass dependent of the position. Now, we will use the quantum-corrected metric to study their thermodynamic properties.

III. THERMODYNAMICS OF THE ASG IMPROVED BLACK STRING

Now, we will check the ASG corrections in the thermodynamics of a black string. To facilitate the analysis, we rewrite the improved function $f(r)$ (eq.(5)) in the form

$$f(r) = g(r) - mh(r),$$

where

$$g(r) = \frac{r^2}{l^2},$$

$$h(r) = \frac{1}{r^{d-3}} \left( 1 - \frac{\xi}{r^2} \right).$$

The Hawking temperature is defined by

$$T_H = \frac{f'(r_+)}{4\pi},$$

where $r_+$ is the horizon radius of the black string (remember that the black string can be interpreted as a cylindrical black hole). Firstly, note that the horizon radius is defined as $f(r_+) = 0$. This allows us to write the density of mass $m$ in terms of $r_+$:

$$m(r_+) = \frac{g(r_+)}{h(r_+)} = \frac{r_+^{d-1}}{l^2 \left( 1 - \frac{\xi}{r_+^2} \right)},$$

where we can clearly see that in the general relativity limit, $\xi \to 0$, we recover the classical expression for the mass in function of the horizon radius. Using the eq.(10), the Hawking temperature can be rewritten as

$$T_H = \frac{h(r_+)}{4\pi} \frac{d}{dr_+} \frac{g(r_+)}{h(r_+)},$$

which give us

$$T_H = \frac{(d-1)r_+^2 - (d+1)r_+\xi}{4\pi l^2 (r_+^2 - \xi)}.$$

We can note that the Hawking temperature tends to their usual expression for the black string in $d$-dimensions in the limit $\xi \to 0$, that is, linear in the horizon radius. Furthermore, the Hawking temperature reaches $T_H = 0$ for a non-null value of the horizon radius, and consequently, a non-zero mass. This means that the ASG corrections provides a remnant mass when the black string completely evaporates. By the expression (12), the horizon radius of the remnant mass is given by $r_+ = \sqrt{\frac{(d+1)\xi}{d-1}}$, which corresponds to the respective value of the remnant mass

$$m_{rem} = \frac{d + 1}{2l^2} \left( \sqrt{\frac{(d+1)\xi}{d-1}} \right)^{d-1}.$$
This is in contrast with the usual theory, where the mass of the black string disappears when it evaporates completely. However, it is important to emphasize that for the charged black string case, we will also have a remnant mass in the general relativity limit and the ASG theory provides only small corrections in the value of the horizon radius of the remnant mass.

The behavior of Hawking temperature for the improved black string can be seen in Fig. 1. As we can see, the ASG corrections matches with the usual case for large values of \( r_+ \), once in the limit \( \xi \to 0 \) we have a linear function of \( r_+ \) for the temperature. Indeed, we expect that the quantum effects become relevant in small scales of distance. The deviations due to quantum gravity corrections become relevant only as the black string evaporates, until it completely evaporates to a non-null value of the horizon radius. For the Schwarzschild black hole, the existence of a non-null value of mass when the temperature reaches zero is also observed \[49\]. We can note that the behavior of the Hawking temperature is similar in extra dimensions. The only difference is that at higher dimensions the temperature increases more rapidly with the increasing of the horizon radius.

![FIG. 1: Hawking temperature as a function of the horizon radius of the improved static charged black string. We have set \( \xi = 0.05 \) and \( l = 1 \).](image)

Now, we turn to entropy. The linear density of entropy \( s \) can be calculated using the first law of thermodynamics \( ds = dm/T_H \). Using \( m = g(r_+)/h(r_+) \), we can rewrite the density of entropy in function of \( r_+ \), which give us

\[
 ds = \frac{4\pi dr_+}{h(r_+)} = \frac{4\pi r_+^{d-3} dr_+}{1 - \frac{\xi}{r_+^2}}. \tag{14}
\]

As the minimum value of the horizon radius can reach is non-null (the radius of the remnant mass), we can integrate the above equation without worrying with the pole in \( r_+ = 0 \). Integrating (14), we have

\[
 s = -\frac{r_+^d}{d} \frac{2F_1 \left(1, \frac{d}{2}, \frac{2+d}{2}, \xi \right)}{\xi}. \tag{15}
\]
where \( _2F_1 \) is the hypergeometric function. In the limit \( \xi \to 0 \) we recover the usual expression for the entropy of a black string. In general, for even dimensions we will have logarithmic corrections of the entropy while for odd dimensions we will have corrections due to the inverse hyperbolic tangent. In particular, for the usual \( d = 4 \) spacetime, the logarithmic corrections to the entropy, which is a feature of quantum gravity corrections to the entropy of 4D black holes, have been found in other black hole solutions using other quantum gravity approaches, such as GUP and noncommutative geometry \([54–62]\). The logarithmic correction is also observed in the Schwarzschild metric corrected by the ASG \([53]\). The behavior of the entropy for different dimensions of spacetime can be seen in Fig.(2). As well as the temperature, the behavior of the entropy is similar for different dimensions of the spacetime. However, we have the peculiar feature of the ASG corrections is that the entropy is always negative and has an asymptote for a certain value of \( r_+ \). This means that the black string emits more heat than absorbs, which is not observed for the usual case.

![Fig. 2: Entropy as a function of the horizon radius of a black string in different dimensions. We have set \( \xi = 0.1 \) and \( l = 1 \).](image)

Finally, we study the thermodynamic stability of this system. For this, we have to compute the heat capacity, defined by the expression

\[
C_V = \frac{dm}{dT_H}.
\]

Using the eqs.(10) and (12), we obtain the following expression for the heat capacity

\[
C_V = \frac{4\pi r_+^d \left( (d - 1)r_+^2 - (d + 1)\xi \right)}{(d + 1)\xi^2 + (d - 1)r_+^2 - 2(d - 2)\xi r_+^2}.
\]

The equation for the heat capacity (17) tends to the usual expression in the limit \( \xi \to 0 \), that is, being proportional to \( r_+^{d-2} \). One more time we can see that we have \( C_V = 0 \) for \( r_+ = \sqrt{\frac{(d+1)\xi}{d-1}} \), emphasizing that the black string will have a remnant mass given by the eq.(13) in the final of your evaporation process. The behavior for the ASG corrected heat capacity for a black string can be seen in Fig.(3). We can see that both
ASG and the usual theory provides a stable system, because the heat capacity is always positive in both cases. The heat capacity has a similar behavior in higher dimensions, as well as the other thermodynamic quantities. However, note that the heat capacity for small values of the horizon radius will have the same independently of the dimension of the spacetime.

![Heat capacity graph](image)

**FIG. 3**: Heat capacity as a function of the horizon radius for the improved black string. We have set $\xi = 0.1$ and $l = 1$.

The heat capacity diverges only for complex or negative values of $r_+$, or by a value of $r_+$ which is smaller than the radius of the remnant mass. Therefore, the phase transition is not observed in the improved black string metric by ASG. On other hand, the ASG allows a phase transition in the spherical case, where for the improved Schwarzschild metric, the phase transition is studied in [53].

### IV. CONCLUSION

In this paper, for the first time, black strings were considered in the ASG scenario. To make the improvement, we have considered the quantum corrections from the asymptotic safety gravity with higher derivatives in the infrared limit. These corrections are implemented directly in the metric, via improvement of Newton’s constant, and, can be interpreted as the mass of the system now depending on the position.

The corrected temperature and entropy match with these quantities in the usual case for the large values of the horizon radius $r_+$. However, the temperature goes to zero for a non-null value of $r_+$, indicating that will be a remnant mass, whose value depends on the dimension of the spacetime, when the black string completely evaporates, in contrast to what is expected from the usual theory, where all the mass disappears. For the corrected entropy we have a logarithmic correction for even dimensions and inverse hyperbolic tangent corrections for odd dimensions, which is characteristic of quantum gravity corrections. The ASG corrections allow a negative entropy for a certain value of the horizon radius, while for the usual case the entropy is always positive.
The heat capacity in both cases the is positive, therefore, the black string is a stable system with ASG corrections as well as in the usual case. Also, the heat capacity does not diverge, indicating that there is no phase transition in the improved black string metric. This is very different from the Schwarzschild case, where phase transitions are present. Finally, we should point out that this manuscript is the first description of black strings in the ASG context and therefore many future studies can be done: to consider other improvements, to study particle trajectories, theories with higher curvatures, and so on. Therefore, this is the first of a series of papers about the subject.

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