Aspects of D-brane actions

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A couple of issues concerning the effective dynamics of D-branes in string theory are discussed. Primarily, I am concerned with linearization of the actions by introduction of non-propagating fields and a full super- and \(\kappa\)-symmetric description of D-branes. This contribution summarizes the work of refs. [1–3].

1. BACKGROUND

D-branes, or Dirichlet branes, play a central rôle in non-perturbative superstring theory [4–6]. They occur as soliton-like solutions of the low-energy effective actions for the massless sector of type II string theories, and provide hypersurfaces of Dirichlet boundary conditions for open strings. D-branes are the objects in string theory carrying Ramond–Ramond (RR) charges, i.e., they couple to the antisymmetric tensor fields in the massless RR sector of type II string theory. We call the D-brane configurations “soliton-like” because their rôle seems to be intermediate between fundamental and truly solitonic excitations in that they do not act as fundamental excitation in any dual picture of the theory.

One part of the effective actions for D-branes, namely the “kinetic term”, containing the couplings to the massless fields in the NS–NS sector, has been known for quite some time, and was obtained by demanding that the \(\beta\)-functions corresponding to these fields for an open string ending on a D-brane vanishes [4]. Boundary contributions are cancelled by the classical variation of the effective D-brane action, which turns out to be of Dirac–Born–Infeld (DBI) form:

\[
I_{\text{DBI}} = -\int d^{p+1}\xi e^{-\phi} \sqrt{-\det(g_{ij} + \mathcal{F}_{ij})}.
\]

This action contains the dilaton \(\phi\), the target space metric \(g_{mn}\) via its pullback to the world-volume \(g_{ij}\) and the antisymmetric tensor field \(B_{mn}\) via \(\mathcal{F}_{ij} = \frac{\alpha'}{2\pi} F_{ij} - B_{ij}\). The bosonic world-volume degrees of freedom are thus the embedding together with a world-volume U(1) potential \(A\), \(F = dA\). At this point, D-branes differ from ordinary \(p\)-branes in that there are bosonic world-volume fields in non-scalar representations, a fact that becomes important when one wants to write down supersymmetric actions. The DBI action thus contains the kinetic part of the action for the bosonic degrees of freedom, as well as the coupling to the massless fields in the NS–NS sector.

Another piece of information concerning the actions is the form of the coupling to the RR antisymmetric tensor potentials. To its full extent, this was first formulated in refs. [8,9]. To the DBI action should be added a Wess-Zumino (WZ) term of the form

\[
I_{\text{WZ}} = \int e^\Phi C,
\]

where \(C\) is the cochain of RR potentials together with the potentials for the dual field-strengths. The products (including the exponential) are understood to be wedge products of forms. Since \(\mathcal{F}\) is a two-form, we see that \(Dp\)-branes of odd \(p\) couple to even potentials, i.e., those of type IIB superstring theory, while those of even \(p\) couple to the odd potentials of type IIA. This of course agrees with the supergravities in which these branes are found as solutions.

Both the formulation of a supersymmetric action and considerations concerning duality prop-
erties demand a delicate interplay between the DBI and WZ parts of the actions.

So, while there has been quite some information about the effective dynamics of D-branes, and about their role in non-perturbative string theory, important pieces of information have been missing. In this contribution, I would like to address two issues. The first of these is the question of whether it is possible to rewrite the DBI action in a more tractable form, using auxiliary non-propagating fields on the world-volume [1]. In string theory, the corresponding procedure takes us from the Nambu–Goto action to the Brink–DiVecchia–Howe action [11], which is much more useful in that the coordinates simply become free scalars. We do not expect the same dramatic simplification as in string theory, of course, since an auxiliary world-volume metric cannot be gauge-fixed to the same extent as in the case $p = 1$, but such a formulation might still be illuminating, especially with respect to symmetries. The second issue is the inclusion of fermionic degrees of freedom on the world-volume [2,3]. We know that D-brane configurations are BPS states, conserving half of the supersymmetry. In order to address this question, we need to understand the mechanism of $\kappa$-symmetry for D-branes. This fermionic gauge symmetry removes half of the fermionic variables on the world-volume, so that the physical ones may generate the appropriate number of states for a BPS-saturated multiplet.

At the time of the conference, there were only some partial results on supersymmetric D-branes. Some of the key ideas used in refs. [2,3] were developed during this meeting, and since one of the main points of the talk was supersymmetrization, I found it appropriate rather to include results obtained after the date of the talk than to pretend that I still do not know them. Three other papers [12–14] by other authors have also appeared since, that derive parts of the results of refs. [2,3].

I want to take the opportunity to thank the organizers of the 30th Ahrenshoop symposium for an excellent conference, and my collaborators Alexander von Gussich, Aleksandar Miković, Bengt E. W. Nilsson, Per Sundell and Anders Westerberg, whose joint work I report on here.

2. “LINEARIZED” ACTIONS

In string theory, it has been extremely useful to work with a formulation where the area action given in terms of the pullback of a background metric, i.e., the Nambu–Goto action

$$I_{NG} = -\int d^2 \xi \sqrt{-\det g_{ij}},$$

is replaced by an equivalent action containing an auxiliary non-propagating world-sheet metric $\gamma$,

$$I = -\frac{1}{2} \int d^2 \xi \sqrt{-\gamma \gamma^{ij} \partial_i X^m \partial_j X^n g_{mn}}.$$  

The embedding coordinates become ordinary free scalars on world-sheet, and this allows for the extremely powerful machinery of conformal field theory. Although the possible gauge choices for higher-dimensional objects are not expected to lead to as dramatic simplifications as for strings, it might still be relevant to ask whether a similar reformulation is possible. A formulation where the dynamical degrees of freedom behave as free fields would in many respects, such as supersymmetrization, be preferable. We refer to action containing the auxiliary metric as Brink–DiVecchia–Howe–Tucker (BDHT) type actions [11,15].

It is well known that any $p$-brane involving no other bosonic world-volume degrees of freedom than scalars allows for such a formulation. It is a straightforward generalization of the string action, namely

$$I = -\frac{1}{2} \int d^{p+1} \xi \sqrt{-\gamma} \left\{ \gamma^{ij} \partial_i X^m \partial_j X^n g_{mn} \right\}.$$  

Elimination of $\gamma$ via its (algebraic) equations of motion yields the equivalent action

$$I = -\int d^{p+1} \xi \sqrt{-\det g_{ij}}.$$  

where $g$ is the pullback of the background metric, $g_{ij} = \partial_i X^m \partial_j X^n g_{mn}$. The solution of the equations of motion for $\gamma$ is simply $\gamma = g$.

In ref. [1], we investigate the generalization of this procedure to D-branes, where as mentioned one has a world-volume vector potential in addition to the embedding coordinates. The hope
was that, by the introduction of an auxiliary world-volume metric, the Dirac–Born–Infeld action would become more “linear”, and that this would be the appropriate setting for addressing issues like supersymmetrization. The basic assumption that was made was that the embedding coordinates should continue to behave as free fields on the world-volume. We were encouraged by the already known fact [16] that for $p = 2n - 1$ and $p = 2n$. One such set of invariants consists of $\text{tr}X^{2k}$, $k = 1 \ldots n$, and, for odd $p$, $\text{det}X$. There is also a matrix identity, expressing $X^{p+1}$ in terms of lower powers of $X$ and these invariants. Due to symmetry properties, the solution of $u$ in terms of $X$ only contains even powers, and a general Ansatz may be written as $u(X) = \sum_{k=0}^{n} u_k X^{2k}$, where the $u_k$’s are scalar functions of the invariants.

An even more useful basis is one where the matrix $X^2$ is diagonalized. The general structure is that for $p = 2n$ there will be $n$ non-zero eigenvalues with multiplicity two, and a non-degenerate zero eigenvalue, while for $p = 2n - 1$ there will be $n$ non-zero eigenvalues with multiplicity two. When one restricts to the subspace spanned by the even powers of $X$, the degeneracy disappears. The “eigenvectors” $v_0$, $\{v_i\}_{i=1}^n$ are conveniently normalized so that they become projection operators on the appropriate linear subspaces. The expansion we will use is

$$u(X) = u_0 v_0 + \sum_{i=1}^{n} u_i v_i ,$$

(11)

where the $u_i$’s are scalar functions of the invariant eigenvalues $\{\lambda_i\}$ and the first term is present only for even values of $p$.

The observation that simplifies the calculations in this basis is that the unknown function $\varphi$ has the argument $uX$, and this only enters with even powers. Any solution $u(X)$ commutes with $X$, so $\varphi$ only depends on the combination $X^2 u^2 = \sum_{i=1}^{n} \lambda_i u_i^2 v_i$, so that it is a function of the scalars $\{\lambda_i u_i^2\}_{i=1}^n$. It is then straightforward work to write down the equations of motion for the auxiliary metric, and the condition that resubstituting its solution yield the Dirac–Born–Infeld action.

and the solution is obviously $u = 1$. Plugged back into the action, it gives the Nambu–Goto action (3), as promised.

In order to incorporate the tensor field, and to treat all values of $p$, we make the following observation. Due to the antisymmetry of $F$, the number of independent scalar invariants equals the rank of the world-volume Lorentz group, i.e., $n$ for $p = 2n - 1$ and $p = 2n$. One such set of invariants consists of $\text{tr}X^{2k}$, $k = 1 \ldots n$, and, for odd $p$, $\text{det}X$. There is also a matrix identity, expressing $X^{p+1}$ in terms of lower powers of $X$ and these invariants. Due to symmetry properties, the solution of $u$ in terms of $X$ only contains even powers, and a general Ansatz may be written as $u(X) = \sum_{k=0}^{n} u_k X^{2k}$, where the $u_k$’s are scalar functions of the invariants.

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These conditions together become:
\[ p \text{ odd:} \]
\[ 0 = n - 1 - \sum_{j \neq i} u_j - \frac{1}{2} \varphi + \frac{1}{2} u_i \frac{\partial \varphi}{\partial u_i}, \]
\[ i = 1, \ldots, n, \]
\[ \sum u_i + \frac{1}{2} \varphi - (n - 1) = \prod_i (1 - \lambda_i)^{1/2} u_i. \]
\[ p \text{ even:} \]
\[ u_0 = -(2n - 1) + 2 \sum u_i + \varphi, \]
\[ 0 = -u_0 + u_i + \frac{1}{2} u_i \frac{\partial \varphi}{\partial u_i}, \]
\[ i = 1, \ldots, n, \]
\[ u_0 = \prod_i (1 - \lambda_i) u_i^2. \] (12)

To arrive at these equations, we have used the properties of the eigenvectors/projection operators together with the determinant relations det \( u = u_0 \prod_{i=1}^n u_i^2 \) (first factor only present for even \( p \)), det \((1 + X) = \prod_{i=1}^n (1 - \lambda_i)\).

At this point, one would like to look for solutions to these equations. Such solutions provide the BDHT actions, as well as explicit relations between the auxiliary and induces metrics. The solution becomes increasingly difficult with increasing world-volume dimensionality. We will illustrate by starting with the membrane and moving up in \( p \).

The first issue is the (already well known [13]) D2-brane. \( X^2 \) is readily diagonalized with
\[ \lambda_0 = 0, \quad v_0 = 1 - \frac{2}{\sqrt{4X^2}} X^2, \]
\[ \lambda_1 = \frac{1}{2} \sqrt{4X^2}, \quad v_1 = \frac{1}{\sqrt{4X^2}} X^2. \] (13)

(there is no reason to worry about objects that are singular as \( F \to 0 \), they will not occur in the final expressions). When this is inserted in eq. (12), the solution is easily found to be
\[ u_0 = 1 - \lambda_1, \quad u_1 = 1, \quad \varphi = -\lambda_1 u_1^2, \] (14)
or in terms of the variables of the original actions,
\[ \gamma^{-1} g = 1 - \frac{1}{2} \text{tr}(g^{-1} F) + (g^{-1} F)^2, \]
\[ \varphi = -\frac{1}{2} \text{tr}(\gamma^{-1} F)^2. \] (15)

Thus, the very non-linear Dirac–Born–Infeld action becomes the ordinary action for a free Maxwell field when the auxiliary metric is introduced.

For the three-brane, the diagonalization yields
\[ \lambda_\pm = \frac{1}{2} \text{tr} X^2 \pm \sqrt{-\Delta}, \]
\[ v_\pm = \frac{1}{2} \left( 1 \pm \frac{1}{\sqrt{-\Delta}} \left( X^2 - \frac{1}{4} \text{tr} X^2 \right) \right), \] (16)
where \( \Delta(X) = \text{det} X - \frac{1}{16} (\text{tr} X^2)^2 \). Here, it takes some guesswork to find the solution, and we refer to ref. [1] for details. It is
\[ u_\pm = \frac{\sqrt{1 - \Delta_{\pm}}}{1 - \lambda_\pm}, \]
\[ \varphi = 2 \left\{ 1 - \sqrt{\left( 1 + \lambda_+ u_+^2 \right) \left( 1 + \lambda_- u_-^2 \right)} \right\} \]
\[ = 2 \left\{ 1 - \sqrt{1 + \frac{1}{4} \text{tr}(uX)^2 + \text{det}(uX)} \right\}. \] (17)

The situation is clearly more non-linear than for the membrane, and the hope that the presence of an auxiliary metric would simplify the action does not seem to be fulfilled. One may make two curious remarks here. The first one is that although the induced and auxiliary metrics are related via an equation that is non-polynomial in the field strength, their determinants coincide (this follows from \( u_+ u_- = 1 \)). We have not really understood why this should happen, and it may be a clue to some interesting structure. The second one is that if \( \Delta = 0 \), the entire non-linearity dissolves, and one again gets the free Maxwell action. This observation might be interesting for branes with signature (2,2), where such a constraint is equivalent to selfduality or anti-selfduality (in signature (1,3) it is too strong, and makes all the \( F \) contributions to the action vanish). It should also be pointed out that the occurrence of a square root is not directly related to the square root in the Dirac–Born–Infeld action, but rather peculiar to \( p = 3 \).

For the case \( p = 4 \), we are in the peculiar situation that we have been able to derive the solution of the field equations for \( \gamma \), but not a closed form for the action they descend from. The non-zero eigenvalues are \( \lambda_\pm = \frac{1}{2} \text{tr} X^2 \pm \left[ \frac{1}{4} \text{tr} X^4 - \frac{1}{16} (\text{tr} X^2)^2 \right]^{1/2} \), and the relation between the auxiliary and the induced metric is given by
\[ u_\pm = (1 - \lambda_\pm)^{-2/3} (1 - \lambda_\pm)^{1/3}. \] The function \( \varphi \) has as its arguments \( t_\pm = \lambda_\pm u_\pm^2 \), and it is given
implicitly by
\[ t_\pm = u_\pm^2 - \frac{1}{u_\pm}, \]
\[ \phi = 3 + \frac{1}{u_+ u_-} - 2 (u_+ + u_-) . \]  
(18)

Ironically enough, elimination of \( u_\pm \) from these equations amounts to solving a fifth order equation.

To conclude this section, we have devised a systematic procedure for introducing the auxiliary metric and writing down the BDHT action equivalent to the DBI action. A set of equations for general value of \( p \) has been written down, and it has been solved for \( p=2,3 \) and 4. This program started out with the hope that there would be simplifications as compared to the Dirac–Born–Infeld actions, but so far rather the contrary has happened. A positive attitude would perhaps be that the correct set of auxiliary fields still has to be found. As a consequence, the work described in the following section, where the super- and \( \kappa \)-symmetric D-brane actions are formulated, is performed entirely with the use of Dirac–Born–Infeld kinetic actions.

### 3. ACTIONS FOR SUPERSYMMETRIC D-BRANES

This section reviews the work of refs. [2,3]. In the first section of this contribution (eqs. (1) and (2)), we described the action for a purely bosonic D-brane, containing couplings to the massless bosonic fields of type II string theories. The most efficient way of writing down the full supersymmetric action is to replace all background fields by the corresponding superfields, and view the bosonic D-brane world-volume as propagating in a superspace, i.e., the pull-backs are performed by \( \partial_i Z^M \), where \( Z^M = (X^m, \theta^\alpha) \) are coordinates for the appropriate superspace. In such a formulation, supersymmetry will be manifest. We will however perform a little manipulation on the expressions of section 1. The reason for this is that the metric entering eq. (1) is the string metric, and not the Einstein metric. To this end we absorb a dilaton factor in the metric, and write the action as
\[ I = -\int d^{p+1}\xi e^{-\frac{p+2}{2}\phi} \sqrt{-\det(g_{ij} + e^{-\frac{1}{2}\phi}F_{ij})} + \int e^\phi C . \]  
(19)

The relevant superspaces are those used in the superspace formulation of the massless sectors of type IIA and IIB superstrings, i.e., of type IIA and IIB supergravity. The type IIA superspace contains two ten-dimensional Majorana-Weyl spinors of opposite chirality, while in type IIB the chiralities are equal. In either case, we count to a number of 32 real fermionic coordinates, which from a world-volume point of view are propagating fields. This number is cut to half by the fields equations. If we compare to the bosonic sector, containing the 9-\( p \) transverse components of the embedding and the \( p-1 \) of the U(1) potential, there is an obvious mismatch of a factor two. The D-brane world-volume field theory can not have a matching number of bosons and fermions unless there is a mechanism that reduces the number of fermions to eight. This is the role of \( \kappa \)-symmetry. It is the fermionic gauge symmetry that gauges away half a spinor. The necessity of \( \kappa \)-symmetry can also be understood from the observation that D-branes are BPS-saturated states, breaking half the supersymmetry. A BPS-saturated multiplet is a “multiplet” (the same size as a massless one), and \( \kappa \)-symmetry leaves the correct number of fermions to generate such a multiplet. The main issue in the formulation of supersymmetric D-branes is to prove that the action (19) really possesses such a fermionic gauge symmetry. Here, this construction will only be sketched, and the reader is referred to refs. [2,3] for the full details.

In order to make clear the details of the \( \kappa \)-symmetry, it is instructive to begin with the flat superspace case. There, one has coordinates \( (X^a, \theta^\alpha) \) (these letters are used throughout for inertial frame indices, and collectively denoted by \( A \)), and while a rigid supersymmetry transformation on the coordinates is
\[ \delta_\varepsilon \theta^\alpha = \varepsilon^\alpha , \]
\[ \delta_\varepsilon X^a = -i(\bar{\theta}^\alpha \varepsilon) , \]  
(20)
a $\kappa$-transformation takes the form
\begin{align*}
\delta_{\kappa}\theta^\alpha &= \kappa^\alpha, \\
\delta_{\kappa}X^a &= i(\bar{\theta}^a\kappa).
\end{align*}
\hfill (21)

The only formal difference is the sign in the transformation of the bosonic coordinate (and the fact that $\kappa$ is a local parameter). Thus, while the generator of supersymmetry is the ordinary supersymmetry generator, a generator of $\kappa$-symmetry is a covariant derivative, and due to the fact that $\kappa$ is projected down to half a spinor, only half a covariant derivative. The use of chiral superfields in four dimensions is a way to eliminate this redundancy.

When we move to arbitrary curved superspace, the picture is equally simple. In terms of the coordinates, the transformations read
\begin{equation}
\delta_{\kappa}Z^M = \kappa^\mu E_\mu^M \equiv \kappa^M.
\end{equation}
\hfill (22)

It is easily verified that this expression reduces to the ones above for the choice of flat vielbein
\begin{align*}
E_m^a &= \delta_m^a, \quad E_m^\alpha = 0, \\
E_\mu^a &= i\delta_\mu^\alpha(\gamma^a)_{\alpha\beta}\theta^\beta\delta_\nu^\beta, \quad E_\mu^\alpha = \delta_\mu^\alpha.
\end{align*}
\hfill (23)

When investigating the transformation properties of the action (19), we need the induced transformations of the background fields, and also the transformation rule for the U(1) potential. It follows straightforwardly from the coordinate transformations that the induced transformations of any background field $\Omega$ is $\delta_{\kappa}\Omega = \delta_{\kappa}Z^M\partial_M\Omega = \kappa^M\partial_M\Omega$. When dealing with pullbacks to the world-volume, we also have to take into account the transformation of the the pullback vielbeins $E_i^M = \partial_iZ^M$, and obtain
\begin{align*}
\delta_{\kappa}\phi_{ij} &= 2E_{(i}^\alpha E_{j)}^B \kappa^\alpha T_{\alpha B} \equiv 2E_{(i}^\alpha E_{j)}^B \kappa^\alpha T_{\alpha B} \equiv 2E_{(i}^\alpha E_{j)}^B \kappa^\alpha T_{\alpha B} \equiv 2E_{(i}^\alpha E_{j)}^B \kappa^\alpha T_{\alpha B} , \\
\delta_{\kappa}C &= \mathcal{L}_{\kappa}C = i_{\kappa}dC + di_{\kappa}C ,
\end{align*}
\hfill (24)

the latter equation valid for the pullback of any form ($T$ is the torsion tensor). The transformation of the U(1) potential is of course not a priori given, it must be constructed so that the action is invariant. It turns out that the correct transformation is
\begin{equation}
\delta_{\kappa}A = i_{\kappa}B ,
\end{equation}
\hfill (25)

which means that $\mathcal{F} = F - B$ transforms as
\begin{equation}
\delta_{\kappa}\mathcal{F} = i_{\kappa}dB = i_{\kappa}H .
\end{equation}
\hfill (26)

This also implies that the transformation of the WZ part of the action is (modulo boundary terms, which will not be considered here)
\begin{equation}
\delta_{\kappa}I_{\text{WZ}} = \int e^\mathcal{F}i_{\kappa}R ,
\end{equation}
\hfill (27)

where the RR curvature cochain is
\begin{equation}
R = e^B d(e^{-B}C) .
\end{equation}
\hfill (28)

In order to write down the action (19) more explicitly, we need to examine which background fields enter this expression. These are the bosonic fields of type II supergravity, i.e., the metric $g_{mn}$, the dilaton $\phi$ and the antisymmetric tensor field $B_{mn}$ from the NS-NS sector of the corresponding superstring theory, together with the antisymmetric tensors of the RR sector. In a superspace formulation, these fields are all superfields. They are not independent, but subject to constraints that fix some of the components and relate other to each other, in order to bring the total number of fields down to the physical content mentioned. Here, we will display only part of these constraints, namely those needed for proving $\kappa$-symmetry of the action. From the transformations of the background fields above, it is clear that only components with at least one spinor index are affected. This allows us to consider only fields of dimension 0 or 1/2.

Before giving the constraints, it is necessary to digress on the relevant superspaces. In type IIA, the fermions are a pair of Majorana–Weyl spinors of opposite chirality. They can be put together into a Majorana spinor, which is identical to a Majorana spinor in eleven dimensions. Besides the gamma matrices, there is an invariant matrix $\gamma_{11}$ which squares to $1$ and anticommutes with the gamma matrices. In type IIB, the two Majorana–Weyl spinors have the same chirality, and can be considered as a (complex) Weyl spinor. However, this is not convenient for our purposes. The division of the fields into the NS-NS and RR sectors, which in this context corresponds to couplings in the kinetic and WZ terms, respectively, roughly
means splitting complex fields in real and imaginary components. We therefore stay with a real formulation, and in addition to the gamma matrices there are three invariant real 2×2-matrices I, J and K spanning the Lie algebra of SL(2). Of these, J and K are symmetric and square to 1 and I antisymmetric and squares to −1. We also use IJ = K and all three matrices mutually anticommuting.

Our constraints are

\[ T_{\alpha\beta}^{\gamma} = 2i\gamma_{\alpha3}, \quad T_{\alpha\beta}^{c} = 0, \]

IIA: \( T_{\alpha\beta}^{\gamma} = \frac{2}{3} \epsilon_{\alpha}^{\gamma} \Lambda_{\beta} + 2(\gamma_{11})_{\alpha}^{\gamma}(\gamma \gamma \Lambda)_{\beta} + \frac{1}{2} \epsilon_{\alpha}^{\gamma} \Lambda_{\beta} + \frac{1}{2} \epsilon_{\alpha}^{\gamma} \Lambda_{\beta} \)

III: \( T_{\alpha\beta}^{\gamma} = -(J)_{\alpha}^{\gamma}(J \Lambda)_{\beta}, \quad H_{\alpha\beta\gamma} = 0, \quad H_{ab\gamma} = e^{\frac{1}{4} \phi} (\gamma \gamma \Lambda)_{\gamma}, \quad H_{ab\gamma} = e^{\frac{1}{4} \phi} (\gamma \gamma \Lambda)_{\gamma}, \)

\[ R_{(n)\alpha\beta\gamma A_1...A_{n-1}} = 0, \]

IIA: \( R_{(n)\alpha\beta\gamma A_1...A_{n-2}} = 2i \epsilon_{\alpha}^{\gamma} (\gamma_{11} \gamma_{a}^{(\gamma \gamma \Lambda)}_{\beta} + \frac{1}{2} \epsilon_{\alpha}^{\gamma} \Lambda_{\beta} - \frac{1}{2} \epsilon_{\alpha}^{\gamma} \Lambda_{\beta} \)

III: \( R_{(n)\alpha\beta \gamma A_1...A_{n-2}} = 2i \epsilon_{\alpha}^{\gamma} (\gamma_{11} \gamma_{a}^{(\gamma \gamma \Lambda)}_{\beta} - \frac{1}{2} \epsilon_{\alpha}^{\gamma} \Lambda_{\beta} - \frac{1}{2} \epsilon_{\alpha}^{\gamma} \Lambda_{\beta} \)

\[ \Lambda_{\alpha} = \frac{1}{2} \partial_{\alpha} \phi. \quad (29) \]

Constraints are formulated in terms of gauge-invariant quantities. When a constraint is imposed on a set of curvatures, one has to check that the Bianchi identities are still satisfied. This involves a number of Fierz identities valid in the different superspaces, and these nor the Bianchi identities themselves will be restated here. This is done in full in refs. [17,18]. For the original formulations of the type II supergravities we refer to refs. [17,18], where the superspace conventions however are quite different from ours, that are streamlined for D-brane calculations.

The constraints are to a certain extent a matter of convention. One such convention is that the dimension-1/2 component \( T_{\alpha}^{b} \) of the torsion tensor is set to zero. The other dimension-1/2 component, \( T_{\alpha}^{\gamma} \) can not consistently be set to zero, it is needed for a full treatment of curved superspace. Note also that \( \Lambda_{\alpha} \), the superfield that contains the physical fermion as its leading component, does not vanish in a general background, since it contains physical fields in its higher components. It should also be noted that the constraints put the supergravity theories on-shell.

The task is now to demonstrate that the action (19) is invariant under \( \kappa \)-symmetry. This proof contains essentially two steps. The first one consists of finding a projection operator that can be used to project the transformation parameter to half its original number of components, and the second one of performing the actual transformations and to confirm that the contributions from the Dirac–Born–Infeld and Wess–Zumino terms cancel.

The projection operator \( \Pi \) must have rank 16, since the superspaces have 32 fermionic directions. We write it as \( \Pi_{\pm} = \frac{1}{2} (I \mp \Gamma) \), where \( \Gamma \) is a traceless matrix that squares to 1. It then follows from the fact that the eigenvalues of \( \Gamma \) are \( \pm 1 \) that \( \Pi \) is a projection operator, and from the tracelessness that the number of positive and negative eigenvalues are equal, so that the rank is correct. We will use the + sign, but this is just a matter of convention.

Some guidance can be obtained from the known properties of type I branes, where no antisymmetric world-volume tensor is present, and \( \Gamma \) takes the simple form

\[ \Gamma = (p+1)(p+2) \frac{2 \epsilon_{i_{1}...i_{p+1}}}{(p+1)!\sqrt{-\det g}} \gamma_{i_{1}...i_{p+1}}. \quad (30) \]
Here, world-volume gamma matrices are of course pullbacks of the space-time ones. They depend on the fermionic coordinates even in the flat case, through the projection vielbeins $E_i^M$. One must be careful with the “sign factor” in this expression; for certain values of $p$ it will be imaginary, which is unacceptable when the spinors are real. In the D-brane case, this will be compensated by appropriate factors of $I$ (which squares to $-\mathbb{1}$) or $\gamma_{11}$ (which anticommutes with gamma matrices). We note that this must essentially be the leading part of the D-brane $\Gamma$’s, obtained when $\mathcal{F}$ and the dilaton are set to zero. We also note the “form character” of $\Gamma$, that makes it possible to rewrite it as (disregarding the sign factor)

$$d^{p+1}\xi \Gamma \sim -\frac{1}{2} \chi \gamma \ ,$$

with $\gamma$ being the natural gamma matrix $(p+1)$-form

$$\gamma = \frac{1}{(p+1)!} d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_p} \gamma_{i_1 \ldots i_{p+1}} .$$

The projection operator $\Pi$ acts exactly as a chirality projection operator.

We can now state the form of the matrix $\Gamma$ for D-branes. It is

$$\begin{align*}
&d^{p+1}\xi \Gamma \\
&= -\frac{\tilde{\varepsilon}^{(p-3)\phi}}{Z_{DBI}} \exp \left(e^{-\frac{1}{2} \phi} \mathcal{F} \right) \wedge X |_{\text{vol}} ,
\end{align*}$$

with

$$X = \bigoplus_n \gamma_{(2n+q)} P^{n+q}$$

and

\[ \Pi A: \quad P = \gamma_{11} , \quad Y = \mathbb{1} , \quad q = 1 , \]

\[ \Pi B: \quad P = K , \quad Y = I , \quad q = 0 . \]

In this expression, the $(p+1)$-form in the right hand side should be singled out. To actually show that $\Gamma$ squares to $\mathbb{1}$ is quite complicated, and we refer to ref. [8] for the proof. The tracelessness is trivial. The picture maybe becomes more clear from an example. For the D3-brane, $\Gamma$ takes the form

$$\begin{align*}
\Gamma &= \frac{e^{ijkl}}{\sqrt{-\det(g+\mathcal{F})}} \\
&\quad \times \left( \frac{1}{4\pi^2} \gamma_{ijkl} I + \frac{1}{4} \mathcal{F}_{ij} \gamma_{kl} I + \frac{1}{8} \mathcal{F}_{ij} \mathcal{F}_{kl} I \right) ,
\end{align*}$$

and a similar pattern is followed by all the $\Gamma$’s.

It now only remains to demonstrate the invariance of the action (19). This is straightforward but tedious work. It of course makes use of the fact stated above that all variations may be expressed in terms of the dimension-0 and $1/2$ background field components, and uses the solutions of the Bianchi identities already put forward. The conclusion is that we have proven that eq. (19) correctly describes the dynamics of the supersymmetric D-branes, provided they propagate in a background that solves the equations of motion for the effective action of the massless fields of the appropriate superstring theory.

4. CONCLUSIONS

In this contribution, we have mainly discussed two aspects of actions for D-branes, the possibility of simplifying them with the help of an auxiliary non-propagating world-volume metric, and the formulation of supersymmetric and $\kappa$-symmetric D-brane actions.

The investigation of the first of these issues resulted in partly negative results — although we were able to determine the form of these actions for lower values of $p$, we were not able to detect a clear pattern of the solutions possible to generalize to all $p$. We also detected increasing complication as $p$ was increased. Still, there should be a reservation for the possibility that some additional auxiliary fields might simplify the situation. The status of the problem is not conclusive.

One of the main motivations behind this first investigation was the hope that supersymmetrization might be made easier in such a framework. We were however able to solve that second problem without resorting to any auxiliary fields. The presentation of that solution was the other main issue addressed here. We were able to write down the general action for a supersymmetric D-brane and demonstrate the mechanism of $\kappa$-symmetry in a general supergravity background. Consistency of the D-brane action required the supersymmetry equations of motion to be satisfied (actually, the actions for higher $p$ require inclusion of both the ordinary supergravity fields and their duals, which is only possible on-shell).
There are of course many questions to address in the future in connection to the present work. Some of the most interesting are related to M-theory in eleven dimensions \[1\] \[23\], which in a fundamental and surprising way seems to be related to D-branes \[23\]. We would like to apply the techniques to the eleven-dimensional fivebrane, whose action is not even known in a bosonic truncation. Closely connected is the problem of treating configurations of multiple coinciding D-branes \[24\], which seems to go outside the realm of a classical space-time, into a non-commutative geometry. This provides a concrete realization of what for a long time generally has been suspected to be necessary if one hopes to get hold of what non-perturbative string theory really is.

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