1. Introduction

We classify in the present paper Poisson brackets on modules over a semisimple complex Lie algebra which are based on classical $r$-matrices. We then quantize these Poisson structures in the spirit of the recent joint paper [6] with A. Berenstein and show that we recover many well known examples of quantized coordinate rings of classical varieties.

Let us briefly discuss the main results in the case of a simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a simple complex Lie algebra and let $\{E_\alpha | \alpha \in R_+\}$ (where $R_+$ is the set of positive roots of $\mathfrak{g}$) be the standard basis of $\mathfrak{n}_+$ and $\{F_\alpha | \alpha \in R_+\}$ the standard basis of $\mathfrak{n}_-$. Recall that $r = \sum_{\alpha \in R_+} E_\alpha \otimes F_\alpha \in \mathfrak{g} \otimes \mathfrak{g}$ is a classical $r$-matrix and $r^\pm = \sum_{\alpha \in R_+} E_\alpha \otimes F_\alpha - F_\alpha \otimes E_\alpha \in \mathfrak{g} \wedge \mathfrak{g}$ the antisymmetrized $r$-matrix. For
each $\mathfrak{g}$-module $V$ define a quadratic bracket $\{\cdot,\cdot\}$ on the symmetric algebra $S(V)$ by the formula:

$$\{a, b\} = r^-(a \wedge b) = \sum_{\alpha \in R_+} E_\alpha(a) F_\alpha(b) - E_\alpha(b) F_\alpha(a)$$

for $a, b \in S(V)$. In particular, if $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, then $\{a, b\} = E(a)F(b) - E(b)F(a)$.

The bracket is, by construction, skew-commutative and satisfies the Leibniz rule. To determine whether it is Poisson for a pair $(\mathfrak{g}, V)$ one has to verify whether it satisfies the Jacobi identity. Our first main result is the following theorem.

**Main Theorem 1.1.** (Theorem 3.12) Let $V$ be a simple finite-dimensional $\mathfrak{g}$-module. Assume that $(\mathfrak{g}, V) \neq (\mathfrak{sp}_{2n}(\mathbb{C}), V_{\omega_1})$. Then the following are equivalent:

(a) The bracket (1.1) on $S(V)$ is Poisson.

(b) $c(\Lambda^3V) = \{0\}$, where $c$ the canonical $\mathfrak{g}$-invariant in $\Lambda^3\mathfrak{g}$ corresponding (under the identification $\mathfrak{g}^* \cong \mathfrak{g}$) to the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$.

(c) $V$ is geometrically decomposable as defined by R. Howe (18).

(d) $\text{Hom}_\mathfrak{g}(\Lambda^3V, S^3V) = \{0\}$.

(e) $\Lambda^2V$ is simple.

If $(\mathfrak{g}, V) = (\mathfrak{sp}_{2n}(\mathbb{C}), V_{\omega_1})$, then parts (a), (b), and (d) of Theorem 1.1 hold, but parts (c) and (e) fail. In Theorem 3.13 we classify all simple modules $V$ over a semisimple Lie algebra for which the bracket (1.1) on $S(V)$ is Poisson. The only, nontrivial, example of a simple module over a semisimple Lie algebra with this property is the natural module of $\mathfrak{g} = \mathfrak{sl}_n \times \mathfrak{sl}_m(\mathbb{C})$ for arbitrary $m, n \in \mathbb{Z}_{\geq 0}$.

All pairs $(\mathfrak{g}, V)$ for which the bracket (1.1) on $S(V)$ is Poisson are classified in (11).

We then continue to show that the deformation quantization of the $r$-matrix Poisson structure on a $\mathfrak{g}$-module $V_{\lambda}$ recovers the braided symmetric algebras of the $U_q(\mathfrak{g})$-module $V_{\lambda}^q$. The braided symmetric algebra $S_q(V_{\lambda}^q)$ is quadratic $q$-deformations of the symmetric algebra of the $U(\mathfrak{g})$-module $V_{\lambda}$ (see 6 or Section 4.2) which are $U_q(\mathfrak{g})$-module algebras. An important problem in [6] is the question, for which $U_q(\mathfrak{g})$-module $V^q$ the deformation is flat; i.e. one has $\dim((S_q(V^q))_n) = (\dim(V)+n-1)$ for all graded components $(S_q(V^q))_n$. The following result completely classifies such flat modules.

**Main Theorem 1.2.** (Theorem 4.24) A $U_q(\mathfrak{g})$-modules $V^q$ is flat, if and only if the bracket (1.1) defines a Poisson structure on the symmetric algebra of the classical limit $V$ of $V^q$.

If $\mathfrak{g}$ is of type $A_n$, $B_n$ or $D_n$, then the braided symmetric algebras of the flat modules are the quantized coordinate rings of the classical varieties such as the quantum $m \times n$-matrices, the quantum Euclidean space (see e.g. 32), quantum symmetric and quantum antisymmetric matrices (see e.g. 31 and 32). The braided exterior powers of the flat natural modules of quantized enveloping algebras of types $B_n$, $C_n$ and $D_n$ agree with the $q$-wedge modules constructed by Jing, Misra and Okado in 22. Our approach, thus, provides a natural unifying construction for these objects. Moreover, following the arguments in 17 Ch. 5] one obtains that the braided symmetric algebras are the quantizations of equivariant Poisson structures on partial flag varieties.

Theorem 1.1 shows that there is an apparent relation between $r$-matrix Poisson structures, flat modules and maximal parabolic with Abelian or Heisenberg type
radicals and classical invariant theory as studied by Howe in [18]. We use this connection to give the following explicit construction of braided symmetric algebras of flat simple modules.

**Theorem 1.3.** (Theorem 5.4, Theorem 5.9) In the notation of Theorem 1.1, if the bracket (1.1) is Poisson (including the case \((g, V) = (\text{sp}_{2n}(\mathbb{C}), V_{\nu_1})\)), then:

(a) There exists a unique simple Lie algebra \(g'\) and a maximal parabolic subalgebra \(p \subset g'\) such that \(g\) is the semisimple part of the Levi factor of \(p\) and \(V\) is isomorphic (as a \(g\)-module) to the nil-radical \(\text{rad}_p\) of \(p\).

(b) The associated graded of the quantized enveloping algebra \(U_q(\text{rad}_p)\) is the Kontsevich deformation \(S_q(V^q)\) of the Poisson algebra \(S(V)\) and carries a natural \(U_q(g)\)-module structure.

The paper is organized as follows: In Section 2 we study quadratic Poisson algebras. We introduce the notions of decorated space and bracketed algebras and show that the categories of decorated spaces and bracketed algebras are symmetric monoidal. We show how decorated spaces define bracketed algebras and show under which conditions the bracket satisfies the Jacobi identity (Theorem 2.21) extending well known results of Gelfand and Fokas [16]. We also show that if the bracketed algebra associated to a tensor product of two decorated spaces is Poisson, then each of the factors must define a Poisson algebra (Theorem 2.25).

We use these results in Section 3 to classify all simple Poisson modules over simple Lie algebras (Theorem 3.12) and finally all simple Poisson modules over semisimple Lie algebras (Theorem 3.13). Since the proof of Theorem 3.12 is rather long we present it in Section 6.

Section 4 is devoted to the classification of flat simple modules (Theorem 4.24). For the convenience of the reader we provide brief introductions to the quantized enveloping algebras \(U_q(g)\), the category of their finite-dimensional modules (Section 4.1), the definition and basic properties of braided symmetric and exterior algebras and powers (Section 4.2) and the classical limit (Section 4.3).

In Section 5 we construct the braided symmetric algebras as the quantized enveloping algebra of nilradicals (Theorem 5.4), respectively their associated graded algebras (Theorem 5.9). In Section 5.2 we prove a PBW-type theorem for quantum Schubert cells and study Levi actions on quantized nilradicals.

The results in this paper open up many questions and suggest connections between the theory of braided symmetric algebras, cluster algebras, geometric crystals, equivariant Poisson structures and classical invariant theory. Appropriately, the paper concludes with a section on open questions and conjectures (Section 7).

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### 2. Quadratic Poisson Brackets

#### 2.1. Decorated Spaces and Bracketed Algebras.

In this section we will introduce the notion of a decorated space and relate it to bracketed algebras, which are commutative algebras with a skew-commutative bracket satisfying the Leibniz rule, but not necessarily the Jacobi identity. Consider a linear tensor category \(\mathcal{C}\) over a field \(k\) of characteristic \(\text{char}(k) = 0\). We will view \(\mathcal{C}\) as a symmetric tensor category with the braiding given by the permutation of factors. Define a decorated
Lemma 2.1. The category $D(C)$ is a braided symmetric category. More precisely we have the following:

(a) The unit object is $(k,0)$, where $0: k \otimes k \to 0 \in k$ is the trivial map.

(b) The tensor product $(V \otimes V', \Phi'') = (V, \Phi) \otimes (V', \Phi')$ of two decorated spaces $(V, \Phi)$ and $(V', \Phi')$ is defined for all $u, v \in V$ and $u', v' \in V'$ via

\[
\Phi'' = \Phi_{13} + \Phi_{24},
\]

where $\Phi_{13} = \tau_{23} \circ (\Phi \otimes \text{Id}_{\otimes 2}) \circ \tau_{23}$ and $\Phi_{24} = \tau_{23} \circ (\text{Id}_{\otimes 2} \otimes \Phi') \circ \tau_{23}$.

(c) The symmetric braiding is given by the morphisms $D\tau: (V, \Phi) \otimes (V', \Phi') \to (V', \Phi') \otimes (V, \Phi)$ such that $D\tau(V \otimes V') = V' \otimes V$ and $D\tau(\Phi_{13} + \Phi_{24}) = \Phi_{13} + \Phi_{24}$.

(d) Moreover, the direct sum $(V, \Phi) \oplus (V', \Phi') = (V \oplus V', \Phi + \Phi')$ of two decorated spaces $(V, \Phi)$ and $(V', \Phi')$ is a decorated space.

Proof. It is easy to see that the tensor product $(V, \Phi) \otimes (V', \Phi')$ of two decorated spaces is a decorated space, and that the tensor product is indeed associative. One can now show that $(k,0)$ satisfies the axioms of the unit. Parts (a) and (b) are proved. Parts (c) and (d) are obvious. The lemma is proved.

Lemma 2.2. Let $C'$ be a symmetric linear category and let $F: C \to C'$ be a covariant monoidal-functor compatible with the braiding. Then, $F$ defines a covariant functor $D,F: D(C) \to D(C')$, which takes an object $(V, \Phi)$ to $(F(V), F(\Phi))$.

Proof. It suffices to show that for every object $V$ of $C$ and every $\Phi \in \text{End}(V \otimes V)$ such that $\tau \circ \Phi = -\Phi \circ \tau$ one has $\tau \circ F(\Phi) = - (F(\Phi) \circ \tau_{F(V),F(V)})$. Since $F$ is compatible with the braiding we compute:

\[
\tau_{F(V),F(V)} \circ F(\Phi) = F(\tau_{V',V} \circ \Phi) = F(-\Phi \circ \tau_{V',V}) = - (F(\Phi) \circ \tau_{F(V),F(V)}).
\]

The lemma is proved.

We next introduce bracketed and Poisson algebras in the category $C$.

Definition 2.3. (a) A bracketed algebra is a pair $(A, \{\cdot, \cdot\})$, where $A$ is a commutative algebra in $C$ and $\{\cdot, \cdot\}$ is a a structure preserving bilinear map $\{\cdot, \cdot\}: A \otimes A \to A$ satisfying:

(i) anti-commutativity

\[
\{a, b\} + \{b, a\} = 0,
\]

for any $a, b \in A$.

(ii) the Leibniz rule

\[
\{a, bc\} = \{a, b\}c + b \cdot \{a, c\}
\]

for any $a, b, c \in A$.

(b) A bracketed algebra is called Poisson, if $\{\cdot, \cdot\}$ satisfies the Jacobi identity:

\[
\{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0
\]

for all $a, b, c \in A$. 

Lemma 2.5. The Poisson algebras form a full subcategory of $BA\text{lg}(C)$, the category of Poisson algebras.

We need the following fact.

Lemma 2.6. The category $BA\text{lg}(C)$ is a symmetric monoidal category. More precisely:

(a) The tensor product $(A \otimes B, \{\cdot, \cdot\}) := (A, \{\cdot, \cdot\}A) \otimes (B, \{\cdot, \cdot\}B)$ of two objects $(A, \{\cdot, \cdot\}A)$ and $(B, \{\cdot, \cdot\}B)$ is defined in the following way: for all $a, a' \in A$, $b, b' \in B$ define,

$$\{a \otimes b, a' \otimes b'\} = ((1 \otimes \mu_B) \circ (\{a, a'\} \otimes b \otimes b') + (\mu_A \otimes 1) \circ (a \otimes a \otimes \{b, b'\})),$$

where $\mu_A$ denotes the multiplication in $A$ and $\mu_B$ the multiplication in $B$.

(b) Its unit element is $(k, 0)$.

(c) The symmetric braiding is defined as

$$\sigma((A, \{\cdot, \cdot\}A) \otimes (B, \{\cdot, \cdot\}B) = (B, \{\cdot, \cdot\}B) \otimes (A, \{\cdot, \cdot\}A).$$

Proof. To prove part (a) we have to show that the tensor product is associative. We compute:

$$\{a \otimes (b \otimes c), a' \otimes (b' \otimes c')\}$$

$$= (\{\cdot, \cdot\} \otimes \mu_B \otimes \mu_C + \mu_A \otimes \{\cdot, \cdot\} \otimes \mu_C + \mu_A \otimes \mu_B \otimes \{\cdot, \cdot\} \otimes \sigma_{23,45}(a \otimes (b \otimes c) \otimes a' \otimes (b' \otimes c')))$$

where $\sigma_{23,45} = Id_{A \otimes A \otimes B} \otimes \sigma_{C, B} \otimes Id_C$ and $\sigma_{23,4} = Id^A \otimes \sigma_{B \otimes C, A} \otimes Id_{B \otimes C}$.

Similarly, we obtain that

$$\{(a \otimes b) \otimes c, (a' \otimes b') \otimes c'\}$$

$$= (\{\cdot, \cdot\} \otimes \mu_B \otimes \mu_C + \mu_A \otimes \{\cdot, \cdot\} \otimes \mu_C + \mu_A \otimes \mu_B \otimes \{\cdot, \cdot\} \otimes \sigma_{23} \circ \sigma_{34,5}(a \otimes (b \otimes c) \otimes a' \otimes (b' \otimes c')))$$

where $\sigma_{23} = Id_A \otimes \sigma_{B, A}$ and $\sigma_{34,5} = Id_{A \otimes B} \otimes \sigma_{C, A \otimes B} \otimes Id_C$. Since $\sigma$ is a braiding we have $\sigma_{23,4} = \sigma_{23} \circ \sigma_{34}$ and $\sigma_{34,5} = \sigma_{45} \circ \sigma_{34}$ with $\sigma_{34} = Id_{A \otimes B} \otimes \sigma_{C, A} \otimes Id_{B \otimes C}$.

Since $\sigma_{23} \circ \sigma_{34} = \sigma_{45} \circ \sigma_{23}$ it is now easy to verify that $\sigma_{23} \circ \sigma_{34,5} = \sigma_{45} \circ \sigma_{23,4}$ and hence, that the tensor product is indeed associative.

Part (a) is proved, and Parts (b) and (c) are obvious. Lemma 2.6 is proved. □

We have the following obvious facts.

Lemma 2.7. (a) The assignment $A \rightarrow (A, 0)$ defines a faithful tensor functor from the category of commutative algebras in $C$ to the category of bracketed algebras.

(b) The assignment $(A, \{\cdot, \cdot\}) \rightarrow A$ defines a forgetful functor from $BA\text{lg}(C)$ to the category of algebras in $C$. 

2.2. Symmetric Algebras and Bracketed Symmetric Algebras. Let $V$ be an object of $\mathcal{C}$. The category $\mathcal{C}$ is Abelian, hence $Id_{V \otimes V} + \sigma_{V,V}$ is a morphism in $\mathcal{C}$ and the symmetric square of $S^2 V = \text{Im}(Id_{V \otimes V} + \sigma)$, respectively the exterior square $\Lambda^2 V = \text{Ker}(Id_{V \otimes V} + \sigma)$ are objects in $\mathcal{C}$. Define the $n$-th symmetric power as

$$S^n V = S^2 V \otimes V^{\otimes (n-2)} \cap V \otimes S^2 V \otimes V^{\otimes (n-2)} \cap \ldots \cap V \otimes S^2 V \subset V^{\otimes n}.$$  

Similarly, define the $n$-th exterior power as

$$\Lambda^n V = \Lambda^2 V \otimes V^{\otimes (n-2)} \cap V \otimes \Lambda^2 V \otimes V^{\otimes (n-2)} \cap \ldots \cap V \otimes \Lambda^2 V \subset V^{\otimes n}.$$  

Define the symmetric algebra $S(V) = T(V)/\langle \Lambda^2 V \rangle$ as the quotient of the tensor algebra of $V$ by the two-sided ideal generated by the exterior square.

The following facts are immediate for a symmetric linear category where the braiding is given by the permutation of factors.

**Proposition 2.8.** (a) Let $V$ be an object of $\mathcal{C}$. The symmetric algebra $S(V)$ is a commutative algebra in $\mathcal{C}$.

(b) The assignment $V \mapsto S(V)$ is functorial.

(c) The functor is exponential, i.e. $S(U \oplus V) \cong S(U) \otimes_S S(V)$.

Define morphisms $\sigma_{i,i+1} \in \text{End}_C(V^{\otimes n})$ for all $1 \leq i \leq (n-1)$ by $\sigma_{i,i+1} = Id^{\otimes (i-1)} \otimes \sigma \otimes Id^{\otimes (n-i-1)}$. Define for any permutation $\tau$ in the symmetric group $\mathfrak{S}_n$ the morphism $\sigma_\tau \in \text{End}(V^{\otimes n})$ as $\sigma_\tau = \sigma_{i_1,i_2+1} \circ \ldots \circ \sigma_{i_{\ell},i_{\ell+1}+1}$, where $\tau = (i_{\ell}, i_{\ell+1} + 1) \circ \ldots \circ (i_1, i_1 + 1)$ is a presentation of $\tau$ consisting of simple transpositions and $\sigma_{i,i+1}$ defined as above. Note that $\sigma_\tau$ is well defined, independent of the choice of presentation of $\tau$.

Recall the definition of the $n$-th braided factorial:

$$[n]!_\sigma = [n]!_{V,\sigma} = [n]!_{\sigma} : V^{\otimes n} \to V^{\otimes n}, [n]!_{\sigma} = \sum_{\tau \in \mathfrak{S}_n} \sigma_\tau .$$

By definition $[n]!_{\sigma} \circ \sigma_i = \sigma_i \circ [n]!_{\sigma} = [n]!_{\sigma}$. Similarly, define the $n$-th braided skew-factorial $[n]!_{-\sigma} = [n]!_{-\sigma} : V^{\otimes n} \to V^{\otimes n}, [n]!_{-\sigma} = \sum_{\tau \in \mathfrak{S}_n} (-1)^{\ell(\tau)} \sigma_\tau$, where $\ell(\tau)$ denotes the length of the permutation $\tau$. The following fact is well known.

**Proposition 2.9.** For any object $V$ of $\mathcal{C}$ one has:

(a) $S^n V = \text{Im}([n]!_{V,\sigma})$ and $\Lambda^n V = \text{Im}([n]!_{V,-\sigma})$.

(b) $\langle \Lambda^2 V \rangle_n = \text{Ker} [n]!_{\sigma}$, where $\langle \Lambda^2 V \rangle_n$ is the degree $n$ component of the ideal $\langle \Lambda^2 V \rangle$. Equivalently, the composition $\phi_n : S^n V \hookrightarrow T V \to S(V)$ is an isomorphism between $S^n V$ and $S(V)_n$, where $S(V)_n$ is the $n$-th graded component of $S(V)$.

Let $V$ be an object in the symmetric tensor category $\mathcal{C}$ and let $\Phi : V \otimes V \to V \otimes V$ be a morphism in $\mathcal{C}$. For $1 \leq i < j \leq n$ define $\Phi_{i,j} : V^{\otimes n} \to V^{\otimes n}$ as follows. First, $\Phi_{i,i+1} \in \text{End}(V^{\otimes n}) = Id^{\otimes i-1} \otimes \Phi \otimes Id^{\otimes n-i-1}$ and, recursively, for $i < j - 1$

$$\Phi_{i,j} = \sigma_{(j-1,j)} \circ \Phi_{i,j-1} \circ \sigma_{(j-1,j)} .$$

**Proposition 2.10.** For any $i < j$ and any $\tau \in \mathfrak{S}_n$ such that $\tau(i) < \tau(j)$ one has $\Phi_{\tau(i),\tau(j)} = \sigma_{\tau} \circ \Phi_{i,j} \circ \sigma_{\tau^{-1}}$.

**Proof.** We first need the following fact.

**Lemma 2.11.** If $1 \leq i < j \leq n$, $m \leq n - 1$ and $\tau = (m, m + 1)$, then $\sigma_{m,m+1} \circ \Phi_{i,j} \circ \sigma_{m,m+1} = \Phi_{\tau(i),\tau(j)}$.
Proof. Clearly, we obtain
\begin{equation}
\sigma_{m,m+1} \circ \Phi_{i,j} \circ \sigma_{m,m+1} = \Phi_{i,j},
\end{equation}
if \{m, m+1\} \cap \{i, j\} = \emptyset.

The assertion holds by definition (2.5) if \( m = j \) or \( m = j - 1 \). It remains the case, when \( m = i \) or \( m + 1 = i \). Since \( \sigma_{m,m+1} \) is an involution, it suffices to prove the assertion for \( m + 1 = i \); i.e. we have to show that \( \sigma_{i-1,i} \circ \Phi_{i,j} \circ \sigma_{i-1,i} = \Phi_{i-1,j} \).

We need the following fact.

Lemma 2.12. Let \( \tau = (j-1,j) \circ (j-2,j-1) \circ \ldots \circ (i-1,i) \). Then we have for \( \Phi \in \text{End}(V \otimes V) \) that
\[
\sigma_{\tau} \circ \Phi_{i,j} \circ \sigma_{\tau}^{-1} = \Phi_{i-1,j-1} = \Phi_{\tau(i), \tau(j)}.
\]

Proof. The braid relation of the symmetric group \( S_n \) yields that \( \sigma_{i+1,i} \circ \sigma_{i+1,i} \circ \Phi_{i+1,i} = \Phi_{i+1,i} \circ \sigma_{i+1,i} \circ \sigma_{i+1,i} \). We obtain through repeated application that \( \Phi_{i,j} \circ \sigma_{i,j} = \Phi_{i-1,j-1} \circ \sigma_{\tau} \).

Now, we compute
\[
\sigma_{i-1,i} \circ \Phi_{i,j} \circ \sigma_{i-1,i} = \sigma_{i-1,i} \circ (\sigma_{\tau})^{-1} \circ \Phi_{i-1,j-1} \circ \sigma_{\tau} \circ \sigma_{i-1,i} = \sigma_{i-1,i} \circ \sigma_{i-1,i} = \Phi_{i,j}.
\]

We obtain applying (2.6) multiple times that
\[
\sigma_{i-1,i} \circ \sigma_{i-1,i} \circ \sigma_{i-1,i} = \Phi_{i,j}.
\]

We now have \( \sigma_{i-1,i} \circ \sigma_{i-1,i} = \Phi_{i,j} \). Lemma 2.11 is proved.

We can now prove Proposition 2.10 by induction on the length \( \ell(\tau) \) of \( \tau \). The inductive base is provided by Lemma 2.11. Next, let \( \tau \in S_n \) such that \( \ell(\tau) > 1 \). Write \( \tau = \tau' \circ \sigma_{m,m+1} \) such that \( \ell(\tau') = \ell(\tau) - 1 \). It is easy to see that \( \tau'(i) < \tau'(j) \).

Otherwise we would have \( \tau(i) = m, \tau(j) = m + 1 \), and \( \tau'(i) = m + 1 \) and \( \tau'(j) = m \).

This implies that we would obtain a reduced expression \( \tau'' = \tau'' \circ \sigma_{m',m'+1} \circ \tau'' \)
with \( \ell(\tau') = \ell(\tau'') + \ell(\tau''') + 1 \), such that \( \tau''(i) = m, \tau''(j) = m + 1 \). It is now easy to see that \( \tau = \tau'' \circ \tau''' \), and hence \( \ell(\tau) < \ell(\tau') \). We can now apply the inductive hypothesis to \( \tau'' \) to obtain \( \sigma_{\tau''} \circ \Phi_{i,j} \circ \sigma_{\tau''} = \Phi_{\tau'', \tau''} \).

Note that \( \tau^{-1} = \sigma_{m,m+1} \circ (\tau')^{-1} \).

Lemma 2.11 implies that
\[
\Phi_{m,m+1} \circ \Phi_{\tau^{-1}(i), \tau^{-1}(j)} \circ \sigma_{m,m+1} = \Phi_{\tau(i), \tau(j)}.
\]

Proposition 2.10 is proved.

We now define a map \( \Phi^{m,n} : V^{\otimes(m+n)} \rightarrow V^{\otimes(m+n)} \) by the formula:
\[
\Phi^{m,n} := \sum_{i=1}^{m} \sum_{j=m+1}^{m+n} \Phi_{i,j}.
\]

For any \( a \in S^m V, b \in S^n V \), and \( \hat{a} \in V^{\otimes m} \) and \( \hat{b} \in V^{\otimes n} \) such that \( \frac{1}{m!} [m]!_{\sigma} (\hat{a}) = a \) and \( \frac{1}{n!} [n]!_{\sigma} (\hat{b}) = b \) we define the morphism \( \{ \cdot, \cdot \}^{(m,n)} : S^m V \otimes S^n V \rightarrow S^{m+n} V \) by:
\[
\{a, b\}^{(m,n)} := \frac{1}{(m+n)!} [m+n]!_{\sigma} \Phi^{m,n} (\hat{a} \otimes \hat{b}).
\]

The above morphism is well-defined because of the following result. Recall that \( S_m \times S_n \) embeds naturally in \( S_{m+n} \).
Lemma 2.13. If \( \tau = (\tau_1, \tau_2) \in \mathfrak S_m \times \mathfrak S_n \subset \mathfrak S_{m+n} \), then \( \sigma_\tau \circ \Phi^{(m,n)} = \Phi^{(m,n)} \circ \sigma_\tau \).

Proof. One has by Lemma 2.10 that for any \( \tau \in \mathfrak S_{n+m} \) and any \( 1 \leq i < j \leq m+n \) that \( \sigma_\tau \circ \Phi_{\tau(i),\tau(j)} = \Phi_{\tau(i),\tau(j)} \circ \sigma_\tau \). We compute

\[
\Phi^{(m,n)} \circ \sigma_\tau = \sum_{i=1}^m \sum_{j=m+1}^{m+n} \Phi_{i,j} \circ \sigma_\tau = \sum_{i=1}^m \sum_{j=m+1}^{m+n} \sigma_\tau \circ \Phi_{\tau^{-1}(i),\tau^{-1}(j)} = \sigma_\tau \circ \Phi^{(m,n)} .
\]

The lemma is proved. \( \square \)

We have the following result.

Proposition 2.14. Let \((V, \Phi)\) be a decorated space. The pair \((\mathbb S(V), \{\cdot, \cdot\}_\Phi)\), where \(\{\cdot, \cdot\}_\Phi = \bigoplus_{m,n \in \mathbb Z_{\geq 0}} \{\cdot, \cdot\}^{(m,n)}_\Phi\) is a bracketed algebra in \( \mathcal C \).

Proof. Prove anti-commutativity first. We need the following fact.

Lemma 2.15. (a) Let \((V, \Phi)\) be a decorated space. Then for all \( n \geq 2 \) and \( i < j \leq n \) one has

\[
\sigma_{(i,j)} \circ \Phi_{i,j} = -\Phi_{i,j} \circ \sigma_{(i,j)} .
\]

(b) If \( i < j \leq n \) and \( \tau \in \mathfrak S_n \) such that \( \tau(i) > \tau(j) \), then

\[
\sigma_\tau \circ \Phi_{i,j} \circ \sigma_{\tau^{-1}} = -\Phi_{\tau(j),\tau(i)} .
\]

Proof. Prove (a) first. Note that \( \sigma_{(i,i+1)} \circ \Phi_{i,i+1} \circ \sigma_{(i,i+1)} = -\Phi_{i,i+1} \) and that \( (i,j) = (j-2, j-1) \circ (i, i+1) \circ (i, i+2) \). Denote \( \tau' = (j-1, j) \circ (j-2, j-1) \circ (i, i+1) \circ (i, i+2) \). We compute using Proposition 2.10

\[
\sigma_{(i,j)} \circ \Phi_{i,j} \circ \sigma_{(i,j)^{-1}} = \sigma_{\tau'} \circ \sigma_{(i,i+1)} \circ \sigma_{\tau'^{-1}} \circ \Phi_{i,j} \circ \sigma_{\tau'} \circ \sigma_{(i,i+1)} \circ \sigma_{\tau'^{-1}} = \sigma_{\tau'} \circ \Phi_{i,i+1} \circ \sigma_{\tau'} \circ \Phi_{i,j} .
\]

Part (a) is proved.

Part (b) now. We clearly have \( \tau'(i) < \tau'(j) \) for \( \tau' = (\tau(j), \tau(i)) \). Therefore, we have by Lemma 2.10 and Part (a)

\[
-\Phi_{\tau(j),\tau(i)} = \sigma_{(\tau(j),\tau(i))} \circ \Phi_{\tau(j),\tau(i)} \circ \sigma_{(\tau(j),\tau(i))} = \sigma_{(\tau(j),\tau(i))} \circ \sigma_{\tau'} \circ \Phi_{i,j} \circ \sigma_{\tau'} \circ \sigma_{(\tau(j),\tau(i))} = \sigma_{\tau} \circ \Phi_{i,j} \circ \sigma_{\tau^{-1}} .
\]

Part (b) is proved. The lemma is proved. \( \square \)

Let \( \tau \in \mathfrak S_{m+n} \) be the permutation, which sends \( (1, 2, \ldots, m, m+1, \ldots, n+m) \mapsto (n+1, n+2, \ldots, m+n, 1, 2, \ldots, n) \). Lemma 2.15 yields that for \( i \leq n < j \leq m+n \) one has \( \Phi_{i,j} \circ \sigma_\tau = -\sigma_\tau \circ \Phi_{i,j} \). Therefore, we have

\[
\Phi^{(m,n)} \circ \sigma_\tau = \sum_{i=1}^m \sum_{j=m+1}^{m+n} \Phi_{i,j} \circ \sigma_\tau = -\sigma_\tau \sum_{i=1}^m \sum_{j=m+1}^{m+n} \Phi_{\tau(i),\tau(j)} .
\]

This implies that \( \{a, b\}_\Phi^{(m,n)} = -\{a, b\}_\Phi^{(m,n)} \) for \( a \in S^m V \) and \( b \in S^n V \), hence \( \{a, b\}_\Phi = -\{a, b\}_\Phi \). Anti-commutativity is proved.

It remains to verify the Leibniz identity \( \{a, b\}_\Phi \). Let \( a \in S^m V, b \in S^n V \) and \( c \in S^V \) and let \( \hat a \in V^\otimes n, \hat b \in V^\otimes m \) and \( \hat c \in V^\otimes \ell \) be representatives of \( a, b \) and \( c \), respectively. Denote by \( \tau' \in \mathfrak S_{m+n+\ell} \) the permutation \( \tau'(1, \ldots, n+m+\ell) = (n+1, n+2, \ldots, m+n+\ell) \). We compute
\{a, b \cdot c\}_\Phi = \{a, b \cdot c\}_\Phi^{(n,m+\ell)} = \frac{1}{(n+m+\ell)!}[n+m+\ell]!_\sigma (\hat{a} \otimes \hat{b} \otimes \hat{c})
\frac{1}{(n+m+\ell)!}[n+m+\ell]!_\sigma \left(\frac{[n+m]!_\sigma \otimes [\ell]!_\sigma}{(n+m)!\ell!}\sum_{i=1}^{n+m+\ell} \sum_{j=n+1}^{n+m+\ell} \Phi_{i,j} \right) (\hat{a} \otimes \hat{b} \otimes \hat{c})
+ \frac{1}{(n+m+\ell)!}[n+m+\ell]!_\sigma \left(\frac{[m]!_\sigma \otimes [n+\ell]!_\sigma}{m!(n+\ell)!}\sum_{i=1}^{n+m+\ell} \sum_{j=n+m+1}^{\ell} \Phi_{i,j} \right) (\hat{a} \otimes \hat{b} \otimes \hat{c})
= \{a, b \cdot c\} + b \cdot \{a, c\}.

Therefore the Leibniz rule holds. Proposition \ref{proposition:2.14} is proved. \hfill \square

We will denote by $S(V, \Phi)$ the bracketed algebra $(S(V), \{\cdot, \cdot\}_\Phi)$ from Proposition \ref{proposition:2.13} and refer to it as the \textit{symmetric algebra} of the decorated space $(V, \Phi)$. We have the following result.

**Proposition 2.16.** The correspondence $(V, \Phi) \mapsto S(V, \Phi)$ defines a faithful exponential functor from the category of decorated spaces to the category of bracketed algebras $B\text{Alg}(\mathcal{C})$.

**Proof.** It is easy to verify that the correspondence is functorial and faithful. It remains to check that it is exponential. By Proposition \ref{proposition:2.8} one has $S(V \oplus V') = S(V) \otimes S(V')$, and we obtain the bracket defined by

$$\{u + u', v + v'\}_\Phi = (\Phi + \Phi')( (u + u') \wedge (v + v') )$$

for all $u, v \in V$ and $u', v' \in V'$. The proposition is proved. \hfill \square

2.3. Bracketed Poisson Algebras. Denote by $J : A^{\otimes 3} \to A$ the Jacobian map defined by

\begin{equation}
J = F + F \circ \sigma_{12} \circ \sigma_{23} + F \circ \sigma_{23} \circ \sigma_{12},
\end{equation}

where $F : A^{\otimes 3} \to A, F(a, b, c) = \{a, b, c\}$ and $\sigma_{12} = \sigma \circ Id$ and $\sigma_{23} = Id \circ \sigma$.

The following fact is obvious.

**Lemma 2.17.** (\cite[Definition 2.23]{R}) A bracketed algebra $(A, \{\cdot, \cdot\})$ is Poisson, if and only if $J(A^{\otimes 3}) = 0$.

Define the Jacobian ideal $(J_\Phi)$ as the two-sided (bracketed) ideal in the bracketed algebra $S(V, \Phi)$ generated by the image of the Jacobian map. We call the quotient of $S(V)_{\Phi}$ by $(J_\Phi)$ the Poisson closure. The bracket $\{\cdot, \cdot\}_\Phi$ induces a bracket $\{\cdot, \cdot\}_\Phi$ on $S(V)_{\Phi}$, because $(J_\Phi)$ is by definition closed under the bracket.

**Definition 2.18.** The \textit{reduced symmetric algebra} $S(V, \Phi)$ is the bracketed algebra $S(V, \Phi) = (S(V)_{\Phi}, \{\cdot, \cdot\}_\Phi)$.

We have the following result.
Proposition 2.19. (a) The reduced symmetric algebra $S(V, \Phi)$ is Poisson.
(b) The reduced symmetric algebra $S(V, \Phi)$ has the following universal property: Any homomorphism of bracketed algebras from $S(V, \Phi)$ to a Poisson algebra $P$ factors through $S(V, \Phi)$.
(c) The assignment $(V, \Phi) \mapsto S(V, \Phi)$ defines a functor from the category of decorated spaces to the category of Poisson algebras. Moreover, if $(V'', \Phi'') = (V, \Phi) \oplus (V', \Phi')$, then there exists a surjective homomorphism $S(V, \Phi) \otimes S(V', \Phi') \rightarrow S(V'', \Phi'')$.

Proof. Prove (a) first. Let $\overline{\pi, b, c} \in S(V, \Phi)$ and let $a, b, c \in S(V, \Phi)$ be representatives of the equivalence classes of $\overline{\pi, b, c}$, respectively. Then $J(a, b, c)$ is a representative of the class of $\overline{\rho(\pi, b, c)}$, where $\rho$ is the induced Jacobian map on $S(V, \Phi)$. By definition, $J(a, b, c) \in \langle J_\Phi \rangle$, hence $J(\overline{\pi, b, c}) = 0 \in S(V, \Phi)$. Part (a) is proved.

Prove (b) next. Let $P$ be a Poisson algebra and $\rho : S(V, \Phi) \rightarrow P$ a homomorphism of bracketed algebras. It is easy to see that $(J)$ is contained in the kernel of $\rho$. Hence, $\rho$ factors through $S(V, \Phi)$. Part (b) is proved.

Prove (c) now. Let $(V, \Phi)$ and $(V', \Phi')$ be decorated spaces, $(V'', \Phi'') = (V, \Phi) \oplus (V', \Phi')$, and $(J)$, resp. $(J')$, the Jacobian ideal in $S(V, \Phi)$, resp. $S(V', \Phi')$. It is clear that the ideals generated by $(J)$ and $(J')$ in $S(V'', \Phi'')$ are contained in the Jacobian ideal $(J'')$ of $S(V'', \Phi'')$. Part (c) follows since $S(V'', \Phi'') = S(V, \Phi) \otimes S(V', \Phi')$ by Proposition 2.19. The proposition is proved.

Due to Proposition 2.19 (a) we will sometimes refer to $S(V, \Phi)$ as the Poisson closure of $S(V, \Phi)$ (see also [5, Section 3.1]).

We will now discuss, when $S(V, \Phi)$ is Poisson; i.e., when $S(V, \Phi) = S(V, \Phi)$.

For any $\Phi \in \text{End}(V \otimes V)$ define the Schouten square $[[\Phi, \Phi]] \in \text{End}(V \otimes V \otimes V)$ by:

$$[[\Phi, \Phi]] = [[\Phi_{12}, \Phi_{13}]] + [[\Phi_{12}, \Phi_{23}]] + [[\Phi_{13}, \Phi_{23}]],$$

where $[a, b] = a \circ b - b \circ a$ denotes the usual commutator.

The Schouten square has the following very important property.

Lemma 2.20. Let $V$ be an object of $\mathcal{C}$ and let $\Phi \in \text{End}(V \otimes V)$ such that $\Phi \circ \sigma = -\sigma \circ \Phi$. Then, $\sigma_{i,i+1} \circ [[\Phi, \Phi]] \circ \sigma_{i,i+1} = -[[\Phi, \Phi]]$ for $i = 1, 2$.

Proof. Straightforward computation yields:

$$\sigma_{12} \circ [[\Phi, \Phi]] \circ \sigma_{12} = \sigma_{12} \circ [[\Phi_{12}, \Phi_{13}]] + [[\Phi_{12}, \Phi_{23}]] + [[\Phi_{13}, \Phi_{23}]] \circ \sigma_{12}$$

$$= -[[\Phi_{12}, \Phi_{23}]] - [[\Phi_{12}, \Phi_{13} + \Phi_{23}, \Phi_{13}]] = -[[\Phi, \Phi]].$$

Similarly, we compute

$$\sigma_{23} \circ [[\Phi, \Phi]] \circ \sigma_{23} = [[\Phi_{13}, \Phi_{12}]] - [[\Phi_{13}, \Phi_{23}]] - [[\Phi_{12}, \Phi_{23}]] = -[[\Phi, \Phi]].$$

The lemma is proved.

We call a decorated space $(V, \Phi)$ Poisson, if the symmetric algebra $S(V, \Phi)$ of the decorated space $(V, \Phi)$ is Poisson.

Theorem 2.21. Let $(V, \Phi)$ be a decorated space. The following are equivalent:

(a) $(V, \Phi)$ is Poisson
(b) $\Phi$ satisfies the equation

$$[3]!_\sigma \circ [[\Phi, \Phi]] = 0.$$

Proof. \hfill $\square$
(c) \( \Phi \) satisfies the equation
\[
[[\Phi, \Phi]] \circ [3]! \cdot \sigma = 0.
\]

(d) \( \Phi \) satisfies
\[
[[\Phi, \Phi]]|_{\Lambda^3V} = 0,
\]
where \( |_{\Lambda^3V} \) denotes the restriction to \( \Lambda^3V \).

**Proof.**

The equivalence of (a) and (b) is well known and proved in [16, Theorem 3.1]. For the convenience of the reader we nevertheless prove here that (a) equivalent (b).

We need the following fact.

**Lemma 2.22.** One has
\[
\text{im}(J) \cap S^3(V) = [3]! \cdot \sigma \circ [[\Phi, \Phi]].
\]

**Proof.** Define the lifted Jacobian \( J' : V^{\otimes 3} \to V^{\otimes 3} \) by
\[
J'(x, y, z) := G + G \circ \sigma_{12} \circ \sigma_{23} + G \circ \sigma_{23} \circ \sigma_{12},
\]
where \( G : V^3 \to V^{\otimes 3} \) is the morphism given by
\[
G = \Phi_{12} \circ (1 \otimes 1 \otimes 1 + \sigma_{23}) \circ \Phi_{32} = (\Phi_{23} \circ \Phi_{12} + \sigma_{12} \circ \Phi_{13} \circ \Phi_{12}).
\]

By definition, \( J(x, y, z) = [3]! \sigma(J'(x, y, z)) \) for all \( x, y, z \in S(V) \) and all \( x, y, z \in T(V) \) such that \( [3]! \sigma(x) = 3! \cdot x, [3]! \sigma(y) = 3! \cdot y \) and \( [3]! \sigma(z) = 3! \cdot z \).

One has \( [3]! \sigma \circ \sigma_{i,j} = [3]! \sigma \) for all \( i, j = 1, 2, 3 \). Therefore, we obtain:
\[
[3]! \sigma \circ G = [3]! \sigma \circ (\Phi_{23} \circ \Phi_{12} + \sigma_{12} \circ \Phi_{13} \circ \Phi_{12})
\]
\[
\quad = [3]! \sigma \circ (\Phi_{23} \circ \Phi_{12} + \Phi_{13} \circ \Phi_{12}).
\]

Similarly,
\[
[3]! \sigma \circ G \circ \sigma_{12} \circ \sigma_{23} = [3]! \sigma \circ (-\sigma_{12} \circ \sigma_{23} \circ \Phi_{12} \circ \Phi_{13} + \sigma_{23} \circ \Phi_{23} \circ \Phi_{13})
\]
\[
\quad = [3]! \sigma \circ (-\Phi_{12} \circ \Phi_{13} + \Phi_{23} \circ \Phi_{13}).
\]

Combining these equations, we obtain :
\[
[3]! \sigma \circ (G + G \circ \sigma_{12} \circ \sigma_{23} + G \circ \sigma_{23} \circ \sigma_{12}) = -[3]! \sigma \circ [[\Phi, \Phi]]
\]
and thus:
\[
(2.9) \quad 3! \cdot J(x, y, z) = -([3]! \sigma \circ [[\Phi, \Phi]])(x \otimes y \otimes z).
\]

for all \( x, y, z \in V \). The lemma is proved.

\( \square \)

We need the following fact which generalizes [6, Lemma 3.7].

**Lemma 2.23.** Let \((A, \{\cdot, \cdot\})\) be a bracketed \( \mathbb{Z}_{\geq 0} \)-graded algebra in \( C \) generated by \( A_1 \) and such that \( A_0 \cong k \). Then \( A \) is Poisson if and only if the Jacobian (see (2.7)) vanishes on \((A_1)^3\).
Proof. We proceed by induction in homogeneity degrees \( \ell = n + m + k \) of monomials \( u \cdot v \cdot w \) with \( u \in A_n, v \in A_m, w \in A_k \). We start with the base of induction, which is the assumption: suppose that for all \( u', v', w' \in A_1 \) one has \( J(u', v', w') = 0 \). Now let \( u \in A_n, v \in A_m, w \in A_k, z \in A_1 \) and assume that the assertion holds for \( \ell = n + m + k \). We compute using the inductive hypothesis and the Leibniz rules \([2.4]\) and \([2.3]\):

\[
J(a, b, c \cdot d) = J(a, b, c) \cdot d + c(J(a, b, d)) = 0.
\]

Since for all \( u, v, w, z \in A \) one has \( J(u, v, w) = J(w, u, v) = J(v, w, u) \), the assertion holds for \( n + m + k = \ell + 1 \). This implies that \( A \) is indeed Poisson. The lemma is proved. \( \square \)

The above lemma implies that \( J(S(V)^3) = 0 \) if and only if \( \Phi \) satisfies \([2.8]\). Therefore, (a) and (b) are equivalent.

We will now prove the equivalence of (b) and (c). It follows from Lemma \([2.20]a\) that \([3]!_\sigma \circ [\Phi, \Phi] = ([\Phi, \Phi] \circ [3]!_{-\sigma}). Therefore, \([3]!_\sigma \circ [\Phi, \Phi] = 0, \) and only if \([\Phi, \Phi] \circ [3]!_{-\sigma} = 0 \), and (b) and (c) are equivalent.

Parts (c) and (d) are clearly equivalent. Theorem \([2.21]\) is proved. \( \square \)

We will now employ Theorem \([2.21]\) to study Poisson structures on subspaces and tensor products of decorated spaces.

First, note the following fact.

Proposition 2.24. Let \( V, V' \) and \( V'' \) be objects of \( \mathcal{C} \) such that \( V = V' \oplus V'' \). Let additionally, \( (V, \Phi) \) and \( (V', \Phi') \) be decorated spaces such that for all \( v_1 \otimes v_2 \in V' \otimes V' \)

one has \( \Phi'(v_1 \otimes v_2) = \pi_{V', V'} \circ \Phi(v_1 \otimes v_2) \), where \( \pi_{V', V'} : V \otimes V \to V' \otimes V' \) denotes the canonical projection. If \( (V, \Phi) \) is Poisson, then \( (V', \Phi') \) is Poisson.

Proof. One has \( \Lambda^3 V = \bigoplus_{i=0}^3 \Lambda^i V' \otimes \Lambda^{3-i} V'' \), and \( S^3 V = \bigoplus_{i=0}^3 S^i V' \otimes S^{3-i} V'' \). Clearly, \([\Phi, \Phi'] : \Lambda^3 V' \to S^3 V' \) and it follows from our assertion that \([\Phi, \Phi'] : \Lambda^3 V' \to S^3 V \subset S^3 V \). If \( (V, \Phi) \) is Poisson, then \([\Phi, \Phi'] : \Lambda^3 V' \to S^3 V \) is Poisson, and hence \( (V', \Phi') \) is Poisson. \( \square \)

The following result relates Poisson structures and tensor products.

Theorem 2.25. Let \( (U, \Phi) \) and \( (V, \Phi') \) be decorated spaces. If their tensor product \( (U \otimes V, \Phi'') = (U, \Phi) \otimes (V, \Phi') \) is Poisson, then \( (U, \Phi) \) and \( (V, \Phi') \) are Poisson.

Proof.

Let \( \tilde{\sigma} \) be the "shuffle" \( U^\otimes 3 \otimes V^\otimes 3 \approx (U \otimes V)^3 \). Abbreviating

\[
U_{\otimes 3} = S^3 U = (S^2 U \otimes U) \cap (U \otimes S^2 U), U_{\otimes 2} = (S^2 U \otimes U) \cap (U \otimes S^2 U), U_{\otimes 1} = (S^2 U \otimes U) \cap (U \otimes S^2 U)
\]

and the same for \( V \), we have the following containments for \( \Lambda^3 (U \otimes V) \) and \( S^3 (U \otimes V) \):

\[
\Lambda^3 (U \otimes V) \supseteq \bigoplus_{i+j=3} \tilde{\sigma}(U^{i,j} \otimes V^{j,i}),
\]

\[
S^3 (U \otimes V) \supseteq \bigoplus_{i+j=3} \tilde{\sigma}(U^{i,j} \otimes V^{j,i}).
\]

Since \( \Phi'' = \Phi_{13} + \Phi_{24} \) and \( [\Phi_{13}, \Phi_{24}] = 0 \in End((U \otimes V)^\otimes 2) \) we have that

\[
[[\Phi'', \Phi'']] = [[\Phi_{13} + \Phi_{24}, \Phi_{13} + \Phi_{24}]] = [[\Phi_{13}, \Phi_{13}]] + [[\Phi_{24}, \Phi_{24}]].
\]
Note that $\Phi(\Lambda^2 U) \subseteq S^2 U$ (resp. $\Phi(\Lambda^2 V) \subseteq S^2 V$ and $\Phi(S^2 U) \subseteq \Lambda^2 U$ (resp. $\Phi(S^2 U) \subseteq \Lambda^2 U$). Hence
$$[[\Phi_{13}, \Phi_{13}]](\tilde{\sigma}(U^{i,j} \otimes V^{j,i})) \subseteq \tilde{\sigma}(U^{i,j} \otimes V^{j,i}), [[\Phi_{24}, \Phi_{24}]](\tilde{\sigma}(U^{i,j} \otimes V^{j,i})) \subseteq \tilde{\sigma}(U^{i,j} \otimes V^{j,i})$$
for all $i + j = 3$. This implies that
$$[[\Phi''’, \Phi’’’]](\tilde{\sigma}(U^{i,j} \otimes V^{j,i})) \subseteq \tilde{\sigma}(U^{i,j} \otimes V^{j,i} + U^{i,j} \otimes V^{j,i}).$$

Theorem 2.29 now follows as the special cases $i = 3, j = 0$ and $i = 0, j = 3$ from the following more general obvious result.

**Lemma 2.26.** If $(U, \Phi) \otimes (V, \Phi')$ is Poisson, then
$$[[\Phi'', \Phi''']](\tilde{\sigma}(U^{i,j} \otimes V^{j,i})) = \{0\} \subseteq \tilde{\sigma}(U^{i,j} \otimes V^{j,i} + U^{i,j} \otimes V^{j,i})$$
for all $i + j = 3$.

Theorem 2.29 is proved.

The following example shows that the converse of Theorem 2.29 does not hold.

**Example 2.27.** Let $V = V' = C^2$ with standard basis $\{e_1, e_2\}$, and let $\Phi(e_i \otimes e_j) = \text{sign}(i-j)(e_j \otimes e_i)$ and $\Phi'(e_i \otimes e_j) = \lambda \cdot \text{sign}(i-j)(e_j \otimes e_i)$. Clearly, both $(V, \Phi)$ and $(V, \Phi')$ are Poisson, because $\Lambda^3 C^2 = \{0\}$, but straightforward calculation shows that $(V, \Phi) \otimes (V, \Phi')$ is Poisson, if and only if $\lambda = \pm 1$.

We conclude this section with an apparently well known and useful observation regarding a general operator $\Phi : V \otimes V \to V \otimes V$ that satisfies the identity $[[\Phi, \Phi]] = 0$, the classical Yang-Baxter-Equation. However, for the reader's convenience we give a proof.

**Proposition 2.28.** Let $\Phi$ be an operator such that $[[\Phi, \Phi]] = 0$, and define $\Phi^+ = \frac{1}{2}(\Phi + \tau(\Phi))$ and $\Phi^- = \frac{1}{2}(\Phi - \tau(\Phi))$. One has
$$[[\Phi^-, \Phi^-]] = -[\Phi_{12}, \Phi_{23}].$$

**Proof.** First note the following fact.

**Lemma 2.29.** Let $\Phi : V \otimes V \to V \otimes V$ satisfy $[[\Phi, \Phi]] = 0$. Then $\Phi^{op} = \sigma \circ \Phi$ satisfies $[[\Phi^{op}, \Phi^{op}]] = 0$.

**Proof.** If $[[\Phi, \Phi]] = 0$, then also $\sigma_{13} \circ [[\Phi, \Phi]] \circ \sigma_{13} = 0$. Hence,
$$0 = [\Phi_{32}, \Phi_{31}] + [\Phi_{23}, \Phi_{21}] + [\Phi_{31}, \Phi_{21}] = -[[\Phi^{op}, \Phi^{op}]].$$
The lemma is proved.

We need the following lemma.

**Lemma 2.30.** Let $\Phi$ be an operator such that $[[\Phi, \Phi]] = 0$. In the notation of Proposition 2.28 one has the following identity: $[[\Phi^-, \Phi^-]] = -[[\Phi^+, \Phi^+]]$.

**Proof.** Since $[[\Phi, \Phi]] = [[\Phi^{op}, \Phi^{op}]] = 0$ we obtain that $[[\Phi^-, \Phi^-]] = \frac{1}{2}[[\Phi, -\Phi^{op}]] = -\frac{1}{2}[[\Phi, \Phi^{op}]]$ and $[[\Phi^+, \Phi^+]] = \frac{1}{2}[[\Phi, \Phi^{op}]]$. The lemma is proved.

By Lemma 2.20 we obtain that $[[\Phi^-, \Phi^-]] = -\sigma_{13} \circ [[\Phi^-, \Phi^-]] \circ \sigma_{13} = 2[[\Phi^-, \Phi^-]]$. Using Lemma 2.30 we obtain
$$2[[\Phi^-, \Phi^-]] = -[[\Phi^+, \Phi^+]] + \sigma_{13} \circ [[\Phi^+, \Phi^+]] \circ \sigma_{13} = -2[\Phi_{12}, \Phi_{23}].$$

Proposition 2.28 is proved.
3. Poisson Modules over Lie Algebras

3.1. Definition and Basic Properties of Poisson Modules. Let \((\mathfrak{g}, (\cdot, \cdot))\) be a quadratic complex Lie algebra; i.e. a complex Lie algebra \(\mathfrak{g}\) with a symmetric invariant bilinear form \((\cdot, \cdot) : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}\). Clearly \((\cdot, \cdot) \in (\mathfrak{g} \otimes \mathfrak{g})^\ast \cong \mathfrak{g}^* \otimes \mathfrak{g}^*\). The form \((\cdot, \cdot)\) defines an isomorphism between \(\mathfrak{g}^*\) and \(\mathfrak{g}\), and under this isomorphism we can identify the form with a symmetric \(\mathfrak{g}\)-invariant element \((\cdot, \cdot) = c \in S^2(\mathfrak{g}) \subset \mathfrak{g} \otimes \mathfrak{g}\). In the case when \(\mathfrak{g}\) is semisimple and \((\cdot, \cdot)\) is the Killing form, \(c\) is known as the Casimir element. Similarly, note that the Lie bracket \([\cdot, \cdot] : \mathfrak{g} \land \mathfrak{g} \to \mathfrak{g}\) defines an element \([\cdot, \cdot] : \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}\) and we obtain under the isomorphism above the canonical element \(c = [\cdot, \cdot] \in \mathfrak{g}^3\). Observe the following facts.

Lemma 3.1. (a) The canonical element \(c\) is \(\mathfrak{g}\)-invariant and totally skew symmetric; i.e., \(c \in (\Lambda^3 \mathfrak{g})^g\).
(b) The elements \(c \in S^2 \mathfrak{g}\) and \(c \in \Lambda^3 \mathfrak{g}\) are related by

\[
(3.1) \quad c = -[c_{12}, c_{23}] ,
\]

where \(c_{12} = c \otimes 1\) and \(c_{23} = 1 \otimes c\).

Proof. Prove (a) first. By definition \(c = c_{(1)} \otimes c_{(1)} \otimes [c_{(2)}, c_{(2)}] = [c_{13}, c_{23}]\), where \(c = c_{(1)} \otimes c_{(2)}\). Note that \([c_{(2)}, c_{(2)}] \neq 0\), despite Sweedler’s notation being very suggestive.

Since \(c\) is \(\mathfrak{g}\)-invariant, it is easy to see that \(c\) is \(\mathfrak{g}\)-invariant, as well. We have to prove that \(c\) is anti-symmetric. We will show first that \(c\) indeed anti-commutes with the permutation \(\sigma_{13}\); i.e., \([c_{13}, c_{12}] = -[c_{13}, c_{23}]\). Since \(c\) is \(\mathfrak{g}\)-invariant we have

\[
[c, g \otimes 1] = -[c, 1 \otimes g]
\]

for all \(g \in \mathfrak{g}\). Now let \(c = c_{(1)} \otimes c_{(2)}\). We obtain,

\[
[c_{13}, c_{12}] = [c_{(1)}, c_{(1)}] \otimes c_{(2)} \otimes c_{(2)} = -c_{(1)} \otimes c_{(1)} \otimes [c_{(2)}, c_{(2)}] = -[c_{13}, c_{23}] .
\]

We can show analogously that \(c\) anti-commutes with \(\sigma_{23}\), as well. Part (a) is proved and (b) follows immediately. The lemma is proved.

Note that \(c\) defines for each finite-dimensional \(\mathfrak{g}\)-module \(V\) a \(\mathfrak{g}\)-module homomorphism \(c : \Lambda^3 V \to S^3 V\). We make the following definition, and then explain, how it is connected to the Poisson decorated spaces introduced in Section 2.

Definition 3.2. Let \((\mathfrak{g}, (\cdot, \cdot))\) be a Lie algebra with a symmetric invariant bilinear form. We say that a finite-dimensional \(\mathfrak{g}\)-module \(V\) is Poisson, if

\[
c(\Lambda^3 V) = \{0\} \in S^3 V .
\]

We immediately obtain the following sufficient condition guaranteeing that a \(\mathfrak{g}\)-module \(V\) is Poisson.

Proposition 3.3. Let \(V\) be a finite-dimensional \(\mathfrak{g}\)-module. If \(\text{Hom}_\mathfrak{g}(\Lambda^3 V, S^3 V) = \{0\}\), then \(V\) is Poisson.

We make the following definition.

Definition 3.4. (a) Let \(\mathfrak{g}\) be a complex semisimple Lie algebra. An element \(r \in \mathfrak{g} \otimes \mathfrak{g}\) is called a classical \(r\)-matrix if \(r\) satisfies the (i) the classical Yang-Baxter-equation, i.e.

\[
[r, r] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0
\]

(ii) and \(r + \tau(r) = 2c\), where \(c\) is the Casimir element of \(\mathfrak{g}\).
Example 3.5. Let \( g = sl_2(\mathbb{C}) \). The standard \( r \)-matrix is \( r = E \otimes F \), where \( E, F, H \) are the elements of the standard basis of \( sl_2(\mathbb{C}) \).

The classical \( r \)-matrices have been classified in the celebrated paper [2] in terms of Belavin-Drinfeld triples.

Consider a classical \( r \)-matrix \( r \) and its antisymmetric \( r \)-matrix \( r^- = \frac{1}{2}(r-\tau(r)) \), and a finite-dimensional \( g \)-module \( V \). The element \( r^- \in g \otimes g \) acts on \( V \otimes V \) and the corresponding decorated space \( (V, r^-) \) defines a bracket on the symmetric \( g \)-module algebra \( S(V) \) defined as \( \{u, v\}_{r^-} = r^-(u \wedge v) \) on all \( u, v \in S(V) \) as constructed in Proposition 2.21. We have the following result.

Proposition 3.6. Let \( g \) be a complex semisimple Lie algebra, \( (\cdot, \cdot) \) the Killing form and \( r \) a classical \( r \)-matrix, and let \( V \) be a finite-dimensional \( g \)-module. The decorated space \( (V, r^-) \) is Poisson if and only if \( V \) is Poisson.

Proof. We have to prove that \((V, r^-)\) is Poisson, if and only if \( c(\Lambda^3 V) = \{0\} \in S^3 V \). We obtain from Proposition 2.28 that
\[
[r^-, r^-] = [r^+_{12}, r^+_{23}] = [c_{12}, c_{23}] = c.
\]
The assertion now follows from Theorem 2.21.

We note the following facts.

Lemma 3.7. Let \((g, (\cdot, \cdot))\) be a quadratic algebra and let \( V = V^g \) be a trivial \( g \)-module. Then \( V \) is Poisson.

Proof. Obvious.

Lemma 3.8. Let \((g_1, (\cdot, \cdot)_1)\) and \((g_2, (\cdot, \cdot)_2)\) be quadratic Lie algebras and let \( c_1 \in S^2 g_1 \) and \( c_2 \in S^2 g_2 \) be the elements corresponding to \((g_1, (\cdot, \cdot)_1)\) and \((g_2, (\cdot, \cdot)_2)\). Then \((g, (\cdot, \cdot))\) is a quadratic Lie algebra with
\[
(g_1 + g_2, g'_1 + g'_2) = (g_1, g'_1)_1 + (g_2, g'_2)_2
\]
for \( g_1, g'_1 \in g_1 \) and \( g_2, g'_2 \in g_2 \). Moreover, one has \( c = c_1 + c_2 \).

Proof. The assertion follows from the fact that the subalgebras \((g_1, 0) \in g\) and \((0, g_2) \in g\) commute.

The following technical result will be of particular importance for the classification of Poisson modules over a semisimple Lie algebra \( g \), as it allows to restrict to certain good subalgebras, such as Levi subalgebras (see Proposition 3.4).

Proposition 3.9. Let \((g, (\cdot, \cdot))\) be a quadratic Lie-algebra. Denote by \( c \in S^2(g) \) the \( g \)-invariant element corresponding to \((\cdot, \cdot)\). Let \( g_{sub} \subset g \) be a subalgebra such that \( g_{sub} \cap g_{sub} = \{0\} \). Denote by \( c_{sub} \in S^2(g_{sub}) \) the \( g \)-invariant element corresponding to \((\cdot, \cdot)_{g_{sub}}\).

(a) One has \( c = c_{sub} + c'' \), where \( c'' \in g_{sub}^+ \wedge g \wedge g \) in the notation of Appendix [X].

(b) Let \( V \) be a \( g \)-module and \( V_1 \subset V\) a \( g_{sub} \)-module such that \( g_{sub}^+ (V_1) \cap V_1 = \{0\} \). If the canonical element \( c \in \Lambda^3 g \) defined in [X.1] satisfies \( c(\Lambda^3 V) = \{0\} \), then the element \( c_{sub} \in \Lambda^3 g_{sub} \) satisfies \( c_{sub}(\Lambda^3 V_1) = \{0\} \).
Proof. Prove (a) first. By definition we can express the element $c \in g \otimes g$ corresponding to $(\cdot, \cdot)$ as $c = c_{\text{sub}} + c_{\text{rest}}$, where $c_{\text{sub}} \in g_{\text{sub}} \otimes g_{\text{sub}}$ and $c_{\text{rest}} \in g_{\text{sub}}^\perp \otimes g + g \otimes g_{\text{sub}}^\perp$. We obtain from (3.1) that

$$c = [(c_{\text{sub}} + c_{\text{rest}})_{12}, (c_{\text{sub}} + c_{\text{rest}})_{23}] = c_{\text{sub}} + c'' ,$$

where $c'' \in g_{\text{sub}}^\perp \wedge g \wedge g$. Part (a) is proved.

Prove (b) now. Recall that $S^3 V \cong \bigoplus_{i=0}^{3} S^i V_1 \otimes S^{3-i} V_2$. One has $c_{\text{sub}}(\Lambda^3 V_1) \subset S^3 V_1$. Clearly,

$$c''(V_1 \otimes V_1 \otimes V_1) \subset (S^3 V_1)^c ,$$

in the notation of Appendix 8. If $V$ is Poisson, then $c(\Lambda^3 V_1) = \{0\}$, and hence $c_{\text{sub}}(\Lambda^3 V_1) = \{0\}$ and $c''(\Lambda^3 V_1) = \{0\}$. This implies directly that $V_1$ is Poisson as a $g_{\text{sub}}$-module.

Proposition 3.10 is proved. \qed

If $g$ is a reductive Lie algebra we have the following fact.

**Proposition 3.10.** Let $g$ be a reductive Lie algebra and $\kappa(x, y) = \text{tr}(\text{ad}(x)\text{ad}(y))$. The Lie algebra $g$ splits as $g = g' \oplus \mathfrak{z}$ into a semisimple part $g'$ and a central subalgebra $\mathfrak{z}$. A finite-dimensional $(g, \kappa)$-module $V$ is Poisson, if and only if $V$ is Poisson under the restriction to $(g', \kappa_{g'})$, where $\kappa_{g'}$ denotes the restriction of $\kappa$ to $g'$, the Killing-form.

**Proof.** Since $\mathfrak{z} V = \{0\}$ we obtain that $V$ is Poisson only if $V$ is Poisson as a $g'$-module by applying Proposition 3.9 (b) to $V = V'$. In order to prove the other direction note that since $g$ and $\mathfrak{z}$ commute we have $(g, \kappa) = (g', \kappa_{g'}) \oplus (\mathfrak{z}, \kappa_{\mathfrak{z}})$ and can apply Lemma 3.8 to obtain that $c = c_{g'} + c_{\mathfrak{z}}$. The Lie algebra $\mathfrak{z}$ is Abelian, and therefore, the form vanishes on $\mathfrak{z} \otimes \mathfrak{z}$ and $c_{\mathfrak{z}} = 0$. This implies that $c = c_{g'}$. The assertion now follows immediately. \qed

### 3.2. Classification of Poisson Modules over Semisimple Lie Algebras.

In this section we will classify all simple Poisson modules over a semisimple Lie algebra $g$. By Proposition 3.10 we immediately obtain a classification of all simple modules over reductive Lie algebras. First we will introduce some notation. Choose a Borel subalgebra $\mathfrak{b} \subset g$ and denote by $\mathfrak{h}$ and $\mathfrak{n}^+$ the corresponding Cartan and upper nilpotent subalgebras, and, similarly, by $\mathfrak{b}^-$ and $\mathfrak{n}^-$ the lower Borel and nilpotent subalgebras. By $W(g)$ we shall denote the Weyl group of $g$ and by $(\cdot, \cdot)_\mathfrak{b}$ and $(\cdot, \cdot)_{\mathfrak{h}}$ the standard inner product on $\mathfrak{h}$ and $\mathfrak{h}^*$, which we identify via the inner product. Denote by $R(g) \subset \mathfrak{h}^*$ the set of roots, by $R^+(g)$ (resp. $R^-(g)$) the set of positive (resp. negative) roots and by $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots. Denote by $E_\alpha$ for $\alpha \in R(g)$ and $H_\alpha \subset \mathfrak{h}$, $\alpha \in R^+(g)$ the standard generators of $g$ with the property that $[E_\alpha, E_{-\alpha}] = H_\alpha = \dot{\alpha} = 2 \frac{\alpha}{(\alpha, \alpha)} \in \mathfrak{h} \subset g$. We will also use the notation $P(g)$ for the weight-lattice of $g$ and $\omega_i$ for the $i$-th fundamental weight.

We now introduce the notion of *geometrically decomposable* modules following [18, ch.4]. Let $g$ be a reductive Lie algebra and $V$ a $g$-module and $U \subset V$ be a $\mathfrak{b}$-module. Denote by $\det(U)$ the one-dimensional subspace $\det(U) = \Lambda^0 \cap U$ of $\Lambda(U)$. Clearly, $\det(U)$ is a $\mathfrak{b}$-submodule of $\Lambda(V)$, therefore, every $u \in \det(U)$ is a highest weight vector in $\Lambda(V)$. In analogy to [18, ch. 4.6], we call a highest weight vector $v \in \Lambda(V)$ geometric, if $v \in \det(U)$ for some $\mathfrak{b}$-module $U \subset V$.

**Definition 3.11.** [18, ch. 4.6] A $g$-module $V$ is called geometrically decomposable, if $\Lambda V$ is generated as a $g$-module by geometric highest weight vectors.
The following result is the first main theorem of this section.

**Main Theorem 3.12.** Let \( g \) be a simple complex Lie algebra, and let \( V \) be a non-trivial simple finite-dimensional \( g \)-module. Then the following are equivalent:

(a) The module \( V \) is Poisson.

(b) The decorated space \( (V, r^-) \) is Poisson for any classical \( r \)-matrix \( r \in g \otimes g \).

(c) \( \text{Hom}_g(\Lambda^3 V, S^3 V) = \{0\} \).

(d) \( \Lambda^2 V \) is simple or \((g, V) = (sp_{2n}(\mathbb{C}), \mathbb{C}^{2n}) \) for some \( n \).

(e) The module \( V \) is a geometrically decomposable \( g \)-module or \((g, V) = (sp(2n), V_{\omega_1}) \).

(f) The pair \((g, V) \) is one of the following:

(i) \((sl_n(\mathbb{C}), V_\lambda) \) where \( \lambda \in \{\omega_1, 2\omega_1, \omega_2, \omega_{n-2}, \omega_{n-1}, 2\omega_{n-1}\} \).

(ii) \((so(n), V_{\omega_1}), (so(5), V_{\omega_{2}}), (so(8), V_{\omega_3}), (so(10), V_{\omega_4}) \) and \((so(10), V_{\omega_5}) \).

(iii) \((sp(2n), V_{\omega_1}) \) and \((sp(4), V_{\omega_2}) \).

(iv) \((E_6, V_{\omega_1}) \) and \((E_6, V_{\omega_2}) \).

We prove the theorem using the following strategy. The equivalence of (a) and (b) is proved in Proposition 3.6. The implication (c) implies (a) follows from Proposition 3.3. To prove that (a) yields (c) and (f) and, we will give necessary conditions for a dominant weight \( \lambda \in P^+(g) \) to have a \( g \)-module of highest weight \( (\lambda_1, \ldots, \lambda_n) \in P(g_1) \oplus \cdots \oplus P(g_n) \cong P(g) \). Denote by the support \( \text{supp}_g(V) \) of \( g \)-module \( V \) the product of all simple factors \( g_i \) for which \( g_i(V) \neq \{0\} \). We have the following classification result.

**Theorem 3.13.** Let \( g \) be a semisimple Lie algebra and \( V \) a simple \( g \)-module. The following are equivalent:

(a) \( V \) is Poisson.

(b) The pair \((\text{supp}_g(V), V) \) is listed in Theorem 3.12 (f) or \((\text{supp}_g(V), V) = (sl_m \times sl_n, V_{\omega_1, \omega_1}) \) where \( V_{\omega_1, \omega_1} \) is the natural \( sl_m \times sl_n \)-module.

**Proof.** Recall that the \( m \times n \)-matrices \( \text{Mat}_{m \times n}((\mathbb{C}) \) can be given a \( gl_m \times gl_n \)-module such that \( \text{Mat}_{m \times n}((\mathbb{C}) \cong V_{\omega_m, \omega_n} \cong V_{\omega_1, \omega_1}^* \) with \( gl_m \) acting on the left and \( gl_n \) acting on the right. This action yields a \( gl_m \times gl_n \)-module algebra structure on \( \mathbb{C}[\text{Mat}_{m \times n}] = S(V_{\omega_1, \omega_1}) \). It is well known that the \( r \)-matrix bracket defines a Poisson structure on the algebra \( \mathbb{C}[\text{Mat}_{m \times n}] = S(V_{\omega_1, \omega_1}) \) via

\[
 r^- (x_{ij} \otimes x_{kl}) = (\text{sign}(i-k) + \text{sign}(j-l)) x_{kj} x_{il} .
\]

It remains to show that if \( \text{supp}_g(V) \) is non-simple and \((\text{supp}_g(V), V) \neq (sl_m \times sl_n, V_{\omega_1, \omega_1}) \), then \( V \) is not Poisson.

Let \( r_1, \ldots, r_n \) be classical \( r \)-matrices for \( g_1, \ldots, g_n \). It is easy to see that \( r = r_1 + \cdots + r_n \) is a classical \( r \)-matrix for \( g \). Recall that as a vector space \( V \) can be decomposed as a tensor product \( V = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \), where \( V_{\lambda_i} \) is a simple \( g_i \)-module. The decorated space \((V, r^-) \) decomposes as a tensor product \((V, r^-) =\)
(V_{\lambda_1}, r_1) \otimes \ldots \otimes (V_{\lambda_n}, r_n). \) It now follows from Theorem 2.25 that if a simple \( \mathfrak{g} \)-module \( V \) is Poisson, then each \( V_{\lambda_i} \) is Poisson as a \( \mathfrak{g}_i \)-module as are all the products \( V_{\lambda_i} \otimes V_{\lambda_{i+1}} \), as \( \mathfrak{g}_i \otimes \mathfrak{g}_{i+1} \)-modules. It therefore suffices to show the following. First, let \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) be simple Lie algebras and let \( V_{\lambda_1} \) and \( V_{\lambda_2} \) be simple Poisson \( \mathfrak{g}_1 \)- (resp. \( \mathfrak{g}_2 \))-modules and and \((\mathfrak{g}_2, V_{\lambda_2}) \neq (\mathfrak{sl}_k, V_{\varpi_1})\) for some \( k \geq 2 \). Then \( V_{\lambda_1}, \lambda_2 \) is not Poisson. Second, we have to prove that the natural \( \mathfrak{sl}_\ell \times \mathfrak{sl}_m \times \mathfrak{sl}_n \)-module \( V_{\omega_1, \omega_2, \omega_3} \) is not Poisson for all \( \ell, m, n \geq 2 \).

We can further reduce the list of cases to investigate by considering the embedding of some Levi subalgebra in \( \mathfrak{g} \) and making use of Proposition 6.6. We need the following result.

**Proposition 3.14.** (a) If \( \mathfrak{g} = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \), then \( V = V_{2,1} \) is not Poisson, where \( V_i \) denotes the \( n + 1 \)-dimensional simple \( \mathfrak{sl}_2 \)-module.

(b) If \( \mathfrak{g} = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2 \), then \( V_{1,1,1} \) is not Poisson.

(c) If \( \mathfrak{g} = \mathfrak{sl}_2 \times \mathfrak{sp}(4) \), then \( V = V_{1,0} \) is not Poisson.

**Proof.** Let \( \mathfrak{g} = \mathfrak{g}_1 \otimes \mathfrak{g}_2 \) be a semisimple Lie algebra. We have \( c = c_1 + c_2 \), where \( c, c_1 \) and \( c_2 \) are the Casimir elements of \( \mathfrak{g}, \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \), respectively and \( c = c_1 + c_2 \).

We will first prove case (a). Let \( \mathfrak{g} = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \) and let \( V = V_{\lambda, 2i} \), \( i \geq 1 \), be a simple \( \mathfrak{g} \)-module. Denote by \( V_i \) and \( V_2 \) the corresponding simple \( \mathfrak{sl}_2 \)-modules. Since \( V \cong V_i \otimes V_2 \) as vector spaces we can choose non-zero vectors \((u \otimes v), (u' \otimes v') \in V_i \otimes V_2 \subset V \) such that \( u \in V_i(i), u' = F(u) \) and \( v \in V_2(2), v' = F(v) \) and \( v'' \in V_2(-2), \) where \( V_k(\ell) \) denotes the \( k \)-weight space of \( V_k \). Abbreviate \( uv = (u \otimes v) \wedge (u' \otimes v') \wedge (u \otimes v'') \in \Lambda^3 V \).

Note that if \( \mathfrak{g} = \mathfrak{sl}_2 \), then \( c_{\mathfrak{sl}_2} = E \wedge F \wedge H \) and \( c_{\mathfrak{sl}_2 \times \mathfrak{sl}_2} = E_1 \wedge F_1 \wedge H_1 + E_2 \wedge F_2 \wedge H_2 \). It is easy to verify that

\[
E_2 \wedge F_2 \wedge H_2 \cdot (uv) \in V(i, 2) \cdot V(i, -2) - V(i, 0) \cdot V(i, 0) \cdot V(i, -2) \subset S^3 V .
\]

Similarly we obtain that

\[
E_1 \wedge F_1 \wedge H_1 \cdot (uv) \in (V(i, 2) \cdot V(i, -2) - V(i, 0) \cdot V(i, 0) \cdot V(i, -2)) \cdot (uv) \subset S^3 V .
\]

Hence, \( c(uv) \neq 0 \) and \( V \) is not Poisson. Part (a) is proved.

Prove (b) next. Denote by \( V_{111} \) the 8-dimensional natural \( \mathfrak{sl}_2(\mathbb{C}) \)-module. Choose a basis of \( V_{1,1,1} \) with basis vectors \( x_{j_1,j_2,j_3}, j_i \in \{0,1\} \), such that \( x_{j_1,j_2,j_3} \) is a weight vector of weight \((1 - 2j_1)\) of each subalgebra \( \mathfrak{g}_i \), the \( i \)-th copy of \( \mathfrak{sl}_2(\mathbb{C}) \) in \( \mathfrak{g} \). Moreover, we can choose the basis such that \( F_1(x_{ijk}) = \delta_{i,0} x_{1,j,k} \) and \( E_1(x_{ijk}) = \delta_{i,1} x_{0,j,k} \), and analogously for \( E_2, F_2, E_3 \) and \( F_3 \).

We have \( c = \sum_{i=1}^3 -E_i \wedge H_i \wedge F_i \).

It is easy to compute that

\[
c(x_{111} \wedge x_{000} \wedge x_{100}) = -x_{111}x_{000} + x_{110}x_{100} - x_{111}x_{000} - x_{100}x_{110} + x_{110}x_{001}x_{100} - x_{110}x_{000}x_{100} \neq 0 .
\]

Therefore \( c(\Lambda^3 V_{111}) \neq \{0\} \), hence \( V_{111} \) is not Poisson. Part(b) is proved.

It remains to prove part (c). Let \( V_{\omega_1} \) be the four-dimensional natural \( \mathfrak{sp}(4) \)-module and let \( V_{1,\omega_1} \) be the natural \( \mathfrak{sl}_2 \times \mathfrak{sp}(4) \)-module. As in the proof of parts (a) and (b) choose \( u, u' \in V_1 \) such that \( u \in V_1(1) \) and \( u' \in V_1(-1) \) and \( v \in V_{\omega_1}(\omega_1) \), \( v' = F_\alpha(v) \) and \( v'' \in V_{\omega_1}(\omega_1) \). Denote \( uv = (u \otimes v) \wedge (u' \otimes v') \wedge (u \otimes v'') \in \Lambda^3 V \).

We have by [6.2] \( c = E \wedge F \wedge H + \sum_{\alpha, \beta \in R(\mathfrak{sp}(4))} \frac{[\alpha, \beta]}{4} E_\alpha \wedge E_\beta \wedge [E_{-\alpha}, E_{-\beta}] \).

We obtain that
\[ E_{\alpha_1} \wedge E_{-\alpha_1} \wedge H_{\alpha_1}(uv) \in V(-1, \omega_1) \cdot V(1, \omega_1 - \alpha_1) \cdot V(1, -\omega_1) \]
and observe that indeed
\[
c = \frac{(\alpha_1, \alpha_1)^2}{4} E_{\alpha_1} \wedge E_{-\alpha_1} \wedge H_{\alpha_1}(uv) \in (V(-1, \omega_1) \cdot V(1, \omega_1 - \alpha_1) \cdot V(1, -\omega_1))^c.
\]
This implies that \( c(uv) \neq 0 \) and that \( V_{1, \omega_1} \) is not Poisson. Part (c) and the proposition are proved.

We now return to the proof of Theorem 3.13. Now let \( g = g_1 \oplus g_2 \) such that \( g_1 \) and \( g_2 \) are two simple Lie algebras, and \( V_{\lambda_1} \) and \( V_{\lambda_2} \) simple Poisson \( g_1 \), respectively \( g_2 \)-modules and assume that \((g, V) \neq (sl_m \times sl_m, V_{\omega_1, \omega_1})\). We will list the semisimple part \( g' \subset g \) of the Levi subalgebra and the corresponding simple \( g' \)-module \( V' \subset V_{\lambda_1, \lambda_2} \) verifying that \( V_{\lambda_1, \lambda_2} \) is not Poisson. First, note the following fact.

**Proposition 3.15.** Let \( g \) be a semisimple Lie algebra and \( V \) a simple \( g \)-module such that \( \text{supp}_g(V) \) has at least three simple factors. Then \( V \) is not Poisson.

**Proof.**

Note the following fact.

**Lemma 3.16.** Let \( g = sl_2 \) and let \( V = V_{i,j,k} \) be a simple finite-dimensional \( g \)-module with \( 0 \notin \{i, j, k\} \). Then \( V \) is not Poisson.

**Proof.** Since \( V \) is not Poisson if \( \ell \geq 3 \) by Theorem 3.12 (f), we obtain from Theorem 2.25 that if \( V_{i,j,k} \) is Poisson, then \( i, j, k \leq 2 \). If \( i = j = k = 1 \), then the assertion of the lemma agrees with the assertion of Proposition 3.14 (b). Now suppose, without loss of generality, that \( j = 1 \). Then \( V_{i,j,k} \) is not Poisson by Proposition 3.14 (b) and \( V = V_{i,j,k} \) is not Poisson by Theorem 2.25. The lemma is proved.

Suppose \( \text{supp}_g(V) \) has at least three simple factors. We can find a Levi subalgebra \( g' \cong sl_2 \) such that a highest weight vector \( v \in V \) generates a \( g' \)-submodule \( V' \cong V_{i,j,k} \) with \( i, j, k \geq 1 \). Hence \( V \) is not Poisson by the previous lemma and Proposition 6.6. The proposition is proved.

Now we are able to complete the proof of Theorem 3.13. We assume that \( \text{supp}_g(V) \) has two simple factors. In order to deal with most cases, it suffices to exhibit a Levi subalgebra \( g' \subset g \) and a simple module \( V' \subset g' \) \( V \) such that \((g', V') \in \{(sl_2 \times sl_2, V_{i,2}), (sl_2 \times sp(4), V_{1,\omega_1})\}\) to show that \((V, g)\) is not Poisson by Proposition 6.6. Since the choice is obvious in a large number of cases, and a complete list would, therefore, be rather long, we will list only the non-obvious choices. All these special cases except for the first one require to us to consider Levi subalgebras with three simple factors.

(a) If \( g = so(2n+1) \oplus g_2 \) and \( V = V_{\omega_1, \lambda} \) choose \( g' = sl_2 \times sl_2 \) generated by the second node of the Dynkin diagram of \( so(2n+1) \), resp. a node \( i \) of the diagram associated to \( g_2 \) such that \((\lambda, \alpha_i) \geq 1 \). Note that \( E_{-\alpha_1}(uv_{\omega_1}) \in V_{\omega_1} \) generates a three-dimensional simple \( sl_2 \)-module for the subalgebra corresponding to the second node of the Dynkin diagram. Hence, we find a \( sl_2 \times sl_2 \)-submodule \( V' \cong V_{2,1} \subset V_{\omega_2, \lambda} \) and \( V \) is not Poisson by Proposition 3.14 (a).

(b) If \( g = sl_n \times g_2 \), \( n \geq 4 \) and \( V = V_{\omega_2, \lambda} \), (resp. \( V_{\omega_{n-2}, \lambda} \)) choose \( g' = (sl_2 \times sl_2) \times g_2 \) generated by the first and third nodes of the Dynkin diagram \( A_{n-1} \) (resp. the last and third to last nodes) and \( g_2 \). Let \( uv \) be a highest weight vector in \( V_{\omega_2} \). Note
that $E_{-\alpha_i}(v_{\omega_j}) \in V_{\omega_j}$ generates a four-dimensional simple $sl_2 \times sl_2$-module $V_{1,1}$.

Hence, we find a $sl_2 \times sl_2 \times g_2$ submodule $V' \cong V_{1,1}\lambda \subset V_{\omega_2}$, and $V$ is not Poisson by Proposition 3.15. Similarly we obtain that $V_{\omega_{n-2},\lambda}$ is not Poisson.

(c) If $g = so(2n) \oplus g_2$, and $V = V_{\omega_1,\lambda}$, consider the Levi subalgebra of $so(2n)$, isomorphic to $sl_4$, generated by the $(n-2)$nd, $(n-1)$st and $n$th nodes of the Dynkin diagram $D_n$. It can be easily observed that if $v \in V_{\omega_1}(\omega_1)$ is a highest weight vector, then $v' = E_{\alpha_{n-2}} \circ \ldots \circ E_{\alpha_1}(v)$ generates a simple $sl_4$-module $V_{\omega_2}$. We obtain that $V$ is not Poisson by applying the argument in case (b).

(d) If $g = so(8) \oplus g_2$ and $V = V_{\omega_i,\lambda}$, $i = 3,4$ or $g = so(10) \oplus g_2$ and $V = V_{\omega_i,\lambda}$, $i = 4,5$ we argue analogous to case (c).

(e) If $g = E_6 \oplus g_2$ and $V = V_{\omega_1,\lambda}$ consider the Levi subalgebra $sl_4 \subset E_6$ generated by the second, third and fourth nodes of the Dynkin diagram $E_6$ (in the notation of [7]). If $v \in V_{\omega_1}(\omega_1)$ is a highest weight vector, then $v' = E_{\alpha_3} \circ E_{\alpha_2}(v)$ generates a simple $sl_4$-module $V_{\omega_2}$. We obtain that $V$ is not Poisson by applying the argument in case (b).

The proof of Theorem 4.13 is now complete. □

4. Quantum Symmetric Algebras

4.1. The Quantum Group $U_q(g)$ and its Modules. We start with the definition of the quantized enveloping algebra associated with a complex reductive Lie algebra $g$ (our standard reference here will be [3]). Let $h \subset g$ be a Cartan subalgebra, $P(g)$ the weight lattice, as introduced above, and let $A = (a_{ij})$ be the Cartan matrix for $g$. Additionally, let $(\cdot, \cdot)$ be the standard non-degenerate symmetric bilinear form on $h$.

The quantized enveloping algebra $U$ is a $\mathbb{C}(q)$-algebra generated by the elements $E_i$ and $F_i$, for $i \in [1,r]$, and $K_\lambda$ for $\lambda \in P(g)$, subject to the following relations:

$$
E_i F_j - F_j E_i = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q^{d_i} - q^{-d_i}}
$$

for $i, j \in [1,r]$, where $d_i = \frac{\langle a_i, a_n \rangle}{2}$; and the quantum Serre relations

$$
\sum_{p=0}^{1-a_{ij}} (-1)^p E_i^{(1-a_{ij}-p)} E_j E_i^{(p)} = 0, \quad \sum_{p=0}^{1-a_{ij}} (-1)^p F_i^{(1-a_{ij}-p)} F_j F_i^{(p)} = 0
$$

for $i \neq j$, where the notation $X_i^{(p)}$ stands for the divided power

$$
X_i^{(p)} = \frac{X_i^p}{(1_i \cdots _i)} = \frac{q^{kd_i} - q^{-kd_i}}{q^{a_i} - q^{-a_i}}.
$$

The algebra $U$ is a $q$-deformation of the universal enveloping algebra of the reductive Lie algebra $g$, so it is commonly denoted by $U = U_q(g)$. It has a natural structure of a bialgebra with the co-multiplication $\Delta : U \rightarrow U \otimes U$ and the co-unit homomorphism $\varepsilon : U \rightarrow \mathbb{Q}(q)$ given by

$$
\Delta(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_{-\alpha_i} + 1 \otimes F_i, \quad \Delta(K_\lambda) = K_\lambda \otimes K_\lambda,
$$

$$
\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_\lambda) = 1.
$$
In fact, $U$ is a Hopf algebra with the antipode anti-homomorphism $S : U \to U$ given by

\begin{equation}
S(E_i) = -K_{-\alpha_i} E_i, \quad S(F_i) = -F_i K_{\alpha_i}, \quad S(K_\lambda) = K_{-\lambda}.
\end{equation}

Let $U^-$ (resp. $U^0$, $U^+$) be the $\mathbb{Q}(q)$-subalgebra of $U$ generated by $F_1, \ldots, F_r$ (resp. by $K_\lambda (\lambda \in P)$; by $E_1, \ldots, E_r$). It is well-known that $U = U^- \cdot U^0 \cdot U^+$ (more precisely, the multiplication map induces an isomorphism $U^- \otimes U^0 \otimes U^+ \to U$).

We will consider the full sub-category $\mathcal{O}_f$ of the category $U_q(\mathfrak{g}) - \text{Mod}$. The objects of $\mathcal{O}_f$ are finite-dimensional $U_q(\mathfrak{g})$-modules $V^q$ having a weight decomposition

\[ V^q = \bigoplus_{\mu \in P} V^q(\mu), \]

where each $K_\lambda$ acts on each weight space $V^q(\mu)$ by the multiplication with $q^{(\lambda | \mu)}$ (see e.g., [8, I.6.12]). The category $\mathcal{O}_f$ is semisimple and the irreducible objects $V^q_\lambda$ are generated by highest weight spaces $V^q_\lambda(\lambda) = \mathbb{C}(q) \cdot v_\lambda$, where $\lambda$ is a dominant weight, i.e., $\lambda$ belongs to $P^+ = \{ \lambda \in P : (\lambda | \alpha_i) \geq 0 \ \forall \ i \in [1, r]\}$, the monoid of dominant weights.

By definition, the universal $R$-matrix $R \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) R$ has can be decomposed as

\begin{equation}
R = R_0 R_1 = R_1 R_0
\end{equation}

where $R_0$ is "the diagonal part" of $R$, and $R_1$ is unipotent, i.e., $R_1$ is a formal power series

\begin{equation}
R_1 = 1 \otimes 1 + (q - 1)x_1 + (q - 1)^2 x_2 + \cdots,
\end{equation}

where all $x_k \in U^+_k \otimes_{\mathbb{C}[q, q^{-1}]} U^+_k$, where $U^+_+$ (resp. $U^+\cdot$) is the integral form of $U^+$, i.e., $U^+_+$ is a $\mathbb{C}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by all $F_i$ (resp. by all $E_i$) and $U^+_+$ is the $k$-th graded component under the grading $\text{deg}(F_i) = 1$ (resp. $\text{deg}(E_i) = 1$).

By definition, for any $U^q, V^q$ in $\mathcal{O}_f$ and any highest weights elements $u_\lambda \in U^q(\lambda)$, $v_\mu \in V^q(\mu)$ we have $R_0(u_\lambda \otimes v_\mu) = q^{(\lambda | \mu)} u_\lambda \otimes v_\mu$.

Let $R^{\text{op}}$ be the opposite element of $R$, i.e., $R^{\text{op}} = \tau(R)$, where $\tau : U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is the permutation of factors. Clearly, $R^{\text{op}} = R_0 R_1^{\text{op}} = R_1^{\text{op}} R_0$.

Following [13, Section 3], define $D \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ by

\begin{equation}
D := R_0 \sqrt{R_1^{\text{op}} R_1} = \sqrt{R_1^{\text{op}} R_1} R_0.
\end{equation}

Clearly, $D$ is well-defined because $R_1^{\text{op}} R_1$ is also unipotent as well as its square root. By definition, $D^2 = R^{\text{op}} R$, $D^{\text{op}} R = R D$.

Furthermore, define

\begin{equation}
\hat{R} := R^{\text{op}} R = (D^{\text{op}})^{-1} R = R_1 \left( \sqrt{R_1^{\text{op}} R_1} \right)^{-1}
\end{equation}

It is easy to see that

\begin{equation}
\hat{R}^{\text{op}} = \hat{R}^{-1}
\end{equation}

According to [13, Proposition 3.3], the pair $(U_q(\mathfrak{g}), \hat{R})$ is a coboundary Hopf algebra.

The braiding in the category $\mathcal{O}_f$ is defined by $\mathcal{R}_{U_q^q, V_q} : U^q \otimes V^q \to V^q \otimes U^q$, where

\[ \mathcal{R}_{U_q^q, V_q}(u \otimes v) = \tau R(u \otimes v) \]
for any \( u, v \in U^q \), \( \tau : U^q \otimes V^q \to V^q \otimes U^q \) is the ordinary permutation of factors.

Denote by \( C \in Z(U_q(\mathfrak{g})) \) the quantum Casimir element which acts on any irreducible \( U_q(\mathfrak{g}) \)-module \( V^q \) in \( \mathcal{O}_f \) by the scalar multiple \( q^{(\lambda | \lambda + 2\rho)} \), where \( 2\rho \) is the sum of positive roots.

The following fact is well-known.

**Lemma 4.1.** One has \( \mathcal{R}^2 = \Delta(C^{-1}) \circ (C \otimes C) \). In particular, for each \( \lambda, \mu, \nu \in \mathbb{P} \), the restriction of \( \mathcal{R}^2 \) to the \( \nu \)-th isotypic component \( I^\nu_{\lambda, \mu} \) of the tensor product \( V^q_\lambda \otimes V^q_\mu \) is scalar multiplication by \( q^{(\lambda | \lambda + (\mu | \mu) - (\nu | \nu)) + (2\rho | \lambda + \mu - \nu)} \).

This allows to define the diagonalizable \( \mathbb{C}(q) \)-linear map \( D_{U^q, V^q} : U^q \otimes V^q \to U^q \otimes V^q \) by \( D_{U^q, V^q}(u \otimes v) = D(u \otimes v) \) for any objects \( U^q \) and \( V^q \) of \( \mathcal{O}_f \). It is easy to see that the operator \( D_{V^q_\lambda, V^q_\mu} : V^q_\lambda \otimes V^q_\mu \to V^q_\lambda \otimes V^q_\mu \) acts on the \( \nu \)-th isotypic component \( I^\nu_{\lambda, \mu} \) in \( V^q_\lambda \otimes V^q_\mu \) by the scalar multiplication with \( q^{(\lambda | \lambda + (\mu | \mu) - (\nu | \nu)) + (\mu | \lambda + \mu - \nu)} \).

For any \( U^q \) and \( V^q \) in \( \mathcal{O}_f \) define the normalized braiding \( \sigma_{U^q, V^q} \) by

\[
\sigma_{U^q, V^q}(u \otimes v) = \tau \hat{R}(u \otimes v),
\]

Therefore, we have by \ref{eq:4.10}:

\[
\sigma_{U^q, V^q} = D_{U^q, V^q}^{-1} \mathcal{R}_{U^q, V^q} = \mathcal{R}_{U^q, V^q} D_{U^q, V^q}^{-1}.
\]

We will sometimes write \( \sigma_{U^q, V^q} \) in a more explicit way:

\[
\sigma_{U^q, V^q} = \sqrt{\mathcal{R}_{V^q, U^q}^{-1} \mathcal{R}_{U^q, V^q}^{-1} \mathcal{R}_{U^q, V^q} \mathcal{R}_{V^q, U^q}} = \mathcal{R}_{U^q, V^q} \sqrt{\mathcal{R}_{V^q, U^q}^{-1} \mathcal{R}_{U^q, V^q}^{-1} \mathcal{R}_{V^q, U^q}}.
\]

The following fact is an obvious corollary of \ref{eq:4.11}.

**Lemma 4.2.** \( \sigma_{V^q, U^q} \circ \sigma_{U^q, V^q} = id_{U^q \otimes V^q} \) for any \( U^q, V^q \) in \( \mathcal{O}_f \). That is, \( \sigma \) is a symmetric commutativity constraint.

We also have the following coboundary relation (even though we will not use it).

**Lemma 4.3.** [13, section 3] Let \( A^q, B^q, C^q \) be objects of \( \mathcal{O}_f \). Then, the following diagram commutes:

\[
\begin{array}{ccc}
A^q \otimes B^q \otimes C^q & \xrightarrow{\sigma_{12,3}} & C^q \otimes A^q \otimes B^q \\
\downarrow{\sigma_{1,23}} & & \downarrow{\sigma_{23}} \\
B^q \otimes C^q \otimes A^q & \xrightarrow{\sigma_{12}} & C^q \otimes B^q \otimes A^q
\end{array}
\]

where we abbreviated

\[
\sigma_{12,3} := \sigma_{A^q \otimes B^q, C^q} : (A^q \otimes B^q) \otimes C^q \to C^q \otimes (A^q \otimes B^q),
\]

\[
\sigma_{1,23} := \sigma_{A^q, B^q \otimes C^q} : A^q \otimes (B^q \otimes C^q) \to (B^q \otimes C^q) \otimes A^q.
\]

**Remark 4.4.** If one replaces the braiding \( \mathcal{R} \) of \( \mathcal{O}_f \) by its inverse \( \mathcal{R}^{-1} \), the symmetric commutativity constraint \( \sigma \) will not change.
4.2. Braided Symmetric and Exterior Powers. In this section we will use the notation and conventions of Section 4.1.

For any morphism \( f : V^q \otimes V^q \to V^q \otimes V^q \) in \( \mathcal{O}_f \) and \( n > 1 \) we denote by \( f^{i,i+1}, i = 1, 2, \ldots, n - 1 \) the morphism \( V^q \otimes \cdots \otimes V^q \to V^q \otimes \cdots \otimes V^q \) which acts as \( f \) on the \( i \)-th and the \( i + 1 \)-st factors. Note that \( \sigma^{i,i+1} \) is always an involution on \( V^q \otimes \cdots \otimes V^q \).

**Definition 4.5.** For an object \( V^q \) in \( \mathcal{O}_f \) and \( n \geq 0 \) define the braided symmetric power \( S^q_n V^q \subset V^q \otimes \cdots \otimes V^q \) and the braided exterior power \( \Lambda^q_n V^q \subset V^q \otimes \cdots \otimes V^q \) by:

\[
S^q_n V^q = \bigcap_{1 \leq i \leq n-1} \langle \text{Ker } \sigma_{i,i+1} - \text{id} \rangle = \bigcap_{1 \leq i \leq n-1} \langle \text{Im } \sigma_{i,i+1} + \text{id} \rangle,
\]

\[
\Lambda^q_n V^q = \bigcap_{1 \leq i \leq n-1} \langle \text{Ker } \sigma_{i,i+1} + \text{id} \rangle = \bigcap_{1 \leq i \leq n-1} \langle \text{Im } \sigma_{i,i+1} - \text{id} \rangle,
\]

where we abbreviated \( \sigma^{i,i+1} = \sigma^{i+1,i} \).

**Remark 4.6.** Clearly, \( -R \) is also a braiding on \( \mathcal{O}_f \) and \( -\sigma \) is the corresponding normalized braiding. Therefore, \( \Lambda^q_n V^q = S^q_{-\sigma} V^q \) and \( S^q_n V^q = \Lambda^q_{-\sigma} V^q \). That is, informally speaking, the symmetric and exterior powers are mutually "interchangeable".

**Remark 4.7.** Another way to introduce the symmetric and exterior squares involves the well-known fact that the braiding \( R_{V^q,V^q} \) is a semisimple operator \( V^q \otimes V^q \to V^q \otimes V^q \), and all the eigenvalues of \( R_{V^q,V^q} \) are of the form \( \pm q^r \), where \( r \in \mathbb{Z} \). Then positive eigenvectors of \( R_{V^q,V^q} \) span \( S^q_2 V^q \) and negative eigenvectors span \( \Lambda^q_2 V^q \). Clearly, \( S^q_0 V^q = \mathbb{C} \langle q \rangle \), \( S^q_1 V^q = V^q \), \( S^q_{-1} V^q = \mathbb{C} \langle q \rangle \), \( \Lambda^q_0 V^q = V^q \), and \( S^q_2 V^q = \{ v \in V^q \otimes V^q \mid \sigma_{V^q,V^q}(v) = v \} \), \( \Lambda^q_2 V^q = \{ v \in V^q \otimes V^q \mid \sigma_{V^q,V^q}(v) = -v \} \).

The following fact is obvious.

**Proposition 4.8.** For each \( n \geq 0 \) the association \( V^q \mapsto S^q_n V^q \) is a functor from \( \mathcal{O}_f \) to \( \mathcal{O}_f \) and the association \( V^q \mapsto \Lambda^q_n V^q \) is a functor from \( \mathcal{O}_f \) to \( \mathcal{O}_f \). In particular, an embedding \( U^q \hookrightarrow V^q \) in the category \( \mathcal{O}_f \) induces injective morphisms

\[ S^q_i U^q \hookrightarrow S^q_n V^q, \quad \Lambda^q_i U^q \hookrightarrow \Lambda^q_n V^q. \]

**Definition 4.9.** For any \( V^q \in \text{Ob}(\mathcal{O}) \) define the braided symmetric algebra \( S_\sigma(V^q) \) and the braided exterior algebra \( \Lambda_\sigma(V^q) \) by:

\[
S_\sigma(V^q) = T(V^q)/\langle \Lambda^q_2 V^q \rangle, \quad \Lambda_\sigma(V^q) = T(V^q)/\langle S^q_2 V^q \rangle,
\]

where \( T(V^q) \) is the tensor algebra of \( V^q \) and \( \langle I \rangle \) stands for the two-sided ideal in \( T(V^q) \) generated by a subset \( I \subset T(V^q) \).

Note that the algebras \( S_\sigma(V^q) \) and \( \Lambda_\sigma(V^q) \) carry a natural \( \mathbb{Z}_{\geq 0} \)-grading:

\[
S_\sigma(V^q) = \bigoplus_{n \geq 0} S_\sigma(V^q)_n, \quad \Lambda_\sigma(V^q) = \bigoplus_{n \geq 0} \Lambda_\sigma(V^q)_n,
\]

since the respective ideals in \( T(V^q) \) are homogeneous.

Denote by \( \mathcal{O}_{gr,f} \) the sub-category of \( U_q(\mathfrak{g}) - \text{Mod} \) whose objects are \( \mathbb{Z}_{\geq 0} \)-graded:

\[ V^q = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V^q_n, \]
where each $V^n_q$ is an object of $\mathcal{O}_f$; and morphisms are those homomorphisms of $U_q(\mathfrak{g})$-modules which preserve the $\mathbb{Z}_{\geq 0}$-grading.

Clearly, $\mathcal{O}_{gr,f}$ is a tensor category under the natural extension of the tensor structure of $\mathcal{O}_f$. Therefore, we can speak of algebras and co-algebras in $\mathcal{O}_{gr,f}$.

By the very definition, $S_\sigma(V^q)$ and $\Lambda_\sigma(V^q)$ are algebras in $\mathcal{O}_{gr,f}$.

**Proposition 4.10.** The assignments $V^q \mapsto S_\sigma(V^q)$ and $V^q \mapsto \Lambda_\sigma(V^q)$ define functors from $\mathcal{O}_f$ to the category of algebras in $\mathcal{O}_{gr,f}$.

We conclude the section with two important features of braided symmetric exterior powers and algebras.

**Proposition 4.11.** [6] Prop.2.11 and Eq. 2.3] Let $V^q$ be an object of $\mathcal{O}_f$ and $V^*$ its dual in $\mathcal{O}_f$. We have the following $U_q(\mathfrak{g})$-module isomorphisms.

$$(4.17) \quad (S^n_qV^{q,*})^* \cong S^n_q(V^q), \quad (\Lambda^n_qV^{q,*})^* \cong \Lambda^n_q(V^q).$$

**Proposition 4.12.** [6] Prop.2.13] For any $V^q$ in $\mathcal{O}_f$ each embedding $V^n_q \hookrightarrow V^q$ defines embeddings $V^n_{q\lambda} \hookrightarrow S^n_q V^q$ for all $n \geq 2$. In particular, the algebra $S_\sigma(V^q)$ is infinite-dimensional.

### 4.3. The Classical Limit of Braided Algebras

In this section we will discuss the specialization of the braided symmetric and exterior algebras at $q = 1$, the classical limit. All of the results in this section are either well known or proved in [6]. For a more detailed discussion of the classical limit we refer the reader to [6] Section 3.2.

We will first introduce the notion of an almost equivalence of categories:

**Definition 4.13.** We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an almost equivalence of $\mathcal{C}$ and $\mathcal{D}$ if:

(a) for any objects $c, c'$ of $\mathcal{C}$ an isomorphism $F(c) \cong F(c')$ in $\mathcal{D}$ implies that $c \cong c'$ in $\mathcal{C}$;

(b) for any object in $\mathcal{D}$ there exists an object in $\mathcal{C}$ such that $F(c) \cong d$ in $\mathcal{D}$.

Denote by $\overline{\mathcal{O}}_f$ the full (tensor) sub-category of $U(\mathfrak{g}) - Mod$, whose objects $\overline{V}$ are finite-dimensional $U(\mathfrak{g})$-modules having a weight decomposition $\overline{V} = \oplus_{\mu \in \mathcal{P}} \overline{V}(\mu)$. The following fact will be the first result of this section.

**Proposition 4.14.** [6] Cor.3.22] The categories $\mathcal{O}_f$ and $\overline{\mathcal{O}}_f$ are almost equivalent. Under this almost equivalence a simple $U_q(\mathfrak{g})$-module $V_\lambda$ is mapped to the simple $U(\mathfrak{g})$-module $\overline{V}_\lambda$.

Let $V \cong \bigoplus_{i=1}^{n} V_{\lambda_i} \in \mathcal{O}_f$. We call $\overline{V} \cong \bigoplus_{i=1}^{n} \overline{V}_{\lambda_i} \in \overline{\mathcal{O}}_f$ the classical limit of $V$ under the above almost equivalence.

**Proof.** First, we have to introduce the notion of $(k, A)$-algebras and investigate their properties. Let $k$ be a field and $A$ be a local subring of $k$. Denote by $m$ the only maximal ideal in $A$ and by $k$ the residue field of $A$, i.e., $k := A/m$.

We say that an $A$-submodule $L$ of a $k$-vector space $V$ is an $A$-lattice of $V$ if $L$ is a free $A$-module and $k \otimes_A L = V$, i.e., $L$ spans $V$ as a $k$-vector space. Note that for any $k$-vector space $V$ and any $k$-linear basis $B$ of $V$ the $A$-span $L = A \cdot B$ is an $A$-lattice in $V$. Conversely, if $L$ is an $A$-lattice in $V$, then any $A$-linear basis $B$ of $L$ is also a $k$-linear basis of $V$. 
Denote by \((k, A) - Mod\) the category whose objects are pairs \(V = (V, L)\) of a \(k\)-vector space \(V\) and an \(A\)-lattice \(L \subset V\) of \(V\); an arrow \((V, L) \to (V', L')\) is any \(k\)-linear map \(f : V \to V'\) such that \(f(L) \subset L'\).

Clearly, \((k, A) - Mod\) is an Abelian category. Moreover, \((k, A) - Mod\) is \(A\)-linear because each \(\text{Hom}(U, V)\) in \((k, A) - Mod\) is an \(A\)-module.

It can be easily verified that \((k, A) - Mod\) is a symmetric tensor category \([6, \text{Lemma 3.14}]\). We have the following fact.

**Lemma 4.15.** \([6, \text{Lemma 3.12}]\) The forgetful functor \((k, A) - Mod \to \hat{k} - Mod\) given by \((V, L) \mapsto V\) is an almost equivalence of symmetric tensor categories.

Define a functor \(\mathcal{F} : (k, A) - Mod \to \hat{k} - Mod\) by:

\[
\mathcal{F}(V, L) = L/mL
\]

for any object \((V, L)\) of \((k, A) - Mod\) and for any morphism \(f : (V, L) \to (V', L')\) we define \(\mathcal{F}(f) : L/mL \to L'/mL'\) to be a natural \(k\)-linear map.

**Lemma 4.16.** \([6, \text{Lemma 3.14}]\) \(\mathcal{F} : (k, A) - Mod \to \hat{k} - Mod\) is a tensor functor and almost equivalence.

Let \(U\) be a \(k\)-Hopf algebra and let \(U_A\) be a Hopf \(A\)-subalgebra of \(U\). This means that \(\Delta(U_A) \subset U_A \otimes_A U_A\) (where \(U_A \otimes_A U_A\) is naturally an \(A\)-sub-algebra of \(U \otimes_k U\), \(\varepsilon(U_A) \subset A\), and \(S(U_A) \subset U_A\). We will refer to the above pair \(U = (U, U_A)\) as to \((k, A)\)-Hopf algebra (please note that \(U_A\) is not necessarily a free \(A\)-module, that is, \(U\) is not necessarily a \((k, A)\)-module).

Given \((k, A)\)-Hopf algebra \(U = (U, U_A)\), we say that an object \(V = (V, L)\) of \((k, A) - Mod\) is a \(U\)-module if \(V\) is a \(U\)-module and \(L\) is an \(U_A\)-module.

Denote by \(U - Mod\) the category which objects are \(U\)-modules and arrows are those morphisms of \((k, A)\)-modules which commute with the \(U\)-action.

Clearly, for \((k, A)\)-Hopf algebra \(U = (U, U_A)\) the category \(U - Mod\) is a tensor (but not necessarily symmetric) category.

For each \((k, A)\)-Hopf algebra \(U = (U, U_A)\) we define \(U^\prime := U_A/mU_A\). Clearly, \(U^\prime\) is a Hopf algebra over \(\hat{k} = A/m\).

The following fact is obvious.

**Lemma 4.17.** \([6, \text{Lemma 3.15}]\) In the notation of Lemma 4.16, for any \((k, A)\)-Hopf algebra \(U\) the functor \(\mathcal{F}\) naturally extends to a tensor functor

\[
\mathcal{U} : U - Mod \to \mathcal{U}^\prime - Mod .
\]

Now let \(k = \mathbb{C}(q)\) and \(A\) be the ring of all those rational functions in \(q\) which are defined at \(q = 1\). Clearly, \(A\) is a local PID with maximal ideal \(m = (q - 1)A\) (and, moreover, each ideal in \(A\) is of the form \(m^n = (q - 1)^nA\)). Therefore, \(k := A/m = \mathbb{C}\).

Recall from Section 4.1.1 the definition of the quantized universal enveloping algebra \(U_q(g)\). Denote \(h_\lambda = \frac{k_{\lambda - 1}}{q - 1}\) and let \(U_A(g)\) be the \(A\)-algebra generated by all \(h_\lambda, \lambda \in P\) and all \(E_i, F_i\).

Denote by \(U_q(g)\) the pair \((U_q(g), U_A(g))\).

**Lemma 4.18.** \((a)\) The pair \(U_q(g) = (U_q(g), U_A(g))\) is a \((k, A)\)-Hopf algebra \([6, \text{Lemma 3.16}]\).

\((b)\) We have \(\mathcal{U}(g) = U(g)\) \([6, \text{Lemma 3.17}]\).

Let \(V_\lambda \in \text{Ob}(O_f)\) be an irreducible \(U_q(g)\)-module with highest weight \(\lambda \in P^+\) and let \(v_\lambda \in V_\lambda\) be a highest weight vector. Define \(L_{v_\lambda} = U_A(g) \cdot v_\lambda\).
Lemma 4.19. [6, Lemma 3.18] \((V_\lambda, L_{V_\lambda}) \in \mathcal{O}_f(U_q(\mathfrak{g})).\)

The following fact is obvious and, apparently, well-known.

Lemma 4.20. [6, Lemma 3.19]
(a) Each object \((V_\lambda, L_{V_\lambda})\) is irreducible in \(\mathcal{O}_f(U_q(\mathfrak{g}))\); and each irreducible object of \(\mathcal{O}_f(U_q(\mathfrak{g}))\) is isomorphic to one of \((V_\lambda, L_{V_\lambda})\).
(b) The category \(\mathcal{O}_f(U_q(\mathfrak{g}))\) is semisimple.
(c) The forgetful functor \((V, L) \mapsto V\) is an almost equivalence of tensor categories \(\mathcal{O}_f(U_q(\mathfrak{g})) \rightarrow \mathcal{O}_f\).

We also have the following fact.

Lemma 4.21. [6, Lemma 3.21]
(a) The restriction of the functor \(U_q(\mathfrak{g}) \rightarrow \mathcal{M}od \rightarrow U(\mathfrak{g}) \rightarrow \mathcal{M}od\) defined by (4.18) to the sub-category \(\mathcal{O}_f(U_q(\mathfrak{g}))\) is a tensor functor (4.19) \(\mathcal{O}_f(U_q(\mathfrak{g})) \rightarrow \mathcal{O}_f\).
(b) The functor (4.19) is an almost equivalence of categories.

Combining Lemma 4.20 and Lemma 4.21 we obtain Proposition 4.14. □

The following result relates the classical limit of braided symmetric algebras and Poisson algebras.

Theorem 4.22. [6, Theorem 2.29] Let \(V\) be an object of \(\mathcal{O}_f\) and let \(\sigma(V)\) in \(\mathcal{O}_f\) be the classical limit of \(V\). Then:

The classical limit \(S_\sigma(V)\) of the braided symmetric algebra \(S_\sigma(V)\) is a quotient of the symmetric algebra \(S(V)\). In particular, \(\dim_{\mathbb{C}(q)}S_\sigma(V)_n = \dim_{\mathbb{C}}(S_\sigma(V)) \leq \dim_{\mathbb{C}}(S(V))_n\).

Moreover, \(S_\sigma(V)\) admits a Poisson structure defined by \(\{u, v\} = r^-(u \wedge v)\), where \(r^-\) is an anti-symmetrized \(r\)-matrix.

4.4. Flat Modules over Reductive Lie Algebras. In [6] we introduce the notion of flatness of a \(U_q(\mathfrak{g})\)-module. In this section we will recall the definition and basic properties of flat modules and then proceed to classify all flat modules over \(U_q(\mathfrak{g})\), where \(\mathfrak{g}\) is any semisimple Lie algebra.

We view \(S_\sigma(V^q)\) and \(\Lambda_\sigma(V^q)\) as deformations of the quadratic algebras \(S(V)\) and \(\Lambda(V)\) respectively, where \(V\) denotes the classical limit of \(V^q\). In [6] we show that

\[
\dim S_\sigma^n V^q = \dim S_\sigma(V^q)_n \leq \binom{\dim V^q + n - 1}{n}
\]

for all \(n\).

Therefore, it is natural to make the following definition.

Definition 4.23. A finite dimensional \(U_q(\mathfrak{g})\)-module is flat, if and only if

\[
\dim S_\sigma^n V^q = \binom{\dim V^q + n - 1}{n}
\]

for all \(n \geq 0\); i.e., the braided symmetric power \(S_\sigma^n V^q\) is isomorphic (as a vector space) to the ordinary symmetric power \(S^n V^q\).

The following theorem is our main result.
Main Theorem 4.24. Let \( \mathfrak{g} \) be a semisimple Lie algebra and \( U_q(\mathfrak{g}) \) its quantized enveloping algebra. A simple \( U_q(\mathfrak{g}) \)-module \( V \) is flat if and only if its classical limit \( \mathcal{V} \) is Poisson as a \( U(\mathfrak{g}) \)-module.

Proof.

The ”only if” assertion follows immediately from the following result.

Proposition 4.25. \([6] \text{ Theorem 2.29}\) If \( V^q \) is an object of \( \mathcal{O}_f \) and \( V^q \) is flat, then \( S(V) \) is Poisson.

Proof. \([4,22]\) asserts that the classical limit of \( S_q(V^q) \) is a Poisson algebra. If \( V^q \) is flat then \( (S(V), r^-) \) is a Poisson algebra and \( V \) Poisson by Proposition 3.6.

It therefore, remains to show that if a simple \( \mathfrak{g} \)-module \( V \) is Poisson, then \( V^q \) is a flat \( U_q(\mathfrak{g}) \)-module. Following the strategy of Section 3.2 we will first consider the case when \( \mathfrak{g} \) is a simple Lie algebra and \( V \) a simple \( \mathfrak{g} \)-module. First assume that \( \mathfrak{g} \) is a simple Lie algebra and \( V \) a simple \( \mathfrak{g} \)-module. Recall from Corollary \([6,33]\) that a simple \( \mathfrak{g} \)-module \( V_\lambda \) is Poisson, if and only if \( V_\lambda \) is rigid, hence the assertion of Theorem 1.24 follows immediately from the following result.

Proposition 4.26. Let \( \mathfrak{g} \) be a simple Lie algebra and \( V_\lambda \) a simple \( \mathfrak{g} \)-module. The \( U_q(\mathfrak{g}) \)-module \( V^q_\lambda \) is flat if \( V_\lambda \) is rigid.

Proof. Indeed \([6] \text{ Theorem 2.36}\) asserts that \( V^q \) is flat, if \( V \) is rigid: We have \( \text{dim } S^0_q V^q = \binom{\text{dim } V^q + 1}{n} \) and \( S^0_q V^q \cong S^0 V \) for \( n = 0, 1, 2 \). Employing a well known result by Drinfeld \([14] \text{ Theorem 1}\) about quadratic algebras it is shown in \([6] \text{ Proposition 2.33}\) that \( V^q \) is flat if and only if \( S^3_q V^q = \binom{\text{dim } V^q + 2}{3} \). Since dequantization is an almost equivalence of the tensor categories \( \mathcal{O}_f \) and \( \mathcal{O}_f(\mathfrak{g}) \)(Lemma 3.20(c)), we obtain that in the notation around Lemma 6.19 the multiplicity of \( V^q \) in \( S^3_q V^q \) (resp. \( \Lambda^3_q V^q \)) is \( c^q_{\lambda, \mu} \) (resp. \( c^q_{\lambda, \mu} \)).

Denote by \( c^3_{\lambda, \mu} \) (resp. \( c^3_{\lambda, \mu} \)) the multiplicity of \( V^q_\mu \) in \( S^3_q V^q_\lambda \) (resp. of \( V_\mu \) in \( S^3 V_\lambda \)). We derive, arguing analogously to the proof of Lemma 6.19 that \( d^3_{\lambda} \leq c^3_{\lambda, \mu} \) for all \( \mu \in P^+(\mathfrak{g}) \). Since \( c^3_{\lambda, \mu} \leq c^3_{\lambda, \mu} \) for all \( \mu \in P^+(\mathfrak{g}) \), we obtain that if \( V_\lambda \) is rigid and \( d^3_{\lambda} = c^3_{\lambda, \mu} \), then \( c^3_{\lambda, \mu} = c^3_{\lambda, \mu} \) for all \( \mu \in P^+ \) and hence \( V^q_\lambda \) is flat. The proposition is proved.

Now consider the case when \( \mathfrak{g} \) is semisimple and \( V \) a simple \( \mathfrak{g} \)-module. Theorem 3.13 asserts that if \( \text{supp}(\mathfrak{g}) \) is not simple and \( V \) is Poisson, then \( \text{supp}(\mathfrak{g}) \cong \text{sl}_n \times \text{sl}_m \) for some \( m, n \geq 1 \) and \( V \) isomorphic to the natural module \( V_{01,01} \). We show in \([6] \text{ Proposition 2.38}\) that the natural \( U_q(\text{sl}_n \times \text{sl}_m) \)-module is flat, its braided symmetric algebra isomorphic to the algebra of quantum \( m \times n \)-matrices. Theorem 4.24 is proved.

Remark 4.27. A straightforward argument shows that if \( \mathfrak{g} \) is a reductive Lie algebra, then a \( U_q(\mathfrak{g}') \)-module \( V^q \) is flat if \( V^q |_{U_q(\mathfrak{g})} \) is a flat \( U_q(\mathfrak{g}') \)-module, where \( \mathfrak{g}' \subset \mathfrak{g} \) is the maximal semisimple subalgebra of \( \mathfrak{g} \).

5. Deformations of Symmetric Algebras of Poisson Modules

In this section we will explicitly construct the braided symmetric algebras of flat modules, employing the relationship between geometrically decomposable modules and Abelian nil-radicals.
5.1. Quantum Radicals as Symmetric Algebras. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra corresponding to a Lie algebra $\mathfrak{g}$ introduced in Section 4.1. Denote, as above, by $W$ the Weyl group of $\mathfrak{g}$ generated by the simple reflections $s_i$ for $i \in [1, r]$. Corresponding to each $i \in [1, r]$ there exist maps $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined on the generators of $U_q(\mathfrak{g})$ in the following way:

\begin{equation}
T_i(E_i) = -F_iK_{\alpha_i}, \quad T_i(F_i) = -K_{\alpha_i}^{-1}E_i, \quad T_i(E_j) = \sum_{k=0}^{\alpha_{i,j}} (-1)^k a_{i,j} q_i^{-k} E_i^{-a_{i,j} - k} E_j^{a_{i,j} - k},
\end{equation}

\[T_i(F_j) = \sum_{k=0}^{\alpha_{i,j}} (-1)^k a_{i,j} q_i^{k} F_i^{k} F_j F_i^{-a_{i,j} - k}, \quad T_i(K_{\lambda}) = K_{\sigma_i(\lambda)}.\]

For every element $w \in W$ with presentation $w = s_{i_1} \ldots s_{i_k}$ we define $T_w$ as $T_w = T_{s_{i_1}} \ldots T_{s_{i_k}}$. We will need the following well known fact.

Lemma 5.1. [21, 8.18] If $w \in W$, then $T_w$ is independent of the choice of reduced expression; i.e. if $w = s_{i_1} \ldots s_{i_k}$ and $w = s_{j_1} \ldots s_{j_k}$ are reduced expressions of $w \in W$, then

\[T_{s_{i_1}} \ldots T_{s_{i_k}} = T_{s_{j_1}} \ldots T_{s_{j_k}}.\]

Recall from Section 8 that $U^+$ denotes the subalgebra of $U_q(\mathfrak{g})$ generated by the $E_i$ for $i \in [1, r]$, $U^-$ the subalgebra of $U_q(\mathfrak{g})$ generated by the $F_i$ for $i \in [1, r]$ and $U_q(\mathfrak{b}_{-})$ the subalgebra of $U_q(\mathfrak{g})$ generated by all $K_{\lambda}$ and all $F_i$.

Recall (see e.g. [21] ch. 8]) that we can associate to each reduced expression of the longest element $w_0 \in W$ a PBW-basis of $U_q(\mathfrak{g})$ in the following way: Let $w_0 = \sigma_{i_1} \ldots \sigma_{i_k}$ be a presentation of the longest word in $W$. It is well known that the set of positive roots $R^+$ of the Lie algebra $\mathfrak{g}$ can be ordered in the following way:

\[\alpha_{(1)} = \alpha_{i_1} < \alpha_{(2)} = s_{i_1} \circ \alpha_{2} < \ldots < \alpha_{(k)} = s_{i_1} \ldots s_{i_{k-1}} \alpha_{i_k},\]

where $\alpha_{j}$ denotes the $j$-th simple root.

We define for each presentation of $w_0$ a set of positive roots spanning $U^+$ following [21] ch.8:

\[E_{\alpha_{(1)}} = E_{i_1}, \quad E_{\alpha_{(2)}} = T_{i_1}(E_{i_2}), \quad \ldots, \quad E_{\alpha_{(k)}} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}).\]

Similarly, we define a set of negative roots spanning $U^-$ by:

\[F_{\alpha_{(1)}} = F_{i_1}, \quad F_{\alpha_{(2)}} = T_{i_1}(F_{i_2}), \quad \ldots, \quad F_{\alpha_{(k)}} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}).\]

The following is the key definition for this section.

Definition 5.2. For every element $w \in W$ in the Weyl group we define the quantum Schubert cell $U(w)$ as

\[U(w) = T_{w^{-1}}(U_q(\mathfrak{b}^-)) \cap U^+.\]

We also have the following alternative description of quantum Schubert cells.

Lemma 5.3. Let $w \in W$ and $w_0$ be the longest element in $W$. Denote $w' = w_0 w$. We have

\[U(w) = T_{(w')^{-1}}(U_q(\mathfrak{b}^+)) \cap U^+.\]
Proof. We have $T_{w_0}^{-1} T = T_{w_0} T = T_{w_0}^{-1} T_{w_0} T^{-1} = T_{w_0} T^{-1}$, since $w_0^{-1} = w_0$. Note that $T_{w_0}(U_q(b^+)) = (U_q(b^-))$. Therefore,

$$U(w) = T_{w^{-1}}(U_q(b^-)) \cap U^+ = T_{w^{-1}w_0}(U_q(b^+)) \cap U^+ .$$

We will now consider quantum Schubert cells $U(w_{\Delta})$ where $w_{\Delta} \in W$ corresponds to subsets $\Delta \subset [1, r]$ in the following way. Let $W_{\Delta}$ be the subgroup of Delta generated by the simple reflections $s_i$ for $i \in \Delta$ and denote by $w_{0, \Delta}$ the longest element of $W_{\Delta}$. The element $w_{\Delta} = w_{0, \Delta} w_0$ is commonly referred to as a parabolic element of $W$. If $p_{\Delta}$ is the standard parabolic subalgebra of $g$ associated with $\Delta$, then $w_{0, \Delta}$ is the longest element of its Levi subalgebra $l_{\Delta}$. Denote the nil-radical by $rad_{\Delta}$. Recall that $p_{\Delta}$ splits as a semi-direct product $p_{\Delta} \cong l_{\Delta} \ltimes rad_{\Delta}$ (see e.g. [20]). Additionally recall that any Hopf algebra $H$ algebra acts on itself via the adjoint action:

\begin{equation}
\text{ad}(a).b = a_{(1)} b S(a_{(2)})
\end{equation}

The following theorem is the first main result of this section.

**Main Theorem 5.4.** (a) Let $g'$ be a reductive Lie algebra, $p_{\Delta}$ a parabolic subalgebra with Levi $l_{\Delta}$ and radical $rad_{\Delta}$. If $rad_{\Delta}$ is an Abelian Lie algebra, then $U(w_{\Delta})$ is a flat quadratic $q$-deformation of the symmetric algebra $S(rad_{\Delta})$.

(b) The quantum Schubert cell $U(w_{\Delta})$ is a $\mathbb{Z}_{\geq 0}$-graded $U_q(l_{\Delta})$ module algebra and $U(w_{\Delta})$ is the braided symmetric algebra of the $U_q(l_{\Delta})$-module $U(w_{\Delta})_1$.

(c) Moreover, let $g_{\Delta}$ be the maximal semisimple submodule of $l_{\Delta}$. Then, $U(w_{\Delta})$ is a $\mathbb{Z}_{\geq 0}$-graded $U_q(g_{\Delta})$ module algebra and $U(w_{\Delta})$ is the braided symmetric algebra of the $U_q(g_{\Delta})$-module $U(w_{\Delta})_1$.

**Proof.** In order to prove Theorem 5.4(a) we have to show that the classical limit $q \rightarrow 1$ of $U(w_{\Delta})$ is $S(rad_{\Delta})$. We call a root $\alpha \in R(g)$ radical, if $\alpha \notin R(g') \cap \text{span}_{\mathbb{Z}}(\Delta)$. Recall that $rad_{\Delta}$ is spanned by $E_{\alpha}$ where $\alpha$ is radical. We obtain the following well known characterization of Abelian radicals.

**Lemma 5.5.** Let $g'$ be a reductive Lie algebra, $p_{\Delta}$ a parabolic subalgebra with Levi $l_{\Delta}$ and radical $rad_{\Delta}$. The radical $rad_{\Delta}$ is Abelian, if and only if all radical roots are of the form $\alpha = \alpha_i + \sum_{j \neq i} c_j \alpha_j$.

Theorem 5.4(a) yields that $U(w_{\Delta})$ is generated as an algebra by the $E_{\alpha}$ for which $\alpha \in R^+(g)$ is a radical root.

We need the following well-known fact.

**Lemma 5.6.** ([11] Lemma 2.2) Let $R_{\Delta} = w(R^-) \cap R^+$. Then,

\begin{equation}
\text{ad}(E_{\alpha})(E_{\beta}) = [E_{\alpha}, E_{\beta}]_q = E_{\alpha} E_{\beta} - q^{(\alpha, \beta)} E_{\beta} E_{\alpha} \in \text{span}(E_{\gamma_1} \ldots E_{\gamma_k}) ,
\end{equation}

where $\alpha < \gamma_1 \leq \ldots \leq \gamma_k < \beta$ and $\gamma_1 + \ldots + \gamma_k = \alpha + \beta$.

We now obtain from Lemma 5.5 and Lemma 5.6 that if $rad_{\Delta}$ is Abelian, then $U(w_{\Delta})$ is a quadratic algebra and its classical limit is $S(rad_{\Delta})$. Theorem 5.4(a) is proved.
Let us now prove part (b). Note first that \( U_q(I_\Delta) \) acts adjointly on \( U(w_\Delta) \) by Theorem 5.18. We now obtain that if \( rad_\Delta \) is Abelian, then Lemma 5.5 and Lemma 5.6 imply that \( ad(U_q(I_\Delta))U(w_\Delta) \subset (U(w_\Delta))_i \), and that, hence, \( U(w_\Delta) \) is a graded \( U_q(I_\Delta) \)-module algebra. Denote by \( \pi_\Delta \) the canonical \( U_q(I_\Delta) \)-module homomorphism \( \pi_\Delta : T(U(w_\Delta))_1 \to U(w_\Delta) \). We obtain from Theorem 5.4 that the classical limit of the kernel \( \ker(\pi) \) is equal to the \( U(I_\Delta) \)-ideal generated by \( \Lambda^2 rad_\Delta \). Howe proves in [18, ch. 4.6] that \( rad_\Delta \) is weight-multiplicity-free and simple as a \( I_\Delta \)-module, as well as a \( g_\Delta \)-module, that means all weight-spaces are one-dimensional. Recall the following well-known fact.

**Lemma 5.7.** Let \( g \) be a reductive Lie algebra. Then, \( \dim(\text{Hom}_g(V_\lambda \otimes V_\lambda, V_\mu)) \leq \dim V_\lambda(\mu - \lambda) \).

The lemma implies that \( rad_\Delta \otimes rad_\Delta \) is multiplicity-free as a \( I_\Delta \)-module. Employing Lemma 4.21 we obtain that \( (U(w_\Delta))_1 \otimes (U(w_\Delta))_1 \) contains a unique \( U_q(I_\Delta) \)-submodule \( \text{Ext}_q^2(rad_\Delta) \) such that its classical limit is isomorphic to \( \Lambda^2 rad_\Delta \). This implies that \( \Lambda_q^2(U(w_\Delta))_1 = \text{Ext}_q^2(rad_\Delta) = \ker(\pi) \cap (U(w_\Delta))_1 \otimes (U(w_\Delta))_1 \). Therefore, \( U_q(rad_\Delta) = S_\pi(U(w_\Delta))_1 \). Theorem 5.4 (b) is proved.

Part (c) can be proved analogously to part (b). Theorem 5.4 is proved. \( \square \)

Call a \( U_q(g) \)-module **geometrically decomposable** if its classical limit is geometrically decomposable as a \( U(g) \)-module. Theorem 5.4 (c) has the following consequence.

**Corollary 5.8.** Let \( g \) be a semisimple Lie algebra and let \( V^q \) be a simple geometrically decomposable \( U_q(g) \)-module. There exists a simple Lie algebra \( g' \) and a parabolic element \( w_\Delta \in W(g') \) such that \( U_q(g) \cong U_q(g_\Delta) \) and the braided symmetric algebra \( S_\pi V^q \cong U(w_\Delta) \) as \( U_q(g_\Delta) \)-modules.

**Proof.** The corollary follows immediately from Theorem 5.4 and the description of geometrically decomposable modules as Abelian radicals in Section 5.2. \( \square \)

Many of the braided symmetric algebras obtained by the construction of Theorem 5.4 are well known examples of quantized coordinate rings of classical varieties. Our theory presents a unifying construction of these important examples. We have the following list according to [17, ch. 5]:

- If \( g = \mathfrak{sl}_k \) and \( \Delta = \{1, \ldots, n\}/\{i\} \), then \( U(w_\Delta) = \mathbb{C}[\text{Mat}_{i \times (n-i)}] \), the algebra of quantum \( i \times (n-i) \)-matrices.
- If \( g = \mathfrak{so}(2n + 1) \) and \( \Delta = \{2, \ldots, n\} \), then \( U(w_\Delta) \) is the algebra of the odd-dimensional Euclidean space \( \mathbb{O}^{2N-1}_2(\mathbb{C}) \) introduced in [32] (see also [30]).
- If \( g = \mathfrak{sp}(2n) \) and \( \Delta = \{2, \ldots, n\} \), then \( U(w_\Delta) \) is the algebra of quantum symmetric matrices introduced in [31] Theorem 4.3 and Proposition 4.4 and by [23].
- If \( g = \mathfrak{so}(2n) \) and \( \Delta = \{2, \ldots, n\} \), then \( U(w_\Delta) \) is the algebra of the even-dimensional Euclidean space \( \mathbb{O}^{2N-2}_4(\mathbb{C}) \) introduced in [32] (see also [30]).
- If \( g = \mathfrak{so}(2n) \) and \( \Delta = 1, \ldots, n-1 \) or \( \Delta = \{1, \ldots, n-2, n\} \), then \( U(w_\Delta) \) is the algebra of quantum antisymmetric matrices introduced in [35, Section 1].
Lemma 5.11. Let $\mathfrak{g} = E_6$ and $\Delta = \{2, \ldots, 6\}$, resp. $\Delta = \{1, \ldots, 5\}$ or $\mathfrak{g} = E_7$ and $\Delta = \{1, \ldots, 6\}$, then we obtain quantum algebras $U(w_\Delta)$, which apparently have not been studied previously.

We will now extend the result of Theorem 5.4 to some subalgebras, when $\text{rad}_\Delta$ is of Heisenberg type; i.e., the derived subalgebra $[\text{rad}_\Delta, \text{rad}_\Delta] \subseteq \text{rad}_\Delta$ is one-dimensional. Recall that an algebra $\mathcal{U}$ is called filtered, if $\mathcal{U} = \bigcup_{i=0}^{\infty} \mathcal{U}_i$ with $\mathcal{U}_i \subseteq \mathcal{U}_{i+1}$ and $\mathcal{U}_i \cdot \mathcal{U}_j \subseteq \mathcal{U}_{i+j}$. The associated graded algebra $\text{gr}(\mathcal{U})$ of $\mathcal{U}$ is defined as $\text{gr}(\mathcal{U}) = \bigoplus_{i=0}^{\infty} \mathcal{U}_i/\mathcal{U}_{i-1}$, where we set $\mathcal{U}_{-1} = \{0\}$. The following result is the second main result of this section.

Theorem 5.9. Let $\Delta \subseteq [1, r]$ and let $\mathfrak{p}_\Delta$ be the corresponding parabolic subalgebra, and $\text{rad}_\Delta$ its nil-radical. If $U(w_\Delta)$ is a filtered $U_q(\mathfrak{l}_\Delta)$-module algebra, (a) then $\text{gr}(U(w_\Delta))$ is a $U_q(\mathfrak{l}_\Delta)$-module algebra and a flat $q$-deformation of $S(\text{rad}_\Delta)$, (b) and $\text{gr}(U(w_\Delta))$ is a $U_q(\mathfrak{g}_\Delta)$-module algebra, where $\mathfrak{g}_\Delta$ is the maximal semisimple subalgebra of $\mathfrak{l}_\Delta$.

Proof.

The following fact is well known.

Lemma 5.10. (a) If $\mathcal{U}$ is a filtered $k$-algebra, there are isomorphisms of vector spaces $\phi_n : \mathcal{U}_n \rightarrow \bigoplus_{i=0}^{n} \hat{\mathcal{U}}_i$, where $\hat{\mathcal{U}}_i = \mathcal{U}_i/\mathcal{U}_{i-1}$.
(b) The isomorphisms $\phi_n$ induce an isomorphism of $k$-algebras $\phi : \mathcal{U} \rightarrow \text{gr}(\mathcal{U})$.

Another well known and important fact is the following.

Lemma 5.11. Let $\mathcal{U}$ be a Hopf algebra, and $\mathcal{V}$ be a filtered $\mathcal{A}$-module algebra. Then $\phi : \mathcal{U} \rightarrow \text{gr}(\mathcal{V})$ is an isomorphism of $\mathcal{A}$-modules.

Recall that an algebra $\mathcal{A}$ is called quadratic-linear, if $\mathcal{A}$ is the quotient of a free algebra $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ by an ideal generated by elements of $\bigoplus_{i=0}^{2}(\mathbb{C}\langle x_1, \ldots, x_n \rangle)_i$, where $(\mathbb{C}\langle x_1, \ldots, x_n \rangle)_i$ denotes the $i$-th graded component. The following proposition is a key step to proving Theorem 5.9.

Proposition 5.12. Let $\Delta \in [1, r]$. If $U(w_\Delta)$ is quadratic linear, i.e. $[E_\alpha, E_\beta]_q \in (U(w_\Delta))_2$ for all radical roots $\alpha, \beta$ and $U(w_\Delta)$ is a filtered $U_q(\mathfrak{l}_\Delta)$-module algebra, then $\text{gr}(U(w_\Delta))$ is a $U_q(\mathfrak{l}_\Delta)$-module algebra.

Proof. If $U(w_\Delta)$ is quadratic linear, then $U(w_\Delta)$ is a filtered Hopf algebra. Hence, $\text{gr}(U(w_\Delta))$ is a $U_q(\mathfrak{l}_\Delta)$-module algebra by Lemma 5.11. The proposition is proved.

Note first that if $U(w_\Delta)$ is filtered, then $U(w_\Delta)$ must be quadratic linear. Hence, $\text{gr}(U(w_\Delta))$ is quadratic, and Theorem 5.12 and Lemma 5.10 yield that its classical limit is $S(\text{rad}_\Delta)$. Theorem 5.9 (a) is proved.

Part (b) can be proved analogously. Theorem 5.9 is proved.

We now obtain the construction of the braided symmetric algebra of the natural module of $U_q(sp(2n))$.

Corollary 5.13. Let $\mathfrak{g} = sp(2n)$, $\mathfrak{g}' = sp(2n+2)$ and $\Delta = \{2, \ldots, n+1\}$. The braided symmetric algebra $S_\alpha(V_{\omega_1})$ of the natural $U_q(sp(2n))$-module $V_{\omega_1}$ is isomorphic to $U(w_\Delta)$ as a $U_q(sp(2n))$-module.
Proof. Note that the maximal semisimple subalgebra $\mathfrak{g}_\Delta$ of $\mathfrak{l}_\Delta$ is isomorphic to $sp(2n)$. Using Theorem 5.9 we have to show that $U(w_\Delta)$ is a filtered $U_q(\mathfrak{l}_\Delta)$-module algebra. The radical roots corresponding to $\Delta$ are of the form $\alpha_1 + \sum_{i=2}^j \alpha_i$, $\alpha_1 + \ldots + \alpha_j + 2\alpha_{j+1} + \ldots + 2\alpha_n + \alpha_{n+1}$ for $j \leq n$, and $\alpha_{\text{max}} = 2\alpha_1 + \ldots + 2\alpha_n + \alpha_{n+1}$. For convenience we will fix a reduced expression of $w_0$ such that the roots of $sp(2n)$ are ordered as

\[(5.4) \alpha_1 < \alpha_1 + \alpha_2 < \ldots < \sum_{i=1}^{n+1} \alpha_i < \sum_{i=1}^{n-1} \alpha_i + 2\alpha_n + \alpha_{n+1} < \ldots < \alpha_{\text{max}} < \ldots .\]

We need the following fact.

Lemma 5.14. The quantum Schubert cell $U(w_\Delta)$ is quadratic linear.

Proof. It is easy to see from (5.4) that if $\alpha, \beta \leq \alpha_{\text{max}}$ one cannot find $\alpha < \gamma_1 \leq \ldots \leq \gamma_k < \beta$ with $k \geq 3$ such that $\alpha + \beta = \gamma_1 + \ldots + \gamma_k$. The assertion follows. \(\blacksquare\)

The lemma implies that $U(w_\Delta)$ is filtered. It remains to investigate the $U_q(\mathfrak{l}_\Delta)$-action.

Lemma 5.15. Let $\mathfrak{g}' = sp(2n + 2)$ and $\Delta = \{2, 3, \ldots, n + 1\} \subset [1, n + 1]$. Then, $U(w_\Delta)$ is a filtered $U_q(\mathfrak{l}_\Delta)$-module algebra.

Proof.

It is obvious from the definition of the adjoint action (5.2) and the defining relations of $U_q(\mathfrak{g}')$ (see Section 4.1) that $ad(F_i)(U(w_\Delta)_m) \subset U(w_\Delta)_m$ for all $m \in \mathbb{Z}_{\geq 0}$ and $i \in \Delta$ as well as $ad(K_\lambda)(U(w_\Delta)_m) \subset U(w_\Delta)_m$ for all $\lambda \in P(\mathfrak{g}')$. It remains to check that $ad(E_i)(E_{\alpha_1} \ldots E_{\alpha_m}) \in U(w_\Delta)_m$ for $i \in \Delta$ and radical roots $\alpha_1, \ldots, \alpha_m$. We prove this by induction on $m$.

Let $m = 1$. Note that $ad(E_i)(E_\alpha) \subset U(w_\Delta)$ for all radical roots $\alpha$ and all $i \in [2, n + 1]$ by Theorem 5.18 (b). If $\alpha = \alpha_{\text{max}}$, then and (5.3) and (5.4) imply that $ad(E_i)(E_\alpha) = 0 \in U(w_\Delta)$. If $\alpha < \alpha_{\text{max}}$, then $(\alpha, \omega_1) = 1$ and hence one cannot find $\gamma_1 \leq \ldots \leq \gamma_k \leq \alpha_{\text{max}}$ with $k \geq 2$ such that $\alpha + \alpha_i = \gamma_1 + \ldots + \gamma_k$, since $(\alpha + \alpha_i, \omega_1) = 1$ and $(\gamma_1 + \ldots + \gamma_k, \omega_1) \geq k$. We obtain that $ad(E_i)(U(w_\Delta)_1 \subset U(w_\Delta)_1$.

Let $m > 1$ and note that

$ad(E_i)(E_{\alpha_1} \ldots E_{\alpha_m}) = ad(E_i)(E_{\alpha_1})E_{\alpha_2} \ldots E_{\alpha_m} + q^r E_{\alpha_1} ad(E_i)(E_{\alpha_2} \ldots E_{\alpha_m})$, for some $r \in \mathbb{Z}$. The assertion follows immediately from this hypothesis. The lemma is proved. \(\blacksquare\)

Note that $r^\text{ad}_\Delta \cong V_{w_1} \oplus V_0$ as a $sp(2n)$-module. Hence, $(gr(U(w_\Delta)))_1 \cong V_0^q \oplus V_0^q$ as $U_q(sp(2n))$-modules. Therefore $gr(U(w_\Delta))$ is a flat deformation of $S(V_{w_1} \oplus V_0)$ by Theorem 5.3.

Denote by $\text{Sym}_2$ the $U_q(sp(2n))$-module homorphism $\text{Sym}_2 : (V^q_{w_1} \oplus V^q_0)^{\otimes 2} \rightarrow gr(U(w_\Delta))_2$, given by the relations defining the quadratic algebra $gr(U(w_\Delta))$. Note that

$\ker(\text{Sym}_2) \subset V^q_{w_1} \otimes V^q_{w_1} \oplus V^q_0 \otimes V^q_0 \oplus V^q_{w_1} \otimes V^q_0 \otimes V^q_{w_1} \otimes V^q_0 \subset (V^q_{w_1} \oplus V^q_0)^{\otimes 2}$.

We have $\ker(\text{Sym}_2) \cong V^q_{w_2} \oplus V^q_0 \oplus V^q_0$ and obtain that

$\ker(\text{Sym}_2) \cap (V^q_{w_1} \otimes V^q_{w_1}) \cong V^q_{w_2} \oplus V^q_0$. 


Since \( V_{\omega_1} \) is multiplicity-free and weight-multiplicity-free and \( V_{\omega_1} \oplus V_{\omega_1}^q \cong \Lambda_0^2 V_{\omega_1}^q \), we obtain from Lemma \[5.7\] that, analogous to the proof of Theorem \[5.4\], indeed \( \Lambda_0^2 V_{\omega_1} = \ker(Sym_1) \). Thus, the subalgebra of \( gr(U(w_\Delta)) \) generated by \( V_{\omega_1}^q \) is the braided symmetric algebra \( S_\sigma(V_{\omega_1}^q) \).

Corollary \[5.13\] is proved. 

\[\Box\]

**Remark 5.16.** The braided symmetric algebra \( S_\sigma(V_{\omega_1}^q) \) of the natural \( U_q(sp(2n)) \)-module can be obtained, by an argument similar to the proof of Corollary \[5.7\], as the quotient of \( gr(U(w_\Delta)) \) by the two-sided ideal generated by the copy of the trivial module \( V_0 \) in \( gr(U(w_\Delta))_1 \).

**Problem 5.17.** Describe the \( q \)-deformed symmetric algebras associated to radicals of Heisenberg type, where the quantum radical \( U(w_\Delta) \) satisfies the assumptions of Theorem \[5.9\]. These algebras cannot be braided symmetric algebras, but should provide examples for some more general concept of quantum symmetric algebra.

### 5.2. Quantum Schubert Cells: PBW-Theorem and Levi action.

Let \( g \) be a complex reductive Lie algebra, and let \( W \) be the Weyl group of \( g \). In this section we prove a PBW-type theorem for quantum Schubert cells \( U(w) \subset U_q(g) \) associated to \( w \in W \) and show that if \( w_\Delta \) is a parabolic element of the \( W \), then the Hopf subalgebra \( U_q(l_\Delta) \subset U_q(g) \) acts adjointly on \( U(w_\Delta) \) (for definitions see the beginning of Section \[5.1\]).

The following theorem is the main result of this section.

**Theorem 5.18.** (a) Let \( w \in W \) be an element of the Weyl group \( W \) and \( U(w) \) the corresponding quantum Schubert cell. The monomials \( E_{[\alpha(1)]} \ldots E_{[\alpha(k)]} \), satisfying \( \ell(i) = 0 \) if \( \alpha(i) \in R^- \), form a \( \mathbb{C}(q) \)-linear basis of \( U(w) \).

(b) If \( w = w_\Delta \in W \) is parabolic, then \( U_q(l_\Delta) \) acts adjointly on \( U(w_\Delta) \).

**Proof.**

Prove (a) first. Recall the PBW-theorem for \( U_q(g) \).

**Proposition 5.19.** \[21, 8.24\]

(a) The monomials \( E_{\alpha(1)} \ldots E_{\alpha(k)} F_{\beta(1)} \ldots F_{\beta(\ell)} \), with \( \ell(i) \in \mathbb{Z}_{\geq 0} \), form a \( \mathbb{C}(q) \)-linear basis of \( U_q(g) \).

(b) The monomials \( E_{\alpha(1)} \ldots E_{\alpha(k)} \) with \( \ell(i) \in \mathbb{Z}_{\geq 0} \) form a \( \mathbb{C}(q) \)-linear basis of \( U^+ \).

Similarly, the monomials \( F_{\alpha(1)} \ldots F_{\alpha(k)} \) with \( m(i) \in \mathbb{Z}_{\geq 0} \) form a \( \mathbb{C}(q) \)-linear basis of \( U^- \).

Denote by \( \ell(w) \) the length of an element \( w \in W \). The following fact relates quantum Schubert cells.

**Proposition 5.20.** (a) \( U(w_0) = U^+ \).

(b) Let \( w, w' \in W \) such that \( w_0 = ww' \) and \( \ell(w) + \ell(w') = \ell(w_0) \). Then

\[ U^+ = U(w_0) = U(w)T_w(U(w')) , U(w) \cap T_w(U(w')) = \mathbb{C}(q) \cdot 1 \subset U_q(g) \].

**Proof.** We need the following fact.

**Lemma 5.21.** Let \( \alpha \in R^+(g) \) and let \( w \in W \).

1. If \( \alpha(\alpha) \in R^+(g) \), then \( T_w(E_\alpha) \in U^+ \).
2. If \( \alpha(\alpha) \in R^-(g) \), then \( T_w(E_\alpha) \in U_q(b^-) \).
Proof. Note first the following fact.

Lemma 5.22. (a) Let $\beta \in R^+(\mathfrak{g})$. Then, $T_1(E_\beta) \in U^+$ if $s_1(\beta) \in R^+(\mathfrak{g})$.
(b) Let $\beta \in R^+(\mathfrak{g})$. Then, $T_1(E_\beta) \in U(b^-)$ if $s_1(\beta) \in R^-(\mathfrak{g})$. Moreover, $T_1(E_\beta) = -F_1K_\alpha$.

(c) Let $\beta \in R^+(\mathfrak{g})$. Then $T_1(F_\beta K_\lambda) \in U_q(b^-)$.

Proof. Prove (a) first. Let $\beta = s_{i_1} \ldots s_{i_k}(\alpha_j)$ and let $w = s_{i_1}s_{i_2} \ldots s_{i_k}$ (not necessarily reduced). One has $w(\alpha_j) = s_{i_1}(\beta) \in R^+(\mathfrak{g})$. It is well known that $w(\alpha_k) \in R^+(\mathfrak{g})$ implies that $\ell(ws_k) = \ell(w) + 1$. Hence there exist $w, w' \in W$ such that $w_0 = ws_jw'$ with $\ell(w_0) = \ell(w) + \ell(w') + 1$. That implies that for some choice of reduced expression, $E_w(\alpha_j) = T_w(E_j) = T_1(E_\alpha) \in U^+$. Part (a) is proved.

Prove (b) now. Note that if $\beta \in R^+(\mathfrak{g})$ and $s_1(\beta) \in R^-(\mathfrak{g})$, then $\beta = \alpha_i$. The assertion follows from the definition of the $T_1$ in (5.1). Part (b) is proved.

In order to prove part (c) note first that the $T_1$ are algebra homomorphisms and that $T_1(K_\lambda) \in U_q(b^-)$. The assertion now follows from an argument analogous to the proof of (a). The lemma is proved.

Now, let $w = s_{i_1} \ldots s_{i_k}$ be a reduced expression of $w$. If $w(\alpha) \in R^+(\mathfrak{g})$, then $s_{i_1} \ldots s_{i_k}(\alpha) \in R^+(\mathfrak{g})$ for all $1 \leq j \leq k$. Indeed if $s_{i_1} \ldots s_{i_k}(\alpha) \in R^-(\mathfrak{g})$ for some $1 \leq j \leq k$, then there exist $j_1 \geq 1$ such that $s_{i_{j_1+1}} \ldots s_{i_k}(\alpha) = \alpha_j$ and $s_{i_{j_1}} = e_j$. Hence, $s_{i_1} \ldots s_{i_{j_1-1}}(-\alpha_j) = w(\alpha) \in R^+(\mathfrak{g})$. Recall the well known exchange property of the Weyl group: Let $\tilde{w} = s_{i_m} \ldots s_{i_{k-1}}s_{i_k}$ be a reduced expression of $w \in W$. If $w(\alpha_i) \in R^-(\mathfrak{g})$, then there exists $m \leq r \leq k$ such that

$s_{i_r} \ldots s_{i_{k+1}} = s_{i_{r+1}} \ldots s_{i_k}s_i$.

In our case we obtain that $s_{i_1} \ldots s_{i_{j_1-1}}$ has a reduced expression $s_{i_1} \ldots s_{i_{j_1-1}} = s_{m_1} \ldots s_{m_{j_1-2}}s_i$, hence $w$ has an expression

$w = s_{m_1} \ldots s_{m_{j_1-2}}s_is_{i_{j_1+1}} \ldots s_{j_k} = s_{m_1} \ldots s_{m_{j_1-2}}s_{i_{j_1+1}} \ldots s_{i_k}$,

contradicting the assumption that $w = s_{i_1} \ldots s_{i_k}$ was reduced. It follows now inductively from Lemma 5.22 that $T_w(E_\alpha) \in U^+$. Part (a) is proved.

Prove (b) now. If $w(\alpha) \in R^-(\mathfrak{g})$, and $w = s_{i_1} \ldots s_{i_k}$ is a reduced expression, then we can find, as in part (a) $1 \leq j \leq k$ such that $w = s_{i_1} \ldots s_{i_{j-1}}s_is_{i_{j+1}} \ldots s_{i_k}$, $s_{i_{j+1}} \ldots s_{i_k}(\alpha) = (\alpha_j)$. Employing the exchange property as in part (a) we have $s_{i_1} \ldots s_{i_{j-1}}(-\alpha_j) \in R^-(\mathfrak{g})$ and $s_{i_2} \ldots s_{i_k}(\alpha) \in R^+(\mathfrak{g})$ for $1 \leq j_1 < j < j_2 \leq k$. Arguing as in the proof of Lemma 5.22 (a) we obtain that $T_{s_{j_1} \ldots s_{j_2}}(E_\alpha) = E_{\alpha_j}$, hence Lemma 5.22(b) and (c) yield that $T_w(E_\alpha) \in U_q(b^-)$. Lemma 5.21 is proved.

Now we are ready to complete the proof of Proposition 5.20. Part(a) follows directly from Lemma 5.21(b), since $w_\alpha(\alpha) \in R^-(\mathfrak{g})$ for all $\alpha \in R^+(\mathfrak{g})$.

Prove (b) now. Let $\alpha = w\alpha'$ and $\ell(w_0) = \ell(w) + \ell(w')$. It follows from Lemma 5.21 that $T_{w_0}(E_\alpha) \in U^+$ or $T_{w_0}(E_\alpha) \in U_q(b^-)$. Since $T_{w_0}(E_\alpha) \in U_q(b^-)$ we obtain that $T_{w_0}(E_\alpha) \in U(w')$ and $E_\alpha \in T_{w_0}(U(w'))$, if $T_{w_0}(E_\alpha) \in U^+$. Similarly we obtain that $E_\alpha \in U(w)$ if $T_{w_0}(E_\alpha) \in U_q(b^-)$. The fact that the $T_i$ are algebra homomorphisms and the PBW-theorem (Proposition 5.14(b)) now imply that $U^+ = U(w_0) = U(w)T_w(U(w'))$. 
It is easy to see that $U(w) \cap T_w(U(w')) = \mathbb{C}(q) \cdot 1$, because

$$\mathbb{C}(q) \cdot 1 \subset T_{w^{-1}}(U(w)) \cap T_w(U(w')) \subset U_q(\mathfrak{b}^-) \cap U^+ = \mathbb{C}(q) \cdot 1.$$ 

Part (b) and Proposition are proved.

We can now complete the proof of Theorem 5.18 (a). Note that if $w^{-1}(\alpha_j) \in R^-(\mathfrak{g})$, then $T_{w^{-1}}(E_{\alpha_j}) \in U_q(\mathfrak{b}^-)$. Since $T_{w^{-1}}$ is an algebra automorphism we obtain that the monomials $E_{\alpha_1}^{\ell_1} \cdots E_{\alpha_k}^{\ell_k}$ with $\ell_i = 0$ if $w^{-1}(\alpha_i) \in R^+(\mathfrak{g})$ are elements of $U(w)$. $U(w)$ is an algebra, hence the linear span of the above monomials is contained in $U(w)$. The monomials are linearly independent by Proposition 5.19 hence it remains to show that they span $U(w)$. Choose $w' \in W$ such that $w'w = w_0$ and $\ell(w') + \ell(w) = \ell(w_0)$. We showed in the proof of Proposition 5.20 that $E_{\alpha_1}^{\ell_1} \cdots E_{\alpha_k}^{\ell_k} \in T_w(U(w'))$ if $\ell_i = 0$ whenever $w^{-1}(\alpha_i) \in R^-(\mathfrak{g})$. Hence we can write each $u \in U^+$ by Proposition 5.17 as $u = \sum_{i=1}^k u_i u'_i$, where the $u_i \in U(w)$ are linearly independent and $u'_i \in T_w(U(w'))$. It follows immediately that $T_{w^{-1}}(u) \in U_q(\mathfrak{b}^-)$, if and only if $T_{w^{-1}}(u'_i) \in U_q(\mathfrak{b}^-)$; i.e., if $u'_i \in \mathbb{C}(q) \cdot 1$ for all $i$ by Proposition 5.20(b).

Theorem 5.18 (a) is proved.

We will now prove Theorem 5.18 (b). It suffices to show that the $E_i, F_i$, $i \in \Delta$ and $K_i, \lambda \in P(\mathfrak{g})$ which generate $U(\Delta)$ act adjointly on $U(w_\Delta)$.

**Proposition 5.23.** (a) Let $w \in W$. Then $K_\lambda, \lambda \in P(\mathfrak{g})$ acts adjointly on $U(w)$.

(b) Let $w_\Delta \in W$ be parabolic and let $w_0 = w_{0,\Delta} w_\Delta$. If $i \in \Delta$, then $E_i$ and $F_i$ act adjointly on $U(w_\Delta)$.

**Proof.** Prove (a) first. Let $w_0 = w'w$. In order to prove the assertion it suffices by Lemma 5.24 to show that $T_{w^{-1}}(a_d(x)(K_\lambda)) \in U^+$ for all $x \in U^+$ and $u \in U^w$.

We obtain that $K_\lambda, \lambda \in P(\mathfrak{g})$ acts on $U(w)$ since

$$T_{w^{-1}}(ad(K_\lambda)) = T_{w^{-1}}(K_\lambda)T_{w^{-1}}(u)T_{w^{-1}}(K_\lambda) \in U^+$$

for all $u \in U(w)$. Part (a) is proved.

Prove (b) now. We need the following fact.

**Lemma 5.24.** Let $w \in W$ be an element of the Weyl group $W$, and $\alpha_i, \alpha_j$ simple roots such that $w(\alpha_i) = -\alpha_j$. Then, $T_w(E_i) = -F_j K_{\alpha_j}$ and $T_w(F_i) = -K_{-\alpha_j} E_i$.

**Proof.**

Recall that by the exchange property (see proof of Lemma 5.21), we can choose a reduced expression $w = w's_1 \cdots s_{k} w$ for $w$ such that $s_i \cdots s_k (\alpha_i) \in R^+(\mathfrak{g})$. Note that if $w'(\alpha_j) = \alpha_j$, then $T_w(E_i) = E_j$ and $T_w(F_i) = F_j$. We compute using (5.1)

$$T_w(E_i) = T_w'(T_1(E_i)) = T_w'(F_j K_{\alpha_j}) = -F_j K_{\alpha_j},$$

$$T_w(F_i) = -T_w'(T_i(F_i)) = T_w'(K_{-\alpha_j} E_i) = -K_{-\alpha_j} E_j.$$

The lemma is proved.

Suppose that $i \in \Delta$. Note that $w_{0,\Delta}^{-1}(\alpha_i) = -\alpha_j$ with $j \in \Delta$, and hence $T_{w_{0,\Delta}^{-1}}(F_i) = K_{\alpha_j} E_j$ by Lemma 5.24.

Let $u \in U(w_\Delta)$. We show that $ad(F_i)u \in U^w$, if $u \in U(w)$ by computing

$$T_{w_{0,\Delta}^{-1}}(ad(F_i)u) = T_{w_{0,\Delta}^{-1}}(F_i)T_{w_{0,\Delta}^{-1}}(u)T_{w_{0,\Delta}^{-1}}(K_{-\alpha_j}) - T_{w_{0,\Delta}^{-1}}(u)T_{w_{0,\Delta}^{-1}}(F_i)T_{w_{0,\Delta}^{-1}}(K_{\alpha_j}) = -K_{-\alpha_j} E_j T_{w_{0,\Delta}^{-1}}(u) K_{\alpha_j} - T_{w_{0,\Delta}^{-1}}(u) K_{-\alpha_j} E_j K_{\alpha_j} \in U_q(\mathfrak{n}^+).$$
because $K_{-\alpha} m K_{\alpha} = q^m m$ for every monomial $m$ in $U(w)$. Note that the proof does not require $w_\Delta$ to be a parabolic element. However, the assumption will be needed to prove the assertion for the action of $E_i$.

Choose $i \in \Delta$ and choose a reduced expression $w_0 = w_{0,\Delta} w$ such that $E_{w_0,\Delta} = E_i$. To complete the proof of the proposition it suffices by Theorem \ref{thm:5.18} (a) to show that $ad(E_i)(E_{\alpha_{i_1}} \ldots E_{\alpha_{i_r}}) \in U(w)$, if $i \in \Delta$ and $E_{\alpha_{ij}} \in U(w)$ for $j \in [1, \ell]$. We use induction on $\ell$.

Consider the case $\ell = 1$; i.e., we have to show that $ad(E_i)(E_\alpha) \in U(w)$ if $i \in \Delta$ and $E_\alpha \in U(w)$. Note that by our choice of $w_0$ we have that if $\alpha_i < \alpha$ for some root $\alpha$, then $E_\alpha \in U(w)$ by Theorem \ref{thm:5.18} (a). Lemma \ref{lem:5.6} yields that

$$ad(E_i)(E_\alpha) \in \text{span}(E_{\gamma_1} \ldots E_{\gamma_k}),$$

where $\alpha < \gamma_1 \leq \ldots \leq \gamma_k < \beta$, and therefore $ad(E_i)(E_\alpha) \in U(w)$ as desired.

Now consider the case when $\ell > 1$. Note that by a straightforward calculation

$$ad(E_i)(E_{\alpha_{i_1}} \ldots E_{\alpha_{i_r}}) = ad(E_i)(E_{\alpha_{i_1}} (E_{\alpha_{i_2}} \ldots E_{\alpha_{i_r}}) + q^r E_{\alpha_{i_1}} ad(E_i)(E_{\alpha_{i_2}} \ldots E_{\alpha_{i_r}})$$

for some $r \in \mathbb{Z}$, and hence $ad(E_i)(E_{\alpha_{i_1}} \ldots E_{\alpha_{i_r}}) \in U(w)$ by the inductive hypothesis. Part (b) is proved. Proposition \ref{prop:5.23} is proved.

Theorem \ref{thm:5.18} (b) is proved.

\section{Proof of Theorem \ref{thm:3.12}}

\subsection{Necessary Conditions.} In this section we establish necessary conditions a weight $\lambda \in P(g)$ has to satisfy if the simple module $V_\lambda$ is Poisson; i.e., we prove the "only if" assertion of the equivalence of (a) and (f) in Theorem \ref{thm:3.12}. Before we proceed with the proof of Theorem \ref{thm:3.12} we will have introduce some convenient notation. Since any finite-dimensional module $V$ over a semisimple Lie algebra $g$ splits as a direct sum of weight spaces $V = \bigoplus_{\mu \in P(g)} V(\mu)$, we will use the abbreviation $V(\mu)^c = \bigoplus_{\nu \neq \mu} V(\nu)$, as the "standard" complement of $V(\mu)$. Additionally we will use the notation and results from Appendix \ref{app:5}.

First we have to calculate $c$ explicitly.

\begin{lemma}
Let $g$ be a semisimple Lie algebra of rank $r$ and $c$ its Casimir element. Then (up to a constant multiple)
\begin{equation}
(6.1) \quad c = [c_{12}, c_{23}] = \sum_{\alpha, \beta \in R^+} \frac{(\alpha, \alpha)(\beta, \beta)}{4} E_\alpha \wedge [E_{-\alpha}, E_\beta] \wedge E_{-\beta}.
\end{equation}
\end{lemma}

\begin{proof}
Choose a basis $H_1, \ldots, H_r$ for $\mathfrak{h}$ which is orthonormal with respect to the Killing form. It is well known that the Casimir element $c$ is up to a constant $c = \sum_{\alpha \in R}(\alpha, \alpha) E_\alpha \otimes E_{-\alpha} + \sum_{i=1}^r H_i \otimes H_i$. We calculate
\begin{equation}
(6.2) \quad c = [c_{12}, c_{23}] = \sum_{\alpha, \beta \in R} (\alpha, \alpha)(\beta, \beta) E_\alpha \otimes [E_{-\alpha}, E_\beta] \otimes E_{-\beta}
\quad + \sum_{\alpha \in R, i=1}^r (\alpha, \alpha) (E_\alpha \otimes [E_{-\alpha}, H_i] \otimes H_i + (\alpha, \alpha) H_i \otimes [H_i, E_\alpha] \otimes E_{-\alpha}).
\end{equation}
\end{proof}
It is easy to see that for all the summands $X \otimes Y \otimes Z$ we have $\{X, Y, Z\} \cap \{E_{\alpha} : \alpha \in R^+\} \neq \emptyset$ and $\{X, Y, Z\} \cap \{E_{-\alpha} : \alpha \in R^+\} \neq \emptyset$. The element $c \in \mathfrak{g}^{\otimes 3}$ is skew-symmetric by Lemma 3.1(a), hence we can write $c = \sum_{\alpha, \beta \in R^+} E_{\alpha} \wedge X_{\alpha, \beta} \wedge E_{-\beta}$.

It follows from (6.2) that or all $\alpha, \beta \in R^+$:

$$E_{\alpha} \wedge X_{\alpha, \beta} \wedge E_{-\beta} = (\alpha, \alpha)(\beta, \beta)E_{\alpha} \otimes [E_{-\alpha}, E_{\beta}] \otimes E_{-\beta} + \text{other terms}.$$ 

This yields that $E_{\alpha} \wedge X_{\alpha, \beta} \wedge E_{-\beta} = 6 \sum_{\alpha, \beta \in R^+} (\alpha, \alpha)(\beta, \beta)E_{\alpha} \wedge [E_{-\alpha}, E_{\beta}] \wedge E_{-\beta}$. Rescaling shows that the lemma is proved. □

The following result will now allow us to observe that large classes of simple modules are “too big” to be Poisson.

**Lemma 6.2.** Let $\mathfrak{g}$ be a complex simple Lie algebra, $P(\mathfrak{g})$ its weight-lattice and $R(\mathfrak{g}) \subset P(\mathfrak{g})$ the corresponding root-system with basis $S = \{\alpha_1, \ldots, \alpha_n\}$. Denote by $w_0 \in W$ the longest element of the Weyl group $W$. Let $\lambda \in P^+(\mathfrak{g})$ be a dominant weight, such that $w_0(\lambda) = -\lambda$. If $V_\lambda$ is Poisson, then $(2\lambda - \alpha_i) \in R(\mathfrak{g}) \cup \{0\}$ for all $\alpha_i$ such that $(\lambda, \alpha_i) \neq 0$.

**Proof.** Let $v_\lambda \in V_\lambda(\lambda)$ be a highest weight vector, and suppose that $2\lambda - \alpha_i \notin R(\mathfrak{g}) \cup \{0\}$. Let $\alpha_i$ be a simple root such that $(\lambda, \alpha_i) \neq 0$. Since $\lambda$ is dominant, $(\lambda, \alpha_i) > 0$. Set $v' = F_{-\alpha_i}(v) \neq 0$. We have, by assumption, $V_\lambda(-\lambda) \neq 0$, since $w_0(\lambda) = -\lambda$, and clearly $(\alpha_i|\lambda) < 0$. Therefore, we obtain for all $v'' \in V_\lambda(-\lambda)$

$$E_{\alpha_i} \wedge F_{\alpha_i} \wedge R_{\alpha_i}(v \wedge v' \wedge v'') = cv \cdot v' \cdot v'' + \varpi,$$

where $\varpi \in (V(\lambda) \cdot V(-\lambda))$. If $2\lambda - \alpha_i \notin R(\mathfrak{g}) \cup \{0\}$, then $E_{\alpha} (v'') \notin V(\lambda - \alpha_i)$ for all $\alpha \in R(\mathfrak{g})$. Additionally, if $2\lambda - \alpha_j \notin R(\mathfrak{g}) \cup \{0\}$ for all $j \in [1, n]$, then $2\lambda$ is not a root, either, and hence $E_{\alpha} (v'') \notin V(\lambda - \alpha_j)$ for all $\alpha \in R(\mathfrak{g})$. Therefore, $c(v \wedge v' \wedge v'') = cv \cdot v' \cdot v'' + \varpi$, where $\varpi \in (V(\lambda) \cdot V(-\lambda))$, as defined in the Appendix in Lemma 8.1 and (8.1). We obtain that $c(v \wedge v' \wedge v'') \neq 0$ and, hence, $V_\lambda$ is not Poisson. The lemma is proved. □

Lemma 6.2 has the following consequence.

**Proposition 6.3.** Let $\mathfrak{g}$ be a complex simple Lie algebra, not isomorphic to $E_8$ or $sl_n(\mathbb{C})$, and let $\lambda \in P^+(\mathfrak{g})$. If $V_\lambda$ is Poisson, then $2\lambda - \alpha_i \in R(\mathfrak{g})$ for all simple roots $\alpha_i$ such that $(\lambda, \alpha_i) \neq 0$.

**Proof.**

Recall the following well known fact.

**Lemma 6.4.** Let $\mathfrak{g}$ be a simple Lie algebra of rank $r$ and let $P(\mathfrak{g})$ be its weight lattice, spanned by the fundamental weights $\omega_i$, $i = 1, \ldots, r$, labeled according to [7] Tables]. Denote by $W(\mathfrak{g})$ the Weyl group and by $w_0$ the longest element of $W(\mathfrak{g})$. We have $w_0^2 = 1 \in W(\mathfrak{g})$ and:

(a) if $\mathfrak{g} = sl_{r+1}(\mathbb{C})$, then $w_0(\omega_i) = -\omega_{r-i}$.

(b) if $\mathfrak{g} = E_6$, then $w_0(\omega_1) = -\omega_6$, $w_0(\omega_2) = -\omega_5$ and $w_0(\omega_3) = -\omega_2$ and $w_0(\omega_3) = -\omega_3$.

(c) if $\mathfrak{g} \neq sl_{r+1}, E_6$, then $w_0(\omega_i) = -\omega_i$ for all $i$.

Lemma 6.3 yields that the assumptions of Lemma 6.2 are satisfied for all $\lambda \in P^+(\mathfrak{g})$, if $\mathfrak{g}$ is not isomorphic to either $sl_n$ or $E_6$. Therefore, Proposition 6.3 now follows from Lemma 6.2. □

Another very useful result is the following.
Lemma 6.5. Let \( g = sl_2 \) and \( V_\ell \) be a \( \ell + 1 \)-dimensional simple \( sl_2 \)-module. It is Poisson if and only if \( \ell \leq 2 \).

Proof. If \( \ell = 0,1 \), then \( V_\ell \) is Poisson, because \( \Lambda^3 V_\ell = 0 \). If \( \ell = 2 \), then \( \Lambda^3 V_2 \cong V_0 \oplus V_2 \) we obtain that \( Hom_g(\Lambda^3 V_2, \Lambda^3 V_2) = \{0\} \) and hence \( V_2 \) is Poisson. Now, suppose that \( \ell \geq 3 \). Let \( E,F,H \) be a standard basis of \( sl_2 \). Since \( \Lambda^3(sl_2) = span(E \wedge F \wedge H) \) we have \( c = E \wedge F \wedge H \). Choose a weight basis \( v_0, \ldots, v_\ell \) of \( V_\ell \) such that \( H(v_i) = (\ell - 2i)v_i, E(v_i) = iv_{i-1} \) and \( F(v_i) = (\ell - i)v_{i+1} \). If, however, \( \ell \geq 3 \), then it is easy to see that

\[
    c(v_1 \wedge v_0 \wedge v_\ell) = E \wedge F \wedge H(v_1 \wedge v_0 \wedge v_\ell) = (-\ell^2)v_1 \cdot v_0 \cdot v_\ell \neq 0.
\]

Hence, \( V_\ell \) is not Poisson, if \( \ell \geq 3 \). The lemma is proved. \( \square \)

Another, very powerful tool will be a special case of Proposition 3.9. Recall that a parabolic subalgebra \( p \) of a semisimple Lie algebra \( g \) is a Lie subalgebra containing a Borel subalgebra of \( g \), and that \( p \) splits as a semidirect product \( g = l \ltimes n \) of a reductive Lie-algebra \( l \), the Levi subalgebra and a nilpotent Lie algebra \( n \), the nilradical. Let \( l \cong g' \oplus z \), where \( g' \) is a semisimple Lie algebra and \( z \) is the center of \( l \). Recall that if \( W \) is a simple \( g \)-module, then the \( W \)-isotypic component of a \( g \)-module \( V \) is the submodule of \( V \) isomorphic to \( (Hom_g(W, V)) \otimes W \).

Proposition 6.6. Let \( g \) be a semisimple Lie algebra, and let \( p \) be a parabolic subalgebra with Levi subalgebra \( l \cong g' \oplus z \), where \( g' \) is semisimple and \( z \) is the center of \( l \). Let \( V \) be a \( g \)-module, and let \( V \) split as an \( l \)-module into a direct sum of isotypic components \( V \cong \bigoplus V_i \). If \( V \) is a Poisson \( g \)-module, then each \( V_i \) must be a Poisson \( l \)-module. Moreover, \( V_i \) must be Poisson as a \( g' \)-module.

Proof. The Lie algebra \( g \) splits, in the notation of Proposition 3.9 as a vector space into \( g = g' \oplus z \), where \( g' \) is semisimple and \( z \) is the center of \( l \). Let \( V \) be a \( g \)-module, and let \( V \) split as an \( l \)-module into a direct sum of isotypic components \( V \cong \bigoplus V_i \). If \( V \) is a Poisson \( g \)-module, then each \( V_i \) must be a Poisson \( l \)-module. Moreover, \( V_i \) must be Poisson as a \( g' \)-module.

Lemma 6.7. Let \( g \) and \( g' \) be as assumed in Proposition 6.6. Let \( V \) be a finite-dimensional \( g \)-module, and \( u \subset V \) an isotypic \( g' \)-module component of \( V \). Let \( \beta \in \mathfrak{B} \) and \( u(\beta) \in U \) be a weight vector. Then \( E_\alpha(v_\beta) \in V_\mathfrak{B} \setminus \{0\} \) implies that \( \alpha \in R(g') \).

Proof. Let \( \alpha \in R(g) \), and \( u_\beta \in U(\beta) \) for some \( \beta \in \mathfrak{B} \). Then, \( E_\alpha(v_\beta) \in V(\alpha + \beta) \). Clearly \( (\alpha + \beta) \in \mathfrak{B} \) implies that \( \alpha = (\alpha + \beta) - \beta \) lies in the \( Z \)-linear span of \( R(g') \), or, equivalently, in the \( Z \)-linear span of a basis of \( R(g') \). Since a root-system is determined uniquely by its basis, this implies that \( \alpha \in R(g') \). Therefore, \( E_\alpha(U) \subset V_\mathfrak{C} \), if \( \alpha \in (R(g) \setminus R(g')) \). Lemma 6.7 is proved. \( \square \)

We can now apply Proposition 3.9 by choosing \( g_{sub} = l \) and \( V_1 = V \). Therefore \( V \) is Poisson if and only if \( V_1 \) is Poisson as a \( l \)-module. The \( l \)-module \( V_1 \) is Poisson if and only if \( V_1 \) is Poisson as a \( g' \)-module (Lemma 6.10). Proposition 6.6 is proved. \( \square \)

We can now derive the following criterion, which allows us to reduce the classification-problem to a few cases.

Lemma 6.8. Let \( g \) be a simple Lie algebra, and let \( \lambda \in P^+(g) \) be a dominant weight. If \( V_\lambda \) is Poisson, then \( (\lambda|\alpha) \leq 2 \) for all roots \( \alpha \in R^+(g) \).
Proof. Consider the subalgebra $\mathfrak{g}_\alpha = E_\alpha \oplus E_{-\alpha} \oplus \mathfrak{h} \subset \mathfrak{g}$. Clearly, $\mathfrak{g}_\alpha$ is isomorphic to $\mathfrak{sl}_2 \oplus \mathbb{C}^{\text{rank}(\mathfrak{g})-1}$. Denote by $\mathfrak{g}_\alpha^+$ the vector space complement of $\mathfrak{g}_\alpha$ spanned by the $E_\beta$ for $\alpha \neq \pm \beta \in R(\mathfrak{g})$. Let $v_\lambda \in V_\lambda(\lambda)$ be a highest weight vector in $V_\lambda$. Then, $v$ generates a simple $(\mathfrak{h}^\perp)$-module $V_\lambda$ and $V_\lambda = V_\ell \oplus V_\ell^+$ as $\mathfrak{g}_\alpha$-modules. We have $\mathfrak{g}_\alpha^+(V_\ell) \subset V_\ell^+$ by Lemma 6.7. We can now apply Proposition 3.9 and obtain that if $V$ is Poisson, then $V_\ell$ is Poisson by Proposition 6.10 as a $\mathfrak{g}^\perp$-module. Hence, $V_\ell$ is Poisson as a $\mathfrak{sl}_2$-module by Proposition 6.10. This implies that $(\lambda|\alpha) + 1) \leq 3$ by Lemma 6.8. Part (a) is proved. The lemma is proved. □

We will now address the necessary conditions on $\lambda \in P^+(\mathfrak{g})$ by type of Lie algebra.

6.1.1. The case of $\mathfrak{g} = \mathfrak{sl}_n$.

Claim 6.9. Let $\mathfrak{g} = \mathfrak{sl}_n$. If $V_\lambda$ is Poisson then $\lambda \in \{\omega_1, 2\omega_1, \omega_2, \omega_{n-2}, \omega_{n-1}, 2\omega_{n-1}\}$

Proof. Lemma 6.8 has the following consequence.

Lemma 6.10. Let $\mathfrak{g} = \mathfrak{sl}_n$ and let $\lambda = \sum_{i=1}^{n-1} \ell_i \omega_i$ be a dominant weight. If $V_\lambda$ is Poisson, then $\sum_{i=1}^{n-1} \ell_i \leq 2$.

Proof. Recall that if $\mathfrak{g} = \mathfrak{sl}_n$, i.e. of type $A_{n-1}$, then the highest root $\alpha_{\text{max}} = \sum_{i=1}^{n-1} \alpha_i$. Since $(\alpha_i, \omega_j) = \delta_{ij}$ for all $i, j \in [1, n-1]$, we obtain that $(\alpha_{\text{max}}|\lambda) = \sum_{i=1}^{n-1} \ell_i$. The assertion now follows from Lemma 6.8(b).

It remains to prove that $V_\lambda$ is not Poisson if $\lambda = \omega_i + \omega_j$, $i \neq j$, $\lambda = 2\omega_k$, $2 \leq k \leq n-2$ or $\lambda = \omega_\ell$ with $3 \leq \ell \leq n-3$. We will consider them case by case.

Lemma 6.11. (a) Let $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 3$ and let $\lambda = \omega_1 + \omega_{n-1}$. Then, $V_\lambda$, the adjoint module, is not Poisson.
(b) Let $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 3$, and let $\lambda = \omega_i + \omega_j$, $1 \leq i < j \leq n-1$. Then $V_\lambda$ is not Poisson.

Proof. Prove (a) first. Denote by $w_0$ the longest element of the Weyl group $W$. It is well known that $w_0(\omega_i) = -\omega_{n-i}$, and hence $w_0(\lambda) = -\lambda$. We know that $\lambda = \omega_1 + \omega_{n-1} = \alpha_{\text{max}}$, the highest root, since $V_\lambda$ is the adjoint module. It is easy to see that $2\alpha_{\text{max}} - \alpha_i$ is not a root for all $i \in [1, n-1]$. Therefore, $V_\lambda$ is not Poisson by Lemma 6.2. Part (a) is proved.

Prove (b) next. Consider the Levi subalgebra $\mathfrak{t}_i$ of $\mathfrak{g}$ obtained by removing the first $i-1$ nodes and the last $n-j-1$ nodes from the Dynkin diagram $A_{n-1}$. Clearly, $\mathfrak{t}_i \cong \mathfrak{sl}_j \oplus \mathbb{C}^{n-j-i+1}$, where $\mathbb{C}$ denotes the trivial $\mathfrak{sl}_n$-module. Any vector $0 \neq v_\lambda \in V_\lambda(\lambda)$ generates an adjoint $\mathfrak{sl}_j \oplus \mathbb{C}^{n-j-i+1}$-module, which is not Poisson by part (a). Therefore, $V_\lambda$ is not Poisson by Proposition 6.3. The lemma is proved. □

Now, consider simple modules of highest weight $\omega_i$ and $2\omega_i$ for $3 \leq i \leq n-3$.

Lemma 6.12. (a) Let $\mathfrak{g} = \mathfrak{sl}_6$ and $\lambda = \omega_3$. The simple module $V_{k\omega_3}$ is not Poisson for all $k \geq 1$.
(b) Let $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 6$ and let $3 \leq i \leq n-3$. If $V_\lambda$ is Poisson, then $\lambda \neq k\omega_i$ for $3 \leq i \leq n-3$.
Claim 6.14. The case of is proved. The highest root is Poisson by part (a). Therefore, \( V(V_{\omega}) \) resp. \( V(0) \) is not Poisson by Proposition 6.6. The lemma is proved.

The following lemma addresses the last case.

Lemma 6.13. (a) If \( g = sl_4, \) then \( V_{\omega} \) is not Poisson. (b) If \( g = sl_n, n \geq 4, \) then \( V_{\omega} \) and \( V_{\omega_n-2} \) are not Poisson.

Proof. Prove (a) first. We have \( 2\omega_3 = \alpha_1 + 2\alpha_2 + \alpha_3 \) and \( w_0(2\omega_2) = -2\omega_2 \) by Lemma 6.4. It is easy to see that \( 4\omega_3 - \alpha_1 \) is not a root for all \( i \in [1, 3], \) since the highest root is \( \alpha_{\max} = \sum_{i=1}^{3} \alpha_i. \) Part (a) is proved.

Prove (b) now. Consider the Levi subalgebra \( \mathfrak{sl}_{3} \) and \( \mathfrak{sl}_{n-3,n-1} \) of \( g \) obtained by removing the last \( n-3 \) nodes (resp. the first \( n-3 \)) from the Dynkin diagram \( A_{n-1}. \) Clearly, \( \mathfrak{sl}_{3} \cong \mathfrak{sl}_{n-3,n-1} \cong \mathfrak{sl}_3 \oplus \mathfrak{c}^{n-3}. \) Any vector \( 0 \neq v_{\omega_i} \in V_{\omega_i}(k\omega_i) \) generates an \( \mathfrak{sl}_3 \)-module isomorphic to \( V_{\omega_i} \), which is not Poisson by part (a). Therefore, \( V_{\omega_i} \) is not Poisson by Proposition 6.6. The lemma is proved.

Claim 6.9 is proved.

6.1.2. The case of \( g = so(2n+1). \)

Claim 6.14. Let \( g = so(2n+1). \) If \( V_\lambda \) is Poisson, then \( \lambda = \omega_1 \) or \( (g, V_\lambda) = (so(5), V_\omega). \)

Proof. Consider first the case of \( g = so(5). \) Let \( \{\alpha_1, \alpha_2\} \) be a basis of the rootsystem \( R(g). \) The fundamental weights are \( \omega_1 = \alpha_1 + \alpha_2 \) and \( \omega_2 = \frac{\alpha_1 + \alpha_2}{2}. \) The highest root is \( \alpha_{\max} = \alpha_1 + 2\alpha_2 = 2\omega_2. \) It is easy to verify that if \( \lambda \in P^+(g) \) and the weight \( 2\lambda - \alpha_i \in R(g) \) for some \( i = 1, 2 \) imply that \( \lambda \in \{\omega_1, \omega_2\}. \) Employing Proposition 6.3 we obtain immediately that if \( \lambda \) is Poisson, then \( \lambda \in \{\omega_1, \omega_2\}. \)

We now consider \( g = so(2n+1), n \geq 3. \) Let \( R(g) \) be the corresponding root system and let \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) be a basis. The fundamental weights are, for \( 1 \leq i \leq n-1, \)

\[
\omega_i = \alpha_1 + 2\alpha_2 + \ldots + (i-1)\alpha_{i-1} + i(\alpha_i + \ldots + \alpha_n)
\]

\[
\omega_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \ldots + n\alpha_n).
\]

The highest root is \( \alpha_{\max} = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_n. \) It is easy to verify that for \( \lambda \in P^+(g) \) and \( 2\lambda - \alpha_i \in R(g) \) for some \( i \in [1, n] \) imply that \( \lambda = \omega_1 \) or, in the case \( n = 3, \lambda = \omega_3. \) We now obtain immediately from Proposition 6.3 that if \( \lambda \) is Poisson and \( n \geq 4 \) then \( \lambda = \omega_1. \)

Consider the case \( n = 3. \) We have to show that \( V_{\omega_3} \) is not Poisson. Note that \( \lambda = \omega_3 = \frac{1}{2}\alpha_1 + \alpha_2 + \frac{3}{2}\alpha_3 \) and \( \alpha_{\max} = \alpha_1 + 2\alpha_2 + 2\alpha_3. \) Let \( v \in V_{\omega_3}(\omega_3), \)

\( v' = E_{\alpha_3}(v) \) and \( v'' = E_{\alpha_{\max}}(v'). \) Clearly \( v' \neq 0, v'' \neq 0, v' \in V_{\omega_3}(\alpha_1 + \alpha_2 + \frac{3}{2}\alpha_3) \)

\hfill \Box
and \( v'' \in V_{\omega_3}(-\omega_3) \). Consider roots, \( \alpha, \beta \in R^+(g) \). We have for all \( \alpha, \beta \in R(g) \), \( w(\gamma_i) \in V(\gamma_i) \),

\[
E_{\alpha} \land [E_{-\alpha}, E_{\beta}] \land E_{-\beta}(v \land v' \land v'') = \sum_{i=1}^{k} w(\gamma_i) \cdot w(\gamma'_i) \cdot w(\gamma''_i),
\]

where \( w(\gamma_i) \in V(\gamma_i) \), \( w(\gamma'_i) \in V(\gamma'_i) \), and \( w(\gamma''_i) \in V(\gamma''_i) \) such that \( \gamma_i + \gamma'_i + \gamma''_i = \frac{1}{2} \alpha_1 + \alpha_2 + \frac{1}{2} \alpha_3 \). We obtain that \( (\gamma_i, \gamma'_i, \gamma''_i) = (\omega_3, \frac{1}{2} \alpha_1 + \alpha_2 + \frac{1}{2} \alpha_3, -\omega_3) \), if and only if \( (\alpha, \beta) \in \{(\alpha_3, \alpha_3), (\alpha_{max}, \alpha_{max})\} \). Denote by

\[
c' = \frac{(\alpha_3, \alpha_3)^2}{4} E_{\alpha_3} \land \hat{\alpha}_3 \land E_{-\alpha_3} + \frac{(\alpha_{max}, \alpha_{max})^2}{4} E_{\alpha_{max}} \land \hat{\alpha}_{max} \land E_{-\alpha_{max}}.
\]

We obtain that

\[
(c - c')(v \land v' \land v'') \in \left( V_{\omega_3}(\omega_3) \cdot V_{\omega_3}(\frac{1}{2} \alpha_1 + \alpha_2 + \frac{1}{2} \alpha_3) \cdot V_{\omega_3}(-\omega_3) \right)^c \subset S^3 V_{\omega_3}.
\]

A straightforward calculation shows that \( c'(v \cdot v' \cdot v'') \neq 0 \). This implies that \( c(v \land v' \land v'') \neq 0 \) and proves that \( V_{\omega_3} \) is not Poisson. Claim 6.13 is proved.

6.1.3. Type \( g = so(2n) \).

Claim 6.15. Let \( g = so(2n) \). If \( V_\lambda \) is Poisson, then \( \lambda = \omega_1 \) or \( n \in \{4, 5\} \) and \( \lambda \in \{\omega_n, \omega_{n-1}\} \).

Proof. Let \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) be a basis of \( R(g) \). The fundamental weights have the form

\[
\omega_1 = \alpha_1 + 2\alpha_2 + (i-1)\alpha_{i-1} + i(\alpha_i + \ldots + \alpha_{n-2}) + \frac{1}{2} i(\alpha_{n-1} + \alpha_n),
\]

\[
\omega_{n-1} = \frac{1}{2} \left( \alpha_1 + 2\alpha_2 + \ldots + (n-2)\alpha_{n-2} + \frac{1}{2} n\alpha_{n-1} + \frac{1}{2} (n-2)\alpha_n \right),
\]

\[
\omega_n = \frac{1}{2} \left( \alpha_1 + 2\alpha_2 + \ldots + (n-2)\alpha_{n-2} + \frac{1}{2} (n-2)\alpha_{n-1} + \frac{1}{2} n\alpha_n \right).
\]

The highest root is \( \alpha_{max} = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \). It is easy to verify that if the weight \( 2\lambda - \alpha_i \in R(g) \) for some \( \lambda \in P^+(g) \), then \( \lambda = \omega_1 \) or, in the case of \( n = 4, 5, \lambda = \omega_n \) and \( \lambda = \omega_{n-1} \). Therefore, we obtain immediately from Proposition 6.3 that if \( V_\lambda \) is Poisson, then \( \lambda = \omega_1 \), or \( n \in \{4, 5\} \) and \( \lambda \in \{\omega_n, \omega_{n-1}\} \).

Claim 6.15 is proved.

6.1.4. Type \( g = sp(2n) \).

Claim 6.16. Let \( g = sp(2n) \). If \( V_\lambda \) is Poisson, then \( \lambda = \omega_1 \) or \( (g, V) = (sp(4), V_{\omega_2}) \).

Proof. If \( n = 2 \), we have \( sp(4) \cong so(5) \) and we obtain from Section 6.1.2 that \( V_\lambda \) is not Poisson, unless \( \lambda = \omega_i \) for \( i = 1, 2 \).

Now, let us continue with the case of \( g = sp(2n), n \geq 3 \). Let \( \{\alpha_1, \ldots, \alpha_n\} \) be a basis of \( R(g) \). The fundamental weights have the form

\[
\omega_i = \alpha_1 + 2\alpha_2 + \ldots + (i-1)\alpha_{i-1} + i(\alpha_i + \ldots + \alpha_{n-1}) + \frac{1}{2} i(\alpha_n),
\]

for \( i \leq n \), and the highest root is \( \alpha_{max} = 2\alpha_1 + 2\alpha_2 + \ldots + \alpha_n \). It is easy to verify that if \( \lambda \in P^+(g) \) and \( 2\lambda - \alpha_i \in R(g) \), then \( \lambda = \omega_1 \). Therefore, we obtain
Claim 6.17. Let $\mathfrak{g}$ be an exceptional complex simple Lie algebra. If $V_\lambda$ is a non-trivial simple Poisson module, then $\mathfrak{g} = E_6$ and $\lambda \in \{\omega_1, \omega_6\}$.

**Proof.** The Dynkin diagram $E_6$ contains two subdiagrams of type $D_5$ and one of type $A_5$ and Levi subalgebras isomorphic to $so(10) \oplus \mathbb{C}$ and $sl_6 \oplus \mathbb{C}$. Let $\lambda = \sum_{i=1}^6 \ell_i \omega_i \in P^+ (\mathfrak{g})$. We obtain that $v_\lambda \in V_\lambda (\lambda)$ generates a $so(10)$-module $V_\lambda'$, where $\lambda' = \ell_1 \omega_1 + \ell_2 \omega_2 + \ell_4 \omega_4 + \ell_5 \omega_5$ and an $sl_6$-module $V_\lambda''$, where $\lambda'' = \ell_1 \omega_1 + \sum_{i=2}^5 \ell_{i+1} \omega_i$. Note that we are only adjusting notations from [7] for the various root systems $A_5$, $D_5$ and $E_6$. Applying Proposition 6.3 to the corresponding Levi subalgebras and the results of Sections 6.1 and 6.3 we obtain that if $V_\lambda$ is Poisson, then $\lambda = \omega_i$, if $i = 1, 2, 6$. When considering the case $E_6$, we cannot apply Proposition 6.3 to all weights $\lambda$, because the longest word in the Weyl-group associated to $E_6$ does not send all dominant weights $\lambda \in P^+ (\mathfrak{g})$ to $-\lambda \in P (\mathfrak{g})$. Since $\omega_2 = \alpha_{\text{max}}$ we immediately see that $V_{\omega_2}$ is the adjoint module. Clearly, $2\omega_{\text{max}} - \alpha_j$ is not a root for all $j \in [1, 6]$, and hence $V_{\omega_2}$ is not Poisson by Proposition 6.3.

Next, let $\mathfrak{g} = E_7$. Denote by $R(\mathfrak{g})$ the corresponding root system and by $P(\mathfrak{g})$ the weight lattice. It is easy to derive from the tables in [7] that there exists no $\lambda \in P^+ (\mathfrak{g})$ such that $2\lambda - \alpha_i \in R(\mathfrak{g})$ for any $i \in [1, 7]$. Therefore, Proposition 6.3 yields that there is no $\lambda \in P^+ (\mathfrak{g})$ such that $V_\lambda$ is Poisson.

Now, let $\mathfrak{g} = E_8$. Denote by $R(\mathfrak{g})$ the corresponding root system and by $P(\mathfrak{g})$ the weight lattice. It is easy to derive from the tables in [7] that there exists no $\lambda \in P^+ (E_7)$ such that $2\lambda - \alpha_i \in R(\mathfrak{g})$ for any $i \in [1, 8]$. Therefore, Proposition 6.3 yields that there is no $\lambda \in P^+ (\mathfrak{g})$ such that $V_\lambda$ is Poisson.

As the second to last case, we will consider the case of $\mathfrak{g} = F_4$. Denote by $R(\mathfrak{g})$ the corresponding root system and by $P(\mathfrak{g})$ the weight lattice. It is easy to derive from the tables in [7] that there exists no $\lambda \in P^+ (\mathfrak{g})$ such that $2\lambda - \alpha_i \in R(\mathfrak{g})$ for any $i \in [1, 4]$. Therefore, Proposition 6.3 yields that there is no $\lambda \in P^+ (\mathfrak{g})$ such that $V_\lambda$ is Poisson.

Finally, let $\mathfrak{g} = G_2$. Denote by $R(\mathfrak{g})$ the corresponding root system and by $P(\mathfrak{g})$ the weight lattice, $\{\alpha_1, \alpha_2\}$ a basis of $R(\mathfrak{g})$. The fundamental weights are $\omega_1 = 2\alpha_1 + \alpha_2$ and $\omega_2 = 3\alpha_1 + 2\alpha_2$, and the highest root is $\alpha_{\text{max}} = \omega_2 = 3\alpha_1 + 2\alpha_2$. It is easy to verify that if $\lambda \in P^+ (\mathfrak{g})$ and $2\lambda - \alpha_i \in R(\mathfrak{g})$ for some $i \in [1, 2]$ then $\lambda = \omega_1$. Therefore, if $V_\lambda$ is Poisson then $\lambda = \omega_1$ by Proposition 6.3.

Now, let $\lambda = \omega_1$. Let $v \in V_{\omega_1} (\omega_1)$ and $v' = E_{\alpha_1} (v) \in V_{\omega_1} (\alpha_1 + \alpha_2)$ and $v'' = E_{3\alpha_1 + 2\alpha_2} (v') \in V_{\omega_1} (-\omega_1)$. It is easy to see that $v' \neq 0$ and $v'' \neq 0$. As in the discussion of the $so(7)$-case we have for all $\alpha, \beta \in R(\mathfrak{g})$, $w(\gamma_i) \in V(\gamma_i)$,

$$E_\alpha \wedge [E_{-\beta}, E_\beta] \wedge E_{-\beta} (v \wedge v' \wedge v'') = \sum_{i=1}^k w(\gamma_i) \cdot w(\gamma'_i) \cdot w(\gamma''_i),$$

and $w(\gamma_i) \in V(\gamma_i)$, $w(\gamma'_i) \in V(\gamma''_i)$ such that $\gamma_i + \gamma'_i + \gamma''_i = \alpha_1 + \alpha_2$. We obtain that $(\gamma_i, \gamma'_i, \gamma''_i) = (\omega_1, \alpha_1 + \alpha_2, -\alpha_1 - \alpha_2)$ if and only if $(\alpha, \beta) \in \{\{\alpha, \alpha_1\}, \{\alpha_{\text{max}}, \alpha_{\text{max}}\}\}$. Denote by

$$c' = \frac{(\alpha_1, \alpha_1)^2}{4} E_{\alpha_1} \wedge \tilde{\alpha}_1 \wedge E_{-\alpha_1} + \frac{\alpha_{\text{max}}, \alpha_{\text{max}})^2}{4} E_{\alpha_{\text{max}}} \wedge \tilde{\alpha}_{\text{max}} \wedge E_{-\alpha_{\text{max}}}.$$
This implies that
\[ c - c' \in (V_{\omega_1}(\omega_1) \cdot V_{\omega_2}(\alpha_1 + \alpha_2) \cdot V_{\omega_1}(-\omega_1))^c \subset S^3V_{\omega_3}. \]

A straightforward computation yields that \( c(v \wedge v' \wedge v'') \neq 0 \). Hence, \( V_{\omega_1} \) is not Poisson. That concludes our discussion of the case \( g = G_2 \).

Claim \ref{claim:6.17} is proved. \hfill \square

6.2. **Sufficient Conditions and Rigidity.** In order to prove the remaining assertions of Theorem \ref{theorem:6.12}, it now suffices to show that for all the modules \( V_\lambda \) listed in Theorem \ref{theorem:6.12} (f), we have that \( Hom_g(\Lambda^3V_\lambda, S^3V_\lambda) = \{0\} \), since this implies that \( V_\lambda \) is Poisson by Proposition \ref{proposition:5.3}. The decomposition of the symmetric and exterior powers of the geometrically decomposable modules and the \( sp(2n) \)-module \( V_{\omega_1} \) are well known. As a reference see (e.g. \cite{18}). In particular, in the case of \( g = sl_n \), the decompositions of symmetric powers of the geometrically decomposable modules \( V_\lambda \) are well known results from classical invariant theory, in the remaining cases the symmetric powers have been computed by multiple authors (see e.g. \cite{34}).

The decomposition of the exterior powers of the simple geometrically decomposable modules and the \( sp(2n) \)-module \( V_{\omega_1} \) are computed in a beautiful way by Stembridge in \cite{50}; the decomposition however was already well known through the calculations of Lie algebra cohomology of nilradicals by Kostant (\cite{26}). One immediately obtains from these results that \( Hom_g(\Lambda^3V_\lambda, S^3V_\lambda) = \{0\} \).

We are, however, interested in proving a stronger result which will be useful in the classification of flat modules in Section 4.4. In \cite{6} we introduced a lower bound for the dimension of the symmetric and exterior cube, and correspondingly we construct a minimal submodule contained in the symmetric and exterior cubes. We will use the notation
\[
V^\mu
\]
for the space of highest weight vectors of weight \( \mu \) in a finite-dimensional \( g \)-module \( V \). For dominant \( \lambda, \mu, \nu \in P^+(g) \) denote \( c^\mu_{\lambda,\mu} = \dim(V_\lambda \otimes V_\mu)^\nu - \dim_{C}(Hom_g(V_{\lambda}, V_\mu \otimes V_\nu)) \); i.e., \( c^\mu_{\lambda,\mu} \) is the tensor product multiplicity. And for any \( \lambda, \mu \in P^+(g) \) denote \( c^\mu_{\lambda,\mu} = \dim(S^2V_\lambda)^\mu \) and \( c^\mu_{\lambda,\mu} = \dim(\Lambda^2V_\lambda)^\mu \), so that \( c^+_{\lambda,\mu} + c^-_{\lambda,\mu} = c^\mu_{\lambda,\mu} \). Ultimately, define:
\[
d^\mu_{\lambda} := \sum_{\nu \in P^+_+} (c^+_{\lambda,\nu} - c^-_{\lambda,\nu}) c^\mu_{\nu,\lambda}.
\]

We need the following definition.

**Definition 6.18.** We will call the \( g \)-module \( S^3_{low}V_\lambda = \bigoplus_{\mu \in P^+_+} \mathbb{C}^{\max(d^\mu_{\lambda,\mu},0)} \otimes V_\mu \) the \"lower symmetric cube\" of \( V_\lambda \), and similarly \( \Lambda^3_{low}V_\lambda = \bigoplus_{\mu \in P^+_+} \mathbb{C}^{\max(-d^\mu_{\lambda,\mu},0)} \otimes V_\mu \) the \"lower exterior cube\".

The definition is motivated by the following fact.

**Lemma 6.19.** There exist injective homomorphisms of \( g \)-modules from \( S^3_{low}V_\lambda \hookrightarrow S^3V_\lambda \) and \( \Lambda^3_{low}V_\lambda \hookrightarrow \Lambda^3V_\lambda \).

**Proof.** By definition of \( S^3V_\lambda \), one has: \( (S^3V_\lambda)^\mu = (S^2V_\lambda \otimes V_\lambda)^\mu \cap (V_\lambda \otimes S^2V_\lambda)^\mu \). Therefore, we obtain the inequality:
\[
\dim(S^3V_\lambda)^\mu \geq \dim(S^2V_\lambda \otimes V_\lambda)^\mu + \dim(V_\lambda \otimes S^2V_\lambda)^\mu - \dim(V_\lambda \otimes V_\lambda \otimes V_\lambda)^\mu
\]
\[
= \dim(S^2V_\lambda \otimes V_\lambda)^\mu - \dim(V_\lambda \otimes \Lambda^2V_\lambda)^\mu = d^\mu_{\lambda}\]
The existence of injective homomorphisms of \( g \)-modules from \( S^3_{\text{low}} V_\lambda \rightarrow S^3 V_\lambda \) follows and the assertion for \( \Lambda^3 V_\lambda \) can be proved analogously.

We say that a \( g \)-module \( V_\lambda \) is rigid, if \( S^3_{\text{low}} V_\lambda \cong S^3 V_\lambda \) and \( \Lambda^3_{\text{low}} V_\lambda \cong \Lambda^3 V_\lambda \). The following theorem is the main result of this section.

**Theorem 6.20.** Let \( g \) be a simple Lie algebra. A simple \( g \)-module is rigid, if and only if \((g, V)\) is one of the pairs listed in Theorem 6.12 (f).

**Proof.** We will first prove the "only if" assertion. Note the following fact connecting rigid and Poisson modules.

**Proposition 6.21.** If a simple \( g \)-module \( V_\lambda \) is rigid, then \( \text{Hom}_g(\Lambda^3 V_\lambda, S^3 V_\lambda) = \{0\} \) and \( V_\lambda \) is Poisson.

**Proof.** It is easy to see that

\[
\dim(\text{Hom}_g(\Lambda^3 V_\lambda, S^3 V_\lambda)) = \sum_{\mu \in P^+} \max(d_{\lambda, \mu}^a, 0) \cdot \max(-d_{\lambda, \mu}^b, 0) = 0.
\]

We immediately obtain \( \text{Hom}_g(\Lambda^3 V_\lambda, S^3 V_\lambda) = \{0\} \), and hence, that \( V_\lambda \) is Poisson.

We obtain from Proposition 6.21 and the arguments in Section 6.11 that if \( V \) is rigid, then \((g, V)\) must be one of the pairs of Theorem 6.12 (f).

The proof of the converse will consist of the following steps for each of the listed simple modules \( V_\lambda \).

1. Determine the decomposition \( S^2 V_\lambda \) and \( \Lambda^2 V_\lambda \), as well as \( S^3 V_\lambda \) and \( \Lambda^3 V_\lambda \). Recall from above that the splittings of the symmetric and exterior powers of the modules in question are well known (see e.g. [18, chapter 4]).

2. Suppose \( V_\mu \) appears with multiplicity one in \( S^3 V_\mu \). Show that \( d_{\lambda, \mu}^a = 1 \). Similarly, if \( V_\mu \) appears with multiplicity one in \( \Lambda^3 V_\mu \). Show that \( d_{\lambda, \mu}^b = -1 \).

In order to further simplify our computations note the following fact. Recall that if \( g \) is a semisimple Lie algebra and \( \tau \) a graph automorphism of the Dynkin diagram, then \( \tau \) induces automorphisms \( \tau_g : g \rightarrow g, \tau_P : P(g) \rightarrow P(g) \).

**Lemma 6.22.** Let \( g \) be a semisimple Lie algebra and let \( \tau \) be a graph automorphism of the Dynkin diagram of \( g \).

(a) We have for the tensor product multiplicities \( c_{\lambda, \mu}^a = c_{\tau(\lambda), \tau(\mu)}^a \). Similarly, if \( S^n V_\lambda = \bigoplus_i V_{\nu_i} \) then \( S^n V_{\tau(\lambda)} = \bigoplus_i V_{\tau(\nu_i)} \) and if \( \Lambda^n V_\lambda = \bigoplus_i V_{\nu_i} \), then \( \Lambda^n V_{\tau(\lambda)} = \bigoplus_i V_{\tau(\nu_i)} \).

(b) If the simple \( g \)-module \( V_\lambda \) is rigid, then \( V_{\tau(\lambda)} \) is rigid.

**Proof.** Part (a) is well known, and (b) follows immediately from (a).

In order to accomplish Step (2) we will need a generalization of the tensor product stabilization of the Littlewood-Richardson rule to other types of simple Lie algebras by Kleber and Vishwanath ([24] and [38]). Let \( g \) be a complex simple Lie algebra of type \( X_n \), \( X \in \{A, B, C, D\} \). We denote for a triple \( \lambda, \mu, \nu \in P^+(g) = P^+(X_n) \) of dominant weights the tensor multiplicity \( c_{\lambda, \mu}^\nu(X_n) \).

We first recall the Littlewood-Richardson rule for \( g = sl_{n+1} \mathbb{C} \); i.e. the Dynkin diagram of \( g \) is of type \( A_n \): For dominant weights \( \lambda = \sum_{i=1}^n \ell_i \omega_i, \mu = \sum_{i=1}^n m_i \omega_i \) and \( \nu = \sum_{i=1}^n n_i \omega_i \) one has \( c_{\lambda, \mu}^\nu(A_n) = c_{\lambda, \mu}^\nu(A_m) \) for all \( m \geq n \) if
This phenomenon is commonly referred to as tensor product stabilization (see e.g. [24]).

Next, let \( \mathfrak{g} \) be a complex simple Lie algebra of type \( X_n \), \( X \in \{ B, C, D \} \). Following ideas of \([38] \) Ch. 6, Corollary 3] we call a weight \( \gamma \in P(\mathfrak{g}) = P(X_n) \) \( A \)-supported, if \( \gamma = \sum_{i=1}^{\omega_n} c_i \omega_i \). This means that \( \gamma \) is supported entirely in the \( A_k \)-part of the Dynkin diagram \( X_n \). Note that if a weight \( \gamma = \sum_{i=1}^{\omega_n} c_i \omega_i \) is \( A \)-supported in \( P(X_n) \), then it is \( A \)-supported in \( P(X_m) \) for all \( m \geq n \). One obtains the following fact.

**Proposition 6.23.** \([38] \) Ch. 6, Corollary 3, Remark 10.1] Let \( \mathfrak{g} \) be a complex simple Lie algebra of type \( X_n \) where \( X \in \{ B, C, D \} \). If \( \lambda, \mu, \nu \in P^+(X_n) \) are \( A \)-supported, then

\[
c^\nu_{\lambda, \mu}(X_n) = c^\nu_{\lambda, \mu}(X_m) , \ m \geq n .
\]

We will now show, case by case, that the modules in question are indeed rigid.

Since the computations are rather long but straightforward, we include the complete proof in only one nontrivial case ((\( \mathfrak{g}, V \) = (\( \mathfrak{sl}_n, V_{\omega_1} \))). Complete calculations can be found in \([41] \).

### 6.2.1. The case \( \mathfrak{g} = \mathfrak{sl}_n \).

**Proposition 6.24.** If \( \mathfrak{g} = \mathfrak{sl}_n \), then the simple module \( V = V_\lambda \) is rigid, if \( \lambda \in \{ \omega_1, 2\omega_1 \omega_2, \omega_n-2, \omega_n-1, 2\omega_n-1 \} \).

**Proof.** By Lemma 6.22 it suffices to consider the case of \( \lambda \in \{ \omega_1, 2\omega_1 \omega_2 \} \).

**Lemma 6.25.** Let \( \mathfrak{g} = \mathfrak{sl}_n \), \( n \geq 7 \). We have the following:

(a) Let \( V = V_{\omega_1} \). We have \( S^2 V = V_{2\omega_1} \), \( S^3 V = V_{3\omega_1} \), \( \Lambda^2 V = V_{\omega_2} \) and \( \Lambda^3 V = V_{\omega_3} \).

(b) \([18] \) Theorem 3.1, Theorem 4.4.2] Let \( V = V_{2\omega_1} \). We have \( S^2 V \cong V_{4\omega_1} \oplus V_{2\omega_2} \), \( \Lambda^2 V \cong V_{2\omega_1 + \omega_2} \), \( S^3 V \cong V_{6\omega_1} \oplus V_{2\omega_1 + 2\omega_2} \oplus V_{2\omega_2 - 3} \) and \( \Lambda^3 V \cong V_{3\omega_1 + \omega_3} \oplus V_{3\omega_2} \).

(c) \([18] \) Theorem 3.8.1, Theorem 4.4.4] Let \( V = V_{\omega_2} \). We have \( S^2 V \cong V_{2\omega_2} \oplus V_{\omega_4} \), \( \Lambda^2 V \cong V_{\omega_1 + \omega_3} \), \( S^3 V \cong V_{3\omega_1} \oplus V_{\omega_2 + \omega_4} \oplus V_{2\omega_3} \).

It remains to show the following.

**Lemma 6.26.** Let \( \mathfrak{g} = \mathfrak{sl}_n \), \( n \geq 7 \).

(a) If \( \lambda = \omega_1 \), then \( d^\lambda_{\omega_1} = 1 \) if \( \lambda = 3\omega_1 \), and \( d^\lambda_{\omega_1} = -1 \) if \( \lambda = \omega_3 \).

(b) If \( \lambda = 2\omega_1 \), then \( d^\lambda_{2\omega_1} = 1 \) if \( \lambda \in \{ 6\omega_1, 2\omega_1 + 2\omega_2, 2\omega_3 \} \) and \( d^\lambda_{2\omega_1} = -1 \) if \( \lambda \in \{ 3\omega_1 + \omega_2, 3\omega_2 \} \).

(c) If \( \lambda = \omega_2 \), then \( d^\lambda_{2\omega_1} = 1 \), if \( \lambda \in \{ 3\omega_2, \omega_2 + \omega_4, \omega_6 \} \) and \( d^\lambda_{2\omega_1} = -1 \), if \( \lambda \in \{ 2\omega_1 + \omega_4, 2\omega_3 \} \).

**Proof.**

Computing the decompositions manually (or using the computer algebra system LIE \([37] \) for \( n = 7 \), and applying the Littlewood Richardson rule for all \( n \geq 7 \) we obtain

We have for \( S^2 V_{\omega_1} \otimes V_{\omega_1} \):

\[
c^\omega_{2\omega_1, \omega_1} = 1 , c^\omega_{2\omega_1, \omega_1} = 0,
\]
Lemma 6.29. The assertion for the remaining cases of $\lambda \neq \omega_1$. Part (a) is proved.

We have for the factors of $S^2 V_{2\omega_1}$:

$$V_{2\omega_1} \otimes V_{2\omega_1} : c_{2\omega_1, 2\omega_1} = 1, c_{2\omega_1 + 2\omega_2, 2\omega_1} = 1, c_{2\omega_1 + 2\omega_3, 2\omega_1} = 1, c_{2\omega_1 + 2\omega_4, 2\omega_1} = 1,$$

for $\lambda \in P^+(g)$, unless $(\lambda, \omega_i) > 0$ for some $i \geq 7$.

If $\lambda \in P^+(g)$, unless $(\lambda, \omega_i) > 0$ for some $i \geq 7$.

For $\lambda \in P^+(g)$, unless $(\lambda, \omega_i) > 0$ for some $i \geq 7$. Part (b) is proved.

Proceed similarly to prove part (c). We can now read off the $d^u_\lambda$ mentioned in the Lemma. The lemma is proved.

We obtain that the simple $sl_n$-modules $V_{\omega_1}$, $V_{2\omega_1}$ and $V_{\omega_2}$ are rigid for $n \geq 7$.

In case of $n < 7$ one can prove the assertion of the proposition by direct computation (e.g. using LIE). Proposition 6.24 is proved.

6.2.2. The case $g = \mathfrak{so}(k)$. Next, we will proceed with the case when, $g = \mathfrak{so}(k)$.

Proposition 6.27. Let $g = \mathfrak{so}(k)$ and $V = V_{\omega_1}$. Then $V$ is rigid. Moreover, $V = V_\lambda$ is rigid, if $g = \mathfrak{so}(8)$ and $\lambda = \{\omega_3, \omega_4\}$ or if $g = \mathfrak{so}(10)$ and $\lambda = \{\omega_4, \omega_5\}$.

Proof.

Lemma 6.28. [34] [36] Let $g = \mathfrak{so}(k)$, $k \geq 9$. Then $S^2 V_{\omega_1} \cong V_{2\omega_1} \oplus V_0$, $\Lambda^2 V_{\omega_1} \cong V_{\omega_2}$, $S^3 V_{\omega_1} \cong V_{3\omega_1} \oplus V_{\omega_1}$ and $\Lambda^3 V_{\omega_1} \cong V_{\omega_3}$.

Lemma 6.29. Let $g = \mathfrak{so}(k)$, $k \geq 9$. If $\mu \in \{3\omega_1, \omega_3\}$, then $d^u_\mu = 1$. If $\mu = \omega_3$, then $d^u_\mu = -1$.

Proof. Analogous to the proof of Lemma 6.26.

We obtain the assertion of Proposition 6.27 for $k \geq 9$ from Lemma 6.28 and Lemma 6.29. The assertion for the remaining cases of $k \leq 8$ and the case $k = 10$, $\lambda \in \{\omega_4, \omega_5\}$ can be verified directly (e.g. using LIE). Proposition 6.27 is proved.
6.2.3. The case \( g = \text{sp}(2n) \).

**Proposition 6.30.** Let \( g = \text{so}(2n) \) and \( V = V_{\omega_1} \). Then \( V \) is rigid.

**Proof.**

**Lemma 6.31.** \([18], [39]\) Let \( g = \text{sp}(2n) \) with \( n \geq 4 \) and let \( V = V_{\omega_1} \). We have \( S^2V \cong V_{2\omega_1} \) and \( \Lambda^2V \cong V_{\omega_2} \oplus V_0 \). Moreover, \( S^3V \cong V_{3\omega_1} \) and \( \Lambda^3V \cong V_{\omega_1} \oplus V_{\omega_3} \).

Moreover we have the following fact.

**Lemma 6.32.** Let \( g = \text{sp}(2n) \), \( n \geq 4 \). We have that \( d^{3}\omega_1 = 1 \), if \( \mu = 3\omega_1 \) and \( d^{\mu}_{\omega_1} = -1 \), if \( \mu \in \{\omega_3, \omega_1\} \).

**Proof.** Analogous to the proof of Lemma 6.26.

This proves the proposition in the case \( n \geq 4 \). The assertion for the remaining cases \( n \leq 3 \) can be verified directly using LIE. Proposition 6.30 is proved.

6.2.4. **Proof of Theorem 6.20 and the proof of Theorem 3.12** The assertion of Theorem 6.20 in the cases of \( g = E_6 \) and \( \lambda = \{\omega_1, \omega_6\} \) can also be verified through direct computation (e.g. using LIE). Theorem 6.20 then follows from Propositions 6.21 and 6.30.

Theorem 6.20 and Proposition 6.21 imply that (f) yields (c) and (a). Thus, we have so far proved the equivalence of parts (a–c), (e) and (f) of Theorem 3.12. Additionally, we obtain that (f) implies (d) from the explicit computation of the exterior squares in the proof of Theorem 6.20. We complete the proof of Theorem 3.12 by showing that (d) implies (a).

**Proposition 6.33.** Let \( g \) be a semisimple Lie algebra and let \( V_\lambda \) be a simple \( g \)-module. If \( \Lambda^2V_\lambda \) is simple, then \( V_\lambda \) is Poisson.

**Proof.** Suppose that \( V_\lambda \) is simple. Then \( c \) acts as multiplication by a constant \( \mu \) on \( \Lambda^2V_\lambda \). Therefore, \( c_{12}(\varpi) = \mu \varpi \) for all \( \varpi \in \Lambda^2V_\lambda \oplus V_\lambda \) and \( c_{23}(\varpi) = \mu \varpi \) for all \( \varpi' \in V_\lambda \otimes \Lambda^2V_\lambda \). This implies that

\[
[c_{12}, c_{23}](\varpi) = (\mu^2 - \mu^2)(\varpi) = 0
\]

for all \( \varpi \in \Lambda^2V_\lambda \otimes V_\lambda \cap V_\lambda \otimes \Lambda^2V_\lambda = \Lambda^3V_\lambda \) and hence, \( V_\lambda \) is Poisson. Proposition 6.33 is proved.

Proposition 6.33 completes the proof of Theorem 3.12.

Theorem 3.12 and Theorem 6.20 yield the following corollary, which will play an important role in the classification of flat modules in Section 4.

**Corollary 6.34.** Let \( g \) be a simple Lie algebra and \( V \) a simple \( g \)-module. \( V \) is Poisson, if and only if \( V \) is rigid.
7. Open Questions and Conjectures

In this final chapter we will present a number of conjectures and questions which will be interesting for future research in the area of braided symmetric algebras. The classification of flat modules lets us expect that the following conjecture holds.

**Conjecture 7.1.** Let $g$ be a reductive Lie algebra and $V^g$ a finite dimensional $U_q(g)$-module. The braided symmetric algebra $S_n(V^g)$ is a flat deformation of the reduced symmetric algebra $S(V, r^{-})$ of the classical limit $V$ of $V^g$.

This conjecture is of particular interest because it opens the possibility to address the following not yet investigated question.

**Problem 7.2.** Quantize a commutative Poisson algebra $A$; e.g. a reduced symmetric algebra.

While this conjecture gives rise to the question of quantization of manifolds with a bracketed structure, the classification of flat modules in Theorem [4.21] and our computation of the braided symmetric and exterior cubes of simple $U_q(sl_2)$-modules in $[8]$ Theorem 2.40] suggest that braided symmetric and exterior cubes are rigid in the following sense.

**Conjecture 7.3.** Let $g$ be a simple Lie algebra and $V^g$ a simple $U_q(g)$-module and $V$ its classical limit. The classical limit of $S^3V^g$ (resp. $\Lambda^3V^g$) is isomorphic to $S^3_{low}V$ (resp. $\Lambda^3_{low}V$) as a $U(g)$-module.

It is easy to see that the assertion does not hold for nonsimple $V^g$ or $g$ semisimple, as the example of the natural $U_q(sl_n \times sl_n)$-module shows.

8. Appendix

We develop in this appendix notation for products and complements of weight-spaces in symmetric and exterior powers of vectorspaces. Let $V$ be a finite-dimensional vector space, and let $V_1, V_2, \ldots, V_n$ be subspaces of $V$. We define $V_1 \cdot V_2 \cdot \ldots \cdot V_n \subset S^nV$ and $V_1 \wedge V_2 \wedge \ldots \wedge V_n \subset \Lambda^nV$ the subspaces generated by elements $v_1 \cdot v_2 \cdot \ldots \cdot v_n$, resp. $v_1 \wedge v_2 \wedge \ldots \wedge v_n$, where $v_i \in V_i$ for $i \in [1, n]$.

**Lemma 8.1.** Let $V$ be a finite-dimensional vector space and let $V_1, \ldots, V_n$ be subspaces such that $V \cong \bigoplus_{i=1}^n V_i$. Denote by $\mathcal{P}(m)$ the set of all increasing $m$-element sequences $\sigma = (p_1, \ldots, p_m), 1 \leq p_1 \leq \ldots \leq p_m \leq n$ in $[1, n]$. The $m$-th symmetric and exterior powers now admit the following decomposition:

$$S^mV \cong \bigoplus_{\sigma \in \mathcal{P}_m} V_{p_1} \cdot \ldots \cdot V_{p_m},, \Lambda^mV \cong \bigoplus_{\sigma \in \mathcal{P}_m} V_{p_1} \wedge \ldots \wedge V_{p_m}.$$  

**Proof.**

Recall that if $V \cong V_1 \oplus V_2$, then $S^mV \cong \bigoplus_{i=0}^m S^iV_1 \boxtimes S^{m-i}V_2$, resp. that $\Lambda^mV \cong \bigoplus_{i=0}^m \Lambda^iV_1 \boxtimes \Lambda^{m-i}V_2$. We can now use induction in $m$ to prove that for $V \cong \bigoplus_{i=1}^n V_i$ we have

$$S^mV \cong \bigoplus_{\ell \in \mathcal{L}_m} \bigotimes_{i=1}^n S^{\ell_i}V_i, \Lambda^mV \cong \bigoplus_{\ell \in \mathcal{L}_m} \bigotimes_{i=1}^n \Lambda^{\ell_i}V_i,$$

where $\mathcal{L}_m$ denotes the set of of $n$-element sequences $(\ell_1, \ldots, \ell_n)$ such that $\ell_1 + \ldots + \ell_n = m$. 


Using the notation $V \cdot \ldots \cdot V = S^m V$ and $V \wedge \ldots \wedge V = \Lambda^m V$ and the fact that $S^1 V_1 \otimes S^1 V_2 \cong S^1 V_1 \cdot S^1 V_2$, resp. $\Lambda^1 V_1 \otimes \Lambda^1 V_2 \cong \Lambda^1 V_1 \wedge \Lambda^1 V_2$, we obtain the desired result. The lemma is proved. □

We will use the notation $(V_{p_1} \cdots V_{p_m})^c \subset S^m V$ to denote

$$\left(\bigoplus_{\ell' \neq \ell \in P} V(\lambda_{p_{\ell'}}) \wedge \ldots \wedge V(\lambda_{p_{m}})\right).$$

REFERENCES

[1] Y. Bazlov, Nichols-Woronowicz algebra model for Schubert Calculus on Coxeter groups, math/0409206.
[2] A. Belavin and V. Drinfeld, Triangle equations and simple Lie algebras, Soviet Sci. Rev. Sect. C, Math. Phys. Rev. 4 (1984), 93–165.
[3] A. Berenstein and D. Kazhdan, Geometric and unipotent crystals, Geom. Funct. Anal., Special Volume, Part I, (2000), 188–236.
[4] A. Berenstein and D. Kazhdan, Geometric and unipotent crystals II: from geometric crystals to crystal bases, to appear in Cont. Math., math.QA/0601391.
[5] A. Berenstein, A. Zelevinsky, Quantum cluster algebras, Adv. Math. 195 (2005), no.2, 405–455.
[6] A. Berenstein and S. Zwicknagl, Braided Symmetric and Exterior Algebras, to appear in Trans. of the Amer. Math. Soc., math.QA/0504155.
[7] N. Bourbaki, Groupes et Algebres de Lie, Masson, 1981.
[8] K. Brown and K. Goodearl, Lectures on algebraic quantum groups, Birkhäuser, 2002.
[9] J. Burndan, Kazhdan-Lusztig Polynomials and Character Formulæ for the Lie Superalgebra q(n), Adv. Math. 182 (2004), 28–77.
[10] K. Brown, K. Goodearl, M. Yakimov, Poisson structures on affine spaces and flag varieties, I. Matrix Affine Spaces, to appear in Adv. Math., [math.QA/0501109].
[11] C. De Concini, V.G. Kac, C. Procesi, Some Quantum Analogues of Solvable Lie Groups, Geometry and Analysis (Bombay, 1992), Tata Inst. Fund. Res. Bombay, 1995, 41–65.
[12] J. Donin, Double quantization on the coadjoint representation of sl(n), Quantum groups and integrable systems, Part I (Prague, 1997), Czech. J. Phys. 47 (1997), no. 11, 1115–1122.
[13] V. Drinfel’d, Quasi-Hopf Algebras, Leningrad Math. Journal, 1 (1990), no. 6, 1419–1457.
[14] V. Drinfel’d, Commutation Relations in the Quasi-Classic Case, Sel. Math. Sov., 11, no.4 (1992), 317–326.
[15] M. Durdevic, Z. Oziewicz, Clifford Algebras and Spinors for Arbitrary Braids, Differential geometric methods in theoretical physics (Ixtapa-Zihuatanejo, 1993), Adv. Appl. Clifford Algebras, 4 (1994), Suppl. 1, 461–467.
[16] A. Fokas, I. Gel’fand, Quadratic Poisson algebras and their infinite dimensional extensions, J. Math. Phys. 35 no. 6 (June 1994).
[17] K.R. Goodearl, M. Yakimov, Poisson structures on affine spaces and flag varieties. II. general case, math.QA0509075.
[18] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, Israel Math. Conf. Proc. 8 1995, 1–182.
[19] R. Howe, private communication.
[20] J. Humphreys, Introduction to Lie algebras and representation theory, Graduate texts in Mathematics 9, Springer, 1972.
[21] J.C. Jantzen, An Introduction to Quantum Groups, Graduate Studies in Mathematics, American Mathematical Society, 1996.
[22] N. Jing, K. Misra, M. Okado, q-Wedge modules for quantized enveloping algebras of classical type, J. Algebra 230 (2000), no.2, 518–539.
[23] A. Kamita, Quantum Deformations of Certain Prehomogeneous Vector Spaces III, Hiroshima Math. J. 30 (2000), 79–105.
[24] M. Kleber, S. Viswanath, Tensor product stabilization in Kac-Moody Lie algebras, to appear in Adv. Math., [math.RT/0405444].
[25] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (2003) no. 3, 157–216.
[26] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem, Ann. Math. 74 (1961), 329–387.
[27] C. Lecouvey, An algorithm for computing the global basis of a finite dimensional irreducible $U_q(sl_{2n+1})$ or $U_q(sl_{2n})$-module, Comm. Algebra 32 (2004), no. 5, 1969–1996.
[28] G. Lusztig, Canonical bases arising from quantized enveloping algebras. J. Amer. Math. Soc. 3 (1990), no. 2, 447–498.
[29] G. Lusztig, Quantum Groups at Roots of 1, Geometriae Dedicata 35 (1990) 89–114.
[30] I. Musson, Ring theoretic properties of the coordinate rings of quantum symplectic and Euclidean space, Ring Theory, Proc. Biennial Ohio State-Denison Conf. 1992 (S.K. Jain and S.T.Rizvi, eds.), World Scientific, Singapore, 1993, 248–258.
[31] M. Noumi, Macdonald’s Symmetric Polynomials as Zonal Spherical Functions on some Quantum Homogeneous Spaces, Adv. Math., 123 (1996), 16–77.
[32] N. Reshitikhin, L. A. Takhtadzhyan and L. D. Fadeev, Quantization of Lie groups and Lie algebras, Leningrad Math. J., 1 (1990), 193–225.
[33] O. Rossi-Doria, A $U_q(sl(2))$-representation with no quantum symmetric algebra. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 10 (1999), no. 1, 5–9.
[34] W. Schmid, Die Randwerte holomorpher Funktionen auf hermiteschen symmetrischen Räumen, Inv. Math.91 (1969-70), 61-80.
[35] E. Strickland, Classical Invariant Theory for the Quantum Symplectic Group, Adv. Math. 123 (1996), 78–90.
[36] J. Stembridge, On the Classification of Multiplicity-Free Exterior Algebras, Int. Math. Res. Not., no. 40 (2003), 2181–2191.
[37] M. VanLieuwen, LiE, A software package for Lie group computations, http://young.sp2mi.univ-poitiers.fr~marc/LiE/
[38] S. Vishwanath, Dynkin diagram sequences and stabilization phenomena.math.RT/0505616
[39] S. Woronowicz, Differential Calculus on Matrix Pseudogroups (Quantum Groups), Comm. in Math. Phys., 122, (1989), 125–170.
[40] R. Zhang, Howe Duality and the Quantum General Linear Group, Proc. Amer. Math.Soc., Amer. Math. Soc., Providence, RI, 131, no. 9, 2681–1692.
[41] S. Zwicknagl, Equivariant Poisson Algebras and their Deformations, PhD thesis, University of Oregon, Dec. 2006.

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