RECOVERY OF THE ABSORPTION COEFFICIENT IN RADIATIVE TRANSPORT FROM A SINGLE MEASUREMENT

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Abstract. In this paper, we investigate the recovery of the absorption coefficient from boundary data assuming that the region of interest is illuminated at an initial time. We assume that the initial state of radiation is sufficiently strong and isotropic, but otherwise unknown. This result is part of an effort to reconstruct optical properties using unknown illumination embedded in the unknown medium.

This problem can be posed as an inverse source problem, where we seek to simultaneously recover a source of the form $\sigma(t, x, \theta)f(x)$ (with $\sigma$ known) and an isotropic initial condition $u_0(x)$, using the single measurement induced by these data. More precisely, based on exact boundary controllability, we derive a system of equations for the unknown terms $f$ and $u_0$. The system is shown to be Fredholm if $\sigma$ satisfies a certain positivity condition. We show that for generic term $\sigma$ and weakly absorbing media, the inverse source problem is uniquely solvable with a stability estimate.

1. Introduction. The radiative transport equation (RTE) models physical phenomena in various scientific disciplines including medical imaging, semiconductors, astrophysics, nuclear reactors, etc. The mathematical treatment of some of these applications is found in [1, 2, 3, 4, 5, 6]. We are mainly motivated by medical imaging, and we refer the reader to [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and references therein for descriptions of such problems.

The transport of radiation is modeled by the following equation,

$$\partial_t w + (\theta \cdot \nabla)w + \mu_a w + \mu_s (I - K)w = 0$$

augmented by an initial condition $w_0$ and an in-flow profile. The optical properties of the medium are the absorption coefficient $\mu_a$, the scattering coefficient $\mu_s$ and the scattering operator $K$. The solution $w = w(t, x, \theta)$ represents the density of radiation at time $t \in [0, \tau]$, position $x \in \Omega \subset \mathbb{R}^n$, moving in the direction $\theta \in S$ at unit speed.

An inverse problem for optical imaging consists of reconstructing one or more optical coefficients. In this paper, we focus on the recovery of the absorption coefficient $\mu_a$ and we assume that the scattering properties of the medium are known. We

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let $\tilde{w}$ be the solution of the radiative transport problem with absorption coefficient $\tilde{\mu}_a$ and initial condition $\tilde{w}_0$. Then the following functions

$$u = w - \tilde{w}, \quad \sigma = \tilde{w}, \quad f = \mu_a - \tilde{\mu}_a \quad \text{and} \quad u_0 = w_0 - \tilde{w}_0$$

(1)
satisfy the initial boundary value problem (7)-(9) defined in Section 2. As usual, the system (7)-(9) has the aspect of a linearized problem where $\mu_a$ represents a reference absorption and we seek to reconstruct the contrast $f = \mu_a - \tilde{\mu}_a$ between the true and reference coefficients from knowledge of the boundary measurements of the field $u$.

Some of the most fundamental results in the recovery of coefficients for the RTE are based on knowledge of the so-called albedo operator which maps all possible inflow illuminations to corresponding outflow measurements. For reviews of these results see [10, 18] and reference therein. In the present work, however, we focus on the recovery of a coefficient from a single measurement. To the best of our knowledge, all works in the literature for similar single-measurement problems aim at the recovery of $f$ given full knowledge of the initial condition. Hence, it is usually assumed that $u_0 = 0$. In this category of assumptions, we find the recent works of Klibanov and Pamyatnykh [19], and Machida and Yamamoto [20]. Both of them are based on ingenious Carleman estimates leading to uniqueness in the recovery of $f$ in [19], and global Lipschitz stability in [20]. These estimates are patterned after similar results for the wave equation in the general form of [21, Thm 8.2.2]. Unfortunately, assuming precise knowledge of a non-vanishing initial condition is not realistic in most applications. The main contribution of this paper is a strategy to partially bypass that assumption.

One of the standing questions in imaging applications is the following: Can we use unknown sources (such as ambient radiative noise) to illuminate a region and recover its optical properties? In this paper, the illuminating source is represented by the initial state of radiation $\sigma|_{t=0} = \tilde{w}|_{t=0}$, and we seek to show that $f = 0$ if the boundary outflow is zero. The purpose of our work is to provide sufficient conditions for the unique and stable recovery of $f$ from knowledge of the outflow profile of radiation $u$ at the boundary of $\Omega$, but without full knowledge of $u_0$ (see Definition 2.4 below). In plain words, our main result is that if the initial condition $u_0$ is isotropic and the region $\Omega$ is properly illuminated, then both $f$ and $u_0$ can be reconstructed in a stable manner. The precise statement of our result is presented in Section 2. Our proof, found in Section 3, for the recovery of $f$ is primarily based on boundary controllability for the RTE recently obtained in [22, 23].

2. Background and statement of main results. In this section we state the direct problem for transient radiative transport and also the exact boundary controllability property. We also review some preliminary facts in order to state our main results in the proper mathematical ground.

We assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded convex domain with smooth boundary $\partial \Omega$. The unit sphere in $\mathbb{R}^n$ is denoted by $S$. The outflow $(\cdot)$ and inflow $(\cdot)$ parts of the boundary are

$$(\partial \Omega \times S)_\pm = \{(x, \theta) \in \partial \Omega \times S : \pm \nu(x) \cdot \theta > 0\}$$

where $\nu$ is the outward unit normal vector on $\partial \Omega$. Without loss of generality, it is assumed that the particles travel at unit speed. The spatial scale of the problem is given by

$$l = \text{diam}(\Omega).$$
Now we define the appropriate Hilbert spaces over which the radiative transport problem is well-posed. First, we denote by $\mathcal{V}^0$ and $\mathcal{V}^1$ the completion of $C^1(\overline{\Omega} \times S)$ with respect to the norms associated with the following inner products,
\begin{align}
\langle u, w \rangle_{\mathcal{V}^0} &= \langle u, w \rangle_{L^2(\Omega \times S)} \\
\langle u, w \rangle_{\mathcal{V}^1} &= l^2(\theta \cdot \nabla u, \theta \cdot \nabla w)_{\mathcal{V}^0} + \langle u, w \rangle_{\mathcal{V}^0} + l\langle \nu \cdot \theta |u, w \rangle_{L^2(\partial \Omega \times S)}
\end{align}
where $\theta \cdot \nabla$ denotes the weak directional derivative. Now, denote by $\mathcal{T}$ the trace space defined as the completion of $C(\partial \Omega \times S)$ with respect to the norm associated with the following inner product,
\begin{equation}
\langle u, w \rangle_{\mathcal{T}} = l\langle \nu \cdot \theta |u, w \rangle_{L^2(\partial \Omega \times S)}.
\end{equation}
We also have the spaces $\mathcal{T}_\pm$ denoting the restriction of functions in $\mathcal{T}$ to the in- and out-flow portions of the boundary $\partial \Omega \times S$, respectively. Functions in $\mathcal{V}^1$ have well-defined traces on the space $\mathcal{T}$ as asserted by the following lemma whose proof is found in [3, 4, 24, 25].

**Lemma 2.1.** The trace mapping $u \mapsto u|_{\partial \Omega}$ defined for $C^1(\overline{\Omega} \times S)$ can be extended to a bounded operator $\gamma : \mathcal{V}^1 \to \mathcal{T}$. Moreover, $\gamma : \mathcal{V}^1 \to \mathcal{T}$ is surjective. Same claims hold for the partial trace maps $\gamma \pm : \mathcal{V}^1 \to \mathcal{T}_\pm$.

In addition, we have the following definition for traceless closed subspaces of $\mathcal{V}^1$,
\begin{equation}
\mathcal{V}^1_\pm = \text{null}(\gamma \pm) = \{ v \in \mathcal{V}^1 : \gamma \pm v = 0 \},
\end{equation}
as well as the following integration-by-parts formula or Green’s identity for functions $u, v \in \mathcal{V}^1$,
\begin{equation}
\int_{\Omega \times S} (\theta \cdot \nabla u)v = \int_{\partial \Omega \times S} (\theta \cdot \nu)uv - \int_{\Omega \times S} (\theta \cdot \nabla v)u.
\end{equation}

The following transient radiative transport problem for general heterogeneous, scattering media is well-posed.

**Definition 2.2** (Direct Problem). Given forcing terms 
\begin{equation*}
f \in L^2(\Omega) \quad \text{and} \quad \sigma \in C^1([0, \tau]; L^\infty(\Omega \times S)),
\end{equation*}
and initial condition
\begin{equation*}
u_0 \in \mathcal{V}^1_-, \end{equation*}
find a solution $u \in C^1([0, \tau]; \mathcal{V}^0) \cap C([0, \tau]; \mathcal{V}^1)$ to the following initial boundary value problem
\begin{align}
\dot{u} + (\theta \cdot \nabla)u + \mu_\sigma u + \mu_s (I - K)u &= \sigma f & \text{in } [0, \tau] \times (\Omega \times S), \\
u &= u_0 & \text{on } \{ t = 0 \} \times (\Omega \times S), \\
\gamma_- u &= 0 & \text{on } [0, \tau] \times (\partial \Omega \times S)_-.
\end{align}
\begin{equation}
(Ku)(x, \theta) = \int_S \kappa(x, \theta, \theta') u(x, \theta') dS(\theta'),
\end{equation}
where $\kappa$ is known as the scattering kernel.

Throughout the paper we will make the following assumptions concerning the regularity of the absorption and scattering coefficients, and the scattering kernel. These assumptions ensure the well-posedness of the direct problem 2.2 using semigroup theory as described in [4, 26, 27], and they allow for heterogeneous media...
modeled by coefficients with low regularity. First, we have non-negative absorption
\( \mu_a \in L^\infty(\Omega) \) and scattering \( \mu_s \in L^\infty(\Omega) \) coefficients. We denote their norms as follows,
\[
\| \mu_a \|_{L^\infty(\Omega)} \quad \text{and} \quad \| \mu_s \|_{L^\infty(\Omega)}.
\]
(11)
We also consider a scattering kernel \( 0 \leq \kappa \in L^\infty(\Omega, L^2(S \times S)) \). It is assumed that
the scattering operator is conservative in the following sense,
\[
\int_S \kappa(x, \theta, \theta') dS(\theta') = 1, \quad \text{for a.a. } (x, \theta) \in \Omega \times S.
\]
(12)
In addition, we assume a reciprocity condition on the scattering kernel given by
\[
\kappa(x, \theta, \theta') = \kappa(x, -\theta', -\theta), \quad \text{for a.a. } (x, \theta, \theta') \in \Omega \times S \times S.
\]
(13)
This means that the scattering events are reversible in a local sense at each point \( x \in \Omega \).

2.1. Tools from control theory. Here we proceed to define the boundary controllability for the RTE. This is the major tool to prove our main result. Hence, we consider the following adjoint transport problem with prescribed outflow data. Given \( \eta \in L^2([0, \tau]; \mathbb{T}_+) \), find a mild solution \( \psi \in C([0, \tau]; \mathbb{V}^0) \) of the following problem
\[
\psi + (\theta \cdot \nabla) \psi - \mu_a \psi - \mu_s (I - \mathcal{K}^*) \psi = 0 \quad \text{in } [0, \tau] \times (\Omega \times S),
\]
(14)
\[
\psi = 0 \quad \text{on } \{ t = \tau \} \times (\Omega \times S),
\]
(15)
\[
\gamma_+ \psi = \eta \quad \text{on } [0, \tau] \times (\partial \Omega \times S)_+.
\]
(16)

Given arbitrary \( \phi \in \mathbb{V}^0 \), the goal of the control problem is to find an outflow
control condition \( \eta \in L^2([0, \tau]; \mathbb{T}_+) \) to drive the solution \( \psi \in C([0, \tau]; \mathbb{V}^0) \) of the problem (14)-(16) from \( \psi(\tau) = 0 \) to \( \psi(0) = \phi \). The well-posedness of the control problem is described in the following theorem which is a direct consequence of [22].

**Theorem 2.3.** Assume that \( |\mu_a| e^{1(3\mu_s + 5\mu_s)} < e^{-1} \). Then there exists a steering time
\( \tau < \infty \) such that for any initial state \( \phi \in \mathbb{V}^0 \), there exists outflow control \( \eta \in L^2([0, \tau]; \mathbb{T}_+) \) so that the mild solution \( \psi \in C([0, \tau]; \mathbb{V}^0) \) of the problem (14)-(16)
satisfies \( \psi(0) = \phi \). Among all such controls there exists \( \eta_{\min} \), with minimum norm,
which is uniquely determined by \( \phi \) and satisfies the following stability condition
\[
\| \eta_{\min} \|_{L^2([0, \tau]; \mathbb{T}_+)} \leq C \| \phi \|_{\mathbb{V}^0}
\]
for some positive constant \( C = C(\|\mu_a\|, \|\mu_s\|, \tau) \).

Hence, the mapping \( \phi \mapsto \eta_{\min} \) described in theorem 2.3 defines a bounded control operator,
\[
C : \mathbb{V}^0 \rightarrow L^2([0, \tau]; \mathbb{T}_+).
\]
(17)

We also define a bounded solution operator
\[
S : \mathbb{V}^0 \rightarrow L^2([0, \tau]; \mathbb{V}^0)
\]
(18)
mapping \( \phi \mapsto \psi \) where \( \psi \) is the solution of (14)-(16) with \( \eta = C\phi \).

We wish to point out that the condition \( |\mu_a| e^{1(3\mu_s + 5\mu_s)} < e^{-1} \) can be avoided using
the control result of Klibanov and Yamamoto [23] if we assume sufficiently regular coefficients \( \mu_a, \mu_s \) and \( \kappa \).
2.2. Main result for the inverse problem. Now we state the inverse source problem for transient transport along with our main result. Our proof, presented in Section 3, is based on tools from control theory developed in [22, 23]. Our main goal is to provide a constructive proof that the recovery of the forcing term $f$ and an isotropic initial condition $u_0$ can be reduced to a Fredholm system of equations.

If we fix the source term $\sigma$, we may see the solution $u$ of the direct problem 2.2 as dependent on the other source term $f \in L^2(\Omega)$ and the initial condition $u_0 \in \mathbb{V}_1$. The outflowing boundary measurements are modeled by the operator $\Lambda : L^2(\Omega) \times \mathbb{V}_1 \rightarrow H^1([0, \tau]; \mathbb{T}_+)$ defined as

$$\Lambda(f, u_0) = \gamma_+ u,$$

where $\gamma_+ : \mathbb{V}_1 \rightarrow \mathbb{T}_+$ is the outflowing trace operator defined in lemma 2.1, and $u$ is the solution to the direct problem 2.2.

Notice that $\sigma f \in C^1([0, \tau]; \mathbb{V}^0)$ which implies that $\dot{u} \in C([0, \tau]; \mathbb{V}^0)$ is the mild solution of the same transport equation with $\sigma f \in C([0, \tau]; \mathbb{V}^0)$ as the forcing term and an initial condition in $\mathbb{V}^0$. For the existence of mild solutions in small group theory, see the standard references [26, 27]. Now, using the concept of generalized traces for mild solutions, we can show that the measurement mapping $\Lambda : L^2(\Omega) \times \mathbb{V}_1 \rightarrow H^1([0, \tau]; \mathbb{T}_+)$ is (extends as) a bounded operator. The treatment of generalized traces for mild solutions can be found in [23, Section 2], [4, Section 14.4] or Cessenat [25, 24].

With this notation we define the inverse problem as follows.

**Definition 2.4** (Inverse Problem). Let $u$ be the solution to the direct problem 2.2 for unknown source term $f$ and unknown initial condition $u_0$. The inverse source problem is, given the out-flowing measurement $\Lambda(f, u_0)$, find $f$ and $u_0$.

In order to state and prove our main result, we define the following angular-averaging operator $P_\theta : \mathbb{V}^0 \rightarrow L^2(\Omega)$ given by

$$(P_\theta v)(x) = \frac{1}{|S|} \int_S v(x, \theta) dS(\theta),$$

where $|S|$ is the surface area of the unit sphere $S = \{x \in \mathbb{R}^n : |x| = 1\}$. The well-known velocity averaging lemmas developed in [28] imply the compactness of this operator when defined as $P_\theta : \mathbb{V}^1 \rightarrow L^2(\Omega)$. We also define a time integral operator $P_t : L^2([0, \tau]; \mathbb{V}^0) \rightarrow \mathbb{V}^0$ as follows,

$$(P_t v)(x, \theta) = \int_0^\tau v(t, x, \theta) dt,$$

which is clearly bounded.

Our main result concerning this inverse problem is the following.

**Theorem 2.5.** Let $\tau < \infty$ be the steering time for exact controllability from theorem 2.3. If

(i) there exists a constant $\delta > 0$ such that $|(P_\theta \sigma)(0, x)| \geq \delta$ for a.a. $x \in \Omega$,

(ii) $\sigma \in C^1([0, \tau]; L^\infty(\Omega \times S))$, and

(iii) the unknown initial condition is isotropic, i.e., $u_0 = P_\theta u_0$,

then the inverse problem 2.4 for $(f, u_0)$ can be reduced to the following Fredholm system on $L^2(\Omega) \times L^2(\Omega)$,

$$\begin{bmatrix} (P_\theta \sigma_0) + (P_\theta P_t \sigma S)^* & \mu_a \\ (P_\theta P_t \sigma S)^* & I \end{bmatrix} \begin{bmatrix} f \\ u_0 \end{bmatrix} = \begin{bmatrix} P_\theta C^* \bar{m} \\ P_\theta C^* \bar{m} \end{bmatrix}$$

(22)
where \( m = \Lambda(f, u_0) \), \( \sigma_0 = \sigma|_{t=0} \), and \( P_\theta P_\sigma S \) and \( P_\theta P_\sigma S \) are compact operators.

Hence, the lack of uniqueness (if there is any) in the recovery of \((f, u_0)\) is restricted to a finite-dimensional subspace. Moreover, there exists an open and dense subset of \((i) \cap (ii)\) such that for each \( \sigma \) in this set, if \( \mathcal{T}_a \) is sufficiently small, then (22) is uniquely solvable and the following stability estimate

\[
\| f \|_{L^2(\Omega)} + \| u_0 \|_{L^2(\Omega)} \leq C \| \Lambda(f, u_0) \|_{H^1([0, \tau]; T^+)}
\]

holds for some positive constant \( C = C(\mathcal{T}_a, \mathcal{T}_a, l, \tau, \sigma) \).

Recall the definition (1) of the functions \( u \), \( \sigma \) and \( f \). We may interpret \( \tilde{\mu}_a \) as the true optical absorption, and \( \mu_a \) as a reference value. These coefficients define respective radiation fields \( \tilde{w} \) and \( w \) with the same in-flowing profile. Then, in physical terms, the above theorem says that we can recover \( \tilde{\mu}_a \) from a single boundary measurement if the following conditions are met:

- The initial states of radiation \( w_0 \) and \( \tilde{w}_0 \) are isotropic, that is, we have illuminated the region \( \Omega \) during time \( t < 0 \) as to produce a diffuse state at time \( t = 0 \).
- The region \( \Omega \) is properly illuminated by the initial state \( \sigma_0 = \tilde{w}|_{t=0} \).
- The medium is weakly absorbing.

Notice however, that the initial state \( \tilde{w}_0 \) does not have to be fully known a-priori, that is, we do not assume \( u_0 = w|_{t=0} - \tilde{w}|_{t=0} = 0 \). In other words, we may uniquely identify the optical absorption even if we use an unknown, but sufficiently strong and diffuse, illuminating field \( \sigma_0 = \tilde{w}|_{t=0} \).

The smallness of \( \mu_a \) is only a technical condition which guarantees the invertibility of (22). We are not satisfied with this condition since we do not find it to have a physical meaning or relevance.

Finally, we also note that the governing operator in (22) depends on \( \sigma \), which is not known in practice. This dependence is due to the fact that the recovery of the absorption coefficient is indeed a nonlinear problem. Hence, this paper only provides a statement of unique identifiability rather than an explicit reconstruction procedure. However, the system (22) could be employed when the inverse problem is linearized.

3. The inverse problem. In this section we reduce the inverse problem 2.4 to an equation of Fredholm form. First, in order to simplify the notation, we introduce the following transport operators \( A, A^* : \mathbb{V}^1 \rightarrow \mathbb{V}^0 \) given by

\[
Au = (\theta \cdot \nabla)u + \mu_a u + \mu_s (I - K) u
\]

\[
A^* \psi = - (\theta \cdot \nabla) \psi + \mu_a \psi + \mu_s (I - K^*) \psi
\]

which behave as formal adjoints of each other with respect to the \( \mathbb{V}^0 \)-inner-product.

Now we are ready to reduce the inverse problem to a Fredholm equation using duality. To make things simple, we momentarily suppose that input data such \( \tilde{\phi} \) and \( \eta \) in the adjoint (14)-(16) are sufficiently smooth leading to a strong solution \( \psi \). Following the usual density arguments, we would take limits and use the continuity of appropriate operators to extend the meaning of the main equations to less regular data. In what follows, we will evaluate the duality pairing between the terms in equation (7) against \( \psi \) and \( \tilde{\psi} \) to obtain a system of two equations. The system will then be shown to have Fredholm form which is the main result of the paper.
Proof of Theorem 2.5. Let $m = \Lambda(f, u_0) \in H^1([0, \tau]; \mathbb{T}_+)$ and $\sigma_0 = \sigma|_{t=0}$, and consider
\[
\langle \sigma f, \psi \rangle_{L^2([0, \tau]; \mathbb{V}^0)} = \langle \dot{u} + Au, \psi \rangle_{L^2([0, \tau]; \mathbb{V}^0)}
\]
\[
= \langle u(\tau), \psi(\tau) \rangle_{\mathbb{V}^0} - \langle u(0), \psi(0) \rangle_{\mathbb{V}^0} - \langle u, \psi - A^* \psi \rangle_{L^2([0, \tau]; \mathbb{V}^0)} + \langle (\nu \cdot \theta) u, \psi \rangle_{L^2([0, \tau]; \partial \Omega \times \mathbb{S})}
\]
\[
= -\langle u_0, \phi \rangle_{\mathbb{V}^0} + \langle m, C\phi \rangle_{L^2([0, \tau]; \mathbb{T}_+)}
\]
where the boundary term appeared from use of the Green’s identity (6). The above identity holds for all $\phi \in \mathbb{V}^0$, but we use it only for $\phi \in L^2(\Omega)$ because we are considering isotropic source term $f$ and isotropic initial condition $u_0$. Hence, we obtain
\[
u_0 + (\mu f P_\sigma \sigma^* f) = P_\sigma C^* m,
\]
where we view $\sigma : L^2([0, \tau]; \mathbb{V}^0) \to L^2([0, \tau]; \mathbb{V}^0)$ as a pointwise multiplicative operator mapping $v \mapsto \sigma v$.

Now we proceed to derive a second equation. Consider,
\[
\langle \sigma f, \psi \rangle_{L^2([0, \tau]; \mathbb{V}^0)} = \langle \dot{u} + Au, \psi \rangle_{L^2([0, \tau]; \mathbb{V}^0)}
\]
\[
= \langle u(\tau), \psi(\tau) \rangle_{\mathbb{V}^0} - \langle u(0), \psi(0) \rangle_{\mathbb{V}^0} - \langle u, \psi - A^* \psi \rangle_{L^2([0, \tau]; \mathbb{V}^0)} + \langle (\nu \cdot \theta) u, \psi \rangle_{L^2([0, \tau]; \partial \Omega \times \mathbb{S})}
\]
leading to
\[
\langle \sigma f, \phi \rangle_{\mathbb{V}^0} + \langle \dot{f} f, \psi \rangle_{L^2([0, \tau]; \mathbb{V}^0)} + \langle u_0, A^* \phi \rangle_{\mathbb{V}^0} = \langle m, C\phi \rangle_{L^2([0, \tau]; \mathbb{T}_+)}
\]
valid for all sufficiently smooth $\phi \in \mathbb{V}^0$. But again we restrict to all smooth $\phi \in L^2(\Omega)$ to obtain,
\[
\left[ (\mu f P_\sigma) + (\mu f P_\sigma \sigma^* f)^* \right] f + \mu_\alpha u_0 = P_\sigma C^* m.
\]
(26)
Here again, the choice of isotropic functions $f$ and $u_0$ leads to an advantageous structure for the above equation. In particular, the action of the angular-averaging operator $P_\sigma$ renders desired compactness (see lemma 3.1 below) as well as the following fact already employed to obtain (26). If $u_0, \phi \in L^2(\Omega)$ are sufficiently smooth then
\[
\langle u_0, A^* \phi \rangle_{\mathbb{V}^0} = |S| \langle u_0, P_\sigma A^* \phi \rangle_{L^2(\Omega)} = |S| \langle u_0, \mu_\alpha \phi \rangle_{L^2(\Omega)}.
\]
The last equality is due to $P_\sigma (\theta \cdot \nabla) \phi = 0$ when $\phi$ is independent of $\theta$, and $P_\sigma (I - K^*) = 0$ due to the conservative nature of the scattering operator $K$. We emphasize that the above equality is a subtle but crucial fact employed in the proof of theorem 2.5.

Equations (25) and (26) constitute the focus of this paper. We already expressed them in operator-valued matrix notation in (22). Notice that the governing operator of the system (22) can be expressed as follows,
\[
\left[ \begin{array}{c|c} (P_\sigma \sigma) & \mu_\alpha \mu_\alpha \sigma^* \\ \hline 0 & I \end{array} \right] + \left[ \begin{array}{c|c} (P_\sigma P_\sigma \sigma^* f)^* & 0 \\ \hline (P_\sigma P_\sigma \sigma^* f)^* & 0 \end{array} \right],
\]
(27)
where the first term is boundedly invertible on $L^2(\Omega) \times L^2(\Omega)$ provided that $|\langle P_\sigma \sigma \rangle| \geq \delta > 0$ and the second term is a compact operator on $L^2(\Omega) \times L^2(\Omega)$ as asserted by lemma 3.1 below. Hence, we obtain a Fredholm system.

Now we prove the existence of an open and dense set for $\sigma \in C^1([0, \tau]; \mathbb{V}^1) \cap C^2([0, \tau]; L^\infty(\Omega \times \mathbb{S})) \cap (i)$ on which (27) is boundedly invertible. First, standard
perturbation shows that the set of \( \sigma \)'s over which the (27) is invertible in \( L^2(\Omega) \times L^2(\Omega) \) is open. To show denseness, consider replacing \( \sigma \) with

\[
\rho(\lambda) = \lambda \sigma + (1 - \lambda) \sigma_0.
\]

Now notice that the first term in (27) remains unchanged for any choice of \( \lambda \in \mathbb{C} \), and the second term remains compact and analytic with respect to \( \lambda \in \mathbb{C} \). If we set \( \lambda = 0 \), then the governing operator (27) becomes

\[
\begin{bmatrix}
(P_0 \sigma_0) & \mu_a \\
(P_0 P_0 \sigma_0^* S) & I
\end{bmatrix}
\]

which is boundedly invertible provided that \( \mu_a \) is sufficiently small. By the analytic Fredholm theorem [29], then the system is boundedly invertible for all but a discrete set of \( \lambda \)'s. In particular, this holds for values arbitrarily close to \( \lambda = 1 \). This shows the desired denseness. The other claims of theorem 2.5 are well-known consequences of Fredholm-Riesz-Schauder theory.

Before going into lemma 3.1, we wish to make some remarks. Notice that if \( f \) and \( u_0 \) were not isotropic, then we would have gotten the following governing operator

\[
\begin{bmatrix}
\sigma_0 + (P_t \sigma S)^* & A \\
(P_t \sigma S)^* & I
\end{bmatrix}
\]

However, this operator does not have a favorable form. In other words, it is the isotropy of both \( f \) and \( u_0 \) what leads to the replacement of \( A \) by \( \mu_a \), and to the appearance of the angular-averaging operator \( P_0 \) which renders the needed compactness. If only one of the unknowns (\( f, u_0 \)) is assumed isotropic, we do not obtain a favorable structure either as the reader can easily check. In practical applications it is usually acceptable to assume \( f \) is independent of \( \theta \in \mathbb{S} \). However, assuming that \( u_0 \) is isotropic constitutes the most restrictive assumption needed for our approach to work.

Now we proceed to prove a lemma already employed in the proof of theorem 2.5.

**Lemma 3.1.** If \( \sigma \in C^1([0, \tau]; L^\infty(\Omega \times \mathbb{S})) \) then both \( P_0 P_0 \sigma S : L^2(\Omega) \to L^2(\Omega) \) and \( P_0 P_0 \sigma S : L^2(\Omega) \to L^2(\Omega) \) are compact operators.

**Proof.** First, we consider \( \sigma \in C^\infty([0, \tau] \times \overline{\Omega} \times \mathbb{S}) \), keeping in mind that this condition will be relaxed later.

Let \( \phi \in L^2(\Omega) \) and \( \eta = C\phi \) and \( \psi = S\phi \). We proceed with a density argument by having \( \{\eta^\varepsilon\}_{\varepsilon > 0} \subset C([0, \tau]; T_+) \) be a family of functions such that \( \eta^\varepsilon(\tau) = 0 \) and \( \eta^\varepsilon \to \eta \) in the norm of \( L^2([0, \tau]; T_+) \) as \( \varepsilon \to 0 \). Let also \( \psi^\varepsilon \in C^1([0, \tau]; V^0) \cap C([0, \tau]; V^1) \) be the unique strong solution of (14)-(16) with \( \eta^\varepsilon \) as the prescribed outflow boundary condition. Notice that \( \psi^\varepsilon \to \psi \) in the norm of \( C([0, \tau]; V^0) \) because (14)-(16) is well-posed in a mild sense.

Now, let \( \varphi^\varepsilon = \sigma \psi^\varepsilon \in C^1([0, \tau]; V^0) \cap C([0, \tau]; V^1) \). Notice that \( \varphi^\varepsilon \) satisfies (in a strong sense) the following problem,

\[
\varphi^\varepsilon + (\theta \cdot \nabla) \varphi^\varepsilon = F^\varepsilon \quad \text{in} \quad [0, \tau] \times (\Omega \times \mathbb{S}),
\]

\[
\varphi^\varepsilon = 0 \quad \text{on} \quad \{t = \tau\} \times (\Omega \times \mathbb{S}),
\]

\[
\gamma_+ \varphi^\varepsilon = G^\varepsilon \quad \text{on} \quad [0, \tau] \times (\partial \Omega \times \mathbb{S}^+),
\]

where \( G^\varepsilon = \gamma_+ (\sigma \psi^\varepsilon) \in C([0, \tau]; T_+) \) and \( F^\varepsilon = \psi^\varepsilon (\sigma + (\theta \cdot \nabla) \sigma) \) and \( (\mu_a + \mu_b (I - K^*)) \psi^\varepsilon \in C([0, \tau]; V^0) \).
Hence, the time-integral $\varphi^\epsilon = P_t \varphi^\epsilon$ satisfies a stationary problem of the form
\begin{equation}
(\theta \cdot \nabla) \varphi^\epsilon = P_t F^\epsilon + \sigma(0) \psi^\epsilon(0) \quad \text{in } (\Omega \times S),
\varphi^\epsilon \mid_{\partial \Omega \times S} = P_t G^\epsilon \quad \text{on } (\partial \Omega \times S)_{+}.
\end{equation}
The latter is a well-posed stationary adjoint problem (see for instance [3]) with prescribed outflow condition $P_t G^\epsilon \in T_+ \text{ and forcing term } P_t F^\epsilon + \sigma(0) \psi^\epsilon(0) \in V^0$. The solution $\varphi^\epsilon \in V^1$ depends continuously on the input data in the appropriate norms. Therefore,
\begin{equation}
\| \varphi^\epsilon \| _{V^1} \leq C \left( \| \varphi^\epsilon \| _{C([0,\tau];V^0)} + \| \eta^\epsilon \| _{L^2([0,\tau];T_+)} \right) \leq \tilde{C} \| \eta^\epsilon \| _{L^2([0,\tau];T_+)}.
\end{equation}

for all $\epsilon > 0$ where $\eta^\epsilon \to \eta = C\phi$ in the norm of $L^2([0,\tau];T_+)$. This in turn implies that the mapping $\phi \mapsto P_t \sigma S \phi$ extends as a bounded operator from $L^2(\Omega)$ to $V^1$.

Finally, from well-known averaging lemmas [28], we obtain that $P_0 P_t \sigma S : L^2(\Omega) \to H^{1/2}(\Omega)$ is bounded. Our claim follows due to the compact Sobolev embedding of $H^{1/2}(\Omega)$ into $L^2(\Omega)$. The above proof is valid for a sufficiently smooth $\sigma$. However, it is straightforward to show that
\begin{equation}
\| P_0 P_t (\sigma_1 - \sigma_2) S \| \leq C \| \sigma_1 - \sigma_2 \| _{C([0,\tau];L^\infty(\Omega \times S))}.
\end{equation}

Since the subspace of compact operators is closed, then we conclude that $P_0 P_t \sigma S : L^2(\Omega) \to L^2(\Omega)$ is compact for $\sigma \in C([0,\tau];L^\infty(\Omega \times S))$.

The proof of compactness for $P_0 P_t \sigma S : L^2(\Omega) \to L^2(\Omega)$ is the same due to our assumption that $\sigma \in C^1([0,\tau];L^\infty(\Omega \times S))$. \qed

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