ON INTEGRAL CLASS FIELD THEORY FOR VARIETIES
OVER $p$-ADIC FIELDS

THOMAS H. GEISSER AND BAPTISTE MORIN

Abstract. Let $K$ be a finite extension of the $p$-adic numbers $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, $X$ a regular scheme, proper, flat, and geometrically irreducible over $\mathcal{O}_K$ of dimension $d$, and $X_K$ its generic fiber. We show, under some assumptions on $X_K$, that there is a reciprocity isomorphism of locally compact groups $H^{2d-1}_{\text{et}}(X_K, \mathbb{Z}(d)) \simeq \pi_1^{ab}(X_K)_W$ from the cohomology theory defined in [9] to an integral model $\pi_1^{ab}(X_K)_W$ of the abelianized geometric fundamental groups $\pi_1^{ab}(X_K)_\text{geo}$. After removing the contribution from the base field, the map becomes an isomorphism of finitely generated abelian groups. The key ingredient is the duality result in [9].

1. Introduction

Let $K$ be a finite extension of the $p$-adic numbers $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, $X$ a regular scheme, proper, flat, and geometrically irreducible over $\mathcal{O}_K$ of dimension $d$, and $X_K$ its generic fiber. Classically, the abelianized fundamental group $\pi_1^{ab}(X_K)$ was studied using the reciprocity map

$$\rho : SK_1(X_K) \simeq H^{2d-1}_{\text{et}}(X_K, \mathbb{Z}(d)) \to \pi_1^{ab}(X_K).$$

Removing the contribution from the base field we let $SK_1(X_K)^0$ and $\pi_1^{ab}(X_K)^{\text{geo}}$ be the kernels of the norm map $SK_1(X_K) \to K^\times$ and the natural surjection $\pi_1^{ab}(X_K) \to G_K$ to the Galois group of $K$, respectively. If $X_K$ is a curve, then S. Saito [18] showed that $\rho$ has a divisible kernel, and a cokernel isomorphic to $\hat{\mathbb{Z}}^r$, and the resulting map

$$SK_1(X_K)^0 \to \pi_1^{ab}(X_K)^{\text{geo}}$$

has divisible kernel, finite image, and a cokernel isomorphic to $\mathbb{Z}/r\mathbb{Z}$. More generally, Yoshida [25] showed that in arbitrary dimension $\pi_1^{ab}(X_K)^{\text{geo}}$ has finite torsion, and its torsion free quotient is isomorphic to $\mathbb{Z}/r\mathbb{Z}$, where $r$ is the $K$-rank of the special fiber of the Néron model of the Albanese variety of $X_K$. If $X_K$ is a surface, then Sato [20] gave an example showing that the kernel of $\rho$ need not be divisible. On the other hand, Szamuely [23] showed that the kernel of $\rho$ is $l$-divisible if $H^2_{\text{et}}(X_K, \mathbb{Q}_l)$ vanishes, and Jannsen-Saito [12] showed that it is a direct sum of a finite group and a group divisible by all primes different from $p$. Forré [4] generalized this statement to arbitrary dimension, and Yamazaki

2020 Mathematics Subject Classification. Primary: 14G45; Secondary: 11G25, 14G20, 14F42.

Key words and phrases. Geometric class field theory; Local fields.

The first named author is supported by JSPS Grant-in-Aid (C) 18K03258, and the second named author by grant ANR-15-CE40-0002.
showed that the kernel of $\rho$ is divisible for a product of curves all but one of which have potentially good reduction.

In this paper, we study the abelianized fundamental group $\pi_1^{ab}(X_K)$ using the cohomology theory $R\Gamma_{ar}(X_K, \mathbb{Z}(d))$ consisting of locally compact groups defined in [9]. We recall the construction of the cohomology theory, and define an integral integral model $\pi_1^{ab}(X_K)_{W}$ of $\pi_1^{ab}(X_K)$, similar to the construction in [10]. Our main result is that the resulting reciprocity map is an isomorphism under some mild assumptions. The main tool is the duality theorem of [9, Cor. 5.13], which is inspired by the Pontryagin duality [3, Thm. 4.9] of the Weil-Arakelov cohomology defined by Flach and the second named author in [3].

The advantage of $\Gamma_{ar}$ is that it is expected to satisfy a duality with integral coefficients, whereas étale motivic cohomology with integral coefficients does not satisfy duality [8].

More precisely, let $X$ be a regular, connected scheme of Krull dimension $d$, which is proper and flat over the ring of integers $O_K$ of $K$. The geometric fibers of $X/\mathcal{O}_K$ are connected, and we assume that they are reduced. We denote by $X_K$ and $\mathcal{X}_s$ the generic and special fiber of $\mathcal{X}$, respectively. Following [6], we denote by $\mathcal{Z}^c(0)$ Bloch’s cycle complex on $X_s$, and by $H^i_W(\mathcal{X}_s, \mathbb{Z})$ the $i$th homology group of the complex $R\Gamma_{W}(\mathcal{X}_s, \mathcal{Z}^c(0))[1]$. Let $\Pi_1^{ab}(X_s)$ the abelian enlarged fundamental group of the closed fiber, a constant pro-group, let $\Pi_1^{ab}(X_s)_W := \Pi_1^{ab}(X_s) \times_{G_{\mathcal{s}}(s)} W_{\mathcal{s}}$, and define

$$\pi_1^{ab}(X_K)_W := \pi_1^{ab}(X_K) \times_{\pi_1^{ab}(X_s)} \Pi_1^{ab}(X_s)_W.$$  

**Theorem 1.1** (Thm. [3.8]). Assume that $X$ has good or strictly semi-stable reduction and that $R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}^c(0))$ is a perfect complex of abelian groups. Then there exists a unique functorial isomorphism of locally compact groups

$$\text{rec} : H^{2d-1}_{ar}(X_K, \mathbb{Z}(d)) \sim \pi_1^{ab}(X_K)_W$$

inducing the classical isomorphism

$$H^{2d-1}_{et}(X_K, \mathbb{Z}(d)) \sim \pi_1^{ab}(X_K)$$

after profinite completion.

For $X = \mathcal{O}_K$, the map $\text{rec}$ is the isomorphism

$$K^X = H^1_{ar}(X_K, \mathbb{Z}(1)) \sim \pi_1^{ab}(X_K)_W = W^{ab}_K$$

of Weil’s local class field theory, where $W_K$ is the Weil group of $K$. The Theorem can be refined as follows. Setting $\pi_1^{ab}(X_K)^{geo}$ and $H^{2d-1}_{ar}(X_K, \mathbb{Z}(d))^0$ to be the kernel of the map $\pi_1^{ab}(X_K)_W \to W^{ab}_K$ and $H^{2d-1}_{ar}(X_K, \mathbb{Z}(d))^0 \to K^X$, respectively, we obtain:

**Corollary 1.2** (Cor. [3.14]). The reciprocity map (2) induces an isomorphism of finitely generated abelian groups

$$\text{rec} : H^{2d-1}_{ar}(X_K, \mathbb{Z}(d))^0 \sim \pi_1^{ab}(X_K)^{geo}$$

of rank

$$r = \text{rank}_\mathbb{Z}(H^{2d-1}_{ar}(X_K, \mathbb{Z}(d))^0) = \text{rank}_\mathbb{Z}(H^1_W(X_s, \mathbb{Z})) - 1.$$
In the last three sections of this paper, we compare our reciprocity map with the classical reciprocity map \(1\). For \(X\) of arbitrary dimension, we obtain some results on the kernel of \(\rho\) in Sections 4 and 5. In Section 6 we specialize these results to obtain the following:

**Theorem 1.3** (Thm. 6.2). \(\) If \(X_K\) is a curve with good or strictly semi-stable reduction, then \(1\) factors as follows:

\[
SK_1(X_K)^0 \to H^3_{et}(X_K, \mathbb{Z}(2))^0 \xrightarrow{\sim} \pi_1^{ab}(X_K)^{geo} \to \pi_1^{ab}(X_K)^{geo}.
\]

The left map has kernel the maximal divisible subgroup of \(SK_1(X_K)^0\) and its image is the torsion subgroup of the finitely generated abelian group \(H^3_{et}(X_K, \mathbb{Z}(2))^0\). The right map is the inclusion of the finitely generated group \(\pi_1^{ab}(X_K)^{geo}\) into its profinite completion \(\pi_1^{ab}(X_K)^{geo}\).

2. Review of arithmetic cohomology with Tate twists 0 and \(d\).

Let \(p\) be a prime number, let \(K/\mathbb{Q}_p\) be a finite extension, and let \(\bar{K}/K\) be a maximal unramified extension of \(K\) inside \(\bar{K}\), by \(O_K\) and \(O_{K^un}\) the rings of integers in \(\bar{K}\) and \(K^un\) respectively, and by \(s\) and \(\bar{s}\) the closed points of \(\text{Spec}(O_K)\) and \(\text{Spec}(O_{K^un})\) respectively. Let \(X/\mathcal{O}_{\bar{K}}\) be a \(d\)-dimensional, connected, regular scheme, proper and flat over \(\mathcal{O}_{\bar{K}}\) and by \(\bar{X}_s\) (resp. \(\bar{X}_K\)) be its special (resp. generic) fiber. We consider the following diagram:

\[
\begin{array}{ccc}
X_{\bar{K}^un} & \xrightarrow{j} & X_{\bar{O}_{\bar{K}^un}} \\
\downarrow & & \downarrow \\
X_{\bar{K}} & \xrightarrow{j} & X_{\bar{s}} \\
\end{array}
\]

We define by \(\mathbb{Z}(d)\) Bloch’s cycle complex, which we consider as a complex of étale sheaves. We denote by \(G_{\kappa(s)} \simeq \hat{\mathbb{Z}}\) and by \(W_{\kappa(s)} \simeq \mathbb{Z}\) the Galois group and the Weil group of the finite field \(\kappa(s)\). We define

\[
\begin{align*}
R\Gamma_W(X_{\bar{K}}, \mathbb{Z}(d)) & := R\Gamma(W_{\kappa(s)}, R\Gamma_{et}(X_{\bar{O}_{\bar{K}^un}}, \mathbb{Z}(d))), \\
R\Gamma_W(X, \mathbb{Z}(d)) & := R\Gamma(W_{\kappa(s)}, R\Gamma_{et}(X_{\bar{O}_{\bar{K}^un}}, \mathbb{Z}(d))), \\
R\Gamma_W(X_s, R\tilde{i}^d\mathbb{Z}(d)) & := R\Gamma(W_{\kappa(s)}, R\Gamma_{et}(X_s, R\tilde{i}^d\mathbb{Z}(d))).
\end{align*}
\]

There is a fiber sequence

\((3)\) \(R\Gamma_{et}(X_s, R\tilde{i}^d\mathbb{Z}(d)) \to R\Gamma_{et}(X_{\bar{O}_{\bar{K}^un}}, \mathbb{Z}(d)) \to R\Gamma_{et}(X_{\bar{K}^un}, \mathbb{Z}(d)).\)

Applying the natural transformation \(R\Gamma(G_{\kappa(s)}, -) \to R\Gamma(W_{\kappa(s)}, -)\) to \((3)\), we obtain the morphism of fiber sequences in \(D(\text{Ab})\)

\[
\begin{array}{ccccccc}
R\Gamma_{et}(X_s, R\tilde{i}^d\mathbb{Z}(d)) & \rightarrow & R\Gamma_{et}(X, \mathbb{Z}(d)) & \rightarrow & R\Gamma_{et}(X_{\bar{K}}, \mathbb{Z}(d)) \\
\downarrow & & \downarrow & & \downarrow \\
R\Gamma_W(X_s, R\tilde{i}^d\mathbb{Z}(d)) & \rightarrow & R\Gamma_W(X, \mathbb{Z}(d)) & \rightarrow & R\Gamma_W(X_{\bar{K}}, \mathbb{Z}(d))
\end{array}
\]
The complex $R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))$ is not known to be bounded below. However the complex

$$R\Gamma_W(\mathcal{X}, \mathbb{Q}/\mathbb{Z}(d)) \simeq R\Gamma_{et}(\mathcal{X}, \mathbb{Q}/\mathbb{Z}(d))$$

is bounded, as can be seen by duality, hence the cohomology groups $H^i_W(\mathcal{X}, \mathbb{Z}(d))$ are $\mathbb{Q}$-vector spaces for $i << 0$. In particular, for $a < b << 0$ the map

$$\tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to \tau^{>b} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))$$

induces an equivalence

$$(\tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))) \hat{\otimes} \mathbb{Z} \sim (\tau^{>b} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))) \hat{\otimes} \mathbb{Z}.$$ 

The same observation applies to $R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d))$. Here the functor

$$(-) \hat{\otimes} \mathbb{Z} : D^b(\text{Ab}) \to D^b(\text{LCA})$$

is defined in [9, Section 2.3], where $D^b(\text{Ab})$ (resp. $D^b(\text{LCA})$) denotes the derived $\infty$-category of bounded complexes of abelian groups (respectively of bounded complexes of locally compact abelian groups), see [11] and [9, Section 2.2].

**Definition 2.1.** Let $a \ll 0$. We define

$$R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d)) := (\tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))) \hat{\otimes} \mathbb{Z}.$$ 

If $R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0))$ is cohomologically bounded, we define $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d))$ as the cofiber of the composite map

$$R\Gamma_W(\mathcal{X}_s, R\Gamma^1(\mathbb{Z}(d))) \to \tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d))$$

in $D^b(\text{LCA})$. Similarly, we define

$$R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d)) := (\tau^{>a} R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d))) \hat{\otimes} \mathbb{Z}.$$ 

and $R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}(d))$ as the cofiber of the composite map

$$R\Gamma_{et}(\mathcal{X}_s, R\Gamma^1(\mathbb{Z}(d))) \to \tau^{>a} R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d)) \to R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d))$$

in $D^b(\text{LCA})$.

**Remark 2.2.** On the connected, $d$-dimensional and regular scheme $\mathcal{X}$, we have $\mathbb{Z}(d)^\mathcal{X} = \mathbb{Z}^c(0)^{\mathcal{X}}[-2d]$ by definition. By [7, Cor. 7.2], we have $R\Gamma^1(\mathbb{Z}^c(0)^{\mathcal{X}})^\mathcal{X}_s = \mathbb{Z}^c(0)^{\mathcal{X}}$, hence $R\Gamma^1(\mathbb{Z}(d))^{\mathcal{X}} = \mathbb{Z}^c(0)^{\mathcal{X}}[-2d]$. Moreover, the following assertions are equivalent:

- $R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0))$ is cohomologically bounded;
- $R\Gamma_W(\mathcal{X}_s, \mathbb{Q}^c(0))$ is cohomologically bounded;
- $R\Gamma_{et}(\mathcal{X}_s, \mathbb{Z}^c(0))$ is cohomologically bounded;
- $R\Gamma_{et}(\mathcal{X}_s, \mathbb{Q}^c(0))$ is cohomologically bounded.

Indeed, both $H^i_W(\mathcal{X}_s, \mathbb{Z}^c(0))$ and $H^i_{et}(\mathcal{X}_s, \mathbb{Z}^c(0))$ are $\mathbb{Q}$-vector spaces for $i << 0$. Moreover all these complexes are bounded above. Hence the result follows from the direct sum decomposition

$$R\Gamma_W(\mathcal{X}_s, \mathbb{Q}^c(0)) \simeq R\Gamma_{et}(\mathcal{X}_s, \mathbb{Q}^c(0)) \oplus R\Gamma_{et}(\mathcal{X}_s, \mathbb{Q}^c(0))[-1].$$
If $R\Gamma_W(\mathcal{A}, \mathbb{Z}^c(0))$ is cohomologically bounded, then we have a commutative diagram in $D^b(\text{LCA})$:

\[
\begin{array}{c}
\begin{array}{cccc}
R\Gamma_{et}(\mathcal{A}_s, R^i\mathbb{Z}(d)) & R\Gamma_{et}(\mathcal{A}, \mathbb{Z}(d)) & R\Gamma_{et}(\mathcal{A}_K, \mathbb{Z}(d)) \\
\downarrow & \downarrow & \downarrow \\
R\Gamma_{et}(\mathcal{A}_s, R^i\mathbb{Z}(d)) & R\Gamma_{et}(\mathcal{A}, \mathbb{Z}(d)) & R\Gamma_{et}(\mathcal{A}_K, \mathbb{Z}(d)) \\
\downarrow & \downarrow & \downarrow \\
R\Gamma_W(\mathcal{A}_s, R^i\mathbb{Z}(d)) & R\Gamma_{ar}(\mathcal{A}, \mathbb{Z}(d)) & R\Gamma_{ar}(\mathcal{A}_K, \mathbb{Z}(d))
\end{array}
\end{array}
\]

where the rows are fiber sequences, and the top row consists of discrete objects in $D^b(\text{LCA})$ in the sense of [9, Def. 2.7].

**Proposition 2.3.** The map

\[R\Gamma_{et}(\mathcal{A}, \mathbb{Z}(d)) \to R\Gamma_{ar}(\mathcal{A}, \mathbb{Z}(d))\]

is an equivalence. If $R\Gamma_W(\mathcal{A}_s, \mathbb{Z}^c(0))$ is cohomologically bounded, then we have a canonical cofiber sequence

\[(4) \quad R\Gamma_{et}(\mathcal{A}_s, R^i\mathbb{Q}(d))[-1] \to R\Gamma_{et}(\mathcal{A}_K, \mathbb{Z}(d)) \to R\Gamma_{ar}(\mathcal{A}_K, \mathbb{Z}(d)).\]

**Proof.** In view of the equivalence

\[(\tau^{>a} R\Gamma_{et}(\mathcal{A}, \mathbb{Z}(d))) \otimes^L \mathbb{Z}/m\mathbb{Z} \xrightarrow{\sim} (\tau^{>a} R\Gamma_W(\mathcal{A}, \mathbb{Z}(d))) \otimes^L \mathbb{Z}/m\mathbb{Z}\]

for any $m$, the first assertion follows from [9, Remark 2.12].

If $R\Gamma_W(\mathcal{A}_s, \mathbb{Z}^c(0))$ is cohomologically bounded, then we have a commutative square

\[
\begin{array}{c}
\begin{array}{cccc}
R\Gamma_{et}(\mathcal{A}_s, R^i\mathbb{Z}(d)) & R\Gamma_W(\mathcal{A}_s, R^i\mathbb{Z}(d)) \\
\downarrow & \downarrow \\
R\Gamma_{et}(\mathcal{A}, \mathbb{Z}(d)) & \sim R\Gamma_{ar}(\mathcal{A}, \mathbb{Z}(d))
\end{array}
\end{array}
\]

in $D^b(\text{LCA})$ and a fiber sequence

\[R\Gamma(\mathcal{A}_s, R^i\mathbb{Q}(d))[-2] \to R\Gamma_{et}(\mathcal{A}_s, R^i\mathbb{Z}(d)) \to R\Gamma_W(\mathcal{A}_s, R^i\mathbb{Z}(d))\]

of discrete objects in $D^b(\text{LCA})$ in the sense of [9, Def. 2.7]. We obtain the following diagram

\[
\begin{array}{c}
\begin{array}{cccc}
R\Gamma_{et}(\mathcal{A}_s, R^i\mathbb{Q}(d))[-2] & R\Gamma_{et}(\mathcal{A}_s, R^i\mathbb{Z}(d)) & R\Gamma_W(\mathcal{A}_s, R^i\mathbb{Z}(d)) \\
\downarrow & \downarrow & \downarrow \\
0 & R\Gamma_{et}(\mathcal{A}, \mathbb{Z}(d)) & \sim R\Gamma_{ar}(\mathcal{A}, \mathbb{Z}(d)) \\
\downarrow & \downarrow & \downarrow \\
R\Gamma_{et}(\mathcal{A}_s, R^i\mathbb{Q}(d))[-1] & R\Gamma_{et}(\mathcal{A}_K, \mathbb{Z}(d)) & R\Gamma_{ar}(\mathcal{A}_K, \mathbb{Z}(d))
\end{array}
\end{array}
\]

with exact rows and columns. The lower horizontal cofiber sequence is (4). \qed
Notation 2.4. We set
\[R\Gamma_{et}(\mathcal{X}, \hat{\mathbb{Z}}(d)) := R\lim (R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d)) \otimes^L \mathbb{Z}/m\mathbb{Z})\]
\[R\Gamma_{et}(\mathcal{X}_K, \hat{\mathbb{Z}}(d)) := R\lim (R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}(d)) \otimes^L \mathbb{Z}/m\mathbb{Z})\]
in \(D^b(Ab)\).

Recall from [9] Prop. 2.6 that we have an adjunction
\[\iota : D^b(Ab) \rightleftarrows D^b(LCA) : \text{disc}\]
We sometimes denote \((-)^{\delta} := \text{disc}(-)\) for brevity and suppress any mention of the functor \(\iota\).

Proposition 2.5. We have equivalences
\[\text{(5)} \quad \text{disc}(R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d))) \simeq \text{disc}(R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d))) \simeq R\Gamma_{et}(\mathcal{X}, \hat{\mathbb{Z}}(d))\]
in \(D^b(Ab)\). If \(R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0))\) is cohomologically bounded, then we have a canonical map
\[\text{(6)} \quad \text{disc}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d))) \to R\Gamma_{et}(\mathcal{X}_K, \hat{\mathbb{Z}}(d))\]
in \(D^b(Ab)\), which induces an equivalence after applying \((-)^{\delta}\).

Proof. The equivalences of (5) are given by Proposition 2.3 and [9] Remark 4.12 respectively. If \(R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0))\) is cohomologically bounded, then we have a morphism of cofiber sequences
\[
\begin{array}{ccc}
R\Gamma_W(\mathcal{X}_s, R^i\mathbb{Z}(d)) \otimes^L \hat{\mathbb{Z}} & \xrightarrow{\simeq} & R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d))^{\delta} \otimes^L \hat{\mathbb{Z}} \\
& \simeq & R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^{\delta} \otimes^L \hat{\mathbb{Z}} \\
R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}(d)) \otimes^L \hat{\mathbb{Z}} & \xrightarrow{\simeq} & R\Gamma_{et}(\mathcal{X}_K, \hat{\mathbb{Z}}(d)) \\
& \simeq & R\Gamma_{et}(\mathcal{X}_K, \hat{\mathbb{Z}}(d))
\end{array}
\]
where the left and middle vertical maps are equivalences. Hence the right vertical map is an equivalence as well.  

\[\square\]

Remark 2.6. Using the push-forward map of [7] Cor. 7.2, it is easy to check that the complexes \(R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d))\) and \(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d))\) are covariantly functorial for proper maps \(\mathcal{X} \to \mathcal{Y}\), and that the equivalences (5) and the map (6) are functorial.

Example 2.7. We have \(R\Gamma_{et}(\text{Spec}(\mathcal{O}_K), \mathbb{Z}(1)) \simeq R\Gamma(G_{\kappa(s)}, \mathcal{O}_K^\times) \simeq \mathcal{O}_K^\times[-1]\) in \(D^b(Ab)\). We obtain
\[R\Gamma_{ar}(\text{Spec}(\mathcal{O}_K), \mathbb{Z}(1)) \simeq R\hat{\Gamma}_{et}(\text{Spec}(\mathcal{O}_K), \mathbb{Z}(1)) \simeq (\tau^{-a} R\Gamma_{et}(\text{Spec}(\mathcal{O}_K), \mathbb{Z}(1)) \otimes^L \hat{\mathbb{Z}} \simeq (\mathcal{O}_K^\times[-1] \otimes^L \hat{\mathbb{Z}} \simeq \mathcal{O}_K^\times[-1] \]
where \(\mathcal{O}_K^\times\) is endowed with its profinite topology. Here we use Proposition 2.3 and the fact that \(\mathcal{O}_K^\times\) is derived profinite complete. Moreover, we have \(R^i\mathbb{Z}(1) \simeq \mathbb{Z}[-2], H^i_W(s, \mathbb{Z}) = \mathbb{Z} \text{ for } i = 0, 1, \text{ and } H^i_W(s, \mathbb{Z}) = 0 \text{ for } i \neq 0, 1\). On the other
hand, we have $H^i_{et}(s,\mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Q}/\mathbb{Z}$ for $i = 0, 1, 2$, and $H^i_{et}(s,\mathbb{Z}) = 0$ for $i \neq 0, 2$. We obtain
\[
H^i_{ar}(\text{Spec}(K), \mathbb{Z}(1)) \simeq K^\times, \mathbb{Z} \text{ for } i = 1, 2,
\]
\[
\tilde{H}^i_{et}(\text{Spec}(K), \mathbb{Z}(1)) \simeq K^\times, 0, \mathbb{Q}/\mathbb{Z} \text{ for } i = 1, 2, 3
\]
where $K^\times$ is endowed with its natural topology, $H^i_{ar}(\text{Spec}(K), \mathbb{Z}(1)) = 0$ for $i \neq 1, 2$, and $\tilde{H}^i_{et}(\text{Spec}(K), \mathbb{Z}(1)) = 0$ for $i \neq 1, 3$.

2.1. **Weight 0.** Recall that for a finite field $k$ with algebraic closure $\bar{k}$ and $W_k$ be its Weil-group, the Weil-étale and $Wh$-cohomology of a proper scheme $Y$ over $k$ of the constant sheaf $\mathbb{Z}$ are defined to be
\[
R\Gamma_W(Y, \mathbb{Z}) := R\Gamma(W_k, R\Gamma_{et}(Y \times_k \bar{k}, \mathbb{Z})),
\]
\[
R\Gamma_{Wh}(Y, \mathbb{Z}) := R\Gamma(W_k, R\Gamma_{eh}(Y \times_k \bar{k}, \mathbb{Z})).
\]
By [9, Prop. 3.2, Prop. 3.5], $R\Gamma_W(Y, \mathbb{Z})$ is a perfect complex of abelian groups if $Y^{red}$ is a strict normal crossing scheme, and $R\Gamma_{Wh}(Y, \mathbb{Z})$ is a perfect complex of abelian groups under resolution of singularities.

**Notation 2.8.** For the rest of the paper we will write $R\Gamma_{ar}(Y, \mathbb{Z})$ in both of the above cases. Since $R\Gamma_W(Y, \mathbb{Z}) \simeq R\Gamma_{Wh}(Y, \mathbb{Z})$ if $Y^{red}$ is a strict normal crossing scheme under resolution of singularities, there is no conflict in the intersection of both cases.

If $R\Gamma_{ar}(Y, \mathbb{Z})$ is a perfect complex of abelian groups, it belongs to $D^b(FLCA)$, where $D^b(FLCA) \subseteq D^b(LCA)$ is the full stable subcategory consisting of bounded complexes of locally compact abelian groups of finite ranks in the sense of [11]. Hence we can define
\[
R\Gamma_{ar}(\mathcal{X}, \mathbb{R}/\mathbb{Z}) := R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \boxtimes^{L} \mathbb{R}/\mathbb{Z} \in D^b(FLCA).
\]
Recall from [9, Section 4.3, Notation 4.7] the definition of the complexes $R\Gamma_{ar}(\mathcal{X}, \mathbb{Z})$ and $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z})$: If $\mathcal{X}^{red}$ is a simple normal crossing scheme in the sense of [9, Def. 3.1] (in particular when $\mathcal{X}$ has strictly semi-stable reduction in the sense of [9, Def. 4.15]), or if we assume resolution of singularities, we define $R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) = R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \in D^b(LCA)$, and $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \in D^b(LCA)$ by the fiber sequences
\[
R\Gamma_{FCA}(\mathcal{X}, R^i\mathbb{Z}) \rightarrow R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \rightarrow R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z})
\]
in $D^b(LCA)$.

The following result is [9, Prop. 4.13], where $D^b(FLCA) \subseteq D^b(LCA)$ is the full stable subcategory consisting of bounded complexes of locally compact abelian groups of finite ranks in the sense of [11].

**Proposition 2.9.** 1) Assume resolution of singularities for schemes over $\kappa(s)$ of dimension $\leq d - 1$, or assume that $\mathcal{X}^{red}$ is a simple normal crossing scheme. Then $R\Gamma_{ar}(\mathcal{X}, \mathbb{Z})$ and $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z})$ belong to $D^b(FLCA)$.

2) Assume that $R\Gamma_{W}(\mathcal{X}_s, \mathbb{Z}(0))$ is a perfect complex of abelian groups. Then $R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d))$ and $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d))$ belong to $D^b(FLCA)$. 

If $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z})$ belong to $D^b(FLCA)$, then we define $[9]$:

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}) := R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \otimes^{L} \mathbb{R} \in D^b(FLCA)$$

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) := R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \otimes^{L} \mathbb{R}/\mathbb{Z} \in D^b(FLCA)$$

where $\otimes^{L}$ denotes the derived tensor product in $D^b(FLCA)$. We have a fiber sequence

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \rightarrow R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}) \rightarrow R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}).$$

The cohomology groups $H^i_{ar}(\mathcal{X}_K, A) \in \mathcal{LH}(FLCA)$ for $A = \mathbb{Z}, \mathbb{R}, \mathbb{R}/\mathbb{Z}$ are defined using the $t$-structure on the stable $\infty$-category $D^b(FLCA)$, whose heart is the abelian category $\mathcal{LH}(FLCA)$. A fiber sequence in $D^b(FLCA)$ therefore yields a long exact sequence in $\mathcal{LH}(FLCA)$.

3. Class Field Theory

From now on, $\mathcal{X}/\mathcal{O}_K$ denotes a flat, proper, separable $\mathcal{O}_K$-scheme with geometrically connected fibers. We assume that $\mathcal{X}$ is regular of Krull dimension $d$.

3.1. The abelian Weil-étale fundamental group. We denote by $\pi_1^{ab}(S)$ the abelianization of the classical étale profinite fundamental group of a scheme $S$. We have morphisms

$$\pi_1^{ab}(\mathcal{X}_K) \rightarrow \pi_1^{ab}(\mathcal{X}) \xrightarrow{\sim} \pi_1^{ab}(\mathcal{X}_s)$$

where the map on the left map is surjective because $\mathcal{X}$ is normal [17] V, Prop. 8.2 and the right map is an isomorphism by [17] X, Thm 2.1. We denote by $\Pi_1^{ab}(\mathcal{X}_s)$ the abelian enlarged fundamental group of the closed fiber. The pro-group $\Pi_1^{ab}(\mathcal{X}_s)$ is isomorphic to a constant abelian group.

**Definition 3.1.** We define

$$\Pi_1^{ab}(\mathcal{X}_s)_W := \Pi_1^{ab}(\mathcal{X}_s) \times_{G_{n(s)}} W_{n(s)}$$

and

$$\pi_1^{ab}(\mathcal{X}_K)_W := \pi_1^{ab}(\mathcal{X}_K) \times_{\pi_1^{ab}(\mathcal{X}_s)} \Pi_1^{ab}(\mathcal{X}_s)_W.$$
Example 3.2. If $X = \text{Spec}(O_K)$ then 
$$\pi_1^{ab}(X_K)_W \simeq G_K^{ab} \times G_\kappa W_\kappa \simeq W_K^{ab}.$$ 

Remark 3.3. The group $\Pi_1^{ab}(X_\kappa)$ is the abelianization of the fundamental group of the small étale topos $X_{s,\text{et}}$. It follows from [2] Theorem 3.1, Corollary 3 that $\Pi_1^{ab}(X_\kappa)_W$ is the abelianization of the fundamental group of the Weil-étale topos $X_{s,W}$. A geometrical interpretation of the fundamental group $\pi_1^{ab}(X_K)_W$ is unknown yet.

Notation 3.4. If $A, B$ are locally compact abelian groups, then we denote by $\text{Hom}(A, B)$ the abelian group of continuous morphisms from $A$ to $B$, endowed with the compact-open topology. For any locally compact abelian group $A$, we denote by $A^D := \text{Hom}(A, \mathbb{R}/\mathbb{Z})$ the Pontryagin dual of $A$.

Lemma 3.5. 1) If $Y$ is a normal scheme of finite type over $O_K$, then $R\Gamma_{et}(Y, \mathbb{Z})$ is in $D^b(\text{FLCA})$ and $H^i_{et}(Y, \mathbb{Q}/\mathbb{Z}) \simeq H^i_{et}(Y, \mathbb{R}/\mathbb{Z})$ for all $i \geq 1$.

2) If $T$ is a scheme of finite type over a finite field such that $R\Gamma_{Wh}(T, \mathbb{Z})$ belongs to $D^b(\text{FLCA})$, then 
$$H^i_{Wh}(T, \mathbb{Q}/\mathbb{Z}) \hookrightarrow H^i_{Wh}(T, \mathbb{R}/\mathbb{Z})$$
is injective for any $i$.

Proof. 1) Since $Y$ is normal, then $H^0_{et}(Y, \mathbb{Z}) = \mathbb{Z}$, and for $i \geq 1$ we have $H^i_{et}(Y, \mathbb{Q}) = 0$. Thus $H^i_{et}(Y, \mathbb{Z})$ is the image of the $H^{i-1}_{et}(Y, \mathbb{Q}/\mathbb{Z})$, a group of cofinite type [1 Thm. 1.1]. Hence $R\Gamma_{et}(Y, \mathbb{Z})$ is in $D^b(\text{FLCA})$ and $H^i_{et}(Y, \mathbb{R}) = 0$ for $i \geq 1$. We can conclude with the map of long exact coefficient sequences

$$
\cdots \rightarrow H^1_{et}(Y, \mathbb{Z}) \rightarrow H^1_{et}(Y, \mathbb{Q}) \rightarrow H^1_{et}(Y, \mathbb{Q}/\mathbb{Z}) \rightarrow H^{1+1}_{et}(Y, \mathbb{Z}) \cdots
$$

2) Comparing the analog coefficient sequences, we see that the kernel of $H^1_{Wh}(T, \mathbb{Q}) \rightarrow H^1_{Wh}(T, \mathbb{R})$ surjects onto the kernel in question, but this map is injective because the groups are finitely generated. □

Proposition 3.6. Under resolution of singularities for schemes of dimension at most $\dim T$, we have 
$$H^1_W(T, \mathbb{R}/\mathbb{Z}) \simeq H^1_{Wh}(T, \mathbb{R}/\mathbb{Z})$$
for any $T$ of finite type over a finite field.

Proof. Both sides do not change if we replace $T$ by $T^{\text{red}}$, so we assume that $X_s$ is reduced. We first claim that the result holds for normal $T$. We first show that for an integral scheme $S$ over a field, one has $H^1_{et}(S, \mathbb{Z}) = 0$. Consider the Hochschild-Serre spectral sequence $H^j_{et}(S, R\mathcal{f}_*\mathcal{Z}) \Rightarrow H^{j+1}_{et}(\eta, \mathbb{Z})$ associated to the inclusion of the generic point $j : \eta \rightarrow S$. Since $j_*\mathcal{Z} \cong \mathbb{Z}$, we obtain an inclusion $H^1_{et}(S, \mathbb{Z}) \hookrightarrow H^1_{et}(\eta, \mathbb{Z})$, but the latter group vanishes because every abstract blow-up cover of a field splits, so that this group agrees with the Galois cohomology of $\mathbb{Z}$, which is zero. Now we obtain 
$$Z^{\pi_0(T)} \simeq H^0_{et}(\bar{T}, \mathbb{Z})_W \simeq H^1_W(T, \mathbb{Z})$$
and similarly $H^1_{Wh}(T, \mathbb{Z}) \simeq \mathbb{Z}^{\pi_0(T)}$. Thus there is short exact sequences

$$
0 \longrightarrow (\mathbb{Q}/\mathbb{Z})^{\pi_0(T)} \longrightarrow H^1_W(T, \mathbb{Q}/\mathbb{Z}) \longrightarrow \text{Tor} H^2_W(T, \mathbb{Z}) \longrightarrow 0
$$



The proper base-change theorem implies that middle map, hence that the right hand map is an isomorphism. As $H^2_W(T, \mathbb{Z})$ is finitely generated discrete, the kernel of $H^2_W(T, \mathbb{Z}) \to H^2(W, R) \simeq H^2_W(T, \mathbb{Z}) \otimes R$ is the torsion subgroup of $H^2_W(T, \mathbb{Z})$ as an abelian group, and similarly for $H^1_W(T, \mathbb{Z})$. Thus we obtain a map of short exact sequences

$$
0 \longrightarrow (\mathbb{R}/\mathbb{Z})^{\pi_0(T)} \longrightarrow H^1_W(T, \mathbb{R}/\mathbb{Z}) \longrightarrow \text{Tor} H^2_W(T, \mathbb{Z}) \longrightarrow 0
$$

and conclude that the middle map is an isomorphism.

In the general case let $T' \to T$ be the normalization, $Y$ the closed subscheme with the reduced structure of smaller dimension where $T'' \to T$ is not an isomorphism, and $Y' = Y \times_T T'$. Consider the abstract blow-up sequence of $[9]$ Lemma 3.4,

$$
H^i_W(T, \mathbb{R}/\mathbb{Z}) \longrightarrow H^i_W(T', \mathbb{R}/\mathbb{Z}) \oplus H^i_W(Y, \mathbb{R}/\mathbb{Z}) \longrightarrow H^i_W(Y', \mathbb{R}/\mathbb{Z})
$$

and proceed by induction on dim $T$, using that $H^0_W(S, \mathbb{R}/\mathbb{Z}) \simeq H^0_W(S, \mathbb{R}/\mathbb{Z}) \simeq (\mathbb{R}/\mathbb{Z})^{\pi_0(S)}$ for any scheme $S$ of finite type over a finite field. □

**Proposition 3.7.** Assume resolution of singularities for schemes over $\kappa(s)$ of dimension $\leq d - 1$, or assume that $\mathcal{X}$ has good or strictly semi-stable reduction. Then we have a canonical isomorphism of locally compact groups

$$
H^1_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \simeq \text{Hom}(\pi_1^{	ext{ab}}(\mathcal{X}_K)_W, \mathbb{R}/\mathbb{Z}).
$$

**Proof.** Assume resolution of singularities for schemes over $\kappa(s)$ of dimension $\leq d - 1$. By the remark after Proposition 3.5, $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z})$ is well defined. One has a morphism of fiber sequences

$$
R\Gamma_{et}(\mathcal{X}_s, R^i\mathbb{Z}) \longrightarrow R\Gamma_{et}(\mathcal{X}, \mathbb{Z}) \longrightarrow R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z})
$$

where the left vertical map is an equivalence since $R^i\mathbb{Z}$ is torsion. It follows that the square on the right is a push-out square of (bounded) discrete complexes. The complexes of the top row are of finite ranks, since their cohomology is torsion in degrees $> 0$ and finite modulo $l$ for all $l$. The complex of abelian groups $R\Gamma_{ar}(\mathcal{X}_s, \mathbb{Z})$ is perfect by [9] Prop. 3.2, 3.5. Hence the right square
above is a push-out in $D^b(FLCA)$. Applying $(-)\otimes^L R/\mathbb{Z}$, we obtain the push-out square

$$
\begin{array}{ccc}
R\Gamma_{et}(\mathcal{X}, \mathbb{R}/\mathbb{Z}) & \longrightarrow & R\Gamma_{et}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \\
\downarrow & & \downarrow \\
R\Gamma_{ar}(\mathcal{X}_s, \mathbb{R}/\mathbb{Z}) & \longrightarrow & R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z})
\end{array}
$$

hence an exact sequence in $\mathcal{LH}(FLCA)$

$$
\cdots \longrightarrow H^1_{et}(\mathcal{X}, \mathbb{R}/\mathbb{Z}) \xrightarrow{\iota} H^1_{et}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \oplus H^1_{ar}(\mathcal{X}_s, \mathbb{R}/\mathbb{Z}) \longrightarrow H^2_{et}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \oplus H^2_{ar}(\mathcal{X}_s, \mathbb{R}/\mathbb{Z}).
$$

We have $H^i_{et}(\mathcal{X}, \mathbb{Q}) = 0$ for $i \geq 1$ since $\mathcal{X}$ is normal. It follows that $H^i_{et}(\mathcal{X}, \mathbb{R}) := H^i(R\Gamma_{et}(\mathcal{X}, \mathbb{R})\otimes^L \mathbb{R}) = 0$ for $i \geq 1$, hence

$$
H^i_{et}(\mathcal{X}, \mathbb{R}/\mathbb{Z}) \cong H^{i+1}_{et}(\mathcal{X}, \mathbb{Z}/\mathbb{Z}) \cong H^i_{et}(\mathcal{X}, \mathbb{Q}/\mathbb{Z})
$$

for any $i \geq 1$. Moreover the morphism

$$
H^i_{ar}(\mathcal{X}_s, \mathbb{Q}/\mathbb{Z}) \hookrightarrow H^i_{ar}(\mathcal{X}_s, \mathbb{R}/\mathbb{Z})
$$

is injective since the groups $H^i_{ar}(\mathcal{X}_s, \mathbb{Z})$ are finitely generated. In view of the isomorphisms

$$
H^i_{et}(\mathcal{X}, \mathbb{Z}/m\mathbb{Z}) \sim H^i_{et}(\mathcal{X}_s, \mathbb{Z)/m\mathbb{Z}) \sim H^i_{ar}(\mathcal{X}_s, \mathbb{Z}/m\mathbb{Z})
$$

given by proper base change, we see that the morphism

$$
H^i_{et}(\mathcal{X}, \mathbb{R}/\mathbb{Z}) \sim H^i_{et}(\mathcal{X}, \mathbb{Q}/\mathbb{Z}) \sim H^i_{ar}(\mathcal{X}_s, \mathbb{Q}/\mathbb{Z}) \hookrightarrow H^i_{ar}(\mathcal{X}_s, \mathbb{R}/\mathbb{Z})
$$

is injective for $i \geq 1$. We obtain an exact sequence in $\mathcal{LH}(LCA)$

$$
(7) \quad 0 \longrightarrow H^1_{et}(\mathcal{X}, \mathbb{R}/\mathbb{Z}) \xrightarrow{\iota} H^1_{et}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \oplus H^1_{ar}(\mathcal{X}_s, \mathbb{R}/\mathbb{Z}) \longrightarrow H^2_{et}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \longrightarrow 0.
$$

On the other hand, by definition of $\pi^1_{ab}(\mathcal{X}_K)_W$, we have a strictly exact sequence

$$
0 \longrightarrow \pi^1_{ab}(\mathcal{X}_K)_W \xrightarrow{s} \pi^1_{ab}(\mathcal{X}_K) \oplus \Pi^1_{ab}(\mathcal{X}_s)_W \xrightarrow{\iota} \pi^1_{ab}(\mathcal{X}) \longrightarrow 0
$$

since $\iota$ is a closed embedding by definition, and since $s$ is an open surjection (as $\pi^1_{ab}(\mathcal{X}_K) \to \pi^1_{ab}(\mathcal{X})$ is an open surjection), see [11, Section 1]. The dual of a strictly exact sequence in LCA is also strictly exact, and the fully faithful functor $LCA \hookrightarrow \mathcal{LH}(LCA)$ sends a strictly exact sequence in LCA to an exact sequence in the abelian category $\mathcal{LH}(LCA)$, see [21, Cor. 1.2.28]. Hence we have an exact sequence

$$
0 \longrightarrow \text{Hom}(\pi^1_{ab}(\mathcal{X}), \mathbb{R}/\mathbb{Z}) \xrightarrow{s^\triangleright} \text{Hom}(\pi^1_{ab}(\mathcal{X}_K), \mathbb{R}/\mathbb{Z}) \oplus \text{Hom}(\Pi^1_{ab}(\mathcal{X}_s)_W, \mathbb{R}/\mathbb{Z}) \longrightarrow \text{Hom}(\pi^1_{ab}(\mathcal{X}_K)_W, \mathbb{R}/\mathbb{Z}) \longrightarrow 0.
$$

in $\mathcal{LH}(LCA)$.

By [10, Prop. 2.4] and Proposition 4.6 we have canonical isomorphisms of discrete abelian groups

$$
\text{Hom}(\Pi^1_{ab}(\mathcal{X}_s)_W, A) \xrightarrow{\sim} H^1_{W}(\mathcal{X}_s, A) \xrightarrow{\sim} H^1_{WH}(\mathcal{X}_s, A)
$$

for $A = \mathbb{Z}, \mathbb{R}, \mathbb{R}/\mathbb{Z}$. Moreover, the map $\text{Hom}(\Pi^1_{ab}(\mathcal{X}_s)_W, \mathbb{R}) \to H^1_{W}(\mathcal{X}_s, \mathbb{R})$ is an $\mathbb{R}$-linear map between finite dimensional $\mathbb{R}$-vector spaces, hence bicontinuous with respect with the Euclidean topology. This implies that $\text{Hom}(\Pi^1_{ab}(\mathcal{X}_s)_W, \mathbb{R}) \to$
$H^1_{ar}(X_s, \mathbb{R})$ is an isomorphism of topological groups and then it follows that $\text{Hom}(\Pi^1_{ab}(X_s)_W, \mathbb{R}/\mathbb{Z}) \rightarrow H^1_{ar}(X_s, \mathbb{R}/\mathbb{Z})$ is continuous as well. We obtain a canonical isomorphism $H^1_{ar}(X_s, \mathbb{R}/\mathbb{Z}) \simeq \text{Hom}(\Pi^1_{ab}(X_s)_W, \mathbb{R}/\mathbb{Z})$ of compact abelian groups. Moreover, the map $\iota$ in (7) is canonically identified with $s^D$. We obtain a canonical isomorphism between their cokernels:

\[
\text{Hom}(\pi^1_{ab}(X_K)_W, \mathbb{R}/\mathbb{Z}) \sim H^1_{ar}(X_K, \mathbb{R}/\mathbb{Z}).
\]

\[\square\]

### 3.2. Class field theory.

**Theorem 3.8.** Assume that $X$ has good or strictly semi-stable reduction and that $R\Gamma_W(X_s, \mathbb{Z}^c(0))$ is a perfect complex of abelian groups. Then there exists a unique functorial isomorphism of locally compact groups

\[
\text{rec}: H^{2d-1}_{ar}(X_K, \mathbb{Z}(d)) \sim \pi^1_{ab}(X_K)_W
\]

inducing

\[
H^{2d-1}_{et}(X_K, \hat{\mathbb{Z}}(d)) \sim \pi^1_{ab}(X_K)
\]

after profinite completion.

**Proof.** By [9, Cor. 5.13], Proposition 3.7 and Pontryagin duality, we have isomorphisms of locally compact groups

\[
H^{2d-1}_{ar}(X_K, \mathbb{Z}(d)) \sim \text{Hom}(H^1_{ar}(X_K, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z})
\]

\[
\simeq \text{Hom}(\text{Hom}(\pi^1_{ab}(X_K)_W, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z})
\]

\[
\simeq \pi^1_{ab}(X_K)_W.
\]

To prove uniqueness, we set $\langle - \rangle^\delta := \text{disc}(\langle - \rangle)$ for brevity. Recall from Proposition 2.5 that we have canonical morphisms of discrete abelian groups

\[
H^i_{ar}(X_K, \mathbb{Z}(d))^\delta \rightarrow H^i_{et}(X_K, \hat{\mathbb{Z}}(d)).
\]

The commutative square of [9, Prop. 5.8] (respectively the proof of Proposition 3.7) shows that the left square (respectively the right square) of the following diagram of discrete abelian groups

\[
H^{2d-1}_{ar}(X_K, \mathbb{Z}(d))^\delta \rightarrow H^{2d-1}_{ar}(X_K, \mathbb{Z}(d))^\delta
\]

\[
\rightarrow \text{Hom}(H^1_{ar}(X_K, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z})^\delta \rightarrow \pi^1_{ab}(X_K)_W^\delta
\]

\[
H^{2d-1}_{et}(X_K, \hat{\mathbb{Z}}(d))^\delta \rightarrow H^{2d-1}_{et}(X_K, \hat{\mathbb{Z}}(d))^\delta
\]

\[
\rightarrow \text{Hom}(H^1_{et}(X_K, \mathbb{Q}/\mathbb{Z}), \mathbb{R}/\mathbb{Z})^\delta \rightarrow \pi^1_{ab}(X_K)^\delta
\]

commutes. Here the right vertical maps is injective, the horizontal maps are all isomorphisms and the composition of the upper horizontal maps is $\text{rec}^\delta$. The unicity of the map $\text{rec}^\delta$ follows, as well as the unicity of the map $\text{rec}$ since the functor $\langle - \rangle^\delta : \text{LCA} \rightarrow \text{Ab}$ is faithful.
Finally, we check that \( \text{rec} \) is functorial. A proper map \( \mathcal{X} \to \mathcal{Y} \) induces a commutative square

\[
\begin{array}{ccc}
H_{ar}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d\mathcal{X})) & \longrightarrow & H_{ar}^{2d-1}(\mathcal{Y}_K, \mathbb{Z}(d\mathcal{Y})) \\
\downarrow & & \downarrow \\
H_{et}^{2d-1}(\mathcal{X}_K, \hat{\mathbb{Z}}(d\mathcal{X})) & \longrightarrow & H_{et}^{2d-1}(\mathcal{Y}_K, \hat{\mathbb{Z}}(d\mathcal{Y}))
\end{array}
\]

as mentioned in Remark 2.6. Considering the reciprocity maps to the corresponding (discrete) fundamental groups, we obtain a 8-terms diagram. A look at this 8-terms diagram shows the result, since \( H_{et}^{2d-1}(\mathcal{X}_K, \hat{\mathbb{Z}}(d)) \to \pi_1^{ab}(\mathcal{X}_K) \) is functorial for proper maps, \( \pi_1^{ab}(\mathcal{Y}_K)_W \to \pi_1^{ab}(\mathcal{Y}_K) \) is injective and since \((-)^d\) is faithful.

**Example 3.9.** If \( \mathcal{X}_K = \text{Spec}(K) \), we recover Weil’s isomorphism of locally compact abelian groups

\[
K^\times = H_{ar}^1(\mathcal{X}_K, \mathbb{Z}(1)) \xrightarrow{\sim} \pi_1^{ab}(\mathcal{X}_K)_W = W_K^{ab}.
\]

**Notation 3.10.** We define the locally compact groups

\[
\begin{align*}
H_{ar}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d))^0 & := \ker \left( H_{ar}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d)) \to K^\times \right); \\
H_{ar}^{2d-1}(\mathcal{X}, \mathbb{Z}(d))^0 & := \ker \left( H_{ar}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)) \to \mathcal{O}_K^\times \right); \\
\pi_1^{ab}(\mathcal{X}_K)_{\text{geo}} & := \ker \left( \pi_1^{ab}(\mathcal{X}_K)_W \to W_K^{ab} \right).
\end{align*}
\]

We define similarly the discrete abelian group

\[
H_{et}^{2d-1}(\mathcal{X}, \hat{\mathbb{Z}}(d))^0 := \ker \left( H_{et}^{2d-1}(\mathcal{X}, \hat{\mathbb{Z}}(d)) \to \mathcal{O}_K^\times \right).
\]

**Lemma 3.11.** The cokernel of the push-forward maps from

\[
H_{\mathcal{M}}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)), H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)), H_{et}^{2d-1}(\mathcal{X}, \hat{\mathbb{Z}}(d)),
\]

\[
\tilde{H}_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)), H_{ar}^{2d-1}(\mathcal{X}, \mathbb{Z}(d))
\]

to \( \mathcal{O}_K^\times \) are finite, where \((-)^\sim\) denotes the profinite completion. Similarly, the cokernels of the push-forward from

\[
H_{\mathcal{M}}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d)), H_{et}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d)), \tilde{H}_{et}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d)), H_{ar}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d))
\]

to \( K^\times \) are finite.

**Proof.** We first note that for \( \mathcal{X} = \text{Spec}(\mathcal{O}_K) \) the first groups are isomorphic to \( \mathcal{O}_K^\times \), and for \( X = \text{Spec}(K) \) the second groups are isomorphic to \( K^\times \). In view of Proposition 2.5 the push-forward maps \( H_{\mathcal{M}}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)) \to \mathcal{O}_K^\times \) and \( H_{\mathcal{M}}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d)) \to K^\times \) factor through the other groups, so that it suffices to prove the Lemma for these maps.

Let \( L/K \) be a finite extension such that \( \mathcal{X}_K \) has a \( L \)-rational point. By properness, we obtain a finite flat morphism \( \text{Spec}(\mathcal{O}_L) \to \mathcal{X} \) over \( \text{Spec}(\mathcal{O}_K) \). Then the composite maps

\[
\mathcal{O}_K^\times \subseteq \mathcal{O}_L^\times \simeq H_{\mathcal{M}}^1(\mathcal{O}_L, \mathbb{Z}(1)) \to H_{\mathcal{M}}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)) \to H_{\mathcal{M}}^1(\mathcal{O}_K, \mathbb{Z}(1)) \simeq \mathcal{O}_K^\times
\]
Proposition 3.12. Assume that $\mathcal{X}$ has good or strictly semi-stable reduction. Then the group $H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d))^0 \simeq H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d))^0$ is finite.

Proof. By [9, Proof of Theorem 4.16], we have isomorphisms $H_{et}^{2d-1}(\mathcal{X}, \mathbb{Q}_p(d)) \simeq H_f^1(G_K, H^{2d-2}(\mathcal{X}_K, \mathbb{Q}_p(d)))$ and $H_{et}^1(\mathcal{O}_K, \mathbb{Q}_p(1)) \simeq H_f^1(G_K, \mathbb{Q}_p(1))$. The push-forward map $H_{et}^{2d-1}(\mathcal{X}, \mathbb{Q}_p(d)) \to H_{et}^1(\mathcal{O}_K, \mathbb{Q}_p(1))$ can be identified with the map $H_f^1(G_K, H^{2d-2}(\mathcal{X}_K, \mathbb{Q}_p(d))) \to H_f^1(G_K, \mathbb{Q}_p(1))$ induced by the trace map $H^{2d-2}(\mathcal{X}_K, \mathbb{Q}_p(d)) \simeq \mathbb{Q}_p(1)$, which is an isomorphism since $\mathcal{X}_K$ is geometrically connected. Hence the kernel

$$H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}_p(d))^0 := \text{Ker} \left( H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}_p(d)) \to H_{et}^1(\mathcal{O}_K, \mathbb{Z}_p(1)) \right)$$

is torsion. Since $H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}_p(d))$ is a finitely generated $\mathbb{Z}_p$-module, $H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}_p(d))^0$ is finite. Moreover, the group $H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}_l(d)) \simeq H_{et}^{2d-1}(\mathcal{X}_s, \mathbb{Z}_l(d))$ is finite for all $l \neq p$ and vanishes for almost all $l$, see [9, Lemma 4.17]. Hence $H_{et}^{2d-1}(\mathcal{X}, \hat{\mathbb{Z}}(d))^0$ is finite as well. Since we have $\text{disc}(H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d))^0) = H_{et}^{2d-1}(\mathcal{X}, \hat{\mathbb{Z}}(d))^0$ by Proposition 2.5, the locally compact group $H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d))^0$ is finite hence discrete, and may therefore be identified with $H_{et}^{2d-1}(\mathcal{X}, \hat{\mathbb{Z}}(d))^0$. 

Proposition 3.13. Assume that $R\Gamma_W(\mathcal{X}_s, \mathbb{Z}(0))$ is cohomologically bounded. Then we have a diagram with exact rows and columns

$$
\begin{array}{cccccc}
H_2^W(\mathcal{X}_s, \mathbb{Z}) & \longrightarrow & H_{ar}^{2d-1}(\mathcal{X}, \mathbb{Z}(d))^0 & \longrightarrow & H_{ar}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d))^0 & \longrightarrow & H_1^W(\mathcal{X}_s, \mathbb{Z})^0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2^W(\mathcal{X}_s, \mathbb{Z}) & \longrightarrow & H_{ar}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)) & \longrightarrow & H_{ar}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d)) & \longrightarrow & H_1^W(\mathcal{X}_s, \mathbb{Z}) & \longrightarrow & 0 \\
\downarrow & & \downarrow \delta & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & K^\times & & \mathbb{Z} & & 0 \\
\end{array}
$$

Moreover, we have an injective map $\text{coker}(f) \hookrightarrow \text{coker}(s)$ of finite groups.

Proof. We have by duality

$$H_{ar}^{2d}(\mathcal{X}, \mathbb{Z}(d))^0 \simeq H_{et}^{2d}(\mathcal{X}, \hat{\mathbb{Z}}(d)) \simeq H_f^1(\mathcal{O}_K, \mathbb{Q}/\mathbb{Z})^D$$

$$\simeq \left( \text{Ker}(\pi_{ab}^0(\mathcal{X})^D) \to \pi_{ab}^0(\mathcal{X}_K)^D \right)^D = 0$$

since $\pi_{ab}^0(\mathcal{X}_K) \to \pi_{ab}^0(\mathcal{X})$ is surjective because $\mathcal{X}$ is normal [17, V, Prop. 8.2]. Hence the middle row is the localization sequence for arithmetic cohomology. The fact that the lower squares commute follows from the functoriality of localization sequences. The top row is defined by taking kernels; it is easily seen to be exact. The injective map $\text{coker}(f) \hookrightarrow \text{coker}(s)$ is given by the snake lemma. The fact that $\text{coker}(s)$ is finite is Lemma in [3.11].
Corollary 3.14. Assume that $\mathcal{X}$ has good or strictly semi-stable reduction and that $\text{HT}_W(\mathcal{X}, \mathbb{Z}^\wedge(0))$ is a perfect complex of abelian groups. Then the map rec induces an isomorphism
\[
\text{rec}^0 : H^{2d-1}_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d))^0 \cong \pi_1^{ab}(\mathcal{X}_K)_W^{\text{geo}}
\]
of finitely generated abelian groups of ranks
\[
\text{rank}_\mathbb{Z}(H^{2d-1}_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d))^0) = \text{rank}_\mathbb{Z}(\text{HT}_1^W(\mathcal{X}_s, \mathbb{Z})) - 1.
\]

Proof. The fact that $\text{rec}^0$ is an isomorphism follows from the fact that isomorphism rec is functorial. Finite generation and the statement about the rank of $H^{2d-1}_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d))^0$ follows from Proposition 3.12 and Proposition 3.13 because the map $\text{HT}_1^W(\mathcal{X}_s, \mathbb{Z}) \to H^1_W(s, \mathbb{Z})$ has finite cokernel, as can be seen by considering a closed point of $\mathcal{X}_s$. □

4. Comparison to the classical reciprocity map

In this section we study the composition $H^{2d-1}_{\text{et}}(\mathcal{X}_K, \mathbb{Z}(d)) \to \text{HT}^{2d-1}_{\text{et}}(\mathcal{X}_K, \mathbb{Z}(d)) \to H^{2d-1}_{\text{ar}}(\mathcal{X}_K, \mathbb{Z}(d))$, generalizing the reciprocity map of Saito [18] for curves.

4.1. The map $H^{2d-1}_{\text{et}}(\mathcal{X}_K, \mathbb{Z}(d)) \to \text{HT}^{2d-1}_{\text{et}}(\mathcal{X}_K, \mathbb{Z}(d))$. The following lemma follows by an easy diagram chase.

Lemma 4.1. Consider a diagram with exact rows of abelian groups
\[
\begin{array}{cccccccc}
A^i & \longrightarrow & B^i & \longrightarrow & C^i & \longrightarrow & A^{i+1} & \longrightarrow & B^{i+1} & \longrightarrow & C^{i+1} & \longrightarrow & A^{i+2} \\
\scriptstyle \cong & & \scriptstyle \cong & & \scriptstyle \cong & & \scriptstyle \cong & & \scriptstyle \cong & & \scriptstyle \cong & & \\
A^0 & \longrightarrow & B^0 & \longrightarrow & C^0 & \longrightarrow & A^1 & \longrightarrow & B^1 & \longrightarrow & C^1 & \longrightarrow & A^2,
\end{array}
\]

where the maps $A^j \to A^{j+1}$ are isomorphisms. Then there is an induced exact sequence
\[
0 \to \text{coker}(f^i) \to \text{coker}(g^i) \to \ker(f^{i+1}) \to \ker(g^{i+1}) \to 0.
\]

Lemma 4.2. Assume that the group $H^{2d-1}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d))^0$ is finite. Then one has $\text{TH}^{2d}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d)) = 0$ and the map
\[
H^{2d-1}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d)) \to H^{2d-1}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d)) \wedge
\]
is surjective, where $(-)\wedge$ denotes the (naive) profinite completion. In particular, $H^{2d-1}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d)) \to H^{2d-1}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d))$ is surjective.

Proof. Consider the morphism of short exact sequences:
\[
\begin{array}{cccccccc}
0 & \longrightarrow & H^{2d-1}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d)) & \longrightarrow & H^{2d-1}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d)) & \longrightarrow & \text{TH}^{2d}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d)) & \longrightarrow & 0 \\
\scriptstyle a & & \scriptstyle b & & \scriptstyle 0 & & \scriptstyle 0 & & \scriptstyle 0 \\
0 & \longrightarrow & H^1_{\text{et}}(\mathcal{O}_K, \mathbb{Z}(1)) & \longrightarrow & H^1_{\text{et}}(\mathcal{O}_K, \mathbb{Z}(1)) & \longrightarrow & 0 & & \longrightarrow & 0
\end{array}
\]
The group $\text{coker}(a)$ is finite by Lemma 3.11. Since $\ker(b)$ is finite by assumption, the Tate module $\text{TH}^{2d}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d))$ is finite by the snake lemma. Since $\text{TH}^{2d}_{\text{et}}(\mathcal{X}, \mathbb{Z}(d))$ is also torsion-free, it is in fact trivial.
We obtain an isomorphism of finite groups $\ker(a) \simeq \ker(b)$. Now we consider the morphism of exact sequences

$$
\begin{array}{ccccccccc}
0 & \rightarrow & H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d))^0 & \rightarrow & H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)) & \rightarrow & \mathcal{O}_K^\times & \rightarrow & C & \rightarrow & 0 \\
0 & \rightarrow & \ker(a) & \rightarrow & H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)) & \rightarrow & \mathcal{O}_K^\times & \rightarrow & C' & \rightarrow & 0 \\
\end{array}
$$

where $C$ and $C'$ are finite by Lemma 4.1. Since $\ker(a)$ is finite, it follows from Lemma 4.3 that $\ker(a)$ is finite as well. But $\ker(A \rightarrow \hat{A})$ is divisible for any abelian group $A$. Hence $\ker(a)$ vanishes. 

**Lemma 4.3.** One has $CH^d(\mathcal{X}_{\mathcal{O}_{K_{un}}}) = 0$.

*Proof.* We set $\mathcal{X}^un := \mathcal{X}_{\mathcal{O}_{K_{un}}}$. We denote the closed fiber of $\mathcal{X}^un$ by $\mathcal{X}_s$. It suffices to show that the image of every closed point $x$ of $\mathcal{X}^un$ in $CH_0(\mathcal{X}_{\mathcal{O}_{K_{un}}})$ vanishes. Note that $\mathcal{O}_{K_{un}}$ is an henselian discrete valuation ring. Since $\mathcal{X}^un$ is proper over $\mathcal{O}_{K_{un}}$, $x$ lies in the closed fiber $\mathcal{X}_s$. By [19] Lemma 7.2, there exists a regular integral closed subscheme $Z$ of $\mathcal{X}$ of dimension 1 containing $x$ which meets the generic fiber. Now $Z$ is proper over the base $\mathcal{O}_{K_{un}}$, hence finite. Since the base is henselian, this implies that $Z$ is the spectrum of an henselian discrete valuation ring. But for the spectrum $Z$ of a discrete valuation ring with function field $K$ and closed point $x$ the map $CH_0(K, 1) \simeq K^\times \rightarrow \mathbb{Z} = CH_0(x)$ in the localization sequence is surjective, hence its cokernel $CH_0(Z)$ vanishes. Thus the map $\mathbb{Z} = CH_0(x) \rightarrow CH_0(\mathcal{X}_{\mathcal{O}_{K_{un}}})$ is the zero map, since it factors through $CH_0(Z)$.

**Theorem 4.4.** Assume that the group $H_{et}^{2d-1}(\mathcal{X}, \hat{\mathbb{Z}}(d))^0$ is finite. Then there are exact sequences of abelian groups

$$
\begin{array}{cccccc}
0 & \rightarrow & D & \rightarrow & H_{et}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d)) & \rightarrow & \hat{H}_{et}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d))^\delta & \rightarrow & 0 \\
0 & \rightarrow & D & \rightarrow & H_{et}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d))^0 & \rightarrow & \hat{H}_{et}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d))^{0\delta} & \rightarrow & 0, \\
\end{array}
$$

where $D$ is divisible, and

$$
\text{Tor}D \simeq \text{coker}(H_{et}^{2d-2}(\mathcal{X}_K, \mathbb{Z}(d)) \rightarrow \hat{H}_{et}^{2d-2}(\mathcal{X}_K, \mathbb{Z}(d))^{\delta}) \otimes \mathbb{Q}/\mathbb{Z},
$$

Note that by Proposition 3.2 the hypothesis of the Theorem is satisfied if $\mathcal{X}_K$ has good or strictly semi-stable reduction. If $A$ is an abelian group, we denote by $UA := \cap_m m \cdot A$ the subgroup of all divisible elements in $A$, and by $A_{div}$ the maximal divisible subgroup of $A$.

*Proof.* Let $\ker^i(\mathcal{X})$ and $\text{coker}^i(\mathcal{X})$ be the kernel and cokernel of the map $H_{et}^i(\mathcal{X}, \mathbb{Z}(d)) \rightarrow \hat{H}_{et}^i(\mathcal{X}, \mathbb{Z}(d))^{\delta}$, respectively. We define similarly $\ker^i(\mathcal{X}_K)$ and $\text{coker}^i(\mathcal{X}_K)$. The morphism of long exact sequences

$$
\begin{array}{ccccccc}
\cdots & \rightarrow & H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)) & \rightarrow & H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)) & \rightarrow & H_{et}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d)) & \rightarrow & H_{et}^{2d}(\mathcal{X}, \mathbb{Z}(d)) & \cdots \\
\simeq & & & & & & & & & \\
\cdots & \rightarrow & H_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d)) & \rightarrow & \hat{H}_{et}^{2d-1}(\mathcal{X}, \mathbb{Z}(d))^{\delta} & \rightarrow & \hat{H}_{et}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d))^{\delta} & \rightarrow & H_{et}^{2d}(\mathcal{X}_K, \mathbb{Z}(d)) & \cdots
\end{array}
$$
yields by Lemma 4.11 the exact sequence

\[ 0 \rightarrow \text{coker}^{2d-1}(\mathcal{X}) \rightarrow \text{coker}^{2d-1}(\mathcal{X}_K) \rightarrow \ker^{2d}(\mathcal{X}) \rightarrow \ker^{2d}(\mathcal{X}_K) \rightarrow 0. \]

First we note that \( \ker^{2d}(\mathcal{X}) = 0 \) because \( H^{2d}_{et}(\mathcal{X}, \mathbb{Z}(d)) \rightarrow H^{2d}_{et}(\mathcal{X}, \hat{\mathbb{Z}}(d)) \simeq \hat{H}^{2d}_{et}(\mathcal{X}, \mathbb{Z}(d)) \delta \) is surjective by Lemma 4.2 and Proposition 2.5.

Since the map \( H^{2d}_{et}(\mathcal{X}, \mathbb{Z}(d)) \rightarrow H^{2d}_{et}(\mathcal{X}, \hat{\mathbb{Z}}(d)) \) is injective, we have \( \ker^{2d}(\mathcal{X}) = UH^{2d}_{et}(\mathcal{X}, \mathbb{Z}(d)) \). By [16, Cor. 4], the group \( UH^{2d}_{et}(\mathcal{X}, \mathbb{Z}(d)) \) is uniquely divisible because \( TH^{2d}_{et}(\mathcal{X}, \mathbb{Z}(d)) = 0 \) and \( H^{2d}(\mathcal{X}, \mathbb{Z}(d))/m \subset H^{2d}_{et}(\mathcal{X}, \mathbb{Z}/m(d)) \) is finite for any \( m \). Moreover, \( H^{2d}_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}}, \mathbb{Z}(d)) \simeq H^{2d}_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}}, \mathbb{Z}(d)) \simeq CH^{2d}(\mathcal{X}_{\mathcal{O}_{K^{un}}}) = 0 \) by Lemma 4.3, hence the Hochschild-Serre spectral sequence gives a short exact sequence

\[ d_2^{b,2d-1} \rightarrow H^2(G_s, H^{2d-2}_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}}, \mathbb{Z}(d))) \rightarrow H^{2d}_{et}(\mathcal{X}, \mathbb{Z}(d)) \rightarrow H^1(G_s, H^{2d-1}_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}}, \mathbb{Z}(d))) \rightarrow 0 \]

since \( G_s \) has strict cohomological dimension 2. Since Galois cohomology is torsion in degrees \( > 0 \), the abelian group \( H^{2d}_{et}(\mathcal{X}, \mathbb{Z}(d)) \) is torsion as well. Therefore, \( \ker^{2d}(\mathcal{X}) = UH^{2d}_{et}(\mathcal{X}, \mathbb{Z}(d)) \) is both uniquely divisible and torsion, hence it vanishes. In view of (10), we obtain \( \text{coker}^{2d-1}(\mathcal{X}_K) = 0 \).

To obtain the second exact sequence it suffices to consider the kernels of the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & D & \rightarrow & H^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d)) & \rightarrow & \hat{H}^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d)) & \delta & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 0 & \rightarrow & K^\times & \rightarrow & K^\times & \rightarrow & 0
\end{array}
\]

Applying Lemma 4.11 to the coefficient sequences induced by \( 0 \rightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0 \), and using that

\[ R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}(d)) \otimes^L \mathbb{Z}/m \mathbb{Z} \rightarrow R\hat{\Gamma}_{et}(\mathcal{X}_K, \mathbb{Z}(d)) \delta \otimes^L \mathbb{Z}/m \mathbb{Z} \]

is an equivalence by definition, we obtain an exact sequence

\[ 0 \rightarrow \text{coker}^{i-1}(\mathcal{X}_K) \xrightarrow{\times m} \text{coker}^{i-1}(\mathcal{X}_K) \rightarrow \ker^i(\mathcal{X}_K) \xrightarrow{\times m} \ker^i(\mathcal{X}_K) \rightarrow 0, \]

which implies that \( \ker^i(\mathcal{X}_K) \) is divisible and and \( \ker^i(\mathcal{X}_K) \simeq \text{coker}^{i-1}(\mathcal{X}_K)/m \) for all \( i \) and \( m \). In particular, \( D \) is divisible and in the colimit we obtain \( \text{Tor} D \simeq \text{coker}^{2d-2}(\mathcal{X}_K) \otimes \mathbb{Q}/\mathbb{Z} \).

4.2. The maps \( \hat{H}^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d)) \rightarrow H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d)) \).

**Proposition 4.5.** Assume that \( \mathcal{X} \) has good or strictly semi-stable reduction. Then the group \( \hat{H}^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d)) \delta \) is torsion.
Proof. Replacing $R\Gamma_{ar}(-, \mathbb{Z}(d))$ with $R\hat{\Gamma}_{et}(-, \mathbb{Z}(d))$ in the diagram of Proposition \[\text{3.13}\] we obtain an exact sequence

$$
\hat{H}^{2d-1}_{et}(\mathcal{X}, \mathbb{Z}(d))^0 \to \hat{H}^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d))^0 \to H^0_{et}(\mathcal{X}, \mathbb{Z}^c(0))^0
$$

where $H^0_{et}(\mathcal{X}, \mathbb{Z}^c(0))^0$ is the kernel of the push-forward map $H^0_{et}(\mathcal{X}_K, \mathbb{Z}^c(0)) \to H^0_{et}(s, \mathbb{Z}^c(0)) \cong \mathbb{Z}$. Since $\hat{H}^{2d-1}_{et}(\mathcal{X}, \mathbb{Z}(d))^0 \cong H^{2d-1}_{ar}(\mathcal{X}, \mathbb{Z}(d))^0$ is finite by Proposition \[\text{3.12}\] it is enough to show that $H^0_{et}(\mathcal{X}_s, \mathbb{Z}^c(0))^0$ is torsion, i.e. that the degree map $CH_0(\mathcal{X}_s)_\mathbb{Q} \cong H^0(\mathcal{X}_s, \mathbb{Q}^c(0)) \to \mathbb{Q} \cong H^0(s, \mathbb{Q}^c(0))$ is an isomorphism. But every element of $CH_0(\mathcal{X}_s)_\mathbb{Q}$ is supported on a curve, and in this case the result is well-known, see for example [5] Thm. 3.1, Prop. 6.2].

\[\square\]

**Proposition 4.6.** Assume that $\mathcal{X}$ has good or strictly semi-stable reduction and and $R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0))$ is a perfect complex of abelian groups. Then there is an exact sequence

$$0 \to \hat{D} \to \hat{H}^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d))^0 \xrightarrow{\tau} \text{Tor}H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^0 \to 0$$

with $\hat{D}$ is the maximal divisible subgroup of $\hat{H}^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d))^0$, a torsion quotient of $CH_0(\mathcal{X}_s, 2)_\mathbb{Q}$.

**Proof.** Since $H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^0$ is finitely generated by Corollary \[\text{3.14}\] the maximal divisible subgroup of $\hat{H}^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d))^0$ is contained in ker $\tau$. Moreover, $\hat{H}^{2d-1}_{et}(\mathcal{X}_s, \mathbb{Z}(d))^0$ is torsion by Proposition \[\text{4.5}\] so that $\hat{D}$ is torsion and the image of $\tau$ is contained in the torsion subgroup $\text{Tor}H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^0$. In order to show that $\tau$ is surjective it suffices to show that $\text{coker}\tau$ is torsion free, and to determine ker $\tau$ it suffices to show that it is a quotient $CH_0(\mathcal{X}_s, 2)_\mathbb{Q}$. Consider the morphism of short exact sequences of discrete abelian groups:

$$
\begin{array}{cccc}
0 & \to & \hat{H}^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d))^0 & \xrightarrow{\delta} & \hat{H}^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d))^0 & \to & \mathbb{K}^\times \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^0 & \xrightarrow{\delta} & H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^0 & \to & \mathbb{K}^\times. \\
\end{array}
$$

We see that the kernels of the left two maps are isomorphic, and that the cokernel of the left vertical map is subgroup of the cokernel of the middle vertical map. Hence the long exact sequence associated with the cofiber sequence \[\text{4.1}\]

$$CH_0(\mathcal{X}_s, 2)_\mathbb{Q} \to \hat{H}^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d))^0 \to H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^0 \to CH_0(\mathcal{X}_s, 1)_\mathbb{Q}
$$

shows that the map $\hat{H}^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d))^0 \to H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^0$ has kernel a quotient of $CH_0(\mathcal{X}_s, 2)_\mathbb{Q}$ and torsion free cokernel.

\[\square\]

**Corollary 4.7.** Under the hypothesis of the theorem there is an exact sequence of abelian groups

$$0 \to \mathcal{K} \to H^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d))^0 \to \text{Tor}H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^0 \to 0$$

where $\mathcal{K} \cong \hat{D} \oplus \hat{D}$ is the maximal divisible subgroup of $H^{2d-1}_{et}(\mathcal{X}_K, \mathbb{Z}(d))^0$ and Tor$\mathcal{K}$ is the direct sum of $\text{coker}(H^{2d-2}_{et}(\mathcal{X}_K, \mathbb{Z}(d))) \to \hat{H}^{2d-2}_{et}(\mathcal{X}_K, \mathbb{Z}(d))) \otimes \mathbb{Q}/\mathbb{Z}$ and a torsion quotient of $CH_0(\mathcal{X}_s, 2)_\mathbb{Q}$. 

\[\square\]
Proof. This follows from Theorem 1.4 and Proposition 1.6 because for a composition $g \circ f$ of surjections we have an exact sequence $0 \to \ker f \to \ker(g \circ f) \to \ker g \to 0$, which splits because $D = \ker f$ is divisible. The divisible group $K$ is maximal divisible because $H_{ar}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d))^0$ is finitely generated. \hfill \square

5. Comparison to the motivic reciprocity map

In this section we compare the composition $H_{\mathcal{M}}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d)) \to H_{ar}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d))$ to the classical reciprocity map.

5.1. Kato-homology. Only for this section we are using the work of Jannsen-Saito and Kerz-Saito on Kato-homology in order to be able to make statements about $H_{\mathcal{M}}^{2d-1}(\mathcal{X}_K, \mathbb{Z}(d))$, see [13, 15, and 7].

For $\mathcal{X}$ a separated scheme of finite type over $\mathcal{O}_K$, we define the Kato complex $KC(\mathcal{X})$ to be $\text{cone}(\mathbb{Z}^c(\mathcal{X}) \to R\Gamma_{et}(\mathcal{X}, \mathbb{Z}^c))[-1]$, where $\mathbb{Z}^c = \mathbb{Z}^c(0)$ is Bloch’s cycle complex of cycles of relative dimension 0 viewed as a complex of étale sheaves. For an abelian group $A$ we let $KH_i(\mathcal{X}, A)$ be the homology of $KC(\mathcal{X}) \otimes^L A$ so that there is an exact sequence

$$\cdots \to KH_{i+2}(\mathcal{X}, A) \to CH_i(\mathcal{X}, i, A) \to H^i_{et}(\mathcal{X}, A) \to KH_{i+1}(\mathcal{X}, A) \to \cdots.$$  

By [7, Cor. 5.2, Cor. 5.5, Cor. 7.7], $KC(\mathcal{X}) \otimes^L \mathbb{Z}/m$ is quasi-isomorphic to the Kato-complexes discussed in [13 and 15]. Kerz-Saito proved the following theorem [15, Thm. 8.1], which was conjectured by Kato [13, Conj. 0.3, 5.1],

**Theorem 5.1.** 1) If $Y$ is regular and connected scheme, proper over a finite field, then $KH_i(Y, \mathbb{Z}/m) = 0$ for $i > 0$ and $KH_0(Y, \mathbb{Z}/m) \cong \mathbb{Z}/m$ if either $m$ is prime to $p$, or if $i \leq 4$ and $Y$ is projective.

2) If $\mathcal{X}$ is regular and connected scheme, proper over $\mathcal{O}_K$, then $KH_i(\mathcal{X}, \mathbb{Z}/m) = 0$ for all $i$ if either $m$ prime to $p$, or if $i \leq 4$ and $\mathcal{X}$ is projective.

Since $\mathbb{Z}^c(\mathcal{X}) \otimes \mathbb{Q} \to R\Gamma_{et}(\mathcal{X}, \mathbb{Z}^c) \otimes \mathbb{Q}$ is a quasi-isomorphism, this implies:

**Corollary 5.2.** If $\mathcal{X}$ is a regular and connected scheme, projective over $\mathcal{O}_K$, then $KH_i(\mathcal{X}, \mathbb{Z}) = 0$ for $i \leq 3$. In particular, $H_{\mathcal{M}}^i(\mathcal{X}, \mathbb{Z}(d)) \cong H^i_{et}(\mathcal{X}, \mathbb{Z}(d))$ for $i \geq 2d-1$, and $H_{\mathcal{M}}^{2d-2}(\mathcal{X}, \mathbb{Z}(d)) \to H^i_{et}(\mathcal{X}, \mathbb{Z}(d))$ is surjective.

Now let $\mathcal{X}$ be a regular scheme, proper scheme over $\mathcal{O}_K$ with generic fiber $\mathcal{X}_K$. Note that $KC(\mathcal{X}_K)$ is $\text{cone}(\mathbb{Z}^c(-1)(\mathcal{X}_K) \to R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}^c(-1))[-1])$ because a scheme of dimension $n$ over $\mathcal{O}_K$ has generic fiber of dimension $n-1$ over $K$.

We obtain an exact sequence

$$\cdots \to KH_{i+1}(\mathcal{X}_K, A) \to CH_i(\mathcal{X}_K, i, A) \to H^i_{et}(\mathcal{X}_K, A(-1)) \to KH_i(\mathcal{X}_K, A) \to \cdots$$

or equivalently, for $d = \dim \mathcal{X}$,

$$\cdots \to KH_{i+1}(\mathcal{X}_K, A) \to H^{2d-i}_{\mathcal{M}}(\mathcal{X}_K, A(d)) \to H^{2d-i}_{et}(\mathcal{X}_K, A(d)) \to KH_{i}(\mathcal{X}_K, A) \to \cdots.$$  

From the localization sequences we obtain a long exact sequence

$$\cdots \to KH_{i+1}(\mathcal{X}, A) \to KH_{i}(\mathcal{X}_K, A) \to KH_{i}(\mathcal{X}_s, A) \to KH_{i}(\mathcal{X}, A) \to \cdots$$

and the Corollary implies that for $\mathcal{X}$ a regular and connected scheme, projective over $\mathcal{O}_K$, we have

$$KH_i(\mathcal{X}_K, A) \cong KH_i(\mathcal{X}_s, A).$$
Example 5.3. Let $\mathcal{X}$ be Spec $O_K$. Then the exact sequence for Spec $k$ becomes

$$CH_0(k) \cong H^0_\text{et}(k,\mathbb{Z}) \cong \mathbb{Z}, \quad H^2_\text{et}(k,\mathbb{Z}) \cong KH_0(k,\mathbb{Z}) = 0,$$

and

$$H^2_\text{et}(k,\mathbb{Z}) \cong KH_{-1}(k,\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}.$$  

For $X_K$ we obtain $H^1_\text{et}(K,\mathbb{Z}(1)) \cong H^1_\text{et}(K,\mathbb{Z}(1)) \cong K^\times$, $H^2_\text{et}(K,\mathbb{Z}(1)) \cong H^2_\text{et}(K,\mathbb{Z}(1)) = 0$, and $H^3_\text{et}(K,\mathbb{Z}(1)) \cong KH_{-1}(K,\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$.

5.2. Homology of the closed fiber. For a variety $Y$ over a finite field, the groups $H^i_W(Y,\mathbb{Z}^c) = H^{1-i}R\Gamma_W(Y,\mathbb{Z}^c)$ have been studied in [5], and for proper $Y$ in [10]. They are finitely generated for all $i$ and $Y$ if and only if the groups $CH_0(X, i)$ are torsion for all $i > 0$ and all smooth and proper $X$ [5].

If $Y$ is connected and proper, then the degree map induces an isomorphism $H^1_W(Y,\mathbb{Z}^c) \cong \mathbb{Z}$ [5]. The homology groups $H^i_K(Y, A)$ of the complex $C^K(Y) \otimes A$ defined in [5] Def. 5.1 are isomorphic to $KH_i(Y,\mathbb{Z}/m)$ for $A = \mathbb{Z}/m$.

For curves finite generation is known:

Proposition 5.4. [5] Thm. 6.2, Prop. 6.3] If $C$ is a curve, then $H^0_W(C,\mathbb{Z}^c) \cong \mathbb{Z}^{\pi_0(C)}$, there is a short exact sequence of finitely generated groups

$$0 \to CH_0(C) \to H^1_W(C,\mathbb{Z}^c) \to H^1_K(C,\mathbb{Z}) \to 0,$$

an isomorphism $CH_0(C, 1) \cong H^2_W(C,\mathbb{Z}^c)$ of finitely generated groups, and $H^i_W(C,\mathbb{Z}^c)$ vanishes for $i > 2$. In particular, $R\Gamma_W(C,\mathbb{Z}^c(0))$ is perfect.

We are also using the following finite generation statement for Chow groups:

Proposition 5.5. For any proper scheme $Y$ over a finite field, the group $CH_0(Y)^0 = \ker(CH_0(Y) \to \mathbb{Z}^{\pi_0(Y)})$ is finite.

Proof. For an abstract blow-up diagram

$$
\begin{array}{ccc}
T' & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
T & \longrightarrow & Y
\end{array}
$$

we obtain a map of exact sequences

$$
CH_0(T') \longrightarrow CH_0(T) \oplus CH_0(Y') \longrightarrow CH_0(Y) \longrightarrow 0
$$

Since the vertical maps have finite cokernel, finiteness of the kernel of the right map. Using Chow’s Lemma and then normalizing to obtain $Y'$, we can argue by induction on the dimension of $Y$, to assume that $Y$ is normal and projective, and then that $Y$ is connected. Since the abelianized geometric fundamental group $\pi^b_1(Y)^{geom}$ is finite for normal $Y$, it suffices to show that $CH_0(Y) \to \pi^b_1(Y)^{geom}$ is injective. Now an argument of Colliot-Thélène [11] §9, using [22] to produce hyperplane sections, shows that it suffices to prove this for surfaces, in which case we can use a resolution of singularities $Y' \to Y$ to reduce to the smooth and proper case, which is known by Kato-Saito [13] Thm. 1].

□
5.3. **The reciprocity map** $H^{2d-1}_M(\mathcal{X}_K, \mathbb{Z}(d)) \to H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))$.

**Theorem 5.6.** If $\mathcal{X}_s$ is a strict normal crossing scheme and $R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0))$ is a perfect complex of abelian groups, then the reciprocity map

$$H^{2d-1}_M(\mathcal{X}_K, \mathbb{Z}(d))^0 \to H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^0$$

has finite image, and the prime to $p$ torsion part of the kernel is, up to finite groups, the image of $CH_0(\mathcal{X}_s, 1) \to H^{2d-1}_M(\mathcal{X}, \mathbb{Z}(d))$ tensored with $\mathbb{Q}/\mathbb{Z}'$.

The groups $CH_0(Y, i)$ are expected to be finitely generated for any variety $Y$ over a finite field, see Prop. 5.3 for the case of curves. In this case, we obtain that the prime to $p$ torsion part of the kernel has corank at most the rank of $CH_0(\mathcal{X}_s, i)$.

**Proof.** We have the following commutative diagram:

$$
\begin{array}{cccccc}
CH_0(\mathcal{X}_s, 1) & \xrightarrow{w} & H^{2d-1}_M(\mathcal{X}, \mathbb{Z}(d))^0 & \longrightarrow & H^{2d-1}_M(\mathcal{X}_K, \mathbb{Z}(d))^0 & \longrightarrow & CH_0(\mathcal{X}_s)^0 \\
\downarrow f_1 & & \downarrow s & & \downarrow r & & \downarrow f_0 \\
H^2_W(\mathcal{X}_s, \mathbb{Z}^c) & \longrightarrow & H^{2d-1}_{ar}(\mathcal{X}, \mathbb{Z}(d))^0 & \longrightarrow & H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^0 & \overset{f}{\longrightarrow} & H^W_1(\mathcal{X}_s, \mathbb{Z}^c)^0 \\
\end{array}
$$

The lower row is exact and $f$ has finite cokernel by Proposition 3.13. The group $CH_0(\mathcal{X}_s)^0$ is finite by Proposition 5.5. The group $H^{2d-1}_{ar}(\mathcal{X}, \mathbb{Z}(d))^0$ is finite by Proposition 3.12 and $H^{2d-1}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^0$ is finitely generated of the same rank as that of $H^W_1(\mathcal{X}_s, \mathbb{Z}^c)^0$ by Corollary 3.14. It follows that $r$ has finite image, and that modulo the Serre subcategory of finite group, there is an exact sequence

$$
CH_0(\mathcal{X}_s, 1) \xrightarrow{w} H^{2d-1}_M(\mathcal{X}, \mathbb{Z}(d))^0 \to \ker r \to 0.
$$

The Tor$_i(-, \mathbb{Q}/\mathbb{Z})$ sequence becomes (away from $p$)

$$
\text{Tor}_i H^{2d-1}_M(\mathcal{X}, \mathbb{Z}(d))^0 \to \text{Tor} \ker r \to \text{im} w \otimes \mathbb{Q}/\mathbb{Z} \to H^{2d-1}_M(\mathcal{X}, \mathbb{Z}(d))^0 \otimes \mathbb{Q}/\mathbb{Z}.
$$

By [9] Lemma 4.16, the group $H^i_{et}(\mathcal{X}, \mathbb{Z}(d))$ is finite for all $i$, and using the coefficient sequence this implies that $H^i_{et}(\mathcal{X}, \mathbb{Q}/\mathbb{Z}(d))$ is finite, hence the divisible subgroup $H^i_{et}(\mathcal{X}, \mathbb{Z}(d)) \otimes \mathbb{Q}/\mathbb{Z}$ vanishes and the prime to $p$-torsion of $H^i_{et}(\mathcal{X}, \mathbb{Z}(d))$ is finite. By Cor. 5.2, $H^i_{et}(\mathcal{X}, \mathbb{Z}(d)) \cong H^i_{et}(\mathcal{X}, \mathbb{Z}(d))$ for any $i \geq 2d - 1$, hence the same is true for motivic cohomology. Thus the outer terms in the short exact sequence are finite, and the prime to $p$ torsion of $\ker r$ is isomorphic to the im $w \otimes \mathbb{Q}/\mathbb{Z}$.

6. **Curves**

Suppose that $\mathcal{X}/\mathcal{O}_K$ is a relative curve with good or strictly semi-stable reduction. By Proposition 5.3, the hypothesis of Proposition 4.6 is satisfied and we have $CH_0(\mathcal{X}_s, 2)_\mathbb{Q}$; hence Proposition 4.6 gives an isomorphism

$$
\tilde{H}^3_{et}(\mathcal{X}_K, \mathbb{Z}(2))^0 \cong \text{Tor}H^3_{ar}(\mathcal{X}_K, \mathbb{Z}(2))^0,
$$

and the groups are finite because $H^3_{ar}(\mathcal{X}_K, \mathbb{Z}(2))^0$ is finitely generated. We use this to relate our results to the result of Saito [18].

Theorem 6.1. Let $\mathcal{X}/\mathcal{O}_K$ be a relative curve with has good or strictly semi-stable reduction. Then the reciprocity map
$$\operatorname{rec}: SK_1(\mathcal{X}_K) \to \pi_1^{ab}(\mathcal{X}_K)$$
factors as follows:
$$SK_1(\mathcal{X}_K) \simeq H^3_{et}(\mathcal{X}_K, \mathbb{Z}(2)) \xrightarrow{s} \hat{H}^3_{et}(\mathcal{X}_K, \mathbb{Z}(2)) \xrightarrow{i} H^3_{ar}(\mathcal{X}_K, \mathbb{Z}(2)) \rightarrow \tilde{\pi}_1^{ab}(\mathcal{X}_K)W \to \pi_1^{ab}(\mathcal{X}_K).$$
The map $i$ is injective, $s$ is surjective, and the kernel of $s$ is the maximal divisible subgroup of $SK_1(\mathcal{X}_K)$. Its torsion is isomorphic to
$$\operatorname{coker}(H^2_{et}(\mathcal{X}_K, \mathbb{Z}(2)) \rightarrow \hat{H}^2_{et}(\mathcal{X}_K, \mathbb{Z}(2)))^{\delta} \otimes \mathbb{Q}/\mathbb{Z}.$$  
Proof. We have canonical isomorphisms
$$SK_1(\mathcal{X}_K) \simeq H^3_{et}(\mathcal{X}_K, \mathbb{Z}(2)) \simeq H^3_{et}(\mathcal{X}_K, \mathbb{Z}(2))$$
and the reciprocity map $\operatorname{rec}$ coincides with the composition
$$SK_1(\mathcal{X}_K) \simeq H^3_{et}(\mathcal{X}_K, \mathbb{Z}(2)) \rightarrow H^3_{et}(\mathcal{X}_K, \hat{\mathbb{Z}}(2)) \rightarrow \tilde{\pi}_1^{ab}(\mathcal{X}_K)$$
where the last isomorphism is given by étale duality with finite coefficients. The middle map factors as
$$H^3_{et}(\mathcal{X}_K, \mathbb{Z}(2)) \rightarrow \hat{H}^3_{et}(\mathcal{X}_K, \mathbb{Z}(2)) \rightarrow H^3_{ar}(\mathcal{X}_K, \hat{\mathbb{Z}}(2)) \rightarrow H^3_{et}(\mathcal{X}_K, \hat{\mathbb{Z}}(2))$$
by construction, where the maps are continuous if $H^3_{et}(\mathcal{X}_K, \mathbb{Z}(2))$ is endowed with the discrete topology and both $H^3_{et}(\mathcal{X}_K, \hat{\mathbb{Z}}(2))$ and $\pi_1^{ab}(\mathcal{X}_K)$ are endowed with their natural profinite topology. As in the proof of Theorem 3.8 we have a commutative square
$$\begin{array}{ccc}
H^3_{ar}(\mathcal{X}_K, \mathbb{Z}(2)) & \xrightarrow{\sim} & \tilde{\pi}_1^{ab}(\mathcal{X}_K)W \\
\downarrow & & \downarrow \\
H^3_{et}(\mathcal{X}_K, \hat{\mathbb{Z}}(2)) & \xrightarrow{\sim} & \pi_1^{ab}(\mathcal{X}_K)
\end{array}$$
which we now consider as a commutative square of locally compact groups. The factorization of the reciprocity map claimed in the theorem follows. The surjectivity of $s$ and the description of its kernel is Theorem 4.4. The injectivity of the map $\hat{H}_3^{et}(\mathcal{X}_K, \mathbb{Z}(2)) \rightarrow H^3_{ar}(\mathcal{X}_K, \mathbb{Z}(2))$ follows from [12].

Removing the contribution coming from the base, we can obtain a version with finitely generated groups. Define
$$SK_1(\mathcal{X}_K)^0 := \operatorname{Ker}(SK_1(\mathcal{X}_K) \rightarrow K^\times);$$
and similarly
$$\pi_1^{ab}(\mathcal{X}_K)^{\text{geo}} := \operatorname{Ker}(\pi_1^{ab}(\mathcal{X}_K) \rightarrow G_1^{ab}).$$

Theorem 6.2. Assume that $\mathcal{X}$ is a relative with good or strictly semi-stable reduction. Then the reciprocity map
$$SK_1(\mathcal{X}_K)^0 \to \pi_1^{ab}(\mathcal{X}_K)^{\text{geo}}$$
factors as
\[ SK_1(X_K)^0 \overset{s^0}{\longrightarrow} \tilde{H}^3_{el}(X_K, \mathbb{Z}(2))^0 \overset{\iota^0}{\longrightarrow} H^3_{ar}(X_K, \mathbb{Z}(2)) \overset{\sim}{\longrightarrow} \pi_1^{ab}(X_K)^{geo} \to \pi_1^{ab}(X_K)^{geo} \]
where

- \( H^3_{ar}(X_K, \mathbb{Z}(2)) \) is finitely generated.
- \( s^0 \) is surjective and \( \ker s^0 \) the maximal divisible subgroup of \( SK_1(X_K)^0 \).
- \( \iota^0 \) is the inclusion of the torsion subgroup.
- The last map is the inclusion of a finitely generated group into its profinite completion.

**Proof.** This follows by combining Theorems 4.4, 6.1 and Proposition 4.6. □

**References**

[1] Deligne, P: Théorèmes de finitude en cohomologie l-adique. Cohomologie étale, 233–261, Lecture Notes in Math., 569, Springer, Berlin, 1977.
[2] Flach, M.; Morin, B.: On the Weil-étale Topos of Regular Arithmetic Schemes. Doc. Math. 17 (2012), 313–399.
[3] Flach, M; Morin, B: Weil-étale cohomology and Zeta-values of proper regular arithmetic schemes. Doc. Math. 23 (2018), 1425–1560.
[4] Foreè, P.: The kernel of the reciprocity map of varieties over local fields. J. Reine Angew. Math. 698 (2015), 55–69.
[5] Geisser, T.: Arithmetic homology and an integral version of Kato’s conjecture. J. Reine Angew. Math. 644 (2010), 1–22.
[6] Geisser, T.: Arithmetic cohomology over finite fields and special values of \( \zeta \)-functions. Duke Math. J. 133 (2006), no. 1, 27–57.
[7] Geisser, T.: Duality via cycle complexes. Ann. of Math. (2) 172 (2010), 1095–1126
[8] Geisser, T. H.: Duality of integral étale motivic cohomology. K-Theory–Proceedings of the International Colloquium, Mumbai, 2016, 195-209, Hindustan Book Agency, New Delhi, 2018.
[9] Geisser, T.; Morin, B.: Pontryagin duality for varieties over \( p \)-adic fields. Preprint (2021). arXiv:2108.01849
[10] Geisser, T.; Schmidt, A.: Tame Class Field Theory for Singular Varieties over Finite Fields. Preprint (2017). arXiv:1405.2752
[11] Hoffmann, N.; Spitzweck, M.: Homological algebra with locally compact abelian groups. Adv. Math. 212 (2007), no. 2, 504–524.
[12] Jannsen, U.; Saito, S.: Kato homology of arithmetic schemes and higher class field theory over local fields. Kazuya Kato’s fiftieth birthday. Doc. Math. 2003, Extra Vol., 479–538.
[13] Kato, K.: A Hasse principle for two-dimensional global fields. With an appendix by Jean-Louis Colliot-Thélène. J. Reine Angew. Math. 366 (1986), 142–183.
[14] Kato, K.; Saito, S.: Unramified class field theory of arithmetical surfaces. Ann. of Math. (2) 118 (1983), no. 2, 241–275.
[15] Kerz, K.; Saito, S.: Cohomological Hasse principle and motivic cohomology for arithmetic schemes. Publ. Math. Inst. Hautes ‘Études Sci. 115 (2012), 123–183.
[16] Milne, J. S.: Addendum to “Values of zeta functions of varieties over finite fields” [MR0833860]. Amer. J. Math. 137 (2015), no. 6, 1703–1712.
[17] SGA1: Revêtements étalés et groupe fondamental (SGA 1). Séminaire de géométrie algébrique du Bois Marie 1960–61. Directed by A. Grothendieck. Lecture Notes in Mathematics, Vol. 224. Springer-Verlag, Berlin-New York, 1971.
[18] Saito, S.: Class field theory for curves over local fields. J. Number Theory 21 (1985), no. 1, 44–80.
[19] Saito, S; Sato, K.: A finiteness theorem for zero-cycles over \( p \)-adic fields. With an appendix by Uwe Jannsen. Ann. of Math. (2) 172 (2010), no. 3, 1593–1639.
[20] Sato, K.: *Non-divisible cycles on surfaces over local fields*. J. Number Theory 114 (2005), no. 2, 272–297.

[21] Schneiders, J-P.: *Quasi-abelian categories and sheaves*. Mém. Soc. Math. Fr. (N.S.), 76:1–134, 1999.

[22] Seidenberg, A.: *The hyperplane sections of normal varieties*. Trans. Amer. Math. Soc. 69 (1950), 357–386.

[23] Szamuely, T., *Sur la théorie des corps de classes pour les variétés sur les corps p-adiques*. J. Reine Angew. Math. 525 (2000), 183–212.

[24] Yamazaki, T.: *Class field theory for a product of curves over a local field*. Math. Z. 261 (2009), no. 1, 109–121.

[25] Yoshida, T.: *Finiteness theorems in the class field theory of varieties over local fields*. J. Number Theory 101 (2003), no. 1, 138–150.

Department of Mathematics, Rikkyo University, Ikebukuro, Tokyo, Japan
Email address: geisser@rikkyo.ac.jp

Department of Mathematics, Université de Bordeaux, Bordeaux, France
Email address: Baptiste.Morin@math.u-bordeaux.fr