Towards effective Lagrangians for adelic strings

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Abstract

$\mathbf{p}$-Adic strings are important objects of string theory, as well as of $\mathbf{p}$-adic mathematical physics and nonlocal cosmology. By a concept of adelic string one can unify and simultaneously study various aspects of ordinary and $\mathbf{p}$-adic strings. By this way, one can consider adelic strings as a very useful instrument in the further investigation of modern string theory. It is remarkable that for some scalar $\mathbf{p}$-adic strings exist effective Lagrangians, which are based on real instead of $\mathbf{p}$-adic numbers and describe not only four-point scattering amplitudes but also all higher ones at the tree level. In this work, starting from $\mathbf{p}$-adic Lagrangians, we consider some approaches to construction of effective field Lagrangians for $\mathbf{p}$-adic sector of adelic strings. It yields Lagrangians for nonlinear and nonlocal scalar field theory, where spacetime nonlocality is determined by an infinite number of derivatives contained in the operator-valued Riemann zeta function. Owing to the Riemann zeta function in the dynamics of these scalar field theories, obtained Lagrangians are also interesting in themselves.

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1
1 Introduction

It is well known that string theory is the best candidate for a unification of fundamental forces and elementary particles. People is mainly interested in ordinary strings, which description is based on real (and complex) numbers. In 1987, some $p$-adic analogs [1] of ordinary open and closed strings were introduced. Description of these $p$-adic strings employs, more or less, $p$-adic numbers. The most popular $p$-adic strings have only world sheet parameterized by $p$-adic numbers, while all other ingredients of the scattering amplitude use real (or complex) numbers. The starting point is the usual crossing symmetric Veneziano amplitude

\[ A_\infty(a, b) = g_\infty^2 \int_\mathbb{R} |x|^{a-1}_\infty |1 - x|^{b-1}_\infty \, dx \stackrel{\text{(1)}}{=} g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)} , \]

where $a = -\alpha(s) = -\frac{s}{2} - 1$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $a + b + c = 1$, i.e. $s + t + u = -8$. In (1), $| \cdot |_\infty$ denotes the ordinary absolute value, $\mathbb{R}$ is the field of real numbers, kinematic variables $a, b, c \in \mathbb{C}$, and $\zeta$ is the Riemann zeta function. The corresponding Veneziano amplitude for scattering of $p$-adic strings was introduced [2] as $p$-adic analog of the integral form of (1), i.e.

\[ A_p(a, b) = g_p^2 \int_{\mathbb{Q}_p} |x|^{a-1}_p |1 - x|^{b-1}_p \, dp \, x , \]

where $\mathbb{Q}_p$ is the field of $p$-adic numbers, $| \cdot |_p$ is $p$-adic absolute value and $dp \, x$ is the Haar measure on $\mathbb{Q}_p$. In (2), kinematic variables $a, b, c$ maintain their complex values and condition $a + b + c = 1$. Thus $x$, which is related to the string world sheet, is real-valued argument in (1) and $p$-adic one in (2). Performing integration in (2) one obtains

\[ A_p(a, b) = g_p^2 \frac{1 - p^{a-1}}{1 - p^{-a}} \frac{1 - p^{b-1}}{1 - p^{-b}} \frac{1 - p^{c-1}}{1 - p^{-c}} , \]

where $p$ is any prime number. Recall the definition of the Riemann zeta function

\[ \zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \quad s = \sigma + i\tau , \quad \sigma > 1 , \]

with $\sigma, \tau \in \mathbb{R}$.
which has analytic continuation to the entire complex $s$ plane, excluding the point $s = 1$, where it has a simple pole with residue 1. According to (4) one has

$$\prod_p A_p(a, b) = \frac{\zeta(a)}{\zeta(1-a)} \frac{\zeta(b)}{\zeta(1-b)} \frac{\zeta(c)}{\zeta(1-c)} \prod_p g_p^2,$$

what gives a nice simple product formula [3]

$$A_\infty(a, b) \prod_p A_p(a, b) = g_\infty^2 \prod_p g_p^2,$$

which to be finite requires $g_\infty^2 \prod_p g_p^2 = \text{const.}$ From (6) follows that the ordinary Veneziano amplitude, which is rather complex, can be expressed as product of all inverse $p$-adic counterparts, which are much more simpler. Moreover, expression (6) gives rise to consider it as the amplitude for an adelic string, which is composed of the ordinary and $p$-adic ones.

Various $p$-adic structures have been observed and investigated not only in string theory but also in many other models of modern mathematical physics and related topics (see [1] for a review of early developments, and [6] for the most recent short overview of $p$-adic mathematical physics). For instance, in [7] was shown that degeneration of the genetic code has a $p$-adic structure. For a basic information on $p$-adic numbers one can refer to [4] and [5].

It is worth pointing out that for an open scalar $p$-adic string exists an effective Lagrangian, which is based on real instead of $p$-adic numbers and describes all scattering amplitudes at the tree level [8, 9].

The adelic approach connects $p$-adic phenomena with the related ordinary ones. In addition to product formula for string amplitudes [3], adelic modelling has been successfully applied to quantum mechanics [10], Feynman path integral [11], quantum cosmology [12], summation of divergent series [13], and dynamical systems [14].

The present paper can be regarded as a result of some attempts to construct an effective Lagrangian for entire $p$-adic sector of an adelic scalar string. We use two approaches, which we call additive and multiplicative. In the additive approach, we start with the exact Lagrangian for the effective field of $p$-adic tachyon string, then extend prime number $p$ to arbitrary natural number $n$, and perform various summations of such Lagrangians over $n$. It leads us to the possibility to apply the summation form (4) of the zeta function. The multiplicative approach enables to use the product over primes form (4) to introduce the zeta function.
2 Lagrangian for a $p$-adic open string

The exact tree-level Lagrangian of the effective scalar field $\varphi$, which describes the open $p$-adic string tachyon, is \[8, 9\]

$$L_p = m_p^D \frac{p^2}{g_p^2} \left[ -\frac{1}{2} \varphi p^{-\frac{\Box}{2m_p^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \right], \quad (7)$$

where $p$ is a prime, $\Box = -\partial_t^2 + \nabla^2$ is the $D$-dimensional d’Alembertian and we adopt the metric with signature $(- + ... +)$. At the first glance it may look strange that there is nothing $p$-adic in Lagrangian (7). However, (7) can be rewritten in the form

$$L_p = m_p^D \frac{1}{g_p^2 |p|_p (1 - |p|_p)} \left[ -\frac{1}{2} \varphi |p|_p^{-\frac{\Box}{2m_p^2}} \varphi + \frac{|p|_p}{|p|_p+1} \varphi^{|p|_p+1} \right], \quad (8)$$

where prime $p$ is treated as a $p$-adic number. Using $p$-adic norm of $p$ in (8) gives real prime in (7). Note that similarly the Riemann zeta function can be introduced as

$$\zeta(s) = \prod_p \frac{1}{1 - |p|_p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1, \quad (9)$$

where its origin in $p$-adics is evident. In the sequel we shall use the usual real form (7) and (4).

An infinite number of spacetime derivatives follows from the expansion

$$p^{-\frac{\Box}{2m_p^2}} = \exp \left( -\frac{1}{2m_p^2} \log p \Box \right) = \sum_{k=0}^{+\infty} \left( -\frac{\log p}{2m_p^2} \right)^k \frac{1}{k!} \Box^k.$$

The equation of motion for (7) is

$$p^{-\frac{\Box}{2m_p^2}} \varphi = \varphi^p, \quad (10)$$

and its properties have been studied by many authors (see, e.g. \[15, 16, 17\] and references therein).

Based on (7), many aspects of $p$-adic string dynamics were considered and compared with the dynamics of ordinary strings (see, e.g. \[18, 15, 19\] and references therein) and cosmology \[20, 21\].
3 Zeta-nonlocal Lagrangians

Starting from (7), we explore two approaches to the extension of \( L_p \) that lead to introduction of the Riemann zeta function. We will first review additive approach and then introduce a multiplicative one.

3.1 Additive approach

It is worth noting that prime number \( p \) in (7) and (8) can be replaced by any natural number \( n \geq 2 \) and such expressions also make sense.

Now we want to introduce a Lagrangian which incorporates all the above Lagrangians (7), with \( p \) replaced by \( n \in \mathbb{N} \). To this end, we take the sum of all Lagrangians \( L_n \) in the form

\[
L = \sum_{n=1}^{\infty} C_n L_n = \sum_{n=1}^{\infty} C_n \frac{m_n^D}{g_n^2} \frac{n^2}{n-1} \left[ -\frac{1}{2} \phi n \frac{n}{2m_n} \phi + \frac{1}{n+1} \phi^{n+1} \right],
\]

whose explicit realization depends on particular choice of coefficients \( C_n \), string masses \( m_n \) and coupling constants \( g_n \). To avoid a divergence in \( 1/(n-1) \) when \( n = 1 \) one has to take that \( C_n m_n^D / g_n^2 \) is proportional to \( n-1 \). Here we consider some cases when coefficients \( C_n \) are proportional to \( n-1 \), while masses \( m_n \) as well as coupling constants \( g_n \) do not depend on \( n \), i.e. \( m_n = m, g_n = g \). To differ this new field from a particular \( p \)-adic one, we use notation \( \phi \) instead of \( \varphi \).

We have considered three cases for coefficients \( C_n \) in (11): (i) \( C_n = \frac{n-1}{n^2 h} \), where \( h \) is a real parameter; (ii) \( C_n = \frac{n^2-1}{n^2} \); and (iii) \( C_n = \mu(n) \frac{n-1}{n^2} \), where \( \mu(n) \) is the Möbius function.

Case (i) was considered in [22]. Obtained Lagrangian is

\[
L_h = \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \zeta \left( \frac{\Box}{2m^2} + h \right) \phi + \mathcal{AC} \sum_{n=1}^{\infty} \frac{n-h}{n+1} \phi^{n+1} \right],
\]

where \( \mathcal{AC} \) denotes analytic continuation.

Case (ii) was investigated in [23] and the corresponding Lagrangian is

\[
L = \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \left\{ \zeta \left( \frac{\Box}{2m^2} - 1 \right) + \zeta \left( \frac{\Box}{2m^2} \right) \right\} \phi + \frac{\phi^2}{1 - \phi} \right].
\]
Case with the Möbius function $\mu(n)$ is presented in [24]. Recall that the Möbius function is defined for all positive integers and has values 1, 0, −1 depending on factorization of $n$ into prime numbers $p$. Its explicit definition as follows:

$$
\mu(n) = \begin{cases} 
0, & n = p^2 m, \\
(-1)^k, & n = p_1 p_2 \cdots p_k, \ p_i \neq p_j, \\
1, & n = 1, \ (k = 0).
\end{cases}
$$

Since the inverse Riemann zeta function can be defined as

$$
\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + i\tau, \quad \sigma > 1,
$$

then the corresponding Lagrangian is

$$
L = \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \frac{1}{\zeta(\frac{\Box}{2m^2})} \phi + \int_0^\phi \mathcal{M}(\phi) \, d\phi \right],
$$

where $\mathcal{M}(\phi) = \sum_{n=1}^{+\infty} \mu(n) \phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \ldots.$

### 3.2 Multiplicative approach

Let us now consider a new approach, which is not based on a summation of $p$-adic Lagrangians, but the Riemann zeta function will emerge through its product form. Our starting point is again $p$-adic Lagrangian (7) with equal masses, i.e. $m_p^2 = m^2$ for every $p$. It is useful to rewrite (7), first in the form,

$$
\mathcal{L}_p = \frac{m^D}{g^2_p} \frac{p^2}{p^2 - 1} \left\{ -\frac{1}{2} \phi \left[ p^{-\frac{\Box}{2m^2} + 1} + p^{-\frac{\Box}{2m^2}} \right] \phi + \phi^{p+1} \right\}
$$

and then, by addition and substraction of $\phi^2$, as

$$
\mathcal{L}_p = \frac{m^D}{g^2_p} \frac{p^2}{p^2 - 1} \left\{ \frac{1}{2} \phi \left[ \left( 1 - p^{-\frac{\Box}{2m^2} + 1} \right) + \left( 1 - p^{-\frac{\Box}{2m^2}} \right) \right] \phi - \phi^2 \left( 1 - \phi^{p-1} \right) \right\}.
$$

Now we introduce a Lagrangian for the entire $p$-adic sector by taking products

$$
\prod_p g_p^2 = C, \quad \prod_p \frac{1}{1 - p^2}, \quad \prod_p \left( 1 - p^{-\frac{\Box}{2m^2} + 1} \right), \quad \prod_p \left( 1 - p^{-\frac{\Box}{2m^2}} \right) \quad \text{and} \quad \prod_p \left( 1 - \phi^{p-1} \right)
$$

(19)
in (18) at the corresponding places. Then this new Lagrangian becomes

$$L = \frac{m^D}{C} \frac{1}{\zeta(2)} \left\{ \frac{1}{2} \phi \left[ \zeta^{-1} \left( \frac{\Box}{2m^2} - 1 \right) + \zeta^{-1} \left( \frac{\Box}{2m^2} \right) \right] \phi - \phi^2 \prod_p \left( 1 - \phi^{p-1} \right) \right\}, \quad (20)$$

where $\zeta^{-1}(s) = 1/\zeta(s)$ and new scalar field is denoted by $\phi$. For the coupling constant $g_p$ there are two interesting possibilities: (1) $g_p^2 = \frac{p^2}{p^2 - 1}$, what yields $\zeta(2)/C = 1$ in (20), and (2) $g_p = |r|_p$, where $r$ may be any non zero rational number and it gives $|r|_\infty \prod_p |r|_p = 1$ (this possibility was considered in [25]). Both these possibilities are consistent with adelic product formula (6). For simplicity, in the sequel we shall take $C = \zeta(2)$. It is worth noting that having Lagrangian (20) one can easily reproduce its $p$-adic ingredient (17). Namely, one can use the opposite procedure to the construction of (20) from (17).

Let us rewrite (20) in the simple form

$$L = \frac{1}{2} \phi \left[ \zeta^{-1} \left( \frac{\Box}{2} - 1 \right) + \zeta^{-1} \left( \frac{\Box}{2} \right) \right] \phi - \phi^2 \Phi(\phi), \quad (21)$$

with $m = 1$ and $\Phi(\phi) = \mathcal{A}C \prod_p (1 - \phi^{p-1})$, where $\mathcal{A}C$ denotes analytic continuation of infinite product $\prod_p (1 - \phi^{p-1})$, which is convergent if $|\phi|_\infty < 1$. One can easily see that $\Phi(0) = 1$ and $\Phi(1) = \Phi(-1) = 0$.

For (21), the corresponding equation of motion is

$$\left[ \zeta^{-1} \left( \frac{\Box}{2} - 1 \right) + \zeta^{-1} \left( \frac{\Box}{2} \right) \right] \phi = 2 \phi \Phi(\phi) + \phi^2 \Phi'(\phi), \quad (22)$$

and has $\phi = 0$ as a trivial solution. In the weak-field approximation ($\phi(x) \ll 1$), equation (22) becomes

$$\left[ \zeta^{-1} \left( \frac{\Box}{2} - 1 \right) + \zeta^{-1} \left( \frac{\Box}{2} \right) \right] \phi = 2 \phi. \quad (23)$$

Note that the above operator-valued zeta function can be regarded as a pseudodifferential operator. Then (22) and (23) are transformed to the integral form.

Mass spectrum of $M^2$ is determined by solutions of equation

$$\zeta^{-1} \left( \frac{M^2}{2} - 1 \right) + \zeta^{-1} \left( \frac{M^2}{2} \right) = 2. \quad (24)$$
There are infinitely many tachyon solutions, which are below largest one \( M^2 \approx -3.5 \).

The potential follows from \(-\mathcal{L}\) at \( \Box = 0 \), i.e.

\[
V(\phi) = [7 + \Phi(\phi)] \phi^2, \tag{25}
\]

since \( \zeta(-1) = -1/12 \) and \( \zeta(0) = -1/2 \). This potential has local minimum \( V(0) = 0 \) and values \( V(\pm 1) = 7 \). To explore behavior of \( V(\phi) \) for all \( \phi \in \mathbb{R} \) one has first to investigate properties of the function \( \Phi(\phi) \).

### 4 Concluding remarks

We have obtained a new Lagrangian (21), which incorporates all \( p \)-adic Lagrangians (7) by a multiplicative way. It contains information on the entire \( p \)-adic sector. To compare it with \( p \)-adic sector of the adelic string one has to derive four-point scattering amplitude. It will be considered elsewhere. There are also many classical properties which should be explored, e.g. possible solutions of the equation of motion (22).

It is worth noting that a Lagrangian similar to (21) can be obtained by an additive approach. Namely, let us start from (17) and take coupling constant \( g_p^2 = p^2/p^2-1 \) and mass \( m = 1 \). Then let us extend prime \( p \) to natural number \( n \) and take infinite sum

\[
L = -\sum_{n=1}^{+\infty} \mu(n) L_n = \frac{1}{2} \phi \left[ \sum_{n=1}^{+\infty} \mu(n) p^{-\frac{n-1}{2}} + \sum_{n=1}^{+\infty} \mu(n) p^{-\frac{n-3}{2}} \right] \phi - \sum_{n=1}^{+\infty} \mu(n) \phi^{n+1}, \tag{26}
\]

where \( \mu(n) \) is the above Möbius function defined by (14). Introducing zeta function in (26), one can rewrite it in the form

\[
L = \frac{1}{2} \phi \left[ \zeta^{-1}(\frac{1}{2}) - 1 \right] \phi - \phi^2 F(\phi), \tag{27}
\]

where \( F(\phi) = \sum_{n=1}^{+\infty} \mu(n) \phi^{n-1} \). The difference between (21) and (27) is only in functions \( \Phi(\phi) \) and \( F(\phi) \). Since \( \Phi(\phi) = (1 - \phi)(1 - \phi^2)(1 - \phi^4)... = 1 - \phi - \phi^2 + \phi^3 - \phi^4 + ... \) and \( F(\phi) = 1 - \phi - \phi^2 - \phi^4 + ... \), it follows that these functions have the same behavior for \( |\phi| \ll 1 \). Hence, Lagrangians (21) and (27) have the same mass spectrum and in weak-field approximation describe the same scalar field theory.
Note that an interesting approach to the foundation of a field theory and cosmology based on the Riemann zeta function was proposed in [26].

Acknowledgements

This paper was supported in part by the Ministry of Science and Technological Development, Serbia (Contract No. 144032D). The author would like to thank organizers of the 4-th RTN “Forces-Universe” EU Network Workshop in Varna, 11-17 September 2008, for invitation to present preliminary results of this work as well as for kind hospitality. The author also thanks I. Ya. Aref’eva and I. V. Volovich for useful discussions.

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