The Aharonov-Anandan phase of a classical dynamical system seen mathematically as a quantum dynamical system

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It is shown that the non-adiabatic Hannay’s angle of an integrable non-degenerate classical hamiltonian dynamical system may be related to the Aharonov-Anandan phase it develops when it is looked mathematically as a quantum dynamical system.

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I. A QUANTUM DYNAMICAL SYSTEM SEEN MATHEMATICALLY AS A CLASSICAL DYNAMICAL SYSTEM

Let us start from the following:

**DEFINITION I.1**

quantum dynamical system:

a couple \((\mathcal{H}, \hat{H})\) such that:

- \(\mathcal{H}\) is an Hilbert space
- \(\hat{H} \in \mathcal{L}_{s.a.}(\mathcal{H})\) (called the quantum hamiltonian)

where \(\mathcal{L}_{s.a.}(\mathcal{H})\) denotes the set of all the self-adjoint linear operators over \(\mathcal{H}\).

Given a quantum dynamical system \(QDS = (\mathcal{H}, \hat{H})\) let us introduce the following:

**DEFINITION I.2**

classical hamiltonian dynamical system associated to \(QDS\):

\[
CDS[QDS] := ((P(\mathcal{H}), \omega_{\text{Kahler}[g_{\text{Fubini-Study}}]}), H)
\]

where:

- \(P(\mathcal{H}) := \mathcal{H}/\mathbb{C}\) is the projective Hilbert space associated to \(\mathcal{H}\) i.e. the set of equivalence classes in \(\mathcal{H}\) with respect to the following equivalence relation:

  \[|\psi\rangle \sim |\phi\rangle : \exists c \in \mathbb{C} : |\psi\rangle = c|\phi\rangle\]

- \(\omega_{\text{Kahler}[g_{\text{Fubini-Study}}]}\) is the Kähler form of the Fubini-Study metric over \(P(\mathcal{H})\)

- \(H \in C^\infty(P(\mathcal{H}))\) is the hamiltonian defined by:

  \[
  H(P_\psi) := Tr(\hat{H}P_\psi)
  \]

where \(P_\psi := |\psi\rangle <\psi|\) being the projector associated to a \(|\psi\rangle \in S(\mathcal{H})\), where \(S(\mathcal{H}) := \{|\psi\rangle \in \mathcal{H} : <\psi|\psi\rangle = 1\}\) is the unit sphere in \(\mathcal{H}\) and where we have used the fact that:

\[
P(\mathcal{H}) \simeq_{\text{diff}} \{P_\psi, |\psi\rangle \in S(\mathcal{H})\}
\]

**Remark I.1**

Since \((P(\mathcal{H}), g_{\text{Fubini-Study}})\) is a Kähler manifold, its Kähler form \(\omega_{\text{Kahler}[g_{\text{Fubini-Study}}]}\) is in particular a symplectic form, so that \(CDS[QDS]\) is indeed a classical hamiltonian dynamical system.

**Remark I.2**

The classical hamiltonian dynamical system \(CDS[QDS]\) has not to be confused with the classical dynamical system obtained taking the classical limit of \(QDS\) since obviously:

\[
CDS[QDS] \neq \lim_{\hbar \to 0} QDS \forall QDS
\]

\(CDS[QDS]\) is the quantum dynamical system \(QDS\) seen mathematically as a classical hamiltonian dynamical system.

**Remark I.3**

The projective unitary group \(U(P(\mathcal{H})) := \frac{U(\mathcal{H})}{U(1)}\) of \(\mathcal{H}\) acts on \(P(\mathcal{H})\) by isometries of \(g_{\text{Fubini-Study}}\) that are symplectomorphisms of the symplectic manifold \((P(\mathcal{H}), \omega_{\text{Kahler}[g_{\text{Fubini-Study}}]})\).

Let \(\rho : G \to P(\mathcal{H})\) be a projective unitary representation on \(\mathcal{H}\) of a Lie group \(G\). The associated momentum map \(J : PD_G \to L(G)^*\) (where \(L(G)^*\) denotes the dual of the Lie algebra \(L(G)\) of \(G\) and where \(PD_G\) denotes the essential \(G\)-smooth part of \(P(\mathcal{H})\)) is equivariant.

Let us consider a quantum dynamical system \(QDS = (\mathcal{H}, \hat{H})\) such that the associated classical dynamical system \(CDS[QDS]\) is integrable.

**Remark I.4**

It has been shown in [4] that the Aharonov-Anandan phase (i.e. the non-adiabatic quantum Berry’s phase) of \(QDS\) may be related to the non-adiabatic Hannay angle (i.e. the holonomy of the Hannay-Berry connection [6], [4]) of \(CDS[QDS]\).
II. A CLASSICAL HAMILTONIAN DYNAMICAL SYSTEM SEEN MATHEMATICALLY AS A QUANTUM DYNAMICAL SYSTEM

Let us start from the following [7]:

**DEFINITION II.1**

*continuous-time classical dynamical system:*

a couple \((X, \sigma, \mu), \{T_t\}_{t \in \mathbb{R}}\) such that:

- \((X, \sigma, \mu)\) is a classical probability space
- \{\(T_t\)\}_{t \in \mathbb{R}}\ is a one-parameter family of automorphisms of \((X, \sigma, \mu)\), i.e.:
  \[ \mu \circ T^{-1}_t = \mu \quad \forall t \in \mathbb{R} \]

We have seen in the previous section particular instances of the following notion:

**DEFINITION II.2**

*classical hamiltonian dynamical system* 

a couple \(((M, \omega), H)\) such that:

- \((M, \omega)\) is a symplectic manifold
- \(H \in C^\infty(M)\)

One has that:

**Theorem II.1**

**HP:**

\(((M, \omega), H)\) classical hamiltonian dynamical system such that \(M\) is compact and orientable

**TH:**

\(((M, \omega), H)\) is a continuous-time classical dynamical system

**PROOF:**

Let us introduce the classical probability space \((M, \sigma_{\text{Borel}}, \mu_{\text{Liouville}})\), where \(\sigma_{\text{Borel}}\) is the Borel-\(\sigma\)-algebra of \(M\) and where:

\[
\mu_{\text{Liouville}} := \frac{\bigwedge_{i=1}^{\dim M} \omega}{\int_M \bigwedge_{i=1}^{\dim M} \omega} \tag{2.1}
\]

is the normalized Liouville measure over \((M, \sigma_{\text{Borel}})\).

The hamiltonian flow \(\{T_t^{(H)}\}_{t \in \mathbb{R}}\) generated by \(H\) is a one-parameter family of symplectomorphisms of \((M, \omega)\) and hence:

\[
\mu_{\text{Liouville}} \circ (T_t^{(H)})^{-1} = \mu_{\text{Liouville}} \quad \forall t \in \mathbb{R} \tag{2.2}
\]

Given a continuous-time classical dynamical system \(CDS := ((X, \sigma), \{T_t\}_{t \in \mathbb{R}})\) we can adopt Koopman’s formalism to introduce the following:

**DEFINITION II.3**
quantum dynamical system associated to CDS

\[ QDS[CDS] := (H, \hat{H}) \]  \hspace{1cm} (2.3)

where:

- \( H := L^2(X, \mu) \)
- \( \hat{H} \) is the generator (defined by Stone’s Theorem) of the strongly-continuous unitary group \( \{ \hat{U}_t = \exp(it\hat{H}) \}_{t \in \mathbb{R}} \) such that:
  \[
  (\hat{U}_t\psi)(x) := (\psi \circ T_t)(x) \quad \psi \in \mathcal{H}, t \in \mathbb{R}
  \]  \hspace{1cm} (2.4)

**Remark II.1**

QDS [CDS] has not to be confused with the quantum dynamical system obtained quantizing CDS since obviously:

\[
\lim_{\hbar \to 0} QDS[CDS] \neq CDS \quad \forall CDS
\]  \hspace{1cm} (2.5)
In the remark I.4 we saw that the Aharonov-Anandan phase of a quantum dynamical system QDS may be related to
the non-adiabatic Hannay angle of CDS\[QDS\].

In this section we will show that also the opposite occurs, i.e. that the non-adiabatic Hannay’s angle of an integrable
classical hamiltonian dynamical system CDS may be related to the Aharonov-Anandan phase of QDS[CDS]

Given an integrable hamiltonian classical dynamical system \((M,\omega, H)\) with \((\dim M = 2n)\) Liouville’s theorem
states that a compact and connected level set of \(n\) independent first integrals in involution is diffeomorphic to
an \(n\)-dimensional torus \(T^n\) on which the dynamics can be expressed in the action-angle canonical (i.e. such that
\(\omega = dI \wedge d\Phi\)) variables \((I = (I_1, \cdots, I_n), \Phi = (\Phi_1, \cdots, \Phi_n))\) as:

\[
\dot{I} = 0 \tag{3.1}
\]

\[
\dot{\Phi} = \Omega(I) \tag{3.2}
\]

where:

\[
\Omega(I) := \frac{\partial H(I)}{\partial I} \tag{3.3}
\]

As it is well known there are two cases:

- if \((k \cdot \Omega := \sum_{i=1}^{n} k_i \Omega_i = 0 \Rightarrow k = 0)\) \(\forall k \in \mathbb{Z}^n\) then the torus \(T^n\) is said nonresonant and the dynamics on it is quasi-periodic
- if \((k \cdot \Omega = 0 \Rightarrow k = 0)\) \(\forall k \in \mathbb{Z}^n\) then the torus \(T^n\) is said resonant and the dynamics on it is periodic

We will assume that CDS is everywhere non-degenerated, i.e.:

\[
det \frac{\partial \Omega}{\partial I} \neq 0 \tag{3.4}
\]

**Remark III.1**

In general the canonical coordinates \((I, \Phi)\) are defined only locally.

This means that considered two different level sets of the \(n\) independent first-integrals in involution one obtains two
different local charts \(A := (U_A, \chi_A)\) and \(B := (U_B, \chi_B)\) such that :

\[
\chi_A(y) = (I_A, \Phi_A) : \omega(y) = dI_A \wedge d\Phi_A \forall y \in U_A \tag{3.5}
\]

\[
\chi_B(y) = (I_B, \Phi_B) : \omega(y) = dI_B \wedge d\Phi_B \forall y \in U_B \tag{3.6}
\]

and where the map \(\psi_{A,B} : \chi_B(U_A \cap U_B) \mapsto \chi_A(U_A \cap U_B)\):

\[
\psi_{A,B} := \chi_A \circ \chi_B^{-1} \tag{3.7}
\]

is infinitely differentiable.

Since the consideration of a symplectic atlas of charts on \((M,\omega)\) defining a collection of different action-angle
variables simply complicates the situation without adding any further insight (at least for the matter we are going to
discuss) we will assume that the canonical action-angle coordinates \((I, \Phi)\) can be extended globally over \((M,\omega)\).

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1 The Aharonov-Anandan phase of QDS[CDS] was first proposed in \[9\] by the author as the definition of a non adiabatic analogou s of Hannay’s angle. At that time I was unaware that non-adiabatic Hannay’s angle was a notion already existing \[5\]. I strongly apologize for such an error. In this paper non-adiabatic Hannay’s angle refers to the notion discovered in \[5\] mathematically expressed by the holonomy of the Hannay-Berry connection \[6\], \[4\], \[10\].
Clearly one has that:

\[ QDS[CDS] = (\mathcal{H}, \hat{H}) \]  

(3.8)

where:

\[ \mathcal{H} = L^2(T^n, \frac{\delta \Phi}{(2\pi)^n}) \]  

(3.9)

while the strongly continuous unitary group \( \{\exp(i\hat{H}t)\}_{t \in \mathbb{R}} \) is specified by its action on the following basis:

\[ E := \{|n> := \exp(i\mathbf{n} \cdot \Phi) , \mathbf{n} \in \mathbb{Z}^n\} \]  

(3.10)

given by:

\[ \exp(i\hat{H}t)|n> = \exp(i\mathbf{n} \cdot \Omega t)|n> \quad \forall t \in \mathbb{R} \]  

(3.11)

Considered the \( U(1) \)-principal bundle \( \mathcal{S}(\mathcal{H})(\mathcal{P}(\mathcal{H}),U(1)) \) it is well-known that the Aharonov-Anandan geometric phase is given by the holonomy of the following natural connection one-form \( A \in T^*\mathcal{P}(\mathcal{H}) \otimes \mathcal{L}[U(1)] \) (where we denote by \( \mathcal{L}[G] \) the Lie algebra of a Lie group \( G \)):

\[ A_{\psi}(X) := iIm \psi X > \psi \in \mathcal{S}(\mathcal{H}), X \in T_{\psi} \mathcal{S}(\mathcal{H}) \subset \mathcal{H} \]  

(3.12)

A curve \( t \mapsto |\psi(t)> \in \mathcal{S}(\mathcal{H}) \) is horizontal with respect to \( A \) if and only if:

\[ <\psi(t)|\dot{\psi}(t)> = 0 \quad \forall t \]  

(3.13)

So the Aharonov-Anandan geometric phase acquired by QDS[CDS] when it is subjected to a loop \( \gamma : [0,1] \mapsto \mathcal{P}(\mathcal{H}) \) such that \( \gamma(0) = \gamma(1) = P_{\psi} \), \(|\psi> \in \mathcal{S}(\mathcal{H}) \) is the holonomy \( \tau^\gamma_{\mathcal{A}}(|\psi>) \).

Let us now consider a family of integrable classical hamiltonian dynamical systems \( CDS_x := ((M, \omega), H_x) \) where \( x \) is a parameter taking values on a parameters’ connected differentiable manifold \( P \) such that \( H_x \) depends smoothly by \( x \) and it there exists a point \( x_0 \in P \) such that \( CDS_{x_0} = CDS \).

Let us then introduce the family of quantum dynamical systems:

\[ QDS[CDS_x] = (\mathcal{H}, \hat{H}_x) \quad x \in P \]  

(3.14)

Let us suppose that the parameter \( x \) evolves adiabatically realizing a loop \( \gamma : [0,1] \mapsto P : \gamma(0) = \gamma(1) = x_0 \) in \( P \).

The adiabatic limit under which the Aharonov-Anandan phase of QDS[CDS] reduces to the adiabatic Berry phase of such a quantum dynamical system may be simply implemented through a suitable pullback \cite{11}.

In the adiabatic limit the basis:

\[ E_x := \{|n, x> : n \in \mathbb{Z}^n, x \in P\} \]  

(3.15)

continues to be formed by eigenvectors of \( \hat{U}_t \).

Let us assume that the eigenvalue corresponding to \(|n, x> \) is non-degenerate for every \( x \in P \).

Given \( n \in \mathbb{Z}^n \) let us then introduce the following map \( f_n : P \mapsto \mathcal{P}(\mathcal{H}) \):

\[ f_n(x) := P_{|n,x>} = |n,x> \quad \forall n \in \mathbb{Z}^n \]  

(3.16)

Let us then introduce the pullback-bundle \( f_n^* \mathcal{S}(\mathcal{H}) \) of the \( U(1) \)-bundle \( \mathcal{S}(\mathcal{H})(\mathcal{P}(\mathcal{H}),U(1)) \) by \( f_n \) and let us denote by \( f_n^*A \) the connection on the principal bundle \( f_n^* \mathcal{S}(\mathcal{H}) \) induced by the connection \( A \) through the pull-back operation; clearly such a connection is the Berry-Simon connection.

The adiabatic Berry phase developed by QDS[CDS] after the adiabatic evolution \( \gamma \) is then the holonomy \( \tau^\gamma_{f_n^*A} \) of the connection \( f_n^*A \) along the loop \( \gamma \).

Let us now consider the Hannay angles of the classical hamiltonian dynamical system CDS.

At this purpose let us introduce \( E := M \times P \) and the trivial bundle \( E \overset{\pi_E}{\rightarrow} P \) where clearly \( \pi_P : M \times P \mapsto P \) is such that:

\[ \pi_P(y,x) := x \quad y \in M, x \in P \]  

(3.17)

and let us introduce also the other canonical projection \( \pi_M : M \times P \mapsto P \) defined as:

\[ \pi_M(y,x) := y \quad y \in M, x \in P \]  

(3.18)
Let us observe that the restriction of the pullback $\pi^*_M \omega$ to each fibre $E_x := \pi_P^{-1}(x)$ is a symplectic form on such a fibre.

Introduced the natural splitting of the total exterior derivative on $M \times P$ of a function $f \in C^\infty(M \times P)$:

$$df = d_M f + d_P f \quad (3.19)$$

meaning that, if $(y^1, \cdots, y^{2n})$ are local coordinates on $M$ and $(x^1, \cdots, x^m)$ are local coordinates on $P$, then:

$$d_M f = \sum_{i=1}^{2n} \frac{\partial f}{\partial y^i} dy^i \quad (3.20)$$

$$d_P f = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i} dx^i \quad (3.21)$$

let us introduce the following:

**DEFINITION III.1**

*Fibrewise Hamiltonian vector field $X_f$ corresponding to $f$*:

$$i_{X_f}(\pi^*_M \omega) = d_M f \quad (3.22)$$

Note that $X_f$ is tangent to each fibre $\pi_P^{-1}(x)$ and hence defines an Hamiltonian vector field on $\pi_P^{-1}(x)$ in the usual sense.

Given a Lie group $G$:

**DEFINITION III.2**

*Family of Hamiltonian $G$-actions on $E$*

a smooth left action $\Upsilon : G \times E \to E$ of $G$ on $E$ such that:

- each fibre $E_x$ is invariant under the action
- the action, restricted to each fibre $E_x$, is symplectic
- it admits a smooth family of momentum maps $J : M \times P \to L(G)^*$, i.e., for any $x \in P$, the map $J(\cdot, x) : M \to L(G)^*$ is a momentum map in the usual sense for every $x \in P$.

Given a family $\Upsilon : G \times E \to E$ of Hamiltonian $G$-actions on $E$ and an arbitrary tensor $T$ on $E$ let us introduce the following:

**DEFINITION III.3**

*G-average of $T$*:

$$< T > := \frac{1}{|G|} \int_G \Upsilon_g^* T dg \quad (3.23)$$

where $dg$ is the Haar measure on $G$ and where $|G| := \int_G dg$.

Let us now observe that since $CDS_x$ is integrable for every $x \in P$ there exists, due to Liouville theorem, a set of local $x$-dependent action variables $I(\cdot, x) := (I_1(\cdot, x), \cdots, I_n(\cdot, x))$.

For the same reasons exposed in the remark III.1 we will assume, from here and beyond, that this system is globally defined on $E$ and, furthermore, that is everywhere non-degenerated, i.e.:

$$\det \frac{\partial \Omega}{\partial I} \neq 0 \quad (3.24)$$

Let us now look at the $n$-torus $T^n$ as an abelian Lie group; we have clearly that:

$$L(T^n)^* = L(T^n) = \mathbb{R} \quad (3.25)$$

Under the assumed hypotheses it results defined a family of Hamiltonian $T^{(n)}$-actions $\Upsilon : T^n \times E \to E$ on $E$ whose associated smooth family of momentum maps is $J = I : E \to \mathbb{R}^n$. 

Remark III.2

Let us observe that chosen at random an initial condition on M the probability of getting into a resonant torus is zero.

Since the quasi-periodic dynamics on a non-resonant torus is ergodic, the $T^n$-average and the temporal averages are equal.

Let us introduce the following:

**DEFINITION III.4**

Hannay-Berry connection on $E \to \mathbb{P}$

the connection $\mathcal{B}$ on $E \to \mathbb{P}$ such that:

$$\text{hor}_\mathcal{B}(X) := \langle 0, X \rangle \quad \forall X \in T_x \mathbb{P}$$

(3.26)

where $\text{hor}_\mathcal{B}(X) \in T_x \mathbb{P} \times T_y M$ is the horizontal lift of a vector $X \in T_x \mathbb{P}$ induced by the connection $\mathcal{B}$.

Let $\mu \in \mathbb{R}^n$ be a regular value of the momentum map $J(x) : M \to \mathbb{R}^n$ and let us introduce the following sets:

$$E_x^\mu := J^{-1}(\mu) \cap \pi^{-1}_P(x) = T^n$$

(3.27)

$$E^\mu := \bigcup_{x \in \mathbb{P}} E_x^\mu$$

(3.28)

Introducing also the projection:

$$\pi_\mu := \pi_P|E^\mu$$

(3.29)

one has that $E^\mu(P,T^n)$ is a torus-bundle over $M^2$.

Let us finally introduce the following:

**DEFINITION III.5**

Hannay-Berry connection on $E^\mu(P,T^n)$

the restriction of $\mathcal{B}$ to $E^\mu$.

Let us suppose that the parameter $x$ evolves realizing a loop $\gamma : [0,1] \to \mathbb{P}$: $\gamma(0) = \gamma(1) = x_0$ in $\mathbb{P}$.

The Hannay angle of CDS is then the holonomy $\tau^\mathcal{B}_\gamma$.

Let us now compare the Aharonov-Anandan phase of QDS[CDS] and the Hannay angle of CDS.

As we saw the former is the holonomy $\tau^A_\gamma$ over the $U(1)$-bundle $S(\mathcal{H})(\mathcal{P}(\mathcal{H}),U(1))$ while the latter is the holonomy $\tau^\mathcal{B}_\gamma$ over the $T^n$-bundle $E^\mu(P,T^n)$.

Let us first of all make the passage to the Simon’s spectral bundle considering, for each $n \in \mathbb{Z}^n$, the map $f_n : \mathbb{P} \to \mathcal{P}(\mathcal{H})$:

$$f_n(x) := P_{|n,x>} = |n,x > | n,x$$

(3.30)

and taking into account the spectral bundle $F := f_n^*S(\mathcal{H})$ previously introduced:

such a $U(1)$-bundle has the same base space, i.e. $\mathbb{P}$, of the $T^n$-bundle $E^\mu(P,T^n)$ while its fibre $F_x$ in $x \in \mathbb{P}$ is:

$$F_x = \{ \exp(i\alpha)|n,x > , \alpha \in \mathbb{R}\}$$

(3.31)

Given $n \in \mathbb{Z}^n$ let us now introduce the following:

**DEFINITION III.6**

The first intuitive idea of the fact that the adiabatic Hannay angle should have been given by the holonomy of a connection on such a torus-bundle was first proposed in [12]
map of relation between the Hannay angle of CDS and the Aharonov-Anandan phase of QDS[CDS]:

the map $R_n : Hol_B \mapsto Hol_{f_n^*A}$:

$$R_n(\tau^B_\gamma) = \tau^{f_n^*A}_\gamma \quad \forall \gamma \in C_{x_0}(P)$$ (3.32)

where $Hol_B$ is the holonomy group of the connection $B$, where $Hol_{f_n^*A}$ is the holonomy group of the connection $f_n^*A$ and where:

$$C_{x_0}(P) := \{ \gamma : [0, 1] \mapsto P : \gamma(0) = \gamma(1) = x_0 \}$$ (3.33)

is the set of loops in $P$ based at $x_0$.

**Remark III.3**

Let us observe that $R_n$ maps the Hannay angle of CDS into the adiabatic Berry phase of QDS[CDS].

Since the adiabatic Berry phase of QDS[CDS] is a particular case of the Aharonov-Anandan phase related to it by the pull-back $f_n^*$ we can see $R$ as a map relating the Hannay angle of CDS and the Aharonov-Anandan phase of QDS[CDS].

The function $R_n$ maps the holonomy of $B$ associated to a loop $\gamma$ into the holonomy of $f_n^*A$ associated to the same loop.

Since $\tau^B_\gamma \in T^n$ while $\tau^{f_n^*A}_\gamma \in U(1)$ the map $R_n$ has to be of the form:

$$R_n(\tau^B_\gamma) = \exp[iS(\mathbf{n} \cdot \tau^B_\gamma)]$$ (3.34)

for some $S : \mathbb{R} \mapsto \mathbb{R}$.

Considering the case in which $\bar{\gamma}$ is the constant loop $\bar{\gamma}(t) := x_0 \ \forall t \in [0, 1]$ one has that since $\tau^B_{\bar{\gamma}} = \text{id}_{T^n}$ and $\tau^{f_n^*A}_{\bar{\gamma}} = \text{id}_{U(1)}$ it follows that:

$$S(0) = 0$$ (3.35)
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