STRONGLY QUASIPOSITIVE LINKS ARE
CONCORDANT TO INFINITELY MANY STRONGLY
QUASIPOSITIVE LINKS

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Abstract. We show that every strongly quasipositive link other than an un-
link is smoothly concordant to infinitely many (pairwise non-isotopic) strongly
quasipositive links. In contrast to our result, it was conjectured by Baker that
smoothly concordant strongly quasipositive fibered knots are isotopic. Our
construction uses a satellite operation whose companion is a slice knot with
maximal Thurston-Bennequin number -1.

1. Introduction

We study links in the 3-sphere $S^3$, i.e. non-empty, oriented, closed, smooth
1-dimensional submanifolds of $S^3$, up to (ambient) isotopy. By a fundamental
theorem of Alexander [Ale23], every link in $S^3$ can be represented as the closure
of an $n$-braid for some positive integer $n$. An $n$-braid is an element of the braid
group on $n$ strands, denoted $B_n$, whose classical presentation with $n-1$ generators
$\sigma_1, \ldots, \sigma_{n-1}$ and relations

\[ \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2 \quad \text{and} \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \]

was introduced by Artin [Art25]. For a more detailed account on braids and their
closures, we refer the reader to [BB05]. An $n$-braid is called quasipositive if it is
a (finite) product of conjugates of positive Artin generators $\sigma_i$ of $B_n$. It is called
strongly quasipositive if the product consists only of positive band words $\sigma_{i,j}$, where

\[ \sigma_{i,j} = (\sigma_i \cdots \sigma_{j-2}) \sigma_{j-1} (\sigma_i \cdots \sigma_{j-2})^{-1} \quad \text{for } 1 \leq i < j \leq n. \]

Note that $\sigma_{i,i+1} = \sigma_i$. A link is called (strongly) quasipositive if it arises as the
 closure of a (strongly) quasipositive $n$-braid for some $n \geq 1$, respectively.

Two (ordered) links $L_0 = L_0^1 \cup \cdots \cup L_0^r$ and $L_1 = L_1^1 \cup \cdots \cup L_1^r$ of $r$ components
are called (smoothly) concordant if there exists a smoothly and properly embedded
oriented submanifold $A = A_1 \cup \cdots \cup A_r$ of $S^3 \times [0,1]$ that is diffeomorphic to a disjoint
union of $r$ annuli $S^1 \times [0,1]$ such that $\partial A_i = L_0^i \times \{0\} \cup L_1^i \times \{1\}, i \in \{1, \ldots, r\}$,
and such that the induced orientation on $\partial A$ agrees with the orientation of $L_0$, but
is the opposite one on $L_1$.

Our main result is the following.

Theorem 1. Every strongly quasipositive link that is not an unlink is smoothly
concordant to infinitely many (pairwise non-isotopic) strongly quasipositive links.

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To discuss the context of this result, we will focus on knots in the rest of the introduction. Knots are links with one connected component. A knot is called slice if it is concordant to the unknot.

Quasipositive knots arise as transverse intersections of smooth algebraic curves in $\mathbb{C}^2$ with the 3-sphere $S^3 \subset \mathbb{C}^2$, which provides a geometric characterization of these knots \cite{Rud98, BO01}. In the context of smooth concordance, (strongly) quasipositive knots are special. For example, it follows from Rudolph's slice-Bennequin inequality \cite{Rud93} that not every knot is concordant to a quasipositive knot, and this is contrary to the behavior in the topological category \cite{BF19}.

In contrast to our result and generalizing earlier results by Stoimenow \cite{Sto08, Sto15}, Baader, Dehornoy, and Liechti \cite{BDL17} showed that every knot is (topologically and thus also smoothly) concordant to at most finitely many (pairwise non-isotopic) positive knots. Recall that positive knots — knots admitting a diagram in which all crossings are positive — form a subset of the set of strongly quasipositive knots \cite{Nak00, Rud99}. Considering an even smaller subset, it was shown by Litherland \cite{Lit79} that algebraic knots, which are knots of isolated singularities of complex algebraic plane curves, are isotopic if they are concordant. Algebraic knots arise as closures of positive braids and are therefore positive knots; see e.g. \cite{BK86, Theorem 12 in Section 8.3}.

In fact, by a result of Baker \cite{Bak16}, either smoothly concordant, fibered, strongly quasipositive knots are isotopic or not every slice knot is a ribbon knot — a knot bounding an immersed disk in $S^3$ with only ribbon singularities. A pair of smoothly concordant, but non-isotopic, fibered, strongly quasipositive knots would thus provide a counterexample to the so-called slice-ribbon conjecture, which goes back to a question by Fox \cite{Fox62}.

Remark 1. In \cite{Bak16}, Baker explains a strategy personally communicated to him by Hedden which directly shows that, contrary to the conjectured result for strongly quasipositive fibered knots, there are (infinitely many) pairs of (ribbon) concordant strongly quasipositive knots that are not isotopic. Indeed, the positive $k$-twisted Whitehead doubles of two concordant, non-isotopic knots provide examples of such pairs for $k$ sufficiently negative.

This project began with the observation that, using the above idea but being careful about the choice of $k$, it is not difficult to construct an infinite family of concordant, pairwise non-isotopic strongly quasipositive knots by taking the positive $(-1)$-twisted Whitehead doubles of an infinite family of concordant, pairwise non-isotopic knots, all of which have maximal Thurston-Bennequin number $TB$ equal to $-1$. For instance, for a slice knot $C$ with $TB(C) = -1$ (see Section 2.1 for an example of such a knot), the connected sums of $m$ copies of $C$ for $m \geq 1$ can serve as the latter infinite family. In Remark 4 we explain this in more detail.

Note that the statement of Theorem 1 is stronger, since it shows that each strongly quasipositive knot other than the unknot is concordant to infinitely many strongly quasipositive knots.

Remark 2. The nontriviality assumption in Theorem 1 is necessary, since there exists only one strongly quasipositive slice knot: the unknot. This follows from the fact that for strongly quasipositive knots the genus and the smooth 4-genus coincide \cite{Ben83, Rud93}.

Remark 3.
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2. Proof of Theorem 1

In this section we will prove Theorem 1. To that end, we will first establish some notations and definitions regarding quasipositive Seifert surfaces and study some examples of such surfaces in Section 2.1. In Section 2.2, we will construct from two quasipositive Seifert surfaces $F_1$ and $F_2$ for links $\partial F_1$ and $\partial F_2$ a third one which has as boundary a link which is concordant, but not isotopic to $\partial F_2$. The surface $F_1$ will be one of the quasipositive annuli from Section 2.1. We will finally prove Theorem 1 in Section 2.3, leaving the proof of the technical Lemma 1 for Section 3.

2.1. Quasipositive Seifert Surfaces and Particular Quasipositive Annuli

We first define quasipositive Seifert surfaces. Let $L$ be a link that arises as the closure $\hat{\beta}$ of a strongly quasipositive braid $\beta \in B_n$, $n \geq 1$, which is a product of $m \geq 0$ positive band words $\sigma_{i,j}$. We refer to such a product as a strongly quasipositive braid word, and — despite minor abuse of notation — also denote it by $\beta$. There is a canonical Seifert surface of Euler characteristic $n - m$ for $L$ associated to the braid word $\beta$ that consists of $n$ copies of disjoint parallel disks and $m$ half-twisted bands connecting these disks [Rud92a]; see Figure 2. A Seifert surface (for $L$) is a compact, oriented surface in $S^3$ (with oriented boundary $L$) without closed components. We will denote the canonical Seifert surface associated to $L = \hat{\beta}$ by $F(\beta)$. We call any Seifert surface $F$ for a link $L = \partial F$ quasipositive if, for some strongly quasipositive braid word $\beta \in B_n$, $n \geq 1$, the surface $F$ is ambient isotopic to $F(\beta)$. We will be particularly interested in certain quasipositive annuli.

Let $C$ be a nontrivial (i.e. not the unknot) slice knot with maximal Thurston-Bennequin number $TB(C) = -1$, e.g. the mirror of the knot $9_{46}$ from Rolfsen’s knot table [Ng01, LM22] — denoted by $m(9_{46})$. For our purposes, we could use any such knot $C$, but for the sake of concreteness of our illustrations we will fix $C = m(9_{46})$ in the entire text.

Recall that every knot $K$ has a Legendrian representative (which is at each point in $S^3$ tangent to the 2-planes of the standard contact structure on $S^3$) and its maximal Thurston-Bennequin number $TB(K)$ is defined as

$$TB(K) = \max\{tb(\mathcal{L}) \mid \mathcal{L} \text{ is a Legendrian representative of } K\}.$$ 

Here, for a Legendrian knot $\mathcal{L}$, $tb(\mathcal{L})$ denotes the Thurston-Bennequin number of that Legendrian knot; see e.g. [Etn05] for a definition.

Figure 1 shows the front projection of a Legendrian representative $\mathcal{L}$ of $m(9_{46})$ with $tb(\mathcal{L}) = -1$. There is a Lagrangian concordance between the Legendrian
unknot $U$ with $tb(U) = -1$ and the Legendrian representative $L$ of $m(9_{46})$; see [Cha15, Figure 4]. In particular, the knot $m(9_{46})$ is slice, and since $TB(K) \leq -1$ for every slice knot $K$ [Rud95], this implies $TB(m(9_{46})) = -1$.

![Figure 1](image)

**Figure 1.** Front projection of a Legendrian representative $L$ of $m(9_{46})$ with $tb(L) = -1$; cf. [Cha15, Figure 1].

For every knot $K$ and every integer $k$, following Rudolph’s notation [Rud92b], let $A(K, k)$ denote an annulus of type $K$ with $k$ full twists, i.e. $A(K, k)$ is an annulus in $S^3$ with $K \subset \partial A(K, k)$ and such that the linking number $lk(K, \partial A(K, k) \setminus K) = -k$. By [Rud95, Proposition 1], we have

$$\sup \{ k \mid A(K, k) \text{ is quasipositive} \} = TB(K).$$

Hence, for every knot $K$ with $TB(K) = -1$, the annulus $A(K, -1)$ is quasipositive, in particular for $K = C = m(9_{46})$. This is the key observation of this subsection. It implies the existence of a strongly quasipositive braid word $\alpha \in B_m$ for some $m \geq 1$ such that $A(C, -1)$ is ambient isotopic to $F(\alpha)$. For example, we can choose $\alpha$ as in (1) from Section 2.1. Let $\beta = \prod_{i,k=1}^k \sigma_{i,k}$. We can put the surfaces $F(\alpha)$ and $F(\beta)$ in split position in $S^3$ as sketched in Figure 4(a). Concretely, we can take $F(\alpha)$ to lie in the

2.2. Tying Knots into Bands of Quasipositive Seifert Surfaces Preserving Quasipositivity

Let $F$ be a quasipositive Seifert surface for a link $L$ that is not an unlink and let $C = m(9_{46})$, so that the annulus $A(C, -1)$ is quasipositive (see Section 2.1). As mentioned in Section 2.1 for $C$ we could also use every other nontrivial slice knot with maximal Thurston-Bennequin number $TB(C) = -1$.

In this section, starting from the quasipositive Seifert surfaces $A(C, -1)$ and $F$, we will define a new quasipositive Seifert surface $F'$ that will have as boundary a link which is concordant, but not isotopic to $L = \partial F$. To that end, for both $A(C, -1)$ and $F$, choose strongly quasipositive braid words $\alpha \in B_m$ and $\beta \in B_n$ for $m, n \geq 1$, respectively, such that $A(C, -1)$ is ambient isotopic to $F(\alpha)$ and $F$ is ambient isotopic to $F(\beta)$. For example, we can and will choose $\alpha$ as in (1) from Section 2.1. Let $\beta = \prod_{i,k=1}^k \sigma_{i,k}$. We can put the surfaces $F(\alpha)$ and $F(\beta)$ in split position in $S^3$ as sketched in Figure 4(a). Concretely, we can take $F(\alpha)$ to lie in the
lower hemisphere and \( F(\beta) \) to lie in the upper hemisphere of \( S^3 \), respectively. Then we can choose a cylinder \( Z \subset S^3 \) such that the bands \( B_\alpha \) of \( F(\alpha) \) and \( B_\beta \) of \( F(\beta) \) corresponding to the positive band words \( \sigma_{4,7} \) and \( \sigma_{i_1,j_1} \), respectively, intersect \( Z \) as indicated in the upper part of Figure 3 (ignoring the red curve for now). More precisely, we can choose a cylinder \( Z \subset S^3 \) and an orientation-preserving diffeomorphism \( \varphi: Z \to D^2 \times [0, 1] \) such that \( Z \cap F(\alpha) = Z \cap B_\alpha \), \( Z \cap F(\beta) = Z \cap B_\beta \) and \( \varphi \) maps

\[\begin{align*}
Z \cap B_\alpha & \xrightarrow{\varphi} \left[ -\frac{2}{3}, -\frac{1}{3} \right] \times [0, 1], \\
Z \cap \partial B_\alpha & \xrightarrow{\varphi} \left\{ -\frac{2}{3}, -\frac{1}{3} \right\} \times [0, 1], \\
Z \cap B_\beta & \xrightarrow{\varphi} \left[ \frac{1}{3}, \frac{2}{3} \right] \times [0, 1], \\
Z \cap \partial B_\beta & \xrightarrow{\varphi} \left\{ \frac{1}{3}, \frac{2}{3} \right\} \times [0, 1],
\end{align*}\]

where \( X \xrightarrow{\varphi} Y \) indicates an orientation-preserving diffeomorphism. Here, \([a, b]\) denotes the straight line segment connecting \( a \) and \( b \) in the closed unit disk \( D^2 \) in \( \mathbb{C} \). We choose the orientations on \( D^2 \times [0, 1] \) and \([a, b] \times [0, 1]\) induced by the standard orientations on \( \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3 \subset S^3 \) and \( \mathbb{R}^2 \), respectively; the orientation on \([a, b] \times [0, 1]\) also induces an orientation on \([a, b] \times [0, 1]\).

We claim that we can choose \( Z \) and \( \varphi: Z \xrightarrow{\varphi} D^2 \times [0, 1] \) such that \( \varphi \) satisfies \( \Box \) and such that there exists a simple closed curve \( \gamma \) in \( S^3 \setminus F(\beta) \) that goes once around the band \( B_\beta \) of \( F(\beta) \) corresponding to \( \sigma_{i_1,j_1} \) and is not null-homotopic in
**Figure 3.** Top: The triple \((D^2 \times [0, 1], [-\frac{2}{3}, -\frac{3}{3}], [0, 1] \cup [\frac{1}{3}, \frac{2}{3}] \times [0, 1], \{\pm \frac{1}{3}, \pm \frac{2}{3}\} \times [0, 1])\), which is mapped to \((Z, Z \cap (B_\alpha \cup B_\beta), Z \cap \partial (B_\alpha \cup B_\beta))\) via \(\varphi\). The red curve depicts \(\varphi(\gamma) \subset D^2 \times [0, 1]\); see (3). Bottom: \(B' \subset D^2 \times [0, 1]\) for \(B'\) as defined in (4).

\[S^3 \setminus \partial F(\beta)\]. More precisely, we claim that we can choose \(Z\) and \(\varphi\) such that

\[\gamma = \varphi^{-1}\left(C_\beta\left(\frac{1}{2}\right) \times \left\{\frac{1}{2}\right\}\right) \subset S^3 \setminus F(\beta)\]

is a simple closed curve that is not null-homotopic in \(S^3 \setminus \partial F(\beta)\), where \(C_\beta\left(\frac{1}{2}\right) \subseteq D^2\) denotes the circle with center \(\frac{1}{2}\) and radius \(\frac{1}{3}\). The situation is shown in the upper part of Figure 3 with \(\varphi(\gamma) \subset D^2 \times S^1\) in red. The above claim follows from the following lemma.

**Lemma 1.** Let \(F(\beta)\) denote the canonical Seifert surface associated to a strongly quasipositive braid word \(\beta\) such that \(\partial F(\beta)\) is not an unlink. Then we can choose a cylinder \(Z' \subset S^3\) and an orientation-preserving diffeomorphism \(\varphi'\) of triples of manifolds with corners

\[\varphi': (Z', Z' \cap F(\beta), Z' \cap \partial F(\beta)) \\sim \to \left(D^2 \times [0, 1], \left[-\frac{1}{2}, \frac{1}{2}\right] \times [0, 1], \left\{-\frac{1}{2}, \frac{1}{2}\right\} \times [0, 1]\right)\]

such that \(B_\beta = Z' \cap F(\beta)\) is a band of \(F(\beta)\) corresponding to a positive band word of \(\beta\) and such that \(\gamma = (\varphi')^{-1}\left((\partial D^2) \times \left\{\frac{1}{2}\right\}\right)\) is a simple closed curve in \(S^3 \setminus F(\beta)\) which is not null-homotopic in \(S^3 \setminus \partial F(\beta)\). Moreover, either the two components of \(\partial B_\beta \cap F(\beta)\) belong to two different components of the link \(\partial F(\beta)\) or we can assume that there exists some quasipositive Seifert surface \(G\) that is a connected component of \(F(\beta)\) with boundary a knot \(J = \partial G\) (which is one of the components of \(F(\beta)\)) such that \(B_\beta \subset G\) and \(\gamma\) is not null-homotopic in \(S^3 \setminus J\).

For the proof of Lemma 1 we refer the reader to Section 3.

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1 Note that up to conjugation of \(\beta\) or ambient isotopy of \(F(\beta)\), that is, up to a different choice of \(Z\) and \(\varphi\), we can choose any positive band word of \(\beta\) to be the first one \(\sigma_{1,1}\).
Now, let \( B = Z \cap (B_\alpha \cup B_\beta) \) for bands \( B_\alpha \) and \( B_\beta \) of \( F(\beta) \) and a cylinder \( Z \) as in (2) and define \( F' = (F(\alpha) \cup F(\beta)) \setminus B \cup \varphi^{-1}(B') \), where \( B' \) is as in the lower part of Figure 3. More precisely, we let

\[
B' = \left\{ (a + t, t) \mid t \in [0, 1], \quad a \in \left[ -\frac{2}{3}, -\frac{1}{3} \right] \right\} 
\cup \left\{ (a - t + ti, t) \mid t \in \left[ 0, \frac{1}{2} \right], \quad a \in \left[ \frac{1}{3}, \frac{2}{3} \right] \right\} 
\cup \left\{ (a - t + (1 - t)i, t) \mid t \in \left[ \frac{1}{2}, 1 \right], \quad a \in \left[ \frac{1}{3}, \frac{2}{3} \right] \right\} \subseteq D^2 \times [0, 1].
\]

(4)

In the definition of \( B' \) in (4) (and only there), \( i \in \mathbb{C} \) denotes the imaginary unit.

We smooth the corners of \( \varphi^{-1}(B') \) to obtain a smooth surface \( F' \) and claim the following.

**Lemma 2.** Let \( Z \) and \( \varphi \) be defined as above such that (2) is satisfied and such that \( \gamma \) as in (3) is a simple closed curve in \( S^3 \setminus F(\beta) \) that goes once around the band \( B_\beta \) of \( F(\beta) \) and is not null-homotopic in \( S^3 \setminus \partial F(\beta) \). Moreover, assume that either the two components of \( \partial B_\beta \cap \partial F(\beta) \) belong to two different components of the link \( \partial F(\beta) \) or there exists a connected component \( G \) of \( F(\beta) \) with boundary a knot \( J = \partial G \) such that \( B_\beta \subset G \) and \( \gamma \) is not null-homotopic in \( S^3 \setminus J \). Then the surface \( F' = (F(\alpha) \cup F(\beta)) \setminus B \cup \varphi^{-1}(B') \) with smoothed corners (where \( B = Z \cap (B_\alpha \cup B_\beta) \) and \( B' \) is defined as in (4)) is a quasipositive Seifert surface for a link \( \partial F' \) that is concordant, but not isotopic to \( \partial F(\beta) \).

**Proof of Lemma 2.** We will show the following two claims separately.

**Claim 1.** The surface \( F' \) is quasipositive.

**Claim 2.** The boundary of \( F' \) is concordant, but not isotopic to \( \partial F(\beta) \).

**Proof of Claim 1.** The quasipositivity of \( F' \) can be shown using an isotopy as depicted in Figures 4(b) to 4(d). A strongly quasipositive braid word \( \delta \) for a quasipositive Seifert surface \( F(\delta) \) that is ambient isotopic to \( F' \) can then be read off in Figure 4(d).

**Proof of Claim 2.** Observe that the surface \( F' \) is obtained from \( F(\beta) \) by tying the knot \( C \) with framing 0 into the band \( B_\beta \) of \( F(\beta) \); see Figure 5. This amounts to realizing the boundary of \( F' \) as a satellite with pattern \( \partial F(\beta) \) and companion \( C \).

We explain this in detail. The link \( \partial F(\beta) \) can be viewed as a link in the solid torus \( S^3 \setminus \nu(\gamma) \) given by the complement of an open tubular neighborhood \( \nu(\gamma) \) of \( \gamma \) in \( S^3 \).

We can identify this solid torus with \( V = D^2 \times S^1 \subset S^3 \) by an orientation-preserving diffeomorphism that takes the preferred longitude of \( S^3 \setminus \nu(\gamma) \) to \( \{1\} \times S^1 \subset V \).

Then \( \partial F' \) arising as a satellite link with pattern \( \partial F(\beta) \) and companion \( C \) means that it is the image of \( \partial F(\beta) \subset S^3 \setminus \nu(\gamma) \cong V \) under an orientation-preserving embedding \( h: V = D^2 \times S^1 \to S^3 \) that maps \( \{0\} \times S^1 \) to \( C \) and \( \{1\} \times S^1 \) to a curve that has linking number 0 with \( h (\{0\} \times S^1) \). For more details on satellite constructions, and in particular on the terms used here, see Rolfsen’s Sections 2E and 4D. Note that our choices ensure that \( h \) is faithful in Rolfsen’s terminology and that the companion is really \( C \) and not the reverse of \( C \).

Now, it is a standard fact from concordance theory that, since \( C \) is concordant to the unknot \( U \), there is a concordance between \( \partial F' \), the satellite with pattern \( \partial F(\beta) \) and companion \( C \), and \( \partial F(\beta) \), the satellite with same pattern but companion \( U \).
Figure 4. Quasipositivity of the surface $F'$ defined from $F(\alpha)$ and $F(\beta)$ (see Lemma 2). The surface $F(\alpha)$ is shown in black, the surface $F(\beta)$ in blue, and $Z \cap (F(\alpha) \cup F(\beta))$ in grey. Subfigure 4(a) shows $F(\alpha)$ and $F(\beta)$ together with $\gamma$ as in Lemma 2. Subfigure 4(b) shows the surface $F'$ which is ambient isotopic to the canonical quasipositive Seifert surface in 4(d); an intermediate stage of such an isotopy is shown in 4(c).

Indeed, if $C$ and $U$ are concordant via an annulus $A \cong S^1 \times [0, 1] \subset S^3 \times [0, 1]$, then we can identify $(S^3 \setminus \nu(\gamma)) \times [0, 1]$ with a tubular neighborhood of $A$ in $S^3 \times [0, 1]$ and the image of $(\partial F(\beta)) \times [0, 1]$ in $S^3 \times [0, 1]$ under this identification provides us with a concordance between the two satellite links.

On the other hand, we claim that since $C$ is not isotopic to $U$, the satellite links $\partial F'$ and $\partial F(\beta)$ are not isotopic. To prove this claim, we distinguish two cases. Note that the two components of $\partial B_\beta \cap \partial F(\beta)$ do not necessarily belong to the same component of the link $\partial F(\beta)$.

We first assume that they do, which is the case, for example, if $\partial F(\beta)$ is a knot; and we can further assume that $\gamma$ is not null-homotopic in the complement of this
component $J$ of $\partial F(\beta)$ in $S^3$ (see the assumptions in the lemma). The claim then follows from work of Kouno and Motegi [KM94, Theorem 1.1] since in this case our satellite operation modifies up to ambient isotopy only the component $J$ of $\partial F(\beta)$ by applying a satellite operation with companion $C$ and pattern $J$; that $\partial F'$ and $\partial F(\beta)$ are not isotopic follows from the fact that this satellite operation on the non-isotopic knots $C$ and $U$ produces non-isotopic components of $\partial F'$ and $\partial F(\beta)$. Here we need the assumptions on $\gamma$, which imply that the pattern $J$ we use in the satellite construction has geometric winding number (wrapping number in Kouno–Motegi’s terminology) strictly greater than 1. The geometric winding number $\omega_V(P)$ of a pattern $P$ in the solid torus $V = D^2 \times S^1$ is the minimal geometric intersection number of $P$ and a generic meridional disk of $V$. Recall that we can consider $\partial F(\beta)$ and hence also its component $J$ as a link in the solid torus $S^3 \setminus \nu(\gamma)$, which we identify with $V$ by an orientation-preserving diffeomorphism that takes the preferred longitude of $S^3 \setminus \nu(\gamma)$ to $\{1\} \times S^1$. Then $\gamma$ not being null-homotopic in $S^3 \setminus J$ implies that $J$ geometrically intersects nontrivially every meridional disk in $S^3 \setminus \nu(\gamma) \cong V$, so $\omega_V(J) \neq 0$. Since the algebraic winding number of $J$ in $S^3 \setminus \nu(\gamma) \cong V$ is zero (thus even), we get $\omega_V(J) > 1$.

Now, suppose that the two components of $\partial B_\beta \cap \partial F(\beta)$ belong to two different components $L_1$ and $L_2$ of the link $\partial F(\beta)$. The satellite operation then has the effect of tying $C$ into both of these components (up to orientation), i.e. the resulting link has components $L_1 \# C$ and $L_2 \# C'$, where $C'$ denotes $C$ with the reversed
orientation, and all other components unchanged. Note that for our particular choice \( C = m(9_{46}) \) we have \( C = C' \) [LM22]. This clearly produces a link \( \partial F' \) that is not isotopic to \( \partial F(\beta) \).

This concludes the proof of Lemma 2. □

2.3. Proof of Theorem 1

Recall the statement of Theorem 1: Every strongly quasipositive link that is not an unlink is smoothly concordant to infinitely many (pairwise non-isotopic) strongly quasipositive links.

*Proof of Theorem 1.* Let \( F \) be a quasipositive Seifert surface for a link \( L \) that is not an unlink, and let \( C = m(9_{46}) \) so that the annulus \( A(C, -1) \) is quasipositive (see Section 2.2). Let \( \alpha \) be as in 1 from Section 2.1 so that \( A(C, -1) \) is ambient isotopic to \( F(\alpha) \), and as in Section 2.2 choose a strongly quasipositive braid word \( \beta \in B_n, n \geq 1 \), such that \( F \) is ambient isotopic to \( F(\beta) \).

The statement of Theorem 1 will follow from an iterative application of the operation defined in Section 2.2. Given two quasipositive Seifert surfaces \( F(\alpha) \) and \( F(\beta) \) for links \( \partial F(\alpha) \) and \( \partial F(\beta) \), respectively, using Lemma 1 and Lemma 2 we can construct a quasipositive Seifert surface \( F' \) with boundary that is concordant, but not isotopic to \( \partial F(\beta) \). We will denote this surface by \( F(\alpha) \oplus F(\beta) := F' \). We define \( F_0 = F(\beta), F_1 = F' = F(\alpha) \oplus F(\beta) \) and, inductively, \( F_{i+1} = F(\alpha) \oplus F_i \) for all \( i \geq 1 \). The links \( \{\partial F_i\}_{i \geq 0} \) are then all in the same concordance class (the class of \( L = \partial F_0 = \partial F(\beta) \)), but pairwise non-isotopic. Let us make this more precise.

Recall that we constructed the surface \( F_1 = F' \) by tying the knot \( C \) into a specific band \( B_\beta \) of \( F(\beta) \) (see Section 2.2) which implied that we obtained \( \partial F_1 \) as a satellite with pattern \( \partial F(\beta) \) and companion \( C \) (see the proof of Lemma 2). The surfaces \( F(\beta) \) and \( F_1 \) are both quasipositive Seifert surfaces (see Lemma 2) that can again be put in a position where we can choose a cylinder \( Z_1 \subset S^3 \) and an orientation-preserving diffeomorphism \( \varphi_1 : Z_1 \to D^2 \times [0,1] \) that satisfies a condition equivalent to the one in 2 from Section 2.2 for \( Z \) and \( \varphi \). By Lemma 1 and Lemma 2 we can choose \( Z_1 \) and \( \varphi_1 \) such that the surface \( F_2 \) obtained from \( F_1 \) by tying the knot \( C \) into a specific band of \( F_1 \) is quasipositive and has as boundary \( \partial F_2 \) a link that is concordant, but not isotopic to \( \partial F_1 \). Inductively, \( F_{i+1} \) is obtained from \( F_i \) by tying \( C \) into a band of \( F_i \) such that \( \partial F_{i+1} \) is concordant, but not isotopic to \( \partial F_i \). However, up to an ambient isotopy of \( F_1 \) and \( F_2 \), we can assume that for \( F_2 \) we tie \( C \) into the “same band” of \( F_1 \) as \( B_\beta \) of \( F(\beta) \).\(^2\) Note that the additional assumptions in Lemma 2 about this band will still be satisfied.

As in the proof of Claim 2 in the proof of Lemma 2 we now distinguish two cases. If the two components of \( \partial B_\beta \cap \partial F(\beta) \) belong to the same component \( J \) of the link \( \partial F(\beta) \), then we actually obtain \( \partial F_2 = P_J(C \# C) \), the satellite with pattern \( P_J = J \), but companion \( C \# C \). We see that inductively

\[
\partial F_i = P_J(C \# C \ldots \# C),
\]

Since \( C \) is not isotopic to the unknot, the connected sums of \( i \) and \( k \) copies of \( C \), respectively, are not isotopic for \( i \neq k \) (e.g. by arguing with the additivity of the

\(^2\)To make the term “same band” more precise, we could fix an abstract embedding of the surface \( F(\beta) \) throughout.
nonzero genus of $C$). It thus follows from [KM94] Theorem 1.1 that

$$\partial F_i = P_j(C \# C \ldots \# C)$$

and

$$\partial F_k = P_j(C \# C \ldots \# C)$$

are not isotopic if $i \neq k$. Again, it is important that the pattern $J$ has geometric winding number strictly greater than 1 under the assumptions of Lemma 2 for $F(\beta)$.

If the two components of $\partial B_\beta \cap \partial F(\beta)$ belong to different components $L_1$ and $L_2$ of the link $\partial F(\beta)$, then, by induction, the link $\partial F_i$ has components

$$L_1 \# C \# C \ldots \# C$$

and

$$L_2 \# C \# C \ldots \# C$$

and so again $\partial F_i$ and $\partial F_k$ are not isotopic if $i \neq k$.  

\[\square\]

Remark 3. A careful generalization of our proof of Theorem \ref{thm:main} and in particular Lemma \ref{lem:main} shows the following slightly stronger statement. Let $F$ be a quasipositive Seifert surface for a link $L$ other than an unlink. Then there exists an infinite family $\{\Sigma_i \times [0,1] \}_{i \geq 1}$ of smoothly and properly embedded 3-manifolds $\Sigma_i \times [0,1]$ in $S^3 \times [0,1]$ where each $\Sigma_i$ is a surface such that $\partial (\Sigma_i \times [0,1]) = F \times \{0\} \cup \Sigma_i \times \{1\}$ for some quasipositive Seifert surface $\Sigma_i$ with boundary $\partial \Sigma_i'_{\beta}$ non-isotopic to $\partial F = L$ and such that the boundaries $\partial \Sigma_i'_{\beta}$ and $\partial \Sigma_j'_{\beta}$ are non-isotopic for $i \neq j$.

3. PROOF OF LEMMA \ref{lem:main}

In this section we prove Lemma \ref{lem:main} which we recall here for the reader’s convenience.

**Lemma \ref{lem:main}**. Let $F(\beta)$ denote the canonical Seifert surface associated to a strongly quasipositive braid word $\beta$ such that $\partial F(\beta)$ is not an unlink. Then we can choose a cylinder $Z \subset S^3$ and an orientation-preserving diffeomorphism $\varphi$ of triples of manifolds with corners

$$\varphi: (Z, Z \cap F(\beta), Z \cap \partial F(\beta))$$

such that $B_\beta = Z \cap F(\beta)$ is a band of $F(\beta)$ corresponding to a positive band word of $\beta$ and such that $\gamma = \varphi^{-1} \left( (\partial D^2) \times \left\{ \frac{1}{2} \right\} \right)$ is a simple closed curve in $S^3 \setminus F(\beta)$ which is not null-homotopic in $S^3 \setminus \partial F(\beta)$. Moreover, either the two components of $\partial B_\beta \cap \partial F(\beta)$ belong to two different components of the link $\partial F(\beta)$ or we can assume that there exists some quasipositive Seifert surface $G$ that is a connected component of $F(\beta)$ with boundary a knot $J = \partial G$ (which is one of the components of $F(\beta)$) such that $B_\beta \subset G$ and $\gamma$ is not null-homotopic in $S^3 \setminus J$.

**Proof of Lemma \ref{lem:main}** Let $\beta = \prod_{k=1}^{\ell} \sigma_{i_k,j_k} \in B_n$ for some $n \geq 1$. We claim that one of the following is true.

Case 1: There exists a half-twisted band in $F(\beta)$ corresponding to one of the positive band words $\sigma_{i_k,j_k}$, $k \in \{1, \ldots, \ell\}$, of $\beta$ such that the boundary of this band intersected with $\partial F(\beta)$ has two components that belong to two different components of the link $\partial F(\beta)$.

Case 2: $F(\beta)$ is a disjoint union of quasipositive Seifert surfaces each of which has only one boundary component.
Here is the argument why: If the half-twisted bands in $F(\beta)$ are such that for each of them the boundary of the band intersected with $\partial F(\beta)$ has two components that belong to the same component of the link $\partial F(\beta)$, then for each of the disks in $F(\beta)$, there is a component of $\partial F(\beta)$ such that the entire boundary of the disk belongs to that component. All the bands emanating from a disk must belong to the same component of $\partial F(\beta)$ as the boundary of that disk, and so each connected component of $F(\beta)$ must have only one component of $F(\beta)$ (a knot) as its boundary.

Let us first assume that we are in Case 2, such that $F(\beta)$ is a disjoint union of quasipositive Seifert surfaces each of which has as boundary a knot. By assumption, $F(\beta)$ is not a union of disks. Let $G$ be one of the connected components of $F(\beta)$ that is not a disk. We claim that we can choose $Z$ and $\varphi$ as in (5) such that $B_\beta = Z \cap F(\beta)$ is a band of $G \subset F(\beta)$ corresponding to a positive band word of $\beta$ and such that $\gamma = \varphi^{-1}(\partial D^2 \times \{1\})$ is a simple closed curve in $S^3 \setminus F(\beta)$ which is not null-homotopic in $S^3 \setminus G$. The claim of Lemma 3 in this case will then follow from a more general statement provided in Lemma 4 below, which can be shown using a standard innermost circle argument. Note that quasipositive Seifert surfaces are of minimal genus [Rud93, KM93] and thus incompressible. For the reader’s convenience, we will prove Lemma 4 below.

**Lemma 4.** Let $F$ be an incompressible Seifert surface for a link $L$ and let $\gamma \subset S^3 \setminus F$ be a simple closed curve. If there exists a disk in $S^3 \setminus L$ with boundary $\gamma$, then there also exists a disk in $S^3 \setminus F$ with boundary $\gamma$.

So if we find $Z$ and $\varphi$ as in (5) such that $\gamma = \varphi^{-1}(\partial D^2 \times \{1\})$ is not null-homotopic in $S^3 \setminus G$, it is also not null-homotopic in $S^3 \setminus \partial G$. To conclude the proof of Lemma 3 in Case 2, it remains to show that this is always possible.

To that end, we claim that there exists a positive band word $\sigma_{i_3,j_3}$ in $\beta$ which fulfills the following condition: the core of the half-twisted band $B_\beta$ of $F(\beta)$ associated to $\sigma_{i_3,j_3}$ together with an arc in $G$ the interior of which misses $B_\beta$ unite to a simple closed curve $\eta$ in $G$ so that $\eta$ and a meridian of $B_\beta$ have linking number $\pm 1$. Under a diffeomorphism $\varphi: Z \to D^2 \times S^1$ as in (5), we can identify any of the half-twisted bands in $G$ with $[-\frac{1}{2}, \frac{1}{2}] \subset D^2 \times [0,1]$ for an appropriately chosen cylinder $Z \subset S^3$. A meridian of a band $B_\beta$ of $G$ for us is then a simple closed curve in $S^3 \setminus G$ which is isotopic to $\varphi^{-1}(\partial D^2 \times \{\frac{1}{2}\})$ under this identification and the core of $B_\beta$ is $\varphi^{-1}(\{0\} \times [0,1])$. If we find a band $B_\beta$ with the above requirements, the condition on the linking number of $\eta$ and the meridian $\gamma = \varphi^{-1}(\partial D^2 \times \{\frac{1}{2}\})$ of $B_\beta$ will imply that $\gamma$ cannot be null-homotopic in $S^3 \setminus G$.

The quasipositive Seifert surface $G$ deformation retracts onto a graph $\Gamma$ in $S^3$ where vertices of $\Gamma$ correspond to the disks of $G$ and edges of $\Gamma$ correspond to the bands of $G$, respectively. Since $G$ is not disk, $\Gamma$ is not a tree, hence there must exist an edge $e$ of $\Gamma$ such that $\Gamma \setminus e$ is not disconnected. This edge $e$ together with a path in $\Gamma$ connecting the vertices of $e$, but missing the interior of $e$, forms a simple closed curve in $\Gamma$ which has linking number $\pm 1$ with its meridian in $S^3 \setminus \Gamma$. For the desired positive band word $\sigma_{i_3,j_3}$, we can take the one corresponding to the edge $e$.

In summary, we have shown that in Case 2 we can choose $Z$ and $\varphi$ as in (5) such that $B_\beta = Z \cap F(\beta)$ is a band of $F(\beta)$ corresponding to a positive band word of $\beta$ that is contained in one of these Seifert surfaces $G$ and such that
\[ \gamma = \varphi^{-1} \left( (\partial D^2) \times \left\{ \frac{1}{2} \right\} \right) \] is a simple closed curve in \( S^3 \setminus F(\beta) \) that is not null-homotopic in \( S^3 \setminus G \) and therefore by Lemma 4 not null-homotopic in \( S^3 \setminus \partial G \).

Now suppose that we are in Case 1, so we can choose a cylinder \( Z \subset S^3 \) and an orientation-preserving diffeomorphism \( \varphi \) as in \( \mathcal{A} \) such that \( B_\beta = Z \cap F(\beta) \) is a band of \( F(\beta) \) corresponding to a positive band word of \( \beta \) where the two components of \( \partial B_\beta \cap \partial F(\beta) \) belong to two different components of the link \( \partial F(\beta) \). We claim that in this case \( \gamma = \varphi^{-1} \left( (\partial D^2) \times \left\{ \frac{1}{2} \right\} \right) \) is a simple closed curve in \( S^3 \setminus F(\beta) \) which is not null-homotopic in \( S^3 \setminus F(\beta) \) and thus, by Lemma 4, not null-homotopic in \( S^3 \setminus \partial F(\beta) \).

Similar as in the argument in Case 2 above, the quasipositive Seifert surface \( F(\beta) \) deformation retracts onto a graph \( \Gamma \) in \( S^3 \) where vertices of \( \Gamma \) correspond to the disks of \( F(\beta) \) and edges of \( \Gamma \) correspond to the bands of \( F(\beta) \), respectively. Consider the edge \( e \) of \( \Gamma \) that corresponds to the band \( B_\beta \). Since the two components of \( \partial B_\beta \cap \partial F(\beta) \) belong to two different components of \( \partial F(\beta) \), this edge must be part of a cycle in \( \Gamma \). This cycle is a simple closed curve in \( \Gamma \) which has linking number \( \pm 1 \) with its meridian in \( S^3 \), so the core of the band \( B_\beta \) together with a certain arc in \( F(\beta) \) unite to form a simple closed curve in \( F(\beta) \) which has linking number \( \pm 1 \) with the meridian of \( B_\beta \) and the claim follows.

\[ \square \]

**Proof of Lemma 4.** Let \( D \subset S^3 \setminus L \) be a disk with \( \partial D = \gamma \subset S^3 \setminus F \) and suppose that \( D \) intersects \( F \) nontrivially. Up to an ambient isotopy we can assume that \( D \) and \( F \) intersect transversally in \( S^3 \) [GP10]. Then \( D \cap F \) is a one-dimensional compact manifold, so a finite collection of simple closed curves. Using the 2-dimensional Schoenflies theorem [Rol03, Section 2A], each of these simple closed curves bounds a disk in \( D \). Let \( C \) be one of the simple closed curves in \( D \cap F \) which is innermost in the sense that the interior of the disk \( D' \) bounded by \( C \) in \( D \) misses \( F \). Since \( F \) is incompressible, \( C \) must also bound a disk \( D'' \) in \( F \). The union of \( D' \) and \( D'' \) forms a 2-sphere which, by the 3-dimensional Schoenflies theorem [Rol03, Section 2F], bounds a ball in \( S^3 \). We can push \( F \) along this ball to obtain a Seifert surface \( F' \) for \( L \) which is ambient isotopic to \( F \) and intersects \( D \) in less simple closed curves than \( F \). We repeat this process until we obtain a Seifert surface \( F'' \) for \( L \) which is ambient isotopic to \( F \) and disjoint from \( D \). In summary, up to an ambient isotopy we found the desired disk in \( S^3 \setminus F \). \[ \square \]

We conclude with the promised details on the construction in Remark 1.

**Remark 4.** We elaborate on how to construct an infinite family of concordant, pairwise non-isotopic strongly quasipositive knots using Whitehead doubles as outlined in Remark 1. To that end, let \( C \) be a nontrivial slice knot with maximal Thurston-Bennequin number \( \text{TB}(C) = -1 \), e.g. \( C = m \langle 9_{46} \rangle \) (see Section 2.1). For \( m \geq 1 \), let \( K_m \) be the connected sum of \( m \) copies of \( C \). Then for every \( m \geq 1 \), the knot \( K_m \) is slice (since \( C \) is) and by inductively using the formula \( \text{TB}(L_1 \# L_2) = \text{TB}(L_1) + \text{TB}(L_2) + 1 \) for any knots \( L_1, L_2 \) [EH03, LT03], we have \( \text{TB}(K_m) = -1 \). Note that \( K_m \) and \( K_n \) are not isotopic for \( m \neq n \), since \( K \) is nontrivial. Using the notation from [Hed07], we now define \( J_m := D_2(\langle K_m, -1 \rangle) \) as the positive \((1 \text{-})\)-twisted Whitehead double of \( K_m \). Then \( \{J_m\}_{m \geq 1} \) is the desired infinite family. Indeed, using \( \text{TB}(K_m) \geq -1 \), each \( J_m \) is strongly quasipositive by work of Rudolph (see e.g. [Rud05, 102.4]). Moreover, as \( K_m \) and \( K_n \) are not isotopic for \( m \neq n \), the knots \( J_m \) and \( J_n \) are not isotopic either for such \( m \) and \( n \).
On the other hand, $J_m$ and $J_n$ are concordant for every $m \neq n$ as $K_m$ and $K_n$ are. Indeed, as noted in the proof of Lemma 2 the satellite operation induces a well-defined map on the concordance group of which taking the positive twisted Whitehead double is a special case.

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