Graded Monads and Behavioural Equivalence Games

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ABSTRACT
The framework of graded semantics uses graded monads to cap-
ture behavioural equivalences of varying granularity, for example as found in the linear-time / branching-time spectrum, over general system types. We describe a generic Spoiler-Duplicator game for graded semantics that is extracted from the given graded monad, and may be seen as playing out an equational proof; instances include standard pebble games for simulation and bisim-
ulation as well as games for trace-like equivalences and coalgebraic behavioural equivalence. Considerations on an infinite vari-
ant of such games lead to a novel notion of infinite-depth graded semantics. Under reasonable restrictions, the infinite-depth graded semantics associated to a given graded equivalence can be char-
acterized in terms of a determinization construction for coalgebras under the equivalence at hand.

KEYWORDS
games, graded monads, semantics, behavioural equivalence, linear-
time/branching-time spectrum

1 INTRODUCTION
The classical linear-time / branching-time spectrum [20] organizes a plethora of notions of behavioural equivalence on labelled transition systems at various levels of granularity ranging from (strong) bisimulation to trace equivalence. Similar spectra appear in other system types, e.g. on probabilistic systems, again ranging from branching-time equivalence such as probabilistic bisimilarity to linear-time ones such as probabilistic trace equivalence [27]. While the variation in system types (nondeterministic, probabilistic, etc.) is captured within the framework of universal coalgebra [37], the variation in the granularity of equivalence, which we shall generally refer to as the semantics of systems, has been tackled, in coalgebraic generality, in a variety of approaches [22, 25, 26, 30]. One

setting that manages to accommodate large portions of the linear-
time / branching-time spectrum, notably including also intermedi-
ate equivalences such as ready similarity, is based on graded mon-
ads [15, 18, 35].

An important role in the theoretical and algorithmic treatment of a behavioural equivalence is classically played by equivalence games [20, 40], e.g. in partial-order techniques [24] or in on-the-fly equivalence checking [23]. In the present work, we contri-
but to graded semantics in the sense indicated above by showing that, under mild conditions, we can extract from a given graded monad a Spoiler-Duplicator game [40] that characterizes the respective equivalence, i.e. ensures that two states are equivalent under the semantics iff Duplicator wins the game.

As the name suggests, graded monads provide an algebraic view on system equivalence; they correspond to graded theories, i.e. algebraic theories equipped with a notion of depth on their operations. It has been noticed early on [35] that many desirable properties of a semantics depend on this theory being depth-1, i.e. having only equations between terms that are uniformly of depth 1. Standard examples include distribution of actions over non-deterministic choice (trace semantics) or monotonicity of actions w.r.t. the choice ordering (similarity) [15]. Put simply, our generic equivalence game plays out an equational proof in a depth-1 equational theory in a somewhat nontraditional manner: Duplicator starts a round by playing a set of equational assumptions she claims to hold at the level of successors of the present state, and Spoiler then challenges one of these assumptions.

In many concrete cases, the game can be rearranged in a straightforward manner to let Spoiler move first as usual; in this view, the equational claims of Duplicator roughly correspond to a short-term strategy determining the responses she commits to playing after Spoiler’s next move. In particular, the game instantiates, after such rearrangement, to the standard pebble game for bisimilarity. We analyse additional cases, including similarity and trace equivalence, in more detail. In the latter case, several natural variants of the game arise by suitably restricting strategies played by Duplicator.

It turns out that the game is morally played on a form of pre-determinization of the given coalgebra, which lives in the Eilenberg-Moore category of the zero-th level of the graded monad,
and as such generalizes a determinization construction that applies in certain instances of coalgebraic language semantics of automata [26]. Under suitable conditions on the graded monad, this pre-determinization indeed functions as an actual determinization, i.e. it turns the graded semantics into standard coalgebraic behavioural equivalence for a functor that we construct on the Eilenberg-Moore category. This construction simultaneously generalizes, for instance, the standard determinization of serial labelled transition systems for trace equivalence and the identification of similarity as behavioural equivalence for a suitable functor on posets [28] (specialized to join semilattices).

While graded semantics has so far been constrained to apply only to finite-depth equivalences (finite-depth bisimilarity, finite trace equivalence, etc.), we obtain, under the mentioned conditions on the graded monad, a new notion of infinite-depth equivalence induced by a graded semantics, namely via the (pre-)determinization. It turns out the natural infinite version of our equivalence game captures precisely this infinite-depth equivalence. This entails a fixpoint characterization of graded semantics on finite systems, giving rise to perspectives for a generic algorithmic treatment.

Related Work. Game characterizations of process equivalences are an established theme in concurrency theory; they tend to be systematic but not generic [14, 20]. Work on games for spectra of quantitative equivalences is positioned similarly [16, 17]. The idea of developing (bi)simulation games in coalgebraic generality goes back to work on branching-time simulations based on relations [9]. There is recent highly general work, conducted in a fibrational setting, on so-called codensity games for various notions of bisimilarity [31]. The emphasis in this work is on generality w.r.t. the measure of bisimilarity, covering, e.g. two-valued equivalences, metrics, pre-orders, and topologies, while, viewed through the lens of spectra of equivalences, the focus remains on branching time. The style of the codensity game is inspired by modal logic, in the spirit of coalgebraic Kantorovich liftings [8, 41]; Spoiler plays predicates thought of as arguments of modalities. Work focused more specifically on games for Kantorovich-style coalgebraic behavioural equivalence and behavioural metrics [32] similarly concentrates on the branching-time case. A related game-theoretic characterization is implicit in work on Λ-(bi)similarity [21], also effectively limited to branching-time. Comonadic game semantics [1, 2, 36] proceeds in the opposite way compared to the mentioned work and ours: It takes existing games as the point of departure, and then aims to develop categorical models.

Graded semantics was developed in a line of work mentioned above [15, 18, 35]. The underlying notion of graded monad stems from algebro-geometric work [39] and was introduced into computer science (in substantially higher generality) in work on the semantics of effects [29]. Our pre-determinization construction relates to work on coalgebras over algebras [7].

Organization. We discuss preliminaries on categories, coalgebras, graded monads, and games in Section 2. We recall the key notions of graded algebra and canonical graded algebra in Section 4, and graded semantics in Section 3. We introduce our pre-determinization construction in Section 5, and finite behavioural equivalence games in Section 6. In Section 7, we consider the infinite version of the game, relating it to behavioural equivalence on the pre-determinization. We finally consider specific cases in detail in Section 8.

2 PRELIMINARIES

We assume basic familiarity with category theory [4]. We will review the necessary background on coalgebra [37], graded monads [35, 39], and the standard bisimilarity game [46].

The category of sets. Unless explicitly mentioned otherwise, we will work in the category Set of sets and functions (or maps), which is both complete and cocomplete. We fix a terminal object 1 = {∗} and use 1x (or just ! if confusion is unlikely) for the unique map X → 1.

In the subsequent sections, we will mostly draw examples from (slight modifications of) the following (endo-)functors on Set. The powerset functor \( \mathcal{P} \) sends each set X to its set of subsets \( \mathcal{P}X \), and acts on a map \( f: X \to Y \) by taking direct images, i.e. \( \mathcal{P} f(S) := f[S] \) for \( S \in \mathcal{P}X \). We write \( \mathcal{P} \) for the finitary powerset functor which sends each set to its set of finite subsets; the action of \( \mathcal{P} \) on maps is again given by direct images. Similarly, \( \mathcal{P}^+ \) denotes the non-empty powerset functor \( (\mathcal{P}^+(X) = \{ Y \in \mathcal{P}(X) \mid Y \neq \emptyset \}) \), and \( \mathcal{P}^*_f \) its finitary subfunctor \( (\mathcal{P}^*_f(X) = \{ Y \in \mathcal{P}_f(X) \mid Y \neq \emptyset \}) \).

We write DX for the set of distributions on a set X: maps \( \mu: X \to [0, 1] \) such that \( \sum_{x \in X} \mu(x) = 1 \). A distribution \( \mu \) is \emph{finitely supported} if the set \( \{ x \in X \mid \mu(x) \neq 0 \} \) is finite. The set of finitely supported distributions on X is denoted \( D_fX \). The assignment \( X \mapsto DX \) is the object-part of a functor: given \( f: X \to Y \), the map \( Df: DX \to DY \) assigns to a distribution \( \mu \in DX \) the image distribution \( Df(\mu): Y \to [0, 1] \) defined by \( Df(\mu)(y) = \sum_{x \in X} f(x) = y \mu(x) \). Then, \( Df(\mu) \) is finitely supported if \( \mu \) is, so \( D_f \) is functorial as well.

Coalgebra. We will review the basic definitions and results of universal coalgebra [37], a categorical framework for the uniform treatment of a variety of reactive system types.

Definition 2.1. For an endofunctor \( G: \mathcal{E} \to \mathcal{E} \) on a category \( \mathcal{E} \), a \( G \)-coalgebra (or just coalgebra) is a pair \((X, \gamma)\) consisting of an object \( X \) in \( \mathcal{E} \) and a morphism \( \gamma: X \to GX \). A (coalgebra) morphism from \((X, \gamma)\) to a coalgebra \((Y, \delta)\) is a morphism \( h: X \to Y \) such that \( \delta \cdot h = Fh \cdot \gamma \).

Thus, for \( \mathcal{E} = \text{Set} \), a coalgebra consists of a set \( X \) of states and a map \( \gamma: X \to GX \), which we view as a transition structure that assigns to each state \( x \in X \) a structured collection \( \gamma(x) \in GX \) of successors in \( X \).

Example 2.2. We describe some examples of functors on \( \text{Set} \) and their coalgebras for consideration in the subsequent. Fix a finite set \( \mathcal{A} \) of actions.

1. Coalgebras for the functor \( G = \mathcal{P}(\mathcal{A} \times -) \) are just \( \mathcal{A} \)-labelled transition systems (LTS): Given such a coalgebra \((X, \gamma)\), we can view the elements \( (a, y) \in \gamma(x) \) as the \( a \)-successors of \( x \). We call \((X, \gamma)\) finitely branching (resp. serial if \( \gamma(x) \) is finite (resp. non-empty)) for all \( x \in X \). Finitely branching (resp. serial) LTS are coalgebras for the functor \( G = \mathcal{P}((\mathcal{A} \times -)) \) (resp. \( \mathcal{P}^*(\mathcal{A} \times -)) \).
(2) A coalgebra \((X, y)\) for the functor \(G = D(\mathcal{A} \times -)\) is a \((\text{generative})\) probabilistic transition system (PTS): The transition structure \(y\) assigns to each state \(x \in X\) a distribution \(y(x)\) on pairs \((a, y) \in \mathcal{A} \times X\). We think of \(y(x)(a, y)\) as the probability of executing an \(a\)-transition to state \(y\) while sitting in state \(x\). A PTS \((X, y)\) is \(\text{finitely branching}\) if \(y(x)\) is finitely supported for all \(x \in X\); then, finitely branching PTSs are coalgebras for \(D_f(\mathcal{A} \times -)\).

Given coalgebras \((X, y)\) and \((Y, \delta)\) for an endofunctor \(G\) on \(\text{Set}\), states \(x, y \in X\) and \(y \in Y\) are \(G\)-\text{behaviourally equivalent} if there exist coalgebra morphisms

\[
(f, g) : (X, y) \rightarrow (Z, \zeta) \leftarrow (Y, \delta)
\]

such that \(f(x) = g(y)\). Behavioural equivalence can be approximated via the \((\text{initial} \omega\text{-segment of})\) the \(n\)th \(G\)-chain \((G^n1)_{n \in \omega}\), where \(G^n\) denotes \(n\)-fold application of \(G\). The canonical cone of a coalgebra \((X, y)\) is then the family of maps \(\gamma_n : X \rightarrow G^n1\) defined inductively for \(n \in \omega\) by

\[
y_0 = (X \xrightarrow{1} 1), \text{ and } \gamma_{n+1} = (X \xrightarrow{y} GX \xrightarrow{G\gamma_n} G^{n+1}1 = Gn+11).
\]

States \(x, y \in X\) are \(\text{finite-depth behaviourally equivalent}\) if \(\gamma_n(x) = \gamma_n(y)\) for all \(n \in \omega\).

**Remark 2.3.** It follows from results of Worrell [42] that behavioural equivalence and finite-depth behavioural equivalence coincide for finitary functors on \(\text{Set}\), where a functor \(G\) on \(\text{Set}\) is \(\text{finitary}\) if it preserves filtered colimits. Equivalently, for every set \(X\) and each \(x \in GX\) there exists a finite subset \(Y \subseteq X\) such that \(x = G(iY)\), where \(i : Y \hookrightarrow X\) is the inclusion map [5, Cor. 3.3].

**Bisimilarity games.** We briefly recapitulate the classical \(\text{bismilarity game}\), a two-player graph game between the players Duplicator (D) and Spoiler (S); player D tries to show that two given states are bisimilar, while S tries to refute this. \(\text{Configurations of the game are pairs}\ (x, y) \in X \times X\ \text{of states in a LTS} (X, \gamma). The game proceeds in rounds, starting from the \(\text{initial configuration},\) which is contested by the \(\text{initial pair of states}. In each round, starting from a configuration \((x, y)\), S picks one of the sides, say, \(x,\) and then selects an action \(a \in \mathcal{A}\) and an \(a\)-successor \(x'\) of \(x;\) player D then selects a corresponding successor on the other side, in this case an \(a\)-successor \(y'\) of \(y. The game then reaches the new configuration \((x', y')\). If a player gets stuck, the play is \(\text{winning}\) for their opponent, whereas any infinite play is \(\text{winning}\) for D. It is well known (e.g. [40]) that D has a winning strategy in the bisimilarity game at a configuration \((x, y)\) iff \((x, y)\) is a pair of bisimilar states. Moreover, for finitely branching LTSs, an equivalent formulation may be given in terms of the \(n\)th \(\text{bismilarity game}: the rules of the \(n\)th-game are the same as those above, only now D wins as soon as at most \(n\) rounds have been played. In fact, a configuration \((x, y)\) is a bisimilar pair precisely if D has a winning strategy in the \(n\)-round bisimilarity game for all \(n \in \omega\).

We mention just one obvious variation of this game that characterizes a different spot on the \(\text{linear-time/branching-time spectrum}: the \text{mutual-simulation game}\) is set up just like the bisimilarity game, except that S may only choose his side once, in the first round, and then has to move on that side in all subsequent rounds (in the bisimulation game, he can switch sides in every round if he desires). It is easily checked that states \(x, y\) are mutually similar iff S wins the position \((x, y)\) in the mutual-simulation game. We will see that both these games (and many others) are obtained as instances of our generic notion of graded equivalence game.

**Graded monads.** We now review some background material on graded monads [35, 39].

**Definition 2.4.** A graded monad \(\mathcal{M}\) on a category \(\mathcal{C}\) is a triple \((M, \eta, \mu)\) where \(M\) is a family of functors \(M_n : \mathcal{C} \rightarrow \mathcal{C}\) (\(n \in \omega\)), \(\eta : \text{id} \rightarrow M_0\) is a natural transformation (the \(\text{unit} ), \text{and } \mu\ is a family of natural transformations

\[
\mu^n_k : M_nM_k \rightarrow M_{n+k}\quad(n, k \in \omega)
\]

(\(the multiplication\)) such that the following diagrams commute for all \(n, m, k \in \omega\):

\[
\begin{array}{ccc}
M_nM_k & \xrightarrow{\mu^{n,k}} & M_{n+k} \\
\downarrow \mu^{n+k,m} & & \downarrow \mu^{n+k,m} \\
M_{n+k}M_m & \xrightarrow{\mu^{m,k}M_n} & M_{n+k+m}
\end{array}
\]

We refer to (2.1) and (2.2) as the \(\text{unit}\) and \(\text{associative laws of } \mathcal{M}\), respectively. We call \(\mathcal{M}\) \(\text{finitary}\) if all the functors \(M_n : \mathcal{C} \rightarrow \mathcal{C}\) are finitary.

The above notion of graded monad is due to Smirnov [39], Katsumata [29], Fuji et al. [19], and Mellies [34] consider a more general notion of graded (or \text{parametrized}) monad given as a lax monoidal action of a monoidal category \(\mathcal{M}\) (representing the system of grades) on a category \(\mathcal{C}\). Graded monads in the above sense are recovered by taking \(\mathcal{M}\) to be the (discrete category induced by the) monoid \((\mathbb{N}, +, 0)\).

The graded monad laws imply that the \(\text{triple}\ (M_0, \eta, \mu^{0,0})\) is a \(\text{(plain) monad}\) on the base category \(\mathcal{C}\); we use this freely without further mention.

**Example 2.5.** We review some salient constructions [35] of graded monads on \(\text{Set}\) for later use.

(1) Every endofunctor \(G\) on \(\text{Set}\) induces a graded monad \(\mathcal{M}_G\) with underlying endofunctors \(M_n = G^n\) (the \(n\)-fold composite of \(G\) with itself); the unit \(\eta_X : X \rightarrow G^0X = X\) and multiplication \(\mu^{n,k}_X : G^nG^kX \rightarrow G^{n+k}X\) are all identity maps. We will later see that \(\mathcal{M}_G\) captures (finite-depth) \(G\)-behavioural equivalence.

(2) Let \((T, \eta, \mu)\) be a monad on \(\text{Set}\), let \(F\) be an endofunctor on \(\text{Set}\), and let \(\lambda : FT \rightarrow TF\) be a natural transformation such that

\[
\lambda \cdot F\eta = \eta F\quad\text{and}\quad\lambda \cdot F\mu = \mu F \cdot T\lambda \cdot \lambda T
\]

(i.e. \(\lambda\) is a \text{distributive law of the functor} \(F\text{ over the monad} T\)). For each \(n \in \omega\), let \(\lambda^n : FT^n \rightarrow TF^n\) denote the natural transformation defined inductively by

\[
\lambda^0 := \text{id}_T; \quad \lambda^{n+1} := \lambda^nF \cdot F^n\lambda.
\]
We obtain a graded monad with $M_n := \mathcal{T}^n$, unit $\eta$, and components $\mu^{n,k}$ of the multiplication given as the composites

\[ \mathcal{T}^n\mathcal{T}^k \xrightarrow{T^n\mu^{n,k}} T^n\mathcal{T}^{n+k} = T^n\mathcal{T}^{n+k} \]

Such graded monads relate strongly to Kleisli-style coalgebraic trace semantics [22].

(3) We obtain (by instance of the example above) a graded monad $\mathcal{M}_T(\mathcal{A})$ with $M_n = T(\mathcal{A}^n \times -)$ for every monad $T$ on $\mathcal{Set}$ and every set $\mathcal{A}$. Thus, $\mathcal{M}_T$ is a graded monad for traces under effects specified by $T$; e.g. for $T = \mathcal{D}$, we will see that $\mathcal{M}_T(\mathcal{A})$ captures probabilistic trace equivalence on PTS.

(4) Similarly, given a monad $T$, an endofunctor $F$, both on the same category $\mathcal{C}$, and a distributive law $\lambda: TF \to FT$ of $T$ over $F$, we obtain a graded monad with $M_n := F^n T$, unit and multiplication given analogously as in item (2) above (see [35, Ex. 5.2.6]). Such graded monads relate strongly to Eilenberg-Moore-style coalgebraic language semantics [12].

Graded variants of Kleisli triples have been introduced and proved equivalent to graded monads (in a more general setting) by Katsumata [29]:

**Notation 2.6.** We will employ the graded Kleisli star notation: for $n \in \omega$ and a morphism $f: X \to M_k Y$, we write

\[ f^n := (M_n X \xrightarrow{\lambda^n} M_n M_k \xrightarrow{\mu^{n,k}} M_{n+k} Y). \]

In this way, we obtain a morphism satisfying the following graded variants [29, Def. 2.3] of the usual laws of the Kleisli star operation for ordinary monads: for every $m \in \omega$ and morphisms $f: X \to M_n Y$ and $g: Y \to M_k Z$ we have:

\[ f^n \circ \eta_X = f, \]

\[ (\eta^m)_* \lambda = \text{id}_{M_n X}, \]

\[ (\eta^m \circ f)_n = g_{m+n} \cdot f^n. \]

**Graded theories.** Graded theories, in a generalized form in which arities of operations are not restricted to be finite, have been proved equivalent to graded monads on $\mathcal{Set}$ (the finitary case was implicitly covered already by Smirnov [39]). We work primarily with the finitary theories below; we consider infinitary variants of such theories only when considering infinite-depth equivalences (Section 7).

**Definition 2.7.** (1) A graded signature is a set $\Sigma$ of operations $f$ equipped with a finite arity $\text{ar}(f) \in \omega$ and a finite depth $d(f) \in \omega$. An operation of arity 0 is called a constant.

(2) Let $X$ be a set of variables and let $n \in \omega$. The set $\mathcal{F}_{\Sigma,n}(X)$ of $\Sigma$-terms of uniform depth $n$ with variables in $X$ is defined inductively as follows: every variable $x \in X$ is a term of uniform depth 0 and, for $f \in \Sigma$ and $t_1, \ldots, t_{\text{ar}(f)} \in \mathcal{F}_{\Sigma,k}(X), f(t_1, \ldots, t_{\text{ar}(f)})$ is a $\Sigma$-term of uniform depth $k + d(f)$. In particular, a constant $c$ has uniform depth 0 for all $k \geq d(c)$.

(3) A graded $\Sigma$-theory is a set $E$ of uniform-depth equations: pairs $(s, t)$, written ‘$s = t$’, such that $s, t \in \mathcal{F}_{\Sigma,n}(X)$ for some $n \in \omega$; we say that $(s, t)$ is depth-$n$. A theory is depth-$n$ if all of its equations and operations have depth at most $n$.

**Notation 2.8.** A uniform-depth substitution is a map $\sigma: X \to \mathcal{F}_{\Sigma,k}(Y)$, where $k \in \omega$ and $X, Y$ are sets. Then $\sigma$ extends to a family of maps $\bar{\sigma}_n: \mathcal{F}_{\Sigma,n}(X) \to \mathcal{F}_{\Sigma,k+n}(Y)$ ($n \in \omega$) defined recursively by

\[ \bar{\sigma}_n(f(t_1, \ldots, t_{\text{ar}(f)})) = f(\bar{\sigma}(t_1), \ldots, \bar{\sigma}(t_{\text{ar}(f)})), \]

where $ti \in \mathcal{F}_{\Sigma,m}$ and $d(f) + m = n$. For a term $t \in \mathcal{F}_{\Sigma,k}(X)$, we also write $\bar{\sigma}_\tau := \bar{\sigma}_n(t)$ if confusion is unlikely.

Given a graded theory $T = (\Sigma, E)$, we have essentially the standard notion of equation derivation (sound and complete over graded algebras, cf. Section 4), restricted to uniform-depth equations. Specifically, the system includes the expected rules for reflexivity, symmetry, transitivity, and congruence, and moreover allows substituted introduction of axioms: if $s = t$ is in $E$ and $\sigma$ is a uniform-depth substitution, then derive the (uniform-depth) equation $\sigma s = \sigma t$. (A substitution rule that more generally allows uniform-depth substitution into derived equations is then admissible.) For a set $Z$ of uniform-depth equations, we write

\[ Z \vdash s = t \]

if the uniform-depth equation $s = t$ is derivable from equations in $Z$ in this system; note that unlike the equational axioms in $E$, the equations in $Z$ cannot be substituted into in such a derivation (they constitute assumptions on the variables occurring in $s, t$).

We then see that $T$ induces a graded monad $\mathcal{M}_T$ with $M_n X$ being the quotient of $\mathcal{F}_{\Sigma,n}(X)$ modulo derivable equality under $E$; the unit and multiplication of $\mathcal{M}_T$ are given by the inclusion of variables as depth-0 terms and the collapsing of layered terms, respectively. Conversely, every graded monad arises from a graded theory in this way [35].

We will restrict attention to graded monads presented by depth-1 graded theories.

**Definition 2.9.** A presentation of a graded monad $\mathcal{M}$ is a graded theory $T$ such that $\mathcal{M} \cong \mathcal{M}_T$, in the above notation. A graded monad is depth-1 if it has a depth-1 presentation.

**Example 2.10.** Fix a set $\mathcal{A}$ of actions. We describe depth-1 graded theories associated (via the induced behavioural equivalence, Section 3) to standard process equivalences on LTS and PTS [15].

(1) The graded theory $\text{JSL}(\mathcal{A})$ of $\mathcal{A}$-labelled join semilattices has as depth-1 operations all formal sums

\[ \sum_{i=1}^{n} a_i(-), \quad \text{for } n \geq 0 \text{ and } a_1, \ldots, a_n \in \mathcal{A} \]

(and no depth-0 operations); we write 0 for the empty formal sum. The axioms of $\text{JSL}(\mathcal{A})$ consist of all depth-1 equations $\sum_{i=1}^{n} a_i(x_i) = \sum_{i=1}^{m} b_j(y_j)$ (where the $x_i$ and $y_j$ are variables, not necessarily distinct) such that \{(a_i, x_i) \mid 1 \leq i \leq n\} = \{(b_j, y_j) \mid 1 \leq j \leq m\}. The graded monad induced by $\text{JSL}(\mathcal{A})$ is $\mathcal{M}_G$ for $G = \mathcal{P}(\mathcal{A} \times (-))$ (cf. Example 2.5.1).

(2) The graded theory of probabilistic traces, $\text{PT}(\mathcal{A})$, has a depth-0 convex sum operation

\[ \sum_{i=1}^{n} p_i \cdot (-) \quad \text{for all } p_1, \ldots, p_n \in [0,1] \text{ such that } \sum_{i=1}^{n} p_i = 1 \]

and unary depth-1 operations $a(-)$ for all actions $a \in \mathcal{A}$. As depth-0 equations, we take the usual equational axiomatisation of convex algebras, which is given by the equation $\sum_{i=1}^{n} \delta_{ij} \cdot x_j = x_i$ (where
δ_{ij} denotes the Kronecker delta function) and all instances of the equation scheme
\[ \sum_{i=0}^{n} p_i \cdot \sum_{j=1}^{m} q_{ij} \cdot x_j = \sum_{j=1}^{m} \left( \sum_{i=1}^{n} p_i q_{ij} \right) \cdot x_j. \]

We further impose depth-1 equations stating that actions distribute over convex sums:
\[ a\left( \sum_{i=1}^{n} p_i \cdot x_i \right) = \sum_{i=1}^{n} p_i \cdot a(x_i). \]

The theory \( \mathcal{PT}(\mathcal{A}) \) presents \( M_{\mathcal{D}_I}(\mathcal{A}) \), where \( \mathcal{D}_I \) is the finitely supported distribution monad (cf. Example 2.5(3)).

Example 3.3. We recall [15, Section 4] several graded equivalences, restricting primarily to LTS and PTS.

Given a \( G \)-coalgebra \( (X, y) \), the graded semantics \( (\alpha, \mathcal{M}) \) induces a sequence of maps \( \nu^{(n)} : X \rightarrow M_{\mathcal{M}}^{n} \) inductively defined by
\[ \nu^{(0)} := (X \xrightarrow{\eta_X} M_{\psi} X) \xrightarrow{M_{\psi}} M_{\psi} \]
\[ \nu^{(n+1)} := (X \xrightarrow{\alpha_X . y} M_{\psi} X \xrightarrow{M_{\psi}} M_{\psi} \xrightarrow{M_{\psi}} M_{\psi} \xrightarrow{M_{\psi}} M_{\psi} \xrightarrow{M_{\psi}} M_{\psi}) \]
(or, using the graded Kleisli star, \( \nu^{(n+1)} = (\nu^{(n)} \cdot \alpha_X . y) \)). We call \( \nu^{(n)}(x) \in M_{\mathcal{M}}^{n} \) the \( n \)-step \((\alpha, \mathcal{M})\)-behaviour of \( x \) in \( X \).

Definition 3.2 (Graded behavioural equivalence). States \( x \in X, y \in Y \) in \( G \)-coalgebras \( (X, y) \) and \( (Y, \delta) \) are depth-\( n \) behaviourally equivalent under \((\alpha, \mathcal{M})\) if \( \nu^{(n)}(x) = \delta^{(n)}(y) \), and \((\alpha, \mathcal{M})\)-behaviourally equivalent if \( \nu^{(n)}(x) = \delta^{(n)}(y) \) for all \( n \in \omega \). We refer to \((\alpha, \mathcal{M})\)-behavioural equivalence as a graded behavioural equivalence or just a graded equivalence.

Example 3.3. We recall [15, Section 4] several graded equivalences, restricting primarily to LTS and PTS.
Definition 4.1 (Graded algebra). Let $k \in \omega$ and let $\mathcal{M}$ be a graded monad on a category $\mathcal{C}$. An $M_k$-algebra $A$ consists of a family of $\mathcal{C}$-objects $(A_n)_{n \leq k}$ (the carriers) and a family of $\mathcal{C}$-morphisms

$$a^{n,m} : M_n A_m \to A_{n+m} \quad (n + m \leq k)$$

(the structure) such that $a^{0,n} \cdot \eta_{A_n} = \text{id}_{A_n} (n \leq k)$ and

$$\mu^{n,m} \cdot M_n a^{m,r} \cdot M_m a^{r,n} \Rightarrow a^{n,m+r} \quad (4.1)$$

for all $n, m, r \in \omega$ such that $n + m + r \leq k$. The $i$-part of an $M_k$-algebra $A$ is the $M_0$-algebra $(A_i, \alpha^{0,i})$.

A homomorphism from $A$ to an $M_k$-algebra $B$ is a family of $\mathcal{C}$-morphisms $h_n : A_n \to B_n$ ($n \leq k$) such that

$$h_{n+m} \cdot a^{n,m} = h_n \cdot a^{m,n} : M_n h_m \quad \text{for all } n, m, \in \omega \text{ s.th. } n + m \leq k.$$

We write $\text{Alg}_{M_k} (\mathcal{M})$ for the category of $M_k$-algebras and their homomorphisms.

We define $M_\omega$-algebras (and their homomorphisms) similarly, by allowing the indices $n, m, r$ to range over $\omega$.

Remark 4.2. The above notion of $M_\omega$-algebra corresponds with the concept of graded Eilenberg-Moore algebras introduced by Fujita et al. [19]. Intuitively, $M_\omega$-algebras are devices for interpreting terms of unbounded uniform depth. We understand $M_k$-algebras [35] as a refinement of $M_\omega$-algebras which allows the interpretation of terms of uniform depth at most $k$. Thus, $M_k$-algebras serve as a formalism for specifying properties of states exhibited in $k$ steps. For example, $M_1$-algebras are used to interpret one-step modalities of characteristic logics for graded semantics [15, 18]. Moreover, for a depth-1 graded monad, its $M_\omega$-algebras may be understood as complete blocks of $M_1$-algebras [35], and a depth-1 graded monad can be reconstructed from its $M_1$-algebras.

We will be chiefly interested in $M_0$- and $M_1$-algebras.

Example 4.3. Let $\mathcal{M}$ be a graded monad on $\text{Set}$.

(1) An $M_0$-algebra is just an Eilenberg-Moore algebra for the monad $(M_0, \eta, \mu^{0,1})$. It follows that $\text{Alg}_{M_0} (\mathcal{M})$ is complete and cocomplete, in particular has coequalizers.

(2) An $M_1$-algebra is a pair $((A_0, a^{0,0}), (A_1, a^{0,1}))$ of $M_0$-algebras – often we just write the carriers $A_1$ to also denote the algebras, by abuse of notation – equipped with a main structure map $a^{1,0} : M_1 A_0 \to A_1$ satisfying two instances of (4.1). One instance states that $a^{0,1}$ is an $M_0$-algebra homomorphism from $(M_1 A_0, \mu^{1,1})$ to $(A_1, a^{0,1})$ (homomorph); the other expresses that $a^{1,0} \cdot \mu^{1,0} = a^{1,0} \cdot \mu^{0,0}$ (coequalization):

$$M_1 M_0 A_0 \xrightarrow{\mu^{0,0}} M_0 M_1 A_0 \xrightarrow{a^{1,0}} A_1. \quad (4.2)$$

Remark 4.4. The free $M_0$-algebra on a set $X$ is formed in the expected way, in particular has carriers $M_0 X, \ldots, M_0 X$, see [35, Prop. 6.3].

Canonical algebras. We are going to review the basic definitions and results on canonical $M_1$-algebras [15]. Fix a graded monad $\mathcal{M}$ on $\text{Set}$.

We write $(-) : \text{Alg}_{M_1} (\mathcal{M}) \to \text{Alg}_{M_0} (\mathcal{M})$, $i = 0, 1$, for the functor which sends an $M_1$-algebra $A$ to its $i$-part $A_i$ and sends a homomorphism $h : A \to B$ to $h_i : A_i \to B_i$.

Definition 4.5. An $M_1$-algebra $A$ is canonical if it is free over its $0$-part with respect to $(-) : \text{Alg}_{M_1} (\mathcal{M}) \to \text{Alg}_{M_0} (\mathcal{M})$.

Remark 4.6. The universal property of a canonical algebra $A$ is the following: for every $M_1$-algebra $B$ and every $M_0$-algebra homomorphism $h : A_0 \to B_0$, there exists a unique $M_1$-algebra homomorphism $h^\#: A \to B$ such that $(h^\#)_0 = h_0$.

Lemma 4.7 [15, Lem. 5.3]. An $M_1$-algebra $A$ is canonical if and only if (4.2) is a coequalizer in $\text{Alg}_{M_0} (\mathcal{M})$.

Example 4.8. Let $X$ be a set and let $\mathcal{M}$ be a depth-1 graded monad on $\text{Set}$. For each $k \in \omega$, we may view $M_k X$ as an $M_\omega$-algebra with structure $\mu^{0,k}$. For the $M_1$-algebra $(M_k X, M_{k+1} X)$ (with main structure map $\mu^{1,k}$), the instance of Diagram (4.2) required by Lemma 4.7 is a coequalizer by Lemma 2.11; that is, $(M_k X, M_{k+1} X, \mu^{1,k})$ is canonical.

5 PRE-DETERMINIZATION IN EILENBERG-MOORE

We describe a generic notion of pre-determinization (the terminology will be explained in Remark 5.2) for coalgebras of an endofunctor $G$ on $\text{Set}$ with respect to a given depth-1 graded semantics $(\sigma, \mathcal{M})$, generalizing the Eilenberg-Moore-style coalgebraic determination construction by Silva et al. [38]. The behavioural equivalence game introduced in the next section will effectively be played on the pre-determinization of the given coalgebra. We will occasionally gloss over issues of finite branching in the examples.

We first note that every $M_0$-algebra $A$ extends (uniquely) to a canonical $M_1$-algebra $EA$ (with $0$-part $A$), whose $1$-part and main structure are obtained by taking the coequalizer of the pair of morphisms in (4.2) (canonicity then follows by Lemma 4.7). This construction forms the object part of a functor $\text{Alg}_{M_0} (\mathcal{M}) \to \text{Alg}_{M_1} (\mathcal{M})$ which sends a homomorphism $h : A \to B$ to its unique extension $E h := h^\#: EA \to EB$ (cf. Remark 4.6). We write $\overline{M}_1$ for the endofunctor on $\text{Alg}_{M_0} (\mathcal{M})$ given by

$$\overline{M}_1 := (\text{Alg}_{M_0} (\mathcal{M}) \xrightarrow{E} \text{Alg}_{M_1} (\mathcal{M}) \xrightarrow{(-)} \text{Alg}_{M_1} (\mathcal{M})), \quad (5.1)$$

where $(-)_1$ is the functor taking $1$-parts. Thus, for an $M_0$-algebra $A$, $\overline{M}_1 (A)$ is the vertex of the coequalizer (4.2).

By Example 4.8, we have

$$\overline{M}_1 (M_k X, \mu^{0,k}) = (M_{k+1} X, \hat{\mu}^{0,k+1}) \quad (5.2)$$

every set $X$ and every $k \in \omega$. In particular,

$$U \overline{M}_1 F = M_1 \quad (5.3)$$

where $F : \text{Alg}_{M_0} (\mathcal{M}) \to \text{Set}$ is the canonical adjunction of the Eilenberg-Moore category of $M_0$ – that is, $U$ is the forgetful functor, and $F$ takes free $M_0$-algebras, so $FX = (M_0 X, \mu^{0,0})$. For an $M_1$-coalgebra $f : X \to M_1 X = U \overline{M}_1 FX$, we therefore obtain a homomorphism $f^\#: FX \to \overline{M}_1 FX$ (in $\text{Alg}_{M_0} (\mathcal{M})$) via adjoint transposition. This leads to the following pre-determinization construction:
Definition 5.1. Let $(\alpha, \mathcal{M})$ be a depth-1 graded semantics on G-coalgebras. The \textit{pre-determinization} of a G-coalgebra $(X, \gamma)$ under $(\alpha, \mathcal{M})$ is the $\mathcal{M}_1$-coalgebra
\[(\alpha_\chi \cdot \gamma)^k : FX \to \mathcal{M}_1FX.\] (5.4)

Remark 5.2. (1) We call this construction a \textit{pre-determinization} because it will serve as a \textit{determinization} in the expected sense that the underlying graded equivalence transforms into behavioural equivalence on the determinization – only under additive conditions. Notice that given a G-coalgebra $(X, \gamma)$, (finite-depth) behavioural equivalence on the $\mathcal{M}_1$-coalgebra $(\alpha_\chi \cdot \gamma)^0$ is given by the canonical cone into the final chain
\[1 \leftarrow \mathcal{M}_1 \leftarrow \mathcal{M}_1^1 \leftarrow \mathcal{M}_1^2 \leftarrow \cdots \]
while graded behavioural equivalence on $(X, \gamma)$ is given by the maps $\gamma^{(k)}$ into the sequence $M_0, M_1, M_2, \ldots$, equivalently given as homomorphisms $(\gamma^{(k)})^0 : FX \to (M_1, P_{0,1}^{0,k})$, whose codomains can, by (5.2), be written as the sequence
\[F1, \mathcal{M}_1, \mathcal{M}_1^1, \ldots\]
of $M_0$-algebras. The two sequences coincide in case $M_1 = 1$, and indeed one easily verifies that in this case, finite-depth behavioural equivalence on $\mathcal{M}_1$-coalgebras coincides with $(\alpha, \mathcal{M})$-behavioural equivalence. For instance, this holds in the case of probabilistic trace equivalence (Example 3.3.(3)), where $M_0 = D$, so $M_1 = 1$. In the case of trace equivalence (Example 3.3.(2)), $M_1 = 1$ can be ensured by restricting to serial labelled transition systems, which, as noted in Example 2.2.(1), are coalgebras for $\mathcal{P}(\mathcal{A} \times -)$ with $\mathcal{P}$ denoting non-empty powerset, so that in the corresponding variant of the graded monad for trace semantics, we have $M_0 = 1$, and hence $M_0 = 1$.

On the other hand, the condition $M_0 = 1$ fails for trace equivalence of unrestricted systems where we have $M_0 = \mathcal{P}$, in fact constitutes a radical example where behavioural equivalence on the pre-determinization is strictly coarser than the given graded equivalence. In this case, since the actions preserve the bottom 0, we in fact have $\mathcal{M}_1 = 1$: it follows that all states in $\mathcal{M}_1$-coalgebras are behaviourally equivalent (as the unique coalgebra structure on 1 is final).

(2) Using (5.3), we see that the underlying map of the pre-determinization of a coalgebra $(X, \gamma)$ is $(\alpha_\chi \cdot \gamma)^0 : M_0X \to M_1X = U_0\mathcal{M}_1F_0X$ (written using graded Kleisli star as per Notation 2.6).

Indeed, one easily shows that $(\alpha_\chi \cdot \gamma)^0$ is an $M_0$-algebra morphism $(M_0X, \mu_X^{0,0}) \to \mathcal{M}_1(M_0X, \mu_X^{0,0}) = (M_1X, \mu_X^{0,1})$ satisfying $(\alpha_\chi \cdot \gamma)^0 : \mu_X = \alpha_\chi \cdot \gamma$. Thus, this is the adjoint transpose in (5.4).

(3) As indicated above, pre-determinization captures the Eilenberg-Moore style generalized determinization by Silva et al. [38] as an instance. Indeed, for a monad $T$ and an endofunctor $F$, both on the category $\mathcal{C}$, one considers a coalgebra $\gamma : X \to FTX$. Assuming that $FTX$ carries the structure of an Eilenberg-Moore algebra for $T$ (e.g. because the functor $F$ lifts to the category of Eilenberg-Moore algebras for $T$), one obtains an $F$-coalgebra $\gamma^F : TX \to FTX$ by taking the unique homomorphic extension of $\gamma$. Among the concrete instances of this construction are the well-known powerset construction of non-deterministic automata (take $T = \mathcal{P}$ and $F = 2 \times (-)^d$), the non-determinization of alternating automata and that of Markov decision processes [26].

To view this as an instance of pre-determinization, take the graded monad with $M_0 = F^dT$ (Example 2.5.(4)), let $G = FT$, and let $\alpha = idFT$. Using (5.3), we see that $(\alpha_\chi \cdot \gamma)^0$ in (5.4) is the generalized determinization $\gamma^F$ above.

(4) We emphasize that the construction applies completely universally; e.g. we obtain as one instance a ‘determinization’ of serial labelled transition systems modulo similarity, which transforms a coalgebra $X \to \mathcal{P}^d(\mathcal{A} \times X)$ into an $\mathcal{M}_1$-coalgebra $\mathcal{P}^d(X) \to \mathcal{P}^d(\mathcal{A} \times \mathcal{P}^d(X))$ (Example 2.10.(4)); instantiating the observations in item (1), we obtain that finite-depth behavioural equivalence on $\mathcal{M}_1$-coalgebras (see Example 5.3 for the description of $\mathcal{M}_1$) coincides with finite-depth mutual similarity.

Example 5.3. We give a description of the functor $\mathcal{M}_1$ on $M_0$-algebras constructed above in some of the running examples.

(1) For graded monads of the form $\mathcal{M}_G$, which capture finite-depth behavioural equivalence (Example 2.5.(1)), we have $M_0 = Id$, so $M_0$-algebras are just sets, and under this correspondence, $\mathcal{M}_1$ is the original functor $G$.

(2) Trace semantics of LTS (Example 2.10.(3)): Distribution of actions over the join semilattice operations ensures that depth-1 terms over a join semilattice $\mathcal{A}$ can be normalized to sums of the form $\sum_{a \in \mathcal{A}} a(x_a)$, with $x_a \in X$ (possibly $x_a = 0$). It follows that $\mathcal{M}_1$ is simply given by $\mathcal{M}_1(X) = X^\mathcal{A}$ ($\mathcal{A}$-th power, where $\mathcal{A}$ is the finite set of labels). Other forms of trace semantics are treated similarly.

(3) In the graded theory for simulation (Example 2.10.(4)), the description of the induced graded monad [15] extends analogously to $\mathcal{M}_1$, yielding that $\mathcal{M}_1B$ is the join semilattice of finitely generated downwards closed subsets of $\mathcal{A} \times B$ where, again, $\mathcal{A}$ carries the discrete ordering.

Remark 5.4. The assignment $\mathcal{M} \mapsto \mathcal{M}_1$ exhibits the category $\mathcal{K}$ of depth-1 graded monads whose 0-part is the monad $(\mathcal{M}_0, \eta, \mu^{0,0})$ as a coreflective subcategory (up to isomorphism) of the category $\text{Fun}(\text{Set}^{\mathcal{M}_0})$ of all endofunctions on the Eilenberg-Moore category of that monad.

Indeed, given an endofunctor $H$ on $\text{Set}^{\mathcal{M}_0}$ we form the 6-tuple $(M_0UHF, \eta, \mu^{0,0}, \mu^{0,1}, \mu^{1,0})$, where the latter two natural transformations arise from the counit $\epsilon : FU \to Id$ of the canonical adjunction $F + U : \text{Alg}_{\mathcal{M}_0}(M_0) \to \text{Set}$:
\[
\mu^{0,1} = (M_0UHF = UUFUHF \xrightarrow{UHF} UHF); \quad \mu^{1,0} = (UHF\mathcal{M}_0 = UHUFUHF \xrightarrow{UHF} UHF).
\] It is not difficult to check that this data satisfies all applicable instances of the graded monad laws. Hence, it specifies a depth-1 graded monad $R(H)$ [15, Thm. 3.7]. This assignment is the object part of a functor $R : \text{Fun}(\text{Set}^{\mathcal{M}_0}) \to \mathcal{K}$.

In the other direction, we have for each depth-1 graded monad $\mathcal{M}$ with 0-part $M_0$ the endofunctor $I(\mathcal{M}) = \mathcal{M}_1$. By (5.3), we have $RI(\mathcal{M}) = \mathcal{M}$. Now, given a depth-1 graded monad $\mathcal{M}$ and an endofunctor $H$ on $\text{Set}^{\mathcal{M}_0}$, consider $\mathcal{M}_1 = IR(H)$ (so that $M_1 = UHF$). We obtain for every algebra $(A, a)$ in $\text{Set}^{\mathcal{M}_0}$ a homomorphism
c_{(A,a)} : \overline{M}_1(A,a) \rightarrow H(A,a) by using the coequalizer defining \( \overline{M}_1(A,a) \) (cf. Lemma 4.7):

\[
\begin{array}{c}
M_1 M_0 A \xrightarrow{\mu^0_A \mu} M_1 A = HFA \\
\xrightarrow{H a} \overline{M}_1(A,a)
\end{array}
\]

Note that \( M_1 M_0 A \) is the carrier of the Eilenberg-Moore algebra \( HFA = H(M_0 A, \mu^0_A) \) and similarly for the middle object (in both cases we have omitted the algebra structures given by \( \mu^0_A \) and \( \mu^0_M \) coming from the graded monad \( I(H) \)). It is easy to see that the homomorphism \( Ha \) merges the parallel pair, and therefore we obtain the dashed morphism such that the triangle commutes, yielding the components of a natural transformation \( c : \overline{M}_1 \rightarrow H \) which is couniversal: for each depth-1 graded monad \( N \)

\[ \text{categorical definition. Given a coalgebra}\]

\[ \text{is a pair of depth-0 terms over}\]

\[ \text{phisms}\]

\[ \ell(\_ \cdot \_)(m) = h. \text{This shows that } I \vdash R.\]

6 BEHAVIOURAL EQUIVALENCE GAMES

Let \( S = (\alpha, \mathcal{M}) \) be a depth-1 graded semantics for an endofunctor \( G \) on \( \mathbf{Set} \). We are going to describe a game for playing out depth-\( n \) behavioural equivalence under \( S \)-semantics on states in \( G \)-coalgebras.

We first give a description of the game in the syntactic language of graded equational reasoning, and then present a more abstract categorical definition. Given a coalgebra \( (X, \gamma) \), we will see the states in \( X \) as variables, and the map \( \alpha X : \gamma \) as assigning to each variable \( x \) a depth-1 term over \( X \); we can regard this assignment as a (uniform-depth) substitution \( \sigma \). A configuration of the game is a pair of depth-0 terms over \( X \); to play out the equivalence of states \( x, y \in X \), the game is started from the initial configuration \( (x, y) \). Each round of the game then proceeds in two steps: First, Duplicator plays a set \( Z \) of equalities between depth-0 terms over \( X \) that she claims to hold under the semantics. This move is admissible in the configuration \( (s, t) \) if \( Z + sa = ta \). Then, Spoiler challenges one of the equalities claimed by Duplicator, i.e. picks an element \( (s', t') \in Z \), which then becomes the next configuration. Any player who cannot move, loses. After \( n \) rounds have been played, reaching the final configuration \( (s, t) \), Duplicator wins if \( s \sigma = t \sigma \) is a valid equality, where \( \sigma \) is a substitution that identifies all variables. We refer to this last check as calling the bluff. Thus, the game plays out an equational proof between terms obtained by unfolding depth-0 terms according to \( \sigma \), cutting off after \( n \) steps.

We introduce some technical notation to capture the notion of admissibility of \( Z \) abstractly:

Notation 6.1. Let \( Z \subseteq M_0 X \times M_0 X \) be a relation, and let \( c_Z : M_0 X \rightarrow C_Z \) be the coequalizer in \( \mathbf{Alg}(\mathcal{M}) \) of the homomorphisms \( c_\ell, c_r : M_0 Z \rightarrow M_0 X \) given by applying the Kleisli star (2.3) to the projections \( \ell, r : Z \rightarrow M_0 X \). We define a homomorphism \( Z : M_0 X \rightarrow M_1 C_Z \) in \( \mathbf{Alg}(\mathcal{M}) \) by

\[
Z = (M_0 X \xrightarrow{\alpha_X \gamma} M_1 M_0 X \xrightarrow{\overline{M}_1(C_Z)} \overline{M}_1 C_Z)
\]

(6.1)

(omitting algebra structures, and again using the Kleisli star).

Remark 6.2. Using designators as in Notation 6.1, we note:

1. By the universal property of \( \eta_Z : Z \rightarrow M_0 Z \), an \( M_0 \)-algebra homomorphism \( h : M_0 X \rightarrow A \) merges \( \ell \), \( r \) if it merges \( c_\ell, c_r \). This implies that the coequalizer \( M_0 X \xrightarrow{\epsilon_Z} C_Z \) quotients the free \( M_0 \)-algebra \( M_0 X \) by the congruence generated by \( Z \). Also, it follows that in case \( Z \) is already an \( M_0 \)-algebra and \( \ell, r : Z \rightarrow M_0 X \) are \( M_0 \)-algebra homomorphisms (e.g. when \( Z \) is a congruence), one may take \( c_Z : M_0 X \rightarrow C_Z \) to be the coequalizer of \( \ell, r \).

2. The map \( Z : M_0 X \rightarrow \overline{M}_1 C_Z \) associated to the relation \( Z \) on \( M_0 X \) may be understood as follows. As per the discussion above, we view the states of the coalgebra \( (X, \gamma) \) as variables, and the map \( \gamma \xrightarrow{\alpha_X} GX \xrightarrow{\overline{M}_1} M_1 X \) as a substitution mapping a state \( x \in X \) to the equivalence class of depth-1 terms encoding the successor structure \( \gamma(x) \). The second factor \( \overline{M}_1 C_Z \) in (6.1) then essentially applies the relations given by the closure of \( Z \) under congruence w.r.t. depth-0 operations, embodied in \( C_Z \) as per (1), under depth-1 operations in (equivalence classes of) of depth-1 terms in \( M_1 X \); to sum up, \( \overline{M}_1 C_Z \) merges a pair of equivalence classes \( \{t, t'\} \) iff \( Z + t + t' = t' \) in a depth-1 theory presenting \( \mathcal{M} \) (in notation as per Section 2).

Definition 6.3. For \( n \in \omega \), the \( n \)-round \( S \)-behavioural equivalence game \( G_n(y) \) on a \( G \)-coalgebra \((X, \gamma)\) is played by Duplicator (D) and Spoiler (S). Configurations of the game are pairs \((s, t) \in M_0 X \times M_0 X \). Starting from an initial configuration designated as needed, the game is played for \( n \) rounds. Each round proceeds in two steps, from the current configuration \((s, t)\): First, D chooses a relation \( Z \subseteq M_0 X \times M_0 X \) such that \( Z(s) = Z(t) \) (for \( Z \) as per Notation 6.1). Then, S picks an element \((s', t') \in Z \), which becomes the next configuration. Any player who cannot move at his turn, loses. After \( n \) rounds have been played, D wins if \( M_0!(s_n) = M_0!(t_n) \); otherwise, S wins.

Remark 6.4. By the description of \( Z \) given in Remark 6.2.(2), the categorical definition of the game corresponds to the algebraic one given in the lead-in discussion. The final check whether \( M_0!(s_n) = M_0!(t_n) \) corresponds to what we termed calling the bluff. The apparent difference between playing either on depth-0 terms or on elements of \( M_0 X \), i.e. depth-0 terms modulo derivable equality, is absorbed by equational reasoning from \( Z \), which may incorporate also the application of depth-0 equations.

Remark 6.5. A pair of states coming from different coalgebras \((X, \gamma)\) and \((Y, \delta)\) can be treated by considering those states as elements of the coproduct of the two coalgebras:

\[ X + Y \xrightarrow{\epsilon_\Lambda} GX + GY \xrightarrow{[\text{GIn}][\text{Gimr}]} G(X + Y), \]

where \( \xrightarrow{\text{inl}} X + Y \xrightarrow{\text{inr}} Y \) denote the coproduct injections. There is an evident variant of the game played on two different coalgebras \((X, \gamma)\), \((Y, \delta)\), where moves of D are subsets of \( M_0 X \times M_0 Y \). However, completeness of this version depends on additional assumptions on \( \mathcal{M} \), to be clarified in future work. For instance, if we instantiate the graded monad for traces with effects specified by \( T \) (Example 2.5.(3)) to \( T \) being the free real vector space monad, and a state \( x \in X \) has successor structure \( 2 \cdot x' = 2 \cdot x'' \), then D
can support equivalence between $x$ and a deadlock $y \in Y$ (with successor structure 0) by claiming that $x' = x''$, but not by any equality between terms over $X$ with terms over $Y$. That is, in this instance, the variant of the game where $D$ plays relations on $M_0 X \times M_0 Y$ is not complete.

Soundness and completeness of the game with respect to $S$-behavioural equivalence is stated as follows.

**Theorem 6.6.** Let $(a, \mathcal{M})$ be a depth-1 graded semantics for a functor $G$ such that $M_1 \mathcal{M}$ preserves monomorphisms, and let $(X, \gamma)$ be a $G$-coalgebra. Then, for all $n \in \omega$, $D$ wins $(s, t)$ in $\mathcal{G}_n(\gamma)$ if and only if $(\gamma^n)_0(s) = (\gamma^n)_0(t)$.

**Corollary 6.7.** States $x, y$ in a $G$-coalgebra $(X, \gamma)$ are $S$-behaviourally equivalent if and only if $D$ wins $(\eta(x), \eta(y))$ for all $n \in \omega$.

**Remark 6.8.** In algebraic terms, the condition that $M_1 \mathcal{M}$ preserves monomorphisms amounts to the following: In the derivation of an equality of depth-1 terms $s, t$ over $X$ from depth-0 relations over $X$ (i.e. from a presentation of an $M_0$-algebra by relations on generators $X$), if $X$ is included in a larger set $Y$ of variables with relations that conservatively extend those on $X$, i.e. do not imply additional relations on $X$, then it does not matter whether the derivation is conducted over $X$ or more liberally over $Y$. Intuitively, this property is needed because not all possible $n$-step behaviours, i.e. elements of $Y = M_0 A$, are realized by states in a given coalgebra on $X$.

Preservation of monos by $M_1 \mathcal{M}$ is automatic for graded monads of the form $\mathbb{M}_G$ (Example 2.5.(1)), since $M_0 = \text{Id}$ in this case. In the other running examples, preservation of monos is by the respective descriptions of $M_1 \mathcal{M}$ given in Example 5.3.

**Example 6.9.** We take a brief look at the instance of the generic game for the case of bisimilarity on finitely branching LTS (more extensive examples are in Section 8), i.e. we consider the depth-1 graded semantics $(\text{id}, \mathbb{M}_G)$ for the functor $G = P((\mathcal{A} \times (-)))$. In this case, $M_0 = \text{id}$, so when playing on a coalgebra $(X, \gamma)$, $D$ plays relations $Z \subseteq X \times X$. If the successor structures of states $x, y$ are represented by depth-1 terms $\sum_i a_i(x_i) + \sum_j b_j(y_j)$, respectively, in the theory $\mathcal{JSL(\mathcal{A})}$ (Example 2.10.(1)), then $D$ is allowed to play $Z$ iff the equality $\sum_i a_i(x_i) = \sum_j b_j(y_j)$ is entailed by $Z$ in $\mathcal{JSL(\mathcal{A})}$. This, in turn, holds iff for each $i$, there is $j$ such that $a_i = b_j$ and $(x_i, y_j) \in Z$, and symmetrically. Thus $Z$ may be seen as a pre-announced non-deterministic winning strategy for $D$ in the usual bisimilarity game where $S$ moves first (Section 2); $D$ announces that if $S$ moves from, say, $x$ to $x_i$, then she will respond with some $y_j$ such that $a_i = b_j$ and $(x_i, y_j) \in Z$.

7 INFINITE-DEPTH BEHAVIOURAL EQUIVALENCE

We have seen in Section 5 that in case $M_0 l = 1$, $(a, \mathcal{M})$-behavioural equivalence on $G$-coalgebras coincides, via a determinization construction, with finite-depth behavioural equivalence on $M_1 \mathcal{M}$-coalgebras for a functor $M_1 \mathcal{M}$ on $M_0$-algebras constructed from $\mathcal{M}$. If $G$ is finitary, then finite-depth behavioural equivalence coincides with full behavioural equivalence (Remark 2.3), but in general, finite-depth behavioural equivalence is strictly coarser. Previous treatments of graded semantics stopped at this point, in the sense that for non-finitary functors (which describe infinitely branching systems), they did not offer a handle on infinite-depth equivalences such as full bisimilarity. In case $M_0 l = 1$, a candidate for a notion of infinite-depth equivalence induced by a graded semantics arises via full behavioural equivalence of $M_1 \mathcal{M}$-coalgebras. We fix this notion explicitly:

**Definition 7.1.** States $x, y$ in a $G$-coalgebra $(X, \gamma)$ are infinite-depth $(a, \mathcal{M})$-behaviourally equivalent if $\eta(x)$ and $\eta(y)$ are behaviourally equivalent in the pre-determinization of $(X, \gamma)$ as described in Section 6.

We hasten to re-emphasize that this notion in general only makes sense in case $M_0 l = 1$. We proceed to show that infinite-depth equivalence is in fact captured by an infinite variant of the behavioural equivalence game of Section 6.

Since infinite depth-equivalences differ from finite-depth ones only in settings with infinite branching, we do not assume in this section that $G$ or $\mathcal{M}$ are finitary, and correspondingly work with generalized graded theories where operations may have infinite arities [35]; we assume arities to be cardinal numbers. We continue to be interested only in depth-1 graded monads and theories, and we fix such a graded monad $\mathcal{M}$ and associated graded theory for the rest of this section. The notion of derivation is essentially the same as in the finitary case, the most notable difference being that the congruence rule is now infinitary, as it has one premise for each argument position of a given possibly infinitary operator. We do not impose any cardinal bound on the arity of operations; if all operations have arity less than $\kappa$ for a regular cardinal $\kappa$, then we say that the monad is $\kappa$-ary.

**Remark 7.2.** One can show using tools from the theory of locally presentable categories that $M_1 \mathcal{M}$ has a final coalgebra if $\mathcal{M}$ is $\kappa$-ary in the above sense. To see this, first note that $\text{Alg}_0(\mathcal{M})$ is locally $\kappa$-presentable if $M_0 l$ is $\kappa$-accessible [6, Remark 2.78]. Using a somewhat similar argument one can prove that $\text{Alg}_1(\mathcal{M})$ is also locally $\kappa$-presentable. Moreover, the functor $\mathcal{M}_1 \mathcal{M}$ is $\kappa$-accessible, being the composite (5.1) of the left adjoint $E: \text{Alg}_0(\mathcal{M}) \to \text{Alg}_1(\mathcal{M})$ (which preserves all colimits) and the 1-part functor $(\cdot): \text{Alg}_1(\mathcal{M}) \to \text{Alg}_0(\mathcal{M})$, which preserves $\kappa$-filtered colimits since those are formed componentwise. It follows that $M_1 \mathcal{M}$ has a final coalgebra [6, Exercise 2]). Alternatively, existence of a final $M_1 \mathcal{M}$-coalgebra will follow from Theorem 7.12 below.

Like before, we assume that $M_1 \mathcal{M}$ preserves monomorphisms.

**Example 7.3.** We continue to use largely the same example theories as in Example 2.10, except that we allow operations to be infinitary. For instance, the graded theory of complete join semilattices over $\mathcal{A}$ has as depth-1 operations all formal sums $\sum_{i \in I} a_i(-)$ where $I$ is now some (possibly infinite) index set; the axioms are then given in the same way as in Example 2.10.(1), and all depth-1 equations

$$\sum_{i \in I} a_i(x_i) = \sum_{j \in J} b_j(y_j)$$

such that $(\{a_i(x_i)\} \mid i \in I) = (\{b_j(y_j)\} \mid j \in J)$. This theory presents the graded monad $M_G$ for $G = \mathcal{P}(\mathcal{A} \times (-))$.

The infinite game may then be seen as defining a notion of derivable equality on infinite-depth terms by playing out a non-standard, infinite-depth equational proof; we will make this view
explicit further below. In a less explicitly syntactic version, the game is defined as follows.

**Definition 7.4** (Infinite behavioural equivalence game). The infinite \((\alpha, \beta, \alpha)\)-behavioural equivalence game \(G_\infty(y)\) on a \(G\)-coalgebra \((X, y)\) is played by Spoiler (S) and Duplicator (D) in the same way as the finite behavioural equivalence game (Definition 6.3) except that the game continues forever unless one of the players cannot move. Any player who cannot move, loses. Infinite matches are won by D.

As indicated above, this game captures infinite-depth \((\alpha, \beta)\)-behavioural equivalence (under the running assumption that \(M_1\) preserves monomorphisms):

**Theorem 7.5.** Given a \(G\)-coalgebra \((X, y)\), two states \(s, t\) in the pre-determination of \(y\) are behaviourally equivalent iff D wins the infinite \((\alpha, \beta)\)-behavioural equivalence game \(G_\infty(y)\) from the initial configuration \((s, t)\).

**Corollary 7.6.** Two states \(x, y\) in a \(G\)-coalgebra \((X, y)\) are infinite-depth \((\alpha, \beta)\)-behaviourally equivalent iff D wins the infinite \((\alpha, \beta)\)-behavioural equivalence game \(G_\infty(y)\) from the initial configuration \((\eta(x), \eta(y))\).

**Remark 7.7.** Like infinite-depth \((\alpha, \beta)\)-behavioural equivalence, the infinite \((\alpha, \beta)\)-behavioural equivalence game is sensible only in case \(M_0\) is 1. For instance, as noted in Section 5, in the graded monad for trace semantics (Example 2.10(2)), which does not satisfy this condition, behavioural equivalence of \(M_1\)-coalgebras is trivial. In terms of the game, D wins every position in \(G_\infty(y)\) by playing \(Z = \{(t, 0) \mid t \in M_0X\}\) since the actions preserve the bottom element 0, this is always an admissible move. In the terminology introduced at the beginning of Section 6, the reason that D wins in this way is that in the infinite game, her bluff is never called \((M_0!t(0)\) will in general not equal \(M_0!0\). However, see Example 7.8(1) below.

**Example 7.8.** (1) As noted in Remark 5.2(1), the graded monad for trace semantics can be modified to satisfy the condition \(M_0 = 1\) by restricting to well-founded transition systems. By modification of Example 5.3(2), we obtain that in this setting, \(M_1X\) consists of partial maps \(A \to X\) that are not everywhere undefined. It follows that in this case, infinite-depth \((\alpha, \beta)\)-behavioural equivalence is just finite trace equivalence, and thus coincides with plain \((\alpha, \beta)\)-behavioural equivalence.

(2) In the case of graded monads \(\mathcal{M}_G\) (Example 2.5(1)), which so far were used to capture finite-depth behavioural equivalence in the standard (branching-time) sense, we have \(M_0 = M_1\), in particular, \(M_0 = 1\). In this case, the infinite-depth behavioural equivalence game instantiates to a game that characterizes full behavioural equivalence of \(G\)-coalgebras. Effectively, a winning strategy of D in the infinite game \(G_\infty(y)\) on a \(G\)-coalgebra \((X, y)\) amounts to a relation \(R \subseteq X \times X\) (the positions of D actually reachable when D follows her winning strategy) that is a precongruence on \((X, y)\).

**Remark 7.9** (Fixpoint computation). Via its game characterization (Theorem 7.5), infinite-depth \((\alpha, \beta)\)-behavioural equivalence can be cast as a greatest fixpoint, specifically of the monotone function \(F\) on \(\mathcal{P}(M_0X \times M_0X)\) given by

\[
F(Z) = \{(s, t) \in M_0X \times M_0X \mid \mathcal{Z}(s) = \mathcal{Z}(t)\}.
\]

If \(M_0\) preserves finite sets, then this fixpoint can be computed on a finite coalgebra \((X, y)\) by fixpoint iteration; since \(F(Z)\) is clearly always an equivalence relation, the iteration converges after at most \(|M_0X|\) steps, e.g., in exponentially many steps in case \(M_0 = \mathcal{P}\). In case \(M_0X\) is infinite (e.g., if \(M_0 = \mathcal{D}\)), then one will need to work with finite representations of subspaces of \(M_0X \times M_0X\). We leave a more careful analysis of the algorithmics and complexity of solving infinite \((\alpha, \beta)\)-behavioural equivalence games to future work. We do note that on finite coalgebras, we may assume w.l.o.g. that both the coalgebra functor \(G\) and graded monad \(M_1\) are finitary, as we can replace them with their finitary parts if needed (e.g., the powerset functor \(\mathcal{P}\) and the finite powerset functor \(\mathcal{P}_1\) have essentially the same finite coalgebras). If additionally \(M_0 = 1\), then \((\alpha, \beta)\)-behavioural equivalence coincides with infinite-depth \((\alpha, \beta)\)-behavioural equivalence, so that we obtain also an algorithmic treatment of \((\alpha, \beta)\)-behavioural equivalence. By comparison, such a treatment is not immediate from the finite version of the game, in which the number of rounds is effectively chosen by Spoiler in the beginning.

Assume from now on that \(\beta\) is \(\kappa\)-ary. We note that in this case, we can describe the final \(M_1\)-coalgebra in terms of a syntactic variant of the infinite game that is played on infinite-depth terms, defined as follows.

**Definition 7.10** (Infinite-depth terms). Recall that we are assuming a graded signature \(\Sigma\) with operations of arity less than \(\kappa\). A (uniform) infinite-depth \((\Sigma)\)-term is an infinite tree with ordered branching where each node is labelled with an operation \(\sigma \in \Sigma\), and then has as many children as given by the arity of \(\sigma\); when there is no danger of confusion, we will conflate nodes with (occurrences of) operations. We require moreover that every infinite path in the tree contains infinitely many depth-1 operations (finite full paths necessarily end in constants). We write \(T_{\Sigma, \infty}\) for the set of infinite-depth \(\Sigma\)-terms. By cutting off at the top-most depth-1 operations, we obtain for every \(t \in T_{\Sigma, \infty}\) a top-level decomposition \(t = t_1\sigma\) into a depth-1 term \(t_1 \in T_{\Sigma, 1}(X)\), for some set \(X\), and a substitution \(\sigma: X \to T_{\Sigma, \infty}\).

**Definition 7.11.** The syntactic infinite \((\alpha, \beta)\)-behavioural equivalence game \(G_\text{syn}^\infty\) is played by S and D. Configurations of the game are pairs \((s, t)\) of infinite-depth \(\Sigma\)-terms. For such \((s, t)\), we can, by the running assumption that \(M_1\) preserves monomorphisms, that the top level decompositions \(s = s_1\sigma, t = t_1\sigma\) are such that \(s_1, t_1 \in T_{\Sigma, 1}(X)\), \(\sigma: X \to T_{\Sigma, \infty}\) for the same \(X, \sigma\). Starting from a designated initial configuration, the game proceeds in rounds. In each round, starting from a current such configuration \((s, t)\), D first chooses a relation \(R \subseteq T_{\Sigma, 0}(X) \times T_{\Sigma, 0}(X)\) such that \(R \vdash s_1 = t_1\) in the graded theory that presents \(\mathcal{M}\) (cf. Section 2). Then, S selects an element \((u, v)\) in \(Z\), upon which the game reaches the new configuration \((u\sigma, v\sigma)\). The game proceeds forever unless a player cannot move. Again, any player who cannot move, loses, and infinite matches are won by D. We write \(s \sim_G t\) if D wins \(G_\text{syn}^\infty\) from position \((s, t)\).
We construct an $\mathcal{M}_T$-coalgebra on the set $U = \mathcal{T}_\infty \sim g$ of infinite-depth terms modulo the winning region of $D$ as follows. We make $U$ into an $\mathcal{M}_D$-algebra by letting depth-0 operations act by term formation. We then define the coalgebra structure $\zeta : U \to \mathcal{M}_T U$ by $\zeta(q(t_1)) = \mathcal{M}_T((q \cdot \sigma^0_1)(t_1))$ (using Kleisli stars as per Notation 2.6) where $t_1 \sigma$ is a top-level decomposition of an infinite-depth term, with $t_1 \in \mathcal{T}_1(X)$:

\[-: \mathcal{T}_1(X) \to \mathcal{M}_1 \mathcal{M}_0 X \quad \text{and} \quad q : \mathcal{T}_\infty \sim g \to U \text{ denote canonical quotient maps. These data are well-defined.}

**Theorem 7.12.** The coalgebra $(U, \zeta)$ is final.

### 8 CASE STUDIES

We have already seen (Example 6.9) how the standard bisimulation game arises as an instance of our generic game. We elaborate on some further examples.

**Simulation equivalence.** We illustrate how the infinite $(\alpha, \emptyset \emptyset \emptyset)$-behavioural equivalence game can be used to characterise simulation equivalence [20] on serial LTS. We have described the graded theory of simulation in Example 3.3.(4). Recall that it requires actions to be monotone, via the depth-1 equation $a(x+y) = a(x)+a(y)$. When trying to show that depth-1 terms $\sum_{i \in I} a_i(t_i)$ and $\sum_{j \in J} b_j(s_j)$ are equal, $D$ may exploit that over join semantics, inequalities can be expressed as equalities ($x \leq y$ iff $x+y = y$), and instead endeavour to show inequalities in both directions. By the monotonicity of actions, $\sum_{i \in I} a_i(t_i) \leq \sum_{j \in J} b_j(s_j)$ is implied by $D$ claiming, for each $i$, that $t_i \leq s_j$ for some $j$ such that $a_i = b_j$; symmetrically for $\geq$ (and by the description of the relevant graded monad as per Example 2.10.(4), this proof principle is complete). Once $S$ challenges either a claim of the form $t_i \leq s_j$ or one of the form $t_i \geq s_j$, the direction of inequalities is fixed for the rest of the game; this corresponds to the well-known phenomenon that in the standard pebble game for similarity, $S$ cannot switch sides after the first move. Like for bisimilarity (Example 6.9), the game can be modified to let $S$ move first: $S$ first picks, say, one of the terms $t_i$, and $D$ responds with an $s_j$ such that $a_i = b_j$, for which she claims $t_i \leq s_j$. Overall, the game is played on positions in $\mathcal{P}^*(X) \times \mathcal{P}^*(X)$, but if started on two states $x, y$ of the labelled transition systems, i.e. in a position of the form $(\{x\}, \emptyset)$, the game forever remains in positions where both components are singletons, and thus is effectively played on pairs of states. Summing up, we recover exactly the usual pebble game for mutual similarity. Variants such as complete, failure, or ready simulation are captured by minor modifications of the graded semantics [15].

**T-structured trace equivalence.** Fix a set $\mathcal{A}$ and a finitary monad $T$ on $\mathcal{S}$. We are going to consider the $(id, \mathcal{M}_T(\mathcal{A}))$-behavioural equivalence game on coalgebras for the functor $T(\mathcal{A} \times \cdot)$ (cf. Example 2.5.(3)).

**Notation 8.1.** Fix a presentation $(\Sigma', E')$ of $T$ (i.e. an equational theory in the sense of universal algebra). We generalize the graded trace theory described in Example 2.10.(3) to a graded theory $T = (\Sigma, \varepsilon)$ for $\mathcal{M}_T(\mathcal{A})$ as follows: $(\Sigma', E')$ forms the depth-0 part of $T$ and, at depth-1, $T$ has unary actions $a(-)$ which distribute over all operations $f \in \Sigma'$:

$a(f(s_1, \ldots, s_{\sigma(f)})) = f(a(s_1), \ldots, a(s_{\sigma(f)}))$

The arising theory $T$ presents $\mathcal{M}_T(\mathcal{A})$.

Recall from Remark 6.2.(1) that since, in this setting, $\mathcal{M}_0 = T$, a legal move for $D$ in position $(s, t) \in TX \times TX$ is a relation $Z$ on $TX$ such that equality of the respective successors $((s \cdot \gamma)^\emptyset_0(s)$ and $(t \cdot \gamma)^\emptyset_0(t)$, viewed as (equivalence classes of) depth-1 terms, is derivable in the theory $T$ under assumptions $Z$.

**Remark 8.2.** A natural question is whether there exist algorithms for deciding if a pair $(\alpha, \gamma)^\emptyset_0(s)$, $(\alpha, \gamma)^\emptyset_0(t)$ sits in the congruence closure of $Z$. In fact, there are algorithms to check congruence closure of depth-0 terms for the powerset monad $T = \mathcal{P}_T[11]$ and for certain semiring monads [10]. The idea behind those algorithms is to obtain rewrite rules from pairs in $Z$, and two elements are in the congruence closure if and only if they can be rewritten to the same normal form. Applying depth-1 equations to normal forms could potentially yield a method to check automatically whether a given pair of $M_1$-terms lies in the congruence closure of $Z$.

**Finite-trace equivalence.** More concretely, we examine the behavioural equivalence game for trace equivalence on finitely branching LTS (i.e. $(id, \mathcal{M}_\mathcal{P}(\mathcal{A}))$-semantics as per Example 3.3.(2)).

**Example 8.3.** Consider the following process terms representing a coalgebra $\gamma$ (in a fragment of CCS):

$p_1 \equiv a_1 p_1' + b_1 p_2' + p_3' \quad p_2 \equiv a_2 p_1' + b_2 p_2' + b_3 p_3' \quad p_3 \equiv b_3 p_3'$

where $p_1', p_2', p_2'', p_3'$ are deadlocked. It is easy to see that $s = \{p_1, p_2\}$ and $t = \{p_2, p_3\}$ are trace equivalent: In particular, $s, t$ have the same traces of length 1. We show that $D$ has a winning strategy in the 1-round $(id, \mathcal{M}_\mathcal{P}(\mathcal{A}))$-behavioural equivalence game at $(s, t)$. Indeed, the relation $Z = \{p_1' + p_2' = p_2', p_2' + p_3'' = p_3''\}$ is admissible at $(s, t)$: We must show that equality of $(\alpha, \gamma)^\emptyset(s) = a(p_1') + a(p_1') + b(p_2')$ and $(\alpha, \gamma)^\emptyset(t) = a(p_3') + b(p_2') + b(p_3')$ is entailed by $Z$. To see this, note that

$Z \vdash a(p_1') + a(p_1') = a(p_1')$ and $Z \vdash b(p_2') = b(p_2') + b(p_3')$.

Moreover, the pairs $\{(p_1', p_2'), (p_3')\}$ and $\{(p_2', p_3'), (p_3')\}$ are both identified by $M_0!$ (all terms are mapped to $\{e\}$ when $1 = \{e\}$). That is, $Z$ is a winning move for $D$.

In general, admissible moves of $D$ can be described via a normalisation of depth-1 terms as follows:

**Proposition 8.4.** In $(id, \mathcal{M}_\mathcal{P}(\mathcal{A}))$, every depth-1 term is derivably equal to one of the form $\sum_{a \in A}(a(t_a))$, with depth-0 terms (i.e. finite, possibly empty, sets) $t_a$. Over serial LTS (i.e. $T = \mathcal{P}_T$), every depth-1 term has a normal form of the shape $\sum_{a \in A}(a(t_a))$ with $B \in \mathcal{P}_T^* A$ (where the $t_a$ are now finite and non-empty).

**Proposition 8.5.** Let $\rho = \sum_{a \in A} a(t_a)$ be depth-1 terms over $X$ in normal form, for $\rho \in (s, t)$. Then a relation $Z \subseteq \mathcal{P}_X \times \mathcal{P}_X$ is a legal move of $D$ in position $(s, t)$ iff the following conditions hold for all $a \in A$, in the notation of Proposition 8.4:

1. $\forall x \in s_a. \exists t' \quad (t' \subseteq t_a \land Z \cap x \subseteq t')$
2. $\forall y \in t_a. \exists s' \quad (s' \subseteq s_a \land Z \cap y \subseteq s')$

where, again, $s, t \approx t$ abbreviates $s + t = t$. Over serial LTS (i.e. $T = \mathcal{P}_T$), and for normal forms $\rho = \sum_{a \in B_a} a(t_a)$, a relation $Z \subseteq \mathcal{P}_X \times \mathcal{P}_X$ is a legal move of $D$ in position $(s, t)$ iff $B_a = B_t$ and the above conditions hold for all $a \in B_a$. 


To explain terminology, we note at this point that by the above, in particular $Z = \{(x,0) \mid x \in X\} \cup \{(0,y) \mid y \in X\}$ is always admissible. Playing $Z$, D is able to move in every round, bluffing her way through the game; but this strategy does not win in general, as her bluff is called at the end (cf. Section 6). More reasonable strategies work as follows.

On the one hand, D can restrict herself to playing the bisimulation relation on the determined transition system because the term $s'$ (resp. $t'$) can be taken to be exactly $s_0$ (resp. $t_0$) in Condition 2 (resp. Condition 1). This form of the game may be recast as follows. Each round consists just of S playing some $a \in \mathcal{A}$ (or $a \in B_1$ in the serial case), moving to $(s_0, t_0)$ regardless of any choice by D. In the non-serial case, the game runs until the bluff is called after the last round. In the serial case, D wins if either all rounds are played or as soon as $B_s = B_t = 0$, and $S$ wins as soon as $B_s \neq B_t$.

On the other hand, D may choose to play in a more fine-grained manner, playing one inequality $x \leq t'$ for every $x \in s_2$ and one inequality $s' \geq y$ for every $y \in t_0$. Like in the case of simulation, the direction of inequalities remains fixed after S challenges one of them, and the game can be rearranged to let S move first, picking, say, $x \in s_2$ (or symmetrically), which D answers with $t' \leq t_0$, reaching the new position $x \leq t'$. The game thus proceeds like the simulation game, except that D is allowed to play sets of states.

**Probabilistic traces.** These are treated similarly as traces in non-deterministic LTS: Every depth-1 term can be normalized into one of the form $\sum A \cdot a \cdot (t_0)$, where $\sum A \cdot a = 1$ and the $t_0$ are depth-0 terms. To show equality of two such normal forms $\sum A \cdot a \cdot a \cdot (t_0)$ and $\sum A \cdot a \cdot (t_2)$ (arising as successors of the current configuration), D needs to have $p_0 = q_0$, and then claim $t_1 = t_2$ for all $a \in \mathcal{A}$. Thus, the game can be rearranged to proceed like the first version of the trace game described above: $S$ selects $a \in \mathcal{A}$, and wins if $p_a \neq q_a$ (and the game then reaches the next configuration $(t_1, s_1)$ without intervention by D).

**Failure equivalence.** Let $\mathcal{A} \times X \to \mathcal{P}(\mathcal{A} \times X)$ be an LTS. A tuple $(w, b) \in \mathcal{A} \times \mathcal{P}(\mathcal{A} \times \mathcal{A})$ is a failure pair of a state $x$ if there is a $w$-path from $x$ to a state $x' \in X$ such that $x' \not\mathcal{X} x'$ fails to perform some action $b \in b$ (the failure set). Two states are failure equivalent if they have the same set of failure pairs.

The **graded theory of failure semantics** [15] extends the graded theory of traces by depth-1 constants $A$ for each $A \in \mathcal{P}(\mathcal{A})$ (failure sets) and depth-1 equations $A + (A \cup B) = A$ for each $A, B \in \mathcal{P}(\mathcal{A})$ (failure sets are downwards closed). The resulting graded monad [15] has $M_0 X = \mathcal{P}_1 X$ and $M_1 X = \mathcal{P}_1 \mathcal{A} \times X + \mathcal{P}_1 \mathcal{A}$, where $\mathcal{P}_1 \mathcal{A}$ is ordered by inclusion, $\mathcal{A} \times X$ carries the discrete order, and $\mathcal{P}_1$ denotes the finitary downwards-closed powerset. It is clear that $\mathcal{P}_1$ still preserves monos since we have only expanded the theory of traces by constants. The game in general is then described similarly as the one for plain traces above; the key difference is that now $S$ is able to challenge whether a pair of failure sets are matched up to downwards closure.

**Example 8.6.** Consider the following process terms with $\mathcal{A} = \{a, b, c\}$: $p_1 \equiv (a, 0), p_2 \equiv (a, 0, 0), p_3 \equiv b, 0$.

Clearly, the states $s = \{p_1, p_2, p_3\}$ and $t = \{p_1, p_3\}$ in the pre-determinized system are failure equivalent. To reason this through our game $\mathcal{G}_0(\gamma)$. Duplicator starts with the relation $Z = \{(0,0)\}$. From $Z$, we derive

$$(\alpha \cdot (a) \cdot (b)) = a \equiv b \equiv (a, 0) \equiv (b, 0) \equiv (a) \equiv (b) \equiv (a, c) \ni (a).$$

Thus, $Z$ is admissible at $(s, t)$ and the game position advances to $(0, 0)$ from where Duplicator has a winning strategy.

## 9 CONCLUSIONS AND FUTURE WORK

We have shown how to extract characteristic games for a given graded behavioural equivalence, such as similarity, trace equivalence, or probabilistic trace equivalence, from the underlying graded monad, effectively letting Spoiler and Duplicator play out an equational proof. The method requires only fairly mild assumptions on the graded monad; specifically, the extension of the first level of the graded monad to algebras for the zero-th level needs to preserve monomorphisms. This condition is not completely for free but appears to be unproblematic in typical application scenarios. In case the zero-th level of the graded monad preserves the terminal object (i.e. the singleton set), it turns out that the induced graded behavioural equivalence can be recast as standard coalgebraic behavioural equivalence in a category of Eilenberg-Moore algebras, and is then characterized by an infinite version of the generic equivalence game. A promising direction for future work is to develop the generic algorithms and complexity theory of the infinite equivalence game, which has computational content via the implied fixpoint characterization. Moreover, we will extend the framework to cover further notions of process comparison such as behavioural preorder [18] and, via a graded version of quantitative algebra [33], behavioural metrics.

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A OMITTED PROOFS AND DETAILS

A.1 Proofs for Section 6

Lemma A.1. Let \( \mathcal{M} \) be a depth-1 graded monad and let \( X, Y \) be sets. Then for every function \( f : X \to M_k Y \) we have

\[
\overline{f}(f_0^*) = f_1^*.
\]

Proof. It follows by equational reasoning using the associativity laws of \( \mathcal{M} \) and naturality of \( \mu^{0,n} \) that \( f_0^* \) is a homomorphism of \( M_0 \)-algebras from \( (M_0 X, \mu^{0,n}) \) to \( (M_0 M_0 X, \mu^{0,(n+m)}) \). For the homomorphism \( f_0^* : M_0 X \to M_0 X \) there exists, by canonicity of \( (M_0 X, M_0 X) \) (see Example 4.8), a unique homomorphism \( h : M_1 X \to M_0 X \) making \( (f_1^*, h) \) a homomorphism of \( M_0 \)-algebras. Thus, it is enough to show that \( (f_0^*, f_1^*) \) is an \( M_1 \)-algebra homomorphism. Indeed, we will then follow that \( f_1^* \) is the \( i \)-part of the free extension of \( f_0^* \) to an \( M_0 \)-algebra homomorphism, i.e. \( M_1(f_0^*) = f_1^* \), as required. We conclude the proof by showing that \( (f_0^*, f_1^*) \) satisfies the remaining interchange law stating that

\[
\mu_{M_1,k}^l \cdot M_1(f_0^*) = (f_0^*)_1^* \cdot \mu_X^l
\]

making it an \( M_1 \)-algebra homomorphism:

\[
\mu_{M_1,k}^l \cdot M_1(f_0^*) = (f_0^*)_1^* \quad \text{def. of } (\cdot)_1^* \\
= (f_0^* \cdot \text{id}_{M_0 X})_1^* \\
= f_1^* \cdot (\text{id}_{M_0 X})_1^* \quad \text{by (2.6)} \\
= f_1^* \cdot \mu_X^l \quad \text{def. of } (\cdot)_1^*.
\]

Hence, the proof is complete.

Example A.2. Let \( (\alpha, \mathcal{M}) \) be a depth-1 graded semantics on \( G \)-coalgebras and let \( \gamma \) be a \( G \)-coalgebra. We have

\[
\overline{\gamma} \circ (\gamma)^{k+1}_0 = (\gamma)^{k+1}_0 \quad (k \in \omega)
\]

by instantiation of Lemma A.1 to the map \( \gamma : X \to M_k X \).

Proof of Theorem 6.6.

Remark A.3. Whenever \( c : X \to C \) is the coequalizer of the kernel pair \( p, q : Z \to X \) of a morphism \( f : X \to Y \), then there is a unique morphism \( m : C \to Y \) such that \( m \cdot c = f \). Recall (e.g. from the proof of Borceux [13, Thm. 2.1.3]) that if \( C \) is a regular category, then \( m \) is monic; in particular, this holds when \( C \) is the category of algebras for a monad on \( \mathbf{Set} \).

Proof (Theorem 6.6). Let us denote the winning region for \( D \) in \( \mathcal{G}_n(\gamma) \) by \( \mathcal{W}_n(\gamma) \) (\( D \) (leaving the coalgebra \( X, \gamma \) implicit in the notation). We are going to show that \( (s,t) \in \mathcal{W}_n(\gamma) \) if and only if \( (\gamma)^{(n)}_0 \) merges \( s \) and \( t \) by induction on \( n \in \omega \). For \( n = 0 \), we must show that \( M_0 (s) = M_0 (t) \) if and only if \( (\gamma^{(0)}_0) s \) merges \( s \) and \( t \). In fact, we show that \( M_0 (s) = (\gamma^{(0)}_0) s \). First note that the following diagram commutes due to the unit law (2.1) of \( \mathcal{M} \) (instantiated to \( n = 0 \)) and naturality of \( \mu^{0,0} \):

\[
\begin{array}{ccc}
M_0 X & \xrightarrow{\eta_X M_0} & M_0 M_0 X \\
\downarrow{\mu_X^0} & & \downarrow{\mu_{M_0 X}^0} \\
M_0 M_0 X & \xrightarrow{\mu_{M_0 M_0 X}^0} & M_0 M_0 M_0 M_0 X \\
\end{array}
\]

Then unfold the definition of \((\gamma^{(0)}_0)^* \) and compute

\[
(\gamma^{(0)}_0)^* M_0 (\eta_X) = \mu_1^0 \cdot M_0 (\eta_X)
\]

where the last step uses that the outside of the previous diagram commutes.

Now, inductively assume that \( (s,t) \in \mathcal{W}_n(\gamma) \) if and only if \( (\gamma^{(n)}_0) s \) merges \( s \) and \( t \). We proceed to show that the equivalence carries through to \( n = k + 1 \):

\((=)\) Let \( (s,t) \in \mathcal{W}_n(\gamma) \) and \( (\gamma^{(n+1)}_0) s \) merges \( s \) and \( t \); we must show that \( D \) has a winning strategy in the \( (k+1) \)-round \( S \)-behavioural equivalence game at \( (s,t) \). Define

\[
Z := \ker((\gamma^{(k)}_0)^*).
\]

That is, \( Z \) is the pullback, computed in \( \mathbf{Set} \), as in the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{(\gamma^{(k)}_0)^*} & M_0 X \\
\downarrow{t} & & \downarrow{t} \\
M_0 X & \xrightarrow{(\gamma^{(k)}_0)^*} & M_0 X \\
\end{array}
\]

It is clear that \( (\gamma^{(k)}_0)^* \) is a winning move for \( D \) at \( (s,t) \) provided that it is admissible. Indeed, for every \( z \in Z \) we have that \( (\gamma^{(k)}_0)^* s \) merges \( f(z) \) and \( r(z) \) by construction; so, by induction, \( Z \subseteq \mathcal{W}_k(\gamma) \), as claimed. Thus, to conclude the proof of \( \mathcal{W}_n(\gamma) \), it suffices to show that \( Z \) is admissible at \( (s,t) \), i.e. we want to show that the homomorphism \( \overline{Z} : M_0 X \to \overline{M}_1 C Z \) from (6.1) merges \( s \) and \( t \).

Let \( m : C Z \to M_k 1 \) be the unique (mono-)morphism such that \( m \cdot C Z = (\gamma^{(k)}_0)^* \) according to Remark A.3, and consider the following diagram of \( M_0 \)-algebra homomorphisms:

\[
\begin{array}{ccc}
\mathcal{M}_1 C Z & \xrightarrow{\overline{m}} & \overline{M}_1 1 \\
\downarrow{\overline{M}_1 (\gamma^{(k)}_0)^*} & & \downarrow{\overline{M}_1 m} \\
\mathcal{M}_1 (\gamma^{(k+1)}_0)^* & \xrightarrow{\overline{m}} & \overline{M}_1 1 \\
\end{array}
\]

The above diagram commutes: the upper part is the definition of \( \overline{Z} \), the right hand triangle commutes since \( m \cdot C Z = (\gamma^{(k)}_0)^* \), and the left-hand triangle commutes by the following:

\[
\overline{M}_1 (\gamma^{(k+1)}_0)^* \cdot (m \cdot C Z) = (\gamma^{(k+1)}_0)^* \cdot (m \cdot C Z) \quad \text{by Lemma A.1}
\]

\[
= (\gamma^{(k+1)}_0)^* \cdot (m \cdot C Z) \quad \text{by Eq. (2.6)}
\]

\[
= (\gamma^{(k+1)}_0)^* \quad \text{def. of } (\gamma^{(k+1)}_0)^*.
\]

Thus

\[
\overline{M}_1 m \cdot \overline{Z} = (\gamma^{(k+1)}_0)^* \quad \text{as desired}
\]

Since \( m \) is a monomorphism and \( \overline{M}_1 \) preserves monomorphisms, we conclude that \( \overline{Z} = \overline{Z} \) holds. Hence \( Z \) is admissible at \( (s,t) \), as desired.
(⇒) Suppose that \((s, t) \in \text{Win}_{n+1}(\mathcal{D})\) so that there exists \(Z \subseteq \text{Win}_k(\mathcal{D})\) which is admissible at \((s, t)\); we proceed to show that \((\gamma^{(k+1)})_0^*\) merges \(s\) and \(t\), as required. By admissibility of \(Z\), it is then known that \(Z = M_0X \rightarrow M_1C_Z\) from (6.1) merges \(s\) and \(t\).

Next we see that \((\gamma^{(k)})_0^*\) merges \(s\) and \(t\), as required. By admissibility of \(Z\), it is then known that \(Z = M_0X \rightarrow M_1C_Z\) from (6.1) merges \(s\) and \(t\).

Note that the upper part commutes by the definition (6.1) of \(Z\). The left-hand triangle commutes by Lemma A.1 as before. We conclude that \((\gamma^{(k+1)})_0^*\) merges \(s\) and \(t\), as desired, since so does \(Z\). □

### A.2 Proofs for Section 7

**Details for Remark 7.2.** We will prove that for every accessible graded monad \(M\) on a locally presentable category, the category \(\text{Alg}_n(M)\), \(n \leq \omega\), is locally presentable. Furthermore, we show that the functor \(M_1 : \text{Alg}_n(M) \rightarrow \text{Alg}_n(M)\) is accessible and therefore has a final coalgebra.

Our proof makes use of several facts from the theory of accessible and locally presentable categories which we tersely recall from Adámek and Rosický’s book [6].

As before, we fix a regular cardinal \(\kappa\). A diagram \(D : \mathcal{D} \rightarrow \mathcal{C}\) is \(\kappa\)-filtered if its diagram scheme \(\mathcal{D}\) is a \(\kappa\)-filtered category, that is a category in which every subcategory with less than \(\kappa\) objects and morphisms has a cocone. (In particular, \(\mathcal{D}\) is then non-empty since a cocone for the empty subcategory is some object of \(\mathcal{D}\).) A functor \(F : \mathcal{C} \rightarrow \mathcal{D}\) is \(\kappa\)-accessible if it preserves \(\kappa\)-filtered colimits, and a monad is \(\kappa\)-accessible if so is its underlying functor. An object \(X\) of a category \(\mathcal{C}\) is \(\kappa\)-presentable if its hom-functor \(\mathcal{C}(X, -) : \mathcal{C} \rightarrow \text{Set}\) is \(\kappa\)-accessible.

**Remark A.4.** We shall need the following facts about \(\kappa\)-presentable objects and left adjoints.

1. A left adjoint \(L : \mathcal{C} \rightarrow \mathcal{D}\) with a \(\kappa\)-accessible right adjoint \(R\) preserves \(\kappa\)-presentable objects. Indeed, for every \(\kappa\)-presentable object \(C\) in \(\mathcal{C}\) and every \(\kappa\)-filtered diagram \(D\) in \(\mathcal{D}\) we have the following chain of natural isomorphisms

\[
\mathcal{D}(LC, \text{colim } D) \cong (C, R(\text{colim } D)) \cong (C, \text{colim } RD) \cong \text{colim } \mathcal{D}(C, RD).
\]

2. Recall (e.g. [4, Def. 7.74]) that an epimorphism \(e\) is extremal if whenever \(e = m \cdot f\) for a monomorphism \(m\), then \(m\) is an isomorphism. Let \(L : \mathcal{C} \rightarrow \mathcal{D}\) be an adjunction where the right adjoint \(R\) is faithful and reflects isomorphisms. Then the adjoint transpose of every extremal epimorphism \(e : C \rightarrow RD\) is an extremal epimorphism, too.

Indeed, to see that the adjoint transpose \(\tilde{e} : LC \rightarrow D\) is epic take a pair of morphisms \(f, g : D \rightarrow D'\) such that \(f \cdot \tilde{e} = g \cdot \tilde{e}\). Then \(Uf \cdot e = Ug \cdot e\) by adjoint transposition, whence \(UF \neq UG\) since \(e\) is epic, and therefore \(f \neq g\) since \(U\) is faithful.

Now suppose that \(\tilde{e} = m \cdot f\) for some monomorphism \(m\) in \(\mathcal{D}\). Then \(m = Um \cdot f\) in \(\mathcal{C}\), where \(Um\) is monic since right adjoints preserve monos. Thus, \(Um\) is an isomorphism in \(\mathcal{C}\), whence \(m\) is an isomorphism in \(\mathcal{D}\) since \(U\) reflects isomorphisms.

A category \(\mathcal{C}\) is \(\kappa\)-accessible if it has \(\kappa\)-filtered colimits and a set of \(\kappa\)-presentable objects such that every object of \(\mathcal{C}\) is a \(\kappa\)-filtered colimit of object from that set. The category \(\mathcal{C}\) is locally \(\kappa\)-accessible if it is locally \(\kappa\)-accessible and cocomplete. We say that a category is accessible (locally presentable) if it is \(\kappa\)-accessible (locally \(\kappa\)-presentable) for some regular cardinal \(\kappa\).

**Remark A.5.** We now collect several facts used in the proof of Theorem A.6 below.

1. A category is locally \(\kappa\)-presentable iff is is \(\kappa\)-accessible and complete [6, Cor. 2.47].

2. A strong generator in a cocomplete category \(\mathcal{C}\) is a set \(\mathcal{G}\) of objects such that every object is an extremal quotient of a coproduct of objects from \(\mathcal{G}\). A cocomplete category is locally \(\kappa\)-presentable iff it has a strong generator formed by \(\kappa\)-presentable objects [6, Thm. 1.20].

3. For a \(\kappa\)-accessible monad \(T\) on a locally \(\kappa\)-presentable category \(\mathcal{C}\), the category \(\text{Alg}(T)\) of all Eilenberg-Moore algebras is locally \(\kappa\)-presentable [6, Thm. & Rem. 2.78].

4. For every locally \(\kappa\)-presentable category \(\mathcal{C}\) the product category \(\mathcal{C} \times \mathcal{C}\) is locally \(\kappa\)-presentable. Indeed, first \(\mathcal{C} \times \mathcal{C}\) is \(\kappa\)-accessible [6, Prop. 2.67]. In addition, the product is (co)complete since so is \(\mathcal{C}\) and (co)limits are formed componentwise in the product category. Moreover, the product projection functors are clearly \(\kappa\)-accessible.

5. Let \(\mathcal{C}\) and \(\mathcal{X}\) be locally \(\kappa\)-presentable, and let \(F, G : \mathcal{X} \rightarrow \mathcal{C}\) be \(\kappa\)-accessible. The inserter category of \(F, G\) has objects \((K, f)\) where \(K\) is an object of \(\mathcal{X}\) and \(f : FK \rightarrow GK\) is a morphism of \(\mathcal{C}\); morphisms \(h : (K, f) \rightarrow (K', f')\) are morphisms \(h : K \rightarrow K'\) of \(\mathcal{X}\) such that \(f' \cdot Fk = Gk \cdot f\). The inserter category is \(\lambda\)-accessible for some regular cardinal \(\lambda \geq \kappa\) [6, Thm. 2.72]. (Note that, in general, \(\lambda \neq \kappa\).)

6. Let \(\mathcal{C}\) and \(\mathcal{X}\) be \(\kappa\)-accessible, and let \(F, G : \mathcal{X} \rightarrow \mathcal{C}\) be \(\kappa\)-accessible functors. The equiffer of a pair \(\alpha, \beta : F \rightarrow G\) of natural transformations is the full subcategory of \(\mathcal{X}\) given by all objects \(K\) such that \(\alpha_K = \beta_K\). This category is \(\kappa\)-accessible and its inclusion functor into \(\mathcal{X}\) is \(\kappa\)-accessible. The same holds, more generally, for any set of pairs \(F^i, G^i : \mathcal{X} \rightarrow \mathcal{C}\) (\(i \in I\)) and any set of pairs \(\alpha^i, \beta^i : F^i \rightarrow G^i\) [6, Lem. 2.76].

We say that a graded monad \(M\) on \(\mathcal{C}\) is \(\kappa\)-accessible if every of its endofunctors \(M_n\) (\(n \in \omega\)) is \(\kappa\)-accessible.
Theorem A.6. Let \( \mathcal{M} \) be a \( \kappa \)-accessible graded monad on a locally \( \kappa \)-presentable category \( \mathcal{C} \). Then the category \( \text{Alg}_\kappa(\mathcal{M}) \) is locally \( \kappa \)-presentable for every \( n \leq \omega \).

Remark A.7. The proof is essentially a more involved variation on the proof of the result on categories \( \text{Alg}(T) \) of Eilenberg-Moore algebras mentioned in Remark A.5.(3). We provide the details for the convenience of the reader.

Proof. For \( n = 0 \) we are done since \( \text{Alg}_0(\mathcal{M}) \) is simply the Eilenberg-Moore category for the \( \kappa \)-accessible monad \( \mathcal{M}_0 \). We now give an explicit proof for \( n = 1 \); the general case is then an easy exercise.

1. Let \( \mathcal{X} = \text{Alg}_0(\mathcal{M}) \times \text{Alg}_0(\mathcal{M}) \) be the product of the Eilenberg-Moore category for the \( \kappa \)-accessible monad \( \mathcal{M}_0 \). Then \( \mathcal{X} \) is locally \( \kappa \)-presentable by Remark A.5.(4).

2. Let \( P_0, P_1 : \mathcal{X} \to \mathcal{C} \) be the functors obtained by composing the product projections with the forgetful functor \( U_0 : \text{Alg}_0(\mathcal{M}) \to \mathcal{C} \). By Remark A.5.(5), the inserter category \( \mathcal{I} \) of the pair \( M_0 P_0, P_1 \) of \( \kappa \)-accessible functors is \( \lambda \)-accessible for some regular cardinal \( \lambda \geq \kappa \).

3. Note that objects of the inserter category \( \mathcal{I} \) are 5-tuples \( A = (A_0, A_1, (a_{i,j})_{i+j \leq 1}) \), where \( a_{i,j} : M_0 A_j \to A_{i+j} \) is a morphism of \( \mathcal{C} \) (i.e. the same data as objects of \( \text{Alg}_0(\mathcal{M}) \)) such that \( (A_0, a_{0,0}) \) and \( (A_1, a_{1,1}) \) are Eilenberg-Moore algebras for the monad \( \mathcal{M}_0 \) (the remaining axioms, homorphy and coequalization, of \( M_0 \)-algebras need not hold). Let \( V : \mathcal{I} \to \mathcal{X} \) be the functor forgetting the structure morphism \( a_{0,0} \).

4. We prove that \( V \) reflects isomorphisms. First note that morphisms in \( \mathcal{I} \) from \( (A_0, A_1, (a_{i,j})_{i+j \leq 1}) \) to \( (B_0, B_1, (b_{i,j})_{i+j \leq 1}) \) are pairs \( h_i : A_i \to B_i, i = 0, 1 \), of \( M_0 \)-algebra homomorphisms such that, in addition, we have \( h_1 \cdot a_{0,0} = b_{0,0} \cdot M_0 h_0 \). Now given such a pair such that \( V(h_0, h_1) \) is an isomorphism in \( \mathcal{X} \), that means that \( h_1 \) is an isomorphism of \( M_0 \)-algebras for \( i = 0, 1 \), we need to show that the inverses \( h_1^{-1} \) form a morphism in \( \mathcal{I} \). To this end we make use of the following diagram:

\[
\begin{array}{c}
M_1 B_0 \xrightarrow{h_1} B_1 \\
M_1 A_0 \xrightarrow{a_{0,0}} A_1 \\
M_1 B_0 \xrightarrow{h_1} B_1
\end{array}
\]

Indeed, the outside, left- and right-hand parts and the lower square commute. Thus so does the desired upper square when postcomposed by the isomorphism \( h_1 \). Thus, this square commutes.

5. We now prove that \( V \) creates limits and \( \kappa \)-filtered colimits. Observe first that limits and \( \kappa \)-filtered colimits of \( M_0 \)-algebras are formed at the level of their carrier objects since the forgetful functor \( U_0 : \text{Alg}_0(\mathcal{M}) \to \mathcal{C} \) creates limits and \( \kappa \)-filtered colimits (the latter since \( M_0 \) preserves them). Further note that all limits and colimits in \( \mathcal{X} \) are formed componentwise.

For limits, given a diagram of objects \( A^j (j \in J) \) in \( \mathcal{I} \), let \( A \) be its limit in \( \mathcal{X} \) with the limit projections \( p^j : A \to A^j \). Then we obtain a unique morphism \( a^{1,0} : M_1 A_0 \to A_1 \) such that every \( p^j \) (\( j \in J \)) is a morphism in \( \mathcal{I} \), i.e. the following squares commute (throughout we put the index \( j \) of the structure morphisms of \( A^j \) in the subscript):

\[
\begin{array}{c}
M_1 A_0 \xrightarrow{a_{0,0}} A_1 \\
M_1 p_0^j \xrightarrow{\eta^j_0} A^j \\
M_1 A_0 \xrightarrow{a_{0,0}} A_1
\end{array}
\]

Indeed, the morphisms \( a_{0,0} \cdot M_1 p_0^j \) form a cone (in \( \mathcal{C} \)) on the diagram yielding the component \( A_1 \) of the colimit \( A \); for every morphism \( h = (h_0, h_1) : A^j \to A^k \) in the given diagram scheme we have a commutative diagram

\[
\begin{array}{c}
M_1 A_0 \xrightarrow{a_{0,0}} A_1 \\
M_1 p_0^j \xrightarrow{\eta^j_0} A^j \\
M_1 A_0 \xrightarrow{a_{0,0}} A_1
\end{array}
\]

Thus, \( A \) carries a unique structure of an \( \mathcal{I} \)-object such that the limit projections \( p^j \) are morphisms of \( \mathcal{I} \).

We still need to prove that \( A \) is the limit of the \( A^j \) in \( \mathcal{I} \). Given any cone \( h^j : B \to A^j (j \in J) \) in \( \mathcal{I} \), we know that there is a unique morphism \( h : B \to A \) of \( \mathcal{X} \) such that \( p^j \cdot h = h^j \) for every \( j \in J \). It suffices to show that \( h \) is a morphism in \( \mathcal{I} \). For this consider the following diagram:

\[
\begin{array}{c}
M_1 B_0 \xrightarrow{h_1} B_1 \\
M_1 B_0 \xrightarrow{a_{0,0}} A_1 \\
M_1 B_0 \xrightarrow{h_1} B_1
\end{array}
\]

The outside commutes since \( h^j \) is a morphism of \( \mathcal{I} \). The left- and right-hand parts commute by the definition of \( h \), the lower part commutes by the definition of \( a_{0,0} \) and the upper part trivially does. Thus, the middle square commutes when postcomposed by every limit projection \( p^j \), whence this square commutes, as desired.

For \( \kappa \)-filtered colimits, given a \( \kappa \)-filtered diagram of \( M_1 \)-algebras \( A^j (j \in J) \), let \( A \) be its colimit in \( \mathcal{I} \) with colimit injections \( \text{in}^j : A^j \to A \). We use that \( M_1 \) preserves the colimit in the first component; that is, we have a \( \kappa \)-filtered colimit with the injections \( M_1 \text{in}^j_0 : M_1 A_0 \to M_1 A_0 (j \in J) \) in \( \text{Alg}_0(\mathcal{M}) \) (whence in \( \mathcal{C} \)). Then we obtain a unique algebra structure \( a^{1,0} : M_1 A_0 \to A_1 \) such that every \( \text{in}^j (j \in J) \) is a morphism in \( \mathcal{I} \), i.e. the following squares commute:

\[
\begin{array}{c}
M_1 A_0 \xrightarrow{a_{0,0}} A_1 \\
M_1 \text{in}^j_0 \xrightarrow{\text{in}^j_0} A^j \\
M_1 A_0 \xrightarrow{a_{0,0}} A_1
\end{array}
\]

for every \( j \in J \).

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Indeed, the morphisms $a^j_i: M_1 A^j_i \rightarrow A^j_i$ form a cocone of the above diagram formed by the $M_0$-algebras $M_1 A^j_i$, for every morphism $h = (h_0, h_1): A^j \rightarrow A^k$ in the diagram scheme we have a commutative diagram

$$
\begin{array}{c}
M_1 A^j_i \\
M_0 h_0 \\
M_1 M_1 A^j_k \\
M_1 A^j_i \\
\end{array}
\xymatrix{
A^j_i \ar[r]^{a^j_i} & A^j_k \ar[d]^{h_1} & A^j_0 \\
A^j_0 & A^j_1 \\
\ar[u]^{in^j_0} & \ar[u]^{in^j_1} & \ar[u]^{h_0}
}\]

Thus, $A$ carries a unique structure of an $\mathcal{F}$-object such that the colimit injections in $\mathcal{F}$ are morphisms of $\mathcal{F}$.

We still need to prove that $A$ is the colimit of the $A^j_i$ in $\mathcal{F}$. Given any cocone $h^j: A^j \rightarrow B$ ($j \in J$) in $\mathcal{F}$ we know that there is a unique morphism $h: A \rightarrow B$ of $\mathcal{F}$ such that $h \cdot \cdot = h^j$ for every $j \in J$. It suffices to show that $h$ is a morphism in $\mathcal{F}$. For this consider the following diagram:

$$
\begin{array}{c}
M_1 A^j_i \\
M_0 h_0 \\
M_1 B_0 \\
\end{array}
\xymatrix{
A^j_i \ar[r]^{a^j_i} & A^j_k \ar[d]^{h_1} & B_1 \\
B_0 & B_1 \\
\ar[u]^{in^j_0} & \ar[u]^{in^j_1} & \ar[u]^{h_0}
}\]

The outside commutes since $h^j$ is a morphism in $\mathcal{F}$. The left- and right-hand parts commute by the definition of $h$, the upper part commutes by the definition of $a^j_i$, and the lower part trivially does. Thus, the inner square commutes when precomposed by every colimit injection $M_1 in^j_i: A^j_i \rightarrow A^j_i$.

The missing laws of an $M_1$-algebra (homomorphism and coequalization) can be expressed by equifiers of the following pairs of natural transformations

$$
\begin{array}{c}
\phi_{ij}^0, \mu_{ij}^0: M_1 \rightarrow Q_1, \\
\phi_{ij}^1, \mu_{ij}^1: M_0 \rightarrow Q_1
\end{array}
$$

Consequently, $\mathcal{Alg}_1(\mathcal{M})$ is $k$-accessible and the inclusion functor $I: \mathcal{Alg}_0(\mathcal{M}) \hookrightarrow \mathcal{F}$ is $k$-accessible by Remark A.5.(6).

(8) To see that $\mathcal{Alg}_1(\mathcal{M})$ is locally $k$-presentable, we prove that $\mathcal{Alg}_1(\mathcal{M})$ is closed under limits and $k$-filtered colimits in $\mathcal{F}$. This is a slight variation of the standard argument that the Eilenberg-Moore algebras for a $k$-accessible monad $T$ is closed under limits and $k$-filtered colimits.

For limits, given a diagram of $M_1$-algebras $\mathcal{A}^j (j \in J)$, let $A$ be its limit in $\mathcal{F}$ with limit projections $p^j: A \rightarrow A^j$. We show that $A$ is an $M_1$-algebra, that means that it satisfies homomorphism and coequalization. For the latter consider the following diagram (structure morphisms of $A^j$ have their index as a subscript, say, $a^j_i: M_1 A^j_i \rightarrow A^j_i$):

$$
\begin{array}{c}
M_1 M_0 A^j_i \\
M_1 M_0 A^j_0 \\
M_1 A^j_0 \\
\end{array}
\xymatrix{
A^j_i \ar[r]^{a^j_i} & A^j_k \ar[d]^{h_1} & A^j_0 \\
A^j_0 & A^j_1 \\
\ar[u]^{\mu_{ij}^0} & \ar[u]^{\mu_{ij}^1} & \ar[u]^{p^j}
}\]

The outside commutes due to coequalization of $A^j$ and the four inner trapezoids by the naturality of $\mu_{ij}^0$ and since $p^j = (p^j_0, p^j_1)$ is a morphism in $\mathcal{F}$. Thus, the desired middle square commutes when postcomposed by the projection $p^j_i$. Since limits in $\mathcal{F}$ are formed componentwise, the $p^j_i (j \in J)$ form a limit cone whence a jointly monic family. This, implies that the desired inner square commutes. That $A$ satisfies homomorphism is shown analogously.

For $k$-filtered colimits, given a $k$-filtered diagram of $M_1$-algebras $\mathcal{A}^j (j \in J)$, let $A$ be its colimit in $\mathcal{F}$ with colimit injections $in^j_i: A^j_i \rightarrow A^j_i$. We show that $A$ is an $M_1$-algebra, that means that it satisfies homomorphism and coequalization. For the former, observe first that since $M_0 M_1$ is $k$-accessible and colimits in $\mathcal{F}$ are formed componentwise we have a colimit cocone $M_0 M_1 in^j_i: A^j_0 \rightarrow A^j_0$. Now consider the following diagram (again we put the index of
Now suppose further that $F: \mathcal{C} \to \mathcal{C}$ preserves monomorphisms. Then every coalgebra morphism $h: (A, \gamma) \to (B, \delta)$ can be factorized into a coalgebra morphism carried by a regular epi and one carried by a monomorphism in $\mathcal{C}$. Indeed, take the factorization $h$ into a regular epimorphism $e: A \to C$ followed by a monomorphism $m: C \to B$ in $\mathcal{C}$, and observe that the unique diagonal fill-in property yields a unique coalgebra structure $\alpha: C \to FC$ such that $e$ and $m$ are coalgebra morphisms (here one uses that $Fm$ is a monomorphism):

\[
\begin{aligned}
&\begin{array}{c}
A \\
\downarrow e \\
C \\
\downarrow m \\
B
\end{array} &\xmapsto{\gamma} &\begin{array}{c}
Y \\
\downarrow Fe \\
FC \\
\downarrow Fm \\
FB
\end{array} \\
\end{aligned}
\]

Towards the proof of Theorem 7.5, we generalize the game to be played on any $\mathcal{M}_1$-coalgebra $(A, \gamma)$. Positions of $D$ then are pairs in $A \times A$; positions of $S$ are relations $Z \subseteq A \times A$. A move from $(x, y) \in A \times A$ to such a $Z$ is admissible for $D$ if the relations in $Z$ entail equality of $\gamma(x)$ and $\gamma(y)$; in categorical formulation, this means that $\mathcal{M}_1\mathcal{C}Z(\gamma(x)) = \mathcal{M}_1\mathcal{C}Z(\gamma(y))$ where $\mathcal{C}Z: A \to \mathcal{C}Z$ is the coequalizer of the pair $\eta^*_0, \eta^*_1: M_0Z \to A$ for the projections $\ell, r: Z \to A$. Again, $S$ just picks from $Z$; any player who cannot move, loses, and infinite matches are won by $D$. We then claim, generalizing the claim of the theorem, that $x, y \in A$ are behaviourally equivalent iff $D$ wins $(x, y)$.

**Proof (Theorem 7.5).** $(\Rightarrow)$ We denote the winning region of $D$ by $\sim_D$. It is easy to see that $\sim_D$ is a congruence on the $M_0$-algebra $A$ (this is similar as in the proof of Theorem 7.12), so we have a unique $M_0$-algebra structure on $A/\sim_D$ such that the quotient map $e: A \to A/\sim_D$ is an $M_0$-algebra homomorphism. We write elements of $A/\sim_D$ in the form $[x]D$ for $x \in A$. We shall define a coalgebra structure $\gamma_D: A/\sim_D \to \mathcal{M}_1(A/\sim_D)$ such that the square below commutes in $\text{Alg}_{\mathcal{B}}(\mathcal{B})$:

\[
\begin{aligned}
&\begin{array}{c}
A \\
\downarrow e \\
A/\sim_D
\end{array} &\xmapsto{\gamma} &\begin{array}{c}
\mathcal{M}_1A \\
\downarrow \mathcal{M}_1e \\
\mathcal{M}_1(A/\sim_D)
\end{array}
\end{aligned}
\]

That means that it suffices to show that $\gamma_D([x]D) = \mathcal{M}_1e(\gamma(x))$ for every $x \in A$ is well-defined; indeed, since $\mathcal{M}_1 e \cdot \gamma$ is an $M_0$-algebra homomorphism and $\sim_D$ a congruence, $\gamma$ is then clearly an $M_0$-algebra homomorphism, too, and the above diagram commutes so that $e$ becomes a coalgebra morphism that identifies all $x, y$ such that $D$ wins $(x, y)$.

So let $D$ win $(x, y)$; we have to show that $\mathcal{M}_1e(\gamma(x)) = \mathcal{M}_1e(\gamma(y))$.

Let $Z$ be the winning move of $D$ at $(x, y)$ (that is, $Z$ is $D$’s first move of the corresponding match). Since $S$ can pick any element of $Z$, $D$ wins on every element of $Z$, so that $e$ factors through the above coequalizer $\mathcal{C}Z: A \to \mathcal{C}Z$; there is a unique $M_0$-algebra homomorphism $h: C_Z \to A/\sim_D$ such that $h \cdot c_Z = e$. Hence, $\mathcal{M}_1 e$ factors...
through \( \overline{M}_1c_Z \) via \( M_0h \). Since \( Z \) is an admissible move, \( \overline{M}_1c_Z \) identifies \( y(x) \) and \( y(y) \); hence, so does \( M_1z \), as required.

\((\Rightarrow)\) Let \( h : (A, y) \to (B, \delta) \) be an \( \overline{M}_1 \)-coalgebra morphism; it suffices to show that the kernel

\[
\ker h = \{(x, y) \in A \times A \mid h(x) = h(y)\}
\]

of \( h \) is contained in \( \sim \). To see this, it suffices to show that \( D \) can maintain the invariant \( \ker h \), i.e. ensure that her positions always remain in \( \ker h \) if the game starts in a position in \( \ker h \). But when at a position \( (x, y) \in \ker h \), she can clearly ensure that the next position is still in \( \ker h \) by just playing \( Z := \ker h \).

We proceed to show that \( Z \) is admissible. Let \( \ell, r : Z \to A \) be the obvious projection maps. Since the kernel is clearly a congruence, \( Z \) is an \( M_0 \)-algebra, and \( \ell, r : Z \to A \) are \( M_0 \)-algebra homomorphisms. (In fact, \( Z \) is the pullback of \( h \) along itself, and the forgetful functor \( \text{Alg}(A) \to \text{Set} \) creates limits.) Using Remark 6.2.1 (and noting that it holds for arbitrary algebras \( A \) in lieu of \( M_0X \)) we take the coequalizer \( c_Z : A \to C \) of \( \ell, r \), and we now prove that \( \overline{M}_1c_Z \cdot y \) merges the pair \( \ell, r \). By Remark A.3, the unique morphism \( m : c_Z \to B \) such that \( h = m \cdot c_Z \) is monoic. By Remark A.8 we know that \( C \to A \) carries a unique coalgebra structure \( \alpha : C \to \overline{M}_1c_Z \) such that \( c_Z \) and \( m \) are coalgebra morphisms. Using the former fact and that \( c_Z \) merges \( \ell \) and \( r \) we obtain the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{y} & FA \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
C & \xrightarrow{\mu} & \overline{M}_1c_Z \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
A & \xrightarrow{y} & FA \\
\end{array}
\]

The commutativity of its outside is the desired equality. \( \square \)

**Proof of Theorem 7.12.** It is straightforward to see that \( \sim \) is an equivalence relation. For instance, transitivity is seen as follows. Assume that \( D \) wins on \((s, t)\) and on \((t, u)\), with winning (first) moves \( Z, Z' \), respectively. Then \( Z \cup Z' \) is a winning move for \( D \) on \((s, u)\), where we exploit that by the assumption that \( \overline{M}_1 \) preserves monos, we do not need to worry about \( Z \cup Z' \) possibly using more variables than needed in the top-level decompositions of \( s \) and \( u \) (cf. Remark 6.8). Symmetry and reflexivity are easier. We write \([t]^D = q(t)\). The \( M_0 \)-algebra structure of \( U \) is then given by

\[
f_U([((t_i)^D)_{i \in I}]) = [f((t_i)_{i \in I})]^D
\]

for a depth-0 operation \( f \) of arity \( I \). Well-definedness is seen similarly as transitivity above.

For well-definedness of \( \zeta \), suppose that \( t_1\sigma, s_1\sigma \) are top-level decompositions of infinite-depth terms such that \( D \) wins on \((t_1, s_1, \sigma)\), again w.l.o.g. with \( t_1, s_1 \in \mathcal{T}_{\Sigma,1}(X) \) for the same % \( X \), and with the same \( \sigma \). Let \( Z \subseteq \mathcal{T}_{\Sigma,1}(X) \times \mathcal{T}_{\Sigma,1}(X) \) be \( D \)'s winning move. Then \( Z \) being an admissible move of \( D \), entails \( t_1 = s_1 \). Moreover, since \( S \) can pick any element \((u, v) \in Z \) as a response, \( D \) wins on each of the subsequent positions \((u, \sigma, \sigma)\), so \( u \) and \( \sigma \) are identified under \( q \cdot \sigma \). It follows that \( M_1(q \cdot \sigma)(\{t_1\}) = M_1(q \cdot \sigma)(\{s_1\}) \). This shows well-definedness; it is then clear by construction that \( \zeta \) is an \( M_0 \)-homomorphism.

Finally, let \( \gamma : A \to M_1A \) be an \( \overline{M}_1 \)-coalgebra on an \( M_0 \)-coalgebra \( A \); we abuse \( A \) to denote also the carrier of \( A \). For each \( x \in A \), the successor structure \( y(x) \in \overline{M}_1A \) is represented as a term \( g_x \in \mathcal{T}_{\Sigma,1}(A) \). Infinite unrolling of \( A \) thus produces an infinite-depth term \( h_x \) for each \( x \in A \). We claim that the map \( h : A \to U \) given by \( h(x) = [h_x]^D \) is the unique \( \overline{M}_1 \)-coalgebra morphism from \((A, y)\) to \((U, \zeta)\).

First note that the map \( h \) is independent of the choice of representing terms \( g_x \). This is seen by letting \( D \) play, in \( G^{\text{sym}} \), a strategy based on maintaining the invariant that the present state is a pair of unrollings, under differently chosen representing terms (where the choice of representing terms may change during the unrolling process), of the same state in \( A \). Like in the ‘only if’ direction of the proof of Theorem 7.5, \( D \) can maintain the invariant by playing it in every move. To see that \( h \) is an \( M_0 \)-homomorphism, let \( f \) be a depth-0 operation of arity 1; then \( \gamma(f(x_i)_{i \in I}) = \gamma(f(y(x_i)_{i \in I})) \) has \( f(g(x_i)_{i \in I}) \) as a representing term, so the unrolling \( h(f(x_i)_{i \in I}) \) arises by applying \( f \) to the unrollings \( h(x_i) \).

Finally, being a coalgebra morphism \( A \to U \) amounts precisely to the unrolling property; so \( h \) is a coalgebra morphism, and unique as such. \( \square \)

**Proof of Proposition 8.5.**

Proof. We first show that \( Z \) is admissible at \((s, t)\), i.e. \( Z \vdash \gamma(a \cdot \gamma y)^t_s = (a \cdot \gamma y)^t_s \), where we write \( Z \vdash \) to indicate equational derivability from \( Z \). This follows from the construction of \( Z \) since Condition 1 ensures that

\[
Z \vdash s_u + t_u = \sum_{x \in s_u} x + t_u = t_u.
\]

Likewise Condition 2 ensures that \( Z \vdash s_u + t_u = s_u \). Thus, \( Z \vdash s_u = t_u \), whence, \( Z \vdash (a \cdot \gamma y)^t_s = (a \cdot \gamma y)^t_s \).

Moreover, Spoiler thanks to Condition 1 (resp. Condition 2) can play the positions either of the form \((x \lor t', t')\) or \((y \lor s', s')\). In either cases, \( M_0! \) will map the left and right term to 1 since \( M_01 = 1 \). So, in hindsight, if \((s, t)Z(s', t')\) is a winning match for Duplicator, then \( s' \lor Z t' \). Thus, repeating the above argument \( n \)-times gives the desired result. Lastly, \( Z \) is also a winning strategy in the infinite game because \((s', t') \in Z \) whenever \((s, t)Z(s', t')\) is a match. \( \square \)