ON THE HOMEOMORPHISM GROUPS OF MANIFOLDS AND THEIR UNIVERSAL COVERINGS

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ABSTRACT. Let \( \mathcal{H}(M) \) stand for the path connected identity component of the group of all compactly supported homeomorphisms of a manifold \( M \). It is shown that \( \mathcal{H}(M) \) is perfect and simple under mild assumptions on \( M \). Next, conjugation-invariant norms on \( \mathcal{H}(M) \) are considered and the boundedness of \( \mathcal{H}(M) \) and its subgroups is investigated. Finally, the structure of the universal covering group of \( \mathcal{H}(M) \) is studied.

1. Introduction

Let \( M \) be a topological metrizable manifold of dimension \( n \geq 1 \), possibly with boundary, and let \( \mathcal{H}(M) \) (resp. \( \mathcal{H}_c(M) \)) be the path connected identity component of the group of all (resp. compactly supported) homeomorphisms of a manifold \( M \) endowed with the compact-open topology. In this paper we will deal with algebraic properties of the group \( \mathcal{H}_c(M) \) and of its universal covering.

Recall that a group \( G \) is called perfect if it is equal to its own commutator subgroup \([G,G]\). That is, \( H_1(G) = 0 \). The following basic fact is probably well-known but we have not found it explicitly proven in the literature.

**Theorem 1.1.** Assume that either \( M \) is compact (possibly with boundary), or \( M \) admits a compact exhaustion, i.e. there is a sequence of compact submanifolds with boundary \((M_\ell)_{\ell=1}^\infty \) with \( \dim M_\ell = \dim M = n \) such that \( M_1 \subset M_2 \subset \ldots \) and \( M = \bigcup_{\ell=1}^\infty M_\ell \). Then the group \( \mathcal{H}_c(M) \) is perfect.

The proof of the perfectness is a consequence of Mather’s paper [14] combined with Edwards and Kirby [7], Corollary 1.3. In the case \( n = 1 \) and \( M \) with boundary the proof requires an additional argument. See section 3. A special case of Theorem 1.1 was already proved by Fisher [8] (see also Anderson [2]). Observe that McDuff in [16] proved that \( \mathcal{H}(M) \) is perfect provided...
$M$ is the interior of a compact manifold with boundary. There exist some generalizations of Theorem 1.1 (see, e.g., Fukui and Imanishi [10], and Rybicki [18]).

If $M$ is a smooth manifold then Theorem 1.1 has its smooth analogue. Let $\mathcal{D}(M)$ be the identity component of the group of all compactly supported $C^\infty$-diffeomorphisms of $M$. Thurston proved that $\mathcal{D}(M)$ is perfect and simple (see [23], [4]). Also Mather in [15] proved the same in the class of $C^r$-diffeomorphisms unless $r = \dim M + 1$. Analogous results for classical groups of diffeomorphisms are also known ([3], [4], [11], [20]).

In the case of a manifold with boundary $M$ we denote by $M^o$ the interior of $M$, and by $\partial M$ the boundary of $M$. We will consider the following groups:

$$\mathcal{H}_c(M^o) \leq \mathcal{H}_c^\partial(M) \leq \mathcal{H}_c(M) \leq \mathcal{H}(M^o).$$

Here $h \in \mathcal{H}_c^\partial(M)$ if there is a compactly supported isotopy $h_t$ connecting $h_0 = \text{id}$ with $h_1 = h$ such that $h_t = \text{id}$ on $\partial M$ for all $t$. Moreover, $\mathcal{H}_c(M)$ identifies with a subgroup of $\mathcal{H}(M^o)$ by restricting elements of $\mathcal{H}_c(M)$ to $M^o$.

**Theorem 1.2.** If the boundary $\partial M$ is compact then $\mathcal{H}_c^\partial(M)$ is a perfect group.

Concerning the simplicity of $\mathcal{H}_c(M)$ we have the following

**Corollary 1.3.** Let $M$ be connected and satisfy the hypothesis of Theorem 1.1. Then $M$ is boundaryless (i.e. $\partial M = \emptyset$) if and only if $\mathcal{H}_c(M)$ is simple.

The proof will be given in section 4 together with further comments on the simplicity by using some ideas of Ling [13].

Conjugation-invariant norms related to homeomorphism groups on $M$ are considered in section 5. Recall that a group is bounded if every conjugation-invariant norm is bounded on it. Following an argument from [6] we will prove in section 6 the following

**Theorem 1.4.** Under the assumption of Theorem 1.1 on $M$, $\mathcal{H}_c(M)$ is bounded if and only if $\text{frag}_{M}$ is bounded, where $\text{frag}_{M}$ is the fragmentation norm on $M$ with respect to homeomorphisms. In particular, $\mathcal{H}_c(\mathbb{R}^n)$ is bounded.

Also we have the following boundedness theorem.

**Theorem 1.5.** Let $\partial M$ be compact. If the group $\mathcal{H}_c(M)$ is bounded then $\mathcal{H}(\partial M)$ is bounded also. Moreover, if the group $\mathcal{H}_c(M^o)$ is bounded then so is the group $\mathcal{H}_c^\partial(M)$.

Observe that $\mathcal{H}_c(M^o)$ is bounded if $M^o$ is portable (sect. 6). In [20] the second-named author proved that $\mathcal{H}(M^o)$ is bounded provided so is $\mathcal{H}_c(M^o)$.

The last part of the paper is devoted to the structure of the universal coverings of some homeomorphism groups. Let $\mathcal{H}_c(M)$ denote the universal covering group of $\mathcal{H}_c(M)$.
**Theorem 1.6.** Let \( n = \dim M \geq 2 \) or \( \partial M = \emptyset \). The group \( \mathcal{H}_c(M)^\sim \) is perfect. Moreover, the groups \( \mathcal{H}_c(\mathbb{R}^n)^\sim \) and \( \mathcal{H}_c(\mathbb{R}^+)^\sim \) are acyclic, where \( \mathbb{R}^+ = \{ x \in \mathbb{R}^n : x_n \geq 0 \} \) is the half-space.

The proof will be given in section 7. We will also study the problem of boundedness.

**Theorem 1.7.** Let \( \text{frag}_M^{iso} \) be the isotopy fragmentation norm on the universal covering group \( \mathcal{H}_c(M)^\sim \) (c.f. sect.7). Suppose that \( \dim M \geq 2 \) or \( \partial M = \emptyset \). Then \( \mathcal{H}_c(M)^\sim \) is bounded if and only if \( \text{frag}_M^{iso} \) is bounded.

We emphasize that many facts presented in this paper are specific for the topological category, that is there are no longer true in the smooth (or even Lipschitz) category. See, e.g., Remark 3.4 and Prop. 6.3.

2. Fragmentation property and isotopy extension theorem

The results of this paper depend essentially on the deformation properties for the spaces of imbeddings obtained by Edwards and Kirby in [7]. See also Siebenmann [22]. Let us recall basic notions and facts from [7].

Given a subset \( S \subset M \), by \( \mathcal{H}_S(M) \) we denote the path connected identity component of the subgroup of all elements of \( \mathcal{H}(M) \) with compact support contained in \( S \). By a ball (resp. half-ball) \( B \) we mean rel. compact open ball (resp. half-ball with \( \partial B = \partial B \cap \partial M \)) embedded in \( M \) with its closure. By \( \mathcal{B} \) we denote the family of all balls and half-balls in \( M \).

Using the Alexander trick, we have that \( \mathcal{H}(\mathbb{R}^n) \) coincides with the group of all compactly supported homeomorphisms of \( \mathbb{R}^n \). In fact, if \( \text{supp}(g) \) is compact, we define an isotopy \( g_t : \mathbb{R}^n \rightarrow \mathbb{R}^n, t \in I, \) from the identity to \( g \), by

\[
g_t(x) = \begin{cases} 
t x & \text{for } t > 0 \\
x & \text{for } t = 0.
\end{cases}
\]

In particular, for every ball \( B \) in \( M \) the group \( \mathcal{H}_B(M) \) consists of all homeomorphisms compactly supported in \( B \). Observe that the Alexander trick is no longer true in the smooth category.

Let us formulate the fragmentation property in the following stronger way.

**Definition 2.1.** Let \( \mathcal{U} \) be an open covering of \( M \). A subgroup \( G \leq \mathcal{H}(M) \) is *locally continuously factorizable* if for any finite subcovering \( (U_i)_{i=1}^d \) of \( \mathcal{B} \), there exist a neighborhood \( \mathcal{P} \) of \( \text{id} \in G \) and continuous mappings \( \sigma_i : \mathcal{P} \rightarrow G, i = 1, \ldots, d \), such that for all \( f \in \mathcal{P} \) one has

\[
f = \sigma_1(f) \ldots \sigma_d(f), \quad \text{supp}(\sigma_i(f)) \subset U_i, \forall i.
\]

Throughout for a topological group \( G \) by \( \mathcal{P}G \) we will denote the totality of paths \( \gamma : I \rightarrow G \) with \( \gamma(0) = e \) (where \( I = [0,1] \)). Observe that Def. 2.1 can also be formulated for \( \mathcal{P}G \) rather than \( G \), where \( G \leq \mathcal{H}(M) \).
From now on $M$ is a metrizable topological manifold. If $U$ is a subset of $M$, a proper imbedding of $U$ into $M$ is an imbedding $h : U \to M$ such that $h^{-1}(\partial M) = U \cap \partial M$. An isotopy of $U$ into $M$ is a family of imbeddings $h_t : U \to M, t \in I$, such that the map $h : U \times I \to M$ defined by $h(x, t) = h_t(x)$ is continuous. An isotopy is proper if each imbedding in it is proper. Now let $C$ and $U$ be subsets of $M$ with $C \subseteq U$. By $I(U, C; M)$ we denote the space of proper imbeddings of $U$ into $M$ which equal the identity on $C$, endowed with the compact-open topology.

Suppose $X$ is a space with subsets $A$ and $B$. A deformation of $A$ into $B$ is a continuous mapping $\varphi : A \times I \to X$ such that $\varphi|_{A \times 0} = \text{id}_A$ and $\varphi(A \times 1) \subseteq B$. If $\mathcal{P}$ is a subset of $I(U; M)$ and $\varphi : \mathcal{P} \times I \to I(U; M)$ is a deformation of $\mathcal{P}$, we may equivalently view $\varphi$ as a map $\varphi : \mathcal{P} \times I \times U \to M$ such that for each $h \in \mathcal{P}$ and $t \in I$, the map $\varphi(h, t) : U \to M$ is a proper imbedding.

If $W \subseteq U$, a deformation $\varphi : \mathcal{P} \times I \to I(U; M)$ is modulo $W$ if $\varphi(h, t)|_W = h|_W$ for all $h \in \mathcal{P}$ and $t \in I$.

Suppose $\varphi : \mathcal{P} \times I \to I(U; M)$ and $\psi : Q \times I \to I(U; M)$ are deformations of subsets of $I(U; M)$ and suppose that $\varphi(\mathcal{P} \times 1) \subseteq Q$. Then the composition of $\psi$ with $\varphi$, denoted by $\psi \star \varphi$, is the deformation $\psi \star \varphi : \mathcal{P} \times I \to I(U; M)$ defined by

$$
\psi \star \varphi (h, t) = \begin{cases} 
\varphi(h, 2t) & \text{for } t \in [0, 1/2] \\
\psi(\varphi(h, 1), 2t - 1) & \text{for } t \in [1/2, 1].
\end{cases}
$$

The main result of [7] is the following

**Theorem 2.2.** Let $M$ be a topological manifold and let $U$ be a neighborhood in $M$ of a compact subset $C$. For any neighborhood $Q$ of the inclusion $i : U \subseteq M$ in $I(U; M)$ there are a neighborhood $\mathcal{P}$ of $i \in I(U; M)$ and a deformation $\varphi : \mathcal{P} \times I \to Q$ into $I(U, C; M)$ which is modulo the complement of a compact neighborhood of $C$ in $U$ and such that $\varphi(i, t) = i$ for all $t$. We have also that if $D_i \subset V_i$, $i = 1, \ldots, q$, is a finite family of closed subsets $D_i$ with their neighborhoods $V_i$, then $\varphi$ can be chosen so that the restriction of $\varphi$ to $(\mathcal{P} \cap I(U, U \cap V_i; M)) \times I$ assumes its values in $I(U, U \cap D_i; M)$ for each $i$.

Moreover, if $M$ has compact boundary $\partial M$ then $\varphi$ restricted to $(\mathcal{P} \cap I(U, \partial M \cap U; M)) \times I$ takes its values into $I(U, \partial M \cap U; M)$.

The first part coincides with Theorem 5.1[7]. The second part is specified in Remark 7.2 in [7].

We can derive from Theorem 2.2 the following fragmentation theorem.

**Theorem 2.3.** Let $M$ be a compact manifold, possibly with boundary. Then the groups $\mathcal{H}_c(M)$, $\mathcal{H}_c^d(M)$, $\mathcal{PH}_c(M)$ and $\mathcal{PH}_c^d(M)$ are locally continuously factorizable, i.e., they satisfy Def. 2.1.

**Proof.** (See also [7].) We will consider only the case of $\mathcal{H}_c(M)$, the remaining ones being analogous. First we have to shrink the cover $(U_i)^d_{i=1}$ $d$ times, that
is we choose an open $U_{i,j}$ for every $i = 1, \ldots, d$ and $j = 0, \ldots, d$ with $U_{i,0} = U_i$ such that $\bigcup_{j=1}^d U_{i,j} = M$ for all $j$ and such that $\text{cl}(U_{i,j+1}) \subset U_{i,j}$ for all $i, j$.

We make use of Theorem 2.2 $d$ times with $q = 1$. Namely, for $i = 1, \ldots, d$ we have a neighborhood $\mathcal{P}_i$ of the identity in $I(M, \bigcup_{\alpha=1}^{i-1} U_{\alpha,i-1}; M)$ and a deformation $\varphi_i : \mathcal{P}_i \times I \to \mathcal{H}_c(M)$ which is modulo $M \setminus U_{i,0}$ and which takes its values in $I(M, \bigcup_{\alpha=1}^i \text{cl}(U_{\alpha,i}); M)$ and such that $\varphi_i(\text{id}, t) = \text{id}$ for all $t$. Here we apply Theorem 2.2 with $\mathcal{P} = \text{cl}(U_{i,i}), U = U_{i,0}, D_1 = \bigcup_{\alpha=1}^{i-1} \text{cl}(U_{\alpha,i})$ and $V_1 = \bigcup_{\alpha=1}^{i-1} U_{\alpha,i-1}$. Taking a neighborhood $\mathcal{P}$ of $\text{id}$ small enough, we have that $\varphi_d \cdots \varphi_1$ restricted to $\mathcal{P} \times I$ is well defined. For every $h \in \mathcal{P}$ we set $h_0 = h$ and $h_i = \varphi_i \cdots \varphi_1(h, 1), i = 1, \ldots, d$. It follows that $h_d = \text{id}$ and $h = \prod_{i=1}^d h_i h_{i-1}^{-1}$. It suffices to define $\sigma_i : \mathcal{P} \to \mathcal{H}_c(M)$ by $\sigma_i(h) = h_i h_{i-1}^{-1}$ for all $i$.

**Corollary 2.4.** Let $h_t : M \to M, t \in I$, be an isotopy of a compact manifold $M$ with $h_0 = \text{id}$, and let $(U_i)_{i=1}^d$ be an open cover of $M$. Then $h_t$ can be written as a composition of isotopies $h_t = h_{k,t} h_{k-1,t} \cdots h_{1,t}$, where each isotopy $h_{j,t} : M \to M$ is supported by some $U_j$. Moreover, if $h_t|_{\partial M} = \text{id}$ for all $t$, then $h_{j,t}|_{\partial M} = \text{id}$ for all $j$ and $t$. The same is true for homeomorphisms instead of isotopies. □

Another important consequence of Theorem 2.2 is the following Isotopy Extension Theorem.

**Theorem 2.5.** Let $f_t$ be an isotopy in $\mathcal{H}(M)$ and let $C \subset M$ be a compact set. Then for any open neighborhood $U$ of the track of $C$ by $f_t$ given by $\bigcup_{t \in [0,1]} f_t(C)$ there is an isotopy $g_t$ in $\mathcal{H}_c(M)$ such that $g_t = f_t$ on $C$ and $\text{supp}(g_t) \subset U$.

3. Perfectness of $\mathcal{H}_c(M)$ and $\mathcal{H}_c^0(M)$

The goal of this section is to give the proof of Theorem 1.1. We begin with the following fact, with a straightforward proof, which plays a basic role in studies on homeomorphism groups.

**Lemma 3.1.** (Basic lemma) Let $B \subset M$ be a ball and $U \subset M$ be an open subset such that $\overline{B} \subset U$. Then there are $\varphi \in \mathcal{H}_U(M)$ and a homomorphism $S : \mathcal{H}_B(M) \to \mathcal{H}_U(M)$ such that $h = [S(h), \varphi]$ for all $h \in \mathcal{H}_B(M)$.

**Proof.** First choose a larger ball $B'$ such that $\overline{B} \subset B' \subset \overline{B'} \subset U$. Next, fix $p \in \partial B'$ and set $B_0 = B$. There exists a sequence of balls $(B_k)_{k=1}^\infty$ such that $\text{cl}(B_k) \subset B'$ for all $k$, where the family $(B_k)_{k=0}^\infty$ is pairwise disjoint, locally finite in $B'$, and $B_k \to p$ as $k \to \infty$. Choose a homeomorphism $\varphi \in \mathcal{H}_U(M)$ such that $\varphi(B_{k-1}) = B_k$ for $k = 1, 2, \ldots$. Here we use the fact that $\mathcal{H}_U(M)$ acts transitively on the family of balls in $B'$, c.f. [12].
Next we define a homomorphism \( S : \mathcal{H}_B(M) \to \mathcal{H}_U(M) \) by the formula
\[
S(h) = \varphi^k h \varphi^{-k} \quad \text{on } B_k, \ k = 0, 1, \ldots
\]
and \( S(h) = \text{id} \) outside \( \bigcup_{k=0}^\infty B_k \). It is clear that \( h = [S(h), \varphi] \), as required. \( \square \)

The above reasoning appeared in Mather’s paper [14]. Actually Mather proved also the acyclicity of \( \mathcal{H}(\mathbb{R}^n) \). It is easily seen that [14] and Lemma 3.1 are no longer true for \( C^1 \) homeomorphisms. However, Tsuboi gave an excellent improvement of this reasoning and adapted it for \( C^r \)-diffeomorphisms with small \( r \), see [24].

**Corollary 3.2.** Assume that either

1. \( \partial M \neq \emptyset \) with \( \text{dim } M \geq 2 \), and \( B, U \subset M \) are such that \( B \) is a half-ball, and \( U \) is open with \( \overline{B} \subset U \); or
2. \( M = N \times \mathbb{R} \), where \( N \) is a manifold, and \( B = N \times I, \ U = N \times J \) where \( I, J \subset \mathbb{R} \) are open intervals with the closure of \( I \) contained in \( J \).

Then there are \( \varphi \in \mathcal{H}_U(M) \) and a homomorphism \( S : \mathcal{H}_B(M) \to \mathcal{H}_U(M) \) such that \( h = [S(h), \varphi] \) for all \( h \in \mathcal{H}_B(M) \). Moreover, in the case (1), if \( h \in \mathcal{H}_B(M) \) satisfies \( h = \text{id} \) on \( \partial M \) then \( S(h) = \text{id} \) on \( \partial M \).

The proof is analogous to that of Lemma 3.1.

Suppose that \( \{U_i\}_{i \in \mathbb{N}} \) is a pairwise disjoint, locally finite family of open sets of \( M^0 \). Put \( U = \bigcup_i U_i \). Let \( \mathcal{H}_{[U]}(M) \) (resp. \( \mathcal{H}^{\partial}_{[U]}(M) \)) denote the group of all homeomorphisms from \( \mathcal{H}_c(M) \) (resp. \( \mathcal{H}_c^\partial(M) \)) supported in \( U \) such that for the decomposition \( h = h_1 h_2 \ldots \) resulting from the partition \( U = \bigcup_i U_i \) one has \( h_i \in \mathcal{H}_{U_i}(M) \) for all \( i \).

**Corollary 3.3.** Let \( \overline{B_i} \subset U_i, \ i \in \mathbb{N} \), and let the pair \((B_i, U_i)\) be such as in Lemma 3.1 or Corol. 3.2. Then any element \( h \in \mathcal{H}_{[B]}(M) \) (where \( B = \bigcup_i B_i \)) is expressed as \( h = \tilde{h} \bar{h} \), where \( \tilde{h}, \bar{h} \in \mathcal{H}_{[U]}(M) \). Moreover, we can arrange so that if \( h \in \mathcal{H}^\partial_{[B]}(M) \) then \( \tilde{h}, \bar{h} \in \mathcal{H}^\partial_{[U]}(M) \).

In fact, we can glue together \( S(h_i) \) and \( \varphi_i \) obtained for particular \( U_i \).

**Proof of Theorem 1.1 for \( n > 1 \) or \( \partial M = \emptyset \).** For \( M \) compact it follows from Corol. 2.4, Lemma 3.1, Corol. 3.2(1) and, for \( \partial M \neq \emptyset \) and \( \text{dim } M = 1 \), from the fact that \( \mathcal{H}([0,1]) \) is perfect. The proof of the latter fact will follow from the proof of Theorem 1.2 below. Suppose now that \( M \) admits a compact exhaustion. If \( h \in \mathcal{H}_c(M) \) then there are \( j \in \mathbb{N} \) and an isotopy \( h_t \) such that \( h_0 = \text{id}, \ h_1 = h, \) and \( \supp(h_t) \subset M_j \) for all \( t \). In view of Corol. 2.2 it follows that \( h|_{M_j} \) can be written as \( h|_{M_j} = h_d \ldots h_1 \) such that \( h_i \in \mathcal{H}_{B_i}(M_j) \), where \( B_i \) is a ball or half-ball of \( M_i \) for \( i = 1, \ldots, d \). Moreover, we have \( h_i = \text{id} \) on \( \partial M_i \) for all \( i \). Then due to Corol. 3.2(1) we have \( h_i = [S_i(h_i), \varphi_i] \) and \( S_i(h_i) = \text{id} \) on \( \partial M_i \) for each \( i \). Is is easily seen that \( \varphi_i \) may be defined as an element of...
Proof of Theorem 1.1 for $n = 1$ and $\partial M \neq \emptyset$, and of Theorem 1.2. Let $M$ be a manifold with boundary $\partial$. By a collar neighborhood of $\partial$ we mean a set $P = \partial \times [0, 1]$ embedded in $M$, where $\partial \times \{0\}$ identifies with $\partial$. It is well-known that such a neighborhood exists.

In the case of 1.1 we have $\partial = \{0\}$. In view of Theorem 2.3 it suffices to consider $H_c(\partial \times \mathbb{R}_+)$, where $\mathbb{R}_+ = [0, \infty)$. For any $f \in H_c(\partial \times \mathbb{R}_+)$ there is a sequence $(0, 1)$

$$(3.1) \quad 1 > b_1 > \bar{b}_1 > a_1 > b_2 > \ldots > b_k > \bar{b}_k > a_k > \ldots > 0,$$

tending to 0, and $h \in H_c(\partial \times \mathbb{R}_+)$ such that

$$(3.2) \quad h = f \quad \text{on} \quad \partial \times \bigcup_{k=1}^{\infty} [a_k, \bar{b}_k].$$

Moreover, setting $A_k := \partial \times (a_k, b_k)$ and $A := \bigcup_{k=1}^{\infty} A_k$, we may also have that

$$(3.3) \quad \text{supp}(h) \subset A,$$

and that for the decomposition $h = h_1 h_2 \ldots$ resulting from the partition $A = \bigcup_{k=1}^{\infty} A_k$ and from (3.3) we have

$$(3.4) \quad h_k \in H_{A_k}(\partial \times \mathbb{R}_+) \quad \text{for all} \ k.$$

The condition (3.4) means that we exclude any twisting of $h_k$.

In order to show the above statements we apply Theorem 2.5 for $M_0$. This enables us to define recurrently $b_k > \bar{b}_k > a_k > \partial$ and $h|_{\partial \times [a_k, b_k]}$ for $k = 1, 2, \ldots$. In fact, let $f_t$ be an isotopy in $H_c(\partial \times \mathbb{R}_+)$ connecting $f$ with the identity. Suppose we have defined $1 > b_1 > \ldots > a_{k-1}$ and $g \in H_c(\partial \times \mathbb{R}_+)$ such that $g = f$ on $\partial \times \bigcup_{i=1}^{k-1} [a_i, b_i]$, supp$(g) \subset \bigcup_{i=1}^{k-1} A_i$, and $g_i \in H_{A_i}(\partial \times \mathbb{R}_+)$ for all $i \leq k - 1$. Now it suffices to take $a_{k-1} > b_k > \bar{b}_k > a_k > \partial$ in such a way that $\partial \times (0, b_k)$ is disjoint with $\bigcup_{t \in [0, 1]} f_t^{-1}(\partial \times [a_k, 1])$. In view of Theorem 2.5 we get an isotopy $h_t$ such that $h_t = f_t$ on $\partial \times [a_k, b_k]$ and supp$(h_t) \subset \partial \times (a_k, b_k)$. Next we define $h$ on $\partial \times [a_k, 1]$ by gluing together $g$ and $h_t$. Continuing this procedure we define $h \in H^0(\partial \times \mathbb{R}_+)$ fulfilling (3.2), (3.3) and (3.4). Here we put $h(x, 0) = (x, 0)$ for all $x \in \partial$.

Next we set $h' := h^{-1} f$, that is $f = hh'$. It follows that $h'$ also enjoys the properties (3.2), (3.3) and (3.4) with a suitably chosen sequence similar to (3.1).

Let $U_k = (\bar{a}_k, \bar{b}_k)$, where $\bar{a}_k, \bar{b}_k \in (0, 1)$, $k = 1, 2, \ldots$, are such that $\bar{a}_{k-1} > \bar{b}_k > b_k > a_k > \bar{a}_k$ for all $k$ ($\bar{b}_0 = 1$). Now in view of Corollaries 3.2 and 3.3 with $M = \partial \times \mathbb{R}_+$ $h$ belongs to the commutator subgroup of the group $H^0(U)(\partial \times \mathbb{R}_+)$, where $U = \bigcup_{k} U_k$. More precisely $h = [\bar{h}, h]$ for $\bar{h}, h \in H^0(U)(\partial \times \mathbb{R}_+)$. It is
easily seen that \( \tilde{h}, \tilde{h} \in \mathcal{H}_c^0(\partial \times \mathbb{R}_+) \). The same is true for \( h' \). Thus \( \mathcal{H}_c^0(\partial \times \mathbb{R}_+) \) is a perfect group. \( \square \)

**Remark 3.4.** (1) Tsuboi gave another proof of the perfectness of \( \mathcal{H}(\mathbb{R}_+) \) in [25]. He did not use [7] in it.

(2) Given a smooth manifold with boundary \( M \) of dimension \( \geq 2 \), it is known that the group \( \mathcal{D}(M) \) is perfect (see Rybicki [17]; also Abe and Fukui [1] by using a different method). For \( n = 1 \) \( \mathcal{D}(M) \) is not perfect. In particular, Fukui in [9] calculated that \( \mathcal{H}_1(\mathcal{D}(\mathbb{R}_+)) = \mathbb{R} \).

(3) Let \( \mathbb{R}^n_+ = [0, \infty) \times \mathbb{R}^{n-1} \) be the half-space and \( 0 \leq s \leq \infty \). Let \( \mathcal{D}_s(\mathbb{R}^n_+) \) be the compactly supported identity component of the subgroup of all elements of \( \mathcal{D}(\mathbb{R}^n_+) \) which are \( s \)-tangent to the identity on \( \partial \mathbb{R}^n_+ \). Here \( f \) is \( 0 \)-tangent to \( \text{id} \) means that \( f = \text{id} \) on \( \partial \mathbb{R}^n_+ \). If \( 0 \leq s < \infty \) then \( \mathcal{D}_s(\mathbb{R}^n_+) \) is not perfect. In fact, for any diffeomorphisms \( f, g \in \mathcal{D}_s(\mathbb{R}^n_+) \) we have

\[
D^{s+1}(fg)(0) = D^{s+1}f(0) + D^{s+1}g(0), \quad D^{s+1}f^{-1}(0) = -D^{s+1}f(0).
\]

Therefore if we choose \( h \in \mathcal{D}_s(\mathbb{R}^n_+) \) such that \( D^{s+1}h(0) \neq 0 \), the above equalities yield that \( h \) cannot be in the commutator subgroup.

Finally, let us indicate further perfectness result concerning homeomorphism groups. Let \( M \) be a compact manifold with boundary \( \partial \). Let \( \partial = \partial_i, i = 1, \ldots, k, \) be the family of all connected components of the boundary \( \partial \) of \( M \), that is \( \partial = \partial_1 \cup \ldots \cup \partial_k \). Let \( K = \{1, \ldots, k\} \). For any \( J \subset K \) let \( \mathcal{H}(M^\circ, J) \) denote all the elements of \( \mathcal{H}(M^\circ) \) that can be joined with the identity by an isotopy which stabilizes near \( \partial_J \), where \( \partial_J := \bigcup_{i \in J} \partial_i \). In particular, \( \mathcal{H}(M^\circ) = \mathcal{H}(M^\circ, \emptyset) \) and \( \mathcal{H}_c(M^\circ) = \mathcal{H}(M^\circ, K) \). Then we have

**Theorem 3.5.** [16] The groups \( \mathcal{H}(M^\circ, J) \), where \( J \subset K \), are perfect.

For the proof, see also [20]. The proof is no longer valid if we drop the assumption that \( M^\circ \) is the interior of a manifold with boundary, e.g. if \( M \) is the cylinder \( \mathbb{S}^1 \times \mathbb{R} \) with attached infinitely many handles.

4. On the simplicity of \( \mathcal{H}_c(M) \)

The following result is related to Ling’s paper [13].

**Proposition 4.1.** Under the hypothesis of 1.1, there does not exist any fixed point free normal subgroup of \( \mathcal{H}_c(M) \).

**Proof.** Suppose that \( G \) is a fixed point free normal subgroup of \( \mathcal{H}_c(M) \). It follows that if \( M \) has boundary then \( \dim M \geq 2 \). Choose a cover \( \mathcal{U} \subset \mathcal{B} \) such that for any \( U \in \mathcal{U} \) there is \( f \in G \) such that \( U \) and \( f(U) \) are disjoint. Take a cover \( \mathcal{V} \) which is starwise finer than \( \mathcal{U} \). This is possible since \( M \) is metrizable,
so paracompact. We may assume that \( \mathcal{H}_c(M) \) is factorizable with respect to \( V \) (see Def. 5.1(1) and Prop. 5.2 below). In view of the commutator equalities

\[
[fg, h] = f[g, h]f^{-1}[f, h], \quad [f, gh] = [f, g][f, h]g^{-1}
\]

and Theorem 1.1, it follows that

\[
\mathcal{H}_c(M) = [\mathcal{H}_c(M), \mathcal{H}_c(M)] = \prod_{U \in \mathcal{U}} [\mathcal{H}_U(M), \mathcal{H}_U(M)].
\]

Let \([h_1, h_2] \in [\mathcal{H}_U(M), \mathcal{H}_U(M)]\) with \(U \in \mathcal{U}\) and \(f \in G\) such that \(U \cap f(U) = \emptyset\). Then \([h_1, h_2] = [[h_1, f], h_2] \in G\). Thus \(\mathcal{H}_c(M) \subset G\) as required.

**Proof of Corol. 1.3.** (\(\Rightarrow\)) It follows from a theorem of Ling [13] since \(\mathcal{H}_c(M)\) is factorizable (Prop. 5.2 below) and transitively inclusive. The latter means that for any \(U, V \in \mathcal{B}\) there is \(h \in \mathcal{H}_c(M)\) such that \(h(U) \subset V\).

(\(\Leftarrow\)) \(\mathcal{H}_c(M^o)\) is a normal subgroup of \(\mathcal{H}_c(M)\).

**Corollary 4.2.** If \(\partial M \neq \emptyset\), \(\mathcal{H}_c^o(M)\) is not simple

In fact, \(\mathcal{H}_c(M^o)\) is a proper normal subgroup of \(\mathcal{H}_c^o(M)\).

5. **Conjugation-invariant norms**

The notion of the conjugation-invariant norm is a basic tool in studies on the structure of groups. Let \(G\) be a group. A *conjugation-invariant norm* (or *norm* for short) on \(G\) is a function \(\nu : G \to [0, \infty)\) which satisfies the following conditions. For any \(g, h \in G\)

1. \(\nu(g) > 0\) if and only if \(g \neq e\);
2. \(\nu(g^{-1}) = \nu(g)\);
3. \(\nu(gh) \leq \nu(g) + \nu(h)\);
4. \(\nu(hgh^{-1}) = \nu(g)\).

Recall that a group is called *bounded* if it is bounded with respect to any bi-invariant metric. It is easily seen that \(G\) is bounded if and only if any conjugation-invariant norm on \(G\) is bounded.

Let \(g \in [G, G]\). The *commutator length* of \(g\), \(\text{cl}_G(g)\), is the least integer \(r\) such that \(g\) can be expressed by

\[
(5.1) \quad g = [h_1, \bar{h}_1] \ldots [h_r, \bar{h}_r]
\]

for some \(h_i, \bar{h}_i \in G, i = 1, \ldots, r\). Observe that the commutator length \(\text{cl}_G\) is a conjugation-invariant norm on \([G, G]\). In particular, if \(G\) is a perfect group then \(\text{cl}_G\) is a conjugation-invariant norm on \(G\). Then \(G\) is called *uniformly perfect* if \(G = [G, G]\) and the norm \(\text{cl}_G\) is bounded.

**Definition 5.1.** Let \(G\) be a subgroup of \(\mathcal{H}(M)\) and let \(\mathcal{B}\) be the family of all balls and half-balls of \(M\).
(1) $G$ is called factorizable (resp. with respect to a cover $U \subset \mathcal{B}$) if for any $g \in G$ there are $d \in \mathbb{N}$, $B_1, \ldots, B_d \in \mathcal{B}$ (resp. $B_1, \ldots, B_d \in \mathcal{U}$) and $g_1, \ldots, g_d \in G$ such that
\begin{equation}
(5.2) \quad g = g_1 \ldots g_d \quad \text{with } g_i \in G_{B_i}
\end{equation}
for all $i$. Here $G_B$ is the subgroup of $G$ of all elements that can be connected to the identity by an isotopy in $G$ compactly supported in $B$.

(2) Next, a topological group $G$ is continuously factorizable if there exist $d \in \mathbb{N}$, $B_1, \ldots, B_d \in \mathcal{B}$, and continuous mappings $S_i : G \to G_{B_i}$, $i = 1, \ldots, r$, such that for all $g \in G$
\begin{equation}
(5.3) \quad g = S_1(g) \ldots S_r(g).
\end{equation}

**Proposition 5.2.** Under the assumption of Theorem 1.1 on $M$, the groups $H_c(M)$ and $H^\partial_c(M)$ are factorizable with respect to any cover $U \subset \mathcal{B}$. The same is true for the isotopy groups $PH_c(M)$ and $PH^\partial_c(M)$.

**Proof.** If $M$ is compact, it follows from Theorem 2.3. If $M$ admits a compact exhaustion, the reasoning is similar to that in the proof of 1.1. □

For any $g \in H_c(M), g \neq id$, denote by frag$_M(g)$ the smallest $d$ such that (5.2) holds. By definition frag$_M(id) = 0$. Clearly frag$_M$ is a norm on $H_c(M)$. Likewise we define frag$_M^{iso}$ on the isotopy group $PH_c(M)$. Clearly frag$_M(f) \leq$ frag$_M^{iso}(f)$ if $f$ is an isotopy connecting $f$ with the identity.

The significance of frag$_M$ is illustrated by Theorem 1.4.

**Definition 5.3.** (1) A topological group $G$ is continuously perfect if there exist $r \in \mathbb{N}$ and continuous mappings $S_i : G \to G$, $\bar{S}_i : G \to G$, $i = 1, \ldots, r$, satisfying the equality
\begin{equation}
(5.3) \quad g = [S_1(g), \bar{S}_1(g)] \ldots [S_r(g), \bar{S}_r(g)]
\end{equation}
for all $g \in G$.

(2) Let $H$ be a subgroup of $G$. $H$ is said to be continuously perfect in $G$ if there exist $r \in \mathbb{N}$ and continuous mappings $S_i : H \to G$, $\bar{S}_i : H \to G$, $i = 1, \ldots, r$, satisfying the equality (5.3) for all $g \in H$. Then $r_{H,G}$ denotes the smallest $r$ as above.

Of course, every continuously perfect group is uniformly perfect.

**Proposition 5.4.** Suppose that the closure of $B$ is included in $U$, where $B$ is a ball (or a half-ball and $n \geq 2$) and $U$ is open in $M$. Then $\mathcal{H}_B(M)$ is continuously perfect in $\mathcal{H}_U(M)$ with $r_{\mathcal{H}_B(M),\mathcal{H}_U(M)} = 1$.

**Proof.** It suffices to observe that in the proof of Lemma 3.1 the homomorphism $S : \mathcal{H}_B(M) \to \mathcal{H}_U(M)$ is continuous, and the mapping $\bar{S}$ is a constant depending on $B$ and $U$. □
The following fact is a consequence of Prop. 5.4.

**Proposition 5.5.** If $\mathcal{H}_c(M)$ is continuously factorizable then it is also continuously perfect.

**Proof.** If $B_1, \ldots, B_d \in \mathcal{B}$ is as in Def. 5.1(2), then choose any open subsets $U_1, \ldots, U_d$ with $\overline{B}_i \subset U_i$. Then we use Prop. 5.4 to each pair $(B_i, U_i)$. \hfill \Box

However we do not know whether some homeomorphism groups $\mathcal{H}_c(M)$ are continuously factorizable. See also [21] about locally continuously perfect groups of homeomorphisms.

Burago, Ivanov and Polterovich proved in the [6] that $\mathcal{D}(M)$ is bounded (and a fortiori uniformly perfect) for many manifolds. We will need some preparatory notions and results from [6]. A subgroup $H$ of $G$ is called strongly $m$-displaceable if there is $f \in G$ such that the subgroups $H, fHf^{-1}, \ldots, f^m H f^{-m}$ pairwise commute. Then we say that $f$ $m$-displaces $H$. Fix a conjugation-invariant norm $\nu$ on $G$ and assume that $H \subset G$ is strongly $m$-displaceable. Then $e_m(H) := \inf \nu(f)$, where $f$ runs over the set of elements of $G$ that $m$-displaces $H$, is called the order $m$ displacement energy of $H$.

**Theorem 5.6.** [6] Given a group $G$ equipped with a conjugation-invariant norm $\nu$ and given $H \subset G$, if there exists $g \in G$ that $m$-displaces $H$ for every $m \geq 1$ then for all $h \in [H, H]$

1. $\text{cl}_G(h) \leq 2$; and
2. $\nu(h) \leq 14\nu(g)$.

It follows from (1) a weaker version of Lemma 3.1.

**Corollary 5.7.** Suppose that $B$ is a ball and $\overline{B} \subset U$, where $U$ is open. Then any homeomorphism supported in $B$ can be written as a product of two commutators of elements of $\mathcal{H}_U(M)$.

However, contrary to Lemma 3.1, the method based on Theorem 5.6(1) is still true in the smooth category.

6. **Boundedness of $\mathcal{H}_c(M)$ and $\mathcal{H}_c^0(M)$**

The proof of the following theorem is essentially in [6].

**Theorem 6.1.** Let $B$ be a ball or a half-ball in $M$ (in the latter case we assume $n \geq 2$). Then $\mathcal{H}_B(M)$ is bounded.

For the proof we need the following

**Proposition 6.2.** [6] Suppose that $U, V$ are open disjoint subsets of $M$ such that there is $f \in \mathcal{H}_c(M)$ satisfying $f(U \cup V) \subset V$. Then $f$ $k$-displaces $\mathcal{H}_U(M)$ for all $k \geq 1$. 

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Proof. Indeed, this follows from the relation \( f^k(U) \subset f^{k-1}(V) \setminus f^k(V) \) for all \( k \geq 1 \).

Proof of Theorem 6.1 We can choose an open subset \( V \) of \( M \) disjoint with \( B \) and a homeomorphism \( f \in \mathcal{H}_c(M) \) such that \( f(B \cup V) \subset V \). In view of Prop. 6.2 \( f \) \( k \)-displaces \( \mathcal{H}_B(M) \) for all \( k \). Therefore Theorems 1.1 and 5.7(2) imply the assertion. \( \square \)

Proof of Theorem 1.4 The part only if is trivial. Conversely, the proof is an immediate consequence of Prop. 5.2 and Theorem 6.1 except for the case \( n = 1 \) and \( \partial M \neq \emptyset \) (see the proof of 1.5). \( \square \)

Now we turn to the proof of Theorem 1.5. Let \( \mathbb{R}_+ = [0, \infty) \). We begin with the following

**Proposition 6.3.** For any decreasing sequence in \((0,1)\) of the form
\[
1 > b_1 > a_1 > b_2 > a_2 > \ldots > b_k > a_k > \ldots > 0
\]
converging to 0, there exist \( f_1, f_2 \in \mathcal{H}(\mathbb{R}_+) \) such that for \( k = 1, 2, \ldots \) one has
\[
f_1([a_{2k-1}, b_{2k-1}] \cup [a_{2k}, b_{2k}]) \subset (a_{2k}, b_{2k}),
\]
\[
f_2([a_{2k}, b_{2k}] \cup [a_{2k+1}, b_{2k+1}]) \subset (a_{2k+1}, b_{2k+1}).
\]
Moreover, if we have another sequence
\[
1 > \bar{b}_1 > \bar{a}_1 > \bar{b}_2 > \bar{a}_2 > \ldots > \bar{b}_k > \bar{a}_k > \ldots > 0,
\]
then there is an element of \( \psi \in \mathcal{H}(\mathbb{R}_+) \) with \( \psi(a_k) = \bar{a}_k \) and \( \psi(b_k) = \bar{b}_k \) for \( k = 1, 2, \ldots \).

Proof. In order to prove the first assertion it suffices to choose \( f_1 \) (and similarly \( f_2 \)) of the form \( \varphi = \bigcup_{k=1}^{\infty} \varphi_k \) with \( \varphi_k([a_{2k-1}, b_{2k-1}] \cup [a_{2k}, b_{2k}]) \subset (a_{2k}, b_{2k}) \) for all \( k \) and with \( \text{supp}(\varphi_k) \) mutually disjoint. The \( \psi \) in the second assertion is obtained by gluing together linear homeomorphisms on the consecutive intervals \([a_1,1],[b_1,a_1]\), and so on. \( \square \)

Proof of Theorems 1.4 (for \( n = 1 \) and \( \partial M \neq \emptyset \)) and 1.5. Let \( P = \partial \times [0,1] \) be a collar neighborhood embedded in \( M \) such that \( \partial \) identifies with \( \partial \times \{0\} \).

Since \( \partial = \partial M \) is compact, in view of Theorem 2.5 the restriction mapping
\[
\mathcal{H}_c(M) \ni f \mapsto f|_{\partial} \in \mathcal{H}(\partial M)
\]
is an epimorphism. It follows from Lemma 1.10 in [3] that \( \mathcal{H}_c(\partial M) \) is bounded. Thus it suffices to show the second assertion of Theorem 1.5. Let \( g \in \mathcal{H}_c(\partial \times \mathbb{R}_+) \). Arguing as in the proof of Theorem 1.2, there is a sequence, converging to 0, of the form
\[
1 > b_1 > \bar{b}_1 > \bar{a}_1 > a_1 > b_2 > \ldots > b_k > \bar{b}_k > \bar{a}_k > a_k > \ldots > 0
\]
and homeomorphisms \( h_1, h_2 \in \mathcal{H}_c(\partial \times \mathbb{R}_+) \) such that

\[
h_1 = g \quad \text{on} \quad \bigcup_{k=1}^{\infty} \partial \times [\bar{a}_{2k-1}, \bar{b}_{2k-1}], \quad \text{supp}(h_1) \subset U_1 := \bigcup_{k=1}^{\infty} \partial \times (a_{2k-1}, b_{2k-1}),
\]

\[
h_2 = g \quad \text{on} \quad \bigcup_{k=1}^{\infty} \partial \times [\bar{a}_{2k}, \bar{b}_{2k}], \quad \text{supp}(h_2) \subset U_2 := \bigcup_{k=1}^{\infty} \partial \times (a_{2k}, b_{2k}).
\]

Continuing the reasoning from the proof of Theorem 1.2 for \( h' = h^{-1} g \), it can be checked that \( g \) admits a decomposition of the form

\[
g = h_1 h_2 h_3 h_4,
\]

where

\[
h_3 = g \quad \text{on} \quad \bigcup_{k=1}^{\infty} \partial \times [b_k, a_{2k-1}], \quad \text{supp}(h_3) \subset U_3 := \bigcup_{k=1}^{\infty} \partial \times (b_k, a_{2k-1}),
\]

\[
h_4 = g \quad \text{on} \quad \bigcup_{k=0}^{\infty} \partial \times [b_{2k+1}, a_{2k}], \quad \text{supp}(h_4) \subset U_4 := \bigcup_{n=0}^{\infty} \partial \times (b_{2k+1}, a_{2k}),
\]

and where \( a_0, \bar{a}_0 \) satisfy \( 1 > \bar{a}_0 > a_0 > b_1 \). Furthermore, \( h_j \) satisfy conditions analogous to (3.4) for \( j = 1, 2, 3, 4 \).

In view of Prop. 6.3 there exist \( \bar{f}_j \in \mathcal{H}^0_c(\partial \times \mathbb{R}_+) \) of the form \( \bar{f}_j = \text{id} \times f_j \) such that \( \mathcal{H}_{U_j}(M) \) is \( m \)-displaceable by \( \bar{f}_j \) for \( j = 1, 2, 3, 4 \) and for all \( m \geq 1 \).

Let \( \nu \) be a conjugation-invariant norm on \( \mathcal{H}^0_c(M) \). In view of Theorem 5.6(2) and the invariance of \( \nu \) we have

\[\nu(g) \leq \nu(h_1) + \cdots + \nu(h_4) \leq 14(\nu(\bar{f}_1) + \cdots + \nu(\bar{f}_4)).\]

Observe that the sets \( U_1, \ldots, U_4 \) depend on \( g \). Nevertheless, in view of the second assertion of Prop. 6.3 and the invariance of \( \nu \), the norms \( \nu(\bar{f}_j) \) are independent of \( g \). It follows that \( \nu(g) \) is bounded, as required. \( \square \)

**Definition 6.4.** A connected open manifold \( M \) is called **portable (in the wider sense)** if there are disjoint open subsets \( U, V \) of \( M \) such that there is \( f \in \mathcal{H}_c(M) \) with \( f(U \cup V) \) contained in \( V \). Furthermore, for every compact subset \( K \subset M \) there is \( h \in \mathcal{H}_c(M) \) satisfying \( h(K) \subset U \).

**Remark 6.5.** The notion of a portable manifold has been introduced in [6] for smooth open manifolds. The definition there is specific for smooth category and a bit stronger than Def. 6.4 (a definition similar to 6.4 is also mentioned in [6]).

The class of portable manifolds comprises the euclidean spaces \( \mathbb{R}^n \), the manifolds of the form \( M \times \mathbb{R}^n \), or the manifolds admitting an exhausting Morse function with finite numbers of critical points such that all their indices
are less than $\frac{1}{2} \dim M$. In particular, every three-dimensional handlebody is a portable manifold.

**Theorem 6.6.** If $M$ is portable that $\mathcal{H}_c(M)$ is bounded.

The proof is a consequence of Prop. 5.2, and is completely analogous to that for diffeomorphisms (Theorem 1.7 in [6]).

**Corollary 6.7.** If $M^o$ is portable then $\mathcal{H}_c^o(M)$ is bounded.

The proof follows from Theorems 1.5 and 6.6. In contrast, for diffeomorphism groups we have the following

**Proposition 6.8.** Let $M$ be a smooth manifold with boundary and let $D^\partial(M)$ be the subgroup of all $f \in D(M)$ such that there exists a compactly supported isotopy $f_t$ with $f_0 = \text{id}$ and $f_1 = f$ satisfying $f_t|_{\partial M} = \text{id}$ for all $t$. Then $D^\partial(M)$ is an unbounded group.

*Proof.* Choose a chart at $p \in \partial M$. Then there is the epimorphism $\mathcal{D}^\partial(M) \ni f \mapsto \text{Jac}_p(f) \in \mathbb{R}_+$, where $\text{Jac}_p(f)$ is the Jacobian of $f$ at $p$ in this chart. In view of Prop. 1.3 in [6], an abelian group is bounded if and only if it is finite. Therefore $\mathbb{R}_+$ is unbounded. Now Lemma 1.10 in [6] implies that $D^\partial(M)$ is unbounded. $\square$

**Example 6.9.** Let $\bar{B}^{n+1} \subset \mathbb{R}^{n+1}$ be the closed ball and $S^n = \partial \bar{B}^{n+1}$. Then $\mathcal{H}_c(S^n)$ is bounded by an argument similar to that of Theorem 1.11(ii) in [6] stating that $\mathcal{D}(S^n)$ is bounded. Next, $\mathcal{H}_c(B^{n+1})$ is bounded in view of Theorem 6.6, where $B^{n+1}$ is the interior of $\bar{B}^{n+1}$. Hence, due to Theorem 1.5 the group $\mathcal{H}_c^\partial(B^{n+1})$ are bounded.

7. **The universal covering groups of $\mathcal{H}_c(M)$ and $\mathcal{H}_c^\partial(M)$**

Let $G$ be a topological group. The symbol $\tilde{G}$ will stand for the universal covering group of $G$, that is $\tilde{G} = \mathcal{P}G/\sim$, where $\sim$ denotes the relation of the homotopy relatively endpoints.

We introduce the following two operations on the space of paths $\mathcal{P}G$. Let $\mathcal{P}^*G = \{ \gamma \in \mathcal{P}G : \gamma(t) = e \ \text{for} \ t \in [0, \frac{1}{2}] \}$. For all $\gamma \in \mathcal{P}G$ we define $\gamma^*$ as follows:

$$\gamma^*(t) = \begin{cases} e & \text{for} \ t \in [0, \frac{1}{2}] \\ \gamma(2t - 1) & \text{for} \ t \in [\frac{1}{2}, 1] \end{cases}$$

Then $\gamma^* \in \mathcal{P}^*G$ and the subgroup $\mathcal{P}^*G$ is the image of $\mathcal{P}G$ by the mapping $\ast : \gamma \mapsto \gamma^*$. The elements of $\mathcal{P}^*G$ are said to be *special* paths in $G$. It is important that the group of special paths is preserved by conjugations,
i.e. for each \( g \in \mathcal{P}G \) we have \( \text{conj}_g(\mathcal{P}^*G) \subset \mathcal{P}^*G \) for every \( g \in \mathcal{P}G \), where \( \text{conj}_g(h) = ghg^{-1}, h \in \mathcal{P}G \).

Next, let \( \mathcal{P}^\square G = \{ \gamma \in \mathcal{P}G : \gamma(t) = \gamma(1) \text{ for } t \in [\frac{1}{2}, 1] \} \). For all \( \gamma \in \mathcal{P}G \) we define \( \gamma^\square \) by

\[
\gamma^\square(t) = \begin{cases} 
\gamma(2t) & \text{for } t \in [0, \frac{1}{2}] \\
\gamma(1) & \text{for } t \in [\frac{1}{2}, 1]
\end{cases}
\]

As before \( \gamma^\square \in \mathcal{P}^\square G \) and the subgroup \( \mathcal{P}^\square G \) coincides with the image of \( \mathcal{P}G \) by the mapping \( \square : \gamma \mapsto \gamma^\square \).

**Lemma 7.1.** For any \( \gamma \in \mathcal{P}G \) we have \( \gamma \sim \gamma^* \) and \( \gamma \sim \gamma^\square \).

**Proof.** We have to find a homotopy \( \Gamma \) rel. endpoints between \( \gamma \) and \( \gamma^* \). For all \( s \in I \) define \( \Gamma^* \) as follows:

\[
\Gamma^*(t,s) = \begin{cases} 
e & \text{for } t \in [0, \frac{s}{2}] \\
\gamma(\frac{2t-s}{2}) & \text{for } t \in (\frac{s}{2}, 1]
\end{cases}
\]

It is easy to check that such \( \Gamma^* \) fulfils all the requirements.

For the second claim define \( \Gamma^\square \) as follows: for any \( s \in I \)

\[
\Gamma^\square(t,s) = \begin{cases} 
\gamma(\frac{s}{2-s}) & \text{for } t \in [0, \frac{2-s}{2-s}] \\
\gamma(1) & \text{for } t \in (\frac{2-s}{2-s}, 1]
\end{cases}
\]

\( \square \)

Given a group \( G \) recall the definition of homology groups of \( G \). The usual construction of homology groups proceeds by defining a *standard chain complex* \( C(G) \). Its homology is the homology of \( G \).

The complex \( C(G) \) is defined as follows. For any integer \( r \geq 0 \) denote

\[
C_r(G) = \text{free abelian group on the set of all } r\text{-tuples } (g_1, \ldots, g_r),
\]

where \( g_i \in G \). Next introduce the *boundary operator* \( \partial : C_r(G) \rightarrow C_{r-1}(G) \) by the formula

\[
\partial(g_1, \ldots, g_r) = (g_1^{-1}g_2, \ldots, g_{r-1}^{-1}g_r) + \sum_{i=1}^{r} (-1)^i (g_1, \ldots, \hat{g}_i, \ldots, g_r).
\]

Then \( \partial^2 = 0 \). Let \( Z_r(G) = \{ c \in C_r(G) : \partial(c) = 0 \} \) and \( B_r(G) = \{ c \in C_r(G) : (\exists b \in C_{r+1}(G)), \partial(b) = c \} \). The symbol \( H_r(G) = Z_r(G)/B_r(G) \) will stand for the \( r \)-th homology group of the above chain complex. It is well known that

\[
H_1(G) = G/[G,G],
\]

that is, the first homology group is equal to the abelianization of \( G \). For any \( g \in G \) the conjugation mapping \( \text{conj}_g : G \rightarrow G \) induces an identity so \( (\text{conj}_g)_*(h) = h \) for any \( h \in H_r(G) \), c.f. [5].
For any \( g \in \mathcal{P}G \) denote \( \tilde{g} := [g]_{\sim} \in \tilde{G} \) and for any \( c \in C_r(\mathcal{P}G) \) of the form \( c = \sum k_j(g_{1j}, \ldots, g_{rj}) \), where \( k_j \in \mathbb{Z} \), denote by \( \tilde{c} := \sum k_j(\tilde{g}_{1j}, \ldots, \tilde{g}_{rj}) \) the corresponding element of \( C_r(\tilde{G}) \). Then it is easily checked that

\[
(7.1) \quad \tilde{\partial} \tilde{c} = [\partial c]_{\sim} = \tilde{\partial} c,
\]

where \( \tilde{\partial} \) is the differential in the chain complex \( C_r(\tilde{G}) \). That is, (7.1) can serve as a definition of \( \tilde{\partial} \).

In order to compute \( H_r(\mathcal{H}_c(\mathbb{R}^n)_{\sim}) \) we fix notation. Let \( c = \sum k_j(g_{1j}, \ldots, g_{rj}) \), where \( k_j \in \mathbb{Z} \), be a chain from \( C_r(\mathcal{P}H_c(\mathbb{R}^n)) \). We define the support of \( c \) by

\[
\text{supp}(c) := \bigcup_{i,j} \text{supp}(g_{ij}),
\]

where \( \text{supp}(g) := \bigcup_{i \in I} \text{supp}(g_i) \), for \( g : I \ni t \mapsto g_t \in \mathcal{H}_c(\mathbb{R}^n) \). Thus \( \text{supp}(c) \subset U \) iff \( \text{supp}(g_{ij}) \subset U \) for each \( i, j \), or \( (g_{ij}), \in \mathcal{H}_U(\mathbb{R}^n) \) for each \( i, j, t \).

**Theorem 7.2.** Let \( G \) be either \( \mathcal{H}_c(\mathbb{R}^n) \), or \( \mathcal{H}_c(\mathbb{R}^n_{+}) \) (in the cases \( \mathbb{R}^n_{+} \) we assume \( n \geq 2 \)). For \( r \geq 1 \) one has \( H_r(\tilde{G}) = 0 \). In particular, \( \tilde{G} \) is a perfect group.

Let \( B \subset \mathbb{R}^n \) be a ball or \( B \subset \mathbb{R}^n_{+} \) be a half-ball. By \( i : \mathcal{H}_B(\mathbb{R}^n)_{\sim} \rightarrow \mathcal{H}_c(\mathbb{R}^n)_{\sim} \) we denote the inclusion, and \( i_* : H_r(\mathcal{H}_B(\mathbb{R}^n)_{\sim}) \rightarrow H_r(\mathcal{H}_c(\mathbb{R}^n)_{\sim}) \) is the corresponding map on the homology level.

**Lemma 7.3.** \( i_* \) is an isomorphism.

**Proof.** First we show that \( i_* \) is surjective. Let \( h \in H_r(\mathcal{H}_c(\mathbb{R}^n)_{\sim}) \) and let \( h = \tilde{c} \), where \( c = \sum k_j(g_{1j}, \ldots, g_{rj}) \) be a cycle representing \( h \). According to Lemma 7.1 we can assume that \( g_{ij} \in \mathcal{P}^* \mathcal{H}_c(\mathbb{R}^n) \). Then \( C = \text{supp}(c) \) is compact. We can find \( \tilde{\varphi} \in \mathcal{P} \mathcal{H}_c(\mathbb{R}^n) \) such that \( \varphi_t(C) \subset B \). Define \( \varphi := \tilde{\varphi} \in \mathcal{P} \mathcal{H}_c(\mathbb{R}^n) \). Since any conjugation induces the identity on homology, \( (\text{conj}_\varphi)_*(h) = h \). But, in view of (7.1), \( (\text{conj}_\varphi)_*(h) \) is represented by the cycle \( \text{conj}_\varphi(c) \). It is easily seen that for \( 0 \leq t \leq \frac{1}{2} \) \( \text{conj}_\varphi(c)_t = \text{id} \), and for \( \frac{1}{2} \leq t \leq 1 \) \( \text{conj}_\varphi(c)_t \) is supported in \( B \). Hence \( \text{conj}_\varphi(c) \) is a the cycle representing homology \( h' \) of the group \( \mathcal{H}_B(\mathbb{R}^n)_{\sim} \) such that \( i_*h' = h \).

In order to show injectivity let \( h \in \ker(i_*) \). As above let \( c \) be a cycle from \( \mathcal{P} \mathcal{H}_c(\mathbb{R}^n) \) representing \( h \). Since \( i_*(h) = 0 \), there is a cycle \( c' \in C_{r+1}(\mathcal{P} \mathcal{H}_c(\mathbb{R}^n)) \) such that \( \partial c' = c \). In view of Lemma 7.1 we may assume that \( c' \in C_{r+1}(\mathcal{P}^* \mathcal{H}_c(\mathbb{R}^n)) \). We choose \( \varphi \in \mathcal{P} \mathcal{H}_c(\mathbb{R}^n) \) such that \( \varphi_1(\text{supp}(c')) \subset B \). We then have \( \partial(\text{conj}_\varphi(c')) = \text{conj}_\varphi(\partial c') = \text{conj}_\varphi(c) = c \). This means that \( c \) is the boundary of an element from \( C_{r+1}(\mathcal{P} \mathcal{H}_B(\mathbb{R}^n)) \). Consequently, \( h = 0 \). \( \square \)

**Proof of Theorem 7.2.** (See also [14].) By Lemma 7.3 it suffices to consider \( \mathcal{H}_B(\mathbb{R}^n) \) (resp. \( \mathcal{H}_B(\mathbb{R}^n_{+}) \)), where \( B \subset \mathbb{R}^n \) is a ball (resp. \( B \subset \mathbb{R}^n_{+} \) is a half-ball). As in the proof of Lemma 3.1 we define \( B_0 = B \) and we choose a locally finite, pairwise disjoint sequence of balls (resp. half-balls) \( (B_k)_{k=0}^\infty \) converging
Proof of Theorem 1.7. It follows from Prop. 5.2 and Theorem 7.4.

□

Now we proceed by the induction on \( n \). Let \( \{c\} \in H_\ast(H_B(\mathbb{R}^n)\sim) = 0 \) for \( 1 \leq s \leq r-1 \). Then by the Kunneth formula we get

\[
H_\ast(H_B(\mathbb{R}^n)\sim \times H_B(\mathbb{R}^n)\sim) = H_\ast(H_B(\mathbb{R}^n)\sim) \oplus H_\ast(H_B(\mathbb{R}^n)\sim).
\]

Now choose arbitrarily \( \{c\} \in H_r(H_B(\mathbb{R}^n)\sim) \). Then \( \Delta_\ast\{c\} = \{c\} \oplus \{c\} \) by (7.3). It follows by (7.2) and (7.3) that

\[
\psi_0\ast\{c\} = \eta_\ast\Delta_\ast\{c\} = \iota_\ast\{c\} + \psi_1\ast\{c\} = \iota_\ast\{c\} + \psi_0\ast\{c\}.
\]

Thus \( \iota_\ast\{c\} = 0 \), and \( \{c\} = 0 \) by Lemma 7.3, as required. In the case of \( H_\ast(\mathbb{R}^n_+) \) the proof is the same. □

Proof of Theorem 1.6. The second claim coincides with Theorem 7.2. The first claim is a consequence of Prop. 5.2 for \( \mathcal{P}H_\ast(M) \), and of the second claim. □

**Theorem 7.4.** Let \( B \) be a ball or a half-ball in \( M \) (in the latter case we assume \( n \geq 2 \)). Then \( H_B(M)\sim \) is bounded.

**Proof.** Let \( f \in H_\ast(M) \) be as in Prop. 6.2. We choose an isotopy \( f_t \in \mathcal{P}H_\ast(M) \) joining \( f \) with the identity. Next we observe that, due to Theorem 1.6 and Lemma 7.1 any class from \( H_B(M)\sim \) can be represented as a product of commutators of elements from \( \mathcal{P}H_\ast(M) \). The proof is now analogous to that of Theorem 6.1. □

Proof of Theorem 1.7. It follows from Prop. 5.2 and Theorem 7.4. □

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