Relations between Average Distance, Heterogeneity and Network Synchronizability

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Abstract

By using the random interchanging algorithm, we investigate the relations between average distance, standard deviation of degree distribution and synchronizability of complex networks. We find that both increasing the average distance and magnifying the degree deviation will make the network synchronize harder. Only the combination of short average distance and small standard deviation of degree distribution that ensures strong synchronizability. Some previous studies assert that the maximal betweenness is a right quantity to estimate network synchronizability: the larger the maximal betweenness, the poorer the network synchronizability. Here we address an interesting case, which strongly suggests that the single quantity, maximal betweenness, may not give a comprehensive description of network synchronizability.

Key words: Synchronizability, Complex Networks, Average Distance, Heterogeneity
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1 Introduction

A variety of systems in nature can be described by complex networks and the most important statistical features of complex networks are the small-world effect and scale-free property[1,2,3,4]. Networks that have small average distance as random networks and large clustering coefficient as regular ones are called small-world networks[5]. And the scale-free property means the degree distribution of networks obeys the power-law form[6]. One of the ultimate goals of researches on complex networks is to understand how the structure of complex networks affects the dynamical process taking place on them, such as traffic flow[7,8,9,10,11], epidemic spread[12,13,14,15,16,17], cascading behavior[18,19,20], and so on.

The large networks of coupled dynamical systems that exhibit synchronized state are subjects of great interest. Previous studies have demonstrated that scale-free and small-world networks are much easier to synchronize than regular lattice[21,22,23,24]. Then what makes complex networks synchronize so easily? It is intuitively believed that shorter average distance predicts better synchronizability[22,23,25]. However, it is found that to decrease average distance will make some complex networks synchronize even harder[26]. More bewilderingly, a very recent work suggests that on some synchronization systems, the synchronizability is independent of the average distance[27]. Some authors also addressed that the homogeneous distribution of degree will lead to better synchronizability. Hong et al.[28] investigate the relationship between network synchronizability and various topological ingredients, including average distance, heterogeneity, and betweenness of Watts-Strogatz (WS) networks[5]. They suggest the maximal betweenness a right indicator for synchronizability. This tentative conclusion has been widely accepted now[29,30,31]. Recently, several researchers examine the effect of clustering coefficient on the synchronization by using Kuramoto model[29] or master stability function[32] and find that increasing clustering coefficient will hinder the global synchronization. All the four topological ingredients, average distance, heterogeneity (measured by the standard deviation of degree distribution), betweenness and clustering coefficient, may reflect the networks synchronizability to some extent, but which one or ones indicate the network synchronizability simply and exactly?

There is another problem need to be mentioned. When investigating the relations between various topological ingredients and network synchronizability, some parameters, such as rewiring probability $p$ of WS networks[28] or power-law exponent $\gamma$ of the degree distribution in scale-free networks[26], are adjusted to modulate other topological ingredients, like average distance or heterogeneity of degree distribution. However, in this process all the topological ingredients keep changing with the adjusting of these parameters. It is impossible to get clear relation between an ingredient and synchronizability
Here in this paper, we try to discuss the relationship between these ingredients and the synchronization of complex networks precisely. This paper is organized as follows. In section 2, we give a brief review on how to measure the network synchronizability. And then, in section 3, the so-called random interchanging algorithm\cite{33,34,35} is introduced, which allows one to manipulate the clustering coefficient while keeping the network’s degree distribution unchanged. The main simulations are shown in section 4, and an interesting case is laid out and discussed in section 5. Finally, in section 6, we sum up this paper and discuss the relevance of our work to the real world.

2 Network Synchronizability

In this section, we will introduce a generic model of coupled oscillators on networks and a useful measure\cite{36}, which is often used to test the stability of the global synchronized states. Consider $N$ identical dynamical systems (oscillators) with the same output function, which are located on the vertices of a network and coupled linearly and symmetrically with neighbors. The coupling fashion ensures the synchronization manifold an invariant manifold, and the dynamics can be locally linearized near the synchronous state. The state of the $i$th oscillator is denoted by $x^i$, we get the set of equations of motion governing the dynamics of the $N$ coupled oscillators:

$$\dot{x}^i = F(x^i) + \eta \sum_{j=1}^{N} G_{ij} H(x^j), \quad (1)$$

where $\dot{x}^i = F(x^i)$ governs the dynamics of individual oscillator, $H(x^i)$ is the output function and $\eta$ the coupling strength. The $N \times N$ Laplacian $G$ is given
by

\[
G_{ij} = \begin{cases} 
  k_i & \text{for } i = j \\
  -1 & \text{for } j \in \Lambda_i, \\
  0 & \text{otherwise}
\end{cases}
\]  

(2)

where \(\Lambda_i\) denotes the neighbor set of node \(i\). Because of the positive semidefinite of \(G\), all the eigenvalues of it are nonnegative reals and the smallest eigenvalue \(\theta_0\) is always zero, for the rows of \(G\) have zero sum. If all the nodes are connected, there is only one zero eigenvalue. Thus, the eigenvalues can be ranked as \(\theta_0 < \theta_1 \leq \cdots \leq \theta_{N-1}\). The ratio of the maximum eigenvalue \(\theta_{N-1}\) to the smallest nonzero one \(\theta_1\) is widely used to measure the synchronizability of the network[36], if the eigenratio \(R = \theta_{N-1}/\theta_1\) satisfies

\[
R < \alpha_2/\alpha_1,
\]  

we say the network is synchronizable. The right-hand side \(\alpha_2/\alpha_1\) of this inequality depends on the dynamics of individual oscillator and the output function (one can see ref. [37] for details), while the eigenratio \(R\) depends only on the Laplacian \(G\). \(R\) indicates the synchronizability of the network, the smaller it is the better synchronizability and vice versa. In this paper, for universality, we will not address a particular dynamical system, but concentrate on how the network topology affects the eigenratio \(R\).

3 The Random Interchanging Algorithm

To investigate the structural effects on network synchronizability, we use random interchanging algorithm [33,34,35] to adjust clustering coefficient while keeping degree distribution unchanged. The procedure is as follows:

(1) Randomly pick two existing edges \(e_1 = x_1x_2\) and \(e_2 = x_3x_4\), such that \(x_1 \neq x_2 \neq x_3 \neq x_4\) and there is no edge between \(x_1\) and \(x_4\) as well as \(x_2\) and \(x_3\).

(2) Interchange these two edges, that is, connect \(x_1\) and \(x_4\) as well as \(x_2\) and \(x_3\), and remove the edges \(e_1\) and \(e_2\).

(3) Ensure the network is still connected and compute whether this interchange increases/decreases the network clustering coefficient. If it does, accept the new configuration, else recover the old one.
(4) Repeat step (1) unless the desired clustering coefficient is achieved.

Since this algorithm only rewires connections and does not change the degree of any node, the degree distribution as well as the degree sequence is fixed. Figure 1 provides a sketch maps of random interchanging algorithm, which may help us understanding the program flow.

4 Simulations

In the random interchanging process, operations that bring nonlocal couplings will reduce network average distance $L$ [23,38], and at the same time the clustering coefficient $C$ will be reduced[39]. Figure 2 and 3 exhibit the relationship between $L$ and $C$. In figure 2, the original networks are the WS networks with size $N = 2000$, average degree $< k > = 4$, and standard deviations of degree distributions $\sigma = 0.2$, 0.5, 0.6 and 0.87, respectively. In figure 3, the original networks are the extensional BA networks[40,41] with $N = 2000$, average degree $< k > = 12$ and standard deviations of degree distributions $\sigma = 18.43, 19.26, 20.65$ and 21.26, respectively. Here, the different standard deviations for WS networks and extensional BA networks are obtained by adjusting the rewiring probability $p$ and the power-law exponent $\gamma$, respectively. Clearly, the trends of $L$ and $C$ are qualitatively the same. We have checked that the positive correlation between $L$ and $C$ is not sensitive to the network size, the average degree and the standard deviation of degree distribution. In this paper, we examine the relation between average distance and synchronizability, and the relation between clustering coefficient and synchronizability can be obtained easily.

The eigenratios for WS and extensional BA model are obtained numerical and their behaviors with the average distance $L$ at different standard deviations of degree distributions are exhibited in Figs. 4 and 5 respectively. From each curve in Fig. 4 it can be seen that with the increasing of average distance $L$, the eigenratio $R$ grows, which means shorter average distance predicts better synchronizability. The similar result is obtained for extensional BA network shown in Fig. 5. The present result is consistent with the very recent result[29,32] that the larger clustering coefficient will inhibit global synchronization in scale-free network.

It can also be seen from Figs. 4 and 5 , at equal average distance $L$ the larger the standard deviation of degree $\sigma$ is, the larger the eigenratio $R$ will be, indicating networks with a homogeneous distribution of connectivity are more synchronizable than heterogeneous ones when average distance $L$ keeps constant.
Average distance and heterogeneity of connections are topological ingredients for network synchronization. Shortening average distance and making the connections more homogeneous solely will increase network synchronizability, however, only their combination could make the network have strong synchronizability. The average distance of star coupling network is very short, \( L \to 2 \) as \( N \to \infty \), while the standard deviation of degree is very large, \( \sigma \sim \sqrt{N} \), and the eigenratio \( R \to N \), suggesting that it is hard to synchronize when the network size is large. In one-dimensional ring lattice, all the nodes have equal degree, thus \( \sigma = 0 \). However, the average distance \( L \sim N/4z \) is too large (here \( z \) denotes the coordination number\[42\]), thus it is also very hard to synchronize with the increasing network size\[43\].

For many scale-free network models, heterogeneous distribution of connectivity tends to reduce the average network distance\[44,26\], WS small-world network obeys the same law\[28\]. Thus, in this two kind of networks, the increasing of heterogeneity will diminish the average distance. While, for WS small-world network the standard deviation of degree is very small, the network synchronizability is mainly determined by average distance, with \( L \)'s
Fig. 3. (Color online) The relationship between average distance $L$ and clustering coefficient $C$ when the original networks are the extensional BA networks. The black squares, red circles, green up-triangles and blue down-triangles represent the cases of $\sigma =18.43$, 19.26 and 21.26, respectively. All the data are the average over 20 different realizations.

decreasing, although $\sigma$ increases, $R$ will still be diminished, the network becomes more synchronizable. For some real-life scale-free networks, because of their ultra-small feature\cite{44,45} ($L \sim \ln \ln N$ or even shorter) and large degree deviation, $R$ strongly depends on heterogeneity, the more heterogeneous is the harder to synchronize. Therefore, shortening average distance is an effective way to enhance small-world network synchronizability and diminishing the heterogeneity of connectivity can make scale-free network synchronize easier.

5 An Interesting Case: Smaller Maximal Betweenness may not Indicate Better Synchronizability

Some previous studies suggest that the maximal betweenness centrality $B_{\text{max}}$ is a suitable indicator for predicting synchronizability on complex networks\cite{28,29,30}; the larger $B_{\text{max}}$ is, the poorer the synchronizability. The betweenness centrality of node $n$ is defined as the probability that a randomly selected shortest
Fig. 4. (Color online) The eigenratio $R$ vs average distance $L$ when the original networks are the WS networks. Eigenratio $R$ shows positive correlation with average distance $L$ when standard deviation of degree $\sigma$ is fixed, and at a fixed value of $L$, $R$ will increase with the rising of $\sigma$. For a variety of chaotic oscillators, $\alpha_2/\alpha_1$ ranges from 5 to 100\[23\], so we only investigate the situations with eigenratios less than about 100. All the data are the average over 20 different realizations.

path of a randomly picked pair of nodes contains the node $n$\[46,47\]

$$B_n := \frac{1}{(N-1)(N-2)} \sum_{i \neq j \neq n} \frac{g_{ij}(n)}{g_{ij}},$$

(4)

where $g_{ij}$ is the number of shortest paths between nodes $i$ and $j$, and $g_{ij}(n)$ is the number of those paths passing through node $n$. From the definition of betweenness centrality, it is easy to get the relationship between the average betweenness centrality $< B >$ and the average distance $L$

$$< B > = \frac{N(N-1)(L-1)}{N(N-1)(N-2)} = \frac{L-1}{N-2}.$$  (5)

Previous studies indicate that there exists strongly positive correlation between degree and betweenness centrality\[48,49\], that is to say, the node with larger degree will statistically have higher betweenness centrality. Therefore,
Fig. 5. (Color online) The eigenratio $R$ vs average distance $L$ when the original networks are the extensional BA networks. Eigenratio $R$ shows positive correlation with average distance $L$ when standard deviation of degree $\sigma$ is fixed, and at a fixed value of $L$, $R$ will increase with the rising of $\sigma$. All the data are the average over 20 different realizations.

The insets of figure 6 and 7 respectively show the changes of maximal betweenness with average distance at different standard deviation of degree. For WS network, the increasing of average distance or degree deviation solely always induces the increasing of maximal betweenness. It is consistent with the former analysis. However, for extensional BA network, maximal betweenness is not sensitive to average distance but increases with the increasing of degree deviation clearly.

By running the random interchanging algorithm, we computing $R$ and $B_{max}$ for different configurations, figure 6 and 7 show the correlation between them at fixed $\sigma$ for WS network and extensional BA network, respectively. When
the original network is a WS network, there exist strongly positive correlation between $R$ and $B_{\text{max}}$ at fixed $\sigma$ (see figure 6), which support the previous conclusion[28]. But for different degree deviation, networks shows different $R$ at the same $B_{\text{max}}$, although the number of nodes and edges are the same. When the original network is an extensional BA network at fixed $\sigma$, the positive correlation between $R$ and $B_{\text{max}}$ vanishes. The simulation results show that the single quantity, maximal betweenness $B_{\text{max}}$, may not give a comprehensive description of network synchronizability.

6 Conclusion Remarks

In conclusion, with the help of random interchanging algorithm we show that shorter average distance and homogeneity solely will lead to better synchronizability, but only their combination could make the network easy to synchronize.
Fig. 7. (Color online) The eigenratio $R$ vs maximal betweenness $B_{\text{max}}$ when the
original network the extensional BA networks. Clearly, no positive correlation be-
tween $R$ and $B_{\text{max}}$ can be observed under this case. All the data are the average
over 20 different realizations.

Some Numerical studies have been done to check if the maximal betweenness
$B_{\text{max}}$ is a proper quantity to estimate network synchronizability. The simu-
lation results strongly suggest that the single quantity, $B_{\text{max}}$, may not give a
comprehensive description of network synchronizability.

It is worthwhile to emphasize that this work is not only of theoretical interest,
but also of practical value. The clear picture of topological effects on network
synchronizability may provide us a guideline to design algorithm aiming at
enhancing or reducing the network synchronizability[50].

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