THE SUPERPOTENTIAL $XYZ + XZY + \frac{c}{3}(X^3 + Y^3 + Z^3)$

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Abstract. The motivic Donaldson-Thomas series associated to an elliptic Sklyanin algebra corresponding to a point of order two differs from the conjectured series in [5, Conjecture 3.4].

1. Introduction

A 3-dimensional elliptic Sklyanin algebra $S = S_{a, b, c}$ is a quotient of the free algebra $\mathbb{C}\langle X, Y, Z \rangle$ modulo the graded ideal generated by the three quadratic relations

$$\begin{cases} aXY + bYX + cZ^2 = 0 \\ aYZ + bZY + cX^2 = 0 \\ aZX + bXZ + cY^2 = 0 \end{cases}$$

If $abc \neq 0$ and $3(abc)^3 \neq (a^3 + b^3 + c^3)^3$ these algebras have excellent ring-theoretic and homological properties, as proved by M. Artin, J. Tate and M. Van den Bergh in [1], [2]. They are determined by the plane elliptic curve

$$E_{pt} : (a^3 + b^3 + c^3)XYZ - abc(X^3 + Y^3 + Z^3) = 0 \subset \mathbb{P}^2$$

and translation by the point $\tau = [a : b : c] \in E_{pt}$ on it. The tools of noncommutative projective algebraic geometry have been used to classify the finite dimensional simple representations of $S_{a, b, c}$ in case $\tau \in E_{pt}$ is a point of finite order, see [18], [7], and more recently [19]. We recall these result in section 2 and make them explicit in the case when $\tau$ has order two, using the theory of Clifford algebras.

The Sklyanin algebra $S_{a, b, c}$ can also be realized as the Jacobi algebra associated to the superpotential

$$W = aXYZ + bXZY + \frac{c}{3}(X^3 + Y^3 + Z^3)$$

That is, if $\partial_V$ denotes the cyclic derivative with respect to the variable $V$, then

$$S_{a, b, c} = \frac{\mathbb{C}\langle X, Y, Z \rangle}{(\partial_X(W), \partial_Y(W), \partial_Z(W))}$$

$Tr(W)$ determines the Chern-Simons functional $M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \rightarrow \mathbb{C}$ and for every $\lambda \in \mathbb{C}$ we will denote by $M_n^W(\lambda)$ the fiber $Tr(W)^{-1}(\lambda)$. Because the degeneracy locus of $Tr(W)$ coincides with the scheme of $n$-dimensional representations of $S_{a, b, c}$ it is conjectured in [8] that the motivic Donaldson-Thomas series

$$U_W(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{[M_n^W(0)] - [M_n^W(1)]}{[GL_n]} \right] t^n$$
is determined by the virtual motives of simple representations of $S_{a,b,c}$. If $\tau$ has order $n$ and $(n,3) = 1$ it is known that apart from the trivial 1-dimensional representation all finite dimensional simple representations of $S_{a,b,c}$ have dimension $n$ and [5] Conjecture 3.4 conjectures that in this case we have

$$U_W(t) = \text{Exp}(\frac{t}{1-t} - \frac{t^n}{1-t})$$

with $M_1 = L^{-\frac{3}{2}}([X_{DT} = 1, \mu_3] - [X_{DT} = 0])$ where $X_{DT}$ is the cubic in $k^3$.

$$X_{DT} = (a + b)xyz + \frac{c}{3}(x^3 + y^3 + z^3)$$

and where $M_n = L^{1/2}([\mathbb{P}^2] - [E_c])$ where $E_c$ is the plane elliptic curve $E_{pt}/\langle \tau \rangle$ isogenous to $E_{pt}$ by dividing out the cyclic subgroup generated by $\tau$.

In [12] we developed a method to verify such conjectures inductively by calculating the motives of certain Brauer-Severi schemes. In this paper we will compute the second term of $U_W(t)$ for the Sklyanin algebra $S_{1,1,c}$, that is when $\tau$ is a point of order two. By [5] Conjecture 3.4 one would expect this coefficient to involve the motives of at least two different elliptic curves $[E_c]$ and $[E_{DT}]$ (which have different $j$-invariants). However, the computed term only involves the motif $[E_{DT}]$.

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2. Simple representations of Sklyanin algebras

The elliptic curve associated to the Sklyanin algebra $S_{a,b,c}$

$$E_{pt} : (a^3 + b^3 + c^3)XYZ - abc(X^3 + Y^3 + Z^3) = 0$$

is the locus of all point modules of $S_{a,b,c}$, that is, graded (critical) left-modules $A/(Al_1 + Al_2)$ with the $l_i$ linear in $X,Y,Z$ (and hence $l_1,l_2$ determine a point in $\mathbb{P}^2$) such that its Hilbert series is $(1-t)^{-1}$. Addition by the point $p = [a : b : c] \in E_{pt}$ describes the automorphism on point modules given by the shift-by-1 functor. A line module of $S_{a,b,c}$ is a graded (critical) left-module $A/Al$ with $l$ linear and Hilbert series $(1-t)^{-2}$. As $S_{a,b,c}$ is a domain, line modules correspond to lines in $\mathbb{P}^2$.

We are particularly interested in elliptic Sklyanin algebras which are finite modules over their centers. S. P. Smith and J. Tate [16] proved that this is the case if and only if $\tau \in E_{pt}$ is a point of finite order $n$. In this case $S_{a,b,c}$ is a maximal order in a division algebra of dimension $n^2$ over its center and the center of $S_{a,b,c}$ is isomorphic to

$$Z_{a,b,c} = \frac{\mathbb{C}[u_1, u_2, u_3, c_3]}{\Phi(u_1, u_2, u_3) - c_3^{n^2}}$$

where the $u_i$ are central elements of degree $n$, $c_3$ is a central element of degree 3 and $\Phi$ is a homogeneous polynomial of degree 3 in the $u_i$ describing the isogenous elliptic curve $E_c = E_{pt}/(\tau)$. In [18] and [7] it is shown that when $(n,3) = 1$ all finite dimensional simple representations of $S_{a,b,c}$ (apart from the trivial 1-dimensional simple) are of dimension $n$ and correspond to the smooth points of the central variety, which has an isolated singularity at the top.
In principle, one can give an explicit description of the triple of $n \times n$ matrices describing the simple $n$-dimensional representation $M_q$, corresponding to the maximal (non-graded) ideal $m_q$ of $R_{a,b,c}$ using the isogeny $E_{pt} \longrightarrow E_c$, see [11] or [7]. If $c_3$ does not vanish in $q$, the ruling from the top-singularity through $q$ determines a point $\mathfrak{q}$ in $\text{Proj}(R_{a,b,c}) = \mathbb{P}^2 = \mathbb{P}(u_1^{*}, u_2^{*}, u_3^{*})$ not lying on the elliptic curve $E_c$. Write $\mathfrak{q}$ as the intersection of two lines $L_1$ and $L_2$ in $\mathbb{P}^2$ and lift $L_1$ through the isogeny to a line $L$ in $\mathbb{P}^2 = \mathbb{P}(X^*, Y^*, Z^*)$, then $\mathfrak{q}$ determines the fat point of multiplicity $n$, that is, the graded (critical) left-module with Hilbert series $n/(1-t)$

$$F_{\mathfrak{q}} = \frac{A}{Al + A_2}$$

where $l$ is the linear form in $X, Y, Z$ determining $L$ and $l_2$ the degree $n$ central element which is the linear form in $u_1, u_2, u_3$ determining $L_2$. The central localization of $S_{a,b,c}$ at $c_3$ has a central element $t$ of degree 1 and the simple representation $M_q$ is then the quotient of $F_{\mathfrak{q}}$ by $t - \lambda$ where $\lambda$ is the evaluation of $t$ in $q$. If $c_3$ is zero in $q$, the ruling determines a point $\mathfrak{q} \in E_c$ which lifts through the isogeny to a point on $E_{pt}$ which form a $\tau$-orbit. The coordinates of the corresponding $n$ points on $E_{pt}$ can then be used to give explicit $n \times n$ matrices of the corresponding simple representation $M_q$, see [7] §3.1.

Clearly, this approach is only as effective as we have explicit formulas for lifting through the isogeny $E_{pt} \longrightarrow E'$, that is for small $n$. Next, we give explicit matrices describing the simple representations in the case when $n = 2$, that is when $a = b = 1$, not using the isogeny but the fact that in this case the Sklyanin algebras $S_c = S_{1,1,-c}$ can be viewed as Clifford algebras of ternary symmetric bilinear forms and we can apply the theory of quadratic forms to describe its simple 2-dimensional representations.

In a recent paper [11] D.J. Reich and C. Walton describe a Maple algorithm to obtain explicit representations of 3-dimensional Sklyanin algebras associated to a point of order two. Here we give a pen-and-paper approach, using classical quadratic form theory.

Let $A = (a_{ij})_{i,j} \in M_3(\mathbb{C})$ be a symmetric $3 \times 3$ matrix of rank $\geq 2$. The associated Clifford algebra $\text{Cliff}_{\mathbb{C}}(A)$ is the 8-dimensional $\mathbb{C}$-algebra generated by three elements $x_1, x_2$ and $x_3$ with defining relations

$$x_i x_j + x_j x_i = a_{ij} \quad \text{for all } 1 \leq i, j \leq 3$$

The symmetric bilinear form on $V = \mathbb{C}x_1 + \mathbb{C}x_2 + \mathbb{C}x_3$ defined by $A$ coincides with $\langle v, w \rangle = Tr(v, w)$ for all $v, w \in V$, where the product is taken in the Clifford algebra. The structure of Clifford algebras is well-known, see for example [7].

$$\text{Cliff}_{\mathbb{C}}(A) \simeq \begin{cases} M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) & \text{if } \text{rk}(A) = 3 \\ M_2(\mathbb{C}) \otimes \mathbb{C}[t] & \text{if } \text{rk}(A) = 2 \end{cases}$$

That is, $\text{Cliff}_{\mathbb{C}}(A)$ has two distinct simple 2-dimensional representations $\psi_{\pm}$, which coincide when $\text{det}(A) = 0$. We want to describe these explicitly, that is determine the $2 \times 2$ matrices $\psi_{\pm}(x_i)$. There is an invertible matrix $P \in GL_3(\mathbb{C})$ such that

$$P^T A P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1,1,1) \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (1,1,0)$$
The **Pauli matrices** describe the simple representations of $\text{Cliff}_C((1,1,\delta))$. If

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

then we have

$$\psi_\pm(u_1) = \sigma_1, \quad \psi_\pm(u_2) = \sigma_2 \quad \text{and} \quad \psi_\pm(u_3) = \pm \delta \sigma_3$$

for the new basis $(u_1, u_2, u_3)^T = P(x_1, x_2, x_3)^T$ of $V$. But then, if $P^{-1} = (q_{ij})_{i,j}$ we have:

**Lemma 1.** The simple 2-dimensional representation(s) of $\text{Cliff}_C(A)$ are given by

$$\psi_\pm(x_i) = \sum_{j=1}^{3} q_{ji} \psi_\pm(u_j) = q_{11} \sigma_1 + q_{22} \sigma_2 + \pm q_{33} \delta \sigma_3$$

The 3-dimensional quaternion **Sklyanin algebra** $S_c = S_{1,1,-c}$ is the $\mathbb{C}$-algebra generated by three elements $X = x_1, Y = x_2, Z = x_3$ with defining quadratic relations

$$XY + YX = cZ^2, \quad YZ + ZY = cX^2 \quad \text{and} \quad ZX + XZ = Y^2$$

It follows that $u = X^2, v = Y^2$ and $Z^2 = w$ are central elements and hence that $S_c$ is the Clifford algebra over $R = \mathbb{C}[u,v,w]$ as in [3] associated with the ternary symmetric bilinear form on the free module $V = Rx_1 \oplus Rx_2 \oplus Rx_3$ determined by the symmetric matrix in $M_3(R)$

$$Q = \begin{bmatrix} 2u & cw & cv \\ cw & 2v & cu \\ cv & cu & 2w \end{bmatrix}$$

Evaluating the entries of $Q$ in a point $p = (\alpha, \beta, \gamma) \in \mathbb{A}^3_C$ we obtain a symmetric matrix $A = Q(p) \in M_3(\mathbb{C})$ which is of rank at least two if and only if $p \neq (0,0,0)$. Lemma 1 gives us explicit representations of the two (or one) simple 2-dimensional representations $\psi_\pm(p)$ of $S_c$ lying over the point $p$.

It follows from [10] or [16] that the center $Z(S_c) = R \oplus R Tr(x_1 x_2 x_3)$ where $Tr(x_1 x_2 x_3)^2 = D = \text{det}(Q)$. As a result $\text{max}(Z(S_c))$ is a two-fold cover of $\mathbb{A}^3_C = \text{max}(R)$ ramified along the surface where $D$ vanishes. By the above, points of $\text{max}(Z(S_c))$ (apart from the unique point lying over $0 = (0,0,0)$) are in one-to-one correspondence with the isomorphism classes of 2-dimensional simple representations of $S_c$.

We will now construct families of explicit representations as in [14]. The idea is to diagonalize $Q$ over $\mathbb{A}^3_C - \{0\}$ and to keep track of the base-change matrix $P \in M_3(\mathbb{C}[u,v,w])$. For this we apply the classical diagonalization algorithm which in this case involves the choice of just two pivots.

As $p \neq (0,0,0)$ we may assume (after permuting the variables $x_i$ if necessary) that $2u \neq 0$ which will be our first pivot. One starts off with the $3 \times 6$ matrix $(Q|I_3)$ and uses the pivot to obtain zeroes in positions 2,3 of the first column and positions 2,3 in the first row by the usual trick of adding suitable multiples of rows and columns. The row-operations also have an effect on the right-hand side $3 \times 3$ matrix. After this step one obtains the matrix

$$\begin{bmatrix} 2u & 0 & 0 \ 0 & 2u(4uv - c^2 w^2) & 2u(2cu^2 - c^2 vw) \ 0 & 2u(2cu^2 - c^2 vw) & 2u(4uw - c^2 v^2) \end{bmatrix}$$
Case 1: If $A = 4w - c^2w^2 \neq 0$ (or, after permuting the variables, $4uw - c^2v^2 \neq 0$) use this as pivot. After this step one obtains the diagonal matrix $\Delta$ and the base-change matrix $P$

\[
(\Delta | P^*) = \begin{bmatrix}
2u & 0 & 0 & 1 & 0 & 0 \\
0 & 2uA & 0 & -cw & 2u & 0 \\
0 & 0 & 4u^2AD & 2cuB & 2cuC & 2uA
\end{bmatrix}
\]

where $B = cw - 2v^2$ and $C = cv - 2u^2$. Clearly, $P$ is invertible on the open set where $uA \neq 0$.

Case 2: If $4uv - c^2w^2 = 0 = 4uw - c^2v^2$, we have $2cu^2 - c^2vw \neq 0$. In this case we add the third row to the second and the third column to the second, use the resulting $(2, 2)$-entry as pivot in order to arrive at

\[
(\Delta | P^*) = \begin{bmatrix}
2u & 0 & 0 & 1 & 0 & 0 \\
0 & -2uL & 0 & -cv - cw & 2u & 2u \\
0 & 0 & -16u^4LD & 4cu^2Q_0 & 4u^2Q_1 & -4u^2Q_2
\end{bmatrix}
\]

where

\[
\begin{align*}
Q_0 &= (w - v)(2w + 2v + cu) \\
Q_1 &= c^2vw - 4uw + c^2v^2 - 2cu^2 \\
Q_2 &= c^2w^2 + c^2vw - 4uw - 2cu^2
\end{align*}
\]

and $L = Q_1 + Q_2$. The determinant of the basechange matrix is $-8u^2L$. In a point where $4uv - c^2w^2 = 0 = 4uw - c^2v^2$, $L$ is equal to $-2(2cu^2 - c^2vw)$ so $P$ is invertible in those points. Observe that these two cases cover all points in $\max(Z(S_c))$ where $u \neq 0$.

Lemma 2. With notations as above, let $\Delta = \text{diag}(D_1, D_2, D_3)$ and $P^{-1} = (Q_{ij})_{i,j}$. Then, the maps (remember that $x_1 = X, x_2 = Y$ and $x_3 = Z$)

\[
\psi_{\pm}(x_1) = Q_{11}\sqrt{D_1}\sigma_1 + Q_{12}\sqrt{D_2}\sigma_2 \pm \sqrt{D_3}\sigma_3
\]

give a family of explicit representations of $S_c$, with a unique representative for all simple 2-dimensional representations on the open set of $\max(Z(S_c))$ where $u \neq 0$. Here we take the matrices of the first case if $uA \neq 0$ and those of the second case on the locus where $4uv - c^2w^2 = 0 = 4uw - c^2v^2$. Permuting the variables covers the entire Azumaya-locus of $S_c$ which is $\max(Z(S_c))$ with the unique isolated singularity lying over $(0, 0, 0)$ removed.

For example, on the open set where $uA \neq 0$ we have the following explicit matrix-representations:

\[
\psi_\pm(X) = \begin{bmatrix}
0 & \sqrt{2u} \\
\sqrt{2u} & 0
\end{bmatrix}
\]

\[
\psi_\pm(Y) = \frac{c\sqrt{u}}{2u} \begin{bmatrix}
0 & \sqrt{2u} \\
\sqrt{2u} & 0
\end{bmatrix} \pm \frac{i}{\sqrt{2uA}} \begin{bmatrix}
0 & -i\sqrt{2uA} \\
i\sqrt{2uA} & 0
\end{bmatrix}
\]

\[
\psi_\pm(Z) = \frac{c\sqrt{u}}{2u} \begin{bmatrix}
0 & \sqrt{2u} \\
\sqrt{2u} & 0
\end{bmatrix} - \frac{cw}{2uA} \begin{bmatrix}
0 & -i\sqrt{2uA} \\
i\sqrt{2uA} & 0
\end{bmatrix} \pm \frac{1}{2uA} \begin{bmatrix}
2u\sqrt{AD} & 0 \\
0 & -2u\sqrt{AD}
\end{bmatrix}
\]
3. Superpotentials and motives

Consider the cubic superpotential \( W = aXYZ + bXYZ + \zeta(X^3 + Y^3 + Z^3) \) in the noncommutative variables \( X, Y \) and \( Z \). For every dimension \( n \geq 1 \), the superpotential \( W \) determines the Chern-Simons functional

\[
Tr(W) : M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \to \mathbb{C}
\]

obtained by replacing \( X, Y \) and \( Z \) by the first, second resp. third component matrix and taking the trace. The representation theoretic interest of the degeneracy locus \( \{ dTr(W) = 0 \} \) of this functional is that it coincides with the scheme of \( n \)-dimensional representations \( \text{rep}_n(R_W) \) of the associated Jacobi algebra

\[
R_W = \frac{\mathbb{C}(X,Y,Z)}{(\partial_X(W), \partial_Y(W), \partial_Z(W))}
\]

where the \( \partial_V \) are the cyclic derivative with respect to the variables \( V \), which in the case of the above superpotential \( W \) gives us the defining equations of \( S_{a,b,c} \). That is, the degeneracy locus of the superpotential \( W \)

\[
\{ dTr(W) = 0 \} = \text{rep}_n(S_{a,b,c})
\]

By the Denef-Loeser theory of motivic nearby cycles, see [3], the motive of this degeneracy locus can often be computed as the difference of the motives of the general fiber and the zero-fiber of the functional. For this reason we are interested in the (naive, equivariant) motive of the \( \lambda \)-fiber of the functional \( Tr(W) \) which we denote by \( \mathbb{M}_n^W(\lambda) = Tr(W)^{-1}(\lambda) \).

Recall that to each isomorphism class of a complex variety \( X \) (equipped with a good action of a finite group of roots of unity) we associate its naive equivariant motive \( [X] \) which is an element in the ring \( K_0^a(\text{Var}_C)[L^{-1/2}] \) (see [6] or [4]) and is subject to the scissor- and product-relations

\[
[X] - [Z] = [X - Z] \quad \text{and} \quad [X][Y] = [X \times Y]
\]

whenever \( Z \) is a Zariski subvariety of \( X \). A special element is the Lefschetz motive \( \mathbb{L} = [k_1, id] \) and we recall from [13, Lemma 4.1] that \( GL_n = \prod_{k=0}^{n-1}(L^n - L^k) \) and from [5, 2.2] that \( [k_\mu, \mu_k] = L^n \) for a linear action of \( \mu_k \) on \( k_\mu \). This ring is equipped with a plethystic exponential \( \text{Exp} \), see for example [4] and [6].

As \( W \) is homogeneous it follows from [9, Thm. 1.3] that the virtual motive of the degeneracy locus is equal to

\[
[dTr(W) = 0]_{\text{virt}} = [\text{rep}_n(S_{a,b,c})]_{\text{virt}} = \mathbb{L}^{-\frac{2\pi^2}{\mu}}([\mathbb{M}_n^W(0)] - [\mathbb{M}_n^W(1)])
\]

where \( \mu \) acts via \( \mu_d \) on \( \mathbb{M}_n^W(1) \) and trivially on \( \mathbb{M}_n^W(0) \). These virtual motives can be packaged together into the motivic Donaldson-Thomas series

\[
U_W(t) = \sum_{n=0}^{\infty} \mathbb{L}^{-\frac{2\pi^2}{\mu} \frac{[\mathbb{M}_n^W(0)] - [\mathbb{M}_n^W(1)]}{[GL_n]} t^n
\]

By the Jordan-Hölder theorem, the sequence \( [\text{rep}_n(S_{a,b,c})]_{\text{virt}} \) is expected to jump at every dimension \( n \) where \( S_{a,b,c} \) has simple \( n \)-dimensional representations. For this reason A. Cazzaniga, A. Morrison, B. Pym and B. Szendrői conjecture in [5] that the generating sequence \( U_W(t) \) has an exponential expression involving rational functions of virtual motives connected to the simple representations of the
The superpotential $XYZ + XYZ + \frac{4}{3}(X^3 + Y^3 + Z^3)$

Jacobi algebra $S_{a,b,c}$. Explicitly, their conjecture [5, Conjecture 3.4] asserts that in case $\tau \in E_{pt}$ has infinite order that then

$$U_W(t) = \text{Exp}(- \frac{M_1}{L^\tau - L^\frac{1}{2}} \frac{t}{1 - t})$$

where $M_1 = L^{-3/2}([X_{DT} = 1] - [X_{DT} = 0])$ where $X_{DT}$ is the cubic function in the three commuting variables $x, y, z$

$$X_{DT} = (a + b)xyz + \frac{c}{3}(x^3 + y^3 + z^3)$$

which gives $Tr(W)$ for $n = 1$. Note that $X_{DT}$ determines an elliptic curve in $\mathbb{P}^2$, usually with a different $j$-invariant than $E_{pt}$ and $E_c$. If however $\tau \in E_{pt}$ is a point of finite order $n$ and $(n, 3) = 1$ one expects another term in the exponential expression coming from the simples in dimension $n$. In [5, Conjecture 3.4] it is conjectured that in this case

$$U_W(t) = \text{Exp}(- \frac{M_1}{L^\tau - L^\frac{1}{2}} \frac{t}{1 - t} - \frac{M_n}{L^\tau - L^{-1/2}} \frac{t^n}{1 - t^n})$$

where $M_n = L^{1/2}([\mathbb{P}^2] - [E_c])$. Observe already from section 2 that this term only encodes the simple $n$-dimensional representations determined by points $q \in \text{Spec}(Z_{a,b,c})$ not lying on the cone over $E_c$.

**Lemma 3.** If we denote with

$$N_1 = (L - 1)[E_{DT}] + 1 - [S_{DT}, \mu_3] \quad \text{and} \quad N_2 = [E_c] - [\mathbb{P}^2]$$

then the coefficient of $t^2$ in the conjectured series $U_W(t)$ is equal to

$$\frac{L(L^2 - 1)N_2 + L^{-2}N_1^2 + L^{-1}(L^2 - 1)N_1 + L^{-2}(L - 1)\sigma_2(N_1)}{(L^2 - 1)(L - 1)}$$

**Proof.** With these notations, the conjecture [5, Conjecture 3.4] can be rewritten as

$$U_W(t) = \text{Exp}\left(\frac{L(L^2 - 2)N_1}{L - 1} \frac{t}{1 - t}\right) \cdot \text{Exp}\left(\frac{LN_2}{L - 1} \frac{t^2}{1 - t^2}\right)$$

The second term is equal to

$$\text{Exp}\left(\sum_{k \geq 1} \sum_{j \geq 0} L^{-j} N_2 t^{2k}\right) = \prod_{k \geq 1} \prod_{j \geq 0} \text{Exp}(L^{-j} N_2 t^{2k}) = \prod_{k \geq 1} \prod_{j \geq 0} \left(\sum_{n \geq 0} \sigma_n(L^{-j} N_2 t^{2k})\right) = \prod_{k \geq 1} \prod_{j \geq 0} \left(\sum_{n \geq 0} L^{-nj} \sigma_n(N_2) t^{2kn}\right)$$

As we are only interested in the coefficient of $t^2$ we need only consider the term in the first product where $k = 1$ and then get

$$(1 + N_2 t^2 + \ldots)(1 + L^{-1} N_2 t^2 + \ldots)(1 + L^{-2} N_2 t^2 + \ldots) \ldots = 1 + \frac{N_2}{1 - L^{-1} t^2} + \ldots$$

For the first term, we get likewise

$$\text{Exp}\left(\sum_{k \geq 1} \sum_{j \geq 2} L^{-j} N_1 t^{k}\right) = \prod_{k \geq 1} \prod_{j \geq 2} \text{Exp}(L^{-j} N_1 t^{k}) = \prod_{k \geq 1} \prod_{j \geq 2} \left(\sum_{n \geq 0} \sigma_n(L^{-j} N_1 t^{k})\right) = \prod_{k \geq 1} \prod_{j \geq 2} \left(\sum_{n \geq 0} L^{-nj} \sigma_n(N_1) t^{kn}\right)$$

As we only want the coefficient of $t^2$ we have to consider three contributions:
In two brackets with \( j_2 > j_1 \geq 2 \) this gives

\[
\sum_{2 \leq j_1 < j_2} N_{j_1}^2 \mathbb{L}^{-(j_1+j_2)} = \sum_{j \geq 2} \sum_{k \geq 0} \mathbb{L}^{-2j-k} N_1^2 \]

\[
\mathbb{L}^{-5} N_1^2 \sum_{j \geq 2} \sum_{k \geq 0} \mathbb{L}^{-2j} = \frac{\mathbb{L}^{-5} N_1^2}{(1 - \mathbb{L}^{-2})(1 - \mathbb{L}^{-1})} = \frac{\mathbb{L}^{-2} N_1^2}{(\mathbb{L}^2 - 1)(\mathbb{L} - 1)}
\]

\[k = 2, n = 1\] in one bracket and \( n = 0 \) in all others. This gives

\[
\sum_{j \geq 2} \mathbb{L}^{-j} N_1 = \frac{\mathbb{L}^{-2} N_1}{1 - \mathbb{L}^{-1}} = \frac{\mathbb{L}^{-1} N_1}{\mathbb{L} - 1}
\]

\[k = 1, n = 2\] in one bracket and \( n = 0 \) in all others. Then we get

\[
\sum_{j \geq 2} \mathbb{L}^{-2j} \sigma_2(N_1) = \frac{\mathbb{L}^{-5}\sigma_2(N_1)}{1 - \mathbb{L}^{-2}} = \frac{\mathbb{L}^{-2}\sigma_2(N_1)}{\mathbb{L}^2 - 1}
\]

Summing up all terms gives the claimed expression. \( \square \)

4. Brauer-Severi motives

In [12] an inductive method was proposed to compute the coefficients of the series \( \mathcal{U}_n(t) \) inductively. For every \( n \geq 1 \) and every \( \lambda \in \mathbb{C} \) introduce the following quotient of the trace ring \( T_{3,n} \) of 3 generic \( n \times n \) matrices

\[
\mathbb{T}_n^W(\lambda) = \frac{T_{3,n}}{(Tr(W) - \lambda)}
\]

The reason being that the \( \lambda \)-fiber \( Tr(W)^{-1}(\lambda) \) is the scheme of \( n \)-dimensional trace preserving representations of \( T_n^W(\lambda) \)

\[
Tr(W)^{-1}(\lambda) = \text{trep}_n(\mathbb{T}_n^W(\lambda))
\]

Now, consider the associated Brauer-Severi scheme in the sense of M. Van den Bergh [17]. That is, consider the open subscheme \( \mathcal{U}_n^W \) of \( \text{tre}p_n(\mathbb{T}_n^W(\lambda)) \times \mathbb{C}^n \) consisting of couples

\[
\mathcal{U}_n^W(\lambda) = \{(\phi, v) \in \text{tre}p_n(\mathbb{T}_n^W(\lambda)) \times \mathbb{C}^n \mid \phi(\mathbb{T}_n^W(\lambda)), v \in \mathbb{C}^n \}
\]

on which \( GL_n \) acts freely and let the Brauer-Severi scheme be the corresponding quotient variety \( \text{BS}_n^W(\lambda) = \mathcal{U}_n^W(\lambda) / GL_n \). Then it is shown in [12] Prop. 5] that one can compute the fiber-motives at \( n \) from knowledge of the Brauer-Severi-motives for all dimensions \( k \leq n \) and the fiber-motives at all \( k < n \). Explicitly,

\[
(L^n - 1) \frac{[M_{n-1}^W(0)] - [M_{n-1}^W(1)]}{[GL_n]}
\]

is equal to

\[
([\text{BS}_n^W(0)] - [\text{BS}_n^W(1)]) + \sum_{k=1}^{n-1} \frac{L^{2k(n-k)}}{[GL_{n-k}]} (\text{BS}_k^W(0) - [\text{BS}_k^W(1)])([M_k^W(0)] - [M_k^W(1)])
\]

We will next compute the first two terms in \( \mathcal{U}_W(t) \) and for \( n = 2 \) the previous formula reduces to

\[
(L^2 - 1) \frac{[M_2^W(0)] - [M_2^W(1)]}{[GL_2]} = \text{BS}_2^W(0) - \text{BS}_2^W(1) + \frac{L^2}{(L - 1)} ([M_1^W(0)] - [M_1^W(1)])^2
\]
and we have already that
\[
[M^W_1(1)] = [X_{DT} = 1] \quad \text{and} \quad [M^W_1(0)] = [X_{DT} = 0] = (L - 1) [E_{DT}] + 1
\]
so it remains to compute the difference of the Brauer-Severi motives \([BS^W_2(0)] - [BS^W_2(1)]\).

From \cite{15} we deduce that \(BS_2(T_{3,2})\) has a cellular decomposition as \(\mathbb{A}^{10} \sqcup \mathbb{A}^8 \sqcup \mathbb{A}^8\) where the three cells have representatives

\[
\begin{align*}
\text{cell}_1 : v &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & p \\ 1 & r \end{bmatrix}, \quad Y = \begin{bmatrix} s & t \\ u & v \end{bmatrix}, \quad Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\
\text{cell}_2 : v &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} n & p \\ 0 & r \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & t \\ 1 & v \end{bmatrix}, \quad Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\
\text{cell}_3 : v &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} n & p \\ 0 & r \end{bmatrix}, \quad Y = \begin{bmatrix} s & t \\ 0 & v \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & x \\ 1 & z \end{bmatrix}
\end{align*}
\]

It follows that \(BS^W_{3,2}(\lambda)\) decomposes as \(S_1(\lambda) \sqcup S_2(\lambda) \sqcup S_3(\lambda)\) where the subschemes \(S_i(\lambda)\) of \(\mathbb{A}^{11-i}\) have defining equations

\[
\begin{align*}
S_1(\lambda) & : (C + Q_u u + Q_y y + Q_y)|_{n=0} = \lambda \\
S_2(\lambda) & : (C + Q_y y + Q_u)|_{n=0} = \lambda \\
S_3(\lambda) & : (C + Q_y)|_{n=0} = \lambda \\
\end{align*}
\]

where

\[
\begin{align*}
C &= \frac{c}{3} (n^3 + r^3 + s^3 + v^3 + w^3 + z^3) + (a + b)(rvz + nsw) \\
Q_u &= a(tz + sx) + b(vx + tw) + cp(r + n) \\
Q_y &= a(rx + pu) + b(pz + nx) + ct(v + s) \\
Q_y &= a(pw + nt) + b(rt + ps) + cx(z + w)
\end{align*}
\]

Note that in using the cellular decomposition, we set a variable equal to 1. So, in order to retain a homogeneous form we let \(G_m\) act on \(n, s, w, r, v, z\) with weight one, on \(q, u, y\) with weight two and on \(x, t, p\) with weight zero. Thus, we need a slight extension of \cite{6} Thm. 1.3 as to allow \(G_m\) to act with weight two on certain variables.

We will restrict to the case of a Sklyanin algebra with a point of order two, that is the case when \(a = b\), which we may assume to be equal to 1, and with \(c \neq 0\).

**Lemma 4.** With \(a = b = 1\) and \(c \neq 0\) we have

\[
\begin{align*}
[S_3(0)] &= L^7 + L^5 - L^4 \\
[S_3(1)] &= L^7 - L^4
\end{align*}
\]

and therefore \([S_3(0)] - [S_3(1)] = L^5\).

**Proof.** The defining equation of \(S_3(\lambda)\) in \(\mathbb{A}^8\) is

\[
c/3 (n^3 + r^3 + s^3 + v^3 + w^3 + z^3) + 2rvz + (v + s)p + (n + r)t + czx = \lambda
\]
(1) : If \( v + s \neq 0 \) we can eliminate \( p \) from the equation and get a contribution \( L^5(L^2 - L) \) as there are five free variables and \( [v + s \neq 0]_{\lambda^2} = L^2 - L \). Note that this is independent of the value of \( \lambda \).

(2) : If \( v + s = 0 \) we get the equation
\[
\frac{c}{3}(n^3 + r^3 + z^3) + 2rvz + (n + r)t + czx = \lambda
\]
If we assume that in addition \( n + r \neq 0 \) we can eliminate \( t \), then by an argument as above we obtain a contribution \( L^4(L^2 - L) \), again independent of the value of \( \lambda \).

(3) : If \( v + s = 0 \) and \( n + r = 0 \) we get as equation \( \frac{s}{3}z^3 + 2rvz + czx = \lambda \). So, if \( z \neq 0 \) we can eliminate \( x \) and get a term \( L^4(L - 1) \), independent of \( \lambda \).

(4) : If \( v + s = 0 \), \( n + r = 0 \) and \( z = 0 \) we get the equation \( 0 = \lambda \). Hence, if \( \lambda = 1 \) this gives no contribution, but if \( \lambda = 0 \) we get a contribution \( L^5 \).

Summing up we get the claimed motives. \( \square \)

As we are only interested in the differences \( [S_k(0)] - [S_k(1)] \) we will in the remaining computations only determine the difference of the motives in those subcases where the result can depend on the value of \( \lambda \).

**Lemma 5.** With \( a = b = 1 \) and \( c \neq 0 \) we have
\[
[S_2(0)] - [S_2(1)] = L^6 + L^3.[\mu_4].([X_0] - [X_1])
\]
where \( X_{\lambda} \) is the locally closed subset in \( \mathbb{A}^3 \) (with variables \( x, y, z \)) defined by
\[
X_{\lambda} = \begin{cases} x \neq 0 \\ x(3pcz^2 - 3p^2cxz + 6yz + (c^4 + 2c)x^2 - 3pc^2xy + 3p^2c^2y^2) = 3\lambda \end{cases}
\]
and \( \rho^3 = 1 \).

**Proof.** The defining equation of \( S_2(\lambda) \) in \( \mathbb{A}^9 \) is
\[
\frac{c}{3}(n^3 + r^3 + v^3 + w^3 + z^3) + 2rvx + (vp + (n + r)t + c(z + w)x)y + ((r + n)x + (w + z)p + cvt) = \lambda
\]

(1) : If \( vp + (n + r)t + c(z + w)x \neq 0 \) we can eliminate \( y \) from the equation, independent of \( \lambda \).

(2) : If \( vp + (n + r)t + c(z + w)x = 0 \) and \( v \neq 0 \) we have
\[
p = -\frac{n+r}{v}t - c\frac{z+w}{v}x
\]
and after substitution the equation becomes
\[
\frac{c}{3}(n^3 + r^3 + v^3 + w^3 + z^3) + 2rvx + ((r + n) - c\frac{(z + w)^2}{v})x + (cv - (n + r)(w + z))t = \lambda
\]
If \( v(r + n) - c(w + z)^2 \neq 0 \) we can eliminate \( x \) from the equation, and the remaining motive to consider, that is,
\[
[vp + (n + r)t + c(z + w)x = 0, \forall \neq 0, v(r + n) - c(w + z)^2 \neq 0]_{\lambda^2}
\]
does not depend on \( \lambda \).

If \( v(r + n) - c(w + z)^2 = 0 \) but \( cv^2 - (n + r)(w + z) \neq 0 \) we can eliminate \( t \), and again the resulting motive independent of \( \lambda \), so does not contribute.
(3) : We arrive at the first subcase which depends on $\lambda$. The defining equations of the locally closed subset of $\mathbb{A}^5$ (we have eliminated $p$ and the variables $y, x$ and $t$ are free) are

\[
\begin{align*}
&v \neq 0 \\
&v(r + n) - c(w + z)^2 = 0 \\
&cv^2 - (n + r)(w + z) = 0 \\
&\frac{c}{(n^3 + r^3 + w^3 + z^3)} + 2rvz = \lambda
\end{align*}
\]

From the first equation we obtain $r + n = \frac{c(w + z)^2}{v}$, and substituting this in the second equation gives

\[
v^3 = (w + z)^3 \quad \text{whence} \quad \begin{cases} 
 w = \rho v - z \\
 n = c\rho^2 v - r 
\end{cases}
\]

for $\rho^3 = 1$, so we have three subcases to consider which are clearly isomorphic, giving a factor $[\mu_3]$. If we substitute the obtained equations in the last equation, we obtain the locally closed subset in $\mathbb{A}^3$ (with remaining coefficients $r, v, z$)

\[
X_\lambda = \begin{cases} 
 v \neq 0 \\
 v(3p\rho z^2 - 3\rho^2 cvz + 6rz + (c^4 + 2c)v^2 - 3pc^3 rv + 3\rho^2 c^2 r^2) = 3\lambda
\end{cases}
\]

Therefore, this subcase contributes a term equal to

\[
\mathbb{L}^3.[\mu_3].([X_0] - [X_1])
\]

(4) : We have exhausted the $v \neq 0$ case, so from now on $v = 0$ and we have to solve in $\mathbb{A}^7$

\[
\begin{cases} 
 (n + r)t + c(z + w)x = 0 \\
 \frac{c}{(n^3 + r^3 + w^3 + z^3)} + (r + n)x + (w + z)p = \lambda
\end{cases}
\]

If $w + z \neq 0$ we can eliminate $x$ from the first equation, substitute it in the second and eliminate $p$ from the second, all this independent of $\lambda$.

(5) : If $w + z = 0$ we have

\[
\begin{cases} 
 (n + r)t = 0 \\
 \frac{c}{(n^3 + r^3)} + (r + n)x = \lambda
\end{cases}
\]

So, if $r + n \neq 0$ we must have that $t = 0$ and can eliminate $x$ from the second equation, independent of $\lambda$.

(6) : The remaining case is when $y, x, t$ and $p$ are free variables and we have

\[
\begin{cases} 
 v = 0 \\
 w + z = 0 \\
 r + n = 0
\end{cases}
\]

and the remaining equation is $0 = \lambda$. So, for $\lambda = 1$ we get no contribution, whereas for $\lambda = 0$ we get a contribution $\mathbb{L}^6$. \qed
Lemma 6. With $a = b = 1$ and $c \neq 0$ we have

$$[S_3(0)] - [S_3(1)] = L^7 + L^3, [\mu_3], ([X_0] - [X_1])$$

where $X_\lambda$ is the locally closed subset in $\mathbb{A}^3$ (with variables $x, y, z$) defined by

$$X_\lambda = \begin{cases} x \neq 0 \\ x(3pcz^2 - 3p^2cxz + 6yz + (c^4 + 2c)x^2 - 3pc^3xy + 3p^2c^2y^2) = 3\lambda \\ \end{cases}$$

and $\rho^3 = 1$.

Proof. The defining equation of $S_3(\lambda)$ in $\mathbb{A}^{10}$ is equal to

$$\frac{c}{3}(r^3 + s^3 + v^3 + w^3 + z^3) + 2rvz + ((w + z)p + c(v + s)t + rx)u +$$

$$(u+s)p + rt + c(z + w)x) + (crp + (z + w)t + (s + v)x) = \lambda$$

Again, we will split the computations into subcases and only work out those for which the difference of motives may depend on $\lambda$.

(1) : If $(w + z)p + c(v + s)t + rx \neq 0$ we can eliminate $u$ from the equation, independent of the value of $\lambda$.

(2) : If $(w + z)p + c(v + s)t + rx = 0$, $u$ is a free variable and the equation becomes

$$\frac{c}{3}(r^3 + s^3 + v^3 + w^3 + z^3) + 2rvz + ((w + z)p + rt + c(z + w)x)y + (crp + z + w)t + (s + v)x = \lambda$$

If $ry + (z + w) \neq 0$ we can eliminate $t$ from the equation, independent of $\lambda$.

(3) : If $(w + z)p + c(v + s)t + rx = 0$ and $ry + (z + w) = 0$ and $r \neq 0$, then we have the equations

$$\begin{cases} y = -\frac{r + w}{r}p - \frac{c(v + s)}{r}t \\ x = -\frac{w + z}{r}p - \frac{c(v + s)}{r}t \\ \end{cases}$$

and substitution gives us the equation

$$\frac{c}{3}(r^3 + s^3 + v^3 + w^3 + z^3) + 2rvz - \frac{z + w}{r}((w + z)p + c(z + w))(-\frac{w + z}{r}p - \frac{c(v + s)}{r}t) +$$

$$(c + s + v)(-\frac{w + z}{r}p - \frac{c(v + s)}{r}t)) = \lambda$$

The coefficient of $t$ is equal to $-\frac{c(v + s)^2 + (z + w)^2(v + s) - r(v + s)^2 \neq 0$ we can eliminate $t$ from the equation, independent of $\lambda$.

(4) : If $r \neq 0$, $(w + z)p + c(v + s)t + rx = 0$ and $ry + (z + w) = 0$ and $c(z + w)^2(v + s) - r(v + s)^2 = 0$, the equation becomes

$$\frac{c}{3}(r^3 + s^3 + v^3 + w^3 + z^3) + 2rvz + \frac{c(v + s)^3}{r^2} - 2 \frac{(z + w)(s + v)}{r} + cr)p = \lambda$$

That is, if $c(z + w)^3 - 2r(z + w)(s + v) + cr^3 \neq 0$ we can eliminate $p$, independent of $\lambda$.

(5) : The first subcase dependent on $\lambda$ is now that $u, p$ and $t$ are free variables and we have the following locally closed subset of $\mathbb{A}^5$ (in the remaining variables
Lemma 7. For the Brauer-Severi motives we have

\[
\begin{align*}
&\begin{cases}
  r \neq 0 \\
  c(z + w)^2(v + s) - r(v + s)^2 = 0 \\
  c(z + w)^3 - 2r(z + w)(s + v) + cr^3 = 0 \\
  \frac{c}{3}(r^3 + s^3 + v^3 + w^3 + z^3) + 2rvz = \lambda
\end{cases}
\end{align*}
\]

If \( v + s \neq 0 \) we have \( r(v + s) = c(z + w)^2 \) and substituting in the third equation gives \( r^3 = (z + w)^3 \) whence \( z + w = \rho r \) for \( \rho^3 = 1 \), but then also \( c\rho^2 r = v + s \). If we substitute

\[
\begin{cases}
  w = \rho r - z \\
  s = c\rho^2 - v
\end{cases}
\]

in the last equation, we get the locally closed subset in \( \mathbb{A}^3 \), isomorphic to \( X_\lambda \) of the previous case (interchanging the variables \( r \) and \( v \))

\[
X_\lambda = \begin{cases}
  r \neq 0 \\
  r(3\rho cz^2 + 6\rho z - 3\rho^2 crz + 3\rho^2 c^2 v^2 - 3\rho c^2 rv + (c^4 + 2c)r^2) = \lambda
\end{cases}
\]

Therefore, this subcase contributes a term equal to

\[
\mathbb{L}^3, [\mu_3], ([X_0] - [X_1])
\]

(6) : From now on we may assume that \( r = 0 \), together with \((w + z)p + c(v + s)t + rx = 0 \) and \( ry + (z + w) = 0 \). But then, \( z + w = 0 \) and the conditions are equivalent to the following system of equations in \( \mathbb{A}^5 \) (in the variables \( s, t, v, p, x, y \)). Observe that we have \( u \) and \( w \) as extra free variables

\[
\begin{cases}
  c(s + v)t = 0 \\
  \frac{c}{3}(s^3 + v^3) + (s + v)py + (s + v)x = \lambda
\end{cases}
\]

If \( s + v \neq 0 \) we have \( t = 0 \) and can eliminate \( x \) from the last equation, independent of \( \lambda \).

(7) : If \( s + v = 0 \) we have \( u, w, t, p, y, x, s \) as free variables and the remaining condition is \( 0 = \lambda \). That is, if \( \lambda = 1 \) there is no contribution and for \( \lambda = 0 \) we get a term \( \mathbb{L}^7 \).

Summing up the three contributions, we have:

**Lemma 7.** For the Brauer-Severi motives we have

\[
[B_{W_2}^W(0)] - [B_{W_2}^W(1)] = L^7 + L^6 + L^5 + 2L^3[\mu_3]([X_0] - [X_1])
\]

Therefore, the coefficient of \( t^2 \) in the series \( U_W(t) \) is equal to

\[
\frac{L^{-4} [M_{W_2}^W(0)] - [M_{W_2}^W(1)]}{[GL_2]} = \frac{L(L^3 - 1) + 2[\mu_3]([X_0] - [X_1])L^{-1}(L - 1) + L^{-2}N^2}{(L^2 - 1)(L - 1)}
\]

Remains to compute the motives \([X_\lambda]\) where

\[
X_\lambda = \begin{cases}
  x \neq 0 \\
  x(\rho cz^2 - \rho^2 c^2 xz + 2yz + \frac{c^4 + 2c}{3}x^2 - \rho c^3 xy + \rho^2 c^2 y^2) = \lambda
\end{cases}
\]
After performing the linear change of variables
\[
\begin{align*}
X &= \sqrt{\frac{c^4}{12}} x + i \frac{\sqrt{(c^3 - 1)^2}}{c} z \\
Y &= -\frac{c^2}{2} x + \rho cy + \frac{c^2}{2} z \\
Z &= \sqrt{\frac{a^4 + 8c}{12}} x - i \frac{\sqrt{(c^3 - 1)^2}}{c} z
\end{align*}
\]
we can express
\[
X_\lambda = \begin{cases} 
X + Z \neq 0 \\
(X + Z)(Y^2 + XZ) = \lambda 
\end{cases}
\]

**Lemma 8.** With notations as above we have
\[
[X_0] = (L - 1)^2 \quad \text{and} \quad [X_1] = (L - 1)^2 + [\mu_3]L
\]

**Proof.** We have \([X_0] = [Y^2 + XZ = 0]_{\Lambda^3} - [Y^2 + XZ = 0, X + Z = 0]_{\Lambda^3}\) which equals
\[
[Y^2 + XZ = 0]_{\Lambda^3} - [(X + Y)(X - Y) = 0]_{\Lambda^2} = L^2 - (2L - 1)
\]
As for \(X_1\), we have for every \(X + Z = a \neq 0\)
\[
[Y^2 - X^2 + aX = 1]_{\Lambda^2} = \begin{cases} 
L - 1 & \text{if } a^3 \neq 4 \\
2L - 1 & \text{if } a^3 = 4
\end{cases}
\]
as this is the affine part of a quadric \(Y^2 - X^2 + aXU - \frac{1}{a} U^2 = 0\) in \(\mathbb{P}^2\), having two points at infinity \(U = 0\), for every \(a \neq 0\). The quadric has a unique singular point \([1 : 0 : 1]\) if and only if \(a^3 = 4\). Therefore,
\[
[X_1] = (L - 1 - [\mu_3])(L - 1) + [\mu_3](2L - 1).
\]

**Theorem 1.** For the quaternionic Sklyanin algebra \(S_{1,1,c}\), we have that the coefficient of the second term in the motivic Donaldson-Thomas series \(U_W(t)\) is equal to
\[
L^{-4} \frac{[M_2^W(0)] - [M_2^W(1)]}{[GL_2]} = \frac{L^2(L^3 - 1) - 2[\mu_3]^2(L - 1) + L^{-2}N^2_1}{(L^2 - 1)(L - 1)}
\]

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