LOCAL DETECTION OF STRONGLY IRREDUCIBLE HEEGAARD SPLITTINGS VIA KNOT EXTERIORS

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1. Introduction

Let \( T \) be a compressible torus in an irreducible 3-manifold \( M \) other than \( S^3 \). It is easy to see that either: 1. \( T \) bounds a solid torus, or: 2. \( T \) bounds a submanifold homeomorphic to the exterior of a non-trivial knot in \( S^3 \), where the compressing disk for \( T \) is a meridian disk of the knot.

The intersection of a strongly irreducible Heegaard surface with solid tori was analyzed by Y.Moriah and H.Rubinstein [4] and M.Scharlemann [8], where it was shown that such intersection can only occur in a very restricted manner. The purpose of this paper is to analyze the intersection of a strongly irreducible Heegaard surface with the knot exterior in the situation 2 above. Precisely, let \( M \) be a 3-manifold with a strongly irreducible Heegaard splitting \( H_1 \cup \Sigma H_2 \). Let \( X \) be a 3-dimensional submanifold such that:

1. \( X \) is homeomorphic to the exterior of a non-trivial knot in \( S^3 \), and—
2. there is a compressing disk, say \( D_X \), of \( \partial X \) such that \( \partial D_X \) is a meridian curve of \( X \).

Note that \( N(X \cup D_X) \) is a 3-ball, hence \( X \) embeds in any manifold, and there are several ways \( X \) can intersect \( \Sigma \). These are shown in Figure 1. In Figure 1(a) \( \Sigma \) intersects \( T \) in simple closed curves which are essential in \( T \) but inessential in \( \Sigma \), and in Figure 1(b) \( \Sigma \) intersects \( T \) in simple closed curves which are essential in \( \Sigma \) but inessential in \( T \). More interesting is Figure 1(c), where all curves of intersection are essential in both \( T \) and \( \Sigma \). (The part of the Heegaard surface shown there is a cylinder, which is a neighborhood of the boundary of the shaded meridian disk of \( H_1 \).) Note that in Figure 1(c), the slope of \( \Sigma \cap T \) is meridional, and each component of \( \Sigma \cap X \) is an annulus. We call such an annulus a meridional annulus. A meridional annulus in \( X \) is either boundary parallel, or a decomposing annulus in the exterior of a composite knot.

The main result of this paper is as follows.

Theorem 1.1. Let \( M \) be a 3-manifold other than the 3-sphere \( S^3 \) with a strongly irreducible Heegaard splitting \( H_1 \cup \Sigma H_2 \). Let \( X \) be a 3-dimensional submanifold of \( M \) such that:

1. \( X \) is homeomorphic to the exterior of a non-trivial knot in \( S^3 \), and—
2. there is a compressing disk, say \( D_X \), of \( \partial X \) such that \( \partial D_X \) is a meridian curve of \( X \).

Suppose that \( \partial X \cap \Sigma \) consists of a non-empty collection of simple closed curves which are essential in both \( \partial X \) and \( \Sigma \). Then we have:

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Figure 1.

1. the closure of some component of $\Sigma \setminus \partial X$ is an annulus and is parallel to an annulus in $\partial X$, and—
2. each component of $\Sigma \cap X$ is a (possibly boundary parallel) meridional annulus.

Remarks 1.2. 1. The annulus in conclusion 1 of Theorem 1.1 may be contained in $X$ or in $\text{cl}(M \setminus X)$.
2. In [4] and [8], the intersection of a strongly irreducible Heegaard surface with a ball was considered, and it is shown that if the boundary of the ball is incompressible in the handlebodies then the intersection of the Heegaard surface with the ball is an unknotted planar surface.

Before finishing this section, we bring its main application. It is concerned with a pair of strongly irreducible Heegaard surfaces that intersect essentially and spindly. In [7] J.H.Rubinstein and Scharlemann showed that such intersection can always be obtained, if we allow a single trivial simple closed curve. In [8] Rieck and Rubinstein show that the trivial simple closed curve can be avoided.

Corollary 1.3. Let $M$ be an irreducible, a-toroidal manifold, let $\Sigma_1$, $\Sigma_2 \subset M$ be strongly irreducible Heegaard surfaces that intersect essentially and spindly.

If $T \subset \Sigma_1 \cup \Sigma_2$ is a torus then $T$ bounds a solid torus.

We prove the corollary assuming the theorem:

Proof of Corollary 1.3. Suppose $T$ does not bound a solid torus, then $T$ bounds a knot exterior $X$ as in Theorem 1.1. Since $\Sigma_1 \cup \Sigma_2$ is a finite complex we may pass to an innermost counterexample to the corollary, i.e. we may assume there
does not exist $T' \subset \Sigma_1 \cup \Sigma_2$ bounding a non-trivial knot exterior $X'$ so that $X'$ is strictly contained in $X$. We shrink $X$ slightly to obtain the knot exterior $\hat{X}$ and the torus $T = \partial \hat{X}$, so that $T$ is transverse to $\Sigma_1$ and $\Sigma_2$. By essentiality all the curves of $\Sigma_1 \cap \Sigma_2$ are essential in $\Sigma_1$, and since any curve of $\Sigma_1 \cap \hat{T}$ is parallel to some such curve in $\Sigma_1$, it must also be essential in $\Sigma_1$. Furthermore, if a curve of $\Sigma_1 \cap \hat{T}$ is inessential in $\hat{T}$ it is parallel to a curve of $\Sigma_1 \cap \Sigma_2$ on $T$ that is inessential there, contradicting essentiality. Hence the conditions of Theorem [1.1] are satisfied. If $\Sigma_1 \cap \text{int}(X) \neq \emptyset$ any component of that intersection yields (by Theorem [1.1]) a meridional annulus in $\hat{X}$. Since an annulus that decomposes a non-trivial knot exterior into two solid tori is not meridional we can use the meridional annulus and an annulus from $T$ to get a torus $T'$ that contradicts our choice of $X$. ($T'$ would not bound a solid torus on either side: on the side contained in $B$ as we just saw, and on the other side it bounds a piece in which a punctured copy of $M$ is embedded.) Hence $\Sigma_1 \cap \text{int}(X) = \emptyset$ and similarly $\Sigma_2 \cap \text{int}(X) = \emptyset$. Thus $X$ is a component of $M$ cut open along $\Sigma_1 \cup \Sigma_2$ but in $\hat{X}$ Rieck showed that every such component is a handlebody (it is here that we use the spinality assumption), a contradiction. □

2. Preliminaries

Throughout this paper, we work in the differentiable category. For a submanifold $H$ of a manifold $M$, $N(H, M)$ denotes a regular neighborhood of $H$ in $M$. When $M$ is well understood we abbreviate $N(H, M)$ to $N(H)$. For the definitions of standard terms in 3-dimensional topology, we refer to [2] or [3].

A 3-manifold $C$ is a compression body if there exists a compact, connected (not necessarily closed) surface $F$ such that $C$ is obtained from $F \times [0, 1]$ by attaching 2-handles along mutually disjoint simple closed curves in $F \times \{1\}$ and capping off the resulting 2-sphere boundary components which are disjoint from $F \times \{0\}$ by 3-handles. The subsurface of $\partial C$ corresponding to $F \times \{0\}$ is denoted by $\partial_+ C$. Then $\partial_- C$ denotes the subsurface $\text{cl}(\partial C - (\partial F \times [0, 1] \cup \partial_+ C))$ of $\partial C$. A compression body $C$ is called a handlebody if $\partial_- C = \emptyset$. A compressing disk $D(\subset C)$ of $\partial_+ C$ is called a meridian disk of the compression body $C$.

Remarks 2.1. The following properties are known for handlebodies

1. Let $F$ be an incompressible surface in a handlebody. Then either $F$ is boundary compressible or a meridian disk.
2. Let $F$ be an incompressible surface in a solid torus (i.e. genus one handlebody). Then $F$ is either a meridian disk or a boundary parallel annulus.
3. Every incompressible surface in a handlebody cuts the handlebody into handlebodies.

Let $N$ be a cobordism rel $\partial$ between two surfaces $F_1$, $F_2$ (possibly $F_1 = \emptyset$ or $F_2 = \emptyset$), i.e., $F_1$ and $F_2$ are mutually disjoint surfaces in $\partial N$ with $\partial F_1 \cong \partial F_2$ such that $\partial N = F_1 \cup F_2 \cup (\partial F_1 \times [0, 1])$, and $F_i \cap (\partial F_1 \times [0, 1]) = \partial F_i$ ($i = 1, 2$).

Definition 2.2. We say that $C_1 \cup_P C_2$ (or $C_1 \cup C_2$) is a Heegaard splitting of $(N, F_1, F_2)$ (or simply, $N$) if it satisfies the following conditions.
1. $C_i$ ($i = 1, 2$) is a compression body in $N$ such that $\partial_- C_i = F_i$,
2. $C_1 \cup C_2 = N$, and
3. $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = P$.

The surface $P$ is called a Heegaard surface of $(N, F_1, F_2)$ (or, $N$).
Definitions 2.3.
1. A Heegaard splitting $C_1 \cup_P C_2$ is reducible if there exist meridian disks $D_1$, $D_2$ of the compression bodies $C_1$, $C_2$ respectively such that $\partial D_1 = \partial D_2$.
2. A Heegaard splitting $C_1 \cup_P C_2$ is weakly reducible if there exist meridian disks $D_1$, $D_2$ of the compression bodies $C_1$, $C_2$ respectively such that $\partial D_1 \cap \partial D_2 = \emptyset$. If $C_1 \cup_P C_2$ is not weakly reducible, then it is called strongly irreducible.

A spine of a handlebody $H$ is a 1-complex $L$ embedded in $\text{int}H$ such that $L$ is a deformation retract of $H$. A cycle of the spine $L$ is a simple closed curve embedded in $L$. Then the following is proved by C.Frohman [4], and will be used in the proof of Theorem 1.1.

Lemma 2.4 (Frohman’s Lemma). Let $H_1 \cup_S H_2$ be a Heegaard splitting of a closed irreducible 3-manifold $M$, and $Y$ a spine of $H_j$ ($j = 1$ or 2). Suppose that there is a 3-ball $B^3$ in $M$ such that some cycle of $Y$ is contained in $B^3$. Then $H_1 \cup_S H_2$ is reducible.

The next lemma proved by Scharlemann [8] is also used in the proof.

Lemma 2.5 (No Nesting Lemma). Suppose that $H_1 \cup_S H_2$ is a strongly irreducible Heegaard splitting of a 3-manifold $M$, and $F$ a disk in $M$ transverse to $S$ with $\partial F \subset S$. Then $\partial F$ also bounds a disk in $H_j$ ($j = 1$ or 2).

Example 2.6. The $(1, 1)$ curve on the standard torus in $S^3$ shows that the transversality assumption is needed.

The next lemma must be well known, but for the convenience of the reader, we bring it with a proof.

Lemma 2.7. Let $N$ be a 3-manifold with a toral boundary component $T$. Let $S$ be a 2-sided surface properly embedded in $N$ such that $S \cap T$ consists of essential simple closed curves in $T$. Suppose that there is a boundary compressing disk $\Delta$ for $S$ such that $\Delta$ compresses $S$ into $T$, i.e., $\Delta \cap \partial N = \partial \Delta \cap T$ is an arc, say $\alpha$, and $\Delta \cap S = \partial \Delta \cap S$ is an essential arc in $S$, say $\beta$, such that $\alpha \cup \beta = \partial \Delta$. Then we have have either one of the following.

1. $S$ is compressible. Moreover if $S$ is separating in $N$, then the compression occurs in the same side as the boundary compression.
2. $S$ is an annulus; moreover, when $N$ is irreducible, $S$ is boundary parallel.

Proof. Case 1. $\partial \alpha$ is contained in a single component, say $\ell$, of $S \cap T$.

Since $S$ is 2-sided, neighborhoods of both endpoints of $\alpha$ are contained in the same side of $\ell$. Then there is a subarc, say $\alpha'$, of $\ell$ such that $\alpha \cup \alpha'$ bounds a disk $D$ in $T$. We isotope $\Delta$ by moving $\alpha$ to $\alpha'$ along $D$ to obtain $\Delta'$ such that $\partial \Delta' \subset S$. Since $\Delta \cap S$ is an essential arc in $S$, we see that $\Delta'$ is a compressing disk for $S$, and this gives the conclusion 1.

Case 2. $\partial \alpha$ is contained in different components, say $\ell_1$ and $\ell_2$, of $S \cap T$.

Let $A(\subset T)$ be the annulus bounded by $\ell_1 \cup \ell_2$ such that $\alpha \subset A$. Let $D$ be a disk obtained from $A$ by boundary compressing along $\Delta$, hence $\partial D \subset S$. If $\partial D$ is essential in $S$, then we have the conclusion 1. If $\partial D$ bounds a disk in $S$, then we see that $S$ is an annulus. If in addition $N$ is irreducible, the sphere obtained by
compressing $S \cup A$ along $\Delta$ bounds a ball, and we easily see that $S$ is boundary parallel; hence conclusion 2 holds.

**Definition 2.8.** A surface properly embedded in a handlebody is called *essential* if it is incompressible and not boundary parallel.

**Lemma 2.9.** Let $A$ be separating essential annulus properly embedded in a handlebody $H$. Then there is a spine $Y$ of $H$ such that $Y$ intersect $A$ in one point, and that each component of $Y \setminus A$ contains a cycle of $Y$.

**Proof.** By 1 of Remarks 2.1 $A$ is boundary compressible, and let $D$ be a disk obtained from $A$ by a boundary compression. Since $A$ is essential, $D$ is a meridian disk of $H$. Since $A$ is separating in $H$, $D$ is also separating in $H$. Hence we can find a spine $Y$ of $H$ such that $Y$ intersects $D$ in one point, and that each component of $Y \setminus D$ contains a cycle of $Y$. Note that $A$ is recovered from $D$ by adding a band. We may suppose that the band is disjoint from $Y$, hence $Y$ gives the conclusion of the lemma.

3. **Proof of Theorem 1.1**

Let $M$, $H_1 \cup_\Sigma H_2$, $X$, and $D_X$ be as in Theorem 1.1. Let $T = \partial X$, $B = N(X \cup D_X)$, and $M_X = \text{cl}(M \setminus X)$.

![Figure 2. X and M_X.](image)

Note that $B$ is a 3-ball in $M$, which contains $X$. Note also that $M_X \cong (D^2 \times S^1)\#M$, where the sphere $\partial B$ defines the connect sum structure, and the disk $D_X$ is a meridian disk for $D^2 \times S^1$. See Figure 2. We always assume $\partial B \cap D_X = \emptyset$. Recall that $X(\subset B)$ is in fact a knot exterior in $S^3$ and the slope defined by $\partial D_X$ on its boundary is the slope of the trivial filling. We refer to $X$ as the *knot exterior*. The slope of $\partial D_X$ plays a crucial role in our game and is called the *meridian* slope; $D_X$ is called the meridian disk. Any other slope on $T$ is called *longitudinal* if it intersects the meridional slope once, *cabled* otherwise. Finally, we note that since $X$ is (by assumption) a non-trivial knot exterior, $\partial X$ is incompressible in $X$, and on the boundary of $(D^2 \times S^1)\#M$ only one slope compresses. Thus $D_X \subset M$ is the unique compressing disk for $T$ (up-to isotopy relative to $T$), and the only slope that compresses is the meridional slope.

**Proof of Theorem 1.1.** We divide the proof of the theorem into three steps. The first (and main) step is:
Step 1: the slope of $\Sigma \cap T$ is meridional.

Assume, for contradiction, that the slope is not meridional. Note that each component of $T \cap H_i \ (i = 1, 2)$ is an annulus.

Claim 1. Each component of $T \cap H_i \ (i = 1, 2)$ is incompressible in $H_i$.

Proof. Assume that there is a component $A$ of $T \cap H_j \ (j = 1 \text{ or } 2)$ such that $A$ is compressible in $H_j$. By using innermost disk arguments, we may suppose that $\text{int} D \cap T = \emptyset$. This shows that $D \subset M_X$, and $\partial D$ is a meridional slope. Hence the slope of $\Sigma \cap T$ is meridional, contradicting the assumption of the proof of Step 1.

Claim 2. By applying an isotopy, if necessary, we may suppose that no component of $\Sigma$ cut along $T$ is an annulus which is boundary parallel in $X$ or $M_X$.

Proof. Suppose there is such a component. Using it to guide an isotopy of $\Sigma$ we reduce $|\Sigma \cap T|$ by two. Repeat the procedure as much as possible. If we come to the situation that $\Sigma \cap T$ becomes empty. Then $\Sigma$ is pushed into $X$ or $M_X$. However the former is absurd ($\Sigma$ is contained in the 3-ball $B$). Hence $\Sigma$ is pushed into $M_X$. Note that prior to the last isotopy $\Sigma \cap T$ consists of two simple closed curves, and we analyze this configuration. Then $T \cap H_i \ (i = 1, 2)$ consists of an annulus, say $A_1$, and $A_j \ (j = 1 \text{ or } 2)$ is boundary parallel in $H_j$.

Since the argument is symmetric, we may suppose that $A_1$ is boundary parallel in $H_1$.

Subclaim 2.1. $A_2$ is not boundary parallel in $H_2$.

Proof. Assume that $A_2$ is boundary parallel in $H_2$. Then either $T$ bounds a solid torus (if $A_1$ and $A_2$ are parallel to the same annulus in $\Sigma$), or $T$ is isotopic to $\Sigma$ (if $A_1$ and $A_2$ are parallel to different annuli in $\Sigma$), contradiction either way.

This together with Claim 1 shows that $A_2$ is an essential annulus in $H_2$ and by Lemma 2.9 there is a cycle of a spine of $H_2$ on each side of it. But $A_2$ separates $H_2$ into $X \cap H_2$ and $M_X \cap H_2$ and so one of these cycles is contained in $X$ and hence in $B$, and by Frohman’s Lemma (2.4) $\Sigma$ reduces, contradiction.

This completes the proof of Claim 2.

Note. The argument in the proof of Claim 2 is a warm-up case of the proof of Step 1, where we drive to find a cycle in $X$ (and hence in $B$) violating Frohman’s Lemma.

Notation: We denote $\Sigma \cap M_X$ by $\Sigma_{M \setminus X}$ and $\Sigma \cap X$ by $\Sigma_X$.

Claim 3. By retaking $X$, if necessary, we may suppose that no component of $\Sigma_{M \setminus X}$ is an annulus.

Proof. Suppose that there is an annulus component, say $A$, in $\Sigma_{M \setminus X}$.

Subclaim 3.1. $A$ is incompressible in $M_X$.

Proof. Assume that $A$ compressible in $M_X$. By compressing $A$, we obtain two compressing disks for $\partial M_X$. This shows that $\partial A$ is a meridional slope of $X$, contradicting the assumption of the proof of Step 1.
Recall that $M_X \cong (D^2 \times S^1) \# M$. Hence, by Subclaim 3.1, 2 of Remarks 2.1 and Claim 2, we see that $A$ together with an annulus in $T$, say $A'$, bounds a piece $P$ homeomorphic to $(D^2 \times S^1) \# M$, where the slope of $\partial A$ is longitudinal in that solid torus. Consider a torus, say $T'$, obtained by slightly pushing $\partial P$ ($= A \cup A'$) into $P$. Let $P'$ be the submanifold bounded by $T'$ which is contained in $P$.

Subclaim 3.2. $T' \cap \Sigma$ consists of non-empty collection of simple closed curves which are essential in both $T'$ and $\Sigma$.

**Proof.** Since $P'$ contains a punctured copy of $M$, $\Sigma \cap P' \neq \emptyset$. Note that the annulus $A$ is contained in the exterior of $P'$. Since $\Sigma$ is connected, $T' \cap \Sigma \neq \emptyset$. By the construction, it is clear that each component of $T' \cap \Sigma$ is essential in $T'$ (and, moreover, the slope of $\partial A$ is longitudinal in $P' \cong (D^2 \times S^1) \# M$). Since the intersection $\Sigma \cap T'$ can be regarded as a subset of $\Sigma \cap T$, we see that each component of $T' \cap \Sigma$ is essential in $\Sigma$.

Let $X' = \text{cl}(M - P')$.

Subclaim 3.3. The submanifold $X'$ satisfies the following.

1. $X'$ is homeomorphic to the exterior of a non-trivial knot in $S^3$, and
2. there is a compressing disk, say $D_{X'}$, of $\partial X'$ such that $\partial D_{X'}$ is a meridian curve of $X'$.

**Proof.** Recall that $T'$ bounds $P' \cong (D^2 \times S^1) \# M$. By the construction, we see that $T'$ bounds a manifold homeomorphic to $X \cup \text{cl}(M_X \setminus P)$ on the other side. Since the slope of $\partial A$ is not meridional in $X$, we see that $X \cup \text{cl}(M_X \setminus P)$ is homeomorphic to the exterior of the same knot for $X$ (if the slope of $\partial A$ is longitudinal) or, the exterior of a cable knot of the knot for $X$ (if the slope of $\partial A$ is not longitudinal). In either case, the knot for $X'$ is non-trivial. It is clear that a meridian disk for the solid torus factor of $P' \cong (D^2 \times S^1) \# M$ can be taken as $D_{X'}$.

By Subclaims 3.2 and 3.3, we see that we may take $X'$ for $X$. The procedures described above and in the proof of Claims 2 may repeated, if necessary, and the process terminates since each application reduces $|\Sigma \cap T|$.

**Claim 4.** $\Sigma_{M \setminus X}$ compresses in $M_X$ into both $H_1 \cup M_X$ and $H_2 \cup M_X$, and $\Sigma_X$ is essential in $X$.

**Proof.** We first show the following:

Subclaim 4.1. For each $i = 1, 2$, there is a meridian disk $D_i$ of $H_i$ such that $D_i \cap T = \emptyset$.

**Proof.** Let $D \subset H_i$ be a meridian disk which minimizes $|D \cap T|$ among all meridian disks. If $D \cap T = \emptyset$, then we are done. Suppose that $D \cap T \neq \emptyset$. By the minimality of $|D \cap T|$ no component of $D \cap T$ is a simple closed curve. Then each outermost disk in $D$ gives a boundary compression of $\Sigma_X(\subset X)$ or $\Sigma_{M \setminus X}(\subset M_X)$. Hence by Claims 2 and 3 and Lemma 2.7, we see that there is a compressing disk $D_i$ for $\Sigma_X(\subset X)$ or $\Sigma_{M \setminus X}(\subset M_X)$ such that $D_i$ is contained in $H_i$, however, this contradicts the minimality of $D$. 


Let $D_1, D_2$ be as in Subclaim 4.1. If one of $D_1, D_2$ is contained in $M_X$, and the other in $X$, then the pair $\{D_1, D_2\}$ gives a weak reduction for $\Sigma$, a contradiction. Hence either $\Sigma_{M\setminus X}$ compresses into both sides or $\Sigma_X$ does.

Subclaim 4.2. $\Sigma_{M\setminus X}$ compresses into both sides.

**Proof.** Recall that $B = N(X \cup D_X)$. Then we minimize $|\Sigma \cap \partial B|$ via isotopy rel $X$. If $|\Sigma \cap \partial B| = 0$, then $\Sigma$ is pushed into $B$, which is absurd. By using an innermost disk argument, essentiality of the intersection and irreducibility of $M$, we can show that each component of $\Sigma \cap B$ is essential in $\Sigma$. Let $D^\ast (\subset \partial B)$ be an innermost disk. Then $D^\ast$ is a meridian disk of $H_j (j = 1 \text{ or } 2)$ contained in $M_X$. This together with the strong irreducibility of $\Sigma$ shows that both $D_1, D_2$ are contained in $M_X$. Hence $\Sigma_{M\setminus X}$ compresses into both sides as desired. 

By comment before Subclaim 4.2, we see that $\Sigma_X$ is incompressible in $X$. By Claim 2, we see that $\Sigma_X$ is not boundary parallel in $X$. Hence $\Sigma_X$ is essential in $X$. 

**Claim 5.** $\Sigma_{M\setminus X}$ is connected.

**Proof.** Since $\Sigma$ is strongly irreducible, the compressions for $\Sigma_{M\setminus X}$ (Claim 4) occurs on the same component of $\Sigma_{M\setminus X}$. Assume that there exists another component of $\Sigma_{M\setminus X}$, say $F$. If $F$ compresses in $M_X$, then by the No Nesting Lemma (2.3) the curve on $F$ that is compressed bounds a meridian disk, say $D'$ in $H_j (j = 1 \text{ or } 2)$. Then $D'$ together with $D_{3-j}$ gives a weak reduction for $\Sigma$, a contradiction. Hence $F$ is incompressible in $M_X \cong (D^2 \times S^1) \# M$. By 2 of Remark 2.4, we see that $F$ is an annulus, contradicting Claim 3. Hence $\Sigma_{M\setminus X}$ is connected. 

Let $S_i^\ast (i = 1, 2)$ be a surface properly embedded in $M_X$ obtained from $\Sigma_{M\setminus X}$ by compressing into $H_i$-side maximally.

**Claim 6.** Each $S_i^\ast$ is incompressible in $M_X$.

**Proof.** Assume that there is a compressing disk $D$ for $S_i^\ast$ in $M_X$. By isotopy of $D$ near its boundary we may assume that $\partial D$ is contained in $\Sigma_{M\setminus X}$. By No Nesting Lemma(2.5) we may suppose that $D$ is contained in $H_1$ or $H_2$. Since $S_i^\ast$ is obtained from $\Sigma_{M\setminus X}$ by compressing into $H_i$-side maximally, $D$ must be contained in $H_{3-i}$. However this shows that $\Sigma$ is weakly reducible, a contradiction. 

By Claim 6, 2 of Remarks 2.4, and the assumption of Step 1, each component of $S_i^\ast$ (and $S_k^\ast$) is an annulus.

**Claim 7.** All the annuli of $T \cap H_i (i = 1, 2)$ are simultaneously boundary compressible into $H_i \cap M_X$, but not into $X$.

**Proof.** We will show the existence of boundary compressions into $M_X$. This together with Claim 4, strong irreducibility and Lemma 2.7 then implies there are no boundary compressions into $X$.

Since the argument is symmetric, it enough to prove Claim 7 for $T \cap H_1$. Recall that $S_i^\ast$ is an incompressible surface in $M_X \cong (D^2 \times S^1) \# M$ obtained from $\Sigma_{M\setminus X}$ by simultaneously compressing into $H_1$ side. Hence the tubings for retrieving $\Sigma_{M\setminus X}$ from $S_i^\ast$ are all done to the same side of $S_i^\ast$ and the tubes are not nested. Connectedness of $\Sigma_{M\setminus X}$ implies that there is a unique $Z$, which is the closure of
a component of \( M_X \setminus S'_1 \), and within which all the tubings are performed. (\( Z \) was obtained by attaching 2-handles to \( M_X \cap H_2 \).

By using innermost disk argument, we may assume that \( S'_1 \) and the 2-sphere giving the connected sum structure \((D^2 \times S^1)\# M\) are disjoint. For the analysis of the situation, we temporarily ignore \( M \) in \( M_X \sim (D^2 \times S^1)\# M \). Then each component of \( S'_1 \) is a boundary parallel annulus in \( D^2 \times S^1 \). We possibly have the following cases.

Case 1. The components of \( S'_1 \) are not nested in \( D^2 \times S^1 \).

Case 2. The components of \( S'_1 \) are nested in \( D^2 \times S^1 \).

Suppose first that Case 1 holds. If \( S'_1 \) is a single longitudinal annulus, it boundary compresses into both sides, and since retrieving \( \Sigma_{M \setminus X} \) is done via tubing into one side only, Claim 7 clearly holds. Else, let \( Q \) be the union of the parallelisms between the components of \( S'_1 \) and mutually disjoint annuli in \( T \). Then let \( R = \text{cl}(D^2 \times S^1) \setminus Q \). Now recall punctured \( M \) in \( M_X \). We have the following two subcases.

Case 1.1. Punctured \( M \) is contained in \( R \) (i.e. \( Z = R \)).

Note that \( \Sigma_{M \setminus X} \) is retrieved from the simultaneously boundary parallel annuli \( S'_1 \) by adding tubes along mutually disjoint arcs properly embedded in \( R \). This gives the conclusion of Claim 7.

Case 1.2. Punctured \( M \) is contained in \( Q \) (i.e. \( Z \subset Q \)).

Note that \( \Sigma_{M \setminus X} \) is retrieved from \( S'_1 \) by adding tubes along mutually disjoint arcs properly embedded in \( Q \). Hence by Claim 3, we see that \( S'_1 \) consists of exactly one annulus (and \( Z = Q \)). This implies that \( R \cap T \) is an annulus properly embedded in \( H_1 \). Denote this annulus by \( A^* \). By Claims 1, and 2 we see that \( A^* \) is an essential annulus in \( H_1 \). By Lemma 2.3, there is a cycle of a spine of \( H_1 \) in each side of \( A^* \), in particular in \( H_1 \cap X \). This cycle is contained in \( X \) and hence in the ball \( B \), so Frohman’s Lemma (2.4) shows that \( \Sigma \) is reducible, a contradiction. This shows that Case 1.2 does not occur.

Suppose now that Case 2 holds (see Figure 3).

![Figure 3. Case 2.](attachment:figure3.png)
Since $\Sigma_X$ is connected, and tubings for $S_1'$ for retrieving $\Sigma_X$ is performed in one side of $S_1'$, we see that the depth of the nesting is two (i.e. there exist outermost and second outermost annuli in $S_1'$, but no third outermost annulus), and the tubings are performed along a system of mutually disjoint arcs, say $\alpha$, properly embedded in the region between outermost and second outermost components. Moreover connectedness of $\Sigma_{M \setminus X}$ implies that there exists exactly one second outermost component, say $A^*$. Then the closures of the components of $(D^2 \times S^1) \setminus A^*$ consists of two components, say $P$ and $R$, such that $P$ is a parallelism between $A^*$ and an annulus in $T$, and $R$ a solid torus which contains a core of $D^2 \times S^1$. (Note that $Z \subset P$.) Again by the connectedness of $\Sigma_{M \setminus X}$, we see that every outermost component of $S_1'$, say $A_1, \ldots, A_n$, is contained in $P$.

Let $A_1', \ldots, A_n'$ be annuli in $T$ such that $\partial A_k' = \partial A_k$, and $A_k'$ and $A_k$ are parallel in $P$ ($k = 1, \ldots, n$). (Hence $A_1', \ldots, A_n'$ are mutually disjoint.) By simultaneously boundary compressing $A_1', \ldots, A_n'$ into $\Sigma_{M \setminus X}$ in $P$, we obtain disks, say $D_1, \ldots, D_n$, properly embedded in $H_1$. Let $D^*$ be a disk properly embedded in $R$ such that $\partial D^*$ is a simple closed curve in $A^*$ which bounds a disk (in $A^*$) containing the points $\alpha \cap A^*$. We may regard $D^*$ as a disk properly embedded in $H_1$, and it is clear from the definition that $D_1 \cup \cdots \cup D_n \cup D^*$ cuts off a handlebody corresponding to $H_1 \cap \{\text{the region between } A_1 \cup \cdots \cup A_n \text{ and } A^*\}$. Note that the exterior of this handlebody in $H_1$ is a non-trivial handlebody, since it contains simple closed curves $\partial A^*$ which are essential in $\Sigma$. This shows that there is a cycle of a spine of $H_1$ that is contained in the exterior of the punctured $M$. Hence by Frohman’s Lemma (2.4) we see that $\Sigma$ is reducible, a contradiction. This shows that Case 2 does not occur.

This completes the proof of Claim 7.

Completion of the proof of Step 1.

By Claim 7, we see that $T \cap H_1$ consists of annuli that are simultaneously boundary compressible into $M_X$ side. Let $K$ be the closure of a component of $H_1 \setminus T$ to which the boundary compressions are not performed. By Claim 7, we see that $K$ is contained in $X$. By performing the boundary compressions on $T \cap H_1$, we obtain a union of mutually disjoint meridian disks (say $\tilde{D}$) in $H_1$. Let $K'$ be the closure of the component of $H_1 \setminus \tilde{D}$ such that $K' \supset K$. Since we obtain $A^*$ from $\tilde{D}$ by banding into $K'$, we see that $K'$ is not a ball. Hence $K'$ contains a cycle of a spine of $H_1$, and $K$ contains the same cycle. By Frohman’s Lemma (2.4), we see that $\Sigma$ is reducible. This contradiction completes the proof of Step 1.

**Step 2:** $T \cap H_j$ ($j = 1$ or 2) contains a boundary parallel annulus.

Recall that the assertion of Step 2 is conclusion 1 of Theorem 1.1.

Claim 8. There is an annulus component of $T \cap H_j$ ($j = 1$ or 2) which is compressible in $H_j$.

**Proof.** Let $\gamma$ be a component of $T \cap \Sigma$. Since $\gamma$ defines a meridional slope (Step 1), it bounds a disk such that a neighborhood of $\gamma$ in the disk is embedded in one of the handlebodies $H_1$ or $H_2$. By the No Nesting Lemma (2.3), $\gamma$ bounds a disk that is entirely in $H_1$ or $H_2$. By innermost disk argument applied to the intersection of this disk with $T$, we see that some annulus of $T \setminus \Sigma$ compresses in some $H_j$. \qed

Let $A$ be the annulus obtained in Claim 8. Without loss of generality, we may suppose that $A \subset H_1$. Let $A'$ be an annulus component of $T \cap H_2$ adjacent to
Step 2. Assume that $A \subset H_j$ is compressible in $H_j$ (Claim 8). Then either $A \cup A_\Sigma$ bounds a parallelism $P$ contained in $H_j$. Hence $H_j \cup \Sigma H_2$ is reducible, a contradiction.

This completes the proof of Step 2.

Step 3: Completion of the proof.

Finally we induct on $|T \cap \Sigma|/2$ to show that every component of $\Sigma_X$ is a meridional annulus. Let $A' \subset H_j$ be an annulus obtained in Step 2, and $A_\Sigma$ the annulus in $\Sigma$ such that $A \cup A_\Sigma$ bounds a parallelism $P$ contained in $H_j$. Suppose that $|T \cap \Sigma|/2 = 1$. Then either $\Sigma_X = A_\Sigma$ or $\Sigma_M \setminus X = A_\Sigma$. However if $\Sigma_M \setminus X = A_\Sigma$, then $\Sigma$ can be isotoped into the 3-ball $B$, a contradiction. Hence $\Sigma_X = A_\Sigma$, which gives the conclusion 2 of Theorem 1.1.

Suppose that $|T \cap \Sigma|/2 > 1$. By passing to outermost one, if necessary, we may suppose that $\text{int} P \cap \Sigma = \emptyset$. We have the following two cases.

Case 1. $A_\Sigma \subset X$.

In this case, we push $A_\Sigma$ along the parallelism $P$ out of $X$. Then by induction, we see that the image of $\Sigma$ intersects $X$ in meridional annuli. Note that $\Sigma_X$ is the union of these annuli and $A_\Sigma$. Hence each component of $\Sigma_X$ is an annulus, and their slope is meridional by Step 1.

Case 2. $A_\Sigma \subset M_X$.

In this case, we push $A_\Sigma$ along the parallelism $P$ into $X$. Then by induction, we see that the image of $\Sigma$ intersects $X$ in meridional annuli. To retrieve $\Sigma_X$ we push a core curve of one of these annuli out of $X$. Thus this annulus breaks into two annuli. Again each component of $\Sigma_X$ is a meridional annulus.

This completes the proof of Theorem 1.1. \qed

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