SYMOMETRY NON RESTORATION AND
INVERSE SYMMETRY BREAKING ON THE LATTICE

G. Bimonte
Departamento de Física Teórica, Facultad de Ciencias
Universidad de Zaragoza, 50009 Zaragoza, SPAIN

and

G. Lozano
Imperial College, Theoretical Physics
Prince Consort Road, SW7 2BZ London, ENGLAND

Abstract

We study the finite temperature symmetry behaviour of $O(N_1) \times O(N_2)$ scalar models on the lattice and we prove that at sufficiently high temperatures and in arbitrary dimensions their full symmetry is always restored or, equivalently, that the phenomenon of Symmetry Non Restoration which, according to lowest order perturbation theory, takes place in the continuum version of these models, does not occur on the lattice.

\textsuperscript{1} E-mail addresses: bimonte@napoli.infn.it, g.lozano@ic.ac.uk
The high temperature behaviour of relativistic field theories has been the subject of an intense research since the early works by Kirzhnitz and Linde [1], Weinberg [2] and Dolan and Jackiw [3]. The results of these investigations show that in a "typical" case the symmetry of the vacuum increases when the temperature is raised and thus, in spontaneously broken theories, internal symmetries are gradually restored when the system is heated up. Thus, in the context of Grand Unified Theories and Cosmology the fact that the universe might have been in phases with different symmetry properties at different stages of its evolution, undergoing a series of phase transitions in the process of cooling down, would have significant consequences, like for instance the creation of topological defects via the Kibble mechanism [4].

Nevertheless, as noticed by Weinberg [2], $O(N_1) \times O(N_2)$ theories may present an “atypical” symmetry behaviour, where by atypical, we mean that either the symmetry is not restored at high temperatures or that an exact symmetry of the low temperature theory becomes broken at higher temperatures. These phenomena, known as Symmetry Non Restoration (SNR) and Inverse Symmetry Breaking (ISN) are in fact two aspects of the same problem, the only difference between them being whether or not the symmetry is broken in the zero temperature theory. To lowest order in perturbation theory, the existence of SNR or ISB is related to the possibility of having negative Debye masses. This can be achieved in multi-scalar theories, provided that some of the fourth-order couplings are taken negative and large enough in absolute value (but also small enough as to produce a bounded potential). As on the other hand, the scalar sector of most extensions of the Standard Model and Grand Unified Theories is rather undetermined, it turns out that SNR and ISB are not so atypical as one would first suppose.

Indeed, it has been recognised that the ideas of SNR and ISB can have interesting phenomenological implications. Very recently for instance, the phenomenon of SNR has been used by Dvali, Melfo and Senjanović [5] to suggest that the monopole problem might not exist in some GUT’s, by arguing that the monopole-producing phase transition might have never occurred (for an earlier implementation of this idea see [6]). Also in connection to monopoles, ISB is the basis of the proposal by Langacker and Pi [7], which states that a period of broken $U(1)_{em}$ can cure the monopole problem. Other no less interesting
applications of SNR and ISB concern the breaking of the CP symmetry \[8\], the domain wall problem \[9\], Baryogenesis \[10\], Inflation \[11\] and P, Strong CP, and Peccei-Quinn Symmetries \[12\]. ISB and SNR have also been considered in \[13\].

Appealing as the idea of SNR and ISB may be, no consensus has yet been reached on whether they really correspond to a true physical effect or rather to an artifact of (may be, lowest order?) perturbation theory. This second point of view stems from the fact that when non-perturbative approximations are used to study the symmetry behaviour of these models \[14\], it is found that symmetry is invariably restored at high temperature. Moreover, even staying within the realms of perturbation theory, it has been shown \[15\] that the inclusion of next to leading order effects in the calculation of thermal masses tends to reduce the region of parameter space in which SNR and ISB occur. Recently \[16\], the gap equations used in \[17\] have been rederived by a more detailed analysis based in the Cornwall-Jackiw-Tamboulis effective potential \[17\] at finite temperature \[18\]. An independent study, which also encodes some non perturbative information through the Effective Action Technique \[19\], has been carried out \[20\]. For the small values of the scalar self-couplings which were considered in \[20\], it was found that the corrections to the lowest order perturbative computation are small. While this is not in disagreement with the results of \[13\], which also predict small corrections when these couplings are small, we think that an extension of the analysis of \[20\] to include large couplings may be unavoidable, since, as pointed out in \[13\], realistic models may require large scalar self-couplings, due to the presence of gauge interactions which conspire against SNR and ISB.

Due to the conflicting results which emerge from the perturbative, semi perturbative and non-perturbative methods we mentioned above, we think that Lattice Field Theory might result to be a useful setting to study SNR and ISB. Some work along this lines has been already done in \[21\] where it is shown that, when an approximation based on the constraint effective potential \[22\] is made, symmetry is always restored in less than four dimensions.

In this letter, we will study SNR and ISB on the Lattice without making approximations and in arbitrary dimensions and we will prove that symmetry is always restored, at sufficiently high temperatures,
for $O(N_1) \times O(N_2)$ models. In doing this, we will closely follow the steps of a theorem by King and Yaffe on symmetry restoration for $O(N)$ models on the Lattice. It is worth mentioning that while King and Yaffe’s result for $O(N)$ models confirms the result obtained from perturbation theory, the result we present here exactly contradicts the lowest order perturbative calculation.

The continuum model we initially consider is a global $O(N_1) \times O(N_2)$ scalar theory in $(d+1)$ euclidean dimensions described by the action,

$$S = \int d^{d+1}x \left\{ \sum_{i=1,2} \left[ \frac{1}{2} \partial_\mu \phi_i \right|^2 + \frac{1}{2} m_i^2 \left| \phi_i \right|^2 + \lambda_i \left( \left| \phi_i \right|^2 \right)^2 \right\} - \lambda \left| \phi_1 \right|^2 \left| \phi_2 \right|^2, \right\},$$

(1)

where $\phi_i \equiv (\phi_i^{(1)}, \cdots, \phi_i^{(N_i)})$ is an $N_i$-component real scalar and $\left| \phi_i \right|^2 = \sum_{j=1}^{N_i} (\phi_i^{(j)})^2$. The condition of boundedness for the potential constrains the coupling constants to satisfy the relations:

$$\lambda_i > 0, \quad 4\lambda_1 \lambda_2 > \lambda^2.$$  

(2)

This condition allows for positive values of $\lambda$ and, as it immediately follows from the one-loop computation [2], if $\lambda$ is such that,

$$\lambda > \frac{4 + 2N_2}{N_1} \lambda_2$$

(3)

the $O(N_1) \times O(N_2)$ symmetry is necessarily broken to $O(N_1) \times O(N_2 - 1)$ at high $T$. The symmetry at lower temperatures depends instead on the signs and magnitudes of the masses $m_i^2$.

We will now prove that when the model (1) is defined on a discrete lattice of points, the full $O(N_1) \times O(N_2)$ symmetry is restored at sufficiently high temperatures, for all values of the parameters in the action.

We consider an anisotropic hypercubic $(d+1)$-dimensional lattice $\Lambda \equiv \mathcal{T} \times \Sigma$. Here, $\Sigma$ is an infinite $d$-dimensional hypercubic lattice accounting for space, while $\mathcal{T}$ is a finite one-dimensional lattice consisting of $N_\tau$ points, accounting for the finite euclidean time axis. We assume a priori distinct spacings $\Delta x$ and $\Delta \tau$ for $\Sigma$ and $\mathcal{T}$ respectively. Thus:

$$\Lambda \equiv \mathcal{T} \times \Sigma = \{ x = (x_0, \bar{x}) : x_0 = n_0 \Delta \tau \quad \bar{x} = (n_1 \Delta x, \cdots, n_d \Delta x) ; \}

n_0 = 1, \cdots, N_\tau; \quad n_i \in \mathbb{Z} \quad i = 1, \cdots, d \} .$$

(4)
The temperature of the system is then given by:

\[ T = \frac{1}{N \tau \Delta \tau}. \]  

(5)

The lattice version of the model is described by the action:

\[ S = \sum_{x \in \Lambda} \Delta \tau (\Delta x)^d \left\{ \sum_i \left[ \frac{1}{2} \left( \frac{1}{(\Delta \tau)^2} | \hat{\phi}_i(x + e_x \Delta \tau) - \hat{\phi}_i(x) |^2 + \sum_{k=1}^d \frac{1}{(\Delta x)^2} | \hat{\phi}_i(x + e_k \Delta x) - \hat{\phi}_i(x) |^2 \right) \\
+ \frac{1}{2} \ddot{m}_i^2 | \hat{\phi}_i(x) |^2 + \ddot{\lambda}_i | \hat{\phi}_i(x) |^2 \right] - \ddot{\lambda} | \hat{\phi}_1(x) |^2 | \hat{\phi}_2(x) |^2 \right\} \right. \]  

(6)

where \( e_x \) and \( e_k \) are unit vectors pointing along the time and \( k \)-th spatial directions respectively and the fields \( \hat{\phi}_i(x) \) satisfy periodic boundary conditions in the time direction:

\[ \hat{\phi}_i(x_0 + \Delta \tau N \tau, \bar{x}) = \hat{\phi}_i(x_0, \bar{x}). \]  

(7)

It is convenient to introduce the dimensionless quantities:

\[ \hat{\phi}_i(x) = (\Delta x)^\frac{\Delta \tau}{a} \phi_i(x), \quad a = \frac{\Delta \tau}{\Delta x}, \]

\[ m_i^2 = (\Delta x)^2 \ddot{m}_i^2, \quad \lambda_i = (\Delta x)^3 - d \ddot{\lambda}_i, \quad \lambda = (\Delta x)^3 - d \ddot{\lambda}. \]  

(8)

In terms of them the action \((6)\) reads:

\[ S = \sum_{x \in \Lambda} \Delta \tau (\Delta x)^d \left\{ \sum_i \left[ \frac{1}{2} \left( \frac{1}{a} | \phi_i(x + e_x \Delta \tau) - \phi_i(x) |^2 + \sum_{k=1}^d a | \phi_i(x + e_k \Delta x) - \phi_i(x) |^2 \right) \right. \\
+ \frac{1}{2} \ddot{m}_i^2 | \phi_i(x) |^2 + \ddot{\lambda}_i | \phi_i(x) |^2 \right] - \ddot{\lambda} | \phi_1(x) |^2 | \phi_2(x) |^2 \right\} \]  

(9)

It is also convenient to measure the temperature in units of the inverse of the lattice spacing \( \Delta x \). We thus define the “lattice temperature”

\[ T_L = (\Delta x) T = \frac{1}{N a} \equiv \frac{1}{\beta_L}. \]  

(10)

A way to test the symmetry of the system is to consider the two-point functions:

\[ \langle \phi_j(z) \cdot \phi_j(w) \rangle \equiv Z^{-1} \int \left( \prod_{x \in \Lambda} \prod_{i=1,2} d\phi_i(x) \right) \phi_j(z) \cdot \phi_j(w) \exp(-S), \]  

(11)
where \( Z \) is the partition function and \( z \) and \( w \) are two spacelike separated points of \( \Lambda \). In the broken phases, one or both of the correlators (11) have a non-vanishing limit \( M_j^2 \) for infinite separations:

\[
M_j^2 = \lim_{|z-w| \to \infty} \langle \phi_j(z) \cdot \phi_j(w) \rangle \neq 0 \quad \text{for some } j
\]  

(12)

while in the symmetric phase both these limits vanish. Now, we will prove that, for all values of the mass parameters and coupling constants in (9), and in any dimension \( d \), there exists a temperature \( T^*_L \) above which the correlators (11) decay exponentially with the separation \( |z-w|\):

\[
| \langle \phi_j(z) \cdot \phi_j(w) \rangle | \leq c \sqrt{\frac{T_L}{\nu}} \exp\left[-M(T_L)|z-w|\right] \quad T_L \geq T^*_L.
\]  

(13)

In this equation \( c \) is a numerical constant, \( M(T_L) \) is a function such that:

\[
M(T_L) > 0 \quad \text{for } T_L > T^*_L; \quad \lim_{T_L \to \infty} M(T_L) = \infty
\]  

(14)

and \( \nu \) is a constant independent on \( T_L \). Once (13) is proved, it then will follow that above \( T^*_L \) both limits \( M_j^2 \) vanish and the full \( O(N_1) \times O(N_2) \) symmetry is restored.

Before proceeding further, a few comments are in order:

a) the temperature \( T^*_L \) provides only an upper bound for the true critical temperature \( T^*_L \) of the symmetry-restoring phase transition. As pointed out already in [23], this bound is not expected to have the correct dependence on the bare coupling constants, for large values of the latter. Consequently, based on this bound, no conclusions can be drawn for the critical temperature in the continuum limit, whenever this limit exists.

b) the bound we are going to derive, like the one in [23] for the \( O(N) \) models, depends on \( a \) only through the temperature \( T_L = aN_\tau \). This will allow us to take the continuum limit in the time direction \( a \to 0, \ aN_\tau \to \text{const} \) in a straightforward way.

We now turn to the proof, which is, as we mentioned before, a simple extension of King and Yaffe theorem on symmetry restoration for \( O(N) \) models [23]. The intuitive idea behind it is that, at high temperature, the system can be thought of as a collection of oscillators sitting on the sites of the spatial lattice \( \Sigma \), such that no order is possible. Guided by this idea, we will write the action, the partition function and the correlators in a way which will turn to be useful to derive the bound (13).
We will start by rewriting the action \( \Phi \) as:

\[
S = \sum_{\bar{x} \in \Sigma} S_{\bar{x}} - \sum_{l \in \Sigma^*} V(l) ,
\]

where \( \Sigma^* \) is the set of links in \( \Sigma \) and

\[
S_{\bar{x}} = \sum_{x_0 \in T} \sum_{i} \left[ \frac{1}{2} a \left| \phi_i(x_0 + \Delta \tau, \bar{x}) - \phi_i(x_0, \bar{x}) \right|^2 + \frac{1}{2} a \left( m_i^2 + 2d \right) \left| \phi_i(x_0, \bar{x}) \right|^2 + a \lambda_i \left( \left| \phi_i(x_0, \bar{x}) \right|^2 \right)^2 \right] +
\]
\[
- \sum_{x_0 \in T} a \lambda \left| \phi_1(x_0, \bar{x}) \right|^2 \left| \phi_2(x_0, \bar{x}) \right|^2 ,
\]

while

\[
V(l) = \frac{1}{2} \sum_{x_0 \in T} \sum_{i=1,2} a \left| \phi_i(x_0, \bar{x}) + \phi_i(x_0, \bar{y}) \right|^2 ,
\]

where \( \bar{x} \) and \( \bar{y} \) are the end points of the link \( l \). Now, the first term on the r.h.s. of (15) precisely describes a set of uncoupled oscillators located at the sites of \( \Sigma \), while the second sum provides an interaction among them. Notice also that the \( V(l) \) is positive, a property which will be important in what follows.

Upon defining the measure

\[
d\mu = \prod_{\bar{x} \in \Sigma} d\mu_{\bar{x}}
\]

where

\[
d\mu_{\bar{x}} = z^{-1} \prod_{x_0 \in T} \prod_{i=1,2} d\phi_i(x_0, \bar{x}) \exp[-S_{\bar{x}}] .
\]

and \( z \) is a coefficient that normalises \( d\mu_{\bar{x}} \) to one, the partition function and the correlators can be written as

\[
Z = \int d\mu \exp \left[ \sum_{l \in \Sigma^*} V(l) \right] ,
\]

\[
\langle \phi_j(z) \cdot \phi_j(w) \rangle \equiv Z^{-1} \int d\mu \overline{\phi_j(z)} \cdot \phi_j(w) \exp \left[ \sum_{l \in \Sigma^*} V(l) \right] .
\]

If we now define:

\[
\exp V(l) = 1 + \rho(l) ,
\]

and

\[
K(Y) = \int d\mu \prod_{l \in Y} \rho(l) ,
\]
then
\[ Z = \int d\mu \prod_{l \in \Sigma^*} (1 + \rho(l)) = \sum_{Y \subset \Sigma^*} K(Y) \]  
(23)

In the same way, defining
\[ K_j(Y) = \int d\mu \overline{\phi_j(z)} \cdot \phi_j(w) \prod_{l \in Y} \rho(l) \],
(24)
we then have:
\[ \langle \overline{\phi_j(z)} \cdot \phi_j(w) \rangle \equiv Z^{-1} \sum_{Y \subset \Sigma^*} K_j(Y) \].
(25)

Here and in (23) the sum is over all subsets \( Y \) of \( \Sigma^* \).

Let now \( W \) be the connected component of \( Y \) that contains \( w \) and let \( X = Y - W \). As the action \( S_\bar{x} \) is invariant under \( \phi_i \to -\phi_i \), the only non-vanishing terms in the sum (25) are those for which \( z \) is in \( W \).

Together with the fact that the measures \( d\mu_\bar{x} \) are normalised to one, this allows us to write:
\[ \langle \overline{\phi_j(z)} \cdot \phi_j(w) \rangle = \sum_{W \subset \Sigma^* \atop z, w \in W} K_j(W) \sum_{X \subset \Sigma^* - W} \frac{1}{Z} K(X) 
(26) \]
where \( \overline{W} \) denotes the closure of \( W \), namely the set of links in \( \Sigma^* \) that share an end-point with some link in \( W \).

Up to here, we have only rewritten \( Z \) and the correlation functions in a convenient way. We shall now start to look for bounds for these quantities. The positivity of \( V(l) \) comes now into play in a crucial way, as it immediately implies:
\[ \frac{1}{Z} \sum_{X \subset \Sigma^* - W} K(X) = \int d\mu \exp \left[ \sum_{l \in \Sigma^* - W} V(l) \right] \leq 1 \quad \forall W \subset \Sigma^* \],
(27)
which in turn implies the bound:
\[ \langle \overline{\phi_j(z)} \cdot \phi_j(w) \rangle \leq \sum_{W \subset \Sigma^* \atop z, w \in W} K_j(W) \].
(28)

The next step is then to find a bound for \( K_j(W) \). First of all, we observe that:
\[ | \rho(l) | = | \exp V(l) - 1 | \leq | V(l) | \exp V(l) \].
(29)
Moreover, by Schwarz’s inequality, we have

\[ V(l) \leq \sum_{x_0 \in T} \sum_{i=1,2} \left[ a \left| \phi_i(x_0, \bar{x}) \right|^2 + a \left| \phi_i(x_0, \bar{y}) \right|^2 \right], \tag{30} \]

which implies the other bound

\[ \prod_{l \in W} V(l) \leq \sum_{\{q(x)\} \in \Omega} \prod_{x_0 \in T} \left\{ \sum_{x_0 \in T} \sum_{i=1,2} a \left| \phi_i(x_0, \bar{x}) \right|^2 \right\}^{q(\bar{x})}, \tag{31} \]

Here the sum is over all possible choices of the non-negative integer \( q(\bar{x}) \leq 2d \) which vanish for all \( \bar{x} \notin W \) and are such that

\[ \sum_{\bar{x} \in \Omega} q(\bar{x}) = |W| \tag{32} \]

where \(|W|\) denotes the number of links in \( W \). Use of the inequality:

\[ |\phi_j(z) \cdot \phi_j(w)| \leq |\phi_j(z)|^2 + |\phi_j(w)|^2 \tag{33} \]

finally implies the bound:

\[ |K_j(W)| \leq \sum_{\{q(x)\} \in \Omega} \int d\mu \left( |\phi_j(z)|^2 + |\phi_j(w)|^2 \right) \times \prod_{\bar{x} \in \Omega} \left\{ \sum_{x_0 \in T} \sum_{i=1,2} a \left| \phi_i(x_0, \bar{x}) \right|^2 \right\}^{q(\bar{x})} \exp \left[ p(\bar{x}) \sum_{x_0 \in T} \sum_{i=1,2} a \left| \phi_i(x_0, \bar{x}) \right|^2 \right]. \tag{34} \]

The factor \( p(\bar{x}) \) in the above formula represents the number of links in \( W \) that have \( \bar{x} \) as endpoint. Obviously

\[ p(\bar{x}) \leq 2d, \quad \forall \bar{x}. \tag{35} \]

The important feature of eq. (34) is that \( K_j(W) \) is now bounded by a sum of products of independent one-dimensional integrals. The next move is to bound the latter by Gaussian integrals. To this purpose, we define:

\[ d\nu_z = z'^{-1} \prod_{x_0 \in T} \prod_{i=1,2} d\phi_i(x_0, \bar{x}) \exp \left\{ -\frac{1}{a} \sum_{x_0 \in T} \sum_{i=1,2} \frac{1}{a} \left| \phi_i(x_0 + \Delta \tau, \bar{x}) - \phi_i(x_0, \bar{x}) \right|^2 + a \mu^2 \left| \phi_i(x_0, \bar{x}) \right|^2 \right\}. \tag{36} \]

In this equation, \( z' \) is a normalisation factor, while \( \mu^2 \) represents a variational parameter, whose value will be fixed at the end such as to provide the best bound. For simplicity, we have introduced a common
parameter $\mu^2$ for both fields $\phi_i$, while in principle a better bound could be obtained by letting distinct values. The derivation of the bound would in this case be slightly more involved but the final bound would not be qualitatively different from the one obtained with one parameter only.

Let us now go back to (34): each term in the sum is, as we said, a product of independent one-dimensional integrals, $I_\bar{x}$, one for each $\bar{x} \in W$. There are now two cases to be considered: whether $\bar{x}$ is distinct from $z$ and $w$ or not. In the first case, with the aid of (36), we can write the corresponding one-dimensional integral $I_{\bar{x}}$ as:

$$I_{\bar{x}} = \frac{N_{\bar{x}}}{D_{\bar{x}}}$$

with

$$N_{\bar{x}} = \int \nu_{\bar{x}} \left( \sum_{x_0 \in T} \sum_{i=1,2} a \phi_i(x_0, \bar{x}) |^2 \right)^{q(\bar{x})} \exp T_{\bar{x}}[p(\bar{x})]$$

and

$$D_{\bar{x}} = \int \nu_{\bar{x}} \exp T_{\bar{x}}[p = 0]$$

and

$$T_{\bar{x}}[p(\bar{x})] = \sum_{x_0 \in T} \sum_{i=1,2} \left[ \left( \frac{1}{2} \mu^2 - \frac{1}{2} m_1^2 + p(\bar{x}) - 2d \right) a \phi_i(x_0, \bar{x}) |^2 - a \lambda_i (\phi_i(x_0, \bar{x}) |^2)^2 \right] +$$

$$+ \sum_{x_0 \in T} a \lambda | \phi_1(x_0, \bar{x}) |^2 | \phi_2(x_0, \bar{x}) |^2 .$$

Now, it is trivial to check that $T_{\bar{x}}[p(\bar{x})]$ is bounded by:

$$T_{\bar{x}}[p(\bar{x})] \leq \frac{\beta L}{4 \lambda_1 \lambda_2 - \lambda_2} \left[ \lambda_2 \left( \frac{1}{2} \mu^2 - \frac{1}{2} m_1^2 + p(\bar{x}) - 2d \right)^2 + \lambda_1 \left( \frac{1}{2} \mu^2 - \frac{1}{2} m_2^2 + p(\bar{x}) - 2d \right)^2 + \right.$$  

$$+ \lambda \left( \frac{1}{2} \mu^2 - \frac{1}{2} m_1^2 + p(\bar{x}) - 2d \right) \left( \frac{1}{2} \mu^2 - \frac{1}{2} m_2^2 + p(\bar{x}) - 2d \right) \right] \equiv A ,$$

where we have used $\sum_{x_0 \in T} a = \beta L$. We then find that:

$$N_{\bar{x}} \leq \exp(A) \int \nu_{\bar{x}} \left( \sum_{x_0 \in T} \sum_{i=1,2} a \phi_i(x_0, \bar{x}) |^2 \right)^{q(\bar{x})} .$$

To proceed further, we will need to use the following bounds for the moments of a Gaussian distribution:

$$\int \nu \phi_i(x_0^{(1)}, \bar{x}) |^2 \cdots | \phi_i(x_0^{(p)}, \bar{x}) |^2 \leq (2p - 1)! \left[ \frac{N_i \mu^2 + \beta L}{\beta L \mu} \right]^p$$

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Using (45) together with (44) we obtain:

\[
\int d\nu_\bar{x} \left| \phi_1(x_0, \bar{x}) \right|^2 \geq \frac{N_i}{\beta_L \mu^2}, \\
\int d\nu_\bar{x} \left| \phi_1(x_0, \bar{x}) \right|^2 \left| \phi_2(x_0, \bar{x}) \right|^2 \geq \frac{N_1N_2}{\beta_L^2 \mu^4},
\]

to write:

\[
N_\bar{x} \leq \exp(\mathcal{A}) \sum_{r=0}^{q(\bar{x})} \left( \begin{array}{c} r \\ q(\bar{x}) \end{array} \right) \int d\nu_\bar{x} \left( \sum_{x_0 \in \mathcal{T}} a \left| \phi_1(x_0, \bar{x}) \right|^2 \right)^r \left( \sum_{x_0 \in \mathcal{T}} a \left| \phi_2(x_0, \bar{x}) \right|^2 \right)^{q(\bar{x})-r}
\]

\[
\leq (4d - 1)!! \sum_{r=0}^{q(\bar{x})} \left( \begin{array}{c} r \\ q(\bar{x}) \end{array} \right) \left[ N_1 \left( \frac{1}{\mu^2} + \frac{\beta_L}{\mu} \right) \right]^r \left[ N_2 \left( \frac{1}{\mu^2} + \frac{\beta_L}{\mu} \right) \right]^{q(\bar{x})-r} \exp \mathcal{A} = (4d - 1)!! \left( N_1 + N_2 \right) \left( \frac{1}{\mu^2} + \frac{\beta_L}{\mu} \right)^{q(\bar{x})} \exp \mathcal{A} .
\]

As for $D_\bar{x}$, we can use Jensen’s inequality (see [24]) to bound it as:

\[
D_\bar{x} \geq \exp \int d\nu_\bar{x} T_\bar{x}[0] \geq \exp \left\{ \sum_{i=1,2} \left[ \frac{N_i}{\mu^2} \left( \frac{1}{2} \mu^2 - \frac{1}{2} m_i^2 - 2d \right) - 3\lambda_i \beta_L \left( \frac{N_i}{\beta_L \mu^2} + \frac{N_i}{\mu} \right)^2 \right] + \frac{\lambda N_1N_2}{\beta_L \mu^4} \right\} \equiv \exp(-\mathcal{B}) .
\]

Using (45) together with (44) we obtain:

\[
I_\bar{x} \leq (4d - 1)!! \left( N_1 + N_2 \right) \left( \frac{1}{\mu^2} + \frac{\beta_L}{\mu} \right)^{(4d - 1)！！} \exp(A + B)
\]

If instead $\bar{x}$ coincides with either $z$ or $w$ the previous bound has to be multiplied by a factor of

\[
(4d + 1) \frac{N_i}{\beta_L} \left( \frac{1}{\mu^2} + \frac{\beta_L}{\mu} \right).
\]

Since now the only sites $\bar{x}$ for which $p(\bar{x})$ or $q(\bar{x})$ are different from zero are those which are endpoints of some link in $W$ and since there are less than $2 | W |$ such sites, we can use the previous bounds and eq.(32) to say

\[
| K_j(W) | \leq 2(4d+1) \frac{N_j}{\beta_L} \left( \frac{1}{\mu^2} + \frac{\beta_L}{\mu} \right) \sum_{\{q(\bar{x})\}} \left[ \left( N_1 + N_2 \right) \left( \frac{1}{\mu^2} + \frac{\beta_L}{\mu} \right) \right]^{(4d-1)!!} \exp(2 | W | (A+B)) .
\]

The number of terms in the sum above is less than $(2d)^2|W|$ (see [23] for a proof) and thus we have:

\[
| K_j(W) | \leq 2(4d+1) \frac{N_j}{\beta_L} \left( \frac{1}{\mu^2} + \frac{\beta_L}{\mu} \right) \left[ \bar{c}(N_1 + N_2) \left( \frac{1}{\mu^2} + \frac{\beta_L}{\mu} \right) \right]^{(4d-1)!!} \exp(2 | W | (A+B)) ,
\]
where \( \hat{c} = 4d^2((4d - 1)!!)^2 \). If we now take:
\[
\mu^2 = \sqrt{\frac{\nu}{\beta_L}},
\]
(48)
where \( \nu \) is independent on \( \beta_L \), there will be a \( \bar{\beta}_L \) such that, for \( \beta_L < \bar{\beta}_L \)
\[
\mu \beta_L < 1 \quad \mu^2 \geq |m_i^2| \quad i = 1, 2.
\]
(49)
These relations, together with (35) then imply:
\[
|K_j(W)| \leq 4(4d + 1) \frac{N_j}{\sqrt{\beta_L \nu}} \left[ 2\hat{c}(N_1 + N_2)\sqrt{\frac{\beta_L}{\nu}} \right]^{W_j} \times \exp \left\{ 2 |W| \left[ 2d(N_1 + N_2)\sqrt{\frac{\beta_L}{\nu}} + \frac{\lambda_1 + \lambda_2 + \lambda}{4\lambda_1\lambda_2 - \lambda^2} \frac{1}{\nu}(12\lambda_1 N_1^2 + 12\lambda_2 N_2^2) \right] \right\}
\equiv e^{\frac{N_j}{\sqrt{\beta_L \nu}}} \exp \left[ - |W| M'(\beta_L) \right].
\]
(50)
It is clear that \( \lim_{\beta_L \to 0} M'(\beta_L) = \infty \) and thus there exists a \( \beta_L^* \leq \bar{\beta}_L \) such that, for \( \beta_L < \beta_L^* \), \( M'(\beta_L) > 0 \).

The steps to derive the final bound (13) from eq.(50) are exactly the same as in [23] and we omit repeating the short proof here. \( (\beta_L^*)^{-1} \) thus represent our bound for the true critical temperature of the symmetry-restoring phase transition:
\[
T_c^* < \frac{1}{\beta_L^*}.
\]
(51)
\( \beta_L^* \) depends of course on \( \nu, d, \lambda_i, \lambda \) and \( m_i^2 \) and one might further exploit the freedom in the choice of \( \nu \) in order to get the best bound; as the analogue best bound for \( \beta_L^* \) in the \( O(N) \) case [23] does not have the right behaviour for large values of the bare couplings and as we believe that the same will occur in our case, we will not explicitly show it here.

The above proof can be extended to the case when the \( O(N_1) \times O(N_2) \) symmetry is gauged (partly or completely), along the lines of [23], and we refer the reader to that paper for the details.

We have proved that, in arbitrary dimensions and at sufficiently high temperatures, the full \( O(N_1) \times O(N_2) \) symmetry is restored, or equivalently, that the phenomenon of Symmetry Non Restoration does not occur on the lattice. Notice, though, that the implications of our proof regarding Inverse Symmetry Breaking are weaker, in the sense that we can not exclude the possibility of this phenomenon taking
place at intermediate temperatures. We can only state that if an ISB phase transition occurs at a given temperature, then the system will necessarily undergo a symmetry restoring phase transition at higher temperatures.

Whether these results are relevant for the continuum, when the continuum limit exists, depends on the behaviour of the critical temperature of the symmetry-restoring phase transition, when the lattice spacing $\Delta x$ is taken to zero. Analogously to the $O(N)$ case, our bound for $T_c$ diverges when the bare couplings become large and thus it is not possible to say, based on this bound, if $T_c$ remains finite in the continuum limit. We would like to stress, once more, that this behaviour of the bound is not peculiar to the case examined here, but appears also in the $O(N)$ case in [23], for which one knows that $T_c$ has a finite continuum limit. An answer to this question using Monte Carlo simulations is under current investigation.

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