Duality of deconfined quantum critical point in two dimensional Dirac semimetals

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In this paper we discuss the Néel and Kekulé valence bond solids quantum criticality in graphene Dirac semimetal. Considering the quartic four-fermion interaction $g(\bar{\psi}_i \Gamma_{ij} \psi_j)^2$ that contains spin, valley, and sublattice degrees of freedom in the continuum field theory, we find the microscopic symmetry is spontaneously broken when the coupling $g$ is greater than a critical value $g_c$. The symmetry breaking gaps out the fermion and leads to semimetal-insulator transition. All possible quartic fermion-bilinear interactions give rise to the uniform critical coupling, which exhibits the multicritical point for various orders and the Landau-forbidden quantum critical point. We also investigate the typical critical point between Néel and Kekulé valence bond solid transition when the symmetry is broken. The quantum criticality is captured by the Wess-Zumino-Witten term and there exist a mutual-duality for Néel-Kekulé VBS order. We show the emergent spinon in the Néel-Kekulé VBS transition, from which we conclude the phase transition is a deconfined quantum critical point between the Néel and Kekulé VBS orders. Additionally, the connection between the index theorem and zero energy mode bounded by the topological defect in the Kekulé VBS phase is studied to reveal the Néel-Kekulé VBS duality.

I. INTRODUCTION

The study of quantum phases of matter and the phase transition in strongly correlated systems is one of the central issues of modern condensed matter physics1–5. In past seven years, the semimetal phases have attracted extensive attentions. In topological Weyl/Dirac semimetals, the conduction and valence bands cross at zero-dimensional nodal points6–13, but for topological nodal line semimetals, it crosses at one-dimensional nodal line (see Refs. 14, 15 and references therein). Recently, based on the 2D and 3D semimetal, the fermion-boson mixed system was proposed to describe the Landau-forbidden quantum phase transition, which gives rise to Fermion-induced quantum critical points (FIQCP)16–19 and fluctuation-induced continuous phase transition with hidden degrees of freedom20–22. The FIQCP is type-II Landau-forbidden transitions and the type-I is deconfined quantum critical point in quantum magnet23. In addition, the fermion-boson mixed system have been shown to provide us well understanding of space-time supersymmetry at quantum critical point (QCP)24–26.

Graphene is a Dirac semimetal phase, and the low energy quasiparticle excitations are described by (2+1) dimensions field theory of free Dirac fermions. Due to the special relativistic dispersion, the graphene-like structure provides a platform for realization of many quantum electrodynamics predictions (e.g., Klein tunneling27, Zitterbewegung effect28). When suffering from perturbations or electron correlation, the semimetal opens a fermion gap and leads to rich phases of matter. For example, the integer quantum Hall state is obtained by introducing the next-nearest neighbor hopping with complex phases but without net magnetic flux threads each hexagonal plaquette29. $\mathbb{Z}_2$ quantum spin Hall effect with time reversal symmetry could be obtained in a similar way30. The charge density wave, Kekulé dimerization (Kekulé valence-bond-solid), and spin density wave are known orders that gap out the fermion spectrum31,32. Also, the fermionic coupling gaps out the fermion and leads to the surprising fractional charge33–35. It’s shown that the semimetal exhibits exotic topological phases36,37 and the non-Landau-Ginzburg-Wilson transition19,20,38 with electron correlation and emergent fluctuation. The fluctuation with emergent degree of freedom gives rise to the rich quantum critical behavior and the various QCP is shown to belong to the Gross-Neveu-Yukawa(GNY) universality class39–41. Meanwhile, recent large-scale quantum Monte Carlo simulations have confirmed that the quantum criticality in interacting Dirac semimetals is consistent with the Gross-Neveu universality class42–44.

In this paper, by using the nonlinear sigma model field theory45–50, we understand the Néel and Kekulé valence-bond-solid (VBS) quantum criticality in Dirac semimetals. Our motivation is stated as follows. Recently, Ref.16 have studied the semimetal-Kekulé VBS transition, and shown the evidence of a deconfined quantum critical point between the Kekulé-VBS and antiferromagnetic phases of $SU(N)$ fermions model on the honeycomb lattice ($N = 2$ is the flavors of four-component Dirac fermions). All the Néel and Kekulé VBS are gapped quantum phases, the induced mass-gap corresponds to the modulation of the fermion bilinear. The coupling between the Néel-Kekulé VBS fermion bilinear and the fermion fields leads to the purely bosonic phases for competing Néel-Kekulé VBS orders. From the corresponding dynamical of bosonic fields, the quantum criticality will be easily identified. The phase transition is shown to associate with a emergent spinon and gauge field at the critical point, so the Néel-Kekulé VBS transition indeed a deconfined QCP. The mass-gap equivalent to the dynamical symmetry breaking-induced massive fermion and the (2+1) GNY theory is crucial for the mass-gap generation.
In correlated many-body systems, we often encounter the Yukawa terms that driven by short range fermion-fermion interacting. For mass generation and competing phase in interacting systems, the instability of interacting play a key role. Apart from the short range Coulomb interaction, the general four-body interaction in graphene semimetals includes the quartic fermion bilinear that mixed the spin, valley, and sublattice. When the symmetry is broken, the general fermion bilinear generates various mass matrices that anticommute with each other. The physical explanation of these mass matrices have been established in Ref.\textsuperscript{51}. We will show there exists multicritical point for possible orders, and the multicritical point only depend on the dimensions of the mass matrix. The emergence of two symmetry-unrelated orders at a critical point display the Landau-beyond quantum criticality, and the effective action for a set of anticommutating orders may be derived upon integrating over the fermion fields\textsuperscript{52}. Under such formalism, the quantum criticality and the quantum phase transition between different phases can be readily understood.

As shown in the text, the Néel and Kekulé VBS orders at the critical point are dual to each other, and the topological excitation or topological defect also share the duality in two phases. The topological defect bounds midgap fermion spectrum (i.e. zero energy mode) in the core is well known for fermi-boson mixed field theory. Jackiw and Rossi have studied the fermion-vortex system with fermion-scalar field interaction, and point that the Dirac equation possesses $|n|$ zero-eigenvalue modes in the $n$ -vorticity background field\textsuperscript{53}. This result has been proven associate with a index theorem\textsuperscript{54}. The fermion zero mode has caused great interests in various condensed matter systems\textsuperscript{55–58}. Here, we study the zero energy mode bounded by the topological texture in Kekulé VBS phase as a complementary study. And more importantly, we study the index of the topological excitation beyond any symmetries in the sense of Néel-Kekulé VBS duality.

The paper is organized as follows. The dynamical gap equation is established in sec. II. From the gap equation, we solve uniform critical coupling constant for possible broken orders. In Sec. III, the typical Néel-Kekulé VBS transition is studied. We derive the Wess-Zumino-Witten action for five-component Néel-Kekulé VBS order vector and discuss the Néel-Kekulé VBS mutual duality. In Sec. IV, we study the index theorem of zero energy mode in Kekulé VBS state to reveal the Néel-Kekulé VBS mutual duality. In the end, we give a summary in Sec. V.

II. GAP GENERATION AND SYMMISTRY

The hexagonal lattice of graphene semimetal contains two sets of triangular sublattices, which denoted by $a$ and $b$. Near two inequivalent Dirac points ($k, k'$) of momentum space, The kinetic part of spinless graphene is described by the Hamiltonian $H = \sum_k \psi_k^\dagger h(k) \psi_k$, where the fermion annihilation operators $\psi_k$ around the Dirac nodes is given by

$$\psi_k = (c_{k\alpha}, c_{k'\beta}, c_{k\beta}, c_{k'\alpha})^T.$$  \hspace{1cm} (1)

The single-particle Hamiltonian $h(k)$ in momentum space reads (setting the Fermion velocity as unit)

$$h(k) = \sum_{i=1,2} \alpha_i k_i, \quad \alpha_1 = -\tau_3 \sigma_2, \quad \alpha_2 = \tau_3 \sigma_1,$$  \hspace{1cm} (2)

where the $2 \times 2$ Pauli matrices $\tau_i$ act on the valley index, and $\sigma_i$ act on the sublattice index. Any mass matrix $M$ entering the single particle Hamiltonian obeys the relations

$$[\alpha_{1,2}, M]_\tau = 0, \quad M^2 = I \Delta,$$  \hspace{1cm} (3)

and the mass matrix $M$ gapped the Fermionic spectral with the form $E_k = \pm \sqrt{k^2 + M^2}$. To describe more electron ground states within the uniform framework of Dirac equation, we can introduce another set of Pauli matrix $s_{\sigma i}$, which act on the spin index. Taking spin, valley and sublattice into account, the matrices $\alpha_i (i = 1, 2)$ are $8 \times 8$ Dirac matrices. The uniform descriptions of different order parameter with gap generation will be constructed by defining the following matrices,

$$\alpha_1 = s_0 (-\tau_3) \sigma_2, \quad \alpha_2 = s_0 \tau_3 \sigma_1.$$  \hspace{1cm} (4)

Any mass matrix of the form $M_{\mu \nu \lambda} = s_\mu \tau_\nu \sigma_\lambda$ must anticommute with matrices $\alpha_i$, sixteen matrices match this property. In this paper, we define $\beta = s_0 \tau_0 \sigma_3$ as Dirac beta matrix, which realize mass-gap for quantum Hall effect rather than charge density wave as usual.

A. Gap generation

The microscopic symmetry may be spontaneously broke and lead to semimetal-insulator transition in graphene-like Dirac semimetal. Thus, started from a four-fermions interaction, the insulator can be builded from a nonvanishing fermion bilinear at mean field level. The nonzero fermion bilinear will lead to the mass-gap of the fermion ground states within the uniform framework of Dirac equation. We derive the Wess-Zumino-Witten action for five-component Néel-Kekulé VBS order vector and discuss the Néel-Kekulé VBS mutual duality. In Sec. IV, we study the index theorem of zero energy mode in Kekulé VBS state to reveal the Néel-Kekulé VBS mutual duality. In the end, we give a summary in Sec. V.
mass generation of the gauged vector mesons in particle physics\textsuperscript{32}. The gap generation or order profiles can be studied by rewriting the functional integrals as

\[ Z = \int D[\bar{\psi}, \psi] e^{iS(\bar{\psi}, \psi, \phi)}, \]

\[ S(\bar{\psi}, \psi, \phi) = \int d^d x \left[ \bar{\psi}(\gamma^\mu i\partial_\mu - \phi \Gamma_a)\psi - \frac{N\phi^2}{2g} \right], \quad (7) \]

The decoupling process of four-fermion interaction by introducing auxiliary boson field in the functional integral is known as Hubbard-Stratonovich transformation. Integrating out the fermion fields at large-N limit, we get the effective potential for boson field as

\[ V_{\text{eff}}(\phi) = \frac{N\phi^2}{2g} + \frac{iN}{VT} \ln \left| \det(\gamma^\mu i\partial_\mu - \phi \Gamma_a) \right|. \quad (8) \]

Eq.(7) show that \( \phi \Gamma_a \) play the role of mass matrix and corresponds to an order pattern. In order to study the ground state behavior, we consider the static Higgs field. Then, the effective potential is given by

\[ V_{\text{eff}}(\phi) = \frac{N\phi^2}{2g} + \frac{iN}{(2\pi)^3} \int d^3k \ln(k^2 - \phi^2)^{d/2}. \]

\[ = \frac{N\phi^2}{2g} - \frac{Nd}{4\pi^2} \int d\phi \left( \Lambda - \frac{\phi^2}{2} \ln \frac{\Lambda^2 + \phi^2}{\phi^2} \right), \]

where \( d \) is the dimension of mass matrix and \( \Lambda \) is an ultraviolet cut-off in the regularization. Here we have used the relation \( \det(\gamma^\mu i\partial_\mu - \phi \Gamma_a) = (k^2 - \phi^2)^{d/2} \), which only determined by the dimension of the matrix. By minimizing the potential \( \partial V_{\text{eff}} / \partial \phi = 0 \), the nonzero vacuum immediately gives the fermion mass due to spontaneous symmetry breaking. The gap equation reads

\[ \frac{1}{g(\Lambda)} = \frac{4}{\pi^2} \int_0^\Lambda \frac{k^2 dk_E}{k_E^2 + \phi^2}, \quad (9) \]

from which we get the critical coupling strength,

\[ g_c = \frac{\pi^2}{4\Lambda}. \quad (10) \]

The mass-gap generate only if \( g(\Lambda) > g_c \), and lead to an insulating phase. To confirm the fermion mass, setting \( \phi = \phi(\Lambda, g) \), we immediately find the flow equation,

\[ \frac{\partial g(\Lambda)}{\partial \Lambda} = -\frac{4g^2(\Lambda)}{\pi^2} \left( 1 - \frac{\Lambda\phi^2}{\Lambda^2 + \phi^2} \right). \quad (11) \]

Moreover, the mass-gap satisfies

\[ \left( \Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} \right) \phi = 0, \quad \beta(g) = \Lambda \frac{\partial g}{\partial \Lambda}. \quad (12) \]

This means that the mass-gap is irrelevant to the ultraviolet cut-off \( \Lambda \) and the running coupling \( g \), this should be, because the fermion mass term must be a renormalization group invariant.

B. Mass matrix and symmetry classification

When the symmetry is spontaneously broken, the matrix \( \phi \Gamma_a \) opens a fermion gap at two Dirac points. As shown above, the kinetic part allow 16 mass matrices, and each corresponds to an insulating phase. To find the physical explanation of these matrices, it’s useful to define the following chiral symmetry time reversal symmetry. For the Hamiltonian with lattice fermion residing on different sublattice, flipping the sign of one sublattice(for example sublattice-b) will reverse the overall sign of the Hamiltonian. The mapping of postive energy to negative energy can be implemented by the operator \( \Theta = s_0 \tau_3 \sigma_3 \), and this symmetry named sublattice symmetry(SLS) or chiral symmetry. If a chiral order parameter is presented, the Hamiltonian preserve SLS when satisfy

\[ \Theta H(k) \Theta^{-1} = -H(k). \quad (13) \]

The time reversal is defined as \( Tc_\uparrow = c_\downarrow, Tc_\downarrow = -c_\uparrow \), and which can be implemented by the matrix \( T = is_2 \tau_1 \sigma_1 \). The Hamiltonian preserve time reverse symmetry(TRS) when satisfy

\[ TH^*(-k)T^{-1} = H(k). \quad (14) \]

The more symmetry classification of mass matrix was presented in Ref. \textsuperscript{52}, and one may to the reference for details.

The Kekulé dimerization corresponds to the fermion bilinear residing on the opposite ends of nearest bond with a wave vector connecting two Dirac points, which can be realized in terms of \( \tau_1 \sigma_0 \) and \( \tau_2 \sigma_0 \). Such modulation preserve SLS, the associated mass matrix named Kekulé-VBS order pattern in analogy to the quantum dimer model. Since there are four spin matrices(Three Pauli matrix and a identity matrix), we have total eight Kekulé-VBS mass matrices(see table-I). Any modulation with lattice fermion sitting on the same sublattice does not preserve SLS, the electron patter corresponds to chage density wave, spin density wave, quantum Hall effect, and spin-orbital coupling driven quantum spin Hall effect. The total 16 mass matrices and the corresponding physical explanation were listed in table-I.

III. COMPETITION ORDERS, WZW TERM AND DUALITY

Near the two sides of the quantum critical point(QCP), the two ordered phases break different symmetry and the continuous phase transition associate with two competing orders. Once suppressing one of the orders, which necessarily leads to the emergence of another order. The above features may embodies in the isotropic nonlinear sigma model with a topological term (for example Wess-Zumino-Witten term and mutual Chern-Simions term\textsuperscript{39,60}). We will see that the crucial feature of the theory is the existence of anticommutating mass terms...
and mutual duality, which may play key role in the understand of Landau-forbidden quantum criticality.

In this section, we discuss the typical Néel-Kekulé VBS quantum phase transition. To describe the competition between the antiferromagnetic and Kekulé VBS state, we include the five components Néel-Kekulé VBS unit vector \( \vec{n} = \vec{\phi}/|\vec{\phi}| \), with the constraint \( |\vec{n}|^2 = 1 \).

The \( O(3) \times U(1) \) vector (defined on \( S^4 \) surface) in (2+1)d spacetime do not have topological nontrivial configurations for \( \pi_3(S^4) = 0 \), but it support perturbative topological term due to non-triviality of the homotopy group \( \pi_4(S^4) = Z \), which is known as WZW term. Considering the action

\[
S[\psi, \bar{\psi}, \vec{n}] = \int d^3x \bar{\psi}(i\gamma^\mu \partial_\mu - m\gamma^5 n^I)\psi, \tag{16}
\]

the non-linear sigma model theory and topological WZW terms can be restored perturbatively by integrating out the gapped fermions in the presence of the slowly varying \( \vec{n} \). Follows the spirit Abanov and Wiegmann, the effective action \( S[\vec{n}] \) for Néel-Kekulé-VBS vector is obtained as

\[
e^{iS[\vec{n}]} = \int D[\psi, \bar{\psi}] e^{-S[\bar{\psi}, \psi, \vec{n}]}, \quad S[\vec{n}] = -\text{tr} \ln (i\gamma^\mu \partial_\mu - m\gamma^5 n^I) \tag{17}
\]

The \( 1/m \) expansion leads to the following remarkable action, \( S[\vec{n}] = S_0[\vec{n}] + S_{WZW}^E[\vec{n}] \),

\[
S_0[\vec{n}] = \frac{1}{g} \int d^3x (\partial_\mu n^I) \partial^\mu n^J, \quad S_{WZW}^E[\vec{n}] = \frac{4\pi}{4!\text{Area}(S^4)}\epsilon^{\alpha\beta\gamma\rho} \epsilon^{abcde} \int d^2xd\tau \int_0^1 d\rho \times (\partial_\alpha n^a)(\partial_\beta n^b)(\partial_\gamma n^c)(\partial_\rho n^d)n_z. \tag{19}
\]

The details of the derivation of Eq.(19) are included in the appendix A. The appearance of the WZW topological term is the consequence of the duality between Néel and Kekulé-VBS order, implying that the topological objects of one phase carries the quantum number of another phase (see follow for details). From above derivations, the Néel-Kekulé-VBS QCP is obtained by incorporating the WZW term to the SO(5) nonlinear sigma model.

The Néel order breaks the \( SU(2) \) spin rotation to \( U(1) \) symmetry, then the \( O(3) \) vector \( \vec{n} = (n_1, n_2, n_3) \) for Néel order should be defined on \( SU(2)/U(1) = S^2 \). The Kekulé-VBS breaks \( C_6 \) lattice rotation or lattice translational symmetry, if one consider the complex Kekulé-VBS order parameter, the discrete symmetry may be enlarged to a enhanced \( U(1) \) symmetry. As discussed in superfluid phase transition or XY model, for the symmetry unrelated phase transition, each ordered phase can be driven by the condensation of the topological defect of the ordered phase. With the CP(1) parametrization: \( n_a = z^a |\sigma_a z, z = (z_1, z_2)^i \) is a complex two-component spinor with the constraint \( |z_1|^2 + |z_2|^2 = 1 \) and \( z_1, z_2 \) are fractionalized “spinon” fields, we observe the Skyrmion charge in the Néel order as a total gauge flux

\[
q_s = \frac{1}{2\pi} \int d^2xe^{i\beta}a_j, \quad a_i = -\frac{i}{2} \epsilon^{\beta\gamma\delta} \frac{\partial_\gamma z - \partial_\delta z + \partial_\beta z}{2}. \tag{20}
\]

The internal gauged \( U(1) \) symmetry appears as \( a_j \rightarrow a_j + \partial_j f \), so the Skyrmion number is conserved. The Skyrmion is precisely mapped to a “magnetic” flux quantum. Since the Kekulé-VBS order parameter preserve \( U(1) \), it support topological defect in the form of vortex. In the dual description of vortex defect in the Kekulé-VBS order, the vorticity play the role of “electric” charge

\[
q_v = \int dl \partial_\beta c_i = \int dl \epsilon^{\beta\gamma\delta} \partial_\gamma a_0, \tag{21}
\]

The scalar field \( a_0 \) is phase field in this case, therefore the Néel and Kekulé-VBS dual to each other due to electromagnetic duality.

Because \( n_a \) preserve a \( U(1) \) gauge degree of freedom after \( SU(2) \) is broken, the spinon field \( z_a \) will be coupled to the dynamical \( U(1) \) gauge field \( a_\mu \), we obtain the CP(1) model \( S_a = \int d^3xd\tau L_z \), that describes the deconfined QCP between Néel and Kekulé-VBS order

\[
L_z = |(\partial_\mu - ia_\mu)z_\alpha|^2 + s_z |z_\alpha|^2 + r_z |z_\alpha|^4 + \epsilon^{\mu\nu\lambda}z_\alpha \partial_\nu a_\lambda \tag{22}
\]

The study of critical point can be aided by introducing a matter fields and gauge field \( b_\mu \) for vortex defect in Kekulé-VBS order. Once the matter fields is condensed, the gauge field acquires a mass term \( b_\mu^2 \) via Higgs mechanism. Then, we derive the Maxwell term in \( L_z \) by incorporating the mutual BF Chern-Simons(CS) term

\[
L_{MC} = i\sqrt{2}\epsilon^{\mu\nu\lambda}b_\mu \partial_\nu a_\lambda, \tag{23}
\]
after integrating out $b_{\mu}$. The mutual CS term is the concentrated reflection of mutual duality between Néel and Kekulé-VBS order. Under the gauge transformation $b_{\mu} \rightarrow b_{\mu} + \partial_{\mu} f^{b}$, the current $j^{a}_{\mu} = i\sqrt{2\pi} \epsilon^{a\mu\lambda} \partial_{\mu} \partial_{\lambda} f^{b}$, and the time component as the vortex when $f^{a}$ sweep across the vortex singularity in Kekulé-VBS order. Thus, the charge-vortex duality is captured by the mutual CS term, or WZW term in terms of order vector.

Now, let us turn to describe the phase transition. As shown in Eq.(20), the Skyrmion is conserved, it correspond to configuration of $a_{\mu}$ at which creates a $2\pi$ flux. Once the quantum flux is condensed, the extra term induced by mutual CS is non-vanished, i.e., $\int d^{2}x d\tau \epsilon^{\mu\nu\lambda} \partial_{\mu} f^{b} \langle \partial_{\nu} a_{\lambda} \rangle \neq 0$. The flux condensation destroys the Skyrmion number, meanwhile, it spontaneously breaks the $U(1)_{b}$ symmetry and leads to the Kekulé-VBS order. The condensation of vortex in Kekulé-VBS order gives $\langle b_{\mu} \rangle \neq 0$, such condensation is irrelevent to the gauge condition: $\int d^{2}x d\tau \epsilon^{\mu\nu\lambda} \partial_{\nu} \partial_{\lambda} f^{a} \langle b_{\mu} \rangle = 0$, it does not destroy conservation law

$$\partial_{\mu} j^{a}_{\mu} = 0. \quad (24)$$

The spinon fields $z_{\alpha}$ preserve the $U(1)_{a}$ symmetry and whose condensation defined on $SU(2)/U(1) = S^{2}$, which is equivalent to Néel order.

As discussed by Xu in Ref. 61, the Skyrmion of Néel order carries the quantum number of Kekulé-VBS order, the condensate of Skyrmion breaks Néel order and it also induces Kekulé-VBS order. To make the Néel-Kekulé-VBS duality explicitly, we parametrize the order vector $\vec{n}$ as

$$\vec{n} = [\phi(r)\vec{\sigma}, \sqrt{1 - \phi^{2}(r)}\vec{s}(\tau, \rho)], \quad (25)$$

with $\phi(0) = 0$ and $\phi(\infty) = 1$. Integrating over the space lead to the effective action near the core of vortex defect,

$$S^{\nu} = i\frac{1}{4} \int d\tau d\rho \epsilon^{\alpha\beta\gamma} \epsilon^{abc} s_{a}(\tau, \rho) \partial_{\beta} s_{b}, \quad (26)$$

which is right the WZW term for a 1/2-spin in (0+1)D, that is 1/2-spinon. Therefore, the vortex defect carries the 1/2-spinon quantum number. It’s known that the Kekulé-VBS order enjoy the spectral reflection, symmetry and the vortex defect host single fermionic midgap zero energy mode in the core. If the system preserve the time-reversal symmetry, the Kramers conjugation ensure that there are two zero energy levels, each carries up 1/2-spinon quantum number and down 1/2-spinon quantum number, repectively. The occupation of these zero modes leads to fractionalization of electrons, which gives the following four types of states:

$$f(+1/2 \uparrow, +1/2 \downarrow)(\Delta Q = 1, S_{z} = 0),$$

$$f(-1/2 \uparrow, -1/2 \downarrow)(\Delta Q = -1, S_{z} = 0),$$

$$f(+1/2 \uparrow, -1/2 \downarrow)(\Delta Q = 0, S_{z} = 1/2),$$

$$f(-1/2 \uparrow, +1/2 \downarrow)(\Delta Q = 0, S_{z} = -1/2). \quad (27)$$

The states are the right the spin-charge separation holon, chargeon, and the two spinon states studied in quantum spin Hall effect, and all the viewed particles are bosons. The self duality also result to spin-charge separation. Moreover, the condenses of these bosons will help us understand the rich phase transition like spin liquid-magnetic, superconductor-magnetic transition, etc.

By using sign-free Majorana quantum Monte Carlo simulations, Ref. 16 show the evidences of deconfined QCP between antiferromagnetic and Kekulé-VBS transition. Based on present understands, the WZW term or mutual CS term guarantee the duality between Néel and Kekulé-VBS order, and the phase transition is described by the fractionalized fields $z_{\alpha}$. Therefore, we conclude that the QCP between Néel-Kekulé-VBS transition is deconfined, and such QCP may be generized to other multiple-components order parameter.

IV. THE INDEX THEOREM AND NEEL-KEKULE-VBS DUALITY

The WZW term exhibits mutual duality between Néel and Kekulé-VBS order, we now turn our attention to how the duality relate to index of the topological excitation. For the system with chiral symmetry, there exist a index theorem relate the topological stablility to the number of the zero modes hosting by the topological excitation. Here we first briefly review the Jackiw-Rossi-like mode and the index theorem. And beyond any symmetry, the topological index for twisted order is discussed in the sense of Néel-Kekulé-VBS duality.

A. The nontrivial twisted order pattern

The Kekulé-VBS order pattern corresponds to the modulation of fermion bilinear operator with different sublattice and Dirac point, and thus preserve chiral symmetry. Without considering the spin degrees of freedom, if $\psi$ enter as the Kekulé-VBS mass matrix, the matrix fields $V_{ab} = \psi^{\dagger} M_{ab} \psi$ condenses, for instance

$$\langle V_{10} \rangle = |\Delta| \langle \psi^{\dagger} \gamma_{1} \sigma_{0} \psi \rangle \neq 0, \quad (28)$$

such condenses gapped out the nodal quasiparticles. In order to demonstrate the connection between the twist Kekulé-VBS order pattern and midgap zero energy states at the Dirac equation level, it’s useful to suppose that the order pattern could be slowly varying on the scale of the lattice spacing, and the nontrivial background topology will be realized by the complex valued mass matrix field. In continuum limit, the complex Kekulé-VBS mass matrix as $V = V_{Re} + iV_{Im}$, with $V_{Re} = \Delta \langle \psi^{\dagger} \gamma_{1} \sigma_{0} \psi \rangle$ and $V_{Im} = \Delta \langle \psi^{\dagger} \gamma_{2} \sigma_{0} \psi \rangle$. Thus, the fermion interacting with the twist Kekulé-VBS order according to the Lagrangian

$$L_{\text{VBS}} = \bar{\psi} \gamma^{\mu} (i\partial_{\mu} + \gamma_{5} \partial_{\mu}) \psi - \bar{\psi} \beta \eta \psi, \quad (29)$$

$$V = \beta_{1} \cos \theta + \beta_{2} \sin \theta,$$
where $\beta = \tau_0\sigma_3, \gamma_1 = \tau_1\sigma_0, \beta_2 = \tau_2\sigma_0$, and the gauge potential $\psi_{\mathbf{x}}$ choose with vanishing temporal component $a_0 = 0$. The phase twist in such Kekulé-VBS mass matrix realize domain wall or vortex at two dimensional spatial, indeed, the background topology for the Lagrangian equal to the Jackiw-Rossi model that describe charged fermion interacting with the scalar fields of the two dimensional Abelian Higgs model.

It’s known that the soliton excitations are associated with many profound physical phenomena like Fractionalization, topological degeneracy and zero energy quasiparticle. The occupation of zero energy soliton lead to spin-charge separation, before discussing the wavefunction of the zero mode solutions, we first make some discussion about the Lagrangian $L_{\text{VBS}}$ and the Fu-Kane model that describe the surface of topological insulator with a conventional superconductor proximity to it in terms of the Hamiltonian

$$H_{FK} = \tau_2I(\sigma, \mathbf{k} - \mu) + \Delta(\beta_1\cos\theta + \beta_2\sin\theta).$$

$H_{FK}$ does not breaking time reversal symmetry, and the Pauli matrix $\tau_2$ mixed the particle and hole part. When the chemical potential satisfy $\mu > |\Delta|$, due to the $Z_2$ topological index, there is at least one topologically protected zero mode localized on the odd vorticity vortex. When $\mu = 0$, the massive vortex (non-trivial twist of background field) also guarantee the existence of zero modes and now the model equal to $L_{\text{VBS}}$.

Many literatures have studied the single-valued and normalizable zero energy solution in the presence of nontrivial twist fields. For $|n|$-twisted order parameter, there are $|n|$ independent normalizable zero energy states, the property and the special form for each zero energy solution may differ. The Hamiltonian subjected to the Kekulé-VBS order pattern is given by

$$H = \sigma_i(-i\partial_i - \gamma_5a_i) + \Delta(r)(\beta_1\cos\theta + \beta_2\sin\theta),$$

and the normalizable zero energy solutions have been discussed in appendix B. When the degree of twisting for the Kekulé-VBS order equal to $2k$, there are $2k$ normalizable zero modes, and each zero energy mode spinors characterized by two phase dependence $\psi_{\mathbf{k}a} = f e^{i\theta_1} + g e^{i\theta_2}\theta (l_1 \neq l_2$ or $m_1 \neq m_2)$: When the whole twist of the Kekulé-VBS order equal to $2k + 1$, there are $2k + 1$ zero modes, among which only one for single phase dependence($l_1 = l_2$) and the others characterized by two phase dependence. The normalizable single phase dependence wave function for zero mode on sublattice-a can be easily obtained as

$$\psi_{\mathbf{k}a} = N_a e^{-\int_0^r (\Delta(r) + \frac{2\pi|n-1|}{2\pi} dr) e^{i\frac{(n-1)\theta}{2}}}.$$

Since $\Delta(r)$ and $a(r)$ vanish at small $r$, the wave function to be normalizable require $n \leq -1$. Similarly, the normalizable single-phase dependence zero mode on sublattice-b as

$$\psi_{\mathbf{k}b} = N_b e^{-\int_0^r (\Delta(r) + \frac{2\pi(n+1)}{2\pi} dr) e^{i\frac{(n+1)\theta}{2}}},$$

$$\psi_{\mathbf{k}'b} = -i\psi_{\mathbf{k}b}.$$

The wave function at small $r$ to be normalizable require $n \geq 1$, and same condition can be seen from Ref. 33. Since the sublattice symmetry (or named chiral symmetry), the zero mode wave function on sublattice-a also hold on sublattice-b with the substitutions $\psi_{\mathbf{k}a} \rightarrow \psi_{\mathbf{k}b}$ and $n \rightarrow -n$. The chiral operation is defined as $\Gamma_5 = \tau_3\sigma_3$, which gives $\Gamma_5\psi_{\mathbf{k}a/b} = \pm 1$. The sublattice symmetry ensure that the energy eigenstate $\pm E$ come into pairs, thus the zero modes bound to the VBS vortex are classified by $Z_2$. Only for odd twisting, there is at least a topologically protected Fermion zero mode. If the system possess particle-hole symmetry, which will lead to Majorana zero mode, such as topological superconductor studied in Ref. 57.

B. The topological index and the Néel-Kekulé VBS duality

The index theorem bridge the total number and the topological stability of the zero modes, which relate the analytical index of a Dirac operator to the the winding number of background scalar field in 2D spatial. It’s easily show that the Hamiltonian in Eq. (31) connect with an elliptic differential operator for vanishing energy. For the system with chiral symmetry, the mid-gap states are always come into pairs, and the difference between the number of zero mode with opposite chirality give the analytical index $\text{ind}H = n_+ - n_-$. The eigen function for vanishing energy is

$$\sigma_2(i\partial_1 + a_1)\psi - \sigma_1(i\partial_2 + a_2)\psi + \Delta\sigma_2^*\psi^* = 0.$$

By writing $\psi(x)$ as two real function $\psi(x) = \psi_1(x) + i\psi_2(x)$, and expressing $\Delta = \Delta_1 + i\Delta_2$, the corresponding differential operator is of the form

$$D = (-\partial_2 - i\gamma_2\partial_1) + (a_1 - i\gamma_2a_2) + (\gamma_3\Delta_1 + \gamma_1\Delta_2).$$

The index theorem for $D$ gives the analytical index

$$\text{ind}D = \frac{1}{2\pi} \int d^2x \text{Tr} (D^{-1}).$$

The index theorem is applicable only when the system preserve chiral symmetry. Beyond chiral symmetry, the Néel-Kekulé VBS duality provide us an alternative way to derive topological index via dual topological excitation. For present $(2+1)$D Dirac fermion interacting with the background scalar fields(complex Kekulé-VBS order), the dual target space can be obtained by considering the extended order parameter $\phi = (\text{Re}\Delta, \text{Im}\Delta, h)$ according to the Lagrangian

$$L = \frac{1}{4}i\gamma_5(\partial_\mu - i\gamma_5a_\mu)\psi - \psi(\sum_{k=1}^3 \Gamma_k\phi_k)\psi.$$

Since $\Delta(r)$ and $a(r)$ vanish at small $r$, the wave function to be normalizable require $n \leq -1$. Similarly, the normalizable single-phase dependence zero mode on sublattice-b as

$$\psi_{\mathbf{k}b} = N_b e^{-\int_0^r (\Delta(r) + \frac{2\pi(n+1)}{2\pi} dr) e^{i\frac{(n+1)\theta}{2}}},$$

$$\psi_{\mathbf{k}'b} = -i\psi_{\mathbf{k}b}.$$
As shown in appendix C, the topological current

$$J^\mu = -\frac{1}{8\pi\phi^3}e^{\mu\nu\lambda}e^{abc}\phi_\alpha\partial_\nu\phi_\beta\partial_\lambda\phi_c,$$  \hspace{1cm} (35)

and the topological number $Q = \int d^2x J^0$ classifies all the homotopy classes $\tau_2(S^2)$, which is equivalent to the degree of mapping from $T^2$ to $S^2$. We suppose the VBS vortex $\phi_c = (\text{Re}\Delta, \text{Im}\Delta)$ lie on the plane where $h = 0$. The half-degrees of mapping from $T^2$ to $S^2(h > 0)$ can be separated into two parts,

$$\frac{\Omega}{2} = \int d^2x J^0(a = 3) + \int d^2x J^0(a \neq 3).$$  \hspace{1cm} (36)

One can easily show that (by direct computations), the second term vanishes when $h$ is regularized to zero. Now the half-degrees of mapping then becomes

$$\frac{\Omega}{2} = -\int d^2x e^{ij}e^{ab}h\partial_i\phi_a\partial_j\phi_b/8\pi\phi^3$$
$$= -\int h e^{ab}\partial_a \wedge \partial_b /4\pi\phi^3$$  \hspace{1cm} (37)

Writing $\phi = \Delta(r)\phi(\theta)$, so $d\phi_a \wedge d\phi_b = \partial_i\phi_a\partial_j\phi_b dr \wedge d\theta$, now the half-degrees of mapping is given by

$$\frac{\Omega}{2} = -\frac{1}{4\pi}\int_0^\infty \frac{h d\Delta^2}{(\Delta^2 + h^2)^{3/2}} \int d\theta e^{ab}e^{ab}\Delta\partial_a\Delta\partial_b$$
$$= \frac{1}{4\pi}\int d\theta e^{ab}e^{ab}\Delta\partial_a\Delta\partial_b$$  \hspace{1cm} (38)

Thus we obtain the well known result

$$\Omega = \frac{1}{2\pi}\int d\theta e^{ab}e^{ab}\Delta\partial_a\Delta\partial_b,$$  \hspace{1cm} (39)

this quantum number gives the difference between the number of zero mode with opposite chirality, from which the number of zero modes can be identified. Thus, the analytic index of a Dirac operator relate to a topological defect of mapping in its dual topological defect, at least valid for present case. The Néel-Kekulé-VBS duality imply that there exist an invariant connects the following two theories:

$$L_1 = \bar{\psi}i\gamma \cdot (\partial - i\gamma_5a)\psi + m(r)e^{i\gamma_0}\bar{\psi}\psi + \cdots,$$  \hspace{1cm} (40)
$$L_2 = \bar{\psi}i\gamma \cdot (\partial - ib)\psi + \bar{\psi}i\gamma_5\partial\psi + \cdots.$$  \hspace{1cm} (41)

**V. CONCLUSIONS AND DISCUSSION**

In conclusion, motivated by the predictable conclusion that the quantum critical point between Kekulé valence-bond-solid (VBS) and antiferromagnetic is deconfined in Ref. 16, we have studied the dynamical spontaneous symmetry breaking in two dimensional graphene Dirac semimetal, and discussed the Néel-Kekulé VBS mutual duality and the quantum transition between them. We consider the general quartic fermion-bilinear interactions that will generate rich order parameters when the dynamical symmetry is broken. One feature in our study is the appearance of multicritical point for various orders, which is shown to irrelevant to the special form of quartic fermion-bilinear but only depends on the dimensions of the mass matrix (each mass matrix corresponds to an order parameter). The multicritical point exhibits the Landau-forbidden quantum phase transition, and the key ingredient is the anticommutativity among these mass matrices. Very recently, we observed that Ref. 38 have numerically shown the multicritical and the continuous Néel-Kekulé VBS transition, which is consistent with our results.

Further, we have shown that the typical Néel-Kekulé VBS quantum criticality can be understood in term of the mutual duality between them. By using the 'super-spin' non-linear sigma field theory, the Néel-Kekulé VBS quantum criticality is captured by the WZW action or self BF Chern-Simons field theory, which reveal the mutual duality for Néel and Kekulé VBS order. Statically speaking, the mutual duality in the sense that the topological defect in either phase carries the quantum number of another phase. Dynamically, the mutual duality in the sense that the order parameter in either phases carries the symmetry of another phase, which embodies in the Chern-Simons action. The breaking of symmetry in one phase meanwhile breaks the dual symmetry and lead to the emergence of another order. Since the transition is described by the CP(1) fields coupled with $U(1)$ gauge field, the quantum critical point for Néel-Kekulé transition is deconfined. We also saw that the mutual duality lead to the profound phenomenon of spin-charge separation, and the concept of duality may help us understand more rich physics.

We now make some comments about the concept of duality. The Néel-Kekulé VBS duality in present paper means self-duality near two sides of the critical point rather than duality between two seemingly different theories. Various recent study concentrate on the duality between the theoretical fields theory in high-energy physics and the quantum criticality in condensed matter physics. For example, the $N_f = 2$ noncompact QED$_3$ and the easy-plane $U(1)$ noncompact CP$_1$ model$^{64,65}$ describing the Néel-VBS transition in magnets with XY spin symmetry, the QED$_3$-Gross-Neveu model and SU(2) noncompact CP$_1$ model$^{66}$. The duality in the sense that both two theories are described by the same effective low-energy field theory and are characterized by same critical behavior at criticality. The hidden duality relate the interacting driven topological quantum phase transition and the quantum criticality, and which will deepen our consensus for the fundament.

Finally, let us give a summary and prospect for Landau-forbidden quantum criticality, which features the follows: (i) Two phases are symmetry-incompatible. (ii) There exist mutual duality between the two phases, as stated in the above. (iii) The space-time topological ex-
citations must be involved in the phase transition process. (iv) The phase transition from phase-B to phase-A may destroy some structure (like entanglement or interacting symmetry protected topological phase) belong to the phase-B. The study of FIQC\textsuperscript{16} and symmetric mass generation (many-body gap\textsuperscript{19}) give some new insights for us, and the duality might shed some light on this topic. More, the surface realization of Landau-beyond quantum criticality for strong correlated topological states in high dimensional is also an interesting issue to be explored.

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Appendix A: Derivation of nonlinear sigma model with WZW term

This appendix is aimed to derive the low energy effective theory for the Néel-Kekulé VBS SO(5) order parameters. The perturbation which used in the derivation named gradient expansion or large-m expansion, and such procedure had been extensively adopted. In the presence of slowly varying background, the total action included Fermi and Boson fields reads

\[ S[\psi, \bar{\psi}, \bar{n}] = \int d^3x \bar{\psi}(i\gamma^\mu \partial_\mu - m \Upsilon l_n)\psi, \]

where $\Upsilon^1 = s_1 \gamma_1 I, \Upsilon^2 = s_2 \gamma_2 I, \Upsilon^3 = s_3 \gamma_3 I, \Upsilon^4 = I \gamma_1 \sigma_3, \Upsilon^5 = \gamma_2 \sigma_3 I$ and $I$ denotes the $2 \times 2$ identity matrix.

Following the spirit of Abanov and Wiegmann\textsuperscript{53}, the superspin nonlinear sigma model and the WZW term can be derived by integrating over the fermions. When gapped fermions are integrated out, the effective action for the boson fields is obtained as

\[ e^{iS_e[\bar{n}]} = \int D[\psi, \bar{\psi}] e^{-S[\psi, \bar{\psi}, \bar{n}]} \]

\[ S_e[\bar{n}] = -i tr \ln(i\gamma^\mu \partial_\mu - m \Upsilon l_n). \]

The variation $\delta S_e[\bar{n}]$ respect to $n$ gives

\[ \delta S_{e(n)}^{(2)}[\bar{n}] = -i tr((i\gamma^\alpha \partial_\alpha - m \Upsilon^a n_a)^{-1}(-m \Upsilon l_n) \]

\[ = -i \sum_n (-1)^n tr\left\{\frac{1}{2}(\partial^2 + m^2)^{-1}(-im \Upsilon^c \gamma^\alpha \partial_\alpha n_c)\right\} \]

\[ \times \partial^2 + m^2)^{-1}(-i\gamma^\beta \partial_\beta - m \Upsilon^a n_a)(-m \Upsilon l_n), \]

where the trace represents the trace over the spatial and momenta coordinates. It’s easily observed that the $\delta S_{e(1)}^{(1)}[\bar{n}]$ is non-vanished,

\[ \delta S_{e(1)}^{(1)}[\bar{n}] = \frac{i m^2 tr[I]}{2\pi} \int d^3x \bar{\psi}(m^2 - k^2)^{-1} \partial_\alpha n_\alpha \delta \partial_\delta n_\delta |^{(m^2 - k^2)^{-1} \delta \partial_\delta n_\delta}, \]

\[ \]

where the integrand

\[ \]

\[ N = \int \frac{d^3k}{(2\pi)^3} \frac{m^5}{(m^2 - k^2)^4} = \frac{2\pi i}{4! \text{Area}(S^4)}. \]

The trace in the right of Eq. (A5) indeed a product of two anti-symmetrical tensors, which gives by

\[ \text{tr}(\Upsilon^a \gamma^\alpha \Upsilon^b \gamma^\beta \Upsilon^c \gamma^\tau \Upsilon^d \gamma^\tau) = 8i\epsilon^\alpha\beta\gamma\epsilon^{abcde}. \]

Let $\bar{n}(x, \rho)$ distort continuously such that $\bar{n}(x, 0) = (n_1 = 1, \bar{0})$ and $\bar{n}(x, \rho = 1) = \bar{n}(x)$, the $\bar{n}(x)$ is thus defined on the four dimensional spacetime disk $D^4$ with boundary $\partial D^4 = S^3$. Therefore, the field $\bar{n}(x)$ defines a map from $S^3$ to $S^3$. We can introduce an auxiliary variable $\rho \in [0, 1]$, then $\delta S_{e(3)}^{(3)}[\bar{n}]$ is rewritten as

\[ \delta S_{e(3)}^{(3)}[\bar{n}] = 8i N \epsilon^\alpha\beta\gamma\epsilon^{abcde} \int d^5\rho \times \]

\[ \partial_\rho \{[\partial_\alpha n_\alpha(\partial_\beta n_\beta(+\partial_\gamma n_\gamma(+\partial_\delta n_\delta(\delta n_\delta)))\}, \]

which indeed the functional derivative of the WZW term. The fields variable only defined on the boundary, note that the term

\[ \epsilon^\alpha\beta\gamma\epsilon^{abcde} \int d^3x \int_0^1 d\rho \times \partial_\rho \{[\partial_\alpha n_\alpha(\partial_\beta n_\beta(+\partial_\gamma n_\gamma(+\partial_\delta n_\delta(\delta n_\delta)))\}, \]

need to vanish, since for the small changes, $\bar{n} \rightarrow \bar{n} + \delta \bar{n}$, $\delta n_\alpha$ is perpendicular to $n_\alpha$. Combined all these results lead to the Euclidean WZW action $S_{\text{WZW}}[\bar{n}] = 2\pi i S$, \[ S = \frac{2}{4! \text{Area}(S^4)} \epsilon^\alpha\beta\gamma\epsilon^{abcde} \int d^2x d\tau \int_0^1 d\rho \times \]

\[ (\partial_\nu n_\nu)(\partial_\beta n_\beta)(\partial_\gamma n_\gamma(\partial_\delta n_\delta)n_\delta). \]
Appendix B: Derivation of the zero mode under twist order

In this appendix, we discuss the zero-energy modes for fermion-vortex system with n-twist background fields. The Hamiltonian subjected to a complex Kekulé VBS order parameter(VBS order pattern with a real and imaginary part) is given by

$$H = \alpha_i(-i\partial_t - \gamma_3a_i) + \Delta(\beta_1 \cos \theta + \beta_2 \sin \theta), \quad (B1)$$

where $\alpha_1 = -\gamma_3a_2, \alpha_2 = \gamma_3a_1$, and the eigenvalue equation reads $H\psi_0 = 0$ for zero energy modes, with

$$\psi_0 = (\psi_1, \sigma_2 \psi_1^t)^t.$$ \quad (B2)

We assume the system is isotropic and set $\psi_1 = [f(r, \theta), g(r, \theta)]^t$, then the vanishing energy solutions is determined by

$$[\partial_+ + (a_2 - ia_1)]f - i\Delta f^* = 0,$n$$

$$[\partial_+ - (a_2 + ia_1)]g + i\Delta g^* = 0,$$ \quad (B3)

where $\partial_\pm = e^{i\theta}(\partial_r \pm i\partial_\theta/2)$. We consider a n-twists of the scalar field, so the decomposition of $\Delta$ in polar coordinates is $\Delta = \Delta(r)e^{in\theta}$, $\Delta(r)$ vanishes at the origin and tend to a constant at infinity. The static axial gauge field $\vec{\alpha}$ integral over a closed loop that encircles the origin yields the degree of twisting, which we can parameterize as

$$a_j = -n\varepsilon_{ij}r_ja(r)/r^2, a(r \to \infty) = 1/2.$$ \quad (B4)

The existence of gauge field only play the role of changing the profile of background but not the zero energy eigenvalue. Using the polar coordinates, the eigen equation for vanishing energy take the form

$$e^{i\theta}[\partial_+ + (a_2 - ia_1)]f - i\Delta f^* = 0,$$ \quad (B5)

$$e^{-i\theta}[\partial_+ - (a_2 + ia_1)]g + i\Delta g^* = 0.$$

The general ansatz for the zero energy eigenvectors may be chosen as

$$f(r, \theta) = e^{-i\theta}[f_1(r)e^{i\eta_1} + f_2(r)e^{i\eta_2}],$$

$$g(r, \theta) = e^{i\theta}[g_1(r)e^{im_1\theta} + g_2(r)e^{im_2\theta}],$$ \quad (B6)

where both $f_i(r)$ and $g_i(r)$ are real. The radial equations for $f_i(r)$ then becomes

$$\left(\partial_r - \frac{l_1}{r} + \frac{na(r)}{r}\right)f_1 + \Delta(r)f_2 = 0,$$ \quad (B7)

$$\left(\partial_r - \frac{l_2}{r} + \frac{na(r)}{r}\right)f_2 + \Delta(r)f_1 = 0,$$

the corresponding compatibility condition satisfies $l_1 + l_2 = n - 1$. And the radial equations for $g_i(r)$ becomes

$$\left(\partial_r + \frac{m_1}{r} - \frac{na(r)}{r}\right)g_1 + \Delta(r)g_2 = 0,$$ \quad (B8)

$$\left(\partial_r + \frac{m_2}{r} - \frac{na(r)}{r}\right)g_2 + \Delta(r)g_1 = 0,$$

the corresponding compatibility condition is $m_1 + m_2 = n + 1$.

The index theorem state that the analytical index is identical to the the number of zero modes, and also identical to the winding number of the background fields. Here the normalizable zero mode is given by $\psi_0 = (\psi_{k\alpha}, \psi_{k\beta}, \psi_{k'\alpha})^t$. According to the compatibility condition, if the whole twist of the Kekulé-VBS order is even, all the normalizable zero modes are characterized by two-phase dependence($l_1 \neq l_2$ or $m_1 \neq m_2$); If the whole twist of the order is odd, one of normalizable zero modes is characterized by single-phase dependence($l_1 = l_2$), and the others $n - 1$ normalizable modes are two-phase dependence. The single phase dependence wave function for zero mode on sublattice-a is

$$\psi_{k\alpha} = N_a e^{-\int_0^\infty \left(\frac{2\pi a(n-1)}{2} - \frac{\pi a n}{2}\right)dr}e^{i\frac{x_1a(n-1)}{2} + \frac{x_1a n}{2}},$$ \quad (B9)

$$\psi_{k\beta} = i\bar{\psi}_{k\alpha}.$$ \quad (B10)

The wave function at small $r$ to be normalizable require $n \leq -1$. Similarly, the normalizable single-phase dependence zero mode on sublattice-b as

$$\psi_{k\beta} = N_b e^{-\int_0^\infty \left(\frac{\pi a(n-1)}{2} - \frac{\pi a n}{2}\right)dr}e^{i\frac{x_1a(n-1)}{2} + \frac{x_1a n}{2}}.$$ \quad (B11)

$$\psi_{k\alpha} = -i\bar{\psi}_{k\beta}.$$ \quad (B12)

The wave function at small $r$ to be normalizable require $n \geq 1$. With the substitutions $\psi_{k\alpha} \to \psi_{k\beta}$ and $n \to -n$, the zero mode wave function on sublattice-a also hold on sublattice-b.

Appendix C: Derivation of the topological current

This appendix detailed the topological current in the presence of extended complex VBS order parameter,

$$\vec{\phi} = (Re \Delta, Im \Delta, h).$$ \quad (C1)

We consider the Lagrangian

$$L = \bar{\psi}i\gamma^\mu(\partial_\mu - i\gamma_3a_\mu)\psi - \bar{\psi}\left(\sum_{k=1}^3 \Gamma_k \phi_k\right)\psi,$$ \quad (C2)

where we assume $h > 0$, and

$$\Gamma_1 = \beta \beta_1 = \tau_1 \sigma_3,$$ \quad (C3)

$$\Gamma_2 = \beta \beta_2 = \tau_2 \sigma_3,$$ \quad (C4)

$$\Gamma_3 = \beta \beta_3 = \tau_0 \sigma_3.$$ \quad (C5)

We should note that the generalized order parameter breaks chiral symmetry. Integrating over fermion fields, the effective Euclidean action $S_{eff}[\vec{a}, \mu]$ is obtained via

$$\int d\bar{\psi}d\psi e^{-\int d^3xL(\bar{\psi}, \psi, \vec{\alpha})} = e^{-S_{eff}[\vec{a}, \mu]},$$ \quad (C6)
the current is defined in terms of $J^\mu = \delta S^E_{\mu\nu}/\delta a_\mu|_{a_\nu=0}$, Again, using the $1/m$-perturbation expansion
\begin{equation}
J^\mu = -\frac{\delta S^E_{\mu\nu}}{\delta a_\mu} = -\text{Tr}\{\Gamma_5 \gamma^\nu G_0^{-1} [G_0^{-1}(G_0^{-1})^{-1}] \},
\end{equation}
where $G_0^{-1} = i\gamma^\nu \partial_\mu - \Gamma_k \phi_k$. The large-$\phi$ expansion lead to the current
\begin{equation}
J^\mu = -\sum_{l=0}^{\infty} \text{Tr}\{\Gamma_5 \gamma^\nu \mu \frac{i\gamma^\nu \partial_\mu - \Gamma_k \phi_k}{-\partial^2 + \phi^2} \left( \frac{-i\Gamma_5 \gamma^\nu \partial_\mu \phi_k}{-\partial^2 + \phi^2} \right)^l \}.
\end{equation}
It’s easily check that both $l = 0$ and $l = 1$ term vanish and the dominant term is
\begin{align*}
J^\mu &= \text{Tr}\{\Gamma_5 \gamma^\nu \mu \frac{\Gamma_k \phi_k}{-\partial^2 + \phi^2} \left( \frac{-i\gamma^\nu \Gamma_k \phi_k}{-\partial^2 + \phi^2} \right)^2 \} \\
&= \int \frac{d^2 k}{(k^2 + \phi^2)^3} \text{Tr}\{\gamma^\nu \gamma^\nu \gamma^i \Gamma_{5a} \Gamma_b \Gamma_{c} \phi^a \phi^b \phi^c \phi_d \partial_\mu \phi^c \} \\
&= -\frac{1}{8\pi} e^{\mu \lambda \gamma i a b c} \phi^a \phi^b \phi^c \phi_d \phi^d \phi^c \gamma^i \partial_\mu \phi^c \phi^c \\
&= \frac{1}{Q} \int d^2 x J^0.
\end{align*}

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