CONSTRUCTING INFINITELY MANY HALF-ARC-TRANSITIVE
COVERS OF TETRAVALENT GRAPHS

PABLO SPIGA AND BINZHOU XIA

Abstract. We prove that, given a finite graph \( \Sigma \) satisfying some mild conditions, there exist infinitely many tetravalent half-arc-transitive normal covers of \( \Sigma \). Applying this result, we establish the existence of infinite families of finite tetravalent half-arc-transitive graphs with certain vertex stabilizers, and classify the vertex stabilizers up to order \( 2^8 \) of finite connected tetravalent half-arc-transitive graphs. This sheds some new light on the longstanding problem of classifying the vertex stabilizers of finite tetravalent half-arc-transitive graphs.

Key words: half-arc-transitive; vertex stabilizer; normal quotient; normal cover; concentric group
MSC2010: 20B25, 05C20, 05C25

1. Introduction

Let \( \Gamma \) be a graph and let \( G \) be a subgroup of the automorphism group \( \text{Aut}(\Gamma) \) of \( \Gamma \). We say that \( G \) is vertex-transitive, edge-transitive or arc-transitive if \( G \) acts transitively on the vertex set, edge set or the set of ordered pairs of adjacent vertices, respectively, of \( \Gamma \). If \( G \) is vertex-transitive and edge-transitive but not arc-transitive, then we say that \( G \) is half-arc-transitive. The graph \( \Gamma \) is said to be half-arc-transitive if \( \text{Aut}(\Gamma) \) is half-arc-transitive.

Numerous papers have been published on half-arc-transitive graphs over the last half a century (see the survey papers [5, 11]), most of which are on those of valency 4, the smallest valency of half-arc-transitive graphs. However, somewhat surprisingly, not so many examples of tetravalent half-arc-transitive graphs are known in the literature (see [16]), compared with the considerable attention they have received.

For a graph \( \Gamma \) and a group \( N \) such that \( N \) is normal in \( G \) for some vertex-transitive subgroup \( G \) of \( \text{Aut}(\Gamma) \), the normal quotient \( \Gamma/N \) is the graph whose vertex set \( V(\Gamma/N) \) is the set of \( N \)-orbits on the vertex set \( V(\Gamma) \) of \( \Gamma \), with an edge of \( \Gamma/N \) between vertices \( \Delta \) and \( \Omega \) if and only if there is an edge of \( \Gamma \) between \( \alpha \) and \( \beta \) for some \( \alpha \in \Delta \) and \( \beta \in \Omega \). Such a graph \( \Gamma \) is called a normal cover of the graph \( \Gamma/N \). Broadly speaking, in this paper, given a graph \( \Sigma \) satisfying some mild conditions, we establish the existence of infinitely many tetravalent half-arc-transitive graphs that are normal covers of \( \Sigma \).

Let \( p \) be a prime number. For a positive integer \( m \), denote the largest power of \( p \) dividing \( m \) by \( m_p \). Moreover, given a group \( X \), let \( O_p(X) \) denote the largest normal \( p \)-subgroup of \( X \). Our main result is as follows.

Theorem 1.1. Let \( \Sigma \) be a finite connected tetravalent graph and let \( T \) be a nonabelian simple half-arc-transitive subgroup of \( \text{Aut}(\Sigma) \). Then, for each prime number \( p \), such that \( p > |T|_2 \) and \( p \) is coprime to \( |T| \), there exists a finite connected tetravalent graph \( \Gamma \) satisfying the following:

(a) \( \Gamma \) is half-arc-transitive;
(b) \( \text{Aut}(\Gamma)\) has vertex stabilizer isomorphic to that of \( T\);
(c) \( O_p(\text{Aut}(\Gamma)) \neq 1, \text{Aut}(\Gamma)/O_p(\text{Aut}(\Gamma)) \cong T\) and \( \Gamma/O_p(\text{Aut}(\Gamma)) \cong \Sigma \).

Although it is not hard to construct a graph \( \Gamma \) with a half-arc-transitive group \( G \) of automorphisms, it is in general not known whether \( \text{Aut}(\Gamma) \) is larger than \( G \) to possibly make \( \text{Aut}(\Gamma) \) arc-transitive on \( \Gamma \). In this sense, the significance of Theorem 1.1 is asserting the existence (under some mild conditions) of infinitely many half-arc-transitive graphs which are normal covers of a given connected tetravalent graph, even if the given graph is not itself half-arc-transitive. Thus, with the help of Theorem 1.1, one can construct infinitely many connected tetravalent half-arc-transitive graphs with some exotic vertex stabilizers, and we will present some examples in this paper.

For a half-arc-transitive graph \( \Gamma \), the vertex stabilizer in \( \text{Aut}(\Gamma) \) will be called the \textit{vertex stabilizer} of \( \Gamma \). It is not hard to construct half-arc-transitive graphs with abelian vertex stabilizers, and we will present some examples in this paper.

For a half-arc-transitive graph \( \Gamma \), the vertex stabilizer in \( \text{Aut}(\Gamma) \) will be called the \textit{vertex stabilizer} of \( \Gamma \). It is not hard to construct half-arc-transitive graphs with abelian vertex stabilizers, and we will present some examples in this paper.

In Example 3.3 we construct a finite connected tetravalent graph \( \Sigma_m \) for every integer \( m \geq 4 \) such that \( \Sigma_m \) admits a half-arc-transitive action of the alternating group \( A_{2m} \) with vertex stabilizer \( D_8 \times C_2^{m-3} \). Then, by applying Theorem 1.1 to the graphs in Example 3.3 and to the graphs in [18], we obtain the following result:

\textbf{Theorem 1.2.} For every integer \( m \geq 4 \), there exist infinitely many finite connected tetravalent half-arc-transitive graphs with vertex stabilizer \( D_8 \times C_2^{m-3} \) and, for every integer \( m \geq 7 \), there exist infinitely many finite connected tetravalent half-arc-transitive graphs with vertex stabilizer \( D_8 \times D_8 \times C_2^{m-6} \).

A group \( H = \langle a_1, \ldots, a_m \rangle \) is said to be \textit{concentric} if \( |\langle a_i, \ldots, a_j \rangle| = 2^{j-i+1} \) for all \( 1 \leq i < j \leq m \) and there exists a group isomorphism

\[ \varphi : \langle a_1, \ldots, a_{m-1} \rangle \to \langle a_2, \ldots, a_m \rangle \]

such that \( a_i^2 = a_{i+1} \) for \( i = 1, \ldots, m - 1 \). The study of concentric groups dates back to Glauberman [8, 9] about 50 years ago and was made systematic by Marušič and Nedela [13] in 2001. It was proved in [13] that a group \( H \) is concentric if and only if there exist a connected tetravalent graph \( \Gamma \) and a subgroup \( G \) of \( \text{Aut}(\Gamma) \) such that \( G \) is half-arc-transitive with vertex stabilizer \( H \). Moreover, Marušič and Nedela gave a characterization of concentric groups in terms of their defining relations [13, Theorem 5.5] and determined the concentric groups of order up to \( 2^8 \) [13, Theorem 6.3]. Let

\[ \mathcal{H}_7 = \langle a_1, \ldots, a_7 \rangle | a_i^2 = 1 \text{ for } i \leq 7, \ (a_i a_j)^2 = 1 \text{ for } |i - j| \leq 4, \ (a_1 a_6)^2 = a_3, \ (a_2 a_7)^2 = a_4, \ (a_1 a_7)^2 = a_5 \].

\textbf{Theorem 1.3} (Marušič-Nedela). The following are precisely the concentric groups of order at most \( 2^8 \):

\[ C_2^m \text{ for } 1 \leq m \leq 8, \quad D_8 \times C_2^{m-3} \text{ for } 3 \leq m \leq 8, \]
\[ D_8 \times D_8 \times C_2^{m-6} \text{ for } 6 \leq m \leq 8, \quad \mathcal{H}_7 \times C_2^{m-7} \text{ for } 7 \leq m \leq 8. \]
Marušić [12] has shown that every nontrivial elementary abelian 2-group is the vertex stabilizer of a connected tetravalent half-arc-transitive graph. Similar results have been proved for $D_8$ by Conder and Marušić [4] and for $D_8 \times C_2$ by Conder, Potočnik and Šparl [5]. Moreover, the first author showed in [17] that $D_8 \times D_8$ and $H_7$ are both vertex stabilizers of connected tetravalent half-arc-transitive graphs in a response to a problem posed in [13], and the second author recently proved in [18] that $D_8 \times D_8 \times C_2 m−6$ is the vertex stabilizer of a connected tetravalent half-arc-transitive graph for every integer $m \geq 7$. In light of these results and Theorem 1.2, we see that the only concentric group of order at most $2^8$ that is not known to be the vertex stabilizer of a connected tetravalent half-arc-transitive graph is $H_7 \times C_2$. In Example 3.2, we apply Theorem 1.1 to construct connected tetravalent half-arc-transitive graphs with vertex stabilizer $H_7 \times C_2$. This leads to the next theorem.

**Theorem 1.4.** Every concentric group of order at most $2^8$ is the vertex stabilizer of infinitely many finite connected tetravalent half-arc-transitive graphs.

We prove Theorem 1.1 in Section 2. Then in Section 3 we construct some connected tetravalent graphs admitting a half-arc-transitive nonabelian simple group action with vertex stabilizer $H_7 \times C_2$ and $D_8 \times C_2^{m−3}$ for $m \geq 3$, respectively, which will be used in Section 4 to prove Theorems 1.2 and 1.4. In Section 5 we briefly discuss the relevance of our work and a conjecture of Džambić-Jones and Conder concerning faithful amalgams. We also include a natural open problem at the end of Section 5.

**2. Proof of Theorem 1.1**

For a group $X$, let $\text{Soc}(X)$ denote the socle of $X$ and let $\text{Rad}(X)$ denote the maximal normal solvable subgroup of $X$. Let $\Gamma$ be a graph, let $G$ be a vertex-transitive subgroup of $\text{Aut}(\Gamma)$ and let $N$ be a normal subgroup of $G$. Then the group $G$ induces a vertex-transitive subgroup of $\text{Aut}(\Gamma/N)$. Denote by $\alpha^N$ and $\beta^N$ the $N$-orbits containing the vertices $\alpha$ and $\beta$, respectively, of $\Gamma$. If $\alpha^N$ and $\beta^N$ are adjacent in $\Gamma/N$, then each vertex in $\alpha^N$ is adjacent to the same number of vertices in $\beta^N$ (because $N$ is transitive on both sets). Moreover, the stabilizer in $G$ of the vertex $\alpha^N$ in $\Gamma/N$ is $G_{\alpha N}$.

**Proof of Theorem 1.1.** Let $\Sigma$ and $T$ be as in Theorem 1.1 and let $p$ be a prime number such that $p > |T|_2$ and $p$ is coprime to $|T|$. Viewing [15, Corollary 8] and applying [15, Theorem 6] with the prime $p$, the graph $\Sigma$ and the group of automorphisms $T$, we obtain a regular covering projection $\phi : \Gamma \to \Sigma$ such that the following hold:

(i) $\Gamma$ is finite;
(ii) the maximal group that lifts along $\phi$ is $T$;
(iii) the group of covering transformations of $\phi$ is a $p$-group.

Let $A = \text{Aut}(\Gamma)$, let $G$ be the subgroup of $A$ that $T$ lifts to along $\phi$, and let $P$ be the group of covering transformations of $\phi$. Then conclusion (iii) shows that $P$ is a $p$-group, and $G/P \cong T$ is nonabelian simple. Since $P$ is a normal solvable subgroup of $G$, it follows that $P = \text{Rad}(G)$. Moreover, we deduce from conclusion (ii) and [15, Lemma 1] that

$$N_A(P) = G.$$  \hspace{1cm} (1)

Since $P = \text{Rad}(G)$ is characteristic in $G$, we derive that $P$ is normal in $N_A(G)$, that is, $N_A(G) \leq N_A(P)$. Thus it follows from (1) that

$$N_A(G) = G.$$  \hspace{1cm} (2)
We aim to prove that \( A = G \), from which the proof of Theorem 1.1 immediately follows. Assume for a contradiction that \( A > G \). Then \( G < B \) for some subgroup \( B \) of \( A \) such that \( G \) is maximal in \( B \).

Let \( \alpha \) be a vertex of \( \Gamma \). Since \( T \) is half-arc-transitive on \( \Sigma \), the group \( G \) is half-arc-transitive on \( \Gamma \). This implies that \( G_\alpha \) is a 2-group and \( B = GB_\alpha \) is edge-transitive on \( \Gamma \). It follows that \( |B : G| = |GB_\alpha : G| = |B_\alpha : G_\alpha| \) divides \( |B_\alpha| \). As \( B_\alpha \) is a \( \{2, 3\} \)-group and \( p > |T|_2 \geq 5 \), we infer that \( p \) is coprime to \( |B : G| \). Since \( p \) is coprime to \( |T| = |G/P| \), we see that \( P \) is a Sylow \( p \)-subgroup of \( B \). According to Sylow’s theorem, the number of Sylow \( p \)-subgroups of \( B \) is \( |B : N_B(P)| \equiv 1 \) (mod \( p \)) and so \( p \) divides \( |B : N_B(P)| - 1 \).

By (3) we have \( N_B(P) = G \). Hence

\[
p \mid (|B : G| - 1).
\]

Let \( K \) be the core of \( G \) in \( B \). Then \( K \leq B, K \leq G \), and the action of \( B/K \) on the set \( \Omega \) of right cosets of \( G/K \) in \( B/K \) is faithful and primitive of degree \( |B : G| \). Since both \( K \) and \( P \) are normal in \( G \), the group \( KP \) is normal in \( G \), which implies that \( KP/P \) is normal in \( G/P \). As \( G/P \cong T \) is a simple group, we deduce that either \( G = KP \) or \( K \leq P \).

**Case 1.** \( G = KP \).

In this case, \( P \cap K \) is a normal subgroup of \( K \) with

\[
K/(P \cap K) \cong KP/P = G/P \cong T.
\]

nonabelian simple. Since \( P \cap K \) is solvable, we conclude that

\[
P \cap K = \text{Rad}(K)
\]

is characteristic in \( K \). As \( K \) is normal in \( B \), it follows that

\[
P \cap K \leq B.
\]

Note that \( |G/K| = |KP/K| = |P/(P \cap K)| \) is a power of \( p \) and \( G \neq K \) by (2). We have

\[
|G/K| = p^n
\]

for some positive integer \( n \).

Suppose that \( |B_\alpha| \) is divisible by 3. Then \( B_\alpha \) is 2-transitive on the neighborhood of \( \alpha \) in \( \Gamma \), and so it follows from a result of Gardiner (see for instance [11, Lemma 2.3]) that \( |B_\alpha| \) divides \( 2^4 \cdot 3^6 \). Now \( B/K \) is a primitive group of degree \( |B : G| = |B_\alpha : G_\alpha| \) dividing \( 2^4 \cdot 3^6 \) such that the point stabilizer \( G/K \) is a \( p \)-group. We deduce from [10] that \( B/K \) is an affine group of degree \( 3^k \) with \( 3 \leq k \leq 6 \), and \( \text{Soc}(B/K) \) is the unique Sylow 3-subgroup of \( B/K \). Since \( B/K = (G/K)/(B_\alpha K/K) \) and \( |G/K| \) is coprime to 3, it follows that \( \text{Soc}(B/K) \leq B_\alpha K/K \cong B_\alpha/K_\alpha \). Note that

\[
|\text{Soc}(B/K)| = 3^k = |B : G| = |B_\alpha : G_\alpha| = |B_\alpha|_3
\]

as \( G_\alpha \) is a 2-group. We conclude that the Sylow 3-subgroup of \( B_\alpha \) is elementary abelian of order \( 3^k \geq 3^3 \). The structure of the vertex stabilizer \( B_\alpha \) is described in [11, Table 1], which shows that \( B_\alpha \) cannot have an elementary abelian Sylow 3-subgroup of order at least \( 3^3 \), a contradiction. Thus \( B_\alpha \) is a 2-group, and so \( |B : G| = |B_\alpha : G_\alpha| \) is a power of 2, say

\[
|B : G| = 2^k.
\]
Note that $\ell > 1$ by (2).

Since $|B : G| = 2^\ell$ and $|G : K| = p^n$, we see that $B/K$ has order $2^\ell p^n$ and thus is solvable. Moreover, as $B/K$ is a primitive group of degree $2^\ell$, it follows that $H/K = \text{Soc}(B/K)$ is an elementary abelian group of order $2^\ell$. The reader may find Figure 1 useful at this point.

\[ \begin{array}{c}
B \\
| \phantom{B} \\
G & H \\
| \phantom{G} & | \\
P & K \\
\phantom{P} & | \\
P \cap K
\end{array} \]

Figure 1. The structure of $B$

Let $\overline{B} = B/(P \cap K)$, $\overline{H} = H/(P \cap K)$, $\overline{K} = K/(P \cap K)$ and $\overline{C} = C_{\overline{P}}(\overline{K})$. Then $\overline{K} \cong T$, and both $\overline{H}$ and $\overline{K}$ are normal in $\overline{B}$. It follows that $\overline{C} = \overline{H} \cap C_{\overline{P}}(\overline{K}) \leq \overline{B}$, and

$$\overline{H}/\overline{C} \leq \text{Aut}(\overline{K}) \cong \text{Aut}(T).$$

Moreover,

$$\overline{C}\overline{K}/\overline{C} \cong \overline{K}/(\overline{K} \cap \overline{C}) \cong \text{Inn}(\overline{K}) \cong \text{Inn}(T). \quad (4)$$

Thus $\overline{H}/(\overline{C}\overline{K}) \leq \text{Out}(T)$. Let $C$ be the subgroup of $H$ containing $P \cap K$ such that $C/(P \cap K) = \overline{C}$. Then

$$C \leq B$$

and $H/(CK) \leq \text{Out}(T)$. Now $CK \leq B$ and so $CK/K \leq B/K$. As $CK/K \leq H/K$ and $H/K = \text{Soc}(B/K)$ is a minimal normal subgroup of the affine primitive group $B/K$, it follows that either $CK/K = 1$ or $CK/K = H/K$. If $CK/K = 1$, then the elementary abelian 2-group $H/K = H/(CK)$ is isomorphic to a subgroup of $\text{Out}(T)$, which implies that

$$|B : G| = 2^\ell = |H/K| \leq |\text{Out}(T)|_2 \leq |T|_2 < p,$$

contradicting (3). (Observe that the inequality $|\text{Out}(T)|_2 \leq |T|_2$ follows by inspecting the list of finite simple groups.) Therefore, $CK/K = H/K$ and hence $H = CK$. This in turn with (4) implies that

$$H/C = CK/C \cong \overline{C}\overline{K}/\overline{C} \cong T.$$

Note that $T$ is the unique nonsolvable composition factor of $H$ as $H/K$ is solvable and $K$ is a $p$-group extended by $T$. We then conclude that

$$C = \text{Rad}(H).$$

Consequently,

$$C \cap K = \text{Rad}(H) \cap K = \text{Rad}(K) = P \cap K$$

and so

$$|C/(P \cap K)| = |C/(C \cap K)| = |CK/K| = |H/K| = 2^\ell.$$

The reader may find Figure 2 useful at this point.

Consider the quotient graph $\Gamma/C$. Let $N$ be the kernel of $B$ acting on $V(\Gamma/C)$. Since $H$ is a normal subgroup of $B$ with index $p^n$ odd and $B_\alpha$ is a 2-group, we have $B_\alpha \leq H$. The reader may find Figure 3 useful at this point.
Consequently, $N = CN_\alpha \leq CB_\alpha \leq H$. Moreover, $N = CN_\alpha$ is a $\{2, p\}$-group and thus is solvable. Hence $N \leq \text{Rad}(H) = C$. This shows that the action of $B/C$ on $V(\Gamma/C)$ is faithful. Suppose that $C_\alpha \neq 1$. Then the number of orbits of $C_\alpha$ on the neighborhood of $\alpha$ in $\Gamma$ is less than 4. It follows that the valency of $\Gamma/C$ is less than 4 and so must be 1 or 2, being a divisor of 4. Thereby we conclude that $B/C \leq \text{Aut}(\Gamma/C)$ is solvable, a contradiction. Thus $C_\alpha = 1$.

As $C_\alpha = 1$, the orbits of $C$ on $V(\Gamma)$ have size $|C|$. Since $C$ is normal in $B$ and $B$ is transitive on $V(\Gamma)$, it follows that $|C|$ divides $|V(\Gamma)|$. Hence $|C|$ divides $|G|$ as $G$ is transitive on $V(\Gamma)$. In particular, $|C|_2 \leq |G|_2$. As $|C|_2 = |C/(P \cap K)|_2 = 2^\ell$ and $|G|_2 = |G/P|_2 = |T|_2$, we then obtain $2^\ell \leq |T|_2$. This together with (3) implies that $p < |B : G| = 2^\ell \leq |T|_2$, contradicting our choice of $p$.

**Case 2.** $K \leq P$.

Let $\overline{B} = B/K$, $\overline{G} = G/K$, $\overline{P} = P/K$ and $\overline{H} = H/K = \text{Soc}(\overline{B})$. Recall that $\overline{B}$ acts primitively and faithfully on the set of right cosets of $\overline{G}$ in $\overline{B}$, and

$$|\overline{B} : \overline{G}| = |B : G| = |GB_\alpha : G| = |B_\alpha : G|.$$

As $B_\alpha$ is a $\{2, 3\}$-group, we obtain $|\overline{B} : \overline{G}| = 2^\ell 3^k$ for some nonnegative integers $\ell$ and $k$. If $|B_\alpha|$ is divisible by 3, then $B_\alpha$ is 2-transitive on the neighborhood of $\alpha$ in $\Gamma$ and so [1] Lemma 2.3] shows that $|B_\alpha|$ divides $2^\ell 3^6$. Consequently, either $\ell \leq 3$ and $1 \leq k \leq 6$, or $k = 0$.

Since $K$ is normal in $B$, we deduce from (1) that $K \neq P$. Hence $K < P$ and so $\overline{P}$ is a nontrivial $p$-group. This shows that $\overline{G}$ is a nontrivial $p$-group extended by the nonabelian simple group $G/P \cong T$. Then as $\overline{G}$ is a point stabilizer of the primitive group $\overline{B}$ of degree $|\overline{B} : \overline{G}| = 2^\ell 3^k$, it follows from [10] that $k = 0$ and $\overline{B}$ is an affine primitive group of degree $2^\ell$. Hence $|\overline{H}| = 2^\ell$, and so $H$ is a $\{2, p\}$-group.

Let $R = PH$. Then $R$ is a $\{2, p\}$-group and thus is solvable. Moreover, $R \leq B$, and as $P \leq G$, we have $B = HG = HPG = RG$. Hence

$$B/R = RG/R \cong G/(G \cap R) \cong (G/P)/(G \cap R)/P.$$
Since $G/P \cong T$ is simple, it follows that either $B/R = 1$ or $B/R \cong T$. Clearly, $B \neq R$ as $R$ is solvable and $B$ is nonsolvable. Thus $B/R \cong T$ is nonabelian simple, which implies $R = \text{Rad}(B)$.

Consider the quotient graph $\Gamma/R$. Let $M$ be the kernel of $B$ acting on $V(\Gamma/R)$. Then $M = RM_a$. Since $M_a \leq B_a$ is a 2-group, we see that $M$ is a $\{2, p\}$-group as $R$ is a $\{2, p\}$-group. Accordingly, $M$ is solvable, and so $M \leq \text{Rad}(B) = R$. This shows that the action of $B/R$ on $V(\Gamma/R)$ is faithful. Suppose that $R_a \neq 1$. Then the number of orbits of $R_a$ on the neighborhood of $\alpha$ in $\Gamma$ is less than 4. It follows that the valency of $\Gamma/R$ is less than 4 and so must be 1 or 2 as it divides 4. Thereby we conclude that $B/R \leq \text{Aut}(\Gamma/R)$ is solvable, a contradiction. Thus $R_a = 1$.

As $R_a = 1$, the orbits of $R$ on $V(\Gamma)$ have size $|R|$. Since $R$ is normal in $B$ and $B$ is transitive on $V(\Gamma)$, it follows that $|R|$ divides $|V(\Gamma)|$. Hence $|R|$ divides $|G|$ as $G$ is transitive on $V(\Gamma)$. In particular, $|R|^2 \leq |G|^2$. As $|R|^2 = |PH|^2 = |H|^2 = |H/K|^2 = 2^\ell$ and $|G|^2 = |G/P|^2 = |T|^2$, we then obtain $2^\ell \leq |T|^2$. This in conjunction with (3) implies that $p < |G : B| = |B : G| = 2^\ell \leq |T|^2$, contradicting our choice of $p$.

3. Examples

Recall the standard construction of the coset graph $\text{Cos}(X,Y,YSY)$ for a group $X$ with a subgroup $Y$ and a subset $S$ such that $Y \cap S = \emptyset$ and $YSY$ is finite and inverse-closed. Such a graph has vertex set $[X : Y]$, the set of right cosets of $Y$ in $X$, and edge set $\{\{yt, ys\} \mid t \in X, s \in YSY\}$. It is easy to see that $\text{Cos}(X,Y,YSY)$ has valency $|YSY|/|Y|$, and $X$ acts by right multiplication on $[X : Y]$ as a group of automorphisms of $\text{Cos}(X,Y,YSY)$. Moreover, $\text{Cos}(X,Y,YSY)$ is connected if and only if $X = \langle Y, S \rangle$.

3.1. Example $D_8$. Let $G = A_{10}$ and

$$H = \langle (1, 2, 3, 4)(5, 6, 7, 8), (1, 4)(2, 3)(5, 7)(9, 10) \rangle < G.$$ 

Clearly, $H \cong D_8$. Let

$$s = (1, 8, 10)(2, 7, 4, 6, 9, 3, 5) \in G.$$ 

It can be checked immediately by the computational algebra system Magma [1] that

$$\langle H, s \rangle = G, \quad |H : s^{-1}Hs| = 2 \quad \text{and} \quad s^{-1} \notin HsH.$$ 

Then letting

$$\Sigma = \text{Cos}(G, H, H\{s, s^{-1}\}H),$$ 

we see that

- $\Sigma$ is a connected tetravalent graph;
- $G$ acts faithfully and half-arc-transitively on $\Sigma$;
- the vertex stabilizer in $G$ is $H \cong D_8$.

3.2. Example $H_7 \times C_2$. Let

$$H = \langle a_1, \ldots, a_8 \mid a_i^2 = 1 \text{ for } i \leq 8, \ (a_ia_j)^2 = 1 \text{ for } |i - j| \leq 5, \ (a_1a_7)^2 = a_3, \ (a_2a_8)^2 = a_4, \ (a_1a_8)^2 = a_6 \rangle.$$ 

Then $H = \langle a_1, a_2, a_3, a_4, a_6, a_7, a_8 \rangle \times \langle a_5 \rangle \cong H_7 \times C_2$. Let

$$B = \langle a_1, \ldots, a_7 \rangle, \quad C = \langle a_2, \ldots, a_8 \rangle$$
and let $\varphi : B \to C$ be the group isomorphism defined by
\[ a_i^2 = a_{i+1} \quad \text{for} \quad i = 1, \ldots, 7. \]
Then $H = B \cup a_8 B = C \cup a_1 a_2 C$. Let $x$ be the permutation on $H$ defined by
\[ b^x = b^\varphi \quad \text{and} \quad (a_8 b)^x = a_1 a_2 b^\varphi \quad \text{for} \quad b \in B. \]
Denote the right regular representation of $H$ by $R : H \to \text{Sym}(H)$. It can be checked easily by the computational algebra system MAGMA [1] that
\[ \langle R(H), x \rangle = \text{Alt}(H), \quad x^{-1} R(H) x = R(C) \quad \text{and} \quad x^{-1} \notin R(H) x R(H). \]
Then letting
\[ \Pi = \text{Cos}(\text{Alt}(H), R(H), R(H) \{x, x^{-1}\} R(H)), \quad (6) \]
we see that
- $\Pi$ is a connected tetravalent graph;
- $\text{Alt}(H)$ acts faithfully and half-arc-transitively on $\Pi$;
- the vertex stabilizer in $\text{Alt}(H)$ is $R(H) \cong H \cong H_7 \times C_2$.

### 3.3. Example $D_8 \times C_2^{m-3}$.

Let $m \geq 4$ be an integer,
\[ H = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle \times \langle c_1 \rangle \times \cdots \times \langle c_{m-3} \rangle, \]
where $c_1, \ldots, c_{m-3}$ are involutions. Clearly, $H \cong D_8 \times C_2^{m-3}$. Let $h = a \prod_{i=0}^{[(m-5)/2]} c_{2i+1}$ and
\[ K = \langle a^2, b, c_1, \ldots, c_{m-3} \rangle = \langle a^2 \rangle \times \langle b \rangle \times \langle c_1 \rangle \times \cdots \times \langle c_{m-3} \rangle. \]
Then $K \cong C_2^{m-1}$ and $H = K \cup aK = K \cup hK$. For convenience, put $c_i = 1$ for $i \leq 0$. Define $x \in \text{Aut}(H)$ by letting
\[ a^x = a^{-1}, \quad b^x = ab, \quad c_{2i+1}^x = c_{2i+1} \quad \text{and} \quad c_{2i+2}^x = a^2 c_{2i+1} c_{2i+2} \]
for $0 \leq i \leq [(m-5)/2]$ and letting $c_{m-3}^x = a^2 c_{m-3}$ in addition if $m$ is even. Define $\tau \in \text{Aut}(K)$ by letting
\[ (a^2)^\tau = b, \quad b^\tau = a^2, \quad c_{2i+1}^\tau = c_{2i-1} c_{2i+2} \quad \text{and} \quad c_{2i+2}^\tau = c_{2i-1} c_{2i} c_{2i+1} \]
for $0 \leq i \leq [(m-5)/2]$ and letting $c_{m-3}^\tau = c_{m-3}$ in addition if $m$ is even.

Note that $x$ and $\tau$ are automorphisms of $H$ and $K$ respectively as the images of generators under $x$ and $\tau$ are generators of $H$ and $K$ satisfying the defining relations. Let $y$ be the permutation of $H$ such that $g^y = g^\tau$ and
\[ (hg)^y = \begin{cases} h g^\tau & \text{if} \ m \ \text{is odd,} \\ h g^\tau c_{m-3} & \text{if} \ m \ \text{is even,} \end{cases} \]
for $g \in K$. Denote the right regular representation of $H$ by $R : H \to \text{Sym}(H)$. It follows from [2] Lemmas 2.1 and 2.3 that $x$ and $y$ are both involutions and $\langle x, y, R(H) \rangle \leq \text{Alt}(H)$. Let
\[ \Sigma_m = \text{Cos}(\text{Alt}(H), R(H), R(H) \{xy, yx\} R(H)). \quad (7) \]

Fix the notation of $H$, $R$, $x$ and $y$ in this subsection, and let
\[ z = \begin{cases} R(h)y R(h^{-1}) & \text{if} \ m \ \text{is odd,} \\ R(h) y R(h^{-1}c_{m-3}) & \text{if} \ m \ \text{is even.} \end{cases} \]
According to [2, Lemma 2.1], the permutation $z$ is an involution. Since $z \in \langle x, y, R(H) \rangle$, it follows from $\langle x, y, R(H) \rangle \leq \text{Alt}(H)$ that $z \in \text{Alt}(H)$. As $x$, $y$ and $z$ all fix $1 \in H$, we may also view them as elements of $\text{Alt}(H \setminus \{1\})$ when they cause no confusion. Use $\sqcup$ to denote a disjoint union of sets.

**Lemma 3.1.** The following hold:

(a) $R(H)xyR(H) = R(H)xy \sqcup R(H)xz$;
(b) $R(H)yxR(H) = R(H)yx \sqcup R(H)zx$;
(c) $R(H)\{xy, yx\}R(H) = R(H)xy \sqcup R(H)yx \sqcup R(H)xz \sqcup R(H)zx$.

**Proof.** Note that $x$, $y$ and $z$ are all involutions. It is straightforward to verify that $x$ and $y$ normalize $R(H)$ and $(K)$, respectively. If $yR(H)x \cap R(H) = R(H)$, then

$$\langle x, y, R(H) \rangle \leq N_{\text{Alt}(H)}(R(H)) < \text{Alt}(H),$$

contrary to [2, Lemmas 3.6 and 3.11]. Thus

$$yxR(H)xy \cap R(H) = yR(H)y \cap R(H) \neq R(H).$$

Since

$$yxR(H)xy \cap R(H) = yR(H)y \cap R(H) \geq yR(K)y \cap R(K) = R(K)$$

and $R(K)$ has index 2 in $R(H)$, we then deduce that $yxR(H)xy \cap R(H) = R(K)$. In particular, $yxR(H)xy$ has index 2 in $R(H)$, whence

$$\frac{|R(H)xyR(H)|}{|R(H)|} = \frac{|R(H)|}{|yxR(H)xy \cap R(H)|} = 2.$$

Consequently,

$$|R(H)yxR(H)| = |(R(H)yxR(H))^{-1}| = |R(H)xyR(H)| = 2|R(H)|$$

and thus

$$|R(H)\{xy, yx\}R(H)| \leq |R(H)xyR(H)| + |R(H)yxR(H)| = 4|R(H)|.$$

Note from the definition of $z$ that

$$xz \in xR(H)yR(H) = R(H)xyR(H).$$

Hence $R(H)xz \subseteq R(H)xyR(H)$ and $R(H)zx \subseteq R(H)yxR(H)$. It is direct to verify that

$$(a^2)^{xy} = b, \quad (a^2)^{yx} = ab, \quad (a^2)^{xz} = a^2b, \quad (a^2)^{zx} = a^3b,$$

which shows that $xy$, $yx$, $xz$ and $zx$ are pairwise distinct. Then as $xy, yx, xz, zx \in \text{Alt}(H)_1$ and $\text{Alt}(H)_1$ forms a right transversal of $R(H)$ in $\text{Alt}(H)$, it follows that $R(H)xy$, $R(H)yx$, $R(H)xz$ and $R(H)zx$ are pairwise disjoint. Therefore,

$$R(H)xyR(H) \supseteq R(H)xy \sqcup R(H)xz,$$

$$R(H)yxR(H) \supseteq R(H)yx \sqcup R(H)zx$$

and

$$R(H)\{xy, yx\}R(H) \supseteq R(H)xy \sqcup R(H)yx \sqcup R(H)xz \sqcup R(H)zx.$$

This combined with (8) and (9) yields the lemma. \qed

**Proposition 3.2.** Let $m \geq 4$ be an integer and let $\Sigma_m$ be the graph defined in (7). Then $\Sigma_m$ is a connected tetravalent graph admitting a half-arc-transitive action of $A_2 \times C_2^{m-3}$. 




**Proof.** Let \( S = \{xy, yx, xz, zx\} \subset \text{Alt}(H \setminus \{1\}) \). According to [2, Lemmas 4.1 and 4.2], \( \text{Cay}(\text{Alt}(H \setminus \{1\}), \{x, y, z\}) \) is connected, which means that \( \text{Alt}(H \setminus \{1\}) = \langle x, y, z \rangle \). Consider the subgroup \( W \) of even words of the generators \( x, y \) and \( z \) in \( \text{Alt}(H \setminus \{1\}) \). Then \( W \) has index 1 or 2 in \( \text{Alt}(H \setminus \{1\}) \). Since \( \text{Alt}(H \setminus \{1\}) \) is simple, it follows that \( W = \text{Alt}(H \setminus \{1\}) \). Moreover, as \( x, y \) and \( z \) are involutions, we have

\[
W = \langle xy, xz, yz \rangle = \langle xy, xz, (xy)^{-1}(xz) \rangle = \langle xy, xz \rangle.
\]

Thus \( \text{Alt}(H \setminus \{1\}) = \langle xy, xz \rangle \), and so \( \text{Cay}(\text{Alt}(H \setminus \{1\}), S) \) is connected.

Let \( \varphi: g \mapsto R(H)g \) be the mapping from \( \text{Alt}(H \setminus \{1\}) \) to the vertex set of \( \Sigma_m \). Since \( \text{Alt}(H \setminus \{1\}) \) forms a right transversal of \( R(H) \) in \( \text{Alt}(H) \), \( \varphi \) is bijective. Moreover, for any \( u \) and \( v \) in \( \text{Alt}(H \setminus \{1\}) \), \( u \) is adjacent to \( v \) in \( \text{Cay}(\text{Alt}(H \setminus \{1\}), S) \) if and only if

\[
vu^{-1} \in S = \{xy, yx, xz, zx\},
\]

which is equivalent to

\[
R(H)vu^{-1} \in \{R(H)xy, R(H)yx, R(H)xz, R(H)zx\}.
\]

By Lemma 3.1, this means that \( u \) and \( v \) are adjacent in \( \text{Cay}(\text{Alt}(H \setminus \{1\}), S) \) if and only if

\[
R(H)vu^{-1} \subseteq R(H)S = R(H)\{xy, yx\}R(H),
\]

or equivalently, \( R(H)u \) is adjacent to \( R(H)v \) in \( \Sigma_m \). Therefore, \( \varphi \) is a graph isomorphism from \( \text{Cay}(\text{Alt}(H \setminus \{1\}), S) \) to \( \Sigma_m \). As a consequence, \( \Sigma_m \) is a connected tetravalent graph.

Finally, Lemma 3.1 implies that \( \{R(H)xy, R(H)xz\} \) and \( \{R(H)yx, R(H)zx\} \) are the two orbits of the group \( R(H) \) acting on the neighborhood of the vertex \( R(H) \) in \( \Sigma_m \). Thereby we conclude that the right multiplication action of \( \text{Alt}(H) \) on \( \Sigma_m \) is half-arc-transitive. Since \( R(H) \cong H \cong D_8 \times C_2^{m-3} \) and \( \text{Alt}(H) \cong A_{|H|} = A_{2^m} \), this completes the proof. \( \square \)

4. Proof of Theorem 1.2 and Theorem 1.4

**Proof of Theorem 1.2.** Let \( m \geq 4 \) be an integer and let \( \Sigma_m \) be the graph defined by (7). Then as Proposition 3.2 asserts, \( \Sigma_m \) is a connected tetravalent graph admitting a half-arc-transitive action of \( A_{2^m} \) with vertex stabilizer \( D_8 \times C_2^{m-3} \). Thus, by Theorem 1.1 there exist infinitely many finite connected tetravalent half-arc-transitive graphs with vertex stabilizer \( D_8 \times C_2^{m-3} \).

Let \( m \geq 7 \) be an integer and let \( \Gamma_m \) be the graph defined in [18, Section 3]. Then [18, Theorem 1.2] asserts that \( \Gamma_m \) is a connected tetravalent half-arc-transitive graph whose automorphism group is isomorphic to \( A_{2^m} \) with vertex stabilizer \( D_8 \times D_8 \times C_2^{m-6} \). Thus, by Theorem 1.1 there exist infinitely many finite connected tetravalent half-arc-transitive graphs with vertex stabilizer \( D_8 \times D_8 \times C_2^{m-6} \). \( \square \)

**Proof of Theorem 1.4.** According to [12, Theorem 1.1] and Theorem 1.2 each of the groups

\[
C_2^m \text{ with } 1 \leq m \leq 8, \quad D_8 \times C_2^{m-3} \text{ with } 4 \leq m \leq 8, \quad D_8^2 \times C_2^{m-6} \text{ with } 7 \leq m \leq 8
\]

is the vertex stabilizer of infinitely many finite connected tetravalent half-arc-transitive graphs. By Example 3.1, [4], [17] and Example 3.2 each of the groups

\[
D_8, \quad D_8 \times D_8, \quad \mathcal{H}_7, \quad \mathcal{H}_7 \times C_2
\]
is the vertex stabilizer of a half-arc-transitive nonabelian simple group acting on a finite connected tetravalent graph. This implies that each of these groups is the vertex stabilizer of infinitely many finite connected tetravalent half-arc-transitive graphs by Theorem 1.1. Hence every group in the list of Theorem 1.3 is the vertex stabilizer of infinitely many finite connected tetravalent half-arc-transitive graphs, and so Theorem 1.4 is true. □

5. Concluding remarks

The reader may have noticed that our key ingredient in this article is [15]. The main results of [15] are rather general and apply to most actions of groups on graphs. Our application of [15] in our work is rather successful, in our opinion, as we consider normal covers of graphs (†) admitting a nonabelian simple group of automorphisms.

Under this extra hypothesis, the results in [15] can be combined with rather strong group-theoretic results based on CFSG and, as a consequence, we are able to obtain infinite families of graphs having exotic vertex stabilizers. As far as we are aware, this type of constructions is a novelty.

In light of the following conjecture of Džambić and Jones [6], the hypothesis (†) does not seem strong. This suggests that Theorem 1.1 could be applied to show the existence of tetravalent half-arc-transitive graphs with other vertex stabilizers as well, and thus sheds light on classifying the vertex stabilizers of finite tetravalent half-arc-transitive graphs.

Conjecture 5.1. If $A$ and $B$ are finite groups with intersection $C$ of index at least 2 in $A$ and at least 3 in $B$, then all but finitely many alternating groups are homomorphic images of the amalgamated free product $A *_C B$.

Remark. This conjecture was also presented by Marston Conder in several conference talks. In fact, Marston Conder has a stronger conjecture [3].

We conclude this section by giving a natural generalization of the example in Subsection 3.2. Let $m \geq 7$ be an integer and let

$$H = \langle a_1, \ldots, a_m \mid a_i^2 = 1 \text{ for } i \leq m, \ (a_i a_j)^2 = 1 \text{ for } |i - j| \leq m - 3,$$

$$(a_1 a_{m-1})^2 = a_3, \ (a_2 a_m)^2 = a_4, \ (a_1 a_m)^2 = a_{m-2} \rangle.$$

Then $H = \langle a_1, a_2, a_3, a_4, a_{m-2}, a_{m-1}, a_m \rangle \times \langle a_5, \ldots, a_{m-3} \rangle \cong \mathcal{H}_7 \times C_2^{m-7}$. Let

$$B = \langle a_1, \ldots, a_{m-1} \rangle, \quad C = \langle a_2, \ldots, a_m \rangle$$

and let $\varphi : B \to C$ be the group isomorphism defined by

$$a_i^x = a_{i+1} \quad \text{for} \quad i = 1, \ldots, m - 1.$$

Then $H = B \cup a_m B = C \cup a_1 a_2 C$. Let $x$ be the permutation on $H$ defined by

$$b^x = b^\varphi \quad \text{and} \quad (a_m b)^x = a_1 a_2 b^\varphi \quad \text{for} \quad b \in B.$$

Denote the right regular representation of $H$ by $R$. Inspired by [17] and the results in Subsection 3.2 we make the following conjecture.

Conjecture 5.2. Let $H$, $R$ and $x$ be as above. Then

$$\text{Cos(Alt}(H), R(H), R(H) \{x, x^{-1}\} R(H))$$

is a connected tetravalent graph on which the right multiplication action of $\text{Alt}(H)$ is half-arc-transitive.
If this conjecture is true then Theorem 1.1 will imply that for every integer $m \geq 7$ there exist infinitely many finite connected tetravalent half-arc-transitive graphs with vertex stabilizer $H_7 \times C_{2^{m-7}}$.

REFERENCES

[1] W. Bosma, J. Cannon, C. Playoust, The magma algebra system I: The user language, *J. Symbolic Comput.*, 24 (1997), no. 3-4, 235–265.
[2] J. Chen, B. Xia, J.-X. Zhou, An infinite family of cubic nonnormal Cayley graphs on nonabelian simple groups, *Discrete Math.* 341 (2018), no. 5, 1282–1293.
[3] M. D. E. Conder, Simple group actions on arc-transitive graphs with prescribed transitive local action, *2017 MATRIX annals*, 327–335, MATRIX Book Ser., 2, Springer, Cham, 2019.
[4] M. D. E. Conder, D. Marušič, A tetravalent half-arc-transitive graph with non-abelian vertex stabilizer, *J. Combin. Theory Ser. B* 88 (2003), no. 1, 67–76.
[5] M. D. E. Conder, P. Potočnik, P. Šparl, Some recent discoveries about half-arc-transitive graphs, *Ars Math. Contemp.* 8 (2015), no. 1, 149–162.
[6] A. Džambić, G. A. Jones, p-adic Hurwitz groups, *J. Algebra* 379 (2013), 179–207.
[7] X. G. Fang, C. H. Li, M. Y. Xu, On edge-transitive Cayley graphs of valency four, *European J. Combin.* 25 (2004), no. 7, 1107–1116.
[8] G. Glauberman, Normalizers of $p$-subgroups in finite groups, *Pacific J. Math.* 29 (1969), 137–144.
[9] G. Glauberman, Isomorphic subgroups of finite $p$-groups. I, *Canad. J. Math.* 23 (1971), 983–1022.
[10] C. H. Li, X. Li, On permutation groups of degree a product of two prime-powers, *Comm. Algebra* 42 (2014), no. 11, 4722–4743.
[11] D. Marušič, Recent developments in half-transitive graphs, *Discrete Math.* 182 (1998), no. 1–3, 219–231.
[12] D. Marušič, Quartic half-arc-transitive graphs with large vertex stabilizers, *Discrete Math.* 229 (2005), no. 1-3, 180–193.
[13] D. Marušič, R. Nedela, On the point stabilizers of transitive groups with non-self-paired suborbits of length 2, *J. Group Theory* 4 (2001), no. 1, 19–43.
[14] P. Potočnik, A list of 4-valent 2-arc-transitive graphs and finite faithful amalgams of index (4, 2), *European J. Combin.* 30 (2009), no. 5, 1323–1336.
[15] P. Potočnik, P. Spiga, Lifting a prescribed group of automorphisms of graphs, *Proc. Amer. Math. Soc.* 147 (2019), no. 9, 3787–3796.
[16] P. Potočnik, P. Spiga, G. Verret, A census of 4-valent half-arc-transitive graphs and arc-transitive digraphs of valence two, *Ars Math. Contemp.* 8 (2015), no. 1, 133–148.
[17] P. Spiga, Constructing half-arc-transitive graphs of valency four with prescribed vertex stabilizers, *Graphs Combin.* 32 (2016), no. 5, 2135–2144.
[18] B. Xia, Tetravalent half-arc-transitive graphs with unbounded nonabelian vertex stabilizers, https://arxiv.org/abs/1908.09361.

(SPIGA) DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITY OF MILANO-BICOCCA, VIA COZZI 55, 20125 MILANO, ITALY
E-mail address: pablo.spiga@unimib.it

(XIA) SCHOOL OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF MELBOURNE, PARKVILLE, VIC 3010, AUSTRALIA
E-mail address: binzhoux@unimelb.edu.au