Quantitative uniform distribution results for geometric progressions

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Abstract

By a classical theorem of Koksma the sequence of fractional parts $(\{x^n\})_{n \geq 1}$ is uniformly distributed for almost all values of $x > 1$. In the present paper we obtain an exact quantitative version of Koksma’s theorem, by calculating the precise asymptotic order of the discrepancy of $(\{\xi x^n\})_{n \geq 1}$ for typical values of $x$ (in the sense of Lebesgue measure). Here $\xi > 0$ is an arbitrary constant, and $(s_n)_{n \geq 1}$ can be any increasing sequence of positive integers.

1 Introduction and statement of results

A sequence $(x_n)_{n \geq 1}$ of real numbers from the unit interval is called uniformly distributed modulo 1 (u.d. mod 1) if for any $0 \leq a < b \leq 1$

$$\frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{(a,b)}(x_n) \to b - a \quad \text{as } N \to \infty.$$ (1)

In other words, a sequence is u.d. mod 1 if the relative number of elements of the sequence contained in an interval $[a, b] \subset [0, 1)$ always converges to the length (or Lebesgue measure) of this interval. Here the length of such an interval can be interpreted as the expected value for the relative number of elements of a random sequence contained in it, and with regard to the Glivenko–Cantelli theorem a uniformly distributed sequence can be considered as a sequence showing random behavior. There exist many sequences which are u.d. mod 1, for example the sequence $(\{nx\})_{n \geq 1}$ whenever $x \notin \mathbb{Q}$ (here, and in the sequel, $\{\cdot\}$ denotes the fractional part).

The speed of convergence in (1) is measured by the discrepancy and the star-discrepancy of the sequence $(x_n)_{n \geq 1}$. For a finite sequence $(x_1, \ldots, x_N)$ the (extremal) discrepancy $D_N$ and the star-discrepancy $D_N^*$ are defined as

$$D_N(x_1, \ldots, x_N) = \sup_{0 \leq a < b \leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{(a,b)}(x_n) - (b - a) \right|$$

$$D_N^*(x_1, \ldots, x_N) = \sup_{0 \leq a < b \leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[a,b]}(x_n) - \mu_{a,b} \right|$$

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and
\[
D_N(x_1, \ldots, x_N) \sup_{0 < a \leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[0,a)}(x_n) - a \right|.
\]
For simplicity, we will write \(D_N(x_n)\) and \(D^*_N(x_n)\) for the discrepancy resp. star-discrepancy of the first \(N\) elements of a (finite or infinite) sequence. For an introduction to the theory of uniform distribution modulo 1 and discrepancy theory the reader is referred to the monographs [27, 47].

By a remarkable result of Weyl [66] for any sequence of distinct integers \((s_n)_{n \geq 1}\) the sequence \((\{s_n x\})_{n \geq 1}\) is u.d. mod 1 for almost all \(x\) (in the sense of Lebesgue measure). This is equivalent to the fact that
\[
D_N(\{s_n x\}) \to 0 \quad \text{as } N \to \infty \quad \text{for almost all } x.
\]
Precise results are only known in a few special cases. For example, when \(s_n = n, \ n \geq 1,\) we have
\[
\frac{N D_N(\{nx\})}{\log N \log \log N} \to \frac{2}{\pi^2} \quad \text{in measure}
\]
due to Kesten [44] (see also [58]). Exact results of this type are possible since there is an intimate connection between the discrepancy of \((\{nx\})_{n \geq 1}\) and the continued fraction expansion of \(x\). The second class of sequences for which precise metric results are known are sequences satisfying the Hadamard gap condition
\[
\frac{s_{n+1}}{s_n} \geq q > 1, \quad n \geq 1.
\]
In this case Philipp [56] proved the bounded law of the iterated logarithm (LIL)
\[
\frac{1}{4} \leq \frac{\sqrt{N D_N(\{s_n x\})}}{\sqrt{\log \log N}} \leq C_q \quad \text{a.e.},
\]
where \(C_q\) depends only on the growth factor \(q\) (the lower bound follows from an older result of Erdős and Gál [30] and Koksma’s inequality). For sub-exponentially growing \((s_n)_{n \geq 1}\) the LIL [30] generally fails, unless \((s_n)_{n \geq 1}\) satisfies some strong number-theoretic conditions (see for example [11, 20]). For sequences of the special form \(s_n = \beta^n, \ n \geq 1,\) for some \(\beta > 1,\) Fukuyama [35] recently proved the precise LIL
\[
\frac{\sqrt{N D_N(\{\beta^n x\})}}{\sqrt{\log \log N}} = \sigma_\beta \quad \text{a.e.},
\]
where \(\sigma_\beta\) is a constant depending on the number-theoretic properties of \(\beta\) in a very complicated and interesting way. In particular
\[
\frac{\sqrt{N D_N(\{2^n x\})}}{\sqrt{\log \log N}} = \frac{2\sqrt{21}}{9} \quad \text{a.e.},
\]
and
\[
\frac{\sqrt{N D_N(\{\beta^n x\})}}{\sqrt{\log \log N}} = \frac{1}{\sqrt{2}} \quad \text{a.e.}
\]
if $\beta$ is a number for which $\beta^r \not\in \mathbb{Q}$ for all $r \geq 1$. These results should be compared to the Chung–Smirnov law of the iterated logarithm for independent, identically $[0,1]$-uniformly distributed random variables $(X_n)_{n \geq 1}$, which states that

$$\frac{\sqrt{N} D_N(X_n)}{\sqrt{\log \log N}} = \frac{1}{\sqrt{2}} \quad \text{a.s.}$$

In this specific form the law of the iterated logarithm for the discrepancy (in the language of probability theory: for the Kolmogorov–Smirnov statistic) of $(X_n)_{n \geq 1}$ is due to Chung [26] and Cassels [25]: for a general formulation, see e.g. [59, p. 504]. Recall that a number $x$ is a normal number in base $\beta$ if and only if $D_N(\{\beta^n x\}) \to 0$ as $N \to \infty$. Consequently Fukuyama’s result is a precise quantitative version of Borel’s well-known theorem that almost all numbers are normal [21].

To the best of my knowledge the two mentioned classes of sequences (arithmetic progressions and lacunary sequences) are essentially the only two classes of parametric sequences for which the typical (in the sense of Lebesgue measure) asymptotic order of the discrepancy is precisely known. For general sequences $(s_n)_{n \geq 1}$ of distinct integers we only have the upper bounds

$$D_N(\{s_n x\}) = O\left(\frac{(\log N)^{5/2+\varepsilon}}{\sqrt{N}}\right) \quad \text{a.e.}$$

(Erdős and Koksma [32]) and

$$D_N(\{s_n x\}) = O\left(\frac{(\log N)^{3/2+\varepsilon}}{\sqrt{N}}\right) \quad \text{a.e.} \quad \text{if } (s_n)_{n \geq 1} \text{ is increasing} \quad (5)$$

(Baker [12]). It is known that the exponent of the logarithmic term in (5) can in general not be reduced below $1/2$ (Berkes and Philipp [19]), but as (6) shows for a specific sequence $(s_n)_{n \geq 1}$ the typical speed of convergence of $D_N(\{s_n x\})$ can differ from (5) significantly. More details on metric discrepancy theory can be found in the book of Harman [40] and in the survey paper [7].

In 1935, Koksma [46] proved a very general result in uniform distribution theory, which as a special case contains the fact that for any $\xi > 0$ and any sequence $(s_n)_{n \geq 1}$ of distinct positive integers the sequence $(\{\xi x^n\})_{n \geq 1}$ is u.d. mod 1 for almost all $x > 1$. In particular, geometric progressions $(\{x^n\})_{n \geq 1}$ are u.d. mod 1 for almost all $x > 1$. Erdős and Koksma [31] proved that the asymptotic order of the discrepancy of $(\{\xi x^n\})_{n \geq 1}$, in the case of increasing $(s_n)_{n \geq 1}$, satisfies

$$D_N(\{\xi x^n\}) = O\left(\frac{(\log N)^{3/2}(\log \log N)^{1/2+\varepsilon}}{\sqrt{N}}\right) \quad \text{as } N \to \infty \quad \text{for almost all } x > 1. \quad (6)$$

In 1950 this was improved by Cassels [24], who obtained

$$D_N(\{\xi x^n\}) = O\left(\frac{\log N(\log \log N)^{3/2+\varepsilon}}{\sqrt{N}}\right) \quad \text{as } N \to \infty \quad \text{for almost all } x > 1. \quad (7)$$
Since then, no further improvements of \( \{7\} \) have been made. On the other hand, as far as I know, no asymptotic lower bounds for \( D_N(\langle \xi x^n \rangle) \) or \( D_N(\{x^n\}) \) for typical values of \( x \) (in the sense of Lebesgue measure) have ever been proved.

Concerning the asymptotic distribution of \( \{\langle x^n \rangle\}_{n \geq 1} \), it should be mentioned that Niederreiter and Tichy \( [60] \) proved that this sequence is completely uniformly distributed\(^4\) modulo 1 for almost all \( x > 1 \), by this means solving a problem posed by Knuth \( [45] \), who suggested complete uniform distribution as a criterion for pseudorandomness of deterministic sequences. In particular Theorem 1 solves a problem of V.I. Arnold, who in one of his final papers formulated the conjecture that the discrepancy

\[
\limsup_{n \to \infty} \frac{1}{n} |D_n(\{\xi x^n\})| = 0
\]

lower bounds for the discrepancy of such sequences. In particular Theorem 1 solves a problem of V.I. Arnold, who in one of his final papers formulated the conjecture that the discrepancy

\[
D_N(\{\xi x^n\}) \quad \text{is not of order } o(N^{-1/2}) \quad \text{for almost all } x > 1 \quad \text{(see } [10], \text{ p. 36}).
\]

\(^4\)A sequence \( \{x_n\}_{n \geq 1} \) is called completely uniformly distributed modulo 1, if for any \( s \geq 1 \) the \( s \)-dimensional sequence \( \{(x_{n}, \ldots, x_{n+s−1})\}_{n \geq 1} \) is uniformly distributed mod 1 in \([0,1]^s\). See \( [27], [47] \) for details.
Theorem 1 For any strictly increasing sequence \((s_n)_{n \geq 1}\) of positive integers and any number \(\xi > 0\) we have for almost all \(x > 1\)

\[
\limsup_{N \to \infty} \frac{\sqrt{N} D_N(\{\xi x^{s_n}\})}{\sqrt{\log \log N}} = \limsup_{N \to \infty} \frac{\sqrt{N} D_N^*(\{\xi x^{s_n}\})}{\sqrt{\log \log N}} = \frac{1}{\sqrt{2}}.
\]

As an immediate consequence of Theorem 1 we obtain the following corollary for geometric progressions of the form \((\xi x^{s_n})_{n \geq 1}\).

Corollary 1 For any \(\xi > 0\) we have for almost all \(x > 1\)

\[
\limsup_{N \to \infty} \frac{\sqrt{N} D_N(\{\xi x^{n}\})}{\sqrt{\log \log N}} = \limsup_{N \to \infty} \frac{\sqrt{N} D_N^*(\{\xi x^{n}\})}{\sqrt{\log \log N}} = \frac{1}{\sqrt{2}}.
\]

As a byproduct of our proof of Theorem 1 we also get the following central limit theorem.

Theorem 2 Let \(f\) be a function satisfying

\[
f(x) = f(x + 1), \quad \int_0^1 f(x) \, dx = 0, \quad \text{Var}_{[0,1]} f \leq 2. \tag{8}
\]

Then for any sequence \((s_n)_{n \geq 1}\) of distinct positive integers, any number \(\xi > 0\) and any nonempty interval \([A, B] \subset (1, \infty)\) we have

\[
P\left(x \in [A, B]: \frac{1}{\sqrt{N}} \sum_{n=1}^{N} f(\xi x^{s_n}) \leq t\right) \to \Phi(t) \quad \text{as } N \to \infty.
\]

Here \(P\) denotes the normalized Lebesgue measure on \([A, B]\), and \(\Phi\) denotes the standard normal distribution function. The convergence is uniform in \(t \in \mathbb{R}\).

I am not certain if these results are expected or surprising. Of course, it is reasonable to imagine that the functions \(f(\xi x^n)\) and \(f(\xi x^m)\) are “almost independent” if the difference between \(m\) and \(n\) is large, and that therefore the system \((f(\xi x^n))_{n \geq 1}\) and the discrepancy \(D_N(\{\xi x^{s_n}\})\) should show “almost” the same behavior as in the case of an i.i.d. random sequence. For example, Beck [13, p. 55] writes that Koksma’s theorem on the uniform distribution of \((\{x^n\})_{n \geq 1}\) for a.e. \(x\) “was extended later to more delicate results such as the law of the iterated logarithm and the central limit theorem”, although such results have not been proved so far; apparently Beck was convinced that they must be true.

However, comparing the case of sequences of the form \((\xi x^{s_n})_{n \geq 1}\) to the somewhat similar case of lacunary sequences, one sees that it is by no means clear that the (precise) LIL and CLT have to hold for geometric progressions. In the case of lacunary sequences \((s_n x)_{n \geq 1}\), the value of the limsup in the LIL for the discrepancy depends on the precise number-theoretic properties of \((s_n)_{n \geq 1}\) in a very complicated way, and can even be non-constant (see [2, 3]). Therefore, the asymptotic behavior of lacunary sequences can change significantly after a permutation of its terms, see [36, 38]. Similarly, the CLT for lacunary sequences \((s_n)_{n \geq 1}\) is only true if the sequence satisfies certain number-theoretic conditions, and the limit distribution of \(N^{-1/2} \sum_{n=1}^{N} f(s_n x)\) can fail to be Gaussian (see [4]).

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Lacunary sequences and geometric progressions are essentially of the same order of growth, so it could also be imagined that additional number-theoretic conditions (like the Diophantine conditions in the case of lacunary sequences) would be necessary to obtain the precise LIL and CLT for geometric progressions. However, no such additional conditions are necessary, and apparently this is due to the fact that sequences of the form \(\{\xi x^n\}\) by construction necessarily have a more inhomogeneous structure than lacunary sequences, which for example in the case \(\{2^n x\}\) can have a very strong periodic and homogeneous structure with respect to both \(x\) and \(n\). For the relation between the metric discrepancy results for geometric progressions in this paper and similar metric discrepancy results for lacunary series see also the addendum at the end of this paper, which I owe Katusi Fukuyama.

The proof of the main theorem of this paper is based on methods which were developed for lacunary function systems. However, there are several major differences to the case of geometric progressions, which made it necessary to develop a new machinery. The two most significant differences are:

- Lacunary systems \((f(s_n x))_{n \geq 1}\) have a direct connection with Fourier analysis, and can be expanded into a Fourier series in a very simple and natural way. This makes it possible to reduce the calculation of \(L^p\)-norms or exponential norms to counting the number of solutions of Diophantine equations, by utilizing the orthogonality of the trigonometric system. In the case of geometric progressions this is not possible, and instead of orthogonality properties we have to use the fact that a function \(f(\xi x^n)\) is highly oscillatory in comparison with \(f(\xi x^m)\) if \(n \gg m\). While in the lacunary case the orthogonality of the trigonometric system guarantees that in calculating integrals most of the mixed factors vanish, we have to use the van der Corput inequality (see below) instead and take care of a huge number of small quantities.

- For any \(f\) satisfying (8) and any integer \(n\) the function \(f(nx)\) is periodic with period \(n\), which means that the global problem of considering all possible values of \(x\) can often be reduced to considering \(x\) only “locally”, and all values of \(x\) can be treated in the same way. In the present case we have functions of the form \(f(\xi x^n)\), which do not posses this homogeneous structure. On the contrary, the speed of oscillation of \(f(\xi x^n)\) increases as \(x\) increases, which for example makes the martingale approximation in Section 7 much more complicated than in the lacunary case.

The rest of this paper is organized as follows:

- In Section 2 we formulate several auxiliary results which will be necessary for the proofs. In particular this includes the statement of the van der Corput lemma, which is a crucial ingredient in our proof.

- In Section 3 we prove a large deviations bound for \(\sum_{n=M+1}^{M+N} f(\xi x^n)\), which is a consequence of an exponential inequality in the spirit of Takahashi [63] and Philipp [56].

- In Section 4 we prove a maximal version of the large deviations inequality from Section 3. For the proof of this maximal inequality we use a dyadic decomposition of the index set.
In Section 5 we prove a bounded law of the iterated logarithm for functions which are the remainder of a Fourier series of a function satisfying (8). Since the contribution of the remainder function of the $d$-th partial sum of the Fourier series is small, it is sufficient to prove the exact LIL for trigonometric polynomials instead of general functions $f$.

In Section 6 we prove a bounded law of the iterated logarithm for a modified discrepancy, which considers only “small” subintervals of $[0,1]$. We show that the contribution of these small intervals is small, and that the proof of Theorem 1 can be reduced to proving the exact LIL for a single function $f$ instead of a supremum over uncountable many indicator functions.

In Section 7 and Section 8 we prove the exact LIL for trigonometric polynomials. The proof uses an approximation by martingale differences, which has been developed by Berkes and, independently, Philipp and Stout. The main ingredient in the proof is a martingale version of the Skorokhod representation theorem due to Strassen.

In Section 9 the precise LIL for functions satisfying (8) is obtained as a consequence of the results from Section 5 and Section 8.

Finally, in Section 10 we give the proof of Theorem 1. In Section 11 we show how the proof of Theorem 2 can be obtained without much additional effort as a byproduct of the proof of Theorem 1.

2 Preliminaries

We will assume throughout the rest of the paper that the number $\xi > 0$ is fixed. Furthermore, it is sufficient to prove that Theorem 1 holds for almost all $x \in [A,B]$, where $[A,B] \subset (1,\infty)$ is an arbitrary interval. Throughout the rest of the paper, the numbers $A, B$ satisfying $1 < A < B$ will be fixed. We will write $c$ for positive numbers, not always the same, which may only depend on $\xi$ and $A, B$, but not on $N, n, f, d$ or anything else (unless stated otherwise at the beginning of the respective section). In the same sense we will use the symbols “$\ll$” and “$\gg$”. For simplicity of writing we will assume that $B - A = 1$, which means that the interval $[A, B]$, equipped with Borel sets and Lebesgue measure, is a probability space. We will write $\mathbb{P}$ for the Lebesgue measure on $[A, B]$, and $E$ for the expected value with respect to this measure.

Throughout the rest of this paper, we will write $\exp(x)$ for $e^x$. Furthermore, $\log x$ denotes the natural logarithm, and should be interpreted as $\max\{1, \log x\}$. We set

$$||f|| = \left( \int_{A}^{B} (f(x))^2 \, dx \right)^{1/2}$$

and

$$I_{[a,b)}(x) = I_{[a,b]}(x) - (b - a). \tag{9}$$

Then for any $0 \leq a < b \leq 1$ the function $I_{[a,b)}$ satisfies (8).

Lemma 1 ([67, p. 48]) Let $f$ be a function satisfying (8), and write

$$f(x) \sim \sum_{j=1}^{\infty} (a_j \cos 2\pi x + b_j \sin 2\pi x)$$
for its Fourier series. Then
\[ |a_j| \leq \frac{1}{j}, \quad |b_j| \leq \frac{1}{j}, \quad \text{for} \quad j \geq 1. \]

The following Lemma 2 is a special case of the van der Corput lemma. It can be found e.g. in [17, Chapter 1, Section 1, Lemma 2.1] or [60, Chapter VIII, Proposition 2].

**Lemma 2** Suppose that \( \phi(x) \) is real-valued, that \( |\phi'(x)| \geq \gamma \) for some positive \( \gamma \), and that \( \phi' \) is monotonic for all \( x \in (\alpha, \beta) \). Then
\[
\left| \int_{\alpha}^{\beta} e^{2\pi i \phi(x)} \, dx \right| \leq \gamma^{-1}.
\]

Lemma 3 and Lemma 4 below follow directly from Lemma 2.

**Lemma 3** Let \( n \) be a positive integer. Then for any subinterval \( [\alpha, \beta] \) of \( [A, B] \) and any integer \( j \geq 1 \) we have
\[
\left| \int_{\alpha}^{\beta} \cos(2\pi j \xi x^n) \, dx \right| \leq \frac{1}{j \xi n^{\alpha^{-1}} - 1}.
\]

**Lemma 4** Let \( m \neq n \) be positive integers. Then for any positive integers \( j, k \) and any subinterval \( [\alpha, \beta] \) of \( [A, B] \) we have
\[
\left| \int_{\alpha}^{\beta} \cos(2\pi \xi (j x^n + k x^m)) \, dx \right| \leq \frac{1}{\xi \max\{m, n\} \alpha^{\max\{m, n\}^{-1}}.
\]

**Lemma 5** Let \( m < n \) be positive integers. Then for any positive integers \( j, k \) and for any \( \eta > 0 \) there exist three disjoint intervals \( I_1, I_2, I_3 \) (depending on \( j, k, m, n \)) such that
\[
P([A, B] \setminus (I_1 \cup I_2 \cup I_3)) \leq 2B\eta
\]
and such that for any interval \( [\alpha, \beta] \) which is completely contained in one of the intervals \( I_1, I_2 \) or \( I_3 \) we have
\[
\left| \int_{\alpha}^{\beta} \cos(2\pi \xi (j x^n - k x^m)) \, dx \right| \leq \frac{1}{\xi \eta^{\alpha m^{-1}} - 1}.
\]

**Proof of Lemma 5** We want to use Lemma 2 for \( \phi(x) = 2\pi \xi (j x^n - k x^m) \). Obviously this is not directly possible, since it might happen that \( \phi'(x) = 2\pi \xi (j n x^{n-1} - k m x^{m-1}) = 0 \) for some \( x \). We have
\[
\phi'(x) = \xi \left( j n x^{n-1} - k m x^{m-1} \right) = \xi (j n x^{n-m} - k m x^{m-1}).
\]
Clearly, for
\[
x_1 = \left( \frac{km}{jn} \right)^{1/(n-m)}
\]
we have \( \phi'(x_1) = 0 \), and for any other \( x > 1 \) we have \( \phi'(x) \neq 0 \). Furthermore, it is easily seen that for \( x > 1 \) there is only one possible value where \( \phi''(x) = 0 \), namely the value
\[
x_2 = \left( \frac{km(m-1)}{jn(n-1)} \right)^{1/(n-m)}.
\]
Note that \( x_2 < x_1 \). This means that the interval \([\alpha, \beta]\) can be partitioned into at most 3 subintervals, in all of which \( \phi'(x) \) is monotonic, respectively. More precisely, in the interval \((1, x_2]\) the function \( \phi' \) is negative and monotonic decreasing, in \([x_2, x_1]\) it is negative and monotonic increasing, and in \([x_1, \infty)\) it is positive and monotonic increasing.

We have

\[
\frac{\sum_{n=1}^{n} x^{n-1}}{\sum_{m=1}^{m} \lambda^{m-1}} = 1.
\]

Thus for any \( x \) satisfying \( x \geq x_1 (1 + \eta) \) this implies

\[
\frac{\sum_{n=1}^{n} x^{n-1}}{\sum_{m=1}^{m} \lambda^{m-1}} \geq \frac{\sum_{n=1}^{n} (1 + \eta)^{n-m}}{\sum_{m=1}^{m} \lambda^{m-1}} \geq 1 + \eta,
\]

and consequently

\[
\phi'(x) \geq (1 + \eta - 1) \xi \sum_{m=1}^{m} \lambda^{m-1} \geq \xi \eta \alpha^{m-1}.
\]  \( \text{(10)} \)

Similarly, for any \( x \) satisfying \( x \leq x_1 (1 - \eta) \) we have

\[
\frac{\sum_{n=1}^{n} x^{n-1}}{\sum_{m=1}^{m} \lambda^{m-1}} \leq \frac{\sum_{n=1}^{n} (1 - \eta)^{n-m}}{\sum_{m=1}^{m} \lambda^{m-1}} \leq 1 - \eta
\]

and consequently

\[
\left| \phi'(x) \right| \geq \left| (1 - \eta - 1) \xi \sum_{m=1}^{m} \lambda^{m-1} \right| \geq \xi \eta \alpha^{m-1}.
\]  \( \text{(11)} \)

Now we set

\[
E = [x_1 (1 - \eta), x_1 (1 + \eta)]
\]

and

\[
I_1 = [A, x_2] \setminus E, \quad I_2 = [x_2, x_1] \setminus E, \quad I_3 = [x_1, B] \setminus E.
\]

Note that it is possible that some of these three intervals are empty, which is no problem. Whenever \([\alpha, \beta]\) is completely contained in one of the intervals \(I_1, I_2\) or \(I_3\) the derivative of \( \phi(x) \) is monotonic in \([\alpha, \beta]\), and by \(\text{(10)}\) and \(\text{(11)}\) for any \( x \in [\alpha, \beta] \) we have

\[
\left| \phi'(x) \right| \geq \xi \eta \alpha^{m-1}.
\]

Thus in this case by Lemma 2

\[
\left| \int_{\alpha}^{\beta} \cos(2\pi \xi (jx^n - kx^m)) \ dx \right| \leq \frac{1}{\xi \eta \alpha^{m-1}}.
\]

Note also that

\[
\mathbb{P}((A, B) \setminus (I_1 \cup I_2 \cup I_3)) \leq \mathbb{P}(E) \leq 2B\eta.
\]

This proves Lemma 4.
3 Exponential inequality

Lemma 6 Let $f$ be a function satisfying (8). Assume additionally that
\[ \|f\| \geq \frac{N^{-1/4}}{\sqrt{2}}. \tag{12} \]

Then there exist numbers $c_A \geq 1$ and $N_0$ (depending only on $A$) such that for any $M \geq 0$, $N \geq N_0$, and any $\delta > 0$ we have
\[
\mathbb{P} \left( x \in [A, B] : \left| \sum_{n=M+1}^{M+N} f(\xi x s_n) \right| \geq \delta c_A \|f\|^{1/4} \sqrt{N \log \log N} \right) \ll \exp \left( (1 - \delta/2) \|f\|^{-1/2} \log \log N \right) + \delta^{-2} N^{-16}. \tag{17}\n\]

For the proof of Lemma 6 we use a method of Takahashi [63], in a refined form of Philipp [56]. For simplicity of writing we assume that $f$ is an even function, i.e. that it can be expanded into a pure cosine-series
\[ f(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi jx; \]

the proof in the general case is exactly the same. Then by Lemma 1 we have
\[ |a_j| \leq \frac{1}{j}, \quad j \geq 1. \tag{13}\]

For any given $M \geq 0$, we write $(w_1, \ldots, w_N)$ for the sequence $(s_{M+1}, \ldots, s_{M+N})$. Set
\[ g(x) = \sum_{j=1}^{N^{3/8}} a_j \cos 2\pi jx, \quad r(x) = f(x) - g(x). \]

Note that (8) implies
\[ \|f\| \leq \|f\|_\infty \leq 1. \tag{14}\]

Thus
\[ \|g\|_\infty \leq \|f\|_\infty + \text{Var}_{[0,1]} f \leq 3, \tag{15}\]

by Lemma 1 and equations (1.25) and (3.5) of Chapter III of [67].

Lemma 6 will be deduced from Lemma 7 and Lemma 8 below.

Lemma 7 There exists a constant $\tilde{c}_A$ such that for any sufficiently large $N$ (depending only on $A$) and any $\tau > 0$ satisfying
\[ \tau \geq N^{-1/2} \quad \text{and} \quad 12\tau \left( N^{1/8} \right) \leq 1 \tag{16}\]

we have
\[ \int_A^B \exp \left( \tau \sum_{n=1}^{N} g(\xi x w_n) \right) \, dx \ll e^{\tau^2 \tilde{c}_A \|f\| N}. \tag{17}\]
Lemma 8

\[ \int_A^B \left( \sum_{n=1}^N r(\xi x^w_n) \right)^2 \, dx \ll N^{-15}. \]

Corollary 2 Under the assumptions of Lemma 7 we have

\[ \int_A^B \exp \left( \sum_{n=1}^N g(\xi x^w_n) \right) \, dx \ll e^{\tau^2 \tilde{c}_A \|f\|N}. \]

The corollary is obtained by using Lemma 7 also for the function \(-g(x)\) instead of \(g(x)\).

Proof of Lemma 7: We use the inequality

\[ e^z \leq 1 + z + z^2, \quad \text{for } |z| \leq 1. \] (18)

Set

\[ H = \left\lceil N^{1/8} \right\rceil, \] (19)

and

\[ P = \max\{m \in \mathbb{N} : Hm < N\}. \]

For \(1 \leq m < P\) set

\[ U_m(x) = \sum_{n=Hm+1}^{H(m+1)} g(\xi x^w_n), \] (20)

and

\[ U_P(x) = \sum_{n=HP+1}^N g(\xi x^w_n). \]

Then

\[ \sum_{n=1}^N g(\xi x^w_n) = \sum_{m=1}^P U_m(x). \]

By (15) and the first inequality in (16) we have

\[ \int_A^B \exp \left( 4\tau U_P(x) \right) \, dx \leq e^1 \ll e^{\tilde{c}_A \tau^2 N} \] (21)

(the value of \(\tilde{c}_A\) will be chosen later, but we can assume that \(\tilde{c}_A \geq 1\)). By the Cauchy–Schwarz inequality, (17) follows from (21), together with

\[ \int_A^B \exp \left( 4\tau \sum_{m: 1 \leq 2m < P} U_{2m}(x) \right) \, dx \ll e^{\tilde{c}_A \tau^2 N}, \] (22)

and

\[ \int_A^B \exp \left( 4\tau \sum_{m: 1 \leq 2m+1 < P} U_{2m+1}(x) \right) \, dx \ll e^{\tilde{c}_A \tau^2 N}. \] (23)
The main idea of splitting the integral (17) into the parts (22) and (23) is that by separating the functions $U_m$ into two classes (those with even and those with odd index) there is also a separation of the corresponding values of $w_n$ in (24). Consequently, two functions $U_{m_1}$ and $U_{m_2}$ which have both even or both odd index are “almost” independent, and

$$
\int_A^B e^{U_{m_1} e^{U_{m_2}}} \, dx \approx \left( \int_A^B e^{U_{m_1}} \, dx \right) \left( \int_A^B e^{U_{m_2}} \, dx \right).
$$

Furthermore, the sum of summands in the definition of $U_m$ is so small that we can use the approximation (18) for $\tau U_m$.

We will only prove (22); the proof of (23) can be given in exactly the same way. By (15) for which

$$
\int_A^B e^{U_m} \, dx \approx \tau U_m,
$$

is so small that we can use the approximation (18) for $\tau U_m$.

By (18) this implies

$$
\int_A^B \exp \left( 4\tau \sum_{1 \leq 2m < P} U_{2m}(x) \right) \, dx = \int_A^B \prod_{1 \leq 2m < P} \exp \left( 4\tau U_{2m}(x) \right) \, dx
$$

$$
\leq \int_A^B \prod_{1 \leq 2m < P} \left( 1 + 4\tau U_{2m}(x) + 16\tau^2 U_{2m}(x)^2 \right) \, dx.
$$

For any $m$, $1 \leq 2m < P$, using the standard trigonometric identity

$$
\cos x \cos y = \frac{1}{2} (\cos(x + y) + \cos(x - y)),
$$

we have

$$
16\tau^2 U_{2m}^2 = 16\tau^2 \left( \sum_{n=2Hm+1}^{N^{38}} \sum_{j=1}^{N^{38}} a_j \cos \left( 2\pi \xi j x^{w_n} \right) \right)^2
$$

$$
= 16\tau^2 \sum_{n_1,n_2=2Hm+1}^{H(2m+1)} \sum_{j_1,j_2=1}^{N^{38}} a_{j_1} a_{j_2} \cos \left( 2\pi \xi (j_1 x^{w_{n_1}} + j_2 x^{w_{n_2}}) \right)
$$

$$
+ 16\tau^2 \sum_{n_1,n_2=2Hm+1}^{H(2m+1)} \sum_{j_1,j_2=1}^{N^{38}} a_{j_1} a_{j_2} \cos \left( 2\pi \xi (j_1 x^{w_{n_1}} - j_2 x^{w_{n_2}}) \right)
$$

$$
= 16\tau^2 \sum_{n_1,n_2=2Hm+1}^{H(2m+1)} \sum_{j_1,j_2=1}^{N^{38}} a_{j_1} a_{j_2} \cos \left( 2\pi \xi (j_1 x^{w_{n_1}} + j_2 x^{w_{n_2}}) \right)
$$

$$
+ 16\tau^2 \sum_{n_1,n_2=2Hm+1}^{H(2m+1)} \sum_{j_1,j_2=1}^{N^{38}} a_{j_1} a_{j_2} \cos \left( 2\pi \xi (j_1 x^{w_{n_1}} - j_2 x^{w_{n_2}}) \right)
$$

$$
+ 16\tau^2 \sum_{n_1,n_2=2Hm+1}^{H(2m+1)} \sum_{j_1,j_2=1}^{N^{38}} a_{j_1} a_{j_2} \cos \left( 2\pi \xi (j_1 x^{w_{n_1}} - j_2 x^{w_{n_2}}) \right).
$$

Here the symbol “*” in (26) indicates that in this sum only those values of $j_1, j_2$ are considered for which

$$
\max\{j_1, j_2\} \geq \min\{j_1, j_2\} A^{n_1-n_2},
$$
while the symbol "∗∗" in (27) means that this sum is restricted to those $j_1, j_2$ for which
\[
\max\{j_1, j_2\} < \min\{j_1, j_2\} A^{[n_1-n_2]}.
\]
We write $W_{2m}(x)$ for the sum of $4\tau U_{2m}(x)$ plus the expressions in (25) and (26), and $V_{2m}$ for the expression in (27). Then
\[
1 + 4\tau U_{2m}(x) + 16\tau^2 U_{2m}(x)^2 = 1 + W_{2m}(x) + V_{2m}(x).
\] (29)

For $V_{2m}(x)$ we have, using (13), (28), and the Cauchy–Schwarz inequality,
\[
\|V_{2m}\|_\infty \leq 16\tau^2 \sum_{n_1, n_2 = 2Hm+1}^{N^{38}} \frac{\sum_{j_1, j_2 = 1}^{N^{38}} a_{j_1} a_{j_2}}{\max\{j_1, j_2\} \geq \min\{j_1, j_2\} A^{[n_1-n_2]}} \leq 64\tau^2 \sum_{n_1, n_2 = 2Hm+1}^{N^{38}} \frac{\sum_{j_1, j_2 = 1}^{N^{38}} a_{j_1} a_{j_2}}{2}
\]
\[
\leq 64\tau^2 H \sum_{\ell = 0}^{\infty} \sum_{j_2 = 1}^{N^{38}} \sum_{j_1 \geq j_2 A^\ell} \frac{a_{j_1} a_{j_2}}{2}
\]
\[
\leq 64\tau^2 H \left( \sum_{j = 1}^{N^{38}} \left( \frac{a_j}{\sqrt{2}} \right)^2 \right)^{1/2} \sum_{\ell = 0}^{\infty} \left( \sum_{j \geq A^\ell} \frac{1}{j^2} \right)^{1/2}
\]
\[
\leq \tau^2 H \|f\| \tilde{c}_A,
\] (30)

where $\tilde{c}_A$ is a constant depending only on $A$. Thus by (29)
\[
1 + 4\tau U_{2m}(x) + 16\tau^2 U_{2m}(x)^2 \leq 1 + W_{2m}(x) + \tau^2 H \|f\| \tilde{c}_A.
\]

We can write the function $W_{2m}(x)$ as a sum of at most $3H^2 N^{76}$ functions of the form
\[
\cos \left( 2\pi \xi j_1 x^{\ell_1} \right) \quad \text{or} \quad \cos \left( 2\pi \xi (j_1 x^{\ell_1} \pm j_2 x^{\ell_2}) \right),
\] (31)
all of which have coefficients bounded by $\max\{|a_j|, j \geq 1\} \leq 1$. The derivative of the arguments of the cosine-functions in (31) is at most
\[
4\pi \xi N^{38} w_{H(2m+1)} x^{w_{H(2m+1)}-1} \quad \text{for} \quad x \in [A, B],
\] (32)
and, on the other hand, by construction, this derivative is at least
\[
2\pi \xi w_{2Hm+1} x^{w_{2Hm+1}-1} (1 - A^{-1}) \quad \text{for} \quad x \in [A, B]
\] (33)
(since functions having smaller derivative are collected in $V_{2m}$). Furthermore, the second derivative is at least
\[
2\pi \xi w_{2Hm+1} (w_{2Hm+1} - 1) x^{w_{2Hm+1}-2} (1 - A^{-1}) \quad \text{for} \quad x \in [A, B].
\] (34)
By (29) and (30) we have
\[ \int_A^B \prod_{1 \leq 2m < P} \left( 1 + 4\tau U_{2m}(x) + 16\tau^2 U_{2m}(x)^2 \right) dx \leq \int_A^B \prod_{1 \leq 2m < P} \left( 1 + W_{2m}(x) + \tau^2 H \|f\|\bar{c}_A \right) dx. \]

For some \( L \) let \( i_1 \leq \cdots \leq i_L \) be any numbers from the set \( \{m : 1 \leq 2m < P\} \), and let \( h_{i_1}(x), \ldots, h_{i_L}(x) \) be functions of the form (31) from \( W_{2i_1}(x), \ldots, W_{2i_L}(x) \), resp. Then by (32) and (33) the product
\[ \prod_{\ell=1}^L h_{i_\ell} \]
is a sum of cosine-functions with coefficients at most 1, such that the argument of each cosine-function has derivative at least
\[ 2\pi \xi \left( w_{2Hi_{i_l}+1} w_{2Hi_{i_l}+1}^{-1} (1 - A^{-1}) - \sum_{\ell=1}^{L-1} 2N^{38} w_{2H(2i_{\ell+1})} w_{2H(2i_{\ell+1})}^{-1} \right) \]
\[ \geq 2\pi \xi w_{2Hi_{i_l}+1} \left( x_{2Hi_{i_l}+1}^{-1} (1 - A^{-1}) - 2N^{38} \sum_{\ell=1}^{L-1} x_{2H(2i_{\ell+1})}^{-1} \right) \]
\[ \geq 2\pi \xi w_{2Hi_{i_l}+1} \left( x_{2Hi_{i_l}+1}^{-1} \left( (1 - A^{-1}) - 2N^{39} x_{2Hi_{i_l}+1}^{-1} \right) \right) \]
\[ \gg A^{2Hi_{i_l}}, \]
for sufficiently large \( N \), since by (19)
\[ 2N^{39} x^{-H} \leq \frac{(1 - A^{-1})}{2} \]
for sufficiently large \( N \) (depending only on \( A \)). Similarly, using (34), it is seen that the second derivative of the argument of each cosine-function which appears in (35) is positive (for sufficiently large \( N \)). Thus by Lemma 2
\[ \int_A^B \left( \prod_{\ell=1}^L h_{i_\ell} \right) dx \ll A^{-2Hi_{i_l}}. \]

For any fixed \( K \), there are in total at most
\[ (3H^2 N^{76})^{i_L} \]
functions of the form (35) for which \( i_L = K \) (in other words, functions which are composed from one function in \( W_{2K} \) and at most one function from \( W_2, W_4, \ldots, W_{2K-2} \)). Furthermore,
each of them has coefficient at most 1. Thus, using $1 + x \leq e^x$ and
\[
\sum_{K=1}^{[P/2]} \left( \frac{3H^2N^{76}}{A^2H} \right)^{2K} \leq \sum_{K=1}^{\infty} \left( \frac{3H^2N^{76}}{A^2H} \right)^{2K} \ll 1 \quad \text{for sufficiently large } N,
\]
we obtain
\[
\int_A^B \prod_{1 \leq 2m < P} \left( 1 + W_{2m}(x) + \tau^2 H \|f\| \tilde{c}_A \right) \, dx 
\ll \left( \prod_{1 \leq 2m < P} \left( 1 + \tau^2 H \|f\| \tilde{c}_A \right) \right)^2 \left( 1 + \sum_{K=1}^{[P/2]} \left( \frac{3H^2N^{76}}{A^2H} \right)^{2K} \right)
\ll \prod_{1 \leq 2m < P} \exp \left( \tau^2 H \|f\| \tilde{c}_A \right)
\ll \exp \left( \tau^2 \tilde{c}_A \|f\| \right).
\]
This proves Lemma 7.

**Proof of Lemma 8** By Minkowski’s inequality
\[
\left( \int_A^B \left( \sum_{n=1}^{N} r(\xi x_{wn}) \right)^2 \, dx \right)^{1/2} \leq \sum_{n=1}^{N} \left( \int_A^B (r(\xi x_{wn}))^2 \, dx \right)^{1/2}.
\]
(36)
For $w_n$ fixed, using (13), Lemma 3 and the inequality of arithmetic and geometric means we have
\[
\int_A^B (r(\xi x_{wn}))^2 \, dx
= \int_A^B \left( \sum_{j,k=N^{38}+1}^{\infty} \frac{a_j a_k}{2} \left( \cos(2\pi(j + k)\xi x_{wn}) + \cos(2\pi(j - k)\xi x_{wn}) \right) \right) \, dx
\leq \sum_{j,k=N^{38}+1}^{\infty} \frac{1}{2jk} \left| \int_A^B \left( \cos(2\pi(j + k)\xi x_{wn}) + \cos(2\pi(j - k)\xi x_{wn}) \right) \, dx \right|
\leq 2 \sum_{j=N^{38}+1}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{2j^2(j + \ell)} \left( \int_A^B \left( \cos(2\pi(2j + \ell)\xi x_{wn}) + \cos(2\pi\ell\xi x_{wn}) \right) \, dx \right)
\leq 2 \sum_{j=N^{38}+1}^{\infty} \frac{1}{2j^2 + 2j(j + \ell) + \ell(j + \ell)\xi w_n A_{wn}^{-1}}
\leq 2 \sum_{j=N^{38}+1}^{\infty} \frac{1}{j(j + \ell)\xi w_n A_{wn}^{-1}}
\leq \frac{2}{N^{38}} + \frac{2}{\xi} \sum_{j=N^{38}+1}^{\infty} \frac{1}{j(j + \ell)\ell}.
\]
By Lemma 8, Markov’s inequality and (12) we have

\[ \leq \frac{2}{N^{38}} + \frac{1}{\xi} \sum_{j=N^{39}+1}^{\infty} \sum_{\ell=1}^{\infty} \frac{1}{j^{3/2} \ell^{3/2}} \]

\[ \ll N^{-17}. \]

Together with (36) this proves Lemma 8.

**Proof of Lemma 6.** We use Corollary 2 for

\[ \tau = \tilde{c}_A^{-1/2} N^{-1/2} (\log \log N)^{1/2} / f \]

(her \( \tilde{c}_A \) is the constant from the statement of Corollary 2). Then by (12) and (14) we have

\[ \tau \geq \tilde{c}_A^{-1/2} N^{-1/2} (\log \log N)^{1/2} \quad \text{and} \quad \tau \leq 2 \tilde{c}_A^{-1/2} N^{-5/16} (\log \log N)^{1/2}, \]

and condition (16) is satisfied for sufficiently large \( N \). Consequently,

\[ \int_A^B \exp \left( \tilde{c}_A^{-1/2} N^{-1/2} (\log \log N)^{1/2} / f \right)^{-3/4} \sum_{n=M+1}^{M+N} g(\xi x^n) \, dx \]

\[ \ll e^{\|f\|^{-1/2} \log \log N}, \]

which implies that for arbitrary \( \delta > 0 \) we have

\[ \mathbb{P} \left( \left| \sum_{n=M+1}^{M+N} g(\xi x^n) \right| \geq (\delta/2) \sqrt{\tilde{c}_A} \|f\|^{1/4} \sqrt{N \log \log N} \right) \ll e^{(1-\delta/2)\|f\|^{-1/2} \log \log N}. \]  

(37)

By Lemma 8, Markov’s inequality and (12) we have

\[ \mathbb{P} \left( \left| \sum_{n=M+1}^{M+N} r(\xi x^n) \right| \geq (\delta/2) \sqrt{\tilde{c}_A} \|f\|^{1/4} \sqrt{N \log \log N} \right) \ll \|f\|^{-1/2} \delta^{-2} N^{-17} \leq \delta^{-2} N^{-16}. \]

(38)

Combining (37) and (38) we finally obtain

\[ \mathbb{P} \left( \left| \sum_{n=M+1}^{M+N} f(\xi x^n) \right| \geq \delta \sqrt{\tilde{c}_A} \|f\|^{1/4} \sqrt{N \log \log N} \right) \ll e^{(1-\delta/2)\|f\|^{-1/2} \log \log N} + \delta^{-2} N^{-16}, \]

which proves Lemma 6.

**4 Maximal inequality**

**Lemma 9** For any sufficiently large \( m \) (depending only on \( A \)) we have the following: Let \( f \) be a function satisfying (8) and

\[ \|f\| \geq \frac{2^{-m/4}}{\sqrt{2}}. \]  

(39)

Then for the number \( c_A \) from Lemma 7 and any \( \gamma \geq 1 \) we have

\[ \mathbb{P} \left( x \in [A, B] : \max_{1 \leq M \leq 2^m} \left| \sum_{n=1}^{M} f(\xi x^n) \right| \geq 84 \gamma c_A \|f\|^{1/4} \sqrt{2m \log \log 2^m} \right) \]

\[ \ll \exp \left( -\gamma \|f\|^{-1/2} \log \log 2^m \right) + 2^{-3m}. \]
Corollary 3 For any sufficiently large \( N \) (depending only on \( A \)) we have the following: Let \( f \) be a function satisfying (3). Assume additionally that (12) holds. Then for the number \( c_A \) from Lemma 6 and any \( \gamma \geq 1 \) we have

\[
\mathbb{P} \left( x \in [A, B] : \max_{1 \leq M \leq N} \left| \sum_{n=1}^{M} f(\xi x^n) \right| \geq 119 \gamma c_A \|f\|^{1/4} \sqrt{N \log \log N} \right) \\
\ll \exp \left( -\gamma \|f\|^{-1/2} \log \log N \right) + N^{-3}.
\]

Proof of Lemma 7 For the proof of Lemma 6 we use a classical dyadic decomposition method, which is frequently used for proving maximal inequalities in probability theory and probabilistic number theory (see, for example, [11, 39]). By Lemma 6 for the complete sum \( \sum_{n=1}^{2^m} f(\xi x^n) \) we have

\[
\mathbb{P} \left( \sum_{n=1}^{2^m} f(\xi x^n) \geq 84 \gamma c_A \|f\|^{1/4} \sqrt{N \log \log N} \right) \\
\ll \exp \left( (1 - 84 \gamma /2) \|f\|^{-1/2} \log \log 2^m \right) + 2^{-16m} \\
\ll \exp \left( -41 \gamma \|f\|^{-1/2} \log \log 2^m \right) + 2^{-16m}. \tag{40}
\]

Any number \( M < 2^m \) can be written in dyadic representation

\[
M = \varepsilon_0 + 2\varepsilon_1 + 4\varepsilon_2 + \cdots + 2^{m-1}\varepsilon_{m-1} \quad \text{for digits } \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m-1}.
\]

Writing \( S \) for the set of those numbers \( M, \ 1 \leq M \leq 2^m - 1 \), for which \( \varepsilon_0 = 0, \varepsilon_1 = 0, \ldots, \varepsilon_{m/4} = 0 \), then by (39), for sufficiently large \( m \),

\[
\mathbb{P} \left( \max_{1 \leq M \leq 2^m-1} \left| \sum_{n=1}^{M} f(\xi x^n) \right| \geq 84 \gamma c_A \|f\|^{1/4} \sqrt{2^m \log \log 2^m} \right) \\
\leq \mathbb{P} \left( \max_{M \in S} \left| \sum_{n=1}^{M} f(\xi x^n) \right| \geq 83 \gamma c_A \|f\|^{1/4} \sqrt{2^m \log \log 2^m} \right). \tag{41}
\]

For a set \( U(K) \) containing \( 2^K \) consecutive elements of \( \{1, \ldots, 2^m - 1\} \) for some \( K, m/4 \leq K \leq m - 1 \), we have, using the fact that

\[
\log \log 2^m \geq \frac{\log \log 2^m}{2} \quad \text{and} \quad 1 - 5(m - K) \geq 4(m - K)
\]

for sufficiently large \( m \), and using Lemma 6 for \( \delta = 10(m - K) \gamma \),

\[
\mathbb{P} \left( \sum_{n \in U(K)} f(\xi x^n) \right) \geq 10(m - K)2^{(K-m)/2} \gamma c_A \|f\|^{1/4} \sqrt{2^m \log \log 2^m} \right) \\
\leq \mathbb{P} \left( \sum_{n \in U(K)} f(\xi x^n) \right) \geq 10(m - K) \gamma c_A \|f\|^{1/4} \sqrt{2^K \log \log 2^K} \\
\ll \exp \left( (1 - 10(m - K) \gamma /2) \|f\|^{-1/2} \log \log 2^K \right) + 2^{-16K}
\]
provided \( m \) is sufficiently large. To be able to represent every set \( \{1, \ldots, M\} \) for \( M \in \mathcal{S} \) as a disjoint union of at most one set of cardinality \( 2^K \) for each \( K \in \{K : m/4 \leq K \leq m - 1\} \), we need in total \( 2^{m-K} \) sets of cardinality \( 2^K \), for each \( m/4 \leq K \leq m - 1 \). Thus, using

\[
\sum_{k=1}^{\infty} 10k2^{-k/2} \leq 83,
\]

we have

\[
\left| \sum_{n=1}^{M} f(\xi x^n) \right| < \sum_{K=m/4}^{m-1} 10(m - K)2^{(K-m)/2} \gamma c_A \|f\|^{1/4} \sqrt{2^m \log \log 2^m}
\]

\[
< 83 \gamma c_A \|f\|^{1/4} \sqrt{2^m \log \log 2^m}
\]

for all \( M \in \mathcal{S} \), except for a set \( x \in [A, B] \) of measure at most

\[
\ll \sum_{K=m/4}^{m-1} 2^{m-K} \left( 2^{-2(m-K)} \exp \left( -\gamma \|f\|^{-1/2} \log \log 2^m \right) + 2^{-4m} \right)
\]

\[
\ll \exp \left( -\gamma \|f\|^{-1/2} \log \log 2^m \right) + 2^{-3m},
\]

provided \( m \) is sufficiently large. Together with (40) and (41) this proves Lemma 9.

**Proof of Corollary 3** Write \( \hat{N} \) for the smallest number \( \geq N \), which is a power of 2. Then \( 2N > \hat{N} \geq N \). Using Lemma 9 we have

\[
\max_{1 \leq M \leq \hat{N}} \left| \sum_{n=1}^{M} f(\xi x^n) \right| \leq \max_{1 \leq M \leq \hat{N}} \frac{1}{\sqrt{N \log \log \hat{N}}} \left\lfloor \sum_{n=1}^{\hat{N}} f(\xi x^n) \right\rfloor
\]

\[
< 84 \gamma c_A \|f\|^{1/4} \sqrt{N \log \log \hat{N}}
\]

\[
\leq 119 \gamma c_A \|f\|^{1/4} \sqrt{N \log \log N}
\]

for sufficiently large \( N \), except for a set \( x \in [A, B] \) of probability at most

\[
\exp \left( -\gamma \|f\|^{-1/2} \log \log \hat{N} \right) + \left( \hat{N} \right)^{-3} \ll \exp \left( -\gamma \|f\|^{-1/2} \log \log N \right) + N^{-3}.
\]

**5 The law of the iterated logarithm for functions having small \( L^2 \)-norm**

**Lemma 10** Let \( f \) be a function of bounded variation satisfying (8). For some \( d \geq 1 \), let \( p \) denote the \( d \)-th partial sum of the Fourier series of \( f \), and let \( r \) denote the remainder term \( f - p \). Then for the constant \( c_A \) from Lemma 9 we have

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \left| \sum_{n=1}^{N} r(x^n) \right| \leq 238c_Ad^{-1/8} \quad \text{a.e.}
\]
Proof of Lemma 10: Using Lemma 1 we have
\[ \| r \| \leq \sum_{j=d+1}^{\infty} \frac{2}{j^2} \leq \frac{2}{\sqrt{d}}, \]
and, by (8), we also have \( \| r \| \leq 1. \) Setting \( D_m := \left( x \in [A, B] : \max_{1 \leq M \leq 2^m} \left| \sum_{n=1}^{M} r(\xi_x s_n) \right| > 84(2d^{-1/8})c_A \sqrt{2^m \log \log 2^m} \right) \)
and using Lemma 9 for \( \gamma = 2d^{-1/8}\|r\|^{-1/4} \geq 1.6 \) we obtain
\[ \mathbb{P}(D_m) \ll \exp \left( -1.6 \log \log 2^m \right) + 2^{-3m} \ll \frac{1}{m^{1.6}}. \]
Thus
\[ \sum_{n=1}^{\infty} \mathbb{P}(D_m) < \infty, \]
and by the Borel-Cantelli lemma with probability 1 only finitely many events \( D_m \) occur. Thus since \( 238 > \sqrt{2} \cdot 2 \cdot 84 \) there are, also with probability 1, only finitely many \( N \) for which
\[ \left| \sum_{n=1}^{N} r(\xi_x s_n) \right| > 238c_A d^{-1/8} \sqrt{N \log \log N}, \]
which proves the lemma.

6 The law of the iterated logarithm for the discrepancy for small intervals

In the present section we will prove a bounded law of the iterated logarithm for a modified version of the discrepancy, which only takes into account “small” intervals. More precisely, for an integer \( R \geq 1 \) and a sequence \( (z_1, \ldots, z_N) \in [0, 1)^N \) set
\[ D_N^{(\leq 2^{-R})}(z_1, \ldots, z_N) := \sup_{a \in \mathbb{Z}, 0 \leq a < 2^R} \left| \frac{1}{N} \sum_{n=1}^{N} I_{[a2^{-R}, a2^{-R}+b]}(z_n) \right| \]

(42)
(the functions \( I \) were defined in (9)). In other words, the discrepancy \( D_N^{(\leq 2^{-R})} \) considers only “small” intervals (those of length \( \leq 2^{-R} \)), which have their left corner in a point of the form \( a2^{-R} \) for some \( a \in \{0, \ldots, 2^R - 1\} \). Furthermore, we set
\[ D_N^{(\geq 2^{-R})}(z_1, \ldots, z_N) := \max_{a, b \in \mathbb{Z}, 0 \leq a < b \leq 2^R} \left| \frac{1}{N} \sum_{n=1}^{N} I_{[a2^{-R}, b2^{-R}]}(z_n) \right| \]
and
\[ D_N^{(\geq 2^{-R})}(z_1, \ldots, z_N) := \max_{a \in \mathbb{Z}, 0 < a \leq 2^R} \left| \frac{1}{N} \sum_{n=1}^{N} I_{[0, a2^{-R}]}(z_n) \right|. \]
It is easily seen that always
\[ D_{N}(\geq 2^{-R}) \leq D_{N}^{*} \leq D_{N}(\geq 2^{-R}) + 3D_{N}(\leq 2^{-R}). \] (43)

The idea to split the discrepancies \( D_{N} \) and \( D_{N}^{*} \) in this way to obtain precise metric discrepancy results is due to Fukuyama [35].

**Lemma 11** For any positive integer \( R \) we have for almost all \( x \in [A, B] \)
\[
\limsup_{N \to \infty} \frac{\sqrt{N}D_{N}(\geq 2^{-R})(\{\xi x^{n}\})}{\log \log N} \leq 10^{7}c_{A}R^{-1},
\]
where \( c_{A} \) is the constant from Lemma 6.

We use a dyadic decomposition of the unit interval, which was also used in [56]. For simplicity we will only consider the case \( a = 0 \), i.e.
\[
\limsup_{N \to \infty} \sup_{0 \leq b \leq 2^{-R}} \frac{\sum_{n=1}^{N} I_{[0,b)}(\xi x^{n})}{\sqrt{N \log \log N}} \leq 10^{7}c_{A}R^{-1}
\]
for almost all \( x \in [A, B] \). The proof for the other possible values of \( a \), that is for \( 1 \leq a \leq 2^{R} \), can be given in exactly the same way. This means that the exceptional set in Lemma 11 is a finite union of sets of measure zero, and consequently also has zero measure.

For \( N \geq 1 \) we set
\[
E_{N} = \left( \sup_{0 \leq b \leq 2^{-R}} \left| \sum_{n=1}^{N} I_{[0,b)}(\xi x^{n}) \right| \geq 10^{7}c_{A}R^{-1} \sqrt{N \log \log N} \right),
\]
and for \( m \geq 1 \) we set
\[
F_{m} = \left( \max_{1 \leq N \leq 2^{2m}} \sup_{b \in \mathbb{Z}, 1 \leq b \leq 2^{-R}} \left| \sum_{n=1}^{N} I_{[0,b2^{-m})}(\xi x^{n}) \right| \geq 10^{6}c_{A}R^{-1} \sqrt{2^{2m} \log \log 2^{2m}} \right).
\]
Every interval \([0, b), \ 0 \leq b \leq 2^{-R}\), can be written as the union of an interval of the form \([0, j2^{-m})\) for some appropriate \( j \in \mathbb{Z}, \ 1 \leq j < 2^{m-R} \), and an interval \( B \) of length at most \( 2^{-m} \). For any \( x \) from the complement of \( F_{m} \) we have for any \( N, \ 1 \leq N \leq 2^{2m} \), and any such interval \( B \),
\[
\left| \sum_{n=1}^{N} I_{B}(\xi x^{n}) \right| < 2 \cdot 10^{6}c_{A}R^{-1} \sqrt{2^{2m} \log \log 2^{2m}} + N2^{-m}
\]
\[
< (2 \cdot 10^{6} + 1) c_{A}R^{-1} \sqrt{2^{2m} \log \log 2^{2m}},
\]
provided \( m \) is sufficiently large (depending on \( A \) and \( R \)). Consequently for any sufficiently large \( N \) satisfying \( 2^{2m-2} \leq N \leq 2^{2m} \) for some \( m \) we have, for any \( x \) from the complement of \( F_{m} \),
\[
\sup_{0 \leq b \leq 2^{-R}} \left| \sum_{n=1}^{N} I_{[0,b)}(\xi x^{n}) \right| < (3 \cdot 10^{6} + 1)c_{A}R^{-1} \sqrt{2^{2m} \log \log 2^{2m}}
\]
\]
\[ \leq 10^7 c_A R^{-1} \sqrt{N \log \log N}. \]

Thus for sufficiently large \( m \) we have
\[ \bigcup_{2^{m-2} \leq N \leq 2^m} E_N \subset F_m, \]
and hence
\[ \sum_{m=1}^{\infty} F_m < \infty \quad \text{implies} \quad \sum_{N=1}^{\infty} E_N < \infty. \quad (44) \]

Writing \( b \) in binary expansion, it is easily seen that for any possible number \( 1 \leq b < 2^{m-R} \) the interval \([0, b2^{-m})\) can be written as the disjoint union of at most one interval of length \( 2^{-R-1} \), at most one interval of length \( 2^{-R-2} \), etc., and at most one interval of length \( 2^{-m} \). Furthermore, to be able to represent all possible intervals \([0, b2^{-m})\) we need exactly \( 2^{k-R} \) intervals of length \( 2^{-k} \), for any \( k \in \{R+1, \ldots, m\} \). \quad (45)

Let \( f \) be the indicator function of an interval of length \( 2^{-k} \) for some \( k \leq m \). Then
\[ \frac{2^{-k/2}}{\sqrt{2}} \leq \|f\| \leq 2^{-k/2} \]
and
\[ \|f\| \geq \frac{2^{-2m/4}}{\sqrt{2}}, \]
Consequently, using Lemma 9 with \( \gamma = 2k \) we obtain
\[ P \left( \max_{1 \leq N \leq 2^{2m}} \left| \sum_{n=1}^{N} f(\xi x^{an}) \right| \geq 84(2k)2^{-k/8}c_A \sqrt{2^{2m} \log \log 2^{2m}} \right) \]
\[ \ll \exp \left(-2k \log \log 2^{2m}\right) + 2^{-6m} \]
\[ \ll \left( \frac{1}{m} \right)^{2k} + 2^{-6m}. \]

It can be shown that
\[ \sum_{k=R+1}^{m} 84(2k)2^{-k/8} \leq 250000R^{-1}, \quad \text{for any } R \geq 1. \]

Thus, using (45), we see that for any \( b, 1 \leq b < 2^{m-R} \), we have
\[ \max_{1 \leq N \leq 2^{2m}} \left| \sum_{n=1}^{N} I_{[0,b2^{-m})}(\xi x^{an}) \right| \leq \sum_{k=R+1}^{m} 84(2k)2^{-k/8}c_A \sqrt{2^{2m} \log \log 2^{2m}} \]
\[ \leq 250000c_A R^{-1} \sqrt{2^{2m} \log \log 2^{2m}}, \]
except for a set of measure at most
\[ \ll \sum_{k=R+1}^{m} 2^{k-R} \left( \left( \frac{1}{m} \right)^{2k} + 2^{-6m} \right) \]
\[ \ll \sum_{k=R+1}^{m} 2^{k-R} \left( \left( \frac{1}{m} \right)^{2k} + 2^{-6m} \right) \]
Furthermore we also have for the full interval $[0, 2^{-R})$, again by Lemma 9
\[
\mathbb{P} \left( \max_{1 \leq N \leq 2^m} \left| \sum_{n=1}^{N} I_{[0,2^{-R})}(\xi x^n) \right| \geq 250,000 c_A R^{-1} \sqrt{2^{2m} \log \log 2^{2m}} \right) \ll m^{-2}.
\]
Thus
\[
\mathbb{P}(F_m) \ll m^{-2},
\]
which by (44) and the Borel-Cantelli lemma implies that with probability 1 only finitely many events $E_N$ occur. This proves Lemma 11.

7 Martingale approximation

Throughout this section we assume that the number $d$ is fixed; also throughout this section the constants $c$ and the implied constants in “$\ll$” and “$\gg$” may depend on $d$. We will exclude the trivial case $\|p\| = 0$, which is equivalent to $p \equiv 0$.

Lemma 12 Let $p(x)$ be a trigonometric polynomial. Then for all numbers $N$ which can be written in the form
\[
N = \sum_{i=1}^{M} (i^4 + i)
\]
for some $M$ (46) we have
\[
\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( x \in [A, B] : \frac{\sum_{n=1}^{N} p(\xi x^n)}{\|p\| \sqrt{N}} < t \right) - \Phi(t) \right| \ll \frac{\log N}{N^{1/25}},
\]
where $\Phi$ denotes the standard normal distribution function.

Lemma 13 Let $p(x)$ be a trigonometric polynomial, and let $(N_k)_{k \geq 1}$ be the sequence of numbers which can be written in the form (46). Then
\[
\limsup_{k \to \infty} \frac{\left| \sum_{n=1}^{N_k} p(\xi x^n) \right|}{\sqrt{N_k \log \log N_k}} = \sqrt{2\|p\|}, \quad \text{a.e.}
\]

The crucial ingredient in the proofs of Lemma 12 and Lemma 13, which will be given simultaneously, are the following results of Strassen and of Heyde and Brown [41], which are a consequence of a martingale version of the Skorokhod representation theorem due to Strassen [61]. For Lemma 14 (which is used to prove Theorem 1) we use the formulation from [5, Lemma 2.1], for Lemma 15 (which is used to prove Theorem 2) we use the formulation from [18, Theorem A].

Lemma 14 Let $Y_1, Y_2, \ldots$ be a martingale difference sequence with finite fourth moments, let $V_M = \sum_{i=1}^{M} \mathbb{E}(Y_i^2 | Y_1, \ldots, Y_{i-1})$ and assume that $V_1 > 0$ and $V_M \to \infty$. Let $(b_M)_{M \geq 1}$ be any sequence of positive numbers such that
\[
\lim_{M \to \infty} \frac{V_M}{b_M} = 1 \quad \text{a.s.,}
\]
\]

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\[ \sum_{M=1}^{\infty} \frac{(\log b_M)^{10}}{b_M^2} \mathbb{E} Y_M^4 < \infty. \]

Then
\[ \limsup_{M \to \infty} \frac{\sum_{i=1}^{M} Y_i}{\sqrt{b_M \log \log b_M}} = \sqrt{2} \quad \text{a.s.} \]

**Lemma 15** Let \( Y_1, Y_2, \ldots \) be a martingale difference sequence with finite fourth moments, let \( V_M = \sum_{i=1}^{M} \mathbb{E}(Y_i^2|Y_1, \ldots, Y_{i-1}) \) and let \((b_M)_{M \geq 1}\) be any sequence of positive numbers. Then
\[ \sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \frac{Y_1 + \cdots + Y_M}{\sqrt{b_M}} < t \right) - \Phi(t) \right| \leq K \left( \sum_{i=1}^{M} \mathbb{E} Y_i^4 + \mathbb{E} \left( (V_M - b_M)^2 \right) \right)^{1/5}, \]

where \( K \) is an absolute constant.

**Proof of Lemma 12.** We use an argument based on approximation by martingale differences, which was already used in [5, 6]. This method was originally developed by Berkes [15, 16, 17] and Philipp and Stout [57]. Since in our case the functions which we want to approximate are not periodic, we have to construct an increasing sequence of space-inhomogeneous discrete sigma-algebras for the approximation.

For simplicity of writing we assume that \( p \) is an even function; the proof in the general case is exactly the same. Then we can write \( p \) in the form
\[ p(x) = \sum_{j=1}^{d} a_j \cos 2\pi jx. \]

For simplicity of writing we will also assume that \( ||p|| \leq 1 \) and \( |a_j| \leq 1, \ j \geq 1 \).

We subdivide the set of positive integers consecutively into blocks \( \Delta_i \) ("large blocks") and \( \Delta'_i \) ("small blocks"), in such a way that
- the block \( \Delta_i \) contains \( i^4 \) elements, for \( i \geq 1 \).
- the block \( \Delta'_i \) contains \( i \) elements, for \( i \geq 1 \).
- elements of \( \Delta_i \) are smaller than elements of \( \Delta'_i \), for \( i \geq 1 \).
- elements of \( \Delta'_i \) are smaller than elements of \( \Delta_{i+1} \), for \( i \geq 1 \).
- \( \bigcup_{i \geq 1} (\Delta_i \cup \Delta'_i) = \mathbb{N} \).

Assume that \( N \) is of the form \( \lfloor M \rfloor \) for some \( M \). Then by construction
\[ \{1, \ldots, N\} = \bigcup_{i=1}^{M} (\Delta_i \cup \Delta'_i). \]

We write \( \min(i) \) and \( \max(i) \) for the smallest resp. largest element of \( \Delta_i \), and set
\[ m(i) = \left\lfloor \log_2 \left( i^6 w_{\max(i)} A_{w_{\max(i)}} \right) \right\rfloor, \quad 1 \leq i \leq M. \]
Note that
\[ w_{\min(i)} - w_{\max(i-1)} \geq \min(i) - \max(i-1) \geq i - 1, \quad 1 \leq i \leq M. \quad (47) \]

We write \( G_i \) for the set of intervals of the form
\[ H_j^i = \left[ A + j2^{-m(i)}, A + (j + 1)2^{-m(i)} \right], \quad j \in \{0, \ldots, 2^m(i) - 1\}, \quad 1 \leq i \leq M. \]

In other words, \( G_i \) is a partition of \([A, B]\) into \( 2^m(i) \) subintervals of equal length. Write \( x_j^i \) for the smallest number in \( H_j^i \), that is
\[ x_j^i = A + j2^{-m(i)}, \]
and split every interval \( H_j^i \) into
\[ 2^\left[ \log_2 \left( \frac{w_{\max(i)}(x_j^i/A)}{x_j^i} \right) \right] \]
pieces of equal length. Let \( \mathcal{F}_i \) denote the sigma-algebra generated by all these sets. Then \( (\mathcal{F}_i)_{1 \leq i \leq M} \) is an increasing family of sigma-algebras. For any \( i \), any number \( x \in [A, B] \) is contained in an atom of \( \mathcal{F}_i \) which has length between
\[ 2^{-m(i)} \left( \frac{A}{x} \right)^{w_{\max(i)}}, \quad 2^{-m(i)} \left( \frac{A}{x - 2^{-m(i)}} \right)^{w_{\max(i)}}. \quad (48) \]

For \( 1 \leq i \leq M \) we set
\[ T_i = \sum_{n \in \Delta_i} p(\xi x^n), \quad T'_i = \sum_{n \in \Delta'_i} p(\xi x^n) \]
and
\[ Y_i = \mathbb{E} \left( T_i \mid \mathcal{F}_i \right) - \mathbb{E} \left( T_i \mid \mathcal{F}_{i-1} \right). \]

Then
\[ \frac{1}{\sqrt{N}} \sum_{n=1}^N p(\xi x^n) = \frac{1}{N} \sum_{i=1}^M (T_i + T'_i). \]

Furthermore, \( Y_i \) is a discrete function, which by construction is constant on the atoms of \( \mathcal{F}_i \), and
\[ \mathbb{E} \left( Y_i \mid \mathcal{F}_{i-1} \right) = 0. \]

In other words, \( (Y_i)_{1 \leq i \leq M} \) is a martingale difference. Let \([\alpha, \beta]\) be any atom of \( \mathcal{F}_{i-1} \). Then by (48)
\[ \beta - \alpha \geq 2^{-m(i-1)} \left( \frac{A}{\alpha} \right)^{w_{\max(i-1)}} \geq \frac{1}{2(i - 1)\theta w_{\max(i-1)} A^{w_{\max(i-1)}}} \left( \frac{A}{\alpha} \right)^{w_{\max(i-1)}} \]
\[ \gg \frac{1}{\theta w_{\max(i-1)} \alpha^{w_{\max(i-1)}}}. \quad (49) \]
Thus by Lemma 3 and (47)

\[
\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta T_i(x) \, dx \right| \leq \frac{1}{\beta - \alpha} \sum_{n \in \Delta} \sum_{j=1}^d |a_j| \left| \int_\alpha^\beta \cos 2\pi j \xi x_w \, dx \right|
\]

\[
\ll \frac{\nu^{m(i)} \nu^{m(i-1)}}{\nu_{\min(i)}^{m(i)}}
\]

\[
\ll i^6 A^{-i},
\]

and consequently

\[
\mathbb{E}(T_i | \mathcal{F}_{i-1}) \ll i^6 A^{-i}.
\] (50)

Now, let \([\alpha, \beta]\) denote an atom of \(\mathcal{F}_i\). By (48) we have

\[
\beta - \alpha \leq 2^{m(i)} \left( \frac{A}{\alpha - 2^{-m(i)}} \right)^{\nu_{\max(i)}} \ll \frac{1}{i^6 \nu_{\max(i)} \left( \alpha - 2^{-m(i)} \right)^{\nu_{\max(i)}}}.
\] (51)

The derivative of \(p(\xi x_w)\) on \([\alpha, \beta]\) is bounded by

\[
|p'(\xi x_w)| \leq \sum_{j=1}^d |2\pi j \xi w_n \beta_{w_n-1}| \ll \nu_{\max(i)} (\alpha + 2^{-m(i)})^{\nu_{\max(i)}}.
\]

By the definition of \(m(i)\) it is easily seen that

\[
\left( \frac{\alpha + 2^{-m(i)}}{\alpha - 2^{-m(i)}} \right)^{\nu_{\max(i)}} \ll 1.
\]

Thus by (51) for any \(n \in \Delta_i\) the fluctuation of \(p(\xi x_w)\) on \([\alpha, \beta]\) is bounded by \(\ll i^{-6}\). Therefore, together with (50), we obtain

\[
|T_i - Y_i| \ll |\Delta_i|i^{-6} \ll i^{-2}
\] (52)

(here, and in the sequel, we write \(| \cdot |\) for the number of elements of a set).

Next we have to calculate the conditional variances \(\mathbb{E}(Y_i^2 | \mathcal{F}_{i-1})\). By (52)

\[
|\mathbb{E}(Y_i^2 | \mathcal{F}_{i-1}) - \mathbb{E}(T_i^2 | \mathcal{F}_{i-1})| \leq \mathbb{E} \left( \left| (Y_i + T_i)(Y_i - T_i) \right| \mathcal{F}_{i-1} \right)
\]

\[
\ll \|Y_i + T_i\|_{\infty} \|Y_i - T_i\|_{\infty}
\]

\[
\ll |\Delta_i|i^{-2} \ll i^2,
\] (53)

and thus we can reduce the problem to estimating \(\mathbb{E}(T_i^2 | \mathcal{F}_{i-1})\). Using (24), we have

\[
T_i^2 = \left( \sum_{n \in \Delta, j=1}^d a_j \cos(2\pi j \xi x_w) \right)^2
\]

\[
= \frac{1}{2} \sum_{n_1, n_2 \in \Delta_i} \sum_{j_1, j_2=1}^d a_{j_1} a_{j_2} (\cos(2\pi j_1 \xi x_{w_1} + j_2 \xi x_{w_2}) + \cos(2\pi j_1 \xi x_{w_1} - j_2 \xi x_{w_2})).
\]
In the above sum, for \( n_1 = n_2 \) and \( j_1 = j_2 \) we have \( \cos(2\pi\xi(j_1 x^{w_{n_1}} - j_2 x^{w_{n_2}})) = 1 \), and thus

\[
\frac{1}{2} \sum_{n_1, n_2 \in \Delta_i} \sum_{j_1, j_2 = 1}^{d} a_{j_1}a_{j_2} \cos(2\pi\xi(j_1 x^{w_{n_1}} - j_2 x^{w_{n_2}})) = |\Delta_i| \cdot \|p\|^2.
\]

Consequently

\[
|E(T_i^2|F_{i-1}) - |\Delta_i| \cdot \|p\|^2| \leq \sum_{n_1, n_2 \in \Delta_i} \sum_{j_1, j_2 = 1}^{d} \left| E\left( \cos(2\pi\xi(j_1 x^{w_{n_1}} + j_2 x^{w_{n_2}})) \right|F_{i-1} \right| + \sum_{n_1, n_2 \in \Delta_i} \sum_{j_1, j_2 = 1}^{d} \left| E\left( \cos(2\pi\xi(j_1 x^{w_{n_1}} - j_2 x^{w_{n_2}})) \right|F_{i-1} \right|.
\]

(54)

(55)

Let \((\alpha, \beta)\) be any atom of \(F_{i-1}\). Using (47), (49) and Lemma 4, we see that for any function from (54)

\[
\left| E\left( \cos(2\pi\xi(j_1 x^{w_{n_1}} + j_2 x^{w_{n_2}})) \right|F_{i-1} \right| \leq 2^{i6}w_{\max(i-1)}^{\alpha}w_{\max(i-1)}^{\beta} \xi^{\min(i)}\alpha^{\min(i)} \lesssim i^{6} A^{-i}.
\]

(56)

Thus for any function from (54) we have

\[
\left| E\left( \cos(2\pi\xi(j_1 x^{w_{n_1}} - j_2 x^{w_{n_2}})) \right|F_{i-1} \right| \lesssim i^{6} A^{-i},
\]

and consequently the whole double sum in (54) is bounded by

\[
|\Delta_i|^2 d^2 i^{6} A^{-i}
\]

(56)

Now consider any function of the form \(\cos(2\pi\xi(j_1 x^{w_{n_1}} - j_2 x^{w_{n_2}}))\) from \(\Delta_i\). Then, using Lemma 5 with \(\eta = i^{-12}\) we know that there exist three intervals \(I_1, I_2\) and \(I_3\) of total measure at least \(1 - 2B\eta = 1 - 2Bi^{-12}\), such that for any atom \((\alpha, \beta)\) of \(F_{i-1}\) which is completely contained in one of the intervals \(I_1, I_2\) or \(I_3\) we have

\[
\frac{1}{(\beta - \alpha)} \int_{\alpha}^{\beta} \cos(2\pi\xi(j_1 x^{w_{n_1}} - j_2 x^{w_{n_2}})) \, dx \leq \frac{2i^{6}w_{\max(i)}^{\beta}w_{\max(i)}^{\alpha}}{\xi^{\min(i)}\alpha^{\min(i)}} \lesssim i^{18} A^{-i}.
\]

(57)
Since by (48) the length of any atom of $F_{i-1}$ is at most
\[ 2^{-m(i)} \leq A^{-i}, \]
the total measure of those atoms of $F_{i-1}$ which are not completely contained in one of the intervals $I_1, I_2, I_3$ is at most
\[ \mathbb{P}([A, B]) - \mathbb{P}(I_1 \cup I_2 \cup I_3) + 6A^{-i} \ll i^{-12}. \]
Combining this with (57) we obtain for any function from (55)
\[ |E|^{(\cos(2\pi(j_1x^{v_1} - j_2x^{v_2}))|F_{i-1})|} \ll i^{18}A^{-i} + i^{-12} \ll i^{-12}. \]
Thus the double sum in (55) is bounded by
\[ |\Delta_i|^2d^2i^{-12} \ll i^{-4}, \tag{58} \]
and together with (56) we conclude that
\[ |E(T^2_i|F_{i-1}) - |\Delta_i| \cdot ||p||^2| \ll i^{-4}. \tag{59} \]
Consequently, setting
\[ V_M = \sum_{i=1}^{M} E(Y^2_i|F_{i-1}) \quad \text{and} \quad b_M = \sum_{i=1}^{M} |\Delta_i| \cdot ||p||^2, \]
we obtain by (53) and (59) that
\[ |V_M - b_M| \leq \sum_{i=1}^{M} \left( |E(Y^2_i|F_{i-1}) - E(T^2_i|F_{i-1})| + |E(T^2_i|F_{i-1}) - |\Delta_i| \cdot ||p||^2| \right) \]
\[ \ll \sum_{i=1}^{M} (i^2 + i^{-4}) \]
\[ \ll M^3. \tag{60} \]
To prove Lemma 12, we observe that by Lemma 15 we have
\[ \sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \frac{Y_1 + \cdots + Y_M}{\sqrt{b_M}} < t \right) - \Phi(t) \right| \leq K \left( \frac{\sum_{i=1}^{M} EY^4_i + E \left( (V_M - b_M)^2 \right) }{b_M^2} \right)^{1/5}, \tag{61} \]
where $K$ is an absolute constant and $\Phi(t)$ denotes the standard normal distribution function.

Using Lemma 6 with $\delta = 2(\log |\Delta_i|)(\log \log |\Delta_i|)^{-1/2}$ we obtain
\[ \mathbb{P} \left( |T_i| \geq 2c_A\sqrt{|\Delta_i| \log |\Delta_i|} \right) \ll e^{-\log(i^4)} + \frac{1}{i^{64}} \ll i^{-64}. \]
Consequently, since $|T_i| \ll |\Delta_i|$, we have
\[ E(T^4_i) \ll \left( \sqrt{|\Delta_i| \log |\Delta_i|} \right)^4 + i^{-64}|\Delta_i|^4 \ll i^8(\log i)^4, \tag{62} \]
and, by (53), we obtain
\[ \mathbb{E}(Y_i^4) \ll i^8 (\log i)^4 \] (63)
and
\[ \sum_{i=1}^{M} \mathbb{E}(Y_i^4) \ll \sum_{i=1}^{M} i^8 (\log i)^4 \ll M^9 (\log M)^4. \] (64)

On the other hand,
\[ b_M \gg \sum_{i=1}^{M} |\Delta_i| \gg M^5. \] (65)

Combining (60), (61), (62) and (65) we get
\[ \sup_t \left| \mathbb{P} \left( \frac{\sum_{i=1}^{M} Y_i}{\sqrt{b_M}} < t \right) - \Phi(t) \right| \ll \left( \frac{M^9 (\log M)^4 + M^6}{M^{10}} \right)^{1/5} \ll \log M \frac{1}{M^{1/5}}. \]

By (52) we have
\[ \left| \sum_{i=1}^{M} (T_i - Y_i) \right| \ll \sum_{i=1}^{M} i^{-2} \ll 1. \] (66)

Furthermore, we have
\[ \sqrt{N} \|p\| - \sqrt{b_M} \leq \frac{N \|p\|^2 - b_M}{\sqrt{N} \|p\| + \sqrt{b_M}} \ll \frac{\sum_{i=1}^{M} i \|p\|^2}{M^{5/2}} \ll M^{-1/2}, \] (67)
and
\[ \left| T'_1 + \ldots + T'_M \right| \ll \sum_{i=1}^{M} |\Delta'_i| \ll M^2. \] (68)

Note that
\[ \sum_{i=1}^{M} Y_i = \sum_{n=1}^{N} p(\xi x^w n) + \sum_{i=1}^{M} (T_i - Y_i) + \sum_{i=1}^{M} T'_i, \]
and consequently
\[ \sum_{n=1}^{N} p(\xi x^w n) \ll \|p\| \sqrt{N} t \]
is equivalent to
\[ \sum_{i=1}^{M} Y_i \frac{1}{\sqrt{b_M}} \left( \frac{\sqrt{N}}{t \sqrt{b_M}} - \frac{\sum_{i=1}^{M} (T_i - Y_i)}{t \sqrt{b_M}} - \frac{\sum_{i=1}^{M} T'_i}{t \sqrt{b_M}} \right) \ll \hat{t} \]

Using (60), (67) and (68) we get
\[ \hat{t} = t \left( 1 + O \left( M^{-3} + M^{-1/2} t^{-1} \right) \right), \]
and consequently
\[ \left| \Phi(t) - \Phi(\hat{t}) \right| \ll M^{-1/2}. \]
Thus we finally get
\[
\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \frac{\sum_{n=1}^{N} p(\xi x^n)}{\|p\| \sqrt{N}} < t \right) - \Phi(t) \right| \ll \frac{\log M}{M^{1/5}} \ll \frac{\log N}{N^{1/25}},
\]
which proves Lemma 12.

To prove Lemma 13, note that by (60), (63) and (65) we have
\[
\frac{V_M}{b_M} \to 1
\]
and
\[
\sum_{M=1}^{\infty} \frac{(\log b_M)^{10}}{b_M^2} \mathbb{E} Y_M^4 \ll \sum_{M=1}^{\infty} \frac{(\log M)^{10}}{M^{10}} M^8 (\log M)^4 < \infty.
\]
Thus by Lemma 14 we have
\[
\limsup_{M \to \infty} \frac{\sum_{i=1}^{M} Y_i}{\sqrt{2b_M \log \log b_M}} = 1 \quad \text{a.s.}
\]
Since \( \sum_{i=1}^{M} T_i \ll M^2 \ll \sqrt{b_M} \) and
\[
\frac{b_M}{\|p\| \sum_{i=1}^{M} (|\Delta_i| + |\Delta'_i|)} \to 1 \quad \text{as } M \to \infty,
\]
we obtain, using (62), that
\[
\limsup_{M \to \infty} \frac{\sum_{i=1}^{M} (T_i + T'_i)}{\sqrt{\sum_{i=1}^{M} (|\Delta_i| + |\Delta'_i|) \log \log \left( \sum_{i=1}^{M} (|\Delta_i| + |\Delta'_i|) \right)}} = \sqrt{2\|p\|} \quad \text{a.e.}
\]
which is Lemma 13.

8 The law of the iterated logarithm for trigonometric polynomials

In the present section we will prove the exact law of the iterated logarithm for trigonometric polynomials.

Lemma 16 Let \( p \) be a trigonometric polynomial. Then for almost all \( x \in [A, B] \)
\[
\limsup_{N \to \infty} \frac{\left| \sum_{n=1}^{N} p(\xi x^n) \right|}{\sqrt{N \log \log N}} = \sqrt{2\|p\|}.
\]

Proof of Lemma 16: Choose \( \theta > 1 \) ("small"). For any \( k \geq 1 \), let \( N_k \) denote the smallest number of the form
\[
\sum_{i=1}^{M} (i^4 + i) \quad \text{for some } M
\]
(69)
which satisfies

\[ N_k \geq \theta^k. \]  

(70)

Then by Lemma 13 we have

\[ \limsup_{k \to \infty} \frac{\sum_{n=1}^{N_k} p(\xi x^{n})}{\sqrt{N_k \log \log N_k}} = \sqrt{2}\|p\| \quad \text{a.e.} \]  

(71)

Since the sequence of numbers of the form (69) grows polynomially, we have

\[ \frac{N_{k+1}}{N_k} \to \theta \quad \text{and} \quad \frac{N_k}{\theta^k} \to 1 \quad \text{as} \quad k \to \infty. \]  

(72)

Let

\[ V_k = \left( x \in [A, B] : \max_{N_k < M \leq N_{k+1}} \left| \sum_{n=N_k+1}^{M} p(\xi x^{n}) \right| \geq 238c_A \sqrt{(N_{k+1} - N_k) \log \log (N_{k+1} - N_k)} \right), \quad k \geq 1, \]

where \( c_A \) is the constant from Lemma 6. Using Corollary 3 with \( \gamma = 2\|r\|^{-1/4} \) we obtain

\[ P(V_k) \ll \exp(-2 \log \log (N_{k+1} - N_k)) \ll k^{-2}. \]

Thus

\[ \sum_{k=1}^{\infty} P(V_k) < \infty, \]

which implies that with probability one only finitely many events \( V_k \) occur. Hence, by (71) and (72), for almost all \( x \in [A, B] \) we have

\[ \sqrt{2}\|p\| \leq \limsup_{N \to \infty} \frac{\sum_{n=1}^{N} p(\xi x^{n})}{\sqrt{N \log \log N}} \leq \sqrt{2}\|p\| + 238c_A \sqrt{1 - \theta^{-1}}. \]

Since \( \theta > 1 \) can be chosen arbitrarily close to 1, this proves Lemma 16.

9 The law of the iterated logarithm for functions of bounded variation

**Lemma 17** For any function \( f \) satisfying (8) we have for almost all \( x \in [A, B] \) we have

\[ \limsup_{N \to \infty} \frac{\sum_{k=1}^{N} f(\xi x^{n})}{\sqrt{N \log \log N}} = \sqrt{2}\|f\|. \]

Choose \( d \geq 1 \), and split the Fourier series of \( f \) into a trigonometric polynomial of degree \( d \) (which will be denoted by \( p \)) and a remainder function \( r \). Then, since

\[ p(x) - r(x) \leq f(x) \leq p(x) + r(x), \]
by Lemma 10 and Lemma 16 for almost all \(x \in [A, B]\) we have

\[
\sqrt{2}\|p\| - 238c_A d^{-1/8} \leq \limsup_{N \to \infty} \frac{\left| \sum_{k=1}^{N} f(\xi x^n) \right|}{\sqrt{N \log \log N}} \leq \sqrt{2}\|p\| + 238c_A d^{-1/8}.
\]

By Parseval’s identity we have \(\|p\| \to \|f\|\) as \(d \to \infty\). Since \(d\) can be chosen arbitrarily, we obtain

\[
\limsup_{N \to \infty} \frac{\left| \sum_{k=1}^{N} f(\xi x^n) \right|}{\sqrt{N \log \log N}} = \sqrt{2}\|f\|
\]

for almost all \(x \in [A, B]\), which proves the lemma.

### 10 Proof of Theorem 1

First we note that

\[
\sup_{0 < a < b \leq 1} \|I_{[a,b]}\| = \|I_{[0,1/2]}\| = \frac{1}{2}.
\]

Thus by Lemma 17 for any \(R \geq 1\)

\[
\limsup_{N \to \infty} \frac{\sqrt{N} D_N(\xi x^n)}{\sqrt{\log \log N}} \geq \limsup_{N \to \infty} \frac{\sum_{k=1}^{N} I_{[0,1/2]}(\xi x^n)}{\sqrt{\log \log N}} = \frac{1}{\sqrt{2}}
\]

for almost all \(x \in [A, B]\). Together with Lemma 11 and (43) this proves

\[
\frac{1}{\sqrt{2}} \leq \limsup_{N \to \infty} \frac{\sqrt{N} D_N(\xi x^n)}{\sqrt{\log \log N}} \leq \limsup_{N \to \infty} \frac{\sqrt{N} D_N(\xi x^n)}{\sqrt{\log \log N}} \leq \frac{1}{\sqrt{2}} + 3 \cdot 10^7 c_A R^{-1}
\]

for almost all \(x \in [A, B]\). Since \(R\) can be chosen arbitrarily large, this proves Theorem 1.

### 11 Proof of Theorem 2

In this section we give a sketch of the proof of Theorem 2. This theorem is in large parts a consequence of Lemma 12 from Section 7. By the assumptions made at the beginning of Section 2 we consider an interval \([A, B] \subset (1, \infty)\) which is of length 1; however, the proof remains true for intervals of arbitrary (positive) length in exactly the same way.

Let \(\varepsilon > 0\) be given. Let \(N \geq 1\) also be given, and set

\[
\check{N} = \max\{n \leq N : n \text{ is of the form } (46)\}.
\]

Then

\[
N - \check{N} \ll N^{4/5}.
\]

Assume again for simplicity that \(f\) is an even function, set \(J = \lceil \varepsilon^{-1} \rceil\) and

\[
p(x) = \sum_{j=1}^{J} a_j \cos 2\pi jx, \quad r(x) = \sum_{j=J+1}^{\infty} a_j \cos 2\pi jx.
\]
Using the methods from Section 3, we can prove that
\[
\int_A^B \left( \sum_{n=1}^{N} r(\xi x^n) \right)^2 \, dx \ll NJ^{-1}. \quad (74)
\]
Similarly, using (73), we can prove that
\[
\int_A^B \left( \sum_{n=N+1}^{N} f(\xi x^n) \right)^2 \, dx \ll N^{4/5}. \quad (75)
\]
For the distribution of the normalized sum
\[
\sum_{n=1}^{N} p(\xi x^n) \|f\| \sqrt{N} \quad (76)
\]
we have the approximation result from Section 7. We further have
\[
\sum_{n=1}^{N} f(\xi x^n) = \sum_{n=1}^{N} p(\xi x^n) + \sum_{n=N+1}^{N} p(\xi x^n) + \sum_{n=1}^{N} r(\xi x^n). \quad (77)
\]
Now inequalities (74) and (75) tell us that the last two sums on the right-hand side of (77) are "small" with large probability in comparison to the normalizing factor $\sqrt{N}$. Note further that $\|p\| \to \|f\|$ as $\varepsilon \to 0$. Thus the fact that the distribution of (76) is close to the normal distribution tells us that also the distribution of
\[
\frac{\sum_{n=1}^{N} f(\xi x^n)}{\|f\| \sqrt{N}}
\]
is close to the normal distribution, provided $\varepsilon$ is sufficiently small. Arguing as at the end of Section 7, all these results are sufficient to obtain
\[
\left| \mathbb{P} \left( x \in [A, B] : \frac{\sum_{n=1}^{N} f(\xi x^n)}{\|f\| \sqrt{N}} < t \right) - \Phi(t) \right| \ll \varepsilon
\]
for sufficiently large $N$, for all $t \in \mathbb{R}$.

**Addendum**

I thank Katusi Fukuyama for the following argument, which illuminates the relation between Theorem 1 and the corresponding results for the discrepancy of lacunary series. Let $x > 1$. Then, according to (3), we have
\[
\frac{D_N(\{\xi x^n\})}{\sqrt{N} \log \log N} = \frac{1}{\sqrt{2}} \quad \text{a.e. } \xi, \quad (78)
\]
if $x$ is a number for which $x^r \not\in \mathbb{Q}$ for all integers $r \geq 1$. Since the exceptional set of such $x$ is countable (and consequently has vanishing Lebesgue measure), this implies that
\[
\frac{D_N(\{\xi x^n\})}{\sqrt{N} \log \log N} = \frac{1}{\sqrt{2}} \quad \text{a.e. } \xi, \quad \text{for a.e. } x > 1.
\]
Applying Fubini’s theorem we obtain
\[
\frac{D_N(\{\xi x^n\})}{\sqrt{N \log \log N}} = \frac{1}{\sqrt{2}} \quad \text{a.e. } x > 1, \text{ for a.e. } \xi. \tag{79}
\]

On the other hand, Corollary 1 can be written as
\[
\frac{D_N(\{\xi x^n\})}{\sqrt{N \log \log N}} = \frac{1}{\sqrt{2}} \quad \text{a.e. } x > 1, \text{ for all } \xi > 0.
\]
Thus (79) is a significantly weaker version of Corollary 1 which, however, gives a plausible explanation of the appearance of the constant 1/\sqrt{2} in Corollary 1. A result similar to (78), with the sequence \(x^n\) replaced by \(x^{s_n}\) for increasing \((s_n)_{n \geq 1}\), has been proved in [37]; applying again Fubini’s theorem one can conclude that
\[
\frac{D_N(\{\xi x^{s_n}\})}{\sqrt{N \log \log N}} = \frac{1}{\sqrt{2}} \quad \text{a.e. } x > 1, \text{ for a.e. } \xi.
\]
which is a weaker version of Theorem 1.

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