Dynamical Resources

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Quantum channels are quintessential to quantum information, being used in all protocols, and describing how systems evolve in space and time. As such, they play a key role in the manipulation of quantum resources, and they are often resources themselves, called dynamical resources. This forces us to go beyond standard resource theories of quantum states. Here we provide a rigorous foundation for dynamical resource theories, where the resources into play are quantum channels, explaining how to manipulate dynamical resources with free superchannels. In particular, when the set of free superchannels is convex, we present a novel construction of an infinite and complete family of convex resource monotones, giving necessary and sufficient conditions for convertibility under free superchannels. After showing that the conversion problem in convex dynamical resource theories can be solved with conic linear programming, we define various resource-theoretic protocols for dynamical resources. These results serve as the framework for the study of concrete examples of theories of dynamical resources, such as dynamical entanglement theory.

I. INTRODUCTION

The remarkable success of quantum information stems from the fact that quantum objects provide concrete advantages in several tasks. Think, for instance, of entangled states [1], which can be harnessed to implement protocols that have no classical analogue [2–5]. Similar to entanglement, other quantum features are resources, such as coherence in quantum superpositions [6]. The idea of entanglement and other quantum features helping in information-theoretic tasks can be made rigorous with the framework of resource theories [7–17]. This framework is so general and powerful that it can be extended even beyond the quantum case [18–27].

Resource theories have been used to study a great number of physical situations [14], always providing new insights into quantum theory and novel results for quantum information protocols. The basic idea behind them is that an agent operates on a quantum system to perform some task, but they do not have access to the full set of quantum operations. Instead, they can only perform a strict subset of them, called free operations. Similarly, they cannot prepare the full set of quantum states, but only a strict subset of them, the free states. The restriction usually comes from the physical constraints of the task the agent is trying to perform: free operations are those that are easy to implement in the physical scenario the agent operates in. Anything that can help the agent overcome their restriction is regarded as a valuable resource.

The convertibility between two resources under free operations sets up a preorder on the set of resources, whereby a resource is more valuable than another if, from the former, it is possible to reach a larger set of resources. This allows one to introduce the notion of resource monotone, a real-valued function that assigns a “price” to resources according to their preorder. Monotones often have a very important operational and physical meaning (e.g. the entropy or the free energy in quantum thermodynamics [28–30]), for they quantify how well a given task can be performed [14]. Two tasks that are particularly relevant in resource theories are extracting the maximum amount of the maximal resource out of a generic resource (distillation), and minimizing the amount of the maximal resource necessary to produce a given resource (cost) [7, 8, 10, 14, 17, 23]. The distillation and cost of a state obey a Carnot-like inequality, with the distillation always less than or equal to the cost [31].

Resource theories have been studied in great detail when the resources involved are states (also known as static resources) [14]. In this case, one wants to study the conversion between states. This is the usual setting in which, e.g., one studies entanglement theory [1, 32].

Nevertheless, if one looks closely at the first examples where entanglement proved to be a resource (e.g. quantum teleportation [2] and dense coding [3]), one notices they involve the conversion of a state into a particular channel, i.e. a static resource into a dynamical one [33, 34]. Therefore the need to go beyond conversion between static resources is built in the very first protocol showing the value of quantum resources. This is supported by the fact that in physics everything, including a state, can be viewed as a dynamical resource [35–37]. Extending resource theories from states to channels [14, 38–40] has recently gained considerable attention [18, 21, 41–64], because of their relevance in a lot of information-theoretic situations [14, 35, 39, 65]. Moreover, since quantum channels represent the most general ways in which a physical system evolves, for a more effective exploitation of quantum resources, it is essential to understand how they are consumed or produced by evolution.

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In theories of dynamical resources, the agent converts different channels by means of a restricted set of supermaps \[35, 43, 66–70\]. In particular, we focus on supermaps that send quantum channels to quantum channels. They are called superchannels. They are not just abstract entities, but they can be realized in a laboratory with a pre-processing channel and a post-processing channel, connected by a memory system \[35, 43\]. Clearly, if we take the pre- and the post-processing of a superchannel to be free channels (according to some resource theory of states), we have a free superchannel \[18\], which sends free channels to free channels (even in a complete sense). This is the most common approach to constructing free superchannels \[18, 39\].

In this article, which is a companion to Ref. \[59\], we present the general framework of resource theories of quantum processes, which constitutes the mathematical framework for our treatment of dynamical entanglement announced in Ref. \[59\]. We note how for the largest class of free superchannels in a resource theory, which are completely resource non-generating superchannels, it is not clear if they can actually be realized in terms of free pre- and post-processing, and we conjecture that this is not the case.

Then we turn to the conversion problem, showing two ways to solve it in convex dynamical resource theories by means of a conic linear program. In the first approach, we construct a complete family of convex dynamical monotones, which give necessary and sufficient conditions for convertibility under free superchannels. In the second approach, solving the conversion problem becomes equivalent to calculating a particular type of distance—the conversion distance—from one channel to another.

Finally, we present the classic resource-theoretic protocols of cost and distillation both in the single-shot and the asymptotic regime. We note that for dynamical resources such protocols take a new twist from their static counterpart, whereby various dynamical resources can also be applied one after another (and not just in parallel) to create an adaptive strategy \[42, 47, 58, 59, 71–73\].

The article is organized as follows. In section II, we present basic facts on the formalism of superchannels, including a new result on the uniqueness of a superchannel realization in terms of pre- and post-processing. In the same section we give an overview of quantum resource theories as well. Section III is all devoted to the general formalism of resource theories for quantum processes, with a new construction of a complete set of monotones, and precise definitions of several conversion protocols. Conclusions are drawn in section IV.

**II. PRELIMINARIES**

This section contains some basic notions to understand the rest of this article. First we specify the notation we use, and then we move to give a brief overview of the formalism used to manipulate quantum channels, namely supermaps, superchannels, and combs. Here we also prove a new result (theorem 2), concerning the uniqueness of the realization of a superchannel in terms of quantum channels. Finally we give a brief introduction to resource theories.

### A. Notation

Physical systems and their corresponding Hilbert spaces will be denoted by \(A, B, C, \text{ etc.}\), where we will use the notation \(AB\) to mean \(A \otimes B\). Dimensions will be denoted with vertical bars; e.g. the dimension of system \(A\) will be denoted by \(|A|\). The tilde symbol will be reserved to indicate a replica of a system. For example, \(A\) denotes a replica of \(A\), i.e. \(|A| = |A|\). Density matrices acting on Hilbert spaces will be denoted by lowercase Greek letter \(\rho, \sigma, \text{ etc.}\), with one exception for the maximally mixed state (i.e. the uniform state), which will be denoted by \(\rho_{AA} := \frac{1}{|A|^2}I_A\).

The set of all bounded operators acting on system \(A\) is denoted by \(\mathfrak{B}(A)\), the set of all Hermitian matrices acting on \(A\) by \(\text{Herm}(A)\), and the set of all density matrices acting on system \(A\) by \(\mathfrak{D}(A)\). Note that \(\mathfrak{D}(A) \subset \text{Herm}(A) \subset \mathfrak{B}(A)\). We use the calligraphic letters \(\mathcal{D}, \mathcal{E}, \mathcal{F}, \text{ etc.}\) to denote quantum channels, reserving \(\mathcal{N}\) to represent an isometry map. The identity map on a system \(A\) will be denoted by \(\text{id}_A\). The set of all linear maps from \(\mathfrak{B}(A)\) to \(\mathfrak{B}(B)\) is denoted by \(\mathcal{L}(A \to B)\), the set of all completely positive (CP) maps by \(\mathcal{C}(A \to B)\), and the set of quantum channels by \(\mathcal{CPTP}(A \to B)\). Note that \(\mathcal{CPTP}(A \to B) \subset \mathcal{C}(A \to B) \subset \mathcal{L}(A \to B)\). \(\text{Herm}(A \to B)\) will denote the real vector space of all Hermitian-preserving maps in \(\mathcal{L}(A \to B)\). We will write \(\mathcal{N} \geq 0\) to mean that the map \(\mathcal{N} \in \text{Herm}(A \to B)\) is completely positive.

Since in this paper we focus on dynamical resources in the form of quantum channels, unless otherwise specified, it will be convenient to associate two subsystems \(A_0\) and \(A_1\) with every physical system \(A\), referring, respectively, to the input and output of the resource. Hence, any physical system will be comprised of two subsystems \(A = (A_0, A_1)\), even those representing a static resource, in which case we simply have \(|A_0| = 1\). For simplicity, we will denote a channel with a subscript \(A\), e.g. \(\mathcal{N}_A\), to mean that it is an element of \(\mathcal{CPTP}(A_0 \to A_1)\). Similarly, a bipartite channel in \(\mathcal{CPTP}(A_0B_0 \to A_1B_1)\) will be denoted by \(\mathcal{N}_{AB}\). This notation makes the analogy with bipartite states more transparent.

In this setting, when we consider \(A = (A_0, A_1), B = (B_0, B_1)\), etc, comprised of input and output subsystems, the symbol \(\mathcal{L}(A \to B)\) refers to all linear maps from the vector space \(\mathcal{L}(A_0 \to A_1)\) to the vector space \(\mathcal{L}(B_0 \to B_1)\). Similarly, \(\text{Herm}(A \to B) \subset \mathcal{L}(A \to B)\) is a real vector space consisting of all the linear maps that take elements in \(\text{Herm}(A_0 \to A_1)\) to elements in \(\text{Herm}(B_0 \to B_1)\). In other terms, maps
in Herm \((A \to B)\) take Hermitian-preserving maps to Hermitian-preserving maps. Linear maps in \(\mathcal{L}(A \to B)\) and Herm \((A \to B)\) will be called supermaps, and will be denoted by capital Greek letters \(\Theta, \Upsilon, \Omega, \) etc. The identity supermap in \(\mathcal{L}(A \to A)\) will be denoted by \(\mathbb{1}_A\).

We will use square brackets to denote the action of a supermap \(\Theta_{A \to B} \in \mathcal{L}(A \to B)\) on a linear map \(N_A \in \mathcal{L}(A_0 \to A_1)\). For example, \(\Theta_{A \to B}[N_A]\) is a linear map in \(\mathcal{L}(B_0 \to B_1)\) obtained from the action of the supermap \(\Theta\) on the map \(N\). Moreover, for a simpler notation, the identity supermap will not appear explicitly in equations; e.g. \(\Theta_{A \to B}[N_{RA}]\) will mean \((\mathbb{1}_R \otimes \Theta_{A \to B})[N_{RA}]\). Instead, the action of linear map (e.g. quantum channel) \(N_A \in \mathcal{L}(A_0 \to A_1)\) on a matrix \(\rho \in \mathcal{B}(A_0)\) is written with round brackets, i.e. \(N_A(\rho_{A_0} \in \mathcal{B}(A_1)\).

Finally, we adopt the following convention concerning partial traces: when a system is missing, we take the partial trace over the missing system. This applies to matrices as well as to maps. For example, if \(M_{AB}\) is a matrix on \(A_0A_1B_0B_1\), \(M_{AB_0}\) denotes the partial trace on the missing system \(B_1\): \(M_{AB_0} := \text{Tr}_{B_1}[M_{AB}]\).

### B. Supermaps and superchannels

The space \(\mathcal{L}(A_0 \to A_1)\) is equipped with an inner product given by

\[
\langle N_A, M_A \rangle := \sum_{i,j} \langle N_A(|i\rangle\langle j|), M_A(|i\rangle\langle j|)\rangle_{\text{HS}},
\]

where \(\langle X, Y \rangle_{\text{HS}} := \text{tr}[X^\dagger Y]\) is the Hilbert-Schmidt inner product between matrices \(X, Y \in \mathcal{B}(A_1)\). The inner product above can be expressed in terms of the Choi matrices of \(N\) and \(M\). Denote by \(\mathcal{J}^N_A := N_{A_0 \to A_1}(\phi^+_{A_0A_1})\) the Choi matrix of \(N_A\), where \(\phi^+_{A_0A_1} := (|\phi^+\rangle \langle \phi^+|)_{A_0A_1}\) and \(\phi^+_{A_0A_1}\) is the unnormalized maximally entangled state. With this notation, the inner product between \(N_A\) and \(M_A\) can be expressed as [43]

\[
\langle N_A, M_A \rangle = \langle \mathcal{J}^N_A, \mathcal{J}^M_A \rangle_{\text{HS}} = \text{tr}\left[\mathcal{J}^N_A^\dagger \mathcal{J}^M_A\right].
\]

The canonical orthonormal basis (relative to the above inner product) is given by \(\{\mathcal{E}^{ijkl}_{A}\}\), where

\[
\mathcal{E}^{ijkl}_{A}(\rho_{A_0}) := \langle i|\rho_{A_0}|j\rangle \otimes |k\rangle \langle l|_{A_1}, \forall \rho \in \mathcal{B}(A_0).
\]

The space \(\mathcal{L}(A \to B)\) with \(A = (A_0, A_1)\) and \(B = (B_0, B_1)\) is also equipped with the following inner product: given \(\Theta, \Omega \in \mathcal{L}(A \to B)\)

\[
\langle \Theta_{A \to B}, \Omega_{A \to B} \rangle := \sum_{i,j,k,l} \langle \Theta_{A \to B}[\mathcal{E}^{ijkl}_{A}], \Omega_{A \to B}[\mathcal{E}^{ijkl}_{A}] \rangle,
\]

where the inner product on the right-hand side is the inner product between maps as defined in Eq. (1). Similarly to the inner product between maps, the inner product between supermaps can also be expressed in terms of Choi matrices. We define the Choi matrix of a supermap \(\Theta \in \mathcal{L}(A \to B)\) to be [43]

\[
\mathcal{J}^\Theta_{AB} := \sum_{i,j,k,l} J^{\epsilon^{ijkl}}_A \otimes J^{\epsilon^{ijkl}}_B.
\]

Then, with this notation, the inner product between two supermaps \(\Theta, \Omega\) can be expressed as

\[
\langle \Theta_{A \to B}, \Omega_{A \to B} \rangle = \langle \mathcal{J}^\Theta_{AB}, \mathcal{J}^\Omega_{AB} \rangle_{\text{HS}} = \text{tr}\left[[\mathcal{J}^\Theta_{AB}]^\dagger \mathcal{J}^\Omega_{AB}\right].
\]

The Choi matrix of a supermap \(\Theta \in \mathcal{L}(A \to B)\) can also be expressed in other three alternative ways [43]. First, from its definition, \(\mathcal{J}^\Theta_{AB}\) can be expressed as the Choi matrix of the map

\[
\mathcal{P}^\Theta_{AB} := \mathcal{S}_{A \to B}[\Phi^+_{A,A}],
\]

where the map \(\Phi^+_{AB}\) is defined as

\[
\Phi^+_{A,A} := \sum_{i,j,k,l} \mathcal{E}^{ijkl}_{A} \otimes \mathcal{E}^{ijkl}_{A}.
\]

A simple calculation shows that \(\Phi^+_{A,A}\) is completely positive, and acts on \(\rho \in \mathcal{B}(A_0A_0)\) as

\[
\Phi^+_{A,A}(\rho_{A_0A_0}) = \text{tr}[\rho_{A_0A_0} \Phi^+_{A,A}]_{A_1A_1A_1A_1}.
\]

In other terms, the CP map \(\Phi^+_{A,A}\) can be viewed as a generalization of the (unnormalized) maximally entangled state \(\phi^+_{A_0A_0}\).

A supermap \(\Theta \in \mathcal{L}(A \to B)\) can also be characterized by its action on Choi matrices. One can define a linear map \(\mathcal{R}^\Theta : \mathcal{B}(A) \to \mathcal{B}(B)\) as

\[
\mathcal{R}^\Theta_{A \to B}(\rho_A) := \text{tr} \left[ \mathcal{J}^\Theta_{AB}(\rho_{A_0A_0} \otimes I_B) \right], \forall \rho \in \mathcal{B}(A).
\]

With this definition, \(\mathcal{J}^\Theta_{AB}\) can be viewed as the Choi matrix of \(\mathcal{R}^\Theta_{A \to B}\). Note that although \(\mathcal{P}^\Theta_{AB}\) and \(\mathcal{R}^\Theta_{A \to B}\) have the same Choi matrix \(\mathcal{J}^\Theta_{AB}\), \(\mathcal{P}^\Theta_{AB}\) takes systems \(A_0B_0\) to \((A_1, B_1)\), whereas the map \(\mathcal{R}^\Theta\) takes system \(A = (A_0, A_1)\) to system \(B = (B_0, B_1)\). This brings us to the last representation of a supermap in terms of a linear map \(\mathcal{Q}^\Theta : \mathcal{B}(A_0B_0) \to \mathcal{B}(A_0B_1),\) which is defined as the map satisfying

\[
\mathcal{J}^\Theta_{AB} := \mathcal{Q}^\Theta_{A_1B_0 \to A_1B_1}\left(\Phi^+_{A_0A_0} \otimes \mathcal{B}_{B_0B_0}\right),
\]

or as \(\mathcal{Q}^\Theta := I_A \otimes \Theta_{A \to B}[S_A]\), where \(S_A\) is the swap from \(A_1\) to \(A_0\). All these three representations of a supermap, \(\mathcal{P}^\Theta, \mathcal{Q}^\Theta,\) and \(\mathcal{R}^\Theta,\) play a useful role in the study of quantum resource theories, as shown in Ref. [74] in the case of the entanglement of bipartite channels.

A superchannel is a supermap \(\Theta_{A \to B} \in \mathcal{L}(A \to B)\) that takes quantum channels to quantum channels even when tensored with the identity supermap [35, 43, 66–70]. More precisely, \(\Theta_{A \to B} \in \mathcal{L}(A \to B)\) is called a superchannel if it satisfies the following two conditions:
We will also say that a supermap are equivalent.

The third part of the following theorem shows that these superchannels from completely positive maps \([43, 70]\). We will denote the set of maps \([35, 43]\). Therefore, a superchannel is a positive if it takes CP maps to CP maps, and completely positive (CP), if it satisfies the second condition above \([35, 43]\). Therefore, a superchannel is a CP supermap that takes trace-preserving maps to trace-preserving maps \([43, 70]\). We will denote the set of superchannels from \(A\) to \(B\) by \(\mathcal{S}(A \rightarrow B)\). Note that \(\mathcal{S}(A \rightarrow B) \subseteq \mathcal{L}(A \rightarrow B)\).

The above definition is axiomatic and minimalist, in the sense that any physical evolution (or simulation) of a quantum channel must satisfy these two basic conditions. The third part of the following theorem shows that these two conditions are sufficient to ensure that superchannels are indeed physical processes.

**Theorem 1** \([35, 43]\). Let \(\Theta \in \mathcal{L}(A \rightarrow B)\). The following are equivalent.

1. \(\Theta\) is a superchannel.

2. The Choi matrix \(J^{\Theta}_{AB} \geq 0\) of \(\Theta\) has marginals

   \[
   J^{\Theta}_{A_1B_0} = I_{A_1B_0}, \quad J^{\Theta}_{AB} = J^{\Theta}_{A_0B_0} \otimes u_{A_1},
   \]

   where \(u_{A_1}\) is the maximally mixed state (i.e. the uniform state) on system \(A_1\).

3. There exists a Hilbert space \(E\), with \(|E| \leq |A_0B_0|\), and two CPTP maps \(\mathcal{F} \in \mathcal{CPTP}(B_0 \rightarrow E A_0)\) and \(\mathcal{E} \in \mathcal{CPTP}(E A_1 \rightarrow B_1)\) such that for all \(\mathcal{N} \in \mathcal{L}(A_0 \rightarrow A_1)\)

   \[
   \Theta[\mathcal{N}] = \mathcal{E}_{E A_1 \rightarrow B_1} \circ \mathcal{N}_{A_0 \rightarrow A_1} \circ \mathcal{F}_{B_0 \rightarrow E A_0}
   \]

   (see Fig. 1). Furthermore, \(\mathcal{Q}^{\Theta}_{A_1B_0 \rightarrow A_0B_1} = \mathcal{E}_{E A_1 \rightarrow B_1} \circ \mathcal{F}_{B_0 \rightarrow E A_0} \in \mathcal{CPTP}(A_1B_0 \rightarrow A_0B_1)\), and \(\mathcal{F}\) can be taken to be an isometry.

4. For every \(\mathcal{N} \in \mathcal{CPTP}(A_0 \rightarrow A_1)\), the matrix \(R^{\Theta}_{A \rightarrow B}(J^\mathcal{N}_{A})\) is a Choi matrix of a quantum channel. That is,

   \[
   R^{\Theta}_{A \rightarrow B}(J^\mathcal{N}_{A}) \geq 0 \quad \text{and} \quad \text{tr}_{B_1}[R^{\Theta}_{A \rightarrow B}(J^\mathcal{N}_{A})] = I_{B_0}.
   \]

In general, the realization of a superchannel as given in Fig. 1 is not unique. This is due to the presence of a memory system, described in Fig. 1 with the letter \(\mathcal{J}\). To see why, consider an isometry channel \(\mathcal{V}_{E \rightarrow E'}\) defined for all \(\rho \in \mathcal{B}(E)\) by \(\mathcal{V}_{E \rightarrow E'}(\rho) = V \rho V^\dagger\), where \(V : E \rightarrow E'\) is an isometry matrix satisfying \(V^\dagger V = I_E\). Then, this isometry channel has many left inverses given by

\[
\mathcal{V}^{-1}_{E \rightarrow E'}(\sigma_{E'}) = V^\dagger \sigma_{E'} V + \text{tr} [(I_{E'} - VV^\dagger) \sigma_{E'}] \tau_E,
\]

where \(\tau \in \mathcal{D}(E)\) is an arbitrary fixed density matrix. We can easily check that \(\mathcal{V}^{-1} \circ \mathcal{V} = \mathcal{I}\). In Fig. 2 we use this map to show that the realization of a superchannel in terms of pre- and post-processing is not unique. Moreover, there is another way in which the realization of a superchannel can be non-unique, namely by appending a state in the pre-processing, and then discarding it in the post-processing. To see how this works, let \(\mathcal{F}_{B_0 \rightarrow B_0A_0}\) and \(\mathcal{E}_{E A_1 \rightarrow B_1}\) be the pre-processing and the post-processing in a realization of a superchannel \(\Theta \in \mathcal{S}(A \rightarrow B)\), respectively. Now consider the new pre-processing \(\mathcal{F}^\mathcal{J}_{B_0 \rightarrow E' A_0} := \rho_{E'} \otimes \mathcal{F}_{B_0 \rightarrow E A_0}\), where \(\rho \in \mathcal{D}(E')\), and the new post-processing \(\mathcal{E}_{E A_1 \rightarrow B_1} := \mathcal{V} \otimes \mathcal{E}_{E A_1 \rightarrow B_1}\). It is straightforward to check that \(\mathcal{F}^\mathcal{J}\) and \(\mathcal{E}\) realize exactly the same superchannel, \(\Theta\), as \(\mathcal{F}\) and \(\mathcal{E}\).

Although the realization of a superchannel is not unique, if we restrict the dimension of system \(E\) to be the smallest possible, and the map \(\mathcal{F}\) to be an isometry, we can obtain a new uniqueness result, expressed by the following theorem, which subsumes some of the results in Ref. [75].

**Theorem 2** (Uniqueness). Let \(\Theta \in \mathcal{S}(A \rightarrow B)\) be a superchannel, and let \(r := \text{Rank}(J^{\Theta}_{A_0B_1})\). Then, there exists a system \(E\) with \(|E| = r\), an isometry \(\mathcal{F} \in \mathcal{CPTP}(B_0 \rightarrow E A_0)\) and a channel \(\mathcal{E} \in \mathcal{CPTP}(E A_1 \rightarrow B_1)\) such that \(\Theta\) can be realized as in Eq. (4). Furthermore, if there exists a system \(E'\) such that \(|E'| \leq r\), an isometry \(\mathcal{F}' \in \mathcal{CPTP}(B_0 \rightarrow E' A_0)\), and a channel \(\mathcal{E}' \in \mathcal{CPTP}(E' A_1 \rightarrow B_1)\) such that \(\Theta\) can be realized as in Eq. (4) with \(\mathcal{F}'\) and \(\mathcal{E}'\) replacing \(\mathcal{E}\) and...
and \( |E'\rangle = |E\rangle \), and there exists a unitary channel \( \mathcal{U} \in \text{CPTP} (E \rightarrow E') \) such that
\[
\mathcal{E}'_{E'A_1 \rightarrow B_1} = \mathcal{E}_{E A_1 \rightarrow B_1} \circ \mathcal{U}_{E' \rightarrow E}^{-1}
\]
and
\[
\mathcal{F}'_{B_0 \rightarrow E' A_0} = \mathcal{U}_{E \rightarrow E'} \circ \mathcal{F}_{B_0 \rightarrow E A_0}.
\]

**Proof.** The first part of the theorem follows from the proof of Theorem 1 as given in Ref. [43], in which system \( E \) was chosen to be the purifying system of \( \mathcal{J}^{\theta}_{A_0B_0} \) (see also Ref. [75]). Thus \( |E\rangle \) can always be taken to have dimension \( |E| = r \). We only need to prove the uniqueness part.

First note that by Theorem 1 we have that
\[
\mathcal{Q}^{\theta}_{A_0B_1 \rightarrow A_1B_0} = \mathcal{E}'_{E'A_1 \rightarrow B_1} \circ \mathcal{F}_{B_0 \rightarrow E'A_0} = \mathcal{F}_{B_0 \rightarrow A_0} \circ \mathcal{E}_{E A_1 \rightarrow B_1} \circ \mathcal{F}_{B_0 \rightarrow E A_0},
\]
whose Choi matrix is \( \mathcal{J}^{\theta}_{A'B'} \). Therefore, recalling Eq. (2), the marginal \( \mathcal{J}^{\theta}_{A_0B_0} \) can be expressed as
\[
\mathcal{J}^{\theta}_{A_0B_0} = |A_1| \mathcal{F}_{B_0 \rightarrow A_0} \left( \phi^+_{B_0B_0} \right) = |A_1| \mathcal{F}_{B_0 \rightarrow A_0} \left( \phi^+_{B_0B_0} \right).
\]

Now, observe that \( |A_1| \mathcal{F}_{B_0 \rightarrow E'A_0} \left( \phi^+_{B_0B_0} \right) \) is a purification of \( \mathcal{J}^{\theta}_{A_0B_0} \) since \( \mathcal{F}_{B_0 \rightarrow E'A_0} \) is an isometry (where \( |A_1| \mathcal{F}_{B_0 \rightarrow E'A_0} \left( \phi^+_{B_0B_0} \right) \) is a pure state). Therefore, \( |E'| \geq r \) so that \( |E'| = r = |E| \). Moreover, since \( |A_1| \mathcal{F}_{B_0 \rightarrow E'A_0} \left( \phi^+_{B_0B_0} \right) \) and
\[
|A_1| \mathcal{F}_{B_0 \rightarrow E'A_0} \left( \phi^+_{B_0B_0} \right)
\]
are two purifications of \( \mathcal{J}^{\theta}_{A_0B_0} \), they must be related by a unitary \( U_{E \rightarrow E'} \), so \( \mathcal{F}_{B_0 \rightarrow E'A_0} = U_{E \rightarrow E'} \circ \mathcal{F}_{B_0 \rightarrow E A_0} \), as their Choi matrices are the same.

To conclude the proof, set \( \psi_{E A_0B_0} := \mathcal{F}_{B_0 \rightarrow E A_0} \left( \phi^+_{B_0B_0} \right) \). Recalling that \( \mathcal{J}^{\theta}_{A'B'} \) is the Choi matrix of \( \mathcal{Q}^{\theta} \), we get
\[
\mathcal{J}^{\theta}_{A'B'} = \mathcal{E}_{E A_1 \rightarrow B_1} \left( \psi_{E A_0B_0} \otimes \phi^+_{A_1A_1} \right).
\]

Let system \( \tilde{E} \) be the support of \( \psi_{A_0B_0} \) (i.e., it is the Hilbert space spanned by the eigenvectors of \( \psi_{A_0B_0} \) that correspond to non-zero eigenvalues). Hence, \( |\tilde{E}| = |E| = r \). Denoting the restriction of \( \psi_{E A_0B_0} \) to the space \( EE \) by \( \psi_{\tilde{E} E} \), by Eq. (5) we have that
\[
\mathcal{E}_{E A_1 \rightarrow B_1} \left( \psi_{\tilde{E} E} \otimes \phi^+_{A_1A_1} \right).
\]

By definition, the marginal \( \psi_{\tilde{E}} \) is invertible, and we have
\[
\psi_{\tilde{E} E} = \left( I_{\tilde{E}} \otimes \sqrt{\psi_{\tilde{E}} U_{\tilde{E}}} \right) \phi^+_{\tilde{E} E} \left( I_{\tilde{E}} \otimes U_{\tilde{E}}^+ \sqrt{\psi_{\tilde{E}}} \right), \text{ where } U_{\tilde{E}}
\]
is some unitary. Hence, by (Hermite-) conjugating both sides of Eq. (6) above by \( U_{\tilde{E}}^+ \psi_{\tilde{E}}^{-\frac{1}{2}} \), we get that the Choi matrix of \( \mathcal{E}_{E A_1 \rightarrow B_1} \) equals the Choi matrix of \( \mathcal{E}'_{E'A_1 \rightarrow B_1} \circ U_{E \rightarrow E'} \). Consequently we conclude that the channels must be the same.

\[\square\]

### C. Measurements on quantum channels

A quantum instrument is a collection of CP maps \( \{\mathcal{E}_x\} \) such that their sum \( \sum_x \mathcal{E}_x \) is a CPTP map. Note that each \( \mathcal{E}_x \) is trace non-increasing, and that every CP map that is trace non-increasing can be completed to a full quantum instrument. Quantum instruments are used to characterize the most general measurements that can be performed on a physical system, including, as special cases, projective von Neumann measurements, POVMs, and generalized measurements. Therefore, we discuss the generalization of a quantum instrument to a collection of objects that act on quantum channels. We call this generalization a superinstrument [70].

A superinstrument is a collection of supermaps \( \{\Theta_x\} \), where each \( \Theta_x \in \mathcal{L}(A \rightarrow B) \) is CP (i.e. \( \mathcal{J}^{\Theta}_{A_0B_0} \geq 0 \)), and the sum \( \sum_x \Theta_x \) is a superchannel. Similar to the state domain, every \( \Theta_x \) maps quantum channels to CP trace non-increasing maps. However, in the channel domain not every supermap \( \Theta \in \mathcal{L}(A \rightarrow B) \) with a positive semi-definite Choi matrix, and that takes channels to CP trace non-increasing maps, can be completed to a superchannel. In Ref. [70] a counterexample was given, and it was also shown that a CP supermap \( \Theta \in \mathcal{L}(A \rightarrow B) \) can be completed to a superchannel (i.e. there exists a CP supermap \( \Omega \in \mathcal{L}(A \rightarrow B) \) such that \( \Theta + \Omega \) is a superchannel) if and only if for any system \( R \), the supermap \( I_R \otimes \Theta \) takes quantum channels to CP trace non-increasing maps. In Ref. [70] it was shown that this phenomenon is associated with the existence of signaling bipartite channels.

While the above discussion is subtle, it demonstrates (see details in Ref. [70]) that every element \( \Theta_x \) of a superinstrument \( \{\Theta_x\} \) satisfies \( \text{tr} \left[ \mathcal{J}^{\Theta}_{A_0B_0} \alpha_{A_0B_0} \right] \leq 1 \) for every \( \alpha_{A_0B_0} \geq 0 \) such that \( \alpha_{A_0B_0} = I_{A_0} \otimes \rho_{B_0} \), where \( \rho \in \mathcal{D}(B_0) \). Moreover, every superinstrument can be realized as in Fig. 3, with an isometry pre-processing and a quantum instrument as the post-processing [35, 70].

Like quantum instruments, any superinstrument \( \{\Theta_x\} \in \mathcal{L}(A \rightarrow BX) \) can be viewed as a superchannel \( \Theta \in \mathcal{L}(A \rightarrow BX) \), where system \( X = (X_0, X_1) \) has trivial input dimension \( |X_0| = 1 \), and the output system \( X_1 \) is classical. Hence, a superinstrument can be expressed as
\[
\Theta_{A \rightarrow BX} = \sum_x \Theta_{A \rightarrow B} \otimes |x\rangle \langle x|_X,
\]
where \( X \equiv X_1 \). This characterization of a superinstrument is particularly useful in the context of quantum resource theories, since the above relation demonstrates...
that the set of free superinstruments can be viewed as a subset of the set of free superchannels.

D. Quantum combs

Quantum combs are multipartite channels with a well-defined causal structure (see Fig. 4(a)) [65, 66, 76–79]. They generalize the notion of superchannels to objects that take several channels as input, and output a channel (see Refs. [65, 66] for more details, and a for a further generalization where the input and the output of combs are combs themselves). A comb acting on \( n \) channels is depicted in Fig. 4(b). We will denote a comb with \( n \) channel-slots as input by \( \mathcal{E}_n \), and its action on \( n \) channels by \( \mathcal{E}_n [N_1, \ldots, N_n] \). The causal relation between the different slots ensures that each such comb can be realized with \( n + 1 \) channels \( \mathcal{E}_1, \ldots, \mathcal{E}_{n+1} \) as in Fig. 4(b). We therefore associate a quantum channel

\[
Q^{\mathcal{E}_n} := \mathcal{E}_{n+1} \circ \mathcal{E}_n \circ \cdots \circ \mathcal{E}_1
\]

with every comb. Note that the quantum channel \( Q^{\mathcal{E}_n} \) has a causal structure in the sense that the input to \( \mathcal{E}_k \) cannot affect the output of \( \mathcal{E}_{k+1} \) for any \( k = 2, \ldots, n + 1 \). The Choi matrix of the comb is defined as the Choi matrix of \( Q^{\mathcal{E}_n} \). Owing to the causal structure of \( Q^{\mathcal{E}_n} \), the marginals of the Choi matrix of \( \mathcal{E}_n \) satisfy similar relations to Eq. (3) (see Refs. [65, 66] for more details).

Note that there are other ways to manipulate multiple quantum channels where we do not require any causal structure on the different channel-slots [67, 80, 81], but we will not use them in our analysis.

E. Quantum resource theories

For every pair of physical systems \( A \) and \( B \), consider a subset of CPTP maps \( \mathfrak{F} (A \to B) \subset \text{CPTP} (A \to B) \). \( \mathfrak{F} \) identifies a quantum resource theory (QRT) if the following two conditions hold [14]:

1. For every physical system \( A \), the set \( \mathfrak{F} (A \to A) \) contains the identity map \( \text{id}_A \).

2. For any three systems \( A, B, C \), if \( M \in \mathfrak{F} (A \to B) \) and \( N \in \mathfrak{F} (B \to C) \), then \( N \circ M \in \mathfrak{F} (A \to C) \).

The elements in each set \( \mathfrak{F} (A \to B) \) are called free operations. The set \( \mathfrak{F} (A) := \mathfrak{F} (1 \to A) \), where the \( 1 \) stands for the trivial (i.e. 1-dimensional) system, will be used to denote the set of free states.

In any QRT we can consider either static or dynamical inter-conversions. In a static inter-conversion we look for conditions under which a conversion from one resource state (i.e. not in \( \mathfrak{F} (A) \)) to another is possible by free operations. In a dynamical inter-conversion we are interested in the conditions under which a conversion from one resource channel (i.e. not in \( \mathfrak{F} (A \to B) \)) to another is possible with free superchannels. Clearly, static inter-conversions can be viewed as a special type of dynamical ones.

In this article we will consider QRTs that admit a tensor product structure. That is, the set of free operations \( \mathfrak{F} \) satisfies the following additional conditions:

3. Free operations are “completely free”: for any three physical systems \( A, B, \) and \( C \), if \( M \in \mathfrak{F} (A \to B) \) and \( N \in \mathfrak{F} (B \to C) \) then \( \text{id}_C \otimes M \in \mathfrak{F} (CA \to CB) \).

4. Discarding a system (i.e. the trace) is a free operation: for every system \( A \), the set \( \mathfrak{F} (A \to 1) \) is not empty.

The above additional conditions are very natural, and satisfied by almost all QRTs studied in literature [14]. These conditions imply that if \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are free channels, then also \( \mathcal{M}_1 \otimes \mathcal{M}_2 \) is free. In addition, they also imply that appending free states is a free operation; i.e. for any given free state \( \sigma \in \mathfrak{F} (B) \), the CPTP map \( F_\sigma (\rho) := \rho \otimes \sigma \) is a free map, i.e. it belongs to \( \mathfrak{F} (A \to AB) \). This in turn implies that the replacement map \( R_\sigma \) is free, where \( R_\sigma (\rho) = \text{tr} [\rho | \sigma \rangle \langle \sigma | \rho] \), for every density matrix \( \rho \) and some fixed free state \( \sigma \). In the following we will also assume that \( \mathfrak{F} (A \to B) \) is topologically closed for all systems \( A \) and \( B \), as it is natural to assume that arbitrarily good approximations of free operations are free as well.
III. RESOURCE THEORIES OF QUANTUM PROCESSES

In this section we build resource theories of processes, and we present a new construction of a complete set of monotones for convex resource theories of processes. We also give the precise definition of several resource-theoretic protocols.

Similarly to resource theories of quantum states, free superchannels will be a subset of all physical superchannels. If we already have a QRT of static resources, theorem 1 gives us a sufficient condition for free superchannels: a superchannel is free if both the pre-processing and the post-processing are free in the underlying resource theory of states, i.e. if \( \mathcal{F} \in \mathcal{F}_{\mathcal{EA}}(B_0 \rightarrow EA_0) \) and \( \mathcal{E} \in \mathcal{F}(EA_1 \rightarrow B_1) \). We call these free superchannels “freely realizable”. Since we consider QRTs with a tensor product structure, if a superchannel \( \Theta \) is free, then also its map

\[
\mathcal{Q}^{\Theta}_{A_1 B_0 \rightarrow A_0 B_1} := \mathcal{E}_{EA_1 \rightarrow B_1} \circ \mathcal{F}_{B_0 \rightarrow EA_0}
\]  

(7)

is free: \( \mathcal{Q}^{\Theta}_{A_1 B_0 \rightarrow A_0 B_1}, \Theta \in \mathcal{F}(A_1 B_0 \rightarrow A_0 B_1) \). Recall that the mapping \( \Theta \mapsto \mathcal{Q}^{\Theta}_{A_1 B_0 \rightarrow A_0 B_1} \) is a bijection, so that a free superchannel \( \Theta \) corresponds to a unique free map \( \mathcal{Q}^{\Theta}_{A_1 B_0 \rightarrow A_0 B_1} \). However, if \( \mathcal{Q}^{\Theta}_{A_1 B_0 \rightarrow A_0 B_1} \) is a free CPTP map, it does not necessarily mean that there exists a realization of \( \Theta \) in terms of free pre- and post-processing: we only know that their combination is free.

The problem of determining whether a free channel \( \mathcal{Q} \) can be decomposed as in Eq. (7) with both \( \mathcal{E} \) and \( \mathcal{F} \) being free can be very hard to solve, even when the resource theory is relatively simple (that is, even if inclusion in \( \mathcal{F} \) can be determined with an SDP; e.g. in NPT entanglement, see Ref. [59]). Therefore, typically, resource theories of quantum processes can be very hard to handle, even if the corresponding QRT of states is relatively simple. In Ref. [59], we announced that, for NPT dynamical entanglement, if we enlarge the set of free superchannels to include all superchannels for which \( \mathcal{Q} \) is a PPT channel, we obtain a resource theory of NPT dynamical entanglement that is much more manageable. The price we pay is that not all such free superchannels may be freely realizable. In view of this more relaxed definition of free superchannels, let us focus on the minimal requirements for free superchannels. For any two systems \( A \) and \( B \), we denote by \( \text{FREE}(A \rightarrow B) \) the set of all free superchannels in \( \mathcal{F}(A \rightarrow B) \). The minimal requirements the set \( \text{FREE} \) must satisfy are the following (analogous to those satisfied by \( \mathcal{F} \)):

1. \( \mathcal{I}_A \in \text{FREE}(A \rightarrow A) \), where \( \mathcal{I}_A \) is the identity supermap acting on \( \mathcal{E}(A \rightarrow A) \).
2. If \( \Theta_1 \in \text{FREE}(A \rightarrow B) \) and \( \Theta_2 \in \text{FREE}(B \rightarrow C) \), then \( \Theta_2 \circ \Theta_1 \in \text{FREE}(A \rightarrow C) \).

In particular, the second condition also implies that the superchannels in \( \text{FREE} \) are resource non-generating (RNG) [8, 14]. In other words, for every input channel \( \mathcal{M}_A \in \mathcal{F}(A_0 \rightarrow A_1) \) and every free superchannel \( \Theta \in \text{FREE}(A \rightarrow B) \), the output channel \( \Theta[\mathcal{M}_A] \in \mathcal{F}(B_0 \rightarrow B_1) \). Note that we can recover free channels by trivializing the input \( A \) of a free superchannel \( \Theta_{A \rightarrow B} \), i.e. by taking \( A_0 \) and \( A_1 \) to be 1-dimensional.

Moreover, since we consider QRTs that admit a tensor product structure, we require free superchannels to be “completely free”: for any three physical systems \( A = (A_0, A_1), B = (B_0, B_1), \) and \( R = (R_0, R_1) \), if \( \Theta \in \text{FREE}(A \rightarrow B) \), then \( \mathcal{I}_R \otimes \Theta \in \text{FREE}(RA \rightarrow RB) \).

Note that appending free channels is a free operation: it is the tensor product of the identity superchannel with a free channel. Therefore, for any given free channel \( \mathcal{M}_B \in \mathcal{F}(B_0 \rightarrow B_1) \), the superchannel \( \Theta_{\mathcal{M}}[\mathcal{M}_A] := \mathcal{N}_A \otimes \mathcal{M}_B \) is a free superchannel, i.e. it belongs to \( \text{FREE}(A \rightarrow AB) \).

In some important resource theories, e.g. in entanglement theory [82–85], the set of natural free operations can be hard to characterize mathematically [86, 87]. For this reason, it can be convenient to enlarge the set of free operations to work with a less complicated set. A standard enlargement is to consider all resource non-generating (RNG) superchannels [8, 14]:

\[
\text{RNG}(A \rightarrow B) := \{ \Theta \in \mathcal{F}(A \rightarrow B) : \Theta[\mathcal{M}_A] \in \mathcal{F}(B_0 \rightarrow B_1) \}
\]

for all \( \mathcal{M}_A \in \mathcal{F}(A_0 \rightarrow A_1) \). Similarly to the case of states, this is the set of superchannels that transform free channels into free channels. In this setting, since we require free superchannels to be completely free, RNG superchannels are also completely resource non-generating (CRNG) (in general, however, they are two distinct sets, with \( \text{CRNG} \subseteq \text{RNG} \)): \( \Theta \) is CRNG if and only if \( \mathcal{I}_R \otimes \Theta \) is RNG, for all systems \( R = (R_0, R_1) \). In Ref. [59] we consider PPT operations [88, 89] and separable operations [90] as extensions of the LOCC paradigm. Both of these sets are CRNG. Note that, however, a priori, there is no guarantee that CRNG superchannels are freely realizable in terms of CRNG channels in the underlying resource theory of states.

Dynamical resources are quantified by dynamical resource monotones.

Definition 3. Let \( \mathcal{F} \) be a QRT admitting a tensor product structure. Let \( f : \text{CPTP} \rightarrow \mathbb{R} \) be a function on the set of all channels in all dimensions. Then, \( f \) is called a dynamical resource monotone if, for every channel \( \mathcal{N} \in \text{CPTP}(A_0 \rightarrow A_1) \) and every superchannel \( \Theta \in \text{FREE}(A \rightarrow B) \), \( f(\Theta[\mathcal{N}_A]) \leq f(\mathcal{N}_A) \).

It is customary, although not essential, to request that, for any system \( A_0 \) the value of \( f \) on the identity channel \( \text{id}_{A_0} \) is zero; i.e. \( f(\text{id}_{A_0}) = 0 \). This condition implies that \( f \) is non-negative, and satisfies

\[
f(\mathcal{N}_A) = 0 \quad \forall \mathcal{N} \in \mathcal{F}(A_0 \rightarrow A_1),
\]

for every system \( A = (A_0, A_1) \). The above property follows from a combination of the monotonicity property of
channel divergence \(f\) with the fact that the replacement superchannel that takes any channel to a fixed free channel is itself a free superchannel, as it can be realized with free pre- and post-processing. Applying the replacement superchannel preparing \(\mathcal{N} \in \mathcal{F}(A_0 \to A_1)\) to the identity superchannel, we get \(f(\mathcal{N}) \leq f(\text{id}_{A_0}) = 0\). Applying the replacement superchannel preparing the identity channel to \(\mathcal{N}\) instead yields \(f(\mathcal{N}) \geq f(\text{id}_{A_0}) = 0\), whence Eq. (8) follows.

Examples of dynamical monotones that are given in terms of the relative entropy were discussed in Refs. [38–40, 42, 58]. One such example is defined in terms of the channel divergence \(D(f, g)\), which is the relative entropy \(\rho \log \rho\) of the states for the considered channel. The channel divergence is

\[
D(f, g) := \text{tr} [\rho \log \rho] - \text{tr} [\rho \log \sigma]
\]

where \(D(f, g)\) is the relative entropy, \(\rho\) is a reference system, and the supremum is over all \(\rho\) and all density matrices \(\sigma\). The channel divergence is also a way to elevate any static monotone into a dynamical monotone. Given a static monotone \(E\), define

\[
E(N) := \sup_{\rho \in \mathcal{D}(A_0)} E(N(\rho_{RA_0})) - E(\rho_{RA_0}),
\]

for any \(N \in \text{CPTP}(A_0 \to A_1)\). Then, it can be shown that \(E\) is non-increasing under CRNG superchannels [38–40]. This was called amortized extension in Ref. [40]. This definition captures the generating power of the channel \(N\), understood as the maximum amount of static resource \(N\) can generate.

A. A complete family of dynamical monotones

The examples of dynamical monotones presented in the previous subsection are typically very hard to compute due to the optimizations involved. Here for the first time we introduce a family of dynamical resource monotones for convex resource theories that in some cases (e.g. NPT entanglement, Ref. [59]) can be computed with SDPs. Furthermore, each member of the family is convex, and the family itself is complete, in the sense that the monotones provide both necessary and sufficient conditions for the conversion of a dynamical resource into another with free superchannels. In this sense, this family of monotones fully captures the resourcefulness of a dynamical resource.

An example of a complete family of static resource monotones is known for pure-state entanglement theory [93–95]. There, the family of entanglement monotones is given in terms of Ky-Fan norms, and due to Nielsen majorization theorem [93], this family provides both necessary and sufficient conditions for the convertibility of pure bipartite states. The fact that the family consists of a finite number of monotones makes it easy to determine the convertibility of bipartite pure states under LOCC. However, for mixed states it is known that, already in local dimension 4, a finite number of monotones is insufficient to fully determine the exact interconversions between bipartite mixed states [96]. Therefore, in general, one cannot expect to find a finite and complete family for a generic QRT.

**Theorem 4.** Let \(\text{FREE}(A \to B)\) be as above, such that for every two systems \(A = (A_0, A_1)\) and \(B = (B_0, B_1)\), the set \(\text{FREE}(A \to B)\) is convex and topologically closed. For any quantum channel \(P_B \in \text{CPTP}(B_0 \to B_1)\) define

\[
f_P(N) := \max_{\Theta \in \text{FREE}(A \to B)} \langle P_B, \Theta | N_A \rangle,
\]

for every \(N_A \in \text{CPTP}(A_0 \to A_1)\). Let \(M_N \in \text{CPTP}(A_0 \to A_1)\) and \(M_B \in \text{CPTP}(B_0 \to B_1)\) be two quantum channels. Then, \(M_B = \Theta_{A \to B} | N_A \rangle\), for some superchannel \(\Theta \in \text{FREE}(A \to B)\) if and only if

\[
f_P(N_A) \geq f_P(M_B) \quad \forall P \in \text{CPTP}(B_0 \to B_1).
\]

**Proof.** Denote

\[
\mathcal{E}_N := \{\Theta | \mathcal{N} : \Theta \in \text{FREE}(A \to B)\}.
\]

Since we assume that \(\text{FREE}\) is convex and closed, so is \(\mathcal{E}_N\). Therefore, by the supporting hyperplane theorem, \(M_B \notin \mathcal{E}_N\) if and only if there exists a Hermitian-preserving map \(P_B \in \text{Herm}(B_0 \to B_1)\) such that

\[
\langle P_B, M_B \rangle > \max_{\Theta \in \text{FREE}(A \to B)} \langle P_B, \Theta | N_A \rangle.
\]

Alternatively, \(M_B \in \mathcal{E}_N\) if and only if for all Hermitian-preserving maps \(P_B \in \text{Herm}(B_0 \to B_1)\)

\[
\langle P_B, M_B \rangle \leq \max_{\Theta \in \text{FREE}(A \to B)} \langle P_B, \Theta | N_A \rangle.
\]

First we show that the above inequality holds for all Hermitian-preserving maps \(P \in \text{Herm}(B_0 \to B_1)\) if and only if

\[
f_P(M_B) = \max_{\Theta' \in \text{FREE}(B \to B)} \langle P_B, \Theta' | M_B \rangle
\]

\[
\leq \max_{\Theta \in \text{FREE}(A \to B)} \langle P_B, \Theta | N_A \rangle = f_P(N_A)
\]

for all Hermitian-preserving \(P_B \in \text{Herm}(B_0 \to B_1)\). Indeed, if Eq. (10) holds, then take \(\Theta'\) to be the identity superchannel \(I_B\); thus we immediately get Eq. (9) because

\[
\langle P_B, M_B \rangle \leq \max_{\Theta \in \text{FREE}(A \to B)} \langle P_B, \Theta | N_A \rangle.
\]
Conversely, suppose Eq. (9) holds. Then, for any \( \Theta' \in \text{FREE}(B \to B) \) we have
\[
\langle P_B, \Theta' [M_B] \rangle = \langle \Theta'^* [P_B], M_B \rangle \\
\leq \max_{\Theta \in \text{FREE}(A \to B)} \langle \Theta'^* [P_B], \Theta [N_A] \rangle \\
= \max_{\Theta \in \text{FREE}(A \to B)} \langle P_B, (\Theta' \circ \Theta) [N_A] \rangle \\
\leq \max_{\Theta \in \text{FREE}(A \to B)} \langle P_B, \Theta [N_A] \rangle,
\]
where the first inequality follows from assuming Eq. (9), and the last inequality from the property that if \( \Theta \) and \( \Theta' \) are both free, then \( \Theta' \circ \Theta \) is also free. Eq. (10) immediately holds.

It is left to show that it is sufficient to take \( P_B \) to be a CPTP map. To this end, it will be convenient to express the inner products in terms of the Choi matrices. Now, for any Hermitian-preserving map \( P_B \), consider a CPTP map \( \overline{P}_B \) whose Choi matrix is
\[
J_{\overline{P} B} := (1 - \varepsilon) I_B \otimes u_{B_1} + \varepsilon \left( J_P^B + (I_B - J_P^B) \otimes u_{B_1} \right),
\]
where \( \varepsilon > 0 \) is small enough so that \( J_{\overline{P} B} \geq 0 \). Note also that \( J_P^B \) = \( I_B \) so that \( \overline{P}_B \) is a quantum channel. Now, a key observation is that, for any quantum channel \( N_B \), we get
\[
\langle \overline{P}_B, M_B \rangle = \text{tr} \left[ J_{\overline{P} B} J_B^M \right] \\
= (1 - \varepsilon) \left| B_0 \right| \left| B_1 \right| + \varepsilon \text{tr} \left[ J_B^M \right] + \varepsilon \left| B_0 \right| \left| B_1 \right| - \varepsilon \frac{1}{\left| B_1 \right|} \text{tr} \left[ J_P^B \right].
\]
Hence
\[
\langle \overline{P}_B, M_B \rangle = \varepsilon \langle P_B, M_B \rangle + c_P,
\]
where
\[
c_P := \frac{1}{\left| B_1 \right|} \left( \left| B_0 \right| - \varepsilon \text{tr} \left[ J_P^B \right] \right)
\]
is a constant depending only on \( P_B \). Therefore, Eqs. (9) and (10) hold for \( P_B \) if and only if they hold for \( \overline{P}_B \). In other words, it is sufficient to consider CPTP maps \( P_B \).

\textit{Remark 5.} The definition of the functions \( f_P \) makes them convex.

\textit{Remark 6.} Similar families of monotones have been given recently in Refs. [11, 43, 97–99] for static resource theories, and in Ref. [26] in the context of channel discrimination tasks (see also the related discussion in Ref. [16]). The monotones constructed in theorem 4 can be reduced to all the ones introduced in Ref. [11, 16, 26, 43, 97–99], when restricting some of the input/output subsystems to be trivial or classical.

The functions \( f_P \) behave monotonically under free superchannels, therefore also under superchannels that replace any input channel with a fixed free channel. This in turn implies that, for every \( \mathcal{P} \), \( f_P \) take the same value on all free channels \( \mathcal{N} \in \mathcal{F}(A_0 \to A_1) \): if \( \mathcal{N} \in \mathcal{F}(A_0 \to A_1) \) we have
\[
f_P (\mathcal{N}_A) = \max_{\Theta \in \text{FREE}(A \to B)} \langle P_B, \Theta [N_A] \rangle \\
= \max_{M \in \text{FREE}(B_0 \to B_1)} \langle P_B, M_B \rangle \\
\equiv g (P_B).
\]
Therefore, if we want monotones that vanish on free channels, for any \( P \in \text{CPTP} (B_0 \to B_1) \), define
\[
G_P (\mathcal{N}_A) := f_P (\mathcal{N}_A) - g (P_B).
\]
In this way, \( \{ G_P \} \) is a complete set of non-negative resource monotones that vanish on free channels.

The way \( f_P \) were constructed means that they can be expressed in terms of resource witnesses. To see why, denote the set of (free) Choi matrices by
\[
\mathcal{J}_{AB} := \{ J_{AB}^\Theta : \Theta \in \text{FREE}(A \to B) \}.
\]
Since \( \text{FREE}(A \to B) \) is closed and convex, so is \( \mathcal{J}_{AB} \). The monotones \( f_P \) can be expressed as
\[
f_P (\mathcal{N}_A) = \max_{\Theta \in \text{FREE}(A \to B)} \langle P_B, \Theta [N_A] \rangle \\
= \max_{\Theta \in \text{FREE}(A \to B)} \text{tr} \left[ J_{\overline{P} B}^\Theta J_B^{|N_A|} \right] \\
= \max_{\Theta \in \text{FREE}(A \to B)} \text{tr} \left[ J_{AB}^\Theta (J_A^N)^T \otimes J_B^P \right] \\
= \max_{J_{AB}^\Theta \in \mathcal{J}_{AB}} \text{tr} \left[ J_{AB}^\Theta (J_A^N)^T \otimes J_B^P \right].
\]
In other terms, \( f_P (\mathcal{N}_A) \) is the support function of \( \mathcal{J}_{AB} \) evaluated at \( (J_A^N)^T \otimes J_B^P \). Let \( \mathcal{R} \) be the (convex) cone obtained from \( \mathcal{J}_{AB} \) by multiplying its elements by a non-negative number, i.e. \( \mathcal{R} := \mathbb{R}_+ \mathcal{J}_{AB} \). With this definition we can write
\[
f_P (\mathcal{N}_A) = \max_{J_{AB}^\Theta \in \mathcal{R}} \text{tr} \left[ J_{AB}^\Theta (J_A^N)^T \otimes J_B^P \right].
\]
The above optimization problem is a conic linear program. As such, using duality, \( f_P \) can be equivalently expressed as
\[
f_P (\mathcal{N}_A) = |A_1 B_0| \min \left\{ x : xI_{AB} - (J_A^N)^T \otimes J_B^P \in \mathcal{R}^* \right\},
\]
where \( x \in \mathbb{R} \) and \( \mathcal{R}^* \) is the dual cone
\[
\mathcal{R}^* = \{ W \in \text{Herm (AB)} : \text{tr} [WM] \geq 0 \ \forall M \in \mathcal{R} \}.
\]
Since the cone \( \mathcal{R} \) consists of only positive semi-definite matrices, it follows that any positive semi-definite matrix belongs to \( \mathcal{R}^* \). Note also that we must have \( x > 0 \) in the equation above, otherwise \( M := xI_{AB} - (J_A^N)^T \otimes J_B^P < 0 \), and therefore \( M \) would not belong to \( \mathcal{R}^* \).
The cone $\mathcal{R}$ is convex and closed. Therefore, as a consequence of the hyperplane separation theorem, $\mathcal{R}^*$ is $\mathcal{R}$. This in particular implies that $M \in \mathcal{R}$ if and only if $\text{tr} [MW] \geq 0$ for all $W \in \mathcal{R}^*$. Hence the Hermitian matrices (observables) in $\mathcal{R}^*$ that are not positive semidefinite can be viewed as witnesses of supermaps that are not free. However, among them, some will only witness whether or not a matrix $M$ corresponds to a valid superchannel, while others will witness if it corresponds to a non-free superchannel.

B. Single-shot interconversions with conic linear programming

Here we consider single-shot interconversions between resources. For this purpose, following similar ideas to [100], we define the conversion distance for any two channels $\mathcal{N}_A$ and $\mathcal{M}_B$ as

$$d_\delta (\mathcal{N}_A \rightarrow \mathcal{M}_B) := \frac{1}{2} \min_{\Theta \in \text{FREE}(A \rightarrow B)} \| \Theta_{A \rightarrow B} [\mathcal{N}_A] - \mathcal{M}_B \|_\infty.$$ 

If $d_\delta (\mathcal{N}_A \rightarrow \mathcal{M}_B) \leq \varepsilon$, for some small $\varepsilon > 0$, we will say that $\mathcal{N}_A$ can be converted to $\mathcal{M}_B$ by free superchannels up to a small error $\varepsilon$. When $\varepsilon = 0$, the conversion is exact.

Therefore, calculating the conversion distance between two channels becomes equivalent to determining if the former channel can be converted into the latter. As such, an important question is whether the conversion distance can be computed efficiently. First of all, recall that, as far as the diamond norm is concerned, the answer is positive, because it can be expressed as the SDP [101]

$$\frac{1}{2} \| E_B - F_B \|_\infty = \min_{\omega_B \geq 0; \omega_B \geq J_{F - E}^\varepsilon} \| \omega_B \|_\infty,$$

for all $E, F \in \text{CPTP} (B_0 \rightarrow B_1)$. Now, in Ref. [40], it was shown that it can be written also as

$$\frac{1}{2} \| E_B - F_B \|_\infty = \min \{ \lambda : \lambda Q_B \geq E_B - F_B \},$$

where $Q_B \in \text{CPTP} (B_0 \rightarrow B_1)$. Now take $E_B := \Theta_{A \rightarrow B} [\mathcal{N}_A]$ and $F_B := \mathcal{M}_B$; $d_\delta (\mathcal{N}_A \rightarrow \mathcal{M}_B)$ becomes

$$d_\delta (\mathcal{N}_A \rightarrow \mathcal{M}_B) = \min \{ \lambda : \lambda Q_B \geq \Theta_{A \rightarrow B} [\mathcal{N}_A] - \mathcal{M}_B \},$$

where $\Theta \in \text{FREE}(A \rightarrow B)$ and $Q \in \text{CPTP} (B_0 \rightarrow B_1)$. This can be phrased as a conic linear program, so it has a dual (see appendix A for details), by which $d_\delta (\mathcal{N}_A \rightarrow \mathcal{M}_B)$ can also be expressed as

$$d_\delta (\mathcal{N}_A \rightarrow \mathcal{M}_B) = \max \{ t | A_1 B_0 | + \text{tr} [\zeta_B J_B^M] \},$$

subject to the constraints $0 \leq \zeta_B \leq \eta_B \otimes I_{B_1}$, $\text{tr} [\eta_B] = 1$, and $(J_B^N)^T \otimes \zeta_B - tI_{A_1 B_1} \in \mathcal{R}^*$, where the cone $\mathcal{R}^*$ is the dual of the cone generated by the Choi matrices of free superchannels (see Eq. (12)).

C. Definitions of various rates in the asymptotic regime

In the asymptotic regime, we are interested in the asymptotic rates of converting one resource into another by means of the set of free superchannels. The asymptotic rate of conversion from a channel $\mathcal{N} \in \text{CPTP} (A_0 \rightarrow A_1)$ to a channel $\mathcal{M} \in \text{CPTP} (B_0 \rightarrow B_1)$ is defined as

$$R_\delta (\mathcal{N} \rightarrow \mathcal{M}) := \lim_{\varepsilon \rightarrow 0^+} \inf \left\{ \frac{n}{m} : d_\delta (\mathcal{N}_{A}^{\otimes n} \rightarrow \mathcal{M}_{B}^{\otimes m}) \leq \varepsilon; m, n \in \mathbb{N} \right\}.$$ 

If a maximal resource exists, we can also define the asymptotic resource cost and distillation (see also Ref. [38]) respectively as

$$\text{COST}_\delta (\mathcal{N}_A) := \lim_{\varepsilon \rightarrow 0^+} \inf_n \frac{1}{n} \text{COST}^{(1)}_{\delta, \varepsilon} (\mathcal{N}_{A}^{\otimes n}).$$
Remark 7. Note that, since the \( n \) slots of a quantum comb are causally ordered [65–67], it is important to know the order in which the resources are inserted. If the \( n \) channels \( \mathcal{N}_1, \ldots, \mathcal{N}_n \) are all available at the initial time, and we do not know which to plug first into the comb, then we must pick a particular ordering of them. More formally, we need to pick a permutation \( \pi \in S_n \) that fixes the causal ordering between the \( n \) resources, whereby their most general manipulation is \( \mathcal{E}_n \left[ \mathcal{N}_{\pi(1)}, \ldots, \mathcal{N}_{\pi(n)} \right] \).

With this in mind, when a maximal resource exists, we define the single-shot \( \varepsilon \)-resource cost of a channel \( \mathcal{N} \in \text{CPTP} \) (\( A_0 \rightarrow A_1 \)) as

\[
\text{COST}^{(n), \text{Ad}}_{\varepsilon} (\mathcal{N}_A) := \log \min \left\{ |B| : d_\varepsilon \left( \mathcal{E}_n \left[ \Phi_B^{+n} \right] \rightarrow \mathcal{N}_A \right) \leq \varepsilon \right\}.
\]

The single-shot \( \varepsilon \)-resource distillation of a channel \( \mathcal{N} \in \text{CPTP} \) (\( A_0 \rightarrow A_1 \)) is, instead, defined as

\[
\text{DISTILL}^{(n), \text{Ad}}_{\varepsilon} (\mathcal{N}_A) := \log \max \left\{ |B| : d_\varepsilon \left( \mathcal{E}_n \left[ \mathcal{N}_A^m \right] \rightarrow \Phi_B^{+} \right) \leq \varepsilon \right\}.
\]

The asymptotic adaptive rate of conversion from a channel \( \mathcal{N} \in \text{CPTP} \) (\( A_0 \rightarrow A_1 \)) to a channel \( \mathcal{M} \in \text{CPTP} \) (\( B_0 \rightarrow B_1 \)) by free operations is given by

\[
R_{\varepsilon}^{\text{Ad}} (\mathcal{N}_A \rightarrow \mathcal{M}_B) := \lim_{\varepsilon \rightarrow 0^+} \inf \left\{ \frac{n}{m} : d_\varepsilon \left( \mathcal{E}_n \left[ \mathcal{N}_A^m \right] \rightarrow \mathcal{M}_B^m \right) \leq \varepsilon ; m, n \in \mathbb{N} \right\}.
\]

Here by \( \mathcal{M}_B^m \) we denote the channel \( \mathcal{D}_m \left[ \mathcal{M}_B \right] \), i.e. the action of a (possibly non-free) comb \( \mathcal{D}_m \) on \( m \) copies of \( \mathcal{M}_B \) inserted in its \( m \) slots. Again, this also includes the case in which the target resource \( \mathcal{M}_B \) arises in \( m \) parallel copies, i.e. \( \mathcal{M}_B^m \). If a maximal resource exists, we can also define the asymptotic adaptive resource cost and adaptive resource distillation respectively as

\[
\text{COST}^{\text{Ad}}_{\varepsilon} (\mathcal{N}_A) := \lim_{\varepsilon \rightarrow 0^+} \inf \left\{ \frac{1}{m} : \text{COST}^{(n), \text{Ad}}_{\varepsilon} (\mathcal{N}_A^m) \right\}
\]

and

\[
\text{DISTILL}^{\text{Ad}}_{\varepsilon} (\mathcal{N}_A) := \lim_{\varepsilon \rightarrow 0^+} \inf \left\{ \frac{1}{m} : \text{DISTILL}^{(n), \text{Ad}}_{\varepsilon} (\mathcal{N}_A^m) \right\},
\]

where, as above \( \mathcal{N}_A^m \) denotes the action of a (possibly non-free) comb on \( m \) copies of \( \mathcal{N}_A \) (note that \( m \) depends on \( n \)). The adaptive exact resource distillation and resource cost are defined similarly as above.

**IV. CONCLUSIONS**

In this article we presented the general framework for resource theories of quantum processes. In particular, we introduced a new construction of a complete family of monotones governing the simulation of channels by free superchannels, which is valid in all convex resource theories of quantum processes. We showed that the problem of resource interconversion can be turned into a conic linear program, whose hardness depends on the particular resource theory under consideration.

Moreover, we also showed that shifting our focus from states to processes introduces a richer landscape of protocols that can be implemented for resource conversions. This stems from the fact that channels, unlike states, have an input and an output, therefore they can be composed in a variety of ways. Hence the most general manipulation of multiple copies of a resource follows an adaptive scheme, in which the various copies are inserted into the slots of a free circuit (a free comb). This scheme is most general, as it includes the well-known case of the tensor product of many copies. This added layer of complexity makes resource theories of processes far more complicated to study than resource theories of states.
However, this is not the only extra complication. A further difficulty concerns the realization of free superchannels. We saw that a priori there is no guarantee that all free superchannels are also freely realizable, i.e., they can be implemented with free pre-processing and post-processing channels. We conjecture that in fact there exist free superchannels that are not freely realizable. In general, it is hard to determine if a given free superchannel admits a free realization, so we were not able to provide a conclusive answer to this issue. However, the results we announced in Ref. [59] suggest that focusing only on freely realizable superchannels makes the issue of studying resource interconversion much more complicated than considering generic free superchannels.

A further possible generalization is to relax the hypothesis of causal manipulation of multiple dynamical resources. Indeed, when multiple resources are plugged into a free quantum comb to be converted, the order in which they are inserted matters, for the slots of the comb into a free quantum comb to be converted, the order in which they are inserted matters, for the slots of the comb might help us get an advantage on resource manipulation, as we already know this to happen in the case of the quantum switch [67, 103–106].

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The first step to determine the dual of the conic linear program associated with Eq. (13) is to express $d_3 (\mathcal{N} \to \mathcal{M})$ using Choi matrices and the characterization of the diamond norm in Ref. [101]. We have

$$
d_3 (\mathcal{N}_A \to \mathcal{M}_B) = \min \left\{ \lambda : \omega_B \geq \text{tr}_A \left[ J_{AB}^\Theta \left( (J_N^N)^T \otimes I_B \right) \right] - J_{AB}^M ; \lambda I_{B_0} \geq \omega_{B_0}; \Theta \in \text{FREE} (A \to B) ; \omega_B \geq 0 \right\}
$$

$$
= \min \left\{ \lambda : \omega_B \geq \text{tr}_A \left[ \alpha_{AB} \left( (J_N^N)^T \otimes I_B \right) \right] - J_{AB}^M ; \lambda I_{B_0} \geq \omega_{B_0}; \omega_B \geq 0; \alpha_{AB} \in \mathcal{J}_{AB} \right\}, \quad (A1)
$$

where $\mathcal{J}_{AB}$ is the set of the Choi matrices of free superchannels (cf. Eq. (11)). We want to work with the dual problem using conic linear programming, but $\mathcal{J}_{AB}$, albeit convex, is not a cone. Therefore we consider the cone $\mathcal{R}$ generated by $\mathcal{J}_{AB}$ (see subsection III A). Now Eq. (A1) can be rewritten as

$$
d_3 (\mathcal{N}_A \to \mathcal{M}_B) = \min \left\{ \lambda : \omega_B \geq \text{tr}_A \left[ \alpha_{AB} \left( (J_N^N)^T \otimes I_B \right) \right] - J_{AB}^M ; \lambda I_{B_0} \geq \omega_{B_0}; \omega_B \geq 0; \alpha_{AB} \in \mathcal{R}; \text{tr} \left[ \alpha_{AB} \right] = |A_1 B_0| \right\}.
$$

Now, following Ref. [107], consider the two convex cones

$$
\mathcal{R}_1 := \{ (\lambda, \omega_B, \alpha_{AB}) : \lambda \in \mathbb{R}^+; \omega_B \geq 0; \alpha_{AB} \in \mathcal{R} \}
$$

$$
\mathcal{R}_2 := \{ (R_{B_0}, P_B, 0) : R_{B_0} \geq 0; P_B \geq 0 \}.
$$

$\mathcal{R}_1$ is a subset of the vector space $\mathbb{R} \oplus \text{Herm} (B) \oplus \text{Herm} (AB)$, whereas $\mathcal{R}_2$ is a subset of $\text{Herm} (B_0) \oplus \text{Herm} (B) \oplus \mathbb{R}$. These two vector spaces carry an inner product. For $\mathbb{R} \oplus \text{Herm} (B) \oplus \text{Herm} (AB)$ it is

$$
\langle (\lambda, \omega_B, \alpha_{AB}), (\lambda', \omega'_B, \alpha'_{AB}) \rangle = \lambda \lambda' + \text{tr} [\omega_B \omega'_B] + \text{tr} [\alpha_{AB} \alpha'_{AB}];
$$

for $\text{Herm} (B_0) \oplus \text{Herm} (B) \oplus \mathbb{R}$ it is

$$
\langle (\eta_{B_0}, \zeta_B, t), (\eta'_{B_0}, \zeta'_B, t') \rangle = \text{tr} [\eta_{B_0} \eta'_{B_0}] + \text{tr} [\zeta_B \zeta'_B] + tt'.
$$

Now consider the linear map $\mathcal{L} : \mathbb{R} \oplus \text{Herm} (B) \oplus \text{Herm} (AB) \to \text{Herm} (B_0) \oplus \text{Herm} (B) \oplus \mathbb{R}$. Its action on a generic element $X = (\lambda, \omega_B, \alpha_{AB})$ of $\mathcal{R}_1$ is

$$
\mathcal{L} (X) := (\lambda I_{B_0} - \omega_{B_0}, \omega_B - \text{tr}_A \left[ \alpha_{AB} \left( (J_N^N)^T \otimes I_B \right) \right], \text{tr} \left[ \alpha_{AB} \right]) ;
$$

Notice that this specifies $\mathcal{L}$ completely because $\mathcal{R}_1$ spans the whole domain of $\mathcal{L}$. Now consider

$$
H_1 = (1, 0_B, 0_{AB}) \in \mathbb{R} \oplus \text{Herm} (B) \oplus \text{Herm} (AB)
$$

and

$$
H_2 = (0_{B_0}, J_{AB}^M, |A_1 B_0|) \in \text{Herm} (B_0) \oplus \text{Herm} (B) \oplus \mathbb{R}.
$$

With this notation we can write [107]

$$
d_3 (\mathcal{N}_A \to \mathcal{M}_B) = \min \{ \langle X, H_1 \rangle : \mathcal{L} (X) - H_2 \in \mathcal{R}_2; X \in \mathcal{R}_1 \}
$$

$$
= \max \{ \langle Y, H_2 \rangle : H_1 - \mathcal{L}^* (Y) \in \mathcal{R}_1^*; Y \in \mathcal{R}_2^* \}.
$$
where the second equality follows from strong duality. We only need to calculate \( \mathcal{L}^* (Y) \), where \( Y = (\eta_{B_0}, \zeta_B, t) \) is in \( \text{Herm} (B_0) \oplus \text{Herm} (B) \oplus \mathbb{R} \). We have

\[
\mathcal{L}^* (Y) = \left( \text{tr} [\eta_{B_0}], \zeta_B - \eta_{B_0} \otimes I_{B_1}, tI_{AB} - (J_N^N)^T \otimes \zeta_B \right).
\]

Hence,

\[
d_S (\mathcal{N}_A \to \mathcal{M}_B) = \max \left\{ t \left| A_1 B_0 \right| + \text{tr} [\zeta_B J_B^M] : \text{tr} [\eta_{B_0}] \leq 1; 0 \leq \zeta_B \leq \eta_{B_0} \otimes I_{B_1}; (J_N^N)^T \otimes \zeta_B - tI_{AB} \in \mathbb{R}^* \right\}
\]

\[
= \max \left\{ t \left| A_1 B_0 \right| + \text{tr} [\zeta_B J_B^M] : \text{tr} [\eta_{B_0}] = 1; 0 \leq \zeta_B \leq \eta_{B_0} \otimes I_{B_1}; (J_N^N)^T \otimes \zeta_B - tI_{AB} \in \mathbb{R}^* \right\},
\]

where \( \mathbb{R}^* \) is the dual of the cone generated by the Choi matrices of free superchannels. We have obtained Eq. (14).