Stochastic Quadratic BSDE With Two RCLL Obstacles

E. H. Essaky\textsuperscript{1} \quad M. Hassani\textsuperscript{1} \quad Y. Ouknine\textsuperscript{2}

\textsuperscript{1} Université Cadi Ayyad
Faculté Poly-disciplinaire
Département de Mathématiques et d’Informatique
B.P. 4162, Safi, Maroc.
e-mails: essaky@ucam.ac.ma \quad medhassani@ucam.ac.ma

\textsuperscript{2} Université Cadi Ayyad
Faculté des Sciences Semlalia
Département de Mathématiques
B.P. 2390, Marrakech, Maroc.
e-mail: ouknine@ucam.ac.ma

Abstract

We study the problem of existence of solutions for generalized backward stochastic differential equation with two reflecting barriers (GRBSDE for short) under weaker assumptions on the data. Roughly speaking we show the existence of a maximal solution for GRBSDE when the terminal condition $\xi$ is $\mathcal{F}_T$-measurable, the coefficient $f$ is continuous with general growth with respect to the variable $y$ and stochastic quadratic growth with respect to the variable $z$ and the reflecting barriers $L$ and $U$ are just right continuous left limited. The result is proved without assuming any $P$-integrability conditions.

\textbf{Keys Words:} Backward stochastic differential equation; stochastic quadratic growth; comparison theorem; exponential transformation.

\textbf{AMS Classification (1991):} 60H10, 60H20.

1 Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, P)$ be a stochastic basis on which is defined a Brownian motion $(B_t)_{t \leq T}$ such that $(\mathcal{F}_t)_{t \leq T}$ is the natural filtration of $(B_t)_{t \leq T}$ and $\mathcal{F}_0$ contains all $P$-null sets of $\mathcal{F}$. Note that $(\mathcal{F}_t)_{t \leq T}$ satisfies the usual conditions, i.e. it is right continuous and complete.

The notion of BSDE with two reflecting barriers (RBSDE for short) has been first introduced by Civitanic and Karatzas\textsuperscript{1}. A solution for such an equation, associated with a coefficient $f$; terminal

\textsuperscript{1}This work is supported by Hassan II Academy of Science and technology, Action Intégrée MA/10/224 and Marie Curie ITN n° 213841-2.
value \( \xi \) and two barriers \( L \) and \( U \), is a quadruple of processes \((Y, Z, K^+, K^-)\) with values in \( \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+ \) satisfying:

\[
\begin{align*}
(i) \quad & Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T dK_t^+ - \int_t^T dK_t^- - \int_t^T Z_s \, dB_s, \quad t \leq T, \\
(ii) \quad & L_t \leq Y_t \leq U_t, \quad \forall t \leq T, \\
(iii) \quad & \int_0^T (Y_t - L_t) \, dK_t^+ = \int_0^T (U_t - Y_t) \, dK_t^- = 0, \quad P \text{-a.s.,} \\
(iv) \quad & K^+, K^- \text{ are continuous nondecreasing processes with } K_0^+ = K_0^- = 0.
\end{align*}
\]

Here two continuous increasing processes \( K^+ \) and \( K^- \) have been added in order to force the solution \( Y \) to remain in the region enveloped by the lower reflecting obstacle \( L \) and the upper reflecting obstacle \( U \). This is done by the cumulative actions of processes \( K^+ \) and \( K^- \). In the case of a uniformly Lipschitz coefficient \( f \) and a square terminal condition \( \xi \) the existence and uniqueness of a solution have been proved when the barriers \( L \) and \( U \) are either regular or satisfy Mokobodski’s condition. This last condition essentially postulates the existence of a quasimartingale between the barriers \( L \) and \( U \). It has been shown also in \cite{1} that the solution coincides with the value of a stochastic Dynkin game of optimal stopping. The link between obstacle PDEs and RBSDEs has been given in Hamadène and Hassani \cite{3}.

The problem of existence of solutions for generalized BSDE with two reflecting barriers under weaker assumptions on the input data has been studied by Essaky and Hassani \cite{2} (see also \cite{3} for the non-reflected case). The authors have proved the existence of a maximal solution when the terminal condition \( \xi \) is \( \mathcal{F}_T \)-measurable, the coefficient \( f \) is continuous with general growth with respect to the variable \( y \) and stochastic quadratic growth with respect to the variable \( z \) and the reflecting barriers \( L \) and \( U \) are continuous. The result has been proved without assuming any \( P \)-integrability conditions. Applications to the Dynkin game problem as well as to the American game option have been also given.

In this paper we add a term of the form \( \sum_{t<s<T} h(s, Y_{s-}, Y_s) \) in the RBSDE (1.1) where \( h \) is a process with values \( \mathbb{R} \), which can be interpreted as a parameter of jump reflection at barriers \( L \) and \( U \) (see Definition \ref{2.2}). A natural question is then arises : is there any solution for this new RBSDE under the same assumptions as in \cite{2} but when the barriers \( L \) and \( U \) are only right continuous left limited (rcll for short) processes? The present work gives a positive answer to this question. The difficulty here lies in the fact that since the barrier \( L \) and \( U \) are allowed to have jumps then the process \( Y \) is so and then the reflecting processes \( K^+ \) and \( K^- \) are no longer continuous but just rcll. In this case, if \((Y, Z, K^+, K^-)\) is a solution then its size of jumps is given by:

\[
\begin{align*}
K_{t+}^+ - K_{t-}^+ &= \left( L_{t-} - [Y_t + h(t, Y_{t-}, Y_t)] \right)^+ \\
K_{t-}^- - K_{t-}^- &= \left( U_{t-} - [Y_t + h(t, Y_{t-}, Y_t)] \right)^- \\
Y_{t-} &= L_{t-} \lor [Y_t + h(t, Y_{t-}, Y_t)] \land U_{t-}.
\end{align*}
\]

By means of an exponential change, the proof of our main result consists in establishing first a correspondence between our GRBSDE and another GRBSDE whose coefficients are more tractable. We show that the existence of solutions for our initial GRBSDE is equivalent to the existence of solutions for the auxiliary GRBSDE. In order to prove that this auxiliary GRBSDE admits a maximal solution the following four cases are discussed:

1. \( f = h = 0 \).
2. \( f \) is Lipschitz and \( h = 0 \).
3. \( f \) is Lipschitz and there exists a finite family of stopping times \( S_0 = 0 \leq S_1 \leq \ldots \leq S_{p+1} = T \) such that for every \( x, y \in \mathbb{R} \) and \( t \notin \{ S_1, \ldots, S_{p+1} \} \), \( h(t, \omega, x, y) = 0 \).

4. The general case.

In the fourth case, since the integrability conditions on parameters are weaker, we make use of approximations and truncations to establish the existence result for the auxiliary GRBSDE. The final step consists in justifying the passage to the limit and identifying the limit as the solution of the auxiliary GRBSDE.

This paper is organized as follows. In the next section we lay out the notation and the assumptions and state the main result. In Section 3, by means of an exponential change, we show that the existence of solutions for our initial GRBSDE is equivalent to the existence of solutions for an auxiliary GRBSDE whose coefficients are more tractable. Section 4 is devoted to the proof of our main result. A comparison theorem for maximal solutions is proved in Section 5. Finally, in the appendix we prove a comparison theorem for solutions of GRBSDE which plays a crucial role in our proofs.

## 2 Statements and main result for GRBSDE

### 2.1 Notations

Let \( (\Omega, \mathcal{F}, \mathcal{F}_t, t \leq T, P) \) be a stochastic basis on which is defined a Brownian motion \( (B_t)_{t \leq T} \) such that \( (\mathcal{F}_t)_{t \leq T} \) is the natural filtration of \( (B_t)_{t \leq T} \) and \( \mathcal{F}_0 \) contains all \( P \)-null sets of \( \mathcal{F} \). Note that \( (\mathcal{F}_t)_{t \leq T} \) satisfies the usual conditions, i.e. it is right continuous and complete. For simplicity, we omit sometimes dependence on \( \omega \) of some processes or random functions.

Let us now introduce the following notations:

- \( \mathcal{P} \) the sigma algebra of \( \mathcal{F}_t \)-predictable sets on \( \Omega \times [0,T] \).
- \( \mathcal{D} \) the set of \( \mathcal{P} \)-measurable and right continuous with left limits (rcll for short) processes \( (Y_t)_{t \leq T} \) with values in \( \mathbb{R} \).
  - For a given process \( Y \in \mathcal{D} \), we denote : \( Y_{t-} = \lim_{s \nearrow t} Y_s, t \leq T \) \( (Y_0- = Y_0) \) and \( \Delta Y = Y_s - Y_{s-} \) the size of its jump at \( s \).
- \( \mathcal{K} := \{ K \in \mathcal{D} : K \text{ is nondecreasing and } K_0 = 0 \} \).
- \( \mathcal{K}_c := \{ K \in \mathcal{K} : \Delta_t K = 0, \forall t \in [0,T] \} \).
- \( \mathcal{K} - \mathcal{K}_c \) the set of \( \mathcal{P} \)-measurable and rcll processes \( (V_t)_{t \leq T} \) such that there exist \( V^+, V^- \in \mathcal{K} \) satisfying : \( V = V^+ - V^- \). In this case, for each \( \omega \in \Omega \), \( dV_t(\omega) \) denotes the signed measure on \( ([0,T], B_{[0,T]}(\omega)) \) associated to \( V_t(\omega) \) where \( B_{[0,T]} \) is the Borel sigma-algebra on \( [0,T] \).
  - For a given process \( V \in \mathcal{K} - \mathcal{K}_c \), we define : \( \int_a^b dV_s = V_b - V_a = \int_{[a,b]} dV_s \) and \( V^c_t = V_t - \sum_{0<s\leq t} \Delta_s V \).
- \( \mathcal{L}^{2,d} \) the set of \( \mathbb{R}^d \)-valued and \( \mathcal{P} \)-measurable processes \( (Z_t)_{t \leq T} \) such that
  \[ \int_0^T |Z_s|^2 ds < \infty, P - a.s. \]

The following notations are also needed:

- For a stopping time \( \nu \), \( ||\nu|| := \{ (t, \omega) \in [0, T] \times \Omega : \nu(\omega) = t \} \).
- For a set \( B \), we denote by \( B^c \) the complement of \( B \) and \( 1_B \) denotes the indicator of \( B \).
For each \((a, b) \in \mathbb{R}^2\), \(a \wedge b = \min(a, b)\) and \(a \lor b = \max(a, b)\).

For all \((a, b, c) \in \mathbb{R}^3\) such that \(a \leq c\), \(a \lor b \land c = \min(\max(a, b), c) = \max(a, \min(c, b))\).

### 2.2 Definitions

Throughout the paper we introduce the following data:

- \(\xi\) is an \(\mathcal{F}_T\)-measurable one dimensional random variable.
- \(L := \{L_t, \ 0 \leq t \leq T\}\) and \(U := \{U_t, \ 0 \leq t \leq T\}\) are two barriers which belong to \(\mathcal{D}\) such that \(L_t \leq U_t, \ \forall t \in [0, T]\) and assume, without loss of generality, that \(L_T = \xi = U_T\).
- \(f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}\) is a function such that:
  \[
  \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \ (t, \omega) \longmapsto -f(t, \omega, L_t(\omega) \lor y \land U_t(\omega), z) \text{ is } \mathcal{P} - \text{measurable}.
  \]
- \(g : [0, T] \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}\) is a function such that:
  \[
  \forall y \in \mathbb{R}, \ (t, \omega) \longmapsto g(t, \omega, L_t(\omega) \lor y \land U_t(\omega)) \text{ is } \mathcal{P} - \text{measurable}.
  \]
- \(h : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}\) is a function such that:
  \[
  \forall (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \ \omega \longmapsto h(t, \omega, L_t(\omega) \lor x \land U_t(\omega), L_t(\omega) \lor y \land U_t(\omega)) \text{ is } \mathcal{F}_t - \text{measurable}.
  \]
- \(A\) is a process in \(\mathcal{K}^c\).

To give conditions under which solutions to a GRBSDE exist, we should first give the following definitions.

**Definition 2.1.** Let \(K^1\) and \(K^2\) be two processes in \(\mathcal{K}\). We say that:

1. \(K^1\) and \(K^2\) are singular if and only if there exists a set \(D \in \mathcal{P}\) such that
   \[
   \mathbb{E} \int_0^T 1_D(s, \omega)dK^1_s(\omega) = \mathbb{E} \int_0^T 1_D(s, \omega)dK^2_s(\omega) = 0.
   \]
   This is denoted by \(dK^1 \perp dK^2\).

2. \(dK^1 \leq dK^2\) if and only if for each set \(B \in \mathcal{P}\)
   \[
   \mathbb{E} \int_0^T 1_B(s, \omega)dK^1_s(\omega) \leq \mathbb{E} \int_0^T 1_B(s, \omega)dK^2_s(\omega), \ \text{i.e.} \ K^1_t - K^1_s \leq K^2_t - K^2_s, \ \forall s \leq t \ \text{P-a.s.}
   \]
   In this case \(\frac{dK^1}{dK^2}\) denotes a \(\mathcal{P}\)-measurable Radon-Nikodym density of \(dK^1\) with respect to \(dK^2\) which satisfies
   \[
   0 \leq \frac{dK^1}{dK^2}(s, \omega) \leq 1, \ \frac{dK^2_s(\omega)}{d\omega}P(d\omega) - \text{a.e. on } [0, T] \times \Omega.
   \]

Let us now introduce the definition of our GRBSDE with two real obstacles \(L\) and \(U\).
Definition 2.2. 1. We say that \((Y, Z, K^+, K^-) = (Y_t, Z_t, K^+_t, K^-_t)_{t \leq T}\) is a solution of the GRBSDE, associated with the data \((\xi, f, g, h, A, L, U)\), if the following hold:

\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s) dA_s + \sum_{t \leq s \leq T} h(s, Y_s, Y_s) + \int_t^T dK^s_+ - \int_t^T dK^-_s - \int_t^T Z_s dB_s, t \leq T, \\
(ii) & \quad \forall t \in [0, T], \quad L_t \leq Y_t \leq U_t, \\
(iii) & \quad \int_0^T (Y_t - L_t) dK^+_t = \int_0^T (U_t - Y_t) dK^-_t = 0, \text{ a.s.}, \\
(iv) & \quad Y \in \mathcal{D}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}, \\
(v) & \quad dK^+ \perp dK^-.
\end{align*}
\] (2.1)

2. We say that the GRBSDE (2.1) has a maximal (resp. minimal) solution \((Y_t, Z_t, K^+_t, K^-_t)_{t \leq T}\) if \((Y_t, Z_t, K^+_t, K^-_t)_{t \leq T}\) of (2.1) we have for all \(t \leq T, Y_t \geq Y_t, \text{ P-a.s. (resp.} Y_t \leq Y_t, \text{ P-a.s.)}.\)

Remark 2.1. In our definition we introduce a process \(h\) with values \(\mathbb{R}\), which may be interpreted as a parameter of jump reflection at barriers \(L\) and \(U\). Moreover, if \((Y, Z, K^+, K^-)\) is a solution of GRBSDE (2.1) then it satisfies for all \(t \in [0, T]\) (see Lemma 3.1)

\[
\Delta_t K^+ = \left( L_{t-} - [Y_t + h(t, Y_t, Y_t)] \right)^+ \\
\Delta_t K^- = \left( U_{t-} - [Y_t + h(t, Y_t, Y_t)] \right)^- \\
Y_{t-} = L_{t-} \vee [Y_t + h(t, Y_t, Y_t)] \cap U_{t-}.
\]

2.3 Assumptions and remarks

We shall need the following assumptions on \(f, g, h, L\) and \(U\):

(A.1) There exist two processes \(\eta \in L^1(\Omega, L^1([0, T], ds, \mathbb{R}^d))\) and \(C \in \mathcal{D}\) such that:

(a) \(\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad |f(t, \omega, L_t(\omega) \vee y \wedge U_t(\omega), z)| \leq \eta(t, \omega) + C_t(\omega)|z|^2, \quad dtP(d\omega)-a.e.\)

(b) \(dtP(d\omega)-a.e., \text{ the function } (y, z) \mapsto f(t, \omega, L_t(\omega) \vee y \wedge U_t(\omega), z) \text{ is continuous.}\)

(A.2) There exists \(\beta \in L^1(\Omega, L^1([0, T], A(dt), \mathbb{R}^+_d))\) such that:

(a) \(\forall y \in \mathbb{R}, \quad |g(t, \omega, L_t(\omega) \vee y \wedge U_t(\omega))| \leq \beta(t, \omega), \quad A(dt)P(d\omega)-a.e.\)

(b) \(A(dt)P(d\omega)-a.e., \text{ the function } y \mapsto g(t, \omega, L_t(\omega) \vee y \wedge U_t(\omega)) \text{ is continuous.}\)

(A.3)

(a) There exists \(l_0 : [0, T] \times \Omega \to \mathbb{R}_+\) satisfying for each \(t \in [0, T]\), \(l_t\) is \(\mathcal{F}_t\)-measurable and \(P\)-a.s., \(\sum_{0<s\leq T} l_s < +\infty\), such that:

\[P - a.s., \forall (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \quad |h(t, \omega, L_{t-}(\omega) \vee x \wedge U_{t-}(\omega), L_t(\omega) \vee y \wedge U_t(\omega))| \leq l_t(\omega).\]
Remark 2.2. 1. It should be pointed out that conditions (A.1)(a), (A.2)(a) and (A.3)(a) hold if the functions $f$, $g$ and $h$ satisfy the following: $\forall (s, \omega)$, $\forall y \in [L_s(\omega), U_s(\omega)]$, $\forall z \in \mathbb{R}^d$, $\forall x \in [L_{s-}(\omega), U_{s-}(\omega)]$

$$
|f(s, \omega, y, z)| \leq \tilde{\eta}(\omega) \theta(s, \omega, x, y) + \theta(s, \omega, x, y)|z|^2, \\
|g(s, \omega, y)| \leq \tilde{\eta}(\omega) \theta(s, \omega, x, y), \\
|h(s, \omega, x)| \leq \tilde{l}(\omega) \theta(s, \omega, x),
$$

where:

• $\theta : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$ is a function such that for every $t \in [0, T]$:

$$
P - a.s., \quad D_t(\omega) := \sup_{0 \leq s \leq t, \alpha, \delta \in [0, 1]} \theta(s, \omega, \delta L_{s-}(\omega) + (1 - \delta)U_{s-}(\omega), \alpha L_s(\omega) + (1 - \alpha)U_s(\omega)) < +\infty,
$$

and $\mathcal{F}_t$-adapted.

• $\tilde{\eta} \in L^0(\Omega, L^1([0, T], ds, \mathbb{R}_+))$ and $\tilde{\eta} \in L^0(\Omega, L^1([0, T], dA_s, \mathbb{R}_+))$.

• $\tilde{l}_t$ is an $\mathcal{F}_t$-measurable function satisfying $\sum_{0 < s \leq T} \tilde{l}_s < +\infty$, $P$-a.s.

Indeed, we just take in conditions (A.1)(a), (A.2)(a) and (A.3)(a), $\eta$, $C$, $\beta$ and $l$ as follows:

$$
\eta_t = \tilde{\eta}(\omega) D_t(\omega), \\
C_t(\omega) = D_t(\omega), \\
\beta_t(\omega) = \tilde{\eta}(\omega) D_t(\omega), \\
l_t(\omega) = \tilde{l}(\omega) D_t(\omega).
$$

This means that the functions $f$, $g$ and $h$ can have, in particular, a general growth in $(x, y)$ and stochastic quadratic growth in $z$.

2. It is not difficult to see that if $L$ or $U$ are semimartingales, then assumption (A.4) holds. Moreover if the barriers processes $L$ and $U$ are completely separated on $[0, T]$, i.e. $\forall t \in [0, T]$, $L_t < U_t$ and $L_{t-} < U_{t-}$ (this is equivalent to $\inf_{0 \leq t < T} (U_t - L_t) > 0$), then assumption (A.4) holds. Indeed, let

$$
\beta_t = \sup_{s \leq t} \{ |L_s| + |U_s| \}, \\
L'_t = \frac{L_t}{\beta_t} 1_{(t < T)} + \frac{LT - L_t}{\beta_T} 1_{(t = T)}, \\
U'_t = \frac{U_t}{\beta_t} 1_{(t < T)} + \frac{U_T - U_t}{\beta_T} 1_{(t = T)}.
$$

Then, $\forall t \in [0, T]$, $-1 \leq L'_t < U'_t \leq 1$ and $L'_{t-} < U'_{t-}$. It follows then from the work [5] that there exists a semimartingale $S$ such that $L'_t \leq S_t \leq U'_t$, $\forall t \in [0, T]$. Hence, the semimartingale $S_t \beta_t 1_{(t < T)} + \xi 1_{(t = T)}$ is between $L_t$ and $U_t$. 

The following theorem constitute the main result of the paper.

**Theorem 2.1.** If assumptions (A.1)–(A.4) hold then the GRBSDE (2.1) has a maximal solution (resp. minimal solution).

The rest of the paper is devoted to the proof of our main result (Theorem 2.1). By means of an exponential change, the proof of our main result consists in establishing first a correspondence between our GRBSDE and another GRBSDE whose coefficients are more tractable. We show that the existence of solutions for our initial GRBSDE is equivalent to the existence of solutions for the auxiliary GRBSDE. Since the integrability conditions on parameters are weaker, we make use of approximations and truncations to establish the existence result for the auxiliary GRBSDE. The final step consists in justifying the passage to the limit and identifying the limit as the solution of the auxiliary GRBSDE. A useful tool in our considerations is the comparison theorem (see Theorem A.1 in Appendix). Let us start by giving equivalent forms of our GRBSDE.

### 3 Equivalent forms of GRBSDE (2.1)

#### 3.1 First equivalent form of GRBSDE (2.1)

The following lemma shows that the existence of solutions for our initial GRBSDE (2.1) is equivalent to the existence of solutions for another GRBSDE.

**Lemma 3.1.** $(Y, Z, K^+, K^-)$ is a solution of GRBSDE (2.1) if and only if $(Y, Z, K^{+c}, K^{-c})$ is a solution of the following GRBSDE

\[
\begin{align*}
(i) \quad Y_t &= \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s)dA_s - \sum_{t<s\leq T} \Delta Y_s \\
&\quad \quad + \int_t^T dK^{+c}_s - \int_t^T dK^{-c}_s - \int_t^T Z_sdB_s, \quad t \leq T, \\
(ii) \quad \forall t \in [0,T], \quad Y_{t-} &= L_{t-} \vee \{Y_t + h(t,Y_t-;Y_t) \wedge U_{t-}\}, \\
(iii) \quad \int_t^T (Y_t - L_t)dK^{+c}_s &\leq \int_t^T (U_t - Y_t)dK^{-c}_s = 0, \quad a.s., \\
(iv) \quad Y \in \mathcal{D}, \quad K^{+c}, K^{-c} \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}, \\
v) \quad dK^{+c} \perp dK^{-c}. 
\end{align*}
\]

**Proof.** Suppose that $(Y, Z, K^+, K^-)$ is a solution of GRBSDE (2.1) and let $K^{\pm c}$ be the continuous part of $K^\pm$. Clearly for every $t \in [0,T]$

\[
Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s)dA_s - \sum_{t<s\leq T} \Delta_s Y + \int_t^T dK^{+c}_s - \int_t^T dK^{-c}_s - \int_t^T Z_sdB_s,
\]

with $\Delta_t Y = -h(t,Y_{t-},Y_t) + \Delta_t K^- - \Delta_t K^+$. Then for each $t \in [0,T]$,

\[
Y_{t-} = Y_t + h(t,Y_{t-},Y_t) + \Delta_t K^+ - \Delta_t K^-.
\]

We should remark first that

\[
dK^+ \perp dK^- \iff dK^{+c} \perp dK^{-c} \quad \text{and} \quad \Delta K^- \Delta K^+ = 0. \tag{3.2}
\]

Now, we distinguish the following three cases:
1. If $\Delta_iK^+ > 0$, since $dK^+ \perp dK^-$ and $\int_0^T (Y_t - L_t^-)dK_t^+ = 0$ it follows that $\Delta_iK^- = 0$ and $Y_{t-} = L_{t-}$. Consequently $\Delta_iK^+ = \left( L_{t-} - [Y_t + h(t,Y_{t-},Y_t)] \right)^+$ and $L_{t-} > Y_t + h(t,Y_{t-},Y_t)$. Henceforth for each $t \in [0, T]$, 

$$Y_{t-} = L_{t-} \vee [Y_t + h(t,Y_{t-},Y_t)] \land U_{t-}.$$ 

2. If $\Delta_iK^- > 0$, by the same way as above we get $\Delta_iK^- = \left( U_{t-} - [Y_t + h(t,Y_{t-},Y_t)] \right)^-$ and then for each $t \in [0, T]$ 

$$Y_{t-} = L_{t-} \vee [Y_t + h(t,Y_{t-},Y_t)] \land U_{t-}.$$ 

3. If $\Delta_iK^- = \Delta_iK^+ = 0$, we have $Y_{t-} = Y_t + h(t,Y_{t-},Y_t) \in [L_{t-}, U_{t-}]$ and then for each $t \in [0, T]$ 

$$Y_{t-} = L_{t-} \vee [Y_t + h(t,Y_{t-},Y_t)] \land U_{t-}.$$ 

Consequently in the three cases we obtain for each $t \in [0, T]$ 

$$\begin{align*}
\Delta_iK^+ &= \left( L_{t-} - [Y_t + h(t,Y_{t-},Y_t)] \right)^+ \\
\Delta_iK^- &= \left( U_{t-} - [Y_t + h(t,Y_{t-},Y_t)] \right)^- \\
Y_{t-} &= L_{t-} \vee [Y_t + h(t,Y_{t-},Y_t)] \land U_{t-}. 
\end{align*}$$

Hence (ii) of Equation (3.1) is satisfied. Moreover, since $0 \leq (Y_t - L_t)dK_t^{+c} \leq (Y_{t-} - L_{t-})dK_t^+ = 0$, and $0 \leq (U_t - Y_t)dK_t^{-c} \leq (U_{t-} - Y_{t-})dK_t^- = 0$, then we get (iii) of Equation (3.1).

In view of (3.2), we have also (v) of Equation (3.1). Hence $(Y, Z, K^{+c}, K^{-c})$ is a solution of GRBSDE (3.1).

On another hand, suppose now that $(Y, Z, K^{+c}, K^{-c})$ is a solution of GRBSDE (3.1) and set for each $t \in [0, T]$ 

$$\Delta_iK^+ = \left( L_{t-} - [Y_t + h(t,Y_{t-},Y_t)] \right)^+ $$

$$\Delta_iK^- = \left( U_{t-} - [Y_t + h(t,Y_{t-},Y_t)] \right)^- $$

By (ii) of Equation (3.1) we have for each $t \in [0, T]$ 

$$\begin{align*}
-\Delta_iY &= Y_{t-} - Y_t = (L_{t-} - Y_t) \vee h(t,Y_{t-},Y_t) \land (U_{t-} - Y_t), \\
&= h(t,Y_{t-},Y_t) + \left| (U_{t-} - Y_t - h(t,Y_{t-},Y_t)) \land 0 \vee (L_{t-} - Y_t - h(t,Y_{t-},Y_t)) \right| \\
&= h(t,Y_{t-},Y_t) + \Delta_iK^+ - \Delta_iK^- . 
\end{align*}$$

Hence 

$$Y_t = \xi + \int_t^T f(s,Y_s, Z_s)ds + \int_t^T g(s,Y_s)dA_s + \sum_{t < s \leq T} h(s,Y_{s-},Y_s) + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s,$$

where $K^\pm$ is defined by : $K_t^\pm = K_t^{+c} + \sum_{0 < s \leq t} \Delta_iK^\pm, \forall t \in [0, T]$. Moreover it follows that
1. For each \( t \in [0,T] \), \( L_t \leq Y_t \leq U_t \) since \( L_{t-} \leq Y_{t-} \leq U_{t-} \) and \( Y_T = L_T = U_T = \xi \).

2. \( \Delta_t K^+ \Delta_t K^- = 0 \) since \( L_{t-} \leq U_{t-} \) and then \( dK^+ \perp dK^- \).

3. \( \int_0^T (Y_{t-} - L_{t-}) dK_t^+ = \int_0^T (Y_{t-} - L_{t-}) dK_t^+ + \sum_{0 < s \leq t} (Y_{s-} - L_{s-}) \Delta_s K^+ = \sum_{0 < s \leq t} (Y_{s-} - L_{s-}) \Delta_s K^+ = 0 \),

   since if \( Y_{s-} > L_{s-} \) then \( Y_s + h(s, Y_s, Y_t) > L_s \) and hence \( \Delta_s K^+ = 0 \). Similarly it follows also that \( \int_0^T (U_{t-} - Y_{t-}) dK_t^- = 0 \). Therefore \((Y, Z, K^+, K^-)\) is a solution of GRBSDE \((2.1)\).

\[\boxed{\int_0^T (Y_{t-} - L_{t-}) dK_t^+ = \int_0^T (Y_{t-} - L_{t-}) dK_t^+ + \sum_{0 < s \leq t} (Y_{s-} - L_{s-}) \Delta_s K^+ = \sum_{0 < s \leq t} (Y_{s-} - L_{s-}) \Delta_s K^+ = 0,}\]

**Remark 3.1.** The maximal solution \((Y, Z, K^{+\xi}, K^{-\xi})\) of GRBSDE \((3.1)\) satisfies, \(P - a.s.\) \( \forall t \in [0,T] \)

\[Y_{t-} = \max\{ x \in [L_{t-}, U_{t-}] : x = L_{t-} \lor [Y_t + h(t, x, Y_t)] \wedge U_{t-} \}.\]

Indeed, let \( \tau \in [0,T] \) be a stopping time and set

\[\xi = \max\{ x \in [L_{\tau-}, U_{\tau-}] : x = L_{\tau-} \lor [Y_\tau + h(\tau, x, Y_\tau)] \wedge U_{\tau-} \}.\]

Let \((Y', Z', K'^{+\xi}, K'^{-\xi})\) be a solution of GRBSDE \((3.1)\) (which is exists according to our main result) associated to the data: \(f' = 1_{\{s \leq \tau\}} f, \ g' = 1_{\{s \leq \tau\}} g, \ h' = 1_{\{s \leq \tau\}} h, \ \xi = \xi, \ L'_t = L_t 1_{\{t \leq \tau\}} + L_{t-} 1_{\{t > \tau\}} \) and \(U'_t = U_t 1_{\{t \leq \tau\}} + U_{t-} 1_{\{t > \tau\}}\). Set also

\[Y''_t = Y'_t 1_{\{t \leq \tau\}} + Y_t 1_{\{t > \tau\}}, \quad Z''_t = Z'_t 1_{\{t \leq \tau\}} + Z_t 1_{\{t > \tau\}}, \quad dK''_{t\pm} = dK'_{t\pm} 1_{\{t \leq \tau\}} + dK_{t\pm} 1_{\{t > \tau\}}.\]

Clearly \((Y''', Z''', K''^{+\xi}, K''^{-\xi})\) is also a solution of GRBSDE \((3.1)\). Since \((Y, Z, K^{+\xi}, K^{-\xi})\) is a maximal solution of GRBSDE \((3.1)\), then \(Y'''_t \leq Y'_t, \ \forall t \in [0,T]\.\) Henceforth

\[Y''_t = Y'_t = \xi, \quad \leq Y'_t = L_{t-} \lor (Y_\tau + h(\tau, Y_\tau, Y_\tau)) \wedge U_{t-} \quad \leq \max\{ x \in [L_{\tau-}, U_{\tau-}] : x = L_{\tau-} \lor [Y_\tau + h(\tau, x, Y_\tau)] \wedge U_{\tau-} \} = \xi.\]

Then, for every stopping time \( \tau \in [0,T] \) we get

\[Y_{t-} = \max\{ x \in [L_{t-}, U_{t-}] : x = L_{t-} \lor [Y_t + h(t, x, Y_t)] \wedge U_{t-} \}, \quad P - a.s.\]

It therefore follows that, \(P - a.s.\) \( \forall t \in [0,T] \)

\[Y_{t-} = \max\{ x \in [L_{t-}, U_{t-}] : x = L_{t-} \lor [Y_t + h(t, x, Y_t)] \wedge U_{t-} \}.\]

### 3.2 Second equivalent form of GRBSDE \((2.1)\)

In this part, by using an exponential transform, we transform the GRBSDE with two obstacles into another equivalent one whose data satisfy some "good" conditions. This transformation allow us, in particular, to bound the terminal condition and the barriers associated with the transformed GRBSDE.
To begin with, let $m \in \mathcal{K} + \mathbb{R}_+$ and suppose that GRBSDE (2.1) has a solution. It follows then from Itô’s formula that

$$
(Y_t - S_t - m_t)m_t = (\xi - S_T - m_T)m_T + \int_t^T m_s g(s, Y_s, Z_s)ds + \int_t^T m_s g(s, Y_s) dA_s + \int_t^T m_s dK_s^{+c} - \int_t^T m_s dK_s^{-c} - \int_t^T m_s (Z_s - \gamma_s) dB_s + \int_t^T m_s dV_s^c + \int_t^T m_s dm_s^c - \sum_{t<s \leq T} \Delta_s [(Y_t - S_t - m_t)m_t].
$$

Setting $e_t := e^{m_s (Y_t - S_t - m_t)}$, it follows that

$$
e_t = e_T + \int_t^T e_s m_s f(s, Y_s, Z_s)ds + \int_t^T e_s m_s g(s, Y_s) dA_s + \int_t^T e_s m_s K_s^{+c} - \int_t^T e_s m_s K_s^{-c} - \int_t^T e_s m_s (Z_s - \gamma_s) dB_s + \int_t^T e_s m_s dV_s^c + \int_t^T e_s m_s dm_s^c - \sum_{t<s \leq T} \Delta_s (e_i - e^{-m_i^2}).
$$

Since $d(e^{m_t^2}) = 2m_t e^{m_t^2} dm_t^2$, then

$$
e_t - e^{-m_t^2} = e_T - e^{-m_T^2} + \int_t^T e_s m_s \left( f(s, Y_s, Z_s) - \frac{1}{2} m_s |Z_s - \gamma_s|^2 \right) ds + \int_t^T e_s m_s g(s, Y_s) dA_s + \int_t^T e_s m_s K_s^{+c} - \int_t^T e_s m_s K_s^{-c} - \int_t^T e_s m_s (Z_s - \gamma_s) dB_s + \int_t^T e_s m_s dV_s^c + \int_t^T e_s m_s dm_s^c - \sum_{t<s \leq T} \Delta_s (e_i - e^{-m_i^2}).
$$

Let $|V|$ denotes the total variation of the process $V$ and choose the process $m$ as follows: $\forall s \in [0, T],$

$$m_s = 4 \left[ \sup_{t<s} \left( |U_r| + |C_r| + |L_r| \right) + |V| + \int_0^s (1 + \eta_r + |\gamma_r|^2)dr + \int_0^s (1 + \beta_r) dA_r + \sum_{0<r<s} l_r + 1 \right].
$$

Define also for every $s \in [0, T],$

- $\overline{\xi} = e_T - e^{-m_T^2} = e^{m_T \xi - T - m_T} - e^{-m_T^2},$
- $\overline{U_s} = e^{m_s (U_s - S_s - m_s)} - e^{-m_s^2},$
- $\overline{Y_s} = e^{m_s (Y_s - S_s - m_s)} - e^{-m_s^2},$
- $\overline{Z_s} = m_s (\overline{Y}_s + e^{-m_s^2}) (Z_s - \gamma_s),$
- $\overline{dK_s^{+c}} = m_s (\overline{Y}_s + e^{-m_s^2}) dK_s^{+c},$

$$\Delta_s \overline{\xi} = \left( \overline{Y}_s - e^{\left( (Y_{s +} + h(s, Y_{s -}, Y_s) - S_{s -} - m_{s -}) m_{s -} \right)} + e^{-m_s^2} \right),
$$

$$\Delta_s \overline{U_s} = \left( \overline{U}_s - e^{\left( (Y_{s +} + h(s, Y_{s -}, Y_s) - S_{s -} - m_{s -}) m_{s -} \right)} + e^{-m_s^2} \right),
$$

$$\overline{K_s^{+c}} = \overline{K}_s^{+c} + \sum_{0<r<s} \Delta_s \overline{K}_r^{+c}.$$
Remark 3.2. 1. It should be noted that $m$ is $\mathcal{F}_t$-adapted, rcll and increasing process.

2. Since $\mathcal{S}_T = \xi$ then $\xi = 0$.

3. Since, $\forall s \in [0,T]$, $Y_{s-} = L_{s-} \vee [Y_s + h(s, Y_{s-}, Y_s)] \wedge U_{s-}$, then for every $s \in [0,T],\n
\begin{align*}
\mathcal{T}_{s-} &= T_{s-} \wedge \left[ e^{[(Y_s + h(s, Y_{s-}, Y_s) - S_{s-} - m_{s-}) - m_{s-}]} \right] \wedge \mathcal{U}_{s-}.
\end{align*}

Coming back to Equation (3.5), it is clear that the GRBSDE (2.1) can be written as follows:

\begin{align*}
\begin{cases}
(i) & \mathcal{T} = \int_t^T \tilde{f}(s, \mathcal{T}, \mathcal{Z}_s) ds + \int_t^T \tilde{g}(s, \mathcal{T}) dA_s - \sum_{t \leq s \leq T} \Delta_s \mathcal{T} + \int_t^T dK_s \mathcal{T} \quad \forall \mathcal{T} \\
(ii) & \mathcal{T} = \mathcal{T} \vee \mathcal{T}_s \wedge \mathcal{T}_{s-} \wedge \mathcal{U}_{s-} \mathcal{U}_{s-} \\
(iii) & \int_0^T (\mathcal{T} - \mathcal{T}_s) dK_s^c = \int_0^T (\mathcal{U} - \mathcal{T}) dK_t^c = 0, \ a.s. \\
(iv) & \mathcal{T} \in \mathcal{D}, \mathcal{K}^+ \mathcal{K}^c \mathcal{K}^c \in \mathcal{K}^c, \mathcal{Z} \in \mathcal{L}_{2,d} \\
(v) & d\mathcal{T} \bot d\mathcal{T}
\end{cases}
\end{align*}

where $\mathcal{T}, \tilde{g}, \tilde{f}$ and $\tilde{h}$ are given by: for each $s \in [0,T], y \geq \mathcal{T}_s, x \geq \mathcal{T}_{s-}$ and $z \in \mathbb{R}^d$

- $\mathcal{A}_s = 2 \int_0^s e^{-m_r} dm_r,$

- $\tilde{f}(s, y, z) = m_y(y + e^{-m_z}) f(s, \frac{\ln(y + e^{-m_z})}{m_y} + m_s + S_s, \frac{z}{m_s(y + e^{-m_z})} + \gamma_s) - \frac{|z|^2}{2(y + e^{-m_z})},$

- $\tilde{g}(s, y) = m_y(y + e^{-m_z}) \left[ g(s, \frac{\ln(y + e^{-m_z})}{m_y} + m_s + S_s, \frac{dA_s}{dA_s} + \frac{dV^c}{dA_s} + \frac{dZ^c}{dA_s} \right] + 2m_y e^{-m_z} \frac{dA_s}{dA_s} - (y + e^{-m_z}) \frac{\ln(y + e^{-m_z})}{m_y} \frac{dA_s}{dA_s},$

- $\tilde{h}(s, x, y) = \mathcal{h}(s, x, y) - y - e^{-m_z},$ where

\begin{align*}
\mathcal{h}(s, x, y) &= e^{\left[ \ln\left( \frac{\ln(y + e^{-m_z})}{m_y} + S_{s-} + m_{s-} \right) + S_{s-} + m_{s-} + \ln\left( \frac{\ln(y + e^{-m_z})}{m_y} + S_{s-} + m_{s-} \right) + S_{s-} + m_{s-} \right]} \Delta_{s-} S_{s-} m_{s-}.
\end{align*}
It follows from Equation (3.7) and Lemma 3.1 that the GRBSDE (2.1) can be written also as follows:

\[
\begin{align*}
(\text{i}) \quad & \bar{Y}_t = \int_t^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s)ds + \int_t^T \bar{g}(s, \bar{Y}_s)d\bar{A}_s + \sum_{t<s\leq T} \bar{h}(s, \bar{Y}_{s-}, \bar{Y}_s) + \int_t^T d\bar{R}_s \\
& \quad + \int_t^T d\bar{K}^+ - \int_t^T d\bar{K}^- - \int_t^T \bar{Z}_sdB_s, t \leq T, \\
(\text{ii}) \quad & \forall t \leq T, \, \bar{L}_t \leq \bar{Y}_t \leq \bar{U}_t \\
(\text{iii}) \quad & \int_0^T (\bar{Y}_{t+} - \bar{Y}_t-)d\bar{K}^+_t = \int_0^T (\bar{U}_{t+} - \bar{Y}_{t+})d\bar{K}^-_t = 0, \text{ a.s.} \\
(\text{iv}) \quad & \bar{Y}_s \in \mathcal{D}, \, \bar{Y}^+, \bar{Y}^- \in \mathcal{K}, \, \bar{Z} \in \mathcal{L}^{2,d}, \\
(\text{v}) \quad & d\bar{K}^+ \perp d\bar{K}^-,
\end{align*}
\]

where \( f, \, g, \, \bar{h}, \) and \( \bar{R} \) is given by: for each \( s \in [0,T], \, \gamma \in \mathbb{R}, \, \bar{\gamma} \in \mathbb{R} \) and \( \pi \in \mathbb{R}^d \)

- \( \bar{f}(s, \gamma, \bar{\gamma}) = \bar{f}(s, \bar{L}_s \vee \gamma \wedge \bar{U}_s, \bar{\gamma}) - \frac{1}{2} \bar{\gamma}_s, \) where \( \bar{\gamma}_s = 2e^{-m_s} (\gamma_s + \gamma_s^2), \)
- \( \bar{g}(s, \gamma) = \bar{g}(s, \bar{L}_s \vee \gamma \wedge \bar{U}_s) - \frac{1}{2}, \)
- \( \bar{h}(s, \pi, \gamma) = \bar{h}(s, \bar{L}_s \vee \pi \wedge \bar{U}_s - \bar{L}_s \vee \gamma \wedge \bar{U}_s) - e^{3m_s} \Delta s m, \)
- \( \bar{R}_s := \frac{1}{2} \int_0^s d\pi_r + \frac{1}{2} \int_0^s \gamma_r dr + \sum_{0<r \leq s} e^{3m^2} \Delta_r m, \)

Remark 3.3. It is clear now that if \((Y, Z, K^+, K^-)\) is a solution (resp. maximal solution) of GRBSDE (2.1) associated with the data \((\xi, f, g, A, L, U)\) then \((\bar{Y}, \bar{Z}, \bar{K}^+, \bar{K}^-)\), defined by (3.8) and associated with coefficient the data \((\bar{\xi}, \bar{f}, \bar{g}, \bar{A}, \bar{L}, \bar{U})\), is a solution (resp. maximal solution) of GRBSDE (3.8). Conversely, Suppose that there exists a solution (resp. maximal solution) \((\bar{Y}, \bar{Z}, \bar{K}^+, \bar{K}^-)\) for GRBSDE (3.8). Hence, by setting, for all \( t \leq T \)

\[
\begin{align*}
Y_t &= \ln(\bar{Y}_t + e^{-m^2}) + S_t + m_t, \\
Z_t &= \frac{m_t}{m_t(\bar{Y}_t + e^{-m^2})} + \gamma_t, \\
K_t^{\pm} &= \int_0^t \frac{m_t}{m_t(\bar{Y}_s + e^{-m^2})} + \sum_{0<\tau \leq s} \Delta_t K_t^{\pm},
\end{align*}
\]

where \( \Delta_t K^+ = \left( L_{t+} - [Y_t + h(t, Y_{t-}, Y_t)] \right)^+ \) and \( \Delta_t K^- = \left( U_{t+} - [Y_t + h(t, Y_{t-}, Y_t)] \right)^-, \) it is clear that \((Y, Z, K^+, K^-)\) is a solution (resp. maximal solution) for GRBSDE (2.1).

The following proposition states some properties on the data \((\bar{\xi}, \bar{f}, \bar{g}, \bar{A}, \bar{L}, \bar{U})\) of the transformed GRBSDE (3.8).

Proposition 3.1. Assume that assumptions (A.1) – (A.3) hold. Then we have the following:

1. The function \( \bar{f} \) is \( \mathcal{P} \)-measurable and continuous with respect to \((\bar{Y}, \bar{Z})\) satisfying for every \( s \in [0,T], \)
\( \bar{f} \in \mathbb{R} \) and \( \pi \in \mathbb{R}^d \)

\[
- \bar{\gamma}_s - e^{2m^2} |\pi|^2 \leq \bar{f}(s, \bar{Y}, \bar{Z}) \leq 0.
\]
2. \( \int_0^T \eta_t ds \leq \eta_T \leq 1 \).

3. The function \( \eta \) is \( \mathcal{P} \)-measurable and continuous with respect to \( y \) satisfying, for all \( s \in [0, T] \) and \( \eta(y) \in \mathbb{R} \),

\[
-1 \leq \eta(s, \eta) \leq 0.
\]

4. For all \( t \in [0, T] \), \(-1 \leq \eta_t \leq 0 \leq \etaT \leq 1 \).

5. The function \( \eta \) satisfies the following properties

(a) \( P \)-a.s., \( \forall (t, x) \in [0, T] \times \mathbb{R} \), the function \( \eta \mapsto \eta + \eta(t, x, \eta) \) is nondecreasing and continuous on \( \mathbb{R} \).

(b) \( P \)-a.s., \( \forall t \in [0, T] \), \( \forall \eta \in \mathbb{R} \), the function \( \eta \mapsto \eta(t, x, \eta) \) is continuous on \( \mathbb{R} \).

(c) \( P \)-a.s., \( \forall t \in [0, T] \), \( \forall \eta \in \mathbb{R} \), \( \forall \gamma \in \mathbb{R} \),

\[
e^{-3m_2^2 \Delta t m} + \Delta t e^{-m_2} \leq \eta(t, x, \gamma) \leq 0.
\]

**Proof.** 1. It is not difficult to see that \( \eta \) is \( \mathcal{P} \)-measurable and continuous with respect to \( (y, z) \) on \( \mathbb{R} \times \mathbb{R}^d \) for every \( (t, \omega) \in [0, T] \times \Omega \), since \( \eta \) is. Let us prove Inequality (3.10). Let \( s \in [0, T] \), \( \eta \in [\eta_s(\omega), \etaT_s(\omega)] \) and \( \gamma \in \mathbb{R}^d \). By condition (A.1) we get

\[
\tilde{f}(s, \omega, \eta, \gamma) \leq m_s(\eta + e^{-m_2^2}) \left( \eta_s + C_s \left| \eta \right|_{m_s(\eta + e^{-m_2^2})} + \gamma_s \right)^2 - \frac{\left| \gamma \right|^2}{2(\eta + e^{-m_2^2})} + \frac{\gamma_s^2}{2(\eta + e^{-m_2^2})} \leq m_s(\eta + e^{-m_2^2})(\eta_s + 2C_s\gamma_s^2) + 2C_s \frac{m_s(\eta + e^{-m_2^2})}{2m_s(\eta + e^{-m_2^2})} \left| \gamma \right|^2 + \frac{\gamma_s^2}{2(\eta + e^{-m_2^2})} \leq m_s e^{m_s(\eta_s - m_2)}(\eta_s + 2C_s\gamma_s^2) + \frac{m_s^2}{2} e^{-m_2^2} (\eta_s + \gamma_s^2) + 2 \sup_{x \geq 4} (\left| x e^{-x} \right| e^{-m_2^2} (\eta_s + \gamma_s^2) \leq e^{-m_2^2} (\eta_s + \gamma_s^2) = \frac{\eta_s}{2}
\]

where we have used the elementary inequality \( (a + b)^2 \leq 2(a^2 + b^2) \) and the fact that \( m_s - 4C_s \geq 0 \) and \( U_s - S_s - m_s \leq -\frac{m_2}{2}, \quad \forall s \in [0, T] \).

On the other hand, by using condition (A.1), we get also that

\[
\tilde{f}(s, \omega, \eta, \gamma) \geq m_s(\eta + e^{-m_2^2}) \left( -\eta_s + C_s \left| \eta \right|_{m_s(\eta + e^{-m_2^2})} + \gamma_s \right)^2 - \frac{\left| \gamma \right|^2}{2(\eta + e^{-m_2^2})} + \frac{\gamma_s^2}{2(\eta + e^{-m_2^2})} \geq -\frac{1}{2} \eta_s - 2m_s(\eta + e^{-m_2^2}) C_s \frac{m_s(\eta + e^{-m_2^2})}{2m_s(\eta + e^{-m_2^2})^2} \left| \gamma \right|^2 + \frac{\gamma_s^2}{2(\eta + e^{-m_2^2})} \geq -\frac{1}{2} \eta_s - \frac{1}{m_s(\eta + e^{-m_2^2})} \left| \gamma \right|^2 \geq -\frac{1}{2} \eta_s - \frac{\gamma_s^2}{\eta + e^{-m_2}} \geq -\frac{1}{2} \eta_s - \frac{e^{-m_2^2} \left| \gamma \right|^2}{\eta + e^{-m_2}} \geq -\frac{1}{2} \eta_s - \frac{e^{-m_2^2} \left| \gamma \right|^2}{\eta + e^{-m_2}}.
\]
since $\bar{y} + e^{m_2} \geq \bar{T}_s + e^{m_2} = e^{-m_s(L_s - S_s - m_s)}$ and $2C_m + \frac{1}{2} \leq 1$, \( \forall s \in [0, T] \). Consequently

\[-\eta_s - e^{2m^2} |\bar{x}|^2 \leq \mathcal{F}(s, \bar{y}, \bar{z}) \leq 0.\]

2. Since for every $s \in [0, T]$, \( \frac{(\eta_s + \gamma_s^2)ds}{dm_s^c} \leq \frac{1}{4} \), then we have

\[\int_0^T \eta_s ds = 2 \int_0^T e^{-m_1} \frac{(\eta_s + \gamma_s^2)ds}{dm_s^c} dm_s^c \leq \frac{1}{2} \int_0^T e^{-m_1} dm_s^c.\]

Since \( \int_0^T e^{-m_1} dm_s^c = e^{-m_0} - e^{-m_T} + \sum_{0 \leq s \leq T} \Delta e^{-m_s} \leq e^{-m_0} \), then \( \int_0^T \eta_s ds \leq \frac{1}{4} \mathcal{F}_T \leq e^{-m_0} \leq e^{-1} \leq 1. \)

3. By using a similar calculation as above it is easy to prove that for all $s \in [0, T]$ and $\bar{y} \in [\bar{T}_s(\omega), \bar{U}_s(\omega)]$

\[-1 \leq \bar{f}(s, \bar{y}) \leq 0.\]

4. It is not difficult to prove that

\[\forall t \in [0, T], \ -1 \leq \bar{T}_t \leq 0 \leq \bar{U}_t \leq 1.\]

5. (a) and (b) are obvious. Let us prove (c). By definition of process $m$ we have for all $s \in [0, T]$, $\bar{\nu} \in [\bar{L}_s(\omega), \bar{U}_s(\omega)]$ and $\bar{y} \in [\bar{T}_s(\omega), \bar{U}_s(\omega)]$

\[\tilde{h}(s, \bar{\nu}, \bar{y}) \leq e^{\frac{\ln(\bar{y} - e^{-m_2}) - l_s + \Delta_s S + \Delta_s m}{m_s}} - (\bar{y} + e^{-m_2}) - \Delta_s e^{-m_2} + (\bar{y} + e^{-m_2}) \left( e^{\frac{\ln(\bar{y} - e^{-m_2}) - l_s + \Delta_s V + \Delta_s m}{m_s}} - \Delta_s e^{-m_2} \right) \geq \Delta_s e^{-m_2}.\]

On the other hand, by definition of process $m$ we have for all $s \in [0, T]$, $\bar{\nu} \in [\bar{L}_s(\omega), \bar{U}_s(\omega)]$ and $\bar{y} \in [\bar{T}_s(\omega), \bar{U}_s(\omega)]$

\[\tilde{h}(s, \bar{\nu}, \bar{y}) \leq e^{\frac{\ln(\bar{y} - e^{-m_2}) + l_s + \Delta_s S + \Delta_s m}{m_s}} - (\bar{y} + e^{-m_2})
\leq (\bar{y} + e^{-m_2}) \left( e^{\frac{l_s + \Delta_s V + \Delta_s m}{m_s}} - \Delta_s e^{-m_2} \right) - 1
\leq e^{-\frac{e^2}{2}} (e^{3m_s \Delta_s m} - 1)
\leq e^{-\frac{e^2}{2}} \left( 3m_s \Delta_s me^{3m_s \Delta_s m} \right)
\leq 3m_s e^{\frac{e^2}{2} \Delta_s m} e^{3m^2}
\leq \sup_{x \geq 1} (3xe^{-\frac{e^2}{2}}) (\Delta_s m) e^{3m^2}
\leq (\Delta_s m) e^{3m^2}.\]

The proof of Proposition 3.1 is then finished.
3.3 An equivalent result to Theorem 2.1

Now, by taking advantage of the previous analysis, especially Remark 3.3 and Proposition 3.1, our problem is then reduced to find the maximal solution of the following GRBSDE:

\[
\begin{cases}
(i) \quad Y_t = \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s) dA_s + \int_t^T dR_s + \sum_{t < s \leq T} h(s, Y_s, Y_s) \\
\quad \quad \quad \quad \quad + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_sdB_s, t \leq T, \\
(ii) \quad \forall t \in [0, T], L_t \leq Y_t \leq U_t, \\
(iii) \quad \int_0^T (Y_t - L_t) dK_t^+ = \int_0^T (U_t - Y_t) dK_t^- = 0, P - a.s. \\
(iv) \quad Y \in \mathcal{D}, K^+, K^- \in \mathcal{K}, Z \in L^{d,d}, \\
(v) \quad dK^+ \perp dK^-, 
\end{cases}
\]  

(3.11)

under the following assumptions:

(H.0) \(A \in \mathcal{K}\) and \(R \in \mathcal{K} - \mathcal{K}\) such that: \(0 \leq A_T \leq 1\).

(H.1) There exist two processes \(\eta \in L^0(\Omega \times [0, T], \mathcal{F}_+))\) such that \(\int_0^T \eta_s ds \leq 1\) and \(C \in \mathcal{F}_+ + \mathcal{K}\) such that:

1. \(\forall (s, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d, -\eta_s(\omega) - C_s(\omega) |z|^2 \leq f(s, \omega, y, z) \leq 0,\)

2. \(\forall (t, \omega) \in [0, T] \times \Omega, \text{the function } (y, z) \mapsto f(t, \omega, y, z) \text{ is continuous on } \mathbb{R} \times \mathbb{R}^d.\)

(H.2) For each \((s, \omega) \in [0, T] \times \Omega,\)

1. \(\forall y \in \mathbb{R}, -1 \leq g(s, \omega, y) \leq 0,\)

2. the function \(y \mapsto g(s, \omega, y)\) is continuous on \(\mathbb{R}.\)

(H.3) The function \(h\) satisfies the following conditions:

1. \(P\text{-a.s., } \forall t \in [0, T], \forall x \in \mathbb{R}, \text{the function } y \mapsto y + h(t, x, y) \text{ is nondecreasing and continuous on } \mathbb{R}.\)

2. \(P\text{-a.s., } \forall t \in [0, T], \forall y \in \mathbb{R}, \text{the function } x \mapsto h(t, x, y) \text{ is continuous on } \mathbb{R}.\)

3. There exists a nonnegative function \(l : [0, T] \times \Omega \rightarrow \mathbb{R}_+\) satisfying \(\forall t \in [0, T], l_t \text{ is } \mathcal{F}_t\text{-measurable and } \sum_{0 < s \leq T} l_s < +\infty P - a.s. \text{ such that for every } t \in [0, T], x \in \mathbb{R} \text{ and } y \in \mathbb{R}\)

\[-l_t \leq h(t, x, y) \leq 0.\]

(H.4) For each \((s, \omega) \in [0, T] \times \Omega\)

\[-1 \leq L_s(\omega) \leq 0 \leq U_s(\omega) \leq 1.\]

Our main result Theorem 2.1 is equivalent to the following theorem.

**Theorem 3.1.** Let assumptions (H.0)-(H.4) hold. Then the GRBSDE (3.11) has a maximal solution.
4 Proof of Theorem 3.1

For the proof of Theorem 3.1, we distinguish the following four cases.

4.1 Existence of solution for GBSDE (3.11) : the "f = g = h = 0" case

Theorem 4.1. Let assumptions (H.0)-(H.4) hold. Assume moreover that \( f = g = h = 0 \), then the GBSDE (3.11) has a unique solution.

In order to prove Theorem 4.1, we need some preliminary results. To begin with, let \((\tau_n)_{n \geq 0}\) be the family of stopping times defined by

\[
\tau_n = \inf \{ s \geq 0 : |R|_s \geq n \} \wedge T. \tag{4.1}
\]

It is easy seen that \( P[\lim_{n \to 0} (\tau_n = T)] = 1 \). Indeed, let \( \omega \in \cap_{n \geq 0} (\tau_n < T) \) then \( \forall k, \tau_k < T \). Hence \( \forall k, |R|_T \geq k \iff |R|_T = \infty \), which contradict the fact that the total variation of \( R \) is finite.

For all \((i, j) \in \mathcal{N} \times \mathcal{N}\), let us set

\[
\begin{align*}
\xi^i &= L_{T-} \vee (\Delta_T R) \wedge U_{T-}, \\
L_i^j &= L_t 1_{\{t \geq T\}} + L_T 1_{\{t = T\}}, \\
U_i^j &= U_t 1_{\{t < T\}} + U_T 1_{\{t = T\}}, \\
\overline{\xi}^i &= \xi^i + \int_0^t 1_{\{s < \tau_i\}} dR_s^+ - \int_0^t 1_{\{s < \tau_i\}} dR_s^- \\
\overline{L}_t^j &= L_t^i + \int_0^t 1_{\{s < \tau_i\}} dR_s^+ - \int_0^t 1_{\{s < \tau_i\}} dR_s^- \\
\overline{U}_t^j &= U_t^i + \int_0^t 1_{\{s < \tau_i\}} dR_s^+ - \int_0^t 1_{\{s < \tau_i\}} dR_s^-.
\end{align*}
\]

Clearly we have \( \overline{L}_t^j \leq \overline{\xi}^i \leq \overline{U}_t^j \) and then by assumption (H.4)

\[-(1 + j) \leq \overline{L}_t^j \leq \overline{U}_t^j = \int_0^t 1_{\{s < \tau_i\}} dR_s^+ - \int_0^t 1_{\{s < \tau_i\}} dR_s^- \leq \overline{U}_t^j \leq 1 + i. \]

Consider the following BSDE with two reflecting barriers associated with \((\overline{\xi}^i, \overline{L}_t^j, \overline{U}_t^j)\)

\[
\begin{align*}
\text{(i)} & \quad \overline{Y}_t^i = \overline{\xi}^i + \int_t^T dK_t^i, \\
\text{(ii)} & \quad \forall t \leq T, \overline{L}_t^j \leq \overline{Y}_t^j \leq \overline{U}_t^j, \\
\text{(iii)} & \quad \int_0^T (\overline{Y}_t^j - \overline{L}_t^j) dK_t^i + \int_0^T (\overline{Y}_t^j - \overline{U}_t^j) dK_t^i = 0, \text{ a.s.} \tag{4.2} \\
\text{(iv)} & \quad \overline{Y}_t^i \in \mathcal{D}, \; K_t^i, \overline{L}_t^i, \overline{U}_t^i \in \mathcal{K}, \; Z_t^i \in L^{2,d}, \\
\text{(v)} & \quad dK_t^i \bot dK_t^j.
\end{align*}
\]

It follows from Lepeltier and Xu [6] (see Hamadène et al. [5]) that Equation (4.2) has a unique solution. Moreover, for all \( i \) and \( j \)

\[
\mathbb{E} \int_0^T |Z_t^i|^2 \, ds + \mathbb{E}(K_T^i)^2 < +\infty. \tag{4.3}
\]
Set $Y_t^{i,j} = \bar{Y}_t^{i,j} - \int_0^t 1_{\{s<\tau_i\}}dR_s^+ + \int_0^t 1_{\{s<\tau_j\}}dR_s^-$. It follows then that GBSDE (4.2) can be written as follows:

$$
\begin{align*}
(i) & \quad Y_t^{i,j} = \xi^i + \int_0^T 1_{\{s<\tau_i\}}dR_s^+ - \int_0^T 1_{\{s<\tau_j\}}dR_s^- + \int_t^T dK_s^{i,j+} - \int_t^T dK_s^{i,j-} - \int_t^T Z_s^{i,j}dB_s, \quad t \leq T, \\
(ii) & \quad \forall t \leq T, \quad L_t^i \leq Y_t^{i,j} \leq U_t^i, \\
(iii) & \quad \int_0^T (Y_t^{i,j} - L_t^{i,j})dK_t^{i,j+} = \int_0^T (U_t^{i,j} - Y_t^{i,j})dK_t^{i,j-} = 0, \quad \text{a.s.} \\
(iv) & \quad Y_t^{i,j} \in \mathcal{D}, \quad K^{i,j+}, K^{i,j-} \in \mathcal{K}, \quad Z^{i,j} \in \mathcal{L}^{2,d}, \\
(v) & \quad dK^{i,j+} \perp dK^{i,j-}.
\end{align*}
$$

The following result follows easily from the Comparison theorem (Theorem 6.1 in Appendix).

**Proposition 4.1.** The solution $(Y^{i,j}, Z^{i,j}, K^{i,j+}, K^{i,j-})$ of RGBSDE (4.4) satisfies the following:

- **i)** Fix $j \in \mathbb{N}^+$, we get for all $i \geq 1$ and $t \leq T$,
  
  $$-1 \leq L_t^i \leq Y_t^{i,j} \leq U_t^i \leq 1, \quad dK^{i+1,j} \leq dK^{i,j+} \quad \text{and} \quad dK^{i,j-} \leq dK^{i+1,j-}.$$

- **ii)** Fix $i \in \mathbb{N}^+$, we get for all $j \geq 1$ and $t \leq T$
  
  $$L_t^i \leq Y_t^{i+1,j} \leq Y_t^{i,j} \leq U_t^i, \quad dK^{i,j+} \leq dK^{i,j+1} \quad \text{and} \quad dK^{i+1,j-} \leq dK^{i,j-}.$$

Let us set

- $Y^j = \sup_i Y_t^{i,j}$, $Y_{t-}^{i,j} = \sup_i Y_t^{i,j}$,
- $dK^j = \sup_i dK_t^{i,j}$ which is a positive measure,
- $dK^j = \inf_i dK_t^{i,j}$ which is also a positive measure since $K_t^{0,j+} < +\infty$, $P$-a.s.

The following result states the existence of a process $Z^j$ such that process $(Y^j, Z^j, K^{j+}, K^{j-})$ is the unique solution of some RBSDE.

**Proposition 4.2.** Suppose that assumptions of Theorem 4.1 hold. Then we have the following.

1. There exists a process $Z^j \in \mathcal{L}^{2,d}$ such that for all $n \in \mathbb{N}$,

   $$
   \mathbb{E} \int_0^{\tau_n} |Z_s^j - Z_s^j|^2 ds \rightarrow 0, \quad \text{as} \quad i \quad \text{goes to infinity}.
   $$

2. The process $(Y^j, Z^j, K^{j+}, K^{j-})$ is the unique solution of the following BSDE with two reflecting barriers

$$
\begin{align*}
(i) & \quad Y_t^j = \xi^j + \int_0^T 1_{\{s<T\}}dR_s^+ - \int_0^T 1_{\{s<\tau_i\}}dR_s^- + \int_t^T dK_s^{j+} - \int_t^T dK_s^{j-} - \int_t^T Z_s^{j}dB_s, \quad t \leq T, \\
(ii) & \quad \forall t \leq T, \quad L_t^j \leq Y_t^j \leq U_t^j, \\
(iii) & \quad \int_0^T (Y_t^j - L_t^j)dK_t^{j+} = \int_0^T (U_t^j - Y_t^j)dK_t^{j-} = 0, \quad \text{a.s.} \\
(iv) & \quad Y_t^j \in \mathcal{D}, \quad K^{j+}, K^{j-} \in \mathcal{K}, \quad Z^j \in \mathcal{L}^{2,d}, \\
(v) & \quad dK^{j+} \perp dK^{j-}.
\end{align*}
$$
Proof. 1. Let \( i, i' \in \mathbb{N} \) such that \( i, i' \geq n \) and \( t \in [0, \tau_n] \) where \( \tau_n \) is defined by (4.4). Clearly we have

\[
(Y_{t_n}^{i,j} - Y_t^{i',j})^2 + \int_t^{\tau_n} |Z_{s}^{i,j} - Z_{s}^{i',j}|^2 ds
= (Y_{t_n}^{i,j} - Y_t^{i',j})^2 + 2 \int_t^{\tau_n} (Y_{s}^{i,j} - Y_{s}^{i',j})((dK_{s}^{i,j} + - dK_{s}^{i',j}) - (dK_{s}^{i,j} - dK_{s}^{i',j}))
- 2 \int_t^{\tau_n} (Y_{s}^{i,j} - Y_{s}^{i',j})(Z_{s}^{i,j} - Z_{s}^{i',j}) dB_s - \sum_{t < s < \tau_n} (\Delta_s(Y_{s}^{i,j} - Y_{s}^{i',j}))^2.
\]

By using a localization procedure it follows that

\[
\lim_{i, i' \to +\infty} \mathbb{E} \int_0^{\tau_n} |Z_{s}^{i,j} - Z_{s}^{i',j}|^2 ds = 0.
\]

Hence there exists \( Z^j \in \mathcal{L}^2, \) such that for every \( n \in \mathbb{N} \)

\[
\lim_{i \to +\infty} \mathbb{E} \int_0^{\tau_n} |Z_{s}^{i,j} - Z_s|^2 ds = 0.
\]

2. According to Bulkholder-Davis-Gundy inequality there exists a universal constant \( c \geq 0 \) such that

\[
\mathbb{E} \sup_{t < \tau_n} (Y_t^{i,j} - Y_t^j)^2 \leq \mathbb{E}(Y_{\tau_n}^{i,j} - Y_{\tau_n}^j)^2 + 2c \mathbb{E} \left( \int_0^{\tau_n} |Y_{s}^{i,j} - Y_{s}^{i',j}|^2 |Z_{s}^{i,j} - Z_{s}^{i',j}|^2 ds \right)^{\frac{1}{2}}.
\]

Henceforth

\[
\lim_{i \to +\infty} \mathbb{E} \sup_{t < \tau_n} (Y_t^{i,j} - Y_t^j)^2 = 0.
\]

Then \( Y^j \) is rcll and \( Y_t^j = Y_t^{i,j}, \forall t \in [0, T], P\text{-a.s.}, \) since \( P[\bigcup_{n \geq 0}(\tau_n = T)] = 1. \) Moreover, since for every \( (i, j) \in \mathbb{N}^2, \)

\[
\mathbb{E}(K_{t_n}^{i,j})^2 \leq \mathbb{E}(K_{t_n}^{0,j})^2 < +\infty,
\]

it follows then, from Fatou’s lemma, that \( \mathbb{E}(K_{t_n}^{0,j})^2 < +\infty. \)

On the other hand, it follows from BSDE (3.3) that for each \( i \geq n \) and \( j \in \mathbb{N}, \)

\[
K_{\tau_n}^{i,j} = Y_{\tau_n}^{i,j} - Y_{\tau_n}^{i'} + \int_0^{\tau_n} 1_{\{s < \tau_n\}} dB_s - \int_0^{\tau_n} dR_s - \int_0^{\tau_n} Z_{s}^{i,j} dB_s
\leq 2 + n + K_{\tau_n}^{0,j} - \int_0^{\tau_n} Z_{s}^{i,j} dB_s.
\]

Hence

\[
\mathbb{E}K_{\tau_n}^{i,j} \leq 2 + n + \mathbb{E}K_{\tau_n}^{0,j}.
\]

By monotone convergence theorem we have

\[
\mathbb{E}K_{\tau_n}^{i,j} \leq 2 + n + \mathbb{E}K_{\tau_n}^{0,j}.
\]

Then \( P\text{-a.s., for each } n \in \mathbb{N}, K_{\tau_n}^{i,j} < +\infty. \) Therefore, since \( P[\bigcup_{n \geq 0}(\tau_n = T)] = 1 \) we get \( K_{\tau_n}^{j} < +\infty, \)

\( P\text{-a.s.} \) Moreover

\[
\Delta_T K_{\tau_n}^{i,j} = \sup_{i \in \mathbb{N}} \Delta_T K_{\tau_n}^{i,j} = \sup_{i \in \mathbb{N}} (U_{\tau_n}^i - (\xi_i + \Delta_T R_{\tau_n}^{i,j}))) = 0.
\]
Similarly we get also that \( \Delta T K^j = 0 \). Therefore \( K_T^j < +\infty, \) \( P \)-a.s.

Observe now that for every \( n, j \in \mathbb{N}, \) we have \( P \)-a.s.

\[
Y_t^{i,j} = Y_{\tau_n}^{i,j} + \int_{\tau_n}^{\tau_n^-} (1_{s<\tau_n} dR_s^+ - 1_{s<\tau_n} dR_s^-) + \int_{\tau_n^-}^{\tau_n} dK_s^{i,j+} - \int_{\tau_n^-}^{\tau_n} dK_s^{i,j-} - \int_{\tau_n}^{\tau_n} Z_s^{i,j} dB_s
\]

Taking the limit as \( i \) goes to infinity we get for every \( n, j \in \mathbb{N}, \) \( P \)-a.s.

\[
Y_t^j = Y_{\tau_n}^j + \int_{\tau_n}^{\tau_n^-} (dR_s^+ - 1_{s<\tau_n} dR_s^-) + \int_{\tau_n^-}^{\tau_n} dK_s^j - \int_{\tau_n^-}^{\tau_n} dK_s^j - \int_{\tau_n}^{\tau_n} Z_s^j dB_s
\]

Now, since \( P[\cup_{n\geq 0}(\tau_n = T)] = 1, \) letting \( n \) goes to infinity we have for each \( j \in \mathbb{N}, \) \( P \)-a.s.

\[
Y_t^j = Y_{\tau_n}^j + \int_{\tau_n}^{T} (1_{s<T} dR_s^+ - 1_{s<\tau_n} dR_s^-) + \int_{\tau_n}^{T} dK_s^j - \int_{\tau_n}^{T} dK_s^j - \int_{\tau_n}^{T} Z_s^j dB_s
\]

Therefore

\[
Y_t^j = \xi' + \int_{\tau_n}^{T} (1_{s<T} dR_s^+ - 1_{s<\tau_n} dR_s^-) + \int_{\tau_n}^{T} dK_s^j - \int_{\tau_n}^{T} dK_s^j - \int_{\tau_n}^{T} Z_s^j dB_s,
\]

where we have used the fact that \( Y_{\tau_n}^j = \sup_{i,j} Y_{\tau_n}^{i,j} = \sup_{i,j} (L_{\tau_n}^i \lor (\xi' + \Delta T R_i^j) \land U_{\tau_n}^i) = \xi', \) since \( \Delta T R_i^j = 0. \) Then \((Y^j, Z^j, K^{j^+}, K^{j^-})\) satisfies (i) of Equation (4.5).

Let us now proof the minimality conditions. Clearly

\[
0 \leq \int_0^T (Y_t^{i,j} - L_{t-}^i) dK_t^{i,j+} \leq \int_0^T (Y_t^{i,j} - L_{t-}^i) dK_t^{i,j+} = 0.
\]

Hence

\[
\int_0^T (Y_t^{i,j} - L_{t-}^i) dK_t^{i,j+} = 0.
\]

Applying Fatou’s lemma we obtain

\[
0 \leq \int_0^T (Y_t^j - L_{t-}^j) dK_t^{j+} \leq \liminf_i \int_0^T (Y_t^{i,j} - L_{t-}^i) dK_t^{i,j+} = 0.
\]

Henceforth

\[
\int_0^T (Y_t^j - L_{t-}^j) dK_t^{j+} = 0.
\]

Similarly we get also that

\[
\int_0^T (Y_t^j - L_{t-}^j) dK_t^{j-} = 0.
\]

Moreover, since \( dK_T^{j^+} = \inf_i dK_T^{i,j^+}, \) \( dK_T^{j^-} = \sup_i dK_T^{i,j^-} \) and \( dK_T^{i,j^+} \perp dK_T^{i,j^-} \) for each \( i \in \mathbb{N} \) then \( dK_T^{j^+} \perp dK_T^{j^-}. \) Proposition 4.2 is then proved.  

\[\blacksquare\]
In view of passing to the limit in Proposition 4.1 (or in Theorem 6.1 in the Appendix), we get the following.

**Proposition 4.3.** For all $j \geq 1$, we obtain

1. $Y_{j+1}^{i} \leq Y_{j}^{i}$.
2. $dK_{t}^{j+} \leq dK_{t}^{j+1}$ and $dK_{t}^{j+1} - dK_{t}^{j-}$.

Now let us set

- $Y_{j}^{i} = \inf_{j} Y_{j}^{i}$, $Y_{j}^{-} = \inf_{j} Y_{j}^{-}$.
- $dK^{-}_{j}$ is a positive measure since $K_{T}^{0} < +\infty$, $P \text{-a.s.}$
- $dK^{+}_{j}$ is also a positive measure.

**Remark 4.1.** It should be noted that $Y_{j}^{i} = Y_{j}^{-} = \xi'$. The following result states the convergence of the process $Z$ in $L^2([0, \tau_{n}] \times \Omega)$.

**Proposition 4.4.** Suppose that assumptions of Theorem 4.7 hold. Then we have the following:

1. There exists a process $Z \in \mathcal{L}^{2,d}$ such that, for all $n \in \mathbb{N},$
   $$\mathbb{E} \int_{0}^{\tau_{n}} |Z_{s}^{j} - Z_{s}^{i}|^{2} ds \rightarrow 0, \quad \text{as } j \text{ goes to infinity.}$$

2. The process $(Y^{i}, Z^{i}, K^{i+}, K^{i-})$ is the unique solution of the following GRBSDE with two reflecting barriers

   \[
   \begin{align*}
   (i) \quad & Y_{t}^{i} = \xi^{i} + \int_{t}^{T} 1_{\{s < T\}} dR_{s} + \int_{t}^{T} dK_{s}^{i+} - \int_{t}^{T} dK_{s}^{i-} - \int_{t}^{T} Z_{s} dB_{s}, \quad t \leq T, \\
   (ii) \quad & \forall t \leq T, \ L_{t}^{i} \leq Y_{t}^{i} \leq U_{t}^{i}, \\
   (iii) \quad & \int_{0}^{T} (Y_{t}^{i} - L_{t}^{i-}) dK_{t}^{i+} = \int_{0}^{T} (U_{t}^{i+} - Y_{t}^{i}) dK_{t}^{i-} = 0, \quad a.s., \\
   (iv) \quad & Y_{t}^{i} \in \mathcal{D}, \ K^{i+}, K^{i-} \in \mathcal{K}, \ Z \in \mathcal{L}^{2,d}, \\
   (v) \quad & dK^{i+} \perp dK^{i-}.
   \end{align*}
   \]

**Proof.** We just sketch the proof since the result follows by the same way as previously. Let $n \in \mathbb{N}$ and $j, j' \geq n$. Applying Itô’s formula to $(Y_{j}^{i} - Y_{j'}^{i})^{2}$ on $[0, \tau_{n}]$ it follows that there exists $Z \in \mathcal{L}^{2,d}$ such that

$$\mathbb{E} \sup_{1 \leq t \leq \tau_{n}} |Y_{t}^{i} - Y_{t}^{i'}|^{2} + \mathbb{E} \int_{0}^{\tau_{n}} |Z_{s}^{i} - Z_{s}^{i'}|^{2} ds \rightarrow 0, \quad \text{as } j \text{ goes to infinity.}$$

Hence $Y^{i}$ is $\text{rcoll}$ and $Y_{n}^{-} = Y_{n}^{i'}, \ \forall t \in [0, T]$ $P\text{-a.s.}$ By the same way as previously we have also (4.6). ■

**Proof of Theorem 4.4.** Uniqueness of solutions follows easily. Let us focus on the existence. Let $Z$ be the process given by Proposition 4.4 and define

- $Y_{t} = Y_{t}^{i} 1_{\{t < T\}}$
- $K_{t}^{i+} = K_{t}^{i+} + (U_{t}^{i+} - \Delta_{t} R)^{+} 1_{\{t = T\}}$
- $K_{t}^{i-} = K_{t}^{i-} + (L_{t}^{i-} - \Delta_{t} R)^{-} 1_{\{t = T\}}$.

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Observe that for all $t \in [0, T]$
\[
Y_t = (\xi' - \Delta_T R)1_{t < T} + \int_t^T dR_s + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_s dB_s.
\]

Since $Y'$ is rcll it follows also that $Y$ is rcll. Clearly we have
\[
\xi' - \Delta_T R = (L_T - \Delta_T R) \vee 0 \wedge (U_T - \Delta_T R) = (L_T - \Delta_T R)^+ - (U_T - \Delta_T R)^-.
\]
\[
K^+_t - K^-_t = (K^+_t + (L_T - \Delta_T R)^+ - K^+_t)1_{t < T},
\]
\[
K^-_t - K^+_t = K^-_t + (L_T - \Delta_T R)^+ 1_{t < T}.
\]

Consequently
\[
\begin{align*}
Y_t &= \int_t^T dR_s + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_s dB_s, t \leq T,
\end{align*}
\]

Moreover it follows that
\[
\int_0^T (Y_{t^-} - L_{t^-}) dK^+_t = ((L_T - (\Delta_T R) \vee U_T) - L_T)(L_T - \Delta_T R)^+ = 0.
\]

Similarly we get also that
\[
\int_0^T (U_{t^-} - Y_{t^-}) dK^-_t = 0.
\]

Moreover, since $L_T \leq U_T$, we have $(L_T - \Delta_T R)^+(U_T - \Delta_T R)^- = 0$, then $dK^+ \perp dK^-$. The proof of Theorem 4.1 is finished.

### 4.2 Existence of solution for GBSDE (3.11): the "$f$, $g$ are Lipschitz and $h = 0^n$ case"

**Theorem 4.2.** Let assumptions (H.0)-(H.4) hold. Assume moreover that $h = 0$ and $f$ and $g$ are $a-$Lipschitz, then the GRBSDE (3.11) has a unique solution.

**Proof.** The existence proof is based on the Picard’s approximation scheme. Let $(Y^0, Z^0, K^{+,0}, K^{0}) = (0, 0, 0, 0)$ and define $(Y^{n+1}, Z^{n+1}, K^{n+1+}, K^{n+1})$ as the solution (which exists according to the previous subsection) of the following GRBSDE

\[
\begin{align*}
(i) & \quad Y_t^{n+1} = \int_0^T f(s, Y_s^n, Z_s^n) ds + \int_0^T g(s, Y_s^n) dA_s + \int_t^T dR_s + \int_t^T dK^{n+1}_s + \int_t^T dK^{n+1}_s - \int_t^T Z^{n+1}_s dB_s, t \leq T, \\
(ii) & \quad \forall t \in [0, T], L_t \leq Y_t^{n+1} \leq U_t, \\
(iii) & \quad \int_0^T (Y_{t^-}^{n+1} - L_t) dK^{n+1}_t = \int_0^T (U_{t^-} - Y_{t^-}^{n+1}) dK^{n+1}_t = 0, P - a.s. \\
(iv) & \quad Y^{n+1} \in D^n, K^{n+1+} \in K, Z^{n+1} \in L^{2,d}, \quad dK^{n+1} \perp dK^{n+1}.
\end{align*}
\]
Let \( n, m \in \mathbb{N} \). By applying Itô's formula to \((Y_t^{n+1} - Y_t^{m+1})^2 e^{\alpha(t + A_t)}\), \( \alpha \geq 8c^2(T + 1)(1 + 4c^2) \), where \( c \) is a universal constant, coming from Bulkholder-Davis-Gundy inequality, and using standard calculations for RBSDE one can prove that there exists a process \( Y \in \mathcal{D} \) and \( Z \in \mathcal{L}^{2,d} \) such that

\[
\lim_{n \to +\infty} \mathbb{E} \left[ \sup_{t \leq T} (Y^n_t - Y_t)^2 \right] = 0, \quad \lim_{n \to +\infty} \mathbb{E} \int_0^T |Z^n_s - Z_s|^2 ds = 0.
\]

Let \((\overline{Y}, \overline{Z}, K^+, K^-)\) be the solution, which exists according to the previous subsection, of the following GRBSDE

\[
\begin{align*}
(i) \quad \overline{Y}_t &= \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s) A_s + \int_t^T dR_s \\
&\quad + \int_t^T dK^+_s - \int_t^T dB_s, t \leq T, \\
(ii) \quad \forall t \leq T, \quad L_t \leq \overline{Y}_t - U_t, \\
(iii) \quad \int_0^T (\overline{Y}_t - L_t) dB_t = \int_0^T (U_t - \overline{Y}_t) dK^-_t = 0, \quad \text{a.s.} \\
(iv) \quad \overline{Y} \in \mathcal{D}, \quad K^+, K^- \in \mathcal{K}, \quad \overline{Z} \in \mathcal{L}^{2,d}.
\end{align*}
\]

(4.8)

It is not difficult to prove that there exists a constant \( C > 0 \) such that

\[
\mathbb{E} \sup_{t \leq T} (Y_t^{n+1} - \overline{Y}_t)^2 + \mathbb{E} \int_0^T |Z_t^{n+1} - \overline{Z}_t|^2 \, ds \leq C(\mathbb{E} \sup_{t \leq T} (Y_t^n - Y_t)^2 + \mathbb{E} \int_0^T |Z_t^n - Z_t|^2 \, ds).
\]

Hence

\[
\lim_{n \to +\infty} [\mathbb{E} \sup_{t \leq T} (Y_t^n - \overline{Y}_t)^2 + \mathbb{E} \int_0^T |Z_t^{n+1} - \overline{Z}_t|^2 \, ds] = 0.
\]

It follows that

\[
\mathbb{E} \sup_{s \leq T} |Y_s - \overline{Y}_s|^2 = 0, \quad \text{and} \quad \mathbb{E} \int_0^T |Z_s - \overline{Z}_s|^2 \, ds = 0.
\]

Therefore \( Y = \overline{Y} \) and \( Z = \overline{Z} \). The proof of existence is then finished. The uniqueness of solutions follows easily by using standard arguments.

4.3 Existence of solution for GBSDE (3.11) : the "f, g are Lipschitz and there exists \( S_0 = 0 \leq S_1 \leq \ldots \leq S_{p+1} = T \) such that \( \forall x, y \in \mathbb{R}, \forall t \notin \{S_1, ..., S_{p+1}\} \) \( h(t, \omega, x, y) = 0" case.

Theorem 4.3. Let assumptions (H.0)-(H.4) hold. Assume moreover that there exists a finite family of stopping times \( S_0 = 0 \leq S_1 \leq \ldots \leq S_{p+1} = T \) such that for each \( x, y \in \mathbb{R} \) and \( t \notin \{S_1, ..., S_{p+1}\} \) \( h(t, \omega, x, y) = 0 \) and \( f \) and \( g \) are \( a \)-Lipschitz, then the GBSDE (3.11) has a maximal solution.

Proof. For every \( x \in \mathbb{R} \) and \( i \in \{1, 2, \ldots, p + 1\} \), set

\[
\begin{align*}
\tilde{h}(t, \omega, y) &= \max \{ x \in [L_{t-}, U_{t-}] : x = L_{t-} \vee [y + h(t, x, y) + \Delta_t R] \wedge U_{t-} \} \\
L_i' &= L_i 1_{\{t < S_i\}} + L_{S_i-} 1_{\{t \geq S_i\}} \\
U_i' &= U_i 1_{\{t < S_i\}} + U_{S_i-} 1_{\{t \geq S_i\}} \\
\xi^{p+1} &= \tilde{h}(T, \omega, 0) = h(S_{p+1}, \omega, 0).
\end{align*}
\]

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Therefore \( \Delta \) Now we want to construct a solution to our GRBSDE by induction and concatenation. For that reason, \( \text{suppose there exists a solution to our GRBSDE} \)

\[
\begin{align*}
\text{(i)} & \quad Y_t^{p+1} = \xi^{p+1} + \int_t^T 1_{\{s < S_{p+1}\}} f(s, Y_s^{p+1}, Z_s^{p+1}) ds + \int_t^T 1_{\{s < S_{p+1}\}} g(s, Y_s^{p+1}) dA_s \\
& \quad \quad + \int_t^T 1_{\{s < S_{p+1}\}} dR_s + \int_t^T dK_s^{p+1} - \int_t^T dK_s^{p+1-} - \int_t^T Z_s^{p+1} dB_s, t \leq T,
\end{align*}
\]

\[
\begin{align*}
\text{(ii)} & \quad \forall t \leq T, L_t^{p+1} \leq Y_t^{p+1} \leq U_t^{p+1}, \\
\text{(iii)} & \quad \int_0^T (Y_t^{p+1} - L_t^{p+1}) dK_t^{p+1+} = \int_0^T (U_t^{p+1} - Y_t^{p+1}) dK_t^{p+1-} = 0, \ a.s.
\end{align*}
\]

We should remark here, by Lemma [3.1] that

\[
Y_{S_{p+1}}^{-} = L_{S_{p+1}}^{-} \cup Y_{S_{p+1}}^{+} \wedge U_{S_{p+1}}^{-} = Y_{S_{p+1}}^{+} = \xi^{p+1} = \hat{h}(S_{p+1}, \omega, 0).
\]

Therefore \( \Delta_{S_{p+1}} Y^{p+1} = 0 \).

Now we want to construct a solution to our GRBSDE by induction and concatenation. For that reason, suppose there exists a solution to our GRBSDE \( (Y^{i+1}, Z^{i+1}, K^{i+1+}, K^{i+1-}) \) on \([0, T]\), for \( i \in \{1, ..., p\} \).

Let \( \xi^i = \hat{h}(S_i, \omega, Y^{i+1}_{S_i}) \) and define \( (Y^i, Z^i, K^{i+}, K^{i-}) \) as the unique solution of the following GRBSDE

\[
\begin{align*}
\text{(i)} & \quad Y_t^i = \xi^i + \int_t^T 1_{\{s < S_i\}} f(s, Y_s^i, Z_s^i) ds + \int_t^T 1_{\{s < S_i\}} g(s, Y_s^i) dA_s + \int_t^T 1_{\{s < S_i\}} dR_s \\
& \quad \quad + \int_t^T dK_s^{i+} - \int_t^T dK_s^{i-} - \int_t^T Z_s^i dB_s, t \leq T, \\
\text{(ii)} & \quad \forall t \leq T, L_t^i \leq Y_t^i \leq U_t^i, \\
\text{(iii)} & \quad \int_0^T (Y_t^i - L_t^i) dK_t^{i+} = \int_0^T (U_t^i - Y_t^i) dK_t^{i-} = 0, \ a.s.
\end{align*}
\]

We have \( \Delta_S Y^i = 0 \) and \( Y_S^i = Y^i_{S_i} = \xi^i, \forall s \in [S_i, T]\). By setting

\[
\begin{align*}
Y_t &= \sum_{i=0}^p Y_t^{i+1} 1_{[S_i, S_{i+1})}(t) \\
Z_t &= \sum_{i=0}^p Z_t^{i+1} 1_{[S_i, S_{i+1})}(t) \\
K_t^{\pm c} &= \sum_{i=0}^p \int_0^T 1_{[S_i, S_{i+1})}(s) dK_s^{i+1 \pm c}
\end{align*}
\]
Henceforth it follows that
\[
\begin{align*}
(i) & \quad Y_t = \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s) dA_s + \int_t^T dR_s^c - \sum_{t<s\leq T} \Delta Y_s \\
& \quad \quad + \int_t^T dK^{+}_s - \int_t^T dK^{-}_s - \int_t^T Z_s dB_s, t \leq T, \\
(ii) & \quad \int_0^T (Y_{t-} - L_{t-}) dK^{+}_t = \int_0^T (U_{t-} - Y_{t-}) dK^{-}_t = 0, \text{ a.s.,} \\
(iii) & \quad Y \in \mathcal{D}, \quad K^{+}, K^{-} \in \mathcal{K}^{c}, \quad Z \in L^{2,d}, \\
(iv) & \quad dK^{+} \perp dK^{-}.
\end{align*}
\]

Let \( t \in [0,T] \). We discuss the following two cases.
1. If \( t \notin \{s_1, ..., s_{p+1}\} \), then there exists \( i \) such that \( t \in [s_i, s_{i+1}] \) and then, since \( h(t, x, y) = 0 \), for every \( x, y \), we have
\[
Y_{t-} = Y_{t-}^{i+1} = L_{t-}^{i+1} \vee \{Y_{t-}^{i+1} + \Delta_t R \} \wedge U_{t-} \\
= \max\{x \in [L_{t-}, U_{t-}] : x = L_{t-} \vee [Y_t + h(t, x, Y_t) + \Delta_t R] \wedge U_{t-}\}.
\]

2. If there exists \( i \) such that \( t = s_i \), then
\[
Y_{s_i-} = Y_{s_i-}^{i} = Y_{s_i} = \xi^{i} \\
= \max\{x \in [L_{s_i}, U_{s_i}] : x = L_{s_i} \vee [Y_{s_i} + h(s_i, x, Y_{s_i}) + \Delta_s R] \wedge U_{s_i}\}.
\]

Hence for each \( t \in [0,T] \),
\[
Y_{t-} = \max\{x \in [L_{t-}, U_{t-}] : x = L_{t-} \vee [Y_t + h(t, x, Y_t) + \Delta_t R] \wedge U_{t-}\}.
\]

Consequently, by Lemma 3.3 \((Y, Z, K^{+}, K^{-})\) is the unique solution of the following GRBSDE
\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s) dA_s + \int_t^T dR_s + \sum_{t<s\leq T} h(s, Y_s, Y_s) \\
& \quad \quad + \int_t^T dK^{+}_s - \int_t^T dK^{-}_s - \int_t^T Z_s dB_s, t \leq T, \\
(ii) & \quad \forall t \leq T, \quad Y_{t-} = \max\{x \in [L_{t-}, U_{t-}] : x = L_{t-} \vee [Y_t + h(t, x, Y_t) + \Delta_t R] \wedge U_{t-}\}, \\
(iii) & \quad \int_0^T (Y_{t-} - L_{t-}) dK^{+}_t = \int_0^T (U_{t-} - Y_{t-}) dK^{-}_t = 0, \text{ a.s.,} \\
(iv) & \quad Y \in \mathcal{D}, \quad K^{+}, K^{-} \in \mathcal{K}^{c}, \quad Z \in L^{2,d}, \\
(v) & \quad dK^{+} \perp dK^{-}.
\end{align*}
\]

Henceforth \((Y, Z, K^{+}, K^{-})\) is the maximal solution of the GRBSDE \((4.11)\).

\[\blacksquare\]

4.4 Existence of solution for GBSDE \((3.11)\) : the "general" case

\textbf{Theorem 4.4.} Let assumptions \((H.0)-(H.4)\) hold. Then the GRBSDE \((3.11)\) has a maximal solution.

Since the integrability conditions on parameters are weaker, the proof of Theorem 4.4 is based on regularization by sup-convolution techniques and a truncation procedure by means of a family of stopping times. The final step consists in justifying the passage to the limit and identifying the limit as the solution of our GRBSDE \((3.11)\).
4.4.1 Approximations

It is not difficult to prove the following lemma which gives an approximation of continuous functions by Lipschitz functions.

**Lemma 4.1.** Let \((T_i)_{i \geq 1}\) be a sequence of stopping times such that \([|T_i|] \cap [|T_j|] = \emptyset, \forall i \neq j\) and \(\bigcup_{i \geq 1}[|T_i|] = \{(t, \omega) \in [0, T] \times \Omega : t_i > 0\}\). For every \(n \in \mathbb{N}\) define the functions \(f_n, g_n\) and \(h_n\) by:

\[
\begin{align*}
    f_n(t, y, z) &= \sup_{p \in \mathbb{R}, q \in \mathbb{R}^d} \{f(t, p, q) - n|p - y| - n|q - z|\}1_{\{n \geq 1\}}, \\
    g_n(t, y) &= \sup_{p \in \mathbb{R}} \{g(t, p) - n|p - y|\}1_{\{n \geq 1\}}, \\
    h_n(t, x, y) &= h(t, x, y)1_{\{t \in (T_i, \ldots, T_n)\}}1_{\{n \geq 1\}}.
\end{align*}
\]

Assume that assumptions (H.0)-(H.4) hold. Then we have the following:

1. For all \((t, \omega, y, z, n) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{N}\),
\[
    f_0(t, y, z) = 0 \geq f_n(t, y, z) \geq f_{n+1}(t, y, z) \geq f(t, y, z) \geq -n - C|z|^2.
\]
2. For all \((t, \omega, y, n) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{N}\),
\[
    g_0(t, y) = 0 \geq g_n(t, y) \geq g_{n+1}(t, y) \geq g(t, y) \geq -1.
\]
3. For all \((t, \omega, y, x, n) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{N}\),
\[
    h_0(t, x, y) = 0 \geq h_n(t, x, y) \geq h_{n+1}(t, x, y) \geq h(t, x, y) \geq -l_t.
\]
4. \(f_n\) is uniformly \(n\)-Lipschitz with respect to \((y, z)\).
5. \(g_n\) is uniformly \(n\)-Lipschitz with respect to \(y\).
6. For all \((t, \omega) \in [0, T] \times \Omega\), \((f_n(t, y, z))_{n \geq 0}\) converges to \(f(t, y, z)\) as \(n\) goes to \(+\infty\) uniformly on every compact of \(\mathbb{R} \times \mathbb{R}^d\).
7. For all \((t, \omega) \in [0, T] \times \Omega\), \((g_n(t, y))_{n \geq 0}\) converges to \(g(t, y)\) as \(n\) goes to \(+\infty\) uniformly on every compact of \(\mathbb{R}\).
8. For all \((t, \omega) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}\), \((h_n(t, x, y))_{n \geq 0}\) converges to \(h(t, x, y)\) as \(n\) goes to \(+\infty\) uniformly on every compact of \(\mathbb{R} \times \mathbb{R}\).

Let us define for \(i \in \{1, \ldots, n-1\}\):

\[
\begin{align*}
    S^0_n &= 0 \\
    S^t_n &= \min\{T_1, \ldots, T_n\} \\
    S^t_{i+1} &= \min\{(T_i, \ldots, T_n) \backslash \{S^t_i, \ldots, S^t_n\}\} \\
    S^t_{n+1} &= T.
\end{align*}
\]

We note here that for \(i \in \{1, \ldots, n\}\), \(S^t_i\) is a stopping time. Indeed, it is clear that \(S^t_n\) is a stopping time. Suppose that for \(i \in \{1, \ldots, n-1\}\), \(S^t_i\) is a stopping time and prove that \(S^t_{i+1}\) is a stopping time which is evident since for every \(t \in [0, T]\),

\[
\{S^t_{i+1} \leq t\} = \bigcup_{j=1}^n \{S^t_i < T_j \leq t\} \in \mathcal{F}_t.
\]
Observe that
\[ \{T_1, \ldots, T_n\} = \{S^n_1, \ldots, S^n_n\} \] and \( 0 = S^n_0 < S^n_1 < \ldots < S^n_n \leq S^n_{n+1} = T. \)

Henceforth
\[ h_n(t, x, y) = 0, \ \forall t \notin \{S^n_1, \ldots, S^n_{n+1}\}, \ \forall (x, y) \in \mathbb{R} \times \mathbb{R}. \]

Therefore, according to the previous subsection, for each \( n \in \mathbb{N} \) there exists a unique solution \((Y^n, Z^n, K^{n+}, K^{n-})\) to the following GRBSDE

\[
\begin{align*}
(i) & \quad Y^n_t = \xi + \int_t^T f_n(s, Y^n_s, Z^n_s)ds + \int_t^T g_n(s, Y^n_s)dA_s + \int_t^T dR_s \\
& \quad + \sum_{0 < s \leq T} h_n(s, Y^n_{s-}, Y^n_s) + \int_t^T dK^{n+}_s - \int_t^T dK^{n-}_s - \int_s^T Z^n_sdB_s, t \leq T, \\
(ii) & \quad \forall t \leq T, Y^n_t = \max\{x \in [L^{t-}, U^{t-}] : x = L^{t-} \lor Y^n_{t-} + h_n(t, x, Y^n_t) + \Delta_t R \land U^{t-}\}, \quad (4.13) \\
(iii) & \quad \int_0^T (Y^n_t - L^{t-})dK^{n+}_t - \int_0^T (U^{t-} - Y^n_{t-})dK^{n-}_t = 0, \text{ a.s.} \\
(iv) & \quad Y^n \in \mathcal{D}, \ K^{n+}, K^{n-} \in \mathcal{K}, \ Z^n \in \mathcal{L}^{2,d}, \\
(v) & \quad dK^{n+} \perp dK^{n-}.
\end{align*}
\]

### 4.4.2 Convergence of the approximating scheme

By using Comparison Theorem (Theorem 6.1) it is not difficult to prove the following proposition.

**Proposition 4.5.** For all \( n \geq 0 \), we obtain

1. \( L_t \leq Y^{n+1}_t \leq Y^n_t \leq U_t, \ \forall t \in [0, T], P \text{-a.s.} \)
2. \( dK^{n+}_t \leq dK^{n+1}_t \) and \( dK^{n+1}_t \leq dK^{n-}_t \) on \([0, T]\).

Set

- \( Y^n_{t-} = \inf_{n} Y^n_{t-}, \ Y^n_{t-} = \inf_{n} Y^n_{t-} \)
- \( dK^{n-}_t = \inf_{n} dK^{n-}_t \) which is a positive measure on \([0, T]\) since \( K^{0-}_{T} < +\infty \ P\text{-a.s.} \)
- \( dK^{n+}_t = \sup_{j} dK^{n+}_t \) which is a positive measure on \([0, T]\).

Let \((\tau_j)_{j \geq 1}\) be the family of stopping times defined by

\[ \tau_j = \inf\{s \geq 0 : \sum_{r \leq s} l_r + C_s + |R|_s \geq j\} \land T. \quad (4.14) \]

It should be pointed out that \( P[\cup_{j \geq 1} (\tau_j = T)] = 1. \)

The following result states the convergence of the process \( Z^n \) in \( L^2([0, \tau_j] \times \Omega) \).

**Proposition 4.6.** Assume that assumptions \((H.0)-(H.4)\) hold. Then there exists a process \( Z \in \mathcal{L}^{2,d} \) such that, for all \( j \),

\[ \mathbb{E} \int_0^{\tau_j} |Z^n_s - Z_s|^2 ds \longrightarrow 0, \text{ as } n \text{ goes to infinity.} \]
Proof. For $s \in \mathcal{R}$ and $j \in \mathbb{N}$, let us set $\psi(s) = \frac{e^{ajs}}{4j} - s$. We mention that $\psi$ satisfies the following for all $s \in \mathcal{R}$,

$$\psi'(s) = e^{ajs} - 1, \quad \psi''(s) = 4je^{ajs} = 4j\psi'(s) + 4j. \quad (4.15)$$

Let $n, m \in \mathbb{N}$ such that $m \geq n$. Applying Itô’s formula to $\psi(Y^n - Y^m)$, we get for $t < \tau_j$,

$$\psi(Y^n_t - Y^m_t) = \psi(Y^n_\tau_j - Y^m_\tau_j) + \int_t^{\tau_j} (f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s))\psi'(Y^n_s - Y^m_s)ds$$

$$+ \int_t^{\tau_j} (g_n(s, Y^n_s) - g_m(s, Y^m_s))\psi'(Y^n_s - Y^m_s)dA_s$$

$$+ \sum_{t<s<\tau_j} \psi'(Y^n_s - Y^m_s)(h_n(s, Y^n_s, Z^n_s) - h_m(s, Y^m_s, Z^m_s))$$

$$+ \int_t^{\tau_j} \psi'(Y^n_s - Y^m_s)d(K^n_s - K^m_s) - \int_t^{\tau_j} \psi'(Y^n_s - Y^m_s)d(K^n_s - K^n_s)$$

$$- \int_t^{\tau_j} \psi'(Y^n_s - Y^m_s)(Z^n_s - Z^m_s)dB_s - \frac{1}{2} \int_t^{\tau_j} \psi''(Y^n_s - Y^m_s)(Z^n_s - Z^m_s)^2ds$$

$$- \sum_{t<s<\tau_j} \left[ \psi(Y^n_s - Y^m_s) - \psi(Y^n_s - Y^m_s) - \psi'(Y^n_s - Y^m_s)\Delta(Y^n_s - Y^m_s) \right].$$

Since $\psi(0) = 0$, we have

$$\int_t^{\tau_j} \psi'(Y^n_s - Y^m_s)d(K^n_s - K^m_s) = - \int_t^{\tau_j} \psi'(Y^n_s - L_s) dK^m_s \leq 0,$$

and

$$- \int_t^{\tau_j} \psi'(Y^n_s - Y^m_s)d(K^n_s - K^n_s) = - \int_t^{\tau_j} \psi'(U_s - Y^m_s)dK^n_s \leq 0.$$ 

Then

$$\psi(Y^n_t - Y^m_t) = \psi(Y^n_\tau_j - Y^m_\tau_j) + \int_t^{\tau_j} dR_{n,m,j} + \int_t^{\tau_j} (\eta_n + j|Z^n_s|^2)\psi'(Y^n_s - Y^m_s)ds$$

$$+ \int_t^{\tau_j} \psi'(Y^n_s - Y^m_s)dA_s + \sum_{t<s<\tau_j} \psi'(Y^n_s - Y^m_s)I_s - \int_t^{\tau_j} \psi'(Y^n_s - Y^m_s)(Z^n_s - Z^m_s)dB_s$$

$$- 2j \int_t^{\tau_j} \psi'(Y^n_s - Y^m_s)|Z^n_s - Z^m_s|^2ds - 2j \int_t^{\tau_j} |Z^n_s - Z^m_s|^2ds. \quad (4.16)$$

where $dR_{n,m,j}$ is a positive measure depending on $n, m$ and $j$. By taking $n = 0$ in Equation (4.16) and using a localization procedure, we get for all $j \in \mathbb{N}$

$$2j\mathbb{E} \int_0^{\tau_j} |Z^n_s|^2ds$$

$$\leq \psi(2) + \psi'(2)\mathbb{E} \int_0^{T} \eta_s ds + 2j\psi'(2)\mathbb{E} \int_0^{\tau_j} |Z^n_s|^2ds + \psi'(2)\mathbb{E} \sum_{0<s<\tau_j} I_s. \quad (4.17)$$

Hence there exists a positive constant $c_j$ depending only on $j$ such that

$$\mathbb{E} \int_0^{\tau_j} |Z^n_s|^2ds \leq c_j.$$
It follows from subsection 4.1 that for all \( j \geq 1 \)
\[
\mathbb{E} \int_0^{T_j} |Z^n_s|^2 \, ds < +\infty,
\]
then for all \( j \in \mathbb{N}^* \)
\[
\sup_{m \in \mathbb{N}} \mathbb{E} \int_0^{T_j} |Z^m_s|^2 \, ds < +\infty.
\]
Henceforth, there exist a subsequence \( \left( m_k^n \right)_k \) of \( m \) and a process \( \tilde{Z}^j \in L^2(\Omega \times [0, T]; \mathbb{R}^d) \) such that \( Z^m_{S_t} 1_{\{s \leq \tau_j\}} \) converges weakly in \( L^2(\Omega \times [0, T]; \mathbb{R}^d) \) to the process \( \tilde{Z}^j 1_{\{s \leq \tau_j\}} \) as \( k \) goes to infinity.

Now coming back to Equation (4.18) we have for \( k \geq n \) (and then \( m_k^j \geq k \geq n \))
\[
2j \mathbb{E} \int_0^{T_j} \psi'(Y^n_s - Y^{m_k}_s) |Z^n_s - Z^{m_k}_s|^2 \, ds + 2j \mathbb{E} \int_0^{T_j} |Z^n_s - Z^{m_k}_s|^2 \, ds
\leq \mathbb{E} \psi(Y^n_{\tau_{j}^-} - Y^{m_k}_{\tau_{j}^-}) + \mathbb{E} \int_0^{T_j} (\eta_j + j |Z^{m_k}_s|^2) \psi'(Y^n_s - Y^{m_k}_s) \, ds
\leq \mathbb{E} \psi(Y^n_{\tau_{j}^-} - Y^{m_k}_{\tau_{j}^-}) + \mathbb{E} \int_0^{T_j} (\eta_j + 2j |\tilde{Z}^j_s|^2) \psi'(Y^n_s - Y^{m_k}_s) \, ds
+ 2j \mathbb{E} \int_0^{T_j} |Z^{m_k}_s - \tilde{Z}^j|^2 \psi'(Y^n_s - Y^{m_k}_s) \, ds
+ \mathbb{E} \int_0^{T_j} \psi'(Y^n_s - Y^{m_k}_s) \, ds + \mathbb{E} \sum_{0 < s < \tau_j} \psi'(Y^n_s - Y^{m_k}_s) \, ds.
\]

Since \( |Z^n_s - Z^{m_k}_s|^2 = |Z^n_s - \tilde{Z}^j|^2 + 2(\tilde{Z}^j_s - Z^{m_k}_s, Z^n_s - \tilde{Z}^j) + |Z^n_s - Z^{m_k}_s|^2 \), we have
\[
4j \mathbb{E} \int_0^{T_j} \psi'(Y^n_s - Y^{m_k}_s) |(\tilde{Z}^j_s - Z^{m_k}_s, Z^n_s - \tilde{Z}^j_s)| \, ds + 2j \mathbb{E} \int_0^{T_j} |Z^n_s - Z^{m_k}_s|^2 \, ds
\leq \mathbb{E} \psi(Y^n_{\tau_{j}^-} - Y^{m_k}_{\tau_{j}^-}) + \mathbb{E} \int_0^{T_j} (\eta_j + 2j |\tilde{Z}^j_s|^2) \psi'(Y^n_s - Y^{m_k}_s) \, ds
+ \mathbb{E} \int_0^{T_j} \psi'(Y^n_s - Y^{m_k}_s) \, ds + \mathbb{E} \sum_{0 < s < \tau_j} \psi'(Y^n_s - Y^{m_k}_s) \, ds.
\]

Letting \( k \) to infinity, we get
\[
2j \mathbb{E} \int_0^{T_j} |Z^n_s - \tilde{Z}^j|^2 \, ds
\leq 2j \liminf_{k \to +\infty} \mathbb{E} \int_0^{T_j} |Z^n_s - Z^{m_k}_s|^2 \, ds
\leq \mathbb{E} \psi(Y^n_{\tau_{j}^-} - Y^{m_k}_{\tau_{j}^-}) + \mathbb{E} \int_0^{T_j} (\eta_j + 2j |\tilde{Z}^j_s|^2) \psi'(Y^n_s - Y^{m_k}_s) \, ds
+ \mathbb{E} \int_0^{T_j} \psi'(Y^n_s - Y^{m_k}_s) \, ds + \mathbb{E} \sum_{0 < s < \tau_j} \psi'(Y^n_s - Y^{m_k}_s) \, ds.
\]

By dominated convergence theorem it follows that
\[
\lim_{n \to +\infty} \mathbb{E} \int_0^{T_j} |Z^n_s - \tilde{Z}^j|^2 \, ds = 0.
\]
By the uniqueness of the limit we obtain that
\[ \tilde{Z}_s^j(\omega)1_{\{0 \leq s \leq \tau_j(\omega)\}} = \tilde{Z}_s^{j+1}(\omega)1_{\{0 \leq s \leq \tau_j(\omega)\}}, \quad P(\omega)ds - a.e. \]
For \( s \in [0, T] \), let us set \( Z_s(\omega) = \lim_j \tilde{Z}_s^j1_{\{s \leq \tau_j\}} = \tilde{Z}_s^j(\omega) \), where \( j(\omega) \) is such that \( \tau_j(\omega) = T \). Then, for all \( j \in \mathbb{N} \)
\[ \mathbb{E} \int_0^T |Z_s|^2 ds < +\infty. \] Hence, according to the Holder inequality, there exists a universal constant \( C > 0 \) such that
\[ \mathbb{E} \sup_{t < \tau_j} |Y^{n_t}_t - Y^{m}_{t^-}| \leq C \mathbb{E} \int_0^{\tau_j} (\eta_s + j)|Z_s|^2 ds. \]
Hence
\[ \mathbb{E} \sup_{s < \tau_j} |Y^{n}_s - Y_s| = 0. \]
Consequently \( Y_t \) is reg and \( Y_{t^-} = Y_{t-} \).

**Proposition 4.8.** For all \( j \in \mathbb{N}^* \), we have
\[
1. \lim_{n \to +\infty} \mathbb{E} \int_0^{\tau_j} |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| ds = 0.
2. \lim_{n \to +\infty} \mathbb{E} \int_0^{\tau_j} |g_n(s, Y_s^n) - g(s, Y_s)| dA_s = 0.
3. \lim_{n \to +\infty} \mathbb{E} \sum_{0 < s < \tau_j} |h_n(s, Y_{s^-}^n, Y_{s}^n) - h(s, Y_{s^-}, Y_s)| = 0.

**Proof.** In view of (4.10), there exists a subsequence \( (n'_k)_{k} \) of \( n \) such that:
\[
1. \mathbb{E} \int_0^{\tau_j} |Z_s^{n'_k} - Z_s|^2 ds \leq \frac{1}{2^k} \quad \text{and then} \quad \mathbb{E} \int_0^{\tau_j} \sum_{k=0}^{+\infty} |Z_s^{n'_k} - Z_s|^2 ds \leq 2,
\]
2. \( Z_s^{n_k}(\omega) \rightarrow Z_s(\omega) \), a.e. \((s, \omega) \in [0, \tau_j] \times \Omega\), and \( |Z_s^{n_k}(\omega)| \leq h_j^s \), a.e. \((s, \omega) \in [0, \tau_j] \times \Omega\), where 
\[
h_j^s = 1_{(s \leq \tau_j)} \left( 2|Z_s|^2 + 2 \sum_{k=0}^{\infty} |Z_s^{n_k} - Z_s|^2 \right)^{\frac{1}{2}}.
\]

It follows then from Lemma 4.1 that
\[
\mathbb{E} \int_0^{\tau_j} |f_{n_k}(s, Y_s^{n_k}, Z_s^{n_k}) - f(s, Y_s^{n_k}, Z_s^{n_k})| ds \\
= \mathbb{E} \int_0^{\tau_j} f_{n_k}(s, Y_s^{n_k}, Z_s^{n_k}) - f(s, Y_s^{n_k}, Z_s^{n_k})| ds \\
= \mathbb{E} \int_0^{\tau_j} \left( f_{n_k}(s, Y_s^{n_k}, Z_s^{n_k}) - f(s, Y_s^{n_k}, Z_s^{n_k}) \right) 1_{(|Z_s^{n_k} - Z_s| \leq 1)} ds \\
+ \mathbb{E} \int_0^{\tau_j} \left( f_{n_k}(s, Y_s^{n_k}, Z_s^{n_k}) - f(s, Y_s^{n_k}, Z_s^{n_k}) \right) 1_{(|Z_s^{n_k} - Z_s| > 1)} ds \\
\leq \mathbb{E} \int_0^{\tau_j} \sup_{(\eta, z) \in [0,1] \times B(Z,1)} \left( f_{n_k}(s, y, z) - f(s, y, z) \right) ds \\
+ \mathbb{E} \int_0^{\tau_j} \left( \eta_s + 2j \right) |Z_s^{n_k} - Z_s|^2 + 2j |Z_s|^2 \right) 1_{(|Z_s^{n_k} - Z_s| > 1)} ds,
\]
where \( B(Z,1) \) is the closed ball of center \( Z \) and radius 1.

By taking account of Lemma 4.1 one can see that
\[
\sup_{(\eta, z) \in [0,1] \times B(Z,1)} \left( f_{n_k}(s, y, z) - f(s, y, z) \right) \leq \eta_s + j(|Z_s| + 1)^2.
\]

Henceforth, by using Lebesgue’s dominated convergence theorem, we get
\[
\lim_k \mathbb{E} \int_0^{\tau_j} \sup_{(\eta, z) \in [0,1] \times B(Z,1)} \left( f_{n_k}(s, y, z) - f(s, y, z) \right) ds = 0,
\]
and
\[
\lim_k \mathbb{E} \int_0^{\tau_j} \left( \eta_s + j \right) |Z_s^{n_k} - Z_s|^2 + j |Z_s|^2 \right) 1_{(|Z_s^{n_k} - Z_s| > 1)} ds = 0.
\]

Therefore
\[
\lim_k \mathbb{E} \int_0^{\tau_j} \left( f_{n_k}(s, Y_s^{n_k}, Z_s^{n_k}) - f(s, Y_s^{n_k}, Z_s^{n_k}) \right) ds = 0.
\]

It follows also from Lebesgue’s dominated convergence theorem and the continuity of \( f \) that for all \( j \in \mathbb{N} \)
\[
\lim_k \mathbb{E} \int_0^{\tau_j} |f(s, Y_s^{n_k}, Z_s^{n_k}) - f(s, Y_s, Z_s)| ds = 0.
\]

Hence for all \( j \in \mathbb{N} \)
\[
\lim_k \mathbb{E} \int_0^{\tau_j} |f_{n_k}(s, Y_s^{n_k}, Z_s^{n_k}) - f(s, Y_s, Z_s)| ds = 0.
\]
Since the above limit doesn’t depend on the choice of the subsequence \((n_k^j)\) we have for all \(j \in \mathbb{N}\)

\[
\lim_{n} \mathbb{E} \int_{0}^{\tau_j} | f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) | \, ds = 0.
\]

It not difficult also to prove that for all \(j \in \mathbb{N}\)

\[
\lim_{n} \mathbb{E} \int_{0}^{\tau_j} | g_n(s, Y_s^n) - g(s, Y_s) | \, dA_s = 0.
\]

Now

\[
\mathbb{E} \sum_{0 < s < \tau_j} | h_n(s, Y_s^n, Y_s^n) - h(s, Y_s, Y_s) | 
\leq \mathbb{E} \sum_{i=1}^{n} | h(T_i, Y_{T_i}^n, Y_{T_i}^n) - h(s, Y_{T_i}, Y_{T_i}) | 1\{T_i < \tau_j\} + \mathbb{E} \sum_{i=n+1}^{+\infty} l_{T_i} 1\{T_i < \tau_j\}
\]

Since \(|h(T_i, Y_{T_i}^n, Y_{T_i}^n) - h(s, Y_{T_i}, Y_{T_i})| \leq 2l_{T_i}\) and \(\mathbb{E} \sum_{i=1}^{+\infty} l_{T_i} 1\{T_i < \tau_j\} \leq j\), by Lebesgue’s convergence theorem and the continuity of \(h\) it follows that

\[
\lim_{n \to +\infty} \mathbb{E} \sum_{0 < s < \tau_j} | h_n(s, Y_s^n, Y_s^n) - h(s, Y_s, Y_s) | = 0.
\]

\[4.4.3\] Identification of the limit

**Proposition 4.9.** The process \((Y, Z, K^+, K^-)\) defined in Subsection 4.4.2 satisfies, \(P\)-a.s., the following:

1. \(K^+_T < +\infty\).

2. \[
Y_t = \int_{t}^{T} f(s, Y_s, Z_s) ds + \int_{t}^{T} g(s, Y_s) dA_s + \int_{t}^{T} dR_s + \sum_{t < s \leq T} h(s, Y_s, Y_s) + \int_{t}^{T} dK^+_s \\
- \int_{t}^{T} dK^-_s - \int_{t}^{T} Z_s dB_s.
\]

3. \[
\int_{0}^{T} (U_{t-} - Y_{t-}) dK^-_t = \int_{0}^{T} (Y_{t-} - L_{t-}) dK^+_t = 0 \text{ and } dK^+ \perp dK^-.
\]

**Proof.** 1. From Equation (4.13)(i) we obtain, \(\forall j, \sup_n \mathbb{E} K^+_j < +\infty\). It follows then from Fatou’s lemma that for any \(j \in \mathbb{N}\), \(\mathbb{E} K^+_j < +\infty\). Henceforth \(K^+_j < +\infty, P\)-a.s. But, \(P\)-a.s.

\[
\Delta_T K^+ = \lim_{n \to +\infty} \Delta_T K^{n+} \\
= \lim_{n \to +\infty} (L_{T-} - (h_n(T, Y^n_{T-}, 0) + \Delta_T R))^+ \\
= (L_{T-} - (h(T, Y_{T-}, 0) + \Delta_T R))^+ \\
< +\infty,
\]

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then $K_T^+ < +\infty$, $P$-a.s.

2. Since
\[
Y_t^n = Y^n_{\tau_j} + \int_t^{\tau_j} f_n(s, Y^n_s, Z^n_s)ds + \int_t^{\tau_j} g_n(s, Y^n_s) dA_s + \int_t^{\tau_j} dR_s
+ \sum_{t<s<\tau_j} h_n(s, Y^n_s - Y^n_s) + \int_t^{\tau_j} dK^n_s - \int_t^{\tau_j} dK^n_s - \int_t^{\tau_j} Z^n_s dB_s
\]
Passing to the limit as $n$ goes to infinity and using the fact that $\tau_j$ is a stationary stopping time we get

\[
P-a.s. \quad \forall \tau_j \leq T.
\]

By using relation (3.3) we obtain
\[
(Y_T - Y_T - \xi - h(T, Y_T, 0) - \Delta T K^+ + \Delta T K^-) = \lim_{n \to +\infty} (Y^n_T - \xi - h_n(T, Y^n_T, 0) - \Delta T K^n+ + \Delta T K^n-) = 0,
\]
then it follows that, $P$-a.s.

\[
Y_t = \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s) dA_s + \int_t^T dR_s + \sum_{t<s<T} h(s, Y_s, Y_s) + \int_t^T dK^+_s
- \int_t^T dK^-_s - \int_t^T Z_s dB_s.
\]

3. Now let us prove the minimality conditions. We have
\[
\int_0^T (U_{t^-} - Y^n_{t-}) dK^n_- = 0.
\]
Hence, since $dK^- = \inf_n dK^n-$, we get
\[
\int_0^T (U_{t^-} - Y^n_{t-}) dK^- = 0.
\]
It follows then from Fatou’s lemma that
\[
\int_0^T (U_{t^-} - L_{t-}) dK^+ = 0.
\]
On the other hand
\[
\int_0^T (Y^n_{t-} - L_{t-}) dK^n+ = 0.
\]
Hence, since \( Y = \inf_n Y^n \), we obtain
\[
\int_0^T (Y_t - L_t^-)dK_t^+ = 0.
\]
Applying Fatou’s lemma we obtain
\[
\int_0^T (Y_t - L_t^-)dK_t^+ = 0.
\]
Now, since \( dK^+_n = \sup_n dK^{n+}, \ dK^-_n = \inf_n dK^{n-} \) and the measures \( dK^{n+} \) and \( dK^{n-} \) are singular, it follows that \( dK^+ \) and \( dK^- \) are singular.

**Proof of Theorem 4.4** By using Propositions 4.7-4.9 it is not difficult to see that the process \((Y, Z, K^+, K^-)\) satisfies Equation (5.11). It remains to prove \((Y, Z, K^+, K^-)\) is maximal.

Let \((Y', Z', K'^+, K'^-)\) be an another solution to Equation (5.11). By comparison theorem (Theorem 3.1) we have that \( Y' \leq Y^n \) and then \( Y' \leq Y \). The proof of Theorem 4.4 is then finished.

## 5 Comparison theorem for maximal solutions

This section is devoted to show a comparison theorem for maximal solutions. For this reason, suppose that assumptions (A.1)–(A.4) hold and \((Y, Z, K^+, K^-)\) is the maximal solution for the following GRBSDE

\[
\begin{align}
(i) \quad & Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s)dA_s + \sum_{t<s\leq T} h(s,Y_{s-},Y_s) \\
& \quad + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_sdB_s, t \leq T, \\
(ii) \quad & \forall t \in [0,T], \quad L_t \leq Y_t \leq U_t, \\
(iii) \quad & \int_0^T dK^+_t = \int_0^T (U_t - Y_t) - dK^-_t = 0, \text{ a.s.}, \\
(iv) \quad & Y \in \mathcal{D}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}, \\
(v) \quad & dK^+ \perp dK^-.
\end{align}
\]

Let \((Y', Z', K'^+, K'^-)\) be a solution for the following GRBSDE

\[
\begin{align}
(i) \quad & Y'_t = \xi' + \int_t^T f'(s)ds + \int_t^T g'(s)dA_s + \sum_{t<s\leq T} h'(s) + \int_t^T dK'^+_s \\
& \quad - \int_t^T dK'^-_s - \int_t^T Z'_sdB_s, t \leq T, \\
(ii) \quad & \forall t \in [0,T], \quad L'_t \leq Y'_t \leq U'_t, \\
(iii) \quad & \int_0^T (Y'_t - L'_t^-)dK'^+_t = \int_0^T (U'_t - Y'_t) - dK'^-_t = 0, \text{ a.s.}, \\
(iv) \quad & Y' \in \mathcal{D}, \quad K'^+, K'^- \in \mathcal{K}, \quad Z' \in \mathcal{L}^{2,d}, \\
(v) \quad & dK'^+ \perp dK'^-.
\end{align}
\]

Assume moreover that for every \( t \in [0,T] \)

1. \( \xi \leq \xi' \),
2. \( Y'_t \leq U_t \),
3. \( L'_t \leq Y_t \),
4. \( f'(s)ds \leq f(s,Y'_t,Z'_t)ds \) on \([0,T]\),
5. \( g'(s)dA_s \leq g(s,Y'_t)dA_s \) on \([0,T]\),
6. \( h'(s) \leq h(s,Y'_t,Y'_s) \) for every \( s \in ]0,T[\).

The way in which the maximal solution for GRBSDE (2.1) has been constructed allow us to deduce the following comparison theorem.

**Theorem 5.1.** (Comparison theorem for maximal solutions) Assume that the above assumptions hold then we have :

1. \( Y'_t \leq Y_t \), for \( t \in [0,T] \), \( P \)-a.s.
2. \( 1_{\{U'_t = U_t \}} dK'^-_t \leq dK'^-_t \) and \( 1_{\{L'_t = L_t \}} dK'^+_t \leq dK'^+_t \).

**Proof.** By using an exponential change like the one used in Section 3.2 one can suppose assumptions (H.0)–(H.4) instead of (A.1)–(A.4). Moreover by using the approximations scheme of Section 4.4 and defining \( (Y^n,Z^n,K^n,\xi^n) \) as the unique solution of GRBSDE (4.13) it follows from comparison theorem (Theorem 6.1) that for every \( n \in \mathbb{N} \)

1. \( Y'_t \leq Y^n_t \), for \( t \in [0,T] \), \( P \)-a.s.
2. \( 1_{\{U'_t = U_t \}} dK'^-_t \leq dK'^-_t \) and \( 1_{\{L'_t = L_t \}} dK'^+_t \leq dK'^+_t \).

By letting \( n \) to infinity, the result follows by using the convergence result of Section 4.4 Theorem 5.4 is proved.

**Remark 5.1.** It should be noted that the result of Theorem 5.1 remains true if the data \( f',g' \) and \( h' \) of GRBSDE (5.2) depend on \((Y',Z'), Y' \) and \((Y'_t,Y'_s)\) respectively.

## 6 Appendix: Comparison theorem

The comparison theorem for real-valued BSDEs turns out to be one of the classic results of the theory of BSDE. It allows to compare the solutions of two real-valued BSDEs whenever we can compare the terminal conditions and the generators. This section is devoted to show a comparison theorem for the following GRBSDE:

\[
\begin{cases}
(i) & Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s)dA_s + \int_t^T dR_s + \sum_{t<s\leq T} h(s,Y_s,Y_s) \\
& \quad + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_s dB_s, t < T, \\
(ii) & \forall t \in [0,T[; L_t \leq Y_t \leq U_t, \\
(iii) & \int_0^T (Y_t - L_t) dK^+_t = \int_0^T (U_t - Y_t) dK^-_t = 0, \text{ a.s.}, \\
(iv) & \bar{Y} \in \mathcal{D}, K^+, K^- \in \mathcal{K}, Z \in \mathcal{L}^{2,d}, \\
(v) & dK^+ \perp dK^-.
\end{cases}
\]
Let $(Y^i, Z^i, K^{i+}, K^{i0})$ (i = 1, 2) be two solutions of Equation (6.1) associated respectively with $(\xi^1, f^1, g^1, h^1, A^1, L^1, U^1)$ and $(\xi^2, f^2, g^2, h^2, A^2, L^2, U^2)$, such that, for (i = 1, 2) the following assumptions are satisfied:

(D.1) $L^i : [0, T] \times \Omega \rightarrow \mathbb{R}$ and $U^i : [0, T] \times \Omega \rightarrow \mathbb{R}$ are two rcll barriers processes satisfying

$$L_i^1 \leq Y_i^1, \quad Y_i^1 \leq U_i^2, \quad U_i^1 \land U_i^2 - L_i^1 \lor L_i^2 \leq 2 \quad \forall t \in [0, T], \; P - a.s.$$  

(D.2) $0 \leq A_i^2 \leq 1, \quad dR_i^1 \leq dR_i^2$ on $[0, T]$.

(D.3) $f^1$ and $f^2$ are such that:

a. $f^1(s, Y_i^1, Z_i^1) \leq f^2(s, Y_i^1, Z_i^2), \quad dsP(d\omega) - a.e.$.

b. $f^2$ is uniformly Lipschitz with respect to $(y, z)$ with Lipschitz constant $C_1 \geq 0$.

d. $g^1$ and $g^2$ are such that:

a. $g^1(s, Y_i^1)dA_i^1 \leq g^2(s, Y_i^1)dA_i^2$ on $[0, T], \; P(d\omega)-a.s.$.

b. $g^2$ is uniformly Lipschitz with respect to $y$ with Lipschitz constant $C_2 \geq 0$.

(D.5) $h^1$ and $h^2$ are such that:

a. $P$-a.s., $\forall s \leq T$, $h^1(s, Y_i^1, Y_i^2) \leq h^2(s, Y_i^1, Y_i^2)$,

b. $P$-a.s., $\forall (t, x) \in [0, T] \times \mathbb{R}$, the function $y \mapsto y + h^2(t, \omega, L_i^2(\omega) \land x \lor U_i^2(\omega), L_i^2(\omega) \lor y \land U_i^2(\omega))$ is nondecreasing.

c. there exists a family of stopping times $S_0 = 0 \leq S_1 \leq ... \leq S_{p+1} = T$ such that : $\forall s \notin \{S_1, ..., S_p, S_{p+1} = T\} \; h^2(s, x, y) = 0$, for every $(x, y) \in \mathbb{R}^2$ and for each $i \in \{1, ..., p\}$

$$Y_{S_i}^2 = \max\{x \in [L_{S_i}^2, U_{S_i}^2] : x = L_{S_i}^2 \lor [Y_{S_i}^2 + h^2(t, x, Y_{S_i}^2) + \Delta_i R_i^2] \land U_{S_i}^2 \}.$$  

The following comparison theorem plays a crucial role in our proofs.

**Theorem 6.1. (Comparison theorem) Assume that assumptions (D.1) – (D.5) hold. Then**

1. $Y_i^1 \leq Y_i^2$ for $t \in [0, T], \; P-a.s.$

2. $1_{\{L_i^1 = U_i^2\}}dK_i^{1-} \leq dK_i^{2-}$ and $1_{\{L_i^1 = U_i^2\}}dK_i^{2+} \leq dK_i^{1+}.$

In order to prove Theorem 6.1 we need the following lemma.

**Lemma 6.1. Let $\tau \in [0, T]$ be a stopping time. If $Y_i^1 \leq Y_i^2$ then $Y_i^1 = Y_i^2.$**

**Proof.** We distinguish two cases.

1. If $Y_i^1 \leq L_i^- \lor U_i^2$, then it is obvious, by assumption (D.1), that $Y_i^1 \leq Y_i^2$.

2. If $Y_i^1 > L_i^- \lor U_i^2$, by Lemma 5.1 and assumptions (D.2) – (D.5), we have

$$Y_i^1 = [Y_i^1 + h^1(\tau, Y_i^1, Y_i^2) + \Delta_i R_i^1] \land U_i^1 \leq [Y_i^2 + h^2(\tau, Y_i^1, Y_i^2) + \Delta_i R_i^2] \land U_i^2.$$  

Since $Y_i^2 = \max\{x \in [L_i^2, U_i^2] : x = L_i^2 \lor [Y_i^2 + h^2(t, x, Y_i^2) + \Delta_i R_i^2] \land U_i^2 \},$ then $Y_i^1 \leq Y_i^2.$

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Proof of Theorem 6.1. We proceed by induction. We have $Y_{S_1}^1 = \xi_1 \leq Y_{S_1}^2 = \xi_2$. Suppose that for $i \in \{0, \ldots, p\}$, $Y_{S_{i+1}}^1 \leq Y_{S_{i+1}}^2$ $P$–a.s., then by Lemma 6.1 we have $Y_{S_{i+1}}^1 \leq Y_{S_{i+1}}^2$. Let $\tau \in [S_i, S_{i+1}]$ and define

$$\lambda_\tau := \inf\{s > \tau : Y_s^2 \geq Y_s^1\} \wedge S_{i+1}.$$ 

On the set $\{\omega \in \Omega : \tau(\omega) < \lambda_\tau(\omega)\}$, for every $s \in [\tau, \lambda_\tau]$, we have $Y_s^2 < Y_s^1$ and then for every $s \in [\tau, \lambda_\tau]$, we have $Y_s^2 

2 \tau - 1 \int \lambda_t^{-1}f(s, Y_s^1, Z_s^1)ds + \int \lambda_t^{-1}g(s, Y_s^1)dA_s^1 + \int \lambda_t^{-1}g(s, Y_s^2)dA_s^2 

+ \int \lambda_t^{-1}(dR_s^1 - dR_s^2) + \sum_{t < s < \lambda_\tau}(h(s, Y_s^1, Y_s^1) - h(s, Y_s^2, Y_s^2)) 

+ \int \lambda_t^{-1}d(K_s^1 - K_s^2) - \int \lambda_t^{-1}d(K_s^1 - K_s^2) - \int \lambda_t^{-1}(Z_s^1 - Z_s^2)dB_s. 

But for every $t \in [\tau, \lambda_\tau]$, $\int \lambda_t^{-1}dK_s^1 = \int \lambda_t^{-1}1_{s < \lambda_\tau}1_{(Y_s^1 < Y_s^2 \leq 1)}dK_s^1 = 0$ and by the same way $\int \lambda_t^{-1}dK_s^2 = 0$. Therefore

$$Y_t^1 - Y_t^2 = \int \lambda_t^{-1}(f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2))ds + \int \lambda_t^{-1}g(s, Y_s^1)dA_s^1 + \int \lambda_t^{-1}g(s, Y_s^2)dA_s^2 

+ \int \lambda_t^{-1}(dR_s^1 - dR_s^2) + \sum_{t < s < \lambda_\tau}(h(s, Y_s^1, Y_s^1) - h(s, Y_s^2, Y_s^2)) 

- \int \lambda_t^{-1}d(K_s^1 - K_s^2) - \int \lambda_t^{-1}(Z_s^1 - Z_s^2)dB_s.$$ 

Since $f^2$ and $g^2$ are Lipschitz then we can write $f^2(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2) = a_s(Y_s^1 - Y_s^2) + \hat{b}_s(Z_s^1 - Z_s^2)$, $s \leq T$ and $g^2(s, Y_s^1) - g^2(s, Y_s^2) = \tilde{a}_s(Y_s^1 - Y_s^2)$, where $(a_t)_{t \leq T}$, $(\hat{a}_t)_{t \leq T}$ and $(\hat{b}_t)_{t \leq T}$ are bounded $P$-measurable processes, we have

$$Y_t^1 - Y_t^2 = \int \lambda_t^{-1}(a_s(Y_s^1 - Y_s^2) + \hat{b}_s(Z_s^1 - Z_s^2))ds + \int \lambda_t^{-1}\tilde{a}_s(Y_s^1 - Y_s^2)dA_s^2 

- \int \lambda_t^{-1}1_{s < \lambda_\tau}dV_s - \int \lambda_t^{-1}(Z_s^1 - Z_s^2)dB_s,$$

where the process $V \in \mathcal{K}$ is defined by :

$$V_t := \int_0^t(f(s, Y_s^1, Z_s^1) - f(s, Y_s^1, Z_s^1))ds + \int_0^t g^2(s, Y_s^1)dA_s^2 - \int_0^t g^1(s, Y_s^1)dA_s^1 

+ \sum_{0 < s < t}(h^2(s, Y_s^1, Y_s^1) - h^1(s, Y_s^1, Y_s^1)) + R_t^2 - R_t^1 + K_t^2 + K_t^1.$$
Setting \( \Gamma_t = e^{f_t'(a, ds + \tilde{\alpha} dA^c_t)} + f_t'(\tilde{b}_t d\bar{B}_t - \frac{1}{2} f_t''(\tilde{b}_t)^2 ds, \) it follows that

\[
\Gamma_t(Y_t^1 - Y_t^2) = - \int_t^\lambda 1_{\{s < \lambda_t\}} \Gamma_s dV_s - \int_t^\lambda \Gamma_s (Z_s^1 - Z_s^2) dB_s.
\]

Let \( \theta_n \) be a family of stopping times defined by

\[
\theta_n = \inf\{s \geq \tau : \int_\tau^s \Gamma_s |Z_s^1 - Z_s^2|^2 ds \geq n\} \land \lambda_t.
\]

By assumption (D.1) we have

\[
\mathbb{E}1_{\{\tau < \lambda_t\}} (Y_t^1 - Y_t^2) \leq \mathbb{E}1_{\{\tau < \lambda_t\}} \Gamma_\theta_n (Y_{\theta_n}^1 - Y_{\theta_n}^2) \leq \mathbb{E}1_{\{\tau < \lambda_t\}} \Gamma_\lambda_n (U_{\theta_n}^1 \land U_{\theta_n}^2 - L_{\theta_n}^1 \lor L_{\theta_n}^2)^\lambda \leq 2 \mathbb{E}1_{\{\tau < \lambda_t\}} \Gamma_\lambda_n.
\]

Since \( P[\cup_{n \geq 0}(\theta_n = \lambda_t)] = 1 \) and \( \mathbb{E}1_{\lambda_t} < +\infty \), it follows that

\[
\mathbb{E}1_{\{\tau < \lambda_t\}} (Y_t^1 - Y_t^2) \leq 2 \lim_{n \to +\infty} \mathbb{E}1_{\{\tau < \lambda_t\}} \Gamma_\lambda_n = 2 \mathbb{E}1_{\{\tau < \lambda_t\}} \Gamma_\lambda_n = 0.
\]

Hence \( 1_{\{\tau < \lambda_t\}} (Y_t^1 - Y_t^2) = 0, P-a.s. \) and then \( P[\tau < \lambda_t, Y_t^2 < Y_t^1] = 0 \). Hence \( \tau = \lambda_t \) or \( Y_t^1 = Y_t^2 \) \( P-a.s. \). Therefore \( Y_t^1 \geq Y_t^1 \), for every \( \tau \in [S_t, S_{t+1}] \). Then \( Y^1 \leq Y^2 \). The proof of assertion \( I. \) is finished.

2. Let us first point out that if

\[
X_t = X_0 + V_t + \int_0^t \alpha_s dB_s + \sum_{0 < s \leq t} \Delta_s X,
\]

with \( X_0 \in \mathbb{R}, V \in K^c - K^c \) and \( \alpha \in L^2,d. \) Then by using Itô-Tanaka formula, there exists \( \tilde{\tau} \in \mathcal{K} \) such that

\[
X_{\tilde{\tau}}^+ = X_0^+ + \int_0^t 1_{\{X_s > 0\}} dV_s + \int_0^t 1_{\{X_s > 0\}} \alpha_s dB_s + \sum_{0 < s \leq t} \Delta_s X^+ + \tilde{\tau}.
\]

Henceforth, if \( X_t \geq 0 \), then

\[
1_{\{X_t = 0\}} \alpha_s = 0, \ ds P(d\omega) - a.e. \ and \ 1_{\{X_t = 0\}} dV_s = \tilde{d} \textrm{ on } [0, T], P-a.s.
\]

Now, by using the above remark to \( X_t := Y_t^2 - Y_t^1 \geq 0 \), then \( 1_{\{X_t = 0\}} (Z_t^2 - Z_t^1) = 0, \ ds P(d\omega) - a.e. \).

Moreover, there exists \( \tilde{\tau} \in \mathcal{K}^c \) such that

\[
- \tilde{d} = 1_{\{X_t = 0\}} (f^2(s, Y_t^1, Z_t^1) - f^1(s, Y_t^1, Z_t^1)) ds + 1_{\{X_t = 0\}} (g^2(s, Y_t^1) dA_t^2 - g^1(s, Y_t^1) dA_t^1) + 1_{\{X_t = 0\}} (dK_{s+}^2 - dK_{s-}^2).
\]

Therefore

\[
1_{\{Y_t^1 = Y_t^2\}} dK_{s+}^2 \leq 1_{\{Y_t^1 = Y_t^2\}} (dK_{s-}^2) + dK_{s+}^1.
\]

Since \( dK_{s+}^2 \perp dK_{s-}^2 \), we get

\[
1_{\{Y_t^1 = Y_t^2\}} dK_{s+}^2 \leq dK_{s+}^1.
\]
Then
\[ 1_{\{L_2^s = L_1^s\}} dK_s^{2+,c} = 1_{\{L_2^s = L_1^s = Y_2^s = Y_1^s\}} dK_s^{2+,c} \leq dK_s^{1+,c}. \]

Let us now compare the discontinuous parts of the reflecting processes. By formula (3.3) and assumption (D.5), we have for each \( s \in [0, T] \) such that \( \Delta_s K^{2+} > 0 \) and \( L_2^s = L_1^s \) (then \( L_1^s = Y_2^s = Y_1^s = L_2^s \))

\[
\Delta_s K^{2+} = \begin{cases} 
L_2^s - [Y_2^s + h^2(s, Y_2^s, Y_2^s) + \Delta_s R^2] \\
L_1^s - [Y_2^s + h^2(s, Y_1^s, Y_2^s) + \Delta_s R^2] \\
\leq L_1^s - [Y_1^s + h^2(s, Y_1^s, Y_1^s) + \Delta_s R^2] \\
\leq L_2^s - [Y_1^s + h^2(s, Y_1^s, Y_1^s) + \Delta_s R^1] \\
= \Delta_s K^{1+}.
\end{cases}
\]

Hence
\[ 1_{\{L_2^s = L_1^s\}} \Delta_s K^{2+} \leq \Delta_s K^{1+}. \]
Consequently
\[ 1_{\{L_2^t = L_1^t\}} dK_t^{2+} \leq dK_t^{1+}. \]
Similarly we have also that
\[ 1_{\{U_2^t = U_1^t\}} dK_t^{1-} \leq dK_t^{2-}. \]
This completes the proof of Theorem 6.1.

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