Reasonable Space for the $\lambda$-Calculus, Logarithmically

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Abstract

Can the $\lambda$-calculus be considered as a reasonable computational model? Can we use it for measuring the time and space consumption of algorithms? While the literature contains positive answers about time, much less is known about space. This paper presents a new reasonable space cost model for the $\lambda$-calculus, based on a variant over the Krivine abstract machine. For the first time, this cost model is able to account for logarithmic space. Moreover, we study the time behavior of our machine and show how to transport our results to the call-by-value $\lambda$-calculus.

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1 Introduction

Overview. Bounding the amount of resources needed by algorithms or programs is a fundamental problem in computer science. Here we are concerned with sub-linear space. In many applications, say, stream processing or web crawling, linear bounds on computing space are not satisfactory, given the enormous amount of data processed. Theoretically, complexity classes such as L (logarithmic space), although apparently small, are already very interesting for complexity theory, and not even known to be distinct from P.

Dealing with sub-linear space bounds in the $\lambda$-calculus, or in functional programming languages, has always been considered a challenge. A first reason is that the natural notions of time and space in the $\lambda$-calculus have some puzzling properties, as we shall see. But sub-linear space is special, in that, since the $\lambda$-calculus does not distinguish between programs and data, there is also no distinction between input space and work space, and thus no natural notion of sub-linear space.

The literature about the $\lambda$-calculus does offer results about space complexity, but they are all partial, as they either concern variants of the $\lambda$-calculus (Dal Lago and Schöpp [16, 35], Mazza [30] and Ochic [22]), or they are not valid when the bounds in spaces are sub-linear (Forster et al. [19]).

The main contribution of this paper is the first fully-fledged space reasonability result for the pure, untyped $\lambda$-calculus. Precisely, we represent the input space as $\lambda$-terms, and the work space as the space used by a variant of the well-known Krivine’s abstract machine (KAM). Two important aspects of our Space KAM are eager garbage collection and the fact that, for the first time, we distinguish between two forms of sharing usually considered as one: the sharing of sub-terms provided by environments, and the sharing of environments themselves. The Space KAM adopts the former but forbids the latter, which is essential to prove that its space cost model is reasonable. Designing the Space KAM, however, is only half of the story. The other half is the refinement of the encoding of Turing machines into the $\lambda$-calculus: the reference one by Dal Lago and Accattoli [14] uses a linear amount of space to simulate the TM tapes, thus forbidding to preserve logarithmic space. Let us detail all this.

Reasonable Cost Models. According to the seminal work by Slot and van Emde Boas [37], the adequacy of space and time cost models is judged in relationship to whether they reflect the corresponding cost models of Turing machines (shortened to TM), the computational theory from which computational complexity stems. Namely, a cost model for a computational theory $T$ is reasonable if there are mutual simulations of $T$ and TMs (or another reasonable theory) working within:

- for time, a polynomial overhead;
- for space, a linear overhead.

In many cases the two bounds hold simultaneously for the same simulation, but this is not a strict requirement. The aim is to ensure that the basic hierarchy of complexity classes

$$L \subseteq P \subseteq \text{PSPACE} \subseteq \text{EXP}$$

(1)

can be equivalently defined on any reasonable theory, that is, that such classes are robust, or theory-independent. Note a slight asymmetry: while for time the complexity of the required overhead (polynomial) coincides with the smallest
robust time class (P), for space the smallest robust class is logarithmic (L) and not linear space.

**Locked Time and Space.** On TMs, space cannot be greater than time, because using space requires time—we shall then say that space and time are locked. If both the time and space cost models of a computational theory are reasonable, are they also necessarily locked? This seems natural, but it is not what happens in the \( \lambda \)-calculus, at least with respect to its natural cost models.

**Natural Cost Models for the \( \lambda \)-Calculus.** The natural time cost model is the number of \( \beta \)-steps (according to some fixed evaluation strategy), its only notion of computational step. The natural space cost model is the maximum size of \( \lambda \)-terms during reduction. The puzzling point is that, in the \( \lambda \)-calculus, natural space can be exponential in natural time (independently of the strategy), a degeneracy known as size explosion—we shall say that time and space are explosive.

Is the \( \lambda \)-calculus reasonable? This was unclear for a while, because of the intuition that reasonable cost models have to be locked. Note that there is, in principle, an alternative and non-explosive approach to time in the \( \lambda \)-calculus: the time to write down the whole evaluation rather than only the number of evaluation steps. Such low-level time is reasonable, and locked with natural space, but implies that the various \( \beta \)-steps in an evaluation sequence are given different costs, which is theoretically not ideal, and also distant from the practice of programming, that rather adopts and confirms the natural (and uniform) cost model (corresponding to the number of function calls).

**Natural Time.** In the study of natural time, what is delicate is the simulation of the \( \lambda \)-calculus into a reasonable theory, which typically is the one of RAMs rather than TMs. The difficulty stems from the explosiveness of natural time, and requires a slight paradigm shift. To circumvent the exponential explosion in space, \( \lambda \)-terms are usually evaluated up to sharing, that is, in abstract machines with sharing that compute shared representations of the results. These representations can be exponentially smaller than the results themselves: explosiveness is then encapsulated in the sharing unfolding process (which itself has to satisfy some reasonable properties, see [7, 13]). The number of \( \beta \) steps (according to various evaluation strategies) then turns out to be a reasonable time cost model (up to sharing), despite explosiveness. The first such result is for weak evaluation by Blelloch and Greiner [11], then extended to strong evaluation by Accattoli and Dal Lago [7], and very recently transferred to Strong CbV independently by Accattoli et al. [5] and Biernacka et al. [10]. As we shall see, the very sharing mechanism that makes natural time reasonable is unfortunately space unreasonable.

**Natural Space.** For space, the difficult direction is, instead, the simulation of TMs in the \( \lambda \)-calculus. TMs are space-minimalist, as their only data structure, the tape, juxtaposes cells rather then linking them—we shall see that this is one of the key points. Motivated by time-efficiency, all abstract machines for the \( \lambda \)-calculus rely instead on linked data structures, and the linking pointers add a logarithmic factor to the overhead for the simulation of TMs that is space unreasonable. Therefore, reasonable space requires to evaluate without using linked data structures when they are not needed, as it is the case for the encoding of TMs. It is a recent observation by Forster et al. [19] that evaluating without any data structure (via plain rewriting, without sharing) is reasonable for natural space even if unreasonable for natural time (because of explosiveness).

**Pairing up Natural Time and Natural Space.** Forster et al. [19] also show a surprising fact. Given two simulations, one that is reasonable for space but not time, and one that is reasonable for time but not space, there is a smart way of interleaving them as to obtain reasonability for time and space simultaneously. Their result therefore shows that, surprisingly, a computational theory can be reasonable for time and space without being locked.

**Work vs Natural Space.** Now, another puzzling fact is that sub-linear space cannot be measured using the natural space cost model, and is then not covered by Forster et al.’s result. The reason is that if space is the maximum size of terms in an evaluation sequence, the first of which contains the input, then space simply cannot be sub-linear. How could we account for logarithmic reasonable space? One needs log-sensitivity, that is, a distinction between an immutable input space, which is not counted for space complexity (because otherwise the complexity would be at least linear), and a (smaller) mutable work space, that is counted. Moreover, logarithmic space usually requires manipulating pointers to the input (which are of logarithmic size) rather than pieces of the input (which can be linear).

Log-sensitivity thus seems to clash with the natural approach based on the rewriting of \( \lambda \)-terms, which does not distinguish between input and work space and that manipulates actual sub-terms rather than pointers. We shall then model evaluation using \( \lambda \)-terms for the input space and the state space of an abstract machine (computing over the input without ever modifying it, and manipulating sub-term pointers) for the work space.

An interesting aspect of the quest for a logarithmic reasonable work space is that it requires dismissing widespread intuitions about the problem, as we now explain.

**A Wrong Positive Intuition: the Geometry of Interaction.** Mackie’s and Danos & Regnier’s interaction abstract machine (shortened to IAM) [17, 29] is a machine rooted in Girard’s geometry of interaction [23] and Abramsky et
al.’s game semantics [1], based on a log-sensitive approach, and—apparently—parsimonious with respect to space. Since the work of Schöpp in 2006 [34, 35] (with later developments with Dal Lago [15]), who showed how IAM-like mechanisms can be used for dealing with logarithmic space, it was conjectured that the space of the IAM were a reasonable cost model. The belief in the conjecture was re-inforsed by further uses of IAM-like mechanisms for space parsimony related to circuits, by Ghica [22], and for characterizing L, by Mazza [30]. However, in 2021, Accattoli et al. have essentially disproved the conjecture: the space used by the IAM to evaluate the reference encoding of TMs is unreasonable [9] (as well as time inefficient [8]). While one might look for different encodings, the unreasonable behavior of the IAM concerns the modeling of recursion via fix-point combinator, which is hardly avoidable by any encoding.

A Wrong Negative Intuition: Environments. Another misleading intuition was that environment-based abstract machines could not be space reasonable. Environments are data structures used to achieve time reasonability. According to Accattoli and Barras [3], there are two main styles of environments, local and global. Global environments (as in the Milner Abstract Machine, see [3]) are log-insensitive because they work over the input space. Local environments (as in the KAM) are log-sensitive. There are two reasons why they are usually space unreasonable. The first one is that garbage collection is not usually accounted for, which leads to ever increasing space usage, while reasonable space should be re-usable. The second and subtler one is the use of pointers for sharing. Local environments use two types of pointers, handling the two forms of sharing: sub-term pointers, which serve to avoid copying sub-terms, and environment pointers, which both realize their linked list structure and share them. Sub-terms pointers, as mentioned above, are a key aspect of logarithmic space computations, and are thus crucial. Environment pointers which according to Douence and Fradet are the essence of the KAM [18], are instead what makes environments space unreasonable: they introduce a logarithmic pointer overhead that, at best, gives simulations of TMs with a O(n log n) overhead in space, instead of the required O(n) for reasonability. It was then generally concluded that environments cannot provide reasonable space.

Work Space Without Environment Pointers. The literature on abstract machines assume that pointers are used in implementations without however accounting for them in the underlying specification. Here, we are instead very careful with pointers. We design an abstract machine, the Space KAM, using local environments with sub-term pointers, but—crucially—without environment pointers. Similarly to the tapes of TMs, the environments of the Space KAM then are not linked structures, but simple unstructured strings. Consequently, the unreasonable pointer overhead vanishes.

The moral is that the use of pointers is both essential and dangerous for logarithmic space: sub-term pointers, that is, those to the input, are mandatory for log-sensitivity, while the environment ones—those to the working tape, essentially—are space unreasonable.

Garbage Collection and Unchaining. The Space KAM crucially relies also on two optimizations. One is eager garbage collection, to maximize space re-usability. It is implemented in the most naive of ways, because it cannot rely on any pointers or counters, as they would add an unreasonable space overhead. In contrast to common practice, the collection happens eagerly, that is, immediately and not when reaching a threshold. The second optimization is environment unchaining, a folklore tweak for avoiding space leaks.

Encoding of TMs. Despite the fine tuning of the abstract machine, a reasonable simulation of TMs preserving logarithmic space is not yet obtained, as the reference encoding of TMs has some inherent limitations with respect to logarithmic space. We then analyze its shortcomings, concerning how tapes are represented and scrolled in the λ-calculus, and modify it accordingly. The main result of the paper is that the work space of the Space KAM—to be referred to simply as the work space—over the new encoding is a reasonable space cost model accommodating sub-linear space. Our new encoding is carefully designed so as to retain the key indifference property of the reference one, that is, the fact that nothing changes if call-by-value rather than call-by-name evaluation is adopted. We then show that our results smoothly transfer to call-by-value evaluation.

Space KAM and Time. We then study the time behavior of the Space KAM. Disabling linked environments implies giving up environment sharing, which—we show with an example—makes the Space KAM unreasonable for natural time. We also prove that the low-level execution time of the Space KAM (which is less interesting than natural time, that is, the number of β steps) is reasonable. The situation is then a familiar one: natural time and work space are explosive, while low-level time and work space are locked. Work space is in this respect a conservative refinement of natural space.

Sub-Term Property. The techniques for reasonable time and reasonable space seem to be at odds, as they make essential but opposite uses of linked data structures. Both techniques, however, crucially rely on the sub-term property of abstract machines, that is, the fact that duplicated terms are sub-terms of the initial one. For time, it allows one to bound the cost of duplications, while for space it allows one to see sub-terms as (logarithmic) pointers to the input. The sub-term property seems to be the unavoidable ingredient for reasonability in the λ-calculus.

Related Work. The space inefficiency of environment machines is also observed by Krishnaswami et al. [26], who
propose some techniques to alleviate it in the context of functional-reactive programming and based on linear types. The relevance for space of disabling environment sharing is also stressed by Paraskevopoulou and Appel [31] in their cost-aware study of closure conversion. A characterization of PSPACE in the λ-calculus is given by Gaboardi et al. [21], but it relies on alternating time rather than on a notion of space. The already mentioned Dal Lago and Schöpp [16, 35] and Mazza [30] characterize L in variants of the λ-calculus, while Jones characterizes L using a programming language but not based on the λ-calculus [24]. Blieghoq and coauthors study in various papers [11, 12, 38] how to profile (that is, measure) space consumption of functional programs, also done by Sansom and Peyton Jones [33]. They are not interested, however, in the reasonability of the cost models, that is, being equivalent to the space of TMs, which is the difficult part of our work. Finally, there is an extensive literature on garbage collection, as witnessed by [25]. We here need a basic eager form, that need not be time efficient, as the Space KAM is time unreasonable anyway.

Proofs. Proofs are in the Appendix.

2 Reasonable Preliminaries

In the study of reasonable cost models for the λ-calculus, it is customary to show that the λ-calculus simulates Turing machines reasonably, and conversely that the λ-calculus can be simulated reasonably by random access machines (RAMs and TMs being both reasonable models) up to sharing. Since space is more delicate than time, we fix the involved theories and their cost measures carefully.

λ-Calculus. Let \( \mathcal{V} \) be a countable set of variables. Terms of the λ-calculus \( \Lambda \) are defined as follows:

\[
\lambda\text{-terms } t, u, r \seteq x \in \mathcal{V} | \lambda x.t | tu.
\]

Free and bound variables are defined as usual: \( \lambda x.t \) binds \( x \) in \( t \). Terms are considered modulo \( \alpha \)-equivalence. Capture-avoiding (meta-level) substitution of all the free occurrences of \( x \) for \( u \) in \( t \) is noted \( t[x/u] \). The computational rule is β-reduction:

\[
(\lambda x.t)u \rightarrow_\beta t[x/u]
\]

which can be applied anywhere in a \( \lambda \)-term. Here, a strategy \( \rightarrow \) shall be a sub-relation of \( \rightarrow_\beta \). Given a relation \( \rightarrow \), its reflexive-transitive closure is noted \( \rightarrow^* \), and a \( \lambda \)-term \( t \) is \( \rightarrow^* \)-normal if there are no \( u \) such that \( t \rightarrow u \). A \( \rightarrow^* \)-sequence is a pair of \( \rightarrow^* \)-related terms, often noted \( \rho : t \rightarrow^* u \), and it is complete if \( u \) is \( \rightarrow^* \)-normal.

The Size of \( \lambda \)-Terms. The (constructor) size of a \( \lambda \)-term is defined as follows:

\[
|t| \seteq 1 \quad |tu| \seteq |t| + |u| + 1 \quad |\lambda x.t| \seteq |t| + 1
\]

The code size \(|t|\) of a \( \lambda \)-term \( t \) is instead bound by \( O(|t| \cdot \log |t|) \). The idea is that, when terms are explicitly represented, variables are some abstract kind of pointer (de Bruijn indices/levels, names, or actual pointers to the syntax tree), of size logarithmic in the number of constructors \(|t|\) of \( t \). Moreover, the other constructors are also usually represented using pointers to sub-terms, so that a term with \( n \) constructors requires space \( O(n \log n) \) to be represented. For our study, it is important to stress the difference between \(|t|\) and \(|\!\!t\!\!|\), because given a binary input string \( i \), at first sight its encoding \( t_i \) as a \( \lambda \)-term satisfies \(|t_i| = \Theta(|i|) \) and \(|\!\!t_i\!\!| = O(|i| \cdot \log |i|) \), and so \(|\!\!t_i\!\!|\) has an additional (unreasonable) logarithmic factor. In Sect. 6, we shall encode strings in the \( \lambda \)-calculus using the Scott encoding, which has the property that, with respect to some concrete representations of terms, the variable pointers have constant size, so that \(|\!\!t_i\!\!| = |t_i| = \Theta(|i|) \), thus removing the unreasonable pointer overhead due to variables.

Turing Machines. We adopt TMs working on the boolean alphabet \( \mathbb{B} \seteq \{0, 1\} \). For a study of logarithmic space complexity, one has to distinguish input space and work space, and to not count the input space for space complexity. On TMs, this amounts to having two tapes, a read-only input tape on the alphabet \( \mathbb{B}_1 \seteq \{0, 1, \mathbb{L}, \mathbb{R}\} \), where \( \mathbb{L} \) and \( \mathbb{R} \) are delimiter symbols for the start and the end of the input binary string, and an ordinary read-and-write work tape on the boolean alphabet extended with a blank symbol \( \mathbb{B}_W \seteq \{0, 1, \mathbb{D}\} \). To keep things simple, we use TMs without any output tape, the machine rather has two final states \( q_0 \) and \( q_1 \) encoding a boolean output—there are no difficulties in extending our results to TMs with an output tape. Let us call these machines log-sensitive TM.

A log-sensitive TM \( M \) computes the function \( f : \mathbb{B}^* \rightarrow \mathbb{B} \) by a sequence of transitions \( \rho : C_M(i) \rightarrow^n C_M'(f(i)) \) where \( i \in \mathbb{B}^* \), \( C_M(i) \) is an initial configuration of \( M \) with input \( i \) and \( C_M'(f(i)) \) is a final configuration of \( M \) on the final state \( q_{f(i)} \). We define the time of the run \( \rho \) as \( T_M(\rho) := n \) and the space \( S_M(\rho) \) of \( \rho \) as the maximum number of cells of the work tape used during the run \( \rho \).

Random Access Machines. While we shall study in detail the encoding of Turing machines in the \( \lambda \)-calculus, we are not going to lay out the details of the simulation of the \( \lambda \)-calculus on RAMs. We shall provide an abstract machine for the \( \lambda \)-calculus and study its complexity using standard considerations for algorithmic analysis (which are grounded on RAMs), but without giving the details of the simulation. The RAM model we target has a read-only input register the space of which is not counted for space complexity, similarly to TMs. For the sake of completeness, we clarify the RAM cost models of reference: the logarithmic measure for time and Slot and Van Emde Boas’s size, measure for space [37], counting 0 for unused registers and taking into account the logarithm of both the content and the index of used registers. Given a RAM \( R \), we use \( T_{RAM}(\rho) \) and \( S_{RAM}(\rho) \) for the time and space used by \( R \) to reach a final configuration with a sequence of transitions \( \rho \).
Reasonable Cost Models. We give the simulations and the bounds we shall consider for reasonable cost models for the \( \lambda \)-calculus.

Definition 2.1 (Reasonable cost model for the \( \lambda \)-calculus). A reasonable time (resp. space) cost model for the \( \lambda \)-calculus is an evaluation strategy \( \rightarrow \) together with a function \( T_M \) (resp. \( S_M \)) from complete \( \rightarrow \)-sequences \( \rho : t \rightarrow^* u \) to \( \mathbb{N} \) such that:

- There is an encoding of \( \lambda \)-binary strings and Turing machines into the \( \lambda \)-calculus such that if the run \( \sigma \) of a Turing machine \( M \) on input \( i \) ends on a state \( q_b \) with \( b \in \mathbb{B} \), then there is a complete sequence \( \rho : \overline{M}t \rightarrow^* b \) such that \( T_M(\rho) = O(poly(T_{TM}(\sigma), |i|)) \) (resp. space \( S_M(\rho) = O(S_{TM}(\sigma) + \log |i|) \)).
- There is an encoding of \( \lambda \)-terms into binary strings, a RAM \( R \), and a decoding \( \downarrow \) of final configurations for \( R \) such that if \( \rho : t \rightarrow^* u \) is a complete sequence then the execution \( \sigma \) of \( R \) on input \( t \) produces a final configuration \( C \) such that \( C_\downarrow = u \) in time \( T_{RAM}(\sigma) = O(poly(T_M(\rho), |t|)) \) (resp. space \( S_{RAM}(\sigma) = O(S_M(\rho) + \log |t|) \)).

Single Inputs, not Input Lengths. Note that our cost assignments concern runs, thus a single input of a given length, rather than the max over all input of the same length, as it is usually done in complexity. The study of cost models is somewhat finer, the max can be considered afterwards.

3 \( \lambda \)-Calculus and Abstract Machines

A term is closed when there are no free occurrences of variables in it. The operational semantics—that is, the evaluation strategy—that we adopt in most of the paper is weak head evaluation \( \rightarrow_{wh} \), defined as follows:

\[(\lambda x.t)u_1 \ldots u_h \rightarrow_{wh} t[x := u]r_1 \ldots r_h.\]

We further restrict the setting by considering only closed terms, and refer to our framework as Closed Call-by-Name (shortened to Closed CbN). Basic well known facts are that in Closed CbN normal forms are precisely abstractions and that \( \rightarrow_{wh} \) is deterministic.

Abstract Machines Glossary. In this paper, an abstract machine \( M = (Q, \rightarrow, init(-), \cdot) \) is a transition system \( \rightarrow \) over a set of states, noted \( Q \), together with two functions:

- **Compilation.** \( init : \Lambda \rightarrow Q \), turning \( \Lambda \)-terms into states;
- **Decoding.** \( \downarrow : Q \rightarrow \Lambda \), turning states into \( \Lambda \)-terms and such that \( init(t) \downarrow = t \) for every \( \Lambda \)-term.

A state is composed by the (immutable) code \( v_0 \), the active term \( t \), and some data structures. Since the code never changes, it is usually omitted from the state itself, focussing on dynamic states that do not mention it. A state \( q \in Q \) is initial \( t \) if \( init(t) = q \). In this paper, \( init(t) \) is always defined as the state having \( t \) as both the code and the active term and having all the data structures empty. Additionally, the code \( t \) shall always be closed, without further mention.

A state is final if no transitions apply. A run \( \rho : q \rightarrow^* q' \) is a possibly empty sequence of transitions, the length of which is noted \( |\rho| \). If \( a \) and \( b \) are transitions labels (that is, \( \rightarrow_a \) and \( \rightarrow_b \)) then \( \rightarrow_{a,b} := \rightarrow_a \cup \rightarrow_b \). \( |\rho|_{\rightarrow_a} \) is the number of \( a \) transitions in \( \rho \), and \( |\rho|_{\rightarrow_a} \) is the size of transitions in \( \rho \) that are not \( \rightarrow_a \). An initial run \( \rho \) is a run from an initial state \( init(t) \), and it is also called a run from \( t \). A state \( q \) is reachable if it is the target state of an initial run. A complete run is an initial run ending on a final state.

Abstract Machines and Abstract Implementations. Abstract machines do not specify how the (abstract) data structures of the machine are meant to be realized. In general an abstract machine can be implemented in various ways, inducing different, possibly incomparable performances. Therefore, it is not really possible to study the complexity of the machine without some assumptions about the implementation of its data structures. Now, the study of reasonable space requires to take into account the use, and especially the size, of pointers, which is instead usually omitted in the coarser study of reasonable time. In that context, indeed, pointers are assumed to be manipulable in constant time, which is safe because the omitted logarithmic factors are irrelevant for the required polynomial overhead. The more constrained study of space instead requires to clarify them. Switching to such a level of detail, apparently innocent gaps between the specification of a machine and how it is going to be implemented suddenly become relevant.

To account for these subtleties, we specify for every construct of the abstract machine the space that it requires, and for every transition the time that it takes, both asymptotically. The adoption of such an abstract implementations is in our opinion a contribution of this paper towards a more solid theory of abstract machines.

Definition 3.1 (Abstract implementation). Let \( M \) be an abstract machine and \( \rho : init(t_0) \rightarrow^* q \) an initial run for \( M \). An abstract implementation \( I \) for \( M \) is the assignment of asymptotic space costs \( |·|_{sp} \) for every component of \( q \) and of asymptotic time costs \( |·|_{tm} \) for every transition from \( q \).

Assigning costs to the state components provides the space cost \( |q|_{sp} \) of each state \( q \), by summing over all components.

Definition 3.2 (Space and time of runs). Let \( \rho : q_0 \rightarrow^k q_k \) be an initial run of an abstract machine \( M \) and \( I \) an abstract implementation for \( M \).

1. The \( I \)-space cost of \( \rho \) is \( |\rho|_{sp} := \max_{q \in \rho} |q|_{sp} \).
2. The \( I \)-time cost of \( \rho \) is \( |\rho|_{tm} := \sum_{i=0}^{k-1} |q_i \rightarrow q_{i+1}|_{tm} \).

A Technical Remark. Note that abstract implementations do not specify the space cost of transitions \( q \rightarrow q' \). According to the space cost for a run, such a cost has to be the difference \( |q'|_{sp} - |q|_{sp} \) in space between the two involved
states, which can be inferred by the size of the states. Therefore, it needs not be specified by an abstract implementation. Note also a subtlety: implementing a transition might require auxiliary space temporarily exceeding both $|q|_{sp}$ and $|q'|_{sp}$, which we are not accounting for. The point is that for all the machines considered in this paper, such a temporary extra space is bounded by the current space (that is, $|q|_{sp}$), and taking it into account would affect the globally used space only linearly, which is reasonable for space. Therefore, the auxiliary use of space can, and shall, be safely omitted.

4 The Naive KAM

The Krivine abstract machine [27] is a standard environment-based machine for Closed CbN, often defined as in Fig. 1. We refer to it as to the Naive KAM, to distinguish it from forthcoming variants. The machine evaluates closed $\lambda$-terms to weak head normal form via three transitions, the union of which is noted $\rightarrow_{\text{NaKAM}}$:

- $\rightarrow_{\text{seq}}$ looks for redexes descending on the left of topmost applications of the active term, accumulating arguments on the stack;
- $\rightarrow_{\beta}$ fires a $\beta$-redex (given by an abstraction as active term having as argument the first element of the stack) but delays the associated meta-level substitutions, adding a corresponding explicit substitution to the environment;
- $\rightarrow_{\text{sub}}$ is a form of micro-step substitution: when the active term is $x$, the machine lookups the environment and retrieves the delayed replacement for $x$.

The data structures used by the Naive KAM are local environments, closures, and a stack. Local environments, that we shall simply refer to as environments, are defined by mutual induction with closures. The idea is that every (potentially open) term $t$ in a dynamic state comes with an environment $e$ that closes it, thus forming a closure $c = (t, e)$, and, in turn, environments are lists of entries $[x \leftarrow c]$ associating to each open variable $x$ of $t$, a closure $c$ i.e., morally, a closed term. The stack simply collects the closures associated to the arguments met during the search for $\beta$-redexes.

A dynamic state $q$ of the Naive KAM is the pair $(c, \pi)$ of a closure $c$ and a stack $\pi$, but we rather see it as a triple $(t, e, \pi)$ by spelling out the two components of the closure $c = (t, e)$. Initial dynamic states of the Naive KAM are defined as init$(t_0) := (t_0, e, \epsilon)$ (where $t_0$ is a closed $\lambda$-term, and also the code). The decoding of closures and states is as follows:

$\text{Clos. } (t, e) \downarrow := t \quad (t, [x \leftarrow c] \cdot e) \downarrow := (t[x \leftarrow c] \downarrow, e) \downarrow$

$\text{States } (t, e, \pi) \downarrow := (t, e, \pi) \downarrow := (t, e, \pi) \downarrow$

**Basic Qualitative Properties.** Some standard facts about the Naive KAM follow. Let $\rho : \text{init}(t_0) \rightarrow_{\text{NaKAM}}^* q$ be a run.

- **Closures-are-closed invariant:** if the code $t_0$ is closed (that is the only case we consider here) then every closure $(u, E)$ in $q$ is closed, that is, for any free variable $x$ of $u$ there is an entry $[x \leftarrow c]$ in $E$, and recursively so for the closures in $e$. Thus $(u, E) \downarrow$ is a closed term, whence the name closures.
- **Final states:** the previous fact implies that the machine is never stuck on the left of a $\rightarrow_{\text{sub}}$ transition because the environment does not contain an entry for the active variable. Final states then have shape $(\lambda x. u, e, \epsilon)$.

**Theorem 4.1 (Implementation).** The Naive KAM implements Closed CbN, that is, there is a complete $\rightarrow_{\text{wh}}$-sequence $t \rightarrow_{\text{wh}}^n u$ if and only if there is a complete run $\rho : \text{init}(t) \rightarrow_{\text{NaKAM}}^* q$ such that $q \downarrow = u$ and $|\rho|_{\beta} = n$.

The proof of the facts and of the theorem are standard and omitted. Similar statements hold for all the variants of the KAM that we shall see, with similar proofs which shall be omitted as well (we shall also omit their decoding).

The key point is that there is a bijection between $\rightarrow_{\text{wh}}$ steps and $\rightarrow_{\beta}$ transitions, so that we can identify the two. Moreover, the number of $\rightarrow_{\text{wh}}$ steps is a reasonable cost model for time, as first proved by Sands et al. [32].

**Quantitative Properties.** We recall also some less known quantitative facts, for runs as above, from papers by Accatoli and co-authors [2, 3, 6]. The aim is to bound quantities...
relative to the run \( \rho \) and the reachable state \( q \). The bounds are given with respect to two parameters: the size \( |t_0| \) of the code and the number \( |\rho|_\beta \) of \( \beta \)-transitions, which, as mentioned, is an abstract notion of time for Closed CbN.

- **Number of transitions**: the number \( |\rho|_{\text{sub}} \) of sub transitions in \( \rho \) is bounded by \( O(\log |\rho|_\beta^2) \), and there are terms on which the bound is tight. The number \( |\rho|_{\text{sea}} \) of sea transitions is bounded by \( O(\log |\rho|_\beta \cdot |t_0|) \), but on complete runs the bound improves to \( O(|\rho|_\beta) \).

- **Sub-term invariant**: every term \( u \) in every closure \( (u, e) \) of any reachable state reachable is a literal (that is, not up to \( \alpha \)-renaming) sub-term of the code \( t_0 \). Therefore, in particular \( |u| \leq |t_0| \).

- **The length of a single environment**: the number of entries in a single environment is bounded by the size \( |t_0| \) of the code.

- **The number of environments**: the number of distinct environments in \( q \) is bounded only by \( |\rho|_\beta \).

- **The length of the stack**: the length of the stack in \( q \) is bounded by \( O(|\rho|_\beta \cdot |t_0|) \).

**Sub-Term Pointers and Data Pointers.** The Naive KAM is usually implemented using two forms of pointers:

1. **Sub-term pointers**: the initial term \( t_0 \) provides the initial immutable code. The essential sub-term invariant mentioned above allows us to represent the terms \( u \) in every closure \( (u, e) \) of any reachable state with a pointer to \( t_0 \) instead that with a copy of \( u \).

2. **Data (structure) pointers**: to ensure that the duplication of the environment \( e \) in transition \( \to_{\text{sea}} \) can be implemented efficiently (in time), environments are shared so that what is duplicated is just a pointer to an environment, and not the environment itself.

Both kind of pointers shall draw our attention. Sub-term pointers have size \( O(\log |t_0|) \). For the present discussion they are space-friendly, because their size does not depend on the length of the run—we shall inspect them in Sect. 6, where we shall ensure that their number is under control. Data pointers, on the other hand, are space-hostile, because (as recalled above) the number of environments is bounded only by \( |\rho|_\beta \), that is, time. Data pointers have thus size \( O(\log |\rho|_\beta) \), entangling space with time, which is unreasonable for space. Since data pointers are hostile, we want them to be explicitly accounted for by the specification of the machine. Therefore, we consider the Naive KAM as being implemented with sub-term pointers but not data pointers. This fact is expressed by the abstract implementation of the Naive KAM.

**Abstract Implementation of the Naive KAM.** Implementing the Naive KAM without data pointers means that environments cannot be implemented as linked lists, and the same is true for the stack, of which length also depends on the length of the run. The idea then is that they are implemented as unstructured strings, in a linear syntax. We abstract from the actual encoding, what we retain is the abstract implementation in Fig. 1, which captures its essence.

The time cost of all \( \to_{\text{NaKAM}} \) transition depends polynomially on the size of the whole source state \( |q| \), because the lack of data sharing forces to use a new string for the new stack and the new environment. To be precise, one could develop a finer analysis, thus obtaining slightly better bounds, but this would require entering in the details of the implementation and would not give a substantial advantage. As we shall see, indeed, the Naive KAM is unreasonable for time.

**5 The Space KAM**

Here we define a space optimization of the Naive KAM. The Space KAM is derived from the Naive KAM implementing two modifications aimed at space efficiency: namely unchaining and eager garbage collection.

**Unchaining.** It is a folklore optimization for abstract machines bringing speed-ups with respect to both time and space, used e.g. by Sands et al. [32], Wand [40], Friedman et al. [20], and Sestoft [36]. Its first systematic study is by Accattoli and Sacerdoti Coen in [4], with respect to time. The optimization prevents the creation of chains of renamings in environments, that is, of delayed substitutions of variables for variables, of which the simplest shape in the KAM is:

\[
[x_0 \mapsto (x_1, [x_1 \mapsto (x_2, [x_2 \mapsto \ldots)]])]
\]

where the links of the chain are generated by \( \beta \)-redexes having a variable as argument. On some families of terms, these chains keep growing and growing, leading to the quadratic dependency of the number of transitions from \( |\rho|_\beta \).

**Eager Garbage Collection.** Beside the malicious chains connected to unchaining, the Naive KAM is not parsimonious with space also because there is no garbage collection (shortened to GC). In transition \( \to_{\text{sub}} \), the current environment is discarded, so something is collected, but this is not enough. It is thus natural to modify the machine as to maximize GC and space re-usage, that is, as to perform it eagerly.

**The Space KAM.** The Naive KAM optimized with both eager GC and unchaining (both optimizations are mandatory for space reasonability) is here called Space KAM and it is defined in Fig. 2. The data structures, namely closures and (local) environments, are defined as before, the changes concern the machine transitions only. Unchaining is realized by transition \( \to_{\text{sea}} \), while eager garbage collection is realized mainly by transition \( \to_{\text{app}} \), which collects the argument if the variable of the \( \beta \) redex does not occur. Transitions \( \to_{\text{app}} \) and \( \to_{\text{appv}} \) also contribute to implement the GC, by restricting the environment to the occurring variables, when the environment is propagated to sub-terms. As a consequence, we obtain the following invariant.
We now turn to the analysis of the encoding of TMs, taking as reference the one by Dal Lago and Accattoli based on the same abstract implementation of the Naive KAM, as a less consuming scrolling algorithm than the GC has a time cost which however stays within the polynomial (in the size of the states) cost of the transitions. It is mandatory that it is implemented by naively and repeatedly checking whether variables occur, and not via pointers or counters, as they would add an unreasonable space overhead. This fact is implicit in using the same abstract implementation of the Naive KAM, as a less naive GC would alter the space requirements.

### 6 Encoding and Moving over Strings

We now turn to the analysis of the encoding of TMs, taking as reference the one by Dal Lago and Accattoli based over the Scott encoding of strings [14]. The first key step is understanding how to scroll Scott strings.

#### Encoding alphabets

Let \( \Sigma = \{a_1, \ldots, a_n\} \) be a finite alphabet. Elements of \( \Sigma \) are encoded in the \( \lambda \)-calculus in accordance to a fixed (but arbitrary) total order of the elements of \( \Sigma \) as follows:

\[
[a_i]^{\Sigma} := \lambda x_1, \ldots, x_n. x_i.
\]

Note that the representation of an element \([a_i]^{\Sigma}\) requires a number of constructors that is linear (and not logarithmic) in \( |\Sigma| = n \). Since the alphabet \( \Sigma \) shall not depend on the input of the TM, however, the cost in space is actually constant.

#### Encoding strings

A string in \( s \in \Sigma^* \) is represented by a term \( s^{\Sigma} \), defined by induction on \( s \) as follows:

\[
s^{\Sigma} := \lambda x_1, \ldots, x_n. y.y , \quad a_{i}^{\Sigma} := \lambda x_1, \ldots, x_n. y.x_i s^{\Sigma}.
\]

Note that the representation depends on the cardinality of \( \Sigma \). As before, however, the alphabet is a fixed parameter, and so such a dependency is irrelevant.

**Lemma 5.1** (Environment domain invariant). Let \( q \) be a Space KAM reachable state. Then \( \text{dom}(e) = \text{fv}(t) \) for every closure \((t, e)\) in \( q \).

Because of the invariant, which concerns also the closure given by the active term and the local environment of the state, the variable transition simplifies as follows:

| Term | Env | Stack | Term | Env | Stack |
|------|-----|-------|------|-----|-------|
| \( tx \) | \( e \) | \( \pi \) | \( \searrow \) | \( t \) | \( e \cdot \pi \) |
| \( tu \) | \( e \) | \( \pi \) | \( \searrow \) | \( t \) | \( (u, e) \cdot \pi \) |
| \( \lambda x.t \) | \( e \cdot c \cdot \pi \) | | \( \beta_{\mu} \) | \( t \) | \( e \) | \( \pi \) |
| \( \lambda x.t \) | \( e \cdot c \cdot \pi \) | | \( \beta_{\mu} \) | \( t \) | \( [x \cdot c] \cdot e \) | \( \pi \) |
| \( x \) | \( e \) | \( \pi \) | \( \searrow_{\text{sub}} \) | \( u \) | \( e' \) | \( \pi \) |

where \( e \cdot \pi \) denotes the restriction of \( e \) to the free variables of \( t \).

**Figure 2. Transitions of the Space KAM.**

We now explain how to obtain that \( |s^{\Sigma}| = |s^{\Sigma}| \), as announced in Sect. 2. Note that in \( \Sigma^* \) every variable occurrence is bound inside the list of binders immediately preceding the occurrence. Therefore, if de Bruijn indices are used to represent \( \lambda \)-terms, one needs only indices—that is, variable pointers—between 1 and \( |\Sigma| \), that is, of constant size. Note that, similarly, if variables are represented with textual names, again having only \( |\Sigma| \) distinct names is enough if one permits that different sequences of abstractions re-use the same names, that is, if one accepts Barendregt’s convention to be violated. Remarkably, a notable folklore property of the (Space) KAM is that its implementation theorem does not need Barendregt’s convention to hold.

**Recursion and Fix-Points.** The encoding of TMs crucially relies on the use of a fix-point operator to implement recursion. Precisely, fix-points are used to model the transition function, making a copy of the (sub-term encoding the) transition table at each step. It is the only point of the encoding where duplication occurs, and it is thus where the expressive power is encapsulated. The rest of the encoding is affine—note that the representation of strings is affine.

**Fix-Points and Toy Scrolling Algorithms.** To understand the delicate interplay between the space of the KAM and fix-points, we analyze it via simple toy algorithms on strings. The first, simplest one is the consuming scrolling algorithm: going through an input string \( s \) doing nothing and accepting when arriving at the end of the string, without having to preserve the string itself—the aim is just to see the space used for scrolling a string. The toy algorithm is a very rough approximations of the moving of TMs over a tape, which is the most delicate aspect of the space reasonable simulation of TMs in the \( \lambda \)-calculus that we shall develop. It is used to illustrate the key aspects of the problems that arise of their solutions, without having to deal with all the details of the encoding of TMs at once. On TMs, scrolling a string obviously runs in constant space, and on log-sensitive TMs the consuming aspect cannot be modeled—we shall consider non-consuming scrolling later in this section.

We encode the algorithm as a \( \lambda \)-term over Scott strings, where a fix-point combinator is used to iterate over the (term


propagation) the input string \( s \). Since the input string \( s \) is consumed in the process, the normal form would be the encoding of the accepting state \( q_1 \) of the TM, which for simplicity here is simply given by the identity combinator \( I \).

We use Turing’s fix-point combinator and the boolean alphabet \( \mathbb{B} := \{ 0, 1 \} \). Let fix be the term \( \theta \), where \( \theta := \lambda x. \lambda y. y(xxy) \). Given a term \( u \), fix \( u \) is a fix-point of \( u \).

\[
\text{fix } u = (\lambda x. \lambda y. y(xxy))u \\
\quad \rightarrow_{\beta} (\lambda y. y(\theta y))u \\
\quad \rightarrow_{\beta} u(\theta y) = u(\text{fix } u)
\]

Algorithms moving over binary Scott strings always follow the same structure. They are given by the fix-point iteration of a term that does pattern matching on the leftmost character of the string and for each of the possible outcomes (in our case, first character is 0, 1, or the string is empty) does the corresponding action. The general term is fix \( (\lambda f. \lambda z. z A_0 A_1 A_r) \), where \( f \) is the variable for the recursive call and \( A_0, A_1, A_r \) represent the three actions, which in our case are simply given by \( A_0 = A_1 = f \) and \( A_r = I \), using the identity \( I \) as encoding of the accepting state.

**Proposition 6.1.** Let \( s \in \mathbb{B}^* \) and \( \text{toy} := \text{fix } (\lambda f. \lambda z. z f f f l) \).

1. \( \text{toy } s^\infty \rightarrow_{\Theta(|s|)} 1 \).
2. The Naive KAM evaluates \( \text{toy } s^\infty \) in space \( \Omega(2^{|s|}) \).
3. The Space KAM evaluates \( \text{toy } s^\infty \) in space \( \Theta(|s| \log |s|) \).

We can see that the Naive KAM is desperately inefficient for space, while the Space KAM works within reasonable bounds. It turns out, however, that the Space KAM is still not enough in order to obtain a space reasonable simulation of TMs. The problem now concerns the standard encoding of TMs and its managing of the tapes, rather than the use of space by the abstract machine itself. The issues can be explained using further toy algorithms.

**String-Preserving Scrolling.** Consider the same scrolling algorithm as above, except that now the input string \( s \) is not consumed by the moving over \( s \), that is, it has to be given back as output of the \( \lambda \)-term implementing the algorithm. This variant is a step forward towards approximating what happens to the tapes of TMs during the computation: the TM moves over the tapes without consuming them, it is only at the end of the computation that the TM can be seen as throwing them away. There are two ways of implementing the new algorithm:

1. **Local copy**: moving over the string \( s \) while accumulating in a new string \( r \) the characters that have already been visited, returning \( r \).
2. **Global copy**: making a copy \( r \) of the string \( s \), and then moving over \( s \) in a consuming way, returning \( r \).

**Local Copy.** The local approach is the one underlying the reference encoding of TMs. In particular, it is almost affine, as duplication is isolated in the fix-point. The \( \lambda \)-term \( 1 \text{ocpy} \) realizing uses the same fix-point schema as before, but with different, more involved action terms \( A_0, A_1, A_r \).

**Proposition 6.2.** Let \( s \in \mathbb{B}^* \).

1. \( 1 \text{ocpy } s^\infty \rightarrow_{\Theta(|s|)} \text{toy } s^\infty \).
2. The Space KAM evaluates \( 1 \text{ocpy } s^\infty \) in space \( \Theta(|s| \log |s|) \).

The \( \Theta(|s| \log |s|) \) bound in point 2 is problematic for the space reasonable modeling in the \( \lambda \)-calculus of both the input and the work tapes, for different reasons.

**Work Tape and Separate Address Spaces.** For a space reasonable managing of the work tape, the local algorithm should rather work in space \( O(|s|) \). This improvement can be realized by a finer complexity analysis. In Prop. 6.2.2, the cost comes from the use of \( O(|s|) \) sub-term pointers to the code \( 1 \text{ocpy } s^\infty \) used by the Space KAM. These pointers have size \( O(|s| \log |s|) \) because \( |1 \text{ocpy } s^\infty| = O(|s|) \), that is, the size of \( 1 \text{ocpy} \) is independent of \( |s| \) and constant. A close inspection of the Space KAM run in Prop. 6.2.2 shows that, of the \( O(|s|) \) pointers used, only \( O(1) \) of them actually point to \( s^\infty \), while all the others (that is, an \( O(|s|) \) amount) point to \( 1 \text{ocpy} \). Since \( 1 \text{ocpy} \) is of size independent from \( |s| \), if one admits separate address spaces for \( 1 \text{ocpy} \) and \( s^\infty \) then the pointers to \( 1 \text{ocpy} \) have size \( O(1) \). Therefore, one obtains that the space cost is given by

\[
O(|s|) \cdot O(1) + O(1) \cdot O(|s| \log |s|) = O(|s|).
\]

From now on then, the first argument of the code of the Space KAM—that in the encoding of TMs shall represent the input—has a dedicated address space.

**Proposition 6.3** (Linear Space Local-Copy Scrolling). Let \( s \in \mathbb{B}^* \). The Space KAM evaluates \( 1 \text{ocpy } s^\infty \) in space \( O(|s|) \) if \( 1 \text{ocpy} \) and \( s^\infty \) have separate space addresses.

**Input Tape and Global Copy.** For the input tape, a linear space bound for scrolling is unreasonable, if one aims at preserving logarithmic space complexity. For meeting the required \( O(|s| \log |s|) \) bound, we need a more radical solution, which is possible because the tape is read-only.

The first step is the straightforward modification of the consuming scrolling algorithm into a global-copy string-preserving algorithm: it is enough to capture the input at the beginning with an extra abstraction \( \lambda x \) and to give it back at the end with the action \( A_r \). Namely, let \( g1 \text{Lcpy} := \lambda x. (\text{fix } (\lambda f. \lambda s. s f f x x)) \). Clearly, this approach breaks the almost affinity of the encoding, as copying is no longer encapsulated only in the fix-point.

**Proposition 6.4.** Let \( s \in \mathbb{B}^* \).

1. \( g1 \text{Lcpy } s^\infty \rightarrow_{\Theta(|s|)} s^\infty \).
2. The Space KAM evaluates \( g1 \text{Lcpy } s^\infty \) in space \( \Theta(|s| \log |s|) \).

Interestingly, the space cost stays logarithmic, because the global copy of the input in point 1 (note that there actually is a copy for every iteration of the fix-point) is not performed by the Space KAM, which instead copies a pointer.
to it and only once. The second step is refining this scheme as to implement a read-only tape. A slight digression is in order.

**Intrinsic and Mathematical Tape Representations.** A TM tape is a string plus a distinguished position, representing the head. There are two tape representations, dubbed intrinsic and mathematical by van Emde Boas in [39]. The intrinsic one represents both the string $s$ and the current position of the head as the triple $s = s_l \cdot h \cdot s_r$, where $s_l$ and $s_r$ are the prefix and suffix of $s$ surrounding the character $h$ read by the head. This is the representation underlying the local-copy scrolling algorithm (and the reference encoding of TM). The mathematical representation, instead, is simply given by the index $n \in \mathbb{N}$ of the head position, that is, the triple $s_l \cdot h \cdot s_r$ is replaced by the pair $(s, |s| + 1)$.

**Mathematical Input and Global Copy.** Given a mathematical read-only tape $(s, n)$, one can use the global-copy scrolling scheme for a simulation in the $\lambda$-calculus in space $O(\log |s|)$. The idea is to represent $n$ as a binary string $[n]$. Since $n \leq |s|$, we have $[|n|] \leq \log |s|$. Moreover, it is possible to pass from $[n]$ to $[n + 1]$ or $[n - 1]$—which is needed to move the position of the head—in $O(\log |s|)$ space. Then one shows that in the $\lambda$-calculus the following is doable in space $O(\log |s|)$: given $(s, n)$, returning $(s, n)$ plus the $n$-th character $s_n$ of $s$, by making a global copy of the tape and scrolling the current copy of $n$ positions, extracting the head $s_n$ of the obtained suffix, and discarding the tail.

Two remarks. First, this approach works because the tape is read-only, so that one can keep making global copies of the same immutable tape, and only changing the index of the head. Second, there is a (reasonable) time slowdown, because at each read the simulation has to scroll sequentially the input tape to get to the $n$-th character.

7 The Space KAM is Reasonable for Space

We are ready for our main result, which is based on a new variant (in Appendix A) over Dal Lago and Accattoli encoding of TMs into the $\lambda$-calculus [14]. The key points are:

- **Refined TMs:** the notion of TM we work with is log-sensitive TMs with mathematical input tape and intrinsic work tape (the definition is laid out in Appendix A).
- **CPS and indifference:** following [14], the encoding is in continuation-passing style, and carefully designed (by adding some $\eta$-expansions) as to fall into the *deterministic* $\lambda$-calculus $\Lambda_{\text{det}}$, a particularly simple fragment of the $\lambda$-calculus where the right sub-terms of applications can only be variables or abstractions and where, consequently, call-by-name and call-by-value collapse on the same evaluation strategy $\rightsquigarrow_{\text{det}}$. We shall exploit this *indifference* property in Sect. 9.

- **Duplication:** duplication is isolated in the unfolding of fix-points and in the managing of the input tape, all other operations are affine.

**Theorem 7.1** (TM are simulated by the Space KAM in reasonable space). There is an encoding $\gamma$ of log-sensitive TMs into $\Lambda_{\text{det}}$ such that if the run $p$ of the TM $M$ on input $i \in \mathbb{B}^*$:

1. **Termination:** ends in $q_b$ with $b \in \mathbb{B}$, then there is a complete sequence $\bar{M} i \overset{\gamma}{\to}_{\text{det}} q_b$ where $n = \Theta((|T_M| + 1) \cdot |i| \cdot \log |i|)$.
2. **Divergence:** diverges, then $\bar{M} i \overset{\gamma}{\to}_{\text{det}} \cdot$ divergent.
3. **Space KAM:** the space used by the Space KAM to simulate the evaluation of point $i$ is $O(S_M(\rho) + \log |i|)$ if $\bar{M}$ and $\bar{\gamma}$ have separate address spaces.

The previous theorem provides the subtle and important half of the space reasonability result. The first two points establish the qualitative part of the simulation in the $\lambda$-calculus, together with the time bound (with respect to the number of $\beta$ steps). The third point provides the space result for the Space KAM. They are connected by the fact that the Space KAM implements closed call-by-name (that coincides with $\rightarrow_{\text{det}}$ in $\Lambda_{\text{det}}$), via an omitted minor variant of Theorem 4.1.

The other half of the result amounts to show that the Space KAM can be simulated on RAMs within the space costs claimed in Sect. 5. The idea is that it can clearly be simulated reasonably by a multi tape TM using one work tape for the active term, one for the environment, and one for the stack, which in turn can be reasonably simulated by a RAM. We use RAM rather TM in the statement for uniformity with the works on time (which is relevant for the discussions in Sect. 8).

**Theorem 7.2** (Space KAM is simulated by RAMs in reasonable space). Let $t$ be a closed $\lambda$-term. Every Space KAM run $p : \text{init}(t) \rightarrow_{\text{SpKAM}} q$ can be implemented on RAMs in space $O(|p|_{sp})$.

From Theorems 7.1 and 7.2 follows our main result.

**Theorem 7.3** (The Space KAM is reasonable for space). Closed CBN evaluation $\rightarrow_{\text{vh}}$ and the space of the Space KAM provide a reasonable space cost model for the $\lambda$-calculus.

**The Space KAM is not Reasonable for Natural Time.** We complete our study of the Space KAM by analyzing its time behavior. For natural time (in our case, the number of Closed CBN $\beta$ steps), the Space KAM is unreasonable, because simulating Closed CBN at times requires exponential overhead. The number of transitions of the Space KAM is reasonable, while it is the cost of single transitions, thus of the manipulation of data structures, that can explode. The failure stems from the lack of data sharing, which on the other hand we showed being mandatory for space reasonability. Essentially, there are size exploding families such that their Space KAM run produces environments of size
### Proposition 7.4 (Space KAM natural time overhead explosion)

There is a family \(\{t_n\}_{n\in\mathbb{N}}\) of closed \(\lambda\)-terms such that there is a complete evaluation \(p_n : t_n \rightarrow^\omega u_n\) is simulated by Space KAM runs \(\sigma_n\) taking both space and time exponential in \(n\), that is, \(|\sigma_n|_{sp} = |\sigma_n|_{tm} = \Omega(2^n)\).

## 8 Time vs Space

Here we discuss how to obtain, or approximate, reasonability for both space and time.

### Reasonable Low-Level Time.

Changing the time cost model from the number of \(\beta\) steps to the time taken by the Space KAM, which is a low-level notion of time, provides a reasonable time cost model. The key point that is the explosions of Prop. 7.4 never happen on \(\lambda\)-terms encoding TMs.

### Theorem 8.1 (TMs are simulated by the Space KAM in reasonable low-level time).

1. Every TM run \(\rho\) can be simulated by the Space KAM in time \(O(\text{poly}(|\rho|))\).
2. Every Space KAM run \(\rho : \text{init}(t) \rightarrow^\omega_{\text{SpKAM}} q\) can be implemented on RAMs in time \(O(|\rho|_{tm})\).
3. Closed CbN and the time of the Space KAM provide a reasonable time cost model for the \(\lambda\)-calculus.

The drawback of this solution is that one gives up the natural cost model for time. Moreover, the low-level time cost model can be very lax in comparison, as Prop. 7.4 shows.

### The Time KAM.

The Naive KAM can also be optimized for time, rather than space, by adopting data pointers together with unchaining (that for time it is not mandatory for reasonability but it improves the overhead). The definition and the abstract implementation of this machine, named Time KAM, is in Appendix D. Environments and stacks are now implemented as linked lists\(^1\), enabling a sharing mechanism that allows transitions to be implementable in constant time (logarithmic in the size of the code and of \(|\rho|_\beta\), if pointers sizes are accounted for). The drawback is that space cannot be collected anymore, since there is alaisimg i.e. the same memory cell could be referenced more than once. In particular, in transition \(\rightarrow_{\text{sub}}\) now no memory gets freed, in contrast to what happens in the Naive/Space KAM.

### Theorem 8.2. Let \(\rho : \text{init}(t_0) \rightarrow_{\text{TKAM}} q\) be a complete Time KAM run. It can be implemented on RAMs in time and space \(O(|\rho|_\beta \cdot \log(|\rho|_\beta \cdot |t_0|))\).

The theorem states that, considering as time the number of \(\beta\) steps, the Time KAM is reasonable for time, whence its name. It is instead unreasonable for space, since its space consumption (quasi)linearly depends from time.

### The Interleaving Technique.

Forster et al. in [19] show that, given one machine that is reasonable for time but not for space and one machine that is reasonable for space but not for time, it is possible to build a third machine that is reasonable for both space and time, by interleaving the two machines in a smart way. Despite being presented on a specific case, their construction is quite general (in fact it is not even limited to the \(\lambda\)-calculus), and can be adapted to our case (the two starting machines being the Time KAM and the Space KAM). The drawback of this solution is that it admits space exponential in time, as Prop. 7.4 shows.

### Trading Time for Space.

From a practical rather than theoretical point of view, there is a further semi solution that we now sketch. Tweaking the Time KAM with a synchronous reference counting GC one obtains a Shared Space KAM which is reasonable for time and slightly unreasonable for space. One could observe that the Shared Space KAM and the Space KAM are indeed strongly bisimilar and use the same number of closures. Then, the number of data pointers used by the Shared Space KAM is no longer entangled with the number of \(\beta\)-steps (as for the Time KAM), it

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\(^1\)Actually, we use one of the smarter implementations by Accattoli and Barras [3], namely balanced trees or random access lists.
is instead related to the space cost—this is the effect of GC plus unchaining of the Space KAM. Therefore, data pointers add a space overhead that is logarithmic in the space of the Space KAM. Also the GC mechanism, based on reference counting, adds the same logarithmic factor, while certainly costs time, though still remaining polynomial. This way, if the Space KAM uses $O(f(n))$ space, then the Shared Space KAM operates in $O(f(n) \log f(n))$ space. Such an overhead is not reasonable but not too unreasonable, and probably the best compromise between reasonability and efficiency for the practice of implementing functional programs.

### 9 Call-by-Value and Other Strategies

How robust is our space cost model to changes of the evaluation strategy? The short answer is very robust.

**Closed Call-by-Value.** We refer to weak call-by-value evaluation with closed terms as to Closed CbV. Our results smoothly adapt to such a setting, as we now explain.

First, it is easy to adapt the Space KAM to Closed CbV. The LAM (Leroy Abstract Machine) is a right-to-left CbV analogue of the KAM defined by Accattoli et al. in [2] and modeled after Leroy’s ZINC [28] (whence the name). It uses a further data structure, the dump, storing the left sub-terms of applications yet to be evaluated. It is upgraded to the Space LAM in Fig. 3 by removing data pointers and adding GC. Unchaining comes for free in CbV, if one considers values to be only abstractions, see Accattoli and Sacerdoti Coen [4].

The next step is realizing that, because of the mentioned indifference property of the deterministic $\lambda$-calculus $\Lambda_{\text{det}}$ (containing the image of the encoding of TMs), the run of the Space LAM on a term $t \in \Lambda_{\text{det}}$ is almost identical (technically, weakly bisimilar) to the one of the Space KAM on $t$.

**Proposition 9.1.** The Space KAM and the Space LAM are weakly bisimilar when executed on $\Lambda_{\text{det}}$-terms. Moreover, their space consumption is the same.

Since the simulation of the Space LAM on RAMs is as smooth as for the Space KAM, we have the following result.

**Theorem 9.2** (The Space LAM is reasonable for space). Closed CbV evaluation and the space of the Space LAM provide a reasonable space cost model for the $\lambda$-calculus.

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**Open and Strong Evaluation.** Extending CbN/CbV evaluation to deal with open terms or even under abstractions, which is notoriously very delicate in the study of reasonable time, is instead straightforward for space. This is because these extensions play no role in the simulation of TMs, which is the delicate direction for space. Given the absence of difficulties, we refrain from introducing variants of the Space KAM/LAM for open and strong evaluation.

**Call-by-Need.** The only major scheme for which our technique breaks is call-by-need (CbNeed) evaluation. To our knowledge, implementations of CbNeed inevitably rely on a heap and on data pointers similar to those of the Time KAM, to realize the memoization mechanism at the heart of CbNeed. Therefore, they are space unreasonable. This is not really surprising: CbNeed, being a time optimization of CbN, treads space for time, sacrificing space reasonability.

### 10 Conclusions

Via a fine study of abstract machines and of the encoding of Turing machines, we provide the first space cost model for the $\lambda$-calculus accounting for logarithmic space. We have reported our main results in Fig. 4.

Our cost model is given by an external device, the 700th abstract machine for the $\lambda$-calculus, so how canonical is it? The constraints for reasonable logarithmic space are very strict. It seems that there is no room for significant variations in the machine nor in the encoding of TMs. Moreover, our work space has the same relationship to time than natural space, and it smoothly adapts to other evaluation strategies, such as call-by-value. We then dare to say that our space cost model is fairly canonical.

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|                  | Natural Time Reasonable (cost model = num. of trans. = $|\beta|$) | Low-Level Time Reasonable (actual implementation cost) | Space Reasonable (low-level by def.) |
|------------------|------------------------------------------------------------------|------------------------------------------------------|-------------------------------------|
| Naive KAM        | No, Proposition 7.4                                             | No, Proposition 6.1.2                                | No, Proposition 6.1.2               |
| Space KAM        | No, Proposition 7.4                                             | Yes, Theorem 8.1.                                    | Yes, Theorem 6.1.2                  |
| Time KAM         | Yes, Theorem 8.2                                                 | Yes, Theorem 8.2.                                    | No, Theorem 8.2                     |
| Space LAM (CbV)  | No, via Prop. 9.1 and Prop. 7.4                                 | Yes, via Prop. 9.1 and Thm. 8.1                      | Yes, Theorem 9.2                    |

| Figure 4. Summary of the results of the paper. |
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The Appendix is structured as follows: first, we give the encoding of Turing machines into the \( \lambda \)-calculus, and then we prove the results stated in the paper, section by section.

### A Encoding Space Sensitive TMs into the \( \lambda \)-Calculus

#### A.1 Preliminaries

**Deterministic \( \lambda \)-Calculus.** The language and the evaluation contexts of the deterministic \( \lambda \)-calculus \( \Lambda_{\text{det}} \) are given by:

- **Terms**
  \[ t, u, r, w \; ::= \; v \mid tv \]
- **Values**
  \[ v, w, v' \; ::= \; \lambda x.t \mid x \]

**Evaluation Contexts**
\[ E \; ::= \; \langle \cdot \rangle \mid Ev \]

Note that
- **Arguments are values**: the right subterm of an application has to be a value, in contrast to what happens in the ordinary \( \lambda \)-calculus.
- **Weak evaluation**: evaluation contexts are weak, i.e. they do not enter inside abstractions.

Evaluation is then defined by:

- **Rule at top level**
  \[ (\lambda x.t)u \mapsto_\beta t[x\leftarrow u] \]

- **Contextual closure**
  \[ E(t) \mapsto_{\text{det}} E(u) \text{ if } t \mapsto_\beta u \]

**Convention:** to improve readability we omit some parenthesis, giving precedence to application with respect to abstraction. Therefore \( \lambda x.tu \) stands for \( \lambda x.(tu) \) and not for \( (\lambda x.t)u \), that instead requires parenthesis.

The name of this calculus is motivated by the following immediate lemma.

**Lemma A.1.** Let \( t \in \Lambda_{\text{det}} \). There is at most one \( u \in \Lambda_{\text{det}} \) such that \( t \mapsto_\beta u \), and in that case \( t \) is an application.

**Proof.** By induction on \( t \). If \( t \) is a value then it does not reduce. Then assume that \( t \) is an application \( t = uv \). Let’s apply the i.h. to \( u \). Two cases:

1. \( u \) reduces and it is an application: then \( t \) has one redex, the one given by \( u \) (because \( \langle \cdot \rangle v \) is an evaluation context), and no other one, because \( v \) does not reduce and \( u \) is not an abstraction by the i.h.

2. \( u \) does not reduce: if \( u \) is not an abstraction then \( t \) is normal, otherwise \( u = \lambda x.r \) and \( t = (\lambda x.r)v \) has exactly one redex.

\( \square \)

**Fixpoint.** Let \( \theta \) be the term \( \theta \theta \), where
\[ \theta := \lambda x.\lambda y.y(\lambda z.xxyz). \]

Now, given a term \( u \) let us show that \( \text{fix } u \) is a fixpoint of \( u \) up to \( \eta \)-equivalence.

\[ \text{fix } u = (\lambda x.\lambda y.Y(\lambda z.xxyz))\theta u \]
\[ \mapsto_{\text{det}} (\lambda y.Y(\lambda z.\theta uz))u \]
\[ \mapsto_{\text{det}} u(\lambda z.\theta uz) \]
\[ = u(\lambda z.\text{fix } uz) \]
\[ \eta u(\text{fix } u) \]

**Encoding alphabets.** Let \( \Sigma = \{a_1, \ldots, a_n\} \) be a finite alphabet. Elements of \( \Sigma \) are encoded as follows:
\[ [a_i]^\Sigma := \lambda x_1. \ldots \lambda x_n.x_i. \]

When the alphabet will be clear from the context we will simply write \([a_i]\). Note that

1. the representation fixes a total order on \( \Sigma \) such that \( a_i \leq a_j \) iff \( i \leq j \);
2. the representation of an element \([a_i]^\Sigma\) requires space linear (and not logarithmic) in \(|\Sigma|\). But, since \( \Sigma \) is fixed, it actually requires constant space.

**Encoding strings.** A string in \( s \in \Sigma^* \) is represented by a term \( \bar{s}^\Sigma \), defined by induction on the structure of \( s \) as follows:
\[ \bar{\varepsilon}^\Sigma := \lambda x_1. \ldots \lambda x_n.\bar{\varepsilon} \]
\[ \bar{a}^\Sigma := \lambda x_1. \ldots \lambda x_n.\lambda y.y, \]
\[ \bar{a}\bar{a}^\Sigma := \lambda x_1. \ldots \lambda x_n.\lambda y.x\bar{a}^\Sigma. \]

Note that the representation depends on the cardinality of \( \Sigma \). In other words, if \( s \in \Sigma^* \) and \( \Sigma \subset \Delta \), \( \bar{s}^\Sigma \neq \bar{s}^\Delta \). In particular, \(|\bar{s}^\Sigma| = \Theta(|s| \cdot |\Sigma|)\). The size of the alphabet is however considered as a fixed parameter, and so we rather have \(|\bar{s}^\Sigma| = \Theta(|s|)\).
Lemma A.2 (Appending a character in constant time). Let $\Sigma$ be an alphabet and $a \in \Sigma$ one of its characters. There is a term $\text{append}_a^k$ such that for every continuation $k$ and every string $s \in \Sigma^*$,

$$\text{append}_a^k \stackrel{Q(1)}{\longrightarrow}_{\text{det}}^2 k(\text{as}).$$

Proof: Define the term $\text{append}_a^k := \lambda k'. \lambda s'. k'((\lambda x_1, \ldots, \lambda x_{|s|}, \lambda y. x_i s'))$ where $i_a$ is the index of $a$ in the ordering of $\Sigma$ fixed by its encoding, that appends the character $a$ to the string $s'$ relatively to the alphabet $\Sigma$. We have:

$$\text{append}_a^k \stackrel{Q(1)}{\longrightarrow}_{\text{det}}^2 (\lambda k'. \lambda s'. k'((\lambda x_1, \ldots, \lambda x_{|s|}, \lambda y. x_i s'))) k \bar{s} = k(\lambda x_1, \ldots, \lambda x_{|s|}, \lambda y. x_i 0) = k(\text{as}). \quad \Box$$

### A.2 Binary Arithmetic

In order to navigate the input word, we consider a counter (in binary). Moving the head left (respectively right) amounts to decrement (respectively increment) the counter by one. The starting idea is to see a number as its binary string representation and to use the Scott encoding of strings. Since it is tricky to define the successor and predecessor on such an encoding, we actually define an ad-hoc encoding.

The first unusual aspect of our encoding is that the binary string is represented in reverse order, so that the representation of 2 is 01 and not 10. This is done to ease the definition of the successor and predecessor functions as $\lambda$-terms, which have to process strings from left to right, and that with the standard representation would have to go to the end of the string and then potentially back from right to left. With a reversed representation, these functions need to process the string only once from left to right.

The second unusual aspect is that, in order to avoid problems with strings made out of all 0s and strings having many 0s on the right (which are not meaningful), we collapse all suffixes made out of all 0 on to the empty string. A consequence is that the number 0 is then represented with the empty string. Non-rightmost 0 bits are instead represented with the usual Scott encoding.

If $n \in \mathbb{N}$ we write $[n]$ for the binary string representing $n$. Then we have:

- $[0] := \varepsilon$
- $[1] := 1$
- $[2] := 01$
- $[3] := 11$
- $[4] := 001$

And so on. Binary strings are then encoded as $\lambda$-terms using the Scott encoding, as follows:

$$\bar{s} := \lambda x_0. \lambda x_1. \lambda y. y$$
$$0 \bar{s} := \lambda x_0. \lambda x_1. \lambda y. x_0 \bar{s}$$
$$1 \bar{s} := \lambda x_0. \lambda x_1. \lambda y. x_1 \bar{s}$$

**Successor Function.** The successor function $\text{succ}$ on the reversed binary representation can be defined as follows (in Haskell-like syntax):

$$\text{succ} \varepsilon = 1$$
$$\text{succ} 0 \bar{s} = 1 \bar{s}$$
$$\text{succ} 1 \bar{s} = 0 \text{succ} 1 \bar{s}$$

For which we have $\text{succ}([n]) = [n + 1]$.

**Lemma A.3.** There is a $\lambda$-term $\text{succ}$ such that for every continuation $k$ and every natural number $n \in \mathbb{N}$,

$$\text{succ} k [n] \stackrel{Q(|n|)}{\longrightarrow}_{\text{det}}^2 k \text{succ} [n].$$

Proof: Define $\text{succ} := \Theta \text{succaux}$ and $\text{succaux} := \lambda f. \lambda k'. \lambda n'. n' N_0 N_1 f k'$ where:

- $N_0 := \lambda f'. \lambda s'. \lambda k'. \lambda \bar{s}' \text{append}^d k' 0 \bar{s}'$
- $N_1 := \lambda f'. \lambda s'. \lambda k'. f' ((\lambda z. \text{append}^d k' z) s')$
- $N_1 := \lambda f'. \lambda k'. k' \bar{s} \varepsilon$
The first steps of the evaluation of \( \text{succ } k[n] \) are common to all natural numbers \( n \in \mathbb{N} \):

\[
\text{succ } k[n] = \text{fix succaux } k[n] \\
\rightarrow^2_\beta \text{succaux}(\lambda z. \text{succ } z)k[n] \\
= (\lambda f. \lambda k'. \lambda n'. n'N_0N_1N_f)(\lambda z. \text{succ } z)k[n] \\
\rightarrow^3_\beta [n]N_0N_1N_r(\lambda z. \text{succ } z)k
\]

Cases of \( n \):

- **Zero**, that is, \( n = 0 \), \([n] = \varepsilon\), and \([n] = \lambda x_0. \lambda x_1. \lambda y. y\): then

\[
[n]N_0N_1N_r(\lambda z. \text{succ } z)k = (\lambda x_0. \lambda x_1. \lambda y. y)N_0N_1N_r(\lambda z. \text{succ } z)k \\
\rightarrow^3_\beta N_r(\lambda z. \text{succ } z)k \\
= (\lambda f'. \lambda k'. k'\overline{\varepsilon})k \\
\rightarrow_\beta k1\overline{\varepsilon} \\
= k \text{ succ } [0]
\]

- **Not zero**, then there are two sub-cases, depending on the first character of the string \([n]\):
  - **0 character**, i.e. \([n] = 0::s\): then

\[
0::rN_0N_1N_r(\lambda z. \text{succ } z)k = (\lambda x_0. \lambda x_1. \lambda y. y)N_0N_1N_r(\lambda z. \text{succ } z)k \\
\rightarrow^3_\beta N_r(\lambda z. \text{succ } z)k \\
= (\lambda f'. \lambda s'. \lambda k'. \text{append } k' s' \overline{s})(\lambda z. \text{succ } z)k \\
\rightarrow^3_\beta \text{ append } k' \overline{s} \\
(L. A. 2) \rightarrow^\text{O}(1) k \overline{\text{pred } s} \\
= k \text{ succ } [n]
\]

  - **1 character**, i.e. \([n] = 1::s\): then

\[
1::rN_0N_1N_r(\lambda z. \text{succ } z)k = (\lambda x_0. \lambda x_1. \lambda y. y)N_0N_1N_r(\lambda z. \text{succ } z)k \\
\rightarrow^3_\beta N_r(\lambda z. \text{succ } z)k \\
= (\lambda f'. \lambda s'. \lambda k'. f'(\lambda z. \text{append } k' z s') \overline{s})(\lambda z. \text{succ } z)k \\
\rightarrow^3_\beta (\lambda z. \text{succ } z) (\lambda z. \text{append } k z \overline{s}) \\
\rightarrow_\beta \text{ succ } (\lambda z. \text{append } k z \overline{s}) \\
(i.h.) \rightarrow^\text{O}(1) (\lambda z. \text{append } k z \overline{s}) \text{ succ } s \\
\rightarrow^\text{det} \text{ append } k z \text{ succ } s \\
(L. A. 2) \rightarrow^\text{O}(1) k0:(\text{succ } s) \\
= k \text{ succ } (1::s) \\
= k \text{ succ } [n]
\]

\( \square \)

**Predecessor Function.** We now define and implement a predecessor function. We define it assuming that it shall only be applied to the encoding \([n]\) of a natural number \( n \) different from \( 0 \), as it shall indeed be the case in the following. Such a predecessor function \( \text{pred} \) is defined as follows on the reversed binary representation (in Haskell-like syntax):

\[
\begin{align*}
\text{pred} & \quad 0::s = 1::(\text{pred } s) \\
\text{pred} & \quad 1::\varepsilon = \varepsilon \\
\text{pred} & \quad 1::b::s = 0::b::s
\end{align*}
\]

It is easily seen that \( \text{pred}(n) = [n - 1] \) for all \( 0 < n \in \mathbb{N} \). Note that \( \text{pred}(n) \) does not introduce a rightmost 0 bit when it changes the rightmost bit of \([n]\), that is, \( \text{pred } 001 = 11 \) and not 110.

**Lemma A.4.** There is a \( \lambda \)-term \( \text{pred} \) such that for every continuation \( k \) and every natural number \( 1 \leq n \in \mathbb{N} \),

\[
\text{pred } k[n] \rightarrow^\text{O}([n]!) k \text{ pred } [n].
\]

**Proof.** Define \( \text{pred} := \text{fix predaux} \) and \( \text{predaux} := \lambda f. \lambda k'. \lambda n'. n'N_0N_1N_fk' \) where:

- \( N_0 := \lambda r'. \lambda f. \lambda k'. f(\lambda z. \text{append } k' z)r' \);
- \( N_1 := \lambda r'. \lambda f. \lambda r'M_0M_1M_r \), where:
By hypothesis, $n \geq 1$. Then $\overline{n}$ is a non-empty string. Cases of its first character:

- 0 character, i.e. $|n| = 0$ - $r$ is then

$$\overline{n} N_0 N_1 N_c (\lambda z. pred z) k = (\lambda x_0, \lambda x_1, \lambda y, x_0 \overline{r}) N_0 N_1 N_c (\lambda z. pred z) k$$
$$\rightarrow^3_{\beta} N_0 \overline{r} (\lambda z. pred z) k$$
$$= (\lambda r'. \lambda f. \lambda k'. f (\lambda z. append^d k' z) r') \overline{r} (\lambda z. pred z) k$$
$$\rightarrow^3 (\lambda z. pred z) (\lambda z. append^d k' z) \overline{r}$$
$$\rightarrow^2_{\beta} pred (\lambda z. append^d k) \overline{r}$$
$$(i.h.) \rightarrow^3_{\beta} (\lambda z. append^d k) \overline{r} \text{ PRED } \overline{r}$$
$$\rightarrow^2_{\beta} append^d k \overline{r} \text{ PRED } \overline{r}$$
$$(L. A. 2) \rightarrow^3_{\beta} k \overline{r} (\text{ PRED } \overline{r})$$
$$= k \overline{r} \text{ PRED } \overline{r}$$

- 1 character, i.e. $|n| = 1$ - $r$ is $r = \overline{1}$ - Then

$$\overline{n} N_0 N_1 N_c (\lambda z. pred z) k = (\lambda x_0, \lambda x_1, \lambda y, x_0 \overline{r}) N_0 N_1 N_c (\lambda z. pred z) k$$
$$\rightarrow^3_{\beta} N_1 \overline{r} (\lambda z. pred z) k$$
$$= (\lambda r'. \lambda f. \lambda r' f (\lambda z. append^d k' z) r') \overline{r} (\lambda z. pred z) k$$
$$\rightarrow^3 (\lambda z. pred z) (\lambda z. append^d k' z) \overline{r}$$
$$\rightarrow^2_{\beta} \overline{r} (\lambda z. append^d k) \overline{r}$$
$$\rightarrow^2_{\beta} append^d k \overline{r} \text{ PRED } \overline{r}$$
$$(L. A. 2) \rightarrow^3_{\beta} k \overline{r} (\text{ PRED } \overline{r})$$
$$= k \overline{r} \text{ PRED } \overline{r}$$
- r start with 1, that is, \([n] = 1\cdot r = 1\cdot p\). Then:

\[
\begin{align*}
\overline{0\cdot p} M_0M_1M_k &= (\lambda x_0.\lambda x_1.\lambda y.\overline{x_1 p})M_0M_1M_k \\
\rightarrow^3_p M_1\overline{p}k \\
&= (\lambda o.\lambda k.\text{append}^d(\lambda z.\text{append}^d k z) o)\overline{p}k \\
\rightarrow^2_p \text{append}^d(\lambda z.\text{append}^d k z)\overline{p}k \\
\end{align*}
\]

(L. A.2) \(\rightarrow^O_\beta (\lambda z.\text{append}^d k z)\overline{1\cdot p}\)

(L. A.2) \(\rightarrow^O_\beta k\overline{0\cdot 1\cdot p}\)

(\(\rightarrow^O_\beta k\overline{0\cdot r}\))

(\(\rightarrow^O_\beta k\overline{0\cdot \text{pred} [n]}\))

\(\square\)

**Lookup Function.** Given a natural number \(n\), we need to be able to extract the \(n + 1\)-th character from a non-empty string \(s\). The partial function \(\text{lookup}\) can be defined as follows (in Haskell-like syntax):

\[
\begin{align*}
\text{lookup} \ [0] \ (c\cdot s) &= c \\
\text{lookup} \ [n] \ (c\cdot s) &= \text{lookup} (\text{pred} [n]) s \quad \text{if } n > 0
\end{align*}
\]

**Lemma A.5.** There is a \(\lambda\)-term \(\text{lookup}\) such that for every continuation \(k\), every natural number \(n\) and every non-empty string \(i \in \mathbb{B}^+\),

\[
\text{lookup} k[\overline{[n]}i] \rightarrow^{O (n \log n)}_{\text{det}} k[\text{lookup} [n] i].
\]

**Proof:** We can now code the function \(\text{lookup} := \text{fix lookupaux}\) where:

\[
\text{lookupaux} := \lambda f.\lambda k'.\lambda i'.\lambda i'.n'N_0N_1N_kfk'\vec{i}''
\]

where:

- \(N_0 := \lambda p'.\lambda f.\lambda k'.\lambda i'.i'M_0M_0M_pfk'\), where
  - \(M_0 := \lambda r'.\lambda p'.\lambda f.\lambda k'.\text{append}^d(\lambda z''\cdot \text{pred}(\lambda z'.fk'z')z'')p'r';
  - \(M_r\) is whatever closed term.
- \(N_1 := \lambda p'.\lambda f.\lambda k'.\lambda i'.i'M_1M_1M_pfk'\), where
  - \(M_1 := \lambda r'.\lambda p'.\lambda f.\lambda k'.\text{append}^d(\lambda z''.\text{pred}(\lambda z'.fk'z')z'')p'r';
  - \(M_r\) is whatever closed term.
- \(N_k := \lambda f.\lambda k'.\lambda i'.i'O_0O_1O_2O_3\), where
  - \(O_b := \lambda s'.\lambda k'.k'[b];
  - \(O_r\) is whatever closed term.

The first steps of the evaluation of \(\text{lookup} k[\overline{[n]}i]\) are common to all strings \(i \in \mathbb{B}^+\) and natural numbers \(n \in \mathbb{N}\):

\[
\begin{align*}
\text{lookup} k[\overline{[n]}i] &= \text{fix lookupaux} k[\overline{[n]}i] \\
&\rightarrow^2_\beta \text{lookupaux}(\lambda z.\text{lookup} z)k[\overline{[n]}i] \\
&= (\lambda f.\lambda k'.\lambda i'.n'N_0N_1N_kfk'\vec{i}')(\lambda z.\text{lookup} z)k[\overline{[n]}i] \\
&\rightarrow^4_\beta \overline{[n]}N_0N_1N_k(\lambda z.\text{lookup} z)\vec{k}i
\end{align*}
\]

Cases of \(n\):

- \(n = 0\), and so \([n] = e\): then

\[
\begin{align*}
\overline{\tau}N_0N_1N_k(\lambda z.\text{lookup} z)\vec{k}i \\
&= (\lambda x_0.\lambda x_1.\lambda y.\overline{x_1 e})N_0N_1N_k(\lambda z.\text{lookup} z)\vec{k}i \\
&\rightarrow^3_\beta N_i(\lambda z.\text{lookup} z)\vec{k}i \\
&= (\lambda f.\lambda k'.\lambda i'.i'O_0O_1O_2O_3\vec{k}')(\lambda z.\text{lookup} z)\vec{k}i \\
&\rightarrow^3_{\text{det}} \overline{O_0O_1O_2O_3}\vec{k}i
\end{align*}
\]
Let $i$ start with $b \in \mathbb{B}$, that is, $i = b \cdot s$:

\[
\begin{align*}
\overline{b \cdot s} O_0 O_1 O_2 O_3 k &= (\lambda x_0, \lambda x_1, \lambda x_2, x_0 \overline{b}) O_0 O_1 O_2 O_3 k \\
\overset{\beta}{\longrightarrow} (\lambda x_0, \lambda x_1, \lambda x_2, x_0 \overline{b}) O_0 O_1 O_2 O_3 k \\
\overset{\beta}{\longrightarrow} (\lambda x_0, \lambda x_1, \lambda x_2, x_0 \overline{b}) O_0 O_1 O_2 O_3 k \\
\end{align*}
\]

- **Non-empty string starting with 0**, that is, $n > 0$ and $|n| = 0:p$; then

\[
\begin{align*}
|n| N_0 N_1 N_2 (\lambda z \overline{\text{lookup}} z) k T &= (\lambda x_0, \lambda x_1, \lambda x_2, x_0 \overline{b}) N_0 N_1 N_2 (\lambda z \overline{\text{lookup}} z) k T \\
\overset{\beta}{\longrightarrow} N_0 N_1 N_2 (\lambda z \overline{\text{lookup}} z) k T \\
\overset{\beta}{\longrightarrow} (\lambda x_0, \lambda x_1, \lambda x_2, x_0 \overline{b}) N_0 N_1 N_2 (\lambda z \overline{\text{lookup}} z) k T \\
\end{align*}
\]

Let $i$ start with $b \in \mathbb{B}$, that is, $i = b \cdot r$:

\[
\begin{align*}
\overline{b \cdot r} M_0 M_1 M_2 k &= (\lambda x_0, \lambda x_1, \lambda x_2, x_0 \overline{b}) M_0 M_1 M_2 (\lambda z \overline{\text{lookup}} z) k T \\
\overset{\beta}{\longrightarrow} (\lambda x_0, \lambda x_1, \lambda x_2, x_0 \overline{b}) M_0 M_1 M_2 (\lambda z \overline{\text{lookup}} z) k T \\
\overset{\beta}{\longrightarrow} (\lambda x_0, \lambda x_1, \lambda x_2, x_0 \overline{b}) M_0 M_1 M_2 (\lambda z \overline{\text{lookup}} z) k T \\
\end{align*}
\]

- **Non-empty string starting with 1**, that is, $n > 0$ and $|n| = 1:p$; same as the previous one, simply replacing $N_0$ with $N_1$, and thus $M_0$ with $M_1$. In particular, it takes the same number of steps.

\[
\square
\]

### A.3 Encoding Turing Machines

**Turing Machines.** Let $\mathbb{B}_1 := \{0, 1, L, R\}$ and $\mathbb{B}_W := \{0, 1, \square\}$ where $L$ and $R$ delimit the input (binary) string, and $\square$ is our notation for the blank symbol. A deterministic binary Turing machine $M$ with input is a tuple $(Q, q_0, q_T, q_F, \delta)$ consisting of:

- A finite set $Q = \{q_1, \ldots, q_m\}$ of states;
- A distinguished state $q_0 \in Q$, called the initial state;
- Two distinguished states $Q_{fin} := \{q_T, q_F\} \subseteq Q$, called the final states;
- A partial transition function $\delta : \mathbb{B}_1 \times \mathbb{B}_W \times Q \to [-1, +1, 0] \times \mathbb{B}_W \times \{\leftarrow, \rightarrow, \downarrow\} \times Q$ such that $\delta(b, a, q)$ is defined only if $q \notin Q_{fin}$.

A configuration for $M$ is a tuple

\[
(i, n, w_i, a, w_r, q) \in \mathbb{B}_1^* \times \mathbb{N} \times \mathbb{B}_W^* \times \mathbb{B}_W \times \mathbb{B}_W \times Q
\]

where:

- $i$ is the immutable input string and is formed as $i = L \cdot s \cdot R$, $s \in \mathbb{B}^*$;
- $n \in \mathbb{N}$ represents the position of the input head. It is meant to be represented in binary (that is, as an element of $\mathbb{B}^*$), to take space $\log n$, but for ease of reading we keep referring to it as a number rather than as a string;
- $w_i \in \mathbb{B}_W^*$ is the work tape on the left of the work head;
a ∈ ℐ is the element on the cell of the work tape read by the work head;
• w_r ∈ ℐ is the work tape on the right of the work head;
• q ∈ Q is the state of the machine.

For readability, we usually write a configuration \((i, n, w_l, a, w_r, q)\) as \((i, n \mid w_l, a, w_r | q)\), separating the input components, the working components, and the current state.

Given an input string \(i \in ℐ\) (where \(i = L \cdot R\) and \(s \in ℐ^+\)) we define:

- the initial configuration \(C_{in}(i)\) for \(i\) is \(C_{in}(i) = (i, 0 \mid \epsilon, \epsilon \mid q_{in})\).
- the final configuration \(C_{fin} := (s, n \mid w_l, a, w_r | q)\), where \(q \in Q_{fin}\).

For readability, a transition, say, \(δ(i_n, a, q) = (-1, a', \leftarrow \mid q')\), is usually written as \((-1 \mid a', \leftarrow \mid q')\) to stress the three components corresponding to those of configurations (input, work, state).

As in Goldreich, we assume that the machine never scans the input beyond the boundaries of the input. This does not affect space complexity.

An example of transition: if \(δ(i_n, a, q) = (-1 \mid a', \leftarrow \mid q')\), then \(M\) evolves from \(C = (i, n \mid w_a\epsilon', a, w_r | q)\), where the \(n\)th character of \(i\) is \(i_n\), to \(D = (i, n - 1 \mid w_l, a', a'w_r | q')\) and if the tape on the left of the work head is empty, i.e., if \(C = (i, n \mid \epsilon, a, w_r | q)\), then the content of the new head cell is a blank symbol, that is, \(D := (i, n - 1 \mid \epsilon, \epsilon, a'w_r | q')\). The same happens if the tape on the right of the work head is empty. If \(M\) has a transition from \(C\) to \(D\) we write \(C \rightarrow_M D\). A configuration having as state a final state \(q \in Q_{fin}\) is final and cannot evolve.

A Turing machine \((Q, q_{in}, q_T, q_f, δ)\) computes the function \(f : ℐ^+ \rightarrow ℐ\) in time \(T : ℤ \rightarrow ℤ\) and space \(S : ℤ \rightarrow ℤ\) if for every \(i \in ℐ^+\), the initial configuration for \(i\) evolves to a final configuration of state \(q_{f(i)}\) in \(T(|i|)\) steps and using at most \(S(|i|)\) cells on the work tape.

### Encoding configurations.
A configuration \((i, n \mid s, a, r \mid q)\) of a machine \(M = (Q, q_{in}, q_T, q_f, δ)\) is represented by the term

\[
(i, n \mid w_l, a, w_r | q)^M := λx. (x_1^{\#1}[n]^{\#2}w_l^{\#3}w_r^{\#4}[a]^{\#5}w_r^{\#6}w_l^{\#7}[q]^{\#8})
\]

where \(w_i^{\#8}\) is the string \(w_i\) with the elements in reverse order. We shall often rather write

\[
(i, n \mid w_l, a, w_r | q) := λx. (x_1^{\#1}[n]^{\#2}w_l^{\#3}w_r^{\#4}[a]^{\#5}w_r^{\#6}w_l^{\#7}[q]^{\#8}).
\]

Letting the superscripts implicit. To ease the reading, we sometimes use the following notation for tuples \((s, q \mid t, u, r \mid w) := λx. (xsqturw)\), so that \((i, n \mid w_l, a, w_r | q) = (i, [n]^{\#1}w_l^{\#2}w_r^{\#3}[a]^{\#4}w_r^{\#5}w_l^{\#6}[q]^{\#7})\).

### Turning the input string into the initial configuration.
The following lemma provides the term \(init\) that builds the initial configuration.

**Lemma A.6** (Turning the input string into the initial configuration). Let \(M = (Q, q_{in}, q_T, q_f, δ)\) be a Turing machine. There is a term \(init^M\), or simply \(init\), such that for every input string \(i \in ℐ^+\) (where \(i = L \cdot R\) and \(s \in ℐ^+\)):

\[
init \rightarrow_{det} k_{C_{in}(i)}^{θ(i)}
\]

where \(C_{in}(i)\) is the initial configuration of \(M\) for \(i\).

**Proof.** Define

\[
init := (λdλε.λf.λk'.λi'.k'(i',d | ε, [□]^{\#1}w_l^{\#2}w_r^{\#3}[f | [q_{in}]^{\#4})[0]^{\#5}w_l^{\#6}w_r^{\#7}w_k^{\#8})
\]

Please note that the term is not in normal form. This is for technical reasons that will be clear next. Then

\[
init \rightarrow_{det}^k \left\{ \begin{array}{ll}
(λdλε.λf.λk'.λi'.k'(i',d | ε, [□]^{\#1}w_l^{\#2}w_r^{\#3}[f | [q_{in}]^{\#4})[0]^{\#5}w_l^{\#6}w_r^{\#7}w_k^{\#8}) & \text{if } q = q_T \\
(λdλε.λf.λk'.λi'.k'(i',d | ε, [□]^{\#1}w_l^{\#2}w_r^{\#3}[f | [q_{in}]^{\#4})[0]^{\#5}w_l^{\#6}w_r^{\#7}w_k^{\#8}) & \text{if } q = q_F 
\end{array} \right.
\]

### Extracting the output from the final configuration.

**Lemma A.7** (Extracting the output from the final configuration). Let \(M = (Q, q_{in}, q_T, q_f, δ)\) be a Turing machine. There is a term \(final^M\), or simply \(final\), such that for every final configuration \(C\) of state \(q \in Q_{fin}\)

\[
final \rightarrow_{det} k_{C}^{θ(Q)} \left\{ \begin{array}{ll}
k(λx.λy.x) & \text{if } q = q_T \\
k(λx.λy.y) & \text{if } q = q_F
\end{array} \right.
\]



Proof. Define

$$\text{final} = \lambda k'. \lambda C'. C'((\lambda i'. \lambda n'. \lambda w_i'. \lambda a'. \lambda w_i'. \lambda q'. q' N_1 \ldots N_{|Q|} k')$$

where:

$$N_i := \begin{cases} 
\lambda k'. k'((\lambda x. \lambda y. x)) & \text{if } q_i = q_T \\
\lambda k'. k((\lambda x. \lambda y. y)) & \text{if } q_i = q_F \\
\text{whatever closed term (say, the identity)} & \text{otherwise}
\end{cases}$$

Then:

$$\text{final} k C = (\lambda k'. \lambda C'. C'((\lambda i'. \lambda n'. \lambda w_i'. \lambda a'. \lambda w_i'. \lambda q'. q' N_1 \ldots N_{|Q|} k')) k C \rightarrow^{\delta}_{\text{det}}$$

$$= (i, n \mid w_i, a, w_r \mid q)(\lambda i'. \lambda n'. \lambda w_i'. \lambda a'. \lambda w_i'. \lambda q'. q' N_1 \ldots N_{|Q|} k)$$

$$= (\lambda x. xT^{n}_{\text{finite}}[n] w_i^{\text{finite}} a^{\text{finite}} w_r^{\text{finite}} [q][Q])(\lambda i'. \lambda n'. \lambda w_i'. \lambda a'. \lambda w_i'. \lambda q'. q' N_1 \ldots N_{|Q|} k)$$

$$\rightarrow^{\delta}_{\text{det}}$$

$$= (\lambda x_1 \ldots x_{|Q|}, x_{|Q|}) N_1 \ldots N_{|Q|} k$$

$$\rightarrow^{Q}_{\text{det}}$$

$$= N_j k$$

If \( q = q_T \), then:

$$N_j k = (\lambda k'. k'((\lambda x. \lambda y. x))) k \rightarrow^{\delta}_{\text{det}} k(\lambda x. \lambda y. x)$$

If \( q = q_F \), then:

$$N_j k = (\lambda k'. k((\lambda x. \lambda y. y))) k \rightarrow^{\delta}_{\text{det}} k(\lambda x. \lambda y. y)$$

\( \square \)

Simulation of a machine transition. Now we show how to encode the transition function \( \delta \) of a Turing machine as a \( \lambda \)-term in such a way to simulate every single transition in constant? time. This is the heart of the encoding, and the most involved proof.

Lemma A.8 (Simulation of a machine transition). Let \( M = (Q, q_0, \delta) \) be a Turing machine. There is a term \( \text{trans}^M \), or simply \( \text{trans} \), such that for every configuration \( C \) of input string \( i \in \mathbb{B}^* \):

- Final configuration: if \( C \) is a final configuration then \( \text{trans} k C \rightarrow^{O(|i| \log |i|)}_{\text{det}} k C \);
- Non-final configuration: if \( C \rightarrow^M k \) then \( \text{trans} k C \rightarrow^{O(|i| \log |i|)}_{\text{det}} k D \).

Proof. The transition function \( \delta(b, a, q) \) is a 3-dimensional table having for coordinates:

- the current bit \( b \) on the input tape, which is actually retrieved from the input tape \( i \) and the counter \( n \) of the current input position,
- the current character \( a \) on the work tape, and
- the current state \( q \).

The transition function is encoded as a recursive \( \lambda \)-term \( \text{trans} \) taking as argument the encodings of \( i \), and \( n \) to retrieve \( b \) and \( a \). It works as follows:

- It first retrieves \( b \) from \( i \) and \( n \) by applying the lookup function;
- It has a subterm \( A_b \) for the four values of \( b \). The right sub-term is selected by applying the encoding \( [b] \) of \( b \) to \( A_0, A_1, A_L \) and \( A_R \);
- Each \( A_b \) in turn has a sub-term \( B_{b,a} \) for every character \( a \in \mathbb{B}_w \), corresponding to the working tape coordinates. The right sub-term is selected by applying the encoding \( [a] \) of the current character \( a \) on the work tape to \( B_{b,0}, B_{b,1}, B_{b,2} \);
- Each \( B_{b,a} \) in turn has a subterm \( C_{b,a,q} \) for every character \( q \) in \( Q \). The right sub-term is selected by applying the encoding \( [q] \) of the current state \( q \) to \( C_{b,a,q}, \ldots, B_{b,a,q} \);

- The subterm \( C_{b,a,q} \) produces the encoding (of the) next configuration according to the transition function \( \delta \). If \( \delta \) decreases (resp. increases) the counter for the input tape then \( C_{b,a,q} \) applies \( \text{pred} \) (resp. \( \text{succ} \)) to the input counter and then applies a term corresponding to the required action on the work tape, namely:
  - \( S \) (for stay) if the head does not move. This case is easy, \( S \) simply produces the next configuration.
At the level of the number of steps, the main cost is paid at the beginning, by the
subterm $L_w^a$ for each $a'' \in \mathbb{B}_w$ the task of which is to add $a'$ to the right part of the work tape, remove $a''$ from the
left part of the work tape (which becomes $w$), and make $a''$ the character in the work head position.

- $R$ if it moves right. Its structure is similar to the one of $L$.

In order to be as modular as possible we use the definition of $S$, $L$, and $R$ for the cases when the input head moves also
for the cases where it does not move, even if this requires a useless (but harmless) additional update of the counter $n$.

Define

$$\text{transaux} := \lambda x.\lambda k'.\lambda C'.\mathcal{C}'(\lambda i'.\lambda n'.\lambda w'_f,\lambda a'.\lambda w'_r,\lambda q'.\lambda \text{lookup} K' i' n')$$

$$\text{trans} := \text{fix transaux},$$

where:

$$K := \lambda b'.b' A_0 A_1 A_2 A R a' q' x k' i' n' w'_f w'_r$$
$$A_b := \lambda a'.a' B_{b,0} B_{b,1} B_{b,\square}$$
$$B_{b,a} := \lambda q'.q' C_{b,a,q} \ldots C_{b,a,q(\square)}$$

$$\begin{align*}
C_{b,a,q} &:= \lambda x.\lambda k'.\lambda i'.\lambda n'.\lambda w'_f,\lambda w'_r,\lambda q' \{ \\
&k'\langle i',n' | w'_f, [a'],w'_r | [q'] \rangle \quad \text{if } q \in Q_{\text{fin}} \\
&\quad S_n' \quad \text{if } \delta(b,a,q) = (0 | a', \downarrow \downarrow | q') \\
&\quad L_n' \quad \text{if } \delta(b,a,q) = (0 | a', \leftarrow | q') \\
&\quad R_n' \quad \text{if } \delta(b,a,q) = (0 | a', \rightarrow | q') \\
&\quad \text{predSn}' \quad \text{if } \delta(b,a,q) = (-1 | a', \downarrow \downarrow | q') \\
&\quad \text{predLn}' \quad \text{if } \delta(b,a,q) = (-1 | a', \leftarrow | q') \\
&\quad \text{predRn}' \quad \text{if } \delta(b,a,q) = (-1 | a', \rightarrow | q') \\
&\quad \text{succSn}' \quad \text{if } \delta(b,a,q) = (+1 | a', \downarrow \downarrow | q') \\
&\quad \text{succLn}' \quad \text{if } \delta(b,a,q) = (+1 | a', \leftarrow | q') \\
&\quad \text{succRn}' \quad \text{if } \delta(b,a,q) = (+1 | a', \rightarrow | q') \\
\end{align*}$$

$$S := \lambda n'' . x k' \langle i',n'' | w'_f, [a'],w'_r | [q'] \rangle$$
$$L := \lambda n'' , w'_f t'_q R'_a R'_q t''_q R'_\square t''_\square L'_{q,a} k' i' n'' w'_r$$
$$R := \lambda n'' , w'_f R'_a R'_q R'_\square t''_q R''_a k' i' n'' w'_r$$

$$L_{q,a} := \lambda w'_f,\lambda x,\lambda k'.\lambda i'.\lambda n'.\text{append}\text{^d} (\lambda w'_f . x k' \langle i',n' | w'_f, [a'],w'_r | [q'] \rangle)$$
$$R_{q,a} := \lambda x,\lambda k'.\lambda i'.\lambda n'.\text{append}\text{^d}((\lambda d . \lambda w'_f . x k' \langle i',n' | d, [\square],w'_r | [q'] \rangle) \overrightarrow{\mathcal{E}})$$
$$L_{q,\square} := \lambda w'_f,\lambda x,\lambda k'.\lambda i'.\lambda n'.\text{append}\text{^d} (\lambda w'_f . x k' \langle i',n' | w'_f, [a'],w'_r | [q'] \rangle)$$
$$R_{q,\square} := \lambda x,\lambda k'.\lambda i'.\lambda n'.\text{append}\text{^d}((\lambda d . \lambda w'_f . x k' \langle i',n' | w'_f, [\square],d | [q'] \rangle) \overrightarrow{\mathcal{E}})$$

Let $C = (i,n | w_l,a,w_r | q)$. We are now going to show the details of how the $\lambda$-calculus simulates the transition function.

At the level of the number of steps, the main cost is paid at the beginning, by the lookup function that looks up the $n$-th character of the input string $i$. The cost of one such call is $O(n \log n)$, but since $n$ can vary and $n \leq |i|$, such a cost is bound by $O(|i| \log |i|)$. The cases of transition where the position on the input tape does not change have a constant cost. Those where
the input position changes require to change the counter $n$ via pred or succ, which requires $O(\log n)$, itself bound by the
cost $O(|i| \log |i|)$ of the previous look-up.

Now, if $\text{lookup}[n]i = b$ then:
trans $kC = \text{fix transaux}kC$

$\downarrow^2_{det}$
transaux($\lambda z.\text{fix transauxz})kC$

$\downarrow^3_{det}$
$\lambda x.\lambda k'.\lambda C'.(\lambda i'.\lambda n'.\lambda w_i'.\lambda a'.\lambda w_i'.\lambda q'.\text{lookup }Ki'n')(\lambda z.\text{trans }z)kC$

$\lambda x.\lambda k'.\lambda C'.(\lambda i'.\lambda n'.\lambda w_i'.\lambda a'.\lambda w_i'.\lambda q'.\text{lookup }Ki'n')$

$\downarrow^5_{det}$
$\text{lookup }Ki'n'\text{ if }i'\neq i$

$\downarrow^6_{det}$
$\lambda b'.b''A_0A_1A_2A_R[a][q](\lambda z.\text{trans }z)k\bar{T}i'n'w_i'w_i'w_{r'}[a]w_r[q]$

$L. A.5$

Now, consider the following four cases, depending on the value of $\delta(b, a, q)$:

1. **Final state**: if $\delta(b, a, q)$ is undefined, then $q \in Q_{fin}$ and replacing $C_{b,a,q}$ with the corresponding $\lambda$-term we obtain:

$$C_{b,a,q}(\lambda z.\text{trans }z)k\bar{T}i'n'w_i'w_i'w_{r'}$$

2. **The heads do not move**: if $\delta(b, a, q) = (0 | a', \downarrow | q')$, then $D = (i, n | w_i, a', w_r, q')$. The simulation continues as follows:

$$C_{b,a,q}(\lambda z.\text{trans }z)k\bar{T}i'n'w_i'w_i'w_{r'}$$

3. **The input head does not move and the work head moves left**: if $\delta(b, a, q) = (0 | a', \leftarrow | q')$ and $w_I = wa''$ then:

$$C_{b,a,q}(\lambda z.\text{trans }z)k\bar{T}i'n'w_i'w_i'w_{r'}$$

Two sub-cases, depending on whether $w_I$ is an empty or a compound string.

a. $w_I$ is the compound string $wa''$. Then $w_I^a = a''w_a$ and $D = (i, n | w_I, a'', a'w_r | q')$. The simulation continues as follows:
\[ = \overline{\alpha \overline{w_1} t_{\overline{q} \overline{a} \overline{d}} t_{\overline{q} \overline{a} \overline{d}} t_{\overline{q} \overline{a} \overline{d}} t_{\overline{q} \overline{a} \overline{d}} (\lambda z. \text{trans} z) k\overline{[n]} w_r} \]
\[ \rightarrow^4_{\text{det}} t_{\overline{q} \overline{a} \overline{d}} w_{\overline{w}(\lambda z. \text{trans} z) k\overline{[n]} w_r} \]
\[ = (\lambda w_{\overline{q} \overline{a} \overline{d}} x_{\overline{a} \overline{d}} (i', n' | w_{j'}, \overline{a''}, w_r | [q'])) w_{\overline{w}(\lambda z. \text{trans} z) k\overline{[n]} w_r} \]
\[ \rightarrow^5_{\text{det}} (\lambda w_{\overline{q} \overline{a} \overline{d}} (\lambda z. \text{trans} z) k\overline{[n]} w_r [a''', w_r | [q']) a w_r \]
\[ = \text{trans} k\overline{[n]} w_r a w_r | [q') \]
\[ = \text{trans} k \overline{D} \]

b. \( w_1 \) is the empty string \( \varepsilon \). Then \( D = (i, n | \varepsilon, \overline{a} w_r | [q']) \). The simulation continues as follows:
\[ = \overline{\tau r_{\overline{q} \overline{a} \overline{d}} \overline{q} \overline{a} \overline{d}} \overline{\tau r_{\overline{q} \overline{a} \overline{d}} \overline{q} \overline{a} \overline{d}} \overline{\tau r_{\overline{q} \overline{a} \overline{d}} \overline{q} \overline{a} \overline{d}} (\lambda \overline{z. \text{trans} z) k\overline{[n]} w_r} \]
\[ \rightarrow^4_{\text{det}} t_{\overline{q} \overline{a} \overline{d}} w_{\overline{w}(\lambda \overline{z. \text{trans} z) k\overline{[n]} w_r}} \]
\[ = (\lambda x_{\overline{a} \overline{d}} \lambda \overline{x'} \lambda \overline{n'}. \overline{\text{append}^d} ((\lambda \overline{d. \lambda w_{\overline{q} \overline{a} \overline{d}} x_{\overline{a} \overline{d}} (i', n' | d, [\square], w_r | [q']) \overline{\tau}) (\lambda z. \text{trans} z) k\overline{[n]} w_r} \]
\[ \rightarrow^5_{\text{det}} (\lambda \overline{d. \lambda w_{\overline{q} \overline{a} \overline{d}} (\lambda z. \text{trans} z) k\overline{[n]} w_r [a''', w_r | [q']) a w_r \]
\[ = \text{trans} k\overline{[n]} w_r a w_r | [q') \]
\[ = \text{trans} k \overline{D} \]

4. The input head does not move and the work head moves right: if \( \delta(b, a, q) = (0 | a', \overline{a''} | q') \) and \( w_r = a'' w \) then:
\[ C_{b, a, q}(\lambda \overline{z. \text{trans} z) k\overline{[n]} w_r} \]
\[ = (\lambda x_{\overline{a} \overline{d}} \lambda \overline{x'} \lambda \overline{n'}. \overline{\text{append}^d} ((\lambda \overline{d. \lambda w_{\overline{q} \overline{a} \overline{d}} x_{\overline{a} \overline{d}} (i', n' | d, [\square], w_r | [q']) \overline{\tau}) (\lambda z. \text{trans} z) k\overline{[n]} w_r) \]
\[ \rightarrow^6_{\text{det}} (\lambda \overline{\text{append}^d} ((\lambda \overline{d. \lambda w_{\overline{q} \overline{a} \overline{d}} x_{\overline{a} \overline{d}} (i', n' | d, [\square], w_r | [q']) \overline{\tau}) (\lambda z. \text{trans} z) k\overline{[n]} w_r) \]
\[ \rightarrow^\overline{\text{det}}_\overline{\alpha \overline{w_1} t_{\overline{q} \overline{a} \overline{d}} t_{\overline{q} \overline{a} \overline{d}} t_{\overline{q} \overline{a} \overline{d}} t_{\overline{q} \overline{a} \overline{d}} (\lambda z. \text{trans} z) k\overline{[n]} w_r) \]

Two sub-cases, depending on whether \( w_1 \) is an empty or a compound string.

a. \( w_1 \) is the compound string \( a'' \). Then \( D = (i, n | w_{a''}, a'', | q') \). The simulation continues as follows:
\[ = \overline{a'' w_{\overline{w}(\lambda z. \text{trans} z) k\overline{[n]} w_r} \]
\[ \rightarrow^4_{\text{det}} t_{\overline{q} \overline{a} \overline{d}} w_{\overline{w}(\lambda z. \text{trans} z) k\overline{[n]} w_r} \]
\[ = (\lambda \overline{\text{append}^d} ((\lambda \overline{d. \lambda w_{\overline{q} \overline{a} \overline{d}} x_{\overline{a} \overline{d}} (i', n' | w_{j'}, \overline{a''}, w_r | [q']) \overline{\tau}) (\lambda z. \text{trans} z) k\overline{[n]} w_r) \]
\[ \rightarrow^5_{\text{det}} (\lambda \overline{\text{append}^d} ((\lambda \overline{d. \lambda w_{\overline{q} \overline{a} \overline{d}} x_{\overline{a} \overline{d}} (i', n' | d, [\square], w_r | [q']) \overline{\tau}) (\lambda z. \text{trans} z) k\overline{[n]} w_r) \]
\[ = \text{trans} k\overline{[n]} w_r a w_r | [q') \]
\[ = \text{trans} k \overline{D} \]

b. \( w_1 \) is the empty string \( \varepsilon \). Then \( D = (i, n | w_{a''}, \overline{\varepsilon} | q') \). The simulation continues as follows:
\[ = \overline{\tau r_{\overline{q} \overline{a} \overline{d}} \overline{\tau r_{\overline{q} \overline{a} \overline{d}} \overline{\tau r_{\overline{q} \overline{a} \overline{d}} \overline{\tau r_{\overline{q} \overline{a} \overline{d}}}} (\lambda z. \text{trans} z) k\overline{[n]} w_r} \]
\[ \rightarrow^4_{\text{det}} t_{\overline{q} \overline{a} \overline{d}} w_{\overline{w}(\lambda z. \text{trans} z) k\overline{[n]} w_r} \]
\[ = (\lambda x_{\overline{a} \overline{d}} \lambda \overline{x'} \lambda \overline{n'}. \overline{\text{append}^d} ((\lambda \overline{d. \lambda w_{\overline{q} \overline{a} \overline{d}} x_{\overline{a} \overline{d}} (i', n' | d, [\square], w_r | [q']) \overline{\tau}) (\lambda z. \text{trans} z) k\overline{[n]} w_r) \]
\[ \rightarrow^5_{\text{det}} (\lambda \overline{\text{append}^d} ((\lambda \overline{d. \lambda w_{\overline{q} \overline{a} \overline{d}} x_{\overline{a} \overline{d}} (i', n' | d, [\square], w_r | [q']) \overline{\tau}) (\lambda z. \text{trans} z) k\overline{[n]} w_r) \]
\[ = \text{trans} k\overline{[n]} w_r a w_r | [q') \]
\[ = \text{trans} k \overline{D} \]

5. The input head moves left and the work head does not move: if \( \delta(b, a, q) = (-1 | a', \downarrow | q') \), then \( D = (i, n - 1 | w_r, a', w_r, q') \). The simulation continues as follows:
A.4 instead of Lemma A.8

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10. Straightforward inductions on the length of executions provide the following corollaries.

9. Diverging computation: if $D$ is a final configuration reachable from $C$ in $n$ transition steps then there exists a derivation $\rho$ such that $\rho : \text{trans } k[\overline{C}] \vdash_{\det}^{O(n+|\overline{C}| \log |\overline{C}|)} k[D]$.

8. The input head moves right and the work head does not move: exactly as case 5 (input head left, work head does not move) just replacing $\text{pred}$ with $\text{succ}$ and using Lemma A.3 instead of Lemma A.4.

7. The input head moves left and the work head moves right: if $\delta(b, a, q) = (-1, a', \leftarrow, \overline{q'})$ and $w_l = wa''$, then $D = (i, n - 1 | w, a'', a'w_r, q')$. The simulation continues as follows:

6. The input head moves left and the work head moves left: if $\delta(b, a, q) = (-1, a', \leftarrow, \overline{q'})$ and $w_l = wa''$, then $D = (i, n - 1 | w, a', a'w_r, q')$. The simulation continues as follows:

5. Diverging computation: if there is no final configuration reachable from $C$ then $\text{trans } k[\overline{C}]$ diverges.

4. The simulation theorem. We now have all the ingredients for the final theorem of this note.

3. The simulation theorem. Let $f : \mathbb{B}^* \rightarrow \mathbb{B}$ a function computed by a Turing machine $M$ in time $T_M$. Then there is an encoding $\lfloor - \rceil$ into $\lambda_{\text{det}}$ of $\mathbb{B}$, strings, and Turing machines over $\mathbb{B}$ such that for every $i \in \mathbb{B}^+$, there exists $\rho$ such that $\rho : \lfloor M[i] \rceil \rightarrow_{\det}^n \lfloor f(i) \rceil$ where $n = \Theta((T_M(|i|) + 1) \cdot |i| \cdot \log |i|)$. 

Proof. Morally, the term is simply

$$\overline{M} := \text{init}(\text{trans}(\text{final}(\lambda w. w)))$$

Corollary A.9 (Executions). Let $M$ be a Turing machine. Then there exist a term $\text{trans}$ encoding $M$ as given by Lemma A.8 such that for every configuration $C$ of input string $i \in \mathbb{B}^+$.

1. Finite computation: if $D$ is a final configuration reachable from $C$ in $n$ transition steps then there exists a derivation $\rho$ such that $\rho : \text{trans } k[\overline{C}] \vdash_{\det}^{O(|\overline{C}| \log |\overline{C}|)} k[D]$.

2. Diverging computation: if there is no final configuration reachable from $C$ then $\text{trans } k[\overline{C}]$ diverges.

The simulation theorem. We now have all the ingredients for the final theorem of this note.

Theorem A.10 (Simulation). Let $f : \mathbb{B}^* \rightarrow \mathbb{B}$ a function computed by a Turing machine $M$ in time $T_M$. Then there is an encoding $\lfloor - \rceil$ into $\lambda_{\text{det}}$ of $\mathbb{B}$, strings, and Turing machines over $\mathbb{B}$ such that for every $i \in \mathbb{B}^+$, there exists $\rho$ such that $\rho : \lfloor M[i] \rceil \rightarrow_{\det}^n \lfloor f(i) \rceil$ where $n = \Theta((T_M(|i|) + 1) \cdot |i| \cdot \log |i|)$.
where $\lambda w.w$ plays the role of the initial continuation.

Such a term however does not belong to the deterministic $\lambda$-calculus, because the right subterms of applications are not always values. The solution is simple, it is enough to $\eta$-expand the arguments. Thus, define

$$\overline{M} := \text{init}(\lambda y.\text{trans}(\lambda x.\text{final}(\lambda w.w)x)y)$$

Then

$$\overline{M}[i] = \text{init}(\lambda y.\text{trans}(\lambda x.\text{final}(\lambda w.w)x)y)[i] \rightarrow_{\text{det}}^{\Theta(1)} (\lambda y.\text{trans}(\lambda x.\text{final}(\lambda w.w)x)y)[C_{\text{trans}}(M)]$$

(by L. A.6)

$$\text{trans}(\lambda x.\text{final}(\lambda w.w)x)[C_{\text{final}}(M)] \rightarrow_{\text{det}}^{\Theta(\text{I}_M[0]+1\cdot|\cdot\text{log}|i|)}$$

(by Cor. A.9)

$$\text{final}(\lambda w.w)[C_{\text{final}}(f(i))] \rightarrow_{\text{det}}^{\Theta(|\cdot|)}$$

(by L. A.7)

\[ \square \]

## B Proofs of Section 6

In the following we will often use the execution of the fix-point combinator. For this reason, we encapsulate its execution by the Space KAM in a lemma.

**Lemma B.1.** For each term $u$, $(\theta, e, (\theta, e)\cdot(u, e)\cdot\pi) \rightarrow_{\text{SpKAM}}^{\Omega(1)} (u, e, \text{fix}^k\cdot\pi)$ where $\text{fix}^k := (x y, y \rightarrow (u, e)\cdot[x \rightarrow (\theta, e)])$ consuming space $O(|e| + |\pi| + \log(|u|))$.

**Proof.**

| Term | Env | Stack |
|------|-----|-------|
| $\theta := \lambda x.\lambda y.x y$ | $\epsilon$ | $(\theta, e)\cdot(u, e)\cdot\pi$ |
| $\lambda y.y$ | $[x \rightarrow (\theta, e)]$ | $(\theta, e)\cdot(u, e)\cdot\pi$ |
| $y$ | $[y \rightarrow (u, e)]\cdot[x \rightarrow (\theta, e)]$ | $\text{fix}^k\cdot\pi$ |
| $u$ | $e$ | $(x y, y \rightarrow (u, e)\cdot[x \rightarrow (\theta, e)])\cdot\pi$ |

\[ \square \]

We prove the following propositions in a top-down style, i.e. required lemmata are below. This is done because otherwise lemmata statements would seem quite arbitrary.

**Proposition B.2.** Let $s \in \mathbb{B}^*$ and $\text{toy} := \text{fix}(\lambda f.\lambda z.z f f l)$.

1. $\text{toy} \rightarrow_{\text{wh}}^{\Theta(|s|)} 1$.
2. The Naive KAM evaluates $\overline{\text{toy}}$ in space $\Omega(2^{|s|})$.
3. The Space KAM evaluates $\overline{\text{toy}}$ in space $\Theta(\text{log}(|s|))$.

**Proof.**

1. This point follows from the implementation theorem, applied to the sequence of point 3.
2. We prove the statement executing $\overline{\text{toy}}$ with the Naive KAM. We set $\text{toyaux} := \lambda f.\lambda z.z f f l$.

| Term | Env | Stack |
|------|-----|-------|
| $\text{toy}$ | $\epsilon$ | $\epsilon$ |
| $\text{toy} := \text{fix} \text{toyaux}$ | $\epsilon$ | $(\epsilon, \epsilon)$ |
| $\text{fix} := \theta \theta$ | $\epsilon$ | $(\text{toyaux}, \epsilon)\cdot(\epsilon, \epsilon)$ |
| $\theta := \lambda x.\lambda y.y (x y)$ | $[x \rightarrow (\theta, e)]\cdot[y \rightarrow (\text{toyaux}, \epsilon)] := e_0$ | $(\epsilon, \epsilon)$ |
| $y (x y)$ | $\epsilon$ | $e'$ |
| $l$ | $e'_{|s|}$ | $\epsilon$ |

The space bound is proved since $e'_{|s|}$ is exponential in $|s|$.
3. We prove the statement executing toy \( \mathfrak{S} \) with the Space KAM. We set \( \text{toyaux} := \lambda f. \lambda z. z f f 1 \).

| Term | Env | Stack |
|------|-----|-------|
| \( \text{toy} \mathfrak{S} \) | \( \epsilon \) | \( \epsilon \) |
| toy := fix toyaux | \( \epsilon \) | \( (\mathfrak{S}, \epsilon) \) |
| fix := \( \theta \theta \) | \( \epsilon \) | \( (\text{toyaux}, \epsilon) \cdot (\mathfrak{S}, \epsilon) \) |
| \( \theta := \lambda x. \lambda y. y(xxy) \) | \( \epsilon \) | \( (\theta, \epsilon) \cdot (\text{toyaux}, \epsilon) \cdot (\mathfrak{S}, \epsilon) \) |
| \( y(xxy) \) | \([x \mapsto (\theta, \epsilon)] \cdot [y \mapsto (\text{toyaux}, \epsilon)] =: e_0 \) | \( (\mathfrak{S}, \epsilon) \) |
| l | \( \epsilon \) | \( \epsilon \) |

The space bound is proved considering the bound in Lemma B.4.

Let \( s := b_1 \ldots b_n \cdot \epsilon \) be a string of length \( n \geq 0 \). Then, we can define \( e_i, e'_i, e''_i \) for \( 0 \leq i \leq n \) as follows:

\[
e_0 := [x \mapsto (\theta, \epsilon)] \cdot [y \mapsto (\text{toyaux}, \epsilon)] \quad e_{i+1} := [x \mapsto (x, e_i)] \cdot [y \mapsto (y, e_i)]
\]

\[
e'_i := [z \mapsto (b_{i+1} \ldots b_n \cdot \epsilon, e''_i)] \cdot [f \mapsto (xxy, e_i)] \quad e''_i := \epsilon
\]

One can easily notice that the sizes of \( e_i, e'_i, e''_i \) are exponential in \( i \).

**Lemma B.3.** \( (y(xxy), e_1, (\mathfrak{S}, e''_n)) \rightarrow_{\text{NaKAM}^{O(|s|)}} (1, e'_{|s|}, \epsilon) \)

**Proof.** By induction on the structure of \( s \).

**Case** \( s = \epsilon \).

| Term | Env | Stack |
|------|-----|-------|
| \( y(xxy) \) | \( e'_i \) | \( (\mathfrak{S}, e''_n) \) |
| \( y \) | \( \epsilon \) | \( (\text{xyy}, e_i) \cdot (\mathfrak{S}, e''_n) \) unfolding \( e_i \) |
| \( \text{toyaux} := \lambda f. \lambda z. z f f 1 \) | \( e_i \) | \( (\text{xyy}, e_i) \cdot (\mathfrak{S}, e''_n) \) |
| \( z f f 1 \) | \( [z \mapsto (\mathfrak{S}, e''_n)] \cdot [f \mapsto (xxy, e_i)] =: e'_i \) | \( (f, e''_n) \cdot (f, e'_i) \cdot (l, e') \) |
| \( z \) | \( [z \mapsto (\mathfrak{S}, e''_n)] \cdot [f \mapsto (xxy, e_i)] \) | \( (f, e'_i) \cdot (f, e'_i) \cdot (l, e') \) |
| \( \mathfrak{S} \) | \( e'_i \) | \( (f, e'_i) \cdot (f, e'_i) \cdot (l, e') \) |

**Case** \( s = b \cdot r \).

| Term | Env | Stack |
|------|-----|-------|
| \( \mathfrak{S} := \lambda x_0. \lambda x_1. \lambda x_r. x_r \) | \( e''_i \) | \( (f, e''_n) \cdot (f, e'_i) \cdot (l, e') \) |
| \( x_r \) | \( [x_r \mapsto (l, e'_i)] \cdot [x_1 \mapsto (f, e'_i)] \cdot [x_0 \mapsto (f, e'_i)] \cdot [x''_i] =: e''_{i+1} \) | \( \epsilon \) |
| \( l \) | \( e'_i \) | \( \epsilon \) |

**Lemma B.4.** The Space KAM executes the reduction \( (y(xxy), e_0, (\mathfrak{S}, \epsilon)) \rightarrow_{\text{SpKAM}^{O(|s|)}} (1, \epsilon, \epsilon) \) consuming \( O(\log(|s|)) \) space.
Proof. By induction on the structure of \( s \).

\[
\begin{array}{|c|c|c|}
\hline
\text{Term} & \text{Env} & \text{Stack} \\
\hline
y(xxy) & e_0 & (s, e) \\
y & [y \leftarrow (\text{toyaux}, e)] & (xxy, e_0) \cdot (s, e) \\
\text{toyaux} := \lambda f \cdot \lambda x. zf f l & e & (xxy, e_0) \cdot (s, e) \\
zf f l & [z \leftarrow (s, e)] \cdot [f \leftarrow (xxy, e_0)] & (xxy, e_0) \cdot (s, e) \\
z & e & (xxy, e_0) \cdot (xxy, e_0) \cdot (1, e) \\
s & & (xxy, e_0) \cdot (xxy, e_0) \cdot (1, e) \\
\hline
\end{array}
\]

Case \( s = \varepsilon \).

\[
\begin{array}{|c|c|c|}
\hline
\text{Term} & \text{Env} & \text{Stack} \\
\hline
\overline{s} := \lambda x_0, \lambda x_1, \lambda x_2. x_2 & e & (xxy, e_0) \cdot (xxy, e_0) \cdot (1, e) \\
x_2 & [x_2 \leftarrow (1, e)] & e \\
l & e & e \\
\hline
\end{array}
\]

The space bound is proved since there is a static bound (8) on the number of closures during the execution. \( \square \)

**Proposition B.5.** Let \( s \in \mathcal{B}^* \) and \( \text{glCpy} := \lambda z. (\text{fix} (\lambda f \cdot \lambda s'. s' f f z)) z \).

1. \( \text{glCpy} \overline{s^0} \xrightarrow[\text{w.h.}]{\Theta(|s|)} \overline{s^0} \).
2. The space used by the Space KAM to simulate the evaluation of the previous point is \( \Theta(\log |s|) \).

**Proof.** We prove the second point of the statement by directly executing the Space KAM. The first point is then obtained as a corollary using the complexity and correctness properties of the Space KAM. Since \( \mathcal{B} \) is the only alphabet that we are using, we remove all the superscripts. Let us define \( t := \lambda f \cdot \lambda s'. s' f f z \).

\[
\begin{array}{|c|c|c|}
\hline
\text{Term} & \text{Environment} & \text{Stack} \\
\hline
\text{glCpy} \overline{s} & e & e \rightarrow \\
\text{glCpy} \cdot \lambda z. (\text{fix} \ t z) & e & (s, e) := s^k \rightarrow \\
\text{fix} \ t z & [z \leftarrow s^k] & e \rightarrow \\
\text{fix} \ t & [z \leftarrow s^k] & s^k \rightarrow \\
\text{fix} := \theta \theta & e & (t, [z \leftarrow s^k]) \cdot s^k \rightarrow \\
\theta := \lambda x. \lambda y. y(xxy) & e & (\theta, e) \cdot (t, [z \leftarrow s^k]) \cdot s^k \rightarrow \Theta(|s|) \quad \text{(Lemma B.6)} \\
\overline{s} & e & e \rightarrow \\
\hline
\end{array}
\]

\( \square \)

**Lemma B.6.** Let \( s \in \mathcal{B}^* \) and \( t := \lambda f \cdot \lambda s'. s' f f z \). Then \( (\theta, e, (\theta, e) \cdot (t, [z \leftarrow (u, e)]) \cdot s^k) \rightarrow_{\text{SpKAM}}^{\Theta(|s|) \cdot (u, e)} \) and moreover the space used is \( O(|e| + \log |s| + \log |u|) \).

**Proof.** We proceed by induction on the structure of \( s \). The first steps are common to both the base case and the induction step. We define \( \text{fix}^k := (xxy, [y \leftarrow (t, [z \leftarrow (u, e)])]) \cdot [x \leftarrow (\theta, e)] \).

\[
\begin{array}{|c|c|c|}
\hline
\text{Term} & \text{Environment} & \text{Stack} \\
\hline
\theta := \lambda x. \lambda y. y(xxy) & e & (\theta, e) \cdot (t, [z \leftarrow (u, e)]) \cdot s^k \rightarrow \quad \text{(Lemma B.1)} \\
t := \lambda f \cdot \lambda s'. s' f f z & [z \leftarrow (u, e)] & \text{fix}^k \cdot s^k \rightarrow^2 \\
s' f f z & [f \leftarrow \text{fix}^k] \cdot [s' \leftarrow s^k] \cdot [z \leftarrow (u, e)] & \epsilon \rightarrow^3 \\
s & [s' \leftarrow s^k] & \text{fix}^k \cdot \text{fix}^k \cdot (u, e) \rightarrow \\
\overline{s} & e & \text{fix}^k \cdot \text{fix}^k \cdot (u, e) \rightarrow \\
\hline
\end{array}
\]
Base case: \( s = \varepsilon \).

\[
\begin{array}{ccc}
\text{Term} & \text{Environment} & \text{Stack} \\
\overline{s} & \lambda x_0, x_1, x_2, x_3 & \varepsilon \\
u & \varepsilon & \text{fix}^k \cdot \text{fix}^k \cdot (u, e) \\
\end{array}
\]

\( \overline{s} \rightarrow^4 \)

Inductive case: \( s = b \cdot r \) where \( b \in \{0, 1\} \).

\[
\begin{array}{ccc}
\text{Term} & \text{Environment} & \text{Stack} \\
x_0 F & \{ x_b \leftarrow \text{fix}^k \} & \text{fix}^k \cdot \text{fix}^k \cdot (u, e) \\
x_b & \{ x_b \leftarrow \text{fix}^k \} & \varepsilon \\
xx y & \{ y \leftarrow (t, [z \leftarrow (u, e)]) \} \cdot \{ x \leftarrow (\theta, e) \} & \varepsilon \\
x & \{ x \leftarrow (\theta, e) \} & \text{fix}^k \\
\theta & \varepsilon & \text{fix}^k \cdot (\theta, e) \cdot (t, [z \leftarrow (u, e)]) \cdot \text{fix}^k \\
u & \varepsilon & \text{fix}^k \cdot \text{fix}^k \cdot (u, e) \\
\end{array}
\]

About space, it is immediate to see that all computations are constrained in space \( O(|e| + \log |s| + \log |u|) \), since at any point during the computation there is a bounded number of closures, independent from both \( s \) and \( u \).

\( \square \)

\section{Proofs of Section 7}

The first part of this appendix is devoted to the proof of the main theorem of the paper, i.e., the space reasonable simulation of TMs into the \( \lambda \)-calculus (better, the Space KAM). It is a boring proof, where we simply execute the image of the encoding of TMs into the \( \lambda \)-calculus with the Space KAM.

First, we need to understand how a TM configuration is represented in the Space KAM, i.e., how it is mapped to environments and closures.

**Definition C.1.** A configuration \( C \) of a TM is represented as a KAM closure \( C^k \) in the following way:

\[
(i, n, s, a, r, q) := ((f, c, m, [a], d, [q]), [f \leftarrow (\overline{t}, \varepsilon)], [c \leftarrow n^k], [m \leftarrow s^k], [d \leftarrow r^k])
\]

where \( s^k = \begin{cases} 
(\varepsilon), & \text{if } s = \varepsilon \\
(\lambda x_1, \ldots, \lambda x_{|s|}, \lambda y, x_3, z, [z \leftarrow r^k]), & \text{if } s = a_i r
\end{cases} \)

We observe that this representation preserves the space consumption, i.e., it is reasonable.

**Lemma C.2.** Let \( C := (i, n, s, a, r, q) \) be a configuration of a Turing machine and \( |C| := |s| + |r| \) its space consumption. Then \( |C^k| = \Theta(|C| + \log(|i|)) \).

In this lemma, we have already considered that the size of pointers inside \( n^k, s^k, r^k \) is constant and that \( n \leq \log |i| \).

Now we are able to prove the theorem. A series of intermediate lemmata, about the different combinators used in the encoding (\( \text{init}, \text{fin}, \text{trans} \)), are necessary. They are stated and proved below the main statement. By \( \rightarrow^\gamma_f \), we mean that the space consumption of that series of transitions is \( f \).

**Theorem C.3 (TM are simulated by the Space KAM in reasonable space).** There is an encoding \( \gamma \) of log-sensitive TM into \( \Lambda_{det} \) such that if the run \( \rho \) of the TM \( M \) on input \( i \in \mathbb{B}^* \):

1. Termination: ends in \( q_b \) with \( b \in \mathbb{B} \), then there is a complete sequence \( \sigma : \overline{M} T \rightarrow^\gamma \overline{M} T \) where \( n = \Theta((TM(\rho) + 1) \cdot |i| \cdot \log |i|) \).
2. Divergence: diverges, then \( \overline{M} T \) is \( \rightarrow_{det} \)-divergent.
3. Space KAM: the space used by the Space KAM to simulate the evaluation of point 1 is \( O(S_{TM}(\rho) + \log |i|) \) if \( \overline{M} T \) have separate address spaces.

**Proof.** The first two points are proved in the section devoted to the encoding of TMs into the \( \lambda \)-calculus. We concentrate on the third point.

We simply evaluate \( \overline{M} T \) with the Space KAM.
**Init and Final.**

**Lemma C.4.** \( (\text{init } k^\top, e) \rightarrow_{\text{SpKAM}} O^1(1) (k, e, C_{in}(i)^k) \) and consumes space \( \Theta(|i|) \).

**Proof:**

| Term                              | Env | Stack |
|-----------------------------------|-----|-------|
| \( \text{init } k^\top \)        | \( e \) | \( e \rightarrow^* \) \( O(\log(|i|)) \) (Lemma C.4) |
| \( \text{init } k \)             | \( e \) | \( e \rightarrow^* \) \( O(S_{\text{fin}}(\rho) + \log |i|) \) (Lemma C.8) |
| \( \text{final}(\lambda x.x) \)  | \( e \) | \( e \rightarrow^* \) \( O(S_{\text{fin}}(\rho) + \log |i|) \) (Lemma C.5) |

The space bound is immediate by inspecting the execution.

**Lemma C.5.** Let \( C \) be a final configuration, i.e. \( C := (i, n, s, a, r, q_{\text{fin}}) \) where \( q_{\text{fin}} \in Q_{\text{fin}} \). Then

\[
(\text{final}(\lambda x.x), e, C^k) \rightarrow_{\text{SpKAM}} O^1(i) \begin{cases} 
(\lambda x.\lambda y.x, e, e) & \text{if } q_{\text{fin}} = q_T \\
(\lambda x.\lambda y.y, e, e) & \text{if } q_{\text{fin}} = q_F 
\end{cases}
\]

Moreover, the space consumption is \( \Theta(|C|^k) \).

**Proof:** We execute the Space KAM. Let us define \( t := \lambda i'. \lambda n'. \lambda w', \lambda a'. \lambda w', \lambda q'. q' N_1 \ldots N_{|Q|} k' \).

| Term                              | Env | Stack |
|-----------------------------------|-----|-------|
| \( \text{final}(\lambda x.x) \)  | \( e \) | \( e \rightarrow^* \) \( C^k \) (Lemma C.5) |
| \( \text{final} := \lambda k'. \lambda C'. C't \) | \( C' \) | \( C' \rightarrow^* \) \( C'^4 \) |
| \( Ct \)                         | \( [C' \rightarrow C'^4] \) | \( [k' \rightarrow (\lambda x.x, e)] \) |
| \( C^k \)                        | \( [C' \rightarrow C'^k] \) | \( [k' \rightarrow (\lambda x.x, e)] \) |
| \( C'^k := \lambda x.x f c m[a][d[q_{\text{fin}}] \) | \( \{f \rightarrow (\lambda x.x, e), [c \rightarrow n^k], [m \rightarrow s^k], [d \rightarrow r^k] \) | \( \{x \rightarrow (\lambda x.x, e), [k' \rightarrow (\lambda x.x, e)] \} \) |
| \( q' \)                         | \( \{q' \rightarrow (\lambda x.x, e)\} \) | \( \{k' \rightarrow (\lambda x.x, e)\} \) |
| \( [q_{\text{fin}}] := \lambda x_1 \ldots \lambda x_{|Q|}, x_i \) | \( e \) | \( e \rightarrow^* \) \( C^k \) (Lemma C.5) |

Two cases. If \( q_{\text{fin}} = q_T \), then:

| Term                              | Env | Stack |
|-----------------------------------|-----|-------|
| \( \{q_T : = \lambda x_1 \ldots \lambda x_{|Q|}, x_i \} \) | \( e \) | \( (N_{i, e})_{1 \leq i \leq |Q|} (\lambda x.x, e) \rightarrow_{|Q|} \) |

| Term                              | Env | Stack |
|-----------------------------------|-----|-------|
| \( N_{1, e} := \lambda k'. k' (\lambda x.\lambda y.x) \) | \( e \) | \( (\lambda x.x, e) \rightarrow_{|Q|} \) |
| \( k' \)                          | \( [k' \rightarrow (\lambda x.x, e)] \) | \( e \rightarrow^* \) \( C^k \) (Lemma C.5) |
| \( \lambda x.x \)                 | \( e \) | \( (\lambda x.x, e) \rightarrow_{|Q|} \) |
| \( x \)                           | \( [x \rightarrow (\lambda x.\lambda y.x, e)] \) | \( e \rightarrow^* \) \( C^k \) (Lemma C.5) |
| \( \lambda x.\lambda y.x \)      | \( e \) | \( e \rightarrow^* \) \( C^k \) (Lemma C.5) |
If $q_{\text{fin}} = q_F$, then:

| Term | Env | Stack |
|------|-----|-------|
| $[q_F]$ := $\lambda x_1 \ldots \lambda x_{|Q|} \cdot x_i$ | $e$ | $(N_i, e)_{1 \leq i \leq |Q|} \cdot (\lambda x, x, e) \rightarrow |Q|$ |
| $x_i$ | $[x_i \leftarrow (N_i, e)]$ | $(\lambda x, x, e)$ |
| $N_i := \lambda k \cdot k'(\lambda x, x, y)$ | $e$ | $(\lambda x, x, e)$ |
| $k'(\lambda x, y, y)$ | $[k' \leftarrow (\lambda x, x, e)]$ | $e$ |
| $k'$ | $[k' \leftarrow (\lambda x, y, e)]$ | $(\lambda x, y, y, e)$ |
| $\lambda x, x$ | $[x \leftarrow (\lambda x, y, y, e)]$ | $e$ |
| $\lambda x, y, y$ | $e$ | $e$ |

The space bound is immediate by inspecting the execution. □

**Transition Function.**

**Lemma C.6.** \( \text{trans}(k, e, C_{\text{in}}(i)^k) \rightarrow_{\text{SpKAM}} O(1) \) \((\theta, e, (\theta, e) \cdot (\text{transaux}, e) \cdot k^k \cdot C_{\text{in}}(i)^k) \) in space \( O(\log(|i|)) \).

**Proof.**

| Term | Env | Stack |
|------|-----|-------|
| \( \text{trans} \) | $e$ | $C_{\text{in}}(i)^k$ |
| \( \text{fixtransaux} \) | $k^k \cdot C_{\text{in}}(i)^k$ | \( \text{Lemma B.1} \) |
| \( \theta \) | $e$ | $k^k \cdot C_{\text{in}}(i)^k$ |

**Lemma C.7.** Let \( C \) be a Turing machine configuration. Then:

- if \( C \) is a final configuration, then \( (\theta, e, (\theta, e) \cdot (\text{transaux}, e) \cdot k^k \cdot C^k) \rightarrow_{\text{SpKAM}} O(1) \) \((k, e, C^k) \) in space \( O(|C^k|) \);
- otherwise \( C \rightarrow_M D \), then \( (\theta, e, (\theta, e) \cdot (\text{transaux}, e) \cdot k^k \cdot C^k) \rightarrow_{\text{SpKAM}} O(1) \) \((\theta, e, (\theta, e) \cdot (\text{transaux}, e) \cdot k^k \cdot D^k) \) in space \( O(|C^k|) \).

**Proof.** The first part of the proof is common to both points.

Let us define \( \text{tx} := \text{transaux} \) and \( t := \lambda i' \cdot \lambda n' \cdot \lambda w' \cdot \lambda a' \cdot \lambda w' \cdot \lambda q'. \text{lookup Ki'n'} \).
Cases of the transition to apply:

- **No transition**, that is, $C$ is a final configuration, which happens when $q_s \in Q_{\text{fin}}$

  We have $C_{i,j,q_s} := \lambda x.\lambda k'.\lambda i'.\lambda n'.\lambda w'_j.\lambda w'_j.\lambda k'\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle$

| Term  | Env  | Stack |
|-------|------|-------|
| $C_{i,j,q_s}$ | $\varepsilon$ | $E_\varepsilon$ $\rightarrow^6$ $\text{fix}^k \cdot k^2 \cdot (n^k \cdot s^k \cdot r^k)$ |
| $k'(i', n' \mid w'_j, [a_j], w'_j \mid [q_s])$ | $[k' \rightarrow k^2] \cdot [w'_j \rightarrow s^k] \cdot [w'_j \rightarrow r^k] \cdot [i' \rightarrow i'] \cdot [n' \rightarrow n^k]$ | $\varepsilon$ $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2) =: C^k$ |
| $k'$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $k$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |

- **The heads do not move**, that is, $\delta(a_i, a_j, q_s) = (0 \mid a_h, \downarrow \mid q_t)$.

| Term  | Env  | Stack |
|-------|------|-------|
| $C_{i,j,q_s} := \lambda x.\lambda k'.\lambda i'.\lambda n'.\lambda w'_j.\lambda w'_j.\lambda n'\cdot\lambda S'^n$ | $\varepsilon$ | $E_\varepsilon$ $\rightarrow^6$ $\text{fix}^k \cdot k^2 \cdot (n^k \cdot s^k \cdot r^k)$ |
| $S := \lambda n'.\lambda k'(i', n' \mid w'_j, [a_j], w'_j \mid [q_s])$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $xk'(i', n' \mid w'_j, [a_j], w'_j \mid [q_s])$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $\emptyset$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |

- **The heads move right**, that is, $\delta(a_i, a_j, q_s) = (1 \mid a_h, \rightarrow \mid q_t)$.

| Term  | Env  | Stack |
|-------|------|-------|
| $C_{i,j,q_s} := \lambda x.\lambda k'.\lambda i'.\lambda n'.\lambda w'_j.\lambda w'_j.\lambda n'\cdot\text{succ}R^n$ | $\varepsilon$ | $E_\varepsilon$ $\rightarrow^6$ $\text{fix}^k \cdot k^2 \cdot (n^k \cdot s^k \cdot r^k)$ |
| $\text{succ}R^n$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $R := \lambda n'.\lambda k'^n.\lambda s'^n.\lambda i'.\lambda n'.\lambda w'_j.\lambda w'_j.\lambda k'^n.\lambda s'^n.\lambda xk'(i', n' \mid w'_j, [a_j], w'_j \mid [q_s])$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $[w'_j \rightarrow r^k]$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $\emptyset$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |

Two cases.

- $\tau = \varepsilon$. Define $i := (\lambda d.\lambda w'_j.xk'(i', n' \mid w'_j, [\square], d \mid [q_t]))^\tau$

| Term  | Env  | Stack |
|-------|------|-------|
| $w'_j$ | $\varepsilon$ | $E_\varepsilon$ $\rightarrow^5$ $\text{fix}^k \cdot k^2 \cdot (n^k \cdot s^k \cdot r^k)$ |
| $R'_{i'}^n.a.b$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $\text{append}^n t$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $t := (\lambda d.\lambda w'_j.xk'(i', n' \mid w'_j, [\square], d \mid [q_t]))^\tau$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $xk'(i', n' \mid w'_j, [\square], d \mid [q_t])$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $\emptyset$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $x$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $xxy$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $\emptyset$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |

Two cases.

- $\tau = \varepsilon$. Define $i := (\lambda d.\lambda w'_j.xk'(i', n' \mid w'_j, [\square], d \mid [q_t]))^\tau$

| Term  | Env  | Stack |
|-------|------|-------|
| $w'_j$ | $\varepsilon$ | $E_\varepsilon$ $\rightarrow^5$ $\text{fix}^k \cdot k^2 \cdot (n^k \cdot s^k \cdot r^k)$ |
| $R'_{i'}^n.a.b$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $\text{append}^n t$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $t := (\lambda d.\lambda w'_j.xk'(i', n' \mid w'_j, [\square], d \mid [q_t]))^\tau$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $xk'(i', n' \mid w'_j, [\square], d \mid [q_t])$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $\emptyset$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $x$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $xxy$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
| $\emptyset$ | $\varepsilon$ | $(\langle i', n' \mid w'_j, [a_j], w'_j \mid [q_s]\rangle, E_2)$ |
We observe that the size of \(s\) is constant, a complete evaluation

\[
\begin{align*}
&= \lambda x_0.\langle \ldots \rangle (x_0 x_0) \\
&= \lambda x_0.\langle \ldots \rangle (x_0 x_0) + 1
\end{align*}
\]

\[\text{Env} \quad \text{Stack}
\begin{align*}
\text{Term} & \quad w'_t & \quad (u_1, u_2, x_t, x_i) \\
\text{Env} & \quad e & \quad (u_1, u_2, x_t, x_i) \\
\text{Stack} & \quad (t, e, \langle \ldots \rangle (x_0 \ldots x_{n+1})) & \quad (t, e, \langle \ldots \rangle (x_0 \ldots x_{n+1}))
\end{align*}
\]

- All the other cases are almost identical mutatis mutandis.

About the space bound we observe that in the simulations all the pointers except for those related to the input part of the state are pointers to the machine, and not to the input. Moreover, the space overhead of the simulation of one step of the TM is constant, i.e. non input dependent.

**Lemma C.8.** If \(\rho : C \rightarrow^n D\) and \(D\) is final, then \((\text{trans }k, \epsilon, C_n(i)k) \rightarrow \text{SpKAM}(k, \epsilon, C^n)\) in space \(O(S_{TM}(\rho) + \log(|i|))\).

**Proof.** By a simple induction on \(n\), using the two lemmata above, and knowing that \(S_{TM}(\rho) = \max_{C \in \rho} |C|\) (we have also to consider that \(|C| = |C^n|\), by Lemma C.2).

**Proposition C.9** (Space KAM natural time overhead explosion). There is a family \(\{t_n\}_{n \in \mathbb{N}}\) of closed \(\lambda\)-terms such that there is a complete evaluation \(\rho_n : t_n \rightarrow^{\omega} u_n\) is simulated by Space KAM runs \(\sigma_n\) taking both space and time exponential in \(n\), that is, \(|\sigma_n|_{sp} = |\sigma_n|_{tm} = \Omega(2^n)\).

Before the main proof we need some preliminaries.

We define the following data structures:

\[
\begin{align*}
\epsilon_0 & := [x_0 \leftarrow (1, \epsilon)] \\
\pi_0 & := (x_0 x_0, \epsilon_0) \\
\epsilon_{n+1} & := [x_{n+1} \leftarrow \pi_n] \epsilon_n \\
\pi_{n+1} & := (x_0 x_{n+1}, \epsilon_{n+1})
\end{align*}
\]

We observe that the size of \(e_n\) is exponential in \(n\), since \(e_{n+1} := [x_{n+1} \leftarrow \pi_n] e_n = [x_{n+1} \leftarrow (x_0 \ldots x_n, e_n)] e_n\) i.e. \(|e_{n+1}| \geq 2 |e_n|\) and thus \(|e_n| \geq 2^n\).

We define \(C_n\) as follows:

\[
\begin{align*}
C_0 & := \lambda x_0. \langle \cdot \rangle (x_0 x_0) \\
C_{n+1} & := \lambda x_{n+1}. \langle \cdot \rangle (x_0 \ldots x_{n+1})
\end{align*}
\]

**Lemma C.10.** If \(\tau\) contains \(x_0, \ldots, x_n\) free, then \((C_0(C_1(\ldots C_n(t) \cdots))) \rightarrow \text{SpKAM}^{\Theta(n)}(\tau, e_n, \pi_n)\).

**Proof.** Case 0.

Case \(n + 1\).

We observe that \(C_0(C_1(\ldots C_n(C_{n+1}(t)) \ldots)) \in\) can be rewritten to \(C_0(C_1(\ldots C_n(u) \ldots)) \in\) when \(u := C_{n+1}(t)\). Of course \(u\) contains \(x_0, \ldots, x_n\) free. We can thus apply the i.h.

\[
\begin{align*}
\text{Term} & \quad C_0(C_1(\ldots C_n(C_{n+1}(t)) \ldots)) \\
\text{Env} & \quad \epsilon \\
\text{Stack} & \quad \pi_n
\end{align*}
\]

\[
\begin{align*}
\text{Term} & \quad C_0(C_1(\ldots C_n(u) \ldots)) \\
\text{Env} & \quad \epsilon \\
\text{Stack} & \quad \pi_n
\end{align*}
\]

\[
\begin{align*}
\text{Term} & \quad t \quad \pi_{n+1} \\
\text{Env} & \quad \epsilon \\
\text{Stack} & \quad \pi_{n+1}
\end{align*}
\]

\[
\begin{align*}
\text{Term} & \quad t \quad \pi_{n+1} \\
\text{Env} & \quad \epsilon \\
\text{Stack} & \quad \pi_{n+1}
\end{align*}
\]
Now we are able to prove the main Proposition.

**Proposition C.11** (Space KAM natural time overhead explosion). There is a family \( \{t_n\}_{n \in \mathbb{N}} \) of closed \( \lambda \)-terms such that there is a complete evaluation \( p_n : t_n \rightarrow^*_{wh} u_n \) is simulated by Space KAM runs \( \sigma_n \) taking both space and time exponential in \( n \), that is, \( |\sigma_n|_{sp} = |\sigma_n|_{tm} = \Omega(2^n) \).

**Proof.** We define \( t_n \) as follows:

\[
  t_n \coloneqq C_0(C_1(\cdots C_{n-1}(\lambda y.l) \cdots))l
\]

Now, we execute it.

| Term | Env | Stack |
|------|-----|-------|
| \( t_n \) | \( \epsilon \) | \( \epsilon \) |
| \( C_n(\lambda y.l) \coloneqq \lambda x_n.(\lambda y.l)(x_0 \ldots x_n) \) | \( \epsilon_{n-1} \) | \( \pi_{n-1} \) |
| \( \lambda y.l \) | \( \epsilon_n \) | \( \epsilon \) |
| \( l \) | \( \epsilon \) | \( \pi_n \) |

The space consumed (and thus also the low-level time) is exponential in \( n \) because the size of \( \epsilon_n \) is exponential in \( n \). \( \square \)

### D Proofs of Section 8

**Theorem D.1** (TMs are simulated by the Space KAM in reasonable low-level time).

1. Every TM run \( \rho \) can be simulated by the Space KAM in time \( O(\text{poly}(|\rho|)) \).
2. Every Space KAM run \( \rho : \text{init}(t) \rightarrow_{spKAM}^* q \) can be implemented on RAM in time \( O(|\rho|_{tm}) \).
3. Closed \( \text{CbN} \) evaluation \( \rightarrow_{wh} \) and the time of the Space KAM provide a reasonable time cost model for the \( \lambda \)-calculus.

**Proof.** The first point is the only one which is non-trivial. We have already proved that the Space KAM can simulate TMs runs \( \rho \) in a number of transitions which is polynomial in \( |\rho| \). However, this does not necessarily mean that the (low-level) time is also polynomial in \( \rho \), see Proposition 7.4. About the execution of terms which are the image of the encoding of TMs into the \( \lambda \)-calculus, we can say however that the overhead stays polynomial. Indeed, the exponential blowup comes from the fact that environments are duplicated in an uncontrolled way. This does not happen in the execution of the encoding of TMs, where duplication is restrained to the fix-point operator and to the input components of the state. In other words, we duplicate only objects of fixed size, thus confirming the polynomial bound.

There is another, indirect, way of proving the same results. If the (low-level) time were exponential in \( |\rho| \), then the space should be at least linear in \( |\rho|^3 \). But we have proved that this is not the case since space is linearly related with the space consumption of \( \rho \), and not with its length. \( \square \)

#### D.1 Time KAM

First, we provide in Fig. 5 the definition and the abstract implementation of the Time KAM.

**Theorem D.2.** Let \( \rho : \text{init}(t_0) \rightarrow_{TKAM}^* q \) be a complete Time KAM run. It can be implemented on RAM in time and space \( O(|\rho|_{\beta} \cdot \log(|\rho|_{\beta} \cdot |t_0|)) \).

**Proof.** Since the Time KAM is implemented with unchaining, the number of transitions of complete runs is \( O(|\rho|_{\beta}) \). The cost of each transition can be bounded by \( \log(|\rho|_{\beta} \cdot |t_0|) \) since in any transition a constant number of pointers to the code and pointers to data structures are manipulated. About space, new closures/environments, costing \( \log(|\rho|_{\beta} \cdot |t_0|) \), are created in any transition, except for \( \rightarrow_{\text{sub}} \). Thus also space consumption is \( O(|\rho|_{\beta} \cdot \log(|\rho|_{\beta} \cdot |t_0|)) \). \( \square \)

### E Proofs of Section 9

**Proposition E.1.** The SpaceKAM and the SpaceLAM are weakly bisimilar when executed on \( \Lambda_{\det} \)-terms. Moreover, also their space consumption is the same.

**Proof.** The transitions of the Space KAM not dealing with applications are identical to the corresponding ones of the Space LAM (if one ignores the dump, that remains untouched). For the two transitions of the Space KAM dealing with applications, we show that, when the argument is a variable or an abstraction (as in \( \Lambda_{\det} \)), the Space LAM behaves as the Space KAM. If the active term is \( tx \), indeed, the \( \rightarrow_{\text{sea}_r} \) transition of the Space KAM is simulated on the Space LAM by (with \( e(x) = (\lambda y.u, e') \)):

\( \text{This is because space cannot be less than logarithmic in time.} \)
states of the Space KAM and the Space LAM as $R$ the previous reasoning shows that
head).

In particular, these macro steps show that to evaluate TM the re is no need of the dump. Now, by defining a relation $R$ between states of the Space KAM and the Space LAM as

$$q_K \mathrel{R} q_L \iff q_K = (t, e, \pi) \text{ and } q_L = (e, t, \pi)$$

the previous reasoning shows that $R$ is a weak bisimulation preserving time and space complexity (modulo a constant overhead). □

| Closures | Environments | Stacks | Heaps | States |
|----------|--------------|--------|-------|--------|
| $c ::= (t, e)$ | $e$ | $\pi$ | $h ::= e$ | $(\pi ::= c \cdot \pi') \cup h$ | $(e ::= [x \rightarrow c] \cdot e') \cup h$ |
| $tx$ | $e$ | $\pi$ | $h$ | $\rightarrow_{sea_{x}}$ | $t$ | $e$ | $\pi'$ | $(\pi' ::= e(x) \cdot \pi \cup h)$ | $\pi'$ fresh |
| $tu$ | $e$ | $\pi$ | $h$ | $\rightarrow_{sea_{u}}$ | $t$ | $e$ | $\pi'$ | $(\pi' ::= (u, e) \cdot \pi \cup h)$ | $\pi'$ fresh, $u \notin V$ |
| $\lambda x.t$ | $e$ | $\pi$ | $(\pi ::= c \cdot \pi') \cup h$ | $\rightarrow_{\beta}$ | $t$ | $e'$ | $\pi'$ | $(\pi' ::= [x \rightarrow c] \cdot e) \cup h$ | $e'$ fresh |
| $x$ | $e$ | $\pi$ | $h$ | $\rightarrow_{sub}$ | $u$ | $e'$ | $\pi$ | $h$ | if $e(x) = (u, e')$ |

**Figure 5.** Data structures, transitions and abstract implementation of the Time KAM.

$$(e, tx, e, \pi) \rightarrow_{\text{SplAM}} ((t, e|t) \cdot \pi, x, e|x, e)$$

$$\rightarrow_{\text{SplAM}} ((t, e|t) \cdot \pi, \lambda y.u, e', e)$$

$$\rightarrow_{\text{SplAM}} (e, t, e|t, (\lambda y.u, e') \cdot \pi)$$

$$= (e, t, e|t, e(x) \cdot \pi)$$

If the active term instead is $t(\lambda x.u)$, the $\rightarrow_{sea_{u}}$ transition of the Space KAM is simulated on the Space LAM by:

$$(e, t(\lambda x.u), e, \pi) \rightarrow_{\text{SplAM}} ((t, e|t) \cdot \pi, \lambda x.u, e|\lambda x.u, e)$$

$$\rightarrow_{\text{SplAM}} (e, t, e|t, (\lambda x.u, e|\lambda x.u) \cdot \pi)$$

In particular, these macro steps show that to evaluate TM there is no need of the dump. Now, by defining a relation $R$ between states of the Space KAM and the Space LAM as

$$q_K \mathrel{R} q_L \iff q_K = (t, e, \pi) \text{ and } q_L = (e, t, e, \pi)$$

the previous reasoning shows that $R$ is a weak bisimulation preserving time and space complexity (modulo a constant overhead). □