Research article

New delay-range-dependent exponential stability criterion and $H_\infty$ performance for neutral-type nonlinear system with mixed time-varying delays

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Abstract: For a neutral system with mixed discrete, neutral and distributed interval time-varying delays and nonlinear uncertainties, the problem of exponential stability is investigated in this paper based on the $H_\infty$ performance condition. The uncertainties are nonlinear time-varying parameter perturbations. By introducing a decomposition matrix technique, using Jensen’s integral inequality, Peng-Park’s integral inequality, Leibniz-Newton formula and Wirtinger-based integral inequality, utilization of a zero equation and the appropriate Lyapunov-Krasovskii functional, new delay-range-dependent sufficient conditions for the $H_\infty$ performance with exponential stability of the system are presented in terms of linear matrix inequalities. Moreover, we present numerical examples that demonstrate exponential stability of the neutral system with mixed time-varying delays, and nonlinear uncertainties to show the advantages of our method.

Keywords: exponential stability; $H_\infty$ performance; neutral system; time-varying delay

Mathematics Subject Classification: 93B36, 93C43, 93D23

1. Introduction

Neutral time-delay systems contain delays both in the state and in the derivatives of the state which can be found in various dynamic systems, such as chemical reactors, nuclear reactors, biological systems, economical systems, water pipes, population ecology, power systems, etc. [1–14]. On the
other hand, nonlinear uncertainties are commonly encountered because it is very problematic to
derive a certain mathematical model due to slowly varying parameters, environmental noise and so
on. Stability criteria for time-delay systems are classified into two categories: delay-independent and
delay-dependent. In general, the delay-dependent criteria are less conservative than the
delay-independent ones, especially when the size of the delay is small. Therefore, many researchers
have dedicated much effort to studying the delay-dependent stability criteria for neutral time-delay
systems with nonlinear uncertainties in recent years; see, for instance, [1, 5, 6, 10, 15–18].

The stability analysis of neutral-type systems is considered with various inequality techniques and
Lyapunov approaches, which are significant to reduce conservatism. Therefore, many inequality
techniques have been applied in the published literature to estimate the upper bound of the time
derivative of the introduced Lyapunov-Krasovskii functional (LKF). In [1], the new stability
conditions for the neutral delay differential system are derived by applying Jensen’s integral
inequality. In order to reduce the conservatism, Wirtinger’s integral inequality was introduced in [19].
The free weighting matrices were utilized with a new integral inequality lemma in [6] to achieve less
conservative results.

As pointed out in [1, 9–11, 16, 17, 20], the exponential stability problem is also significant since it
can determine the convergence rate of system states to equilibrium points. The problem of
delay-dependent exponential stability criteria for neutral systems with nonlinear uncertainties have
been investigated in [10, 11, 20]. Recently, many researchers have paid a lot of attention to the $H_{\infty}$
control problem in time-delay systems [21–24]. Li and Hu [25] studied the problem of $H_{\infty}$ control for
neutral systems without nonlinear uncertainties. The $H_{\infty}$ control for uncertain neutral systems have
been reported in [26]. The problem of $H_{\infty}$ performance for a neutral system with discrete, neutral and
distributed time-varying delays and nonlinear uncertainties have been investigated in [19]. Their
results are restricted on delay-independent criteria for neutral systems [25] or uncertain neutral
systems without the condition of lower bounds of time-varying delays [19, 26].

Motivated by the above statement, in this paper, the problem of $H_{\infty}$ performance and exponential
stability analysis for a neutral system with interval discrete, neutral and distributed time-varying
delays and nonlinear uncertainties are considered based on Jensen’s integral inequality, the
Wirtinger-based integral inequality, an extended Wirtinger’s integral inequality, Peng-Park’s integral
inequality, the Leibniz-Newton formula, utilization of a zero equation, a decomposition matrix
 technique and the appropriate LKF. In the numerical part, we give some examples to present the
effectiveness of the theorem. The main contributions and highlights of this paper are summarized in
the following key points.

- We consider the problem of exponential stability for a neutral system with interval discrete,
  neutral and distributed time-varying delays and nonlinear uncertainties based on an $H_{\infty}$
  performance condition. It is noted that this work is the first study of the exponential stability and
  $H_{\infty}$ performance for an uncertain neutral system with three (discrete, neutral, and distributed)
  interval time-varying delays.
- We construct the LKFs including single, double, and triple integral terms involving lower and
  upper bounds of time delays and use them to formulate a new delay-range-dependent stability
criterion for a neutral system. In addition, the LKF consists of five new triple integral terms, i.e.,
  $\frac{d^2}{2} \int_{-\Delta_2}^{0} \int_{s+\theta}^{0} e^{2k(u+\theta-t)} \dot{\phi}^T(u)Z_{21}(u) \phi(u) dud\theta ds$, 
  $\lambda_2^2 \int_{-\Delta_2}^{0} \int_{s+\theta}^{0} e^{2k(u+\theta-t)} \dot{\phi}^T(u)Z_{22}(u) \phi(u) dud\theta ds$, 

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We derive new delay-range-dependent sufficient conditions for the exponential stability with $H_{\infty}$ performance (Theorem 1). Moreover, we obtain the improved delay-range-dependent exponential stability criterion of a neutral system with discrete, neutral and distributed time-varying delays, and nonlinear uncertainties. The proposed conditions are less conservative than the other references as shown in Theorem 1.

We present numerical examples to demonstrate the feasibility and effectiveness of the theorem.

The outline of this work is structured as follows. In Section 2, we give the problem statement, definitions and lemmas. We discuss some results for a neutral system and their proofs in Section 3. In Section 4, we give two numerical examples to present the effectiveness of the obtained result. Section 5 shows the conclusion of our results.

Notations: $\mathbb{R}^{n}$ denotes the $n$–dimensional Euclidean space, and $\mathbb{R}^{m \times n}$ is the set of all $m \times n$ real matrices. For a matrix $A$, $A > 0$ means that $A$ is a symmetric positive definite matrix and $\lambda_{\text{min}}(P)$ and $\lambda_{\text{max}}(P)$ denote the minimum and maximum eigenvalues of $A$, respectively. The superscript "$T$" denotes matrix transposition. diag{...} denotes the block diagonal matrix. Sym{$A$} = $A + A^T$.

2. Problem formulation and preliminaries

We introduce the following neutral system with interval time-varying delays and nonlinear uncertainties of the form

\[
\begin{align*}
\dot{\varphi}(t) &= A_1 \varphi(t) + A_2 \varphi(t-\lambda(t)) + A_3 \varphi(t-\sigma(t)) + A_4 \int_{t-\rho(t)}^{t} \varphi(s) ds + Bw(t) \\
&\quad + \zeta_1(t, \varphi(t)) + \zeta_2(t, \varphi(t-\lambda(t))) + \zeta_3(t, \varphi(t-\rho(t))), \quad t \geq 0; \\
\chi(t) &= C_1 \varphi(t) + C_2 \varphi(t-\lambda(t)) + Dw(t), \quad t \geq 0; \\
\varphi(t) &= \phi(t), \quad \forall t \in [-\max\{\lambda_2, \sigma_2, \rho_2\}, 0],
\end{align*}
\]

where $\varphi(t) \in \mathbb{R}^{n}$ is the state of the system, $w(t) \in \mathbb{R}^{p}$ is the disturbance input which belongs to $L_2[0, \infty]$, $\chi(t) \in \mathbb{R}^{q}$ is the controlled output, $\phi(t)$ is the initial condition function that is continuously differentiable on $[-\max\{\lambda_2, \sigma_2, \rho_2\}, 0]$ with $\|\phi\| = \sup_{\varphi\in[-\max\{\lambda_2, \sigma_2, \rho_2\}, 0]} \|\phi(\varphi)\|$, $A_1$, $A_2$, $A_3$, $A_4$, $B$, $C_1$, $C_2$ and $D$ are real constant matrices with appropriate dimensions and $\lambda(t)$, $\sigma(t)$ and $\rho(t)$ are time-varying discrete, neutral and distributed delays, respectively. The delays satisfy the following conditions:

\[
\begin{align*}
0 &\leq \lambda_1 \leq \lambda(t) \leq \lambda_d, \quad 0 \leq \dot{\lambda}(t) \leq \lambda_d, \\
0 &\leq \sigma_1 \leq \sigma(t) \leq \sigma_d, \quad 0 \leq \dot{\sigma}(t) \leq \sigma_d, \\
0 &\leq \rho_1 \leq \rho(t) \leq \rho_d, \quad 0 \leq \dot{\rho}(t) \leq \rho_d,
\end{align*}
\]
where \( \sigma_1, \sigma_2, \sigma_d, \lambda_1, \lambda_2, \lambda_d, \rho_1, \rho_2 \) and \( \rho_d \) are positive real constants and \( \zeta_1(t, \varphi(t)), \zeta_2(t, \varphi(t - \lambda(t))) \) and \( \zeta_3(t, \varphi(t - \sigma(t))) \) are nonlinear uncertainties that are assumed to satisfy the following inequalities

\[
\begin{align*}
\zeta_1^T(t, \varphi(t))\zeta_1(t, \varphi(t)) &\leq \eta_1^2 \varphi^T(t) \varphi(t), \\
\zeta_2^T(t, \varphi(t - \lambda(t)))\zeta_2(t, \varphi(t - \lambda(t))) &\leq \eta_2^2 \varphi^T(t - \lambda(t)) \varphi(t - \lambda(t)), \\
\zeta_3^T(t, \varphi(t - \sigma(t)))\zeta_3(t, \varphi(t - \sigma(t))) &\leq \eta_3 \dot{\varphi}^T(t - \sigma(t)) \dot{\varphi}(t - \sigma(t)),
\end{align*}
\]

where \( \eta_1, \eta_2 \) and \( \eta_3 \) are known positive real constants. We consider the Leibniz-Newton formula of the form

\[
0 = \varphi(t) - \varphi(t - \lambda(t)) - \int_{t-\lambda(t)}^{t} \dot{\varphi}(s) ds.
\]

In order to improve the discrete delay \( \lambda(t) \) in (2.2), let us decompose the constant matrix \( A_2 \) as

\[
A_2 = H_1 + H_2,
\]

where \( H_1, H_2 \in \mathbb{R}^{n \times n} \) are real constant matrices. By (2.8) and (2.9), System (2.1) can be represented in the form

\[
\begin{align*}
\dot{\varphi}(t) &= [A_1 + H_1 + I] \varphi(t) + [H_2 - I] \varphi(t - \lambda(t)) + A_3 \dot{\varphi}(t - \sigma(t)) + A_4 \int_{t-\lambda(t)}^{t} \varphi(s) ds \\
&\quad + Bw(t) + \zeta_1(t, \varphi(t)) + \zeta_2(t, \varphi(t - \lambda(t))) + \zeta_3(t, \varphi(t - \sigma(t))) \\
&\quad - [H_1 + I] \int_{t-\lambda(t)}^{t} \dot{\varphi}(s) ds.
\end{align*}
\]

**Remark 1.** In System (2.1), we assume that the delays in the discrete delay term and the distributed delay term are different but these two delay terms in [19] are the same.

**Definition 1.** [20] If there exist real positive scalars \( \beta \) and \( \kappa \) that satisfy

\[
\|\varphi(t, \phi)\| \leq \beta \|\phi\| e^{-\kappa t}, \quad \forall t \geq 0,
\]

then System (2.1) is exponentially stable.

**Definition 2.** [3] For a given real positive scalar \( \delta \), we say that System (2.1) is exponentially stable with the \( H_\infty \) performance level \( \delta \) if the system is exponentially stable and also satisfies \( \|\chi(t)\|_2 \leq \delta \|w(t)\|_2 \), for all nonzero \( w(t) \in L_2[0, \infty) \) under the zero initial condition.

**Lemma 1.** (Jensen’s inequality [19]). For any positive definite symmetric matrix \( W \in \mathbb{R}^{n \times n} \), \( k_2 \) is a positive scalar and the vector function \( \omega : [-k_2, 0] \rightarrow \mathbb{R}^n \) such that the integrals concerned are well defined; the following inequality holds:

\[
-k_2 \int_{-k_2}^{0} \omega^T(s + t)W\omega(s + t)ds \leq -\left( \int_{-k_2}^{0} \omega(s + t)ds \right)^T W \left( \int_{-k_2}^{0} \omega(s + t)ds \right).
\]
Lemma 2. [2] For any positive definite symmetric matrix $W \in \mathbb{R}^{n \times n}$, $k_2$ is a positive scalar and the vector function $\dot{\omega} : [-k_2, 0] \to \mathbb{R}^n$ such that the integrals concerned are well defined; then,

$$- \int_{-k_2}^{0} \int_{t+s}^{t} \dot{\omega}^T(u)W\dot{\omega}(u)du ds \leq \psi_1^T(t) \Omega_1 \psi_1(t),$$

where

$$\psi_1(t) = \begin{bmatrix} \frac{1}{k_2} \int_{t-k_2}^{t} \omega(s) ds \\ \end{bmatrix}, \quad \Omega_1 = \begin{bmatrix} -2W & 2W \\ * & -2W \end{bmatrix}.$$ 

Lemma 3. [27] For any positive definite symmetric matrix $W \in \mathbb{R}^{n \times n}$, $k_1 < k_2$ are positive scalars and the vector function $\dot{\omega} : [-k_2, -k_1] \to \mathbb{R}^n$ such that the integrals concerned are well defined; then,

$$-k_2 \int_{-k_2}^{-k_1} \dot{\omega}^T(s)W\dot{\omega}(s) ds \leq -\psi_2^T(t) W \psi_2(t),$$

$$- \frac{(k_2^2-k_1^2)}{2} \int_{-k_2}^{-k_1} \int_{t+s}^{t} \dot{\omega}^T(u)W\dot{\omega}(u)du ds \leq -\psi_3^T(t) W \psi_3(t),$$

where

$$\psi_2(t) = \left( \int_{-k_2}^{t} \dot{\omega}(s) ds \right), \quad \psi_3(t) = \left( \int_{-k_2}^{t} \int_{t+s}^{t} \omega(u) du ds \right).$$

Lemma 4. [19] For any positive definite symmetric matrix $W \in \mathbb{R}^{n \times n}$, $k(t)$ is a time-varying delay with $0 < k_1 < k(t) < k_2$, $k_2 \in \mathbb{R}$ and the vector function $\omega : [-k_2, -k_1] \to \mathbb{R}^n$ such that the integrals concerned are well defined; then,

$$-[k_2-k_1] \int_{-k_2}^{-k_1} \dot{\omega}^T(s)W\omega(s) ds \leq -\psi_4^T(t) W \psi_4(t) - \psi_5^T(t) W \psi_5(t),$$

where

$$\psi_4(t) = \int_{-k_1}^{t-k(t)} \omega(s) ds, \quad \psi_5(t) = \int_{-k_2}^{t-k(t)} \omega(s) ds.$$ 

Lemma 5. [19] For any constant matrices $Y_1, Y_2, Y_3 \in \mathbb{R}^{n \times n}$, $Y_1 \geq 0, Y_3 > 0, \begin{bmatrix} Y_1 & Y_2 \\ * & Y_3 \end{bmatrix} \geq 0$, $k(t)$ is a time-varying delay with $0 \leq k_1 \leq k(t) \leq k_2$, $k_1, k_2 \in \mathbb{R}$ and the vector function $\dot{\omega} : [-k_2, -k_1] \to \mathbb{R}^n$ such that the integrals concerned are well defined; then,

$$-(k_2-k_1) \int_{-k_2}^{-k_1} \begin{bmatrix} \omega(s) \\ \dot{\omega}(s) \end{bmatrix}^T \begin{bmatrix} Y_1 & Y_2 \\ * & Y_3 \end{bmatrix} \begin{bmatrix} \omega(s) \\ \dot{\omega}(s) \end{bmatrix} ds \leq \psi_6^T(t) \Omega_2 \psi_6(t),$$

where

$$\psi_6(t) = \begin{bmatrix} \omega(t-k_1) \\ \omega(t-k_1) \\ \omega(t-k(t)) \\ \omega(t-k(t)) \\ \int_{t-k(t)}^{t-k_1} \omega(s) ds \\ \int_{t-k(t)}^{t-k_1} \dot{\omega}(s) ds \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} Y_3 & Y_3 & 0 & -Y_2^T & 0 \\ * & -Y_3 - Y_2^T & Y_3 & Y_2^T & -Y_2^T \\ * & * & -Y_3 & 0 & Y_2^T \\ * & * & * & -Y_1 & 0 \end{bmatrix}.$$
Lemma 6. [19] For any constant matrices $W, Y_i \in \mathbb{R}^{n \times n}$, $i = 4, 5, \ldots, 8$, $k(t)$ is a time-varying delay with $0 \leq k_1 \leq k(t) \leq k_2$, $k_2 \in \mathbb{R}$ and the vector function $\dot{\omega} : [-k_2, -k_1] \to \mathbb{R}^n$ such that the integrals concerned are well defined; then,

$$-\int_{t-k_2}^{t-k_1} \dot{\omega}^T(s)W\dot{\omega}(s)ds \leq \psi_5^T(t)\Omega_5\psi_5(t),$$

where

$$\psi_5(t) = \begin{bmatrix} \omega(t-k_1) \\ \omega(t-k_2) \\ \frac{1}{k_2-k_1} \int_{t-k_2}^{t-k_1} \omega(s)ds \end{bmatrix}, \quad \Omega_5 = \begin{bmatrix} -W & -2W & 6W \\ * & -4W & 6W \\ * & * & -12W \end{bmatrix}.$$ 

Lemma 7. (Wirtinger-based integral inequality [28].) For any positive definite symmetric matrix $W \in \mathbb{R}^{n \times n}$, $k_1 < k_2$ are positive scalars and the vector function $\dot{\omega} : [-k_2, -k_1] \to \mathbb{R}^n$ such that the integrals concerned are well defined; then,

$$-(k_2-k_1) \int_{t-k_2}^{t-k_1} \dot{\omega}^T(s)W\dot{\omega}(s)ds \leq \psi_6^T(t)\Omega_6\psi_6(t),$$

where

$$\psi_6(t) = \begin{bmatrix} \omega(t-k_1) \\ \omega(t-k_2) \end{bmatrix}, \quad \Omega_6 = \begin{bmatrix} -W & W-S & S \\ * & -2W + S^T & W-S \\ * & * & -W \end{bmatrix}.$$ 

Lemma 8. (Peng-Park’s integral inequality [29].) If $W$ and $S$ are real constant matrices such that $[W \quad S \\ * \quad W] \geq 0$, $k(t)$ is a time-varying delay with $0 < k(t) < k_2$, $k_2 \in \mathbb{R}$ and the vector function $\dot{\omega} : [-k_2, 0] \to \mathbb{R}^n$ is well defined; then, the following inequality holds:

$$-k_2 \int_{t-k_2}^{t} \dot{\omega}^T(s)W\dot{\omega}(s)ds \leq \psi_9^T(t)\Omega_9\psi_9(t),$$

where

$$\psi_9(t) = \begin{bmatrix} \omega(t) \\ \omega(t-k(t)) \\ \omega(t-k_2) \end{bmatrix}, \quad \Omega_9 = \begin{bmatrix} -W & W-S & S \\ * & -2W + S^T & W-S \\ * & * & -W \end{bmatrix}.$$ 

Lemma 9. (An extended Wirtinger’s integral inequality [30].) For any positive definite symmetric matrix $W \in \mathbb{R}^{n \times n}$, $k_1$ and $k_2$ are positive scalars and the vector function $\omega : [k_1, k_2] \to \mathbb{R}^n$ such that the integrals concerned are well defined; then,

$$(k_2-k_1) \int_{k_1}^{k_2} \omega^T(u)W\omega(u)du \geq \Omega_7^T W\Omega_7 + 3\Omega_8^T W\Omega_8 + 5\Omega_9^T W\Omega_9,$$  \hspace{1cm} (2.11)
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where

\[ \Omega_7 = \int_{k_1}^{k_2} \omega(u) du, \]
\[ \Omega_8 = \int_{k_1}^{k_2} \omega(u) du - \frac{2}{k_2 - k_1} \int_{k_1}^{k_2} du \int_{k_1}^{u} \omega(r) dr, \]
\[ \Omega_9 = \int_{k_1}^{k_2} \omega(u) du - \frac{6}{k_2 - k_1} \int_{k_1}^{k_2} du \int_{k_1}^{u} \omega(r) dr \]
\[ + \frac{12}{(k_2 - k_1)^2} \int_{k_1}^{k_2} du \int_{k_1}^{u} ds \int_{s}^{u} \omega(r) dr. \]

Lemma 10. [31] For any positive definite symmetric matrix \( W \in \mathbb{R}^{n \times n} \), \( k_1 \) and \( k_2 \) are positive scalars and the vector function \( \dot{\omega} : [k_1, k_2] \rightarrow \mathbb{R}^n \) such that the integrals concerned are well defined; then,

\[ \int_{k_1}^{k_2} \int_{u}^{k_2} \omega^T(s)W\dot{\omega}(s)dsdu \geq 2\Omega_{10}^TW\Omega_{10} + 4\Omega_{11}^TW\Omega_{11} + 6\Omega_{12}^TW\Omega_{12}, \quad (2.12) \]

where

\[ \Omega_{10} = \omega(k_2) - \frac{1}{k_2 - k_1} \int_{k_1}^{k_2} \omega(s) ds, \]
\[ \Omega_{11} = \omega(k_2) + \frac{2}{k_2 - k_1} \int_{k_1}^{k_2} \omega(s) ds - \frac{6}{(k_2 - k_1)^2} \int_{k_1}^{k_2} du \int_{k_1}^{u} \omega(s) ds du, \]
\[ \Omega_{12} = \omega(k_2) - \frac{3}{k_2 - k_1} \int_{k_1}^{k_2} \omega(s) ds + \frac{24}{(k_2 - k_1)^2} \int_{k_1}^{k_2} du \int_{k_1}^{u} \omega(s) ds du \]
\[ - \frac{60}{(k_2 - k_1)^3} \int_{k_1}^{k_2} du \int_{k_1}^{u} \int_{s}^{u} \omega(r) dr ds du. \]

3. Main results

We introduce the following notations for later use:

\[ \sum = [\sum^{(i,j)}]_{27 \times 27}, \quad (3.1) \]

where

\[ \sum^{1,1} = 2Q_1^T I + 2Q_2^T I + 2Q_3^T H_1 + 2Q_4^T I + \lambda_2^2 Z_{19} + \lambda_2^2 e^{-2\epsilon_1 \zeta_1} F_3 + e^{-2\epsilon_1 \zeta_1} F_1 + e^{-2\epsilon_1 \zeta_1} F_1^T + 2Z_1 I \]
\[ - 4\lambda_2^2 e^{-2\epsilon_1 \zeta_1} Z_{18} - e^{-2\epsilon_1 \zeta_1} \mu_2^2 \lambda_2^2 G_1 + \lambda_2^2 G_4 + 2\lambda_2^2 \mu_2^2 G_4 + \lambda_2^2 G_4 - e^{-2\epsilon_1 \zeta_1} G_3 - \lambda_2^2 e^{-2\epsilon_1 \zeta_1} Z_{21} \]
\[ - \lambda_2^2 e^{-4\epsilon_1 \zeta_1} Z_{25} + 2\lambda_2^2 e^{-4\epsilon_1 \zeta_1} Z_{25} - \lambda_2^2 e^{-4\epsilon_1 \zeta_1} Z_{25} - 2\lambda_2^2 e^{-4\epsilon_1 \zeta_1} Z_{24} - 2\epsilon_1 \eta_1^2 I \]
\[ - 2\lambda_2^2 e^{-4\epsilon_1 \zeta_1} Z_{24} + 2Z_1 \mu_2^2 Z_1 + \lambda_2 \mu_2 Z_1 + \lambda_2 \mu_2 Z_1 + \lambda_2 \mu_2 Z_1 + \lambda_2 \mu_2 Z_1 + \lambda_2 \mu_2 Z_1 + \lambda_2 \mu_2 Z_1, \]
\[ \sum^{1,2} = Q_1^T I + Q_2^T I + A_1^T Q_{10} + I Q_{2} + I Q_{6} + H_1^T Q_{10} + Q_9^T H_2 - Q_9^T I + I Q_{10} + \lambda_2 e^{-2\epsilon_1 \zeta_1} F_4 \]
\[ - e^{-2\epsilon_1 \zeta_1} F_1^T + e^{-2\epsilon_1 \zeta_1} F_2 + e^{-2\epsilon_1 \zeta_1} Z_{18} - e^{-2\epsilon_1 \zeta_1} S + Z_1 I + e^{-2\epsilon_1 \zeta_1} G_3, \]

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\[
\sum_{1,3} = Z_1 + I Q_4 + I Q_8 - Q_6^T + \alpha_1^T Q_{12} + H_1^T Q_{12} + I_1^T Q_{12} + \lambda_2^2 G_2 + \lambda_2^1 G_5 = 2 \lambda_1 \alpha_2 G_5 + \lambda_1^2 G_5,
\]
\[
\sum_{1,4} = Q_1^T I + Q_5^T I + I Q_3 + I Q_7 - Q_6^T H_1 - Q_7^T I + \alpha_1^T Q_{11} + H_1^T Q_{11} + I Q_{11} + Z_1 I,
\]
\[
\sum_{1,5} = -2e^{-2x_{kl}} Z_{17} + e^{-2x_{kl}} S, \quad \sum_{1,9} = -e^{-2x_{kl}} G_2^T, \quad \sum_{1,12} = \lambda_2 e^{-2x_{kl}} Z_{21},
\]
\[
\sum_{1,13} = \lambda_2 e^{-2x_{kl}} Z_{25} - \lambda_1 e^{-2x_{kl}} Z_{25}, \quad \sum_{1,14} = 6e^{-2x_{kl}} Z_{17} + 12e^{-2x_{kl}} Z_{23} + 2\lambda_2^2 e^{-2x_{kl}} Z_{22},
\]
\[
\sum_{1,15} = 2\lambda_2 e^{-2x_{kl}} Z_{24}, \quad \sum_{1,18} = -120 e^{-2x_{kl}} Z_{23}, \quad \sum_{1,19} = 360 e^{-2x_{kl}} Z_{23}, \quad \sum_{1,20} = Q_9 A_4,
\]
\[
\sum_{1,25} = Q_9^T, \quad \sum_{1,26} = Q_9, \quad \sum_{1,27} = Q_9^T, \quad \sum_{2,26} = Q_{10}^T, \quad \sum_{2,27} = Q_{10}^T,
\]
\[
\sum_{2,2} = 2Q_1^T I + 2Q_8 I + 2Q_{10}^T H_2 - 2Q_9^T I + \lambda_2 e^{-2x_{kl}} F_3 + \lambda_2 e^{-2x_{kl}} F_6 - \lambda_1 F_8 + e^{-2x_{kl}} F_1
\]
\[
- e^{-2x_{kl}} F_1^T - e^{-2x_{kl}} F_2 - e^{-2x_{kl}} F_2^T + \lambda_2 F_3 + \lambda_2 F_6 - \lambda_1 F_{10} + e^{-2x_{kl}} F_6 + e^{-2x_{kl}} F_6^T
\]
\[
- e^{-2x_{kl}} F_7 - e^{-2x_{kl}} F_7^T - 2e^{-2x_{kl}} Z_{18} + e^{-2x_{kl}} S + e^{-2x_{kl}} S^T - e^{-2x_{kl}} G_3 - e^{-2x_{kl}} G_3^T
\]
\[
- e^{-2x_{kl}} G_6 - e^{-2x_{kl}} G_6^T - \lambda_2 e^{-2x_{kl}} Z_5 + \lambda_2 e^{-2x_{kl}} Z_5 + \lambda_1 e^{-2x_{kl}} Z_5 - \lambda_1 \lambda_2 e^{-2x_{kl}} Z_5 + e_2 \eta_1^2 I,
\]
\[
\sum_{2,3} = I Q_4 + I Q_8 - Q_6^T + H_2^T Q_{12} - I Q_{12}, \quad \sum_{3,2} = Q_9^T I + Q_9^T I - Q_{10} - Q_9^T I + Q_9^T H_2,
\]
\[
\sum_{2,4} = Q_9^T I + Q_9^T I + I Q_3 + I Q_7 - Q_{10}^T H_1 - Q_{10}^T I - I Q_{11} + H_2^T Q_{11}, \quad \sum_{3,26} = Q_9^T I,
\]
\[
\sum_{2,5} = \lambda_2 e^{-2x_{kl}} F_4 - e^{-2x_{kl}} F_6 + e^{-2x_{kl}} F_2 + \lambda_2 F_9 - \lambda_1 F_9 - e^{-2x_{kl}} F_6 + e^{-2x_{kl}} F_7 + e^{-2x_{kl}} Z_{18}
\]
\[
- e^{-2x_{kl}} S + e^{-2x_{kl}} G_3 + e^{-2x_{kl}} G_6,
\]
\[
\sum_{2,6} = \lambda_2 F_9^T - \lambda_1 F_9^T - e^{-2x_{kl}} F_6 + e^{-2x_{kl}} F_7 + e^{-2x_{kl}} G_6^T, \quad \sum_{2,7} = Q_9^T G_4^3, \quad \sum_{2,9} = e^{-2x_{kl}} G_2^T,
\]
\[
\sum_{2,10} = e^{-2x_{kl}} G_3^T, \quad \sum_{2,11} = -e^{-2x_{kl}} G_2^T - e^{-2x_{kl}} G_5^T, \quad \sum_{2,20} = Q_9^T A_4, \quad \sum_{2,25} = Q_9^T,
\]
\[
\sum_{3,3} = -2Q_{12} + \lambda_2 Z_{16} + \lambda_2^2 Z_{17} + \lambda_2 Z_{18} + \lambda_2 Z_{20} - \lambda_2 Z_{20} + \lambda_2^2 G_3 + \lambda_2^2 G_3 - 2\lambda_1 \lambda_2 G_6 + \lambda_1^2 G_6
\]
\[
+ \lambda_2^4 / 4 Z_{21} + \lambda_2^4 / 2 Z_{22} + \lambda_2^4 / 2 Z_{23} + \lambda_2^4 / 4 Z_{24} + \lambda_2^4 / 4 Z_{25} + \lambda_2^4 / 4 Z_{25},
\]
\[
\sum_{3,4} = Q_9^T I + Q_9^T I - Q_{11} - Q_{12}^T H_1 - Q_{11}^T I, \quad \sum_{3,20} = Q_9^T A_4, \quad \sum_{3,25} = Q_9^T,
\]
\[
\sum_{4,4} = 2Q_9^T I + 2Q_9^T I - 2Q_9^T I - 2Q_{11}^T I, \quad \sum_{4,7} = Q_9^T A_3, \quad \sum_{4,20} = Q_9^T A_4,
\]
\[
\sum_{4,25} = Q_{11}^T, \quad \sum_{4,26} = Q_{11}^T, \quad \sum_{4,27} = Q_{11}^T, \quad \sum_{4,27} = Q_{11}^T, \quad \sum_{1,5} = e^{-2x_{kl}} G_2^T + e^{-2x_{kl}} G_5^T,
\]
\[
\sum_{5,5} = \lambda_2 e^{-2x_{kl}} F_3 - e^{-2x_{kl}} F_2 - e^{-2x_{kl}} F_2^T - 4e^{-2x_{kl}} Z_{17} + \lambda_2 F_{10} - \lambda_1 F_{10} - e^{-2x_{kl}} F_7 - e^{-2x_{kl}} F_7^T
\]
\[
- e^{-2x_{kl}} Z_{18} - e^{-2x_{kl}} G_3 - e^{-2x_{kl}} G_6 - e^{-2x_{kl}} Z_2 - e^{-2x_{kl}} Z_4,
\]
\[
\sum_{5,14} = 6e^{-2x_{kl}} Z_{17}, \quad \sum_{6,2} = \lambda_2 F_9 - \lambda_1 F_9 - e^{-2x_{kl}} F_6 + e^{-2x_{kl}} F_7 + e^{-2x_{kl}} G_6,
\]
\[
\sum_{6,6} = \lambda_2 F_8 - \lambda_1 F_8 + e^{-2x_{kl}} F_6 + e^{-2x_{kl}} F_6 - e^{-2x_{kl}} G_6 - e^{-2x_{kl}} Z_3 + e^{-2x_{kl}} Z_4,
\]
\[
\sum_{7,7} = -\sigma_2 e^{-2x_{kl}} Z_6 + \sigma_2^T e^{-2x_{kl}} Z_6 + \sigma_1 e^{-2x_{kl}} Z_6 - \sigma_1^T e^{-2x_{kl}} Z_6 + e_1 \eta_1^2 I,
\]
\[
\sum_{8,8} = \rho_2 e^{-2x_{kl}} Z_7 - e^{-2x_{kl}} Z_7, \quad \sum_{9,9} = -e^{-2x_{kl}} G_4 - e^{-2x_{kl}} Z_{13}, \quad \sum_{6,10} = -e^{-2x_{kl}} G_5^T,
\]
\[
\sum_{10,10} = -e^{-2x_{kl}} G_4 - e^{-2x_{kl}} Z_{15}, \quad \sum_{11,11} = -e^{-2x_{kl}} G_4 - e^{-2x_{kl}} G_4 - e^{-2x_{kl}} Z_{13} - e^{-2x_{kl}} Z_{15},
\]
\[ \sum_{12,12} = -e^{-4k_1Z_{21}}, \quad \sum_{13,13} = -e^{-4k_1Z_{25}} \],
\[ \sum_{14,14} = -12e^{-2k_1Z_{17}} - e^{-2k_1}\lambda_2^2Z_{12} - 9\lambda_2^2e^{-2k_1Z_{19}} - 72e^{-4k_1Z_{23}} - 2\lambda_2^2e^{-4k_1Z_{22}}, \]
\[ \sum_{14,16} = 36\lambda_2e^{-2k_1Z_{19}}, \quad \sum_{14,17} = 60\lambda_2e^{-2k_1Z_{19}}, \quad \sum_{14,18} = 480e^{-4k_1Z_{23}}, \]
\[ \sum_{14,19} = -1080e^{-4k_1Z_{23}}, \quad \sum_{15,15} = -2\lambda_1^2e^{-4k_1Z_{24}} - \lambda_1^2e^{-2k_1Z_{14}}, \]
\[ \sum_{16,16} = -192e^{-2k_1Z_{19}}, \quad \sum_{16,17} = -360e^{-2k_1Z_{19}}, \quad \sum_{17,17} = -720e^{-2k_1Z_{19}}, \]
\[ \sum_{18,18} = -3600e^{-4k_1Z_{23}}, \quad \sum_{18,19} = 8640e^{-4k_1Z_{23}}, \quad \sum_{19,19} = -21600e^{-4k_1Z_{23}}, \]
\[ \sum_{20,20} = -Z_0, \quad \sum_{21,21} = -Z_8, \quad \sum_{22,22} = -Z_{10}, \quad \sum_{23,23} = -Z_0 - Z_{11}, \]
\[ \sum_{24,24} = -Z_{11}, \quad \sum_{25,25} = -e_iI, \quad \sum_{26,26} = -e_iI, \quad \sum_{27,27} = -e_jI, \]
and the other terms are 0;

\[ \sum = [\sum_{i,j}]_{28x28}^T, \tag{3.2} \]

where \(\sum_{i,j} = \sum_{i,j}^T, \quad i, j = 1, 2, 3, ..., 27, \) except

\[ \sum_{1,1} = \quad \sum_{Q_1}^T I + 2Q_1^T I + 2Q_9^T H_1 + 2Q_9^T I + \lambda_2^2Z_{19} + \lambda_2^2e^{-2k_1Z_{19}} + e^{-2k_1Z_{19}}F_3 + e^{-2k_1Z_{19}}F_1 + C_1^T C_1 + e^{-2k_1Z_{19}}F_1^T \]
\[ -4e^{-2k_1Z_{17}} - e^{-2k_1Z_{18}} - \lambda_1^2G_1 + \lambda_1^2G_4 - 2\lambda_1\lambda_2G_4 + 2Z_1 + \lambda_1^2G_4 - e^{-2k_1Z_{19}} + 2k_1Z_1 \]
\[ -\lambda_2^2e^{-4k_1Z_{21}} - \lambda_2^2e^{-4k_1Z_{21}}Z_{23} + 2\lambda_1\lambda_2e^{-4k_1Z_{23}} - 2\lambda_1\lambda_2e^{-4k_1Z_{23}} - 2\lambda_1e^{-4k_1Z_{23}} + \lambda_1\lambda_2F_1 \]
\[ -12e^{-2k_1Z_{23}} - 2\lambda_2e^{-2k_1Z_{23}} + Z_2 + Z_3 + \lambda_2Z_5 - \lambda_1Z_5 + Z_1 + \lambda_2Z_5 + \lambda_2Z_5 + \lambda_2Z_5 + \lambda_2Z_5 + \lambda_2Z_5 + \lambda_2Z_5 + \lambda_2Z_5, \]
\[ \sum_{1,2} = \quad Q_1^T I + Q_1^T I + A_1^T Q_{10} + IQ_2 + IQ_6 + H_{10}^T Q_{10} + Q_6^T H_2 - Q_9^T I + I Q_1^T + C_1^T C_2 \]
\[ + \lambda_2^2e^{-2k_1F_4} - e^{-2k_1F_4} + e^{-2k_1F_4} - e^{-2k_1Z_{18}} - e^{-2k_1Z_{18}} + e^{-2k_1Z_{18}} + e^{-2k_1Z_{18}} + e^{-2k_1Z_{18}} + e^{-2k_1Z_{18}} + e^{-2k_1Z_{18}} + e^{-2k_1Z_{18}} \]
\[ \sum_{2,2} = \quad 2Q_2^T I + 2Q_2^T I + 2Q_{10}^T H - 2Q_{10}^T I + \lambda_2e^{-2k_1Z_{19}}F_3 + \lambda_2e^{-2k_1Z_{19}}F_5 + e^{-2k_1Z_{19}}F_1 - e^{-2k_1Z_{19}}F_1 \]
\[ -e^{-2k_1Z_{23}}F_6 - e^{-2k_1Z_{23}}F_6 + \lambda_2F_8 + \lambda_2F_8 - \lambda_1F_8 - \lambda_1F_8 + e^{-2k_1Z_{23}}F_6 + e^{-2k_1Z_{23}}F_6 \]
\[ -e^{-2k_1Z_{23}}F_7 + e^{-2k_1Z_{23}}F_7 + 2e^{-2k_1Z_{23}}F_7 + 2e^{-2k_1Z_{23}}F_7 + e^{-2k_1Z_{23}}F_7 - e^{-2k_1Z_{23}}F_7 - e^{-2k_1Z_{23}}F_7 - e^{-2k_1Z_{23}}F_7 - e^{-2k_1Z_{23}}F_7 \]
\[ + e^{-2k_1Z_{23}}F_7 + C_1^T C_2, \quad \sum_{4,28} = Q_9^T B + C_1^T D, \]
\[ \sum_{2,28} = Q_{10}^T B + C_2^T D, \quad \sum_{3,28} = Q_1^T B, \quad \sum_{4,28} = Q_1^T B, \quad \sum_{28,28} = D^T D - \delta^2 I, \]
and the other terms are 0.

**Theorem 1.** For a prescribed scalar \( \delta > 0 \), given positive scalars \( \lambda_2, \sigma_2, \rho_2, \lambda_d, \sigma_d \) and \( \rho_d \), the System (2.1) is exponentially stable for a decay rate \( \kappa > 0 \) with the \( H_\infty \) performance \( \delta \); if \( \|A_3\| + \eta_3 < 1 \), there exist positive definite symmetric matrices \( Z_i, \) \( i = 1, 2, ..., 25, \) any appropriate
dimensional matrices $S, Q_j, j = 1, 2, ..., 12, G_l, l = 1, 2, ..., 6,$ and $F_k, k = 1, 2, ..., 10,$ and positive real constants $\varepsilon_n, n = 1, 2, 3,$ such that the following symmetric linear matrix inequalities hold

$$
\begin{bmatrix}
Z_{16} & F_1 & F_2 \\
* & F_3 & F_4 \\
* & * & F_5
\end{bmatrix} \geq 0,
$$

(3.3)

$$
\begin{bmatrix}
Z_{20} & F_6 & F_7 \\
* & F_8 & F_9 \\
* & * & F_{10}
\end{bmatrix} \geq 0,
$$

(3.4)

$$
\begin{bmatrix}
Z_{18} & S \\
* & Z_{18}
\end{bmatrix} \geq 0,
$$

(3.5)

$$
\begin{bmatrix}
G_1 & G_2 \\
* & G_3
\end{bmatrix} \geq 0,
$$

(3.6)

$$
\begin{bmatrix}
G_4 & G_5 \\
* & G_6
\end{bmatrix} \geq 0,
$$

(3.7)

$$
\sum_{j=1}^{10} 0 < 0.
$$

(3.8)

**Proof.** Under the condition of the theorem, we first show the exponential stability of System (2.10). Consider System (2.10) with $w(t) = 0$, that is,

$$
\dot{\varphi}(t) = [A_1 + H_1 + I]\varphi(t) + [H_2 - I]\varphi(t - \lambda(t)) + A_3\varphi(t - \sigma(t)) + A_4 \int_{t-\rho(t)}^{t} \varphi(s) ds
$$

$$
+ \zeta_1(t, \varphi(t)) + \zeta_2(t, \varphi(t - \lambda(t))) + \zeta_3(t, \varphi(t - \sigma(t))) - [H_1 + I] \int_{t-\delta(t)}^{t} \dot{\varphi}(s) ds.
$$

Construct an LKF candidate for the System (2.10) of the form

$$
V(t) = \sum_{i=1}^{9} V_i(t),
$$

(3.9)

where

$$
V_1(t) = \varphi^T(t)Z_1\varphi(t) = \beta_1^T(t)I_0\Psi_1\beta_1(t),
$$

wherein

$$
\beta_1(t) = \begin{bmatrix}
\varphi(t) \\
\varphi(t - \lambda(t)) \\
\int_{t-\lambda(t)}^{t} \dot{\varphi}(s) ds \\
\varphi(t)
\end{bmatrix},
$$

$$
I_0 = \begin{bmatrix}
I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
\Psi_1 = \begin{bmatrix}
Z_1 & 0 & 0 & 0 \\
Q_1 & Q_2 & Q_3 & Q_4 \\
Q_5 & Q_6 & Q_7 & Q_8 \\
Q_9 & Q_{10} & Q_{11} & Q_{12}
\end{bmatrix}.
$$

$$
V_2(t) = \int_{t-\lambda_2}^{t} e^{2\varepsilon(s-t)}\varphi^T(s)Z_2\varphi(s) ds + \int_{t-\lambda_1}^{t} e^{2\varepsilon(s-t)}\varphi^T(s)Z_3\varphi(s) ds + \int_{t-\lambda_2}^{t} e^{2\varepsilon(s-t)}\varphi^T(s)Z_4\varphi(s) ds,
$$

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\begin{align*}
V_3(t) &= (\lambda_2 - \lambda_1) \int_{t-\tau_1}^t e^{2\sigma_1(s-t)} \varphi^T(s) Z_5 \varphi(s) ds,
V_4(t) &= (\sigma_2 - \sigma_1) \int_{t-\tau_1}^t e^{2\varphi(s-t)} \varphi^T(s) Z_6 \varphi(s) ds, \\
V_5(t) &= \int_{t-\tau_1}^t e^{2\varphi(s-t)} \varphi^T(s) Z_7 \varphi(s) ds + \rho_2 \int_{t-\tau_2}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_8 \varphi(\theta) d\theta ds \\
&\quad + \rho_2 \int_{t-\tau_2}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_9 \varphi(\theta) d\theta ds + \rho_1 \int_{t-\tau_1}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{10} \varphi(\theta) d\theta ds \\
&\quad + (\rho_2 - \rho_1) \int_{t-\tau_2}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{11} \varphi(\theta) d\theta ds, \\
V_6(t) &= \lambda_2 \int_{t-\tau_1}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{12} \varphi(\theta) d\theta ds + \lambda_2 \int_{t-\tau_1}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{13} \varphi(\theta) d\theta ds \\
&\quad + \lambda_1 \int_{t-\tau_1}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{14} \varphi(\theta) d\theta ds + \lambda_2 \int_{t-\tau_1}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{15} \varphi(\theta) d\theta \\
&\quad - \lambda_1 \int_{t-\tau_1}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{16} \varphi(\theta) d\theta ds, \\
V_7(t) &= \int_{t-\tau_1}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{17} \varphi(\theta) d\theta ds + \lambda_2 \int_{t-\tau_1}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{18} \varphi(\theta) d\theta ds \\
&\quad + \lambda_2 \int_{t-\tau_1}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{19} \varphi(\theta) d\theta ds + \lambda_2 \int_{t-\tau_1}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{20} \varphi(\theta) d\theta ds \\
&\quad + \int_{t-\tau_1}^t \int_{t+s}^t e^{2\varphi(t-s)} \varphi^T(\theta) Z_{21} \varphi(\theta) d\theta ds, \\
V_8(t) &= \lambda_2 \int_{t-\tau_1}^t \int_{t+s}^t \left[ \varphi(\theta)^T [G_1 \ G_2] \varphi(\theta) \right] \varphi(\theta)^T [G_3] \varphi(\theta) d\theta ds \\
&\quad + (\lambda_2 - \lambda_1) \int_{t-\tau_1}^t \int_{t+s}^t \left[ \varphi(\theta)^T [G_4 \ G_5] \varphi(\theta) \right] \varphi(\theta)^T [G_6] \varphi(\theta) d\theta ds, \\
V_9(t) &= \frac{\lambda_2^2}{2} \int_{t-\tau_2}^t \int_{t+s}^t e^{2\varphi(u+\theta-t)} \varphi^T(u) Z_{23} \varphi(u) dud\theta ds \\
&\quad + \lambda_2^2 \int_{t-\tau_2}^t \int_{t+s}^t \int_{t+\theta}^t e^{2\varphi(u+\theta-t)} \varphi^T(u) Z_{24} \varphi(u) dud\theta ds \\
&\quad + \int_{t-\tau_2}^t \int_{t+s}^t \int_{t+\theta}^t e^{2\varphi(u+\theta-t)} \varphi^T(u) Z_{25} \varphi(u) dud\theta ds \\
&\quad + \frac{(\lambda_2^2 - \lambda_1^2)}{2} \int_{t-\tau_2}^t \int_{t+s}^t \int_{t+\theta}^t e^{2\varphi(u+\theta-t)} \varphi^T(u) Z_{26} \varphi(u) dud\theta ds.
\end{align*}

The time derivative of $V(t)$ along the solution of (2.10) is given by

\begin{equation}
\dot{V}(t) = \sum_{i=1}^9 \dot{V}_i(t). \tag{3.10}
\end{equation}
We compute $\dot{V}_1(t)$, $\dot{V}_2(t)$, $\dot{V}_3(t)$ and $\dot{V}_4(t)$ as

$$
\dot{V}_1(t) = 2 \begin{bmatrix} \varphi(t) \\ \varphi(t - \lambda(t)) \\ \int_{t - \lambda(t)}^{t} \dot{\varphi}(s)ds \\ \dot{\varphi}(t) \end{bmatrix}^T \begin{bmatrix} Z_1 & Q_{7}^T & Q_{9}^T & Q_{10}^T \\ 0 & Q_{2}^T & Q_{6}^T & 0 \\ 0 & Q_{3}^T & Q_{11}^T & 0 \\ 0 & Q_{4}^T & Q_{8}^T & Q_{12}^T \end{bmatrix} \begin{bmatrix} \varphi(t) - \beta_2(t) \\ \varphi(t - \lambda(t)) \\ \int_{t - \lambda(t)}^{t} \dot{\varphi}(s)ds \\ \dot{\varphi}(t) \end{bmatrix} + 2k \varphi^T(t)Z_1 \varphi(t) - 2kV_1(t),
$$

where

$$
\beta_2(t) = I\varphi(t) - I\varphi(t - \lambda(t)) - I \int_{t - \lambda(t)}^{t} \dot{\varphi}(s)ds,
$$

$$
\beta_3(t) = -\dot{\varphi}(t) + [A_1 + H_1 + I]\varphi(t) + [H_2 - I] \varphi(t - \lambda(t)) + A_3 (\varphi(t - \sigma(t)) + A_4 \int_{t - \rho(t)}^{t} \varphi(s)ds 
+ \xi_1(t, \varphi(t)) + \xi_2(t, \varphi(t - \lambda(t))) + \xi_3(t, \varphi(t - \sigma(t))) - [H_1 + I] \int_{t - \lambda(t)}^{t} \dot{\varphi}(s)ds,
$$

$$
\dot{V}_2(t) = \varphi^T(t)(Z_2 + Z_3)\varphi(t) - e^{-2\lambda_1} \varphi^T(t - \lambda_2)(Z_2 + Z_4)\varphi(t - \lambda_2) - e^{-2\lambda_2} \varphi^T(t - \lambda_1)(Z_3 - Z_4)\varphi(t - \lambda_1) - 2kV_2(t),
$$

$$
\dot{V}_3(t) \leq (\lambda_2 - \lambda_1) \varphi^T(t)Z_3\varphi(t) - 2kV_3(t)
- (\lambda_2 - \lambda_1)(1 - \lambda_3)e^{-2\lambda_1} \varphi^T(t - \lambda(t))Z_3\varphi(t - \lambda(t)),
$$

$$
\dot{V}_4(t) \leq (\lambda_2 - \lambda_1) \varphi^T(t)Z_3\varphi(t) - 2kV_4(t)
- (\lambda_2 - \lambda_1)(1 - \lambda_3)e^{-2\lambda_1} \varphi^T(t - \lambda(t))Z_3\varphi(t - \lambda(t)).
$$

By Lemmas 3 and 4, we obtain $\dot{V}_5(t)$ and $\dot{V}_6(t)$ as follows

$$
\dot{V}_5(t) \leq \varphi^T(t)Z_1\varphi(t) + \rho_2^2 \varphi^T(t)Z_3\varphi(t) + \rho_2^2 \varphi^T(t)Z_3\varphi(t) + \rho_1^2 \varphi^T(t)Z_10\varphi(t) - 2kV_5(t)
+ (\rho_2 - 1)e^{-2\lambda_2} \varphi^T(t - \rho(t))Z_7\varphi(t - \rho(t)) + (\rho_2 - \rho_1) \varphi^T(t)Z_{11}\varphi(t)
- \left( \int_{t - \rho(t)}^{t} \varphi(s)ds \right)Z_9 \left( \int_{t - \rho(t)}^{t} \varphi(s)ds \right) - \left( \int_{t - \rho(t)}^{t} \varphi^T(s)ds \right)Z_9 \left( \int_{t - \rho(t)}^{t} \varphi(s)ds \right)
- \left( \int_{t - \rho(t)}^{t} \varphi^T(s)ds \right)Z_9 \left( \int_{t - \rho(t)}^{t} \varphi(s)ds \right) - \left( \int_{t - \rho(t)}^{t} \varphi^T(s)ds \right)Z_9 \left( \int_{t - \rho(t)}^{t} \varphi(s)ds \right),
$$

$$
\dot{V}_6(t) \leq \lambda_2^2 \varphi^T(t)(Z_{12} + Z_{13})\varphi(t) + \lambda_1^2 \varphi^T(t)Z_{14}\varphi(t) + (\lambda_2 - \lambda_1)^2 \varphi^T(t)Z_{15}\varphi(t) - 2kV_6(t)
- \lambda_2^2 e^{-2\lambda_1} \left( \frac{1}{\lambda_2} \int_{t - \lambda_2}^{t} \varphi^T(s)ds \right)Z_{12} \left( \frac{1}{\lambda_2} \int_{t - \lambda_2}^{t} \varphi(s)ds \right)
- \lambda_2 e^{-2\lambda_1} \left( \int_{t - \lambda(t)}^{t} \varphi^T(s)ds \right)Z_{13} \left( \int_{t - \lambda(t)}^{t} \varphi(s)ds \right)
- \lambda_2 e^{-2\lambda_1} \left( \int_{t - \lambda(t)}^{t} \varphi^T(s)ds \right)Z_{13} \left( \int_{t - \lambda(t)}^{t} \varphi(s)ds \right),
$$

where $\lambda_2$ and $\lambda_1$ are the Lyapunov exponents.
\[
-e^{2\kappa T} \left( \int_{t-\lambda_1}^{t} \varphi(s) ds \right) (Z_{13} + Z_{15}) \left( \int_{t-\lambda_2}^{t} \varphi(s) ds \right)
- \lambda_1^2 e^{2\kappa T} \left( \frac{1}{\lambda_1} \int_{t-\lambda_1}^{t} \varphi(s) ds \right) Z_{14} \left( \frac{1}{\lambda_2} \int_{t-\lambda_2}^{t} \varphi(s) ds \right).
\]

Applying Lemmas 6–9, we obtain

\[
\dot{V}_7(t) \leq \lambda_2 \dot{\varphi}^T(t) Z_{16} \dot{\varphi}(t) + \lambda_2^2 \dot{\varphi}^T(t) (Z_{17} + Z_{18}) \dot{\varphi}(t) + e^{2\kappa T} \beta_4^T (t) \Psi \beta_4 (t) + (\lambda_2 - \lambda_1) \dot{\varphi}^T(t) Z_{20} \dot{\varphi}(t)
+ \lambda_2 e^{2\kappa T} \beta_4^T (t) \Psi \beta_4 (t) + \lambda_2^2 \dot{\varphi}^T(t) Z_{19} \varphi(t) + e^{-2\kappa T} \beta_5^T (t) \Psi \beta_5 (t) + (\lambda_2 - \lambda_1) \beta_5^T (t) \Psi \beta_5 (t)
+ e^{-2\kappa T} \beta_5^T (t) \Psi \beta_5 (t) + e^{-2\kappa T} \beta_6^T (t) \Psi \beta_6 (t) - \beta_7^T (t) \Psi \beta_7 (t) - 2\kappa V_7(t),
\]

where

\[
\beta_4(t) = \begin{bmatrix}
\varphi(t) \\
\varphi(t - \lambda_1) \\
\varphi(t - \lambda_2)
\end{bmatrix}, \beta_5(t) = \begin{bmatrix}
\varphi(t - \lambda_1) \\
\varphi(t - \lambda_2) \\
\frac{1}{\lambda_1} \int_{t-\lambda_1}^{t} \varphi(s) ds
\end{bmatrix}, \beta_6(t) = \begin{bmatrix}
\varphi(t) \\
\varphi(t - \lambda_1) \\
\frac{1}{\lambda_2} \int_{t-\lambda_2}^{t} \varphi(s) ds
\end{bmatrix},
\]

\[
\beta_7(t) = \begin{bmatrix}
\frac{1}{\lambda_2} \int_{t-\lambda_2}^{t} \varphi(s) ds \\
\frac{1}{\lambda_1} \int_{t-\lambda_1}^{t} \varphi(s) ds \\
\frac{1}{\lambda_2} \int_{t-\lambda_2}^{t} \varphi(s) ds
\end{bmatrix}, \Psi_2 = \begin{bmatrix}
F_3 & F_4 & 0 \\
* & F_3 + F_4 & F_4 \\
* & * & F_5
\end{bmatrix},
\]

\[
\Psi_3 = \begin{bmatrix}
F_1 + F_1^T & -F_2 & 0 \\
* & F_1 + F_1^T - F_2 - F_2^T & -F_1^T + F_2 \\
* & * & -F_2 - F_2^T
\end{bmatrix}, 
\Psi_4 = \begin{bmatrix}
-4Z_{17} & -2Z_{17} & 6Z_{17} \\
* & -4Z_{17} & 6Z_{17} \\
* & * & -12Z_{17}
\end{bmatrix}, \Psi_5 = \begin{bmatrix}
F_8 & F_9 & 0 \\
* & F_8 + F_{10} & F_9 \\
* & * & F_{10}
\end{bmatrix},
\]

\[
\Psi_6 = \begin{bmatrix}
F_6 + F_6^T & -F_7^T & 0 \\
* & F_6 + F_6^T - F_7 - F_7^T & -F_7^T + F_7 \\
* & * & -F_7 - F_7^T
\end{bmatrix}, 
\Psi_7 = \begin{bmatrix}
-Z_{18} & Z_{18} - S & S \\
* & -2Z_{18} + S + S^T & Z_{18} - S \\
* & * & -Z_{18}
\end{bmatrix},
\]

\[
\Psi_8 = \begin{bmatrix}
-9\lambda_2^2 e^{-2\kappa T} Z_{19} & -36\lambda_2 e^{-2\kappa T} Z_{19} & -60\lambda_2 e^{-2\kappa T} Z_{19} \\
-36\lambda_2 e^{-2\kappa T} Z_{19} & -192 e^{-2\kappa T} Z_{19} & 360 e^{-2\kappa T} Z_{19} \\
-60\lambda_2 e^{-2\kappa T} Z_{19} & 360 e^{-2\kappa T} Z_{19} & -720 e^{-2\kappa T} Z_{19}
\end{bmatrix}.
\]

From Lemma 5, we compute \( \dot{V}_8(t) \) as

\[
\dot{V}_8(t) \leq \lambda_2^2 \left( \varphi(t) \right)^T \begin{bmatrix}
G_1 & G_2 \\
G_3 & G_3
\end{bmatrix} \left( \varphi(t) \right) + (\lambda_2 - \lambda_1)^2 \left( \varphi(t) \right)^T \begin{bmatrix}
G_4 & G_5 \\
G_6 & G_6
\end{bmatrix} \left( \varphi(t) \right)
+ e^{-2\kappa T} \beta_6^T (t) \Psi \beta_6 (t) + e^{-2\kappa T} \beta_6^T (t) \Psi \beta_6 (t) - 2\kappa V_8(t),
\]
Using Lemma 2, Lemma 3 and Lemma 10, \( V_9(t) \) can be estimated as follows

\[
V_9(t) \leq \frac{\lambda_1^4}{4} \hat{\varphi}^T(t)Z_{21}\hat{\varphi}(t) + \frac{\lambda_2^4}{2} \hat{\varphi}^T(t)Z_{22}\hat{\varphi}(t) + \frac{\lambda_3^4}{2} \hat{\varphi}^T(t)Z_{23}\hat{\varphi}(t) + \frac{\lambda_4^4}{2} \hat{\varphi}^T(t)Z_{24}\hat{\varphi}(t) - 2\kappa V_9(t)
\]

\[
+ \frac{(\lambda_1^2 - \lambda_2^2)^2}{4} \hat{\varphi}^T(t)Z_{25}\hat{\varphi}(t) - e^{-4\kappa \lambda_2} \beta_{10}(t)Z_{26}\beta_{10}(t) + e^{-4\kappa \lambda_1} \Psi_{11}(t)\beta_{11}(t)
\]

\[
\Psi_{10} = \begin{bmatrix}
-G_6 & G_6 & 0 & -G^T_5 & 0 \\
* & -G_6 - G^T_6 & G_6 & G^T_6 & -G^T_5 \\
* & * & -G_6 & 0 & G^T_6 \\
* & * & * & * & -G_4 \\
* & * & * & * & -G_4
\end{bmatrix}
\]

where

\[
\beta_{10}(t) = (\lambda_2 - \lambda_1)\varphi(t) - \int_{t-\lambda_2}^{t} \varphi(s)ds, \quad \beta_{11}(t) = \frac{1}{\lambda_2^2} \int_{t-\lambda_2}^{t} \varphi(s)ds
\]

\[
\Psi_{11} = e^{-4\kappa \lambda_1} \begin{bmatrix}
-12Z_{23} & 12Z_{23} & -120Z_{23} & 360Z_{23} \\
12Z_{23} & -72Z_{23} & 480Z_{23} & -1080Z_{23} \\
-120Z_{23} & 480Z_{23} & -3600Z_{23} & 8640Z_{23} \\
360Z_{23} & -1080Z_{23} & 8640Z_{23} & -21600Z_{23}
\end{bmatrix}
\]
Under the zero initial condition, (3.15) becomes

\[
\varepsilon_1 (\eta_1^2 \varphi^T(t) \varphi(t) - \xi_1^T(t, \varphi(t)) \xi_1(t, \varphi(t))) \geq 0, \\
\varepsilon_2 (\eta_2^2 \varphi^T(t - \lambda(t)) \varphi(t - \lambda(t)) - \xi_2^T(t, \varphi(t - \lambda(t))) \xi_2(t, \varphi(t - \lambda(t))) \geq 0, \\
\varepsilon_3 (\eta_3^2 \varphi^T(t - \sigma(t)) \varphi(t - \sigma(t)) - \xi_3^T(t, \varphi(t - \sigma(t))) \xi_3(t, \varphi(t - \sigma(t))) \geq 0.
\]

(3.11)  
(3.12)  
(3.13)

According to (3.10)–(3.13), it is straightforward to see that

\[
\dot{V}(t) + 2\kappa V(t) \leq \xi^T(t) \Sigma \xi(t),
\]

where

\[
\xi(t) = \left[ \varphi(t), \varphi(t - \lambda(t)), \varphi(t), \int_{t-h(t)}^{t} \varphi(s)ds, \varphi(t - \lambda_2), \varphi(t - \lambda_1), \varphi(t - \sigma(t)), \varphi(t - \rho(t)), \int_{t-h(t)}^{t} \varphi(s)ds, \right.
\]

\[
\int_{t-h(t)}^{t} \varphi(s)ds, \int_{t-h(t)}^{t} \varphi(s)ds, \int_{t-h(t)}^{t} \varphi(s)ds, \int_{t-h(t)}^{t} \varphi(s)ds, \frac{1}{\lambda_2} \int_{t-h(t)}^{t} \varphi(s)ds, \frac{1}{\lambda_1} \int_{t-h(t)}^{t} \varphi(s)ds,
\]

\[
\frac{1}{\lambda_2} \int_{t-h(t)}^{t} \varphi(s)ds, \frac{1}{\lambda_1} \int_{t-h(t)}^{t} \varphi(s)ds, \int_{t-h(t)}^{t} \varphi(s)ds, \int_{t-h(t)}^{t} \varphi(s)ds, \int_{t-h(t)}^{t} \varphi(s)ds, \int_{t-h(t)}^{t} \varphi(s)ds,
\]

\[
\left. \int_{t-h(t)}^{t} \varphi(s)ds, \int_{t-h(t)}^{t} \varphi(s)ds, \int_{t-h(t)}^{t} \varphi(s)ds, \int_{t-h(t)}^{t} \varphi(s)ds, \right]
\]

If Conditions (3.3)–(3.8) and \( \Sigma < 0 \) hold, then

\[
\dot{V}(t) + 2\kappa V(t) \leq 0, \quad \forall t \in \mathbb{R}^+.
\]

(3.14)

So, we have

\[
\|\varphi(t, \phi)\| \leq M \|\phi\| e^{-\kappa t}, \quad t \in \mathbb{R}^+,
\]

where \( M, \kappa \in \mathbb{R}^+ \). This means that System (2.1) with \( w(t) = 0 \) is exponentially stable. Next, we shall establish the \( H_\infty \) performance of System (2.1) under the zero initial condition. We now introduce

\[
J(t) = \int_0^t [\chi^T(s) \chi(s) - \delta^2 w^T(s) w(s)] ds, \quad t > 0.
\]

(3.15)

Under the zero initial condition, (3.15) becomes

\[
J(t) = \int_0^t [\chi^T(s) \chi(s) - \delta^2 w^T(s) w(s)] ds
\]

\[
= \int_0^t [\chi^T(s) \chi(s) - \delta^2 w^T(s) w(s)] ds + \int_0^t V(s) ds - V(t) + V(0)
\]

\[
= \int_0^t [\chi^T(s) \chi(s) - \delta^2 w^T(s) w(s) + V(s)] ds - V(t)
\]
where $V(t)$ is defined in (3.9). After some algebraic manipulations, we obtain

$$\chi(t)\chi(t) - 2w^T(t)w(t) + \dot{V}(t) \leq \xi(t) + \sum_j \xi_j(t)$$

(3.17)

where

$$\xi_j(t) = [\phi(t), \phi(t - \lambda(t)), \phi(t), \int_{t - \lambda(t)}^t \phi(s)ds, \phi(t - \lambda_2), \phi(t - \lambda_1), \phi(t - \sigma(t)), \phi(t - \rho(t)), \int_{t - \lambda(t)}^t \phi(s)ds,$$

$$\int_{t - \lambda(t)}^t \phi(s)ds, \int_{t - \lambda(t)}^t \phi(s)ds, \int_{t - \lambda(t)}^t \phi(s)ds, \int_{t - \lambda(t)}^t \phi(s)ds, \frac{1}{\lambda_1} \int_{t - \lambda(t)}^t \phi(s)ds, \frac{1}{\lambda_1} \int_{t - \lambda(t)}^t \phi(s)ds,$$

$$\frac{1}{\lambda_2} \int_{t - \lambda(t)}^t \phi(s)ds, \frac{1}{\lambda_2} \int_{t - \lambda(t)}^t \phi(s)ds, \int_{t - \rho(t)}^t \phi(s)ds, \int_{t - \rho(t)}^t \phi(s)ds, \int_{t - \rho(t)}^t \phi(s)ds,$$

$$\int_{t - \rho(t)}^t \phi(s)ds, \xi_1(t, \phi(t)), \xi_2(t, \phi(t - \lambda(t))), \xi_3(t, \phi(t - \sigma(t))), J(t) < 0,$$

which implies that $\|\chi(t)\|_2 \leq \delta\|w(t)\|_2$ for any nonzero $w(t) \in L_2[0, \infty)$. The proof of the theorem is complete.

4. Numerical examples

Example 1. Consider the uncertain neutral system (2.1) with the following parameters:

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Decompose a matrix $A_2 = H_1 + H_2$, where

$$H_1 = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix}.$$

(4.1)

Let the interval discrete time-varying delay be $\lambda(t) = |\cos(t)|$ and the interval neutral and distributed time-varying delays be $\sigma(t) = \rho(t) = \sin^2(0.6t)$ for $t \in [-1, 0]$. By solving the linear matrix inequality (3.8) in Theorem 1, the maximum allowable upper bounds of $\rho_2$ for Example 1 are listed in Table 1 for various values of $\lambda_2$ and $\sigma_2$. We can see in Table 1 that the upper bound of the distributed
delay $\rho_2$ has an effect on $\lambda_2$. For any given $\lambda_2$, $\sigma_2$ decreases as $\rho_2$ increases. Table 2 presents the maximum allowable upper bounds of $\lambda_2$ for Example 1 with different values of $\kappa$ and $\sigma_2$. It shows that all of the conditions stated in Theorem 1 have been satisfied; hence, System (2.1) with the above given parameters has exponential stability with $H_\infty$ performance.

For the initial condition $\phi(t) = [0.5 \ 1]^T$, $\xi_1(t, \phi(t)) = \eta_1 \phi(t) \sin(\phi(t))$, $\xi_2(t, \phi(t - \lambda(t))) = \eta_2 \sin(\phi(t - \lambda(t))) e^{-2.3 \phi(t - \lambda(t))} \cos(\phi(t - \lambda(t)))$, and $\xi_3(t, \dot{\phi}(t - \rho(t))) = \eta_3 \dot{\phi}(t - \rho(t)) \cos(t)$. Figure 1 shows the trajectories of the solution $\varphi^T(t) = [\varphi_1(t), \varphi_2(t)]$ of the neutral system (2.1) with mixed time-varying delays and nonlinear uncertainties.

![Figure 1](image_url)

**Figure 1.** Trajectories of $\varphi_1(t)$ and $\varphi_2(t)$ of System (2.1) in Example 1.

### Table 1. Maximum allowable upper bounds of $\rho_2$ for Example 1 with different values of $\lambda_2$ and $\sigma_2$ when $\kappa = 1$, $\delta = 2$, $\eta_1 = \eta_2 = \eta_3 = 0.5$, $\lambda_1 = \sigma_1 = \rho_1 = 1$, $\lambda_d = \sigma_d = 0.9$ and $\rho_d = 0.8$.

| $\lambda_2$ | $\sigma_2 = 2.0$ | $\sigma_2 = 3.0$ | $\sigma_2 = 5.0$ | $\sigma_2 = 7.0$ |
|-------------|-----------------|-----------------|-----------------|-----------------|
| 2.0         | 20.3004         | 20.0273         | 20.0252         | 19.0005         |
| 3.0         | 19.6857         | 19.5352         | 18.9964         | 18.8596         |
| 5.0         | 19.0086         | 18.9835         | 18.9793         | 17.9930         |
| 7.0         | 17.4794         | 17.4794         | 16.4491         | 7.5000          |

### Table 2. Maximum allowable upper bounds of $\lambda_2$ for Example 1 with different values of $\lambda_2$ and $\sigma_2$ when $\delta = 1$, $\eta_1 = \eta_2 = \eta_3 = 0.5$, $\lambda_1 = \rho_1 = \sigma_1 = 1$, $\rho_2 = 2$, $\lambda_d = \sigma_d = 0.9$ and $\rho_d = 0.8$.

| $\kappa$ | $\sigma_2 = 2.0$ | $\sigma_2 = 3.0$ | $\sigma_2 = 5.0$ | $\sigma_2 = 7.0$ |
|----------|-----------------|-----------------|-----------------|-----------------|
| 0.3      | 25.5703         | 24.9307         | 24.5997         | 24.2467         |
| 0.5      | 16.4537         | 16.2211         | 14.6854         | 13.9659         |
| 0.7      | 11.5230         | 11.4199         | 10.5807         | 9.6449          |
| 0.9      | 9.0099          | 8.6875          | 7.9842          | 7.2311          |
Example 2. Consider the following neutral system with $w(t) = 0, C_1 = C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$:

$$
\dot{\varphi}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \varphi(t) + \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix} \varphi(t - \lambda(t)) + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \varphi(t - \sigma(t)) + \zeta_1(t, \varphi(t))
$$

$$
+ \zeta_2(t, \varphi(t - \lambda(t))) + \zeta_3(t, \dot{\varphi}(t - \rho(t))) + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \int_{t-\rho(t)}^t \varphi(s) ds
$$

(4.2)

when $\eta_1 = 0.1, \eta_2 = \eta_3 = 0.05, \lambda_d = 0.7, \rho_1 = 0.3, \rho_2 = \rho_d = 0.4, \sigma_1 = 0.3, \sigma_2 = 0.5$ and $\sigma_d = 0.1$. We separate a matrix $A_2$ as $A_2 = H_1 + H_2$, where

$$
H_1 = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix}.
$$

Table 3 shows a comparison of the upper bounds for the exponential stability of System (2.1) by different methods. It can be concluded that our results are less conservative than those in [11]. Figure 2 demonstrates the trajectories of the solution $\varphi_1(t)$ and $\varphi_2(t)$ of the uncertain neutral system (2.1) with mixed time-varying delays when $w(t) = 0$.

**Table 3.** Maximum allowable upper bounds of $\lambda_2$ for Example 2 with different values of $\kappa$ and $\lambda_1$.  

| Method     | $\kappa = 0.1$ | $\kappa = 0.3$ | $\kappa = 0.5$ |
|------------|----------------|----------------|----------------|
| $\lambda_1$ | 0.2            | 1.0            | 0.2            |
| [11]       | 67.21          | 64.05          | 26.05          |
| This Paper | 67.89          | 67.61          | 28.23          |

**Figure 2.** Trajectories of $\varphi_1(t)$ and $\varphi_2(t)$ of System (2.1) with $w(t) = 0$ in Example 2.
Example 3. Consider the System (2.1) with the following parameters:

\[
A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},
\]

\[
A_4 = C_1 = C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad w(t) = 0.
\]

Decompose a matrix \( A_2 = H_1 + H_2 \), where

\[
H_1 = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix}.
\]

Using the Matlab LMI toolbox, we obtain the maximum allowable upper bounds of \( \lambda_2 \), which are listed in Table 4. Table 4 describes the maximum allowable upper bounds of delays that guarantee the asymptotic stability of System (2.1) when \( \kappa = 0 \). It is clear that the obtained results in this study are better than those in [7, 10, 32–34].

Table 4. Maximum allowable upper bounds of \( \lambda_2 \) for Example 3 with different values of \( \lambda_d \) when \( \lambda_1 = \sigma_1 = \rho_1 = 0 \), \( \sigma_2 = 1 \), \( \eta_1 = 0.1 \) and \( \eta_2 = \eta_3 = 0.05 \).

| \( \lambda_d \) | 0.5 | 0.9 | 1.1 | Unknown |
|-----------------|-----|-----|-----|---------|
| \( \sigma_d = 0.6 \) | [7] | -   | -   | -       | 3.9563  |
|                 | [32] | -   | -   | -       | 4.6235  |
|                 | [33] | -   | -   | -       | 4.9423  |
|                 | [34] | -   | -   | -       | 8.7375  |
| This paper      | -   | -   | -   | 8.8193  |
| \( \sigma_d = 0 \) | [10] | 8.975 | 8.820 | -     | -       |
|                 | [33] | 9.646 | 9.225 | -     | -       |
|                 | [32] | 9.975 | 9.756 | 9.685 | -       |
|                 | [34] | -   | -   | -       | 9.7967  |
| This paper      | 10.125 | 10.034 | 9.987 | 9.8023 |

Remark 2. The less conservatism of Theorem 1 benefits from the construction of new LKFs with the application of Jensen’s integral inequality (Lemma 1), Peng-Park’s integral inequality (Lemma 8) and extended Wirtinger’s integral inequalities (Lemmas 9 and 10). These allowed our maximum delay to be greater than those in [7, 10, 11, 32–34] as shown in Tables 3 and 4.

5. Conclusions

In this article, the problem of exponential stability and \( H_\infty \) performance with mixed discrete, neutral and distributed interval time-varying delays and nonlinear uncertainties has been studied. To obtain delay-range-dependent sufficient conditions that can be achieved in the form of linear matrix inequalities for the \( H_\infty \) performance with exponential stability of the system, we have introduced an appropriate LKF and applied a decomposition matrix technique, the Leibniz-Newton formula, a zero equation, Peng-Park’s integral inequality, Jensen’s integral inequality and the Wirtinger-based integral
inequality. Numerical examples have been provided to verify the effectiveness of the presented results, showing that our results are better than the existing results.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

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