DERIVED CATEGORIES OF FAMILIES OF SEXTIC DEL PEZZO SURFACES

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ABSTRACT. We construct a natural semiorthogonal decomposition for the derived category of an arbitrary flat family of sextic del Pezzo surfaces with at worst du Val singularities. This decomposition has three components equivalent to twisted derived categories of finite flat schemes of degrees 1, 3, and 2 over the base of the family. We provide a modular interpretation for these schemes and compute them explicitly in a number of standard families. In the Appendix we prove a symmetric version of homological projective duality for $\mathbb{P}^2 \times \mathbb{P}^2$, $\text{Fl}(1, 2; 3)$, and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

1. Introduction

In this paper we describe the bounded derived category of coherent sheaves on an arbitrary flat family of del Pezzo surfaces of canonical degree 6 with du Val singularities. We expect this result to be useful for description of derived categories of varieties, that admit a structure of such a family. There are at least two interesting examples of this sort.

One example is provided by special cubic fourfolds of discriminant 18. In [AHTVA16] it was shown, that a general such cubic fourfold contains an elliptic scroll, the blowup of which has a structure of a family of sextic del Pezzo surfaces over $\mathbb{P}^2$ (this is quite similar to the case of cubic fourfolds containing a plane, when blowing up the plane one gets a family of two-dimensional quadrics over $\mathbb{P}^2$). Another example is provided by Gushel–Mukai fourfolds ([DK15, KP17]) containing a Veronese surface. The results of this paper should have a direct application in these two cases and provide a description of the derived categories of cubic and Gushel–Mukai fourfolds of these types, and in particular, a geometric interpretation of their K3 categories (see [Kuz10, KP16]).

Our main result (Theorem 5.2) proves, that given a flat family $\mathcal{X} \to S$ all of whose fibers are sextic del Pezzo surfaces with at worst du Val singularities, there are two finite flat morphisms $\mathcal{Z}_2 \to S$ and $\mathcal{Z}_3 \to S$ of degrees 3 and 2 respectively, with Brauer classes $\beta_{\mathcal{Z}_2}$ and $\beta_{\mathcal{Z}_3}$ of order 2 and 3 respectively, and an $S$-linear semiorthogonal decomposition

$$D(\mathcal{X}) = \langle D(S), D(\mathcal{Z}_2, \beta_{\mathcal{Z}_2}), D(\mathcal{Z}_3, \beta_{\mathcal{Z}_3}) \rangle,$$

where the second and the third components are the twisted derived categories. This result can be considered as a generalization of [BSST1] and [AB15], where the case of a smooth sextic del Pezzo surface over a (not necessarily algebraically closed) field was considered.

To construct the semiorthogonal decomposition, we first investigate in detail the case when $S$ is the spectrum of an algebraically closed field $k$, and so $\mathcal{X}$ is just a single sextic del Pezzo surface $X$ over $k$ with du Val singularities. In this case, to describe $D(X)$ we first consider the minimal resolution of singularities $\pi: \tilde{X} \to X$. Here $\tilde{X}$ is a weak del Pezzo surface of degree 6; it has at most three $(-2)$-curves, which in the worst case form two chains of lengths 2 and 1. The category $D(X)$ then can be identified with the localization (or the quotient) of the category $D(\tilde{X})$ by the subcategory generated by sheaves $\mathcal{O}_{\Delta}(-1)$, for $\Delta$ running through the set of all $(-2)$-curves.

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On the other hand, the surface $\widetilde{X}$ can be realized as an iterated blowup of $\mathbb{P}^2$, and so comes with a natural exceptional collection. We mutate this collection slightly (Proposition 3.1) to get a semiorthogonal decomposition $\mathbf{D}(\widetilde{X}) = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ such that for each $(-2)$-curve $\Delta$ the sheaf $\mathcal{O}_{D}(\Delta)$ is contained in one of its components (Lemma 3.6). Then we prove (Theorem 3.5) that $\mathbf{D}(X)$ has a semiorthogonal decomposition $\mathbf{D}(X) = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ with components being the localizations of $\mathcal{A}_i$ by the subcategories generated by the appropriate sheaves $\mathcal{O}_{\Delta}(\Delta)$. An explicit computation shows that the categories $\mathcal{A}_i$ are equivalent to products of derived categories of so-called Auslander algebras, and their localizations $\mathcal{A}_i$ are equivalent to derived categories of zero-dimensional schemes of lengths 1, 3, and 2 respectively.

This approach, however, does not generalize to families of del Pezzo surfaces, since one cannot construct a relative minimal resolution. To deal with this problem, we go back to the case of a single del Pezzo surface $X$ (still over an algebraically closed field), and provide a modular interpretation for the components of the constructed semiorthogonal decomposition. Namely, we show that the zero-dimensional schemes associated with the nontrivial components $\mathcal{A}_2$ and $\mathcal{A}_3$ (the component $\mathcal{A}_1$ is generated by the structure sheaf $\mathcal{O}_X$ and has a natural counterpart in any family) can be thought of as moduli spaces of semistable sheaves on $X$ with Hilbert polynomials $h_d(t) = (3t + d)(t + 1)$ for $d = 2$ and $d = 3$ respectively, see Theorem 4.5. These moduli spaces turn out to be fine, and the corresponding universal families provide fully faithful Fourier–Mukai functors from derived categories of the moduli spaces into $\mathbf{D}(X)$.

This description, of course, can be easily used in a family $\mathcal{X} \to S$. We consider the relative moduli spaces $\mathcal{M}_d(\mathcal{X}/S)$ of semistable sheaves on fibers of $\mathcal{X}$ over $S$ with Hilbert polynomials $h_d(t)$. Now, however, the moduli spaces need not to be fine, so we consider their coarse moduli spaces $\mathcal{Z}_d$ and the Brauer obstruction classes $\beta_{\mathcal{Z}_d}$ on them. Then the universal families are well defined as $\beta_{\mathcal{Z}_d}$-twisted sheaves on $\mathcal{X} \times_S \mathcal{Z}_d$ and define Fourier–Mukai functors from the twisted derived categories $\mathbf{D}(\mathcal{Z}_d, \beta_{\mathcal{Z}_d})$ to $\mathbf{D}(\mathcal{X})$. Using the results over an algebraically closed field described earlier, we show in Theorem 5.2 that these functors are fully faithful, and together with the pullback functor $\mathbf{D}(S) \to \mathbf{D}(\mathcal{X})$ form the required semiorthogonal decomposition.

The question of understanding the derived category of a family $\mathcal{X} \to S$ of sextic del Pezzo surfaces thus reduces to understanding the schemes $\mathcal{Z}_2 \to S$ and $\mathcal{Z}_3 \to S$ together with their Brauer classes. We provide a Hilbert scheme interpretation of these. Namely, we show in Proposition 5.14 that the relative Hilbert scheme $F_2(\mathcal{X}/S)$ of conics in the fibers of $\mathcal{X} \to S$ is a $\mathbb{P}^1$-bundle over $\mathcal{Z}_2$ with associated Brauer class $\beta_{\mathcal{Z}_2}$, and the relative Hilbert scheme $F_3(\mathcal{X}/S)$ of twisted cubic curves is a $\mathbb{P}^2$-bundle over $\mathcal{Z}_3$ with associated Brauer class $\beta_{\mathcal{Z}_3}$. We also prove that the relative Hilbert scheme of lines $F_1(\mathcal{X}/S)$ can be written as $F_1(\mathcal{X}/S) \cong \mathcal{Z}_2 \times_S \mathcal{Z}_3$.

Another useful result is the following smoothness criterion. We show that the total space $\mathcal{X}$ of a flat family $\mathcal{X} \to S$ of sextic del Pezzo surfaces with du Val singularities is regular if and only if the three schemes $\mathcal{S}_2$, $\mathcal{S}_3$, and $\mathcal{S}_3$, associated with it, are all regular (Proposition 5.12). This leads to the following description of the schemes $\mathcal{Z}_2$ and $\mathcal{Z}_3$ in case of regular $\mathcal{X}$ — the schemes $\mathcal{Z}_2$ and $\mathcal{Z}_3$ are isomorphic to the normal closures of their generic fibers over $S$. This shows that to understand $\mathcal{Z}_2$ and $\mathcal{Z}_3$ globally, it is enough to understand them over any dense open subset, or even over the general point of $S$ if $S$ is integral. In particular, if $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ are two families with regular total spaces and $F_d(\mathcal{X}/S)$ is birational (over $S$) to $F_d(\mathcal{X}'/S)$ for some $d \in \{2, 3\}$, then $\mathcal{Z}_d(\mathcal{X}/S) \cong \mathcal{Z}_d(\mathcal{X}'/S)$ and $\beta_{\mathcal{Z}_d(\mathcal{X}/S)} = \beta_{\mathcal{Z}_d(\mathcal{X}'/S)}$ (Corollary 5.10). We expect this property to be very useful in geometric applications mentioned at the beginning of the Introduction.

We finish the paper by an explicit description of the schemes $\mathcal{Z}_2$ and $\mathcal{Z}_3$ for some “standard” families of sextic del Pezzo surfaces.
The first standard family is the family of codimension 2 linear sections of \( \mathbb{P}^2 \times \mathbb{P}^2 \). In this case we show that \( \mathcal{Z}_3 = S \sqcup S \), \( \mathcal{Z}_2 \) is the scheme of “degenerate linear equations” of the fibers of \( \mathcal{Z} \), and both Brauer classes are trivial, see Proposition 6.2.

The second standard family is the family of hyperplane sections of \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \). In this case we show that \( \mathcal{Z}_2 = S \sqcup S \sqcup S \), \( \mathcal{Z}_3 \) is the double cover of \( S \) branched over the divisor of “degenerate linear equations” of the fibers of \( \mathcal{Z} \), and again both Brauer classes are trivial, see Proposition 6.6.

In both cases we deduce the required description of \( D(\mathcal{Z}) \) from a symmetric version of homological projective duality for \( \mathbb{P}^2 \times \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) that we discuss in Appendices C and D. Note that the description via the homological projective duality allows to extend the general description to a wider class of families of sextic del Pezzo surfaces, allowing in particular non-integral degenerations. In these cases the schemes \( \mathcal{Z}_2 \) and \( \mathcal{Z}_3 \) controlling the components of \( D(\mathcal{Z}) \) become non-flat over \( S \) (see Remark 6.3).

We also consider families of relative anticanonical models of the blowups of \( \mathbb{P}^2 \) (resp. of \( \mathbb{P}^1 \times \mathbb{P}^1 \)) in length 3 (resp. length 2) subschemes. We show that in these cases one of the schemes \( \mathcal{Z}_2 \) and \( \mathcal{Z}_3 \) coincides with the family of the blowup centers, while the other is obtained by gluing appropriate number of copies of \( S \), see Propositions 6.8 and 6.9 for details.

Of course, the approach used in this paper can be applied to other del Pezzo families. In case of a single del Pezzo surface over an algebraically closed field one should analyze possible configurations of \((-2)\)-curves on its weak del Pezzo resolution and find a block-exceptional collection such that for any \((-2)\)-curve \( \Delta \) the sheaf \( \mathcal{O}_\Delta(-1) \) is contained in the subcategory generated by one of the blocks. Most probably, (weak del Pezzo analogues of) the three-block collections of Karpov and Nogin [KN98] should be used here. This approach definitely should work for del Pezzo surfaces of degrees \( d \geq 5 \), and we leave it to the readers to check the results it leads to. For \( d \leq 4 \), however, the number of possible degenerations of del Pezzo surfaces becomes quite big, and the approach hard to realize, so probably another idea is needed in this case. Let us also mention that to transfer this approach to families of del Pezzo surfaces, one will also need a moduli space interpretation for the components of the semiorthogonal decomposition, which might be not so easy to find.

The paper is organized as follows. In Section 2 we discuss the geometry of sextic del Pezzo surfaces with du Val singularities over an algebraically closed field and remind some general results about resolutions of rational singularities and Grothendieck duality. In Section 3 we describe the derived category of a single del Pezzo surface with du Val singularities over an algebraically closed field. In Section 4 we provide a moduli space interpretation for this description and discuss the relation of these moduli spaces to Hilbert schemes of curves. In Section 5 we prove the main result of the paper — the semiorthogonal decomposition of the derived category for a family of sextic del Pezzo surfaces with du Val singularities, and discuss some properties of this decomposition. In particular, we relate regularity of the total space \( \mathcal{Z} \) of the family to that of \( S \), \( \mathcal{Z}_2 \), and \( \mathcal{Z}_3 \). In Section 6 we describe the schemes \( \mathcal{Z}_2 \) and \( \mathcal{Z}_3 \) for standard families of sextic del Pezzo surfaces. In Appendix A we discuss the derived categories of Auslander algebras and their relation to derived categories of zero-dimensional schemes. In Appendix B we show that the moduli stack of sextic del Pezzo surfaces is smooth. Finally, in Appendices C and D we describe the symmetric homological projective duality for \( \mathbb{P}^2 \times \mathbb{P}^2 \), \( \text{Fl}(1, 2; 3) \), and \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) respectively.

Conventions. Throughout the paper we work over a filed \( k \), whose characteristic is assumed to be coprime to 6. In sections 2, 3 and 4 we assume that \( k \) is algebraically closed, while in sections 5 and 6 we leave this assumption. For a scheme \( X \) we denote by \( D(X) \) the bounded derived category of coherent sheaves on \( X \), and unless something else is specified explicitly, this is what we mean by a derived category. All the functors we consider are derived — for instance \( \otimes \) stands for the derived tensor product, \( f^* \) and \( f_* \) stand for the derived pullback and pushforward functors. If we want to consider the classical pullback or pushforward, we write \( L_0 f^* \) and \( R^0 f_* \) respectively (and similarly for other classical derived functors). We think of the Brauer group of a scheme as of the group of Morita-equivalence classes of Azumaya algebras.
on it. For a Brauer class $\beta$ on a scheme $X$ we denote by $D(X, \beta)$ the twisted bounded derived category of coherent sheaves. We refer to [Huy06] for an introduction into derived categories, and to [Kuz14] and references therein for the introduction into semiorthogonal decompositions.

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2. Preliminaries

2.1. Sextic del Pezzo surfaces. For purposes of this paper we adopt the following definition.

Definition 2.1. A sextic du Val del Pezzo surface is a normal integral projective surface $X$ over a field $k$ with at worst du Val singularities and ample anticanonical class such that $K_X^2 = 6$.

Recall that du Val surface singularities are just canonical singularities or, equivalently, rational double points. In particular, any surface $X$ with du Val singularities is Gorenstein, so $\omega_X$ is a line bundle, $K_X$ is a Cartier divisor, and its square is well-defined.

Let $\pi : \tilde{X} \to X$ be the minimal (in particular crepant) resolution of singularities of $X$. It is well-known (see, e.g. [Dol12 Section 8.4.2]) that the surface $\tilde{X}$ is rational and can be obtained from $\mathbb{P}^2$ by a sequence of three blowups of a point, i.e., we have a diagram

$$X \xleftarrow{\pi} \tilde{X} = X_3 \to X_2 \to X_1 \to X_0 = \mathbb{P}^2,$$

where each map $X_i \to X_{i-1}$ is the blowup of a point $P_i \in X_{i-1}$. We denote by $h$ the hyperplane class on $\mathbb{P}^2$ and its pullback to $\tilde{X}$. We denote by $E_i \subset \tilde{X}$ the pullback (i.e., the total preimage) to $\tilde{X}$ of the exceptional divisor of $X_i \to X_{i-1}$ and by $e_i$ its class in $\text{Pic}(\tilde{X})$. The following result is standard.

Lemma 2.2. We have $\text{Pic}(\tilde{X}) \cong \mathbb{Z}(h, e_1, e_2, e_3)$, with $h^2 = 1$, $e_i^2 = -1$, $he_i = e_ie_j = 0$ for all $i \neq j$. Moreover,

$$(1) \quad K_{\tilde{X}} = -3h + e_1 + e_2 + e_3 = \pi^*K_X.$$

The surface $X$ is the anticanonical model of $\tilde{X}$. In other words, $X$ is obtained from $\tilde{X}$ by contraction of all $(-2)$-curves. By [Dol12 Section 8.4.2] there are six possibilities for configurations of the blowup centers and $(-2)$-curves on $\tilde{X}$ (see Table 1 below for a picture).

Type 0: Neither of $P_i$ lies on the exceptional divisor in $X_{i-1}$ and their images in $\mathbb{P}^2$ do not lie on a common line. Then $\tilde{X}$ contains no $(-2)$-curves and $X = \tilde{X}$ is smooth.

Type 1: Neither of $P_i$ lies on the exceptional divisor in $X_{i-1}$ but their images in $\mathbb{P}^2$ lie on a common line. Then $\tilde{X}$ contains a unique $(-2)$-curve (the strict transform $\Delta_{123}$ of that line) and $X$ has one $A_1$ singularity.

Type 2: The point $P_2$ lies on the exceptional divisor of $X_1 \to X_0$, the point $P_3$ is away of the exceptional divisors, and the line through $P_1$ in the direction of $P_2$ on $\mathbb{P}^2$ does not pass through the image of $P_3$. Then $\tilde{X}$ contains a unique $(-2)$-curve (the strict transform $\Delta_{12}$ of the exceptional divisor of $X_1 \to X_0$) and $X$ has one $A_1$ singularity.

Type 3: The point $P_2$ lies on the exceptional divisor of $X_1 \to X_0$, the point $P_3$ is away of the exceptional divisors, but the line through $P_1$ in the direction of $P_2$ on $\mathbb{P}^2$ passes through the image of $P_3$. Then $\tilde{X}$ contains two disjoint $(-2)$-curves (the strict transforms $\Delta_{123}$ and $\Delta_{12}$ of the line on $\mathbb{P}^2$ and of the exceptional divisor of $X_1 \to X_0$) and $X$ has two $A_1$ singularities.

Type 4: The point $P_2$ lies on the exceptional divisor of $X_1 \to X_0$, the point $P_3$ lies on the exceptional divisor of $X_2 \to X_1$, and the strict transform $L_{12}$ of the line through $P_1$ in the direction of $P_2$ does not contain $P_3$. Then $\tilde{X}$ contains a 2-chain of $(-2)$-curves (the strict transforms $\Delta_{12}$ and $\Delta_{23}$ of the exceptional divisors of $X_1 \to X_0$ and $X_2 \to X_1$) and $X$ has one $A_2$ singularity.
**Type 5:** The point $P_2$ lies on the exceptional divisor of $X_1 \to X_0$, the point $P_3$ lies on the exceptional divisor of $X_2 \to X_1$, but the strict transform of the line through $P_1$ in the direction of $P_2$ contains $P_3$. Then $\tilde{X}$ contains a 2-chain of $(-2)$-curves and one more $(-2)$-curve disjoint from the chain (the strict transforms $\Delta_{123}$, $\Delta_{12}$ and $\Delta_{23}$ of the line and the exceptional divisors of $X_1 \to X_0$ and $X_2 \to X_1$ respectively) and $X$ has one $A_2$ singularity and one $A_1$ singularity.

For readers convenience we draw the configurations of exceptional curves on sextic del Pezzo surfaces of all types. Red thick lines are the $(-2)$-curves, while the thin lines are $(-1)$-curves. We denote by $\Delta = \Delta(X)$ the set of all $(-2)$-curves on $X$. Note that the $(-2)$ curves (when they exist) on $\tilde{X}$ are contained in the following linear systems:

$$\Delta_{12} = E_1 - E_2 \in |e_1 - e_2|,$$
$$\Delta_{23} = E_2 - E_3 \in |e_2 - e_3|,$$
$$\Delta_{123} \in |h - e_1 - e_2 - e_3|.$$

We denote by $L_{ij}$ the strict transform of the line connecting (the images on $\mathbb{P}^2$ of) the points $P_i$ and $P_j$.

| Type | $\Delta$ | Configurations |
|------|------|----------------|
| 0    | $\emptyset$ | $\Delta_{123}$ |
| 2    | $\{\Delta_{12}\}$ | $\Delta_{12}$, $\Delta_{23}$ |
| 3    | $\{\Delta_{12}, \Delta_{123}\}$ | $\Delta_{12}$, $\Delta_{123}$ |
| 4    | $\{\Delta_{12}, \Delta_{23}\}$ | $\Delta_{12}$, $\Delta_{23}$ |
| 5    | $\{\Delta_{12}, \Delta_{23}, \Delta_{123}\}$ | $\Delta_{12}$, $\Delta_{23}$, $\Delta_{123}$ |

Table 1. Configurations of exceptional curves on sextic del Pezzo surfaces

In each of these types there is a unique (up to an isomorphism) sextic del Pezzo surface. Moreover, the surfaces of types 0, 2, 3, and 5 are toric (in particular, the surface of type 5 is just the weighted projective plane $\mathbb{P}(1, 2, 3)$, see [Kaw15, Example 5.7] for a description of its derived category). The surfaces of type 1 and 4 are not toric (because they have three special points on the curves $\Delta_{123}$ and $\Delta_{23}$ respectively).

### 2.2. Resolutions of rational surface singularities.

In the next section we investigate the derived category of a singular del Pezzo surface $X$ through its minimal resolution $\tilde{X}$. In this subsection we collect some facts about resolutions of surface singularities we are going to use.

Let $X$ be a normal surface with rational singularities and let $\pi: \tilde{X} \to X$ be its resolution. The derived categories of $X$ and $\tilde{X}$ are related by the (derived) pushforward functor $\pi_*: D(\tilde{X}) \to D(X)$. 
The (derived) pullback functor does not preserve boundedness, but is well defined on the bounded from above derived category $\pi^*: \mathbf{D}^- (X) \to \mathbf{D}^- (\tilde{X})$. Denote by $\Delta$ the set of irreducible components of the exceptional divisor of $\pi$; each of these is a rational curve on $\tilde{X}$.

**Lemma 2.3.** Let $X$ be a normal surface with rational singularities and let $\pi: \tilde{X} \to X$ be its resolution. The functor $\pi^*: \mathbf{D}^- (X) \to \mathbf{D}^- (\tilde{X})$ is fully faithful. The functor $\pi_*: \mathbf{D}^- (\tilde{X}) \to \mathbf{D}^- (X)$ is its right adjoint, preserves boundedness, and there is an isomorphism of functors

\[
(2) \quad \pi_* \circ \pi^* \cong \text{id}.
\]

Moreover,

\[
\text{Im} \pi^* = \langle \mathcal{O}_\Delta (-1) \rangle_{\Delta \in \Delta} \quad \text{and} \quad \text{Ker} \pi_* = \langle \mathcal{O}_\Delta (-1) \rangle_{\Delta \in \Delta}
\]

where $\langle - \rangle^\flat$ denotes the minimal triangulated subcategory closed under infinite direct sums defined in $\mathbf{D}^-$. 

**Proof.** The pullback-pushforward adjunction is standard. By the projection formula we have

\[
\pi_* (\pi^* \mathcal{F}) \cong \mathcal{F} \otimes \pi_* \mathcal{O}_{\tilde{X}},
\]

and since $X$ has rational singularities, the canonical morphism $\mathcal{O}_X \to \pi_* \mathcal{O}_{\tilde{X}}$ is an isomorphism, hence (2) holds. By adjunction it follows that $\pi^*$ is fully faithful. Finally, by [BB15] Lemma 2.1 (see also [Bri02, Lemma 3.1] and Lemma 2.3 below) and [BB15] Proposition 9.14 and Theorem 9.15 the category $\text{Ker} \pi_*$ is generated by sheaves $\mathcal{O}_\Delta$. The description of $\text{Im} \pi^*$ follows by adjunction. \qed

The following Bridgeland’s spectral sequence argument is quite useful, so we remind it here.

**Lemma 2.4** ([Bri02] Lemma 3.1). Let $\mathcal{F}$ be a possibly unbounded complex of quasicoherent sheaves on $\tilde{X}$ and let $H^i (\mathcal{F})$ be its cohomology sheaf in degree $i$. The spectral sequence $R^i \pi_* (H^j (\mathcal{F})) \Rightarrow H^{i+j} (\pi_* (\mathcal{F}))$ degenerates at the second page, and gives for each $i$ an exact sequence

\[
0 \to R^1 \pi_* H^{i-1} (\mathcal{F}) \to H^i (\pi_* (\mathcal{F})) \to R^0 \pi_* H^i (\mathcal{F}) \to 0.
\]

In particular, if $H^i (\pi_* (\mathcal{F})) = 0$ for $i \leq p$ for some integer $p$, then $\pi_* (H^i (\mathcal{F})) = 0$ for $i \leq p - 1$ and $\pi_* \mathcal{F} \cong \pi_* (\tau^\geq p \mathcal{F})$, where $\tau$ stands for the truncation functor with respect to the canonical filtration.

**Proof.** The fibers of $\pi$ are at most 1-dimensional, hence $R^{\geq 2} \pi_* = 0$, and the second page of the spectral sequence looks like

\[
\begin{array}{cccccccc}
\cdots & 0 & \leftarrow & R^1 \pi_* H^{i-2} (\mathcal{F}) & d_2 & R^1 \pi_* H^{i-1} (\mathcal{F}) & d_2 & R^1 \pi_* H^i (\mathcal{F}) & d_2 & R^1 \pi_* H^{i+1} (\mathcal{F}) & \cdots \\
\cdots & R^0 \pi_* H^{i-2} (\mathcal{F}) & \leftarrow & R^0 \pi_* H^{i-1} (\mathcal{F}) & R^0 \pi_* H^i (\mathcal{F}) & R^0 \pi_* H^{i+1} (\mathcal{F}) & \cdots \\
\end{array}
\]

It follows that the spectral sequence degenerates at the second page, and gives the required exact sequences. The vanishing of $\pi_* (H^i (\mathcal{F}))$ for all $i \leq p - 1$ follows immediately from the exact sequences, and in its turn implies $\pi_* (\tau^\geq p - 1 \mathcal{F}) = 0$. Applying the pushforward to the canonical truncation triangle $\tau^\leq p - 1 \mathcal{F} \to \mathcal{F} \to \tau^\geq p \mathcal{F}$ we obtain the required isomorphism. \qed

The following consequences of this observation will be used later.

**Corollary 2.5.** The functor $\pi_*: \mathbf{D}(\tilde{X}) \to \mathbf{D}(X)$ is essentially surjective.

**Proof.** Let $\mathcal{F} \in \mathbf{D}(X)$ and assume that $p$ is such that $\tau^\leq p (\mathcal{F}) = 0$. Then $\mathcal{F} \cong \pi_* (\pi^* \mathcal{F}) \cong \pi_* (\tau^\geq p \pi^* \mathcal{F})$, and clearly $\tau^\geq p \pi^* \mathcal{F} \in \mathbf{D}(\tilde{X})$. \qed
2.3. Grothendieck and Serre duality. Let \( f: X \to Y \) be a proper morphism. The Grothendieck duality is a bifunctorial isomorphism

\[
\text{RHom}(f_* \mathcal{F}, \mathcal{G}) \cong \text{RHom}(\mathcal{F}, f^! \mathcal{G}),
\]

where \( f^! \) is the twisted pullback functor (if \( \mathcal{G} \) is perfect, \( f^! \mathcal{G} \cong \mathcal{G} \otimes \omega_{X/Y}^\bullet \), where \( \omega_{X/Y}^\bullet = f^! \mathcal{O}_Y \) is the relative dualizing complex). In other words, the twisted pullback functor is right adjoint to the (derived) pushforward.

Grothendieck duality has many consequences. One of them — Serre duality for Gorenstein schemes — will be very useful for our purposes.

**Proposition 2.6.** Let \( X \) be a projective Gorenstein \( k \)-scheme of dimension \( n \). If either \( \mathcal{F} \) or \( \mathcal{G} \) is a perfect complex, there is a natural Serre duality isomorphism

\[
\text{Ext}^i(\mathcal{F}, \mathcal{G})^\vee \cong \text{Ext}^{n-i}(\mathcal{G}, \mathcal{F} \otimes \omega_X).
\]

**Proof.** Let \( f: X \to \text{Spec}(k) \) be the structure morphism. If \( \mathcal{F} \) is a locally free sheaf, then we have \( \text{Ext}^i(\mathcal{F}, \mathcal{G}) \cong H^i(X, \mathcal{F}^\vee \otimes \mathcal{G}) \), and this is a cohomology group of \( f_*(\mathcal{F}^\vee \otimes \mathcal{G}) \). By Grothendieck duality

\[
\text{RHom}(f_*(\mathcal{F}^\vee \otimes \mathcal{G}), k) \cong \text{RHom}(\mathcal{F}^\vee \otimes \mathcal{G}, f^! (k)).
\]

Since \( X \) is Gorenstein, \( f^! (k) \cong \omega_X^*[n] \), hence the right hand side equals \( \text{RHom}(\mathcal{F}^\vee \otimes \mathcal{G}, \omega_X^*[n]) \). Since \( \mathcal{F} \) is locally free, this can be rewritten as \( \text{RHom}(\mathcal{G}, \mathcal{F} \otimes \omega_X)[n] \). Computing the cohomology groups in degree \(-i\), we deduce the required duality isomorphism. For arbitrary perfect \( \mathcal{F} \) the Serre duality follows by using the stupid filtration.

Finally, when \( \mathcal{G} \) is perfect, we replace \( \mathcal{F} \) by \( \mathcal{G}, \mathcal{G} \) by \( \mathcal{F} \otimes \omega_X \), and \( i \) by \( n-i \), and deduce the required isomorphism from the previous case. \( \square \)

Let us also discuss a contravariant duality functor \( \mathcal{RHom}(-, \omega_{X/k}^\bullet) \) for a projective \( k \)-scheme \( X \). It follows from sheafified Grothendieck duality it is an equivalence of categories \( \mathcal{D}(X)^\text{opp} \sim \text{D}(X) \).

In case when the scheme \( X \) is Gorenstein, the dualizing complex \( \omega_{X/k}^\bullet \) is a shift of the canonical line bundle, \( \omega_{X/k}^\bullet \cong \omega_X[\text{dim } X] \), and it follows that the usual duality functor

\[
\mathcal{F} \mapsto \mathcal{F}^\vee := \mathcal{RHom}(\mathcal{F}, \mathcal{O}_X) \cong \mathcal{RHom}(\mathcal{F}, \omega_X^\bullet) \otimes \omega_X^{-1}[\text{dim } X]
\]

is also an equivalence of categories \( \mathcal{D}(X)^\text{opp} \sim \text{D}(X) \).

3. Derived category of a single sextic del Pezzo surface

Let \( X \) be a sextic del Pezzo surface (Definition 2.1) over an algebraically closed field \( k \), and let \( \pi: \tilde{X} \to X \) be its minimal resolution of singularities. We use freely notation introduced in Section 2.1

3.1. Derived category of the resolution. We start by describing the derived category of \( \tilde{X} \).

**Proposition 3.1** (cf. [KN98] Proposition 4.2(3)). Let \( X \) be a sextic del Pezzo surface over an algebraically closed field \( k \) and let \( \pi: \tilde{X} \to X \) be its minimal resolution of singularities. There is a semiorthogonal decomposition

\[
\mathcal{D}(\tilde{X}) = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle,
\]

whose components are generated by the following exceptional collections

\[
\mathcal{A}_1 = \langle \mathcal{O}_{\tilde{X}} \rangle,
\]

\[
\mathcal{A}_2 = \langle \mathcal{O}_{\tilde{X}}(h-e_1), \mathcal{O}_{\tilde{X}}(h-e_2), \mathcal{O}_{\tilde{X}}(h-e_3) \rangle,
\]

\[
\mathcal{A}_3 = \langle \mathcal{O}_{\tilde{X}}(h), \mathcal{O}_{\tilde{X}}(2h-e_1-e_2-e_3) \rangle.
\]
Proof. We start with one of the standard exceptional collections of \( \mathbb{P}^2 \):

\[
\mathbf{D}(\mathbb{P}^2) = \langle \mathcal{O}_{\mathbb{P}^2}(-h), \mathcal{O}_{\mathbb{P}^2}(h) \rangle.
\]

Using the blowup formula, we obtain an exceptional collection on \( \tilde{X} \):

\[
\mathbf{D}(\tilde{X}) = \langle \mathcal{O}_{\tilde{X}}(-h), \mathcal{O}_{\tilde{X}}(h), \mathcal{O}_{E_1}, \mathcal{O}_{E_2}, \mathcal{O}_{E_3} \rangle
\]

where the last three sheaves are the structure sheaves of the total preimages to \( \tilde{X} \) of the exceptional divisors of the blowups (so, with our numbering conventions the first two of them might be reducible but connected curves).

Next, mutate the last three terms to the left of \( \mathcal{O}_{\tilde{X}}(h) \). The twists of the standard exact sequences

\[0 \to \mathcal{O}_{\tilde{X}}(-e_i) \to \mathcal{O}_{\tilde{X}}(h) \to \mathcal{O}_{E_i} \to 0\]

show that the resulting exceptional collection is

\[
\mathbf{D}(\tilde{X}) = \langle \mathcal{O}_{\tilde{X}}(-h), \mathcal{O}_{\tilde{X}}(h - e_1), \mathcal{O}_{\tilde{X}}(h - e_2), \mathcal{O}_{\tilde{X}}(h - e_3), \mathcal{O}_{\tilde{X}}(h) \rangle.
\]

Finally, mutate \( \mathcal{O}_{\tilde{X}}(-h) \) to the far right. Thus, it will get twisted by \( -K_{\tilde{X}} = 3h - e_1 - e_2 - e_3 \), so in the end we get an exceptional collection

\[
\mathbf{D}(\tilde{X}) = \langle \mathcal{O}_{\tilde{X}}, \mathcal{O}_{\tilde{X}}(h - e_1), \mathcal{O}_{\tilde{X}}(h - e_2), \mathcal{O}_{\tilde{X}}(h - e_3), \mathcal{O}_{\tilde{X}}(2h - e_1 - e_2 - e_3) \rangle.
\]

Splitting it into three blocks as indicated in the statement of the proposition, we obtain the required semiorthogonal decomposition.

If \( X \) is smooth (hence \( \tilde{X} = X \)) the exceptional line bundles in each of the components \( \tilde{A}_i \) are pairwise orthogonal. However, for singular \( X \) this is no longer true. We describe the structure of the categories \( \tilde{A}_i \) below, but before that we observe a self-duality property of \( \mathbf{D}(\tilde{X}) \). When applied to \( \text{(1)} \) it produces another semiorthogonal decomposition

\[
\text{(6)} \quad \mathbf{D}(\tilde{X}) = \langle \tilde{A}_3^\vee, \tilde{A}_2^\vee, \tilde{A}_1^\vee \rangle.
\]

It turns out that it is also the right mutation-dual of \( \text{(1)} \), i.e., is obtained from \( \text{(1)} \) by a standard sequence of mutations. Below we denote by \( \mathbb{L}_\mathcal{A} \) the left mutation functor through an admissible subcategory \( \mathcal{A} \).

Lemma 3.2. The semiorthogonal decomposition \text{(6)} is right mutation-dual to \text{(1)}, i.e.,

\[
\tilde{A}_1^\vee = \tilde{A}_1, \quad \tilde{A}_2^\vee = \mathbb{L}_{\tilde{A}_1}(\tilde{A}_2), \quad \text{and} \quad \tilde{A}_3^\vee = \mathbb{L}_{\tilde{A}_2}(\tilde{A}_3) = \tilde{A}_3 \otimes \omega_{\tilde{X}}.
\]

Proof. The claim is trivial for the first component, since \( \tilde{A}_1^\vee = \langle \mathcal{O}_{\tilde{X}} \rangle = \langle \mathcal{O}_{\tilde{X}} \rangle = \tilde{A}_1 \), and is easy for the last component, since \( \mathbb{L}_{\tilde{A}_1}(\mathbb{L}_{\tilde{A}_2}(\tilde{A}_3)) = \tilde{A}_3 \otimes \omega_{\tilde{X}} \) and

\[
\begin{align*}
\mathcal{O}_{\tilde{X}}(h) \otimes \omega_{\tilde{X}} & \cong \mathcal{O}_{\tilde{X}}(-2h + e_1 + e_2 + e_3) \cong \mathcal{O}_{\tilde{X}}(2h - e_1 - e_2 - e_3)^\vee, \\
\mathcal{O}_{\tilde{X}}(2h - e_1 - e_2 - e_3) \otimes \omega_{\tilde{X}} & \cong \mathcal{O}_{\tilde{X}}(-h) \cong \mathcal{O}_{\tilde{X}}(h)^\vee.
\end{align*}
\]

Finally, for the second component we have \( \mathbb{L}_{\tilde{A}_1}(\tilde{A}_2) = \tilde{A}_1^\perp \cap \tilde{A}_2^\perp = (\tilde{A}_1^\perp)^\perp \cap (\tilde{A}_3 \otimes \omega_{\tilde{X}}) = \tilde{A}_2^\vee \). □

The structure of the components \( \tilde{A}_i \) of the decomposition \text{(1)} depends on the type of \( X \). We explain this dependence in the lemma below. For \( m = 2 \) and \( m = 3 \) we denote by \( \tilde{R}_m \) the Auslander algebra defined by \text{(10)}, and refer to Appendix \text{A} for its basic properties, especially note the definition \text{(64)} of the standard exceptional modules and Proposition \text{A.2}.
### Proposition 3.3.
The components $\tilde{A}_i$ of (4) are equivalent to products of derived categories of Auslander algebras as indicated in the next table:

| Type of $X$ | $\tilde{A}_1$ | $\tilde{A}_2$ | $\tilde{A}_3$ |
|-------------|---------------|---------------|---------------|
| 0           | $D(k)$        | $D(k) \times D(k) \times D(k)$ | $D(k) \times D(k)$ |
| 1           | $D(k)$        | $D(k) \times D(k) \times D(k)$ | $D(\tilde{R}_2)$ |
| 2           | $D(k)$        | $D(\tilde{R}_2) \times D(k)$ | $D(\tilde{R}_2)$ |
| 3           | $D(k)$        | $D(\tilde{R}_2) \times D(k)$ | $D(\tilde{R}_2)$ |
| 4           | $D(k)$        | $D(\tilde{R}_3)$ | $D(\tilde{R}_2)$ |
| 5           | $D(k)$        | $D(\tilde{R}_3)$ | $D(\tilde{R}_2)$ |

This equivalence associates exceptional line bundles in (5) with the standard exceptional modules over the corresponding Auslander algebra.

**Proof.** The component $\tilde{A}_1$ is generated by a single exceptional object, hence is equivalent to $D(k)$. So, in view of Proposition [A.2] to prove the proposition it is enough to compute Ext-spaces between the exceptional line bundles forming the components $\tilde{A}_2$ and $\tilde{A}_3$ of the semiorthogonal decomposition (4).

First of all, we have

$$\text{Ext}^p(\mathcal{O}(h - e_i), \mathcal{O}(h - e_j)) \cong H^p(\tilde{X}, \mathcal{O}(e_i - e_j)).$$

Assuming $i \neq j$ and using exact sequences (note that $e_i \cdot e_j = 0$ and $e_i^2 = -1$ by Lemma [2.2])

$$0 \rightarrow \mathcal{O}(e_i - e_j) \rightarrow \mathcal{O}(e_i) \rightarrow \mathcal{O}(E_i) \rightarrow 0, \\
0 \rightarrow \mathcal{O}(e_i) \rightarrow \mathcal{O}(e_i - e_j) \rightarrow \mathcal{O}(E_i(-1)) \rightarrow 0,$$

we obtain an exact sequence

$$0 \rightarrow H^0(\tilde{X}, \mathcal{O}(e_i - e_j)) \rightarrow k \rightarrow k \rightarrow H^1(\tilde{X}, \mathcal{O}(e_i - e_j)) \rightarrow 0,$$

where the middle map is given by the restriction to $E_j$ of the equation of $E_i$. This restriction vanishes if and only if $E_j$ is a component of $E_i$—in this case we deduce that $\text{Ext}^p(\mathcal{O}(h - e_i), \mathcal{O}(h - e_j)) \cong k \oplus k[-1]$, and otherwise $\text{Ext}^p(\mathcal{O}(h - e_i), \mathcal{O}(h - e_j)) = 0$. By Proposition [A.2] this gives the required description of $\tilde{A}_2$ in types from 0 to 3. In the last two types (4 and 5) it remains to check that multiplication map

(8) $\text{Ext}^p(\mathcal{O}(h - e_i), \mathcal{O}(h - e_j)) \otimes \text{Ext}^q(\mathcal{O}(h - e_2), \mathcal{O}(h - e_3)) \rightarrow \text{Ext}^{p+q}(\mathcal{O}(h - e_1), \mathcal{O}(h - e_3))$

is an isomorphism when $p = 0$ or $q = 0$. For this consider exact sequences

(9) $0 \rightarrow \mathcal{O}(h - e_1) \otimes \mathcal{O}(h - e_2) \rightarrow \mathcal{O}(h - e_2) \rightarrow \mathcal{O}(E_1(-1)) \rightarrow 0,$

(10) $0 \rightarrow \mathcal{O}(h - e_2) \otimes \mathcal{O}(h - e_3) \rightarrow \mathcal{O}(E_3(-1)) \rightarrow 0,$

where $\Delta_{12} = E_1 - E_2$ and $\Delta_{23} = E_2 - E_3$. Using Lemma [2.2] we compute

$$\text{Ext}^p(\mathcal{O}(h - e_1), \mathcal{O}(h - e_3)) \cong H^p(\Delta_{23}, \mathcal{O}(e_1 - h)) \otimes \mathcal{O}(\Delta_{23}(-1)) = H^p(\Delta_{23}, \mathcal{O}(\Delta_{23}(-1))) = 0.$$

Applying the functor $\text{Ext}^p(\mathcal{O}(h - e_1), -)$ to (10), we deduce that (9) is an isomorphism for $q = 0$. Similarly, by Serre duality $\text{Ext}^p(\mathcal{O}(\Delta_{12}(-1), \mathcal{O}(h - e_3))$ is dual to $\text{Ext}^p(\mathcal{O}(e_1 + e_2 - 2h), \mathcal{O}(\Delta_{12}(-1)))$, and a computation similar to the above shows it is zero. Applying the functor $\text{Ext}^p(\mathcal{O}(h - e_3))$ to (9), we deduce that (8) is an isomorphism for $p = 0$. 

To describe $\tilde{A}_3$ we only need to compute
\[
\text{Ext}^p(\theta_{\tilde{X}}(h), \theta_{\tilde{X}}(2h-e_1-e_2-e_3)) \cong H^p(\tilde{X}, \theta_{\tilde{X}}(h-e_1-e_2-e_3)).
\]
Using exact sequences
\[
0 \to \theta_{\tilde{X}}(h-e_1) \to \theta_{\tilde{X}}(h) \to \theta_{E_1} \to 0,
\]
\[
0 \to \theta_{\tilde{X}}(h-e_1-e_2) \to \theta_{\tilde{X}}(h-e_1) \to \theta_{E_2} \to 0,
\]
\[
0 \to \theta_{\tilde{X}}(h-e_1-e_2-e_3) \to \theta_{\tilde{X}}(h-e_1-e_2) \to \theta_{E_3} \to 0,
\]
we conclude that $H^*(\tilde{X}, \theta_{\tilde{X}}(h-e_1-e_2-e_3)) = 0$ if the three centers of the blowups are contained on a line (i.e., in types 1, 3, and 5), and $H^0(\tilde{X}, \theta_{\tilde{X}}(h-e_1-e_2-e_3)) = H^1(\tilde{X}, \theta_{\tilde{X}}(h-e_1-e_2-e_3)) = k$ otherwise. By Proposition A.2 this describes $\tilde{A}_3$.

The nontrivial morphisms between the line bundles $\theta_{\tilde{X}}(h-e_1), \theta_{\tilde{X}}(h-e_2), \theta_{\tilde{X}}(h-e_3)$ when exist are realized by exact sequences (9) and (10). Analogously, the nontrivial morphism between the line bundles $\theta_{\tilde{X}}(h)$ and $\theta_{\tilde{X}}(2h-e_1-e_2-e_3)$ is realized by the exact sequence
\[
0 \to \theta_{\tilde{X}}(h) \xrightarrow{\Delta_{123}} \theta_{\tilde{X}}(2h-e_1-e_2-e_3) \to \theta_{\Delta_{123}}(-1) \to 0.
\]

Remark 3.4. Comparing exact sequences (65) of standard exceptional modules over an Auslander algebra with the exact sequences (9), (10) and (11), and taking into account that the equivalence of Proposition 3.3 takes the exceptional line bundles generating $\tilde{A}_i$ to the standard exceptional modules over the corresponding Auslander algebras, we conclude that the sheaves $\theta_{\Delta}(-1)$ go to the corresponding simple modules. For instance, for a surface of type 5, when we have the maximal number of $(-2)$-curves on $\tilde{X}$, and when $\tilde{A}_2 \cong \mathbf{D}(\tilde{R}_3), \tilde{A}_3 \cong \mathbf{D}(\tilde{R}_2)$, the sheaves $\theta_{\Delta_{12}}(-1)$ and $\theta_{\Delta_{23}}(-1)$ go to $S_1$ and $S_2$ in $\mathbf{D}(\tilde{R}_3)$, and the sheaf $\theta_{\Delta_{123}}(-1)$ goes to $S_1$ in $\mathbf{D}(\tilde{R}_2)$.

3.2. Semiorthogonal decomposition for a sextic del Pezzo surface. Now we apply the above computations to describe the derived category of a sextic du Val del Pezzo surface $X$. The main result of this section is the next theorem.

Theorem 3.5. Let $X$ be a sextic du Val del Pezzo surface over an algebraically closed field $k$, and let $\pi: \tilde{X} \to X$ be its minimal resolution of singularities. Then there is a unique semiorthogonal decomposition
\[
\mathbf{D}(X) = \langle A_1, A_2, A_3 \rangle,
\]
such that $\pi_*(\tilde{A}_i) = A_i$ and $\pi^*(A_i \cap \mathbf{D}^{\text{perf}}(X)) \subset \tilde{A}_i$, where $\tilde{A}_i$ are the components of (4). The components $A_i$ are admissible and their projection functors have finite cohomological amplitude. These components are equivalent to products of derived categories of finite-dimensional algebras as indicated in the next table:

| Type of $X$ | $A_1$ | $A_2$ | $A_3$ |
|-------------|-------|-------|-------|
| 0           | $\mathbf{D}(k)$ | $\mathbf{D}(k) \times \mathbf{D}(k) \times \mathbf{D}(k)$ | $\mathbf{D}(k) \times \mathbf{D}(k) \times \mathbf{D}(k)$ |
| 1           | $\mathbf{D}(k)$ | $\mathbf{D}(k) \times \mathbf{D}(k) \times \mathbf{D}(k)$ | $\mathbf{D}(k) \times \mathbf{D}(k) \times \mathbf{D}(k)$ |
| 2           | $\mathbf{D}(k)$ | $\mathbf{D}(k[t]/t^2) \times \mathbf{D}(k)$ | $\mathbf{D}(k) \times \mathbf{D}(k) \times \mathbf{D}(k)$ |
| 3           | $\mathbf{D}(k)$ | $\mathbf{D}(k[t]/t^2) \times \mathbf{D}(k)$ | $\mathbf{D}(k) \times \mathbf{D}(k) \times \mathbf{D}(k)$ |
| 4           | $\mathbf{D}(k)$ | $\mathbf{D}(k[t]/t^2)$ | $\mathbf{D}(k) \times \mathbf{D}(k)$ |
| 5           | $\mathbf{D}(k)$ | $\mathbf{D}(k[t]/t^2)$ | $\mathbf{D}(k[t]/t^2)$ |

The categories $A_i^{\text{perf}} := A_i \cap \mathbf{D}^{\text{perf}}(X)$ form a semiorthogonal decomposition of the perfect category
\[
\mathbf{D}^{\text{perf}}(X) = \langle A_1^{\text{perf}}, A_2^{\text{perf}}, A_3^{\text{perf}} \rangle.
\]
The proof takes the rest of this subsection. We start with some preparations.

Consider the semiorthogonal decomposition (4). Since $X$ is smooth, every component $\tilde{A}_i$ of $D(\tilde{X})$ is admissible, hence (4) is a strong semiorthogonal decomposition in sense of [Kuz11 Definition 2.6]. Therefore, by [Kuz11 Proposition 4.2] it extends to a semiorthogonal decomposition of the unbounded from below derived category

$$D^-(\tilde{X}) = \langle \tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \rangle.$$  

(13)

The next lemma describes the intersections of its components with the kernel category of the pushforward functor $\pi_*$. Recall that $\Delta$ denotes the set of all $(-2)$-curves defined on $\tilde{X}$. We denote by $\Delta_2 \subset \Delta$ the subset formed by those of the curves $\Delta_{12}$ and $\Delta_{23}$ that are defined on $X$ and by $\Delta_3 \subset \Delta$ its complement. Recall the notation of Lemma 2.3

Lemma 3.6. We have

$$\tilde{A}_1 \cap \ker \pi_* = 0, \quad \tilde{A}_2 \cap \ker \pi_* = (\mathcal{O}_\Delta(-1))^\oplus_{\Delta \in \Delta_2}, \quad \text{and} \quad \tilde{A}_3 \cap \ker \pi_* = (\mathcal{O}_\Delta(-1))^\oplus_{\Delta \in \Delta_3}.$$  

Moreover, for any $\mathcal{F} \in \ker \pi_* \subset D^-(\tilde{X})$ there is a canonical direct sum decomposition

$$\mathcal{F} = \mathcal{F}_2 \oplus \mathcal{F}_3$$

with $\mathcal{F}_j \in \tilde{A}_j \cap \ker \pi_*$.  

Proof. By Lemma 2.3 an object $\mathcal{F} \in D^-(\tilde{X})$ is in $\ker \pi_*$ if and only if every its cohomology sheaf $\mathcal{H}^j(\mathcal{F})$ is an iterated extension of sheaves $\mathcal{O}_\Delta(-1)$ where $\Delta$ run over the set $\Delta$ of all $(-2)$-curves on $\tilde{X}$. It remains to note that $\mathcal{O}_\Delta(-1) \in \tilde{A}_2$ for $\Delta \in \Delta_2$ by (9) and (10), while $\mathcal{O}_\Delta(-1) \in \tilde{A}_3$ for $\Delta \in \Delta_3$ by (11). Moreover, the subcategories generated by the sheaves $\mathcal{O}_\Delta(-1)$ with $\Delta \in \Delta_2$ and $\Delta \in \Delta_3$ are completely orthogonal, since the supports of these sheaves do not intersect. This last observation shows that any $\mathcal{F} \in \ker \pi_*$ decomposes as $\mathcal{F}_2 \oplus \mathcal{F}_3$ with the required properties (just take $\mathcal{F}_2$ and $\mathcal{F}_3$ to be the components of $\mathcal{F}$ supported on the union of the curves $\Delta$ from $\Delta_2$ and $\Delta_3$ respectively). \qed

Another useful observation is the following.

Lemma 3.7. We have $\text{Hom}(\tilde{A}_i^-, \tilde{A}_j^- \cap \ker \pi_*) = 0$ for any $i \neq j$.

Proof. Since $\tilde{A}_i^-$ is generated by an exceptional collection of line bundles and $\tilde{A}_j^- \cap \ker \pi_*$ is generated by sheaves $\mathcal{O}_\Delta(-1)$ for $\Delta \in \Delta_j$, it is enough to check that any of the line bundles generating $\tilde{A}_i^-$ restricts trivially to any curve $\Delta$ from $\Delta_j$.

For $i = 1$ this is evident; for $i = 2$ we have $(h - e_k) \cdot \Delta_{123} = (h - e_k)(h - e_1 - e_2 - e_3) = 1 - 1 = 0$, and for $i = 3$ we have $h \cdot (e_k - e_l) = 0$ and $2h - e_1 - e_2 - e_3 \cdot (e_k - e_l) = 1 - 1 = 0$. \qed

Denote by $\tilde{\alpha}_i$ the projection functors of the decomposition (13). By [Kuz11 Proposition 4.2] the projection functors of (4) are given by the restrictions of $\tilde{\alpha}_i$ to $D(X)$.

Proposition 3.8. The subcategories $\mathcal{A}_i^- = \{ \mathcal{F} \in D^-(X) \mid \pi^*(\mathcal{F}) \in \tilde{A}_i^- \} \subset D^-(X)$ form a semiorthogonal decomposition

$$D^-(X) = \langle \mathcal{A}_1^-, \mathcal{A}_2^-, \mathcal{A}_3^- \rangle$$

with projection functors given by

$$\alpha_i = \pi_* \circ \tilde{\alpha}_i \circ \pi^*.$$  

(14)

Moreover, we have

$$\pi_*(\tilde{A}_i^-) = \mathcal{A}_i^-.$$  

(15)
Proof. By Lemma 2.3 the functor $\pi^*: \text{D}^-(X) \to \text{D}^-(\tilde{X})$ is fully faithful, hence the subcategories $\mathcal{A}^-_i$ are semiorthogonal.

Let us prove (15). For this take any object $\mathcal{F} \in \mathcal{A}^-_1$ and consider the standard triangle

$$\pi^*(\pi_*\mathcal{F}) \to \mathcal{F} \to \mathcal{G}.$$ 

Then, of course, $\mathcal{G} \in \text{Ker} \pi_*$. By Lemma 3.6 we have $\mathcal{G} = \mathcal{G}_2 \oplus \mathcal{G}_3$ with $\mathcal{G}_j \in \mathcal{A}^-_j$. If $j \neq i$ then $\text{Ext}^*(\mathcal{F}, \mathcal{G}_j) = 0$ by Lemma 3.7 and $\text{Ext}^*(\pi^*(\pi_*\mathcal{F}), \mathcal{G}_j) = 0$ since $\mathcal{G}_j \in \text{Ker} \pi_*$. It follows from the above triangle that $\text{Hom}(\mathcal{G}, \mathcal{G}_j) = 0$, hence $\mathcal{G}_j = 0$ since it is a direct summand of $\mathcal{G}$. Therefore we have $\mathcal{G} \in \mathcal{A}^-_i$, hence $\pi^*(\pi_*\mathcal{F}) \in \mathcal{A}^-_i$, hence $\pi_*\mathcal{F} \in \mathcal{A}^-_i$ by definition of the latter. This proves an inclusion $\pi_*\mathcal{A}^-_i \subset \mathcal{A}^-_i$. The other inclusion follows from (12).

Now let us decompose any $\mathcal{F} \in \text{D}^-(X)$. For this take $\mathcal{F}' := \pi^*(\mathcal{F}) \in \text{D}^-(\tilde{X})$ and consider its decomposition with respect to (13). It is given by a filtration

$$0 = \mathcal{F}'_3 \to \mathcal{F}'_2 \to \mathcal{F}'_1 \to \mathcal{F}'_0 = \mathcal{F},$$

whose cones are $\tilde{\alpha}_i(\mathcal{F}') \in \mathcal{A}^-_i$. Pushing it forward to $X$, we obtain a filtration

$$0 = \pi_*\mathcal{F}_3 \to \pi_*\mathcal{F}_2 \to \pi_*\mathcal{F}_1 \to \pi_*\mathcal{F}_0 = \pi_*\mathcal{F},$$

whose cones are $\pi_*\tilde{\alpha}_i(\mathcal{F}') \equiv \pi_*\tilde{\alpha}_i(\pi^*(\mathcal{F})) \in \pi_*\mathcal{A}^-_i = \mathcal{A}^-_i$. This proves the semiorthogonal decomposition and shows that its projection functors are given by (14). □

Proposition 3.9. The subcategories $\mathcal{A}_i := \mathcal{A}^-_i \cap \text{D}(X) = \{ \mathcal{F} \in \text{D}(X) \mid \pi^*(\mathcal{F}) \in \mathcal{A}^-_i \} \subset \text{D}(X)$ provide a semiorthogonal decomposition (12); its projection functors $\alpha_i$ are given by (14). They preserve boundedness and have finite cohomological amplitude. Finally, $\mathcal{A}_i = \pi_*\mathcal{A}_i$.

Proof. For the first claim it is enough to check that the projection functors $\alpha_i$ preserve boundedness. Take any $\mathcal{F} \in \text{D}^{[a,b]}(X)$ and consider $\pi^*(\mathcal{F}) \in \text{D}^{(-\infty,b]}(\tilde{X})$. By projection formula $\pi_*\pi^*(\mathcal{F}) \cong \mathcal{F}$, hence by Lemma 2.3 we have $\tau^\leq a-1(\pi^*(\mathcal{F})) \in \text{Ker} \pi_*$. Consider the triangle

$$\tilde{\alpha}_i(\tau^\leq a-1(\pi^*(\mathcal{F}))) \to \alpha_i(\pi^*(\mathcal{F})) \to \alpha_i(\tau^\geq a(\pi^*(\mathcal{F})))$$

obtained by applying the projection functor $\tilde{\alpha}_i$ to the canonical truncation triangle. By Lemma 3.6 the functor $\tilde{\alpha}_i$ preserves $\text{Ker} \pi_*$, hence the first term of the triangle is in $\text{Ker} \pi_*$. Therefore, applying the pushforward we obtain an isomorphism $\alpha_i(\mathcal{F}) \equiv \pi_*\tilde{\alpha}_i(\pi^*(\mathcal{F})) \cong \pi_*\alpha_i(\tau^\geq a(\pi^*(\mathcal{F})))$. So, it remains to note that $\tau^\geq a(\pi^*(\mathcal{F})) \in \text{D}^{[a,b]}(\tilde{X})$, hence $\pi_*\tilde{\alpha}_i(\tau^\geq a(\pi^*(\mathcal{F})))$ is bounded, since both $\tilde{\alpha}_i$ and $\pi_*$ preserve boundedness. Moreover, if the cohomological amplitude of $\tilde{\alpha}_i$ is $(p,q)$ (it is finite since $\tilde{X}$ is smooth, see [Kuz08 Proposition 2.5]), then

$$\alpha_i(\mathcal{F}) \equiv \pi_*\tilde{\alpha}_i(\tau^\geq a(\pi^*(\mathcal{F}))) \in \text{D}^{[a+p,b+q+1]}(X).$$

In particular, $\alpha_i$ has finite cohomological amplitude.

Let us prove the last claim. By (15) we have $\pi_*\mathcal{A}_i \subset \mathcal{A}^-_i$, and since $\pi_* \pi_*$ preserves boundedness, we have $\pi_*\mathcal{A}_i \subset \mathcal{A}_i$. To check that this inclusion is an equality, take any $\mathcal{F} \in \mathcal{A}_i$. By Corollary 2.5 there exists $\mathcal{F} \in \text{D}(\tilde{X})$ such that $\mathcal{F} \cong \pi_*\mathcal{F}$. Let $\mathcal{G}$ be the cone of the natural morphism $\pi^*\mathcal{F} \to \mathcal{F}$. Then $\mathcal{G} \in \text{Ker} \pi_*$. Moreover, $\tilde{\alpha}_i(\mathcal{F}) \in \text{Ker} \pi_*$. Hence applying the functor $\pi_* \circ \tilde{\alpha}_i$ to the distinguished triangle $\pi^*\mathcal{F} \to \mathcal{F} \to \mathcal{G}$, we deduce an isomorphism $\mathcal{F} \cong \alpha_i(\mathcal{F}) \equiv \pi_*\tilde{\alpha}_i(\mathcal{F})$, and it remains to note that $\tilde{\alpha}_i(\mathcal{F}) \equiv \mathcal{A}^-_i$. □

Next, we identify the categories $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ with the corresponding products of $\text{D}(k[t]/t^m)$. By Proposition 3.3 each category $\mathcal{A}_i$ is equivalent to a product of derived categories $\text{D}(\tilde{R}_m)$ of Auslander algebras $\tilde{R}_m$. Take one of these and denote by $\tilde{\gamma}: \text{D}(\tilde{R}_m) \to \text{D}(\tilde{X})$ its embedding functor. Let

$$\pi_{m*}: \text{D}^-(\tilde{R}_m) \to \text{D}^-(k[t]/t^m) \quad \text{and} \quad \pi_{m}^*: \text{D}^-(k[t]/t^m) \to \text{D}^-(\tilde{R}_m)$$

...
be the functors described in Appendix A, see equation (17).

**Proposition 3.10.** The functor
\[ \gamma := \pi_* \circ \tilde{\gamma} \circ \pi_m^* : D^{-}(k[t]/t^{m}) \to D^{-}(X). \]

is fully faithful and preserves boundedness. Moreover, the diagrams
\[ \begin{align*}
&D^{-}(\tilde{R}_m) \xrightarrow{\gamma} D^{-}(X) \\
&D^{-}(\tilde{R}_m) \xrightarrow{\tilde{\gamma}} D^{-}(\tilde{X})
\end{align*} \]

are both commutative.

**Proof.** By definition $\gamma \circ \pi_{m*} = \pi_* \circ \tilde{\gamma} \circ \pi_m^* \circ \pi_{m*}$, so for commutativity of the first diagram it is enough to check that for any $M \in D^{-}(\tilde{R}_m)$ the cone of the canonical morphism $\pi_m^*(\pi_{m*}M) \to M$ is killed by the functor $\pi_* \circ \tilde{\gamma}$. But this cone, by Proposition [A.4] is contained in the subcategory generated by the simple modules $S_1, \ldots, S_{m-1}$ over $\tilde{R}_m$. By Remark 3.4 the functor $\tilde{\gamma}$ takes these modules to sheaves $\mathcal{O}(\Delta(-1))$ for appropriate $(-2)$-curves $\Delta$ on $\tilde{X}$, and the functor $\pi_*$ kills every $\mathcal{O}(\Delta(-1))$.

For commutativity of the second diagram note that by Proposition [A.4] the image of the functor $\pi_m^*$ is the left orthogonal $\perp \langle S_1, \ldots, S_{m-1} \rangle \subset D^{-}(\tilde{R}_m)$ of the subcategory generated by the simple modules. Therefore, using Lemma 3.7 and Remark 3.4 we deduce that the image of $\pi_m^* \circ \tilde{\gamma}$ is contained in the left orthogonal $\perp \langle \mathcal{O}(\Delta(-1)) \rangle \subset D^{-}(\tilde{X})$ of the subcategory generated by all $(-2)$-curves $\Delta$. Hence, by Lemma 2.3 it is contained in the image of $\pi^*$: $D^{-}(X) \to D^{-}(\tilde{X})$. Thus $\gamma \circ \pi_m^* \cong \pi_* \circ \pi_* \circ \pi_* \cong \pi_* \circ \gamma$, so the second diagram commutes.

Let us show that $\gamma$ is fully faithful. Indeed, by commutativity of the second diagram in (17), this follows from full faithfulness of the functors $\pi_m^*$ (Proposition [A.4]), $\tilde{\gamma}$ (Proposition 3.3), and $\pi^*$ (Lemma 2.3).

Finally, let us show that $\gamma$ preserves boundedness. Indeed, take any $N \in D(k[t]/t^{m})$. By Proposition [A.4] there exists $M \in D(\tilde{R}_m)$ such that $N \cong \pi_{m*}(M)$. Then $\gamma(N) = \gamma(\pi_{m*}(M))$ and by commutativity of the first diagram this is the same as $\pi_*(\tilde{\gamma}(M))$. We know that $\tilde{\gamma}(M)$ is bounded by Proposition 3.3 hence so is $\pi_*(\tilde{\gamma}(M)) = \gamma(N)$. \hfill \Box

Now we are ready to describe the components $A_i$ of (12). Let $\tilde{A}_i = D(\tilde{R}_{m_1}) \times \cdots \times D(\tilde{R}_{m_s})$ be the corresponding component of $A_i$. Let $\tilde{\gamma}_j : D(\tilde{R}_{m_j}) \to D(\tilde{X})$ be the corresponding embedding functors and let $\gamma_j : D(k[t]/t^{m_j}) \to D(X)$ be the embeddings constructed in Proposition 3.10.

**Proposition 3.11.** The fully faithful functors $\gamma_1, \ldots, \gamma_s$ induce a completely orthogonal decomposition
\[ A_i = D(k[t]/t^{m_1}) \times \cdots \times D(k[t]/t^{m_s}). \]

**Proof.** By definition $\gamma_j = \pi_* \circ \tilde{\gamma}_j \circ \pi_{m_j}^*$ and its image is contained in $\pi_* A_i \subset A_i$. Moreover, since $\gamma_j$ preserves boundedness, it actually is in $A_i$.

To see that the images of $\gamma_j$ are orthogonal, it is enough to check orthogonality of the images of functors $\pi^* \circ \gamma_j = \tilde{\gamma}_j \circ \pi_{m_j}^*$, which follows immediately from Proposition 3.3.

Finally, let us show the generation. Assume $\mathcal{F} \in A_i$ and let \mathcal{G} \in \tilde{A}_i be such that $\pi_* \mathcal{G} \cong \mathcal{F}$ (Proposition 3.3). Then $\mathcal{G}$ has a direct sum decomposition $\mathcal{G} \cong \oplus \mathcal{G}_j$, where $\mathcal{G}_j \in \tilde{\gamma}_j(D(\tilde{R}_{m_j}))$ (again by Proposition 3.3). Therefore, $\mathcal{F} \cong \pi_* (\mathcal{G}) \cong \oplus \pi_* (\mathcal{G}_j)$. Moreover, $\pi_* (\mathcal{G}_j) \in \pi_* (\tilde{\gamma}_j(D(\tilde{R}_{m_j}))) = \gamma_j(D(k[t]/t^{m_j}))$, hence objects $\mathcal{F}_j = \pi_* (\mathcal{G}_j)$ give the required decomposition of $\mathcal{F}$. \hfill \Box

**Lemma 3.12.** The subcategories $A_i \subset D(X)$ in (12) are admissible.
Proof. Consider the semiorthogonal decompositions

\[ D(\tilde{X}) = \langle \tilde{A}_3 \otimes \omega_{\tilde{X}}, \tilde{A}_1, \tilde{A}_2 \rangle \quad \text{and} \quad D(\tilde{X}) = \langle \tilde{A}_2 \otimes \omega_{\tilde{X}}, \tilde{A}_3 \otimes \omega_{\tilde{X}}, \tilde{A}_1 \rangle, \]

obtained from (14) by mutations. Using the fact that \( \omega_{\tilde{X}} \cong \pi^* \omega_X \) and \( \omega_X \) is a line bundle on \( X \), we see that the arguments of this subsection also prove semiorthogonal decompositions

\[ D(X) = \langle A_3 \otimes \omega_X, A_1, A_2 \rangle \quad \text{and} \quad D(X) = \langle A_2 \otimes \omega_X, A_3 \otimes \omega_X, A_1 \rangle. \]

Since the twist by \( \omega_X \) is an autoequivalence of \( D(X) \), these decompositions together with (12) show that each \( A_i \) is admissible. Indeed, (up to a twist) each \( A_i \) appears in one of the three decompositions on the left, and in one on the right.

Proof of Theorem 3.5. The semiorthogonal decomposition (12) is constructed in Proposition 3.9. The equality \( A_i = \pi_*(\tilde{A}_i) \), which implies uniqueness of the decompositions, and finiteness of cohomological amplitude of the projection functors \( \alpha_i \) is also proved there. Admissibility of \( A_i \) is proved in Lemma 3.12. The structure of the components \( A_i \) is described in Proposition 3.11. The required semiorthogonal decomposition of \( D^{\text{perf}}(X) \) is obtained by [Kuz11, Proposition 4.1] and the embedding \( \pi^*(A_i^{\text{perf}}) \subset A_i \) is evident from the definition.

For further convenience, we would like to rewrite the semiorthogonal decomposition (12) geometrically.

Corollary 3.13. For every sextic du Val del Pezzo surface \( X \) over an algebraically closed field \( k \) there are zero-dimensional schemes \( Z_1, Z_2, Z_3 \) of lengths 1, 3, and 2 respectively, such that \( A_i \cong D(Z_i) \) and

\[ D(X) = \langle D(Z_1), D(Z_2), D(Z_3) \rangle. \]

The scheme structure of \( Z_i \) depends on the type of \( X \) as follows:

| Type of \( X \) | \( Z_1 \) | \( Z_2 \) | \( Z_3 \) |
|----------------|------------|------------|------------|
| 0              | Spec(\( k \)) | Spec(\( k \) \( \sqcup \) Spec(\( k \) \( \sqcup \) Spec(\( k \))) | Spec(\( k \) \( \sqcup \) Spec(\( k \))) |
| 1              | Spec(\( k \)) | Spec(\( k \) \( \sqcup \) Spec(\( k \)) \( \sqcup \) Spec(\( k \))) | Spec(\( k \) \( \sqcup \) Spec(\( k \))) |
| 2              | Spec(\( k \)) | Spec(\( k \) \( \sqcup \) Spec(\( k \)) \( \sqcup \) Spec(\( k \)) \( \sqcup \) Spec(\( k \))) | Spec(\( k \) \( \sqcup \) Spec(\( k \))) |
| 3              | Spec(\( k \)) | Spec(\( k \) \( \sqcup \) Spec(\( k \)) \( \sqcup \) Spec(\( k \)) \( \sqcup \) Spec(\( k \))) | Spec(\( k \) \( \sqcup \) Spec(\( k \))) |
| 4              | Spec(\( k \)) | Spec(\( k \) \( \sqcup \) Spec(\( k \)) \( \sqcup \) Spec(\( k \))) | Spec(\( k \) \( \sqcup \) Spec(\( k \)) \( \sqcup \) Spec(\( k \))) |
| 5              | Spec(\( k \)) | Spec(\( k \) \( \sqcup \) Spec(\( k \)) \( \sqcup \) Spec(\( k \))) | Spec(\( k \) \( \sqcup \) Spec(\( k \)) \( \sqcup \) Spec(\( k \))) |

Since \( X \) is Gorenstein, we can produce yet another semiorthogonal decomposition by dualization.

Proposition 3.14. If \( X \) is a sextic del Pezzo surface over an algebraically closed field, there is a semiorthogonal decomposition

\[ D(X) = \langle A_3^\vee, A_2^\vee, A_1^\vee \rangle, \]

where

\[ A_i^\vee := \{ F \in D(X) \mid F^\vee \in A_i \} \cong D(Z_i), \]

and \( A_i \) are the components of (12). Moreover, this semiorthogonal decomposition is right mutation-dual to (12), i.e., \( A_1^\vee = A_1, A_2^\vee = L_{A_1}(A_2) \) and \( A_3^\vee = L_{A_1}(L_{A_2}(A_3)) \).

Proof. As we discussed in Section 2.3, the functor \( F \mapsto F^\vee \) is an equivalence \( D(X)^{\text{opp}} \rightarrow D(X) \) (since \( X \) is Gorenstein). When applied to the semiorthogonal decomposition of Corollary 3.13, it gives (20) with \( A_i^\vee \cong D(Z_i)^{\text{opp}} \). Since \( Z_1, Z_2 \) and \( Z_3 \) are themselves Gorenstein, we have \( D(Z_i)^{\text{opp}} \cong D(Z_i) \).

Finally, let us prove mutation-duality of (20) and (12). It follows from (14) that \( L_{A_i} \circ \pi_* \cong \pi_* \circ L_{\tilde{A}_i} \) and \( L_{\tilde{A}_i} \circ \pi^* \cong \pi^* \circ L_{A_i} \). So, the required result follows easily from Lemma 3.2.
3.3. Generators of the components. In this section we construct generators of the subcategories $\mathcal{A}_2$ and $\mathcal{A}_3$ of $D(X)$ and describe the equivalences $D(Z_2) \xrightarrow{\sim} \mathcal{A}_2$ and $D(Z_3) \xrightarrow{\sim} \mathcal{A}_3$ of Corollary 3.13 as Fourier–Mukai functors. In what follows for a connected component $Z$ of $Z_i$ we denote by $\gamma_Z$ the embedding functor (constructed in Proposition 3.10)

$$
\gamma_Z : D(Z) \hookrightarrow D(X),
$$

Then we see that $(\mathcal{F}, t) \in \text{coh}(Y, O_Y[t]/t^m)$ considered as a sheaf on $Y \times Z$ is flat only if all the maps in the next chain of epimorphisms

$$(21)\quad \mathcal{F}/(t\mathcal{F}) \xrightarrow{t} (t^2\mathcal{F}) \xrightarrow{t} \ldots \xrightarrow{t} (t^{m-1}\mathcal{F})/(t^m\mathcal{F})$$

are isomorphisms.

**Proof.** The first part follows immediately from an identification $Y \times Z = \text{Spec}_Y(O_Y[t]/t^m)$. To verify flatness, we should compute the derived pullback functors for the embedding $Y \to Y \times Z$ corresponding to the unique closed point of $Z$. Using the standard resolution

$$\ldots \xrightarrow{t} k[t]/t^m \xrightarrow{t} k[t]/t^m \xrightarrow{t} k[t]/t^m \xrightarrow{t} k[t]/t^m \to k \to 0$$

we see that $(\mathcal{F}, t)$ is flat only if the complex

$$\ldots \xrightarrow{t} \mathcal{F} \xrightarrow{t} \mathcal{F} \xrightarrow{t} \mathcal{F} \xrightarrow{t} \mathcal{F} \xrightarrow{t} \ldots$$

is exact. This means that $\text{Ker}(\mathcal{F} \xrightarrow{t} \mathcal{F}) = t^{m-1}\mathcal{F}$ and $\text{Ker}(\mathcal{F} \xrightarrow{t} \mathcal{F}) = t\mathcal{F}$. The second equality implies that the composition of all maps in (21) is an isomorphism, hence so is each of them. On the other hand, if all the maps in (21) are isomorphisms then both equalities above easily follow. This proves the required criterion of flatness.

In the next lemma we describe the images in $D(X)$ of the structure sheaves of points of the schemes $Z_d$ under their embeddings $\gamma_{Z_d}$. We use the convention of Section 2.1 on numbering the blowup centers.

**Lemma 3.15.** Let $Z = \text{Spec}(k[t]/t^m)$. For any scheme $Y$ we have an equivalence

$$\text{coh}(Y \times Z) \cong \text{coh}(Y, O_Y[t]/t^m),$$

where the right hand side is the category of coherent sheaves $\mathcal{F}$ on $Y$ with an operator $t: \mathcal{F} \to \mathcal{F}$ such that $t^m = 0$. An object $(\mathcal{F}, t) \in \text{coh}(Y, O_Y[t]/t^m)$ is flat if and only if all the maps in the next chain of epimorphisms

$$(21)\quad \mathcal{F}/(t\mathcal{F}) \xrightarrow{t} (t^2\mathcal{F}) \xrightarrow{t} \ldots \xrightarrow{t} (t^{m-1}\mathcal{F})/(t^m\mathcal{F})$$

are isomorphisms.

**Proof.** The first part follows immediately from an identification $Y \times Z = \text{Spec}_Y(O_Y[t]/t^m)$. To verify flatness, we should compute the derived pullback functors for the embedding $Y \to Y \times Z$ corresponding to the unique closed point of $Z$. Using the standard resolution

$$\ldots \xrightarrow{t} k[t]/t^m \xrightarrow{t} k[t]/t^m \xrightarrow{t} k[t]/t^m \xrightarrow{t} k[t]/t^m \to k \to 0$$

we see that $(\mathcal{F}, t)$ is flat only if the complex

$$\ldots \xrightarrow{t} \mathcal{F} \xrightarrow{t} \mathcal{F} \xrightarrow{t} \mathcal{F} \xrightarrow{t} \mathcal{F} \xrightarrow{t} \ldots$$

is exact. This means that $\text{Ker}(\mathcal{F} \xrightarrow{t} \mathcal{F}) = t^{m-1}\mathcal{F}$ and $\text{Ker}(\mathcal{F} \xrightarrow{t} \mathcal{F}) = t\mathcal{F}$. The second equality implies that the composition of all maps in (21) is an isomorphism, hence so is each of them. On the other hand, if all the maps in (21) are isomorphisms then both equalities above easily follow. This proves the required criterion of flatness.

In the next lemma we describe the images in $D(X)$ of the structure sheaves of points of the schemes $Z_d$ under their embeddings $\gamma_{Z_d}$. We use the convention of Section 2.1 on numbering the blowup centers.

**Lemma 3.16.** (1) If $z$ is the unique closed $k$-point of $Z_i$ then $\gamma_{Z_1}(O_z) \cong O_X \cong \pi_* O_{\tilde{X}}$.

(2) Let $z \in Z_2$ be a closed $k$-point and let $\mathcal{E}_z := \gamma_{Z_2}(O_z) \in D(X)$ be the corresponding object. Then

(a) if the component of $Z_2$ containing $z$ is reduced and corresponds to the $i$-th blowup center, then

$$\mathcal{E}_z \cong \pi_* O_{\tilde{X}}(h - e_i);$$

(b) if the component of $Z_2$ containing $z$ has length 2 then

$$\mathcal{E}_z \cong \pi_* O_{\tilde{X}}(h - e_1) \cong \pi_* O_{\tilde{X}}(h - e_2);$$

(c) if the component of $Z_2$ containing $z$ has length 3, then

$$\mathcal{E}_z \cong \pi_* O_{\tilde{X}}(h - e_1) \cong \pi_* O_{\tilde{X}}(h - e_2) \cong \pi_* O_{\tilde{X}}(h - e_3).$$
(3) Let \( z \in Z_3 \) be a closed \( k \)-point and let \( \mathcal{E}_z := \gamma_Z(\mathcal{O}_z) \in \mathcal{D}(X) \) be the corresponding object. Then

(a) if the component of \( Z_3 \) containing \( z \) is reduced then
\[
\mathcal{E}_z \cong \pi_\ast \mathcal{O}_{\tilde{X}}(2h - e_1 - e_2 - e_3);
\]

(b) if the component of \( Z_3 \) containing \( z \) has length 2 then
\[
\mathcal{E}_z \cong \pi_\ast \mathcal{O}_{\tilde{X}}(2h - e_1 - e_2 - e_3).
\]

In all these cases, the sheaf \( \mathcal{E}_z \) is globally generated, and if \( z \in Z_d \) then
\[
\dim H^0(X, \mathcal{E}_z) = d \quad \text{and} \quad H^{\neq 0}(X, \mathcal{E}_z) = 0.
\]

Proof. The first part (in all three cases) follows directly from the first diagram in (17), isomorphism \( \pi_{m_\ast}(S_0) \cong \mathcal{E}_z \) proved in Proposition [A.4] and the fact that the simple module \( S_0 \) coincides with the standard exceptional module \( E_0 \) (see (65)), so that by Proposition 3.3 the functor \( \tilde{\gamma} \) takes it to one of the line bundles in (33). Going over the possible cases gives the first isomorphisms. The other isomorphisms of sheaves in case of non-reduced components follow from exact sequences (9), (10) and (11) respectively.

In cases (1), (2a) and (3a) the global generation is easy (the corresponding sheaves are already globally generated on \( \tilde{X} \); extending their evaluation homomorphisms to exact sequences and pushing them forward to \( X \) it is easy to see that their pushforwards are also globally generated). In cases (2b), (2c), and (3b) the same argument shows that \( \pi_\ast \mathcal{O}_{\tilde{X}}(h - e_1), \pi_\ast \mathcal{O}_{\tilde{X}}(h - e_3) \) and \( \pi_\ast \mathcal{O}_{\tilde{X}}(h) \) are globally generated, and for the remaining two sheaves we can use the corresponding isomorphisms.

The cohomology computation reduces to a computation on \( \tilde{X} \) which is straightforward. □

Next we determine the images \( \mathcal{E}_Z := \gamma_Z(\mathcal{O}_Z) \in \mathcal{D}(X) \) of the structure sheaves of connected components of \( Z_d \). Below we identify them as pushforwards of certain vector bundles \( \tilde{\mathcal{E}}_Z \) on \( \tilde{X} \).

First, assume \( Z \subset Z_2 \). If \( Z \) is reduced and corresponds to a line bundle \( \mathcal{O}_{\tilde{X}}(h - e_i) \), we define the bundle \( \tilde{\mathcal{E}}_Z = \mathcal{O}_{\tilde{X}}(h - e_i) \). If \( Z \) has length 2 we define \( \tilde{\mathcal{E}}_Z \) as an extension
\[
0 \to \mathcal{O}_{\tilde{X}}(h - e_2) \to \tilde{\mathcal{E}}_Z \to \mathcal{O}_{\tilde{X}}(h - e_1) \to 0.
\]

If \( Z \) has length 3 we define \( \tilde{\mathcal{E}}_Z \) as an iterated extension
\[
0 \to \mathcal{O}_{\tilde{X}}(h - e_3) \to \tilde{\mathcal{E}}_Z \to 0, \quad \text{where} \quad 0 \to \mathcal{O}_{\tilde{X}}(h - e_2) \to \tilde{\mathcal{E}}_Z \to \mathcal{O}_{\tilde{X}}(h - e_1) \to 0.
\]

Similarly, assume \( Z \subset Z_3 \). If \( Z \) is reduced we define \( \tilde{\mathcal{E}}_Z \) as the corresponding line bundle (either \( \mathcal{O}_{\tilde{X}}(h) \) or \( \mathcal{O}_{\tilde{X}}(2h - e_1 - e_2 - e_3) \)), and if \( Z \) has length 2 we define \( \tilde{\mathcal{E}}_Z \) as an extension
\[
0 \to \mathcal{O}_{\tilde{X}}(2h - e_1 - e_2 - e_3) \to \tilde{\mathcal{E}}_Z \to \mathcal{O}_{\tilde{X}}(h) \to 0.
\]

In all cases \( \tilde{\mathcal{E}}_Z \) is a vector bundle on \( \tilde{X} \) of rank equal to \( \ell(Z) \), the length of the finite scheme \( Z \).

Proposition 3.17. Let \( Z \subset Z_d \) with \( d \in \{2, 3\} \) be a connected component and \( \mathcal{E}_Z := \gamma_Z(\mathcal{O}_Z) \). Then
\[
\mathcal{E}_Z \cong \pi_\ast \tilde{\mathcal{E}}_Z \quad \text{and} \quad \pi_\ast \mathcal{E}_Z \cong \mathcal{E}_Z.
\]

Moreover, \( \mathcal{E}_Z \) is a vector bundle on \( X \) of rank \( \ell(Z) \) and can be represented as an iterated extension of \( \ell(Z) \) copies of the sheaf \( \mathcal{E}_z \), where \( z \) is the closed point of \( Z \). In particular, the bundle \( \mathcal{E}_Z \) is globally generated with \( \dim H^0(X, \mathcal{E}_Z) = d\ell(Z) \) and \( H^{\neq 0}(X, \mathcal{E}_Z) = 0 \).

Proof. Let \( \ell(Z) = m \), so that \( Z \cong \text{Spec}(k[z]/p^m) \), and consider the corresponding diagram (17). By Proposition A.4 we have \( \mathcal{O}_Z \cong \pi_{m_\ast}(P_0) \), where \( P_0 \) is the projective module of \( \tilde{R}_m \), hence
\[
\mathcal{E}_Z = \gamma_Z(\mathcal{O}_Z) \cong \gamma_Z(\pi_{m_\ast}(P_0)) \cong \pi_\ast(\tilde{\gamma}_Z(P_0)).
\]

By [61] the module \( P_0 \) has a filtration with factors being all standard exceptional modules of \( \tilde{R}_m \). By Proposition 3.3 the functor \( \tilde{\gamma}_Z \) sends these exceptional modules to the corresponding line bundles on \( \tilde{X} \).
Therefore, it takes the exact sequences (22), (23) and (24) to the corresponding sequences (64) (in each case there is a unique non-trivial extension), so in the end we conclude that

\[(26) \quad \tilde{e}_Z(P_0) \cong \tilde{E}_Z.\]

Combining this with the above isomorphism, we obtain an isomorphism \(e_Z \cong \pi^* \tilde{E}_Z\) and \(E_Z\) is a vector bundle. For this it is enough to verify that the bundle \(\tilde{E}_Z\) on \(\tilde{X}\) restricts trivially to each \((-2)\)-curve in \(\tilde{X}\). The cases when \(\ell(Z) = 1\) are trivial, so assume that \(\ell(Z) \geq 2\).

First, assume that \(Z = Z_3, \ell(Z) = 2\), so that \(\tilde{E}_Z\) is defined by the exact sequence (24). The restrictions of its first and last terms to the curves \(\Delta_{12}\) and \(\Delta_{23}\) (when these curves exist on \(\tilde{X}\)) are trivial, hence so is \(\tilde{E}_Z\). On the other hand, the sequence (24) restricted to the curve \(\Delta_{123}\) has form

\[0 \to \mathcal{O}_{\Delta_{123}}(-1) \to \tilde{E}_Z|_{\Delta_{123}} \to \mathcal{O}_{\Delta_{123}}(1) \to 0,\]

and it remains to check that this extension is nontrivial. For this we have to check that the restriction map \(H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(h-e_1-e_2-e_3)) \to H^1(\Delta_{123}, \mathcal{O}_{\Delta_{123}}(-2))\) is an isomorphism (the source space is generated by the extension class of (24) and the map corresponds to the restriction of extensions). This, however, follows immediately from the cohomology sequence of (11) twisted by \(\mathcal{O}_{\tilde{X}}(h)\).

The other cases are treated similarly. In all cases \(\tilde{E}_Z\) is an iterated extension of line bundles on \(\tilde{X}\) such that either all of them restrict trivially to the \((-2)\)-curve \(\Delta\) in question, or there are two of them that restrict as \(\mathcal{O}_{\Delta}(1)\) and \(\mathcal{O}_{\Delta}(-1)\), and we only have to check that the corresponding extension is nontrivial. Again, the question reduces to the restriction map on the first cohomology, which can be described by an appropriate twist of one of the sequences (9) or (10).

Finally, \(E_Z\) is a vector bundle since its pullback to \(X\) is; and to show that \(E_Z\) is globally generated, just note that pushing forward to \(X\) exact sequences (22), (23) and (24) we represent these are globally generated vector bundles on \(X\) of ranks 1, 3, and 2 respectively. By Proposition 3.17 these are globally generated vector bundles on \(X\) of ranks 1, 3, and 2 respectively. We denote by \(p_X: X \times Z_d \to X\) the natural projection.

**Proposition 3.18.** There is a sheaf

\[E_d \in \text{coh}(X \times Z_d),\]

flat over \(Z_d\), such that \(p_{X,*} E_d \cong E_{Z_d}\), and the equivalence \(\gamma_{Z_d}: D(Z_d) \to A_d \subset D(X)\) of Corollary 3.13 can be written as

\[\gamma_{Z_d} \cong \Phi_{E_d}: D(Z_d) \to D(X),\]

where \(\Phi_{E_d}\) is the Fourier–Mukai functor with kernel \(E_d\). Moreover, the equivalence \(D(Z_d) \cong A_d^\vee \subset D(X)\) of Proposition 3.14 is given by the Fourier–Mukai functor \(\Phi_{E_d}^\vee\) with kernel \(E_d^\vee\).

**Proof.** Let \(Z \subset Z_d\) be a connected component. We apply Lemma 3.15 to construct a sheaf \(E_Z\) on \(X \times Z\) such that \(E_Z \cong p_{X,*} E_d\). For this, first, note that by (26) the sheaf \(E_Z\) has a natural action of the ring \(\text{End}(P_0) \cong k[t]/t^m\). Clearly, the filtration on \(E_Z\), associated with this action by Lemma 3.15 is the one, coming from the defining exact sequences (22), (23) and (24) of \(E_Z\), and the morphisms between the quotients \((t^i E_Z)/(t^{i+1} E_Z)\) of this filtration induced by \(t\) are the morphisms from exact sequences (9),
where $z$ is the closed point of $Z$, hence the epimorphisms between these quotients induced by $t$ are isomorphisms. Therefore, $E$ is flat over $Z$ by Lemma 3.15.

Next, we define $E_d$ as the sum $E_d := \bigoplus E$ over connected components $Z \subset Z_d$. Then the first part of the proposition holds. It remains to show that the functors $\gamma_d$ are Fourier–Mukai.

The functor $\gamma_d : D(R_1) \to D(X)$ by construction (which is given in Proposition 3.3 and is based on Proposition 3.20) is a Fourier–Mukai functor. By (10), the functor $\gamma_d : D(Z) \to D(X)$ is also a Fourier–Mukai functor. Since the scheme $Z$ is affine, the kernel object defining the functor $\gamma_d$ can be identified with the sheaf $\gamma_d(\mathcal{O}_Z) \cong E \in D(X)$ with its natural module structure over $k[Z] = k[t]/t^m$, which corresponds to the sheaf $E_d$ by its definition above.

The last claim follows from this by dualization (note that Grothendieck duality implies that the equivalence of Lemma 3.15 is compatible with dualization, since the dualizing complex of $Z$ is trivial). \hfill □

**Corollary 3.19.** The components $A_2$ and $A_3$ of $D(X)$ are compactly generated by the bundles $E_{Z_2}$ and $E_{Z_3}$ respectively. In particular, we have $A_2 = \mathcal{O}_X \cap E_{Z_2}$ and $A_3 = \mathcal{O}_X \cap E_{Z_3}$.

**Proof.** Indeed, the derived category of an affine scheme $Z_d$ is compactly generated by its structure sheaf $\mathcal{O}_{Z_d}$. Therefore, the component $A_d$ of $D(X)$ is compactly generated by the bundle $E_{Z_d} \cong \gamma_{Z_d}(\mathcal{O}_{Z_d})$. For the second claim use (12). \hfill □

As we will see in the next section, for verification of the orthogonality the following numerical result is useful. Denote by $\chi(\mathcal{F}, \mathcal{G})$ the alternating sum of the dimensions of the spaces $\text{Ext}^i(\mathcal{F}, \mathcal{G})$ (when this sum is finite, e.g., when one of the arguments is perfect), and by $r(\mathcal{F})$ the alternating sum of the ranks of the cohomology sheaves of $\mathcal{F}$. Note that $r(\mathcal{F}) = \chi(\mathcal{O}_P, \mathcal{F})$ for a smooth point $P$ of $X$.

**Lemma 3.20.** For any $\mathcal{F} \in D(X)$ we have
\[
\chi(E_{Z_2}, \mathcal{F}) = 2\chi(\mathcal{O}_X, \mathcal{F}) + \chi(\mathcal{O}_X, \mathcal{F}(K_X)) - 3r(\mathcal{F}),
\]
\[
\chi(E_{Z_3}, \mathcal{F}) = \chi(\mathcal{O}_X, \mathcal{F}) + \chi(\mathcal{O}_X, \mathcal{F}(K_X)) - 2r(\mathcal{F}).
\]

**Proof.** By Corollary 2.5 we can write $\mathcal{F} = \pi_* \tilde{\mathcal{F}}$ for some $\tilde{\mathcal{F}} \in D(\tilde{X})$, hence by adjunction
\[
\chi(E_{Z_2}, \mathcal{F}) = \chi(E_{Z_2}, \pi_* \tilde{\mathcal{F}}) = \chi(\pi^* E_{Z_2}, \tilde{\mathcal{F}}) = \chi(\mathcal{F}_{Z_2}, \mathcal{F}).
\]
By definition, $E_{Z_2}$ is an extension (possibly trivial) of line bundles $\mathcal{O}_X(h - e_i)$ for $i = 1, 2, 3$, hence
\[
\chi(E_{Z_2}) = \sum \chi(\mathcal{O}_X(h - e_i)) = 3 + (3h - e_1 - e_2 - e_3) = 3 - K_X = 2\chi(\mathcal{O}_X) + \chi(\mathcal{O}_X, \mathcal{F}) - 3\chi(\mathcal{O}_P, \mathcal{F}),
\]
where $\tilde{P}$ is a general point on $\tilde{X}$. Therefore, by Riemann–Roch and adjunction we have
\[
\chi(\mathcal{F}_{Z_2}, \mathcal{F}) = 2\chi(\mathcal{O}_X, \mathcal{F}) + \chi(\mathcal{O}_X, \mathcal{F}(K_X)) - 3\chi(\mathcal{O}_P, \mathcal{F}) = 2\chi(\mathcal{O}_X, \mathcal{F}) + \chi(\mathcal{O}_X, \mathcal{F}) - 3\chi(\mathcal{O}_P, \mathcal{F}),
\]
which gives the first equality. Similarly, for $E_{Z_3}$ we have
\[
\chi(E_{Z_3}) = \chi(\mathcal{O}_X(h)) + \chi(\mathcal{O}_X(2h - e_1 - e_2 - e_3)) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_P),
\]
and using Riemann–Roch and adjunction in the same way as before, we finish the proof. \hfill □

Denote by $p_2 : X \times Z_2 \to Z_3$, $p_3 : X \times Z_3 \to Z_3$, and $p_{23} : X \times Z_2 \times Z_3 \to Z_2 \times Z_3$ the projections. Similarly, consider the projections $p_{X_2} : X \times Z_2 \times Z_3 \to X \times Z_2$ and $p_{X_3} : X \times Z_2 \times Z_3 \to X \times Z_3$. All this structure, of course, is inherited by the vector bundle $E_{Z} \cong \pi_* (\mathcal{E}_Z)$, hence $E_{Z}$ can be considered as a sheaf on $X \times Z$. Moreover, the pushforwards to $X$ of (9), (10) and (11) show that for all $0 \leq i \leq m - 1$ we have
\[
\left(\pi^* E_{Z} \right) / \left(\pi^* E_{Z} \right) \cong E_{Z},
\]
where $z$ is the closed point of $Z$, hence the epimorphisms between these quotients induced by $t$ are isomorphisms. Therefore, $E_{Z}$ is flat over $Z$ by Lemma 3.15.
Lemma 3.21. The pushforwards $p_{Z_2}^*E_{Z_2}$ and $p_{Z_3}^*E_{Z_3}$ are vector bundles on $Z_2$ and $Z_3$ of rank 2 and 3 respectively. Similarly, the pushforward $p_{Z_2}^*(p_{X_2}^*E_{X_2}^\vee \otimes p_{X_3}^*E_{Z_3})$ is a line bundle on $Z_2 \times Z_3$.

Proof. Let $Z \cong \text{Spec}(k[t]/t^m)$ be a connected component of $Z_d$ with $d \in \{2, 3\}$. Consider the commutative square

$$
\begin{array}{ccc}
X \times Z & & Z \\
\downarrow p_Z & & \downarrow Z \\
X & & \text{Spec}(k)
\end{array}
$$

The pushforward to Spec$(k)$ of $p_Z^*E_Z$ equals the pushforward of $p_{X_2}^*(E_Z) \cong E_Z$, i.e. $H^*(X, E_Z)$ with its natural $k[t]/t^m$-module structure. So, by Lemma 3.15 it is enough to show that the natural epimorphisms between the quotient spaces $(t^iH^0(X, E_Z))/(t^{i+1}H^0(X, E_Z))$ are isomorphisms (the other cohomology groups of $E_Z$ vanish by Proposition 3.17). By (28) all these quotients are isomorphic to $H^0(X, E_Z)$, hence $d$-dimensional by Lemma 3.16 hence the epimorphisms $t$ between them are isomorphisms, hence $p_{Z_i}^*E_Z$ is flat over $Z_i$, i.e., locally free. The above argument shows that its rank equals $d$.

For the second statement, let $Z \subset Z_2$ and $Z' \subset Z_3$ be connected components, $Z \cong \text{Spec}(k[t]/t^m)$, $Z' \cong \text{Spec}(k[t']/t'^m)$, with closed points $z$ and $z'$ respectively. The sheaf $p_{Z_2}^*(p_{X_2}^*E_{Z_2}^\vee \otimes p_{X_3}^*E_{Z_3})$ corresponds to the vector space $H^*(X, E_{Z_2}^\vee \otimes E_{Z_3})$ with its natural bifiltration under an analogue of Lemma 3.15. So, we have to check that the operators $t$ and $t'$ induce isomorphisms between the quotients of this bifiltration. Since

$$H^*(X, E_{Z_2}^\vee \otimes E_{Z_3}) \cong \text{Ext}^*(E_{Z_2}, E_{Z_3}) \cong \text{Ext}^*(E_{Z_2}, \pi^*E_{Z_3}) \cong \text{Ext}^*(\pi^*E_{Z_2}, \pi^*E_{Z_3}),$$

and the bifiltration is induced by the defining exact sequences (22), (23) and (24) of $E_Z$ and $E_{Z'}$, it is enough to compute Ext-spaces between the corresponding line bundles on $\tilde X$. A direct computation gives

$$\text{Ext}^*(\mathcal O_{\tilde X}(h-e_1), \mathcal O_{\tilde X}(h)) = H^*(\tilde X, \mathcal O_{\tilde X}(e_i)) = k,$$

$$\text{Ext}^*(\mathcal O_{\tilde X}(h-e_1), \mathcal O_{\tilde X}(2h-e_1-e_2-e_3)) = H^*(\tilde X, \mathcal O_{\tilde X}(h-e_j-e_k)) = k,$$

therefore, the epimorphisms $t$ and $t'$ between them are isomorphisms, and the sheaf $p_{Z_2}^*(p_{X_2}^*E_{Z_2}^\vee \otimes p_{X_3}^*E_{Z_3})$ is locally free of rank 1 on $Z \times Z'$.

Summing up over all connected components completes the proof of the lemma. \qed

Remark 3.22. The above lemma can be interpreted as a computation of the gluing bimodules (cf. [KL15 Section 2.2]) between the components of (12). It says that $D(X)$ is the gluing of $D(k)$, $D(Z_2)$, and $D(Z_3)$ with the gluing bimodules being $k[Z_2]^{\oplus 2}$, $k[Z_2 \times Z_3]$, and $k[Z_3]^{\oplus 3}$.

4. Moduli spaces interpretation

In this section we provide a modular interpretation for the finite length schemes $Z_2$ and $Z_3$ that appeared in the semiorthogonal decomposition (19) of $D(X)$ and for the Fourier–Mukai kernels $E_{Z_2}$ and $E_{Z_3}$ of Proposition 3.18. This interpretation is essential for the description of the derived category of a family of sextic del Pezzo surfaces in Section 5.

All through this section $X$ is a sextic du Val del Pezzo surface (as defined in Definition 2.1) over an algebraically closed field $k$, and we use freely the notation introduced in Section 2.1.

4.1. Moduli of rank 1 sheaves. For a sheaf $\mathcal F$ on $X$ we denote by

$$h_{\mathcal F}(t) := \chi(\mathcal F(-tK_X)) \in \mathbb Z[t],$$

the Hilbert polynomial of $\mathcal{F}$ with respect to the anticanonical polarization of $X$. This is a quadratic polynomial with the leading coefficient equal to $r(\mathcal{F}) \cdot K_X^2 / 2 = 3r(\mathcal{F})$. Note that
\begin{equation}
 h_{\mathcal{E}_X}(t) = 3t(t + 1) + 1. \tag{29}
\end{equation}

For each $d \in \mathbb{Z}$ we consider a polynomial
\begin{equation}
 h_d(t) := (3t + d)(t + 1) \in \mathbb{Z}[t]. \tag{30}
\end{equation}

Note that
\begin{equation}
 h_4(t) > h_3(t) > h_2(t) > h_{\mathcal{E}_X}(t) > h_4(t - 1) > h_3(t - 1) > h_2(t - 1) \quad \text{for all } t \gg 0. \tag{31}
\end{equation}
Indeed, the leading monomial in all the cases is $3t^2$, and the next one is $7t$, $6t$, $5t$, $3t$, $t$, $0t$, and $-t$ respectively.

Below we will be interested in semistable sheaves on $X$ with Hilbert polynomial $h_d(t)$.

**Lemma 4.1.** A sheaf $\mathcal{F}$ on $X$ with Hilbert polynomial $h_d(t)$ is Gieseker semistable if and only if it is Gieseker stable and if and only if it is torsion-free.

**Proof.** As we observed above, the leading monomial of $h_d(\mathcal{F})$ being $3t^2$ means that $r(\mathcal{F}) = 1$. Therefore, it could (and will) be destabilized only by a subsheaf of rank 0, i.e., by a torsion sheaf. Thus, $\mathcal{F}$ is (semi)stable if and only if it is torsion-free. \hfill $\square$

Recall the sheaves $\mathcal{E}_z$ and $\mathcal{E}_{Z_d}$ on $X$ introduced in Lemma 3.16 and in (27) respectively.

**Lemma 4.2.** Let $d \in \{2, 3\}$. For any closed point $z \in Z_d$ the sheaf $\mathcal{E}_z$ is stable with $h_{\mathcal{E}_z}(t) = h_d(t)$. Its derived dual $\mathcal{E}_z^\vee$ is a stable sheaf with $h_{\mathcal{E}_z^\vee}(t) = h_d(-t - 1) = h_{\mathcal{E}_z}(t - 1)$. Furthermore, the sheaf $\mathcal{E}_{Z_d}$ is semistable with $h_{\mathcal{E}_{Z_d}} = \ell(Z_d)h_d$.

**Proof.** As we already mentioned, $r(\mathcal{E}_z) = 1$ implies the leading monomial of $h_{\mathcal{E}_z}$ equals $3t^2$. So, to show an equality of polynomials $h_{\mathcal{E}_z} = h_d$ it is enough to check that they take the same values at points $t = 0$ and $t = -1$. In other words, we have to check that
\begin{equation}
 \chi(X, \mathcal{E}_z) = h_d(0) = d \quad \text{and} \quad \chi(X, \mathcal{E}_z \otimes \omega_X) = h_d(-1) = 0.
\end{equation}

The first is proved in Lemma 3.16 and the second follows from $\mathcal{E}_z \in \mathcal{A}_d$ and decompositions (13).

The sheaf $\mathcal{E}_z$ is torsion free since it is the direct image of a line bundle under a dominant map $\pi$, hence is stable by Lemma 4.1 and it follows from Proposition 3.17 that $\mathcal{E}_{Z_d}$ is semistable with the same reduced Hilbert polynomial.

To show that $\mathcal{E}_z^\vee$ is a sheaf, note that $\mathcal{E}_z$ by definition is the pushforward of a line bundle from $\bar{X}$. Denoting this line bundle by $\mathcal{L}$ and using the fact that $\pi$ is crepant, we obtain
\begin{equation}
 \mathcal{E}_z^\vee \cong R\mathcal{H}om(\pi_*\mathcal{L}, \mathcal{O}_X) \cong \pi_* R\mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \cong \pi_* (\mathcal{L}^\vee).
\end{equation}

It remains to show that $R^1\pi_* (\mathcal{L}^\vee) = 0$. For this just note that the line bundle $\mathcal{L}$ restricts to each of the curves $\Delta \in \Delta$ as $\mathcal{O}_\Delta$ or $\mathcal{O}_\Delta(\pm 1)$. Consequently, the same is true for the dual bundle $\mathcal{L}^\vee$, and since we have $H^1(\Delta, \mathcal{L}^\vee|_\Delta) = 0$, it follows that $R^1\pi_* (\mathcal{L}^\vee) = 0$.

Thus the derived dual $\mathcal{E}_z^\vee$ is a sheaf. Its stability is proved in the same way as that of $\mathcal{E}_z$; and the fact that its Hilbert polynomial equals $h_d(-t - 1)$ follows easily from Serre duality (Proposition 2.6). \hfill $\square$

Denote by
\begin{equation}
 \mathcal{M}_d(X) := \mathcal{M}_{X, -K_X}(h_d) \tag{32}
\end{equation}
the moduli space of Gieseker semistable sheaves on $X$ with Hilbert polynomial $h_d(t)$ (with respect to the anticanonical polarization of $X$). We aim at description of these moduli spaces for $d \in \{2, 3, 4\}$.

We start by describing their closed points.
Lemma 4.4. Let $\mathcal{F}$ be a torsion free sheaf on $X$ whose Hilbert polynomial is $h_d(t)$ with $d \in \{2, 3, 4\}$.  

(i) If $d \in \{2, 3\}$ there is a unique closed point $z \in Z_d$ such that $\mathcal{F} \cong \mathcal{E}_z$.  
(ii) If $d \in \{3, 4\}$ there is a unique closed point $z \in Z_{6-d}$ such that $\mathcal{F} \cong \mathcal{E}_z^\vee \otimes \omega_X^{-1}$.

Proof. First, assume $d = 2$. Let us show that $\mathcal{F} \in \mathcal{A}_2$. By Corollary 3.19 for this we should check that $\text{Ext}^* (\mathcal{F}, \mathcal{O}_X) = \text{Ext}^* (\mathcal{E}_Z, \mathcal{F}) = 0$. By Lemma 4.1 the sheaf $\mathcal{F}$ is stable. By Lemma 4.2 and (31), we have $\frac{1}{h(\mathcal{F})} h_{\mathcal{E}_Z}(t) > h_\mathcal{F}(t) > h_{\mathcal{O}_X}(t)$, hence by semistability

$$\text{Hom}(\mathcal{E}_Z, \mathcal{F}) = \text{Hom}(\mathcal{F}, \mathcal{O}_X) = 0.$$  

Similarly, $\frac{1}{h(\mathcal{F})} h_{\mathcal{E}_Z}(t - 1) < h_{\mathcal{F}}(t) < h_{\mathcal{O}_X}(t + 1)$, hence

$$\text{Hom}(\mathcal{F}, \mathcal{E}_Z(K_X)) = \text{Hom}(\mathcal{O}_X(-K_X), \mathcal{F}) = 0.$$  

By Serre duality on $X$ (Proposition 2.6) we deduce

$$\text{Ext}^2(\mathcal{E}_Z, \mathcal{F}) = \text{Ext}^2(\mathcal{F}, \mathcal{O}_X) = 0.$$  

Since (again by Serre duality and local freeness of $\mathcal{O}_X$ and $\mathcal{E}_Z$) we can have nontrivial $\text{Ext}^p$ only for $p \in \{0, 1, 2\}$, it remains to check that

$$\chi(\mathcal{E}_Z, \mathcal{F}) = \chi(X, \mathcal{F}(K_X)) = 0.$$  

The second equality here just follows from $h_{\mathcal{F}}(-1) = h_2(-1) = 0$, and the first follows from Lemma 3.20 and $h_2(0) + h_2(-1) - 2 = 0$.

Thus, we have shown that $\mathcal{F} \in \mathcal{A}_2$. Since $\mathcal{A}_2 \cong \mathcal{D}(Z_2)$ and $Z_2$ is a zero-dimensional scheme, any object in $\mathcal{D}(Z_2)$ is isomorphic to an iterated extension of shifts of structure sheaves of closed points in $Z_2$. By Lemma 3.16 these sheaves correspond to sheaves $\mathcal{E}_z \in \mathcal{A}_2$, hence $\mathcal{F}$ is an extension of shifts of those. Since $\mathcal{F}$ is a pure sheaf of rank equal to 1 (the latter follows from its Hilbert polynomial), we conclude that $\mathcal{F} \cong \mathcal{E}_z$ for a closed point $z \in Z_2$.

The case $d = 3$ is treated similarly. We first check that any sheaf $\mathcal{F}$ with $h_{\mathcal{F}} = h_3$ is an object of $\mathcal{A}_3$. By using the characterization of Corollary 3.19 this is equivalent to $\text{Ext}^* (\mathcal{F}, \mathcal{O}_X) = \text{Ext}^* (\mathcal{F}, \mathcal{E}_Z) = 0$. From semistability and inequalities (31) of Hilbert polynomials, we conclude that Hom spaces vanish. Using Serre duality we then deduce that also $\text{Ext}^2$ spaces vanish and the only possibly nontrivial Ext-spaces are spaces $\text{Ext}^1$. Then we use Lemma 3.20 to show that these spaces also vanish, and conclude that $\mathcal{F} \cong \mathcal{E}_z$ for a closed point $z \in Z_3$. This proves part (i).

The same argument with $\mathcal{A}_2$ and $\mathcal{A}_3$ replaced by $\mathcal{A}_2^\vee \otimes \omega_X^{-1}$ and $\mathcal{A}_3^\vee \otimes \omega_X^{-1}$ and (12) replaced by a twist of (20) proves part (ii).

Below we consider families of objects parameterized by a scheme $S$. Let $p_S: M \to S$ be a morphism of schemes. For each geometric point $s \in S$ we denote by $M_s$ the scheme fiber of $M$ over $s$ and by $i_{M_s}: M_s \to M$ its embedding. The embedding $s \mapsto S$ is denoted simply by $i_s$.

Lemma 4.4. Assume $\mathcal{F} \in \mathcal{D}^-(M)$ and $M$ is proper over $S$, and in (ii) and (iii) that $M$ is flat over $S$.

(i) If $i_{M_s}^*(\mathcal{F}) \in \mathcal{D}^{\leq p_0}(M_s)$ for some integer $p_0$ and all geometric points $s \in S$, then $\mathcal{F} \in \mathcal{D}^{\leq p_0}(M)$. In particular, if $i_{M_s}^*(\mathcal{F}) = 0$ for all geometric points $s \in S$, then $\mathcal{F} = 0$.

(ii) If for all geometric points $s \in S$ the pullback $i_{M_s}^*(\mathcal{F})$ is a sheaf such that $H^{\neq 0}(M_s, i_{M_s}^* \mathcal{F}) = 0$ and $\dim H^0(M_s, i_{M_s}^* \mathcal{F}) = r$ for a constant $r$, then $p_{S*}(\mathcal{F})$ is a vector bundle of rank $r$ on $S$.

(iii) If for any geometric point $s \in S$ there is a point $m \in M_s$ such that $i_{M_s}^*(\mathcal{F}) \cong \mathcal{O}_m$, then there is a unique section $\varphi: S \to M$ of the projection $p_S$ and a line bundle $\mathcal{L} \in \text{Pic}(S)$ such that $\mathcal{F} \cong \varphi^* \mathcal{L}$. 

□
implies that \( H^q(M_s, \mathcal{F}) \neq 0 \), hence \( k \leq p_0 \).

(ii) Denote \( \mathcal{G} := p_{S*}(\mathcal{F}) \). By base change we have an isomorphism

\[
H^*(M_s, i^*_M(\mathcal{F})) \cong i^*_s(\mathcal{G}).
\]

By assumption, the left hand side is a vector space of dimension \( r \) in degree 0. Applying part (i) of the lemma to \( \mathcal{G} \) we conclude that \( \mathcal{G} \in \mathbb{D}^{-0}(S) \). Moreover the spectral sequence of part (i) with \( M = S \)

\[
L_0 i^*_M \mathcal{H}^p(\mathcal{F}) \Rightarrow \mathcal{H}^{p-q}(i^*_M \mathcal{F})
\]

shows that \( L_0 i^*_M \mathcal{H}^0(\mathcal{G}) \cong H^0(M, i^*_M \mathcal{F}) \) and \( L_1 i^*_M \mathcal{H}^0(\mathcal{G}) = 0 \). Hence by Serre’s criterion, we conclude that \( \mathcal{H}^0(\mathcal{G}) \) is a vector bundle of rank \( r \). Looking again at the spectral sequence, we see that the canonical map \( \mathcal{G} \to \mathcal{H}^0(\mathcal{G}) \) induces an isomorphism \( i^*_s \mathcal{G} \to i^*_s(\mathcal{H}^0(\mathcal{G})) \) for each \( s \in S \), hence by part (i) we have an isomorphism \( \mathcal{G} \cong \mathcal{H}^0(\mathcal{G}) \). Thus \( p_{S*}(\mathcal{F}) = \mathcal{G} \) is a vector bundle of rank \( r \).

(iii) By part (i) we have \( \mathcal{F} \in \mathbb{D}^{-0}(M) \) and by part (ii) we know that \( p_{S*}(\mathcal{F}) \) is a line bundle. Denote it by \( L \). Then replacing the object \( \mathcal{F} \) by \( \mathcal{F} \otimes p_S^* L^{-1} \), we may assume that \( p_{S*}(\mathcal{F}) \cong \mathcal{O}_S \). We have by adjunction a map \( \mathcal{O}_M = p_S^* \mathcal{O}_S \to \mathcal{F} \to \mathcal{H}^0(\mathcal{F}) \) that restricts to the fiber \( M_s \) over a geometric point \( s \in S \) as the natural map \( \mathcal{O}_{M_s} \to \mathcal{O}_m \). Therefore, it is fiberwise surjective, hence by part (i) applied to its cone it is surjective on \( M \). So, \( \mathcal{H}^0(\mathcal{F}) \cong \mathcal{O}_S \) is the structure sheaf of a closed subscheme \( \Gamma \subset M \).

By the assumption for each geometric point \( s \in S \) we have \( i^*_s \mathcal{O}_\Gamma \cong \mathcal{O}_m \), where \( m \) is the point of \( M_s \) corresponding to the geometric point \( s \in S \). Therefore the map \( p_S|_{\Gamma} : \Gamma \to S \) is finite and flat of degree 1. By (ii) we have \( p_{S*} \mathcal{O}_\Gamma \cong \mathcal{O}_S \), so it follows that the map \( p_S|_{\Gamma} : \Gamma \to S \) is an isomorphism. Therefore, \( \Gamma \) is the image of a section \( \varphi : S \to M \) of the morphism \( p_S \).

The above argument proves \( \mathcal{H}^0(\mathcal{F}) \cong \mathcal{O}_\Gamma \cong \varphi_* \mathcal{O}_S \). Restricting the triangle \( \tau_{\leq -1} \mathcal{F} \to \mathcal{F} \to \mathcal{H}^0(\mathcal{F}) \) to an arbitrary fiber \( M_s \) we deduce that \( i^*_M(\tau_{\leq -1} \mathcal{F}) = 0 \) for any \( s \in S \), hence \( \tau_{\leq -1} \mathcal{F} = 0 \) by part (i). Therefore \( \mathcal{F} \cong \varphi_* \mathcal{O}_S \).

Now we return to the sextic del Pezzo surface \( X \) and the moduli spaces \( \mathcal{M}_d(X) \) defined by \( \text{(52)} \). Recall the sheaves \( \mathcal{E}_{Z_d} \) on \( X \times Z_d \) constructed in Proposition \( \text{3.18} \).

**Theorem 4.5.** Let \( X \) be a sextic du Val del Pezzo surface over an algebraically closed field \( k \). The moduli spaces \( \mathcal{M}_2(X), \mathcal{M}_3(X), \text{ and } \mathcal{M}_4(X) \) are fine moduli spaces. Moreover,

(i) \( \mathcal{M}_2(X) \cong \mathcal{M}_4(X) \cong \mathcal{Z}_2 \) and the sheaves \( \mathcal{E}_{Z_2} \) and \( \mathcal{E}_{Z_2}^X \otimes \omega_X^{-1} \) are the corresponding universal families;

(ii) \( \mathcal{M}_3(X) \cong \mathcal{Z}_3 \) and the sheaves \( \mathcal{E}_{Z_3} \) and \( \mathcal{E}_{Z_3}^X \otimes \omega_X^{-1} \) are two universal families for this moduli problem.

**Proof.** Let \( S \) be an arbitrary base scheme and \( \mathcal{F} \) a coherent sheaf on \( X \times S \) flat over \( S \) and such that for any geometric point \( s \in S \) the restriction \( \mathcal{F}_s = i^*_s(\mathcal{F}) \) is a semistable (i.e., torsion free) sheaf with Hilbert polynomial \( h_d(t) \) for \( d \in \{2, 3\} \). By Lemma \( \text{4.3} \) it follows that

\[
\mathcal{F}_s \cong \mathcal{E}_s
\]

for a unique closed point \( z \in Z_d \).

Consider the semiorthogonal decomposition

\[
\text{(33)} \quad \mathbf{D}(X \times S) = (\mathcal{A}_{1S}, \mathcal{A}_{2S}, \mathcal{A}_{3S}) = (\mathbf{D}(S), \mathbf{D}(Z_2 \times S), \mathbf{D}(Z_3 \times S)),
\]

obtained by base change (\cite[Theorem 5.6]{Kuz11}) from \( \text{(19)} \) (the second equality follows from \cite[Theorem 6.4]{Kuz11} and Proposition \( \text{3.18} \)). Since by Proposition \( \text{3.18} \) the embedding functors of \( \text{(19)} \) are
the Fourier–Mukai functors given by the sheaves $\mathcal{E}_{Z_d}$, the embedding functors of $\mathcal{E}_{Z_d}$ are given by the pullbacks $\mathcal{E}_{Z_d} \boxtimes \mathcal{O}_S$ of $\mathcal{E}_{Z_d}$ via the maps $X \times Z_d \times S \to X \times Z_d$.

For any $\mathcal{F} \in D(X \times S)$, a geometric point $s \in S$, and each $d' \in \{1, 2, 3\}$ we have by [Kuz11, (11)] an isomorphism

$$i_{Z_d \times S}'(\alpha_{d'}\mathcal{F})(z) \cong \alpha_{d'}(i_{X \times S}'(\mathcal{F}))(z) \in D(Z_d),$$

where $\alpha_{d'} : D(X) \to D(Z_d)$ and $\alpha_{d'} : D(X \times S) \to D(Z_d \times S)$ are the projection functors of the semiorthogonal decompositions $\mathcal{O}_S$ and $\mathcal{O}_S$ respectively. Using this isomorphism for the sheaf $\mathcal{F}$ we started with, and taking into account Lemma 3.16, we conclude that

$$i_{Z_d \times S}'(\alpha_{d'}\mathcal{F}) \cong \alpha_{d'}(\mathcal{F}) \cong \begin{cases} \mathcal{O}_S, & \text{if } d' = d, \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 4.3(i) it follows that $\alpha_{d'}\mathcal{F} = 0$ when $d' \neq d$, hence

$$\mathcal{F} \cong \Phi_{\mathcal{E}_{Z_d} \boxtimes \mathcal{O}_S}(\mathcal{F}_d)$$

for some object $\mathcal{F}_d \in D(Z_d \times S)$. Moreover, the object $\mathcal{F}_d$ is such that for any geometric point $s \in S$ we have $i_{Z_d \times S}'(\mathcal{F}_d) \cong \mathcal{O}_S$ for some closed point $z \in Z_d$. By Lemma 4.3(iii) it follows that $\mathcal{F}_d \cong \Gamma_{f_d}^*\mathcal{L}$, where $f_d : S \to Z_d$ is a morphism, $\Gamma_{f_d} : S \to Z_d \times S$ is its graph, and $\mathcal{L}$ is a line bundle on $S$. Therefore

$$\mathcal{F} \cong \Phi_{\mathcal{E}_{Z_d} \boxtimes \mathcal{O}_S}(\mathcal{L}) \cong (\Gamma_{f_d}^*\mathcal{E}_{Z_d} \boxtimes \mathcal{L}).$$

This precisely means that the moduli functor $\mathcal{M}_3(X)$ we are interested in is represented by the scheme $Z_d$, and that $\mathcal{E}_{Z_d}$ is the corresponding universal sheaf.

The same argument applied to the dual semiorthogonal decomposition $\mathcal{O}_S$ instead of $\mathcal{O}_S$ gives the other two statements.

**Corollary 4.6.** There is an automorphism $\sigma : \mathcal{M}_3(X) \to \mathcal{M}_3(X)$ such that $\mathcal{E}_{Z_3}^\vee \cong \sigma^*\mathcal{E}_{Z_3} \boxtimes \omega_X$.

**Proof.** Both $\mathcal{E}_{Z_3}$ and $\mathcal{E}_{Z_3}^\vee \boxtimes \omega_X^{-1}$ are universal families on $\mathcal{M}_3(X) \times X$, hence they differ only by an automorphism of $\mathcal{M}_3(X)$ and a twist by a line bundle on it. But since $\mathcal{M}_3(X)$ is zero-dimensional, it has no non-trivial line bundles.

4.2. Hilbert scheme interpretation. Now consider the polynomial

$$h_d(t) = dt + 1 \in \mathbb{Z}[t].$$

Clearly, this is the Hilbert polynomial of a rational normal curve of degree $d$.

**Lemma 4.7.** Let $C \subset X \subset \mathbb{P}^6$ be a subscheme with Hilbert polynomial $h_C(t) = h_d(t)$ and $1 \leq d \leq 3$. Then $C$ is a connected arithmetically Cohen–Macaulay curve.

**Proof.** In cases $d = 1$ and $d = 2$ this is standard (see, e.g., [KPS16, Lemma 2.1.1]). In case $d = 3$ the only other possibility for $C$ (see [LLSV15, §1] and references therein) would be a union of a plane cubic curve with a point (possibly embedded). But $X$ is an intersection of quadrics by [HWS1] Theorem 4.4 and contains no planes, so this is impossible.

**Lemma 4.8.** (i) If $C \subset X$ is a curve with Hilbert polynomial $h_d(t)$ with $d \in \{2, 3\}$ then there is a unique closed point $z \in Z_d$ and an exact sequence

$$0 \to \mathcal{E}_z^\vee \to \mathcal{O}_X \to \mathcal{O}_C \to 0.$$  

(ii) Similarly, if $L \subset X$ is a line, i.e., a curve with Hilbert polynomial $h_1(t)$ then there are unique closed points $z_2 \in Z_2$ and $z_3 \in Z_3$ and an exact sequence

$$0 \to \mathcal{E}_{z_2} \to \mathcal{E}_{z_3} \to \mathcal{O}_L \to 0.$$
Conversely, any nonzero morphism $\mathcal{E}^\vee \to \mathcal{O}_X$ is injective with cokernel isomorphic to $\mathcal{O}_C$, and any nonzero morphism $\mathcal{E}_2 \to \mathcal{E}_3$ is injective with cokernel isomorphic to $\mathcal{O}_L$.

**Proof.** For (i), by Lemma 4.8(iii) it is enough to show that $I_C \otimes \omega_X^{-1}$ is stable with Hilbert polynomial $h_{n-d}(t)$. Stability is clear by Lemma 4.1 and the Hilbert polynomial evidently equals

$$h_{\mathcal{O}_X}(t+1) - h_d(t+1) = (3(t+1)(t+2) + 1) - (d(t+1) + 1) = (3t + 6 - d)(t+1) = h_{n-d}(t).$$

For (ii) first note that by Serre duality $\text{Ext}^i(\mathcal{O}_L, \mathcal{O}_X) \cong \text{Ext}^i(\mathcal{O}_L(K_X)^\vee = H^i(L, \mathcal{O}_L(-1)) = 0$ since $L \cong \mathbb{P}^1$ and $L \cdot K_X = -1$. Thus by Theorem 3.5 the sheaf $\mathcal{O}_L$ is contained in the subcategory $\langle A_2, A_3 \rangle$ of $D(X)$. Next, again by Serre duality

$$\text{Ext}^i(\mathcal{O}_L, \mathcal{E}_Z) \cong \text{Ext}^{2-i}(\mathcal{E}_Z, \mathcal{E}_L(K_X))^\vee.$$

Since $\mathcal{E}_Z$ is a vector bundle and $L$ is a curve, the right hand side is zero unless $i \in \{1, 2\}$. On the other hand, $\mathcal{E}_Z$ is globally generated by Proposition 3.17 and (27), while $\mathcal{O}_L(K_X) \cong \mathcal{O}_L(-1)$ has no global sections, hence for $i = 2$ the right hand side is also zero. On the other hand, by Lemma 3.21 we have

$$\chi(\mathcal{E}_Z, \mathcal{O}_L(K_X)) = 2\chi(X, \mathcal{O}_L(K_X)) + \chi(X, \mathcal{O}_L(2K_X)) - 3r(\mathcal{O}_L(K_X)) = 2 \cdot h_1'(-1) + h_1'(-2) = -1,$$

and we conclude that $\text{Ext}^i(\mathcal{O}_L, \mathcal{E}_Z) = \mathbb{k}[-1]$. This means that there is a unique closed point $z_2 \in Z_2$ and a unique extension

$$0 \to \mathcal{E}_2 \to \mathcal{F} \to \mathcal{O}_L \to 0$$

such that $\mathcal{F} \cong A_3$. It remains to note that $\mathcal{F}$ is a sheaf of rank 1, hence there is a unique $z_3 \in Z_3$ such that $\mathcal{F} \cong \mathcal{E}_3$. □

Consider the Hilbert scheme

$$F_d(X) := \text{Hilb}_{X,-K_X}(h_d')$$

describing subschemes of $X$ with Hilbert polynomial $h_d'(t)$. Thus, $F_1(X)$ is the Hilbert scheme of lines, $F_2(X)$ is the Hilbert scheme of conics, and $F_3(X)$ is the Hilbert scheme of generalized twisted cubic curves on $X$.

Recall the notation $p_2, p_3, p_{23}, p_X$ introduced before Lemma 3.21. By Lemma 3.21 the sheaves $\mathcal{E}_Z, \mathcal{E}_Z, \mathcal{E}_Z$ and $p_{23}(p_X \mathcal{E}_Z \otimes p_X \mathcal{E}_Z)$ are locally free of ranks 2, 3 and 1 on $Z_2, Z_3$ and $Z_2 \times Z_3$ respectively.

**Proposition 4.9.** We have natural isomorphisms of Hilbert schemes

$$F_1(X) \cong Z_2 \times Z_3, \quad F_2(X) \cong \mathbb{P}_{Z_2}(p_{23} \mathcal{E}_Z), \quad \text{and} \quad F_3(X) \cong \mathbb{P}_{Z_2}(p_{34} \mathcal{E}_Z).$$

**Proof.** Assume $d \in \{2, 3\}$. Let $\mathcal{C} \subset X \times S$ be a flat $S$-family of subschemes in $X$ with Hilbert polynomial $h_d'(t)$. Consider the decomposition of the structure sheaf $\mathcal{O}_\mathcal{C} \in D(X \times S)$ with respect to the semiorthogonal decomposition

$$D(X \times S) = \langle A_{2}^{\mathcal{C}}, A_{1}^{\mathcal{C}}, A_{1}^{\mathcal{C}} \rangle,$$

obtained from the decomposition of Proposition 3.11 by a base change $S \to \text{Spec}(K)$ via [Kuz11, Theorem 5.6]. The argument of Theorem 1.5 together with the result of Lemma 3.8(i) shows that there is a morphism $f_d : S \to Z_d$, a line bundle $\mathcal{F}_d$ on $S$, and an exact sequence

$$0 \to (\text{id}_X \times f_d)^*(\mathcal{E}_Z^\vee) \otimes p_S^*(\mathcal{F}_d) \to \mathcal{O}_{X \times S} \to \mathcal{O}_\mathcal{C} \to 0$$

The left arrow corresponds to a global section of the bundle $(\text{id}_X \times f_d)^*(\mathcal{E}_Z^\vee) \otimes p_S^*(\mathcal{F}_d^\vee)$, i.e., to a morphism from $p_S^*(\mathcal{F}_d)$ to $(\text{id}_X \times f_d)^*(\mathcal{E}_Z^\vee)$. We have

$$\text{Hom}(p_S^*(\mathcal{F}_d), (\text{id}_X \times f_d)^*(\mathcal{E}_Z^\vee)) \cong \text{Hom}(\mathcal{F}_d, p_S^*(\text{id}_X \times f_d)^*(\mathcal{E}_Z^\vee)) \cong \text{Hom}(\mathcal{F}_d, f_d^*p_{23}(\mathcal{E}_Z)), \quad (\ast)$$

The result follows since $\text{Hom}(\mathcal{F}_d, f_d^*p_{23}(\mathcal{E}_Z)) \cong \mathbb{P}_{Z_2}(p_{23}(\mathcal{E}_Z))$. □
where the first equality follows from adjunction and the second is the base change for the diagram

\[
\begin{array}{ccc}
X \times S & \xrightarrow{id_X \times f_d} & X \times Z_d \\
\downarrow p_S & & \downarrow p_d \\
S & \xrightarrow{f_d} & Z_d
\end{array}
\]

The flatness over \( S \) of the cokernel \( \mathcal{O}_S \) of the morphism \((id_X \times f_d)^*(E_d^\vee) \otimes p_2^*(\mathcal{F}_d) \rightarrow \mathcal{O}_{X \times S}\) is equivalent to the corresponding morphism \( \mathcal{F}_d \rightarrow f_d^*p_{ds}\mathcal{E}_Z\) being nonzero for every closed point \( s \in S \). Thus the Hilbert scheme functor \( F_d(X) \) is isomorphic to the functor that associates to a scheme \( S \) a morphism \( f_d: S \rightarrow Z_d \) and a line subbundle in \( f_d^*p_{ds}\mathcal{E}_Z\), hence is represented by the projective bundle \( \mathbb{P}_{Z_d}(p_{ds}\mathcal{E}_Z) \).

Now assume \( d = 1 \) and let \( \mathcal{C} \subset X \times S \) be a flat family of lines. Decomposing \( \mathcal{O}_C \) with respect to \( \mathcal{C}_d \) and using the argument of Theorem 4.5 together with the result of Lemma 4.8(ii), we conclude that there are morphisms \( f_2: S \rightarrow Z_2 \) and \( f_3: S \rightarrow Z_3 \), line bundles \( \mathcal{F}_2 \) and \( \mathcal{F}_3 \) on \( S \), and an exact sequence

\[
0 \rightarrow (id_X \times f_2)^*(\mathcal{E}_2) \otimes p_2^*(\mathcal{F}_2) \rightarrow (id_X \times f_3)^*(\mathcal{E}_3) \otimes p_2^*(\mathcal{F}_3) \rightarrow \mathcal{O}_S \rightarrow 0
\]

Moreover, the pushforward to \( S \) of this sequence gives (again by base change) an exact sequence

\[
0 \rightarrow f_2^*(p_{2s}\mathcal{E}_Z) \otimes \mathcal{F}_2 \rightarrow f_3^*(p_{3s}\mathcal{E}_Z) \otimes \mathcal{F}_3 \rightarrow \mathcal{O}_S \rightarrow 0.
\]

Since by Lemma 3.21 we know that \( f_1^*(p_{ds}\mathcal{E}_Z) \) is a vector bundle of rank \( d \) on \( S \), the comparison of determinants gives a relation

\[
(\mathcal{F}_2^3 \otimes \mathcal{F}_3)^\vee \cong f_2^*(\text{det}(p_{2s}\mathcal{E}_Z)) \otimes f_3^*(\text{det}(p_{3s}\mathcal{E}_Z)^{-1}).
\]

Denoting by \( f_{23}: S \rightarrow Z_2 \times Z_3 \) the map induced by \( f_2 \) and \( f_3 \), we deduce by adjunction and base change

\[
\text{Hom}(p_S^*(\mathcal{F}_2 \otimes \mathcal{F}_3^\vee), (id_X \times f_{23})^*(p_{X23}\mathcal{E}_Z \otimes p_{X3}\mathcal{E}_Z)) \cong \text{Hom}(\mathcal{F}_2 \otimes \mathcal{F}_3^\vee, f_{23}^*(p_{X23}\mathcal{E}_Z \otimes p_{X3}\mathcal{E}_Z))
\]

and note that \( f_{23}^*(p_{X23}\mathcal{E}_Z \otimes p_{X3}\mathcal{E}_Z) \) is a line bundle (again by Lemma 3.21). The flatness over \( S \) of the cokernel \( \mathcal{O}_S \) of the morphism of (33) is equivalent to the pointwise injectivity of the corresponding morphism \( \mathcal{F}_2 \otimes \mathcal{F}_3^\vee \rightarrow f_{23}^*(p_{X23}\mathcal{E}_Z \otimes p_{X3}\mathcal{E}_Z) \), i.e., to this map being an isomorphism. On the other hand, any line bundle on \( S \) can be written as the tensor product of line bundles \( \mathcal{F}_2 \otimes \mathcal{F}_3^\vee \) satisfying (37) in a unique way. This shows that the Hilbert scheme functor is isomorphic to the functor that associates to a scheme \( S \) a pair of morphisms \( f_2: S \rightarrow Z_2 \) and \( f_3: S \rightarrow Z_3 \), hence is represented by \( Z_2 \times Z_3 \).

\[
\square
\]

5. Families of sextic del Pezzo surfaces

Starting from this section we assume that \( k \) is an arbitrary field of characteristic distinct from 2 and 3, unless something else is specified explicitly. We keep in mind the notation introduced in previous sections.

5.1. Semiorthogonal decomposition. The main result of this section is a description of the derived category of a du Val family (as defined below) of sextic del Pezzo surfaces.

Definition 5.1. A family \( f: \mathcal{X} \rightarrow S \) is a du Val family of sextic del Pezzo surfaces, if \( f \) is a flat projective morphism such that for every geometric point \( s \in S \) the fiber \( \mathcal{X}_s \) of \( \mathcal{X} \) over \( s \) is a sextic du Val del Pezzo surface (Definition 2.1), i.e., a normal integral surface with at worst du Val singularities such that \(-K_{\mathcal{X}_s}\) is an ample Cartier divisor and \( K_{\mathcal{X}_s}^2 = 6 \).

Note that this definition is much weaker than the notion of a good family used in [AHTVA16], since we do not assume any transversality. Note also that by definition all fibers of \( f: \mathcal{X} \rightarrow S \) are Gorenstein, hence the relative dualizing complex \( \omega_{\mathcal{X}/S} \) when restricted to any fiber of \( f \) is an invertible sheaf, hence by Lemma 4.4 it is an invertible sheaf on the total space \( \mathcal{X} \).
If \( f : \mathcal{X} \to S \) is a du Val family of sextic del Pezzo surfaces then for any base change \( S' \to S \) the induced family \( f' : \mathcal{X}' = \mathcal{X} \times_S S' \to S' \) is still a du Val family. So, du Val families of sextic del Pezzo surfaces form a stack over \((\text{Sch}/k)\). In Appendix \( \text{[3]} \) we prove that this stack is smooth of finite type over \( k \) (but not separated), so in most arguments of this section one may safely assume that the base \( S \) of the family is smooth (and then deduce the necessary results for any family by a base change argument).

The main result of this section (and of this paper) is the following

**Theorem 5.2.** Assume \( f : \mathcal{X} \to S \) is a du Val family of sextic del Pezzo surfaces. Then there is an \( S \)-linear semiorthogonal decomposition (compatible with any base change)

\[
D(\mathcal{X}) = (D(S), D(\mathcal{Z}_2, \beta_{\mathcal{Z}_2}), D(\mathcal{Z}_3, \beta_{\mathcal{Z}_3})),
\]

where \( \mathcal{Z}_2 \to S \) and \( \mathcal{Z}_3 \to S \) are finite flat morphisms of degree 2 and 3 respectively, \( \beta_{\mathcal{Z}_2} \) and \( \beta_{\mathcal{Z}_3} \) are Brauer classes of order 2 and 3 respectively on them, and the corresponding components are the twisted derived categories.

Base change compatibility means that for a du Val family \( \mathcal{X}' \to S' \) of sextic del Pezzo surfaces obtained from a family \( \mathcal{X} \to S \) by a base change, the decomposition of the theorem coincides with the decomposition obtained from \( \text{[3]} \) by \( \text{[Kuz11]} \) Theorem 5.6.

The proof of the theorem takes all Section 5.1, and in Section 5.2 we discuss some properties of this semiorthogonal decomposition.

Let \( f : \mathcal{X} \to S \) be a du Val family of sextic del Pezzo surfaces. For \( d \in \{2, 3, 4\} \) let \( \mathcal{M}_d(\mathcal{X}/S) \) denote the relative moduli stack of semistable sheaves on fibers of \( \mathcal{X} \) over \( S \) with Hilbert polynomial \( h_d(t) \) defined in \( \text{[30]} \).

**Proposition 5.3.** For \( d \in \{2, 3, 4\} \) the stack \( \mathcal{M}_d(\mathcal{X}/S) \) is a \( G_m \)-gerbe over its coarse module space

\[
\mathcal{Z}_d(\mathcal{X}/S) := (\mathcal{M}_d(\mathcal{X}/S))_{\text{coarse}}
\]

with an obstruction given by a Brauer class \( \beta_{\mathcal{Z}_d(\mathcal{X}/S)} \) of order

\[
\operatorname{ord} \beta_{\mathcal{Z}_d(\mathcal{X}/S)} = \begin{cases} 2, & \text{if } d = 2 \text{ or } d = 4, \\ 3, & \text{if } d = 3. \end{cases}
\]

Moreover, there is an isomorphism of the coarse moduli spaces \( \sigma_{2,4} : \mathcal{Z}_4(\mathcal{X}/S) \cong \mathcal{Z}_2(\mathcal{X}/S) \) and an automorphism \( \sigma_{3,3} : \mathcal{Z}_3(\mathcal{X}/S) \cong \mathcal{Z}_3(\mathcal{X}/S) \), such that

\[
\beta_{\mathcal{Z}_4(\mathcal{X}/S)} = \sigma_{2,4}^*(\beta_{\mathcal{Z}_2(\mathcal{X}/S)}^{-1}) \quad \text{and} \quad \beta_{\mathcal{Z}_3(\mathcal{X}/S)} = \sigma_{3,3}^*(\beta_{\mathcal{Z}_3(\mathcal{X}/S)}^{-1}).
\]

**Proof.** By Lemma \( \text{[41]} \) all sheaves parameterized by the moduli stack \( \mathcal{M}_d(\mathcal{X}/S) \) are strictly stable. Therefore, by \( \text{[HL97]} \) Theorem 4.3.7, the coarse moduli space \( \mathcal{Z}_d(\mathcal{X}/S) \) exists and there is a quasiuniversal family \( \mathcal{E}_{\mathcal{Z}_d(\mathcal{X}/S)} \) on the fiber product \( \mathcal{X} \times_S \mathcal{Z}_d(\mathcal{X}/S) \) which has a module structure over a sheaf of Azumaya algebras \( \mathcal{B}_{\mathcal{Z}_d(\mathcal{X}/S)} \) (defined up to a Morita equivalence) of order equal to the greatest common divisor of the values of the Hilbert polynomial \( h_d(t) \).

This means that \( \mathcal{M}_d(\mathcal{X}/S) \) is a \( G_m \)-gerbe over \( \mathcal{Z}_d(\mathcal{X}/S) \). Its obstruction class is given by the Brauer class \( \beta_{\mathcal{Z}_d(\mathcal{X}/S)} \) of Azumaya algebra \( \mathcal{B}_{\mathcal{Z}_d(\mathcal{X}/S)} \), and a simple computation gives \( \text{[39]} \).

By Theorem \( \text{[45]} \) the sheaf \( \mathcal{E}_{\mathcal{Z}_d(\mathcal{X}/S)}^\vee \otimes \omega_{\mathcal{X}/S}^{-1} \) on \( \mathcal{X} \times_S \mathcal{Z}_d(\mathcal{X}/S) \) provides a family of stable sheaves with Hilbert polynomial \( h_4(t) \) and the sheaf \( \mathcal{E}_{\mathcal{Z}_d(\mathcal{X}/S)}^\vee \otimes \omega_{\mathcal{X}/S}^{-1} \) on \( \mathcal{X} \times_S \mathcal{Z}_4(\mathcal{X}/S) \) provides a family of stable sheaves with Hilbert polynomial \( h_2(t) \). Therefore, the moduli stacks \( \mathcal{M}_2(\mathcal{X}/S) \) and \( \mathcal{M}_4(\mathcal{X}/S) \) are isomorphic. Denoting by \( \sigma_{2,4} \) the induced isomorphism of the coarse moduli spaces, we conclude that the family \( \sigma_{2,4}^*(\mathcal{E}_{\mathcal{Z}_2(\mathcal{X}/S)}^\vee \otimes \omega_{\mathcal{X}/S}^{-1}) \) is a quasiuniversal family on \( \mathcal{X} \times_S \mathcal{Z}_4(\mathcal{X}/S) \). It follows that the
pullback of the opposite Azumaya algebra $\mathcal{B}_{\mathcal{A}}(X/S)$ to $\mathcal{Z}(X/S)$ is Morita-equivalent to the Azumaya algebra $\mathcal{B}_{\mathcal{A}}(X/S)$, hence $\beta_{\mathcal{Z}}(X/S) = \sigma_{2,4}^{-1}(\beta_{\mathcal{A}}(X/S))$.

The second isomorphism $\sigma_{3,3}$ is constructed in the same way, and with its construction we also get an isomorphism of quasiuniversal families and an equality of Brauer classes.

When there is no risk of confusion, we abbreviate $\mathcal{M}_d(X/S)$ and $\mathcal{Z}_d(X/S)$ to $\mathcal{M}_d$ and $\mathcal{Z}_d$. Both the stack $\mathcal{M}_d$ and the scheme $\mathcal{Z}_d$ are proper over $S$. We denote by

$$f_d: \mathcal{Z}_d \rightarrow S$$

the natural projection. We replace the Azumaya algebra $\mathcal{B}_{\mathcal{A}}$ by its Brauer class $\beta_{\mathcal{A}} \in \text{Br}(\mathcal{Z}_d)$ and consider the quasiuniversal family as a $\beta_{\mathcal{A}}^{-1}$-twisted family of sheaves on $X \times S \mathcal{Z}_d$:

$$\mathcal{E}_{\mathcal{Z}_d} \in \text{coh}(X \times S \mathcal{Z}_d, \beta_{\mathcal{A}}^{-1}).$$

We will use these sheaves to construct Fourier–Mukai functors in (43). Note that the relation between the quasiuniversal families discussed in Proposition 5.3 in terms of twisted universal families means that there are line bundles $\mathcal{L}_{2,4}$ and $\mathcal{L}_{3,3}$ on $\mathcal{Z}_d(X/S)$ and $\mathcal{Z}_3(X/S)$ respectively, such that

$$\mathcal{E}_{\mathcal{A}}(X/S) = \sigma_{2,4}^*(\mathcal{E}_{\mathcal{A}}(X/S) \otimes \omega_{X/S}^{-1}) \otimes \mathcal{L}_{2,4} \quad \text{and} \quad \mathcal{E}_{\mathcal{A}}(X/S) = \sigma_{3,3}^*(\mathcal{E}_{\mathcal{A}}(X/S) \otimes \omega_{X/S}^{-1}) \otimes \mathcal{L}_{3,3}.$$

Let $\phi: S' \rightarrow S$ be a base change and denote

$$\mathcal{X}':= X \times_S S', \quad \mathcal{Z}_d':= \mathcal{Z}_d(X'/S').$$

We denote by $\phi_{\mathcal{X}}: \mathcal{X}' \rightarrow \mathcal{X}$, $\phi_{\mathcal{Z}_d}: \mathcal{Z}_d' \rightarrow \mathcal{Z}_d$, and $\phi_{\mathcal{X} \times_S \mathcal{Z}_d}: \mathcal{X}' \times_S \mathcal{Z}_d' \rightarrow \mathcal{X} \times_S \mathcal{Z}_d$ the maps induced by the base change $\phi$.

**Lemma 5.4.** We have

$$\mathcal{M}_d(\mathcal{X}'/S') \cong \mathcal{M}_d(\mathcal{X}/S) \times_S S' \quad \text{and} \quad \mathcal{Z}_d(\mathcal{X}'/S') \cong \mathcal{Z}_d(\mathcal{X}/S) \times_S S'.$$

Moreover, isomorphisms (40) are compatible with the universal families, i.e.,

$$\beta_{\mathcal{Z}_d} = \phi_{\mathcal{Z}_d}^*(\beta_{\mathcal{A}}) \quad \text{and} \quad \mathcal{E}_{\mathcal{Z}_d} \cong \phi_{\mathcal{X} \times_S \mathcal{Z}_d}^* \mathcal{E}_{\mathcal{A}}.$$

**Proof.** The first isomorphism in (40) is clear from the definition of a relative moduli space, and the second follows from the GIT construction of the coarse moduli space. Since the pullback of a universal family is a $\phi_{\mathcal{Z}_d}^*(\beta_{\mathcal{A}})$-twisted family of stable sheaves on fibers of $\mathcal{X}'$ over $S'$, the equality of the Brauer classes and the isomorphism of universal families follow. □

Considering base changes to geometric points of $S$ and using Theorem 1.5 we obtain

**Corollary 5.5.** For any geometric point $s \in S$ there are isomorphisms

$$\mathcal{M}_d(X/s) \cong \mathcal{M}_d(X), \quad \mathcal{Z}_d(X/s) \cong \mathcal{Z}_d \quad \text{and} \quad i_{\mathcal{Z}_d}^*(\mathcal{E}_{\mathcal{A}}) \cong \mathcal{E}_{\mathcal{Z}_d},$$

where $X = \mathcal{X}_s$ is the fiber of $\mathcal{X}$ over $s$ and $i_{\mathcal{Z}_d}: \mathcal{Z}_d \cong \mathcal{Z}_d(X/S) \hookrightarrow \mathcal{Z}_d(X/S)$ is the natural embedding.

In the next Lemma we consider the coarse moduli spaces $\mathcal{Z}_2$ and $\mathcal{Z}_3$. Analogous result for $\mathcal{Z}_4$ follows via the isomorphism $\sigma_{2,4}$ of Proposition 5.3

**Lemma 5.6.** The maps $f_2: \mathcal{Z}_2 \rightarrow S$ and $f_3: \mathcal{Z}_3 \rightarrow S$ are finite flat maps of degree 3 and 2 respectively. The relative dualizing complexes $\omega_{\mathcal{Z}_2/S}$ and $\omega_{\mathcal{Z}_3/S}$ are line bundles.
Proof. Since flatness can be verified on an étale covering (note that formation of schemes $\mathcal{Z}_d$ is compatible with base changes by (40)) and the moduli stack of du Val sextic del Pezzo surfaces is smooth by Theorem 13.1, we may assume that $S$ is smooth over $k$, hence reduced. Since $\mathcal{Z}_d$ is proper over $S$, it is enough to show that the (scheme-theoretic) fiber of $\mathcal{Z}_d$ over any geometric point $s \in S$ is zero-dimensional of length $6/d$. But as it was mentioned earlier, the fiber $(\mathcal{Z}_d)_s$ is identified with the zero-dimensional scheme $Z_d$ associated with the surface $X = \mathcal{Z}_s$, and hence its length is $6/d$ by Corollary 3.13.

Since $f_d$ is flat, to check that the relative dualizing complex is a line bundle, it is enough to check that each fiber of $f_d$ is a Gorenstein scheme, which holds true by (42) and definition of $Z_d$ in Corollary 3.13 $\square$

For convenience we also define $\mathcal{Z}_1 = S$, set $f_1 : \mathcal{Z}_1 \to S$ to be the identity, set $\beta_{\mathcal{Z}_1}$ to be the trivial Brauer class, and $\mathcal{E}_{\mathcal{Z}_1} = O_X$. Let $p : \mathcal{X} \times_S \mathcal{Z}_d \to \mathcal{X}$ and $p_d : \mathcal{X} \times_S \mathcal{Z}_d \to \mathcal{Z}_d$ be the projections. Then we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} \times_S \mathcal{Z}_d & \xrightarrow{p_d} & \mathcal{Z}_d \\
p \downarrow & & \downarrow f_d \\
\mathcal{X} & \xrightarrow{f} & S
\end{array}
\]

In what follows we work with Fourier–Mukai functors between twisted derived categories. Since the Brauer classes we consider come from Azumaya algebras, one can consider those twisted derived categories as derived categories of Azumaya varieties, as defined in [Kuz06, Appendix A]; for instance, as explained in loc. cit., we have all standard functors between these varieties and all standard functorial isomorphisms.

**Lemma 5.7.** For each $d \in \{1, 2, 3\}$ the sheaf $\mathcal{E}_{\mathcal{Z}_d}$ is flat over $\mathcal{Z}_d$ and has finite Ext-amplitude over $\mathcal{X}$.

*Proof.* For $d \in \{2, 3\}$ the sheaf $\mathcal{E}_{\mathcal{Z}_d}$ is flat over $\mathcal{Z}_d$ since it comes from a quasiuniversal family for a moduli problem. Since $p$ is a finite map (Lemma 5.6), to check that $\mathcal{E}_{\mathcal{Z}_d}$ has finite Ext-amplitude over $\mathcal{X}$, it is enough (Kuz06, Lemma 10.40) to show that $p_*\mathcal{E}_{\mathcal{Z}_d}$ has finite Ext-amplitude on $\mathcal{X}$. But this is in fact a locally free sheaf by Lemma 4.1 and Proposition 3.17.

In case $d = 1$ both properties are evident. $\square$

For each $d \in \{1, 2, 3\}$ we consider the Fourier–Mukai functor whose kernel is the universal family $\mathcal{E}_{\mathcal{Z}_d}$, considered as a $\beta^{-1}_{\mathcal{Z}_d}$-twisted sheaf on $\mathcal{X} \times_S \mathcal{Z}_d$:

\[
\Phi_d = \Phi_{\mathcal{E}_{\mathcal{Z}_d}} := p_* (\mathcal{E}_{\mathcal{Z}_d} \otimes p^*_d (-)) : D(\mathcal{Z}_d, \beta_{\mathcal{Z}_d}) \to D(\mathcal{X}).
\]

By Lemma 5.7 and [Kuz06, Lemma 2.4] it preserves boundedness and perfectness, and has a right adjoint functor $\Phi^1_d : D(\mathcal{X}) \to D(\mathcal{Z}_d, \beta_{\mathcal{Z}_d})$ defined by

\[
\Phi^1_d(\mathcal{G}) \cong p_{d*} R\mathcal{H}om(\mathcal{E}_{\mathcal{Z}_d}, p^! \mathcal{G}).
\]

Our goal is to show that the functors $\Phi_d$ are fully faithful and that

\[
D(\mathcal{X}) = \langle \Phi_1(D(\mathcal{Z}_1, \beta_{\mathcal{Z}_1})), \Phi_2(D(\mathcal{Z}_2, \beta_{\mathcal{Z}_2})), \Phi_3(D(\mathcal{Z}_3, \beta_{\mathcal{Z}_3})) \rangle
\]

is an $S$-linear semiorthogonal decomposition (this is a more precise version of (48)).

It is convenient to rewrite the functors $\Phi^1_d$ in a Fourier–Mukai form.

**Lemma 5.8.** We have an isomorphism of functors

\[
\Phi^1_d(\mathcal{G}) \cong p_{d*} (\mathcal{E}^\vee_{\mathcal{Z}_d} \otimes p^*_d \omega_{\mathcal{Z}_d/S} \otimes p^*(\mathcal{G})),
\]

where $\mathcal{E}^\vee_{\mathcal{Z}_d} := R\mathcal{H}om(\mathcal{E}_{\mathcal{Z}_d}, \mathcal{O}_{\mathcal{X} \times S \mathcal{Z}_d})$ is a coherent sheaf flat over $\mathcal{Z}_d$.

Moreover, if $\phi : S' \to S$ is a base change, then

\[
\phi^*_{\mathcal{X} \times S \mathcal{Z}_d} (\mathcal{E}^\vee_{\mathcal{Z}_d} \otimes p^*_d \omega_{\mathcal{Z}_d/S}) \cong \mathcal{E}^\vee_{\mathcal{Z}_d'} \otimes p^*_d' \omega_{\mathcal{Z}_d'/S'}.
\]
Proof. For the first part it is enough to show that
\[
\text{RHom}(\mathcal{E}_{\mathcal{X}_d}, p^!\mathcal{G}) \cong \mathcal{E}^\vee_{\mathcal{X}_d} \otimes p_d^* \omega_{\mathcal{X}_d/S} \otimes p^*(\mathcal{G}).
\]
For this we use an argument of Neeman from [Nee96, Theorem 5.4].

First, the functor in the right hand side of (47) commutes with arbitrary direct sums since the pullback and the tensor product functors do. The functor in the left hand side is right adjoint to \( p_* (\mathcal{E}_d \otimes (-)) \), hence by [Nee96, Theorem 5.1] for its commutation with direct sums it is enough to check that the latter functor preserves perfectness. This property is local over \( \mathcal{X} \), hence it is enough to check that \( p_* (\mathcal{E}_d) \) is perfect. But as we have noticed in the proof of Lemma 5.7, it is a vector bundle.

Further, if \( \mathcal{G} \) is a perfect complex, then \( p^! (\mathcal{G}) \cong p^* (\mathcal{G}) \otimes p_d^* \omega_{\mathcal{X}_d/S} \). By Lemma 5.6 the dualizing complex \( \omega_{\mathcal{X}_d/S} \) is a line bundle, hence \( p^! (\mathcal{G}) \) is a perfect complex, and (47) in this case follows.

Following Neeman’s argument, to deduce an isomorphism (47) from this, we note that there is a natural transformation from the functor in the right hand side of (47) to the functor in the left hand side. The subcategory of objects on which this transformation is an isomorphism is a triangulated subcategory of unbounded derived category of \( \mathcal{X} \) which contains all perfect complexes and is closed under arbitrary direct sums, hence is the whole category. This proves (47) for all \( \mathcal{G} \).

The fact that \( \mathcal{E}_{\mathcal{X}_d}^\vee \) is a sheaf flat over \( \mathcal{Z}_d \) follows from the isomorphisms of Proposition 5.3 since \( \omega_{\mathcal{X}/S} \) is a line bundle. Using Lemma 5.4 we deduce the required base change isomorphism. \( \square \)

Given a base change \( \phi : S' \to S \), we abbreviate the Fourier–Mukai functor \( \Phi_{\mathcal{X}_d'} : \mathcal{D}(\mathcal{X}'_d) \to \mathcal{D}(\mathcal{X}) \) as \( \Phi_d' \). We have the following property.

Lemma 5.9. The functors \( \Phi_d \) and \( \Phi_d' \) are \( S \)-linear and compatible with base changes, i.e.,
\[
\begin{align*}
\Phi_d' \circ \phi_{\mathcal{X}_d} &\cong \phi_{\mathcal{X}} \circ \Phi_d, \\
\Phi_d' \circ \phi_{\mathcal{X}} &\cong \phi_{\mathcal{X}_d} \circ \Phi_d', \\
\Phi_{\mathcal{X}_d'} &\cong \Phi_d \circ \phi_{\mathcal{X}_d} \circ \Phi_d'.
\end{align*}
\]

Proof. For a Fourier–Mukai functor compatibility with base changes are proved in [Kuz06, Lemma 2.42] (the assumption of finiteness of Tor-dimension of the base change morphism is only used in the second half of that lemma). It remains to note that both \( \Phi_d \) and \( \Phi_d' \) are Fourier–Mukai functors (the first by definition (44) and the second by Lemma 5.8). \( \square \)

Now we are ready to prove the theorem.

Proof of Theorem 5.2. Take \( d_1, d_2 \in \{1, 2, 3\} \) and consider the diagram
\[
\begin{array}{ccc}
\mathcal{Z}_{d_1} \times_S \mathcal{X} \times_S \mathcal{X} & \xrightarrow{pr_{1,2}} & \mathcal{Z}_{d_1} \times_S \mathcal{Z}_{d_2} \\
\downarrow \quad & & \downarrow \quad pr_{1,3} & & \quad \downarrow \quad pr_{2,3} \\
\mathcal{Z}_{d_1} \times_S \mathcal{X} & & & & \mathcal{X} \times_S \mathcal{Z}_{d_2}
\end{array}
\]
A standard computation shows that the composition of functors \( \Phi_{d_2}^! \circ \Phi_{d_1} \) is a Fourier–Mukai functor given by the object
\[
\text{pr}_{1,3*}(\text{pr}_{2,3}^* (\mathcal{E}_{\mathcal{X}_d}^\vee \otimes p_{d_2}^* \omega_{\mathcal{X}_d/S}) \otimes \text{pr}_{1,2}^* (\mathcal{E}_{\mathcal{X}_d})) \in \mathcal{D}(\mathcal{Z}_{d_1} \times_S \mathcal{Z}_{d_2}, \beta_{\mathcal{X}_d}^{-1} \otimes \beta_{\mathcal{X}_d}).
\]

Let us prove that \( \Phi_d \) is faithful for each \( d \in \{1, 2, 3\} \). For this it is enough to check that the object (48) in case \( d_1 = d_2 = d \) is isomorphic (via the unit of adjunction morphism) to the structure sheaf of the diagonal. Since the unit of the adjunction is induced by a morphism of kernels [ALT12], by Lemma 4.4 it is enough to check that for any geometric point \( s \in S \) we have
\[
i_{\mathcal{Z}_d \times \mathcal{Z}_d}^* (\text{pr}_{1,3*}(\text{pr}_{2,3}^* (\mathcal{E}_{\mathcal{X}_d}^\vee \otimes p_{d_2}^* \omega_{\mathcal{X}_d/S}) \otimes \text{pr}_{1,2}^* (\mathcal{E}_{\mathcal{X}_d})) ) \cong \delta_* \mathcal{O}_{\mathcal{Z}_d} \in \mathcal{D}(\mathcal{Z}_d \times \mathcal{Z}_d)
\]
with $\delta: Z_d \to Z_d \times Z_d$ being the diagonal. Using base change and isomorphisms of Lemma 5.4 and Lemma 5.8 we can rewrite the left hand side of (49) as the Fourier–Mukai kernel of the functor

$$D(Z_d) \xrightarrow{\Phi_{dZ_d}} D(X) \xrightarrow{\Phi'_d Z_d} D(Z_d),$$

where the functor $\Phi_{dZ_d}$ is defined in Proposition 3.18 and $\Phi'_d Z_d$ is its right adjoint. This composition is isomorphic to the identity, since the functor $\Phi_{dZ_d}$ is fully faithful, hence we have (49). Therefore, the functor $\Phi_d$ is fully faithful.

Next, let us prove that the subcategories $\Phi_d(D(\mathcal{Z}_d, \beta_{\mathcal{Z}_d})) \subset D(\mathcal{X})$ for $1 \leq d \leq 3$ are semiorthogonal. For this it is enough to check that the object (48) in case $d_1 < d_2$ is zero. Again, using Lemma 5.4 and base change isomorphisms of Lemma 5.4 and Lemma 5.8 we reduce to the case of $S = \text{Spec}(k)$ with algebraically closed $k$. In the latter case, the required vanishing follows from semiorthogonality of the subcategories $\Phi_{dZ_d}(D(Z_d))$ in $D(X)$.

Finally, let us prove that the subcategories $\Phi_d(D(\mathcal{Z}_d, \beta_{\mathcal{Z}_d})) \subset D(\mathcal{X})$ for $1 \leq d \leq 3$ generate $D(\mathcal{X})$. Take any $\mathcal{G} \in D(\mathcal{X})$ and set

$$\mathcal{G}_3 := \text{Cone}(\Phi_3 \Phi_3^! \mathcal{G} \to \mathcal{G}), \quad \mathcal{G}_1 := \text{Cone}(\Phi_3 \Phi_2 \mathcal{G}_2 \to \mathcal{G}_2), \quad \text{and} \quad \mathcal{G}_0 := \text{Cone}(\Phi_1 \Phi_1^! \mathcal{G}_1 \to \mathcal{G}_1).$$

From full faithfulness and semiorthogonality it easily follows that $\Phi_1^! (\mathcal{G}_0) = \Phi_2^! (\mathcal{G}_0) = \Phi_3^! (\mathcal{G}_0) = 0$. Then Lemma 5.9 implies that for any geometric point $s \in S$ setting $\mathcal{G}_0|_{X_s} = i_X^*(\mathcal{G}_0) \in D(\mathcal{X})$ to be the restriction of $\mathcal{G}_0$ to the fiber $X = \mathcal{X}_s$, we have $\Phi_d^! (\mathcal{G}_0|_{X_s}) = 0$ for all $d$. By semiorthogonal decomposition (10) this means that $\mathcal{G}_0|_{X_s} = 0$. Hence by Lemma 1.1 (i) we have $\mathcal{G}_0 = 0$. Thus we have a chain of morphisms

$$\mathcal{G} =: \mathcal{G}_3 \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G}_0 = 0$$

with $\text{Cone}(\mathcal{G}_d \to \mathcal{G}_{d-1}) \in \Phi_d(D(\mathcal{Z}_d, \beta_{\mathcal{Z}_d}))$, which proves the required semiorthogonal decomposition.

This semiorthogonal decomposition is $S$-linear by Lemma 5.9 and its compatibility with base changes follows from [Kuz11, Theorem 6.4] together with Lemma 5.4 and Lemma 5.9.

5.2. Some properties of the semiorthogonal decomposition. Now we list some properties of the semiorthogonal decomposition of Theorem 5.2.

**Proposition 5.10.** The components of the semiorthogonal decomposition (38) are admissible and their projection functors have finite cohomological amplitude. Moreover, the functors $\Phi_d$ preserve perfectness and induce a semiorthogonal decomposition

$$D^\text{perf}(\mathcal{X}) = \langle \Phi_1(D^\text{perf}(S)), \Phi_2(D^\text{perf}(\mathcal{Z}_2, \beta_{\mathcal{Z}_2})), \Phi_3(D^\text{perf}(\mathcal{Z}_3, \beta_{\mathcal{Z}_3})) \rangle.$$  

**Proof.** The functor $\Phi_d$ has finite cohomological amplitude because the sheaf $\mathcal{E}_{\mathcal{X}_d}$ is flat over $\mathcal{Z}_d$ by Lemma 5.7 (in fact, since $p: \mathcal{X} \times_S \mathcal{Z}_d \to \mathcal{X}$ is finite, the functor $\Phi_d$ is exact). The functor $\Phi'_d$ has finite cohomological amplitude because the sheaf $\mathcal{E}_{\mathcal{X}_d}$ has finite Ext-amplitude over $\mathcal{Z}_d$ by Lemma 5.7 (in fact, since $p_d: \mathcal{X} \times_S \mathcal{Z}_d \to \mathcal{Z}_d$ is flat of relative dimension 2, the cohomological amplitude of the functor $\Phi'_d$ equals $(0, 2)$). Thus $\Phi_d \circ \Phi'_d$ has finite cohomological amplitude. Since the projection functors of (38) can be expressed as combinations of these functors and their counit morphisms to the identity functor (see the proof of Theorem 5.2), it follows that the projection functors also have finite cohomological amplitude.

To show that the components of (38) are admissible, note that we have two more decompositions

$$D(\mathcal{X}) = D(\mathcal{Z}_3, \beta_{\mathcal{Z}_3}) \otimes \omega_{\mathcal{X}/S}, D(S), D(\mathcal{Z}_2, \beta_{\mathcal{Z}_2}),$$

$$D(\mathcal{X}) = D(\mathcal{Z}_2, \beta_{\mathcal{Z}_2}) \otimes \omega_{\mathcal{X}/S}, D(\mathcal{Z}_3, \beta_{\mathcal{Z}_3}) \otimes \omega_{\mathcal{X}/S}, D(S).$$

Indeed, they can be established by the same argument as (38) starting with the semiorthogonal decompositions (18) instead of (12). Since $\omega_{\mathcal{X}/S}$ is invertible, the two above decompositions together with (38)
prove admissibility of all components (each of them appears on the left in one decomposition and on the right in another).

The functors \( \Phi_d \) preserve perfectness because their right adjoints \( \Phi'_d \) commute with arbitrary direct sums by Lemma 5.8. It remains to prove (50). For this we have to check that if \( \mathcal{G} \in D^{\text{perf}}(\mathcal{Z}) \) then all its components in (46) are perfect. Let

\[
0 = \mathcal{G}_3 \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G}_0 = \mathcal{G}
\]

be the filtration of \( \mathcal{G} \) with \( \text{Cone}(\mathcal{G}_d \to \mathcal{G}_{d-1}) \cong \Phi_d(\mathcal{F}_d) \) and \( \mathcal{F}_d \in D(\mathcal{Z}_d, \beta_{\mathcal{X}_d}) \). We use an induction on \( d \) to prove that all \( \mathcal{F}_d \) are perfect complexes. Assume that for \( d' < d \) all \( \mathcal{F}_{d'} \) are perfect. Since \( \Phi_{d'} \) preserve perfectness, it follows that \( \mathcal{G}_d \) is perfect. Note that \( \mathcal{F}_d \) is equal to the image of \( \mathcal{G}_d \) under the left adjoint of the functor

\[
\Phi_d: D(\mathcal{Z}_d, \beta_{\mathcal{X}_d}) \to (\Phi_d(D(\mathcal{Z}_d, \beta_{\mathcal{X}_d})), \ldots, \Phi_3(D(\mathcal{Z}_3, \beta_{\mathcal{X}_3})))
\]

The functor \( \Phi_d \) commutes with arbitrary direct sums, hence by [Nee96, Theorem 5.1] its left adjoint preserves perfectness. This means that \( \mathcal{F}_d \) is perfect and proves the induction step.

Since the morphism \( f: \mathcal{X} \to S \) is Gorenstein, the relative dualizing complex \( \omega_{\mathcal{X}/S} \) is (up to a shift) a line bundle. Therefore, the relative duality functor

\[
D_{\mathcal{X}/S}(\mathcal{F}) := \mathcal{R} \mathcal{H}om(\mathcal{F}, f^* \omega_{\mathcal{X}}^*)
\]

is an anti-autoequivalence of the category \( D(\mathcal{X}) \) (see Section 2.3).

**Proposition 5.11.** The relative duality functor gives a semiorthogonal decomposition

\[
D(\mathcal{X}) = (D_{\mathcal{X}/S}(D(\mathcal{Z}_3, \beta_{\mathcal{X}_3})), D_{\mathcal{X}/S}(D(\mathcal{Z}_2, \beta_{\mathcal{X}_2})), D_{\mathcal{X}/S}(D(S)), D(\mathcal{Z}_1, \beta_{\mathcal{X}_1}), D(\mathcal{Z}_2, \beta_{\mathcal{X}_2}), D(\mathcal{Z}_3, \beta_{\mathcal{X}_3})),
\]

whose components are equivalent to \( D_{\mathcal{X}/S}(D(\mathcal{Z}_3, \beta_{\mathcal{X}_3})), D_{\mathcal{X}/S}(D(\mathcal{Z}_2, \beta_{\mathcal{X}_2})), D_{\mathcal{X}/S}(D(S)) \) respectively. Moreover, this decomposition is right mutation-dual to (38).

**Proof.** Since \( D_{\mathcal{X}/S} \) is an anti-autoequivalence, (51) is a semiorthogonal decomposition. Consider also the right mutation-dual decomposition

\[
D(\mathcal{X}) = (D(\mathcal{X}), D(\mathcal{L}_1(\Phi_3(D(\mathcal{Z}_3, \beta_{\mathcal{X}_3}))), D(\mathcal{L}_1(\Phi_2(D(\mathcal{Z}_2, \beta_{\mathcal{X}_2})))), \Phi_1(D(S))
\]

where \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are the mutation functors through the first and the second components of (46) (they are well-defined because the components of the decomposition are admissible). By base change together with Proposition 3.14 and Lemma 4.4, these two decompositions coincide. \( \square \)

The following result is very useful.

**Proposition 5.12.** Let \( \mathcal{X} \to S \) be a du Val family of sextic del Pezzo surfaces. The total space \( \mathcal{X} \) of the family is regular if and only if all three schemes \( S, \mathcal{Z}_2 \) and \( \mathcal{Z}_3 \) are regular.

**Proof.** Consider decompositions (46) and (50). If \( \mathcal{X} \) is regular then \( D^{\text{perf}}(\mathcal{X}) = D(\mathcal{X}) \), hence it follows that \( D^{\text{perf}}(S) = D(S) \) and \( D^{\text{perf}}(\mathcal{Z}_d, \beta_{\mathcal{X}_d}) = D(\mathcal{Z}_d, \beta_{\mathcal{X}_d}) \) for \( d = 2, 3 \), which means that \( S \) and \( \mathcal{Z}_d \) are regular. The other implication is analogous. \( \square \)

**Corollary 5.13.** Let \( \mathcal{X} \to S \) and \( \mathcal{X}' \to S \) be two du Val families of sextic del Pezzo surfaces with regular total spaces \( \mathcal{X} \) and \( \mathcal{X}' \). Assume there is a dense open subset \( S_0 \subset S \) such that for some \( d \in \{2, 3\} \) there is an \( S_0 \)-isomorphism \( \varphi_0: \mathcal{Z}_d(\mathcal{X}_0/S_0) \overset{\sim}{\longrightarrow} \mathcal{Z}_d(\mathcal{X}'_0/S_0) \), where \( \mathcal{X}_0 = \mathcal{X} \times_S S_0 \) and \( \mathcal{X}'_0 = \mathcal{X}' \times_S S_0 \). Then \( \varphi_0 \) extends to an isomorphism \( \varphi: \mathcal{Z}_d(\mathcal{X}/S) \overset{\sim}{\longrightarrow} \mathcal{Z}_d(\mathcal{X}'/S) \) over \( S \). Moreover, if we have an equality of Brauer classes \( \beta_{\mathcal{Z}_d(\mathcal{X}_0/S_0)} = \varphi_0^* \beta_{\mathcal{Z}_d(\mathcal{X}'_0/S_0)} \) then it extends to \( \beta_{\mathcal{Z}_d(\mathcal{X}/S)} = \varphi^* \beta_{\mathcal{Z}_d(\mathcal{X}'/S)} \).
Proof. For the first part note, that $\mathcal{Z}_d(\mathcal{X}/S)$ is regular by Proposition 5.12 and in particular normal. Hence it is isomorphic to the normal closure of $S$ in the field of rational functions on $\mathcal{Z}_d(\mathcal{X}_0/S_0)$. The same argument works for $\mathcal{Z}_d(\mathcal{X}'/S)$, hence $\varphi_0$ extends to an isomorphism $\varphi$.

The second claim is evident because the restriction morphisms $\text{Br}(\mathcal{Z}_d(\mathcal{X}/S)) \to \text{Br}(\mathcal{Z}_d(\mathcal{X}_0/S_0))$ and $\text{Br}(\mathcal{Z}_d(\mathcal{X}'/S)) \to \text{Br}(\mathcal{Z}_d(\mathcal{X}'_0/S_0))$ of the Brauer groups are injective. □

We also have a Hilbert scheme interpretation for the semiorthogonal decomposition. Let $F_d(\mathcal{X}/S)$ be the relative Hilbert scheme of subschemes in the fibers of $\mathcal{X}$ over $S$ with Hilbert polynomial $h_d(t)$ defined by (34). Thus, $F_1(\mathcal{X}/S)$ is the relative Hilbert scheme of lines, $F_2(\mathcal{X}/S)$ is the relative Hilbert scheme of conics, and $F_3(\mathcal{X}/S)$ is the relative Hilbert scheme of twisted cubic curves.

**Proposition 5.14.** For each $1 \leq d \leq 3$ the scheme $F_d(\mathcal{X}/S)$ is flat over $S$, and

- $F_1(\mathcal{X}/S) \cong \mathcal{Z}_2 \times_S \mathcal{Z}_3$,
- $F_2(\mathcal{X}/S)$ is an étale locally trivial $\mathbb{P}^1$-bundle over $\mathcal{Z}_2$, and
- $F_3(\mathcal{X}/S)$ is an étale locally trivial $\mathbb{P}^2$-bundle over $\mathcal{Z}_3$.

Moreover, the projective bundles $F_2(\mathcal{X}/S) \to \mathcal{Z}_2$ and $F_3(\mathcal{X}/S) \to \mathcal{Z}_3$ are Severi–Brauer varieties associated with the Brauer classes $\beta_{\mathcal{Z}_2}$ and $\beta_{\mathcal{Z}_3}$ respectively.

Proof. Assume $d \in \{2, 3\}$. The construction of the morphism $F_d(\mathcal{X}) \to \mathcal{M}_d(X)$ from Proposition 4.9 works well in an arbitrary du Val family of sextic del Pezzo surfaces and provides a morphism of functors $F_d(\mathcal{X}/S) \to \mathcal{M}_d(\mathcal{X}/S)$. Moreover, it identifies $F_d(\mathcal{X}/S)$ (considered as a stack) with the projectivization of the (twisted) vector bundle $p_d^{\ast}\mathcal{E}_{\mathcal{X}_d}$, where $\mathcal{E}_{\mathcal{X}_d}$ is the (twisted) universal sheaf on the product $\mathcal{X} \times_S \mathcal{Z}_d$ and $p_d : \mathcal{X} \times_S \mathcal{Z}_d \to \mathcal{Z}_d$ is the projection. This is equivalent to the statement of the proposition.

For $d = 1$ also a relative version of the argument of Proposition 4.9 works. □

**Corollary 5.15.** The total space of $\mathcal{X}$ is regular if and only if $S$, $F_2(\mathcal{X}/S)$ and $F_3(\mathcal{X}/S)$ are regular.

Proof. By Proposition 5.14 if $d \in \{2, 3\}$, the morphism $F_d(\mathcal{X}/S) \to \mathcal{Z}_d$ is smooth, hence the scheme $F_d(\mathcal{X}/S)$ is regular if and only if $\mathcal{Z}_d$ is regular. So, Proposition 5.12 applies. □

**Corollary 5.16.** Let $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ be two du Val families of sextic del Pezzo surfaces with regular total spaces $\mathcal{X}$ and $\mathcal{X}'$. Assume for some $d \in \{2, 3\}$ there is a birational $S$-isomorphism $\psi : F_d(\mathcal{X}/S) \to F_d(\mathcal{X}'/S)$. Then it induces a birational isomorphism $\varphi : \mathcal{Z}_d(\mathcal{X}/S) \cong \mathcal{Z}_d(\mathcal{X}'/S)$ over $S$ and we have $\beta_{\mathcal{Z}_d(\mathcal{X}/S)} = \varphi^{\ast}\beta_{\mathcal{Z}_d(\mathcal{X}'/S)}$.

Proof. Recall that $\mathcal{Z}_d(\mathcal{X}/S)$ is the base of the maximal rationally connected fibration for the morphism $F_d(\mathcal{X}/S) \to S$. Therefore, the birational isomorphism $\psi$ of Hilbert schemes induces a birational isomorphism $\varphi_0$ over $S$ of $\mathcal{Z}_d(\mathcal{X}/S)$ and $\mathcal{Z}_d(\mathcal{X}'/S)$. Using Corollary 5.13 we deduce that it extends to an isomorphism $\varphi : \mathcal{Z}_d(\mathcal{X}/S) \cong \mathcal{Z}_d(\mathcal{X}'/S)$. Furthermore, since $F_d(\mathcal{X}/S)$ is a Severi–Brauer variety associated with the Brauer class $\beta_{\mathcal{Z}_d(\mathcal{X}/S)}$, the birational isomorphism of Hilbert schemes implies equality of the Brauer classes. □

6. Standard families

In this section we discuss some standard families of sextic del Pezzo surfaces and their special features. Throughout this section the base field $k$ is an arbitrary field of characteristic coprime to 2 and 3.
6.1. Linear sections of $\mathbb{P}^2 \times \mathbb{P}^2$. The simplest way to construct a sextic del Pezzo surface is by considering an intersection of $\mathbb{P}^2 \times \mathbb{P}^2$ with a linear subspace of codimension 2 in the Segre embedding. More generally, one can consider a family of such surfaces.

We denote by $W_1$ and $W_2$ a pair of vector spaces of dimension 3 and let

$$\mathbb{P}(W_1) \times \mathbb{P}(W_2) \hookrightarrow \mathbb{P}(W_1 \otimes W_2)$$

be the Segre embedding. To give a linear section of codimension 2, we need a two-dimensional subspace $K \subset W_1^\vee \otimes W_2^\vee$. We denote by $K^\perp \subset W_1 \otimes W_2$ its codimension 2 annihilator, and set

$$(52) \quad X_K := (\mathbb{P}(W_1) \times \mathbb{P}(W_2)) \cap \mathbb{P}(K^\perp) \subset \mathbb{P}(W_1 \otimes W_2),$$

to be the corresponding linear section.

Lemma 6.1. Assume the base field $k$ is algebraically closed. The intersection $X_K$ defined by (52) is a sextic du Val del Pezzo surface if and only if

- the line $\mathbb{P}(K)$ does not intersect the dual Segre variety $\mathbb{P}(W_1^\vee) \times \mathbb{P}(W_2^\vee) \subset \mathbb{P}(W_1^\vee \otimes W_2^\vee)$, and
- the line $\mathbb{P}(K)$ is not contained in the discriminant cubic hypersurface $D_{W_1,W_2} \subset \mathbb{P}(W_1^\vee \otimes W_2^\vee)$.

Furthermore, $X_K$ is smooth if and only if the line $\mathbb{P}(K)$ is transversal to $D_{W_1,W_2}$.

Proof. First, let us show that the condition is necessary. If $\mathbb{P}(K)$ intersects $\mathbb{P}(W_1^\vee) \times \mathbb{P}(W_2^\vee)$ then $K$ contains a bilinear form $b \in W_1^\vee \otimes W_2^\vee$ of rank 1, i.e., $b = \varphi_1 \otimes \varphi_2$, where $\varphi_i \in W_i^\vee$. The zero locus of $b$ on $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$ is equal to $(\mathbb{P}(\varphi_1^\perp) \times \mathbb{P}(W_2)) \cup (\mathbb{P}(W_1) \times \mathbb{P}(\varphi_2^\perp))$. Consequently, $X_K$ is its hyperplane section, hence is a union of two cubic scrolls, hence is not an integral surface.

Next, assume that $\mathbb{P}(K)$ is contained in $D_{W_1,W_2}$. Then there are three possibilities: either

- $K \subset w_1^\perp \otimes W_2^\vee$ for some $w_1 \in W_1$, or
- $K \subset W_1^\vee \otimes w_2^\perp$ for some $w_2 \in W_2$, or
- $K \subset \ker(W_1^\vee \otimes W_2^\vee \rightarrow U_1^\vee \otimes U_2^\vee)$ for some 2-dimensional subspaces $U_1 \subset W_1$ and $U_2 \subset W_2$.

In the first case we have $\{w_1\} \times \mathbb{P}(W_2) \subset X_K$, in the second case $\mathbb{P}(W_1) \times \{w_2\} \subset X_K$, and in the third case $\mathbb{P}(U_1) \times \mathbb{P}(U_2) \subset X_K$, so in all these cases the surface $X_K$ is not integral.

Now, let us show that the conditions are sufficient. So, assume $K \subset W_1^\vee \otimes W_2^\vee$ is such that the line $\mathbb{P}(K)$ is not contained in $D_{W_1,W_2}$ and does not intersect $\mathbb{P}(W_1^\vee) \times \mathbb{P}(W_2^\vee)$. Let $b_0 \in K$ be a bilinear form of rank 3 (it exists since $\mathbb{P}(K)$ is not contained in $D_{W_1,W_2}$). Then $b_0$ identifies $W_2$ with $W_1^\vee$ and under this identification $b_0$ corresponds to the identity in $\mathbb{P}(W_1^\vee \otimes W_1) \cong \mathbb{P}(\text{End}(W_1))$, so its zero locus is isomorphic to the flag variety $\text{Fl}(1,2;W_1) \subset \mathbb{P}(W_1) \times \mathbb{P}(W_1^\vee)$. The subspace $K$ is then determined by an operator $b \in \text{End}(W_1)$ defined up to a scalar multiple of the identity, and the condition that $\mathbb{P}(K)$ does not intersect $\mathbb{P}(W_1^\vee) \times \mathbb{P}(W_2^\vee)$ can be rephrased by saying that the pencil $\{b + t \text{id}\}$ does not contain operators of rank 1. From the Jordan Theorem it is clear that there are only three types of such $b$:

$$(0) \quad b \text{ is diagonal with three distinct eigenvalues;}$$
$$(1) \quad b \text{ has two Jordan blocks of sizes 2 and 1 with distinct eigenvalues;}$$
$$(2) \quad b \text{ has one Jordan block of size 3.}$$

It is easy to see that in case (0) the surface $X_K$ is smooth (hence of type 0), in case (1) it has one $A_1$ singularity (and is of type 2), and in case (3) it has one $A_2$ singularity (and is of type 4). It remains to note that case (0) happens precisely when the line $\mathbb{P}(K)$ is transversal to $D_{W_1,W_2}$. □

Consider the universal family of codimension 2 linear sections of $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$ that are sextic du Val del Pezzo surfaces. By Lemma 6.1 it can be described as follows. Consider the Grassmannian $\text{Gr}(2,W_1^\vee \otimes W_2^\vee)$ parameterizing all two-dimensional subspaces $K \subset W_1^\vee \otimes W_2^\vee$, and its open subset parameterizing subspaces satisfying conditions of Lemma 6.1

$$S := \{K \in \text{Gr}(2,W_1^\vee \otimes W_2^\vee) \mid \mathbb{P}(K) \not\subset D_{W_1,W_2} \quad \text{and} \quad \mathbb{P}(K) \cap (\mathbb{P}(W_1^\vee) \times \mathbb{P}(W_2^\vee)) = \emptyset\}.$$
Let $\mathcal{X} \subset W_1' \otimes W_2' \otimes O_S$ be the tautological rank 2 bundle, $\mathcal{X}^\perp \subset W_1 \otimes W_2 \otimes O_S$ its rank 7 annihilator, and let
\begin{equation}
\mathcal{X} := (\mathbb{P}(W_1) \times \mathbb{P}(W_2)) \times_{\mathbb{P}(W_1 \otimes W_2)} \mathbb{P}_S(\mathcal{X}^\perp)
\end{equation}
be the corresponding du Val family of sextic del Pezzo surfaces.

**Proposition 6.2.** Let $\mathcal{X} \to S$ be the du Val family of sextic del Pezzo surfaces defined by (53). Then
\begin{equation}
\mathcal{Z}_2 = \mathbb{P}_S(\mathcal{X}) \times_{\mathbb{P}(W_1' \otimes W_2')} \mathcal{D}_{W_1, W_2}, \quad \mathcal{Z}_3 = S \sqcup S,
\end{equation}
and the Brauer classes $\beta_{\mathcal{Z}_2}$ and $\beta_{\mathcal{Z}_3}$ are both trivial.

**Proof.** Using homological projective duality for $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$, see Theorem [C.1], we obtain a semiorthogonal decomposition
\begin{equation}
\mathcal{D}(\mathcal{X}) = \langle \Phi_{\mathcal{X}}(\mathcal{D}(\mathbb{P}_S(\mathcal{X}) \times_{\mathbb{P}(W_1' \otimes W_2')} \mathbb{Y}_2)), \mathcal{D}(S) \otimes O_X(1, 0), \mathcal{D}(S) \otimes O_X(0, 1), \mathcal{D}(S) \otimes O_X(1, 1) \rangle,
\end{equation}
where $\mathbb{Y}_2 \to \mathcal{D}_{W_1, W_2}$ is the resolution of singularities defined in (69), and $\mathcal{E}_2$ is the derived pullback of the sheaf $\mathcal{E}_2$ defined by (73) with respect to the natural map
\begin{equation}
\mathcal{X} \times_S (\mathbb{P}_S(\mathcal{X}) \times_{\mathbb{P}(W_1' \otimes W_2')} \mathbb{Y}_2) \to \mathcal{Z}_2 \times_{\mathbb{P}(W_1' \otimes W_2')} \mathbb{Y}_2
\end{equation}
(we will discuss this map below), where $\mathcal{Z}_2$ is the universal hyperplane section of $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$. Mutating the last component to the far left, we get
\begin{equation}
\mathcal{D}(\mathcal{X}) = \langle \mathcal{D}(S) \otimes O_X, \Phi_{\mathcal{X}}(\mathcal{D}(\mathbb{P}_S(\mathcal{X}) \times_{\mathbb{P}(W_1' \otimes W_2')} \mathbb{Y}_2)), \mathcal{D}(S) \otimes O_X(1, 0), \mathcal{D}(S) \otimes O_X(0, 1) \rangle.
\end{equation}
We claim that this decomposition agrees with the general decomposition (46) of a du Val family of sextic del Pezzo surfaces. Indeed, the last two components of (55) can be considered as the derived category of $S \sqcup S$ (the trivial double covering of $S$) embedded via the Fourier–Mukai functor with kernel
\begin{equation}
\mathcal{E}_3 := O_X(1, 0) \sqcup O_X(0, 1) \in \mathcal{D}(S) \times S (S \sqcup S),
\end{equation}
and the second component is the derived category of
\begin{equation}
\mathbb{P}_S(\mathcal{X}) \times_{\mathbb{P}(W_1' \otimes W_2')} \mathbb{Y}_2 = \mathbb{P}_S(\mathcal{X}) \times_{\mathbb{P}(W_1' \otimes W_2')} \mathcal{D}_{W_1, W_2}
\end{equation}
(recall that $\mathbb{Y}_2$ maps birationally onto $\mathcal{D}_{W_1, W_2}$ and $\mathbb{P}_S(\mathcal{X})$ by definition of $S$ does not touch the indeterminacy locus of that birational isomorphism), which is a flat degree 3 covering of $S$ (since $\mathcal{D}_{W_1, W_2}$ is a hypersurface of degree 3), and it is embedded via the Fourier–Mukai functor with kernel $\mathcal{E}_2$.

Let us check that both $\mathcal{E}_2$ and $\mathcal{E}_3$ are flat families of torsion-free rank 1 sheaves on the fibers of $\mathcal{X}$ over $S$ with Hilbert polynomials $h_2(t)$ and $h_3(t)$ respectively, parameterized by the schemes
\begin{equation}
\mathcal{Z}_2 := \mathbb{P}_S(\mathcal{X}) \times_{\mathbb{P}(W_1' \otimes W_2')} \mathbb{Y}_2 \quad \text{and} \quad \mathcal{Z}_3 := S \sqcup S.
\end{equation}
For the second family flatness is clear and Hilbert polynomial computation is straightforward, so we skip it. For the first family we note that the map (51) is flat. Indeed, by (53) we have
\begin{equation}
\mathcal{X} \times_S (\mathbb{P}_S(\mathcal{X}) \times_{\mathbb{P}(W_1' \otimes W_2')} \mathbb{Y}_2) = \left( (\mathbb{P}(W_1) \times \mathbb{P}(W_2)) \times_{\mathbb{P}(W_1 \otimes W_2)} \mathbb{P}_S(\mathcal{X}^\perp) \right) \times_S \left( \mathbb{P}_S(\mathcal{X}) \times_{\mathbb{P}(W_1' \otimes W_2')} \mathbb{Y}_2 \right),
\end{equation}
while
\begin{equation}
\mathcal{X} \times_{\mathbb{P}(W_1' \otimes W_2')} \mathbb{Y}_2 = (\mathbb{P}(W_1) \times \mathbb{P}(W_2)) \times_{\mathbb{P}(W_1 \otimes W_2)} Q \times_{\mathbb{P}(W_1' \otimes W_2')} \mathbb{Y}_2,
\end{equation}
where $Q \subset \mathbb{P}(W_1 \otimes W_2) / \mathbb{P}(W_1' \otimes W_2')$ is the incidence quadric, so the map (51) is induced by the natural map $\mathbb{P}_S(\mathcal{X}^\perp) \times S \mathbb{P}_S(\mathcal{X}) \to Q = \text{Fl}(1, 8; W_1' \otimes W_2')$. This map factors as
\begin{equation}
\mathbb{P}_S(\mathcal{X}^\perp) \times S \mathbb{P}_S(\mathcal{X}) \hookrightarrow \mathbb{P}_{\text{Gr}(2, W_1' \otimes W_2')} (\mathcal{X}^\perp) \times_{\text{Gr}(2, W_1' \otimes W_2')} \mathbb{P}_{\text{Gr}(2, W_1' \otimes W_2')} (\mathcal{X}) \to Q.
\end{equation}
The first map is an open embedding, while the last map is a $\mathbb{P}^8$-bundle, so the composition is flat (and even smooth).
Recall the embedding of the Weil divisor $\mathbb{Y}_2 \times \mathbb{P}(W_2) \hookrightarrow \mathcal{X}_2 \times_{\mathbb{P}(W_1) \times \mathbb{P}(W_2)} \mathbb{Y}_2$ defined by (71). Its preimage under the map (54) is a Weil divisor in $\mathcal{X} \times_S (\mathbb{P}(\mathcal{X}) \times_{\mathbb{P}(W_1) \times \mathbb{P}(W_2)} \mathbb{Y}_2)$ such that for any geometric point $(K, b, w_1)$ of $\mathcal{X}_2 := \mathbb{P}(\mathcal{X}) \times_{\mathbb{P}(W_1) \times \mathbb{P}(W_2)} \mathbb{Y}_2$ its fiber is the Weil divisor $L := X_K \times_{\mathbb{P}(W_1)} \{w_1\} \subset X_K$, i.e., a line on the sextic del Pezzo surface $X_K$ contracted by the projection $X_K \to \mathbb{P}(W_1)$ to the point $\{w_1\} \in \mathbb{P}(W_1)$. Thus, by (73), the sheaf we are interested in is the twisted ideal $E_{\mathcal{X}_2}$ of $\mathcal{X}_2$.

By Proposition 5.14 there is a pair of rank 3 vector bundles $E_2$ and $E_3$ such that the Brauer classes $\beta_{E_2}$ and $\beta_{E_3}$ are trivial and $E_2$ and $E_3$ are isomorphic (up to twists by line bundles on $\mathcal{X}_2$ and $\mathcal{X}_3$ respectively) to the pullbacks of the universal bundles $\mathcal{E}_{\mathcal{X}_2}$ and $\mathcal{E}_{\mathcal{X}_3}$. This means that for $d \in \{2, 3\}$ we have isomorphisms of Fourier–Mukai functors $\Phi_{E_d} \cong \Phi_{E_{\mathcal{X}_d}} \circ \mu_{d*} \circ T_d : D(\mathcal{X}_d) \to D(\mathcal{X})$, where $T_d$ is a line bundle twist in $D(\mathcal{X}_d)$.

Since both $\Phi_{E_{\mathcal{X}_d}}$ and $\Phi_{E_d}$ are fully faithful, so is the composition $\mu_{d*} \circ T_d$. Moreover, comparing semiorthogonal decompositions given by (10) and (55), we conclude that $\mu_{d*} \circ T_d$ is essentially surjective, i.e., an equivalence of categories. Since $T_d$ is also an equivalence, we conclude that $\mu_{d*}$ is an equivalence, hence $\mu_d$ is an isomorphism. Thus $\mathcal{X}_d \cong \mathcal{X}_d(\mathcal{X}/S)$ and the Brauer classes $\beta_{\mathcal{X}_2}$ and $\beta_{\mathcal{X}_3}$ on $\mathcal{X}_2(\mathcal{X}/S)$ and $\mathcal{X}_3(\mathcal{X}/S)$ vanish.

Remark 6.3. Considering the family $\mathcal{X} \subset \mathbb{P}(W_1) \times \mathbb{P}(W_2)$ of all codimension 2 linear sections of $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$ over $S := \text{Gr}(2, W_1^\vee \otimes W_2^\vee)$ and applying the semiorthogonal decomposition of Theorem C.1 we obtain $D(\mathcal{X}) = (D(S), D(\mathcal{X}_2), D(\mathcal{X}_3))$, where $\mathcal{X}_2 = \mathbb{P}(\mathcal{X}) \times_{\mathbb{P}(W_1^\vee \otimes W_2^\vee)} \mathbb{Y}_2$ and $\mathcal{X}_3 = S \sqcup S$. Note that the map $\mathcal{X}_2 \to S$ is not flat — its non-flat locus $S \setminus S$ is equal to the non-integral locus of the family $\mathcal{X} \to S$.

The statement of Proposition 6.2 can be inverted as follows.

Lemma 6.4. Let $\mathcal{X} \to S$ be a du Val family of sextic del Pezzo surfaces. If $\mathcal{X}_3(\mathcal{X}/S) = S \sqcup S$ and the Brauer class $\beta_{\mathcal{X}_3}$ is trivial, then Zariski locally over $S$ the family $\mathcal{X} \to S$ can be represented as a family of codimension 2 linear sections of $\mathbb{P}^2 \times \mathbb{P}^2$. In particular, $\beta_{\mathcal{X}_2}$ is trivial.

Proof. By Proposition 5.14 there is a pair of rank 3 vector bundles $\mathcal{W}_1$ and $\mathcal{W}_2$ on $S$ such that $F_3(\mathcal{X}/S) = \mathbb{P}(\mathcal{W}_1) \sqcup \mathbb{P}(\mathcal{W}_2)$.

Moreover, the restrictions of the universal sheaf $\mathcal{E}_{\mathcal{X}_3}$ on $\mathcal{X} \times_S \mathcal{X}_3 = \mathcal{X} \sqcup \mathcal{X}$ to the two components are line bundles defining regular maps $\mathcal{X} \to \mathbb{P}(\mathcal{W}_1)$ and $\mathcal{X} \to \mathbb{P}(\mathcal{W}_2)$ respectively. Thus, the induced map $\mathcal{X} \to \mathbb{P}(\mathcal{W}_1) \times_S \mathbb{P}(\mathcal{W}_2)$ is a closed embedding. Zariski locally the bundles $\mathcal{W}_1$ and $\mathcal{W}_2$ are trivial, hence we obtain the required local presentation of $\mathcal{X}$.

A slightly more general family of sextic del Pezzo surfaces can be obtained by replacing the projective spaces $\mathbb{P}(W_1)$ and $\mathbb{P}(W_2)$ by a pair of projectively dual (i.e., corresponding to mutually inverse Brauer classes) Severi–Brauer planes. In this way one can obtain a family of sextic del Pezzo surfaces with a nontrivial Brauer class $\beta_{\mathcal{X}_3}$ (however, this class will be “constant in a family”). Another possible
generalization, is to consider a double covering \( \tilde{S} \to S \) and a Brauer class \( \beta \) on \( \tilde{S} \) of order 3 such that \( \beta = \sigma^* (\beta^{-1}) \), where \( \sigma : \tilde{S} \to S \) is the involution of the double covering (cf. the isomorphism of Brauer classes in Proposition 5.3). Then one can apply the Weil restriction of scalars to obtain an étale locally trivial fibration over \( S \) with fibers \( \mathbb{P}^2 \times \mathbb{P}^2 \), and then consider its linear section of codimension 2.

6.2. Hyperplane sections of \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \). Another simple way to construct a sextic del Pezzo surface is by considering a hyperplane section of \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) in the Segre embedding.

We denote by \( V_1, V_2, \) and \( V_3 \) three vector spaces of dimension 2 and let

\[
\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3) \to \mathbb{P}(V_1 \otimes V_2 \otimes V_3)
\]

be the Segre embedding. For a trilinear form \( b \in V_1^\vee \otimes V_2^\vee \otimes V_3^\vee \) we denote by

\[
X_b := (\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)) \cap \mathbb{P}(b^\perp) \subset \mathbb{P}(V_1 \otimes V_2 \otimes V_3)
\]

the corresponding hyperplane section, where \( b^\perp \subset V_1 \otimes V_2 \otimes V_3 \) is the annihilator hyperplane of \( b \).

Recall that the group \( G := (\text{PGL}(V_1) \times \text{PGL}(V_2) \times \text{PGL}(V_3)) \rtimes \mathfrak{S}_3 \) acts on \( \mathbb{P}(V_1^\vee \otimes V_2^\vee \otimes V_3^\vee) \) with four orbits. The orbits closures are:

- \( O_3 = \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3^\vee) \);
- \( \overline{O}_4 = (\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee \otimes V_3^\vee)) \cup (\mathbb{P}(V_2^\vee) \times \mathbb{P}(V_1^\vee \otimes V_3^\vee)) \cup (\mathbb{P}(V_3^\vee) \times \mathbb{P}(V_1^\vee \otimes V_2^\vee));\)
- \( \overline{O}_6 = (\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3^\vee))^\vee \) is the projectively dual quartic hypersurface; and
- \( \overline{O}_7 = \mathbb{P}(V_1^\vee \otimes V_2^\vee \otimes V_3^\vee) \)

(we use the dimensions of the orbits as indices). The equation of the quartic hypersurface \( \overline{O}_6 \) is given by the Cayley’s hyperdeterminant, see [GKZ08, 14.1.7].

Lemma 6.5. Assume the base field \( k \) is algebraically closed. The hyperplane section \( X_b \) defined by \( b \) is a sextic du Val del Pezzo surface if and only if \( b \in \mathbb{P}(V_1^\vee \otimes V_2^\vee \otimes V_3^\vee) \setminus \overline{O}_4 \). Furthermore, \( X_b \) is smooth if and only if \( b \in \mathbb{P}(V_1^\vee \otimes V_2^\vee \otimes V_3^\vee) \setminus \overline{O}_6 \).

Proof. Indeed, choosing a representative of each orbit, it is easy to see that \( X_b \) is smooth, if \( b \in O_7 \); has one \( A_1 \) singularity (and is of type 1), if \( b \in O_6 \); is a union of a smooth quadric and a quartic scroll, if \( b \in O_4 \); and is a union of three quadrics, if \( b \in O_3 \). \( \square \)

Consider the universal family of hyperplane sections of \( \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3) \) that are sextic du Val del Pezzo surfaces. By Lemma 6.5 it can be described as follows. Consider the open subset

\[
S := O_6 \cup O_7 = \mathbb{P}(V_1^\vee \otimes V_2^\vee \otimes V_3^\vee) \setminus \overline{O}_4 \subset \mathbb{P}(V_1^\vee \otimes V_2^\vee \otimes V_3^\vee).
\]

Let \( \mathcal{L} \subset V_1^\vee \otimes V_2^\vee \otimes V_3^\vee \otimes \mathcal{O}_S \) be the tautological line bundle, let \( \mathcal{L}^\perp \subset V_1 \otimes V_2 \otimes V_3 \otimes \mathcal{O}_S \) be its rank 7 annihilator, and let

\[
\mathcal{S} := (\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)) \times_{\mathbb{P}(V_1 \otimes V_2 \otimes V_3)} \mathbb{P}(\mathcal{L}^\perp)
\]

be the corresponding du Val family of sextic del Pezzo surfaces.

Denote by \( \mathcal{P}_{V_1, V_2, V_3} \to \mathbb{P}(V_1^\vee \otimes V_2^\vee \otimes V_3^\vee) \) the double covering branched along the Cayley quartic hypersurface \( \overline{O}_6 \).

Proposition 6.6. Let \( \mathcal{S} \to S \) be the du Val family of sextic del Pezzo surfaces defined by \( \mathcal{L} \). Then

\[
\varnothing_2 = S \cup S \cup S, \quad \varnothing_3 = S \times_{\mathbb{P}(V_1^\vee \otimes V_2^\vee \otimes V_3^\vee)} \mathcal{P}_{V_1, V_2, V_3},
\]

and the Brauer classes \( \beta_{\varnothing_2} \) and \( \beta_{\varnothing_3} \) are both trivial.
Proof. The proof is parallel to that of Proposition 6.2.

Using homological projective duality for $\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)$, see Theorem D.1, we obtain a semiorthogonal decomposition

$$\mathbf{D}(\mathscr{X}) = \langle \Phi_{\mathcal{E}_3}(\mathbf{D}(S \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3)), \mathbf{D}(S) \otimes \mathcal{O}_X(1, 1, 1), \mathbf{D}(S) \otimes \mathcal{O}_X(2, 1, 1), \mathbf{D}(S) \otimes \mathcal{O}_X(1, 2, 1), \mathbf{D}(S) \otimes \mathcal{O}_X(1, 1, 2) \rangle,$$

where $\mathbb{P}(V_3) \to \mathcal{D}_{V_1, V_2, V_3}$ is the resolution of singularities defined in (76) and where $\mathcal{E}_3$ is the derived pullback of the sheaf $\mathcal{E}_3$ defined by (80) with respect to the natural map

$$\mathcal{X} \times_S (S \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3)) = \mathcal{X}_3 \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3)$$

(we will discuss this map below). Mutating the last four components to the far left, we get

$$\mathbf{D}(\mathcal{X}) = \langle \mathbf{D}(S) \otimes \mathcal{O}_X, \mathbf{D}(S) \otimes \mathcal{O}_X(1, 0, 0), \mathbf{D}(S) \otimes \mathcal{O}_X(0, 1, 0), \mathbf{D}(S) \otimes \mathcal{O}_X(0, 0, 1), \Phi_{\mathcal{E}_3}(\mathbf{D}(S \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3))) \rangle,$$

We claim that this decomposition agrees with the general decomposition (46) for a du Val family of sextic del Pezzo surfaces. Indeed, its second, third and fourth components can be considered as the derived category of $S \sqcup S \sqcup S$ (the trivial triple covering of $S$) embedded via the Fourier–Mukai functor with kernel

$$\mathcal{E}_2 := \mathcal{O}_X(1, 0, 0) \sqcup \mathcal{O}_X(0, 1, 0) \sqcup \mathcal{O}_X(0, 0, 1) \in \mathbf{D}(\mathcal{X} \sqcup \mathcal{X} \sqcup \mathcal{X}) = \mathbf{D}(\mathcal{X} \times_S (S \sqcup S \sqcup S)),$$

while the last component is the derived category of $S \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3) \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathcal{D}_{V_1, V_2, V_3}$ (recall that $\mathbb{P}(V_3)$ maps birationally onto $\mathcal{D}_{V_1, V_2, V_3}$ and $S$ by definition does not touch the image of the indeterminacy locus of that birational isomorphism), which is a flat degree 2 covering of $S$ (since $\mathcal{D}_{V_1, V_2, V_3}$ is flat over the complement of $O_3$), and it is embedded via the Fourier–Mukai functor with kernel $\mathcal{E}_3$.

Let us check that both $\mathcal{E}_2$ and $\mathcal{E}_3$ are flat families of torsion-free rank 1 sheaves on the fibers of $\mathcal{X}$ over $S$ with Hilbert polynomials $h_2(t)$ and $h_3(t)$ respectively, parameterized by

$$\mathcal{X}_2^0 := S \sqcup S \sqcup S \quad \text{and} \quad \mathcal{X}_3^0 := S \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3).$$

For the first flatness is clear and Hilbert polynomial computation is straightforward, so we skip it. For the second we note that the map (58) is flat. Indeed,

$$\mathcal{X}_3 \times_S (S \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3)) = \left( (\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)) \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3) \right) \times_S \left( \mathbb{P}(V_3) \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3) \right),$$

while

$$\mathcal{X}_3 \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3) = \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3) \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3) \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} Q \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3),$$

where $Q \subset \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3) \times \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)$ is the incidence quadric, so the map (58) is induced by the map $\mathbb{P}(S(\mathcal{L})) \times_S \mathbb{P}(\mathcal{L}) \to Q = \text{Fl}(1, 7; V_1 \otimes V_2 \otimes V_3)$ which is an open embedding.

Recall the embedding of the Weil divisor $\mathbb{P}(V_3) \to \mathcal{X}_3 \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3)$ defined by (75). The preimage of $\mathbb{P}(V_3)$ under (58) is a Weil divisor in $\mathcal{X} \times_S (S \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3))$ such that for any closed point $(b, v_1, v_2)$ of $\mathcal{X}_3 \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3)} \mathbb{P}(V_3)$ its fiber is the Weil divisor

$$L := X_b \times_{\mathbb{P}(V_1) \times \mathbb{P}(V_2)} \{ (v_1, v_2) \} \subset X_b,$$

i.e., a line on the sextic del Pezzo surface $X_b$ contracted by the projection $X_b \to \mathbb{P}(V_1) \times \mathbb{P}(V_2)$ to the point $\{ (v_1, v_2) \} \subset \mathbb{P}(V_1) \times \mathbb{P}(V_2)$. Thus, by (80), the sheaf we are interested in is the twisted ideal $\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}(V_1) \times \mathbb{P}(V_2)}(1, 1)$. In particular, it is torsion-free, and its Hilbert polynomial equals $h_3(t)$.

The rest of the proof repeats the argument of Proposition 6.2.\hfill\Box

The statement of Proposition 6.6 can be inverted as follows (and the proof repeats the proof of Lemma 6.4).
Lemma 6.7. Let $\mathcal{X} \to S$ be a du Val family of sextic del Pezzo surfaces. If $\mathcal{Z}_2(\mathcal{X}/S) = S \sqcup S \sqcup S$ and the Brauer class $\beta_{\mathcal{Z}_2}$ is trivial, then Zariski locally over $S$ the family $\mathcal{X} \to S$ can be represented as a family of hyperplane sections of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In particular, $\beta_{\mathcal{Z}_3}$ is trivial.

6.3. Blowup families. For each length 3 subscheme $Y \subset \mathbb{P}^2$, consider the blowup

$$\hat{X} := \text{Bl}_Y(\mathbb{P}^2).$$

Unless $Y$ is the second neighborhood of a point (i.e., is given by the square of the maximal ideal of a point), it is a weak del Pezzo surface of degree 6. In particular, the anticanonical class of $\hat{X}$ is nef and big and the anticanonical model of $\hat{X}$ (i.e., the image of $\hat{X}$ under the anticanonical map) is a sextic du Val del Pezzo surface.

This construction can be also performed in a family. Let

$$S := \langle \mathbb{P}^2 \rangle^3 \setminus \mathbb{P}^2$$

be the open subset of the Hilbert cube of the plane parameterizing length 3 subschemes in $\mathbb{P}^2$ avoiding second neighborhoods of points. Let $\mathcal{Y} \subset \mathbb{P}^2 \times S$ be the corresponding family of subschemes. Let

$$\tilde{\mathcal{X}} := \text{Bl}_{\mathcal{Y}}(\mathbb{P}^2 \times S)$$

be the blowup and $\mathcal{X} \to S$ its relative anticanonical model. In particular, we have a morphism

$$\tilde{\pi} : \tilde{\mathcal{X}} \to \mathcal{X}.$$

Let $S_1 \subset S$ be the divisor parameterizing subschemes in $\mathbb{P}^2$ contained in a line. It is easy to see that over $S_1$ there is a $\mathbb{P}^1$-bundle $\Delta \to S_1$ (formed by strict transforms of lines supporting the subschemes) and an embedding $\Delta \hookrightarrow \tilde{\mathcal{X}}$, such that $\mathcal{X}$ is the contraction of $\Delta$ to $S_1$. In particular, over $S_0 := S \setminus S_1$ the map $\tilde{\pi}$ is an isomorphism. Thus $\tilde{\mathcal{X}}$ is a small resolution of singularities of $\mathcal{X}$ (we will see below that the scheme $\tilde{\mathcal{X}}$ is smooth over $S$).

Proposition 6.8. Let $S$ be defined by (60) and let $\mathcal{X} \to S$ be the relative anticanonical model of the blowup $\tilde{\mathcal{X}} = \text{Bl}_{\mathcal{Y}}(\mathbb{P}^2 \times S) \to S$. Then

$$\mathcal{Z}_2 \cong \mathcal{Y} \quad \text{and} \quad \mathcal{Z}_3 \cong S \sqcup_{S_1} S,$$

and both Brauer classes $\beta_{\mathcal{Z}_2}$ and $\beta_{\mathcal{Z}_3}$ are trivial.

In this example both $\mathcal{X}$ and $\mathcal{Z}_3$ are not regular.

Proof. Note that $\mathcal{Y}$ is smooth over $\mathbb{P}^2$. Indeed, the fiber of the projection $\mathcal{Y} \to \mathbb{P}^2$ over a point $P \in \mathbb{P}^2$ is an open subset in the Hilbert square (Bl$_P(\mathbb{P}^2))^2$ (we exclude subschemes supported on the exceptional divisor of the blowup). It follows that $\tilde{\mathcal{X}}$ is also smooth.

Using the blowup formula, we obtain a semiorthogonal decomposition

$$\mathbf{D}(\tilde{\mathcal{X}}) = \langle \mathbf{D}(\mathbb{P}^2 \times S), \mathbf{D}(\mathcal{Y}) \rangle.$$

The relative version of the sequence of mutations described in the proof of Proposition 3.1 gives

$$\mathbf{D}(\tilde{\mathcal{X}}) = \langle \mathbf{D}(S), L_{\mathbf{D}(S) \otimes \mathcal{O}_{\mathcal{Y}}(h)}(\mathbf{D}(\mathcal{Y})), \mathbf{D}(S) \otimes \mathcal{O}_{\tilde{\mathcal{X}}}(h), \mathbf{D}(S) \otimes \mathcal{O}_{\tilde{\mathcal{X}}}(2h - e) \rangle,$$

where $e$ is the class of the exceptional divisor $E$ of the blowup $\tilde{\mathcal{X}} = \text{Bl}_{\mathcal{Y}}(\mathbb{P}^2 \times S)$.

For every geometric point $s \in S$ the fiber $\tilde{\mathcal{X}}_s$ is the partial contraction of the minimal resolution $\tilde{\mathcal{X}}_s$ of $\mathcal{X}_s$. In fact, we have a chain of contractions

$$\tilde{\mathcal{X}}_s \to \mathcal{X}_s \to \mathcal{X},$$
the first map contracts the (−2)-curves $\Delta_{12}$ and $\Delta_{23}$ (if they exist), and the second map contracts $\Delta_{13}$. So, it is natural to expect that the pushforward functor $\tilde{\pi}_*: D(\mathcal{X}) \to D(\mathcal{Y})$ induces an equivalence $\mathbb{L}D(S) \otimes \mathcal{O}_{\mathcal{X}}(h) \cong D(\mathcal{X}, \beta_{\mathcal{X}})$ and a localization $(D(S) \otimes \mathcal{O}_{\mathcal{X}}(2h-e)) \to D(\mathcal{X}, \beta_{\mathcal{X}})$. Below we show that this is indeed the case.

Denote by $E$ the exceptional divisor of the blowup $\tilde{\pi}: \mathcal{X} \to \mathbb{P}^2 \times S$. Consider the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\pi} & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathbb{P}^2 \times S & \xrightarrow{\tilde{\pi}} & \mathcal{Y} \\
& & \downarrow \\
& & \mathcal{X}.
\end{array}
$$

Consider $E$ as a subscheme in $\mathcal{X} \times_S \mathcal{Y}$ and let $\tilde{\pi}_*: \mathcal{X} \times_S \mathcal{Y} \to \mathcal{X} \times_S \mathcal{Y}$ be the natural projection. Then the embedding of the second component of $(61)$ is given by the Fourier–Mukai functor with the kernel

$$
\mathcal{F}_2 \cong \tilde{\pi}_*(\mathcal{I}_{E, \mathcal{X} \times_S \mathcal{Y}}(h)) \in D(\mathcal{X} \times_S \mathcal{Y}).
$$

Take a geometric point $y \in \mathcal{Y}$, let $s \in S$ be its image in $S$. Denote $X = \mathcal{X}_s$ and let $Z_2$ and $Z_3$ be the corresponding schemes of length 3 and 2 respectively. Using the base change, it is easy to see that the sheaf $(\mathcal{F}_2)_y$ is isomorphic to $\mathcal{E}_z$ for some $z \in Z_2$. Thus $\mathcal{F}_2$ is a family of semistable sheaves on fibers of $\mathcal{X}$ over $S$ with Hilbert polynomial $h_2(t)$. It follows that there is a map $\mu_2: \mathcal{Y} \to \mathcal{X} \times_S \mathcal{Y}$ (over $S$) such that $\mathcal{F}_2 \cong \mu_2^*\mathcal{E}_{\mathcal{X}} \otimes \mathcal{L}_2$ for a line bundle $\mathcal{L}_2$ on $\mathcal{Y}$, and $\mu_2^*\beta_{\mathcal{X}}$ is trivial. It is easy to see that over each geometric point $s \in S$ the morphism $\mu_2$ is an isomorphism, hence it is an isomorphism everywhere (since both $\mathcal{Y}$ and $\mathcal{F}_2(\mathcal{X}/S)$ are flat of degree 3 over $S$).

Furthermore, consider the sheaves $\mathcal{O}_{\mathcal{X}}(h)$ and $\tilde{\pi}_*(\mathcal{I}_{E, \mathcal{X}}(2h))$ on $\mathcal{X}$. It is easy to check that over the divisor $S_1$ there is a morphism $\mathcal{O}_{\mathcal{X}}(h) \to \mathcal{I}_{E, \mathcal{X}}(2h)$ whose cone is a sheaf supported on $\Delta$ and which restricts as $\mathcal{O}(-1)$ to each fiber of $\Delta$ over $S_1$. Therefore,

$$
\mathcal{O}_{\mathcal{X}}(h)|_{\mathcal{X}_1} \cong \tilde{\pi}_*(\mathcal{I}_{E, \mathcal{X}}(2h))|_{\mathcal{X}_1},
$$

where $\mathcal{X}_1 := \mathcal{X} \times_S S_1$, which allows to glue these two sheaves on $\mathcal{X} \cup_{\mathcal{X}_1} \mathcal{X} = \mathcal{X} \times_S \left(S \cup_{S_1} S\right)$ into a single sheaf $\mathcal{F}_3$. A similar argument to the above shows that $\mathcal{F}_3$ is a family of semistable sheaves on fibers of $\mathcal{X}$ over $S$ with Hilbert polynomial $h_3(t)$, hence there is a map $\mu_3: S \cup_{S_1} S \to \mathcal{X} \times_S \mathcal{Y}$ (over $S$) such that $\mathcal{F}_3 \cong \mu_3^*\mathcal{E}_{\mathcal{X}_1} \otimes \mathcal{L}_3$ for a line bundle $\mathcal{L}_3$ on $S \cup_{S_1} S$, and $\mu_3^*\beta_{\mathcal{X}}$ is trivial. It is easy to see that over each geometric point $s \in S$ the morphism $\mu_3$ is an isomorphism, hence it is an isomorphism everywhere (since both $S \cup_{S_1} S$ and $\mathcal{F}_3(\mathcal{X}/S)$ are flat of degree 2 over $S$).

A similar argument applies to blowups of $\mathbb{P}^1 \times \mathbb{P}^1$.

**Proposition 6.9.** Let $S := (\mathbb{P}^1 \times \mathbb{P}^1)[2]$ be the Hilbert square of $\mathbb{P}^1 \times \mathbb{P}^1$ and let $\mathcal{Y} \subset (\mathbb{P}^1 \times \mathbb{P}^1) \times S$ be the universal family of subschemes. Let $\mathcal{X} := \text{Bl}_{\mathcal{Y}}(\mathbb{P}^1 \times \mathbb{P}^1) \times S$ be the blowup and let $\mathcal{X} \to S$ be its relative anticanonical model. Let $S_{1,0} \subset S$ and $S_{0,1} \subset S$ be the divisors parameterizing subschemes contained in a horizontal or a vertical ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ respectively. Then

$$
\mathcal{F}_2 \cong S \cup_{S_{1,0}} S \cup_{S_{0,1}} S \quad \text{and} \quad \mathcal{F}_3 \cong \mathcal{Y}
$$

and both Brauer classes $\beta_{\mathcal{X}}$ and $\beta_{\mathcal{Y}}$ are trivial.

**Appendix A. Auslander algebras**

Let $Z = \text{Spec}(k[t]/t^m)$ be a non-reduced zero-dimensional scheme. The Auslander algebra $\tilde{R}_m$ defined below provides a categorical resolution of the derived category $D(Z)$, see [KL15, Section 5] for details.
The algebra $\tilde{R}_m$ is defined as the path algebra of a quiver with relations:

$$\tilde{R}_m = \mathbb{k}\left\langle \bullet \overset{\alpha_1}{\underset{\beta_1}{\overset{\alpha_2}{\underset{\beta_2}{\cdots}}}} \bullet \cdots \overset{\alpha_{m-1}}{\underset{\beta_{m-1}}{\bullet}} \bigg| \beta_i\alpha_i = \alpha_{i+1}\beta_{i+1} \text{ for } 0 \leq i \leq m-2, \beta_{m-1}\alpha_{m-1} = 0 \right\rangle.$$ 

Alternatively, it can be written as a matrix algebra

$$\tilde{R}_m = \bigoplus_{i,j=0}^{m-1} (\tilde{R}_m)_{ij}, \quad (\tilde{R}_m)_{ij} = \begin{cases} k[t]/t^{m-i}, & \text{if } i \geq j, \\ t^{-i}k[t]/t^{m-i}, & \text{if } i \leq j, \end{cases}$$

with multiplication induced by the natural maps $(\tilde{R}_m)_{ij} \otimes (\tilde{R}_m)_{jk} \to (\tilde{R}_m)_{ik}$. We identify the category of representations of the quiver with the category of left modules over its path algebra.

We denote by $\epsilon_i$ the $i$-th vertex idempotent in $\tilde{R}_m$ (in terms of (63) it is the unit in $(\tilde{R}_m)_{ii} \cong k[t]/t^{m-i}$).

For every $\tilde{R}_m$-module $M$ we have

$$M = \bigoplus_{i=0}^{m-1} \epsilon_i M.$$ 

We call $M_i := \epsilon_i M$ the $i$-th component of $M$, and the vector $(\dim M_0, \dim M_1, \ldots) \in \mathbb{Z}^m$ the dimension vector of $M$.

For each $0 \leq i \leq m-1$ we denote by $S_i$ the simple module of $i$-th vertex of the quiver (its dimension vector is $(0, \ldots, 0, 1, 0, \ldots, 0)$) and by

$$P_i = \tilde{R}_m \epsilon_i$$

its projective cover (projective module of $i$-th vertex).

The algebra $\tilde{R}_m$ has finite global dimension (it is bounded by $2m-2$, see [KL15 Proposition A.14]) and its derived category $D(\tilde{R}_m)$ is generated by an exceptional collection consisting of representations $E_i$ with dimension vectors

$$\dim(E_i) = (1, \ldots, 1, 0, \ldots, 0)$$

and with $\beta$-arrows acting by zero and $\alpha$-arrows acting by identity. We call the $E_i$ standard exceptional modules. These exceptional modules have simple projective resolutions

$$0 \to P_{i+1} \overset{\beta_{i+1}}{\to} P_i \to E_i \to 0,$$

with the maps induced by the right $\beta_{i+1}$-multiplication. Using these, it is easy to compute the derived endomorphism algebra of the exceptional collection.

**Lemma A.1.** The collection $E_0, E_1, \ldots, E_{m-1}$ is exceptional and $\Ext^i(E_i, E_j) = \mathbb{k} \oplus \mathbb{k}[-1]$ for all $i < j$. Moreover, the multiplication map

$$\Ext^p(E_i, E_j) \otimes \Ext^q(E_j, E_k) \to \Ext^{p+q}(E_i, E_k), \quad i < j < k$$

is an isomorphism when $p = 0$ or $q = 0$.

**Proof.** Using (64) we see that $\Ext^i(E_i, E_j)$ is computed by the complex

$$(E_j)i \overset{\beta_{i+1}}{\to} (E_j)i+1.$$

If $i > j$ both spaces are zero and if $i = j$ the first is $\mathbb{k}$ and the second is zero, hence the collection is exceptional. Similarly, if $j > i$ both spaces are $\mathbb{k}$ and the arrow is zero, hence $\Hom(E_i, E_j) = \Ext^1(E_i, E_j) = \mathbb{k}$.

For the second statement, note that we have an exact sequence

$$0 \to E_{i-1} \overset{\alpha_i}{\to} E_i \to S_i \to 0,$$

and...
which shows that for $j \leq k$ the natural map $E_j \to E_k$ is injective and its cokernel is an extension of simple modules $S_l$ with $j + 1 \leq l \leq k$. On the other hand, (64) implies that $\text{Ext}^i(E_i, S_l) = 0$ as soon as $l \geq i + 2$, hence the map

$$\text{Ext}^i(E_i, E_j) \to \text{Ext}^i(E_i, E_k)$$

induced by the embedding $E_j \to E_k$ is an isomorphism. This proves the case when $q = 0$.

Similarly, merging (65) with (64) we obtain for $l \geq 1$ a projective resolution

$$0 \to P_l \xrightarrow{(-\alpha_{l+1}, \beta_l)} P_{l+1} \oplus P_{l-1} \xrightarrow{(\beta_{l+1}, \alpha_l)} P_l \to S_l \to 0$$

of the simple module $S_l$. It follows that $\text{Ext}^i(S_l, E_k)$ for $1 \leq l < k$ are computed by the complex

$$(E_k)_l \xrightarrow{(0,1)} (E_k)_{l+1} \oplus (E_k)_{l-1} \xrightarrow{(-1,0)} (E_k)_l,$$

hence for $1 \leq l < k$ we have $\text{Ext}^i(S_l, E_k) = 0$. On the other hand, the cokernel of the embedding $E_i \to E_j$ is an extension of simple modules $S_l$ with $1 \leq l \leq j$, hence the map

$$\text{Ext}^i(E_j, E_k) \to \text{Ext}^i(E_i, E_k)$$

induced by this embedding is an isomorphism. This proves the case when $p = 0$. \hfill \Box

The following characterization of the categories $D(\tilde{R}_2)$ and $D(\tilde{R}_3)$ is quite useful.

**Proposition A.2.** Assume $\mathcal{T}$ is a triangulated category admitting a DG enhancement.

(a) If $\mathcal{T}$ is generated by an exceptional pair $(\mathcal{L}_0, \mathcal{L}_1)$ such that $\text{Ext}^i(\mathcal{L}_0, \mathcal{L}_1) \cong k \oplus k[-1]$ then there is an equivalence of categories $\mathcal{T} \cong D(\tilde{R}_2)$ taking $\mathcal{L}_i$ to $E_i$.

(b) If $\mathcal{T}$ is generated by an exceptional triple $(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ such that $\text{Ext}^i(\mathcal{L}_0, \mathcal{L}_j) \cong k \oplus k[-1]$ for all $i < j$ and the multiplication map $\text{Ext}^p(\mathcal{L}_0, \mathcal{L}_1) \otimes \text{Ext}^q(\mathcal{L}_1, \mathcal{L}_2) \to \text{Ext}^{p+q}(\mathcal{L}_0, \mathcal{L}_2)$ is an isomorphism when $p = 0$ or $q = 0$, then there is an equivalence of categories $\mathcal{T} \cong D(\tilde{R}_3)$ taking $\mathcal{L}_i$ to $E_i$.

**Proof.** Since the category $\mathcal{T}$ is enhanced, there is an equivalence of $\mathcal{T}$ with the derived category of the DG algebra $\text{RHom}_{\mathcal{T}}(\oplus \mathcal{L}_i, \oplus \mathcal{L}_i)$. By the assumption, its cohomology is isomorphic (as a graded algebra) to the graded algebra $\text{Ext}^i_{D(\tilde{R}_m)}(\oplus E_i, \oplus E_i)$. To get the desired equivalence, it remains to check that the latter algebra (considered as a DG algebra with trivial differential) is formal.

But formality is clear, since any higher $A_{\infty}$-operation $m_i$ (with $i \geq 3$) requires at least three non-trivial arguments, so one needs the quiver to have at least four vertices to admit such an operation. So, for $\tilde{R}_2$ and $\tilde{R}_3$ all higher operations vanish and the algebra is formal. \hfill \Box

**Remark A.3.** Let $m \geq 4$ and assume $\mathcal{T}$ is an enhanced triangulated category generated by an exceptional collection $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{m-1}$ satisfying the properties of Lemma A.1. To establish an equivalence of $\mathcal{T}$ with $D(\tilde{R}_m)$ one should additionally check that higher $A_{\infty}$-operations in $\mathcal{T}$ vanish when one of the arguments is contained in the space $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1}) = k$.

The endomorphism algebra of the projective module $P_0$ is

$$\text{End}_{\tilde{R}_m}(P_0) = \epsilon_0 \tilde{R}_m \epsilon_0 = (\tilde{R}_m)_{00} = k[t]/t^m.$$ 

This defines an adjoint pair of functors

$$\pi_{m*}: \tilde{R}_m \text{-mod} \to (k[t]/t^m) \text{-mod}, \quad M \mapsto \text{Hom}_{\tilde{R}_m}(P_0, M) = \epsilon_0 M,$$

$$\pi^*_m: (k[t]/t^m) \text{-mod} \to \tilde{R}_m \text{-mod}, \quad N \mapsto P_0 \otimes_{k[t]/t^m} N.$$ 

Following our convention, we denote in the same way their derived functors. Note that the functor $\pi_{m*}: D(\tilde{R}_m) \to D(k[t]/t^m)$ preserves boundedness, while its left adjoint $\pi^*_m$ does not. Recall that $(-)^{\oplus}$ denotes the minimal triangulated subcategory of $D^-(\tilde{R}_m)$ closed under infinite direct sums.
Proposition A.4. The functor $\pi_m^*: \mathbf{D}^-(k[t]/t^m) \to \mathbf{D}^-(\hat{\mathcal{R}}_m)$ is fully faithful, and its right adjoint functor $\pi_{m*}: \mathbf{D}(\hat{\mathcal{R}}_m) \to \mathbf{D}(k[t]/t^m)$ is essentially surjective. We have

$$\text{Ker } \pi_{m*} = \langle S_1, \ldots, S_{m-1} \rangle^\circ \quad \text{and} \quad \text{Im } \pi_{m*} = \langle P_0 \rangle^\circ = \langle S_1, \ldots, S_{m-1} \rangle^\circ.$$  
Moreover, $\pi_{m*}(S_0) = k$ and $\pi_{m*}(P_0) = k[t]/t^m$.

Proof. The functor $\pi_m^*$ is fully faithful by [KL15, Theorem 5.23] and $\pi_{m*} \circ \pi_m^* \cong \text{id}$ by [KL15, (48)]. Furthermore, $\pi_{m*}$ is exact by definition, hence for any $N \in \mathbf{D}(k[t]/t^m)$, taking $M := \pi_m^*\pi_{m*}(N)$ (the canonical truncation) with $p \ll 0$, we obtain $N \cong \pi_{m*}M$, hence $\pi_{m*}$ is essentially surjective.

The image of $\pi_{m*}$ by definition equals the subcategory $\langle P_0 \rangle^\circ \subset \mathbf{D}^-(\hat{\mathcal{R}}_m)$ and this category is evidently equal to the orthogonal of the simple modules $S_1, \ldots, S_{m-1}$. Since the functor $\pi_{m*}$ is the right adjoint of $\pi_m^*$, we have $\text{Ker } \pi_{m*} = (\text{Im } \pi_m^*)^\perp = P_0^\perp = \langle S_1, \ldots, S_{m-1} \rangle^\circ$. Finally, applying (67) we easily get $\pi_{m*}(S_0) = \epsilon_0 S_0 = k$ and $\pi_{m*}(P_0) = \epsilon_0 P_0 = \epsilon_0 \hat{\mathcal{R}}_m \epsilon_0 = (\hat{\mathcal{R}}_m)_{00} = k[t]/t^m$. □

Appendix B. Moduli stack of sextic du Val del Pezzo surfaces

The moduli stack $\mathcal{D}_6$ of sextic du Val del Pezzo surfaces is the fibered category over $(\text{Sch}/k)$ whose fiber over a $k$-scheme $S$ is the groupoid of all du Val $S$-families of sextic del Pezzo surfaces $f: \mathcal{X} \to S$ (in the sense of Definition [5.1]). A morphism from $f': \mathcal{X}' \to S'$ to $f: \mathcal{X} \to S$ is a fiber product diagram

$$\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{f'} & \mathcal{X} \\
\downarrow f & & \downarrow f \\
S' & \to & S
\end{array}$$

The main result of this section was communicated by Jenya Tevelev.

Theorem B.1. The moduli stack $\mathcal{D}_6$ is a smooth Artin stack of finite type over $k$.

Proof. By Hilbert scheme argument, the stack $\mathcal{D}_6$ is an Artin stack of finite type. So, by deformation theory it is enough to check that deformations of a sextic du Val del Pezzo surface $X$ over an algebraically closed field are unobstructed, i.e., that $\text{Ext}^p(\Omega_X, \mathcal{E}_X) = 0$ for $p \geq 2$. On the other hand, since $X$ has isolated hypersurface singularities, we have locally a resolution

$$0 \to \mathcal{A}_{X/M}^\vee \to \Omega_{M}|_X \to \Omega_X \to 0$$

(here $M$ is a smooth variety, in which $X$ sits as a hypersurface) which shows that $\text{Ext}^p(\Omega_X, \mathcal{E}_X) = 0$ for $p \geq 2$ and $\mathcal{E}_X|_X$ has zero-dimensional support. By the local-to-global spectral sequence this means that

$$\text{Ext}^{\geq 2}(\Omega_X, \mathcal{E}_X) = H^{\geq 2}(X,T_X).$$

Since $X$ is a surface, we just have to check that $H^2(X,T_X) = 0$. This holds by [HI10, Proposition 3.1]. □

Remark B.2. The stack $\mathcal{D}_6$ is not separated — one can construct two du Val families $\mathcal{X} \to \mathbb{A}^1$ and $\mathcal{X}' \to \mathbb{A}^1$ that are isomorphic over $\mathbb{A}^1 \setminus \{0\}$, but differ by a small birational transformation over the whole base. The simplest example is to consider the blowup $\mathcal{X} = \text{Bl}_{Y}(\mathbb{P}^2 \times \mathbb{A}^1) \to \mathbb{A}^1$, where $Y = \left\{(1,0,0)\right\} \times \mathbb{A}^1 \sqcup \left\{(0,1,0)\right\} \times \mathbb{A}^1 \sqcup \left\{((1,t),t) \mid t \in \mathbb{A}^1\right\}$, and define $\mathcal{X} \to \mathbb{A}^1$ as the family of relative anticanonical models of $\mathcal{X}$. Then $\mathcal{X} \to \mathbb{A}^1$ is smooth over $\mathbb{A}^1 \times \{0\}$ and degenerates to a singular del Pezzo surface (of type 1) at point $t = 0$. On the other hand, the family $\mathcal{X} \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) \to \mathbb{A}^1 \setminus \{0\}$ is isomorphic to the trivial family, hence can be extended to a trivial family $\mathcal{X}' \to \mathbb{A}^1$. Thus, the families $\mathcal{X}$ and $\mathcal{X}'$ are isomorphic over $\mathbb{A}^1 \setminus \{0\}$, but have different fibers over 0.
Appendix C. Homological projective duality for $\mathbb{P}^2 \times \mathbb{P}^2$ and $\text{Fl}(1, 2; 3)$

We refer to [Kuz07, Kuz14, Tho15] for the definition and a review of homological projective duality. In this section we construct the homologically projective dual variety for $\mathbb{P}^2 \times \mathbb{P}^2$ and the flag variety $\text{Fl}(1, 2; 3)$ with respect to a certain symmetric rectangular Lefschetz decomposition.

Denote by $W_1$ and $W_2$ two three-dimensional vector spaces, consider the product

$$X_2 := \mathbb{P}(W_1) \times \mathbb{P}(W_2),$$

its Segre embedding $X_2 \hookrightarrow \mathbb{P}(W_1 \otimes W_2) =: \mathbb{P}(\mathcal{W})$, and the standard exceptional collection on $X_2$:

$$\mathbf{D}(X_2) = \langle \mathcal{O}_{X_2}, \mathcal{O}_{X_2}(1, 0), \mathcal{O}_{X_2}(2, 0), \mathcal{O}_{X_2}(2, 1), \mathcal{O}_{X_2}(1, 1), \mathcal{O}_{X_2}(2, 2) \rangle.$$

We modify this collection slightly to turn it into a symmetric rectangular Lefschetz decomposition with respect to the line bundle $\mathcal{O}_{X_2}(1, 1)$. For this we mutate $\mathcal{O}_{X_2}(2, 0)$ and $\mathcal{O}_{X_2}(0, 2)$ to the far left. An easy computation shows that the result is the following exceptional collection

$$\mathbf{D}(X_2) = \langle \mathcal{O}_{X_2}(0, -1), \mathcal{O}_{X_2}(-1, 0), \mathcal{O}_{X_2}, \mathcal{O}_{X_2}(1, 0), \mathcal{O}_{X_2}(0, 1), \mathcal{O}_{X_2}(1, 1), \mathcal{O}_{X_2}(2, 1), \mathcal{O}_{X_2}(1, 2), \mathcal{O}_{X_2}(2, 2) \rangle.$$

Clearly, this is a rectangular Lefschetz collection with respect to $\mathcal{O}_{X_2}(1, 1)$ with three blocks equal to

$$(68) \quad \mathcal{A}_{X_2}^\text{sym} = \langle \mathcal{O}_{X_2}(0, -1), \mathcal{O}_{X_2}(-1, 0), \mathcal{O}_{X_2} \rangle.$$

It is symmetric with respect to the transposition of factors.

The homological projective duality of $X_2$ with respect to the standard Lefschetz decomposition

$$\mathbf{D}(X_2) = \langle \mathbf{D}(\mathbb{P}(W_1)), \mathbf{D}(\mathbb{P}(W_1)) \otimes \mathcal{O}_{X_2}(1, 1), \mathbf{D}(\mathbb{P}(W_1)) \otimes \mathcal{O}_{X_2}(2, 2) \rangle$$

is described in [BBF16]. For this the linear homological projective duality argument [Kuz07, Section 8] is used. Indeed, the scheme $X_2$ can be considered as a projectivization of a vector bundle

$$X_2 \cong \mathbb{P}(\mathcal{W}_2), \quad \mathcal{W}_2 := W_2 \otimes \mathcal{O}_{\mathbb{P}(W_1)}(-1) \subset \mathcal{W} \otimes \mathcal{O}_{\mathbb{P}(W_1)}.$$

Consequently, by [Kuz07, Corollary 8.3] the homological projectively dual of $X_2$ with respect to the Lefschetz decomposition with the first block $\mathbf{D}(\mathbb{P}(W_1))$ is

$$(69) \quad \mathcal{Y}_2 := \mathbb{P}(\mathbb{P}(W_1)(\mathcal{W}_2^\perp)),$$

where $\mathcal{W}_2^\perp := \text{Ker}(\mathcal{W}^\vee \otimes \mathcal{O}_{\mathbb{P}(W_1)} \rightarrow \mathcal{W}_2^\vee) \cong W_2^\vee \otimes \Omega_{\mathbb{P}(W_1)}(1)$ is a rank 6 vector bundle on $\mathbb{P}(W_1)$. In the next theorem we show that the result of homological projective duality with respect to (68) is the same.

**Theorem C.1.** The variety $\mathcal{Y}_2$ is homologically projectively dual to the variety $X_2$ with respect to the Lefschetz decomposition of $\mathbf{D}(X_2)$ with first block (68).

**Proof.** Let $\mathcal{X}_2 \subset X_2 \times \mathbb{P}(\mathcal{W}^\vee)$ be the universal hyperplane section of $X_2$. By [Kuz07, Theorem 8.2], there is a semiorthogonal decomposition

$$(70) \quad \mathbf{D}(\mathcal{X}_2) = \langle i_* \phi^*(\mathbf{D}(\mathcal{Y}_2)), \mathbf{D}(\mathbb{P}(W_1) \times \mathbb{P}(\mathcal{W}^\vee)) \otimes \mathcal{O}_{\mathcal{X}_2}(1, 1), \mathbf{D}(\mathbb{P}(W_1) \times \mathbb{P}(\mathcal{W}^\vee)) \otimes \mathcal{O}_{\mathcal{X}_2}(2, 2) \rangle,$$

where the morphisms $i$ and $\phi$ are defined by the Cartesian diagram

$$(71) \quad \begin{array}{ccc}
\mathbb{P}(W_1) & \xrightarrow{i} & \mathcal{X}_2 \\
\phi \downarrow & & p_{\mathcal{X}_2} \\
\mathbb{P}(W_1) \times \mathbb{P}(\mathcal{W}^\vee) & \xrightarrow{p_{\mathcal{Y}_2}} & \mathbb{P}(\mathcal{W}^\vee)
\end{array}$$

and the map $p_{\mathcal{Y}_2}$ is induced by the embedding $\mathcal{W}_2^\perp \hookrightarrow \mathcal{W}^\vee \otimes \mathcal{O}_{\mathbb{P}(W_1)}$. We modify (70) by a sequence of mutations to change it to the form we need.
First, using the standard exceptional collection \( \mathbf{D}(\mathbb{P}(W_1)) = (\mathcal{O}_{\mathbb{P}(W_1)}(-1), \mathcal{O}_{\mathbb{P}(W_1)}, \mathcal{O}_{\mathbb{P}(W_1)}(1)) \), decomposition (70) can be rewritten as

\[
\mathbf{D}(\mathscr{X}_2) = \langle i_* \phi^*(\mathbf{D}(\mathbb{Y}_2)), \\
\mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(0, 1), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(1, 1), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(2, 1), \\
\mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(1, 2), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(2, 2), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(3, 2) \rangle.
\]

Mutating the last component to the far left and taking into account that \( \omega_{\mathscr{X}_2} \cong \mathcal{O}_{\mathscr{X}_2}(-2, -2) \) up to a line bundle pulled back from \( \mathbb{P}(\mathbb{W}^\vee) \), we obtain a semiorthogonal decomposition

\[
\mathbf{D}(\mathscr{X}_2) = \langle \Phi(\mathbf{D}(\mathbb{Y}_2)), \\
\mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(1, 0), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(0, 1), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(1, 1), \\
\mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(2, 1), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(1, 2), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(2, 2) \rangle.
\]

Next, mutating the second component one step to the left, we get

\[
(72) \quad \mathbf{D}(\mathscr{X}_2) = \langle \Phi(\mathbf{D}(\mathbb{Y}_2)), \\
\mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(1, 0), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(0, 1), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(1, 1), \\
\mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(2, 1), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(1, 2), \mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(2, 2) \rangle.
\]

where \( \Phi = \text{L}_{\mathbf{D}(\mathbb{P}(\mathbb{W}^\vee)) \otimes \mathcal{O}_{\mathscr{X}_2}(1, 0)} \circ i_* \circ \phi^*: \mathbf{D}(\mathbb{Y}_2) \to \mathbf{D}(\mathscr{X}_2) \).

This almost proves the result. The only small thing left is to show that the functor \( \Phi \) is a Fourier–Mukai functor with the kernel supported on the fiber product \( D(72) \).

\[\text{It follows from (71) that } \Phi_{p^*_y}(\mathcal{O}_{\mathbb{P}(W_1)}(1)) \cong \mathcal{O}_{\mathbb{P}(W_1)}(1).\]

In other words, this composition is a Fourier–Mukai functor with kernel \( p^*_y(\mathcal{O}_{\mathbb{P}(W_1)}(1)) \otimes p^*_y(\mathcal{O}_{\mathbb{P}(W_1)}(1)) \) on \( \mathbb{Y}_2 \times_{\mathbb{P}(W^\vee)} \mathscr{X}_2 \). Therefore, the functor \( \Phi \) fits into a distinguished triangle

\[
\Phi_{p^*_y}(\mathcal{O}_{\mathbb{P}(W_1)}(1)) \otimes p^*_y(\mathcal{O}_{\mathbb{P}(W_1)}(1)) \to \Phi_{j_* \mathcal{O}_{\mathbb{P}(W_1)}}, \to \Phi.
\]

It follows from (71) that

\[
j^*(p^*_y(\mathcal{O}_{\mathbb{P}(W_1)}(1)) \otimes p^*_y(\mathcal{O}_{\mathbb{P}(W_1)}(1))) \cong \mathcal{O}_{\mathbb{P}(W_1)}(1),
\]

hence the above triangle of functors is induced by (a rotation of the twist of) the triangle associated with the standard exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}(W_1)}(1) \otimes \mathcal{O}_{\mathbb{P}(W_1)}(1) \to \mathcal{O}_{\mathbb{P}(W_1)}(1) \to \mathcal{O}_{\mathbb{P}(W_1)}(1) \to 0.
\]

It finally proves, that up to an irrelevant shift, the functor \( \Phi \) is indeed a Fourier–Mukai functor with the kernel

\[
(73) \quad \mathbb{E}_2 := \mathcal{O}_{\mathbb{P}(W_1)}(1) \otimes \mathcal{O}_{\mathbb{P}(W_1)}(1) \in \mathbf{D}(\mathbb{Y}_2 \times_{\mathbb{P}(W^\vee)} \mathscr{X}_2),
\]

and thus completes the proof of the theorem. \( \square \)
Remark C.2. We could, of course, exchange the role of $W_1$ and $W_2$ in the construction. Then we would get a slightly different homological projectively dual variety as the result, that is

$$
\mathcal{Y}'_2 = \mathbb{P}_{F(W_2)}(W_1^\vee \otimes \Omega_{F(W_2)}(1)).
$$

Note that the natural maps $\mathcal{Y}_2 \to \mathbb{P}(W^\vee)$ and $\mathcal{Y}'_2 \to \mathbb{P}(W^\vee)$ are birational onto the same discriminant cubic hypersurface $\mathcal{D}_{W_1,W_2} \subset \mathbb{P}(W^\vee) = \mathbb{P}(W_1^\vee \otimes W_2^\vee)$ and provide two small resolutions of singularities of $\mathcal{D}_{W_1,W_2}$, related to each other by a flop, identifying their derived categories. So, the homological projectively dual of $\mathcal{X}_2$ as a category is unambiguously defined, but has two different geometric models $\mathcal{Y}_2$ and $\mathcal{Y}'_2$, breaking down its inner symmetry.

One can also use the above result to construct a symmetric homological projective duality for the flag variety $\text{Fl}(W) = \text{Fl}(1,2;W)$ of a three-dimensional vector space $W$. Note that $\text{Fl}(W)$ is a smooth hyperplane section of $\mathbb{P}(W) \times \mathbb{P}(W^\vee)$ corresponding to the natural bilinear pairing $b_0$ between $W$ and $W^\vee$. Since $b_0$ is nondegenerate, it does not lie in the image of $\mathcal{Y}_2$ in $\mathbb{P}(W^\vee \otimes W)$, hence by homological projective duality

$$
\mathcal{D}(\text{Fl}(W)) = \langle \mathcal{O}_{\text{Fl}(W)}(0,-1), \mathcal{O}_{\text{Fl}(W)}(-1,0), \mathcal{O}_{\text{Fl}(W)}, \mathcal{O}_{\text{Fl}(W)}(1,0), \mathcal{O}_{\text{Fl}(W)}(0,1), \mathcal{O}_{\text{Fl}(W)}(1,1) \rangle.
$$

Clearly, this is a Lefschetz collection with respect to $\mathcal{O}_{\text{Fl}(W)}(1,1)$ with two blocks equal to

$$
\mathcal{A}_{\text{Fl}(W)}^{\text{sym}} = \langle \mathcal{O}_{\text{Fl}(W)}(0,-1), \mathcal{O}_{\text{Fl}(W)}(-1,0), \mathcal{O}_{\text{Fl}(W)} \rangle.
$$

On the other hand, the linear projection with center at $b_0$ defines a regular map $\mathcal{Y}_2 \to \mathbb{P}((W^\vee \otimes W)/b_0)$.

**Corollary C.3.** The variety $\mathcal{Y}_2$ defined by (63) is homological projectively dual to the flag variety $\text{Fl}(W)$ with respect to the Lefschetz decomposition of $\mathcal{D}(\text{Fl}(W))$ with first block (74).

**Proof.** Follows from Theorem C.1 by [CT15, Theorem 3.6].

**Appendix D. Homological projective duality for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$**

In this section we construct the homologically projective dual variety for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with respect to a certain symmetric rectangular Lefschetz decomposition.

Denote by $V_1$, $V_2$, and $V_3$ three two-dimensional vector spaces, consider the product

$$
\mathcal{X}_3 := \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3),
$$

its Segre embedding $\mathcal{X}_3 \hookrightarrow \mathbb{P}(V_1 \otimes V_2 \otimes V_3) =: \mathbb{P}(V)$, and the standard exceptional collection on $\mathcal{X}_3$:

$$
\mathcal{D}(\mathcal{X}_3) = \langle \mathcal{O}_{\mathcal{X}_3}, \mathcal{O}_{\mathcal{X}_3}(1,0,0), \mathcal{O}_{\mathcal{X}_3}(0,1,0), \mathcal{O}_{\mathcal{X}_3}(1,1,0), \mathcal{O}_{\mathcal{X}_3}(0,0,1), \mathcal{O}_{\mathcal{X}_3}(1,0,1), \mathcal{O}_{\mathcal{X}_3}(0,1,1) \rangle.
$$

We modify this collection slightly to turn it into a symmetric rectangular Lefschetz decomposition with respect to the line bundle $\mathcal{O}_{\mathcal{X}_3}(1,1,1)$. For this we mutate $\mathcal{O}_{\mathcal{X}_3}(1,1,0)$, $\mathcal{O}_{\mathcal{X}_3}(1,0,1)$ and $\mathcal{O}_{\mathcal{X}_3}(0,1,1)$ to the far right. An easy computation shows that what we get is an exceptional collection

$$
\mathcal{D}(\mathcal{X}_3) = \langle \mathcal{O}_{\mathcal{X}_3}, \mathcal{O}_{\mathcal{X}_3}(1,0,0), \mathcal{O}_{\mathcal{X}_3}(0,1,0), \mathcal{O}_{\mathcal{X}_3}(0,0,1), \mathcal{O}_{\mathcal{X}_3}(1,1,1), \mathcal{O}_{\mathcal{X}_3}(2,1,1), \mathcal{O}_{\mathcal{X}_3}(1,2,1), \mathcal{O}_{\mathcal{X}_3}(1,1,2) \rangle.
$$

Clearly, this is a rectangular Lefschetz collection with respect to $\mathcal{O}_{\mathcal{X}_3}(1,1,1)$ with two blocks equal to

$$
\mathcal{A}_{\mathcal{X}_3}^{\text{sym}} = \langle \mathcal{O}_{\mathcal{X}_3}, \mathcal{O}_{\mathcal{X}_3}(1,0,0), \mathcal{O}_{\mathcal{X}_3}(0,1,0), \mathcal{O}_{\mathcal{X}_3}(0,0,1) \rangle.
$$

It is symmetric with respect to the action of the group $\mathfrak{S}_3$ by permutations of factors.

As in Appendix C we use a small modification of linear homological projective duality. Again, the scheme $\mathcal{X}_3$ can be represented as a projectivization of a vector bundle

$$
\mathcal{X}_3 \cong \mathbb{P}_{\mathbb{P}(V_1) \times \mathbb{P}(V_2)}(\mathcal{Y}_3), \quad \mathcal{Y}_3 := V_3 \otimes \mathcal{O}_{\mathbb{P}(V_1) \times \mathbb{P}(V_2)}(-1,-1) \subset \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}(V_1) \times \mathbb{P}(V_2)}.$$

Consequently, by [Kuz07, Corollary 8.3] the homological projectively dual of $X_3$ with respect to the Lefschetz decomposition with the first block $D(P(V_1) \times P(V_2))$ is

\[
Y_3 := P_{P(V_1) \times P(V_2)}(\mathcal{Y}_3^\perp),
\]

where $\mathcal{Y}_3^\perp := \text{Ker}(V^\vee \otimes \mathcal{O}_{P(V_1) \times P(V_2)} \to \mathcal{Y}_3^\vee) \cong V_3^\vee \otimes \Omega_{P(V_1) \times P(V_2)}(1)|_{P(V_1) \times P(V_2)}$ is a rank 6 vector bundle on $P(V_1) \times P(V_2)$.

**Theorem D.1.** The variety $Y_3$ is homologically projectively dual to the variety $X_3$ with respect to the Lefschetz decomposition of $D(X_3)$ with first block \((75)\).

**Proof.** Let $\mathcal{Y}_3 \subset X_3 \times P(V^\vee)$ be the universal hyperplane section of $X_3$. By [Kuz07, Theorem 8.2], there is a semiorthogonal decomposition

\[
D(\mathcal{Y}_3) = \langle i_* \phi^*(D(Y_3)), D(P(V_1) \times P(V_2) \times P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 1, 1) \rangle,
\]

where the morphisms $i$ and $\phi$ are defined by the commutative diagram

\[
\begin{array}{ccc}
Y_3 \times P(V_3) & \xrightarrow{i} & \mathcal{X}_3 \\
\phi \downarrow & & \downarrow p_{\mathcal{X}_3} \\
Y_3 \atop \mathcal{Y}_3 \atop \mathcal{Y}_3 \leftrightarrow P(V_1) \times P(V_2) \times P(V^\vee) \rightarrow P(V^\vee)
\end{array}
\]

and the map $p_{\mathcal{X}_3}$ is induced by the embedding $\mathcal{Y}_3^\perp \hookrightarrow \mathcal{W}^\vee \otimes \mathcal{O}_{P(V_1)}$. We modify \((74)\) by a sequence of mutations to change it to the form we need.

First, using the standard exceptional collection

\[
D(P(V_1) \times P(V_2)) = \langle \mathcal{O}_{P(V_1) \times P(V_2)}, \mathcal{O}_{P(V_1) \times P(V_2)}(1, 0), \mathcal{O}_{P(V_1) \times P(V_2)}(0, 1), \mathcal{O}_{P(V_1) \times P(V_2)}(1, 1) \rangle
\]

in $D(P(V_1) \times P(V_2))$, we rewrite \((74)\) as

\[
D(\mathcal{X}_3) = \langle i_* \phi^*(D(Y_3)), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 1, 1), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(2, 1, 1), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 2, 1) \rangle.
\]

Mutating the last component to the far left, and taking into account that $\omega_{\mathcal{X}_3} \cong \mathcal{O}_{X_3}(-1, -1, -1)$ up to a line bundle pulled back from $P(V^\vee)$, we get a semiorthogonal decomposition

\[
D(\mathcal{X}_3) = \langle D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 1, 0), i_* \phi^*(D(Y_3)), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 1, 1), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(2, 1, 1), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 2, 1) \rangle.
\]

Next, mutating the second component to the left, we get

\[
D(\mathcal{X}_3) = \langle \Phi(D(Y_3)), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 1, 0), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 1, 1), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(2, 1, 1), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 2, 1) \rangle.
\]

where $\Phi = L_{D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 1, 0)} \circ i_* \circ \phi^*: D(Y_3) \to D(\mathcal{X}_3)$.

Finally, mutating the second component to the far right and using the pullback of the standard exact sequence $0 \to \mathcal{O}_{P(V)}(-1) \to V_3 \otimes \mathcal{O}_{P(V)} \to \mathcal{O}_{P(V_3)}(1) \to 0$, we obtain

\[
D(\mathcal{X}_3) = \langle \Phi(D(Y_3)), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 1, 1), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(2, 1, 1), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 2, 1), D(P(V^\vee)) \otimes \mathcal{O}_{X_3}(1, 2, 1) \rangle.
\]
As before, this almost proves the result. The only small thing left is to show that the functor $\Phi$ is a Fourier–Mukai functor with the kernel supported on the fiber product $\mathbb{Y}_3 \times_{\mathbb{P}(V^\vee)} \mathcal{X}_3$. The same computations as in the proof of Theorem C.1 show that the functor $\Phi$ fits into a distinguished triangle
$$
\Phi \circ \rho_{\mathbb{Y}_3 \times \mathbb{P}(V_3)}(\mathbb{O}(1)) \to \Phi \circ \rho_{\mathbb{Y}_3 \times \mathbb{P}(V_3)}(\mathbb{O}(1)) \to \Phi \circ \rho_{\mathbb{Y}_3 \times \mathbb{P}(V_3)}(\mathbb{O}(1)),
$$
with notation introduced in (78). It follows that this triangle is associated (up to a twist and a rotation) with the standard exact sequence
$$
0 \to \mathcal{J}_{\mathbb{Y}_3 \times \mathbb{P}(V_3), \mathbb{Y}_3 \times \mathbb{P}(V^\vee)} \mathcal{X}_3 \to \mathcal{O}_{\mathbb{Y}_3 \times \mathbb{P}(V^\vee)} \mathcal{X}_3 \to \Phi \circ \rho_{\mathbb{Y}_3 \times \mathbb{P}(V_3)} \to 0,
$$
hence the functor $\Phi$ is indeed a Fourier–Mukai functor with kernel
$$
E_3 := \mathcal{J}_{\mathbb{Y}_3 \times \mathbb{P}(V_3), \mathbb{Y}_3 \times \mathbb{P}(V^\vee)} \mathcal{X}_3 \otimes (\Phi_{\mathbb{O}(1)}(\mathbb{O}(1), -1)) \boxtimes \Phi_{\mathbb{O}(1)}(\mathbb{O}(1), 1)),
$$
and thus completes the proof of the Theorem.

**Remark D.2.** The natural map $\mathbb{Y}_3 \to \mathbb{P}(V^\vee)$ is generically finite of degree 2, with the branch divisor being the Cayley quartic $\overline{O}_6$ (see Section 6.2). Moreover, the fiber of this morphism is isomorphic to $\mathbb{P}^1$ over $\mathbb{Y}_3$, and to a reducible conic over $\mathbb{O}_3$. In particular, $\mathbb{Y}_3$ provides a resolution of singularities of the double cover of $\mathbb{P}(V^\vee)$ branched over $\overline{O}_6$.

**Remark D.3.** We could, of course, exchange the role of $V_i$ in the construction. Then we would get a slightly different homological projectively dual variety as the result, that is for each permutation $w \in S_3$ we would get
$$
\mathbb{Y}_3^w = \mathbb{P}(\mathbb{O}(1) \times \mathbb{P}(\mathbb{O}(1))) \left( \mathbb{V}_w^3 \otimes \mathbb{O}(1) \mathbb{P}(\mathbb{O}(1) \otimes \mathbb{P}(\mathbb{O}(1)) \right).
$$
Note that for each $w$ the natural map $\mathbb{Y}_3^w \to \mathbb{P}(V^\vee)$ factors through a birational morphism onto the double cover $\mathcal{D}_{V_1,V_2,V_3}$ of $\mathbb{P}(V^\vee)$ branched over $\overline{O}_6$. They provide six small resolutions of singularities of $\mathcal{D}_{V_1,V_2,V_3}$, related to each other by flops, identifying their derived categories. So, the homological projectively dual of $X_3$ as a category is unambiguously defined, but has six different geometric models $\mathbb{Y}_3^w$, $w \in S_3$, breaking down its inner symmetry.

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