QUANTUM SUPERGROUPS I. FOUNDATIONS

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Abstract. In this part one of a series of papers, we introduce a new version of quantum covering and super groups with no isotropic odd simple root, which is suitable for the study of integrable modules, integral forms and the bar involution. A quantum covering group involves parameters \( q \) and \( \pi \) with \( \pi^2 = 1 \), and it specializes at \( \pi = -1 \) to a quantum supergroup. Following Lusztig, we formulate and establish various structural results of the quantum covering groups, including a bilinear form, quasi-\( R \)-matrix, Casimir element, character formulas for integrable modules, and higher Serre relations.

CONTENTS

Introduction 1
1. The algebra \( f^3 \) 12
2. The quantum covering and super groups 21
3. The quasi-\( R \)-matrix and the quantum Casimir 27
4. Higher Serre relations 30
References 30

Introduction

Quantum groups have been ubiquitous in Lie theory, mathematical physics, algebraic combinatorics and low-dimensional topology since their introduction by Drinfeld and Jimbo [Dr, Jim]. We refer to the books of Lusztig and Jantzen [Lu, Jan] for a systematic development of the structure and representation theory of quantum groups.

In a recent paper [HW] by two of the authors, the spin nilHecke and quiver Hecke algebras (see Wang [Wa], Kang-Kashiwara-Tsuchioka [KKT], Ellis-Khovanov-Lauda [EKL]) were shown to provide a categorification of quantum covering groups with a quantum parameter \( q \) and a second parameter \( \pi \) satisfying \( \pi^2 = 1 \) (we refer to loc. cit. for more references on categorification); a quantum covering group specializes at \( \pi = -1 \) to half of a quantum supergroup with no isotropic odd simple roots, and to half of the Drinfeld-Jimbo quantum group at \( \pi = 1 \).

In the rank one case, a version of the full quantum covering and super group for \( \mathfrak{osp}(1|2) \) suitable for constructing an integral form, as well as integrable modules over \( \mathbb{Q}(q) \) corresponding to each nonnegative integer, was formulated by two of the authors [CW]. In particular, the structure and representation theories of quantum \( \mathfrak{sl}(2) \) and quantum \( \mathfrak{osp}(1|2) \) were shown to be in a complete agreement, also see [Zou] (in contrast to the classical fact that there are “fewer” integrable modules for \( \mathfrak{osp}(1|2) \) than for \( \mathfrak{sl}(2) \)).

The goal of this paper is to lay the foundations of quantum covering and super groups with no isotropic odd simple roots, following Lusztig [Lu Part I] as a blueprint. We define a
new version of quantum covering and super groups with no isotropic odd simple root, which is suitable for the study of integrable modules for all possible dominant integral weights, exactly as for the Drinfeld-Jimbo quantum groups. We formulate and establish various structural results of the quantum covering and super groups, including a bilinear form, twisted derivations, integral forms, bar-involution, quasi-\( \mathcal{R} \)-matrix, Casimir, characters for integrable modules, and quantum (higher) Serre relations.

The results of this paper on quantum covering groups reduce to Lusztig's quantum group setting [Lu] when specializing the parameter \( \pi \) to 1, and on the other hand, reduce to quantum supergroup setting when specializing the parameter \( \pi \) to \(-1\). For this reason, we work almost exclusively with quantum covering groups. Even if one is mainly interested in the super case, writing \( \pi \) systematically for the super sign \(-1\) offers a conceptual explanation for various formulas and constructions. For earlier definitions of quantum supergroups, we refer to Yamane [Ya], Musson-Zou [MZ], Benkart-Kang-Melville [BKM].

Let us describe the main results in detail. As in [Kac], a super Cartan datum is a Cartan datum \((I, \cdot)\) with a partition \( I = I_0 \sqcup I_1 \) subject to some natural conditions; also see [HW]. Note the only finite type super Cartan datum is of type \( B(0, n) \), for \( n \geq 1 \). In Section 1, we formulate the definition of half a quantum covering group associated to a super Cartan datum. We develop the properties of a bilinear form (and a dual version) and twisted derivations on half the quantum covering group systematically. Then we provide a new proof using twisted derivations of a theorem in [HW] (also cf. Yamane [Ya] and Geer [Gr]) that the existence of a non-degenerate bilinear form implies the quantum Serre relations.

Motivated by the rank one construction in [CW], we formulate in Section 2 a new version of quantum super and covering groups with generators \( E_i, F_i, K_\mu \), and additional generators \( J_\mu \), for \( i \in I \) and \( \mu \in Y \) (the co-weight lattice). The new generators \( J_i \) play a crucial role in formulating the notion of integrable modules of a quantum supergroup for all dominant integral weights. A study of all such representations was not possible before (cf. [Kac, BKM]).

In Section 3, we formulate the quasi-\( \mathcal{R} \)-matrix for quantum covering or super groups and establish its basic properties. This generalizes the construction in the rank one case in [CW]. Then we construct the quantum Casimir and use it to prove the complete reducibility of the integrable modules. We show that the simple integrable modules are parametrized by \( \pi = \pm 1 \) and the dominant integral weights (in contrast to [BKM, Kac]), and their character formulas coincide with their counterpart for quantum groups (which was established by Lusztig [Lu1]).

This character formula (in case \( \pi = -1 \)) is shown to hold for the irreducible integrable modules under some “evenness” restrictions on highest weights as in [BKM] (where a definition of quantum supergroups without operators \( J_i \) was used), deforming the construction in [Kac].

The higher Serre relations for quantum covering groups are then established in Section 4.

This paper lays the foundation for further studies of quantum covering and super groups. In a sequel [CHW], we will construct the canonical basis, à la Lusztig and Kashiwara, of quantum covering groups and of integrable modules. In yet another paper, a braid group action on a quantum covering group and its integrable modules will be studied in depth.

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1. The algebra $f$

In this section, starting with the super Cartan datum and root datum, we formulate half a quantum covering group $f$ in terms of a bilinear form on a free superalgebra $\tilde{f}$, and show that the $(q, \pi)$-Serre relations are satisfied in $f$.

1.1. Super Cartan datum. A Cartan datum is a pair $(I, \cdot)$ consisting of a finite set $I$ and a symmetric bilinear form $\nu, \nu' \mapsto \nu \cdot \nu'$ on the free abelian group $\mathbb{Z}[I]$ with values in $\mathbb{Z}$ satisfying

(a) $d_i = \frac{i \cdot i}{2} \in \mathbb{Z}_{>0}$;
(b) $2\frac{i \cdot j}{i \cdot i} \in -\mathbb{N}$ for $i \neq j$ in $I$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

If the datum can be decomposed as $I = I_0 \coprod I_1$ such that

(c) $I_1 \neq \emptyset$,
(d) $2\frac{i \cdot j}{i \cdot i} \in 2\mathbb{Z}$ if $i \in I_1$,

then it is called a super Cartan datum.

The $i \in I_0$ are called even, $i \in I_1$ are called odd. We define a parity function $p : I \rightarrow \{0, 1\}$ so that $i \in I_{p(i)}$. We extend this function to the homomorphism $p : \mathbb{Z}[I] \rightarrow \mathbb{Z}$. Then $p$ induces a $\mathbb{Z}_2$-grading on $\mathbb{Z}[I]$ which we shall call the parity grading. We define the height of $\nu = \sum_{i \in I} \nu_i \in \mathbb{Z}[I]$ by $ht(\nu) = \sum \nu_i$. (Note we use different notation than [Lu], where the same quantity is denoted by $\text{tr}(\nu)$.)

A super Cartan datum $(I, \cdot)$ is said to be of finite (resp. affine) type exactly when $(I, \cdot)$ is of finite (resp. affine) type as a Cartan datum (cf. [Lu, §2.1.3]). In particular, from (a) and (d) we see that the only super Cartan datum of finite type is the one corresponding to the Lie superalgebras of type $B(0, n)$ for $n \geq 1$.

A super Cartan datum is called bar-consistent or simply consistent if it satisfies

(e) $d_i \equiv p(i) \mod 2$, $\forall i \in I$.

We note that (e) is almost always satisfied for super Cartan data of finite or affine type (with one exception). A super Cartan datum is not assumed to be (bar-)consistent unless specified explicitly below. (Roughly speaking, the “bar-consistent” condition is imposed whenever a bar involution is involved later on.)

Note that (d) and (e) imply that

(f) $i \cdot j \in 2\mathbb{Z}$ for all $i, j \in I$.

1.2. Root datum. A root datum associated to a super Cartan datum $(I, \cdot)$ consists of

(a) two finitely generated free abelian groups $Y, X$ and a perfect bilinear pairing $\langle \cdot, \cdot \rangle : Y \times X \rightarrow \mathbb{Z}$;
(b) an embedding $I \subset X$ ($i \mapsto i'$) and an embedding $I \subset Y$ ($i \mapsto i$) satisfying
(c) $\langle i, j' \rangle = \frac{2i \cdot j}{i \cdot i'}$ for all $i, j \in I$.

We will always assume that the image of the imbedding $I \subset X$ (respectively, the image of the imbedding $I \subset Y$) is linearly independent in $X$ (respectively, in $Y$).

Let $X^+ = \{\lambda \in X \mid \langle i, \lambda \rangle \in \mathbb{N} \text{ for all } i \in I\}$. Note that there are no additional “evenness” assumptions for $X^+$.

Let $\pi$ be a parameter such that

$$\pi^2 = 1.$$
For any \( i \in I \), we set
\[
q_i = q^{i/2}, \quad \pi_i = \pi^{p(i)}.
\]
Note that when the datum is consistent, \( \pi_i = \pi^{\frac{m}{2}} \); by induction, we therefore have \( \pi^{p(\nu)} = \pi^{\nu \cdot \nu/2} \) for \( \nu \in \mathbb{Z}[I] \). We extend this notation so that if \( \nu = \sum \nu_i i \in \mathbb{Z}[I] \), then
\[
q_\nu = \prod_i q_i^{\nu_i}, \quad \pi_\nu = \prod_i \pi_i^{\nu_i}.
\]
For any ring \( R \) we define a new ring \( R^n = R[\pi/((\pi^2 - 1)] \) (with \( \pi \) commuting with \( R \)). We shall need \( Q(q)^n \) below.

1.3. Braid group and Weyl group. Assume a Cartan (super) datum \((I, \cdot)\) is given. For \( i \neq j \in I \) such that \( \langle i, j' \rangle > 0 \), we define an integer \( m_{ij} \in \mathbb{Z}_{\geq 2} \) by \( \cos^2 \frac{\pi}{m_{ij}} = \frac{1}{4} \langle i, j' \rangle \langle j, i' \rangle \) if it exists, and set \( m_{ij} = \infty \) otherwise. We have
\[
\begin{array}{c|cccccc}
(\langle i, j' \rangle \langle j, i' \rangle) & 0 & 1 & 2 & 3 & \geq 4 \\
m_{ij} & 2 & 3 & 4 & 6 & \infty
\end{array}
\]

The braid group (associated to \( I \)) is the group generated by \( s_i \) \((i \in I)\) subject to the relations (whenever \( m_{ij} < \infty \)):
\[
\underbrace{s_is_js_i \cdots}_{m_{ij}} = \underbrace{s_js_is_j \cdots}_{m_{ij}}.
\] (1.1)

The Weyl group \( W \) is defined to be the group generated by \( s_i \) \((i \in I)\) subject to relations (1.1) and additional relations \( s_i^2 = 1 \) for all \( i \).

For \( i \in I \), we let \( s_i \) act on \( X \) (resp. \( Y \)) as follows: for \( \lambda \in X, \lambda' \in Y \),
\[
s_i(\lambda) = \lambda - \langle i, \lambda \rangle i', \quad s_i(\lambda') = \lambda' - \langle \lambda', i' \rangle i.
\]
This defines actions of the Weyl group \( W \) on \( X \) and \( Y \).

1.4. The algebras \( t' \) and \( t \). Define \( t' \) to be the free associative \( Q(q)^n \)-superalgebra with 1 and with even generators \( \theta_i \) for \( i \in I_T \) and odd generators \( \theta_i \) for \( i \in I_T \). We abuse notation and define the parity grading on \( t' \) by \( p(\theta_i) = p(i) \). We also have a weight grading \(| \cdot |\) on \( t' \) defined by setting \(|\theta_i| = i\).

The tensor product \( t' \otimes t' \) as a \( Q(q)^n \)-superalgebra has the multiplication
\[
(x_1 \otimes x_2)(x_1' \otimes x_2') = q^{\langle x_2 | x_1' \rangle - p(x_2)p(x_1')} x_1 x_1' \otimes x_2 x_2'.
\]

Here and below, in all displayed formulas, we will implicitly assume the elements involved are \( \mathbb{N}[I] \times \mathbb{Z}_2 \)-homogeneous.

There is a similar multiplication formula in \( t' \otimes t' \otimes t' \):
\[
(x_1 \otimes x_2 \otimes x_3)(x_1' \otimes x_2' \otimes x_3')
= q^{\langle x_2 | x_1' + | x_3 | x_2' \rangle} \pi p(x_2)p(x_1') + p(x_3)p(x_2') + p(x_3)p(x_1') x_1 x_1' \otimes x_2 x_2' \otimes x_3 x_3'.
\]

We will take \( r : t' \rightarrow t' \otimes t' \) to be an algebra homomorphism such that \( r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i \) for all \( i \in I \). One checks the following co-associativity holds:
\[
(r \otimes 1)r = (1 \otimes r)r : t' \rightarrow t' \otimes t' \otimes t'; \text{ this is an algebra homomorphism.}
\]

**Proposition 1.4.1.** There exists a unique bilinear form \( (\cdot, \cdot) \) on \( t' \) with values in \( Q \) such that \((1,1) = 1 \) and
\[
\begin{align*}
(a) \quad & (\theta_i, \theta_j) = \delta_{ij}(1 - \pi_i q_i^{-2})^{-1} \quad (\forall i, j \in I); \\
(b) \quad & (\theta_i, t') = (t', \theta_i) = 0, \quad (t', t') = 0;
\end{align*}
\]
(b) \((x, y'y'') = (r(x), y' \otimes y'')\) \((\forall x, y', y'' \in \mathcal{F})\);

(c) \((xx', y'') = (x \otimes x', r(y''))\) \((\forall x, x', y'' \in \mathcal{F})\).

Moreover, this bilinear form is symmetric.

Here, the induced bilinear form \((\mathcal{F} \otimes \mathcal{F}) \times (\mathcal{F} \otimes \mathcal{F}) \rightarrow \mathbb{Q}(q)\) is given by

\[
(x_1 \otimes x_2, x'_1 \otimes x'_2) := (x_1, x'_1)(x_2, x'_2),
\]

for homogeneous \(x_1, x_2, x'_1, x'_2 \in \mathcal{F}\).

This is basically [HIV] Proposition 3.3], where \((\theta_i, \theta_j) = \delta_{ij}(1 - \pi_i q_i^2)^{-1}\) was imposed (note a different sign on the exponent for \(q_i^2\)). These two cases do not exactly match under the bar-involution (which sends \(q \mapsto \pi q^{-1}\)), and so we redo a careful proof here.

**Proof.** We follow [L1] 1.2.3] to define an associative algebra structure on \(\mathcal{F}^* := \oplus \nu \mathcal{F}^*_\nu\) by transposing the “coproduct” \(r : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}\). In particular, for \(g, h \in \mathcal{F}^*\), we define \(gh(x) := (g \otimes h)(r(x))\), where \((g \otimes h)(y \otimes z) = g(y)h(z)\).

Let \(\xi_i \in \mathcal{F}^*_i\) be defined by \(\xi_i(\theta_i) = (1 - \pi_i q_i^{-2})^{-1}\). Let \(\phi : \mathcal{F} \rightarrow \mathcal{F}^*\) be the unique algebra homomorphism such that \(\phi(\theta_i) = \xi_i\) for all \(i\). The map \(\phi\) preserves the \(\mathbb{N}[I] \times \mathbb{Z}_2\)-grading.

Define \((x, y) = \phi(y)(x), \) for \(x, y \in \mathcal{F}\). The properties (a) and (b) follow directly from the definition.

Clearly \((x, y) = 0\) unless (homogeneous) \(x, y\) have the same weight in \(\mathbb{N}[I]\) and the same parity. All elements involved below will be assumed to be homogeneous.

It remains to prove (c). Assume that (c) is known for \(y''\) replaced by \(y\) or \(y'\) and for any \(x, x'\). We then prove that (c) holds for \(y'' = yy'\). Write

\[
\begin{align*}
\quad r(x) &= \sum x_1 \otimes x_2, \quad r(x') = \sum x'_1 \otimes x'_2, \\
\quad r(y) &= \sum y_1 \otimes y_2, \quad r(y') = \sum y'_1 \otimes y'_2.
\end{align*}
\]

Then

\[
\begin{align*}
\quad r(xx') &= \sum q^{[x_2]}(1-p(x_2))p(x'_1)x_1x'_1 \otimes x_2x'_2, \\
\quad r(yy') &= \sum q^{[y_2]}(1-p(y_2))p(y'_1)y_1y'_1 \otimes y_2y'_2.
\end{align*}
\]

We have

\[
\begin{align*}
\quad (xx', yy') &= (\phi(y)\phi(y'))(xx') = (\phi(y) \otimes \phi(y'))(r(xx')) \\
\quad &= \sum q^{[x_2]}(1-p(x_2))p(x'_1)(x_1x'_1, y)(x_2x'_2, y) \\
\quad &= \sum q^{[x_2]}(1-p(x_2))p(x'_1)(x_1 \otimes x'_1, r(y))(x_2 \otimes x'_2, r(y')) \\
\quad &= \sum q^{[x_2]}(1-p(x_2))p(x'_1)(x_1, y_1)(x'_1, y_2)(x_2, y'_1)(x'_2, y'_2).
\end{align*}
\]

On the other hand,

\[
\begin{align*}
\quad (x \otimes x', r(yy')) &= \sum q^{[y_2]}(1-p(y_2))p(y'_1)(x \otimes x', y_1 y'_1 \otimes y_2 y'_2) \\
\quad &= \sum q^{[y_2]}(1-p(y_2))p(y'_1)(x, y_1 y'_1)(x', y_2 y'_2) \\
\quad &= \sum q^{[y_2]}(1-p(y_2))p(y'_1)(r(x), y_1 \otimes y'_1)(r(x'), y_2 \otimes y'_2) \\
\quad &= \sum q^{[y_2]}(1-p(y_2))p(y'_1)(x_1, y_1)(x'_1, y_2)(x_2, y'_1)(x'_2, y'_2).
\end{align*}
\]
For a summand to make nonzero contribution, we may assume that each of the four pairs 
\{x_1, y_1\}, \{x_1', y_1'\}, \{x_2, y_2\}, \{x_2', y_2'\} have the same weight in \(\mathbb{N}[I]\) and the same parity. One checks that the powers of \(q\) and \(\pi\) in (1.3) and (1.4) match perfectly. Hence the two sums in (1.3) and (1.4) are equal, and whence (c). \(\square\)

We set \(J\) to denote the radical of \((\cdot, \cdot)\). As in [HW], this radical is a 2-sided ideal of \(\mathfrak{f}\).

Let \(\mathfrak{f} = \mathfrak{f}/J\) be the quotient algebra of \(\mathfrak{f}\) by its radical. Since the different weight spaces

are orthogonal with respect to this inner product, the weight space decomposition descends to a decomposition \(\mathfrak{f} = \bigoplus_{\nu} \mathfrak{f}_\nu\) where \(\mathfrak{f}_\nu\) is the image of \(\mathfrak{f}_\nu\). Each weight space is finite dimensional. The bilinear form descends to a bilinear form on \(\mathfrak{f}\) which is non-degenerate on each weight space.

Note that the notation of \(\mathfrak{f}\) and \(\mathfrak{f}\) in this paper corresponds to the notation of \(\mathfrak{f}^\pi\) and \(\mathfrak{f}^\nu\) in [HW].

The map \(r : \mathfrak{f} \to \mathfrak{f} \otimes \mathfrak{f}\) satisfies \(r(J) \subset J \otimes \mathfrak{f} + \mathfrak{f} \otimes J\) (the proof being entirely the same

as in [HW] §1.2.6), whence it descends to a well-defined homomorphism \(r : \mathfrak{f} \to \mathfrak{f} \otimes \mathfrak{f}\).

Let \(r : \mathfrak{f} \to \mathfrak{f} \otimes \mathfrak{f}\) be the composition of \(r\) with the permutation map

\[ x \otimes y \mapsto y \otimes x \]

of \(\mathfrak{f} \otimes \mathfrak{f}\) to itself. (To have the signs work out below, the tensor permutation cannot be signed.)

The anti-involution \(\sigma : \mathfrak{f} \to \mathfrak{f}\) satisfies \(\sigma(\theta_i) = \theta_i\) for each \(i \in I\) and

\[ \sigma(xy) = \sigma(y)\sigma(x). \]

**Lemma 1.4.2.**

(a) We have \(r(\sigma(x)) = (\sigma \otimes \sigma)^r(x)\), for all \(x \in \mathfrak{f}\).

(b) We have \((\sigma(x), \sigma(x')) = (x, x')\) for all \(x, x' \in \mathfrak{f}\).

**Proof.** Since (b) will follow immediately from (a), it suffices to prove that \(r(\sigma(x)) = (\sigma \otimes \sigma)^r(x)\), for all \(x \in \mathfrak{f}\). This is obviously true for \(x \in \{\theta_i : i \in I\}\).

Suppose that \(r(\sigma(x')) = (\sigma \otimes \sigma)^r(x')\) and \(r(\sigma(x'')) = (\sigma \otimes \sigma)^r(x'')\). Let \(x'' = \sum x_1' \otimes x_2'\) and \(x'' = \sum x''_1 \otimes x''_2\). Then \(r(x'x'') = \sum q^{\lvert x_1' \lvert \lvert x_2' \rvert} \pi^p(x_1') x_1' x_2' \otimes x_1' x_2'_i\) and we have

\[ r(\sigma(x'x'')) = r(\sigma(x'))r(\sigma(x'')) \]

\[ = \left( \sum \sigma(x'') \otimes \sigma(x''_1) \right) \left( \sum \sigma(x'') \otimes \sigma(x''_1) \right) \]

\[ = \sum \pi^p(x''_1) q^{\lvert x_1' \lvert \lvert x_2' \rvert} \sigma(x''_1) \otimes \sigma(x''_1) = \sigma \otimes \sigma(\mathfrak{f} x''_i). \]

The lemma is proved. \(\square\)

We note that \(\sigma\) descends to \(\mathfrak{f}\) and shares the above properties.

Let \(\overline{\pi} : \mathbb{Q}(q)^\pi \to \mathbb{Q}(q)^\pi\) be the unique \(\mathbb{Q}\)-algebra involution (called the bar involution) satisfying \(\overline{\pi} = \pi q^{-1}\) and \(\overline{\pi} = \pi\).

Assume the super Cartan datum is consistent. Then

\[ \overline{\theta}_i = \pi_i q_i^{-1}. \]

We define a bar involution \(\overline{\pi} : \mathfrak{f} \to \mathfrak{f}\) such that \(\overline{\theta_i} = \theta_i\) for all \(i \in I\) and \(\overline{f}x = \overline{\pi}x\) for \(f \in \mathbb{Q}(q)^\pi\) and \(x \in \mathfrak{f}\).

Let \(\mathfrak{f} \otimes \mathfrak{f}\) be the \(\mathbb{Q}(q)^\pi\)-vector space \(\mathfrak{f} \otimes \mathfrak{f}\) with multiplication given by

\[ (x_1 \otimes x_2)(x_1' \otimes x_2') = (\pi q^{-1})^{\lvert x_2 \lvert \lvert x_1' \rvert} \pi^p(x_1') x_1' x_2' \otimes x_2 x_1'. \]
Define \( \tau \) still by \( \tau(x) = \overline{r(\omega x)} \). Then \( \tau : \mathfrak{f} \to \mathfrak{f} \otimes \mathfrak{f} \) is an algebra homomorphism, being a composition of homomorphisms.

The co-associativity holds for \( \tau \):

\[
(\tau \otimes 1)(\tau(x)) = (r \otimes 1)r(\tau(x)) = (1 \otimes r)r(\tau(x)) = (1 \otimes \tau)(\tau(x)).
\]

By checking on the algebra generators \( \theta_i \), it is an easy computation to see that this is an algebra homomorphism.

Let \( \{ \cdot, \cdot \} : \mathfrak{f} \times \mathfrak{f} \to \mathbb{Q}(q) \) be the symmetric bilinear form defined by

\[
\{ x, y \} = \overline{\langle x, y \rangle}.
\]

It satisfies: \( \{ 1, 1 \} = 1 \), and

\[
\{ \theta_i, \theta_j \} = \delta_{i,j}(1 - \pi_i q_j^2)^{-1};
\]

\[
\{ x, y'y'' \} = \{ \tau(x), y' \otimes y'' \}, \text{ for all } x, y', y'' \in \mathfrak{f}.
\]

**Lemma 1.4.3.** Assume the super Cartan datum is consistent.

(a) Let \( r(x) = \sum x_1 \otimes x_2 \). We have

\[
\tau(x) = \sum (\pi q)^{-|x_1 + x_2|} \pi p(x_1) p(x_2) x_2 \otimes x_1.
\]

(b) \( \{ x, y \} = (-1)^{|x||y|} \pi p(x) p(y) q^{-|x||y|/2} q^{-|x|}(x, \sigma(y)) \).

**Proof.** It is straightforward to check both claims are true when \( x = \theta_i \) and \( y = \theta_j \) for some \( i, j \in I \).

Assume (a) holds for \( x \) replaced by \( x' \) and by \( x'' \). We shall prove the claim for \( x = x'x'' \).

Recall \( \overline{\theta} = \pi q^{-1} \), and \( r(\theta) = \overline{\tau(x)} \). Write

\[
r(x') = \sum x'_1 \otimes x'_2, \quad r(x'') = \sum x''_1 \otimes x''_2,
\]

\[
r(x'x'') = \sum q^{|x'_1| - |x'_2|} \pi p(x'_1) p(x'_2) x'_1 x''_1 \otimes x'_2 x''_2. \quad \tag{1.6}
\]

By assumption, we have

\[
r(\overline{x'}) = \sum q^{|x'_1| - |x'_2|} \pi p(x'_1) p(x'_2) \overline{x'_1} \otimes \overline{x'_2},
\]

\[
r(\overline{x''}) = \sum q^{|x''_1| - |x''_2|} \pi p(x''_1) p(x''_2) \overline{x''_1} \otimes \overline{x''_2}.
\]

Hence,

\[
r(\overline{x'})r(\overline{x''}) = \sum q^{|x'_1| - |x'_2|} \pi p(x'_1) p(x'_2) q^{|x''_1| - |x''_2|} \pi p(x''_1) p(x''_2) (\overline{x'_1} \otimes \overline{x'_2}) (\overline{x''_1} \otimes \overline{x''_2})
\]

\[
= \sum q^{|x'_1| - |x'_2| + |x''_1| - |x''_2|} \pi p(x'_1) p(x'_2) + p(x''_1) p(x''_2) + p(x'_1) p(x''_2) + p(x'_2) p(x''_1) q^{|x'_1| - |x'_2|} x'_2 x''_2 \otimes x'_1 x''_1.
\]

Then,

\[
\tau(x'x'') = \overline{r(x')r(x'')}
\]

\[
= \sum (\pi q)^{-|x'_1| - |x'_2| + |x''_1| - |x''_2|} \pi p(x'_1) p(x'_2) + p(x''_1) p(x''_2) + p(x'_1) p(x''_2) + p(x'_2) p(x''_1) q^{|x'_1| - |x'_2|} x'_2 x''_2 \otimes x'_1 x''_1.
\]

Now, since the datum is consistent, \( |x''_1| \cdot |x'_2| \in 2\mathbb{Z} \), and hence we have

\[
\tau(x'x'') = \sum (\pi q)^{-|x'_1| - |x'_2| + |x''_1| - |x''_2|} \pi p(x'_1) p(x'_2) + p(x''_1) p(x''_2) q^{|x'_1| - |x'_2|} x'_2 x''_2 \otimes x'_1 x''_1. \quad \tag{1.7}
\]

Comparing (1.6) and (1.7), we see that (a) holds.
Let $S$ be the set of $y \in \mathfrak{f}$ such that (b) holds for all $x \in \mathfrak{f}$. Let $y', y'' \in S$; we will show $y = y'y'' \in S$. Let $x \in \mathfrak{f}$ and write $r(x) = \sum x' \otimes x''$ with $x, x''$ homogeneous. Then

\[
\{ x, y'y'' \} = \{ \pi(x), y \otimes y'' \} = \left\{ \sum (\pi q)^{-|x'|-|x''|} \pi^{p(x)p(x''')} x'' \otimes x', y' \otimes y'' \right\}
\]

\[
= \sum (-1)^{ht|x'| + ht|x''|} q^{-|y''|-|x''|} q^{x' x''} \{ x', y'' \}
\]

\[
= \sum (-1)^{|x| + |x''|} q^{-|x''|-|x''|} q^{x' x''} \{ x', y'' \}
\]

\[
= (-1)^{|x| + |y''|} q^{-|x''|-|x''|} q^{x' x''} + \pi^{p(x)p(x''')} q^{x' x''} \left( x' \otimes x'', \sigma(y) \otimes \sigma(y) \right)
\]

where the equality (1) follows from the observation that the nonzero terms in the sum only occur when the each of the pairs $\{ x', y'' \}$ and $\{ x'', y' \}$ are of the same weight and parity. Therefore we see $y \in S$. Since the algebra generators lie in $S$, the claim is proved. \qed \qed

In particular, we observe the following corollary.

**Corollary 1.4.4.** Assume the super Cartan datum is consistent. Then $- \sigma$ descends to an involution on $\mathfrak{f}$.

**1.5. The maps $r_i$ and $i_r$.** Let $i \in I$. Clearly there are unique $\mathbb{Q}(q)^\pi$-linear maps $r_i, i_r : \mathfrak{f} \rightarrow \mathfrak{f}$ such that $r_i(1) = i_r(1) = 0$ and $r_i(\theta_j) = i_r(\theta_j) = \delta_{ij}$ satisfying

\[
i_r(xy) = i_r(x) y + \pi^{p(x)p(i)} q^{|x| i} x_i r(y)
\]

\[
r_i(xy) = \pi^{p(y)p(i)} q^{|y| i} r(x) y + x r_i(y)
\]

for homogeneous $x, y \in \mathfrak{f}$, see [K]. We see that if $x \in \mathfrak{f}_\nu$, then $i_r(x), r_i(x) \in \mathfrak{f}_{\nu - i}$ and moreover that

\[
r(x) = r_i(x) \otimes \theta_i + \theta_i \otimes i_r(x) + (\ldots)
\]

(1.8)

where (\ldots) stands in for other bi-homogeneous terms $x' \otimes x''$ with $|x'| \neq i$ and $|x''| \neq i$. Therefore, we have

\[
(\theta_i y, x) = (\theta_i, \theta_i)(y, i_r(x)), \quad (y \theta_i, x) = (\theta_i, \theta_i)(y, r_i(x))
\]

(1.9)

for all $x, y \in \mathfrak{f}$, so $i_r(J) \cup r_i(J) \subseteq J$. Hence, both maps descend to maps on $\mathfrak{f}$. It is also easy to check that

\[
r_i \sigma = \sigma i_r.
\]

Indeed, this is trivially true for the generators, and if this holds for $x, y \in \mathfrak{f}$, then

\[
\begin{align*}
 r_i \sigma(xy) &= r_i(\sigma(y) \sigma(x)) = \pi^{p(i)p(x)} q^{|x| i} r_i(\sigma(y)) \sigma(x) + \sigma(y) r_i(\sigma(x)) \\
 &= \sigma(\pi^{p(i)p(x)} q^{|x| i} x_i r(y)) + i_r(x) y = \sigma i_r(xy).
\end{align*}
\]

**Lemma 1.5.1.** Assume $(\hat{I}, \cdot)$ is consistent. For any homogeneous $x \in \mathfrak{f}$, we have

\[
r_i(x) = \pi^{p(x)p(i) - p(i)p(i)} q^{|x| - i - i} x_i r_i(x).
\]
Proof. This is trivial when $x = \theta_i$. Now assume this is true for $x, y \in \mathfrak{f}$. Then
\[
ir(x) = ir(\pi) + \pi(x)p(i)(\pi q)^{-|x|}ir(\pi y)
\]
\[
= \pi^{-p(x)}p(i)p(i)(\pi q)^{-|x|+i}i r_i(x)y \\
+ \pi^{-p(y)}p(i)p(i)(\pi q)^{-|y|+i}i p(x)p(i)(\pi q)^{-|x|+i} x r_i(y)
\]
\[
= \pi^{-p(x)+y}p(i)p(i)(\pi q)^{-|x+y|+i}i \left( \pi^{-p(y)}p(i)(\pi q)^{-|y|+i} x r_i(x)y + x r_i(y) \right)
\]
\[
= \pi^{-p(x+y)}p(i)p(i)(\pi q)^{-|x+y|+i}i x r_i(xy).
\]
The lemma is proved. \qed \qed

Lemma 1.5.2. Let $x \in \mathfrak{f}$, where $\nu \in \mathbb{N}[I]$ is nonzero.
(a) If $r_i(x) = 0$ for all $i \in I$, then $x = 0$.
(b) If $r(x) = 0$ for all $i \in I$, then $x = 0$.

Proof. Suppose that $r_i(x) = 0$ for all $i$. Using (1.9), this means that $(y \theta_i, x) = 0$ for all $y \in \mathfrak{f}$ and all $i \in I$. But since $\mathfrak{f}$ is spanned by monomials in the $\theta_i$, this implies $x \in \mathfrak{f}$, and so $x = 0$ in $\mathfrak{f}$. The proof of (b) proceeds similarly. \qed \qed

1.6. Gaussian $(q, \pi)$-binomial coefficients. Let $A = \mathbb{Z}[q, q^{-1}]$, and let $A^\pi$ be as in §1.1.3. For $a \in \mathbb{Z}$ and $t \in \mathbb{N}$, we define the $(q, \pi)$-binomial coefficients to be
\[
\begin{bmatrix} a \\ t \end{bmatrix}_i = \frac{\prod_{s=0}^{t-1} (\pi_i q_i)^{a-s} - q_i^{s-a}}{\prod_{s=1}^{t} (\pi_i q_i)^{s} - q_i^{-s}}.
\]
We have
\[
\begin{bmatrix} a \\ t \end{bmatrix}_i = (-1)^t \pi_i^{ta-\binom{t}{2}} t-a-1 \binom{t}{i}, \quad (1.10)
\]
\[
\begin{bmatrix} a \\ t \end{bmatrix}_i = 0 \text{ if } 0 \leq a < t, \quad (1.11)
\]
\[
\prod_{j=0}^{a-1} (1 + \pi_i q_i^2)^t z = \sum_{t=0}^{a} \pi_i^{\binom{t}{2}} q_i^{t(a-1)} \begin{bmatrix} a \\ t \end{bmatrix}_i z^t \text{ if } a \geq 0. \quad (1.12)
\]
Here $z$ is another indeterminate. From (1.10) and (1.12) we deduce that
\[
\begin{bmatrix} a \\ t \end{bmatrix}_i \in A. \quad (1.13)
\]
If $a', a''$ are integers and $t \in \mathbb{N}$, then
\[
\begin{bmatrix} a' + a'' \\ t \end{bmatrix}_i = \sum_{t'+t''=t} \pi_i^{t'+a't'' - a''t'} q_i^{a't'} \left[ \begin{bmatrix} a' \\ t' \end{bmatrix}_i \begin{bmatrix} a'' \\ t'' \end{bmatrix}_i. \quad (1.14)
\]
We have \[
\begin{bmatrix} -1 \\ t \end{bmatrix}_i = (-1)^t \pi_i^{t+1} \text{ for any } t \geq 0, i \in I.
\]
For $(q, \pi)$-integers we shall denote
\[
[n]_i = \begin{bmatrix} n \\ 1 \end{bmatrix}_i = \frac{\pi_i q_i^n - q_i^{-n}}{\pi_i q_i - q_i^{-1}} \text{ for } n \in \mathbb{Z},
\]
and with this notation we have
\[
\begin{bmatrix} a \\ t \end{bmatrix}_i = \frac{[a]_i^t}{[a]_i![a-t]_i} \text{ for } 0 \leq t \leq a.
\]
Note that the \((q, \pi)\)-integers \([n]_i^t\) and the \((q, \pi)\)-binomial coefficients in general are not necessarily bar-invariant unless the super Cartan datum is consistent; see (1.13).

If \(a \geq 1\), then we have
\[
\sum_{t=0}^{a} (-1)^t \pi_i^{(1)} q_i^{(a-1)} \begin{bmatrix} a \\ t \end{bmatrix}_i = 0
\]
which follows from (1.12) by setting \(z = -1\).

If \(x, y\) are two elements in a \(\mathbb{Q}(q)^\pi\)-algebra such that \(xy = \pi_i q_i^2 xy\), then for any \(a \geq 0\), we have the quantum binomial formula:
\[
(x + y)^a = \sum_{t=0}^{a} q_i^{(a-t)} \begin{bmatrix} a \\ t \end{bmatrix}_i y^t x^{a-t}.
\]

1.7. Quantum Serre relations. For any \(n \in \mathbb{Z}\), let the divided powers \(\theta_i^{(n)}\) (in \(f\) or \(f^i\)) be defined as \(\theta_i/[n]_i^t\) if \(n \geq 0\) and 0 otherwise.

Lemma 1.7.1. For any \(n \in \mathbb{Z}\) we have
(a) \(r(\theta_i^{(n)}) = \sum_{t+t' = n} q_i^{t'} \theta_i^{(t)} \otimes \theta_i^{(t')}\),
(b) \(\tau(\theta_i^{(n)}) = \sum_{t+t' = n} (\pi_i q_i)^{-t} \theta_i^{(t)} \otimes \theta_i^{(t')}\).

Proof. By the quantum binomial formula (1.16) applied to \(x = 1 \otimes \theta_i\) and \(y = \theta_i \otimes 1\), the formula follows. \(\square\)

Lemma 1.7.2. For any \(n \geq 0\), we have
\[
(\theta_i^{(n)}, \theta_i^{(n)}) = \prod_{s=1}^{n} \frac{\pi_i^{s-1}}{1 - (\pi_i q_i^{-2})^s} = \pi_i^n (\alpha_i q_i^{-2}) (\pi_i q_i - q_i^{-1})^{-n} ([n]_i^1)^{-1}.
\]

Proof. We prove by induction on \(n\). The lemma is true by definition for \(n = 0, 1\). For general \(n\), it follows by Lemma 1.7.1(a) that
\[
\left(\theta_i^{(n)}, \theta_i^{(n)}\right) = [n]_i^{-1} \left(\theta_i^{(n-1)} \otimes \theta_i, \tau \left(\theta_i^{(n)}\right)\right)
= [n]_i^{-1} \left(\theta_i^{(n-1)} \otimes \theta_i, \sum_{t+t' = n} q_i^{t'} \theta_i^{(t)} \otimes \theta_i^{(t')}\right)
= [n]_i^{-1} \left(\theta_i^{(n-1)} \otimes \theta_i, q_i^{n-1} \theta_i^{(n-1)} \otimes \theta_i\right)
= q_i^{n-1}[n]_i^{-1} (\theta_i, \theta_i) \left(\theta_i^{(n-1)}, \theta_i^{(n-1)}\right).
\]
Hence by the induction hypothesis, we have
\[
\left(\theta_i^{(n)}, \theta_i^{(n)}\right) = q_i^{n-1}[n]_i^{-1} \left(1 - \pi_i q_i^{-2}\right) \prod_{s=1}^{n-1} \left(\alpha_i q_i^{-2}\right) (\pi_i q_i - q_i^{-1})^{-n+1} ([n-1]_i^1)^{-1}
= \pi_i^n (\alpha_i q_i^{-2}) (\pi_i q_i - q_i^{-1})^{-n} ([n]_i^1)^{-1}.
\]
The lemma is proved. □ □

**Proposition 1.7.3** (Quantum Serre relation). The generators \( \theta_i \) of \( \mathfrak{f} \) satisfy the relations

\[
\sum_{n+n'=1} (-1)^{n'} \pi_i^{n'p(j)+(\binom{n'}{2})} \theta_i^{(n)} \theta_j \theta_i^{(n')} = 0
\]

for any \( i \neq j \) in \( I \).

Proposition 1.7.3 appeared as [HW, Theorem 3.8]. We shall give a new and simpler proof of Proposition 1.7.3 below after some preparation.

**Lemma 1.7.4.** Let \( N \in \mathbb{N} \) and \( a, a' \in \mathbb{N} \) with \( N = a + a' \). Let \( i, j, k \in I \) be pairwise distinct. Then

(a) \( r_k(\theta_i^{(a)} \theta_j \theta_i^{(a')}) = 0 \),

(b) \( r_j(\theta_i^{(a)} \theta_j \theta_i^{(a')}) = q_i^{a'(i,j)} \pi_i^{a'p(j)} \left[ \frac{N}{a'} \right] \theta_i^{(N)} \),

(c) \( r_i(\theta_i^{(a)} \theta_j \theta_i^{(a')}) = q_i^{a'+(N+(i,j)-1)p} \theta_i^{(a-1)} \theta_j \theta_i^{(a')-1} + q_i^{a'-1} \theta_i^{(a)} \theta_j \theta_i^{(a')-1} \).

**Proof.** Part (a) is clear from definitions. By (1.8) and Lemma 1.7.1(a) we have

\[ r_j(\theta_i^{(a)} \theta_j \theta_i^{(a')}) = \delta_{i,j} q_i^{-1} \theta_i^{(a-1)} \] .

Parts (b) and (c) follow from this and noting

\[ r_i(cba) = cbr_i(a) + \pi^p(i)p(a) q_i^{|a|} cr_i(b)a + \pi^p(i)p(a) + p(i)p(b) q_i^{|a|+|b|} r_i(c)ba . \]

The lemma is proved. □ □

1.7.5. **Proof of Proposition 1.7.3** Let \( N = 1 - \langle i, j' \rangle \). By the previous lemma, we have

\[
r_k \left( \sum_{n+n'=N} (-1)^{n'} \pi_i^{n'p(j)+(\binom{n'}{2})} \theta_i^{(n)} \theta_j \theta_i^{(n')} \right) = 0 , \quad \text{for } k \neq i, j .
\]

In addition, we have

\[
r_j \left( \sum_{n+n'=N} (-1)^{n'} \pi_i^{n'p(j)+(\binom{n'}{2})} \theta_i^{(n)} \theta_j \theta_i^{(n')} \right)
\]

\[
= \sum_{n+n'=N} (-1)^{n'} \pi_i^{n'p(j)+(\binom{n'}{2})} q_i^{n'(i,j)} \pi_i^{n'p(j)} \left[ \frac{N}{n'} \right] \theta_i^{(N)}
\]

\[
= \theta_i^{(N)} \sum_{t=0}^{N} (-1)^t \pi_i^{(\binom{t}{2})} (q_i)^{t(1-N)} \left[ \frac{N}{t} \right] .
\]

By Condition 1.1(e), \( 1 - N \in 2\mathbb{Z} \) if \( i \) is odd, so in any case, the right-hand side of the last equation is

\[
= \theta_i^{(N)} \sum_{t=0}^{N} (-1)^t \pi_i^{(\binom{t}{2})} (q_i)^{t(N-1)} \left[ \frac{N}{t} \right] = 0 .
\]
where the last equality follows from (1.15). Finally,

\[
\tau_i \left( \sum_{n+n' = N} (-1)^n' \pi_i^{n'p(j)+(n')} q_i^p \theta_i^{(n)} \theta_i^{(n')} \right) \\
= \sum_{n+n' = N} (-1)^n' \pi_i^{n'p(j)+(n')} q_i^p \theta_i^{(n-1)} \theta_i^{(n')} \\
+ \sum_{n+n' = N} (-1)^n' \pi_i^{n'p(j)+(n')} q_i^p \theta_i^{(n'-1)} \\
= \sum_{t=0}^{N-1} (-1)^t \pi_i^{tp(j)+p(j)+(t+1)} q_i^t \theta_i^{(N-1-t)} \theta_i^{(t)} \\
- \sum_{t=0}^{N-1} (-1)^t \pi_i^{tp(j)+(t+1)} q_i^t \theta_i^{(N-1-t)} \theta_i^{(t)} \\
= 0.
\]

Now Proposition 1.7.3 follows by Lemma 1.5.2.

Note that the bar map \(\pi\) on \(f\) may not be well-defined when the datum is not consistent. For example, consider the case \((I, \cdot)\) has \(i, j \in I_\pi\) with \(i \cdot j = -1\), hence \(d_i = d_j = 1\). Then the calculations above hold; that is, \(s(\theta_i, \theta_j) := \theta_i^{(2)} \theta_j - \theta_i \theta_j \theta_i + \theta_j \theta_i^{(2)} = 0\); however, since \([2] = \pi[2],\) it is easy to see that \(s(\theta_i, \theta_j) \notin \mathcal{I}\).

Let \(\mathcal{A} f\) be \(\mathcal{A}^{\pi}\)-subalgebra of \(f\) generated by the elements \(\theta_i^{(s)}\) for various \(i \in I\) and \(s \in \mathbb{Z}\). Since the generators \(\theta_i^{(s)}\) are homogeneous, we have \(\mathcal{A} f = \bigoplus_{\nu} \mathcal{A} f_{\nu}\) where \(\nu\) runs over \(\mathbb{N}[I]\) and \(\mathcal{A} f_{\nu} = \mathcal{A} f \cap f_{\nu}\).

2. THE QUANTUM COVERING AND SUPER GROUPS

In this section we give the definition of the quantum covering group \(U\) as a Hopf superalgebra, which specializes at \(\pi = -1\) to a new variant of a quantum supergroup. We show that \(U\) admits a triangular decomposition \(U = U^+ U^0 U^-\) with positive/negative parts isomorphic to the algebra \(f\). The novelty here is that \(U^0\) contains some new generators \(J_i (i \in I)\) which allow us to construct integrable modules in full generality.

2.1. The algebras \(U\) and \(U\). Assume that a root datum \((Y, X, \langle , \rangle)\) of type \((I, \cdot)\) is given. Consider the associative \(\mathbb{Q}(q)^\pi\)-superalgebra \(\mathcal{U}\) (with 1) defined by the generators

\[
E_i \quad (i \in I), \quad F_i \quad (i \in I), \quad J_\mu \quad (\mu \in Y), \quad K_\mu \quad (\mu \in Y),
\]

where the parity is given by \(p(E_i) = p(F_i) = p(i)\) and \(p(K_\mu) = p(J_\mu) = 0\), subject to the relations (a)-(f) below for all \(i, j \in I, \mu, \mu' \in Y:\)

\[
K_0 = 1, \quad K_\mu K_{\mu'} = K_{\mu+\mu'}, \quad (a)
\]
\[
J_{2\mu} = 1, \quad J_\mu J_{\mu'} = J_{\mu+\mu'}, \quad (b)
\]
\[
J_\mu K_{\mu'} = K_{\mu'} J_\mu, \quad (c)
\]
\[
K_\mu E_i = q^{(\mu, i')} E_i K_\mu, \quad J_\mu E_i = \pi^{(\mu, i')} E_i J_\mu, \quad (d)
\]
\[
K_\mu F_i = q^{-(\mu, i')} F_i K_\mu, \quad J_\mu F_i = \pi^{-(\mu, i')} F_i J_\mu, \quad (e)
\]
\[ E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{\bar{J}_i \bar{K}_i - \bar{K}_{-i}}{\pi i q_i - q_i}, \]  

where for any element \( \nu = \sum_i \nu_i i \in \mathbb{Z}[I] \) we have set \( \bar{K}_\nu = \prod_i K_{d_i \nu_i}, \bar{J}_\nu = \prod_i J_{d_i \nu_i} \). In particular, \( \bar{K}_i = K_{d_i}, \bar{J}_i = J_{d_i} \). (Under Condition \( \Pi \)(e), \( \bar{J}_i = 1 \) for \( i \in I_\tau \) while \( \bar{J}_i = J_i \) for \( i \in I'_\tau \)).

We also consider the associative \( \mathbb{Q}(q)^\pi \)-algebra \( U \) (with 1) defined by the generators

\[ E_i \ (i \in I), \ F_i \ (i \in I), \ J_\mu \ (\mu \in Y), \ K_\mu \ (\mu \in Y) \]

and the relations (a)-(f) above, together with the additional relations

\[ f(\theta_i : i \in I) \in J, f(E_i : i \in I) = f(F_i : i \in I) = 0. \]  

The algebra \( U \) will be called the quantum covering group of type \((I, \cdot)\).

From (g), we see that there are well-defined algebra homomorphisms \( f \rightarrow U, x \mapsto x^+ \) (with image denoted by \( U^+ \)) and \( f \rightarrow U, x \mapsto x^- \) (with image denoted by \( U^- \)) such that \( E_i = \theta_i^+ \) and \( F_i = \theta_i^- \) for all \( i \in I \). Clearly, there are well defined algebra homomorphisms \( f \rightarrow U \) with the aforementioned properties.

(In terms of standard notations used in some other quantum group literature, it is understood that \( K_\mu = q^\mu \) and \( K_i = q^{\mu_i} \). It is instructive to see our new generators \( J \)'s can be understood in the same vein as \( J_\mu = \pi^\mu \) and \( J_i = \pi^\mu \).

For any \( p \geq 0 \), we set \( E_i^{(p)} = (\theta_i^{(p)})^+ \) and \( F_i^{(p)} = (\theta_i^{(p)})^- \).

**Example 2.1.1.** In the case \( I = I_\tau = \{ I \} \), we can identify \( Y = X = \mathbb{Z} \) with \( i = 1 \in Y, i' = 2 \in X \), and \( \langle \mu, \lambda \rangle = \mu \lambda \). Then \( U \) is the \( \mathbb{Q}(q)^\pi \)-algebra generated by \( E, F, K, J \) such that

\[ JK = KJ, \quad JE = EJ, \quad JF = FJ, \quad J^2 = 1, \]

\[ K E K = q^2 E, \quad K F K = q^{-2} F, \]

\[ E F - \pi F E = \frac{J K - K^{-1}}{\pi q - q^{-1}}. \]

Note that the quotient algebras \( U/(\langle J \pm 1 \rangle U) \) are isomorphic to the two variants of the quantum group \( U_q(osp(1|2)) \) defined in [CW].

**2.2. Properties of \( U \).** By inspection, there is a unique algebra automorphism (of order 4) \( \omega : U \rightarrow U \) such that

\[ \omega(E_i) = \pi_i \tilde{J}_i F_i, \quad \omega(F_i) = E_i, \quad \omega(K_\mu) = K_{-\mu}, \quad \omega(J_\mu) = J_\mu \]

for \( i \in I, \mu \in Y \). We have \( \omega(x^+) = \pi_{|x|} \tilde{J}_{|x|} x^- \) and \( \omega(x^-) = x^+ \) for all \( x \in f \), and thus the same formula defines a unique algebra automorphism \( \omega : U \rightarrow U \).

Similarly, there is a unique isomorphism of \( \mathbb{Q}(q)^\pi \)-vector spaces \( \sigma : U \rightarrow U \) such that

\[ \sigma(E_i) = E_i, \quad \sigma(F_i) = \pi_i \tilde{J}_i F_i, \quad \sigma(K_\mu) = K_{-\mu}, \quad \sigma(J_\mu) = J_\mu \]

for \( i \in I, \mu \in Y \) such that \( \sigma(uu') = \sigma(u')\sigma(u) \) for \( u, u' \in U \). We have

\[ \sigma(x^+) = \sigma(x)^+, \quad \sigma(x^-) = \pi_{|x|} \tilde{J}_{|x|} \sigma(x)^- \], \quad \forall x \in f. \]  

Again, this implies that the same formula defines a unique algebra automorphism \( \sigma : U \rightarrow U \). Note that \( \sigma \) on \( U^+ \) matches exactly \( \sigma \) on \( f \), but \( \sigma \) on \( U^- \) looks quite different from \( \sigma \) on \( f \) (in contrast to the quantum group setting [Lu]).
Lemma 2.2.1 (Comultiplication). There is a unique algebra homomorphism $\Delta: \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ (resp. $\Delta: \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$) where $\mathcal{F} \otimes \mathcal{F}$ (resp. $\mathcal{U} \otimes \mathcal{U}$) is regarded as a superalgebra in the standard way, defined by

$$
\Delta(E_i) = E_i \otimes 1 + \overline{J_i} \overline{K_i} \otimes E_i \quad (i \in I),
$$
$$
\Delta(F_i) = F_i \otimes \overline{K_{-i}} + 1 \otimes F_i \quad (i \in I),
$$
$$
\Delta(K_\mu) = K_\mu \otimes K_\mu \quad (\mu \in Y),
$$
$$
\Delta(J_\mu) = J_\mu \otimes J_\mu \quad (\mu \in Y).
$$

Proof. The relations (a)-(c) are trivial to verify. For the relation (d), we have

$$
\Delta(E_i) \Delta(F_j) = E_i F_j \otimes \overline{K}_{-j} + \overline{J}_i \overline{K}_i \otimes E_i F_j + E_i \otimes F_j + \pi^{p(i)p(j)} \overline{J}_i \overline{K}_i F_j \otimes E_i \overline{K}_{-j},
$$
$$
\Delta(F_j) \Delta(E_i) = F_j E_i \otimes \overline{K}_{-j} + \overline{J}_i \overline{K}_i \otimes F_j E_i + \pi^{p(i)p(j)} E_i \otimes F_j + F_j \overline{J}_i \overline{K}_i \otimes \overline{K}_{-j} E_i.
$$

So using the fact that $F_j \overline{K}_i \otimes \overline{K}_{-j} E_i = \overline{K}_i F_j \otimes \overline{K}_{-j} E_i$, we have

$$
\Delta(E_i) \Delta(F_j) - \pi^{p(i)p(j)} \Delta(F_j) \Delta(E_i) = (E_i F_j - \pi^{p(i)p(j)} F_j E_i) \otimes \overline{K}_{-j} + \overline{J}_i \overline{K}_i \otimes (E_i F_j - \pi^{p(i)p(j)} F_j E_i)
$$
$$
= \delta_{i,j} \left( \frac{\overline{J}_i \overline{K}_i - \overline{K}_{-i}}{\pi q_i - q_i^{-1}} \right) \otimes \overline{K}_{-j} + \overline{J}_i \overline{K}_i \otimes \left( \delta_{i,j} \frac{\overline{J}_i \overline{K}_i - \overline{K}_{-i}}{\pi q_i - q_i^{-1}} \right)
$$
$$
= \delta_{i,j} \frac{\Delta(\overline{J}_i) \Delta(\overline{K}_i) - \Delta(\overline{K}_{-i})}{\pi q_i - q_i^{-1}}.
$$

Finally, define maps $j^\pm : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ given by

$$
j^+(x \otimes y) = x^+ \overline{J}_{|y|} \overline{K}_{|y|} \otimes y^+, \quad j^-(x \otimes y) = x^- \otimes \overline{K}_{-|x|} y^-.
$$

Then by construction, these maps are algebra homomorphisms, and satisfy

$$
j^+(r(x)) = \Delta(x^+), \quad j^-(r(x)) = \Delta(x^-).
$$

Since $r, \overline{r}$ factor through $f$, so do $j^+ r$ and $j^- \overline{r}$ implying that

$$
f(\Delta(E_i)) = f(\Delta(F_i)) = 0
$$

for all $f(\theta_i : i \in I) \in J$. \hfill \Box \hfill \Box

The previous proof shows that $j^+(r(x)) = \Delta(x^+)$ and $j^-(r(x)) = \Delta(x^-)$, so in particular we have

$$
\Delta(x^+) = \sum x_i^+ \overline{J}_{|x_2|} \overline{K}_{|x_2|} \otimes x_2^+,
$$
$$
\Delta(x^-) = \sum \pi^{p(x_1)p(x_2)} (\pi q)^{-|x_1|} |x_2| \overline{K}_{-|x_2|} x_2^- \otimes \overline{K}_{-|x_2|} x_1^-,
$$

for $r(x) = \sum x_1 \otimes x_2$. In particular, this yields the formulas

$$
\Delta(E_i^{p}) = \sum q_i^{p_1 p_2} \overline{J}_i^{p_1} \overline{K}_i^{p_2} E_i^{p_1} \overline{K}_i^{p_2} \otimes E_i^{p_2},
$$
$$
\Delta(F_i^{p}) = \sum (\pi q_i)^{-p_1 p_2} F_i^{p_1} \overline{K}_i^{p_2} \otimes \overline{K}_i^{p_2} F_i^{p_2}.
$$

Proposition 2.2.2. For $x \in \mathcal{F}$ and $i \in I$, we have $(\in \mathcal{F})$
(a) \( x^+ F_i - \pi_i^{p(x)} F_i x^+ = \frac{r_i(x)^+ J_i \tilde{K}_i - \tilde{K}_{-i} \pi_i^{p(x)-p(i)} \pi r_i(x)^+}{\pi_i q_i - q_i^1} \),

(b) \( E_i x^- - \pi_i^{p(x)} x^- E_i = \frac{\tilde{J}_i \tilde{K}_i \pi r_i(x)^- - \pi_i^{p(x)-p(i)} r_i(x)^- \tilde{K}_{-i}}{\pi_i q_i - q_i^1} \).

Proof. Assume that (a) is known for \( x' \) and \( x'' \); we shall show it holds for \( x = x' x'' \). Let \( y' = (x')^+ \), \( i y' = i r(x')^+ \) and similarly for \( r_i \), \( x'' \) and \( x \).

\[
y F_i = \pi_i^{p(x'')} y' F_i y'' + \frac{y' y'' \tilde{J}_i \tilde{K}_i - y' \tilde{K}_{-i} \pi_i^{p(x'')-p(i)} i y''}{\pi_i q_i - q_i^1} \\
= \pi_i^{p(x' x'')} F_i y + \frac{y' y'' \tilde{J}_i \tilde{K}_i - y' \tilde{K}_{-i} \pi_i^{p(x'')-p(i)} i y''}{\pi_i q_i - q_i^1} \\
+ \frac{y' y'' \tilde{J}_i \tilde{K}_i - y' \tilde{K}_{-i} \pi_i^{p(x'')-p(i)} i y''}{\pi_i q_i - q_i^1} \\
= \pi_i^{p(x' x'')} F_i y + \frac{y_j \tilde{J}_i \tilde{K}_i - \tilde{K}_{-i} \pi_i^{p(x'')-p(i)} i y}{\pi_i q_i - q_i^1}.
\]

Since (a) holds for the generators, it holds for all \( x \in \mathfrak{f} \).

If we apply \( \omega^{-1} \), we obtain

\[
\pi_i \tilde{J}_i x^- E_i - \pi_i^{p(x)-p(i)} \tilde{J}_i E_i x^- = \frac{r_i(x)^- \tilde{J}_i \tilde{K}_{-i} - \tilde{K}_i \pi_i^{p(x)-p(i)} i r(x)^-}{\pi_i q_i - q_i^1},
\]

and multiplying both sides by \( \pi_i^{p(x)-p(i)} \tilde{J}_i \) establishes (b). \( \square \) \( \square \)

We record the following formulas for further use.

**Lemma 2.2.3.** [CW] Lemma 2.8] For any \( N, M \geq 0 \) we have in \( \mathbf{U} \) or \( \mathbf{U}' \)

\[
E_i^{(N)} F_i^{(M)} = \sum_t \pi_i^{MN - \left( \frac{N+1}{2} \right)} \left[ \tilde{K}_i; 2t - M - N \right]_t F_i^{(M-t)} E_i^{(N-t)},
\]

\[
F_i^{(N)} E_i^{(M)} = \sum_t (-1)^t \pi_i^{(M-t)(N-t)-t^2} \left[ \tilde{K}_i; M + N - (t+1) \right]_t E_i^{(N-t)} F_i^{(M-t)},
\]

\[
E_i^{(N)} F_j^{(M)} = \pi_i^{MN \pi(p(j)} F_j^{(M)} E_i^{(N)} \text{ if } i \neq j,
\]

where

\[
\left[ \tilde{K}_i; a \right]_t = \prod_{s=1}^{t} \frac{(\pi_i q_i)^s - a-s+1 \tilde{J}_i \tilde{K}_i - q_i^{s-a-1} \tilde{K}_{-i}}{(\pi_i q_i)^s - q_i^{s-a}}.
\]

The coproduct \( \Delta \) is coassociative; the verification is the same as in the non-super case.

There is a unique algebra homomorphism \( e : \mathbf{U} \rightarrow \mathbb{Q}(q)^{\pi} \) satisfying \( e(E_i) = e(F_i) = 0 \) and \( e(J_\mu) = e(K_\mu) = 1 \) for all \( i, \mu \).

Recall the bar involution \( \overline{\cdot} \) on \( \mathbb{Q}(q)^{\pi} \) from [15]. This extends to a unique homomorphism of \( \mathbb{Q} \)-algebras \( \overline{\cdot} : \mathbf{U} \rightarrow \mathbf{U} \) such that

\[
\overline{E_i} = E_i, \quad \overline{F_i} = F_i, \quad \overline{J_\mu} = J_\mu, \quad \overline{K_\mu} = J_\mu K_{-\mu},
\]

and \( \overline{f} = \overline{f} \) for all \( f \in \mathbb{Q}(q)^{\pi} \) and \( x \in \mathbf{U} \).
Let $A U^\pm$ be the images of $A f$ defined at the end of [17]. We define $A U$ to be the $A^\pi$-subalgebra of $U$ generated by $E_i^{(t)}, F_i^{(t)}, \left[ K_i, a \right]_t, J_\mu$ and $K_\mu$, for all $i \in I$, $\mu \in Y$ and positive integers $a \geq t$.

2.3. Triangular decompositions for $U$ and $U$. If $M', M$ are two $U$-modules, then $M' \otimes M$ is naturally a $U \otimes U$-module; hence by restriction to $U$ under $\Delta$, it is a $U$-module.

Lemma 2.3.1. Let $\lambda \in X$. There is a unique $U$-module structure on the $Q(q)^{\pi}$-module $'f$ such that for any homogeneous $z \in 'f$, and $\mu \in Y$ and any $i \in I$, we have

$$K_\mu \cdot z = q^{(\mu - |z|)}z, \quad J_\mu \cdot z = \pi^{(\mu - |z|)}z, \quad F_i \cdot z = \theta_i z, \quad E_i \cdot 1 = 0.$$  

Proof. The uniqueness is immediate. To prove the existence, define

$$E_i \cdot z = \frac{-q_i^{(i, \lambda)} r_i(z) + \pi_{i, \lambda}(\theta + i) \pi_{i, \lambda}(\theta + i)}{\pi_i - q_i}.$$  

Note that this is essentially the formula prescribed by Proposition 2.2.2. A straightforward computation shows that this, along with the desired formulas for the $F$ and $K$ actions define a $U$-module structure on $'f$. \qed

We denote this $U$-module by $M_\lambda$ (which is a free $Q(q)^{\pi}$-module). Similarly, to an element $\lambda \in X$, we associate a unique $U$-module structure on $'f$ such that for any homogeneous $z \in 'f$, any $\mu \in Y$ and any $i \in I$, we have

$$K_\mu \cdot z = q^{(\mu - |z|)}z, \quad J_\mu \cdot z = \pi^{(\mu - |z|)}z, \quad E_i \cdot z = \theta_i z, \quad F_i \cdot 1 = 0.$$  

We denote this $U$-module by $M'_\lambda$ (which is again a free $Q(q)^{\pi}$-module). We form the $U$-module $M'_\lambda \otimes M_\lambda$; we denote the unit element of $'f = M_\lambda$ by 1 and that of $'f = M'_\lambda$ by 1'. Thus, we have the canonical element $1' \otimes 1 \in M'_\lambda \otimes M_\lambda$. We emphasize that $M'_\lambda \otimes M_\lambda$ is again free as a $Q(q)^{\pi}$-module.

Proposition 2.3.2. Let $U^0$ be the associative $Q(q)^{\pi}$-algebra with 1 defined by the generators $K_\mu$, $J_\mu$ ($\mu \in Y$) and the relations in [2.1] (a), (b). Then $U^0$ is isomorphic to the group algebra of $Y \times (Y/2Y)$ over $Q(q)^{\pi}$. Moreover, 

(a) The $Q(q)^{\pi}$-linear map $'f \otimes U^0 \otimes 'f \rightarrow 'U$ given by $u \otimes J_\mu K_\mu \otimes w \mapsto u^{-J_\mu K_\mu}w^+$ is an isomorphism. 

(b) The $Q(q)^{\pi}$-linear map $'f \otimes U^0 \otimes 'f \rightarrow 'U$ given by $u \otimes J_\mu K_\mu \otimes w \mapsto u^{+J_\mu K_\mu}w^-$ is an isomorphism.

Proof. Note that (b) follows from (a) by applying $\omega$. As a $Q(q)^{\pi}$-module, $U$ is spanned by words in the $E_i$, $F_i$, $K_\mu$, and $J_\mu$. By using the defining relations, we can rewrite any word as a linear combination of words where the $F_i$ come before the $J_\mu$ and $K_\mu$, which come before the $E_i$, thus the given map is surjective.

To prove the map is injective, let $\lambda, \lambda' \in X$, and consider the module $M'_\lambda \otimes M_\lambda$ described before. There is a $Q(q)^{\pi}$-linear map $\phi : U \rightarrow M'_\lambda \otimes M_\lambda$ given by $\phi(u) = u \cdot 1' \otimes 1$. Pick a $Q(q)^{\pi}$-basis of $'f$ consisting of homogeneous elements containing 1. Assume that in $'f$ there is some relation of the form $\sum b'_{\lambda, \mu} c_{\lambda', \mu, b} b'^{-J_\mu K_\mu b^+} = 0$ and let $N$ be the largest integer such that $\text{ht}[b'] = N$ and $c_{\lambda', \mu, b} \neq 0$ for some $\mu, b$. 


Then
\[ 0 = \phi \left( \sum_{b', \mu, \nu, b} c_{b', \mu, \nu, b} b' - J_\nu K_\mu b^+ \right) = \sum_{b', \mu, \nu, b} c_{b', \mu, \nu, b} \Delta(b' - J_\nu K_\mu b^+) \cdot 1 \otimes 1. \]

Now
\[ \Delta(b') = \sum_{b_1', b_2'} g'(b', b_1', b_2') b_1' \otimes K_{-b_1'} b_2', \]
\[ \Delta(b^+) = \sum_{b_1, b_2} g(b, b_1, b_2) b_1^+ J_{[b_2]} K_{[b_2]} \otimes b_2^+, \]
so we have
\[ 0 = \sum \pi(b_3') \pi(b_1) c_{b', \mu, \nu, b} g(b, b_1, b_2) g'(b', b_1', b_2') b_1' - \times J_\nu K_\mu b_1^+ \cdot 1' \otimes K_{-b_1'} b_2^+ \cdot 1. \]
If \( b_2 \neq 1 \), then \( b_2^+ \cdot 1 = 0 \) so we must have \( b_2 = 1 \) and thus \( b_1 = b \). Therefore, the expression reduces to
\[ 0 = \sum \pi(b_3') \pi(b_1) c_{b', \mu, \nu, b} g'(b', b_1', b_2') b_1' - J_\nu K_\mu b^+ \cdot 1' \otimes K_{-b_1'} b_2^+ \cdot 1. \]

By the definition of the module structure, this becomes
\[ 0 = \sum \pi(b_3') \pi(b_1) c_{b', \mu, \nu, b} g'(b', b_1', b_2') \pi(\nu, \lambda - \lambda' + |b|) q^{(\nu, \lambda - \lambda' + |b|)} b_1' \cdot b \otimes K_{-b_1'} b_2'. \]
We can now project this equality onto the summand \( M_{\lambda'} \otimes f_{\nu} \) where \( h \nu = N \). Then by construction, \( |b_2| \leq |b| \) and \( h \nu |b_2| = N \). Since \( c_{b', \mu, b} = 0 \) if \( h \nu |b'| > N \), we must have \( |b| = |b_2| \) and thus \( b' = b_2, b_1' = 1 \), so
\[ \sum \pi(b_3') \pi(b_1) c_{b', \mu, \nu, b} \pi(\nu, \lambda - \lambda' + |b|) q^{(\nu, \lambda - \lambda' + |b|)} b \otimes b' = 0. \]
It follows that
\[ \sum \pi(b_3') \pi(b_1) c_{b', \mu, \nu, b} \pi(\nu, \lambda - \lambda' + |b|) q^{(\nu, \lambda - \lambda' + |b|)} = 0 \]
for all choices of \( \lambda, \lambda', \mu, b \) and \( b' \) with \( h \nu |b'| = N \). Therefore, \( c_{b', \mu, \nu, b} = 0 \) for any \( b' \) with \( h \nu |b'| = N \), contradicting the choice of \( N \). \( \square \)

**Corollary 2.3.3.**
(a) The \( \mathbb{Q}(q)^{-}\)-linear map \( f \otimes U^0 \otimes f \to U \) given by \( u \otimes J_\nu K_\mu w \mapsto u^{-} J_\nu K_\mu w^+ \) is an isomorphism.

(b) The \( \mathbb{Q}(q)^{-}\)-linear map \( f \otimes U^0 \otimes f \to U \) given by \( u \otimes K_\nu w \mapsto u^+ J_\nu K_\mu w^{-} \) is an isomorphism.

**Proof.** Once again (b) follows from (a) by applying the involution \( \omega \). Let \( J_{\pm} \) be the two-sided ideal of \( U \) generated by \( J = \{ x^\pm : x \in J \} \). Then \( U = \frac{U}{J_+ + J_-} \). Now from Proposition 2.2.2 iterated, we see that
\[ \left( \frac{U^+}{J_-} \right) \left( \frac{U^-}{J_+} \right) \subseteq \frac{J^- U^-}{J_- U^0} \left( \frac{U^+}{J_-} \right); \quad \frac{J^+ (U^-)}{J_- (U^+)} \subseteq \left( \frac{U^+}{J_-} \right) U^0 J_- \]
Using the triangular decomposition of \( U \), we have \( J_- = \left( \frac{U^-}{J_-} \right) \subseteq J^- U^- \left( \frac{U^+}{J_-} \right) \subseteq J_- \), hence \( J_- = J^- U^- \left( \frac{U^+}{J_-} \right) \). Similarly, \( J_+ = \left( \frac{U^-}{J_-} \right) U^0 J_+ \). Therefore,
\[ U = \frac{U^- \otimes U^0 \otimes U^+}{J_- \otimes J_+ + J_0 \otimes U^0 \otimes U^+} = \frac{U^- \otimes U^0 \otimes U^+}{J_- \otimes J_+}, \]
from which (a) follows. \( \square \)
Corollary 2.3.4. The maps $\pm : f \to U^\pm$, $x \mapsto x^\pm$, are $\mathbb{Q}(q)^\pi$-algebra isomorphisms, and $U^0 \to U$ is a $\mathbb{Q}(q)^\pi$-algebra embedding.

For $\nu \in \mathbb{N}[I]$, we shall denote the image $f^\pm_\nu$ by $U^\pm_\nu$.

Proposition 2.3.5. Let $x \in f_\nu$ where $\nu \in \mathbb{N}[I]$ is nonzero.

(a) If $x^+ F_i = \pi_i^{(x)} F_i x^+$ for all $i \in I$ then $x = 0$.
(b) If $x^- E_i = \pi_i^{(x)} E_i x^-$ for all $i \in I$ then $x = 0$.

Proof. It follows from Proposition 2.2.2 and the linear independence of $r_i(x)^+ \tilde{J}_i \tilde{K}_i$ (respectively, the linear independence of $\tilde{J}_i \tilde{K}_{-i} r(x)^+$) that $r_i(x)^+ = i r(x)^+ = 0$ for all $i$. Hence $x = 0$ by Lemma 1.5.2 \qed

2.4. Antipode. For $\nu \in \mathbb{N}[I]$, write $\nu = \sum_i \nu_i i$ and $\nu = \sum_{a=1}^{\text{ht}} \nu_a i_a$ for $i_a \in I$. Then we set
\[
\begin{align*}
\nu(x) &= \nu \cdot \nu/2 - \sum_i \nu_i i \cdot i/2 \in \mathbb{Z}, \\
\nu(x) &= \sum_{a<b} p(i_a)p(i_b) \in \mathbb{Z}.
\end{align*}
\]

Lemma 2.4.1. Let $\nu \in \mathbb{N}[I]$.

(a) There is a unique $\mathbb{Q}(q)^\pi$-linear map $S : U \to U$ such that
\[
S(E_i) = -\tilde{J}_i \tilde{K}_i E_i, \quad S(F_i) = -\tilde{F}_i \tilde{K}_i, \quad S(K_\mu) = K_\mu, \quad S(J_\nu) = J_\nu,
\]
and $S(xy) = \pi^{p(x)p(y)} S(y) S(x)$ for all $x, y \in U$.

(b) For any $x \in f_\nu$, we have
\[
S(x^+) = (-1)^{\text{ht}_\nu} \pi^{\nu(x)} (\pi q)^{\nu(x)} \tilde{J}_\nu \tilde{K}_\nu \sigma(x)^+,
S(x^-) = (-1)^{\text{ht}_\nu} \pi^{\nu(x)} (\pi q)^{\nu(x)} \tilde{J}_\nu \tilde{K}_\nu \sigma(x^-).
\]

(c) There is a unique $\mathbb{Q}(q)^\pi$-linear map $S' : U \to U$ such that
\[
S'(E_i) = -E_i \tilde{J}_i \tilde{K}_i, \quad S'(F_i) = -\tilde{F}_i F_i, \quad S'(K_\mu) = K_\mu, \quad S'(J_\nu) = J_\nu,
\]
and $S'(xy) = \pi^{p(x)p(y)} S'(y) S'(x)$ for all $x, y \in U$.

(d) For any $x \in f_\nu$, we have
\[
S'(x^+) = (-1)^{\text{ht}_\nu} \pi^{\nu(x)} (\pi q)^{\nu(x)} \tilde{J}_\nu \tilde{K}_\nu \sigma(x)^+,
S'(x^-) = (-1)^{\text{ht}_\nu} \pi^{\nu(x)} (\pi q)^{\nu(x)} \tilde{J}_\nu \tilde{K}_\nu \sigma(x^-).
\]

(e) We have $SS' = S'S = 1$.

(f) If $x \in f_\nu$, then $S(x^+) = (\pi q)^{-f(\nu)} S'(x^-)$ and $S(x^-) = q^{f(\nu)} S'(x^-)$ where $f(\nu) = \sum_i \nu_i i \cdot i$.

The map $S$ (resp. $S'$) is called the antipode (resp. the skew-antipode) of $U$. Note that
\[
\begin{align*}
S(E_i^{(n)}) &= (-1)^n (\pi_i q_i^{(2)}) \tilde{J}_n \tilde{K}_n E_i^{(n)}, \\
S'(E_i^{(n)}) &= (-1)^n (\pi_i q_i^{(2)}) \tilde{J}_n \tilde{K}_n E_i^{(n)}, \\
S(F_i^{(n)}) &= (-1)^n (\pi_i q_i^{(2)}) \tilde{F}_n \tilde{K}_n F_i^{(n)}, \\
S'(F_i^{(n)}) &= (-1)^n (\pi_i q_i^{(2)}) \tilde{K}_n \tilde{F}_n F_i^{(n)}.
\end{align*}
\]
2.5. **Specializations of \( U \) at \( \pi = \pm 1 \).** The *specialization at \( \pi = 1 \) (respectively, at \( \pi = -1 \)) of a \( \mathbb{Q}(q)\pi \)-algebra \( R \) is understood as \( \mathbb{Q}(q) \otimes_{\mathbb{Q}(q)^{\pi}} R \), where \( \mathbb{Q}(q) \) is the \( \mathbb{Q}(q)\pi \)-module with \( \pi \) acting as 1 (respectively, as \(-1\)).

Let \( \mathcal{J} \) be the (2-sided) ideal of \( U \) generated by \( \{J_{\mu} - 1 | \mu \in Y\} \).

The specialization at \( \pi = -1 \) of the algebra \( U/\mathcal{J} \) is naturally identified with a quantum group associated to the Cartan datum \((I, \cdot)\) (cf. \([Lu]\)). The specialization at \( \pi = 1 \) of the algebra \( U \), denoted by \( U|_{\pi=1} \), is a variant of this quantum group, with some extra (harmless) central elements \( J_{\mu} \). Specialization at \( \pi = 1 \) for the rest of the paper essentially reduces our results to those of Lusztig \([Lu]\).

The specialization at \( \pi = 1 \) of the superalgebra \( U/\mathcal{J} \) is identified with a quantum supergroup associated to the super Cartan datum \((I, \cdot, \pi)\) considered in the literature; cf. \([BKM, Yaj]\). The specialization at \( \pi = -1 \) of \( U \), denoted by \( U|_{\pi=-1} \), will also be referred to as a *quantum supergroup* of type \((I, \cdot, \pi)\), and the extra generators \( J_{\mu} \) allow us to formulate integrable modules \( V(\lambda) \) for all \( \lambda \in X^+ \), which was not possible before.

**All constructions and results in the remainder of this paper clearly afford specializations at \( \pi = -1 \), which provide new constructions and new results for quantum supergroups and their representations.**

2.6. **The categories \( \mathcal{C} \) and \( \mathcal{O} \).** In the remainder of this paper, by a representation of the algebra \( U \) we mean a \( \mathbb{Q}(q)\pi \)-module on which \( U \) acts. Note we have a direct sum decomposition of the \( \mathbb{Q}(q)\pi \)-module \( \mathbb{Q}(q)^{\pi} = (\pi+1)\mathbb{Q}(q) \oplus (\pi-1)\mathbb{Q}(q) \), where \( \pi \) acts as 1 on \((\pi+1)\mathbb{Q}(q)\) and as \(-1\) on \((\pi-1)\mathbb{Q}(q)\).

We define the category \( \mathcal{C} \) (of weight \( U \)-modules) as follows. An object of \( \mathcal{C} \) is a \( \mathbb{Z}_2 \)-graded \( U \)-module \( M = M_{\pi}^{\pi} \oplus M_{\pi}^{\bar{\pi}} \), compatible with the \( \mathbb{Z}_2 \)-grading on \( U \), with a given weight space decomposition

\[
M = \bigoplus_{\lambda \in X} M_{\lambda}, \quad M_{\lambda} = \left\{ m \in M \mid K_{\mu} m = q^{(\mu,\lambda)} m, J_{\mu} m = \pi^{(\mu,\lambda)} m, \forall \mu \in Y \right\},
\]

such that \( M_{\lambda} = M_{\lambda}^{\pi} \oplus M_{\lambda}^{\bar{\pi}} \) where \( M_{\lambda}^{\pi} = M_{\lambda} \cap M_{\pi}^{\pi} \) and \( M_{\lambda}^{\bar{\pi}} = M_{\lambda} \cap M_{\pi}^{\bar{\pi}} \). The \( \mathbb{Z}_2 \)-graded structure is only particularly relevant to tensor products, and will generally be suppressed when irrelevant. We have the following \( \mathbb{Q}(q)\pi \)-module decomposition for each weight space: \( M_{\lambda} = (\pi+1)M_{\lambda} \oplus (\pi-1)M_{\lambda} \); accordingly, we have \( M = M_{\pi} \oplus M_{\bar{\pi}} \) as \( U \)-modules, where \( M_{\pm} := \oplus_{\lambda \in X^\pm} M_{\lambda} \) is an \( U \)-module on which \( \pi \) acts as \pm 1, i.e. a \( U|_{\pi=\pm 1} \)-module. Hence the category \( \mathcal{C} \) decomposes into a direct sum \( \mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_- \), where \( \mathcal{C}_\pm \) can be identified with categories of weight modules over the specializations \( U|_{\pi=\pm 1} \).

**Lemma 2.6.1.** A simple \( U \)-module is a simple module of either \( U|_{\pi=1} \) or \( U|_{\pi=-1} \).

Let \( M \in \mathcal{C} \) and let \( m \in M_{\lambda} \). The formulas below follow from Lemma 2.2.3.

(a) \( E_i^{(N)} F_i^{(M)} m = \sum_{t} \pi_{\pi}^{MN-\langle i, \lambda \rangle} \left[ N - M + \langle i, \lambda \rangle \right] F_i^{(M-t)} E_i^{(N-t)} m; \)

(b) \( F_i^{(M)} E_i^{(N)} m = \sum_{t} \pi_{\pi}^{M-t(\langle i, \lambda \rangle - t)} \left[ M - N - \langle i, \lambda \rangle \right] E_i^{(N-t)} F_i^{(M-t)} m; \)

(c) \( E_i^{(M)} E_j^{(N)} m = E_j^{(N)} E_i^{(M)} m, \text{ for } i \neq j; \)

(d) \( \left[ \tilde{K}_{i}; a \right] m = \left[ \langle i, \lambda \rangle + a \right] m. \)
Note that $U \otimes U$ is a superalgebra with multiplication $(a \otimes b)(c \otimes d) = \pi^{(b)p(c)}ac \otimes bd$. A tensor product of $U$-modules $M \otimes N$ is naturally a $U \otimes U$-module with the obvious diagonal grading under the action $(x \otimes y)(m \otimes n) = \pi^{(y)p(m)}xn \otimes yn$.

The tensor product of modules is naturally a $U$-module under the coproduct action. Moreover, $C$ is closed under tensor products. Note that for $a \in \mathbb{Z}_{>0}$, $M', M'' \in C$, $m', m'' \in M'^{\lambda'}$ and $m'' \in M''^{\lambda''}$, we have

$$E_i^{(a)}(m' \otimes m'') = \sum_{a'+a''=a} \pi_i^{a''p(m')}a'a''_i q_i^{a''+a''(i,\lambda')} E_i^{(a')}m' \otimes E_i^{(a''')}m''.$$  

$$F_i^{(a)}(m' \otimes m'') = \sum_{a'+a''=a} \pi_i^{a''p(m')}a'a''_i q_i^{a''-a'(i,\lambda'')} F_i^{(a')}m' \otimes F_i^{(a''')}m''.$$  

To any $M \in C$, we can define a new $U$-module structure via $u \cdot m = \omega(u)m$; we denote this module by $\omega M$. By definition, note that $\omega^{M^\lambda} = M^{-\lambda}$.

Let $\lambda \in X$. Then there is a unique $U$-module structure on $\mathfrak{f}$ such that for any $y \in \mathfrak{f}$, $\mu \in Y$ and $i \in I$ we have $K_{\mu,y} = q^{(\mu,\lambda-\mu|y)}y$, $J_{\mu,y} = \pi^{(\mu,\lambda-\mu|y)}y$, $F_iy = \theta_iy$, and $E_i1 = 0$. As in the non-super case, this follows readily from the triangular decomposition. This module will be called a Verma module and denoted by $M(\lambda)$. The parity grading on $\mathfrak{f}$ induces a parity grading on $M(\lambda)$ where $p(1) = 0$. As before, we have a $U$-module decomposition $M(\lambda) = M(\lambda)_+ \oplus M(\lambda)_-$, where $M(\lambda)_\pm$ can be identified as the Verma module of $U|_{\pi=\pm 1}$ (which is a $\mathbb{Q}(q)$-vector space).

For any $M \in C$ and an element $m \in M^\lambda$ such that $E_i m = 0$ for all $i$, there is a unique $U$-homomorphism $M(\lambda) \to M$ via $1 \mapsto m$. This can be proved as in [Lu, 3.4.6] using now Lemma 2.2.3.

Let $\mathcal{O}$ be the full subcategory of $C$ such that for any $M$ in $\mathcal{O}$ and $m \in M$, there exists an $n \geq 0$ such that $x^n m = 0$ for all $x \in \mathfrak{f}_\nu$ with $\mathrm{ht}\nu \geq n$. Note that $M(\lambda)$ and its quotient $U$-modules belong to $\mathcal{O}$.

2.7. Category $\mathcal{C}_{\text{int}}$ of integrable modules. An object $M \in C$ is said to be integrable if for any $m \in M$ and any $i \in I$, there exists $n_0 \geq 1$ such that $E_i^{(n)} m = E_i^{(n-1)} m = 0$ for all $n \geq n_0$. Let $\mathcal{C}_{\text{int}}$ be the full subcategory of $C$ whose objects are the integrable $U$-modules.

For $M, M', M'' \in \mathcal{C}_{\text{int}}$, we have $\omega M, M' \otimes M'' \in \mathcal{C}_{\text{int}}$. The proof of the following lemma proceeds as in the non-super case; see [Lu, Lemma 3.5.3].

**Lemma 2.7.1.** For $(a_i), (b_i) \in \mathbb{N}^I$ and $\lambda \in X$, let $M$ be the quotient of $U$ by the left ideal generated by the elements $F_i^{a_i+1}, E_i^{b_i+1}, K_\mu - q^{(\mu,\lambda)}(\mu,\lambda)$ with $\mu \in Y$, and $J_\nu - q^{(\nu,\lambda)}(\nu,\lambda)$ with $\nu \in Y$. Then $M$ is an integrable $U$-module.

The proof of the following proposition proceeds as in the non-super case; see [Lu, Proposition 3.5.4 and 23.3.11].

**Proposition 2.7.2.** If $u \in U$ such that $u$ acts as zero on every integrable module, then $u = 0$.

**Proposition 2.7.3.** Let $\lambda \in X^+$.

(a) Let $\mathcal{T}$ be the left ideal of $\mathfrak{f}$ generated by the elements $\theta_i^{(i,\lambda)+1}$ for all $i \in I$. Then $\mathcal{T}$ is a $U$-submodule of the Verma module $M(\lambda)$.

(b) The quotient $U$-module $V(\lambda) := M(\lambda)/\mathcal{T}$ is integrable.
The proof is as in the non-super case [Lu, Proposition 3.5.6]. As usual \( V(\lambda) = V(\lambda)_+ \oplus V(\lambda)_- \), and \( \mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_- \); moreover we have the identification \( V(\lambda)_{\pm} = M(\lambda)_{\pm}/\mathcal{F}_{\pm} \).

We denote the image of 1 in \( V(\lambda) \) by \( v^1_\lambda \) when convenient. This module has an induced parity grading from the associated Verma module by setting \( p(v^1_\lambda) = 0 \). When considering the image of 1 in the module \( ^*V(\lambda) \), we will denote this vector by \( v^1_{\lambda} \).

**Proposition 2.7.4.** Let \( M \) be an object of \( \mathcal{C}_{\text{int}} \) and let \( m \in M^\lambda \) be a non-zero vector such that \( E_i m = 0 \) for all \( i \). Then \( \lambda \in X^+ \) and there is a unique morphism (in \( \mathcal{C}_{\text{int}} \)) \( t' : V(\lambda) \to M \) sending \( v^1_{\lambda} \) to \( m \).

The proof is as in the non-super case [Lu, Proposition 3.5.8].

3. **The quasi-\( \mathcal{R} \)-matrix and the quantum Casimir**

In this section, we introduce the quasi-\( \mathcal{R} \)-matrix as well as the quantum Casimir for \( \mathcal{U} \) and establish their basic properties. Using the Casimir element, we show that the category \( \mathcal{O}_{\text{int}} \) is semisimple and classify its simple object by dominant integral weights.

3.1. **The quasi-\( \mathcal{R} \)-matrix \( \Theta \).** Consider the vector spaces

\[
\mathcal{H}_N = \mathcal{U}^+ \mathcal{U}^0 \left( \sum_{ht \nu \geq N} \mathcal{U}_\nu \right) \otimes \mathcal{U} + \mathcal{U} \otimes \mathcal{U}^- \mathcal{U}^0 \left( \sum_{ht \nu \geq N} \mathcal{U}_\nu \right)
\]

for \( N \in \mathbb{Z}_{>0} \). Note that \( \mathcal{H}_N \) is a left ideal in \( \mathcal{U} \otimes \mathcal{U} \); moreover, for any \( u \in \mathcal{U} \otimes \mathcal{U} \), we can find an \( r \geq 0 \) such that \( \mathcal{H}_{N+r} u \subset \mathcal{H}_N \).

Let \( (\mathcal{U} \otimes \mathcal{U})^\wedge \) be the inverse limit of the vector spaces \( (\mathcal{U} \otimes \mathcal{U})/\mathcal{H}_n \). Then the \( \mathbb{Q}(q)^\wedge \)-algebra structure extends by continuity to a \( \mathbb{Q}(q)^\wedge \)-algebra structure on \( (\mathcal{U} \otimes \mathcal{U})^\wedge \), and we have the obvious algebra embedding \( \mathcal{U} \otimes \mathcal{U} \to (\mathcal{U} \otimes \mathcal{U})^\wedge \).

Let \( \overline{-} : \mathcal{U} \otimes \mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \) be the \( \mathbb{Q} \)-algebra homomorphism given by \( \overline{-} \otimes \overline{-} \). This extends to a \( \mathbb{Q} \)-algebra homomorphism on the completion. Let \( \overline{\Delta} : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \) be the \( \mathbb{Q}(q)^\wedge \)-algebra homomorphism given by \( \overline{\Delta}(x) = \overline{\Delta(x)} \).

**Theorem 3.1.1.**

(a) There is a unique family of elements \( \Theta_\nu \in \mathcal{U}_\nu \otimes \mathcal{U}_\nu^+ \) (with \( \nu \in \mathbb{N}[I] \)) such that \( \Theta_0 = 1 \otimes 1 \) and \( \Theta = \sum_\nu \Theta_\nu \in (\mathcal{U} \otimes \mathcal{U})^\wedge \) satisfies \( \Delta(u) \Theta = \Theta \overline{\Delta}(u) \) for all \( u \in \mathcal{U} \) (where this identity is in \( (\mathcal{U} \otimes \mathcal{U})^\wedge \)).

(b) Let \( B \) be a \( \mathbb{Q}(q)^\wedge \)-basis of \( \mathcal{f} \) such that \( B_\nu = B \cap f_\nu \) is a basis of \( f_\nu \) for any \( \nu \). Let \( \{ b^\nu | b \in B_\nu \} \) be the basis of \( \mathcal{f}_\nu \) dual to \( B_\nu \) under \( (,.) \). We have

\[
\Theta_\nu = (-1)^{ht} \pi_{e(\nu)} \pi_{\nu} q_\nu \sum_{b \in B_\nu} b^- \otimes b^+ \in \mathcal{U}_\nu \otimes \mathcal{U}_\nu^+,
\]

where \( e(\nu) \) is defined as in [2.4].

The element \( \Theta \) will be called the quasi-\( \mathcal{R} \)-matrix for \( \mathcal{U} \).

**Proof.** Consider an element \( \Theta \in (\mathcal{U} \otimes \mathcal{U})^\wedge \) of the form \( \Theta = \sum_\nu \Theta_\nu \) with \( \Theta_\nu = \sum_{b,b' \in B_\nu} c_{\nu,b,b'} b^- \otimes b^+ \), \( c_{\nu,b,b'} \in \mathbb{Q}(q)^\wedge \). The set of \( u \in \mathcal{U} \) such that \( \Delta(u) \Theta = \Theta \overline{\Delta}(u) \) is clearly a subalgebra of \( \mathcal{U} \) containing \( \mathcal{U}_0 \). Therefore, it is necessary and sufficient that it contains the \( E_i \) and \( F_i \). This
amounts to showing that
\[
\sum_{b_1, b_2 \in B_{\nu}} c_{b_1, b_2} E_i b_1^- \otimes b_2^{*+} + \sum_{b_3, b_4 \in B_{\nu-1}} \pi_i^{(b_3)} c_{b_3, b_4} J_i \bar{K}_i b_3^- \otimes E_i b_4^{*+} \\
= \sum_{b_1, b_2 \in B_{\nu}} \pi_i^{(b_2)} c_{b_1, b_2} b_1^- E_i \otimes b_2^{*+} + \sum_{b_3, b_4 \in B_{\nu-1}} c_{b_3, b_4} b_3^- \bar{K}_i \otimes b_4^{*+} E_i,
\]
and
\[
\sum_{b_1, b_2 \in B_{\nu}} \pi_i^{(b_1)} c_{b_1, b_2} b_1^- F_i b_2^{*+} + \sum_{b_3, b_4 \in B_{\nu-1}} c_{b_3, b_4} F_i b_3^- \otimes \bar{K}_i b_4^{*+} \\
= \sum_{b_1, b_2 \in B_{\nu}} c_{b_1, b_2} b_1^- \otimes b_2^{*+} F_i + \sum_{b_3, b_4 \in B_{\nu-1}} \pi_i^{(b_1)} c_{b_3, b_4} b_3^- F_i \otimes b_4^{*+} \bar{J}_i \bar{K}_i.
\]
Let \( z \in f \). Then since the inner product is nondegenerate, this equality is equivalent to the equality
\[
\sum_{b_1, b_2 \in B_{\nu}} c_{b_1, b_2} (b_2^*, z)(E_i b_1^- - \pi_i^{(b_2)} b_1^- E_i) \\
+ \sum_{b_3, b_4 \in B_{\nu-1}} c_{b_3, b_4} \left( \pi_i^{(b_3)} (\theta_i b_4^*, z) \bar{J}_i \bar{K}_i b_3^- - (b_4^{*+} \theta_i, z) b_3^- \bar{K}_i \right) = 0,
\]
and
\[
\sum_{b_1, b_2 \in B_{\nu}} c_{b_1, b_2} (b_1^*, z)(\pi_i^{(b_1)} F_i b_2^{*+} - b_2^{*+} F_i) \\
+ \sum_{b_3, b_4 \in B_{\nu-1}} c_{b_3, b_4} \left( (\theta_i b_3^*, z) \bar{K}_i b_4^{*+} - \pi_i^{(b_4)} (b_3 \theta_i, z) b_4^{*+} \bar{J}_i \bar{K}_i \right) = 0.
\]
Note that \( p(b_1) = p(b_2) = p(b_3) + p(i) = p(b_4) + p(i) \). Using Proposition 2.2.2 and the derivations, we have
\[
\sum_{b_1, b_2 \in B_{\nu}} (\pi_i q_i - q_i^{-1})^{-1} c_{b_1, b_2} (b_2^*, z)(\bar{J}_i \bar{K}_i r(b_1)^{-1} - \pi_i^{(b_1)-p(i)} r_i(b_1)^{-1} \bar{K}_i) \\
+ \sum_{b_3, b_4 \in B_{\nu-1}} c_{b_3, b_4} (\theta_i \theta_i) \left( \pi_i^{(b_3)} (b_4^* r(z)) \bar{J}_i \bar{K}_i b_3^- - (b_4^{*+} r_i(z)) b_3^- \bar{K}_i \right) = 0,
\]
and
\[
\sum_{b_1, b_2 \in B_{\nu}} -(\pi_i q_i - q_i^{-1})^{-1} c_{b_1, b_2} (b_1^*, z)(r_i(b_2)^{+} \bar{J}_i \bar{K}_i - \pi_i^{(b_2)-p(i)} \bar{K}_i r(b_2)^{+}) \\
+ \sum_{b_3, b_4 \in B_{\nu-1}} c_{b_3, b_4} (\theta_i \theta_i) \left( (b_3 \theta_i r(z)) \bar{K}_i b_4^{*+} - \pi_i^{(b_4)} (b_3 \theta_i r(z)) b_4^{*+} \bar{J}_i \bar{K}_i \right) = 0.
\]
Using the triangular decomposition, this is equivalent to the equalities

\[ \sum_{b_1, b_2} c_{b_1, b_2} (b_2^*, z) i r(b_1) + \sum_{b_3, b_4} \pi_i q_i \pi_i^{b_4} c_{b_3, b_4} (b_4^*, i r(z)) b_3 = 0, \]  

(3.1)

\[ \sum_{b_1, b_2} c_{b_1, b_2} \pi_i^{p(b_1) - p(i)} (b_2^*, z) r_i(b_1) + \sum_{b_3, b_4} \pi_i q_i c_{b_3, b_4} (b_4^*, r_i(z)) b_3 = 0, \]  

(3.2)

\[ \sum_{b_1, b_2} c_{b_1, b_2} (b_1, z) r_i(b_2) + \sum_{b_3, b_4} \pi_i q_i \pi_i^{b_4} c_{b_3, b_4} (b_3, r_i(z)) b_4^* = 0, \]  

(3.3)

\[ \sum_{b_1, b_2} \pi_i^{p(b_1) - p(i)} c_{b_1, b_2} (b_1, z) i r(b_2) + \sum_{b_3, b_4} \pi_i q_i c_{b_3, b_4} (b_3, i r(z)) b_4^* = 0. \]  

(3.4)

Now when \( c_{b, b'} = (-1)^{\text{ht}(\nu)} \pi^e(\nu) \pi_\nu q_\nu \delta_{b, b'}, \) we have

\[ \sum_b \pi^e(\nu) q_\nu (b^*, z) i r(b) - \sum_{b'} \pi^e(\nu) \pi_\nu q_\nu (b'^*, i r(z)) b' = 0, \]

\[ \sum_b \pi^e(\nu - i) \pi_\nu q_\nu (b^*, z) r_i(b) - \sum_{b'} \pi^e(\nu - i) \pi_\nu q_\nu (b'^*, r_i(z)) b' = 0, \]

\[ \sum_b \pi^e(\nu) \pi_\nu q_\nu (b, z) i r(b) - \sum_{b'} \pi^e(\nu) \pi_\nu q_\nu (b', r_i(z)) b'^* = 0, \]

\[ \sum_b \pi^e(\nu - i) \pi_\nu q_\nu (b, z) i r(b) - \sum_{b'} \pi^e(\nu - i) \pi_\nu q_\nu (b', i r(z)) b'^* = 0. \]

These equalities are easily verified by checking when \( z \) is a basis or dual basis element.

Thus the existence of such a \( \Theta \) is verified. Suppose \( \Theta' \) and \( \Theta' \) also satisfy the conditions in (a). Then \( \Theta - \Theta' = \sum c_{b, b'} b^- \otimes b'^+ \) must satisfy (3.1)-(3.4) and has \( c_{b, b} = 0 \) for \( b \in B_0 \). Suppose \( c_{b, b'} = 0 \) for \( b, b' \in B_0' \) for \( \text{ht}(\nu') < n \) and assume \( \text{ht}(\nu) = n \). Then the second sum in (3.1) is zero, so \( i r(\sum_{b_1, b_2} c_{b_1, b_2} (b^*_2, z) b_1) = 0 \). But then \( \sum_{b_1, b_2} c_{b_1, b_2} (b^*_2, z) b_1 = 0 \), whence \( \sum_{b_2} c_{b_1, b_2} (b^*_2, z) = 0 \) for all \( z \in \mathfrak{g} \). Therefore \( c_{b_1, b_2} = 0 \) for all \( b_1, b_2 \in B_0' \). By induction \( \Theta - \Theta' = 0 \) proving uniqueness. \( \square \)

**Example 3.1.2.** Let \( I = I_\mathfrak{f} = i \) as in Example 2.1.1 and let us determine \( \Theta \) in this case using Theorem 3.1.1(b). The obvious basis to choose is \( B = \{ \theta(n) : n \in \mathbb{N} \} \), and then we see from Lemma 1.7.1 that \( \Theta = \sum_n a_n F^{(n)} \otimes E^{(n)} \), where \( a_n = (-1)^n (\pi^q)^{-\binom{n}{2}} \) and \( \Theta \) is consistent. \( \square \)

Recall that the bar involution on \( U \) makes sense under the assumption that the super Cartan datum is consistent.

**Corollary 3.1.3.** Assume that the super Cartan datum is consistent. We have \( \overline{\Theta} \Theta = \Theta \overline{\Theta} = 1 \otimes 1 \) with equality in the completion.

**Proof.** First note that by construction \( \Theta \) is invertible. We have \( \Delta(u) \Theta = \Theta \Sigma(u) \), so \( \Theta \Delta(\overline{u}) = \Theta \Sigma(\overline{u}) = \Theta \Delta(u) \). Now applying the bar involution to both sides and rearranging, we get

\[ \overline{\Theta}^{-1} \Delta(u) = \Delta(u) \overline{\Theta}^{-1}. \]

By uniqueness, \( \overline{\Theta}^{-1} = \Theta \). \( \square \)
We can specialize the identity $\Delta(u) \Theta = \Theta \Delta(u)$ to deduce

$$(E_i \otimes 1) \Theta_{\nu} + (\tilde{J}_i \tilde{K}_i \otimes E_i) \Theta_{\nu-i} = \Theta_{\nu}(E_i \otimes 1) + \Theta_{\nu-i}(\tilde{K}_{-i} \otimes E_i),$$

$$(1 \otimes F_i) \Theta_{\nu} + (F_i \otimes \tilde{K}_{-i}) \Theta_{\nu-i} = \Theta_{\nu}(1 \otimes F_i) + \Theta_{\nu-i}(F_i \otimes \tilde{J}_i \tilde{K}_i).$$

Setting $\Theta_{\leq p} = \sum_{ht\nu=p} \Theta_{\nu}$, we obtain that

$$(E_i \otimes 1 + \tilde{J}_i \tilde{K}_i \otimes E_i) \Theta_{\leq p} - \Theta_{\leq p}(E_i \otimes 1 + \tilde{K}_{-i} \otimes E_i)$$

$$= \sum_{ht\nu=p} (\tilde{J}_i \tilde{K}_i \otimes E_i) \Theta_{\nu} - \sum_{ht\nu=p} \Theta_{\nu}(\tilde{K}_{-i} \otimes E_i),$$

$$(3.5) \quad \text{and} \quad (F_i \otimes \tilde{K}_{-i} + 1 \otimes F_i) \Theta_{\leq p} - \Theta_{\leq p}(F_i \otimes \tilde{J}_i \tilde{K}_i + 1 \otimes F_i)$$

$$= \sum_{ht\nu=p} (F_i \otimes \tilde{K}_{-i}) \Theta_{\nu} - \sum_{ht\nu=p} \Theta_{\nu}(F_i \otimes \tilde{J}_i \tilde{K}_i),$$

$$(3.6)$$

### 3.2. The quantum Casimir

Let $B, B_{\nu}$ be as in Theorem 3.1.1. Let $S$ be the antipode and $m: U \otimes U \to U$ be the multiplication map $u \otimes u' \mapsto uu'$. Applying $m(S \otimes 1)$ to the identities (3.5) and (3.6), we obtain that, for any $p \geq 0$,

$$\sum_{ht\nu=p} \sum_{b \in B_{\nu}} (-1)^{ht\nu} \pi_\nu q_{\nu} \left( S(E_i b^-) b^{*-} + \pi_i^{p(\nu)} S(\tilde{J}_i \tilde{K}_i b^-) E_i b^{*-} \right)$$

$$- \pi_i^{p(\nu)} S(b^- E_i) b^{*-} - S(b^- b_{-i}) b^{*-} + E_i \right)$$

$$= \sum_{ht\nu=p} \sum_{b \in B_{\nu}} (-1)^{ht\nu} \pi_\nu q_{\nu} \left( \pi_i^{p(\nu)} S(\tilde{J}_i \tilde{K}_i b^-) E_i b^{*-} - S(b^- \tilde{K}_{-i}) b^{*-} + E_i \right),$$

and

$$\sum_{ht\nu=p} \sum_{b \in B_{\nu}} (-1)^{ht\nu} \pi_\nu q_{\nu} \left( \pi_i^{p(\nu)} S(b^- F_i b^{*-}) + S(F_i b^-) \tilde{K}_{-i} b^{*-} \right)$$

$$- S(b^- b^{*-} F_i) - \pi_i^{p(\nu)} S(b^- F_i) b^{*-} \tilde{J}_i \tilde{K}_i \right)$$

$$= \sum_{ht\nu=p} \sum_{b \in B_{\nu}} (-1)^{ht\nu} \pi_\nu q_{\nu} \left( S(F_i b^-) \tilde{K}_{-i} b^{*-} - \pi_i^{p(\nu)} S(b^- F_i) b^{*-} \tilde{J}_i \tilde{K}_i \right).$$

Now set $\Omega_{\leq p} = \sum_{ht\nu,p} \sum_{b \in B_{\nu}} (-1)^{ht\nu} \pi_\nu q_{\nu} S(b^-) b^{*-}$. Then observing that

$$S(E_i b^-) b^{*-} + \pi_i^{p(\nu)} S(\tilde{J}_i \tilde{K}_i b^-) E_i b^{*-}$$

$$= \pi_i^{p(\nu)} S(b^-) (\tilde{J}_i \tilde{K}_i E_i) b^{*-} + \pi_i^{p(\nu)} S(b^-) \tilde{J}_i \tilde{K}_i E_i b^{*-} = 0,$$

we have

$$\tilde{J}_i \tilde{K}_i E_i \Omega_{\leq p} - \tilde{K}_i \Omega_{\leq p} E_i$$

$$= \sum_{ht\nu,p} \sum_{b \in B_{\nu}} (-1)^{ht\nu} \pi_\nu q_{\nu} \left( \pi_i^{p(\nu)} S(\tilde{K}_i b^-) E_i b^{*-} - S(b^- \tilde{K}_{-i}) b^{*-} + E_i \right),$$

$$\Omega \leq p F_i - F_i \tilde{K}_i \Omega \leq p \tilde{J}_i \tilde{K}_i$$

$$= - \sum_{ht\nu,p} \sum_{b \in B_{\nu}} (-1)^{ht\nu} \pi_\nu q_{\nu} \left( S(F_i b^-) \tilde{K}_{-i} b^{*-} - \pi_i^{p(\nu)} S(b^- F_i) b^{*-} \tilde{J}_i \tilde{K}_i \right).$$
Example 3.2.1. Let $I = I_1 = i$ as in Examples 2.1.1 and let $\Theta$ be as defined in Example 3.1.2. Then using (2.4),

$$\Omega_{\leq p} = \sum_{1 \leq n \leq p} a_n (\pi q^2)^{-\binom{n}{2}} F^{(n)} K^n E^{(n)}.$$

We note that though this is a rather different construction than the Casimir-type element in [CW], it will nevertheless be used toward a similar purpose.

Let $M \in \mathcal{O}$. Then for any $m \in M$ we have that $\Omega(m) = \Omega_{\leq p} m$ is independent of $p$ when $p$ is large enough. We can write

$$\Omega(m) = \sum_b (-1)^{ht(b)} \pi_{|b|}^{|b|} S(b^+) b^+ m$$

Then we have

$$\tilde{J}_i K_i \Omega E_i \tilde{J}_i = K_i \Omega E_i, \quad \Omega F_i = F_i K_i \Omega \tilde{J}_i, \quad \Omega K_\mu = K_\mu \Omega,$$

as operators on $M$. Therefore for $m \in M^\lambda$, we have

$$\Omega E_i m = (\pi q^2)^{-\binom{i,\lambda+i'}{2}} E_i \Omega m, \quad \Omega F_i = (\pi q^2)^{\binom{i,\lambda+i'}{2}} F_i \Omega m.$$

This can be rephrased in terms of the antipode. Define the $Q(q)^+$-linear map $\overline{\mathcal{S}} : U \to U$ by $\overline{\mathcal{S}}(u) = \overline{\mathcal{S}(u)}$. Then $\Omega \overline{\mathcal{S}}(u) = S(u) \Omega : M \to M$ for $u \in U$.

Let $C$ be a fixed coset of $X$ with respect to $\mathbb{Z}[I] \leq X$. Let $G : C \to \mathbb{Z}$ be a function such that

$$G(\lambda) - G(\lambda - i) = \frac{i(i)}{2} \langle i, \lambda \rangle \quad \text{for all } \lambda \in C, \ i \in I. \quad (3.7)$$

Clearly such a function exists and is unique up to addition of a constant function.

Lemma 3.2.2. Let $\lambda, \lambda' \in C \cap X^+$. If $\lambda \geq \lambda'$ and $G(\lambda) = G(\lambda')$, then $\lambda = \lambda'$.

Let $M \in \mathcal{O}$. For each $\mathbb{Z}[I]$-coset $C$ in $X$, define $M_C = \bigoplus_{\lambda \in C} M^\lambda$. It is clear that

$$M = \bigoplus_{C \in X/\mathbb{Z}[I]} M_C. \quad (3.8)$$

Proposition 3.2.3. Let $M \in \mathcal{O}$, and let $\Omega : M \to M$ be as above.

(a) Assume there exists $C$ as above such that $M = M_C$. Let $G : C \to \mathbb{Z}$ be a function satisfying (3.7). We define a linear map $\Xi : M \to M$ by $\Xi(m) = (\pi q^2)^{G(\lambda)} m$ for all $\lambda \in C$ and $m \in M^\lambda$. Then $\Omega \Xi$ is a locally finite $U$-module homomorphism.

(b) Assume that $M$ is a quotient of $M(\lambda')$. Then $\Omega \Xi$ acts as $(\pi q^2)^{G(\lambda')}$ on $M$.

(c) For $M$ as in (a), the eigenvalues of $\Omega \Xi$ are of the form $(\pi q^2)^c$ for $c \in \mathbb{Z}$.

The operator $\Omega \Xi$ is called the Casimir element of $U$ (though note that the Casimir element formally lives in a completion of $U$).

Proof. We compute that for $m \in M^\lambda$,

$$\Omega \Xi E_i m = \Omega (\pi q^2)^{G(\lambda+i')} E_i m = (\pi q^2)^{G(\lambda+i') - G(\lambda) - s(i, \lambda+i')} E_i \Omega m = E_i \Omega \Xi m.$$

A similar argument applies to the $F_i$, and clearly $\Omega \Xi$ commutes with $K_\mu, J_\mu$ proving the first assertion of (a). The local finiteness claim is a standard category $\mathcal{O}$ type argument. Parts (b) and (c) follow now easily. \qed
3.3. The complete reducibility in $\mathcal{O}_{\text{int}}$. Recall the categories $\mathcal{O}$ and $\mathcal{C}_{\text{int}}$ from \[2.6\]-\[2.7\]. Form another category $\mathcal{O}_{\text{int}} := \mathcal{O} \cap \mathcal{C}_{\text{int}}$.

**Lemma 3.3.1.** Let $M \in \mathcal{C}$. Assume that $M$ is a nonzero quotient of the Verma module $M(\lambda)$ and that $M$ is integrable. Then

(a) $\lambda \in X^+$;

(b) $M_+$ and $M_-$ are either simple or zero.

**Proof.** It is clear that (a) holds by some rank one consideration. An argument similar to that for \[Lu\] Lemma 6.2.1 shows that if $\dim_{\mathbb{Q}(q)} M^\lambda = 1$ then $M$ is simple; in this case, $M$ must be equal to either $M_+$ or $M_-$. Otherwise, $\dim_{\mathbb{Q}(q)} M^\lambda = 2$, then $\dim_{\mathbb{Q}(q)} M^\lambda_+ = \dim_{\mathbb{Q}(q)} M^\lambda_1 = 1$, and we repeat the argument above for the integrable $U$-module $M^\lambda_\pm$. \[ \square \]

**Theorem 3.3.2.** Let $M$ be a $U$-module in $\mathcal{O}_{\text{int}}$. Then $M$ is a sum of simple $U$-submodules.

**Proof.** Note that as discussed in \[2.6\] we may assume that $M = M_+$ or $M = M_-$. Since the case for $M_+$ follows from \[Lu\] Theorem 6.2.2, it is enough to prove the theorem for $M = M_-$. Virtually the same argument as in loc. cit. holds, which we will now sketch.

Using \[3.8\], we may further assume there is a coset $C$ of $\mathbb{Z}[I]$ in $X$ such that $M = M_C$. Then we may pick a function $G$ satisfying \[3.7\] and avail ourselves of Proposition \[3.2.3\]. Since the Casimir element commutes with the action, we may further assume that $M$ lies in a generalized eigenspace of the Casimir element.

Consider the set of singular vectors of $M$(that is, the set of vectors $m \in M$ for which $E_im = 0$ for all $i \in I$) and let $M'$ be the submodule they generate. Then each homogeneous singular vector generates a simple submodule by virtue of Lemma \[3.5.1\] so $M'$ is a sum of simple modules.

It remains to show that $M = M'$, so take $M'' = M/M'$ and suppose $M'' \neq 0$. Then there is a maximal weight $\lambda \in C$ such that $M''^\lambda \neq 0$. Then the Casimir element acts on the submodule generated by a nonzero $m_1 \in M''^\lambda$ by $(-q^2)^{G(\lambda)}$ by Proposition \[3.2.3\] and so in particular $M$ must lie in the generalized $(-q^2)^{G(\lambda)}$-eigenspace of the Casimir element.

On the other hand, $m$ is the image of a vector $\tilde{m} \in M \setminus M'$, the $U^+$-module generated by $\tilde{m}$ contains a singular vector $m_2$ of weight $\eta \geq \lambda$, and the Casimir element acts on the module generated by $m_2$ as $(-q^2)^{G(\eta)}$. Then $G(\eta) = G(\lambda)$ and $\eta \geq \lambda$, so by Lemma \[3.2.2\] $\eta = \lambda$. But the $\tilde{m}$ is a singular vector, contradicting our choice of $m_1$ was nonzero. \[ \square \]

**Corollary 3.3.3.** (a) For $\lambda \in X^+$, the $U$-modules $V(\lambda)_+$ and $V(\lambda)_-$ are simple objects of $\mathcal{O}_{\text{int}}$.

(b) For $\lambda, \lambda' \in X^+$, the $U$-modules $V(\lambda)_+$ and $V(\lambda')_+$, and respectively $V(\lambda)_-$ and $V(\lambda')_-$, are isomorphic if and only if $\lambda = \lambda'$. (Clearly, $V(\lambda)_+$ and $V(\lambda)_-$ are non-isomorphic.)

(c) Any integrable module in $\mathcal{O}$ is a direct sum of simple modules of the form $V(\lambda)_{\pm}$ for various $\lambda \in X^+$.

**Proof.** The argument in \[Lu\] Corollary 6.2.3 holds using our Lemma \[3.3.1\] above. \[ \square \]

3.4. Character formula. Denote by $\rho \in X$ such that $\langle i, \rho \rangle = 1$ for all $i \in I$. We claim the following character formula of $V(\lambda)$ for every $\lambda \in X^+$:

$$ch V(\lambda)_\pm = \frac{\sum_{w \in \mathcal{W}} (-1)^{\ell(w)} e^{w(\lambda - \rho) - \rho}}{\sum_{w \in \mathcal{W}} (-1)^{\ell(w)} e^{w(\rho) - \rho}}. \quad (3.9)$$
This is equivalent to claiming $V(\lambda)$ is always a $\mathbb{Q}(q)^X$-free module for each $\lambda \in X^+$. This character formula holds for $V(\lambda)_+$ with $\lambda \in X^+$ by a theorem of Lusztig [Lus]. A proof of this formula for $V(\lambda)_-$ is possible, but requires techniques outside the scope of this paper.

Assume now that $\lambda \in X^+$ satisfies an evenness condition

$$\langle i, \lambda \rangle \in 2\mathbb{Z}_+, \quad \forall i \in I_\gamma. \tag{3.10}$$

Then the action of $U$ on $V(\lambda)$ factors through an action of the algebra $U/\mathcal{J}$ (see §2.5), and (3.9) holds by [BKM, Theorem 4.9] on the characters of integrable modules of the usual quantum groups. The irreducible integrable modules of the corresponding Kac-Moody superalgebras were known [Kac] to be parametrized by highest weights $\lambda \in X^+$ satisfying (3.10). Hence, for $\lambda \in X^+$ which does not satisfy (3.10), the usual $q$-deformation argument cannot be applied directly to $V(\lambda)_-$. Therefore, the same type of arguments as in [Lus, Chapter 33] show that the algebra $F$ and hence $U$ admit the following equivalent formulations.

**Proposition 3.4.1.** The algebra $F$ is isomorphic to the algebra generated by $\theta_i, i \in I$, subject to the quantum Serre relation as in Proposition 1.7.3.

**Proposition 3.4.2.** The algebra $U$ is isomorphic to the algebra generated by $E_i, F_i (i \in I)$ and $J_\mu, K_\mu (\mu \in Y)$, subject to the relations (3.7) (a)-(f) and the quantum Serre relations for $E_i$’s as well as for $F_i$’s (in place of $\theta_i$’s in Proposition 1.7.3).

As a consequence of (3.9) and Proposition 3.4.1 the character of $U^-$ is given by

$$\text{ch} U^- = \frac{1}{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho) - \rho}} = \prod_{\alpha > 0} (1 - e^{-\alpha})(-1)^{1 + p(\alpha)} \dim g_\alpha, \tag{3.11}$$

where $g$ denotes the Kac-Moody superalgebra of type $(I, \cdot)$ (cf. [Kac]), “$\alpha > 0$” denotes positive roots of $g$, $p(\cdot)$ denotes the parity function, and $g_\alpha$ denotes the $\alpha$-root space.

## 4. Higher Serre relations

In this section we formulate and establish the higher Serre relations, which will be instrumental in determining the action of a braid group on a quantum covering group and integrable modules in a future work.

### 4.1. Higher Serre elements

For $i, j \in I$, and $n, m \geq 0$, set

$$p(n, m; i, j) = mn p(i)p(j) + \binom{m}{2} p(i).$$

For $i \neq j$, define the elements

$$e_{i,j;n,m} = \sum_{r+s=m} (-1)^r \pi_i^{p(n, r; i, j)} (\pi_i q_i)^{-r(n(i,j')+m-1)} E_i^{(r)} E_j^{(n)} E_i^{(s)}, \tag{4.1}$$

$$e'_{i,j;n,m} = \sum_{r+s=m} (-1)^r \pi_i^{p(n, r; i, j)} (\pi_i q_i)^{-r(n(i,j')+m-1)} E_i^{(s)} E_j^{(n)} E_i^{(r)}, \tag{4.2}$$

$$f_{i,j;n,m} = \sum_{r+s=m} (-1)^r \pi_i^{p(n, r; i, j)} d_i^{r(n(i,j')+m-1)} F_i^{(s)} F_j^{(n)} F_i^{(r)}, \tag{4.3}$$

$$f'_{i,j;n,m} = \sum_{r+s=m} (-1)^r \pi_i^{p(n, r; i, j)} d_i^{r(n(i,j')+m-1)} F_i^{(r)} F_j^{(n)} F_i^{(s)}. \tag{4.4}$$
When there is no confusion by fixing \( i \) and \( j \), we will abbreviate \( e_{i,j;n,m} = e_{n,m}, e'_{i,j;n,m} = e'_{n,m}, f_{i,j;n,m} = f_{n,m}, f'_{i,j;n,m} = f'_{n,m} \). Note that we have the equalities

\[
e'_{n,m} = \sigma(e_{n,m}), \quad f'_{n,m} = \sigma \omega^2(f_{n,m}), \quad e_{n,m} = \omega(f_{n,m}), \quad e'_{n,m} = \omega(f_{n,m}). \tag{4.5}
\]

### 4.2. Commutations with divided powers.

**Lemma 4.2.1.** The following hold:

(a) \(-q^{-n(i,j')-2m}_{i} m^{i+nq(j)} E_{i} e_{n,m} + e_{n,m} E_{i} = [m + 1] e_{n,m+1} \)

(b) \(-F_{i} e_{n,m} + \pi_{i}^{m+nq(j)} e_{n,m} F_{i} = [-n (i, j') - m + 1] \pi_{i}^{q(j') + 1} K_i^{-1} e_{n,m-1} \)

**Proof.** When \( i \in I_i \) this is [Lemma 7.1.2](#). We therefore assume \( i \in I_i \). Then, \( (i, j') \in 2Z \) by (d). The left hand side of (a) is

\[
\sum_{r+s=m} (-1)^{r} \pi_{i}^{p(n, r, i, j)} (\pi_{i} q_{i})^{-r(n(i, j') + m - 1)} [s + 1] E_{i}^{(n)} E_{j}^{(s+1)} + \sum_{r+s=m} (-1)^{r+1} \pi_{i}^{p(n, r, i, j)+nq(j)+m} (\pi_{i} q_{i})^{-r(n(i, j') + m - 1) - n(i, j') - 2m} \times [r+1] E_{i}^{(r+1)} E_{j}^{(n)} E_{i}^{(s)} = \sum_{r+s=m+1} (-1)^{r} \pi_{i}^{p(n, r, i, j)} (\pi_{i} q_{i})^{-r(n(i, j') + m)} \times \left( (\pi_{i} q_{i})^{r-(m+1)} \pi_{i}^{r-1+m} [r]_{i} + (\pi_{i} q_{i})^{r} [s]_{i} \right) E_{i}^{(s)} E_{j}^{(n)} E_{i}^{(s)},
\]

where we have used

\[
p(n, r-1; i, j) \equiv p(n, r; i, j) + nq(i)p(j) + (r-1)p(i) \pmod{2} \tag{4.6}
\]

in the last line. Part (a) now follows from the computation

\[
(\pi_{i} q_{i})^{-r(m+1)} \pi_{i}^{r-1+m} [r]_{i} + (\pi_{i} q_{i})^{r} [s]_{i} = q_{i}^{-s} [r]_{i} + (\pi_{i} q_{i})^{r} [s]_{i} = [r + s]_{i} = [m + 1]_{i}.
\]

To prove (b), observe that

\[
F_{i} E_{i}^{(r)} E_{j}^{(n)} E_{i}^{(s)} = \pi_{i}^{r+nq(j)+s} E_{i}^{(r)} E_{j}^{(n)} E_{i}^{(s)} F_{i} - \pi_{i}^{r+nq(j)+1} E_{i}^{(r)} E_{j}^{(n)} E_{i}^{(s-1)} [K_{i}; r-1] - \pi_{i} E_{i}^{(s-1)} [K_{i}; r-1] E_{j}^{(n)} E_{i}^{(s)}.
\]

Therefore,

\[
-F_{i} e_{n,m} + \pi_{i}^{m+nq(j)} e_{n,m} F_{i} = \sum_{r+s=m} (-1)^{r} \pi_{i}^{p(n, r, i, j)+r+nq(j)+1} (\pi_{i} q_{i})^{-r(n(i, j') + m - 1)} E_{i}^{(r)} E_{j}^{(n)} E_{i}^{(s-1)} [K_{i}; s-1] + \sum_{r+s=m} (-1)^{r+1} \pi_{i}^{p(n, r, i, j)+1} (\pi_{i} q_{i})^{-r(n(i, j') + m - 1)} E_{i}^{(r-1)} [K_{i}; r-1] E_{j}^{(n)} E_{i}^{(s)} = \sum_{r+s=m-1} (-1)^{r} \pi_{i}^{p(n, r+1, i, j)+1} (\pi_{i} q_{i})^{-r(n(i, j') + m - 1)} E_{i}^{(r)} [K_{i}; r] E_{j}^{(n)} E_{i}^{(s)} + \sum_{r+s=m-1} (-1)^{r-1} \pi_{i}^{p(n, r+1, i, j)+1} (\pi_{i} q_{i})^{-r(n(i, j') + m - 1)} E_{i}^{(r+1)} [K_{i}; r] E_{j}^{(n)} E_{i}^{(s)}.
\]
where, using (4.6) we compute

\[(\star) = (\pi_i q_i)^{-r}[\tilde{K}_i; -s - n \langle i, j'r \rangle - 2r] - (\pi_i q_i)^{-n(i,j')-m+1-r}[\tilde{K}_i; -r]
= (\pi_i q_i)^{-r}(\pi_i q_i)^{-m+1-n(i,j')-r} - (\pi_i q_i)^{-n(i,j')-m+1-r}(\pi_i q_i)^{-r}J_i\tilde{K}_i
\]

\[+ (\pi_i q_i)^{-n(i,j')-m+1-r}q_i^r - (\pi_i q_i)^{-r}q_i^{n(i,j')+m-1}'\tilde{K}_i^{-1}.
\]

This proves (b). □ □

The next result, which is a \(\pi\)-analogue of [Liu, Lemma 7.1.3], follows by a straightforward induction argument.

**Lemma 4.2.2.** The following formulas hold:

(a)

\[E_i^{(N)} e_{n,m} = \sum_{k=0}^{N} (-1)^k q_i^{N(n(i,j')+2m)+(N-1)k} \frac{\pi_i}{\pi_i - q_i^{-1}} \left(\begin{array}{c} m+k \\ k \end{array}\right) e_{n,m+k} E_i^{(N-k)};
\]

(b)

\[F_i^{(M)} e_{n,m} = \sum_{h=0}^{M} (-1)^h q_i^{-(M-1)h} \frac{\pi_i}{\pi_i - q_i^{-1}} \left(\begin{array}{c} M(m+np(j))+(M-m)h \\ h \end{array}\right) \tilde{K}_i^{-h} e_{n,m-h} F_i^{(M-h)}.
\]

**Lemma 4.2.3.** Let \(m = 1 - n \langle i, j' \rangle\). Then

\[F_j e_{n,m} - \pi_j^{mp(i)+n} e_{n,m} F_j = \pi_j^n \left(\begin{array}{c} q_j^{n-1} \\ \pi_j q_j - q_j^{-1} \end{array}\right) e_{n-1,m} - J_j\tilde{K}_j \frac{q_j^{1-n}}{\pi_j q_j - q_j^{-1}} e_{n-1,m}.
\]

**Proof.** To begin, if \(r + s = m\), then

\[F_j E_i^{(r)} E_j^{(n)} E_i^{(s)} = \pi_j^{mp(i)+n} E_i^{(r)} E_j^{(n)} F_j - \pi_j^{mp(i)+1} E_i^{(r)} E_j^{(n-1)}[\tilde{K}_j, n-1]E_i^{(s)}.
\]

Since \(m = 1 - \langle i, j' \rangle\), the exponent of \(\pi_i q_i\) in \(e_{n,m}\) is 0; see (4.1). Therefore,

\[F_j e_{n,m} - \pi_j^{mp(i)+n} e_{n,m} F_j
= -\pi_j \sum_{r+s=m} (-1)^r \pi_i^{p(n-1,r;i,j)}[\tilde{K}_j; 1 - n - ra_{ij}] E_i^{(r)} E_j^{(n-1)} E_i^{(s)}
\]

\[= -\pi_j \sum_{r+s=m} (-1)^r \pi_i^{p(n-1,r;i,j)} q_i^{-r(i,j')} (\pi_j q_j)^{1-n} \tilde{K}_j E_i^{(r)} E_j^{(n-1)} E_i^{(s)}
\]

\[+ \pi_j \sum_{r+s=m} (-1)^r \pi_i^{p(n-1,r;i,j)} (\pi_j q_j)^{r(i,j')} (\pi_j q_j)^{n-1} \tilde{K}_j^{-1} E_i^{(r)} E_j^{(n-1)} E_i^{(s)}.
\]
We have used \( p(n, r; i, j) = p(n - 1, r; i, j) + rp(i)p(j) \) to simplify the second line, and 
\[
q^{-r(i', j')} = q^{-r(j', i')} \quad \text{and} \quad \pi^{-r(i', j')} = 1 = \pi^{-r(j', i')}
\]
in the two subsequent lines. Since \((n - 1) \langle i, j' \rangle + m - 1 = -\langle i, j' \rangle\), the result follows. □

As a consequence of the previous lemmas we obtain a generalization of the quantum Serre relations.

**Proposition 4.2.4** (Higher Serre Relations). Let \( i, j \in I \) be distinct. If \( m > -n \langle i, j' \rangle \), then \( e_{i,j;n,m} = 0 \).

**Proof.** As before, fix \( i \) and \( j \) and write \( e_{n,m} = e_{i,j;n,m} \). Note that \( e'_{1,1-\langle i, j' \rangle} = \sigma(e_{1,1-\langle i, j' \rangle}) \) is just the usual quantum Serre relations (see Proposition [1,7,3]). Using Lemma 4.2.1(a), it follows by induction on \( m \) that \( e_{1,m} = 0 \) for \( m \geq 1 - \langle i, j' \rangle \). Now, let \( n > 1 \) and assume that \( e_{n-1,m} = 0 \) for all \( m > (1 - n) \langle i, j' \rangle \). By Lemma 4.2.1(b), \( e_{n-1,\langle i, j' \rangle} \) supercommutes with \( F_1 \), and by Lemma [1,2,3] and induction, it supercommutes with \( F_1 \) (note that \( m = 1 - n \langle i, j' \rangle > (1 - n) \langle i, j' \rangle \)). It trivially supercommutes with \( F_k \) for \( k \neq i, j \). Therefore, by Proposition 2.335 we deduce that \( e_{n-1,\langle i, j' \rangle} = 0 \). Again, using Lemma 4.2.1(a) and induction on \( m \), the \( e_{n,m} = 0 \) for \( m \geq 1 - n \langle i, j' \rangle \). □

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