DUALITY AND LOCAL GROUP COHOMOLOGY

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Abstract. Recently, Meierfrankenfeld has published three theorems on the cohomology of a finitary module. They cover the local determination of complete reducibility; the local splitting of group extensions; and the representation of locally split extensions in the double dual. In this note we derive all three by combining a certain duality between homology and cohomology with the continuity of homology.

Local Cohomology

We describe a general framework for three recent results of Meierfrankenfeld’s [M], on the cohomology of finitary modules. It turns out that Theorem A below — which began life as an attempt to understand the third of this trio — includes as corollaries all three. Moreover, it implies some new corollaries, and covers all cohomological degrees.

Throughout, $G$ is a group, $k$ is a commutative field, and $L$ is a local system — an upwardly directed collection of subgroups whose union is $G$. Modules are $k$-spaces. If $V$ is a module, then $V^\vee$ denotes its dual, and $\langle \| \rangle$ denotes their pairing.

We will explore the relationship between the cohomology of $G$, $H^*(G,V)$, with the local cohomology, $H^*(L,V)$. This latter is defined by taking the limit, with respect to restriction maps, of the cohomology groups $H^*(L,V)$, for $L$ in $L$:

$$H^*(L,V) := \lim_{\leftarrow} H^*(L,V).$$

We can define analogous limits, $C^*(L,V)$, $Z^*(L,V)$, and $B^*(L,V)$, for cochains, cocycles, and coboundaries, respectively. What we find is that there are canonical isomorphisms $C^*(L,V) \cong C^*(G,V)$ and $Z^*(L,V) \cong Z^*(G,V)$. Indeed, the localization $C^*(G,V) \rightarrow C^*(L,V)$ is defined by restriction to the local subgroups, $\phi \mapsto \{ \phi|_L \}$. In the reverse direction, the map $C^*(L,V) \rightarrow C^*(G,V)$ splices an $L$-sequence of $n$-cochains $\{ \phi_L \}$ into a cochain on $G$: $g_1, \ldots, g_n \mapsto \phi_L(g_1, \ldots, g_n)$, where $L$ is any member of $L$ that contains all the $g_i$. This is well-defined, since in the inverse limit $C^*(L,V)$, $\phi_L|_{L'} = \phi_{L'}$, whenever $L \supseteq L'$.

These maps respect coboundary, and so they induce isomorphisms for the cocycle groups. However, splicing an $L$-sequence of coboundaries need not yield a coboundary in $G$. In general we obtain only an embedding $B^*(G,V) \hookrightarrow B^*(L,V)$.

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Now consider the localization $H^*(G,V) \to H^*(L,V)$. An element of $H^i(L,V)$ is an inverse system of affine flats $\{\phi_L + B'(L,V)\}$, where $\phi_L|_{L'} \equiv \phi_{L'} \mod B'(L',V)$ whenever $L \supset L'$. The image of $H^i(G,V)$ consists of those whose inverse limit is nonempty. Similarly, an element in the kernel of the map $H^{i+1}(G,V) \to H^{i+1}(L,V)$ is an inverse system of affine flats $\{\psi_L + Z^i(L,V)\}$, where $\delta \psi_L|_{L'} = \delta \psi_{L'}$ whenever $L \supset L'$ viewed modulo the collection of those inverse systems that have a nonempty limit.

The simple ‘compactness’ proof of [M, Thm 2] applies in arbitrary cohomological degree, to give the following generalization.

**Proposition.** Let $i$ be a positive integer.

a. If $\dim_k B'(L,V) < \infty$ for all $L$ in $\mathcal{L}$, then localization $H^i(G,V) \to H^i(L,V)$ is surjective in degree $i$.

b. If moreover $\dim_k Z^i(L,V) < \infty$ for all $L$ in $\mathcal{L}$, then additionally localization $H^{i+1}(G,V) \to H^{i+1}(L,V)$ is injective in degree $i + 1$. □

We omit the proof because we will not be using this result. The problem with this generalization is that it concerns finitary modules only for $H^2(G,V)$.

We will write $\text{Ext}^*_G(X,Y)$ instead of $H^*(G,\text{Hom}_k(X,Y))$, and $\text{Tor}^*_G(X,Y)$ instead of $H_*(G,X \otimes_k Y)$. In addition to local cohomology, we could also define local homology: $\text{Tor}^*_L(X,Y) = \varinfty \oplus \text{Tor}^*_L(X,Y)$. However, this yields nothing new.

**Lemma 1.** If $X$ and $Y$ are $G$-modules, then $\text{Tor}^*_L(X,Y) = \varinfty \oplus \text{Tor}^*_L(X,Y)$.

**Proof.** First of all, note that if $P_* = C_*(G,Z)$, and $V$ is any $G$-module, then $\varinfty \oplus L V = P_* \otimes G V$. To see this, observe that $P_* \otimes G V = (P_* \otimes V)/[P_* \otimes G V]$ and $[P_* \otimes V,G] = \sum_L [P_* \otimes V,L]$.

Since the maps $P_* \otimes_L V \to P_* \otimes_G V$ are surjective, we obtain that the boundaries of the limit are limits of boundaries. Now if an element maps to 0 in a direct limit then it maps to 0 at some finite stage. This tells us that the cycles of the limit are limits of cycles. □

Although local cohomology can differ from global cohomology, we will see that they are the same for the dual of a module.

**Main Theorem**

In this section we first record a duality between homology and cohomology. We then exploit this duality, together with the continuity of homology, to obtain a theorem on the local detection of cohomology, valid in all degrees.

**Lemma 2.** If $f: V \to V$ is a linear endomorphism, then

$$\begin{align*}
\text{ann}_V (\text{im}(f^\vee)) &= \ker(f) \\
\text{ann}_V (\ker(f^\vee)) &= \text{im}(f)
\end{align*}$$

**Proof.** The first two equalities follow from the identity $(f(x) | \lambda) = \langle x | f^\vee(\lambda) \rangle$. The third follows from the first, since any subspace of $V$ is the annihilator of its annihilator. The last follows from composition of the canonical isomorphisms $\text{ann}_V(\ker(f)) = \text{im}(f) = f^\vee$. □

The following is a special case of [B, Prop 2.8.5], but we include a direct proof.
**Lemma 3.** If $X$ and $Y$ are $L$-modules, then $\text{Ext}^*_L(X, Y^\vee) = \text{Tor}^*_L(X, Y^\vee)$.

*Proof.* Let $P_* = C_*(G, \mathbb{Z})$, and note the canonical isomorphism

$$\text{Hom}(P_*, \text{Hom}_k(X, Y^\vee)) = (P_* \otimes X \otimes_k Y^\vee)^\vee.$$ 

This isomorphism can be described as follows: a functional $\lambda$ in $(P_* \otimes X \otimes_k Y^\vee)^\vee$ is paired to an additive map $f: P_* \to \text{Hom}_k(X, Y^\vee)$ when they satisfy the relation

$$\langle y | f(c)(x) \rangle = \langle c \otimes x \otimes y | \lambda \rangle.$$ 

Now if we take fixed points for $L$ in this isomorphism, we obtain that $\text{Hom}_L(P_*, \text{Hom}_k(X, Y^\vee)) = (P_* \otimes_L (X \otimes_k Y))^\vee$.

To finish, apply Lemma 2 to the boundary of $P_* \otimes_L (X \otimes_k Y)$. □

**Theorem A.** If $X$ and $Y$ are $G$-modules, then localization $\text{Ext}^*_G(X, Y^\vee) \to \text{Ext}^*_L(X, Y^\vee)$ is an isomorphism.

*Proof.* Apply Lemmas 1 and 3, and the fact that the dual of a direct limit of $k$-spaces is the inverse limit of the duals of those spaces. □

**Corollary 1.** Locally trivial coclasses are trivial in the double dual.

*Proof.* Extension of scalars from $Y$ to $Y^{\vee\vee}$ factors through localization:

$$\begin{array}{ccc}
\text{Ext}^*_G(X, Y) & \to & \text{Ext}^*_G(X, Y^{\vee\vee}) \\
\downarrow & & \| \\
\text{Ext}^*_L(X, Y) & \to & \text{Ext}^*_L(X, Y^{\vee\vee}).
\end{array}$$ □

**Corollary 2.** If $X$ is an arbitrary $G$-module and $Y$ is a finite-dimensional $G$-module, then $\text{Ext}^*_G(X, Y)$ is determined locally. □

In the next section we derive several more corollaries of this theorem, including all three theorems from [M].

**Finitary Cohomology**

We now turn to the cohomology of finitary modules. The following characterization of finitary groups — due to Meierfrankenfeld — is the key to understanding their cohomology.

**Lemma 4.** The finitary group on $V$ is exactly the centralizer of $V^{\vee\vee}/V$.

*Proof.* Let $x$ be a transformation of $V$. Apply Lemma 2 to $x-1$ twice: between $V$ and $V^\vee$, and also between $V^\vee$ and $V^{\vee\vee}$. Next, use the fact that a space equals its double dual if and only if it is finite dimensional. □

In light of this characterization, consider the long exact sequence associated to the extension $V \hookrightarrow V^{\vee\vee}$:

$$\cdots \to \text{Ext}^{i-1}_G(U, V^{\vee\vee}/V) \to \text{Ext}^i_G(U, V) \to \text{Ext}^i_G(U, V^{\vee\vee}) \to \cdots.$$
**Theorem B.** If $U$ is an arbitrary $G$-module and $V$ is a finitary $G$-module, then the locally trivial portion of $\text{Ext}^1_G(U,V)$ lies in the image of $\text{Ext}^1_G(U,V^\vee/V)$.

**Proof.** Apply Theorem A and Lemma 4 to (i). □

**Corollary 3.** Let $U$ be an arbitrary module, and $V$ a finitary module.

a. If $U = [U,G]$, then localization $\text{Ext}^1_G(U,V) \to \text{Ext}^1_G(U,V)$ is injective.

b. If $\text{Hom}(G,k) = 0$, then localization $H^2(G,V) \to H^2(L,V)$ is injective.

**Proof.** If $U = [U,G]$ then $\text{Ext}^1_G(U,V^\vee/V) = \text{Hom}_{kG}(U,V^\vee/V) = 0$, whence we obtain part a. If $\text{Hom}(G,k) = 0$ then $\text{Ext}^1_G(k,V^\vee/V) = \text{Hom}(G,V^\vee/V) = 0$, which yields part b. □

**Corollary 4.** [M, Thm 1] If $V$ is locally completely reducible, then $[V,G]$ is completely reducible.

**Proof.** Let $W < [V,G]$. We show that $W$ is a direct summand. Set $U = [V,G]/W$. Since $[V,G] = [V,G,G]$, $\text{Hom}(U,W^\vee/W) = 0$. Now apply Corollary 3a. □

**Corollary 5.** [M, Thm 2] If $H^1(L,V)$ and $C_V(L)$ are finite-dimensional for every $L$ in $L$, then localization $H^2(G,V) \to H^2(L,V)$ is injective.

**Proof.** If $V$ is infinite dimensional, apply Corollary 2. If not, the exact sequence

$$
\cdots \to V/C_V(L) \to H^1(L,C_V(L)) \to H^1(L,V) \to \cdots
$$

tells us that $H^1(L,k) = 0$, since otherwise $H^1(L,V)$ would be infinite dimensional. Hence $H^1(G,k) = 0$. Now apply Corollary 3b. □

**Corollary 6.** [M, Thm 3] If $V \to W$ is a locally split extension of $V$ by a trivial $G$-module, then there is a canonical injection $W/C_W(G) \to [V^\vee,G]^\vee$.

**Proof.** Note that $[V^\vee,G]^\vee = V^\vee/C_V^\vee(G)$, and apply Theorem A. □

**Two Examples**

For the first example, let $\Omega$ be a set of some infinite cardinal $\aleph$, let $G$ be the finitary symmetric group on $\Omega$, let $k$ be of characteristic 2, and let $V$ be the natural permutation module $k\Omega$. The Eckmann-Shapiro Lemma (see [B, Cor 2.8.4]) tells us that $H^1(G,V^\vee) = H^1_G(G,V^\vee) = k$. Now $V$ embeds in $V^\vee$, with $G$-trivial quotient (it is a quotient of $V^\vee/V$). The associated long exact sequence contains the fragment

$$
\cdots \to H^1(G,V^\vee) \to H^1(G,V^\vee/V) \to H^2(G,V) \to H^2(G,V^\vee) \to \cdots
$$

Thus $\dim_k H^2(G,V) = \dim_k H^1(G,V^\vee/V) = 2^\aleph$. On the other hand, if we express $G$ as a union of finite symmetric groups, then we have that $H^1(L,V) = k$. To see this, let $\Gamma$ and $\Gamma'$ be cofinite subsets, with $\Gamma \supset \Gamma'$. Set $F = C_G(\Omega)$, $F' = C_G(\Omega')$, and $\Delta = \Omega - \Omega'$. We have that $H^2(F,V) = k \oplus H^2(F,k\Delta) \oplus H^2(F,k\Omega')$, while $H^2(F',V) = k \oplus H^2(F',k\Omega')$. Thus, taking successively smaller $\Gamma'$, we see that none of the classes in $H^2(F,V)$ that take values in $k\Gamma$ survive in the limit.

Although very little of the 2-cohomology of the symmetric group is detected locally, all of the 2-cohomology for the alternating subgroup is detected locally, by Corollary 3. Moreover, all of the 2-cohomology for the symmetric group is detected locally when we replace $k$ by a field of any other characteristic.
For the second example, let $k$ have odd characteristic, let $V$ be an infinite-dimensional $k$-space with a nondegenerate symplectic form, and let $G$ be the finitary symplectic group. The form gives an embedding of $V$ in $V^\vee$. Again, the quotient $V^\vee/V$ is $G$-trivial. The extension $V \to V^\vee$ is nonsplit since $C_{V^\vee}(G) = 0$. However, if we let $\mathcal{L}$ be the collection of symplectic groups of the finite-dimensional, nondegenerate subspaces of $V$, then we find that the extension is locally split. (See [CPS, Tbl 4.5, pp 186–187].) By Corollary 5, any $G$-trivial extension of $V^\vee$ is split.

**Skew Fields**

Of the three theorems in [M], Meierfrankenfeld assumes that $k$ is commutative only in the first. The results above can be extended to the case of a skew field $k$, by carefully tracking whether $k$ should multiply from the left or right — or both. The guiding principle is the duality of Lemma 3, and so we must ensure that all the homology and cohomology groups are $k$-spaces. For Lemma 1 we need only take $X$ to be a right $k$-space and $Y$ a left $k$-space. However, for Lemma 3 and Theorem A we must further assume that $X$ is a $k$-$kG$-bimodule. This is tantamount to requiring that the $G$-action on $X$ be defined over the center of $k$. Note that Lemma 3 no longer follows from [B, Prop 2.8.5] when $k$ is not commutative. However, the direct proof given above works in general, if assume that $X$ is a bimodule.

In the applications to finitary modules, we must assume similarly that $U$ is a bimodule. In most of the corollaries this causes no difficulty. In particular, since we are taking $U = k$ in the application to [M, Thm 2] and [M, Thm 3], we find that these are subsumed by Theorem B even when $k$ is not commutative. However, in trying to apply Theorem B to [M, Thm 1], we find that we must assume that the $G$-action on $V$ is defined over the center of $k$ — which in effect echoes Meierfrankenfeld’s hypothesis for this theorem.

Recently — and independently — Wehrfritz proves Corollary 4 assuming only that $k$ is finite-dimensional over its center $[W_1]$. In fact Theorem A can be extended to this situation, by tensoring with a splitting field. If $\zeta$ is the center of $k$, and $\mu$ is a maximal commutative subfield of $k$, then $\text{Hom}_k(X,Y^\vee) \otimes_{\zeta} \mu = \text{Hom}_\mu(X,Y^\vee)$. So, $\text{Ext}^*_G(X,Y^\vee) \otimes_{\zeta} \mu = \text{Ext}^*_\zeta(X,Y^\vee) \otimes_{\zeta} \mu$, whence $\text{Ext}^*_G(X,Y^\vee) = \text{Ext}^*_\zeta(X,Y^\vee)$, since field extension is faithfully flat.

The example in [W_2] shows that, without some hypothesis on the field, Corollary 4 would be false. (See also [ShW, Ex 1.18].)

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