Ensemble Averages when $\beta$ is a Square Integer

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Abstract

We give a hyperpfaffian formulation of partition functions and ensemble averages for Hermitian and circular ensembles when $L$ is an arbitrary integer and $\beta = L^2$ and when $L$ is an odd integer and $\beta = L^2 + 1$.

Keywords: Random matrix theory, partition function, Pfaffian, hyperpfaffian, Selberg integral

For many ensembles of random matrices, the joint probability density of eigenvalues has the form

$$\Omega_N(\lambda) = \frac{1}{Z_N N!} \left( \prod_{n=1}^{N} w(\lambda_n) \right) \prod_{m<n} |\lambda_n - \lambda_m|^\beta, \quad \lambda \in \mathbb{R}^N \quad (0.1)$$

where, $\beta > 0$, $w : \mathbb{R} \to [0, \infty)$ is a weight function and $Z_N$ is a normalization constant called the partition function. The parameter $\beta$ is often taken to be 1, 2 or 4, since for these values the correlation functions (viz. marginal probabilities) have a determinantal or Pfaffian form.

The partition function is given explicitly by

$$Z_N = Z_N^\nu = \frac{1}{N!} \int_{\mathbb{R}^N} \prod_{m<n} |\lambda_n - \lambda_m|^\beta d\nu_N(\lambda), \quad (0.2)$$

where $d\nu(\lambda) = w(\lambda)d\lambda$ and $\nu_N$ is the resulting product measure on $\mathbb{R}^N$. However, for now, we will take $\nu$ to be arbitrary, with the restriction that $0 < Z_N < \infty$, and we will suppress the notational dependence of $Z_N$ on $\nu$ except when necessary to allay confusion.

Ensemble averages of functions which are multiplicative in the eigenvalues can be expressed as integrals of the form (0.2) where $\nu$ is determined by the function on the eigenvalues, the weight function of the ensemble and the underlying reference measure.

In the case where $\beta = 2\gamma$ and

$$d\nu(\lambda) = \lambda^{a-1}(1 - \lambda)^{b-1}1_{[0,1]}(\lambda) d\lambda; \quad a, b, \gamma \in \mathbb{C},$$

$S_N(\gamma,a,b) = N!Z_N$ is the (now) famous Selburg integral [19, 20], and in this case

$$S_N(\gamma,a,b) = \frac{1}{\Gamma(a + b + (N + n - 1)\gamma)} \frac{\Gamma(a + n\gamma) \Gamma(b + n\gamma) \Gamma(1 + (n + 1)\gamma)}{\Gamma(a + b + (N + n - 1)\gamma) \Gamma(1 + \gamma)}.$$

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The case where
\[ d\nu(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2} \quad \text{and} \quad \beta = 2\gamma, \]

\[ F_N(\gamma) = N! Z_N \]

is called the Mehta integral [15, 17], and has the evaluation

\[ F_N(\gamma) = \prod_{n=1}^{N} \frac{\Gamma(1 + n\gamma)}{\Gamma(1 + \gamma)}. \]

This evaluation follows the same basic idea as that of the Selberg integral. For an interesting history of these evaluations, see [9].

The purpose of this note is not to provide further explicit evaluations of for particular choices of \( \nu \), but rather to show that when \( \beta \) is a square integer (or adjacent to a square integer) that \( Z_N \) can be expressed as a hyperpfaffian. Expressing ensemble averages as a determinant or Pfaffian is the first step to demonstrating the solvability of an ensemble; that is writing the correlation functions, gap probabilities and other quantities of interest in terms of determinants or Pfaffians formed from a kernel associated to the particulars of the ensemble [22, 21, 5]. Pfaffian formulations for \( Z_N \) are known when \( \beta = 1 \) and \( \beta = 4 \), and the results here generalize those formulas. To be specific, we show that when \( L \) is odd and \( \beta = L^2 \) or \( L^2 + 1 \), then \( Z_N \) can be written as the hyperpfaffian of a 2\( L \)-form, the coefficients of which are formed from double integrals of products of \( L \times L \) Wronskians of monic polynomials in a manner which generalizes the well known \( \beta = 1 \) and \( \beta = 4 \) cases. When \( L \) is even, \( Z_N \) can be written as a hyperpfaffian of an \( L \)-form, the coefficients of which are integrals (with respect to the measure \( \nu \)) of Wronskians of any complete family of monic polynomials. This generalizes the situation when \( \beta = 4 \). Similar results for \( L \) even were given in [14].

We will also give a similar characterization of ensemble averages over circular ensembles.

1 Pfaffians, Hyperpfaffians and Wronskians

1.1 Pfaffians and Hyperpfaffians

Suppose \( V = \mathbb{R}^{NL} \) with basis \( e_1, e_2, \ldots, e_{NL} \). For each increasing function \( t \), from \( L = \{1, 2, \ldots, L\} \) into \( NL = \{1, 2, \ldots, NL\} \), we write

\[ \epsilon_t = e_{t(1)} \wedge e_{t(2)} \wedge \cdots \wedge e_{t(L)}, \]

so that \( \{ \epsilon_t \mid t : L \nrightarrow NL \} \) is a basis for \( \Lambda^L V \). The volume form of \( \mathbb{R}^{NL} \) is given by

\[ \epsilon_{\text{vol}} = e_1 \wedge e_2 \wedge \cdots \wedge e_{NL}. \]

The \( N \)-fold wedge product of an \( L \)-form \( \omega \) is a constant times \( \epsilon_{\text{vol}} \). This constant is (up to a factor of \( N! \)) the hyperpfaffian of \( \omega \). Specifically, for \( \omega \in \Lambda^L V \), we define \( \text{PF} \omega \) by

\[ \frac{\omega^{\wedge N}}{N!} = \frac{1}{N!} \underline{\omega \wedge \cdots \wedge \omega} = \text{PF} \omega \cdot \epsilon_{\text{vol}}. \]

When \( L = 2 \), the hyperpfaffian generalizes the notion of the Pfaffian of an antisymmetric matrix: If \( A = [a_{m,n}] \) is an antisymmetric \( 2N \times 2N \) matrix, then the Pfaffian of \( A \) is given
by
\[ \text{Pf } A = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \text{sgn } \sigma \prod_{n=1}^{N} a_{\sigma(2n-1), \sigma(2n)}. \]

We may identify \( A \) with the 2-form
\[ \alpha = \sum_{m<n} a_{m,n} e_m \wedge e_n. \]

It is an easy exercise to show that \( \text{Pf } A = \text{PF } \alpha \). There exists a formula for the hyperpfaffian as a sum over the symmetric group, and in fact, the original definition of the hyperpfaffian was given as such a sum [14, 3].

### 1.2 Wronskians

A complete family of monic polynomials is an \( NL \)-tuple of polynomials \( p = (p_n)_{n=1}^{NL} \) such that each \( p_n \) is monic and \( \deg p_n = n - 1 \). Given an increasing function \( t : \mathbb{L} \overset{\rightarrow}{\rightarrow} NL \),
we define the \( L \)-tuple \( p_t = (p_{t(\ell)})_{\ell=1}^{L} \). And, given \( 0 \leq \ell < L \) we define the modified \( \ell \)th differentiation operator by
\[ D^\ell f(x) = \frac{1}{\ell!} \frac{d^\ell f}{dx^\ell}. \]

The Wronskian of \( p_t \) is then defined to be
\[ \text{Wr}(p_t; x) = \det \left[ D^{\ell-1}-1 p_{t(n)}(x) \right]_{n,\ell=1}^{L}. \]

The Wronskian is often defined without the \( \ell! \) in the denominator of (1.1); this combinatorial factor will prove convenient in the sequel. The reader has likely seen Wronskians in elementary differential equations, where they are used to test for linear dependence of solutions.

### 2 Statement of Results

For each \( x \in \mathbb{R} \), we define the \( L \)-form \( \omega(x) \in \Lambda^L V \) by
\[ \omega(x) = \sum_{t: \mathbb{L} \overset{\rightarrow}{\rightarrow} NL} \text{Wr}(p_t; x) \epsilon_t. \]

This form clearly depends on the choice of \( p_t \), though we will suppress this dependence. We may arrive at an \( L \)-form with constant coefficients by integrating the coefficients of the form with respect to \( dv \). In this situation we will write
\[ \int_{\mathbb{R}} \omega(x) \, dv(x) = \sum_{t: \mathbb{L} \overset{\rightarrow}{\rightarrow} NL} \left\{ \int_{\mathbb{R}} \text{Wr}(p_t; x) dv(x) \right\} \epsilon_t. \]

Notice that we are not integrating the form in the sense of integration on manifolds, but rather formally extending the linearity of the integral over the coefficient functions of \( \omega(x) \) to arrive at an \( L \)-form with constant coefficients.

To deal with the \( N \) odd case, we set \( V' = \mathbb{R}^{(N+1)L} \) and define the basic \( L \)-form \( \epsilon' = e_{NL+1} \wedge e_{NL+2} \wedge \cdots \wedge e_{NL+L} \).
Theorem 2.1. Suppose $L$ and $N$ are positive integers, and let $\omega(x)$ be the $L$-form given as in (2.1) for any complete family of monic polynomials $p$. Then,

1. if $\beta = L^2$ is even,
   \[ Z_N = \text{PF} \left( \int_{\mathbb{R}} \omega(x) \, d\nu(x) \right); \]

2. if $\beta = L^2$ is odd and $N$ is even,
   \[ Z_N = \text{PF} \left( \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \text{sgn}(y - x) \, d\nu(x) \, d\nu(y) \right); \]

3. if $\beta = L^2$ is odd and $N$ is odd,
   \[ Z_N = \text{PF} \left( \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \text{sgn}(y - x) \, d\nu(x) \, d\nu(y) + \int_{\mathbb{R}} \omega(x) \wedge \epsilon \, d\nu(x) \right); \]

4. if $\beta = L^2 + 1$ is even and $N = 2M$ is even,
   \[ Z_N = \text{PF} \left( \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \left[ \frac{(y^M - x^M)^2}{y - x} \right] \, d\nu(x) \, d\nu(y) \right). \]

More explicitly, when $\beta = L^2$ is even we have
\[
\int_{\mathbb{R}} \omega(x) \, d\nu(x) = \sum_{t: L \not\equiv 2N} \int_{\mathbb{R}} \text{Wr}(p_t; x) \, d\nu(x) \epsilon_t
\]
In the case where $\beta = 4$, we have
\[
\text{Wr}(p_t; x) = p_{t(1)}(x)p_{t(2)}'(x) - p_{t(1)}'(x)p_{t(2)}(x); \quad t \not\equiv 2N,
\]
and
\[
\int_{\mathbb{R}} \omega(x) \, d\nu(x) = \sum_{t: 2 \not\equiv 2N} \left\{ \int_{\mathbb{R}} [p_{t(1)}(x)p_{t(2)}'(x) - p_{t(1)}'(x)p_{t(2)}(x)] \, d\nu(x) \right\} \epsilon_{t(1)} \wedge \epsilon_{t(2)}.
\]
This is exactly the 2-form associated to the antisymmetric matrix
\[
\mathbf{W} = \left[ \int_{\mathbb{R}} [p_m(x)p'_n(x) - p'_m(x)p_n(x)] \, d\nu(x) \right]_{m,n=1}^{2N},
\]
and $Z_N = \text{Pf} \mathbf{W}$, as is well known [7, 16].

When $\beta = L^2$ is odd and $N$ is even,
\[
\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \text{sgn}(y - x) \, d\nu(x) \, d\nu(y)
= \sum_{t,u: L \not\equiv 2N} \left\{ \frac{1}{2} \int_{\mathbb{R}} \text{Wr}(p_t; x)\text{Wr}(p_u; y) \, d\nu(x) \, d\nu(y) \right\} \epsilon_t \wedge \epsilon_u.
\]
In the case where \( L = \beta = 1 \), each \( t : \mathbb{N} \to \mathbb{N} \) selects a single integer between 1 and \( N \), and \( \text{Wr}(p_{t(x)}; x) = p_{t(\{x\})}(x) \). It follows that

\[
\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \omega(y) \text{sgn}(y - x) \, d\nu(x) \, d\nu(y)
\]

\[
= \sum_{m,n=1}^{N} \left\{ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} p_m(x)p_n(y) \text{sgn}(y - x) \, d\nu(x) \, d\nu(y) \right\} e_m \wedge e_n
\]

This is the 2-form associated to the antisymmetric matrix

\[
U = \left[ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} p_n(x)p_m(y) \text{sgn}(y - x) \, d\nu(x) \, d\nu(y) \right]_{n,m=1}^{N},
\]

and deBruijn’s identities in this case have that \( Z_N = \text{Pf} U \).

When \( L = 1 \) and \( N \) is odd, the additional term on the right hand side of (3) correspond to bordering the matrix \( U \) to produce

\[
U' = \left[ \begin{array}{cccc}
\int_{\mathbb{R}} p_1(x) \, d\nu(x) \\
\vdots \\
-\int_{\mathbb{R}} p_1(x) \, d\nu(x) & \cdots & -\int_{\mathbb{R}} p_N(x) \, d\nu(x) & 0
\end{array} \right],
\]

and in this situation, \( Z_N = \text{Pf} U' \) as is well known [1].

The case where \( \beta = L^2 + 1 \) is even and \( N = 2M \) does not seem to generalize any known situation, though it is apparently applicable when \( \beta = 2 \). The partition functions of \( \beta = 2 \) ensembles are traditionally described in terms of determinants. And, while every determinant can be written trivially as a Pfaffian, the Pfaffian expression which appears for the partition function here does not seem to arise from such a trivial modification. At any rate, when \( \beta = 2 \) we have that

\[
Z_{2M} = \text{Pf} Y,
\]

where

\[
Y = \left[ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} p_m(x)p_n(y) \left( \frac{(y^M - x^M)^2}{y - x} \right) \, d\nu(x) \, d\nu(y) \right]_{m,n=1}^{2M}.
\]

\[
(2.2)
\]

### 2.1 For Circular Ensembles

The proof of Theorem 2.1 is mostly formal, and similar results can be had for other ensembles. For instance, the joint density of Dyson’s circular ensembles [8] is given by

\[
Q_N(\lambda) = \frac{1}{C_N N!} \prod_{m<n} |e^{i\theta_n} - e^{i\theta_m}|^\beta; \quad \theta_1, \theta_2, \ldots, \theta_N \in [-\pi, \pi)
\]

where

\[
C_N = \frac{1}{N!} \int_{[-\pi, \pi)^N} \prod_{m<n} |e^{i\theta_n} - e^{i\theta_m}|^\beta \, d\theta_1 \, d\theta_2 \cdots d\theta_N.
\]

By [16, (11.3.2)],

\[
|e^{i\theta_n} - e^{i\theta_m}| = -ie^{-i(\theta_n + \theta_m)/2} \text{sgn}(\theta_n - \theta_m)(e^{i\theta_n} - e^{i\theta_m}),
\]

\[
|e^{i\theta_n} - e^{i\theta_m}| = -ie^{-i(\theta_n + \theta_m)/2} \text{sgn}(\theta_n - \theta_m)(e^{i\theta_n} - e^{i\theta_m}),
\]
and thus, if we define,
\[ d\mu(\theta) = (-ie^{-i\theta})^{(N-1)/2} \, d\theta, \tag{2.3} \]
then
\[ C_N = \frac{1}{N!} \int_{[-\pi, \pi]^N} \left\{ \prod_{m<n} (e^{i\theta_n} - e^{i\theta_m}) \text{sgn}(\theta_n - \theta_m) \right\}^\beta d\mu_N(\theta). \]

Of course the \( \text{sgn}(\theta_n - \theta_m) \) terms can be ignored when \( \beta \) is even.

An explicit evaluation of \( C_N \) in the case where \( d\mu(\theta) = d\theta/2\pi \), conjectured in [8] and proved by various means in [11, 10, 2, 23].

**Theorem 2.2.** Suppose \( L \) and \( N \) are positive integers with \( \beta = L^2 \) and let \( \omega(x) \) be the L-form given as in (2.1) and suppose \( \mu \) is given as in (2.3). Then, for any complete family of monic polynomials \( p \),

1. if \( \beta = L^2 \)
\[ C_N = \text{PF} \left( \int_{-\pi}^{\pi} \omega(e^{i\theta}) \, d\mu(\theta) \right); \]
2. if \( \beta = L^2 \) is odd and \( N \) is even
\[ C_N = \text{PF} \left( \frac{1}{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \omega(e^{i\theta}) \wedge \omega(e^{i\psi}) \text{sgn}(\psi - \theta) \, d\mu(\theta) \, d\mu(\psi) \right); \]
3. if \( \beta = L^2 \) is odd and \( N \) is odd
\[ C_N = \text{PF} \left( \frac{1}{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \omega(e^{i\theta}) \wedge \omega(e^{i\psi}) \text{sgn}(\psi - \theta) \, d\mu(\theta) \, d\mu(\psi) + \int_{-\pi}^{\pi} \omega(e^{i\theta}) \wedge e^\prime \, d\mu(\theta) \right); \]
4. if \( \beta = L^2 + 1 \) is even and \( N = 2M \) is even,
\[ C_N = \text{PF} \left( \frac{1}{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \omega(e^{i\theta}) \wedge \omega(e^{i\psi}) \left[ \frac{(e^{iM\psi} - e^{iM\theta})^2}{e^{i\psi} - e^{i\theta}} \right] \, d\mu(\theta) \, d\mu(\psi) \right). \]

The proof of this theorem is the same, mutatis mutandis, as that of Theorem 2.1.

### 2.2 In Terms of Moments

For \( j, k \geq 0 \), let the \( k \)th moment of \( \nu \) be given by
\[ M(k) = \int_{\mathbb{R}} x^k \, d\nu(x), \]
and the \( j, k \)th skew-moment of \( \nu \) to be
\[ M(j, k) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} x^j y^k \text{sgn}(y-x) \, d\nu(x) \, d\nu(y). \]

If we set \( p \) to be just the monomials, and define
\[ \Delta t = \prod_{j<k} (t(k) - t(j)) \quad \text{and} \quad \Sigma t = \sum_{\ell=1}^{L} t(\ell), \]
then the Wronskian of $p_t$ is given by [6]

$$\text{Wr}(p_t; x) = \Delta t \left\{ \prod_{\ell=1}^{L} \frac{1}{(t(\ell) - 1)!} \right\} x^{\Sigma t - L(L+1)/2}.$$ 

It follows that

$$\omega(x) = \sum_{t: L > N_L} \Delta t \left\{ \prod_{\ell=1}^{L} \frac{1}{(t(\ell) - 1)!} \right\} x^{\Sigma t - L(L+1)/2} \epsilon_t,$$

$$\int_{\mathbb{R}} \omega(x) d\nu(x) = \sum_{t: L > N_L} \Delta t \left\{ \prod_{\ell=1}^{L} \frac{1}{(t(\ell) - 1)!} \right\} M \left( \Sigma t - \frac{L(L+1)}{2} \right) \epsilon_t.$$

$$\frac{1}{2} \int_{\mathbb{R}^2} \omega(x) \omega(y) \text{sgn}(y - x) d\nu(x) d\nu(y)$$

$$= \sum_{t,u} \Delta t \Delta u \left\{ \prod_{\ell=1}^{L} \frac{1}{(t(\ell) - 1)!(u(\ell) - 1)!} \right\} M \left( \Sigma t - \frac{L(L+1)}{2}, \Sigma u - \frac{L(L+1)}{2} \right) \epsilon_t \wedge \epsilon_u.$$

Similarly,

$$\frac{1}{2} \int_{\mathbb{R}^2} \omega(x) \omega(y) \text{sgn}(y - x) d\nu(x) d\nu(y) + \int_{\mathbb{R}} \omega(x) \wedge \epsilon' d\nu(x)$$

$$= \sum_{t,u} \Delta t \Delta u \left\{ \prod_{\ell=1}^{L} \frac{1}{(t(\ell) - 1)!(u(\ell) - 1)!} \right\} M \left( \Sigma t - \frac{L(L+1)}{2}, \Sigma u - \frac{L(L+1)}{2} \right) \epsilon_t \wedge \epsilon_u$$

$$+ \sum_{t: L > N_L} \Delta t \left\{ \prod_{\ell=1}^{L} \frac{1}{(t(\ell) - 1)!} \right\} M \left( \Sigma t - \frac{L(L+1)}{2} \right) \epsilon_t \wedge \epsilon'.$$

3 A Remark on Correlation Functions

We consider the joint density $\Omega_N(\lambda)$ given as in (0.1), though the results here easily extend to circular and other related ensembles. We will define the measure $d\mu(\lambda) = w(\lambda) d\lambda$ and define the $n$th correlation function is given by

$$R_n(x_1, x_2, \ldots, x_n) = \frac{1}{Z_N^0(N-n)!} \Omega_N(x_1, \ldots, x_n, y_1, \ldots, y_{N-n}) dy_1 \cdots dy_{N-n}.$$

It shall be convenient to abbreviate things so that

$$\Delta(\lambda) = \prod_{m<n} (\lambda_n - \lambda_m); \quad \lambda \in \mathbb{R}^N,$$

and if $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{N-n}$ then

$$x \vee y = (x_1, \ldots, x_n, y_1, \ldots, y_{N-n}).$$
Thus, 

\[ R_n(x) = \frac{1}{Z_N^n(N-n)!} \int_{\mathbb{R}^{N-n}} |\Delta(x \lor y)|^\beta d\mu^{N-n}(y). \]

The \( n \)th correlation function is clearly a renormalized version of the \( n \)th marginal density of \( \Omega_N \). However, this observation belies the importance of correlation functions in random matrix theory, point processes and statistical physics (see for instance [12, 4]).

Given indeterminants \( c_1, c_2, \ldots, c_N \) and real numbers \( x_1, x_2, \ldots, x_N \) we define the measure \( \nu \) on \( \mathbb{R} \) by

\[ d\nu(\lambda) = w(\lambda) \sum_{n=1}^{N} c_n d\delta(\lambda - x_n), \]

where \( \delta \) is the probability measure with unit mass at 0. It is then a straightforward exercise in symbolic manipulation to show that \( Z_{\nu + \mu}^N / Z_{\mu}^N \) is the generating function for the correlation functions. That is, \( R_N(x_1, \ldots, x_n) \) is the coefficient of \( c_1 c_2 \cdots c_n \) in \( Z_{\nu + \mu}^N / Z_{\mu}^N \).

When \( \beta = 1 \) or 4, this observation and the fact that \( Z_{\nu + \mu}^N / Z_{\mu}^N \) is the first step in showing that the correlation functions have a Pfaffian formulation. The simplest derivation is given by Tracy and Widom [22] using the fact that if \( A \) and \( B \) are (perhaps rectangular) matrices for which both \( AB \) and \( BA \) are square, then

\[ \det(I + AB) = \det(I + BA). \]

(3.1)

A similar fact is true for Pfaffians [18] which streamlines the proof [5, 21].

The existence of a hyperpfaffian representation of \( Z_{\nu + \mu}^N \) when \( \beta \) is a square is suggestive of a hyperpfaffian formulation of the correlation functions, however the necessary analog of (3.1) for hyperpfaffians remains unknown.

## 4 The Proof of Theorem 2.1

### 4.1 Case: \( \beta = L^2 \) even

We define the confluent \( NL \times NL \) Vandermonde matrix by first defining the \( NL \times L \) matrix

\[ V(x) = \left[ D^{\ell-1} p_n(x) \right]_{n, \ell=1}^{NL,L}, \]

and then defining

\[ V(\lambda) = \left[ V(\lambda_1) \ V(\lambda_2) \ \cdots \ V(\lambda_N) \right]. \]

The confluent Vandermonde identity has that

\[ \det V(\lambda) = \prod_{m<n} (\lambda_n - \lambda_m)^{L^2} = \prod_{m<n} |\lambda_n - \lambda_m|^\beta. \]

It follows that

\[ Z_N = \frac{1}{N!} \int_{\mathbb{R}^N} \det V(\lambda) d\nu^N(\lambda). \]

We may use Laplace expansion to write \( \det V(\lambda) \) as a sum of \( N \)-fold products of \( L \times L \) determinants, where each determinant appearing in the products is a function of exactly
one of the variables $\lambda_1, \lambda_2, \ldots, \lambda_N$. That is, given $t: L \not\rightarrow NL$, define $V_t(\lambda)$ to be the $L \times L$ matrix,

$$V_t(\lambda) = [D^{L-1}p(t)_{\lambda}]_{n,\ell=1}^L.$$ 

Then, we may write $\det V(\lambda)$ as an alternating sum over products of the form

$$\det V_{t_1}(\lambda_1) \cdot \det V_{t_2}(\lambda_2) \cdot \ldots \cdot \det V_{t_N}(\lambda_N),$$

where $t_1, t_2, \ldots, t_N: L \not\rightarrow NL$ have ranges which are mutually disjoint (or equivalent, the disjoint union of their ranges is all of $NL$). The sign of each term in the sum can be specified by defining $\text{sgn}(t_1, t_2, \ldots, t_N)$ via

$$\epsilon_{t_1} \wedge \epsilon_{t_2} \wedge \cdots \wedge \epsilon_{t_N} = \text{sgn}(t_1, t_2, \ldots, t_N) \cdot \epsilon_{\text{Vol}}.$$ 

We remark that, if $t_1, t_2, \ldots, t_N: L \not\rightarrow NL$ do not have disjoint ranges, then $\text{sgn}(t_1, t_2, \ldots, t_N) = 0$. Thus we can write

$$\det V(\lambda) = \sum_{(t_1, \ldots, t_N)} \text{sgn}(t_1, t_2, \ldots, t_N) \det V_{t_1}(\lambda_1) \cdot \det V_{t_2}(\lambda_2) \cdot \ldots \cdot \det V_{t_N}(\lambda_N), \quad (4.1)$$

and

$$\int_{\mathbb{R}^N} \det V(\lambda) d\nu^N(\lambda) = \sum_{(t_1, \ldots, t_N)} \text{sgn}(t_1, t_2, \ldots, t_N) \prod_{n=1}^N \int_{\mathbb{R}} \det V_{t_n}(\lambda) d\nu(\lambda)$$

$$= \sum_{(t_1, \ldots, t_N)} \text{sgn}(t_1, t_2, \ldots, t_N) \prod_{n=1}^N \int_{\mathbb{R}} \text{Wr}(p_{t_n}; x) d\nu(x)$$

It follows that

$$\frac{1}{N!} \left\{ \int_{\mathbb{R}^N} \det V(\lambda) d\nu^N(\lambda) \right\} \epsilon_{\text{Vol}}$$

$$= \frac{1}{N!} \sum_{(t_1, \ldots, t_N)} \left\{ \prod_{n=1}^N \int_{\mathbb{R}} \text{Wr}(p_{t_n}; x) d\nu(x) \right\} \text{sgn}(t_1, t_2, \ldots, t_N) \epsilon_{\text{Vol}}$$

$$= \frac{1}{N!} \sum_{(t_1, \ldots, t_N)} \left\{ \bigwedge_{n=1}^N \int_{\mathbb{R}} \text{Wr}(p_{t_n}; x) d\nu(x) \epsilon_{t_n} \right\}.$$ 

Finally, exchanging the sum and the wedge product, we find

$$\frac{1}{N!} \left\{ \int_{\mathbb{R}^N} \det V(\lambda) d\nu^N(\lambda) \right\} \epsilon_{\text{Vol}} = \frac{1}{N!} \left\{ \sum_{t_1, \ldots, t_N: NL} \int_{\mathbb{R}} \text{Wr}(p_t; x) d\nu(x) \epsilon_t \right\}^\wedge_N,$$

which proves this case of the theorem.

**4.2 Case: $\beta = L^2$ odd, $N$ even**

When $L$ is odd, the situation is complicated by the fact that

$$\prod_{m < n} |\lambda_n - \lambda_m|^2 = |\det V(\lambda)|.$$
(Note the absolute values). Thus, in this case

\[ Z_N = \frac{1}{N!} \int \mathbb{R}^N \| \det \mathbf{V}(\lambda) \| d\nu^N(\lambda). \]

This complication is eased by the observation, [7, Eq. 5.3], that if

\[ T(\lambda) = [\text{sgn}(\lambda_n - \lambda_m)]_{m,n=1}^N, \quad \text{then} \quad \text{Pf} \ T(\lambda) = \prod_{m<n} \text{sgn}(\lambda_n - \lambda_m), \quad (4.2) \]

and thus

\[ Z_N = \frac{1}{N!} \int \mathbb{R}^N \det \mathbf{V}(\lambda) \text{Pf} \ T(\lambda) d\nu^N(\lambda). \]

We remark, it is at this point that we require \( N \) be even, since the Pfaffian of \( T(\lambda) \) is not defined when \( N \) is odd.

Using (4.1), we have

\[ Z_N = \frac{1}{N!} \sum_{t_1, t_2, \ldots, t_N} \text{sgn}(t_1, t_2, \ldots, t_N) \int \mathbb{R}^N \left\{ \prod_{n=1}^N \det V_{t_n}(\lambda_n) \right\} \text{Pf} \ T(\lambda) d\nu^N(\lambda), \]

where as before the sum is over \( N \)-tuples of functions \( L^N \). Now, if \( N = 2M \), then

\[ \text{Pf} \ T(\lambda) = \frac{1}{M!2^M} \sum_{\sigma \in S_N} \text{sgn} \sigma \prod_{m=1}^M \text{sgn}(\lambda_{\sigma(2m)} - \lambda_{\sigma(2m-1)}), \]

and

\[ \prod_{n=1}^N \det V_{t_n}(\lambda_n) = \prod_{m=1}^M \det V_{t_{\sigma(2m-1)}}(\lambda_{\sigma(2m-1)}) \det V_{t_{\sigma(2m)}}(\lambda_{\sigma(2m)}). \]

Thus,

\[ Z_N = \frac{1}{N!M!2^M} \sum_{\sigma \in S_N} \sum_{t_1, t_2, \ldots, t_N} \text{sgn} \sigma \text{sgn}(t_1, t_2, \ldots, t_N) \]

\[ \times \int \mathbb{R}^N \left\{ \prod_{m=1}^M \det V_{t_{\sigma(2m-1)}}(\lambda_{\sigma(2m-1)}) \det V_{t_{\sigma(2m)}}(\lambda_{\sigma(2m)}) \right\} \times \text{sgn}(\lambda_{\sigma(2m)} - \lambda_{\sigma(2m-1)}) d\nu^N(\lambda) \]

\[ = \frac{1}{N!M!} \sum_{\sigma \in S_N} \sum_{t_1, t_2, \ldots, t_N} \text{sgn} \sigma \text{sgn}(t_1, t_2, \ldots, t_N) \]

\[ \times \left\{ \prod_{m=1}^M \frac{1}{2} \int \mathbb{R}^2 \det V_{t_{\sigma(2m-1)}}(x) \det V_{t_{\sigma(2m)}}(y) \text{sgn}(y - x) d\nu(x) d\nu(y) \right\}. \]

Next, we notice that \( S_N \) acts on the \( N \)-tuples \( (t_1, t_2, \ldots, t_N) \) by permutation, and if

\[ (u_1, u_2, \ldots, u_N) = (t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(N)}), \]

then

\[ \text{sgn}(u_1, u_2, \ldots, u_N) = \text{sgn}(\sigma) \text{sgn}(t_1, t_2, \ldots, t_N). \]
Moreover, the action of each $\sigma$ produces a bijection on the set of $N$-tuples, and thus

$$Z_N = \frac{1}{N!M!} \sum_{\sigma \in S_N} \sum_{u_1, u_2, \ldots, u_N} \text{sgn}(u_1, u_2, \ldots, u_N)$$

$$\times \left\{ \prod_{m=1}^M \frac{1}{2} \int_{\mathbb{R}^2} \det \mathbf{V}_{u_{2m-1}}(x) \det \mathbf{V}_{u_{2m}}(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\}.$$  

The summand is independent of $\sigma$, and thus

$$Z_N = \frac{1}{M!} \sum_{v_1, v_2, \ldots, v_M} \text{sgn}(u_1, u_2, \ldots, u_N)$$

$$\times \left\{ \prod_{m=1}^M \frac{1}{2} \int_{\mathbb{R}^2} \det \mathbf{V}_{u_{2m-1}}(x) \det \mathbf{V}_{u_{2m}}(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\}.$$  

Now, every $N$-tuple $(u_1, u_2, \ldots, u_N)$ can be obtained from an $M$-tuple $(v_1, v_2, \ldots, v_M)$ of functions $2L \nearrow 2LM$ and an $M$-tuple $(w_1, w_2, \ldots, w_M)$ of functions $L \nearrow 2L$ specified by

$$u_{2m-1} = v_m \circ w_m \quad \text{and} \quad u_{2m} = v_m \circ w'_m,$$

where $w'_m$ is the unique function $L \nearrow 2L$ whose range is disjoint from $w_m$. Defining the sign of each of the $w_m$ by

$$\epsilon_{w_m} \land \epsilon_{w'_m} = \text{sgn} w_m (e_1 \land e_2 \land \cdots \land e_{2L}),$$

we have

$$\text{sgn}(u_1, u_2, \ldots, u_N) = \text{sgn}(v_1, v_2, \ldots, v_M) \prod_{m=1}^M \text{sgn} w_m.$$  

Using this decomposition,

$$Z_N = \frac{1}{M!} \sum_{v_1, v_2, \ldots, v_M} \text{sgn}(v_1, v_2, \ldots, v_M)$$

$$\times \left\{ \prod_{m=1}^M \frac{1}{2} \int_{\mathbb{R}^2} \det \mathbf{V}_{v_m \circ w_m}(x) \det \mathbf{V}_{v_m \circ w'_m}(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\}$$

$$= \frac{1}{M!} \sum_{v_1, v_2, \ldots, v_M} \text{sgn}(v_1, v_2, \ldots, v_M)$$

$$\times \left\{ \prod_{m=1}^M \frac{1}{2} \sum_{w_m \nearrow 2L} \text{sgn} w_m \int_{\mathbb{R}^2} \det \mathbf{V}_{v_m \circ w_m}(x) \det \mathbf{V}_{v_m \circ w'_m}(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\}.$$
and consequently,

\[ Z_N\epsilon_{\text{vol}} = \frac{1}{M!} \sum_{v_1, v_2, \ldots, v_M} \left\{ \bigwedge_{m=1}^M \left[ \frac{1}{2} \sum_{\nu \in L \not\supset 2L} \text{sgn } \nu \int_{\mathbb{R}^2} \det V_{v_m \circ \nu}(x) \det V_{v_m \circ \nu'}(y) \text{sgn}(y - x) d\nu(x) d\nu(y) \right] \epsilon_{v_m} \right\} \]

\[ = \frac{1}{M!} \left( \frac{1}{2} \sum_{\nu \in L \not\supset 2L} \sum_{\nu' \in L \not\supset 2L} \text{sgn } \nu \int_{\mathbb{R}^2} \det V_{v \circ \nu}(x) \det V_{v \circ \nu'}(y) \text{sgn}(y - x) d\nu(x) d\nu(y) \right)^M. \]

Thus, substituting \( \text{Wr}(p_{\text{com}}; x) = \det V_{\text{com}}(x) \) (and likewise for \( \text{Wr}(p_{\text{com}'}; y) \)) we have \( Z_N \) equal to

\[ \text{PF} \left( \frac{1}{2} \sum_{\nu \in L \not\supset 2LM} \sum_{\nu' \in L \not\supset 2LM} \text{sgn } \nu \int_{\mathbb{R}^2} \text{Wr}(p_{\text{com}}; x) \text{Wr}(p_{\text{com}'}; y) \text{sgn}(y - x) d\nu(x) d\nu(y) \right) \epsilon_{v}. \]

Next, we use the fact that \( \epsilon_v = \text{sgn } \nu \epsilon_{\text{com}} \land \epsilon_{\text{com}'} \) to write

\[ \frac{1}{2} \sum_{\nu \in L \not\supset 2LM} \sum_{\nu' \in L \not\supset 2LM} \text{sgn } \nu \int_{\mathbb{R}^2} \text{Wr}(p_{\text{com}}; x) \text{Wr}(p_{\text{com}'}; y) \text{sgn}(y - x) d\nu(x) d\nu(y) \right] \epsilon_v \]

\[ = \frac{1}{2} \sum_{\nu \in L \not\supset 2LM} \sum_{\nu' \in L \not\supset 2LM} \left( \int_{\mathbb{R}^2} \text{Wr}(p_{\text{com}}; x) \text{Wr}(p_{\text{com}'}; y) \text{sgn}(y - x) d\nu(x) d\nu(y) \right) \epsilon_{\text{com}} \land \epsilon_{\text{com}'} \cdot \]

Finally, we see that as \( v \) varies over \( 2L \not\supset 2LM \) and \( \nu \) varies over \( L \not\supset 2L \), \( t = v \circ \nu \) and \( u = v \circ \nu' \) vary over pairs in \( L \not\supset 2LM \) with disjoint ranges. However, since \( \epsilon_t \land \epsilon_u = 0 \) if the ranges of \( t \) and \( u \) are not disjoint, we may replace the double sum in (4.3) with a double sum over \( L \not\supset 2LM \). That is, (4.3) equals

\[ \frac{1}{2} \sum_{t \in L \not\supset 2LM} \sum_{u \in L \not\supset 2LM} \left( \int_{\mathbb{R}^2} \text{Wr}(p_t; x) \text{Wr}(p_u; y) \text{sgn}(y - x) d\nu(x) d\nu(y) \right) \epsilon_t \land \epsilon_u \]

\[ = \frac{1}{2} \int_{\mathbb{R}^2} \left( \sum_{t \in L \not\supset 2LM} \sum_{u \in L \not\supset 2LM} \text{Wr}(p_t; x) \text{Wr}(p_u; y) \text{sgn}(y - x) \epsilon_t \land \epsilon_u \right) d\nu(x) d\nu(y) \]

\[ = \frac{1}{2} \int_{\mathbb{R}^2} \omega(x) \land \omega(y) \text{sgn}(y - x) d\nu(x) d\nu(y), \]

as desired.
4.3 Case: $\beta = L^2$ odd, $N$ odd

In this section we will assume that $N = 2K - 1$ is odd and for $\lambda \in \mathbb{R}^N$ we will introduce the $2K \times 2K$ matrix $T'(\lambda)$ by

$$T'(\lambda) = \begin{bmatrix} T(\lambda) & 1 \\ 1 & -T(\lambda) \end{bmatrix}.$$ 

That is

$$T'(\lambda) = T(\lambda \lor \infty) = \lim_{t \to \infty} T(\lambda \lor t),$$

where, for instance, $(\lambda \lor t) = (\lambda_1, \ldots, \lambda_N, t) \in \mathbb{R}^{N+1}$. From (4.2),

$$\text{PF } T'(\lambda) = \prod_{1 \leq m < n \leq N} \text{sgn}(\lambda_n - \lambda_m).$$

Here we will write

$$\Pi_{2K} = \{ \sigma \in S_{2K} : \sigma(2k-1) < \sigma(2k) \text{ for } k = 1, 2, \ldots, K \},$$

and expand the Pfaffian of $U(\lambda)$ as

$$\text{PF } U(\lambda) = \frac{1}{K!} \sum_{\sigma \in \Pi_{2K}} \text{sgn } \prod_{k=1}^{K} t_{\sigma(2k-1), \sigma(2k)}.$$

For each $\sigma \in \Pi_{2K}$ there exists $k_\sigma \in K$ such that $\sigma(2k_\sigma) = 2K$, and hence, $t_{\sigma(2k_\sigma)-1, \sigma(2k_\sigma)} = 1$. That is,

$$\text{PF } U(\lambda) = \frac{1}{K!} \sum_{\sigma \in \Pi_{2K}} \text{sgn } \prod_{k=1}^{K} \text{sgn}(\lambda_{\sigma(2k)} - \lambda_{\sigma(2k-1)}).$$

Using this, and following the outline of the $L$ odd $N$ even case, we find

$$Z_N = \frac{1}{N!K!} \sum_{\sigma \in \Pi_{2K}} \sum_{t_1, t_2, \ldots, t_N} \text{sgn } \text{sgn}(t_1, t_2, \ldots, t_N) \int_{\mathbb{R}^N} \text{Wr}(p_{\tau_{\sigma(2k-1)}}; \lambda_{\sigma(2k-1)})$$

$$\times \left\{ \prod_{k=1}^{K} \text{Wr}(p_{\tau_{\sigma(2k-1)}}; \lambda_{\sigma(2k-1)}) \text{Wr}(p_{\tau_{\sigma(2k)}}; \lambda_{\sigma(2k)}) \text{sgn}(\lambda_{\sigma(2k)} - \lambda_{\sigma(2k-1)}) \right\} d\nu^N(\lambda),$$

where, as before, $(t_1, t_2, \ldots, t_N)$ is an $N$-tuple of functions $L \not\supset NL$. Fubini’s Theorem then yields

$$Z_N = \frac{1}{N!K!} \sum_{\sigma \in \Pi_{2K}} \sum_{t_1, t_2, \ldots, t_N} \text{sgn } \text{sgn}(t_1, t_2, \ldots, t_N) \int_{\mathbb{R}} \text{Wr}(p_{\tau_{\sigma(2k-1)}}; x) d\nu(x)$$

$$\times \left\{ \prod_{k=1}^{K} \int_{\mathbb{R}} \text{Wr}(p_{\tau_{\sigma(2k-1)}}; x) \text{Wr}(p_{\tau_{\sigma(2k)}}; y) \text{sgn}(y - x) d\nu(x) d\nu(y) \right\}.$$
Now, given $\sigma \in \Pi_{2K}$, let $\tilde{\sigma} \in \Pi_{2K}$ be the permutation given by
\[
\tilde{\sigma}(n) = \begin{cases} 
\sigma(n) & \text{if } n \leq 2k_{\sigma}; \\
\sigma(n + 2) & \text{if } 2k_{\sigma} + 2 \leq n \leq 2K - 2; \\
\sigma(2k_{\sigma} - 1) & \text{if } n = 2K - 1; \\
2K & \text{if } n = 2K.
\end{cases}
\]
That is, $\tilde{\sigma}$ is the permutation whose range as an ordered set is equal to that formed from the range of $\sigma$ by 'moving' $\sigma(2k_{\sigma} - 1)$ and $\sigma(2k_{\sigma})$ to the 'end.' Clearly $\text{sgn } \tilde{\sigma} = \text{sgn } \sigma$, and each $\tilde{\sigma}$ corresponds to exactly $K$ distinct $\sigma \in \Pi_{2K}$. It follows that
\[
Z_N = \frac{1}{N!K!} \sum_{\sigma \in \Pi_{2K}} \sum_{t_1, t_2, \ldots, t_N} \text{sgn } \tilde{\sigma} \text{ sg}(t_1, t_2, \ldots, t_N) \int_{\mathbb{R}} \text{Wr}(p_{t_2(2k-1)}; x) d\nu(x)
\times \left\{ \prod_{k=1}^{K-1} \int_{\mathbb{R}^2} \text{Wr}(p_{t_2(2k-1)}; x) \text{Wr}(p_{t_1(2k)}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\}
\]
Next, denote the transpositions $\tau_k = (2k - 1, 2k)$ for $k = 1, 2, \ldots, K - 1$ and let $G_{2K-2}$ be the group of permutations generated by the $\tau_k$. Clearly the cardinality of $G_{2K-2}$ is $2^{K-1}$, and moreover the map
\[
\Pi_{2K} \times G_{2K-2} \to \{ \pi \in S_{2K} : \pi(2K) = 2K \}
\]
given by $(\sigma, \tau) \mapsto \tilde{\sigma} \circ \tau$ is a $K$ to one map. Clearly, the right hand set is in correspondence with $S_N$. Now, if $\pi = \tilde{\sigma} \circ \tau$ for some $\sigma \in \Pi_{2K}$ and $\tau \in G_{2K-2}$ then,
\[
\text{sgn } \pi \left\{ \prod_{k=1}^{K-1} \int_{\mathbb{R}^2} \text{Wr}(p_{t_2(2k-1)}; x) \text{Wr}(p_{t_1(2k)}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\}
\int_{\mathbb{R}} \text{Wr}(p_{t_1(2K-1)}; x) d\nu(x)
\times \int_{\mathbb{R}} \text{Wr}(p_{t_{\tau(2K-1)}}; x) d\nu(x),
\]
the factors of $-1$ introduced into $\pi$ by the transpositions in $\tau$ being compensated by the fact that the double integral swaps sign when the arguments are swapped. It follows that we may replace the sum over $\Pi_{2K}$ with a sum over $S_N$ so long as we compensate for the cardinality of $G_{2K-2}$ and the fact that the map is $K$ to 1. That is,
\[
Z_N = \frac{1}{N!(K-1)!} \sum_{\pi \in S_N} \sum_{t_1, t_2, \ldots, t_N} \text{sgn } \pi \text{ sg}(t_1, t_2, \ldots, t_N)
\times \left\{ \prod_{k=1}^{K-1} \int_{\mathbb{R}^2} \text{Wr}(p_{t_2(2k-1)}; x) \text{Wr}(p_{t_{\pi(2k)}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\}
\times \int_{\mathbb{R}} \text{Wr}(p_{t_{\pi(K)}}; x) d\nu(x).
\]
As before
\[ \text{sgn } \pi \text{ sgn}(t_1, t_2, \ldots, t_N) = \text{sgn}(t_{\pi(1)}, t_{\pi(2)}, \ldots, t_{\pi(N)}) , \]
and the action of any particular \( \pi \in S_N \) on the \( N \)-tuple \((t_1, t_2, \ldots, t_N)\) is a bijection on the set of such \( N \)-tuples. Thus, we may eliminate the sum over \( S_N \) so long as we compensate by \( N! \). That is,
\[ Z_N = \frac{1}{(K-1)!} \sum_{t_1, t_2, \ldots, t_N} \text{sgn}(t_1, t_2, \ldots, t_N) \]
\[ \times \left\{ \prod_{k=1}^{K-1} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(p_{L_{2k-1}}; x) \text{Wr}(p_{L_{2k}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\} \]
\[ \times \int_{\mathbb{R}} \text{Wr}(p_{L_N}; x) d\nu(x) . \]

With a slight modification of the argument in the \( L \) odd \( N \) even case, we set \( t = t_N \) and write \((t_1, t_2, \ldots, t)\) as
\[ (t' \circ v_1 \circ w_1, t' \circ v_1 \circ w'_1, \ldots, t' \circ v_{K-1} \circ w_{K-1}, t' \circ v_{K-1} \circ w'_{K-1}, t) , \quad (4.6) \]
where \((v_1, v_2, \ldots, v_{K-1})\) is a \( K-1 \)-tuple of functions \( 2L \not> (N-1)L \) the union of ranges of which is all of \( L(N-1) \), and \((w_1, w_2, \ldots, w_{K-1})\) is a \( K-1 \)-tuple of functions \( L \not> 2L \). In words, (4.6) is formed by first forming a partition of \((N-1)L\) into \( L \) disjoint subsets given by the ranges of the \( v \)'s. The \( w \)'s serve to divide each of these sets into two equal sized sets giving a partition of \((N-1)L\) into \( 2L \) disjoint subsets. Finally, \( t' \) is the function from \( N(L-1) \not> NL \) whose range is disjoint from \( t_N \). Each \((t_1, t_2, \ldots, t_N)\) has a unique representation of this form, and
\[ \text{sgn}(t_1, t_2, \ldots, t_{N-1}, t) = \text{sgn} t' \text{ sgn}(v_1, v_2, \ldots, v_{K-1}) \prod_{k=1}^{K-1} \text{sgn } w_k . \]

With these observations, we may rewrite (4.5) as
\[ Z_N = \frac{1}{(K-1)!} \sum_{t' \in L} \int_{\mathbb{R}^2} \text{Wr}(p_t; x) d\nu(x) \sum_{v_1, \ldots, v_{K-1}} \sum_{w_1, \ldots, w_{K-1}} \text{sgn}(v_1, v_2, \ldots, v_{K-1}) \]
\[ \times \left\{ \prod_{k=1}^{K-1} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(p_{t' \circ v_{K-1} \circ w_k}; x) \text{Wr}(p_{t' \circ v_{K-1} \circ w'_k}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\} , \]
and thus, since \( \epsilon_{\text{vol}} = \text{sgn } t' \epsilon_t \wedge \epsilon_t , \)
\[ Z_N \epsilon_{\text{vol}} = \frac{1}{(K-1)!} \sum_{t' \in L} \int_{\mathbb{R}^2} \text{Wr}(p_t; x) d\nu(x) \sum_{v_1, \ldots, v_{K-1}} \sum_{w_1, \ldots, w_{K-1}} \text{sgn}(v_1, v_2, \ldots, v_{K-1}) \]
\[ \times \prod_{k=1}^{K-1} \left\{ \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(p_{t' \circ v_{K-1} \circ w_k}; x) \text{Wr}(p_{t' \circ v_{K-1} \circ w'_k}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\} \epsilon_{t'} \wedge \epsilon_t , \]
Now,
\[ \epsilon_{t'} = \text{sgn}(v_1, v_2, \ldots, v_{K-1}) \left\{ \prod_{k=1}^{K-1} \text{sgn } w_k \right\} \bigwedge_{k=1}^{K-1} \epsilon_{t' \circ v_{K-1} \circ w_k} \wedge \epsilon_{t' \circ v_{K-1} \circ w'_k} . \]
and thus,

\[ Z_N \epsilon_{\text{vol}} = \frac{1}{(K-1)!} \sum_{t:L \not\supseteq NL} \left[ \sum_{b_1, \ldots, b_K \in \mathbb{R}} \sum_{m_1, \ldots, m_K = 1}^{K-1} \left\{ \sum_{k=1}^{K-1} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(p_{t+b_k m_k}; x) \text{Wr}(p_{t+b_k m_k}; y) \right. \right. \]

\[ \times \text{sgn}(y-x) d\nu(x) d\nu(y) \epsilon_{t+b_k m_k} \wedge \epsilon_{t+b_k m_k}' \left. \left. \right\} \right\} \wedge \int_{\mathbb{R}} \text{Wr}(p_t; x) d\nu(x) \epsilon_t \right].\]

Using more-or-less the same maneuvers as in the \( L \) odd, \( N \) even case, we may exchange the sums and the \( K-1 \)-fold wedge product to find

\[ Z_N \epsilon_{\text{vol}} = \sum_{t:L \not\supseteq NL} \frac{1}{(K-1)!} \left( \sum_{s,u:L \not\supseteq (N-1)L} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(p_{s+u}; x) \text{Wr}(p_{s+u}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \epsilon_{s+u} \wedge \epsilon_{s+u}' \right)^{(K-1)} \]

\[ \wedge \int_{\mathbb{R}} \text{Wr}(p_t; x) d\nu(x) \epsilon_t, \]

and thus,

\[ Z_N \epsilon_{\text{vol}}' = \sum_{t:L \not\supseteq NL} \frac{1}{(K-1)!} \left( \sum_{s,u:L \not\supseteq (N-1)L} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(p_{s+u}; x) \text{Wr}(p_{s+u}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \epsilon_{s+u} \wedge \epsilon_{s+u}' \right)^{(K-1)} \]

\[ \wedge \int_{\mathbb{R}} \text{Wr}(p_t; x) d\nu(x) \epsilon_t \wedge \epsilon', \]

Next, we notice that we may extend the sum in the \( K-1 \)-wedge product to all basic forms of the form \( \epsilon_s \wedge \epsilon_u \), since if the range of \( s \) or \( u \) has nontrivial intersection with that of \( t \), wedging by \( \epsilon_t \wedge \epsilon' \) will cause that term to vanish is the final expression. That is,

\[ Z_N \epsilon_{\text{vol}}' = \sum_{t:L \not\supseteq NL} \frac{1}{(K-1)!} \left( \sum_{s,u:L \not\supseteq (N-1)L} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(p_{s+u}; x) \text{Wr}(p_{s+u}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \epsilon_s \wedge \epsilon_u \right)^{(K-1)} \]

\[ \wedge \int_{\mathbb{R}} \text{Wr}(p_t; x) d\nu(x) \epsilon_t \wedge \epsilon'. \]

Now, since the \( K-1 \)-fold wedge product is independent of \( t \) we may factor it out of the
sum to write $Z_N \epsilon_{vol}$ as

$$
\frac{1}{(K-1)!} \left( \sum_{s,u \in \mathcal{L} \cap N \mathcal{L}} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(p_s; x) \text{Wr}(p_u; y) \text{sgn}(y - x) \, d\nu(x) \, d\nu(y) \right) \epsilon_s \wedge \epsilon_u \wedge (K-1)
\wedge \sum_{t \in \mathcal{L} \cap N \mathcal{L}} \int_{\mathbb{R}} \text{Wr}(p_t; x) \epsilon_t \wedge \epsilon'.
$$

$$= \frac{1}{(K-1)!} \left( \frac{1}{2} \int_{\mathbb{R}^2} \omega(x) \wedge \omega(y) \text{sgn}(y - x) \, d\nu(x) \, d\nu(y) \right) \wedge \int_{\mathbb{R}} \omega(x) \wedge \epsilon' \, d\nu(x),$$

where the last equation follows from the definition of $\omega$.

Finally, using the binomial theorem,

$$\frac{1}{K!} \left( \frac{1}{2} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \text{sgn}(y - x) \, d\nu(x) \, d\nu(y) + \int_{\mathbb{R}} \omega(x) \wedge \epsilon' \, d\nu(x) \right) \wedge (K-k)
\wedge \left( \int_{\mathbb{R}} \omega(x) \wedge \epsilon' \, d\nu(x) \right)^k.$$
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