A CONSTRUCTION OF
A COMPLETE BOUNDED NULL CURVE IN $\mathbb{C}^3$

L. FERRER, F. MARTÍN, M. UMEHARA, AND K. YAMADA

Abstract. We construct a complete bounded immersed null holomorphic
curve in $\mathbb{C}^3$, which is a recovery of the previous paper of the last three authors
on this subject.

Introduction

The study of global properties of complete complex null-curves is interesting from
different points of view. Firstly, the real and imaginary part of such a curve are
complete minimal surfaces in $\mathbb{R}^3$. Secondly, there exists a close relationship between
null curves in $\mathbb{C}^3$ and constant mean curvature $H = 1$ surfaces in hyperbolic 3-space.

An important problem in the global theory of complete null curves is the so called
Calabi-Yau problem, which deals with the existence of complete null-curves inside
a ball of $\mathbb{C}^3$. This problem was approached firstly in [5, Theorem A] using similar
ideas to those used by Nadirashvili in [7] to solve the Calabi-Yau conjecture in $\mathbb{R}^3$.
Unfortunately, the paper [5] has a mistake, and the first examples of complete
bounded null curves in $\mathbb{C}^3$ were provided using other approximation, by Alarcón
and López [3]. Very recently, Alarcón and Forstnerič have got the most general
results in this line (see [1,2]).

The purpose of this paper is to show that similar ideas to those given in [5]
can be used to produce examples of complete bounded null holomorphic disks in
a ball of $\mathbb{C}^3$: In [5], Martín, Umehara and Yamada tried to construct a bounded
holomorphic curve in $\text{SL}(2, \mathbb{C})$ and used this example to get the desired bounded
disk in $\mathbb{C}^3$. However, in this paper, we construct the bounded null curves directly
in $\mathbb{C}^3$. In this aspect, our strategy is similar to that used by Alarcón and López
in [3]. Although, as we mentioned before, these examples have been generalized
in Alarcón and Forstnerič [4] by using different (and powerful) methods, we think
that the arguments and techniques exhibited in this paper is different from [3,1,2],
and might be of use in the solution of other questions related to the Calabi-Yau
problem in different settings.

As applications of Theorem A in [5], the following objects were constructed;
(1) complete bounded minimal surfaces in the Euclidean 3-space $\mathbb{R}^3$ ([5, Theorem A]),
(2) complete bounded holomorphic curves in $\mathbb{C}^2$ ([5, Corollary B]),
(3) weakly complete bounded maximal surfaces in the Lorentz-Minkowski 3-
space $\mathbb{R}^3_1$ ([5, Corollary D]),
(4) complete bounded null curves in $\text{SL}(2, \mathbb{C})$ ([5, Theorem C]),
(5) complete bounded constant mean curvature one surfaces in the hyperbolic
3-space $H^3$ ([5, Theorem C]).

We also constructed higher genus examples of the first three objects in [6]. All of
these applications in [5] and [6] are correct as a consequence.
1. The Main Theorem and the Key Lemma

We denote by $(\cdot,\cdot)$ (resp. $(\cdot,\cdot)_R$) the $\mathbb{C}$-bilinear inner product (resp. the Hermitian inner product) of $\mathbb{C}^3$:

\begin{equation}
(x,y) := x_1y_1 + x_2y_2 + x_3y_3, \quad (x,y)_R := (x,\overline{y}),
\end{equation}

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{C}^3$, and $\overline{y}$ denotes the complex conjugate of $y$. We identify an element of $\mathbb{C}^3$ with a column vector when the matrix product is used. The Hermitian norm of $\mathbb{C}^3$ is denoted by $|x| := \sqrt{(x,x)}$ for $x \in \mathbb{C}^3$. In particular, it holds that

\begin{equation}
|x| = |(x,y)| = |(x,\overline{y})| \leq |x||y|.
\end{equation}

Let $M(3,\mathbb{C})$ (resp. $M(3,\mathbb{R})$) be the set of complex (resp. real) $(3 \times 3)$-matrices. Moreover, we will use the following notation for the set of complex (resp. special) orthogonal matrices

\begin{equation}
O(3,\mathbb{C}) := \{ A \in M(3,\mathbb{C}) ; A^tA = \text{id} \}, \quad \text{resp. } \quad SO(3) := \{ A \in M(3,\mathbb{R}) ; A^tA = \text{id}, \det A = 1 \},
\end{equation}

where $A^t$ means the transposed matrix of $A$. As usual, we denote $U(3) := \{ A \in M(3,\mathbb{C}) ; A^tA = \text{id} \}$, where $A^*$ is the conjugate transposed matrix of $A$. For each $A \in M(3,\mathbb{C})$, we define the matrix norm as

\begin{equation}
||A|| := \sup_{x \in \mathbb{C}^3 \setminus \{0\}} \frac{|Ax|}{|x|}.
\end{equation}

If $A \in M(3,\mathbb{C})$ is a non-singular matrix,

\begin{equation}
\frac{1}{||A^{-1}||}|x| \leq |Ax| \leq ||A|| |x|
\end{equation}

holds. It is well-known that

\begin{equation}
||A|| = \sqrt{\max \{\mu_1, \mu_2, \mu_3\}} \quad (A \in M(3,\mathbb{C}))
\end{equation}

holds, where $\mu_j \in \mathbb{R}$ ($j = 1,2,3$) are the eigenvalues of positive semi-definite Hermitian matrix $A^*A$.

A holomorphic map $F : D \to \mathbb{C}^3$, defined on a domain $D \subset \mathbb{C}$, is a null immersion if and only if

\begin{equation}
(\varphi_F, \varphi_F) = 0 \quad \text{and} \quad |\varphi_F|^2 \equiv (\varphi_F, \varphi_F) > 0, \quad \text{where} \quad \varphi = \varphi_F := \frac{dF}{dz},
\end{equation}

where $z$ is the canonical complex coordinate of $\mathbb{C}$. In this case the pull-back of the Hermitian metric of $\mathbb{C}^3$ by $F$ is expressed as

\begin{equation}
ds_F^2 := (dF,dF) = |\varphi_F|^2 |dz|^2,
\end{equation}

which is called the induced metric of $F$. For a holomorphic null immersion $F : D \to \mathbb{C}^3$, the first equality of (1.6) implies that there exist a meromorphic function $g$ and a holomorphic function $\eta$ such that

\begin{equation}
\varphi_F = \frac{1}{2} \left( 1 - g^2, i(1 + g^2), 2g \right) \eta \quad (i = \sqrt{-1}).
\end{equation}

We call $(g,\eta)$ the Weierstrass data of $F$. Using these data, the induced metric (1.7) is expressed as

\begin{equation}
ds_F^2 = \frac{1}{2} \left( 1 + |g|^2 \right)^2 |\eta|^2 |dz|^2.
\end{equation}
Throughout this paper, we denote the open (resp. closed) disc on \( \mathbb{C} \) centered at 0 with radius \( r \) by
\[
D_r := \{ z \in \mathbb{C} ; |z| < r \}, \quad (\text{resp. } \overline{D}_r := \{ z \in \mathbb{C} ; |z| \leq r \}) \quad (r > 0).
\]

The goal of this paper is to prove the following

**Theorem 1.1** (The Main Theorem). There exists a holomorphic null immersion \( X : \overline{B}_1 \rightarrow \mathbb{C}^3 \) such that the induced metric \( ds_X^2 \) is complete, and the image \( X(\overline{B}_1) \) is bounded in \( \mathbb{C}^3 \).

The main theorem can be proved by the following assertion in the same way as [7][4]:

**Proposition 1.2.** Let \( X : \overline{B}_1 \rightarrow \mathbb{C}^3 \) be a holomorphic null immersion of the closed disc \( \overline{B}_1 \subset \mathbb{C} \) into \( \mathbb{C}^3 \). Suppose that there exists positive numbers \( p \) and \( r \) such that

- (X-1) \( X(0) = 0 \),
- (X-2) \( \overline{B}(r) \) contains the geodesic disc centered at 0 and of radius \( r \),
- (X-3) \( |X| \leq r \) holds on \( \partial B_1 \).

Then, for an arbitrary given positive numbers \( \epsilon \) and \( s \), there exists a holomorphic null immersion \( Y : \overline{B}_1 \rightarrow \mathbb{C}^3 \) satisfying

- (Y-1) \( |\varphi_Y - \varphi_X| < \epsilon \) and \( |Y - X| < \epsilon \) hold on \( B_{1-\epsilon} \), where \( \varphi_X = dX/dz \) and \( \varphi_Y = dY/dz \),
- (Y-2) \( \overline{B}(s) \) contains the geodesic disc \( \partial \mathcal{D} \) centered at 0 with radius \( p + s \),
- (Y-3) \( |Y| \leq \sqrt{r^2 + s^2 + \epsilon} \).

This proposition is a consequence of the following Key Lemma. (The proof of Proposition 1.2 is given in Section [4].) To explain it, we define three constants

\[
N = N(p, r, \mu, \nu, s, \epsilon), \quad C_1 = C_1(p, r, \mu, \nu, s, \epsilon) \quad \text{and} \quad C_2 = C_2(p, r, \mu, \nu, s, \epsilon)
\]

depending on six positive constants \( p, r, \mu, \nu, s, \epsilon \). Here \( p, r \) and \( s, \epsilon \) have been already given in [X-2] and [X-3]; we will fix \( \mu, \nu \) in the statement of Lemma 1.3. The remaining two constants \( s, \epsilon \) are arbitrary in the statement of Lemma 1.3 but will coincide with the corresponding constants as in Proposition 1.2.

The constants \( C_1 \) and \( C_2 \) are set as
\[
C_1 := \frac{\nu}{5}, \quad C_2 := 6(\mu^2 + 2\mu + 2).
\]

Next, we set
\[
c_1 := 6\mu^2 + 12\mu + 8, \quad c_2 := 3\mu + \frac{2\epsilon(\rho + s)}{C_1},
\]
\[
c_3 := \frac{s\alpha + s^2}{\sqrt{r^2 + s^2}} + 2r\epsilon + 2\epsilon^2 \left( \alpha := c_2 + 5\epsilon + (r + 2\epsilon)\sqrt{2C_2} \right).
\]

We then choose an integer \( N \) so that it satisfies the following four inequalities:

\[
N \geq \max \left\{ 36, \frac{2\epsilon}{\nu}, \epsilon, (12\mu)^2, \left( \frac{2\epsilon(\rho + s)}{\nu} \left( 2 + \frac{6\mu + \epsilon}{3\nu} \right) \right)^4 \right\},
\]
\[
N \geq \max \left\{ \frac{3}{\epsilon}, \left( \frac{2\epsilon}{C_1} \right)^4, \left( \frac{1}{\nu} \left( \epsilon + \frac{C_1}{2} \right) \right)^{4/3}, \left( \frac{2(\rho + s)}{C_1} \right)^4 \right\},
\]
\[
N \geq \max \left\{ \left( \frac{c_3 + 2\epsilon}{\epsilon} \right)^4, \left( \frac{1 + c_2 + 6\mu + 3\epsilon}{\epsilon} \right)^4 \right\}.
\]
Lemma 1.3 (The Key Lemma). Assume a holomorphic null immersion $X : \mathbb{D}_1 \to \mathbb{C}^3$ and positive real numbers $\rho$ and $r$ satisfy \([X-1] (X-3)\) We set

\begin{align}
(1.16) \quad &\nu := \min \{\varphi_X \} > 0, \quad \mu := \max \left\{ 1 + \max_{\mathbb{D}_1} \varphi_X, \max_{\mathbb{D}_1} \varphi'_X \right\},
\end{align}

where $\varphi_X := X' = dX/dz$ and $\varphi'_X := d\varphi_X/dz$. For an arbitrary positive number $\varepsilon$ and $s$, we take positive constant $C_1, C_2$ and positive integer $N$ as in \([1.11], (1.13) - (1.15)\). Then there exist a sequence $\{F_j\}_{j=0,\ldots,2N}$ of holomorphic null immersions $F_j : \mathbb{D}_1 \to \mathbb{C}^3$ and a sequence $\{v_j\}_{j=1,\ldots,2N}$ of unit vectors in $\mathbb{C}^3$ which satisfy the following assertions, where the compact set $\omega_j \subset \mathbb{C}$, an open neighborhood $\omega_j$ of $\omega_j$ and the “base point” $\zeta_j$ of $\omega_j$ are as in \([1.8] \) and \([A.9] \) in Appendix A and

\begin{align}
(1.17) \quad &\varphi_l := \frac{dF_l}{dz} \quad (l = 0, \ldots, 2N)
\end{align}

(K-0) $F_0 = X$.
(K-1) $F_l(0) = 0 \ (l = 0, \ldots, 2N)$.
(K-2) $|\varphi_l - \varphi_{l-1}| \leq \frac{C_2}{\sqrt{N}}$ holds on $\mathbb{D}_1 \setminus \omega_l$ for each $l = 1, \ldots, 2N$.
(K-3) The inequality

\begin{align}
|\varphi_l| \geq \begin{cases} C_1 N^{9/4} & \text{on } \omega_l, \\ C_1 N^{-3/4} & \text{on } \omega_l \end{cases}
\end{align}

holds for each $l = 1, \ldots, 2N$.
(K-4) $|\langle v_l, v_l \rangle | \geq 1/N^{1/4}$ for each $l = 1, \ldots, 2N$.
(K-5) $|F_{l-1}(\zeta_l)| < 1/\sqrt{N}$, or

\begin{align}
|\left\langle \frac{F_{l-1}(p)}{|F_{l-1}(p)|}, v_l \right\rangle | \geq 1 - \frac{C_2}{\sqrt{N}} \quad (\text{on } \omega_l)
\end{align}

holds for each $l = 1, \ldots, 2N$.
(K-6) $\langle F_{l-1}, v_l \rangle = \langle F_l, v_l \rangle$ holds on $\mathbb{D}_1$ for each $l = 1, \ldots, 2N$.

In the proof of the Key Lemma \([1.3] \) we use the notion of Gauss maps of holomorphic null immersions: Let $F : D \to \mathbb{C}^3$ be a holomorphic null immersion. Then both the real part $\Re F$ and the imaginary part $\Im F$ give conformal minimal immersions into $\mathbb{R}^3$ with the same Gauss map. So we call the Gauss map $G : D \to S^2$ of both $\Re F$ and $\Im F$ the Gauss map of $F$, where $S^2 \subset \mathbb{R}^3$ is the unit sphere. Then $G$ is expressed as

\begin{align}
(1.18) \quad &G = \frac{-i}{|\varphi|^2} (\varphi \times \bar{\varphi}) : D \to S^2 \subset \mathbb{R}^3 \quad \left( \varphi = \frac{dF}{dz} \right),
\end{align}

because \([1.6] \) implies that $|\varphi \times \bar{\varphi}| = |\varphi|^2$, where “$\times$” denotes the complexification of the vector product of $\mathbb{R}^3$. Using the Weierstrass data \([1.8] \), $G$ is expressed as

\begin{align}
(1.19) \quad &G = \left( \frac{2 \Re g}{1 + |g|^2}, \frac{2 \Im g}{1 + |g|^2}, \frac{|g|^2 - 1}{1 + |g|^2} \right).
\end{align}

That is, $g = \pi_S \circ G$, where $\pi_S : S^2 \to \mathbb{C} \cup \{\infty\}$ is the stereographic projection from the north pole.

2. Preliminary estimates

Let $F_0 = X : \mathbb{D}_1 \to \mathbb{C}^3$ be a holomorphic null immersion as in the assumption of the Key Lemma \([1.3] \). Here, we prepare some basic properties of $\{F_j\}_{j=0,\ldots,2N}$ in the conclusion of the Key Lemma \([1.3] \).
**Lemma 2.1.** If [(K-1)] and [(K-2)] in the Key Lemma (A) are satisfied for \( l \in \{1, \ldots, 2N\} \) then

\[
|F_l - F_{l-1}| \leq \frac{\varepsilon}{N^2} \quad \text{on } \overline{D}_1 \setminus \varpi_l.
\]

**Proof.** Let \( p \in \overline{D}_1 \setminus \varpi_l \). Then there exists a path \( \gamma \) in \( \overline{D}_1 \setminus \varpi_l \) joining 0 and \( p \) whose Euclidean length is not greater than \( 1 + \frac{\pi}{N^2} \) (see Lemma (A.2) in Appendix A). Thus, we have

\[
|F_l(p) - F_{l-1}(p)| = \left| \int_{\gamma} (\varphi_l(z) - \varphi_{l-1}(z)) \, dz \right| \leq \int_{\gamma} \left| \varphi_l(z) - \varphi_{l-1}(z) \right| \, |dz| \leq \text{Length}_C(\gamma) \frac{\varepsilon}{2N^2} \quad \text{(by (K-2))}
\]

\[
\leq \left( 1 + \frac{\pi}{N} \right) \frac{\varepsilon}{2N^2} \leq 2 \cdot \frac{\varepsilon}{2N^2} = \frac{\varepsilon}{N^2} \quad \text{(by (1.13))},
\]

where \( \text{Length}_C(\gamma) \) is the length of \( \gamma \) with respect to the metric \( |dz|^2 \) on \( \mathbb{C} \). \( \square \)

**Lemma 2.2.** Fix an integer \( j \) \((1 \leq j \leq 2N)\). If \( F_0, F_1, \ldots, F_{j-1} \) satisfy [(K-0)] and [(K-2)] of the Key Lemma (A). Then

\[
|\varphi_{j-1}| \geq \frac{\nu}{2} \quad \text{and} \quad |\varphi_{j-1}| \leq \mu \quad \text{(on } \overline{D}_1 \setminus (\varpi_1 \cup \cdots \cup \varpi_{j-1})\text{)},
\]

hold, where \( \mu \) and \( \nu \) are constants defined in (1.16).

**Proof.** By [(K-0)] [(K-2)] (1.16) and (1.13),

\[
|\varphi_{j-1}| \geq |\varphi_0| - |\varphi_1 - \varphi_0| - \cdots - |\varphi_{j-1} - \varphi_{j-2}|
\]

\[
\geq \min_{\overline{D}_1} |\varphi_0| - \frac{(j-1)\varepsilon}{2N^2} \geq \nu - \frac{\varepsilon}{N} \geq \frac{\nu}{2}
\]

holds on \( \overline{D}_1 \setminus (\varpi_1 \cup \cdots \cup \varpi_{j-1}) \). On the other hand, we have

\[
|\varphi_{j-1}| \leq |\varphi_0| + |\varphi_1 - \varphi_0| + \cdots + |\varphi_{j-1} - \varphi_{j-2}|
\]

\[
\leq \max_{\overline{D}_1} |\varphi_0| + \frac{(j-1)\varepsilon}{2N^2} \leq \max_{\overline{D}_1} |\varphi_0| + \frac{\varepsilon}{N} \leq \max_{\overline{D}_1} |\varphi_0| + 1 \leq \mu. \qedhere
\]

**Lemma 2.3.** Fix an integer \( j \) \((1 \leq j \leq 2N)\). If \( F_0, F_1, \ldots, F_{j-1} \) satisfy [(K-0)] and [(K-2)] of the Key Lemma (A). Then for each \( q \in \varpi_j \), it holds that

\[
|F_{j-1}(q) - F_{j-1}(\zeta_j)| \leq \frac{6\mu}{N}, \quad |\varphi_{j-1}(q) - \varphi_{j-1}(\zeta_j)| \leq \frac{6\mu + 2\varepsilon}{N},
\]

where \( \zeta_j \) is the “base point” of \( \varpi_j \), see (A.9) in Appendix A.

**Proof.** By Lemma (A.3) in Appendix A, there exists a path \( \gamma \) in \( \varpi_j \) joining \( \zeta_j \) and \( q \) such that \( \text{Length}_C(\gamma) \leq 6/N \). Since the image of \( \gamma \) lies on \( \overline{D}_1 \setminus (\varpi_1 \cup \cdots \cup \varpi_{j-1}) \), Lemma (A.2) implies that we have

\[
|F_{j-1}(q) - F_{j-1}(\zeta_j)| \leq \int_\gamma |\varphi_{j-1}(z)| \, |dz| \leq \mu \cdot \text{Length}_C(\gamma) \leq \frac{6\mu}{N}.
\]

On the other hand,

\[
|\varphi_{j-1}(q) - \varphi_{j-1}(\zeta_j)|
\]

\[
\leq |\varphi_{j-1}(q) - \varphi_j(\zeta_j)| + \cdots + |\varphi_1(q) - \varphi_0(q)| + |\varphi_0(q) - \varphi_0(\zeta_j)|
\]

\[
+ |\varphi_{j-1}(\zeta_j) - \varphi_j(\zeta_j)| + \cdots + |\varphi_1(\zeta_j) - \varphi_0(\zeta_j)|
\]

\[
\leq \frac{2(j-1)\varepsilon}{2N^2} + |\varphi_0(q) - \varphi_0(\zeta_j)| \leq \frac{2\varepsilon}{N} + \left| \int_\gamma \varphi_0'(z) \, dz \right| \quad \text{(by (K-2))}
\]

\[
\leq \frac{2\varepsilon}{N} + \int_\gamma |\varphi_0'(z)| \, |dz| \leq \frac{2\varepsilon}{N} + \mu \cdot \text{Length}_C(\gamma) \leq \frac{2\varepsilon}{N} + \frac{6\mu}{N} \quad \text{(by (1.10))}. \qedhere
\]
We fix \(j (1 \leq j \leq 2N)\) and assume \(F_0, F_1, \ldots, F_{j-1}\) are already constructed and satisfy \((K-0), (K-6)\). From now on, we give a recipe of construction of \(F_j\) and \(v_j\) as an inductive procedure:

**Lemma 2.4.** There exists a unit vector \(u \in \mathbb{C}^3\) (i.e. \(|u| = 1\)) such that

1. \(\delta^2 := |\langle u, u \rangle| \geq 1/N^{1/4}\).
2. If

\[
|F_{j-1}(\zeta_j)| \geq \frac{1}{\sqrt{N}},
\]

it holds that

\[
\left| \frac{F_{j-1}(p)}{|F_{j-1}(p)|} \cdot u \right| \geq 1 - \frac{c_1}{\sqrt{N}} \quad (p \in \mathbb{R}_j),
\]

where \(c_1\) is the constant as in \((1.12)\).

**Proof.** When \(|F_{j-1}(\zeta_j)| < 1/\sqrt{N}\), the unit vector \(u = (0, 0, 1)\) satisfies the conclusions. (Note that the conclusion \([2]\) is empty in this case.)

Now, we assume \((2.1)\), and set

\[
u_0 := \frac{F_{j-1}(\zeta_j)}{|F_{j-1}(\zeta_j)|}.
\]

By Lemma 2.3,

\[
|F_{j-1}(p) - F_{j-1}(\zeta_j)| \leq \frac{6\mu}{N}
\]

holds for each \(p \in \mathbb{R}_j\). Then, for \(p \in \mathbb{R}_j\), it holds that

\[
|F_{j-1}(p)| \geq |F_{j-1}(\zeta_j)| - |F_{j-1}(p) - F_{j-1}(\zeta_j)| \geq |F_{j-1}(\zeta_j)| - \frac{6\mu}{N} \quad \text{(by (2.3))}
\]

\[
\geq |F_{j-1}(\zeta_j)| \left(1 - \frac{6\mu}{N|F_{j-1}(\zeta_j)|}\right) \geq |F_{j-1}(\zeta_j)| \left(1 - \frac{6\mu}{N \sqrt{N}}\right) \quad \text{(by (2.1))}
\]

\[
= |F_{j-1}(\zeta_j)| \left(1 - \frac{6\mu}{N \sqrt{N}}\right) \geq \frac{1}{2} |F_{j-1}(\zeta_j)| \quad \text{(by (1.14))}.
\]

Thus, using \((2.1)\) again, we have

\[
|F_{j-1}(p)| \geq \frac{1}{2} |F_{j-1}(\zeta_j)| \geq \frac{1}{2 \sqrt{N}} \quad (p \in \mathbb{R}_j).
\]

Then by the relationship of the arithmetic mean and the geometric mean, we have

\[
\frac{(6\mu)^2}{N^2} \geq |F_{j-1}(p) - F_{j-1}(\zeta_j)|^2 \quad \text{(by (2.3))}
\]

\[
= |F_{j-1}(p)|^2 + |F_{j-1}(\zeta_j)|^2 - 2 \text{Re} \langle F_{j-1}(p), F_{j-1}(\zeta_j) \rangle
\]

\[
\geq |F_{j-1}(p)| |F_{j-1}(\zeta_j)| \left(1 - \frac{2 \text{Re} \langle F_{j-1}(p), F_{j-1}(\zeta_j) \rangle}{|F_{j-1}(p)| |F_{j-1}(\zeta_j)|}\right)
\]

\[
\geq 2 |F_{j-1}(p)| |F_{j-1}(\zeta_j)| \left(1 - \text{Re} \frac{F_{j-1}(p)}{|F_{j-1}(p)|} \cdot \frac{F_{j-1}(\zeta_j)}{|F_{j-1}(\zeta_j)|}\right)
\]

\[
\geq \frac{1}{N} \left(1 - \text{Re} \left(\frac{F_{j-1}(p)}{|F_{j-1}(p)|} \cdot \frac{F_{j-1}(\zeta_j)}{|F_{j-1}(\zeta_j)|}\right)\right) \quad \text{(by (2.4), (2.1))}
\]

\[
\geq \frac{1}{N} \left(1 - \left|\frac{F_{j-1}(p)}{|F_{j-1}(p)|} \cdot \frac{F_{j-1}(\zeta_j)}{|F_{j-1}(\zeta_j)|}\right|\right).
\]
Hence, by (1.13), we have
\[
\left| \frac{F_{j-1}(p)}{|F_{j-1}(p)|}, u_0 \right| = \left| \frac{F_{j-1}(\zeta_j)}{|F_{j-1}(\zeta_j)|} \right| \geq 1 - \frac{(6\mu)^2}{N} = 1 - \frac{6\mu^2}{\sqrt{N}} \geq 1 - \frac{6\mu^2}{\sqrt{N}}.
\]

(2.5)

Case A. We consider the case \(|(u_0, u_0)| \geq 1/N^{1/4}\). In this case, we set \(u = u_0\). Then the unit vector \(u\) satisfies (1.1) trivially. Moreover, (2.5) implies the assertion (2) because \(c_1 \leq 1\).

Case B. We next consider the case \(|(u_0, u_0)| < 1/N^{1/4}\). In this case, set
\[
u := \frac{\tilde{u}}{|\tilde{u}|}, \quad \text{where} \quad \tilde{u} := u_0 + \frac{2}{N^{1/4}} u_0.
\]

To show (1) and (2) we set
\[
\delta_0^2 := |(u_0, u_0)| \left( < \frac{1}{N^{1/4}} \right), \quad (\delta_0 \geq 0).
\]

Since \(u_0\) is a unit vector, (2.6) yields
\[
|\tilde{u}|^2 = |u_0|^2 + \frac{4}{N^{1/4}} |\tilde{u}_0|^2 + \frac{4}{N^{1/4}} \text{Re}(u_0, \tilde{u}_0) = 1 + \frac{4}{N^{1/4}} + \frac{4}{N^{1/4}} \text{Re}(u_0, u_0) \leq 1 + \frac{4}{N^{1/4}} + \frac{4}{N^{1/4}} |(u_0, u_0)| \leq 1 + \frac{4}{N^{1/4}} + \frac{4}{N^{1/4}} \leq 1 + \frac{8}{N} \leq \frac{7}{3}.
\]

Then (1) holds because of (2.8) and (2.9).

Finally, we prove (2). Let \(p \in \mathbb{F}_j\). Then we have
\[
|(F_{j-1}(p), u_0)| = |(F_{j-1}(p) - F_{j-1}(\zeta_j), u_0)| + |(F_{j-1}(\zeta_j), u_0)| \leq |(F_{j-1}(p) - F_{j-1}(\zeta_j), u_0)| + |(F_{j-1}(\zeta_j), u_0)| \leq |F_{j-1}(p) - F_{j-1}(\zeta_j)| \cdot |u_0| + |(F_{j-1}(\zeta_j), u_0)| \leq \frac{6\mu}{N} + |F_{j-1}(\zeta_j)| \leq \frac{6\mu}{N} + \frac{2\delta_1^2}{N} |F_{j-1}(\zeta_j)| \leq \frac{6\mu}{N} + \frac{2\delta_1^2}{N} \leq \frac{6\mu}{N} + \frac{6\mu}{N} \frac{|F_{j-1}(\zeta_j)|}{N^{1/4}} \quad \text{(by (1.2), (2.2))}
\]

(2.10)
On the other hand, since
\[(2.11) \quad \frac{1}{\sqrt{1 + x}} \geq 1 - \frac{x}{2} \quad (0 \leq x \leq 2),\]
we have
\[
\left\| \frac{F_{j-1}(p)}{|F_{j-1}(p)|} u \right\| = \frac{1}{|u|} \left\| \frac{F_{j-1}(p)}{|F_{j-1}(p)|} u_0 \right\| + \frac{2}{N^{1/4}} \left\| \frac{F_{j-1}(p)}{|F_{j-1}(p)|} u_0 \right\| \quad \text{(by (2.6))}
\]
\[
\geq \frac{1}{\sqrt{1 + \frac{2}{N}}} \left\| \frac{F_{j-1}(p)}{|F_{j-1}(p)|} u_0 \right\| + \frac{2}{N^{1/4}} \left\| \frac{F_{j-1}(p)}{|F_{j-1}(p)|} u_0 \right\| \quad \text{(by (2.9))}
\]
\[
\geq \left( 1 - \frac{4}{\sqrt{N}} \right) \left\| \left( \frac{F_{j-1}(p)}{|F_{j-1}(p)|} u_0 \right) \right\| - \frac{2}{N^{1/4}} \left\| \frac{F_{j-1}(p)}{|F_{j-1}(p)|} u_0 \right\| \quad \text{(by (2.11))}
\]
\[
\geq \left( 1 - \frac{4}{\sqrt{N}} \right) \left[ \left( 1 - \frac{(6\mu)^2}{N} \right) - \frac{2}{N^{1/4}} \left\| \frac{F_{j-1}(p)}{|F_{j-1}(p)|} u_0 \right\| \right] \quad \text{(by (2.5))}
\]
\[
\geq \left( 1 - \frac{4}{\sqrt{N}} \right) \left[ \left( 1 - \frac{(6\mu)^2}{N} \right) - \frac{2}{N^{1/4}} \left\| \frac{F_{j-1}(p)}{|F_{j-1}(p)|} u_0 \right\| \right] \quad \text{(by (2.10))}
\]
\[
\geq \left( 1 - \frac{4}{\sqrt{N}} \right) \left[ \left( 1 - \frac{(6\mu)^2}{N} \right) - \frac{4}{\sqrt{N}} (1 + 3\mu) \right] \quad \text{(by (2.4))}
\]
\[
\geq \left( 1 - \frac{4}{\sqrt{N}} \right) \left[ \left( 1 - \frac{(6\mu)^2}{N} \right) + \frac{4}{\sqrt{N}} (3\mu + 1) \right] \quad \text{(by (2.13))}
\]
\[
\geq \left( 1 - \frac{4}{\sqrt{N}} \right) \left( 6\mu^2 + 12\mu + 4 \right) \quad \text{(by (1.12))}
\]
\[
\geq 1 - \frac{1}{\sqrt{N}} (6\mu^2 + 12\mu + 8) + \frac{4}{N} (6\mu^2 + 12\mu + 4)
\]
\[
\geq 1 - \frac{1}{\sqrt{N}} (6\mu^2 + 12\mu + 8) = 1 - \frac{c_1}{\sqrt{N}}. \quad \text{(by (1.12))}
\]
Thus we have the conclusion. \(\square\)

3. The Proof of the Key Lemma 1.3

To continue the procedure of the iterational construction of \(F_j\), we prepare the following lemma:

**Lemma 3.1.** For a unit vector \(u \in \mathbb{C}^3\), there exists \(P \in \text{SO}(3)\) and \(\tau \in \mathbb{R}\) such that
\[(3.1) \quad e^{-i\tau} Pu = \begin{pmatrix} 0 \\ i \sin \theta \\ \cos \theta \end{pmatrix} \quad (i = \sqrt{-1}).\]

Here, \(\theta\) is a real number such that
\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = |(u, u)| \quad \left(0 \leq \theta \leq \frac{\pi}{4}\right).
\]

**Proof.** Write \(u = x + iy\) \((x, y \in \mathbb{R}^3)\), and let \(\tau \in \mathbb{R}\) be
\[
\tau = \begin{cases} 
\frac{1}{2} \arctan \frac{2 \langle x, y \rangle}{|x|^2 - |y|^2} & \text{when } |x| \neq |y| \\
\frac{\pi}{4} & \text{when } |x| = |y|.
\end{cases}
\]
Then \( \tilde{\mathbf{u}} := e^{-i\tau} \mathbf{u} \) satisfies \( \langle \text{Re} \tilde{\mathbf{u}}, \text{Im} \tilde{\mathbf{u}} \rangle = 0 \). Moreover, replacing \( \tau \) with \( \tau + \frac{\pi}{2} \) if necessary, we may assume
\[
|\text{Re} \tilde{\mathbf{u}}| \geq |\text{Im} \tilde{\mathbf{u}}|\tag{3.2}
\]
without loss of generality. In particular, since \( \|	ilde{\mathbf{u}}\| = 1 \), it holds that \( |\text{Re} \tilde{\mathbf{u}}| > 0 \).

Hence there exists a matrix \( P_1 \in \text{SO}(3) \) such that
\[
P_1(\text{Re} \tilde{\mathbf{u}}) = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \quad (t > 0).
\]

Since \( P_1 \) is a real matrix, \( \text{Im}(P_1 \tilde{\mathbf{u}}) \) is orthogonal to \( \text{Re}(P_1 \tilde{\mathbf{u}}) \). Hence, we have
\[
P_1(\tilde{\mathbf{u}}) = \begin{pmatrix} iu_1 \\ iu_2 \\ t \end{pmatrix} \quad (u_1, u_2, t \in \mathbb{R}, t > 0, (u_1)^2 + (u_2)^2 + t^2 = 1).
\]

Moreover, \( t \geq \sqrt{(u_1)^2 + (u_2)^2} \) holds because (3.2). Next, choose a real number \( s \) such that
\[
\begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix} \quad u = \sqrt{(u_1)^2 + (u_2)^2} \geq 0,
\]
and set
\[
P := \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \\ 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot P_1 \in \text{SO}(3).
\]

Then
\[
e^{-i\tau} P \mathbf{u} = P \tilde{\mathbf{u}} = \begin{pmatrix} 0 \\ iu \\ t \end{pmatrix} \quad (u, t \in \mathbb{R}, t \geq u \geq 0, u^2 + t^2 = 1).
\]

Hence there exists \( \theta \in [0, \frac{\pi}{4}] \) such that \( u = \sin \theta, t = \cos \theta \). In particular,
\[
|\langle \mathbf{u}, \mathbf{u} \rangle| = t^2 - u^2 = \cos^2 \theta - \sin^2 \theta = \cos 2\theta
\]
holds and thus we have the conclusion. \( \square \)

We set
\[
A := \begin{pmatrix} \sqrt{\cos 2\theta} & 0 & 0 \\ 0 & \cos \theta & -i \sin \theta \\ 0 & i \sin \theta & \cos \theta \end{pmatrix},
\]
where \( \theta \in [0, \frac{\pi}{4}] \). Then \( A \) is non-singular if and only if \( \theta \neq \frac{\pi}{4} \). In this case,
\[
A \in \delta \cdot \mathbb{O}(3, \mathbb{C}) \quad (\delta := \sqrt{\cos 2\theta}),
\]
and
\[
A^{-1} = \frac{1}{\delta^2} \begin{pmatrix} \sqrt{\cos 2\theta} & 0 & 0 \\ 0 & \cos \theta & i \sin \theta \\ 0 & -i \sin \theta & \cos \theta \end{pmatrix}.
\]

**Lemma 3.2.** Let \( \theta \in [0, \frac{\pi}{4}] \) be a real number. Then the matrix \( A \) in (3.3) satisfies
\[
\|A\| = \cos \theta + \sin \theta \leq \sqrt{2}, \quad \|A^{-1}\| = \frac{\cos \theta + \sin \theta}{\cos 2\theta} \leq \frac{\sqrt{2}}{\delta^2}, \quad (\delta = \sqrt{\cos 2\theta}).
\]
Proof. Since the eigenvalues of the matrix
\[ A^*A = \begin{pmatrix} \cos 2\theta & 0 & 0 \\ 0 & 1 & -i\sin 2\theta \\ 0 & i\sin 2\theta & 1 \end{pmatrix} \]
are \((\cos \theta - \sin \theta)^2, \cos 2\theta,\) and \((\cos \theta + \sin \theta)^2\), \[\|A\| = \cos \theta + \sin \theta = \sqrt{2} \sin \left(\theta + \frac{\pi}{4}\right) \leq \sqrt{2}.\] On the other hand, the eigenvalues of \((A^{-1})^*A^{-1}\) are
\[ \frac{(\cos \theta - \sin \theta)^2}{\cos^2 2\theta}, \frac{1}{\cos 2\theta}, \text{ and } \frac{(\cos \theta + \sin \theta)^2}{\cos^2 2\theta}. \]
Hence
\[ \|A^{-1}\| = \frac{\cos \theta + \sin \theta}{\cos 2\theta} \leq \frac{\sqrt{2}}{\delta^2}, \]
which is the conclusion. \[\square\]

We return to the construction of \(F_j\): Take \(u\) as in Lemma 2.4 and take \(P \in \text{SO}(3)\) and \(\tau \in \mathbb{R}\) as in Lemma 3.1, where \(\theta \in [0, \frac{\pi}{4}]\) is given by \(\cos 2\theta = |(u, u)|\).
Observe that by (1) of Lemma 3.1 we have
\[ (3.1) \quad \delta := \sqrt{\cos 2\theta} \geq \frac{1}{N^{1/4}} \]
and therefore \(\theta \in [0, \frac{\pi}{4}]\). We set
\[ (3.7) \quad F := e^{i\tau}PF_{j-1}, \quad \varphi := \varphi_F := \frac{dF}{dz} = e^{i\tau}P\varphi_{j-1}. \]
Since \(P \in \text{SO}(3) \subset O(3, \mathbb{C})\), \(F\) is a holomorphic null immersion. On the other hand, since \(P \in \text{SO}(3) \subset U(3)\), \(F\) is congruent to \(F_{j-1}\) in \(\mathbb{C}^3\). In particular,
\[ (3.8) \quad |\varphi| = |\varphi_{j-1}|, \quad |\varphi(q) - \varphi(p)| = |\varphi_{j-1}(q) - \varphi_{j-1}(p)| \]
hold for \(p, q \in \mathbb{S}_1\).

Taking into account \[\delta,\] we consider the matrix
\[ (3.9) \quad A = (a^{(1)}, a^{(2)}, a^{(3)}) := \begin{pmatrix} \delta & 0 & 0 \\ 0 & \cos \theta & -i\sin \theta \\ 0 & i\sin \theta & \cos \theta \end{pmatrix} \in \delta \cdot O(3, \mathbb{C}). \]
In particular, by \[\delta,\] it holds that
\[ (3.10) \quad a^{(3)} = e^{-i\tau}Pu = e^{i\tau}Pu. \]
By Lemma 3.2 and \[\delta,\] it holds that
\[ (3.11) \quad \|A\| \leq \sqrt{2}, \quad \|A^{-1}\| \leq \frac{\sqrt{2}}{\delta^2} \leq \sqrt{2}N^{1/4}. \]
Using the matrix \(A\) in \[\delta,\] we set
\[ (3.12) \quad E = (E^{(1)}, E^{(2)}, E^{(3)}) := A^{-1}F = e^{i\tau}A^{-1}PF_{j-1}, \quad \psi := \frac{dE}{dz} = A^{-1}\varphi = e^{i\tau}A^{-1}P\varphi_{j-1}. \]
Since \(A \in \delta \cdot O(3, \mathbb{C})\), \(E\) is a holomorphic null immersion although it is not necessarily congruent to \(F_{j-1}\). Moreover, by \[\delta,\] \[\delta,\] \[\delta,\] \[\delta,\] and \[\delta,\] we have
\[ (3.13) \quad |\psi| = |A^{-1}\varphi| \geq \frac{1}{\|A\|} |\varphi| \geq \frac{|\varphi|}{\sqrt{2}} = \frac{|\varphi_{j-1}|}{\sqrt{2}}, \]
\[ (3.14) \quad |\psi(q) - \psi(p)| = |A^{-1}(\varphi(q) - \varphi(p))| \leq \|A^{-1}\| |\varphi(q) - \varphi(p)| \leq \sqrt{2}N^{1/4} |\varphi_{j-1}(q) - \varphi_{j-1}(p)|. \]
Lemma 3.3. Let $G = G_E: \mathbb{H}_1 \to S^2$ be the Gauss map of $E$ as in (1.18). Then there exists a real matrix $Q$

\begin{equation}
Q = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \Theta & -\sin \Theta \\
0 & \sin \Theta & \cos \Theta
\end{pmatrix} \in \text{SO}(3), \quad |\Theta| \leq \frac{4}{\sqrt{N}}
\end{equation}

such that

\begin{equation}
\text{dist}_{S^2}(QG(p), \pm e_3) \geq \frac{1}{\sqrt{N}} \quad (e_3 = (0, 0, 1))
\end{equation}

holds for each point $p \in \mathbb{H}_j$, where dist$_{S^2}$ is the canonical distance function of the unit sphere $S^2$. In particular, the matrix $Q$ commutes with $A^{-1}$ as in (3.9), that is,

\begin{equation}
AQ^{-1} = Q^{-1}A,
\end{equation}

and

\begin{equation}
\|Q^{-1} - \text{id}\| \leq |\Theta| \leq \frac{4}{\sqrt{N}}
\end{equation}

holds.

Proof. By (3.9) and (3.14), (3.17) is trivial. Moreover, since

\begin{equation}
Q^{-1} - \text{id} = \begin{pmatrix}
0 & \cos \Theta - 1 & 0 \\
0 & -\sin \Theta & \cos \Theta \\
0 & \sin \Theta & \cos \Theta - 1
\end{pmatrix} = -2 \sin \frac{\Theta}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & \sin \frac{\Theta}{2} & -\cos \frac{\Theta}{2} \\
0 & \cos \frac{\Theta}{2} & \sin \frac{\Theta}{2}
\end{pmatrix},
\end{equation}

the maximum eigenvalue of $(Q^{-1} - \text{id})^*(Q^{-1} - \text{id})$ is $(2 \sin \frac{\Theta}{2})^2$. Hence by (1.15),

\begin{equation}
\|Q^{-1} - \text{id}\| = 2 \sin \frac{\Theta}{2} \leq |\Theta|.
\end{equation}

holds, and thus we have (3.18).

So it is sufficient to show (3.16) for suitable $Q$. The Euclidean distance between $G(p)$ and $G(\zeta_j)$ in $\mathbb{R}^3$ can be estimated as

\begin{equation}
|G(p) - G(\zeta_j)| = \left| \psi(p) \times \psi(p) - \psi(\zeta_j) \times \psi(\zeta_j) \right| = \left| \frac{\psi(p) \times \psi(p)}{|\psi(p)|^2} - \frac{\psi(\zeta_j) \times \psi(\zeta_j)}{|\psi(\zeta_j)|^2} \right| \quad (\text{by (1.18)})
\end{equation}

\begin{equation}
= \frac{1}{|\psi(p)|^2 |\psi(\zeta_j)|^2} \left| \left( \psi(p) \times \psi(p) \right) |\psi(\zeta_j)|^2 - \left( \psi(\zeta_j) \times \psi(\zeta_j) \right) |\psi(p)|^2 \right|
\end{equation}

\begin{equation}
= \frac{1}{|\psi(p)|^2 |\psi(\zeta_j)|^2} \left| \left( \psi(p) \times \psi(p) \right) |\psi(\zeta_j)|^2 - \left( \psi(\zeta_j) \times \psi(\zeta_j) \right) |\psi(p)|^2 \right|
\end{equation}

\begin{equation}
+ \left| \psi(p) \times \psi(p) \right| |\psi(p)|^2 - \left( \psi(\zeta_j) \times \psi(\zeta_j) \right) |\psi(p)|^2
\end{equation}

\begin{equation}
\leq \frac{|\psi(p)|^2 \left( |\psi(\zeta_j)|^2 - |\psi(p)|^2 \right) + |\psi(\zeta_j)| \times \psi(\zeta_j) \times \psi(p)|^2
\end{equation}

\begin{equation}
= \frac{1}{|\psi(\zeta_j)|^2} \left( |\psi(p)|^2 \left( |\psi(\zeta_j)|^2 - |\psi(p)|^2 \right) + |\psi(p) \times \psi(p) - \psi(\zeta_j) \times \psi(\zeta_j) \right)
\end{equation}

\begin{equation}
\leq \frac{1}{|\psi(\zeta_j)|^2} \left( |\psi(\zeta_j)| - |\psi(p)| \right) \left( |\psi(\zeta_j)| + |\psi(p)| \right)
\end{equation}
\[
\psi(p) \times \psi(p) - \psi(p) \times \psi(\zeta) + \psi(p) \times \psi(\zeta) - \psi(p) \times \psi(\zeta) - \psi(p) \times \psi(\zeta)
\]

\[
\leq \frac{1}{|\psi(\zeta)|^2} \left( |\psi(\zeta) - \psi(p)| (|\psi(\zeta)| + |\psi(p)|) + |\psi(p) - \psi(\zeta)| + \left( \psi(p) - \psi(\zeta) \right) \times \psi(\zeta) \right)
\]

\[
\leq \frac{2}{|\psi(\zeta)|^2} |\psi(p) - \psi(\zeta)| (|\psi(p) - \psi(\zeta)| + 2|\psi(\zeta)|)
\]

\[
\leq \frac{2}{|\psi(\zeta)|} |\psi(p) - \psi(\zeta)| \left( 2 + \frac{|\psi(p) - \psi(\zeta)|}{|\psi(\zeta)|} \right)
\]

\[
\leq \frac{2\sqrt{2}}{\varphi_j^{-1}(\zeta)} |\psi(p) - \psi(\zeta)| \left( 2 + \frac{\sqrt{2}|\psi(p) - \psi(\zeta)|}{\varphi_j^{-1}(\zeta)} \right)
\] (by (3.13))

\[
\leq \frac{4N^{1/4}|\varphi_j^{-1}(p) - \varphi_j^{-1}(\zeta)|}{|\varphi_j^{-1}(\zeta)|} \left( 2 + \frac{2N^{1/4}|\varphi_j^{-1}(p) - \varphi_j^{-1}(\zeta)|}{|\varphi_j^{-1}(\zeta)|} \right) \] (by (3.14))

\[
\leq \frac{8N^{1/4}|\varphi_j^{-1}(p) - \varphi_j^{-1}(\zeta)|}{\nu} \left( 2 + \frac{4N^{1/4}}{\nu} |\varphi_j^{-1}(p) - \varphi_j^{-1}(\zeta)| \right) \] (Lemma 2.2)

\[
\leq \frac{8N^{1/4}6\mu + 2e}{N} \left( 2 + \frac{4N^{1/4}6\mu + 2e}{N} \right) \] (Lemma 2.3)

\[
\leq \frac{1}{\sqrt{N}} \frac{1}{N^{1/4}} \left( \frac{16(3\mu + \epsilon)}{\nu} \left( 2 + \frac{4(6\mu + \epsilon)}{N^{3/4}\nu} \right) \right) \] (by (1.13))

\[
\leq \frac{1}{\sqrt{N}} \frac{1}{N^{1/4}} \left( \frac{16(3\mu + \epsilon)}{\nu} \left( 2 + \frac{4(6\mu + \epsilon)}{3\nu} \right) \right) \leq \frac{1}{2\sqrt{N}} \] (by (1.13)).

Then we have

\[
(3.19) \quad \text{dist}_{S^2}(G(p), G(\zeta)) = 2 \arcsin \left( \frac{1}{2} |G(p) - G(\zeta)| \right)
\]

\[
\leq \frac{\pi}{2} |G(p) - G(\zeta)| \leq 2|G(p) - G(\zeta)| \leq \frac{1}{\sqrt{N}}
\]

Here we used the inequality \( \arcsin x \leq \pi x / 2 \) (0 ≤ x ≤ 1). In particular, \( G(\varpi_j) \) is contained in the geodesic disc in \( S^2 \) centered at \( G(\zeta) \) with radius \( 1/\sqrt{N} \).

**Case 1:** Assume both \( \text{dist}_{S^2}(G(\zeta), e_1) \geq 2/\sqrt{N} \) and \( \text{dist}_{S^2}(G(\zeta), -e_3) \geq 2/\sqrt{N} \) hold. Then for each \( p \in \varpi_j \), (3.19) implies that

\[
\text{dist}_{S^2}(G(p), e_3) \geq \text{dist}_{S^2}(G(\zeta), e_3) - \text{dist}_{S^2}(G(p), G(\zeta)) \geq \frac{2}{\sqrt{N}} - \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}}.
\]

Similarly \( \text{dist}_{S^2}(G(p), -e_3) \geq 1/\sqrt{N} \) holds. Then we have the conclusion (3.10) for \( Q = \text{id} \) and \( \Theta = 0 \).

**Case 2:** Assume

\[
(3.20) \quad \text{dist}_{S^2}(G(\zeta), e_3) < \frac{2}{\sqrt{N}}
\]

In this case, take the matrix \( Q \) as in (3.15) with

\[
(3.21) \quad \Theta := \frac{4}{\sqrt{N}}
\]
Then
\[
\text{dist}_{S^2}(QG(p), e_3) \geq \text{dist}_{S^2}(QG(p), QG(\zeta)) - \text{dist}_{S^2}(QG(\zeta), Qe_3)
\]
\[
= \frac{4}{\sqrt{N}} - \text{dist}_{S^2}(QG(p), QG(\zeta)) - \text{dist}_{S^2}(QG(\zeta), Qe_3) \quad \text{(by 3.21)}
\]
\[
= \frac{4}{\sqrt{N}} - \text{dist}_{S^2}(G(p), G(\zeta)) - \text{dist}_{S^2}(G(\zeta), e_3) \quad \text{(Q \in SO(3))}
\]
\[
\geq \frac{4}{\sqrt{N}} - \frac{1}{\sqrt{N}} - \text{dist}_{S^2}(G(\zeta), e_3) > \frac{1}{\sqrt{N}} \quad \text{(by 3.19, 3.20)}.
\]

On the other hand,
\[
\text{dist}_{S^2}(QG(p), e_3) \leq \text{dist}_{S^2}(QG(p), QG(\zeta)) + \text{dist}_{S^2}(QG(\zeta), Qe_3) + \text{dist}_{S^2}(Qe_3, e_3)
\]
\[
= \text{dist}_{S^2}(QG(p), QG(\zeta)) + \text{dist}_{S^2}(QG(\zeta), Qe_3) + \frac{4}{\sqrt{N}} \quad \text{(by 3.21)}
\]
\[
= \text{dist}_{S^2}(G(p), G(\zeta)) + \text{dist}_{S^2}(G(\zeta), e_3) + \frac{4}{\sqrt{N}} \quad \text{(Q \in SO(3))}
\]
\[
\leq \frac{1}{\sqrt{N}} + \text{dist}_{S^2}(G(\zeta), e_3) + \frac{4}{\sqrt{N}} \quad \text{(by 3.19)}
\]
\[
< \frac{1}{\sqrt{N}} + \frac{2}{\sqrt{N}} + \frac{4}{\sqrt{N}} = \frac{7}{\sqrt{N}} \quad \text{(by 3.20)}
\]
and then,
\[
\text{dist}_{S^2}(QG(p), -e_3) = \pi - \text{dist}_{S^2}(QG(p), e_3) \geq 3 - \frac{7}{6} \geq \frac{1}{\sqrt{N}}
\]
because of (1.13). Thus, we have the conclusion (3.16).

Case 3: If \(\text{dist}_{S^2}(G(\zeta), -e_3) < 2/\sqrt{N}\) holds, then we have the conclusion by the same way as in the previous case. \(\square\)

Using \(P \in SO(3), \tau \in \mathbb{R}\) in (3.11), \(A \in \delta \cdot O(3, \mathbb{C})\) in (3.9) and \(Q \in SO(3)\) in (3.13), we define
\[
(3.22) \quad \tilde{E} := QE = B^{-1}F_{j-1}, \quad \tilde{\psi} := \frac{d\tilde{E}}{dz} = Q\psi,
\]
where
\[
(3.23) \quad B = (b^{(1)}, b^{(2)}, b^{(3)}) := (e^{i\tau QA^{-1}P})^{-1} \in (e^{-ir}\delta) \cdot O(3, \mathbb{C}).
\]
Then \(\tilde{E}\) is a holomorphic null immersion which is congruent to \(E\) in (3.12). Denote by \((g, \eta)\) the Weierstrass data (cf. (1.3)) of \(\tilde{E}\):
\[
(3.24) \quad \tilde{\psi} = \frac{1}{2}(1 - g^2, i(1 + g^2), 2g)\eta, \quad |\tilde{\psi}|^2 = \frac{1}{2}(1 + |g|^2)^2|\eta|^2.
\]
Then we have

**Lemma 3.4.** The meromorphic function \(g\) as in (3.24) satisfies
\[
\frac{1}{2\sqrt{N}} \leq |g| \leq 2\sqrt{N} \quad \text{and} \quad \frac{|g|}{1 + |g|^2} \geq \frac{2\sqrt{N}}{1 + 4N} \quad \text{(on \(\delta^2\)).}
\]
Proof. The Gauss map \( \tilde{G} \) of \( \tilde{E} \) is obtained by

\[
\tilde{G} = QG = \frac{1}{1 + |g|^2} \begin{pmatrix}
2 \text{Re} \, g \\
2 \text{Im} \, g \\
|g|^2 - 1
\end{pmatrix}.
\]

Here, since \( \tilde{G} = QG \) satisfies (3.19) on \( \omega_j \), it holds that

\[
\text{dist}_{S^2}(\tilde{G}, e_3) = \arccos \left( \tilde{G} \cdot e_3 \right) = \arccos \left( \frac{|g|^2 - 1}{|g|^2 + 1} \right) \geq \frac{1}{\sqrt{N}},
\]

(3.25)

\[
\text{dist}_{S^2}(\tilde{G}, -e_3) = \arccos \left( \tilde{G} \cdot (-e_3) \right) = \arccos \left( \frac{1 - |g|^2}{|g|^2 + 1} \right) \geq \frac{1}{\sqrt{N}}
\]

(3.26)
on \( \omega_j \), where \( \langle , \rangle \) denotes the canonical inner product of \( \mathbb{R}^3 \). Since (3.25) implies

\[
\frac{|g|^2 - 1}{|g|^2 + 1} \leq \cos \frac{1}{\sqrt{N}},
\]

we have

\[
|g|^2 \leq \frac{1 + \cos \frac{1}{\sqrt{N}}}{1 - \cos \frac{1}{\sqrt{N}}} = \cot^2 \frac{1}{2\sqrt{N}} \leq (2\sqrt{N})^2.
\]

Similarly, by (3.26), we have

\[
|g|^2 \geq \tan^2 \frac{1}{2\sqrt{N}} \geq \left( \frac{1}{2\sqrt{N}} \right)^2.
\]

Thus, we have the first inequality of the conclusion. The second inequality is obtained immediately by the first inequality. \( \square \)

We set

\[
v_j := \overline{b^{(3)}},
\]

(3.27)

where \( b^{(3)} \) is the third column of the matrix \( B \) as in (3.23).

Lemma 3.5. The vector \( v_j \) in (3.27) is a unit vector satisfying \( |(v_j, v_j)| \geq 1/N^{1/4} \). Moreover, when (2.11) holds, that is, \( |F_{j-1}(\zeta)| \geq 1/\sqrt{N} \), it holds that

\[
\left| \left\langle \frac{F_{j-1}(p)}{|F_{j-1}(p)|}, v_j \right\rangle \right| \geq 1 - \frac{C_2}{\sqrt{N}} \quad \text{for } p \in \omega_j,
\]

where \( C_2 \) is the constant in (2.11).

Proof. Let \( e_3 = (0, 0, 1) \). Since the matrix \( A \) and \( Q^{-1} \) commute (cf. (3.17)), the third column of the matrix \( B \) is obtained as

\[
b^{(3)} = Be_3 = e^{-ir}P^{-1}AQ^{-1}e_3 = e^{-ir}P^{-1}Q^{-1}Ae_3 \quad \text{(by (3.27), (3.17))}
\]

\[
e^{-ir}P^{-1}Q^{-1}a^{(3)} = e^{-ir}P^{-1}Q^{-1}(e^{ir}P\bar{u}) \quad \text{(by (3.9), (3.10))}
\]

\[
= P^{-1}Q^{-1}P\bar{u}.
\]

Taking into account that \( P \) and \( Q \) are real matrices, (3.27) implies that \( v_j = P^{-1}Q^{-1}P\bar{u} \). Then by Lemma 2.4, we have \( |v_j| = 1, \ |(v_j, v_j)| \geq 1/N^{1/4} \), because \( P, Q \in \text{SO}(3) \). Moreover, when \( |F_{j-1}(\zeta)| \geq 1/\sqrt{N} \) (i.e. (2.11) holds),

\[
\left| \left\langle \frac{F_{j-1}(p)}{|F_{j-1}(p)|}, v_j \right\rangle \right| = \left| \left\langle \frac{F_{j-1}(p)}{|F_{j-1}(p)|}, P^{-1}Q^{-1}P\bar{u} \right\rangle \right|
\]

\[
= \left| \left\langle \frac{F_{j-1}(p)}{|F_{j-1}(p)|}, u + P^{-1}(Q^{-1} - \text{id})P\bar{u} \right\rangle \right|
\]

\[
\geq \left| \left\langle \frac{F_{j-1}(p)}{|F_{j-1}(p)|}, u \right\rangle \right| - \left| \left\langle \frac{F_{j-1}(p)}{|F_{j-1}(p)|}, P^{-1}(Q^{-1} - \text{id})P\bar{u} \right\rangle \right|
\]
Lemma 3.6. There exists a holomorphic function \( h \) on \( \mathbb{C} \) which does not vanish on \( \mathbb{C} \) and satisfies

\[
\begin{align*}
&|h - 1| \leq \varepsilon_1 \quad (\text{on } \overline{\mathbb{D}} \setminus \mathbb{D}) \\
&|h - T| \leq 1 \quad (\text{on } \omega_j)
\end{align*}
\]

where

\[
(3.28) \quad \varepsilon_1 = \frac{\varepsilon}{\varepsilon + 4\sqrt{2}\mu_{j-1}N^{3/4}}, \quad \mu_{j-1} = \max \frac{|\varphi_{j-1}|}{\varphi_1}, \quad T = 4N^{7/2} + 1.
\]

Using the function \( h \) in Lemma 3.6 as a Lópe-Ros parameter, we produce new Weierstrass data as follows:

\[
(3.29) \quad \hat{g} := \frac{g}{h}, \quad \hat{\eta} := h\eta, \quad \hat{\psi} := \frac{1}{2}(1 - \hat{g}^2, i(1 + \hat{g}^2), 2\hat{g})\hat{\eta}.
\]

We denote

\[
(3.30) \quad \hat{E}(z) := \int_0^z \hat{\psi}(z) \, dz, \quad F_j := B\hat{E},
\]

where \( B \) is the matrix as in (3.23). By definition (3.29), \( \gamma\eta = \hat{g}\hat{\eta} \) holds. Thus, if we write

\[
\hat{\psi} = (\hat{\psi}^{(1)}, \hat{\psi}^{(2)}, \hat{\psi}^{(3)}) \quad \text{and} \quad \hat{\psi} = (\hat{\psi}^{(1)}, \hat{\psi}^{(2)}, \hat{\psi}^{(3)}),
\]

then

\[
(3.31) \quad \hat{\psi}^{(3)} = \hat{\psi}^{(3)}
\]

holds.

Now, the construction procedure of \( F_j \) is accomplished. Thus, we obtain a sequence \( \{F_j\}_{j=1,\ldots,2N} \) of holomorphic null immersions and a sequence \( \{v_j\}_{j=1,\ldots,2N} \) of unit vectors.

In this subsection, we shall prove that \( \{F_j\} \) and \( \{v_j\} \) satisfy the conclusions (K-0) and (K-6) of the Key Lemma 1.3.

Lemma 3.7. For each \( j = 1, \ldots, 2N \), \( \langle F_j, v_j \rangle = \langle F_{j-1}, v_j \rangle \) holds.

Proof. By (3.31), we have

\[
(3.32) \quad \langle \hat{E}, e_3 \rangle = \int_0^\infty \hat{\psi}^{(3)}(w) \, dw = \int_0^\infty \hat{\psi}^{(3)}(w) \, dw = \langle \hat{E}, e_3 \rangle \quad (e_3 = (0,0,1)).
\]
Since $B \in (e^{-ir}\delta) \cdot O(3, \mathbb{C})$,
\begin{equation}
(Bx, By) = e^{-2ir}\delta^2 \langle x, y \rangle
\end{equation}
holds. Then
\begin{align*}
(F_j, v_j) &= (F_j, \psi_j) \\
&= \left( B\hat{E}, \psi_j \right) = \left( B\hat{E}, b^{(3)} \right) = \left( \hat{B}\hat{E}, Be_3 \right) & \text{by (3.30), (3.27), (3.23)} \\
&= e^{-2ir}\delta^2 \left( \hat{E}, e_3 \right) = e^{-2ir}\delta^2 \left( \hat{E}, e_3 \right) = (F_{j-1}, v_j) & \text{by (3.33), (3.32)}.
\end{align*}

The properties [(K-4), (K-5), (K-6)] in the Key Lemma 1.3 for $l = j$ hold by Lemmas 3.5 and 3.7. The property [(K-1)] holds trivially because of (3.30). So we shall prove that $F_j$ satisfies [(K-2), (K-3)] of the Key Lemma 1.3.

Lemma 3.8. The holomorphic null immersion $F_j$ as in (3.30) satisfies
\begin{align}
|\varphi_j - \varphi_{j-1}| &\leq \frac{\epsilon}{2N^2} \quad \text{on } \overline{\mathbb{D}_1 \setminus \mathbb{D}_j}, \\
|\varphi_j| &\geq \frac{C_1}{N^{3/4}} \quad \text{on } \overline{\mathbb{D}_j}, \\
|\varphi_j| &\geq C_1 N^{9/4} \quad \text{on } \omega_j.
\end{align}
where $C_1$ is given in (1.11).

Proof. By the definitions (3.32), (3.22) and (3.30), and noticing that $Q^{-1}$ and $A$ commute (cf. (3.21)), we have
\begin{align*}
\varphi_{j-1} &= e^{-ir}P^{-1}AQ^{-1}\hat{\psi} = e^{-ir}P^{-1}Q^{-1}A\hat{\psi}, \\
\varphi_j &= e^{-ir}P^{-1}AQ^{-1}\hat{\psi} = e^{-ir}P^{-1}Q^{-1}A\hat{\psi}.
\end{align*}
Then
\begin{equation}
|\varphi_{j-1}| = |A\hat{\psi}|, \quad |\varphi_j| = |A\hat{\psi}|, \quad |\hat{\psi}| = |A^{-1}P\varphi_{j-1}|, \quad |\hat{\psi}| = |A^{-1}P\varphi_j|,
\end{equation}
hold because $P, Q \in SO(3)$. By (1.4) and (3.11),
\begin{align}
|\varphi_j - \varphi_{j-1}| &= |A(\hat{\psi} - \hat{\psi})| \leq \frac{1}{\|A\|} |\hat{\psi} - \hat{\psi}| \leq \sqrt{2} |\hat{\psi} - \hat{\psi}| \\
|\varphi_j - \varphi_{j-1}| &= |A(\hat{\psi} - \hat{\psi})| \geq \frac{1}{\|A^{-1}\|} |\hat{\psi} - \hat{\psi}| \geq \frac{1}{\sqrt{2N^{1/4}}} |\hat{\psi} - \hat{\psi}|
\end{align}
hold. Here, by (3.24), (3.29) and (3.31), we have
\begin{align*}
|\hat{\psi} - \hat{\psi}| &= \frac{1}{2} \left| \left( 1 - g^2 \right) \eta - (1 - g^2)\eta i (1 + g^2)\eta - i(1 + g^2)\eta \right| \\
&= \frac{1}{2} \left| \left( 1 - \frac{g^2}{h^2} \right) h\eta - (1 - g^2)\eta i \left( 1 + \frac{g^2}{h^2} \right) h\eta - i(1 + g^2)\eta \right| \\
&= \frac{1}{2} \left| (h - 1) \left( 1 + \frac{g^2}{h^2} \right) i \left( 1 - \frac{g^2}{h^2} \right) \eta \right| \\
&= \frac{1}{2} |h - 1| |\eta| \left( 1 + \frac{|g^2|}{h^2} \right) \left( 1 + \frac{|g^2|}{h^2} \right) \leq \frac{1}{2} |h - 1| |\eta| \left( 1 + \frac{|g^2|}{h} \right) \left( 1 + \frac{|g^2|}{h} \right) \\
&\leq |h - 1| |\eta| \left( 1 + \frac{|g^2|}{h^2} \right) \leq |h - 1| |\eta| \left( 1 + \frac{|g^2|}{h - 1} \right) \\
&\leq |h - 1| \frac{1 + |g^2| |\eta|}{1 - |h - 1|} = \sqrt{2} |\hat{\psi}| \frac{|h - 1|}{1 - |h - 1|}.
\end{align*}
Since $h$ is taken as in Lemma 3.6 and $P \in SO(3)$, we have
\[
|\tilde{\psi} - \psi_j| \leq \sqrt{2} |\tilde{\psi}| \frac{\varepsilon_1}{1 - \varepsilon_1} = \frac{\varepsilon_1}{4\sqrt{2}M_{j-1}N^{9/4}} \tag{Lemma 3.6 (3.25)}
\]
\[
= |A^{-1}P\phi_{j-1}| \frac{\varepsilon_1}{4\sqrt{2}M_{j-1}N^{9/4}} \leq \|A^{-1}\| |P\phi_{j-1}| \frac{\varepsilon}{4M_{j-1}N^{9/4}} \tag{by (3.37), (1.4)}
\]
\[
\leq \sqrt{2}N^{1/4} |\phi_{j-1}| \frac{\varepsilon}{4M_{j-1}N^{9/4}} \tag{by (3.11)}
\]
\[
\leq \sqrt{2}N^{1/4}M_{j-1}^{-1} \frac{\varepsilon}{4M_{j-1}N^{9/4}} = \frac{\varepsilon}{2\sqrt{2}N^2} \tag{by (3.28)}
\]
holds on $\partial U \setminus \omega_j$. Thus, by (3.35) $|\phi_j - \phi_{j-1}| \leq \sqrt{2} |\tilde{\psi} - \psi_j| \leq \varepsilon/(2N^2)$, which is (3.36).

Next, on $\omega_j$, it holds that
\[
|\phi_j| = |A\phi_j| \geq \|A^{-1}\| |\phi_j| \geq \frac{1}{\sqrt{2}N^{1/4}} |\phi_j| \tag{by (3.37), (1.4), (3.11)}
\]
\[
= \frac{1}{N^{1/4}} |\phi_j| = \frac{1}{N^{1/4}} |g\eta| \tag{by (3.23)}
\]
\[
= \sqrt{2}N^{1/4} |\phi_j| \frac{1 + |g|^2|\eta|}{1 + |g|^2} = \frac{\sqrt{2}|\psi|}{N^{1/4}} \frac{|g|}{1 + |g|^2} \tag{by (3.37), (1.4), $P \in SO(3)$)
\]
\[
= \frac{|\phi_{j-1}|}{N^{1/4}} \frac{|g|}{1 + |g|^2} \geq \frac{1}{N^{3/4}} \frac{1}{1 + |g|^2} \tag{by (3.11), Lemma 3.4}
\]
\[
\geq \frac{\nu}{2N^{1/4}} \frac{1}{1 + 4N} = \frac{\nu}{N^{3/4}} \frac{1}{4 + 1/N} \geq \frac{\nu}{5N^{3/4}} = C_1 \frac{1}{N^{3/4}} \tag{Lemma 2.1 (1.11)}
\]
Thus, we have (3.35).

Finally, on $\omega_j$, we have
\[
|\phi_j| \geq \frac{1}{\sqrt{2}N^{1/2}} (1 + |g|^2|\eta|)
\]
by the same way as in the previous argument. Then
\[
|\phi_j| \geq \frac{1}{2N^{1/4}} (1 + |g|^2|\eta|) \geq \frac{1}{2N^{1/4}} |\phi_j|
\]
\[
= \frac{|h|}{\sqrt{2}N^{1/4}} |\phi_j| \frac{1}{1 + |g|^2} = \frac{|h|}{\sqrt{2}N^{1/4}} |A^{-1}P\phi_{j-1}| \frac{1}{1 + |g|^2} \tag{by (3.37)}
\]
\[
\geq \frac{|h|}{\|A\|\sqrt{2}N^{1/4}} |P\phi_{j-1}| \frac{1}{1 + |g|^2} \geq \frac{|h|}{2N^{1/4}} \frac{1}{1 + |g|^2} \tag{by (1.4), (3.11)}
\]
\[
\geq \frac{|h|}{2N^{1/4}} \frac{1}{1 + 4N} \geq \frac{|h|}{2N^{1/4}} \frac{1}{2 + 4N} \tag{P \in SO(3), Lemmas 3.4, 5.22}
\]
\[
\geq \frac{\nu}{4N^{3/4}} \frac{1}{20N^{3/4}} |h| = \frac{\nu}{20N^{5/4}} (T - |h| - T) \geq \frac{\nu}{20N^{5/4}} \frac{1}{4N^{7/2}} \geq C_1 \frac{1}{N^{9/4}} \tag{Lemma 4.1 (1.11)}
\]
Hence we have (3.36). □

Thus we have \( \{F_j\} \) and \( \{v_j\} \) satisfying properties (K-0)–(K-6) in Lemma 1.3.

4. A PROOF OF PROPOSITION 1.2

In this section, we prove Proposition 1.2. We take the sequences \( \{F_j\} \) and \( \{v_j\} \) as in the Key Lemma 1.3, and set

\[
Y := F_{2N}.
\]

Recall that \( X = F_0 \) by (K-0). Then we shall prove (Y-1)–(Y-3) in Proposition 1.2.

Lemma 4.1. It holds that

\[
|\varphi_Y - \varphi_X| \leq \frac{\varepsilon}{N}, \quad \text{and} \quad |Y - X| \leq \frac{2\varepsilon}{N} \quad \text{on} \quad D_1 \setminus (\varpi_1 \cup \cdots \cup \varpi_{2N}).
\]

Proof. By (K-2) of the Key Lemma 1.3,

\[
|\varphi_Y - \varphi_X| = |\varphi_{2N} - \varphi_0| \leq |\varphi_{2N} - \varphi_{2N-1}| + \cdots + |\varphi_1 - \varphi_0| \leq 2N \cdot \frac{\varepsilon}{2N^2} = \frac{\varepsilon}{N}
\]

holds on \( D_1 \setminus (\varpi_1 \cup \cdots \cup \varpi_{2N}) \). On the other hand, by Lemma 2.1,

\[
|Y - X| = |F_{2N} - F_0| \leq |F_{2N} - F_{2N-1}| + \cdots + |F_1 - F_0| \leq 2N \cdot \frac{\varepsilon}{N^2} = \frac{2\varepsilon}{N}
\]

holds on \( \varpi_1 \cup \cdots \cup \varpi_{2N} \).

□

Corollary 4.2 (the conclusion (Y-1)). It holds that

\[
|\varphi_Y - \varphi_X| < \varepsilon \quad \text{and} \quad |Y - X| < \varepsilon \quad \text{on} \quad D_1 - \varepsilon.
\]

Proof. Note that we take the labyrinth as in Appendix A. Here, by (1.14),

\[
\frac{2}{N} + \frac{1}{8N^3} = \frac{1}{N} \left( 2 + \frac{1}{8N^2} \right) < \frac{3}{N} \leq \varepsilon
\]

holds. Then by (2) of Lemma A.1 in Appendix A, we have that

\[
D_{1-\varepsilon} \subset D_1 \setminus (\varpi_1 \cup \cdots \cup \varpi_{2N}).
\]

Thus, by Lemma 4.1 and (1.13), it holds on \( D_{1-\varepsilon} \) that

\[
|\varphi_Y - \varphi_X| = |\varphi_{2N} - \varphi_0| \leq \frac{\varepsilon}{N} \leq \varepsilon, \quad |Y - X| = |F_{2N} - F_0| \leq \frac{2\varepsilon}{N} \leq \varepsilon
\]

□

Lemma 4.3. The function \( \varphi_Y = \varphi_{2N} \) satisfies

\[
|\varphi_Y| \geq \begin{cases} 
C_1 2^{N^{9/4}} & \text{on} \ \omega_1 \cup \cdots \cup \omega_{2N} \\
C_1 2^{N^{3/4}} & \text{on} \ \varpi_1 
\end{cases}
\]

Proof. On \( \omega_j \),

\[
|\varphi_Y| = |\varphi_{2N}| \geq |\varphi_j| - |\varphi_{2N} - \varphi_{2N-1}| - \cdots - |\varphi_{j+1} - \varphi_j|
\]

\[
\geq C_1 2^{N^{9/4}} - \frac{(2N - j + 1)\varepsilon}{2N^2} \geq C_1 2^{N^{9/4}} - \frac{\varepsilon}{N^3} \geq C_1 2^{N^{9/4}} \left( 1 - \frac{\varepsilon}{N^{9/4}} \right) \geq \frac{C_1}{2} N^{9/4}
\]

(by (K-3) [K-2])

(by (1.14)).
On the other hand, on \( \varpi_j \), we have
\[
|\varphi_Y| = |\varphi_{2N}| \geq |\varphi_j| - |\varphi_{2N} - \varphi_{2N-1}| - \cdots - |\varphi_{j+1} - \varphi_j| \\
\geq \frac{C_1}{N^{3/4}} \left( \frac{2N - j + 1}{2N^2} \right) \varepsilon \quad \text{(by [K-3], [K-2])}
\]
\[
\geq \frac{C_1}{N^{3/4}} \cdot \frac{\varepsilon}{N} = \frac{1}{N^{3/4}} \left( C_1 - \frac{\varepsilon}{N^{1/4}} \right) \geq \frac{C_1}{2N^{3/4}} \quad \text{(by (1.14)).}
\]

Finally, on \( \mathbb{D}_1 \setminus (\varpi_1 \cup \cdots \cup \varpi_{2N}) \),
\[
|\varphi_Y| = |\varphi_{2N}| \geq |\varphi_0| - |\varphi_{2N} - \varphi_{2N-1}| - \cdots - |\varphi_1 - \varphi_0| \\
\geq |\varphi_0| - 2N \cdot \frac{\varepsilon}{2N^2} \geq \frac{\varepsilon}{N} \quad \text{(by (K-2), (1.16))}
\]
\[
\geq \nu - \frac{\varepsilon}{N^{3/4}} \geq \frac{C_1}{2N^{3/4}} \quad \text{(by (1.14)).}
\]

Hence we have the conclusion.

\[\square\]

**Corollary 4.4** (the conclusion [Y-2].) *The disc \( \mathbb{D}_1, ds_Y^2 \) contains a geodesic disc \( \mathcal{D} \) centered at 0 with radius \( \rho + s \).*

**Proof.** The induced metric \( ds_Y^2 \) is expressed as
\[
ds_Y^2 = |\varphi_Y|^2 |dz|^2.
\]

Consider a Riemannian metric
\[
ds^2 := \left( \frac{2N^{3/4}}{C_1} \right)^2 ds_Y^2 = \lambda^2 |dz|^2, \quad \left( \lambda := \frac{2N^{3/4}}{C_1} |\varphi_Y| \right).
\]

Then by Lemma 4.3, \( ds^2 \) satisfies the assumptions of Lemma A.4 in Appendix A.

Thus, we have
\[
dist_{ds^2}(0, \partial \mathbb{D}_1) \geq N,
\]
where \( dist_{ds^2} \) denotes the distance function with respect to \( ds^2 \). Then by (1.14), we have
\[
dist_{ds_Y^2}(0, \partial \mathbb{D}_1) \geq \frac{C_1}{2N^{3/4}} N = \frac{C_1N^{1/4}}{2} \geq \rho + s.
\]

Hence we have the conclusion.

\[\square\]

By Corollary 4.4, one can take a geodesic disc \( \mathcal{D} \) of \( (\mathbb{D}_1, ds_Y^2) \) centered at the origin with radius \( \rho + s \). Fix \( p \in \partial \mathcal{D} \), and prove [Y-3] of the Key Lemma 1.3. First, we assume \( p \in \varpi_j \) for some \( j \in \{1, \ldots, 2N\} \) (otherwise, the proof of [Y-3] is rather easy). Since \( p \in \partial \mathcal{D} \), there exists a \( ds_Y^2 \)-geodesic \( \gamma \) joining 0 and \( p \) with length \( \rho + s \). Since \( ds_Y^2 \) is a Riemannian metric of non-positive Gaussian curvature,

(4.3) an arbitrary subarc of \( \gamma \) is the shortest geodesic.

Hence the image of \( \gamma \) is contained in \( \mathcal{D} \).

**Lemma 4.5.** *The Euclidean length of \( \gamma \) satisfies*
\[
\text{Length}_C(\gamma) \leq \frac{2(\rho + s)}{C_1} N^{3/4}.
\]

**Proof.** Since he \( ds_Y^2 \)-arc-length of \( \gamma \) is \( \rho + s \), Lemma 4.3 implies that
\[
\rho + s = \int_\gamma |\varphi_Y| |dz| \geq \int_\gamma \frac{C_1}{2N^{3/4}} |dz| = \frac{C_1}{2N^{3/4}} \text{Length}_C(\gamma).
\]

Hence we have the conclusion.

\[\square\]

Now, take points \( \bar{p}, \tilde{p} \in \mathcal{D} \) on the arc \( \gamma \) such that

- \( \bar{p} \in \partial \varpi_j \) and the subarc of \( \gamma \) joining \( \bar{p} \) and \( p \) is contained in \( \varpi_j \), namely, \( \tilde{p} \) is the final point where \( \gamma \) meets \( \partial \varpi_j \),
and the subarc of $\gamma$ joining 0 and $\tilde{p} \in \partial D_1^\gamma - \frac{2}{N} - \frac{8}{N^3}$ contained in $\overline{\partial D_1^\gamma - \frac{2}{N} - \frac{8}{N^3}}$; namely, $\tilde{p}$ is the first point where $\gamma$ meets $\partial D_1^\gamma - \frac{2}{N} - \frac{8}{N^3}$.

See Figure 1

Lemma 4.6. It holds that

\begin{align}
|F_l(\tilde{p})| &\leq r + \frac{2\varepsilon}{N} \quad (l = 0, \ldots, 2N), \quad (4.4) \\
|F_{j-1}(p)| &\leq r + \frac{2\varepsilon}{N}, \quad (4.5) \\
|F_{2N}(p) - F_j(p)| &\leq \frac{2\varepsilon}{N}, \quad |F_{2N}(\tilde{p}) - F_j(\tilde{p})| \leq \frac{2\varepsilon}{N}, \quad (4.6) \\
|F_{2N}(p) - F_{2N}(\tilde{p})| &\leq s + \frac{c_2}{N^{1/4}}, \quad (4.7)
\end{align}

where $C_1$ and $c_2$ are defined by (1.11) and (1.12), respectively.

Proof. Since $\tilde{p} \not\in \varpi_1 \cup \cdots \cup \varpi_{2N}$, Lemma 2.1 and the assumption [X-3] of the Proposition 1.2 imply

\begin{align*}
|F_l(\tilde{p})| &\leq |F_0(\tilde{p})| + |F_1(\tilde{p}) - F_0(\tilde{p})| + \cdots + |F_l(\tilde{p}) - F_{l-1}(\tilde{p})| \\
&\leq r + \frac{le}{N^2} \leq r + \frac{2\varepsilon}{N}.
\end{align*}

Hence we have (4.4). A similar reasoning proves (4.5).

Since $p \not\in \varpi_{j+1} \cup \cdots \cup \varpi_{2N}$, Lemma 2.1 implies

\begin{align*}
|F_{2N}(p) - F_j(p)| &\leq |F_{2N}(p) - F_{2N-1}(p)| + \cdots + |F_{j+1}(p) - F_j(p)| \\
&\leq \frac{(2N - j)\varepsilon}{N^2} < \frac{2\varepsilon}{N}.
\end{align*}

Then the first inequality of (4.6) holds. Similarly, we have the second inequality of (4.6).

Let $\gamma_1$ be the subarc of the geodesic $\gamma$ joining 0 and $\bar{p}$, and let $\gamma_2$ be the line segment joining $\bar{p}$ and $\partial D_1$ which is contained in the line $0\bar{p}$, see Figure 1. Since $\gamma_1 \cup \gamma_2$ is a path joining 0 and $\partial D_1$, the assumption [X-2] and [K-0] implies that

\begin{align}
\text{Length}_{d_{\delta^2}}(\gamma_1 \cup \gamma_2) = \int_{\gamma_1 \cup \gamma_2} |\phi_0(z)||dz| &\geq \text{dist}_{d_{\delta^2}}(0, \partial D_1) \geq \rho, \quad (4.8)
\end{align}
Then, with respect to the Hermitian inner product \( (4.11) \) \( \Pi \) orthogonal projection

Lemma 4.7.

Case 1: \( p \in \mathbb{R} \) and \( |F_{j-1}(\mathbf{r})| > 1/\sqrt{N} \).

Lemma 4.7. When \( p \in \mathbb{R} \) and \( |F_{j-1}(\mathbf{r})| > 1/\sqrt{N} \),

\[ |F_j(p)| \leq \sqrt{r^2 + s^2 + \frac{c_3}{N^{1/4}}} \]

holds.

Proof. Let \( v_j \in \mathbb{C}^3 \) be the unit vector as in \([K-4]\) \([K-6]\) and denote

\[ (v_j)^\perp := (\text{the orthogonal complement of } v_j \text{ with respect to } \langle \cdot, \cdot \rangle) \]

Then \( (v_j)^\perp \) is a (complex) 2-dimensional subspace of \( \mathbb{C}^3 \). Denote by \( \Pi_j \) the orthogonal projection

\[ \Pi_j : \mathbb{C}^3 \ni x \mapsto x - \langle x, v_j \rangle v_j \in (v_j)^\perp \]

with respect to the Hermitian inner product \( \langle \cdot, \cdot \rangle \). Then for any vector \( x \in \mathbb{C}^3 \),

\[ |x|^2 = |\langle x, v_j \rangle|^2 + |\Pi_j x|^2 \]
holds. Thus, we have

\[ |\Pi_j F_j(p)| \leq |\Pi_j F_j(p) - \Pi_j F_j(\bar{p})| + |\Pi_j F_j(\bar{p})| \]

\[ \leq |\Pi_j (F_j(p) - F_j(\bar{p}))| + |\Pi_j F_j(\bar{p})| \]

\[ \leq |F_j(p) - F_j(\bar{p})| + |\Pi_j F_j(\bar{p})| \]

(by (4.12))

\[ \leq |F_j(p) - F_j(\bar{p})| + |\Pi_j (F_j(p) - F_j(\bar{p}))| + |\Pi_j F_j(\bar{p})| \]

\[ \leq |F_j(p) - F_j(\bar{p})| + |F_j(\bar{p}) - F_{j-1}(\bar{p})| + |\Pi_j F_j(\bar{p})| \]

(by (4.12))

\[ \leq |F_j(p) - F_{j-1}(\bar{p})| + |\Pi_j F_{j-1}(\bar{p})| \]

\[ \leq |F_j(p) - F_{j-1}(\bar{p})| + |\Pi_j F_{j-1}(\bar{p})| \]

\[ \leq \left( s + \frac{c_2}{N^{1/4}} \right) + \frac{2\varepsilon}{N} + \frac{2\varepsilon}{N} + |F_j(\bar{p}) - F_{j-1}(\bar{p})| + |\Pi_j F_{j-1}(\bar{p})| \]

(Lemma 4.6)

\[ \leq \left( s + \frac{c_2}{N^{1/4}} \right) + \frac{4\varepsilon}{N} + \frac{\varepsilon}{N^2} + |\Pi_j F_{j-1}(\bar{p})| \]

(Lemma 2.1)

\[ \leq s + \frac{1}{N^{1/4}} \left( c_2 + \frac{4\varepsilon}{N^{3/4}} + \frac{\varepsilon}{N^{1/2}} \right) + |\Pi_j F_{j-1}(\bar{p})| \]

\[ \leq s + \frac{c_2 + 5\varepsilon}{N^{1/4}} + |\Pi_j F_{j-1}(\bar{p})|. \]

Hence we have

(4.13) \[ |\Pi_j F_j(p)| \leq s + \frac{c_2 + 5\varepsilon}{N^{1/4}} + |\Pi_j F_{j-1}(\bar{p})|. \]

Here, we assume \( F_{j-1}(\bar{p}) \neq 0 \). Since \( \bar{p} \in \mathcal{P}_j \), we have

\[ |\Pi_j F_{j-1}(\bar{p})| = \sqrt{|F_{j-1}(\bar{p})|^2 - |(F_{j-1}(\bar{p}), v_j)|^2} \]

(by (4.12))

\[ = |F_{j-1}(\bar{p})| \sqrt{1 - \left| \frac{F_{j-1}(\bar{p})}{|F_{j-1}(\bar{p})|} \cdot v_j \right|^2} \]

\[ \leq |F_{j-1}(\bar{p})| \sqrt{1 - \left( 1 - \frac{C_2}{\sqrt{N}} \right)^2} \]

(by (K-5))

\[ = |F_{j-1}(\bar{p})| \sqrt{\frac{2C_2}{\sqrt{N}} - \frac{C_2^2}{N}} \leq |F_{j-1}(\bar{p})| \sqrt{\frac{2C_2}{\sqrt{N}}} \]

\[ = |F_{j-1}(\bar{p})| \sqrt{\frac{2C_2}{N^{1/4}}} \leq \left( r + \frac{2\varepsilon}{N} \right) \cdot \sqrt{\frac{2C_2}{N^{1/4}}} \]

(by (4.1) in Lemma 4.6)

\[ \leq \frac{(r + 2\varepsilon) \cdot \sqrt{2C_2}}{N^{1/4}}. \]

Then by (4.13), we have

(4.14) \[ |\Pi_j F_j(p)| \leq s + \frac{\alpha}{N^{1/4}} \]

\[ \left( \alpha := c_2 + 5\varepsilon + (r + 2\varepsilon) \sqrt{2C_2} \right) \]

when \( F_{j-1}(\bar{p}) \neq 0 \). Otherwise, namely when \( F_{j-1}(\bar{p}) = 0 \), (4.14) holds trivially. Thus,

\[ |F_j(p)| = \sqrt{|(F_j(p), v_j)|^2 + |\Pi_j F_j(p)|^2} \]

(by (4.12))

\[ = \sqrt{|(F_{j-1}(p), v_j)|^2 + |\Pi_j F_j(p)|^2} \]

(by (K-6))

\[ \leq \sqrt{|F_{j-1}(p)|^2 + |\Pi_j F_j(p)|^2} \]
Proof. Case 2: The case that holds. When Lemma 4.9.

Under the assumption of Lemma 4.7, we have holds, which is the conclusion. □

Corollary 4.8. Under the assumption of Lemma 4.7 we have

\[ |Y(p)| = |F_{2N}(p)| \leq \sqrt{r^2 + s^2} + \varepsilon \quad \text{for } p \in (\partial D \cap \omega_j). \]

Proof.

\[ |F_{2N}(p)| \leq |F_j(p)| + |F_{2N}(p) - F_j(p)| \leq |F_j(p)| + \frac{2\varepsilon}{N} \quad \text{(by (4.6) in Lemma 4.6)} \]

\[ \leq \sqrt{r^2 + s^2} + \frac{c_3}{N^{1/4}} + \frac{2\varepsilon}{N} \quad \text{(Lemma 4.7)} \]

\[ = \sqrt{r^2 + s^2} + \frac{1}{N^{1/4}} \left( c_3 + \frac{2\varepsilon}{N^{3/4}} \right) \]

\[ \leq \sqrt{r^2 + s^2} + \frac{1}{N^{1/4}} (c_3 + 2\varepsilon) \leq \sqrt{r^2 + s^2} + \varepsilon \quad \text{(by (4.13)).} \quad \Box \]

Case 2: The case that \( p \in \omega_j \) and \( |F_{j-1}(\zeta_j)| \leq 1/\sqrt{N} \).

Lemma 4.9. When \( |F_{j-1}(\zeta_j)| \leq 1/\sqrt{N} \) and \( p \in (\partial D \cap \omega_j) \),

\[ |F_{2N}(p)| \leq \sqrt{r^2 + s^2} + \varepsilon \]

holds.

Proof. Since \( \bar{p} \in \partial \omega_j \),

\[ |F_{2N}(p)| \leq |F_{2N}(p) - F_{2N}(\bar{p})| + |F_{2N}(\bar{p})| \leq \left( s + \frac{c_2}{N^{1/4}} \right) + |F_{2N}(\bar{p})| \quad \text{(by (4.7))} \]

\[ \leq \left( s + \frac{c_2}{N^{1/4}} \right) + |F_{2N}(\bar{p})| + |F_j(\bar{p})| \quad \text{(by (4.6))} \]

\[ \leq s + \frac{1}{N^{1/4}} \left( c_2 + \frac{2\varepsilon}{N^{3/4}} \right) + |F_j(\bar{p}) - F_{j-1}(\bar{p})| + |F_{j-1}(\bar{p})| \quad \text{(Lemma 2.1)} \]

\[ \leq s + \frac{1}{N^{1/4}} \left( c_2 + \frac{2\varepsilon}{N^{3/4}} + \frac{\varepsilon}{N^{7/4}} \right) + |F_{j-1}(\bar{p}) - F_{j-1}(\zeta_j)| + |F_{j-1}(\zeta_j)| \quad \text{(Lemma 2.3)} \]

\[ \leq s + \frac{1}{N^{1/4}} \left( c_2 + \frac{2\varepsilon}{N^{3/4}} + \frac{\varepsilon}{N^{7/4}} + \frac{6\mu}{N^{3/4}} \right) + |F_{j-1}(\zeta_j)| \]
we refer the reader to [7] or [4].

\[
\text{Since } \{1.15\} \quad \{4.15\}
\]

than we have the conclusion. \(\square\)

Case 3: \(p \in \overline{D}_1 \setminus (\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{2N})\). The remaining case is that

\[
|F_{2N}(p)| \leq \sqrt{r^2 + s^2 + \varepsilon}
\]

Lemma 4.10. If \(p\) satisfies \(\{4.13\}\), then

\[
|F_{2N}(p)| \leq \sqrt{r^2 + s^2 + \varepsilon}
\]

Proof. By the assumption \(\{X-3\}\) \(|X(p)| = |F_0(p)| \leq r\) holds. Then by Lemma 4.11 and \(\{1.13\}\), we have

\[
|F_{2N}(p)| \leq |F_0(p)| + |F_{2N}(p) - F_0(p)| \leq r + \frac{2\mu}{N} \leq r + \varepsilon \leq \sqrt{r^2 + s^2 + \varepsilon}. \quad \square
\]

Summing up, Corollary 4.8 and Lemmas 4.9 and 4.10 implies \(\{Y-3\}\) of the Proposition 1.2

APPENDIX A. LABYRINTH

For the sake of completeness, we recall Nadirashvili’s labyrinth (for further details we refer the reader to [7] or [4]).

For each number \(j = 0, 1, 2, \ldots, 2N^2\), we set

\[
(A.1) \quad r_j := 1 - \frac{j}{N^2} \quad \left( r_0 = 1, r_1 = 1 - \frac{1}{N}, \ldots, r_{2N^2} = 1 - \frac{2}{N} \right),
\]

and take a sequence of domains

\[
(A.2) \quad \mathbb{D}_{r_j} = \{ z \in \mathbb{C} \, | \, |z| < r_j \} \quad (j = 0, \ldots, 2N^2).
\]

Since \(\{r_j\}\) is decreasing in \(j\), it holds that

\[
\mathbb{D}_1 = \mathbb{D}_{r_0} \supset \mathbb{D}_{r_1} \supset \cdots \supset \mathbb{D}_{r_{2N^2}} = \mathbb{D}_{1-\frac{2}{N}}.
\]

We denote the boundaries of \(\mathbb{D}_{r_j}\) by

\[
(A.3) \quad S_{r_j} = \partial \mathbb{D}_{r_j} = \{ z \in \mathbb{C} \, | \, |z| = r_j \}.
\]

We set

\[
(A.4) \quad \mathcal{A} := \overline{\mathbb{D}_1} \setminus \mathbb{D}_{r_{2N^2}} = \overline{\mathbb{D}_1} \setminus \mathbb{D}_{1-\frac{2}{N}}
\]

\[
A := \bigcup_{j=0}^{N^2-1} (\mathbb{D}_{r_{2j}} \setminus \mathbb{D}_{r_{2j+1}}) = (\mathbb{D}_{r_0} \setminus \mathbb{D}_{r_1}) \cup (\mathbb{D}_{r_2} \setminus \mathbb{D}_{r_3}) \cup \cdots \cup (\mathbb{D}_{r_{2N^2-2}} \setminus \mathbb{D}_{r_{2N^2-1}}),
\]

\[
\tilde{A} := \bigcup_{j=0}^{N^2-1} (\mathbb{D}_{r_{2j+1}} \setminus \mathbb{D}_{r_{2j+2}}) = (\mathbb{D}_{r_1} \setminus \mathbb{D}_{r_2}) \cup (\mathbb{D}_{r_3} \setminus \mathbb{D}_{r_4}) \cup \cdots \cup (\mathbb{D}_{r_{2N^2-1}} \setminus \mathbb{D}_{r_{2N^2}}).
\]

Next, let

\[
(A.5) \quad L := \left( \bigcup_{j=0}^{N-1} l_{2j}^{\mathbb{D}_{r_j}} \right) \cap \mathcal{A}, \quad \tilde{L} := \left( \bigcup_{j=0}^{N-1} l_{(2j+1)}^{l_{2j+1}} \right) \cap \tilde{A}.
\]
where $l_t := \{re^{it}; r \geq 0\}$, and set

\[
H := L \cup \tilde{L} \cup S \quad \left( S = \bigcup_{j=0}^{2N^2} \partial D_r \right).
\]

We define

\[
\Omega = A \setminus U \left[ \frac{1}{4N^3} \right] (H),
\]

where $U[\varepsilon](B)$ denotes the $\varepsilon$-neighborhood of the subset $B \subset \mathbb{C}$ (in the Euclidean distance). Note that each connected component of $\Omega$ has the width $1/(2N^3)$.

For each number $j = 1, \ldots, 2N$, we set

\[
\omega_j := \left( \frac{1}{\sqrt{N^3}} \cap A \right) \cup \text{(the connected components of $\Omega$ intersecting with $l_{\frac{1}{\sqrt{N^3}}}$)}
\]

and

\[
\omega_j := U \left( \frac{1}{8N^3} \right) (\omega_j) = \text{(the $\frac{1}{8N^3}$-neighborhood of $\omega_j$)}.
\]
Finally we denote by $\zeta_j$ the “base point” of $\varpi_j$:

\[(A.9) \quad \zeta_j := \left(1 - \frac{2}{N} - \frac{1}{8N^3}\right)e^{i\pi j/N} \in \partial \varpi_j \quad (j = 1, \ldots, 2N)\]

(Figure 3).

By definition, we have

**Lemma A.1.**  
(1) For each $j = 1, \ldots, 2N$, both $\omega_j$ and $\overline{D}_1 \setminus \varpi_j$ are disjoint compact subsets of $\mathbb{C}$ such that $\mathbb{C} \setminus (\omega_j \cup (\overline{D}_1 \setminus \varpi_j))$ is connected.

(2) It holds that

\[\overline{D}_1 \setminus \frac{\varpi_j}{\varpi_j \setminus C} \supset \varpi_1 \cup \cdots \cup \varpi_{2N}.\]

**Lemma A.2.** Let $j \in \{1, \ldots, 2N\}$. Then for each $p \in \overline{D}_1 \setminus \varpi_j$, there exists a path $\gamma$ in $\overline{D}_1 \setminus \varpi_j$ joining 0 and $p$ whose length (with respect to the Euclidean metric of $\mathbb{C}$) is not greater than $1 + \pi/N$.

**Proof.** By a rotation and a reflection on $\mathbb{C} = \mathbb{R}^2$, we assume $j = 2N$ and $p = re^{i\theta}$ \((0 \leq r \leq 1, 0 \leq \theta \leq \pi)\) without loss of generality.

If $\pi/N < \theta < \pi$, the line segment $\gamma$ joining 0 and $p$ does not intersect with $\varpi_{2N}$. Then $\gamma$ is the desired path.

Otherwise, both the line segment $\gamma_1$ joining 0 and $p_0 := re^{i\pi/N}$ and the circular arc $\gamma_2$ joining $p_0$ and $p$ centered at 0 do not intersect with $\varpi_{2N}$. Then the path $\gamma := \gamma_1 \cup \gamma_2$ is the desired one.

**Lemma A.3.** Let $j \in \{1, \ldots, 2N\}$. Then for each $p \in \varpi_j$, there exists a path $\gamma$ in $\varpi_j$ joining the base point $\zeta_j$ and $p$ whose length (with respect to the Euclidean metric of $\mathbb{C}$) is not greater than $6/N$.

**Proof.** We write $p = re^{i\theta} \in \varpi_j$, where

\[\frac{1 - 2}{N} - \frac{1}{8N^3} \leq r \leq 1, \quad \frac{\pi(j-1)}{N} \leq \theta \leq \frac{\pi(j+1)}{N}.\]

Then the line segment $\gamma_1$ joining $\zeta_j$ and $p_1 := re^{i\pi j/N}$ lies in $\varpi_j$, and its Euclidean length does not exceed $\frac{\pi}{N} + \frac{1}{8N^3}$. On the other hand, the length of the circular arc $\gamma_2$ centered at the origin joining $p_1$ and $p$ does not exceed $\pi/N$. Then the path $\gamma = \gamma_1 \cup \gamma_2$ joins $\zeta_j$ and $p$ in $\varpi_j$, whose length does not exceed

\[\frac{2}{N} + \frac{1}{8N^3} + \frac{\pi}{N} = \frac{1}{N} \left(2 + \frac{1}{8N^2} + \pi\right) \leq \frac{1}{N} \left(2 + \frac{1}{8} + \pi\right) \leq \frac{6}{N}.\]

Hence we have the conclusion.

**Lemma A.4.** Assume $N \geq 4$, and let $\Omega \subset D_1$ be the set as in (A.7). Note that $\Omega \subset \omega_1 \cup \cdots \cup \omega_{2N}$.

Consider a Riemannian metric $ds^2 = \lambda^2 |dz|^2$ on $\overline{D}_1$ such that

\[
\begin{cases}
\lambda \geq 1 & \text{(on } \overline{D}_1) \\
\lambda \geq N^3 & \text{(on } \Omega).
\end{cases}
\]

Then for an arbitrary path $\sigma$ in $\overline{D}_1$ joining 0 and $\partial D_1$, it holds that $\int_{\sigma} ds^2 \geq N$.

**Proof.** For $j = 0, \ldots, N^2 - 1$, let $\gamma_j$ be a subarc of $\sigma$ joining $\partial D_{r_{2j}}$ and $\partial D_{r_{2j+2}}$ contained in $\overline{D}_{r_{2j}} \setminus D_{r_{2j+2}}$. It suffices to prove that $\text{Length}_{ds^2}(\gamma_j) \geq \frac{1}{N}$. In this case, since the path $\sigma$ contains at least $N^2$ such paths, we have

\[\text{Length}_{ds^2}(\sigma) = \int_{\sigma} ds \geq N^2 \cdot \frac{1}{N} = N,\]
In order to prove that \( \text{Length}_{d^2}(\gamma_j) \geq \frac{1}{N} \), we distinguish two cases. First we assume that \( \text{Length}_{C}(\gamma_j) \geq \frac{1}{N} \). In this case by the assumption \( \lambda \geq 1 \) we have

\[
\text{Length}_{d^2}(\gamma_j) = \int_{\gamma_j} ds = \int_{\gamma_j} \lambda(z) |dz| \geq \int_{\gamma_j} |dz| \geq \frac{1}{N}
\]

On the contrary, if \( \text{Length}_{C}(\gamma_j) < \frac{1}{N} \) it is not difficult to see that \( \gamma_j \) must be contained in a wedge of \( \mathbb{D}_1 \) of angle bounded by \( \frac{\pi}{N} - \frac{2}{N^2} \). Taking into account the shape of the labyrinth, this implies that \( \gamma_j \) crosses a connected component of \( \Omega \) transversely, and therefore the Euclidean length of \( \gamma_j \cap \Omega \) is greater than \( 1/(2N^3) \). Hence by the assumption,

\[
\text{Length}_{d^2}(\gamma_j) = \int_{\gamma_j} ds \geq \int_{\gamma_j \cap \Omega} ds \geq \int_{\gamma_j \cap \Omega} |dz| \geq N^3 \cdot \frac{1}{2N^3} = \frac{1}{2} > \frac{1}{N}.
\]

\[\square\]

References

[1] A. Alarcón and F. Forstnerič, Every bordered Riemann surface is a complete proper curve in a ball, preprint [arXiv:1210.5634v1].
[2] A. Alarcón and F. Forstnerič, Null curves and directed immersions of open Riemann surfaces, preprint [arXiv:1210.5617v1].
[3] A. Alarcón and F. López, Null Curves in \( C^3 \) and Calabi-Yau Conjectures, Math. Ann., in press, [arXiv:0912.2847].
[4] P. Collin and H. Rosenberg, Notes sur la démonstration de N. Nadirashvili des conjectures de Hadamard et Calabi-Yau, Bull. Sci. Math., 123 (1999), 563–576.
[5] F. Martín, M. Umehara and K. Yamada, Complete bounded null curves immersed in \( C^3 \) and \( SL(2, \mathbb{C}) \), Calculus of Variations and PDE's, 36 (2009), 119–139.
---

Erratum: Complete bounded null curves immersed in \( C^3 \) and \( SL(2, \mathbb{C}) \), Calculus of Variations and PDE's, in press.
[6] F. Martín, M. Umehara and K. Yamada, Complete bounded holomorphic curves immersed in \( C^2 \) with arbitrary genus, Proc. Amer. Math. Soc., 137 (2009), 3437–3450.
[7] N. Nadirashvili, Hadamard’s and Calabi-Yau’s conjectures on negatively curved and minimal surfaces, Invent. Math., 126 (1996), 457–465.
