THE MEAN-SQUARE DICHOTOMY SPECTRUM
AND A BIFURCATION TO A MEAN-SQUARE ATTRACTOR

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\textbf{Abstract.} The dichotomy spectrum is introduced for linear mean-square random dynamical systems, and it is shown that for finite-dimensional mean-field stochastic differential equations, the dichotomy spectrum consists of finitely many compact intervals. It is then demonstrated that a change in the sign of the dichotomy spectrum is associated with a bifurcation from a trivial to a non-trivial mean-square random attractor.

1. Introduction. Mean-square properties are of traditional interest in the investigation of stochastic systems in engineering and physics. This is quite natural since the Ito stochastic calculus is a mean-square calculus. At first sight, it is thus somewhat surprising that the classical theory of random dynamical systems and their spectra is a pathwise theory, although this can be justified by Doss–Sussman-like transformations between stochastic differential equations and path-wise random ordinary differential equations \cite{1}. Such transformations, however, do not apply to mean-field stochastic differential equations, which include expectations of the solution in their coefficient functions \cite{6}.

Mean-square random dynamical systems based on deterministic two-parameter semi-groups from the theory of nonautonomous dynamical systems acting on a state space of random variables or random sets with the mean-square topology were introduced in \cite{7}. These act like deterministic systems with the stochasticity built into the state spaces of mean-square random variables. A mean-square random attractor was defined as a nonautonomous pullback attractor for such systems from the theory of nonautonomous dynamical systems \cite{9}. The main difficulty in applying

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the theory is the lack of useful characterisations of compact sets of such spaces of mean-square random variables.

In this paper, a theory of mean-square exponential dichotomies is presented for linear mean-field stochastic differential equations. (It also applies to classical linear stochastic differential equations). Although the corresponding mean-square random dynamical systems are essentially infinite-dimensional their dichotomy spectrum is given by the union of finitely many intervals. This is applied to analyse a nonlinear mean-field stochastic differential equation, for which it is shown that the trivial solution undergoes a mean-square bifurcation leading to a nontrivial mean-square attractor.

The paper is structured as follows. Section 2 contains the definition of a mean-square random dynamical system, and the notions of mean-square exponential dichotomy and mean-square dichotomy spectrum are introduced. Section 3 explains under which conditions, a mean-field stochastic differential equation generates a mean-square random dynamical system. In Section 4, the spectral theorem is established, which says that the mean-square spectrum of a linear mean-field stochastic differential equation consists of finitely many compact intervals. Finally, in the last section, it is shown that for a one-dimensional mean-field SDE of pitchfork-type, a stability change in the mean-square spectrum is associated with a bifurcation from a trivial to a non-trivial mean-square random attractor.

2. Mean-square random dynamical systems. Consider the time set $\mathbb{R}$, and define $\mathbb{R}^2_\geq := \{(t,s) \in \mathbb{R}^2 : t \geq s\}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$ be a complete filtered probability space satisfying the usual hypothesis, i.e., $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ is an increasing and right-continuous family of sub-$\sigma$-algebras of $\mathcal{F}$, which contain all $\mathbb{P}$-null sets. Essentially, $\mathcal{F}_t$ represents the information about the randomness at time $t \in \mathbb{R}$. Finally, define $\mathcal{X} := L^2(\Omega, \mathcal{F}; \mathbb{R}^d)$ and $\mathcal{X}_t := L^2(\Omega, \mathcal{F}_t; \mathbb{R}^d)$ for $t \in \mathbb{R}$ with the norm $\|X\|_{ms} := \sqrt{\mathbb{E}|X|^2}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$.

**Definition 1.** A mean-square random dynamical system (MS-RDS for short) $\varphi$ on the underlying phase space $\mathbb{R}^d$ with the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$ is a family of mappings

$$
\varphi(t, s, \cdot) : \mathcal{X}_s \to \mathcal{X}_t, \quad \text{for } (t, s) \in \mathbb{R}^2_\geq,
$$

which satisfies:

1. Initial value condition. $\varphi(s, s, X_s) = X_s$ for all $X_s \in \mathcal{X}_s$ and $s \in \mathbb{R}$.
2. Two-parameter semigroup property. For all $X \in \mathcal{X}_w$ and all $(t, s), (s, w) \in \mathbb{R}^2_\geq$

$$
\varphi(t, w, X) = \varphi(t, s, \varphi(s, w, X)).
$$

3. Continuity. $\varphi$ is continuous.

Mean-square random dynamical systems are essentially deterministic with the stochasticity built into or hidden in the time-dependent state spaces.

A MS-RDS $\varphi$ is called linear if for each $(t, s) \in \mathbb{R}^2_\geq$, the map $\varphi_{t,s}(\cdot) := \varphi(t, s, \cdot)$ is a bounded linear operator. It will be denoted by $\Phi_{t,s}$, and $\Phi_{t,s}(X)$ will conventionally be written $\Phi_{t,s}X$. A spectral theory for linear mean-square random dynamical systems can be established based on exponential dichotomies.
Definition 2 (Mean-square exponential dichotomy). Let $\gamma \in \mathbb{R}$. A linear mean-square random dynamical system $\Phi_{t,s} : X_s \to X_t$ is said to admit an exponential dichotomy with growth rate $\gamma$ if there exist positive constants $K, \alpha$ and a time-dependent decomposition

$$X_t = U_\gamma(t) \oplus S_\gamma(t) \quad \text{for } t \in \mathbb{R}$$

such that

$$\|\Phi_{t,s} X_s\|_{\text{ms}} \leq Ke^{(\gamma-\alpha)(t-s)}\|X_s\|_{\text{ms}} \quad \text{for } X_s \in S_\gamma(s) \text{ and } t \geq s,$$

$$\|\Phi_{t,s} X_s\|_{\text{ms}} \geq \frac{1}{K}e^{(\gamma+\alpha)(t-s)}\|X_s\|_{\text{ms}} \quad \text{for } X_s \in U_\gamma(s) \text{ and } t \geq s.$$

A special case of exponential dichotomy, when the growth rate is equal to zero and the space of initial condition consists of the deterministic vectors in $\mathbb{R}^d$, is also investigated in [2, 12], where a Perron-type condition for existence of this exponential dichotomy is established.

Definition 3 (Mean-square dichotomy spectrum). The mean-square dichotomy spectrum for a linear MS-RDS $\Phi$ is defined as

$$\Sigma := \{\gamma \in \mathbb{R} : \Phi \text{ has no exponential dichotomy with growth rate } \gamma\}.$$

The set $\rho := \mathbb{R} \setminus \Sigma$ is called the resolvent set of $\Phi$.

The dichotomy spectrum was first introduced in [11] for nonautonomous differential equations. Dichotomy spectra for random dynamical systems have been discussed recently in [3, 4, 13].

3. Mean-field stochastic differential equations. Mean-field stochastic differential equations of the form

$$dX_t = f(t, X_t, \mathbb{E}X_t) \, dt + g(t, X_t, \mathbb{E}X_t) \, dW_t$$

were introduced in [6]. Here $\{W_t\}_{t \in \mathbb{R}}$ is a two-sided scalar Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $F := (f, g) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$.

Let $\text{Lip}(\mathbb{R}^d)$ denote the set of Lipschitz continuous functions $f : \mathbb{R}^d \to \mathbb{R}^d$, and for each $f \in \text{Lip}(\mathbb{R}^d)$, set

$$\|f(\cdot)\|_{l_{\infty}} := \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{1 + |x|}.$$

Suppose that

(A1) $\Gamma := \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^d} \{\text{Lip}F(t, x, \cdot) + \|F(t,x,\cdot)\|_{l_{\infty}}\} < \infty.$

(A2) For each $R > 0$, there exist a constant $L_R$ and a modulus continuity $\omega_R$ such that

$$\|F(t_1, x_1, \cdot) - F(t_2, x_2, \cdot)\|^2_{l_{\infty}} \leq L_R |x_1 - x_2|^2 + \omega_R(|t_1 - t_2|)$$

for all $(t_k, x_k) \in \mathbb{R} \times \mathbb{R}^d$ with $t_k + |x_k|^2 \leq R$, $k \in \{1, 2\}$.

Let $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ be the natural filtration generated by $\{W_t\}_{t \in \mathbb{R}}$, and define

$$\mathcal{X} := L^2(\Omega, \mathcal{F}; \mathbb{R}^d), \quad \mathcal{X}_t := L^2(\Omega, \mathcal{F}_t; \mathbb{R}^d) \quad \text{for } t \in \mathbb{R}.$$

Given any initial condition $X_s \in \mathcal{X}_s$, $s \in \mathbb{R}$, a solution of (1) is a stochastic process $\{X_t\}_{t \geq s}$ with $X_t \in \mathcal{X}_t$ for $t \geq s$, satisfying the stochastic integral equation

$$X_t = X_s + \int_s^t f(u, X_u, \mathbb{E}X_u) \, du + \int_s^t g(u, X_u, \mathbb{E}X_u) \, dW_u.$$
It was shown in [6] that the SDE (1) has a unique solution and generates a MS-RDS \( \{\varphi_{t,s}\}_{t \geq s} \) on the underlying phase space \( \mathbb{R}^d \) with a probability set-up \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P) \), defined by \( \varphi_{t,s} : \mathcal{X}_s \rightarrow \mathcal{X}_t \) with
\[
\varphi_{t,s}(X_s) = X_t \quad \text{for } X_s \in \mathcal{X}_s.
\]

4. Mean-square dichotomy spectrum for linear mean-field stochastic differential equations. Consider a linear mean-field stochastic differential equation
\[
dX_t = (A(t)X_t + B(t)\mathbb{E}X_t) \, dt + (C(t)X_t + D(t)\mathbb{E}X_t) \, dW_t,
\]
where \( A, B, C, D : \mathbb{R} \rightarrow \mathbb{R}^{d \times d} \) are continuous bounded functions, which generates a linear mean-square random dynamical system \( \Phi_{t,s} \).

Proposition 4 (Equations for the first and second moments). Let \( X_s \in \mathcal{X}_s \), and define \( X_t := \Phi_{t,s} X_s \) for \( t \geq s \). Then \( \frac{d}{dt} \mathbb{E}X_t = (A(t) + B(t)) \mathbb{E}X_t \) and for all \( i, j \in \{1, \ldots, d\} \),
\[
\frac{d}{dt} \mathbb{E}X_i^j t = \sum_{k=1}^d \left( a_{ik}(t) \mathbb{E}X_t^k X_i^j t + a_{jk}(t) \mathbb{E}X_t^k X_i^j t \right)
+ \sum_{m,n=1}^d c_{im}(t)c_{jn}(t) \mathbb{E}X_t^m X_i^n t + \sum_{k=1}^d \left( b_{ik}(t) \mathbb{E}X_t^k X_i^j t + b_{jk}(t) \mathbb{E}X_t^k \mathbb{E}X_i^j t \right)
+ \sum_{m,n=1}^d \left( c_{im}(t)d_{jn}(t) + c_{jn}(t)d_{im}(t) + d_{im}(t)d_{jn}(t) \right) \mathbb{E}X_t^m \mathbb{E}X_i^n.
\]

Proof. From (2),
\[
dX_i^j t = \sum_{k=1}^d \left( a_{ik}(t) X_t^k + b_{ik}(t) \mathbb{E}X_t^k \right) \, dt + \sum_{k=1}^d \left( c_{ik}(t) X_t^k + d_{ik}(t) \mathbb{E}X_t^k \right) \, dW_t
\]
holds for \( t \in \mathbb{R} \). Taking the expectation of variables in both sides gives
\[
\mathbb{E}X_t = \mathbb{E}X_s + \int_s^t \left( A(u) + B(u) \right) \mathbb{E}X_u \, du \quad \text{for } t \geq s,
\]
which proves the first statement. Ito’s product formula [8, Example 3.4.1] and the expectation then yield the second statement. \( \square \)

Corollary 5. Let \( (U_{i,j}(t,s), V_{i,j}(t,s)) \) be the evolution operator of the linear nonautonomous differential equation in \( \mathbb{R}^{d(d+1)} \)
\[
\frac{du_{i,j}}{dt} = \sum_{k=1}^d \left( a_{ik}(t) + b_{ik}(t) \right) u_{k,j} + \left( a_{jk}(t) + b_{jk}(t) \right) u_{k,i}
\]
\[
\frac{dv_{i,j}}{dt} = \sum_{k=1}^d \left( a_{ik}(t) v_{k,j} + a_{jk}(t) v_{k,i} \right) + \sum_{m,n=1}^d c_{im}(t)c_{jn}(t) v_{m,n}
+ \sum_{k=1}^d \left( b_{ik}(t) u_{k,j} + b_{jk}(t) u_{k,i} \right)
+ \sum_{m,n=1}^d \left( c_{im}(t)d_{jn}(t) + c_{jn}(t)d_{im}(t) + d_{im}(t)d_{jn}(t) \right) u_{m,n},
\]
where 1 ≤ i ≤ j ≤ d. Then for any X_s ∈ X_s,

\[ \|\Phi_{t,s}X_s\|_{ms} = \left( \sum_{i=1}^{d} V_{i,i}(t,s)(\pi_s X_s) \right)^{1/2}, \tag{4} \]

where the map \( \pi_s = \pi_s^1 \times \pi_s^2 : X_s → \mathbb{R}^{d(d+1)} × \mathbb{R}^{d(d+1)} \) is defined by

\[ (\pi_s^1 X_s)_{i,j} = E X_s^i E X_s^j \quad \text{and} \quad (\pi_s^2 X_s)_{i,j} = E X_s^i E X_s^j. \]

**Remark 6.** It is interesting to compare the above equations with the ordinary differential equations for the first moment and second moments of linear stochastic differential equations, see e.g. [5, Section 6.2].

The proofs of the following preparatory results are straightforward.

**Lemma 7.** Let \( \gamma ∈ \mathbb{R} \) be such that that the linear MS-RDS \( \Phi_{t,s} : X_s → X_t \) generated by (1) admits an exponential dichotomy with the growth rate \( \gamma \) and a decomposition

\[ X_t = U_\gamma(t) ⊕ S_\gamma(t). \]

Then the subspace \( S_\gamma(t) \) is uniquely determined, i.e., if the linear MS-RDS \( \Phi_{t,s} \) also admits an exponential dichotomy with the growth rate \( \gamma \) and another decomposition

\[ X_t = \hat{U}_\gamma(t) ⊕ \hat{S}_\gamma(t), \]

then \( S_\gamma(t) = \hat{S}_\gamma(t) \) for all \( t ∈ \mathbb{R}. \)

The subspaces \( S_\gamma(t) \) of an exponential dichotomy with the growth rate \( \gamma \) are its stable subspaces. The following lemma provides an inclusion relation between these stable subspaces. Its proof follows directly from the definition of an exponential dichotomy.

**Lemma 8.** Let \( \gamma_1 < \gamma_2 \) be such that the linear MS-RDS admits an exponential dichotomy with the growth rates \( \gamma_1 \) and \( \gamma_2. \) Then \( S_{\gamma_1}(t) ⊂ S_{\gamma_2}(t) \) for all \( t ∈ \mathbb{R}. \)

One of the main results of this paper is the following characterisation of the dichotomy spectrum.

**Theorem 9 (Spectral Theorem).** Suppose that the coefficient functions in the linear mean-field stochastic differential equation (2) satisfy

\[ \max \{|a_{ij}(t)|, |b_{ij}(t)|, |c_{ij}(t)|, |d_{ij}(t)|\} ≤ m \quad \text{for} \ i, j \in \{1, \ldots, d\} \text{ and } t ∈ \mathbb{R} \tag{5} \]

with some \( m > 0. \) Then the dichotomy spectrum \( \Sigma \) is the disjoint union of at most \( d(d+1) \) compact intervals \([a_1, b_1], \ldots, [a_n, b_n]\) with \( a_1 ≤ b_1 < a_2 ≤ b_2 ≤ \cdots < a_n ≤ b_n. \) Furthermore, for each \( s ∈ \mathbb{R}, \) there exists a filtration of subspaces

\[ \{0\} ⊆ V_1(s) ⊆ V_2(s) ⊆ \cdots ⊆ V_n(s) = X_s, \]

which satisfies that for any \( i ∈ \{1, \ldots, n\}, \) a random variable \( X_s ∈ V_i(s) \) if and only if for any \( \varepsilon > 0, \) there exists \( K(\varepsilon) > 0 \) such that

\[ \|\Phi_{t,s}X_s\|_{ms} ≤ K(\varepsilon)e^{(b_i + \varepsilon)(t-s)} \quad \text{for} \ t ≥ s. \]

**Proof.** The proof is divided into several steps.

**Step 1.** First it will be shown that \( (-∞, −\Gamma) ⊂ ρ \) and \( (\Gamma, ∞) ⊂ ρ, \) where \( \Gamma := 2dm + 2d^2m^2. \) Let \( X_s ∈ X_s \) be arbitrary, and define

\[ \alpha(t) := \max_{i,j ∈ \{1, \ldots, d\}} \{E X_s^i E X_s^j, E X_s^i E X_s^j\} \quad \text{for} \ t ≥ s. \]
By the inequalities \(\mathbb{E}XY \leq \sqrt{\mathbb{E}X^2\mathbb{E}Y^2}\) and \((\mathbb{E}X)^2 \leq \mathbb{E}X^2\), it follows that
\[
\alpha(t) = \max_{1 \leq i \leq d} (\mathbb{E}X_i^t)^2 \leq \|\Phi(t, s)X_s\|_{\text{ms}}^2.
\]
Then by Corollary 5,
\[
\alpha(t) \leq \alpha(s) + 2\Gamma \int_s^t \alpha(u) \, du,
\]
and Gronwall’s inequality then yields
\[
\|\Phi(t, s)X_s\|_{\text{ms}}^2 \leq d\alpha(t) \leq de^{2\Gamma(t-s)}\alpha(s) \leq de^{2\Gamma(t-s)}\|X_s\|_{\text{ms}}^2.
\]
This proves that \((\Gamma, \infty) \subset \rho\). Time reversal of the equations in Corollary 5 leads to
\[
\|\Phi(t, s)X_s\|_{\text{ms}}^2 \geq \frac{1}{d}e^{-2\Gamma(t-s)}\|X_s\|_{\text{ms}}^2 \quad \text{for} \; (t, s) \in \mathbb{R}^2_+,
\]
which proves \((-\infty, \Gamma) \subset \rho\).

**Step 2.** It will be shown that for any \(t \in \mathbb{R}\), the set
\[
\{S_{\gamma}(t) : \gamma \in \rho \cap (-\Gamma - 1, \Gamma + 1)\}
\]
consists of at most \(d(d + 1) + 1\) elements. Suppose the contrary, i.e., there exist \(n + 1\) numbers \(\gamma_0 < \gamma_1 < \cdots < \gamma_n\) in \(\rho \cap (-\Gamma - 1, \Gamma + 1)\), where \(n > d(d + 1)\), such that
\[
S_{\gamma_i}(t) \neq S_{\gamma_j}(t) \quad \text{for} \; i \neq j.
\]
Then by Lemma 8,
\[
S_{\gamma_n}(t) \subset S_{\gamma_{n-1}}(t) \subset \cdots \subset S_{\gamma_1}(t).
\]
Thus, there exist \(X_1^t, \ldots, X_n^t\) such that
\[
X_i^t \in S_{\gamma_i}(t), \quad X_i^t \notin S_{\gamma_{i-1}}(t) \quad \text{for} \; i \in \{1, \ldots, n\}.
\]
By definition of the \(\gamma_i\), there exist \(K, \alpha > 0\) and complementary subspaces \(U_{\gamma_i}(t)\) such that \(X_t = U_{\gamma_i}(t) \oplus S_{\gamma_i}(t)\) and
\[
\|\Phi_{\bar{t}, t}X_t\|_{\text{ms}} \leq Ke^{(\gamma_i - \alpha)(\bar{t} - t)}\|X_t\|_{\text{ms}} \quad \text{for} \; X_t \in S_{\gamma_i}(t) \quad \text{and} \; \bar{t} \geq t \tag{6}
\]
and
\[
\|\Phi_{\bar{t}, t}X_t\|_{\text{ms}} \geq \frac{1}{K}e^{(\gamma_i + \alpha)(\bar{t} - t)}\|X_t\|_{\text{ms}} \quad \text{for} \; X_t \in U_{\gamma_i}(t) \quad \text{and} \; t \geq \bar{t}. \tag{7}
\]
Since \(\mathbb{R}^{\frac{d(d+1)}{2}} \times \mathbb{R}^{\frac{d(d+1)}{2}}\) is \(d(d + 1)\)-dimensional, it follows that there exist \(k \leq n\) and \(\alpha_1, \ldots, \alpha_{k-1}\) with \(\alpha_1^2 + \cdots + \alpha_{k-1}^2 \neq 0\) and
\[
\pi_tX_t^k = \alpha_1\pi_tX_t^1 + \cdots + \alpha_{k-1}\pi_tX_t^{k-1}. \tag{8}
\]
Consequently, by Corollary 5,
\[
\|\Phi_{\bar{t}, t}X_t^k\|_{\text{ms}}^2 = \sum_{i=1}^{d} V_{i, i}(\bar{t}, t)(\pi_tX_t^k) = \sum_{i=1}^{d} \sum_{j=1}^{k-1} \alpha_j V_{i, i}(\bar{t}, t)(\pi_tX_t^j) \\
\leq \sum_{j=1}^{k-1} |\alpha_j| \sum_{i=1}^{d} \sum_{j=1}^{k-1} V_{i, i}(\bar{t}, t)(\pi_tX_t^j) \\
= \sum_{j=1}^{k-1} |\alpha_j| (\|\Phi_{\bar{t}, t}X_t^1\|_{\text{ms}}^2 + \cdots + \|\Phi_{\bar{t}, t}X_t^{k-1}\|_{\text{ms}}^2).
\]
By definition of $\gamma_i$ and (6)

$$\|\Phi_{t,\ell}X_{\ell}^k\|_{\text{ms}} \leq (k - 1)K \left( \sum_{j=1}^{k-1} |\alpha_j| \right) \left( \sum_{j=1}^{k-1} \|X_i^j\|_{\text{ms}}^2 \right)^{e(\gamma_k-\alpha)(\ell-t)}.$$

Hence, it follows by (6) and (7) that $X_k^k \in S_{\gamma_k-1}(t)$, which leads to a contradiction.

**Step 3.** As proved in Step 2, for $t \in \mathbb{R}$, let $S_0(t) \subseteq S_1(t) \subseteq \cdots \subseteq S_n(t)$ with $n \leq d(d+1)$ satisfy

$$\{S_i(t) : \gamma \in \rho \cap (-\Gamma - 1, \Gamma + 1)\} = \{S_0(t), S_1(t), \ldots, S_n(t)\}.$$

By Step 1, it follows that $S_0(t) = \{0\}$ and $S_n(t) = X_t$. For each $i \in \{0, \ldots, n\}$, define

$$I_i := \{\gamma \in \rho \cap (-\Gamma - 1, \Gamma + 1) : S_\gamma(t) = S_i(t)\}.$$

It will be shown that $I_i = (b_i, a_{i+1})$, where

$$b_i = \inf\{\gamma : \gamma \in I_i\} \quad \text{and} \quad a_{i+1} = \sup\{\gamma : \gamma \in I_i\} \quad \text{for } i \in \{0, \ldots, n\}.$$

First, let $\gamma \in I_i$ be arbitrary. By the definition of $I_i$, there exist $K, \alpha > 0$ and a decomposition $X_t = U(t) \oplus S_i(t)$ and

$$\|\Phi_{t,\ell}X_{\ell}\|_{\text{ms}} \leq K e^{(\gamma - \alpha)(\ell-t)} \|X_{\ell}\|_{\text{ms}} \quad \text{for } X_{\ell} \in S_i(t) \text{ and } \ell \geq t$$

and

$$\|\Phi_{t,\ell}X_{\ell}\|_{\text{ms}} \geq \frac{1}{K} e^{(\gamma + \alpha)(\ell-t)} \|X_{\ell}\|_{\text{ms}} \quad \text{for } X_{\ell} \in U(t) \text{ and } \ell \geq t.$$

This implies that $(-\Gamma, \alpha, \gamma + \alpha) \subseteq I_i$, so $I_i$ is open. It can be shown similarly that $I_i$ is connected. Hence $I_i = (a_i, b_i)$. Combining this result and Step 1 gives

$$\rho = (-\infty, a_1) \cup (b_1, a_2) \cup \cdots \cup (b_{n-1}, a_n) \cup (b_n, \infty),$$

which implies that

$$\Sigma = [a_1, b_1] \cup \cdots \cup [a_n, b_n].$$

To conclude the proof, the filtration corresponding to the spectral intervals is constructed as follows: for $t \in \mathbb{R}$, $V_0(t) := \{0\}$, $V_n(t) := X_t$, and

$$V_i(t) := S_\gamma(t), \quad \gamma \in (b_i, a_{i+1}), \quad i \in \{1, \ldots, n-1\}.$$

Due to Lemma 7, the definition of $V_i$ is independent of $\gamma \in (b_i, a_{i+1})$ for $i \in \{1, \ldots, n-1\}$. The strict inclusion $V_i \subset V_{i+1}$ for $i \in \{0, \ldots, n-1\}$ follows from the construction of the open interval $(b_i, a_{i+1})$ above. Finally, the dynamical characterisation of $V_i$ follows from the definition of $(b_i, a_{i+1})$ and the definition of exponential dichotomy. This completes the proof. \(\square\)

5. **Bifurcation of a mean-square random attractor.** A mean-square random attractor was defined in [7] as the pullback attractor of the nonautonomous dynamical system formulated as a mean-square random dynamical system.

Specifically, a family $\mathcal{A} = \{A_t\}_{t \in \mathbb{R}}$ of nonempty compact subsets of $X$ with $A_t \subseteq X_t$ for each $t \in \mathbb{R}$ is called a pullback attractor if it pullback attracts all uniformly bounded families $\mathcal{D} = \{D_t\}_{t \in \mathbb{R}}$ of subsets of $\{X_t\}_{t \in \mathbb{R}}$, i.e.,

$$\lim_{s \to -\infty} \text{dist}(\varphi(t, s, D_s), A_t) = 0.$$

Uniformly bounded here means that there is an $R > 0$ such that $\|X\|_{\text{ms}} \leq R$ for all $X \in D_t$ and $t \in \mathbb{R}$.

The existence of pullback attractors follows from that of an absorbing family. A uniformly bounded family $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$ of nonempty closed subsets of $\{X_t\}_{t \in \mathbb{R}}$
is called a pullback absorbing family for a MS-RDS $\varphi$ if for each $t \in \mathbb{R}$ and every uniformly bounded family $D = \{D_t\}_{t \in \mathbb{R}}$ of nonempty subsets of $\{X_t\}_{t \in \mathbb{R}}$, there exists some $T = T(t, D) \in \mathbb{R}^+$ such that

$$\varphi(t, s, D_s) \subseteq B_t \quad \text{for } s \in \mathbb{R} \text{ with } s \leq t - T.$$  

**Theorem 10.** Suppose that a MS-RDS $\varphi$ has a positively invariant pullback absorbing uniformly bounded family $B = \{B_t\}_{t \in \mathbb{R}}$ of nonempty closed subsets of $\{X_t\}_{t \in \mathbb{R}}$ and that the mappings $\varphi(t, s, \cdot) : X_s \to X_t$ are pullback compact (respectively, eventually or asymptotically compact) for all $(t, s) \in \mathbb{R}^2$. Then, $\varphi$ has a unique global pullback attractor $A = \{A_t\}_{t \in \mathbb{R}}$ with its component sets determined by

$$A_t = \bigcap_{s \leq t} \varphi(t, s, B_s) \quad \text{for } t \in \mathbb{R}.$$  

Consider the nonlinear mean-field SDE

$$dX_t = (\alpha X_t + \beta EX_t - X_t E X_t^2) \, dt + X_t \, dW_t \quad \text{(9)}$$

with real-valued parameters $\alpha, \beta$. Note that the theory in Section 3 can be easily extended to include the second moment of the solution in the equation.

This SDE has the steady state solution $\bar{X}(t) \equiv 0$. Linearising along this solution gives the bi-linear mean-field SDE

$$dZ_t = (\alpha Z_t + \beta E Z_t) \, dt + Z_t \, dW_t \quad \text{(10)}.$$  

**Theorem 11.** The dichotomy spectrum of the linear MS-RDS $\Phi$ generated by (10) is given by

$$\Sigma = \left\{ \begin{array}{ll}
\{\alpha + 1/2\} \cup \{\alpha + \beta\} & \text{if } \beta > 1/2, \\
\{\alpha + 1/2\} & \text{if } \beta \leq 1/2.
\end{array} \right.$$  

**Proof.** Taking the expectation of two sides of (10) yields that

$$\frac{d}{dt} EZ_t = (\alpha + \beta) EZ_t,$$

which implies that

$$E \Phi(t, s) Z_s = e^{(\alpha + \beta)(t-s)} EZ_s \quad \text{for } (t, s) \in \mathbb{R}^2_+ \text{ and } Z_s \in X_s.$$

Rito’s formula for the function $U(x) = x^2$ then gives

$$dZ_t^2 = [(2\alpha + 1)Z_t^2 + 2\beta Z_t EZ_t] \, dt + 2Z_t^2 \, dW_t.$$  

Consequently,

$$\frac{d}{dt} EZ_t^2 = (2\alpha + 1)EZ_t^2 + 2\beta(EZ_t)^2.$$  

Thus, using (11) for $(t, s) \in \mathbb{R}^2_+$ and $Z_s \in X_s$, it follows that

$$\begin{align*}
\| \Phi(t, s) Z_s \|_{\text{ms}}^2 &= e^{(2\alpha + 1)(t-s)} \| Z_s \|_{\text{ms}}^2 + 2\beta \int_s^t e^{(2\alpha + 1)(t-u)}(EZ_u)^2 \, du \\
&= e^{(2\alpha + 1)(t-s)} \| Z_s \|_{\text{ms}}^2 + 2\beta(EZ_s)^2 \int_s^t e^{(2\alpha + 1)(t-u)}e^{(2\alpha + 2\beta)(u-s)} \, du \\
&= e^{(2\alpha + 1)(t-s)} \left( \| Z_s \|_{\text{ms}}^2 + 2\beta(EZ_s)^2 \int_s^t e^{(2\beta - 1)(u-s)} \, du \right). \quad \text{(12)}
\end{align*}$$
The assertions of the lemma will be shown for the three cases $\beta < 1/2$, $\beta = 1/2$ and $\beta > 1/2$.

**Case 1 ($\beta < 1/2$).** By (12),
\[
\|\Phi(t, s)Z_s\|^2_{\text{ms}} = e^{(2\alpha+1)(t-s)} \left(\|Z_s\|^2_{\text{ms}} + \frac{2\beta}{1-2\beta} \left(1 - e^{(2\beta-1)(t-s)}\right)(EZ_s)^2\right),
\]
which together with the inequality $(EX)^2 \leq EX^2$ yields
\[
\|\Phi(t, s)Z_s\|^2_{\text{ms}} \leq e^{(2\alpha+1)(t-s)} \left(1 + \frac{2|\beta|}{1-2\beta}\right)\|Z_s\|^2,
\]
and
\[
\|\Phi(t, s)Z_s\|^2_{\text{ms}} \geq \begin{cases} 
\frac{e^{(2\alpha+1)(t-s)}}{1-2\beta} & \text{if } \beta \geq 0, \\
\frac{1}{1-2\beta}e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\text{ms}} & \text{if } \beta < 0.
\end{cases}
\]
This implies that $\Sigma = \{\alpha + 1/2\}$.

**Case 2 ($\beta = 1/2$).** By (12),
\[
\|\Phi(t, s)Z_s\|^2_{\text{ms}} = e^{(2\alpha+1)(t-s)} \left(\|Z_s\|^2_{\text{ms}} + (t-s)(EZ_s)^2\right),
\]
which implies that
\[
\|\Phi(t, s)Z_s\|^2_{\text{ms}} \geq e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\text{ms}} \text{ for } (t, s) \in \mathbb{R}^2_+.
\]
Let $\varepsilon > 0$ be arbitrary. Since $(t-s) \leq \frac{1}{2}e^{\varepsilon(t-s)}$ for $t \geq s$ and $(EX)^2 \leq EX^2$, it follows that
\[
\|\Phi(t, s)Z_s\|^2_{\text{ms}} \leq \left(1 + \frac{1}{2}\right)e^{(2\alpha+1+\varepsilon)(t-s)}\|Z_s\|^2_{\text{ms}}.
\]
Consequently, $\Sigma \subset [\alpha + \frac{1}{2}, \alpha + \frac{1}{2} + \varepsilon]$. The limit $\varepsilon \to 0$ leads to $\Sigma = \{\alpha + 1/2\}$.

**Case 3 ($\beta > 1/2$).** By (12),
\[
\|\Phi(t, s)Z_s\|^2_{\text{ms}} = e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\text{ms}} + \frac{2\beta}{2\beta-1} \left(e^{(2\alpha+2\beta)(t-s)} - e^{(2\alpha+1)(t-s)}\right)(EZ_s)^2.
\]
Together with the inequality $(EX)^2 \leq EX^2$, this implies that
\[
e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\text{ms}} \leq \|\Phi(t, s)Z_s\|^2_{\text{ms}} \leq \left(1 + \frac{2\beta}{2\beta-1}\right)e^{(2\alpha+2\beta)(t-s)}\|Z_s\|^2_{\text{ms}}.
\]
Consequently, $\Sigma \subset [\alpha + \frac{1}{2}, \alpha + \beta]$. Let $\gamma \in (\alpha + \frac{1}{2}, \alpha + \beta)$ be arbitrary. Choose and fix $\varepsilon > 0$ such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subset (\alpha + \frac{1}{2}, \alpha + \beta)$. The aim is to show that $\Phi$ admits an exponential dichotomy with the growth rate $\gamma$ for the decomposition $X_s = U_s \oplus S_s$, where
\[
S_s := \{f \in X_s : Ef = 0\},
\]
\[
U_s := \{f \in X_s : f \text{ is independent of noise}\}.
\]
Obviously, any $X \in X_s$ can be written as $X = (X - EX) + EX$, with $X - EX \in S_s$ and $EX \in U_s$. By (13), for any $Z_s \in S_s$,
\[
\|\Phi(t, s)Z_s\|^2_{\text{ms}} = e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\text{ms}}.
\]
Now $(EZ_s)^2 = EZ_s^2$ for any $Z_s \in U_s$, so by (13), for $t - s \geq 1$,
\[
\|\Phi(t, s)Z_s\|^2_{\text{ms}} \geq \frac{2\beta}{2\beta-1} \left(e^{(2\gamma+2\varepsilon)(t-s)} - e^{(2\gamma-2\varepsilon)(t-s)}\right)\|Z_s\|^2_{\text{ms}}
\]
\[
\geq \frac{4\varepsilon\beta}{2\beta-1}e^{2\gamma(t-s)}\|Z_s\|^2_{\text{ms}}.
\]
Here the inequality \( e^x \geq 1 + x \) for \( x \geq 0 \) has been used. Thus, \( \Phi \) admits an exponential dichotomy with the growth rate \( \gamma \), which means that \( \Sigma \subseteq \{ \alpha + 1/2 \} \cup \{ \alpha + \beta \} \). Considering the decomposition \( S_\alpha \oplus U_\alpha \), it follows that \( \alpha + \frac{1}{2}, \alpha + \beta \in \Sigma \). Thus, \( \Sigma = \{ \alpha + 1/2 \} \cup \{ \alpha + \beta \} \). This completes the proof. \( \square \)

A globally bifurcation of pullback attractor of (9) with \( \beta = 1 \), i.e.,

\[
\frac{dX_t}{dt} = (\alpha X_t + \mathbb{E}[X_t] - X_t\mathbb{E}[X_t]^2) \, dt + X_t \, dW_t,
\]

as \( \alpha \) varies, will be investigated in a series of theorems.

The first and second moment equations of the mean-field SDE (14) are given by

\[
\frac{d}{dt}\mathbb{E}[X_t] = (\alpha + 1)\mathbb{E}[X_t] - \mathbb{E}[X_t]\mathbb{E}[X_t]^2,
\]

\[
\frac{d}{dt}\mathbb{E}[X_t^2] = (2\alpha + 1)\mathbb{E}[X_t^2] + 2(\mathbb{E}[X_t])^2 - 2(\mathbb{E}[X_t]^2)^2,
\]

where Ito’s formula with \( y = x^2 \) was used to derive (16). These can be rewritten as the system of ODEs

\[
\frac{dx}{dt} = x(\alpha + 1 - y) \quad \text{and} \quad \frac{dy}{dt} = (2\alpha + 1)y + 2x^2 - 2y^2, \quad \text{where} \quad x^2 \leq y,
\]

which has a steady state solution \( \bar{x} = \bar{y} = 0 \) for all \( \alpha \) corresponding to the zero solution \( X_t \equiv 0 \) of the mean-field SDE (14). There also exist valid (i.e., with \( y \geq 0 \)) steady state solutions \( \bar{x} = \pm \sqrt{(\alpha + 1)/2}, \bar{y} = \alpha + 1 \) for \( \alpha > -1 \) and \( \bar{x} = 0, \bar{y} = \alpha + \frac{1}{2} \) for \( \alpha \geq -\frac{1}{2} \). It needs to be shown if there are solutions of the SDE (14) with these moments.

**Theorem 12.** The MS-RDS \( \varphi \) generated by (14) has a uniformly bounded positively invariant pullback absorbing family.

**Proof.** Let \( \alpha \) be arbitrary and define

\[
B_t = \{ X \in \mathcal{X}_t : \|X\| \leq \sqrt{1/2} \} \quad \text{for} \quad t \in \mathbb{R}.
\]

Using

\[
(2\alpha + 1)\mathbb{E}[X_t^2] + 2(\mathbb{E}[X_t])^2 \leq (2\alpha + 3)\mathbb{E}[X_t^2] - 2(\mathbb{E}[X_t]^2)^2,
\]

which holds since \( (\mathbb{E}[X_t])^2 \leq \mathbb{E}[X_t^2] \), the second moment equation (16) gives the differential inequality

\[
\frac{d}{dt}\mathbb{E}[X_t^2] \leq (2\alpha + 3)\mathbb{E}[X_t^2] - 2(\mathbb{E}[X_t]^2)^2 \quad (17)
\]

Let \( D = \{ D_t \}_{t \in \mathbb{T}} \) be a uniformly bounded family of nonempty subsets of \( \{ \mathcal{X}_t \}_{t \in \mathbb{R}} \), i.e., \( D_t \subseteq \mathcal{X}_t \) and there exists \( R > 0 \) such that \( \|X\| \leq R \) for all \( X \in D_t \).

Specifically, it will be shown that \( \varphi(t, s, D_s) \subseteq B_t \) for \( t - s \geq T \), where \( T \) is defined by

\[
T := \log \left( \frac{R^2}{|\alpha| + 2} \right).
\]

Pick \( X_s \in D_s \) arbitrarily and \( (t, s) \in \mathbb{R}^2 \) with \( t - s \geq T \). Motivated by the differential inequality (17), consider the scalar system

\[
\dot{y} = (2\alpha + 3)y - 2y^2, \quad \text{where} \quad y(s) \leq R^2.
\]

A direct computation yields

\[
y(t) = y(s) \exp \left( \int_s^t (2\alpha + 3) - 2y(u) \, du \right) \leq R^2 \exp \left( \int_s^t (2\alpha + 3) - 2y(u) \, du \right).
\]
From the definition of $T$ in (18), it follows that $\min_{s \leq u \leq t} y(u) \leq |\alpha| + 2$. Furthermore, $y = 0$ and $y = \alpha + \frac{3}{2}$ are stationary points of the ODE (19). For this reason, $\min_{s \leq u \leq t} y(u) \leq |\alpha| + 2$ implies that $y(t) \leq |\alpha| + 2$. Then from (17), it follows that $y(t) \geq \|\phi(t, s)X_s\|_{ms}^2$. This means that

$$\|\phi_{t,s}X_s\|_{ms}^2 \leq y(t) \leq |\alpha| + 2,$$

i.e., $\phi_{t,s}X_s \in B_t$ for $t - s \geq T$. Hence, $\{B_t\}_{t \in \mathbb{R}}$ is a pullback absorbing family for the MS-RDS $\phi$. It is clear that this family is uniformly bounded and positively invariant for the MS-RDS $\phi$.

**Theorem 13.** The MS-RDS $\phi$ generated by (14) has a pullback attractor with component sets $\{0\}$ when $\alpha < -1$.

**Proof.** Let $\alpha < -1$ be arbitrary. Let $\mathcal{D} = \{D_t\}_{t \in \mathbb{R}}$ be a uniformly bounded family of nonempty subsets of $\{X_t\}_{t \in \mathbb{R}}$ with $\|X\|_{ms} \leq R$ for all $X \in D_t$ where $R > 0$. Let $(X_s)_{s \in \mathbb{R}}$ be an arbitrary sequence with $X_s \in D_s$. The moment equations (15)–(16) can be written as

$$\frac{d}{dt} \mathbb{E}[\phi(t, s)X_s] = \mathbb{E}[\phi(t, s)X_s] \left(\alpha + 1 - \mathbb{E}[\phi(t, s)X_s]^2\right)$$

$$\frac{d}{dt} \mathbb{E}[\phi(t, s)X_s]^2 = (2\alpha + 1)(\mathbb{E}[\phi(t, s)X_s]^2 + 2\mathbb{E}[\phi(t, s)X_s])^2 - 2(\mathbb{E}[\phi(t, s)X_s]^2)^2.$$

Then

$$\mathbb{E}[\phi(t, s)X_s] = \mathbb{E}[X_s] \exp\left(\int_s^t \alpha + 1 - \mathbb{E}[X_u]^2 \, du\right),$$

which implies that

$$(\mathbb{E}[\phi(t, s)X_s])^2 \leq e^{(\alpha+1)(t-s)} R^2.$$

Moreover, by the variation of constants formula,

$$\mathbb{E}[\phi(t, s)X_s]^2 = e^{(2\alpha+1)(t-s)} \mathbb{E}[X_s]^2 + 2 \int_s^t e^{(2\alpha+1)(t-u)} \left(\mathbb{E}[\phi(u, s)X_s]^2 - \mathbb{E}[\phi(u, s)X_s]^2\right) \, du$$

$$\leq e^{(2\alpha+1)(t-s)} R^2 + 2 R^2 \int_s^t e^{(2\alpha+1)(t-u)} e^{(\alpha+1)(u-s)} \, du$$

$$\leq e^{(2\alpha+1)(t-s)} R^2 + 2 e^{(\alpha+1)(t-s)} \int_s^t e^{(\alpha+1)(t-s)} \, du$$

$$\leq e^{(2\alpha+1)(t-s)} R^2 + 2 e^{(\alpha+1)(t-s)} \frac{e^{(\alpha+1)(t-s)} - 1}{\alpha+1},$$

which implies that $\lim_{s \to -\infty} \mathbb{E}[\phi(t, s)X_s] = 0$. Thus $\{0\}$ is the pullback attractor of (14) in this case.

**Theorem 14.** The MS-RDS $\phi$ generated by (14) has a nontrivial pullback attractor when $-1 < \alpha < -\frac{1}{2}$.

**Remark 15.** The idea for the proof is taken from [7, Subsection 4.1], but their result cannot be applied directly, since the Lipschitz constant of the nonlinear terms is at least 1, and $\alpha$ is not less than $-4$. Instead the uniform equicontinuity of the mapping $t \mapsto \mathbb{E}X_t$, and then the positivity of the second moment to obtain a better estimate are used.

**Proof.** In order to apply Theorem 10, it needs to be shown that $\phi$ is pullback asymptotically compact, i.e., given a uniformly bounded family $\mathcal{D} = \{D_t\}_{t \in \mathbb{R}}$ of nonempty subsets of $X_t$ and sequences $\{t_k\}_{k \in \mathbb{N}}$ in $(-\infty, s)$ with $t_k \to -\infty$ as $k \to \infty$ and $\{X_k\}_{k \in \mathbb{N}}$ with $X_k \in D_{t_k} \subset X_{t_k}$ for each $k \in \mathbb{N}$, then the subset
{φ(t, t_k, X_k)}_{k \in \mathbb{N}} \subset \mathcal{X}_t$ is relatively compact. For this purpose, let $\varepsilon > 0$ be arbitrary. A finite cover of $\{\varphi(t, t_k, X_k)\}_{k \in \mathbb{N}}$ with diameter less than $\varepsilon$ will be constructed. Choose and fix $s \in \mathbb{R}$ with $s < t$ such that
\[ 4e^{(2\alpha+1)(t-s)}(|\alpha| + 2)^2 < \frac{\varepsilon^2}{2}, \quad (21) \]
and define $Y_s^k := \varphi(s, t_k, X_k)$. Using (20) it can be assumed without loss of generality that
\[ \mathbb{E}(Y_s^k)^2 \leq |\alpha| + 2 \quad \text{for } k \in \mathbb{N}. \quad (22) \]

For $u \in [s, t]$, let $Y_u^k := \varphi(u, s)Y_s^k$ and consider a family of functions $f_k : [s, t] \to \mathbb{R}$ defined by $f_k(u) := \mathbb{E}Y_u^k$. By Itô’s formula,
\[ Y_t^k = e^{\alpha(t-s)}Y_s^k + \int_s^t e^{\alpha(t-u)} (\mathbb{E}Y_u^k - Y_u^k) \mathbb{E}(Y_u^k)^2 \, du + \int_s^t e^{\alpha(t-u)} Y_u^k \, dW_u, \]
from which it can be shown that $\{f_k\}_{k \in \mathbb{N}}$ is a uniformly equicontinuous sequence of functions. By the Arzelà–Ascoli theorem, for $\delta := \varepsilon \sqrt{\frac{\alpha}{2} + \frac{1}{2}}$, there exists a finite index set $J(\delta) \subset \mathbb{N}$ such that for any $k \in \mathbb{N}$ there exists $n_k \in J(\delta)$ for which
\[ \sup_{u \in [s, t]} |\mathbb{E}Y_u^k - \mathbb{E}Y_{n_k}^k| < \delta. \quad (23) \]

To conclude the proof of asymptotic compactness, it is sufficient to show the inequality $|\|Y_t^k - Y_t^{n_k}\|_\infty \leq \varepsilon$. Indeed, for $u \in [s, t]$, a direct computation gives
\[ \frac{d}{dt} \mathbb{E}(Y_u^k - Y_{n_k}^k)^2 = (1 + 2\alpha)\mathbb{E}(Y_u^k - Y_{n_k}^k)^2 + 2(\mathbb{E}Y_u^k - \mathbb{E}Y_{n_k}^k)^2 - 2(\mathbb{E}(Y_u^k)^2)^2 \]
\[ - 2(\mathbb{E}(Y_{n_k}^k)^2)^2 + 2\mathbb{E}Y_u^k Y_{n_k}^k (\mathbb{E}(Y_u^k)^2 + \mathbb{E}(Y_{n_k}^k)^2). \]

Note that
\[ 2(\mathbb{E}(Y_u^k)^2)^2 + 2(\mathbb{E}(Y_{n_k}^k)^2)^2 - 2\mathbb{E}Y_u^k Y_{n_k}^k (\mathbb{E}(Y_u^k)^2 + \mathbb{E}(Y_{n_k}^k)^2) \geq 0. \]

Hence, by the variation of constant formula,
\[ |\|Y_t^k - Y_t^{n_k}\|_\infty | \leq e^{(2\alpha+1)(t-s)} |\|Y_s^k - Y_{n_k}^k\|_\infty | + 2 \int_s^t e^{(2\alpha+1)(t-u)} (\mathbb{E}Y_u^k - \mathbb{E}Y_{n_k}^k) \, du, \]
which together with (22) and (23) implies that
\[ |\|Y_t^k - Y_t^{n_k}\|_\infty | \leq 4e^{(2\alpha+1)(t-s)}(|\alpha| + 2)^2 + 2s^2 \alpha + 1 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2. \]

Here (21) was used to obtain the preceding inequality. Thus the set $\{\varphi(t, t_k, X_k)\}_{k \in \mathbb{N}}$ is covered by the finite union of open balls with radius $\varepsilon$ centered at $\varphi(t, t_k, X_k)$, where $k \in J(\lambda)$, i.e., is totally bounded.

Hence the MS-RDS is asymptotically compact and by Theorem 10 has a pullback attractor with component subsets $\{A_t\}_{t \in \mathbb{R}}$ that contain the zero solution. It remains to show that the sets $A_t$ also contain other points. Consider a uniformly bounded family of bounded sets defined by
\[ D_t := \left\{ X \in \mathcal{X}_t : \mathbb{E}X = \sqrt{(\alpha + 1)/2}, \; \mathbb{E}X^2 = \alpha + 1 \right\}. \]
Recall that these values are a steady state solution of the moment equations (15)–(16). Hence $\varphi(t, s, X_s) \in D_t$ for all $t > s$ when $X_s \in D_s$. Then by pullback attraction
\[ \text{dist}(\varphi(t, s, X_s), A_t) \leq \text{dist}(\varphi(t, s, D_s), A_t) \to 0 \quad \text{as } s \to -\infty. \]
The convergence here is mean-square convergence, and \( \mathbb{E} \varphi(t, s, X_s)^2 \equiv \alpha + 1 \) for all \( t > s \). Thus, there exists a random variable \( X_t^* \in A_t \cap D_t \) for each \( t \in \mathbb{R} \), i.e., the pullback attractor is nontrivial. Note that by arguments in [10] (see also [9]), it follows that there is in fact an entire solution \( \bar{X}_t \in A_t \cap D_t \) for all \( t \in \mathbb{R} \).

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