EQUIVARIANT COHOMOLOGY OF INCIDENCE HILBERT SCHEMES AND LOOP ALGEBRAS

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ABSTRACT. Let $S$ be the affine plane $\mathbb{C}^2$ together with an appropriate $T = \mathbb{C}^*$ action. Let $S^{[m,m+1]}$ be the incidence Hilbert scheme. Parallel to [LQ], we construct an infinite dimensional Lie algebra that acts on the direct sum

$$\tilde{H}_T = \bigoplus_{m=0}^{+\infty} H_T^{2(m+1)}(S^{[m,m+1]})$$

of the middle-degree equivariant cohomology group of $S^{[m,m+1]}$. The algebra is related to the loop algebra of an infinite dimensional Heisenberg algebra. In addition, we study the transformations among three different linear bases of $\tilde{H}_T$. Our results are applied to the ring structure of the ordinary cohomology of $S^{[m,m+1]}$ and to the ring of symmetric functions in infinitely many variables.

1. Introduction

Let $S$ be the affine plane $\mathbb{C}^2$ together with the $T = \mathbb{C}^*$ action

$$a(w, z) = (aw, a^{-1}z), \quad a \in T$$

on the coordinate functions $w$ and $z$ of $S$. This $T$-action on $S$ induces a $T$-action on the Hilbert scheme $S^{[n]}$ of $n$-points on $S$. The $T$-fixed points in $S^{[n]}$ are of the form $\xi_\lambda$ where $\lambda$ denotes partitions of $n$. In [Na2, Na3, Vas, LQW1, LQW2], the equivariant cohomology $H_T^*(S^{[n]})$ of the Hilbert scheme $S^{[n]}$ has been studied via representation theory. A generalization of Nakajima’s work [Na1] to the equivariant cohomology $H_T^*(S^{[n]})$ shows in [Vas] that the space

$$\mathbb{H}_T = \bigoplus_{n=0}^{+\infty} H_T^{2n}(S^{[n]})$$

is an irreducible representation of a Heisenberg algebra generated by the linear operators $a_n^T$, $n \in \mathbb{Z}$ in $\text{End}(\mathbb{H}_T)$. As a consequence, it induces a linear isomorphism

$$\Phi : \mathbb{H}_T \to \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$$

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where $\Lambda$ is the vector space of symmetric functions in infinitely many variables (see p.19 of [Mac]). More specifically, let $C = C^z$ be the $z$-axis of $S$. The homomorphism $\Phi$ maps $a^T_{-x}[0]$ and $[L^x C]$ (defined in (3.29) and (4.1)) to the power-sum symmetric function $p_\lambda$ and the monomial symmetric function $m_\lambda$ respectively. Here $[\cdot]$ denotes the equivariant fundamental cohomology class.

A new feature of the equivariant setup is the existence of the $\mathbb{T}$-fixed points. By the localization theorem, the ring structure of $H^*_T(S^{[n]})$ is easy to describe using the fixed points $\xi_\lambda \in S^{[n]}$. Note that $\Lambda$ is a ring as well by the usual multiplication of functions. However, $\Phi$ is not a ring isomorphism. On the other hand, if we define a new ring structure on $\Lambda$ by requiring $s_\lambda \cdot s_\mu = \delta_{\lambda,\mu} h(\lambda) s_\lambda$ for the Schur functions $s_\lambda$ and $s_\mu$, then $\Phi$ is a ring isomorphism from $\mathbb{H}^\mathbb{T}$ to $(\Lambda, \cdot)$. Here $h(\lambda)$ denotes the hook number of the Young diagram associated to $\lambda$. In fact, $\Phi$ maps the fixed point class $(-1)^{|\lambda|/h(\lambda)} \cdot [\lambda]$ to the Schur function $s_\lambda$. This is an extra property gained by going to equivariant cohomology (see [Vas]).

In this paper, we study the equivariant cohomology $H^*_T(S^{[n,n+1]}))$ of the incidence Hilbert scheme $S^{[n,n+1]}$ which is defined by

$$S^{[n,n+1]} = \{ (\xi, \xi') | \xi, \xi' \in S^{[n]} \times S^{[n+1]} \}.$$  

It is known from [Chl, Tik] that the incidence Hilbert scheme $S^{[n,n+1]}$ is irreducible, smooth and of dimension $2(n+1)$. Following [LQ], we construct the Heisenberg operators $\tilde{a}^T_n, n \in \mathbb{Z}$ and the translation operator $t^T$ on the space

$$\mathbb{H}^\mathbb{T} = \bigoplus_{n=0}^{+\infty} H^*_{\mathbb{T}}(S^{[n,n+1]}).$$

Let $\tilde{h}^\mathbb{T}$ be the Heisenberg algebra generated by the operators $\tilde{a}^T_n, n \in \mathbb{Z}$. The loop algebra of $\tilde{h}^\mathbb{T}$ is the space $\mathbb{C}[u, u^{-1}] \otimes_{\mathbb{C}} \tilde{h}^\mathbb{T}$ together with the Lie bracket

$$[u^n \otimes g_1, u^m \otimes g_2] = u^{m+n} \otimes [g_1, g_2].$$

**Theorem 1.1.** The space $\mathbb{H}^\mathbb{T}$ is a representation of the Lie algebra $\mathbb{C}[u^{-1}] \otimes_{\mathbb{C}} \tilde{h}^\mathbb{T}$ with a highest weight vector being the vacuum vector

$$|0\rangle = [C^z] \in H^*_{\mathbb{T}}(S^{[0,1]}) = H^2_T(S) = H^2_T(\mathbb{C}^2)$$

where $u^{-1}$ acts via $t^T$, and $C^z$ denotes the $z$-axis of $S = \mathbb{C}^2$.

It follows that a linear basis of the space $\mathbb{H}^\mathbb{T}$ is given by

$$\mathcal{B}_2 = \left\{ (t^T)^i \tilde{a}^T_{\nu-\nu} | 0 \right\}_{i \geq 0, \nu}.$$  

On the other hand, the $\mathbb{T}$-fixed points of $S^{[n,n+1]}$ are of the form $\xi_{\lambda,\mu} = (\xi_\lambda, \xi_\mu)$ where $\lambda$ and $\mu$ denote partitions of $n$ and $(n+1)$ respectively, and the Young diagram of $\lambda$ is contained in the Young diagram of $\mu$. Such a pair $(\lambda, \mu)$ of partitions is defined to be an incidence pair. For an incidence pair $(\lambda, \mu)$, let

$$[\lambda, \mu] = t^{-(n+1)} \cup [\xi_{\lambda,\mu}] \in H^*_{\mathbb{T}}(S^{[n,n+1]}).$$
where \( t \) is the character associated to the 1-dimensional standard module \( \theta \) of \( T \) on which \( a \in T \) acts as multiplication by \( a \). By the localization theorem, the ring structure of \( H^2_{T}(S^{[n,n+1]}) \) (and hence of \( \bar{H}_T \)) can be easily described in terms of the classes \([\lambda, \mu]\). In addition, these classes form another linear basis of \( \bar{H}_T \):

\[
\bar{B}_1 = \{ [\lambda, \mu] \}_{(\lambda, \mu) \text{ incidence}}.
\]

**Theorem 1.2.** There exists an algorithm to express each element \((\bar{t}_T^i \bar{a}^{-v}_T)|0\rangle\) in the linear basis \(\bar{B}_2\) as a linear combination of the elements in the linear basis \(\bar{B}_1\).

This theorem implies that the ring structure of \( \bar{H}_T \) can also be described (implicitly) in terms of the elements in the linear basis \(\bar{B}_2\). The main idea in proving Theorem 1.2 is to introduce a third linear basis of the space \(\bar{H}_T\):

\[
\bar{B}_3 = \{ [\bar{L}^{\lambda, \mu}C] \}
\]

where \(\bar{L}^{\lambda, \mu}C\) is defined by (4.5). We show that there exist algorithms to express every element in \(\bar{B}_2\) as a linear combination of the elements in \(\bar{B}_3\) and to express every element in \(\bar{B}_3\) as a linear combination of the elements in \(\bar{B}_1\).

There are two applications of our results. The first is to describe the ordinary cohomology ring \( H^*(S^{[n,n+1]}) \) of the incidence Hilbert scheme \( S^{[n,n+1]} \). The second is to the ring of symmetric functions. Indeed, define a linear isomorphism

\[
\bar{\Phi} : \bar{H}_T \rightarrow \Lambda \otimes_{\mathbb{Z}} C[v]
\]

by sending \((\bar{t}_T^i \bar{a}^{-v}_T)|0\rangle\) to \(p^i \otimes v^i\). Then the ring structure on \( \bar{H}_T \) induces a ring structure on \( \Lambda \otimes_{\mathbb{Z}} C[v] \) such that \( \Lambda \otimes_{\mathbb{Z}} \subset C[v] \) becomes a subring of \( \Lambda \otimes_{\mathbb{Z}} C[v] \). Moreover, we have a commutative diagram of ring homomorphisms:

\[
\begin{array}{ccc}
H_T & \xrightarrow{\Phi} & \Lambda \otimes_{\mathbb{Z}} C \\
\downarrow & & \downarrow \iota \\
\bar{H}_T & \xrightarrow{\bar{\Phi}} & \Lambda \otimes_{\mathbb{Z}} C[v]
\end{array}
\]

respecting the Heisenberg algebra actions on \( H_T \) and \( \bar{H}_T \), where \( \iota \) denotes the inclusion map. It is natural for us to ask what the induced ring structure on \( \Lambda \otimes_{\mathbb{Z}} C[v] \) is in the realm of symmetric functions.

The paper is organized as follows. In §2, we study the equivariant aspects of the incidence Hilbert scheme \( S^{[n,n+1]} \), including a description of the \( T \)-fixed points, the generating function for the Betti numbers, a \( T \)-invariant cell decomposition, the equivariant Zariski tangent spaces at the fixed points, and a bilinear pairing. In §3, we construct the loop algebra action on the space \( \bar{H}_T \), and compare it with the Heisenberg algebra action on the space \( H_T \). In §4, we study the transformations among the three linear bases \( \bar{B}_1, \bar{B}_2 \) and \( \bar{B}_3 \) of \( \bar{H}_T \). In §5, the two applications mentioned above are addressed. In §6 (the Appendix), we prove Lemma 2.8.
Conventions. We use $\lambda$ and $\mu$ to denote partitions of $n$ and $(n+1)$ respectively. The sign $\sim$, in the case of cohomology and operators, is for the incidence Hilbert schemes $S^{[n,n+1]}$. The sign $'$, in the case of equivariant cohomology, is for the localized equivariant cohomology.

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2. The equivariant setup for incidence Hilbert schemes

When a smooth algebraic variety $X$ admits a torus $\mathbb{C}^*$ action, one can study its equivariant cohomology $H^*_{\mathbb{C}^*}(X)$. It is known that the localized $H^*_{\mathbb{C}^*}(X)'$ has extra properties coming from the fixed points. In the case of Hilbert scheme $S^{[n]}$ of points on a surface $S$, this provides a much richer structure on the equivariant cohomology of $S^{[n]}$ than the ordinary cohomology [Vas, Na3, LQW1, LQW2].

Besides the Hilbert scheme of points $S^{[n]}$ for a surface $S$, the incidence Hilbert scheme $S^{[n,n+1]}$ for a surface $S$ is the only class of (generalized or nested) Hilbert schemes of points on smooth varieties of dimension bigger than one which are smooth for all $n$ (see [Ch1]). It has a nice generating function of Betti numbers. When the surface $S$ is $\mathbb{C}^2$, the torus $\mathbb{C}^*$ action on $S^{[n,n+1]}$ was studied in details in [Ch1]. In this section, we follow Cheah’s approach to the equivariant tangent spaces of the fixed points. We calculate the generating function of the Betti numbers of $S^{[n,n+1]}$, study in details the equivariant tangent spaces, and hence determine the ring structure of the localized equivariant cohomology $H^*_{\mathbb{C}^*}(S^{[n,n+1]})'$ in terms of the fixed points. It turns out that it is more natural to work on a modified cohomology ring, as illustrated in [Vas], which will be the material in the last subsection. We draw a special attention to three different torus actions on $S = \mathbb{C}^2$ in (2.10), (2.11) and (2.12) which serve for different purposes.

2.1. The equivariant homology and cohomology.

Let $\mathbb{T} = \mathbb{C}^*$, and let $\theta$ be the 1-dimensional standard module of $\mathbb{T}$ on which $a \in \mathbb{T}$ acts as multiplication by $a$, and let $t$ be the associated character. Then the representation ring $\mathcal{R}(\mathbb{T})$ is isomorphic to $\mathbb{Z}[t, t^{-1}]$.

Let $X$ be an algebraic variety acted by $\mathbb{T}$. Let $H^*_T(X)$ and $H_*^T(X)$ be the equivariant cohomology and the equivariant homology with $\mathbb{C}$-coefficient respectively. Note that $H^*_T(pt) = H^*(B\mathbb{T}) = \mathbb{C}[t]$. Then there exist bilinear maps

$$\cup : H^*_T(X) \otimes H_*^T(X) \to H^*_T(X),$$

$$\cap : H^*_T(X) \otimes H_*^T(X) \to H^*_T(X).$$

If $X$ is of pure dimension, then there exists a linear map

$$D : H^*_T(X) \to H^*_T(X).$$

If $X$ is smooth of pure dimension, then $D$ is an isomorphism. When $f : Y \to X$ is a $\mathbb{T}$-equivariant and proper morphism of varieties, we have a Gysin homomorphism

$$f_! : H^*_T(Y) \to H^*_T(X).$$
of equivariant homology. Moreover, when both $Y$ and $X$ are smooth of pure dimension, we have the Gysin homomorphism

$$D^{-1} f; D : H^*_T(Y) \to H^*_T(X)$$

of equivariant cohomology, which will still be denoted by $f$.

2.2. Incidence Hilbert schemes of points on surfaces.

Let $S$ be a smooth complex surface, and $S^{[n]}$ be the Hilbert scheme of points in $S$. An element in $S^{[n]}$ is represented by a length-$n$ 0-dimensional closed subscheme $\xi$ of $S$. For $\xi \in S^{[n]}$, let $I_\xi$ be the corresponding sheaf of ideals. It is well known that $S^{[n]}$ is a nonsingular complex variety of dimension $2n$. Sending an element in $S^{[n]}$ to its support in the symmetric product $\text{Sym}^n(S)$, we obtain the Hilbert-Chow morphism $\pi_n : S^{[n]} \to \text{Sym}^n(S)$, which is a resolution of singularities. Let

$$Z_n = \{(\xi, s) \in S^{[n]} \times S | s \in \text{Supp}(\xi)\}$$

be the universal codimension-2 subscheme in $S^{[n]} \times S$.

Fix a point $s \in S$. For $m \geq 0$ and $n > 0$, we define two closed subsets:

$$M_m(s) = \{\xi \in S^{[m]} | \text{Supp}(\xi) = \{s\}\}, \quad (2.1)$$

$$M_{m,m+n}(s) = \{(\xi, \xi') | \xi \subset \xi' \} \subset M_m(s) \times M_{m+n}(s). \quad (2.2)$$

It is known that $M_{m,m+1}(s)$ and $M_{m+1}(s)$ are irreducible with

$$\dim M_{m,m+1}(s) = \dim M_{m+1}(s) = m. \quad (2.3)$$

The incidence Hilbert scheme $S^{[n,n+1]}$ is defined by

$$S^{[n,n+1]} = \{(\xi, \xi') | \xi \subset \xi' \} \subset S^{[n]} \times S^{[n+1]} \quad (2.4)$$

It is known from [Ch1, Tik] that the incidence Hilbert scheme $S^{[n,n+1]}$ is irreducible, smooth and of dimension $2(n + 1)$. In fact, we have

$$S^{[n,n+1]} \cong \widetilde{S^{[n]}} \times S \quad (2.5)$$

where $\widetilde{S^{[n]}} \times S$ denotes the blowup of $S^{[n]} \times S$ along the subscheme $Z_n$ (see [ES2]). Note that sending a pair $(\xi, \xi') \in S^{[n,n+1]}$ to the support of $I_\xi/I_{\xi'}$ yields a morphism:

$$\rho_n : S^{[n,n+1]} \to S \quad (2.6)$$

which is also the composition of the isomorphism (2.5) and the projection

$$\widetilde{S^{[n]}} \times S \to S^{[n]} \times S \to S.$$

2.3. The torus action on the incidence Hilbert schemes.

Let $S = \mathbb{C}^2$. Then the 2-dimensional complex torus $T^2 = (\mathbb{C}^*)^2$ acts on the affine coordinate functions $w$ and $z$ of $S$ by

$$(a, b)w = aw, \quad (a, b)z = bz \quad (a, b) \in T^2. \quad (2.7)$$

It induces $T^2$-actions on both $S^{[n]}$ and $S^{[n,n+1]}$. It is known from [ES1] that the $T^2$-fixed points in $S^{[n]}$ are parametrized by the partitions of $n$. Let $\lambda$ be a partition
Lemma 2.1. Let \( \xi_\lambda \) be the \( \mathbb{T}^2 \)-fixed point on \( S^{[n]} \) corresponding to \( \lambda \). If \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r) \) with \( \lambda_1 + \ldots + \lambda_r = n \), then we have

\[
I_{\xi_\lambda} = (w^{\lambda_1},zw^{\lambda_2},zr^{-1}w^{\lambda_r},z^r).
\] (2.8)

The multiplicity of part \( i \) in a partition \( \mu \) is denoted by \( m_i(\mu) \), or simply by \( m_i \) if there is no confusion. Using these multiplicities, we can also express \( \mu \) as:

\[
\mu = (1^{m_1(\mu)}2^{m_2(\mu)}\ldots i^{m_i(\mu)}\ldots) = (1^{m_1}2^{m_2}\ldots i^{m_i}\ldots).
\]

Lemma 2.1. The \( \mathbb{T}^2 \)-fixed points in \( S^{[n,n+1]} \) are of the form \((\xi_\lambda, \xi_\mu)\) where

\[
\begin{align*}
\lambda &= (\lambda_1 \geq \lambda_2 \geq \ldots), \quad \mu = (\mu_1 \geq \mu_2 \geq \ldots) \\
\mu &= (\cdots (i-1)^{m_{i-1}}i^{m_i}(i+1)^{m_{i+1}}\ldots) \vdash (n+1)
\end{align*}
\] (2.9)

for some \( i \geq 1 \) with \( m_i > 0 \) (the parts \((i-2)^{m_{i-2}},(i-1)^{m_{i-1}}\) and \((i-1)^{m_{i-1}}\) do not appear if \( i = 1 \)).

Proof. Since the \( \mathbb{T}^2 \)-fixed points in \( S^{[n]} \) are of the form \( \xi_\lambda \) with \( \lambda \vdash n \), the \( \mathbb{T}^2 \)-fixed points in \( S^{[n,n+1]} \) are of the form \((\xi_\lambda, \xi_\mu)\) where \( \lambda \vdash n, \mu \vdash (n+1) \) and \( I_{\xi_\mu} \subseteq I_{\xi_\lambda} \). Let \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \) and \( \mu = (\mu_1 \geq \mu_2 \geq \ldots) \). By (2.8),

\[
\mu_1 \geq \lambda_1, \quad \mu_2 \geq \lambda_2, \ldots.
\]

Since \( \sum_j \lambda_j = n \) and \( \sum_j \mu_j = n+1 \), there exists \( j_0 \) satisfying \( \mu_{j_0} = \lambda_{j_0} + 1 \) and \( \mu_j = \lambda_j \) whenever \( j \neq j_0 \). This is equivalent to (2.9). \( \square \)

Definition 2.2. (i) The step length \( s(\mu) \) of a partition \( \mu \) is defined to be

\[
s(\mu) = \#\{i \mid m_i(\mu) > 0\};
\]

(ii) If \((\xi_\lambda, \xi_\mu) \in S^{[n,n+1]} \), then \((\lambda, \mu)\) is defined to be an incidence pair. Put

\[
\xi_{\lambda,\mu} = (\xi_\lambda, \xi_\mu).
\]

Fix \( \mu \vdash (n+1) \). By Lemma 2.1, the number of \( \mathbb{T}^2 \)-fixed points in \( S^{[n,n+1]} \) which are of the form \((\xi_\lambda, \xi_\mu)\) is precisely equal to the step length of the partition \( \mu \).

Let \( T_+ = T_- = \mathbb{C}^* \). Consider the three actions on coordinate functions of \( S \):

\[
\begin{align*}
a(w, z) &= (aw, a^{-1}z), \quad a \in T, \quad \text{(2.10)} \\
a(w, z) &= (a^uw, a^{v}z), \quad a \in T_+, \quad \text{(2.11)} \\
a(w, z) &= (a^{-v}w, a^{v}z), \quad a \in T_-, \quad \text{(2.12)}
\end{align*}
\]

where \( 0 < u \ll v \). We regard them as three 1-dimensional subgroups of \( \mathbb{T}^2 \). The fixed points in \( S^{[n,n+1]} \) under the action of \( T \) (respectively, \( T_+ \) and \( T_- \)) are exactly the same as those given by Lemma 2.1.
2.4. The generating function for the Betti numbers.

When the surface $S$ is projective, the generating function for the Betti numbers of the incidence Hilbert schemes $S^{[n,n+1]}$ has been determined by Cheah [Ch2]:

$$
\sum_{n=0}^{+\infty} \left( \sum_{i} (-1)^i b_i(S^{[n,n+1]}) z^i \right) q^n
= \left( \sum_{i} (-1)^i b_i(S) z^i \right) \cdot \frac{1}{1-z^2} \cdot \prod_{n=1}^{+\infty} \prod_{i} \left( \frac{1}{1-z^{2n-2}q^n} \right)^{(-1)^i b_i(S)}
$$

(2.13)

where $b_i(\cdot)$ denotes the $i$-th Betti number, i.e., the rank of the $i$-th ordinary (co)homology with $\mathbb{C}$-coefficients. For a general smooth quasi-projective surface $S$, it is unclear whether the above formula still holds. In the following, we let $S = \mathbb{C}^2$ and show that the above formula holds for $S = \mathbb{C}^2$.

**Proposition 2.3.**

$$
\sum_{n=0}^{+\infty} \left( \sum_{i} (-1)^i b_i(S^{[n,n+1]}) z^i \right) q^n = \frac{1}{1-z^2} \cdot \prod_{n=1}^{+\infty} \prod_{i} \frac{1}{1-z^{2n-2}q^n}.
$$

**Proof.** Let $O$ be the origin of $S = \mathbb{C}^2$. The $\mathbb{T}_+$-action on $S^{[n,n+1]}$ gives rise to a cell decomposition $C^n_+$ of the punctual incidence Hilbert scheme:

$$
S^{[n,n+1]}_O = \{ (\xi, \xi') \in S^{[n,n+1]} \mid \text{Supp}(\xi') = \{O\} \}.
$$

By the Proposition 2.6.4 in [Ch1], the dimension of the positive part of the tangent space of $S^{[n,n+1]}$ at a $\mathbb{T}_+$-fixed point $\xi_{\lambda,\mu} = (\lambda, \mu)$ is equal to

$$(n+1) - \mu_1,
$$

(2.14)

where $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r)$. By the Theorem 3.3.3 (5) of [Ch1],

$$
\sum_{n=0}^{+\infty} \left( \sum_{i} (-1)^i b_i^{\mathbb{T}_+}(S^{[n,n+1]}_O) z^i \right) q^n = \frac{1}{1-z^2} \cdot \prod_{n=1}^{+\infty} \frac{1}{1-z^{2n-2}q^n}
$$

(2.15)

where $b_i^{\mathbb{T}_+}(\cdot)$ stands for the rank of the Borel-Moore homology group $H_i^{\mathbb{T}_+}(\cdot)$. On the other hand, the $\mathbb{T}_-$-action on $S^{[n,n+1]}$ gives rise to a cell decomposition $C^n_-$ of $S^{[n,n+1]}$ itself. The positive part of the tangent space of $S^{[n,n+1]}$ at a $\mathbb{T}_-$-fixed point $\xi_{\lambda,\mu}$ is precisely the negative part of the tangent space of $S^{[n,n+1]}$ at the same $\mathbb{T}_+$-fixed point $\xi_{\lambda,\mu}$. Since $S^{[n,n+1]}$ is of dimension $2(n+1)$, we see from (2.15) that

$$
\sum_{n=0}^{+\infty} \left( \sum_{i} (-1)^i b_i^{\mathbb{T}_-}(S^{[n,n+1]}) z^i \right) q^n = \frac{1}{1-z^2} \cdot \prod_{n=1}^{+\infty} \frac{1}{1-z^{2n-2}q^n}
$$

Again, since $S^{[n,n+1]}$ is smooth, there are natural isomorphisms:

$$
H_i^{\mathbb{T}_+}(S^{[n,n+1]}) \cong H_i(S^{[n,n+1]})^* \cong H^i(S^{[n,n+1]})
$$

(2.16)

where $H_i(S^{[n,n+1]})^*$ is the dual of $H_i(S^{[n,n+1]})$. So we obtain the desired formula. □
2.5. A $\mathbb{T}$-invariant cell decomposition.

In the rest of this section, we let $S = \mathbb{C}^2$. Recall that the conjugate of a partition $\mu$ is the partition $\mu'$ whose Young diagram is the transpose of that of $\mu$. Then

$$\ell(\mu) = \mu'_1$$

(2.17)

if $\mu' = (\mu'_1 \geq \mu'_2 \geq \ldots \geq \mu'_r)$. In addition, $\mu$ and $\mu'$ have the same step length.

**Proposition 2.4.** Let $S = \mathbb{C}^2$. Then $S^{[n,n+1]}$ admits a cell decomposition

$$S^{[n,n+1]} = \bigcup_{\mu \vdash (n+1), m_i(\mu') > 0} S_{\mu,i}$$

(2.18)

such that $(\xi_{(\mu')^c}, \xi_{\mu'}) \in S_{\mu,i} \cong \mathbb{C}^{(n+1) + \ell(\mu)}$ and every cell $S_{\mu,i}$ is $\mathbb{T}$-invariant.

**Proof.** From the proof of Proposition [2.3] we see that the $\mathbb{T}_-$-action on $S^{[n,n+1]}$ gives rise to a cell decomposition $C^\mu_n$ of $S^{[n,n+1]}$. Let $C_{\mu',i}$ be the cell corresponding to the $\mathbb{T}_-$-fixed point $(\xi_{(\mu')^c}, \xi_{\mu'})$ where $m_i(\mu') > 0$. By (2.14) and (2.17),

$$\dim(C_{\mu',i}) = 2(n + 1) - [(n + 1) - \mu'_1] = (n + 1) + \ell(\mu).$$

Define $S_{\mu,i} = C_{\mu',i}$ for every partition $\mu \vdash (n+1)$ and $m_i(\mu') > 0$. Then we have the cell decomposition (2.18) with $(\xi_{(\mu')^c}, \xi_{\mu'}) \in S_{\mu,i} \cong \mathbb{C}^{(n+1) + \ell(\mu)}$.

To show that $S_{\mu,i} = C_{\mu',i}$ is $\mathbb{T}$-invariant, let $(\eta, \eta') \in C_{\mu',i}$. Then,

$$\lim_{b \to 0} b(\eta, \eta') = (\xi_{(\mu')^c}, \xi_{\mu'})$$

where $b \in \mathbb{T}_-$. Let $a \in \mathbb{T}$. To show $a(\eta, \eta') \in C_{\mu',i}$, it suffices to verify

$$\lim_{b \to 0} b(a(\zeta)) = \lim_{b \to 0} b(\zeta)$$

(2.19)

for every $\zeta \in S^{[n]}$. Let $f = f(w, z) \in I_\zeta \subset \mathbb{C}[w, z]$. Then the contribution of $f \in I_\zeta$ to the limiting ideal $\lim_{b \to 0} I_{b(\zeta)}$ is equal to $w^j(f) z^i(f)$ where

$$j(f) = \max \{j \mid \text{for some } i, w^i z^j \text{ is a term in } f\},$$

$$i(f) = \max \{i \mid w^i z^j(f) \text{ is a term in } f\}.$$  

Since $j(a(f)) = j(f)$ and $i(a(f)) = i(f)$, we conclude that the contribution of $a(f) \in I_{a(\zeta)}$ to $\lim_{b \to 0} I_{b(a(\zeta))}$ is also $w^j(f) z^i(f)$. This proves (2.19). \[\square\]

**Corollary 2.5.** (i) $H^2(k(S^{[n,n+1]})) \cong H^2_{\mathbb{T}}(S^{[n,n+1]}(\mathbb{C}[w, z])$ when $k \geq (n + 1)$;

(ii) There exists a ring isomorphism $H^*_T(S^{[n,n+1]})/t \cong H^*(S^{[n,n+1]})$.

**Proof.** (i) Let $\overline{S_{\mu,i}}$ be the closure of $S_{\mu,i}$ in $S^{[n,n+1]}$. By Proposition 2.4,

$$H^*_T(S^{[n,n+1]}) = \bigcup_{\mu \vdash (n+1), m_i(\mu') > 0} \mathbb{C}^{2} \cup \bigcup_{\mu \vdash (n+1), m_i(\mu') > 0} \mathbb{C}^{2} \bigcup \overline{S_{\mu,i}}$$

(2.20)

Here and below, $[\cdot]$ denotes the equivariant fundamental cycle or its associated equivariant cohomology class. Now (i) follows immediately.

(ii) There is the forgetful map $H^*_T(S^{[n,n+1]}) \to H^*(S^{[n,n+1]})$ which is a ring homomorphism. By Proposition 2.4 and (2.16), a $\mathbb{C}$-linear basis of $H^*(S^{[n,n+1]})$ consists
of the (ordinary) fundamental cohomology classes of the closures \( S_{\mu,i} \). Combining this with (2.20), we obtain \( H^*_T(S^{[n,n+1]}/(t) \cong H^*(S^{[n,n+1]}). \)

\[ \square \]

2.6. The equivariant Zariski tangent spaces of \( S^{[n]} \).

The first study of the equivariant Zariski tangent space of the Hilbert scheme \( S^{[n]} \) at the fixed points was carried out in [ES1]. Here we review an approach due to Cheah in [Ch1]. It will be used in the next subsection for the equivariant Zariski tangent space of the incidence Hilbert scheme \( S^{[n,n+1]} \).

Let \( \lambda \vdash n \). Then the \( T \)-invariant ideal \( I_{\xi_\lambda} \subset R = \mathbb{C}[w,z] \) is given by:

\[ I_{\xi_\lambda} = (w^{\lambda_1}, zw^{\lambda_2}, \ldots, z^{r-1}w^{\lambda_r}, z^r). \]

The Zariski tangent space \( T_{\xi_\lambda}S^{[n]} \) at \( \xi_\lambda \) is canonically isomorphic to the space \( \text{Hom}(I_{\xi_\lambda}, R/I_{\xi_\lambda}) \). To obtain a pure-weight linear basis of the \( T \)-invariant space \( \text{Hom}(I_{\xi_\lambda}, R/I_{\xi_\lambda}) \), we represent \( I_{\xi_\lambda} \) by a Young diagram \( D_\lambda \) as in [Ch1].

**Example 2.6.** The ideal \( I = \langle w^5, zw^4, z^4w^2, z^6 \rangle \) is represented by the diagram:

![Diagram](image1)

If we look at the Young diagram of \( I = I_{\xi_\lambda} \), then the corners of its complement (that is, the shaded boxes in the second diagram of Figure 2) represent the unique minimal set of monomials that generate the ideal \( I_{\xi_\lambda} \). Denote this set by \( A \), and let \( B \) be the set of monomials not in \( I_{\xi_\lambda} \):

![Diagram](image2)
The elements (called the canonical generators) in the set $A$ are:

\[
\begin{align*}
\alpha_0 & := w^{\lambda_0}, \\
\alpha_1 & := z^{p_0} w^{\lambda_{p_0} + p_1}, \\
\vdots & \quad \vdots \\
\alpha_{m-1} & := z^{p_0 + p_1 + \ldots + p_{m-2}} w^{\lambda_{p_0 + p_1 + \ldots + p_{m-2}}}, \\
\alpha_m & := z^{p_0 + p_1 + \ldots + p_{m-1}} = z^r.
\end{align*}
\]

(2.23)

Define $q_m = \lambda_{p_0 + p_1 + \ldots + p_{m-2} + p_{m-1}}$. For $1 \leq i \leq m-1$, define

\[
q_i = \lambda_{p_0 + p_1 + \ldots + p_i} - \lambda_{p_0 + p_1 + \ldots + p_{i-1} + p_i}.
\]

(2.24)

Note that $p_i$ is the vertical distance between the cells representing $\alpha_i$ and $\alpha_{i+1}$ and that $q_i$ is the horizontal distance between the cells representing $\alpha_i$ and $\alpha_{i-1}$.

For $\alpha = \alpha_i \in A$, let $P_\alpha$ be the subset of $B$ consisting of the elements $b$ satisfying

(i) $b$ lies to the left of $\alpha$ in the Young diagram,

(ii) $z^{p_i} b \in I_{\xi_\alpha},$

and let $Q_\alpha$ be the subset of $B$ consisting of the elements $b$ satisfying

(i) $b$ lies above $\alpha$ in the Young diagram,

(ii) $w^{q_i} b \in I_{\xi_\alpha}.$

Let $S$ be the subset of $\text{Hom}(I_{\xi_\alpha}, R/I_{\xi_\alpha})$ consisting of elements of pure weight which take canonical generators in $A$ either to zero or to monomials in $B$ modulo $I_{\xi_\alpha}$. For $\beta \in P_\alpha \cup Q_\alpha$, define $f_{\alpha, \beta} \in S$ to be the unique element satisfying

(i) $f_{\alpha, \beta}(\alpha) = \beta,$

(ii) $f_{\alpha, \beta}$ takes the largest number of canonical generators to zero.

The conditions (i) and (ii) imply that $f_{\alpha_i, \beta}(\alpha_j) = 0$ if $\beta \in P_\alpha$, and $j > i$, and that $f_{\alpha_i, \beta}(\alpha_j) = 0$ if $\beta \in Q_\alpha$, and $j < i$. By the Proposition 2.5.4 of [Ch1], a pure weight basis of the tangent space $T_{\xi_\alpha} S[\alpha] \cong \text{Hom}(I_{\xi_\alpha}, R/I_{\xi_\alpha})$ of $S[\alpha]$ at $\xi_\alpha$ is

\[
\{f_{\alpha, \beta} \mid \alpha \in A, \beta \in P_\alpha \cup Q_\alpha\}.
\]

When $\beta \in Q_\alpha$, we have $\beta = \alpha \cdot w^{i_1}/z^{i_2}$ for some integers $i_1 \geq 0$ and $i_2 > 0$, and the weight of $f_{\alpha, \beta}$ is equal to $(i_1 + i_2)$, which is the hook length $h(\square)$ of certain cell $\square$ in the Young diagram $D_{\lambda}$. As $\beta$ runs in the set $Q_\alpha$, $\square$ runs over all the cells in $D_{\lambda}$ exactly once. Similarly, when $\beta \in P_\alpha$, we have $\beta = \alpha \cdot z^{i_2}/w^{i_1}$ for some integers $i_1 > 0$ and $i_2 \geq 0$, and the weight of $f_{\alpha, \beta}$ is equal to $-(i_1 + i_2)$, where $(i_1 + i_2)$ is
the hook length \( h(\square) \) of certain cell \( \square \) in \( D_\lambda \). Again, as \( \beta \) runs in \( Q_\alpha \), \( \square \) runs over all the cells in \( D_\lambda \) exactly once. Hence, there exists a \( \mathbb{T} \)-equivariant identification:

\[
T_{\xi_\lambda}S^{[n]} = \bigoplus_{\square \in D_\lambda} (\theta^{h(\square)} \oplus \theta^{-h(\square)}).
\]

(2.25)

It follows that the \( \mathbb{T} \)-equivariant Euler class of the tangent space is

\[
e_T(T_{\xi_\lambda}S^{[n]}) = (-1)^n \cdot \prod_{\square \in D_\lambda} h(\square)^2 \cdot t^{2n} = (-1)^n \cdot h(\lambda)^2 \cdot t^{2n}
\]

(2.26)

where \( h(\lambda) \) is the product of all the hook lengths \( h(\square), \square \in D_\lambda \).

2.7. The equivariant Zariski tangent spaces of \( S^{[n,n+1]} \).

Let \((\lambda, \mu)\) be an incidence pair of partitions with \( \lambda \vdash n \). Then, \((\xi_\lambda, \xi_\mu)\) is a \( \mathbb{T} \)-fixed point in \( S^{[n,n+1]} \). There are \( \mathbb{T} \)-equivariant maps:

\[
\phi : \text{Hom}(I_{\xi_\lambda}, R/I_{\xi_\lambda}) \to \text{Hom}(I_{\xi_\mu}, R/I_{\xi_\lambda}),
\]

(2.27)

\[
\psi : \text{Hom}(I_{\xi_\mu}, R/I_{\xi_\mu}) \to \text{Hom}(I_{\xi_\mu}, R/I_{\xi_\lambda}).
\]

(2.28)

From pages 42-43 in [Ch1], we see that the Zariski tangent space of \( S^{[n,n+1]} \) at the point \((\xi_\lambda, \xi_\mu)\) is canonically isomorphic to \( \ker(\phi - \psi) \) where

\[
(\phi - \psi) : \text{Hom}(I_{\xi_\lambda}, R/I_{\xi_\lambda}) \oplus \text{Hom}(I_{\xi_\mu}, R/I_{\xi_\mu}) \to \text{Hom}(I_{\xi_\mu}, R/I_{\xi_\lambda})
\]

is defined by letting \((\phi - \psi)(a, b) = \phi(a) - \psi(b)\). By the Lemma 2.6.2 in [Ch1], \((\phi - \psi)\) is surjective. Therefore, there is a \( \mathbb{T} \)-equivariant exact sequence:

\[
0 \to \ker(\phi - \psi) \to \text{Hom}(I_{\xi_\lambda}, R/I_{\xi_\lambda}) \oplus \text{Hom}(I_{\xi_\mu}, R/I_{\xi_\mu}) \to \text{Hom}(I_{\xi_\mu}, R/I_{\xi_\lambda}) \to 0.
\]

(2.29)

We keep using the notations \( A, B, p_i, \alpha_i, q_i \) associated to \( \lambda \) from §2.6 and let \( A', B' \) be the corresponding notations associated to \( \mu \). Put

\[
A' = \{\alpha'_0, \alpha'_1, \ldots, \alpha'_s\}.
\]

The Young diagram of \( I_{\xi_\mu} \) is obtained from that of \( I_{\xi_\lambda} \) by adding one of the cells which represents a canonical generator in \( A \). Let

\[
\alpha_k \in A
\]

(2.30)

be this canonical generator. Then, \( \alpha_k \in B' \) and so \( \alpha_k \not\in A' \). Note that

\[
\alpha_k \in P_{\alpha'_i} \cup Q_{\alpha'_i}
\]

for all \( 0 \leq i \leq s \), and that the homomorphism \( f_{\alpha'_i, \alpha_k} \in \text{Hom}(I_{\xi_\mu}, R/I_{\xi_\mu}) \) maps \( \alpha'_i \) to \( \alpha_k \) and all the other canonical generators in \( A' \) to zero. Moreover, \( f_{\alpha'_0, \alpha_k}, \ldots, f_{\alpha'_s, \alpha_k} \) form a basis of \( \ker(\psi) \subset \ker(\phi - \psi) \), i.e., we have

\[
\ker(\phi - \psi) \supset \ker(\psi) = \bigoplus_{i=0}^s \mathbb{C}f_{\alpha'_i, \alpha_k}.
\]

(2.31)

**Definition 2.7.** Let \((\lambda, \mu)\) be an incidence pair, and \( k \) be from (2.30).
For $0 \leq i \leq k - 1$, let $\square_{k,i}$ be the cell in the Young diagram $D_\lambda$ which is directly to the left of $\alpha_i$ and directly above $\alpha_k$, and let $\square'_{k,i}$ be the cell which is the $(p_i - 1)$-th cell directly under $\square_{k,i}$ ($\square'_{k,i} = \square_{k,i}$ if $p_i = 1$). For $k + 1 \leq i \leq m = s(\lambda)$, let $\square_{k,i}$ be the cell which is directly above $\alpha_i$ and directly to the left of $\alpha_k$, and let $\square'_{k,i}$ be the cell which is the $(q_i - 1)$-th cell directly to the right of $\square_{k,i}$ ($\square'_{k,i} = \square_{k,i}$ if $q_i = 1$).

(ii) Define $k(\lambda, \mu)$ to be the integer $k$, and define

$$h(\lambda, \mu) = h(\lambda)^2 \cdot \prod_{0 \leq i \leq s(\lambda)} \frac{1 + h(\square_{k(\lambda, \mu), i})}{h(\square'_{k(\lambda, \mu), i})}.$$  

(2.32)

The following lemma is the key step for determining the ring structure of the equivariant cohomology ring of the incidence Hilbert scheme $S^{[n,n+1]}$. Since the proof is a bit technical, we place it in the Appendix.

**Lemma 2.8.** Let $(\xi_\lambda, \xi_\mu) \in S^{[n,n+1]}$ be a $\mathbb{T}$-fixed point. Then the $\mathbb{T}$-equivariant Euler class of the tangent space of $S^{[n,n+1]}$ at $(\xi_\lambda, \xi_\mu)$ is equal to

$$e_\mathbb{T} = (-1)^{n+1} h(\lambda, \mu) \cdot t^{2(n+1)}.$$ 

(2.33)

**2.8. A bilinear pairing.**

Recall that the $\mathbb{T}$-fixed points in $S^{[n,n+1]}$ are of the form $\xi_{\lambda, \mu} = (\xi_\lambda, \xi_\mu)$ where $(\lambda, \mu)$ denotes incidence pairs of partitions. Let

$$t_{\lambda, \mu} : \xi_{\lambda, \mu} = (\xi_\lambda, \xi_\mu) \hookrightarrow S^{[n,n+1]}$$

(2.34)

be the inclusion map. Let $1_{\xi_{\lambda, \mu}} \in H^0_\mathbb{T}(\xi_{\lambda, \mu})$ be the unit. Thus,

$$[\xi_{\lambda, \mu}] = (t_{\lambda, \mu})!(1_{\xi_{\lambda, \mu}}) \in H^4_{\mathbb{T}}(S^{[n,n+1]}).$$

Denote by $\mathbb{C}[t]'$ the localization of the ring $\mathbb{C}[t]$ at the ideal $(t - 1)$, and denote

$$\tilde{t}_n = \bigoplus_{(\lambda, \mu) \text{ incidence}} t_{\lambda, \mu} : (S^{[n,n+1]})^\mathbb{T} \to S^{[n,n+1]}.$$  

We define $H^*_\mathbb{T}((S^{[n,n+1]})^\mathbb{T})' = H^*_\mathbb{T}((S^{[n,n+1]})^\mathbb{T} \otimes \mathbb{C}[t]' \otimes \mathbb{C}[t]'$ and define $H^*_\mathbb{T}(S^{[n,n+1]})'$ similarly. Then we have the induced Gysin map:

$$\tilde{t}_n^* : H^*_\mathbb{T}((S^{[n,n+1]})^\mathbb{T})' \longrightarrow H^*_\mathbb{T}(S^{[n,n+1]})'.$$

By the localization theorem, $\tilde{t}_n^*$ is an isomorphism. The inverse $(\tilde{t}_n^*)^{-1}$ is given by

$$\tilde{A} \mapsto \left( \frac{(t_{\lambda, \mu})^*(\tilde{A})}{e_\mathbb{T}(T_{\xi_{\lambda, \mu}} S^{[n,n+1]})} \right)_{(\lambda, \mu) \text{ incidence}}.$$  

Therefore, we conclude from Lemma 2.8 on the Euler class $e_\mathbb{T}$ that

$$(\tilde{t}_n^*)^{-1}(\tilde{A}) = \left( \frac{(t_{\lambda, \mu})^*(\tilde{A})}{(-1)^{n+1} h(\lambda, \mu) \cdot t^{2(n+1)}} \right)_{(\lambda, \mu) \text{ incidence}}.$$  

(2.35)
Next, we define a bilinear pairing on \( H^*_T(S^{[n,n+1]})' \) by:

\[
\langle -,- \rangle : H^*_T(S^{[n,n+1]})' \otimes \mathbb{C}[t] \to \mathbb{C}[t]',
\]

\[
\langle A, B \rangle = (-1)^{n+1} \pi_n(t_n)^{-1}(A \cup B) \tag{2.37}
\]

where \( \pi_n \) is the projection of the set \((S^{[n,n+1]})^T\) of \(T\)-fixed points to a point. This induces a bilinear pairing, again denoted by \( \langle -,- \rangle \), on the space:

\[
\tilde{\mathbb{H}}_T' = \bigoplus_{n=0}^{+\infty} H^*_T(S^{[n,n+1]})'.
\tag{2.38}
\]

2.9. A new ring \( \tilde{\mathbb{H}}_{T,n} \) and its linear basis from the fixed points.

For \( n \geq 0 \), let \( \tilde{\mathbb{H}}_{T,n} = H^{2(n+1)}_T(S^{[n,n+1]}) \) be the middle-degree equivariant cohomology of the incidence Hilbert scheme \( S^{[n,n+1]} \). By Corollary 2.5 (i),

\[
H^{2(n+1)}_T(S^{[n,n+1]}) = t^{(n+1)} \cup H^{2(n+1)}_T(S^{[n,n+1]}).
\]

Note that \( H^*_T(S^{[n,n+1]}) \) is \( \mathbb{C}[t] \)-torsion free. Define a product \( \hat{\star} \) on \( \tilde{\mathbb{H}}_{T,n} \) by:

\[
t^{n+1} \cup [\lambda, \mu] = [\xi_{\lambda,\mu}]
\tag{2.39}
\]

for \( \tilde{A}, \tilde{B} \in \tilde{\mathbb{H}}_{T,n} \). Then, we see that \((\tilde{\mathbb{H}}_{T,n}, \hat{\star})\) is a ring.

Next, we construct a linear basis of \( \tilde{\mathbb{H}}_{T,n} \). Define the class \( [\lambda, \mu] \in \tilde{\mathbb{H}}_{T,n} \) by

\[
t^{n+1} \cup [\lambda, \mu] = [\xi_{\lambda,\mu}]
\tag{2.40}
\]

since \( [\xi_{\lambda,\mu}] \in H^{2(n+1)}_T(S^{[n,n+1]}) \). Note that we have

\[
[\xi_{\lambda,\mu}] \cup [\xi_{\lambda,\mu}] = (t_{\lambda,\mu})!(1_{\xi_{\lambda,\mu}}) \cup (t_{\lambda,\mu}^*)(1_{\xi_{\lambda,\mu}})
\]

\[
= (t_{\lambda,\mu})!(1_{\xi_{\lambda,\mu}}) \cup (t_{\lambda,\mu}^*)(1_{\xi_{\lambda,\mu}})
\]

\[
= \delta(\lambda,\mu) e_T(T_{\xi_{\lambda,\mu}} S^{[n,n+1]})[\xi_{\lambda,\mu}]
\]

\[
= \delta(\lambda,\mu)(\lambda,\mu) (-1)^{n+1} h(\lambda,\mu) t^{(n+1)}[\xi_{\lambda,\mu}]
\tag{2.41}
\]

by the projection formula and Lemma 2.8. Thus we obtain

\[
[\lambda, \mu] \hat{\star} [\lambda, \mu] = \delta(\lambda,\mu) (-1)^{n+1} h(\lambda,\mu) [\lambda, \mu].
\tag{2.42}
\]

Combining this with the localization theorem, we see that the classes

\[
[\lambda, \mu]
\]

where \( \mu \vdash (n+1) \) and \( (\lambda, \mu) \) is an incidence pair, form a linear basis of \( \tilde{\mathbb{H}}_{T,n} \).

In addition, we obtain from (2.37) and (2.41) that

\[
\langle [\lambda, \mu], [\lambda, \mu] \rangle = (-1)^{n+1} \pi_n(t_n)^{-1}([\lambda, \mu] \cup [\lambda, \mu])
\]

\[
= (-1)^{n+1} \pi_n(t_n)^{-1}([-2(\lambda,\mu) \cup [\xi_{\lambda,\mu}])]
\]

\[
= \delta(\lambda,\mu)(\lambda,\mu) h(\lambda,\mu) \cdot \pi_n(t_n)^{-1}([\xi_{\lambda,\mu}])
\]

\[
= \delta(\lambda,\mu)(\lambda,\mu) h(\lambda,\mu).
\tag{2.43}
\]
It follows that the restriction to $\tilde{H}_{T,n}$ of the bilinear form $\langle -, - \rangle$ on the space $H^*_T(S^{[n,n+1]})'$ is a nondegenerate bilinear form:

$$\langle -, - \rangle : \tilde{H}_{T,n} \times \tilde{H}_{T,n} \rightarrow \mathbb{C}$$  \hspace{1cm} (2.44)

This induces a nondegenerate bilinear form, denoted again by $\langle -, - \rangle$, on:

$$\tilde{H}_T = \bigoplus_{n=0}^{+\infty} \tilde{H}_{T,n}.$$  \hspace{1cm} (2.45)

Note that a $\mathbb{C}$-linear basis of the vector space $\tilde{H}_T$ is given by

$$\tilde{B}_1 = \{ \{\lambda, \mu\} \} \text{ incidence}.$$  \hspace{1cm} (2.46)

3. The loop algebra action on $\tilde{H}_T$

One of the most important features of the Hilbert schemes $S^{[n]}$ of points on a surface $S$ is a Heisenberg algebra action on the direct sum of the cohomology groups of $S^{[n]}$ over all $n$ discovered by Nakajima and Grojnowski [Gro, Na1]. It lays the foundation for a new method in the study of the cohomology ring of the Hilbert scheme $S^{[n]}$. Without much difficulty, one can transport the Heisenberg algebra action to the equivariant cohomology of $S^{[n]}$ when $S = \mathbb{C}^2$.

A loop algebra of a Heisenberg algebra was found in [LQ] to act on the direct sum of the cohomology groups of incidence Hilbert schemes $S^{[n,n+1]}$. It is generated by a Heisenberg algebra and a translation operator. In this section, we transport the results in [LQ] to the equivariant cohomology of $S^{[n,n+1]}$ when $S = \mathbb{C}^2$.

3.1. The Heisenberg operators.

In the rest of this section, let $S = \mathbb{C}^2$. Let $C^w$ and $C^z$ be the $w$-axis and $z$-axis of $S = \mathbb{C}^2$ respectively. By the localization theorem, we have

$$[C^w] = t = -t^{-1}[O], \quad [C^z] = -t = t^{-1}[O]$$  \hspace{1cm} (3.1)

in $H^2_T(S)$, where $O \in S$ is the origin. In particular, $[C^w] = -[C^z]$ in $H^*_T(S)$.

Next, let $Y = C^w$ or $C^z$. Then, $Y$ is $T$-invariant. For $m \geq 0$ and $n > 0$, define the closed subset $\tilde{Q}_Y^{[m,n,m]}$ of $S^{[m+n,m+n+1]} \times S^{[m,m+1]}$:

$$\tilde{Q}_Y^{[m+n,m]} = \{ ((\xi, \xi'), (\eta, \eta')) | \xi \supset \eta, \xi' \supset \eta', \text{ Supp}(I_\eta/I_\xi) = \{ s \} \subset Y, \text{ Supp}(I_{\xi'/I_{\xi'}}) = \text{ Supp}(I_{\eta'/I_{\eta}}) \}.$$  \hspace{1cm} (3.2)

Then $\tilde{Q}_Y^{[m+n,m]}$ is $T$-invariant. Define the linear operator $\tilde{a}_{-n}( [Y] ) \in \text{End}(\tilde{H}_T)$ by

$$\tilde{a}_{-n}( [Y] )(\tilde{A}) = D^{-1} \tilde{p}_1 ! \left( \tilde{p}_2 \tilde{A} \cap [\tilde{Q}_Y^{[m+n,m]}] \right)$$  \hspace{1cm} (3.2)

for $\tilde{A} \in H^*_T(S^{[m,m+1]})'$, where $\tilde{p}_1$ and $\tilde{p}_2$ are the two projections of $S^{[m+n,m+n+1]} \times S^{[m,m+1]}$. Note that the restriction of $\tilde{p}_1$ to $\tilde{Q}_Y^{[m+n,m]}$ is proper. Define $\tilde{a}_n( [Y] ) \in$
End(\(\widetilde{\mathbb{H}}_T^r\)) to be the adjoint operator of \(\tilde{a}_{-n}([Y])\) with respect to the bilinear form \(\langle\cdot,\cdot\rangle\) on the linear space \(\widetilde{\mathbb{H}}_T\). Alternatively, we have

\[
\tilde{a}_{-n}([Y])(\widetilde{A}) = D^{-1}(\tilde{p}_1 H(I_{\mathbb{H}_{\mathbb{T}}^{m^{[m+n,m+n+1]} \times \mathbb{H}_m})^{-1} \left(\tilde{p}_2 \tilde{A} \cap [\widetilde{Q}_{Y}^{m+n,m}]\right), \tag{3.3}
\]

\[
\tilde{a}_{n}([Y])(\widetilde{B}) = (-1)^n D^{-1}(\tilde{p}_2 H(I_{\mathbb{H}_{\mathbb{T}}^{m+n+m+1}}) \cap \tilde{B} \cap [\widetilde{Q}_{Y}^{m+n,m}] \tag{3.4}
\]

for \(\tilde{A} \in H_T^s(S^{[m,m+n+1')}\) and \(\tilde{B} \in H_T^s(S^{[m,m+n+1]}')\), where \(\tilde{p}_1, \tilde{p}_2\) are the projections:

\[
\tilde{p}_1 : S^{[m,m+n+1]} \times (S^{[m,m+1]})^\mathbb{T} \to S^{[m,m+n+1]},
\]

\[
\tilde{p}_2 : (S^{[m,m+n+1]})^\mathbb{T} \times S^{[m,m+1]} \to S^{[m,m+1]},
\]

Let \(n > 0\). From the definition of \(\tilde{a}_{-n}([Y]) \in \text{End}(\widetilde{\mathbb{H}}_T)\), we see that

\[
\tilde{a}_{-n}([Y])(\widetilde{A}) \in \mathbb{H}_{\mathbb{T},m+n}
\]

if \(\tilde{A} \in \mathbb{H}_{\mathbb{T},m} = H_T^{2(m+1)}(S^{[m,m+1]}) \subset H_T^s(S^{[m,m+1]}')\). Hence the restriction of \(\tilde{a}_{-n}([Y])\) to the space \(\mathbb{H}_T\) gives a linear operator in \(\text{End}(\mathbb{H}_T)\), denoted by \(\tilde{a}_{-n}([Y])\) as well. Recall from (2.44) that there is a bilinear form

\[
\langle\cdot,\cdot\rangle : \mathbb{H}_T \otimes \mathbb{H}_T \to \mathbb{C},
\]

which is the restriction of the bilinear form \(\langle\cdot,\cdot\rangle\) on \(\mathbb{H}_T^r\). Thus, the restriction of \(\tilde{a}_n([Y])\) to \(\mathbb{H}_T\) is the adjoint operator of \(\tilde{a}_{-n}([Y])\) with respect to the bilinear form \(\langle\cdot,\cdot\rangle\) on \(\mathbb{H}_T\), and hence is an operator in \(\text{End}(\mathbb{H}_T)\) which will again be denoted by \(\tilde{a}_n([Y])\). Finally, we define \(\tilde{a}_0([Y]) = 0 \in \text{End}(\mathbb{H}_T)\).

**Proposition 3.1.** The operators \(\tilde{a}_n([C^z])\), \(n \in \mathbb{Z}\), acting on the space \(\mathbb{H}_T^r\) satisfy the following Heisenberg commutation relation:

\[
[\tilde{a}_n([C^z]), \tilde{a}_m([C^z])] = n\delta_{n-m} \text{Id}_{\mathbb{H}_T^r}. \tag{3.5}
\]

**Proof.** Since \(C^w = -[C^z]\), we have \(\tilde{Q}_{C^w}^{m+n,m} = -\tilde{Q}_{C^z}^{m+n,m}\). It follows from the definition that \(\tilde{a}_m([C^w]) = -\tilde{a}_m([C^z])\). Hence (3.5) is equivalent to

\[
[\tilde{a}_n([C^w]), \tilde{a}_m([C^z])] = -n\delta_{n-m} \text{Id}_{\mathbb{H}_T^r}. \tag{3.6}
\]

Note that \(C^w\) and \(C^z\) intersect transversely at the origin. By (3.3) and (3.4), the commutation relation (3.6) is reduced to the intersections between certain cycles related to various \(\tilde{Q}_{C^w}^{[\ell_1,\ell_2]}\) and certain cycles related to various \(\tilde{Q}_{C^z}^{[\ell_3,\ell_4]}\). Therefore, an argument similar to the one used in the proof of the Proposition 3.5 in LQ (for smooth projective surfaces) works in our situation. This proves (3.6). \(\square\)

### 3.2. The translation operator.

For \(m \geq 0\), define \(\tilde{Q}_m \subset S^{[m+1,m+2]} \times S^{[m,m+1]}\) to be the closed subset:

\[
\tilde{Q}_m = \{((\xi', \xi''), (\xi, \xi'))|\text{Supp}(I_{\xi'}/I_{\xi''}) = \text{Supp}(I_{\xi'}/I_{\xi''})\}.
\]

Then, the subset \(\tilde{Q}_m\) is \(\mathbb{T}\)-invariant, and \(\dim \tilde{Q}_m = (2m + 3)\).
Definition 3.2. Define the linear operator \( \tilde{t}^\dagger \in \text{End}(\check{H}_T) \) by
\[
\tilde{t}^\dagger(\tilde{A}) = D^{-1}\tilde{p}_1(\tilde{p}_2^\dagger \tilde{A} \cap [\check{Q}_m])
\]
for \( \tilde{A} \in \check{H}_{T,m} \), where \( \tilde{p}_1, \tilde{p}_2 \) are the two projections of \( S^{[m+1.m+2]} \times S^{[m,m+1]} \).

Proposition 3.3. (i) The adjoint operator \((\tilde{t}^\dagger)^\dagger\) is the left inverse of \(\tilde{t}^\dagger\);
(ii) \(\tilde{t}^\dagger\) and \((\tilde{t}^\dagger)^\dagger\) commute with the Heisenberg operators \(\check{a}_{-n}(|C^z|)\).

Proof. Note that \( \tilde{t} \) and its adjoint operator \( \tilde{t}^\dagger \) are also given by
\[
(\tilde{t}^\dagger)(\tilde{B}) = -D^{-1}(\tilde{p}_2^\dagger)(\tilde{B} \cap [\check{Q}_m])^{-1},
\]
for \( \tilde{A} \in \check{H}_{T,m} \) and \( \tilde{B} \in \check{H}_{T,m+1} \), where \( \tilde{p}_1^\dagger \) and \( \tilde{p}_2^\dagger \) are the projections:
\[
\tilde{p}_1^\dagger: \quad S^{[m+1.m+2]} \times (S^{[m,m+1]})^\dagger \to S^{[m+1,m+2]},
\]
\[
\tilde{p}_2^\dagger: \quad (S^{[m+1,m+2]})^\dagger \times S^{[m,m+1]} \to S^{[m,m+1]}.
\]

So our results follow from arguments similar to the proofs of the Lemma 4.2 (i) and Proposition 4.3 in \([\text{LQ}]\) for smooth projective surfaces. \(\square\)

3.3. The loop algebra action.

Definition 3.4. Let \( \check{a}_n^\dagger = \check{a}_n(|C^z|) \) for all \( n \in \mathbb{Z} \). Define \( \check{h}_T \) to be the Heisenberg algebra generated by the operators \( \check{a}_n^\dagger, n \in \mathbb{Z} \), and the identity operator \( \text{Id}_{\check{H}_T} \).

The loop algebra of a Lie algebra \( \mathfrak{g} \) is \( \mathbb{C}[u,u^{-1}] \otimes \mathbb{C} \mathfrak{g} \) with
\[
[u^m \otimes g_1, u^n \otimes g_2] = u^{m+n} \otimes [g_1, g_2].
\]

Theorem 3.5. The space \( \check{H}_T \) is a representation of the Lie algebra \( \mathbb{C}[u^{-1}] \otimes \mathbb{C} \check{h}_T \) with a highest weight vector being the vacuum vector
\[
|0\rangle = |C^z\rangle \in \check{H}_{T,0} = H^2_\mathcal{T}(S^{[0,1]}) = H^2_\mathcal{T}(S) = H^2_\mathcal{T}(\mathbb{C}^2)
\]
where \( u^{-1} \) acts via \( \tilde{t}^\dagger \), and \( C^z \) denotes the z-axis of \( S = \mathbb{C}^2 \).

Proof. Follows from Proposition 2.3, Proposition 3.1 and Proposition 3.3. \(\square\)

For a partition \( \nu = (1^{m_1}2^{m_2}\cdots) \), we establish the notations:
\[
\check{a}_{-\nu}^\dagger = \prod_j (\check{a}_{-j}^\dagger)^{m_j},
\]
\[
\check{z}_\nu = \prod_j (j^{m_j}m_j!).
\]

By Theorem 3.5 a \( \mathbb{C} \)-linear basis of the space \( \check{H}_T \) is given by
\[
\check{B}_2 = \left\{ (\tilde{t}^\dagger)^i \check{a}_{-\nu}^\dagger|0\rangle \right\}_{i \geq 0, \nu}.
\]
Lemma 3.6. \( \langle (\bar{t}^T)i_\nu^+ a_{\nu}^\pm |0\rangle, (\bar{t}^T)j_\nu^+ a_{\nu}^\pm |0\rangle \rangle = 3^\nu \delta_{i,j} \delta_{\nu,\bar{\nu}}. \)

Proof. By Proposition 3.3 (ii) and Proposition 3.1, we obtain
\[
\langle \bar{t}^T \bar{a}_n, \bar{t}^T \bar{b} \rangle = \langle \bar{a}_n, (\bar{t}^T)^{-1} \bar{t}^T \bar{b} \rangle = \langle \bar{a}_n, \bar{b} \rangle,
\]
\[
\langle a_{\nu}^j \bar{a}_n, \bar{a}_{\nu}^j \bar{b} \rangle = \langle \bar{a}_n, a_{\nu}^j \bar{a}_{\nu}^j \bar{b} \rangle = j \langle \bar{a}_n, \bar{b} \rangle + \langle \bar{a}_n, \bar{a}_{\nu}^j a_{\nu}^j \bar{b} \rangle.
\]
Now the lemma follows from repeatedly applying these two formulas and
\[
\langle |0\rangle, |0\rangle \rangle = (\langle |C^\nu\rangle, [C^\nu] \rangle = \langle -t, -t \rangle = 1. \qed
\]

3.4. Relations with the Heisenberg operators on \( \bigoplus_n H_T^{2n}(S^{[n]}) \).

For \( n \geq 0 \), let \( H_{T,n} = H_T^{2n}(S^{[n]}) \). Define the infinite dimensional space
\[
H_T = \bigoplus_{n=0}^{+\infty} H_{T,n}. \tag{3.13}
\]

In [Vas] (see also [LQW2]), an irreducible representation of a Heisenberg algebra on the space \( H_T \) was constructed. The Heisenberg algebra is generated by the linear operators \( a_n^\pm := a_n([C^\nu]) \) in \( \text{End}(H_T) \) and the identity operator \( \text{Id}_{H_T} \).

The operators \( a_n([C^\nu]) \) were defined similarly as in \( \mathbb{C} \). Let \( Y = C^m \) or \( C^2 \). For \( m \geq 0 \) and \( n > 0 \), define the closed subset \( Q_Y^{m+n,m} \) of \( S^{[m+n]} \times S^{[m]} \):
\[
Q_Y^{m+n,m} = \{ (\xi, \eta) \mid \xi \supset \eta, \text{Supp}(I_{\eta}/I_{\xi}) = \{ s \} \subset Y \}.
\]

Define \( a_0([Y]) = 0 \in \text{End}(H_T) \), and define \( a_{-n}([Y]), a_n([Y]) \in \text{End}(H_T) \) by
\[
a_{-n}([Y]) (A) = D^{-1}(p_1')^t (\text{Id}_{S^{m+n}} \times \iota_m)^{-1} \left( p_2^* A \cap [Q_Y^{m+n,m}] \right),
\]
\[
a_n([Y]) (B) = (-1)^n D^{-1}(p_1')^t (\iota_{m+n} \times \text{Id}_{S^{m}})^{-1} \left( p_1^* B \cap [Q_Y^{m+n,m}] \right)
\]
for \( A \in H_{T,m} = H_T^{2m}(S^{[m]}) \) and \( B \in H_{T,m+n}, \) where \( p_1', p_2' \) are the projections:
\[
p_1' : S^{[m+n]} \times (S^{[m]})^T \rightarrow S^{[m+n]},
p_2' : (S^{[m+n]})^T \times S^{[m]} \rightarrow S^{[m]},
\]
p_1, p_2 are the two projections on \( S^{[m+n]} \times S^{[m]} \), and \( \iota_m \) is the inclusion map:
\[
\iota_m : (S^{[m]})^T \rightarrow S^{[m]}.
\]

It was proved in [Vas] (see [LQW2] for the correct sign) that
\[
[a_n^+, a_m^+] = n \delta_{n,-m} \text{Id}_{H_T}. \tag{3.14}
\]

Recall the morphism \( \rho_m : S^{[m,m+1]} \rightarrow S^{[m]} \) from (2.6). In addition, there are two natural morphisms from \( S^{[m,m+1]} \) to \( S^{[m]} \) and \( S^{[m+1]} \) respectively:
\[
S^{[m,m+1]} \xrightarrow{\beta_{m+1}} S^{[m+1]}
\]
\[
\downarrow f_m \quad S^{[m+1]}.
\]
These morphisms $\rho_m$, $f_m$ and $g_{m+1}$ are $T$-equivariant, and there are induced maps:

$$ t \cup f_m^* : \mathbb{H}_{T,m} \to \tilde{\mathbb{H}}_{T,m}, \quad (3.15) $$

$$ g_{m+1}^* : \mathbb{H}_{T,m+1} \to \tilde{\mathbb{H}}_{T,m}. \quad (3.16) $$

We remark that $H^2_T(S) = H^2_T(C^2) = \mathbb{C}$. Therefore, $t \cup f_m^*$ is essentially the only way to come up with a map $\mathbb{H}_{T,m} \to \tilde{\mathbb{H}}_{T,m}$ based on the pullback $f_m^*$.

**Proposition 3.7.** (i) For every $n \in \mathbb{Z}$, there is a commutative diagram:

$$
\begin{array}{ccc}
\mathbb{H}_{T,m} & \xrightarrow{a^*_n} & \mathbb{H}_{T,m+n} \\
\downarrow{t \cup f_m^*} & & \downarrow{t \cup f_{m+n}^*} \\
\tilde{\mathbb{H}}_{T,m} & \xrightarrow{\tilde{a}^*_n} & \tilde{\mathbb{H}}_{T,m+n}.
\end{array}
\quad (3.17)
$$

(ii) Let $n > 0$. Then we have a commutative diagram:

$$
\begin{array}{ccc}
\mathbb{H}_{T,m+1} & \xleftarrow{a^*_n} & \mathbb{H}_{T,m+n+1} \\
\downarrow{g_{m+1}^*} & & \downarrow{g_{m+n+1}^*} \\
\tilde{\mathbb{H}}_{T,m} & \xleftarrow{\tilde{a}^*_n} & \tilde{\mathbb{H}}_{T,m+n}.
\end{array}
\quad (3.18)
$$

(iii) Let $n > 0$ and $A \in \mathbb{H}_{T,m+1}$. Then, we have

$$
g_n^* a^*_n[A] = n \cdot (t^n)^{n-1}[0], \quad (3.19)
$$

$$
g_{m+n+1}^* a^*_m[A] = \tilde{a}^*_m (g_{m+1}^* A) - n \cdot (t^n)^{n-1} (t \cup f_{m+1}^* (A)). \quad (3.20)
$$

**Proof.** Note that $a^*_n = a_n([C^*])$ and $\tilde{a}^*_m = \tilde{a}_m([C^*])$. Regard the two operators $a^*_n$ and $\tilde{a}^*_m$ as operators on the vector spaces

$$
\mathbb{H}_T^* = \bigoplus_{m=0}^{+\infty} H^*_T(S^{[m]}), \quad \tilde{\mathbb{H}}_T^* = \bigoplus_{m=0}^{+\infty} H^*_T(S^{[m,m+1]}),
$$

respectively. Then similar arguments as in the proofs of the Lemma 3.4, Lemma 4.4 and Proposition 4.6 in [LQ] prove the results. One needs to note that $\tilde{a}^*_m$ is $\mathbb{C}[t]$-linear and the sign discrepancy in (3.20) comes from the fact $[C^*] = \chi = 0$. \(\square\)

### 3.5. Further properties of $t \cup f_n^*$ and $g_{n+1}^*$

We recall some results from [Vas] and [LQW2]. First. As in Corollary 2.5 (i),

$$ H^n_T(S^{[n]}) = t^n \cup H^2_T(S^{[n]}). $$

Also, $H^n_T(S^{[n]})$ is a free $\mathbb{C}[t]$-module. A ring product $*$ on $\mathbb{H}_{T,n}$ is defined by

$$ t^n \cup (A \ast B) = A \cup B. \quad (3.22) $$

For $\lambda \vdash n$, define the class $[\lambda] \in \mathbb{H}_{T,n} = H^2_T(S^{[n]})$ by

$$ t^n \cup [\lambda] = [\xi, \lambda] \quad (3.23) $$
(note that our $[\lambda]$ differs that in [Vas, LQW2] by a scalar). Then the classes $[\lambda]$, $\lambda \vdash n$
form a linear basis of $\mathbb{H}_{T,n}$. Let $\iota_\lambda : \xi_\lambda \hookrightarrow S^{[n]}$ be the inclusion map, and let
$$
\iota_n = \bigoplus_{\lambda \vdash n} \iota_\lambda : (S^{[n]})^T \rightarrow S^{[n]}.
$$
The inverse of the induced Gysin map $\iota_n^* : H^*_T((S^{[n]})^T)' \rightarrow H^*_T(S^{[n]})'$ is given by
$$
(\iota_n)^{-1}(A) = \left( \frac{(\iota_\lambda)^*(A)}{(-1)^n h(\lambda)^2 t^{2n}} \right)_{\lambda \vdash n}.
$$
(3.24)

Define a bilinear pairing on the localization $H^*_T(S^{[n]})'$ by:
$$
\langle -,- \rangle : H^*_T(S^{[n]})' \otimes_C H^*_T(S^{[n]})' \rightarrow H^*_T(S^{[n]})',
$$
$$
\langle A, B \rangle = (-1)^n \pi_n(\iota_n)^{-1}(A \cup B)
$$
(3.26)
where $\pi_n$ is the projection of the set $(S^{[n]})^T$ of $T$-fixed points to a point. This induces a $C$-valued bilinear pairing, again denoted by $\langle -,- \rangle$, on $\mathbb{H}_T$ with
$$
\langle [\lambda], [\bar{\lambda}] \rangle = \delta_{\lambda, \bar{\lambda}} h(\lambda)^2.
$$
(3.27)

Then the operator $a_\lambda^T$ with $n > 0$ is the adjoint of $a_{-\lambda}^T$, and
$$
\langle a_{-\lambda}^T|0\rangle, a_{\bar{\lambda}}^T|0\rangle \rangle = \delta_{\lambda, \bar{\lambda}}
$$
(3.28)
where for a partition $\lambda = (1^{m_1} 2^{m_2} \cdots)$, $a_{-\lambda}^T$ is defined by
$$
a_{-\lambda}^T = \prod_j (a_{-j}^T)^{m_j}.
$$
(3.29)

**Proposition 3.8.** With the product $\ast$ on $\mathbb{H}_{T,n}$ and the product $\check{\ast}$ on $\mathbb{H}_{T,n}$, the linear maps $t \cup f_n^*$ and $g_{n+1}^*$ are ring homomorphisms.

**Proof.** Let $A, B \in \mathbb{H}_{T,n}$. By (2.39) and (3.22), we have
$$
t^{n+1} \cup ((t \cup f_n^* A) \check{\ast} (t \cup f_n^* B)) = (t \cup f_n^* A) \cup (t \cup f_n^* B),
$$
t^{n} \cup (A \ast B) = A \cup B.
Since $f_n^* : H^*_T(S^{[n]}) \rightarrow H^*_T(S^{[m,n+1]})$ is a ring homomorphism with respect to the cup products and also a $C[t]$-module homomorphism, we obtain
$$
t^{n+1} \cup ((t \cup f_n^* A) \check{\ast} (t \cup f_n^* B)) = t^2 \cup f_n^* A \cup f_n^* B
$$
$$
= t^2 \cup f_n^* (A \cup B)
$$
$$
= t^2 \cup f_n^* (t^n \cup (A \ast B))
$$
$$
= t^{n+2} \cup f_n^* (A \ast B).
$$
So $(t \cup f_n^* A) \check{\ast} (t \cup f_n^* B) = t \cup f_n^* (A \ast B)$, and $t \cup f_n^*$ is a ring homomorphism.
By a similar argument, we see that $g_{n+1}^*$ is a ring homomorphism. \qed
Remark 3.9. (i) We can also show that the linear map $t \cup f_n^*$ preserves bilinear forms, and that $\langle g_{n+1}^* A, g_{n+1}^* B \rangle = (n + 1) \langle A, B \rangle$ for $A, B \in \mathbb{H}_{T,n+1}$. Moreover,

$$t \cup f_n^* \lambda = - \sum_{(\lambda, \mu) \text{ incidence}} \frac{h(\lambda)^2}{h(\lambda, \mu)} [\lambda, \mu],$$

$$g_n^* \mu = \sum_{(\lambda, \mu) \text{ incidence}} \frac{h(\mu)^2}{h(\lambda, \mu)} [\lambda, \mu].$$

(ii) It follows from (i) that the number $h(\lambda, \mu)$ satisfies some interesting identities:

$$\sum_{(\lambda, \mu) \text{ incidence}} \frac{h(\lambda)^2}{h(\lambda, \mu)} = \sum_{k=0}^{s(\lambda)} \prod_{0 \leq s(k)} \frac{h(\square_{k,i})}{1 + h(\square_{k,i})} = 1,$$

$$\sum_{(\lambda, \mu) \text{ incidence}} \frac{h(\mu)^2}{h(\lambda, \mu)} = |\mu|. $$

4. Transformations among various linear bases of $\widetilde{H}_T$

Recall that the vector space $\widetilde{H}_T$ has two linear bases:

$$\tilde{B}_1 = \{ [\lambda, \mu] \}_{(\lambda, \mu) \text{ incidence}}, \quad \tilde{B}_2 = \left\{ (i^T)^i a_{-\nu}^T | 0 \right\}_{i \geq 0, \nu}.$$

The ring structure of $\widetilde{H}_{T,n}$ is easily described in terms of the linear basis $\tilde{B}_1$ coming from the fixed points. However, the basis $\tilde{B}_1$ doesn’t exist in the ordinary cohomology of $S^{[n,n+1]}$, while the second basis $\tilde{B}_2$ survives. Since we are interested in the ring structure of the ordinary cohomology $H^*(S^{[n,n+1]})$, it is important to know the linear transformation between these two bases. In this section, we give an algorithm to express $(i^T)^i a_{-\nu}^T | 0$ as a linear combination of the elements in $\tilde{B}_1$. This allows us to describe (implicitly) the ring structure of $\widetilde{H}_{T,n}$ in terms of the linear basis $\tilde{B}_2$. The method is to introduce the subvarieties $\tilde{L}^{(\lambda,\mu)}(C^z)$ on $S^{[n,n+1]}$ and study the actions of the loop algebra and the Heisenberg algebra on them. As a consequence, the classes $[\tilde{L}^{(\lambda,\mu)}(C^z)]$ provide a link between the bases $\tilde{B}_1$ and $\tilde{B}_2$.

4.1. The subvariety $L^{(\lambda)}(C^z)$ on $S^n$.

Let $C = C^z$ be the $z$-axis of $S = \mathbb{C}^2$. For $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \vdash n$, let

$$S_{\lambda}^n C = \left\{ \sum_i \lambda_i s_i \in \text{Sym}^n(S) | s_i \in C \text{ and the } s_i \text{'s are distinct} \right\}.$$ 

Recall the Hilbert-Chow morphism $\pi_n : S^{[n]} \to \text{Sym}^n(S)$. Let

$$L^{(\lambda)} C = \text{Closure of } (\pi_n)^{-1}(S_{\lambda}^n C). \quad (4.1)$$

The subvariety $L^{(\lambda)} C$ was first introduced in [Gro], and was studied intensively in [Na2, Na3]. Note that $L^{(\lambda)} C$ is irreducible, of dimension $n$ and $\mathbb{T}$-invariant.
By the results in [Na2, Na3, Vas], $\xi_\lambda$ is a smooth point of $L^\lambda C$, $\xi_\lambda \in L^\lambda C$ if and only if $\lambda \leq \lambda$ where $\leq$ denotes the dominance partial ordering, and

$$a^T_{-m}[L^\lambda C] = \sum_\nu a_{\lambda,\nu}[L^\nu C],$$

(4.2)

where the summation is over partitions $\nu$ of $|\lambda| + m$, which are obtained as follows:

(i) add $m$ to a term in $\lambda$, say $\lambda_k$ (possibly 0), and then

(ii) arrange it in descending order.

The coefficient $a_{\lambda,\nu}$ is the number of $\ell$ with $\nu_\ell = \lambda_k + m$. Denote the above $\nu$ by:

$$\nu = \lambda(\lambda_k, m).$$

(4.3)

Then, formula (4.2) can be rewritten as

$$a^T_{-m}[L^\lambda C] = \sum_{\text{distinct } \lambda_k} a_{\lambda,\lambda(\lambda_k,m)}[L^{\lambda(\lambda_k,m)} C].$$

(4.4)

4.2. The subvariety $\tilde{L}^{\lambda,\mu}(C^z)$ on $S^{[n,n+1]}$.

Again, let $C = C^z$ be the $z$-axis of $S = \mathbb{C}^2$. Let $(\lambda, \mu)$ be an incidence pair:

$$\lambda = (\cdots (i - 1)^{m_{i-1}}i^m(i + 1)^{m_{i+1}}\cdots) \vdash n,$$

$$\mu = (\cdots (i - 1)^{m_{i-1}}i^{m_{i-1}}(i + 1)^{m_{i+1}+1}(i + 2)^{m_{i+2}}\cdots)$$

where the parts $(i - 1)^{m_{i-1}}, i^m$, and $(i + 1)^{m_{i+1}}$ do not appear when $i = 0$. We fix this $i$ throughout the subsection. Let $\tilde{L}^{\lambda,\mu} C$ be the closed subset of $S^{[n,n+1]}$ defined by

$$\tilde{L}^{\lambda,\mu} C = \{(\xi, \xi')|\xi \in L^\lambda C, \xi' \in L^\mu C, \xi \subset \xi'\}.$$  

(4.5)

Then, $\tilde{L}^{\lambda,\mu} C$ is irreducible, of dimension $(n + 1)$ and $T$-invariant.

**Lemma 4.1.** $\overline{\{\tilde{L}^{\lambda,\mu} C\}} = [\tilde{L}^{\lambda,\mu} C]$ where the partition $\nu$ is defined by

$$\nu = (\cdots (i - 1)^{m_{i-1}}i^{m_{i-1}}(i + 1)^{m_{i+1}}(i + 2)^{m_{i+2}+1}(i + 3)^{m_{i+3}}\cdots).$$

**Proof.** Let $\tilde{p}_1, \tilde{p}_2$ be the two projections of $S^{[n+1,n+2]} \times S^{[n,n+1]}$. Then,

$$\overline{\{\tilde{L}^{\lambda,\mu} C\}} = D^{-1}\tilde{p}_1* \left(\tilde{p}_2^*[\tilde{L}^{\lambda,\mu} C] \cap [\tilde{Q}_n]\right).$$

An element $((\xi', \xi''), (\xi, \xi'))$ in $\tilde{Q}_n \cap (\tilde{p}_2)^{-1}\tilde{L}^{\lambda,\mu} C$ is of the form

$$\xi = \xi_0 + \xi_s \in L^\lambda C, \quad \xi' = \xi_0 + \xi'_s \in L^\mu C, \quad \xi'' = \xi_0 + \xi''_s$$

where $\xi_s \subset \xi'_s \subset \xi''_s$, $\text{Supp}(\xi'') = \{s\} \subset C$, and $s \notin \text{Supp}(\xi_0) \subset C$. The image

$$p_1 \left(\tilde{Q}_n \cap (\tilde{p}_2)^{-1}\tilde{L}^{\lambda,\mu} C\right)$$

consists of elements of the form

$$\xi' = \xi_0 + \xi'_s, \quad \xi'' = \xi_0 + \xi''_s.$$ 

Let $\ell(\xi_s) = \ell$. Choose a local coordinate $(w_s, z)$ of $S$ near $s$ such that $C$ is given locally near $s$ by $w_s = 0$. Now there are two cases:

**Case 1:** the element $\xi''_s \in M_{\ell+2}(s)$ is not generic. Since $M_{\ell+1,\ell+2}(s)$ is irreducible and has dimension $(\ell + 1)$, the corresponding element $((\xi', \xi''), (\xi_0 + \xi'_s, \xi_0 + \xi''_s)$ forms a subset of dimension less than $(n + 2)$.
**Case 2:** the element $\xi'' \in M_{\ell+2}(s)$ is generic. Then $\xi''_s$ is of the form:

$$I_{\xi''_s} = (w_s^{\ell+2}, z + b_1 w_s + \ldots + b_{\ell+1} w_s^{\ell+1})$$

where $b_1, \ldots, b_{\ell+1} \in C$. The corresponding $\xi'_s$ and $\xi_s$ must be of the form

$$I_{\xi'_s} = (w_s^{\ell+1}, z + b_1 w_s + \ldots + b_{\ell} w_s^{\ell}),$$

$$I_{\xi_s} = (w_s^\ell, z + b_1 w_s + \ldots + b_{\ell-1} w_s^{\ell-1}).$$

By the definition of $L^{\mu}C$, we must have $\ell = i$.

It follows that $p_1 \left( \tilde{Q}_n \cap (\tilde{p}_2)^{-1} \tilde{L}^{\lambda,\mu}C \right)$ has exactly one irreducible component of the expected dimension $(n + 2)$, which is $\tilde{L}^{\mu,\nu}C$, and possibly other components with smaller dimension. Note that the intersection $\tilde{Q}_n \cap (\tilde{p}_2)^{-1} \tilde{L}^{\lambda,\mu}C$ is transversal along the element $((\xi', \xi''), (\xi, \xi'))$ if $\xi''_s$ is from Case 2 and $\xi_0$ is generic. Hence

$$\tilde{t}^T [\tilde{L}^{\lambda,\mu}C] = [\tilde{L}^{\mu,\nu}C].$$

**Lemma 4.2.** Let $m_0 = 2$. Let $\lambda(\lambda_k, m)$ and $a_{\lambda, \lambda(\lambda_k, m)}$ be from (4.4). Then,

$$\tilde{a}^T_{-m} [\tilde{L}^{\lambda,\mu}C] = \sum_{\text{distinct } \lambda_k \neq i} a_{\lambda, \lambda(\lambda_k, m)} [\tilde{L}^{\lambda(\lambda_k, m), \mu(\lambda_k, m)}C]$$

$$+ (1 - \delta_{m, -1, 0}) a_{\lambda, \lambda(\lambda, m)} [\tilde{L}^{\lambda(i, m), \mu(i, m)}C] + [\tilde{L}^{\lambda(i, m), \mu(i+1, m)}C].$$

**Proof.** Let $n = |\lambda|$. Recall that $\tilde{a}^T_{-m} = \tilde{a}_{-m}([C])$. By (3.2), we have

$$\tilde{a}^T_{-m}([C]) [\tilde{L}^{\lambda,\mu}C] = D^{-1} \tilde{p}_1 \left( \tilde{p}_2^T [\tilde{L}^{\lambda,\mu}C] \cap [\tilde{Q}_C^{n+m, m}] \right).$$

Let $\xi_{0,1} = \xi_{0,2} = \emptyset$. Then a generic element $(\xi, \xi') \in \tilde{L}^{\lambda,\mu}C$ is of the form:

$$\xi = \sum_{r \geq 0} \sum_{1 \leq j \leq m_r} \xi_{r,j},$$

$$\xi' = \xi - \xi_{i,m_i} + \xi_{i,m_i}$$

where $i \geq 0$, $\ell(\xi_{r,j}) = r$, $\text{Supp}(\xi_{r,j}) = \{s_{r,j}\} \subset C$ for $r \geq 1$, the points $s_{r,j}$ are distinct, $(\xi_{i,m_i}, \xi'_{i,m_i}) \in M_{i+1}(s_{i,m_i})$ when $i > 0$, and when $i = 0$, $\xi'_{i,m_i}$ is a point in $C$ different from the points $s_{r,j}$ with $r \geq 1$.

The effect of the action of $\tilde{a}^T_{-m}([C])$ on $(\xi, \xi') \in \tilde{L}^{\lambda,\mu}C$ has two types:

**Type 1:** the action results in generic elements $(\eta, \eta')$ of the form:

$$\eta = \xi - \xi_{r_0,j_0} + \eta_{r_0+m},$$

$$\eta' = \xi' - \xi_{r_0,j_0} + \eta_{r_0+m}$$

where $(r_0, j_0) \neq (i, m_i)$, $\ell(\eta_{r_0+m}) = r_0 + m$, $\text{Supp}(\eta_{r_0+m}) = \{s_{r_0+m}\} \subset C$, and

$$s_{r_0+m} \not\in \text{Supp}(\xi').$$

This type of action is similar to the action of $a^T_{-m}$ on $[L^{\lambda}C]$. It follows that $\tilde{a}^T_{-m} [\tilde{L}^{\lambda,\mu}C]$ contains $a_{\lambda, \lambda(\lambda_k, m)} [\tilde{L}^{\lambda(\lambda_k, m), \mu(\lambda_k, m)}C]$ when $\lambda_k \neq i$, and contains

$$(1 - \delta_{m, -1, 0}) a_{\lambda, \lambda(i, m)} [\tilde{L}^{\lambda(i, m), \mu(i, m)}C]$$
Type 2: the action results in generic elements \((\eta, \eta')\) of the form:

\[
\eta = \xi - \xi_{i,m} + \eta_{i+m}, \\
\eta' = \xi - \xi_{i,m} + \eta'_{i+m}
\]

where \(i \geq 0\), \((\eta_{i+m}, \eta'_{i+m}) \in M_{i+m, i+m+1}(s_{i+m})\) for some \(s_{i+m} \in C\) with

\[s_{i+m} \notin \text{Supp}(\xi)\].

It follows that \(\tilde{a}^T_{-m}[\tilde{L}^{\lambda, \mu}C]\) contains \(a[\tilde{L}^{\lambda(|i,m), \mu(|i+1,m)}C]\) for certain multiplicity \(a\). As at the end of the proof of Lemma 4.1, the intersection multiplicity \(a\) is 1. \(\Box\)

**Proposition 4.3.** Let \((\lambda, \mu)\) be an incidence pair. Then, there exists an algorithm to express \([\tilde{L}^{\lambda, \mu}C]\) as a linear combination of the elements in the linear basis \(\tilde{B}_2\).

**Proof.** Use induction on \(|\lambda|\). When \(|\lambda| = 0\), we have

\[
[\tilde{L}^{\lambda, \mu}C] = [C] = [C^2] = [0].
\] (4.6)

So the conclusion holds. Next, let \(n \geq 1\) and assume that the conclusion holds for \([\tilde{L}^{\lambda, \mu}C]\) whenever \(|\lambda| < n\). In the following, let \(|\lambda| = n\).

Use a second induction on \(\ell(\lambda)\). When \(\ell(\lambda) = 1\), \(\lambda = (n)\), and either \(\mu = (1, n)\) or \(\mu = (n + 1)\). When \(\mu = (n + 1)\), we see from Lemma 4.1 that

\[
[\tilde{L}^{\lambda, \mu}C] = [\tilde{L}^{(n), (n+1)}C] = ([t^T]^n[0]),
\]

and so the conclusion holds in this case. When \(\mu = (1, n)\), applying the operator \(\tilde{a}^T_{-n}\) to (4.6) and using Lemma 4.2, we conclude that

\[
[\tilde{L}^{(n), (n+1)}C] + [\tilde{L}^{(n), (1, n)}C] = \tilde{a}^T_{-n}[0].
\]

Hence, the conclusion holds for \([\tilde{L}^{(n), (1, n)}C]\) as well.

Let \(|\lambda| = n\) and \(\ell(\lambda) > 1\). In this case, we can choose a part \(m > 0\) of \(\lambda\) so that \(m\) is also a part of \(\mu\). Let \(\lambda^{[m]}\) and \(\mu^{[m]}\) be the partitions of \(n - m\) obtained from \(\lambda\) and \(\mu\) respectively by deleting a copy of part \(m\). Let

\[
\begin{align*}
\lambda &= (\cdots (i-1)^{m_{i-1}}i^{m_i}(i+1)^{m_{i+1}}\cdots) + n, \\
\mu &= (\cdots (i-1)^{m_{i-1}}i^{m_{i-1}}(i+1)^{m_{i+1}}(i+2)^{m_{i+2}}\cdots).
\end{align*}
\]

Apply the formula in Lemma 4.2 to \(\tilde{a}^T_{-m}[\tilde{L}^{\lambda^{[m]}, \mu^{[m]}}C]\), for which the conclusion holds by the induction hypothesis on the size of partitions since \(|\lambda^{[m]}| < |\lambda|\). There are three types of terms on the right hand side of the formula. One is \([\tilde{L}^{\lambda^{[m]}, \mu^{[m]}}C]\) with a non-zero coefficient. The second type is \([\tilde{L}^{\tilde{\lambda}, \tilde{\mu}}C]\) with \(|\tilde{\lambda}| = |\lambda| = n\) and \(\ell(\tilde{\lambda}) < \ell(\lambda)\), for which the conclusion holds by the induction hypothesis on the length of partitions. The third type is the term \([\tilde{L}^{\lambda^{[m]}(i,m), \mu^{[m]}(i+1,m)}C]\) = \((\tilde{t}^T)^m[\tilde{L}^{\lambda^{[m]}, \mu^{[m]}}C]\), for which the conclusion holds by induction on the size again since \(|\lambda^{[m]}| < |\lambda|\). It follows immediately that the conclusion holds for \([\tilde{L}^{\lambda, \mu}C]\). \(\Box\)
4.3. Transformations between the linear bases $\tilde{B}_1$ and $\tilde{B}_2$.

Note that the $T$-action on $S^{[n,n+1]}$ induces a cell decomposition of
\[
\tilde{L}^{n,n+1}C := \bigcup_{\lambda,\mu} \tilde{L}^{\lambda,\mu}C.
\]

Let $\xi, \xi' \in \tilde{L}^{\lambda,\mu}C$ be a generic point. Then we see that
\[
\lim_{a \to 0} a(\xi, \xi') = (\xi_\lambda, \xi_\mu) \quad (4.7)
\]
for $a \in \mathbb{T}$. Since $\dim \tilde{L}^{\lambda,\mu}C = \dim \tilde{L}^{n,n+1}C = n + 1$, we conclude that the cell $C^{\lambda,\mu}$ corresponding to the fixed point $(\xi_\lambda, \xi_\mu)$ is isomorphic to $\mathbb{C}^{n+1}$ and $\tilde{L}^{\lambda,\mu}C$ is the closure of $C^{\lambda,\mu}$. In particular, the fixed point $(\xi_\lambda, \xi_\mu)$ is a smooth point of $\tilde{L}^{\lambda,\mu}C$.

Recall the notation $k(\lambda, \mu)$ from Definition 2.7 (ii). Put
\[
h_+(\lambda, \mu) = h(\lambda) \cdot \prod_{k(\lambda, \mu) + 1 \leq i \leq s(\lambda)} \frac{1 + h(\square_{k(\lambda, \mu), i})}{h(\square_{k(\lambda, \mu), i})}. \quad (4.8)
\]

**Lemma 4.4.** Let $(\xi_\lambda, \xi_\mu) \in S^{[n,n+1]}$ be a $T$-fixed point. Then the $T$-equivariant Euler class of the tangent space of $\tilde{L}^{\lambda,\mu}C$ at $(\xi_\lambda, \xi_\mu)$ is equal to $h_+(\lambda, \mu)t^{n+1}$.

**Proof.** Since $\tilde{L}^{\lambda,\mu}C$ is the closure of the cell $C^{\lambda,\mu}$ corresponding to the fixed point $(\xi_\lambda, \xi_\mu)$, it suffices to compute the $T$-equivariant Euler class of the tangent space of $C^{\lambda,\mu}$ at $(\xi_\lambda, \xi_\mu)$. By (4.7), the tangent space of $C^{\lambda,\mu}$ at $(\xi_\lambda, \xi_\mu)$ is the positive part of the tangent space of $S^{[n,n+1]}$ at $(\xi_\lambda, \xi_\mu)$. The positive part of the tangent space of $S^{[n,n+1]}$ at $(\xi_\lambda, \xi_\mu)$ can be read from the detailed study of the equivariant Zariski tangent space in the Appendix. Hence we see that the $T$-equivariant Euler class of the tangent space of $C^{\lambda,\mu}$ (and hence of $\tilde{L}^{\lambda,\mu}C$) at $(\xi_\lambda, \xi_\mu)$ is $h_+(\lambda, \mu)t^{n+1}$. \(\square\)

Note that $(\xi_\lambda, \xi_\mu) \in \tilde{L}^{\lambda,\mu}C$ only if $\lambda \geq \bar{\lambda}$ and $\mu \geq \bar{\mu}$. When $\lambda \geq \bar{\lambda}$, $\mu \geq \bar{\mu}$ and $(\lambda, \mu) \neq (\bar{\lambda}, \bar{\mu})$, we define $(\lambda, \mu) > (\bar{\lambda}, \bar{\mu})$.

**Lemma 4.5.** Let $(\lambda, \mu)$ be an incidence pair. Then, we have
\[
[\tilde{L}^{\lambda,\mu}C] = h_+(\lambda, \mu)^{-1} [\lambda, \mu] + \sum_{(\lambda, \mu) > (\bar{\lambda}, \bar{\mu})} d_{(\lambda, \mu), (\bar{\lambda}, \bar{\mu})} [\bar{\lambda}, \bar{\mu}] \quad (4.9)
\]
for some constants $d_{(\lambda, \mu), (\bar{\lambda}, \bar{\mu})} \in \mathbb{Q}$. Moreover, there exists an algorithm to compute all the constants $d_{(\lambda, \mu), (\bar{\lambda}, \bar{\mu})} \in \mathbb{Q}$.

**Proof.** It is known from [Bri] that if $X, Y$ are $T$-equivariant equidimensional varieties such that $Y \subset X$ is closed and $X^T$ is finite, then
\[
[Y] = \sum_{y \in Y^T} c_y(Y)t^{-\dim Y} \cup [y] \in H^*_T(X')
\]
for some constants $c_y(Y) \in \mathbb{Q}$. Moreover, if $y \in Y^T$ is a smooth point of $Y$, then $c_y(Y) \neq 0$ and $c_y(Y)^{-1}t^{\dim Y}$ is the $T$-equivariant Euler class of the tangent space.
of $Y$ at $y$. Apply this formula to $X = S^{[|\lambda|,|\lambda|+1]}$ and $Y = L^{\lambda,\mu}C$. We see that (4.9) follows from Lemma 4.4 and the definition of the class $[\lambda, \mu]$.

By Proposition 4.3 and Lemma 3.6, there exists an algorithm to compute the pairings among the classes $[\tilde{L}^{\lambda,\mu}C]$. By (4.9), the classes $[\tilde{L}^{\lambda,\mu}C]$ are related to the classes $[\lambda, \mu]$ via an upper triangular matrix. The diagonal entries of the matrix are $h_+([\lambda, \mu])^{-1}$, which are nonzero, and the entries above the diagonal are the constants $d_{(\lambda, \mu), (\tilde{\lambda}, \tilde{\mu})}$. Since the pairings between the fixed point classes $[\lambda, \mu]$ are already computed in (2.43), this upper triangular matrix can be determined. Therefore, there exists an algorithm to compute the constants $d_{(\lambda, \mu), (\tilde{\lambda}, \tilde{\mu})}$.

Theorem 4.6. There exists an algorithm to express each element $(\tilde{t}_i^T)^i \tilde{a}_{-\nu}^T|0\rangle$ in the linear basis $\tilde{B}_2$ as a linear combination of the elements in the linear basis $\tilde{B}_1$.

Proof. By Lemma 4.5, there exists an algorithm to express each element $[\lambda, \mu]$ in $\tilde{B}_1$ as a linear combination of the elements in the third linear basis

$$\tilde{B}_3 = \left\{ [\tilde{L}^{\lambda,\mu}C] \right\}_{(\lambda, \mu) \text{ incidence}}$$

of $\tilde{H}$. Note that the transition matrix between these two linear bases can be arranged to be lower triangular. By Proposition 4.3, there exists an algorithm to express each element $[\lambda, \mu]$ in $\tilde{B}_1$ as a linear combination of the elements in $\tilde{B}_2$. Therefore, we conclude that there exists an algorithm to express each element $(\tilde{t}_i^T)^i \tilde{a}_{-\nu}^T|0\rangle$ in $\tilde{B}_2$ as a linear combination of the elements in $\tilde{B}_1$.

5. Applications

5.1. Application to the ring structure of $H^*(S^{[n,n+1]})$.

In §5.2 of [LQ], we showed that the infinite dimensional space

$$\tilde{H}_S = \bigoplus_{n=0}^{+\infty} H^*(S^{[n,n+1]})$$

is a representation of the Lie algebra $\mathbb{C}[u^{-1}] \otimes_\mathbb{C} \tilde{h}_S$ with a highest weight vector being the vacuum vector

$$|0\rangle = 1_S \in H^0(S^{[0,1]}) = H^0(S).$$

Here, $1_S \in H^0(S)$ is the fundamental cohomology class of $S = \mathbb{C}^2$, $u^{-1}$ acts via a translation operator $\tilde{t}$ similarly defined as in §3.2 and $\tilde{h}_S$ is the Heisenberg algebra generated by $\text{Id}_{\tilde{H}_S}$ and the Heisenberg operators $\tilde{a}_n$ with $n \in \mathbb{Z}$. When $n > 0$, the creation Heisenberg operator $\tilde{a}_{-n} = \tilde{a}_{-n}(1_S)$ is defined similarly as in §3.1. In particular, the elements in $\tilde{H}_S$ are of the form:

$$\tilde{t}^{i_1} \tilde{a}_{-n_1} \cdots \tilde{a}_{-n_k} |0\rangle$$

where $k \geq 0$, $i \geq 0$, $i_1, \ldots, i_k > 0$, and $n_1, \ldots, n_k > 0$. 

For a partition $\nu = (1^{n_1} 2^{n_2} \cdots)$, we introduce the notation:

$$\tilde{a}_{-\nu} = \prod_j \tilde{a}_{-j}^{m_j}. $$

Recall the ring isomorphism in Corollary 2.5 (ii) induced by the forgetful map

$$\Psi : H^*_T(S^{[n,n+1]}) \to H^*(S^{[n,n+1]})$$

Let $\nu$ be a partition with $i + |\nu| = n$. As in §4 of [LQW3], we have

$$\Psi \left( (-t)^{-t(\nu)-1} (t\bar{\nu})^\lambda C^{\nu} |0\rangle \right) = \bar{t}^i \tilde{a}_{-\nu} |0\rangle$$

noting $[C^2] = -t^{-1}[C^2]$ in $H^*_T(S^2) = H^*_T(S)$, $\tilde{a}_n^T = \tilde{a}_n([C^2])$, and the Heisenberg commutation relation (3.3). It follows that the cup products of the classes $\bar{t}^i \tilde{a}_{-\nu} |0\rangle$ can be reduced, by using (5.3) and Theorem 4.6, to computations in terms of the linear basis $\tilde{B}_i$ of the fixed points. The cup products of the elements in $\tilde{B}_i$ are already determined in §2.9. This gives the ring structure of $H^*(S^{[n,n+1]})$.

### 5.2. Application to the ring of symmetric functions.

Let $\Lambda$ be the space of symmetric functions in infinitely many variables (see p.19 of [Mac]). For a partition $\lambda$, let $p_\lambda, m_\lambda$ and $s_\lambda$ be the power-sum symmetric function, the monomial symmetric function and the Schur function associated to $\lambda$ respectively. Define a ring structure on $\Lambda$ by requiring $s_\lambda \cdot s_\mu = \delta_{\lambda,\mu} h(\lambda) s_\lambda$ for the Schur functions $s_\lambda$ and $s_\mu$. Note that $\Lambda$ already has a natural ring structure of the multiplication of functions. To avoid the confusion, we use $(\lambda, \cdot)$ to denote $\Lambda$ with the new ring structure, and always refer to this new ring structure when we mention the ring structure of $\Lambda \otimes \mathbb{C}$.

Let $C = C^2 \subset S = \mathbb{C}^2$. By the Proposition B of [Vas], there is a ring isomorphism

$$\Phi : H_T \to (\Lambda \otimes \mathbb{C}, \cdot)$$

which sends the classes $a_{\lambda} T |0\rangle$, $[\lambda^a C]$ and $(-1)^n / h(\lambda) \cdot [\lambda]$ to the symmetric functions $p_\lambda, m_\lambda$ and $s_\lambda$ respectively. Moreover, under this isomorphism, the operator $a_{\lambda} T_n$ with $n > 0$ on $H_T$ corresponds to multiplication by $p_{(n)}$ on $\Lambda \otimes \mathbb{C}$.

Extending the map $\Phi$, we define a linear isomorphism

$$\tilde{\Phi} : \tilde{H}_T \to \Lambda \otimes \mathbb{C} [v]$$

by sending $(\bar{t}^i)^\lambda \tilde{a}_{\lambda} T |0\rangle$ to $p_\lambda \otimes v^i$. Under this linear isomorphism, the operators $\bar{t}^i$ and $\tilde{a}_{\lambda} T_n$ with $n > 0$ on $\tilde{H}_T$ correspond to multiplications by $1 \otimes v$ and $p_{(n)} \otimes 1$ on $\Lambda \otimes \mathbb{C} [v]$ respectively. Moreover, the ring structure on $\tilde{H}_T$ induces a ring structure on $\Lambda \otimes \mathbb{C} [v]$ such that $\Lambda \otimes \mathbb{C} \subset \Lambda \otimes \mathbb{C} [v]$ is a subring of $\Lambda \otimes \mathbb{C} [v]$. Let

$$\nu : \Lambda \otimes \mathbb{C} \hookrightarrow \Lambda \otimes \mathbb{C} [v]$$

be the inclusion map. Next, recall from Proposition 3.8 that there exists a ring homomorphism $t \cup f_m^* : \tilde{H}_T \to \tilde{H}_T \otimes m$. It induces a ring homomorphism:

$$t \cup f^*_m : H_T \to \tilde{H}_T.$$
By Proposition 3.7 (i), we obtain a commutative diagram of ring homomorphisms:

\[
\begin{array}{ccl}
\mathbb{H}_T & \xrightarrow{\phi} & \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \\
\downarrow \iota \cup f^* & & \downarrow \iota \\
\widehat{\mathbb{H}}_T & \xrightarrow{\tilde{\phi}} & \Lambda \otimes_{\mathbb{Z}} \mathbb{C}[v],
\end{array}
\]

which respects the Heisenberg algebra actions on \( \mathbb{H}_T \) and \( \widehat{\mathbb{H}}_T \).

It is natural to ask what the induced ring structure on \( \Lambda \otimes_{\mathbb{Z}} \mathbb{C}[v] \) is. It should provide an interesting feature on \( \Lambda \otimes_{\mathbb{Z}} \mathbb{C}[v] \) in the realm of symmetric functions.

6. Appendix: the proof of Lemma 2.8

For simplicity, we denote \( \square_{k(\lambda,\mu),i} = \square_{k,i} \) and \( \square'_{k(\lambda,\mu),i} = \square'_i \) by \( \square_i \) and \( \square'_i \) respectively. Note from (2.22) and Definition 2.7 (ii) that (2.33) is the same as

\[
e_T = (-1)^{n+1} \cdot h(\lambda)^2 \cdot \prod_{0 \leq i \leq m, i \neq k} \frac{1 + h(\square_i)}{h(\square'_i)} \cdot t^{2(n+1)}.
\]

To prove (6.1), we follow the setup in [Ch1]. There are four separate cases.

Case 1a: \( q_k = 1 \) but \( p_k \neq 1 \). Then \( s = m, \alpha'_k \notin A, \) and \( \alpha'_i = \alpha_i \) if \( 0 \leq i \leq s \) and \( i \neq k \). For \( 0 \leq i \leq (k-2) \), we have \( p'_i = p_i \) and let \( \beta_i \in B \) be the \( p_i \)-th cell directly above \( \alpha_k \). Then, \( \beta_i \in P_{\alpha_i} \). For \( k \leq i \leq (s-1) \), we have \( q'_{i+1} = q_{i+1} \) and let \( \beta_{i+1} \in B \) be the \( g_{i+1} \)-th cell directly to the left of \( \alpha_k \). Then, \( \beta_{i+1} \in Q_{\alpha_{i+1}} \). By the formula (2.6.1) in [Ch1], \( \text{Hom}(I_{\xi_{\mu}}, R/I_{\xi_{\lambda}}) \) is equal to

\[
\text{im}(\psi) \bigoplus \left( \bigoplus_{i=0}^{k-2} \mathbb{C}\phi(f_{\alpha_i,\beta_i}) \right) \bigoplus \left( \bigoplus_{i=k}^{s-1} \mathbb{C}\phi(f_{\alpha_{i+1},\beta_{i+1}}) \right).
\]

Combining this with (2.29) and (2.31), we obtain an exact sequence

\[
0 \rightarrow \ker(\phi - \psi) \bigoplus_{i=0}^{k-2} \mathbb{C}f_{\alpha'_i,\alpha_k} \rightarrow \text{Hom}(I_{\xi_{\lambda}}, R/I_{\xi_{\lambda}}) \oplus \text{im}(\psi) \rightarrow \text{im}(\psi) \bigoplus \left( \bigoplus_{i=0}^{k-2} \mathbb{C}\phi(f_{\alpha_i,\beta_i}) \right) \bigoplus \left( \bigoplus_{i=k}^{s-1} \mathbb{C}\phi(f_{\alpha_{i+1},\beta_{i+1}}) \right) \rightarrow 0.
\]  

If \( 0 \leq i \leq (k-1) \), then the weight of \( f_{\alpha'_i,\alpha_k} \) is \(-1 - h(\square_i); \) the weight of \( f_{\alpha'_i,\alpha_k} \) is \( 1 \); if \( (k+1) \leq i \leq s \), then the weight of \( f_{\alpha'_i,\alpha_k} \) is \( 1 + h(\square_i) \). Note that the weight of \( \phi(f_{\alpha_i,\beta_i}) \) is the same as the weight of \( f_{\alpha_i,\beta_i} \). Hence if \( 0 \leq i \leq (k-2) \), then the weight of \( \phi(f_{\alpha_i,\beta_i}) \) is \((p_i - 1) - h(\square_i) = -h(\square'_i) \); if \( k \leq i \leq s - 1 \), then the weight
of $\phi(f_{a_{i+1},b_{i+1}})$ is $-(q_{i+1} - 1) + h(\square_{i+1}) = h(\square_{i+1}')$. By (6.3) and (2.26),
\[
\begin{align*}
    e_T &= (-1)^{n+1} \cdot h(\lambda)^2 \cdot \prod_{0 \leq i \leq m, i \neq k-1, k} \frac{1 + h(\square_i)}{h(\square_i')} \cdot [1 + h(\square_{i-1})] \cdot t^{2(n+1)} \\
    &= (-1)^{n+1} \cdot h(\lambda)^2 \cdot \prod_{0 \leq i \leq m, i \neq k} \frac{1 + h(\square_i)}{h(\square_i')} \cdot t^{2(n+1)} \quad (6.4)
\end{align*}
\]
where we have used the observation that $h(\square_{k-1}) = 1$.

**Case 1b:** $p_k = 1$ but $q_k \neq 1$. Then $s = m$, $\alpha'_k \notin A$, and $\alpha'_i = \alpha_i$ if $0 \leq i \leq s$ and $i \neq k$. For $0 \leq i \leq (k - 1)$, let $\beta_i \in B$ be the $p_i$-th cell directly above $\alpha_k$. Then, $\beta_i \in P_{a_i}$. For $(k + 1) \leq i \leq (s - 1)$, let $\beta_{i+1} \in B$ be the $q_{i+1}$-th cell directly to the left of $\alpha_k$. Then, $\beta_{i+1} \in Q_{a_{i+1}}$. As in Case 1a, there is an exact sequence
\[
0 \rightarrow \ker(\phi - \psi) \rightarrow \bigoplus_{i=0}^{k-1} \mathbb{C}f_{a_i',\alpha_k} \rightarrow \text{Hom}(I_{\xi}, R/I_{\xi}) \oplus \text{im}(\psi)
\]
\[
\rightarrow \text{im}(\psi) \bigoplus \left( \bigoplus_{i=0}^{k-1} \mathbb{C}f_{a_i',\alpha_i} \right) \bigoplus \left( \bigoplus_{i=k+1}^{s-1} \mathbb{C}f_{a_{i+1},\beta_{i+1}} \right) \rightarrow 0. \quad (6.5)
\]
If $0 \leq i \leq (k - 1)$, then the weight of $f_{a_{i}'',\alpha_k}$ is $-1 - h(\square_i)$; the weight of $f_{a_{i}',\alpha_k}$ is $-1$; if $(k + 1) \leq i \leq s$, then the weight of $f_{a_{i}',\alpha_k}$ is $1 + h(\square_i)$. If $0 \leq i \leq (k - 1)$, then the weight of $\phi(f_{a_{i}',\beta_i})$ is $-1 - h(\square_i)$; if $k + 1 \leq i \leq s - 1$, then the weight of $\phi(f_{a_{i+1},\beta_{i+1}})$ is $h(\square_i')$. Combining with $h(\square_{k+1}') = 1$, we obtain
\[
\begin{align*}
    e_T &= (-1)^{n+1} \cdot h(\lambda)^2 \cdot \prod_{0 \leq i \leq m, i \neq k, k+1} \frac{1 + h(\square_i)}{h(\square_i')} \cdot [1 + h(\square_{k+1})] \cdot t^{2(n+1)} \\
    &= (-1)^{n+1} \cdot h(\lambda)^2 \cdot \prod_{0 \leq i \leq m, i \neq k} \frac{1 + h(\square_i)}{h(\square_i')} \cdot t^{2(n+1)}. \quad (6.6)
\end{align*}
\]

**Case 2:** $k = 0$ and $p_0 > 1$, or $k = m$ and $q_m > 1$, or $0 < k < m$ and $p_k, q_k > 1$. Then $s = m + 1$, $\alpha'_k, \alpha'_{k+1} \notin A$, $\alpha'_i = \alpha_i$ if $0 \leq i \leq (k - 1)$, and $\alpha'_i = \alpha_{i-1}$ if $(k + 2) \leq i \leq s$. For $0 \leq i \leq (k - 1)$, we have $p_i' = p_i$ and let $\beta_i \in B$ be the $p_i$-th cell directly above $\alpha_k$. Then, $\beta_i \in P_{a_i}$. For $(k + 1) \leq i \leq (s - 1)$, we have $q_{i+1} = q_i$ and $\alpha'_{i+1} = \alpha_i$. Let $\beta_i \in B$ be the $q_i$-th cell directly to the left of $\alpha_k$. Then, $\beta_i \in Q_{a_{i}}$. There is an exact sequence
\[
0 \rightarrow \ker(\phi - \psi) \rightarrow \bigoplus_{i=0}^{k-1} \mathbb{C}f_{a_i',\alpha_k} \rightarrow \text{Hom}(I_{\xi}, R/I_{\xi}) \oplus \text{im}(\psi)
\]
\[
\rightarrow \text{im}(\psi) \bigoplus \left( \bigoplus_{i=0}^{k-1} \mathbb{C}f_{a_i',\alpha_i} \right) \bigoplus \left( \bigoplus_{i=k+1}^{s-1} \mathbb{C}f_{a_{i+1},\beta_{i+1}} \right) \rightarrow 0. \quad (6.7)
\]
If $0 \leq i \leq (k - 1)$, then the weight of $f_{a_{i}'',\alpha_k}$ is $-1 - h(\square_i)$; the weight of $f_{a_{i}',\alpha_k}$ is $-1$; the weight of $f_{a_{k+1}'',\alpha_k}$ is $1$; if $(k + 2) \leq i \leq s$, then the weight of $f_{a_{i}',\alpha_k}$
is $1 + h(□_{i-1})$. If $0 \leq i \leq (k - 1)$, then the weight of $\phi(f_{α, β})$ is $-h(□'_i)$; if $k + 1 \leq i \leq s - 1$, then the weight of $\phi(f_{α, β})$ is $h(□''_i)$. Hence (6.1) holds.

**Case 3:** $p_k = q_k = 1$. Then, $s = m - 1$, $α'_i = α_i$ if $0 \leq i \leq (k - 1)$, and $α'_i = α_{i+1}$ if $k \leq i \leq s$. For $0 \leq i \leq (k - 2)$, we have $p'_i = p_i$ and let $β_i \in B$ be the $p_i$-th cell directly above $α_k$. Then, $β_i \in P_{α_k}$. For $k \leq i \leq (s - 1)$, we have $q'_{i+1} = q_{i+2}$ and $α'_{i+1} = α_{i+2}$. Let $β_{i+2} \in B$ be the $q_{i+2}$-th cell directly to the left of $α_k$. Then, $β_{i+2} \in Q_{α_{i+2}}$. There is an exact sequence

$$0 \rightarrow \frac{\text{ker}(\phi - \psi)}{\bigoplus_{i=0}^s C f_{α'_i, α_k}} \rightarrow \text{Hom}(I_{ξ_λ}, R/I_{ξ_λ}) \oplus \text{im}(ψ) \rightarrow \text{im}(ψ) \bigoplus \left(\bigoplus_{i=0}^{k-2} C φ(f_{α_i, β_i})\right) \bigoplus \left(\bigoplus_{i=k}^{s-1} C φ(f_{α_{i+2}, β_{i+2}})\right) \rightarrow 0. \quad (6.8)$$

If $0 \leq i \leq (k - 1)$, then the weight of $f_{α'_i, α_k}$ is $-1 - h(□_i)$; if $k \leq i \leq s$, then the weight of $f_{α'_i, α_k}$ is $1 + h(□_{i+1})$. If $0 \leq i \leq (k - 2)$, then the weight of $φ(f_{α_i, β_i})$ is $-h(□'_i)$; if $k \leq i \leq s - 1$, then the weight of $φ(f_{α_{i+2}, β_{i+2}})$ is $h(□''_{i+1})$. Hence

$$(-1)^{n+1} \cdot h(λ)^2 \cdot \prod_{0 \leq i \leq m \atop i \neq k-1, k, k+1} \frac{1 + h(□_i)}{h(□'_i)} \cdot [1 + h(□_{k-1})] \cdot [1 + h(□_{k+1})] \cdot t^{2(n+1)}$$

noting that $h(□_{k-1}) = h(□'_{k+1}) = 1$. Therefore, (6.1) holds.

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