Minimal digraph obstructions for small matrices*  

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Abstract  

Given a \{0, 1, *\}-matrix \( M \), a minimal \( M \)-obstruction is a digraph \( D \) such that \( D \) is not \( M \)-partitionable, but every proper induced sub-digraph of \( D \) is. In this note we present a list of all the \( M \)-obstructions for every \( 2 \times 2 \) matrix \( M \).

Notice that this note will be part of a larger paper, but we are archiving it now so we can cite the results.

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1 Introduction

Given an $m \times m$ matrix $M$ over $0, 1, *$ (a pattern), an $M$-partition of a digraph $G$ is a partition of the vertices into parts $V_1, V_2, \ldots, V_m$ such that two distinct vertices in $V_i$ are non-adjacent if $M_{i,i} = 0$, and adjacent in both directions if $M_{i,i} = 1$ ($M_{i,i} = *$ represents no restriction). Similarly, each vertex in $V_i$ must (respectively must not) dominate each vertex in $V_j$ if $M_{i,j} = 1$ (respectively $M_{i,j} = 0$).

Given a pattern $M$, the $M$-partition problem is the decision problem of determining whether a digraph admits an $M$-partition. Notice that if we regard (undirected) graphs as digraphs in which every arc is a digon, the $M$-partition problem is also defined for graphs when $M$ is a symmetric matrix. Many well known problems in graph theory can be posed as $M$-partition problems, e.g., a \( \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \)-partition is just a bipartition, and a \( \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix} \) is a split partition.

Observe that having an $M$-partition is an hereditary property, and thus, $M$-partitionable digraphs can be characterized through a set of forbidden induced subdigraphs. A minimal obstruction to $M$-partition, or minimal $M$-obstruction for short, is a digraph that does not admit an $M$-partition but such that every proper induced subdigraph does. Clearly, if a pattern $M$ has only a finite number of minimal obstructions, then the $M$-partition problem is polynomial time solvable. Nonetheless, there are patterns $M$ with infinitely many minimal $M$-obstructions and with a polynomial time solvable $M$-partition problem, e.g. the bipartition problem.

There are two main problems associated with the concept of an $M$-partition.

**Problem 1** (The Characterization Problem). Which patterns $M$ have the property that the number of minimal $M$-obstructions is finite.

**Problem 2** (The Complexity Problem). Which patterns $M$ have the property that the $M$-partition problem can be solved by a polynomial time algorithm?

We refer the reader to [5] for a survey on the subject.

The main goal of this note is to give a full list of minimal $M$-obstructions for every $2 \times 2$ pattern $M$. Although it is already known that the $M$-partition problem for such patterns is polynomial time solvable, it is useful to have the exact list of minimal $M$-obstructions. Such list has been already used in [6] (where it was meant to be originally included), and recently in [4].
We refer the reader to [1] and [2] for general concepts. In this work, \( D = (V_D, A_D) \) will be a digraph with the vertex set \( V_D \) and the arc set \( A_D \), without loops of multiple arcs in the same direction. If \( A_D = \emptyset \) we say that \( D \) is an empty digraph. The dual of \( D \) is the digraph \( \overrightarrow{D} \) obtained from \( D \) by reversing each of its arcs. We will denote the underlying graph of \( D \) by \( G_D \). The complement of \( D \) is the digraph \( \overrightarrow{\neg D} \) obtained from \( D \) by reversing each of its arcs. We will denote the underlying graph of \( D \) by \( G_D \). The complement of \( D \) is the digraph \( \overrightarrow{\neg D} \) obtained from \( D \) by reversing each of its arcs. We will denote the underlying graph of \( D \) by \( G_D \). The complement of \( D \) is the digraph \( \overrightarrow{\neg D} \) obtained from \( D \) by reversing each of its arcs. We will denote the underlying graph of \( D \) by \( G_D \).

When \((x, y) \in A_D ((x, y) \notin A(D))\) we will denote it by \( x \to y \) (\( x \not\to y \)). We will say that an arc \((x, y)\) is a digon if \( y \to x \); otherwise, we will say that \((x, y)\) is an asymmetric arc. If \( x \to y \) we say that \( x \) is an in-neighbour of \( y \) and \( y \) is an out-neighbour of \( x \). The in-neighbourhood (out-neighbourhood) of a vertex \( v \), \( N^-(v) \) (\( N^+(v) \)) is the set of all its in-neighbours (out-neighbours). The neighbourhood of \( v \), \( N(v) \) is defined as \( N(v) = N^-(v) \cup N^+(v) \). For vertices \( x \) and \( y \), we say that \( x \) is adjacent to \( y \) if \( y \in N(x) \).

Given a graph \( G \), a superorientation of \( G \) is obtained by replacing each edge \( xy \) in \( G \) with \((x, y)\), \((x, y)\) or both of them. An orientation of \( G \) is a superorientation of \( G \) without digons. A biorientation of \( G \) is a superorientation of \( G \) where every arc is a digon; the (unique up to isomorphism) biorientation of \( G \) is denoted by \( \overrightarrow{\neg G} \). A subset \( S \) of \( V_D \) is a strong clique if it induces a biorientation of a complete graph in \( D \). We will often abuse language and say that a strong clique on two vertices is a digon. A digraph is strict split if \( V_D \) admits a partition \((V_0, V_1)\) such that \( V_0 \) is an independent set and \( V_1 \) is a strong clique.

The disjoint union of \( D_1 \) and \( D_2 \) is denoted by \( D_1 + D_2 \).

2 Main results

It follows from [3] that for every two by two matrix \( M \) the recognition of \( M \)-partitionable digraphs is possible in polynomial time by reducing the problem to 2-SAT. If \( M \) has an asterisk on the main diagonal, every digraph has an \( M \)-partition. To reduce the possibilities when there are no asterisks on the main diagonal, we present two simple results.

Let \( \overline{M} \) denote the pattern obtained from \( M \) by replacing each entry 0 by 1 and vice versa. The following result is easy to verify.

**Proposition 1.** A partition of \( V_D \) is an \( \overline{M} \)-partition of \( D \) if and only if it is an \( M \)-partition of \( \overrightarrow{D} \).
A similar result can be obtained for $\overline{D}$ and the transpose $M^t$ of $M$.

**Proposition 2.** A partition of $V_D$ is an $M^t$-partition of $D$ if and only if it is an $M$-partition of $\overline{D}$.

It follows from Proposition 1 that a digraph $D$ is a minimal $M$-obstruction if and only if $\overline{D}$ is a minimal $M^t$-obstruction. Analogously, it follows from Proposition 2 that a digraph $D$ is a minimal $M$-obstruction if and only if $\overline{D}$ is a minimal $M^t$-obstruction.

There are 36 different $2 \times 2$ patterns with the required properties. Nonetheless, it follows from Propositions 1 and 2 and a simple additional analysis when $M_{11} = 0$ and $M_{22} = 1$, that there are exactly 10 such patterns with essentially different sets of minimal obstructions,

$$
M_1 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ * & 0 \end{pmatrix}, \\
M_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
M_7 = \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}, \quad M_8 = \begin{pmatrix} 0 & 0 \\ * & 1 \end{pmatrix}, \quad M_9 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \\
M_{10} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Thus, it suffices to analyze the minimal $M_i$-obstructions for $1 \leq i \leq 10$. We begin by analyzing the case when our pattern has only zeros in the main diagonal. Notice first that a digraph admits an $M_1$-partition if and only if it is bipartite, which happens if and only if its underlying graph is bipartite. Thus, the digraph minimal $M_1$-obstructions are all the possible superorientations of every (undirected) odd cycle. We will show in the following theorem that this is the only $2 \times 2$ matrix with zero diagonal and infinitely many minimal obstructions. The sets of minimal $M_i$-obstructions for $2 \leq i \leq 6$ are depicted in Figure 1. In this figure, an edge between a pair of vertices means that an arc must be present between them, and it can be oriented either way or as a digon. For each $i \in \{2, \ldots, 6\}$, let us refer to the digraphs corresponding to $M_i$ in Figure 1 as $F_i$.

**Theorem 3.** Suppose $M$ is a two by two matrix with zero diagonal different from $M_1$. There are a finite number of minimal $M$-obstructions, which are depicted in Figure 1 for every possible $M$. 

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**Proof.** It is easy to verify that members of $\mathcal{F}_i$ are minimal $M_i$-obstructions for $2 \leq i \leq 6$. It is clear as well that every other digraph on two or three vertices is not a minimal $M_i$-obstruction.

Suppose that $D$ is a minimal $M_2$-obstruction on at least 4 vertices. Since a digon is a minimal $M_2$-obstruction, we can assume that $D$ is an oriented graph. Let $v$ be an arbitrary vertex of $D$, and let $(V_1, V_2)$ be an $M_2$-partition of $D - v$. Observe that the neighbourhood of $v$ must be an independent set, otherwise, there would be a tournament on three vertices properly contained in $D$, which is already a minimal $M_2$-obstruction. Since the directed path of length 2 is also a minimal $M_2$-obstruction, then either $N^+(v) = \emptyset$ or $N^-(v) = \emptyset$.

![Figure 1: Minimal obstructions for matrices in Theorem 3.](image)

| Matrix | Diagram |
|--------|---------|
| \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) | ![Diagram](image) |
| \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) | ![Diagram](image) |
| \( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) | ![Diagram](image) |
| \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) | ![Diagram](image) |
| \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) | ![Diagram](image) |
∅. It cannot be the case that the neighbourhood of \( v \) is empty, because \( (V_1 \cup \{v\}, V_2) \) would be an \( M_2 \)-partition of \( D \). Suppose first that \( N^+(v) = \emptyset \). If \( N^-(v) \subseteq V_2 \), then \( (V_1 \cup \{v\}, V_2) \) is an \( M_2 \)-partition, a contradiction. Hence, \( N^-(v) \cap V_1 \neq \emptyset \). If some vertex in \( N^-(v) \cap V_1 \) has positive indegree, then there is an induced directed path of length 2 in \( D \), contradicting the minimality of \( D \). Thus, \( V'_2 = V_2 \cup (N^-(v) \cap V_1) \) is an independent set. If \( V'_1 = (V_1 \cup \{v\}) \setminus N^-(v) \), then \( (V'_1, V'_2) \) is an \( M_2 \)-partition of \( D \), a contradiction. Thus, \( N^+(v) \neq \emptyset \). A very similar argument shows that we also reach a contradiction when \( N^-(v) = \emptyset \). Since the contradiction comes from assuming that there exists a minimal \( M_2 \)-obstruction of order greater than three, we conclude that all the minimal \( M_2 \)-obstructions have order at most three, and hence, they are precisely the digraphs in \( \mathcal{F}_2 \).

Clearly, an \( M_5 \)-partitionable digraph is either an empty digraph or a biorientation of a complete bipartite graph. Thus, it suffices to show that a non-empty graph which contains neither \( K_3 \), nor \( K_1 + K_2 \) as an induced subgraph, is a complete bipartite graph. Let \( G \) be such a graph. Since \( G \) does not contain \( K_1 + K_2 \) as an induced subgraph, it is clear that \( G \) does not contain any cycle of length greater than 4 as an induced subgraph. Thus, \( G = (X, Y) \) is a bipartite graph. If \( |X| = 1 \) or \( |Y| = 1 \), it is direct to verify that \( G \) is complete bipartite. Since \( G \) is non-empty, we can choose an edge \( xy \) of \( G \) with \( x \in X \) and \( y \in Y \). Let \( x' \in X \) and \( y' \in Y \) be arbitrarily chosen. The edges \( x'y \) and \( xy' \) must be present in \( G \), otherwise, a \( K_1 + K_2 \) would be an induced subgraph of \( G \). But then, the edge \( x'y' \) must also be present in \( G \), else, \( \{x, x', y'\} \) would induce a \( K_1 + K_2 \) in \( G \). Since the choice of \( x' \) and \( y' \) is arbitrary, we conclude that \( G \) is a complete bipartite graph.

Let \( D \) be an \( \mathcal{F}_4 \)-free digraph. Since the digon is an element of \( \mathcal{F}_4 \), \( D \) is an oriented graph. Moreover, the underlying graph of \( D \), \( G_D \), contains neither \( K_3 \) nor \( K_1 + K_2 \), because \( \mathcal{F}_4 \) contains the two tournaments on three vertices and the digraph on three vertices consisting of an isolated vertex and an arc. Thus, following the argument used in the previous case, we conclude that \( D \) is either an empty digraph, or \( G_D = (X, Y) \) is a complete bipartite graph. In the latter case, every arc of \( D \) is oriented, without loss of generality, from \( X \) to \( Y \), otherwise, there would be a directed path of length 2 as an induced subdigraph of \( D \), but \( D \) is \( \mathcal{F}_4 \)-free. Thus, \( (X, Y) \) is an \( M_4 \)-partition of \( D \).

If \( D \) is an \( \mathcal{F}_3 \)-free digraph, then, as in the two previous cases, we derive that either \( D \) is an empty digraph, or the underlying graph of \( D \), \( G_D = (X, Y) \), is a complete bipartite graph. If all the arcs of \( D \) are digons, then \( (X, Y) \) is an \( M_3 \)-partition, and we are done. Assume without loss of
generality that \( x \to y \) and \( y \not\to x \) for some \( x \in X \) and \( y \in Y \). Suppose for a contradiction that there are \( x' \in X \) and \( y' \in Y \) such that \( y' \to x' \) and \( x' \not\to y' \). Since \( D \) does not contain directed paths of length 2 as induced subdigraphs, we have \( x \neq x' \) and \( y \neq y' \); with the same argument we conclude that the arcs \((x, y')\) and \((x', y)\) are digons. But, \( \{x, x', y, y'\} \) induces the only digraph on four vertices in \( F_3 \), a contradiction. Therefore, all the arcs from \( X \) to \( Y \) are present in \( D \), and thus, \((X, Y)\) is an \( M_3 \)-partition of \( D \).

When \( M = M_6 \), an \( M \)-partitionable digraph \( D \) is just an empty graph. Hence, it is clear that the only minimal \( M_6 \)-obstructions are an asymmetric arc and a digon, this is, \( F_6 \).

We conclude this note with the analysis of the patterns having a 0 and a 1 in the main diagonal. The pattern \( M_7 \) has been already studied in \cite{7}; an \( M_7 \)-partition corresponds to a strict split partition, this is, a partition \((V_0, V_1)\) with \( V_0 \) an independent set and \( V_1 \) a strong clique. Refer to \cite{7}, for the complete list of minimal \( M_7 \)-obstructions. Together with the aforementioned result for \( M_7 \), our next result shows that for these kind of patterns, there are always finitely many minimal obstructions.

For \( 8 \leq i \leq 10 \), it is clear that \( \vec{K}_3 \) and its complement are \( M_i \)-partitionable. If \( i \in \{8, 9\} \), then it is not hard to verify that every other digraph on three vertices is a minimal \( M_i \)-obstruction, except for the digraphs depicted in Figure 2. If \( i = 10 \), then clearly the asymmetric arc is a minimal \( M_i \)-obstruction, hence, every other obstruction is a biorientation of some (undirected) graph. Define \( F_i \) in the following way.

- \( F_8 \) consists of the aforementioned minimal \( M_8 \)-obstructions on 3 vertices, together with all the superorientations of \( 2K_2 \).
- \( F_9 \) consists of the aforementioned minimal \( M_9 \)-obstructions on 3 vertices.
- \( F_{10} \) consists of the asymmetric arc, \( \vec{2K}_2 \), and the biorientations of all graphs on three vertices, except for \( K_1 + K_2 \).

It is direct to verify that every digraph in \( F_i \) is indeed a minimal \( M_i \)-obstruction, for \( 8 \leq i \leq 10 \). It comes as no surprise that these are the only minimal \( M_i \)-obstructions.

**Theorem 4.** If \( i \in \{8, 9, 10\} \), then the minimal \( M_i \)-obstructions are precisely the digraphs in \( F_i \).
Proof. It is not hard to verify that every minimal $M_7$-obstruction is an element of $\mathcal{F}_i$, or properly contains an element of $\mathcal{F}_i$ as an induced subdigraph, for $8 \leq i \leq 10$. Thus, every $\mathcal{F}_i$-free digraph is a strict split digraph, for $8 \leq i \leq 10$.

Let $D$ be an $\mathcal{F}_8$-free digraph with strict split partition $(V_0, V_1)$. If $V_D$ is a strong clique or an independent set, then $D$ is $M_8$-partitionable. Also, every digraph on 2 vertices is $M_8$-partitionable, so let us assume that $|V_D| \geq 3$ and $V_0 \neq \emptyset \neq V_1$. If $|V_1| = 1$, then it follows from the fact that $D$ is $\mathcal{F}_8$-free that either all the arcs between $V_0$ and $V_1$ are oriented towards $V_1$, or there is an unique arc $(v_1, v_0)$ between them from $V_1$ to $V_0$. In this case $(V_0 \cup \{v_1\}, \{v_0\})$ is an $M_8$-partition of $D$. Otherwise, $|V_1| \geq 2$, and it is easy to observe that every arc between $V_0$ and $V_1$ must be oriented from $V_1$ to $V_0$, because $D$ is $\mathcal{F}_8$-free. Thus, $(V_0, V_1)$ is an $M_8$-partition of $D$.

Let $D$ be an $\mathcal{F}_9$-free digraph with split partition $(V_0, V_1)$. Since the underlying graph of $D$ is $(K_1 + K_2)$-free, and $D$ is $\mathcal{F}_9$-free, all the arcs from $V_0$ to $V_1$ must be present in $D$, and no arc from $V_1$ to $V_0$ can exist. Therefore, $(V_0, V_1)$ is an $M_9$-partition of $D$.

Finally, let $D$ be an $\mathcal{F}_{10}$-free digraph with split partition $(V_0, V_1)$ maximizing the size of $V_1$. Notice that the asymmetric arc is an element of $\mathcal{F}_{10}$.
hence, $D$ is a biorientation of its underlying graph $G_D$. Thus, it suffices to notice that $G_D$ is a split graph where the path of length 2 is forbidden. It is easy to verify that $(V_0, V_1)$ is an $M_{10}$-partition of $D$, unless there is a vertex in $V_0$ which is adjacent to every vertex in $V_1$. But this cannot happen as it would contradict the choice of $(V_0, V_1)$. \qed

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