Action-angle coherent states for quantum systems with cylindric phase space

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Received 4 March 2012, in final form 13 June 2012
Published 26 July 2012
Online at stacks.iop.org/JPhysA/45/335302

Abstract
Quantum versions of cylindric phase space, like for the motion of a particle on a circle, are obtained through different families of coherent states. The latter are built from various probability distributions of the action variable. The method is illustrated with Gaussian distributions and uniform distributions on intervals, and resulting quantizations are explored.

1. Introduction

In most of the introductory references in the literature devoted to quantum mechanics, the quantum versions of two simple models are presented, namely the motion of a particle on the circle and on an interval (e.g. the infinite square well potential). Experience with the harmonic oscillator suggests that the concept of coherent states (CS) [1] would be an important tool for the better understanding of the periodic motion of a quantum particle. It is well known, essentially since Klauder [2, 3] and Berezin [4] (see also the stochastic quantum mechanics and quantum spacetime program based on Prugovecki’s views and comprehensively presented in [5]), that one can easily achieve canonical quantization of the classical phase space by using standard CS. The CS quantization with its various generalizations reveals itself as an efficient tool for quantizing physical systems. Recently, this method has been implemented on various simple systems with phase spaces like the complex plane with CS different of the standard ones [6, 7], the cylinder [8], an infinite strip in the plane [9–11] or yet more exotic phase spaces like the finite set $\mathbb{Z}_d \times \mathbb{Z}_d$ [12] or paragrassmann algebras [13]. Starting from a solution to a version of the Stieltjes moment problem [14], a family of CS has also been used to construct a Fock–Bergmann representation related to the particle quantization which takes into account the circle topology of the classical motion. In [15], canonical and CS quantizations of a particle moving in a magnetic field have been compared in the case of the non-commutative plane and semi-classical aspects have been explored. Other examples are given in [16] in which the method is explained at length and more complete references are given.
For the particular case of a quantum particle moving on a circle, i.e. when the phase space is topologically a cylinder, CS constructions have been independently proposed in [17–23]. All these constructions lead to certain types of CS: they are Gaussian in the sense that, as superpositions of angular momentum eigenstates $|n\rangle$, $n \in \mathbb{Z}$ (and not $\mathbb{N}$!), their Fourier coefficients involve Gaussian functions centered at $n$. It is noticeable that similar states were proposed earlier by Chang and Chi [24] for the treatment of a generalized quantum Chirikov map under the rational resonance condition $2\pi \hbar = M/N$ (see also Życzkowski [25]).

The content of this paper also concerns the motion of a quantum particle on a circle, and more generally systems for which the phase space has cylindric topology. Our work lies in the continuation of those quoted above and is also based on developments elaborated in [26–28]. We present families of CS built from various probability distributions of the action variable for the motion on the circle. Our results might be particularly relevant to recent models in superconducting circuit QED [29, 30] for which the longstanding question is raised again (see for instance [31–33]) of commutation relation between phase operator (analogous to the angle) and number operator (analogous to the angular momentum), more precisely excess Cooper pair number operator with spectrum $\mathbb{Z}$ and not just $\mathbb{N}$. They could also offer new perspectives in the study of time behavior of quantum-chaotic phenomena.

The paper is organized as follows. In section 2, we give a short account of the CS quantization procedure with respect to a set $X$ of parameters equipped with a measure $\mu$, and the statistical aspects leading to a Bayesian duality related to these states are summarized in appendix A. Section 3 is devoted to the construction of families of CS for the cylinder viewed as a phase space and are associated with various distributions of the action (or angular momentum) variable. In section 4, the quantization of classical observables based on these various CS families is analyzed on a general level. In section 5, the instructive although quite elementary case of uniform distributions on intervals is worked out, and some semi-classical aspects are examined. Some further extensions of this work are investigated in section 6.

### 2. CS quantization: the general setting

Let $X$ be a set of parameters equipped with a measure $\mu$ and let $L^2(X, \mu)$ be its associated Hilbert space of complex-valued square integrable functions with respect to $\mu$. Let us choose in $L^2(X, \mu)$ a finite or countable orthonormal set $O = \{\phi_n, n \in \mathcal{F}\}$, with $\mathcal{F}$ some countable set ($\sim \mathbb{N}$ or $\sim \mathbb{Z}$ ...),

$$\langle \phi_m | \phi_n \rangle = \int_X \phi_m(x) \phi_n(x) \mu(dx) = \delta_{mn},$$

(1)

obeying the (crucial) condition

$$0 < \sum_n |\phi_n(x)|^2 \overset{\text{def}}{=} N(x) < \infty \quad \text{a.e.}$$

(2)

Let $\mathcal{H}$ be a separable complex Hilbert space with an orthonormal basis $\{ |e_n\rangle, n \in \mathcal{F}\}$, in one-to-one correspondence with the elements of $O$. In particular, it can be chosen as the Hilbert subspace $K_O \overset{\text{def}}{=} \text{span}(O)$ in $L^2(X, \mu)$ itself. We then define the family of states $\mathcal{F}_\mathcal{H} = \{ |x\rangle, x \in X \}$ in $\mathcal{H}$ as

$$|x\rangle = \frac{1}{\sqrt{N(x)}} \sum_n \phi_n(x) |e_n\rangle.$$

(3)
From conditions (1) and (2), these ‘coherent’ states are normalized, \( \langle x | x \rangle = 1 \), and resolve the identity in \( \mathcal{H} \):

\[
\int_X \mu(dx) \mathcal{N}(x) |x\rangle \langle x| = 1_{\mathcal{H}}.
\]  

(4)

Relation (4) allows us to implement a coherent state or frame quantization of the set of parameters \( X \) by associating with a function \( X \ni x \mapsto f(x) \) that satisfies appropriate conditions the following operator in \( \mathcal{H} \):

\[
f(x) \mapsto A_f \equiv \int_X \mu(dx) \mathcal{N}(x) f(x) |x\rangle \langle x|.
\]  

(5)

The matrix elements of \( A_f \) with respect to the basis \( |e_n\rangle \) are given by

\[
(A_f)_{mn} = \langle e_n | A_f | e_m \rangle = \int_X \mu(dx) f(x) \phi_n(x) \phi_m(x).
\]  

(6)

The operator \( A_f \) is symmetric if \( f(x) \) is real-valued, bounded if \( f(x) \) is bounded and self-adjoint if \( f(x) \) is real semi-bounded (through Friedrich’s extension). In order to view the ‘upper’ symbol \( f \) of \( A_f \) as a quantizable object (with respect to the family \( \mathcal{F}_{\mathcal{H}} \)), a reasonable requirement is that the so-called lower symbol of \( A_f \), defined as

\[
\hat{f}(x) \equiv \langle x | A_f | x \rangle = \int_X \mu(dx) \mathcal{N}(x) f(x) |\langle x | x \rangle|^2,
\]  

(7)

be a smooth function on \( X \) with respect to some topology assigned to the set \( X \). In appendix A, we review some interesting statistical aspects of the above construction.

3. The cylinder as a phase space for the motion on the circle

Quantization of the motion of a particle on the circle (like the quantization of polar coordinates in the plane) is an old question with so far mildly evasive answers. A large literature exists concerning the subject, more specifically devoted to the problem of angular localization and related Heisenberg inequalities [31–33]. Let us apply our scheme of CS quantization to this particular problem. The observation set \( X \) is the phase space of a particle moving on the circle, precisely the cylinder \( X = S^1 \times \mathbb{R} = \{ (\varphi, J), \ 0 \leq \varphi < 2\pi, \ J \in \mathbb{R} \} \), equipped with the measure \( \mu(dx) = \frac{1}{2\pi} d\varphi dJ \).

We now introduce a probability distribution on the range of the variable \( J \). It is a non-negative, even, well-localized and normalized integrable function,

\[
\mathbb{R} \ni J \mapsto \sigma^\sigma(J), \quad \sigma^\sigma(J) = \sigma^\sigma(-J), \quad \int_{-\infty}^{+\infty} dJ \sigma^\sigma(J) = 1,
\]  

(8)

where \( \sigma > 0 \) is a kind of width parameter. This function must obey the following conditions:

(i) \( 0 < \mathcal{N}^\sigma(J) \equiv \sum_{n \in \mathbb{Z}} \sigma^\sigma_n(J) < \infty \) for all \( J \in \mathbb{R} \), where \( \sigma^\sigma_n(J) \equiv \hat{\sigma}_n^\sigma(2\pi J) \); (ii) the Poisson summation formula is applicable to \( \sigma^\sigma(J) \):

\[
\mathcal{N}^\sigma(J) = \sum_{n \in \mathbb{Z}} \sigma^\sigma_n(J) = \sqrt{2\pi} \sum_{n \in \mathbb{Z}} e^{-2\pi i n J} \hat{\sigma}_n^\sigma(2\pi n),
\]  

(9)

where \( \hat{\sigma}^\sigma \) is the Fourier transform of \( \sigma^\sigma \); (iii) its limit at \( \sigma \to 0 \), in a distributional sense, is the Dirac distribution:

\[
\sigma^\sigma(J) \to_\sigma \delta(J);
\]  

(10)
(iv) the limit at $\sigma \to \infty$ of its Fourier transform is proportional to the characteristic function of the singleton $[0]$:

$$\hat{\sigma}_\sigma(k) \to \frac{1}{\sigma \to \infty} \frac{1}{2\pi} \delta_{k0};$$

(11)

(v) considering the overlap matrix of the two distributions $J \mapsto \sigma^\sigma_n(J)$, $J \mapsto \sigma^\sigma_m(J)$ with matrix elements,

$$\sigma^\sigma_{n,m} = \int_{-\infty}^{+\infty} \delta J \sqrt{\sigma^\sigma_n(J) \sigma^\sigma_m(J)} \leq 1,$$

we impose the two conditions

$$\sigma^\sigma_{n,m} \to 0 \quad \text{as} \quad n - n' \to \infty \quad \text{at fixed } \sigma,$$

(13)

$$\exists N_0 \geq 1 \quad \text{such that} \quad \forall n, n' \quad \text{such that} \quad |n - n'| \leq N_0.$$  

(14)

Properties (ii) and (iv) entail that $N^\sigma(J) \to 1$. Also note the properties of the overlap matrix elements $\sigma^\sigma_{n,m}$ due to the properties of $\sigma^\sigma$:

$$\sigma^\sigma_{n,m} = \sigma^\sigma_{m,n} = \sigma^\sigma_{0,0} = \sigma^\sigma_{-n,-n}, \quad \sigma^\sigma_{n,n} = 1 \quad \forall n, n' \in \mathbb{Z}.$$  

(15)

The most immediate (and historical) choice for $\sigma^\sigma(J)$ is Gaussian, i.e. $\sigma^\sigma(J) = \frac{1}{\sqrt{\pi \sigma^2}} e^{-\frac{J^2}{\sigma^2}}$ (for which $N_0$ in (14) is $\infty$), as it appears under various forms in the existing literature on the subject [17–23]. In appendix B, we recall a few features of CS issued from such a choice.

Let us now introduce the weighted Fourier exponentials

$$\phi_n(x) = \sqrt{\sigma^\sigma_n(J)} e^{i\sigma\phi}, \quad n \in \mathbb{Z}.$$  

(16)

These functions form the countable orthonormal system in $L^2(X, \mu(dx))$ needed to construct CS in agreement with the procedure explained in section 2. In consequence, the correspondent family of CS on the circle reads as

$$|J, \phi\rangle = \frac{1}{\sqrt{N^\sigma(J)}} \sum_{n \in \mathbb{Z}} \sqrt{\sigma^\sigma_n(J)} e^{-i\phi |e_n\rangle}.$$  

(17)

As expected, these states are normalized and resolve the unity. They overlap as

$$\langle J, \phi | J', \phi' \rangle = \frac{1}{\sqrt{N^\sigma(J) N^\sigma(J')}} \sum_{n \in \mathbb{Z}} \sqrt{\sigma^\sigma_n(J) \sigma^\sigma_n(J')} e^{-i\phi|\phi'\rangle}.$$  

(18)

As explained in appendix A, the function $\sigma^\sigma(J)$ gives rise to a double probabilistic interpretation [16, 26].

- For all $J$ viewed as a shape parameter, there is the discrete distribution,

$$\mathbb{Z} \ni n \mapsto |\langle e_n | \phi \rangle|^2 = \frac{\sigma^\sigma_n(J)}{N^\sigma(J)}.$$  

(19)

This probability, of genuine quantum nature, concerns experiments performed on the system described by the Hilbert space $\mathcal{HH}$ within some experimental protocol, say $\mathcal{E}$, in order to measure the spectral values of the self-adjoint operator acting in $\mathcal{HH}$ and having the discrete spectral resolution $\sum_{a \in \mathcal{A}} a_0 |e_n\langle e_n|$. For $a_0 = n$, this operator is the quantum angular momentum, as will be shown in the next section.

- For each $n$, there is the continuous distribution on the cylinder $X$ (resp. on $\mathbb{R}$) equipped with its measure $dJ/2\pi$ (resp. $d\phi$),

$$X \ni J, \phi \mapsto |\langle \phi_n(J, \phi) | \phi \rangle|^2 = \sigma^\sigma_n(J) \quad \text{resp.} \quad \mathbb{R} \ni J \mapsto \sigma^\sigma_n(J).$$  

(20)

This probability, of classical nature and uniform on the circle, determines the CS quantization of functions of $J$, as will be seen in the following section.
4. Quantization of classical observables with CS on the circle

4.1. General setting

By virtue of the CS quantization scheme described in section 2, the quantum operator (acting on \( \mathcal{H} \)) associated with the classical observable \( f(x) \) is obtained through

\[
A_f := \int_X f(x) |x\rangle \langle x| \mathcal{N}(x) \mu(dx) = \sum_{n,n'} (A_f)_{nn'} |e_n\rangle \langle e_{n'}|,
\]

where

\[
(A_f)_{nn'} = \int_{-\infty}^{+\infty} dJ \sqrt{\sigma_\mu^\sigma(J) \sigma_\mu^\sigma(J')} 1 \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{-i(n-n')\varphi} f(J, \varphi).
\]

The lower symbol of \( f \) is given by

\[
\tilde{f}(J, \varphi) = \langle J, \varphi | A_f | J, \varphi \rangle = \int_{-\infty}^{+\infty} dJ' \int_0^{2\pi} d\varphi' \sqrt{\mathcal{N}^\sigma(J) f(J', \varphi') \mathcal{N}^\sigma(J')} (J, \varphi') |\langle J, \varphi | J', \varphi' \rangle|^2.
\]

If \( f \) depends on \( J \) only, \( f(x) \equiv f(J) \), then \( A_f \) is diagonal with matrix elements that are \( \sigma^\sigma \) transforms of \( f(J) \):

\[
(A_{f(J)})_{nn'} = \delta_{nn'} \int_{-\infty}^{+\infty} dJ \sigma_\mu^\sigma(J) f(J) = \delta_{nn'} \langle f \rangle_{n\equiv},
\]

where \( \langle \cdot \rangle_{\equiv} \) designates the mean value w.r.t. the distribution \( J \mapsto \sigma_\mu^\sigma(J) \). For the most basic case, \( f(J) = J \), our assumptions on \( \sigma^\sigma \) give

\[
A_J = \int_X \mu(dx) \mathcal{N}^\sigma(J) J |J, \varphi\rangle \langle J, \varphi| = \sum_{n \in \mathbb{Z}} n |e_n\rangle \langle e_n|.
\]

This is nothing but the angular momentum operator (in unit \( \hbar = 1 \)), which reads \( A_J = -i\partial/\partial \theta \) in angular position representation, i.e. when \( \mathcal{H} \) is chosen as \( L^2(S^1, d\theta/2\pi) \) with the orthonormal basis \( |e_n\rangle \equiv e^{in\theta} \) (Fourier series). The covariance property of the CS with respect to rotations is a direct consequence of (24):

\[
e^{iA_J}|J, \varphi\rangle = |J, \varphi - \theta\rangle.
\]

The quantization of \( f(J) = J^2 \), i.e. the kinetic energy of the particle in suitable units, produces a quantum spectrum which behaves like \( n^2 \):

\[
A_{J^2} = \int_X \mu(dx) \mathcal{N}^\sigma(J) J^2 |J, \varphi\rangle \langle J, \varphi| = c 1_{\mathcal{H}} + \sum_{n \in \mathbb{Z}} n^2 |e_n\rangle \langle e_n|,
\]

where

\[
c = \int_{-\infty}^{+\infty} dJ J^2 \sigma^\sigma(J) \equiv \langle J^2 \rangle_{n\equiv}.
\]

We can understand through this quantization procedure the probabilistic origin of the elementary quantum of energy, i.e. the difference between the classical zero energy point and the quantum ‘vacuum energy’.

If \( f \) depends on \( \varphi \) only, \( f(x) \equiv f(\varphi) \), we have

\[
A_{f(\varphi)} = \int_X \mu(dx) \mathcal{N}^\sigma(J) f(\varphi) |J, \varphi\rangle \langle J, \varphi| = \sum_{n,n'} \sigma_\mu^\sigma_{n,n'} c_{n-n'}(f) |e_n\rangle \langle e_{n'}|,
\]

where \( c_n(f) \) is the \( n \)th Fourier coefficient of \( f \). At a first look at (30), one understands that the more the distributions overlap, the more the non-commutativity is enhanced. In particular, we have
the self-adjoint ‘angle’ operator corresponding to the $2\pi$-periodic saw function $B(\varphi)$ defined by the periodic extension of $B(\varphi) = \varphi$ for $0 \leq \varphi < 2\pi$, and abusively denoted in the following by $\varphi$:

$$A_\varphi = \pi J_\mathcal{H} + i \sum_{n \neq n'}^{n} \frac{\sigma_{n,n'}^\sigma}{n - n'} |e_n\rangle \langle e_{n'}|.$$  \hspace{1cm} (31)

- the operator Fourier fundamental harmonics corresponding to the elementary Fourier exponential,

$$A_{\pm}^{\varphi} = \sigma_{1,0}^\varphi \sum_n |e_{n+1}\rangle \langle e_n|, \quad A_{\pm}^{\varphi} = A_{\pm}^{\varphi}.$$  \hspace{1cm} (32)

We remark that $A_{\pm}^{\varphi} A_{\pm}^{\varphi} = A_{\pm}^{\varphi} A_{\pm}^{\varphi} = (\sigma_{1,0}^\varphi)^2 1_{\mathcal{H}}$. Therefore, this operator fails to be unitary. It is ‘almost’ unitary at large $\sigma$ since the factor $(\sigma_{1,0}^\varphi)^2$ can be made arbitrarily close to 1 at large $\sigma$ as a consequence of requirement (14). In the Fourier series realization of $\mathcal{H}$, for which the kets $|e_n\rangle$ are the Fourier exponentials $e^{\pm i n \varphi}$, the operators $A_{\pm}^{\varphi}$ are the multiplication operator by $e^{\pm \varphi}$ up to the factor $\sigma_{1,0}^\varphi$.

4.2. Elementary commutators and classical limit

The commutation rules

$$[A_J, A_{\pm}^{\varphi}] = \pm A_{\pm}^{\varphi}$$  \hspace{1cm} (33)

are canonical in the sense that they are in exact correspondence with the classical Poisson brackets

$$[J, e^{\pm \varphi}] = \pm ie^{\pm \varphi}.$$  \hspace{1cm} (34)

(For other non-trivial commutators having this exact correspondence in the Gaussian case, see [35].) In consequence, our CS quantization based on a choice of $\sigma$ fulfilling conditions (i)–(v) respects the underlying symmetry $SO(2) \times \mathbb{R}^2$ of the cylinder viewed as a phase space. Indeed, after introducing a positive constant $\lambda$, we deduce from (33) the commutation rules

$$[A_J, A_{\lambda,\cos \varphi}] = iA_{\lambda, \sin \varphi}, \quad [A_J, A_{\lambda, \sin \varphi}] = -iA_{\lambda, \cos \varphi}, \quad [A_{\lambda, \cos \varphi}, A_{\lambda, \sin \varphi}] = 0.$$  \hspace{1cm} (35)

They are those verified by generators of a unitary representation of the Euclidean group of the plane.

One could be puzzled by commutators of the type

$$[A_J, A_{f(\varphi)}] = \sum_{n,n'} (n - n') \sigma_{n,n'}^\varphi e_{n-n'} (f) |e_n\rangle \langle e_{n'}|,$$  \hspace{1cm} (36)

and, in particular, for the angle operator itself:

$$[A_J, A_{\varphi}] = i \sum_{n,n'} \sigma_{n,n'}^\varphi |e_n\rangle \langle e_{n'}|,$$  \hspace{1cm} (37)

to be compared with the classical $[J, \varphi] = 1$. One observes that the overlap matrix completely encodes the basic commutator of quantized canonical action and angle variables.

Because of the required properties of the distribution $\sigma$, the departure of the rhs of equation (37) from the canonical rhs $-i1_{\mathcal{H}} \varphi$ can be bypassed by examining the behavior of the lower symbols at large $\sigma$. For an original function depending on $\varphi$ only, we have the Fourier series

$$\tilde{f}(J_0, \varphi_0) = \langle J_0, \varphi_0 | A_{f(\varphi)} | J_0, \varphi_0 \rangle = c_0(f) + \sum_{m \neq 0} d^m_m (J_0) \sigma_{0,m}^\varphi c_m(f) e^{im \varphi_0},$$  \hspace{1cm} (38)
with

\[ d^\sigma_m(J) = \frac{1}{N}\sum_{r=-\infty}^{+\infty} \sqrt{\sigma^\alpha_r(J)\sigma^\alpha_{m+r}(J)} \leq 1, \tag{39} \]

the last inequality resulting from condition (i) and Cauchy–Schwarz inequality. If we further impose the condition that $d^\sigma_m(J) \to 1$ uniformly as $\sigma \to \infty$, then the lower symbol $f(J_0, \varphi_0)$ tends to the Fourier series of the original function $f(\varphi)$. A similar result is obtained for the lower symbol of commutator (36):

\[ \langle J_0, \varphi_0 | [A_f, A_{f(\varphi)}] | J_0, \varphi_0 \rangle = \sum_{m \neq 0} d^\sigma_m(J_0) \sigma^\alpha_{0,m} m c_m(f) e^{im\varphi_0}, \tag{40} \]

and in particular,

\[ \langle J_0, \varphi_0 | [A_f, A_p] | J_0, \varphi_0 \rangle = i \sum_{m \neq 0} d^\sigma_m(J_0) \sigma^\alpha_{0,m} e^{im\varphi_0}. \tag{41} \]

Therefore, with the condition that $d^\sigma_m(J) \to 1$ uniformly as $\sigma \to \infty$, we obtain at this limit

\[ \langle J_0, \varphi_0 | [A_f, A_p] | J_0, \varphi_0 \rangle \to -i + i \sum_{m} \delta(\varphi_0 - 2\pi m). \tag{42} \]

So we asymptotically (almost) recover the classical canonical commutation rule except for the singularity at the origin mod $2\pi$, a logical consequence of the discontinuities of the saw function $B(\varphi)$ at these points.

4.3. Other semi-classical aspects

We have tested in the previous section a few semi-classical features of the CS (3) by studying how lower symbols of the operators $A_f$ approach the original $f(J, \varphi)$. When the latter is semi-bounded from below, another possible test consists in evaluating, at large $\sigma$, the relative error function $\text{rerr}_C(J, \varphi; f)$:

\[ \text{rerr}_C(J, \varphi; f) \overset{\text{def}}{=} \left| \frac{\langle J, \varphi | A_f | J, \varphi \rangle - f(J, \varphi)}{f(J, \varphi) + C} \right|, \tag{43} \]

where the constant $C$ has to be added to $f$ in order that the denominator does not cancel. Of course, $C$ should not be chosen too large. It is also possible to avoid such a precaution by restricting the study of this function to the positive part of the range of $f$.

Another interesting exploration is the temporal behavior of the lower symbol, given some classical Hamiltonian function $H(J, \varphi)$ and its quantum version $A_H$, once the initial condition $(J_0, \varphi_0)$ has been chosen in the phase space. Thus, we can explore analytically and numerically expressions of the type

\[ \langle J_0, \varphi_0 | e^{-itA_f}A_f e^{itA_f} | J_0, \varphi_0 \rangle \tag{44} \]

\[ \langle J_0, \varphi_0 | e^{-itA_p}A_p e^{itA_p} | J_0, \varphi_0 \rangle. \tag{45} \]

Moreover, the resolution of identity (4) allows for the probability distribution $\langle J, \sigma^\alpha \rangle \mapsto N(\sigma) |\langle J, \varphi | J_0, \varphi_0 \rangle|^2 \equiv \rho_{(|J_0, \varphi_0\rangle)}(J, \varphi)$ on the cylindric phase space. It is natural to consider this distribution as a localization measure on the phase space. Hence, given a Hamiltonian $H(J, \varphi)$, it is also natural to explore analytically and numerically the time evolution of such a distribution,

\[ t \mapsto \rho_{e^{-itA_f}J_0, \varphi_0}(J, \varphi) = N(\sigma) |\langle J, \varphi | e^{-itA_f} | J_0, \varphi_0 \rangle|^2, \tag{46} \]
and to compare it with the classical phase space trajectory on the cylinder. For instance, in the Gaussian case, and with the Hamiltonian $H = J^2$ of the free motion on the circle, we can study the time evolution of the following series obtained either from equation (B.8) or equation (B.9) in appendix B:

$$\rho_{e^{-i\sigma\varphi}}(J, \varphi) = \frac{\rho_{e^{-i\sigma\varphi}}(J_0, \varphi)}{2\pi \sigma^2 N^\sigma(J_0)} \left| \sum_{n \in \mathbb{Z}} e^{-\frac{1}{2\pi^2} (J-n)^2} \left| e^{i \pi \sigma (n \varphi - n^2 t)} \right|^2 \right|^2.$$  \hspace{1cm} (47)

We now apply the above material to the (apparently trivial) case of a uniform distribution on an interval centered at the origin and with length $2\sigma$:

$$(\sigma \sigma)(J) = \frac{1}{2\sigma} \chi_{[-\sigma, \sigma]}(J).$$  \hspace{1cm} (49)

A first limitation on the range of $\sigma$ is necessary because of condition (i):

$$\sigma \geq \frac{1}{2}.$$  \hspace{1cm} (50)

Indeed, the normalization function reads in the present case

$$N^\sigma(J) = \frac{1}{2\sigma} \left[ 1 + \sum_{n \in \mathbb{Z}} \chi_{[n+1-\sigma, n+\sigma]}(J) \right].$$  \hspace{1cm} (51)

and would vanish for all $J \in \cup_{n \in \mathbb{Z}} (n+1-\sigma, n+\sigma)$ if $\sigma < 1/2$. This periodic crenel function, period 1, assumes only two values, $1/(2\sigma)$ and $1/\sigma$. In particular, $N^\sigma(n) = 1/(2\sigma)$ and $N^\sigma(n+1/2) = 1/\sigma$ for all $n \in \mathbb{Z}$. For $\sigma = 1/2$ or $\sigma = 1$, it is equal a.e. to 1.

As is well known, the Fourier transform of $\sigma \sigma(J)$ is a cardinal sine,

$$\widetilde{\sigma \sigma}(k) = \frac{1}{\sqrt{2\pi}} \sin(\sigma k),$$  \hspace{1cm} (52)

which is at $k = 0$ equal to $1/\sqrt{2\pi}$ for any $\sigma$. It is then clear that conditions (i)–(iv) are fulfilled.

Let us impose the supplementary limitation on the choice of $\sigma$,

$$\sigma \leq 1.$$  \hspace{1cm} (53)

Such a choice prevents us from examining limits at large $\sigma$. On the other hand, it makes analytic computations easier. Thus, the equation

$$\sqrt{\sigma_n \sigma_n}(J) = \frac{1}{2\sigma} \left[ \delta_{n0} \chi_{[n-\sigma, n+\sigma]}(J) + \delta_{n+1} \chi_{[n-\sigma, n+1+\sigma]}(J) + \delta_{n-1} \chi_{[n+1-\sigma, n+\sigma]}(J) \right].$$  \hspace{1cm} (54)
shows that only nearest-neighbors overlap:

$$\sigma_{n,n'}^{\sigma} = \delta_{n,n'} + \left(1 - \frac{1}{2\sigma}\right)[\delta_{n,n'+1} + \delta_{n,n'-1}]. \quad (55)$$

which simply means that $$\sigma_{n,0}^\sigma = 1 - 1/(2\sigma)$$ and $$\sigma_{n,n'}^\sigma = 0$$ for all $$n, n'$$, such that $$|n - n'| > 1$$. Note that the overlap is lost at the lowest limit $$\sigma = 1/2$$. Increasing the lowest upper bound in (53) would enlarge the overlap.

The quantization of any locally integrable function $$f(J, \varphi)$$ produces a tridiagonal matrix $$A_f$$ (which is Jacobi if $$f$$ is real):

$$(A_f)_{nn'} = \frac{1}{2\sigma} \left[ \delta_{nn'} \int_{n-\sigma}^{n+\sigma} \frac{dJ}{2\pi} \int_0^{2\pi} d\varphi f(J, \varphi) 
+ \delta_{n+1,n'} \int_{n-\sigma}^{n+\sigma-1+\sigma} \frac{dJ}{2\pi} \int_0^{2\pi} d\varphi e^{-i\varphi} f(J, \varphi) 
+ \delta_{n-1,n'} \int_{n+1-\sigma}^{n+\sigma} \frac{dJ}{2\pi} \int_0^{2\pi} d\varphi e^{i\varphi} f(J, \varphi) \right]. \quad (56)$$

5.1. Quantization of elementary observables

We now specify the procedure to the most elementary functions $$f(J, \varphi)$$, noting that this specification from the general case is straightforward.

(i) Angular momentum or action operator: it is given by the expression

$$A_J = \int\mu(dx) N^\sigma (J) |J, \varphi\rangle \langle J, \varphi| = \sum_{n \in \mathbb{Z}} n |e_n\rangle\langle e_n|, \quad (57)$$

which coincides with (25).

(ii) Energy operator: it is expressed as

$$A_{J^2} = \int\mu(dx) N^\sigma (J) J^2 |J, \varphi\rangle \langle J, \varphi| = \frac{\sigma^2}{3} 1_H + \sum_{n \in \mathbb{Z}} n^2 |e_n\rangle\langle e_n| \quad (58)$$

as it should from relation (27) defined for an arbitrary distribution. The first right-hand-side term $$\sigma^2/3$$ indeed represents the average of the classical energy with respect to the uniform probability distribution on the interval $$[n-\sigma, n+\sigma]$$, $$n \in \mathbb{Z}$$.

(iii) Elementary Fourier harmonic operator: the operator ‘Fourier fundamental harmonics’ is defined by

$$A_{\cos} = \left(1 - \frac{1}{2\sigma}\right) \sum_{n \in \mathbb{Z}} |e_{n+1}\rangle\langle e_n|. \quad (59)$$

It becomes null operator at the lowest limit $$\sigma = 1/2$$, whereas it is one-half of the expected one at the upper limit $$\sigma = 1$$.

(iv) Angle operator: it is provided by

$$A_\varphi = \pi 1_H + i \left(1 - \frac{1}{2\sigma}\right) \sum_{n \in \mathbb{Z}} |e_n\rangle\langle e_{n-1}| - |e_n\rangle\langle e_{n+1}| = A_{\pi - 2\sin \varphi}. \quad (60)$$

Hence, it amounts to replace the angle function by the first two terms of its Fourier series,

$$B(\varphi) = \pi - 2 \sum_{n \neq 1} \frac{\sin(n\varphi)}{n}. \quad (61)$$

Note that the same operator can correspond to more than one classical observable and that for $$\sigma = 1/2$$, the angle operator reduces to the classical angle average, namely $$\pi$$. 

9
5.2. Some commutators

(i) Commutator of action and angle operators: the evaluation of the commutator of the action and angle operators is just proportional to the ‘free’ infinite tridiagonal Jacobi matrix [36],

\[
[A_J, A_\phi] = i \left(1 - \frac{1}{2\sigma}\right) \sum_{n \in \mathbb{Z}} [\langle e_n | e_{n-1} \rangle \langle e_{n+1} | e_n \rangle] = iA_2 \cos \phi, \tag{62}
\]

an expression which is consistent with (60) and the fact that \(A_J\) acts as \(-i \partial / \partial \phi\) in the space \(L^2(S^1, d\phi / 2\pi)\). This expression has to be compared with the classical Poisson bracket \(\{ J, \phi \} = 1\). It is well known that the spectral measure of the Jacobi matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & \cdots \\
0 & 1 & 0 & 1 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\end{pmatrix}
\]

is supported by \([-2, 2]\), and so the spectrum of commutator (62) is continuous and equal to the interval \(i [-2 - 1/\sigma, 2 - 1/\sigma]\).

(ii) Commutator of energy-angle operators: similarly, we obtain for the commutator of energy-angle operators

\[
[A_{J^2}, A_\phi] = i \left(1 - \frac{1}{2\sigma}\right) \sum_{n \in \mathbb{Z}} [(2n - 1)e_n \langle e_{n-1} | e_n \rangle + (2n + 1)e_n \langle e_{n+1} | e_n \rangle] = 2iA_2 \cos \phi + A_2 \sin \phi, \tag{63}
\]

an expression to be compared with the classical Poisson bracket \(\{ J^2, \phi \} = 2J\).

(iii) Commutator of action-elementary harmonics operators: we have

\[
[A_J, A_{e^{i\phi}}] = \pm \left(1 - \frac{1}{2\sigma}\right) \sum_{n \in \mathbb{Z}} e_{n} \langle e_{n} | e_{n} \rangle = \pm A_{e^{i\phi}}, \tag{64}
\]

to be compared with \(\{ J, e^{i\phi} \} = \pm ie^{i\phi}\).

(iv) Commutator of energy-elementary harmonics operators:

\[
[A_{J^2}, A_{e^{i\phi}}] = A_{e^{i\phi}} \pm 2 \left(1 - \frac{1}{2\sigma}\right) \sum_{n \in \mathbb{Z}} n e_{n} \langle e_{n} | e_{n} \rangle, \tag{65}
\]

to be compared with \(\{ J^2, e^{i\phi} \} = \pm 2Je^{i\phi}\).

5.3. Some lower symbols

The mean values of the above-described operators with respect to the CS for the distribution \(\sigma_n^\sigma\) are obtained as linear superposition of distributions \(\sigma_n^\sigma, n \in \mathbb{Z}\), as follows.

(i) Angular momentum or action: the lower symbol of the angular momentum operator \(A_J\) is given by

\[
\langle J_0, \phi_0 | A_J | J_0, \phi_0 \rangle = \frac{1}{N^\sigma (J_0)} \sum_{n \in \mathbb{Z}} n \sigma_n^\sigma (J_0) = \frac{1}{N^\sigma (J_0)} \sum_{n \in \mathbb{Z}} \frac{n}{2\sigma} \chi_{[n-\sigma,n+\sigma]}(J_0). \tag{66}
\]
(ii) Energy:

\[ \langle J_0, \varphi_0 | A_{J'} | J_0, \varphi_0 \rangle = \frac{\sigma^2}{3} + \frac{1}{N^{\sigma}(J_0)} \sum_{n \in \mathbb{Z}} \frac{n^2}{2\sigma} \chi_{[n-\sigma, n+\sigma]}(J_0). \]  

(67)

In the case of non-correlation \( \sigma = 1/2 \), i.e. in the commutative situation, the intervals 
\([n-\sigma, n+\sigma]\) are those separating successive half-integers \([n+1/2], n \in \mathbb{Z}\). Then, 
if \( J_0 \) is a half-integer, there exists an integer \( n_0 \in \mathbb{Z} \) such that \( \chi_{[n_0+1/2]}(J_0) = 1 \) and 
\( \chi_{[n_0+1/2]}(J_0) = 0 \) for \( n \neq n_0 \). Therefore, we arrive at

\[ \langle J_0, \varphi_0 | A_{J'} | J_0, \varphi_0 \rangle = \frac{1}{N^{\sigma=1/2}(J_0)} n_0. \]  

(68)

\[ \langle J_0, \varphi_0 | A_{J'} | J_0, \varphi_0 \rangle = \frac{1}{N^{\sigma=1/2}(J_0)} n_0. \]  

(69)

If \( J_0 \) is not a half-integer, for \( n \in \mathbb{Z} \), we obtain

\[ \langle J_0, \varphi_0 | A_{J'} | J_0, \varphi_0 \rangle = 0. \]  

(70)

Besides, for \( 1/2 < \sigma \leq 1 \), if there exists \( n_0 \in \mathbb{Z} \) such that \( J_0 \in [n_0-\sigma, n_0+\sigma] \), then

\[ \langle J_0, \varphi_0 | A_{J'} | J_0, \varphi_0 \rangle = \frac{1}{N^{\sigma}(J_0)} n_0, \quad \langle J_0, \varphi_0 | A_{J'} | J_0, \varphi_0 \rangle = \frac{\sigma^2}{3} + \frac{1}{N^{\sigma}(J_0)} \frac{n_0^2}{2\sigma}. \]  

(71)

(iii) Angle: the lower symbol of the angle operator is given by

\[ \langle J_0, \varphi_0 | A_{\chi} | J_0, \varphi_0 \rangle = \frac{1}{2} \left( 1 - \frac{1}{2\sigma} \right) \frac{1}{N^{\sigma}(J_0)} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\theta} \chi_{[n-\sigma, n+\sigma]}(J_0) - e^{-in\theta} \chi_{[n+1-\sigma, n+\sigma]}(J_0). \]  

(72)

In the lowest limit \( \sigma = 1/2 \), we obtain for \( J_0 \) a.e. \( J_0, \varphi_0 | A_{\chi} | J_0, \varphi_0 \rangle = \pi. \)

(iv) Fourier exponentials: the respective lower symbols of the elementary Fourier exponentials 
are given by

\[ \langle J_0, \varphi_0 | A_{e^{i\omega n}} | J_0, \varphi_0 \rangle = \left( 1 - \frac{1}{2\sigma} \right) \frac{1}{N^{\sigma}(J_0)} \frac{1}{2\sigma} e^{i\omega n} \sum_{n \in \mathbb{Z}} \chi_{[n+1-\sigma, n+\sigma]}(J_0). \]  

(73)

(v) Commutator action-angle: the lower symbol of the commutator \([A_J, A_{\chi}]\) is given by

\[ \langle J_0, \varphi_0 | [A_J, A_{\chi}] | J_0, \varphi_0 \rangle = i \left( 1 - \frac{1}{2\sigma} \right) \frac{1}{N^{\sigma}(J_0)} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\theta} \chi_{[n-\sigma, n+\sigma]}(J_0) + e^{-in\theta} \chi_{[n+1-\sigma, n+\sigma]}(J_0). \]  

(74)

(vi) Commutator energy-Fourier exponentials: the same computation for the commutator 
\([A_{J'}, A_{e^{i\omega n}}]\) gives

\[ \langle J_0, \varphi_0 | [A_{J'}, A_{e^{i\omega n}}] | J_0, \varphi_0 \rangle = \left( 1 - \frac{1}{2\sigma} \right) \frac{e^{i\omega n}}{2\sigma N^{\sigma}(J_0)} \sum_{n \in \mathbb{Z}} (2n + 1) \chi_{[n+1-\sigma, n+\sigma]}(J_0). \]  

(75)

\[ \langle J_0, \varphi_0 | [A_{J'}, A_{e^{i\omega n}}] | J_0, \varphi_0 \rangle = \left( 1 - \frac{1}{2\sigma} \right) \frac{e^{-i\omega n}}{2\sigma N^{\sigma}(J_0)} \sum_{n \in \mathbb{Z}} (1 - 2n) \chi_{[n-\sigma, n-1+\sigma]}(J_0). \]  

(76)
Note that we can now use the normalization function obtained in (51) to simplify some of the expressions of the above lower symbols as follows:

\[
\frac{1}{2\sigma} \sum_{n \in \mathbb{Z}} x_{n} I_{n+1,0} I_{n+\sigma,0} = \mathcal{N}^{\sigma}(J_{0}) - \frac{1}{2\sigma},
\]

(77)

\[
\langle J_{0}, \phi_{0}|A_{\pm}|J_{0}, \phi_{0} \rangle = e^{\pm i\phi_{0}} \left( 1 - \frac{1}{2\sigma} \right) \left( 1 - \frac{1}{2\sigma N^{\sigma}(J_{0})} \right).
\]

(78)

\[
\langle J_{0}, \phi_{0}|A_{\phi}|J_{0}, \phi_{0} \rangle = \pi - 2 \left( 1 - \frac{1}{2\sigma} \right) \left( 1 - \frac{1}{2\sigma N^{\sigma}(J_{0})} \right) \sin \phi_{0}.
\]

(79)

We note that for \(\phi_{0} = \pi\), the lower symbol (79) is equal to the classical average \(\pi\).

Moreover, we obtain

\[
\langle J_{0}, \phi_{0}|[A_{J}, A_{\phi}]|J_{0}, \phi_{0} \rangle = 2i \left( 1 - \frac{1}{2\sigma} \right) \left( 1 - \frac{1}{2\sigma N^{\sigma}(J_{0})} \right) \cos \phi_{0}.
\]

(80)

Thus, we recover the canonical commutation rule up to a multiplicative factor. Actually, for \(J_{0}\) such that \(N^{\sigma}(J_{0}) = 1/(2\sigma)\) the above expression is 0, whereas in the other case \(N^{\sigma}(J_{0}) = 1/(\sigma)\), it is equal to \(i(1 - 1/(2\sigma)) \cos \phi_{0}\). It is also 0 for \(\phi_{0} = \pi/2\) or \(3\pi/2\).

6. Conclusion

We have reviewed the general procedure of quantization for a given set \(X\) of parameters equipped with a measure \(\mu\) and studied some relevant statistical features by taking into account the interplays between discrete and continuous probability distributions. The motion of a particle on a circle has been studied by considering the cylinder as the corresponding phase space which, in this context, plays the role of the set \(X\). We have constructed various families of CS which are determined by probability distributions on the cylinder. These distributions are requested to obey a minimal set of properties which still leave a large spectrum of possibilities. The resulting quantization of classical observables has been implemented. The relations between the derived quantum operators together with their respective commutators have been analyzed either directly from the properties of the corresponding operators or through the behavior of their respective lower symbols. The method has been illustrated in more detail with the particular case of uniform distributions on intervals. An interesting feature of our formalism lies in the possibility of applications to models encountered in nanophysics like circuit QED (see [30] and references therein). Indeed, the question of validity of a precise choice of probability distribution could be experimentally tested in such a context. Another domain of applications where such tests are possible is the so-called quantum chaos appearing in systems with cylindric phase space, e.g. the kicked pendulum or rotator for which the consistency between CS and standard quantizations of the Chirikov standard map [24, 37] (and references therein) should be fully validated. Finally, note that it should be interesting to deepen the Euclidean symmetry (equations (26) and (35)) preserved by CS quantization in the spirit of Perelomov’s group theoretical methods for building generalized CS [38]. All these questions require the elaboration of appropriate theoretical framework which is now under investigation and will be in the core of a forthcoming paper.

Acknowledgments

JPG expresses his gratitude to the ICMPA-UNESCO Chair and University of Abomey-Calavi for their financial support and hospitality, and to the French Ministry of Foreign Affairs for financial support.
Appendix A. Statistical aspects of CS quantization

First, the transform \( f \mapsto \hat{f} \) is built from the nonnegative kernel \(|\langle x | x' \rangle|^2\) which is also a family of probability distributions, indexed by \( x \in X \), on the set \( X \) equipped with the measure \( \mathcal{N}(x')\mu(dx') \). Hence, the function \( x \mapsto \hat{f}(x) \) is the average of \( f \) with respect to the latter. Here is encountered a sort of regularization of the original \( f \) (depending of course on the topology affected to \( X \)).

Second, there is also an interplay between two probability distributions [16, 26].

For almost each \( x \), a discrete distribution,

\[
\mu_n = |\langle e_n | x \rangle|^2 = |\phi_n(x)|^2 \mathcal{N}(x),
\]

(A.1)

Within a quantum physics framework, this probability could be considered as concerning experiments performed on the system described by the Hilbert space \( \mathcal{H} \) within some experimental protocol, say \( \mathcal{E} \), in order to measure the spectral values of a certain self-adjoint operator, a ‘quantum observable’, \( A \), acting in \( \mathcal{H} \) and having the discrete spectral resolution \( A = \sum_n a_n |e_n\rangle\langle e_n| \).

For each \( n \), a ‘continuous’ distribution on \( (X, \mu) \),

\[
X \ni x \mapsto |\phi_n(x)|^2.
\]

(A.2)

Here, we observe a Bayesian duality typical of CS [26]. There are two interpretations: the resolution of the unity verified by the ‘coherent’ states \(|x\rangle\) introduces a preferred prior measure on the set \( X \), which is the set of parameters of the discrete distribution, with this distribution itself playing the role of the likelihood function. The associated discretely indexed continuous distributions become the related conditional posterior distribution.

Hence, a probabilistic approach to experimental observations concerning \( A \) should serve as a guideline in choosing the set of the \( \phi_n(x) \)’s.

We note that the continuous prior distribution will be relevant for the quantization, whereas the discrete posterior one characterizes the measurement of the physical spectrum from which is built the ‘coherent’ superposition of quantum states \(|e_n\rangle\).

Appendix B. Normal law CS for the motion on the circle

The functions \( \phi_n(x) \) forming the orthonormal system needed to construct CS are chosen as Gaussian weighted Fourier exponentials:

\[
\phi_n(x) = \left( \frac{1}{2\pi \sigma^2} \right)^{1/4} e^{-\frac{1}{4\sigma^2}(J-n)^2} e^{i\phi}, \quad n \in \mathbb{Z},
\]

(B.1)

where \( \sigma > 0 \) is a regularization parameter that can be arbitrarily small. The CS [17–19] read as

\[
|x\rangle = |J, \psi\rangle = \frac{1}{\sqrt{\mathcal{N}(J)}} \left( \frac{1}{2\pi \sigma^2} \right)^{1/4} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{4\sigma^2}(J-n)^2} e^{-i\phi} |e_n\rangle,
\]

(B.2)

where the states \(|e_n\rangle\)’s, in one-to-one correspondence with the \( \phi_n \)’s, form an orthonormal basis of some separable Hilbert space \( \mathcal{H} \). For instance, they can be considered as the Fourier exponentials \( e^{i\phi} \) forming the orthonormal basis of the Hilbert space \( L^2(S^1, d\theta/2\pi) \cong \mathcal{H} \).

They would be the spatial or angular modes in this representation. In this representation, the CS read as the following Fourier series:

\[
\zeta_{J,\psi}(\theta) = \frac{1}{\sqrt{\mathcal{N}(J)}} \left( \frac{1}{2\pi \sigma^2} \right)^{1/4} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{4\sigma^2}(J-n)^2} e^{i\phi} e^{i\theta - \psi}.
\]

(B.3)
The normalization factor is a periodic train of normalized Gaussians which can be written as an elliptic theta function [34]:

\[ N^\alpha(J) = \sqrt{\frac{1}{2\pi\sigma^2}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{2\pi\sigma^2}(J-n)^2} = \delta_\alpha(J, 2\pi i\sigma^2) = \text{Poisson} \sum_{n \in \mathbb{Z}} e^{2\pi i\sigma J} e^{-2\pi^2\sigma^2 n^2}. \] (B.4)

Its asymptotic behavior at small and large values of the parameter \( \sigma \) is given by

\[ \lim_{\sigma \to 0} N^\alpha(J) = \sum_{n \in \mathbb{Z}} \delta(J-n) \quad \text{(Dirac comb)}, \] (B.5)

\[ \lim_{\sigma \to \infty} N^\alpha(J) = 1. \] (B.6)

We also note that \( \lim_{\sigma \to 0} \sqrt{2\pi\sigma^2}N^\alpha(J) = 1 \) if \( J \in \mathbb{Z} \) and = 0 otherwise.

By construction, states (B.2) are normalized and resolve the identity in the Hilbert space \( \mathcal{H} \):

\[ \int_{-\infty}^{+\infty} dJ \int_{0}^{2\pi} d\varphi \frac{2}{2\pi} N^\alpha(J) |J, \varphi\rangle \langle J, \varphi| = 1_{\mathcal{H}}. \] (B.7)

They overlap as

\[ \langle x|\chi' \rangle = \frac{e^{-\frac{1}{2\pi\sigma^2}(J-J')^2}}{\sqrt{\frac{2}{2\pi\sigma^2}N^\alpha(J)N^\alpha(J')}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{2\pi\sigma^2}(2\varphi-n)^2} e^{i\varphi_0(x-\varphi')} \] (B.8)

\[ = \text{Poisson} \frac{e^{-\frac{1}{2\pi\sigma^2}(J-J')^2}}{\sqrt{\frac{2}{2\pi\sigma^2}}N^\alpha(J)} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{2\pi\sigma^2}(\varphi_0(\varphi'-2\pi n)^2} e^{-i\varphi_0(j+j')}. \] (B.9)

These expressions stand for the representation of the CS \( |\chi' \rangle \) as a function of \( x = (J, \varphi) \). It is interesting to explore the two possible limits of the Gaussian width:

\[ \lim_{\sigma \to 0} \langle x|\chi' \rangle = \begin{cases} 0 & \text{if } J \notin \mathbb{Z} \text{ or } J' \notin \mathbb{Z} \\ \delta_{JJ'} e^{i\varphi_0(x-\varphi')} & \text{if } J \in \mathbb{Z} \end{cases}, \] (B.10)

\[ \lim_{\sigma \to \infty} \langle x|\chi' \rangle = \begin{cases} 0 & \text{if } \varphi - \varphi' \notin 2\pi \mathbb{Z} \\ 1 & \text{if } \varphi - \varphi' \in 2\pi \mathbb{Z}, \end{cases} \] (B.11)

where \( \delta_{JJ'} \) is the Kronecker symbol, i.e. = 0 if \( J \neq J' \) and = 1 if \( J = J' \). Therefore, from (B.10), the CS tend to be orthogonal at small \( \sigma \) if \( J \notin \mathbb{Z} \) or if \( J \neq J' \) whatever the value of the difference \( \varphi - \varphi' \) is. On the other hand, from (B.11), the CS tend to become orthogonal at large \( \sigma \) if \( \varphi - \varphi' \notin 2\pi \mathbb{Z} \), whatever the value of the difference \( J - J' \) is. We have here an interesting duality in semi-classical aspects of these states, the term ‘semi-classical’ being used for both limits of the parameter \( \sigma \). We discuss this important point at the end of section 4.

References

[1] Klauder J R and Skagerstam B S 1985 Coherent States—Applications in Physics and Mathematical Physics (Singapore: World Scientific)
[2] Klauder J R 1963 Continuous-representation theory: I. Postulates of continuous-representation theory J. Math. Phys. 4 1055–8
[3] Klauder J R 1995 Quantization without quantization Ann. Phys. 237 147–60
[4] Berezin F A 1975 General concept of quantization Commun. Math. Phys. 40 153–74
[5] Ali S T 1985 Stochastic localization, quantum mechanics on phase space and quantum space-time Riv. Nuovo Cimento 8 1–128
[6] Cotfas N, Gazeau J P and Górska K 2010 Complex and real Hermite polynomials and related quantizations J. Phys. A: Math. Theor. 43 305304

[7] Gazeau J P and Szafirneci F H 2011 Holomorphic Hermite polynomials and a non-commutative plane J. Phys. A: Math. Theor. 44 495201

[8] Gazeau J P and Piechocki W 2004 Coherent state quantization of a particle in de Sitter space J. Phys. A: Math. Gen. 37 6977

[9] Garcia de Leon P, Gazeau J P and Quèva J 2008 The infinite well revisited: coherent states and quantization Phys. Lett. A 372 3597

[10] Garcia de Leon P, Gazeau J P, Gitman D and Quèva J 2009 Infinite quantum well: on the quantization problem Quantum Wells: Theory, Fabrication and Applications (Hauppauge, NY: Nova Science Publishers)

[11] Bergeron H, Gazeau J P, Siegl P and Youssef A 2010 Semi-classical behavior of Pöschl–Teller coherent states Eur. Phys. Lett. 92 60003

[12] Cotfas N, Gazeau J P and Vourdas A 2011 Finite quantum systems and frame quantization J. Phys. A: Math. Theor. 44 175303

[13] ElBaz M, Gazeau J P, Fresneda R and Hassouni Y 2010 Coherent state quantization of paragrassmann algebras J. Phys. A: Math. Theor. 43 385202

[14] Baldiotti M, Gazeau J P and Gitman D M 2009 Coherent states of a particle in magnetic field and Stieltjes moment problem Phys. Lett. A 373 1916–20

[15] Baldiotti M, Gazeau J P and Gitman D M 2009 Phys. Lett. A 373 2600 (erratum)

[16] Gazeau J P 2009 Coherent States in Quantum Physics (Berlin: Wiley-VCH)

[17] De Bièvre S and González J A 1993 Semiclassical behaviour of coherent states on the circle Quantization and Coherent States Methods in Physics ed A Odzijewicz et al (Singapore: World Scientific)

[18] Kowalski K, Rembielinski J and Papaloucas L C 1996 Coherent states for a quantum particle on a circle J. Phys. A: Math. Gen. 29 4119

[19] González J A and del Olmo M A 1998 Coherent states on the circle J. Phys. A: Math. Gen. 31 8841

[20] Kowalski K and Rembielinski J 2002 Exotic behaviour of a quantum particle on a circle Phys. Lett. A 293 109

[21] Kowalski K and Rembielinski J 2002 On the uncertainty relations and squeezed states for the quantum mechanics on a circle J. Phys. A: Math. Gen. 35 1405

[22] Kowalski K and Rembielinski J 2003 Reply to the ‘Comment on ‘On the uncertainty relations and squeezed states for the quantum mechanics on a circle’ J. Phys. A: Math. Gen. 36 5695

[23] Hall B C and Mitchell J J 2002 Coherent states on spheres J. Math. Phys. 43 1211

[24] Chang S-J and Shi K-J 1985 Time evolution and eigenstates of a quantum iterative system Phys. Rev. Lett. 55 269–72

[25] Chang S-J and Shi K-J 1986 Evolution of exact eigenstates of a resonant quantum system Phys. Rev. A 34 7–22

[26] Zyczkowski K 1989 Squeezed states in a quantum chaotic system J. Phys. A: Math. Gen. 22 L1147–51

[27] Ali S T, Gazeau J P and Heller B 2008 Coherent states and Bayesian duality J. Phys. A: Math. Theor. 41 365302

[28] Gazeau J P and Kanamoto R 2012 Quantization with action-angle coherent states J. Phys.: Conf. Ser. 343 012038 (arXiv:1110.6678v1 [quant-ph])

[29] Bagrov V G, Gazeau J P, Gitman D and Levine A 2012 Coherent states and related quantizations for unbounded motions J. Phys. A: Math. Theor. 45 125306

[30] Bouchiat V, Vion D, Joyez P, Esteve D and Devoret M H 1998 Quantum coherence with a single Cooper pair Phys. Scr. T 76 165–70

[31] Carruthers P and Nieto M M 1968 Rev. Mod. Phys. 40 411–40

[32] Magnus W, Oberhettinger F and Soni R P 1966 Formulas and Theorems for the Special Functions of Mathematical Physics (Berlin: Springer)
[35] Rabeie A, Huguet E and Renaud J 2007 Wick ordering for coherent state quantization in $1 + 1$ de Sitter space
Phys. Lett. A 370 123

[36] Killip R and Simon B 2003 Sum rules for Jacobi matrices and their applications to spectral theory Ann. Math. 158 253–321

[37] Haake F and Shepelyansky D L 1988 The kicked rotator as a limit of the kicked top Europhys. Lett. 5 671–6

[38] Perelomov A M 1986 Generalized Coherent States, Applications in Physics and Mathematical Physics (Berlin: Springer)