Solutions of the Boundary Yang-Baxter Equation for $A–D–E$ Models

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Abstract

We present the general diagonal and, in some cases, non-diagonal solutions of the boundary Yang-Baxter equation for a number of related interaction-round-a-face models, including the standard and dilute $A_L$, $D_L$ and $E_{6,7,8}$ models.

1. Introduction

A two-dimensional lattice spin model in statistical mechanics can be considered as solvable with periodic boundary conditions if its bulk Boltzmann weights satisfy the Yang-Baxter equation [1], and as additionally solvable with certain non-periodic boundary conditions if it admits boundary weights which satisfy the boundary Yang-Baxter equation [2].

Many such models are now known. Restricting our attention to interaction-round-a-face models, these are the $A_{\infty}$ or eight-vertex solid-on-solid model [3], the $A_L^{(1)}$ or cyclic solid-on-solid models [4], the $A_L$ or Andrews-Baxter-Forrester models [5, 6, 7], the dilute $A_L$ models [8], and certain higher-rank models associated with $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $A_n^{(2)}$ [9]. Here, we present general forms of the boundary weights for some of these previously-considered models and for some additional, related models.

We begin, in Section 2, by outlining the standard relations, including the Yang-Baxter equation and the boundary Yang-Baxter equation, which may be satisfied by the bulk and boundary weights of an interaction-round-a-face model, and we define two important types of boundary weight, diagonal and non-diagonal. In Section 3, we consider certain intertwining properties which may be satisfied by the bulk and boundary weights of two appropriately-related interaction-round-a-face models. In Sections 4–9, we obtain boundary weights, mostly of the diagonal type, which represent general solutions of the boundary Yang-Baxter equation for various models. These models, some of their relationships and the sections in which they are considered are indicated in Figure 1. We conclude, in Section 10, with a discussion of general techniques for solving the boundary Yang-Baxter equation.

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2. Interaction-Round-a-Face Models

An interaction-round-a-face model is generally associated with an adjacency graph $G$ whose nodes correspond to the model’s spin values and whose bonds specify pairs of spin values which are allowed on neighbouring lattice sites.

The adjacency matrix $A$ associated with $G$ is defined by

$$A_{ab} = \text{number of bonds of } G \text{ which connect } a \text{ to } b$$

for any spin values $a$ and $b$. Here, we consider cases in which $G$ contains only bidirectional, single bonds, implying that $A$ is symmetric and that each of its entries is either 0 or 1. However the formalism can be generalised straightforwardly to accommodate directional or multiple bonds.

For interaction-round-a-face models with non-periodic boundary conditions, we associate a bulk weight $W$ with each set of spin values $a, b, c, d$ satisfying $A_{ab} A_{cb} A_{dc} A_{da} = 1$ and a boundary weight $B$ with each set of spin values $a, b, c$ satisfying $A_{ba} A_{bc} = 1$ [10, 6]. These weights are assumed to depend on a complex spectral parameter $u$ and are denoted

$$W \left( \begin{array}{cc} d & c \\ a & b \end{array} \right) = \begin{array}{c} d \\ u \\ c \\ a \\ \end{array}$$

$$B \left( \begin{array}{cc} a & b \\ c & \end{array} \right)

\text{ for } a, b, c \text{ satisfying } A_{ba} A_{bc} = 1 \text{ [10, 6].}$$
and

\[ B\left( b \begin{array}{c} a \\ c \end{array} \mid u \right) = b \begin{array}{c} a \\ c \end{array} \]. \quad (2.2)\]

Relations which may be satisfied by these weights are:

- The Yang-Baxter equation,

\[
\sum_{A_f g A_{ab} A_{gd} = 1} W\left( f \begin{array}{c} g \\ b \end{array} \mid u-v \right) W\left( g \begin{array}{c} d \\ c \end{array} \mid u \right) W\left( f \begin{array}{c} e \\ g \end{array} \mid v \right) = \\
\sum_{A_{ac} A_{eg} A_{gc} = 1} W\left( a \begin{array}{c} g \\ b \end{array} \mid v \right) W\left( f \begin{array}{c} e \\ a \end{array} \mid u \right) W\left( c \begin{array}{c} d \\ g \end{array} \mid u-v \right) \] \quad (2.3)

for all \( u \) and \( v \) and all \( a, b, c, d, e, f \) satisfying \( A_{ab} A_{bc} A_{cd} A_{de} A_{fe} A_{fa} = 1 \).

- The boundary Yang-Baxter equation,

\[
\sum_{A_{cf} A_{fe} A_{gd} = 1} W\left( c \begin{array}{c} f \\ d \end{array} \mid u-v \right) B\left( f \begin{array}{c} g \\ e \end{array} \mid u \right) W\left( c \begin{array}{c} b \\ g \end{array} \mid u+v \right) B\left( b \begin{array}{c} a \\ g \end{array} \mid v \right) = \\
\sum_{A_{cf} A_{fa} A_{fg} A_{dg} = 1} B\left( d \begin{array}{c} g \\ e \end{array} \mid v \right) W\left( c \begin{array}{c} f \\ d \end{array} \mid u+v \right) B\left( f \begin{array}{c} a \\ g \end{array} \mid u \right) W\left( c \begin{array}{c} b \\ a \end{array} \mid u-v \right) \] \quad (2.4)

for all \( u \) and \( v \) and all \( a, b, c, d, e \) satisfying \( A_{ba} A_{cb} A_{cd} A_{de} = 1 \).

- The initial condition,

\[ W\left( d \begin{array}{c} c \\ a \end{array} \mid 0 \right) = \delta_{ac} \] \quad (2.5)
for all $a, b, c, d$ satisfying $A_{ab} A_{cb} A_{dc} A_{da} = 1$.

- The boundary initial condition,

$$B\left( \begin{array}{c} b \\ a \\ c \end{array} \right) = \chi_a \delta_{ac}$$

for all $a, b, c$ satisfying $A_{ba} A_{bc} = 1$, where $\chi_a$ are fixed, model-dependent factors.

- Invariance of the bulk weights under a symmetry transformation $Z$ of the graph $G$,

$$W\left( \begin{array}{c} d \\ a \\ b \\ c \\ u \end{array} \right) = W\left( \begin{array}{c} Z(d) \\ Z(a) \\ Z(b) \\ Z(c) \end{array} \right)$$

for all $u$ and all $a, b, c, d$ satisfying $A_{ab} A_{cb} A_{dc} A_{da} = 1$.

- Invariance of the boundary weights under a symmetry transformation $Z$ of the graph $G$,

$$B\left( \begin{array}{c} b \\ a \\ c \end{array} \right) = B\left( \begin{array}{c} Z(b) \\ Z(a) \\ Z(c) \end{array} \right)$$

for all $u$ and all $a, b, c$ satisfying $A_{ba} A_{bc} = 1$.

- Reflection symmetry of the bulk weights,

$$W\left( \begin{array}{c} d \\ a \\ b \\ u \end{array} \right) = W\left( \begin{array}{c} b \\ a \\ c \\ d \\ u \end{array} \right) = W\left( \begin{array}{c} d \\ a \\ c \\ b \\ u \end{array} \right)$$

for all $u$ and all $a, b, c, d$ satisfying $A_{ab} A_{cb} A_{dc} A_{da} = 1$.

- Reflection symmetry of the boundary weights,

$$B\left( \begin{array}{c} b \\ a \\ c \end{array} \right) = B\left( \begin{array}{c} b \\ c \\ a \end{array} \right)$$

for all $u$ and all $a, b, c$ satisfying $A_{ba} A_{bc} = 1$.

- Crossing symmetry,

$$W\left( \begin{array}{c} d \\ a \\ b \\ c \\ u \end{array} \right) = \left( \frac{S_d S_c}{S_a S_b} \right)^{1/2} W\left( \begin{array}{c} a \\ d \\ c \\ b \\ \lambda - u \end{array} \right)$$

for all $u$ and all $a, b, c, d$ satisfying $A_{ab} A_{cb} A_{dc} A_{da} = 1$, where the crossing parameter $\lambda$ and crossing factors $S_a$ are fixed for a particular model.

- Boundary crossing symmetry,

$$\eta_a(u) B\left( \begin{array}{c} b \\ a \\ c \end{array} \right) = \sum_{d, A_{da} A_{dc} = 1} \left( \frac{S_d^2}{S_a S_c} \right)^{1/2} W\left( \begin{array}{c} b \\ a \\ c \\ d \\ 2u \end{array} \right) B\left( \begin{array}{c} a \\ c \\ d \end{array} \right) \left( \lambda - u \right)$$

$$\eta_a(u) = \left( \frac{S_d^2}{S_a S_c} \right)^{1/2}$$
for all $u$ and all $a, b, c$ satisfying $A_{ba} A_{bc} = 1$, where $\lambda$ and $S_a$ are the same as in (2.11) and $\eta_a$ are fixed, model-dependent functions.

• The inversion relation,

$$
\sum_{A_{de} A_{eb} = 1} W\left( \begin{array}{cc} d & e \\ a & b \end{array} \right) W\left( \begin{array}{cc} d & c \\ e & b \end{array} \right) = \rho(u) \rho(-u) \delta_{ac}
$$

(2.13)

for all $u$ and all $a, b, c, d$ satisfying $A_{ab} A_{cb} A_{dc} = 1$, where $\rho$ is a fixed, model-dependent function.

• The boundary inversion relation,

$$
\sum_{A_{bd} = 1} B\left( \begin{array}{cc} b & d \\ c & u \end{array} \right) B\left( \begin{array}{cc} b & a \\ d & u \end{array} \right) = \hat{\rho}_a(u) \delta_{ac}
$$

(2.14)

for all $u$ and all $a, b, c$ satisfying $A_{ba} A_{bc} = 1$, where $\hat{\rho}_a$ are fixed, model-dependent functions.

These relations are not all independent. For example, it can be shown that the boundary inversion relation is a consequence of the initial condition and the boundary Yang-Baxter equation, and that boundary crossing symmetry is a consequence of the initial condition, bulk crossing symmetry and the boundary Yang-Baxter equation.

We note that the boundary Yang-Baxter equation used here differs from the left and right reflection equations introduced in [6]. However if crossing symmetry is satisfied then the equations are effectively equivalent.

The bulk and boundary weights can be used to define matrices whose rows and columns are labelled by elements of $\{(a_0, \ldots, a_{N+1}) | A_{a_0 a_1} A_{a_1 a_2} \ldots A_{a_N a_{N+1}} = 1\}$, the set of paths of length $N+1$. The entries of the $j$th bulk face transfer matrix, for $1 \leq j \leq N$, are defined by

$$
X_j(u)(a_0 \ldots a_{N+1}, b_0 \ldots b_{N+1}) = \prod_{k=0}^{j-1} \delta_{a_k b_k} W\left( \begin{array}{cc} a_{j-1} & b_j \\ a_j & a_{j+1} \end{array} \right) \prod_{k=j+1}^{N+1} \delta_{a_k b_k}
$$

(2.15)
and the entries of the boundary face transfer matrix are defined by

\[ K(u)_{(a_0 \ldots a_{N+1}), (b_0 \ldots b_{N+1})} = \prod_{k=0}^{N} \delta_{a_k b_k} B\left( a_N \ b_{N+1} \left| \ a_{N+1} \ u \right. \right). \]  

(2.16)

In terms of these matrices, the Yang-Baxter equation is

\[ X_j(u-v) X_{j+1}(u) X_j(v) = X_{j+1}(v) X_j(u) X_{j+1}(u-v) \]  

(2.17)

and the boundary Yang-Baxter equation is

\[ X_N(u-v) K(u) X_N(u+v) K(v) = K(v) X_N(u+v) K(u) X_N(u-v). \]  

(2.18)

In the study of exactly solvable interaction-round-a-face models with non-periodic boundary conditions, we generally begin with bulk weights which satisfy the Yang-Baxter equation and then attempt to solve the boundary Yang-Baxter equation for corresponding boundary weights. Two classes of solution which are of particular interest, since they are needed for fixed and free boundary conditions respectively, are diagonal solutions, for which

\[ B\left( b \ a \ c \left| \ u \right. \right) = 0 \quad \text{whenever} \quad a \neq c \]  

(2.19)

and non-diagonal solutions, for which

\[ B\left( b \ a \ c \left| \ u \right. \right) \neq 0 \quad \text{for all} \quad a, b, c. \]  

(2.20)

We note that if a diagonal solution satisfies the boundary initial condition, then \( \chi \) is given by

\[ \chi_a = B\left( b \ a \left| \ 0 \right. \right) \]  

(2.21)

for any \( b \) satisfying \( A_{ab} = 1 \), and if a diagonal solution satisfies the boundary inversion relation, then \( \hat{\rho} \) is given by

\[ \hat{\rho}_a(u) = B\left( b \ a \left| -u \right. \right) B\left( b \ a \left| \ u \right. \right) \]  

(2.22)

for any \( b \) satisfying \( A_{ab} = 1 \).

### 3. Intertwiners

In this section we outline the formalism of intertwiners \[ [1], [2], [3], [4] \]. We are considering two interaction-round-a-face models, with respective finite adjacency graphs \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \), adjacency matrices \( A \) and \( \tilde{A} \), spin values \( a, b, \ldots \) and \( \bar{a}, \bar{b}, \ldots \), crossing factors \( S_a \) and \( \tilde{S}_{\bar{a}} \), bulk weights \( W \) and \( \overline{W} \), and boundary weights \( B \) and \( \overline{B} \).
We now assume that there exists an intertwiner graph $G'$ each of whose bonds connects a node of $G$ to a node of $\bar{G}$, and whose adjacency matrix $C$, defined by

$$C_{a\bar{a}} = \text{number of bonds of } G' \text{ which connect } a \text{ to } \bar{a}$$

for any spin values $a$ and $\bar{a}$, satisfies the intertwining relation

$$AC = C\bar{A}. \quad (3.1)$$

Here, we consider cases in which $G'$ contains only single bonds, so that each entry of $C$ is either 0 or 1.

We now associate an interwiner cell $I$ with each set of spin values $a, b, \bar{a}, \bar{b}$ satisfying $A_{ba} A_{\bar{a}\bar{b}} C_{a\bar{a}} C_{b\bar{b}} = 1$,

$$I\left(\begin{array}{c} b \\ a \end{array} \right) = \frac{b}{a} \quad \left(\begin{array}{c} \bar{b} \\ \bar{a} \end{array} \right). \quad (3.2)$$

Relations which may be satisfied by the intertwiner cells are:

- The first intertwiner inversion relation,

$$\sum_{b, \bar{b}} A_{ba} I\left(\begin{array}{c} a \\ \bar{b} \end{array} \right) I\left(\begin{array}{c} b \\ \bar{b} \end{array} \right) = \delta_{a\bar{c}} \quad (3.3)$$

for all $a, \bar{a}, \bar{b}, \bar{c}$ satisfying $A_{a\bar{a}} A_{\bar{b}\bar{c}} C_{a\bar{a}} C_{\bar{b}\bar{c}} = 1$.

- The second intertwiner inversion relation,

$$\sum_{b, \bar{b}} S_{ab} S_{\bar{b}a} I\left(\begin{array}{c} b \\ \bar{b} \end{array} \right) I\left(\begin{array}{c} b \\ a \end{array} \right) = \delta_{a\bar{c}} \quad (3.4)$$

for all $a, \bar{a}, \bar{b}, \bar{c}$ satisfying $A_{a\bar{a}} A_{\bar{b}\bar{c}} C_{a\bar{a}} C_{\bar{b}\bar{c}} = 1$.

- The bulk weight intertwining relation,

$$\sum_{b, \bar{b}} A_{ad} W\left(\begin{array}{c} a \\ d \end{array} \right) I\left(\begin{array}{c} a \\ \bar{b} \end{array} \right) I\left(\begin{array}{c} d \\ b \end{array} \right) = \delta_{a\bar{c}} \quad (3.5)$$
for all $u$ and all $a, b, c, \bar{a}, \bar{b}, \bar{c}$ satisfying $A_{ab} A_{bc} \bar{A}_{\bar{a}\bar{b}} \bar{A}_{\bar{b}\bar{c}} C_{ac} C_{dc} = 1$.

- The boundary weight intertwining relation,

$$\sum_{A_{a'b'c'd'=1}} B\left(\begin{array}{c|c} a & b \\ c & d \end{array} \right) I\left(\begin{array}{c|c} a & b \\ c & \bar{a} \end{array} \right) = \sum_{\bar{A}_{\bar{a}'\bar{b}'\bar{c}'\bar{d}'=1}} I\left(\begin{array}{c|c} a & \bar{b} \\ b & \bar{c} \end{array} \right) \bar{B}\left(\begin{array}{c|c} \bar{b} & \bar{a} \\ \bar{c} & \bar{d} \end{array} \right)$$

for all $u$ and all $a, b, \bar{a}, \bar{b}$ satisfying $A_{ab} \bar{A}_{\bar{b}a} C_{\bar{a}b} = 1$.

The bulk and boundary weight intertwining relations can be combined with the intertwiner inversion relations to give expressions for the weights of one model in terms of those of the other model. For example, assuming the first intertwiner inversion relation, we find that the bulk weight intertwining relation is equivalent to

$$\mathcal{W}\left(\begin{array}{c|c} \bar{d} & \bar{c} \\ \bar{a} & \bar{b} \end{array} \right) \delta_{\bar{d}\bar{d}'} = \sum_{abc} A_{ab} A_{bc} A_{\bar{a}d} C_{ac} C_{\bar{b}d} \mathcal{W}\left(\begin{array}{c|c} d & c \\ a & b \end{array} \right)$$

for all $u$ and all $d, \bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{d}'$ satisfying $\bar{A}_{\bar{a}\bar{b}} \bar{A}_{\bar{b}\bar{c}} \bar{A}_{\bar{c}\bar{d}} C_{\bar{d}\bar{d}'} = 1$. Similarly, assuming the second intertwiner inversion relation, we find that the bulk weight intertwining relation is equivalent to

$$\mathcal{W}\left(\begin{array}{c|c} \bar{d} & \bar{c} \\ \bar{a} & \bar{b} \end{array} \right) \delta_{\bar{b}\bar{b}'} = \sum_{acd} A_{ab} A_{bc} A_{\bar{d}a} C_{ac} C_{\bar{d}d} \mathcal{W}\left(\begin{array}{c|c} d & c \\ a & b \end{array} \right)$$

for all $u$ and all $d, \bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{d}'$ satisfying $\bar{A}_{\bar{a}\bar{b}} \bar{A}_{\bar{b}\bar{c}} \bar{A}_{\bar{c}\bar{d}} C_{\bar{d}\bar{d}'} = 1$. Similarly, assuming the second intertwiner inversion relation, we find that the bulk weight intertwining relation is equivalent to

$$\mathcal{W}\left(\begin{array}{c|c} \bar{d} & \bar{c} \\ \bar{a} & \bar{b} \end{array} \right) \delta_{\bar{b}\bar{b}'} = \sum_{acd} A_{ab} A_{bc} A_{\bar{d}a} C_{ac} C_{\bar{d}d} \mathcal{W}\left(\begin{array}{c|c} d & c \\ a & b \end{array} \right)$$
for all \(u\) and all \(b, \bar{a}, \bar{b}, \bar{c}, \bar{d}\) satisfying \(\bar{A}_{\bar{a} \bar{b}} \bar{A}_{\bar{c} \bar{d}} \bar{A}_{\bar{a} \bar{d}} C_{\bar{b} \bar{c}} C_{\bar{b} \bar{d}} = 1\), and that the boundary weight intertwining relation is equivalent to

\[
B\left(\bar{b} \left| \bar{a}, \bar{c}\right. \right) = \sum_{\bar{a} \bar{b}} \frac{S_{\bar{b}} \bar{S}_{\bar{c}}}{S_{\bar{c}} \bar{S}_{\bar{b}}} I\left(\bar{b} \left| \bar{a}, \bar{c}\right. \right) I\left(\bar{b} \left| \bar{c}, \bar{c}\right. \right) B\left(\bar{b} \left| \bar{a}, \bar{c}\right. \right)
\]

(3.9)

for all \(u\) and all \(c, \bar{a}, \bar{b}, \bar{c}\) satisfying \(\bar{A}_{\bar{a} \bar{b}} \bar{A}_{\bar{c} \bar{c}} C_{\bar{c} \bar{c}} = 1\). We note that the right sides of (3.7), (3.8) and (3.9) must be independent of \(d, b\) and \(c\) respectively.

The importance of the intertwiner relations (3.3)–(3.6) is that they imply that if the Yang-Baxter equation and boundary Yang-Baxter equation are satisfied by the weights of one model, then they are also satisfied by the weights of the other model. For if, on each side of the Yang-Baxter equation for the weights of one model, we introduce three intertwiner cells, apply (3.5) three times successively, introduce a further three intertwiner cells, and apply (3.3) three times, then we obtain the Yang-Baxter equation for the weights of the other model. Similarly, if, on each side of the boundary Yang-Baxter equation for the weights of one model, we introduce two intertwiner cells, apply (3.5) twice and (3.6) twice, introduce a further two intertwiner cells, and apply (3.4) twice, then we obtain the boundary Yang-Baxter equation for the weights of the other model.
4. **$A_\infty$ Model**

4.1 **Bulk Weights**

Throughout this and subsequent sections, we shall use the elliptic theta functions,

\[
\vartheta_1(u, q) = 2 q^{1/4} \sin u \prod_{n=1}^{\infty} \left( 1 - 2q^{2n} \cos 2u + q^{4n} \right) \left( 1 - q^{2n} \right)
\]

\[
\vartheta_2(u, q) = 2 q^{1/4} \cos u \prod_{n=1}^{\infty} \left( 1 + 2q^{2n} \cos 2u + q^{4n} \right) \left( 1 - q^{2n} \right)
\]

\[
\vartheta_3(u, q) = \prod_{n=1}^{\infty} \left( 1 + 2q^{2n-1} \cos 2u + q^{4n-2} \right) \left( 1 - q^{2n} \right)
\]

\[
\vartheta_4(u, q) = \prod_{n=1}^{\infty} \left( 1 - 2q^{2n-1} \cos 2u + q^{4n-2} \right) \left( 1 - q^{2n} \right)
\]

(4.1)

Since the nome $q$ is generally fixed, we shall abbreviate

\[
\vartheta_i(u) = \vartheta_i(u, q).
\]

(4.2)

We now consider the $A_\infty$ or eight-vertex solid-on-solid model [15]. The spins in this model take values from the set of all integers and the adjacency graph is

\[
A_\infty = \begin{array}{cccccc}
-2 & -1 & 0 & 1 & 2 & -
\end{array}
\]

(4.3)

The bulk weights are

\[
W \left( \begin{array}{c|c}
  a \pm 1 & a \\
  a & a \perp \end{array} \right| u \right) = \frac{\vartheta_1(\lambda - u)}{\vartheta_1(\lambda)}
\]

\[
W \left( \begin{array}{c|c}
  a & a \pm 1 \\
  a \perp & a \\
\end{array} \right| u \right) = \left( \frac{\vartheta_1((a-1)\lambda + w_0)}{\vartheta_1((a+1)\lambda + w_0)} \right)^{1/2} \frac{\vartheta_1(u)}{\vartheta_1(\lambda)}
\]

\[
W \left( \begin{array}{c|c}
  a & a \pm 1 \\
  a \perp & a \\
\end{array} \right| u \right) = \frac{\vartheta_1(a\lambda + w_0 \pm u)}{\vartheta_1(a\lambda + w_0)}
\]

(4.4)

where $\lambda$ and $w_0$ are arbitrary.

These weights satisfy the initial condition, reflection symmetry, crossing symmetry with crossing parameter $\lambda$ and crossing factors

\[
S_a = \vartheta_1(a\lambda + w_0),
\]

(4.5)

the inversion relation with

\[
\rho(u) = \frac{\vartheta_1(\lambda - u)}{\vartheta_1(\lambda)},
\]

(4.6)

and the Yang-Baxter equation.

The bulk weights for the critical $A_\infty$ model are obtained by taking $q \rightarrow 0$ so that all of the $\vartheta_1$ functions in (4.4) become sin functions.
4.2 Diagonal Boundary Weights

Diagonal boundary weights for the $A_\infty$ model are

\[
B \begin{pmatrix} a \pm 1 & a \mid u \\ a & a \end{pmatrix} = (x_1(a) \vartheta_1(u) \vartheta_1(u + a\lambda - w_0) + x_2(a) \vartheta_4(u) \vartheta_4(u + a\lambda + w_0)) f(a, u)
\]

\[
B \begin{pmatrix} a & a \mid u \\ a \pm 1 & a \end{pmatrix} = 0
\]

where $x_1$, $x_2$ and $f$ are arbitrary.

We now prove that these boundary weights represent the general diagonal solution of the boundary Yang-Baxter equation for the $A_\infty$ model. Having set $B \begin{pmatrix} a \pm 1 & a \mid u \end{pmatrix} = 0$, we find that the only spin assignments in the boundary Yang-Baxter equation (2.4) which lead to non-trivial equations are $a = c = e$, $b = a \pm 1$, $d = a \mp 1$. These equations can be written

\[
B_a(u)^t \mathcal{R}_a(u, v) B_a(v) = 0
\]

where

\[
B_a(u) = \begin{pmatrix} B \begin{pmatrix} a+1 & a \mid u \end{pmatrix} \\ B \begin{pmatrix} a-1 & a \mid u \end{pmatrix} \end{pmatrix}
\]

and

\[
\mathcal{R}_a(u, v) = \begin{pmatrix} \vartheta_1(u-v) \vartheta_1(u+v+a\lambda+w_0) & -\vartheta_1(u+v) \vartheta_1(u-v+a\lambda+w_0) \\ -\vartheta_1(u+v) \vartheta_1(u-v-a\lambda-w_0) & \vartheta_1(u-v) \vartheta_1(u+v-a\lambda-w_0) \end{pmatrix}.
\]

Decomposing each entry of $\mathcal{R}_a(u, v)$ using a standard elliptic addition identity, we find that

\[
\vartheta_4(0) \vartheta_4(a\lambda+w_0) \mathcal{R}_a(u, v) = S_a(u)^t Q S_a(v)
\]

where

\[
S_a(u) = \begin{pmatrix} \vartheta_4(u) \vartheta_4(u+a\lambda+w_0) & -\vartheta_4(u) \vartheta_4(u-a\lambda-w_0) \\ -\vartheta_4(u) \vartheta_4(u+a\lambda+w_0) & \vartheta_4(u) \vartheta_4(u-a\lambda-w_0) \end{pmatrix}
\]

and

\[
Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

We therefore obtain, from (4.8),

\[
K(u)^t Q K(v) = 0 \quad \text{or} \quad K_1(u) K_2(v) - K_2(u) K_1(v) = 0
\]

where $K(u) = \begin{pmatrix} K_1(u) \\ K_2(u) \end{pmatrix} = S_a(u) B_a(u)$. The general solution of (4.9) is $K(u) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f(u)$, where $x_1$ and $x_2$ are arbitrary constants and $f$ is an arbitrary function, and therefore the general solution of (4.8) is

\[
B_a(u) = S_a(u)^{-1} \begin{pmatrix} x_1(a) \\ x_2(a) \end{pmatrix} \tilde{f}(a, u)
\]

\[
= \begin{pmatrix} \vartheta_1(u) \vartheta_1(u-a\lambda-w_0) & \vartheta_4(u) \vartheta_4(u-a\lambda-w_0) \\ \vartheta_1(u) \vartheta_1(u+a\lambda+w_0) & \vartheta_4(u) \vartheta_4(u+a\lambda+w_0) \end{pmatrix} \begin{pmatrix} x_1(a) \\ x_2(a) \end{pmatrix} f(a, u)
\]
where \( \tilde{f}(a, u) = \det \mathcal{S}_a(u) \) \( f(a, u) = -\vartheta_4(0) \vartheta_1(a\lambda + w_0) \vartheta_4(a\lambda + w_0) \vartheta_1(2u) f(a, u) \), and \( x_1, x_2 \) and \( f \) are arbitrary. This solution matches (4.7) and concludes our proof.

The \( A_\infty \) diagonal boundary weights also satisfy the boundary initial condition and the boundary inversion relation, with \( \chi \) and \( \tilde{\rho} \) given by (2.21) and (2.22), and boundary crossing symmetry with

\[
\eta_a(u) = \frac{\vartheta_1(2\lambda-2u)}{\vartheta_1(\lambda)} \frac{f(a, \lambda-u)}{f(a, u)}.
\]

If we set

\[
(x_1(a), x_2(a)) = \frac{1}{\vartheta_4(0) \vartheta_4(a\lambda + w_0)} \left( \vartheta_4(\xi_a) \vartheta_4(a\lambda + w_0 + \xi_a), -\vartheta_1(\xi_a) \vartheta_1(a\lambda + w_0 + \xi_a) \right)
\]

\( f(a, u) = 1 \)

where \( \xi_a \) are arbitrary, then the weights can be expressed in terms of \( \vartheta_1 \) functions only as

\[
B\left( a \pm 1 \begin{array}{c} a \\ a \end{array} | u \right) = \vartheta_1(u \mp a\lambda \mp w_0 \mp \xi_a) \vartheta_1(u) \vartheta_1(u \mp a\lambda \mp w_0 \mp \xi_a) \vartheta_1(u) \vartheta_1(u) \vartheta_1(u).
\]

The general diagonal solution of the boundary Yang-Baxter equation for the critical \( A_\infty \) model can be obtained from (1.7) by replacing \( x_1(a) \) by \( x_1(a)/q^{u/2} \) and taking \( q \to 0 \), giving

\[
B\left( a \pm 1 \begin{array}{c} a \\ a \end{array} | u \right) = (x_1(a) \sin u \sin(u \mp a\lambda \mp w_0) + x_2(a)) f(a, u).
\]

### 4.3 Non-Diagonal Boundary Weights

We now proceed to the case of non-diagonal boundary weights. Using a method similar to that for the derivation of the diagonal boundary weights, we have found that the general non-diagonal solution of the boundary Yang-Baxter equation for the \( A_\infty \) model is

\[
B\left( a \pm 1 \begin{array}{c} a \\ a \end{array} | u \right) = K(a, \pm u) f^{\pi_a}(u)
\]

\[
B\left( a \mp 1 \begin{array}{c} a \\ a \end{array} | u \right) = \kappa_{\pm}(a) k(u) f^{\pi_a}(u)
\]

where \( \pi_a \) is the parity (even or odd) of \( a \),

\[
K(a, u) = \frac{1}{\vartheta_1(a\lambda + w_0)} \times
\]

\[
\left( x_1^{\pi_a} \vartheta_1(u + a\lambda + w_0) \vartheta_2(u) \vartheta_3(u) \vartheta_4(u) + x_2^{\pi_a} \vartheta_2(u + a\lambda + w_0) \vartheta_1(u) \vartheta_3(u) \vartheta_4(u) + x_3^{\pi_a} \vartheta_3(u + a\lambda + w_0) \vartheta_1(u) \vartheta_2(u) \vartheta_4(u) + x_4^{\pi_a} \vartheta_4(u + a\lambda + w_0) \vartheta_1(u) \vartheta_2(u) \vartheta_3(u) \right)
\]

\[
= \frac{\vartheta_2(0) \vartheta_3(0) \vartheta_4(0) \vartheta_1(2u)}{2 \vartheta_1(a\lambda + w_0)} \sum_{i=1}^{4} x_i^{\pi_a} \vartheta_i(u + a\lambda + w_0) \vartheta_i(u)
\]

\[
\kappa_{+}(a) = \kappa_{-}(a) = (K(a, -\sigma^{\pi_a}) K(a, \sigma^{\pi_a}))^{1/2}
\]

\[
k(u) = \frac{\vartheta_1(2u)}{\vartheta_1(2\sigma^{\pi_a})},
\]

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and $x_1, x_2, x_3, x_4, \sigma$ and $f$ are arbitrary. In fact, $\kappa_- (a)$ and $\kappa_+ (a)$ are only defined up to their product, which must be $K(a, -\sigma) K(a, \sigma)$, however the choice (4.16) has been made so that the weights satisfy reflection symmetry. These weights also satisfy the boundary initial condition with

$$\chi_{a+1} = x_1^{\pi\alpha} \vartheta_2(0) \vartheta_3(0) \vartheta_4(0) f^{\pi\alpha}(0),$$

boundary crossing symmetry with

$$\eta_{a+1}(u) = \frac{\vartheta_1(2\lambda-2u)}{\vartheta_1(\lambda)} \frac{f^{\pi\alpha}(\lambda-u)}{f^{\pi\alpha}(u)},$$

and the boundary inversion relation with

$$\hat{\rho}_{a+1}(u) = \vartheta_1(\sigma^{\pi\alpha} - u) \vartheta_1(\sigma^{\pi\alpha} + u) \left( \frac{\vartheta_2(u) \vartheta_3(u) \vartheta_4(u) x_1^{\pi\alpha}}{\vartheta_1(\sigma^{\pi\alpha})} \right)^2 - \left( \frac{\vartheta_1(u) \vartheta_3(u) \vartheta_4(u) x_2^{\pi\alpha}}{\vartheta_2(\sigma^{\pi\alpha})} \right)^2$$

$$+ \left( \frac{\vartheta_1(u) \vartheta_2(u) \vartheta_4(u) x_3^{\pi\alpha}}{\vartheta_3(\sigma^{\pi\alpha})} \right)^2 - \left( \frac{\vartheta_1(u) \vartheta_2(u) \vartheta_3(u) x_4^{\pi\alpha}}{\vartheta_4(\sigma^{\pi\alpha})} \right)^2 \frac{f^{\pi\alpha}(-u)}{f^{\pi\alpha}(u)}. $$

The $A_\infty$ non-diagonal boundary weights include those which are obtained in [3] by using known boundary weights for the eight-vertex model and vertex-face intertwiners.

The weights can also be written in terms of

$$\tilde{\vartheta}_i(u) = \vartheta_i(u, q^2)$$

as

$$K(a, u) = y_1^{\pi\alpha} \frac{\tilde{\vartheta}_1(2u + a\lambda + w_0) \tilde{\vartheta}_2(2u)}{\tilde{\vartheta}_1(a\lambda + w_0)} + y_2^{\pi\alpha} \frac{\tilde{\vartheta}_4(2u + a\lambda + w_0) \tilde{\vartheta}_4(2u)}{\tilde{\vartheta}_4(a\lambda + w_0)}$$

$$+ y_3^{\pi\alpha} \frac{\tilde{\vartheta}_4(2u + a\lambda + w_0) \tilde{\vartheta}_1(2u)}{\tilde{\vartheta}_1(a\lambda + w_0)} + y_4^{\pi\alpha} \frac{\tilde{\vartheta}_1(2u + a\lambda + w_0) \tilde{\vartheta}_1(2u)}{\tilde{\vartheta}_4(a\lambda + w_0)}$$

$$k(u) = \frac{\tilde{\vartheta}_1(2u) \tilde{\vartheta}_4(2u)}{\tilde{\vartheta}_1(2\sigma^{\pi\alpha}) \tilde{\vartheta}_4(2\sigma^{\pi\alpha})}$$

where

$$(y_1^{\pi\alpha}, y_2^{\pi\alpha}, y_3^{\pi\alpha}, y_4^{\pi\alpha}) = \frac{\vartheta_2(0) \tilde{\vartheta}_4(0)}{2} (x_1^{\pi\alpha} + x_2^{\pi\alpha}, x_1^{\pi\alpha} - x_2^{\pi\alpha}, x_4^{\pi\alpha}, x_4^{\pi\alpha} - x_3^{\pi\alpha}),$$

or in terms of

$$\tilde{\vartheta}_i(u) = \vartheta_i(u, q^{\sqrt{2}})$$

as

$$K(a, u) = z_1^{\pi\alpha} \frac{\tilde{\vartheta}_1(u + \frac{a\lambda + w_0}{2}) \tilde{\vartheta}_2(u)}{\tilde{\vartheta}_1(\frac{a\lambda + w_0}{2})} + z_2^{\pi\alpha} \frac{\tilde{\vartheta}_2(u + \frac{a\lambda + w_0}{2}) \tilde{\vartheta}_2(u)}{\tilde{\vartheta}_2(\frac{a\lambda + w_0}{2})}$$

$$k(u) = \frac{\tilde{\vartheta}_1(2u) \tilde{\vartheta}_2(2u)}{\tilde{\vartheta}_1(2\sigma^{\pi\alpha}) \tilde{\vartheta}_2(2\sigma^{\pi\alpha})}$$
\[ + z_3^{\gamma_a} \frac{\hat{\vartheta}_2(u + a + u \nu_{1})}{\vartheta_1(\frac{a + u \nu_{1}}{2})} \vartheta_1(u) + z_4^{\gamma_a} \frac{\hat{\vartheta}_1(u + a + u \nu_{1})}{\vartheta_2(\frac{a + u \nu_{1}}{2})} \vartheta_1(u) \] (4.24)

\[ k(u) = \frac{\vartheta_1(u) \vartheta_2(u)}{\vartheta_1(\sigma_{\gamma_a}) \vartheta_2(\sigma_{\gamma_a})} \]

where

\[ (z_1^{\gamma_a}, z_2^{\gamma_a}, z_3^{\gamma_a}, z_4^{\gamma_a}) = \frac{\vartheta_4(0) \vartheta_2(0)}{4} (x_1^{\gamma_a}, x_2^{\gamma_a}, x_3^{\gamma_a}, x_4^{\gamma_a}) \]

Furthermore, if we set

\[ x_1^{\gamma_a} = \frac{(-1)^{i+1}}{\vartheta_1(\xi^{\gamma_a}_1 - \nu^{\gamma_a}_1) \vartheta_1(\xi^{\gamma_a}_2 + \nu^{\gamma_a}_2) \vartheta_1(\xi^{\gamma_a}_2 + \nu^{\gamma_a}_2)}{(\vartheta_2(0) \vartheta_3(0) \vartheta_4(0))} \]

\[ \sigma^{\gamma_a} = \xi^{\gamma_a}_1 + \nu^{\gamma_a}_1 \]

\[ f^{\gamma_a}(u) = 1 \]

(4.25)

where \( \xi^{\gamma_a}_1, \nu^{\gamma_a}_1, \xi^{\gamma_a}_2, \nu^{\gamma_a}_2 \) are arbitrary, then the weights can be expressed in terms of \( \vartheta_1 \) functions only as

\[ B(a \left| a \pm 1 \right| u) = \frac{1}{\vartheta_1(a \lambda + w_0) \vartheta_1(2 \nu^{\gamma_a}_1)} \times \]

\[ \left( \vartheta_1(\frac{a \lambda + w_0}{2} + \nu^{\gamma_a}_1 - \nu^{\gamma_a}_2) \vartheta_1(\frac{a \lambda + w_0}{2} + \nu^{\gamma_a}_1 + \nu^{\gamma_a}_2) \vartheta_1(\pm u - \xi^{\gamma_a}_1 + \nu^{\gamma_a}_1) \vartheta_1(\pm u + \xi^{\gamma_a}_1 + \nu^{\gamma_a}_1) \times \right) \]

\[ \vartheta_1(\pm u + \frac{a \lambda + w_0}{2} - \xi^{\gamma_a}_1 - \nu^{\gamma_a}_1) \vartheta_1(\pm u + \frac{a \lambda + w_0}{2} + \xi^{\gamma_a}_1 - \nu^{\gamma_a}_1) \]

(4.26)

The general non-diagonal solution of the boundary Yang-Baxter equation for the critical \( A_{\infty} \) model can be obtained from (1.22) by replacing \( y_1^{\gamma_a} \) by \( y_1^{\gamma_a}/q \) and taking \( q \rightarrow 0 \), giving

\[ K(a, u) = y_1^{\gamma_a} \frac{\sin(2u + a \lambda + w_0)}{\sin(a \lambda + w_0)} + y_2^{\gamma_a} + y_3^{\gamma_a} \frac{\sin(2u)}{\sin(a \lambda + w_0)} + y_1^{\gamma_a} \sin(2u + a \lambda + w_0) \sin(2u) \]

\[ k(u) = \frac{\sin(2u)}{\sin(2 \sigma^{\gamma_a})} \]

(4.27)

which matches the trigonometric weights obtained in [3].
5. \( A_L \) Models

5.1 Bulk Weights

We now consider the \( A_L \), or Andrews-Baxter-Forrester, models \([16]\), which can be regarded as restricted cases of the \( A_\infty \) model. There is one such model for each integer \( L \geq 2 \). The spins in this model take values from the set \( \{1, 2, \ldots, L\} \) and the adjacency graph is

\[
A_L = \begin{array}{c}
1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
2 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\ddots \\
L-1 \quad \cdot \quad \cdot \\
L \end{array}
\]

(5.1)

The \( A_L \) bulk weights are obtained from (4.4) with \( w_0 = 0 \) as

\[
W(a \pm 1 \left| a \right| u) = \frac{\vartheta_1(\lambda - u)}{\vartheta_1(\lambda)} , \quad a = 2, \ldots, L-1
\]

\[
W(a \pm 1 \left| a \right| u) = \left( \frac{\vartheta_1((a-1)\lambda) \vartheta_1((a+1)\lambda)}{\vartheta_1(a\lambda)^2} \right)^{1/2} \frac{\vartheta_1(u)}{\vartheta_1(\lambda)} , \quad a = 2, \ldots, L-1
\]

\[
W(a \pm 1 \left| a \right| u) = \frac{\vartheta_1(a\lambda \pm u)}{\vartheta_1(a\lambda)} , \quad \begin{cases} +, & a = 1, \ldots, L-1 \\ -, & a = 2, \ldots, L \end{cases}
\]

(5.2)

where

\[
\lambda = \frac{\pi}{L+1} . \quad (5.3)
\]

These weights satisfy the Yang-Baxter equation, the initial condition, invariance under the symmetry transformation

\[
Z(a) = L + 1 - a , \quad (5.4)
\]

reflection symmetry, crossing symmetry with crossing parameter (5.3) and crossing factors

\[
S_a = \vartheta_1(a\lambda) , \quad (5.5)
\]

and the inversion relation with \( \rho \) given by (4.6).

5.2 Diagonal Boundary Weights

The general diagonal solution of the boundary Yang-Baxter equation for the \( A_L \) models is obtained from (4.7) as

\[
B(a \pm 1 \left| a \right| u) = (x_1(a) \vartheta_1(u) \vartheta_1(u \mp a\lambda) + x_2(a) \vartheta_4(u) \vartheta_4(u \mp a\lambda)) f(a, u) , \quad \begin{cases} +, & a = 1, \ldots, L-1 \\ -, & a = 2, \ldots, L \end{cases}
\]

\[
B(a \pm 1 \left| a \right| u) = 0 , \quad a = 2, \ldots, L-1
\]

(5.6)

where \( x_1, x_2 \) and \( f \) are arbitrary.

These weights satisfy the boundary initial condition, the boundary inversion relation and boundary crossing symmetry, with \( \chi, \hat{\rho} \) and \( \eta \) given by (2.21), (2.22) and (4.10). The
weights are also invariant under the transformation (5.4) if the arbitrary parameters satisfy appropriate conditions, such as

\[ x_1(a) = -x_1(L+1-a), \quad x_2(a) = x_2(L+1-a), \quad f(a, u) = f(L+1-a, u). \]  

(5.7)

By using (4.11) with \( w_0 = 0 \), we find that these weights match those obtained in [6].

### 5.3 Non-Diagonal Boundary Weights

To obtain non-diagonal boundary weights for the \( A_L \) models, we set

\[
B\begin{pmatrix} a & a \pm 1 \\ a & a \pm 1 \end{pmatrix} = K(a, \pm u) f^\pi(u), \quad \{+, a = 1, \ldots, L-1 \}
\]

\[
B\begin{pmatrix} a & a \pm 1 \\ a & a \pm 1 \end{pmatrix} = \kappa_\pm(a) k(u) f^\pi(u), \quad a = 2, \ldots, L-1
\]

(5.8)

where \( f \) is arbitrary and \( K, \kappa \) and \( k \) are given by (4.11)–(4.17) with \( w_0 = 0 \). We then find that in order for the boundary Yang-Baxter equation to be satisfied for spin assignments which include the spin values 1 or \( L \), the constants \( x_1, x_2, x_3, x_4 \) and \( \sigma \) must satisfy

\[
x_{1,even}^\pi + x_{2,even}^\pi + x_{3,even}^\pi + x_{4,even}^\pi = 0
\]

\[
x_1^{\pi L+1} + x_2^{\pi L+1} - x_3^{\pi L+1} - x_4^{\pi L+1} = 0
\]

\[
(\gamma_{1,-} x_1^{\pi} + \gamma_{2,-} x_2^{\pi} + \gamma_{3,-} x_3^{\pi} + \gamma_{4,-} x_4^{\pi})(\gamma_{1,+} x_1^{\pi} + \gamma_{2,+} x_2^{\pi} + \gamma_{3,+} x_3^{\pi} + \gamma_{4,+} x_4^{\pi}) = 0
\]

where

\[
\gamma_{1,\pm} = \frac{\partial_i(\sigma^{\pi,\pm})}{\partial_i(\sigma^{\pi})}.
\]

The \( A_L \) non-diagonal boundary weights satisfy the boundary initial condition, reflection symmetry, boundary crossing symmetry and the boundary inversion relation, with \( \chi, \eta \) and \( \hat{\rho} \) given by (4.18)–(4.20). The weights are also invariant under the transformation (5.4) if the arbitrary parameters satisfy appropriate conditions.

By using (4.25), we find that these weights include those obtained in [7].

### 6. \( D_L \) Models

#### 6.1 Bulk Weights

We now consider the \( D_L \) models [17]. There is one such model for each integer \( L \geq 3 \). The spins in this model take values from the set \( \{1, 2, \ldots, L\} \) and the adjacency graph is

\[
D_L = \begin{array}{c}
1 & 2 & \cdot & \cdot & \cdot & L-3 & L-2 & L-1 & L
\end{array}
\]

(6.1)
The $D_L$ bulk weights are
\[
W\left(\begin{array}{c} a \pm 1 \\ a \\ a \mp 1 \end{array} \right| u \right) = \frac{\vartheta_1(\lambda-u)}{\vartheta_1(\lambda)}, \quad a = 2, \ldots, L-3
\]
\[
W\left(\begin{array}{c} L-1 \\ L-2 \\ L-3 \end{array} \right| u \right) = W\left(\begin{array}{c} L-3 \\ L-2 \\ L-1 \end{array} \right| u \right) = W\left(\begin{array}{c} L \\ L-2 \\ L-3 \end{array} \right| u \right) = W\left(\begin{array}{c} L-3 \\ L-2 \\ L \end{array} \right| u \right) = \frac{\vartheta_1(\lambda-u)}{\vartheta_1(\lambda)}
\]
\[
W\left(\begin{array}{c} L \\ L-2 \\ L-1 \end{array} \right| u \right) = W\left(\begin{array}{c} L-1 \\ L-2 \\ L \end{array} \right| u \right) = \frac{\vartheta_1(\lambda) \vartheta_2(u) - \vartheta_2(\lambda) \vartheta_1(u)}{\vartheta_2(0) \vartheta_1(\lambda)}
\]
\[
W\left(\begin{array}{c} a \pm 1 \\ a \mp 1 \end{array} \right| u \right) = \left( \frac{\vartheta_1((a-1)\lambda) \vartheta_1((a+1)\lambda)}{\vartheta_1(a\lambda)^2} \right)^{1/2} \frac{\vartheta_1(u)}{\vartheta_1(\lambda)}, \quad a = 2, \ldots, L-3
\]
\[
W\left(\begin{array}{c} L-2 \\ L-3 \\ L-2 \end{array} \right| u \right) = W\left(\begin{array}{c} L-2 \\ L-1 \\ L-2 \end{array} \right| u \right)
\]
\[
W\left(\begin{array}{c} L-2 \\ L-1 \\ L-2 \end{array} \right| u \right) = W\left(\begin{array}{c} L-2 \\ L \\ L-2 \end{array} \right| u \right) = \frac{1}{2} \left( \frac{\vartheta_2(\lambda-u)}{\vartheta_2(\lambda)} - \frac{\vartheta_1(\lambda-u)}{\vartheta_1(\lambda)} \right)
\]
\[
W\left(\begin{array}{c} L-2 \\ L-1 \\ L-2 \end{array} \right| u \right) = W\left(\begin{array}{c} L-2 \\ L \\ L-2 \end{array} \right| u \right) = \frac{1}{2} \left( \frac{\vartheta_2(\lambda-u)}{\vartheta_2(\lambda)} + \frac{\vartheta_1(\lambda-u)}{\vartheta_1(\lambda)} \right)
\]
\[
W\left(\begin{array}{c} L-1 \\ L-2 \\ L-1 \end{array} \right| u \right) = W\left(\begin{array}{c} L-1 \\ L-2 \\ L \end{array} \right| u \right) = \frac{\vartheta_1(\lambda) \vartheta_2(u) + \vartheta_2(\lambda) \vartheta_1(u)}{\vartheta_2(0) \vartheta_1(\lambda)}
\]

where
\[
\lambda = \frac{\pi}{2(L-1)}. \quad (6.3)
\]

These weights satisfy the Yang-Baxter equation, the initial condition, invariance under the symmetry transformation
\[
Z(a) = \begin{cases} a, & a = 1, \ldots, L-2 \\ L, & a = L-1 \\ L-1, & a = L \end{cases} \quad (6.4)
\]
reflection symmetry, crossing symmetry with crossing parameter (6.3) and crossing factors
\[
S_a = \begin{cases} \vartheta_1(a\lambda), & a = 1, \ldots, L-2 \\ \vartheta_2(0)/2, & a = L-1, L \end{cases} \quad (6.5)
\]
and the inversion relation with $\rho$ given by (4.6).
6.2 Diagonal Boundary Weights

We have found, using a method similar to that for the $A_L$ models, that the general diagonal solution of the boundary Yang-Baxter equation for the $D_L$ models is

$$
B(a \pm 1 \begin{array}{c} a \\ a \end{array} | u) = (x_1(a) \vartheta_1(u) \vartheta_1(u \mp a \lambda) + \\
\; x_2(a) \vartheta_4(u) \vartheta_4(u \mp a \lambda)) f(a, u),
$$

$$
\begin{align*}
&\quad \begin{cases}
  +, a = 1, \ldots, L-3 \\
  -, a = 2, \ldots, L-3
\end{cases} \\
\end{align*}
$$

$$
B\left(\begin{array}{c} a \\ L-2 \\ L-2 \end{array} | u\right) = k_i(a, u) f(L-2, u), \quad a = L-3, L-1, L
$$

$$
B\left(\begin{array}{c} L-2 \\ a \\ u \end{array} | u\right) = \vartheta_3(u) \vartheta_4(u) f(a, u), \quad a = L-1, L
$$

$$
B\left(\begin{array}{c} a \pm 1 \\ a \mp 1 \end{array} | u\right) = 0, \quad a = 2, \ldots, L-2
$$

$$
B\left(\begin{array}{c} L-2 \\ L-2 \\ L-3 \\ L \end{array} | u\right) = B\left(\begin{array}{c} L-2 \\ L-1 \\ L-3 \\ L \end{array} | u\right) = B\left(\begin{array}{c} L-2 \\ L-1 \\ L \\ L-1 \end{array} | u\right) = 0
$$

where

$$
\begin{align*}
k_1(L-3, u) &= x_1(L-2) \vartheta_1(u) \vartheta_2(u - \lambda) + x_2(L-2) \vartheta_4(u) \vartheta_3(u - \lambda) \\
k_1(L-1, u) &= k_1(L, u) = k_1(L-3, -u) \\
k_2(L-3, u) &= \left(x_1(L-2) \vartheta_3(u - \lambda) - x_2(L-2) \vartheta_4(u - \lambda)\right) \\
&\quad \times \left(x_1(L-2) \vartheta_3(u - \lambda) + x_2(L-2) \vartheta_4(u - \lambda)\right) \vartheta_3(u) \vartheta_4(u) \\
k_2(L-1, u) &= \left(x_1(L-2) \vartheta_3(u - \lambda) - x_2(L-2) \vartheta_4(u - \lambda)\right) \\
&\quad \times \left(x_1(L-2) \vartheta_3(u + \lambda) + x_2(L-2) \vartheta_4(u + \lambda)\right) \vartheta_3(u) \vartheta_4(u) \\
k_2(L, u) &= k_2(L-1, -u),
\end{align*}
$$

$x_1, x_2$ and $f$ are arbitrary, and $i$ may be chosen arbitrarily as 1 or 2.

These weights satisfy the boundary initial condition, the boundary inversion relation and boundary crossing symmetry, with $\chi, \hat{\rho}$ and $\eta$ given by (2.21), (2.22) and (4.10). The weights are also invariant under the transformation (6.4) if the arbitrary parameters satisfy appropriate conditions, such as

$$
i = 1, \quad f(L-1, u) = f(L, u).
$$

6.3 $A_{2L-3} - D_L$ Intertwiner

The $A_{2L-3}$ and $D_L$ models can be related by intertwiner cells [13, 14, 12, 13, 14]. The intertwiner graph, whose adjacency matrix satisfies the intertwining relation, is
and the intertwiner cells are
\[
I \left( \begin{array}{cc} a \pm 1 & a \pm 1 \\ a & a \end{array} \right) = I \left( \begin{array}{cc} 2L-2 \pm 1 & a \\ 2L-2 - a & a \end{array} \right) = 1, \quad \{ +, a = 1, \ldots, L-3 \}
\]
\[
I \left( \begin{array}{cc} L-1 & L-1 \\ L & L-2 \end{array} \right) = I \left( \begin{array}{cc} L-1 & L \\ L-2 & L \end{array} \right) = I \left( \begin{array}{cc} L-1 & L \\ L-2 & L \end{array} \right) = \frac{1}{\sqrt{2}}, \quad I \left( \begin{array}{cc} L-1 & L \\ L-2 & L \end{array} \right) = -\frac{1}{\sqrt{2}}.
\]
These intertwiner cells satisfy the intertwiner inversion relations and the bulk weight intertwining relation. They also satisfy the boundary weight intertwining relation together with the $A_{2L-3}$ and $D_L$ diagonal boundary weights, provided that the arbitrary parameters satisfy appropriate conditions, such as
\[
i = 1, \quad x_1^A(a) = -x_1^A(2L-2-a) = x_1^D(a), \quad x_2^A(a) = x_2^A(2L-2-a) = x_2^D(a),
\]
\[
f^A(a, u) = f^A(2L-2-a, u) = f^D(a, u), \quad a = 1, \ldots, L-2
\]
\[
x_1^A(L-1) = 0, \quad x_2^A(L-1) f^A(L-1, u) = f^D(L-1, u) = f^D(L, u).
\]

7. **Temperley-Lieb Models**

7.1 **Bulk Weights**

We now consider the Temperley-Lieb models \([19, 20]\). The adjacency graph $G$ for these models can be any finite, connected graph which has only bidirectional, single bonds. By the Perron-Frobenius theorem, the adjacency matrix of $G$ has a unique, positive maximum eigenvalue $\Lambda$, with an associated eigenvector $(S_1, S_2, \ldots)$ which has all positive entries.

The Temperley-Lieb bulk weights are
\[
W \left( \begin{array}{cc} d & c \\ a & b \end{array} \right) u = \frac{s(\Lambda-u)}{s(\Lambda)} \delta_{ac} + \frac{(S_a S_e)^{\frac{1}{2}}}{S_b} \frac{s(u)}{s(\Lambda)} \delta_{bd}
\]
where $\Lambda$ is any solution of
\[
\Lambda = 2 c(\Lambda)
\]
and
\[
s(u) = \begin{cases} 
\sin u, & \Lambda < 2 \\
u, & \Lambda = 2 \\
\sinh u, & \Lambda > 2
\end{cases} \quad c(u) = \begin{cases} 
cos u, & \Lambda < 2 \\
1, & \Lambda = 2 \\
cosh u, & \Lambda > 2
\end{cases}
\]

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These weights satisfy the Yang-Baxter equation, the initial condition, reflection symmetry, crossing symmetry with crossing parameter $\lambda$ and crossing factors $S_a$, and the inversion relation with
\[
\rho(u) = \frac{s(\lambda-u)}{s(\lambda)} .
\]
From (2.15) and (7.1), we obtain the Temperley-Lieb bulk face transfer matrices,
\[
X_j(u) = \frac{s(\lambda-u)}{s(\lambda)} I + \frac{s(u)}{s(\lambda)} e_j
\]
where
\[
e_j (a_0...a_{N+1},b_0...b_{N+1}) = \prod_{k=0}^{j-1} \delta_{a_k b_k} \left(\frac{S_{a_j} S_{b_j}}{S_{a_j-1}} \right)^{1/2} \delta_{a_{j-1} a_{j+1}} \prod_{k=j+1}^{N+1} \delta_{a_k b_k} .
\]
The matrices $e_j$ satisfy
\[
e_i e_j - e_j e_i = 0, \quad |i-j| > 1
\]
\[
e_j e_{j\pm1} e_j = e_j
\]
\[
e_j^2 = \Lambda e_j
\]
and therefore form a representation of the Temperley-Lieb algebra [21]. In fact, it can be shown straightforwardly that if $e_j$ are any matrices which satisfy (7.7), then $X_j(u)$, defined by (7.2), (7.3) and (7.5) alone, satisfy the Yang-Baxter equation (2.17). We also note that (7.5) immediately implies commutation of the bulk face transfer matrices,
\[
X_j(u) X_j(v) = X_j(v) X_j(u) .
\]
It is known that the only simple, connected graphs with $\Lambda < 2$ are $A_L$, $D_L$, and $E_6$, $E_7$ and $E_8$,
\[
E_6 = \begin{align*}
1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 \\
\end{align*}
\]
\[
E_7 = \begin{align*}
1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 \\
\end{align*}
\]
\[
E_8 = \begin{align*}
1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 \\
\end{align*}
\]
It can be shown that for $A_L$ and $D_L$, $\lambda$ is given by (5.3) and (6.3), while for $E_6, E_7, E_8$,
\[
\lambda = \begin{cases} 
\pi/12, & E_6 \\
\pi/18, & E_7 \\
\pi/30, & E_8 . 
\end{cases}
\]
For these graphs, it is also known that we may set

\[ S_a = \begin{cases} 
\sin\alpha, & a = 1, \ldots, L; \\
\sin\alpha, & a = 1, \ldots, L-2 \\
1/2, & a = L-1, L \\
\sin(L-1)\alpha / (2\cos\alpha), & a = L-2 \\
2\cos(L-2)\alpha \sin\alpha, & a = L-1 \\
\sin(L-3)\alpha / (2\cos\alpha), & a = L 
\end{cases} \quad (7.11) \]

From (7.1) and (7.11) we find that the bulk weights for the Temperley-Lieb models with adjacency graphs \( A_L \) and \( D_L \) match those for the critical \( A_L \) and \( D_L \) models, obtained by taking \( q \to 0 \) in (7.2) and (6.2).

### 7.2 Diagonal Boundary Weights

Diagonal boundary weights for the Temperley-Lieb models are

\[
B(b \ a \mid u) = \begin{cases} 
\left[ x_1(a) s(u) \left( s(u+\lambda) - \sum_{d \in \nu(a)} S_d / S_a s(u) \right) + x_2(a) \right] f(a, u) \delta_{ac}, & b \in \nu(a) \\
\left[ -x_1(a) s(u) \left( s(u+\lambda) - \sum_{d \in \nu'(a)} S_d / S_a s(u) \right) + x_2(a) \right] f(a, u) \delta_{ac}, & b \in \nu'(a) 
\end{cases} \quad (7.12)
\]

where, for each \( a, \nu(a) \) and \( \nu'(a) \) are any non-intersecting sets whose union is the set of neighbours of \( a \), and \( x_1, x_2 \) and \( f \) are arbitrary.

We now prove that these boundary weights represent the general diagonal solution of the boundary Yang-Baxter equation for the Temperley-Lieb models. Having set \( B(b \ a \mid u) = B_a(b, u) \delta_{ac} \), we find, using the Temperley-Lieb bulk weights, that the only spin assignments in the boundary Yang-Baxter equation (2.4) which lead to non-trivial equations are those in which \( a = c = e \), and \( b \) and \( d \) are distinct neighbours of \( a \). These equations are

\[
0 = \mathcal{E}_a(b, d) = s(u+v) s(u-v-\lambda) \left( B_a(b, u) B_a(d, v) - B_a(d, u) B_a(b, v) \right) \\
- s(u-v) s(u+v-\lambda) \left( B_a(b, u) B_a(b, v) - B_a(d, u) B_a(d, v) \right) \\
+ s(u-v) s(u+v) \sum_{c \in \mathcal{N}(a)} S_c B_a(c, u) / S_a \left( B_a(b, v) - B_a(d, v) \right)
\]

where \( \mathcal{N}(a) \) is the set of neighbours of \( a \).

We shall from now on treat \( a \) as fixed. If \( a \) has \( n \) neighbours, then there are, since \( \mathcal{E}_a(b, d) = -\mathcal{E}_a(d, b) \), \( n(n-1)/2 \) distinct equations for the \( n \) boundary weights \( B_a(b, u) \).
Throughout this proof, we shall also use the eigenvector equation
\[ \sum_{b \in \mathcal{N}(a)} S_b/S_a = 2c(\lambda). \]

We now observe that
\[ 0 = \mathcal{E}_a(b, c) + \mathcal{E}_a(c, d) + \mathcal{E}_a(d, b) = \]
\[ s(u+v) s(u-v-\lambda) \det \begin{pmatrix} 1 & 1 & 1 \\ \mathcal{B}_a(b, u) & \mathcal{B}_a(c, u) & \mathcal{B}_a(d, u) \\ \mathcal{B}_a(b, v) & \mathcal{B}_a(c, v) & \mathcal{B}_a(d, v) \end{pmatrix}. \]

The general solution of this system of equations is
\[ \mathcal{B}_a(b, u) = y(b) g(u) + h(u) \]
where \( y(b) \) are arbitrary constants and \( g \) and \( h \) are arbitrary functions. Using this solution, we obtain
\[ 0 = \mathcal{E}_a(b, d) = (y(d) - y(b)) \left( s(\lambda) s(2v) g(u) h(v) - s(\lambda) s(2u) h(u) g(v) \right) \]
\[ + s(u-v) \left( (y(b) + y(d)) s(u+v-\lambda) - \sum_{c \in \mathcal{N}(a)} y(c) S_c/S_a s(u+v) \right) g(u) g(v) \].

These equations are satisfied if \( y(b) \) are equal for all \( b \in \mathcal{N}(a) \). In order to obtain the remaining solutions, we assume \( y(\tilde{b}) \neq y(\tilde{d}) \) for particular \( \tilde{b} \) and \( \tilde{d} \). We now transform
\[ g(u) = s(\lambda) s(2u) \tilde{g}(u) \]
\[ h(u) = \left( (y(\tilde{b}) + y(\tilde{d})) s(u) s(u-\lambda) - \sum_{c \in \mathcal{N}(a)} y(c) S_c/S_a s(u)^2 \right) \tilde{g}(u) + \tilde{h}(u) \]
with \( \tilde{g} \) and \( \tilde{h} \) arbitrary, which gives
\[ 0 = \mathcal{E}_a(b, d) = (y(d) - y(b)) s(\lambda)^2 s(2u) s(2v) (\tilde{g}(u) \tilde{h}(v) - \tilde{h}(u) \tilde{g}(v)) \]
\[ + (y(b) - y(\tilde{b}) + y(d) - y(\tilde{d})) s(u-v) s(u+v-\lambda) \tilde{g}(u) \tilde{g}(v) \].

The general solution of \( \mathcal{E}_a(\tilde{b}, \tilde{d}) = 0 \) is
\[ \tilde{g}(u) = \tilde{x}_1 f(u), \quad \tilde{h}(u) = \tilde{x}_2 f(u) \]
where \( \tilde{x}_1, \tilde{x}_2 \) and \( f \) are arbitrary. The remaining cases of \( \mathcal{E}_a(b, d) = 0 \) now imply that
\[ y(b) = \begin{cases} \tilde{g}, & b \in \nu(a) \\ \tilde{g}', & b \in \nu'(a) \end{cases} \]
where \( \tilde{g} \) and \( \tilde{g}' \) are arbitrary, and \( \nu(a) \) and \( \nu'(a) \) are non-intersecting sets whose union is \( \mathcal{N}(a) \) and which contain \( \tilde{b} \) and \( \tilde{d} \) respectively. The previous case in which all \( y(b) \) are
equal is included if we also allow $\nu(a) = N(a)$, $\nu'(a) = \emptyset$. This now leads to the general solution (7.12), where $x_1(a) = (\tilde{y} - \tilde{y}') \tilde{x}_1$, $x_2(a) = \tilde{x}_2$ and $f(a, u) = f(u)$, and concludes our proof.

The Temperley-Lieb diagonal boundary weights satisfy the boundary initial condition and the boundary inversion relation, with $\chi$ and $\hat{\rho}$ given by (2.21) and (2.22), and boundary crossing symmetry with

$$\eta_a(u) = \frac{s(2\lambda - 2u)}{s(\lambda)} \frac{f(a, \lambda - u)}{f(a, u)}. \quad (7.13)$$

If, in (7.12), we set $x_1(a) = 0$, $x_2(a) = 1$ and $f(a, u) = 1$ for each $a$, then we obtain boundary weights whose boundary face transfer matrix is the identity matrix. That the identity satisfies (2.18) is equivalent to the commutation of the bulk face transfer matrices (7.8).

The diagonal boundary weights for the critical $A_L$ and $D_L$ models, obtained by taking $q \to 0$ in (5.6) and (6.6), match those for the Temperley-Lieb models with adjacency graphs $A_L$ and $D_L$ for appropriate choices of the arbitrary parameters. In particular, for $i = 2$ in (5.6) we should set $x_1(L-3) = \vartheta_3(0) (\vartheta_4(0)^2 \tilde{x}_1 + \vartheta_3(0)^2 \tilde{x}_2)/q^{v_2}$ and $x_2(L-3) = \pm \vartheta_4(0) (\vartheta_3(0)^2 \tilde{x}_1 + \vartheta_4(0)^2 \tilde{x}_2)/q^{v_2}$, with $\tilde{x}_1$, $\tilde{x}_2$ and the $+$ or $-$ arbitrary, which gives

$$k_2(L-3, u) \to - (\tilde{x}_1 + \tilde{x}_2)^2 \sin(u - \lambda)^2 + 2 (\tilde{x}_1 + \tilde{x}_2) \tilde{x}_2$$

$$k_2(L-1, u) = k_2(L, -u) \to - (\tilde{x}_1 + \tilde{x}_2)^2 \sin(u \mp \lambda)^2 + 2 (\tilde{x}_1 + \tilde{x}_2) \tilde{x}_2.$$

The Temperley-Lieb models with adjacency graphs $E_6$, $E_7$ and $E_8$ can be related to those with adjacency graphs $A_{11}$, $A_{17}$ and $A_{29}$ respectively by intertwiner cells [11, 14]. These cells satisfy the intertwiner inversion relations and the bulk weight intertwining relation. However, due to the absence of certain symmetries in the $E$ graphs, we find that the only diagonal boundary weights which satisfy the boundary weight intertwining relation are those which effectively correspond to the identity solution.

8. Dilute $A_L$ Models

8.1 Bulk Weights

We now consider the dilute $A_L$ models [22]. There is one such model for each integer $L \geq 2$. The spins in this model take values from the set \{1, 2, \ldots, L\} and the adjacency graph is

\[
A'_L = \begin{array}{c}
\circ \quad \circ \\
1 & 2 \\
\cdots & \cdots \\
L-1 & L
\end{array}
\]
The dilute $A_L$ bulk weights are

$$W\left( \begin{array}{c} a \ a \\ a \ a \end{array} \right) = \frac{\vartheta_1(6\tilde{\lambda} - u) \vartheta_1(3\tilde{\lambda} + u)}{\vartheta_1(6\lambda) \vartheta_1(3\lambda)} - \frac{S_{a+1}}{S_a} \frac{\vartheta_4((2a-5)\tilde{\lambda})}{\vartheta_4((2a+1)\tilde{\lambda})} \frac{\vartheta_4((2a+5)\tilde{\lambda})}{\vartheta_4((2a-1)\tilde{\lambda})} \frac{\vartheta_1(u) \vartheta_1(3\tilde{\lambda} - u)}{\vartheta_1(6\lambda) \vartheta_1(3\lambda)}, \quad a = 1, \ldots, L$$

$$W\left( \begin{array}{c} a \pm 1 \ a \\ a \ a \end{array} \right) = W\left( \begin{array}{c} a \ a \\ a \ a \pm 1 \end{array} \right) = \frac{\vartheta_1(3\tilde{\lambda} - u) \vartheta_4((\pm 2a+1)\tilde{\lambda} - u)}{\vartheta_1(3\lambda) \vartheta_4((\pm 2a+1)\tilde{\lambda})},$$

$$W\left( \begin{array}{c} a \pm 1 \ a \\ a \ a \end{array} \right) = W\left( \begin{array}{c} a \ a \pm 1 \\ a \ a \end{array} \right) = \left( \frac{S_{a+1}}{S_a} \right)^{\frac{1}{2}} \frac{\vartheta_4((\pm 2a+3)\tilde{\lambda}) \vartheta_4((\pm 2a-1)\tilde{\lambda})}{\vartheta_1((\pm 2a+1)\tilde{\lambda})^2} \frac{\vartheta_1(u) \vartheta_1(3\tilde{\lambda} - u)}{\vartheta_1(2\lambda) \vartheta_1(3\lambda)},$$

$$W\left( \begin{array}{c} a \pm 1 \ a \\ a \ a \pm 1 \end{array} \right) = W\left( \begin{array}{c} a \ a \pm 1 \\ a \ a \pm 1 \end{array} \right) = \left( \frac{S_{a+1}}{S_a} \right)^{\frac{1}{2}} \frac{\vartheta_1(3\tilde{\lambda} - u) \vartheta_1((\pm 4a+2)\tilde{\lambda} + u)}{\vartheta_1(3\lambda) \vartheta_1((\pm 4a+2)\tilde{\lambda})} + \frac{S_{a+1}}{S_a} \frac{\vartheta_1((\pm 4a-1)\tilde{\lambda} + u)}{\vartheta_1(3\lambda) \vartheta_1((\pm 4a+2)\lambda)}$$

$$\times \begin{cases} +, & a = 1, \ldots, L-1 \\ -, & a = 2, \ldots, L \end{cases},$$

$$W\left( \begin{array}{c} a \pm 1 \ a \\ a \ a \pm 1 \end{array} \right) = \frac{\vartheta_1(2\tilde{\lambda} - u) \vartheta_1(3\tilde{\lambda} - u)}{\vartheta_1(2\lambda) \vartheta_1(3\lambda)},$$

$$W\left( \begin{array}{c} a \ a \pm 1 \\ a \ a \pm 1 \end{array} \right) = - \left( \frac{S_{a-1} S_{a+1}}{S_a^2} \right)^{\frac{1}{2}} \frac{\vartheta_1(u) \vartheta_1(\tilde{\lambda} - u)}{\vartheta_1(2\lambda) \vartheta_1(3\lambda)}, \quad a = 1, \ldots, L-2$$

where

$$\tilde{\lambda} = \frac{L}{L + 1} \frac{\pi}{4} \text{ or } \frac{L + 2}{L + 1} \frac{\pi}{4} \quad (8.3)$$

and

$$S_a = (-1)^a \frac{\vartheta_1(4a\tilde{\lambda})}{\vartheta_4(2a\tilde{\lambda})}.$$  

(8.4)

These weights satisfy the Yang-Baxter equation, the initial condition, reflection symmetry, crossing symmetry with crossing parameter $\lambda = 3\tilde{\lambda}$ and crossing factors $S_a$, and the inversion relation with

$$\rho(u) = \frac{\vartheta_1(2\tilde{\lambda} - u) \vartheta_1(3\tilde{\lambda} - u) \vartheta_1(2\tilde{\lambda} + u) \vartheta_1(3\tilde{\lambda} + u)}{\vartheta_1(2\lambda)^2 \vartheta_1(3\lambda)^2}. \quad (8.5)$$

Invariance under the symmetry transformation (7.4) is satisfied for $L$ even, but not for $L$ odd (assuming $q \neq 0$).
8.2 Diagonal Boundary Weights

We have found that the general diagonal solution of the boundary Yang-Baxter equation for the dilute $A_L$ models is

$$B\left(a \ a \ u \bigg| \ u \right) = \vartheta_{i(a)}(\frac{5\lambda}{2}-u) \vartheta_{i(a)}(\frac{3\lambda}{2}+u) \vartheta_{j(a)}((2a-\frac{1}{2})\lambda+u) \vartheta_{j(a)}((2a+\frac{1}{2})\lambda-u) \ f(a, u),$$

$$a = 1, \ldots, L$$

$$B\left(a \pm 1 \ a \ u \bigg| \ u \right) = \vartheta_{i(a)}(\frac{5\lambda}{2}-u) \vartheta_{i(a)}(\frac{3\lambda}{2}+u) \vartheta_{j(a)}((2a-\frac{1}{2})\lambda+u) \vartheta_{j(a)}((2a+\frac{1}{2})\lambda-u) \ f(a, u).$$

$$B\left(a \ a \pm 1 \ u \bigg| \ u \right) = 0,$$  \hspace{1cm} \{+, a = 1, \ldots, L-1 \}

$$- , a = 2, \ldots, L$$

$$B\left(a \ a \pm 1 \ u \bigg| \ u \right) = 0, \hspace{1cm} a = 2, \ldots, L-1$$

where, for each $a$, $(i(a), j(a))$ may be chosen arbitrarily as $(1, 4)$, $(2, 3)$, $(3, 2)$ or $(4, 1)$ and $f$ is arbitrary. These weights match those obtained in [8], and their derivation is similar to that of the dilute Temperley-Lieb diagonal boundary weights which is given in the next section. They satisfy the boundary initial condition and the boundary inversion relation, with $\chi$ and $\hat{\rho}$ given by (2.21) and (2.22), and boundary crossing symmetry with

$$\eta_{a}(u) = \frac{\vartheta_{1}(6\lambda-2u)}{\vartheta_{1}(2\lambda)} \frac{\vartheta_{1}(2u-\lambda)}{\vartheta_{1}(3\lambda)} \ f(a, 3\lambda-u) \ f(a, u).$$

(8.7)

For $L$ even, the weights can be made invariant under the transformation (5.4) for appropriate choices of the arbitrary parameters. However, this is not possible for $L$ odd (assuming $q \neq 0$, $f(a, u) \neq 0$).

9. Dilute Temperley-Lieb Models

9.1 Bulk Weights

We now consider the dilute Temperley-Lieb models [23, 22, 24]. The adjacency graph $G$ for these models can be any finite, connected graph which has only bidirectional, single bonds and in which each node is connected to itself.

The dilute Temperley-Lieb bulk weights are

$$W\left(\begin{array}{c} d \\ a \\ c \\ b \end{array} \bigg| \ u \right) = \rho_{1}(u) \delta_{abcd} + \rho_{2}(u) \delta_{abc} \tilde{A}_{ad} + \rho_{3}(u) \delta_{acd} \tilde{A}_{ab} +$$

$$\left(\frac{S_{a}}{S_{b}}\right)^{\gamma/2} \rho_{4}(u) \delta_{bcd} \tilde{A}_{ab} + \left(\frac{S_{c}}{S_{a}}\right)^{\gamma/2} \rho_{5}(u) \delta_{abd} \tilde{A}_{ac} + \rho_{6}(u) \delta_{ab} \delta_{cd} \tilde{A}_{ac} + \rho_{7}(u) \delta_{ad} \delta_{bc} \tilde{A}_{ab} +$$

$$\rho_{8}(u) \delta_{ac} \tilde{A}_{ab} \tilde{A}_{ad} + \left(\frac{S_{a} S_{c}}{S_{b} S_{d}}\right)^{\gamma/2} \rho_{9}(u) \delta_{bd} \tilde{A}_{ab} \tilde{A}_{bc}$$

(9.1)
where \( \delta_{a_1 \ldots a_m} = \prod_{j=1}^{m-1} \delta_{a_j a_{j+1}} \) and

\[
\begin{align*}
\rho_1(u) &= 1 + \frac{\sin u \sin(3\tilde{\lambda} - u)}{\sin 2\tilde{\lambda} \sin 3\tilde{\lambda}} \\
\rho_2(u) &= \rho_3(u) = \frac{\sin(3\tilde{\lambda} - u)}{\sin 3\tilde{\lambda}} \\
\rho_4(u) &= \rho_5(u) = \frac{\sin u}{\sin 3\tilde{\lambda}} \\
\rho_6(u) &= \rho_7(u) = \frac{\sin u \sin(3\tilde{\lambda} - u)}{\sin 2\tilde{\lambda} \sin 3\tilde{\lambda}} \\
\rho_8(u) &= \frac{\sin(2\tilde{\lambda} - u) \sin(3\tilde{\lambda} - u)}{\sin 2\tilde{\lambda} \sin 3\tilde{\lambda}} \\
\rho_9(u) &= -\frac{\sin u \sin(\tilde{\lambda} - u)}{\sin 2\tilde{\lambda} \sin 3\tilde{\lambda}}.
\end{align*}
\] (9.2)

Furthermore, \( \tilde{\mathbb{A}} = \mathbb{A} - I \) is the adjacency matrix of the graph \( \tilde{\mathbb{G}} \) obtained from \( \mathbb{G} \) by removing the bonds connecting each node to itself, \( \Lambda \) is its maximum eigenvalue, \( (S_1, S_2, \ldots) \) is the associated eigenvector with all positive entries, and \( \tilde{\lambda} \) is any solution of

\[
\Lambda = -2 \cos 4\tilde{\lambda}.
\] (9.3)

These weights satisfy the Yang-Baxter equation, the initial condition, reflection symmetry, crossing symmetry with crossing parameter \( \lambda = 3\tilde{\lambda} \) and crossing factors \( S_a \), and the inversion relation with

\[
\rho(u) = \frac{\sin(2\tilde{\lambda} - u) \sin(3\tilde{\lambda} - u) \sin(2\tilde{\lambda} + u) \sin(3\tilde{\lambda} + u)}{\sin^2 2\tilde{\lambda} \sin^2 3\tilde{\lambda}}.
\] (9.4)

From (9.1) and (7.11), we see that the bulk weights for the dilute Temperley-Lieb models with adjacency graphs \( \mathbb{A}_L' \) match those for the critical dilute \( \mathbb{A}_L \) models, obtained by taking \( q \to 0 \) in (8.2).

The dilute Temperley-Lieb bulk face transfer matrices can be expressed in terms of matrices \( e^1_{j}, \ldots, e^9_{j} \) defined by (9.1) and

\[
X_j(u) = \sum_{n=1}^{9} \rho_n(u) e^n_j.
\] (9.5)

We then find that the matrices \( e^n_j \) form a representation of the dilute Temperley-Lieb algebra [25] and that the Yang-Baxter equation (2.17) is satisfied through the relations of this algebra alone. We also note that, in contrast with the Temperley-Lieb bulk face transfer matrices (7.5), the dilute Temperley-Lieb bulk face transfer matrices (9.5) do not commute.

### 9.2 Diagonal Boundary Weights

Diagonal boundary weights for the dilute Temperley-Lieb models are

\[
B(b \ a \ c \| u) = \begin{cases} 
\sin(\xi(a) - \frac{\tilde{\lambda}}{2} + u) \sin(\xi(a) + \frac{\tilde{\lambda}}{2} - u) f(a, u) \delta_{ac}, & b = a \\
\sin(\xi(a) - \frac{\tilde{\lambda}}{2} + u) \sin(\xi(a) + \frac{\tilde{\lambda}}{2} + u) f(a, u) \delta_{ac}, & b \in \nu(a) \\
\sin(\xi(a) - \frac{\tilde{\lambda}}{2} - u) \sin(\xi(a) + \frac{\tilde{\lambda}}{2} - u) f(a, u) \delta_{ac}, & b \in \nu'(a)
\end{cases}
\] (9.6)
where \( f \) is arbitrary and, for each \( a, \nu(a) \) and \( \nu'(a) \) are any non-intersecting sets whose union is the set of neighbours of \( a \) on \( \tilde{G} \) and \( \xi(a) \) is any solution of

\[
\tan 2\xi(a) = \frac{\sin 4\bar{\lambda}}{\cos 4\bar{\lambda} + \sum_{d \in \nu(a)} S_d/S_a}
\]  

(9.7)

We now prove that these boundary weights represent the general diagonal solution of the boundary Yang-Baxter equation for the dilute Temperley-Lieb models. Having set \( B(b^a|u) = B_a(b, u) \delta_{a\nu} \), we find, using the dilute Temperley Lieb bulk weights, that the only classes of spin assignments in (2.4) which lead to non-trivial equations are \( a = d = e \) and \( b = c \) with \( b \in \mathcal{N}(a) \), \( a = c = d = e \) with \( b \in \mathcal{N}(a) \), and \( a = c = e \) with \( b \in \mathcal{N}(a) \) and \( d \in \mathcal{N}(a) \), where \( \mathcal{N}(a) \) is the set of neighbours of \( a \) on \( \tilde{G} \). These give, respectively,

\[
0 = \mathcal{E}_a^1(b) = \sin(u-v) (B_a(a, u) B_a(a, v) - B_a(b, u) B_a(b, v)) + \sin(u+v) (B_a(b, u) B_a(a, v) - B_a(a, u) B_a(b, v)) ,
\]  

(9.8a)

\[
0 = \mathcal{E}_a^2(b) = \rho_4(u-v) \rho_1(u+v) B_a(a, u) B_a(a, v) - \rho_4(u-v) \rho_4(u+v) B_a(a, u) B_a(b, v)
+ \rho_8(u-v) \rho_4(u+v) B_a(b, u) B_a(a, v) - \rho_4(u-v) \rho_8(u+v) B_a(b, u) B_a(b, v)
+ \sum_{c \in \mathcal{N}(a)} S_c B_a(c, u) S_a (\rho_9(u-v) \rho_4(u+v) B_a(a, v) - \rho_4(u-v) \rho_9(u+v) B_a(b, v)) ,
\]  

(9.8b)

and

\[
0 = \mathcal{E}_a^3(b, d) = \rho_9(u-v) \rho_8(u+v) (B_a(b, u) B_a(b, v) - B_a(d, u) B_a(d, v)) + \rho_8(u-v) \rho_9(u+v) (B_a(d, u) B_a(b, v) - B_a(b, u) B_a(d, v)) + \rho_9(u-v) \rho_9(u+v) \sum_{c \in \mathcal{N}(a)} S_c B_a(c, u) S_a (B_a(b, v) - B_a(d, v)) .
\]  

(9.8c)

We shall from now on treat \( a \) as fixed. If \( a \) has \( n \) neighbours on \( \tilde{G} \) then (9.8a) and (9.8c) each provide \( n \) equations and (9.8d) provides \( n(n-1)/2 \) equations for the \( n + 1 \) boundary weights, \( B_a(a, u) \) and \( B_a(b, u) \) with \( b \in \mathcal{N}(a) \).

Using a method similar to that for solving (4.8), we find that the general solution of a single case of (9.8a) can be written as

\[
B_a(a, u) = (x_1 \cos(u-\chi) + x_2 \sin(u-\chi)) f(u)
\]

\[
B_a(b, u) = (x_1 \cos(u+\chi) - x_2 \sin(u+\chi)) f(u)
\]

where \( x_1 \), \( x_2 \) and \( f \) are arbitrary and \( \chi \) may be set to any fixed value, which here we shall take as \( \chi = \bar{\lambda}/2 \). Therefore, the general solution of the system of equations (9.8d) is
\[ B_a(a, u) = \prod_{c \in \mathcal{N}(a)} (x_1(c) \cos(u-\frac{\lambda}{2}) + x_2(c) \sin(u-\frac{\lambda}{2})) f(u) \]  
\[ B_a(b, u) = (x_1(b) \cos(u+\frac{\lambda}{2}) - x_2(b) \sin(u+\frac{\lambda}{2})) \prod_{c \in \mathcal{N}(a)-\{b\}} (x_1(c) \cos(u-\frac{\lambda}{2}) + x_2(c) \sin(u-\frac{\lambda}{2})) f(u) \]

where \( x_1, x_2 \) and \( f \) are arbitrary.

We also observe that

\[ 0 = \sin(u-v-\tilde{\lambda}) (E_a^2(b) - E_a^2(d)) + \sin 2\tilde{\lambda} E_a^3(b, d) = \sin(u+v) \sin(u-v-2\tilde{\lambda}) \sin(u-v-3\tilde{\lambda})/ (\sin 2\tilde{\lambda} \sin 3\tilde{\lambda}) E_a^4(b, d) \]

where

\[ E_a^4(b, d) = \sin(u-v+\tilde{\lambda}) B_a(a, u) (B_a(b, v) - B_a(d, v)) + \sin(u-v-\tilde{\lambda}) (B_a(b, u) - B_a(d, u)) B_a(a, v) - \sin(u+v-\tilde{\lambda}) (B_a(b, u) B_a(d, v) - B_a(d, u) B_a(b, v)). \]

Using (9.9), we now find that

\[ 0 = E_a^4(b, d) = \prod_{c \in \mathcal{N}(a)-\{b,d\}} (x_1(c) \cos(u-\frac{\lambda}{2}) + x_2(c) \sin(u-\frac{\lambda}{2})) (x_1(c) \cos(v-\frac{\lambda}{2}) + x_2(c) \sin(v-\frac{\lambda}{2})) \times (x_1(b) x_2(d) - x_1(d) x_2(b)) (x_1(b) x_2(d) + x_1(d) x_2(b)) \sin 2u \sin 2v \sin(u-v) f(u) f(v). \]

The general solution of this system of equations is

\[ x_1(b) = \bar{x}_1 y(b), \quad x_2(b) = \begin{cases} \bar{x}_2 y(b), & b \in \nu(a) \\ -\bar{x}_2 y(b), & b \in \nu'(a) \end{cases} \]

where \( \bar{x}_1, \bar{x}_2 \) and \( y \) are arbitrary and \( \nu(a) \) and \( \nu'(a) \) are any non-intersecting sets whose union is \( \mathcal{N}(a) \). This gives

\[ B_a(a, u) = (\bar{x}_1 \cos(u-\frac{\lambda}{2}) + \bar{x}_2 \sin(u-\frac{\lambda}{2})) (\bar{x}_1 \cos(u+\frac{\lambda}{2}) - \bar{x}_2 \sin(u+\frac{\lambda}{2})) \tilde{f}(u) \]
\[ B_a(b, u) = \begin{cases} (\bar{x}_1 \cos(u+\frac{\lambda}{2}) - \bar{x}_2 \sin(u+\frac{\lambda}{2})) (\bar{x}_1 \cos(u-\frac{\lambda}{2}) - \bar{x}_2 \sin(u-\frac{\lambda}{2})) \tilde{f}(u), & b \in \nu(a) \\ (\bar{x}_1 \cos(u+\frac{\lambda}{2}) + \bar{x}_2 \sin(u+\frac{\lambda}{2})) (\bar{x}_1 \cos(u-\frac{\lambda}{2}) + \bar{x}_2 \sin(u-\frac{\lambda}{2})) \tilde{f}(u), & b \in \nu'(a) \end{cases} \]

where

\[ \tilde{f}(u) = (\bar{x}_1 \cos(u-\frac{\lambda}{2}) + \bar{x}_2 \sin(u-\frac{\lambda}{2}))^{(|\nu(a)|-1)} (\bar{x}_1 \cos(u-\frac{\lambda}{2}) - \bar{x}_2 \sin(u-\frac{\lambda}{2}))^{(|\nu'(a)|-1)} \prod_{c \in \mathcal{N}(a)} y(c) f(u). \]
We now see that (1.8d) is automatically satisfied for \( b, d \in \nu(a) \), or \( b, d \in \nu'(a) \), while for \( b \in \nu(a) \) and \( d \in \nu'(a) \) we have

\[
0 = \mathcal{E}_a^2(b, d) = 2 \sin 2u \sin 2v \sin(u-v) \sin(u+v) \sin(u-v-\tilde{\lambda}) \sin(u+v-\tilde{\lambda})/(\sin 2\tilde{\lambda} \sin 3\tilde{\lambda})^2 \times \tilde{f}(u) \tilde{f}(v) \mathcal{P}
\]

\[0 = \mathcal{E}_a^2(b) = (\tilde{x}_1 \cos(u-\frac{\tilde{\lambda}}{2}) - \tilde{x}_2 \sin(u-\frac{\tilde{\lambda}}{2})) (\tilde{x}_1 \cos(v-\frac{\tilde{\lambda}}{2}) - \tilde{x}_2 \sin(v-\frac{\tilde{\lambda}}{2})) \times \sin 2u \sin 2v \sin(u-v) \sin(u+v)/(\sin 2\tilde{\lambda} \sin 2\tilde{3}\tilde{\lambda}) \tilde{f}(u) \tilde{f}(v) \mathcal{P}
\]

\[0 = \mathcal{E}_a^2(d) = (\tilde{x}_1 \cos(u-\frac{\tilde{\lambda}}{2}) + \tilde{x}_2 \sin(u-\frac{\tilde{\lambda}}{2})) (\tilde{x}_1 \cos(v-\frac{\tilde{\lambda}}{2}) + \tilde{x}_2 \sin(v-\frac{\tilde{\lambda}}{2})) \times \sin 2u \sin 2v \sin(u-v) \sin(u+v)/(\sin 2\tilde{\lambda} \sin 2\tilde{3}\tilde{\lambda}) \tilde{f}(u) \tilde{f}(v) \mathcal{P}
\]

where

\[\mathcal{P} = \sin 4\tilde{\lambda} (\tilde{x}_1^2 - \tilde{x}_2^2) - 2 \left( \cos 4\tilde{\lambda} + \sum_{d \in \nu(a)} S_d/S_a \right) \tilde{x}_1 \tilde{x}_2 ,\]

and we have used the eigenvector equation

\[
\sum_{b \in \mathcal{N}(a)} S_b/S_a = -2 \cos 4\tilde{\lambda} .
\]

We must therefore set \( \mathcal{P} = 0 \), the general solution of which is

\[
\tilde{x}_1 = z \sin \xi , \quad \tilde{x}_2 = -z \cos \xi
\]

where \( z \) is arbitrary and \( \xi \) is any solution of

\[
\tan 2\xi = \frac{\sin 4\tilde{\lambda}}{\cos 4\tilde{\lambda} + \sum_{d \in \nu(a)} S_d/S_a} = -\frac{\sin 4\tilde{\lambda}}{\cos 4\tilde{\lambda} + \sum_{d \in \nu'(a)} S_d/S_a} .
\]

This now leads to the general solution (9.4), where \( \xi(a) = \xi \) and \( f(a, u) = z^2 \tilde{f}(u) \), and concludes our proof.

The dilute Temperley-Lieb diagonal boundary weights satisfy the boundary initial condition and the boundary inversion relation, with \( \chi \) and \( \hat{\rho} \) given by (2.21) and (2.23), and boundary crossing symmetry with

\[
\eta_a(u) = \frac{\sin(6\tilde{\lambda}-2u) \sin(2u-\tilde{\lambda}) f(a, 3\tilde{\lambda}-u)}{\sin 2\tilde{\lambda} \sin 3\tilde{\lambda}} \frac{f(a, u)}{f(a, u)} . \quad (9.10)
\]

We also see that the diagonal boundary weights for the critical dilute \( A_L \) models, obtained by replacing \( f(a, u) \) by \( f(a, u)/q^{v/2} \) and taking \( q \to 0 \) in (8.6), match those for the dilute Temperley-Lieb models with adjacency graphs \( A'_L \).

If, in (9.6), we set \( \nu(a) \) equal to the set of neighbours of \( a \), \( \nu'(a) = \emptyset \), \( f(a, u) = 1 \) and \( \xi(a) = -2\tilde{\lambda} \) or \( \xi(a) = -2\tilde{\lambda} + \pi/2 \), for all \( a \), then we obtain

\[
B \left( b \begin{array}{c} a \end{array} \begin{array}{c} c \end{array} \right) = \hat{\rho}_1(u) \delta_{abc} + \hat{\rho}_2(u) \delta_{ac} \tilde{A}_{ab} . \quad (9.11)
\]
where

\[ \hat{\rho}_1(u) = \sin\left(\frac{5\lambda}{2} - u\right) \sin\left(\frac{3\lambda}{2} + u\right), \quad \hat{\rho}_2(u) = \sin\left(\frac{5\lambda}{2} - u\right) \sin\left(\frac{3\lambda}{2} - u\right) \]

or

\[ \hat{\rho}_1(u) = \cos\left(\frac{5\lambda}{2} - u\right) \cos\left(\frac{3\lambda}{2} + u\right), \quad \hat{\rho}_2(u) = \cos\left(\frac{5\lambda}{2} - u\right) \cos\left(\frac{3\lambda}{2} - u\right). \]  

(9.12)

We have found that the boundary face transfer matrices obtained from (9.11) satisfy the boundary Yang-Baxter equation (2.18) through the relations of the dilute Temperley-Lieb algebra alone.

10. Discussion

We have obtained general solutions, mostly of the diagonal type, of the boundary Yang-Baxter equation for a number of related interaction-round-a-face models. The boundary weights all involve arbitrary parameters, some of which may take any complex value, while others of which may take only finitely many values.

The solutions were derived by direct consideration of the relevant equations for each model. In some cases our solutions include boundary weights which can also be obtained by indirect means. These alternative methods include the consideration of algebraic relations associated with the model, the construction of new weights from known, simpler weights using fusion [7], and the generation of weights for one model from known weights for another model using vertex-face intertwiners [3] or face-face intertwiners. Such means are useful for establishing the existence of solutions and for efficiently deriving particular solutions which are adequate for many purposes. However it seems that a direct approach is still needed in order to obtain general solutions, and thereby to identify the exact number and nature of associated arbitrary parameters.

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