THE ASYMPTOTIC INFINITESIMAL DISTRIBUTION
OF A REAL WISHART RANDOM MATRIX

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ABSTRACT. Let $X_N$ be a $N \times N$ real Wishart random matrix with aspect ratio $M/N$. The limit eigenvalue distribution of $X_N$ is the Marchenko-Pastur law with parameter $c = \lim N M/N$. The limit moments $\{m_n\}$ are given by $m_n = \sum \pi c^{\#(\pi)}$ where the sum runs over $NC(n)$.

Let $m'_n$ be the limit of $N(E(tr(X_N^*)^n)) - m_n$. These are the asymptotic infinitesimal moments of a real Wishart matrix. We show that $m'_n$ can be written as a sum over planar diagrams with two terms, $\sum \pi c'(\#(\pi) - 1)c^{\#(\pi) - 1}$, and $\sum_{\pi \in S^{NC}_N(n, -n)} c^{\#(\pi)/2}$, where $S^{NC}_N(n, -n)$ is a set of non-crossing annular permutations satisfying a symmetry condition. Moreover we present a recursion formula for the second term which is related to one for higher order freeness.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In 1974 G. ’t Hooft [11] considered gauge theory with colour gauge group $U(N)$ and quarks having a colour index from 1 to $N$. He showed that in the limit $N \to \infty$ “only planar diagrams with quarks at the edges dominate”. In 1991 D. Voiculescu [27] showed that, also in the limit $N \to \infty$, free independence described the behaviour of independent and unitarily invariant matrix ensembles. In 1994 R. Speicher [21] tied these together when he showed that free independence could be described with planar diagrams.

We introduce a new class of planar diagrams that lie between the non-crossing partitions that connect moments to free cumulants and the annular diagrams used by Nica and Mingo [16] to describe the global fluctuations of random matrices. We show that these diagrams describe the infinitesimal laws for some orthogonally invariant ensembles.

Given a random matrix ensemble $\{X_N\}_{N=1}^\infty$ where $X_N$ is $N \times N$ self-adjoint random matrix, we say the ensemble has a limit distribution (or a limit eigenvalue distribution) if the random probability measure $\mu_N$ on $\mathbb{R}$ which has a mass of weight $1/N$ at each eigenvalue of $X_N$ converges, as $N \to \infty$, to a non-random probability measure, $\mu$. We will study the averaged moments $E(\int t^n \, d\mu_N(t)) = E(tr(X_N^n))$. The existence of a limit

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law comes to showing that for each $n \geq 1$ we have
\[
\lim_{N} E(\text{tr}(X_N^n)) = \int t^n \, d\mu(t).
\]
We let $m_n(N) = E(\text{tr}(X_N^n))$ and $m_n = \int t^n \, d\mu(t)$. If we have as above that $m_n = \lim_N m_n(N)$, then we can consider the difference quotient
\[
m'_n(N) = \frac{m_n(N) - m_n}{N-1 - 0}.
\]
If the limit $\lim_{N \to \infty} m'_n(N)$ exists for each $n$, we say that the ensemble $\{X_N\}_N$ has a limit infinitesimal distribution and denote the resulting moments by $m'_n = \lim_N m'_n(N)$. The existence of infinitesimal laws were demonstrated by Johansson [12] (in the Hermite case), Mingo and Nica [16] and Dumitriu and Edelman [6] (in the Laguerre case).

In [2], Belinschi Shlyakhtenko introduced infinitesimal freeness as stronger form of freeness for random variables. In [8] Février and Nica showed the connection of infinitesimal freeness to freeness of type B. In [25] Shlyakhtenko has shown how infinitesimal freeness can be used to analyze spike models in random matrix theory. This paper introduces a class of planar objects extending those in [15] that can be used to analyze infinitesimal freeness when the matrices only have orthogonal invariance.

For a real or complex Wishart matrix the limit distribution is the Marchenko-Pastur law. This is a probability measure $\mu_c$ on $[0, \infty)$, which depends on a shape parameter $c > 0$. The moments of $\mu_c$ are given by simple combinatorial formula
\[
m_n = \sum_{\pi \in NC(n)} c^{\#(\pi)}
\]
where $NC(n)$ is the set of non-crossing partitions of $[n] = \{1, 2, 3, \ldots, n\}$ and for a partition $\pi$, $\#(\pi)$ denotes the number of blocks of $\pi$. This is equivalent to showing that for each $n$, $\kappa_n = c$, where $\kappa_n$ is the $n^{th}$ free cumulant of $\mu_c$, see [20] Def. 12.12 & Ex. 22.18.

If we suppose that $X_N$ is a complex Wishart random matrix, $\lim_N M/N = c$, and in addition that $c' := \lim_N M - cN$ exists then it shown in [16] Cor. 9.4] that
\[
m'_n = \sum_{\pi \in NC(n)} c'^{\#(\pi)c^{\#(\pi)}-1}.
\]
This can also be expressed by saying that the infinitesimal free cumulant $\kappa'_n = c'$ for all $n$. In addition in [15] Thm. 3.1] a signed measure $\nu'_1$ depending on $c$ and $c'$, was given so that $m'_n = \int t^n \, d\nu'_1(t)$. In this paper we consider the case where $X_N$ is a real Wishart matrix. We find that when $\lim_N M/N = c$, and $c' := \lim_N M - cN$ exists there is also a limit infinitesimal distribution:
\[
m'_n = \sum_{\pi \in NC(n)} c'^{\#(\pi)c^{\#(\pi)}-1} + \sum_{\pi \in S^4_{NC(n,n-n)}} c^{\#(\pi)/2}
\]
Figure 1. The first 6 moments and infinitesimal moments of the Marchenko-Pastur law, assuming \( c’ = 0 \). The coefficient of \( c^k \) in \( m_n \) is \( \frac{1}{n} \binom{n}{k-1} \binom{n}{k} \). The coefficient of \( c^k \) in \( m'_n \) is \( \frac{1}{2} \left( \left( \binom{2n}{2k} - \binom{n}{k}^2 \right) \right) \).

where \( S_{NC}(n, -n) \) is a new combinatorial object whose study is one of the main objects of this paper. The first term, let’s call it the shape term, does not appear in the tridiagonal model of Dumitriu and Edelman [6]. The second term is the same as was found in the tridiagonal model of Dumitriu and Edelman.

If we look at the second term of equation (2) and let (3)

\[
    m'_n = \sum_{\pi \in S_{NC}(n, -n)} c^{\#(\pi)/2}
\]

we can then show that these moments \( \{m'_n\}_n \) satisfy a recursion relation

\[
    m'_n = (n-1)m_{n-1} + (1 + c)m'_{n-1} + 2 \sum_{k=2}^{n-2} m'_k m_{n-k-1}
\]

which is independent of \( c’ \) but depends on the moments \( \{m_n\}_n \). See Figure 1 above for values of \( m_n \) and \( m'_n \) for small values of \( n \). In the case \( c = 1 \), this recursion relation has already appeared in the work of Merlini, Sprugnoli, and Verri [14] where \( m'_{n+1} \) is the sum of the areas under Dyck paths of length \( 2n \). The same recursion appeared in Féray [7, Lemma 3.12], where \( m'_n \) appears in the \( 1/N \) expansion of the orthogonal Weingarten function as a coefficient of a subleading term. There is a simple relation connecting \( S_{NC}(n, -n) \) to the orthogonal Weingarten calculus, but we prefer to leave this for a later discussion. We obtain our recursion from a simple ‘pinching’ construction that is the same as that which gives the recursion for Catalan numbers; see Figure 7.

One should also compare the recursion above with the one above Theorem 5.24 in [5]

\[
    (-1)^m \mu(1_{m+n}, \gamma_{m,n}) = m \cdot \mu(1_{m+n}, \gamma_{m+n})
    + \sum_{1 \leq k \leq n-1} \left\{ \mu(1_{m+k}, \gamma_{m,k}) \mu(1_{n-k}, \gamma_{n-k}) + \mu(1_{m+n-k}, \gamma_{m,n-k}) \mu(1_k, \gamma_k) \right\},
\]

where \( \mu(1_n, \gamma_n) \) is the Möbius function for one cycle with \( n \) points and \( \mu(1_{m+n}, \gamma_{m+n}) \) is the Möbius function for 2 cycles, one with \( m \) points and
one with \( n \) points. Recently higher order generalizations have been found by Borot, Charbonnier, Garcia-Failde, Leid, and Shadrin in [4].

We show that the Cauchy transform of \( \{m'_n\} \) is

\[
g(z) = \sum_{n=0}^{\infty} \frac{m'_n}{z^{n+1}} = \frac{-c}{z\sqrt{P(z)}} \frac{(1-c)^2 - (1+c)z - (1-c)\sqrt{P(z)}}{\sqrt{P(z)} + z - 1 + c} + \frac{1}{2} \left\{ \frac{1}{z-a} + \frac{1}{z-b} \right\} - \frac{1}{\sqrt{(z-a)(z-b)}}.
\]

To complete this introduction let us recall the relation between the fluctuation moments of Wishart matrices and non-crossing annular permutations, as the sum in equation (3) is over a subset of the non-crossing annular permutations. Let us recall that \( S_{NC}(p,q) \) denotes the set of non-crossing permutations of a \((p,q)\)-annulus. These are permutations of \([p+q]\) such that the cycles can be drawn in an annulus, with \( p \) points on the outer circle and \( q \) points on the inner circle, in such a way that the cycles do not cross, see the left hand figure in Figure 2. See [18, §5.1] for a full definition and
examples. In [16] we showed that for a complex Wishart random matrix, $X_N$, where $c = \lim_N M/N$ we have

$$\lim_{M} \text{cov}(\text{Tr}(X_N^p), \text{Tr}(X_N^q)) = \sum_{\pi \in S_{NC}(p,q)} c^\#(\pi).$$

In [23, Remark 5.13] this was extended to the case of real Wishart matrices, provided that the right hand side of (4) is multiplied by 2. The factor of 2 is necessary because in the real case we need both possible orientations of the inside circle. Indeed, for the fluctuation moments one needs the orientation in Figure 2 where the two circles have opposite orientations, in addition, we need the orientation where the two circles have the same orientation. When working with the infinitesimal moments we shall only need the case where the two circles have the same orientation. We let $\pm n = \{\pm 1, \pm 2, \ldots, \pm n\}$, $S_{\pm n}$ be the permutations of $\pm n$ and $\delta \in S_{\pm n}$ be given by $\delta(k) = -k$. If $\sigma \in S_{\pm n}$ we say that $\sigma \delta$ is a pairing mean that all cycles of $\sigma \delta$ have 2 elements. This, for elements of $S_{NC}^\delta(n, -n)$, is equivalent to requiring that $\delta \sigma \delta^{-1} = \sigma^{-1}$ and that no cycle of $\sigma$ can contain both $k$ and $-k$ for any $k \in [n]$. It also means that for elements of $S_{NC}^\delta(n, -n)$ the cycles always occur in pairs: $c$ and $c'$, where $c'$ is obtained from $c$ by reversing the order and flipping the sign of each entry; see Remark 4.

The main combinatorial object of this paper will be

$$S_{NC}^\delta(n, -n) = \{\sigma \in S_{NC}(n, -n) \mid \sigma \delta \text{ is a pairing}\}.$$

Outline of the Paper. After this introduction the sections of this paper are as follows. In Section 2 we will set out the matrix model we will be using and establish the notation to be used in the rest of the paper. In Section 3 we will write the infinitesimal moments in terms of certain pairings. In Section 4 we will show how this sum can be written a sum over $S_{NC}^\delta(n, -n)$. In Section 5 we will examine the multi-matrix case and show how this leads to a new kind of independence as found in [15, Thm. 37]. We are grateful to Alexandru Nica for suggesting this case. In Section 6 we establish some preliminaries that will be necessary for Sections 7, 8, and 9. In Section 7 we divide $S_{NC}^\delta(n, -n)$ into the three parts, $S_I$, $S_{II}$, and $S_{III}$, which are necessary for the recursion in Sections 8 and 9. In two brief concluding
sections, §§10 and §11, we make some comments on connecting our results with those in the recent paper of Arizmendi, Garza-Vargas, and Perales [1].

2. NOTATION AND PRELIMINARIES

In this section we shall describe the matrix model we are using and set up our notation for calculations in the symmetric group.

**Definition 1.** $G = (g_{ij})_{ij}$ with $\{g_{ij}\}_{ij}$ independent identically distributed $\mathcal{N}(0,1)$ random variables, and $1 \leq i \leq N$, $1 \leq j \leq M$. Let $X_N = \frac{1}{N}GG^t$. This is what we mean by a real Wishart random matrix. Note that $G$ depends on $M$ and $N$ but we shall suppress this dependency.

When expanding powers of $X_N$ we shall get products of $G$ and $G^t$, the transpose of $G$. For convenience we shall adopt the following convention: $G^{(1)} = G$ and $G^{(-1)} = G^t$.

$S_{\pm n}$ is the symmetric group on $[\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. We embed $S_n$ into $S_{\pm n}$ by making $\pi \in S_n$ act trivially on $\{-1, -2, -3, \ldots, -n\}$. We shall denote by $\gamma$ the permutation in $S_n$ with the long cycle $(1, 2, \ldots, n)$. In this notation $\gamma$ depends on $n$; in case it isn’t clear which $n$ is meant we shall write $\gamma_n$.

$P(n)$ is the set of partitions of $[n]$; $P_2(2n)$ will denote the pairings of $[2n]$, i.e. a partition with all blocks of size 2. Every pairing will be thought of as a permutation where each block of size 2 becomes a transposition. Every permutation can be thought of as the partition whose blocks are the cycles of the permutation. $\#(\pi)$ is the number of cycles of the permutation $\pi$, for any $\pi$ and $\sigma$, $\#(\pi \sigma) = \#(\sigma \pi)$.

For permutations $\pi$ and $\sigma$ in $S_n$ such that the subgroup $\langle \pi, \sigma \rangle$ acts transitively on $[n]$ there is an integer $g > 0$ (the genus of a certain surface) such that

$$\#(\pi) + \#(\pi^{-1} \sigma) + \#(\sigma) = n + 2(1 - g).$$

When $g = 0$ we say that $\pi$ is non-crossing relative to $\sigma$. When $\sigma = \gamma$, we say that $\pi$ is non-crossing and the set of such permutations is denoted $NC(n)$. See [20, Lect. 9] for a full discussion. We shall also be interested in the case when $\sigma$ has two cycles. If $n = p+q$ and $\sigma = (1, 2, \ldots, p)(p+1, \ldots, p+q)$ and $\pi$ is non-crossing relative to $\sigma$ we say that $\pi$ is a non-crossing permutation of the $(p,q)$-annulus and denote by $S_{NC}(p,q)$ the set of such permutations. See [16] or [18, Ch. 5] for further discussion. An example when $p = 6$ and $q = 4$ is shown on the left hand side of Figure 2.

The pairing $\delta \in S_{\pm n}$ given by $\delta(k) = -k$ will be central to all of our constructions. When $\sigma = \gamma \delta \gamma^{-1} \delta$ and $\pi \in S_{\pm n}$ is non-crossing relative to $\sigma$ we say that $\pi$ is a non-crossing annular permutation with the reversed orientation and denote by $S_{NC}(n,-n)$ the set of such permutations. The $-n$ is to indicate that the orientation has been reversed on the inner circle. Both the sets $S_{NC}(p,q)$ and $S_{NC}(p,-q)$ figured in the work of Redelmeier on real second order freeness [23].
The main focus of this paper will be on the subset $S_{NC}^6(n, -n)$ of $S_{NC}(n, -n)$ consisting of those permutations $\pi$ that satisfy the symmetry condition that $\pi\delta$ is a pairing. Examples when $n = 6$ can be found on the right hand side of Figure 2, the left hand side of Figure 7, the top left of Figure 8, the NC crossing pairing and the set of such pairings is denoted $\{\pi\}$. It was shown that these describe the infinitesimal moments of the Gaussian orthogonal ensemble.

We will also need the pairing $\omega \in NC_2(2n)$ given by $\omega = (1, 2)(3, 4) \cdots (2n - 1, 2n)$. Again both $\delta$ and $\omega$ depend on $n$. One uses $\omega$ to describe the bijection: $NC_2(2n) \ni \pi \mapsto \omega\pi|_E \in NC(E) \cong NC(n)$ where $E = \{2, 4, 6, \ldots, 2n\}$. See Remark 8. The annular version of this is illustrated in Figure 5.

We set $Z_2 = \{-1, 1\}$, let $\epsilon \in Z_2^{2n}$ be given by $\epsilon = (1, -1, \ldots, 1, -1)$. We shall also regard $\epsilon$ as a permutation in $S_{\pm 2n}$ as follows.

(a) If $k \in [\pm 2n]$ we let $\epsilon(k) = \epsilon_{[k]}k$. As permutations $\epsilon$ and $\delta$ commute. For notational convenience we shall write:

(b) $\tilde{\omega} = \omega\epsilon\omega\delta = (1, 2)(-1, -2) \cdots (2n - 1, 2n)(-2n - 1, -2n) = \epsilon\omega\epsilon\omega\epsilon,$

(c) $\tilde{\gamma} = \gamma\delta\gamma^{-1}\delta$.

We are using the convention that a permutation $\pi \in S_n$ is considered a permutation in $S_{\pm n}$ where $\pi$ acts trivially on negative numbers. Note that

(d) $\epsilon\gamma\delta\gamma^{-1}\epsilon = \gamma\omega\gamma^{-1}\delta\omega\delta$.

Given an $n$-tuple $(j_1, j_2, \ldots, j_n)$ with $j_k \in [N]$ for some integer $N$ and $1 \leq k \leq n$, we consider $j$ to be a function from $[n]$ to $[N]$ and its kernel, ker($j$) to be the partition on $[n]$ such that $r \sim_{ker(j)} s$ if and only if $j_r = j_s$. We use the usual ordering on partitions, namely $\pi \leq \sigma$ means every block of $\pi$ is contained in some block of $\sigma$. For example

(e) ker$(j) \geq \gamma\delta\gamma^{-1}$ \iff $j_1 = j_2, j_3 = j_4, \ldots, j_{2n} = j_1$.

When $\pi$ and $\sigma$ are permutations and we write $\pi \vee \sigma$ we considering them to be partition and $\pi \vee \sigma$ is the smallest partition in $\mathcal{P}(n)$ larger than or equal to $\pi$ and $\sigma$.

3. Expansion of the Trace of a Power of $X_N$

In this section we shall write $E(\text{tr}(X_N^k))$ as a sum over pairings of $[2n]$. This is similar to the calculation done in [11] and [22], but in order to extract the $1/N$ term we repeat it here as we need to have it expressed in our notation.

Remark 2. Given a $4n$-tuple $j = (j_{\pm j}, \ldots, j_{\pm 2n})$ where $1 \leq j_1, \ldots, j_{2n} \leq N$ and $1 \leq j_1, \ldots, j_{2n} \leq M$, let $t = j \circ \epsilon$ where $\epsilon$ is as in (a) above. Then $(G^{(\epsilon)})_{j_{\pm j}, \pm k} = g_{j, k, \pm k}$. Also if ker$(j) \geq \gamma\delta\gamma^{-1}$, then ker$(t) \geq \epsilon\gamma\delta\gamma^{-1}\epsilon = \gamma\omega\gamma^{-1}\delta\omega\delta$. In the equations below we use the convention that
Lemma 3.

Remark 4. We have to decide for a pairing $\pi$ what the maximum value of $\#(\gamma \omega \gamma^{-1} \vee \pi) + \#(\omega \vee \pi) - (n + 1)$ is. Recall that if $p$ and $q$ are pairings and $p \vee q$ denotes the join as partitions then $2(p \vee q) = \#(pq)$. Moreover we can write the cycle decomposition of $pq$ as $pq = c_1c'_1 \cdots c_kc'_k$ where $c'_i = qc_i^{-1}q$ and the blocks of $p \vee q$ are $\{c_1 \cup c'_1, \ldots, c_k \cup c'_k\}$; see [17, Lemma 2].

Remark 5. In the proof of Lemma 3 we have established the following expansion, writing $\overline{\pi} = \gamma^{-1}\pi\gamma$,

$$E(\text{Tr}(X^n_N)) = N^{-n} \sum_{\pi \in P_2(2n)} M \#(\omega \vee \pi) N \#(\gamma \omega \gamma^{-1} \vee \pi) + \#(\omega \vee \pi).$$
Figure 3. We let $I_1 = \{2, 3, 4, 5\}$, $I_2 = \{9, 10, 11, 12\}$, $I_3 = \{17, 18, 19, 20, 21, 22\}$ and then $J = \{1, 6, 7, 8, 13, 14, 15, 16\}$ is the complement of $I_1 \cup I_2 \cup I_3$. To make a pairing $\pi$ such that $\#(\omega \lor \pi) + \#(\omega \land \pi) = 22/2$ we take the pairing of $J$: $(1, 13)$, $(6, 14)$, $(7, 15)$, and $(8, 16)$ of $J$, shown by ht 4 thick lines above. Then we choose non-crossing pairings of $I_1$, $I_2$, and $I_3$, shown by the thin lines. This produces $\pi = \{(1, 13), (2, 3), (4, 5), (6, 14), (7, 15), (8, 16), (9, 12), (10, 11), (17, 20), (18, 19), (21, 22))\}$. On the top row we have $\omega = \{(1, 2), (3, 4), \ldots, (21, 22)\}$. We have $\#(\omega \lor \pi) = 5$.

Figure 4. On the top row we have $\omega = \gamma^{-1}\omega\gamma$. We have $\#(\omega \lor \pi) = 6$. Thus $\#(\omega \lor \pi) + \#(\omega \land \pi) = 11$. So this $\pi$ contributes to the $N^{-1}$ term, i.e the infinitesimal term. See Remark 6.

For a complex Wishart matrix, $Y_N$, we have

$$E(\text{Tr}(Y_N^N)) = N^{-n} \sum_{\sigma \in S_n} M^{\#(\sigma)} N^{\#(\sigma^{-1}\gamma)}.$$  

This suggests that $\pi$ plays the role of the Kreweras complement of $\pi$. Indeed, the leading term of equation (6) is given by those $\pi$’s for which $\#(\omega \lor \pi) + \#(\omega \land \pi) = n+1$; these are exactly the non-crossing pairings. The subleading term of equation (6) must be exactly those $\pi$’s for which $\#(\omega \lor \pi) + \#(\omega \land \pi) = n$. These $\pi$’s are not planar on the circle, but will be planar when drawn in an annulus. See Figure 5 for an example, and Notation 23 for additional comments on the Kreweras complement.

**Remark 6.** There is a simple way to generate all possible examples of $\pi \in P_2(2n)$ such that $\#(\omega \lor \pi) + \#(\omega \land \pi) = n$. Given $n$, choose disjoint intervals, each of even length, $I_1, \ldots, I_k \subseteq [2n]$ such that the cardinality of $J$, the complement of $\cup_{i=1}^k I_k$, is divisible by 4. Let $J = \{j_1, \ldots, j_m\}$ with $m = 2p$, and consider the pairs $(j_1, j_{p+1}), \ldots, (j_t, j_{p+t}), \ldots, (j_p, j_{2p})$. Choose
a non-crossing pairing for each interval $I_1, \ldots, I_k$. Together these will give a $\pi$ satisfying $\#(\omega \vee \pi) + \#(\omega \vee \pi) = n$. This is illustrated in Figures 3 and 4.

**Lemma 7.** (a) For $\pi \in \mathcal{P}_2(2n)$ we have $\#(\gamma \omega \gamma^{-1} \vee \pi) + \#(\omega \vee \pi) - (n + 1) \leq 0$, with equality only if for all $(r, s) \in \pi$ we have $\epsilon_r = -\epsilon_s$ and $\pi$ is a non-crossing pairing of $[2n]$.

(b) If there is $(r, s) \in \pi$ with $\epsilon_r = \epsilon_s$ then $\#(\gamma \omega \gamma^{-1} \vee \pi) + \#(\omega \vee \pi) - (n + 1) \leq -1$, with equality only if $\epsilon \pi \delta \pi \epsilon$ is a non-crossing pairing of a $(2n, -2n)$-annulus.

**Proof.** In the equations below we use ‘·’ to separate two expressions being multiplied, just to make the reading easier.

\[
2(\#(\gamma \omega \gamma^{-1} \vee \pi) + \#(\omega \vee \pi)) = \#(\delta \gamma \omega \gamma^{-1} \delta \cdot \delta \pi \delta) + \#(\omega \pi \delta \gamma \omega \gamma^{-1} \delta \cdot \delta \pi \delta) = \#(\epsilon \gamma \delta \gamma^{-1} \epsilon \cdot \pi \delta \pi \delta) = \#(\gamma \delta \gamma^{-1} \delta \cdot \epsilon \pi \delta \pi \epsilon).
\]

In particular $\#(\gamma \omega \gamma^{-1} \vee \pi) + \#(\omega \vee \pi) = \#(\epsilon \gamma \delta \gamma^{-1} \epsilon \cdot \pi \delta \pi \delta)$.

Now $\gamma \delta \gamma^{-1} \delta$ has 2 cycles and $\epsilon \pi \delta \pi \epsilon$ is a pairing. Thus $\#(\gamma \delta \gamma^{-1} \delta) = 2$ and $\#(\epsilon \pi \delta \pi \epsilon) = 2n$.\]

Next we consider two cases. In the first case for all $(r, s) \in \pi$ we have $\epsilon_r = -\epsilon_s$. Then $\epsilon \pi \delta \pi \epsilon = \pi \delta \pi \delta$. In this case
\[
\#(\gamma \delta \gamma^{-1} \delta \epsilon \pi \delta \pi \epsilon) = \#(\gamma \delta \gamma^{-1} \delta \pi \delta \pi \delta) = 2 \#(\gamma \pi) = 2 \#(\gamma \pi).
\]

So by Equation 5 we have for some integer $g \geq 0$
\[
\#(\pi) + \#(\gamma \pi) + \#(\gamma) = 2n + 2(1 - g).
\]

So
\[
\#(\gamma \pi) = n + 1 - 2g.
\]

Thus
\[
\#(\epsilon \gamma \delta \gamma^{-1} \epsilon \cdot \pi \delta \pi \delta) - (n + 1) = \#(\gamma \pi) - (n + 1) = -2g \leq 0
\]

So in the first case we get $\#(\gamma \omega \gamma^{-1} \vee \pi) + \#(\omega \vee \pi) - (n + 1) = 0$ only when $\pi$ is non-crossing and $-2$ or less when $\pi$ has a crossing. This proves (a).

In the second case there is some $(r, s) \in \pi$ such that $\epsilon_r = \epsilon_s$. In this case $\langle \gamma \delta \gamma^{-1} \delta, \epsilon \pi \delta \pi \epsilon \rangle$ acts transitively on $[\pm 2n]$. So again by Equation 5 we have for some integer $g' \geq 0$
\[
\#(\gamma \delta \gamma^{-1} \delta \epsilon \pi \delta \pi \epsilon) + \#(\epsilon \pi \delta \pi \epsilon) + \#(\gamma \delta \gamma^{-1} \delta) = 4n + 2(1 - g')
\]

Thus
\[
\#(\epsilon \gamma \delta \gamma^{-1} \epsilon \cdot \pi \delta \pi \delta) - (n + 1) = -1 - 2g' \leq -1.
\]

Moreover $g' = 0$ only if $\epsilon \pi \delta \pi \epsilon$ is a non-crossing pairing of a $(2n, -2n)$-annulus. This proves (b).
Remark 8. We have identified the leading term as all the non-crossing pairings \( \pi \) where \( \epsilon_r = -\epsilon_s \) for all \((r, s) \in \pi \). But the second condition is automatic for a non-crossing pairing. Thus the \( N^0 \) term is exactly the part of the sum where \( \pi \in NC_2(2n) \). Recall that there is a bijection from \( NC_2(2n) \ni \pi \to \sigma_\pi \in NC(n) \) and under this bijection \( \#(\omega \lor \pi) = \#(\sigma_\pi) \).

The map is given explicitly as follows.

For the purposes of this remark let \( E = \{2, 4, 6, \ldots, 2n\} \) and \( O = \{1, 3, 5, \ldots, 2n - 1\} \); this use is at variance with what we use in Notation 13 but the discussion will be illustrative. Note that \( \gamma^2 \) leaves \( O \) and \( E \) invariant, we denote by \( \gamma_E \) the restriction of \( \gamma^2 \) to \( E \).

Both \( \pi \) and \( \omega \) map \( E \) onto \( O \) and vice versa. Thus \( \omega \pi \) leaves both \( E \) and \( O \) invariant. Let \( \sigma_\pi \) be the restriction of \( \omega \pi \) to \( E \). Since both \( \pi \) and \( \omega \) are pairings we have \( \#(\omega \lor \pi) = \#(\sigma_\pi) \). Moreover we have the following.

\[
\omega \gamma|_E = \gamma_0|_E, \quad \pi \omega \gamma^2|_E = \pi \omega \gamma|_E = \pi \gamma|_E. 
\]

Also, as maps from \( O \) to \( E \),

\[
\omega \gamma|_O = \gamma|_O, \quad \pi \omega \gamma^2|_O = \pi \omega \gamma|_O = \pi \gamma|_O. 
\]

Hence \( \#(\pi \sigma_1 \gamma_E) = \#(\pi \omega \gamma^2|_E) = \#(\pi \gamma|_E) = \#(\gamma \omega \gamma^{-1}|_E) = \#(\gamma \omega \gamma^{-1} \lor \pi) \). Since \( \#(\pi \lor \omega) + \#(\pi \lor \gamma \omega^{-1}) = n + 1 \) we have that \( \#(\sigma_\pi) + \#(\sigma_\pi^{-1} \gamma_E) = n + 1 \) and thus, \( \sigma_\pi \) is non-crossing, and \( \#(\omega \lor \pi) = \#(\sigma_\pi) \).

Conversely given \( \sigma \in NC(E) \), let \( \pi_\sigma = \sigma^{-1} \omega \sigma \) be a pairing of \([2n]\). Here we use the convention that \( \sigma \) acts trivially on \( O \). Then \( \omega \pi_\sigma|_E = \omega \sigma^{-1} \omega \sigma|_E = \sigma \). Also we have \( \#(\omega \lor \pi_\sigma) = \#(\sigma) \) and \( \#(\gamma \omega \gamma^{-1} \lor \pi_\sigma) = \#(\sigma^{-1} \gamma_E) \). Thus \( \#(\omega \lor \pi_\sigma) + \#(\gamma \omega \gamma^{-1} \lor \pi_\sigma) = n + 1 \). So \( \pi_\sigma \) is non-crossing. This gives the bijection between the non-crossing pairings of \([2n]\) and the non-crossing partitions of \( E \). In Proposition 16 we shall extend this construction to the annular case.

Corollary 9. The only pairings \( \pi \) that can contribute to the coefficient of \( N^{-1} \) in \([6]\) are those for which there is at least one \((r, s) \in \pi \) such that \( \epsilon_r = \epsilon_s \).

We have already seen that for such a pair we have the subgroup \( \langle \gamma \delta \gamma^{-1} \delta, \epsilon \pi \delta \pi \epsilon \rangle \) acts transitively on \([\pm 2n]\) and so we may apply Equation 5 to conclude

\[
\#(\gamma \delta \gamma^{-1} \delta \epsilon \pi \delta \pi \epsilon) + \#(\epsilon \pi \delta \pi \epsilon) + \#(\gamma \delta \gamma^{-1} \delta) = 4n + 2(1 - g')
\]

for some \( g' \geq 0 \). Hence

\[
\#(\epsilon \gamma \delta \gamma^{-1} \epsilon \lor \pi \delta \pi \delta) - (n + 1) = -1 - g'
\]

So we seek all those pairings \( \pi \) such that \( g' = 0 \). This means that \( \epsilon \pi \delta \pi \epsilon \) is non-crossing with respect to \( \gamma \delta \gamma^{-1} \delta = (1, 2, \ldots, 2n)(-2n, -(2n - 1), \ldots, -2, -1) \). Thus \( \epsilon \pi \delta \pi \epsilon \) must be a non-crossing annular pairing (see [16] Thm. 6.1 and [15] §4) such that

(i) \( \epsilon \pi \delta \pi \epsilon \) connects the two circles

(ii) \( \epsilon \pi \delta \pi \epsilon \) commutes with \( \delta \)
... through strings of $\rho$ Notation 10. Let $r = 2(\cdot(\cdot(k \pmod{2})$ and $(\cdot(\cdot(a_i \pmod{2})$ with the indices interpreted modulo $k$. This produces a pair of strings that do not connect the two cycles: $\gamma$ and $\delta\gamma^{-1}\delta$.

(iv) For $(r, s) \in \pi$ with $\epsilon_r = \epsilon_s$, $(r, s), (-r, -s)$ are the corresponding pairs of $\epsilon \pi \delta \pi \epsilon$. This produces a pair of strings that do connect the two cycles: $\gamma$ and $\delta\gamma^{-1}\delta$, i.e. through strings.

Notation 10. We denote by $NC_2^B(2n, -2n)$ the set of non-crossing annular pairings that satisfy (i), (ii), (iii), and (iv) above.

Combining the previous results we have $\epsilon = (1 - 1, 1, -1, \ldots, 1, -1) \in Z_2^{2n}$ the following theorem.

Theorem 11. \[ m'_n = \lim_{N \to \infty} N(\text{tr}(X^n)) - \sum\limits_{\pi \in NC(n)} c^\#(\pi) \]
\[ = \sum\limits_{\pi \in NC(n)} c^\#(\pi) e^{\#(\pi)} + \sum\limits_{\pi \in P_2(2n)} c^\#(\pi). \]

4. FROM PAIRINGS TO PERMUTATIONS

In this section we shall focus on the second term in the right hand side of Theorem 11. We want to rewrite this as a sum over $S_{NC}^B(n, -n)$. This is done in Proposition 16. In Proposition 19 we write this term as a polynomial in $c$ and give an explicit formula for each coefficient.

Lemma 12. Let $\pi \in P_2(2n)$ and $\rho = \epsilon \pi \delta \pi \epsilon \in NC_2^B(2n, -2n)$. Suppose the through strings of $\rho$ are $(r_1, -s_1), (r_2, -s_2), \ldots, (r_k, -s_k)$, with $0 < r_1 < r_2 < \cdots < r_k$. Then

(a) $k$ is even; let $l = k/2$;
(b) for $1 \leq i \leq l$, $r_i = s_{i+1}$ and $s_i = t_{i+1}$;
(c) for $1 \leq i < k$, $r_i$ and $r_{i+1}$ have opposite parity;
(d) for all $i$, $r_i$ and $s_i$ have the same parity;
(e) $l$ is even.

Proof. Since $(r_i, -s_i) \in \rho$ we have $(r_i, s_i) \in \pi$, and in particular $r_i \neq s_i$. Also since $(r, -s) \in \rho \iff (s, -r) \in \rho$, we have that for each $i$ there is $j$ such that $r_i = s_j$. Let $1 \leq t \leq k$ be such that $s_1 = r_t$. The through strings of $\rho$ must form a spoke diagram; so $r_2 = r_{t+1}$, $s_3 = t_{t+2}$, \ldots, $s_i = t_{t+i-1}$ with the indices interpreted modulo $k$. Thus for all $i$, $(r_i, -r_{t+i-1}) \in \rho$ and $(r_{t+i-1}, -r_i) \in \rho$. Hence $r_i = s_{t+i-1} = r_{2(t+i-1)}$. Hence $2(t - 1) = 0 \pmod{k}$. We cannot have $t = 1$, otherwise $r_i = s_i$; so we must have $k = 2(t - 1)$ and thus $k = 2l$ with $l = t - 1$. Thus the through strings are $(r_1, -r_{l+1}), (r_2, -r_{l+2}), \ldots, (r_{2l}, -r_l)$. This proves (a) and (b).
Between $r_i$ and $r_{i+1}$ there are blocks of $\rho$ which do not cross any other pairs of $\rho$, hence there is an even number of points in the gap. Thus $r_i$ and $r_{i+1}$ have opposite parity. This proves (c).

If $(r,s) \in \pi$ then $(\epsilon(r), -\epsilon(s)) \in \rho$. If $(\epsilon(r), -\epsilon(s))$ is a through string and $r$ is odd then we need $\epsilon(s) = s$ so $s$ must be odd as well. Likewise if $r$ is even, $s$ must be even. This proves (d). Thus $r_1$ and $r_{l+1}$ will have the same parity so by (c), $l$ must be even. \qed

A converse to the previous lemma is the following construction of all the pairings $\pi \in \mathcal{P}_2(2n)$ that satisfy the condition $\epsilon \pi \delta \pi \epsilon \in NC_2^d(n, -n)$. First, we choose mutually disjoint intervals of even length $I_1, I_2, \ldots, I_m \subset [2n]$ with the constraint that 4 divides $|2n| \setminus \bigcup_{j=1}^m I_j$ and take $\{r_1 < r_2 < \cdots < r_{4l+1} = [2n] \setminus \bigcup_{j=1}^m I_j$. Note that the $r$'s alternate in parity since each $I_j$ has even length. Second, we choose non-crossing parings $\pi_j \in NC_2(I_j)$ for $j = 1, 2, \ldots, m$. Finally, taking

$$\pi = \pi_1 \pi_2 \cdots \pi_m(r_1, r_{2t+1})(r_2, r_{2t+2}) \cdots (r_{2t}, r_{4t}),$$

we obtain a pairing that satisfies $\epsilon \pi \delta \pi \epsilon \in NC_2^d(n, -n)$.

**Notation 13.** Let $E = \{2, 4, 6, \ldots, 2n\} \cup \{-1, -3, -5, \ldots, -(2n-1)\}$ and $O = \{1, 3, 5, \ldots, 2n-1\} \cup \{-2, -4, -6, \ldots, -2n\}$. Recall that $\tilde{\omega} = \omega \delta \omega \delta$ and $\tilde{\gamma} = \gamma \delta \gamma^{-1} \delta$. This extends the notation used in Remark [8]

**Remark 14.** $\tilde{\omega}(O) = E$, $\tilde{\omega}(E) = O$. If $\rho \in NC_2^d(2n, -2n)$ then $\rho(O) = E$ and $\rho(E) = O$. Thus $\tilde{\omega} \rho(E) = E$. We also have that $\tilde{\gamma}(O) = E$ and $\tilde{\gamma}(E) = O$. Moreover $\tilde{\omega}|_O = \tilde{\omega}|_O$ and $\tilde{\gamma}^{-1}|_E = \tilde{\gamma}|_E$.

**Lemma 15.** If $\pi \in \mathcal{P}_2(2n)$ and $\rho = \epsilon \pi \delta \pi \epsilon \in NC_2^d(2n - 2n)$ then

(a) $\#(\omega \pi) = \#(\tilde{\omega} \rho|_E)$,
(b) $\#(\omega \vee \pi) = 1/2 \#(\tilde{\omega} \rho|_E)$,
(c) $\#(\tilde{\omega} \rho|_E) + \#(\rho \omega \cdot \tilde{\gamma}^2|_E) = 2n$.

**Proof.** When we decompose $\tilde{\omega} \rho = c_1 c_1' \cdots c_k c_k'$ we have, by Remark [14], each cycle is either in $E$ or in $O$. Moreover for each pair $\{c_i, c_i'\}$, if one is in $E$ then the other must be in $O$. Also

$$\#(\rho \tilde{\omega}) = \#(\epsilon \pi \delta \pi \epsilon \cdot \omega \delta \omega \delta) = \#(\epsilon \omega \delta \omega \pi \delta \pi \epsilon) = \#(\omega \delta \omega \pi \delta \pi \epsilon) = 2 \#(\omega \pi).$$

Putting these together we get $\#(\omega \pi) = 1/2 \#(\rho \tilde{\omega}) = \#(\rho \tilde{\omega}|_E)$. Finally we have $\#(\omega \vee \pi) = 1/2 \#(\omega \pi) = 1/2 \#(\rho \tilde{\omega}|_E)$. Thus $\#(\rho \tilde{\omega}|_E) = \#(\rho \tilde{\omega}|_O) = 1/2 \#(\rho \tilde{\omega})$. This proves (a) and (b).

Because we assumed that $\rho \in NC_2^d(2n, -2n)$ we have $2n = \#(\rho \tilde{\gamma}) = \#(\rho \tilde{\gamma}|_O) + \#(\rho \tilde{\gamma}|_E)$. Since $\tilde{\gamma}|_O = \tilde{\omega}|_O$, we then have $\#(\rho \tilde{\gamma}|_O) = \#(\rho \tilde{\omega}|_O) = \#(\rho \tilde{\omega}|_E)$. Again as $\tilde{\gamma}|_O = \tilde{\omega}|_O$, we have $\tilde{\gamma}^2|_E = \tilde{\omega} \tilde{\gamma}|_E$, thus $\rho \tilde{\omega} \cdot \tilde{\gamma}^2|_E = \rho \tilde{\omega} \cdot \tilde{\omega} \tilde{\gamma}|_E = \rho \tilde{\gamma}|_E$. Hence

$$2n = \#(\rho \tilde{\gamma}|_O) + \#(\rho \tilde{\gamma}|_E) = \#(\rho \tilde{\omega}|_E) + \#(\rho \tilde{\omega} \cdot \tilde{\gamma}^2|_E).$$
Figure 5. On the left we have $\rho \in NC_2^3(8, -8)$ and on the right we have $\sigma = \tilde{\omega}\rho|_E$. This gives the bijection demonstrated in the proof of Proposition 16. From a topological point of view we squeeze together the pairs of $\tilde{\omega} = (1, 2)(3, 4)(5, 6)(7, 8)$ $(-1, -2)(-3, -4)(-5, -6)(-7, -8)$ to produce the points of $E = \{2, 4, 6, 8, -1, -3, -5, -7\}$. This gives the embedding of $S_N^2(2n, -2n)$ into $NC_2^3(2n, -2n)$.

This proves (c).

Proposition 16.

$$
\sum_{\pi \in P_2(2n)} c^{#(\omega \lor \pi)} = \sum_{\sigma \in S_{NC}^b(n, -n)} c^{#(\sigma)/2}.
$$

Proof. Given $\pi \in P_2(2n)$ such that $\rho = \epsilon\pi\delta\epsilon \in NC_2^3(2n, -2n)$, let $\sigma = \tilde{\omega}\rho|_E$. We have shown, Lemma 15 (b), that $#(\omega \lor \pi) = \#(\sigma)/2$.

Recall that $\gamma^2(E) = E$, so let $\gamma_E = \tilde{\gamma}^2|_E$. By Lemma 15 (c) we have that $#(\sigma) + #(\sigma^{-1}\gamma_E) = 2n$. With our notation

$$
\gamma_E = (2, 4, 6, \ldots, 2n - 2, 2n)(-2n - 1), -(2n - 3), \ldots, -3, -1.
$$

Let $\delta_E$ be the pairing of $E$ given by $\delta_E(2k) = \tilde{\omega}\delta(2k) = -(2k - 1)$. For the purposes of this proof we shall use the following notation. Let $S_{NC}^b(E)$ be the set of permutations, $\sigma$, of $E$ such that $#(\sigma) + #(\sigma^{-1}\gamma_E) = 2n$, $\sigma\delta_E$ is a pairing, and $\sigma$ connects the two cycles of $\gamma_E$. We shall show that the map $\pi \mapsto \sigma$ is a bijection from $\{\pi \in P_2(2n) \mid \epsilon\pi\delta\epsilon \in NC_2^3(2n, -2n)\}$ to $S_{NC}^b(E)$.

We shall begin by showing that $\sigma$ connects the cycles of $\gamma_E$. Let $(r, -s)$ be a through cycle of $\rho$. Suppose first that $r$ is even; then by Lemma 12 (c), $s$ is also even and so $-(s - 1) \in E$. Moreover $\sigma(r) = \tilde{\omega}(-s) = -(s - 1)$. 
Thus in this case $\sigma$ connects the two cycles of $\gamma_E$. Next, let us assume that $r$ is odd. Then again by Lemma 12(c), we have $r + 1, -s \in E$. Moreover $\sigma(-s) = \tilde{\omega}(r) = r + 1$. Thus again, $\sigma$ connects the two cycles of $\gamma_E$. We also have

- $\delta_E = e \tilde{\omega} e|_E$, and
- $\sigma = e \omega \pi \delta \omega \pi e|_E$.

As

$\tilde{\omega} \rho \cdot e \tilde{\omega} e = e \cdot \omega \pi \omega \cdot \delta \omega \pi \omega \delta \cdot e$

is a pairing, we have that $\sigma \delta_E$ is a pairing. By Lemma 15 we have $\#(\sigma) + \#(\sigma^{-1} \gamma_E) = 2n$, so $\sigma \in S^3_{NC}(E)$.

Conversely let $\sigma \in S^3_{NC}(E)$ be given and let $\rho = \sigma^{-1} \tilde{\omega} \sigma$. As we have conjugated the pairing $\tilde{\omega}$, $\rho$ is a pairing such that $\tilde{\omega} \rho|_E = \tilde{\omega} \sigma^{-1} \tilde{\omega} |_E = \sigma$. As $\sigma$ connects the cycles of $\gamma_E$, $\rho$ connects the cycles of $\gamma$. Next we shall show that $\rho \gamma|_E = \sigma^{-1} \gamma_E$ and $\rho \gamma|_O = \tilde{\omega} \sigma \tilde{\omega}$. This will show that $\#(\rho \gamma) = 2n$, so $\rho \in NC^3_2(2n, -2n)$. For $k > 0$ we have

$$\rho \gamma(2k) = \sigma^{-1} \tilde{\omega} \sigma(2k + 1) = \sigma^{-1} \tilde{\omega}(2k + 1) = \sigma^{-1}(2k + 1) = \sigma^{-1} \gamma_E(2k),$$

and

$$\rho \gamma(-2k - 1) = \sigma^{-1} \tilde{\omega} \sigma(-2k) = \sigma^{-1} \tilde{\omega}(-2k) = \sigma^{-1}(-(2k + 1)) = \sigma^{-1} \gamma_E(-(2k + 1)).$$

This shows that $\rho \gamma|_E = \sigma^{-1} \gamma_E$. Again for $k > 0$

$$\rho \gamma(2k - 1) = \sigma^{-1} \tilde{\omega} \sigma(2k) = \tilde{\omega} \sigma(2k) = \tilde{\omega} \sigma \tilde{\omega}(2k - 1),$$

and

$$\rho \gamma(-2k) = \sigma^{-1} \tilde{\omega} \sigma(-2k - 1) = \tilde{\omega} \sigma(-2k - 1) = \tilde{\omega} \sigma \tilde{\omega}(-2k).$$

Finally let us show that $\rho = e \pi \delta \pi e$ for some pairing $\pi$; or equivalently that for $k > 0$ we have $\epsilon \rho e(k) < 0$. For $k > 0$ we have

$$\epsilon \rho e(2k) = \epsilon \sigma^{-1} \tilde{\omega} \sigma(-2k) = \epsilon \sigma^{-1} \tilde{\omega}(-2k) = \epsilon \sigma^{-1}(-2k - 1) < 0$$

because if $\sigma^{-1}(-2k - 1) < 0$ it must be odd and if $\sigma^{-1}(-2k - 1) > 0$ it must be even. Also

$$\epsilon \rho e(2k - 1) = \epsilon \sigma^{-1} \tilde{\omega} \sigma(2k - 1) = \epsilon \sigma^{-1} \tilde{\omega}(2k - 1) = \epsilon \sigma^{-1}(2k) < 0$$

because if $\sigma^{-1}(2k) < 0$ it must be odd and if $\sigma^{-1}(2k) > 0$ it must be even. Hence $\rho = e \pi \delta \pi e$ where $\pi = \delta \epsilon \rho e|_{[2n]}$. This completes the proof of the claimed bijection.

If we let $\psi : E \to [\pm n]$ be the bijection $\psi(2k) = k$ and $\psi(-(2k - 1)) = -k$, for $k \in [n]$, we see that $\psi$ conjugates $\gamma_E$ to $\gamma' = (1, \ldots, n)(-n, \ldots, -1)$, and $\delta_E$ to $\delta|_{[\pm n]}$ and $\sigma$ to a permutation $\sigma' \in S_{\pm n}$ such that

- $\sigma' \delta$ is a pairing;
- $\#(\sigma') + \#(\sigma'^{-1} \gamma') = 2n$, and
- $\sigma'$ connects the two cycles of $\gamma'$.
Thus $\sigma' \in S_{NC}^\delta(n, -n)$. Hence
\[
\sum_{\pi \in \mathcal{P}_2(2n)} e^\#(\omega \triangleright \pi) = \sum_{\sigma \in S_{NC}^\delta(n, -n)} e^\#(\sigma)/2.
\]

Remark 17. The proof of Proposition 16 shows that $|S_{NC}^\delta(n, -n)| = |\{\rho \in NC^\delta_2(2n - 2n) | \epsilon \rho \epsilon = \pi \delta \pi \text{ for some pairing } \pi \in \mathcal{P}_2(2n)\}|$. We shall show, in Corollary 36, that $|S_{NC}^\delta(n, -n)| = 4^{n-1} - \frac{1}{2}\binom{2n}{n}$, whereas in [15, Lemma 23], we showed that $|NC^\delta_2(2n, -2n)| = \frac{1}{2}(4^n - \binom{2n}{n})$. Thus
\[
|NC^\delta_2(2n, -2n)| = 4^{n-1} + |S_{NC}^\delta(n, -n)|.
\]

The conclusions of Theorem 11 and Proposition 16 give us the following corollary.

Corollary 18.
\[
m'_n = \lim_{N \to \infty} N \left( \text{tr}(X^n) \right) - \sum_{\pi \in NC(n)} e^\#(\pi)
= \sum_{\pi \in NC(n)} e^\#(\pi) e^\#(\pi) - 1 + \sum_{\sigma \in S_{NC}^\delta(n, -n)} e^\#(\sigma)/2.
\]

Proposition 19.
\[
\sum_{\sigma \in S_{NC}^\delta(n, -n)} e^\#(\sigma)/2 = \sum_{l=1}^{[n/2]} \sum_{j=0}^{n-2l} \binom{n}{j} \binom{n}{j + 2l} e^{j+l} = \sum_{k=1}^{n-1} a_k e^k
\]

where $a_k = \sum_{i=1}^{\min\{k, n-k\}} \binom{n}{k-i} \binom{n}{k+i} = \frac{1}{2} \left( \binom{2n}{2k} - \binom{n}{k} \binom{n}{k} \right)$.

Proof. The condition $\delta \sigma \delta = \sigma^{-1}$ for any element $\sigma \in S_{NC}^\delta(n, -n)$ implies that the number of through cycles and the number of non-through cycles in $\sigma$ are both even numbers. Thus, since each $\sigma \in S_{NC}^\delta(n, -n)$ must have at least two through cycles, the sum $\sum_{\sigma \in S_{NC}^\delta(n, -n)} e^\#(\sigma)/2$ can be rewritten as
\[
\sum_{l=1}^{[n/2]} \sum_{j=0}^{n-2l} |S_{NC}^\delta(n, -n)_{j,l}| e^{j+l}
\]

where $S_{NC}^\delta(n, -n)_{j,l}$ denotes the set of all permutations from $S_{NC}^\delta(n, -n)$ with exactly $2j$ non-through cycles and $2l$ through cycles. The condition $\delta \sigma \delta = \sigma^{-1}$ also implies that each permutation $\sigma \in S_{NC}^\delta(n, -n)_{j,l}$ is completely and uniquely determined by the non-crossing circular half-permutation with $j$ closed blocks and $2l$ open blocks resulting from restricting $\sigma$ to the set $[n]$, see Figure 6. It then follows from [13, Theorem 25] that
Figure 6. On the left we have an element of $S_{NC}^{\delta}(6, -6)$. We convert it to a non-crossing circular half-permutation (in the language of [13 §6 Def. 2]), which is shown on the right. Then open blocks are those with a zigzag edge; the closed blocks have a smooth edge. We will always get an even number of open blocks. See [13 §6 Def. 2] for details. We shall use this in the proof of Proposition 19.

$|S_{NC}^{\delta}(n, -n)_{j,l}|$ is given by $\binom{n}{j} \binom{n}{j+2l}$, and hence, we obtain

$$\sum_{\sigma \in S_{NC}^{\delta}(n, -n)} c^{\#(\sigma)/2} = \sum_{l=1}^{[n/2]} \sum_{j=0}^{n-2l} \binom{n}{j} \binom{n}{j+2l} c^{j+l}.$$  

See [13 Figure 16] for an explicit example of how the counting works. Applying the change of variable $k = j + l$ in the sum in the right-hand side of the equality above, and regrouping its terms with respect to $c^k$, yields

$$\sum_{l=1}^{[n/2]} \sum_{j=0}^{n-2l} \binom{n}{j} \binom{n}{j+2l} c^{j+l} = \sum_{k=1}^{n-1} c^k \sum_{l=1}^{\min\{k, n-k\}} \binom{n}{k-l} \binom{n}{k+l}.$$  

Note that the sum $\sum_{l=1}^{\min\{k, n-k\}} \binom{n}{k-l} \binom{n}{k+l}$ is invariant under the transformation $k' = n - k$, so in evaluating the sum we may assume that $k \leq n - k$.

Then we have

$$\sum_{l=1}^{k} \binom{n}{k-l} \binom{n}{k+l} (k-l) \sum_{l=0}^{k-1} \binom{n}{l} \binom{n}{2k-l}$$

$$= \frac{1}{2} \left( \sum_{l=0}^{k-1} \binom{n}{l} \binom{n}{2k-l} + \sum_{l=k+1}^{2k} \binom{n}{2k-l} \binom{n}{l} \right)$$
= \frac{1}{2} \left( \sum_{l=0}^{2k} \binom{n}{l} \left( \binom{n}{2k-l} - \binom{n}{k} \right)^2 \right) = \frac{1}{2} \left( \binom{2n}{2k} - \binom{n}{k}^2 \right)

where the last equality is by Vandermonde convolution (see e.g. [10, Eq. (5.22)]).

5. The case of independent Wishart matrices

In this section we extend Corollary 18 to the case of a family of independent Wishart matrices. The conclusion is exactly what happens at the first order; the blocks of \(\pi\) in the first term, or \(\sigma\) in the second can only connect a matrix with itself, see [16, Cor. 9.4].

So to this end let \(X_{1,N}, \ldots, X_{s,N}\) be \(s\) independent Wishart matrices as in Definition 1. Let \(l_1, \ldots, l_n \in [s]\) and \(\ker(l) \in \mathcal{P}(n)\) be the kernel of \(l\). We know from [16, Cor. 9.4] that

\[
\lim_{N \to \infty} E(\text{tr}(X_{l_1,N} \cdots X_{l_n,N})) = \sum_{\pi \in \mathcal{NC}(n)} c^\#(\pi).
\]

Given any partition \(\tau\) of \([n]\) we get a partition \(\tilde{\tau}\) of \([\pm n]\) by setting, for \(r, s \in [\pm n]\), \(r \sim_{\tilde{\tau}} s\) if and only if \(|r| \sim_{\tau} |s|\). Here, \(|r|\) denotes the absolute value of \(r\).

We now turn to the multi-matrix version of Equation (6) of Lemma 3. Given our \(l_1, \ldots, l_n \in [s]\), let \(k_1, k_2, \ldots, k_{2n-1}, k_{2n} \in [s]\) be given by \(k_{2r-1} = k_{2r} = l_r\) for \(1 \leq r \leq n\). Then \(\ker(k) \in \mathcal{P}(2n)\). By repeating the proof of Lemma 3 we have

\[
(8) \quad E(\text{tr}(X_{l_1,N} \cdots X_{l_n,N})) = \sum_{\pi \in \mathcal{P}_2(2n)\atop \pi \subseteq \ker(k)} \left( \binom{M}{\pi} \right) \#(\omega \vee \pi N)^\#(\gamma \omega \gamma^{-1} \vee \pi) + \#(\omega \vee \pi) - (n+1).
\]

The following Lemma will sort out which \(\sigma\)'s can appear in the second term on the right hand side of Equation (7).

Lemma 20. Suppose \(\pi \in \mathcal{P}_2(2n)\) and \(\rho = \varepsilon \pi \delta \varepsilon \in \mathcal{NC}_2^\delta(2n, -2n)\). Suppose in addition that \(\sigma \in S^\delta_{\mathcal{NC}}(n, -n)\) is the permutation produced in the bijection of Proposition 16. Then

\[
\sigma \preceq \ker(l) \iff \pi \preceq \ker(k).
\]

Proof. Suppose first that \(\sigma \preceq \ker(l)\), and we will show that \(\pi \preceq \ker(k)\); i.e. if \((r, s) \in \pi\) then \(k_r = k_s\). We break this into two cases.

Case (a): \((r, -s)\) is a through string of \(\rho = \varepsilon \pi \delta \varepsilon\). Then by Lemma 12(d), \(r\) and \(s\) have the same parity. In the even case we have \(\sigma(r/2) = s/2\) and thus \(k_s = l_{s/2} = l_{\sigma(r/2)} = l_{r/2} = k_r\). In the odd case we have \(\sigma(-(s+1)/2) = (r+1)/2\). Hence \(k_r = k_{r+1} = l_{(r+1)/2} = l_{(s+1)/2} = k_{s+1} = k_s\). In either case \(k_r = k_s\).
Case (b): $(r, s)$ is a pair of $\rho = \epsilon \pi \delta \epsilon e$. Then by Lemma \[12\] $r$ and $s$ have opposite parities; suppose $r$ is even. Then $\sigma(r/2) = \sigma((s + 1)/2)$. Hence $k_r = l_{r/2} = l_{(s+1)/2} = k_{s+1} = k_s$.

In the opposite direction, suppose that $\pi \leq \ker(k)$, and we will show that $\sigma \leq \ker(l)$. Suppose that $r, s \in [n]$.

Case (c): $\sigma(r) = s$. Then $\rho(2r) = 2s - 1$; so $\pi(2r) = 2s - 1$. Hence $l_r = k_{2r} = k_{2s-1} = k_{2s} = l_s$.

Case (d): $\sigma(r) = -s$. Then $\rho(2r) = -2s$; so $\pi(2r) = 2s$. Hence $l_r = k_{2r} = k_{2s} = l_s$. With these two cases we conclude that $\sigma \leq \ker(l)$ as claimed.

**Theorem 21.** Suppose $X_{1,N}, \ldots, X_{s,N}$ are independent Wishart matrices with the same shape, i.e. all are obtained using the same $M$. Suppose $\lim_{N} (M - cN) = c'$. Then

$$
\lim_{N} N \left\{ \mathbb{E}(\text{tr}(X_{l_1,N} \cdots X_{l_n,N})) - \sum_{\pi \in \text{NC}(n)} c^\#(\pi) \right\}
$$

$$
= \sum_{\pi \in \text{NC}(n) \atop \pi \leq \ker(l)} c^\#(\pi)c^\#(\pi) - 1 + \sum_{\sigma \in \text{NC}(n,-n) \atop \sigma \leq \ker(l)} c^\#(\sigma)/2.
$$

**Proof.** From Equation (8) we only have to decide which $\pi$’s survive in the large $N$ limit. According to Theorem \[11\] there are two cases: the non-crossing $\pi$’s and those for which $\epsilon \pi \delta \pi e \in \text{NC}_2(2n, -2n)$. That the non-crossing ones produce the first term is proved in \[16\] Cor. 9.4. In the second case we use the bijection in Proposition \[16\] restricted to those $\pi$’s such that $\pi \leq \ker(k)$. The image of this subset is provided by Lemma \[20\].

This produces the second term on the right hand side of the statement.

**Remark 22.** We can lift the assumption that all the Wishart matrices $X_{1,N}, \ldots, X_{s,N}$ have the same shape parameters $c$ and $c'$. For each block of $\pi$ or $\sigma$ we simply use corresponding pair $(c, c')$ of shape parameters. This suggests that these matrices exhibit a new kind of independence, in fact the same as found in \[15\] Thm. 37. We will address this point in a subsequent paper.

6. PRELIMINARIES FOR THE RECURSION FORMULAS FOR $|S^6_{NC}(n, -n)|$

In this section we will present some preliminaries for the recursion formula for $|S^6_{NC}(n, -n)|$ given below. In Proposition \[26\] we give a criterion for dividing $S^6_{NC}(n, -n)$
The formula will be proved in Theorem 34 of Section 8. In the formula \( m_n = |NC(n)| \) and \( \overline{m}_n' = |S^\delta_{NC}(n, -n)| \):

\[
\overline{m}'_n = (n - 1)m_{n - 1} + 2 \sum_{k=1}^{n-1} m_{k-1} \overline{m}'_{n-k}.
\]

This is the same recursion as the one for the areas under Dyck paths, see [14] or [7, Lemma 3.12]. While the formula above is a special case of Equation (9), we need the results here to obtain (9). When we put \( c = 1 \) in Figure 1 above we get for \( n = 1, \ldots, 7 \), the numbers 0, 1, 6, 29, 130, 562, 2380. This is sequence A 008 549 in [21].

Notation 23. Let \( n \geq 1 \) be fixed. Let us recall our notation: \( \gamma = (1, 2, 3, \ldots, n) \in S_n \subset S_{\pm n} \). Let \( \delta \in S_{\pm n} \) be given by \( \delta(k) = -k \). \( S_{NC}(n, -n) = \{ \pi \in S_{\pm n} \mid \pi \) connects the cycles of \( \gamma \delta \gamma^{-1} \delta \) and \( \#(\pi) + \#(\pi^{-1} \gamma \delta \gamma^{-1} \delta) = 2n \}. \) \( S^\delta_{NC}(n, -n) = \{ \pi \in S_{NC}(n, -n) \mid \pi \delta \) is a pairing \}. \) For \( \pi \in S^\delta_{NC}(n, -n) \), let \( K(\pi) = \delta \gamma^{-1} \delta \pi^{-1} \gamma \) be the Kreweras complement of \( \pi \). Note that \( K(\pi) \in S^\delta_{NC}(n, -n) \). This is the same Kreweras complement as used in [22].

Lemma 24. Suppose, \( \pi \in S^\delta_{NC}(n, -n) \), \( 1 \leq j \leq n \), and \( \pi^{-1}(j) = -k \).

Let

\[
\hat{\gamma} = \gamma \delta \gamma^{-1} \delta \cdot (-j, \gamma^{-1}(k)) \cdot (-k, \gamma^{-1}(j)).
\]

Then

\[
\hat{\gamma} = (1, 2, \ldots, j - 1, -(k - 1), -(k - 2), \ldots, -j, k, k + 1, \ldots, n)
\]

\[
\times (-n, -(n - 1), \ldots, -k, j, j + 1, \ldots, k - 1, -(j - 1), \ldots, -1),
\]

and \( \pi \) is non-crossing with respect to \( \hat{\gamma} \). By this we mean that each cycle of \( \pi \) is contained in one of the two cycles of \( \hat{\gamma} \) and \( \#(\pi) + \#(\pi^{-1} \hat{\gamma}) = 2n + 2 \).

Proof.

\[
\hat{\gamma} = \gamma \delta \gamma^{-1} \delta (-j, \gamma^{-1}(k)) (-k, \gamma^{-1}(j))
\]

\[
= \gamma \cdot (-\gamma^{-1}(j), \gamma^{-1} k) \cdot (-\gamma^{-1}(k), \gamma^{-1}(j)) \cdot \delta \gamma^{-1} \delta
\]

so

\[
\#(\pi^{-1} \hat{\gamma}) = \#((-\gamma^{-1}(k), \gamma^{-1}(j)) \cdot \delta \gamma^{-1} \delta \cdot \pi^{-1} \gamma (-\gamma^{-1}(j), \gamma^{-1}(k)))
\]

\[
= \#((-\gamma^{-1}(k), \gamma^{-1}(j)) \cdot K(\pi) \cdot (-\gamma^{-1}(j), \gamma^{-1}(k))).
\]

Since \( \pi^{-1} \hat{\gamma}(\gamma^{-1}(j)) = -k \) and \( \pi^{-1} \hat{\gamma}(\gamma^{-1}(k)) = -j \) we have that \( -\gamma^{-1}(k) \) and \( \gamma^{-1}(j) \) are in the same cycle of \( K(\pi) \) and \( \gamma^{-1}(k) \) and \( -\gamma^{-1}(j) \) are in the same cycle of \( K(\pi) \); but not in the same cycle as \( -\gamma^{-1}(k) \) and \( \gamma^{-1}(j) \).

Hence

\[
\#((-\gamma^{-1}(k), \gamma^{-1}(j)) \cdot K(\pi) \cdot (-\gamma^{-1}(j), \gamma^{-1}(k))) = \#(K(\pi)) + 2.
\]

Thus

\[
\#(\pi) + \#(\pi^{-1} \hat{\gamma}) = \#(\pi) + \#(K(\pi)) + 2 = 2n + 2.
\]
Let \( \langle \pi, \hat{\gamma} \rangle \) be the subgroup generated by \( \pi \) and \( \hat{\gamma} \). If \( \langle \pi, \hat{\gamma} \rangle \) acts transitively on \([\pm n]\) then there is an integer \( g \geq 0 \) such that
\[
\#(\pi) + \#(\pi^{-1} \hat{\gamma}) + \#(\hat{\gamma}) = 2n + 2(1 - g).
\]
Then we would have
\[
2m + 2 = \#(\pi) + \#(\pi^{-1} \hat{\gamma}) = 2n - 2g,
\]
which is impossible. Hence \( \langle \pi, \hat{\gamma} \rangle \) does not act transitively on \([\pm n]\), this means that no cycle of \( \pi \) meets both of the cycles of \( \hat{\gamma} \). Hence each cycle of \( \pi \) is contained in one of the cycles of \( \hat{\gamma} \).

**Lemma 25.** Suppose \( 1 \leq j < k \leq n \) and \( \hat{\gamma} \) is as in Lemma 24. Let \( \pi \in S_{\pm n} \) be such that (i) \( \pi \delta \) is a pairing; (ii) each cycle of \( \pi \) is contained in one of the two cycles of \( \hat{\gamma} \), (iii) \( \#(\pi) + \#(\pi^{-1} \hat{\gamma}) = 2n + 2 \), and (iv) \( \pi \lor \gamma \delta \gamma^{-1} \delta = 1_{\pm n} \). Then \( \pi \in S_{NC}^\delta (n, -n) \).

**Proof.** Let us consider the cycle of \( \hat{\gamma} \) containing 1:
\[
(1, 2, \ldots, (j - 1), -(k - 1), -(k - 2), \ldots, -j, k, k + 1, \ldots, n).
\]
If \( -j \) and \( \gamma^{-1}(j) \) are in the same block of \( \pi^{-1} \hat{\gamma} \) then \( \pi \) cannot have a cycle connecting a point in \( \{-k - 1, -k - 2, \ldots, -j\} \) to points outside, violating (iv). Likewise for \( -k \) and \( \gamma^{-1}(k) \), they cannot be in the same block of \( \pi^{-1} \hat{\gamma} \).

In addition \( -j \) and \( \gamma^{-1}(j) \) are in the cycle of \( \hat{\gamma} \) above and \( -k \) and \( \gamma^{-1}(k) \) are in the other cycle of \( \hat{\gamma} \):
\[
(-n, -(n - 1), \ldots, -k, j, j + 1, \ldots, k - 1, -(j - 1), -(j - 2), \ldots, -1).
\]
So we have that both pairs \( (-j, \gamma^{-1}(k)) \) and \( (-k, \gamma^{-1}(j)) \) are in different cycles of \( \pi^{-1} \hat{\gamma} \). Hence
\[
\#(K(\pi)) = \#(\pi^{-1} \gamma \delta \gamma^{-1} \delta) = \#(\pi^{-1} \hat{\gamma} \cdot (-j, \gamma^{-1}(k)) \cdot (-k, \gamma^{-1}(j)) = \#(\pi^{-1} \hat{\gamma}) - 2.
\]
Thus \( \#(\pi) + \#(K(\pi)) = \#(\pi) + \#(\pi^{-1} \hat{\gamma}) - 2 = 2n \). Hence \( \pi \in S_{NC}(n, -n) \). As we also have (i) we get that \( \pi \in S_{NC}^\delta (n, -n) \).

**Proposition 26.** Let \( \pi \in S_{NC}^\delta (n, -n) \) and suppose \( \pi^{-1}(1) = j \in [m] \). Let \( k \geq 2 \) be the smallest integer such that \( \pi^{-1}(k) \in [-n] \). Let \( I_1 = \{1, 2, \ldots, j\} \) and \( I_2 = \{j + 1, \ldots, n\} \). Then either \( I_1 \) or \( I_2 \), but not both, meets a through cycle of \( \pi \).

**Proof.** Let \( l = \pi^{-1}(k) \). Note that \( \pi^{-1}(l) = -k \); so \( k < l \). As in Notation 23 let \( \hat{\gamma} \) be the permutation
\[
(1, 2, \ldots, k - 1, -(l - 1), \ldots, -k, l, l + 1, \ldots, n)
\]
\[
\times (-n, \ldots, -(l + 1), -l, k, \ldots, l - 1, -(k - 1), \ldots, -1).
\]
Let us consider the cycle of \( \hat{\gamma} \) containing 1:
\[
(1, 2, \ldots, k - 1, -(l - 1), \ldots, -k, l, l + 1, \ldots, n);
\]
On the left we have a partition in $S_{NC}^δ(6, -6)$, with $\pi^{-1}(1) = 3$ and $\pi(-1) = -3$. We shrink to a point the line joining 1 to $\pi^{-1}(1)$ and the line joining $-1$ to $\pi(1)$. This breaks off two circles. We will label $\pi_1$ the non-crossing partition of the circle we have produced after we remove $\pi^{-1}(1)$ from the block containing 1. This shrinking procedure shows a step half way between the left and right figures in Figure 8 infra.

the cycle of $\pi$ containing 1 must, by Lemma 24, be contained in this cycle, and in particular we have

$$j \in (1, 2, \ldots, k - 1, -(l - 1), \ldots, -k, l, l + 1, \ldots, n).$$

So either $j \leq k - 1$ or $j \geq l$.

Suppose $j \leq k - 1$. Then $I_1 \subset \{1, 2, \ldots, k - 1\}$ and so no through block meets $I_1$ because the first through block is at $k$; but then $k$ is in $I_2$, so a through block meets $I_2$. On the other hand, suppose $j \geq l$. Then $I_2 \subset \{l + 1, \ldots, n\}$. If a through block of $\pi$ were to meet $I_2$, then the whole cycle would, by Lemma 24, lie in $I_2$; but this impossible because $I_2 \subset [m]$. However $k < l \leq j$ so $k \in I_1$ and thus $I_1$ does meet a through block.

7. THE SUBSETS $S_I$, $S_{II}$, AND $S_{III}$

We shall write $S_{NC}^δ(n, -n)$ as the disjoint union of three subsets according to the value of $\pi^{-1}(1)$. By Proposition 26, for each $\pi \in S_{NC}^δ(n, -n)$ exactly one of the three cases (a), (b), or (c) below holds.

**Definition 27.** Let

(a) $S_I = \{\pi \in S_{NC}^δ(n, -n) \mid \pi^{-1}(1) \in [-n]\}$;
(b) $S_{II} = \{\pi \in S_{NC}^δ(n, -n) \mid \pi^{-1}(1) \in [n] \text{ and the interval } I_1 = [1, \pi^{-1}(1)] \text{ does not meet a through cycle of } \pi\}$;
(c) $S_{III} = \{\pi \in S_{NC}^δ(n, -n) \mid \pi^{-1}(1) \notin [-n] \text{ and } \pi^{-1}(1) \notin [n]\}$.

![Figure 7](image_url)
(c) \(S_{III} = \{\pi \in S^\delta_{NC}(n,-n) \mid \pi^{-1}(1) \in [n]\} \) and the interval \(I_1 = [1, \pi^{-1}(1)]\) does meet a through cycle of \(\pi\).

Let

(d) \(T_I = \{(k, \pi) \mid k \in [n-1] \text{ and } \pi \in NC(n-1)\}\);

(e) \(T_{II} = \bigcup_{k=1}^{n-1} NC(k-1) \times S^\delta_{NC}(n-k,-(n-k))\);

(f) \(T_{III} = \bigcup_{k=2}^{n} S^\delta_{NC}(k-1,-(k-1)) \times NC(n-k)\).

To simplify the notation, we have adopted the convention that the cardinality of \(NC(0)\) is 1. On the other hand, the cardinality of \(S^\delta_{NC}(1,-1)\) is 0.

We shall exhibit bijections from \(S_I\) to \(T_I\), from \(S_{II}\) to \(T_{II}\), and from \(S_{III}\) to \(T_{III}\) in Notations 28, 30, and 32 respectively. In each case, \(k = \pi^{-1}(1)\).

**Remark 28.** Let \(k \in [n-1]\) and \(\sigma \in NC(n-1)\) be given. After relabelling we consider \(\sigma\) to be a non-crossing partition of the \(n - 1\) points \(\{1, 2, 3, \ldots, k-1, -(n-1), \ldots, -(k+1)\}\). By this we mean we relabel the point according to the map:

\[
j \mapsto \begin{cases} 
  j & 1 \leq j < k \\
  -(n+k-j) & k \leq j \leq n-1
\end{cases}
\]

We let \(\pi_1\) be the partition of \(\{1, 2, 3, \ldots, k-1, -(n-1), \ldots, -(k+1), -k\}\) obtained by joining \(-k\) to the block containing 1. Since 1 and \(-k\) are cyclically adjacent \(\pi_1\) will be non-crossing. Then we let \(\pi = \pi_1 \delta \pi_1^{-1} \delta \in S_{\pm n}\).

By construction \(\pi^{-1}(1) = -k\). Also, as the two cycles of \(\hat{\gamma}\) are

\[
c_1 = (1, 2, 3, \ldots, k-1, -(n-1), \ldots, -(k+1), -k)\]

\[
c_2 = (k, k+1, \ldots, n, -(k-1), -(k-2), \ldots, -2, -1)
\]

we have that \(\pi\) is non-crossing with respect to \(\hat{\gamma}\).

Note that \(\pi^{-1}(\hat{\gamma}(-k)) = \pi^{-1}(1) = \pi_1^{-1}(1) = -k\) and \(\pi^{-1}(\hat{\gamma}(-1)) = \pi^{-1}(k) = \delta \pi_1 \delta(k) = \delta \pi_1(-k) = -1\). So \(-k\) and \(-1\) are singletons of \(\pi^{-1}(\hat{\gamma})\). Also \(k-1\) and \(m\) are in different cycles of \(\pi^{-1}(\hat{\gamma})\). Thus the four points \(-1, -k, k-1\) and \(m\) are all in different cycles of \(\pi^{-1}(\hat{\gamma})\). Thus \(\#(\pi^{-1} \gamma \delta \gamma^{-1} \delta) = \#(\pi^{-1}(\hat{\gamma})) = 2\).

Hence \(\#(\pi) + \#(K(\pi)) = 2n\).

**Theorem 29.** The map in Remark 28 is a bijection from \(T_I\) to \(S_I\).

**Proof.** We need to give the inverse map, see Figure 8. Given \(\pi \in S^\delta_{NC}(n,-n)\) we let \(k = -\pi^{-1}(1)\). By Lemma 24 with \(j = 1\), we know that each cycle of \(\pi\) is contained in one of the two cycles of \(\hat{\gamma}\). Let \(\pi_1\) be the cycles of \(\pi\) in the cycle \(c_1\) of \(\hat{\gamma}\):

\[
(1, 2, 3, \ldots, k-1, -(n-1), \ldots, -(k+1), -k).
\]

By construction 1 and \(-k\) are in the same block of \(\pi_1\). Let \(\sigma\) be the partition of \(\{1, 2, 3, \ldots, k-1, -(n-1), \ldots, -(k+1)\}\) obtained from \(\pi_1\) by removing \(-k\) from the block containing 1. Then \(\sigma\) is a non-crossing partition in \(NC(\{1, 2, 3, \ldots, k-1, -(n-1), \ldots, -(k+1)\})\) which we identify with \(NC(m-1)\), using the inverse of the labelling map above. So from this we
Figure 8. On the top left we have \( \pi = (1, 2, -5)(3, 4)(-1, 5, -2)(-3, -4)(6)(-6) \in S_{NC}^6(6, -6) \). In the terminology of Definition 27 \( \pi \in S_I \). On the top right we have cut open the annulus, the cycles of \( \pi \) form a non-crossing partition of this disc. In the notation of Lemma 24 \( \hat{\gamma} = (-6, -5, 1, 2, 3, 4)(5, 6, -4, -3, -2, -1) \).

We remove \( \pi^{-1}(1) \) from the block containing 1 and \( \pi^{-1}(-1) \) from the block containing \(-1\) to obtain two copies of the figure on the bottom, the second copy (not shown) is the mirror image of the first.

In the notation of Definition 7 (\( d \)) and Theorem 29, \( k = 5 \) and \( \sigma = (1, 2)(3, 4)(-6) \), shown in the lower figure.

get the pair \((k, \sigma) \in T_I \). And by applying the construction in Notation 28 we get back to \( \pi \). Thus the map is a bijection.

Remark 30. Given \( \pi \in S_{NC}^6(n, -n) \) with \( \pi^{-1}(1) \in [n] \). Let \( j = \pi^{-1}(1) \) and \( k \) be as in Proposition 26 i.e. \( k > 1 \) is the smallest integer such that \( \pi^{-1}(k) \in [-n] \). Suppose that we are in the case where \( I_1 = \{1, 2, \ldots, j\} \) does not meet a through block of \( \pi \), but \( I_2 \) does. So \( 1 \leq j < n - 1 \). Thus \( \pi|_{I_1} \) is a non-crossing partition of \( I_1 \), and 1 and \( j \) are in the same block of \( \pi|_{I_1} \). Let
\[ \pi_1 \in NC(j - 1) \] be the partition obtained from \( \pi|_{I_1} \) obtained by removing \( j \) from the block containing \( 1 \). Also \( \delta \pi_1^{-1} \delta \) is a non-crossing partition of \( \left[ -(j - 1) \right] \). Let \( \pi_2 = \pi|_{I_2 \cup -I_2} \). Since this a restriction of a non-crossing annular permutation, it is itself a non-crossing annular permutation of \( (I_2, -I_2) \). If we identify the points of \( I_2 \) with \([n - j]\) we have \( \pi_2 \in S_{NC}(n - j, -(n - j)) \).

Since \( \pi_2 \) is a pairing \( \pi_2 \delta|_{I_2} \) is also a pairing. Thus \( \pi_2 \in S_{NC}(n - j, -(n - j)) \).

Hence we get a pair \((\pi_1, \pi_2) \in NC(j - 1) \times S_{NC}^\delta(n - j, -(n - j)) \). For \( j > 1 \) we have \( \#(\pi) = 2\#(\pi_1) + \#(\pi_2) \). When \( j = 1 \) something special happens: \((1) \) and \((-1) \) are singletons of \( \pi \). Hence \( \pi_1 \) is the unique partition of the empty set. In this case \( \#(\pi) = \#(\pi_2) + 2 \). If \( j = n - 1 \) or \( n \) then, as we are assuming that \( I_1 \) does not meet a through block, \( \pi \) could not have a through block. As this is not possible the largest possible value of \( j \) is \( n - 2 \).

**Theorem 31.** The map in Remark 32 is a bijection from \( T_{II} \) to \( S_{II} \).

**Proof.** We need to give the inverse map, see Figure 10. Let \((\pi_1, \pi_2) \in NC(j - 1) \times S_{NC}^\delta(n - j, -(n - j)) \) be given. If we identify the points of \( I_2 \) with \([n - j]\) we have \( \pi_2 \in S_{NC}(I_2, -I_2) \). We add \( j \) to block of \( \pi_1 \) containing 1 and denote this non-crossing partition \( \pi_1 \). Then, set \( \pi = \pi_1 \delta \pi_1^{-1} \delta \pi_2 \).

Because we have inserted \( \pi_1 \) into the gap between \( n \) and \( j + 1 \), we get a non-crossing annular permutation; similarly with \( \delta \pi_1^{-1} \delta \). Hence \( \pi \in S_{NC}(n, -n) \).

Since \( (\pi_1 \delta \pi_1^{-1} \delta) \delta = \pi_1 \delta \pi_1^{-1} \) is a pairing we have that \( \pi \delta \) is a pairing and \( \pi \in S_{NC}^\delta(n, -n) \).

**Remark 32.** Given \( \pi \in S_{NC}^\delta(n, -n) \) with \( \pi^{-1}(1) \in [n] \). Let \( j = \pi^{-1}(1) \) and \( k \) be as in Proposition 26 i.e. \( k \) is the smallest integer such that \( \pi^{-1}(k) \in [-n] \). We suppose that we are in the case where \( I_2 = \{j + 1, \ldots, n\} \) does not meet a through block of \( \pi \), but \( I_1 \) does. So \( 2 \leq j \leq n - 1 \). Thus \( \pi_2 = \pi|_{I_2} \) is a non-crossing partition of \( I_2 \). Next, 1 and \( j \) are in the same block of \( \pi|_{I_1 \cup -I_1} \), as are \(-1\) and \(-j\). Let \( \pi_1 \) be the partition obtained from \( \pi|_{I_1 \cup -I_1} \) obtained by removing \( j \) from the block containing 1, and \(-j \) from the block containing \(-1\). Since \( \pi_1 \) a restriction of a non-crossing annular permutation, it is itself a non-crossing annular permutation of \( (j - 1, -(j - 1)) \). Since \( \pi \delta \) is a pairing we have \( \delta \pi \delta = \pi^{-1} \) and hence \( \delta \pi_1 \delta = \pi_1^{-1} \). If \( \pi_1 \delta \) had a singleton then so would \( \pi \). Thus \( \pi_1 \in S_{NC}(j - 1, -(j - 1)) \). If we identify the points of \( I_2 \) with \([n - j]\) we have \( \pi_2 \in NC(n - j) \). Hence we get a pair \((\pi_1, \pi_2) \in NC(j - 1, -(j - 1)) \times NC(n - j) \). We cannot have \( j = 1 \) or \( 2 \) as if either occurred \( \pi \) could not have any through blocks. So \( 3 \leq j \leq n \).

**Theorem 33.** The map in Remark 32 is a bijection from \( T_{III} \) to \( S_{III} \).

**Proof.** We need to give the inverse map, see Figure 10. Let \((\pi_1, \pi_2) \in S_{NC}(j - 1, -(j - 1)) \times NC(n - j) \) be given. We identify the points of \( I_2 = \{j + 1, \ldots, n\} \) with \([n - j]\) we have \( \pi_2 \in NC(I_2) \). We add \( j \) to block of \( \pi_1 \) containing 1 and \(-j \) to the block of \( \pi_1 \) containing \(-1\), and denote this non-crossing partition \( \pi_1 \) i.e. \( \pi_1 = \pi_1(j, \pi_1^{-1}(1))(-j, \pi_1(-1)) \). Set \( \pi =
Figure 9. On the left we have $\pi = (1, 2, 3)(-3, -2, -1)(4, -6)$ $(-4, 6)(5)(-5) \in S_{NC}(6, -6)$. $\pi^{-1}(1) = 3$, so $I_1 = \{1, 2, 3\}$ which does not meet any through blocks. So $\pi \in S_{II}$. On the right we have squeezed off two circles $(1, 2, 3)$ and $(-3, -2, -1)$. Next, $\pi^{-1}(1) = 3$ is removed from the block containing 1 and $\pi(-1) = -3$ is removed from the block containing $-1$.

Figure 10. On the left we have $\pi = (1, 4)(-1, -4)(2, -3)(3, -2)$ $(5, 6)(-5, -6) \in S_{NC}(6, -6)$. In this example $\pi^{-1}(1) = 4$ and $I_1 = \{1, 2, 3, 4\}$ does meet a through block. So $\pi \in S_{III}$. On the right we have $\pi_1 = (1)(-1)(2, -3)(-2, 3) \in S_{NC}(3, -3)$ and $\pi_2 = (5, 5) \in NC(2)$. 
\( \tilde{\pi}_1 \pi_2 2\pi_2^{-1} \delta \). Because we have inserted \( \pi_2 \) into the gap between \( j + 1 \) and \( n \), we get a non-crossing annular permutation. Hence \( \pi \in S_{NC}(n, -n) \). Since 

\[(\pi_2 2\pi_2^{-1} \delta) = \pi_2 2\pi_2^{-1} \delta \]

is a pairing we have that \( \pi \delta \) is a pairing and thus \( \pi \in S_{NC}(n, -n) \).

\[\square\]

8. **The recursion formula for \( |S_{NC}^D(n, -n)| \)**

Let \( m_k = |NC(k)| \) and \( m'_k = |S_{NC}^D(k, -k)| \). We adopt the convention that there is one partition of the empty set, so \( m_0 = 1 \). Also \( S_{NC}^D(1, -1) \) is empty as there must be a through block, which has to be \( (1, -1) \), but then \( (1, -1) \delta \) is not a pairing, so \( m'_1 = 0 \).

**Theorem 34.**

\[
m'_n = (n - 1)m_{n-1} + \sum_{k=1}^{n} \{m_{k-1} m'_{n-k} + m'_{k-1} m_{n-k}\}.\]

**Proof.** When \( n = 1 \), \( m'_1 = 0 \). On the other hand \( (n - 1)m_{n-1} = 0 \) and the sum is empty; thus the identity holds for \( n = 1 \). Now fix \( n > 1 \). The cardinality of \( T_1 \) is \( (n - 1)m_{n-1} \). From Definition \( 27 \) \((e)\), the cardinality of \( T_{II} \) is \( \sum_{k=1}^{n} m_{k-1} m'_{n-k} \), keeping in mind that the term for \( k = n - 1 \) is 0. The cardinality of \( T_{III} \) is \( \sum_{k=2}^{n} m'_{k-1} m_{n-k} = \sum_{k=3}^{n} m'_{k-1} m_{n-k} = \sum_{k=1}^{n-2} m_{k-1} m'_{n-k} \). Adding these up we get the claim in the theorem. \( \square \)

**Theorem 35.** Let \( M(z) = \sum_{n=0}^{\infty} m_n z^n \) and \( m(z) = \sum_{n=1}^{\infty} m'_n z^n \). Then

\[
m(z) = \frac{z^2 M'(z)}{1 - 2z M(z)} = \frac{1 - 2z - \sqrt{1 - 4z}}{2(1 - 4z)}.\]

**Proof.** We begin by treating \( m \) and \( M \) as formal power series, so we may differentiate under the summation sign to obtain

\[
\sum_{n=1}^{\infty} (n - 1)m_{n-1} z^n = z^2 \sum_{n=1}^{\infty} m_{n-1}(n - 1) z^{n-2} = z^2 \frac{d}{dz} \sum_{n=1}^{\infty} m_{n-1} z^{n-1} = z^2 M'(z).
\]

Next, observe that

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{n} m_{k-1} m'_{n-k} z^n = z \sum_{k=1}^{\infty} m_{k-1} z^{k-1} \sum_{n=k+1}^{\infty} m'_{n-k} z^{n-k} = z \sum_{k=1}^{\infty} m_{k-1} z^{k-1} \sum_{n=k+1}^{\infty} m'_{n-k} z^{n-k} = z M(z) \sum_{n=1}^{\infty} m'_{n} z^{n} = zm(z) M(z).
\]

Thus the recursion equation in Theorem \( 34 \) gives us

\[
m(z) = z^2 M'(z) + 2z m(z) M(z).
\]
This gives us the first claimed equality. Now we have 
\[ M(z) = \frac{1 - \sqrt{1 - 4z}}{2z}, \]
has a radius of convergence of \(1/4\) so \(M\) is analytic on \(D\), the open disc with centre 0 and radius 1/4. As \(1 - 2zM(z) = \sqrt{1 - 4z}\) we have that \(1 - 2zM(z) \neq 0\) on \(D\). Hence \(m\) is analytic on \(D\). Finally we have 
\[ z^2M'(z) = \frac{1 - 2z - \sqrt{1 - 4z}}{2\sqrt{1 - 4z}}, \]
this gives us the second claimed equality.

**Corollary 36.** For \(n \geq 1\)

\[ \bar{m}'_n = 4^{n-1} - \frac{1}{2} \binom{2n}{n}. \]

**Proof.** We have
\[
\sum_{n=1}^{\infty} 4^{n-1}z^n = \frac{z}{1 - 4z} \quad \text{and} \quad \sum_{n=1}^{\infty} \binom{2n}{n} z^n = \frac{1 - \sqrt{1 - 4z}}{\sqrt{1 - 4z}}.
\]
So
\[
\sum_{n=1}^{\infty} \left[ 4^{n-1} - \frac{1}{2} \binom{2n}{n} \right] z^n = \frac{z}{1 - 4z} - \frac{1 - \sqrt{1 - 4z}}{2\sqrt{1 - 4z}} = m(z).
\]

9. **The Recursion Formula for the Infinitesimal Moments of a Real Wishart Matrix**

Let recall the free cumulants of the Marchenko-Pastur law with parameter \(c\): \(\kappa_n = c\) for \(n \geq 1\). This means that the moments are given by
\[ m_n = \sum_{\pi \in NC(n)} c^{\#(\pi)}. \]
From this one gets the following recursion for the moments:
\[ m_n = (c - 1)m_{n-1} + \sum_{k=1}^{n} m_{k-1}m_{n-k}. \]

Our goal in this section is to establish the recurrence
\[ \bar{m}'_n = (n - 1)m_{n-1} + (c - 1)\bar{m}'_{n-1} + \sum_{k=1}^{n} \{ \bar{m}'_{k-1}m_{n-k} + m_{k-1}\bar{m}'_{n-k} \}. \]
for the infinitesimals \(\{\bar{m}'_{n}\}_{n=1}^{\infty}\) and \(\{\bar{m}'_{n}\}_{n=1}^{\infty}\) are sequences of polynomials in \(c\) with integer coefficients, see Proposition 19. When \(c = 1\), equation 9 is the usual recurrence for the Catalan numbers. When \(c = 1\), equation 10 gives the recurrence in Theorem 34. As the method for getting the second recurrence is the annular version of the recurrence for the Catalan numbers, we shall review the method here.

For the purposes of this section we shall regard \(NC(n)\) as a subset of the symmetric group \(S_n\), where the cycles of the permutation are the blocks of the partition. As usual we embed \(S_n\) into \(S_{n+1}\) by making it act trivially
on \{n+1\}. From the point of view of partitions this means adding to a partition of [n] the one element block \(n+1\) to get a partition of \([n+1]\).

If we let \(\tau_n = (1,n)\) and \(\sigma \in S_{n-1}\) then \(\tau_n \sigma \in S_n\) is the permutation whose cycles are unchanged except the one containing 1. If this cycle was \((i_1, \ldots, i_k)\) with \(i_1 = 1\), the it becomes \((n, i_1, \ldots, i_k)\) in \(\tau_n \sigma\). If \(\sigma\) was non-crossing then \(\tau_n \sigma\) is also non-crossing; and \((\tau_n \sigma)^{-1}(1) = n\).

Conversely if \(\sigma \in NC(n)\) and \(\sigma^{-1}(1) = n\) then \(\tau_n \sigma\) leaves \(n\) fixed and so \(\tau_n \sigma \in S_{n-1}\); moreover \(\tau_n \sigma \in NC(n-1)\) because we cannot introduce crossing by removing a block.

**Lemma 37.** The map \(\sigma \mapsto \tau_n \sigma\) maps \(NC(n-1)\) bijectively onto \(\{\pi \in NC(n) | 1 \text{ and } n \text{ are in the same block}\}\).

**Remark 38.** Let \(NC(n)_k\) denote the set of non-crossing partitions of \([n]\) such that \(k\) is the largest element in the block containing 1. Then we have \(NC(n) = \bigcup_{k=1}^{n} NC(n)_k\). In addition \(NC(n)_k \ni \sigma \mapsto \sigma_1 \times \sigma_2 \in NC(k-1) \times NC(n-k)\) is a bijection where \(\sigma_1\) is the element of \(NC(k-1)\) produced by restricting \(\sigma\) to \([k]\) and then applying Lemma 37. We get \(\sigma_2\) by restricting \(\sigma\) to \([k+1, n]\).

For the Marchenko-Pastur law with parameter \(c\) we have

\[
(11) \quad m_n = \sum_{\pi \in NC(n)} c^{\#(\pi)}.
\]

Thus by Lemma 37

\[
m_n = \sum_{\pi \in NC(n)} c^{\#(\pi)} = \sum_{k=1}^{n} \sum_{\pi \in NC(n)_k} c^{\#(\pi)}
= c \sum_{\pi \in NC(n-1)} c^{\#(\pi)} + \sum_{k=2}^{n} \sum_{\pi_1 \in NC(k-1) \text{ and } \pi_2 \in NC(n-k)} c^{\#(\pi_1)} c^{\#(\pi_2)}
= cm_{n-1} + \sum_{k=2}^{n} m_{k-1} m_{n-k} = (c - 1) m_{n-1} + \sum_{k=1}^{n} m_{k-1} m_{n-k}.
\]

This gives us the recurrence for the moments of the Marchenko-Pastur law: equation (9).

Now let us turn to the infinitesimal law of a real Wishart matrix. We have that using the notation of Definition 27

\[
m'_n = \sum_{\pi \in S_{NC(n,n-n)}^k} c^{\#(\pi)/2}
= \sum_{\pi \in S_I} c^{\#(\pi)/2} + \sum_{\pi \in S_{II}} c^{\#(\pi)/2} + \sum_{\pi \in S_{III}} c^{\#(\pi)/2}.
\]

Now let us separately find an expression for each term.
Lemma 39. \[ \sum_{\pi \in S_I} c^{\#(\pi)/2} = (n - 1)m_{n-1}. \]

Proof. By Theorem 29
\[ \sum_{\pi \in S_I} c^{\#(\pi)/2} = (n - 1) \sum_{\pi_1 \in NC(n-1)} c^{\#(\pi_1)} \]
because when we go from \( \pi \in S_I \) we let \( k = -\pi^{-1}(1) \) and we get \( \pi_1 \) by restricting \( \pi \)
\( (1, 2, 3, \ldots, k - 1, -n, -(n - 1), \ldots, -(k + 1), -k). \)
and then we get \( \pi \in NC(\{1, 2, 3, \ldots, k - 1, -n, -(n - 1), \ldots, -(k + 1), -k\}) \)
by removing \(-k\) from the cycle containing 1. Since there are exactly the
same number of cycles of \( \pi \) in the other cycle of \( \hat{\gamma} \)
\( (k, k + 1, \ldots, n - 1, n, -(k - 1), \ldots, -3, -2, -1) \)
we have \( \#(\pi) = 2\#(\pi_1). \) Now sum over \( k \), noting that we cannot have
\( k = 1 \), this gives the factor of \( (n - 1). \)

Lemma 40. \[ \sum_{\pi \in S_{II}} c^{\#(\pi)/2} = (c - 1)m'_{n-1} + \sum_{j=1}^{n-2} m'_{j-1}m'_{n-j}. \]

Proof. Let \( j = \pi^{-1}(1) \). We will break the proof into two parts; the first part
is when \( j = 1 \) and the second part is when \( 2 \leq j \leq n - 2 \). As noted in
Remark 30, the largest \( j \) can be is \( n - 2 \).

When \( j = 1 \) then \( \pi(1) = 1 \). Thus \( \pi_1 \) is the empty partition with 0 blocks,
so \( c^{\#(\pi_1)} = 1 \). Thus \( \#(\pi_2) = \#(\pi) - 2 \). Hence the contribution for \( j = 1 \) is
\( c m'_{n-1}. \)

When \( j \geq 2 \) we have \( \#(\pi) = 2\#(\pi_1) + \#(\pi_2) \). Thus the contribution for
\( j \geq 2 \) is \( m_{j-1}m'_{n-j} \). Summing we have, as \( m_0 = 1 \),
\[ \sum_{j=1}^{n-2} \sum_{\pi \in S_{II} \atop \pi^{-1}(1) = j} c^{\#(\pi)/2} = cm'_{n-1} + \sum_{j=2}^{n-2} m'_{j-1}m'_{n-j} \]
\[ = (c - 1)m'_{n-1} + \sum_{j=1}^{n-2} m'_{j-1}m'_{n-j}. \]

Lemma 41. \[ \sum_{\pi \in S_{III}} c^{\#(\pi)/2} = \sum_{j=1}^{n-2} m'_{j-1}m'_{n-j}. \]
Proof. Let $j = \pi^{-1}(1)$. By Remark 32 we have $3 \leq j \leq n$. Thus

$$\sum_{\pi \in S_{III}} c^{\#(\pi)/2} = \sum_{j=3}^{n} m'_{j-1} m_{n-j} = \sum_{j=1}^{n-2} m_{j-1} m'_{n-j}$$

Theorem 42. Let $m'_n = \sum_{\pi \in S_{NC}(n, -n)} c^{\#(\pi)/2}$. Then $m'_1 = 0$ and for $n \geq 2$ we have

$$m'_n = (n-1)m_{n-1} + (c-1)m'_{n-1} + 2 \sum_{k=1}^{n-2} m_{k-1} m'_{n-k}.$$ 

Proof. For $n = 1$, $m'_1 = 0$ because $S_{NC}(1, -1)$ is empty. For $n \geq 2$ we have

$$\sum_{\pi \in S_{NC}(n, -n)} c^{\#(\pi)/2} = \sum_{\pi \in S_{I}} c^{\#(\pi)/2} + \sum_{\pi \in S_{II}} c^{\#(\pi)/2} + \sum_{\pi \in S_{III}} c^{\#(\pi)/2}$$

$$= (n-1)m_{n-1} + (c-1)m'_{n-1} + 2 \sum_{k=1}^{n-2} m_{k-1} m'_{n-k}.$$ 

Let $a = (1 - \sqrt{c})^2$, $b = (1 + \sqrt{c})^2$, and $\{m_n\}_{n=1}^{\infty}$ be the moments of the Marchenko-Pastur law with parameter $c$, (see equation 11). Recall that the moment generating function $M(z) = 1 + \sum_{n=1}^{\infty} m_n z^n$ is given by

$$M(z) = \frac{1 - (1 + c)z - \sqrt{(1 - az)(1 - bz)}}{2z}$$

Theorem 43. The generating function $\bar{m}(z) = \sum_{n=2}^{\infty} m'_n z^n$ is given by

$$\bar{m}(z) = \frac{z^2 M'(z)}{1 + z(1 - c) - 2zM(z)}.$$ 

Remark 44. When $c = 1$ we get the formula of Theorem 35.

Proof.

$$\bar{m}(z) = \sum_{n=2}^{\infty} (n-1)m_{n-1} z^n + (c-1)\sum_{n=2}^{\infty} m'_{n-1} z^n$$

$$+ 2 \sum_{n=2}^{\infty} \sum_{k=1}^{n-2} m_{k-1} m'_{n-k} z^n$$

and we shall do each sum separately.

As in the proof of Theorem 35 we have $\sum_{n=2}^{\infty} (n-1)m_{n-1} z^n = z^2 M'(z)$. Also $\sum_{n=2}^{\infty} m'_{n-1} z^n = z\bar{m}(z)$. Finally

$$\sum_{n=2}^{\infty} \sum_{k=1}^{n-2} m_{k-1} m'_{n-k} z^n = z \sum_{k=1}^{\infty} \sum_{n=k+2}^{\infty} m_{k-1} m'_{n-k} z^{n-1}$$
Thus
\[
m(z) = z^2 M'(z) + (c - 1) z m(z) + 2 z m(z) M(z).
\]

Let \(P(z) = (z-a)(z-b)\) and \(G\) be the Cauchy transform of the Marchenko-Pastur law with parameter \(c\). Recall that
\[
G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{m_n}{z^{n+1}} = \frac{z + 1 - c - \sqrt{P(z)}}{2z}.
\]
Let \(g(z) = \frac{1}{z} m\left(\frac{1}{z}\right)\) be the (infinitesimal) Cauchy transform of the moment sequence \(\{m'_n\}_{n=1}^{\infty}\), see e.g. [15, §2]. The next theorem shows that the second term of Equation (2) gives the same distribution as in Dumitriu and Edelman [6]. We choose the branch of \(\sqrt{P(z)}\) as in [15, Ex. 3.6].

**Theorem 45.**

(12) \(g(z) = \frac{zG(z) - 1}{P(z)} = \frac{1}{2} \left\{ \frac{1}{z-a} + \frac{1}{z-b} \right\} - \frac{1}{\sqrt{(z-a)(z-b)}} \).

**Proof.** Since \(M(z^{-1}) = zG(z)\) we have \(-z^{-2} M'(z^{-1}) = G(z) + z G'(z)\). It is routine to check that \(G(z) + z G'(z) = \frac{1 - z G(z)}{\sqrt{P(z)}}\). Hence

\[
g(z) = \frac{1}{z} m\left(\frac{1}{z}\right) = \frac{1}{z} \frac{z^{-2} M'(z^{-1})}{1 + z^{-1}(1-c) - 2z^{-1} M(z^{-1})} = \frac{z^{-2} M'(z^{-1})}{z + (1-c) - 2 z G(z)} = \frac{z^{-2} M'(z^{-1})}{\sqrt{P(z)}} = z G(z) - 1 \frac{1}{P(z)} = \frac{1}{2} \left\{ \frac{1}{z-a} + \frac{1}{z-b} \right\} - \frac{1}{\sqrt{(z-a)(z-b)}} \).
\]

**Remark 46.** If we let \(\nu_1\) be the signed measure:

(13) \(d\nu_1(x) = -c' \left\{ \begin{array}{ll}
\delta_0 - \frac{x+1-c}{2\pi x \sqrt{(b-x)(x-a)}} dx & c < 1 \\
\frac{1}{2} \delta_0 - \frac{1}{2\pi x \sqrt{(b-x)}} dx & c = 1 \\
- \frac{x+1-c}{2\pi x \sqrt{(b-x)(x-a)}} dx & c > 1 
\end{array} \right.\)

where \(a = (1 - \sqrt{c})^2\) and \(b = (1 + \sqrt{c})^2\) and

(14) \(d\nu_2(t) = \frac{1}{2} \left( \frac{1}{2} (\delta_a + \delta_b) - \frac{1}{\pi \sqrt{(b-t)(t-a)}} \right),\)
and \( \mu' = \nu_1 + \nu_2 \) then \( m'_n = \int t^n \, d\mu'(t) \). Combining the conclusion of
Theorem 45 with [15, Thm. 31] we have the infinitesimal Cauchy transform for \( \{m'_n\}_n \) is
\[
g(z) = \sum_{n=0}^{\infty} \frac{m'_n}{z^{n+1}}
= \frac{-c' \sqrt{P(z)} - (1 + c)z - (1 - c)\sqrt{P(z)}}{z \sqrt{P(z)}}
+ \frac{1}{2} \left\{ \frac{1}{z - a} + \frac{1}{z - b} \right\} - \frac{1}{\sqrt{(z - a)(z - b)}} \}
\]

10. INFINITESIMAL \( R \)-TRANSFORM

In Arizmendi, Garza-Vargas, and Perales [1, Example 5.8] gave the infinitesimal \( R \)-transform of negative of the measure in Equation (14). Let us recall that given two moment sequences \( \{m_n\}_{n \geq 1} \) and \( \{m'_n\}_{n \geq 1} \) the infinitesimal cumulants defined by Février and Nica [8, §1.2] are given by formally differentiating the moment cumulant relation
\[
m_n = \sum_{\pi \in NC(n)} \kappa_\pi \quad \text{to get} \quad m'_n = \sum_{\pi \in NC(n)} \partial \kappa_\pi
\]
where \( \partial \kappa_\pi \) is computed using the Leibnitz rule:
\[
\partial \kappa_\pi = \sum_{\pi \in NC(n)} \sum_{V \in \pi} \kappa'_{|V|} \prod_{W \in \pi \setminus V} \kappa_\pi|W|.
\]

There is an equivalent operator valued version of this relation as follows. Let
\[
M_n = \begin{bmatrix} m_n & m'_n \\ 0 & m_n \end{bmatrix} \quad \text{and} \quad K_n = \begin{bmatrix} \kappa_n & \kappa'_n \\ 0 & \kappa_n \end{bmatrix}.
\]
Then Equation (15) becomes
\[
M_n = \sum_{\pi \in NC(n)} K_\pi \quad \text{where} \quad K_\pi = \prod_{V \in \pi} K_{|V|}.
\]
Now let \( Z = \begin{bmatrix} z & w \\ 0 & z \end{bmatrix} \), \( G(Z) = \sum_{n=0}^{\infty} M_n Z^{-(n+1)} \) and \( R(Z) = \sum_{n=1}^{\infty} K_n Z^{n-1} \). Expanding out the series we have
\[
G(Z) = \begin{bmatrix} G(z) & wG'(z) + g(z) \\ 0 & G(z) \end{bmatrix} \quad \text{and} \quad R(Z) = \begin{bmatrix} R(z) & wR'(z) + r(z) \\ 0 & R(z) \end{bmatrix}.
\]
Then, as all these \( 2 \times 2 \) matrices commute, Equation (15) implies the Voiculescu relation
\[
Z^{-1} + R(G(Z)) = Z
\]
which then implies that \( r(z) = -g(K(z))K'(z) \) where \( K = G^{(-1)} \) denotes the compositional inverse of \( G \), \( K' \) is the derivative of \( K \) with respect to \( z \), \( g(z) = \sum_{n=1}^{\infty} m'_n z^{-(n+1)} \), and \( r(z) = \sum_{n=1}^{\infty} \kappa'_n z^{n-1} \), is the infinitesimal \( R \)-transform. See Theorem 2 from [15].

Here we will briefly show that one can also do a direct computation of \( r \) using \( r(z) = -g(K(z))K'(z) \).
We shall let $G(z) = \frac{z + 1 - c\sqrt{(z - a)(z - b)}}{2z}$ be the Cauchy transform of the Marchenko-Pastur law, $g$ be the Cauchy transform in Equation (12). Then, one lets $c_1 = (1 - \sqrt{c})^{-1}$, $c_2 = (1 + \sqrt{c})^{-1}$, and then checks the following steps:

- $K(z) = \frac{1}{z} + \frac{c}{1 - z}$
- $\sqrt{P(K(z))} = K(z) + 1 - c - 2zK(z) = (c - 1)\frac{(z - c_1)(z - c_2)}{z(z - 1)}$
- $K'(z) = (c - 1)\frac{(z - c_1)(z - c_2)}{z^2(z - 1)^2}$
- $g(K(z))K'(z) = (c - 1)^{-1}\frac{cz}{1 - z} \frac{1}{(z - c_1)(z - c_2)}$.

Upon doing a partial fraction expansion we then get

$r(z) = \frac{1}{2}\left\{\frac{1}{c_1 - z} + \frac{1}{c_2 - z} - \frac{2}{1 - z}\right\}$ which gives $\kappa'_n = \frac{1}{2}\{c_1^{-n} + c_2^{-n} - 2\}$. This produces the following table.

| $n$ | $\kappa_n$ | $\kappa'_n$ |
|-----|------------|-------------|
| 1   | $c$        | 0           |
| 2   | $c$        | $c$         |
| 3   | $3c$       | $3c$        |
| 4   | $c^2 + 6c$ | $3c$        |
| 5   | $5c^2 + 10c$ | $15c^2 + 21c$ |
| 6   | $c^3 + 15c^2 + 15c$ | $28c^3 + 28c$ |
| 7   | $7c^3 + 35c^2 + 21c$ | $70c^3 + 70c^2 + 28c$ |
| 8   | $c^4 + 28c^3 + 70c^2 + 28c$ | $210c^3 + 210c^2 + 140c$ |

This enables to come full circle and check agreement with Figure 1. For example Equation (15) gives us that $m_3 = \kappa_3 + 3\kappa_1\kappa_2 + \kappa_3^2$ and $m'_3 = \kappa'_3 + 3\kappa_1'\kappa_2 + 3\kappa_1\kappa_2 + 3\kappa_1^2\kappa'_2$. Using that $\kappa_n = c$ for all $n$ and $\kappa'_1 = 0$ we have $m'_3 = \kappa'_3 + 2\kappa_1\kappa'_2 = 3c + 3c^3$ as claimed in Figure 1.

### 11. Infinitesimal Distributions of Some Orthogonal Polynomials

Given one of the classical ensembles of orthogonal polynomials (see e.g. [26, §7.6]), we can create a sequence of probability measures by putting a mass of $1/n$ at each of the $n$ zeros of $p_n$, the polynomial in the ensemble of degree $n$. In the case of the Hermite and Laguerre polynomials the corresponding measures converge to the semi-circle law and the Marchenko-Pastur law respectively. Moreover the convergence is such that there are infinitesimal laws as well. In the case of the Hermite polynomials we get the negative of the infinitesimal law of the GOE: $\mu' = \nu_1 - \nu_2$ where $\nu_1 = \frac{1}{2}(\delta_{-2} + \delta_2)$ is the Bernoulli law and $d\nu_2(t) = \text{on the interval } [-2, 2] \text{ is the arcsine law, see Arizmendi, Garza-Vargas, and Perales in [1, Example 5.6]. In the case of the Laguerre polynomials it was shown by Arizmendi, Garza-Vargas, and}
Perales in [11] that one gets the negative of the measure in Equation (14).
Indeed, from [1, Example 5.8] we see that the negative of Cauchy transform of the infinitesimal distribution of the zeros of the Laguerre polynomials is

\[
g_{\text{mp}}(z) = \frac{zG(z) - 1}{P(z)} = \frac{1}{2} \left\{ \frac{1}{z - a} + \frac{1}{z - b} \right\} - \frac{1}{\sqrt{(z - a)(z - b)}}. \tag{16} \]

From [15, Remark 28] we see that the Cauchy transform of the infinitesimal measure obtained from the GOE is

\[
g_{\text{sc}}(z) = \frac{1}{2} \left\{ \frac{1}{2} \left( \frac{1}{z - 2} + \frac{1}{z + 2} \right) - \frac{1}{\sqrt{z^2 - 4}} \right\}. \]

If we let \( w = (z - (1 + c))/\sqrt{c} \), then we have \( g_{\text{mp}}(z) = g_{\text{sc}}(w)/\sqrt{c} \). Thus up to a change of variable and rescaling these distributions are the same.

References

[1] O. Arizmendi, J. Garza-Vargas, and D. Perales, Finite Free Cumulants: Multiplicative Convolutions, Genus Expansion and Infinitesimal Distributions, arXiv:2108.08489
[2] S. Belinschi and D. Shlyakhtenko, Free Probability of Type B: Analytic Interpretation and Applications, Amer. J. Math. 134 (2012), 193-234.
[3] P. Biane, F. Goodman, and A. Nica, Non-crossing Cumulants of Type B, Trans. Amer. Math. Soc. 355 (2003), 2263-2303.
[4] G. Borot, S. Charbonnier, E. Garcia-Failde, F. Leid, S. Shadrin Analytic theory of higher order free cumulants, arXiv:2112.12184.
[5] B. Collins, J. A. Mingo, P. Śniady, and R. Speicher, Second Order Freeness and Fluctuations of Random Matrices: III. Higher Order Freeness and Free Cumulants, Documenta Math., 12 (2007), 1-70.
[6] I. Dumitriu and A. Edelman. Global spectrum fluctuations for the \( \beta \)-Hermite and \( \beta \)-Laguerre ensembles via matrix models. J. Math. Phy., 47(063302), 2006, 36pp.
[7] V. Féray, On Complete Functions in Jucys-Murphy Elements, Ann. Comb. 16 (2012), 677-707.
[8] M. Février and A. Nica, Infinitesimal non-crossing cumulants and free probability of type B, J. Funct. Anal. 258 (2010), 2983-3023.
[9] P. Graczyk, G. Letac, and H. Massam, The Hyperoctahedral group, symmetric group representations and the moments of the real Wishart distribution, J. Theoret. Probab. 18 (2005), 1-42.
[10] R. Graham, D. Knuth, and O. Patashnik, Concrete Mathematics, 2nd ed., Addison-Wesley, Reading, MA, 1994.
[11] G. ’t Hooft, A planar diagram theory theory for strong interactions, Nuclear Phy. B 72 (1974), 461-473.
[12] K. Johansson, On fluctuations of random Hermitian matrices, Duke Math. J. 91 (1998) 151-203.
[13] T. Kusak, J. A. Mingo, and R. Speicher, Orthogonal polynomials and fluctuations of random matrices, J. Reine Angew. Math., 604 (2007), 1 - 46.
[14] D. Merlini, R. Sprugnoli, and M. C. Verri, The area determined by underdiagonal lattice paths, in Trees in Algebra and Programming—CAAP’96, H. Kirchner, (ed.), Springer-Verlag, London (1996), 59-71.
[15] J. A. Mingo, Non-crossing annular pairings and the infinitesimal distribution of the goe, J. Lond. Math. Soc. 100 (2019), 987-1012.
[16] J. A. Mingo and A. Nica, Annular Non-crossing Permutations and Partitions, and Second Order Asymptotics for Random Matrices, IMRN 2004, no. 28, 1413-1460.
[17] J. A. Mingo and M. Popa, Real second order freeness and Haar orthogonal matrices, 
J. Math. Phy. 54 (2013), 051701, 1-35.
[18] J. A. Mingo and R. Speicher, Free Probability and Random Matrices, Springer, New York, 2017.
[19] A. Nica and I. Oancea, Posets of annular non-crossing partitions of types B and D, 
Discrete Math. 309 (2009), 1443-1466.
[20] A. Nica and R. Speicher, Lectures on the Combinatorics of Free Probability, Cambridge University Press, Cambridge, 2006.
[21] R. Speicher, Multiplicative functions on the lattice of non-crossing partitions and free convolution, Math. Annalen 298 (1994), 611-628.
[22] C. Emily I. Redelmeier, Genus expansion for real Wishart matrices, J. Theoret. Probab., 24 (2011), 1044–1062.
[23] C. E. I. Redelmeier, Real second-order freeness and the asymptotic real second-order freeness of several real matrix ensembles, Int. Math. Res. Not. 2014, no. 12, pp. 3353-3395.
[24] N. J. A. Sloan and the OEIS Foundation, The on-line encyclopedia of integer sequences. http://oeis.org/.
[25] D. Shlyakhtenko, Free Probability of Type B and Asymptotics of Finite Rank Perturbations of Random Matrices, Indiana Univ. Math. J. 67 (2018), 971-991.
[26] G. Szegő, Orthogonal Polynomials, 4th ed., Amer. Math. Soc., Providence R. I., 1975.
[27] D. Voiculescu, Limit laws for random matrices and free products, Invent. Math. 104 (1991), 201-220.

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