Upper Bound for Lebesgue Constant of Bivariate Lagrange Interpolation Polynomial on the Second Kind Chebyshev Points

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1.Introduction

Chebyshev polynomials play an important role in modern developments, including orthogonal polynomials, polynomial approximation, numerical integration, and spectral methods for partial differential equations (cf. [1]). Especially, the zeros of Chebyshev polynomials are often used in the studies of one-variable Lagrange interpolation polynomials. Many good approximation properties have been obtained over the past decades (cf. [2]). Since multivariate Lagrange interpolation polynomials are difficult to express concretely, many scholars are interested to study them (cf. [3–15]).

Let $K \subset \mathbb{R}^d$ be a nonempty compact set and $V$ be a subspace of $\Pi_n^d$, where $\Pi_n^d$ denotes the space of polynomials with $d$ variables whose degrees do not exceed $n$ and the dimension $\dim V = N$. Then, based on the nodes $X := \{x_k\}_{k=1}^N \subset K$, the Lagrange interpolation problem related to $V$ and $X$ can be described as follows: for any function $f \in C(K)$, where $C(K)$ represents the continuous function space on $K$, we can find a unique polynomial $p \in V$ to satisfy the equation

$$p(x_k) = f(x_k), \quad k = 1, \ldots, N. \quad (1)$$

This polynomial is the so-called Lagrange interpolation polynomial and can be expressed as

$$L_n(f, x) = \sum_{k=1}^N f(x_k)l_k(x), \quad (2)$$

where $l_k(x)$ are the Lagrange interpolation basis functions that satisfy the following formula:

$$l_k(x_j) = \delta_{kj}. \quad (3)$$

The mapping $f \mapsto L_n(f)$ can be regarded as an operator from $C(K)$ to itself, and the norm of the operator is defined as

$$\lambda_n = \|L_n\| = \max_{x \in K} \sum_{k=1}^N |l_k(x)|, \quad (4)$$

which is called the Lebesgue constant. We know that the uniform convergence of $L_n(f, x)$ for $f \in C(K)$ is closely related to the Lebesgue constant.
The univariate Lagrange interpolation polynomial and its Lebesgue constant have been extensively studied (cf. [2, 16]). Specially, for $K = [-1, 1]$ and $V = \prod_{n}^d$, the Lebesgue constant $L_n \geq C \log n$ and the order of the Lebesgue constant is $O(\log n)$ when the Chebyshev points are taken as the nodes (cf. [16]).

There are relatively few research results on multivariate Lagrange interpolation polynomials. In [3], from Berman’s Theorem, it is shown that for $K = B^d$, the unit ball in $R^d$, $d \geq 2$, and $V = \prod_{n}^d$, the order of the Lebesgue constant is $O((n^{(d-1)/2})$.

It is well known that the Lagrange interpolation polynomial is closely related to cubature formula. Möller (cf. [4]) stated that for centrally symmetric weight functions, the Lebesgue constant of the Xu-type Lagrange interpolation polynomial is $O(n/(\log n)^{(d-1)/2})$.

Our result gives that the growth order of the Lebesgue constant of Xu-type Lagrange interpolation polynomial on the second kind Chebyshev polynomial on the square $[-1, 1]^2$ is $O((n + 2)^2)$. Obviously, it is different from the Lebesgue constant on the disk $B^2$, the growth order of which is $O(\sqrt{n})$, and is different from the Lebesgue constant of Xu-type Lagrange interpolation polynomial on the first kind Chebyshev polynomial on $[-1, 1]^2$, the growth order of which is $O((\log n)^2)$.

### 2. The Lebesgue Constant of Xu-Type Lagrange Interpolation Polynomial on the Second Kind Chebyshev Polynomial

In order to prove Theorem 1, by using reproducing kernel, we give the expression of the Lebesgue constant $\lambda_n$ in this section.

First, we briefly introduce the Xu-type Lagrange interpolation polynomial on the second kind Chebyshev polynomial in [10].

Let $\mathbb{N}_0$ denote the set of nonnegative integers. For $n \in \mathbb{N}_0$, Chebyshev polynomial of the second kind $U_n(x)$ (cf. [17]) is defined by

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin \theta}, \quad x = \cos \theta, \quad \theta \in [0, \pi],$$

and they are orthogonal polynomials with respect to the second kind Chebyshev weight $w_1(x) = (2/n)\sqrt{1-x^2}$.

The product Chebyshev polynomial of the second kind of degree $n$ on $[-1, 1]^2$ (cf. [5]) is defined by

$$P^2_n(x, y) = U_{n-k}(x)U_k(y), \quad (x, y) \in [-1, 1]^2, \quad k = 0, 1, \ldots, n, \quad n \in \mathbb{N}_0,$$

and correspondingly, the product Chebyshev weight function of the second kind is

$$W_1(x, y) = w_1(x)w_1(y) = \frac{4}{n^2} \sqrt{1-x^2} \sqrt{1-y^2}, \quad (x, y) \in [-1, 1]^2.$$

For $x, y \in [-1, 1]^2$, the reproducing kernel of the product Chebyshev polynomials is defined by

$$K_n(x, y) = \sum_{k=0}^{n-1} \sum_{j=0}^{k} P^2_j(x)P^2_j(y), \quad n = 1, 2, \ldots$$

### Theorem 1

For $K = [-1, 1]^2$, the upper bound estimate of the Lebesgue constant of Xu-type Lagrange interpolation polynomial on the second kind Chebyshev polynomial in [10] is

$$\lambda_n \leq 160(\sqrt{2}(n + 2)^2).$$
\[ L_n(f, x) = \sum_{k=1}^{N} f(x_k) l_k(x) \]

\[ = \sum_{k=1}^{N} f(x_k) \frac{K^*_n(x, x_k)}{K^*_n(x_k, x_k)} \]

\[ = \sum_{i=1}^{[(n+1)/2]} \sum_{j=0}^{[n/2]} \lambda_{2i,2j+1} f(x_{2i,2j+1}) K^*_n(x, x_{2i,2j+1}) + \sum_{i=1}^{[n/2]} \sum_{j=0}^{[(n+1)/2]} \lambda_{2i+1,2j} f(x_{2i+1,2j}) K^*_n(x, x_{2i+1,2j}), \]

where \( x = (x, y) \in [-1, 1]^2 \), \( K^*_n(x, y) = (1/2) [K_{n+1}(x, y) + K_n(x, y)] \).

\[ \lambda_{2i,2j+1} = \left[ K^*_n(x_{2i,2j+1}, x_{2i,2j+1}) \right]^{-1} \]

\[ = \frac{8}{(n+2)^2} \left( \sin \theta_{2j} \right)^2, \]

\[ \lambda_{2i+1,2j} = \left[ K^*_n(x_{2i+1,2j}, x_{2i+1,2j}) \right]^{-1} \]

\[ = \frac{8}{(n+2)^2} \left( \sin \theta_{2i+1} \right)^2, \quad i, j = 0, 1, \ldots, \left[ \frac{n+1}{2} \right]. \]

Obviously, the node number \( N \) of formula (11) is

\[ 2 \left\lceil (n + 1)/2 \right\rceil \left\lceil [n/2] + 1 \right\rceil. \]

When \( n = 2m \), \( N = n(n + 2)/2 \), which reaches the lower bound \( \dim \Pi_{n-1} + [n/2] \). When \( n = 2m - 1 \), \( N = (n + 1)^2/2 \), which is one more than the lower bound (cf. [4]).

\[ \lambda_n = \max_{x \in [-1, 1]^2} \sum_{k=1}^{N} \frac{K^*_n(x, x_k)}{K^*_n(x, x_k)} \]

is called Lebesgue constant of Xu-type Lagrange interpolation polynomial on the second kind Chebyshev polynomial. Writing...
where $\varepsilon_k = (-1)^k$, $k = 1, 2$, $x = (\cos \varphi_1, \cos \varphi_2)$, and $y = (\cos \phi_1, \cos \phi_2)$. Then,

$$F_n(r_1, r_2) = \frac{[\cos (n + 2)r_1 - \cos (n + 2)r_2] + [\cos (n + 1)r_1 - \cos (n + 1)r_2]}{4(\cos r_1 - \cos r_2)}. \quad (18)$$

**Lemma 1.** If $\phi_1 = \theta_{2i}$ and $\phi_2 = \theta_{2j+1}$, $i = 1, 2, \ldots$, $[n/2]$ and $j = 0, 1, \ldots, [n/2]$, then

$$\sin (n + 2) \frac{\varphi_1 + \varepsilon_k \varphi_1 + \varphi_2 + \varepsilon_l \varphi_2}{2} = (-1)^{i_0 + j_0} \varepsilon_k \cos (n + 2) \frac{\varphi_1 + \varphi_2}{2}, \quad (19)$$

$$\sin (n + 2) \frac{\varphi_1 + \varepsilon_k \varphi_1 - \varphi_2 - \varepsilon_l \varphi_2}{2} = (-1)^{i_0 - j_0} (-\varepsilon_l) \cos (n + 2) \frac{\varphi_1 - \varphi_2}{2}. \quad (20)$$

If $\phi_1 = \theta_{2i+1}$ and $\phi_2 = \theta_{2j}$, $i = 0, 1, 2, \ldots, [n/2]$ and $j = 1, 2, \ldots, [n + 1)/2]$, then

$$\sin (n + 2) \frac{\varepsilon_k \varphi_1 + \varphi_2 + \varepsilon_l \varphi_2}{2} = (-1)^{i_0 + j_0} \varepsilon_k \cos (n + 2) \frac{\varphi_1 + \varphi_2}{2}, \quad (21)$$

$$\sin (n + 2) \frac{\varepsilon_k \varphi_1 - \varphi_2 - \varepsilon_l \varphi_2}{2} = (-1)^{i_0 - j_0} \varepsilon_k \cos (n + 2) \frac{\varphi_1 - \varphi_2}{2}.$$

where $\varepsilon_k = (-1)^k$, $k = 1, 2$, and $\varepsilon_l = (-1)^l$, $l = 1, 2$.

We only prove formula (19); other formulae can be proved similarly. For $\phi_1 = \theta_{2i}, \phi_2 = \theta_{2j+1}, \varepsilon_k = (-1)^k$, and $\varepsilon_l = (-1)^l$, we have

$$\cos (n + 2) \frac{\varepsilon_k \varphi_1 + \varphi_2}{2} = 0. \quad (22)$$

Hence,

$$\sin (n + 2) \frac{\varphi_1 + \varepsilon_k \varphi_1 + \varphi_2 + \varepsilon_l \varphi_2}{2} = \cos (n + 2) \frac{\varphi_1 + \varphi_2}{2} \sin \left[\left(i_0 + j_0\right)\pi + \frac{\pi}{2} \varepsilon_l\right]$$

$$= (-1)^{i_0 + j_0} \varepsilon_k \cos (n + 2) \frac{\varphi_1 + \varphi_2}{2}. \quad (23)$$

**Lemma 2.** If $\phi_1 = \theta_{2i}$ and $\phi_2 = \theta_{2j+1}, i = 1, 2, \ldots, [n + 1)/2] and j = 0, 1, \ldots, [n/2]$, then

$$\sum_{k=1}^{2} \sum_{l=1}^{2} \varepsilon_k \varepsilon_l \left[F_n(\varphi_1 + \varepsilon_k \varphi_1, \varphi_2 + \varepsilon_l \varphi_2) + F_{n+1}(\varphi_1 + \varepsilon_k \varphi_1, \varphi_2 + \varepsilon_l \varphi_2)\right]$$

$$= \sum_{k=1}^{2} \sum_{l=1}^{2} \varepsilon_k \varepsilon_l \cos^2 \frac{\varphi_1 + \varepsilon_k \varphi_1}{2} U_{n+1} \left(\frac{\varphi_1 + \varepsilon_k \varphi_1 + \varphi_2 + \varepsilon_l \varphi_2}{2}\right) \times U_{n+1} \left(\frac{\varphi_1 + \varepsilon_k \varphi_1 - \varphi_2 - \varepsilon_l \varphi_2}{2}\right) \quad (24)$$

$$= \sum_{k=1}^{2} \sum_{l=1}^{2} \varepsilon_k \varepsilon_l \frac{2 \cos^2 \left(\frac{\varphi_1 + \varepsilon_k \varphi_1}{2}\right) \cos (n + 2)(\varphi_1 + \varphi_2) \cos (n + 2)(\varphi_1 - \varphi_2)}{\cos (\varphi_1 + \varepsilon_k \varphi_1) - \cos (\varphi_2 + \varepsilon_l \varphi_2)}. \quad (24)$$
If \( \phi_1 = \theta_{2^i+1} \) and \( \phi_2 = \theta_{2^j} \), \( i = 0, 1, 2, \ldots, [n/2] \) and \( j = 1, 2, \ldots, [(n+1)/2] \), then

\[
\sum_{k=1}^{2} \sum_{l=1}^{2} \epsilon_k \epsilon_l [F_n(\phi_1 + \epsilon_k \phi_1, \phi_2 + \epsilon_l \phi_2) + F_{n+1}(\phi_1 + \epsilon_k \phi_1, \phi_2 + \epsilon_l \phi_2)]
\]

\[
= \sum_{k=1}^{2} \sum_{l=1}^{2} \epsilon_k \epsilon_l \cos^2 (\phi_1 + \epsilon_k \phi_1 + \phi_2 + \epsilon_l \phi_2) \times U_{n+1}(\phi_1 + \epsilon_k \phi_1 - \phi_2 - \epsilon_l \phi_2)
\]

\[
= -\sum_{k=1}^{2} \sum_{l=1}^{2} \epsilon_k \epsilon_l 2 \cos^2 ((\phi_1 + \epsilon_k \phi_1)/2) \cos(n+2)((\phi_1 + \phi_2)/2) \cos(n+2)((\phi_1 - \phi_2)/2)
\]

\[
\cos(\phi_1 + \epsilon_k \phi_1) - \cos(\phi_2 + \epsilon_l \phi_2)
\]

where \( \epsilon_k = (-1)^k, k = 1, 2 \), and \( \epsilon_l = (-1)^l, l = 1, 2 \). By (18), we have

\[
F_n(\tau_1, \tau_2) + F_{n+1}(\tau_1, \tau_2)
\]

\[
= \frac{\cos(\tau_1/2)(\cos((2n+3)/2)\tau_1 + \cos((2n+5)/2)\tau_2) - \cos(\tau_2/2)(\cos((2n+3)/2)\tau_1 + \cos((2n+5)/2)\tau_2)}{4 \sin((\tau_1 + \tau_2)/2)\sin((\tau_1 - \tau_2)/2)}
\]

\[
= -\frac{\cos(n+2)\tau_1 \cos^2(\tau_1/2) - \cos(n+2)\tau_2 \cos^2(\tau_2/2)}{2 \sin((\tau_1 + \tau_2)/2)\sin((\tau_1 - \tau_2)/2)}
\]

\[
= \frac{[\cos(n+2)\tau_1 - \cos(n+2)\tau_2] \cos^2(\tau_1/2) + \cos(n+2)\tau_2 \cos^2(\tau_2/2) - \cos^2(\tau_2/2)}{2 \sin((\tau_1 + \tau_2)/2)\sin((\tau_1 - \tau_2)/2)}
\]

\[
= \frac{\cos^2 \tau_1/2 U_{n+1}(\tau_1 + \tau_2/2) U_{n+1}(\tau_1 - \tau_2/2)}{2} + \frac{1}{2} \cos(n+2)\tau_2.
\]

From (19) and (20), we can obtain (24). And, we can similarly prove (25). □

**Lemma 3.** The following relation holds:

\[
I = \frac{\sum_{k=1}^{2} \sum_{l=1}^{2} \epsilon_k \epsilon_l \cos^2 ((\phi_1 + \epsilon_k \phi_1)/2)}{\cos(\phi_1 + \epsilon_k \phi_1) - \cos(\phi_2 + \epsilon_l \phi_2)}
\]

\[
= 4 \sin \phi_1 \sin \phi_2 \sin \phi_2,
\]

\[
\prod_{k=1}^{2} \prod_{l=1}^{2} \left[ \cos(\phi_1 + \epsilon_k \phi_1) - \cos(\phi_2 + \epsilon_l \phi_2) \right],
\]

\[\left\{ - \cos^2 ((\phi_1 - \phi_1)/2) \left[ \cos(\phi_1 + \phi_1) - \cos(\phi_2 - \phi_2) \right] - \cos^2 ((\phi_2 - \phi_2)/2) \left[ \cos(\phi_1 - \phi_1) - \cos(\phi_2 + \phi_2) \right] \right\},
\]

\[
\prod_{k=1}^{2} \prod_{l=1}^{2} \left[ \cos(\phi_1 + \epsilon_k \phi_1) - \cos(\phi_2 + \epsilon_l \phi_2) \right],
\]

\[\left\{ - \cos^2 ((\phi_1 - \phi_1)/2) \left[ \cos(\phi_1 + \phi_1) - \cos(\phi_2 - \phi_2) \right] - \cos^2 ((\phi_2 - \phi_2)/2) \left[ \cos(\phi_1 - \phi_1) - \cos(\phi_2 + \phi_2) \right] \right\},
\]
where $\varepsilon_k = (-1)^k, k = 1, 2,$ and $\varepsilon_l = (-1)^l, l = 1, 2$. We have

\[
I = \frac{2}{\cos(\phi_1 + \varepsilon_k \phi_1) - \cos(\phi_2 + \varepsilon_k \phi_2)} \sum_{k=1}^{2} \varepsilon_k \varepsilon_l \left( \cos^2 \left( \frac{\phi_1 + \varepsilon_k \phi_1}{2} \right) - \cos(\phi_2 - \phi_2) + \cos(\phi_2 + \phi_2) \right)
\]

\[
= \cos^2 \phi_1 - \phi_1 - \frac{\cos(\phi_2 - \phi_2) + \cos(\phi_2 + \phi_2)}{2} \frac{\cos(\phi_1 + \phi_1) - \cos(\phi_2 + \phi_2)}{[\cos(\phi_1 + \phi_1) - \cos(\phi_2 + \phi_2)] [\cos(\phi_1 + \phi_1) - \cos(\phi_2 + \phi_2)]}
\]

\[
= \frac{2 \sin \phi_2 \sin \phi_1 \sin \phi_2}{2} \left[ \frac{\cos^2 \left( \frac{\phi_1 + \phi_1}{2} \right)}{\cos(\phi_1 + \phi_1) - \cos(\phi_2 + \phi_2)} \right]
\]

\[
= \frac{4 \cos^2 \left( \frac{\phi_1 - \phi_1}{2} \right) \cos \phi_1 \cos \phi_1 - \cos \phi_2 \cos \phi_2}{\prod_{k=1}^{2} \prod_{l=1}^{2} \cos(\phi_1 + \varepsilon_k \phi_1) - \cos(\phi_2 + \varepsilon_l \phi_2)}
\]

\[
= \frac{4 \sin \phi_1 \sin \phi_1 \sin \phi_2}{\prod_{k=1}^{2} \prod_{l=1}^{2} \cos(\phi_1 + \varepsilon_k \phi_1) - \cos(\phi_2 + \varepsilon_l \phi_2)}
\]

By Lemmas 2 and 3, the following result can be obtained.

Lemma 4. If $x = (\cos \phi_1, \cos \phi_2), y = (\cos \phi_1, \cos \phi_2), \phi_1 = \theta_2, \phi_2 = \theta_2 + i, i = 1, 2, \ldots, [(n + 1)/2]$ and $j = 0, 1, \ldots, [n/2]$, then

\[
K_n^*(x, y) = \frac{\cos(n + 2) \left( (\phi_1 + \phi_2)/2 \right) \cos(n + 2) \left( (\phi_1 - \phi_2)/2 \right) - \cos^2 \left( \frac{(\phi_1 + \phi_2)}{2} \right)}{4 \sin \phi_1 \sin \phi_2 \sin \phi_1 \sin \phi_2} I
\]

\[
= \cos(n + 2) \frac{\phi_1 + \phi_2}{2} \cos(n + 2) \frac{\phi_1 - \phi_2}{2}
\]

\[
= \frac{\left\{ - \cos^2 \left( \frac{(\phi_1 - \phi_1)}{2} \right) \left[ \cos(\phi_1 + \phi_1) - \cos(\phi_2 + \phi_2) \right] \right\}}{\prod_{k=1}^{2} \prod_{l=1}^{2} \cos(\phi_1 + \varepsilon_k \phi_1) - \cos(\phi_2 + \varepsilon_l \phi_2)}
\]
If $\phi_1 = \theta_{2j+1}$ and $\phi_2 = \theta_{2j}$, $i = 0, 1, 2, \ldots$, $[n/2]$ and $j = 1, 2, \ldots, [(n + 1)/2]$, then

$$K^*_n(x, y) = \frac{-\cos(n + 2)((\phi_1 + \phi_2)/2)\cos(n + 2)((\phi_1 - \phi_2)/2)}{4 \sin \phi_1 \sin \phi_2 \sin \phi_1 \sin \phi_2}$$

$$= \cos(n + 2)\frac{\phi_1 + \phi_2}{2} \cos(n + 2)\frac{\phi_1 - \phi_2}{2}$$

$$\left\{ \cos^2((\phi_1 - \phi_2)/2)[\cos(\phi_1 + \phi_1) - \cos(\phi_1 - \phi_2)] + \cos^2((\phi_2 - \phi_2)/2)[\cos(\phi_1 - \phi_1) - \cos(\phi_2 + \phi_2)] \right\}$$

$$\prod_{k=1}^{n/2} \prod_{j=1}^{(n-1)/2} \cos(\phi_1 + \epsilon_k \phi_1) - \cos(\phi_2 + \epsilon_k \phi_2)$$

Furthermore, we can obtain the following lemma.

Lemma 5. Let $x = (\cos \phi_1, \cos \phi_2)$ and $y = (\cos \phi_1, \cos \phi_2)$. If $\phi_1 = \theta_2$ and $\phi_2 = \theta_{2j+1}$, $i = 1, 2, \ldots, [(n + 1)/2]$ and $j = 0, 1, \ldots, [n/2]$, then

$$\Lambda^*_n(x) = \sum_{i=1}^{[n+1]/2} \sum_{j=0}^{[n]/2}$$

$$\left( \frac{(\sin \phi_1 \sin \phi_2)^2}{(n + 2)^2} \cos(n + 2)\frac{\phi_1 + \phi_2}{2} \cos(n + 2)\frac{\phi_1 - \phi_2}{2} \right)$$

$$\sin((\phi_1 + \phi_1 + \phi_2 + \phi_2)/2) \sin((\phi_1 + \phi_1 - \phi_2 - \phi_2)/2) \sin((\phi_1 + \phi_1 - \phi_2 - \phi_2)/2)$$

$$\sin((\phi_1 + \phi_1 + \phi_2 + \phi_2)/2) \sin((\phi_1 + \phi_1 - \phi_2 - \phi_2)/2) \sin((\phi_1 + \phi_1 - \phi_2 - \phi_2)/2)$$

$$\cos^2((\phi_1 - \phi_2)/2)$$

$$\cos^2((\phi_2 - \phi_2)/2)$$

$$\cos^2((\phi_1 - \phi_1)/2)$$

$$\prod_{k=1}^{n/2} \prod_{j=1}^{(n-1)/2} \cos(\phi_1 + \epsilon_k \phi_1) - \cos(\phi_2 + \epsilon_k \phi_2)$$

If $\phi_1 = \theta_{2j+1}$ and $\phi_2 = \theta_{2j}$, $i = 0, 1, \ldots, (n/2)$ and $j = 1, 2, \ldots, [(n + 1)/2]$, then

$$\Lambda^*_n(x) = \sum_{i=0}^{[n]/2} \sum_{j=1}^{(n+1)/2}$$

$$\left( \frac{(\sin \phi_1 \sin \phi_2)^2}{(n + 2)^2} \cos(n + 2)\frac{\phi_1 + \phi_2}{2} \cos(n + 2)\frac{\phi_1 - \phi_2}{2} \right)$$

$$\sin((\phi_1 + \phi_1 + \phi_2 + \phi_2)/2) \sin((\phi_1 + \phi_1 - \phi_2 - \phi_2)/2) \sin((\phi_1 + \phi_1 - \phi_2 - \phi_2)/2)$$

$$\sin((\phi_1 + \phi_1 + \phi_2 + \phi_2)/2) \sin((\phi_1 + \phi_1 - \phi_2 - \phi_2)/2) \sin((\phi_1 + \phi_1 - \phi_2 - \phi_2)/2)$$

$$\cos^2((\phi_1 - \phi_2)/2)$$

$$\cos^2((\phi_2 - \phi_2)/2)$$

$$\cos^2((\phi_1 - \phi_1)/2)$$

$$\prod_{k=1}^{n/2} \prod_{j=1}^{(n-1)/2} \cos(\phi_1 + \epsilon_k \phi_1) - \cos(\phi_2 + \epsilon_k \phi_2)$$
3. Proof of Theorem 1

The proof of Theorem 1 is given in this section. And, since the estimates of $\Lambda_2^q(x)$ and $\Lambda_{n}^q(x)$ are similar, we need to only estimate $\Lambda_{n}^1(x)$. Setting $\tau_1 = (\phi_1 + \phi_2)/2 \in [0, \pi]$, $\tau_2 = (\phi_1 - \phi_2)/2 \in [-\pi/2, (\pi/2)]$, we have

$$
\frac{K_n^*(x, x_{j_1+1})}{K_n^*(x_{i+1}, x_{i+2})} = A_{i,j}^{(1)} + B_{i,j}^{(1)} = A_{i,j}^{(11)} A_{i,j}^{(12)} + B_{i,j}^{(11)} B_{i,j}^{(12)},
$$

where

$$
A_{i,j}^{(11)} = \frac{\sin \theta_{i+1} \sin \theta_{j+1}}{n + 2} \cdot \frac{\cos(n + 2) \sin(\phi_{1} - \theta_{i+1})}{\sin(\tau_{2} + (\theta_{i+1} - \theta_{j+1})/2) \sin(\tau_{2} + (\theta_{i+1} + \theta_{j+1})/2) \sin(\tau_{2} - (\theta_{i+1} - \theta_{j+1})/2)}.
$$

$$
A_{i,j}^{(12)} = \frac{\sin \theta_{i+1} \sin \theta_{j+1}}{n + 2} \cdot \frac{\cos(n + 2) \sin(\phi_{1} - \theta_{i+1})}{\sin(\tau_{1} + (\theta_{i+1} + \theta_{j+1})/2) \sin(\tau_{1} - (\theta_{i+1} - \theta_{j+1})/2) \sin(\tau_{1} - (\theta_{i+1} + \theta_{j+1})/2)}.
$$

$$
B_{i,j}^{(11)} = \frac{\sin \theta_{i+1} \sin \theta_{j+1}}{n + 2} \cdot \frac{\cos(n + 2) \sin(\phi_{2} - \theta_{i+1})}{\sin(\tau_{2} + (\theta_{i+1} - \theta_{j+1})/2) \sin(\tau_{2} - (\theta_{i+1} - \theta_{j+1})/2) \sin(\tau_{2} - (\theta_{i+1} + \theta_{j+1})/2)}.
$$

$$
B_{i,j}^{(12)} = \frac{\sin \theta_{i+1} \sin \theta_{j+1}}{n + 2} \cdot \frac{\cos(n + 2) \sin(\phi_{2} - \theta_{i+1})}{\sin(\tau_{1} + (\theta_{i+1} - \theta_{j+1})/2) \sin(\tau_{1} + (\theta_{i+1} + \theta_{j+1})/2) \sin(\tau_{1} - (\theta_{i+1} + \theta_{j+1})/2)}.
$$

To prove Theorem 1, we first prove some lemmas.

**Lemma 6.** If $|\tau_2| \in [0, (\theta_1/2)]$ and $\tau_1 \in [0, (\theta_1/2)] \cup [\pi - (\theta_1/2), \pi]$, we have the following:

1. For $0 \leq j \leq i - 2$ or $i + 1 \leq j \leq [n/2]$,

$$
|A_{i,j}^{(11)}| \leq \frac{4\sin \theta_{i+1} \cos(n + 2) \tau_{2} \sin \theta_{j+1}}{(n + 2) \sin((\theta_{i+1} - \theta_{j+1})/2))}.
$$

(35)

For $j = i$ or $j = i - 1$,

$$
|A_{i,j}^{(11)}| \leq 2(n + 2).
$$

(36)

2. For $0 \leq j \leq i - 2$,

$$
|A_{i,j}^{(12)}| \leq \frac{8\cos(n + 2) \tau_{1} \sin \theta_{j+1}}{(n + 2) \sin((\theta_{i+1} - \theta_{j+1})/2))}.
$$

(37)

For $i + 1 \leq j \leq [n/2]$,

$$
\sin\left(\frac{\theta_{i+1} + \theta_{j+1}}{2} - \tau_{1}\right) \geq \min\left\{ \sin\left(\frac{\theta_{i+1} + \theta_{j+1}}{2}\right), \sin\left(\frac{\theta_{i+1} + \theta_{j+1}}{2} - \frac{\theta_{1}}{2}\right) \right\} \geq \frac{1}{2} \sin\left(\frac{\theta_{i+1} + \theta_{j+1}}{2}\right) \geq \frac{1}{2} \sqrt{\sin \theta_{i+1} \sin \theta_{j+1}}.
$$

(41)

For $0 \leq j \leq i - 2$, $\tau_2 = (\theta_{i+1} - \theta_{j+1})/2 - \tau_1 \leq (\theta_{i+1} + \theta_{j+1})/2 - \tau_1 \leq (\theta_{i+1} - \theta_{j+1})/2 \leq \pi/2$, then

$$
\sin\left(\frac{\theta_{i+1} - \theta_{j+1}}{2} - \tau_{1}\right) \geq \sin\left(\frac{\theta_{i+1} - \theta_{j+1}}{2} - \frac{\theta_{1}}{2}\right) \geq \frac{1}{2} \sin\left(\frac{\theta_{i+1} - \theta_{j+1}}{2}\right) \geq \frac{1}{2} \sqrt{\sin \theta_{i+1} \sin \theta_{j+1}}.
$$

(42)
By (40)–(42), we can obtain (35).

\[
\sin\left(\frac{\theta_{2j+1} - \theta_{2i} + \tau_j}{2}\right) \geq \sin\left(\frac{\theta_{2j+1} - \theta_{2i}}{2}\right) \geq \frac{1}{2} \sin\left(\frac{\theta_{2j+1} - \theta_{2i}}{2}\right). \tag{43}
\]

For \((\theta_{2j+1} - \theta_{2i})/2 \leq (\theta_{2j+1} - \theta_{2i})/2 + \tau_j \leq (\theta_{2j+1} - \theta_{2i})/2 + \theta_j/2 \leq \theta_j \leq \pi/2\), we have

\[
\sin\left(\frac{\theta_{2j+1} - \theta_{2i} + \tau_j}{2}\right) \geq \sin\left(\frac{\theta_{2j+1} - \theta_{2i}}{2}\right). \tag{44}
\]

and combining (41), we can obtain (35).

\[
\sin\left(\frac{\theta_{2i} + \theta_{2j+1} + \tau_j}{2}\right) \geq \min\left\{ \sin\left(\frac{\theta_{2i} + \theta_{2j+1}}{2}\right), \sin\left(\frac{\theta_{2i} + \theta_{2j+1} + \theta_j}{2}\right) \right\} \geq \frac{1}{2} \sin\left(\frac{\theta_{2i} + \theta_{2j+1}}{2}\right) \geq \frac{1}{2} \sin\theta_{2i} \sin\theta_{2j+1}. \tag{45}
\]

And, we have

\[
\sin\left(\frac{\theta_{2i} + \theta_{2j+1} + \tau_j}{2}\right) \geq \frac{1}{2} \sqrt{\sin \theta_{2i} \sin \theta_{2j+1}}, \tag{46}
\]

so we obtain (37).

For \(i + 1 \leq j \leq [n/2]\), considering

\[
\sin\left(\frac{\theta_{2i} + \theta_{2j+1} + \tau_j}{2}\right) \geq \frac{1}{2} \sqrt{\sin \theta_{2i} \sin \theta_{2j+1}}, \tag{47}
\]

we have (38).

For \(j = i \) or \(j = i - 1\), it is easy to prove that \(|A_{i,j}^{(12)}| \leq 4\).

When \(\tau_1 \in [\pi - (\theta_j/2), \pi]\), setting \(\tau'_1 = \pi - \tau_1 \in [0, (\theta_j/2)]\), similar to the case of \(\tau_1 \in [0, (\theta_j/2)]\), the estimation of \(|A_{i,j}^{(12)}|\) can be obtained.

In the same way, we can obtain the following estimates of \(|B_{i,j}^{(11)}|\) and \(|B_{i,j}^{(12)}|\).

**Lemma 7.** If \(|\tau_j| \leq |(\theta_j/2)|\) and \(\tau_1 \in [0, (\theta_j/2)]\) \(\cup \) \(\{\pi - (\theta_j/2), \pi]\), then we have the following:

1. For \(0 \leq j \leq i - 2\) or \(i + 1 \leq j \leq [n/2]\),

\[
|B_{i,j}^{(11)}| \leq 4 \frac{\sin \theta_{2i} \sin \theta_{2j+1} \cos(n+2)\tau_j}{(n+2) \sin^2 \left(\theta_{2i} - \theta_{2j+1}/2\right)}. \tag{48}
\]

2. For \(0 \leq j \leq i - 2\),

\[
|B_{i,j}^{(12)}| \leq \frac{4 \cos(n+2)\tau_1}{(n+2) \sin \left(\theta_{2i} - \theta_{2j+1}/2\right)} \tag{49}
\]

3. For \(i + 1 \leq j \leq [n/2]\),

\[
|B_{i,j}^{(12)}| \leq \frac{8 \cos(n+2)\tau_1}{(n+2) \sin \left(\theta_{2i} - \theta_{2j+1}/2\right)} \tag{50}
\]

4. For \(j = i \) or \(j = i - 1\), \(|B_{i,j}^{(12)}| \leq 4\sqrt{2}\).

**Lemma 8.** If \(\tau_1 \in [(\theta_j/2), (\pi/4)] \cup [(3\pi/4), \pi - (\theta_j/2)]\), then \(|A_{i,j}^{(12)}| \leq 4\sqrt{2}, |B_{i,j}^{(11)}| \leq 4\sqrt{2}\).

1. We first prove the case of \(\tau_1 \in [(\theta_j/2), (\pi/4)]\). If \(0 < (\theta_{2i} + \theta_{2j+1})/2 < \pi/2\), we have

\[
\sin\left(\frac{\theta_{2i} + \theta_{2j+1} + \tau_j}{2}\right) \geq \frac{\sqrt{2}}{2} \sin\left(\frac{\theta_{2i} + \theta_{2j+1}}{2}\right) \geq \frac{\sqrt{2}}{4} \sin \theta_{2i}, \tag{51}
\]

2. For \((\pi/2) \leq (\theta_{2i} + \theta_{2j+1})/2 \leq \pi\), we obtain

\[
\cos(n+2)\tau_1 \sin \theta_{2i+1} \sin \left(\frac{\theta_{2i} - \theta_{2j+1}/2}{\sin \left(\tau_1 - \left(\theta_{2i} + \theta_{2j+1}/2\right)\right)} \right) \leq \cos(n+2)\tau_1 \cos\left(\tau_1 - \left(\theta_{2i} + \theta_{2j+1}/2\right)\right) \leq 2(n+2). \tag{52}
\]
Lemma 9. If \( \tau_1 \in [(0, \pi/2), (\pi/2), \pi - (\pi/2)] \), by setting \( \tau_1 = \pi - \tau_1 \in [\pi, 2\pi] \), the conclusion of the lemma can be proved.

Similar to Lemma 8, we can obtain the following conclusion.

\[
\sin\left(\frac{\theta_{2i} + \theta_{2j+1} - \tau_1}{2}\right) \geq \frac{\sqrt{2}}{2} \sin\left(\frac{\theta_{2i} + \theta_{2j+1}}{2}\right) \geq \frac{\sqrt{2}}{4} \sin \theta_{2j+1},
\]

(53)

\[
\cos(n + 2\tau_1) \sin \theta_{2j} \
\sin(\tau_1 + (\theta_{2i} + \theta_{2j+1})/2) \sin(\tau_1 - (\theta_{2i} - \theta_{2j+1})/2) \leq 2(n + 2).
\]

(54)

In summary, we can obtain \(|A_{i,j}^{(1)}| \leq 4\sqrt{2}\).

(2) If \( \tau_1 \in [(3\pi/4), \pi - (\pi/2)] \), by setting \( \tau_1 = \pi - \tau_1 \in [\pi, 2\pi] \) and the same as the case of \( \tau_1 \in [(\pi/2), (\pi/4)] \), the conclusion of the lemma can be proved.

Lemma 10. If \( |\tau_2| \in [(\pi/2), (\pi/4)] \), then \( |A_{i,j}^{11}| \leq 4, |B_{i,j}^{11}| \leq 4 \).

The estimates of \(|A_{i,j}^{11}| \) and \(|B_{i,j}^{11}| \) are similar, so we only take \(|A_{i,j}^{11}| \) as an example.

We first prove the case of \( \tau_2 \in [(\pi/2), (\pi/4)] \). For every \( i \), there is \( j_0 \) so that \( |\tau_2 - (\theta_{2i} + \theta_{2j+1})/2| \leq (\pi/2) \) holds, that is, \( (\theta_{2i} + \theta_{2j+1})/2 - (\pi/2) \leq \tau_2 \leq (\theta_{2i} + \theta_{2j+1})/2 + (\pi/2) \).

(i) For \( j = j_0 \), we have \( \theta_{2i} - (\theta_{2i}/2) \leq \tau_2 + (\theta_{2i} - \theta_{2j+1})/2 \leq \theta_{2i} + (\theta_{2i}/2); \) then,

\[
\sin\left(\frac{\theta_{2i} + \theta_{2j+1} - \tau_2}{2}\right) \geq \min \left\{ \sin\left(\frac{\theta_{2i} + \theta_{1}}{2}\right), \sin\left(\frac{\theta_{2i} - \theta_{1}}{2}\right) \right\} \geq \frac{1}{2} \sin \theta_{2i}.
\]

(55)

Because of \( \theta_{2j+1} - (\theta_{2i}/2) \leq \tau_2 - (\theta_{2i} - \theta_{2j+1})/2 \leq \theta_{2j+1} + (\theta_{2i}/2) \), we have

\[
\sin\left(\frac{\theta_{2i} - \theta_{2j+1}}{2}\right) \geq \min \left\{ \sin\left(\frac{\theta_{2j+1} + \theta_{1}}{2}\right), \sin\left(\frac{\theta_{2j+1} - \theta_{1}}{2}\right) \right\} \geq \frac{1}{2} \sin \theta_{2j+1},
\]

(56)

and on account of \( (\cos(n + 2\tau_2)/\sin(\tau_2 - (\theta_{2i} + \theta_{2j+1})/2)) \leq n + 2 \), we obtain \(|A_{i,j}^{11}| \leq 4 \).

(ii) The remaining part will be discussed in two situations: \( j_0 \leq i - 1 \) and \( j_0 \geq i \).

(a) The case \( j_0 \leq i - 1 \).

For \( j \leq j_0 - 1 \leq i - 2 \), since \( (\pi/2) \geq \tau_2 - (\theta_{2i} - \theta_{2j+1})/2 \geq \theta_{2j+1} \), we have \( \sin(\tau_2 - (\theta_{2i} - \theta_{2j+1})/2) \geq \sin \theta_{2j+1} \). And, because of

\[
\sin(\theta_{2i} \cos(n + 2\tau_2)/\sin(\tau_2 + (\theta_{2i} + \theta_{2j+1})/2) \sin(\tau_2 - (\theta_{2i} - \theta_{2j+1})/2) \leq 2(n + 2),
\]

(57)

we obtain \(|A_{i,j}^{11}| \leq 2 \).

For \( j_0 + 1 \leq j \leq i - 1 \leq \left( (n + 1)/2 \right) - 1 \), considering that \( \theta_i \leq \tau_2 + (\theta_{2i} - \theta_{2j+1})/2 < \theta_{2i} \), we have

\[
\sin(\tau_2 + (\theta_{2i} - \theta_{2j+1})/2) \geq \min \{ \sin \theta_{2i}, \sin \theta_i \},
\]

so

\[
\sin(\theta_{2j+1} \cos(n + 2\tau_2)/\sin(\tau_2 - (\theta_{2j+1} - \theta_{2i})/2) \sin(\tau_2 - (\theta_{2i} - \theta_{2j+1})/2) \leq 2(n + 2),
\]

(58)
And, using the following (61), similar to (57), we obtain \( |A_{ij}^{(11)}| \leq 4 \).

\[
\sin \left( \frac{\theta_{2j+1} - \theta_{2j}}{2} \right) \geq \min \left\{ \sin \frac{\theta_{2j+1}}{2}, \sin \frac{\theta_{2j}}{2} \right\}.
\]

Therefore,

\[
\left| \frac{\sin \theta_{2j+1}}{\sin \left( \tau_2 + (\theta_{2j+1} - \theta_{2j})/2 \right)} \right| \leq 2.
\]

(60)

And, on account of (57), we have \( |A_{ij}^{(11)}| \leq 4 \).

\[
\sin \theta_{2j+1} \cos(n + 2)\tau_2 \leq 2(n + 2),
\]

(61)

we obtain \( |A_{ij}^{(11)}| \leq 2 \).

For \( i \leq j \leq j_0 - 1 \), on account of \( \pi/4 \geq \tau_2 - (\theta_{2j+1} - \theta_{2j})/2 > \theta_{2j} \), we have \( \sin(\tau_2 - (\theta_{2j+1} - \theta_{2j})/2) \geq \sin \theta_{2j} \). Since \( \theta_{2j+1} \leq \tau_2 + (\theta_{2j+1} - \theta_{2j})/2 < (\pi/2) \), we obtain \( \sin(\tau_2 + (\theta_{2j+1} - \theta_{2j})/2) \geq \sin \theta_{2j+1} \). And, considering that \( |\cos(n + 2)\tau_2/ \sin(\tau_2 - (\theta_{2j+1} + \theta_{2j})/2)| \leq n + 2 \), we obtain \( |A_{ij}^{(11)}| \leq 1 \).

Thus,

\[
\left| \frac{\sin \theta_{2j+1}}{\sin \left( \tau_2 + (\theta_{2j+1} - \theta_{2j})/2 \right)} \right| \leq \max \left\{ \frac{\sin \theta_{2j+1}}{\sin \theta_{2j+1}}, \frac{\sin \theta_{2j+1}}{\sin(\theta_{2j+1}/2)} \right\} \leq 2.
\]

(63)

If \( \tau_2 \in (-\pi/2, -\pi/4) \), by setting \( \tau' = \tau_2 \in (\pi/4), (\pi/2) \), we can similarly prove the conclusion.

To sum up, when \( |\tau_2| \in [(\pi/4), (\pi/2)] \), we have \( |A_{ij}^{(11)}| \leq 4\sqrt{2} \).

Lemma 11. If \( |\tau_2| \in [(\pi/4), (\pi/2)] \), then \( |A_{ij}^{(11)}| \leq 4\sqrt{2} \).

The estimates of \( A_{ij}^{(11)} \) and \( B_{ij}^{(11)} \) are similar, so we only take \( A_{ij}^{(11)} \) as an example.

For \( j \geq i - 1 \), we have

\[
\sin \left( \frac{\theta_{2j} - \theta_{2j+1}}{2} \right) \geq \frac{\sqrt{2}}{2} \cos \frac{\theta_{2j} - \theta_{2j+1}}{2} \geq \frac{\sqrt{2}}{4} \sin \theta_{2j}.
\]

(64)

And, using (61), we obtain \( |A_{ij}^{(11)}| \leq 4\sqrt{2} \).

For \( i \geq j \), we have \( \sin(\tau_2 + (\theta_{2j+1} - \theta_{2j})/2) \geq (\sqrt{2}/4)\sin \theta_{2j+1} \) and (57). So, we obtain \( |A_{ij}^{(11)}| \leq 4\sqrt{2} \).

Next, we will discuss each case separately.
Case 1. If \( |r_2| \in [0, (\theta_1/2)], r_1 \in [0, (\theta_1/2)] \cup [\pi - (\theta_1/2), \pi] \), then

\[
\Lambda_n^1(x) \leq 28(n + 2)^2. \tag{65}
\]

By Lemma 6, we can obtain

\[
\sum_{i=1}^{[n/2]} \sum_{j=i+1}^{[n/2]} |A_{i,j}^{(1)}| \leq \sum_{i=1}^{[n/2]} \sum_{j=i+1}^{[n/2]} 16 \left( \sin \frac{\theta_{2i} \sin \theta_{2j+1}}{(n + 2)^2} \right) \leq 16 \left( \frac{2(n + 2)}{n} \right) \leq 2(n + 2)^2. \tag{66}
\]

From (66)–(68), we can obtain

\[
\sum_{i=1}^{[n/2]} |A_{i,i}^{(1)}| \leq \sum_{i=1}^{[n/2]} 8(n + 2) \leq 4(n + 2)^2. \tag{68}
\]

Similarly, by Lemma 7, we have

\[
\sum_{i=1}^{[n/2]} \sum_{j=0}^{[n/2]} |A_{i,j}^{(1)}| \leq 14(n + 2)^2, \quad |r_2| \in \left[ 0, \frac{\theta_1}{2} \right], r_1 \in \left[ 0, \frac{\theta_1}{2} \right] \cup \left[ \pi - \frac{\theta_1}{2}, \pi \right]. \tag{69}
\]

Therefore,

\[
\Lambda_n^1(x) \leq 28(n + 2)^2, \quad |r_2| \in \left[ 0, \frac{\theta_1}{2} \right], r_1 \in \left[ 0, \frac{\theta_1}{2} \right] \cup \left[ \pi - \frac{\theta_1}{2}, \pi \right]. \tag{71}
\]

Case 2. If \( |r_2| \in [0, (\theta_1/2)] \) and \( r_1 \in ((\theta_1/2), (\pi/4)] \cup [(3\pi/4), \pi - (\theta_1/2)] \), then

\[
\Lambda_n^1(x) \leq 32\sqrt{2}(n + 2)^2. \tag{72}
\]

By Lemmas 6 and 8, it can be proved that
Case 4. If \( |r_2| \in [(\theta_1/2), (\pi/4)] \) and \( r_1 \in [(\theta_1/2), (\pi/3)] \cup [(2\pi/3), \pi - (\theta_1/2)] \), then
\[
\Lambda_n^{(1)}(x) \leq 16(n + 2)^2. 
\] (82)

By Lemmas 9 and 10, it is easy to prove this conclusion.

Case 5. If \( |r_2| \in [(\theta_1/2), (\pi/4)] \) and \( r_1 \in [(\pi/3), (2\pi/3)\pi] \), then
\[
\Lambda_n^{(1)}(x) \leq 4(n + 2)^2. 
\] (83)
If \( |r_2| \in [(\theta/2), (\pi/4)] \) and \( r_1 \in [\pi/3, 2\pi/3] \), since \( \sin \varphi_1 \sin \varphi_2 = \sin^2 r_1 - \sin^2 r_2 \geq (1/4) \), then similar to (81), we obtain \( A_n^{(1)}(x) \leq 4(n + 2)^2 \).

Case 6. If \( |r_2| \in [(\theta/2), (\pi/4)] \) and \( r_1 \in [\pi - (\theta/2), \pi] \), then

\[ A_n^{(1)}(x) \leq 8(n + 2)^2. \]  

If \( r_1 \in [\pi - (\theta/2), \pi] \), it is easy to prove that \( |A_{i,j}^{(12)}| \leq 4 \). Combining the results of Lemma 10, we can obtain

\[ \sum_{i=1}^{[n/2]} \sum_{j=0}^{[n/2]} |A_{i,j}^{(1)}| \leq \sum_{i=1}^{[n/2]} \sum_{j=0}^{[n/2]} 16 \leq 4(n + 2)^2. \]  

Similarly, we can obtain

\[ \sum_{i=1}^{[n/2]} \sum_{j=0}^{[n/2]} |B_{i,j}^{(1)}| \leq \sum_{i=1}^{[n/2]} \sum_{j=0}^{[n/2]} 16 \leq 4(n + 2)^2. \]  

Thus, \( A_n^{(1)}(x) \leq 8(n + 2)^2 \), \( |r_2| \in [(\theta/2), (\pi/4)] \), and \( r_1 \in [\pi - (\theta/2), \pi] \).

Case 7. If \( |r_2| \in [(\pi/4), (\pi/2)] \) for \( r_1 \in [(\pi/4), (\pi/2) - (\theta/2)] \cup [(\pi/2) + (\theta/2), (3\pi/4)\pi] \), then

\[ A_n^{(1)}(x) \leq 80\sqrt{2}(n + 2)^2. \]  

By Lemma 11, we know \( |A_{i,j}^{(11)}| \leq 4\sqrt{2} \) and \( |r_2| \in [(\pi/4), (\pi/2)]. \) Next, let us estimate \( |A_{i,j}^{(12)}| \), for \( r_1 \in [(\pi/4), (\pi/2) - (\theta/2)]. \) For every \( j \), there is \( i_0 \) such that \( |r_1 - (\theta_{2i_0} + \theta_{2j+1}/2)| \leq (\theta/2) \) holds; that is, \( (\theta_{2i_0} + \theta_{2j+1})/2 - \theta_{1/2} \leq r_1 \leq (\theta_{2i_0} + \theta_{2j+1})/2 + (\theta/2). \)

(i) For \( i = i_0 \), since \( (\pi/2) - (\theta/2) \leq r_1 + (\theta_{2i_0} + \theta_{2j+1})/2 \leq \pi - (\theta/2) \), then

\[ \sin\left(\theta_{1/2} + \theta_{2j+1}/2\right) \leq \sin\left(\theta_{1/2} - \frac{\theta_1}{2}\right) \geq \frac{\theta_1}{2} \geq \frac{1}{n + 2}. \]  

And, because of \( \theta_{2j+1} + (\theta/2) \leq r_1 + (\theta_{2i_0} - \theta_{2j+1})/2 \leq \theta_{2j+1} + (\theta/2), \) we can obtain

\[ \sin\left(\theta_{1/2} + \theta_{2j+1}/2\right) \leq \sin\left(\theta_{1/2} - \frac{\theta_1}{2}\right) \geq \frac{\theta_1}{2} \sin\theta_{2j+1}. \]  

Furthermore, considering that \( |\cos(n + 2)r_1/\sin(r_1 - (\theta_{2i_0} + \theta_{2j+1}/2))| \leq n + 2 \), we have \( |A_{i,j}^{(12)}| \leq 2(n + 2). \) Thus,

\[ \sum_{j=0}^{[n/2]} |A_{i,j}^{(1)}| \leq 8\sqrt{2}(n + 2) \leq 4\sqrt{2}(n + 2)^2. \]  

(ii) For \( 0 \leq i \leq i_0 - 1 \), the two cases \( j \leq i + 1 \) and \( i \geq j \) are discussed separately.

(a) If \( j \leq i - 1 \), for \( \theta_{2j+1} \leq r_1 - (\theta_{2i_0} - \theta_{2j+1})/2 \leq (\theta_{2i_0} + \theta_{2j+1})/2, \) we have

\[ \sin\left(\theta_{1/2} + \theta_{2j+1}/2\right) \geq \sin\left(\theta_{1/2} - \frac{\theta_1}{2}\right) \geq \frac{1}{2} \sin\theta_{2j+1}, \]  

so \( |(\sin\theta_{2i_2}\sin\theta_{2j+1}/\sin(r_1 - (\theta_{2i_2} - \theta_{2j+1}/2))| \leq 2. \)

And, considering that

\[ \sin\left(\frac{\theta_{2i_2} + \theta_{2j+1}}{2}\right)\sin\left(\frac{\theta_{1/2} - \theta_{2j+1}}{2}\right) = \sin^2 r_1 - \sin^2\left(\frac{\theta_{2i_2} + \theta_{2j+1}}{2}\right) = \frac{\sin^2(r_1 - (\theta_{2j+1}/2)}{2} - \frac{\sin^2(r_1 + (\theta_{2j+1})}{4}, \]  

and

\[ \sin\left(\frac{\theta_{1/2} - \theta_{2j+1}}{2}\right) = \left(\sin\frac{\theta_{2j+1}}{2}\right)e^{-(\theta_{2j+1})}, \]  

so

\[ \sin\left(\theta_{1/2} + \theta_{2j+1}/2\right) \geq \sin\left(\theta_{1/2} - \frac{\theta_1}{2}\right) \geq \frac{1}{2} \sin\theta_{2j+1}. \]  

Therefore, we have

\[ |\sin\theta_{2i_2}\sin\theta_{2j+1}/\sin(r_1 - (\theta_{2i_2} - \theta_{2j+1}/2))| \leq 2. \]
we obtain
\[
|A_{i,j}^{(12)}| \leq \frac{2}{(n + 2)\sin^2\left(\frac{\tau_i}{2} - \left(\theta_{2i} + \theta_{2j+1}\right)/4\right)}
\]  
(93)

\[
\sin\left(\frac{\tau_1 + \theta_{2j+1} - \theta_{2i}}{2}\right) \geq \sin\frac{\pi}{4}\cos\frac{\theta_{2j+1} - \theta_{2i}}{2} \geq \frac{\sqrt{2}}{2} \sqrt{\sin \theta_{2j+1} \sin \theta_{2i}}.
\]
(94)

And, considering that
\[
\sin\left(\frac{\tau_1 + \theta_{2i} + \theta_{2j+1}}{2}\right) \sin\left(\frac{\tau_1 - \theta_{2i} + \theta_{2j+1}}{2}\right) = \left(\sin \frac{\tau_1 + \theta_{2i} + \theta_{2j+1}}{2}\right)\left(\sin \frac{\tau_1 - \theta_{2i} + \theta_{2j+1}}{2}\right)
\]
\[
\geq \sqrt{2}\sqrt{\sin \theta_{2i} \sin \theta_{2j+1}\sin^2\left(\frac{\tau_1 - \theta_{2i} + \theta_{2j+1}}{4}\right)},
\]
(95)

we obtain
\[
|A_{i,j}^{(12)}| \leq \frac{1}{(n + 2)\sin^2\left(\frac{\tau_i}{2} - \left(\theta_{2i} + \theta_{2j+1}\right)/4\right)}.
\]
(96)

\[
\sum_{j=0}^{[n/2]} \sum_{i=1}^{[n/2]} |A_{i,j}^{(1)}| \leq \sum_{j=0}^{[n/2]} \sum_{i=1}^{[n/2]} \frac{4\sqrt{2}}{n + 2} \cdot \frac{2}{\sin^2\left(\frac{\tau_i}{2} - \left(\theta_{2i} + \theta_{2j+1}\right)/2\right)}
\]
\[
\leq \sum_{j=0}^{[n/2]} \sum_{i=1}^{[n/2]} \frac{32\sqrt{2}(n + 2)}{2(i_0 - i - 1)^2} \leq 32\sqrt{2}(n + 2) \sum_{j=0}^{[n/2]} \left(1 + \int_1^{i_0-1} \frac{1}{(2x - 1)^2} \, dx\right)
\]
\[
\leq 24\sqrt{2}(n + 2)^2.
\]
(97)

(iii) For \(i_0 + 1 \leq i \leq \lceil (n + 1)/2 \rceil\), the following two cases are discussed separately.

(a) If \(j \geq i\), from (96), we have
\[
\sum_{j=0}^{[n/2]} \sum_{i=0}^{i_0} |A_{i,j}^{(1)}| \leq \sum_{j=0}^{[n/2]} \sum_{i=0}^{i_0} \frac{4\sqrt{2}}{n + 2} \cdot \frac{1}{\sin^2\left(\theta_{2i} + \theta_{2j+1}\right)/4 - \tau_i/2} \leq \sum_{j=0}^{[n/2]} \sum_{i=0}^{i_0} \frac{16\sqrt{2}(n + 2)^2}{2(i_0 - i - 1)^2} \leq 12\sqrt{2}(n + 2)^2.
\]
(98)
(b) If \( j \leq i - 1 \), for every \( j \), there is \( i_1 \) such that \(|r_1 - (\theta_{2i_1} - \theta_{2j+1})/2|\leq (\theta_i/2)\) holds, that is,
\[
\frac{\theta_{2i_1} - \theta_{2j+1}}{2} - \frac{\theta_i}{2} \leq r_1 \leq \frac{\theta_{2i_1} - \theta_{2j+1}}{2} + \frac{\theta_i}{2}.
\]  
(99)

\[
\sin\left(\frac{r_1 + \frac{\theta_{2i} + \theta_{2j+1}}{2}}{2}\right) \geq \min\left\{\sin\left(\frac{\theta_{2i_1} + \theta_i}{2}\right), \sin\left(\frac{\theta_{2i_1} - \theta_i}{2}\right)\right\} \geq \frac{1}{2}\sin\theta_{2i_1}.
\]  
(100)

On account of \(-\theta_{2j+1} - (\theta_i/2) \leq r_1 - (\theta_{2i_1} + \theta_{2j+1})/2 \leq -\theta_{2j+1} + (\theta_i/2)\), we obtain

\[
\sin\left(\frac{r_1 - \frac{\theta_{2i} + \theta_{2j+1}}{2}}{2}\right) \geq \min\left\{\sin\left(\frac{\theta_{2j+1} - \theta_i}{2}\right), \sin\left(\frac{\theta_{2j+1} + \theta_i}{2}\right)\right\} \geq \frac{1}{2}\sin\theta_{2j+1}.
\]  
(101)

And because of \(|\cos(n+2)r_1/\sin(r_1 - (\theta_{2i} - \theta_{2j+1})/2)| \leq n + 2\), we obtain \(|A_{1,j}^{(12)}| \leq 4\).

\[
\left|\frac{\cos(n+2)r_1 \sin \theta_{2j+1}}{\sin(r_1 - (\theta_{2i} - \theta_{2j+1})/2) \sin(r_1 - (\theta_{2i} + \theta_{2j+1})/2)}\right| \leq 2(n + 2),
\]  
(103)

we obtain \(|A_{1,j}^{(12)}| \leq 4\).

Thus, for \( j \leq i - 1 \), we have
\[
\sum_{j=0}^{[n/2]} \sum_{i=i_1+1}^{[n+1]/2} |A_{1,j}^{(1)}| \leq 4\sqrt{2}(n + 2)^2.
\]  
(104)

Based on the above results, if \(|r_2| \in [\pi/4, \pi/2]\) and \(r_1 \in [\pi/4, \pi/2 - \theta_i/2]\), we can obtain
\[
\sum_{j=0}^{[n/2]} \sum_{i=1}^{[n+1)/2} |A_{1,j}^{(1)}| \leq 40\sqrt{2}(n + 2)^2.
\]  
(105)

Similarly, if \(|r_2| \in [\pi/4, \pi/2]\) and \(r_1 \in [\pi/4, \pi/2 - \theta_i/2]\), we can obtain
\[
\sum_{j=0}^{[n/2]} \sum_{i=1}^{[n+1)/2} |B_{1,j}^{(1)}| \leq 40\sqrt{2}(n + 2)^2.
\]  
(106)

Therefore, we obtain \(\Lambda_1^{(1)}(x) \leq 80\sqrt{2}(n + 2)^2\), \(|r_2| \in [(\pi/4), (\pi/2)]\), and \(r_1 \in [(\pi/4), (\pi/2) - (\theta_i/2)]\).

And, we can similarly prove the case of \(|r_2| \in [(\pi/4), (\pi/2)]\) and \(r_1 \in [(\pi/2) - (\theta_i/2), (\pi/2)]\).

Case 8. If \(|r_2| \in [(\pi/4), (\pi/2)]\) and \(r_1 \in [(\pi/2) - (\theta_i/2), (\pi/2)]\), then
\[
\Lambda_1^{(1)}(x) \leq 66\sqrt{2}(n + 2)^2.
\]  
(107)

If \(r_1 \in [(\pi/2) - (\theta_i/2), (\pi/2)]\), for every \(j\), there is \(i_0 = [(n + 1)/2] - j\) such that \(|r_1 - (\theta_{2i} + \theta_{2j+1})/2| \leq (\theta_i/2)\) holds.

(i) For \(1 \leq i \leq i_0 - 1\), the following two cases are discussed separately:

(a) If \(j \leq i - 1\), since \(\theta_{2j+1} \leq r_1 - (\theta_{2i} - \theta_{2j+1})/2 < (\pi/2)\), then \(\sin(r_1 - (\theta_{2i} - \theta_{2j+1})/2) \geq \sin\theta_{2j+1}\).
And, because of \((\pi/2) + \theta_i \leq r_i + (\theta_{2i} + \theta_{2j+1})/2 \leq \pi - (\theta_i/2)\), we have

\[
\sin \left( r_1 + \frac{\theta_{2i} + \theta_{2j+1}}{2} \right) \geq \sin \left( \frac{\pi}{2} + \frac{\theta_{2i} + \theta_{2j+1}}{2} \right) \geq 1 - \frac{2i + 2j + 1}{n + 2}. \tag{108}
\]

And, considering that \((\theta_i/2) \leq r_1 - (\theta_{2i} + \theta_{2j+1})/2 < (\pi/2)\), we obtain

\[
\sin \left( r_1 - \frac{\theta_{2i} + \theta_{2j+1}}{2} \right) \geq \sin \left( \frac{\pi}{2} - \frac{\theta_{2i} + \theta_{2j+1}}{2} \right) \geq 1 - \frac{2i + 2j + 2}{n + 2}. \tag{109}
\]

Hence,

\[
\left| A_{i,j}^{(12)} \right| \leq \frac{n + 2}{(n - 2i - 2j)^2}. \tag{110}
\]

(b) If \(j \geq i\), we have

\[
\sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{i=1}^{\lfloor n/2 \rfloor - j} \left| A_{i,j}^{(1)} \right| \leq 8 \sqrt{2} (n + 2) \sum_{j=0}^{\lfloor (n+1)/2 \rfloor - j} \sum_{i=1}^{\lfloor (n+1)/2 \rfloor - j} \frac{1}{(n - 2i - 2j)^2} \leq 8 \sqrt{2} (n + 2) \sum_{j=0}^{\lfloor (n+1)/2 \rfloor - j} \left[ \frac{1}{(n - 2j - 2x)^2} \right] \leq 6 \sqrt{2} (n + 2)^3. \tag{113}
\]

(ii) For \(i_0 + 2 \leq i \leq \lfloor (n + 1)/2 \rfloor\), the following two cases are discussed separately:

(a) If \(j \leq i - 1\), since \((\pi/2) - \theta_i \leq r_1 - (\theta_{2i} + \theta_{2j+1})/2 \leq \pi - (\theta_i/2)\), then

\[
\frac{\sin \theta_{2i} \sin \theta_{2j+1}}{\sin (r_1 - (\theta_{2i} - \theta_{2j+1})/2)} \leq \frac{\sin \theta_{2i} \sin \theta_{2j+1}}{\sin (\theta_i + (\theta_i/2)) \sin (\theta_j + (\theta_j/2))} \leq 4. \tag{114}
\]

Because of \(0 \leq \theta_{j-i} - (\theta_j/2) \leq (\theta_{2i} + \theta_{2j+1})/2 - r_1 \leq \theta_{i-i} + (\theta_i/2) < (\pi/2)\), we can obtain

\[
\sin \left( \frac{\theta_{2i} + \theta_{2j+1}}{2} - r_1 \right) \geq \sin \left( \frac{\theta_{2i} + \theta_{2j+1}}{2} \right) \geq 2 \frac{2i + 2j + 1}{n + 2} - 1. \tag{115}
\]
Owing to $\theta_1/2 \leq \tau_1 + (\theta_{2i} + \theta_{2j+1})/2 - \pi \leq \pi/2 - 3\theta_1/2$, we have

$$\sin \left( \tau_1 + \frac{\theta_{2i} + \theta_{2j+1}}{2} - \pi \right) \geq \sin \left( \frac{\theta_{2i} + \theta_{2j+1}}{2} - \frac{\theta_1}{2} \cdot \frac{\pi}{2} \right) \geq \frac{2i + j - \pi}{n + 2} - 1. \quad (116)$$

Then,

$$|A_{i,j}^{(11)}| \leq \frac{4(n + 2)}{(2i + 2j - n - 1)(2(i + j) - (n + 2))} \quad (117)$$

(b) If $j \geq i$, from (111), (115), (116), we know that

$$\sum_{j=0}^{[n/2]} \sum_{i=n+j}^{[n+1/2]} |A_{i,j}^{(1)}| \leq \sum_{j=0}^{[n/2]} \sum_{i=n+j}^{[n+1/2]} \frac{4\sqrt{2}}{n + 2} \cdot \frac{4(n + 2)}{(2i + 2j - n - 1)(2(i + j) - (n + 2))} \leq \sum_{j=0}^{[n/2]} \sum_{i=n+j}^{[n+1/2]} \frac{16\sqrt{2}(n + 2)}{(2i + 2j - n - 2)^2} \leq 12\sqrt{2}(n + 2)^2. \quad (120)$$

Based on the above conclusions, we can obtain

$$\sum_{i=1}^{[n/2]} \sum_{j=0}^{[n/2]} |A_{i,j}| \leq 26\sqrt{2}(n + 2)^2. \quad (121)$$

Similarly, we can prove that

$$\sum_{i=1}^{[n+1/2]} \sum_{j=0}^{[n/2]} |B_{i,j}| \leq 40\sqrt{2}(n + 2)^2. \quad (122)$$

Then, $A_n^i(x) \leq 66\sqrt{2}(n + 2)^2, |r_1| \in [(\pi/4), (\pi/2)], and \tau_1 \in [(\pi/2) - (\theta_1/2), (\pi/2)].$

**Case 9.** If $|r_2| \in [\pi/4, \pi/2] and \tau_1 \in [(\pi/2), (\pi/2) + (\theta_1/2)], then $A_n^i(x) \leq 38\sqrt{2}(n + 2)^2. \quad (123)$

If $|r_1| \in [(\pi/4), (\pi/2)] and \tau_1 \in [(\pi/2), (\pi/2) + (\theta_1/2)], similar to the estimates of (121) and (122), we can obtain

$$\sum_{i=1}^{[n/2]} \sum_{j=0}^{[n/2]} |A_{i,j}| \leq 14\sqrt{2}(2 + n)^2, \quad (124)$$

$$\sum_{i=1}^{[n+1/2]} \sum_{j=0}^{[n/2]} |B_{i,j}| \leq 24\sqrt{2}(n + 2)^2. \quad (125)$$

Therefore, $A_n^i(x) \leq 38\sqrt{2}(n + 2)^2, |r_2| \in [(\pi/4), (\pi/2)], and \tau_1 \in [(\pi/2), (\pi/2) + (\theta_1/2)].$

**Case 10.** If $|r_2| \in [\pi/4, \pi/2] and \tau_1 \in [3\pi/4, \pi - \theta_1/2], then $A_n^i(x) \leq 16(n + 2)^2. \quad (125)$

By Lemma 8 and 11, we can easily obtain the conclusion.

**Case 11.** If $|r_2| \in [\pi/4, \pi/2] and \tau_1 \in [\pi - \theta_1/2, \pi], then $A_n^i(x) \leq 8\sqrt{2}(n + 2)^2. \quad (126)$

Proof: If $\tau_1 \in [\pi - \theta_1/2, \pi]$, it is easy to prove that $|A_n^{(1)}| \leq 4, |B_n^{(1)}| \leq 4. And, by Lemma 11, we can obtain the conclusion.

Based on the conclusion of Case 1–Case 11, we obtain $A_n^i(x) \leq 80\sqrt{2}(n + 2)^2, |r_2| \in [0, \pi/2], \tau_1 \in [0, \pi].$

And similarly, we have $A_n^2(x) \leq 80\sqrt{2}(n + 2)^2, |r_1| \in [0, \pi/2], and \tau_1 \in [0, \pi].$

Then, the proof of Theorem 1 is completed. □
Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that there are no conflicts of interest.

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