Higgs mass prediction in the MSSM at three-loop level in a pure DR context

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Abstract The impact of the three-loop effects of order $\alpha s^2$ on the mass of the light CP-even Higgs boson in the MSSM is studied in a pure DR context. For this purpose, we implement the results of Kant et al. (JHEP 08:104, 2010) into the C++ module Himalaya and link it to FlexibleSUSY, a Mathematica and C++ package to create spectrum generators for BSM models. The three-loop result is compared to the fixed-order two-loop calculations of the original FlexibleSUSY and of FeynHiggs, as well as to the result based on an EFT approach. Aside from the expected reduction of the renormalization scale dependence with respect to the lower-order results, we find that the three-loop contributions significantly reduce the difference from the EFT prediction in the TeV-region of the SUSY scale $M_{\tilde{g}}$. Himalaya can be linked also to other two-loop DR codes, thus allowing for the elevation of these codes to the three-loop level.

1 Introduction

The measurement of the Higgs boson mass at the Large Hadron Collider (LHC) represents a significant constraint on the viability of supersymmetric (SUSY) models. Given a particular SUSY model, the mass of the Standard Model-like Higgs boson is a prediction, which must be in agreement with the measured value of $(125.09 \pm 0.21 \pm 0.11)$ GeV [2]. Noteworthy, the experimental uncertainty on the measured Higgs mass has already reached the per-mille level. Theory predictions in SUSY models, however, struggle to reach the same level of accuracy. The reason is that the Higgs mass receives large higher-order corrections, dominated by the top Yukawa and the strong gauge coupling [3–5]. Both of these two couplings are comparatively large, leading to a relatively slow convergence of the perturbative series. Furthermore, the scalar nature of the Higgs implies corrections proportional to the square of the top-quark mass, on top of the top-mass dependence due to the Yukawa mass, which enters the loop corrections quadratically. On the other hand, corrections from SUSY particles are only logarithmic in the SUSY particle masses due to the assumption of only soft SUSY-breaking terms. If the SUSY particles are not too far above the TeV scale [6,7], the SUSY Higgs mass can be obtained from a fixed-order calculation of the relevant one- and two-point functions with external Higgs fields. In this case, higher-order corrections up to the three-loop level are known in the Minimal Supersymmetric Standard Model (MSSM) [1,5,8–23].

There are plenty of publicly available computer codes which calculate the Higgs pole mass(es) in the MSSM at higher orders: CPsuperH [24–26], FeynHiggs [9,27–31], FlexibleSUSY [32,33], H3m [1,20], ISASUSY [34], MhEFT [35], SARAH/SPheno [36–42], SOFTSUSY [43,44], SUspect [45] and SusyHD [46]. FeynHiggs adopts the on-shell scheme for the renormalization of the particle masses, while all other codes express their results in terms of MS/DR parameters. All these schemes are formally equivalent up to higher orders in perturbation theory, of course. The numerical difference between the schemes is one of the sources of theoretical uncertainty on the Higgs mass prediction, however. All of these programs take into account one-loop corrections, most of them also leading two-loop corrections. H3m is the only one which includes three-loop corrections of order $\alpha s^2\alpha t^2$, where $\alpha t$ is the squared top Yukawa and $\alpha s$ is the strong coupling. It combines these terms with the on-shell two-loop result of FeynHiggs after transforming the $O(\alpha t)$ and $O(\alpha s\alpha t)$ terms from there to the DR scheme.

Here we present an alternative implementation of the $O(\alpha t^2)$ contributions of Refs. [1,20] for the light CP-even Higgs mass in the MSSM into the framework of FlexibleSUSY [32], referring to the combination as FlexibleSUSY+Himalaya in what follows. This allows us to study the effect of the three-loop contributions in a pure
DR scheme, i.e. without the trouble of combining the corrections with an on-shell calculation. The three-loop terms are provided in the form of a separate C++ package, named Himalaya, which one should be able to include in any other DR code without much effort. The Himalaya package and the dedicated version of FlexibleSUSY, which incorporates the three-loop contributions from Himalaya, can be downloaded from Refs. [47,48], respectively. In this way, we hope to contribute to the on-going effort of improving the precision of the Higgs mass prediction in the MSSM.

In the present paper we study the impact of the three-loop corrections for low and high SUSY scales and compare our results to the two-loop calculations of the public spectrum generators of FlexibleSUSY and FeynHiggs. By quantifying the size of the three-loop corrections, we also provide a measure for the theoretical uncertainty of the DR fixed-order calculation.

As will be shown below, the implementation of the $\alpha_t \alpha_s^2$ corrections also applies to the terms of order $\alpha_s \alpha_t^2$, where $\alpha_t$ is the bottom Yukawa coupling. Therefore, Himalaya will take such terms into account, and we will refer to the sum of top- and bottom-Yukawa induced supersymmetric QCD (SQCD) corrections as $\mathcal{O}(\alpha_t \alpha_s^2 + \alpha_s \alpha_t^2)$ in what follows. However, it should be kept in mind that this does not include effects of order $\alpha_t^2 \sqrt{\alpha_t \alpha_t}$, which arise from three-loop Higgs self energies involving both a top/stop and a bottom/bottom triangle. The results of Himalaya are thus unreliable in the (rather exotic) case where $\alpha_t$ and $\alpha_b$ are comparable in magnitude.

The remainder of this paper is structured as follows. Section 2 describes the form in which the three-loop contributions of order $(\alpha_t + \alpha_b)\alpha_s^2$ are implemented in Himalaya. Its input parameters are to be provided in the DR scheme at the appropriate perturbative order. Section 3 details how this input is prepared in the framework of FlexibleSUSY. It also summarises all the contributions that enter the final Higgs mass prediction in FlexibleSUSY+Himalaya. Section 4 analyses the impact of various three-loop contributions on this prediction as well as the residual renormalization scale dependence, and it compares the results obtained with FlexibleSUSY+Himalaya to existing fixed-order and resummed results for the light Higgs mass. In particular, this includes a comparison to the original implementation of the three-loop effects in H3m. Our conclusions are presented in Sect. 5. Technical details of Himalaya, its link to FlexibleSUSY, and run options are collected in the appendix.

## 2 Higgs mass prediction at the three-loop level in the MSSM

The results for the three-loop $\alpha_t \alpha_s^2$ corrections to the Higgs mass in the MSSM have been obtained in Refs. [1,20] by a Feynman diagrammatic calculation of the relevant one- and two-point functions with external Higgs fields in the limit of vanishing external momenta. The dependence of these terms on the squark and gluino masses was approximated through asymptotic expansions, assuming various hierarchies among the masses of the SUSY particles. For details of the calculation we refer to Refs. [1,20].

### 2.1 Selection of the hierarchy

A particular set of parameters typically matches several of the hierarchies mentioned above. In order to select the most suitable one, Ref. [1] suggested a pragmatic approach, namely the comparison of the various asymptotic expansions to the exact expression at two-loop level. Himalaya also adopts this approach, but introduces a few refinements in order to further stabilise the hierarchy selection (see also Ref. [49]).

In a first step the Higgs pole mass $M_h$ is calculated at the two-loop level at order $\alpha_t \alpha_s$ using the result of Ref. [12] in the form of the associated FORTRAN code provided by the authors. We refer to this quantity as $M_{h}^{DSZ}$ in what follows. Subsequently, for all hierarchies $i$ which fit the given mass spectrum, $M_h$ is calculated again using the expanded expressions of Ref. [1] at the two-loop level, resulting in $M_{h,i}$. In the original approach of Ref. [1], the hierarchy is selected as the value of $i$ for which the difference

$$
\delta_{i}^{2L} = \left| M_{h,i}^{DSZ} - M_{h,i} \right|
$$

is minimal. However, we found that this criterion alone causes instabilities in the hierarchy selection in regions where several hierarchies lead to similar values of $\delta_{i}^{2L}$. We therefore refine the selection criterion by taking into account the quality of the convergence in the respective hierarchies, quantified by

$$
\delta_{i}^{\text{conv}} = \sqrt{\sum_{j=1}^{n} \left( M_{h,i} - M_{h,i}^{(j)} \right)^2}.
$$

While $M_{h,i}$ includes all available terms of the expansion in mass (and mass difference) ratios, in $M_{h,j}^{(j)}$ the highest terms of the expansion for the mass (and mass difference) ratio $j$ are dropped. We then define the “best” hierarchy to be the one which minimises the quadratic mean of Eqs. (1) and (2),

$$
\delta_{i} = \sqrt{\left( \delta_{i}^{2L} \right)^2 + \left( \delta_{i}^{\text{conv}} \right)^2}.
$$

The relevant analytical expressions for the three-loop terms of order $\alpha_t \alpha_s^2$ to the CP-even Higgs mass matrix in the various mass hierarchies are quite lengthy. However, they are accessible in Mathematica format in the framework of the publicly available program H3m. We have transformed
these formulas into C++ format and implemented them into Himalaya.

The hierarchies defined in H3m equally apply to the top and the bottom sector of the MSSM, so that the results of Ref. [1] can also be used to evaluate the corrections of order \( \alpha_t \alpha_s^2 \rightarrow 2.2 \) Modified DR scheme

By default, all the parameters of the calculation are renormalised in the DR scheme. However, in this scheme, one finds artificial “non-decoupling” effects [12], meaning that the two- and three-loop result for the Higgs mass depends quadratically on a SUSY particle mass if this mass gets much larger than the others. Such terms are avoided by transforming the stop masses to a non-minimal scheme, named MDR (modified DR) in Ref. [1], which mimics the virtue of the on-shell scheme of automatically decoupling the heavy particles.

If the user wishes to use this scheme rather than pure DR, Himalaya writes the Higgs mass matrix as

\[
\tilde{M}(\tilde{m}_t) = \tilde{M}^{\text{tree}} + \tilde{M}^{(\alpha)}(\tilde{m}_t) + \tilde{M}^{(\alpha \alpha_s)}(\tilde{m}_t) \\
+ \tilde{M}^{(\alpha \alpha_s^2)}(\tilde{m}_t) + \ldots \\
= \tilde{M}^{\text{tree}} + \tilde{M}^{(\alpha)}(m_t) + \tilde{M}^{(\alpha \alpha_s)}(m_t) \\
+ \delta M(m_t, \tilde{m}_t) + \tilde{M}^{(\alpha \alpha_s^2)}(\tilde{m}_t) + \ldots, \tag{4}
\]

where \( M_0 \) and \( \tilde{M} \) are the Higgs mass matrices in the DR and the MDR scheme, respectively, \( \tilde{M}^{\text{tree}} = \tilde{M}^{\text{tree}} \) is the tree-level expression, and the superscript \( (\cdot) \) denotes the term of order \( x \in \{ \alpha_t, \alpha_s, \alpha_s \alpha_t, \ldots \} \). The ellipsis in Eq. (4) symbolises any terms that involve coupling constants other than \( \alpha_t \) or \( \alpha_s \), or higher orders of the latter. For brevity we suppress the stop mass indices “1” and “2” here. Himalaya provides the numerical results for \( \tilde{M}^{(\alpha \alpha_s^2)}(\tilde{m}_t) \) as well as

\[
\delta M(m_t, \tilde{m}_t) \equiv \left( \tilde{M}^{(\alpha)}(\tilde{m}_t) + \tilde{M}^{(\alpha \alpha_s)}(\tilde{m}_t) \right) \\
- \left( M^{(\alpha)}(m_t) + M^{(\alpha \alpha_s)}(m_t) \right), \tag{5}
\]

where the MDR stop mass \( \tilde{m}_t \) is calculated from its DR value \( m_t \) by the conversion formulas through \( \mathcal{O}(\alpha_t^2) \), provided in Ref. [1]. Note that these conversion formulas depend on the underlying hierarchy, and may be different for \( m_{t,1} \) and \( m_{t,2} \).

Even if the result is requested in the MDR scheme, the output of Himalaya can thus be directly combined with pure DR results through \( \mathcal{O}(\alpha_t \alpha_s) \) according to Eq. (4) in order to arrive at the mass matrix at order \( \alpha_t \alpha_s^2 \). Of course, one may also request the plain DR result from Himalaya, in which case it will simply return the numerical value for \( \tilde{M}^{(\alpha \alpha_s^2)}(m_t) \), which can be directly added to any two-loop DR result.

In any case, the difference between the DR and MDR result is expected to be quite small unless the mass splitting between one of the stop masses and other, heavier, strongly interacting SUSY particles becomes very large. As a practical example, in Fig. 1 we show the difference of the lightest Higgs mass at the three-loop level calculated in the DR and MDR scheme. All DR soft-breaking mass parameters, the \( \mu \) parameter of the MSSM super-potential, and the running CP-odd Higgs mass are set equal to \( M_0 \) here. The running trilinear couplings, except \( A_t \), are chosen such that the sfermions do not mix. The DR stop mixing parameter \( X_t = A_t - \mu / \tan \beta \) is left as a free parameter. For this scenario we find that the difference between the DR and MDR scheme is below 100 MeV for different values of the stop mixing parameter.

Note that, for all terms in the Higgs mass matrix except \( \alpha_t \), \( \alpha_s \alpha_t \), and \( \alpha_s \alpha_t^2 \), it is perturbatively equivalent to use either the DR or the MDR stop mass as defined above. Predominantly, this concerns the electroweak contributions as well as the terms of order \( \alpha_s^2 \). In this paper, we use the DR stop mass for these contributions.

3 Implementation into FlexibleSUSY

3.1 Determination of the MSSM DR parameters

FlexibleSUSY determines the running DR gauge and Yukawa couplings as well as the running vacuum expectation value of the MSSM along the lines of Ref. [50] by setting the scale to the Z-boson pole mass \( M_Z \). In this approach, the following Standard Model (SM) input parameters are used:

\[
\alpha_{em}^{SM(5)}(M_Z), \alpha_s^{SM(5)}(M_Z), G_F, M_Z, \\
M_e, M_\mu, M_\tau, m_{d,d,s}(2 \text{ GeV}), m_c^{SM(4), \overline{\text{MS}}}(m_c), \\
m_b^{SM(5), \overline{\text{MS}}}(m_b), M_t, \tag{6}
\]

where \( \alpha_{em}^{SM(5)}(M_Z) \) and \( \alpha_s^{SM(5)}(M_Z) \) denote the electromagnetic and strong coupling constants in the \( \overline{\text{MS}} \) scheme in the Standard Model with five active quark flavours, and \( G_F \) is the Fermi constant. \( M_e, M_\mu, M_\tau, \) and \( M_t \) denote the pole masses of the electron, muon, tau lepton, and top quark, respectively. The input masses of the up, down and strange quark are
Fig. 1 Difference between the lightest Higgs pole mass calculated in the DR scheme and the MDR scheme as a function of the SUSY scale $M_S$ for $\tan \beta = 5$. In the left panel the soft-breaking stop and gluino mass parameters are set equal to $M_S$. In the right panel, we use $m_{\tilde{g}} = 2M_S$.

defined in the $\overline{\text{MS}}$ scheme at the scale 2 GeV. The charm and bottom quark masses are defined in the $\overline{\text{MS}}$ scheme at their scale in the Standard Model with four and five active quark flavours, respectively.

The MSSM DR gauge couplings $g_1, g_2$ and $g_3$ are given in terms of the DR parameters $\alpha_{\text{em}}^{\text{MSSM}}(M_Z)$ and $\alpha_s^{\text{MSSM}}(M_Z)$ in the MSSM as

$$g_1(M_Z) = \sqrt{\frac{\sqrt[4]{4\pi \alpha_{\text{em}}^{\text{MSSM}}(M_Z)}}{\cos \theta_w(M_Z)}}, \quad (7)$$

$$g_2(M_Z) = \sqrt{\frac{\sqrt[4]{4\pi \alpha_{\text{em}}^{\text{MSSM}}(M_Z)}}{\sin \theta_w(M_Z)}} \quad (8)$$

$$g_3(M_Z) = \sqrt{\frac{\sqrt[4]{4\pi \alpha_s^{\text{MSSM}}(M_Z)}}{\sin \theta_w(M_Z)}} \quad (9)$$

The couplings $\alpha_{\text{em}}^{\text{MSSM}}(M_Z)$ and $\alpha_s^{\text{MSSM}}(M_Z)$ are calculated from the corresponding input parameters as

$$\alpha_{\text{em}}^{\text{MSSM}}(M_Z) = \frac{\alpha_{\text{em}}^{\text{SM}(5)}(M_Z)}{1 - \Delta \alpha_{\text{em}}(M_Z)}, \quad (10)$$

$$\alpha_s^{\text{MSSM}}(M_Z) = \frac{\alpha_s^{\text{SM}(5)}(M_Z)}{1 - \Delta \alpha_s(M_Z)}, \quad (11)$$

where the threshold corrections $\Delta \alpha_{\text{em}}(M_Z)$ and $\Delta \alpha_s(M_Z)$ have the form

$$\Delta \alpha_{\text{em}}(M_Z) = \frac{\alpha_{\text{em}}}{2\pi} \left( 1 - \frac{16}{9} \log \frac{m_t}{M_Z} \right) - \frac{4}{9} \sum_{i=1}^{6} \log \frac{m_{\tilde{u}_i}}{M_Z} - \frac{1}{9} \sum_{i=1}^{6} \log \frac{m_{\tilde{d}_i}}{M_Z}$$

We have cut off curves with non-zero $X_t$ around or below the TeV scale, where the DR CP-even Higgs mass becomes tachyonic at the electroweak scale.

$$- \frac{4}{3} \sum_{i=1}^{2} \log \frac{m_{\tilde{\chi}_i^+}}{M_Z} - \frac{6}{3} \sum_{i=1}^{6} \log \frac{m_{\tilde{\chi}_i^0}}{M_Z}$$

$$- \frac{1}{3} \log \frac{m_{H^+}}{M_Z}, \quad (12)$$

$$\Delta \alpha_s(M_Z) = \frac{\alpha_s}{2\pi} \left[ 1 - 2 \log \frac{m_{\tilde{g}}}{M_Z} - \frac{2}{3} \log \frac{m_t}{M_Z} \right] - \frac{1}{6} \sum_{i=1}^{6} \left( \log \frac{m_{\tilde{u}_i}}{M_Z} + \log \frac{m_{\tilde{d}_i}}{M_Z} \right) \quad (13)$$

The $\overline{\text{DR}}$ weak mixing angle in the MSSM, $\theta_w$, is determined at the scale $M_Z$ from the Fermi constant $G_F$ and the Z pole mass via the relation

$$\sin^2 \theta_w \cos^2 \theta_w = \frac{\pi \alpha_{\text{em}}^{\text{MSSM}}}{\sqrt{2} M_Z^2 G_F (1 - \delta_\rho)}, \quad (14)$$

where

$$\delta_\rho = \frac{\Re \left[ \Sigma_{W,T}(0) \right]}{M_W^2} - \frac{\Re \left[ \Sigma_{Z,T}(M_Z^2) \right]}{M_Z^2} + \delta_{\text{VB}} + \delta_{\rho}^{(2)}, \quad (15)$$

$$\hat{\rho} = \frac{1}{1 - \Delta \hat{\rho}},$$

$$\Delta \hat{\rho} = \Re \left[ \frac{\Sigma_{Z,T}(M_Z^2)}{\hat{\rho}^2 M_Z^2} - \frac{\Sigma_{W,T}(M_W^2)}{M_W^2} \right] + \Delta \hat{\rho}^{(2)} \quad (16)$$

Here, $\Sigma_{V,T}(p^2)$ denotes the transverse part of the $\overline{\text{DR}}$-renormalised one-loop self-energy of the vector boson $V$ in
the MSSM. The vertex and box contributions $\delta_{V\nu}$ as well as the two-loop contributions $\delta_{ij}^{(2)}$ are taken from Ref. [50]. The $\overline{\text{DR}}$ vacuum expectation values of the up- and down-type Higgs doublets are calculated by

$$v_u(M_Z) = \frac{2m_Z(M_Z) \sin \beta(M_Z)}{\sqrt{3/5}g_1^2(M_Z) + g_2^2(M_Z)}, \quad (17)$$

$$v_d(M_Z) = \frac{2m_Z(M_Z) \cos \beta(M_Z)}{\sqrt{3/5}g_1^2(M_Z) + g_2^2(M_Z)}, \quad (18)$$

where $\tan \beta(M_Z)$ is an input parameter and $m_Z(M_Z)$ is the $Z$ boson $\overline{\text{DR}}$ mass in the MSSM, which is calculated from the $Z$ pole mass at the one-loop level as

$$m_Z^2 = M_Z^2 + \text{Re} \Sigma_{Z,t,R}(M_Z^2). \quad (19)$$

In our approach, we relate the $\overline{\text{DR}}$ top mass to the top pole mass $M_t$ at the scale $M_Z$ as

$$m_t(M_Z) = M_t + \text{Re} \Sigma_t(M_t^2, M_Z) + M_t \left[ \text{Re} \Sigma_t^L(M_t^2, M_Z) + \text{Re} \Sigma_t^R(M_t^2, M_Z) + \Delta m_t^{(1),\text{SQCD}}(M_Z) + \Delta m_t^{(2),\text{SQCD}}(M_Z) \right], \quad (21)$$

where the $\Sigma_t^{S,L,R}(p^2, Q)$ denote the scalar (superscript $S$), and the left- and right-handed parts ($L$, $R$) of the $\overline{\text{DR}}$ renormalised one-loop top self-energy without the gluon, stop, and gluino contributions, and $\Delta m_t^{(1),\text{SQCD}}$ and $\Delta m_t^{(2),\text{SQCD}}$ are the full one- and two-loop SQCD corrections taken from Refs. [51,52],

$$\Delta m_t^{(1),\text{SQCD}} = \frac{\alpha_s}{4\pi} C_F \left[ \frac{m_t m_Z^2}{m_t} \right] \left( \frac{m_t m_Z^2}{m_t^2 - m_Z^2} \right) + \frac{m_t^4}{2(m_t^2 - m_Z^2)^2} \right) \log \frac{m_t^2}{Q^2} \right] \left( \frac{m_t m_Z^2}{m_t^2 - m_Z^2} \right) + \frac{m_t^4}{2(m_t^2 - m_Z^2)^2} \right) \log \frac{m_t^2}{Q^2} \right] + \frac{m_t^4}{2(m_t^2 - m_Z^2)^2} \right) \log \frac{m_t^2}{Q^2} \right] \left( \frac{m_t m_Z^2}{m_t^2 - m_Z^2} \right) + \frac{m_t^4}{2(m_t^2 - m_Z^2)^2} \right) \log \frac{m_t^2}{Q^2} \right] + \frac{m_t^4}{2(m_t^2 - m_Z^2)^2} \right), \quad (22)$$

$$m_t^2 = M_t^2 + \text{Re} \Sigma_{Z,t,R}(M_t^2). \quad (19)$$

In order to calculate the Higgs pole mass in the $\overline{\text{DR}}$ scheme at the three-loop level $O(\alpha_s \alpha_t^2 + \alpha_t^2 \alpha_t^2)$, the $\overline{\text{DR}}$ top and bottom Yukawa couplings must be extracted from the input parameters $M_t$ and $m_b^{\text{SM}(5),\overline{\text{MS}}}(m_b)$ at the two-loop level at $O(\alpha_t^2)$. In order to achieve that, we make use of the known two-loop SQCD contributions to the top and bottom Yukawa couplings of Refs. [51–54], as described in the following: We calculate the $\overline{\text{DR}}$ Yukawa couplings $y_t$ at the scale $M_Z$ from the $\overline{\text{DR}}$ top mass $m_t$ and the $\overline{\text{DR}}$ up-type VEV $v_u$ as

$$y_t(M_Z) = \sqrt{2} \frac{m_t(M_Z)}{v_u(M_Z)}, \quad (20)$$

$$\Delta m_t^{(2),\text{SQCD}} = \left( \Delta m_t^{(1),\text{SQCD}} \right)^2 - \Delta m_t^{(2),\text{dec}}. \quad (23)$$

In Eq. (22), it is $C_F = 4/3$ and $s_{2\theta_t} = \sin 2\theta_t$, with $\theta_t$ the stop mixing angle. The two-loop term $\Delta m_t^{(2),\text{dec}}$ is given in Ref. [51] for general stop, sbottom, and gluino masses.

The MSSM $\overline{\text{DR}}$ bottom-quark Yukawa coupling $y_b$ is calculated from the $\overline{\text{DR}}$ bottom-quark mass $m_b$ and the down-type VEV at the scale $M_Z$ as

$$y_b(M_Z) = \sqrt{2} \frac{m_b(M_Z)}{v_d(M_Z)}, \quad (24)$$

We obtain $m_b(M_Z)$ from the input $\overline{\text{MS}}$ mass $m_b^{\text{SM}(5),\overline{\text{MS}}}(m_b)$ in the Standard Model with five active quark flavours by first
evolving $m_b^{\text{SM(5),MS}}(m_h)$ to the scale $M_Z$, using the one-loop QED and three-loop QCD renormalization group equations (RGEs). Afterwards, $m_b^{\text{SM(5),MS}}(M_Z)$ is converted to the \overline{\text{DR}} mass $m_b^{\text{SM(5),\overline{\text{DR}}}}(M_Z)$ by the relation

$$m_b^{\text{SM(5),\overline{\text{DR}}}}(M_Z) = m_b^{\text{SM(5),MS}}(M_Z) \times \left(1 - \frac{\alpha_s}{3\pi} + \frac{3g_\gamma^2}{128\pi^2} + \frac{13g_\gamma^2}{1152\pi^2}\right).$$

Finally, the MSSM \overline{\text{DR}} bottom mass $m_b(M_Z)$ is obtained from $m_b^{\text{SM(5),\overline{\text{DR}}}}(M_Z)$ via

$$m_b(M_Z) = \frac{m_b^{\text{SM(5),\overline{\text{DR}}}}(M_Z)}{1 + \Delta m_b^{(1)} + \Delta m_b^{(2)}},$$

$$\Delta m_b^{(1)} = -\text{Re} \left(\Sigma_b^S \left((m_b^{\text{SM(5),MS}})^2 , M_Z\right) \right) / m_b$$

$$-\text{Re} \left(\Sigma_b^L \left((m_b^{\text{SM(5),MS}})^2 , M_Z\right) \right)$$

$$-\text{Re} \left(\Sigma_b^R \left((m_b^{\text{SM(5),MS}})^2 , M_Z\right) \right),$$

$$\Delta m_b^{(2)} = \Delta m_b^{(2),\text{dec}} - \frac{\alpha_s}{3\pi} \Delta m_b^{(1)},$$

where $\Sigma_b^{S,L,R}(p^2, Q)$ are the scalar, left- and right-handed parts of the \overline{\text{DR}} renormalised one-loop bottom quark self-energy in the MSSM, in which all Standard Model particles, except the bottom quark, the top quark and the $W$, $Z$, and Higgs bosons, are omitted. In Eq. (28) $\Delta m_b^{(2),\text{dec}}$ denotes the two-loop decoupling relation of order $O(\alpha_s^2)$ between the \overline{\text{MS}} bottom mass $m_b^{\text{SM(5),MS}}$ and the \overline{\text{DR}} bottom mass in the MSSM calculated in Refs. [53, 54].

Note that the matching of the SM to the MSSM leads to large logarithmic contributions in the MSSM \overline{\text{DR}} parameters in the case of a heavy SUSY particle spectrum. These contributions can be resummed in a so-called EFT approach [31, 33, 46, 55, 56].

3.2 Calculation of the CP-even Higgs pole masses

FlexibleSUSY calculates the two CP-even Higgs pole masses $M_h$ and $M_H$ by diagonalising the loop-corrected mass matrix

$$M = M^{\text{tree}} + M^{1L}(p^2) + M^{2L} + M^{3L}$$

1 We do not distinguish between \overline{\text{DR}} and $\overline{\text{MS}}$ parameters here, and drop the hat over $M$ introduced in Eq. (4) for simplicity.

2 FlexibleSUSY+Himalaya provides a flag to calculate the corrections of order $O(\alpha_s(1 + \alpha_s + \alpha_s^2) + \alpha_s(1 + \alpha_s + \alpha_s^2))$ in the $\overline{\text{MS}}$ scheme, as described in Sect. 2.2. See “Appendix C” for more details.
to the Higgs pole mass (black dotted line) is again positive and around +2 GeV for $M_S \approx 1$ GeV. This is consistent with the findings of Ref. [1], of course. As a result, the sum of these three three-loop effects (red solid line) leads to a net positive shift of the Higgs mass relative to the two-loop result without all these corrections.

The size of the individual three-loop contributions depends on the stop mixing parameter $X_t/M_S$, as can be seen from the r.h.s. of Fig. 2: between minimal ($X_t/M_S = 0$) and maximal stop mixing ($X_t/M_S \approx \sqrt{6}$) the size of the individual three-loop contributions changes by 1–2 GeV. For maximal (minimal) mixing, their impact is maximal (minimal). The direction of the shift is independent of $X_t/M_S$.

Note that the nominal two-loop result of the original FlexibleSUSY (i.e., without Himalaya) includes by default the one-loop threshold correction to $\alpha_s$ and the SM QCD two-loop contributions to the top Yukawa coupling [32, 33]. This means that the two-loop Higgs mass as evaluated by the original FlexibleSUSY already incorporates partial three-loop contributions. As a result, the two-loop result of the original FlexibleSUSY does not correspond to the zero-line in Fig. 2, but it is rather close to the blue dashed line. This implies that, compared to the two-loop result of the original FlexibleSUSY, the effect of the remaining $\alpha_s^2$ contributions in the Higgs mass prediction is negative.

4.2 Scale dependence of the three-loop Higgs pole mass

To estimate the size of the missing higher-order corrections, Fig. 3 shows the renormalization scale dependence of the one-, two- and three-loop Higgs pole mass for the scenario defined in Sect. 2.2 with $\tan \beta = 5$ and $X_t = 0$. The one- and two-loop calculations correspond to the original FlexibleSUSY. In the one-loop calculation the threshold corrections to $\alpha_s$ and $\gamma_t$ are set to zero, and in the two-loop calculation the one-loop threshold corrections to $\alpha_s$ and the two-loop QCD corrections to $\gamma_t$ are taken into account. The three-loop result of FlexibleSUSY+Himalaya includes all three-loop contributions at $(\alpha_s + \alpha_t)\alpha_s^2$ discussed above, i.e. the one-loop threshold correction to $\alpha_s$, the full two-loop SQCD corrections to $\gamma_t$, and the genuine three-loop correction to the Higgs pole mass from Himalaya. In addition, the Higgs mass predicted at the two-loop level in the pure EFT calculation of HSSUSY is shown as the black dotted line, see Sect. 4.3. The bands show the corresponding variation of the Higgs pole mass when the renormalization scale is varied using the three-loop renormalization group equations [57–63] for all parameters except for the vacuum expectation values, where the $\beta$-functions are known only up to the two-loop level [64,65]. In FlexibleSUSY and FlexibleSUSY+Himalaya, the renormalization scale is varied in the full MSSM within the interval $[M_S/2, 2M_S]$, while in HSSUSY it is varied in the Standard Model within the interval $[M_t/2, 2M_t]$, keeping the matching scale fixed at $M_S$. The plot shows that the successive inclusion of higher-order corrections reduces the scale dependence, as expected. In particular, the three-loop corrections to the Higgs mass reduce the scale dependence by around a factor two, compared to the two-loop calculation. The scale dependence of HSSUSY is almost independent of $M_S$, because scale variation is done within the SM after integrating out all SUSY particles at $M_S$. Note that the variation of the renormalization scale only serves as an indicator of the theoretical uncertainty due to missing higher-order effects.

![Fig. 2 Influence of different three-loop contributions to the Higgs pole mass. In the left panel we show the shift in the Higgs pole mass with respect to $M^2_{t, b} (\alpha_s^{1L}, \alpha_s^{2L})$ for $\tan \beta = 5$ and $X_t = 0$ as a function of the SUSY scale. In the right panel we fix $\tan \beta = 5$ and $M_S = 2$ TeV and vary the relative stop mixing parameter $X_t/M_S$.](image-url)
Fig. 3 Variation of the Higgs pole mass when the renormalization scale is varied by a factor two at which the Higgs pole mass is calculated, for \( \tan \beta = 5 \) and \( X_t = 0 \).

Fig. 4 Comparison of Higgs mass predictions between two- and three-loop fixed-order programs and a two-loop EFT calculation as a function of the SUSY scale for \( \tan \beta = 5 \) and \( X_t = 0 \). In the left panel the absolute Higgs pole mass and in the right panel the difference w.r.t. the three-loop calculation is shown (FS = FlexibleSUSY, FS+H = FlexibleSUSY+Himalaya, FH = FeynHiggs).

4.3 Comparison with lower-order and EFT results

In Figs. 4, 5, we compare the three-loop calculation of FlexibleSUSY+Himalaya (red) with other MSSM spectrum generators. As input we use \( M_t = 173.34 \) GeV, \( \alpha_{em}^{SM}(M_Z) = 1/127.95, \alpha_s^{SM}(M_Z) = 0.1184 \) and \( G_F = 1.1663787 \times 10^{-5} \) GeV\(^{-2}\). All \( \overline{\text{DR}} \) soft-breaking mass parameters as well as the \( \mu \) parameter of the super-potential in the MSSM, and the running CP-odd Higgs mass are set equal to \( M_S \). The running trilinear couplings, except for \( A_t \), are chosen such that there is no sfermion mixing. The stop mixing parameter \( X_t = A_t - \mu/\tan \beta \) is defined in the \( \overline{\text{DR}} \) scheme and left as a free parameter. The lightest CP-even Higgs pole mass is calculated at the scale \( Q = \sqrt{m_{\tilde{t}_1} m_{\tilde{t}_2}} \).

**FlexibleSUSY 1.7.4** The blue dashed line shows the original two-loop calculation with FlexibleSUSY 1.7.4 [32]. Note that, by construction of FlexibleSUSY, this result coincides exactly with the one of SOFTSUSY 3.5.1. As described above, it includes the one-loop threshold corrections to \( \alpha_s \) and the two-loop QCD contributions to \( \gamma_1 \), and it uses the three-loop RGEs of the MSSM [57,58]. FlexibleSUSY 1.7.4 (and SOFTSUSY) use the explicit two-loop Higgs pole mass contribution of order \( \mathcal{O}(\alpha_s (\alpha_t + \alpha_b) + (\alpha_t + \alpha_b)^2 + \alpha_t^2) \) of Refs. [12–16].
HSSUSY 1.7.4 The black dotted line has been obtained using the pure two-loop effective field theory (EFT) calculation of HSSUSY [48]. HSSUSY is a spectrum generator from the FlexibleSUSY suite, which implements the two-loop threshold correction for the quartic Higgs coupling of the Standard Model at $\mathcal{O}(\alpha_t(\alpha_t + \alpha_s))$ when integrating out the SUSY particles at a common SUSY scale [46,55]. Renormalization group running is performed down to the top mass scale using the three-loop RGEs of the Standard Model [59–63] and, finally, the Higgs mass is calculated at the two-loop level in the Standard Model at order $\mathcal{O}(\alpha_t(\alpha_t + \alpha_s))$. In terms of the implemented corrections, HSSUSY is equivalent to SusyHD [46], and resums large logarithms up to NNLL level while neglecting terms of order $v^2/M_S^2$. The $\mathcal{O}(v^2/M_S^2)$ corrections calculated in Ref. [66] have not been taken into account here.

FeynHiggs 2.13.0-beta The green dash-dotted line shows the Higgs mass prediction using FeynHiggs 2.13.0-beta without large log resummation [9,27–31]. FeynHiggs 2.13.0-beta includes the two-loop contributions of order $\mathcal{O}(\alpha_t\alpha_s + \alpha_b\alpha_s + \alpha_s^2 + \alpha_t\alpha_b)$.

Consider first Fig. 4. The left panel shows the Higgs mass prediction as a function of $M_S$ according to three codes discussed above, together with the FlexibleSUSY+Himalaya result (solid red). The stop mixing parameter $X_t$ is set to zero. The right panel shows the difference of $M_S = 2\,\text{TeV}$ in the left panel the absolute Higgs pole mass and in the right panel the difference w.r.t. the three-loop calculation is shown these curves to the latter. Note that the resummed result of HSSUSY neglects terms of order $v^2/M_S^2$, and thus forfeits reliability towards lower values of $M_S$. The deviation from the fixed-order curves below $M_S \approx 400\,\text{GeV}$ clearly underlines this. In contrast, the fixed-order results start to suffer from large logarithmic contributions toward large $M_S$, which on the other hand are properly resummed in the HSSUSY approach. From Fig. 4, we conclude that the fixed-order $\overline{\text{DR}}$ result loses its applicability once $M_S$ is larger than a few TeV, while the deviation between the non-resummed on-shell result of FeynHiggs and HSSUSY increases more rapidly above $M_S \approx 1\,\text{TeV}$. Note that the good agreement of FlexibleSUSY with HSSUSY above the few-TeV region is accidental, as shown in Ref. [33].

The effect of the three-loop $\alpha_t\alpha_s^2$ terms on the fixed-order result is negative, as discussed in Sect. 4.1, and amounts to a few hundred MeV in the region where the fixed-order approach is appropriate. They significantly improve the agreement between the fixed-order and the resummed prediction for $M_h$ in the intermediate region of $M_S$, where both approaches are expected to be reliable. Between $M_S$ of about 500 GeV and 5 TeV, our three-loop curve from FlexibleSUSY+Himalaya deviates from the HSSUSY result by less than 300 MeV. This corroborates the compatibility of the two approaches in the intermediate region. Considering the current estimate of the theoretical uncertainty in the Higgs mass prediction [28,33,46,55,67], our observation even legitimizes a naive switching between the fixed-order and the resummed approach at $M_S \approx 1\,\text{TeV}$, instead of a more sophisticated matching procedure along the lines of Refs. [31,56]. Nevertheless, the latter is clearly desirable through order $\alpha_t\alpha_s^2$, in particular in the light of the observa-

---

[^3]: We use the SLHA input interface of FeynHiggs, which performs a conversion of the $\overline{\text{DR}}$ input parameters to the on-shell scheme. Resummation is disabled, as it would lead to an inconsistent result in combination with the $\overline{\text{DR}}$ to on-shell conversion of FeynHiggs [56]. We call FeynHiggs with the flags 4002020110.
underlines this by setting $X_t = -\sqrt{6}M_S$ and varying $M_S$. The kink in the three-loop curve originates from a change of the optimal hierarchy chosen by Himalaya. The red band shows the uncertainty $\delta_i$ as defined in Eq. (3), which is used to select the best fitting hierarchy. We find that $\delta_i$ is comparable to the size of the kink, which indicates a reliable treatment of the hierarchy selection criterion.

4.4 Comparison with other three-loop results

The three-loop $O(\alpha_t \alpha_s^2)$ corrections to the light MSSM Higgs mass discussed in this paper were originally implemented in the Mathematica code $H^3m$. We checked that the implementation of the $\alpha_t$ and $\alpha_s \alpha_t$ terms in Himalaya leads to the same numerical results as in $H^3m$, if the same set of DR parameters is used as input. Since the $\alpha_t \alpha_s^2$ terms of Himalaya are derived from their implementation in $H^3m$, it is not surprising that they also result in the same numerical value if the same set of input parameters is given and the same mass hierarchy is selected. But since Himalaya has a slightly more sophisticated way of choosing this hierarchy (see Sect. 2.1), its numerical $\alpha_t \alpha_s^2$ contribution does occasionally differ slightly from the one of $H^3m$.

In Fig. 7 we compare our results to the three-loop calculation presented in Ref. [68], assuming the input parameters for the “heavy sfermions” scenario defined in detail in the example folder of Ref. [69]. In the left panel the blue circles show the $H^3m$ result, including only the terms of $O(\alpha_t + \alpha_s \alpha_t + \alpha_t \alpha_s^2)$, where the MSSM DR top mass is calculated using the “running and decoupling” procedure described in Ref. [68]. The black crosses show the same result, except that the DR top mass

![Fig. 6 Comparison of the lightest Higgs pole mass calculated at the two- and three-loop level with FlexibleSUSY, FlexibleSUSY+Himalaya and HSSUSY as a function of the SUSY scale $M_S$ for $\tan \beta = 5$ and $X_t = -\sqrt{6}M_S$. The red band shows the size of the hierarchy selection criterion $\delta_i$. In the fixed-order calculations of FlexibleSUSY and FlexibleSUSY+Himalaya the Higgs mass becomes tachyonic for $M_S \lesssim 350$ GeV.](image)

![Fig. 7 Comparison of the lightest Higgs pole mass calculated at the one-, two- and three-loop level with FlexibleSUSY, FlexibleSUSY+Himalaya, $H^3m$ and HSSUSY as a function of the SUSY scale for the “heavy sfermions” scenario of Ref. [68]. The horizontal orange band shows the measured Higgs mass $M_h = (125.09 \pm 0.32)$ GeV including its experimental uncertainty.](image)
prediction obtained with \textit{HSSUSY}. The MSSM parameters are defined in the \textit{DR} scheme and are chosen in the style of Ref. [70].\footnote{The scenario of Ref. [70] appears to be not fully defined; in particular, $M_A$ and the sfermion mixing parameters other than $X_t$ remain unspecified.} The soft-breaking mass parameters of the left- and right-handed stops are set equal at the SUSY scale $M_S = \sqrt{m_{\tilde{t}_1} m_{\tilde{t}_2}}$, i.e. $m_{\tilde{t}}(M_S) = m_{\bar{t}}(M_S)$. All other soft-breaking sfermion mass parameters are set to $m_{\tilde{f}}(M_S) = m_{\tilde{f}_R}(M_S) + 1 \text{ TeV}$. Stop mixing is disabled, $X_t(M_S) = 0$, and the remaining trilinear couplings are set to zero at the scale $M_S$. The gaugino mass parameters, the super-potential $\mu$ parameter and the CP-odd \textit{DR} Higgs mass are set to $M_1(M_S) = M_2(M_S) = M_3(M_S) = 1.5 \text{ TeV}$, $\mu(M_S) = 200 \text{ GeV}$ and $m_A(M_S) = M_S$, respectively, and we fix $\tan\beta(M_Z) = 20$. As opposed to the results shown in Fig. 1 of Ref. [70],\footnote{Note that, in contrast to Ref. [70], we are using a logarithmic scale in Fig. 8.} we observe a reduction of $M_h$ towards higher loop orders, thus leading to the opposite conclusion of a heavy SUSY spectrum in this scenario, given the current experimental value for the Higgs mass. Reassuringly, the higher-order corrections move the fixed-order result closer to the resummed result, leading to agreement between the two at the level of about 1 GeV even at comparatively large SUSY scales.

5 Conclusions

We have presented the implementation \textit{Himalaya} of the three-loop $\mathcal{O}(\alpha_t \alpha_t^2 + \alpha_b \alpha_b^2)$ terms of Refs. [1,20] for the light CP-even Higgs mass in the MSSM, and its combination with the \textit{DR} spectrum generator framework \textit{FlexibleSUSY}. These three-loop contributions have been available in the public program \textit{H3m} before, where they were combined with the on-shell calculation of \textit{FeynHiggs}. With the implementation into \textit{FlexibleSUSY} presented here, we were able to study the size of the three-loop contributions within a pure \textit{DR} environment. Despite the fact that the genuine $\mathcal{O}(\alpha_t \alpha_t^2)$ corrections are positive\cite{1}, the combination with the two-loop decoupling terms in the top Yukawa coupling lead to an overall reduction of the Higgs mass prediction relative to the “original” two-loop \textit{FlexibleSUSY} result by about 2 GeV, depending on the value of the stop masses and the stop mixing. This moves the fixed-order prediction for the Higgs mass significantly closer to the result obtained from a pure EFT calculation in the region where both approaches are expected to give sensible results. Contributions of order $\mathcal{O}(\alpha_b \alpha_b^2)$ are found to be negligible in all scenarios studied here.
To indicate the remaining theory uncertainty due to higher-order effects, we have varied the renormalization scale which enters the calculation by a factor two. The results show that the inclusion of the three-loop contributions reduces the scale uncertainty of the Higgs mass by around a factor two, compared to a calculation without the genuine three-loop effects. We conclude that our implementation leads to an improved CP-even Higgs mass prediction relative to the two-loop results. Our implementation of the three-loop terms should be useful also for other groups that aim at a high-precision determination of the Higgs mass in SUSY models.

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Appendix A: Installation of Himalaya
Himalaya can be downloaded as a compressed package from [47]. After the package has been extracted, Himalaya can be configured and compiled by running

```
cd $HIMALAY_PATH
mkdir build
cd build
make
```

where $HIMALAY_PATH$ is the path to the Himalaya directory. When the compilation has finished, the build directory will contain the Himalaya library libHimalaya.a. For convenience, a library named libDSZ.a is created in addition, which contains the two-loop $O(a_t a_s)$ corrections from Ref. [12].

Appendix B: Installation of FlexibleSUSY with Himalaya
We provide a dedicated version of FlexibleSUSY 1.7.4, which uses Himalaya to calculate the Higgs pole mass at the three-loop level. This package contains three pre-generated MSSM models:

- **MSSMNoFVHimalaya** This model represents the MSSM without (s)fermion flavour violation, where $\tan \beta$ is fixed at the scale $M_Z$ and the other SUSY parameters are fixed at a user-defined input scale. The parameters $\mu$ and $B\mu$ are fixed by the electroweak symmetry breaking conditions. The SUSY mass spectrum, including the Higgs pole masses, is calculated at the scale $Q = \sqrt{m_{\tilde{t}_1} m_{\tilde{t}_2}}$, where $m_{\tilde{t}_i}$ are the two DR stop masses.

- **MSSMNoFVatMGUTHimalaya** This is the same model as the MSSMNoFVHimalaya, except that the input scale is the GUT scale $M_X$, defined to be the scale where $g_1(M_X) = g_2(M_X)$.

- **NUHMSSMNoFVHimalaya** This is the same model as the MSSMNoFVHimalaya, except that the soft-breaking Higgs mass parameters $m_{H_u}^2$ and $m_{H_d}^2$ are fixed by the electroweak symmetry breaking conditions.

The package FlexibleSUSY-1.7.4-Himalaya.tar.gz can be downloaded from Ref. [48]. To extract the package at the command line, run

```
tar -xf FlexibleSUSY-1.7.4-Himalaya.tar.gz
```

After the extraction, FlexibleSUSY must be configured and compiled by running

```
./configure \
--with-himalaya-incdir=$HIMALAY_PATH/source/include \n--with-himalaya-libdir=$HIMALAY_PATH/build
make
```
See ./configure --help for more options. One can use -j<N> to speed-up the compilation if <N> CPU cores are available. When the compilation has finished, the MSSM spectrum generators can be run from the command line as

```
models/MSSMNoFVHimalaya/run_MSSMNoFVHimalaya.x
--slha-input-file=models/MSSMNoFVHimalaya/LesHouches.in.MSSMNoFVHimalaya
--slha-output-file=LesHouches.out.MSSMNoFVHimalaya
```

The file LesHouches.out.MSSMNoFVHimalaya will then contain the SUSY particle spectrum in SLHA format. Alternatively, the Mathematica interface of FlexibleSUSY can be used:

```
math -run "<< "models/MSSMNoFVHimalaya/run_MSSMNoFVHimalaya.m"
```

For each model an example SLHA input file and an example Mathematica script can be found in models/<model>/.

### Appendix C: Configuration options to calculate the Higgs mass at three-loop level with FlexibleSUSY

To calculate the CP-even Higgs pole masses at order $O(\alpha_t \alpha^2_s + \alpha_b \alpha^2_s)$ at the scale $Q = M_S$, the top and bottom Yukawa couplings $y_t(M_S)$ and $y_b(M_S)$ must be extracted from the input parameters at the appropriate loop level.

#### Strong coupling constant
To calculate $M_h$ at the three-loop level at $O(\alpha_t \alpha^2_s + \alpha_b \alpha^2_s)$ correctly, $\alpha_s(M_S)$ must be extracted at the one-loop level from the input value $\alpha_s^{\text{SM}}(M_Z)$ as described in Sect. 3.1. To achieve that in FlexibleSUSY, the global threshold correction loop order (EXTPAR[7]) must be set to 1 (or higher) and the specific threshold correction loop order for $\alpha_s$ (3rd digit from the right in EXTPAR[24]) must be set to 1 (or higher) in the SLHA input file. See the next paragraph for an example.

In the Mathematica interface of FlexibleSUSY these two settings are controlled using the `thresholdCorrectionsLoopOrder` and `thresholdCorrections` symbols:

```
handle = FS<model>OpenHandle[
  fsSettings -> {
    thresholdCorrectionsLoopOrder -> 2,
    thresholdCorrections -> 122111121
  }
];
```

Here, `<model>` is the used FlexibleSUSY model from above, i.e. either MSSMNoFVHimalaya, MSSMNoFVatMGUTHimalaya or NUHMSMNoFVHimalaya.
Three-loop corrections to the CP-even Higgs mass To use
the three-loop corrections of order \( O(\alpha_t \alpha_t^2 + \alpha_b \alpha_b^2) \) to
the light CP-even Higgs mass in the MSSM from Refs. [1,20],
the pole mass and EWSB loop orders must be set to 3
in the SLHA input file. In addition, the individual three-loop
corrections should be switched on, by setting the flags 26
and 27 to 1. The user can select between the DR and MDR
scheme for the three-loop corrections by setting the flag 25
to 0 or 1, respectively:

| Block FlexibleSUSY |
|---------------------|
| 4 3 # pole mass loop order |
| 5 3 # EWSB loop order |
| 25 0 # ren. scheme for Higgs 3-loop corrections O(\alpha_t \alpha_t^2) |
| 26 1 # Higgs 3-loop corrections O(\alpha_b \alpha_b^2) |
| 27 1 # Higgs 3-loop corrections O(\alpha_t \alpha_t^2) |

In the Mathematica interface of FlexibleSUSY the
pole mass and EWSB loop orders are controlled using the
poleMassLoopOrder and ewsbLoopOrder symbols, respectively. The individual three-loop corrections can be switched on/off by using the higgs3loopCorrectionAtAsAs and
higgs3loopCorrectionAbAsAs symbols. The renormalization scheme is controlled by higgs3loopCorrectionRenScheme.
The above shown SLHA input settings read in FlexibleSUSY’s Mathematica interface

```mathematica
definition
handle = FS<model>OpenHandle[
   fsSettings -> {
   poleMassLoopOrder -> 3,  
   ewsbLoopOrder -> 3,  
   higgs3loopCorrectionRenScheme -> 0,  
   higgs3loopCorrectionAtAsAs -> 1,  
   higgs3loopCorrectionAbAsAs -> 1
   }
];
```

Three-loop renormalization group equations Optionally,
the known three-loop renormalization group equations
can be used to evolve the MSSM DR parameters from \( M_Z \)
to \( M_S \) [57,58]. To activate the three-loop RGEs, the \( \beta 
function loop order must be set to 3 in the SLHA input file:

| Block FlexibleSUSY |
|---------------------|
| 6 3 # beta-functions loop order |

In the Mathematica interface of FlexibleSUSY the
\( \beta 
function loop order is controlled using the betaFunctionLoopOrder symbol:

```mathematica
handle = FS<model>OpenHandle[
   fsSettings -> {
   betaFunctionLoopOrder -> 3
   }
];
```

Recommended configuration options for FlexibleSUSY+
Himalaya We recommend to run FlexibleSUSY+
Himalaya with the following SLHA configuration options:

| Block FlexibleSUSY |
|---------------------|
| 4 3 # pole mass loop order |
| 5 3 # EWSB loop order |
| 6 3 # beta-functions loop order |
| 7 2 # threshold corrections loop order |
| 24 12211121 # individual threshold correction loop orders |
| 25 0 # ren. scheme for 3L corrections O(\alpha_t \alpha_t^2) |
| 26 1 # Higgs 3-loop corrections O(\alpha_t \alpha_t^2) |
| 27 1 # Higgs 3-loop corrections O(\alpha_b \alpha_b^2) |
At the Mathematica level we recommend to use:

```mathematica
handle = FSCmodelOpenHandle[
  fsSettings -> {
    poleMassLoopOrder -> 3,
    ewslLoopOrder -> 3,
    betaFunctionLoopOrder -> 3,
    thresholdCorrectionsLoopOrder -> 2,
    thresholdCorrections -> 122111121,
    higgs3loopCorrectionRenScheme -> 0,
    higgs3loopCorrectionAtAsks -> 1,
    higgs3loopCorrectionAbAsks -> 1
  }
]; ...
```

### Appendix D: Himalaya interface

**Input parameters** To calculate the three-loop corrections to the light CP-even Higgs pole mass at order $O(\alpha_t^2 \alpha_s^2 + \alpha_b \alpha_s^2)$ with Himalaya, the set of DR parameters is needed, which is shown in the following code snippet. The parameters are stored in the `struct Parameters` which contains the following members:

```c
typedef Eigen::Matrix<double,2,1> V2;
typedef Eigen::Matrix<double,2,2> RM22;
typedef Eigen::Matrix<double,3,3> RM33;

struct Parameters {
  // DR-bar parameters
  double scale{}; // renormalization scale
  double mu{}; // mu parameter
  double g3{}; // gauge coupling g3 SU(3)
  double vd{}; // VEV of down Higgs
  double vu{}; // VEV of up Higgs
  RM33 mq2(RM33::Zero()); // soft-breaking squared left-handed squark
  RM33 md2(RM33::Zero()); // soft-breaking squared right-handed
  RM33 mu2(RM33::Zero()); // soft-breaking squared down-squark mass parameters
  double At{}; // trilinear stop-Higgs coupling
  double Ab{}; // trilinear sbottom-Higgs coupling
  // DR-bar masses
  double MG{}; // gluino
  double MW{}; // W
  double MZ{}; // Z
  double Mt{}; // top quark
  double Mb{}; // down quark
  double MA{}; // CP-odd Higgs
  V2 MSt(nan, nan); // stops
  V2 MSb(nan, nan); // sbottoms
  // DR-bar mixing angles
  double s2t(nan); // sine of 2 times the stop mixing angle
  double s2b(nan); // sine of 2 times the sbottom mixing angle
};
```

All these parameters are given at the scale stored in the `scale` variable, which is typically the SUSY scale. The input values of the stop/sbottom masses and their associated mixing angle are optional, so their default value is set to `nan` (std::numeric_limits<T>::quiet_NaN()). If no input is provided, the DR stop masses will be calculated by diagonalising the stop mass matrix,

$$
\mathcal{M}_t = \begin{pmatrix} (m_{\tilde{Q}}^2)_{33} + m_t^2 + g_t M_Z^2 c_{2\beta} & \tilde{X}_t \\
\tilde{X}_t & (m_{\tilde{u}}^2)_{33} + m_t^2 + Q_t s_W^2 M_Z^2 c_{2\beta} \end{pmatrix}.
$$

(30)

Here, $(m_{\tilde{Q}}^2)_{33}$ is the left third generation scalar quark mass parameter, $g_t = 1/2 - Q_t s_W^2$, $\tilde{X}_t = m_t (A_t - \mu \cot \beta)$, $(m_{\tilde{u}}^2)_{33}$ the right scalar top mass parameter, $Q_t = 2/3$, $s_W$ the sine of the weak mixing angle and $c_{2\beta} = \cos(2\beta)$. The sbottom mass matrix is obtained by replacing $t \to b$ and $\tilde{u} \to \tilde{d}$ in (30) with $g_b = -(1/2 + Q_b s_W^2)$, $\tilde{X}_b = m_b (A_b - \mu \tan \beta)$ and $Q_b = -1/3$. 
Table 1 Description of the member functions of the HierarchyObject class

| Function name               | Returned value                                                                 |
|-----------------------------|---------------------------------------------------------------------------------|
| getIsAlphab()               | Returns the bool isAlphab                                                        |
| getSuitableHierarchy()      | Returns the suitable hierarchy as an int                                          |
| getAbsDiff2L()              | Returns the double $\delta_{i0}^{2L}$ for the suitable hierarchy                |
| getRelDiff2L()              | Returns the double $\delta_{i0}^{2L}/M_{h}^{DSZ}$ for the suitable hierarchy    |
| getExpUncertainty(int loops)| Returns the uncertainty of the expansion at the given loop order (cf. Sect. 2.1) |
| getDMh(int loops)           | Returns the Higgs mass matrix proportional to $\alpha_t$ or $\alpha_b$ at the given loop order. Note that at the two-loop level only corrections of order $O(\alpha_t \alpha_s)$ are considered. |
| getDRToMDRShift()           | Returns the loop correction to the Higgs mass matrix to convert from the DR to MDR scheme, according to Eq. (5). The MDR corrections are of order $O(\alpha_s + \alpha_b^2)$ by convention. |
| getMDRMasses()              | Returns the vector of MDR masses $(\tilde{m}_{\tilde{t},1}, \tilde{m}_{\tilde{t},2})$ $(\tilde{m}_{\tilde{b},1}, \tilde{m}_{\tilde{b},2})$, if isAlphab is false (true). |

Calculation of the three-loop corrections All the functions which are required for the calculation of the three-loop corrections are implemented as methods of the class HierarchyCalculator.

In the context of Himalaya, the procedure described in Sect. 2 is implemented by the member function

```cpp
calculateDMh3L(bool isAlphab, int mdrFlag);
```

Here, the integer mdrFlag is optional and can be used to switch between the DR- (0) and the MDR-scheme (1). The DR-scheme is chosen as default. The returned object holds all information of the hierarchy selection process, such as the best fitting hierarchy, or the relative error $\delta_{i0}^{2L}/M_{h}^{DSZ}$, the selection method described in Sect. 2 is also applied to the (s)bottom contributions by replacing $t \rightarrow b$, so that only terms of order $O(\alpha_b \alpha_s)$ are considered in the comparison. By setting the Boolean parameter isAlphab to false (true) the calculateDMh3L function returns the HierarchyObject for the loop corrections proportional to $\alpha_t$ ($\alpha_b$).
Example  Function calls for the benchmark point SPS2:

```cpp
#include "HierarchyCalculator.hpp"
#include "HierarchyObject.hpp"

h3m::Parameters setupSPS2()
{
    h3m::Parameters pars;
    pars.scale = 1.11090135E+03;
    pars.mu = 3.73337018E+02;
    pars.g3 = 1.0618716E+00;
    pars.vd = 2.51008404E+01;
    pars.vu = 2.41869332E+02;
    pars.mq2 << 2.36646981E+06, 0, 0,
                0, 2.36644973E+06, 0,
                0, 0, 1.63230152E+06;
    pars.md2 << 2.35612778E+06, 0, 0,
                0, 2.35610884E+06, 0,
                0, 0, 2.31917415E+06;
    pars.mu2 << 2.35685097E+06, 0, 0,
                0, 2.35682945E+06, 0,
                0, 0, 9.05923409E+05;
    pars.Ab = -784.3356416708631;
    pars.At = -527.8746242245387;
    pars.MA = 1.48446235E+03;
    pars.MG = 6.69045022E+02;
    pars.MW = 8.04001915E+01;
    pars.MZ = 8.97608307E+01;
    pars.Mt = 1.47885846E+02;
    pars.Mb = 2.38918959E+00;
    pars.MSt << 9.57566721E+02, 1.28878643E+03;
    pars.MSb << 1.27884964E+03, 1.52314587E+03;
    pars.s2t = sin(2*asin(1.13197339E-01));
    pars.s2b = sin(2*asin(-9.99883015E-01));
    return pars;
}

int main() {
    h3m::HierarchyCalculator hc(setupSPS2());
    // get the HierarchyObject with entries proportional to alpha_t
    // in the DR scheme
    auto hoTop = hc.calculateDMh3L(false);
    // get the 3-loop correction O(alpha_t * alpha_s^2)
    auto DMh_top_3L = hoTop.getDMh(3);
}
```

Estimation of the uncertainty of the expansion  In addition to the relative error of the hierarchy choice $\delta_{\log}^{L}/M_{DZ}$ (see above), we provide a member function which returns a measure for the quality of convergence of the expansion at a given loop order, given by $\delta_{\text{conv}}^{i_0}$ defined in Eq. (2), where again $i_0$ labels the “optimal” hierarchy. It can be called with

```cpp
Eigen::Matrix2d HierarchyCalculator::getExpansionUncertainty(
    HierarchyObject ho, const Eigen::Matrix2d& massMatrix
    int oneLoopFlag, int twoLoopFlag, int threeLoopFlag);
```
Its arguments are a `HierarchyObject`, the Higgs mass matrix `MhHiggs` up to the loop order of interest, and three flags (oneLoopFlag, twoLoopFlag, threeLoopFlag) to define the desired loop orders. Using the member function `calculateDMh`, the returned `HierarchyObject` provides the user with the quantity $\delta^{\text{conv}}_h$ at two and three loops by default.

**Example** For the benchmark point SPS2 one could estimate the uncertainty by calling

```
... // get the HierarchyObject with entries proportional to alpha_t
// in the DR scheme
auto hoTop = hc_.calculateDMh3L(false);

// get the expansion uncertainty for the
// 3-loop correction O(alpha_t * alpha_s^-2)
auto expansionUncertainty3LTop = hoTop.getExpUncertainty(3);

// calculate the expansion uncertainty for the
// 1-loop correction O(alpha_t)
auto expansionUncertainty1LTop = hc_.getExpansionUncertainty(hoTop,
    ho_.getDMh(0), 1, 0, 0);
```

**References**

1. P. Kant, R.V. Harlander, L. Mihaila, M. Steinhauser, Light MSSM Higgs boson mass to three-loop accuracy. JHEP 08, 104 (2010). arXiv:1005.5709
2. ATLAS, CMS collaboration, G. Aad et al., Combined measurement of the Higgs boson mass in pp collisions at $\sqrt{s} = 7$ and 8 TeV with the ATLAS and CMS experiments. Phys. Rev. Lett. 114, 191803 (2015). arXiv:1503.07589
3. H.E. Haber, R. Hempfling, Can the mass of the lightest Higgs boson of the minimal supersymmetric model be larger than $m_Z$? Phys. Rev. Lett. 66, 1815–1818 (1991)
4. J.R. Ellis, G. Ridolfi, F. Zwirner, Radiative corrections to the masses of supersymmetric Higgs bosons. Phys. Lett. B 257, 83–91 (1991)
5. S. Heinemeyer, W. Hollik, G. Weiglein, QCD corrections to the masses of the neutral CP-even Higgs bosons in the MSSM. Phys. Rev. D 58, 091701 (1998). arXiv:hep-ph/9803277
6. The ATLAS collaboration, Search for a scalar partner of the top quark in the jets+Etmiss final state at $\sqrt{s} = 13$ TeV with the ATLAS detector, ATLAS-CONF-2017-020 (2017)
7. The ATLAS collaboration, Search for direct top squark pair production in events with a Higgs or Z boson, and missing transverse momentum in $\sqrt{s} = 13$ TeV pp collisions with the ATLAS detector, ATLAS-CONF-2017-019 (2017)
8. S. Heinemeyer, W. Hollik, G. Weiglein, Precise prediction for the mass of the lightest Higgs boson in the MSSM. Phys. Lett. B 440, 296–304 (1998). arXiv:hep-ph/9807423
9. S. Heinemeyer, W. Hollik, G. Weiglein, The masses of the neutral CP-even Higgs bosons in the MSSM: Accurate analysis at the two loop level. Eur. Phys. J. C 9, 343–366 (1999). arXiv:hep-ph/9812472
10. R.-J. Zhang, Two loop effective potential calculation of the lightest CP even Higgs boson mass in the MSSM. Phys. Lett. B 447, 89–97 (1999). arXiv:hep-ph/9808299
11. J.R. Espinosa, R.-J. Zhang, MSSM lightest CP even Higgs boson mass to $O(\alpha_t, \alpha_s)$: The effective potential approach. JHEP 03, 026 (2000). arXiv:hep-ph/9912236
12. G. Degrassi, P. Slavich, F. Zwirner, On the neutral Higgs boson masses in the MSSM for arbitrary stop mixing. Nucl. Phys. B 611, 403–422 (2001). arXiv:hep-ph/0105096
13. A. Brignole, G. Degrassi, P. Slavich, F. Zwirner, On the $O(\alpha_t^2)$ two loop corrections to the neutral Higgs boson masses in the MSSM. Nucl. Phys. B 631, 195–218 (2002). arXiv:hep-ph/0112177
14. A. Dedes, P. Slavich, Two loop corrections to radiative electroweak symmetry breaking in the MSSM. Nucl. Phys. B 657, 333–354 (2003). arXiv:hep-ph/0212132
15. A. Brignole, G. Degrassi, P. Slavich, F. Zwirner, On the two loop sbottom corrections to the neutral Higgs boson masses in the MSSM. Nucl. Phys. B 643, 79–92 (2002). arXiv:hep-ph/0206101
16. A. Dedes, G. Degrassi, P. Slavich, On the two loop Yukawa corrections to the MSSM Higgs boson masses at large tan $\beta$. Nucl. Phys. B 672, 144–162 (2003). arXiv:hep-ph/0305127
17. J.R. Espinosa, R.-J. Zhang, Complete two loop dominant corrections to the mass of the lightest CP even Higgs boson in the minimal supersymmetric standard model. Nucl. Phys. B 586, 3–38 (2000). arXiv:hep-ph/0003246
18. S. Heinemeyer, W. Hollik, H. Rzehak, G. Weiglein, High-precision predictions for the MSSM Higgs sector at $O(\alpha_t, \alpha_s)$. Eur. Phys. J. C 39, 465–481 (2005). arXiv:hep-ph/0411114
19. S.P. Martin, Complete two loop effective potential approximation to the lightest Higgs scalar boson mass in supersymmetry. Phys. Rev. D 67, 095012 (2003). arXiv:hep-ph/0211366
20. R.V. Harlander, P. Kant, L. Mihaila, M. Steinhauser, Higgs boson mass in supersymmetry to three loops. Phys. Rev. Lett. 100, 191602 (2008). arXiv:0803.0672
21. S. Borowka, T. Hahn, S. Heinemeyer, G. Heinrich, W. Hollik, Momentum-dependent two-loop QCD corrections to the neutral Higgs-boson masses in the MSSM. Eur. Phys. J. C 74, 2994 (2014). arXiv:1404.7074
63. A.V. Bednyakov, A.F. Pikelner, V.N. Velizhanin, Higgs self-coupling beta-function in the Standard Model at three loops. Nucl. Phys. B 875, 552–565 (2013). arXiv:1303.4364

64. M. Sperling, D. Stöckinger, A. Voigt, Renormalization of vacuum expectation values in spontaneously broken gauge theories. JHEP 07, 132 (2013). arXiv:1305.1548

65. M. Sperling, D. Stöckinger, A. Voigt, Renormalization of vacuum expectation values in spontaneously broken gauge theories: Two-loop results. JHEP 01, 068 (2014). arXiv:1310.7629

66. E. Bagnaschi, J. Pardo Vega, P. Slavich, Improved determination of the Higgs mass in the MSSM with heavy superpartners. Eur. Phys. J. C77, 334 (2017). arXiv:1703.08166

67. B.C. Allanach, A. Djouadi, J.L. Kneur, W. Porod, P. Slavich, Precise determination of the neutral Higgs boson masses in the MSSM. JHEP 09, 044 (2004). arXiv:hep-ph/0406166

68. D. Kunz, L. Mihaila, N. Zerf, $\mathcal{O}(\alpha_s^3)$ corrections to the running top-Yukawa coupling and the mass of the lightest Higgs boson in the MSSM. JHEP 12, 136 (2014). arXiv:1409.2297

69. https://www.ttp.kit.edu/Progdata/ttp10/ttp10-23/H3m-v1.3/. Accessed 22 Nov 2017

70. J.L. Feng, P. Kant, S. Profumo, D. Sanford, Three-loop corrections to the Higgs Boson mass and implications for supersymmetry at the LHC. Phys. Rev. Lett. 111, 131802 (2013). arXiv:1306.2318