Pairing in asymmetric two-component fermion matter

J. Mur-Petit*, A. Polls, and H.-J. Schulze
Departament d’Estructura i Constituents de la Matèria, Universitat de Barcelona,
Av. Diagonal 647, E-08028 Barcelona, Spain

Abstract

We analyze the possibilities of pairing between two different fermion species in asymmetric matter at low density. While the direct interaction allows pairing only for very small asymmetries, the pairing mediated by polarization effects is always possible, with a pronounced maximum at finite asymmetry. We present analytical results up to second order in the low-density parameter $k_F a$.

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*Corresponding author. Address: Dept. ECM, Facultat de Física, Avda. Diagonal 647, E-08028 Barcelona (Spain). Phone: (+34)3 4021185. Fax: (+34)3 4021198. E-mail: jordim@ecm.ub.es
The recent experimental achievement of trapping fermionic alkali atoms at very low density and temperature \([1]\) demands theoretical estimates of the characteristic size of the pairing gaps \(\Delta\) that might be observable in those systems. The canonical case of an attractive \(s\)-wave interaction leads (including polarization effects) to the well-known low-density result \([2–4]\)

\[
\frac{\Delta}{\mu} = \frac{1}{(4e)^{1/3}} \frac{8}{e^2} \exp \left[ \frac{\pi}{2k_F a} \right],
\]

(1)

where \(\mu = k_F^2 / 2m\) is the chemical potential and \(a < 0\) the \(s\)-wave scattering length. However, under certain circumstances direct \(s\)-wave pairing is not possible: in the case of repulsion, \(a > 0\), or, e.g., in spin-polarized one-component Fermi systems.

A similar, particular system has recently been advocated for experimental study. It is a spin-polarized alkali gas composed of two different hyperfine levels of \(^{6}\)Li \([5]\). In this specific environment an (attractive) \(s\)-wave interaction exists only between the atoms at different levels, whereas atoms at like levels can only interact via \(p\)-waves. This report is dedicated to the quantitative study of the principal possibilities of pairing in such a system, and in particular of the dependence of pairing on the asymmetry of the system, which evidently is an important experimental parameter that influences strongly the magnitude of the observable gaps.

More generally, we will assume a fermionic system composed of two distinct species 1 and 2 (carrying definite spin orientations) of equal mass \(m\) and with densities \(\rho_1\) and \(\rho_2\), or equivalently total density \(\rho = \rho_1 + \rho_2\) and asymmetry \(\alpha = (\rho_1 - \rho_2) / (\rho_1 + \rho_2)\). We also introduce the notation \(k_i \equiv k_F^{(i)}\), \((i = 1, 2)\) for the Fermi momenta \(k_i = (6\pi^2 \rho_i)^{1/3}\), and \(\mu_i = k_F^2 / 2m\) for the two chemical potentials at low density.

For the sake of presentation we will for the moment consider an idealized system without direct interaction between like particles 1-1 and 2-2, whereas 1 and 2 are interacting via a potential \(V\) with a \(s\)-wave scattering length \(a\). We are only interested in the situation at very low density, \(k_i |a| \ll 1\), where the pairing properties are completely determined by the scattering length, or, equivalently, the low-momentum \(s\)-wave \(T\)-matrix \(T_0 = 4\pi a / m\). We will now analyze the pairing gaps generated by this interaction.

In the case of attraction, \(a < 0\), and for very small asymmetries clearly by far the dominant process is the pairing generated by the direct \(s\)-wave interaction between different species, see Fig. [1][a]. The BCS theory generalized to asymmetric matter \([3, 4]\) yields the basic coupled equations for the determination of the gap function \(\Delta_k\), total density \(\rho\), and density difference \(\delta \rho\),

\[
\Delta_{k'} = -\sum_k V_{kk'} \frac{1 - f(E^-_k) - f(E^+_k)}{2E_k} \Delta_k,
\]

(2)

\[
\rho = \rho_1 + \rho_2 = \sum_k \left[ 1 - \frac{\epsilon_k}{E_k} \left[ 1 - f(E^-_k) - f(E^+_k) \right] \right],
\]

(3)

\[
\delta \rho = \rho_1 - \rho_2 = \sum_k \left[ f(E^-_k) - f(E^+_k) \right],
\]

(4)

with the Fermi function \(f(E) = [1 + \exp(\beta E)]^{-1}\) and
\[ E_k^\pm = E_k \pm \delta \mu, \quad E_k = \sqrt{\epsilon_k^2 + \Delta_k^2}, \quad \epsilon_k = e_k - \mu, \quad e_k = k^2/2m. \] (5)

The chemical potentials of the species 1 and 2 are \( \mu_{1,2} = \mu \pm \delta \mu \) and \( V \) is the bare potential acting between them. Throughout this report we will only determine gaps at zero temperature, where one has \( f(E) = \theta(-E) \) and therefore

\[
1 - f(E^-_k) - f(E^+_k) = \theta(E^-_k),
\]

\[
f(E^-_k) - f(E^+_k) = \theta(-E^-_k),
\]

(6)

(7)
i.e., the unpaired particles are concentrated in the energy interval \([\mu - \delta e, \mu + \delta e]\), \( \delta e = \sqrt{\delta \mu^2 - \Delta^2} \), which does not contribute to the pairing interaction. This situation is illustrated in Fig. 2 that sketches the BCS momentum distributions of the species 1 and 2, according to Eqs. (3) and (4), for a positive asymmetry. The excess of particles 1 is located in the interval around \( \mu \), Pauli-blocking the gap equation. This leads to a rapid decrease of the resulting gap when increasing the size \( 2\delta e \) of the interval, i.e., the asymmetry.

In the weak-coupling case, \( \Delta \ll \delta e \ll \mu \), which is adequate in the low-density limit, the momentum distributions of the two species are very sharp and one obtains from Eqs. (3) and (4):

\[
\alpha = \frac{\delta \rho}{\rho} \approx \frac{3 \delta e}{2 \mu} \ll 1,
\]

(8)
i.e., the asymmetry \( \alpha \) is directly proportional to \( \delta e \). Analyzing also the gap equation in the context of a weak-coupling approximation, one obtains for the dependence of the gap on the parameter \( \delta e \) [5, 6],

\[
\delta \mu + \delta e = \text{const.} = \Delta_0 \iff \Delta^2 = \Delta_0^2 - 2\Delta_0 \delta e,
\]

(9)

where \( \Delta_0 \) is the gap in symmetric matter of the same total density. Consequently the gap as a function of asymmetry is:

\[
\frac{\Delta}{\Delta_0} = \sqrt{1 - \frac{4\mu}{3\Delta_0^2} \alpha}.
\]

(10)
The gap vanishes at \( \alpha_{\text{max}} = 3\Delta_0/4\mu \), which at low density is indeed an extremely small number, cf. Eq. (11). Therefore, for very small asymmetries already, pairing generated by the direct interaction between different species becomes impossible.

For larger asymmetry only \( p \)-wave pairing between like species can take place, which in leading order in density is given by the polarization diagram shown in Fig. 1(b). We discuss in the following the gap of species 1 mediated by the polarization interaction due to species 2. Clearly the results are invariant interchanging 1 and 2. Quantitatively the relevant interaction kernel reads at low density [2, 4, 8, 9]

\[
\langle k' | \Gamma_1 | k \rangle = \frac{\Pi_2(|k' - k|)}{2} T_0^2,
\]

(11)

with the static Lindhard function (pertaining to the species 2)
\[ \Pi_2(q) = \frac{-mk_2}{\pi^2} \left[ \frac{1}{2} + \frac{1-x^2}{4x} \ln \left| \frac{1+x}{1-x} \right| \right], \quad x = \frac{q}{2k_2}. \] 

(12)

The factor 1/2 in Eq. (11) corrects for the fact that conventionally the Lindhard function contains a factor two for the spin orientations, which is not present in our case. It should be noted that it is the absence of exchange diagrams that renders this low-density polarization interaction attractive in contrast to the case of one species, where the polarization effects reduce the s-wave BCS gap by a factor \((4e)^{1/3}\) in the low-density limit \([2,3]\), see Eq. (1). It is also remarkable that the polarization interaction is always attractive, depending only on the square of the scattering length \(a\) \([10]\).

Projecting out the \(L = 1\) partial-wave interaction, one obtains in particular \([11,12]\)

\[ \Gamma_1(k_1, k_1) = \frac{T^2}{2} \int_{-1}^{1} dz \, \Pi_2(\sqrt{2(1-z)k_1}) \quad z = \hat{k}' \cdot \hat{k} \] 

(13)

\[ = \frac{-8a^2k_2}{m} \frac{2}{5} \ln 2 - \frac{1}{g\left(\frac{k_1}{k_2}\right)}, \] 

(14)

with

\[ g(y) = \frac{-1}{6(2 \ln 2 - 1)y^4} \left[ (4 - 10y^2) \ln |1 - y^2| - (5 + y^2)y^3 \ln \left| \frac{1+y}{1-y} \right| + 4y^2 + 2y^4 \right]. \] 

(15)

This function is normalized in symmetric matter, \(g(1) = 1\), and plotted in Fig. 3(a). The numerical factor \((2 \ln 2 - 1)/5\) in Eq. (14) is often replaced by the approximate value \(1/13\). Using now the general result for the (angle-averaged) \(L\)-wave pairing gap \([10,13,14]\),

\[ \Delta_L(k_1) \to c_L \mu_1 \exp \left[ \frac{2\pi^2}{mk_1T_L(k_1, k_1)} \right], \] 

(16)

and considering that for a pure polarization interaction \(\Gamma_L\), the leading order in density is \(T_L = \Gamma_L\), one obtains

\[ \Delta_1(k_1) = c_1 \frac{k_1^2}{2m} \exp \left[ -\frac{13(\pi/2)^2}{\alpha^2k_2g(k_1/k_2)} \right]. \] 

(17)

Taking into account the dependence of this expression on the two Fermi momenta \(k_1\) and \(k_2\), the final result for the variation of the pairing gap with asymmetry \(\alpha\) for fixed total density \(\rho\) is therefore

\[ \frac{\Delta(\rho, \alpha)}{\Delta(\rho, 0)} = (1 + \alpha)^{2/3} \exp \left[ u(\rho)h(\alpha) \right] \] 

(18)

with

\[ h(\alpha) = 1 - \frac{1}{(1 - \alpha^2)^{1/3} g\left(\left((1 + \alpha)/(1 - \alpha)\right)^{1/3}\right)} \] 

(19)

and the density parameter
\[ u(\rho) = 13 \left( \frac{\pi}{2k_Fa} \right)^2, \quad k_F \equiv (3\pi^2 \rho)^{1/3}. \]  

The function \( h(\alpha) \) is displayed in Fig. 3(b), where \( \alpha = 1 \) corresponds to pure 1-matter. One notes a maximum at \( \alpha \approx 0.478 \) with an expansion \( h(\alpha) \approx 0.465 - 1.343(\alpha - 0.478)^2 \). This means that the gap at this asymmetry is enhanced by a factor \( e^{0.46u} \approx 10^{0.2u} \) compared to the symmetric case \( [h(0) = 0] \). Evidently, in the low-density limit \( u \to \infty \) this represents an enormous amplification at finite asymmetry. Around this peak, the variation of the gap with asymmetry is well described by a Gaussian with width \( \sigma = 1/(1.64\sqrt{u}) \). As an illustration, Fig. 3(c) shows the ratio, Eq. (18), for a value of the density parameter \( u = 100 \). A peak of the order of \( 10^{20} \) is observed, that becomes rapidly more pronounced and narrower with decreasing density \( \rho \) (increasing \( u \)), although of course at the same time the absolute magnitude of the gap decreases strongly with decreasing density.

Let us now briefly discuss the next-to-leading-order effects, namely additional contributions of order \((k_i a)^3\) in the denominator of the exponent of Eq. (17). There are two principal sources of such effects, which are shown diagramatically in Fig. 4. The first one, Fig. 4(a), is the direct \( p \)-wave interaction \([15]\) between like species that we have neglected before. Parametrizing the low-density \( p \)-wave \( T \)-matrix in the standard form \( T_1(k, k') \approx 4\pi a_1^3 k k' / m \), where \( a_1 \) is the \( p \)-wave scattering length, leads to a BCS gap \([15]\)

\[ \frac{\Delta}{\mu} = \frac{8}{e^{8/3}} \exp\left[ \frac{\pi}{2(k_Fa_1)^3} \right]. \]  

The second type of third-order contributions are polarization effects involving the \( s \)-wave scattering length \( a \). In contrast to the case of a one-component system, where there are several relevant diagrams, in the present two-component system only one diagram exists. It is drawn in Fig. 4(b). Unfortunately it can only be computed numerically, which will not be attempted here.

Finally, to fourth order, there is a large number of diagrams contributing to the interaction kernel. Apart from that, at this order it is also necessary to take into account retardation effects, i.e., the energy dependences of gap equation, interaction kernel, and self-energy need to be considered \([16]\). All this can only be done numerically and was performed in Ref. \([12]\) for the case of a one-component system.

In any case, the existence of higher-order corrections will not alter the main conclusions drawn so far, namely the presence of a strongly peaked Gaussian variation of the gap with asymmetry. They will, however, shift this peak to a different density-dependent location, and also modify the absolute size of the gap. Apart from that, the perturbative approach that is followed here, is clearly limited to the low-density range \( k_F |a| < 1 \). For larger density, different theoretical methods have to be used \([17]\), which is still a difficult field of current investigation.

In conclusion, we have studied the possibility of pairing in asymmetric fermion matter composed of two distinct species. In the low-density limit, the direct \( s \)-wave interaction between different species produces a gap \( \sim \exp[\pi/2k_F a] \) only for very small asymmetries of the order of \( \Delta_0 / \mu \), Eq. (14), whereas for larger asymmetries the polarization-induced \( p \)-wave attraction between two like species produces a much smaller gap \( \sim \exp[-13(\pi/2k_F a)^2] \), which extends however in principle over the whole range of asymmetry. In practice a sharp
maximum at $\alpha \approx 0.478$ ($\rho_1/\rho_2 \approx 2.83$, $k_1/k_2 \approx 1.41$) appears. Explicit expressions for the variation of these gaps with asymmetry were given. Higher-order corrections will only quantitatively modify these particular features of pairing in the low-density regime. Clearly the experimental observation of both types of pairing is supposedly difficult, in the first case due to the nearly perfect symmetry that is required, in the second one due to the extremely small size of the resulting gap.

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FIGURES

FIG. 1. The two possible lowest-order pairing interactions: (a) Direct $s$-wave interaction between different species. (b) Polarization-induced $p$-wave interaction between like species. $V_0$ and $T_0$ are the $s$-wave ($L = 0$) bare potential and $T$-matrix between species 1 and 2, respectively.

FIG. 2. BCS momentum distributions of the species 1 and 2 in asymmetric superfluid matter.

FIG. 3. (a,b) The functions $g$ and $h$, appearing in Eqs. (15) and (19), respectively. (c) The variation of the gap with asymmetry, Eq. (18), for a density parameter $u = 100$.

FIG. 4. Third-order diagrams: (a) Direct $p$-wave interaction, (b) Polarization contribution.
FIG. 3
FIG. 4