1. INTRODUCTION

Ordinary differential equations play an important role in the study of physical systems. There are various approaches for the study of the properties of physical systems described by differential equations as well as to construct and determine exact and analytic solutions. Relativistic astrophysics is an active area employing various solution-generating techniques to solve highly nonlinear systems of differential equations. These equations arise in the modeling of static and non-static stellar objects within the framework of general relativity (or in modifications like Einstein-Gauss-Bonnet gravity, \( f(R) \) gravity, and Brans-Dicke theory, to name a few) where solutions of governing differential equations have played a key role.

The end states of gravitational collapse of bounded configurations have held the attention of astrophysicists since the pioneering work of Oppenheimer and Snyder\(^1\) in which they studied idealized collapse of a dust sphere. The Weak Cosmic Censorship Conjecture, first articulated by Penrose, forbids the existence of naked singularities arising from continued gravitational collapse. However, there have been many counterexamples put forward within the framework of Einstein's...
classical general relativity. The confirmation of classical general relativity as a cornerstone of gravitational theory was borne out in 2019 when the photograph of the shadow of a black hole was obtained, heralding a new frontier of theoretical predictions and observations. The discovery of the Vaidya solution paved the way for researchers to study dissipative collapse of stars. The boundary of the collapsing object divides spacetime into two distinct regions, $\mathcal{M}^+$, the interior region and $\mathcal{M}^+$, the exterior region described by the Vaidya solution. Early work on dissipative gravitational collapse can be attributed to Herrera et al. (see Bonnor et al and Herrera and Santos and references therein). In their investigations, they studied spherically symmetric, shear-free stellar objects undergoing dissipative gravitational collapse in the form of a radial heat flux. The junction conditions for the smooth matching of the interior spacetime to the exterior Vaidya solution were derived by Santos. The Santos junction conditions demonstrated that the pressure at the boundary of a collapsing, radiating star is nonzero and is proportional to the magnitude of the outgoing radial heat flux. This junction condition represents the conservation of momentum across the boundary of the collapsing star. Recently, the Santos junction conditions have been extended to include a dynamically unstable core with a general energy-momentum tensor describing an imperfect fluid with heat flux and null radiation with the exterior being described by the generalized Vaidya solution. Over the next two decades, the study of radiating stars has provided us with a rich insight into the end states of continued gravitational collapse, particularly with regards to time of formation of the horizon, temperature profiles, and relaxational effects related to causal heat flux. The shear-free models were subsequently extended to include the effects of shear viscosity. It has been demonstrated that the Chandrasekhar stability criterion for isotropic fluid spheres is modified in the presence of shear viscosity. Furthermore, in both the Newtonian and relativistic regimes, the shear viscosity decreases the instability of the stellar fluid.

The inclusion of shear, anisotropy, electromagnetic field, and rotation in the slow approximation have been fruitful in studying the thermodynamics of such systems. The impact of shear on the kinematics and dynamics of the collapse process has been addressed by several authors. The instability of the shear-free condition has been demonstrated by Herrera et al in which they showed that the shear-free condition may hold for a limited epoch of the collapse process. The presence of pressure anisotropies, density inhomogeneities, and dissipative fluxes can mimic shear-like effects. The inclusion of shear during dissipative collapse has led to interesting results when contrasted to the shear-free case. Shearing effects lead to higher core temperatures; it has been shown that horizon formation is delayed when shear is present. In an attempt to model shearing, radiating stars, Ivanov introduced the so-called horizon function which simplifies the boundary condition representing the temporal behavior of the model. The horizon function has a physical attribute in the sense that it is directly related to the surface redshift of the collapsing sphere. Once this function is determined from the boundary condition, the end state or possible outcome of continued gravitational collapse can be studied. The avoidance of the singularity was demonstrated by using a simple model of shear-free collapse in which the rate of collapse is balanced by the rate of energy emission to the exterior spacetime. This so-called horizon-free model, or Banerjee, Chatterjee, and Dadhich (BCD) model, forms the basis of the work contained in this paper. The horizon-free collapse in the presence of shear was studied in various models in which the gravitational potentials were highly simplified. The so-called Euclidean stars, in which the areal radius is equal to the proper radius, were shown to undergo horizon-free collapse. The BCD model has two inherent assumptions: The interior spacetime is shear free, and the gravitational potentials are separable in space and time. Works by Chan et al. attempted to generalize the shear-free model to include shear by demanding that the metric functions be separable. These models have several drawbacks such as the proper radius being independent of time or the resulting boundary condition rendered solvable via numerical methods.

In this piece of work, we investigate the dynamics for the master equation of the temporal equation of radiating stars. The equation that we are interested describes the Einstein field equations for a spherically symmetric line element

$$ds^2 = -A^2(r, t)dt^2 + B^2(r, t) \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right),$$

for a pressure isotropy fluid with an exterior solution the Vaidya metric such that to describe a radiative solution. Banerjee et al. suggested the metric ansatz $A(t, r) = 1 + \zeta_0 r^2, B(t, r) = R(t)$ where $\zeta_0$ is a positive constant. In this case, the field equations reduce to the second-order differential equation:

$$2R(t)\ddot{R}(t) + \dot{R}(t)^2 + a\dot{R}(t) = \beta,$$

where $a$ and $\beta$ are constants and the dot means derivative with respect to the time $t$. A special solution of the latter equation is the solution $R = -Ct$ where $C > 0$ is a constant, which describes a collapsing star. Recently, in Paliathanasis...
et al., new families of exact solutions for the master equation (2) were found using Lie symmetries. This body of work forms part of several investigations about dissipative collapse via Lie symmetries. Radiating stars with shear, in which the particle trajectories within the stellar fluid were geodesics, were studied using Lie symmetries. Several new solutions were obtained while other solutions were reduced to well-known cases studied earlier in the literature. The geodesic case with shear was extended to the most general shearing matter distribution radiating energy to the exterior spacetime. The method of Lie symmetries proved useful in obtaining four new classes of solutions, two of which included the horizon function and the Euclidean condition.

In the following, we study the dynamics and the stability properties for the master equation (2), and also, we show in detail how the method of Lie point symmetries and the singularity analysis can be used for the derivation of new analytic and exact solution for Equation (2).

The theory of Lie symmetries is a powerful approach for the investigation of invariant functions for differential equations. The novelty of Lie theory is that the infinitesimal representations of the finite transformations of continuous groups are considered, by moving from the group to a local algebraic representation, and to studying the invariance properties under them. The resulting invariant functions can be used for the reduction of order for a given ordinary differential equation and consequently for the derivation of solutions. These exact solutions that follow from the application of Lie symmetries are known as similarity solutions. The theory of Lie symmetries cover a range of applications in physical science and gravitational theory; see, for instance, previous studies and references therein.

The singularity analysis, which is mainly associated with the school of Painlevé, that is why also it is called Painlevé analysis, is another powerful mathematical approach for the derivation of solutions of differential equations. When a differential equation passes the singularity the analysis, we say that the differential equation possesses the Painlevé property. Nowadays, the application of the singularity analysis is summarized in ARS algorithm, from the initials of Ablowitz, Ramani, and Segur who established a systematic method for investigation of analytic solutions, inspired by the approach applied by Kovalevski for the determination of the third integrable case of Euler’s equations for a spinning top. The basic characteristic for a differential equation to possesses the Painlevé property is the existence of at least a movable singularity. In the singularity analysis, the analytic solution for a given differential equation is expressed by Painlevé series, and specifically, in our consideration, we shall write the analytic solution in the Puiseux series.

On the other hand, stability analysis provides us with an important tool to investigate the evolution of the given dynamical system. Indeed, from the stability analysis, we can investigate if a given solution is stable or not, while we can determine families of initial conditions such that specific behavior for the dynamical system to be stable. In addition, we can define constraints for the free parameters of the differential equations according to the stability of exact solutions. Hence, we can extract important information about the nature of the free parameters. In our consideration, for Equation (2), we can understand how the free parameters $\alpha$ and $\beta$ affect the dynamics. The plan of the paper is as follows.

In Section 2, we investigate the stability properties for the master equation (2) for the exact solution found in Banerjee et al., while we perform a detailed study of the global dynamics for the master equation. From the dynamical analysis, a new family of exact solutions is constructed. In Section 3, we investigate Equation (2) by applying Lie’s theory, as we study if Equation (2) possesses the Painlevé property. We find that the differential equation admits two Lie point symmetries that form the $A_{2,2}$ Lie algebra. Consequently, the application of the Lie symmetries provides two similarity transformations. Moreover, the application of the ARS algorithm indicates that the master equation (2) satisfies the Painlevé test, and the analytic solution is expressed by a right Painlevé series. However, for a specific relation between the two free variables $\alpha$, $\beta$, a new exact solution is determined. Finally, in Section 4, we discuss our results and draw our conclusions.

## 2 STABILITY ANALYSIS AND GLOBAL DYNAMICS

For the master equation (2), by convenience, we assume $\alpha > 0$. A special and simple solution to this equation is $R = -Ct$ defined for $-\infty < t \leq 0$, where $C > 0$ is a constant given by the positive roots of

$$-\beta + C^2 - aC = 0. \quad (3)$$

That is, for $\beta > 0$, we have only one positive root $C = \frac{1}{2} \left( \sqrt{\alpha^2 + 4\beta} + \alpha \right)$. For $\beta < 0$ and $\alpha^2 + 4\beta > 0$, we have two different positive roots: $C_- = \frac{1}{2} \left( \alpha - \sqrt{\alpha^2 + 4\beta} \right)$ and $C_+ = \frac{1}{2} \left( \sqrt{\alpha^2 + 4\beta} + \alpha \right)$.

To avoid ambiguities, we prefer to use the relation $\beta = C^2 - Ca$, and in the analysis, separate the cases $\beta < 0$ and $\beta \geq 0$. 

For the analysis of stability of the scaling solution \( R(t) = -Ct \), with \( C = \frac{1}{2} \left( \sqrt{\alpha^2 + 4\beta + \alpha} \right) > 0 \) in the interval \(-\infty < t < 0\), we use similar methods as in Liddle and Scherrer\(^{39}\) and Uzan\(^{40}\). For this purpose, the new logarithmic time variable \( \tau \) related to \( t \) through
\[
t = -e^{-\tau}, \quad -\infty < \tau < \infty, \quad (4)
\]
such that \( t \to -\infty \) as \( \tau \to -\infty \) and \( t \to 0 \) as \( \tau \to +\infty \) is defined.

Then, replacing the relation between \( t \) and \( \tau \) given by (4) in the expression \( R(t) \), we obtain a function of \( \tau \) from \( R(t) \) denoted and defined by
\[
\bar{R}(\tau) = R(-e^{-\tau}). \quad (5)
\]

Using again the transform (4), the scaling solution \( R(t) = -Ct \) can be re-expressed as a function of \( \tau \) as \( R_s(\tau) = Ce^{-\tau} \).

To compare a generic solution \( \bar{R}(\tau) \) with the scaling solution \( R_s(\tau) = Ce^{-\tau} \), we define a dimensionless function of \( \tau \) as the ratio
\[
u(\tau) = \frac{\bar{R}(\tau)}{R_s(\tau)}. \quad (6)
\]

Thus, the solution \( R_s(\tau) \) will correspond to the equilibrium point \( u = 1 \) in the new formulation of the master equation (2). The physical region corresponds to \( u \geq 0 \). Recall the physical solution \( R = -Ct \) is defined for \(-\infty < t \leq 0\), where \( C > 0 \) is a constant.

To reformulate the master equation (2) in terms of \( u(\tau) \), and its derivatives with respect to \( \tau \), we use the chain rule to obtain several differentiation rules. Let the time derivative with respect to \( \tau \) be denoted by a prime and the time derivative with respect to \( t \) be denoted by a dot. That is, for \( f(\tau) \), let
\[
f'(\tau) \equiv \frac{df(\tau)}{d\tau}. \quad (7)
\]

and for \( g(t) \), let
\[
g'(t) \equiv \frac{dg(t)}{dt}. \quad (8)
\]

Then,
\[
\bar{R}(t) = \frac{d\tau(t)}{dt} \bar{R}'(\tau) = e^\tau \bar{R}'(\tau), \quad \bar{R}(t) = e^{2\tau} \left( \bar{R}''(\tau) + \bar{R}'(\tau) \right), \quad (9)
\]

and
\[
R_s'(\tau)/R_s(\tau) = -1, \quad R_s(\tau) = Ce^{-\tau}. \quad (10)
\]

Therefore, by using the rules (9), and substituting expression (4), Equation (2) becomes
\[
-\beta + ae^\tau \bar{R}'(\tau) + e^{2\tau} \left( \bar{R}'(\tau)^2 + 2\bar{R}(\tau) \left( \bar{R}''(\tau) + \bar{R}'(\tau) \right) \right) = 0. \quad (11)
\]

The next step is to rewrite Equation (11) in terms of \( \nu(\tau) \) and its derivatives.

Solving Equation (6) for \( \bar{R} \), and then using the derivatives rules for products of functions and the chain rule, we obtain, subsequently,
\[
\bar{R}(\tau) = Ce^{-\tau}u(\tau), \quad (12)
\]
\[
\bar{R}'(\tau) = Ce^{-\tau} \left( u'(\tau) - u(\tau) \right), \quad (13)
\]
\[
\bar{R}''(\tau) = Ce^{-\tau} \left( u''(\tau) - 2u'(\tau) + u(\tau) \right). \quad (14)
\]

Substituting (12), (13), and (14) in Equation (11), and dividing by the overall factor \( e^{-\tau} \), we obtain
\[
2C^2u(\tau)u''(\tau) + C^2u'(\tau)^2 + Cu'(\tau)(\alpha - 4Cu(\tau)) + C(u(\tau) - 1)(-\alpha + Cu(\tau) + C) = 0, \quad (15)
\]

where we have used the relation \( \beta = C^2 - Ca \).

Defining the new function
\[
v(\tau) = u'(\tau), \quad (16)
\]
we obtain the dynamical system
\begin{equation}
    u'(t) = v(t),
\end{equation}
\begin{equation}
    v'(t) = -\left(\frac{(u(t) - 1)(-a + Cu(t) + C)}{2Cu(t)} + v(t)\right) \left(2 - \frac{a}{2Cu(t)}\right) - \frac{v(t)^2}{2u(t)}.
\end{equation}

The scaling solution \( R_{s}(t) = Ce^{-\tau} \) corresponds to the equilibrium point \( u = 1, v = 0 \).

Defining \( \epsilon \) through \( u = 1 + \epsilon \), we obtain the final dynamical system
\begin{equation}
    \epsilon'(t) = v(t),
\end{equation}
\begin{equation}
    v'(t) = \frac{\epsilon(t)(a - Ce(\epsilon(t) - 2C))}{2C(\epsilon(t) + 1)} + v(t)\left(2 - \frac{a}{2C\epsilon(t) + 2C}\right) - \frac{v(t)^2}{2(\epsilon(t) + 1)},
\end{equation}

where the equilibrium point is translated to the origin.

The linearization matrix is defined by
\begin{equation}
    J(\epsilon, v) = \begin{pmatrix}
    \frac{1}{2} \left(\frac{(\epsilon+1)(C-C+a)}{C(\epsilon+1)^2} - 1\right) & -1 + \frac{a}{2C} \\
    2 - \frac{a}{2C} & \frac{1}{2C} - 1 - \frac{a}{2C}
    \end{pmatrix}.
\end{equation}

Then, linearizing around the equilibrium point, \( \epsilon = 0, v = 0 \), we obtain
\begin{equation}
    \begin{pmatrix}
    \epsilon'(t) \\
    v'(t)
    \end{pmatrix} = \begin{pmatrix}
    0 \\
    -1 + \frac{a}{2C} \\
    2 - \frac{a}{2C}
    \end{pmatrix} \begin{pmatrix}
    \epsilon(t) \\
    v(t)
    \end{pmatrix}.
\end{equation}

The linearization matrix
\begin{equation}
    J(0, 0) = \begin{pmatrix}
    0 \\
    \frac{a}{2C} - 1 - \frac{a}{2C}
    \end{pmatrix}
\end{equation}

has eigenvalues \( \left\{ 1, 1 - \frac{a}{2C} \right\} \).

Assume first \( \beta \geq 0 \). In this case, the origin is always unstable as \( \tau \to +\infty \) due to \( 2C = a + \sqrt{a^2 + 4\beta} > a \). That is, the origin is stable as \( \tau \to -\infty \).

An additional equilibrium point is
\begin{equation}
    \epsilon = \frac{a}{C} - 2 < 0, \ u = \frac{a}{C} - 1, \ v = 0.
\end{equation}

Evaluating the linearization matrix \( J \left( \frac{a}{C} - 2, 0 \right) \), we obtain the eigenvalues \( \left\{ 1, \frac{2C-a}{2(C-a)} \right\} \). Due to \( 2C-a > 0 \), it is unstable as \( \tau \to \infty \). Indeed for \( 0 < \frac{a}{C} < C < a \), it is a saddle, whereas for \( C > a > 0 \), it is an unstable node. The last conditions is forbidden due to the physical condition \( u \geq 0 \) evaluated at \( (u, v) = \left( \frac{a}{C} - 1, 0 \right) \) implies \( a \geq C \).

Now, let us study the case \( \beta < 0, a < -2\sqrt{-\beta} \) or \( \beta < 0, a > 2\sqrt{-\beta} \). Henceforth, we have two solutions
\begin{equation}
    R_{s\pm}(t) = -C_{s\pm}t(t) = C_{s\pm}e^{-\tau},
\end{equation}

where
\begin{equation}
    2C_{s\pm} = a \pm \sqrt{a^2 - 4|\beta|}.
\end{equation}

Observe that \( 2C_{s} > a \) implies that \( R_{s}(t) \) is an unstable solution (unstable node) as \( \tau \to \infty \). Due to \( a > 2C_{s} > 0 \), \( R_{s}(t) \) is an unstable (saddle) solution.

Figure 1 shows the phase plot of system (17), (18) for some \( a > 0, \beta > 0 \) and the choice \( 2C := a + \sqrt{a^2 + 4\beta} > a \). The origin represents the solution \( R_{s}(t) = Ce^{-\tau} \), which is unstable (node).

Figure 2 shows the phase plot of system (17), (18) for some \( a > 0, \beta < 0 \) and the choice \( 2C_{s} := a - \sqrt{a^2 - 4|\beta|} \). The origin represents the solution \( R_{s}(t) = C_{s}e^{-\tau} \), which is unstable (saddle).

Figure 3 shows the phase plot of system (17), (18) for some \( a > 0, \beta < 0, a^2 + 4\beta \geq 0 \) and the choice \( 2C_{s} := a + \sqrt{a^2 - 4|\beta|} \). The origin represents the solution \( R_{s}(t) = C_{s}e^{-\tau} \), which is unstable (node).

As can be seen in Figures 1–3, there are nontrivial dynamics as \( (u, v) \) are unbounded.
2.1 | Dynamics as \((u, v)\) are unbounded

Assume that there are \(u_0 > 0\), and a coordinate transformation \(\phi = h(u)\), with inverse \(h^{-1}(\phi)\), which maps the interval \([u_0, \infty)\) onto \((0, \delta]\), where \(\delta = h(u_0)\), satisfying \(\lim_{u \to \infty} h(u) = 0\), and has the following additional properties:

1. \(h\) is \(C^{k+1}\) and strictly decreasing,
2. 
   \[
   \bar{h}'(\phi) = \begin{cases} 
   h'(h^{-1}(\phi)), & \phi > 0, \\
   \lim_{\phi \to \infty} h'(\phi), & \phi = 0
   \end{cases}
   \]
   \(27\)

   is \(C^k\) on the closed interval \([0, \delta]\), and
3. \(\frac{dh}{d\phi}(0)\) and higher derivatives \(\frac{d^m h}{d\phi^m}(0)\) satisfy
   \[
   \frac{d\bar{h}'}{d\phi}(0) = \frac{d^m h'}{d\phi^m}(0) = 0.
   \]
   \(28\)

   It can be proved using the above conditions that

   \[
   \lim_{\phi \to 0} \frac{1}{h^{-1}(\phi)} = 0,
   \]
   \(29\)

   \[
   \lim_{\phi \to 0} \frac{h'(h^{-1}(\phi))}{\phi} = 0,
   \]
   \(30\)

   \[
   \lim_{\phi \to 0} \frac{h''(h^{-1}(\phi))}{h'(h^{-1}(\phi))} = 0.
   \]
   \(31\)
In the following, we say that \( g \) is well behaved at infinity (WBI) of exponential order \( N \), if there is \( N \) such that

\[
\lim_{u \to \infty} \left( \frac{g'(u)}{g(u)} - N \right) = 0. \tag{32}
\]

Let \( g \) be a WBI function of exponential order \( N \), and then, exponential dominated means, for all \( \lambda > N \),

\[
\lim_{u \to \infty} e^{-\lambda u}g(u) = 0. \tag{33}
\]

From

\[
\lim_{\phi \to 0} \frac{h''(h^{-1}(\phi))}{h'(h^{-1}(\phi))} = 0, \tag{34}
\]

it follows that \( g(u) = \frac{1}{h'(u)} \) is WBI of exponential order 0, that is, \( \lim_{u \to \infty} \frac{g'(u)}{g(u)} - N = 0 \) for \( N = 0 \), and hence, it is exponential dominated. This implies in turn that \( \frac{1}{h(u)} \) is also exponential dominated. The function \( h(u) \) must therefore obey the following condition: For all \( k > 0 \),

\[
\lim_{u \to \infty} \frac{\phi^k}{h'(u)} = \lim_{u \to \infty} \frac{\phi^k}{h(u)} = 0. \tag{35}
\]

In general, we can obtain functions \( \phi = h(u) \) satisfying the above Conditions 1–3 and previously commented facts if we demand the existence of \( n > 1 \) such that the functions

\[
\frac{1}{h'(h^{-1}(\phi))}, \frac{h'(h^{-1}(\phi))}{\phi}, \frac{h''(h^{-1}(\phi))}{h'(h^{-1}(\phi))} \tag{36}
\]
FIGURE 3  Phase plot of (17), (18) for some $\alpha > 0, \beta < 0$ and the choice $2C_+ := \alpha + \sqrt{\alpha^2 - 4|\beta|}$. The origin represents the solution $R_s(\tau) = C_+ e^{-\tau}$, which is unstable (node). The physical region corresponds to $u \geq 0$ [Colour figure can be viewed at wileyonlinelibrary.com]

behave as $\mathcal{O}(\phi^m)$, and

$$h^{(m)}(h^{-1}(\phi)) \sim \mathcal{O}(\phi^{(m+1)}), \text{ } m \in \mathbb{N}, \text{ } m \geq 1,$$

as $\phi \to 0$, where the superscript $(m)$ means $m$th derivative with respect the argument. Let be defined

$$\theta = 1 - u + v.$$  \hfill (38)

Then, we obtain

$$\phi' = \overline{h'}(\phi)(h^{-1}(\phi) + \theta - 1),$$  \hfill (39)

$$\theta' = \frac{(2\alpha - u)\theta}{2Ch^{-1}(\phi)} - \frac{\theta^2}{2h^{-1}(\phi)},$$  \hfill (40)

where $\overline{h'}(\phi)$ is defined in (27) and $2C - \alpha > 0$. The system (39), (40) defines a flow in the phase region

$$\Omega_{\delta} := \{ (\phi, \theta) \in \mathbb{R}^2 : 0 < \phi < h(\delta^{-1}), \theta \in K \},$$  \hfill (41)

where $K$ is a compact set, such that $\Omega_{\delta}$ is a positive invariant set for large $\tau$.

The system (39), (40) admits a curve of equilibrium points $L : (\phi, \theta) = (0, \theta^*)$ parametrized by $\theta^*$ that is approached as $\tau \to \infty$ (for bounded $\theta$). This curve of equilibrium points is represented by a red dashed line in Figures 4–6.
The linearization matrix of system (39) and (40) at a generic point \((\phi, \theta)\) is

\[
J(\phi, \theta) = \begin{pmatrix}
1 + \left(\frac{\theta h^{-1}(\phi)}{h(h^{-1}(\phi))} \frac{h''(h^{-1}(\phi))}{h(h^{-1}(\phi))} - \frac{\alpha + 2C(\theta - 1)}{2Ch^{-1}(\phi)} \right) \\
1 + \left(\frac{\theta h^{-1}(\phi)}{h(h^{-1}(\phi))} \frac{h''(h^{-1}(\phi))}{h(h^{-1}(\phi))} + \mathcal{O}(\phi^n) \frac{h'(h^{-1}(\phi))}{\frac{\alpha + 2C(\theta - 1)}{2Ch^{-1}(\phi)}} \right) \\
1 + \left(\frac{\theta h^{-1}(\phi)}{h(h^{-1}(\phi))} \frac{h''(h^{-1}(\phi))}{h(h^{-1}(\phi))} + \mathcal{O}(\phi^n) \frac{\mathcal{O}(\phi^n)}{\mathcal{O}(\phi^n)} \right)
\end{pmatrix}
\]

as \(\phi \to 0\).

We have,

\[
J(\phi, \theta) = \begin{pmatrix}
1 + \left(\frac{\theta h^{-1}(\phi)}{h(h^{-1}(\phi))} \frac{h''(h^{-1}(\phi))}{h(h^{-1}(\phi))} + \mathcal{O}(\phi^n) \frac{h'(h^{-1}(\phi))}{\frac{\alpha + 2C(\theta - 1)}{2Ch^{-1}(\phi)}} \right) \\
\end{pmatrix}
\]
has characteristic polynomial

\[

t(\phi)h'(\phi) - \frac{h^{(-1)}(\phi)h''(h^{(-1)}(\phi))}{h'(h^{(-1)}(\phi))} - \lambda + O(\phi^n) - \frac{(\alpha + C(\theta - 2))\theta}{2Ch^{(-1)}(\phi)2} = 0. \tag{44}
\]

That is,

\[

\lambda \left( -1 - \frac{h^{(-1)}(\phi)h''(h^{(-1)}(\phi))}{h'(h^{(-1)}(\phi))} \right) + O(\phi^n) = 0, \tag{45}
\]

with eigenvalues \( \left\{ 1 + \frac{h^{(-1)}(0)h''(h^{(-1)}(0))}{h'(h^{(-1)}(0))}, 0 \right\} \) as \( \phi \to 0 \). Therefore, the line \( L \) of equilibrium points is normally hyperbolic.

A set of nonisolated equilibrium points is said to be normally hyperbolic if the eigenvalues with zero real part correspond to eigenvectors that are tangent to the set. By definition, any point on a set of nonisolated singular points will have at least one eigenvalue zero. Then, all points in the set are nonhyperbolic. However, the stability of a normally hyperbolic set can be completely classified by considering the signs of eigenvalues in the remaining directions (i.e., for a curve, in the remaining \( n - 1 \) directions) (see Aulbach,\(^4^1\) p. 36). Therefore, the stability condition will be verified as \( \phi \to 0 \) is \( h^{(-1)}(0)h''(h^{(-1)}(0)) < -1 \).

Setting, for example, \( h(u) = u^{-1/2} \), with \( n > 1 \), which satisfies the previous Conditions 1–3, we obtain

\[

\phi' = -\frac{\phi}{n} + \left( \frac{1 - \theta}{n} \right) \phi^{n+1}, \tag{46}
\]

\[

\theta' = \phi^n \left( 1 - \frac{\alpha}{2C} \right) \theta - \frac{\theta^2}{2}. \tag{47}
\]

The curve of equilibrium points \( L : (\phi, \theta) = (0, \theta^*) \) as \( \tau \to \infty \) for bounded \( \theta \) has eigenvalues \( \left\{ -\frac{1}{n}, 0 \right\} \). Therefore, it is normally hyperbolic and stable.

### 2.2 Global dynamics

Defining the compact variables

\[

\Phi = \frac{2 \arctan(\phi)}{\pi}, \quad \Theta = \frac{2 \arctan(\theta)}{\pi},
\]

we obtain

\[

\phi' = \sin(\pi \Phi) \left( -\left( \tan \left( \frac{\pi \phi}{2} \right) - 1 \right) \right),
\]

\[

\phi' = \tan^{\pi n} \left( \frac{\pi \phi}{2} \right) (2C - \alpha) \sin(\pi \Theta) + C(\cos(\pi \Theta) - 1) \tag{49}
\]

\[

\theta' = \frac{\tan^{\pi n} \left( \frac{\pi \phi}{2} \right) (2C - \alpha) \sin(\pi \Theta)}{2\pi C}. \tag{50}
\]

In this coordinates, the points with \( \Phi = 0 \) correspond to \( u \to \infty \). The points with \( \Phi = \pm 1 \) are representations of \( u \to 0^+ \) or \( u \to 0^- \), respectively. Moreover, \( \Theta = \pm 1 \) are representations of \( \theta = 1 - u + v \to \pm \infty \). As before, the physical region corresponds to \( \Phi \geq 0 \) (corresponding to \( u \geq 0 \)).

Finally, if \( \theta \) is bounded as \( \tau \to \infty \) (which we will have so), we would have from (46) and (47) that

\[

\phi' = -\frac{\phi}{n} + O(\phi^{n+1}), \tag{51}
\]

\[

\theta' = \phi^n \left( 1 - \frac{\alpha}{2C} \right) \theta - \frac{\theta^2}{2} + O(\phi^{n+1}). \tag{52}
\]

The asymptotic equations (51), (52) as \( \phi \to 0 \) are integrable with solution

\[

\begin{pmatrix}
\phi(\tau) \\
\theta(\tau)
\end{pmatrix} = \begin{pmatrix}
\frac{e^{-\tau}c_1}{2C} \\
\frac{c - e^{-(\tau + \frac{\alpha^2}{2C}) - 2c_2}}{2C}
\end{pmatrix}, \tag{53}
\]
converging to $L : (\phi, \theta) = (0, \theta^*)$ as $\tau \to \infty$ for bounded $\theta$.

Figure 4 shows the phase plot of system (49), (50) for some $\alpha > 0, \beta > 0, n = 2$, and the choice $2C := \alpha + \sqrt{\alpha^2 + 4\beta}$. The red dashed line is a stable line of equilibrium points $L : (\phi, \theta) = (0, \theta^*)$.

Figure 5 shows the phase plot of system (49), (50) for some $\alpha > 0, \beta > 0, n = 2$, and the choice $2C_- := \alpha - \sqrt{\alpha^2 - 4|\beta|}$. The red dashed line is a stable curve of equilibrium points $L : (\phi, \theta) = (0, \theta^*)$.

Figure 6 shows the phase plot of system (49), (50) for some $\alpha > 0, \beta > 0, n = 2$, $\alpha^2 + 4\beta \geq 0$ and the choice $2C_+ := \alpha + \sqrt{\alpha^2 - 4|\beta|}$. The red dashed line is a stable curve of equilibrium points $L : (\phi, \theta) = (0, \theta^*)$.

All these plots illustrate our analytical findings. These are (i) the solution of (2) $R = -Ct$ defined for $-\infty < t \leq 0$, where $C > 0$ is a fixed constant, is unstable. (ii) The curve of equilibrium points $L : (\phi, \theta) = (0, \theta^*)$ (i.e., $u \to \infty$, and $v \to \infty$, in such a way that $1 - u + v \to \theta^*$) is stable as $\tau \to \infty$ for bounded $\theta$.

Result (ii) means that, as $\tau \to \infty$, we have

$$1 - u(\tau) + u'(\tau) = \theta^*, \ u(\infty) = \infty. \quad (54)$$

The solution of (54) is

$$u(\tau) = c_1 e^{\tau} - \theta^* + 1, \ c_1 \neq 0. \quad (55)$$

Then,

$$\dot{R}(\tau) = R_0(\tau)u(\tau) = Ce^{-\tau}u(\tau) = C(c_1 - (\theta^* - 1)e^{-\tau}). \quad (56)$$

Hence,

$$R(\tau) = C(c_1 + (\theta^* - 1)t). \quad (57)$$

Substituting back in (2), we have that in order of (57) to be an exact solution for (2), we must impose the compatibility condition:

$$C\theta^*(\alpha + C(\theta^* - 2)) = 0. \quad (58)$$

We have some specific solutions when $\theta^* \in \left\{ 0, 2 - \frac{\alpha}{C} \right\}$. 

![Figure 5](https://wileyonlinelibrary.com)
However, recall that $\theta^*$ is an arbitrary constant value by definition of line $L$. So the natural condition is

$$C = \frac{\alpha}{2 - \theta^*}. \tag{59}$$

Then, the solution of (2) given by

$$R(t) = \frac{ac_1}{2 - \theta^*} + \frac{\alpha(\theta^* - 1)t}{2 - \theta^*}, \ c_1 \neq 0, \tag{60}$$

defined in the semi-infinite interval $-\infty < t \leq 0$, is stable as $t \to 0^-$ ($\tau \to +\infty$). Finally,

$$\lim_{t \to 0^-} R(t) = \frac{ac_1}{2 - \theta^*} \neq 0 \tag{61}$$

by construction.

3 | LIE SYMMETRIES AND SINGULARITY ANALYSIS

In the following, we discuss the application of Lie's theory and the singularity analysis for the derivation of exact and analytic solutions for the master equation (2).

3.1 | Lie symmetries

Consider the function $\Phi$ that describes the map of a one-parameter point transformation such as $\Phi(R(t)) = R(t)$ with infinitesimal transformation

$$t' = t + \epsilon \xi(t, R), \tag{62}$$

$$R' = R + \epsilon \eta(t, R), \tag{63}$$

and generator $X = \frac{\partial \epsilon}{\partial \xi} \partial_t + \frac{\partial \epsilon}{\partial R} \partial_R$, where $\epsilon$ is the parameter of smallness, $t$ is the independent variable, and $R(t)$ is the dependent variable.
Assume $R(t)$ is a solution for the differential equation $H \left( t, R, \dot{R}, \ddot{R} \right) = 0$. Then, under the one-parameter map $\Phi$, function $R' \left( t' \right) = \Phi \left( R(t) \right)$ is a solution for the differential equation $H$, if and only if the differential equation is also invariant under the action of the map, $\Phi$, that is, the following condition holds:\(^{27}\)

$$\Phi \left( H \left( t, R, \dot{R}, \ddot{R} \right) \right) = 0. \quad (64)$$

For every map $\Phi$ in which condition (64) holds, it means that the generator $X$ is a Lie point symmetry for the differential equation, while the following condition is true:

$$X^{[2]} \left( H \left( t, R, \dot{R}, \ddot{R} \right) \right) = 0. \quad (65)$$

Vector field $X^{[1]}$ describes the first extension of the symmetry vector in the jet-space of variables, $\{ t, R, \dot{R}, \ddot{R} \}$ defined as

$$X^{[2]} = X + \eta^{[1]} \partial_{\dot{R}} + \eta^{[2]} \partial_{R}, \quad (66)$$

with $\eta^{[1]} = \dot{\eta} - \dot{R} \xi$ and $\eta^{[2]} = \ddot{\eta}^{[1]} - \ddot{R} \xi$.

The existence of a Lie symmetry for a given differential equation indicates the associated Lagrange's system, $\frac{dt}{\xi} = \frac{dR}{\eta}$, in which the solution of this system provides the invariant functions that can be used to reduce the order of the ordinary differential equation.

Let us now turn our attention to the boundary condition (2). By applying Lie's theory, it follows that Equation (2) is invariant under the infinitesimal transformation:\(^{27}\)

$$t \rightarrow t + \varepsilon (a_1 + a_2 t), \quad (67)$$

$$R \rightarrow R + \varepsilon (a_2 R), \quad (68)$$

where $\varepsilon$ is the infinitesimal parameter, that is, $\varepsilon^2 \approx 0$.

Hence, Equation (2) admits as Lie point symmetries the vector fields $X_1 = \partial_t$ and $X_2 = t \partial_t + R \partial_R$, with commutator $[X_1, X_2] = X_1$. The Lie symmetries $\{ X_1, X_2 \}$ form the $A_{2,2}$ Lie algebra in the Morozov–Mubarakzyanov classification scheme.\(^{42}\) We continue by applying the corresponding Lie invariants to reduce the order of the master equation (2).

From the Lie point symmetry $X_1$, we define the differential invariants $y = \dot{R}$, $x = R$. Thus, by assuming $x$ to be the new independent variable and $y = y(x)$, Equation (2) is written as

$$2xy \frac{dy(x)}{dx} + y^2(x) + a y(x) - \beta = 0. \quad (69)$$

Equation (69) admits the Lie symmetry vector $L^1 = \left( y(x) + a y - \frac{\beta}{y(x)} \right) \partial_y$ which provides the Lie's integration factor $\mu = \left( 2x \left( y^2(x) + a y - \beta \right) \right)^{-1}$; hence, by multiplying Equation (69) with it, we end with the equation

$$\int \frac{y}{(y^2 + ay - \beta)} \, dy = - \frac{dx}{2x}, \quad (70)$$

that is,

$$\ln \left( x - x_0 \right) = - \ln \left( y^2 + ay - \beta \right) - \frac{2a}{\sqrt{a^2 + 4\beta}} \arctanh \left( \frac{2y + a}{\sqrt{a^2 + 4\beta}} \right). \quad (71)$$

When $\beta = -\frac{a^2}{4}$, solution (71) is written:

$$\ln \left( 2 + y + a \right) + \frac{a}{2y + a} = - \frac{1}{2} \ln \left( x - x_0 \right). \quad (72)$$
Moreover, application of the symmetry vector $X_2$ in (2) provides the first-order ordinary differential equation:

$$2z(y(z) - z) \frac{dy(z)}{dz} + y^2(z) + ay(z) - \beta = 0. \quad (73)$$

The latter equation belongs to the family of Abel equations of the second kind. Equation (73) can be integrated similarly with Equation (69) with the derivation of Lie’s integration factor. For a constant value $y = y_0$, with $y_0^2 + ay_0 - \beta = 0$, it is clear that the exact solution of Banerjee et al.\textsuperscript{23} is recovered.

### 3.2 Singularity analysis

The ARS algorithm\textsuperscript{35-37} has three basic steps that we briefly discuss. The first step is based on the determination of the leading-order behavior, at least in terms of the dominated term. The coefficient of the leading-order term may or may not be explicit. This indicates the existence of movable singularities for the given differential equation. A second step is the determination of the exponents at which the arbitrary constants of integration enter. This step provides information about the existence of integration constants for the differential equation. Finally, the third step is called the consistency test, where we substitute an expansion up to the maximum resonance into the full equation to check if solves the equation.

For the singularity analysis to work, the exponents of the leading-order term need to be a negative integer or a nonintegral rational number, while the resonances should be rational numbers, while one of the resonances should be $-1$, which warrant the singularity is a movable pole. Excluding the generic resonance $-1$, the analytic solution is expressed by a right Painlevé series if the rest of the resonances are nonnegative, for a left Painlevé series, the resonances must be nonpositive while for a full Laurent expansion, the resonances have to be mixed. Clearly, for a second-order ordinary equation, the possible Laurent expansions are left or right Painlevé series.

We apply the ARS algorithm for Equation (2); from the first step, we determine the leading-order term to be $R_{\text{leading}}(t) = R_0(t - t_0)^{\frac{3}{2}}$, where $t_0$ indicates the location of the movable singularity and $R_0$ is arbitrary. From the second step, the resonances are derived to be $s_1 = -1$ and $s_4 = 4$, which means that the analytic solution of (2) can be expressed in terms of the right Painlevé series

$$R(t) = R_0(t - t_0)^{\frac{3}{2}} + R_1(t - t_0) + R_2(t - t_0)^{\frac{5}{2}} + R_3(t - t_0)^{\frac{7}{2}} + \ldots . \quad (74)$$

We replace in (2) from where we find that

$$R_1 = -\frac{3a}{4}, R_2 = \frac{9}{320R_0} \left(3a^2 + 16\beta\right), R_2 = \frac{3a}{320R_0^2} \left(3a^2 + 16\beta\right), \ldots . \quad (75)$$

In the special case in which $(3a^2 + 16\beta) = 0$, that is, $\beta = -\frac{3a^2}{16}$, we find that $R_I = 0, I > 1$; thus, we end with the closed-form solution

$$R_i(t) = R_0(t - t_0)^{\frac{3}{2}} - \frac{3a}{4}(t - t_0), \quad \beta = -\frac{3a^2}{16}, \quad (76)$$

which can be seen as extension of the exact solution of Banerjee et al.\textsuperscript{23} Indeed for large values of $t - t_0$, $R_i(t) \approx \frac{3a}{4}(t - t_0)$, however, for small values of $(t - t_0)$, the term $(t - t_0)^{\frac{3}{2}}$ dominates such that $R_i(t) \approx R_0(t - t_0)^{\frac{3}{2}}$.

It is clear that from the symmetry analysis and the singularity analysis, new exact solutions were found for the master equation (2).

### 4 CONCLUSIONS

We performed a detailed study for the master equation of the temporal equation of radiating stars by investigating the global dynamics and the Lie point symmetries and applying the ARS algorithm of singularity analysis. From the analysis of the global dynamics of the master equation, we were able to find a new asymptotic behavior that corresponds to a new exact solution for a radiating star spacetime, as also to extract important information to infer about the stability and the physical properties of the asymptotic solutions.

We have new analytical findings. The solution of (2) $R = -Cc t$ defined for $-\infty < t \leq 0$, where $c_1 > 0$ is a fixed constant, is unstable. The curve of equilibrium points $L: (\phi, \theta) = (0, 0^o)$ has associated a family of solutions of (2) given by (60), defined in the semi-infinite interval $-\infty < t \leq 0$, which is stable as $t \to 0^-$ ($t \to +\infty$). Furthermore, the Lie symmetry analysis provides that the master equation admits two Lie point symmetries that form the $A_{2,2}$ Lie algebra. Each Lie
symmetry was applied to reduce the order for the ordinary differential equation into a first-order differential equation, while new exact similarity solutions were found.

Finally, from the application of the ARS algorithm, we found that the master equation possesses the Painlevé property, and we were able to write the analytic solution for the radiating stars in order of Painlevé series (74), where $R_0$ is arbitrary, $R_1$ is function of $\alpha$, and the $R_j$, $j \geq 2$, are functions of parameters $\alpha$ and $\beta$. Additionally, we have obtained the closed-form solution (76) for $\beta = -\frac{3\alpha^2}{16}$. To summarize, we have investigated the global dynamics for the given master ordinary differential equation to understand the evolution of solutions for various initial conditions as also to investigate the existence of asymptotic solutions. Moreover, with the application of Lie’s theory, we have reduced the order of the master differential equation, while an exact similarity solution was determined. Finally, the master equation possesses the Painlevé property. Therefore, the analytic solution can be expressed in terms of a Laurent expansion. Such analyses are essential for understanding the physical properties of spacetimes that describe radiating stars.

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CONFLICT OF INTEREST

The authors declare to have no conflict of interest.

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REFERENCES

1. Oppenheimer JR, Snyder H. On continued gravitational contraction. *Phys Rev*. 1939;56:455-459. doi:10.1103/PhysRev.56.455
2. Guo JQ, Zhang L, Chen Y, Joshi PS, Zhang H. Strength of the naked singularity in critical collapse. *Eur Phys J C*. 2020;80(10):924. doi:10.1140/epjc/s10052-020-08486-7
3. Ong YC. Space–time singularities and cosmic censorship conjecture: a review with some thoughts. *Int J Mod Phys A*. 2020;35(14):14. doi:10.1142/S0217751X20300070
4. Shaikh R, Joshi PS. Can we distinguish black holes from naked singularities by the images of their accretion disks? *J Cosmol Astropart Phys*. 2019;10:064. doi:10.1088/1475-7516/2019/10/064
5. Wagh SM, Maharaj SD. Naked singularity of the Vaidya-de Sitter space-time and cosmic censorship conjecture. *Gen Rel Grav*. 1999;31:975-982. doi:10.1023/A:1026675313562
6. The Event Horizon Telescope Collaboration, Akiyama K, Alberdi A, et al. First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole. *J Astrophys*. 2019;875:L1. doi:10.3847/2041-8213/ab0ec7
7. Vaidya PC. The gravitational field of a radiating star. *Proc Indian Acad Sci (Math Sci)*. 1951;33:264. doi:10.1007/BF03173260
8. Bonnor WB, de Oliveira AKG, Santos NO. Radiating spherical collapse. *Phys Rep*. 1989;181:269. doi:10.1016/0370-1573(89)90069-0
9. Herrera L, Santos NO. Local anisotropy in self-gravitating systems. *Phys Rept*. 1997;286:53-130. doi:10.1016/S0370-1573(96)00042-7
10. Santos NO. Non-adiabatic radiating collapse. *Mon Not R Astron Soc*. 1985;216:403-410. doi:10.1093/mnras/216.2.403
11. Maharaj SD, Braxiles B. Radiating stars with composite matter distributions. *Eur Phys J C*. 2021;81:366. doi:10.1140/epjc/s10052-021-09163-z
12. Naidu NF, Bogadi RS, Kaisavelu A, Govender M. Stability and horizon formation during dissipative collapse. *Gen Relativ Grav*. 2020;52:79. doi:10.1007/s10714-020-02728-5
13. Herrera L, di Prisco A, Fuenmayor E, Troconis O. Dynamics of viscous dissipative gravitational collapse: a full causal approach. *Int J Modern Phys D*. 2009;18:129. doi:10.1142/S0218271809014285
14. Chan R, Herrera L, Santos NO. Dynamical instability for shearing viscous collapse. *Society Mon Not R Astron Soc*. 1994;267:637-646. doi:10.1093/mnras/267.3.637
15. Herrera L, Di Prisco A, Ospino J. On the stability of the shear-free condition. *Gen Relativ Gravit*. 2010;42:1585-1599. doi:10.1007/s10714-010-0931-6
16. Govender M. Nonadiabatic spherical collapse with a two-fluid atmosphere,. *Int J Mod Phys D*. 2013;22:1350049. doi:10.1142/S0218271813500491
17. Ivanov BV. A different approach to anisotropic spherical collapse with shear and heat radiation. *Int J Mod Phys D*. 2015;25:1650049. doi:10.1142/S0218271816500498

18. Ivanov BV. Generating solutions for charged geodesic anisotropic spherical collapse with shear and heat radiation. *Eur Phys J C*. 2015;79:255. doi:10.1140/epjc/s10052-015-6722-4

19. Banerjee A, Chatterjee S, Dadhich N. Spherical collapse with heat flow and without horizon. *Mod Phys Lett A*. 2002;17:2335-2340. doi:10.1142/S0217732302008320

20. Govinder KS, Govender M. A general class of Euclidean stars. *Gen Relativ Gravit*. 2012;44:147-156. doi:10.1007/s10714-011-1268-5

21. Chan R. Radiating gravitational collapse with shear viscosity. *Mon Not R Astron Soc*. 2000;308:588. doi:10.1046/j.1365-8711.2000.3547.x

22. Pinheiro G, Chan R. Radiating gravitational collapse with shear motion and bulk viscosity revisited. *Int J Mod Phys D*. 2012;19:11. doi:10.1142/S0218271810018050

23. Banerjee A, DuttaChoudhury SB, Bhui BK. Conformally flat solution with heat flux. *Phys Rev D*. 1989;40:670. doi:10.1103/PhysRevD.40.670

24. Paliaathanasis A, Govender M, Leon G. Temporal evolution of a radiating star via Lie symmetries. *Eur Phys J C*. 2021;81(8):718. doi:10.1140/epjc/s10052-021-09521-x

25. Maharaj SD, Tiwari AK, Mohanlal R, Narain R. Riccati equations for bounded radiating systems. *J Math Phys*. 2016;57:092501. doi:10.1063/1.4961929

26. Abebe GZ, Maharaj SD, Govinder KS. Generalized Euclidean stars with equation of state. *Gen Relativ Gravit*. 2014;46:1733. doi:10.1007/s10714-013-1650-6

27. Bluman GW, Kumei S. *Symmetries and Differential Equations*. Springer-Verlag; 1989.

28. Kaur L, Wazwaz AM. Einstein's vacuum field equation: Painlevé analysis and Lie symmetries. *Waves Random Complex Media*. 2021;31(2):199-206. doi:10.1080/17455030.2019.1574410

29. Christodoulakis T, Dimakis N, Terzis PA. Lie point and variational symmetries in minisuperspace Einstein gravity. *J Phys A*. 2014;47:095202. doi:10.1088/1751-8113/47/9/095202

30. Cotsakis S, Leach PGL, Pantazi H. Symmetries of homogeneous cosmologies. *Grav Cosmol*. 1998;4:314-325. https://ui.adsabs.harvard.edu/abs/1998GrCo....4..314C

31. Mohanlal R, Maharaj SD, Tiwari AK, Narain R. Radiating stars with exponential Lie symmetries. *Gen Rel Grav*. 2016;48:87. doi:10.1007/s10714-016-2081-y

32. Abebe GZ, Maharaj SD. Charged radiating stars with Lie symmetries. *Eur Phys J C*. 2019;79(10):849. doi:10.1140/epjc/s10052-019-7383-2

33. Tsamparlis M, Paliaathanasis A. Symmetries of differential equations in cosmology. *Symmetry*. 2018;10(7):233. doi:10.3390/sym10070233

34. Conte R, Musette M. *The Painlevé Handbook*. Springer; 2008.

35. Ablowitz MJ, Ramani A, Segur H. Nonlinear evolution equations and ordinary differential equations of Painlevé type. I. *J Math Phys*. 1980;21:715-721. doi:10.1063/1.524491

36. Ablowitz MJ, Ramani A, Segur H. A connection between nonlinear evolution equations and ordinary differential equations of P-type. I. *J Math Phys*. 1980;21:1006-1015. doi:10.1063/1.524548

37. Kowalevski S. Sur la problème de la rotation d’un corps solide autour d’un point fixe. *Acta Math*. 1889;12:177-232.

38. Liddle AR, Scherrer RJ. A classification of scalar field potentials with cosmological scaling solutions. *Phys Rev D*. 1999;59:023509. doi:10.1103/PhysRevD.59.023509

39. Uzan JP. Cosmological scaling solutions of nonminimally coupled scalar fields. *Phys Rev D*. 1999;59:123510. doi:10.1103/PhysRevD.59.123510

40. Aulbach B. *Continuous and Discrete Dynamics Near Manifolds of Equilibria*. Lecture Notes in Mathematics No. 1058: Springer; 1984.

41. Morozov VV. Classification of six-dimensional nilpotent Lie algebras. *Izvestia Vysshikh Uchebn Zavedenii, Matematika*. 1958;5:161.