Supplementary Information

Supplementary Note 1: Effective Hamiltonian for dissipative coupling

The dissipative interaction is based on the indirect coupling between two spatially separated nanomechanical resonators mediated by a common optical field in a two-membrane-in-the-middle cavity optomechanical system. The cavity is driven by a laser field with amplitude modulation. The total Hamiltonian of such a system in the rotating frame of the driving laser frequency can be written as ($\hbar = 1$) [1-3]

$$
\hat{H} = -\Delta \hat{a}^\dagger \hat{a} + \omega_1 \hat{b}_1^\dagger \hat{b}_1 + \omega_2 \hat{b}_2^\dagger \hat{b}_2 - g_1 \hat{a}^\dagger \hat{a} \left( \hat{b}_1^\dagger + \hat{b}_1 \right) - g_2 \hat{a}^\dagger \hat{a} \left( \hat{b}_2^\dagger + \hat{b}_2 \right) + i\eta(t) \left( \hat{a}^\dagger - \hat{a} \right). 
$$

(S1)

Here $\hat{a}$ and $\hat{b}_{1,2}$ are the annihilation operators of the cavity mode and the mechanical oscillators, respectively. $\Delta = \omega_2 - \omega_c$ is the frequency detuning between the driving laser and the cavity resonance. $\omega_{1,2}$ is the intrinsic frequency of the mechanical oscillator. $g_{1,2}$ is the optomechanical coupling strength. $\eta(t) = \sqrt{P(t) \kappa_a / \hbar \omega_c}$ is the driving strength. $P(t) = P_0 \left( 1 + M \cos \omega_d t \right)$ is the input laser power with a constant power $P_0$, modulation depth $M$, and modulation frequency $\omega_d$. $\kappa_a$ is the loss of the cavity input mirror.

According to Eq. (S1), the quantum Langevin equations can be written as follows

$$
\partial_t \hat{a} = \left[ -\kappa/2 + i\Delta + i \sum_{j=1}^2 g_j \left( \hat{b}_j + \hat{b}_j^\dagger \right) \right] \hat{a} + \eta(t) + \sqrt{\kappa} \hat{a}_a, 
$$

(S2a)

$$
\partial_t \hat{b}_j = - \left( \gamma_j/2 + i\omega_j \right) \hat{b}_j + ig_j \hat{a}^\dagger \hat{a} + \sqrt{\gamma_j} \hat{b}_{j, a}, \quad (j = 1, 2) 
$$

(S2b)

where $\kappa$ is the total cavity decay rate and $\gamma_j$ is the mechanical damping rate. $\hat{a}_a$ and $\hat{b}_{j, a}$ are the Markovian quantum noise operators of cavity mode and $j^{th}$ mechanical mode, respectively. In our experiment, $\gamma_j$ is $\sim 120$Hz, which is much smaller than the mechanical frequency $\omega_j \sim 400$kHz. The cavity-enhanced optomechanical coupling strength $g_j \sqrt{\pi_{\text{cav}}} \left( \pi_{\text{cav}} = \langle \hat{a}^\dagger \hat{a} \rangle \right.$ is the average intracavity photon number) is $\sim 4$kHz, which is smaller than $\kappa \sim 330$kHz. Therefore, the
mechanical decay rate and the optomechanical coupling strength are not strong enough to consider the non-Markovian effect and the system-bath correlation \[1,4,5\]. When the cavity is driven by a classical field, we can linearly decompose the cavity mode into the average field \( \langle \hat{a} \rangle = \alpha \) and the corresponding fluctuation \( \delta \hat{a} \), i.e., \( \hat{a} \to \alpha + \delta \hat{a} \). When the modulation depth \( M \) of the driven laser is much less than one, we can retain the first-order expansion of the driving strength \( \eta(t) \approx \sqrt{P_0 \kappa_{in}/\hbar \omega_L} \left( 1 + e^{i\omega_d t} M/4 + e^{-i\omega_d t} M/4 \right) \). Then, the average intracavity amplitude can be obtained as \( \alpha(t) = \alpha_0 + \alpha_1 e^{i\omega_d t} + \alpha_2 e^{-i\omega_d t} \), where \( \alpha_0 = \eta_0 \chi_c(0) \), \( \alpha_1 = \eta_0 \chi_c(-\omega_d) M/4 \), and \( \alpha_2 = \eta_0 \chi_c(\omega_d) M/4 \), with the susceptibility function of cavity field \( \chi_c(\omega) = \left[ \kappa/2 - i (\Delta + \omega) \right]^{-1} \) and \( \eta_0 = \sqrt{P_0 \kappa_{in}/\hbar \omega_L} \). By assuming \( \omega_d \ll \Delta \), the intracavity amplitude is approximately as \( \alpha(t) \approx \eta_0 \chi_c(0) \left( 1 + e^{i\omega_d t} M/4 + e^{-i\omega_d t}/4 \right) \).

In order to obtain the explicit expression of the effective interaction between two mechanical modes, we utilize the transformations \( \hat{b}_j \to \hat{b}_j e^{-i\alpha_j t} \) and \( \hat{b}_j \to \hat{b}_j e^{-i\alpha_j t} \). Thus, the dynamics of intracavity field fluctuation \( \delta \hat{a} \) and the mechanical mode \( \hat{b}_j \) can be obtained as

\[
\partial_t \delta \hat{a} = (-\kappa/2 + i\Delta) \delta \hat{a} + \sum_{j=1}^{2} g_j \left( \hat{b}_j e^{-i\alpha_j t} + \hat{b}_j^\dagger e^{i\alpha_j t} \right) + \sqrt{\kappa} \hat{a}_{in}, \quad \text{(S3a)}
\]

\[
\partial_t \hat{b}_j = -\left( \gamma_j/2 + i \left( \omega_j - \Omega_j \right) \right) \hat{b}_j + i g_j \left( \alpha^* \delta \hat{a} + \alpha \delta \hat{a}^\dagger \right) e^{i\alpha_j t} + \sqrt{\gamma_j} \hat{b}_{in}, \quad \text{(S3b)}
\]

Due to the fact that the cavity damping rate is much larger than the mechanical loss, the intracavity field adiabatically follows the dynamics of the mechanical modes \[6,7\]. From Eq. (S3a), we can obtain the formal solution of intracavity fluctuation on long time scales compared with \( \kappa^{-1} \) as follows

\[
\delta \hat{a}(t) = \sqrt{\kappa} \hat{f}_{in}(t) + i \sum_{j=1}^{2} g_j \left[ A_j(\omega_d) \hat{b}_j e^{-i\alpha_j t} + B_j(\omega_d) \hat{b}_j^\dagger e^{i\alpha_j t} \right], \quad \text{(S4)}
\]

where

\[
A_j(\omega_d) = \left[ \alpha_0 \chi_c(\Omega_j) + \alpha_1 \chi_c(\Omega_j - \omega_d) e^{i\omega_d t} + \alpha_2 \chi_c(\Omega_j + \omega_d) e^{-i\omega_d t} \right],
\]

\[
B_j(\omega_d) = \left[ \alpha_0 \chi_c(\Omega_j) + \alpha_1 \chi_c(\Omega_j - \omega_d) e^{-i\omega_d t} + \alpha_2 \chi_c(\Omega_j + \omega_d) e^{i\omega_d t} \right].
\]
\[ B_j(\omega_d) = \left[ \alpha_0 \chi_e(-\Omega_j) + \alpha_{-1} \chi_e(-\Omega_j - \omega_d) e^{i\omega_d t} + \alpha_1 \chi_e(-\Omega_j + \omega_d) e^{-i\omega_d t} \right], \]

and \( \hat{f}_m(t) = \int_0^t d\tau e^{(i\alpha - \kappa/2)(t-\tau)} \hat{a}_m(\tau) \) is the noise operator.

By substituting Eq. (S4) into Eq. (S3b) and neglecting all the noise operator terms including \( \hat{b}_m \) and \( \hat{b}_m \), we can obtain the time-dependent coupled-mode equations of mechanical oscillators

\[ i\hbar \dot{\hat{h}}_1 = (-i\gamma_1/2 + (\omega_1 - \Omega_1)) \hat{h}_1 + \Sigma_{11} \hat{h}_1 + \Sigma_{12} \hat{h}_2 e^{i(\Omega_1 - \Omega_2) t}, \]  

(S5a)

\[ i\hbar \dot{\hat{h}}_2 = \Sigma_{21} \hat{h}_1 e^{i(\Omega_2 - \Omega_1) t} + (-i\gamma_2/2 + (\omega_2 - \Omega_2)) \hat{h}_2 + \Sigma_{22} \hat{h}_2, \]  

(S5b)

where \( \Sigma_{kk} = -ig_j g_k (\alpha A_k - \alpha B_k^*) \) and

\[ (\alpha^* A_k - \alpha B_k^*) = \]

\[ \alpha_0^* \alpha_0 \left( \chi_e(\omega_1) - \chi_e^*(-\omega_1) \right) + \alpha_1^* \alpha_{-1} \left( \chi_e(\omega_1 - \omega_2) - \chi_e^*(-\omega_1 + \omega_2) \right) + \alpha_1^* \alpha_1 \left( \chi_e(\omega_1 + \omega_2) - \chi_e^*(-\omega_1 + \omega_2) \right) \]

\[ + e^{i\omega_d t} \left[ \alpha_0^* \alpha_{-1} \left( \chi_e(\omega_1 - \omega_d) - \chi_e^*(-\omega_1 + \omega_d) \right) + \alpha_1^* \alpha_0 \left( \chi_e(\omega_1) - \chi_e^*(-\omega_1 + \omega_d) \right) \right] \]

\[ + e^{-i\omega_d t} \left[ \alpha_0^* \alpha_{-1} \left( \chi_e(\omega_1 - \omega_d) - \chi_e^*(-\omega_1 + \omega_d) \right) + \alpha_1^* \alpha_0 \left( \chi_e(\omega_1) - \chi_e^*(-\omega_1 + \omega_d) \right) \right] \]

Further, we drop the fast oscillating terms contained in \( \Sigma_{11}, \Sigma_{22}, \Sigma_{12} e^{i(\Omega_2 - \Omega_1) t} \) and \( \Sigma_{21} e^{i(\Omega_2 - \Omega_1) t} \), and Eq. (S5) is simplified to be

\[ i\hbar \dot{\hat{h}}_1 = (-i\gamma_1/2 + (\omega_1 - \Omega_1)) \hat{h}_1 + \Sigma_{11} \hat{h}_1 + \Sigma_{12} \hat{h}_2 e^{i(\Omega_2 - \Omega_1) t}, \]  

(S6a)

\[ i\hbar \dot{\hat{h}}_2 = \Sigma_{21} \hat{h}_1 e^{i(\Omega_2 - \Omega_1) t} + (-i\gamma_2/2 + (\omega_2 - \Omega_2)) \hat{h}_2 + \Sigma_{22} \hat{h}_2, \]  

(S6b)

where

\[ \sigma_{11} = -ig_j^2 \left[ \alpha_0^* \alpha_0 \left( \chi_e(\omega_1) - \chi_e^*(-\omega_1) \right) + \alpha_1^* \alpha_{-1} \left( \chi_e(\omega_1 - \omega_d) - \chi_e^*(-\omega_1 - \omega_d) \right) + \alpha_1^* \alpha_1 \left( \chi_e(\omega_1 + \omega_d) - \chi_e^*(-\omega_1 + \omega_d) \right) \right], \]

\[ \sigma_{22} = -ig_j^2 \left[ \alpha_0^* \alpha_0 \left( \chi_e(\omega_2) - \chi_e^*(-\omega_2) \right) + \alpha_1^* \alpha_{-1} \left( \chi_e(\omega_2 - \omega_d) - \chi_e^*(-\omega_2 - \omega_d) \right) + \alpha_1^* \alpha_1 \left( \chi_e(\omega_2 + \omega_d) - \chi_e^*(-\omega_2 + \omega_d) \right) \right], \]

\[ \sigma_{12} = -ig_j g_k \left[ \alpha_0^* \alpha_{-1} \left( \chi_e(\omega_2 - \omega_d) - \chi_e^*(-\omega_2) \right) + \alpha_1^* \alpha_0 \left( \chi_e(\omega_2) - \chi_e^*(-\omega_2 + \omega_d) \right) \right], \]
\[ \sigma_{21} = -i g_2 g_1 \left[ \alpha_\ast \alpha_1 \left( \chi_\varepsilon \left( \omega_1 + \omega_d \right) - \chi_\varepsilon^* \left( -\omega_1 \right) \right) + \alpha_\ast \alpha_0 \left( \chi_\varepsilon \left( \omega_1 \right) - \chi_\varepsilon^* \left( -\omega_1 - \omega_d \right) \right) \right]. \]

Thus, the effective time-dependent Hamiltonian of mechanical modes can be extracted from Eq. (S6) as follows

\[ H_D (t) = \begin{pmatrix} -i \gamma_1 / 2 + (\omega_1 - \Omega_1) + \sigma_{11} & \sigma_{12} e^{i(\omega_1 - (\Omega_1 - \Omega_2))t} \\ \sigma_{21} e^{-i(\omega_1 - (\Omega_1 - \Omega_2))t} & -i \gamma_2 / 2 + (\omega_2 - \Omega_2) + \sigma_{22} \end{pmatrix}. \] (S7)

Using the unitary transformation

\[ U(t) = \begin{pmatrix} e^{i \delta_d t / 2} & 0 \\ 0 & e^{-i \delta_d t / 2} \end{pmatrix} \]

with \( \delta_d = \omega_d - (\Omega_2 - \Omega_1) \), \( \hat{H}_D (t) \) can be transformed into the following time-independent Hamiltonian \( \hat{H}_D^{\text{rot}} = U^{-1} \hat{H}_D (t) U - iU^{-1} \dot{U} U \), i.e.,

\[ \hat{H}_D^{\text{rot}} = \begin{pmatrix} \delta_d / 2 + (\omega_1 - \Omega_1) - i \gamma_1 / 2 + \sigma_{11} & \sigma_{12} \\ \sigma_{21} & -\delta_d / 2 + (\omega_2 - \Omega_2) - i \gamma_2 / 2 + \sigma_{22} \end{pmatrix}. \] (S8)

By assuming \( \omega_d \ll \omega_{1,2} \), we have \( \chi_\varepsilon \left( \omega_{1,2} \right) \approx \chi_\varepsilon \left( \omega_{1,2} \pm \omega_d \right) \). Thus, \( \sigma_{jk} \) can be simplified as \( \sigma_{11} \approx g_2^2 \chi_{\text{eff}} \), \( \sigma_{22} \approx g_2^2 \chi_{\text{eff}} \), and \( \sigma_{12} \approx \sigma_{21} \approx g_1 g_2 \chi_{\text{eff}} M / 2 \). Here \( \chi_{\text{eff}} = -i \left[ \eta L \right] \left( \chi_\varepsilon \left( \omega_m \right) - \chi_\varepsilon^* \left( -\omega_m \right) \right) \) is effective susceptibility introduced by the intracavity field with \( \omega_m = (\omega_1 + \omega_2) / 2 \), i.e.,

\[ \chi_{\text{eff}} = \frac{P_0 \kappa_{\text{m}} / \hbar \omega_1}{\kappa^2 / 4 + \Delta^2} \left( -i \kappa / 2 + (\Delta + \omega_m) \right) + \frac{i \kappa / 2 + (\Delta - \omega_m)}{\kappa^2 / 4 + (\Delta + \omega_m)^2}, \] (S9)

which results from the dynamical backaction with the optomechanical spring and damping effects.

Thus, Eq. (S7) can be shown as

\[ \hat{H}_D^{\text{rot}} = \begin{pmatrix} \delta_d / 2 - i \gamma_1 / 2 + g_2^2 \chi_{\text{eff}} & g_1 g_2 \chi_{\text{eff}} M / 2 \\ g_1 g_2 \chi_{\text{eff}} M / 2 & -\delta_d / 2 - i \gamma_2 / 2 + g_2^2 \chi_{\text{eff}} \end{pmatrix}. \] (S10)
with complex off-diagonal elements $g_1 g_2 \chi_{\text{eff}}^M / 2$ describing the coherent (real part) and dissipative (imaginary part) interactions between two mechanical oscillators, which depends on $P_0$, $\Delta$, and $M$.

The real $(\text{Re}(\chi_{\text{eff}}))$ and imaginary $(\text{Im}(\chi_{\text{eff}}))$ parts of the effective susceptibility, and the absolute value of $\text{Im}(\chi_{\text{eff}})/\text{Re}(\chi_{\text{eff}})$ are plotted as a function of the cavity detuning $\Delta/\omega_m$ in both unresolved- and resolved-sideband regimes in Fig. S1. It is straightforward to see that the resolved-sideband regime is desired for the purely imaginary interaction.

![Image](https://via.placeholder.com/150)

**FIG. S1.** The real and imaginary parts of $\chi_{\text{eff}}$, and the absolute value of $\text{Im}(\chi_{\text{eff}})/\text{Re}(\chi_{\text{eff}})$ as a function of cavity detuning $\Delta/\omega_m$. (a,b) The unresolved-sideband regime ($\kappa=3\omega_m$) and (c,d) the resolved-sideband regime ($\kappa=0.8\omega_m$).

In the resolved-sideband regime, $\chi_{\text{eff}}$ is dominantly imaginary if the cavity detuning is close to the mechanical frequency $\left(\Delta \approx \pm \omega_{1,2}\right)$, as shown in Fig. S1d. Consequently, the effective interaction between two mechanical modes can be treated as purely dissipative. Moreover, if we let $\omega_1=\Omega_1$ and $\omega_2=\Omega_2$ in Eq. (S10), the effective two-mode Hamiltonian for dissipative coupling $\hat{H}_{\text{eff}}$ between two mechanical modes can be shown as
\[ \hat{H}_{\text{eff}} = \begin{pmatrix} \frac{\delta_1}{2} - i\gamma_1/2 + i g_1^2 \text{Im}(\chi_{\text{eff}}) & i g_2 g_1 \text{Im}(\chi_{\text{eff}}) M/2 \\ ig_2^2 g_1 \text{Im}(\chi_{\text{eff}}) M/2 & -\frac{\delta_2}{2} - i\gamma_2/2 + i g_2^2 \text{Im}(\chi_{\text{eff}}) \end{pmatrix} \]  

(S11)

**Supplementary Note 2: Coexistence of dissipative and coherent interactions**

In the main text, we mainly discuss the purely dissipative interaction. Here, we briefly investigate the situation when the coherent interaction \( g_1 g_2 \text{Re}(\chi_{\text{eff}}) M/2 \) between two membranes cannot be ignored, which is realized by controlling the cavity detuning.

![Supplementary Note 2](image)

**FIG. S2.** The measured mechanical power spectral density of the dissipatively and coherently coupled system as a function of effective frequency detuning \( \delta_j \) for the different ratios of (a) \( \left| \text{Im}(\chi_{\text{eff}})/\text{Re}(\chi_{\text{eff}}) \right| \approx 3 \) and (d) \( \left| \text{Im}(\chi_{\text{eff}})/\text{Re}(\chi_{\text{eff}}) \right| \approx 0.3 \). (b,e) and (c,f) are the corresponding
eigenfrequencies $\omega_\pm$ and linewidths $\gamma_\pm$ of the normal mode, respectively. The dots are the experimental measurements and the solid curves are the theoretical simulations. The system parameters are $\gamma_1'/2\pi \approx 110\text{Hz}$, $\gamma_2'/2\pi \approx 290\text{Hz}$, and $\Gamma/2\pi \approx 100\text{Hz}$ for (a-c), and $\gamma_1'/2\pi \approx 150\text{Hz}$, $\gamma_2'/2\pi \approx 250\text{Hz}$, and $\Gamma/2\pi \approx 47\text{Hz}$ for (d-f).

The mechanical power spectral densities of membrane are plotted in Fig. S2 as a function of $\delta_d$. The level attraction of normal modes can be observed when the coherent interaction is relatively small compared to the dissipative coupling strength with a ratio $\left|\text{Im}(\chi_{\text{eff}})/\text{Re}(\chi_{\text{eff}})\right| \approx 3$, as shown in Fig. S2a and Fig. S2b. The level repulsion of normal modes can be observed when the coherent interaction is relatively large with a ratio $\left|\text{Im}(\chi_{\text{eff}})/\text{Re}(\chi_{\text{eff}})\right| \approx 0.3$, as shown in Fig. S2d and Fig. S2e. The asymmetry of the linewidths with respect to $\delta_d$, as shown in Figs. S2c and S2f, is due to the unbalanced linewidths of bare modes and the non-negligible coherent coupling. The phonon lasing is also observed in spite of the mixed coherent interaction, as shown in Fig. S2c.

**Supplementary Note 3: Experimental setup**

The experiment is performed in an optomechanical system with two spatially separated membranes inside an optical cavity in the resolved-sideband regime [3]. The optical cavity consists of two mirrors with reflectivity $\sim 0.9998$ (at the wavelength of 1064nm) with a cavity length of 36 mm. Two stoichiometric silicon nitride membranes with a size $1\times1\text{mm}^2$ and a thickness of 50nm are placed inside the optical cavity. The cavity finesse with two membranes in the middle is about 12000. We can precisely control the natural frequency and position of each membrane by the attached piezos. A narrow linewidth 1064nm Nd: YAG laser is split into two beams. One weak locking beam passes through an electro-optic modulator (EOM) to stabilize the cavity frequency via the Pound-Drever-Hall technique. The other strong driving beam goes through two-cascade acousto-optic modulators (AOM) to control the frequency detuning between the driving laser and optical cavity. The whole two-membrane-in-the-middle cavity optomechanical system is placed inside a vacuum chamber. The cavity driving field is amplitude modulated via the AOM. The mechanical motion of two membranes can be detected by a weak 795nm probe laser reflecting from two membranes (not resonant with the cavity). A spectrum analyzer and a lock-in amplifier are used for the measurements of membranes’ motions.
Supplementary Note 4: Data analysis

In this note, we first describe the data fitting for extracting the eigenvalues from the measured mechanical spectrum. According to the effective Hamiltonian, the noise power spectrum of each phonon mode can be obtained as follows

\[
S_n^{(a)}(\omega) = \frac{\left[ \gamma_2^{\prime 2}/4 + (\omega + \delta_2)^2 \right] n_1^{th} + \Gamma^2 \gamma_1^{nh}}{\left( \gamma_1^{\prime 2}/4 - \Gamma^2 - (\omega + \delta_2)(\omega + \delta_1) \right)^2 + \left( (\omega + \delta_1) \gamma_2^{\prime 2}/2 + (\omega + \delta_2) \gamma_1^{\prime 2}/2 \right)^2},
\]

(S12a)

\[
S_n^{(b)}(\omega) = \frac{\left[ \gamma_1^{\prime 2}/4 + (\omega + \delta_1)^2 \right] n_2^{th} + \Gamma^2 \gamma_1^{nh}}{\left( \gamma_2^{\prime 2}/4 - \Gamma^2 - (\omega + \delta_2)(\omega + \delta_1) \right)^2 + \left( (\omega + \delta_1) \gamma_2^{\prime 2}/2 + (\omega + \delta_2) \gamma_1^{\prime 2}/2 \right)^2},
\]

(S12b)

where \( n_1^{th} \) is the average effective thermal phonon number and \( \delta_{1,2} \) is the effective frequency of two oscillators. Notably, Eqs. (S12a) and (S12b) are only valid for the nonlasing regime. Further, Eqs. (S12a) and (S12b) can be written in a general expression with the superposition of two Lorentz-dispersive lineshapes as

\[
S_{n,\pm}^{(a)}(\omega) = \frac{A_+ (\omega - \omega_{\pm}) \gamma_{\pm} + B_+ \gamma_{\pm}^2}{(\omega - \omega_{\pm})^2 + \gamma_{\pm}^2 / 4} + \frac{A_- (\omega - \omega_{\pm}) \gamma_{\pm} + B_- \gamma_{\pm}^2}{(\omega - \omega_{\pm})^2 + \gamma_{\pm}^2 / 4}.
\]

(S13)

Here \( \omega_{\pm} \) and \( \gamma_{\pm} \) are the frequency and linewidth of normal modes of the coupled system. \( A_\pm \) and \( B_\pm \) are real constants. Therefore, \( \omega_{\pm} \) and \( \gamma_{\pm} \) can be extracted by fitting the measured noise power spectrum.

Figure S3 presents the measured noise power spectra for three distinct regimes and their corresponding data fitting. Figure S3a shows a typical noise power spectrum in the region before the EP, where the real components are distinguishable. Figure S3b is for the region between the EP and lasing threshold, where the real components coalesce and the imaginary components are nondegenerated. We use Eq. (S13) to fit the spectra in these two cases, and the fitting parameters are \( \gamma_+ / 2\pi (\gamma_- / 2\pi) = 82.6(75.6) \) Hz and \( \omega_+ / 2\pi (\omega_- / 2\pi) = -13.0(-976.4) \) Hz for Fig. S3a, and \( \gamma_+ / 2\pi (\gamma_- / 2\pi) = 40.0(130.0) \) Hz and \( \omega_+ / 2\pi (\omega_- / 2\pi) = -207.4(-210.6) \) Hz for Fig. S3b. Figure
S3c is for the region where the phonon lasing occurs. In this region, we use a single Lorentz function to fit the data, which has $\gamma/2\pi = 3.5$ Hz and $\omega/2\pi = -58.1$ Hz.

FIG. S3. Data fitting of the measured noise power spectra with coupling strength $\Gamma/2\pi = 164$ Hz. The blue circles are the experiment data and the red curves are the corresponding fitting. (a) shows the case before the EP ($\delta_\nu/2\pi \approx -1$ kHz). (b) is between the EP and lasing threshold ($\delta_\nu/2\pi \approx -300$ Hz). (c) is for the phonon lasing regime ($\delta_\nu/2\pi \approx 0$ Hz).

Supplementary Note 5: Second-order correlation function

In order to analyze the phonon statistics for the generation of phonon lasing, we introduce the second-order phonon correlation function, which is defined as

$$g_j^{(2)}(\tau) = \frac{\langle N_j(t) N_j(t+\tau) \rangle}{\langle N_j(t) \rangle \langle N_j(t+\tau) \rangle},$$

(S14)

where $N_j(t) = \beta_j^*(t) \beta_j(t)$ denotes the phonon number of jth membrane and $\beta_j(t) = \{\hat{b}_j\}$ is the complex amplitude of the mechanical vibration. Please note that the definition of $g_j^{(2)}(\tau)$ is consistent with the classical definition of the second-order correlation function in quantum optics [8]. For zero time delay,

$$g_j^{(2)}(0) = \frac{\langle N_j^2(t) \rangle}{\langle N_j(t) \rangle^2}.$$  

(S15)

It is known that $\langle N_j^2(t) \rangle = \langle N_j(t) \rangle + 2 \langle N_j(t) \rangle^2$ for a thermal state and $\langle N_j^2(t) \rangle = \langle N_j(t) \rangle + \langle N_j(t) \rangle^2$ for a coherent state [8]. Therefore, it is straightforward to obtain that
For the thermal state, which corresponds to a Boltzmann distribution, \( g_j^{(2)}(0) = 2 \) for the coherent state, which corresponds to a Poisson distribution. For a classical state, \( g_j^{(2)}(0) \) lies in the range between 1 and 2.

Next, we present how to calibrate phonon numbers and obtain the second-order correlation function. In the experiment, the mechanical motions of membranes are detected by the probe light, which contain the information regarding the phonon number. With the real-time measurements of the lock-in amplifier, the phonon number of each mechanical mode can be extracted directly.

According to the equipartition theorem, i.e., \( k_B T_j = \frac{m \omega_j^2 \langle x_j^2 \rangle}{2} \) (j=1,2), the thermal noise spectrum of the fundamental mode at room temperature is used to calibrate the effective phonon number of each mechanical mode. With the lock-in amplifier, the mechanical displacement \( x_j \) of the \( j^{th} \) membrane can be reconstructed by the quadrature components \( X_j \) and \( Y_j \). Thus, the real-time phonon number of \( j^{th} \) membrane can be obtained as

\[
N_j(t) = A_j \left( X_j(t)^2 + Y_j(t)^2 \right) / 2.
\)

(S16)

Here \( A_j = 2 k_B T_{room} / h \omega_j \langle X_j^2 + Y_j^2 \rangle_{room} \) is the calibrated coefficient with \( \langle X_j^2 + Y_j^2 \rangle_{room} / 2 \) being the statistical average of thermal noise signal of fundamental mode at room temperature \( T_{room} \). Thus, by using Eqs. (S14) and (S16), the second-order correlation function can be measured in the experiment.

Supplementary References

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