EXPLICIT RENAMING OF BOUND VARIABLES

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ABSTRACT. We present the lambda calculus $\lambda W$ with explicit substitutions and named variables. The characteristic feature of this calculus is as follows: renaming of bound variables when performing substitutions is done using special reductions and may be delayed.

1. Introduction

There is a gap between lambda calculi with explicit substitutions using De Bruijn indices and lambda calculi with explicit substitutions using ordinary (named) variables. The first follow the spirit of category theory. The second attempt to reflect the “real way to work with bound variables”. We clarify this with an example. Simultaneous substitution will be denoted by

\[ [x_1/N_1, x_2/N_2, \ldots, x_k/N_k] \]

Let’s call this substitution $s$. Suppose the variable $x$ is different from all $x_1, x_2, \ldots, x_k$. By $[s, x/N]$ denote the substitution

\[ [x_1/N_1, x_2/N_2, \ldots, x_k/N_k, x/N] \]

According to [5], the substitution $s$ moves under a binder this way

\[(\lambda x. M)[s] \rightarrow \lambda y. (M[s, x/y])\]

where $y$ is a “fresh” variable. The similar reduction for categorical combinators is

\[ \Lambda(M) \circ s \rightarrow \Lambda(M \circ (s \circ F, S)) \]

where $F$ denotes the first projection and $S$ denotes the second projection. A significant difference is that in the latter case the substitution $s$ is multiplied by the first projection. Abadi, Cardelli, Curien, and Levy in [1] suggested to use the substitution $\uparrow$, corresponding to the first projection, together with named variables. They have obtained the equality

\[(\lambda x. M)[s] = \lambda x. (M[(x/x) \cdot (s \circ \uparrow)])\]

We rewrite this equality as

\[(\lambda x. M)[s] \rightarrow \lambda x. (M[s \circ \uparrow, x/x])\]

Abadi, Cardelli, Curien, and Levy write “In this notation, intuitively, $x[\uparrow]$ refers to $x$ after the first binder.” To clarify this point, consider some typed calculus with contexts, where contexts are finite lists of the form $x_1 : A_1, x_2 : A_2, \ldots, x_k : A_k$, where $A_1, A_2, \ldots, A_k$ are types and repetitions
of variables are permitted. A judgement of the form $\Gamma \vdash x : A$ means “the rightmost occurrences of the variable $x$ in the context $\Gamma$ has type $A$.” For example, the judgement $x : A, x : B \vdash x : B$ is true, but the judgement $x : A, x : B \vdash x : A$ is not true. But the judgement $x : A, x : B \vdash x[^\uparrow] : A$ is true. The crucial idea is this: if we allow repetitions of identical variables as in $\lambda x.\lambda x.M$, then we must allow repetitions in contexts too. In this way we will obtain some lambda calculus with explicit substitutions and named variables such that:

1. It is close to the calculi of categorical combinators;
2. It is convenient to work;
3. Renaming of bound variables when performing substitutions is done using special reductions and may be delayed.

Now we must introduce a convenient notation. To give a definition of free variables it is much more convenient to use the notation $[s]M$ than $M[s]$. Substitutions should be on the same side where contexts and binders are. Composition of substitutions also will be written in the reverse order (we will write $q \circ s$ where it was written $s \circ q$). For example, the rewrite rule

$$M[s][q] \rightarrow M[s \circ q]$$

will now look like this

$$[q][s]M \rightarrow [q \circ s]M$$

Now we can write far fewer parentheses. For example, $[s]\lambda x.[q]\lambda y.M$ is uniquely deciphered as $[s](\lambda x.([q](\lambda y.M)))$. I chose the notation $s \circ M$ instead of $[s]M$, because $s \circ \lambda x.q \circ \lambda y.M$ is easy to read, this notation is close to the notation of category theory, and we can now use angle brackets to denote ordered pairs and nothing else ($id \circ M$ looks better than $(id)M$).

After some doubts I have replaced the symbol $\uparrow$ by $\mathcal{W}$. We will have to supply this symbol with a subscript, and $\langle \mathcal{W}_x \circ \mathcal{W}_y \mathcal{Z}, \mathcal{W}_z \circ \mathcal{Z} \mathcal{Z} \rangle$ is much easier to read than $\langle \uparrow_x \circ \uparrow_y, \uparrow_z \circ \mathcal{Z} \mathcal{Z} \rangle$. The symbols $\mathcal{W}_x$ correspond to $\mathcal{W}_x$ from [3] to some extent, but are not the same.

The sets of untyped terms and substitutions are defined inductively as follows:

$$M, N :: = x | MN | \lambda x.M | s \circ M$$

$$s, q :: = id | \mathcal{W} | \langle s, N \setminus x \rangle | s \circ q$$

where the symbol $x$ denotes an arbitrary variable.

The sets of typed terms and substitutions are defined inductively as follows:

$$M, N :: = x | MN | \lambda x^A.M | s \circ M$$

$$s, q :: = id | \mathcal{W} | \langle s, N \setminus x \rangle | s \circ q$$

where $A$ is an arbitrary type.

A usual simultaneous substitution

$$[x_1/N_1, x_2/N_2, \ldots, x_k/N_k]$$
in the new notation looks like
\[ \langle \ldots \langle \text{id}, N_1 \backslash x_1 \rangle, N_2 \backslash x_2 \rangle, \ldots N_k \backslash x_k \rangle \]

For brevity, we will write
\[ \langle \text{id}, N_1 \backslash x_1, N_2 \backslash x_2, \ldots, N_k \backslash x_k \rangle \]

But now any two (or more) of the variables \( x_1, \ldots, x_k \) may coincide (as in contexts).

A judgement is an expression of the form \( \Gamma \vdash M : A \) or of the form \( \Gamma \vdash s \triangleright \Delta \), where \( \Gamma \) and \( \Delta \) are contexts, \( A \) is a type, \( M \) is a term, and \( s \) is a substitution.

**Definition 1.1.** (Typing rules).

(i) \( \Gamma, x : A \vdash x : A \)

(ii) \[ \frac{\Gamma \vdash x : A}{\Gamma, y : B \vdash x : A} \quad (x \neq y) \]

(iii) \[ \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \]

(iv) \[ \frac{\Gamma \vdash \lambda x^A.M : A \rightarrow B}{\Gamma \vdash s \triangleright \Delta \quad \Delta \vdash M : A}{\Gamma \vdash s \circ M : A} \]

(vi) \[ \Gamma \vdash id \triangleright \Gamma \]

(vii) \[ \frac{\Gamma, x : A \vdash W \triangleright \Gamma}{\Gamma \vdash s \triangleright \Delta \quad \Delta \vdash N : A}{\Gamma \vdash \langle s, N \backslash x \rangle \triangleright \Delta, x : A} \]

(viii) \[ \frac{\Gamma \vdash \langle s, N \backslash x \rangle \triangleright \Delta, x : A}{\Gamma \vdash s \triangleright \Delta \quad \Delta \vdash q \triangleright \Sigma}{\Gamma \vdash s \circ q \triangleright \Sigma} \]

The restriction in the rule (ii) is necessary because \( \Gamma \vdash x : A \) means “the rightmost occurrences of the variable \( x \) in the context \( \Gamma \) has type \( A \).”

**Example 1.2.**
\[ x : A, x : B \vdash x : B \]
\[ x : A, x : B, y : C \vdash x : B \]

**Example 1.3.**
\[ x : A, x : B \vdash W \triangleright x : A \]
\[ x : A \vdash x : A \]
\[ x : A, x : B \vdash W \circ x : A \]
Example 1.4.
\[ x : A, x : B, y : C \vdash W \triangleright x : A, x : B \quad x : A, x : B \vdash W \triangleright x : A \]

\[ x : A, x : B, y : C \vdash W \circ W \triangleright x : A \]

\[ x : A \vdash x : A \]

Example 1.5.
\[ x : A, x : B \vdash x : B \quad x : A \vdash \lambda x^B.x : B \rightarrow B \]

\[ \vdash \lambda x^A.\lambda x^B.x : A \rightarrow (B \rightarrow B) \]

There are no weakening rules except the rule (ii). But now we have an explicit weakening. For example, we can derive \( \Gamma, y : B \vdash W \circ M : A \) from \( \Gamma \vdash M : A \)

Example 1.6.
\[ \vdash \Gamma, y : B \vdash W \triangleright \Gamma \quad \Gamma \vdash M : A \]

\[ \Gamma, y : B \vdash W \circ M : A \]

If the variable \( y \) does not occur in the context \( \Gamma \), then \( W \circ M \) reduces to \( M \) in some sense (more precisely, \( W \circ M \) and \( M \) have a common reduct).

The typing rules have a pleasant property: every derivable judgement has a unique derivation. This is not true for the usual typing rules because of weakening rules. This pleasant property allows us to determine uniquely the value of any judgement in some cartesian closed category by induction over the derivation. Assume that some objects are assigned to types. To each context of the form
\[ x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n \]
we assign the object
\[ (\ldots (1 \times A_1) \times A_2) \times \ldots \times A_n) \]
where \( 1 \) is the (canonical) terminal object.

Denote by \( A \xrightarrow{f \circ g} C \) the composition of \( A \xrightarrow{f} B \) and \( B \xrightarrow{g} C \).

To any derivable judgement of the form \( \Gamma \vdash M : A \) we put in correspondence some arrow from \( \Gamma \) to \( A \).

To any derivable judgement of the form \( \Gamma \vdash s \triangleright \Delta \) we put in correspondence some arrow from \( \Gamma \) to \( \Delta \).

Definition 1.7. \((\Gamma \vdash M : A) \Rightarrow \Gamma \xrightarrow{f} A\) is shorthand for “the arrow \( \Gamma \xrightarrow{f} A \) corresponds to the judgement \( \Gamma \vdash M : A \).”

\((\Gamma \vdash s \triangleright \Delta) \Rightarrow \Gamma \xrightarrow{f} \Delta\) is shorthand for “the arrow \( \Gamma \xrightarrow{f} \Delta \) corresponds to the judgement \( \Gamma \vdash s \triangleright \Delta \).”
Definition 1.8. (Values of derivable judgements in cartesian closed categories).

\( (i) \quad (\Gamma, x : A \vdash x : A) \Rightarrow \Gamma \times A \overset{pr_2}{\rightarrow} A \)

\( (ii) \quad (\Gamma \vdash x : A) \Rightarrow \Gamma \overset{f}{\rightarrow} A \)

\( (\Gamma, y : B \vdash x : A) \Rightarrow \Gamma \times B \overset{pr_1 \circ f}{\rightarrow} A \)

\( (iii) \quad (\Gamma \vdash M : A \rightarrow B) \Rightarrow \Gamma \overset{f}{\rightarrow} B^A \)

\( (\Gamma \vdash N : A) \Rightarrow \Gamma \overset{g}{\rightarrow} A \)

\( (\Gamma \vdash M N : B) \Rightarrow \Gamma \overset{(f, g) \circ Ev}{\rightarrow} B \)

\( (iv) \quad (\Gamma, x : A \vdash M : B) \Rightarrow \Gamma \overset{f}{\rightarrow} B \)

\( (\Gamma \vdash \lambda x^A.M : A \rightarrow B) \Rightarrow \Gamma \overset{A(f)}{\rightarrow} B^A \)

\( (v) \quad (\Gamma \vdash s \triangleright \Delta) \Rightarrow \Gamma \overset{f}{\rightarrow} \Delta \)

\( (\Delta \vdash M : A) \Rightarrow \Delta \overset{g}{\rightarrow} A \)

\( (\Gamma \vdash s \circ M : A) \Rightarrow \Gamma \overset{f \circ g}{\rightarrow} A \)

\( (vi) \quad (\Gamma \vdash id \triangleright \Gamma) \Rightarrow \Gamma \overset{id}{\rightarrow} \Gamma \)

\( (vii) \quad (\Gamma, x : A \vdash W \triangleright \Gamma) \Rightarrow \Gamma \times A \overset{pr_1}{\rightarrow} \Gamma \)

\( (viii) \quad (\Gamma \vdash s \triangleright \Delta) \Rightarrow \Gamma \overset{f}{\rightarrow} \Delta \)

\( (\Gamma \vdash N : A) \Rightarrow \Gamma \overset{g}{\rightarrow} A \)

\( (\Gamma \vdash s \circ q \triangleright \Sigma) \Rightarrow \Gamma \overset{f \circ g}{\rightarrow} \Sigma \)

\( (\Gamma \vdash s \circ q \triangleright \Sigma) \Rightarrow \Gamma \overset{f \circ g}{\rightarrow} \Sigma \)

Now we can write some equations (untyped for simplicity).
**Definition 1.9.** (The calculus of equations).

\( (\text{Beta}) \) \( \lambda x. M N = (id, N \setminus x) \circ M \)

\( (\text{Abs}) \) \( s \circ \lambda x. M = \lambda x. (W \circ s, x \setminus x) \circ M \)

\( (\text{App}) \) \( s \circ (MN) = (s \circ M)(s \circ N) \)

\( (\text{ConsVar}) \) \( \langle s, N \setminus x \rangle \circ x = N \)

\( (\text{New}) \) \( \langle s, N \setminus x \rangle \circ y = s \circ y \quad (x \neq y) \)

\( (\text{IdVar}) \) \( id \circ x = x \)

\( (\text{Clos}) \) \( s \circ q \circ M = (s \circ q) \circ M \)

\( (\text{Ass}) \) \( s \circ q \circ r = (s \circ q) \circ r \)

\( (\text{IdR}) \) \( s \circ id = s \)

\( (\text{IdShift}) \) \( id \circ W = W \)

\( (\text{ConsShift}) \) \( \langle s, N \setminus x \rangle \circ W = s \)

\( (\text{Map}) \) \( s \circ \langle q, N \setminus x \rangle = \langle s \circ q, s \circ N \setminus x \rangle \)

\( (\alpha) \) \( \lambda x. M = \lambda y. (W, y \setminus x) \circ M \quad (x, y \text{ are arbitrary}) \)

Here \( s, q, r \) are substitutions,

\( s \circ q \circ M \) is shorthand for \( s \circ (q \circ M) \)

\( s \circ q \circ r \) is shorthand for \( s \circ (q \circ r) \)

\( \langle s \circ q, s \circ N \setminus x \rangle \) is shorthand for \( \langle (s \circ q), (s \circ N) \setminus x \rangle \)

The names of the equations are taken from \[1\], but partially reversed (\text{ConsVar} instead of \text{VarCons} and so on) because of the reversed notation. The equations \text{New} and \alpha are new\[4\].

When a substitution is applied to a variable, the rightmost occurrence of this variable works. See the following example

\[ \langle id, M \setminus x, N \setminus x, L \setminus y \rangle \circ x = \text{New} \langle id, M \setminus x, N \setminus x \rangle \circ x = \text{ConsVar} N \]

I want to stress that there is no restriction on the variables in \( (\alpha) \). For example, we can write

\[ \lambda x. M = \lambda x. (W, x \setminus x) \circ M \]

The following special case of \( (\alpha) \) is important

\[ \lambda x. y = \lambda x. (W, x \setminus x) \circ y \quad (x \neq y) \]

Applying \text{New} to the right part, we obtain

\[ (W) \quad \lambda x. y = \lambda x. W \circ y \quad (x \neq y) \]

Now we can compute lambda-terms.

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1. After the article was written, Johan G. Granström pointed me to his PhD thesis \[2\] with a very similar calculus in Chapter V.
Example 1.10.

\[(\lambda xy.x) y\]
\[= (\langle id, y\backslash x \rangle \circ \lambda y.x) \quad \text{Beta}\]
\[= \lambda y.\langle W \circ (\langle id, y\backslash x \rangle, y\backslash y \rangle \circ x \quad \text{Abs}\]
\[= \lambda y.\langle W \circ (\langle id, y\backslash x \rangle) \circ x \quad \text{New}\]
\[= \lambda y.\langle W \circ id, W \circ y\backslash x \rangle \circ x \quad \text{Map}\]
\[= \lambda y.W \circ y \quad \text{ConsVar}\]
\[= \lambda z.\langle W, z\backslash y \rangle \circ W \circ y \quad \alpha\]
\[= \lambda z.(\langle W, z\backslash y \rangle \circ W) \circ y \quad \text{Clos}\]
\[= \lambda z.W \circ y \quad \text{ConsShift}\]
\[= \lambda z.y \quad W\]

We show by examples how to define free variables of terms. Our calculus has an unexpected feature: the variable \(x\) can occur freely in a term of the form \(\lambda x.M\). To each free occurrence of \(x\) in the term \(x\) assign its level (it is not the De Bruijn level), which is a natural number \(\geq 1\). The only occurrence of \(x\) in the term \(x\) has level 1. We can immediately bind this occurrence and get \(\lambda x.x\). The only occurrence of \(x\) in the term \(W \circ x\) has level 2. The rightmost occurrence of \(x\) in the term \(\lambda x.W \circ x\) is free and has level 1. The rightmost occurrence of \(x\) in the term \(\lambda x.\lambda x.W \circ x\) is bound. The only occurrence of \(x\) in the term \(W \circ W \circ x\) has level 3. The rightmost occurrence of \(x\) in the term \(\lambda x.W \circ W \circ x\) is free and has level 2. The rightmost occurrence of \(x\) in the term \(\lambda x.\lambda x.W \circ W \circ x\) is free and has level 1. The rightmost occurrence of \(x\) in the term \(\lambda x.\lambda x.\lambda x.W \circ W \circ x\) is bound.

Because the rightmost occurrence of \(y\) in the term \(\lambda y.W \circ y\) is free, this term is \(\alpha\)-equal to the term \(\lambda z.W \circ y\). This renaming of the bound variable is done in Example 1.10.

\[\lambda y.W \circ y\]
\[= \lambda z.\langle W, z\backslash y \rangle \circ W \circ y \quad \alpha\]
\[= \lambda z.(\langle W, z\backslash y \rangle \circ W) \circ y \quad \text{Clos}\]
\[= \lambda z.W \circ y \quad \text{ConsShift}\]

I do not have good rewrite rules for the calculus 1.9, hence I propose a different approach. We change the language. Now each symbol \(W\) is equipped with a variable as a subscript (\(W_x, W_y, W_z\ldots\)). The sets of untyped terms and substitutions are defined inductively as follows:

\[M, N ::= x \mid MN \mid \lambda x.M \mid s \circ M\]
\[s, q ::= id \mid W_x \mid \langle s, N\backslash x \rangle \mid s \circ q\]

where the symbol \(x\) denotes an arbitrary variable.
The sets of typed terms and substitutions are defined inductively as follows:

\[ M, N :: = x \mid MN \mid \lambda x^A.M \mid s \circ M \]

\[ s, q :: = id \mid W_x \mid \langle s, N \setminus x \rangle \mid s \circ q \]

Rule (vii) of Definition 1.1 is changed to

\( (vii) \quad \Gamma, x : A \vdash W_x \triangleright \Gamma \)

Rule (vii) of Definition 1.8 is changed to

\( (vii) \quad (\Gamma, x : A \vdash W_x \triangleright \Gamma) \Rightarrow \Gamma \times A \vdash \Gamma \)

Example 1.11.

\[ x : A, x : B \vdash W_x \triangleright x : A \quad x : A \vdash x : A \]

\[ x : A, x : B \vdash W_x \circ x : A \]

Example 1.12.

\[ x : A, x : B, y : C \vdash W_y \triangleright x : A, x : B \quad x : A, x : B \vdash W_x \triangleright x : A \]

\[ x : A, x : B, y : C \vdash W_y \circ W_x \triangleright x : A \quad x : A \vdash x : A \]

\[ x : A, x : B, y : C \vdash (W_y \circ W_x) \circ x : A \]

The calculus $\mathcal{L}_W$ is a draft. We write a similar calculus in the new language, this is $\lambda W$.

**Definition 1.13.** (The calculus $\lambda W$ without several rules).

- (Beta) \( (\lambda x.M)N \rightarrow \langle id, N \setminus x \rangle \circ M \)
- (Abs) \( s \circ \lambda x.M \rightarrow \lambda x.\langle W_x \circ s, x \setminus x \rangle \circ M \)
- (App) \( s \circ (MN) \rightarrow (s \circ M)(s \circ N) \)
- (ConsVar) \( \langle s, N \setminus x \rangle \circ x \rightarrow N \)
- (New) \( \langle s, N \setminus x \rangle \circ y \rightarrow s \circ y \) \((x \neq y)\)
- (IdVar) \( id \circ x \rightarrow x \)
- (Clos) \( s \circ q \circ M \rightarrow (s \circ q) \circ M \)
- (Ass) \( s \circ q \circ r \rightarrow (s \circ q) \circ r \)
- (IdR) \( s \circ id \rightarrow s \)
- (IdShift) \( id \circ W_x \rightarrow W_x \)
- (ConsShift) \( \langle s, N \setminus x \rangle \circ W_x \rightarrow s \)
- (Map) \( s \circ \langle q, N \setminus x \rangle \rightarrow \langle s \circ q, s \circ N \setminus x \rangle \)
- (W1) \( W_x \circ y \rightarrow y \) \((x \neq y)\)
- (W2) \( (s \circ W_x) \circ y \rightarrow s \circ y \) \((x \neq y)\)
- (α1) \( \lambda x.M \rightarrow \lambda y.\langle W_y, y \setminus x \rangle \circ M \) \((*)\)

where \((*)\) is some restriction on the variables: if the variable \(x\) occurs freely in $\lambda x.M$, we can rename \(x\) to a “good” variable.

Example 1.10 now looks like this:
Example 1.14.

\[(\lambda xy.x) y\]
\[\to \langle id, y \卤 x \rangle \circ \lambda y.x \quad \text{Beta}\]
\[\to \lambda y.\langle W_y \circ \langle id, y \卤 x \rangle, y \卤 y \rangle \circ x \quad \text{Abs}\]
\[\to \lambda y.\langle W_y \circ id, W_y \circ y \卤 x \rangle \circ x \quad \text{Map}\]
\[\to \lambda y.W_y \circ y \quad \text{ConsVar}\]
\[\to \lambda z.\langle W_z, z \卤 y \rangle \circ W_y \circ y \quad \alpha_1\]
\[\to \lambda z.\langle W_z \circ z \卤 y \rangle \circ W_y \circ y \quad \text{Clos}\]
\[\to \lambda z.W_z \circ y \quad \text{ConsShift}\]
\[\to \lambda z.y \quad W_1\]

We were able to apply \(\alpha_1\) because \(y\) occurs freely in \(\lambda y.W_y \circ y\) (the right-most occurrence is free).

Let’s try to write the formal analogue of Definition 1.14 for untyped terms and substitutions. Contexts are now simply finite lists of variables with multiplicity (i.e., repetitions are permitted).

A *judgement* is now an expression of the form \(\Gamma \vdash M\) or of the form \(\Gamma \vdash s \triangleright \Delta\), where \(\Gamma\) and \(\Delta\) are contexts, \(M\) is a term, and \(s\) is a substitution. \(\Gamma \vdash M\) means that \(M\) is a well-formed term in the context \(\Gamma\). \(\Gamma \vdash s \triangleright \Delta\) means that \(s\) is a well-formed substitution for \(\Delta\) over \(\Gamma\).

**Definition 1.15.** (Well-formed terms and substitutions).

1. \(\Gamma, x \vdash x\)
2. \(\Gamma \vdash x, y \vdash x\) \((x \not\equiv y)\)
3. \(\Gamma \vdash M, \Gamma \vdash N \quad \Gamma \vdash MN\)
4. \(\Gamma, x \vdash M \quad \Gamma \vdash \lambda x.M\)
5. \(\Gamma \vdash s \triangleright \Delta, \Delta \vdash M \quad \Gamma \vdash s \circ M\)
6. \(\Gamma \vdash id \triangleright \Gamma\)
7. \(\Gamma, x \vdash W_x \triangleright \Gamma\)
8. \(\Gamma \vdash s \triangleright \Delta, \Gamma \vdash N \quad \Gamma \vdash \langle s, N \卤 x \rangle \triangleright \Delta, x\)
9. \(\Gamma \vdash s \triangleright \Delta, \Delta \vdash q \triangleright \Sigma \quad \Gamma \vdash s \circ q \triangleright \Sigma\)
Example 1.16.

\[
\begin{align*}
  &x, x \vdash x \\
  &x, x, y \vdash x
\end{align*}
\]

Example 1.17.

\[
\begin{align*}
  &x, x \vdash W_x \triangleright x \quad x \vdash x \\
  &x, x \vdash W_x \circ x
\end{align*}
\]

Example 1.18.

\[
\begin{align*}
  &x, x, y \vdash W_y \triangleright x, x \quad x, x \vdash W_x \triangleright x \\
  &x, x, y \vdash W_y \circ W_x \triangleright x \\
  &x, x \vdash (W_y \circ W_x) \circ x
\end{align*}
\]

Example 1.19.

\[
\begin{align*}
  &x, x \vdash x \\
  &x \vdash \lambda x. x \\
  &\vdash \lambda x. \lambda x. x
\end{align*}
\]

All usual \( \lambda \)-terms (without explicit substitutions) are well-formed. But there are some restrictions on subscripts of the symbols \( W_x \). For example, a term of the form \( \lambda x. W_x \circ M \) is well-formed if \( M \) is well-formed

\[
\begin{align*}
  &\vdots \\
  &\Gamma, x \vdash W_x \triangleright \Gamma \quad \Gamma \vdash M \\
  &\Gamma, x \vdash W_x \circ M \\
  &\Gamma \vdash \lambda x. W_x \circ M
\end{align*}
\]

but a term of the form \( \lambda x. W_y \circ M \) is never well-formed

\[
\begin{align*}
  &\vdots \\
  &\Gamma, y \vdash W_y \triangleright \Gamma \quad \Gamma \vdash M - (\text{?}) \\
  &\Gamma, x \vdash W_y \circ M \\
  &\Gamma \vdash \lambda x. W_y \circ M
\end{align*}
\]

Reducts of well-formed terms and substitutions are well-formed, hence reducts of usual \( \lambda \)-terms are well-formed. We will work only with well-formed terms and substitutions.

But there is a problem: we can not reduce such term as \( W_y \circ y \). We can reduce \( \lambda y. W_y \circ y \) (to \( \lambda z. y \)), but not \( W_y \circ y \). It is unpleasant to have such normal forms. Hence we introduce a new idea. So far we have one step reductions \( M_1 \rightarrow M_2 \) and \( s_1 \rightarrow s_2 \) defined on the sets of terms and substitutions respectively. We introduce also a one step reduction \( \Gamma_1 \vdash M_1 \rightsquigarrow \Gamma_2 \vdash M_2 \) defined on the set of judgements of the form \( \Gamma \vdash M \).

Really we need only derivable judgements in the sense of Definition 1.15.
Definition 1.20. (Compatible closure).

\[
\frac{M_1 \rightarrow M_2}{\lambda x. M_1 \rightarrow \lambda x. M_2}
\]

\[
\frac{M_1 \rightarrow M_2}{M_1 N \rightarrow M_2 N}
\]

\[
\frac{N_1 \rightarrow N_2}{M N_1 \rightarrow M N_2}
\]

\[
\frac{N_1 \rightarrow N_2}{M_1 \rightarrow M_2}
\]

\[
\frac{s \rightarrow s_2}{s_1 \circ M \rightarrow s_2 \circ M}
\]

\[
\frac{s_1 \rightarrow s_2}{s_1 \circ q \rightarrow s_2 \circ q}
\]

\[
\frac{\Gamma \vdash M_1 \rightsquigarrow \Gamma \vdash M_2}{M_1 \rightarrow M_2}
\]

At last, we add one more rewrite rule (called \(\alpha_2\)), which can be applied to a judgement of the form \(\Gamma \vdash M\) and renames a variable in the context \(\Gamma\). For example, the term \(\mathcal{W}_y \circ y\) can be well-formed only in a context of the form \(\Delta, y\). We can apply \(\alpha_2\) to the judgement \(\Delta, y \vdash \mathcal{W}_y \circ y\) and obtain the judgement \(\Delta, z \vdash \mathcal{W}_z \circ y\), which then reduces to \(\Delta, z \vdash y\). We denote by \(\Lambda\mathcal{W}\) the set of derivable judgements of the form \(\Gamma \vdash M\). For \(\lambda\mathcal{W}\) this set is like \(\Lambda\) for \(\lambda\beta\) and \(\rightsquigarrow\) is the main one step reduction.

The rest of the paper is organized as follows. Section 2 defines the sets of contexts, terms, and substitutions. Section 3 provides a definition of free variables. Section 4 introduces the calculus \(\lambda\mathcal{W}\). Section 5 proves Subject reduction. Section 6 proves several useful properties of \(\lambda\mathcal{W}\). Section 7 compares \(\lambda\mathcal{W}\) with \(\lambda\sigma\) from \([1]\). Section 8 defines the \(\alpha\)-equivalence. Section 9 proves that \(\lambda\mathcal{W}\) is confluent. Section 10 shows that any computation without \(\text{Beta}\) is strongly normalized.
2. Terms and substitutions

For accuracy, we will use metavariables for variables. For example, beta-reduction rule would be written as:

\[(\lambda a. M)N \rightarrow (id, N \backslash a) \circ M,\]

where \(a\) is a metavariable for variables, \(M\) and \(N\) are metavariables for terms. Replacing \(a\) by the variable \(x\), \(M\) by the term \(xx\), and \(N\) by the term \(y\), we obtain the following concrete example of beta-reduction:

\[(\lambda x.xx)y \rightarrow (id, y \backslash x) \circ (xx)\]

For simplicity we will work with the untyped calculus. However, we will use contexts.

Definition 2.1. The symbols \(x, y, z, \ldots\) are variables. The symbols \(M, N, L\) range over terms, \(s, q, r\) range over substitutions, and \(a, b, c\) range over variables (they are metavariables). The sets of terms and substitutions are defined inductively as follows:

\[
M, N ::= a \mid MN \mid \lambda a. M \mid s \circ M
\]

\[
s, q ::= id \mid W \mid a \mid \langle s, N \backslash a \rangle \mid s \circ q
\]

Note that

- \(s \circ M\) corresponds to \(M[s]\) from \([1]\);
- \(s \circ q\) corresponds to \(q \circ s\) from \([1]\);
- \(\langle s, N \backslash a \rangle\) corresponds to \(N \cdot s\) from \([1]\);
- \(W_a\) corresponds to \(\uparrow\) from \([1]\).

Convention 2.2. Outermost parentheses are not written. Outermost parentheses around \(s\) in \(\langle s, N \backslash a \rangle\) are not written. Outermost parentheses around \(N\) in \(\langle s, N \backslash a \rangle\) are not written.

Convention 2.3.

\[
MN_1 \ldots N_k \quad \lambda a_1 \ldots a_k. M \quad s \circ N_1 \ldots N_k \quad \lambda a. M \quad s \circ \lambda a. M \quad s_1 \circ \ldots \circ s_k \circ M
\]

is shorthand for

\[
((MN_1) \ldots)N_k \quad \lambda a_1 (\ldots (\lambda a_k. M)) \quad s \circ (MN_1 \ldots N_k) \quad \lambda a. (s \circ M) \quad s \circ (\lambda a. M) \quad s_1 \circ (\ldots \circ (s_k \circ M))
\]

\[
\langle s, N_1 \backslash b_1, \ldots, N_n \backslash b_n \rangle \quad \langle \langle \ldots \langle s, N_1 \backslash b_1, N_2 \backslash b_2, \ldots \rangle, \ldots, N_n \backslash b_n \rangle
\]

is shorthand for

\[
\langle \langle \ldots \langle s, N_1 \backslash b_1, N_2 \backslash b_2, \ldots \rangle, \ldots, N_n \backslash b_n \rangle
\]

Example 2.4. \(id \circ id \circ x\) is shorthand for \(id \circ (id \circ x)\)

Example 2.5. \(\lambda x. id \circ y\) is shorthand for \(\lambda x. (id \circ y)\)
Example 2.6. \( \text{id} \circ x(yz) \) is shorthand for \( \text{id} \circ (x(yz)) \)

Example 2.7. 
\( \langle \text{id} \circ \text{id}, \text{id} \circ y \backslash x \rangle \) is shorthand for \( \langle (\text{id} \circ \text{id}) , (\text{id} \circ y) \backslash x \rangle \)

Example 2.8. 
\( \text{id} \circ \lambda x.\mathcal{W}_x \circ \lambda y.\mathcal{Z} \) is shorthand for \( \text{id} \circ (\lambda x.(\mathcal{W}_x \circ (\lambda y.\mathcal{Z}))) \)

Example 2.9. 
\( \langle \text{id} , y \backslash x , z \backslash x \rangle \) is shorthand for \( \langle \langle \text{id} , y \backslash x \rangle , z \backslash x \rangle \)

For a more precise definition of terms and substitutions see Section 11.

Definition 2.10. A context is a possibly empty, finite list of variables with multiplicity (i.e., repetitions are permitted). The symbols \( \Gamma, \Delta, \Sigma, \Psi \) range over contexts.

Example 2.11. The list \( x, x, y \) is a context.

Definition 2.12. A judgement is an expression of the form \( \Gamma \vdash M \) or of the form \( \Gamma \vdash s \triangleright \Delta \).

A judgement of the form \( \Gamma \vdash M \) means “\( M \) is a well-formed term in the context \( \Gamma \).” A judgement of the form \( \Gamma \vdash s \triangleright \Delta \) means “\( s \) is a well-formed substitution for \( \Delta \) over \( \Gamma \).”

Definition 2.13. (The inference rules for judgements).

(i) \( \Gamma, a \vdash a \)

(ii) \( \frac{\Gamma \vdash a \quad \Gamma, b \vdash a}{\Gamma, b \vdash a} \quad (a \neq b) \)

(iii) \( \frac{\Gamma \vdash M \quad \Gamma \vdash N}{\Gamma \vdash MN} \)

(iv) \( \frac{\Gamma, a \vdash M}{\Gamma \vdash \lambda a. M} \)

(v) \( \frac{\Gamma \vdash s \triangleright \Delta \quad \Delta \vdash M}{\Gamma \vdash s \circ M} \)

(vi) \( \Gamma \vdash id \triangleright \Gamma \)

(vii) \( \frac{\Gamma, a \vdash \mathcal{W}_a \triangleright \Gamma \quad \Gamma \vdash s \triangleright \Delta \quad \Gamma \vdash N}{\Gamma \vdash (s, N \backslash a) \triangleright \Delta, a} \)

(viii) \( \frac{\Gamma \vdash s \triangleright \Delta \quad \Delta \vdash q \triangleright \Sigma}{\Gamma \vdash s \circ q \triangleright \Sigma} \)

Here \( a \neq b \) means that \( a \) and \( b \) denote distinct variables.
Example 2.14. 
\[
\frac{x, x \vdash W x \triangleright x}{x, x, y \vdash W y \triangleright x, x} \quad \frac{x, x \vdash W x \circ x}{x, x, y \vdash W y \circ W x \circ x}
\]

Example 2.15. 
\[
\vdots \\
\frac{\Gamma, x \vdash M}{\Gamma \vdash \lambda x. M} \\
\frac{\Gamma \vdash N}{\Gamma \vdash (\lambda x. M) N}
\]

Example 2.16. 
\[
\vdots \\
\frac{\Gamma \vdash id \triangleright \Gamma}{\Gamma \vdash N} \\
\frac{\Gamma \vdash N}{\Gamma \vdash \langle id, N \setminus x \rangle \triangleright \Gamma, x} \\
\frac{\Gamma, x \vdash M}{\Gamma \vdash \langle id, N \setminus x \rangle \circ M}
\]

**Lemma 2.17** (Generation lemma). 
Each derivation of \( \Gamma, a \vdash a \) is an application of the rule (i).
Each derivation of \( \Gamma, b \vdash a \) (where \( a \neq b \)) is an application of the rule (ii) to some derivation of \( \Gamma \vdash a \).
Each derivation of \( \Gamma \vdash MN \) is an application of the rule (iii) to some derivations of \( \Gamma \vdash M \) and \( \Gamma \vdash N \).
Each derivation of \( \Gamma \vdash \lambda a. M \) is an application of the rule (iv) to some derivation of \( \Gamma, a \vdash M \).
Each derivation of \( \Gamma \vdash s \circ M \) is an application of the rule (v) to some derivations of \( \Gamma \vdash s \triangleright \Delta \) and \( \Delta \vdash M \) for some \( \Delta \).
Each derivation of \( \Gamma \vdash id \triangleright \Delta \) is an application of the rule (vi), where \( \Delta \) coincides with \( \Gamma \).
Each derivation of \( \Delta \vdash W a \triangleright \Gamma \) is an application of the rule (vii), where \( \Delta \) coincides with \( \Gamma, a \).
Each derivation of \( \Gamma \vdash \langle s, N \setminus a \rangle \triangleright \Sigma \) is an application of the rule (viii) to some derivations of \( \Gamma \vdash s \triangleright \Delta \) and \( \Delta \vdash N \) for some \( \Delta \), where \( \Sigma \) coincides with \( \Delta, a \).
Each derivation of \( \Gamma \vdash s \circ q \triangleright \Sigma \) is an application of the rule (ix) to some derivations of \( \Gamma \vdash s \triangleright \Delta \) and \( \Delta \vdash q \triangleright \Sigma \) for some \( \Delta \).

**Proof.** The proof is straightforward. \( \square \)

**Lemma 2.18.** If a judgement of the form \( \Gamma \vdash s \triangleright \Delta \) is derivable, then \( \Delta \) is uniquely defined for given \( \Gamma \) and \( s \).

**Proof.** The proof is by induction over the structure of \( s \) (see Definition 2.1). 
Case 1: \( s \) is \( id \). This implies that \( \Delta \) coincides with \( \Gamma \).
Case 2: \( s \) has the form \( W_a \) for some \( a \). This implies that \( \Gamma \) coincides with \( \Delta \).
\( \Delta, a \).

Case 3: \( s \) has the form \( \langle q, N \setminus a \rangle \) for some \( q, N, a \). By Generation lemma, we can derive \( \Gamma \vdash q \triangleright \Sigma \) for some \( \Sigma \). By the induction hypothesis, \( \Sigma \) is uniquely defined for \( \Gamma \) and \( q \). Then \( \Delta \) coincides with \( \Sigma, a \).

Case 4: \( s \) has the form \( q \circ r \) for some \( q, r \). By Generation lemma, we can derive \( \Gamma \vdash q \triangleright \Sigma \) for some \( \Sigma \). By the induction hypothesis, \( \Sigma \) is uniquely defined for \( \Gamma \) and \( q \). By Generation lemma, we can derive \( \Sigma \vdash r \triangleright \Delta \), where \( \Delta \) is uniquely defined for \( \Sigma \) and \( r \). \( \square \)

**Lemma 2.19.** For any derivable judgement, there is a unique derivation.

**Lemma 2.20.** The problem of derivability for judgements is decidable.

*Proof.* We try to construct a derivation from the bottom up. \( \square \)

**Example 2.21.** Not each term is well-formed in any context. A term of the form \( \lambda a.W_b \circ M \) is not well-formed in any context if \( a \neq b \).

\[
\begin{array}{c}
\Gamma, y \vdash W_y \triangleright \Gamma \\
\Gamma, x \vdash W_y \circ M \\
\Gamma \vdash \lambda x.W_y \circ M
\end{array}
\]

**Example 2.22.** A term of the form \((W_a \circ M)(W_b \circ N)\) is not well-formed in any context if \( a \neq b \).

\[
\begin{array}{c}
\Gamma, x \vdash W_x \triangleright \Gamma \\
\Gamma, y \vdash W_y \circ M \\
\Gamma \vdash \lambda x.W_y \circ M
\end{array}
\]

\[
\begin{array}{c}
\Gamma, y \vdash W_y \triangleright \Gamma \\
\Gamma, x \vdash W_x \circ M \\
\Gamma \vdash \lambda x.W_y \circ M
\end{array}
\]

\[
? \vdash (W_x \circ M)(W_y \circ N)
\]

**Example 2.23.** A substitution of the form \( \langle s, N \setminus a \rangle \circ W_b \) is not well-formed in any context if \( a \neq b \).

\[
\begin{array}{c}
\Gamma \vdash s \triangleright \Delta \\
\Gamma \vdash \langle s, N \setminus x \rangle \triangleright \Delta, x \\
\Delta, y \vdash W_y \triangleright \Delta
\end{array}
\]

\[
\Gamma \vdash \langle s, N \setminus x \rangle \circ W_y \triangleright \Delta \quad (?)
\]
3. Free variables

Consider some term \( M \) and some variable \( a \). To each free occurrence of \( a \) in \( M \) assign its level, which is a natural number \( \geq 1 \). The only occurrence of \( x \) in the term \( x \) has level 1. We can immediately bind this occurrence and get \( \lambda x.x \). The only occurrence of \( x \) in the term \( Wy \circ x \) has level 2. We can write the term \( \lambda x.Wy \circ x \), but this term is not well-formed (see Example 2.21). If we want to bind this occurrence and get a well-formed term, we must write \( \lambda xy.Wy \circ x \), hence the level is 2. The only occurrence of \( x \) in the term \( Wz \circ Wy \circ x \) has level 3. The simplest way to bind this occurrence and get a well-formed term is \( \lambda xyz.Wz \circ Wy \circ x \). Subscripts of the symbols \( Wa \) are not considered as free occurrences.

Definition 3.1. The symbols \( A, B \) range over infinite sequences of sets

\[
\langle A_1, A_2, A_3, \ldots \rangle \in \text{Sets}^\omega
\]

\[
\langle B_1, B_2, B_3, \ldots \rangle \in \text{Sets}^\omega
\]

By \( A \cup B \) denote

\[
\langle A_1 \cup B_1, A_2 \cup B_2, A_3 \cup B_3, \ldots \rangle
\]

In fact, we need only finite sets whose elements are variables. To each term \( L \) we assign an infinite sequence of sets

\[
FV(L) \equiv \langle FV_1(L), FV_2(L), FV_3(L), \ldots \rangle \in \text{Sets}^\omega
\]

The variables from the set \( FV_i(L) \) have free occurrences of level \( i \) in \( L \). The set of free variables of \( L \) is \( \bigcup_{i \geq 1} FV_i(L) \).

Definition 3.2. (Free variables of terms). By definition, put

(i) \( FV(a) = \{ \{ a \}, \emptyset, \emptyset, \ldots \} \)

(ii) \( FV(MN) = FV(M) \cup FV(N) \)

(iii) \( FV(\lambda a.M) = O_{\lambda a}(FV(M)) \)

(iv) \( FV(s \circ M) = O_s(FV(M)) \),

where

(v) \( O_{\lambda a}(A) = \langle (A_1 \setminus \{ a \}) \cup A_2, A_3, A_4, \ldots \rangle \)

(vi) \( O_{id}(A) = A \)

(vii) \( O_{Wa}(A) = \langle \emptyset, A_1, A_2, \ldots \rangle \)

(viii) \( O_{sq}(A) = O_s(O_q(A)) \)

(ix) \( O_{(s, N \setminus a)}(A) = O_s(O_{\lambda a}(A)) \cup FV(N) \)
Corollary 3.3.

\( FV_1(\lambda a. M) = (FV_1(M) \setminus \{a\}) \cup FV_2(M) \)
\( FV_{n+1}(\lambda a. M) = FV_{n+2}(M) \quad (n \geq 1) \)
\( FV(id \circ M) = FV(M) \)
\( FV_1(W_a \circ M) = \emptyset \)
\( FV_{n+1}(W_a \circ M) = FV_n(M) \quad (n \geq 1) \)
\( FV((s, N \setminus a) \circ M) = O_s(O_{\lambda a}(FV(M))) \cup FV(N) \)

Example 3.4.

\( FV(x) = (\{x\}, \emptyset, \emptyset, \ldots) \)

Example 3.5.

\( FV(W_y \circ x) = (\emptyset, \{x\}, \emptyset, \ldots) \)

Example 3.6.

\( FV(W_z \circ W_y \circ x) = (\emptyset, \emptyset, \{x\}, \emptyset, \ldots) \)

Example 3.7.

\( FV(\lambda z. W_z \circ W_y \circ x) = (\emptyset, \{x\}, \emptyset, \ldots) \)

Example 3.8.

\( FV(\lambda yz. W_z \circ W_y \circ x) = (\{x\}, \emptyset, \emptyset, \ldots) \)

Example 3.9.

\( FV(\lambda x y z. W_z \circ W_y \circ x) = (\emptyset, \emptyset, \emptyset, \ldots) \)

Example 3.10.

\( FV(x (W_z \circ W_y \circ x)) = (\{x\}, \emptyset, \{x\}, \emptyset, \ldots) \)

Example 3.11.

\( FV(\lambda z. x(W_z \circ W_y \circ x)) = (\{x\}, \{x\}, \emptyset, \ldots) \)

Example 3.12.

\( FV(\lambda yz. x(W_z \circ W_y \circ x)) = (\{x\}, \emptyset, \emptyset, \ldots) \)

Example 3.13.

\( FV(\lambda x y z. x(W_z \circ W_y \circ x)) = (\emptyset, \emptyset, \emptyset, \ldots) \)

Warning! May be that \( a \in \bigcup_{i \geq 1} FV_i(\lambda a. M) \).

Example 3.14. \( FV(W_x \circ x) = (\emptyset, \{x\}, \emptyset, \emptyset, \ldots) \)

Example 3.15. \( FV(\lambda x. W_x \circ x) = (\{x\}, \emptyset, \emptyset, \ldots) \)

In fact, the term \( \lambda x. W_x \circ x \) is \( \alpha \)-equal to \( \lambda y. W_y \circ x \).

Lemma 3.16. \( (s \circ q) \circ M \) and \( s \circ q \circ M \) have the same \( FV \).

Proof. The proof is straightforward. \( \square \)

Lemma 3.17. \( (s, N \setminus a) \circ M \) and \( (s \circ \lambda a. M)N \) have the same \( FV \).
Proof. The proof is straightforward. □

Convention 3.18. Since \( O_{Wa} \) and \( O_{Wb} \) are the same for any \( a, b \), we will simply write \( O_W \).

Definition 3.19. \( A \subseteq B \) is shorthand for “\( A_i \subseteq B_i \) for all \( i \geq 1 \).”

Lemma 3.20. \( O_{\lambda a} \) and \( O_{W} \) are monotone operators with respect to \( \subseteq \) (for any \( a \)).

Proof. The proof is straightforward. □

Corollary 3.21. \( O_s \) is monotone with respect to \( \subseteq \) for any \( s \).

Definition 3.22. We define \( \lambda \Gamma. M \) as follows:

\[
\lambda \text{nil}. M \equiv M \\
\lambda \Sigma, a. M \equiv \lambda \Sigma. (\lambda a. M),
\]

where \( \text{nil} \) is the empty context. For example,

\[
\lambda x, y, z. M \equiv \lambda xyz. M
\]

Definition 3.23. (Free variables of judgements). By definition, put

\[
FV(\Gamma \vdash M) = FV(\lambda \Gamma. M)
\]
Definition 4.1. We define $\uparrow_\Delta (s)$ as follows:

- $\uparrow_{\text{nil}} (s) \equiv s$
- $\uparrow_{\Sigma, a} (s) \equiv \langle W_a \circ \uparrow_{\Sigma} (s), a \backslash a \rangle$

where $\text{nil}$ is the empty context. For example,

$$\uparrow_{x,y,z} (s) \equiv \langle W_z \circ (W_y \circ (W_x \circ s \backslash x), y \backslash y), z \backslash z \rangle$$

Note that $\uparrow_{\Sigma, a} (s) \equiv \uparrow_{a} (\uparrow_{\Sigma} (s))$.

Convention 4.2.

$\uparrow_\Delta (s, N \backslash a)$ is shorthand for $\uparrow_\Delta (\langle s, N \backslash a \rangle)$

Now we introduce several one-step reductions: two reductions with the same name $\rightarrow$ defined on the sets of terms and substitutions, and the reduction $\leadsto$ defined on the set of judgements of the form $\Gamma \vdash M$.

Definition 4.3. (The calculus $\lambda W$).

\[
\begin{align*}
\frac{M_1 \rightarrow M_2}{\lambda a. M_1 \rightarrow \lambda a. M_2} \\
\frac{M_1 \rightarrow M_2}{M_1 N \rightarrow M_2 N} \\
\frac{s_1 \rightarrow s_2}{s_1 \circ M \rightarrow s_2 \circ M} \\
\frac{s_1 \rightarrow s_2}{\langle s_1, N \backslash a \rangle \rightarrow \langle s_2, N \backslash a \rangle} \\
\frac{s_1 \rightarrow s_2}{s_1 \circ q \rightarrow s_2 \circ q} \\
\frac{M_1 \rightarrow M_2}{\Gamma \vdash M_1 \leadsto \Gamma \vdash M_2}
\end{align*}
\]
(Beta) \((\lambda a. M) N \rightarrow (\mathit{id}, N \setminus a) \circ M\)

(Abs) \(s \circ \lambda a. M \rightarrow \lambda a. (\mathcal{W}_a \circ s, a \setminus a) \circ M\)

(App) \(s \circ MN \rightarrow (s \circ M)(s \circ N)\)

(ConsVar) \(\langle s, N \setminus a \rangle \circ a \rightarrow N\)

(New) \(\langle s, N \setminus a \rangle \circ b \rightarrow s \circ b\) \((a \neq b)\)

(IdVar) \(\mathit{id} \circ a \rightarrow a\)

(Clos) \(s \circ q \circ M \rightarrow (s \circ q) \circ M\)

(Ass) \(s \circ q \circ r \rightarrow (s \circ q) \circ r\)

(IdR) \(s \circ \mathit{id} \rightarrow s\)

(IdShift) \(id \circ \mathcal{W}_a \rightarrow \mathcal{W}_a\)

(ConsShift) \(\langle s, N \setminus a \rangle \circ \mathcal{W}_a \rightarrow s\)

(Map) \(s \circ (q, N \setminus a) \rightarrow (s \circ q, s \circ N \setminus a)\)

(\(\mathcal{W}_1\)) \(\mathcal{W}_a \circ b \rightarrow b\) \((a \neq b)\)

(\(\mathcal{W}_2\)) \((s \circ \mathcal{W}_a) \circ b \rightarrow s \circ b\) \((a \neq b)\)

(\(\alpha_1\)) \(\lambda a. M \rightarrow \lambda b. (\mathcal{W}_b, b \setminus a) \circ M\) \((\ast)\)

(\(\alpha_2\)) \(\Gamma, a, \Delta \vdash M \rightsquigarrow \Gamma, b, \Delta \vdash \uparrow\Delta (\mathcal{W}_b, b \setminus a) \circ M\) \((\ast\ast)\)

where the side conditions are as follows:

\((\ast)\) \(a \in \bigcup_{i \geq 1} \mathit{FV}_i(\lambda a. M); \ b \notin \bigcup_{i \geq 1} \mathit{FV}_i(\lambda a. M)\)

\((\ast\ast)\) \(a \in \bigcup_{i \geq 1} \mathit{FV}_i(a, \Delta \vdash M); \ b \notin \bigcup_{i \geq 1} \mathit{FV}_i(a, \Delta \vdash M)\)

Recall that \(s \circ MN\) is shorthand for \(s \circ (MN)\).

Note that (Abs) can be written as

(Abs) \(s \circ \lambda a. M \rightarrow \lambda a. \uparrow\Delta (s) \circ M\)

**Definition 4.4.** By \(\rightarrow\) denote the reflexive transitive closure of \(\rightarrow\).
By \(\rightsquigarrow\) denote the reflexive transitive closure of \(\rightsquigarrow\).

**Lemma 4.5.**
If \(a \neq b_1, \ldots, a \neq b_k\), then \(\langle s, N \setminus a, N_1 \setminus b_1, \ldots, N_k \setminus b_k \rangle \circ a \rightarrow\) \(N\).

**Proof.** We use New (repeatedly), then we use ConsVar. \(\square\)

**Example 4.6.**
\[
\langle \mathit{id}, N \setminus x, L \setminus y \rangle \circ x \\
\rightarrow \langle \mathit{id}, N \setminus x \rangle \circ x \quad \text{New} \\
\rightarrow N \quad \text{ConsVar}
\]

**Example 4.7.**
\[
\langle \mathit{id}, N \setminus x, L \setminus y \rangle \circ y \\
\rightarrow L \quad \text{ConsVar}
\]

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Example 4.8.

\[
\langle \text{id}, N \backslash x, L \backslash y \rangle \circ z \\
\rightarrow \langle \text{id}, N \backslash x \rangle \circ z \quad \text{New} \\
\rightarrow \text{id} \circ z \quad \text{New} \\
\rightarrow z \quad \text{IdVar}
\]

Example 4.9.

\[
\langle \text{id}, N \backslash x, L \backslash x \rangle \circ x \\
\rightarrow L \quad \text{ConsVar}
\]

Example 4.10.

\[
\langle \text{id}, N \backslash x, L \backslash x \rangle \circ W_x \circ x \\
\rightarrow (\langle \text{id}, N \backslash x, L \backslash x \rangle \circ W_x) \circ x \quad \text{Clos} \\
\rightarrow \langle \text{id}, N \backslash x \rangle \circ x \quad \text{ConsShift} \\
\rightarrow N \quad \text{ConsVar}
\]

Example 4.11.

\[
\langle W_x, N \backslash y, L \backslash y \rangle \circ z \\
\rightarrow \langle W_x, N \backslash y \rangle \circ z \quad \text{New} \\
\rightarrow W_x \circ z \quad \text{New} \\
\rightarrow z \quad W_1
\]

Example 4.12.

\[
\langle W_x, N \backslash y, L \backslash y \rangle \circ x \\
\rightarrow \langle W_x, N \backslash y \rangle \circ x \quad \text{New} \\
\rightarrow W_x \circ x \quad \text{New}
\]

where \( W_x \circ x \) is a normal form.

Example 4.13. \( FV(\lambda x. W_x \circ x) = \langle \{x\}, \emptyset, \emptyset, \ldots \rangle \)

Example 4.14.

\[
\lambda x. W_x \circ x \\
\rightarrow \lambda y. \langle W_y, y \backslash x \rangle \circ W_x \circ x \quad \alpha_1 \\
\rightarrow \lambda y. (\langle W_y, y \backslash x \circ W_x \rangle \circ x \quad \text{Clos} \\
\rightarrow \lambda y. W_y \circ x \quad \text{ConsShift} \\
\rightarrow \lambda y. x \quad W_1
\]

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Example 4.15.

\[(\lambda xy.x) y\]
\[\to \langle id, y \setminus x \rangle \circ \lambda y.x\]  \hspace{1cm} Beta
\[\to \lambda y.(W_y \circ \langle id, y \setminus x \rangle) \circ y \circ x\]  \hspace{1cm} Abs
\[\to \lambda y.(W_y \circ \langle id, y \setminus x \rangle) \circ x\]  \hspace{1cm} New
\[\to \lambda y.(W_y \circ id, W_y \circ y \setminus x) \circ x\]  \hspace{1cm} Map
\[\to \lambda y.W_y \circ y\]  \hspace{1cm} ConsVar
\[\to \lambda z.(W_z, z \setminus y) \circ y\]  \hspace{1cm} α₁
\[\to \lambda z.(W_z, z \setminus y) \circ W_y \circ y\]  \hspace{1cm} Clos
\[\to \lambda z.W_z \circ y\]  \hspace{1cm} ConsShift
\[\to \lambda z.y\]  \hspace{1cm} W₁

Example 4.16.

\[(\lambda xy.x) y\]
\[\to \langle id, y \setminus x \rangle \circ \lambda y.x\]  \hspace{1cm} Beta
\[\to \lambda y.(W_y \circ \langle id, y \setminus x \rangle) \circ y \circ x\]  \hspace{1cm} Abs
\[\to \lambda y.(W_y \circ \langle id, y \setminus x \rangle) \circ x\]  \hspace{1cm} New
\[\to \lambda y.(W_y \circ id, W_y \circ y \setminus x) \circ x\]  \hspace{1cm} Map
\[\to \lambda y.(W_y \circ id, W_y \circ y \setminus x) \circ x\]  \hspace{1cm} New
\[\to \lambda y.W_y \circ y\]  \hspace{1cm} ConsVar
\[\to \lambda z.(W_z, z \setminus y) \circ y\]  \hspace{1cm} α₁
\[\to \lambda z.(W_z, z \setminus y) \circ W_y \circ y\]  \hspace{1cm} Clos
\[\to \lambda z.W_z \circ y\]  \hspace{1cm} ConsShift
\[\to \lambda z.y\]  \hspace{1cm} W₁

Example 4.17. \(\text{FV}(x \vdash W_x \circ x) = \text{FV}(\lambda x.W_x \circ x) = \{x\}, \emptyset, \emptyset, \ldots\)

Example 4.18.

\[x, x \vdash W_x \circ x \sim_{\alpha_2} x, y \vdash \langle W_y, y \setminus x \rangle \circ W_x \circ x\]

Further,

\[\langle W_y, y \setminus x \rangle \circ W_x \circ x\]
\[\to (\langle W_y, y \setminus x \rangle \circ W_x) \circ x\]  \hspace{1cm} Clos
\[\to W_y \circ x\]  \hspace{1cm} ConsShift
\[\to x\]  \hspace{1cm} W₁

We see that

\[x, x \vdash W_x \circ x\]
\[\sim_{\alpha_2} x, y \vdash \langle W_y, y \setminus x \rangle \circ W_x \circ x\]
\[\sim_{\text{Clos}} x, y \vdash (\langle W_y, y \setminus x \rangle \circ W_x) \circ x\]
\[\sim_{\text{ConsShift}} x, y \vdash W_y \circ x\]
\[\sim_{W_1} x, y \vdash x\]
Example 4.19. $FV(\lambda xz.W_z \circ W_x \circ x) = \langle \{x\}, \emptyset, \emptyset, \ldots \rangle$

Example 4.20.

\[
\begin{align*}
\lambda x x z. W_z \circ W_x \circ x \\
\to \lambda x y. (W_y, y \setminus x) \circ \lambda z. W_z \circ W_x \circ x & \quad \alpha_1 \\
\to \lambda x y z. \uparrow z \langle W_y, y \setminus x \rangle \circ W_z \circ W_x \circ x & \quad \text{Abs} \\
\equiv \lambda x y z. (W_z \circ (W_y, y \setminus x), z \setminus z) \circ W_z \circ W_x \circ x & \quad \text{Definition} \footnote{4.1} \\
\to \lambda x y z. (W_z \circ (W_y, y \setminus x), z \setminus z) \circ W_z \circ W_x \circ x & \quad \text{Clos} \\
\to \lambda x y z. (W_z \circ W_y) \circ x & \quad \text{ConsShift} \\
\to \lambda x y z. W_z \circ x & \quad W_2 \\
\to \lambda x y z. x & \quad W_1
\end{align*}
\]

Example 4.21. $FV(x, z \vdash W_z \circ W_x \circ x) = FV(\lambda x z. W_z \circ W_x \circ x) = \langle \{x\}, \emptyset, \emptyset, \ldots \rangle$

Example 4.22.

\[
\begin{align*}
& x, x, z \vdash W_z \circ W_x \circ x \\
& \leadsto x, y, z \vdash \uparrow z (W_y, y \setminus x) \circ W_z \circ W_x \circ x & \quad \alpha_2 \\
& \equiv x, y, z \vdash (W_z \circ (W_y, y \setminus x), z \setminus z) \circ W_z \circ W_x \circ x & \quad \text{Definition} \footnote{4.1} \\
& \leadsto x, y, z \vdash (W_z \circ (W_y, y \setminus x), z \setminus z) \circ W_z \circ W_x \circ x & \quad \text{Clos} \\
& \leadsto x, y, z \vdash (W_z \circ W_y) \circ x & \quad \text{ConsShift} \\
& \leadsto x, y, z \vdash (W_z \circ W_y) \circ x & \quad \text{ConsShift} \\
& \leadsto x, y, z \vdash (W_z \circ W_y) \circ x & \quad \text{Clos} \\
& \leadsto x, y, z \vdash (W_z \circ W_y) \circ x & \quad \text{ConsShift} \\
& \leadsto x, y, z \vdash (W_z \circ W_y) \circ x & \quad \text{ConsShift} \\
& \leadsto x, y, z \vdash x & \quad W_2 \\
& \leadsto x, y, z \vdash x & \quad W_1
\end{align*}
\]
Example 4.23. (Some redexes are underlined).

\[(\lambda x y z. x z)(\lambda x y. x)\]
\[
\to \langle id, \lambda x y. x \backslash x \rangle \circ \lambda y z. x z(y z) \tag{Beta}
\]
\[
\to \lambda y. \uparrow y (\langle id, \lambda x y. x \backslash x \rangle \circ \lambda z. x z(y z)) \tag{Abs}
\]
\[
\to \lambda y z. \uparrow y z (\langle id, \lambda x y. x \backslash x \rangle \circ x z(y z)) \tag{Abs}
\]
\[
\to \lambda y z. (\uparrow y z (\langle id, \lambda x y. x \backslash x \rangle \circ x z)(\uparrow y z (id, \lambda x y. x \backslash x) \circ y z)) \tag{App}
\]
\[
\to \lambda y z. (\lambda x y. x)(\uparrow y z (\langle id, \lambda x y. x \backslash x \rangle \circ y z)) \tag{App, Example 4.24 Example 4.25}
\]
\[
\to \lambda y z. (\lambda x y. x)(y z) \tag{App, Example 4.26 Example 4.25}
\]
\[
\to \lambda y z. (\lambda y. \uparrow y (id, z \backslash x) \circ \lambda y. x)(y z) \tag{Beta}
\]
\[
\to \lambda y z. (\lambda y. \uparrow y (id, z \backslash x) \circ x)(y z) \tag{Abs}
\]
\[
\to \lambda y z. (\lambda y z. y z) \tag{Example 4.27}
\]
\[
\to \lambda y z. (\lambda y. \uparrow y (id, z \backslash x) \circ y z) \tag{Beta}
\]
\[
\to \lambda y z. id \circ z \tag{New}
\]
\[
\to \lambda y z. z \tag{IdVar}
\]

Example 4.24.

\[\uparrow y z (id, \lambda x y. x \backslash x) \circ x\]
\[
\equiv (W_z \circ \uparrow y (id, \lambda x y. x \backslash x), y \backslash y) \circ x \tag{Definition 4.1}
\]
\[
\to (W_z \circ \uparrow y (id, \lambda x y. x \backslash x)) \circ x \tag{New}
\]
\[
\equiv (W_z \circ \langle W_y \circ (id, \lambda x y. x \backslash x), y \backslash y \rangle \circ x \tag{Definition 4.1}
\]
\[
\to (W_z \circ W_y \circ (id, \lambda x y. x \backslash x), W_z \circ y \backslash y) \circ x \tag{Map}
\]
\[
\to (W_z \circ W_y \circ (id, \lambda x y. x \backslash x)) \circ x \tag{New}
\]
\[
\to (W_z \circ W_y \circ id, W_z \circ W_y \circ \lambda x y. x \backslash x \circ x \tag{Map}
\]
\[
\to W_z \circ W_y \circ \lambda x y. x \tag{ConsVar}
\]
\[
\to \lambda x y. x \tag{because \lambda x y. x is closed}
\]

Example 4.25.

\[\uparrow y z (id, \lambda x y. x \backslash x) \circ z\]
\[
\equiv (W_z \circ \uparrow y (id, \lambda x y. x \backslash x), z \backslash z) \circ z \tag{Definition 4.1}
\]
\[
\to z \tag{ConsVar}
\]
Example 4.26.

\[
\uparrow_{y,z} \langle id, \lambda xy.x \setminus x \rangle \circ y \\
\equiv \langle W_z \circ \uparrow_y \langle id, \lambda xy.x \setminus x \rangle, z \setminus z \rangle \circ y \\
\rightarrow (W_z \circ \uparrow_y \langle id, \lambda xy.x \setminus x \rangle) \circ y \\
\equiv (W_z \circ \langle W_y \circ \langle id, \lambda xy.x \setminus x \rangle, y \setminus y \rangle \circ y \\
\rightarrow \langle W_z \circ W_y \circ \langle id, \lambda xy.x \setminus x \rangle, W_z \circ y \setminus y \rangle \circ y \\
\rightarrow W_z \circ y \\
\rightarrow y
\]

Example 4.27.

\[
\uparrow_y \langle id, z \setminus x \rangle \circ x \\
\equiv \langle W_y \circ \langle id, z \setminus x \rangle, y \setminus y \rangle \circ x \\
\rightarrow (W_y \circ \langle id, z \setminus x \rangle) \circ x \\
\rightarrow (W_y \circ id, W_y \circ z \setminus x) \circ x \\
\rightarrow W_y \circ z \\
\rightarrow z
\]
5. Subject reduction

**Theorem 5.1** (Subject reduction, part one).

If $\Gamma \vdash M_1$ is derivable and $M_1 \rightarrow M_2$, then $\Gamma \vdash M_2$ is derivable.

If $\Gamma \vdash s_1 \triangleright \Delta$ is derivable and $s_1 \rightarrow s_2$, then $\Gamma \vdash s_2 \triangleright \Delta$ is derivable.

**Proof.** The proof is straightforward, but tedious.

**Case Beta.** $(\lambda a. M)N \rightarrow (id, N\setminus a) \circ M$

\[
\vdots \\
\Gamma, a \vdash M \\
\hline \\
\Gamma \vdash \lambda a. M \\
\Gamma \vdash N \\
\hline \\
\Gamma \vdash (\lambda a. M)N
\]

\[
\vdots \\
\Gamma \vdash id \triangleright \Gamma \\
\Gamma \vdash N \\
\hline \\
\Gamma \vdash (id, N\setminus a) \triangleright \Gamma, a \\
\Gamma, a \vdash M \\
\hline \\
\Gamma \vdash (id, N\setminus a) \circ M
\]

**Case Abs.** $s \circ \lambda a. M \rightarrow \lambda a. (W_a \circ s, a\setminus a) \circ M$

\[
\vdots \\
\Delta, a \vdash M \\
\hline \\
\Gamma \vdash s \triangleright \Delta \\
\Delta \vdash \lambda a. M \\
\hline \\
\Gamma \vdash s \circ \lambda a. M
\]

\[
\vdots \\
\Gamma, a \vdash W_a \triangleright \Gamma \\
\Gamma \vdash s \triangleright \Delta \\
\hline \\
\Gamma, a \vdash W_a \circ s \triangleright \Delta \\
\Gamma, a \vdash a \\
\hline \\
\Gamma, a \vdash (W_a \circ s, a\setminus a) \triangleright \Delta, a \\
\Delta, a \vdash M \\
\hline \\
\Gamma, a \vdash (W_a \circ s, a\setminus a) \circ M \\
\Gamma \vdash \lambda a. (W_a \circ s, a\setminus a) \circ M
\]

**Case $\alpha_1$.** $\lambda a. M \rightarrow \lambda b. (W_b, b\setminus a) \circ M$ (*)

\[
\vdots \\
\Gamma, a \vdash M \\
\hline \\
\Gamma \vdash \lambda a. M
\]
\[
\begin{align*}
\Gamma, b \vdash W_b & \rightarrow \Gamma, b \vdash b \\
\Gamma, b \vdash \langle W_b, b\setminus a \rangle & \rightarrow \Gamma, a \vdash M \\
\therefore \Gamma, b \vdash \langle W_b, b\setminus a \rangle \circ M & \rightarrow \Gamma \vdash \lambda b. \langle W_b, b\setminus a \rangle \circ M
\end{align*}
\]

And so on. Note that we do not use \((*)\) in the proof of the case \(\alpha_1\). □

**Lemma 5.2.** If \(\Gamma \vdash s \circ \lambda a. M\) is derivable, then \(\Gamma \vdash \lambda a. \uparrow_a (s) \circ M\) is derivable.

*Proof.* Theorem 5.1, the case \(Abs\). □

**Lemma 5.3.** If \(\Gamma \vdash s \circ \lambda \Delta. M\) is derivable, then \(\Gamma \vdash \lambda \Delta. \uparrow_\Delta (s) \circ M\) is derivable.

*Proof.* Recall that \(\uparrow_{\Sigma, a} (s) \equiv \uparrow_a (\uparrow_\Sigma (s))\). Now we can use Lemma 5.2 repeatedly. □

**Theorem 5.4** (Subject reduction, part two).

Suppose
\[
\begin{align*}
\Gamma, a, \Delta & \vdash M \text{ is derivable and} \\
\Gamma, a, \Delta & \vdash M \rightsquigarrow_{\alpha_2} \Gamma, b, \Delta \vdash \uparrow_\Delta \langle W_b, b\setminus a \rangle \circ M; \\
\text{then} \\
\Gamma, b, \Delta & \vdash \uparrow_\Delta \langle W_b, b\setminus a \rangle \circ M \text{ is derivable.}
\end{align*}
\]

*Proof.* By Generation lemma \(\Gamma, a, \Delta \vdash M\) is derivable iff \(\Gamma \vdash \lambda a. \lambda \Delta. M\) is derivable. To conclude the proof, it is sufficient to prove the following lemma. □

**Lemma 5.5.** Suppose
\[
\Gamma \vdash \lambda a. \lambda \Delta. M \text{ is derivable; then} \\
\Gamma \vdash \lambda b. \lambda \Delta. \uparrow_\Delta \langle W_b, b\setminus a \rangle \circ M \text{ is derivable.}
\]

*Proof.* We use Theorem 5.1 (the case \(\alpha_1\)) and Lemma 5.3. □
6. Two theorems about normal forms

**Definition 6.1.** By $\sigma W \alpha$ denote $\lambda W$ without Beta.

**Definition 6.2.** By $W_{a_1...a_n}$ denote $(\ldots((W_{a_1} \circ W_{a_2}) \circ \ldots) \circ W_{a_n})$.

**Theorem 6.3.** Suppose $\Gamma \vdash s \sigma \Delta$ is derivable and $s$ is a $\sigma W \alpha$-normal form (with respect to $\rightarrow$); then $s$ has one of the following forms:

- (i) $id$
- (ii) $W_{a_1...a_n}$ ($n \geq 1$)
- (iii) $⟨id, N_1 \backslash b_1, \ldots, N_k \backslash b_k⟩$ ($k \geq 1$)
- (iv) $⟨W_{a_1...a_n}, N_1 \backslash b_1, \ldots, N_k \backslash b_k⟩$ ($n \geq 1, k \geq 1$)

Of course, the terms $N_1, \ldots, N_k$ are not arbitrary, they are $\sigma W \alpha$-normal forms (with respect to $\rightarrow$).

**Proof.** The proof is by induction over the structure of $s$ (see Definition 6.2). The set of substitutions of the forms (i) – (iv) contains $id$ and $W_a$ for any $a$. This set is also closed under $(-, N \backslash b)$ for any $N, b$. To conclude the proof, it is sufficient to prove the following lemma.

**Lemma 6.4.** If $\Gamma \vdash s \sigma q \Delta$ is derivable and both $s, q$ belong to (i), (ii), (iii), (iv), then $s \sigma q$ $\sigma W \alpha$-reduces to one of the forms (i), (ii), (iii), (iv).

**Proof.** Let us consider five cases.

Case 1: $q$ is $id$.
$s \circ id \rightarrow s$

Case 2: $s$ is $id$ and $q$ has the form $W_{c_1...c_m}$.
$id \circ W_{c_1...c_m} \rightarrow W_{c_1...c_m}$

Case 3: $s$ has the form $W_{a_1...a_n}$ and $q$ has the form $W_{c_1...c_m}$.
$(W_{a_1...a_n}) \circ W_{c_1...c_m} \rightarrow W_{a_1...a_n c_1...c_m}$

Case 4: $s$ has the form $⟨r, N_1 \backslash b_1, \ldots, N_k \backslash b_k⟩$ and $q$ has the form $W_{c_1...c_m}$, where $r$ is $id$ or $W_{a_1...a_n}$. Hence, $s \circ q$ has the form $⟨r, N_1 \backslash b_1, \ldots, N_k \backslash b_k⟩ \circ W_{c_1...c_m}$. By Generation lemma, $b_k = c_1$, $b_{k-1} = c_2$, and so on (see Example 2.23).

If $k = m$, then

$⟨r, N_1 \backslash b_1, \ldots, N_k \backslash b_k⟩ \circ W_{c_1...c_m}$ is the same as

$⟨r, N_1 \backslash b_1, \ldots, N_k \backslash b_k⟩ \circ W_{b_k...b_1} \rightarrow r$

If $k > m$, then

$⟨r, N_1 \backslash b_1, \ldots, N_k \backslash b_k⟩ \circ W_{c_1...c_m}$ is the same as

$⟨r, N_1 \backslash b_1, \ldots, N_k \backslash b_k⟩ \circ W_{b_k...b_{k-m+1}} \rightarrow (r, N_1 \backslash b_1, \ldots, N_{k-m} \backslash b_{k-m})$

If $k < m$, then

$⟨r, N_1 \backslash b_1, \ldots, N_k \backslash b_k⟩ \circ W_{c_1...c_m}$ is the same as
\[ \langle r, N_1 \setminus c_k, \ldots, N_k \setminus c_1 \rangle \circ W_{c_1 \ldots c_m} \]

If \( r \) is \( id \), this term reduces to \( W_{c_1 \ldots c_m} \). If \( r \) is \( W_{a_1 \ldots a_n} \), this term reduces to \( W_{a_1 \ldots a_n c_{k+1} \ldots c_m} \).

Case 5: \( q \) has the form \( \langle r, N_1 \setminus b_1, \ldots, N_k \setminus b_k \rangle \), where \( r \) is \( id \) or \( W_{c_1 \ldots c_m} \). 

\[ s \circ \langle r, N_1 \setminus b_1, \ldots, N_k \setminus b_k \rangle \rightarrow \langle s \circ r, s \circ N_1 \setminus b_1, \ldots, s \circ N_k \setminus b_k \rangle \]

Then we use the previous cases to reduce \( s \circ r \).

Note that we do not use \( W_1, W_2, \alpha_1, \alpha_2 \) in this proof.

**Definition 6.5.** A term \( M \) is called pure iff it does not contain sub-terms of the shape \( s \circ N \).

**Theorem 6.6.** If \( \Gamma \vdash M \) is derivable and \( \Gamma \vdash M \) is a \( \sigma W\alpha \)-normal form (with respect to \( \rightsquigarrow \)), then \( M \) is pure.

**Proof.** Suppose \( M \) contain a sub-term of the shape \( s \circ N \); then \( N \) must be a variable (we denote it by \( b \)), else we can apply \( \text{Abs}, \text{App} \) or \( \text{Clos} \). The substitution \( s \) is a \( \sigma W\alpha \)-normal form and must have the form \( W_{a_1 \ldots a_n} \) (see theorem 6.3), else we can apply \( \text{IdVar}, \text{ConsVar} \) or \( \text{New} \). Further, 

\[ a_n \in (W_{a_1 \ldots a_n}) \circ b \quad \text{must coincide with} \quad b, \quad \text{else we can apply} \quad W_1 \quad \text{or} \quad W_2. \]

We see that \( M \) must be constructed from variables and blocks of the form 

\[ (W_{a_1 \ldots a_n b}) \circ b \quad (m \geq 0) \]

by using application and abstraction. To conclude the proof, it is sufficient to prove the following lemma.

**Lemma 6.7.** If \( \Gamma \vdash M \) is derivable and \( M \) is constructed from variables and blocks of the form \( (W_{a_1 \ldots a_n b}) \circ b \quad (m \geq 0) \) by using application and abstraction, then \( M \) is pure (this means that \( M \) does not contain blocks) or we can apply \( \alpha_1 \) or \( \alpha_2 \) to \( \Gamma \vdash M \).

**Proof.** The proof is by induction over the structure of \( M \). Let us consider four cases.

Case 1: \( M \) is a variable. The proof is trivial.

Case 2: \( M \) has the form \( (W_{a_1 \ldots a_n b}) \circ b \). By Generation lemma, \( \Gamma \vdash M \) has the form 

\[ \Delta, b, a_m, \ldots, a_1 \vdash (W_{a_1 \ldots a_n b}) \circ b \]

and we can apply \( \alpha_2 \), because

\[ FV(b, a_m, \ldots, a_1 \vdash (W_{a_1 \ldots a_n b}) \circ b) = (\{b\}, \emptyset, \emptyset, \ldots) \]

Case 3: \( M \) has the form \( \lambda a. N \). By Generation lemma, \( \Gamma, a \vdash N \) is derivable. Suppose \( N \) contains a block of the form \( (W_{a_1 \ldots a_n b}) \circ b \). By induction hypothesis, we can apply \( \alpha_1 \) or \( \alpha_2 \) to \( \Gamma, a \vdash N \). But any application of \( \alpha_1 \) or \( \alpha_2 \) to \( \Gamma, a \vdash N \) corresponds to some application of \( \alpha_1 \) or \( \alpha_2 \) (and, in some cases, \( \text{Abs} \)) to \( \Gamma \vdash \lambda a. N \). For example, 

\[ \Gamma, a \vdash N \rightsquigarrow_{\alpha_2} \Gamma, b \vdash (W_{b}, b \setminus a) \circ N \]

corresponds to

\[ \Gamma \vdash \lambda a. N \rightsquigarrow_{\alpha_1} \Gamma \vdash \lambda b. (W_{b}, b \setminus a) \circ N \]

Case 4: \( M \) has the form \( NL \). By Generation lemma, \( \Gamma \vdash N \) and \( \Gamma \vdash L \).
are derivable. Suppose one of these terms contains a block of the form 
\((W_{a_1 \ldots a_m}a) \circ b\). For clarity, let it be \(N\). By induction hypothesis, we can apply \(\alpha_1\) or \(\alpha_2\) to \(\Gamma \vdash N\). I claim that we can apply \(\alpha_1\) or \(\alpha_2\) to \(\Gamma \vdash NL\). For the case \(\alpha_1\) is nothing to prove, because any \(\alpha_1\)-redex in \(N\) occurs in \(NL\) too. For the case \(\alpha_2\), suppose that \(\Gamma \vdash NL\) has the form \(\Sigma, a, \Delta \vdash NL\).

Recall that \(FV(NL) = FV(N) \cup FV(L)\), hence \(FV(N) \subseteq FV(NL)\). By Lemma 3.20, \(FV(a, \Delta \vdash N) \subseteq FV(a, \Delta \vdash NL)\). Hence if \(a \in \bigcup_{i \geq 1} FV_i(a, \Delta \vdash N)\), then \(a \in \bigcup_{i \geq 1} FV_i(a, \Delta \vdash NL)\). If we can apply \(\alpha_2\) to \(\Sigma, a, \Delta \vdash N\), then we can apply \(\alpha_2\) to \(\Sigma, a, \Delta \vdash NL\). \(\square\)

Warning! In general, the terms \(N_1, \ldots, N_k\) in the statement of Theorem 6.3 are not pure. For example, the judgement 
\(x, x \vdash \langle id, W_x \circ x \\vdash y \rangle \triangleright x, x, y\)
is derivable and the substitution
\(\langle id, W_x \circ x \\vdash y \rangle\)
is a \(\sigma W\alpha\)-normal form.
7. Relation with \( \lambda \sigma \)

**Definition 7.1.** The symbols \( U, V, W \) range over *name-free terms* and the symbols \( u, v, w \) range over *name-free substitutions*. The sets of name-free terms and name-free substitutions are defined inductively as follows:

\[
U, V :: = 1 \mid UV \mid \lambda U \mid u \circ U \\
u, v :: = id \mid W \mid \langle u, V \rangle \mid u \circ v
\]

**Definition 7.2.** (The calculus \( \lambda \sigma \) in the new notation).

\[
\begin{align*}
U_1 \rightarrow U_2 \quad & \lambda U_1 \rightarrow \lambda U_2 \\
U_1 V \rightarrow U_2 V \quad & V_1 \rightarrow V_2 \\
u_1 \circ U \rightarrow u_2 \circ U \quad & u \circ U_1 \rightarrow u \circ U_2 \\
u_1 \rightarrow u_2 \quad & V_1 \rightarrow V_2 \\
\langle u_1, V \rangle \rightarrow \langle u_2, V \rangle \quad & \langle u, V_1 \rangle \rightarrow \langle u, V_2 \rangle \\
u_1 \circ v \rightarrow u_2 \circ v \quad & v_1 \rightarrow v_2
\end{align*}
\]

\((Beta)\) \quad \( (\lambda U)V \rightarrow \langle id, V \rangle \circ U \)

\((Abs)\) \quad \( u \circ \lambda U \rightarrow \lambda \langle W \circ u, 1 \rangle \circ U \)

\((App)\) \quad \( u \circ UV \rightarrow (u \circ U)(u \circ V) \)

\((ConsVar)\) \quad \( \langle u, V \rangle \circ 1 \rightarrow V \)

\((IdVar)\) \quad \( id \circ 1 \rightarrow 1 \)

\((Clos)\) \quad \( u \circ v \circ V \rightarrow (u \circ v) \circ V \)

\((Ass)\) \quad \( u \circ v \circ w \rightarrow (u \circ v) \circ w \)

\((IdR)\) \quad \( u \circ id \rightarrow u \)

\((IdShift)\) \quad \( id \circ W \rightarrow W \)

\((ConsShift)\) \quad \( \langle u, V \rangle \circ W \rightarrow u \)

\((Map)\) \quad \( u \circ \langle v, V \rangle \rightarrow \langle u \circ v, u \circ V \rangle \)

**Definition 7.3.** By \( \sigma \) denote \( \lambda \sigma \) without *Beta*.

By \( \sigma(U) \) denote the \( \sigma \)-normal form of \( U \) (this normal form exists and is uniquely defined because \( \sigma \) is strongly normalizing and confluent).
**Definition 7.4.** By definition, put 

\[ n \equiv ((\ldots (W \circ W) \circ \ldots) \circ W) \circ 1 \quad (n \geq 1) \]

We see that 1 ≡ 1, 2 ≡ W \circ 1, and \( n + 1 \equiv \sigma(W \circ n) \).

**Definition 7.5.** \((\Gamma \vdash M) \Rightarrow U\) is shorthand for “the name-free term \( U \) corresponds to the judgement \( \Gamma \vdash M \).”

\((\Gamma \vdash s \triangleright \Delta) \Rightarrow u\) is shorthand for “the name-free substitution \( u \) corresponds to the judgement \( \Gamma \vdash s \triangleright \Delta \).”

**Definition 7.6.** (The rules of correspondence between judgements and name-free terms/substitutions).

\[(i) \quad (\Gamma, a \vdash a) \Rightarrow 1 \]

\[(ii) \quad (\Gamma \vdash a) \Rightarrow n \quad (a \neq b) \]

\[(iii) \quad (\Gamma \vdash M) \Rightarrow U \quad (\Gamma \vdash N) \Rightarrow V \]

\[ (\Gamma \vdash MN) \Rightarrow UV \]

\[(iv) \quad (\Gamma, \lambda a.M) \Rightarrow \lambda U \]

\[(v) \quad (\Gamma \vdash s \triangleright \Delta) \Rightarrow u \quad (\Delta \vdash M) \Rightarrow U \]

\[ (\Gamma \vdash s \circ M) \Rightarrow u \circ U \]

\[(vi) \quad (\Gamma \vdash id \triangleright \Gamma) \Rightarrow id \]

\[(vii) \quad (\Gamma, a \vdash W \triangleright \Gamma) \Rightarrow W \]

\[(viii) \quad (\Gamma \vdash s \triangleright \Delta) \Rightarrow u \quad (\Gamma \vdash N) \Rightarrow V \]

\[ (\Gamma \vdash (s, N \setminus a) \triangleright \Delta, a) \Rightarrow (u, V) \]

\[(ix) \quad (\Gamma \vdash s \triangleright \Delta) \Rightarrow u \quad (\Delta \vdash q \triangleright \Sigma) \Rightarrow v \]

\[ (\Gamma \vdash s \circ q \triangleright \Sigma) \Rightarrow u \circ v \]

**Example 7.7.**

\[(x \vdash x) \Rightarrow 1 \]

**Example 7.8.**

\[ (x \vdash x) \Rightarrow 1 \]

\[ (x, y \vdash x) \Rightarrow W \circ 1 \]

**Example 7.9.**

\[ (x, y \vdash W \triangleright x) \Rightarrow W \quad (x \vdash x) \Rightarrow 1 \]

\[ (x, y \vdash W \circ y \triangleright x) \Rightarrow W \circ 1 \]
Corollary 7.10. \((\Gamma, a, b_1, \ldots, b_n \vdash a) \Rightarrow n + 1\) if \(a \neq b_1, \ldots, a \neq b_n\).

Lemma 7.11. If \((\Gamma \vdash M) \Rightarrow U\), then \(\Gamma \vdash M\) is derivable.

Proof. The proof is straightforward, see Definition 2.13 and Definition 7.6. \qed

Definition 7.12. We write \((\Gamma \vdash M) \simeq (\Delta \vdash N)\) iff
\((\Gamma \vdash M) \Rightarrow U\) and
\((\Delta \vdash N) \Rightarrow U\), for some \(U\).

Example 7.13. \((x, y \vdash x) \simeq (x, y \vdash W_y \circ x)\)

Definition 7.14. A name-free term \(U\) is called \emph{pure} if it is constructed from the terms \(n\) by using application and abstraction.

Lemma 7.15. If \((\Gamma \vdash M) \Rightarrow U\) and \(M\) is pure, then \(U\) is pure. If \(U\) is pure, then \(U\) is a \(\sigma\)-normal form.

Proof. Each pure term \(M\) is constructed from variables by using application and abstraction. \qed

Definition 7.16. By definition, put
\[\uparrow (s) \equiv \langle W \circ s, 1 \rangle\]
\[\uparrow^n (s) \equiv \uparrow (\ldots (\uparrow (s)) \ldots)\]
\(n\) times

Lemma 7.17. For any nameless term \(U\),
\(\sigma(\langle W, 1 \rangle \circ U) \equiv \sigma(U)\) and
\(\sigma(\uparrow^n (W, 1) \circ U) \equiv \sigma(U)\)

Proof. See [1], Lemma 3.6. \qed

Definition 7.18. By \(\sim\sim\sim\) and \(\Rightarrow\) denote (reflexive and transitive) reductions in a calculus \(T\) (\(T\) may be \(\lambda W, \sigma W\alpha, \lambda\sigma\) or \(\sigma\)).

Theorem 7.19. Suppose
\[\Gamma \vdash M \stackrel{\sigma W\alpha}{\sim\sim\sim} \Sigma \vdash L;\]
\(\Sigma \vdash L\) is a \(\sigma W\alpha\)-normal form (with respect to \(\sim\sim\sim\));
\((\Gamma \vdash M) \Rightarrow U;\)
\((\Sigma \vdash L) \Rightarrow V;\)
then \(V\) is a \(\sigma\)-normal form and \(U \Rightarrow\sigma V\).

Proof. \(\Gamma \vdash M\) is derivable by Lemma 7.11 \(\Sigma \vdash L\) is derivable by Subject reduction. Therefore \(L\) is pure and \(V\) is a \(\sigma\)-normal form (Theorem 6.6, Lemma 7.15). Why \(U \Rightarrow\sigma V\)? It is sufficient to prove that \(\sigma(U) \equiv \sigma(V)\) (\(U\) reduces to its \(\sigma\)-normal form because \(\sigma\) is strongly normalizing and confluent). The proof is by induction over the length of the reduction sequence \(\Gamma \vdash M \stackrel{\sigma W\alpha}{\sim\sim\sim} \Sigma \vdash L.\) If this length is equal to 0, there is nothing to prove.
Otherwise, suppose this sequence has the form
\( \Gamma \vdash M \leadsto \ldots \leadsto \Delta \vdash N \leadsto \Sigma \vdash L \),
where \( (\Delta \vdash N) \Rightarrow W \) and \( \sigma(U) \equiv \sigma(W) \).
Any possible \( \sigma W \alpha \) -reduction step \( \Delta \vdash N \leadsto \Sigma \vdash L \), except \( \text{New} \), \( W_1 \), \( W_2 \), \( \alpha_1 \), and \( \alpha_2 \) corresponds to the same name \( \sigma \)-reduction step of the nameless terms
\( W \rightarrow V \), hence \( \sigma(W) \equiv \sigma(V) \) in these cases. For \( \alpha_1 \) and \( \alpha_2 \) use Lemma 7.17.

(Case \( W_1 \)) \( W_a \circ b \rightarrow b \quad (a \neq b) \)

\[
\begin{array}{c}
\Gamma, a \vdash W_a \triangleright \Gamma \\
\Gamma, a \vdash W_a \circ b \\
\Gamma, a \vdash b
\end{array}
\]

Suppose
\( (\Gamma \vdash b) \Rightarrow \nu; \)
then
\( (\Gamma, a \vdash W_a \circ b) \Rightarrow \sigma(W \circ \nu) \) and
\( (\Gamma, a \vdash b) \Rightarrow n + 1. \)
We see that \( \sigma(W \circ \nu) \equiv \sigma(n + 1) \).

(Case \( W_2 \)) \( (s \circ W_a) \circ b \rightarrow s \circ b \quad (a \neq b) \)

\[
\begin{array}{c}
\Delta \vdash s \triangleright \Gamma, a \\
\Delta \vdash s \circ W_a \triangleright \Gamma \\
\Delta \vdash (s \circ W_a) \circ b \\
\Delta \vdash s \circ b
\end{array}
\]

Suppose
\( (\Delta \vdash s \triangleright \Gamma, a) \Rightarrow \nu; \)
\( (\Gamma \vdash b) \Rightarrow \nu; \)
then
\( (\Delta \vdash (s \circ W_a) \circ b) \Rightarrow (u \circ W) \circ \nu; \)
\( (\Gamma, a \vdash b) \Rightarrow n + 1; \)
\( (\Delta \vdash s \circ b) \Rightarrow u \circ n + 1. \)
We see that
\( \sigma((u \circ W) \circ \nu) \equiv \sigma(u \circ n + 1) \)

(Case \( \text{New} \)) \( \langle s, N \setminus a \rangle \circ b \rightarrow s \circ b \quad (a \neq b) \)

\[
\begin{array}{c}
\Delta \vdash s \triangleright \Gamma \\
\Delta \vdash s \triangleright N \\
\Delta \vdash s \triangleright \langle s, N \setminus a \rangle, a \\
\Delta \vdash s \circ b
\end{array}
\]

Suppose
\( (\Delta \vdash s \triangleright \Gamma) \Rightarrow \nu; \)
\( (\Delta \vdash N) \Rightarrow V; \)
\( (\Gamma \vdash b) \Rightarrow \nu; \)
then
\((\Gamma \vdash a \Rightarrow b) \Rightarrow n + 1;\)
\((\Delta \vdash (s, N \setminus a) \circ b) \Rightarrow (u, V) \circ n + 1;\)
\((\Delta \vdash s \circ b) \Rightarrow u \circ n.\)
We see that
\(\sigma((u, V) \circ n + 1) \equiv \sigma((u, V) \circ W \circ n) \equiv \sigma(u \circ n).\)

**Theorem 7.20.** \(\sigma W\alpha\) is strongly normalizing (on the sets of terms, substitutions, and judgements of the form \(\Gamma \vdash M\)).

**Proof.** The proof is postponed until Section 10.

**Definition 7.21.** (One-step \(\beta\)-reduction on the set of pure name-free terms).
\[
\begin{align*}
U & \rightarrow_{\text{Beta}} V \\
U & \rightarrow_{\beta} \sigma(V) \quad (U \text{ is pure})
\end{align*}
\]
(the compatible closure, of course).

**Lemma 7.22.** If \(U \rightarrow_{\text{Beta}} V\), then \(\sigma(U) \rightarrow^*_{\beta} \sigma(V)\), where \(\rightarrow^*_{\beta}\) is the reflexive closure of \(\rightarrow_{\beta}\).

**Proof.** See [1], Lemma 3.5.

**Theorem 7.23.** Suppose
\(\Gamma \vdash M \Rightarrow L;\)
\((\Gamma \vdash M) \Rightarrow U;\)
\((\Sigma \vdash L) \Rightarrow V;\)
then \(\sigma(U) \rightarrow_{\lambda\sigma} \sigma(V).\)

**Proof.** The proof is by induction over the length of the reduction sequence \(\Gamma \vdash M \Rightarrow L\). If this length is equal to 0, there is nothing to prove. Otherwise, suppose this sequence has the form
\(\Gamma \vdash M \Rightarrow \ldots \Rightarrow \Delta \vdash N \Rightarrow \Sigma \vdash L,\)
where \((\Delta \vdash N) \Rightarrow W\) and \(\sigma(U) \rightarrow_{\lambda\sigma} \sigma(W)\).
If the reduction step \(\Delta \vdash N \Rightarrow \Sigma \vdash L\) belongs to \(\sigma W\alpha\), everything is all right, because \(\sigma(W) \equiv \sigma(V)\) in this case. Indeed, \(\Gamma \vdash M\) is derivable by Lemma 7.11 \(\Delta \vdash N\) and \(\Sigma \vdash L\) are derivable by Subject reduction. Take any \(\sigma W\alpha\)-normal form of \(\Sigma \vdash L\) (this normal form exists by Theorem 7.20 and are derivable too) and use Theorem 7.19 to get \(W \rightarrow^*_{\beta} \sigma(V)\).
If \(\Delta \vdash N \Rightarrow_{\text{Beta}} \Sigma \vdash L\), then \(W \rightarrow_{\text{Beta}} V\), because any \(\text{Beta}\)-redex in \(N\) corresponds to some \(\text{Beta}\)-redex in \(W\), hence \(\sigma(W) \rightarrow^*_{\beta} \sigma(V)\) by Lemma 7.22.

**Theorem 7.24.** Suppose
\((\Gamma \vdash M) \Rightarrow U;\)
\(U \rightarrow_{\lambda\sigma} V;\)
then there is \( \Sigma \vdash L \) such that

\[ \Gamma \vdash M \xrightarrow{\lambda W} \Sigma \vdash L; \]

\( \Sigma \vdash L \) is a \( \sigma W\alpha \)-normal form (with respect to \( \sim \sim \sim \));

\[ (\Sigma \vdash L) \Rightarrow \sigma(V). \]

Proof. The proof is by induction over the length of the reduction sequence \( U \xrightarrow{\lambda} V \).

Case 1: If this length is equal to 0, take any \( \sigma W\alpha \)-normal form of \( \Gamma \vdash M \) as \( \Sigma \vdash L \) and use Theorem 7.19.

Case 2: Suppose this sequence has the form \( U \rightarrow \ldots \rightarrow W \rightarrow V \) and the sequence \( U \rightarrow \ldots \rightarrow W \) satisfies the statement of the theorem, i.e.:

\[ \Gamma \vdash M \xrightarrow{\lambda W} \Delta \vdash N \text{ for some } \Delta \vdash N, \]

where \( \Delta \vdash N \) is a \( \sigma W\alpha \)-normal form and \( (\Delta \vdash N) \Rightarrow \sigma(W) \).

If the reduction step \( W \rightarrow V \) belongs to \( \sigma \), everything is all right, because \( \sigma(W) \equiv \sigma(V) \) in this case and we can use \( \Delta \vdash N \) as \( \Sigma \vdash L \).

If \( W \rightarrow_{Beta} V \), then \( \sigma(W) \rightarrow_{\beta}^* \sigma(V) \) by Lemma 7.22. If \( \sigma(W) \) coincides with \( \sigma(V) \), everything is all right. Otherwise, suppose \( \sigma(W) \rightarrow_{\beta}^* \sigma(V) \) has the form \( \sigma(W) \rightarrow_{Beta} W' \rightarrow_{\beta}^* \sigma(V) \). Any Beta-redex in \( \sigma(W) \) corresponds to some Beta-redex in \( N \). Contracting this redex in \( N \), we obtain \( \Delta \vdash N \xrightarrow{Beta} \Delta \vdash N' \) and \( (\Delta \vdash N') \Rightarrow W' \), for some \( N' \). Take any \( \sigma W\alpha \)-normal form of \( \Delta \vdash N' \) as \( \Sigma \vdash L \), then use Theorem 7.19 to obtain \( (\Sigma \vdash L) \Rightarrow \sigma(V) \).
8. \(\alpha\)-EQUIVALENCE

**Definition 8.1.** Only in this section, we use the following notation: the symbols \(U, V, W\) range over *extended name-free terms* and the symbols \(u, v, w\) range over *extended name-free substitutions*. The sets of extended name-free terms and extended name-free substitutions are defined inductively as follows:

\[
U, V :: = n | UV | \lambda U | u \circ U \\
u, v :: = id | W | \langle u, V \rangle | u \circ v
\]

\((n \in \mathbb{N}, n \geq 1)\)

**Example 8.2.** \(\lambda\lambda(\lambda W \circ 1)\) is an extended nameless term.

**Definition 8.3.** An extended nameless term \(U\) is called *pure* iff it does not contain sub-terms of the form \(u \circ U\).

It is clear that any pure term is constructed from the symbols \(n\) by using application and abstraction.

**Definition 8.4.** A *name-free judgement* is an expression of the form \(m \vdash U\) or of the form \(m \vdash u\), where \(m \in \mathbb{N}, m \geq 0\).

Informally, \(m\) is “the length of an invisible context”.

**Definition 8.5.** \((\Gamma \vdash M) \Rightarrow (m \vdash U)\) is shorthand for “the name-free judgement \(m \vdash U\) corresponds to the judgement \(\Gamma \vdash M\).”

\((\Gamma \vdash s \triangleright \Delta) \Rightarrow (m \vdash u)\) is shorthand for “the name-free judgement \(m \vdash u\) corresponds to the judgement \(\Gamma \vdash s \triangleright \Delta\).”

**Definition 8.6.** By \(|\Gamma|\) denote the length of \(\Gamma\).
Definition 8.7. (The rules of correspondence between judgements and name-free judgements).

(i) \((\Gamma, a \vdash a) \Rightarrow (|\Gamma| \vdash 1)\)

(ii) \((\Gamma \vdash a) \Rightarrow (|\Gamma| \vdash n)\) \((a \neq b)\)

(iii) \((\Gamma \vdash M) \Rightarrow (|\Gamma| \vdash U)\) \((\Gamma \vdash N) \Rightarrow (|\Gamma| \vdash V)\)

(iv) \((\Gamma, a \vdash M) \Rightarrow (|\Gamma, a| \vdash U)\)

(v) \((\Gamma \vdash \lambda a. M) \Rightarrow (|\Gamma| \vdash \lambda U)\)

Corollary 8.8. If \((\Gamma \vdash M) \Rightarrow (m \vdash U)\), then \(m = |\Gamma|\).
If \((\Gamma \vdash s \triangleright \Delta) \Rightarrow (m \vdash u)\), then \(m = |\Gamma|\).

Example 8.9.
\((x \vdash x) \Rightarrow (1 \vdash 1)\)

Example 8.10.
\((x \vdash x) \Rightarrow (1 \vdash 1)\)
\((x, y \vdash x) \Rightarrow (2 \vdash 2)\)

Example 8.11.
\((x, y \vdash W_y \triangleright x) \Rightarrow (2 \vdash W)\) \((x \vdash x) \Rightarrow (1 \vdash 1)\)
\((x, y \vdash W_y \circ x) \Rightarrow (2 \vdash W \circ 1)\)

Definition 8.12. (\(\alpha\)-equivalence).
We say that \((\Gamma \vdash M)\) is \(\alpha\)-equal to \((\Delta \vdash N)\) and write
\((\Gamma \vdash M) \equiv_\alpha (\Delta \vdash N)\) iff
\((\Gamma \vdash M) \Rightarrow (m \vdash U)\) and
\((\Delta \vdash N) \Rightarrow (m \vdash U)\), for some \(m, U\).

Example 8.13.
\((x, y \vdash W_y \circ x) \Rightarrow (2 \vdash W \circ 1)\)
(x, x ⊢ W_x ◦ x) ⇒ (1 ⊢ W ◦ 1)
(x, y ⊢ W_y ◦ x) ≡_α (x, x ⊢ W_x ◦ x)

Example 8.14.
(x ⊢ λy.W_y ◦ x) ⇒ (1 ⊢ λW ◦ 1)
(x ⊢ λx.W_x ◦ x) ⇒ (1 ⊢ λW ◦ 1)
(x ⊢ λy.W_y ◦ x) ≡_α (x ⊢ λx.W_x ◦ x)

Example 8.15.
(0 ⊢ λx.λy.W_y ◦ x) ⇒ (0 ⊢ λλW ◦ 1)
(0 ⊢ λx.λx.W_x ◦ x) ⇒ (0 ⊢ λλW ◦ 1)
(0 ⊢ λx.λy.W_y ◦ x) ≡_α (0 ⊢ λx.λx.W_x ◦ x)

Warning! We can apply W_1 to the term λx.λy.W_y ◦ x, but not to the
term λx.λx.W_x ◦ x (and we can apply α_1 to the term λx.λx.W_x ◦ x, but not
to the term λx.λy.W_y ◦ x).

Example 8.16.
(x, y ⊢ x) ≍ (x, y ⊢ W_y ◦ x)
(x, y ⊢ x) ≉_α (x, y ⊢ W_y ◦ x)

Lemma 8.17. If (Γ ⊢ M) ⇒ (m ⊢ U), then Γ ⊢ M is derivable.

Proof. The proof is straightforward, see Definition 2.13 and Definiton 8.7.

Corollary 8.18. If (Γ ⊢ M) ≡_α (Δ ⊢ N), then Γ ⊢ M and Δ ⊢ N are
derivable.
9. Confluence

Lemma 9.1. Suppose $(\Gamma \vdash M) \equiv_\alpha (\Delta \vdash N); \text{ then } (\Gamma \vdash M) \simeq (\Delta \vdash N)$. Suppose $(\Gamma \vdash M) \simeq (\Delta \vdash N)$, where $|\Gamma| = |\Delta|$ and both $M$ and $N$ are pure; then $(\Gamma \vdash M) \equiv_\alpha (\Delta \vdash N)$.

Proof. The proof of the first part is straightforward, see Definition 7.6 and Definition 8.7. To prove the second part, recall that each pure term is constructed from variables by using application and abstraction. This prevents such counterexamples as Example 8.16.

Theorem 9.2 ($\sigma W_\alpha$ is confluent). Suppose

$(\Gamma_1 \vdash M_1) \equiv_\alpha (\Gamma_2 \vdash M_2)$;

$\Gamma_1 \vdash M_1 \sigma W_\alpha \Delta_1 \vdash N_1$;

$\Gamma_2 \vdash M_2 \sigma W_\alpha \Delta_2 \vdash N_2$;

then there are $\Sigma_1 \vdash L_1$ and $\Sigma_2 \vdash L_2$ such that

$\Delta_1 \vdash N_1 \sigma W_\alpha \Sigma_1 \vdash L_1$;

$\Delta_2 \vdash N_2 \sigma W_\alpha \Sigma_2 \vdash L_2$;

$(\Sigma_1 \vdash L_1) \equiv_\alpha (\Sigma_2 \vdash L_2)$.

Proof. By Lemma 9.1 we have $(\Gamma_1 \vdash M_1) \simeq (\Gamma_2 \vdash M_2)$. Suppose $(\Gamma_1 \vdash M_1) \Rightarrow U$ and $(\Gamma_2 \vdash M_2) \Rightarrow U$.

Let $\Sigma_1 \vdash L_1$ be any $\sigma W_\alpha$-normal form of $\Delta_1 \vdash N_1$ and let $\Sigma_2 \vdash L_2$ be any $\sigma W_\alpha$-normal form of $\Delta_2 \vdash N_2$. By Theorem 7.19 we have $(\Sigma_1 \vdash L_1) \Rightarrow \sigma(U)$ and $(\Sigma_2 \vdash L_2) \Rightarrow \sigma(U)$, hence $(\Sigma_1 \vdash L_1) \simeq (\Sigma_2 \vdash L_2)$. Note that $L_1$ and $L_2$ are pure (Theorem 6.6). Note that $|\Sigma_1| = |\Sigma_2| = |\Gamma_1| = |\Gamma_2|$ (because all reductions preserve lengths of contexts). By Lemma 9.1 we have $(\Sigma_1 \vdash L_1) \equiv_\alpha (\Sigma_2 \vdash L_2)$.

Theorem 9.3 ($\lambda W$ is confluent). Suppose

$(\Gamma_1 \vdash M_1) \equiv_\alpha (\Gamma_2 \vdash M_2)$;

$\Gamma_1 \vdash M_1 \lambda W \Delta_1 \vdash N_1$;

$\Gamma_2 \vdash M_2 \lambda W \Delta_2 \vdash N_2$;

then there are $\Sigma_1 \vdash L_1$ and $\Sigma_2 \vdash L_2$ such that

$\Delta_1 \vdash N_1 \lambda W \Sigma_1 \vdash L_1$;

$\Delta_2 \vdash N_2 \lambda W \Sigma_2 \vdash L_2$;

$(\Sigma_1 \vdash L_1) \equiv_\alpha (\Sigma_2 \vdash L_2)$.

Proof. By Lemma 9.1 we have $(\Gamma_1 \vdash M_1) \simeq (\Gamma_2 \vdash M_2)$. Suppose $(\Gamma_1 \vdash M_1) \Rightarrow U$; $(\Gamma_2 \vdash M_2) \Rightarrow U$; $(\Delta_1 \vdash N_1) \Rightarrow V_1$; and $(\Delta_2 \vdash N_2) \Rightarrow V_2$.

By Theorem 7.23 we have $\sigma(U) \xrightarrow{\lambda \sigma} \sigma(V_1)$ and $\sigma(U) \xrightarrow{\lambda \sigma} \sigma(V_2)$. We know that $\lambda \sigma$ is confluent, hence $\sigma(V_1) \xrightarrow{\lambda \sigma} V$ and $\sigma(V_2) \xrightarrow{\lambda \sigma} V$ for
some $V$. Therefore $V_1 \xrightarrow{\lambda \sigma} V$ and $V_2 \xrightarrow{\lambda \sigma} V$. By Theorem 7.24 we have

$\Sigma_1 \vdash L_1$ and $\Sigma_2 \vdash L_2$ such that

$\Delta_1 \vdash N_1 \overset{\lambda W}{\sim \sim} \Sigma_1 \vdash L_1;$

$\Delta_2 \vdash N_2 \overset{\lambda W}{\sim \sim} \Sigma_2 \vdash L_2;$

$\Sigma_1 \vdash L_1$ is a $\sigma W\alpha$-normal form;

$\Sigma_2 \vdash L_2$ is a $\sigma W\alpha$-normal form;

$(\Sigma_1 \vdash V_1) \Rightarrow \sigma(V);$  

$(\Sigma_2 \vdash V_2) \Rightarrow \sigma(V).$

Hence $(\Sigma_1 \vdash L_1) \equiv (\Sigma_2 \vdash L_2)$. Note that $L_1$ and $L_2$ are pure (Theorem 6.6). Note that $|\Sigma_1| = |\Sigma_2| = |\Gamma_1| = |\Gamma_2|$ (because all reductions preserve lengths of contexts). By Lemma 9.1 we have

$(\Sigma_1 \vdash L_1) \equiv_{\alpha} (\Sigma_2 \vdash L_2).$

□

Definition 9.4. By $\Lambda W$ denote the set of derivable judgements of the form $\Gamma \vdash M$.

We see that $\overset{\lambda W}{\sim \sim}$ and $\overset{\sigma W\alpha}{\sim \sim}$ are confluent (up to $\equiv_{\alpha}$) on the set $\Lambda W$.  

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Definition 10.1. \( A \subseteq B \) is shorthand for “\( A_i \subseteq \bigcup_{j \geq i} B_j \) for all \( i \geq 1 \)”.

Example 10.2. \( \langle \{y\} , \emptyset , \emptyset , \ldots \rangle \subseteq \langle \emptyset , \{y\} , \emptyset , \emptyset , \ldots \rangle \)

Note that \( A \subseteq B \) implies \( A \subseteq B \).

Lemma 10.3. \( O_{\lambda a} \) and \( O_W \) are monotone operators with respect to \( \subseteq \) (for any \( a \)).

Proof. The proof is straightforward. \( \square \)

Lemma 10.4. \( A \cup B \) is monotone in both arguments with respect to \( \subseteq \).

Proof. The proof is straightforward. \( \square \)

Corollary 10.5. \( O_s \) is monotone with respect to \( \subseteq \) for any \( s \).

Recall that \( O_s \) is also monotone with respect to \( \subseteq \) for any \( s \).

Lemma 10.6. If \( M_1 \rightarrow M_2 \), then \( F V(M_2) \subseteq F V(M_1) \). If \( s_1 \rightarrow s_2 \), then \( O_{s_2}(A) \subseteq O_{s_1}(A) \) for any \( A \).

Proof. The proof is straightforward, but tedious. For example, consider

(Abs) \( s \circ \lambda a. M \rightarrow \lambda a.(W_a \circ s , a \setminus a) \circ M \)

\[
F V(\lambda a.(W_a \circ s , a \setminus a) \circ M) \\
= F V(\lambda a.((W_a \circ s) \circ \lambda a.M)a) \quad \text{(Lemma 3.17)} \\
= F V(\lambda a.(W_a \circ s \circ \lambda a.M)a) \quad \text{(Lemma 3.16)} \\
= O_{\lambda a}(O_W(F V(s \circ \lambda a.M)) \cup F V(a)) \\
= O_{\lambda a}((\{a\}, F V_1(s \circ \lambda a.M), F V_2(s \circ \lambda a.M), \ldots)) \\
= F V(s \circ \lambda a.M) \\
\]

(App) \( s \circ M N \rightarrow (s \circ M)(s \circ N) \)

\[
F V(s \circ M N) \\
= O_s(F V(M N)) \\
= O_s(F V(M) \cup F V(N)) \\
\supseteq O_s(F V(M)) \cup O_s(F V(N)) \\
= F V(s \circ M) \cup F V(s \circ N) \\
= F V((s \circ M)(s \circ N)) \\
\]

(ConsVar) \( \langle s , N \setminus a \rangle \circ a \rightarrow N \)

\[
F V(\langle s , N \setminus a \rangle \circ a) \\
= O_{\langle s , N \setminus a \rangle}(F V(a)) \\
= O_s(O_{\lambda a}(F V(a))) \cup F V(N) \\
\supseteq F V(N) \\
\]

(New) \( \langle s , N \setminus a \rangle \circ b \rightarrow s \circ b \quad (a \neq b) \)
\[ FV((s, N\setminus a) \circ b) \\
= O_{(s, N\setminus a)}(FV(b)) \\
= O_s(O_{\lambda a}(FV(b))) \cup FV(N) \\
\supseteq O_s(O_{\lambda a}(FV(b))) \\
= O_s(\{(b), \emptyset, \emptyset, \ldots\}) \\
= O_s(FV(b)) \\
= FV(b) \]
\( FV(\lambda b. (W_b \circ b \backslash a) \circ M) \)
\[= FV(\lambda b. (W_b \circ \lambda a.M)b) \quad \text{(Lemma 3.17)} \]
\[= O_{\lambda b}(O_{W}(FV(\lambda a.M)) \cup FV(b)) \]
\[= O_{\lambda b}((\{b\}, FV_1(\lambda a.M), FV_2(\lambda a.M), \ldots)) \]
\[= FV(\lambda a.M) \]

In addition, it is necessary to prove that all operations from Definition 2.1 are in some sense monotone, but this is not difficult.

**Corollary 10.7.** \( FV(s \circ \lambda a.M) = FV(\lambda a. \uparrow_a (s) \circ M) \)

**Corollary 10.8.** \( FV(s \circ \lambda \Delta.M) = FV(\lambda \Delta. \uparrow_{\Delta} (s) \circ M) \)

**Corollary 10.9.** If \( M \rightarrow N \), then \( \bigcup_{i \geq 1} FV_i(N) \subseteq \bigcup_{i \geq 1} FV_i(M) \).

To prove that \( \sigma W_\alpha \) is strongly normalizing, we consider the following two-sorted term rewriting system \( R \).

**Definition 10.10.** The signature of \( R \) contains:
- \( M, N, L, \ldots \) variables;
- \( s, q, r, \ldots \) variables;
- \( x, y, z, \ldots \) constants;
- \( id, W_x, W_y, W_z, \ldots \) constants;
- \( \lambda x, \lambda y, \lambda z, \ldots \) functional symbols of arity one;
- \( \cdot, \circ \) functional symbols of arity two;
- \( \langle -, - \backslash x \rangle, \langle -, - \backslash y \rangle, \langle -, - \backslash z \rangle, \ldots \) functional symbols of arity two.

We will omit \( \cdot \), which denotes application. The sets of ground terms and ground substitutions of \( R \) are defined inductively as follows:

\[ M, N ::= a \mid MN \mid \lambda a.M \mid \lambda a.M \mid s \circ M \]
\[ s, q ::= id \mid W_a \mid \langle s, N \backslash a \rangle \mid s \circ q \]

We will use the same abbreviations as in Convention 2.2 and Convention 2.3.
Definition 10.11. (The rewriting system \( R \)).

(Abs1) \( s \circ \lambda a.M \rightarrow \lambda a.(W_a \circ s, a \setminus a) \circ M \)

(Abs2) \( s \circ \lambda a.M \rightarrow \lambda a.(W_a \circ s, a \setminus a) \circ M \)

(Abs3) \( s \circ \lambda a.M \rightarrow \lambda a.(W_a \circ s, a \setminus a) \circ M \)

(Abs4) \( s \circ \lambda a.M \rightarrow \lambda a.(W_a \circ s, a \setminus a) \circ M \)

(App) \( s \circ MN \rightarrow (s \circ M)(s \circ N) \)

(ConsVar) \( \langle s, N \setminus a \rangle \circ a \rightarrow N \)

(New) \( \langle s, N \setminus a \rangle \circ b \rightarrow s \circ b \) \( (a \neq b) \)

(IdVar) \( id \circ a \rightarrow a \)

(Clos) \( s \circ q \circ M \rightarrow (s \circ q) \circ M \)

(Ass) \( s \circ q \circ r \rightarrow (s \circ q) \circ r \)

(IdR) \( s \circ id \rightarrow s \)

(IdShift) \( id \circ W_a \rightarrow W_a \)

(ConsShift) \( \langle s, N \setminus a \rangle \circ W_a \rightarrow s \)

(Map) \( s \circ \langle q, N \setminus a \rangle \rightarrow \langle s \circ q, s \circ N \setminus a \rangle \)

(W1) \( W_a \circ b \rightarrow b \) \( (a \neq b) \)

(W2) \( (s \circ W_a) \circ b \rightarrow s \circ b \) \( (a \neq b) \)

(\( \alpha \)) \( \lambda a.M \rightarrow \lambda b.\langle W_b, b \setminus a \rangle \circ M \)

(\( \xi \)) \( \lambda a.M \rightarrow \lambda a.M \)

Definition 10.12. To each term \( M \) we assign \( FV(M) \) as in Definition 12.1 with the additional case:

\[ FV(\lambda a.M) = FV(\lambda a.M) = O_{\lambda a}(FV(M)) \]

Lemma 10.13. If \( M_1 \stackrel{R}{\rightarrow} M_2 \), then \( FV(M_2) \subseteq FV(M_1) \).

Proof. See Lemma 10.6 □

Lemma 10.14. The restriction \( * \) in Definition 4.3 can be written as

\( * \) \( a \in \bigcup_{i \geq 1} FV_i(\lambda a.M); \ b \notin \bigcup_{i \geq 1} FV_i(\lambda b.(W_b, b \setminus a) \circ M) \)

Proof. \( FV(\lambda b.(W_b, b \setminus a) \circ M) = FV(\lambda a.M) \) by Lemma 10.6 (the case \( \alpha_1 \)). □

Definition 10.15. By \( M^* \) denote the term \( M \) in which all sub-terms of the shape \( \lambda a.L \), such that \( a \in \bigcup_{i \geq 1} FV_i(\lambda a.L) \), are replaced by \( \lambda a.L \).

Theorem 10.16. If \( R \) is strongly normalizing on the sets of ground terms and ground substitutions, then \( \sigma W\alpha \) is strongly normalizing (on the sets of terms, substitutions, and judgements of the form \( \Gamma \vdash M \)).

Proof. Suppose we have some infinite \( \sigma W\alpha \)-sequence

\[ M_1 \rightarrow M_2 \rightarrow \ldots \rightarrow M_n \rightarrow \ldots \]
I claim that we can get some infinite \( R \)-sequence

\[(M_1)^* \rightarrow R (M_2)^* \rightarrow R \ldots \rightarrow R (M_n)^* \rightarrow R \ldots \]

The proof is by induction over \( n \). If \( n \) is equal to 1, there is nothing to prove. Else there are three cases.

1) If the reduction step \( M_n \rightarrow M_{n+1} \) is not \( \text{Abs}, \alpha_1, \alpha_2 \), we can apply the \( R \)-reduction of the same name \((M_n)^* \rightarrow (M_{n+1})^*\), but then might need several \( \xi \)-steps, because \( \text{ConsVar}, \text{New}, \text{ConsShift}, W_1, W_2 \) can decrease \( \text{FV} \).

2) If \( M_n \rightarrow \alpha_1 M_{n+1} \), we can apply \( \alpha_1 \): \((M_n)^* \rightarrow \alpha_1 (M_{n+1})^* \) (see Lemma 10.14 and Example 10.18).

3) If \( M_n \rightarrow \text{Abs} M_{n+1} \) and the \( \text{Abs} \)-redex is \( s \circ \lambda a.M \), there are four possible subcases:

Subcase 1. \( a \notin \text{FV}_1(\lambda a.M) \), \( a \notin \text{FV}_1(\lambda a.(W_a \circ s, a \setminus a) \circ M) \);
Subcase 2. \( a \in \text{FV}_1(\lambda a.M) \), \( a \notin \text{FV}_1(\lambda a.(W_a \circ s, a \setminus a) \circ M) \);
Subcase 3. \( a \notin \text{FV}_1(\lambda a.M) \), \( a \in \text{FV}_1(\lambda a.(W_a \circ s, a \setminus a) \circ M) \);
Subcase 4. \( a \in \text{FV}_1(\lambda a.M) \), \( a \notin \text{FV}_1(\lambda a.(W_a \circ s, a \setminus a) \circ M) \);

and we can apply \( \text{Abs}_1, \text{Abs}_2, \text{Abs}_3, \text{Abs}_4 \), respectively. See Examples 10.19, 10.20, 10.21, and 10.22.

The proof is similar for substitutions. For judgements, suppose we have some infinite \( \sigma W \)-sequence

\[\Gamma_1 \vdash M_1 \dashv \Gamma_2 \vdash M_2 \dashv \Gamma_3 \vdash M_3 \dashv \ldots\]

We can obtain the \( \sigma W \)-sequence of terms

\[\Lambda \Gamma_1.M_1 \rightarrow \Lambda \Gamma_2.M_2 \rightarrow \Lambda \Gamma_3.M_3 \rightarrow \ldots\]

where \( \alpha_2 \)-steps are replaced by \( \alpha_1 \) and \( \text{Abs} \). \( \square \)

Example 10.17. The \( \sigma W \)-sequence

\[\lambda x.(id, W_x \circ x \setminus x) \circ y \rightarrow_{\text{New}} \lambda x.id \circ y \rightarrow \ldots\]

becomes the following \( R \)-sequence

\[\lambda x.(id, W_x \circ x \setminus x) \circ y \rightarrow_{\text{New}} \lambda x.id \circ y \rightarrow_{\xi} \lambda x.id \circ y \rightarrow \ldots\]

Example 10.18. The \( \sigma W \)-sequence

\[\lambda x.W_x \circ x \rightarrow_{\alpha_1} \lambda y.(W_y, y \setminus x) \circ W_x \circ x \rightarrow \ldots\]

becomes the following \( R \)-sequence

\[\lambda x.W_x \circ x \rightarrow_{\alpha} \lambda y.(W_y, y \setminus x) \circ W_x \circ x \rightarrow \ldots\]

Example 10.19. The \( \sigma W \)-sequence

\[id \circ \lambda x.x \rightarrow_{\text{Abs}} \lambda x.(W_x \circ id, x \setminus x) \circ x \rightarrow \ldots\]
becomes the following $R$-sequence

$$id \circ \lambda x.x \rightarrow_{Abs} \lambda x.\langle W_x \circ id, x \setminus x \rangle \circ x \rightarrow \ldots$$

**Example 10.20.** The $\sigma W\alpha$-sequence

$$id \circ \lambda x.W_x \circ x \rightarrow_{Abs} \lambda x.\langle W_x \circ id, W_x \circ x \rangle \circ x \rightarrow \ldots$$

becomes the following $R$-sequence

$$id \circ \lambda x.W_x \circ x \rightarrow_{Abs2} \lambda x.\langle W_x \circ id, W_x \circ x \rangle \circ x \rightarrow \ldots$$

**Example 10.21.** The $\sigma W\alpha$-sequence

$$\langle id, x \setminus y \rangle \circ \lambda x.x \rightarrow_{Abs3} \lambda x.\langle W_x \circ \langle id, x \setminus y \rangle, x \setminus x \rangle \circ x \rightarrow \ldots$$

becomes the following $R$-sequence

$$\langle id, x \setminus y \rangle \circ \lambda x.x \rightarrow_{Abs3} \lambda x.\langle W_x \circ \langle id, x \setminus y \rangle, x \setminus x \rangle \circ x \rightarrow \ldots$$

**Example 10.22.** The $\sigma W\alpha$-sequence

$$\langle id, \lambda y.y \setminus x \rangle \circ \lambda x.W_x \circ x \rightarrow_{Abs4} \lambda x.\langle W_x \circ \langle id, \lambda y.y \setminus x \rangle, x \setminus x \rangle \circ W_x \circ x \rightarrow \ldots$$

becomes the following $R$-sequence

$$\langle id, \lambda y.y \setminus x \rangle \circ \lambda x.W_x \circ x \rightarrow_{Abs4} \lambda x.\langle W_x \circ \langle id, \lambda y.y \setminus x \rangle, x \setminus x \rangle \circ W_x \circ x \rightarrow \ldots$$

To prove that $R$ is strongly normalizing on the sets of ground terms and ground substitutions, we use the method of semantic labelling. See [6].

**Definition 10.23.** To each term $M$ and each substitution $s$ we put in correspondence natural numbers $|M|$ and $|s|$ respectively defined as follows:

$$|\lambda a. M| = |M| + 1$$

$$|s \circ M| = |s| + |M|$$

$$|s \circ q| = |s| + |q|$$

$$|MN| = max(|M|, |N|)$$

$$|\langle s, N \setminus a \rangle| = max(|s|, |N|)$$

$$|id| = 0$$

$$|W_a| = 0$$

$$|a| = 0$$

Note that any functional symbol of $R$ now turns to some monotone function of $\mathbb{N}$ to $\mathbb{N}$ or of $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. Consider the following two-sorted term rewriting system $Q$. 

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Definition 10.24. The signature of $Q$ contains:

- $M, N, L, \ldots$ variables;
- $s, q, r, \ldots$ variables;
- $x, y, z, \ldots$ constants;
- $id, W_x, W_y, W_z, \ldots$ constants;
- $\lambda x, \lambda y, \lambda z, \ldots$ functional symbols of arity one;
- $\lambda_i x, \lambda_i y, \lambda_i z, \ldots$ functional symbols of arity one;
- $\cdot, \circ_i$ functional symbols of arity two;
- $\langle -, - \setminus x \rangle, \langle -, - \setminus y \rangle, \langle -, - \setminus z \rangle, \ldots$ functional symbols of arity two;

where $i \in \mathbb{N}, i \geq 0$.

We will omit $\cdot$, which denotes application. The sets of ground terms and ground substitutions of $Q$ are defined inductively as follows:

\[
M, N :: = a \mid MN \mid \lambda a.M \mid \lambda_i a.M \mid s \circ_i M
\]

\[
s, q :: = id \mid W_a \mid \langle s, N \setminus a \rangle \mid s \circ_i q
\]

We will use the same abbreviations as in Convention 2.2 and Convention 2.3.
Definition 10.25. (The rewriting system $Q$).

(Abs1) $s \circ_{i+1} \lambda a. M \rightarrow \lambda a. (W_a \circ_k s \circ_{a\backslash a} \circ_i M$ \hspace{1cm} ($i \geq k$)

(Abs2) $s \circ_{i+1} \lambda_{j+1} a. M \rightarrow \lambda_{i+1} a. (W_a \circ_k s \circ_{a\backslash a} \circ_i M$ \hspace{1cm} ($i = j + k$)

(Abs3) $s \circ_{i+1} \lambda a. M \rightarrow \lambda_{i+1} a. \langle W_a \circ_k s \circ_{a\backslash a} \circ_i M$ \hspace{1cm} ($i \geq k$)

(Abs4) $s \circ_{i+1} \lambda_{j+1} a. M \rightarrow \lambda a. (W_a \circ_k s \circ_{a\backslash a} \circ_i M$ \hspace{1cm} ($i = j + k$)

(App) $s \circ_i MN \rightarrow (s \circ_j M)(s \circ_k N)$ \hspace{1cm} ($i \geq j, i \geq k$)

(ConsVar) $\langle s, N\backslash a \rangle \circ_i a \rightarrow N$

(New) $\langle s, N\backslash a \rangle \circ_i b \rightarrow s \circ_j b$ \hspace{1cm} ($a \neq b, i \geq j$)

(IdVar) $id \circ_0 a \rightarrow a$

(Clos) $s \circ_{i+j+k} q \circ_{j+k} M \rightarrow (s \circ_{i+j} q) \circ_{i+j+k} M$

(Ass) $s \circ_{i+j+k} q \circ_{j+k} r \rightarrow (s \circ_{i+j} q) \circ_{i+j+k} r$

(IdR) $s \circ_i id \rightarrow s$

(IdShift) $id \circ_0 W_a \rightarrow W_a$

(ConsShift) $\langle s, N\backslash a \rangle \circ_i W_a \rightarrow s$

(Map) $s \circ_i \langle q, N\backslash a \rangle \rightarrow \langle s \circ_j q, s \circ_k N\backslash a \rangle$ \hspace{1cm} ($i \geq j, i \geq k$)

(W1) $W_a \circ_0 b \rightarrow b$ \hspace{1cm} ($a \neq b$)

(W2) $s \circ_i W_a \circ_i b \rightarrow s \circ_i b$ \hspace{1cm} ($a \neq b$)

($\alpha$) $\lambda_{i+1} a. M \rightarrow \lambda b. (W_b \circ_{b\backslash a} \circ_i M$

($\xi$) $\lambda_{i+1} a. M \rightarrow \lambda a. M$

(Decr1) $\lambda_{i+1} a. M \rightarrow \lambda_j a. M$ \hspace{1cm} ($i > j$)

(Decr2) $s \circ_i M \rightarrow s \circ_j M$ \hspace{1cm} ($i > j$)

(Decr3) $s \circ_i q \rightarrow s \circ_j q$ \hspace{1cm} ($i > j$)

where $i, j, k \in N$. (Roughly, these are the rewrite rules of $R$, where $\circ$ and $\lambda a$ are labelled by theirs own values).
Theorem 10.26. \( Q \) is strongly normalizing on the sets of ground terms and ground substitutions.

Proof. By choosing the well-founded precedence

\[
\lambda_{i+1}a > o_i > \lambda_ia \quad \text{for all } i, a;
\]
\[
o_i > \lambda a \quad \text{for all } i, a;
\]
\[
o_i > \cdot \quad \text{for all } i;
\]
\[
o_i > \langle -, -, \backslash a \rangle \quad \text{for all } i, a;
\]
\[
o_i > \mathcal{W}_a \quad \text{for all } i, a;
\]
\[
o_i > a \quad \text{for all } i, a;
\]
\[
\lambda_ia > \lambda b \quad \text{for all } i, a, b;
\]
\[
\lambda_ia > \langle -, -, \backslash a \rangle \quad \text{for all } i, a;
\]
\[
\lambda_ia > \mathcal{W}_b \quad \text{for all } i, a, b;
\]
\[
\lambda_ia > b \quad \text{for all } i, a, b;
\]
\[
\lambda_ia > \lambda_ja \quad \text{for } i > j;
\]
\[
o_i > o_j \quad \text{for } i > j;
\]

termination is easily proved by the lexicographic path order. \( \square \)

Theorem 10.27. \( R \) is strongly normalizing on the sets of ground terms and ground substitutions.

Proof. For any infinite \( R \)-sequence

\[
M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow \ldots
\]

we can get some infinite \( Q \)-sequence simply by labelling all symbols \( o \) and \( \lambda a \) by theirs own values. See [6], Theorem 81 for details (see also [6], Example 33). The proof is similar for substitutions. \( \square \)
11. Post canonical system for terms and substitutions

In this section we consider some Post canonical system building the sets of terms and substitutions. The alphabet of the system contains:

symbols

\( x \mid y \mid z \mid \lambda \mid . \mid \circ \mid id \mid W \mid (\mid) \mid (\mid) \)

variables

\( a \mid b \mid M \mid N \mid s \mid q \mid G \)

Term means that \( M \) is a term. Subst means that \( s \) is a substitution. Var means that \( a \) is a variable. App means that \( M \) is a term of the form \( M_1 M_2 \ldots M_n \), where \( n \geq 2 \). Abs means that \( M \) is a term of the form \( \lambda a_1 \ldots a_n. N \), where \( n \geq 1 \). Clos means that \( M \) is a term of the form \( s \circ N \). Cons means that \( s \) is a substitution of the form \( \langle q, N_1 \backslash a_1, \ldots, N_n \backslash a_n \rangle \), where \( n \geq 1 \). Comp means that \( s \) is a substitution of the form \( s_1 \circ s_2 \circ \ldots \circ s_n \), where \( n \geq 2 \). We will write \( W_a \) instead of \( Wa \).

The general rules:

| Var a          | App M       | Abs M   | Clos M   |
|----------------|-------------|---------|----------|
| Term a         | Term M      | Term M  | Term M   |
| Subst id       | Var a       | Cons s  | Comp s   |
| Subst Wa       | Subst s     | Subst s |

For simplicity, we will use only \( x, y, z \).

\( \text{Var } x \quad \text{Var } y \quad \text{Var } z \)

The following rules build terms of the form \( ab, \ a(N_1 N_2 \ldots N_k), \ a(\lambda a_1 \ldots a_k. N), \ a(s \circ N) \)

| Var a          | Var b       |
|----------------|-------------|
| App ab         |             |
| Var a          | App N       |
| App a(N)       |             |
| Var a          | Abs N       |
| App a(N)       |             |
| Var a          | Clos N      |
| App a(N)       |             |
The following rules build terms of the forms
\( M_1 M_2 \ldots M_n a \), \( M_1 M_2 \ldots M_n (N_1 N_2 \ldots N_k) \), \( M_1 M_2 \ldots M_n (\lambda a_1 \ldots a_k. N) \),
and \( M_1 M_2 \ldots M_n (s \circ N) \)

\[
\begin{align*}
&\text{App } M \quad \text{Var } a \\
&\text{App } Ma \\
&\text{App } M \quad \text{App } N \\
&\text{App } M(N) \\
&\text{App } M \quad \text{Abs } N \\
&\text{App } M(N) \\
&\text{App } M \quad \text{Clos } N \\
&\text{App } M(N)
\end{align*}
\]

The following rules build terms of the forms
\( (\lambda a_1 \ldots a_n. M) \ a \), \( (\lambda a_1 \ldots a_n. M)(N_1 N_2 \ldots N_k) \), \( (\lambda a_1 \ldots a_n. M)(\lambda b_1 \ldots b_k. N) \),
and \( (\lambda a_1 \ldots a_n. M)(s \circ N) \)

\[
\begin{align*}
&\text{Abs } M \quad \text{Var } a \\
&\text{App } (M)a \\
&\text{Abs } M \quad \text{App } N \\
&\text{App } (M)(N) \\
&\text{Abs } M \quad \text{Abs } N \\
&\text{App } (M)(N) \\
&\text{Abs } M \quad \text{Clos } N \\
&\text{App } (M)(N)
\end{align*}
\]

The following rules build terms of the forms
\( (s \circ M) \ a \), \( (s \circ M)(N_1 N_2 \ldots N_k) \), \( (s \circ M)(\lambda a_1 \ldots a_k. N) \), \( (s \circ M)(q \circ N) \)

\[
\begin{align*}
&\text{Clos } M \quad \text{Var } a \\
&\text{App } (M)a \\
&\text{Clos } M \quad \text{App } N \\
&\text{App } (M)(N) \\
&\text{Clos } M \quad \text{Abs } N \\
&\text{App } (M)(N) \\
&\text{Clos } M \quad \text{Clos } N \\
&\text{App } (M)(N)
\end{align*}
\]
The following rules build lists of variables:

\[
\begin{align*}
\text{Var } a & \quad \text{list } G \quad \text{Var } a \\
\text{list } a & \quad \text{list } G a
\end{align*}
\]

The following rules build terms of the forms
\[
\lambda a_1 \ldots a_n. a, \quad \lambda a_1 \ldots a_n. M_1 M_2 \ldots M_k, \quad \lambda a_1 \ldots a_n. s \circ M
\]
\[
\begin{align*}
\text{list } G & \quad \text{Var } a \\
& \quad \text{Abs } \lambda G. a \\
\text{list } G & \quad \text{App } M \\
& \quad \text{Abs } \lambda G. M \\
\text{list } G & \quad \text{Clos } M \\
& \quad \text{Abs } \lambda G. M
\end{align*}
\]

The following rules build terms of the forms
\[
id \circ M, \quad W_a \circ M, \quad \langle s, N_1 \setminus a_1, \ldots, N_n \setminus a_n \rangle \circ M, \quad (s_1 \circ s_2 \circ \ldots \circ s_n) \circ M
\]
\[
\begin{align*}
\text{Term } M & \\
\text{Clos } id \circ M & \\
\text{Var } a & \quad \text{Term } M \\
\text{Clos } W_a \circ M & \\
\text{Cons } s & \quad \text{Term } M \\
\text{Clos } s & \quad \text{Term } M \\
\text{Comp } s & \quad \text{Term } M \\
\text{Clos } (s) \circ M
\end{align*}
\]

The following rules build lists of the form
\[
N_1 \setminus a_1, \ldots, N_n \setminus a_n
\]
\[
\begin{align*}
\text{Term } N & \quad \text{Var } a \\
\text{List } N \setminus a
\end{align*}
\]
\[
\begin{align*}
\text{List } G & \quad \text{Term } N \quad \text{Var } a \\
\text{List } G, N \setminus a
\end{align*}
\]

The following rules build substitutions of the forms
\[
\langle id, N_1 \setminus a_1, \ldots, N_n \setminus a_n \rangle, \quad \langle W_a, N_1 \setminus a_1, \ldots, N_n \setminus a_n \rangle,
\]
and
\[
\langle s_1 \circ s_2 \circ \ldots \circ s_k, N_1 \setminus a_1, \ldots, N_n \setminus a_n \rangle
\]
\[
\begin{align*}
\text{List } G & \\
\text{Cons } \langle id, G \rangle
\end{align*}
\]
The following rules build substitutions of the forms

\[ \text{id} \circ \text{id}, \quad W_a \circ \text{id}, \quad \langle s, N_1 \setminus a_1, \ldots, N_n \setminus a_n \rangle \circ \text{id}, \quad (s_1 \circ s_2 \circ \ldots \circ s_n) \circ \text{id} \]

\[ \text{Comp} \text{id} \circ \text{id} \]

\[ \text{Var} \ a \quad \text{List} G \]
\[ \text{Cons} \ (W_a, G) \]
\[ \text{Comp} \ s \quad \text{List} G \]
\[ \text{Cons} \ (s, G) \]

\[ \text{id} \circ \text{id}, \quad W_a \circ \text{id}, \quad \langle s, N_1 \setminus a_1, \ldots, N_n \setminus a_n \rangle \circ \text{id}, \quad (s_1 \circ s_2 \circ \ldots \circ s_n) \circ \text{id} \]

\[ \text{Comp} \text{id} \circ W_a \]
\[ \text{Var} \ a \]
\[ \text{Comp} W_a \circ \text{id} \]
\[ \text{Cons} s \]
\[ \text{Comp} s \circ \text{id} \]
\[ \text{Comp} s \]
\[ \text{Comp} (s) \circ \text{id} \]

\[ \text{id} \circ W_a, \quad W_b \circ W_a, \quad \langle s, N_1 \setminus a_1, \ldots, N_n \setminus a_n \rangle \circ W_a, \quad (s_1 \circ s_2 \circ \ldots \circ s_n) \circ W_a \]

\[ \text{id} \circ \langle q, M_1 \setminus b_1, \ldots, M_k \setminus b_k \rangle, \quad W_a \circ \langle q, M_1 \setminus b_1, \ldots, M_k \setminus b_k \rangle, \quad \langle s, N_1 \setminus a_1, \ldots, N_n \setminus a_n \rangle \circ \langle q, M_1 \setminus b_1, \ldots, M_k \setminus b_k \rangle, \quad (s_1 \circ s_2 \circ \ldots \circ s_n) \circ \langle q, M_1 \setminus b_1, \ldots, M_k \setminus b_k \rangle \]

\[ \text{Cons} q \quad \text{Comp} \text{id} \circ q \]
The following rules build substitutions of the forms
\[ \text{id} \circ q_1 \circ q_2 \circ \ldots \circ q_k, \quad \mathcal{W}_a \circ q_1 \circ q_2 \circ \ldots \circ q_k, \]
\[ \langle s, N_1 \setminus a_1, \ldots, N_n \setminus a_n \rangle \circ q_1 \circ q_2 \circ \ldots \circ q_k, \]
and \((s_1 \circ s_2 \circ \ldots \circ s_n) \circ q_1 \circ q_2 \circ \ldots \circ q_k\)
12. Notes

(1) We can accept Abs in the stronger form
\[ s \circ \lambda a. M \rightarrow \lambda b. (W_b \circ s, b \setminus a) \circ M \] (a, b are arbitrary)

All results of this article remain true. We can also add the following rewrite rules
\[ id \circ M \rightarrow M \]
\[ id \circ s \rightarrow s \]
All results of this article remain true.

(2) It is easy to add some \( \alpha_2 \)-like reduction for substitutions, but that little benefit, because the analogue of Lemma 7.17 is false for substitutions.

(3) It is easy to give the following definitions:

Definition 12.1. (Free variables of substitutions). By definition, put
\begin{align*}
(i) & \quad FV(id) = (\emptyset, \emptyset, \emptyset, \ldots) \\
(ii) & \quad FV(W_a) = (\emptyset, \emptyset, \emptyset, \ldots) \\
(iii) & \quad FV((s, N \setminus a)) = FV(s) \cup FV(N) \\
(iv) & \quad FV(s \circ q) = O_s(FV(q))
\end{align*}

We see that
\[ FV(s) = O_s((\emptyset, \emptyset, \emptyset, \ldots)) \]

Definition 12.2. (\( \alpha \)-equivalence for substitutions).
We say that \( \Gamma \vdash s \triangleright \Delta \) is \( \alpha \)-equal to \( \Sigma \vdash q \triangleright \Psi \) and write
\[ (\Gamma \vdash s \triangleright \Delta) \equiv_\alpha (\Sigma \vdash q \triangleright \Psi) \iff \]
\( (\Gamma \vdash s \triangleright \Delta) \Rightarrow (m \vdash u) \) and
\( (\Sigma \vdash q \triangleright \Psi) \Rightarrow (m \vdash u) \), for some \( m, u \).

Example 12.3.
\begin{align*}
(x, y \vdash W_y \triangleright x) & \Rightarrow (2 \vdash W) \\
(x, x \vdash W_x \triangleright x) & \Rightarrow (2 \vdash W) \\
(x, y \vdash W_y \triangleright x) & \equiv_\alpha (x, x \vdash W_x \triangleright x)
\end{align*}

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