Front motion for phase transitions in systems with memory

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Abstract

We consider the following partial integro-differential equation (Allen-Cahn equation with memory)

\[ \epsilon^2 \phi_t = \int_0^t a(t - t') \left[ \epsilon^2 \Delta \phi + f(\phi) + \epsilon h(t') \right] dt'. \]

where \(\epsilon\) is a small parameter, \(h\) is a constant, \(f(\phi)\) is minus the derivative of a double well potential and the kernel \(a\) is a piecewise continuous, differentiable at the origin,

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scalar-valued function on $(0, \infty)$. The prototype kernels are exponentially decreasing functions of time and they reduce the integrodifferential equation to a hyperbolic one, the damped Klein-Gordon equation. By means of a formal asymptotic analysis we show that to the leading order and under suitable assumptions on the kernels, the integrodifferential equation behave like a hyperbolic partial differential equation obtained by considering prototype kernels: the evolution of fronts is governed by the extended, damped Born-Infeld equation.

We also apply our method to a system of partial integro-differential equations which generalize the classical phase field equations with a non-conserved order parameter and describe the process of phase transitions where memory effects are present,

\[
\begin{align*}
\epsilon^2 \phi_t &= \int_0^t a_1(t - t') \Delta u(t') dt' \\
\epsilon^2 \phi_t &= \int_0^t a_2(t - t') \left( \epsilon^2 \Delta \phi + f(\phi) + \epsilon u \right)(t') dt',
\end{align*}
\]

where $\epsilon$ is a small parameter. In this case the functions $u$ and $\phi$ represent the temperature field and order parameter respectively. The kernels $a_1$ and $a_2$ are assumed to be similar to $a$. For the phase field equations with memory we obtain the same result as for the generalized Klein-Gordon equation or Allen-Cahn equation with memory.

1 Introduction

In this manuscript we consider the following partial integro-differential equation

\[
\epsilon^2 \phi_t = \int_0^t a(t - t') \left[ \epsilon^2 \Delta \phi + f(\phi) + \epsilon u \right](t') dt',
\]

where $\phi(x, t)$ is a field, called an order parameter, defined in $\Omega \times [0, T]$, where $\Omega \subset \mathbb{R}^2$ and $T > 0$, with Dirichlet and Neumann boundary conditions. The parameter $\epsilon$ is assumed to be small, $\epsilon \ll 1$, $f(\phi)$ is a real odd function with a positive maximum equal to $\phi^*$, a negative minimum equal to $-\phi^*$ and precisely three zeros in the closed interval $[-b, b]$ located at 0.
and ±b, where b is a positive constant. For simplicity we will consider b = 1. The operator Δ is the Laplace operator. The kernel a is assumed to be a piecewise continuous, differentiable at the origin and scalar-valued functions on (0, ∞) satisfying additional conditions to be described later.

Specific cases of equation (I) are \( a(t) = \delta(t) \), the Allen-Cahn equation:

\[
e^2 \phi_t = e^2 \Delta \phi + f(\phi) + \epsilon h, \tag{2}
\]

which is the simplest model of phase transition with a non-conserved order parameter, and the damped Klein-Gordon equation (used, e.g., in the theory of long Josephson junctions):

\[
e^2 \phi_{tt} + \gamma e^2 \phi_t = e^2 \Delta \phi + f(\phi) + \epsilon h, \tag{3}
\]

which is obtained for an exponentially decreasing kernel of type [?]

\[
a(t) = e^{-\gamma t}. \tag{4}
\]

Equation (I) can be thought of as a phenomenological equation based on energetic penalization driving the evolution of the system toward equilibrium states. More specifically, let us call \( F_\epsilon = \int_{R^d} \left[ \frac{1}{2} e^2 (\nabla \phi)^2 - V(\phi) \right] d\bar{x} \), with \( f(\phi) + \epsilon h = dV/d\phi \) the free energy of the system. The functional derivative of the free energy, \( \delta F_\epsilon(\phi)/\delta \phi \) is considered as a generalized force indicative of the tendency of the free energy to decay towards a minimum. The equation is obtained by assuming that the response of \( \phi \) to the tendency of the free energy to decay towards a minimum is given by [?, ?],

\[
e^2 \phi_t = - \int_0^t a(t - t') \frac{\delta F_\epsilon}{\delta \phi} (t') \, dt'.
\]

A system of integro-differential equations closely related to (I), the phase field equations with memory, is [?, ?]
\[
\begin{aligned}
&u_t + \epsilon^2 \phi_t = \int_0^t a_1(t-t') \Delta u(t')dt' \\
&\epsilon^2 \phi_t = \int_0^t a_2(t-t'\ [ \epsilon^2 \Delta \phi + f(\phi) + \epsilon \ u ](t')dt'
\end{aligned}
\]  

\text{(5)}

in \(\Omega \times [0, t]\) where \(\Omega \subset \mathbb{R}^2\) and \(T > 0\) with Dirichlet and Neumann boundary conditions for the temperature field, \(u(x, t)\) and the order parameter, \(\phi(x, t)\), respectively and for \(u\) vanishing initially. The parameter \(\epsilon\) and \(f(\phi)\) are as for (1) and the kernels \(a_1\) and \(a_2\) are similar to \(a\); i.e., piecewise continuous, differentiable at the origin, and scalar-valued functions on \((0, \infty)\).

For \(a_1(t) = a_2(t) = \delta(t)\), system (3) gives rise to the classical phase field equations with a non-conserved order parameter

\[
\begin{aligned}
&u_t + \epsilon^2 \phi_t = \Delta u, \\
&\epsilon^2 \phi_t = \epsilon^2 \Delta \phi + f(\phi) + \epsilon \ u.
\end{aligned}
\]  

\text{(6)}

System (3) also generalizes the hyperbolic phase field equations with a non-conserved order parameter:

\[
\begin{aligned}
&u_{tt} + \epsilon^2 \phi_{tt} + \gamma_1 \ u_t + \epsilon^2 \ \gamma_1 \ \phi_t = \alpha \ \Delta u, \\
&\epsilon^2 \phi_{tt} + \epsilon^2 \ \gamma_2 \ \phi_t = \epsilon^2 \ \Delta \phi + f(\phi) + \epsilon \ u,
\end{aligned}
\]  

\text{(7)}

reducing to them when the kernels \(a_1\) and \(a_2\) are exponentially decreasing functions; i.e.,
\(a_i(t) = \alpha_i \ e^{-\gamma_i t}, \ i = 1, 2,\) with \(\alpha_i\) and \(\gamma_i\) are non-negative constants. System (7) is obtained by differentiating both equations in (3), rearranging terms and rescaling by means of the transformation \(t \rightarrow \alpha_2^{-\frac{1}{2}} t, \ \gamma_2 \rightarrow \alpha_2^{\frac{1}{2}} \gamma_2, \ \gamma_1 \rightarrow \alpha_2^{\frac{1}{2}} \gamma_1\) and \(\alpha := \alpha_1/\alpha_2\).
The phase field equations with memory (5) describe the process of phase transitions where memory effects are present. For a readable description of the classical phase field equations see [?]. The first equation in (5) is based on the balance of heat equation for a non-Fourier process in which the expression for the heat flux is given by a convolution in time between the temperature gradient and the kernel \( a_1 \) [?, ?]. The second equation is obtained in a similar way as equation (1) [?].

In what follows we assume that there exists a solution \( \phi(x,t) \) of (1) or \( \{u(x,t),\phi(x,t)\} \) of (5) defined for all small \( \epsilon \), every \( x \in \Omega \) and every \( t \in [0,T] \) which contains an internal layer. We also assume, for such solutions, that for all small \( \epsilon \geq 0 \) and all \( t \in [0,T] \), the domain \( \Omega \) can be divided into two open regions \( \Omega_+(t,\epsilon) \) and \( \Omega_-(t,\epsilon) \) with a curve \( \Gamma(t,\epsilon) \), separating between them. This interface defined by

\[
\Gamma(t,\epsilon) := \{x \in \Omega : \phi(x,t,\epsilon) = 0\},
\]

(8)
is assumed to be smooth, which implies that its curvature and its velocity are bounded independently of \( \epsilon \). The function \( \phi \) is assumed to vary continuously across the interface, far from the interface tending to 1 when \( x \in \Omega_+(t,\epsilon) \), -1 when \( x \in \Omega_-(t,\epsilon) \), with rapid spatial variation close to the interface. The problem is to derive a closed equation for the evolution of the interface asymptotically valid as \( \epsilon \to 0 \).

The latter problem has been studied formerly by means of formal asymptotic analysis for PDEs (2) and (3) [?, ?, ?], as well as for systems (6) and (7) [?, ?]. For (2) and (3), fronts evolve according to the mean curvature flow equation [?, ?, ?],

\[
v = \kappa,
\]

(9)

where \( v \) is the normal velocity of the interface and \( \kappa \) its curvature. If \( y = S(x,t) \) is the Cartesian description of the interface, equation (3) reads
\[ S_t = \frac{S_{xx}}{1 + S_x^2}. \]

For the undamped Klein-Gordon equation, (3) with \( \gamma = h = 0 \), Neu \[7\] proved that the evolution of the interface is governed by the Born-Infeld equation

\[
(1 - S_t^2) S_{xx} + 2 S_x S_t S_{xt} - (1 + S_x^2) S_{tt} = 0. \tag{10}
\]

For (3) with \( \gamma > 0 \) and for (7) Rotstein et. al. \[7, 8\] showed that fronts move according to an extended version of the Born-Infeld equation given by

\[
(1 - \alpha S_t^2) S_{xx} + 2 \alpha S_x S_t S_{xt} - \alpha (1 + S_x^2) S_{tt} - \gamma S_t (1 + S_x^2 - \alpha S_t^2) - \nu (1 + S_x^2 - \alpha S_t^2)^{\frac{3}{2}} = 0, \tag{11}
\]

where the parameter \( \nu \) (proportional to \( h \) in (3)) is defined later.

In local (geometric) coordinates equation (11) reads

\[
\frac{v_t}{1 - \alpha v^2} + \gamma v = \kappa + \nu (1 - \alpha v^2)^{\frac{1}{2}}. \tag{12}
\]

where \( v, v_t \) and \( \kappa \) are the normal velocity, normal acceleration and curvature of the interface respectively. Equation (11) or (12) have been studied in \[9\], and in \[9\] for \( \gamma = \nu = 0 \).

The case of integrodifferential equations (1) and (5) is still less investigated. For (5) and for kernels which are Laplace transforms of suitable functions satisfying

\[
\int_0^\infty a(t) \, dt < \infty, \quad \int_0^\infty \tilde{a}(s) \, ds < \infty \quad \text{and} \quad \int_0^\infty \tilde{a}(s) \, s \, ds < \infty
\]

where \( \tilde{a}(s) \) is the inverse Laplace transform of \( a(t) \), Rotstein et al. \[8, 9\] showed that the equation governing the motion of the interface is also (11) where in this case
\[
\gamma = -\frac{a_2'(0)}{a_2(0)^2}.
\]

In the present manuscript we show that in the asymptotic limit studied and for a certain class of kernels, which will be made clear below, equation (\[\Box\]) is equivalent to (\[\Box\]) from the point of view of fronts motion; i.e., the evolution of interfaces is described by the same equation, (\[\Box\]) with

\[
\alpha = \left(\int_0^\infty \tilde{\alpha}(\tau) \, d\tau\right)^{-1},
\]

where \(\tilde{\alpha}(\tau)\) is the inverse Laplace transform of \(a(t)\), and

\[
\gamma = \alpha^2 \left(\int_0^\infty \tilde{\alpha}(\tau) \, \tau \, d\tau\right).
\]

We also find the equivalence between (\[\Box\]) and (\[\Box\]) [?] for a more general class of kernels, described below. In Section 2 we describe the class of kernels considered in this work, give some examples and explain our strategy in dealing with them. In Section 3 we derive the equation of motion for (\[\Box\]). We present the derivation treating the equation in Cartesian coordinates and representing the kernel as a Laplace transform of a suitable function. For kernels represented as a Fourier transform of suitable functions the derivation is similar. In our derivation we assume that the interface has no oscillations in the sense that there are no points on the interface for which the velocity vanishes; i.e., the front is an advancing front. This assumption is not necessary for the case of exponentially decreasing kernels; i.e., for (\[\Box\]).

In Section 4 we derive the equation of motion for (\[\Box\]) for kernels which are inverse Fourier transforms of suitable functions. Here we treat the problem in local coordinates.

2 Description of the class of kernels considered

In order to deal with the memory integral in deriving the equations of front motion for either (\[\Box\]) or (\[\Box\]) we represent the kernels \(a(t)\) and \(a_i(t), i = 1, 2\), as Laplace transforms of suitable
functions:

\[ a(t) = \int_0^\infty \bar{\alpha}(\tau) \, e^{-\tau \cdot t} \, d\tau, \quad (13) \]

or

\[ a_i(t) = \int_0^\infty \bar{\alpha}_i(\tau) \, e^{-\tau \cdot t} \, d\tau. \quad (14) \]

for \( i = 1, 2 \). Substituting (13) into (1), rearranging terms and defining

\[ \chi(t; \tau) = \int_0^t e^{-\tau \, (t-t')} \left[ \epsilon^2 \Delta \phi + f(\phi) + \epsilon \, h \right](t') \, dt', \quad (15) \]

where it is understood that \( \chi \) is also a function of the space variable \( x \) and it is defined for all \( x \in \Omega \), we obtain

\[
\begin{cases} \\
\epsilon^2 \phi_t = \int_0^\infty \bar{\alpha}(\tau) \, \chi(t; \tau) \, d\tau, \\
\chi(t; \tau) + \tau \, \chi(t; \tau) = \epsilon^2 \Delta \phi + f(\phi) + \epsilon \, h,
\end{cases}
\]

for all \( \tau \in [0, \infty) \). Note that the second equation in (16) satisfies \( \chi(0, \tau) = 0 \) for all \( x \in \Omega \) and for all \( \tau \in [0, \infty) \). We can see that, in spite of the fact that there is an integral involved in the first equation, that integral sums over the parameter \( \tau \) not involving the time variable \( t \) as in (1). For the phase field equations with memory, substituting (14) into (3), rearranging terms and calling

\[ \chi(t; \tau) = \int_0^t e^{-\tau \, (t-t')} \left[ \epsilon^2 \Delta \phi + f(\phi) + \epsilon \, u \right](t') \, dt', \quad (17) \]

and

\[ v(t; \tau) = \int_0^t e^{-\tau \, (t-t')} \Delta u(t') \, dt', \quad (18) \]
where, again, it is understood that \( \chi \) and \( v \) are also functions of the space variable \( x \) and are defined for all \( x \in \Omega \), we obtain

\[
\begin{align*}
  u_t + \epsilon^2 \phi_t &= \int_0^\infty \alpha_1(\tau) \ v(t; \tau) d\tau, \\
  v_t(t; \tau) + \tau v(t; \tau) &= \Delta u, \\
  \epsilon^2 \phi_t &= \int_0^\infty \bar{\alpha}(\tau) \ \chi(t; \tau) \ d\tau, \\
  \chi_t(t; \tau) + \tau \chi(t; \tau) &= \epsilon^2 \Delta \phi + f(\phi) + \epsilon \ u,
\end{align*}
\]

for all \( \tau \in [0, \infty) \). Note that the second and fourth equations satisfy \( v(0, \tau) = \chi(0, \tau) = 0 \) for all \( x \in \Omega \) and for all \( \tau \in [0, \infty) \).

For an exponentially decreasing kernel (4), \( \bar{\alpha}(\tau) = \delta(\tau - \gamma) \). For a kernel which is a linear combination of kernels of type (4), \( a(t) = \sum_{k=1}^n \alpha_k e^{-\gamma_k t} \), for positive constants \( \alpha_k \) and \( \gamma_k \) \( (k = 1, 2, \ldots, n) \), we have \( \bar{\alpha}(\tau) = \sum_{k=1}^n \alpha_k \delta(\tau - \gamma_k) \). In the latter case we can define for \( k = 1, 2, \ldots, n \)

\[
\chi(t; \gamma_k) = \int_0^t e^{-\gamma_k (t-t')} \left[ \epsilon^2 \Delta \phi + f(\phi) + \epsilon \ h \right](t') \ dt',
\]

and substitute into (4) obtaining

\[
\begin{align*}
  \epsilon^2 \phi_t &= \sum_{k=1}^n \alpha_k \chi(t; \gamma_k), \\
  \chi_t(t; \gamma_k) + \gamma_k \chi(t; \gamma_k) &= \epsilon^2 \Delta \phi + f(\phi) + \epsilon \ h,
\end{align*}
\]

for \( k = 1, 2, \ldots, n \). It is clear that (21) is a particular (discrete) case of (16) and the simplest generalization of (3). This discussion can be easily adapted to system (3).

The approach discussed up to now in this section allows us to consider kernels which are not
necessarily exponentially decreasing. Examples are, for positive $\alpha, \beta$ and $\gamma$: $a(t) = 1/(\gamma t + \alpha)$, for which $\bar{\alpha}(\tau) = 1/\gamma e^{-\frac{\alpha}{\gamma} \tau}$ or, more generally $a(t) = 1/(\gamma t + \alpha)^n$ (with $n$ a positive integer), for which $\bar{\alpha}(\tau) = 1/\gamma e^{-\frac{\alpha}{\gamma} \tau n - \frac{\alpha}{\gamma} \tau} e^{-\frac{\alpha}{\gamma} \tau}$ or, more generally $a(t) = 1/(\gamma t + \alpha)^n$ (with $n$ a positive integer), for which $\bar{\alpha}(\tau) = 1/(\gamma n - n! \gamma) \tau_{n - 1} e^{-\frac{\alpha}{\gamma} \tau}$ and $a(t) = (\gamma t + \alpha)/((\gamma t + \alpha)^2 + \beta^2)$ for which $\bar{\alpha}(\tau) = 1/\gamma^2 e^{-\frac{\alpha}{\gamma} \tau} \sin(\beta \tau)$ and $a(t) = 1/\gamma^2 e^{-\frac{\alpha}{\gamma} \tau} \cos(\beta \tau)$ respectively. A class of kernels not included in (13) or (14) is $a(t) = \alpha e^{-\gamma t \cos(\beta t)}$, or equivalently, $a(t) = \alpha e^{-z_1 t + e^{-z_1 t}}$ where $z_1 = \gamma - i \beta$.

From a slightly different point of view, kernels of type $a(t) = \alpha e^{-\gamma t \cos(\beta t)}$ can be represented as Fourier transforms of suitable functions.

$$a(t) = \int_{-\infty}^{\infty} \bar{\alpha}(w) e^{-iwt} dw,$$

(22) or

$$a_i(t) = \int_{-\infty}^{\infty} \bar{\alpha}_i(w) e^{-iwt} dw.$$  

(23)

for $i = 1, 2$ where

$$\bar{\alpha}(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(t) e^{iwt} dt,$$

and

$$\bar{\alpha}_i(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a_i(t) e^{iwt} dt.$$  

for $i = 1, 2$. In this case the functions $a(t)$ and $a_i(t), i = 1, 2$ are understood to be defined for all $t$ and being equal to zero for $t < 0$. Substituting (22) into (1), rearranging terms and defining

$$\chi(t; w) = \int_{0}^{t} e^{-iwt(t-t')} \left[ e^2 \Delta \phi + f(\phi) + \epsilon h \right](t') dt',$$

(24)

where it is understood that $\chi$ is a function of the space variable $x \in \Omega$, we obtain
\[
\begin{cases}
\epsilon^2 \phi_t = \int_{-\infty}^{\infty} \bar{\alpha}(w) \chi(t; w) \, dw, \\
\chi_t(t; w) + i \, w \, \chi(t; w) = \epsilon^2 \Delta \phi + f(\phi) + \epsilon \, h,
\end{cases}
\]  
(25)

for all \( w \in (-\infty, \infty) \). For the phase field equations with memory, substituting (23) into (5), rearranging terms and calling

\[
\chi(t; w) = \int_0^t e^{-i w (t-t')} \left[ \epsilon^2 \Delta \phi + f(\phi) + \epsilon \, u \right](t') \, dt',
\]  
(26)

and

\[
v(t; w) = \int_0^t e^{-i w (t-t')} \Delta u(t') \, dt',
\]  
(27)

where, again, it is understood that \( \chi \) and \( v \) are functions of the space variable \( x \in \Omega \). System (3) becomes, for all \( w \)

\[
\begin{cases}
u_t + \epsilon^2 \phi_t = \int_{-\infty}^{\infty} \bar{a}(w) \, v(t; w) \, dw, \\
v_t(t; w) + i \, w \, v(t; w) = \Delta u, \\
\epsilon^2 \phi_t = \int_{-\infty}^{\infty} \bar{\alpha}(w) \, \chi(t; w) \, dw, \\
\chi_t(t; w) + i w \chi(t; w) = \epsilon^2 \Delta \phi + f(\phi) + \epsilon \, u,
\end{cases}
\]  
(28)

for all \( w \in (-\infty, \infty) \). As we set at the beginning of the introduction, the kernels \( a(t) \), are assumed to be piecewise continuous, differentiable at the origin, scalar-valued functions on \((0, \infty)\). Moreover they are assumed to be independent of \( \epsilon \) and such that

\[
\int_0^{\infty} a(t) \, dt < \infty,
\]

and
\[
\int_0^\infty \bar{\alpha}(s) \, ds < \infty \quad \text{and} \quad \int_0^\infty \bar{\alpha}(s) \, s \, ds < \infty.
\]
when we use the Laplace transform, and
\[
\int_\infty^{-\infty} \bar{\alpha}(s) \, ds < \infty \quad \text{and} \quad \int_\infty^{-\infty} \bar{\alpha}(s) \, s \, ds < \infty.
\]
when we use the Fourier transform. In both cases this is equivalent to \( a(0) < \infty \) and \( a'(0) < \infty \).

3 The Allen-Cahn equation with memory - asymptotic analysis in Cartesian coordinates

For points outside the interface and for \( \epsilon > 0 \) we expand \( \phi \) as follows
\[
\phi = \phi(x, t; \epsilon) = \phi^0(x, t) + \epsilon \phi^1(x, t) + \epsilon^2 \phi^2(x, t) + \mathcal{O}(\epsilon^3),
\]
and substitute into (1) obtaining the \( \mathcal{O}(1) \) and \( \mathcal{O}(\epsilon) \) respectively
\[
\int_0^\tau a(t - t') f(\phi^0) \, dt' = 0,
\]
whose solution is \( f(\phi^0) = 0 \) (or \( \phi^0 = \pm 1 \)), and
\[
\int_0^\tau a(t - t') \left[ f'(\phi^0) \, \phi^1 + h \right] \, dt' = 0,
\]
giving \( f'(\phi^0) \, \phi^1 + h = 0 \). Thus, for points which have not yet been reached by the moving front, we have
\[
\chi^0(t; \tau) = 0, \quad (29)
\]
and
\( \chi^1(t; \tau) = 0, \quad (30) \)

for all \( \tau \in [0, \infty) \).

For the asymptotic analysis using Cartesian coordinates the interface is represented by \( y = S(x, t, \epsilon) \) for \( \epsilon \) sufficiently small and assume \( S_t \neq 0 \) for all \( x \in \Omega \) and all \( t \geq 0 \). We define a new variable

\[
    z := \frac{y - S(x, t, \epsilon)}{\epsilon}
\]

which is assumed to be \( O(1) \) as \( \epsilon \to 0 \) near the interface. We call \( \Phi \) the asymptotic form of \( \phi \) as \( \epsilon \to 0 \) with \( z \) fixed; i.e.,

\[
    \phi = \Phi(z, x, t; \epsilon). \quad (31)
\]

The field equations (16) in \((z, x, t)\) coordinates, after differentiating the first equation with respect to \( t \) and calling \( \chi = \Upsilon(x, z, t; \tau, \epsilon) \), become (see Appendix A)

\[
\begin{align*}
    &\quad \epsilon^3 \Phi_{tt} - 2 \epsilon^2 \Phi_{zt} + \epsilon S_t^2 \Phi_z - \epsilon^2 S_{tt} \Phi = \int_0^\infty \bar{\alpha}(\tau) \left[ \epsilon \Upsilon_t - S_t \Upsilon_z \right] d\tau; \\
    &\quad \epsilon \Upsilon_t - S_t \Upsilon_z + \epsilon \tau \Upsilon = \epsilon^3 \Phi_{xx} - 2 \epsilon^2 S_x \Phi_{zx} + \epsilon (1 + S_x^2) \Phi_{zz} - \epsilon^2 S_{xx} \Phi_z + \epsilon f(\Phi) + \epsilon h,
\end{align*}
\]

for \( \tau \in [0, \infty) \).

The asymptotic expansion of \( \Phi \) is assumed to have the form

\[
    \Phi \sim \Phi^0 + \epsilon \Phi^1 + \epsilon^2 \Phi^2 + O(\epsilon^3), \quad \text{as} \quad \epsilon \to 0. \quad (33)
\]

Substituting into (32) and equating coefficients of the corresponding powers of \( \epsilon \), we obtain the following problems for \( O(1), O(\epsilon) \) and \( O(\epsilon^2) \) respectively:
\[
\left\{ \begin{array}{l}
f_0^\infty \bar{\alpha}(\tau) \ S_t \ \Upsilon_2^0 \ d\tau = 0, \\
S_t \ \Upsilon_2^0 = 0,
\end{array} \right.
\] (34)

\[
\left\{ \begin{array}{l}
S_t^2 \ \Phi_0^0 = f_0^\infty \bar{\alpha}(\tau) \ [ \ \Upsilon_1^t - S_t \ \Upsilon_z^1 \ d\tau], \\
\Upsilon_0^0 - S_t \ \Upsilon_z^1 + \tau \ \Upsilon_0^0 = (1 + S_x^2) \ \Phi_{zz}^0 + f(\Phi^0),
\end{array} \right.
\] (35)

and

\[
\left\{ \begin{array}{l}
-2 \ S_t \ \Phi_{zt}^0 + S_t^2 \ \Phi_{zz}^1 - S_{tt} \ \Phi_2^0 = f_0^\infty \bar{\alpha}(\tau) \ [ \ \Upsilon_1^t - S_t \ \Upsilon_z^2 \ ] \ d\tau, \\
\Upsilon_1^t - S_t \ \Upsilon_z^2 + \tau \ \Upsilon_1^0 = -2 \ S_x \ \Phi_{zz}^0 + (1 + S_x^2) \ \Phi_{zz}^1 - S_{xx} \ \Phi_z^0 + f'(\Phi^0) \ \Phi^1 + h,
\end{array} \right.
\] (36)

for all \( \tau \in [0, \infty) \). From (29) and (30) we have

\[
\lim_{z \to \infty} \ \Upsilon_0^0(x,z,t;\tau) = 0,
\] (37)

and

\[
\lim_{z \to \infty} \ \Upsilon_1^0(x,z,t;\tau) = 0.
\] (38)

From (34) we have \( \Upsilon_0^0(x,z,t;\tau) = 0 \) for all \( \tau \in [0, \infty) \). Integrating with respect to \( z \) and applying condition (37) (assuming \( S_t \neq 0 \)) yields

\[
\Upsilon_0^0 \equiv 0,
\] (39)

for all \( \tau \in [0, \infty) \). Substituting (39) into (35) the second equation becomes

\[
-S_t \ \Upsilon_z^1 = (1 + S_x^2) \ \Phi_{zz}^0 + f(\Phi^0).
\] (40)
We see that $\Upsilon_z^1$ does not depend on $\tau$. We multiply the second equation in (35) by $\alpha(\tau)$ and integrate with respect to $\tau$. We obtain

\[
S_t^2 \Phi_{zz}^0 = \left[ (1 + S_x^2) \Phi_{zz}^0 + f(\Phi^0) \right] \int_0^\infty \bar{\alpha}(\tau) \, d\tau
\]

since $\Phi$ and $S$ do not depend on $\tau$. We call

\[
\alpha = \left( \int_0^\infty \bar{\alpha}(\tau) \, d\tau \right)^{-1},
\]

obtaining

\[
(1 + S_x^2 - \alpha S_t^2)\Phi_{zz}^0 + f(\Phi^0) = 0.
\]

From (40) we have that $S_t \Upsilon_z^1 = -\alpha S_t^2 \Phi_{zz}^0$. Integrating with respect to $z$ and applying condition (38) we obtain

\[
\Upsilon^1 = -\alpha S_t \Phi_{z}^0.
\]

In order to solve (42) we define a new variable

\[
\xi := \frac{z}{(1 + S_x^2 - \alpha S_t^2)^{\frac{1}{2}}},
\]

In terms of $\xi$, equation (42) reads

\[
\Phi_{\xi\xi}^0 + f(\Phi^0) = 0,
\]

whose solution is $\Phi^0 = \Psi(\xi)$, the unique solution of $\Psi'' + f(\Psi) = 0$, $\Psi(\pm\infty) = \phi^\pm$, $\Psi(0) = 0$. Thus

\[
\Phi^0 = \Phi^0 \left( \frac{z}{(1 + S_x^2 - \alpha S_t^2)^{\frac{1}{2}}} \right).
\]
Multiplying the second equation in (36) by α(τ), integrating with respect to τ, replacing (43) into (36), and rearranging terms we obtain

\[
(1 + S_x^2 - \alpha S_t^2)\Phi_{zz}^1 + f'(\Phi^0)\Phi^1 = (S_{xx} - \alpha S_{tt})\Phi^0_z - 2\alpha S_t \Phi^0_{zt} + \\
+ 2 S_x \Phi^0_{zx} - \alpha^2 S_t \left( \int_0^{\infty} \bar{\alpha}(\tau) \tau \, d\tau \right) \Phi^0_z. \tag{47}
\]

Setting

\[
\gamma = \alpha^2 \left( \int_0^{\infty} \bar{\alpha}(\tau) \tau \, d\tau \right). \tag{48}
\]

In terms of \(\xi\), \(x\) and \(t\) equation (47) reads (see Appendix A)

\[
\Phi_{\xi\xi}^1 + f'(\Phi^0)\Phi^1 = \frac{S_{xx} - \alpha S_{tt} - \gamma S_t}{(1 + S_x^2 - \alpha_S S_t^2)^{\frac{3}{2}}} \Phi_x^0 - \frac{2 S_x (S_{xx} - \alpha S_t S_{tt})}{(1 + S_x^2 - \alpha_S S_t^2)^{\frac{5}{2}}} (\xi \Phi_{\xi\xi}^0 + \Phi_{\xi}^0) + \\
+ \frac{2 \alpha S_t (S_x S_{xt} - \alpha_S S_{tt})}{(1 + S_x^2 - \alpha_S S_t^2)^{\frac{3}{2}}} (\xi \Phi_{\xi\xi}^0 + \Phi_{\xi}^0) - h. \tag{49}
\]

It is straightforward to check that \(\Psi'(\xi)\) satisfies the homogeneous equation

\[
\Phi_{\xi\xi}^1 + f'(\Phi^0)\Phi^1 = 0. \tag{50}
\]

This implies that the operator \(\Lambda\) defined by

\[
\Lambda := \frac{\partial^2}{\partial \xi^2} + f'(\Psi'(\xi)) \tag{51}
\]

has a simple eigenvalue at the origin with \(\Psi'(\xi)\) as the corresponding eigenfunction. Now the solvability condition for equation (49) gives

\[
\frac{S_{xx} - \alpha S_{tt} - \gamma S_t}{(1 + S_x^2 - \alpha_S S_t^2)^{\frac{3}{2}}} \int_{-\infty}^{\infty} (\Psi')^2 d\xi - h \int_{-\infty}^{\infty} \Psi' d\xi + \\
\int_{-\infty}^{\infty} (\Psi')^2 d\xi - h \int_{-\infty}^{\infty} \Psi' d\xi + \\
\int_{-\infty}^{\infty} (\Psi')^2 d\xi - h \int_{-\infty}^{\infty} \Psi' d\xi +
\]

16
\[
- \frac{2}{(1 + S_x^2 - \alpha_2 S_t^2)^{\frac{1}{2}}} \frac{S_x (S_x S_{xx} - \alpha S_t S_{xt}) - 2 \alpha S_t (S_x S_{xt} - \alpha S_t S_{tt})}{(1 + S_x^2 - \alpha_2 S_t^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} (\xi \Psi'' + \Psi') \Psi' d\xi = 0. \tag{52}
\]

A simple calculation shows that

\[
\int_{-\infty}^{\infty} \xi \Psi' \Psi'' d\xi = -\frac{1}{2} \int_{-\infty}^{\infty} (\Psi')^2 d\xi. \tag{53}
\]

Defining

\[
\nu := h \frac{\Psi(+\infty) - \Psi(-\infty)}{\int_{-\infty}^{\infty} (\Psi')^2 d\xi}. \tag{54}
\]

Substituting (53) and (54) into (52), multiplying equation (52) by \((1 + S_x^2 - \alpha_2 S_t^2)^{\frac{1}{2}}\) and rearranging terms we obtain (11). Note that for \(f(\phi) = (\phi - \phi^3)/2\) (Ginzburg-Landau theory), \(\Psi(\xi) = \tanh \frac{\xi}{2}\) and \(\nu := 3 \, h\), whereas for \(f(\phi) = \sin \phi\), \(\Psi(\xi) = 4 \tan^{-1} e^\xi - \pi\) and \(\nu := \frac{\pi}{4} \, h\).

4 The phase field equations with memory - asymptotic analysis in local coordinates

4.1 Assumptions and definitions

For the asymptotic analysis, using local coordinates, we set \(x = (x_1, x_2)\) and call \(d(x)\) the distance from \(x\) to \(\Gamma\); i.e., \(d(x) = \text{dist}(x, \Gamma)\). We next define a local orthogonal coordinate system \((r, s)\) in a neighborhood of \(\Gamma\) in the following way

\[
r(x, t; \epsilon) = \begin{cases} 
    d(x) & \text{if } \phi(x) > 0 \\
    -d(x) & \text{if } \phi(x) < 0,
\end{cases} \tag{55}
\]
and $s(x, t; \epsilon)$, a smooth function of $t$, such that on $\Gamma(t; \epsilon)$ it measures arclength from some point which moves normally to $\Gamma$ as $t$ varies. The assumed initial smoothness of $\Gamma(t; \epsilon)$ implies that $r$ is a smooth function, at least, in a sufficiently small neighborhood of $\Gamma$.

The outer expansions of $\phi$ and $u$ are assumed to have the form

$$
\phi = \phi(x, t; \epsilon) = \phi^0(x, t) + \epsilon \phi^1(x, t) + \epsilon^2 \phi^2(x, t) + \mathcal{O}(\epsilon^3) \quad (56)
$$

and

$$
u = u(x, t; \epsilon) = u^0(x, t) + \epsilon u^1(x, t) + \epsilon^2 u^2(x, t) + \mathcal{O}(\epsilon^3). \quad (57)$$

In order to determine the inner expansions we first define the inner variable

$$z(x, t; \epsilon) := \frac{r(x, t; \epsilon)}{\epsilon} \quad (58)$$

and then assume the inner expansions to be given by

$$\phi = \Phi(z, s, t; \epsilon) = \Phi^0(z, s, t) + \epsilon \Phi^1(z, s, t) + \mathcal{O}(\epsilon^2) \quad (59)$$

and

$$u = U(z, s, t; \epsilon) = U^0(z, s, t) + \epsilon U^1(z, s, t) + \mathcal{O}(\epsilon^2). \quad (60)$$

The very definition of $\Gamma$ requires $\Phi(0, s, t; \epsilon) = 0$. In what follows we will use the following notation to refer to any variable $g$ evaluated by approaching $\Gamma$ from either side ($r > 0$ or $r < 0$):

$$
g |_{r^\pm} = \lim_{r \to 0^\pm} g(r, s, t; \epsilon), \quad (61)$$

$$g_r |_{r^\pm} = \lim_{r \to 0^\pm} g_r(r, s, t; \epsilon). \quad (62)$$
The following relations between the inner and outer variables obtained in [?] are assumed to hold as \( \rho \to \pm \infty \).

\[
G^0(\rho, s, t) = g^0(0^\pm, s, t),
\]
\( G^1(\rho, s, t) = g^1(0^\pm, s, t) + \rho g^0_r(0^\pm, s, t). \) (64)

4.2 Derivation of the equations of motion

Let us look at the system (5) assuming that \( u = 0 \) initially and along the boundary.

4.2.1 Outer problems

Substituting (56) and (57) into (5) and equating coefficients of the corresponding powers of \( \epsilon \) we obtain the \( \mathcal{O}(1) \) and \( \mathcal{O}(\epsilon) \) outer problems respectively for points where the interface has not yet arrived:

\[
\begin{align*}
\begin{cases}
  u_1^0_t = a_1 \Delta u^0, \\
  a_2 \cdot f(\phi^0) = 0,
\end{cases} 
\end{align*}
\] (65)

and

\[
\begin{align*}
\begin{cases}
  u_1^1_t = a_1 \Delta u^1, \\
  a_2 \cdot [f'(\phi^0) \phi^1 + u^0] = 0.
\end{cases} 
\end{align*}
\] (66)

The solution of (65), given the assumed initially and boundary conditions for \( u \), is \( u^0 \equiv 0 \), \( \phi^0 = \pm 1 \). The solution of (66) is \( u^1 \equiv 0 \), \( \phi^1 \equiv 0 \).
4.2.2 Inner problems

From the solution of the outer problems (65) and (66) and for points where the moving front has not yet arrived, we have

\[ \chi^0(t; w) = 0, \quad \text{and} \quad v^0(t; w) = 0, \]  

(67)

and

\[ \chi^1(t; w) = 0, \quad \text{and} \quad v^1(t; w) = 0, \]  

(68)

for all \( x \in \Omega \) and \( w \in (-\infty, \infty) \).

System (28) expressed in the \((z, s, t)\) coordinates (see appendix B), after differentiating the first and third equations with respect to \( t \) and calling \( v = V(z, s, t; \epsilon) \) and \( \chi = \Upsilon(z, s, t; \epsilon) \), becomes

\[
\begin{align*}
\rho_t U_{zz} + \epsilon \left[ 2 \rho_t U_{zt} + \rho_{tt} U_z \right] + \epsilon^2 \left[ U_{tt} + 2 s_t U_{st} + s_t^2 U_{ss} + s_{tt} U_s + 2 r_t s_t U_{zs} + r_t^2 \Phi_{zz} \right] &= \\
&= \int_{-\infty}^{\infty} \tilde{\alpha}_1(w) \left[ \epsilon^2 V_t + \epsilon \rho_t V_z + \epsilon^2 s_t V_s \right] d\omega + O(\epsilon^3),
\end{align*}
\]

\[
\epsilon \rho_t V_z + \epsilon^2 \left[ V_t + s_t V_s + i w V \right] = U_{zz} + \epsilon \Delta r U_z + \epsilon^2 \left[ U_{ss} |\nabla s| + U_s \Delta s \right],
\]

\[
\epsilon \rho_t^2 \Phi_{zz} + \epsilon^2 \left[ 2 \rho_t \Phi_{zt} + \rho_{tt} \Phi_z + 2 \rho_t s_t \Phi_{zs} \right] = \int_{-\infty}^{\infty} \tilde{\alpha}_2(w) \left[ \epsilon \Upsilon_t + \epsilon \rho_t \Upsilon_z + \epsilon s_t \Upsilon_s \right] d\omega + O(\epsilon^3),
\]

\[
\rho_t \Upsilon_z + \epsilon \left[ \Upsilon_t + s_t \Upsilon_s + i w \Upsilon \right] = \epsilon \left[ \Phi_{zz} + f(\Phi) \right] + \epsilon^2 \left[ \Delta r \Phi_z + U \right] + O(\epsilon^3),
\]

(69)

for \( w \in (-\infty, \infty) \). From (67) and (68) we have
\[ \lim_{z \to \infty} \Upsilon^0(x, z, t; w) = 0, \quad \text{and} \quad \lim_{z \to \infty} V^0(x, z, t; w) = 0, \quad (70) \]

and

\[ \lim_{z \to \infty} \Upsilon^1(x, z, t; w) = 0, \quad \text{and} \quad \lim_{z \to \infty} V^1(x, z, t; w) = 0. \quad (71) \]

Substituting (59) and (60) into (69) and equating coefficients of the corresponding powers of \( \epsilon \) we obtain the \( O(1) \), \( O(\epsilon) \) and \( O(\epsilon^2) \) problems respectively.

\[ O(1): \]

\[ \begin{cases} 
  r_t^2 U_{zz}^0 = 0, \\
  U_{zz}^0 = 0, \\
  f_{-\infty}^{\infty} \bar{\alpha}_2(w) r_t \Upsilon^0_z dw = 0, \\
  r_t \Upsilon^0_z = 0, 
\end{cases} \quad (72) \]

\[ O(\epsilon): \]

\[ \begin{cases} 
  2 r_t U_{zt}^0 + r_t^2 U_{zz}^1 r_t U_{zz}^0 = f_{-\infty}^{\infty} \bar{\alpha}_1(w) r_t V_z^0 dw = 0, \\
  r_t V_z^0 = U_{zz}^1 + \Delta r U_{zz}^0, \\
  r_t^2 \Phi_{zz}^0 = f_{-\infty}^{\infty} \bar{\alpha}_2(w) \left[ \Upsilon_t^0 + r_t \Upsilon_z^1 + s_t \Upsilon_s^0 \right] dw, \\
  \Upsilon^0_t + r_t \Upsilon_z^1 + s_t \Upsilon_s^0 + iw \Upsilon^0 = \Phi_{zz}^0 + f(\Phi^0), 
\end{cases} \quad (73) \]
for $w \in (-\infty, \infty)$. We call

$$
\alpha_i = \left( \int_{-\infty}^{\infty} \bar{\alpha}_i(w) \, dw \right)^{-1}, \tag{75}
$$

for $i = 1, 2$. A bounded solution of the first two equations in (72) satisfying the matching conditions (63) is $U^1 \equiv 0$. From the third and fourth equations in (72) we have $\Upsilon^0_0(x, z; t; w) = 0$ for $w \in (-\infty, \infty)$. Integrating with respect to $z$ (assuming that $r_t \neq 0$) and applying condition (70) yields

$$
\Upsilon^0 \equiv 0, \tag{76}
$$

for $w \in (-\infty, \infty)$. Substituting $U^1 \equiv 0$ in the first two equations in (73), multiplying the second and fourth equations in (73) by $\bar{\alpha}_1$ and $\bar{\alpha}_2$ respectively, and integrating with respect to $t$ yields
\[
\left\{ \begin{array}{l}
(\alpha_1 - r_t^2) \, U_{zz}^1 = 0, \\
(1 - \alpha_2 \, r_t^2) \, \Phi_{zz}^0 + f(\phi^0) = 0,
\end{array} \right. 
\] (77)

Assuming that \( r_t^2 \neq \alpha_1 \), the bounded solution of the first equation in (77) satisfying the matching conditions (63) is \( U^1 \equiv 0 \). To solve the second equation in (77), we assume that \( r_t^2 \neq \alpha_2 \) and we define the new variable

\[
\xi := \frac{z}{(1 - \alpha_2 \, r_t^2)^{\frac{1}{2}}}.
\] (78)

In terms of \((\xi, s, t)\), equation the second equation in (77) reads

\[
\Phi_{\xi\xi}^0 + f(\Phi^0) = 0,
\] (79)

whose solution is \( \Phi^0 = \Psi(\xi) \), the unique solution of \( \Psi'' + f(\Psi) = 0 \), \( \Psi(\pm \infty) = \pm 1 \), \( \psi(0) = 0 \). Thus

\[
\Phi^0 = \Phi^0 \left( \frac{z}{(1 - \alpha_2 \, r_t^2)^{\frac{1}{2}}} \right),
\] (80)

which satisfies (63). From the fourth equation in (73) we have \( r_t \, \Upsilon_z^1 = \alpha_2 \, r_t^2 \, \Phi_{zz}^0 \). Integrating with respect to \( z \) and applying condition (71) we get

\[
\Upsilon^1 = -\alpha_2 \, r_t^2 \, \Phi_z^0.
\] (81)

Substituting (80) into the third and fourth equations in (74), multiplying the fourth equation by \( \bar{\alpha}_2 \) and integrating with respect to \( w \) yields

\[
(1 - \alpha_2 \, r_t^2) \, \Phi_{zz}^1 + f'(\Phi^0) \, \Phi^1 =
\]

\[
= (\alpha_2 \, r_{tt} + \gamma_2 \, \alpha_2 \, r_t - \Delta r) \, \Phi_z^0 + +2 \, r_t \, \Phi_{zt}^0,
\] (82)
where

\[ \gamma_2 = i \alpha_2^2 \left( \int_{-\infty}^{\infty} \tilde{\alpha}_2(w) \, w \, dw \right). \]  

Equation (82) expressed in the \((\xi, s, t)\) coordinate system reads (see appendix B)

\[ \Phi_{1\xi \xi} + f'(\Phi^0) \Phi^1 = \]

\[ \frac{2 r_t^2 r_{tt}}{(1 - \alpha_2 r_t^2) \frac{1}{2}} \left( \xi \Phi^0_{\xi \xi} + \Phi_0^0 \right) + \frac{r_{tt} + \gamma_2 r_t - \Delta r}{(1 - \alpha_2 r_t^2) \frac{1}{2}} \Phi^0_\xi. \]  

It is straightforward to check that \(\Psi'(\xi)\) satisfies the homogeneous equation \(\Phi_{1\xi \xi} + f'(\Phi^0) \Phi^1 = 0\). That means that the operator \(\Lambda := \frac{\partial^2}{\partial \xi^2} + f'(\Psi'(\xi))\) has a simple eigenvalue at the origin with \(\Psi'\) as the corresponding eigenfunction. The solvability condition for the equation (84) now gives

\[ \frac{2 r_t^2 r_{tt}}{(1 - \alpha_2 r_t^2) \frac{1}{2}} \int_{-\infty}^{\infty} \left( \xi \Psi'' + \Psi' \right) \Psi' \, d\xi + \]

\[ \frac{r_{tt} + \gamma_2 r_t - \Delta r}{(1 - \alpha_2 r_t^2) \frac{1}{2}} \int_{-\infty}^{\infty} (\Psi')^2 \, d\xi = 0. \]  

A simple calculation shows that \(\int_{-\infty}^{\infty} \xi \Psi' \Psi'' \, d\xi = -\frac{1}{2} \int_{-\infty}^{\infty} (\Psi')^2 \, d\xi\). Hence multiplying equation (85) by \((1 - \alpha_2 r_t^2) \frac{1}{2}\) and rearranging terms one obtains

\[ \frac{r_{tt}}{1 - \alpha_2 r_t^2} + \gamma_2 r_t = \kappa. \]  

Taking into consideration that on the interface \(\Delta r = \kappa\), the curvature of the interface, and that \(r_t = -v\), its normal velocity \([?]\), equation (86) becomes (12).
5 Conclusions

In this paper we showed that to the leading order and for a large class of kernels $a$, $a_1$ and $a_2$ under suitable assumptions on them, the law governing the evolution of interfaces for the integro-differential equation (1) is the same as for the differential equation (3). It is easy to see that $\gamma_2$ is given by

$$\gamma = -\frac{a'(0)}{(a(0))^2}.$$  

For equation (5) the result is similar with $\gamma$ and $a$ changed by $\gamma_2$ and $a_2$. Our derivation is only valid for advancing fronts; i.e., non-oscillating fronts in the sense that there exist points on the interface with vanishing velocity. This restriction prevent us from analyzing system where those oscillations may be relevant. This assumption is not necessary when the kernels are exponentially decreasing [?, ?].

To solve the $O(\epsilon)$ problem we have assumed that $|r_t| \neq \sqrt{\alpha_1}$ ($\alpha_1$ being similar to $\alpha_2$). On the other hand, assuming that $1 - \alpha_2 v^2 > 0$ initially and that there is not change of sign in this direction during the evolution, a fact which is true for the circular case, we can easily see from equation (12) that $|r_t| < \frac{1}{\sqrt{\alpha_2}}$. Therefore, if $\alpha_1 \alpha_2 \geq 1$ the former assumption does not add further restrictions on the interfacial motion.

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A Cartesian coordinates

From Cartesian to $(z,x,t)$ coordinates:
To go from Cartesian to $(z, x, t)$ coordinates we transform derivatives as follows

\[ \phi_t = \Phi_t - \frac{1}{\epsilon} S_t \phi_z, \]

\[ \phi_{tt} = \Phi_{tt} - \frac{1}{\epsilon} 2 S_t \Phi_{zt} + \frac{1}{\epsilon^2} S_t^2 \Phi_{zz} - \frac{1}{\epsilon} S_{tt} \Phi_z, \]

\[ \phi_{xx} = \Phi_{xx} - \frac{1}{\epsilon} 2 S_x \Phi_{zx} + \frac{1}{\epsilon^2} S_x^2 \Phi_{zz} - \frac{1}{\epsilon} S_{xx} \Phi_z, \]

\[ \phi_{yy} = \frac{1}{\epsilon^2} \Phi_{zz}. \]

\[ \phi_{xx} = \Phi_{xx} - \frac{1}{\epsilon} 2 S_x \Phi_{zx} + \frac{1}{\epsilon^2} S_x^2 \Phi_{zz} - \frac{1}{\epsilon} S_{xx} \Phi_z, \]

\[ \phi_{yy} = \frac{1}{\epsilon^2} \Phi_{zz}. \]

\[ \phi_{zxt} = \Phi_{zxt} - \frac{1}{\epsilon} S_t \Phi_{zzt} - \frac{1}{\epsilon} 2 S_x \Phi_{zt} + \frac{1}{\epsilon} 2 S_x S_t \Phi_{zzz} + \]

\[ + \frac{1}{\epsilon^2} 2 S_x S_xt \Phi_{zz} + \frac{1}{\epsilon^2} S_x^2 \Phi_{zzt} - \frac{1}{\epsilon^3} S_x^2 S_t \Phi_{zzz} - \frac{1}{\epsilon} S_{xxx} \Phi_z - \frac{1}{\epsilon} S_{xx} \Phi_{zt} + \frac{1}{\epsilon^2} S_{xx} S_t \Phi_{zz}, \]

\[ \phi_{yyt} = \frac{1}{\epsilon^2} \Phi_{zzt} - \frac{1}{\epsilon^3} S_t \Phi_{zzz}. \]

**From $(z, x, t)$ to $(\xi, x, t)$ coordinates:**

To go from $(z, x, t)$ to $(\xi, x, t)$ coordinates, derivatives are transformed as follows

\[ \xi_z = (1 + S_x^2 - \alpha_2 S_t^2)^{-\frac{3}{4}}, \]

\[ \xi_x = -z(1 + S_x^2 - \alpha_2 S_t^2)^{-\frac{3}{4}} (S_x S_{xx} - \alpha S_t S_{xt}), \]

\[ \xi_t = -z(1 + S_x^2 - \alpha_2 S_t^2)^{-\frac{3}{4}} (S_x S_{xt} - \alpha S_t S_{tt}), \]

\[ \xi_{xx} = -(1 + S_x^2 - \alpha_2 S_t^2)^{-\frac{3}{4}} (S_x S_{xx} - \alpha S_t S_{xt}), \]
\[ \xi_{zt} = -(1 + S_x^2 - \alpha_2 S_t^2)^{-\frac{1}{2}} (S_x S_{xt} - \alpha S_t S_{tt}). \]

\[ \Phi_0^0 = \Phi_0^0 \xi_{\xi z} = \frac{1}{(1 + S_x^2 - \alpha_2 S_t^2)^{\frac{1}{2}}} \Phi_0^0 \]  
(87)

\[ \Phi_0^0 = \Phi_0^0 \xi_{\xi z} \xi_t + \Phi_0^0 \xi_{z t} = -\frac{S_x S_{xt} - \alpha S_t S_{tt}}{(1 + S_x^2 - \alpha_2 S_t^2)^{\frac{1}{2}}} (\xi \Phi_0^0 + \Phi_0^0). \]  
(88)

\[ \Phi_0^0 = \Phi_0^0 \xi_{\xi z} \xi_x + \Phi_0^0 \xi_{z x} = -\frac{S_x S_{xx} - \alpha S_x S_{xt}}{(1 + S_x^2 - \alpha_2 S_t^2)^{\frac{1}{2}}} (\xi \Phi_0^0 + \Phi_0^0) . \]  
(89)

B Local coordinates

B.1 From Cartesian to \((r, s, t)\) and \((z, s, t)\) coordinates

To go from Cartesian to \((r, s, t)\) coordinates we transform derivatives as follows (using the fact that \(|\nabla r| \equiv 1\))

\[ \phi_t = \Phi_t + \Phi_r r_t + \Phi_s s_t, \]

\[ \phi_{tt} = \Phi_{tt} + 2\Phi_r r_t + 2\Phi_s s_t + \Phi_{rr} r_t^2 + \Phi_{ss} s_t^2 + \Phi_r r_{tt} + \Phi_s s_{tt} + 2\Phi_{rs} r_t s_t, \]

and

\[ \Delta \phi = \Phi_{rr} + \Phi_r \Delta r + \Phi_{ss} \nabla s^2 + \Phi_s \Delta s, \]

where the operators \(\nabla\) and \(\Delta\) refers only to the spatial variable \(x\). These derivatives expressed in terms of \((z, s, t)\) are
\[ \phi_t = \Phi_t + \frac{1}{\epsilon} \Phi_z r_t + \Phi_s s_t, \]

\[ \phi_{tt} = \Phi_{tt} + 2 \frac{1}{\epsilon} \Phi_{zt} r_t + 2 \Phi_{st} s_t + \frac{1}{\epsilon^2} \Phi_{zz} r_t^2 + \Phi_{ss} s_t^2 + \frac{1}{\epsilon} \Phi_z r_{tt} + \Phi_s s_{tt} + 2 \frac{1}{\epsilon} \Phi_{zs} r_t s_t, \]

and

\[ \Delta \phi = \frac{1}{\epsilon^2} \Phi_{zz} + \frac{1}{\epsilon} \Phi_z \Delta r + \Phi_{ss} |\nabla s|^2 + \Phi_s \Delta s, \]

**B.2 From \((z, s, t)\) to \((\xi, s, t)\) coordinates**

To go from \((z, s, t)\) to \((\xi, s, t)\) coordinates, derivatives are transformed as follows

\[ \xi_z = (1 - \alpha r_t^2)^{-\frac{1}{2}}, \]

\[ \xi_t = z (1 - \alpha r_t^2)^{-\frac{3}{2}} \alpha r_t r_{tt}, \]

\[ \xi_{zt} = (1 - \alpha r_t^2)^{-\frac{3}{2}} \alpha r_t r_{tt}. \]

\[ \Phi_0^0 = \Phi_0^0 \xi_\xi z = \frac{1}{(1 - \alpha r_t^2)^{\frac{1}{2}} \Phi_0^0 \xi.} \] (90)

\[ \Phi_0^0 = \Phi_0^0 \xi_\xi z = \frac{\alpha r_t r_{tt}}{(1 - \alpha r_t^2)^{\frac{3}{2}}} (\xi \Phi_0^0 \xi + \Phi_0^0). \] (91)