A search on Dirac equation

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The solutions, in terms of orthogonal polynomials, of Dirac equation with analytically solvable potentials are investigated within a novel formalism by transforming the relativistic equation into a Schrödinger like one. Earlier results are discussed in a unified framework and certain solutions of a large class of potentials are given.

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1 Introduction

A new algebraic technique for solving Schrödinger and Klein-Gordon equations [1], and the related other works therein, has been introduced recently and used to search many interesting problems in different disciplines of physics. These works have clarified the power of the suggested model when compared the results obtained with those provided by other analytical methods in the literature. Nevertheless, this formalism involves a deficiency in its present form which requires, to express the excited state wave functions, the application of linear operators on the ground state wave function appeared automatically in the mathematical framework. This is indeed a cumbersome procedure though it provides explicit expressions for the state functions having non-zero angular momenta.

To remove this drawback inherent in the formalism used in our previous works [1], we suggest here an alternative scheme, unifying the spirit of the two theoretical models [1, 2], to work out relativistic/non-relativistic quantum mechanical problems analytically in a unified framework. This is the main motivation behind the work presented in this article which in particular focuses on the solution of Dirac equation since recently considerable attention has been paid to exactly solvable Dirac equations.

The arrangement of this article is as follows. In the next section, a brief introduction of the usual Dirac formalism and its treatment within the frame of
new scheme are presented. Third section involves application results. Finally, the results are summarized in the concluding section.

2 Theoretical Consideration

To proceed, let us first remind the mathematical frame of Dirac formalism which is discussed briefly in the following section. Section 2.2 then illustrates the formalism of the new model.

2.1 Background on Dirac equation

Dirac equation for scalar and vector potentials is given by [3] ($\hbar = c = 1$)

$$H\Psi = \{\hat{\alpha}\hat{p} + \beta(m + V_S) + V_V\}\Psi \tag{1}$$

where $\hat{p}$ is the momentum operator, $m$ is the rest mass of the particle, $V_S$ and $V_V$ are scalar and vector potentials respectively and $\hat{\alpha}$, $\hat{\beta}$ are Pauli matrices.

To separate angular part of Eq. (1) from the radial part one follows

$$\Psi_{\ell jm} = \left[ \frac{iG_{\ell j}}{r} \varphi_{jm}^\ell \quad \frac{F_{\ell j}}{r} \varphi_{jm}^\ell \right] \tag{2}$$

where $\hat{\sigma}$ represents the Pauli spin matrices while $G_{\ell j}$ and $F_{\ell j}$ are the radial components

$$G_{\ell j} = \begin{cases} G_j^+ & j = \ell + \frac{1}{2} \\ G_j^- & j = \ell - \frac{1}{2} \end{cases} \quad F_{\ell j} = \begin{cases} F_j^+ & j = \ell + \frac{1}{2} \\ F_j^- & j = \ell - \frac{1}{2} \end{cases}$$

and $\varphi_{jm}^\ell$ is the angular part of the wave function

$$\varphi_{jm}^\ell = \begin{cases} \varphi_{jm}^+ & j = \ell + \frac{1}{2} \\ \varphi_{jm}^- & j = \ell - \frac{1}{2} \end{cases}$$

Then, using the standard relations

$$\hat{\sigma}.\hat{p}\frac{g(r)}{r} \varphi_{jm}^\ell = -i\left( \frac{dg}{dr} + \frac{kg}{r^2} \right) \frac{\hat{\sigma}.\hat{r}}{r^2} \varphi_{jm}^\ell \tag{3}$$

and

$$\hat{\sigma}.\hat{p}\frac{\hat{\sigma}.\hat{r}}{r^2} g(r) \varphi_{jm}^\ell = -i\left( \frac{dg}{dr} - \frac{kg}{r} \right) \varphi_{jm}^\ell \tag{4}$$

where

$$k = \begin{cases} -(\ell + 1) = -(j + 1/2) & j = \ell + 1/2 \\ +\ell = +(j + 1/2) & j = \ell - 1/2 \end{cases}$$

and $g(r)$ is an arbitrary function.

Hence, the radial equations appear as

$$-\frac{dF(r)}{dr} + \frac{k}{r} F(r) = (\varepsilon - m - V_S - V_V)G(r) \tag{5}$$
\[
\frac{dG(r)}{dr} + \frac{k}{r}G(r) = (\varepsilon + m + V_S - V_V)F(r),
\] (6)

where \( \varepsilon \) is the total relativistic energy of the system. From (5) and (6), omitting the derivatives of \( V_S \) and \( V_V \), together with the elimination of one radial component \( (F(r)) \), one obtains a Schrödinger like equation for the other component of the relativistic wave function
\[
\left\{ -\frac{d^2}{dr^2} + \frac{k(k+1)}{r^2} + (V_S^2 - V_V^2) + (2mV_S + 2\varepsilon V_V) \right\} G = (\varepsilon^2 - m^2)G. \tag{7}
\]

### 2.2 Formalism

Exact solutions of systems in physics has a great importance. To provide such solutions recently a new method has been carried out successfully in our earlier works [1], which unfortunately has a considerable algebraic difficulty in the calculation of excited state functions. To overcome this tedious procedure in the calculations, we propose here to use a new scheme involving orthogonal polynomials in order to express all bound state wave functions in an explicit form.

For this purpose, we start from Eq.(7) which can be defined as
\[
\frac{G''(r)}{G(r)} = V(r) - E , \tag{8}
\]

where \( V(r) = \frac{k(k+1)}{r^2} + (V_S^2 - V_V^2) + (2mV_S + 2\varepsilon V_V) \) and \( E = \varepsilon^2 - m^2 \). As is well known, the solution of (8) generally takes the form
\[
G(r) = f(r)F[s(r)]. \tag{9}
\]

The substitution of (9) into (8) yields the second-order differential equation
\[
\left( \frac{f''}{f} + \frac{F''s'^2}{F} + \frac{s''}{F} + \frac{2F's'f'}{Ff} \right) = V - E , \tag{10}
\]

and rearranging (10) for a more useful form, one gets
\[
F'' + \left( \frac{s''}{s'^2} + 2\frac{f'}{s'f} \right) F' + \left( \frac{f''}{s'^2f} + \frac{E - V}{s'^2} \right) F = 0 . \tag{11}
\]

Eq. (11) is in the form of the most familiar second-order differential equations to the hypergeometric type [2],
\[
F''(s) + \frac{\tau(s)}{\sigma(s)}F'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}F(s) = 0 , \tag{12}
\]
where $\sigma$ and $\tilde{\sigma}$ are at most second degree polynomials, and $\tau$ is a first degree polynomial. The form of $\frac{\tau(s)}{\sigma(s)}$ and $\frac{\tilde{\sigma}(s)}{\sigma(s)}$ is well defined for any special function $F(s)$ [2]. From (12), it follows that

$$\frac{s''}{s'^2} + 2\frac{f'}{sf} = \frac{\tau(s)}{\sigma(s)}; \quad \frac{f''}{fs'^2} + \frac{E - V}{s'^2} = \frac{\tilde{\sigma}}{\sigma^2}.$$

(13)

From the previous works in [1], the energy and potential terms in (13) can be decomposed in two pieces, which provides a clear understanding for the individual contributions of the $F$ and $f$ terms to the whole of the solutions, such that $E - V = (E_f + E_F) - (V_f + V_F)$. Therefore, the second equality in (13) is transformed to a couple of equation

$$\frac{f''}{f} = V_f - E_f, \quad -\frac{\tilde{\sigma}}{\sigma^2}s'^2 = V_F - E_F,$$

(14)

where $f$ can be expressed in an explicit form due to the first part in (13)

$$f(r) = (s')^{-1/2} \exp \left[ \frac{1}{2} \int_{s'}^{s(r)} \frac{\tau(s)}{\sigma(s)} ds \right].$$

(15)

Since the corresponding $\sigma$, $\tilde{\sigma}$ and $\tau$ terms are well known for a given polynomial ($F$), the transformation function ($s$) in (14), and afterwards $f$ in (15), are easily defined. So, from (9), the corresponding total wave function is readily obtained for the whole spectrum.

The potential and total energy terms for Dirac equation in this case

$$\frac{f''}{f} = V_f - E_f, \quad V_f = 2mV_S + 2\varepsilon V_V + \frac{k(k + 1)}{r^2},$$

(16)

$$-\frac{\tilde{\sigma}}{\sigma^2}s'^2 = V_F - E_F, \quad V_F = V_S^2 - V_V^2,$$

(17)

and

$$E_f + E_F = \varepsilon^2 - m^2.$$  

(18)

To understand how efficiently this method works, some physically possible potentials are solved in the following section to obtain their eigenvalues and eigenfunctions, within the frame of the present formalism.

3 Application

As illustrative examples, here we deal with the two well known problems of the literature: Dirac oscillator and Dirac-Coulomb problem. The other solutions are shown in Table 1.
In order to get exact solutions for the present consideration, one needs to start with choosing a physically plausible equal magnitudes for the vector and scalar potentials. Otherwise, the system considered becomes quasi-exactly solvable which is out of the scope of the present article.

Setting $V_V = V_S = a\omega r^2$ for the relativistic treatment of oscillator problem right hand sides of equations (16) and (17) gives

$$V_f = 2a(m + \varepsilon)r^2 + \frac{k(k + 1)}{r^2}, \quad V_F = 0$$

(19)

Concentrating on the generalized Laguerre polynomials $L^\alpha_n(s)$ related to confluent hypergeometric functions, one sees that

$$\sigma = s, \quad \tau = \alpha + 1 - s, \quad \tilde{\sigma} = n\sigma$$

(20)

Keeping in mind that the right-hand sides of (14) provide a three dimensional harmonic oscillator potential, one obviously realizes that $\frac{s^2}{s} = 2w$ and consequently $s = \frac{1}{2}\omega r^2$. Then, substituting (20) into (15) it is not hard to see that

$$f = Cr^{\alpha+1/2}e^{-\frac{\omega r^2}{4}},$$

(21)

where $C = \frac{1}{\sqrt{2}}(\frac{w}{\ell})^{\frac{\alpha}{2}}$. This makes possible to predict $V_f$ and $E_f$ as

$$\frac{f''}{f} = V_f - E_f, \quad E_f = (\alpha + 1)\omega,$$

$$V_f = \frac{1}{4}\omega^2 r^2 + \frac{(\alpha - 1/2)(\alpha + 1/2)}{r^2},$$

(22)

where $\alpha = -(k + 1/2) = \ell + \frac{1}{2}$ for the case $j = \ell + \frac{1}{2}$. To find also $V_F$ and $E_F$, one should consider Eq. (17). After some simple algebra we find

$$V_F = 0, \quad E_F = 2n\omega.$$  

(23)

Thus, in the non-relativistic limit, the full energy spectrum and wave functions for the system of interest are gives as

$$E = E_f + E_F = (\alpha + 1 + 2n)\omega = (2n + \ell + 3/2)\omega,$$

$$\Psi = fF = Cs^{\frac{(\ell+1)}{2}}e^{-\frac{a}{2}}L_n^{(\ell+\frac{1}{2})}(s).$$

(24)

Finally, the relativistic energy of Dirac oscillator reads

$$\varepsilon^2 = m^2 + (2n + \ell + 3/2)\omega.$$  

(25)
The results obtained are in agreement with the work of Levai [2] which considers only the non-relativistic case, and also, for proper parameters \((\ell + 1 = \kappa, \ w = 4\alpha^2\zeta)\), these results agree well with the study of Alhaidari [4]. Moreover, the findings justify the excellent discussion in [5] on the confinement properties for Dirac equation with scalar and vector like potentials.

Obviously, from the similarity between Eqs. (19) and (22) it is clear that \(a = \frac{w^2}{8(m+\varepsilon)}\).

It is importantly noted that the choose of equal magnitudes for vector and scalar potentials leads to the non-relativistic limit of Dirac equation, removing the relativistic corrections. For a comprehensive understanding of this interesting point, the reader is referred to the individual works of Gönül and Koçak in [1] regarding the treatment of Klein-Gordon equation. We additionally remark that the present algebraic treatment has been performed only for spin-up case. Clearly, following similar procedure, one can easily repeat the same calculations for the other spinor wave function where now \(k = +\ell\).

The relativistic hydrogen atom is also an exactly solvable system within the frame of Dirac equation where the piece of potentials now should be defined as

\[
\frac{f''}{f} = V_f - E_f, \quad V_f = V_s^2 - V_v^2,
\]

\[
-\frac{\tilde{\sigma}}{\sigma^2}s'^2 = V_F - E_F, \quad V_F = 2mV_s + 2\varepsilon V_v + \frac{k(k+1)}{r^2}.
\]

For again the equal vector and scalar potentials; \(V_v = V_s = -\frac{b}{r}\) one gets

\[
V_f = 0, \quad V_F = -\frac{2(m + \varepsilon)b}{r} + \frac{k(k+1)}{r^2}.
\]

In order to apply the present orthogonal polynomial technique, we choose the most suitable generalized Laguerre polynomial \([F = e^{-s/2} s^{\frac{\alpha+1}{2}} L_n^\alpha(s)]\) where

\[
\sigma = 1, \quad \tau = 0, \quad \tilde{\sigma} = \frac{2n + \alpha + 1}{2s} + \frac{1 - \alpha^2}{4s^2} - \frac{1}{4};
\]

leading to \(s = ar\), and to be in convenience with the literature, we set \(a = \frac{e^2}{n+\ell+1}\). Then, Eq. (15) reads

\[
f = \frac{(n+\ell+1)^\frac{3}{2}}{e}.
\]

This justifies that \(\frac{f''}{f} = V_f - E_f = 0\), while

\[
V_F = -\frac{e^2}{r} + \frac{\ell(\ell+1)}{r^2}, \quad E_F = -\frac{e^4}{4(n+\ell+1)^2},
\]
where \( k = -(\ell + 1) \) and \( \alpha = 2\ell + 1 \). Comparing Eqs. (28) and (31) one sees that \( b = \frac{e^2}{2(n+\ell+1)} \). Thus, for this system, the full energy spectrum and wave functions are

\[
E = E_f + E_F = E_F = -\frac{e^4}{4(n + \ell + 1)^2},
\]

\[
\Psi = fF = Ce^{-s/2}s^{\ell+1}L_n^{2\ell+1}(s),
\]

where \( C = \frac{(n+\ell+1)^{\frac{1}{2}}}{e} \). Finally, the relativistic energy for Dirac-Coulomb problem is

\[
\varepsilon^2 = m^2 - \frac{e^4}{4(n + \ell + 1)^2}.
\]

The results are in agreement with the work of Levai [2] and, with a suitable parameters such that \( 1 - \left[ \frac{e^2}{2(n+\ell+1)} \right]^2 = \left[ 1 + \left( \frac{\alpha Z}{\gamma+n+1} \right)^2 \right]^{-1} \), overlap with those obtained by Alhaidari in [4].

The formalism used here also provides explicit expressions for the relativistic spectra of three more potentials, which are illustrated in Table 1. We stress that, unlike the work in [4], the strategy followed in this article for transforming Dirac equation into a Schrödinger like one does not enlarge the class of exactly solvable potentials as much as it might appear at first sight. Hence, the present brief work supports the individual criticism of the Castro and Vaidya-Rodrigues [6].

It is additionally stressed that one of the potentials listed in Table 1 do not require a restriction as in the examples discussed above in defining equal magnitudes for vector and scalar potentials leading to the exact solvability.

4 Concluding Remarks

We have presented an idea of connecting the methods used in the analysis of exactly solvable potentials in non-relativistic quantum mechanics with the solution procedure of Dirac equation. The suggested formalism systematically recovers known results in a natural unified way and allows one to extend certain results known in particular cases. A straightforward generalization would be the application of the scheme to the other relativistic equations for integer spin cases. Beyond its intrinsic importance as a new solution for a fundamental equation in physics, we also expect that the present simple method would find a widespread application in the study of different quantum mechanical and nuclear scattering systems. Along this line the works are in progress.
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|                  | Oscillator                          | Coulomb                           | Morse                |
|------------------|------------------------------------|-----------------------------------|----------------------|
| $V_S$            | $ar^2$                             | $-b/r$                            | $-Ae^{-ar} + (\sqrt{B^2 + m^2} - m)$ |
| $V_V$            | $ar^2$                             | $-b/r$                            | $-Ce^{-ar}$          |
| $s$              | $\frac{1}{2} \omega r^2$          | $(e^2/(n + \ell + 1))r$           | $(2D/a)e^{-ax}$; $D^2 = A^2 - C^2$ |
| $\sigma$         | $s$                                | $1$                               | $s$                  |
| $\tau$           | $\alpha + 1 - s$                   | $0$                               | $\alpha + 1 - s$     |
| $\tilde{\sigma}$| $ns$                               | $\frac{2n + \alpha + 1}{2s} + \frac{1 - \alpha^2}{4\beta^2} - \frac{1}{4}$ | $ns$                |
| $\varepsilon$    | $(m^2 + (2n + \ell + 3/2)\omega)^{1/2}$ | $(m^2 - \frac{e^2}{4(n + \ell + 1)^2})^{1/2}$ | $(m^2 + B^2 - a^2\alpha^2)\frac{1}{4}$ |
| $\Psi$           | $s\frac{\alpha + 1}{2} e^{-s/2} L_n^{(\alpha)}(s)$ | $s\frac{\alpha + 1}{2} e^{-s/2} L_n^{(\alpha)}(s)$ | $s^{\alpha/2} e^{-s/2} L_n^{(\alpha)}(s)$ |
| $\alpha$         | $\ell + 1/2$                       | $2\ell + 1$                       | $2(2\sqrt{A^2 + m^2 + \varepsilon C}) - 1 - 2n$ |

**Rosen – Morse**

|                  | Eckart                            |
|------------------|-----------------------------------|
| $V_S$            | $(Atanh(ar) + B)^2$                | $(-A\coth(ar) + B)^2$            |
| $V_V$            | $(Atanh(ar) + B)^2$                | $(-A\coth(ar) + B)^2$            |
| $s$              | tanh $ar$                         | $\coth ar$                       |
| $\sigma$         | 1                                 | 1                                 |
| $\tau$           | 0                                 | 0                                 |
| $\tilde{\sigma}$| $\frac{1 - \alpha^2}{4(1-s^2)} + \frac{1 - \beta^2}{4(1+s^2)} + \frac{c_n}{1-s^2}$ | $\frac{1 - \alpha^2}{4(1-s^2)} + \frac{1 - \beta^2}{4(1+s^2)} + \frac{c_n}{1-s^2}$ |
| $\varepsilon$    | $(m^2 + \eta - \frac{a^2(\alpha^2 + \beta^2)}{2})^{1/2}$ | $(m^2 + \zeta - \frac{a^2(\alpha^2 + \beta^2)}{2})^{1/2}$ |
| $\Psi$           | $(1 - s)^{\alpha/2}(1 + s)^{\beta/2} P_n^{(s,\beta)}(s)$ | $(s - 1)^{\alpha/2}(1 + s)^{\beta/2} P_n^{(s,\beta)}(s)$ |
| $\alpha$         | $\gamma - n + \frac{\lambda}{\gamma + n}$ | $-\gamma - n + \frac{\lambda}{\gamma + n}$ |
| $\beta$          | $\gamma - n - \frac{\lambda}{\gamma - n}$ | $-\gamma - n - \frac{\lambda}{\gamma - n}$ |
| $c_n$            | $n(n + \alpha + \beta + 1) + \frac{1}{2}(\alpha + 1)(\beta + 1)$ | $n(n + \alpha + \beta + 1) + \frac{1}{2}(\alpha + 1)(\beta + 1)$ |

Table 1: Relativistic energy and unnormalized eigenfunctions of the five potentials deduced within the present Dirac formalism discussed in section 3. In the treatment of Rosen-Morse and Eckart potentials, the notation carried out in [7] is used.