Quantization of Spin Direction for Solitary Waves in a Uniform Magnetic Field

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It is known that there are nonlinear wave equations with localized solitary wave solutions. Some of these solitary waves are stable (with respect to a small perturbation of initial data) and have nonzero spin (nonzero intrinsic angular momentum in the center of momentum frame). In this paper we consider vector-valued solitary wave solutions to a nonlinear Klein-Gordon equation and investigate the behavior of these spinning solitary waves under the influence of an externally imposed uniform magnetic field. We find that the only stationary spinning solitary wave solutions have spin parallel or anti-parallel to the magnetic field direction.

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I. INTRODUCTION

Stable localized solitary wave solutions of nonlinear wave equations are known to exist ([1], [2], [3], [4], [5], [6]). In particular, the existence of stable solitary wave solutions has been proven for certain nonlinear Klein-Gordon and Schrödinger equations ([2], [3], [4], [5]) and it has also been established that these solutions can have nonzero spin (intrinsic angular momentum) ([2], [3]).

Here we consider the nonlinear Klein-Gordon equation (NLKG)

\[ u_{tt} - \Delta u = \vec{g}(u) \tag{1} \]

where \( u : \mathbb{R}^{3+1} \rightarrow \mathbb{R}^N \) with \( N \) even (see [5]) and the nonlinearity \( \vec{g} : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is defined by \( \vec{g}(y) = g(|y|) \frac{y}{|y|} \), for \( y \neq 0 \), \( g : [0, \infty) \rightarrow \mathbb{R} \) being a continuous function and \( \vec{g}(0) = 0 \).

In this paper we will examine the Noether conserved quantity

\[ \vec{S}[u] \equiv \int_{\mathbb{R}^3} u_t \cdot (\vec{X} \times \vec{\nabla} u) d^3 \vec{X}. \tag{2} \]

called spin, which results from the rotational invariance of NLKG. (We note that for a function \( u : \mathbb{R}^{3+1} \rightarrow \mathbb{R}^N \), we define a counterclockwise rotation of \( u \) about an axis through the origin in \( \mathbb{R}^3 \) through an angle \( \theta \) to be \( u(R_\theta^{-1} \vec{X}, t) \) where \( R_\theta^{-1} \) is a \( 3 \times 3 \) counterclockwise rotation matrix.) This functional, \( \vec{S} \), gives the angular momentum about the origin of a solution \( u \). Our goal will be to find the spin of stationary solitary waves when exposed to an external uniform magnetic field, \( \vec{B} \). The notation \( S_x, S_y \), and \( S_z \) will be used to denote the magnitude of the \( x, y \), and \( z \) components of \( \vec{S} \) respectively.

Equation (1) can be written compactly using relativistic index notation as

\[ \partial^\alpha \partial_{\alpha} u = \vec{g}(u). \tag{3} \]

where \( \partial_{\alpha} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}) \), and \( \partial^\alpha = (-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}, \frac{\partial}{\partial t}) \). We model the imposition of an external uniform magnetic field of strength \( B = |\vec{B}| \) parallel to the \( z \)-axis by making the minimal-coupling substitutions \( \partial^\alpha \rightarrow \partial^\alpha - \sigma A^\alpha \) and \( \partial_{\alpha} \rightarrow \partial_{\alpha} - \sigma A_\alpha \), giving us

\[ (\partial^\alpha - \sigma A^\alpha)(\partial_{\alpha} - \sigma A_\alpha)u = \vec{g}(u) \tag{4} \]

where \( \sigma \) is a fixed \( N \times N \) real skew-symmetric matrix with \( \sigma^2 = -I \) and

\[ A = B^2 \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5} \]

It will be assumed throughout that \( B \) is small. This will allow us to simplify matters by ignoring terms in (4) that involve \( B^2 \). We are going to look for stationary solitary waves solutions of (4) that carry nonzero spin. A class of such solitary waves will be examined and it will be shown that they have spin either parallel or antiparallel to the uniform magnetic field.

We look for standing-wave solutions of the form

\[ u(\vec{X}, t) = e^{\Omega t} \psi(\vec{X}) \] where \( \Omega \) is some \( N \times N \) skew-symmetric matrix that commutes with \( \sigma \). We are interested in solutions that have \( \psi(\vec{X}) = \psi(\vec{X})w(r) \), where \( w : [0, \infty) \rightarrow \mathbb{R} \), with \( r = (x^2 + y^2 + z^2)^{\frac{1}{2}} \), and \( \psi : \mathbb{R}^{3+1} \rightarrow \mathbb{R}^N \) is a unit-vector-valued eigenfunction of the spherical Laplacian. The Laplacian \( \Delta \) can be decomposed into radial and angular components \( \Delta = \Delta_R + \frac{1}{r^2} \Delta_S \). The spherical component \( \Delta_S \), is a second-order derivative operator with only angular derivatives that acts on real-valued functions defined on the unit sphere. It also acts on real-valued functions defined on \( \mathbb{R}^3 \), leaving the radial dependence unchanged. It is known that there exist unit-vector-valued eigenfunctions of \( \Delta_S \) with any of the eigenvalues \( \mu_l = -l(l+1) \), where \( l = 0, 1, 2, 3, \ldots (\mathbb{R}, \mathbb{S}) \). Given a nonnegative integer \( l \), such an eigenfunction \( \psi : S^2 \rightarrow S^{N-1} \) with eigenvalue \( \mu_l \) exists provided that \( N \geq 2l+1 \). So \( \Delta_S \psi = -l(l+1)\psi \),

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where we extend the action of $\Delta_S$ to vector $\mathbb{R}^N$-valued functions by allowing the operator to act componentwise. Thus the coordinate functions of $\Psi$ are eigenfunctions of $\Delta_S$. Substituting the ansatz
\[
uu(\hat{X}, t) = e^{it\hat{\Psi}(\hat{X})}w(r)
\]
into (1) produces the ordinary differential equation
\[w''(r) + \frac{2}{r}w'(r) - \frac{l(l + 1)}{r^2}w(r) + f(w(r)) = 0\]
where $l = 0, 1, 2, 3, \ldots$ and $f(y) = g^2(y) + \omega^2y$ with appropriate conditions on $f$ to ensure the existence of localized radial solutions (see [3]). It is shown in [5] that there then exist functions $w(0, \infty) \to \mathbb{R}$ that are exponentially decreasing far from the origin. Consequently, for such functions $w$, $u(\hat{X}, t) = e^{it\hat{\Psi}(\hat{X})}w(r)$ is a solution to NLKG.

We will prove the following theorems:

**Theorem 1.** Let $B$ be a fixed real number, $\sigma$ an $N \times N$ real skew-symmetric matrix, and $\vec{g}: \mathbb{R}^N \to \mathbb{R}^N$ where $\vec{g}(v) = g(|v|^2)\frac{v}{|v|}$ with $\vec{g}(0) = 0$ for some continuous function $g: [0, \infty) \to \mathbb{R}$. Let $u: \mathbb{R}^{3+1} \to \mathbb{R}^N$ be a solution of
\[u_{tt} - \Delta u - B\sigma(x\partial_y - y\partial_x) = \vec{g}(u)\]
of the form $u(\hat{X}, t) = e^{it\hat{\Psi}(\hat{X})}w(r)$ where $\sigma$ is an $N \times N$ real skew-symmetric matrix with the property $[\Omega, \sigma] = 0$, $\hat{\Psi}$ is a unit-vector valued eigenfunction of the spherical Laplacian and $w$ is a solution to (7) as described above. Define the spin functional $\tilde{S}$ by
\[\tilde{S}[u] \equiv \int_{\mathbb{R}^3} u_t \cdot (\hat{X} \times \hat{\nabla} u) d^3\hat{X}.
\]
If $B \neq 0$ then $S_x[u] = S_y[u] = 0$.

**Theorem 2.** There are solutions with spin parallel (anti-parallel) to $\vec{B}$.

In section II we prove Theorems 1 and 2. Simply put, Theorem 1 states that solutions with the ansatz (6) to (10) cannot have nonzero spin components that are not parallel to the magnetic field unless $\vec{B} = 0$, while Theorem 2 implies that the only solutions of the ansatz to the equation have nonzero spin parallel to the $\vec{B}$ field.

It is important to note that the discussion here is not a quantum mechanical one. Although many of the constructions have analogues in quantum mechanics, the interpretations are different.

**II. PROOFS OF THEOREMS 1 AND 2**

In what follows below, we will denote the $x$, $y$, and $z$ components of $(-i)(\hat{X} \times \hat{\nabla})$ by $L_x$, $L_y$ and $L_z$ respectively. Thus we have:
\[L_x = -i\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}\right)\]
\[L_y = -i\left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}\right)\]
\[L_z = -i\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)\]
Inserting (6) into (7) gives
\[\left[\Omega^2 \hat{\Psi}(\hat{X}) - B\sigma(x\partial_y - y\partial_x)\hat{\Psi}(\hat{X})\right]w(r) - \left(\Delta_R w(r) - \frac{l(l + 1)}{r^2}w(r) + g(w(r))\right)\hat{\Psi}(\hat{X}) = 0\]

Proof of **Theorem 1**: Since both $\Omega$ and $\sigma$ are skew-symmetric, then they are both real. By hypothesis, $[\Omega, \sigma] = 0$. Thus there exists an orthonormal basis of $\mathbb{C}^N$ consisting of vectors which are eigenvectors for both $\Omega$ and $\sigma$ (9). Also since $\Delta_S$ and $L_\sigma$ have a common set of orthonormal eigenfunctions, $\xi_{-l}, \ldots, \xi_{-1}, \xi_1, \ldots, \xi_l$ (10) that span the space of all eigenfunctions of $\Delta_S$ with eigenvalue $-l(l + 1)$, there exists $\alpha_{jm} \in \mathbb{C}, -l \leq m \leq l, 1 \leq j \leq N$, giving the expansion
\[\hat{\Psi}(\hat{X}) = \sum_{j=1}^{N} \sum_{m=-l}^{l} \alpha_{jm} \phi_j \xi_m(\hat{X})\]
where $\{\phi_j\}$ are an orthonormal basis of $\mathbb{C}^N$ of eigenvectors of $\sigma$ and $\Omega$ with $\sigma \phi_j = \epsilon_j \phi_j$ where $\epsilon_j = \pm 1$ and $\Omega \phi_j = i\nu_j \phi_j$ with real $\nu_j$. Substituting (14) into (13) we get
\[\sum_{j=1}^{N} \sum_{m=-l}^{l} \alpha_{jm} \phi_j \xi_m \left\{\left(-\nu_j^2 + B\epsilon_j m\right)w(r) - \eta(r)\right\} = 0\]
where $\eta = \Delta_R w(r) - \frac{l(l + 1)}{r^2}w(r) + g(w(r))$. Therefore, for each $j$ and $m$ either
\[\alpha_{jm} = 0\]
or
\[(-\nu_j^2 + \epsilon_j Bm)w(r) - \eta(r) = 0\]
Next we compute the quantity $S_x[u] + iS_y[u]$. It should be noted that since $u_t$ is real, it is equal to its complex conjugate. This will enable us to simplify things later. From (10) we recall that $(L_x + iL_y)\xi_m = \sqrt{l(l + 1) - m(m + 1)}\xi_{m+1}$. Using this
\[S_x[u] + iS_y[u] = \sum_{j=1}^{N} \sum_{m=-l}^{l} \left(\int_0^\infty w^2(r)r^2 dr\right) \times \alpha_{jm(p+1)} \alpha_{jp} \nu_j \sqrt{l(l + 1) - p(p + 1)}\]
If $S_z[u] + iS_y[u]$ is to be nonzero, then for some $m$, $j$ and $p$, $\alpha_{j(p+1)} \neq 0$ and $\alpha_{jp} \neq 0$. From (15) and (17) then

$$
\left(-\nu_j^2 + \epsilon_j B(p + 1)\right)w(r) - \eta(r) = 0
$$

and

$$
\left(-\nu_j^2 + \epsilon_j Bp\right)w(r) - \eta(r) = 0
$$

Subtract to get

$$
Bw(r) = 0 \quad \Rightarrow B = 0 \quad \text{or} \quad w(r) \equiv 0
$$

So if $B \neq 0$ then $S_z + iS_y = 0$

Proof of Theorem 2: Without loss of generality, we assume $B > 0$ and take

$$
\sigma = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

It is straightforward to show that

$$
\hat{\Psi}(\hat{X}) \equiv \frac{1}{r} \begin{pmatrix}
x \\ y \\ z \\ 0
\end{pmatrix}
$$

is an eigenfunction of $\Delta_x$ with eigenvalue $-l(l+1) = -2$. In (13) we will choose $\Omega$ so that

$$
\Omega^2\hat{\Psi}(\hat{X}) - B\sigma(x\partial_y - y\partial_x)\hat{\Psi}(\hat{X})
$$

is a multiple of $\hat{\Psi}$. Let

$$
\Omega = \begin{pmatrix}
0 & -\omega_1 & 0 & 0 \\
\omega_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -\omega_2 \\
0 & 0 & \omega_2 & 0
\end{pmatrix}
$$

Substituting into

$$
\Omega^2\hat{\Psi}(\hat{X}) - B\sigma(x\partial_y - y\partial_x)\hat{\Psi}(\hat{X})
$$

produces

$$
\begin{pmatrix}
(-\omega_1^2 + B)x \\
(-\omega_2^2 + B)y \\
-\omega_z^2 z \\
0
\end{pmatrix}
$$

This will be a multiple of $\hat{\Psi}$ whenever $\omega_2 = \pm \sqrt{\omega_1 - B}$. So

$$
\Omega = \begin{pmatrix}
0 & -\omega & 0 & 0 \\
\omega & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{\omega^2 - B} \\
0 & 0 & \sqrt{\omega^2 - B} & 0
\end{pmatrix}
$$

We put this form of $\Omega$ into the wave equation to get an ordinary differential equation for $w$:

$$
w'' + \frac{2}{r}w' - \frac{2}{r^2}w + g(w) + (\omega^2 - B)w = 0
$$

This is known to have exponentially localized solutions provided $\lim_{s \to 0} \frac{d \left| g(s) \right|}{ds} \leq 0$ (18). So we require $-\rho^2 + \omega^2 - B \leq 0$ (where $\lim_{s \to 0} \frac{d \left| g(s) \right|}{ds} = \rho^2$) which is the case for all $\omega$ with $0 \leq \omega^2 \leq \rho^2 + B$. Then, according to the results of [5], there are $C^2$ exponentially localized solutions to (30). The corresponding functions $u(\vec{X}, t) = e^{i\Omega(t)}\hat{\Psi}(\hat{X})w(r)$ are solutions to (14).

Using the form of $\Omega$ we now show that a solution of the form $u(\vec{X}, t) = e^{i\Omega(t)}\hat{\Psi}(\hat{X})w(r)$ has nonzero spin parallel to the direction of the magnetic field $B$. The component of the spin parallel to $\vec{B}$ is

$$
S_z[u] = \int_{\mathbb{R}^3} w^2 \hat{\Omega}(\hat{X}) \cdot (x\partial_y - y\partial_x)\hat{\Psi}(\hat{X})d^3\hat{X}
$$

$$
= \frac{\omega}{r^2} \left( y^2 + x^2 \right)
$$

$$
> 0
$$

if $\omega > 0$ (and both $x$ and $y$ are nonzero). So,

$$
S_z[u] = \int_{\mathbb{R}^3} \frac{\omega^2}{r^2} \omega(x^2 + y^2)d^3\hat{X}
$$

is nonzero.

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