State Exchange with Quantum Side Information

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We consider a quantum communication task between two users Alice and Bob, in which Alice and Bob exchange their respective quantum information by means of local operations and classical communication assisted by shared entanglement. Here, we assume that Alice and Bob may have quantum side information, not transferred, and classical communication is free. In this work, we derive general upper and lower bounds for the least amount of entanglement which is necessary to perfectly perform this task, called the state exchange with quantum side information. Moreover, we show that the optimal entanglement cost can be negative when Alice and Bob make use of their quantum side information. We finally provide conditions on the initial state for the state exchange with quantum side information which give the exact optimal entanglement cost.

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Introduction.— In quantum information theory, one of the most traditional research topics has been source coding problems of transmitting Alice’s quantum information to Bob under various situations, with paradigmatic examples including Schumacher compression [1] and quantum teleportation [2]. A decade ago, Oppenheim and Winter devised a new type of a quantum communication task named state exchange [3] — in which Alice and Bob exchange their quantum information with each other by means of local operations and classical communication (LOCC) and shared entanglement — and they studied the least amount of entanglement consumed in the task when free classical communication is allowed.

In the original state exchange task, it is assumed that both Alice and Bob do not have any quantum side information (QSI) transferrable during the protocol. On the other hand, most quantum communication tasks, including state merging [4, 5] and state redistribution [6, 7], begin with the assumption that either Alice or Bob has QSI. For example, in the state merging task, Bob can make use of his QSI for merging Alice’s information to him, and the minimum amount of entanglement needed for merging turns out to be exactly given by the quantum conditional entropy [8] conditioned on Bob’s QSI.

In this work we generalize in the state exchange to an exchanging task allowing Alice’s and Bob’s QSI, which is called the state exchange with quantum side information.

We consider three parties, Alice, Bob, and a referee (R), sharing a pure initial state $|\psi\rangle_{AC_1B_1C_1R}$ of Alice, Bob, and a referee (R), and they have additional systems $C_A$ and $C_B$, exploiting their respective QSI $A$ and $B$. The ancillary systems $E_A^R$ and $E_B^R$ represent an initial entanglement consumed for the exchanging task, while $E_A^{\text{out}}$ and $E_B^{\text{out}}$ indicate entanglement generated from the task.

FIG. 1: Illustration of state exchange protocol $\mathcal{E}$ with QSI. Starting from an initial state $|\psi\rangle_{AC_1B_1C_1R}$ of Alice, Bob, and a referee (R), Alice and Bob exchange their parts $C_A$ and $C_B$, exploiting their respective QSI $A$ and $B$. The ancillary systems $E_A^R$ and $E_B^R$ represent an initial entanglement consumed for the exchanging task, while $E_A^{\text{out}}$ and $E_B^{\text{out}}$ indicate entanglement generated from the task.

We show that in general this strategy does not provide

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the optimal entanglement cost of the state exchange with QSI. However for a specific initial state of the state exchange with QSI, the upper bound shows that the optimal entanglement cost for the state exchange without QSI cannot be negative [3]. More importantly, this implies that the use of Alice’s and Bob’s QSI can significantly reduce the optimal entanglement cost of the exchanging task.

We furthermore consider an idealized situation in which the referee plays a more active role and can help Alice and Bob to exchange their information [3]. By virtue of the referee’s assistance, it is possible for Alice and Bob to more efficiently perform the state exchange with QSI, and this provides us with converse bounds on the optimal entanglement cost, which are lower bounds for any achievable entanglement rate. As an application of our bounds, we present conditions on the initial state for the state exchange with QSI such that the exact optimal entanglement cost can be obtained.

State exchange with quantum side information.— In the task of state exchange $E$ with QSI as described in Fig. 1, the global initial state $\psi_i$ and the global final state $\psi_f$ are given by

$$\psi_i = \psi \otimes \Phi_{E_{\text{in}} A_{\text{in}}}$$

and

$$\psi_f = \psi' \otimes \Phi_{E_{\text{out}} A_{\text{out}}}$$

where $\psi = |\psi\rangle\langle\psi|$, $\Phi_{E_{\text{in}} A_{\text{in}}}$ and $\Phi_{E_{\text{out}} A_{\text{out}}}$ are pure maximally entangled states with Schmidt rank $e^\text{in}(E)$ and $e^\text{out}(E)$, respectively, $\psi' = (1_{ABR} \otimes 1_{C_{A} \rightarrow C'_A} \otimes 1_{C_{B} \rightarrow C'_B}) (\psi)$, and $C'_B = (C_A)$ is Alice’s system (Bob’s system) with $\dim C'_B = \dim C_B$ ($\dim C'_A = \dim C_A$). Then a joint operation

$$E : AC_{A}E_{\text{in}} \otimes BC_{B}E_{\text{in}} \rightarrow AC_{A}E_{\text{out}} \otimes BC_{B}E_{\text{out}}$$

is called state exchange with quantum side information of $\psi'$ with error $\varepsilon$, if it consists of LOCC, and satisfies

$$\| (E \otimes 1_R)(\psi_i) - \psi_f \|_1 \leq \varepsilon,$$

where $\| \cdot \|_1$ is the trace norm.

Let us now consider $n$ independent and identically distributed copies of $|\psi\rangle$, say $|\psi\rangle^\otimes n$. If $E_n$ indicates a state exchange with QSI of $|\psi\rangle^\otimes n$ with error $\varepsilon_n$, then the resource rate $(\log e^\text{in}(E_n) - \log e^\text{out}(E_n))/n$ is called the entanglement rate of the protocol. If there is a sequence $\{E_n\}_{n \in \mathbb{N}}$ of state exchanges $E_n$ with QSI of $|\psi\rangle^\otimes n$ with error $\varepsilon_n$ such that

$$\lim_{n \rightarrow \infty} \frac{\log e^\text{in}(E_n) - \log e^\text{out}(E_n)}{n} = e_r,$$

then the real number $e_r$ is called an achievable entanglement rate for the state exchange with QSI of $|\psi\rangle$. The smallest achievable entanglement rate defines the optimal entanglement cost $e_{\text{opt}}$ for the considered task.

**Merge-and-merge strategy.**— We first present a merge-and-merge strategy which is motivated by the merge-and-send protocol introduced in Ref. [3]. The idea of this strategy is as follows. Firstly, Alice’s part $C_A$ is merged from Alice to Bob by using $BC_B$ as QSI. After finishing merging $C_A$, Bob’s part $C_B$ is merged from Bob to Alice by using Alice’s QSI $A$ so that Alice’s $C_A$ and Bob’s $C_B$ are exchanged. By using the exact formula of the entanglement cost for merging [3], [10], we have that the optimal entanglement costs of merging $C_A$ and merging $C_B$ are the quantum conditional entropies $H(C_A|BC_B)$ and $H(C_B|A)$, respectively, so that the total entanglement cost is $H(C_A|B)+H(C_A|BC_B)$, where the quantum conditional entropy $H(X|Y)$ of a state $\rho_{XY}$ is defined by $H(X|Y) = H(Y) - H(Y|X)$, with $H(X)$ the von Neumann entropy of a state $\rho_X$.

From the merge-and-merge strategy, we obtain the following upper bound for the optimal entanglement cost of the state exchange with QSI.

**Theorem 1.** The optimal entanglement cost $e_{\text{opt}}$ for the state exchange with QSI of $|\psi\rangle$ is upper bounded by

$$e_{\text{opt}} \leq u(\psi) = \min\{u_1(\psi), u_2(\psi)\},$$

where $u_1(\psi) = H(C_B|A) + H(C_A|BC_B)$ and $u_2(\psi) = H(C_A|B) + H(C_B|AC_A)\psi$.

Note that $u_2(\psi)$ in Theorem 1 can be obtained by firstly merging Bob’s part $C_B$ to Alice. We further refer the reader to Appendix A for the rigorous proof of Theorem 1 which fulfills the definition of achievability.

**Optimal strategy.**— Since the merge-and-merge strategy is simple and intuitive, one may guess that the strategy is optimal for any initial state of the exchanging task. However, the following example shows that there can be a more effective strategy than the merge-and-merge one. Let us consider a specific form of the initial state

$$|\tilde{\psi}\rangle_{AC_{A}BC_{B}R} = |\tilde{\phi}\rangle_{AC_{A}BC_{B}R_1} \otimes |\text{GHZ}\rangle_{C_{A}BC_{B}R_2},$$

where systems $C_A = C_A^1C_A^2$, $C_B = C_B^1C_B^2$, $R = R_1R_2$, $|\tilde{\phi}\rangle$ is an arbitrary state on the system $AC_{A}BC_{B}R_1$, and

$$|\text{GHZ}\rangle_{C_{A}BC_{B}R_2} = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |kkk\rangle,$$

is the Greenberger-Horne-Zeilinger state [11] with $d \geq 2$.

In order to exchange $C_A$ and $C_B$ in Eq. (1), it suffices for Alice and Bob to only consider the state exchange with QSI of $|\tilde{\phi}\rangle$, since the state $|\text{GHZ}\rangle$ on the parts $C_B^1$ and $C_B^2$ is symmetric. Then by applying the merge-and-merge strategy on $|\tilde{\phi}\rangle$, we obtain a tighter upper bound $\min\{u_1(\tilde{\phi}), u_2(\tilde{\phi})\}$ for the optimal entanglement cost for the state $|\tilde{\phi}\rangle$ in Eq. (1) as follows:

$$\min\{u_1(\tilde{\phi}), u_2(\tilde{\phi})\} = \min\{u_1(\tilde{\psi}), u_2(\tilde{\psi})\} - \log d.$$  

From the relation between upper bounds in Eq. (2), we remark that there can be an arbitrarily large gap between
the optimal entanglement cost and the upper bound in Theorem 1 implying that the upper bound is not optimal in the general case. This example also shows that there exist tighter upper bounds for the optimal entanglement cost. On this account, we argue that the optimal strategy is of the form $\psi^e_n = |\psi^e_n\rangle\langle\psi^e_n|$, and $e_{\text{opt}}^n$ be total amounts of entanglement between Alice and Bob before and after the state exchange with QSI, respectively. Then they can be expressed as $E_{\text{ent}}^n = nH(AC_BV) - nH(AC_BV)$ and $E_{\text{opt}}^n = nH(AC_BV) + nH(AC_BV)$. Since the total entanglement between Alice and Bob cannot increase under LOCC, we have $E_{\text{ent}}^n \geq E_{\text{opt}}^n$. LOCC, we have $E_{\text{ent}}^n \geq E_{\text{opt}}^n$, that is, 

$$\log e_{\text{ent}}^n(\epsilon_n^R) - \log e_{\text{opt}}^n(\epsilon_n^R) \geq nH(AC_BV) - nH(AC_BV).$$

Let $e_{\text{opt}}^R$ be the optimal entanglement cost for the $R$-assisted state exchange with QSI, then

$$\max_N[H(AC_BV) - H(AC_BV)] \leq e_{\text{opt}}^R.$$

Since any state exchange with QSI can be considered as an $R$-assisted state exchange with QSI (in which the referee trivially does nothing), it holds that $e_{\text{opt}}^R \leq e_{\text{opt}}^R$. This leads us to the following theorem.

**Theorem 2.** The optimal entanglement cost $e_{\text{opt}}^R$ for the state exchange with QSI of $|\psi\rangle$ is lower bounded by

$$l(\psi) = \max_N[H(AC_BV)N(\psi) - H(AC_BV)N(\psi)] \leq e_{\text{opt}}^R,$$

where the maximum is taken over all quantum channels $N: R \rightarrow V$.

In general, it is not easy to calculate the converse bound in Theorem 2 since it involves an optimization over all quantum channels. However, if the referee sends the whole part $R$ to either Alice or Bob without dividing $R$ in Theorem 2, then we obtain the following computable converse bound:

**Corollary 3.** For the state exchange with QSI of $|\psi\rangle$, the optimal entanglement cost $e_{\text{opt}}^R$ satisfies

$$\max\{l_1(\psi), l_2(\psi)\} \leq e_{\text{opt}},$$

where $l_1(\psi) = H(AC_BV) - H(AC_BV)$ and $l_2(\psi) = H(AC_BV) - H(AC_BV)$.

By using the continuity of the von Neumann entropy $E\left[\frac{1}{n}\right]$, we can directly show that $l_1(\psi)$ and $l_2(\psi)$ in Corollary 3 are lower bounds to the optimal entanglement cost for the state exchange with QSI of $|\psi\rangle$. The proof of Corollary 3 can be found in Appendix B.

**Large gap between converse bounds.** — It is obvious that the lower bound presented in Corollary 3 is less tight than the one in Theorem 2. Interestingly, the gap between these two converse bounds can be arbitrarily large. To this end, let us consider the initial state

$$|\psi\rangle_{ACBA} = \frac{1}{\sqrt{2}} (00000) + \frac{1}{\sqrt{2}} (01110),$$

where the reference system $R$ consists of the four subsystems $R_A, R_{CA}, R_B$ and $R_{CB}$, and $|\Phi\rangle$ is a maximally entangled state on the corresponding bipartite system $SR_S$ with $\dim S = \dim R_S$ for $S = A, B, CA$ and $CB$. Then we can readily see that

$$l_1(\psi) = H(BC_B) - H(AC_B) = -l_2(\psi).$$

On the other hand, if a channel $N$ is given by $\rho_R \mapsto \rho_R\rho_{R_CA}$, that is, $\rho = \rho_{R_CA}$, then we obtain

$$l(\psi) \geq H(AC_BV)N(\psi) - H(AC_BV)N(\psi) = H(AC_BVR_{R_CA}) - H(AC_BVR_{R_CA}) - H(AC_BVR_{R_CA}),$$

which means that the converse bound $l(\psi)$ in Theorem 2 can be arbitrarily larger than $\max\{l_1(\psi), l_2(\psi)\}$ in Corollary 3 for the class of initial states in Eq. (3).

**Optimal entanglement cost can be negative.** — We finally address the crucial question: Can the optimal entanglement cost for state exchange with QSI be negative? First of all, let us remark that the optimal entanglement cost for state exchange without QSI of $|\psi\rangle_{ACBC_R}$ cannot be negative. If the optimal cost was negative, then Alice and Bob could generate as much entanglement as they need by repeatedly exchanging their state. This contradicts the basic requirement that the amount of entanglement cannot increase by LOCC.

However, quite remarkably, the optimal entanglement cost $e_{\text{opt}}^R$ for the state exchange with QSI of $|\psi\rangle$ can be negative. This is readily seen since the upper bounds $u_1$ or $u_2$ in Theorem 1 can be negative. For example, $e_{\text{opt}}^R$ is negative for the initial state

$$|\psi\rangle_{ACBA} = \frac{\sqrt{\lambda}}{2} (00000) + \frac{\sqrt{1-\lambda}}{2} (01110)$$

with $\lambda \geq 0.65$, as seen in Fig. 2. Furthermore, this example shows that, in the state exchange with QSI, the optimal entanglement cost can be generally reduced by exploiting the QSI $AB$ for the exchanging task. This reveals the prominent role of the QSI for such a quantum communication primitive.
At this point we remark that the negativity of the optimal entanglement cost for the state exchange with QSI does not lead to a contradiction as follows. Let $e_{\text{opt}}$ be the optimal entanglement cost for a state exchange with QSI of the initial state $|\psi\rangle$, and let $e^{2\text{nd}}_{\text{opt}}$ be the optimal entanglement cost for a state exchange with QSI of the exchanged state $|\psi'\rangle$. Then from Corollary 3,

$$e^{1\text{st}}_{\text{opt}} \geq l_1(\psi) \quad \text{and} \quad e^{2\text{nd}}_{\text{opt}} \geq l_1(\psi') = -l_1(\psi).$$

So in this case we have the inequality $e^{1\text{st}}_{\text{opt}} + e^{2\text{nd}}_{\text{opt}} > 0$. This shows that the total amount of entanglement generated from repeated state exchange protocols with QSI does not repeatedly increase although the entanglement cost can be negative in an individual instance of the protocol.

**Optimal entanglement costs for some special cases.**— We now provide several conditions which allow us to compute the exact optimal entanglement cost $e_{\text{opt}}$ for the state exchange with QSI of $|\psi\rangle$. In fact, the merge-and-merge strategy is optimal under these conditions.

**Corollary 4.** Let $e_{\text{opt}}$ be the optimal entanglement cost of the state exchange with QSI of $|\psi\rangle = \psi_{AC_\Lambda BC_\Phi}$. 

(i) The following conditions on $|\psi\rangle$ give the exact optimal entanglement costs:

$$I(\hat{R}; C_A|A)_\psi = 0 \iff e_{\text{opt}} = u_1(\psi) = l_1(\psi),$$
$$I(\hat{R}; C_B|B)_\psi = 0 \iff e_{\text{opt}} = u_2(\psi) = l_1(\psi),$$
$$I(\hat{R}; C_B|B)_\psi = 0 \iff e_{\text{opt}} = u_1(\psi) = l_2(\psi),$$
$$I(\hat{R}; C_B|B)_\psi = 0 \iff e_{\text{opt}} = u_2(\psi) = l_2(\psi),$$

where $I(X;Y|Z)_\rho$ indicates the quantum conditional mutual information (QDMI) of a quantum state $\rho_{XYZ}$, and $u_1(\psi)$, $u_2(\psi)$, $l_1(\psi)$, and $l_2(\psi)$ are given in Theorem 4 and Corollary 3.

(ii) There exists a quantum channel $\hat{N} : R \rightarrow V$ such that $I(C_B : V|A)_{N(\psi)} = I(C_A : E|AV)_{N(\psi)} = 0$ if and only if $e_{\text{opt}} = u_1(\psi) = l(\psi)$, where $l(\psi)$ is in Theorem 4. Similarly, there exists $\hat{N} : R \rightarrow V$ such that $I(C_A : E|B)_{N(\psi)} = I(C_B : V|BE)_{N(\psi)} = 0$ if and only if $e_{\text{opt}} = u_2(\psi) = l(\psi)$.

(iii) Let $|\psi\rangle_{AC_\Lambda BC_\Phi}$ be a pure initial state shared by Alice and Bob (with no referee), then for the state exchange with QSI of $|\psi\rangle_{AC_\Lambda BC_\Phi}$ one has $e_{\text{opt}} = H(AC_B)_{\psi} - H(AC_A)_{\psi}$.

By combining the aforementioned upper and lower bounds, the conditions for the exact optimal cost in Corollary 4 are directly obtained. We remark that there are no general implications among the four QDMI conditions in Corollary 4 (i), that is, there exists an initial state which only satisfies one of these QDMI conditions. We presents related examples in Appendix C.

**Conclusion.**— In this work, we have considered the state exchange with QSI as a fundamental quantum communication task, and have provided the formal descriptions for the protocol and its optimal entanglement cost. We have derived upper and lower bounds to the optimal entanglement cost. From these bounds, we have exactly evaluated the optimal entanglement cost for several special classes of states, including all pure bipartite states. Furthermore, we have shown that the optimal entanglement cost for the state exchange with QSI can be negative. This is at striking variance with the state exchange without QSI, whose entanglement cost is always non-negative.

By replacing classical communication with quantum communication, we can consider a fully quantum version of the state exchange with QSI of $|\psi\rangle_{AC_\Lambda BC_\Phi}$. Similar to the idea of Theorem 1 this task can be performed by applying the state redistribution protocol [6] twice. For example, if the part $C_\Lambda$ is firstly redistributed from Alice to Bob in this strategy, then its achievable rates $E_r$ and $Q_r$ for ebits and qubit channels are given by

$$E_r = \frac{1}{2}[l_1(\psi) + l_2(\psi)],$$
$$Q_r = \frac{1}{2}u_1(\psi) + \frac{1}{2}[H(C_A|A)_\psi + H(C_B|BC_\Phi)],$$

where $u_1(\psi)$, $l_1(\psi)$, and $l_2(\psi)$ are in Theorem 1 and Corollary 3. However, in this case the achievable region of a resource pair $(E_r, Q_r)$ is completely unknown.

To the best of our knowledge, a protocol exchanging Alice’s and Bob’s information in a single step has not been known, and so in this work we have considered the merge-and-merge strategy, in order to obtain achievable entanglement rates. Hence it would be very meaningful to devise one such a direct exchanging protocol. Moreover, recent results for one-shot quantum state merging [16] and implementing bipartite unitaries [17] may be useful to figure out novel strategies which can provide tighter achievable bounds than those in Theorem 1.

Finally, we expect that studying variations on the state exchange with QSI makes quantum information theory richer. For example, one can assume that Alice and Bob can consume noisy resources [18] instead of noiseless resources, or that Alice or Bob is additionally allowed to make use of a local resource, such as maximally coherent states [20][22], as in the incoherent state merging [22] and [23].

**FIG. 2:** Upper bounds $u_1(\psi_\Lambda)$, $u_2(\psi_\Lambda)$ and lower bounds $l_1(\psi_\Lambda)$, $l_2(\psi_\Lambda)$ to the optimal entanglement cost $e_{\text{opt}}$ for the specific initial state $|\psi_\Lambda\rangle$ of Eq. 4 with $0 \leq \lambda \leq 1$. 

the incoherent state redistribution \cite{23}. Exploring these avenues deserves further investigation.

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[1] B. Schumacher, Phys. Rev. A 51, 2738 (1995).
[2] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[3] J. Oppenheim and A. Winter, arXiv:quant-ph/0511082.
[4] M. Horodecki, J. Oppenheim, and A. Winter, Nature 436, 673 (2005).
[5] M. Horodecki, J. Oppenheim, and A. Winter, Commun. Math. Phys. 269, 107 (2007).
[6] I. Devetak and J. Yard, Phys. Rev. Lett. 100, 230501 (2008).
[7] J. T. Yard and I. Devetak, IEEE Trans. Inf. Theory 55, 5339 (2009).
[8] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, 2013).
[9] J. Oppenheim (2008), arXiv:0805.1065.
[10] Y. Lee and S. Lee, Quantum Inf. Process. 17, 268 (2018).
[11] D. M. Greenberger, M. A. Horne, and A. Zeilinger, *Bells Theorem, Quantum Theory, and Conceptions of the Universe* (Kluwer Academics, Dordrecht, The Netherlands, 1989).
[12] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).
[13] M. Fannes, Commun. Math. Phys. 31, 291 (1973).
[14] K. M. R. Audenaert, J. Phys. A: Math. Theor. 40, 81278136 (2007).
[15] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. 78, 2275 (1997).
[16] H. Yamasaki and M. Murao, arXiv:1806.07875.
[17] E. Wakakawa, A. Soeda, and M. Murao, IEEE Trans. Inf. Theory 63, 5372 (2017).
[18] I. Devetak, A. W. Harrow, and A. Winter, Phys. Rev. Lett. 93, 230504 (2004).
[19] I. Devetak, A. W. Harrow, and A. J. Winter, IEEE Trans. Inf. Theory 54, 4587 (2008).
[20] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. 113, 140401 (2014).
[21] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. 115, 020403 (2015).
[22] A. Streltsov, E. Chitambar, S. Rana, M. N. Bera, A. Winter, and M. Lewenstein, Phys. Rev. Lett. 116, 240405 (2016).
[23] A. Anshu, R. Jain, and A. Streltsov, arXiv:1804.04915.
[24] I. Devetak and A. Winter, Proc. R. Soc. A 461, 207235 (2005).
[25] A. Winter, Commun. Math. Phys. 347, 291313 (2016).
Appendix A: Proof of Theorem

We first show that $u_1$ is achievable. From the definition of the optimal costs for the state merging with QSI [10], there are two sequences $\{F_n\}_{n\in N}$ and $\{S_n\}_{n\in N}$. To be specific, an element $F_n$ of the first sequence $\{F_n\}_{n\in N}$,

$$F_n : (A \otimes^n C_A^{\otimes n} F_A^n) \otimes (B \otimes^n C_B^{\otimes n} F_B^n) \longrightarrow (A \otimes^n F_A^n) \otimes (C_A \otimes^n B \otimes^n C_B \otimes^n F_B^n),$$

is the state merging with QSI of $|\psi\rangle \otimes^n$ with error $\epsilon_n$ which is a LOCC operation satisfying

$$\left\| (F_n \otimes 1_R) (|\psi\rangle \otimes^n \otimes |\Phi\rangle_{FA^n FB^n}) - (|\psi\rangle \otimes^n \otimes |\Phi\rangle_{FA^n FB^n}) \right\|_1 \leq \epsilon_n$$

where $C_A$ is Bob’s system with $\dim C_A = \dim C_A$, $|\psi\rangle$ is a target state defined as $(1_{ABCnR} \otimes 1_{CA→C_A'}) |\psi\rangle$, and $|\Phi\rangle_{FA^n FB^n}$ and $|\Phi\rangle_{FA^n FB^n}$ are maximally entangled states with Schmidt rank $e^{\infty}(F_n)$ and $e^{\out}(F_n)$, respectively. An element $S_n$ of the second sequence $\{S_n\}_{n\in N}$,

$$S_n : (A \otimes^n S_{An}^{in}) \otimes (C_A \otimes^n B \otimes^n S_{Bn}^{in}) \longrightarrow (A \otimes^n C_A^{\otimes n} S_{An}^{out}) \otimes (C_A \otimes^n B \otimes^n S_{Bn}^{out}),$$

is the state merging with QSI of $|\psi\rangle \otimes^n$ with error $\epsilon_n$ which satisfies

$$\left\| (S_n \otimes 1_R) (|\psi\rangle \otimes^n \otimes |\Phi\rangle_{SA^n SB^n}) - (|\psi\rangle \otimes^n \otimes |\Phi\rangle_{SA^n SB^n}) \right\|_1 \leq \epsilon_n$$

where $C_B$ is Alice’s system with $\dim C_B = \dim C_A$, $|\psi\rangle$ is a target state defined as $(1_{ACnR} \otimes 1_{CB→C_B'}) |\psi\rangle$, and $|\Phi\rangle_{SA^n SB^n}$ and $|\Phi\rangle_{SA^n SB^n}$ are maximally entangled states with Schmidt rank $e^{\infty}(S_n)$ and $e^{\out}(S_n)$, respectively. The two sequences also satisfy

$$\lim_{n\to\infty} \frac{\log e^{\infty}(F_n) - \log e^{\out}(F_n)}{n} = H(C_A | B)$$

$$\lim_{n\to\infty} \frac{\log e^{\infty}(S_n) - \log e^{\out}(S_n)}{n} = H(C_B | A)$$

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \epsilon_n = 0.$$

Let us now consider a sequence $\{\epsilon_n\}_{n\in N}$ defined as

$$\epsilon_n = \begin{cases} \hat{S}_n \circ F_n & \text{if } e^{\out}(F_n) \geq e^{\infty}(S_n) \\ S_n \circ \hat{F}_n & \text{otherwise,} \end{cases}$$

where $\hat{S}_n = S_n \otimes 1_{EAn} \hat{E}_{An}$ and $\hat{F}_n = F_n \otimes 1_{EAn} \hat{E}_{An}$ with $\dim \hat{E}_{An} = \dim \hat{E}_{An} = e^{\out}(F_n)/e^{\infty}(S_n)$ and $\dim \hat{E}_{An} = e^{\infty}(S_n)/e^{\out}(F_n)$. If $e^{\out}(F_n) \geq e^{\infty}(S_n)$ then

$$\left\| (\epsilon_n \otimes 1_R) (|\psi\rangle \otimes^n \otimes |\Phi\rangle)_{SA^n SB^n} - (|\psi\rangle \otimes^n \otimes |\Phi\rangle)_{SA^n SB^n} \right\|_1 \leq \epsilon_n + \epsilon_n$$

where $|\Phi\rangle_{EAn} \hat{E}_{An}$ is an maximally entangled states with Schmidt rank $e^{\out}(F_n)/e^{\infty}(S_n)$ and $\epsilon_n = \epsilon_n + \epsilon_n$. The first and second inequalities come from the triangle property and the monotonicity of the trace distance [8]. Similarly,
if \( e^{\text{out}}(F_n^-) < e^{\text{in}}(S_n^-) \) then

\[
\begin{align*}
    \| (E_n^- \otimes 1_R^n)(|\psi\rangle \otimes |\Phi\rangle_{E_{A_n} E_{B_n}}) - (|\psi^\otimes n\rangle \otimes |\Phi\rangle_{S_{A_n}^{\text{out}} S_{B_n}^{\text{out}}}) \|_1 \\
    \leq \| (S_n^- \otimes 1_R^n)(|\psi\rangle \otimes |\Phi\rangle_{E_{A_n} E_{B_n}}) - (S_n^- \otimes 1_R^n)(|\psi^\otimes n\rangle \otimes |\Phi\rangle_{E_{A_n} E_{B_n}}) \|_1 \\
    + \| (S_n^- \otimes 1_R^n)(|\psi\rangle \otimes |\Phi\rangle_{E_{A_n} E_{B_n}}) - (|\psi\rangle \otimes |\Phi\rangle_{S_{A_n}^{\text{out}} S_{B_n}^{\text{out}}}) \|_1 \\
    \leq \| (F_n^- \otimes 1_R^n)(|\psi\rangle \otimes |\Phi\rangle_{E_{A_n} E_{B_n}}) - (|\psi\rangle \otimes |\Phi\rangle_{S_{A_n}^{\text{out}} S_{B_n}^{\text{out}}}) \|_1
\end{align*}
\]

where \( |\Phi\rangle_{E_{A_n} E_{B_n}} \) is an maximally entangled states with Schmidt rank \( e^{\text{in}}(S_n^-)/e^{\text{out}}(F_n^-) \). It follows that a LOCC protocol \( \mathcal{E}_n^- \) is a state exchange with QSI of \(|\psi\rangle \otimes n\) with error \( \varepsilon_n \) together with

\[
\lim_{n \to \infty} \frac{\log e^{\text{in}}(\mathcal{E}_n^-) - \log e^{\text{out}}(\mathcal{E}_n^-)}{n} = \lim_{n \to \infty} \frac{\log e^{\text{in}}(F_n^-) + \log e^{\text{in}}(S_n^-) - \log e^{\text{out}}(F_n^-) - \log e^{\text{out}}(S_n^-)}{n} = \lim_{n \to \infty} \frac{\log e^{\text{in}}(F_n^-) - \log e^{\text{out}}(F_n^-)}{n} + \lim_{n \to \infty} \frac{\log e^{\text{in}}(S_n^-) - \log e^{\text{out}}(S_n^-)}{n}
\]

where the first equality comes from the fact that if \( e^{\text{out}}(F_n^-) \geq e^{\text{in}}(S_n^-) \) then

\[
\log e^{\text{in}}(\mathcal{E}_n^-) - \log e^{\text{out}}(\mathcal{E}_n^-) = \log e^{\text{in}}(F_n^-) - \log e^{\text{out}}(S_n^-) - \log \dim \hat{E}_{A_n},
\]

and if \( e^{\text{out}}(F_n^-) < e^{\text{in}}(S_n^-) \) then

\[
\log e^{\text{in}}(\mathcal{E}_n^-) - \log e^{\text{out}}(\mathcal{E}_n^-) = \log e^{\text{in}}(F_n^-) + \log \dim \hat{E}_{A_n} - \log e^{\text{out}}(S_n^-).
\]

Since

\[
\lim_{n \to \infty} \varepsilon_n = \lim_{n \to \infty} \varepsilon_n^- + \lim_{n \to \infty} \varepsilon_n^+ = 0,
\]

\( u_1 \) is an achievable rate of the state exchange with QSI of \(|\psi\rangle\).

Moreover, by relabeling Alice (Bob) with Bob’ (Alice’), we obtain that \( H(C_A|B) + H(C_B|AC_A) \) is also achievable. Therefore, \( e^{\text{opt}} \leq \min\{u_1, u_2\} \).

**Appendix B: Proof of Corollary 3**

We note that if \( \log e^{\text{in}}(\mathcal{E}_n) = n(H(C_A) + H(C_B)) \) then Alice and Bob can clearly perform the state exchange with QSI by using the Schumacher compression [1] and the standard teleportation [2] on the maximally entangled states with Schmidt rank \( e^{\text{in}}(\mathcal{E}_n) \). Hence we may assume that \( \log e^{\text{in}}(\mathcal{E}_n) \) is not more than \( n(H(C_A) + H(C_B)) \).

We now give a proof of Corollary 3 which employs the continuity of the von Neumann entropy.

**Proof.** Let \( e \) be any achievable rate of the state exchange with QSI of \(|\psi\rangle \equiv |\psi\rangle_{AC_A B C_B R} \). Then from the definition of the achievable entanglement rate, there is a sequence \( \{\mathcal{E}_n\}_{n \in \mathbb{N}} \) of state exchanges \( \mathcal{E}_n \) with QSI of \(|\psi\rangle \otimes n\) with error \( 2\varepsilon_n \) such that

\[
\| \rho_{X'Y'R \otimes n} - \phi_{X'Y'R \otimes n} \|_1 \leq 2\varepsilon_n,
\]

where \( X' = A^\otimes n C_B^\otimes n E_{A_n}^{\text{out}} \) and \( Y' = B^\otimes n C_A^\otimes n F_{B_n}^{\text{out}} \),

\[
\rho_{X'Y'R \otimes n} = (\mathcal{E}_n \otimes 1_R^\otimes n)(|\psi\rangle \otimes \Phi_{E_{A_n} E_{B_n}}^{\text{out}}),
\]

\[
\phi_{X'Y'R \otimes n} = |\psi\rangle \otimes \Phi_{E_{A_n} E_{B_n}}^{\text{out}}.
\]
\( \Phi_{E_{in}} E_{in} \) and \( \Phi_{E_{out}} E_{out} \) are pure maximally entangled states with Schmidt rank \( e^{in}(E_n) \) and \( e^{out}(E_n) \), respectively, 0 ≤ log \( e^{in}(E_n) \) ≤ \( n(H(C_A) + H(C_B)) \), \( \lim_{n \to \infty} \frac{\log e^{in}(E_n) - \log e^{out}(E_n)}{n} = \epsilon_r \), and \( \lim_{n \to \infty} \epsilon_n = 0 \). Then the monotonicity of the trace distance [8] implies

\[
\| \rho_{X'} - \phi_{X'} \|_1 \leq 2\epsilon_n, \tag{B2}
\]

where \( \rho_{X'} = Tr_{Y'R} \cdot [\rho_{X'Y'R}] \) and \( \phi_{X'} = Tr_{Y'R} [\phi_{X'Y'R}] \).

From the continuity of the von Neumann entropy [13, 14] together with Eq. (B2), we obtain the following inequality:

\[
|H(X')_{\rho_{X'}} - H(X')_{\phi_{X'}}| \leq \epsilon_n \log \dim(X') + h(\epsilon_n), \tag{B3}
\]

where \( h(\cdot) \) is the binary entropy. It follows that

\[
H(X')_{\rho_{X'}} \geq H(X')_{\phi_{X'}} - \epsilon_n \log \dim(X') - h(\epsilon_n) = nH(AC_B) + \log e^{out}(E_n) - \epsilon_n \left( n \log \dim(AC_B) + \log e^{out}(E_n) \right) - h(\epsilon_n), \tag{B4}
\]

and the von Neumann entropy \( H(X')_{\rho_{X'}} \) is upper bounded as follows:

\[
H(X')_{\rho_{X'}} \leq E_d(X'; Y'R^{\otimes n})_{\rho_{X'Y'R^{\otimes n}}} + H(X'Y'R^{\otimes n})_{\rho_{X'Y'R^{\otimes n}}} \leq E_d(X; YR^{\otimes n}|\psi) \otimes n|\Phi)_{E^{in}_{A_{n}} E^{in}_{B_{n}}} + H(X'Y'R^{\otimes n})_{\rho_{X'Y'R^{\otimes n}}} \leq E_d(X; YR^{\otimes n}|\psi) \otimes n|\Phi)_{E^{in}_{A_{n}} E^{in}_{B_{n}}} + H(X'Y'R^{\otimes n})_{|\phi_{X'Y'R^{\otimes n}}} + \epsilon_n \log \dim(X'Y'R^{\otimes n}) + h(\epsilon_n), \tag{B5}
\]

where \( E_d(X_1; X_2) \) is the distillable entanglement between \( X_1 \) and \( X_2 \) of a given state, \( X = A_{\otimes n} C_{\otimes n} E_{in}_{A_{n}} \), and \( Y = B_{\otimes n} C_{\otimes n} E_{in}_{B_{n}} \). The first inequality comes from the hashing inequality [24]. Since the distillable entanglement is non-increasing under LOCC, the second inequality holds. The last inequality is obtained from the continuity of the von Neumann entropy [13, 14] together with Eq. (B1). Here, \( H(X'Y'R^{\otimes n})_{|\phi_{X'Y'R^{\otimes n}}} = 0 \), since \( |\phi_{X'Y'R^{\otimes n}} \) is pure. Then the inequality in Eq. (B4) becomes

\[
E_d(X; YR^{\otimes n}|\psi) \otimes n|\Phi)_{E^{in}_{A_{n}} E^{in}_{B_{n}}} \geq nH(AC_B) + \log e^{out}(E_n) - \epsilon_n \left( 2n \log \dim(AC_B) + \log \dim(BC_A R) + \frac{3 \log e^{out}(E_n)}{n} \right) - 2h(\epsilon_n).
\]

Thus it follows that

\[
\frac{\log e^{in}(E_n) - \log e^{out}(E_n)}{n} \geq l_4 - \epsilon_n \left( 2 \log \dim(AC_B) + \log \dim(BC_A R) + \frac{3 \log e^{out}(E_n)}{n} \right) - \frac{2}{n} h(\epsilon_n),
\]

which implies that \( \epsilon_r \geq l_4 \) as \( n \to \infty \), since

\[ 0 \leq \lim_{n \to \infty} \frac{\log e^{out}(E_n)}{n} \leq H(C_A) + H(C_B) + \epsilon_r. \]

Moreover, the second lower bound \( l_2 \) can also be obtained in the same way by replacing \( H(X') \) in Eq. (B3) and \( E_d(X'; Y'R^{\otimes n}) \) in Eq. (B5) with \( H(Y') \) and \( E_d(Y'; X'R^{\otimes n}) \), respectively.

**Remark 5.** By employing the continuities of the quantum conditional entropy [25] and the quantum mutual information [8] instead of the von Neumann entropy then we can get another two lower bounds, \( l_3 = -H(AC_B|BC_A) - H(AC_A) \) and \( l_4 = -H(BC_A|AC_B) - H(BC_B) \), on the optimal entanglement cost for the state exchange with QSI. The lower bounds \( l_3 \) and \( l_4 \) are not tighter than \( l_1 \) and \( l_2 \), respectively.

**Appendix C: Examples**

As mentioned earlier, there are four QCMI conditions on the initial state of the state exchange with QSI which give the exact optimal entanglement cost:

\[ I(R; C_A | A) = 0, \quad I(R; C_A | B) = 0, \quad I(R; C_B | A) = 0, \quad I(R; C_B | B) = 0. \]
Let \( S \) be the set of all pure states on the multipartite system \( AC_ABC_RB \), and define \( S(X;Y|Z) \) as the intersection of \( S \) and the set of all pure states which satisfy a condition \( I(X;Y|Z) = 0 \).

We show that there are no inclusion relations among four sets \( S(R;C_A|A) \), \( S(R;C_A|B) \), \( S(R;C_B|A) \), and \( S(R;C_B|B) \). Consider the following state

\[
|\psi\rangle_{AC_ABC_RB} = \frac{1}{\sqrt{3}}(|00000\rangle + |01100\rangle + |10011\rangle)_{AC_ABC_RB},
\]

then we obtain

\[
I(R;C_A|A) = (H(RA) - H(A)) + (H(C_A A) - H(RCA A)) = 0 + 0 = 0,
\]

\[
I(R;C_A|B) = (H(RB) - H(B)) + H(C_AB) - H(RCA_B) = \frac{2}{3} + H(C_A B) - H(RCA_B)
\approx 0.66666 + 0.550048 - 0.918296 > 0,
\]

\[
I(R;C_B|A) = (H(RA) - H(A)) + H(C_B A) - H(RCB A) = 0 + H(C_B A) - H(RCB A)
\approx 0.918296 - 0.550048 > 0,
\]

\[
I(R;C_B|B) = (H(RB) - H(B)) + (H(C_B B) - H(RCB B)) = \frac{2}{3} + 0 > 0,
\]

since

\[
H(RA) = H(A) = H(B) = H(RCA_B) = \frac{2}{3} \log_2 3 + \frac{1}{3} \log_2 3 \approx 0.918296,
\]

\[
H(C_B B) = H(RCB_B) = H(RB) = \log_2 3 \approx 1.58496,
\]

\[
H(C_AB) = \hat{h} \left( \frac{3 - \sqrt{5}}{6} \right) \approx 0.550048,
\]

where \( \hat{h}(\cdot) \) is the binary entropy. Thus, \( |\psi\rangle \in S(R;C_A|A) \), \( |\psi\rangle \notin S(R;C_A|B) \), \( |\psi\rangle \notin S(R;C_B|A) \), and \( |\psi\rangle \notin S(R;C_B|B) \). Moreover, the other relations are easily shown by relabeling the subsystems of \( |\psi\rangle \).