Maximal Steered Coherence and Its Conversion to Entanglement in Multiple Bosonic Reservoirs

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The remote control of coherence is a crucial step for its application in quantum computation. The maximal steered coherence (MSC) and its conversion to entanglement in multiple bosonic reservoirs are investigated. It is shown that the MSC decays with time in the Markovian regime and behaves as damped oscillations in the non-Markovian regime. The MSC can also be connected directly to the strength of non-Markovianity, through which it is shown that it can be noticeably enhanced by taking full advantage of the non-Markovian effects. Besides, the MSC can be converted completely to entanglement via optimal incoherent operation applied to the steered qubit and an incoherent ancilla which is immune to decoherence, and the generated entanglement is stronger than the shared entanglement in prior. These findings suggest a potential way for remotely generating and manipulating coherence and entanglement in noisy environments.

1. Introduction

As the fundamental characteristics differentiating a quantum system from its classical counterpart, quantum coherence has remained one of the research focuses of the quantum community for over a century.\(^{[1]}\) In the past few decades, quantum coherence further found its application in the emerging fields of quantum computation, quantum communication, and quantum metrology.\(^{[2]}\) Recently, it is attracting growing interest once again, motivated by the formulation of the resource theory of coherence which sets the stage for analyzing quantitatively the decoherence mechanism of the open systems.\(^{[3–5]}\) Within this framework, a number of coherence measures have also been introduced,\(^{[5–11]}\) starting from which researchers further studied explicitly its role in the tasks of state merging,\(^{[12]}\) deterministic quantum computation with one qubit,\(^{[13]}\) phase discrimination,\(^{[9,14]}\) subchannel discrimination,\(^{[10]}\) and the Deutsch–Jozsa algorithm.\(^{[15]}\)

Although quantum coherence is defined for a single-partite system, it is intimately related to different quantum correlations. Their interrelations could be revealed by dividing a composite system into different subsystems and then analyzing distribution of the coherence among these subsystems.\(^{[6,11,16–18]}\) Moreover, the maximal steered coherence (MSC) can be linked to discord-like correlations, for example, it vanishes for the zero-discord states.\(^{[19]}\) Here, by saying the MSC, we mean the steered coherence on \(AB\) maximized over the positive-operator-valued measure (POVM) on \(A\).

Quantum coherence of a system \(S\) could be converted to entanglement via incoherent operations on \(S\) and an ancilla.\(^{[6]}\) Moreover, the amount of coherence in a state can be enhanced by performing a unitary operation on it.\(^{[11–13]}\) Thus, for a bipartite system \(AB\) shared between two participants, one might realize the control of coherence on \(B\) via LOCC and the controlled creation of entanglement between \(B\) and an incoherent ancilla \(C\) by utilizing the MSC (or the MSC further maximized over the unitary transformations) on \(B\). Besides, the ancillary system can be initialized in a state that is immune to decoherence in general, so it may be able to suppress the detrimental effect of decoherence on entanglement.

In this paper, we investigate the MSC of two qubits \(A\) and \(B\) immersed in two groups of multiple bosonic reservoirs, aimed at revealing the (non-)Markovian effects triggered by different physical mechanisms on its behaviors and its conversion to entanglement. We will show that the MSC decays exponentially with time in the Markovian regime and behaves as damped oscillations of time in the non-Markovian regime. Besides, the entanglement converted from the MSC is stronger than the entanglement of \(AB\). This observation manifests efficiency of the active quantum operations on remote control of coherence and entanglement in noisy environments.

This paper is arranged as follows: In Section 2, we recall measures of the MSC; then in Section 3, we give solution of the multiple reservoirs and the non-Markovianity. In Sections 4 and 5, we present an analysis of the decay behaviors of the MSC and a
To quantify the net amount of coherence generated on generate, while $C$ entropy of coherence given by \[5\] where the infimum over \{\xi_i\} is introduced as $\rho_B$ may be degenerate, while $C_{\mu}^{\text{MC}}(\rho_B)$ represents a measure of coherence in $\rho_{B|m}$. In this paper, we will consider two well-accepted measures of coherence, that is, the $\xi_i$ norm of coherence and the relative entropy of coherence given by $C^{\text{RE}}(\rho_{B|m})$.

$$C_{\mu}^{\text{MC}}(\rho_{B|m}) = \sum_{ij} [\langle \xi_i | \rho_{B|m} | \xi_j \rangle]$$

$$C_{\text{RE}}^{\text{MC}}(\rho_{B|m}) = S[\rho_{B|m}|_{\text{diag}}] - S(\rho_{B|m})$$

where $\rho_{B|m}|_{\text{diag}}$ denotes the state obtained by deleting all the off-diagonal elements of $\rho_{B|m}$, and $S(\cdot)$ is the von Neumann entropy of the corresponding state. Of course, the MSC can also be quantified by other coherence measures, but their behaviors are qualitatively the same; thus, we will not present them here.

### 2. Maximal Steered Coherence

We consider the steering of coherence between two participants Alice (A) and Bob (B) who share a two-qubit state $\rho_{AB}$ in prior. First, Alice performs the POVM measurements $M$ on qubit $A$, after which the state of qubit $B$ collapses to

$$\rho_{B|M} = \text{tr}_A(\rho_{AB} \otimes 1)/P_M$$

where $P_M = \text{tr}(\rho_{AB} \otimes 1)$ is the probability of attaining $\rho_{B|M}$. To quantify the net amount of coherence generated on $B$, Bob chooses the reference basis $\{\xi_i\}$ spanned by the eigenbasis of $\rho_B = \text{tr}_A(\rho_{AB})$ in which there is no initial coherence on $B$. Hence, the MSC attainable by Bob will be given by $C^{\text{MC}}$(3)

$$C^{\text{MC}}(\rho_{AB}) = \inf_{\{\xi_i\}} \left\{ \max_{M_{\text{POVM}}} C_{\mu}^{\text{MC}}(\rho_{B|m}) \right\}$$

where the infimum over $\{\xi_i\}$ is introduced as $\rho_B$ may be degenerate, while $C_{\mu}^{\text{MC}}(\rho_{B|m})$ represents a measure of coherence in $\rho_{B|m}$. In this paper, we will consider two well-accepted measures of coherence, that is, the $\xi_i$ norm of coherence and the relative entropy of coherence given by $C^{\text{RE}}(\rho_{B|m})$.

$$C_{\text{RE}}^{\text{MC}}(\rho_{B|m}) = S[\rho_{B|m}|_{\text{diag}}] - S(\rho_{B|m})$$

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### 3. Solution of the Model

We consider a central system consisting of two noninteracting two-level atoms (serve as the qubits) labeled as $A$ and $B$. When they are coupled independently to two sets of multiple bosonic reservoirs as shown in Figure 1, the total Hamiltonian is given by $\hat{H} = \hat{H}_A + \hat{H}_B$, where the single “qubit+reservoir” Hamiltonian $\hat{H}_S$ ($S = A$ or $B$) reads (in units of $\hbar$)

$$\hat{H}_S = \frac{1}{2} \omega_0 \sigma_z + \sum_{n=1}^{N_S} \sum_{i,k} \left[ \omega_{n,k} b_{n,k}^\dagger b_{n,k} + g_{n,i} (b_{n,k} \sigma_x + \text{H.c.}) \right]$$

where $\omega_0$ is the transition frequency between the ground state $|0\rangle$ and the excited state $|1\rangle$ of the two qubits, $\sigma_z = (\sigma_+ + \sigma_-)/2$ are the raising and lowering operators, $\sigma_x$ are the three Pauli operators, and $b_{n,k} (b_{n,k}^\dagger)$ is the annihilation (creation) operator of the $n$th reservoir’s field mode $k$ with frequency $\omega_{n,k}$, while its coupling strength to the qubit is $g_{n,i}$. Moreover, $N_S$ is the number of reservoirs acting on $S$, which may be implemented by $N_S$ pairs of lossy cavity mirrors as sketched in Figure 1.

For the qubit $S$ ($S = A$ or $B$) being prepared initially in the state $\rho_S(0)$ and there is no initial correlation between $S$ and the reservoirs, its evolved state after tracing over the $N_S$ reservoirs can be obtained as[35,36]

$$\rho_S(t) = \left( \rho_S^{11}(0) |p_S(t)|^2 \quad \rho_S^{12}(0) |p_S(t)|^2 \quad 1 - \rho_S^{22}(0) |p_S(t)|^2 \right)$$

where $\rho_S^{ij}(0) = \langle i | \rho_S(0) | j \rangle$, and $p_S(t)$ is a time-dependent parameter determined by the spectra of these reservoirs. We will consider the reservoirs with the Lorentzian spectrum $J_n(\omega)$ for which $p_S(t)$ is analytically solvable. Here, $J_n(\omega) = \gamma_n \omega_n^2 / ((2\pi[\omega - \omega_0]^2 + \gamma_n^2))$, where $\gamma_n$ denotes the spectral width of the $n$th reservoir whose reciprocal determines its characteristic correlation time, while the reciprocal of $\gamma_n$ determines the relaxation time of the qubit.[17] In the following, we focus on the case that all the reservoirs are
for the optimal \(\rho_{\text{opt}}(0)\) (i.e., the eigenstates of \(\sigma_x\)), one has \(\bar{N}\) (r) = \(\rho_{\text{opt}}(0)\), which is the time derivative of the trace distance between \(\rho(t)\) and \(\rho_d(t)\). It increases with the increase of \(N_s\) when \(N_s \geq N_{\text{crit}}\). Besides, one can also quantify the non-Markovianity based on the backbone ratio of information, which we define it as

\[
\mathcal{N}_{\text{BRI}} = \frac{\int_{t_{1/2}}^T \epsilon(t, \rho_{\text{opt}}(0))dt}{\int_{t_{1/2}}^T \epsilon(t, \rho_{\text{opt}}(0))dt}
\]  

(7)

which is also larger than zero when \(N_s \geq N_{\text{crit}}\). Besides, it also increases with an increase in \(N_s\) when \(N_s \geq N_{\text{crit}}\).

4. MSC in Multiple Bosonic Reservoirs

While the authors in ref. [36] considered the issue of a single qubit transversally coupled to the multiple bosonic reservoirs, it is also meaningful to generalize this theoretical model to the two-qubit case for which one can further investigate effects of the multiple bosonic reservoirs on controlling quantum correlations. As a matter of fact, for two qubits coupled independently to two bosonic reservoirs with \(N_s = N_{\text{crit}} = 1\), the dynamical behaviors of entanglement,[35,39–41] discord-like correlations,[42–44] and entropic uncertainty relation,[45] have already been investigated, and it is found that the non-Markovian effect triggered by increasing the coupling strength is beneficial to them. Then, it is natural to ask whether the non-Markovian effect triggered by increasing the number of reservoirs acting on each qubit is beneficial for protecting coherence and entanglement of a state.

In this section, we explore behaviors of MSC for two qubits coupled independently to two groups of multiple reservoirs as shown in Figure 1. We will consider two slightly different cases: the case of \(N_s = N_{\text{crit}}\) for which we call it symmetric reservoirs and the case of \(N_s \neq N_{\text{crit}}\) for which we call it asymmetric reservoirs. We consider the initial Bell-like state \(|\Psi\rangle = \alpha|10\rangle + \beta|01\rangle\) \(|\alpha|^2 + |\beta|^2 = 1\). Note that although the two qubits might be initially quantum correlated depending on the parameters \(\alpha\) and \(\beta\), there is no direct interaction neither between the two qubits nor between the two multiple bosonic reservoirs. Hence, one can derive the evolving state of the two qubits based on the method given in ref. [39]. To be explicit, we write the elements of \(\rho(t)\) of Equation (5) as \(\rho^{\text{opt}}(t) = \sum_{i,j} S_{ij}(t)\rho_{ij}(0)\) (i.e., \(i, j, l, l' \in \{0, 1\}\)) from which one can obtain the nonzero \(S_{ij}^{\text{opt}}(t)\) as

\[
S_{11}^{\text{opt}}(t) = |p_{11}^{\text{opt}}|^2, S_{10}^{\text{opt}}(t) = p_{12}^*, S_{01}^{\text{opt}}(t) = p_{21}^*, S_{00}^{\text{opt}}(t) = |p_{22}^{\text{opt}}|^2
\]  

(9)

where \(S = A\) or \(B\). By substituting these into Equation (5) of ref. [39] and using the fact that \(p_{ij} \in \mathbb{R}\), one can obtain

\[
\rho_{\text{LA}}(t) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & |ap_{A}|^2 & a\beta |p_{APB}| & 0 \\
0 & a\beta |p_{APB}| & |\beta p_{PB}|^2 & 0 \\
0 & 0 & 0 & 1 - |ap_{A}|^2 - |\beta p_{PB}|^2
\end{pmatrix}
\]  

(10)

and we will focus on the case of \(a\beta \neq 0\) unless specifically stated based on the consideration that Alice cannot steer Bob’s coherence at all when \(a\beta = 0\).

For the two-qubit states, as has been explained in ref. [19], one only needs to take the maximization over the set of projective measurements \(M = \{\hat{I} + \hat{m} \cdot \sigma/2\}\), where \(\sigma = (\sigma_x, \sigma_y, \sigma_z)\) and \(\hat{m} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\), with \(\theta\) and \(\phi\) being the polar and azimuth angles, respectively. Then the postmeasurement state of qubit \(B\) can be obtained as

\[
\rho_{\text{M}} = \begin{pmatrix}
|p_{A}^2| \sin^2(\theta/2) & e^{-i\phi}p_{A}^2 \sin \theta \\
e^{-i\phi}p_{A}^2 \sin \theta & p_{M}^2 \\
e^{-i\phi}p_{A}^2 \sin \theta & 2p_{M} e^{-i\phi}p_{A}^2 \sin \theta \\
1 - |p_{A}^2|^2 \sin^2(\theta/2) & p_{M}^2
\end{pmatrix}
\]  

(11)

where \(p_{M} = [1 + (2|ap_{A}|^2 - 1) \cos \theta/2]\), so the \(l_1\) norm of MSC remains zero for \(a\beta = 0\), irrespective of \(\theta\) and \(\phi\). Otherwise, the optimal polar angle is \(\theta_{\text{opt}} = \arccos(1 - 2|ap_{A}|^2)\), while the azimuth angle can take any value. As a result, one has

\[
C_{l_1}^{\text{MSC}}(\rho_{\text{LA}}) = \begin{cases}
\sqrt{1 - |ap_{A}|^2} & \text{if } a\beta \neq 0 \\
0 & \text{if } a\beta = 0
\end{cases}
\]  

(12)
from which one can note that it is monotonic increasing functions of both $|p_A|$ and $|p_B|$. As has been shown in Section 3, the decrease of $|p_A|$ ($S = A$ or $B$) signifies an outflow of information to the reservoirs, while the increase of $|p_B|$ signifies a backflow of information to the system. Hence, for the initial state $|\Psi\rangle$ with $\alpha \beta \neq 0$, the backflow of information to either one of the multiple reservoirs is always beneficial for protecting the $l_1$ norm of MSC, but the decay rates with respect to $|p_A|$ and $|p_B|$ are somewhat different. More specifically, for the symmetric multiple reservoirs, one can obtain from Equation (12) that the decay rate of the $l_1$ norm of MSC with respect to $|p_A|$ ($p_B = p_A$) increases with the increase of $|p_A|$. For the asymmetric multiple reservoirs, it decays linearly with the decrease of $|p_B|$ and the decay rate is independent of $|p_B|$, while for $p_B \neq 0$, it shows a parabolic decrease with the decrease of $|p_B|$, but the decay rate is not a constant function of $|p_B|$ (see Figure 2a).

As for the relative entropy of MSC, it is also independent of the azimuth angle $\phi$, but it is difficult to obtain analytically the optimal polar angle $\theta_0$. So we have to optimize it numerically. Here, $C_{\text{re}}^{\text{msc}}(\rho_{AB})$ can be written as

$$C_{\text{re}}^{\text{msc}}(\rho_{AB}) = \max_{\rho_B} \left\{ S(\rho_{\|M})_{\text{diag}} - S(\rho_{\|M}) \right\}$$

and the corresponding numerical result is shown in Figure 2b, from which one can see that its dependence on $|p_A|$ and $|p_B|$ is similar to that of the $l_1$ norm of MSC. The only difference is that $C_{\text{re}}^{\text{msc}}(\rho_{AB})$ does not decrease linearly with the decreasing value of $|p_B|$.

In the following, we investigate the time dependence of the MSC. First, we consider the case of the symmetric Lorentzian reservoirs. For the initial state $|\Psi\rangle$ with $\alpha = \beta = 1/\sqrt{2}$ (we denote by it $|\Psi^\ast\rangle$ for conciseness of later presentation), we show in from which one can see that its dependence on $|p_A|$ and $|p_B|$ is similar to that of the $l_1$ norm of MSC. The only difference is that $C_{\text{re}}^{\text{msc}}(\rho_{AB})$ does not decrease linearly with the decreasing value of $|p_B|$.

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Next, we consider the time dependence of the MSC in the asymmetric Lorentzian reservoirs. In Figure 4, we show $C_{\text{re}}^{\text{msc}}(\rho_{AB})$ versus $\Delta t$ for $|\Psi^\ast\rangle$ with different $\gamma$. When both $N_A$ and $N_B$ are smaller than $N_{\text{cr}}$, $C_{\text{re}}^{\text{msc}}(\rho_{AB})$ shows a Markovian exponential decay; hence as expected, the MSC also decays exponentially with the time evolves (see the solid black lines displayed in Figure 4). When one set of the multiple reservoirs is the non-Markovian regime, it can be found from Figure 4a,b that for $N_A < N_B$, there are still revivals of the MSC, while for $N_A > N_B$, the MSC decays monotonically. Such a difference is rooted in the asymmetric property of the MSC and could be explained from Equation (12), as the enhancement of $|p_A|$ may not suffice to compensate the loss of $|p_B|$ for the system parameters in Figure 4(b). Of course, one may observe a very weak revival of the MSC by

![Figure 2](image1.png)

**Figure 2.** The MSC $C_{\text{msc}}^{\text{msc}}(\rho_{AB})$ ($\mu = l_1$ or re) versus $|p_A|$ and $|p_B|$ for the initial state $|\Psi^\ast\rangle$ for qubits A and B.

![Figure 3](image2.png)

**Figure 3.** The MSC $C_{\text{msc}}^{\text{msc}}(\rho_{AB})$ ($\mu = l_1$ or re) versus $\Delta t$ for the initial state $|\Psi^\ast\rangle$ with different $\gamma$, $N_A$, and $N_B$ in the symmetric Lorentzian reservoirs.

![Figure 4](image3.png)

**Figure 4.** $C_{\text{re}}^{\text{msc}}(\rho_{AB})$ versus $\Delta t$ for $|\Psi^\ast\rangle$ with different $\gamma$. When both $N_A$ and $N_B$ are smaller than $N_{\text{cr}}$, $C_{\text{re}}^{\text{msc}}(\rho_{AB})$ shows a Markovian exponential decay; hence as expected, the MSC also decays exponentially with the time evolves.
It is always weaker than the MSC of Equation (12) when qubit regime for $p_A$ term is determined by both also behaves as damped oscillations with the time evolves, but quantum-jump-based feedback.\[49\] Besides, the coherence might decrease under the same reservoir. For example, if qubit $A$ is uncorrelated with qubit $B$ (so Alice cannot steer its coherence) and is initialized in the state $|\Psi^+\rangle = |\beta\rangle_1 |\alpha\rangle_2$, the $l_1$ norm of coherence for the time-evolved state of $B$ will be given by $C_l[|\rho_B(t)|] = 2|\alpha^* \beta|$. It is always weaker than the MSC of Equation (12) when qubit $A$ is isolated from the reservoirs (i.e., $p_A = 1$). Even when qubit $A$ is immersed in the bosonic reservoirs, $C_l[|\rho_B(t)|]$ is still weaker than the MSC in the parameter region of $|\alpha|^2 < 1 - (1 - p_A^{1/2})/2p_A^{1/2}$.

5. Converting MSC to Entanglement

Quantum coherence and entanglement are expensive resources for quantum communication such as quantum cryptography\[53\] motivated by which great efforts have been devoted to generating long-lived coherence and entanglement in various physical systems, for example, the optomechanical system.\[54\] ultracold atomic ensembles,\[55,56\] and cold ion.\[57\] Moreover, the coherence measures defined within the resource theoretic framework could be measured experimentally with elaborately designed techniques\[58,59\] and are intimately related to quantum entanglement.\[3,4,60\] In particular, as was shown in ref. [6] by performing the cnot operation $\Lambda_{\text{CNOT}}$ on the qubit $B$ which is coherent and an ancillary qubit $C$ initialized in the incoherent state $|0\rangle$, with $B$ ($C$) being the control (target) qubit, these two qubits will become entangled. If one uses concurrence\[61,62\] as a measure of entanglement, then $C(\Lambda_{\text{CNOT}}|\rho_B \otimes |0\rangle\langle 0|) = C_l(\rho_B)_\gamma$ that is, all the coherence in $\rho_B$ is converted to entanglement of $BC$ via the incoherent operation $\Lambda_{\text{CNOT}}$.

The above finding shows a way to control the entanglement of Bob’s qubits. For the scenario we considered in Figure 1, the ancillary qubit $C$ is initialized in the ground state $|0\rangle$ and will be immune to the reservoirs. Hence, the generated entanglement in $BC$ equals to the $l_1$ norm of coherence in $\rho_{BC\text{M}}$. For the optimal state $\rho_{BC\text{M}},\text{opt}$ (i.e., the state $\rho_{BC\text{M}}$ with $\theta = \theta_0$), one has

$$C(\Lambda_{\text{CNOT}}|\rho_{BC\text{M},\text{opt}} \otimes |0\rangle\langle 0|) = C_{l,\text{opt}}(\rho_{BC\text{M},\text{opt}})$$

that is, the amount of created entanglement in $BC$ (measured by concurrence) equals exactly to the $l_1$ norm of MSC for $\rho_{AB}$. This gives an operational interpretation to the MSC.

From Equation (10) one can obtain the concurrence of $\rho_{AB}$ as $C(\rho_{AB}) = 2|\alpha^* \beta|$. As a result,

$$\frac{C(\rho_{AB})}{C(\Lambda_{\text{CNOT}}|\rho_{BC\text{M},\text{opt}} \otimes |0\rangle\langle 0|)} = 2|\alpha^* \beta| \sqrt{1 - |\alpha^* \beta|^2} \leq 1$$

Thus, the generated entanglement in $BC$ is always stronger than the entanglement of $\rho_{AB}(t)$ whenever the reservoirs are present (i.e., $|\rho_{AB}(t)| < 1$). Even for the ideal case (i.e., $\rho_{AB} = 1$), the equality holds only when $\rho_{AB}(t)$ belongs to one of the Bell states. In ref. [63], a scheme for recovering entanglement via local operations is proposed. Our scheme is different from that in ref. [63] as it suggests an efficient way for remotely, instead of locally, manipulating entanglement in noisy environments.

One might also concern the amount of generated entanglement in $BC$ without Alice’s steering, but for such a situation, one has $C(\Lambda_{\text{CNOT}}|\rho_{AB} \otimes |0\rangle\langle 0|) = 0$ for $\rho_{AB}(t)$ of Equation (10). This consolidates the fact that with the help of the prior shared entanglement, Alice can control efficiently the entanglement of Bob’s qubits.

Of course, the enhancement of the generated entanglement in $BC$ in comparison to $C(\rho_{AB})$ is at the expense of reducing the success probability. In fact, the optimal success probability $p_{\text{M,op}}$ (i.e., the probability of attaining $\rho_{BC\text{M},\text{opt}}$) is given by

$$p_{\text{M,op}} = 2|\alpha^* \beta|^2 \left(1 - |\alpha^* \beta|^2 \right)$$

then one can see that the probability cannot exceed 50%. To maximize the generated entanglement in $BC$, one can choose the initial state $|\Psi^+\rangle$ for which $|\alpha^* \beta|^2 \leq 1/2$. As a consequence, $p_{\text{M,op}}$ decreases with the decrease of $|\alpha^* \beta|$. For the case of qubit $A$ being isolated perfectly from the reservoirs, one has $p_{\text{M,op}} = 50%$.

Note that the coherence of a state could be enhanced by performing a unitary operation $U$ on it. For the single-qubit state $\rho$, the maximal coherence under unitary operations is given by
the explicit time dependences of $p_\alpha$ determined by the reservoir spectral density.

Finally, we remark that it is also of great interest to further consider the finite temperature reservoirs. Although it is hard to obtain the two-qubit density operator for this case due to its complexity, one could give a heuristic analysis by considering the Markovian case with $N_{\alpha B} = 1$ for which $\rho_{\alpha B}(t)$ can be obtained by solving the master equation in the Lindblad form, and $\rho_{\alpha B}(t)$ can be obtained in a similar way to that of Equation (10) (see refs. [65, 66] for more detail). Then for the initial Bell-like states, one can get an MSC behavior qualitatively similar to what we have seen in Sections 4 and 5. The only difference is that the MSC is monotonically decreased with an increase in the reservoir temperature. A general study on the details of the finite temperature effects for the non-Markovian multiple reservoirs is still needed. Besides, it is also worthy to consider the case of two qubits immersed in a common reservoir,[37,44] for which there will be reservoir-mediated interaction between them. The combined and intertwined effects of this indirect interaction and non-Markovianity may induce more rich dynamics of the MSC than that for two independent reservoirs, the details of which will be considered elsewhere.

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Research data are not shared.

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