Theory of the multilayer thin anisotropic shells, based on the asymptotic analysis of the general equations for the elasticity theory

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Abstract. A new theory of thin multilayer anisotropic elastic shells is proposed, based on the application of the asymptotic analysis over small geometric parameter to the general 3-dimensional equations of the elasticity theory for curvilinear coordinates. In deriving the equations of the shell theory no assumptions are made concerning to the distribution of displacements, deformations, or displacements trough thickness. Recurrent consequences of local problems of the elasticity theory for shells are formulated and analytical solutions are obtained. The averaged equations of the asymptotic theory of shells that are of the same type as the classical equations of the Kirchhoff-Love theory of shells are derived. It is shown that the asymptotic theory makes it possible to obtain in explicit form the distributions of all 6 components of the stress tensor over the thickness of the shell. An example of the calculation of a cylindrical shell under axisymmetric bending is considered. The case of monoclinic shell layers, which have no more than 13 independent elastic constants, is considered. An algorithm is proposed for obtaining explicit analytical equations for calculating the distribution of components of the total stress tensor over the cylindrical shell. The effect of the geometric dimensions of the shell on the character of the distribution of displacements and stresses in the shell is analyzed.

1. Introduction

To solve many engineering problems of designing structures from composite materials, it is necessary to accurately calculate the components of the total stress tensor at any point in the structure. Due to low shear strength and transverse shear, three-layer honeycomb structures can be destroyed under certain types of loading precisely because of these interlayer and lateral stresses [1-6]. The classical theories of calculating thin composite shells are based on certain assumptions, the adequacy of which must be confirmed for each class of problems. The method of asymptotic averaging of the equations of the three-dimensional elasticity theory for thin multilayer plates and shells was developed in [7-15], which, without any hypotheses regarding the distribution of displacements or stresses along the thickness, allows obtaining an explicit, mathematically valid expression for all 6 components of the stress tensor. In this paper we present the relations for the total stress tensor obtained with the help of the asymptotic theory for the case of thin cylindrical shells.

2. The equations of three-dimensional theory of elasticity in curvilinear coordinates
In a three-dimensional space $\mathbb{R}^3$ with Cartesian coordinates $x^i$, we consider the surface $\Sigma_0$ given by orthogonal coordinates $q^i(x^i)$ in the form $q^i(x^i) = 0$, where $x^i \in \Sigma_0 \subset \mathbb{R}^2$ is the range of changes in the values of the Cartesian coordinates. Consider the shell - the body to which there corresponds a region $V \subset \mathbb{R}^3$ bounded by the outer $\Sigma_+$ and inner $\Sigma_-$ surfaces, whose equations have the form $q^3 = -h/2$, $q^3 = h/2$, and also the end surface $\Sigma_T$ whose equation in curvilinear orthogonal coordinates $q_k$ has the form $F(q_1, q_2) = 0$.

The relations $q_i = q_i(x^i)$ between curvilinear $q_i$ and Cartesian coordinates $x^i$ will be assumed to be smooth functions. The parameter $h$ is the thickness of the shell, the surface $q_3 = 0$ is the middle surface of the shell, it crosses the end surface along the contour $\partial \Sigma_0$. We will further assume that the coordinate lines $q_1$ and $q_2$ are oriented along the lines of the principal curvatures of the middle surface of the shell, and the line $q_3$ along the normal to this surface. All curvilinear coordinates are assumed to be dimensional quantities - the lengths of the arcs in the corresponding coordinate directions.

In the construction of shell models, we shall consider the initial quasistatic problem of the linear theory of elasticity in the following form [16]:

$$H_\beta H_\gamma \frac{\partial \sigma_{\alpha\alpha}}{\partial q_\alpha} + H_\alpha H_\gamma \frac{\partial \sigma_{\alpha\beta}}{\partial q_\beta} + H_\alpha H_\beta \frac{\partial \sigma_{\alpha\gamma}}{\partial q_\gamma} = 0$$

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} + \tilde{C}_{ijkl} \varepsilon_{kl}, \quad \varepsilon_{ij} = C_{ijkl} \varepsilon_{kl} + \tilde{C}_{ijkl} \varepsilon_{kl},$$

$$\varepsilon_{aa} = \frac{1}{H_\alpha} u_{a,a}, \quad \varepsilon_{12} = \frac{1}{2R} (u_{1,2} + u_{2,1} R), \quad \varepsilon_{33} = \frac{1}{\alpha} u_{3,3}, \quad 2\varepsilon_{a3} = \frac{1}{\alpha} u_{a,3} + \frac{u_{3,a}}{H_\alpha},$$

where (1) are the equilibrium equations, (2) is the generalized Hooke's law, (3) the Cauchy relations.

Let us consider a thin multilayer elastic cylindrical shell with a radius $R$ for which relation $\alpha \varepsilon = h / L \ll 1$, where $h$ - thickness of the shell, $\alpha \varepsilon$ - small parameter, $L$ - length of the shell. We introduce the dimensionless orthogonal coordinates $q_k$ and the local coordinate $\xi$ along the shell thickness: $q_k = q_k / L$, $\xi = q_3 / \alpha \varepsilon$

Let us consider the case when the pressure $\tilde{p}_\pm$ on the outer and inner surfaces of the shell is of the order of smallness $O(\alpha \varepsilon^3)$, $\tilde{p}_\pm = \alpha \varepsilon^3 p_\pm$, where $p_\pm$ is a finite value. This order of pressure, as a rule, corresponds to the real conditions of loading of thin shells.

We will seek the solution in the form of asymptotic expansions in the parameter in the form of functions that depend on global and local coordinates:

$$u_k = u_k^{(0)}(\bar{q}_1) + \sum_{n=1}^{\infty} \alpha^n u_k^{(n)}(\bar{q}_1, \xi) = u_k^{(0)}(\bar{q}_1) + \alpha \nu u_k^{(1)}(\bar{q}_1, \xi) + \alpha^2 u_k^{(2)}(\bar{q}_1, \xi) + \ldots$$

Using these asymptotic expansions, explicit expressions are obtained for the longitudinal $\sigma_{ij}$, shear $\sigma_{33}$, and transverse components of the stress tensor through deformations $\varepsilon^{(0)}_{ij}$ and curvatures of the zero approximation $\eta_{KL}$ of the shell.
The formulas (1) denote tensors that depend only on the reduced elastic modulus tensor of the shell layers $C_{ijkl}^{(0)}$, thicknesses of layers and Lame parameters of the middle surface of the shell. The expressions for $e_{kl}^{(0)}$ and $\eta_{kl}$ in the cylindrical shell have the following form:

\begin{align*}
e_{11}^{(0)} &= u_{1,1}^{(0)}, & e_{22}^{(0)} &= \frac{1}{R}(u_{2,2}^{(0)} + u_{3,3}^{(0)}), & 2e_{12}^{(0)} &= \frac{1}{R}u_{1,2}^{(0)} + u_{2,1}^{(0)}, \\
\eta_{11} &= -u_{3,11}^{(0)}, & \eta_{22} &= -(u_{3,22}^{(0)} + u_{2,2}^{(0)})/R^2, & -\eta_{12} &= (u_{3,12}^{(0)} + u_{2,2}^{(0)})/R.
\end{align*}

For curvatures of the second approximation $\eta_{ij}^{(2)}$ we have the following formulas:

\begin{align*}
\eta_{11}^{(2)} &= 0, & \eta_{22}^{(2)} &= -(1/R^3)(u_{3,22}^{(0)} + u_{2,2}^{(0)})/R^2, & \eta_{12}^{(2)} &= -(1/2R)(u_{3,12}^{(0)} + u_{2,2}^{(0)}).
\end{align*}

To find the displacements $u_k^{(0)}(x_1)$, we obtain a system of averaged equations of the Kirchhoff-Love theory.

3. Example of calculation of stresses in a composite cylindrical shell

Consider the case of axisymmetric loading of a 4-layer composite cylindrical shell, under the influence of internal pressure. Each layer was a unidirectional sol, the system of corners of the composite layers reinforcement $\alpha_i$ was chosen as follows $\alpha_1 = 0^\circ$, $\alpha_2 = 90^\circ$, $\alpha_3 = -90^\circ$, $\alpha_4 = 0^\circ$.

Figure (1) shows the distribution of flexural stress $\sigma_{11}$ along the thickness of the composite sheath. Formulas (1) allow us to calculate the distributions of all 6 stresses in the shell.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure1.png}
\caption{The distribution of flexural stress $\sigma_{11}$ along the thickness of the composite sheath.}
\end{figure}

4. Conclusions

A theory of thin cylindrical composite shells is developed, based on the asymptotic analysis of three-dimensional equations of the theory of elasticity without additional assumptions about the distribution of stresses and displacements along the thickness of the shell. Explicit, mathematically accurate analytical formulas are obtained for calculating the distribution of the components of all 6 components of the stress tensor over the thickness of a cylindrical shell. An example is given of calculating stresses in a cylindrical shell under axisymmetric bending of pressures.
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