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Approximation by (p,q) Szász-beta–Stancu operators

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Abstract Motivated by recent investigations, in this paper we introduce \((p, q)\)-Szász-beta–Stancu operators and investigate their local approximation properties in terms of modulus of continuity. We also obtain a weighted approximation and Voronovskaya-type asymptotic formula.

Mathematics Subject Classification 41A30 · 41A35

1 Introduction

During the period of 1997–2017, the application of quantum-calculus came out as a new area of research in the field of approximation of functions by positive linear operators. Lupas [11] presented the first \(q\)-analogue of the Bernstein polynomials using \(q\)-integers, after a decade Phillips [21] gave another \(q\)-analogue of the Bernstein polynomials. Since then diverse operators have been generalized to their quantum variants and their approximation properties were discussed in [5,10,12,13,22].

Post-quantum calculus is an advanced extension of quantum-calculus and symbolized by \((p, q)\)-calculus. Mursaleen et al. [19] introduced the Bernstein polynomials using \((p, q)\)-calculus, which was further improved in [18].

Gupta and Aral introduced the Durrmeyer-type generalization of \((p, q)\)-Bernstein operators in [9]. The quantum variant and post-quantum variant of Szász–Mirakyan operators were introduced and studied in [14] and [1]. Some approximation properties on \((p, q)\)-analogue of Stancu-type generalization of linear positive operators were studied in [6,15,20].

We also consider some more results on approximation of functions by positive linear operators using \((p, q)\)-calculus given in [2,16,17].

We mention some notations, definitions of \((p, q)\)-calculus as follows (see for details [24,25]).

The \((p, q)\)-bracket is defined as

\[
[d]_{p,q} = \frac{p^d - q^d}{p - q}, \quad d = 0, 1, 2 \ldots, [0]_{p,q} = 0.
\]

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The \((p, q)\)-factorial is defined as
\[
[d]_{p,q}! = \prod_{k=1}^{d} [k]_{p,q}, \quad d \geq 1, \quad [0]_{p,q}! = 1.
\]

The \((p, q)\)-binomial coefficient is given by
\[
\binom{d}{k}_{p,q} = \frac{[d]_{p,q}!}{[d-k]_{p,q}![k]_{p,q}!}, \quad 0 \leq k \leq d.
\]

**Definition 1.1** The \((p, q)\)-power basis is defined below:
\[
(x \oplus y)^d_{p,q} = (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{d-1}x + q^{d-1}y),
\]
\[
(x \ominus y)^d_{p,q} = (x - y)(px - qy)(p^2x - q^2y) \cdots (p^{d-1}x - q^{d-1}y).
\]

**Definition 1.2** \([23]\) For \(d \geq 0\), the \((p, q)\)-gamma function is given as
\[
\Gamma_{p,q}(d + 1) = \frac{(p \ominus q)^d_{p,q}}{(p-q)^d} = [d]_{p,q}!, \quad 0 < q < p.
\]

**Definition 1.3** The \((p, q)\) derivative of the function \(f\) is defined as
\[
D_{p,q} f(y) = \frac{f(py) - f(qy)}{(p-q)y}, \quad y \neq 0
\]
and \(D_{p,q} f(0) = f'(0)\), provided that \(f\) is differentiable at zero.

**Proposition 1.4** \([23]\) The \((p, q)\)-integration by parts is given by
\[
\int_a^b g(px) D_{p,q} h(x) d_{p,q}x = g(b)h(b) - g(a)h(a) - \int_a^b h(qx) D_{p,q} g(x) d_{p,q}x.
\]

The \((p, q)\)-beta function of second kind \([4]\) is given by
\[
B_{p,q}(m, n) = \int_0^\infty \frac{x^{m-1}}{(1 \oplus px)_{p,q}^{m+n}} d_{p,q}x,
\]
where \(m, n \in \mathbb{N}\).

The relation between \((p, q)\)-beta and \((p, q)\)-gamma functions is given as
\[
B_{p,q}(m, n) = q^{\frac{m(m-1)}{2}} \frac{p^{-m(m+1)/2} \Gamma_{p,q} m \Gamma_{p,q} n}{\Gamma_{p,q}(m+n)}.
\]

For \(0 \leq x < \infty, 0 < q < p \leq 1\), Aral and Gupta \([3]\) defined the \((p, q)\)-analogue of Szász-beta operators as follows:
\[
D_n^{p,q}(f; x) = \sum_{k=1}^{\infty} \sum_{n,k}^{p,q} \frac{1}{B_{p,q}(k, n+1)} \int_0^\infty \frac{t^{k-1}}{(1 \oplus pt)_{p,q}^{k+n+1}} f(p^{k+1}qt) d_{p,q}t + \frac{f(0)}{E_{p,q}([n]_{p,q}x)}, \quad (1)
\]
where
\[
\sum_{n,k}^{p,q}(x) = \frac{1}{E_{p,q}([n]_{p,q}x)} q^{\frac{k(k-1)}{2}} ([n]_{p,q}x)^k.
\]
The aim of this paper is to generalize the operator in (1) using Stancu-type parameters, (i.e. assuming $0 \leq \alpha \leq \beta$), we define:

$$D_{n,\alpha,\beta}^{p,q}(f; x) = \sum_{k=1}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_{0}^{\infty} \frac{t^{k-1}}{(1+pt)^{k+n+1}} \left( \frac{p^{k+1}qt[n]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) d_{p,q}t + \frac{f\left(\frac{\alpha}{[n]_{p,q}+\beta}\right)}{E_{p,q}\left([n]_{p,q}x\right)},$$

(2)

where $s_{n,k}^{p,q}(x)$ is given in (1).

In particular case, if $\alpha = \beta = 0$, then the operators $D_{n,\alpha,\beta}^{p,q}(f; x)$ turn out to be the one defined by (1).

2 Moments

**Lemma 2.1** [3] For the operator defined in (1), $x \geq 0$ and for $\alpha = \beta = 0$, the following equalities hold for $0 < q < p \leq 1$

(i) $D_{n,\alpha,\beta}^{p,q}(1; x) = 1$,

(ii) $D_{n,\alpha,\beta}^{p,q}(t; x) = x$,

(iii) $D_{n,\alpha,\beta}^{p,q}(t^2; x) = \frac{[2;2]_{p,q}x + p[n]_{p,q}x^2}{[n-1]_{p,q}}$.

**Lemma 2.2** For $x \in [0, \infty)$, $0 < q < p \leq 1$, then for the operator in (2), we have the following moments.

(i) $D_{n,\alpha,\beta}^{p,q}(1; x) = 1$,

(ii) $D_{n,\alpha,\beta}^{p,q}(t; x) = \frac{n[n]_{p,q}x + \alpha}{[n]_{p,q} + \beta}$,

(iii) $D_{n,\alpha,\beta}^{p,q}(t^2; x) = \frac{1}{([n]_{p,q} + \beta)^2} \left( \frac{p[n]_{p,q}^3}{[n-1]_{p,q}} x^2 + \left( \frac{q[n]_{p,q}[2]_{p,q}}{p[n-1]_{p,q}} + 2\alpha [n]_{p,q} \right) x + \alpha^2 \right)$.

**Proof** By the definition of operator (2) and Lemma 2.1, we have

(i) $D_{n,\alpha,\beta}^{p,q}(1; x) = \sum_{k=1}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_{0}^{\infty} \frac{t^{k-1}}{(1+pt)^{k+n+1}} d_{p,q}t + \frac{1}{E_{p,q}([n]_{p,q}x)}$

$= D_{n,\alpha,\beta}^{p,q}(1; x) = 1$.

(ii) $D_{n,\alpha,\beta}^{p,q}(t; x) = \sum_{k=1}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_{0}^{\infty} \frac{t^{k-1}}{(1+pt)^{k+n+1}} \left( \frac{p^{k+1}qt[n]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) d_{p,q}t$

$+ \frac{\alpha}{[n]_{p,q} + \beta} E_{p,q}([n]_{p,q}x)$

$= \frac{[n]_{p,q}}{[n]_{p,q} + \beta} D_{n,\alpha,\beta}^{p,q}(t; x) + \frac{\alpha}{[n]_{p,q} + \beta} D_{n,\alpha,\beta}^{p,q}(1; x)$

$= \frac{[n]_{p,q}x + \alpha}{[n]_{p,q} + \beta}$.

(iii) $D_{n,\alpha,\beta}^{p,q}(t^2; x) = \sum_{k=1}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_{0}^{\infty} \frac{t^{k-1}}{(1+pt)^{k+n+1}} \left( \frac{p^{k+1}qt[n]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right)^2 d_{p,q}t$

$+ \left( \frac{\alpha}{[n]_{p,q} + \beta} \right)^2 E_{p,q}([n]_{p,q}x)$
The second-order modulus of smoothness of $f$ is given by

\[
D_{n,p,q}^2(f; t; x) = \frac{[n]^2_{p,q}}{([n]_{p,q} + \beta)^2} D_n^{p,q}(t^2; x) + \frac{2\alpha[n]_{p,q}}{([n]_{p,q} + \beta)^2} D_n^{p,q}(t; x) + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} D_n^{p,q}(1; x)
\]

and

\[
D_{n,\alpha,\beta}^2(t - x)^2; x) = \beta_1(n)x^2 + \beta_2(n)x + \beta_3(n),
\]

where

\[
\beta_1(n) = \left( \frac{p[n]^3_{p,q}}{([n]_{p,q} + \beta)^2[n - 1]_{p,q}} - \frac{2[n]_{p,q}}{[n]_{p,q} + \beta} + 1 \right),
\]

\[
\beta_2(n) = \left( \frac{q[2]_{p,q}[n]^2_{p,q}}{([n]_{p,q} + \beta)^2} + \frac{2\alpha[n]_{p,q}}{([n]_{p,q} + \beta)^2} - \frac{2\alpha}{[n]_{p,q} + \beta} \right),
\]

\[
\beta_3(n) = \frac{\alpha^2}{([n]_{p,q} + \beta)^2}.
\]

Assuming $\beta^*(n) = \max[\beta_1(n), \beta_2(n), \beta_3(n)]$, we can write

\[
D_{n,\alpha,\beta}^2(t - x)^2; x) \leq \beta^*(n)(1 + x)^2.
\]

### 3 Local approximation

Let us consider the space of all real valued continuous and bounded functions on $\mathbb{R}_+$ and denote this space by $C_B(\mathbb{R}_+)$ under the norm:

\[
\| f \| = \sup_{x \in \mathbb{R}_+} | f(x) |.
\]

where $\mathbb{R}_+ = [0, \infty)$.

Let $W^2 = \{ s \in C_B(\mathbb{R}_+) : s', s'' \in C_B(\mathbb{R}_+) \}$. Then, Peetre’s K-functional is defined as

\[
K_2(f, \delta) = \inf \{ \| f - s \| + \| s'' \| : s \in W^2 \}.
\]

Then as in [7], there exists a positive constant $C$ such that

\[
K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}), \quad \delta > 0.
\]

The second-order modulus of smoothness of $f \in C_B(\mathbb{R}_+)$ is

\[
\omega_2(f, \sqrt{\delta}) = \sup_{0 \leq h \leq \delta} \sup_{0 \leq x < \infty} | f(x + 2h) - 2f(x + h) + f(x) |,
\]

and the usual modulus of continuity is given by

\[
\omega(f, \delta) = \sup_{0 \leq h \leq \delta} \sup_{0 \leq x < \infty} | f(x + h) - f(x) |.
\]
Lemma 3.1 For \( f \in C_B(\mathbb{R}_+) \) and for \( s \in W^2 \), we have
\[
|D_{n,a,b}^{s,p,q}(s; x) - s(x)| \leq \| s'' \| (\beta^s(n)(1 + x)^2 + \mu_n^2(p, q, x)),
\]
where the auxiliary operators are given by
\[
D_{n,a,b}^{s,p,q}(s; x) = D_{n,a,b}^{p,q}(s; x) + s(x) - s(D_{n,a,b}^{p,q}(t; x)),
\]
and
\[
\mu_n^s(p, q, x) = \frac{\alpha - \beta x}{[n]_{p,q} + \beta}.
\]

Proof By the definition of auxiliary operators, it can be shown that
\[
D_{n,a,b}^{s,p,q}(t - x; x) = 0.
\]
Let \( s \in W^2 \). Then from the Taylor’s expansion, we have
\[
s(t) = s(x) + s'(x)(t - x) + \int_x^t (t - u)s''(u)du.
\]
Operating (8) with (5) and using (7), we get
\[
D_{n,a,b}^{s,p,q}(s; x) = s(x) + D_{n,a,b}^{s,p,q}\left(\int_x^t (t - u)s''(u)du; x\right)
\]
\[
(D_{n,a,b}^{s,p,q}(s; x) - s(x)) = D_{n,a,b}^{p,q}\left(\int_x^t (t - u)s''(u)du; x\right) - D_{n,a,b}^{p,q}(t; x) + (D_{n,a,b}^{p,q}(t; x) - u)s''(u)du
\]
\[
= D_{n,a,b}^{p,q}\left(\int_x^t (t - u)s''(u)du; x\right) - \int_x^t \left(\frac{[n]_{p,q} + \alpha}{[n]_{p,q} + \beta} - u\right)s''(u)du
\]
\[
|D_{n,a,b}^{s,p,q}(s; x) - s(x)| \leq D_{n,a,b}^{p,q}\left(\int_x^t |(t - u)||s''(u)|du; x\right)
\]
\[
+ \int_x^t \left|\frac{[n]_{p,q} + \alpha}{[n]_{p,q} + \beta} - u\right||s''(u)|du
\]
\[
\leq \| s'' \| (\beta^s(n)(1 + x)^2 + \mu_n^2(p, q, x)).
\]
Therefore,
\[
|D_{n,a,b}^{s,p,q}(s; x) - s(x)| \leq \| s'' \| (\beta^s(n)(1 + x)^2 + \mu_n^2(p, q, x)).
\]
Hence the proof is completed.
\[\square\]

Theorem 3.2 For \( f \in C_B(\mathbb{R}_+) \) and \( x \in [0, \infty) \), there exists a constant \( C > 0 \) such that
\[
|D_{n,a,b}^{p,q}(f; x) - f(x)| \leq C\omega(f; \sqrt{\beta^s(n)(1 + x)^2 + \mu_n^2(p, q, x)}) + \omega(f; \frac{\alpha - \beta x}{[n]_{p,q} + \beta}).
\]
Proof Using (2), (5) and Lemma 2.2, we have
\[ |D_{n,a,b}^{p,q}(f; x) - f(x)| \leq 4 \| f \|, \] (10)
for any \( s \in W^2 \) and using (5), (9) and (10), we get
\[ |D_{n,a,b}^{p,q}(f; x) - f(x)| \leq |D_{n,a,b}^{p,q}((f - s); x) - (f - s)(x)| + |D_{n,a,b}^{p,q}(s; x) - s(x)| + |f(x + \frac{\alpha - \beta x}{[n]p,q + \beta}) - f(x)| \]
\[ \leq 4 \| f - s \| + \| f(x + \frac{\alpha - \beta x}{[n]p,q + \beta}) - f(x)\| + s^{\alpha} \| (\beta^*(n)(1 + x)^2 + \mu_n^2(p, q, x)).\]

Now taking infimum on right-hand side over all \( s \in W^2 \) and using (4), we get
\[ |D_{n,a,b}^{p,q}(f; x) - f(x)| \leq C K_2(f; \beta^*(n)(1 + x)^2 + \mu_n^2(p, q, x)) + \omega \left(f; \frac{\alpha - \beta x}{[n]p,q + \beta}\right) \]
\[ \leq C \omega \left(f; \sqrt{\beta^*(n)(1 + x)^2 + \mu_n^2(p, q, x)}\right) + \omega \left(f; \frac{\alpha - \beta x}{[n]p,q + \beta}\right). \]

Hence the proof of the theorem. \( \square \)

Remark If \( f \in C[0, \infty) \), \( 0 \leq x < \infty \) and \( \omega(f; \delta) \) is the modulus of continuity, then
\[ |D_{n,a,b}^{p,q}(f; x) - f(x)| \leq \left[1 + \sqrt{\varphi_n(x)}\right] \omega(f; \delta_{n,\beta}), \]
where
\[ \delta_{n,\beta} = \frac{1}{[n]p,q + \beta} \]
and
\[ \varphi_n(x) = \frac{1}{[n]p,q + \beta} \left[ \frac{p[n]p,q^3}{[n-1]p,q} x^2 + \frac{[n]p,q^2[2]p,q q}{p[n-1]p,q + 2\alpha[n]p,q} x + (\alpha)^2 - 2x(n[p,q x + \alpha (n]p,q + \beta) + ([n]p,q + \beta)^2 x^2 \right]. \]

Proof Let \( f \in C[0, \infty) \) and \( 0 \leq x < \infty \). Then using monotonicity of the operator defined in (2), we can easily obtain for every \( \delta > 0 \) that
\[ |D_{n,a,b}^{p,q}(f; x) - f(x)| \leq \left[1 + \frac{1}{\delta_{n,\beta}} \sqrt{D_{n,a,b}^{p,q}((t - x)^2; x)}\right] \omega(f; \delta_{n,\beta}) \]
\[ \leq \left[1 + \sqrt{\varphi_n(x)}\right] \omega(f; \delta_{n,\beta}), \]
which is obtained using Lemma 2.2 and choosing \( \delta_{n,\beta} = \frac{1}{[n]p,q + \beta} \). Hence we arrive at the result. \( \square \)

If we put \( \alpha = \beta = 0 \), we can find the similar results for the operators defined by (1):
\[ |D_n^{p,q}(f; x) - f(x)| \leq \left[1 + \sqrt{\frac{p[n]p,q^3}{[n-1]p,q} x^2 + \frac{[n]p,q^2[2]p,q q}{p[n-1]p,q x - [n]p,q x^2}}\right] \omega(f; \delta_n), \]
where \( \delta_n = \sqrt{\frac{1}{[n]p,q}} \) and it is observed that \( \delta_{n,\beta} \leq \delta_n \).

Therefore, rate of convergence of \( D_{n,a,b}^{p,q} \) is better than \( D_n^{p,q} \).
4 Weighted approximation

Let us consider the functions in weighted space defined as

1. $H^2(\mathbb{R}^+)$ denotes the set of all functions $f$ defined on $[0, \infty)$, such that $|f(x)| \leq M_f (1 + x^2)$ where $M_f > 0$ depending only on $f$.

2. $C(\mathbb{R}^+)$ be the set of all continuous functions $f$ defined on $[0, \infty)$.

3. $C_2(\mathbb{R}^+)$ denotes the subspace of all continuous functions in $H^2(\mathbb{R}^+)$.

4. $C^*_2(\mathbb{R}^+)$ means the subspace of all functions $f \in C_2(\mathbb{R}^+)$ for which $\lim_{x \to \infty} \frac{f(x)}{1 + x^2}$ is finite.

$H^2(\mathbb{R}^+)$ is a normed vector space under the norm:

$$\|f\|_2 = \sup_{x \geq 0} \frac{|f(x)|}{1 + x^2}.$$

Theorem 4.1 Let $p = p_n$ and $q = q_n$ such that $0 < q_n < p_n \leq 1$ and $p_n \to 1$, $q_n \to 1$, $p_n^\ast \to 1$, $q_n^\ast \to 1$ as $n \to \infty$. Then for each $f \in C^*_2(\mathbb{R}^+)$, we have

$$\lim_{n \to \infty} \|D_{n, \alpha, \beta}^{p_n, q_n}(f; x) - f\|_2 = 0.$$

Proof By ([8], Theorem 4.1.4), it is sufficient to verify the following three conditions:

$$\lim_{n \to \infty} \|D_{n, \alpha, \beta}^{p_n, q_n}(t^\lambda; x) - x^\lambda\|_2 = 0, \quad \lambda = 0, 1, 2.$$  \hfill (11)

Applying Lemma 2.2, (11) is true for $\lambda = 0$.

Next by Lemma 2.2, we have

$$\|D_{n, \alpha, \beta}^{p_n, q_n}(t; x) - x\|_2 \leq \sup_{x \geq 0} \left( \frac{\mu^*_n(p_n, q_n, x)}{1 + x^2} \right) \leq \frac{\alpha}{[n]_{p_n, q_n} + \beta} \sup_{x \geq 0} \frac{1}{1 + x^2} + \frac{\beta}{[n]_{p_n, q_n} + \beta} \sup_{x \geq 0} \frac{x}{1 + x^2};$$

this means (11) is true for $\lambda = 1$.

Finally, considering the same, we have

$$\|D_{n, \alpha, \beta}^{p_n, q_n}(t^2; x) - x^2\|_2 \leq \left( \frac{p_n [n]_{p_n, q_n}^3}{([n]_{p_n, q_n} + \beta)^2 [n - 1]_{p_n, q_n}} - 1 \right) \sup_{x \geq 0} \frac{x^2}{1 + x^2} + \left( \frac{q_n [2]_{p_n, q_n} [n]_{p_n, q_n}^2}{p_n ([n]_{p_n, q_n} + \beta)^2 [n - 1]_{p_n, q_n} + \beta [n]_{p_n, q_n} + \beta} \right) \sup_{x \geq 0} \frac{x}{1 + x^2} + \frac{\alpha^2}{([n]_{p_n, q_n} + \beta)^2} \sup_{x \geq 0} \frac{1}{1 + x^2}.$$ 

Hence (11) holds for $\lambda = 2$

Hence the theorem.
5 Voronovskaya type asymptotic formula

Theorem 5.1 Let \( p = p_n \) and \( q = q_n \) such that \( 0 < q_n < p_n \leq 1 \) and \( p_n \rightarrow 1, q_n \rightarrow 1, p_n'' \rightarrow 1, q_n'' \rightarrow 1 \) as \( n \rightarrow \infty \). Then for any \( f \in C^2_2(\mathbb{R}_+) \), such that \( f', f'' \in C^2_2(\mathbb{R}_+) \), we have

\[
\lim_{n \to \infty} [n]_{p_n, q_n} \left( D_{n, \alpha, \beta}^{p_n, q_n} - f(x) \right) = (\alpha - \beta) \frac{f''(x)}{2} [2x + x^2(1 + A)],
\]

where

\[
A = \lim_{n \to \infty} [n]_{p_n, q_n} \left( \frac{[n]_{p_n, q_n}^2}{([n]_{p_n, q_n} + \beta)^2} - \frac{2[n]_{p_n, q_n}^2}{([n]_{p_n, q_n} + \beta)} + 1 \right),
\]

uniformly on any \([0, L] , L > 0\).

Proof Let \( f, f', f'' \in C^2_2(\mathbb{R}_+) \) and \( x \geq 0 \). Then by Taylor's formula

\[
f(t) = f(x) + f'(x)(t - x) + \frac{1}{2} f''(x)(t - x)^2 + v(t, x)(t - x)^2.\tag{12}
\]

where \( v(t, x) \) is the Peano form of the remainder.

Since \( v(\cdot, x) \in C^2_2(\mathbb{R}_+) \), for sufficiently large \( n \)

\[
\lim_{t \to x} v(t, x) = 0.
\]

Applying the operator (2) on both sides of (12), we get

\[
D_{n, \alpha, \beta}^{p_n, q_n}(f; x) - f(x) = f'(x) D_{n, \alpha, \beta}^{p_n, q_n}((t - x); x) + \frac{1}{2} f''(x) D_{n, \alpha, \beta}^{p_n, q_n}((t - x)^2; x)
+ D_{n, \alpha, \beta}^{p_n, q_n}(v(t, x)(t - x)^2; x).
\]

By Cauchy–Schwarz inequality, we have

\[
D_{n, \alpha, \beta}^{p_n, q_n}(v(t, x)(t - x)^2; x) \leq \sqrt{D_{n, \alpha, \beta}^{p_n, q_n}(v^2(t, x); x)} \sqrt{D_{n, \alpha, \beta}^{p_n, q_n}((t - x)^4; x)}.\tag{13}
\]

We can see that \( v^2(x, x) = 0 \) and \( v^2 \in C^2_2(\mathbb{R}_+) \).

Then using Theorem 4.1, we say that

\[
\lim_{n \to \infty} D_{n, \alpha, \beta}^{p_n, q_n}(v^2(t, x); x) = v^2(x, x) = 0,\tag{14}
\]

uniformly with respect to \( x \in [0, L] \).

Therefore, from (13) and (14), we obtain

\[
\lim_{n \to \infty} [n]_{p_n, q_n} D_{n, \alpha, \beta}^{p_n, q_n}(v(t, x)(t - x)^2; x) = 0.
\]

Thus

\[
\lim_{n \to \infty} [n]_{p_n, q_n} \left( D_{n, \alpha, \beta}^{p_n, q_n}(f; x) - f(x) \right) = f'(x) \lim_{n \to \infty} [n]_{p_n, q_n} D_{n, \alpha, \beta}^{p_n, q_n}((t - x); x)
+ \frac{1}{2} f''(x) \lim_{n \to \infty} D_{n, \alpha, \beta}^{p_n, q_n}((t - x)^2; x).\tag{15}
\]

Now

\[
\lim_{n \to \infty} [n]_{p_n, q_n} \left( D_{n, \alpha, \beta}^{p_n, q_n}(f; x) - f(x) \right) = \lim_{n \to \infty} [n]_{p_n, q_n} \frac{\alpha - \beta x}{[n]_{p_n, q_n} + \beta}
= \alpha - \beta x.\tag{16}
\]
Using the equality \( [k]_{p_n,q_n} = q_n^{k-1} + p_n[k-1]_{p_n,q_n} \), we have

\[
\lim_{n \to \infty} [n]_{p_n,q_n} D_{n,\alpha,\beta}^{p_n,q_n} ((t - x)^2; x) = \lim_{n \to \infty} [n]_{p_n,q_n} \left( \frac{p_n[n]_{p_n,q_n}^3}{([n]_{p_n,q_n} + \beta)^3[n - 1]_{p_n,q_n}} - \frac{2[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} + 1 \right) x^2 \\
+ \left( \frac{q_n[2]_{p_n,q_n} [n^2]_{p_n,q_n}}{([n]_{p_n,q_n} + \beta)^2 P_n[n - 1]_{p_n,q_n}} + \frac{2\alpha[n]_{p_n,q_n}}{([n]_{p_n,q_n} + \beta)^2} \right) x \\
+ \frac{\alpha^2}{([n]_{p_n,q_n} + \beta)^2} \\
= 2x + x^2 + \lim_{n \to \infty} [n]_{p_n,q_n} \left( \frac{[n^2]_{p_n,q_n} p_n^2}{([n]_{p_n,q_n} + \beta)^2} - \frac{2[n]_{p_n,q_n}}{([n]_{p_n,q_n} + \beta) + 1} \right) x^2 \\
= 2x + x^2(1 + A).
\]

(17)

Using (15), (16) and (17), we get the desired result,

\[
\lim_{n \to \infty} [n]_{p_n,q_n} \left( D_{n,\alpha,\beta}^{p_n,q_n} - f(x) \right) = (\alpha - \beta x) f'(x) + \frac{f''(x)}{2} [2x + x^2(1 + A)].
\]

\[\square\]

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