FANO 4-FOLDS WITH A SMALL CONTRACTION

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Abstract. Let $X$ be a smooth complex Fano 4-fold. We show that if $X$ has a small elementary contraction, then $\rho_X \leq 12$, where $\rho_X$ is the Picard number of $X$. This result is based on a careful study of the geometry of $X$, on which we give a lot of information. We also show that in the boundary case $\rho_X = 12$ an open subset of $X$ has a smooth fibration with fiber $\mathbb{P}^1$. Together with previous results, this implies that if $X$ is a Fano 4-fold with $\rho_X \geq 13$, then every elementary contraction of $X$ is divisorial and sends a divisor to a surface. The proof is based on birational geometry and the study of families of rational curves. More precisely the main tools are: the study of families of lines in Fano 4-folds and the construction of divisors covered by lines, a detailed study of fixed prime divisors, the properties of the faces of the effective cone, and a detailed study of rational contractions of fiber type.

1. Introduction

The classification of smooth, complex Fano varieties has been achieved up to dimension 3 and attracts a lot of attention also in higher dimensions. Let us focus on dimension 4, the first open case: the context of this paper is the study of Fano 4-folds with “large” Picard number (e.g. $\rho \geq 6$) by means of birational geometry and families of rational curves, with the aim of gaining a good understanding of the geometry and behaviour of these 4-folds. The main result of this paper is the following.

Theorem 1.1. Let $X$ be a smooth Fano 4-fold and $\rho_X$ its Picard number. If $X$ has a small elementary contraction, then $\rho_X \leq 12$.

The proof of this result is based on a careful study of the geometry of $X$, on which we give a lot of information. We also show an additional property in the boundary case $\rho_X = 12$, see Th. 9.1.

Apart from products of del Pezzo surfaces, to the author’s knowledge all the known examples of Fano 4-folds have $\rho \leq 9$; there is just one known family with $\rho = 9$, and it has small elementary contractions, see [CCF19]. Thus we do not know whether the bound of Th. 1.1 is sharp.

Let us notice that if $X$ has an elementary contraction of fiber type, then $\rho_X \leq 11$ by [Cas08, Cor. 1.2(ii)-(iii)], and if $X$ has an elementary divisorial contraction sending a divisor to a point or to a curve, then $\rho_X \leq 5$ [Cas17, Rem. 2.17(1)]. Therefore we have the following.

Corollary 1.2. Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 13$. Then every elementary contraction of $X$ is divisorial and sends the exceptional divisor to a surface.
This is the behaviour of products of del Pezzo surfaces, indeed we expect that for \( \rho_X \) large enough these should be the only Fano 4-folds.

Let us now explain the content of the paper and strategy of the proof of Th. 1.1. In the sequel \( X \) is a smooth, complex Fano 4-fold.

**Previous work.** Our starting point is the results on the geometry of Fano 4-folds developed in [Cas13a, Cas17, Cas20], in particular: the Lefschetz defect, the classification of fixed prime divisors, and the structure of rational contractions of fiber type.

**The Lefschetz defect.** As usual we denote by \( N_1(X) \) the real vector space of numerical equivalence classes of one-cycles in \( X \) with real coefficients, and for every closed subset \( Z \subset X \), we denote by \( N_1(Z,X) \) the linear subspace of \( N_1(X) \) spanned by classes of curves contained in \( Z \). If \( X \) is Fano, the Lefschetz defect \( \delta_X \) of \( X \), introduced in [Cas12], is defined as follows:

\[
\delta_X := \max \{ \text{codim} N_1(D,X) \mid D \text{ is a prime divisor in } X \}. \tag{1.3}
\]

If \( X \) is not a product of surfaces and \( \delta_X \geq 2 \), then \( \rho_X \leq 12 \) (see Th. 2.6 and 2.7), so that we can reduce to the case \( \delta_X \leq 1 \).

**Fixed prime divisors.** A fixed prime divisor is a prime divisor \( D \) which is the stable base locus of the linear system \( |D| \). Thanks to the bounds on the Lefschetz defect, in [Cas13a] it is shown that when \( \rho_X \geq 7 \) there are four possible types of fixed prime divisors of \( X \), called \( (3,2) \), \( (3,1)^{sm} \), \( (3,0)^{sm} \), and \( (3,0)^Q \), in relation to the associated elementary divisorial contractions; see §4.2 for more details. If \( X \) has a fixed prime divisor of type \( (3,0)^{sm} \), then \( \rho_X \leq 12 \) by [Cas17] (see Th. 4.7), so that we can exclude this case and focus on the remaining ones.

**Rational contractions of fiber type.** A rational contraction of fiber type is a rational map \( f: X \to Y \) that factors as sequence of flips followed by a contraction of fiber type \( f': X' \to Y \) (i.e. a surjective map with connected fibers, with \( Y \) normal and projective, and \( \dim Y \leq 3 \), see Section 2). Rational contractions of fiber type of Fano 4-folds are studied in detail in [Cas20], where in particular it is shown that if \( X \) has a rational contraction onto a 3-fold, and \( X \) is not a product of surfaces, then \( \rho_X \leq 12 \) (see Th. 5.8).

**New results.** The new tools and results used for the proof of Th. 1.1 include: the construction of families of lines and of divisors covered by lines, the properties of the faces of the effective cone, a detailed study of fixed prime divisors of type \( (3,1)^{sm} \) and \( (3,0)^Q \), and a detailed study of rational contractions of fiber type onto surfaces and onto \( \mathbb{P}^1 \).

**Families of lines.** We define a line in \( X \) as a rational curve \( C \) such that \( -K_X \cdot C = 1 \), and a family of lines as a “maximal” irreducible subvariety \( V \) of \( \text{Chow}(X) \) whose general member is a line, see Section 3 we study such families. By standard deformation theory one has \( \dim V \geq 2 \), and we show that if \( \rho_X \geq 6 \), then \( \dim V = 2 \) (Th. 3.7), because a family of lines of larger dimension yields a prime divisor \( D \) with small \( \text{dim} N_1(D,X) \), contradicting the results on the Lefschetz defect. Moreover the curves of the family \( V \) cover a surface or a prime divisor. Divisors covered by lines play a crucial role in the paper; when \( \rho_X \geq 7 \) such a divisor is either nef, or fixed of type \( (3,2) \); moreover the covering family of lines is unique (Lemmas 3.11 and 3.12).
Movable and fixed faces of the effective cone. As usual we denote by $N^1(X)$ the real vector space of numerical equivalence classes of divisors with real coefficients in $X$; the cones $\text{Eff}(X)$ and $\text{Mov}(X)$, of classes of effective and movable divisors respectively, are convex polyhedral cones in $N^1(X)$.

We say that a (proper) face $\tau$ of $\text{Eff}(X)$ is a movable face if the relative interior of $\tau$ intersects the movable cone $\text{Mov}(X)$; there exists a movable face if and only if $X$ has some non-zero, non-big movable divisor. Given a movable face $\tau$ of $\text{Eff}(X)$, we construct a rational contraction of fiber type $f: X \to Y$ such that $\rho_Y = \dim(\tau \cap \text{Mov}(X))$ and $\rho_X \leq \dim \tau + \rho_F$, where $F$ is a general fiber of $f': X' \to Y$ (where as above $f$ factors as a sequence of flips followed by the regular contraction $f'$) and it is smooth, Fano, with $\dim F \leq 3$, so that $\rho_F \leq 10$ by classification (see §5.2).

On the other hand, we say that a face $\tau$ of $\text{Eff}(X)$ is a fixed face if $\tau \cap \text{Mov}(X) = \{0\}$. A fixed face is always simplicial, and is generated by classes of fixed prime divisors; these divisors are in very special relative positions (see §4.1 and §4.2).

Thus when $\text{Eff}(X)$ has a movable face of small dimension, we get a good bound on $\rho_X$; in particular if $\text{Eff}(X)$ has a one-dimensional movable face, then $\rho_X \leq 11$. Therefore we can reduce to the case where every one-dimensional face of $\text{Eff}(X)$ is fixed, and this yields a lot of fixed prime divisors in $X$.

Strategic of the Proof. Let $f: X \to Y$ be a small elementary contraction, and assume that $\rho_X \geq 7$ and $\delta_X \leq 1$. We can also assume that $\text{Eff}(X)$ is generated by classes of fixed prime divisors, and there exists one such divisor $D$ such that $D \cdot C < 0$ for a curve $C \subset X$ contracted by $f$, so that $D$ contains the exceptional locus of $f$.

After the classification of fixed prime divisors, we have four possible types for $D$; we can exclude that $D$ is of type $(3,0)^{sm}$. If $D$ were of type $(3,2)$, $\text{Exc}(f) \subset D$ would yield $\dim \mathcal{N}_1(D,X) \leq 2$, which contradicts our assumptions on $\rho_X$ and $\delta_X$. Therefore $D$ is of type $(3,1)^{sm}$ or $(3,0)^Q$; for this reason we focus on these two types of fixed divisors, which are the main characters of the paper (see Rem. §1.11).

Given a fixed prime divisor $E \subset X$ of type $(3,1)^{sm}$ or $(3,0)^Q$, we construct from $E$ several families of lines, each covering a prime divisor different from $E$; this is a key result that allows to produce many prime divisors covered by lines with several good properties.

Using all these preliminary results, first we show that $\rho_X \leq 12$ when $X$ has a fixed prime divisor of type $(3,1)^{sm}$ (Th. §5.1). The proof is quite articulated and is contained in Section §5 we refer the reader to the beginning of that section for an overview.

Then we are left to consider the case where $X$ has only fixed prime divisors of type $(3,2)$ and $(3,0)^Q$; this is treated in Section §9 and again we refer the reader to the beginning of that section for an overview.

A frequent strategy in the paper is to look for movable, non-big divisors on $X$, in order to construct a rational contraction of fiber type, and then use it to bound $\rho_X$. Such movable, non-big divisors are usually obtained using fixed prime divisors and/or prime divisors covered by lines.

Summary. Section §2 contains the notation and recalls the preliminary results on the birational geometry of Fano 4-folds and on the Lefschetz defect. Section §3 contains the results on families of lines and divisors covered by lines in Fano 4-folds. Section §4 is devoted to fixed prime divisors; in particular in §4.1 and §4.3 we show several properties of fixed prime divisors of type $(3,1)^{sm}$ and $(3,0)^Q$ that are needed in the sequel. Section
Section 5 is devoted to rational contractions of fiber type; in particular in §5.3 we show several results on rational contractions of fiber type onto surfaces and onto $\mathbb{P}^1$, that are needed in the sequel.

Section 6 contains a key construction which, given a fixed prime divisor of type $(3,1)^{sm}$ or $(3,0)^Q$, produces many prime divisors covered by lines. Then we prove some properties of these divisors depending on the different settings. Section 7 contains two results where we manage to construct a rational contraction of fiber type on $X$, and use it to bound $\rho_X$. Finally Sections 8 and 9 contain the actual proof of Th. 1.1 first considering the case where $X$ has a fixed prime divisor of type $(3,1)^{sm}$, and then the case where $X$ has only fixed prime divisors of type $(3,2)$ and $(3,0)^Q$.

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2. Notation and preliminaries

If $\mathcal{N}$ is a finite-dimensional real vector space and $a_1, \ldots, a_r \in \mathcal{N}$, $\langle a_1, \ldots, a_r \rangle$ denotes the convex cone in $\mathcal{N}$ generated by $a_1, \ldots, a_r$. Moreover, for every $a \neq 0$, $a^\perp$ is the hyperplane orthogonal to $a$ in the dual vector space $\mathcal{N}^*$. A facet of a cone is a face of codimension one. If $C \subset \mathcal{N}$ is a convex polyhedral cone, we denote by $C^\vee \subset \mathcal{N}^*$ its dual cone.

We refer the reader to [HK00] for the notion and basic properties of a Mori dream space; we recall that smooth Fano varieties are Mori dream spaces by [BCHM10, Cor. 1.3.2]. We also refer to [Deb01, KM98] for the standard notions in birational geometry.

Let $X$ be a projective, normal, and $\mathbb{Q}$-factorial Mori dream space. If $D$ is a divisor and $C$ is a curve in $X$, we denote by $[D] \in \mathcal{N}^1(X)$ and $[C] \in \mathcal{N}_1(X)$ their classes, and we set $D^\perp := [D]^\perp \subset \mathcal{N}_1(X)$ and $C^\perp := [C]^\perp \subset \mathcal{N}^1(X)$.

For every closed subset $Z \subset X$, we denote by $\mathcal{N}_1(Z,X)$ the linear subspace of $\mathcal{N}_1(X)$ spanned by classes of curves contained in $Z$. We will use the following simple property.

**Remark 2.1.** Let $X$ be a smooth projective variety, $Z \subset X$ a closed subset, and $D \subset X$ a prime divisor disjoint from $Z$. Then $\mathcal{N}_1(Z,X) \subset D^\perp$, because $C \cdot D = 0$ for every curve $C \subset Z$.

A movable divisor is an effective divisor $D$ such that the stable base locus of the linear system $|D|$ has codimension $\geq 2$. A fixed prime divisor is a prime divisor $D$
which is the stable base locus of $|D|$, namely such that $h^0(mD) = 1$ for every $m \in \mathbb{Z}_{>0}$.

We will consider the usual cones of divisors and of curves:

$$\text{Nef}(X) \subseteq \text{Mov}(X) \subseteq \text{Eff}(X) \subset N^1(X), \quad \text{mov}(X) \subseteq \text{NE}(X) \subset N_1(X),$$

where all the notations are standard except $\text{mov}(X)$, which is the convex cone generated by classes of curves moving in a family covering $X$. Since $X$ is a Mori dream space, all these cones are closed, rational and polyhedral. An extremal ray of $\text{NE}(X)$ is a one-dimensional face of this cone.

A contraction $f : X \to Y$ is a surjective map, with connected fibers, where $Y$ is normal and projective. Given a divisor $D$ in $X$, $f$ is $D$-negative if $D \cdot C < 0$ for every curve $C \subset X$ such that $f(C) = \{pt\}$. An extremal ray $R$ of $\text{NE}(X)$ is $D$-negative if $D \cdot \gamma < 0$ for $\gamma \in R, \gamma \neq 0$. We do not assume that contractions or flips are $K$-negative, unless specified.

A small $\mathbb{Q}$-factorial modification (SQM) is a birational map $\varphi : X \dasharrow X'$ which is an isomorphism in codimension one, where $X'$ is a normal and $\mathbb{Q}$-factorial projective variety; then $X'$ is a Mori dream space too, and $\varphi$ can be factored as a finite sequence of flips.

A rational contraction (also called a contracting rational map) is a rational map $f : X \dasharrow Y$ that can be factored as $X \xrightarrow{\varphi} X' \xrightarrow{f'} Y$, where $\varphi$ is a SQM and $f'$ is a contraction. As in the regular case, a rational contraction is: of fiber type if $\dim Y < \dim X$, elementary if $\rho_X - \rho_Y = 1$, and elementary divisorial if $f'$ is an elementary divisorial contraction.

**Remark 2.2.** Let $X$ be a projective, normal, and $\mathbb{Q}$-factorial Mori dream space, and $D$ a divisor such that $|D| \in \text{Mov}(X)$, $|D| \neq 0$. Then there is a prime divisor with class in $\mathbb{R}_{\geq 0}[D]$.

Indeed, there is a SQM $X \dasharrow X'$ such that the transform $D'$ of $D$ in $X'$ is nef, and hence semiample. Therefore $|mD'|$ is base-point-free for $m \in \mathbb{N}$ large and divisible enough; by Bertini the general member of $\divisor{mD'}$ is irreducible, unless the linear system yields a contraction $f : X' \to C$ onto a curve. In this last case we have $\mathbb{R}_{\geq 0}[D'] = f^*N^1(C)$, thus the general fiber of $f$ is an irreducible divisor with class in $\mathbb{R}_{\geq 0}[D']$.

Assume now that $X$ is smooth of dimension 4, and let $f : X \to Y$ be an elementary divisorial contraction. We say that $f$ is:

- of type $(3, 2)$ if $\dim(f(\text{Exc}(f))) = 2$;
- of type $(3, 1)^{sm}$ if $Y$ is smooth and $f$ is the blow-up of a smooth curve;
- of type $(3, 0)^Q$ if $\text{Exc}(f)$ is isomorphic to an irreducible quadric $Q$, $f(\text{Exc}(f))$ is a point, and $\mathcal{N}_{\text{Exc}(f)/X} \cong \mathcal{O}_Q(-1)$;
- of type $(3, 0)^{sm}$ if $Y$ is smooth and $f$ is the blow-up of a point.

An exceptional plane is a closed subset $L \subset X$ such that $L \cong \mathbb{P}^2$ and $\mathcal{N}_{L/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$; we will denote by $C_L \subset L$ a curve corresponding to a line in $\mathbb{P}^2$. An exceptional curve is a closed subset $\ell \subset X$ such that $\ell \cong \mathbb{P}^1$ and $\mathcal{N}_{\ell/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$. We have $-K_X \cdot C_L = 1$ and $-K_X \cdot \ell = -1$.

**Theorem 2.3** ([Kaw89]). Let $X$ be a smooth Fano 4-fold and $f : X \to Y$ a small elementary contraction. Then $\text{Exc}(f)$ is a disjoint union of exceptional planes.
Lemma 2.4 ([Cas13a], Rem. 3.6). Let $X$ be a smooth Fano 4-fold and $\varphi: X \dasharrow \tilde{X}$ a SQM. Then $\tilde{X}$ is smooth and there are pairwise disjoint exceptional planes $L_1, \ldots, L_r \subset X$ and exceptional curves $\ell_1, \ldots, \ell_r \subset \tilde{X}$ such that $\varphi$ factors as:

$$
\xymatrix{ & \tilde{X} \\
X \ar[ur]^f \ar[rr]_{g} \ar[u]_{\varphi} & & \tilde{X} \\
}
$$

where $f$ is the blow-up of $L_1, \ldots, L_r$, $g$ is the blow-up of $\ell_1, \ldots, \ell_r$, and $E_i := f^{-1}(L_i) = g^{-1}(\ell_i) \cong \mathbb{P}^2 \times \mathbb{P}^1$ for every $i = 1, \ldots, r$. Moreover:

(a) if $\tilde{\Gamma} \subset \tilde{X}$ is an irreducible curve different from $\ell_1, \ldots, \ell_r$, with transforms $\Gamma \subset X$ and $\tilde{\Gamma} \subset \tilde{X}$, then $-K_X \cdot \Gamma = -K_X \cdot \tilde{\Gamma} + \sum_{i=1}^r E_i \cdot \tilde{\Gamma} > 0$;

(b) if $\tilde{\Gamma}$ intersects some exceptional curve, then $-K_X \cdot \Gamma < -K_X \cdot \tilde{\Gamma}$, so that $-K_X \cdot \tilde{\Gamma} \geq 2$.

Lemma 2.5 ([Cas13a], Rem. 3.7). Let $X$ be a smooth Fano 4-fold and $f: X \dasharrow Y$ a rational contraction. Then one can factor $f$ as $X \dasharrow X' \rightarrow Y$, where $\varphi$ is a SQM, $X'$ is smooth, and $f'$ is a $K$-negative contraction. Moreover $Y$ has rational singularities.

Finally let us recall the following results on the Lefschetz defect (see [L13]).

Theorem 2.6 ([Cas12], Th. 3.3 and Cor. 1.3). Let $X$ be a smooth Fano 4-fold which is not a product of surfaces. Then $\delta_X \leq 3$, and if $\delta_X = 3$, then $\rho_X \leq 6$.

Theorem 2.7 ([Cas13b], Th. 1.2). Let $X$ be a smooth Fano 4-fold with $\delta_X = 2$. Then $\rho_X \leq 12$, and if $\rho_X \geq 7$, then $X$ has a rational contraction onto a 3-fold.

Remark 2.8 ([Cas12], Ex. 3.1). If $X \cong S_1 \times S_2$ where $S_i$ are del Pezzo surfaces with $\rho_{S_1} \geq \rho_{S_2}$, then $\delta_X = \rho_{S_1} - 1$, and $\rho_X \leq 2\delta_X + 2$.

3. Lines in Fano 4-folds

Let $X$ be a normal projective variety. We denote by $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ the Hilbert scheme of morphisms $\mathbb{P}^1 \rightarrow X$ which are birational onto their image, and by $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)^n$ its normalization; there is a natural morphism $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)^n \rightarrow \text{Chow}(X)$ [Kol96 Cor. I.6.9]. By a family of rational curves in $X$ we mean an irreducible subvariety $V$ of Chow($X$) which is the closure of the image of an irreducible component of $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)^n$, see [Kol96 II.2.11]. For $v \in V$ general, the corresponding cycle is an irreducible and reduced rational curve, and every member of the family is an effective, connected one-cycle with rational components. We denote by $[V] \in \mathcal{N}_1(X)$ the numerical equivalence class of the general curve of the family. There is a universal family:

$$
\xymatrix{ \mathcal{C} \ar[r]^e \ar[d]_{\pi} & X \\
V & }
$$

Moreover Locus$V := e(\mathcal{C}) \subset X$ is the union of the curves of the family; it is an irreducible closed subset, and we say that $V$ is covering if Locus$V = X$. 
Remark 3.2. If $X$ is smooth and $V$ is covering, we have $-K_X \cdot [V] \geq 2$, indeed the general member of $V$ corresponds to a free curve [Deb01, 4.10 and Example 4.7(1)].

Definition 3.3. Let $X$ be a Gorenstein normal projective variety. A family of lines in $X$ is a family of rational curves such that $-K_X \cdot [V] = 1$.

We say that a prime divisor $D \subset X$ is covered by a family of lines if there exists a family of lines $V$ with $\text{Locus } V = D$.

We note that, for a smooth Fano variety $X$, lines are usually defined in terms of the index $r_X$ of $X$: if $-K_X = r_X H$, then $V$ would be a family of lines when $H \cdot [V] = 1$. In this paper we are interested in Fano 4-folds $X$ of index 1, indeed we will typically assume that $r_X \geq 7$, while Fano 4-folds with $r_X > 1$ are classified and have $r_X \leq 4$, see [IP99, Theorems 3.3.1 and 7.2.15]. Thus for simplicity we define lines directly as in Def. 3.3.

If $V$ is a family of lines and $X$ is Fano, then every member of the family is irreducible and reduced, so that $V$ is an unsplit family in the terminology of [Kol96, Def. IV.2.1].

Remark 3.4. Suppose that $X$ is smooth and Fano, and let $V$ be a family of lines in $X$. If $Z \subset X$ is a closed subset, we set $\text{Locus } V_Z$ to be the union of the curves of the family meeting $Z$, namely:

$$\text{Locus } V_Z := e(\pi^{-1}(\pi(e^{-1}(Z)))).$$ (3.5)

Thus $\text{Locus } V_Z$ is a closed subset, and if it is non-empty, then

$$\mathcal{N}_1(\text{Locus } V_Z, X) = \mathcal{N}_1(Z, X) + \mathbb{R}[V],$$

see [ACO04, Lemma 4.1]. If $Z = \{x\}$ is a point, we just write $\text{Locus } V_x$, and set $V_x := \pi(e^{-1}(x)) \subseteq V$ for the subvariety of $V$ parametrizing curves containing $x$.

Example 3.6 (lines in products of surfaces). Let $X = S \times T$ with $S$ and $T$ del Pezzo surfaces, and let $V$ be a family of lines in $X$. It is not difficult to see that up to switching $S$ and $T$, the curves of the family have the form $C \times \{p\}$ where $C \subset S$ is a given rational curve with $-K_S \cdot C = 1$, and $p$ varies in $T$, so that dim $V = 2$, $V \cong T$ and $\text{Locus } V \cong S \times T$.

If $C$ is smooth, then it is a $(-1)$-curve in $S$, and $[V]$ generates an extremal ray of type (3, 2) of $\text{NE}(X)$.

If $C$ is singular, then $\rho_S = 9$, $C \subset |-K_S|$, and $D = \text{Locus } V$ is a nef prime divisor.

Theorem 3.7. Let $X$ be a smooth Fano 4-fold and $V$ a family of lines in $X$. Then $2 \leq \text{dim } V \leq 4$, and moreover:

(a) if $\text{dim } V = 4$, then $\text{dim } \text{Locus } V = 3$ and $\rho_X \leq 3$;
(b) if $\text{dim } V = 3$, then $\text{dim } \text{Locus } V = 3$ and $\rho_X \leq 5$;
(c) if $\text{dim } V = 2$, then one of the following holds:
(i) $\text{dim } \text{Locus } V = 3$ and $V_x$ is finite for $x \in \text{Locus } V$ general;
(ii) $\text{dim } \text{Locus } V = 2$ and $\text{Locus } V = \text{Locus } V_x$ for every $x \in \text{Locus } V$.

If there are two curves of the family which are disjoint, then we are in case (i).

The bounds on $\rho_X$ in (a) and (b) are sharp, see Examples 3.9 and 3.10.

Proof. The inequality $\text{dim } V \geq 2$ follows from the standard lower bound on $\text{dim } \text{Hom}(\mathbb{P}^1, X)$, see [Kol96, Th. II.1.2 and Prop. II.2.11]. We also have

$$\text{dim } \text{Locus } V + \text{dim } \text{Locus } V_x = \text{dim } V + 2$$ (3.8)
for general \( x \in \text{Locus } V \), because the family is unsplit, see [Kol96 Prop. IV.2.5], which yields \( \dim \text{Locus } V \geq 2 \).

On the other hand \( \text{Locus } V \subseteq X \) by Rem. 3.2, thus \( \dim \text{Locus } V \leq 3 \), and (3.8) yields \( \dim V \leq 4 \).

Suppose that \( \dim V = 4 \). By (3.8) for general \( x \in \text{Locus } V \) we have \( \dim \text{Locus } V + \dim \text{Locus } V_x = 6 \), and \( \dim \text{Locus } V \leq 3 \), thus \( D = \text{Locus } V \) is a prime divisor and \( D = \text{Locus } V_x \) for general \( x \in D \). This implies that \( \dim \mathcal{N}_1(D, X) = 1 \) by (3.5), hence \( \rho_X \leq 3 \) by [Cas08 Prop. 3.16].

Suppose that \( \dim V = 3 \). By (3.8) for general \( x \in \text{Locus } V \) we have \( \dim \text{Locus } V + \dim \text{Locus } V_x = 5 \), thus \( D = \text{Locus } V \) is a prime divisor and \( \text{Locus } V_x \) is a surface for general \( x \in D \), while \( \dim \text{Locus } V_x \geq 2 \) for every \( x \in D \).

Choose \( x_0 \) such that \( \text{dim Locus } V_{x_0} = 2 \). Since \( x_0 \in \text{Locus } V_{x_0} \subseteq D \) and \( \dim D = 3 \), we can choose an irreducible curve \( C \subseteq D \) containing \( x_0 \) and not contained in \( \text{Locus } V_{x_0} \).

Then

\[
D = \bigcup_{x \in C} \text{Locus } V_x = \text{Locus } V_C,
\]

thus \( \mathcal{N}_1(D, X) = \mathbb{R}[C] + \mathbb{R}[V] \) by (3.5) and \( \dim \mathcal{N}_1(D, X) \leq 2 \). This implies that \( \delta_X \geq \rho_X - 2 \). Let us notice that \( X \) cannot be a product of surfaces (see Ex. 3.6), so that \( \delta_X \leq 3 \) by Th. 2.6, and hence \( \rho_X \leq 5 \).

Finally suppose that \( \dim V = 2 \). By (3.8) for general \( x \in \text{Locus } V \) we have \( \dim \text{Locus } V + \dim \text{Locus } V_x = 4 \).

If \( \dim \text{Locus } V = 3 \), then for \( x \in \text{Locus } V \) general we have \( \dim \text{Locus } V_x = 1 \) and \( \dim V_x = 0 \).

If \( \dim \text{Locus } V = 2 \), we get \( \dim \text{Locus } V_x = 2 \) for \( x \in \text{Locus } V \) general, and in fact for every \( x \in \text{Locus } V \) by upper semicontinuity. Since \( \text{Locus } V \) is irreducible, we conclude that \( \text{Locus } V = \text{Locus } V_x \) for every \( x \in \text{Locus } V \).

We show that in this last case two curves of the family always meet. Let \( \Gamma \subseteq \text{Locus } V \) be a curve of the family and consider the universal family as in (3.11). Then \( e \) has fibers of dimension at least 1, so that \( e^{-1}(\Gamma) \) has dimension 2. Since \( e(e^{-1}(\Gamma)) = \Gamma \), we have \( \pi(e^{-1}(\Gamma)) = V \). This means that \( e^{-1}(\Gamma) \) meets every fiber of \( \pi \), hence \( \Gamma \) meets every curve of the family \( V \).

\[ \square \]

**Example 3.9** (a case with \( \dim V = 4 \) and \( \rho_X = 3 \)). Let \( Y = \mathbb{P}^3(\mathcal{O} \oplus \mathcal{O}(3)) \), \( G \subset Y \) a section of the \( \mathbb{P}^1 \)-bundle \( Y \to \mathbb{P}^3 \) with normal bundle \( 
abla_{G/Y} \cong \mathcal{O}(3) \), and \( X \) the blow-up of \( Y \) along a plane contained in \( G \cong \mathbb{P}^3 \). The 4-fold \( X \) is toric, Fano, with \( \rho_X = 3 \), and contains a divisor \( D \cong \mathbb{P}^3 \) with normal bundle \( 
abla_{D/X} \cong \mathcal{O}(-3) \); this is \( E_1 \) in [Bat99] and \( X^7_{3,1} \) in [Sec21]. Let \( V \) be the family of lines in \( D \), then \( -K_X \cdot [V] = 1 \) and \( \dim V = 4 \); in fact \( V \cong \text{Gr}(1, \mathbb{P}^3) \). Every curve of the family is smooth with normal bundle \( 
abla(1) \oplus \mathcal{O}(-3) \), and \( D \cdot [V] = -3 \).

More generally, Fano 4-folds with \( \rho_X = 3 \) and containing a prime divisor \( D \) with \( \dim \mathcal{N}_1(D, X) = 1 \) are classified in [Sec21], there are 28 families; these are the possible Fano 4-folds with \( \rho_X = 3 \) and with a family of lines \( V \) such that \( \dim V = 4 \).

**Example 3.10** (a case with \( \dim V = 3 \) and \( \rho_X = 5 \)). Let \( X \) be the toric Fano 4-fold \( K_1 \) in Batyrev’s list [Bat99]; we have \( \rho_X = 5 \) and \( X \) has a smooth fibration onto \( \mathbb{P}^2 \), with fiber the surface obtained by blowing-up \( \mathbb{P}^2 \) at 3 non-collinear points. The 4-fold \( X \) contains a prime divisor \( D \cong \mathbb{P}^1 \times \mathbb{P}^2 \) with normal bundle \( 
abla_{D/X} \cong \mathcal{O}(-1, -2) \). Let
V be the family of lines in \( \{pt\} \times \mathbb{P}^2 \subset D \), then \(-K_X \cdot [V] = 1\) and \( \dim V = 3 \); in fact \( V \cong \mathbb{P}^1 \times \mathbb{P}^2 \). Every curve of the family is smooth with normal bundle \( \mathcal{O}(1) \oplus \mathcal{O}(2) \), and \( D \cdot [V] = -2 \). Let us note that \( D \) is also the locus of another family of lines \( W \), given by the curves \( \mathbb{P}^1 \times \{pt\} \subset D \).

More generally, Fano 4-folds with \( \rho_X = 5 \) and containing a prime divisor \( D \) with \( \dim N_1(D, X) = 2 \) are classified in [CR22], there are 6 families; these are the possible Fano 4-folds with \( \rho_X = 5 \) and with a family of lines \( V \) such that \( \dim V = 3 \).

**Lemma 3.11.** Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \), and \( D \subset X \) a prime divisor covered by a family of lines.

(a) The family of lines \( V \) such that \( D = \text{Locus}_V \) is unique, so that \( D \) and \( V \) determine each other.

(b) If \( D \) contains an exceptional plane \( L \), then \([V] = [C_L]\).

**Proof.** We show (a). Suppose by contradiction that there are two distinct families \( V \), \( W \) of lines with \( D = \text{Locus}_V = \text{Locus}_W \). We prove that this yields \( \dim N_1(D, X) \leq 3 \).

Let \( x \in D \) and consider

\[
S_x := \text{Locus}_W \text{Locus}_V x.
\]

We have \( \dim S_x \geq 2 \) and by (3.3) \( \mathcal{N}_1(S_x, X) = \mathbb{R}[V] + \mathbb{R}[W] \). If \( S_x = D \) for some \( x \), we are done. Otherwise take \( x_0 \in D \) a general point and let \( \Gamma \subset D \) be an irreducible curve through \( x_0 \) such that \( \Gamma \not\subset S_{x_0} \). Consider

\[
T := \bigcup_{x \in \Gamma} S_x = \text{Locus}_W \text{Locus}_V \Gamma.
\]

Then \( T = D \), so that by (3.3) we get

\[
\mathcal{N}_1(D, X) = \mathcal{N}_1(\text{Locus}_V \Gamma, X) + \mathbb{R}[W] = \mathbb{R}[\Gamma] + \mathbb{R}[V] + \mathbb{R}[W]
\]

and \( \dim \mathcal{N}_1(D, X) \leq 3 \). Since \( \rho_X \geq 7 \), this yields \( \delta_X \geq 4 \) and hence \( X \) is a product of surfaces by Th. 2.6, a contradiction (see Ex. 3.6).

The proof of (b) is similar and we leave it to the reader. ■

**Lemma 3.12** ([Cas17], Lemma 2.18). Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \), and \( D \subset X \) a prime divisor covered by a family \( V \) of lines. Then \( D \cdot [V] \geq -1 \), and one of the following holds:

(i) \( D \) is nef;

(ii) \( D \cdot [V] = -1 \) and \([V]\) generates an extremal ray of type (3.2) of \( \text{NE}(X) \).

The following result is similar to [NIM86 Prop. 5.3] on Fano 3-folds.

**Lemma 3.13.** Let \( X \) be a smooth Fano 4-fold and \( V \) a family of lines in \( X \) with \( \dim V = 2 \) and \( D := \text{Locus}_V \) a prime divisor. Suppose that there exists a curve \( C \) belonging to the family \( V \) such that \( C \cong \mathbb{P}^1 \), \( \mathcal{N}_{C/X} \cong \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1) \), and \( C \) does not intersect other curves of the family \( V \). Then \( D \cdot [V] = -1 \).

**Proof.** Since \( C \cong \mathbb{P}^1 \) and \( h^1(C, \mathcal{N}_{C/X}) = 0 \), \( \text{Hilb}(X) \) and \( \text{Chow}(X) \) are smooth and locally isomorphic at the point corresponding to \( C \) [Kol96 Th. I.2.8 and I.6.3, Cor. I.6.6.1], and \( C \) corresponds to a smooth point \( v_0 \) of a unique irreducible component \( V \) of \( \text{Chow}(X) \); consider the universal family (3.1) and set \( C_0 := \pi^{-1}(v_0) \subset C \). Notice that \( C \) is smooth around \( C_0 \), \( \pi \) is a smooth fibration in \( \mathbb{P}^1 \) around \( C_0 \), and \( e|_{C_0} : C_0 \to C \) is an isomorphism.
We show that the differential of $e: C \rightarrow X$ is injective at every point of $C_0$; this is a standard argument. Restricting to the two curves, $de: T_C|C_0 \rightarrow T_X|C$ restricts to an isomorphism between $T_{C_0}$ and $T_C$, and induces $\alpha: N_{C_0/C} \rightarrow N_{C/X}$. Moreover $N_{C_0/C} \cong T_{v_0}V \otimes \mathcal{O}_{C_0} \cong \mathcal{H}^0(C, \mathcal{N}_{C/X}) \otimes \mathcal{O}_{C_0}$, and $\alpha$ is naturally identified with the evaluation of global sections (see e.g. [KPS18, §2.2]), which is injective because $\mathcal{N}_{C/X} \cong \mathcal{O}^\oplus_{2} \oplus \mathcal{O}(1)$.

Therefore $e$ is smooth at every point of $C_0$. If there exist $z_1 \in C_0$ and $z_2 \in C$ such that $e(z_1) = e(z_2)$, then $e(\pi^{-1}(\pi(z_2)))$ is a curve of the family which intersects $C$, so by assumption $e(\pi^{-1}(\pi(z_2))) = C$. This implies that $\pi(z_2) = v_0$, $z_2 \in C_0$ and hence that $z_1 = z_2$.

We conclude that $e: C \rightarrow D$ is birational and that $C_0$ is contained in the open subset where $e$ is an isomorphism. Therefore $D$ is smooth around $C$, $-K_D \cdot C = 2$, and finally $D \cdot C = -1$.

4. Fixed prime divisors and associated contractions

This section is devoted to fixed prime divisors. In §4.1 we introduce the notions of “fixed face” of the effective cone of a Mori dream space, and of “adjacent” fixed prime divisors; both will be very relevant in the rest of the paper. In §4.2 we recall the classification of fixed prime divisors in Fano 4-folds with $\rho \geq 7$, and report some results on them. Finally in §4.3 we focus on fixed prime divisors of type $(3,1)^{sm}$ and $(3,0)^{Q}$, which are the ones that are relevant for the proof of Th. 1.1 (see Rem. 4.11), and prove many properties that we will need in the sequel.

4.1. Fixed faces of the effective cone

**Definition 4.1.** Let $X$ be a projective, normal and $\mathbb{Q}$-factorial Mori dream space. A face $\tau$ of $\text{Eff}(X)$ is called a fixed face if $\tau \cap \text{Mov}(X) = \{0\}$.

A fixed face is generated by classes of fixed prime divisors. There is a bijection between fixed prime divisors of $X$ and one-dimensional fixed faces of $\text{Eff}(X)$, via $D \mapsto \mathbb{R}_{\geq 0}[D]$ (see [Cas13a, Rem. 2.19]).

**Lemma 4.2.** Let $X$ be a projective, normal, and $\mathbb{Q}$-factorial Mori dream space. Every fixed face of $\text{Eff}(X)$ is simplicial.

**Proof.** Let $f: X \dashrightarrow Y$ be a birational (rational) contraction such that $Y$ is $\mathbb{Q}$-factorial, and $\varepsilon_f \subset \mathcal{N}^1(X)$ the convex cone generated by the classes of all exceptional prime divisors of $f$. Then $\varepsilon_f$ is a face of $\text{Eff}(X)$, because if $D_1, D_2$ are effective $\mathbb{Q}$-divisors on $X$ such that $[D_1 + D_2] \in \varepsilon_f$, then $0 = f_*(D_1) + f_*(D_2)$ and $f_*(D_1), f_*(D_2)$ are effective $\mathbb{Q}$-divisors on $Y$; we conclude that $f_*(D_1) = f_*(D_2) = 0$ so that $D_1$ and $D_2$ are exceptional and $[D_1], [D_2] \in \varepsilon_f$. Thus $\varepsilon_f$ is a fixed face, and it is a simplicial cone by [Oka16a, Lemma 2.7].

Consider the cone $\sigma_f := \varepsilon_f + f^*\text{Nef}(Y) \subset \mathcal{N}^1(X)$; then every face of $\sigma_f$ is generated by a face of $\varepsilon_f$ and a face of $f^*\text{Nef}(Y)$. Recall that $\text{Eff}(X)$ is the union of the cones $\sigma_f$ when $f$ varies [HK00, Prop. 1.11(2)], and such cones intersect each other along common faces [Oka16a, Prop. 2.9].

Now if $\tau$ is a fixed face of $\text{Eff}(X)$, there exists some $f$ such that $\sigma_f$ has a face $\eta$ with $\eta \subseteq \tau$ and $\dim \eta = \dim \tau$. Since $\tau$ is fixed, we have $\eta \cap f^*\text{Nef}(Y) = \{0\}$ and $\eta \subseteq \varepsilon_f$, therefore $\eta$ is a simplicial face of $\text{Eff}(X)$ and $\eta = \tau$. ■
Definition 4.3. We say that two fixed prime divisors are adjacent if their classes in $\mathcal{N}^1(X)$ generate a fixed face of $\text{Eff}(X)$.

Lemma 4.4. Let $X$ be a projective, normal, and $\mathbb{Q}$-factorial Mori dream space, $f : X \dasharrow Y$ an elementary divisorial rational contraction, and $D := \text{Exc}(f)$, so that $D$ is a fixed prime divisor.

(a) For every $r \in \mathbb{Z}_{>0}$ there is a bijection between $(r+1)$-dimensional faces of $\text{Eff}(X)$ containing $[D]$, and $r$-dimensional faces of $\text{Eff}(Y)$, given by $\tau \mapsto f_*\tau$;

(b) $\tau$ is fixed if and only if $f_*\tau$ is fixed;

(c) there is a bijection between fixed prime divisors $E_Y \subset Y$ and fixed prime divisors $E_X \subset X$ adjacent to $D$ (and different from $D$); here $E_X$ is the transform of $E_Y$ in $X$.

Proof. [Cas17] Lemma 2.21] shows (a) for $r = 1$, as well as (c). The same proof yields (a) for any $r$, because under the push-forward of divisors $f_* : \mathcal{N}^1(X) \to \mathcal{N}^1(Y)$, the cone $\text{Eff}(Y)$ can be seen as the “quotient cone” of $\text{Eff}(X)$ modulo the one-dimensional face $\langle [D] \rangle$, see for instance [Ewa96, Def. V.2.8 and Th. V.2.9].

For (b), if there exists a non-zero movable divisor $M$ with $[M] \in \tau$, then $f_*M$ is a non-zero movable divisor with class in $f_*\tau$. Conversely, if there exists a non-zero movable divisor $M_Y$ with $[M_Y] \in f_*\tau$, then there exists $\mu \in \mathbb{Q}_{\geq 0}$ such that $M := f^*M_Y - \mu D$ is non-zero and movable, and since $f_*(\langle [M] \rangle) = [M_Y] \in f_*\tau$, there exists $\lambda \in \mathbb{Q}$ such that $[M + \lambda D] \in \tau$. Since $\tau$ is a face of $\text{Eff}(X)$, there exists a class $\gamma \in \text{Eff}(X)^\vee$ such that $\tau = \text{Eff}(X) \cap \gamma^\perp$. Then $M \cdot \gamma = (M + \lambda D) \cdot \gamma - \lambda D \cdot \gamma = 0$ and $[M] \in \text{Eff}(X)$, so that $[M] \in \tau$.

4.2. Fixed prime divisors of Fano 4-folds

Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 7$. After [Cas13a, Cas17] there are four possible types of fixed prime divisors in $X$. In this subsection we recall this classification.

Theorem - Definition 4.5 ([Cas17], Th. 5.1, Def. 5.3, Cor. 5.26, Def. 5.27). Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 7$, and $D$ a fixed prime divisor in $X$.

(a) There exists a unique diagram:

$$X \xrightarrow{\varphi} \tilde{X} \xrightarrow{\sigma} Y$$

where $\varphi$ is a SQM, $\sigma$ is an elementary divisorial contraction with exceptional divisor the transform $\tilde{D}$ of $D$, and $Y$ is Fano (possibly singular);

(b) $\sigma$ is of type $(3,0)^{\text{sm}}$, $(3,0)^Q$, $(3,1)^{\text{sm}}$, or $(3,2)$, and we define $D$ to be of type $(3,0)^{\text{sm}}$, $(3,0)^Q$, $(3,1)^{\text{sm}}$, or $(3,2)$, respectively.

(c) If $D$ is of type $(3,2)$, then $X = \tilde{X}$.

(d) We define $C_D \subset D \subset X$ to be the transform of a general irreducible curve $C_{\tilde{D}} \subset \tilde{X}$ contracted by $\sigma$, of minimal anticanonical degree. Then $C_D \cong \mathbb{P}^1$, $D \cdot C_D = -1$, and $C_D$ is contained in the open subset where $\varphi$ is an isomorphism.

(e) Given a SQM $X \dasharrow X'$ and an elementary divisorial contraction $k : X' \to Y$ with $\text{Exc}(k)$ the transform of $D$, then $k$ has the same type as $\sigma$.

We will frequently use the notations $C_D \subset D$ and $C_{\tilde{D}} \subset \tilde{D}$ introduced above.

Example 4.6. Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 7$. If $X$ is a product of surfaces, then every elementary birational contraction of $X$ has the form $S \times T \to S' \times T'$, where $S$ and $S'$ are surfaces and $T$ and $T'$ are 4-dimensional tori.
where \( S \to S' \) is the blow-up of a point. In particular \( X \) has no small contraction, and every fixed prime divisor of \( X \) is of type \((3,2)\).

Fano 4-folds with a fixed prime divisor of type \((3,0)^{sm}\) have been treated in \([\text{Cas17}]\).

**Theorem 4.7** ([Cas17], Th. 5.40). Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \), having a fixed prime divisor of type \((3,0)^{sm}\). Then \( \rho_X \leq 12 \), and if \( \rho_X = 12 \), then \( X \) has a rational contraction onto a 3-fold.

Concerning fixed prime divisors of type \((3,2)\), we recall the following results, that will be relevant in the sequel.

**Theorem 4.8** ([Cas17], Prop. 5.32). Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \) and \( \delta_X \leq 1 \), having a fixed prime divisor \( D \) of type \((3,2)\) such that \( N_1(D,X) \subseteq N_1(X) \).
Then \( \rho_X \leq 12 \), and if \( \rho_X = 12 \), then \( X \) has a rational contraction onto a 3-fold.

**Remark 4.9** ([Cas17], Rem. 2.17(2)). Let \( X \) be a Fano 4-fold with \( \rho_X \geq 7 \) and \( D \subseteq X \) a fixed prime divisor of type \((3,2)\). Then \( D \) does not contain exceptional planes.

**Lemma 4.10.** Let \( X \) be a Fano 4-fold with \( \rho_X \geq 7 \), \( D \subseteq X \) a fixed prime divisor of type \((3,2)\), \( X \rightarrow \tilde{X} \) a SQM, and \( D \subseteq \tilde{X} \) the transform of \( X \). Then \( \dim N_1(D,X) = \dim N_1(D,\tilde{X}) \).

**Proof.** This follows from Rem. 4.9 and \([\text{Cas13a}]\) Cor. 3.14. \( \blacksquare \)

Let us note that Lemma 4.10 extends the applicability of Th. 4.8 also to SQM’s of \( X \). This is a special property of fixed prime divisors of type \((3,2)\), as for a general prime divisor \( D \) in a Fano 4-fold \( X \), if \( X \rightarrow \tilde{X} \) is a SQM and \( D \subseteq \tilde{X} \) is the transform of \( D \), then \( \dim N_1(D,\tilde{X}) \) may be smaller than \( \dim N_1(D,X) \) (see for instance \([\text{Cas13b}]\)).

**Remark 4.11** (fixed prime divisors and small contractions). Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \). If \( X \) has a fixed prime divisor \( D \) not of type \((3,2)\), then \( X \) has a small elementary contraction, indeed by \([\text{Cas17}]\) Th. 5.1 the map \( \varphi \) in Th.-Def. 4.5 factors as a sequence of at least \( \rho_X - 4 \) \( K \)-negative flips.
Conversely, it is not difficult to show (see Rem. 5.7) that if \( X \) has a small elementary contraction \( f \): \( X \rightarrow Y \), then either \( \rho_X \leq 11 \), or \( X \) has a fixed prime divisor \( D \) such that \( D \cdot \text{NE}(f) < 0 \), and \( D \) cannot be of type \((3,2)\) by Th. 2.3 and Rem. 4.9.
Moreover, if \( D \) is of type \((3,0)^{sm}\), we can apply Th. 4.7.
Therefore Th. 4.4 can be seen as a statement on Fano 4-folds with \( \rho_X \geq 7 \) having a fixed prime divisor of type \((3,1)^{sm}\) or \((3,0)^Q\), and we will focus on these two types.

We will need some further properties of fixed prime divisors.

**Lemma 4.12.** Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \) and \( D_1, D_2 \subseteq X \) two distinct fixed prime divisors.

(a) If \( D_1 \cdot C_{D_2} = 0 \), then \( D_1 \) and \( D_2 \) are adjacent.

(b) If \( D_1 \cdot C_{D_2} > 0 \) and \( D_2 \cdot C_{D_1} > 0 \), then \( D_1 + D_2 \) is movable and \([C_{D_1} + C_{D_2}] \in \text{mov}(X)\).

(c) \( D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 1 \) if and only if \( \dim([[D_1],[D_2]]) \cap \text{Mov}(X) = 1 \); in this case \( [[D_1],[D_2]] \cap \text{Mov}(X) = [[D_1 + D_2]] \), and \( D_1 + D_2 \) is movable and non-big.

**Proof.** This is \([\text{Cas20}]\) Lemma 4.6, except (b) which follows from the same proof. \( \blacksquare \)
Lemma 4.13 ([Cas20], Lemma 4.9). Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 7$ and $D, E$ two adjacent fixed prime divisors.

(a) If $E$ is of type $(3, 2)$, then $D \cdot C_E = 0$;
(b) if $D$ and $E$ are not of type $(3, 2)$, then $D \cdot C_E = E \cdot C_D = 0$, and $D \cap E$ is either empty or a disjoint union of exceptional planes.

Corollary 4.14. Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 7$ and $D_1, D_2 \subset X$ two fixed prime divisors, both of type $(3, 2)$ or neither. If $D_1 \cdot C_{D_2} > 0$, then $D_2 \cdot C_{D_1} > 0$, $D_1$ and $D_2$ are not adjacent, $D_1 + D_2$ is movable, and $[C_{D_1} + C_{D_2}] \in \mov(X)$.

Proof. By Lemma 4.13 $D_1$ and $D_2$ are not adjacent, and the rest follows from Lemma 4.12.

Lemma 4.15. Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 7$ and $E_1, E_2$ two fixed prime divisors of type $(3, 2)$ such that $E_1 \cdot C_{E_2} = 0$ and $E_1 \cap E_2 \neq \emptyset$. Then $E_2 \cdot C_{E_1} = 0$ and every connected component of $E_1 \cap E_2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1)$.

Proof. By Cor. 4.14 we have $E_2 \cdot C_{E_1} = 0$ and $E_1$ and $E_2$ are adjacent. Moreover $\langle [C_{E_i}] \rangle$ is the unique $E_i$-negative extremal ray of $\NE(X)$, for $i = 1, 2$, by [Cas17, Rem. 2.17]. Then $-K_X + E_1 + E_2$ is nef and $(-K_X + E_1 + E_2) \cap \NE(X) = \langle [C_{E_1}], [C_{E_2}] \rangle$ is a face of $\NE(X)$. The associated contraction $f : X \rightarrow Z$ is birational, has exceptional locus $E_1 \cup E_2$, and the general fiber of $f|_{E_1}$ is one-dimensional. The possible two-dimensional fibers of $f$ are classified in [AW97, Th. 4.7] and are $\mathbb{P}^2$ or a (possibly singular/reducible) quadric surface.

If $F$ is an irreducible component of $E_1 \cap E_2$, then $N_1(F, X) = \mathbb{R}[C_{E_1}] \oplus \mathbb{R}[C_{E_2}]$ and $f(F) = \{pt\}$. Thus $F$ cannot be isomorphic to $\mathbb{P}^2$ nor to a quadric cone, and we conclude that $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $F$ is a fiber of $f$. The normal bundle is given in [AW97], and $F$ must be a connected component of $E_1 \cap E_2$.

4.3. Contraction of a Fixed Prime Divisor of Type $(3, 1)^{sm}$ or $(3, 0)^Q$

Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 7$ and $D \subset X$ a fixed prime divisor of type $(3, 1)^{sm}$ or $(3, 0)^Q$. Let us consider the diagram:

\[ X \xrightarrow{\varphi} \tilde{X} \xrightarrow{\sigma} Y \]

given by Th.-Def. 4.5(a), where $\varphi$ is a SQM, $\sigma$ is an elementary divisorial contraction with exceptional divisor the transform $\tilde{D} \subset \tilde{X}$ of $D$, and $Y$ is Fano. We will refer to this diagram as the contraction associated to $D$. Let us sum up here its main properties and fix the related notation; see [Cas17, §5.1] for more details.

Recall from Lemma 2.4 that the indeterminacy locus of $\varphi$ is a disjoint union of exceptional planes.

For every exceptional plane $L \subset X$ in the indeterminacy locus of $\varphi$ we have $L \subset D$, $D \cdot C_L < 0$, and if $\ell \subset \tilde{X}$ is the corresponding exceptional curve, then $D \cdot \ell = -D \cdot C_L > 0$. Conversely, every exceptional plane contained in $D$ is in the indeterminacy locus of $\varphi$, so that the exceptional planes in $D$ are pairwise disjoint. No exceptional line of $\tilde{X}$ is contained in $\tilde{D}$ [Cas17, Rem. 5.6]. We have $-K_X \cdot C_D = -K_{\tilde{X}} \cdot C_{\tilde{D}} = 2$.

If $D$ is of type $(3, 1)^{sm}$, then $Y$ is smooth, $\sigma$ is the blow-up of a smooth irreducible curve $C \subset Y$, $\tilde{D}$ is a $\mathbb{P}^2$-bundle over $C$, and $C_{\tilde{D}} \subset \tilde{D}$ is a line in a fiber of $\sigma$. Every
fiber of the $\mathbb{P}^2$-bundle $\sigma_1D$ meets the union of the exceptional lines of $\tilde{X}$ in at most one point [Cas17, Rem. 5.5].

If $D$ is of type $(3,0)^Q$, then $Y$ has an isolated terminal and locally factorial singularity at $p := \sigma(D)$, and $\tilde{D}$ is either a smooth quadric, or the cone over a smooth 2-dimensional quadric [Cas17, Lemma 2.19]; moreover $C_{\tilde{D}} \subset \tilde{D}$ is a line.

**Remark 4.16** ([Cas17, Lemmas 5.10 and 5.20]). Let $\Gamma \subset Y$ be an irreducible curve such that $\Gamma \cap \sigma(\tilde{D}) \neq \emptyset$ and $\Gamma \neq \sigma(\tilde{D})$. Then $-K_Y \cdot \Gamma \neq 2$, and if $-K_Y \cdot \Gamma = 1$ we have $\Gamma = \sigma(\ell)$, where $\ell \subset \tilde{X}$ is an exceptional curve such that $\tilde{D} \cdot \ell = 1$. If $D$ is $(3,1)^m$, then $C$ cannot meet any exceptional plane.

**Remark 4.17.** In the above setting set

$$m := \dim \mathcal{N}_1(D, X) - \dim \mathcal{N}_1(\tilde{D}, \tilde{X}) = \begin{cases} \dim \mathcal{N}_1(D, X) - 1 & \text{if } D \text{ is of type } (3,0)^Q \\ \dim \mathcal{N}_1(D, X) - 2 & \text{if } D \text{ is of type } (3,1)^m. \end{cases}$$

It follows from [Cas13a, Rem. 3.13 and Cor. 3.14] that $D$ contains at least $m$ exceptional planes $L_1, \ldots, L_m$ such that the classes $[C_{L_1}], \ldots, [C_{L_m}]$ are linearly independent in $\mathcal{N}_1(X)$. Let us note that $X$ cannot be a product of surfaces, thus $\delta_X \leq 2$ by Th. 2.6.

**4.4. Additional properties for the case $(3,1)^m$**

**4.18.** In this subsection $X$ is a smooth Fano 4-fold with $\rho_X \geq 7$ and $D \subset X$ is a fixed prime divisor of type $(3,1)^m$. We keep the same notation as in 4.3.

**Lemma 4.19.** In Setting 4.18 let $\ell \subset \tilde{X}$ an exceptional curve and $\Gamma := \sigma(\ell) \subset Y$. Then $\Gamma$ cannot meet any curve of anticanonical degree one, except possibly $C$ (and $\Gamma$ itself).

**Proof.** Let $C_0 \subset Y$ be an irreducible curve with $-K_Y \cdot C_0 = 1$ and $C_0 \neq C, C_0 \neq \Gamma$, and let $\tilde{C}_0 \subset \tilde{X}$ be its transform. Note that $\tilde{C}_0 \neq \ell$, as $C_0 \neq \Gamma$.

If $C \cap C_0 \neq \emptyset$, then $\tilde{C}_0$ is an exceptional curve by Rem. 4.16 and $\tilde{C}_0 \neq \ell$, so that $\tilde{C}_0$ and $\ell$ are disjoint and meet $\tilde{D}$ in different fibers of $\sigma$. Therefore $\Gamma \cap C_0 = \emptyset$.

If instead $C \cap C_0 = \emptyset$, then $-K_{\tilde{X}} \cdot C_0 = 1$ and $\tilde{D} \cap \tilde{C}_0 = \emptyset$, therefore $\ell \cap \tilde{C}_0 = \emptyset$ by Lemma 2.3(b), and their images stay disjoint in $Y$. $\blacksquare$

**Lemma 4.20.** In Setting 4.18 let $\Gamma \subset Y$ be an integral curve such that $-K_Y \cdot \Gamma = 1$, $\Gamma \cap C \neq \emptyset$, and $\Gamma \neq C$. Then $\Gamma \cong \mathbb{P}^1$ and $\mathcal{N}_{\Gamma/Y} \cong \mathcal{O}(1) \oplus \mathcal{O}(\mathbb{Z})$.

**Proof.** By Rem. 4.16 we have $\Gamma = \sigma(\ell)$ where $\ell \subset \tilde{X}$ is an exceptional curve with $\tilde{D} \cdot \ell = 1$; therefore $\sigma|_{\ell}: \ell \to \Gamma$ is an isomorphism, and $\Gamma \cong \mathbb{P}^1$.

We have $\mathcal{N}_{\ell/X} \cong \mathcal{O}(1)^{\oplus 3}$ and $\mathcal{N}_{\Gamma/Y} \cong \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(-a-b)$ with $a,b \in \mathbb{Z}$. The differential of $\sigma$ induces a morphism of sheaves $\zeta: \mathcal{N}_{\ell/X} \to \mathcal{N}_{\Gamma/Y}$, which is generically an isomorphism. Then $\zeta$ is given by a $3 \times 3$ matrix $(s_{ij})_{i,j=1,2,3}$, where for $j = 1, 2, 3$

$$s_{1j} \in \text{Hom}(\mathcal{O}(0), \mathcal{O}(a)) = H^0(\mathbb{P}^1, \mathcal{O}(a+1)),
$$

$$s_{2j} \in H^0(\mathbb{P}^1, \mathcal{O}(b+1)),
$$

$$s_{3j} \in H^0(\mathbb{P}^1, \mathcal{O}(-a-b)),
$$

and the matrix is generically invertible. Thus for every $i = 1, 2, 3$ there exists $j \in \{1, 2, 3\}$ such that $s_{ij} \neq 0$, and this implies that $a \geq -1$, $b \geq -1$, and $a+b \leq 0$, giving two possibilities for the normal bundle of $\Gamma$: either $\mathcal{O}(1) \oplus \mathcal{O}(\mathbb{Z})$ or $\mathcal{O}(0) \oplus \mathcal{O}(\mathbb{Z})$.
Suppose that $\mathcal{N}_{\Gamma/Y} \cong \mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}(1)$. Up to reordering we have $a = b = -1$ and hence $s_{1j}$ and $s_{2j}$ are constant for every $j$, namely the first and second column of the matrix are constant. Since the matrix is generically invertible, these columns must be linearly independent, so that the rank of $\zeta$ is at least 2 at every point of $\ell$. To rule out this case, we show that for $x \in \bar{D} \cap \ell$, the rank of $\zeta$ at $x$ is 1.

Set $y := \sigma(x) \in C \cap \Gamma$. With a local computation one checks that $d_\Sigma \sigma : T_x \bar{X} \to T_y Y$ has rank 2. On the other hand $x \in \ell$ and $d_\Sigma \sigma(\ell) = T_y \Gamma$, thus $\text{Im } d_\Sigma \sigma$ contains $T_y \Gamma$. Then the image of $\text{Im } d_\Sigma \sigma$ in $(\mathcal{N}_{\Gamma/Y})_y = T_y Y / T_y \Gamma$ is one dimensional, and this is the image of $\zeta$.

Lemma 4.21. In Setting 4.18:

(a) if $Y$ contains a nef prime divisor $H$ covered by a family $V$ of lines, such that $H \cap C \neq \emptyset$, then $C$ is a curve of the family $V$;

(b) if $Y$ is covered by a family $W$ of rational curves with $-K_Y \cdot [W] = 2$, then $C$ is a component of a curve of the family $W$.

Proof. (a) Let $q \in C \cap H$, and let $\Gamma$ be a curve of the family $V$ containing $q$. If $\Gamma = C$, we are done. Otherwise, since $-K_Y \cdot \Gamma = 1$ and $\Gamma \cap C \neq \emptyset$, we have that $\Gamma \cong \mathbb{P}^1$, $\mathcal{N}_{\Gamma/Y} \cong \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 2}$, and $\Gamma$ is the image of an exceptional curve, by Lemma 4.20 and Rem. 4.16. Since $H$ is nef, by Lemma 3.13 $\Gamma$ must intersect some other curve of the family $V$. On the other hand by Lemma 4.19 $\Gamma$ cannot intersect any curve of anticanonical degree one, except $C$; this implies that $C$ belongs to the family $V$.

(b) Let $\Gamma_2$ be a one-cycle of the family $W$ intersecting $C$. If $\Gamma_2$ is irreducible and reduced, then $\Gamma_2 = C$ by Rem. 4.16.

If $\Gamma_2$ is reducible, let $\Gamma_2'$, $\Gamma_2''$ be its irreducible components, both of anticanonical degree one. We can assume that $\Gamma_2' \neq C$ and $\Gamma_2' \cap C \neq \emptyset$. Then $\Gamma_2'' = C$ by Lemma 4.19.

Finally if $\Gamma_2$ is non-reduced, either it is supported on $C$, or $\Gamma_2 = 2\Gamma_2'$ where $-K_Y \cdot \Gamma_2' = 1$, $\Gamma_2' \neq C$, and $\Gamma_2' \cap C \neq \emptyset$. Then again by Rem. 4.16 $\Gamma_2'$ is the image of an exceptional curve of $\bar{X}$. In this last case, $\Gamma_2$ meets $C$ in the finite set $T$ given by the intersection with the images of the finitely many exceptional curves of $\bar{X}$. Thus if $\Gamma_3$ is a one-cycle of the family $W$ meeting $C$ outside $T$, it must have $C$ as a component.

Lemma 4.22. In Setting 4.18, suppose moreover that $\rho_X \geq 8$. Let $E_Y \subset Y$ be a fixed prime divisor, and $E \subset X$ its transform, so that $E$ is a fixed prime divisor adjacent to $D$ (see Lemma 4.14(b)). Then $E$ and $E_Y$ have the same type.

Proof. Suppose first that $E$ is not of type (3,2); then, by Lemma 4.13(b), $D \cap E$ is either empty or a disjoint union of exceptional planes, and for every such $L$, we have $D \cdot C_L < 0$ and $E \cdot C_L < 0$ (see 4.13).

There is a SQM $\psi : X \dasharrow \bar{X}$ whose indeterminacy locus is the union of all exceptional planes contained in $D \cup E$ (this can be obtained by flipping consecutively all small $K$-negative extremal rays of $\text{NE}(X)$ having negative intersection with $D + E$). The transforms $\hat{D}, \hat{E} \subset \bar{X}$ are disjoint, and the SQM $\psi \circ \varphi^{-1} : \tilde{X} \dasharrow \hat{X}$ induces an isomorphism between $\hat{D}$ and $\hat{D}$. Similarly, if $X \xrightarrow{\varphi_E} \bar{X}_E \to Y_E$ is the contraction associated to $E$, the SQM $\psi \circ \varphi^{-1}_E : \bar{X}_E \dasharrow \hat{X}$ induces an isomorphism between $\hat{E} \subset \bar{X}_E$ and $\hat{E}$.
It is not difficult to see that $\hat{D}, \hat{E}$ are the loci of two divisorial extremal rays $R_D, R_E$ of $\text{NE}(\tilde{X})$, such that $R_D + R_E$ is a face of $\text{NE}(\tilde{X})$. In fact, if $m := -K_X \cdot C_E$ ($m \in \{2, 3\}$ depending on the type of $E$), the divisor $-K_\tilde{X} + 2\tilde{D} + m\tilde{E}$ is nef and $(-K_\tilde{X} + 2\tilde{D} + m\tilde{E})^+ \cap \text{NE}(\tilde{X}) = R_D + R_E$. Thus we have a diagram:

$$
\begin{array}{c}
\tilde{X} \\
\sigma_D
\end{array} \begin{array}{c}
\rightarrow \\
\downarrow \sigma_E
\end{array} \begin{array}{c}
\tilde{Y} \\
\downarrow k
\end{array} \\
\begin{array}{c}
Z \\
\rightarrow
\end{array} \begin{array}{c}
W
\end{array}
$$

where $\sigma_D$ and $\sigma_E$ are elementary divisorial contractions with exceptional divisors $\hat{D}$ and $\hat{E}$ respectively, $\tilde{Y}$ is a SQM of $Y$, and $k$ is an elementary divisorial contraction with exceptional divisor the transform of $E_Y$. Since $\tilde{D} \cap \hat{E} = \emptyset$, $\sigma_E$ and $k$ coincide in a neighborhood of their exceptional divisors. Moreover, by Th.-Def. 4.5(e), $\sigma_E$ and $E$ have the same type, and similarly for $k$ and $E_Y$. Therefore $E$ and $E_Y$ have the same type.

Suppose now that $E$ is of type $(3, 2)$; then by Lemma 4.13(a) we have $D \cdot C_E = 0$, so that the general curve $C_E$ is disjoint from $D$, hence it is contained in the open subset where the birational map $\Gamma : Y \dasharrow X$ is an isomorphism. Let $\Gamma \subset E_Y \subset Y$ be the image of the general $C_E$. Then $-K_Y \cdot \Gamma = 1$ and $E_Y \cdot \Gamma = -1$, so that $E_Y$ is covered by a family of lines and it is not nef; then $E_Y$ is of type $(3, 2)$ by Lemma 5.12.

**Lemma 4.23.** Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 7$, and $D$ and $E$ two adjacent fixed prime divisors, of type $(3, 1)^\text{em}$ and $(3, 2)$ respectively, such that $E \cdot C_D > 0$. Then $E \cdot C_D = 1$ and $E \cap L = \emptyset$ for every exceptional plane $L \subset D$.

**Proof.** Let $X \dasharrow \tilde{X} \rightarrow Y$ be the contraction of $D$ as in §4.3 and $E_\tilde{X} \subset \tilde{X}, E_Y \subset Y$ the transforms of $E$, so that $E_Y$ is a fixed prime divisor by Lemma 4.13(c). Since $E_\tilde{X} \cdot C_D = E \cdot C_D > 0$, $E_\tilde{X}$ intersects every non-trivial fiber of $\sigma$, thus $E_Y$ contains $C$. By [Cas17, Lemma 5.11] we deduce that $[C]$ generates an extremal ray of type $(3, 2)$ of $\text{NE}(Y)$, $C \cong \mathbb{P}^1$ is a fiber of the associate contraction, $E_Y$ is a smooth $\mathbb{P}^1$-bundle around $C$, $N_{C/Y} \cong \mathcal{O} \oplus \mathcal{O}$, and $\tilde{D} \cong \mathbb{P}^1(\mathcal{O} \oplus \mathcal{O}(1))$ is isomorphic to the blow-up of $\mathbb{P}^3$ along a line. Moreover $E_\tilde{X}$ and $\tilde{D}$ intersect transversally along a smooth irreducible surface and $\sigma^* E_Y = E_\tilde{X} + \tilde{D}$, so that $E \cdot C_D = E_\tilde{X} \cdot C_D = (\sigma^* E_Y - \tilde{D}) \cdot C_D = 1$.

If $\Gamma \subset \tilde{D}$ is a non-trivial fiber of the blow-up $\tilde{D} \rightarrow \mathbb{P}^3$, it is not difficult to see that $\tilde{D} \cdot \Gamma = 0$, using that $\mathcal{O}_{\tilde{X}}(-\tilde{D})|_{\tilde{D}}$ is the tautological line bundle. Then $-K_\tilde{X} \cdot \Gamma = -K_D \cdot \Gamma = 1$, and $\tilde{E}_\Gamma \cdot \Gamma = \sigma^* E_Y \cdot \Gamma = E_Y \cdot C = -1$, so that $\Gamma \subset E_\tilde{X}$. This implies that $\tilde{D} \cap E_\tilde{X} \cong \mathbb{P}^1 \times \mathbb{P}^1$ is the exceptional divisor of the blow-up $\tilde{D} \rightarrow \mathbb{P}^3$, and it is covered by lines. On the other hand also $E_\tilde{X} \mid \tilde{D} \cong E_Y \cap C$ is covered by lines. Thus $E_\tilde{X}$ cannot meet any exceptional curve of $\tilde{X}$ by Lemma 2.41(b), and $E$ cannot meet any exceptional plane contained in $D$.

**4.5. Additional properties for the case $(3, 0)^Q$**

**4.24.** In this subsection $X$ is a smooth Fano 4-fold with $\rho_X \geq 7$ and $D \subset X$ is a fixed prime divisor of type $(3, 0)^Q$. We keep the same notation as in §4.3.
Remark 4.25. In Setting 4.24, suppose that $\tilde{D} \subset \tilde{X}$ is a quadric cone. Then no exceptional curve can intersect $\tilde{D}$ at the vertex of the cone.

Indeed, suppose otherwise: then for every line $\Gamma \subset \tilde{D}$ through the vertex of the cone, we have $-K_{\tilde{X}} \cdot \Gamma = 2$, so if $\Gamma_X \subset X$ is the transform of $\Gamma$, we have $-K_X \cdot \Gamma_X = 1$ by Lemma 2.4(b). This yields that $D$ is covered by a family of lines, it is not nef and of type $(3,0)^Q$, contradicting Lemma 3.12.

Remark 4.26. Let $B \subset X$ be a prime divisor, different from $D$, and such that $B \cap D \neq \emptyset$. Then either $B \cap D$ is a disjoint union of exceptional planes, or $B \cdot C_D > 0$.

Indeed let $\tilde{B} \subset \tilde{X}$ be the transform of $B$. If $\tilde{B} \cap \tilde{D} = \emptyset$, then $B \cap D$ is contained in the indeterminacy locus of $\varphi : X \dashrightarrow \tilde{X}$, thus it is a disjoint union of exceptional planes. If $B \cap D \neq \emptyset$, then $B \cdot C_D = \tilde{B} \cdot C_D > 0$.

The proof of the next lemma is analogous to that of Lemma 4.19 thus we omit it.

Lemma 4.27. In Setting 4.24, let $\ell \subset \tilde{X}$ be an exceptional curve and $\Gamma := \sigma(\ell) \subset Y$. Then $\Gamma$ cannot meet any curve of anticanonical degree one outside $p$ (except possibly $\Gamma$ itself).

Lemma 4.28. In Setting 4.24, let $E \subset X$ be a fixed prime divisor adjacent to $D$, and $E_Y \subset Y$ its transform. If $N_1(E_Y, Y) \subseteq N_1(Y)$, then one of the following holds:

(i) $X$ has a rational contraction onto a 3-fold;
(ii) $X$ has a fixed prime divisor $G$ of type $(3,2)$, adjacent to $D$, and such that $E \cdot C_G > 0$.

Proof. We recall that $Y$ is Fano and has one isolated terminal and locally factorial singularity. Suppose that $N_1(E_Y, Y) \subseteq N_1(Y)$; we apply [Del14] Th. 3.1 and Lemma 3.3] to $Y$ and $E_Y$. This yields a diagram:

$$Y = Y_0 \xrightarrow{\sigma_0} Y_1 \xrightarrow{\cdots} Y_{k-1} \xrightarrow{\sigma_{k-1}} Y_k \xrightarrow{\psi} Z$$

where each $\sigma_i$ is either an elementary divisorial contraction or a flip, and $\psi$ is an elementary contraction of fiber type [Del14] Th. 3.1(2)]. The divisor $E_Y$ is not exceptional for any $\sigma_i$; let $E_Y^i \subset Y_i$ be its transform and set $c_i := \text{codim} N_1(E_Y^i, Y_i)$. We have $c_{i+1} \in \{c_i, c_i - 1\}$ for $i = 0, \ldots, k - 1$ and $c_k \in \{0, 1\}$ [Del14] Th. 3.1(3)].

If $c_k = 1$, then $\text{NE}(\psi) \not\subseteq N_1(E_Y^k, Y_k)$ and $E_Y^k \cdot \text{NE}(\psi) > 0$ [Del14] Th. 3.1(2) and (3)]. This implies that if $F \subset Y_k$ is a fiber of $\psi$, then $\text{dim}(F \cap E_Y^k) \leq 0$ and $F \cap E_Y^k \neq \emptyset$, thus $\text{dim} F = 1$ and $\text{dim} Z = 3$, and we get (i).

If $c_k = 0$, since $c_0 = \text{codim} N_1(E_Y, Y) > 0$, there exists some $i \in \{0, \ldots, k - 1\}$ such that $c_{i+1} = c_i - 1$. Then $\sigma_i$ is a $K$-negative elementary divisorial contraction of type (3,2) [Del14] Lemma 3.3(2)], and $\text{Exc}(\sigma_i)$ is contained in the open subset where the birational map $Y \dashrightarrow Y_i$ is an isomorphism [Del14] Lemma 3.3(3)]; let $G_Y \subset Y$ and $G \subset X$ be the transforms of $\text{Exc}(\sigma_i) \subset Y_i$.

Then $G_Y$ is covered by irreducible curves $\Gamma$ such that $E_Y \cdot \Gamma > 0$, $-K_Y \cdot \Gamma = 1$, and $G_Y \cdot (\Gamma - \Gamma_Y) = -1$ [Del14] Lemma 3.3(5)]; the general $\Gamma$ does not contain $p = \sigma(\tilde{D})$, so by Lemma 4.26 it is contained in the open subset where the birational map $X \dashrightarrow Y$ is an isomorphism. Therefore $G \subset X$ is covered by a family of lines $V$ such that $G \cdot [V] = -1$ and $E \cdot [V] > 0$; $G$ is a fixed prime divisor of type $(3,2)$ with $[V] = [C_G]$ by Lemma 3.12 and it is adjacent to $D$ by Lemma 4.14(c), so we have (ii).
5. Rational contractions of fiber type in Mori dream spaces

5.1. Rational contractions of fiber type

Let $X$ be a projective, normal, and $\mathbb{Q}$-factorial Mori dream space. We need to introduce two special notions of rational contractions of fiber type, namely “quasi-elementary” and “special” contractions; we refer the reader to [Cas13a §2.2] and [Cas20 §2] respectively for more details.

**Definition 5.1.** Let $f: X \to Z$ be a contraction of fiber type. We say that $f$ is **quasi-elementary** if $Z$ is $\mathbb{Q}$-factorial and for every prime divisor $B \subset Z$ the pull-back $f^* B$ is irreducible (but possibly non-reduced).

We say that $f$ is **special** if $Z$ is $\mathbb{Q}$-factorial and $\text{codim} f(D) \leq 1$ for every prime divisor $D \subset X$. A quasi-elementary contraction is always special.

We will be interested mainly in the cases where $\text{dim} Z \geq 2$: if $Z$ is a curve, then every contraction of fiber type is special. If $Z$ is a surface, then $f$ is special if and only if it is **equidimensional**: indeed if $f$ is special and $\text{dim} Z = 2$, then $f$ must be equidimensional; the converse follows from [Cas20 Lemma 2.7].

Consider now a rational contraction of fiber type $f: X \longrightarrow Z$. We say that $f$ is quasi-elementary, respectively special, if given a factorization of $f$ as a SQM $X \longrightarrow X'$ followed by a regular contraction $f': X' \to Z$, then $f'$ is quasi-elementary, respectively special; this does not depend on the choice of the factorization.

**Lemma 5.2.** Let $X$ be a projective, normal, and $\mathbb{Q}$-factorial Mori dream space, and $f: X \longrightarrow S$ a rational contraction onto a surface. Then $S$ is $\mathbb{Q}$-factorial.

**Proof.** Suppose first that $X$ is a surface, so that $f$ is birational and regular. By dimensional reasons, $\text{Exc}(f)$ is a divisor. If $f$ is elementary, the statement is well known.

In general, we can factor $f$ as $X \xrightarrow{g} X' \xrightarrow{f'} S$, where $\sigma$ is elementary; then $X'$ is a $\mathbb{Q}$-factorial Mori dream space, and we conclude by induction on $\rho_X - \rho_S$.

If $\text{dim} X \geq 3$, then $f$ is of fiber type and by [Cas20 Prop. 2.13] it can be factored as $X \xrightarrow{g} T \xrightarrow{h} S$,

where $g$ is a special rational contraction of fiber type and $h$ is birational. Then $T$ is $\mathbb{Q}$-factorial, so that $S$ is $\mathbb{Q}$-factorial by the first part of the proof. ■

**Lemma 5.3.** Let $X$ be a projective, normal, and $\mathbb{Q}$-factorial Mori dream space. Let $f: X \longrightarrow Z$ be a rational contraction of fiber type, and $\tau$ the smallest face of $\text{Eff}(X)$ containing $f^* \text{Eff}(Z)$. The following are equivalent:

(i) $f$ is special;
(ii) $\tau \cap \text{Mov}(X) = f^* \text{Mov}(Z)$;
(iii) $\dim(\tau \cap \text{Mov}(X)) = \rho_Z$.

**Proof.** We show $(i) \Rightarrow (ii)$. Up to composing with a SQM of $X$, we can assume that $f$ is regular. Let $F \subset X$ be a general fiber of $f$; by [Cas13a Lemma 2.21] we have $\tau = \text{Eff}(X) \cap N_1(F, X)^\perp$, thus $\tau \cap \text{Mov}(X) = \text{Mov}(X) \cap N_1(F, X)^\perp$.

If $B$ is a movable divisor on $Z$, then $f^*(B) \cap F = \emptyset$, thus $f^* B \cdot C = 0$ for every curve $C \subset F$, and $[f^* B] \in N_1(F, X)^\perp$. Moreover if $B_0 \subset Z$ is the stable base locus of the linear system $|B|$, then $\text{codim} B_0 \geq 2$, and since $f$ is special we have $\text{codim} f^{-1}(B_0) \geq 2$.
Since the stable base locus of $|f^*B|$ is contained in $f^{-1}(B_0)$, we have $|f^*B| \subseteq \text{Mov}(X) \cap N_1(F,X)^\perp$. Hence $f^*\text{Mov}(Z) \subseteq \text{Mov}(X) \cap N_1(F,X)^\perp$.

Conversely, let $D$ be a divisor on $X$ with $[D] \in \text{Mov}(X) \cap N_1(F,X)^\perp$. By Rem. 2.2 up to replacing $D$ with a divisor with class in $\mathbb{R}_{\geq 0}[D]$, we can assume that $D$ is a prime divisor. Since $D \cdot C = 0$ for every curve $C \subset F$, we must have $D \cap F = \emptyset$ and $f(D) \subseteq Z$; since $f$ is special, $f(D)$ is a prime divisor, and it is movable. If $f^{-1}(f(D))$ is reducible, then by [Cas20, Cor. 2.18] every irreducible component of $f^{-1}(f(D))$ is a fixed prime divisor; since $D$ is movable and is a component of $f^{-1}(f(D))$, we conclude that $f^{-1}(f(D)) = D$ and $D = \lambda f^*(f(D))$ for some $\lambda \in \mathbb{Q}_{>0}$, so that $[D] \in f^*\text{Mov}(Z)$, and we get (ii).

The implication (ii) $\Rightarrow$ (iii) is clear. We show (iii) $\Rightarrow$ (i). By [Cas20] Prop. 2.13 $f$ can be factored as $X \xrightarrow{\varphi} T \xrightarrow{h} Z$, where $\varphi$ is a special rational contraction of fiber type and $h$ is birational. Up to composing with a SQM of $X$, we can assume that $g$ is regular; then $g$ and $f$ have the same general fiber $F \subset X$, and again $\tau = \text{Eff}(X) \cap N_1(F,X)^\perp$.

By the first part of the proof we have $\text{Mov}(X) \cap N_1(F,X)^\perp = g^*\text{Mov}(T)$, thus

$$\rho_Z = \dim(\tau \cap \text{Mov}(X)) = \dim(\text{Mov}(X) \cap N_1(F,X)^\perp) = \rho_T,$$

therefore $h$ is an isomorphism and $f$ is special. \hfill $\blacksquare$

5.2. MOVABLE FACES OF THE EFFECTIVE CONE

Let $X$ be a projective, normal, and $\mathbb{Q}$-factorial Mori dream space.

Definition 5.4. Let $\tau$ be a proper face of $\text{Eff}(X)$. We say that $\tau$ is a movable face if the relative interior of $\tau$ intersects $\text{Mov}(X)$.

There exists a movable face of $\text{Eff}(X)$ if and only if $\partial \text{Eff}(X) \cap \text{Mov}(X) \neq \{0\}$, if and only if $X$ has some non-zero, non-big movable divisor.

If a (non-zero) face of $\text{Eff}(X)$ is not fixed, it always contains a movable face.

We say that a movable face $\tau$ is minimal if every proper face of $\tau$ is not movable, equivalently if every proper face of $\tau$ is fixed.

We recall that the Mori chamber decomposition of $X$ is a fan in $N_1(X)$, supported on the cone $\text{Mov}(X)$, given by the cones $f^*\text{Nef}(Z)$ for all rational contractions $f : X \twoheadrightarrow Z$; the cones of the Mori chamber decomposition are in bijection with the rational contractions of $X$.

5.5. Let $\tau$ be a movable face of $\text{Eff}(X)$. We associate to $\tau$ a special rational contraction of fiber type of $X$, as follows.

The cone $\tau \cap \text{Mov}(X)$ is a non-zero face of the movable cone, so it is a union of cones of the Mori chamber decomposition of $X$. Let us choose a cone $\eta$ of the Mori chamber decomposition such that $\eta \subseteq \tau \cap \text{Mov}(X)$ and $\dim \eta = \dim(\tau \cap \text{Mov}(X))$; note that $\eta$ must intersect the relative interior of $\tau$, because $\tau$ is movable.

Then $\eta$ yields a rational contraction $f : X \twoheadrightarrow Z$, of fiber type because $\eta$ is contained in the boundary of the effective cone. We have:

$$\eta = f^*\text{Nef}(Z), \quad \rho_Z = \dim \eta = \dim(\tau \cap \text{Mov}(X)), \quad \text{and} \quad \tau \cap \text{Mov}(X) \subseteq f^*N_1(Z).$$

We show that $f^*\text{Eff}(Z) \subseteq \tau$. Indeed since $\tau$ is a face of $\text{Eff}(X)$, there exists $\gamma \in \text{Eff}(X)^\perp$ such that $\tau = \text{Eff}(X) \cap \gamma^\perp$. Then $f^*\text{Nef}(Z) = \eta \subseteq \tau \subseteq \gamma^\perp$, thus $f^*N_1(Z) \subseteq \gamma^\perp$ and $f^*\text{Eff}(Z) \subseteq \gamma^\perp \cap \text{Eff}(X) = \tau$. 

Thus given a prime divisor $D$ does not dominate $Z$ and $F$.

By Remark 5.7.

§ (see [IP99, (5.6)])

Let $D$ be a smooth Fano 4-fold with a rational contraction $g: X \to Y$. Then $\text{NE}(g) \not\subset \text{mov}(X) = \text{Eff}(X)^\vee$, hence there exists a one-dimensional face $\tau$ of $\text{Eff}(X)$ such that $\tau \cdot \text{NE}(g) < 0$.

If $\tau$ is movable, we apply the construction above; by Lemma 2.5 we can take $\tilde{X}$ smooth and $\tilde{f}$ $K$-negative. Then $F$ is smooth, Fano, with $\dim F \leq 3$, therefore $\rho_F \leq 10$ (see [IP99, §12.6]) and $\rho_X \leq 11$ by (5.6).

If instead $\tau$ is fixed, then there exists a fixed prime divisor $D \subset X$ such that $D \cdot \text{NE}(g) < 0$.

5.3. Rational contractions of fiber type of Fano 4-folds

Fano 4-folds with a rational contraction of fiber type have been studied in [Cas20]; in particular we will need the following.

Theorem 5.8 ([Cas20], Th. 6.1). Let $X$ be a smooth Fano 4-fold which is not a product of surfaces. If $X$ has a rational contraction onto a 3-fold, then $\rho_X \leq 12$.

In the rest of this subsection we focus on Fano 4-folds having a rational contraction onto a surface or $\mathbb{P}^1$; in this case we do not have an analog of Th. 5.8 and we will prove several results which we will use in the rest of the article to tackle this situation.

Lemma 5.9. Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 7$ and $f: X \dasharrow S$ a quasi-elementary rational contraction onto a surface. Then $S$ is a smooth del Pezzo surface.

Proof. By [Cas13a, Prop. 4.1 and its proof] we know that $S$ is smooth, rational, and a Mori dream space. We show that $-K_S \cdot \text{NE}(g) > 0$ for every elementary contraction $g: S \to S_1$; this implies that $-K_S$ is ample. If $g$ is of fiber type, then either $S \cong \mathbb{P}^2$, or $S$ is isomorphic to a Hirzebruch surface and $g$ is a $\mathbb{P}^1$-bundle, hence $-K_S \cdot \text{NE}(g) > 0$.

Let assume that $g$ is birational, so that $C := \text{Exc}(g)$ is an irreducible curve, and $D := f^*C$ is a fixed prime divisor, see [Cas13a, Rem. 3.10]. Moreover $S_1$ has (at most) a rational singularity at the point $g(C)$ (see Lemma 2.5). Consider a factorization of $f$ as in Lemma 2.3

$X \xrightarrow{\psi} X' \xrightarrow{f'} S$

where $\psi$ is a SQM. We know that $X \setminus \text{dom} \psi$ is a finite union of exceptional planes (see Lemma 2.4). Then in $X$ there exists a curve $\Gamma \equiv C_D$, $\Gamma \subset D \cap \text{dom} \psi$; this follows from Rem. 2.9 if $D$ is of type $(3, 2)$, and from the description in Th.-Def. 4.5 in the other cases.

Let $\Gamma' \subset D' \subset X'$ be the transforms of $\Gamma \subset D \subset X$. Then

$$-1 = D \cdot \Gamma = D' \cdot \Gamma' = (f')^*C \cdot \Gamma' = C \cdot (f')_*\Gamma'.$$
This implies that \( f'(\Gamma') \) is not a point, thus \( f'(\Gamma') = C \) and \( (f')*, \Gamma' = mC \) with \( m \in \mathbb{Z}_{\geq 1} \). We get \( mC^2 = -1 \), hence \( m = 1 \) and \( C^2 = -1 \). We conclude that \( g \) is the blow-up of a smooth point and \( -K_S \cdot NE(g) > 0 \).

**Lemma 5.10** ([Cas13a], proof of Cor. 3.9). Let \( X \) be a smooth Fano 4-fold, \( \varphi : X \to \tilde{X} \) a SQM, and \( f : \tilde{X} \to Z \) a \( K \)-negative contraction of fiber type. If the general fiber \( F \) of \( f \) is either a del Pezzo surface with \( \rho_F = 9 \), or a product \( S \times \mathbb{P}^1 \) with \( S \) a del Pezzo surface with \( \rho_S = 9 \), then \( \varphi \) is an isomorphism.

**Lemma 5.11.** Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \), \( f : X \to Z \) a rational contraction of fiber type, and \( H \) an effective divisor in \( X \) such that \( |H| \in f^*N^1(Z) \) and \( \text{Supp} \ H \) has a component \( E \) which is a fixed prime divisor of type \((3,2)\). Then one of the following holds:

1. \( N_1(E, X) \subseteq N_1(X) \);  
2. \( \dim Z = 3 \);  
3. \( Z \cong \mathbb{P}^2 \) and \( f \) is equidimensional.

**Proof.** Let \( \tilde{X} \to X \) be a SQM such that the composition \( \tilde{f} : \tilde{X} \to Z \) is regular and \( K \)-negative (see Lemma 2.25), and \( \tilde{E} \subset \tilde{X} \) the transform of \( E \).

We have \( N_1(\tilde{f}(\tilde{E}), Z) = \tilde{f}_*(N_1(\tilde{E}, \tilde{X})) \), thus \( \text{codim} N_1(\tilde{E}, \tilde{X}) \geq \text{codim} N_1(\tilde{f}(\tilde{E}), Z) \). Since \( H \) is the pullback of an effective \( \mathbb{Q} \)-divisor of \( Z \), \( \tilde{f}(\tilde{E}) \subseteq \tilde{f}(\text{Supp} \ H) \subseteq Z \). If \( \dim Z = 1 \), or \( \dim Z = 2 \) and \( \rho_Z > 1 \), this easily implies that \( \text{codim} N_1(\tilde{f}(\tilde{E}), Z) > 0 \), therefore \( \text{codim} N_1(\tilde{E}, \tilde{X}) > 0 \) and we get (i) by Lemma 4.10.

If \( \dim Z = 2 \) and \( f \) is not equidimensional, by [Cas20] Prop. 2.13] we can factor \( f \) as \( X \to T \to Z \) where \( T \) is a surface with \( \rho_T > \rho_Z \), and again \( E \) cannot dominate \( T \), so we get (i) as before.

Finally if \( \dim Z = 2 \), \( \rho_Z = 1 \), and \( f \) is equidimensional, then \( Z \) is smooth by [Cas20] Lemma 4.3], and being a rational surface, we have (iii).

**Lemma 5.12.** Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \) and \( f : X \to S \) an equidimensional rational contraction onto a surface. If \( f \) is not quasi-elementary and \( \rho_S > 1 \), then \( X \) has a fixed prime divisor \( E \) of type \((3,2)\) such that \( N_1(E, X) \subseteq N_1(X) \).

**Proof.** Since \( f \) is not quasi-elementary, there exists an irreducible curve \( \Gamma \subset S \) such that \( f^* \Gamma \) is reducible. By [Cas20] Lemma 5.2 there is a fixed prime divisor \( E \subset X \) of type \((3,2)\) which is a component of \( f^* \Gamma \). Then the statement follows from Lemma 5.11.

**Lemma 5.13.** Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \) and \( f : X \to S \) a rational contraction onto a surface with \( \rho_S = 1 \). Suppose that there is a unique prime divisor \( D \subset X \) contracted to a point by \( f \), and that \( D \) is fixed not of type \((3,2)\). Then one of the following holds:

1. \( X \) has a fixed prime divisor \( E \) of type \((3,2)\) such that \( N_1(E, X) \subseteq N_1(X) \);  
2. \( \rho_X \leq 10 \).

**Proof.** By [Cas20] Prop. 2.13] we can factor \( f \) as \( X \to T \to S \) where \( g \) is equidimensional and \( h \) is birational. The surfaces \( T \) and \( S \) are \( \mathbb{Q} \)-factorial by Lemma 5.2, so that \( \rho_T - \rho_S \) is the number of irreducible components of \( \text{Exc}(h) \).
Let $X' \rightarrow X$ be a SQM such that the composition $g': X' \rightarrow T$ is regular and $K$-negative (see Lemma 2.5), $F \subset X'$ a general fiber of $g'$, and $D' \subset X'$ the transform of $D$. Since $D'$ is the unique prime divisor contracted to a point by $h \circ g': X' \rightarrow S$, we deduce that $D' = (g')^{-1}(\text{Exc}(h))$, $\text{Exc}(h)$ is irreducible, and $\rho_T = 2$.

Let $R$ be an extremal ray of $\text{NE}(X')$ such that $(g')_* R = \text{NE}(h)$ (see [Cas08 §2.5]). Then for every curve $C \subset X'$ with $[C] \in R$ we have $g'(C) = \text{Exc}(h)$, therefore $\text{Locus}(\text{most one prime divisor contracted to a point})$. Moreover if $F \subset X'$ is a non-trivial fiber of the contraction of $R$, $g'$ is finite on $F$, and $g'(F) = \text{Exc}(h)$, so that dim $F = 1$. If $-K_{X'} \cdot R > 0$, then $R$ is of type $(3, 2)$ by [Wis91, Th. 1.2], and $D' = \text{Locus}(R)$, a contradiction by Th.-Def. 4.5(e). Therefore $-K_{X'} \cdot R \leq 0$ and $X'$ is not Fano. This implies that $\rho_F \leq 8$ by Lemma 5.10.

Now if $g$ is quasi-elementary, by [Cas20 Cor. 2.16] we get $\rho_X \leq \rho_F + \rho_T \leq 10$, namely (ii). Instead if $g$ is not quasi-elementary, then we get (i) by Lemma 5.12.

**Lemma 5.14.** Let $X$ be a smooth Fano 4-fold with $\delta_X \leq 1$ and $f: X \rightarrow S$ a contraction onto a surface. Then either $\rho_S = 2$ and $f$ is equidimensional, or $\rho_S = 1$ and there is at most one prime divisor contracted to a point.

**Proof.** For any prime divisor $D \subset X$ such that $f(D) \subset S$ we have

$$1 \geq \text{codim} \mathcal{N}_1(D, X) \geq \text{codim} \mathcal{N}_1(f(D), S) \geq \rho_S - 1,$$

hence $\rho_S \leq 2$. If $\rho_S = 2$, then $f(D)$ must always be a curve, and $f$ is equidimensional.

Suppose that $\rho_S = 1$ and $f$ is not equidimensional, so there exists a prime divisor $D$ contracted to a point.

We note that in our setting $f(D) = \{pt\}$ is equivalent to the pushforward of $D$ (as a cycle) being zero, and this is invariant under linear equivalence. Since there cannot be a positive dimensional linear system of divisors contracted to points, $D$ must be a fixed divisor.

Let us take a general very ample curve $A \subset S$ and consider the prime divisor $f^{-1}(A)$. We have $f^{-1}(A) \cap D = \emptyset$, hence $\mathcal{N}_1(f^{-1}(A), X) \subseteq D^\perp$ (see Rem. 2.1) and codim $\mathcal{N}_1(f^{-1}(A), X) \leq \delta_X \leq 1$, so that $\mathcal{N}_1(f^{-1}(A), X) = D^\perp$.

We show that $D$ is unique. Indeed, if $B \subset X$ is another prime divisor contracted to a point, as above we get $\mathcal{N}_1(f^{-1}(A), X) = B^\perp$, hence $D^\perp = B^\perp$. This means that the classes $[D]$ and $[B]$ are multiples in $\mathcal{N}_1(X)$, and being $D$ a fixed divisor, it implies that $B = D$.

**Lemma 5.15.** Let $X$ be a smooth Fano 4-fold, $X \rightarrow \tilde{X}$ a SQM, and $f: \tilde{X} \rightarrow \mathbb{P}^1$ a $K$-negative contraction with general fiber $F$. Suppose that $F \cong \mathbb{P}^1 \times S$ and that dim $\mathcal{N}_1(F, \tilde{X}) = \rho_F$. Then $X$ has a rational contraction onto a 3-fold.

**Proof.** Set $C := \mathbb{P}^1 \times \{pt\} \subset F$. Then $\mathcal{N}_{C/F} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, $(\mathcal{N}_{F/\tilde{X}})|_C \cong \mathcal{O}_C$, and $\mathcal{N}_{C/\tilde{X}} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$.

This implies that $\text{Hilb}(\tilde{X})$ is smooth of dimension 3 at the point $[C]$; let $W$ be the irreducible component of $\text{Hilb}(\tilde{X})$ containing $[C]$.

Any point in $W$ yields an effective and connected one-cycle $\Gamma$ in $\tilde{X}$ such that $\Gamma \equiv C$. We show that if $\Gamma \cap C \neq \emptyset$, then $\Gamma = C$.

We have $f(\Gamma) = \{pt\}$, and $\Gamma$ intersects the fiber $F$, so that $\Gamma \subset F$. Let us consider the natural linear map $\mathcal{N}_1(F) \rightarrow \mathcal{N}_1(F, \tilde{X}) \subseteq \mathcal{N}_1(\tilde{X})$: since dim $\mathcal{N}_1(F, \tilde{X}) = \rho_F$, this map is injective. Therefore $\Gamma \equiv_F C$, thus $\Gamma$ is supported on a fiber of the projection $F \rightarrow S$. Since $\Gamma$ intersects $C$, we conclude that $\Gamma = C$. 
Let $\tilde{X}_0 \subset \tilde{X}$ be the open subset where the fibers of $f$ satisfy the same assumptions as $F$, and $W_0 \subset W$ the open subset parametrizing the curves $\mathbb{P}^1 \times \{pt\}$ in the fibers of $f|_{\tilde{X}_0}$. Let $C_0 \to W_0$ be the universal family and $e: C_0 \to \tilde{X}$ the natural map. Then $W_0$ and $C_0$ are smooth and $e: C_0 \to \tilde{X}_0$ is bijective, and since $\tilde{X}$ is smooth, we conclude that $e$ is an isomorphism and there is a projective morphism $g_0: \tilde{X}_0 \to W_0$. Then $X$ has a rational contraction onto a 3-fold, see [Cas20] proof of Th. 1.2.

Proposition 5.16. Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 7$ and $\delta_X \leq 1$, $\tau$ a movable face of $\text{Eff}(X)$, and $f: X \dasharrow Z$ an associated rational contraction as in (5.9). Then one of the following holds:

(i) $X$ has a rational contraction onto a 3-fold;

(ii) $X$ has a fixed prime divisor $D$ of type (3,2) such that $\mathcal{N}_1(D, X) \subsetneq \mathcal{N}_1(X)$;

(iii) $\rho_X \leq 8 + \text{dim} \tau$, $Z \cong \mathbb{P}^1$, and $\tau$ does not contain classes of fixed prime divisors of type (3,2);

(iv) $\rho_X = 10$ and $f: X \to \mathbb{P}^1$ is regular with general fiber $S \times \mathbb{P}^1$, where $S$ is a del Pezzo surface with $\rho_S = 9$;

(v) $\rho_X \leq 9 + \text{dim} \tau$, $Z \cong \mathbb{P}^2$, and every fixed prime divisor with class in $\tau$ is of type $(3,2)$ or $(3,1)^{sm}$;

(vi) $\tau$ is not minimal, $f$ is quasi-elementary, and $\text{dim} Z = 2$.

Moreover in cases (i) and (ii) we have $\rho_X \leq 12$, and if $\rho_X = 12$, then $X$ has a rational contraction onto a 3-fold.

Proof. We apply the set-up as in (5.5), and keep the same notation. We factor $f$ as $X \dasharrow \tilde{X} \xrightarrow{\tilde{f}} Z$ where $\tilde{f}$ is $K$-negative (see Lemma [2.5]), so that the general fiber $F \subset \tilde{X}$ is a smooth Fano variety. If $\text{dim} Z = 3$ we get (i).

Suppose that $Z = \mathbb{P}^1$. If there exists a fixed prime divisor $D$ of type (3,2) such that $[D] \in \tau$, then its transform $\tilde{D} \subset \tilde{X}$ is contained in a fiber of $\tilde{f}$. Thus $\mathcal{N}_1(\tilde{D}, \tilde{X}) \subseteq \ker \tilde{f}_* \subsetneq \mathcal{N}_1(\tilde{X})$, and we get (ii) by Lemma (4.10).

Assume that $\tau$ does not contain classes of fixed prime divisors of type (3,2). If $\text{dim} \mathcal{N}_1(F, \tilde{X}) \leq 8$, we get (iii) by (5.6). If $\text{dim} \mathcal{N}_1(F, \tilde{X}) \geq 9$ and $\text{dim} \mathcal{N}_1(F, \tilde{X}) = \rho_F$, then $F \cong \mathbb{P}^1 \times S$ (see [IP99, §12.6]), and we get (i) by Lemma 5.15.

The last possibility is $9 = \text{dim} \mathcal{N}_1(F, \tilde{X}) < \rho_F = 10$, and $F \cong \mathbb{P}^1 \times S$ where $\rho_S = 9$. In this case $X \cong \tilde{X}$ by Lemma 5.10 and $f: X \to \mathbb{P}^1$ is regular. Thus $0 < \text{codim} \mathcal{N}_1(F, X) \leq \delta_X = 1$, and we conclude that $\text{codim} \mathcal{N}_1(F, X) = 1$ and $\rho_X = 10$, so we get (iv).

Finally suppose that $\text{dim} Z = 2$, so that $f$ is an equidimensional rational contraction, and $Z$ is a smooth rational surface by [Cas20] Lemma 4.3].

Suppose first that $\rho_Z > 1$. If $f$ is not quasi-elementary, we get (ii) by Lemma 5.12.

If $f$ is quasi-elementary, then $f^* \text{Eff}(Z)$ is a face of $\text{Eff}(X)$ by [Cas13a, Prop. 2.22], thus $f^* \text{Eff}(Z) = \tau$. On the other hand there is a contraction $Z \to \mathbb{P}^1$, so that the boundary of $\text{Eff}(Z)$ contains some non-zero movable divisor, therefore the boundary of $\tau$ contains some non-zero movable divisor. Hence $\tau$ is not minimal, and we get (vi).

Finally if $\rho_Z = 1$, then $Z \cong \mathbb{P}^2$. If $D \subset X$ is a fixed prime divisor with $[D] \in \tau$, then the image of $D$ in $\mathbb{P}^2$ is an irreducible curve $\Gamma$, and $D$ is a component of $f^* \Gamma$. Since $f^* \Gamma$ is movable, while $D$ is fixed, $f^* \Gamma$ is reducible. By [Cas20] Lemma 4.10] $D$ must
be of type \((3,2)\) or \((3,1)^{sm}\). Moreover \(F\) is a del Pezzo surface, so that \(\rho_F \leq 9\) and \(\rho_X \leq 9 + \dim \tau\) by \([5.6]\), so we get (v).

The last statement follows from Theorems \([5.8]\) and \([4.8]\) (note that \(X\) is not a product of surfaces, see Rem. \([2.8]\)).

6. Constructing divisors covered by lines

Let \(X\) be a smooth Fano 4-fold with \(\rho_X \geq 7\). In this section we show that, given a fixed prime divisor \(D \subset X\) of type \((3,1)^{sm}\) or \((3,0)^Q\), and \(L \subset D\) an exceptional plane, then there exists a prime divisor \(B\) covered by a family of lines \(V\) such that \([V] + C_L \equiv C_D\) (Prop. \([6.1]\)). The construction of this family of lines is based on the explicit geometry of the divisor \(D\), and we also use the results of Section \([3]\). By Rem. \([4.17]\) \(D\) contains at least \(\rho_X - 4\) exceptional planes with distinct classes \([C_L]\), so that we obtain many distinct prime divisors covered by lines. This construction will be important in the rest of the paper. Then we give some properties of \(D\) and \(B\) depending on the different settings.

Proposition 6.1. Let \(X\) be a smooth Fano 4-fold with \(\rho_X \geq 7\), \(D \subset X\) a fixed prime divisor of type \((3,1)^{sm}\) or \((3,0)^Q\), and \(L \subset D\) an exceptional plane. Then there exists a family of lines \(V\) such that \(B := \text{Locus } V\) is a divisor different from \(D\), \([V] + C_L \equiv C_D\), and \(B \cdot C_L > 0\). Moreover \(L' \nsubseteq B\) for every exceptional plane \(L' \subset D\).

Proof. Let \(X \xrightarrow{\varphi} \tilde{X} \xrightarrow{\sigma} Y\) be the contraction associated to \(D\) as in \([4.3]\), and \(\ell \subset \tilde{X}\) the exceptional curve corresponding to \(L\) (see Lemma \([2.4]\)), so that \(\bar{D} \cdot \ell > 0\) and \(\ell \not\subset D\). Let \(q \in \ell \cap \bar{D}\).

If \(D\) is \((3,1)^{sm}\), then \(\sigma\) is the blow-up of a smooth curve, and \(q\) belongs to a fiber \(S_0\) of \(\sigma\). Let \(\Gamma_0\) be a line in \(S_0 \cong \mathbb{P}^2\) through \(q\).

If \(D\) is \((3,0)^Q\), then \(\tilde{D}\) is an irreducible quadric, either smooth or with one singular point, which cannot be \(q\) by Rem. \([4.25]\). Let \(\Gamma_0\) be a line in \(\tilde{D}\) through \(q\), and let \(S_0 \subset \tilde{D}\) be the union of the lines through \(q\), which is a quadric cone (if \(\tilde{D}\) is smooth) or a reducible quadric surface, singular at \(q\) (if \(\tilde{D}\) is singular).

In both cases \(-K_{\tilde{X}} \cdot \Gamma_0 = 2\), so that in the factorization of \(\varphi\) given in Lemma \([2.4]\)

\[
\begin{align*}
X & \xrightarrow{f} \tilde{X} & \xrightarrow{g} \bar{X},
\end{align*}
\]

if \(E \subset \tilde{X}\) is the exceptional divisor over \(L\) and \(\ell\), by Lemma \([2.4]\) (a) we have \(E \cdot \tilde{\Gamma} = 1\) for the transform \(\tilde{\Gamma} \subset \tilde{X}\) of \(\Gamma_0\). This implies that \(\Gamma_0\) intersects \(\ell\) only at \(q\), transversally, and does not intersect other exceptional curves. Moreover the transform \(\Gamma \subset X\) of \(\Gamma_0\) is a smooth rational curve with \(-K_X \cdot \Gamma = 1\), so that there is a family of lines \(V\) in \(X\) containing the general \(\Gamma\). The transform \(S \subset D \subset X\) of \(S_0\) yields a surface contained in \(\text{Locus } V\).

The curve \(C_D \subset D\) is the transform of a general line in \(S_0\) (if \(D\) is \((3,1)^{sm}\)), or of a general line in \(\tilde{D}\) (if \(D\) is \((3,0)^Q\)), so we must have \(C_D \equiv [V] + mC_L\) with \(m \in \mathbb{Z}\). Intersecting with \(-K_X\) we get \(m = 1\) and \(C_D \equiv [V] + C_L\).

\[\small\text{Note in particular that } X \text{ is not a product of surfaces, see Ex. \([4.6]\).}\]
We claim that, varying \( \Gamma_0 \) in \( S_0 \), we can find two disjoint \( \Gamma \)'s in \( S \). Indeed if \( D \) is \((3,1)^{sm} \), then the plane \( T_q S_0 \subset T_q \tilde{X} \) is the union of \( T_q \Gamma_0 \) for all \( \Gamma_0 \)'s, and since \( \Gamma_0 \) and \( \ell \) are transverse at \( q \), \( T_q S_0 \) does not contain \( T_q \ell \), namely \( S_0 \) and \( \ell \) meet transversally at \( q \). Moreover \( S_0 \cap \ell = \{q\} \) and \( S_0 \) does not meet other exceptional curves (see \S 4.3).

Locally \( g: \tilde{X} \to \check{X} \) is the blow-up of \( \ell \), therefore the strict transform of \( S_0 \) is isomorphic to the blow-up of \( S_0 \) at \( q \), and the lines \( \Gamma_0 \) get separated.

If \( D \) is \((3,0)^Q \) and \( \tilde{D} \) is singular, we can just repeat the same argument with each irreducible component of \( S_0 \). Finally if \( D \) is \((3,0)^Q \) and \( \tilde{D} \) is smooth, then \( T_q \tilde{D} \subset T_q \tilde{X} \) is a hyperplane, and when \( \Gamma_0 \) varies, \( T_q \Gamma_0 \) describes a quadric cone surface in \( T_q \tilde{D} \), not containing \( T_q \ell \). If we choose \( \Gamma_0 \) and \( \Gamma_0' \) such that the plane generated by \( T_q \Gamma_0 \) and \( T_q \Gamma_0' \) in \( T_q \tilde{X} \) does not contain \( T_q \ell \), then the curves \( \tilde{\Gamma} \) and \( \tilde{\Gamma}' \) in \( \tilde{X} \) must meet \( g^{-1}(q) \cong \mathbb{P}(T_q \tilde{X}/T_q \ell) \) at different points. Since \( f_{g^{-1}(q)}: g^{-1}(q) \to L \) is an isomorphism, the curves \( \Gamma \) and \( \Gamma' \) are disjoint in \( X \).

By Th. 3.7 we deduce that \( \dim V = 2 \) and \( \dim \text{Locus } V = 3 \).

Since \( D \cdot [V] + D \cdot C_L = D \cdot C_D = -1 \), we have \([V] \neq [C_L] \), and \( L \not\subseteq B := \text{Locus } V \) by Lemma 3.11(b). On the other hand \( B \cap L \neq \emptyset \) because \( S \subset B \), so that \( B \cdot C_L > 0 \). This also implies that \( B \neq D \), because \( D \cdot C_L < 0 \), hence \( D \cdot [V] \geq 0 \). If \( L' \subset D \) is any exceptional plane, we have \( D \cdot C_{L'} < 0 \), thus \([V] \neq [C_{L'}] \), and \( L' \not\subseteq B \) again by Lemma 3.11(b).

**Lemma 6.2.** Notation as in Prop. 6.1. Let \( L' \subset D \) be an exceptional plane with \( C_{L'} \neq C_L \), and \( V' \) the corresponding family of lines given by Prop. 6.1, with locus \( B' \).

Suppose moreover that \( D \) is of type \((3,0)^Q \) and that both \( B \) and \( B' \) are fixed of type (3,2).

Then \( B \cdot [V'] > 0 \) and \( B' \cdot [V'] > 0 \).

**Proof.** Since \( C_D \equiv [V] + C_L \equiv [V'] + C_{L'} \), we have \([V] \neq [V'] \) and hence \( B \neq B' \) by Lemma 3.11(a). Moreover by the same lemma we have \([V] = [C_B] \) and \([V'] = [C_{B'}] \).

Assume by contradiction that \( B \cdot [V'] = 0 \). Then by Lemma 4.15 we have \( B' \cdot [V'] = 0 \) and, if \( B \cap B' \neq \emptyset \), every connected component of \( B \cap B' \) is irreducible.

We keep the same notation as in the proof of Prop. 6.1. Let us consider the transforms \( \tilde{B}, \tilde{B}' \) in \( \tilde{X} \), and let \( \ell' \subset \tilde{X} \) be the exceptional curve corresponding to \( L' \). By construction \( \tilde{B}' \) contains the surface \( S_0' \), given by the union of the lines \( \Gamma_0' \) in \( \tilde{D} \) through a point \( q' \in \ell' \cap D \).

The surfaces \( S_0 \) and \( S_0' \) meet along a plane conic \( \Lambda \), which does contain neither \( q \) nor \( q' \); in fact \( \Lambda \) is contained in the open subset where \( \varphi: X \to \tilde{X} \) is an isomorphism.

If some line \( \Gamma_0 \) is not contained in \( \tilde{B}' \), then its transform \( \Gamma \) in \( X \) is a curve of the family \( V \) such that \( \Gamma \cap B' \neq \emptyset \) and \( \Gamma \not\subset B' \), contradicting \( B' \cdot [V] = 0 \). Therefore every \( \Gamma_0 \) must be contained in \( \tilde{B}' \), so that \( S_0 \subset B' \) and \( S \subset B \cap B' \). Similarly \( S' \subset B \cap B' \), where \( S' \) is the transform of \( S_0 \). Hence we get \( S \cup S' \subset B \cap B' \) and \( S \cap S' \neq \emptyset \); this gives a reducible connected component of \( B \cap B' \), a contradiction.

**Lemma 6.3.** Notation as in Prop. 6.1. Suppose moreover that \( D \) is of type \((3,0)^Q \) and that \( D \cdot C_L = -1 \). Then \( B \cap L' = \emptyset \) for every exceptional plane \( L' \subset D \) with \( C_{L'} \neq C_L \).

**Proof.** Since \( D \cdot C_D = D \cdot C_L = -1 \) and \( C_D \equiv C_L + [V] \), we have \( D \cdot [V] = 0 \), thus every line of the family \( V \) that meets \( D \) must be contained in \( D \). On the other hand \( D \neq B \), thus the general line of the family \( V \) is disjoint from \( D \), and \( V \) yields a family of lines \( V_X \) in \( X \). We have \([V_X] \equiv \ell + C_{\tilde{D}}, \tilde{D} \cdot [V_X] = 0 \), and Locus \( V_X = B \), the transform of \( B \).
Since \(-K_{\tilde{X}}\) is not ample, the family \(V_{\tilde{X}}\) can have reducible members, containing some exceptional curve.

Let us consider all the exceptional planes \(L_1 := L, L_2, \ldots, L_d \subset D\) such that \(C_{L_i} \equiv C_D - [V]\), and let \(\ell_1, \ldots, \ell_d \subset \tilde{X}\) be the corresponding exceptional curves. We show that for every one-cycle \(\Lambda\) of the family \(V_{\tilde{X}}\) one of the following holds:

(i) \(\Lambda\) is integral and disjoint from \(\tilde{D}\) and from every exceptional curve of \(\tilde{X}\);
(ii) \(\Lambda = \Gamma_0 + \ell_i\) for some \(i \in \{1, \ldots, d\}\), where \(\Gamma_0 \subset \tilde{D}\) is a line meeting \(\ell_i\), and \(\Lambda\) does not meet exceptional curves except \(\ell_i\).

Indeed, let us consider \(-K_{\tilde{X}} + 2\tilde{D} = \sigma^*(-K_Y)\), so that \(-K_{\tilde{X}} + 2\tilde{D}\) is nef and

\[( -K_{\tilde{X}} + 2\tilde{D} )^\perp \cap \text{NE}(\tilde{X}) = \mathbb{R}_{\geq 0}[C_{\tilde{D}}].\]

We have \(( -K_{\tilde{X}} + 2\tilde{D} ) \cdot \Lambda = 1\), so we can write \(\Lambda = \Lambda_1 + \Lambda_0\) where \(\Lambda_1\) is an integral curve with \(( -K_{\tilde{X}} + 2\tilde{D} ) \cdot \Lambda_1 = 1\), and \(\Lambda_0\) is an effective one-cycle with \((-K_{\tilde{X}} + 2\tilde{D}) \cdot \Lambda_0 = 0\). In particular \(\Lambda_1 \not\subset \tilde{D}\), while \(\Lambda_0\) is supported in \(\tilde{D}\).

If \(\Lambda_0 = 0\), then \(\Lambda = \Lambda_1\) is integral and disjoint from \(\tilde{D}\); moreover \(\Lambda\) cannot intersect any exceptional curve (see Lemma 2.4(b)), and we get (i).

If \(\Lambda_0 \neq 0\), then \(\tilde{D} \cdot \Lambda_0 < 0\). Moreover we have \(0 = \tilde{D} \cdot \Lambda = \tilde{D} \cdot \Lambda_1 + \tilde{D} \cdot \Lambda_0\), therefore \(\tilde{D} \cdot \Lambda_1 > 0\). Together with \(( -K_{\tilde{X}} + 2\tilde{D} ) \cdot \Lambda_1 = 1\), this implies that \(-K_{\tilde{X}} \cdot \Lambda_1 < 0\), so that \(\Lambda_1\) is an exceptional curve (see Lemma 2.4(a)), \(-K_{\tilde{X}} \cdot \Lambda_1 = -1\), \(\tilde{D} \cdot \Lambda_1 = 1\), and finally \(\tilde{D} \cdot \Lambda_0 = -1\). Hence \(\Lambda_0 \equiv C_{\tilde{D}}\) is integral and is a line in the quadric \(\tilde{D}\); moreover \(\Lambda_1 \equiv [V_{\tilde{X}}] - C_{\tilde{D}} \equiv \ell\), so that \(\Lambda_1 = \ell_i\) for some \(i \in \{1, \ldots, d\}\). Since the 1-cycle \(\Lambda\) is connected, \(\Lambda_0\) must meet \(\ell_i\). Finally \(\Lambda_0\) cannot meet other exceptional curves (as shown in the proof of Prop. 6.1), and neither can \(\Lambda_1\) (by Lemma 2.4(b)), so that we get (ii).

Now let \(L' \subset D\) be an exceptional plane such that \(C_{L'} \not\equiv C_L\), and \(\ell' \subset \tilde{X}\) the corresponding exceptional curve, so that \(\ell' \neq \ell_i\) for every \(i = 1, \ldots, d\). By what precedes, we have \(\ell' \cap \tilde{B} = \emptyset\), thus \(L' \cap B = \emptyset\).

**Proposition 6.4.** Notation as in Prop. 6.1. Suppose moreover that \(B\) is fixed of type (3, 2) and adjacent to \(D\), let \(L' \subset D\) be an exceptional plane with \(C_{L'} \not\equiv C_L\), and \(V'\) the corresponding family given by Prop. 6.1. Then \(D \cdot C_L = -1\), \(D \cdot [V] = 0\), and:

- \(B \cdot C_D = 0\), \(B \cdot C_L = 1\), \(B \cdot [V'] = 0\), if \(D\) is of type (3, 1)\(^\text{sm}\);
- \(B \cdot C_D = 1\), \(B \cdot C_L = 2\), \(B \cdot [V'] = 1\), if \(D\) is of type (3, 0)\(^Q\).

**Proof.** We keep the same notation as in the proof of Prop. 6.1. Note that \([C_B] = [V]\) by Lemma 3.11(a).

6.5. By Lemma 4.13(a) we have \(D \cdot [V] = 0\); since \(C_D \equiv C_L + [V]\), this also yields \(D \cdot C_L = -1\). Thus \(\tilde{D} \cdot \ell = 1\), and \(\tilde{D}\) and \(\ell\) meet transversally at the unique point \(q\). This implies that \(\tilde{D}\) is smooth around \(L\) and \(\varphi_{D} : D \to \tilde{D}\) is regular around \(L\) and is just the blow-up of \(q\), with exceptional divisor \(L\).

The transform \(S \subset D \subset X\) of the surface \(S_0 \subset \tilde{D} \subset \tilde{X}\) is isomorphic to either \(F_1\) (if \(D\) is (3, 1)\(^{sm}\)), or to \(F_2\) (if \(\tilde{D}\) is a smooth quadric), or to the union of two copies of \(F_1\) (if \(\tilde{D}\) is a singular quadric).
6.6. Let \( f : X \to X' \) be the contraction of \( \mathbb{R}_{\geq 0}[C_B] \), so that \( \text{Exc}(f) = B \). \( f \) can have at most finitely many 2-dimensional fibers, and \( B \) is a smooth \( \mathbb{P}^1 \)-bundle outside these fibers.

We show that \( D \) is disjoint from the possible 2-dimensional fibers of \( f \). Indeed if \( F \) is such a fiber and \( F \cap D \neq \emptyset \), since \( D \cdot C_B = 0 \), we have \( D \cdot \Gamma = 0 \) for every curve \( \Gamma \subset F \), so that \( F \subset D \). For every exceptional plane \( L \subset D \) we have \( D \cdot C_L < 0 \), therefore the classes \([C_L]\) and \([C_B]\) cannot be proportional. Since \( \mathcal{N}_1(F, X) = \mathbb{R}[C_B] \) and \( \mathcal{N}_1(\ell', X) = \mathbb{R}[C_L] \), we deduce that \( \dim(L' \cap F) \leq 0 \). Therefore there exists an irreducible curve \( \Gamma_2 \subset F \) disjoint from every exceptional plane \( L' \subset D \). Then \( \Gamma_2 \) is contained in the open subset where \( \varphi : X \to \tilde{X} \) is an isomorphism, and we conclude that \( \Gamma_2 \equiv mC_D \) for some \( m \in \mathbb{Z}_{\geq 0} \), which is impossible because \( [\Gamma_2] \in \text{NE}(f) \).

We deduce that \( B \cap D \) is covered by one-dimensional fibers of \( f \) and \( B \cap D \subset B_{\text{reg}} \). The irreducible components of \( B \cap D \) are \( f^{-1}(K) \) where \( K \) is an irreducible component of the curve \( f(B \cap D) \), and distinct irreducible components intersect at most along fibers of \( f \).

6.7. Suppose that \( D \) is \((3,1)^{sm} \). We show that \( S \) is a connected component of \( B \cap D \). By contradiction, let \( T \) be an irreducible component of \( B \cap D \) such that \( T \neq S \) and \( T \cap S \neq \emptyset \). Then \( T \) must contain some curve \( \Gamma \), and \( \Gamma \cap L \neq \emptyset \), so that \( T \cap L \neq \emptyset \) and \( T \cap L \) is a curve (as \( T \subset D \) and \( L \subset D_{\text{reg}} \)). Since \( f(T) \) is an irreducible curve and \( f[T] \) is a \( \mathbb{P}^1 \)-bundle, we also have \( \rho_T = 2 \) and \( \mathcal{N}_1(T, X) = \mathbb{R}[T \cap L] \oplus \mathbb{R}[\Gamma] \).

If \( \tilde{T} \subset \tilde{D} \) is the transform of \( T \), then \( \tilde{T} \) cannot contain exceptional curves, and \( \rho_{\tilde{T}} = 1 \). We have \( S_0 \cap \tilde{T} \neq \emptyset \), \( S_0 \neq \tilde{T} \), \( S_0 \) is a fiber of the \( \mathbb{P}^2 \)-bundle \( \sigma_{|\tilde{D}|} \), and \( \rho_{\tilde{T}} = 1 \), which yields a contradiction.

Thus every irreducible component of \( \tilde{B} \cap \tilde{D} \) is a fiber of \( \sigma \), therefore \( \tilde{B} \) is disjoint from the general fiber of \( \sigma_{|\tilde{D}|} \); this yields \( B \cdot C_D = \tilde{B} \cdot C_{\tilde{D}} = 0 \). Then \( C_D \equiv C_B + C_L \) and \( B \cdot C_B = 1 \).

Finally set \( B' := \text{Locus} \, V' \). We have \([V'] \neq [V] \), thus \( B' \neq B \) by Lemma 3.11(a), and \( B \cdot [V'] \geq 0 \). On the other hand we also have \( B \cdot C_{L'} \geq 0 \) because \( L' \not\subset B \) by Prop. 6.1 hence \( 0 = B \cdot C_D = B \cdot C_{L'} + B \cdot [V'] \) implies that \( B \cdot C_{L'} = B \cdot [V'] = 0 \).

6.8. Suppose now that \( D \) is \((3,0)^{Q} \). The proof of Lemma 6.3 shows that \( \tilde{B} \cap \tilde{D} = S'_0 \cup \cdots \cup S'_d \), where \( S'_i \) is the union of the lines in \( \tilde{D} \) through the point \( q_i := \tilde{D} \cap \ell_i \) for \( i = 1, \ldots, d \), and \( \ell_1, \ldots, \ell_d \) are all the exceptional curves of \( \tilde{X} \) numerically equivalent to \( \ell = \ell_1 \), so that \( \tilde{D} \cdot \ell_i = 1 \) for every \( i \). In \( X \) we have \( B \cap D = S^1 \cup \cdots \cup S^d \), where \( S^d \) is the transform of \( S'_0 \).

We show that \( d = 1 \). Otherwise, for \( i \neq j \) the surfaces \( S^i_0 \) and \( S^j_0 \) meet along a plane conic contained in the open subset where \( \varphi : X \to \tilde{X} \) is an isomorphism, so that \( S^i \) and \( S^j \) meet along a curve with class in \( \mathbb{R}_{\geq 0}[C_D] \). This is impossible, because they should intersect along fibers of \( f : X \to X' \). We conclude that \( d = 1 \) and that \( D \cap B = S \).

Suppose that \( \tilde{D} \) is a smooth quadric, so that \( S \) is irreducible and contained in \( D_{\text{reg}} \) (see 5.2). We have \( B_{|D} = mS \) for some \( m \in \mathbb{Z}_{\geq 1} \). If \( \Gamma \subset S \) is a curve of the family \( V \), we have \( -1 = B \cdot \Gamma = mS \cdot \Gamma \), where the last intersection is in \( D \). This implies that \( m = 1 \) and \( B \cdot C_L = S \cdot C_L = 2 \) (again the last intersection is in \( D \)), because \( S \cap L \) is a conic in \( L \cong \mathbb{P}^2 \).
Proposition 7.1. Let \( D \) be a quadric cone, with vertex \( v_0 \); then \( S_0 \) is a Cartier divisor in \( \tilde{D} \). In \( D \) we have \((\varphi|_D)^*(S_0) = S + 2L\), so that again \( S \) is Cartier in \( D \). Moreover \( S = H_1 + H_2 \) where \( H_1 \cong \mathbb{F}_1 \) and \( H_1 \cap H_2 \) is a common fiber of the \( \mathbb{P}^1 \)-bundles, which contains the singular point \( v := \varphi^{-1}(v_0) \). Note that \( v \notin L \) and that \( L \cup (S \setminus \{v\}) \subset D_{\text{reg}} \).

Write \( B_{id} = m_1 H_1 + m_2 H_2 \), with \( m_i \in \mathbb{Z}_{\geq 1} \), and let \( \Gamma_1 \subset H_1 \) be a curve of the family \( V \) disjoint from \( H_2 \), so that \( \Gamma_1 \subset D_{\text{reg}} \). Then \( -1 = B \cdot \Gamma_1 = m_1 H_1 \cdot \Gamma_1 = -m_1 \), so that \( m_1 = 1 \). Similarly we see that \( m_2 = 1 \), and finally that \( B_{id} = S \); as before this yields \( B \cdot C_L = 2 \).

We have \( C_D \equiv C_B + C_L \) and \( B \cdot C_B = -1 \), therefore \( B \cdot C_D = 1 \). Finally we have \( D \cap L' = \emptyset \) by Lemma 6.3, thus \( 1 = B \cdot C_D = B \cdot [V'] + B \cdot C_{L'} = B \cdot [V'] \).

**Lemma 6.9.** Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \), \( D \subset X \) a fixed prime divisor of type \((3,1)^{sm}\) or \((3,0)^Q\), and \( E \subset X \) a fixed prime divisor of type \((3,2)\) such that \( D \cap E \neq \emptyset \) and \( D \cdot C_E = 0 \). If \( D \) is of type \((3,1)^{sm}\), assume moreover that \( E \cdot C_D = 0 \).

Then there exists an exceptional plane \( L \subset D \) such that \( D \cdot C_L = -1 \) and \( C_E \) is the family of lines given by \( D \) and \( L \) as in Prop. 6.4.

**Proof.** Let \( X \twoheadrightarrow \tilde{X} \twoheadrightarrow Y \) be the contraction associated to \( D \) as in 4.3 and \( E_{\tilde{X}} \subset \tilde{X} \), \( E_Y \subset Y \) the transforms of \( E \). We have \( D \cap E \neq \emptyset \) and \( E \) does not contain exceptional planes (see Rem. 4.10), therefore \( \tilde{D} \cap E_{\tilde{X}} \neq \emptyset \) and \( \sigma(\tilde{D}) \cap E_Y \neq \emptyset \).

Since \( D \cdot C_E = 0 \), \( D \) is disjoint from the general curve \( C_E \), so that \( Y \) has a family of lines \( V_Y \) with locus \( E_Y \). If \( D \) is \((3,1)^{sm}\) and \([C] = [V_Y] \), then \( C \subset E_Y \) and \( E \cdot C_D = E_{\tilde{X}} \cdot C_{\tilde{D}} > 0 \), against our assumptions. Thus \([C] \neq [V_Y] \).

Let \( \Gamma \subset Y \) be a curve of the family \( V_Y \) such that \( \Gamma \cap \sigma(\tilde{D}) \neq \emptyset \) and \( \Gamma \neq \sigma(\tilde{D}) \). By Rem. 4.10 we have \( \Gamma = \sigma(\ell) \) where \( \ell \subset \tilde{X} \) is an exceptional curve; moreover if \( C_E \subset \tilde{X} \) is the transform of the general \( C_E \subset X \), we have \( C_E \equiv \ell + C_{\tilde{D}} \) in \( \tilde{X} \). Then in \( X \) we get \( C_D \equiv C_L + C_E \) where \( L \subset D \) is the exceptional plane corresponding to \( \ell \). Moreover \( -1 = D \cdot C_D = D \cdot (C_L + C_E) = D \cdot C_L \), and we get the statement.

**Corollary 6.10.** Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \), \( D \) and \( E \) two adjacent fixed prime divisors, of type \((3,0)^Q\) and \((3,2)\) respectively, such that \( D \cap E \neq \emptyset \). Then \( E \cdot C_D = 1 \).

**Proof.** By Lemma 6.9 there exists an exceptional plane \( L \subset D \) such that \( C_E \) is the family of lines given by \( D \) and \( L \) as in Prop. 6.1 so that \( E \cdot C_D = 1 \) by Prop. 6.4.

7. Constructing rational contractions of fiber type

In this section we consider two situations where we can construct a rational contraction of fiber type on \( X \), and then prove that \( \rho_X \leq 12 \) using the results of Section 6.

**Proposition 7.1.** Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 8 \) and \( \delta_X \leq 1 \), \( D \) a fixed prime divisor of type \((3,1)^{sm}\) or \((3,0)^Q\), and \( X \twoheadrightarrow \tilde{X} \twoheadrightarrow Y \) the associated contraction as in 4.3. Suppose that \( Y \) contains a nef prime divisor \( H \) covered by a family of lines and such that \( H \cap \sigma(\text{Exc}(\sigma)) = \emptyset \). Then one of the following holds:

(i) \( X \) has a contraction onto a 3-fold;
(ii) \( X \) contains a fixed prime divisor \( E \) of type \((3,2)\) with \( N_1(E,X) \subset N_1(X) \);
(iii) \( \rho_X \leq 10 \).

Moreover \( \rho_X \leq 12 \), and if \( \rho_X = 12 \), then \( X \) has a rational contraction onto a 3-fold.
Proof. Let \( g : Y \to Z \) be the contraction defined by \( mH \) for \( m \in \mathbb{N}, \ m \gg 0 \), so that \( H = g^*A \) where \( A \subset Z \) is an ample prime Cartier divisor, and \( \text{NE}(g) = \text{NE}(Y) \cap H^\perp \).

If \( \ell \subset \breve{X} \) is an exceptional curve, then by Lemma 4.19 \( \sigma(\ell) \) cannot meet any line in \( Y \) outside \( \sigma(\text{Exc}(\sigma)) \), so that \( \sigma(\ell) \cap H = \emptyset \) and \( \sigma(\ell) \in \text{NE}(g) \). Moreover \( H \) is contained in the open subset of \( Y \) where the birational map \( X \dashrightarrow Y \) is an isomorphism; let \( H_X \subset X \) and \( \breve{H} \subset \breve{X} \) be the transforms of \( H \). Then \( \breve{H} = \sigma^*(g^*A) \), and \( H_X \) is still nef, so that \( h := g \circ \sigma \circ \varphi : X \to Z \) is regular, \( H_X = h^*A \), and \( \text{NE}(h) = \text{NE}(X) \cap H_X^\perp \).

Moreover \( H_X \cap D = \emptyset \), therefore \( h(D) = \{ pt \} \).

\[
X \xrightarrow{\varphi} \breve{X} \xrightarrow{\sigma} Y \\
\downarrow h \quad \quad \quad \quad \quad \quad \quad \quad \downarrow g \\
Z
\]

We show that \( g \) and \( h \) are of fiber type.

If \( D = (3,1)^{rm} \), have \( H \cdot C = 0 \) hence \( [C] \in \text{NE}(g) \). If \( [C] \in \text{mov}(Y) \), then \( g \) is of fiber type. Otherwise, \( [C] \) generates an extremal ray of type \((3,2)\) of \( \text{NE}(Y) \) by Lemma 5.11(3), let \( E_1 \) be its locus. By [Cas17, Prop. 5.8 and its proof], there is fixed prime divisor \( E_2 \subset Y \) of type \((3,2)\), such that \( E_2 \cdot C > 0 \), so that \( [C] + [E_2] \in \text{mov}(X) \) by Cor. 4.14. If \( \Gamma \subset E_2 \) is an irreducible curve such that \( \Gamma \equiv C_{E_2} \) and \( \Gamma \cap C \neq \emptyset \), then by Rem. 4.10 \( \Gamma = \sigma(\ell) \) where \( \ell \subset \breve{X} \) is an exceptional curve, thus \( \Gamma \) is of fiber type. Otherwise there exists a prime divisor \( B \) such that \( B \cdot [V] < 0 \), hence \( B = \text{Locus}(V) \) and by Lemma 5.12 \( [V] \) generates an extremal ray of type \((3,2)\).

By Rem. 4.17 we can choose an exceptional plane \( L' \subset D \) with \( C_{L'} \neq C_L \) and this yields another family of lines \( V' \) with \( [V'] \neq [V] \). As before, \( [V'] \in \text{NE}(h) \), and either \( [V'] \in \text{mov}(X) \) and \( h \) is of fiber type, or \( [V'] \) generates an extremal ray of type \((3,2)\), with locus \( B' \). Then \( B \cdot [V'] > 0 \) and \( B' \cdot [V'] > 0 \) by Lemma 5.12 so that \( [V] + [V'] \in \text{mov}(X) \) by Lemma 5.11(3), and again \( h \) is of fiber type.

If \( \dim Z = 3 \), we get (i).

We show that \( \dim Z > 1 \). Otherwise, let \( F \subset C \) be a general fiber of \( h \). We have \( \mathcal{N}_F(X,X) \subset \ker h_* \subset \mathcal{N}_F \), and since \( \delta_X \leq 1 \), we deduce that \( \mathcal{N}_F(X) = \ker h_* \), and similarly \( \mathcal{N}_F(D,X) = \ker h_* \). On the other hand \( F \cap D = \emptyset \), thus \( \mathcal{N}_F(X) \subset D^\perp \), which is impossible because \( D \cdot C_D = -1 \) and \( [C_D] \in \mathcal{N}_F(D,X) \).

Finally suppose that \( \dim Z = 2 \). Since \( h \) is not equidimensional, by Lemma 5.14 we have that \( \rho_Z = 1 \) and \( D \) is the unique prime divisor contracted to a point. Then Lemma 5.13 yields (ii) or (iii).

The last statement follows from Th. 5.8 and 4.8 (note that \( X \) is not a product of surfaces, see Ex. 4.10 or Rem. 4.8). □

**Lemma 7.2.** Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \) and \( \delta_X \leq 1 \), and \( D \subset X \) a fixed prime divisor of type \((3,0)^2\) and \( E_1, E_2 \), of type \((3,2)\), both adjacent to \( D \). Then \( \rho_X \leq 12 \), and if \( \rho_X = 12 \), then \( X \) has a rational contraction onto a 3-fold.
Proof. Let \( i \in \{1, 2\} \); we have \( D \cdot C_{E_i} = 0 \) by Lemma 4.18(a). If \( E_i \cap D = \emptyset \) for some \( i \in \{1, 2\} \), then \( \mathcal{N}(E_i, X) \subseteq D^\perp \subseteq N(X) \), and we conclude by Th. 4.8. Thus we can assume that \( E_i \cap D \neq \emptyset \) for \( i = 1, 2 \), so that by Lemma 6.9 there exists an exceptional plane \( L_i \subset D \) such that \( C_{E_i} \) is the family of lines given by \( D \) and \( L_i \) as in Prop. 6.1.

Then Prop. 6.4 yields:

\[ E_1 \cdot C_D = E_1 \cdot C_{E_2} = E_2 \cdot C_{E_1} = 1. \]

Consider now \( H := E_1 + E_2 + 2D \); we show that \( H \) is movable. By [Cas17, Lemma 5.29(2)], \([H] \in \text{Mov}(X)\) if and only if \( H \cdot C_G \geq 0 \) for every fixed prime divisor \( G \subset X \).

This is clear if \( G \neq E_1, E_2, D \); moreover \( H \cdot C_{E_1} = H \cdot C_{E_2} = H \cdot C_D = 0 \), therefore \( H \) is movable. We also have \( H \cdot (C_{E_1} + C_{E_2}) = 0 \) and \([C_{E_1} + C_{E_2}] \in \text{mov}(X) = \text{Eff}(X)^V \) by Lemma 6.9(b), so that \([H] \in \text{Eff}(X)\), namely \( H \) is not big.

Let \( f: X \dashrightarrow Z \) be the rational contraction of fiber type defined by \( mH \) for \( m \in \mathbb{N} \), \( m \gg 0 \). Since \( H \cdot C_D = 0 \), \( f \) contracts \( D \) to a point, so that if \( \dim Z = 2 \) \( f \) is not equidimensional. Therefore by Lemma 6.11 we have either \( \dim Z = 3 \), or \( \mathcal{N}(E_1, X) \subseteq N(X) \), and we get the statement by Th. 6.8 and 4.8 respectively (note that \( X \) is not a product of surfaces, see Ex. 4.6 or Rem. 2.8).

\section{The Case \((3, 1)^{sm}\)}

Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \) and \( \delta_X \leq 1 \); in this section we prove the following result, which implies Th. 1.1 when \( X \) has a fixed prime divisor \( D \) of type \((3, 1)^{sm}\).

\begin{theorem}
Let \( X \) be a smooth Fano 4-fold with \( \rho_X \geq 7 \) and \( \delta_X \leq 1 \), having a fixed divisor of type \((3, 1)^{sm}\). Then \( \rho_X \leq 12 \), and if \( \rho_X = 12 \), then \( X \) has a rational contraction onto a 3-fold.
\end{theorem}

Let us give an outline of the proof. We consider the contraction \( X \dashrightarrow Y \) associated to \( D \), and work in the smooth Fano 4-fold \( Y \). We show that we can reduce to the following situation:

\begin{itemize}
\item except possibly one, the fixed prime divisors \( E_1, \ldots, E_r \) of type \((3, 2)\) of \( Y \) satisfy \( E_i \cdot C_{E_j} = 0 \) for \( i \neq j \);
\item \( Y \) contains at most one nef prime divisor covered by lines;
\item \( \text{Eff}(Y) \) is generated by classes of fixed prime divisors, none of type \((3, 0)^{sm}\), and there is at least one not of type \((3, 2)\), say \( B \).
\item The divisors covered by lines given by \( B \) and its exceptional planes, as in Section 3 are all fixed of type \((3, 2)\), except at most one. From the properties of these families of lines we deduce that \( B \) is \((3, 1)^{sm}\), so that every fixed prime divisor of \( Y \) is of type \((3, 2)\) or \((3, 1)^{sm}\).
\end{itemize}

Then we use these fixed prime divisors to construct some movable, non-big divisors in \( Y \), which yield some rational contractions of fiber type on \( Y \) and \( X \); these also allow to describe some facets of \( \text{Eff}(Y) \). Finally we conclude applying the results in 5.3.

\begin{proof}[Proof of Th. 8.1]
We assume that \( \rho_X \geq 8 \), and note that \( X \) is not a product of surfaces (see Ex. 4.6 or Rem. 2.8). We also assume the following:

\begin{enumerate}
\item \( X \) has no rational contraction onto a 3-fold;
\item \( \mathcal{N}(E_X, X) = N(X) \) for every fixed prime divisor \( E_X \subset X \) of type \((3, 2)\);
\item \( X \) has no fixed prime divisor of type \((3, 0)^{sm}\).
\end{enumerate}
Otherwise, the statement follows from Th. 5.8 or 4.7 respectively.

8.2. Let $X \dashrightarrow \tilde{X} \overset{\sigma}{\to} Y$ be the contraction associated to $D$ as in 4.3, so that $Y$ is a smooth Fano 4-fold with $\rho_Y \geq 7$, and $\sigma$ is the blow-up of a smooth irreducible curve $C \subset Y$. We have the following:

(a') $Y$ has no rational contraction onto a 3-fold;
(b') $\mathcal{N}_1(E,Y) = \mathcal{N}_1(Y)$ for every fixed prime divisor $E \subset Y$ of type $(3,2)$;
(c') $Y$ has no fixed prime divisor of type $(3,0)^m$.

Proof. The implication (a) $\Rightarrow$ (a') is clear. For (b) $\Rightarrow$ (b'), let $E \subset Y$ be a fixed prime divisor of type $(3,2)$, and let us consider its transforms $E_X \subset X$ and $E_X \subset \tilde{X}$. Then $E_X$ is a fixed prime divisor of type $(3,2)$ by Lemma 1.22 and $\dim \mathcal{N}_1(E_X,X) = \dim \mathcal{N}_1(E_X,\tilde{X})$ by Lemma 4.10. Thus (a) yields $\mathcal{N}_1(E_X,\tilde{X}) = \mathcal{N}_1(\tilde{X})$. On the other hand $E = \sigma(E_X)$, hence $\mathcal{N}_1(E,Y) = \sigma_*(\mathcal{N}_1(E_X,\tilde{X})) = \mathcal{N}_1(Y)$.

Finally if $Y$ has a fixed prime divisor of type $(3,0)^m$, its transform in $X$ is a fixed prime divisor of type $(3,0)^m$ by Lemma 1.22 therefore (c) $\Rightarrow$ (c').

8.3. We note that $Y$ is not a product of surfaces, because a Fano 4-fold $S_1 \times S_2$ with $\rho_{S_2} > 1$ has a contraction onto $\mathbb{P}^1 \times S_2$, contradicting (a'). Then we have $\delta_Y \leq 1$ by (a') and Theorems 2.6 and 2.7.

8.4. Every fixed prime divisor $E$ of type $(3,2)$ in $Y$ meets $C$.

Indeed if $E \cap C = \emptyset$, then $E$ is disjoint from the images of all exceptional curves in $\tilde{X}$, by Lemma 4.19 thus $E$ is contained in the open where the birational map $X \dashrightarrow Y$ is an isomorphism. Its transform $E_X \subset X$ is a fixed prime divisor of type $(3,2)$ by Lemma 1.22 and it is disjoint from $D$, so that $\mathcal{N}_1(E_X,X) \subseteq D^\perp \subseteq \mathcal{N}_1(X)$ (see Rem. 2.4), contradicting (b).

8.5. We introduce the following notation: if $C$ belongs to a family of lines which cover a divisor, we denote such divisor by $E_0$; by Lemma 3.12 it could be nef, or fixed of type $(3,2)$; in this last case $C_{E_0} \equiv C$.

8.6. If $E \subset Y$ is a fixed prime divisor of type $(3,2)$, then $E \cap C \neq \emptyset$ by 8.4 and applying [Cas17, Lemma 5.11] we know that either $C \not\subset E$ and $E \cdot C > 0$, or $E = E_0$ and $C_{E_0} \equiv C$.

8.7. Let $E_1,\ldots,E_r$ be the fixed prime divisors of $Y$ of type $(3,2)$ with $E_i \cdot C > 0$, so that by 8.6 the fixed prime divisors of type $(3,2)$ of $Y$ are either $r$ (namely $E_1,\ldots,E_r$) or $r+1$ (namely $E_0,E_1,\ldots,E_r$).

We show that $E_i \cdot C_{E_j} = 0$ for every $i,j \in \{1,\ldots,r\}$ with $i \neq j$.

Take for simplicity $i = 1$ and $j = 2$, and let $\Gamma$ be a curve such that $\Gamma \equiv C_{E_2}$ and $C \cap \Gamma \neq \emptyset$. Then by Rem. 4.16 $\Gamma$ is the image of an exceptional curve of $X$, hence by Lemma 4.19 $\Gamma$ cannot intersect any curve of anticanonical degree one (different from $C$ and from $\Gamma$ itself). Therefore $\Gamma \cap E_1 = \emptyset$ and $E_1 \cdot C_{E_2} = 0$.

This implies that the classes $[E_1],\ldots,[E_r]$ are linearly independent in $\mathcal{N}(Y)$, hence $r \leq \rho_Y$. In fact $r < \rho_{X}$, otherwise given an ample divisor $A$, we can write $A \equiv \sum_{i=1}^r \lambda_i E_i$ and $A \cdot C_{E_j} = -\lambda_j > 0$ for every $j = 1,\ldots,\rho_Y$, which gives a contradiction.

8.8. Suppose that $E_0$ is a fixed prime divisor of type $(3,2)$. Then $E_0$ and $E_i$ are not adjacent, $E_0 \cdot C_{E_i} > 0$, and $E_0 + E_i$ is movable, for every $i = 1,\ldots,r$. This follows from Cor. 4.14 because $E_i \cdot C_{E_0} = E_i \cdot C > 0$. 


8.9. Suppose that there exists a movable face \( \tau \) of \( \text{Eff}(Y) \) such that, if \( f: Y \rightarrow Z \) is an associated rational contraction of fiber type as in (3, 2), we have \( Z \cong \mathbb{P}^2 \).

Then either \( \rho_X \leq 10 \), or \( \tau \) is a facet and the general fiber of \( f \) is \( \mathbb{P}^2 \).

**Proof.** Let \( \zeta: \tilde{Y} \rightarrow Y \) be a SQM such that \( \tilde{f} := f \circ \zeta: \tilde{Y} \rightarrow \mathbb{P}^2 \) is regular and \( K \)-negative (see Lemma 2.2), and let \( F \subset \tilde{Y} \) be a general fiber of \( \tilde{f} \), so that \( F \) is a smooth del Pezzo surface. Note that \( F \) is contained in the open subset where \( \zeta \) is an isomorphism, because the indeterminacy locus of \( \zeta \) has dimension at most one, see Lemma 2.4.

If \( \rho_F = 1 \), then \( F \cong \mathbb{P}^2 \) and \( \tau \) is a facet by (5.6). Suppose that \( \rho_F > 1 \). Then \( F \) is covered by rational curves of anticanonical degree 2, so that \( \tilde{Y} \) and \( Y \) have a covering family of rational curves of anticanonical degree 2. By Lemma 4.21(b) \( C \) must be a component of a curve of this family in \( Y \); moreover \( C \) cannot meet any exceptional plane (see Rem. 4.16) and thus it is contained in the open subset where \( \zeta^{-1} \) is an isomorphism and \( f \) is regular, so that \( f(C) \) is a point.

Let us consider now the composition \( X \rightarrow \mathbb{P}^2 \). It contracts \( D \) to a point, and no other prime divisors, because \( f \) is equidimensional (see 5.5). Then Lemma 5.13 and (b) yield \( \rho_X \leq 10 \).

8.10. Suppose that there exists a minimal movable face \( \tau \) of \( \text{Eff}(Y) \) with \( \dim \tau \leq 2 \).

Then either \( \rho_X \leq 10 \), or \( \dim \tau = 2 \) and \( \tau \) does not contain classes of fixed prime divisors of type \((3,2)\).

**Proof.** We apply Prop. 5.10 to \( Y \) and \( \tau \); case (vi) of the Proposition is excluded because \( \tau \) is minimal, and cases (i) and (ii) are excluded by \((a') \) and \((b') \) respectively.

In case (iii) of Prop. 5.10 we get either \( \dim \tau = 1 \), \( \rho_Y \leq 9 \), and \( \rho_X \leq 10 \), or \( \dim \tau = 2 \) and \( \tau \) does not contain classes of fixed prime divisors of type \((3,2)\).

In case (v) we apply 5.9 and get \( \rho_X \leq 10 \), because \( \tau \) is not a facet.

Finally in case (iv) of Prop. 5.10 we have \( \rho_Y = 10 \) and there is a contraction \( f: Y \rightarrow \mathbb{P}^1 \) with general fiber \( S \times \mathbb{P}^1 \), \( S \) a del Pezzo surface with \( \rho_S = 9 \). We show that this last case cannot happen.

Since \( S \times \mathbb{P}^1 \) is covered by rational curves of anticanonical degree 2, as in 8.9 we see that \( C \) must be contained in a fiber of \( f \). Let us consider the composition \( f \circ \sigma: X \rightarrow \mathbb{P}^1 \). There is a SQM \( \psi: \tilde{X} \rightarrow \tilde{X} \) such that \( g := \psi \circ f \circ \sigma: \tilde{X} \rightarrow \mathbb{P}^1 \) is regular and \( K \)-negative (see Lemma 2.2), and \( \tilde{X} \cong X \) by Lemma 5.10, let \( F \subset X \) be the general fiber of \( g \). We have \( g(D) = \{p_0\} \) and \( g^{-1}(p_0) \) has at least another irreducible component \( D_2 \), so that \( F \cap (D \cup D_2) = \emptyset \) and \( N_1(F, X) \subseteq D^2 \cap D^2_2 \) (see Rem. 2.4). On the other hand, since \( D \) is fixed, the classes \([D]\) and \([D_2]\) cannot be proportional in \( N^1(X) \), thus \( D^2 \neq D^2_2 \) and \( \dim N_1(F, X) \leq \rho_X - 2 \), contradicting \( \delta_X \leq 1 \).

Thus we can assume that \( \text{Eff}(Y) \) is generated by classes of fixed prime divisors, and that if \( \tau \) is a 2-dimensional movable face of \( \text{Eff}(Y) \), then \( \tau \) does not contain classes of fixed prime divisors of type \((3,2)\).

8.11. If \( Y \) contains a nef prime divisor \( H \) covered by a family \( V \) of lines, then either \( \rho_X \leq 10 \), or \( H = E_0 \) and \([C] \equiv [V]\).

Indeed if \( H \cap C = \emptyset \), then Prop. 7.1 together with (a) and (b) yields \( \rho_X \leq 10 \). If instead \( H \cap C \neq \emptyset \), then \( C \) is a member of the family \( V \) by Lemma 1.21(a), so that \( H = E_0 \) (see 8.3).
Thus we can assume that if $Y$ contains a nef prime divisor $H$ covered by a family $V$ of lines, then $H = E_0$ and $[C] \equiv [V]$.

**8.12.** We show that $Y$ has some fixed prime divisor not of type $(3,2)$.

By [8.10] $\text{Eff}(Y)$ is generated by classes of fixed prime divisors, so there are at least $\rho_Y$ of them. On the other hand, by [8.7] $Y$ has at most $r + 1 \leq \rho_Y$ fixed prime divisors of type $(3,2)$. If $Y$ has exactly $\rho_Y$ fixed prime divisors of type $(3,2)$, then $E_0$ is fixed of type $(3,2)$ (see [8.5] and [8.7]). By [8.10] any 2-dimensional face of $\text{Eff}(Y)$ containing $[E_0]$ is fixed, and yields a fixed prime divisor $B$ adjacent to $E_0$. Then $B \neq E_i$ for every $i = 1, \ldots, r$ by [8.8] thus $B$ is not of type $(3,2)$.

**8.13.** Let $B \subset Y$ be a fixed prime divisor not of type $(3,2)$; by $(c')$ $B$ is of type $(3,1)^{\text{sm}}$ or $(3,0)^2$. Let $L \subset B$ be an exceptional plane, and let us consider the family of lines $V$ in $Y$ given by $B$ and $L$ as in Prop. [6.1] so that Locus $V$ is a divisor and $[V] \equiv C_B - C_L$.

We show that $[V] \in \{[(C),[C_{E_1}],\ldots,[C_{E_r}]]\}$. Indeed if Locus $V$ is not nef, then it is a fixed prime divisor of type $(3,2)$ by Lemma [3.12] so that by [8.7] Locus $V \in \{E_0,E_1,\ldots,E_r\}$, and $[V] \in \{[(C),[C_{E_1}],\ldots,[C_{E_r}]]\}$. If instead Locus $V$ is nef, then by [8.11] Locus $V = E_0$ and $[V] = [C]$.

**8.14.** Let us vary the exceptional plane $L$ in $B$ and consider all the families of lines that we obtain; we denote by $\eta_1,\ldots,\eta_s \in \mathcal{N}_1(Y)$ their distinct numerical classes. We show that $s \geq \rho_Y - 3$.

Indeed we have $\delta_Y \leq 1$ by [8.3], so $\dim \mathcal{N}_1(B,Y) \geq \rho_Y - 1$. By Rem. [4.17] $B$ contains at least $\rho_Y - 3$ exceptional planes $L_1,\ldots,L_{\rho_Y-3} \subset B$ such that the classes $[C_{L_1}],\ldots,[C_{L_{\rho_Y-3}}]$ are linearly independent. In particular the classes $[C_B - C_{L_i}]$ of the $\rho_Y - 3$ corresponding families are all distinct.

**8.15.** By [8.13] we have $\eta_j \in \{[(C),[C_{E_1}],\ldots,[C_{E_r}]]\}$ for every $j = 1,\ldots,s$, and $s \geq \rho_Y - 3$ by [8.14]. Therefore $r \geq s - 1 \geq \rho_Y - 4 = \rho_X - 5 \geq 3$; moreover up to renumbering we can assume that $\eta_1,\eta_2 \in \{[(C),[C_{E_1}],\ldots,[C_{E_r}]]\}$.

**8.16.** By Lemma [4.22] for every $i = 1,\ldots,r$ the transform $\widetilde{E}_i$ of $E_i$ in $X$ is a fixed prime divisor of type $(3,2)$ adjacent to $D$. Moreover, since $E_i \not\subseteq C$ (see [8.2]), in $\widetilde{X}$ the divisor $\sigma^{-1}(E_i)$ is disjoint from the general fiber of $\sigma$, and hence $\widetilde{E}_i \cdot C_D = \sigma^{-1}(E_i) \cdot C_D = 0$.

We conclude that $D$ has at least $\rho_X - 5$ adjacent fixed prime divisors of type $(3,2)$ and having intersection zero with $C_D$.

In this first part of the proof we have shown the following.

**Proposition 8.17.** Let $X$ be a Fano 4-fold with $\rho_X \geq 8$ and $\delta_X \leq 1$, and $D \subset X$ a fixed prime divisor of type $(3,1)^{\text{sm}}$. Then one of the following holds:

(i) $X$ has a rational contraction onto a 3-fold;
(ii) $X$ has a fixed prime divisor $E_X$ of type $(3,2)$ such that $\mathcal{N}_1(E_X,X) \subsetneq \mathcal{N}_1(X)$;
(iii) $X$ has a fixed prime divisor of type $(3,0)^{\text{sm}}$;
(iv) $\rho_X \leq 10$;
(v) there are at least $\rho_X - 5$ fixed prime divisors of type $(3,2)$ adjacent to $D$ and having intersection zero with $C_D$.

We assume from now on that $\rho_X \geq 9$, so that $\rho_Y \geq 8$. 
8.18. By (8.13) we have \( \eta_1 = [C_{E_{i_1}}] \) and \( \eta_2 = [C_{E_{i_2}}] \) for some \( i_1, i_2 \in \{1, \ldots, r\} \). Then \( E_{i_1} \) is the locus of the family of lines with class \( \eta_1 \), and \( E_{i_1} \cdot \eta_2 = E_{i_1} \cdot C_{E_{i_2}} = 0 \) by (8.7).

By Lemma 8.2 this implies that \( B \) is of type \((3,1)^{sm}\).

We conclude that every fixed prime divisor of \( Y \) is of type \((3,1)^{sm}\) or \((3,2)\).

8.19. Let \( i \in \{1, \ldots, r\} \). We show that \( E_i \) and \( B \) are adjacent if and only if \( B \cdot C_{E_i} = 0 \), if and only if \( E_i \cdot C_B = 0 \).

Indeed the first equivalence follows from Lemmas 4.12(a) and 4.13(a); similarly if \( E_i \cdot C_B = 0 \) then \( E_i \) and \( B \) are adjacent. Conversely, if the two divisors are adjacent, let \( i_1, i_2 \) be as in (8.18) we can assume that \( i_1 \neq i \), so that \( E_{i_1} \cdot C_{E_{i_1}} = 0 \) by (8.7). There exists an exceptional plane \( L_1 \subseteq B \) such that \( C_B = C_{L_1} + \eta_1 \equiv C_{L_i} + C_{E_{i_1}} \), hence \( E_i \cdot C_B = E_i \cdot C_{L_i} \). Now if \( E_i \cdot C_B > 0 \), then \( E_i \cdot C_{L_i} > 0 \), but Lemma 3.23 yields \( E_i \cap L_1 = \emptyset \), a contradiction. Thus \( E_i \cdot C_B = 0 \).

8.20. Suppose that \( E_0 \) is a fixed prime divisor of type \((3,2)\). Then \( E_0 \cdot C_B > 0 \).

Indeed consider again \( i_1 \) and \( L_1 \) as in (8.18) and (8.19) We have \( E_0 \cdot C_{E_{i_1}} > 0 \) by (8.8) and \( E_0 \cdot C_{L_i} > 0 \) (see Rem. 4.9), so that \( E_0 \cdot C_B = E_0 \cdot C_{L_i} + E_0 \cdot C_{E_{i_1}} > 0 \).

8.21. Either \( \rho_X \leq 11 \), or \( B \) is adjacent to some divisor among \( E_1, \ldots, E_r \).

Indeed, let us apply Prop. 8.17 to \( Y \) and \( B \). By \((a')\), \((b')\), and \((c')\), either \( \rho_Y \leq 10 \) and \( \rho_X \leq 11 \), or \( Y \) has a fixed prime divisor of type \((3,2)\) adjacent to \( B \) and having intersection zero with \( C_B \). By (8.20) this divisor must be among \( E_1, \ldots, E_r \).

Thus we can assume that every fixed prime divisor \( B \subseteq Y \) of type \((3,1)^{sm}\) is adjacent to some divisor among \( E_1, \ldots, E_r \).

8.22. Let \( h \in \{1, \ldots, r\} \) be such that \( E_h \) and \( B \) are adjacent. Then \( B \cdot C_{E_h} = E_h \cdot C_B = 0 \) by (8.19) and \( E_h \cap B \neq \emptyset \) by \((b')\) (otherwise \( N_1(E_h, Y) \subseteq B^2 \subseteq N_1(Y) \), see Rem. 2.1), so that we can apply Lemma 6.9. We deduce that there exists an exceptional plane \( L_0 \subseteq B \) such that \( C_B = C_{E_h} + C_{L_0} \) and \( E_h \) is the locus of the family of lines given by \( B \) and \( L_0 \), so that \( [C_{E_h}] = \eta_j \) for some \( j \in \{1, \ldots, s\} \) (see (8.14)).

8.23. We have \( \eta_j \in \{[C_{E_1}], \ldots, [C_{E_r}]\} \) for every \( j = 1, \ldots, s \) (see (8.14)).

Indeed by (8.17) it is enough to show that \( \eta_j \neq [C] \) for every \( j = 1, \ldots, s \). Let \( E_h \) and \( \eta_{j_0} \) be as in (8.22) By Prop. 6.9 we have \( E_h \cdot \eta_j = 0 \) for every \( j = 1, \ldots, s \), \( j \neq j_0 \), moreover \( E_h \cdot \eta_{j_0} = -1 \). On the other hand \( E_h \cdot C > 0 \) (see (8.7)), so that \( \eta_j \neq [C] \) for every \( j = 1, \ldots, s \).

We conclude that for every exceptional plane \( L \subseteq B \) there exists some \( i \in \{1, \ldots, r\} \) such that \( C_B \equiv C_L + C_{E_i} \).

8.24. Suppose that there exists a minimal movable face \( \tau \) of \( \text{Eff}(Y) \) with \( \dim \tau \leq 3 \).

Then either \( \rho_X \leq 11 \), or \( \dim \tau = 3 \) and \( \tau \) does not contain classes of fixed prime divisor of type \((3,2)\).

**Proof.** We apply Prop. 5.16 to \( Y \) and \( \tau \); case \((vi)\) of the Proposition is excluded because \( \tau \) is minimal, and cases \((i)\) and \((ii)\) are excluded by \((a')\) and \((b')\) respectively.

In case \((iii)\) of Prop. 5.16 either \( \dim \tau = 2 \), \( \rho_Y \leq 10 \), and \( \rho_X \leq 11 \), or \( \dim \tau = 3 \) and \( \tau \) does not contain classes of fixed prime divisors of type \((3,2)\). In case \((iv)\) we have \( \rho_Y = 10 \) and \( \rho_X = 11 \). In case \((v)\) we apply (8.9) and get \( \rho_X \leq 10 \), because \( \tau \) is not a facet.
Thus we can assume that every 2-dimensional face of $\text{Eff}(Y)$ is fixed, and that if $\tau$ is a 3-dimensional movable face of $\text{Eff}(Y)$, then $\tau$ does not contain classes of fixed prime divisors of type $(3,2)$.

8.25. Let $L \subset B$ be an exceptional plane. There exists a fixed prime divisor $G \subset Y$ of type $(3,1)^{sm}$, adjacent to $B$, and such that $L \subset G$.

Proof. Consider $Y \dashrightarrow Y' \xrightarrow{\sigma_Z} Z$ the contraction associated to $B$ (see 4.3), so that $Z$ is a smooth Fano 4-fold with $\rho_Z \geq 7$, and let $C_Z \subset Z$ be the center of the blow-up $\sigma_Z$. Let $\ell \subset Z$ be the exceptional curve corresponding to $L$, and $\Gamma \subset Z$ its image.

By [8.24] and Lemma 4.4 the cone $\text{Eff}(Z)$ is generated by classes of fixed prime divisors, thus there exists a fixed prime divisor $G_0 \subset Z$ such that $G_0 \cdot \Gamma > 0$. By Lemma 4.22 the transform $G \subset Y$ of $G_0$ is a fixed prime divisor adjacent to $B$ and having the same type as $G_0$.

If $G_0 \cap C_Z = \emptyset$, then $G_0$ cannot be of type $(3,2)$, otherwise as in 8.4 we get $G \cap B = \emptyset$, contradicting $(b')$. Then $G_0$ and $G$ are of type $(3,1)^{sm}$ by 8.13. Moreover $\sigma_Z^{-1}(G_0) \cdot \ell = G_0 \cdot \Gamma > 0$, thus we conclude that $G \cdot C_L < 0$ and $L \subset G$.

Suppose now that $G_0 \cap C_Z \neq \emptyset$. We show that there is another fixed prime divisor $P_0$ of $Z$ such that $P_0 \cdot \Gamma > 0$ and $P_0 \cap C_Z = \emptyset$.

By [Cas17, Lemma 5.11(1)] $G_0$ must be a fixed prime divisor of type $(3,2)$; let $R$ be the associated extremal ray $R$ of type $(3,2)$ of $\text{NE}(Z)$. Since $G_0 \cdot \Gamma > 0$, $\Gamma \not\in R$ and $\Gamma$ must intersect some curve of anticanonical degree one with class in $R$, and by Lemma 4.19 we conclude that $[C_Z] \in R$, $-K_Z \cdot C_Z = 1$, and $\Gamma \not\in G_0$.

Let us consider the contraction $f : Z \to W$ such that $\text{NE}(f) = R$, so that $G_0 = \text{Exc}(f)$, and set $\Gamma' := f(\Gamma)$.

There is a one-dimensional face $\eta$ of $\text{Eff}(W)$ such that $\eta \cdot \Gamma' > 0$. If $\eta$ is movable, then by Lemma 4.4 $\eta = f_*\eta_Z$ where $\eta_Z$ is a 2-dimensional movable face of $\text{Eff}(Z)$ containing $[G_0]$. Again by Lemma 4.4 we have $\eta_Z = (\sigma_Z)_*\eta_Y$ where $\eta_Y$ is a 3-dimensional movable face of $\text{Eff}(Y)$ containing $[G]$, contradicting 8.24.

Thus $\eta$ is fixed, and there is a fixed prime divisor $P \subset W$ such that $P \cdot \Gamma' > 0$, so that $f^*P \cdot \Gamma > 0$ in $Z$.

If $P_0 \subset Z$ is the transform of $P$, we have $f^*P = P_0 + mG_0$ with $m = P_0 \cdot C_Z$. On the other hand $P_0$ is fixed and adjacent to $G_0$ by Lemma 4.13(c), hence $m = 0$ by Lemma 4.13(a), and $P_0 \cdot \Gamma > 0$. Finally $C_Z \not\subset P_0$ by [Cas17, Lemma 5.11(2)], thus $P_0 \cap C_Z = \emptyset$.

8.26. For every fixed prime divisor $B \subset Y$ of type $(3,1)^{sm}$ let $n_B$ be the number of divisors among $E_1, \ldots, E_r$ which are adjacent to $B$, so that $n_B > 0$ by 8.21. Let us choose $B_1$ with $n_{B_1}$ minimal, namely such that $n_{B_1} \leq n_B$ for every fixed prime divisor $B \subset Y$ of type $(3,1)^{sm}$.

8.27. There exists $i \in \{1, \ldots, r\}$ such that $B_1 + E_i$ is movable and non-big.

Proof. By 8.21 there exists some $h \in \{1, \ldots, r\}$ such that $E_h$ is adjacent to $B_1$, hence $E_h \cdot C_{B_1} = B_1 \cdot C_{E_h} = 0$ by 8.19. Then by 8.22 there exists an exceptional plane $L_0 \subset B_1$ such that $C_{B_1} = C_{L_0} + C_{E_h}$.

We apply 8.25 to $B_1$ and $L_0$, and let $G \subset Y$ be a fixed prime divisor of type $(3,1)^{sm}$ adjacent to $B_1$ and such that $L_0 \subset G$. We have $G \cdot C_{B_1} = B_1 \cdot C_G = 0$ by Lemma 4.13(b).
By (8.28a) there exists \( i \in \{1, \ldots, r\} \) such that \( C_G \equiv C_{L_0} + C_{E_i} \); we have \( i \neq h \), because \( C_{B_1} \neq C_G \). Recall that \( E_i \cdot C_{E_h} = E_h \cdot C_{E_i} = 0 \) by (8.27). We get:

\[
(8.28a) \quad C_{B_1} + C_{E_i} \equiv C_{L_0} + C_{E_h} + C_{E_i} \equiv C_G + C_{E_h}
\]

\[
(8.28b) \quad B_1 \cdot C_{E_i} = B_1 \cdot (C_G + C_{E_h} - C_{B_1}) = 1
\]

\[
(8.28c) \quad G \cdot C_{E_h} = G \cdot (C_{B_1} + C_{E_i} - C_G) = G \cdot C_{E_i} + 1 \geq 1
\]

so that \( E_h \) is not adjacent to \( G \) and \( E_i \) is not adjacent to \( B_1 \) by (8.19).

Let \( j \in \{1, \ldots, r\} \setminus \{i, h\} \). Then \( E_j \cdot C_{E_i} = E_j \cdot C_{E_h} = 0 \) by (8.7), so that \( E_j \cdot C_{B_1} = E_j \cdot C_G \) by (8.28a), therefore \( E_j \) is adjacent to \( B_1 \) if and only if it is adjacent to \( G \), again by (8.19).

We conclude that either \( E_i \) is not adjacent to \( G \) and \( n_G = n_{B_1} - 1 \), or \( E_i \) is adjacent to \( G \) and \( n_G = n_{B_1} \) (see (8.26)). By the minimality of \( n_{B_1} \), we conclude that \( E_i \) must be adjacent to \( G \), so that \( E_i \cdot C_G = 0 \) by (8.19) and finally we get \( B_1 \cdot C_{E_i} = E_i \cdot C_{B_1} = 1 \) by (8.28b) and (8.28c).

By Lemma (4.12)(c), this implies that \( B_1 + E_i \) is movable and non-big.

**8.29.** Assume for simplicity that \( i = 1 \), and let \( f: Y \rightarrow Z \) be the rational contraction of fiber type defined by \( m_1(B_1 + E_1) \) for \( m_1 \in \mathbb{N} \), \( m_1 \gg 0 \). By Lemma 5.11 \((a')\), and \((b')\), we have that \( Z \equiv \mathbb{P}^2 \) and \( f \) is equidimensional.

Let \( \tau \) be the smallest face of \( \text{Eff}(Y) \) containing \( f^* \text{Nef}(\mathbb{P}^2) \), so that \( \tau \) is a movable face, \( f \) is a rational contraction associated to \( \tau \) as in (5.5) and \( \tau \cap \text{Mov}(Y) = f^* \text{Nef}(\mathbb{P}^2) \).

By (8.9) we can assume that \( \tau \) is a facet of \( \text{Eff}(Y) \) and that the general fiber of \( f \) is \( \mathbb{P}^2 \).

Let \( \Gamma_1, \ldots, \Gamma_m \subset \mathbb{P}^2 \) be the irreducible curves such that \( f^* \Gamma_i \) is reducible; we can assume that \( B_1 + E_1 = f^* \Gamma_1 \). By [Cas20] Lemma 5.2 we know that \( f^* \Gamma_i \) has two irreducible components, both fixed divisors, at least one of which of type \((3,2)\). We also have \( m = \rho_f - 2 \) by [Cas20] Cor. 2.16.

Recall from (8.7) that the fixed prime divisors of type \((3,2)\) of \( Y \) are \( E_1, \ldots, E_r \), and possibly \( E_0 \). If \( E_0 \) is a component of \( f^* \Gamma_i \) for some \( i \in \{1, \ldots, \rho_f - 2\} \), then for \( k \neq i \) \( f^* \Gamma_k \) must have as a component \( E_j \) for some \( j \in \{1, \ldots, r\} \). Then \( E_0 \) and \( E_j \) should be adjacent by [Cas20] Cor. 2.18, contradicting (8.8).

Therefore up to renumbering we can assume that \( E_i \) is a component of \( f^* \Gamma_i \), for every \( i = 1, \ldots, \rho_f - 2 \) (in particular \( r \geq \rho_f - 2 \)).

If \( j \in \{1, \ldots, r\} \), \( j \neq i \), we have \( E_j \cdot C_{E_i} = 0 \) by (8.7), so that \( E_i \) and \( E_j \) are adjacent by Lemma (4.12)(a); since \( f^* \Gamma_i \) is movable, the second component \( B_i \) of \( f^* \Gamma_i \) must be a fixed prime divisor of type \((3,1)^m\).

Note that \( \tau \) is generated by the classes of fixed prime divisors that do not dominate \( \mathbb{P}^2 \) under \( f \) (see (5.5)), so that:

\[
\tau = \langle [E_1], \ldots, [E_{\rho_f - 2}], [B_1], \ldots, [B_{\rho_f - 2}] \rangle.
\]

Moreover, by [Cas20] Cor. 2.18, for every partition \( \{1, \ldots, \rho_f - 2\} = I \cup J \) the cone \( \langle [E_i], [B_j] \rangle \) is a fixed face of \( \tau \). In particular \( B_i \) and \( E_j \) are adjacent for every \( i, j \in \{1, \ldots, \rho_f - 2\} \), \( i \neq j \), and \( E_j \cdot C_{B_i} = B_i \cdot C_{E_j} = 0 \) by (8.19).

We also have, for every \( i = 1, \ldots, \rho_f - 2 \), that \( \langle [E_i], [B_i] \rangle \cap \text{Mov}(Y) = f^* \text{Nef}(\mathbb{P}^2) \), therefore \( E_i \cdot C_{B_i} = B_i \cdot C_{E_i} = 1 \) by Lemma (4.12)(c). Hence for every \( j \in \{1, \ldots, \rho_f - 2\} \) we have \( E_j \cdot (C_{E_j} + C_{B_j}) = B_j \cdot (C_{E_j} + C_{B_j}) = 0 \), so that \( \tau \) lies on the hyperplane
(\(C_{E_i} + C_{B_i}\))\(^\perp\), and finally
\[
(8.30) \quad \tau = (C_{E_i} + C_{B_i})\(^\perp\) \cap \text{Eff}(Y).
\]

**8.31.** Every fixed prime divisor \(P \subset Y\) of type \((3,1)^{\text{sm}}\) can be not adjacent to at most two divisors among \(E_1, \ldots, E_r\).

Indeed by \(8.29\) \(B_1\) is adjacent to \(E_2, \ldots, E_{\rho_Y - 2}\), so that \(n_{B_1} \geq \rho_Y - 3\) (see \(8.26\)). By the minimality of \(n_{B_1}\), we deduce that \(n_P \geq \rho_Y - 3\) for every fixed prime divisor \(P \subset Y\) of type \((3,1)^{\text{sm}}\), namely \(P\) is adjacent to at least \(\rho_Y - 3\) divisors among \(E_1, \ldots, E_r\). Since \(r \leq \rho_Y - 1\) (see \(8.7\)), we conclude that \(P\) can be not adjacent to at most two divisors among \(E_1, \ldots, E_r\).

**8.32.** We consider now the fixed face \(\langle [B_1], \ldots, [B_{\rho_Y - 4}], [E_{\rho_Y - 3}], [E_{\rho_Y - 2}] \rangle\) of \(\tau\), of dimension \(\rho_Y - 2\); there exists a facet \(\eta\) of \(\text{Eff}(Y)\) such that
\[
\tau \cap \eta = \langle [B_1], \ldots, [B_{\rho_Y - 4}], [E_{\rho_Y - 3}], [E_{\rho_Y - 2}] \rangle.
\]

Let \(P \subset Y\) be a fixed prime divisor with \([P] \in \eta\setminus\langle [B_1], \ldots, [B_{\rho_Y - 4}], [E_{\rho_Y - 3}], [E_{\rho_Y - 2}] \rangle\); in particular \([P] \notin \tau\).

The possibilities for \(P\) are \(P = E_0\), \(P = E_{\rho_Y - 1}\) (and \(r = \rho_Y - 1\)), or \(P\) of type \((3,1)^{\text{sm}}\).

**8.33.** Either \(P = E_{\rho_Y - 1}\) or \(P\) is of type \((3,1)^{\text{sm}}\).

Indeed if \(P = E_0\), then \(\eta\) contains both \([E_0 + E_{\rho_Y - 3}]\) and \([E_0 + E_{\rho_Y - 2}]\) that are movable by \(8.32\). Let \(\eta_0\) be the minimal face of \(\eta\) containing \([2E_0 + E_{\rho_Y - 3} + E_{\rho_Y - 2}]\), so that \(\eta_0\) is movable and \([E_0 + E_{\rho_Y - 3}], [E_0 + E_{\rho_Y - 2}] \in \eta_0 \cap \text{Mov}(Y)\). Let \(g: Y \to W\) be a rational contraction of fiber type associated to \(\eta_0\) as in \(8.7\) so that \(\rho_W = \text{dim}(\eta_0 \cap \text{Mov}(Y)) \geq 2\) and \([E_0 + E_{\rho_Y - 3}] \in g^{*}N^1(W)\). Then Lemma \(5.11\) \((\alpha')\), and \((\beta')\) give a contradiction.

**8.34.** Up to reordering \(E_1, \ldots, E_{\rho_Y - 4}\), we have \(P \cdot C_{E_i} = 0\) for every \(i = 1, \ldots, \rho_Y - 6\).

This is clear by \(8.7\) if \(P = E_{\rho_Y - 1}\). If \(P\) is of type \((3,1)^{\text{sm}}\), then by \(8.31\) \(P\) can be not adjacent to at most two divisors among \(E_1, \ldots, E_{\rho_Y - 4}\), so up to renumbering we can assume that \(P\) is adjacent to \(E_1, \ldots, E_{\rho_Y - 6}\), and \(P \cdot C_{E_i} = 0\) for every \(i = 1, \ldots, \rho_Y - 6\) by \(8.19\).

**8.35.** We have \([P + B_i] \in \eta \cap \text{Mov}(Y)\) for every \(i = 1, \ldots, \rho_Y - 6\).

Indeed let \(i \in \{1, \ldots, \rho_Y - 6\}\). Clearly \([P + B_i] \in \eta\) because \([P], [B_i] \in \eta\). Moreover since \([P] \notin \tau\) (see \(8.32\)), we must have \(P \cdot (C_{E_i} + C_{B_i}) > 0\) by \(8.33\), and \(P \cdot C_{E_i} = 0\) by \(8.34\) thus \(P \cdot C_{B_i} > 0\). We also have \(B_i \cdot C_P > 0\); this follows from \(8.19\) if \(P = E_{\rho_Y - 1}\) and from Cor. 4.14 if \(P\) is \((3,1)^{\text{sm}}\). Finally \([P + B_i] \in \text{Mov}(Y)\) by Lemma 4.12(b).

**8.36.** Let \(\eta_1\) be the minimal face of \(\eta\) containing \([\rho_Y - 6]P + B_1 + \cdots + B_{\rho_Y - 6}\), so that \(\eta_1\) is movable, contains \([P], [B_1], \ldots, [B_{\rho_Y - 6}]\), and \([P + B_i] \in \eta_1 \cap \text{Mov}(Y)\) for every \(i = 1, \ldots, \rho_Y - 6\). Let \(h: Y \to S\) be a rational contraction of fiber type associated to \(\eta_1\) as in \(8.35\) so that \(\rho_S = \text{dim}(\eta_1 \cap \text{Mov}(Y)) \geq \rho_Y - 6 \geq 2\) and \(\eta_1\) is the smallest face of \(\text{Eff}(Y)\) containing \(h^{*} \text{Eff}(S)\).

Using Prop. 5.19 \((\alpha')\), and \((\beta')\), we see that we must be in case \((vi)\) of the Proposition, namely that \(\dim S = 2\) and \(h\) is quasi-elementary. Then \(h^{*} \text{Eff}(S)\) is a face of \(\text{Eff}(Y)\) by Casagrande Prop. 2.22; thus \(\eta_1 = h^{*} \text{Eff}(S) \cong \text{Eff}(S)\), and \(\rho_S = \text{dim}(\eta_1 \cap \text{Mov}(Y)) = \text{dim} \eta_1\). If \(\rho_S \leq 3\) we get \(\rho_Y \leq 9\) and \(\rho_X \leq 10\), so we can assume that \(\rho_S \geq 4\), and \(S\) is a smooth del Pezzo surface by Lemma 5.9.
8.37. Recall that \([B_1], \ldots, [B_{p_Y-4}] \in \eta \) and \([B_1], \ldots, [B_{p_Y-6}] \in \eta_1\). Up to exchanging \(B_{p_Y-5} \) and \(B_{p_Y-6} \) we can assume that for some \(t \in \{p_Y - 6, p_Y - 5, p_Y - 4\} \) we have \([B_1], \ldots, [B_t] \in \eta \) and that \([B_t] \notin \eta_1 \) for \(t = t + 1, \ldots, p_Y - 4\) (if \(t < p_Y - 4\)). Note that \(\langle [B_1], \ldots, [B_t] \rangle \) is a fixed face of \(\eta_1 \) (see 8.32), hence \(p_5 = \dim \eta_1 \geq t + 1\).

8.38. For every \(i = 1, \ldots, t\) we have \(B_i = h^*C_i \) where \(C_i \subseteq S\) is an irreducible curve. Since \(h^*([C_1], \ldots, [C_t]) = \langle [B_1], \ldots, [B_t] \rangle \) is a fixed face of \(\text{Eff}(Y), \langle [C_1], \ldots, [C_t] \rangle \) must be a fixed face of \(\text{Eff}(S)\), so that \(C_1, \ldots, C_t\) are pairwise disjoint \((-1)\)-curves.

We show that there exists a \((-1)\)-curve \(C' \subseteq S\) different from \(C_{t-1}\) and \(C_t\), and disjoint from \(C_1, \ldots, C_{t-2}\). In particular \(\langle [C'], [C_1], \ldots, [C_{t-2}] \rangle\) is a fixed face of \(\text{Eff}(S)\).

Indeed let \(\alpha: S \to S'\) be the contraction of the \((-1)\)-curves \(C_1, \ldots, C_{t-2}\), so that \(S'\) is a smooth del Pezzo surface with \(\rho_{S'} = \rho_S - (t - 2) \geq 3\). Then \(\alpha(C_{t-1})\) and \(\alpha(C_t)\) are \((-1)\)-curves in \(S'\), and there is a \((-1)\)-curve \(C'' \subseteq S'\) different from \(\alpha(C_{t-1})\) and \(\alpha(C_t)\).

Then \(C'\) is the transform of \(C''\) in \(S\).

8.39. Let us consider \(P' := h^*C'\), so that \(\langle [P'], [B_1], \ldots, [B_{t-2}] \rangle = h^*([C'], [C_1], \ldots, [C_{t-2}])\) is a fixed face of \(\text{Eff}(Y)\). Then \(P'\) is a fixed prime divisor in \(Y\), with class in \(\eta\) distinct from \(B_1\) and \(B_t\), and adjacent to \(B_1, \ldots, B_{t-2}\).

If \(P'\) is of type \((3, 2)\), we apply Lemma 8.11 and get a contradiction with (a') and (b'). Therefore \(P'\) is of type \((3, 1)^{m-}\), and \(P' \cdot C_{B_i} = 0\) for \(i = 1, \ldots, t - 2\) by Lemma 8.18(b). Moreover \(P'\) is different from \(B_1, \ldots, B_t\) by construction, from \(B_{t+1}, \ldots, B_{p_Y-4}\) (if \(t < p_Y - 4\)) because \([P'] \in \eta_1\), and from \(E_{p_Y-3}, E_{p_Y-2}\) because they are of type \((3, 2)\).

Thus \([P'] \notin \langle [B_1], \ldots, [B_{p_Y-4}], [E_{p_Y-3}], [E_{p_Y-2}] \rangle = \tau \cap \eta\), and finally \([P'] \notin \tau\).

By 8.30 we have \(0 < P' \cdot (C_{E_i} + C_{B_i}) = P' \cdot C_{E_i}\) for every \(i = 1, \ldots, t - 2\), so that by 8.19 \(P'\) is not adjacent to \(t - 2\) divisors among \(E_1, \ldots, E_r\). Therefore 8.31 yields \(t - 2 \leq 2, \rho_Y - 6 \leq t \leq 4, \rho_Y \leq 10, \) and \(\rho_X \leq 11\). This concludes the proof of Th. 8.11.

9. The case \((3, 0)^Q\)

In this section we conclude the proof of Th. 8.11 by considering the case of fixed prime divisors of type \((3, 0)^Q\); in fact we prove the following more refined version.

Theorem 9.1. Let \(X\) be a smooth Fano 4-fold having a small elementary contraction. Then \(\rho_X \leq 12\), and if \(\rho_X = 12\), then \(X\) has a rational contraction onto a 3-fold.

First we prove two preliminary lemmas which, given two adjacent fixed prime divisors of type \((3, 0)^Q\), yield either \(\rho_X \leq 12\), or the existence of some special exceptional planes. Then we deal with the final proof; let us give an outline.

Let \(X\) be a smooth Fano 4-fold with \(\rho_X \geq 7\) and having a small elementary contraction. Using the previous results we can assume that every fixed prime divisor of \(X\) is of type \((3, 2)\) or \((3, 0)^Q\) (with at least one \(D\) of type \((3, 0)^Q\)), that every 2-dimensional face of \(\text{Eff}(X)\) is fixed, and finally that we are in one of the following cases:

(a) every fixed prime divisor of \(X\) is of type \((3, 0)^Q\);
(b) \(D\) has one adjacent fixed prime divisor of type \((3, 2)\), and the other divisors adjacent to \(D\) are of type \((3, 0)^Q\).

These two cases will be treated in parallel; for simplicity let us consider now case (a). We work with the fixed prime divisors \(E_1, \ldots, E_r\) which are adjacent to \(D\). We also consider the classes \(\gamma_1, \ldots, \gamma_l \in \mathcal{N}_1(X)\) of all lines in the exceptional planes contained
in $D$. We show that for $h = 1, \ldots , r$ the hyperplane $E_h^{\perp} \subset \mathcal{N}_1(X)$ must contain $\rho_X - 2$ classes among $\gamma_1, \ldots , \gamma_r$, and for $j \neq h$, $E_j^{\perp}$ must contain a different subset of $\rho_X - 2$ classes. Then we develop more conditions on the $\gamma_i$'s and the possible subsets of $\rho_X - 2$ classes contained in the $E_h^{\perp}$'s, in order to get the statement by applying the preliminary lemmas.

**Lemma 9.2.** Let $X$ be a smooth Fano 4-fold with $\rho_X \geq 8$ and $\delta_X \leq 1$, and let $D, E$ be two adjacent fixed prime divisors of type $(3, 0)^Q$ in $X$. Let $L \subset E$ be an exceptional plane such that $L \cap D = \emptyset$. Then one of the following holds:

(i) $\rho_X \leq 12$, and if $\rho_X = 12$, then $X$ has a rational contraction onto a 3-fold;

(ii) there exists an exceptional plane $M \subset D$ such that $C_M + C_E \equiv C_D + C_L$ and $D \cdot C_M = -1$.

**Proof.** Let $V$ be the family of lines in $X$ given by $E$ and $L$ as in Prop. 6.1, so that $C_E \equiv C_L + [V]$ and $B := \text{Locus } V$ is a divisor. Since $L \cap D = \emptyset$ we have $D \cdot C_L = 0$, moreover $D \cdot C_E = 0$ by Lemma 4.13(b), hence $D \cdot [V] = 0$, and the general curve of the family $V$ is disjoint from $D$.

Let $X \xrightarrow{\varphi} \tilde{X} \xrightarrow{\sigma} Y$ be the contraction associated to $D$ as in §3.13, so that $\text{Exc}(\sigma) = \tilde{D}$ and $\sigma(\tilde{D}) = p$. Then the birational map $X \dashrightarrow Y$ is an isomorphism on the general curve of the family $V$, and $V$ yields a family of lines $V_Y$ in $Y$, with locus a prime divisor $B_Y \subset Y$ which is the transform of $B \subset X$; we also consider the transform $\tilde{B}$ of $B$ in $\tilde{X}$.

Recall that for every exceptional plane $L' \subset D$ we have $D \cdot C_{E'} < 0$, while $D \cdot [V] = 0$, hence $[V] \neq [C_{L'}]$ and $L' \not\subset B$ by Lemma 3.11(b). Therefore neither $B \cap D$ nor $\tilde{B} \cap \tilde{D}$, if non-empty, can be contained in the indeterminacy locus of $\varphi$ or $\varphi^{-1}$, and we conclude that $B \cap D \neq \emptyset$ if and only if $\tilde{B} \cap \tilde{D} \neq \emptyset$, if and only if $p \in B_Y$.

Suppose that $B \cap D = \emptyset$. If $B$ is not nef, then $B$ is fixed of type $(3, 2)$ by Lemma 3.12 and $\mathcal{N}_1(B, X) \subset D^{\perp} \subset \mathcal{N}_0(X)$ (see Rem. 2.1), so we get (i) by Th. 1.8. If $B$ is nef, then $B_Y$ is nef too (because $B$ is contained in the open subset where the birational map $X \dashrightarrow Y$ is an isomorphism), and we get again (i) by Prop. 7.1.

Finally suppose that $B \cap D \neq \emptyset$, so that $p \in B_Y$. Let $\Gamma$ be a general curve of the family $V_Y$, and $\Gamma_0$ a curve of the family containing $p$. By Rem. 4.16 we have $\Gamma_0 = \sigma(\ell)$, where $\ell \subset X$ is an exceptional curve such that $D \cdot \ell = 1$; moreover $\tilde{\Gamma} \equiv \ell + C_{\tilde{D}}$, where $\tilde{\Gamma} \subset \tilde{X}$ is the transform of $\Gamma$. Now if $M \subset D \subset X$ is the exceptional plane corresponding to $\ell$, we have $D \cdot C_M = -D \cdot \ell = -1$, and $[V] \equiv \lambda C_M + C_D$ where $\lambda \in \mathbb{R}$; intersecting with $-K_X$ we get $C_D \equiv [V] + C_M$. This yields $C_M + C_E \equiv C_D + C_L$, so we get (ii). \hfill \box

For simplicity we set $\rho := \rho_X$ for the rest of the section.

**Lemma 9.3.** Let $X$ be a smooth Fano 4-fold with $\rho \geq 7$ and $\delta_X \leq 1$, and let $D, E$ be two adjacent fixed prime divisors of type $(3, 0)^Q$. Suppose that one of the following holds:

(a) every fixed prime divisor of $X$ is of type $(3, 0)^Q$;

(b) there is a fixed prime divisor $F$, of type $(3, 2)$, such that $([D], [E], [F])$ is a fixed face of $\text{Eff}(X)$.

Then one of the following holds:

(i) $\rho \leq 12$, and if $\rho = 12$, then $X$ has a rational contraction onto a 3-fold;
(ii) there exist \( L_1, \ldots, L_{\rho - 2} \subset E \) exceptional planes, disjoint from \( D \), such that 
\[ \{ C_E \}, \{ C_{L_1} \}, \ldots, \{ C_{L_{\rho - 2}} \} \] is a basis of \( D^\perp \). In case (b), we can moreover assume that \( C_E \equiv C_F + C_{L_1} \) and \( \{ C_E \}, \{ C_F \}, \{ C_{L_2} \}, \ldots, \{ C_{L_{\rho - 2}} \} \) is a basis of \( D^\perp \).

Proof. We note that \( X \) is not a product of surfaces (see Ex. 4.6 or Rem. 2.28). Let \( X \to \tilde{X} \to Y \) be the contraction associated to \( D \) as in 3.13 and \( E_Y \subset \tilde{X} \), \( E_Y \subset Y \) the transforms of \( E \); we still denote by \( C_E \subset E_{\tilde{X}} \), the transform of \( C_E \). By Lemma 4.13(b) we know that \( D \cdot C_E = 0 \) and \( D \cap E \) is either empty, or a union of exceptional planes; on the other hand \( D \cap E_{\tilde{X}} = \emptyset \) in \( \tilde{X} \).

If \( N_1(E_Y, Y) \not\subseteq N_1(Y) \), we apply Lemma 4.12. Either \( X \) has a rational contraction onto a 3-fold and we get (i) by Th. 3.8, or there is a fixed prime divisor \( G \subset X \), of type (3, 2), adjacent to \( D \), and such that \( E \cdot C_G > 0 \). In particular we are in case (b), and \( G \not\equiv F \) because \( E \cdot C_F = 0 \) by Lemma 4.13(a). Then \( D \) is adjacent to two fixed prime divisors of type (3, 2), and we get again (i) by Lemma 7.2.

Otherwise \( N_1(E_Y, Y) = N_1(Y) \), and since \( N_1(E_Y, Y) = \sigma_* (N_1(E_{\tilde{X}}, \tilde{X})) \), we get \( \dim N_1(E_{\tilde{X}}, \tilde{X}) \geq \rho - 1 \). Since \( \tilde{D} \cap E_{\tilde{X}} = \emptyset \), we have \( N_1(E_{\tilde{X}}, \tilde{X}) = \tilde{D}^\perp \) (see Rem. 2.1). Note that \( \{ C_{\tilde{D}} \} \not\subseteq N_1(E_{\tilde{X}}, \tilde{X}) \) because \( \tilde{D} \cdot C_{\tilde{D}} = -1 \).

There is a SQM \( \tilde{X} \to X \), obtained by considering the \( E_{\tilde{X}} \)-negative small extremal rays of \( \operatorname{NE}(\tilde{X}) \), whose indeterminacy locus is the union of the exceptional planes contained in \( E_{\tilde{X}} \), and that has a factorization as in 2.3. Moreover the transform \( E_{\tilde{X}} \subset \tilde{X} \) is the exceptional locus of an elementary divisorial contraction of type \( (3, 0)^Q \).

Then, after Cas13a, Rem. 3.13], as in Rem. 4.17, we see that \( E_{\tilde{X}} \) contain at least \( \dim N_1(E_{\tilde{X}}, \tilde{X}) = \rho - 2 \) exceptional planes \( L_1, \ldots, L_{\rho - 2} \) such that \( \{ C_E \}, \{ C_{L_1} \}, \ldots, \{ C_{L_{\rho - 2}} \} \) is a basis of \( N_1(E_{\tilde{X}}, \tilde{X}) \), so that \( \{ C_{\tilde{D}} \}, \{ C_E \}, \{ C_{L_1} \}, \ldots, \{ C_{L_{\rho - 2}} \} \) is a basis of \( N_1(\tilde{X}) \). Note that each \( L_i \) is disjoint from \( \tilde{D} \) and also from every exceptional curve of \( \tilde{X} \), by Lemma 2.4(b), hence \( L_i \) is contained in the open subset where \( \varphi : X \to \tilde{X} \) is an isomorphism, and its transform in \( X \) is still an exceptional plane \( L_i \subset E \) such that \( L_i \cap D = \emptyset \). Moreover \( \{ C_E \}, \{ C_{L_1} \}, \ldots, \{ C_{L_{\rho - 2}} \} \) are linearly independent in \( N_1(X) \), and they belong to \( D^\perp \).

In case (b), if \( F \cap E = \emptyset \), then \( N_1(F, X) \not\subseteq N_1(X) \) (see Rem. 2.1), and we get again (i) by Th. 4.3. Suppose that \( F \cap E \neq \emptyset \); we have \( E \cdot C_F = 0 \) by Lemma 4.13(a), and by Lemma 6.4 there exists an exceptional plane \( L \subset E \) such that \( C_E \equiv C_F + C_L \).

Moreover we have \( D \cdot C_F = 0 \) again by Lemma 4.13(a), thus \( 0 = D \cdot C_E = D \cdot (C_F + C_L) = D \cdot C_L \). If \( L \subset D \) we have \( D \cdot C_L < 0 \) (see 4.3), therefore \( L \cap D = \emptyset \) and \( L \) is contained in the open subset where \( \varphi : X \to \tilde{X} \) is an isomorphism. Then we just choose \( L_1 = L \) in the previous construction, and get (ii).

Proof of Theorem 9.1

9.4. We assume that \( \rho \geq 8 \). Since \( X \) has a small elementary contraction, it is not a product of surfaces, and we can assume that \( \delta_X \leq 1 \) by Theorems 2.4 and 2.7. We can also assume that \( X \) has no fixed prime divisor of type \((3, 0)^{sm} \) or \((3, 1)^{sm} \), by Theorems 4.7 and 8.1, so that every fixed prime divisor of \( X \) is of type \((3, 0)^{Q} \) or \((3, 2) \).

By Prop. 6.16 we can assume that every 2-dimensional face \( \tau \) of \( \operatorname{Eff}(X) \) is fixed, and that if \( \tau \) is a 3-dimensional non-fixed face of \( \operatorname{Eff}(X) \), then \( \tau \) is generated by classes of
fixed prime divisors of type $(3, 2)$. In particular $\text{Eff}(X)$ is generated by classes of fixed prime divisors.

By assumption $\text{NE}(X)$ has a small extremal ray $R$, and there exists some fixed prime divisor having negative intersection with $R$; moreover it cannot be of type $(3, 2)$ (see Rem. 1.11), thus there exists at least one fixed prime divisor of type $(3, 0)^Q$. By Lemma 7.2 we can assume that we are in one of the following cases:

(a) every fixed prime divisor of $X$ is of type $(3, 0)^Q$;
(b) there is a fixed prime divisor $D$ of type $(3, 0)^Q$ with one adjacent fixed prime divisor $F$ of type $(3, 2)$, and the other fixed prime divisors adjacent to $D$ are of type $(3, 0)^Q$.

9.5. Let $\tau$ be a 3-dimensional face of $\text{Eff}(X)$ containing the class of some fixed prime divisor of type $(3, 0)^Q$. Then $\tau$ is fixed by 9.4 so it is simplicial (see Lemma 4.2), and by Lemma 7.2 we can assume that $\tau$ contains at most one class of a fixed prime divisor of type $(3, 2)$.

9.6. In case (b), we can assume that every 4-dimensional face $\eta$ of $\text{Eff}(X)$ containing $[D]$ and $[F]$ is fixed and simplicial.

Proof. Let $\tau$ be a facet of $\eta$ containing $[D]$. Then by 9.5 $\tau$ is fixed and contains at most one class of a fixed prime divisor of type $(3, 2)$. Thus every facet $\tau'$ of $\eta$ such that $\dim(\tau \cap \tau') = 2$ still contains the class of a fixed prime divisor of type $(3, 0)^Q$.

Proceeding in this way, we conclude that every facet of $\eta$ is fixed.

Now if $\eta$ is not fixed, it is a minimal movable face, and applying Prop. 5.16 we get the statement. If instead $\eta$ is fixed, it is simplicial too (see Lemma 4.2).

9.7. Let us fix the following notation. In case (a), we fix a $D$ of type $(3, 0)^Q$, and $E_1, \ldots, E_r$ are the fixed prime divisors adjacent to $D$. In case (b), $E_1, \ldots, E_r$ are the fixed prime divisors such that $\langle [D], [E_i], [F] \rangle$ is a fixed face of $\text{Eff}(X)$.

In both cases $E_1, \ldots, E_r$ are all of type $(3, 0)^Q$ (in case (b) we use 9.5). Note that $E_i \cdot C_D = D \cdot C_{E_i} = 0$ for all $i$ by Lemma 4.13(b). Moreover, by 9.4 in case (a) $\langle [D], [E_i] \rangle$ are all the 2-dimensional faces of $\text{Eff}(X)$ containing $[D]$, and in case (b) $\langle [D], [E_i], [F] \rangle$ are all the 3-dimensional faces of $\text{Eff}(X)$ containing $\langle [D], [F] \rangle$.

Suppose that we are in case (b); similarly as above we have $D \cdot C_F = E_i \cdot C_F = 0$ for all $i$. Moreover if $F$ is disjoint from one of the divisors $D, E_1, \ldots, E_r$, then $\mathcal{N}_1(F, X) \subseteq \mathcal{N}_1(X)$ (see Rem. 2.4) and we conclude by Th. 4.8. Thus we can assume that $F$ intersects all divisors $D, E_1, \ldots, E_r$, and by Cor 6.10 we deduce that $F \cdot C_D = F \cdot C_{E_i} = 1$ for all $i$. Summing up we have:

$$D \cdot C_{E_i} = E_i \cdot C_D = D \cdot C_F = E_i \cdot C_F = 0 \quad \text{and} \quad F \cdot C_D = F \cdot C_{E_i} = 1 \quad \text{for all } i.$$  (9.8)

9.9. We have $r \geq \rho$ in case (a), and $r \geq \rho - 1$ in case (b).

Proof. Note that $r \geq \rho - 1$ and $r \geq \rho - 2$ respectively for dimensional reasons, because $\text{Eff}(X)$ is a cone of dimension $\rho$, thus $[D]$ must be contained in at least $\rho - 1$ faces of dimension 2, and similarly in case (b).

In case (a), assume that $r = \rho - 1$. Since every 3-dimensional face of $\text{Eff}(X)$ is fixed and simplicial by 9.4 we conclude that for every $i \neq j$ the cone $\langle [D], [E_i], [E_j] \rangle$ is a face of $\text{Eff}(X)$, in particular $E_i$ and $E_j$ are adjacent, hence $E_i \cdot C_{E_j} = 0$ by Lemma 4.13(b).
Now the classes of $D, E_1, \ldots, E_{\rho-1}$ form a basis of $\mathcal{N}^1(X)$, so that we can write

$$-K_X \equiv a_0D + \sum_{i=1}^{\rho-1} a_iE_i$$

with $a_i \in \mathbb{R}$. Using (9.8), intersecting with $C_D$ we get $a_0 = -2$, and intersecting with $C_{E_j}$ we get $a_j = -2$ for $j = 1, \ldots, \rho - 1$, which gives a contradiction.

Case (b) is similar. Suppose that $r = \rho - 2$, and recall that by (9.6) every 4-dimensional face of $\text{Eff}(X)$ containing $[D]$ and $[F]$ is fixed and simplicial. Then for every $i \neq j$ the cone $\langle [D], [F], [E_i], [E_j] \rangle$ is a face of $\text{Eff}(X)$, in particular $E_i$ and $E_j$ are adjacent, hence $E_i \cdot C_{E_j} = 0$. Now the classes of $D, F, E_1, \ldots, E_{\rho-2}$ form a basis of $\mathcal{N}^1(X)$, so that we can write

$$-K_X \equiv a_0D + \sum_{i=1}^{\rho-2} a_iE_i + bF$$

with $a_i, b \in \mathbb{R}$. Using (9.8), intersecting with $C_F$ we get $b = -1$, intersecting with $C_D$ we get $a_0 = b - 2 = -3$ and intersecting with $C_{E_j}$ we get $a_j = b - 2 = -3$, $j = 1, \ldots, \rho - 2$, again a contradiction.

9.10. Let us consider all the exceptional planes contained in $D$, and let $\gamma_1, \ldots, \gamma_t \in \mathcal{N}_1(X)$ be the distinct classes of all lines in these exceptional planes. We have $D \cdot \gamma_i < 0$ for every $i$ (see (4.3); let us reorder the $\gamma_i$’s in such a way that:

$$(9.11) \quad D \cdot \gamma_i = -1 \text{ for } i = 1, \ldots, s \text{ and } D \cdot \gamma_i \leq -2 \text{ for } i = s + 1, \ldots, t.$$  

In case (b), we can assume that $C_D \equiv C_F + \gamma_1$. Indeed we have $D \cap F \neq \emptyset$ and $D \cdot C_F = 0$ (see (9.7), and by Lemma 9.9 there exists $i \in \{1, \ldots, s\}$ such that $C_D \equiv C_F + \gamma_i$; up to renumbering we can assume that $i = 1$.

9.12. Let us consider $E_h$ with $h \in \{1, \ldots, r\}$; we apply Lemma 9.8 to $D$ and $E_h$ (and $F$ in case (b)). Either we get the statement, or $E_h^\perp$ must be generated $[C_D]$ and by $\rho - 2$ classes among $\gamma_1, \ldots, \gamma_t$.

If we choose some $E_j$ different from $E_h$, then the classes $[E_h]$ and $[E_j]$ cannot be multiples, hence the hyperplanes $E_h^\perp$ and $E_j^\perp$ are different. Thus $E_j^\perp$ is generated by $[C_D]$ and by a different choice of $\rho - 2$ among the classes $\gamma_1, \ldots, \gamma_t$.

In case (a), using (9.9) we get:

$$\rho \leq r \leq \left( \frac{t}{\rho - 2} \right),$$

which implies that $t \geq \rho$.

Similarly, in case (b), since $C_D \equiv C_F + \gamma_1$, by Lemma 9.3 we have also that $E_h^\perp$ must be generated $[C_D], [C_F]$, and by $\rho - 3$ classes among $\gamma_2, \ldots, \gamma_t$. Then we get

$$\rho - 1 \leq r \leq \left( \frac{t - 1}{\rho - 3} \right),$$

which implies again that $t \geq \rho$.

9.13. Now we apply Lemma 9.3 to $E_1$ and $D$ (with the roles interchanged with respect to 9.12), and $F$ in case (b). Either we get the statement, or there exist exceptional planes $L_1, \ldots, L_{\rho-2} \subset E_1$ such that $L_j \cap D = \emptyset$ and the classes $[C_{L_1}], \ldots, [C_{L_{\rho-2}}]$ are all distinct.
9.14. We apply Lemma 9.12 to $D$, $E_1$, and $L_j \subset E_1$, with $j \in \{1, \ldots, \rho - 2\}$. Either we get the statement, or there exists an exceptional plane $M_j \subset D$ such that $C_{M_j} \equiv C_D + C_{L_j} - C_{E_1}$ and $D \cdot C_{M_j} = -1$.

In particular the classes $[C_{M_j}]$, for $j = 1, \ldots, \rho - 2$, are all distinct, and must appear among $\gamma_1, \ldots, \gamma_s$ (see 9.11); we deduce that $s \geq \rho - 2$.

9.15. Fix $i \in \{1, \ldots, s\}$, so that $D \cdot \gamma_i = -1$. Let $P_i \subset D$ be an exceptional plane whose lines have class $\gamma_i$, and let $V_i$ be the family of lines in $X$ given by $D$ and $P_i$ as in Prop. 6.1, with locus a prime divisor $B_i \neq D$, so that $C_D \equiv \gamma_i + [V_i]$, $B_i \cdot \gamma_i > 0$, and $P_i \nsubseteq B_i$. Moreover by Lemma 6.3 $B_i$ cannot meet any exceptional plane $L \subset D$ such that $C_L \neq \gamma_i$, so that $B_i \cdot \gamma_j = 0$ for every $j \in \{1, \ldots, t\}$, $j \neq i$. In particular $B_i \cap D$ is not a union of exceptional planes, and $B_i \cdot C_D > 0$ by Rem. 4.26. Summing up we have:

$$C_D \equiv \gamma_i + [V_i], \ B_i \cdot C_D > 0, \ B_i \cdot \gamma_i > 0, \text{ and } B_i \cdot \gamma_j = 0 \text{ for every } j \in \{1, \ldots, t\}, \ j \neq i.$$

9.16. We deduce that both $\gamma_1, \ldots, \gamma_s \in \mathcal{N}_1(X)$ and $[B_1], \ldots, [B_s] \in \mathcal{N}^1(X)$ are linearly independent, and that

$$\gamma_{s+1}, \ldots, \gamma_t \in B_1 \cap \cdots \cap B_s.$$

Moreover note that $(-K_X + D) \cdot \gamma_i = 0$ for every $i = 1, \ldots, s$, and since $-K_X + D \neq 0$, we have $s \leq \rho - 1$, namely $s \in (\rho - 2, \rho - 1)$ (see 9.14).

9.17. We will need the following estimation. For every $i = 1, \ldots, s$ we have $B_i \cdot [V_i] \geq -1$ (see Lemma 9.12) and $0 < B_i \cdot \gamma_i = B_i \cdot C_D - B_i \cdot [V_i] \leq B_i \cdot C_D + 1$, thus

$$\frac{B_i \cdot C_D}{B_i \cdot \gamma_i} \geq \frac{B_i \cdot C_D}{B_i \cdot C_D + 1} \geq \frac{1}{2} \text{ and } M := \sum_{i=1}^{s} \frac{B_i \cdot C_D}{B_i \cdot \gamma_i} \geq \frac{1}{2^s}.$$

9.18. We show that $s = \rho - 2$.

Otherwise, we have $s = \rho - 1$ by 9.16. Note that $(-K_X + D)^\perp$ is generated by $\gamma_1, \ldots, \gamma_{\rho - 1}$ (see 9.10), and $(-K_X + D) \cdot C_D = 1$, so that $\gamma_1, \ldots, \gamma_{\rho - 1}, [C_D]$ is a basis of $\mathcal{N}_1(X)$. Recall that $t \geq \rho$ by 9.12 and write

$$\gamma_\rho = \sum_{i=1}^{\rho - 1} a_i \gamma_i + a_0 [C_D]$$

with $a_i \in \mathbb{R}$. By intersecting with $B_j$ for $j \in \{1, \ldots, \rho - 1\}$ we get (see 9.15)

$$a_j = -a_0 \frac{B_j \cdot C_D}{B_j \cdot \gamma_j} \text{ and } \gamma_\rho = a_0 \left( -\sum_{i=1}^{\rho - 1} \frac{B_i \cdot C_D}{B_i \cdot \gamma_i} \gamma_i + [C_D] \right).$$

Intersecting with $-K_X$ we get $1 = a_0(-M + 2)$, where as in 9.17 we have $M \geq s/2 = (\rho - 1)/2 \geq 7/2$. Hence

$$\gamma_\rho = -\frac{1}{M - 2} \left( -\sum_{i=1}^{\rho - 1} \frac{B_i \cdot C_D}{B_i \cdot \gamma_i} \gamma_i + [C_D] \right),$$

and finally intersecting with $D$ we get (see 9.11):

$$-2 \geq D \cdot \gamma_\rho = -\frac{M - 1}{M - 2},$$

which yields $M \leq 3$, a contradiction. Therefore $s = \rho - 2$. 

9.19. Let us consider the plane
\[ \pi := B_1^+ \cap \cdots \cap B_{\rho-2}^+ \subset N_1(X). \]
We have \( \gamma_{\rho-1}, \ldots, \gamma_t \in \pi \) (see (9.16)), and since \(-K_X \cdot \gamma_i = 1\) for every \(i\), these classes cannot be proportional. Recall that \(t \geq \rho\) by (9.12) so there are at least two such classes.

Let us consider the 2-dimensional cone \( \langle \gamma_{\rho-1}, \ldots, \gamma_t \rangle \subset \pi\); up to renumbering we can assume that \( \gamma_{\rho-1} \) and \( \gamma_\rho \) generate the cone, that \( D \cdot \gamma_{\rho-1} \leq D \cdot \gamma_\rho \), and that if \( t > \rho \) the remaining classes \( \gamma_{\rho+1}, \ldots, \gamma_t \) belong to the interior of the cone.

9.20. Suppose that \( D \cdot \gamma_{\rho-1} = D \cdot \gamma_\rho \) and set \( m := -D \cdot \gamma_{\rho-1} \geq 2 \) (see (9.11)). Then \((-mK_X + D) \cdot \gamma_{\rho-1} = (-mK_X + D) \cdot \gamma_\rho = 0\), hence \((-mK_X + D) \perp \pi\) and
\[ -mK_X + D \equiv \sum_{i=1}^{\rho-2} \lambda_i B_i \]
with \( \lambda_i \in \mathbb{R} \). Intersecting with \( \gamma_j, j \in \{1, \ldots, \rho - 2\} \), we get \( m - 1 = \lambda_j B_j \cdot \gamma_j \) and
\[ -mK_X + D \equiv (m - 1) \sum_{i=1}^{\rho-2} \frac{1}{B_i} \cdot \gamma_i B_i. \]
Then intersecting with \( C_D \) we get \( 2m - 1 = (m - 1)M \) where \( M \geq s/2 = (\rho - 2)/2 \) by (9.17). Since \( m \geq 2 \), we get \( M \leq 3 \) and \( \rho = 8 \).

We assume from now on that \( D \cdot \gamma_{\rho-1} < D \cdot \gamma_\rho \leq -2 \), so that \( D \cdot \gamma_{\rho-1} \leq -3 \).

9.21. Let \( h \in \{1, \ldots, r\} \). Since \( E_h \) is adjacent to \( D \), \( E_h \cap D \) is either empty or a disjoint union of exceptional planes, by Lemma (4.13)(b). Thus for every \( i \in \{1, \ldots, t\} \), if \( P_i \subset D \) is an exceptional plane whose lines have class \( \gamma_i \), we have either \( P_i \cap E_h = \emptyset \) and \( E_h \cdot \gamma_i = 0 \), or \( P_i \subset E_h \) and \( E_h \cdot \gamma_i < 0 \) (see (8.3)); in any case we have:
\[ E_h \cdot \gamma_i \leq 0. \]

9.22. We show that \( E_h^\perp \) does not contain the plane \( \pi \). Suppose otherwise: then there exist \( \lambda_i \in \mathbb{R} \) such that \( E_h \equiv \sum_{i=1}^{\rho-2} \lambda_i B_i \), and for every \( i \in \{1, \ldots, \rho - 2\} \) we get
\[ 0 \geq E_h \cdot \gamma_i = \lambda_i B_i \cdot \gamma_i \]
thus \( \lambda_i \leq 0 \); this is impossible because \( E_h \) is effective and non-zero.

Therefore \( E_h^\perp \) can contain at most one of the classes \( \gamma_{\rho-1}, \ldots, \gamma_t \in \pi \). On the other hand, since \( E_h \cdot \gamma_i \leq 0 \) for every \( i \), \( E_h^\perp \) must cut the cone \( \langle \gamma_{\rho-1}, \gamma_\rho \rangle \) along a face, and we conclude that \( E_h^\perp \cap \{ \gamma_{\rho-1}, \ldots, \gamma_t \} \) is either empty, or \( \gamma_{\rho-1} \), or \( \gamma_\rho \).

9.23. Recall that by (9.12) \( E_h^\perp \) must contain \( \rho - 2 \) classes among \( \gamma_1, \ldots, \gamma_t \), and in case (b) it must contain \( \rho - 3 \) classes among \( \gamma_2, \ldots, \gamma_t \). Thus in case (a) \( E_h^\perp \) must contain either \( \gamma_1, \ldots, \gamma_{\rho-2} \), or \( \gamma_1, \ldots, \gamma_t \), or \( \gamma_1, \ldots, \gamma_t, \ldots, \gamma_{\rho-2}, \gamma_\rho \), with \( i \in \{1, \ldots, \rho - 2\} \); similarly in case (b) \( E_h^\perp \) must contain either \( \gamma_2, \ldots, \gamma_{\rho-2} \), or \( \gamma_2, \ldots, \gamma_t \), \( \gamma_2, \ldots, \gamma_t, \ldots, \gamma_{\rho-2}, \gamma_\rho \), with \( i \in \{2, \ldots, \rho - 2\} \).

Again by (9.12) the set of \( \gamma_i \)'s contained in \( E_h^\perp \) must vary when \( h \) varies, and by (9.9) we have \( r \geq \rho \) in case (a) and \( r \geq \rho - 1 \) in case (b). Therefore there exists \( h_0 \in \{1, \ldots, r\} \) such that \( E_h_0 \cdot \gamma_{\rho-1} = 0 \).
9.24. Consider an exceptional plane $P_{\rho-1} \subset D$ whose lines have class $\gamma_{\rho-1}$. We have $E_{h_0} \cdot \gamma_{\rho-1} = 0$, therefore $P_{\rho-1} \not\subset E_{h_0}$ (otherwise $E_{h_0} \cdot \gamma_{\rho-1} < 0$, see 4.3) and hence $P_{\rho-1} \cap E_{h_0} = \emptyset$.

We apply Lemma 9.2 to $D$, $E_{h_0}$, and $P_{\rho-1} \subset D$. Either we get the statement, or there exists an exceptional plane $N \subset E_{h_0}$ such that $C_D + C_N \equiv C_{E_{h_0}} + \gamma_{\rho-1}$ (note that $[C_N] \neq \gamma_{\rho-1}$). We show that this last case leads to a contradiction.

9.25. We have $D \cdot C_N = D \cdot (C_{E_{h_0}} + \gamma_{\rho-1} - C_D) = D \cdot \gamma_{\rho-1} + 1 \leq -2$ by (9.8) and so that $N \subset D$ and $[C_N] = \gamma_j$ for some $j \in \{\rho, \ldots, t\}$. Note that all the classes $\gamma_{\rho-1}, \ldots, \gamma_t$ have intersection 1 with $-K_X$, so they belong to the segment from $\gamma_{\rho-1}$ to $\gamma_{\rho}$ in the plane $\pi$, and there exists some $\lambda \in (0, 1]$ such that

$$[C_N] = (1 - \lambda)\gamma_{\rho-1} + \lambda\gamma_{\rho}.$$ 

Intersecting with $D$ we get

$$D \cdot \gamma_{\rho-1} + 1 = D \cdot C_N = (1 - \lambda)D \cdot \gamma_{\rho-1} + \lambda D \cdot \gamma_{\rho} = D \cdot \gamma_{\rho-1} + \lambda(D \cdot \gamma_{\rho} - D \cdot \gamma_{\rho-1})$$

and

$$\lambda = \frac{1}{D \cdot \gamma_{\rho} - D \cdot \gamma_{\rho-1}}$$

so that $[C_N]$ is uniquely determined and $[C_{E_{h_0}}] = [C_D] + [C_N] - \gamma_{\rho-1}$ is uniquely determined too. This means that $E_{h_0}$ is the unique divisor in $\{E_1, \ldots, E_r\}$ such that $E_{h_0} \cdot \gamma_{\rho-1} = 0$.

9.26. In case (a), we deduce that $r = \rho$, and up to renumbering we can assume that $h_0 = \rho - 1$ and:

- $E_{i-}^\perp$ contains $\gamma_1, \ldots, \gamma_i, \ldots, \gamma_{\rho-2}, \gamma_{\rho}$, for $i = 1, \ldots, \rho - 2$;
- $E_{i-1}^\perp$ contains $\gamma_2, \ldots, \gamma_{\rho-2}, \gamma_{\rho-1}$;
- $E_{i-}^\perp$ contains $\gamma_1, \ldots, \gamma_{\rho-2}$.

Then $E_1^\perp$, $E_{\rho-1}^\perp$ and $E_{\rho}^\perp$ all contain the $\rho - 2$ classes $\gamma_2, \ldots, \gamma_{\rho-2}$, $[C_D]$, which are linearly independent, because $\gamma_2, \ldots, \gamma_{\rho-2}$ are linearly independent and belong to $(-K_X + D)^\perp$, while $(-K_X + D) \cdot C_D = 1$. We deduce that the classes $[E_1]$, $[E_{\rho-1}]$ and $[E_{\rho}]$ must be linearly dependent, but this is impossible, because they generate one-dimensional faces of Eff$(X)$.

9.27. Case (b) is similar: we deduce that $r = \rho - 1$ and we find three distinct divisors $E_i$’s that have zero intersection with $C_D$, $C_F$, and $\rho - 4$ among the $\gamma_j$’s; we conclude that these three divisors must have linearly dependent classes, which gives again a contradiction. This concludes the proof of Th. 9.1.■

REFERENCES

[ACO04] M. Andreatta, E. Chierici, and G. Occhetta, Generalized Mukai conjecture for special Fano varieties, Cent. Eur. J. Math. 2 (2004), 272–293.

[AW97] M. Andreatta and J.A. Wiśniewski, A view on contractions of higher dimensional varieties, Algebraic Geometry - Santa Cruz 1995, Proc. Symp. Pure Math., vol. 62, 1997, pp. 153–183.

[Bat99] V.V. Batyrev, On the classification of toric Fano 4-folds, J. Math. Sci. (New York) 94 (1999), 1021–1050.

[BCHM10] C. Birkar, P. Cascini, C.D. Hacon, and J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23 (2010), 405–468.

[Cas08] C. Casagrande, Quasi-elementary contractions of Fano manifolds, Compos. Math. 144 (2008), 1429–1460.
On the Picard number of divisors in Fano manifolds, Ann. Sci. Éc. Norm. Supér. 45 (2012), 363–403.

On the birational geometry of Fano 4-folds, Math. Ann. 355 (2013), 585–628.

Numerical invariants of Fano 4-folds, Math. Nachr. 286 (2013), 1107–1113.

Fano 4-folds, flips, and blow-ups of points, J. Algebra 483 (2017), 362–414.

Fano 4-folds with rational fibrations, Algebra Number Theory 14 (2020), 787–813.

The blow-up of $\mathbb{P}^4$ at 8 points and its Fano model, via vector bundles on a del Pezzo surface, Rev. Mát. Complut. 32 (2019), 475–529.

Classification of Fano 4-folds with Lefschetz defect 3 and Picard number 5, J. Pure Appl. Algebra 226 (2022), no. 3, 13 pp.

Higher-dimensional algebraic geometry, Universitext, Springer-Verlag, 2001.

On the Picard number of singular Fano varieties, Int. Math. Res. Not. 2014 (2014), 955–990.

Combinatorial convexity and algebraic geometry, Graduate Texts in Mathematics, vol. 168, Springer-Verlag, 1996.

Mori dream spaces and GIT, Michigan Math. J. 48 (2000), 331–348.

Algebraic geometry V - Fano varieties, Encyclopaedia Math. Sci. vol. 47, Springer-Verlag, 1999.

Small contractions of four dimensional algebraic manifolds, Math. Ann. 284 (1989), 595–600.

Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998.

Rational curves on algebraic varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 32, Springer-Verlag, 1996.

Hilbert schemes of lines and conics and automorphism groups of Fano threefolds, Jpn. J. Math. 13 (2018), 109–185.

Classification of Fano 3-folds with $b_2 \geq 2$, I, Algebraic and Topological Theories – to the memory of Dr. Takehiko Miyata (Kinosaki, 1984), Kinokuniya, Tokyo, 1986, pp. 496–545.

On images of Mori dream spaces, Math. Ann. 364 (2016), 1315–1342.

Fano 4-folds having a prime divisor of Picard number 1, preprint arXiv:2103.16140 2021.

On contractions of extremal rays of Fano manifolds, J. Reine Angew. Math. 417 (1991), 141–157.