$CP$, $T$ and $CPT$ in the non-perturbative formulation of chiral gauge theories

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Abstract

In spite of significant recent progress on the non-perturbative formulation of chiral gauge theories there remained several unsolved problems. One of them is the puzzle that the left- and right-handed projectors, and so the left- and right-handed actions, break $CP$-symmetry on the lattice. We show in this letter that they break $T$-symmetry also, while $CPT$ remains intact.
Introduction

Chiral symmetry is a longstanding problem on the lattice. Under very general conditions, the chiral transformations defined by simple $\gamma^5$ rotations on the fermion fields cannot leave the lattice action invariant \[1\]. This deadlock was resolved in a rather unexpected way: lattice formulations with a Dirac operator $D$ satisfying the Ginsparg-Wilson relation \[2\]

$D\gamma^5 + \gamma^5 D = D\gamma^5 D$ \hspace{1cm} (1)

have an exact symmetry transformation which depends on the Dirac operator itself and can be considered as a realization of the formal continuum transformation \[3\] \[4\]. The projectors to left- and right-handed fermions also depend on $D$. Actually, the form of the projectors is not unique. One might introduce a continuous parameter $s$ and for each value of $s$ the fermion action is decomposed correctly

$\overline{\psi} D \psi = \overline{\psi_L}^{(s)} D \psi_L^{(s)} + \overline{\psi_R}^{(s)} D \psi_R^{(s)}. \hspace{1cm} (2)$

where

$\psi_{L/R}^{(s)} = P_{L/R}^{(s)} \psi, \hspace{1cm} \overline{\psi_{L/R}}^{(s)} = \overline{\psi_{L/R}}^{(s)} P_{L/R}^{(s)} \hspace{1cm} (3)$

and the projectors are defined as

$P_{L/R}^{(s)} = \frac{1}{2} \left( 1 \mp \Gamma_5^{(s)} \right), \hspace{1cm} \overline{P}_{L/R}^{(s)} = \frac{1}{2} \left( 1 \mp \Gamma_5^{(s)} \right), \hspace{1cm} \hspace{1cm} (4)$

where \[5\]

$\Gamma_5^{(s)} = \left( N^{(s)} \right)^{-1} \gamma^5 (1 - s D), \hspace{1cm} \overline{\Gamma_5^{(s)}} = \left( N^{(s)} \right)^{-1} (1 - (1 - s) D) \gamma^5,$

Beyond being gauge field dependent, the projectors in eq.(4) are asymmetric with respect to $\psi$ and $\overline{\psi}$. One might interpret this asymmetry as a signal for a possible fermion number anomaly in chiral gauge theories \[5\]. Indeed, using the index theorem \[2\], it is easy to connect the fermion number violation to the topological charge of the gauge field configurations. On the other hand, the decomposition in eq.(2) violates $CP$-symmetry also. Under the $CP$ transformation\[1\]

$\psi(n) \rightarrow W \overline{\psi}^T (P n), \hspace{1cm} \overline{\psi}(n) \rightarrow -\psi^T (P n) W^{-1}, \hspace{1cm} (5)$

$U_{\mu}(n) \rightarrow U_{\mu}(C P)(n) \equiv \begin{cases} [U_{\mu}(P n - \mu)]^T & (\mu = 1, 2, 3) \\ U_{\mu}^*(P n) & (\mu = 4) \end{cases}$

\[1\]In our convention $W \doteq \gamma^2$, $P n \doteq (-n^1, -n^2, -n^3, n^4)$ and the transposition $T$, as well as the hermitian conjugation $\dagger$, act always only on the Dirac and gauge indices.
and with the $CP$-symmetry condition for the Dirac operator
\[ W^{-1} D(n, n'; U^{(CP)})^T W = D(Pn', Pn; U) \]  
(6)
the left-handed action in eq. (2) is not invariant \[ \sum_{n,n'} \bar{\psi}^{(s)}_L(n)D(n, n'; U)\psi^{(s)}_L(n') \leftrightarrow \sum_{n,n'} \bar{\psi}^{(1-s)}_L(n)D(n, n'; U)\psi^{(1-s)}_L(n'). \]  
(7)

At \( s = \frac{1}{2} \) the $CP$-symmetry is restored, but just at this value of \( s \) the projectors become non-local: the \( \lambda = 2 \) point, where the normalization operator \( N^{(\frac{1}{2})} \) becomes zero, is an accumulation point of the ultraviolet modes. The significance of this $CP$-symmetry violation on the physical properties of the theory is an interesting, but not yet completely understood issue \[ 8].

We make here a short remark only on this difficult problem. Due to a technical constraint in the treatment of chiral gauge theories on the lattice the theory falls into topological sectors where the relative weight factors remain undetermined. A theory defined on the Gaussian fixed-point has a finite number of free parameters only and so these complex overall factors weighting the partition function in the different topological sectors can not be simply chosen to satisfy the requirement of symmetry. Actually, the topological charge \( Q = 0 \) sector should give informations on the \( Q \neq 0 \) sectors due to the clustering property of separated topological objects.

In this letter we investigate a simpler problem, namely the properties of the left- and right-handed action under time reflection \( T \) and \( CPT \).

**Time reflection in Euclidean space**

Time reflection is represented by an anti-unitary operator in Minkowski space and this feature has its trace in Euclidean space also. We summarize below briefly the transformation of fermion and gauge fields under time reflection in Euclidean path integral formulation.

In Minkowski space the fermion operators are transformed as \( \psi(x) \rightarrow (-\gamma^1\gamma^3)\psi(Tx), \bar{\psi}(x) \rightarrow \bar{\psi}(Tx)\gamma^1\gamma^3 \) and \( T_x = (-x^0, x^1, x^2, x^3) \). Having in mind the path integral formulation it is useful to express the time reflection symmetry on Green’s functions
\[
\langle \Omega, T(\psi_{\alpha_n}(x_n) \cdots \psi_{\alpha_1}(x_1)\bar{\psi}_{\beta_m}(\bar{x}_m) \cdots \bar{\psi}_{\beta_1}(\bar{x}_1))\Omega \rangle = \langle \Omega, T([-\gamma^1\gamma^3\psi(Tx_n)]_{\alpha_n} \cdots [-\gamma^1\gamma^3\psi(Tx_1)]_{\alpha_1} \times \bar{\psi}(T\bar{x}_m)\gamma^1\gamma^3_{\beta_m} \cdots \bar{\psi}(T\bar{x}_1)\gamma^1\gamma^3_{\beta_1})\Omega \rangle^*,
\]  
(8)
where the complex conjugation of the expectation value on the right hand side is a consequence of the anti-unitarity of the time reflection operator. Eq.(8) can
also be written as
\[
(\Omega, T(\psi_{\alpha_n}(x_n) \cdots \psi_{\alpha_1}(x_1) \bar{\psi}_{\beta_m}(\bar{x}_m) \cdots \bar{\psi}_{\beta_1}(\bar{x}_1)) \Omega) = (\Omega, T([\gamma^3 \gamma^1 \gamma^0 \psi(T\bar{x}_1)]_{\beta_1} \cdots [\gamma^3 \gamma^1 \gamma^0 \psi(T\bar{x}_m)]_{\beta_m}) \times \bar{\psi}(T\bar{x}_1)(-\gamma^0 \gamma^3 \gamma^1)_{\alpha_1} \cdots \bar{\psi}(T\bar{x}_n)(-\gamma^0 \gamma^3 \gamma^1)_{\alpha_n}) \Omega).
\]

On a Euclidean lattice these Green’s functions are represented by integrals over the Grassmann valued fields \(\psi(n)\) and \(\bar{\psi}(n)\). The symmetry transformation on these fields corresponds to introducing new integration variables which makes the left and right hand sides of eq.(9) identical. These variable substitutions read
\[
\psi(n) \rightarrow -A^{-1} \psi^T(Tn), \quad \bar{\psi}(n) \rightarrow \psi^T(Tn)A, \quad A = -\gamma^1 \gamma^3 \gamma^4,
\]
where the matrix \(A\) has the property
\[
-A = A^{-1} = A^T = A^\dagger = \gamma^5 A \gamma^5 \quad (11)
\]
and \(Tn = (n_1, n_2, n_3, -n_4)\). The transformation in eq.(10) reflects the 4th axes, rotates the Dirac indices and makes a swap between \(\psi\) and \(\bar{\psi}\). This last feature is a consequence of the anti-unitarity of the time reflection operator.

Considering fermions in interaction with gauge fields the parallel transporter \(U_\mu(n)\), associated with the link \((n, n + \mu)\) of the lattice, enters. A simple way to find the transformation law of \(U_\mu(n)\) under time reflection is as follows. Consider a free fermion theory on the lattice which is symmetric under the transformation in eq.(10) (the standard free Wilson action for example). Then introduce parallel transporters \(U_\mu(n)\) to make it gauge invariant and demand time reflection symmetry for this interacting theory. One obtains
\[
U_\mu(n) \rightarrow U_\mu^T(n) = \begin{cases} U_\mu^*(Tn) & (\mu = 1, 2, 3) \\ U_\mu(Tn - \mu)^T & (\mu = 4) \end{cases}.
\]
(12)
Consider now a general vector gauge theory on the lattice
\[
\sum \bar{\psi}_a(n)D(n, n'; U)_{\alpha\beta}^{ab} \psi_b(n'),
\]
where the sum runs over the space-time \((n, n')\), gauge \((a, b)\) and Dirac \((\alpha, \beta)\) indices. Performing time reflection using the transformation rules in eqs.(10, 12) and demanding symmetry leads to the constraint on the Dirac operator \(D\):
\[
AD(n, n'; U^{(T)})^T A^{-1} \doteq D(Tn', Tn; U).
\]
(13)

**Time reflection for the left-handed action**

We shall assume that the Dirac operator \(D\) satisfies the Ginsparg-Wilson relation eq.(1), has the standard hermiticity property
\[
D(n, n'; U) = \gamma^5 D(n', n; U)^\dagger \gamma^5,
\]
(14)
and obeys eq. (13) which is the condition of time reflection symmetry in a vector theory. It follows then
\[ [D, D^\dagger] = 0, \quad [\gamma^5, D D^\dagger] = 0. \] (15)
Using eqs. (11, 13, 14, 15) and the properties of the matrix \( A \) in eq. (11) it is easy to show that
\[ A \left[ \Gamma_5^{(s)}(n, m; U^{(T)}) \right]^T A^{-1} = -\Gamma_5^{(1-s)}(Tm, Tn; U), \] (16)
\[ A \left[ \Gamma_5^{(s)}(n, m; U^{(T)}) \right]^T A^{-1} = -\Gamma_5^{(1-s)}(Tm, Tn; U), \] (17)
leading to
\[ A[\bar{P}^{(s)}(n, m; U^{(T)})]^T A^{-1} = \bar{P}^{(1-s)}_\pm(Tm, Tn; U), \] (18)
\[ A[P^{(s)}(n, m; U^{(T)})]^T A^{-1} = P^{(1-s)}_\pm(Tm, Tn; U). \] (19)
Eqs. (18, 19) imply that the left-handed action violates time reflection in a similar way as \( CP \) did
\[ \sum \chi_L^{(s)}(n) D(n, n'; U) \psi_L^{(s)}(n') \xrightarrow{T} \sum \chi_L^{(1-s)}(n) D(n, n'; U) \psi_L^{(1-s)}(n'). \] (20)
On the other hand, \( CPT \) is a symmetry of the left-handed action independent of the value of the parameter \( s \), since both transformations swap \( s \) and \((1 - s)\). Combining the \( CP \) and \( T \) transformations of eqs. (5, 10, 12) it is easy to see that the \( CPT \) transformation is a hypercubic rotation on the lattice
\[ \psi(n) \rightarrow \gamma_5\psi(-n), \quad \bar{\psi}(n) \rightarrow \bar{\psi}(-n)\gamma_5, \quad U_\mu(n) \rightarrow U_\mu^\dagger(-n - \mu). \] (21)
Hypercubic symmetry implies then invariance under \( CPT \).

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