Quantum indirect synchronization

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It is well known that a system with two or more levels exists a limit cycle and can be synchronized with an external drive when the system and the drive are directly coupled. One might wonder if a system can synchronize with the external drive when they are not coupled directly. In this paper, we examine this case by considering a composite system consisting of two coupled two-level quantum systems, one of which is driven by an external field, while another couples to the driven one. Due to the decoherence caused by environments, the composite system would stay in a mixed state, and an effective limit cycle is formed, so phase locking could occur. We find the phase locking phenomenon in the phase diagram characterized by Husimi $Q$ function, and the synchronization can be generated consequently that we will refer to indirect synchronization. The $S$ function defined in the earlier study can also be used to measure the strength of synchronization. We claim that indirect synchronization is possible. This result provides us with a method to synchronize a quantum system that coupled to its neighbour without interacting with external drive directly.

As a collective dynamic feature of complex systems, synchronization is an ubiquitous phenomenon in physics. The study of synchronization can be dated back to the 17th century when Huygens discovered an odd kind of sympathy between two pendulum clocks, their oscillations show synchronization phenomenon when coupled to each other via a common support\cite{1, 2, 3, 4}. Huygens then noticed that the oscillations of these pendulum clocks tended to synchronize with each other. Since then, synchronization has been extensively studied and widely applied in classical physics, where researchers find that the phases between the phase locking oscillator and the external signal spontaneously synchronize in their respective phase space trajectories.

The extension of the synchronization concept from classical to quantum physics has been suggested\cite{5, 6, 7, 8, 9, 10, 11}, and widely been applied in the fields of cavity quantum electrodynamics\cite{12, 13}, masers\cite{14}, atomic combination\cite{15, 16, 17}, Kerr-anharmonic\cite{18}, Van der Pol (VDP) oscillator\cite{19, 20, 21}, Bose-Einstein condensate\cite{22}, and superconducting circuit system\cite{23, 24}. Though most works focus on the research of the quantum synchronization of position and momentum, studies on synchronization of the other degrees of freedom become active recently. Roulet and Bruder\cite{25} have recently proved that three or more levels quantum system in pure state can synchronize with the external field, which possesses all the synchronization features that classical system has. The authors further pointed out that a two-level system in pure states is impossible to synchronize because the system has no stable limit cycle. But it can be synchronized in mixed states as\cite{26, 48} shown. They proved the synchronization by solving the Lindblad master equation of the two-level system to simulate self-sustained mixed state. The result shows that it supports a valid limit cycle, and phase locking\cite{25} can be found in this system different from that in classical deterministic systems. They also calculated the synchronization measurement that characterizes the phase locking strength and analyzes the dependence of the signal strength on the natural frequency of the two-level system. Except the finite level systems, synchronization can also be observed, for example, in a two-node network consisting of the
self-contained oscillator of spin-1. It is shown that even if the situation that limit cycle can not synchronize with external semi-classical signals, phase locking still can be established, which may help setting quantum node networks [27].

Regardless of these progresses made in quantum synchronization, the study of synchronization between a drive and its indirectly driven object (we will name as indirect synchronization throughout this paper) is in its infancy so far. Inspiring by the control theory developed in recent years [28, 29, 30, 31, 32, 38, 34, 35, 36, 37, 39, 40], we naturally ask if it is possible to indirectly synchronize a quantum system with a drive. At first sight, this question is trivial, however, extensive examination shows that the features of indirect synchronization are far beyond speculation. Besides, drives may be difficult to directly applied to a system in practice, so adding a drive to the other system coupling strongly to the system of interests to achieve the synchronization might find practical applications.

The paper is organized as follows. In Sec. 1, we introduce our model of two coupled quantum two-level systems where one subsystem is driven by an external field. In Sec. 2, we investigate the synchronization between the drive and the subsystem indirectly coupled to the drive, and discuss the limit cycle of the system. By calculating and examining the Husimi Q function, the features of the synchronization are analyzed. In Sec. 3, we quantify the degree of phase locking and the synchronization. The paper finally concludes in Sec. 4 with a summary of our main results.

1 Model

We consider coupled two-level systems $A$ and $B$ with natural frequency $\omega_A$ and $\omega_B$, respectively, as shown in Fig. 1. For such a system, the free Hamiltonians read

$$\hat{H}_j = \frac{1}{2} \hbar \omega_j \hat{\sigma}_j^z,$$

where $j = A, B$, and $\hat{\sigma}_j^z$ are Pauli matrices.

The interaction between the subsystem $A$ and the subsystem $B$ can be described by

$$\hat{V} = i \hbar \frac{g}{2} (\hat{\sigma}_B^+ \hat{\sigma}_A^- - \hat{\sigma}_B^- \hat{\sigma}_A^+),$$

where $g$ is the coupling strength between the subsystem $A$ and the subsystem $B$, $\hat{\sigma}_j^\pm = \frac{1}{2} (\hat{\sigma}_j^x \pm i \hat{\sigma}_j^y)$ is the raising (lowering) operator of spin $j$ ($j = A, B$) and $\hat{\sigma}_j^x = \frac{1}{2} (\hat{\sigma}_j^x \pm i \hat{\sigma}_j^y)$. We use a classical field with frequency $\omega$ and strength $\epsilon$. In the rotating-wave approximation [41], the Hamiltonian takes

$$\hat{H}_\text{signal} = i \hbar \frac{\epsilon}{2} (e^{i\omega t} \hat{\sigma}_A^- - e^{-i\omega t} \hat{\sigma}_A^+).$$

Figure 1: Illustration of our system. Two two-level subsystems $A$ and $B$ are coupled with coupling constant $g$. The drive is directly exerted on the subsystem $A$ and indirectly coupled to the subsystem $B$ via $A$. Both subsystem $A$ and $B$ are coupled to the environments separately, and we denote $\gamma_j^g$ as the gain rate, while $\gamma_j^d$ as the loss rate ($j = A, B$).

In the frame rotating with $\hbar \omega \hat{\sigma}_z$, the dynamics of the system ($A$ plus $B$) is governed by the master equation

$$\dot{\hat{\rho}}_{AB} = -i [\hat{V}, \hat{\rho}_{AB}] - \frac{i}{2} [\Delta_1 \hat{\sigma}_A^z + \Delta_2 \hat{\sigma}_B^z + \epsilon \hat{\sigma}_A^\dag \hat{\sigma}_A, \hat{\rho}_{AB}] + \sum_{j=A,B} \mathcal{L}_j \hat{\rho}_{AB},$$

here we set $\hbar = 1$, $\hat{\rho}_{AB}$ is the density matrix of the system, $\Delta_{1,2} = \omega_{A,B} - \omega$ is the detuning of the external drive frequency to the natural frequency of the subsystem $A, B$. Lindblad dissipator superoperator $\mathcal{L}_j \hat{\rho} = \frac{\gamma_j^g}{2} \hat{D}[\hat{\sigma}_j^+ \hat{\sigma}_j^\dag] \hat{\rho} + \frac{\gamma_j^d}{2} \hat{D}[\hat{\sigma}_j^- \hat{\sigma}_j^\dag] \hat{\rho}$ and the Markovian master equation is written in the standard Lindblad form [42] as $D[\hat{O}] \hat{\rho} = \hat{O} \hat{\rho} \hat{O}^\dag - \frac{1}{2} \{\hat{O}^\dag \hat{O}, \hat{\rho}\}$, and $\gamma_j^g$ is the gain rate, $\gamma_j^d$ is the loss rate.

2 Husimi Q function

By solving the master equation, we can obtain the steady state density matrix $\hat{\rho}_{AB}^S$ of the system $A$ plus $B$,

$$\hat{\rho}_{AB}^S = \sum_{m,n} \hat{\rho}_m \langle m_A, m_B | n_A, n_B \rangle \langle m_A, m_B | n_A, n_B \rangle.$$

where $\vec{m} = (m_A, m_B)$, $\vec{n} = (n_A, n_B)$, and $m_{A(B)}$ and $n_{A(B)}$ denote the quantum numbers of subsystems $A$ and $B$. For the systems under consideration, $m_{A(B)} = n_{A(B)} = \pm 1/2$.

In order to evaluate the synchronization of the subsystem $B$ with the external signal, we perform a partial trace over the subsystem $A$ to obtain the reduced density operator of $B$, $\hat{\rho}_B = \text{Tr}_A[\hat{\rho}_{AB}^S]$. The density matrix $\hat{\rho}_{AB}^S$ is the steady-state density matrix $\hat{\rho}_B$ for spin $B$. To visualize the behavior of the system, we use the Husimi $Q$ representation (Q function) as a measure to quantify the phase of the spin system [43]. It is a quasi-probability distribution that allows us to represent the two-level system in the phase space. The $Q$ function is defined by [25]

$$Q(\theta, \phi) = \frac{1}{2\pi} \langle \theta, \phi | \hat{\rho}_B | \theta, \phi \rangle,$$  

(6)

This spherical representation is formulated in terms of spin-coherent states $|\theta, \phi\rangle$ [44], which in the case of a two-level system (TLS) are the eigenstates of the spin operator $\sigma_n = n \cdot \vec{\sigma}$, i.e., the spin operator along the direction of the unit vector $n$. Here $\theta$ and $\phi$ are the polar coordinates. Examining $Q$ function, we can find the synchronization features of the system under study.

![Figure 2: The limit cycle. When there is no signal, the subsystem $B$ is isolated. We observe that the distribution is independent of the $\phi$, and the distribution is centered on the $\theta = 0$, corresponding to the coherent state describing the spin precessing around the $z$-axis.](image)

When the coupled strength $g = 0$, the Husimi $Q$ of subsystem $B$ is $Q(\theta, \phi) = (11 + 9 \cos(\phi))/88\pi$, which is shown in Fig. 2. It is independent of $\phi$ and centered around $\theta = 0$, corresponding to coherent states on the equator precessing about the $z$-axis. The signal not only attracts the phases towards to $\theta = 0$, but also makes the system having no $\phi$ phase preference. It is because that the master equation is invariant by rotating operation around the $z$-axis. Because the state has no a definite $\phi$ preferred region, the system is not of phase locking. Then synchronization feature can not be displayed when the $g = 0$ similar to the results when there is no signal ($\epsilon$) as previous research shows [26].

For the steady-state $Q$ function with $g \neq 0, \epsilon \neq 0$ and no detuning between the drive and subsystem $B$, the results are shown in Fig. 3(a). We find from the figure that a peak in the $Q$ function appears around $\theta=0$ and $\phi=0$, indicating that system is phase locking. With non-zero detuning between the drive and subsystem $B$, we find from Fig. 3(b) that the peak of the $Q$ function appears near 0. The detuning shifts the phase locking towards $\phi = \frac{\pi}{2}$. In the next section, we attempt to measure how strong the synchronization is via a synchronization measure.

![Figure 3: (a) The steady-state Q function depending on $\theta$ and $\phi$. The steady-state Q function represents phase locking to a resonant external signal. $\Delta_1 = \Delta_2 = 0$, $\epsilon = 5 \gamma_B$, $g = 8 \gamma_B$, $\gamma_A^g/\gamma_A^d = 10$, $\gamma_B^g/\gamma_B^d = 10$ and $\gamma_A^g = \gamma_B^g$. (b) The steady-state Q function depending on $\theta$ and $\phi$. The steady-state Q function represents phase locking to a non-resonant external signal. $\Delta_1 = 3$, $\Delta_2 = 5$, $\epsilon = 5 \gamma_B$, $g = 8 \gamma_B$, $\gamma_A^g/\gamma_A^d = 10$, $\gamma_B^g/\gamma_B^d = 10$ and $\gamma_A^g = \gamma_B^g$.](image)

### 3 Synchronization

To quantify the synchronization, we will use the $Q$ function [25] as a tool to measure the synchronization. In the last section we find that when the drive $\epsilon = 0$ or the strength of coupling $g = 0$, the steady-state $Q$ function distributes on both sides of $\theta = 0$ and since the dissipative source of energy does not favor any phase $\phi$, there is no phase locking for the subsystem $B$. When the drive is applied to the subsystem $A$, phase locking in sub-
system $B$ occurs. From the phase space Husimi $Q$ representation, we can obtain the phase distribution $S(\phi)$ of the state $\rho_B$ by integrating out the $\theta$ angle. Since the dissipative source of energy does not have any phase preference of the subsystem $B$, the noise leads to phase diffusion, and which leads to the limit-cycle state $\rho_B$ with a united phase distribution $S(\phi) = 1/4\pi$. To quantify the strength of the phase locking or the synchronization for the subsystem $B$, we define a measure $S$ as

$$ S(\phi) = \int_0^\pi d\theta \sin \theta Q(\theta, \phi) - \frac{1}{4\pi} \quad (7) $$

$S(\phi)$ can be seen as a peak height of the phase locking above a flat distribution. It is zero when there is no synchronization. The uneven distribution of the phase will result in a non-zero value of the measure, indicating a locking on the corresponding phase $[18, 17]$.

We look for a signature of synchronization by calculating the phase distribution from the steady-state solution of the master equation. In Fig. 4, we show $S(\phi)$ distribution as a measure of phase locking. The subsystem $B$ shows signatures of synchronization as $S(\phi)$ is not flat. As we take different values of detuning $\Delta_2$, the phase locks at different values. The higher value of $S(\phi)$ is around $\Delta_2 = 0$ and $\phi = 0$.

The relationship between signal strength and synchronization measures is shown in Fig. 5. When $g$ is fixed, we find that $S(\phi)$ tends to be more and more localized as we increase the signal strength $\epsilon$. The synchronization feature is displayed. The synchronization is stronger with a greater value for $\epsilon$, locating at $\phi = 0$. The phenomenon is just like that synchronization in TLS, but a larger $\epsilon$ is needed to achieve the same strength of synchronization. We find that when the signal strength reaches a certain value, the synchronization effect is the best. When the signal strength continues to increase, the synchronization effect becomes weaker and weaker. A excessive strong signal will take the system out of synchronization [49], by definition, the synchronization can be realized only if the strength of the signal $\epsilon$ is small enough so that the limit cycle is only slightly disturbed. When the strength of the signal beyond this range means not only affects the phase of the system oscillation, but also affects its amplitude. Thus, the limit cycle is deformed, so the strength of the signal should be in a proper range that can be synchronized.

Since in our system the drive is subjected to the subsystem $A$, and there is a coupling between subsystems $A$ and $B$, therefore we consider the coupling strength between two systems. In Fig. 6, we consider the effect of the coupling strength on synchronization. It shows that when $\epsilon$ is fixed, increasing the coupling strength $g$ results in more distinct phase locking. Thus, we know that in the control of indirect synchronization, when the driving strength is constant, the system synchronization effect with higher coupling strength is more obvious. We find that when the coupling strength reaches a certain value, the synchronization effect is the best. As the signal strength con-
strengths and detuning values that occur phase locking. To a certain extent, when the detuning is very small and the coupling strength is large enough, it is easy to synchronize. These similar results can be seen in the previous researched direct synchronization scheme [25], and the coupling strength in this system is just like the drive strength of direct synchronization, leading the subsystem B into the region of synchronization, which also imply the validity of the indirect method.

4 Conclusion

In classical physics, the theory of synchronization has been extended from single particle synchronization into the synchronization between multi-particles with different properties. This inspire the present study of indirect quantum synchronization.

Examining two coupled two-level system, we found that an effective limit cycle exist in the system without drive, and the synchronization is feasible. We calculate the Husimi $Q$ function and discuss the phase locking feature of the system, then we analyse the $S$ function defined in the earlier publication to measure the synchronization. We find that indirect synchronization is possible. This result provide us with a method to synchronize a quantum system couples to its neighbour without adding the drive directly. The synchronization of multiple TLS, synchronization of two
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