ON THE SMALL CYCLIC TORSION OF ELLIPTIC CURVES OVER CUBIC NUMBER FIELDS.

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ABSTRACT. Merel’s result on the strong uniform boundedness conjecture made it meaningful to classify the torsion part of the Mordell-Weil groups of all elliptic curves defined over number fields of fixed degree \( d \). In this paper, we discuss the cyclic torsion subgroup of elliptic curves over cubic number fields. For \( N = 49, 40, 25 \) or \( 22 \), we show that \( \mathbb{Z}/N\mathbb{Z} \) is not a subgroup of \( E(K)_{tor} \) for any elliptic curve \( E \) over a cubic number field \( K \).

1. INTRODUCTION

In 1996, Merel [15] finally proved the strong uniform boundedness conjecture for elliptic curves over number fields.

Theorem 1.1 (Merel). For every positive integer \( d \), there exists an integer \( B_d \) such that for every number field \( K \) of degree \( d \) and every elliptic curve \( E \) over \( K \), we always have

\[ |E(K)_{tor}| \leq B_d \]

Merel’s result made it meaningful to classify the torsion part of the Mordell-Weil groups of all elliptic curves defined over number fields of fixed degree \( d \). The case \( d = 1 \) was solved by Mazur [14] and Kubert [11]. The case \( d = 2 \) was solved by Kamienny [7], Kenku and Momose [10].

In [19], we proposed the aim of restricting the size of cyclic torsion subgroup of elliptic curves over cubic number fields. In that paper, six out of twenty four composite integers were ruled out with the help of the Kamienný’s criterion. In this paper, we discuss several cases in the list of 24 which are too small to apply Kamienný’s criterion.

The main result of this paper is the following:

Theorem 1.2. If \( N = 49, 40, 25 \) or \( 22 \), then \( \mathbb{Z}/N\mathbb{Z} \) is not a subgroup of \( E(K)_{tor} \) for any elliptic curve \( E \) over a cubic number field \( K \).

2. PRELIMINARIES

In this section, we omit the background materials which were covered in section 2 of [19]. Readers who are interested may switch there.

Let \( N \) be a positive integer. Let \( X_1(N) \) (resp. \( X_0(N) \)) be the modular curve defined over \( \mathbb{Q} \) associated to the congruence subgroup \( \Gamma_1(N) \) (resp. \( \Gamma_0(N) \)). We denote by \( Y_1(N) = X_1(N) \backslash \{cusps\} \),

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Denote by \( J_1(N) \) (resp. \( J_0(N) \)) the jacobian of \( X_1(N) \) (resp. \( X_0(N) \)).

For a modular curve \( X \), let \( X^{(d)} \) be the \( d \)-th symmetric power of \( X \). Define

\[
\Phi : X^{(d)} \rightarrow J_X
\]

by \( \Phi(P_1 + \cdots + P_d) = [P_1 + \cdots + P_d - d\infty] \) where \( J_X \) is the jacobian of \( X \), and \([ \ ]\) denotes the divisor class. Let \( \text{Gon}(X) \) denote the gonality of \( X \). The following generalization of proposition 1(i) in Frey [3], which was proved in [19], is also necessary in section 3.

**Lemma 2.1** (Frey). Assume that \( \text{Gon}(X) > d \) and \( K \) is a finite extension of \( \mathbb{Q} \). Then \( \Phi|_{X^{(d)}(K)} \) is injective.

In this paper, we are interested in the gonality of the modular curves \( X_1(N) \). Since the 1-gonal curves are precisely the curves of genus 0, then \( X_1(N) \) is 1-gonal if and only if \( N \) is among the eleven values \( N = 1 - 10, 12 \) with genus 0. The complete lists of 2-gonal and 3-gonal ones were determined by Ishii-Momose [5] and Jeon-Kim-Schweizer [6].

**Proposition 2.2** (Ishii-Momose). The modular curve \( X_1(N) \) is 2-gonal if and only if \( N \) is one of the following:

\[
\begin{align*}
N & = 1 - 10, 12 \quad (g = 0); \\
N & = 11, 14, 15 \quad (g = 1); \\
N & = 13, 16, 18 \quad (g = 2).
\end{align*}
\]

**Proposition 2.3** (Jeon-Kim-Schweizer). The modular curve \( X_1(N) \) is 3-gonal if and only if \( N \) is one of the following:

\[
\begin{align*}
N & = 1 - 10, 12 \quad (g = 0); \\
N & = 11, 14, 15 \quad (g = 1); \\
N & = 13, 16, 18 \quad (g = 2); \\
N & = 20 \quad (g = 3).
\end{align*}
\]

Any noncuspidal point of \( X_1(N) \) is represented by \( (E, \pm P) \), where \( E \) is an elliptic curve and \( P \in E \) is a point of order \( N \). Any noncuspidal point of \( X_0(N) \) is represented by \( (E, C) \), where \( E \) is an elliptic curve and \( C \subset E \) is a cyclic subgroup of order \( N \). The map \( \pi : X_1(N) \rightarrow X_0(N) \) sends \( (E, \pm P) \) to \( (E, \langle P \rangle) \), where \( \langle P \rangle \) is the cyclic subgroup generated by \( P \).

Let \( p \) be a prime such that \( p \nmid N \). Igusa’s theorem [4] says that the modular curves \( X_1(N) \) and \( X_0(N) \) have good reduction at prime \( p \). The following theorem of Serre and Milne says that reducing the modular curve is compatible with reducing the modular interpretation.

**Theorem 2.4** (Serre-Milne). [17, Theorem 1] Any point of \( Y_1(N) \) or \( Y_0(N) \), rational over a field \( K \) (of characteristic not dividing \( N \)), is represented by a \( K \)-rational pair (i.e. \( E \) is defined over \( K \), and \( P \) is rational over \( K \), or \( C \) is a group rational over \( K \)), and conversely.
Let $K$ be a number field with ring of integers $\mathcal{O}_K$, $\wp \subset \mathcal{O}_K$ a prime ideal lying above $p$, $k = \mathbb{F}_q = \mathcal{O}_K/\wp$ its residue field. Let $E$ be an elliptic curve over $K$ and $P \in E(K)$ a point of order $N$. Let $\tilde{E}$ be the fibre over $k$ of the Néron model of $E$, and let $\tilde{P} \in \tilde{E}(k)$ be the reduction of $P$. The following well-known but rarely mentioned theorem [19, Proposition 2.5] shows that $\tilde{P}$ has order $N$ when $p \nmid N$.

**Theorem 2.5.** Let $m$ be a positive integer relatively prime to $\text{char}(k)$. Then the reduction map

$$E(K)[m] \rightarrow \tilde{E}(k)$$

is injective.

Let $k = \mathbb{F}_q$ be the finite field with $q = p^n$ elements. Let $E/k$ be an elliptic curve over $k$. Let $|E(k)|$ be the number of points of $E$ over $k$. Then Hasse’s theorem states that

$$||E(k)| - q - 1| \leq 2\sqrt{q}$$

i.e.

$$(1 - \sqrt{p^n})^2 \leq |E(k)| \leq (1 + \sqrt{p^n})^2$$

Let $t = q + 1 - |E(k)|$, $E$ is called ordinary if $(t, q) = 1$, otherwise it is called supersingular. In the range proposed by Hasse’s theorem, all the ordinary $t$ appear, while the supersingular $t$ only appears in restricted case. This was determined by Waterhouse [20, Theorem 4.1]:

**Proposition 2.6** (Waterhouse). The isogeny classes of elliptic curves over $k$ are in one-to-one correspondence with the rational integers $t$ having $|t| \leq 2\sqrt{q}$ and satisfying one of the following conditions:

1. $(t, p) = 1$;
2. If $n$ is even: $t = \pm 2\sqrt{q}$;
3. If $n$ is even and $p \not\equiv 1 \mod 3$: $t = \pm \sqrt{q}$;
4. If $n$ is odd and $p = 2$ or $3$: $t = \pm p^{n+1}$;
5. If either (i) $n$ is odd or (ii) $n$ is even and $p \not\equiv 1 \mod 4$: $t = 0$.

3. **Method**

The following Theorem states that the Jacobian $J_1(N)$ decomposes to a direct sum of modular abelian varieties.

**Theorem 3.1.** [1, Theorem 6.6.6] The Jacobian $J_1(N)$ is isogenous to a direct sum of abelian varieties (over $\mathbb{Q}$) associated to equivalence classes of newforms

$$J_1(N) \rightarrow \bigoplus_f A_f^{n_f}$$

with $f(\tau) = \sum_{n=1}^{\infty} a_n(f)e^{2\pi i n \tau}$ newforms of divisor level.
The $L$-series $L(A_f, s)$ of $A_f$ coincides, up to a finite number of Euler factors, with the product
\[ \prod_\sigma L(f^\sigma, s) = \prod_{\sigma} \sum_{n=1}^{\infty} a_n^\sigma n^{-s} \]
where $\sigma$ runs through embeddings $\sigma : K_f \hookrightarrow \mathbb{C}$ with $K_f = \mathbb{Q}(\{a_n\})$ the number field of $f$ (See [18 §7.5]). The following proposition is a special case of Corollary 14.3 in Kato [8] :

**Proposition 3.2.** Let $A$ be an abelian variety over $\mathbb{Q}$ such that there is a surjective homomorphism $J_1(N) \longrightarrow A$ for some $N \geq 1$. If $L(A, 1) \neq 0$, then $A(\mathbb{Q})$ is finite.

The decomposition of $J_1(N)$ and the non-vanishing of the $L$-series at $s = 1$ of modular abelian varieties can be calculated in Magma [12]. If $L(A_f, 1) \neq 0$ for all $A_f$, then we know $A_f(\mathbb{Q})$ is finite for all $A_f$, therefore $J_1(N)_{/\mathbb{Q}}$ is finite. For the $N \leq 65$ in the list in [19], Table 1 is the result of calculations in Magma. The second column $t$ is the number of non-isogenous modular abelian varieties in the decompositon $J_1(N) = \bigoplus_{i=1}^{t} A_i^{m_i}$. The third column list the dimension $d_i$ and multiplicity $m_i$ of each $A_i$ (we omit $m_i$ if $m_i = 1$). The fourth column verifies vanishing of $L$-series at 1 ($T$ means $L(A_i, 1) = 0$ and $F$ means $L(A_i, 1) \neq 0$). It is easy to see $J_1(N)(\mathbb{Q})$ is finite for the $N$’s in Table 1 except $N = 65, 63$.

| $N$ | $t$ | $d_i(m_i)$ | $L(A, 1) = 0$ |
|-----|-----|-------------|---------------|
| 49  | 5   | 1, 48, 6, 12, 2 | $F, F, F, F, F$ |
| 25  | 2   | 8, 4         | $F, F$        |
| 27  | 2   | 1, 12        | $F, F$        |
| 32  | 4   | 1, 4, 8, 2(2) | $F, F, F, F$ |
| 65  | 19  | 1, 2, 2, 6, 20, 20, 8, 2(2), 8, 2(2), 8, 2(2), 4, 4, 12, 6, 2, 2 | $T, F, F, F, F, F, F, F, F, F, F, F, F$ |
| 39  | 10  | 1, 2, 4, 8, 2, 2(2), 4, 2, 4, 2 | $F, F, F, F, F, F, F, F, F$ |
| 26  | 5   | 1, 1, 2(2), 2, 2 | $F, F, F, F, F$ |
| 55  | 10  | 1, 2, 1(2), 4, 32, 8, 8, 16, 4, 4 | $F, F, F, F, F, F, F, F, F, F$ |
| 33  | 6   | 1, 1(2), 8, 4, 4, 2 | $F, F, F, F, F, F$ |
| 22  | 2   | 1(2), 4 | $F, F$ |
| 35  | 8   | 1, 2, 2, 4, 4, 4, 4 | $F, F, F, F, F, F, F, F$ |
| 63  | 20  | 1, 2, 1(2), 6, 6, 2, 10, 4, 2(2), 2, 10, 2, 2(2), 2, 10, 2, 10, 12, 4 | $F, F, F, F, F, F, F, F, F, F, F, F, F, F$ |
| 28  | 4   | 1(2), 4, 2, 2 | $F, F, F, F$ |
| 45  | 8   | 1, 1(2), 2, 6, 16, 4, 2, 8 | $F, F, F, F, F, F, F, F, F$ |
| 30  | 4   | 1, 1(2), 4, 2 | $F, F, F, F$ |
| 40  | 7   | 1, 1(2), 4, 2(2), 8, 2, 4 | $F, F, F, F, F, F, F, F$ |
| 36  | 5   | 1, 8, 2, 2(2), 2 | $F, F, F, F, F$ |
| 24  | 3   | 1, 2, 2 | $F, F, F$ |

In the proof of Lemma 3.5, we use a specialization lemma in Appendix of Katz [9] and a theorem of Manin [13] and Drinfeld [2].
Lemma 3.3 (Specialization Lemma). Let $K$ be a number field. Let $\wp \subset \mathcal{O}_K$ be a prime above $p$. Let $A/K$ be an abelian variety. Suppose the ramification index $e_{\wp}(K/\mathbb{Q}) < p - 1$. Then the reduction map

$$\Psi : A(K)_{\text{tor}} \longrightarrow A(\overline{\mathbb{F}}_p)$$

is injective.

Theorem 3.4 (Manin-Drinfeld). Let $C \subset \text{SL}_2(\mathbb{Z})/(\pm 1)$ be a congruence subgroup. $x, y \in \mathbb{P}^1(\mathbb{Q})$ and $\overline{x}, \overline{y}$ the images of $x$ and $y$ respectively, on $\mathbb{P}/\mathbb{C}$. Then the class of divisors $(\overline{x}) - (\overline{y})$ on curve $\mathbb{P}/\mathbb{C}$ has finite order.

Lemma 3.5. Suppose $N > 4$ such that $\text{Gon}(X_1(N)) > d$, $J_1(N)(\mathbb{Q})$ is finite, $p > 2$ is a prime not dividing $N$. Let $K$ be a number field of degree $d$ over $\mathbb{Q}$ and $\wp$ a prime of $K$ over $p$. Let $E/K$ be an elliptic curve with a $K$-rational point $P$ of order $N$, i.e. $x = (E, \pm P) \in Y_1(N)(K)$. Then $E$ has good reduction at $\wp$.

**Proof.** Suppose $E$ has additive reduction at $\wp$, then $\tilde{E}(k)^0 \cong G_{a/k}$ with $|G_{a/k}| = p^i, i \leq d$ and $\tilde{E}(k)/\tilde{E}(k)^0 \cong G$ with $|G| \leq 4$. Since $\tilde{P}$ is a $k$-rational point of order $N$ in $\tilde{E}$, then $N$ divides $|\tilde{E}(k)| = |G_{a/k}||G|$, which is impossible under our assumption.

Suppose $E$ has multiplicative reduction at $\wp$, i.e. $x$ specializes to a cusp of $\tilde{X}_1(N)$. Then $\tau_1(K)$ is also a cubic field with prime ideal $\tau_1(\wp)$ over $p$ and residue field $k_i = k$. And $\tau_1(E)$ also has multiplicative reduction at $\tau_1(\wp)$. This means all the images $x_1, \ldots, x_d$ of $x$ specialize to cusps of $\tilde{X}_1(N)$. Let $c_1, \ldots, c_d$ be the cusps such that

$$x_i \otimes \overline{\mathbb{F}}_p = c_i \otimes \overline{\mathbb{F}}_p, \quad 1 \leq i \leq d$$

We know all the cusps of $X_1(N)$ are defined over $\mathbb{Q}(\zeta_N)$ [16]. Let $\wp'$ be a prime in $\mathbb{Q}(\zeta_N)$ over $p$. It is an elementary fact in algebraic number theory that $p$ ramifies in $\mathbb{Q}(\zeta_N)$ if and only if $p \nmid N$, so $e_{\wp'}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) = 1$ under our assumption $p \nmid N$. So by Lemma 3.3, the specialization map

$$\Psi : J_1(N)(\mathbb{Q}(\zeta_N))_{\text{tor}} \longrightarrow J_1(N)(\overline{\mathbb{F}}_p)$$

is injective.

Since $\text{Gon}(X_1(N)) > d$, then by Lemma 2.1, the map

$$\Phi : X_1(N)^{(d)}(\mathbb{Q}(\zeta_N)) \longrightarrow J_1(N)(\mathbb{Q}(\zeta_N))$$

is injective.

We know $x_1 + \cdots + x_d$ is $\mathbb{Q}$-rational and $J_1(N)(\mathbb{Q})$ is finite, so $[x_1 + \cdots + x_d - d\infty]$ is in $J_1(N)(\mathbb{Q}(\zeta_N))_{\text{tor}}$. By Theorem 3.4, the difference of two cusps of $X_1(N)$ has finite order in $J_1(N)$. So $[c_1 + \cdots + c_d - d\infty]$ is also in $J_1(N)(\mathbb{Q}(\zeta_N))_{\text{tor}}$. Therefore $\Psi \circ \Phi(x_1 + \cdots + x_d) = \Psi \circ \Phi(c_1 + \cdots + c_d)$ implies $x_1 + \cdots + x_d = c_1 + \cdots + c_d$ since $\Psi \circ \Phi$ is injective. This is a contradiction because we assume $x$ is a noncuspidal point.

Therefore $E$ has good reduction at $\wp$. \qed
4. PROOF OF THEOREM 1.2

4.1. \(N = 49, 40\). As is seen in Table 1, \(J_1(N)(\mathbb{Q})\) is finite. By Proposition 2.2 and 2.3, we know \(\text{Gon}(X_1(N)) > 3\). Let \(K\) be a cubic field and \(\wp\) a prime of \(K\) over 3. Suppose \(x = (E, \pm P) \in Y_1(N)(K)\). Therefore by Lemma 3.5, \(E\) has good reduction at \(\wp\). By Theorem 2.5, the reduction \(\tilde{P}\) of \(P\) is a \(k\)-rational point of order \(N\) in the elliptic curve \(\tilde{E}\) over \(k = \mathcal{O}_K/\wp\). But \(\tilde{E}(k)\) can not have a point of order \(N\) since \(N > (1 + \sqrt{3})^2 \approx 38.4\). This is a contradiction. So \(\mathbb{Z}/N\mathbb{Z}\) is not a subgroup of \(E(K)_{tor}\).

4.2. \(N = 25, 22\). As is seen in Table 1, \(J_1(N)(\mathbb{Q})\) is finite. By Proposition 2.2 and 2.3, we know \(\text{Gon}(X_1(N)) > 3\). Let \(K\) be a cubic field and \(\wp\) a prime of \(K\) over 3. Suppose \(x = (E, \pm P) \in Y_1(N)(K)\). Therefore by Lemma 3.5, \(E\) has good reduction at \(\wp\). By Theorem 2.5, the reduction \(\tilde{P}\) of \(P\) is a \(k\)-rational point of order \(N\) in the elliptic curve \(\tilde{E}\) over \(k = \mathcal{O}_K/\wp\).

If \(k = \mathbb{F}_3\) or \(\mathbb{F}_{3^2}\), then \(\tilde{E}(k)\) can not have a point of order \(N\) since \(N > (1 + \sqrt{3})^2\). If \(k = \mathbb{F}_{3^3}\), suppose \(\tilde{E}(k)\) has a point of order \(N\), then \(\tilde{E}(k) \cong \mathbb{Z}/N\mathbb{Z}\) since \(Nm > (1 + \sqrt{3})^2\) for any \(m > 1\). But by Theorem 2.6, \(|\tilde{E}(k)| \neq N\) (\(t = 3\) for \(N = 25\), \(t = 6\) for \(N = 22\)). This is a contradiction. So \(\mathbb{Z}/N\mathbb{Z}\) is not a subgroup of \(E(K)_{tor}\).

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