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On separated Carleson sequences in the unit disc of $\mathbb{C}$.

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Abstract

The interpolating sequences for $H^\infty(\mathbb{D})$, the bounded holomorphic function in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$, where characterised by L. Carleson by metric conditions on the points. They are also characterised by "dual boundedness" conditions which imply an infinity of functions. A. Hartmann proved recently that just one function in $H^\infty(\mathbb{D})$ was enough to characterize interpolating sequences for $H^\infty(\mathbb{D})$. In this work we use the "hard" part of the proof of Carleson for the Corona theorem, to extend Hartman’s result and answer a question he asked in his paper.

1 Introduction.

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and $S$ a sequence of points in $\mathbb{D}$. Let

$$d_P(a, b) := \left| \frac{a - b}{1 - \overline{a}b} \right|$$

the pseudo hyperbolic distance and

$$d_H(a, b) := \tanh^{-1}(\delta(a, b))$$

the hyperbolic distance in $\mathbb{D}$.

To say that the sequence $S$ is separated means that there is a $\eta$, such that

$$\forall a, b \in S, \ a \neq b, \ d_H(a, b) \geq \eta \iff d_P(a, b) \geq \tanh \eta.$$  

Equivalently to say that the sequence $S$ is $\delta$-separated means that the discs $\forall a \in S, \ D(a, \delta(1 - |a|))$ are disjoint.

We shall also need the notion of Carleson measure. Let $(\zeta, h) \in \mathbb{T} \times (0, 1)$, and note

$$W(\zeta, h) := \{ z \in \mathbb{D} :: |1 - \zeta z| < h \}$$

the associated Carleson window. If $\nu$ is a borelian measure in $\mathbb{D}$, we shall say that $\nu$ is Carleson if there is a constant $C > 0$ such that

$$\forall \zeta \in \mathbb{T}, \ \forall h \in (0, 1), \ |\nu|(W(\zeta, h)) \leq Ch.$$  

A sequence $S$ will be a Carleson sequence if the canonical measure associated to it:

$$\mu_S := \sum_{a \in S} (1 - |a|) \delta_a,$$

is a Carleson measure.

Definition 1.1 We shall say that $S$ is interpolating (for $H^\infty(\mathbb{B})$) if

$$\forall \lambda \in l^\infty(S), \ \exists f \in H^\infty(\mathbb{B}) :: \forall a \in S, \ f(a) = \lambda_a.$$
L. Carleson characterized these sequences by the conditions [1]:

\[
(C) \quad \inf_{a \in S} \prod_{b \in S \setminus \{a\}} \frac{a - b}{1 - \bar{b}a} > 0.
\]

One can see easily that these conditions are equivalent to the fact that \( S \) is dual bounded in \( H^\infty(\mathbb{D}) \), which means:

\[
\exists C > 0, \forall a \in S, \exists \rho_a \in H^\infty(\mathbb{D}), \|\rho_a\|_\infty \leq C : \forall b \in S, \rho_a(b) = \delta_{ab}.
\]

We just take \( \rho_b(z) := \frac{B_b(z)}{B_b(b)} \) with \( B_b(z) := \prod_{a \in S \setminus \{b\}} \frac{a - z}{1 - \bar{a}z} \).

So the metric conditions \((C)\) which characterize the interpolation are equivalent to the functional characterization namely the existence of an infinity of functions verifying the above conditions.

Another functional characterization is due to D. Sarroste [5]:

**Theorem 1.2** If there are \( 0 < \tau < \eta \) such that for any partition \( S = A \cup B \) there is a function \( f \in H^\infty(\mathbb{D}) \), \( \|f\|_\infty \leq 1 \), with \( \forall a \in A, |f(a)| \leq \tau \) and \( \forall b \in B, |f(b)| \geq \eta \), then \( S \) is \( H^\infty(\mathbb{D}) \) interpolating.

Again one needs an infinity of functions in \( H^\infty(\mathbb{D}) \) to characterize interpolating sequences.

A. Hartmann [3] showed that this can be reduced to a condition on only one function:

**Theorem 1.3** Let \( S \) be a separated Blaschke sequence in the unit disc \( \mathbb{D} \) of \( \mathbb{C} \). There is a partition \((A, B)\) of \( S \) such that if there is a function \( f \in H^\infty(\mathbb{D}) \) with \( f = 0 \) on \( A \) and \( f = 1 \) on \( B \), then \( S \) is interpolating for \( H^\infty(\mathbb{D}) \).

So a natural question after these results is : is it possible to have an analogous result as D. Sarroste’s one but replacing for any partition by there is a partition ?

The aim of this work is to prove that the answer is yes, provided that \( \tau < \eta^\kappa \) for a certain constant \( \kappa > 1 \) introduced by Carleson in his proof of the corona theorem.

We shall need the following notions.

**Definition 1.4** We shall say that the partition \((A, B)\) of the sequence of points \( S \subset \mathbb{D} \) is ”good” if there is \( \varphi : A \to B \) such that \( \forall a \in A, d_H(a, \varphi(a)) = \inf_{c \in S \setminus \{a\}} d_H(a, c) \) and if there is \( \psi : B \to A \) such that \( \forall b \in B, d_H(b, \psi(b)) = \inf_{c \in S \setminus \{b\}} d_H(b, c) \).

We shall need more specific partitions. Let \( \gamma \in ]0,1[ \); we set

\[
C_n = C_n(\gamma) := \{z \in \mathbb{D} : 1 - \gamma^{n+1} < |z| \leq 1 - \gamma^n\}.
\]

**1.0.1 Restricted good partition.**

**Definition 1.5** A restricted good partition of the discrete sequence \( S \) in the disc is a partition \((A, B)\) of \( S \) such that \( \exists \gamma \in ]0,1[ \) and with \( A_n := A \cap C_n(\gamma) \), \( B_n := B \cap C_n(\gamma) \) we have \((A_n, B_n)\) is a good partition of \( S_n := S \cap C_n(\gamma) \) for any \( n \in \mathbb{N} \).

As we shall see later there are always restricted good partition for a discrete sequence \( S \) in the disc.
1.0.2 Hoffman partition.

Let $S$ be a discrete sequence in the disc. (See J. Garnett [2].)

We shall cut $S$ in two parts; for this let

$$D_1 := \{z \in \mathbb{D} : \text{Arg} z \in [0, \pi]\}, \quad D_2 := \{z \in \mathbb{D} : \text{Arg} z \in [\pi, 2\pi]\}.$$  

Now set $S_1 := S \cap D_1$, $S_2 := S \cap D_2$. Because if $S_1$ and $S_2$ are Carleson sequences then $S$ is also a Carleson sequence, it will be enough to deal with one of them, say $S = S_1$.

We start with the point $a_0 = |a_0| e^{i \theta_0}$ in $S_n := C_n \cap S$ with the smallest module and smallest argument; if $\# S_n = 1$, put $a_0$ in $A_n$ and set $B_n := \emptyset$; if not take the next point in $S_n$ with the same argument as $a$, hence with a bigger modulus, if any, or at the right side of $a_0$, i.e. such that its argument $\theta$ is bigger than $\theta_0$. Call it $b_0$ and define $\varphi(a_0) := b_0$.

Now take the next point in $S_n$ at the right side of $b_0$, i.e. the same way as above, and call it $a_1$ etc...

Then each time define $\varphi(a_j) := b_j$. Call $A_n$ the set of all $a_j$s and $B_n$ the set of all $b_j$s. Because $S$ is discrete $A_n$ and $B_n$ are finite.

**Definition 1.6** Let $A := \bigcup_{n \in \mathbb{N}} A_n$ and $B := \bigcup_{n \in \mathbb{N}} B_n$ then $(A, B)$ is the Hoffman partition of $S$.

We see easily that for any $a \in A$, $\varphi(a)$ has always its argument bigger or equal to the argument of $a$.

**Definition 1.7** Let $(A, B)$ be a restricted good partition or a Hoffman partition of the sequence $S \subset \mathbb{D}$. Let $\kappa \geq 1$ be a constant, the sequence $S \subset \mathbb{D}$ is $\kappa$-ultra-separated if $S$ is separated and if there are $0 < \tau < \eta, \tau < \eta^*$ and a function $f$ in $H^\infty(\mathbb{D})$, $\|f\|_\infty \leq 1$, such that $|f| \leq \tau$ on $A$ and $|f| \geq \eta$ on $B$.

Now we can state the theorem.

**Theorem 1.8** There is a constant $\kappa > 1$ such that the sequence $S$ is $H^\infty(\mathbb{D})$ interpolating if and only if it is $\kappa$-ultra-separated.

This constant $\kappa$ was introduced by Carleson in his proof of the corona theorem.

The theorem 1.8 generalizes the result of A. Hartman and answer positively to his question:

if there is a $f \in H^\infty(\mathbb{D})$ such that $\forall a \in A, f(a) = 0, \forall b \in B, |f(b)| \geq \eta > 0$ for a Hoffman partition $(A, B)$ of $S$, and $S$ separated, is $S$ interpolating?

I introduce good partitions for dealing with this problem in the unit ball of $\mathbb{C}^n$. This notion in invariant by automorphisms and hence natural. The result in the ball, not as good as in the disc, will be posted later. It involves complex geometry and the key fact is that the measure $(1 - |z|) |\partial f(z)|^2 \, dm(z)$ is a Carleson measure in the unit ball of $\mathbb{C}^n$.

2 General facts.

**Lemma 2.1** Let $S$ be a discrete sequence in the metric space $(X, d)$, there is a good partition $(S_1, S_2)$ of $S$.  


Proof.

Take a point $O \in X$ and $a_1 \in S$ such that $d(a_1, O)$ is minimal, if $\#S = 1$ set $\varphi(a_1) = a_1$ and $S_1 := \{a_1\} = S$; $S_2 = \emptyset$; then we are done.

If $\#S \geq 2$, then take $b_1 \in S$ a nearest neighbour for the distance $d$ of $a_1$ and define $\varphi(a_1) = b_1$. By the assumption on the cardinality of $S$, $b_1$ exists. Take $a_2$ a nearest neighbour of $b_1$, if it exists, and define $\psi(b_1) := a_2$; if $a_2 = a_1$ we stop at this ”perfect” pair $(a_1, b_1)$ with $\psi(b_1) := a_1$. If not we continue with $b_2$ nearest neighbour of $a_2$ etc... We stop at a perfect pair. This way we get a branch $B_1$ finite or infinite. We put all the ”a” in $S_1$ and all the ”b” in $S_2$.

If it remains points in $S$ we have that the points in $S \setminus B_1$ are far from the points in $B_1$ by construction. We take a point $c$ in $S \setminus B_1$ the nearest from $O$.

A) If all the nearest points from $c$ are in $B_1$, which may happen, we take one of them, $d$, now if $d$ is in $S_1$, we put $c$ in $S_2$ and we set $\psi(c) := d$. If $d$ is in $S_2$, we put $c$ in $S_1$ and we set $\varphi(c) := d$. This completes $B_1$ and we start all again.

B) If $c$ has a nearest neighbour which is not in $B_1$, we start a new branch $B_2$ etc...

A new point may have its nearest neighbour in $B_1$ or in $B_2$, etc... Then we put it in $B_1$ or in $B_2$, ... as in the step A.

We continue this way in order to exhaust $S$.

The $S_1$ part is all the ”a” and $S_2$ is all the ”b”.

Then $S$ is a bipartite graph with components $S_j$, $j = 1, 2$ on which the two applications $\varphi$, $\psi$ are well defined.

\section{Proof of the main theorem.}

Let $S$ be a discrete sequence in the unit disc $D$. Fix any $0 < \gamma < 1$ and $n \in \mathbb{N}$ and recall

$$C_n = C_n(\gamma) := \{z \in D : 1 - \gamma^{n+1} < |z| \leq 1 - \gamma^n\}, \quad S_n := S \cap C_n.$$ 

We shall use the following lemmas.

\textbf{Lemma 3.1} If the number of points in $S_n$ is smaller than $m$, i.e. $\forall a \in S$, $\#S_n \leq m$, then $S$ is a Carleson sequence.

\textbf{Proof.} Let $W = W(\zeta, h)$ be a Carleson window; for $S$ to be Carleson we must have

$$\exists C : \sum_{a \in S \cap W} (1 - |a|) \leq Ch.$$ 

If $a \in W$ we have $1 - |a| \leq h$, hence

$$\sum_{a \in S \cap W} (1 - |a|) \leq \sum_{a \in S, 1 - |a| \leq h} (1 - |a|).$$ 

But $a \in C_n(\gamma) \Rightarrow \gamma^{n+1} < 1 - |a| \leq \gamma^n$, and because there are at most $m$ points in $S \cap C_n(\gamma)$ we have

$$\sum_{a \in S, 1 - |a| \leq h} (1 - |a|) = \sum_{n: \gamma^{n+1} < h} \sum_{a \in C_n(\gamma) \cap S} (1 - |a|) \leq m \sum_{n: \gamma^{n+1} < h} \gamma^n \leq \frac{m}{\gamma(1 - \gamma)} h.$$ 

Hence we have the lemma with $C = m/\gamma(1 - \gamma)$.

\textbf{Lemma 3.2} Let $S$ be a discrete sequence in $D$, then there is a restricted good partition for $S$. 
Proof.
Take any $\gamma \in ]0, 1[$; because $S$ is discrete, $S_n := S \cap C_n(\gamma)$ has only a finite number of points. If $S_n = \emptyset$, we simply set $A_n = B_n := \emptyset$. If its cardinal is bigger than one, we can apply the general lemma 2.1: there is a good partition $(A_n, B_n)$ of $S_n$.

Setting $A = \bigcup_{n \in \mathbb{N}} A_n$, $B = \bigcup_{n \in \mathbb{N}} B_n$, we have that $(A, B)$ is a restricted good partition of $S$. $\blacksquare$

We remark that $\gamma \leq 1 - |a| \leq 1/\gamma$ because $a$ and $\varphi(a)$ belong to the same $C_n(\gamma)$.

Back to the proof of the main theorem.
Let $W = W(\zeta, h)$ be a Carleson window; we have to show that $S$ is Carleson i.e.
\[ \sum_{a \in A \cap W} (1 - |a|) \leq Ch, \]
then, because $S$ is separated, it will be $H^\infty(\mathbb{D})$ interpolating.

We shall cut the set $A \cap W$ in two parts.

4 The points $E_W := a \in W \cap S$ such that $\varphi(a) \notin W$.

4.1 Case of Hoffman partition.
We shall work in the half plane $\mathbb{C}^+$, because the geometry is easiest. This means that $1 - |a|$ is replaced by $\Im a$, the imaginary part of $a$, and $\text{Arg} a$ is replaced by $\Re a$, the real part of $a$. Let $C_a$ be the strip $C_n(\gamma) = \{ z :: \gamma^{n+1} < \Im z \leq \gamma^n \}$ which contains $a$. If we deal with a Hoffman partition, the point $b := \varphi(a)$ is the nearest point in $S \cap C_a$, either with the same real part, hence with a smaller imaginary part, or on the right of $a$; hence if $b \notin W$ this means that there is no points of $A$ between $a$ and the right vertical side of $W$ in the strip $C_n(\gamma)$ containing $a$. So $a$ is the nearest point in $A$ to the right side of $W$ in the strip $C_a$. We shall take the maximum possible of these points which means that we have at most one point in each $C_n(\gamma) \cap W$ and then we have
\[ \sum_{a \in E_W} \Im a \leq Ch, \]
by lemma 3.1.

4.2 Case of restricted good partition.
We shall work again in the half plane because the geometry is easiest. So $W$ is a square with one side on the real axis. Let $c$ be the orthogonal projection of $a$ on the side of $W$ in the direction of $b$.

We define the border strip to be a tube $T(a)$ around the segment $[a, c]$ of width $r \Im a$.

The partition $(A, B)$ being restricted, this means that $b$ belongs to the same strip $C_n(\gamma) := \{ z \in \mathbb{C}^+ :: \gamma^{n+1} < \Im z \leq \gamma^n \}$ as $a$. Let us denote $C_a$ the strip $C_n(\gamma)$ to which $a$ belongs.

Lemma 4.1 Let $(A, B)$ be a restricted good partition of $S$ in $\mathbb{D}$. Let $W = W(\zeta, h)$ be a Carleson window and $a \in A$ and $b := \varphi(a)$ be such that $a \in W$, $b \notin W$. Then the border strip $T(a)$ contains at most a fixed number $m$ of points of $A$.

Proof.
Because $b$ is the nearest point in $S \cap C_a$ we have that there is no point of $S \cap C_a$ in the hyperbolic ball $Q(a, b)$ centered at $a$ and passing through $b$. So the worst case is when $b$ belongs to one of the three sides of $W$ in $\mathbb{C}^+$. Suppose first that $b$ is in the vertical left side of $W$.

We have the worst case when $C_a = C_n(\gamma)$, with $\Im a = \gamma^{n+1}$. Then the border strip is $T(a) := \{z = x + iy \in W : (1 - r)\Im a < y < (1 + r)\Im a, x < \Re a\}$.

Then points of $A \setminus \{a\}$ in $T(a)$ must be in the triangle $bde \cap T(a)$ but $|de| \leq |bd| \leq \gamma^n(1 - \gamma^n)$ and in this triangle there is at most $m = m(\gamma, \delta)$ points in $A$ because $A$ is a $\delta$-separated sequence. A fortiori the number of points in $A$ such that $b \notin W$ is smaller than $m$.

If $b$ is on the right side of $W$, this is the same.

Suppose now that $b$ is on the top of $W$.

Still because $b$ is the nearest point in $S \cap C_a$ to $a$, the points in $(A \setminus \{a\}) \cap T(a) \cap W$ must be in a rectangle $cde'c'$ of sides less than $2r\Im a$. Because $S$ is $\delta$-separated, there is no point of $S \setminus \{b\}$ in the disc $D(b, \delta(1 - |b|))$, hence there is no point of $S$ in this rectangle provided that $r < \delta$, but the point $b$, and the lemma.

Again taking border strips $T(a)$ with half of the width, they become disjoint. Now set $E_W'$ the points in $E_W$ such that the $T(a)$ is based on the vertical sides of $W$ and $E_W''$ the points in $E_W$ such that the $T(a)$ is based on the top of $W$.

We have, by lemma 3.1, that $\sum a \in E_W'' \Im a \lesssim h$.

For $E_W''$ we have that $a \in E_W''$ must belong to the $C_n(\gamma)$ with the smallest $n$ such that $W \cap C_n(\gamma) \neq \emptyset$. So we have $\gamma^n \leq \Im a \leq \gamma^{n-1}$, hence, because the tube $T(a)$ has width $r\Im a$, we have at most $h/r\gamma^n$ such tubes. So finally

$$\sum a \in E_W'' \Im a \leq \gamma^{n-1} \times \frac{h}{r\gamma^n} \leq \frac{1}{r\gamma} h.$$
So adding these two inequalities we get
\[ \sum_{a \in E_W} \Re a \lesssim h, \]
and the right estimate for \( E_W \).

## 5 The points \( F_W := a \in W \cap S \) such that \( \varphi(a) \in W \).

In this part, we shall use the "hard" part of the proof of L. Carleson of the corona theorem, as interpreted by Hörmander [4] (Lemma 11, p 948):

**Lemma 5.1** There exists a constant \( \kappa \) such that if \( 0 < \eta < \frac{1}{2} \) and \( f \in H^\infty(\mathbb{D}) \), \( \sup |f| \leq 1 \), one can find \( \psi \) with \( 0 \leq \psi \leq 1 \) so that \( \frac{\partial \psi}{\partial \bar{z}} dm \) is a Carleson measure in \( \mathbb{D} \) and
\[ \psi(z) = 0 \text{ when } |f(z)| < \eta^\kappa, \quad \psi(z) = 1 \text{ when } |f(z)| \geq \eta. \]

We shall call this \( \kappa \) the Carleson constant. Because \( \psi \) is real valued we also have \( \frac{\partial \psi}{\partial z} dm \) is Carleson hence \( |\text{grad} \psi| \ dm \) is Carleson.

Let \((A, B)\) be the partition of \( S \) associated to the function \( f \in H^\infty(\mathbb{D}) \). Taking eventually a power of \( f \), we can assume that \( \eta < 1/2 \) to fit with the hypotheses of Hörmander’s lemma.

We shall use the following well known facts :
\begin{enumerate}
  \item If \( f \in H^\infty(\mathbb{D}) \) and \( |f(a)| \leq \tau \) then if \( \tau' > \tau \) there is a \( r > 0 \) depending only on \( f, \tau' \) such that : 
    \[ \forall z \in D(a, r(1 - |a|)), \ |f(z)| < \tau'. \]
  \item If \( f \in H^\infty(\mathbb{D}) \) and \( |f(b)| \geq \eta \) then if \( \eta' < \eta \) there is a \( r > 0 \) depending only on \( f, \eta' \) such that :
    \[ \forall z \in D(b, r(1 - |b|)), \ |f(z)| > \eta'. \]
\end{enumerate}

Let again \((A, B)\) the restricted or Hoffman partition of \( S \) and \( f \in H^\infty(\mathbb{D}) \) the function with
\[ \forall a \in A, \ |f(a)| \leq \tau < \eta^\kappa, \ \forall b \in B, \ |f(b)| \geq \eta. \]

By the fact 1 we have
\[ \forall z \in D(a, r_1(1 - |a|)), \ |f(z)| < \tau' \]
and by fact 2
\[ \forall z \in D(b, r_2(1 - |b|)), \ |f(z)| > \eta'. \]
Take \( r = \min(r_1, r_2) \) to have both.

We shall need the following notions. Let \( 0 < \alpha < 1, \ \beta > 0, \ a, b \in \mathbb{D} \) and set \( R(a, b, \alpha, \beta) \) a tube around a smooth curve \( \Gamma(a, b) \) with thickness \( \tau := \alpha \min(1 - |a|, 1 - |b|) \) i.e.
\[ R(a, b, \alpha, \beta) := \bigcup_{c \in \Gamma(a, b)} D(c, \tau). \]
Moreover the Lebesgue measure on \( R(a, b, \alpha, \beta) \) must be \( \beta \)-equivalent to the Lebesgue measure on the product \((-\tau, \tau) \times \Gamma(a, b)\).
Lemma 5.2  Let $(A, B)$ be a restricted or a Hoffman partition of $S$. Then we can make tubes $R(a, \varphi(a), s, \pi/2)$ which are disjoint.

Proof.
If $(A, B)$ is a restricted good partition, then for any $a \in A$ we take the tube of width $r(1 - |a|)$ around the segment $(a, \varphi(a))$. Because $a, b := \varphi(a)$ belong to the same strip $C_n(\gamma)$, we have that $\gamma \leq 1 - |a| \leq 1/\gamma$. Also $S$ being $\delta$-separated, there is no point of $S$ in a disc $D(b, \delta(1 - |b|))$ centered at $b$ and of radius $\delta(1 - |b|)$. Moreover, because $b$ is the nearest point in $S$ to $a$, in the hyperbolic metric, the disc $Q(a, b)$ of all the points in $\mathbb{D}$ nearer to $a$ than $b$ contains no other points of $S$; hence the tube with width $s = \min(\delta, r)$ does not contains any point in $S$ but $a$ and $b$. See the picture:

So we take the tubes with width $s/2$ instead of $s$ to have disjoint tubes.

If $(A, B)$ is a Hoffman partition, again $a$ and $b := \varphi(a)$ are in the same strip $C_n(\gamma)$ hence $1 - |a| \simeq 1 - |b|$. Because again the points in $S$ are $\delta$-separated, we can perturb a little bit the segment $(a, b)$ in order to avoid discs $D(c, \delta(1 - |c|))$, $c \in S \setminus \{a, b\}$ with a curve whose length is less than $\pi/2$ times the length of $(a, b)$.

This means that around this curve we can make a tube of width less than $(\delta/4) \times (1 - |a|)$ and these tubes are still disjoint provided that no segments $(a, \varphi(a))$ and $(a', \varphi(a'))$ cut each other. But by the construction of the Hoffman partition this cannot happen. We have now to take $s = \min(\delta/4, r)$ to have that the tubes $R(a, \varphi(a), s, \pi/2)$ are disjoint.

Hence the lemma. ■

Because $S$ is $\kappa$-ultra-separated, we have
\[ \exists f \in H^\infty(D) : \|f\|_\infty \leq 1, \forall a \in A, |f(a)| \leq \tau < \eta^c; \forall b \in B, |f(b)| \geq \eta. \]

Fix \( \tau' :: \tau < \tau' < \eta^c \) then we have a \( r > 0 \) such that
\[ \forall a \in A, \forall z \in D(a, r(1 - |a|)), |f(z)| \leq \tau' < \eta^c. \]

By lemma 5.2 we have that the length of the curve \( \Gamma(a, \varphi(a)) \) is smaller than \( \pi/2 \) times the length of the segment \( (a, \varphi(a)) \) so we can enlarge a little bit \( W \) say \( W' := W'(\zeta, \pi h/2) \), in order to have that
\[ R(a, \varphi(a), s, \pi/2) \subset W'. \]

See the picture

The Carleson-Hörmander lemma 5.1 gives us: there is a \( \psi \) with \( 0 \leq \psi \leq 1 \) so that \( \text{grad} \psi \) is a Carleson measure in \( D \) and
\[ \psi(z) = 0 \text{ when } |f(z)| < \eta^c, \psi(z) = 1 \text{ when } |f(z)| > \eta. \]

Let \( b := \varphi(a), I_a := [-s(1 - |a|), +s(1 - |a|)] \) and parametrize the tube \( R(a, \varphi(a), s, \pi/2) = I_a \times \Gamma(a, b) \)
then because \( \psi = 0 \) on \( I_a \times \{a\} \) and \( \psi = 1 \) on \( I_a \times \{b\} \), by the known facts 1 and 2 and the construction of the \( R(a, \varphi(a), s, \pi/2) \), we have
\[ \forall t \in I_a, 1 = \psi(t, b) - \psi(t, a) = \int_{\Gamma(a, b)} \text{grad} \psi \cdot ds \Rightarrow 1 \leq \int_{\Gamma(a, b)} |\text{grad} \psi| \cdot ds. \]

Now we integrate with respect to \( t \):
\[ 2s(1 - |a|) = \int_{I_a} 1 dt \leq \int_{I_a} \int_a^b |\text{grad} \psi| ds dt \leq \frac{\pi}{2} \int_{R(a, b, s, \pi/2)} |\text{grad} \psi| dm, \]
because the Lebesgue measure on \( R(a, b, s, \pi/2) \) is \( \pi/2 \) equivalent to the product measure.

This gives the estimate for the points \( a \) in \( W \) such that \( R(a, b, s, \pi/2) \) is in \( W' = W'(\zeta, \pi h/2) \) because
\[ \sum_{a \in F_W} (1 - |a|) \leq \frac{1}{2s} \sum_{a \in F_W} \frac{\pi}{2} \int_{R(a, b, \ldots)} |\text{grad} \psi(z)| dm(z) \leq \frac{\pi}{4s} \int_{W'} |\text{grad} \psi(z)| dm(z) \leq \frac{\pi^2}{8s} C h, \]
the tubes being disjoint by lemma 5.2 and the last inequality because \( |\text{grad} \psi(z)| dm(z) \) is a Carleson measure.

Hence the sequence \( A \) is Carleson and separated so it is \( H^\infty(D) \) interpolating.

For the sequence \( B \) we proceed analogously and we get that \( B \) is still separated and Carleson, hence \( H^\infty(D) \) interpolating. Because the union \( S = A \cup B \) is separated, we get that \( S \) is still \( H^\infty(D) \) interpolating and this finishes the proof of the direct part of the theorem.
For the converse part of the theorem let $S$ be an interpolating sequence for $H^\infty(\mathbb{D})$. Then $S$ is $\delta$-separated, hence discrete, so take any restricted or Hoffman partition $(A, B)$ of $S$. Because $S$ is $H^\infty(\mathbb{D})$ interpolating, there is a $f \in H^\infty(\mathbb{D})$ such that $f = 0$ on $A$, $f = 1$ on $B$ and $\|f\|_\infty \leq C$. This means that $g := f/C$ ultra-separates the sequence $S$ for any $\kappa > 1$, hence the theorem. ■

Now the answer to the question by A. Hartman:

**Corollary 5.3** Let $S = A \cup B$, where $(A, B)$ is a restricted or a Hoffman partition of $S$. Suppose that the Blaschke product $B_A$, precisely zero on $A$, verifies that $\inf_{b \in B} |B_A(b)| \geq \eta > 0$, then $S$ is a $H^\infty(\mathbb{D})$ interpolating sequence.

Proof.
We have that $0 < \eta^\kappa$, where $\kappa$ is the Carleson constant, so we can apply the theorem 1.8. ■

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