Solutions of quantum Yang-Baxter equation related to $U_q(gl(2))$ algebra 
and associated integrable lattice models

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Abstract

A coloured braid group representation (CBGR) is constructed with the help of some modified universal $\mathcal{R}$-matrix, associated to $U_q(gl(2))$ quantised algebra. Explicit realisation of Faddeev-Reshetikhin-Takhtajan (FRT) algebra is built up for this CBGR and new solutions of quantum Yang-Baxter equation are subsequently found through Yang-Baxterisation of FRT algebra. These solutions are interestingly related to nonadditive type quantum $\mathcal{R}$-matrix and have a nontrivial $q \to 1$ limit. Lax operators of several concrete integrable models, which may be considered as some ‘coloured’ extensions of lattice nonlinear Schrödinger model and Toda chain, are finally obtained by taking different reductions of such solutions.

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1. Introduction

Quantum integrable systems have received a lot of interest in recent years, particularly because they represent a class of nontrivial, exactly solvable models with many possible physical applications [1,2]. A common feature of each such model is the presence of some mutually commutating conserved quantities including the Hamiltonian, all of which can be generated from the associated Lax operator. This Lax operator $L(\lambda)$ is usually expressed as a matrix having noncommuting operator valued elements, which are required to satisfy the quantum Yang-Baxter equation (QYBE) given by [2]

$$R(\lambda, \mu) \, L_1(\lambda) \, L_2(\mu) = L_2(\mu) \, L_1(\lambda) \, R(\lambda, \mu),$$

(1.1)

where $L_1(\lambda) = L(\lambda) \otimes 1$, $L_2(\lambda) = 1 \otimes L(\lambda)$, and $\lambda$, $\mu$ are spectral parameters. Though $\lambda$, $\mu$ are often taken as single component objects, in this article we shall also investigate on multicomponent spectral parameter dependent solutions of QYBE. However such solutions may easily be reduced to the single component case, by considering all but one components as some dependent functions of the remaining one. The $c$-number valued elements of the quantum $R(\lambda, \mu)$-matrix appearing in (1.1) act like structure constants and due to the associativity of QYBE they obey the Yang-Baxter equation (YBE), which may be written in the matrix form as

$$R_{12}(\lambda, \mu) \, R_{13}(\lambda, \gamma) \, R_{23}(\mu, \gamma) = R_{23}(\mu, \gamma) \, R_{13}(\lambda, \gamma) \, R_{12}(\lambda, \mu),$$

(1.2)

where we have used the standard notation: $R_{12} = R \otimes 1$, etc. . The QYBE (1.1) plays a crucial role in determining the integrability of a system, as well as solving it exactly through quantum inverse scattering method (QISM) [2]. Therefore the search for obtaining nontrivial solutions of QYBE might be considered as an important step towards constructing such exactly solvable models.

Curiously, quantum group related algebras and their different realisations [3,4] are found to be intimately connected with certain type of solutions of QYBE and associated
integrable models [5]. It is also realised that some basic relations occurring in the well known Faddeev-Reshetikhin-Takhtajan (FRT) approach to quantum group [6], can be used as a convenient tool for making such connection [7]. These algebraic relations might be written in the form

\[ R^+ L_1^{(\pm)} L_2^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} R^+ , \quad R^+ L_1^{(\pm)} L_2^{(-)} = L_2^{(-)} L_1^{(\pm)} R^+ , \]  

(1.3a, b)

where \( L_1^{(\pm)} = L^{(\pm)} \otimes 1 \), \( L_2^{(\pm)} = 1 \otimes L^{(\pm)} \) and \( L^{(\pm)} \) are upper (lower) triangular matrices with operator valued elements. Due to the associativity of FRT algebra (1.3), the upper triangular matrix \( R^+ \) satisfies the condition

\[ R^+_{12} R^+_{13} R^+_{23} = R^+_{23} R^+_{13} R^+_{12} , \]  

(1.4)

which, in turn, leads to a braid group representation (BGR) for the matrix \( \hat{R}^+ = \mathcal{P} R^+ \) (\( \mathcal{P} \) being the permutation operator with the property \( \mathcal{P} A \otimes B = B \otimes A \mathcal{P} \)): \( \hat{R}^+_{12} \hat{R}^+_{23} \hat{R}^+_{12} = \hat{R}^+_{23} \hat{R}^+_{12} \hat{R}^+_{23} \). Nevertheless, for the sake of convenience we would call the \( R^+ \)-matrix itself, satisfying (1.4), as the BGR in what follows. It may be observed that the QYBE (1.1) and FRT algebra (1.3) are much similar in form, though the latter is void of any spectral parameter. However, by properly inserting spectral parameters in the FRT relations through Yang-Baxterisation procedure, it is possible to build up for some particular cases an ‘ancestor’ Lax operator \( L(\lambda) \) satisfying QYBE [7]. Specific realisations of the operator elements corresponding to this \( L(\lambda) \) operator, e.g. , with physical bosonic or q-bosonic modes, may then lead to many concrete integrable models including some new ones [8]. Noteworthy, all these integrable models are associated with additive type quantum \( R \)-matrices, which depend only on the difference of spectral parameters: \( R(\lambda, \mu) \equiv R(\lambda - \mu) \).

Quite recently some generalisations of BGR (1.4) have interestingly appeared in the literature [9-11], which are sometimes called as ‘coloured’ braid group representations (CBGRs). These CBGRs obey the relation

\[ R_{12}^{+(\lambda, \mu)} R_{13}^{+(\lambda, \gamma)} R_{23}^{+(\mu, \gamma)} = R_{23}^{+(\mu, \gamma)} R_{13}^{+(\lambda, \gamma)} R_{12}^{+(\lambda, \mu)} , \]  

(1.5)
where $\lambda$, $\mu$, $\gamma$ are ‘colour’ parameters. Though usually $\hat{R}^{+(\lambda,\mu)} = PR^{+(\lambda,\mu)}$ is defined as the CBGR, we would call the $R^{+(\lambda,\mu)}$-matrix itself as CBGR in analogy with the previous standard case. It may be noticed that the form of eqn. (1.5) is essentially same with YBE (1.2), if one interprets the ‘colour parameters’ as ‘spectral parameters’. Now Yang-Baxterisation of FRT algebra related to this CBGR should lead to new integrable models, associated with more general kind of quantum $R$-matrix. This possibility was actually considered in ref.12 for the case of a particular CBGR given by

$$R^{+(\lambda,\mu)} = \begin{pmatrix} q^{1-(\lambda-\mu)} & q^{\lambda+\mu} (q - q^{-1}) s^{-(\lambda-\mu)} & q^{-(\lambda+\mu)} \\ 0 & q^{1+(\lambda-\mu)} & \end{pmatrix}, \quad (1.6)$$

which might be obtained from the fundamental representation of universal $R$-matrix corresponding to $U_q(gl(2))$ quantised algebra [11]. Note that at $\lambda = \mu = 0$ limit the above CBGR reduces to the wellknown BGR

$$R^{+} = \begin{pmatrix} q & q - q^{-1} \\ 0 & q \end{pmatrix}, \quad (1.7)$$

related to the $U_q(sl(2))$ algebra. Through Yang-Baxterisaion of FRT algebra for the case of CBGR (1.6), it is possible to construct as before an ‘ancestor’ Lax operator as a solution of QYBE. But, contrary to the previous cases, this Lax operator interestingly shares non-additive type quantum $R$-matrix and yields some ‘coloured’ extensions of the integrable systems like lattice sine-Gordon model, Ablowitz-Ladik model etc. [12]. In spite of these achievements one obvious drawback in dealing with the CBGR (1.6) is that, its dependence on the colour parameters $\lambda$, $\mu$ becomes trivial at the naive $q \to 1$ limit. However, Lax operator of many interesting integrable systems like lattice nonlinear Schrödinger (LNLS) model, Toda chain etc. are known to be associated with the rational limit ($q \to 1$) of the BGR (1.7). Consequently, one finds it a bit difficult to construct ‘coloured’ generalisation of this type of integrable models by starting from the CBGR (1.6). To overcome
this problem, we modify the previously obtained co-product and universal $R$-matrix related to $U_q gl(2)$ quantum algebra through a scaling transformation, as will be described in sec.2 of this article. Fundamental representation of such modified universal $R$-matrix then yields a CBGR, which interestingly has a nontrivial colour parameter dependence at $q \to 1$ limit and so is quite suitable for our present purpose. In sec.3 we build up explicit realisations of FRT algebra corresponding to this new CBGR and in sec.4 deal with related Yang-Baxterisation procedure, which leads to new solutions of QYBE. The possibility of generating ‘coloured’ extension of LNLS or Toda chain type integrable models, associated with nonadditive quantum $R$-matrix, is also briefly discussed in sec.4. Sec.5 is the concluding section.

2. Construction of a CBGR from $U_q (gl(2))$ quasi-triangular Hopf algebra

Before attempting to generate the CBGR which has a nontrivial rational limit, we would review a little about the construction of CBGR (1.6) by employing the techniques of quasitriangular Hopf algebra. As it is well known for a quasitriangular Hopf algebra $\mathcal{A}$, there exists an invertible universal $R$-matrix ($R \in \mathcal{A} \otimes \mathcal{A}$) such that it interrelates comultiplications $\Delta, \Delta'$ through $\Delta(a)R = R\Delta'(a)$, where $a \in \mathcal{A}$ and satisfies the following conditions

\[
(id \otimes \Delta)R = R_{13}R_{12}, \quad (\Delta \otimes id)R = R_{13}R_{23}, \quad (S \otimes id)R = R^{-1},
\]

$S$ being the antipode. The above relations also imply that the $R$-matrix would satisfy an equation of the form (1.4).

If one considers now the case of $U_q(gl(2))$ algebra, apart from the usual generators $S_3, S_\pm$ of $U_q(sl(2))$, a central element $\Lambda$ is included in the picture with the commutation relations [13,11]

\[
[S_3, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = \frac{\sin(2\alpha S_3)}{\sin \alpha}, \quad [\Lambda, S_\pm] = [\Lambda, S_3] = 0; \quad q = e^{i\alpha}.
\]
The associated comultiplication may be given by

\[
\Delta(S_+) = S_+ \otimes q^{-S_3} \cdot (qs)^\Lambda + \left(\frac{s}{q}\right)^\Lambda \cdot q^{S_3} \otimes S_+ ,
\]

\[
\Delta(S_-) = S_- \otimes q^{-S_3} \cdot (qs)^{-\Lambda} + \left(\frac{s}{q}\right)^{-\Lambda} \cdot q^{S_3} \otimes S_-, \tag{2.2}
\]

\[
\Delta(S_3) = S_3 \otimes 1 + 1 \otimes S_3 , \quad \Delta(\Lambda) = \Lambda \otimes 1 + 1 \otimes \Lambda ,
\]

where an additional parameter \(s\) is appearing due to the symmetry of the algebra. The other Hopf algebraic structures like co-unit, antipode can be consistently defined and the universal \(\mathcal{R}\)-matrix may also be constructed as [11]

\[
\mathcal{R} = q^{2(S_3 \otimes S_3 + S_3 \otimes \Lambda - \Lambda \otimes S_3)} \cdot \sum_{m=0}^\infty \frac{(1 - q^{-2})^m}{[m, q^{-2}]!} \left(q^{S_3} (qs)^{-\Lambda} S_+ \right)^m \otimes \left(q^{-S_3} \left(\frac{s}{q}\right)^\Lambda S_- \right)^m , \tag{2.3}
\]

where \([m, q] = (1 - q^m)/(1 - q)\) and \([m, q]! = [m, q] \cdot [m - 1, q] \cdot \ldots \cdot 1\).

Denoting now the eigenvalue of the Casimir like operator \(\Lambda\) by \(\lambda\) and the corresponding \(n\)-dimensional irreducible representation of algebra (2.1) as \(\Pi_\lambda^n\), we may obtain the ‘colour’ representation \((\Pi_\lambda^n \otimes \Pi_\mu^n)\mathcal{R}\), giving a finite dimensional CBGR satisfying (1.5). In particular for the two dimensional representation \(\Pi_\lambda^2\) through identity operator and Pauli matrices: \(\Lambda = \lambda \mathbf{1} , \vec{S} = \frac{1}{2} \vec{\sigma}\) one gets the CBGR (1.6).

Now for constructing the CBGR which would be useful for our purpose, we first observe that the \(U_q(gl(2))\) quantised algebra (2.1) remains invariant under a scaling transformation of the generator \(\Lambda : \Lambda \rightarrow \frac{1}{c} \Lambda , c\) being an arbitrary complex number. However the related comultiplication (2.2) interestingly gets modified due to this transformation. Therefore by using such scaling of generator \(\Lambda\), it is possible to associate a somewhat more general co-multiplication structure than (2.2), with the algebra (2.1). After setting \(t = q^{-c}\) and redefining the parameter \(s\) in a suitable way, one can cast this modified structure in the form

\[
\Delta(S_+) = S_+ \otimes q^{-S_3} \cdot (ts)^\Lambda + \left(\frac{s}{t}\right)^\Lambda \cdot q^{S_3} \otimes S_+ ,
\]

\[
\Delta(S_-) = S_- \otimes q^{-S_3} \cdot (ts)^{-\Lambda} + \left(\frac{s}{t}\right)^{-\Lambda} \cdot q^{S_3} \otimes S_-, \tag{2.4}
\]

\[
\Delta(S_3) = S_3 \otimes 1 + 1 \otimes S_3 , \quad \Delta(\Lambda) = \Lambda \otimes 1 + 1 \otimes \Lambda ,
\]
where $q$, $s$ and $t$ are being treated as three independent parameters. By applying now the scaling transformation of generator $\Lambda$ to the expression (2.3), we may easily obtain the universal $R$-matrix related to the co-product (2.4) as

$$R = q^{2S_3} \otimes S_3 \cdot t^2 (S_3 \otimes \Lambda - \Lambda \otimes S_3) \cdot \sum_{m=0}^{\infty} \frac{(1 - q^{-2})^m}{[m, q^{-2}]!} \left( q^{S_3} (ts)^{-\Lambda} S^+_3 \right)^m \otimes \left( q^{-S_3} \left( \frac{s}{t} \right)^\Lambda S^-_3 \right)^m.$$ 

(2.5)

Evidently, for the value $t = q$ the above universal $R$-matrix would reproduce the previous expression (2.3). On the other hand, at the limit $s = 1$ (2.5) curiously coincides with the case which was obtained very recently [14] by using Reshetikhin’s procedure of multiparameter deformation.’

After finding the universal $R$-matrix (2.5), one may take as before its two dimensional representation through identity operator and Pauli matrices. This leads us to the CBGR

$$R^{+(\lambda, \mu)} = \begin{pmatrix} qt^{- (\lambda - \mu)} & t^{\lambda + \mu} & (q - q^{-1}) s^{-(\lambda - \mu)} & t^{-(\lambda + \mu)} & qt^{\lambda - \mu} \\ t^{\lambda + \mu} & 0 & (q - q^{-1}) s^{-(\lambda - \mu)} & t^{-(\lambda + \mu)} & qt^{\lambda - \mu} \\ (q - q^{-1}) s^{-(\lambda - \mu)} & t^{-(\lambda + \mu)} & 0 & (q^{-1} - q) s^{-(\lambda - \mu)} & t^{-(\lambda + \mu)} \\ t^{-(\lambda + \mu)} & (q^{-1} - q) s^{-(\lambda - \mu)} & t^{-(\lambda + \mu)} & 0 & (q^{-1} - q) s^{-(\lambda - \mu)} \\ qt^{\lambda - \mu} & t^{-(\lambda + \mu)} & (q^{-1} - q) s^{-(\lambda - \mu)} & t^{-(\lambda + \mu)} & qt^{\lambda - \mu} \end{pmatrix},$$

(2.6)

which would naturally reduce to (1.6) for the case $t = q$. At this point we may mention about some symmetry transformation of eqn. (1.5) related to the ‘particle conserving’ BGRs [15,12]. By applying such a transformation on the BGR (1.7) it is also possible to obtain the CBGR (2.6), without going into the associated Hopf algebra structures.

It is worth noticing that the colour parameter dependence of the CBGR (2.6) remain nontrivial even at the $q \to 1$ limit. So according to our discussion in sec.1, this newly obtained CBGR should help us in constructing ‘coloured’ extension of integrable models related to the rational case. For this purpose we shall next build up explicit realisation of FRT algebra related to this CBGR.

3. Realisation of FRT algebra related to the CBGR

Due to the presence of extra colour parameters, the more frequently used form of FRT relations (1.3) has to be modified consistently [6,12] for the case dealing with a
CBGR $R^{+}(\lambda, \mu)$: 

\[
R^{+}(\lambda, \mu) L_1^{(\pm)}(\lambda) L_2^{(\pm)}(\mu) = L_2^{(\pm)}(\mu) L_1^{(\pm)}(\lambda) R^{+}(\lambda, \mu)
\]

\[
R^{+}(\lambda, \mu) L_1^{(+)}(\lambda) L_2^{(-)}(\mu) = L_2^{(-)}(\mu) L_1^{(+)}(\lambda) R^{+}(\lambda, \mu),
\]

(3.1a, b)

where $L_1^{(\pm)}(\lambda) = L^{(\pm)}(\lambda) \otimes 1$, $L_2^{(\pm)}(\lambda) = 1 \otimes L^{(\pm)}(\lambda)$ and $L^{(\pm)}(\lambda)$ being upper (lower) triangular matrices. By using this coloured version of FRT algebra, explicit realisations of $L^{\pm}(\lambda)$ matrices were previously built up for the CBGR (1.6) [12]. Now for constructing similar realisation for case of CBGR (2.6), we assume the form of corresponding $L^{\pm}(\lambda)$ matrices as

\[
L^{(\pm)}(\lambda) = \rho^{-2\lambda} \begin{pmatrix} t^A \tau_1^+ & (ts)^{-\lambda} \cdot s^A \tau_{21} \\ 0 & t^{-\lambda} \tau_2^+ \end{pmatrix}, \quad L^{(-)}(\lambda) = \rho^{-2\lambda} \begin{pmatrix} t^A \tau_2^- & 0 \\ (ts)^\lambda \cdot s^A \tau_{12} & t^{-\lambda} \tau_2^- \end{pmatrix}.
\]

(3.2a, b)

Inserting the above form of $L^{\pm}(\lambda)$ matrices as well as CBGR (2.6) in (3.1), we surprisingly find that the coloured version of FRT algebra reduces finally to the following algebraic relations for the operators $\tau_i^{\pm}$, $\tau_{ij}$ ($i, j = 1, 2$), $\rho$ and $\Lambda$, where evidently colour parameters are absent.

\[
\tau_i^{\pm} \tau_j = q^{\pm 1} \tau_{ij} \tau_i^{\pm}, \quad \tau_i^{\mp} \tau_{ji} = q^{\mp 1} \tau_{ji} \tau_i^{\mp}, \quad \rho \tau_{12} = t \tau_{12} \rho, \quad \rho \tau_{21} = t^{-1} \tau_{21} \rho, \quad \rho \tau_{12} \tau_{21} = -(q - q^{-1}) \left( \tau_{1}^+ \tau_{2}^- - \tau_{1}^- \tau_{2}^+ \right), \quad [\Lambda, \rho] = [\Lambda, \tau_i^{\pm}] = [\Lambda, \tau_{ij}] = 0,
\]

(3.3)

with all generators $\rho$, $\tau_i^{\pm}$ commuting among themselves. It may be verified that in addition to the Casimir like element $\Lambda$, there exists other Casimir operators of the above algebra as

\[
D_1 = \tau_1^+ \tau_1^-, \quad D_2 = \tau_2^+ \tau_2^-, \quad D_3 = \tau_1^+ \tau_2^+, \quad D_4 = \rho(\tau_1)^c, \quad D_5 = 2 \cos \alpha \left( \tau_1^+ \tau_2^- + \tau_1^- \tau_2^+ \right) - [\tau_{12}, \tau_{21}]_+.
\]

(3.4)

where $c = -\log_q t$. It is also interesting to notice that the subalgebra of (3.3), containing only the generators $\tau$, coincides with the FRT algebra related to the $U_q(sl(2))$ case [7].

Now it is rather easy to check that the choice of generators $\rho$, $\tau$ in the particular form

\[
\rho = t^{S_1}, \quad \tau_1^\pm = q^{\mp S_3}, \quad \tau_2^\pm = q^{\mp S_3}, \quad \tau_{12} = -(q - q^{-1}) S_+, \quad \tau_{21} = (q - q^{-1}) S_-
\]

(3.5)
reduces algebra (3.3) to the $U_q(gl(2))$ given as (2.1). Consequently, by substituting (3.5) to (3.2a,b) one gets explicit forms of upper (lower ) triangular matrices $L^{(\pm)}(\lambda)$ expressed through the generators $S_{\pm}$, $S_3$, $\Lambda$ of the $U_q(gl(2))$ quantised algebra:

$$L^{(+)}(\lambda) = t^{-2\lambda S_3} \begin{pmatrix} t^\Lambda q^{S_3} & (q - q^{-1}) (ts)^{-\lambda} \cdot s^\Lambda S_- \\ 0 & t^{-\Lambda} q^{-S_3} \end{pmatrix},$$

$$L^{(-)}(\lambda) = t^{-2\lambda S_3} \begin{pmatrix} t^\Lambda q^{-S_3} \\ -(q - q^{-1}) (ts)^{\lambda} \cdot s^{-\Lambda} S_+ \\ 0 \end{pmatrix}.$$  

(3.6)

Notice that if one substitutes $\lambda = 0 = \Lambda$ in the above expressions of $L^{(\pm)}(\lambda)$ matrices, they would reduce to well known $L^{(\pm)}$ matrices associated with the BGR (1.7). On the other hand at the limit $q = t$, they yield the $L^{(\pm)}(\lambda)$ matrices [12] related to the CBGR (1.6). We may note further that similar to the FRT algebra (1.3) its colour counterpart (3.1) also exhibits the symmetry that if $L^{(\pm,1)}(\lambda)$ and $L^{(\pm,2)}(\lambda)$ are two independent solution of the algebra acting on different quantum spaces, then their matrix product $\Delta L^{(\pm)}(\lambda) = (L^{(\pm,1)}(\lambda) \cdot L^{(\pm,2)}(\lambda))$ would also be a solution with the same CBGR. Using this important property and the explicit forms (3.6) of $L^{(\pm)}(\lambda)$ we may easily derive the coproduct structure of the related quantised algebra. Curiously one finds that, though the $L^{(\pm)}(\lambda)$-matrices contain colour parameter $\lambda$ in a rather complicated way, the resultant coproducts for the generators $S_{\pm}$, $S_3$ and $\Lambda$ are free from such parameters and in fact coincides with the modified coproduct (2.4) of $U_q(gl(2))$.

Next we may observe that if $R^{+(\lambda,\mu)}$ be a CBGR satisfying eqn. (1.5), then the matrix $R^{-(\lambda,\mu)}$ given by

$$R^{-(\lambda,\mu)} = \mathcal{P} \left\{ R^{+(\mu,\lambda)} \right\}^{-1} \mathcal{P},$$

(3.7)

( $\mathcal{P}$ being the permutation operator with the property $\mathcal{P} A \otimes B = B \otimes A \mathcal{P}$ ) would also be a solution of the same equation. Moreover if $R^{+(\lambda,\mu)}$ satisfies the FRT relations (3.1) for a certain choice of $L^{(\pm)}(\lambda)$-matrices, then by using (3.7) it is rather easy to verify that the
CBGR $R^{-(\lambda,\mu)}$ would automatically satisfy the following complementary FRT relations for the same choice of $L(\pm)(\lambda)$-matrices:

\[
\begin{align*}
R^{-(\lambda,\mu)} L_1^{(\pm)}(\lambda) L_2^{(\pm)}(\mu) &= L_2^{(\pm)}(\mu) L_1^{(\pm)}(\lambda) R^{-(\lambda,\mu)}, \\
R^{-(\lambda,\mu)} L_1^{(-)}(\lambda) L_2^{(+)}(\mu) &= L_2^{(+)}(\mu) L_1^{(-)}(\lambda) R^{-(\lambda,\mu)}.
\end{align*}
\]  

(3.8a, b)

By applying now the relation (3.7) to the case of particular CBGR (2.6), one gets the corresponding lower triangular CBGR as

\[
R^{-(\lambda,\mu)} = \begin{pmatrix}
q^{-1}t^{-(\lambda-\mu)} & t^{\lambda+\mu} & 0 \\
-t^{\lambda+\mu} & -(q - q^{-1})s^{\lambda-\mu} & t^{-(\lambda+\mu)} \\
0 & t^{-\lambda-\mu} & q^{-1}t^{\lambda-\mu}
\end{pmatrix}.
\]  

(3.9)

So this CBGR naturally satisfies the complementary FRT relations (3.8) and the form of related $L(\pm)(\lambda)$ matrices would again be given by (3.6).

4. Yang-Baxterisation of coloured FRT algebra and related integrable models

As mentioned already, it is possible to construct spectral parameter dependent solutions of QYBE (1.1), through Yang-Baxterisation of FRT algebra related to the standard BGRs [7]. Such solutions may then be employed to build up in a coherent way Lax operators of a class of quantum integrable systems, related to trigonometric quantum $R$-matrix. Moreover by taking $q \to 1$ limit of these solutions, one can also generate another class of integrable systems including well known LNLS model, Toda chain, etc., all of which interestingly share rational and additive type quantum $R$-matrix [8]. The above procedure for constructing integrable models should also be applicable to the case of CBGRs, through Yang-Baxterisation of corresponding FRT relations ((3.1),(3.8)). An initial attempt was made in this direction [12] for the case of particular CBGR (1.6), and consequently some ‘coloured’ extension of lattice sine-Gordon model, Ablowitz-Ladik model etc. were obtained. However due to the apparent difficulty in extracting nontrivial $q \to 1$ limit of the CBGR (1.6), possible ‘coloured’ extension of integrable systems like LNLS model or Toda chain have not been considered so far. At present we aim to fill up this gap by
constructing the Yang-Baxterisation of FRT algebra related to the modified CBGR (2.6) and then properly switching over to the corresponding $q \to 1$ limit. So, in parallel to the approach of ref.12, at first we seek solutions of QYBE (1.1) for the two-component spectral parameter case:

$$R(\lambda, \lambda'; \mu, \mu') \ L_1(\lambda, \lambda') \ L_2(\mu, \mu') = L_2(\mu, \mu') \ L_1(\lambda, \lambda') \ R(\lambda, \lambda'; \mu, \mu'), \quad (4.1)$$

by substituting in it $R(\lambda, \lambda'; \mu, \mu')$ and $L_1(\lambda, \lambda')$ matrices of the form

$$R(\lambda, \lambda'; \mu, \mu') = q^{(\lambda'-\mu')} R^{+}(\lambda, \mu) - q^{-(\lambda'-\mu')} R^{-}(\lambda, \mu), \quad (4.2)$$

$$L(\lambda, \lambda') = q^{\lambda'} \cdot L^{(+)}(\lambda) + q^{-\lambda'} \cdot L^{(-)}(\lambda), \quad (4.3)$$

where the CBGRs $R^{\pm}(\lambda, \mu)$ and the corresponding upper (lower) triangular matrices $L^{(\pm)}(\lambda)$ are given through the expressions ((2.6), (3.9)) and (3.2a,b) respectively. Notice that the $L$-operator (4.3) is depending on two independent spectral parameters $\lambda$ and $\lambda'$, the first one is related to the colour parameter of FRT algebra, while the latter one comes through Yang-Baxterisation scheme. To verify that $R$-matrix (4.2) and $L$-operator (4.3) thus constructed are the solutions of QYBE, we insert them to (4.1) and match the coefficients of different powers in spectral parameters $\lambda'$, $\mu'$. As a result we obtain a set of algebraic relations independent of parameters $\lambda'$, $\mu'$ and observe that all of these relations, except one, coincide with the coloured FRT relations ((3.1),(3.8)) and hence are naturally satisfied by construction. The only remaining equation is

$$R^{+}(\lambda, \mu) \ L_1^{(-)}(\lambda) \ L_2^{(+)}(\mu) - R^{-}(\lambda, \mu) \ L_1^{(+)}(\lambda) \ L_2^{(-)}(\mu) =$$

$$L_2^{(+)}(\mu) \ L_1^{(-)}(\lambda) \ R^{+}(\lambda, \mu) - L_2^{(-)}(\mu) \ L_1^{(+)}(\lambda) \ R^{-}(\lambda, \mu). \quad (4.4)$$

By substituting explicit forms of $L^{(\pm)}(\lambda)$ (3.2a,b) and $R^{\pm}(\lambda, \mu)$ ((2.6),(3.9)) in the above equation and using the operator algebra (3.3), one can directly verify that this remaining equation would also be satisfied. So we may conclude that $R(\lambda, \lambda'; \mu, \mu')$ and $L(\lambda, \lambda')$ matrices, given through expressions (4.2) and (4.3) respectively, are indeed a solution of two component spectral parameter dependent QYBE (4.1).
However, as remarked earlier, single-component dependent solutions of QYBE (1.1) are more frequently used for generating concrete integrable models and to extract such solutions at the present case we may consider the colour parameters $\lambda, \mu$ not as independent ones, but as some functions of other spectral parameters $\lambda', \mu'$. For simplicity one may choose $\lambda = \theta \lambda', \mu = \theta \mu'$, where $\theta$ is a constant parameter and may set $\Lambda = 0$. Evidently for this simple choice the two-component dependent solution (4.2), (4.3) would reduce to a single-component dependent one, which may be explicitly written as

$$R_1(\lambda', \mu') = \begin{pmatrix}
\begin{bmatrix}
t^{-\theta(\lambda'-\mu')} a(\lambda'-\mu') \\
t^{\theta(\lambda'+\mu')} b(\lambda'-\mu') \\
q^{-\lambda'-\mu'} s^{-\theta(\lambda'-\mu')} \\
q^{-\lambda'-\mu'} s^{\theta(\lambda'-\mu')} \\
t^{-\theta(\lambda'+\mu')} b(\lambda'-\mu') \\
t^{\theta(\lambda'+\mu')} a(\lambda'-\mu')
\end{bmatrix}
\end{pmatrix} \quad (4.5)$$

where $a(\lambda) = \frac{\sin \alpha (1+\lambda)}{\sin \alpha}$, $b(\lambda) = \frac{\sin \alpha \lambda}{\sin \alpha}$ and

$$L_1(\lambda') = \rho^{-2\theta \lambda'} \begin{pmatrix}
q^{\lambda'} \tau_1^+ + q^{-\lambda'} \tau_1^- \\
q^{-\lambda'} (st)^{\theta \lambda'} \cdot \tau_{12} \\
q^{\lambda'} (st)^{-\theta \lambda'} \cdot \tau_{21} \\
q^{\lambda'} \tau_2^+ + q^{-\lambda'} \cdot \tau_2^-
\end{pmatrix} \quad (4.6)$$

Now for constructing ‘coloured’ extension of integrable models related to the rational case, we intend to study the $q \to 1$ limit of the above solution. Evidently, at this limit the matrix $R_1(\lambda', \mu')$ (4.5) yields

$$R_2(\lambda', \mu') = \begin{pmatrix}
\begin{bmatrix}
(1+\lambda'-\mu') t^{-\theta(\lambda'-\mu')} \\
(1+\lambda'-\mu') t^{\theta(\lambda'+\mu')} \\
\lambda'-\mu' t^{\theta(\lambda'+\mu')} \\
\lambda'-\mu' t^{-\theta(\lambda'+\mu')} \\
(1+\lambda'-\mu') t^{\theta(\lambda'-\mu')} \\
(1+\lambda'-\mu') t^{-\theta(\lambda'-\mu')}
\end{bmatrix}
\end{pmatrix} \quad (4.7)$$

Notice that though the above $R_2(\lambda', \mu')$ is not rational in form, for the case $\theta = 0$, it interestingly reduces to the well known additive, rational $R$-matrix related to the LNLS model or Toda chain. So one may naturally expect that this $R_2(\lambda', \mu')$ can be linked with some ‘coloured’ extension of such integrable models. However, the extraction of $q \to 1$ limit for the case of $L_1(\lambda')$ operator (4.6) turns out to be a little difficult, since at this limit the associated algebra (3.3) would become almost trivial and cannot be used to produce
any physically interesting integrable model. To bypass this difficulty we consider a new set of operator elements \( \vec{K} \), which can be defined by inverting the relations

\[
\tau_\pm = \frac{K_1 \sin \alpha \pm K_2}{2 \sin \alpha}, \quad \tau_2 = \frac{K_3 \sin \alpha \pm K_4}{2 \sin \alpha}, \quad \tau_{12} = K_+, \quad \tau_{21} = K_-. \quad (4.8)
\]

Now by using (4.8), one can rewrite the algebra (3.3) in terms of the new elements \( \vec{K} \) and \( \rho \). Interestingly, the \( q \to 1 \) limit of such algebra turns out to be a nontrivial one and may be given by

\[
[K_1, K_\pm] = \pm K_\pm K_2, \quad [K_3, K_\pm] = \mp K_\pm K_4, \quad [K_+, K_-] = (K_1 K_4 - K_2 K_3), \quad (4.9)
\]

\[
[K_1, K_\pm] = [\rho, K_3] = [K_1, K_3] = 0, \quad \rho K_+ = t K_+ \rho, \quad \rho K_- = t^{-1} K_- \rho,
\]

where \( K_2, K_4 \) are commuting with all elements of the algebra. Next we also express \( L_1(\lambda') \) operator (4.6) through the new elements appearing in (4.8) and then take the required \( q \to 1 \) limit, which yields

\[
L_2(\lambda') = \rho^{-2\theta \lambda'} \left( \begin{array}{cc}
K_1 + i \lambda' K_2 & (st)^{-\theta \lambda'} K_- \\
(st)^{\theta \lambda'} K_+ & K_3 + i \lambda' K_4
\end{array} \right). \quad (4.10)
\]

Evidently \( R_2(\lambda', \mu') \) (4.7) and \( L_2(\lambda') \) (4.10) form a single-component spectral parameter dependent solution of QYBE (1.1), if the corresponding operator elements satisfy the algebra (4.9).

Now for constructing some concrete integrable models we may observe that, the sub-algebra of (4.9) involving only the elements \( \vec{K} \) would reduce to standard \( sl(2) \) algebra:

\[
[K_1, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = 2 K_1, \quad (4.11)
\]

for the particular case \( K_2 = K_4 = 1 \) and \( K_1 = -K_3 \). So in analogy with the Holstein-Primakoff transformation of \( sl(2) \) generators, one can realise the algebra (4.9) for the above mentioned reduction as

\[
K_1 = s - N, \quad \rho = t^{-N}, \quad K_+ = (2s - N)^{\frac{1}{2}} a, \quad K_- = a^\dagger (2s - N)^{\frac{1}{2}}, \quad (4.11)
\]

where \( s \) is a constant parameter, \( N = a^\dagger a \) and \( a, a^\dagger \) satisfy bosonic commutation relations:

\[
[a, a^\dagger] = 1. \quad (4.11)
\]

By substituting now the above realisation in \( L_2(\lambda') \) (4.10), one will readily
obtain a concrete Lax operator expressed through the physical bosonic modes $a, a^\dagger$. This 
Lax operator would automatically satisfy the QYBE (1.1), if the corresponding quantum $R$-matrix is taken as (4.7). Moreover, for the special case $\theta = 0$ it will surprisingly reduce to the Lax operator of quantum integrable LNLS model [16] after some trivial transformation. So such Lax operator, as obtained through the Holstein-Primakoff type realisation (4.11), is expected to generate some ‘coloured’ extension of familiar LNLS model.

Next we may construct another simple realisation of algebra (4.9) in the form

$$
K_1 = p, \quad K_2 = i, \quad K_3 = K_4 = 0, \quad \rho = e^{\beta p}, \quad K_\pm = e^{\mp u},
$$

where $\beta = -i \ln t$ and the canonical operators $u, p$ satisfy $[u, p] = i$. Substitution of this realisation in (4.10) would again generate a Lax operator, which may be written explicitly as

$$
L(\lambda') = e^{k\lambda' p} \begin{pmatrix}
p - \lambda' & -(st)^{-\theta \lambda'} e^u \\
(st)^{\theta \lambda'} e^{-u} & 0
\end{pmatrix},
$$

where $k = -2\theta \beta$. Notice that at the limit $\theta = 0$, (4.13) reduces to the Lax operator of well known Toda chain [17]. So the Lax operator (4.13), associated with the nonadditive $R$-matrix (4.7), should finally lead to some ‘coloured’ extension of this Toda chain. We may hope that by considering similar other realisations of algebra (4.9), it would be possible to produce various quantum integrable models, through their representative Lax operators.

**Conclusion**

The FRT algebra related to quantum group can be used to generate solutions of QYBE through Yang-Baxterisation procedure. However most of the earlier attempts in this direction were restricted to the case of standard BGRs, which gave rise to additive type quantum $R$-matrices depending on the difference of spectral parameters. With the aim of generating more general solutions of QYBE and related integrable models, in this article we apply the Yang-Baxterisation procedure to the case of a CBGR. For this purpose, we first modify the known universal $\mathcal{R}$-matrix related to $U_q(gl(2))$ quantised algebra through
a scaling transformation of corresponding central element and construct a CBGR from the fundamental representation of that universal $R$-matrix. Interestingly, the colour parameter dependence of the CBGR obtained in this way remain nontrivial even at $q \to 1$ limit. We are able to find also explicit form of elements of the coloured FRT algebra like $L^{(\pm)}(\lambda)$, related to this CBGR. Though these upper or lower triangular matrices manifestly depend on the colour parameter, the underlying quantum algebra and the associated coproduct reproduced by them are found to be standard ones, devoid of colour parameters.

For obtaining new solutions of QYBE, we subsequently Yang-Baxterise the FRT algebra related to the CBGR. Such solutions are found to be associated with non-additive type quantum $R$-matrix, which curiously depends on both difference and summation of spectral parameters. Moreover the $q \to 1$ limit of these solutions turns out to be quite interesting and leads to several concrete Lax operators, which would probably generate some ‘coloured’ extension of LNLS model and Toda chain. The problem of constructing Hamiltonian of these models along with their energy spectrum through QISM have not been carried out by us, and should be studied separately. We may also hope that by applying similar type of Yang-Baxterisation procedure to the case of other CBGRs, it would be possible to find out a rich class of solutions of QYBE and integrable models associated with nonadditive quantum $R$-matrix.
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