ELECTROMAGNETIC RADIATION AND
MOTION OF A PARTICLE

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I would like to dedicate this paper to the memory of my sister
Zuzka Klačková (* 5. 7. 1966 †27. 3. 1991)

Abstract. We consider the motion of uncharged dust grains of arbitrary shape including
the effects of electromagnetic radiation and thermal emission. The resulting relativisti-
cally covariant equation of motion is expressed in terms of standard optical parameters.
Explicit expressions for secular changes of osculating orbital elements are derived in de-
tail for the special case of the Poynting-Robertson effect. Two subcases are considered:
(i) central acceleration due to gravity and the radial component of radiation pressure
independent of the particle velocity, (ii) central acceleration given by gravity and the
radiation force as the disturbing force. The latter case yields results which may be com-
pared with secular orbital evolution in terms of orbital elements for an arbitrarily shaped
dust particle. The effects of solar wind are also presented.

Key words: cosmic dust, electromagnetic radiation, thermal emission, relativity theory,
equation of motion, orbital motion, celestial mechanics
1. Introduction

Poynting (1904) formulated a problem of finding equation of motion for a perfectly absorbing spherical particle under the action of electromagnetic radiation. Poynting did not succeed in finding correct solution. The second case, closely connected with the relativistic equation of motion for a free particle under the action of electromagnetic radiation, was presented by Einstein (1905), who calculated the change of energy and light pressure at arbitrary angle of incidence on a plane mirror. Robertson (1937) subsequently derived a correct equation of motion for a perfectly absorbing spherical particle. This result has been applied to astronomical problems for several decades and is known as the Poynting-Robertson (P-R) effect. Robertson’s case was relativistically generalized by Klačka (1992a), who showed that the generalized P-R effect holds only for the special case where the total momentum of the outgoing radiation per unit time is colinear with the incident radiation (in the proper/rest frame of reference of the particle), which may include radiation normally incident on a plane mirror. Since real particles interact with electromagnetic radiation in a complicated manner and particles of various optical properties exist (e.g., Mishchenko et al. 2002), it is essential to have an equation of motion sufficiently general to cover a wide range of optical parameters, not just the limited cases previously investigated. As an attempt of formulating such a general equation of motion, we can mention Lyttleton (1976), Klačka (1994a), Klačka and Kocifaj (1994), Kocifaj et al. (2000). The last three presentations are of hypothetical character only, since none of them proves correctness of the equation of motion. However, Klačka (2000a) succeeded in deriving relativistically covariant equation of motion. This equation of motion was derived in other possible ways by Klačka (2000b), Klačka and Kocifaj (2001), within the accuracy corresponding to Klačka (1994a). Finally, knowing the results obtained by Klačka and Kocifaj (1994) and Kocifaj et al. (2000) and having in disposal papers by Klačka (2000a, 2000b, 2000c), Kimura et al. (2002) presented a repetition of the equation of motion corresponding to Klačka and Kocifaj (1994) and Kocifaj et al. (2000).

The equations of motion for a moving particle have been constructed under the assumption that thermal emission from a particle is isotropic and does not exert a radiation pressure force on the particle in the particle frame of reference. Recently, Mishchenko (2001) has formulated the radiation pressure on arbitrarily shaped particles arising from an anisotropy of thermal emission. In this paper, we derive the equation of motion for an arbitrarily shaped particle moving in a radiation field taking into account the radiation pressure caused by an anisotropy of thermal emission as well as scattering and absorp-
tion of light. Relativistically covariant equation of motion is presented. Derivation of the equation of motion is physically fully reasoned.

We begin by reviewing in Sec. 2 and 3 the basic physical processes in proper and stationary frames. The equation of motion for simultaneous action of gravity and electromagnetic radiation is presented in Sec. 4. Sec. 5 then applies these results to the calculation of osculating orbital elements, including the special case of the P-R effect, which may be treated analytically. The calculation is carried out in Sec. 6 to first order in $v/c$ and applied to the ejection of a dust particle from a parent body such as a comet or asteroid. Two cases are considered: (i) the disturbing acceleration is given in terms of velocity (Robertson 1937, Wyatt and Whipple 1950), and (ii) the electromagnetic radiation itself is a disturbing function. However, these authors (including Lyttleton 1976) did not obtain the correct expression for the secular change of longitude of pericenter (perihelion). In Sec. 7 we use the equation of motion to second order in $v/c$ for the P-R effect. In particular we obtain a correct expression for the secular change of longitude of pericenter (perihelion). Sec. 8 then finds the secular change in the advancement of perihelion to first order in $v/c$, with gravity as a central acceleration. Next, we briefly discuss, in Sec. 9, the secular evolution of orbital elements for the P-R effect and nearly-circular orbits. Sec. 10 then treats the effect of the solar wind on the secular changes in the orbital elements to the second order in $v/u$, where $u$ is the speed of solar wind particles. Finally, Sec. 11 summarizes our results.

Other theoretical papers on the basic properties of the P-R effect were written during the last decades: Burns et al. (1979), Mediavilla and Buitrago (1989), Mignard (1992), Srikanth (1999), Williams (2002). Some others will be mentioned within the context of the discussed problems. Since some confusion exists in presenting derivations and results in the most cited papers (Robertson 1937; Wyatt and Whipple 1950; Burns et al. 1979; Mignard 1992), we present detailed derivation of the secular changes of orbital elements for heliocentric orbits and the P-R effect.

2. Proper reference frame of the particle – stationary particle

The term “stationary particle” will denote a particle which does not move in a given inertial frame of reference. Primed quantities will denote quantities measured in the proper reference frame of the particle – rest frame of the particle.

The flux density of photons scattered into an elementary solid angle $d\Omega' = \sin \theta' d\theta' d\phi'$ is proportional to $p'(\theta', \phi') d\Omega'$, where $p'(\theta', \phi')$ is the “phase function”. The phase function depends on orientation of the particle with respect to the direction.
of the incident radiation and on the particle characteristics; angles $\theta'$, $\phi'$ correspond to the direction (and orientation) of travel of the scattered radiation, $\theta'$ is the polar angle which vanishes for propagation along the unit vector $e'_1$ of the incident radiation. The phase function fulfills the normalisation condition

$$\int_{4\pi} p'(\theta',\phi') \, d\Omega' = 1 \, .$$  \hspace{1cm} (1)

The momentum of the incident beam of photons which is lost in the process of interaction with the particle is proportional to the cross-section $C'_{\text{ext}}$ (extinction). The part proportional to $C'_{\text{abs}}$ (absorption) is emitted in the form of thermal radiation and the part proportional to $C'_{\text{ext}} - C'_{\text{abs}} = C'_{\text{sca}}$ is scattered. The differential scattering cross section $dC'_{\text{sca}}/d\Omega' \equiv C'_{\text{sca}} p'(\theta', \phi')$ depends on the polarization state of the incident light as well as on the incidence and scattering directions (e. g., Mishchenko et al. 2002).

The momentum (per unit time) of the scattered photons into an elementary solid angle $d\Omega'$ is

$$dp'_{\text{sca}} = \frac{1}{c} S' C'_{\text{sca}} p'(\theta', \phi') \, K' \, d\Omega' \, ,$$  \hspace{1cm} (2)

where the unit vector in the direction of scattering is

$$K' = \cos \theta' \, e'_1 + \sin \theta' \, \cos \phi' \, e'_2 + \sin \theta' \, \sin \phi' \, e'_3 \, .$$  \hspace{1cm} (3)

$S'$ is the flux density of radiation energy (energy flow through unit area perpendicular to the ray per unit time). The system of unit vectors used on the RHS of the last equation forms an orthogonal basis. The total momentum (per unit time) of the scattered photons is

$$p'_{\text{sca}} = \frac{1}{c} S' C'_{\text{sca}} \int_{4\pi} p'(\theta', \phi') \, K' \, d\Omega' \, .$$  \hspace{1cm} (4)

The momentum (per unit time) obtained by the particle due to the interaction with radiation – radiation force acting on the particle – is

$$\frac{d \, p'}{d \, t'} = \frac{1}{c} S' \left\{ C'_{\text{ext}} e'_1 - C'_{\text{sca}} \int_{4\pi} p'(\theta', \phi') \, K' \, d\Omega' \right\} + F'_e(T') \, ,$$  \hspace{1cm} (5)

where the emission component of the radiation force acting on the particle of absolute temperature $T'$ is (Mishchenko et al. 2002, pp. 63-66)

$$F'_e(T') = - \frac{1}{c} \int_0^\infty d\omega' \int_{4\pi} \hat{r}' \, K'_e(\hat{r}', T', \omega') \, d\Omega' \, .$$  \hspace{1cm} (6)

The unit vector $\hat{r}' = r' / r'$ is given by position vector $r'$ of the observation point with origin inside the particle (the emitted radiation in the far-field zone of the particle propagates in the radial direction, i. e., along the unit vector $\hat{r}'$), $\omega'$ is (angular) frequency of radiation,
\[ K'_{b} (\hat{r}', T', \omega') = I'_{b} (T', \omega') \left[ K'_{11} (\hat{r}', \omega') - \int_{4\pi} Z'_{11} (\hat{r}'', \omega') d\hat{r}'' \right], \]  

where \( K'_{11} \) is the (1,1) element of the particle extinction matrix, \( Z'_{11} \) is the (1,1) element of the phase matrix and the Planck blackbody energy distribution is given by the well-known relation

\[ I'_{b} (T', \omega') = \frac{\hbar \omega'^{3}}{4 \pi^{3} c^{2}} \left\{ \exp \left( \frac{\hbar \omega'}{k T'} \right) - 1 \right\}^{-1}. \]  

Thermal emission has to be included in the total interaction of the particle with electromagnetic radiation: if the particle’s absolute temperature is above zero, it can emit as well as scatter and absorb electromagnetic radiation.

For the sake of brevity, we will use dimensionless efficiency factors \( Q'_{x} \) instead of cross sections \( C'_{x} \), where \( A' \) is geometrical cross section of a sphere of volume equal to the volume of the particle. Equation (5) can be rewritten to the form

\[ \frac{d p'}{d \tau} = \frac{1}{c} S' A' \left\{ (Q'_{\text{ext}} - < \cos \theta' > Q'_{\text{sca}}) e'_1 + [ - < \sin \theta' \cos \phi' > Q'_{\text{sca}}] e'_2 + \right. \]

\[ \left. [ - < \sin \theta' \sin \phi' > Q'_{\text{sca}}] e'_3 \right\} + \sum_{j=1}^{3} F'_{e j} e'_j, \]  

(9)

where \( < x' > \equiv \int_{4\pi} x' p'(\theta', \phi') d\Omega' \) and \( F'_{e j} \equiv F'_{e}(T') \cdot e'_j \). As for the energy, we assume that it is conserved: the energy (per unit time) of the incoming radiation \( E'_{i} \), equals to the energy (per unit time) of the outgoing radiation (after interaction with the particle) \( E'_{o} \). We will use the fact that time \( t' = \tau \), where \( \tau \) is proper time.

Summarizing important equations, we can write them in a short form

\[ \frac{d p'}{d \tau} = \sum_{j=1}^{3} \left( \frac{S' A'}{c} Q'_{j} + F'_{e j} \right) e'_j; \quad \frac{d E'}{d \tau} = 0, \]  

(10)

where \( Q'_{1} \equiv Q'_{\text{ext}} - < \cos \theta' > Q'_{\text{sca}} \), \( Q'_{2} \equiv - < \sin \theta' \cos \phi' > Q'_{\text{sca}} \), \( Q'_{3} \equiv - < \sin \theta' \sin \phi' > Q'_{\text{sca}} \). We have added an assumption of equilibrium state when the particle’s mass does not change.

### 2.1. Summary of the important equations

Using the text concerning energy below Eq. (9) and the last Eq. (10), we may describe the total process of interaction in the form of the following equations (energies and momenta per unit time):

\[ E'_o = E'_i = A' S', \]

\[ p'_o = (1 - Q'_{1}) p'_i - (Q'_{2} e'_2 + Q'_{3} e'_3) E'_o / c - \sum_{j=1}^{3} F'_{e j} e'_j, \]

\[ p'_i = (E'_i / c) e'_1, \]  

(11)
The index "i" represents the incoming radiation, beam of photons, the index "o" represents the outgoing radiation. The relation for $p'_o$ represents a generalization of the following cases:

i) $Q'_1 = 1, Q'_2 = Q'_3 = F'_{e1} = F'_{e2} = F'_{e3} = 0$ – Robertson (1937), Robertson and Noonan (1968), Srikanth (1999);

ii) $Q'_1$ arbitrary, $Q'_2 = Q'_3 = F'_{e1} = F'_{e2} = F'_{e3} = 0$ – Klačka (1992a);

iii) $Q'_1, Q'_2, Q'_3$ arbitrary, $F'_{e1} = F'_{e2} = F'_{e3} = 0$ – Klačka (2000a).

The changes of energy and momentum of the particle due to the interaction with electromagnetic radiation are

$$\frac{dE'}{d\tau} = E'_i - E'_o = 0,$$
$$\frac{dp'}{d\tau} = p'_i - p'_o. \quad (12)$$

3. Stationary frame of reference

By the term “stationary frame of reference” (laboratory frame) we mean a frame of reference in which particle moves with a velocity vector $v = v(t)$. The physical quantities measured in the stationary frame of reference will be denoted by unprimed symbols.

Our aim is to derive equation of motion for the particle in the stationary frame of reference. We will use the fact that we know this equation in the proper frame of reference – see Eqs. (11) and (12).

If we have a four-vector $A^\mu = (A^0, A)$, where $A^0$ is its time component and $A$ is its spatial component, generalized special Lorentz transformation yields

$$A'^0 = \gamma \left( A^0 - v \cdot A / c \right),$$
$$A' = A + \left[ (\gamma - 1) v \cdot A / v^2 - \gamma A^0 / c \right] v, \quad (13)$$

with inverse

$$A^0 = \gamma \left( A'^0 + v \cdot A'/c \right),$$
$$A = A' + \left[ (\gamma - 1) v \cdot A'/v^2 + \gamma A'^0/c \right] v, \quad (14)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

As for four-vectors we immediately introduce four-momentum:

$$p^\mu = (p^0, p) \equiv (E/c, p). \quad (15)$$
3.1. Incoming radiation

Applying Eqs. (14) and (15) to quantity \((E'_i/c, p'_i)\) (four-momentum per unit time – proper time is a scalar quantity) and taking into account also Eqs. (11), we can write

\[
E_i = E'_i \gamma (1 + \mathbf{v} \cdot \mathbf{e}'_1/c),
\]
\[
p_i = \frac{E'_i}{c} \left\{ \mathbf{e}'_1 + \left[ (\gamma - 1) \mathbf{v} \cdot \mathbf{e}'_1/v^2 + \gamma/c \right] \mathbf{v} \right\} .
\]

(16)

Using the fact that \(p^\mu = (h \nu, h \nu \mathbf{e}_1)\) for photons, we have

\[
\nu' = \nu w_1,
\]
\[
\mathbf{e}'_1 = \frac{1}{w_1} \left\{ \mathbf{e}_1 + \left[ (\gamma - 1) \mathbf{v} \cdot \mathbf{e}_1/v^2 - \gamma/c \right] \mathbf{v} \right\},
\]

(17)

where

\[
w_1 \equiv \gamma (1 - \mathbf{v} \cdot \mathbf{e}_1/c).
\]

(18)

Inserting the second of Eqs. (17) into Eqs. (18), one obtains

\[
E_i = (1/w_1) E'_i,
\]
\[
p_i = (1/w_1) (E'_i/c) \mathbf{e}_1.
\]

(19)

We have four-vector \(p^\mu_i = (E_i/c, p_i) = (1, \mathbf{e}_1) E_i/c = (1/w_1, \mathbf{e}_1/w_1) w_1 E_i/c \equiv \mathbf{b}^\mu_i w_1 E_i/c\).

For monochromatic radiation the flux density of radiation energy becomes

\[
S' = n' h \nu' c; \quad S = n h \nu c,
\]

(20)

where \(n\) and \(n'\) are concentrations of photons (photon number densities) in the corresponding reference frames. We also have continuity equation

\[
\partial_\mu j^\mu = 0, \quad j^\mu = (c n, c n \mathbf{e}_1),
\]

(21)

with current density \(j^\mu\). Application of Eq. (13) then yields

\[
n' = w_1 n.
\]

(22)

Using Eqs. (17), (20) and (22) we finally obtain

\[
S' = w^2_1 S.
\]

(23)

Eqs. (11), (19) and (23) then together give

\(E_i = w_1 S A', \quad p_i = w_1 S A' \mathbf{e}_1/c\).
3.2. Outgoing radiation

The situation is analogous to that of the preceding subsection. It is only a little more algebraically complicated, since radiation may also spread out in directions given by unit vectors $e_2, e_3$. How can we find transformations for the unit vectors $e'_2$ and $e'_3$? The crucial point is what physics do these unit vectors describe. The vectors $e'_2$ and $e'_3$ can be used to describe directions of propagation of radiation scattered by the particle. Thus, aberration of light also exists for each of these unit vectors. The relations between $e'_2$ and $e_2$, $e'_3$ and $e_3$, are analogous to that presented by the second of Eq. (17):

$$e'_j = \frac{1}{w_j} \left\{ e_j + \left[ (\gamma - 1) \mathbf{v} \cdot e_j / v^2 - \gamma / c \right] \mathbf{v} \right\}, \quad j = 1, 2, 3,$$

(24)

where

$$w_j = \gamma (1 - \mathbf{v} \cdot e_j / c), \quad j = 1, 2, 3.$$

(25)

It is worth mentioning that vectors $\{e'_j; j = 1, 2, 3\}$ form an orthonormal set of vectors, and, unit vectors $\{e_j; j = 1, 2, 3\}$ are not orthogonal unit vectors.

Applying Eqs. (14) and (15) to the quantity $(E'_o/c, p'_o)$ (four-momentum per unit time – proper time is a scalar quantity), we can write

$$E_o = \gamma (E'_o + \mathbf{v} \cdot p'_o),$$

$$p_o = p'_o + \left[ (\gamma - 1) \mathbf{v} \cdot p'_o / v^2 + \gamma \frac{E'_o}{c^2} \right] \mathbf{v}.$$

(26)

Using also $p'_i = E'_i e'_i / c$ and Eqs. (11), (24), (26),

$$E_o = Q'_1 w_1 E_i \gamma + (1 - Q'_1) E_i +
\sum_{j=1}^3 F'_{e_j} \left( c / w_j - \gamma c \right),$$

$$p_o = (1 - Q'_1) \frac{E_i}{c} e_1 + Q'_1 \frac{w_1}{c^2} E_i \gamma \mathbf{v} - \sum_{j=2}^3 Q'_j \frac{w_1}{c^2} E_i (c e_j / w_j - \gamma \mathbf{v}) -
\frac{1}{c} \sum_{j=1}^3 F'_{e_j} \left( c e_j / w_j - \gamma \mathbf{v} \right).$$

(27)

3.3. Equation of motion

In analogy with Eqs. (12), we have for the changes of energy and momentum of the particle due to the interaction with electromagnetic radiation

$$\frac{d E}{d \tau} = E_i - E_o,$$

$$\frac{d p}{d \tau} = p_i - p_o.$$

(28)
Putting Eqs. (27) into Eqs. (28), using also $p_i = (E_i/c) e_1$, one easily obtains

$$\frac{d E/c}{d \tau} = \sum_{j=1}^{3} \left( Q_j \frac{w_1 E_i}{c^2} + \frac{1}{c} F'_{ej} \right) \left( c \frac{1}{w_j} - \gamma c \right),$$

$$\frac{d p}{d \tau} = \sum_{j=1}^{3} \left( Q'_j \frac{w_1 E_i}{c^2} + \frac{1}{c} F'_{ej} \right) \left( c \frac{e_j}{w_j} - \gamma v \right).$$

(29)

Eq. (29) may be rewritten in terms of four-vectors:

$$\frac{d p^\mu}{d \tau} = \sum_{j=1}^{3} \left( Q'_j \frac{w_1 E_i}{c^2} + \frac{1}{c} F'_{ej} \right) (c b^\mu_j - u^\mu),$$

(30)

where $p^\mu$ is four-vector of the particle of mass $m$

$$p^\mu = m \, u^\mu,$$

(31)

four-vector of the world-velocity of the particle is

$$u^\mu = (\gamma c, \gamma v).$$

(32)

We have also other four-vectors

$$b^\mu_j = (1/w_j, e_j/w_j) \quad j = 1, 2, 3.$$  

(33)

It can be easily verified that:

i) the quantity $w E_i$ is a scalar quantity – see first of Eqs. (19);

ii) Eq. (30) reduces to Eq. (10) for the case of proper inertial frame of reference of the particle;

iii) Eq. (30) yields $d m/d \tau = 0$.

We introduce

$$b^0_j \equiv 1/w_j = \gamma (1 + v \cdot e'_j / c),$$

$$b_j = e_j / w_j = e'_j / \gamma + [(\gamma - 1) v \cdot e'_j / v^2 + \gamma / c] v, \quad j = 1, 2, 3$$

(34)

for the purpose of practical calculations. Physics of these relations corresponds to aberration of light.

We have derived an equation of motion for real dust particle under the action of electromagnetic radiation (including thermal emission). It is supposed that the equation of motion is represented by Eqs. (11) and (12) in the proper frame of reference of the particle. The final covariant form is represented by Eq. (30), or using $E_i = w_1 S A'$ (see Eqs. (11), (19) and (23)),

$$\frac{d p^\mu}{d \tau} = \sum_{j=1}^{3} \left( \frac{w^2_j S A'}{c^2} Q'_j + \frac{1}{c} F'_{ej} \right) (c b^\mu_j - u^\mu).$$

(35)
Another form of equation of motion is presented in Appendix A. There is also explained why considerations presented by Kimura et al. (2002) are incorrect.

To first order in $\nu/c$, Eqs. (34)-(35) yield

\[
\frac{d}{dt}v = \frac{S A'}{mc} \sum_{j=1}^{3} Q'_j \left[ \left( 1 - 2 \nu \cdot e_1/c + \nu \cdot e_j/c \right) e_j - v/c \right] + \frac{n}{m} \sum_{j=1}^{3} F'_{e_j} \left[ \left( 1 + \nu \cdot e_j/c \right) e_j - \frac{v}{c} \right],
\]

\[e_j = (1 - \nu \cdot e_j/c) e'_j + v/c, \quad j = 1, 2, 3. \tag{36}\]

It is worth mentioning to stress that the values of $Q'$-coefficients depend on particle’s orientation with respect to the incident radiation – their values are time dependent. A little different derivation of the equation of motion within the accuracy to the first order in $\nu/c$ is presented in Klačka and Kocifaj (2001). Inspite of knowing the correct theoretical results, Kimura et al. (2002) have succeeded in publishing a paper where different set of unit vectors is used (in reality, it is repetition of the older results of Klačka and Kocifaj 1994, Kocifaj et al. 2000). The relation between our set of unit vectors $\{e_j; j = 1, 2, 3\}$ and the set of unit vectors $\{k_j; j = 1, 2, 3\}$ used by Kimura et al. (2002) is following:

\[e_1 = k_1, \quad e_2 = k_2 + (\nu \cdot k_2/c) (k_1 - k_2) + v/c, \quad e_3 = k_3 - (\nu \cdot k_3/c) k_3 + v/c\]

and definition $\nu \cdot k_3 = 0$ is used – this transformation yields Eqs. (1) – (3) in Kimura et al. (2002):

\[e'_1 = (1 + \nu \cdot k_1/c) k_1 - v/c, \quad e'_2 = k_2 + (\nu \cdot k_2/c) k_1, \quad e'_3 = k_3. \]

As for physics of the paper Kimura et al. (2002), we refer to Appendix C.

It can be verified that Eq. (35) (or Eqs. (36) within the accuracy to the first order in $\nu/c$) yields as special cases the situations discussed in Einstein (1905) and Robertson (1937) – Robertson’s case is obtained simply by substituting $Q'_1 = 1, Q'_2 = Q'_3 = 0, F'_{e_1} = F'_{e_2} = F'_{e_3} = 0$, Einstein’s results require a little more calculations (see Appendix B).

3.4. Heuristic derivation

Since we completely understand the physics of Eq. (30), we are able to present a short simple derivation of Eq. (36). We have:

i) $d\nu'/dt = \sum_{i=1}^{3} \left\{ [S'A'/(mc)] Q'_i + F'_{e_j}/m \right\} e'_i$ (see Eq. (10)); $dm/dt = 0$ is supposed;

ii) $S' = S(1 - 2\nu \cdot e_1/c)$ (see Eq. (23)), due to the change of concentration of photons

\[n' = n(1 - \nu \cdot e_1/c) - \text{Eq. (22)} \]

and Doppler effect $\nu' = \nu(1 - \nu \cdot e_1/c) - \text{Eq. (17)};

iii) $e'_j = (1 + \nu \cdot e_j/c) e_j - v/c, \quad j = 1, 2, 3$ (aberration of light – see Eq. (24)).

Taking into account these physical phenomena, we finally obtain Eq. (36). However, only
relativistic covariant formulation proves that \( \{ Q_j', F_{ej}'; j = 1, 2, 3 \} \) behave as relativistically invariant quantities.

A reader may compare this heuristic derivation of the general case with heuristic derivation of the special case \((Q_1' = 1, Q_2' = Q_3' = 0, F_e' = 0)\) presented by Burns et al. (1979, pp. 5-6).

Another instructive text enabling better understanding of physics of the P-R effect can be found in (http://xxx.lanl.gov,astro-ph/0108210), which deals with explanation presented in Harwit (1988; pp. 176-177).

A little different derivation of the relation between \( S' \) and \( S \), based on: a) stress-energy tensor (energy-momentum tensor), b) transformations of electric and magnetic fields, may be found in Klačka (1992a: sections 2.3 and 2.5).

3.5. Continuous distribution of density flux of energy

For a continuous frequency distribution of density flux of energy, we can write

\[
\frac{d p'}{d \tau} = \sum_{j=1}^{3} \left\{ \frac{A'}{c} \int_{0}^{\infty} c h \nu' \frac{\partial n'}{\partial \nu'} Q_j'(\nu') \ d\nu' + F_{ej}' \right\} \ e_j' \equiv \\
\sum_{j=1}^{3} \left( \frac{S'}{c} A' \bar{Q}_j' + F_{ej}' \right) \ e_j' .
\] (37)

Taking into account that concentration of photons fulfills \( \nu' = w_1 n \) (Eq. (22)) and that Doppler effect yields \( \nu' = w_1 \nu \) (Eq. (17)), we have \( \partial n'/\partial \nu' = \partial n/\partial \nu \). Lorentz transformation finally yields

\[
\frac{d p'^\mu}{d \tau} = \sum_{j=1}^{3} \left\{ \frac{w_1^2 A'}{c^2} \int_{0}^{\infty} c h \nu Q_j'(w_1 \nu) \ d\nu + \frac{1}{c} F_{ej}' \right\} (c b'^\mu_j - u'^\mu)
\equiv \\
\sum_{j=1}^{3} \left( \frac{w_1^2 S}{c^2} A' \bar{Q}_j' + \frac{1}{c} F_{ej}' \right) (c b'^\mu_j - u'^\mu) .
\] (38)

As a consequence, \( dm/d\tau = 0 \) (this corresponds to the condition \( dE'/d\tau = 0 \)).

If the particle is not irradiated, then one has to use

\[
\left( \frac{d p'^\mu}{d \tau} \right)_e = \sum_{j=1}^{3} F_{ej}' b'^\mu_j ,
\] (39)

instead of Eq. (38). As a consequence, Eq. (39) yields \( (dm/d\tau)_e = \sum_{j=1}^{3} F_{ej}' / c \). Mass of the particle decreases due to the thermal emission, alone.

4. Gravitation and electromagnetic radiation

The generally covariant equation of motion can be immediately written on the basis of Eq. (38) (see e. g., Landau and Lifshitz 1975):
\[
\frac{D p^\mu}{d \tau} = \sum_{j=1}^{3} \left( \frac{w_j^2 S A'}{c} \bar{Q}_j' + F_{e_j}' \right) \left( b_j' - u^\mu / c \right). \tag{40}
\]

where the operator \( D / d \tau \) is the “total” covariant derivative in the general relativistic sense and includes gravitational effects.

4.1. Gravity and radiation – equation of motion to the first order in \( v/c \)

We can write, on the basis of Eqs. (40), (25) and (33)

\[
\frac{d v}{d t} = - \frac{G M_\odot}{r^2} \mathbf{e}_1 + \frac{G M_\odot}{r^2} \sum_{j=1}^{3} \beta_j \left[ \left( 1 - 2 \frac{v \cdot \mathbf{e}_1}{c} + \frac{v \cdot \mathbf{e}_j}{c} \right) \mathbf{e}_j - \frac{v}{c} \right] + \frac{1}{m} \sum_{j=1}^{3} F_{e_j}' \left[ \left( 1 + \frac{v \cdot \mathbf{e}_j}{c} \right) \mathbf{e}_j - \frac{v}{c} \right], \tag{41}
\]

where

\[
e_j = (1 - v \cdot e'_j/c) e'_j + v/c, \quad j = 1, 2, 3, \]

\[
\beta_1 = \frac{\pi R_\odot^2}{G M_\odot m c} \int_{0}^{\infty} B_\odot(\lambda) \{ C'_{ext}(\lambda/w) - C'_{sca}(\lambda/w) g'_1(\lambda/w) \} d\lambda, \]

\[
\beta_2 = \frac{\pi R_\odot^2}{G M_\odot m c} \int_{0}^{\infty} B_\odot(\lambda) \{ - C'_{sca}(\lambda/w) g'_2(\lambda/w) \} d\lambda, \]

\[
\beta_3 = \frac{\pi R_\odot^2}{G M_\odot m c} \int_{0}^{\infty} B_\odot(\lambda) \{ - C'_{sca}(\lambda/w) g'_3(\lambda/w) \} d\lambda, \]

\[
w = 1 - \frac{v \cdot \mathbf{e}_1}{c}, \tag{42}
\]

if we use Sun as a source of gravitation and radiation. \( R_\odot \) denotes the radius of the Sun and \( B_\odot(\lambda) \) is the solar radiance at a wavelength of \( \lambda \); \( G, M_\odot, \) and \( r \) are the gravitational constant, the mass of the Sun, and the distance of the particle from the center of the Sun, respectively. The asymmetry parameter vector \( g' \) is defined by \( g' = (1/C'_{sca}) \int n'(dC'_{sca}/d\Omega')d\Omega' \), where \( n' \) is a unit vector in the direction of scattering; \( g' = g'_1 \mathbf{e}'_1 + g'_2 \mathbf{e}'_2 + g'_3 \mathbf{e}'_3, \quad e_j = (1 + v \cdot e'_j/c) e_j - v/c, \quad e'_i \cdot e'_j = \delta_{ij}. \) We may mention that \( \beta_1 \equiv \beta, \beta_2 \equiv \beta \bar{Q}_2/\bar{Q}_1', \beta_3 \equiv \beta \bar{Q}_3/\bar{Q}_1' \) correspond to quantities used in Klaczka and Kocifaj (2001).

5. Motion and orbital elements

Equation of motion is given by Eq. (40), or, by Eq. (41) to the first order in \( v/c \). We have to take into account that the non-dimensional parameter, for Solar System

\[
\beta_1 \equiv \beta = \frac{r^2 S A'}{G M_\odot m c} \bar{Q}_1' = \frac{L_\odot A'}{4 \pi G M_\odot m c} \bar{Q}_1' = \frac{0.02868}{12 \pi} \bar{Q}_1' A' \left[ \frac{m^2}{kg} \right]. \tag{43}
\]
may change during the motion. Here $L_\odot$ is the rate of energy outflow from the Sun, the solar luminosity. $\beta$ may change and at each point of the orbit all three parameters $\beta_1$, $\beta_2$ and $\beta_3$ have to be numerically calculated (except for a very special cases, when the $\beta$ parameters are constant during the motion).

If we are interested in orbital evolution in terms of osculating orbital elements, we come to the crucial point: Which type of osculating orbital elements is correct? Mainly, if the “radial radiation pressure” (the dominant term containing parameter $\beta \equiv \beta_1$ in Eq. (75)) has to be considered together with the central gravitational force, or not (compare Klačka 1992b).

As for a definition of osculating orbital elements, we refer to any textbook on celestial mechanics. Brouwer and Clemence (1961) write (p. 273): “As the motion progresses under the influence of the various attracting bodies, the coordinates and velocity components at any instant may be used to obtain a set of six orbital elements. These are precisely the elements of the ellipse that the body would follow if from that particular instant on, the accelerations caused by all “perturbing” bodies ceased to exist.” As for our purposes, it is sufficient to make a small change: “attracting bodies” are replaced by “forces”.

In reality, our perturbing physical force corresponds to the total electromagnetic radiation force. Thus, physics incites us to use gravitational force alone, as the central acceleration. However, it may occur that the dominant term containing parameter $\beta_1 \equiv \beta$ in Eq. (41) is comparable to central gravity term. As a consequence, orbital elements will change very rapidly during the motion. This would suggest to divide physical force into two parts and to use “radial radiation pressure” as a part of central acceleration, i. e. together with central gravitational acceleration. The final problem concerns the changes of $\beta$ (almost random changes) which prevent us to use the term “Keplerian orbit” for an unperturbed problem.

Thus, we have to use gravitational acceleration of the central body (Sun) as an acceleration defining Keplerian orbits – complete electromagnetic radiative acceleration is a disturbing acceleration – in Eq. (41). As for long-term orbital evolution, we may use mean values of orbital elements – they may be defined as time mean values of orbital elements when true anomaly changes in $2\pi$ radians.

Let us consider that dust particle is ejected from a parent body. Orbital elements of the parent body, at the moment of ejection, are: $a_P$, $e_P$, $i_P$, $\Omega_P$, $\omega_P$ and $\Theta_P$, i. e., semi-major axis, eccentricity, inclination, longitude of the ascending node, longitude of pericenter – longitude of perihelion for the case of Solar System – and position angle of the body measured from the ascending node in the direction of the motion of the body.
The velocity vector of the parent body, in a given coordinate system with the origin in the dominant central body (Sun in the case of the Solar System) is:

\[
\vec{v}_P = \vec{v}_R \, \vec{e}_{PR} + \vec{v}_T \, \vec{e}_{PT},
\]

\[
v_R = \sqrt{\frac{GM}{a_p (1 - e_p^2)}} \, e_p \sin f_P,
\]

\[
v_T = \sqrt{\frac{GM}{a_p (1 - e_p^2)}} \left(1 + e_p \cos f_P \right),
\]

\[
r_P = \frac{a_p (1 - e_p^2)}{1 + e_p \cos f_P},
\]

\[
\vec{e}_{PR} = (\cos \Omega_p \cos \Theta_p - \sin \Omega_p \sin \Theta_p \cos i_p, \sin \Omega_p \cos \Theta_p + \cos \Omega_p \sin \Theta_p \cos i_p, \sin \Theta_p \sin i_p),
\]

\[
\vec{e}_{PT} = (- \cos \Omega_p \sin \Theta_p - \sin \Omega_p \cos \Theta_p \cos i_p, - \sin \Omega_p \sin \Theta_p + \cos \Omega_p \cos \Theta_p \cos i_p, \cos \Theta_p \sin i_p),
\]

\[
f_P = \Theta_P - \omega_P.
\]  

Radial and transversal velocity components are \(v_R \) and \(v_T \), unit vector \(\vec{e}_{PT}\) is normal to the radial vector and oriented in orientation of motion. True anomaly \(f_P\) equals 0 for pericenter/perihelion and \(\pi\) for apocenter/aphelion.

The initial conditions for the particle ejected with velocity \(\Delta\) from the parent body are:

\[
r_{in} = r_P,
\]

\[
v_{in} = v_P + \Delta.
\]  

Eqs. (44)-(45) immediately yield that initial orbital elements of the particle are equal to those of the parent body for \(\Delta = 0\):

\[
a_{in} = a_p, \quad e_{in} = e_p, \quad i_{in} = i_p,
\]

\[
\Omega_{in} = \Omega_p, \quad \omega_{in} = \omega_p, \quad \Theta_{in} = \Theta_p.
\]  

(Initial orbital elements for general case represented by Eq. (45) can be calculated from the relations presented in Eq. (47) below.)

Complete equation of motion for dust particle orbiting the central body of mass \(M_{\odot}\) under central’s body electromagnetic radiation and gravity interaction is given by Eq. (41). We can find osculating orbital elements for dust particle according to the following equations:

\[
E = \frac{1}{2} \vec{v}^2 - \frac{GM_{\odot}}{r},
\]
\[ \mathbf{H} = \mathbf{r} \times \mathbf{v} \, , \]
\[ p = \frac{\mathbf{H}^2}{G M_\odot} , \]
\[ e = 1 + \frac{2}{G M_\odot} \frac{E}{p} , \]
\[ a = \frac{p}{1 - e^2} , \quad q = a \left(1 - e^2\right)^{1/2} , \]
\[ i = \arccos\left(\frac{H_z}{|\mathbf{H}|}\right) , \]
\[ \sin \Omega \sin i = H_x/|\mathbf{H}| , \quad -\cos \Omega \sin i = H_y/|\mathbf{H}| , \]
\[ e_R = r/|r| = (x, y, z)/r , \]
\[ e_N = \mathbf{H}/|\mathbf{H}| , \]
\[ e_T = e_N \times e_R , \]
\[ \sin \Theta \sin i = z/r , \]
\[ \cos \Theta \sin i = \left(e_T\right)_z \equiv (y H_x - x H_y)/(r|\mathbf{H}|) , \]
\[ \sin (\Theta - \omega) = \mathbf{v} \cdot e_R/\left(e \sqrt{G M_\odot/p}\right) , \]
\[ \cos (\Theta - \omega) = \mathbf{v} \cdot e_T/\left(e \sqrt{G M_\odot/p}\right) - 1 . \]  

where \( q \) is perihelion distance. If \( i = 0 \), then \( \Theta \) has to be obtained from \( \cos(\Omega + \Theta) = x/r \), \( \sin(\Omega + \Theta) = y/r \), and, we may take \( \Omega \) in an arbitrary way (e. g., the requirement that \( \Omega \) is a continuous function of time may define \( \Omega \) for \( i = 0 \)). As for the value of inclination, we repeat the well-known fact: prograde orbits exist for \( i \in (0, \pi/2) \), retrograde orbits exist for \( i \in (\pi/2, \pi) \). The case \( e > 1 \) describes the case when the particle is ejected from the Solar System.

5.1. Poynting-Robertson effect

Putting \( \tilde{Q}'_2 = \tilde{Q}'_4 = F_{e1}' = F_{e2}' = F_{e3}' \equiv 0 \) in Eq. (40), we obtain Poynting-Robertson effect (Robertson 1937, Klačka 1992a):

\[ \frac{D p'^\mu}{d \tau} = \frac{u'^2}{c} S A' \tilde{Q}'_4 \left(b'^\mu - w'^\mu / c\right) . \]  

Eq. (48) enables us some analytical calculations for changes of osculating orbital elements: it is supposed that \( \tilde{Q}'_4 \) is a constant. We will present them, in the following sections. Section 6 will consider the first order in \( v/c \) of Eq. (48), section 7 will consider also the second order in \( v/c \) of Eq. (48).

6. P-R effect – equation of motion to the first order in \( v/c \)

We can write, on the basis of Eqs. (48), (41) – (43)
\[
\frac{d \mathbf{v}}{dt} = -\frac{\mu}{r^2} \mathbf{e}_R + \frac{\mu}{r^2} \beta \left\{ \left(1 - \frac{\mathbf{v} \cdot \mathbf{e}_R}{c}\right) \mathbf{e}_R - \frac{\mathbf{v}}{c} \right\},
\]  

(49)

where \( \mu \equiv G M_\odot, \mathbf{e}_R \equiv e_1 \) and \( \beta \equiv \beta_1 \) is a non-dimensional parameter (“the ratio of radiation pressure force to the gravitational force”; see also Eq. (43)). Eq. (43) reduces to \( \beta = 5.7 \times 10^{-5} \bar{Q}'_1/\rho[g/cm^3] s[cm] \), for homogeneous spherical particle: \( g \) is mass density and \( s \) is radius of the sphere.

6.1. Secular changes of orbital elements – radiation pressure as a part of central acceleration

We have to use \(-\mu (1 - \beta) \mathbf{e}_R / r^2\) as a central acceleration determining osculating orbital elements if we want to take a time average (\( T \) is time interval between passages through two following pericenters) in an analytical way

\[
\langle g \rangle \equiv \frac{1}{T} \int_0^T g(t) dt = \frac{\sqrt{\mu (1 - \beta)}}{a_\beta^{3/2}} \frac{1}{2\pi} \int_0^{2\pi} g(f_\beta) \left( \frac{df_\beta}{dt} \right)^{-1} df_\beta 
= \frac{\sqrt{\mu (1 - \beta)}}{a_\beta^{3/2}} \frac{1}{2\pi} \int_0^{2\pi} g(f_\beta) \frac{\sqrt{\mu (1 - \beta)}}{p_\beta} df_\beta 
= \frac{1}{a_\beta^2 \sqrt{1 - e_\beta^2}} \frac{1}{2\pi} \int_0^{2\pi} g(f_\beta) r^2 df_\beta ,
\]

(50)

assuming non-pseudo-circular orbits and the fact that orbital elements exhibit only small changes during the time interval \( T \); \( a_\beta \) is semi-major axis, \( e_\beta \) is eccentricity, \( f_\beta \) is true anomaly, \( p_\beta = a_\beta (1 - e_\beta^2) \); the second and the third Kepler’s laws were used: \( r^2 \frac{df_\beta}{dt} = \sqrt{\mu (1 - \beta) p_\beta} \) – conservation of angular momentum, \( a_\beta^3 / T^2 = \mu (1 - \beta) / (4\pi^2) \).

Rewriting Eq. (49) into the form

\[
\frac{d \mathbf{v}}{dt} = -\frac{\mu}{r^2} \mathbf{e}_R + \frac{\mu}{r^2} \beta \left\{ \left(1 - \frac{\mathbf{v} \cdot \mathbf{e}_R}{c}\right) \mathbf{e}_R - \frac{\mathbf{v}}{c} \right\},
\]  

(51)

we can immediately write for components of perturbation acceleration to Keplerian motion:

\[
F_{\beta R} = -2 \beta \frac{\mu}{r^2} \frac{v_\beta R}{c}, \quad F_{\beta T} = -\beta \frac{\mu}{r^2} \frac{v_\beta T}{c}, \quad F_{\beta N} = 0 ,
\]

(52)

where \( F_{\beta R}, F_{\beta T} \) and \( F_{\beta N} \) are radial, transversal and normal components of perturbation acceleration, and two-body problem yields

\[
v_{\beta R} = \sqrt{\frac{\mu (1 - \beta)}{p_\beta}} e_\beta \sin f_\beta ,
\]
\[
v_{\beta T} = \sqrt{\frac{\mu (1 - \beta)}{p_\beta}} (1 + e_\beta \cos f_\beta) .
\]

(53)
The important fact that perturbation acceleration is proportional to \( v/c \ll 1 \) ensures the above mentioned small changes of orbital elements during the time interval \( T \).

Perturbation equations of celestial mechanics yield for osculating orbital elements \((a_\beta, e_\beta, i_\beta, \Omega_\beta, \omega_\beta, \Theta_\beta)\) - semi-major axis; \( e_\beta \) - eccentricity; \( i_\beta \) - inclination (of the orbital plane to the reference frame); \( \Omega_\beta \) - longitude of the ascending node; \( \omega_\beta \) - longitude of pericenter; \( \Theta_\beta \) is the position angle of the particle on the orbit, when measured from the ascending node in the direction of the particle’s motion, \( \Theta_\beta = \omega_\beta + f_\beta \):

\[
\frac{da_\beta}{dt} = \frac{2}{1 - e_\beta^2} \sqrt{\frac{p_\beta}{\mu (1 - \beta)}} \left\{ F_\beta R e_\beta \sin f_\beta + F_\beta T (1 + e_\beta \cos f_\beta) \right\},
\]

\[
\frac{de_\beta}{dt} = \sqrt{\frac{p_\beta}{\mu (1 - \beta)}} \left\{ F_\beta R \sin f_\beta + F_\beta T \left[ \cos f_\beta + \frac{e_\beta + \cos f_\beta}{1 + e_\beta \cos f_\beta} \right] \right\},
\]

\[
\frac{di_\beta}{dt} = \frac{r}{\sqrt{\mu (1 - \beta) p_\beta}} F_\beta N \cos \Theta_\beta,
\]

\[
\frac{d\Omega_\beta}{dt} = \frac{r}{\sqrt{\mu (1 - \beta) p_\beta}} F_\beta N \sin \Theta_\beta \sin i_\beta,
\]

\[
\frac{d\omega_\beta}{dt} = -\frac{1}{e_\beta} \sqrt{\frac{p_\beta}{\mu (1 - \beta)}} \left\{ F_\beta R \cos f_\beta - F_\beta T \frac{2 + e_\beta \cos f_\beta}{1 + e_\beta \cos f_\beta} \sin f_\beta \right\} - \frac{r}{\sqrt{\mu (1 - \beta) p_\beta}} F_\beta N \sin \Theta_\beta \sin i_\beta \cos i_\beta,
\]

\[
\frac{d\Theta_\beta}{dt} = \frac{\sqrt{\mu (1 - \beta) p_\beta} r^2}{p_\beta} - \frac{r}{\sqrt{\mu (1 - \beta) p_\beta}} F_\beta N \sin \Theta_\beta \cos i_\beta, \quad (54)
\]

where \( r = p_\beta/(1 + e_\beta \cos f_\beta) \).

Inserting Eqs. (52) - (53) into Eq. (54), one easily obtains

\[
\frac{da_\beta}{dt} = -\beta \frac{\mu}{r^2} \frac{2a_\beta 1 + e_\beta^2 + 2e_\beta \cos f_\beta + e_\beta^2 \sin^2 f_\beta}{1 - e_\beta^2},
\]

\[
\frac{de_\beta}{dt} = -\frac{\beta \mu}{r^2} \frac{1}{c} (2e_\beta + e_\beta \sin^2 f_\beta + 2 \cos f_\beta),
\]

\[
\frac{di_\beta}{dt} = 0,
\]

\[
\frac{d\Omega_\beta}{dt} = 0,
\]

\[
\frac{d\omega_\beta}{dt} = -\beta \frac{\mu}{r^2} \frac{1}{c e_\beta} (2 - e_\beta \cos f_\beta) \sin f_\beta,
\]

\[
\frac{d\Theta_\beta}{dt} = \frac{\sqrt{\mu (1 - \beta) p_\beta}}{r^2}. \quad (55)
\]

It is worth mentioning that \( da_\beta/dt < 0 \) for any time \( t \).

The set of differential equations Eqs. (55) has to be complemented with initial conditions. If the subscript “\( P \)” denotes orbital elements of the parent body (see Eq. (44)), then

\[
r_{\beta in} = r_P \equiv \frac{p_P}{1 + e_P \cos f_P} e_{PR}
\]

\[
v_{\beta in} = v_P + \Delta, \quad (56)
\]
\[\Delta\] is velocity with respect to the nucleus of the parent body, and

\[
\begin{align*}
\mathbf{r}_\beta \text{ in} &= 
\frac{p_{\beta \text{ in}}}{1 + e_{\beta \text{ in}} \cos f_{\beta \text{ in}}} \mathbf{e}_\beta \text{ in} \mathbf{R}_\text{in}, \\
\mathbf{v}_\beta \text{ in} &= v_\beta \mathbf{R}_\text{in} \mathbf{e}_\beta \text{ in} + v_\beta \mathbf{T}_\text{in} \mathbf{e}_\beta \text{ in}, \\
v_\beta \mathbf{R}_\text{in} &= \sqrt{\frac{\mu}{p_{\beta \text{ in}}}} e_{\beta \text{ in}} \sin f_{\beta \text{ in}}, \\
v_\beta \mathbf{T}_\text{in} &= \sqrt{\frac{\mu}{p_{\beta \text{ in}}}} (1 + e_{\beta \text{ in}} \cos f_{\beta \text{ in}}).
\end{align*}
\]

\[
\begin{align*}
e_{\beta \text{ in}} &= (\cos \Omega_{\beta \text{ in}} \cos \Theta_{\beta \text{ in}} - \sin \Omega_{\beta \text{ in}} \sin \Theta_{\beta \text{ in}} \cos i_{\beta \text{ in}}, \\
&\quad \sin \Omega_{\beta \text{ in}} \cos \Theta_{\beta \text{ in}} + \cos \Omega_{\beta \text{ in}} \sin \Theta_{\beta \text{ in}} \cos i_{\beta \text{ in}}, \sin \Theta_{\beta \text{ in}} \sin i_{\beta \text{ in}}), \\
e_{\beta \text{ in}} &= (- \cos \Omega_{\beta \text{ in}} \sin \Theta_{\beta \text{ in}} - \sin \Omega_{\beta \text{ in}} \cos \Theta_{\beta \text{ in}} \cos i_{\beta \text{ in}}, \\
&\quad - \sin \Omega_{\beta \text{ in}} \sin \Theta_{\beta \text{ in}} + \cos \Omega_{\beta \text{ in}} \cos \Theta_{\beta \text{ in}} \cos i_{\beta \text{ in}}, \cos \Theta_{\beta \text{ in}} \sin i_{\beta \text{ in}}), \\
f_{\beta \text{ in}} &= \Theta_{\beta \text{ in}} - \omega_{\beta \text{ in}}.
\end{align*}
\]

Using Eqs. (44), (56), (58), we finally obtain (compare Gajdošík and Kláška 1999):

\[
\begin{align*}
p_{\beta \text{ in}} &= \frac{p_p}{1 - \beta} \frac{(v_T + \Delta v_T)^2 + (\Delta v_N)^2}{v_T^2}, \\
e_{\beta \text{ in}}^2 &= \left(\frac{1 + e_p \cos f_p}{1 - \beta} \frac{v_{TS}^2}{v_T^2} - 1\right)^2 + \left(1 + e_p \cos f_p \frac{v_{TS}}{v_T} \right)^2 \left(\frac{v_R + \Delta v_R}{v_T}\right)^2, \\
\cos i_{\beta \text{ in}} &= \frac{v_T + \Delta v_T}{v_{TS}} \cos i_p - \frac{\Delta v_N}{v_{TS}} \cos \Theta_p \sin \Theta_p, \\
\sin i_{\beta \text{ in}} \cos \Omega_{\beta \text{ in}} &= \frac{v_T + \Delta v_T}{v_{TS}} \sin i_p \cos \Omega_p + \frac{\Delta v_N}{v_{TS}} \cos \Theta_p \cos i_p \cos \Omega_p - \sin \Theta_p \sin \Omega_p, \\
\sin i_{\beta \text{ in}} \sin \Omega_{\beta \text{ in}} &= \frac{v_T + \Delta v_T}{v_{TS}} \sin i_p \sin \Omega_p + \frac{\Delta v_N}{v_{TS}} \cos \Theta_p \cos i_p \sin \Omega_p + \sin \Theta_p \cos \Omega_p, \\
e_{\beta \text{ in}} \cos f_{\beta \text{ in}} &= \frac{p_{\beta \text{ in}}}{p_p} (1 + e_p \cos f_p) - 1, \\
e_{\beta \text{ in}} \sin f_{\beta \text{ in}} &= \frac{1 + e_p \cos f_p}{1 - \beta} \frac{v_{TS}}{v_T} (v_R + \Delta v_R), \\
\sin i_{\beta \text{ in}} \cos \Theta_{\beta \text{ in}} &= \frac{v_T + \Delta v_T}{v_{TS}} \sin i_p \cos \Theta_p + \frac{\Delta v_N}{v_{TS}} \cos i_p, \\
\sin i_{\beta \text{ in}} \sin \Theta_{\beta \text{ in}} &= \sin i_p \sin \Theta_p, \\
v_{TS}^2 &= (v_T + \Delta v_T)^2 + (\Delta v_N)^2.
\end{align*}
\]
where \( v_R \) and \( v_T \) are given by expressions presented in Eq. (78) – \( p_P = a_P \left(1 - e_P^2\right) \).

For the special case \( \Delta = 0 \) Eq. (59) reduces to

\[
a_{\beta \text{ in}} = a_P \left(1 - \beta\right) \left(1 - 2\beta \frac{1 + e_P \cos f_P}{1 - e_P^2}\right)^{-1},
\]

\[
e_{\beta \text{ in}} = \sqrt{1 - \frac{1 - e_P^2 - 2\beta \left(1 + e_P \cos f_P\right)}{(1 - \beta)^2}},
\]

where \( f_P \equiv \Theta_P - \omega_P \), \( \omega_{\beta \text{ in}} \) has to be obtained from

\[
e_{\beta \text{ in}} \cos \left(\Theta_P - \omega_{\beta \text{ in}}\right) = \frac{\beta + e_P \cos f_P}{1 - \beta},
\]
\[
e_{\beta \text{ in}} \sin \left(\Theta_P - \omega_{\beta \text{ in}}\right) = \frac{e_P \sin f_P}{1 - \beta},
\]

\[\Omega_{\beta \text{ in}} = \Omega_P, \quad i_{\beta \text{ in}} = i_P, \quad \Theta_{\beta \text{ in}} = \Theta_P.\]

Physics of Eqs. (60) – (61) is following: meteoroid escapes from the Solar System when the orbital energy becomes positive and this can happen when the energy due to the radial component of the radiation force is included, without it being necessary for this force to exceed the gravitational attraction (Harwit 1963). Some figures may be found in Klačka (1992b). As an example we may mention that particle of \( \beta = (1 - e_P)/2 \) moves in parabolic orbit if ejected at perihelion of the parent body, and, in an orbit with eccentricity \( e_{\beta \text{ in}} = \left|1 - 3e_P\right|/(1 + e_P) \) if released at aphelion; \( e_{\beta \text{ in}} = 0 \) for \( \beta = e_P = 1/3 \) and aphelion ejection.

By inserting Eqs. (55) into Eq. (50), taking into account that \( e_{\beta \text{ in}} < 1 \) (see Eq. (61)), one can easily obtain for the secular changes of orbital elements:

\[
\langle \frac{da_{\beta}}{dt} \rangle = -\beta \frac{\mu}{c} \frac{2 + 3e_{\beta}^2}{a_{\beta} \left(1 - e_{\beta}^2\right)^{3/2}},
\]

\[
\langle \frac{de_{\beta}}{dt} \rangle = -\frac{5}{2} \beta \frac{\mu}{c} \frac{e_{\beta}}{a_{\beta}^2 \left(1 - e_{\beta}^2\right)^{1/2}},
\]

\[
\langle \frac{d\Theta_{\beta}}{dt} \rangle = \sqrt{\frac{\mu}{a_{\beta}^{3/2}}} \left(1 - \beta\right),
\]

\[
\langle \frac{di_{\beta}}{dt} \rangle = \langle \frac{d\Omega_{\beta}}{dt} \rangle = \langle \frac{d\omega_{\beta}}{dt} \rangle = 0.
\]

As an remark we may mention that a little more simple set of differential equations than that represented by Eqs. (64) – (65) is obtained when semi-major axis \( a_{\beta} \) is replaced by the quantity \( p_{\beta} = a_{\beta} \left(1 - e_{\beta}^2\right) \). It can be easily verified that a new set of differential equations for secular evolution is given by the following equations:

i) \( dp_{\beta}/dt = -2\beta(\mu/c)(1 - e_{\beta}^2)^{3/2}/p_{\beta}, \)
\[\frac{d\beta}{dt} = -\frac{(5/2)\beta}{p}\left(1 - \frac{e^2}{p}\right)^{3/2}/p\beta.\]

These two equations immediately yield \(p = p_{\beta \text{in}} \left(\frac{e_{\beta}}{e_{\beta \text{in}}}\right)^{4/5} \).

The last relation corresponds to the relation between orbital angular momentum and eccentricity of the orbit: \(H_{\beta} = H_{\beta \text{in}} \left(\frac{e_{\beta}}{e_{\beta \text{in}}}\right)^{2/5} \).

Analogously, for secular evolution of osculating orbital energy (per unit mass) holds:

\[E_{\beta} = \frac{v^2}{2} - \mu (1 - \beta)/r \equiv -\mu (1 - \beta)/2p,\]

\[E_{\beta} = E_{\beta \text{in}}\left[\left(1 - \frac{e^2}{p^2}\right)\left(1 - \frac{e^2_{\text{in}}}{p^2_{\text{in}}}\right)\right] \left(\frac{e_{\beta}}{e_{\beta \text{in}}}\right)^{4/5}.\]

Equations i) and ii) enable to write:

\[\frac{dp}{dt} = \frac{2\beta}{p} \left(1 - e^2\right)^{3/2}/p,\]

\[\frac{de}{dt} = \frac{(5/2)\beta}{p} \left(\frac{e_{\beta}}{e_{\beta \text{in}}}\right)^{\frac{8}{5}}/p.\]

These equations point out that near-circular / pseudo-circular orbits \((e_{\beta} \approx 0)\) and orbits with small values of \(p_{\beta}\) are not described by the discussed secular changes of orbital elements and by the considered \(v/c\) approximation for the P-R effect.

6.2. Secular changes of orbital elements – gravitation as a central acceleration

As it was discussed in section 5, it is not wise to use \(-\mu \left(1 - \beta_1\right) e_R / r^2\) as a central acceleration determining osculating orbital elements for the general case represented by Eqs. (40) or (41), since \(\beta_1\) changes almost randomly during a motion. In order to have a comparable results in disposal, we have to find secular changes of orbital elements when central central acceleration is given by \(-\mu e_R / r^2\). Thus, we will use \(-\mu e_R / r^2\) as a central acceleration determining osculating orbital elements.

Taking into account Eq. (49), we take the action of electromagnetic radiation as a perturbation to the two-body problem. We can immediately write for components of perturbation acceleration:

\[F_R = \beta \frac{\mu}{r^2} - 2 \beta \frac{\mu}{r^2} \frac{v_{Rd}}{c}, \quad F_T = -\beta \frac{\mu}{r^2} \frac{v_{Td}}{c}, \quad F_N = 0,\]

where \(F_R, F_T\) and \(F_N\) are radial, transversal and normal components of perturbation acceleration, and two-body problem yields

\[v_{Rd} = \sqrt{\frac{\mu}{p}} e \sin f,\]

\[v_{Td} = \sqrt{\frac{\mu}{p}} \left(1 + e \cos f\right).\]

Perturbation equations of celestial mechanics yield for osculating orbital elements \((a –\text{semi-major axis}; e –\text{eccentricity}; i –\text{inclination (of the orbital plane to the reference frame)}; \Omega –\text{longitude of the ascending node}; \omega –\text{longitude of pericenter}; \Theta = \Theta = \omega + f)\)
is the position angle of the particle on the orbit, when measured from the ascending node in the direction of the particle's motion:

\[
\frac{da}{dt} = \frac{2}{1 - e^2} \sqrt{\frac{p}{\mu}} \left\{ F_R e \sin f + F_T (1 + e \cos f) \right\},
\]

\[
\frac{de}{dt} = \sqrt{\frac{p}{\mu}} \left\{ F_R \sin f + F_T \left[ \cos f + \frac{e + \cos f}{1 + e \cos f} \right] \right\},
\]

\[
\frac{di}{dt} = \frac{r}{\sqrt{\mu p}} F_N \cos \Theta,
\]

\[
\frac{d\Omega}{dt} = \frac{r}{\sqrt{\mu p}} F_N \sin \Theta \sin i,
\]

\[
\frac{d\omega}{dt} = -\frac{1}{e} \sqrt{\frac{p}{\mu}} \left\{ F_R \cos f - F_T \frac{2 + e \cos f}{1 + e \cos f} \sin f \right\} - \frac{r}{\sqrt{\mu p}} F_N \sin \Theta \cos i,
\]

\[
\frac{d\Theta}{dt} = \frac{\sqrt{\mu p}}{r^2} - \frac{r}{\sqrt{\mu p}} F_N \frac{\sin \Theta}{\sin i} \cos i,
\]

(70)

where \( r = p/(1 + e \cos f) \).

Inserting Eqs. (68) – (69) into Eq. (70), one easily obtains

\[
\frac{da}{dt} = \frac{2 \beta}{r^2} \sqrt{\frac{\mu}{p}} a^3 - \beta \frac{2}{r^2} \frac{a}{e} \left( 1 + e^2 + 2e \cos f + e^2 \sin^2 f \right),
\]

\[
\frac{de}{dt} = \beta \sqrt{\frac{\mu}{p}} \sin f - \beta \frac{1}{r^2} \left( 2 + e \sin^2 f + 2 \cos f \right),
\]

\[
\frac{di}{dt} = 0,
\]

\[
\frac{d\Omega}{dt} = 0,
\]

\[
\frac{d\omega}{dt} = -\beta \frac{\sqrt{\mu p}}{r^2} \frac{1}{e} \cos f - \beta \frac{1}{r^2} \frac{1}{c e} (2 - e \cos f) \sin f,
\]

\[
\frac{d\Theta}{dt} = \frac{\mu p}{r^2}.
\]

(71)

It is worth mentioning that \( da/dt < 0 \) for any time \( t \) does not hold (perturbation corresponds to complete nongravitational acceleration, and, thus, Eqs. (55) do not hold).

The set of differential equations Eqs. (71) has to be complemented with initial conditions. If the subscript “\( P \)” denotes orbital elements of the parent body, then the initial orbital elements for the particle ejected with velocity \( \Delta \) are:

\[
p_{in} = p_P \left( \frac{v_T + \Delta v_T}{v_T} \right)^2 + \left( \frac{\Delta v_N}{v_T} \right)^2,
\]

\[
e_{in}^2 = \left[ (1 + e_P \cos f_P) \frac{v_{TS}}{v_T} - 1 \right]^2 + \left[ (1 + e_P \cos f_P) \frac{v_{TS}}{v_T} \right]^2 \left( \frac{v_R + \Delta v_R}{v_T} \right)^2,
\]

\[
\cos i_{in} = \frac{v_T + \Delta v_T}{v_{TS}} \cos i_P - \frac{\Delta v_N}{v_{TS}} \cos \Theta_P \sin i_P,
\]

\[
\sin i_{in} \cos \Omega_{in} = \frac{v_T + \Delta v_T}{v_{TS}} \sin i_P \cos \Omega_P +
\]
\[
\sin i_{in} \sin \Omega_{in} = \frac{\Delta v_N}{v_{TS}} (\cos \Theta_P \cos i_P \cos \Omega_P - \sin \Theta_P \sin \Omega_P),
\]
\[
\sin i_{in} \sin \Omega_{in} = \frac{v_T + \Delta v_T}{v_{TS}} \sin i_P \sin \Omega_P + \frac{\Delta v_N}{v_{TS}} (\cos \Theta_P \cos i_P \sin \Omega_P + \sin \Theta_P \cos \Omega_P),
\]
\[
e_{in} \cos f_{in} = \frac{p_{in}}{p_P} (1 + e_P \cos f_P) - 1,
\]
\[
e_{in} \sin f_{in} = (1 + e_P \cos f_P) \frac{v_{TS} (v_R + \Delta v_R)}{v_r^2},
\]
\[
\sin i_{in} \cos \Theta_{in} = \frac{v_T + \Delta v_T}{v_{TS}} \sin i_P \cos \Theta_P + \frac{\Delta v_N}{v_{TS}} \cos i_P,
\]
\[
\sin i_{in} \sin \Theta_{in} = \sin i_P \sin \Theta_P,
\]
\[
v_T^2 = (v_T + \Delta v_T)^2 + (\Delta v_N)^2, \tag{72}
\]

where \(v_R\) and \(v_T\) are given by expressions presented in Eq. (44) - \(p_P = a_P (1 - e_P^2)\).

For the special case \(\Delta = 0\) Eq. (72) reduces to a simple fact: initial osculating orbital elements of the particle are identical with those of the parent body:

\[
a_{in} = a_P, \quad e_{in} = e_P, \quad i_{in} = i_P,
\]
\[
\Omega_{in} = \Omega_P, \quad \omega_{in} = \omega_P, \quad \Theta_{in} = \Theta_P. \tag{73}
\]

The important fact is that Eqs. (71) contain also terms not proportional to \(v/c (\ll 1)\). These important terms protect us to use procedure analogous to that represented by Eq. (50). While dispersion of osculating orbital elements is very small during a time interval \(T\) for the case when \(\mu (1 - \beta)\) is used in central acceleration, the dispersion of osculating orbital elements may be large during the same time interval for the case when \(\mu\) is used in central acceleration (compare Figs. 1 and 2 in Klačka 1994b). Thus, any formal averaging of Eqs. (71) leading to equations analogous to Eqs. (64)-(67) is not correct.

We have explained that it is not allowed to make a simple time averaging analogous to that described by Eq. (50), when \(- \mu e_R / r^2\) is used as a central acceleration determining osculating orbital elements. However, it is of interest if we have to numerically solve Newtonian vectorial equation of motion (Eq. (49)) and make numerical time averaging (over a time interval between passages through two following pericenters), or if some analytical simplifications can be done – something analogous to Eqs. (64)-(67).

Fortunately, we can make analytical calculations for the purpose of obtaining secular changes of semi-major axis and eccentricity even when \(- \mu e_R / r^2\) is used as a central acceleration determining osculating orbital elements. We will derive the correct equations in the following two subsections.
6.2.1. Radial forces and orbital elements

We will proceed according to Klačka (1994), in this subsection.

Let us consider a gravitational system of two bodies

\[ \dot{v} = -\frac{\mu}{r^2} e_R . \]  

(74)

Let perturbation acceleration exists in the form

\[ a = \beta \frac{\mu}{r^2} e_R , \]  

(75)

\[ 0 \leq \beta < 1. \]  

Thus, the final equation of motion is

\[ \dot{v} = -\frac{\mu (1 - \beta)}{r^2} e_R . \]  

(76)

Eq. (76) yields as a solution the well-known Keplerian motion and the orbit is given by

\[ r = \frac{p_c}{1 + e_c \cos (\Theta - \omega_c) ,} \]  

(77)

where

\[ p_c = a_c (1 - e_c^2) . \]  

(78)

The subscript “c” denotes that orbital elements are constants of motion. If we write

\[ v = v_{cR} e_R + v_{cT} e_T , \]  

(79)

where \( e_T \) is a unit vector transverse to the radial vector \( e_R \) in the plane of the trajectory (positive in the direction of motion), we have

\[ v_{cR} = \sqrt{\mu (1 - \beta)} \frac{p_c^{-1} e_c \sin (\Theta - \omega_c) ,} {1 + e_c \cos (\Theta - \omega_c) ,} \]  

(80)

\[ v_{cT} = \sqrt{\mu (1 - \beta)} \frac{p_c^{-1} [1 + e_c \cos (\Theta - \omega_c) ,]} {1 + e_c \cos (\Theta - \omega_c) ,} \]  

(81)

In principle, we may consider also a new set of orbital elements, which are defined by the central gravitational acceleration. Eqs. (77) – (81) are then of the form

\[ r = \frac{p}{1 + e \cos (\Theta - \omega) ,} \]  

(82)

\[ p = a (1 - e^2) , \]  

(83)

\[ v = v_{dR} e_R + v_{dT} e_T , \]  

(84)

\[ v_{dR} = \sqrt{\mu p^{-1}} e \sin (\Theta - \omega) , \]  

(85)

\[ v_{dT} = \sqrt{\mu p^{-1}} [1 + e \cos (\Theta - \omega) ,] \]  

(86)
the fact that $\Theta$ is unchanged in both sets of orbital elements is used; $e_R = (\cos \Theta, \sin \Theta)$, $e_T = (-\sin \Theta, \cos \Theta)$.

Position vector and velocity vector define a state of the body at any time. Equations (77) and (82) yield then
\[
\frac{p_c}{1 + e_c \cos (\Theta - \omega_c)} = \frac{p}{1 + e \cos (\Theta - \omega)} \tag{87}
\]
Analogously, the other two pairs of equations (Eqs. (80) and (85), and, Eqs. (81) and (86)) give
\[
\sqrt{(1 - \beta)} \ p_c^{-1} e_c \sin (\Theta - \omega_c) = \sqrt{p^{-1}} e \sin (\Theta - \omega) \tag{88}
\]
\[
\sqrt{(1 - \beta)} \ p_c^{-1} \ [1 + e_c \cos (\Theta - \omega_c)] = \sqrt{p^{-1}} \ [1 + e \cos (\Theta - \omega)] \tag{89}
\]
One can easily obtain, using Eqs. (87) and (89),
\[
p_c (1 - \beta) = p \tag{90}
\]
and Eqs. (88)-(89) yield then
\[
(1 - \beta) e_c \sin (\Theta - \omega_c) = e \sin (\Theta - \omega) \tag{91}
\]
\[
(1 - \beta) \ [1 + e_c \cos (\Theta - \omega_c)] = 1 + e \cos (\Theta - \omega) \tag{92}
\]
Eq. (92) may be written in the form
\[
(1 - \beta) e_c \cos (\Theta - \omega_c) - \beta = e \cos (\Theta - \omega) \tag{93}
\]
Eqs. (91) and (93) yield
\[
e^2 = (1 - \beta)^2 \ e_c^2 + \beta^2 - 2 \beta (1 - \beta) e_c \cos (\Theta - \omega_c) \tag{94}
\]
Equation for $\omega$ is given by Eqs. (91) and (93), using also Eq. (94). Finally, Eqs. (78), (83), (79) and (94) yield
\[
a = a_c \left\{1 + \beta \frac{1 + e_c^2 + 2 e_c \cos (\Theta - \omega_c)}{1 - e_c^2}\right\}^{-1} \tag{95}
\]
Eqs. (94)-(95) show that orbital osculating elements $a$ and $e$ change in time, during an orbital revolution – the larger $\beta$, the larger change of $a$ and $e$.

The osculating orbital elements $e$ and $a$ obtain values between their maxima and minima, which can be easily found from Eqs. (94)-(95). One can easily verify that these relations hold:
\[
\begin{align*}
    e_{\min} &= \left|(1 - \beta) e_c - \beta\right| \leq e \leq (1 - \beta) e_c + \beta = e_{\max} , \\
    \frac{a_{\min}}{a_c} &= \frac{1 - e_c}{1 - e_c + \beta (1 + e_c)} \leq \frac{a}{a_c} \leq \frac{1 + e_c}{1 + e_c + \beta (1 - e_c)} = \frac{a_{\max}}{a_c} .
\end{align*}
\]
6.2.2. Mean values of semi-major axis and eccentricity

Eqs. (96) and (97) represent interval of values for eccentricity and semi-major axis, when \(-\mu e_R / r^2\) is used as a central acceleration determining osculating orbital elements. However, we can make time averaging during a period \(T\), which was described by Eq. (50):

\[
\langle e \rangle = \frac{1}{T} \int_0^T e(t) \, dt , \quad \langle a \rangle = \frac{1}{T} \int_0^T a(t) \, dt ,
\]

\[
r_r^2 \frac{d\langle e \rangle}{dt} = \sqrt{\mu} (1 - \beta) \, p_c , \quad \frac{a_r^2}{T^2} = \frac{\mu (1 - \beta)}{4 \pi^2} , \quad r = \frac{p_c}{1 + e_c \cos f_c} . \tag{98}
\]

Eqs. (98) yield

\[
\langle e \rangle = (1 - e_c^2)^{3/2} \frac{1}{2 \pi} \int_0^{2\pi} \frac{e(f_c)}{(1 + e_c \cos f_c)^2} \, df_c ,
\]

\[
\langle a \rangle = (1 - e_c^2)^{3/2} \frac{1}{2 \pi} \int_0^{2\pi} \frac{a(f_c)}{(1 + e_c \cos f_c)^2} \, df_c . \tag{99}
\]

Using Eqs. (94) and (95), we finally obtain

\[
\langle e \rangle = (1 - e_c^2)^{3/2} \frac{1}{2 \pi} \int_0^{2\pi} \frac{(1 - \beta)^2 c^2 + \beta^2 - 2 \beta (1 - \beta) e_c \cos f_c}{(1 + e_c \cos f_c)^2} \, df_c ,
\]

\[
\langle a \rangle = a_c (1 - e_c^2)^{3/2} \frac{1}{2 \pi} \int_0^{2\pi} \frac{[1 + \beta (1 + e_c^2 + 2 e_c \cos f_c) / (1 - e_c^2)]^{-1}}{(1 + e_c \cos f_c)^2} \, df_c . \tag{100}
\]

The following properties can be verified:

i) \(\langle e \rangle \geq \beta, \langle e \rangle = \beta\) for \(e_c = 0\); ii) \(\langle e \rangle / e_c \geq 1, \langle e \rangle = e_c\) for \(\beta = 0\);

iii) \(\partial \langle e \rangle / \partial e_c > 0\); iv) \(\partial \langle e \rangle / \partial \beta > 0\);

v) \(\langle a \rangle \geq a_c / (1 + \beta), \langle a \rangle = a_c / (1 + \beta)\) for \(e_c = 0\);

vi) \(\langle a \rangle / a_c \leq 1, \langle a \rangle = a_c\) for \(\beta = 0\);

vii) \(\partial \langle a \rangle / \partial e_c > 0\); viii) \(\partial \langle a \rangle / \partial a_c > 0\); ix) \(\partial \langle a \rangle / \partial \beta < 0\).

6.2.3. Secular changes of semi-major axis and eccentricity

Summarizing our results, it is possible to calculate secular evolution of eccentricity and semi-major axis, according to the following prescription.

At first, initial conditions for \(a_\beta\) and \(e_\beta\) are calculated:

\[
(a_\beta)_{in} = a_P (1 - \beta) \left(1 - 2 \beta \frac{1 + e_P \cos f_P}{1 - e_P^2}\right)^{-1} ,
\]

\[
(e_\beta)_{in} = \sqrt{1 - \frac{1 - e_P^2 - 2 \beta (1 + e_P \cos f_P)}{(1 - \beta)^2}} , \tag{101}
\]

supposing that particle was ejected with zero ejection velocity from a parent body – quantities with subscript “\(P\)” belongs to the parent body’s trajectory; as for more general case, Eqs. (56), (58) and (59) have to be used, \((a_\beta)_{in} = p_\beta \, in / (1 - e_\beta^2 \, in), (e_\beta)_{in} \equiv e_\beta \, in\).
As the second step, the set of the following differential equations must be solved for the above presented initial conditions:

\[
\frac{da_\beta}{dt} = -\beta \frac{\mu}{c} \frac{2 + 3e_\beta^2}{a_\beta \left(1 - e_\beta^2\right)^{3/2}},
\]

\[
\frac{de_\beta}{dt} = -\frac{5}{2} \beta \frac{\mu}{a_\beta^3} \frac{e_\beta}{\left(1 - e_\beta^2\right)^{1/2}}.
\]

Finally, semi-major axis and eccentricity are calculated from:

\[
a = a_\beta \left(1 - e_\beta^2\right)^{3/2} \frac{1}{2\pi} \int_0^{2\pi} \left[1 + \beta \left(1 + e_\beta^2 + 2e_\beta \cos x\right) / \left(1 - e_\beta^2\right)\right]^{-1} \frac{dx}{(1 + e_\beta \cos x)^2},
\]

\[
e = (1 - e_\beta^2)^{3/2} \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - \beta)^2 e_\beta^2 + \beta^2 - 2\beta (1 - \beta) e_\beta \cos x}{(1 + e_\beta \cos x)^2} dx.
\]

The set of equations represented by Eqs. (101)-(103) fully corresponds to detailed numerical calculations of vectorial equation of motion, if we are interested in secular evolution of eccentricity and semi-major axis (supposing \((e_\beta)_{in} < 1\) and \(e_\beta\) does not correspond to pseudo-circular orbit) for the case when central acceleration is defined by gravity alone.

It is worth mentioning that instantaneous time derivatives of semi-major axis and eccentricity may be both positive and negative, while secular evolution yields that semi-major axis and eccentricity are decreasing functions of time.

We may mention that a little more simple procedure is obtained when semi-major axis \(a_\beta\) is replaced by the quantity \(p_\beta = a_\beta \left(1 - e_\beta^2\right)\). As it was already mentioned behind Eq. (67),

i) \(dp_\beta/dt = -2\beta(\mu/c) \left[1 - e_\beta^2 \left(p_\beta / p_{\beta \text{ in}}\right)^{5/2}\right]^{3/2} / p_\beta\),

ii) \(de_\beta/dt = -(5/2)\beta(\mu/c) \left(e_\beta^{8/5} / p_{\beta \text{ in}}^2\right) \left(1 - e_\beta^2\right)^{3/2} / e_\beta^{3/5}\).

Initial condition for \(p_\beta\) is given by the first relation of Eq. (59), e. g., \(p_{\beta \text{ in}} = p_P / (1 - \beta)\) for zero ejection velocity. The first integral of Eq. (103) is replaced by a simple relation: \(p = p_\beta \left(1 - \beta\right)\) (Eq. (90)).

(\text{It is important to stress that quantities } p \equiv \langle p \rangle, \text{ and } a \equiv \langle a \rangle, e \equiv \langle e \rangle, \text{ present in Eq. (137), do not fulfill relation } p = a(1 - e^2).)

7. P-R effect – equation of motion to the second order in \(v/c\)

We can write, on the basis of Eqs. (48) (Balek and Klačka 2002)

\[
\frac{d}{dt} \frac{v}{c} = -\frac{\mu}{r^2} e_R + \beta \frac{\mu}{r^2} \left\{\left[1 - \frac{v \cdot e_R}{c} - \frac{1}{2} \left(\frac{v}{c}\right)^2\right] e_R - \left(1 - \frac{v \cdot e_R}{c}\right) \frac{v}{c}\right\}
\]

\[
- \frac{\mu}{r^2} \left\{\left(\frac{v}{c}\right)^2 - 4 \frac{\mu}{c^2 r} \right\} e_R - 4 \frac{v \cdot e_R}{c} \frac{v}{c} - 7 \beta \frac{\mu}{r^2} \frac{\mu}{c^2 r} e_R,
\]

(104)
where $\mu \equiv G M_\odot$, $e_R \equiv e_1$ and $\beta \equiv \beta_1$ is a non-dimensional parameter (“the ratio of radiation pressure force to the gravitational force”; see also Eq. (43)). The first term in Eq. (104) is Newtonian gravity for two-body problem, the second term is the Poynting-Robertson effect in flat spacetime, the third term corresponds to Einstein’s correction to Newtonian gravity and the last term is a sort of interference term describing the mixing between the effects of gravity and radiation pressure.

7.1. Secular changes of orbital elements – radiation pressure as a part of central acceleration

We have to use $-\mu (1 - \beta) e_R / r^2$ as a central acceleration determining osculating orbital elements if we want to use a fact that the elements do not change rapidly during a motion described by Eq. (104) – during a time interval $T$, where $T$ is time interval between passages through two following pericenters / perihelia:

$$\frac{d v}{d t} = -\frac{\mu (1 - \beta)}{r^2} e_R - \frac{\beta \mu}{r^2} \left\{ \left[ \frac{v \cdot e_R}{c} + \frac{1}{2} \left( \frac{v}{c} \right)^2 \right] e_R + \left[ 1 - \frac{v \cdot e_R}{c} \right] \frac{v}{c} \right\}$$

$$- \frac{\mu}{r^2} \left\{ \left[ \frac{v}{c} \right]^2 - 4 \frac{\mu}{c^2 r} \right\} e_R - 4 \frac{v \cdot e_R}{c} \frac{v}{c} - 7 \frac{\beta}{r^2} \frac{\mu}{c^2 r} e_R .$$

(105)

We can immediately write for components of perturbation acceleration to Keplerian motion, on the basis of Eq. (105) – $\beta$ is considered to be a constant during the motion:

$$F_{\beta \ R} = -2 \frac{\beta \mu}{r^2} \frac{v_{\beta \ R}}{c} + \beta \frac{\mu}{r^2} \left\{ \frac{1}{2} \left[ \left( \frac{v_{\beta \ R}}{c} \right)^2 - \left( \frac{v_{\beta \ T}}{c} \right)^2 \right] - 7 \frac{\mu}{c^2 r} \right\}$$

$$+ \frac{\mu}{r^2} \left\{ 3 \left( \frac{v_{\beta \ R}}{c} \right)^2 - \left( \frac{v_{\beta \ T}}{c} \right)^2 + 4 \frac{\mu}{c^2 r} \right\} ,$$

$$F_{\beta \ T} = - \beta \frac{\mu}{r^2} \frac{v_{\beta \ T}}{c} + \frac{\mu}{r^2} \left( 4 + \beta \right) \frac{v_{\beta \ R} v_{\beta \ T}}{c^2} ,$$

$$F_{\beta \ N} = 0 ,$$

(106)

where $F_{\beta \ R}, F_{\beta \ T}$ and $F_{\beta \ N}$ are radial, transversal and normal components of perturbation acceleration, and the two-body problem yields

$$v_{\beta \ R} = \sqrt{\frac{\mu (1 - \beta)}{p_\beta}} e_\beta \sin f_\beta ,$$

$$v_{\beta \ T} = \sqrt{\frac{\mu (1 - \beta)}{p_\beta}} \left( 1 + e_\beta \cos f_\beta \right) ,$$

(107)

where $e_\beta$ is osculating eccentricity of the orbit, $f_\beta$ is true anomaly, $p_\beta = a_\beta (1 - e_\beta^2)$, $a_\beta$ is semi-major axis. The important fact that perturbation acceleration is proportional to $v/c (\ll 1)$ ensures the above mentioned small changes of orbital elements during the time interval $T$. 
Perturbation equations of celestial mechanics yield for osculating orbital elements ($a_\beta$ – semi-major axis; $e_\beta$ – eccentricity; $i_\beta$ – inclination (of the orbital plane to the reference frame); $\Omega_\beta$ – longitude of the ascending node; $\omega_\beta$ – longitude of pericenter / perihelion; $\Theta_\beta$ is the position angle of the particle on the orbit, when measured from the ascending node in the direction of the particle’s motion, $\Theta_\beta = \omega_\beta + f_\beta$):

\[
\begin{align*}
\frac{da_\beta}{dt} &= \frac{2}{1-e_\beta} \sqrt{\frac{p_\beta}{\mu (1-\beta)}} \left\{ F_\beta \cos f_\beta + F_\beta T \left( 1 + e_\beta \cos f_\beta \right) \right\}, \\
\frac{de_\beta}{dt} &= \sqrt{\frac{p_\beta}{\mu (1-\beta)}} \left\{ F_\beta \sin f_\beta + F_\beta T \left[ \cos f_\beta + \frac{e_\beta + \cos f_\beta}{1 + e_\beta \cos f_\beta} \right] \right\}, \\
\frac{di_\beta}{dt} &= \frac{r}{\sqrt{\mu (1-\beta) p_\beta}} F_\beta N \sin \Theta_\beta, \\
\frac{d\Omega_\beta}{dt} &= \frac{r}{\sqrt{\mu (1-\beta) p_\beta}} F_\beta N \sin \Theta_\beta \sin i_\beta, \\
\frac{d\omega_\beta}{dt} &= \frac{1}{e_\beta} \sqrt{\frac{p_\beta}{\mu (1-\beta)}} \left\{ F_\beta \cos f_\beta - F_\beta T \frac{2 + e_\beta \cos f_\beta}{1 + e_\beta \cos f_\beta} \sin f_\beta \right\} - \frac{r}{\sqrt{\mu (1-\beta) p_\beta}} F_\beta N \sin \Theta_\beta \sin i_\beta \cos i_\beta, \\
\frac{d\Theta_\beta}{dt} &= \frac{\sqrt{\mu (1-\beta) p_\beta}}{r^2} - \frac{r}{\sqrt{\mu (1-\beta) p_\beta}} F_\beta N \sin \Theta_\beta \sin i_\beta \cos i_\beta, \quad (108)
\end{align*}
\]

where $r = p_\beta/(1 + e_\beta \cos f_\beta)$.

Inserting Eqs. (106) – (107) into Eq. (108), one easily obtains

\[
\begin{align*}
\frac{da_\beta}{dt} &= - \beta \frac{2 a_\beta}{r^2 c} \frac{1 + e_\beta^2 + 2 e_\beta \cos f_\beta + e_\beta^2 \sin^2 f_\beta}{1 - e_\beta^2} + \\
& \quad + \frac{\mu a_\beta}{r^2 c^2} \sqrt{\frac{\mu (1-\beta)}{p_\beta}} \beta \times X_{e_1} + 6 \times X_{e_2}, \\
X_{e_1} &= \left( 1 - \frac{14}{1-\beta} \right) e_\beta \sin f_\beta + \left( 1 - \frac{7}{1-\beta} \right) e_\beta^2 \sin (2f_\beta) + e_\beta^3 \sin f_\beta, \\
X_{e_2} &= \left( 1 + \frac{4}{3} \right) e_\beta \sin f_\beta + \left( 1 + \frac{2}{1-\beta} \right) e_\beta^2 \sin (2f_\beta) + e_\beta^3 \sin f_\beta, \\
\frac{de_\beta}{dt} &= - \beta \frac{1}{r^2 c} \left( 2 e_\beta + e_\beta \sin^2 f_\beta + 2 \cos f_\beta \right) + \\
& \quad + \frac{\mu}{r^2 c^2} \sqrt{\frac{\mu (1-\beta)}{p_\beta}} \left( \frac{1}{2} \beta \times X_{e_1} + X_{e_2} \right), \\
X_{e_1} &= - \left( 1 + \frac{14}{1-\beta} \right) \sin f_\beta + \left( 1 - \frac{7}{1-\beta} \right) e_\beta \sin (2f_\beta) + 3 e_\beta^2 \sin f_\beta, \\
X_{e_2} &= - \left( 1 - \frac{4}{1-\beta} \right) \sin f_\beta + \left( 3 + \frac{2}{1-\beta} \right) e_\beta \sin (2f_\beta) + 7 e_\beta^2 \sin f_\beta, \\
\frac{di_\beta}{dt} &= 0, \\
\frac{d\Omega_\beta}{dt} &= 0, \\
\frac{d\omega_\beta}{dt} &= - \beta \frac{1}{r^2 c e_\beta} \left( 2 - e_\beta \cos f_\beta \right) \sin f_\beta + \\
& \quad + \frac{\mu}{r^2 c} \sqrt{\frac{\mu (1-\beta)}{p_\beta}} \left( \frac{1}{2} \beta \times X_{e_1} + X_{e_2} \right), \\
\end{align*}
\]
\[ \frac{\mu}{r^2} \frac{1}{e_\beta^2} \sqrt{\frac{\mu(1-\beta)}{p_\beta}} \left( \frac{1}{2} \beta \times X_{\omega_1} + X_{\omega_2} \right), \]

\[ X_{\omega_1} = \left( 1 + \frac{14}{1-\beta} \right) \cos f_\beta + 4e_\beta - \left( 2 - \frac{14}{1-\beta} \right) e_\beta \cos^2 f_\beta + e_\beta^2 \cos f_\beta, \]

\[ X_{\omega_2} = \left( 1 - \frac{4}{1-\beta} \right) \cos f_\beta + 8e_\beta - \left( 6 + \frac{4}{1-\beta} \right) e_\beta \cos^2 f_\beta + e_\beta^2 \cos f_\beta, \]

\[ \frac{d\Theta_\beta}{dt} = \frac{\sqrt{\mu(1-\beta)p_\beta}}{r^2}. \]

(109)

We want to find secular changes of osculating orbital elements up to \( 1/c^2 \).

As for the terms proportional to \( 1/c^2 \) in Eq. (109), we may take a time average (\( T \) is time interval between passages through two following pericenters) in an analytical way

\[ \langle g \rangle \equiv \frac{1}{T} \int_0^T g(t)dt = \frac{\sqrt{\mu(1-\beta)}}{a_\beta^{3/2}} \frac{1}{2\pi} \int_0^{2\pi} g(f_\beta) \left( \frac{df_\beta}{dt} \right)^{-1} df_\beta \]

\[ = \frac{\sqrt{\mu(1-\beta)}}{a_\beta^{3/2}} \frac{1}{2\pi} \int_0^{2\pi} g(f_\beta) \frac{r^2}{\sqrt{\mu(1-\beta)p_\beta}} df_\beta \]

\[ = \frac{1}{a_\beta^2} \frac{1}{\sqrt{1-e_\beta^2}} \frac{1}{2\pi} \int_0^{2\pi} g(f_\beta) r^2 df_\beta, \]

assuming non-pseudo-circular orbits and the fact that orbital elements exhibit only small changes during the time interval \( T \); the second and the third Kepler’s laws were used:

\[ r^2 \frac{df_\beta}{dt} = \sqrt{\mu(1-\beta)p_\beta} \] – conservation of angular momentum, \( a_\beta^3/T^2 = \mu(1-\beta)/(4\pi^2) \).

The result is:

\[ \langle \frac{da_\beta}{dt} \rangle_{II} = 0, \]

\[ \langle \frac{de_\beta}{dt} \rangle_{II} = 0, \]

\[ \langle \frac{d\omega_\beta}{dt} \rangle_{II} = \frac{3\mu^{3/2}}{e^2a_\beta^{5/2}} \frac{1-\beta^2/2}{(1-\beta)^{1/2}}. \]

(111)

Taking into account terms proportional to \( 1/c \) in Eq. (109), it is inevitable to use perturbation theory of the second order. Simple averaging represented by Eq. (110) is not sufficient: the last equation of Eq. (109) yields

\[ \frac{df_\beta}{dt} = \frac{\sqrt{\mu(1-\beta)p_\beta}}{r^2} - \frac{d\omega_\beta}{dt} \]

\[ \equiv \frac{H_\beta}{r^2} - \frac{d\omega_\beta}{dt}. \]

(112)

Eq. (146) holds due to the fact, that \( F_\beta N = 0 \) – in reality, equation for \( d\Theta_\beta/dt \) presented in Eq. (108) has to be used. Instead of Eq. (110), we have more precise method of averaging, now. We can write, using Eq. (112):

\[ \langle g \rangle = \frac{1}{T} \int_0^T g(t)dt = \]

\[ = \]
We are interested in secular changes of orbital elements up to the order $1/c^2$. Within this accuracy we can write for the terms on the right-hand side of Eq. (113)

\[
\left( \frac{H_\beta - \frac{d\omega_\beta}{dt}}{r^2} \right)^{-1} \times \int_0^{2\pi} \left( \frac{H_\beta - \frac{d\omega_\beta}{dt}}{r^2} \right)^{-1} g(f_\beta) \, df_\beta.
\]  

(113)

We have to find osculating orbital elements present on the right-hand sides of Eq. (109) in terms of true anomaly $f_\beta$ to the required accuracy in $1/c^2$. Differentiation of the relation $p_\beta = a_\beta \left( 1 - e_\beta^2 \right)$ yields, using Eq. (109) for $da_\beta/df_\beta$ and $de_\beta/df_\beta$,

\[
\frac{dp_\beta}{dt} = -2 \beta \frac{\mu}{c} \frac{p_\beta}{r^2} + \frac{\mu}{c^2} \frac{a_\beta}{r^2} \sqrt{\frac{\mu(1 - \beta)}{p_\beta}} \times
\left\{ (\beta \times X_{a1} + 6 \times X_{a2}) - 2e_\beta \left( \frac{1}{2} \beta \times X_{e1} + X_{e2} \right) \right\},
\]  

(116)

and, using Eqs. (109), (112) and (114), we obtain

\[
\frac{dp_\beta}{df_\beta} = -2 \beta \frac{\mu}{c} \frac{p_\beta}{H_\beta} \left( 1 - \beta \frac{\mu}{c} \frac{1}{H_\beta} \frac{2 - e_\beta \cos f_\beta}{e_\beta} \sin f_\beta \right) + \frac{\mu}{c^2} \frac{a_\beta}{p_\beta} \times
\left\{ (\beta \times X_{a1} + 6 \times X_{a2}) - 2e_\beta \left( \frac{1}{2} \beta \times X_{e1} + X_{e2} \right) \right\},
\]  

(117)

\[
\frac{de_\beta}{df_\beta} = -\beta \frac{\mu}{c} \frac{2e_\beta + e_\beta \sin^2 f_\beta + 2 \cos f_\beta}{H_\beta} \left( 1 - \beta \frac{\mu}{c} \frac{1}{H_\beta} \frac{2 - e_\beta \cos f_\beta}{e_\beta} \sin f_\beta \right) + \frac{\mu}{c^2} \frac{1}{p_\beta} \left( \frac{1}{2} \beta \times X_{e1} + X_{e2} \right).
\]  

(118)

Now, if we restrict ourselves to the first order in $1/c$, Eqs. (117)-(118) reduce to

\[
\frac{dp_\beta}{df_\beta} = -2 \beta \frac{\mu}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}}{c} \sqrt{p_\beta},
\]  

(119)

\[
\frac{de_\beta}{df_\beta} = -\beta \frac{\mu}{\sqrt{1 - \beta}} \frac{2e_\beta + e_\beta \sin^2 f_\beta + 2 \cos f_\beta}{\sqrt{c^2} \sqrt{p_\beta}}.
\]  

(120)
Eqs. (119)-(120) can be easily solved:
\[
\sqrt{\beta} = \sqrt{p_{\beta 0}} - \frac{\beta}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}}{c} f_{\beta},
\]
(121)
\[
e_{\beta} = e_{\beta 0} - \frac{\beta}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}}{c} \frac{1}{\sqrt{p_{\beta 0}}} \left\{ 2 \sin f_{\beta} + \frac{e_{\beta 0}}{2} \left( 5f_{\beta} - \frac{1}{2} \sin 2f_{\beta} \right) \right\},
\]
(122)
where initial values \( p_{\beta 0}, e_{\beta 0} \) correspond to \( f_{\beta 0} = 0 \).

On the basis of Eq. (121), we can immediately write solution of Eq. (117):
\[
p_{\beta} = p_{\beta 0} - 2 \frac{\beta}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}}{c} \sqrt{p_{\beta 0}} f_{\beta} + \frac{\mu}{c^2} \left( \frac{\beta^2}{1 - \beta} \times X_{p1} + X_{p2} \right),
\]
\[
X_{p1} = f_{\beta}^2 + \frac{4}{e_{\beta 0}} (1 - \cos f_{\beta}) - \frac{1}{2} (1 - \cos 2f_{\beta}),
\]
\[
X_{p2} = 2 \left( 4 + \beta \right) e_{\beta 0} (1 - \cos f_{\beta}).
\]
(123)

On the basis of Eqs. (121) – (122), we can rewrite Eq. (118) to the form
\[
d e_{\beta} \over df_{\beta} = - \frac{\beta}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}}{c} \left( 5 - \cos 2f_{\beta} \right) e_{\beta 0}/2 + 2 \cos f_{\beta} \frac{1}{\sqrt{p_{\beta 0}}} - \frac{\beta^2}{1 - \beta} \frac{\mu}{c^2} \frac{1}{p_{\beta 0}} \times X_{e3} +
\]
\[
\frac{\mu}{c^2} \frac{1}{p_{\beta 0}} \left( \frac{1}{2} \beta X_{e1} + X_{e2} \right),
\]
\[
X_{e3} = - \frac{15}{4} e_{\beta 0} f_{\beta} + \frac{3}{4} e_{\beta 0} f_{\beta} \cos(2f_{\beta}) + 2f_{\beta} \sin f_{\beta} - \frac{21}{2} \sin f_{\beta} +
\]
\[
\left( \frac{7}{4} e_{\beta 0} - \frac{2}{e_{\beta 0}} \right) \sin(2f_{\beta}) + \frac{3}{2} \sin(3f_{\beta}) - \frac{5}{16} e_{\beta 0} \sin(4f_{\beta}),
\]
(124)
where \( e_{\beta 0} \) has to be inserted into expressions for \( X_{e1} \) and \( X_{e2} \) (see Eq. (109)) instead of \( e_{\beta} \). Solution of Eq. (124) is:
\[
e_{\beta} = e_{\beta 0} - \frac{\beta}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}}{c} \frac{1}{\sqrt{p_{\beta 0}}} \left\{ 2 \sin f_{\beta} + \frac{e_{\beta 0}}{2} \left( 5f_{\beta} - \frac{1}{2} \sin 2f_{\beta} \right) \right\} +
\]
\[
\frac{\mu}{c^2} \frac{1}{p_{\beta 0}} \left( \frac{1}{2} \beta X_{inte1} + X_{inte2} \right) - \frac{\beta^2}{1 - \beta} \frac{\mu}{c^2} \frac{1}{p_{\beta 0}} \times X_{inte3},
\]
\[
X_{inte1} = - \left( 1 + \frac{14}{1 - \beta} \right) (1 - \cos f_{\beta}) +
\]
\[
\frac{1}{2} \left( 1 - \frac{7}{1 - \beta} \right) e_{\beta 0} (1 - \cos 2f_{\beta}) + 3e_{\beta 0}^2 (1 - \cos f_{\beta}),
\]
\[
X_{inte2} = - \left( 1 - \frac{4}{1 - \beta} \right) (1 - \cos f_{\beta}) +
\]
\[
\frac{1}{2} \left( 3 + \frac{2}{1 - \beta} \right) e_{\beta 0} (1 - \cos 2f_{\beta}) + 7e_{\beta 0}^2 (1 - \cos f_{\beta}),
\]
\[
X_{inte3} = - \frac{15}{8} e_{\beta 0} f_{\beta}^2 - \frac{3}{16} e_{\beta 0} (1 - \cos 2f_{\beta}) + \frac{3}{8} e_{\beta 0} f_{\beta} \sin(2f_{\beta}) + 2 \sin f_{\beta} -
\]
\[
- 2f_{\beta} \cos f_{\beta} - \frac{21}{2} (1 - \cos f_{\beta}) + \left( \frac{7}{8} e_{\beta 0} - \frac{1}{e_{\beta 0}} \right) (1 - \cos 2f_{\beta})
\]
\[
+ \frac{1}{2} (1 - \cos 3f_{\beta}) - \frac{5}{64} e_{\beta 0} (1 - \cos 4f_{\beta}).
\]
(125)
Semi-major axis \( a_{\beta} \) can be obtained from the relation \( p_{\beta} = a_{\beta} (1 - e_{\beta}^2) \), using Eqs. (123) and (125).
Let us calculate secular change of $\omega_\beta$ to the order $1/c^2$, generated by osculating orbital change proportional to $1/c$—compare Eqs. (109) and (111). Putting Eq. (114) into Eq. (113), we can write to the required accuracy,

$$
\left( \frac{d\omega_\beta}{dt} \right)_I = \left\{ \int_0^{2\pi} \frac{H_\beta}{r^2} - \frac{d\omega_\beta}{dt} \right\}^{-1} \int_0^{2\pi} \frac{r^2}{H_\beta} \left( 1 + \frac{r^2}{H_\beta} \frac{d\omega_\beta}{dt} \right) \frac{d\omega_\beta}{dt} \, df_\beta ,
$$

(126)
since time derivative of $\omega_\beta$ is proportional to $1/c$, and, thus, the third term on the right-hand side of Eq. (148) would yield a higher order than $1/c^2$ in the last integral.

The last integral in Eq. (126) consists of two terms:

$$
I_\omega = \int_0^{2\pi} \frac{r^2}{H_\beta} \left( 1 + \frac{r^2}{H_\beta} \frac{d\omega_\beta}{dt} \right) \frac{d\omega_\beta}{dt} \, df_\beta \equiv I_{\omega 1} + I_{\omega 2} ,
$$

(127)

Inserting relation for $d\omega_\beta/dt$ from Eq. (109), we can immediately write for $I_{\omega 1}$, within the required accuracy:

$$
I_{\omega 1} = \left( - \beta \frac{\mu}{c} \right)^2 \int_0^{2\pi} \left\{ \frac{\left( 2 - e_\beta \cos f_\beta \right) \sin f_\beta}{H_\beta e_\beta} \right\}^2 df_\beta = \left( - \beta \frac{\mu}{c} \right)^2 \int_0^{2\pi} \left\{ \frac{\left( 2 - e_{\beta 0} \cos f_\beta \right) \sin f_\beta}{H_{\beta 0} e_{\beta 0}} \right\}^2 df_\beta = \frac{\beta^2}{1 - \beta} \frac{\mu}{c^2} \frac{2\pi}{p_{\beta 0}} \left( \frac{2}{e_{\beta 0}^2} + \frac{1}{8} \right) ,
$$

(128)

where results of Eqs. (121) and (122) were used, and, $H_\beta = \sqrt{\mu(1 - \beta)p_\beta}$.

$$
I_{\omega 2} = - \beta \frac{\mu}{c} \int_0^{2\pi} \frac{\left( 2 - e_\beta \cos f_\beta \right) \sin f_\beta}{H_\beta e_\beta} \, df_\beta \equiv I_{\omega 21} + I_{\omega 22} ,
$$

(129)

As for calculation of $I_{\omega 21}$, we need to consider that $H_\beta = \sqrt{\mu(1 - \beta)p_\beta}$ and Eq. (121).

We obtain:

$$
I_{\omega 21} = \beta \frac{\mu}{c} \int_0^{2\pi} \frac{\cos f_\beta \sin f_\beta}{\sqrt{\mu(1 - \beta)p_\beta}} \, df_\beta = \beta \frac{\mu}{c} \frac{1}{H_{\beta 0}} \int_0^{2\pi} \left( 1 + \frac{\beta}{\sqrt{1 - \beta} \frac{\sqrt{p_{\beta 0}}}{c\sqrt{p_{\beta 0}}}} f_\beta \right) \cos f_\beta \sin f_\beta \, df_\beta =
$$

$$
= - \frac{\beta^2}{1 - \beta} \frac{\mu}{c^2} \frac{\pi/2}{p_{\beta 0}} .
$$

(130)
Calculation of \( I_{\omega 22} \) is analogous to calculation of \( I_{\omega 21} \), but we have to use Eqs. (121) and (122) simultaneously. The result is:

\[
I_{\omega 22} = \frac{2}{p_{30}} \frac{\beta^2}{1 - \beta} \frac{\mu}{c^2} \frac{4\pi}{r^2} \left( \frac{7}{2\epsilon_{\beta 0}} - \frac{1}{e_{\beta 0}^2} \right).
\]  

(Eq. 131)

Eqs. (129)-(131) yield

\[
I_{\omega 2} = \frac{2}{p_{30}} \frac{\beta^2}{1 - \beta} \frac{\mu}{c^2} \frac{4\pi}{r^2} \left( -\frac{1}{8} + \frac{7}{2\epsilon_{\beta 0}} - \frac{1}{e_{\beta 0}^2} \right).
\]  

(Eq. 132)

Eqs. (127), (128) and (132) yield

\[
I_{\omega} = \frac{2}{p_{30}} \frac{\beta^2}{1 - \beta} \frac{\mu}{c^2} \frac{4\pi}{r^2} \left( -\frac{1}{16} + \frac{7}{2\epsilon_{\beta 0}} \right).
\]  

(Eq. 133)

We have calculated the last integral in Eq. (126) – it is represented by Eq. (133).

Since it is proportional to \( 1/c^2 \), the first integral in Eq. (126) has to be proportional to \( 1/c^0 \). Thus, the first integral in Eq. (126) reduces to

\[
I_{\omega D} \equiv \int_{0}^{2\pi} \left( \frac{H_{\beta}}{r^2} - \frac{d\omega_{\beta}}{dt} \right)^{-1} df_{\beta},
\]

\[
I_{\omega D} \rightarrow \int_{0}^{2\pi} \frac{r^2}{H_{30}} df_{\beta} = \frac{1}{H_{30}} \int_{0}^{2\pi} \left\{ \frac{p_{30}}{1 + \epsilon_{30} \cos f_{\beta}} \right\}^2 df_{\beta} =
\]

\[
= \frac{p_{30}^{3/2}}{\sqrt{\mu(1 - \beta)}} \frac{2\pi}{(1 - e_{30}^2)^{3/2}}.
\]  

(Eq. 134)

Finally, Eqs. (126), (133) and (134) lead to

\[
\left\langle \frac{d\omega_{\beta}}{dt} \right\rangle_I = \frac{\beta^2}{\sqrt{1 - \beta}} \frac{\mu^{3/2}}{c^2} \left( \frac{1 - e_{30}^2}{p_{30}^{5/2}} \right)^{3/2} \left( -\frac{1}{8} + \frac{7}{e_{\beta 0}} \right).
\]  

(Eq. 135)

Total shift of pericenter / perihelion is given as a sum of Eqs. (111) and (135). Changing the index \( \beta 0 \) into \( \beta \) and using \( p_{\beta} = a_{\beta} (1 - e_{\beta}^2) \), we finally receive:

\[
\left\langle \frac{d\omega_{\beta}}{dt} \right\rangle = \frac{3\mu^{3/2}}{c^2 a_{\beta}^{5/2}} \frac{1 + \beta^2 (-13/8 + 7/e_{\beta})/3}{(1 - \beta)^{1/2}}.
\]  

(Eq. 136)

It can be easily verified that \( < d\omega_{\beta} / dt > \) is

i) an increasing function of \( \beta \),

ii) the perihelion circulates in a positive direction,

iii) the rate of the advancement of perihelion is not bounded for \( \beta \rightarrow 1 \).

Let us calculate secular change of \( a_{\beta} \) to the order \( 1/c^2 \), generated by osculating orbital change proportional to \( 1/c \) – compare Eqs. (109) and (111). Putting Eq. (114) into Eq. (113), we can write to the required accuracy,

\[
\left\langle \frac{da_{\beta}}{dt} \right\rangle_I = \left\{ \int_{0}^{2\pi} \left( \frac{H_{\beta}}{r^2} - \frac{d\omega_{\beta}}{dt} \right)^{-1} df_{\beta} \right\}^{-1} \int_{0}^{2\pi} \frac{r^2}{H_{\beta}} \left( 1 + \frac{r^2}{H_{\beta}} \frac{d\omega_{\beta}}{dt} \right) \frac{da_{\beta}}{dt} df_{\beta},
\]  

(Eq. 137)
since time derivatives of $a_\beta$ and $\omega_\beta$ are proportional to $1/c$, and, thus, the third term on the right-hand side of Eq. (114) would yield a higher order than $1/c^2$ in the last integral.

The last integral in Eq. (137) consists of two terms:

$$I_\alpha = \int_0^{2\pi} \frac{r^2}{H_\beta} \left( 1 + \frac{r^2}{H_\beta} \frac{d\omega_\beta}{dt} \right) \frac{da_\beta}{dt} df_\beta \equiv I_{a_1} + I_{a_2} ,$$

$$I_{a_1} = \int_0^{2\pi} \frac{r^2}{H_\beta} \frac{d\omega_\beta}{dt} \frac{da_\beta}{dt} df_\beta ,$$

$$I_{a_2} = \int_0^{2\pi} \frac{r^2}{H_\beta} df_\beta .$$

Inserting relations for $da_\beta/dt$ and $d\omega_\beta/dt$ from Eq. (109), we can immediately write for $I_{a_1}$, within the required accuracy:

$$I_{a_1} = \left( \frac{\mu}{c} \right)^2 \int_0^{2\pi} \frac{2}{H_\beta e_\beta} \frac{2a_\beta (1 + e_\beta^2 + 2e_\beta \cos f_\beta + e_\beta^2 \sin^2 f_\beta)}{H_\beta (1 - e_\beta^2)} df_\beta$$

$$= \left( \frac{\mu}{c} \right)^2 \int_0^{2\pi} \frac{2}{H_\beta e_\beta} \frac{2a_\beta (1 + e_\beta^2 + 2e_\beta \cos f_\beta + e_\beta^2 \sin^2 f_\beta)}{H_\beta (1 - e_\beta^2) / (2a_\beta)} df_\beta$$

$$= 0 ,$$

(139)

where results of Eqs. (121) and (122) were used.

$$I_{a_2} = - \frac{\mu}{c} \int_0^{2\pi} \frac{2a_\beta (1 + e_\beta^2 + 2e_\beta \cos f_\beta + e_\beta^2 \sin^2 f_\beta)}{H_\beta (1 - e_\beta^2)} df_\beta .$$

(140)

Considering that $H_\beta = \sqrt{\mu(1 - \beta)p_\beta}$ and Eqs. (121) and (122), we have

$$\frac{1}{H_\beta} = \frac{1}{H_\beta} \left( 1 + \frac{\beta}{\sqrt{1 - \beta} c \sqrt{p_\beta}} f_\beta \right) ,$$

$$e_\beta^2 = e_\beta^2 - 2e_\beta \frac{\beta}{\sqrt{1 - \beta} c \sqrt{p_\beta}} \frac{1}{\sqrt{p_\beta}} \left\{ 2 \sin f_\beta + \frac{e_\beta}{2} \left( 5f_\beta - \frac{1}{2} \sin 2f_\beta \right) \right\} ,$$

$$\frac{1}{1 - e_\beta^2} = \frac{1}{1 - e_\beta^2} \left\{ 1 - \frac{2e_\beta}{1 - e_\beta^2} \frac{\beta}{\sqrt{1 - \beta} c \sqrt{p_\beta}} \frac{1}{\sqrt{p_\beta}} \left[ 2 \sin f_\beta + \frac{e_\beta}{2} \left( 5f_\beta - \frac{1}{2} \sin 2f_\beta \right) \right] \right\} ,$$

$$a_\beta = p_\beta \frac{1}{1 - e_\beta^2} = \left( p_\beta - 2 \frac{\beta}{\sqrt{1 - \beta} c \sqrt{p_\beta}} f_\beta \right) \frac{1}{1 - e_\beta^2}$$

$$= a_\beta \left\{ 1 - \frac{2e_\beta}{1 - e_\beta^2} \frac{\beta}{\sqrt{1 - \beta} c \sqrt{p_\beta}} f_\beta - \frac{2e_\beta}{1 - e_\beta^2} \frac{\beta}{\sqrt{1 - \beta} c \sqrt{p_\beta}} \left[ 2 \sin f_\beta + \frac{e_\beta}{2} \left( 5f_\beta - \frac{1}{2} \sin 2f_\beta \right) \right] \right\} .$$

(141)

Inserting results of Eq. (141) into Eq. (140), one finally obtains

$$I_{a_2} = - \frac{\mu}{c} \frac{4\pi a_\beta}{H_\beta (1 - e_\beta^2)} \left\{ 1 + \frac{3}{2} e_\beta^2 - \frac{\beta}{\sqrt{1 - \beta} c \sqrt{p_\beta}} \right\} .$$
Finally, Eqs. (137) – (140), (142) – (144) yield
\[
\frac{\pi + (10\pi + 3) e_{\beta 0}^2 + (9\pi + 2) e_{\beta 0}^4}{1 - e_{\beta 0}^2}.
\] (142)

We have calculated the last integral in Eq. (137) – it is represented by Eq. (142). Since its dominant part is proportional to \(1/c\), the first integral in Eq. (137) has to be proportional to \(1/c^1\). Thus, the first integral in Eq. (137) is given, according to Eq. (115), as
\[
LHS = \left\{ \int_0^{2\pi} \left( \frac{H_{\beta}}{r^2} - \frac{d\omega_{\beta}}{dt} \right)^{-1} \, df_{\beta} \right\}^{-1},
\]
\[
LHS = \left( \int_0^{2\pi} \frac{r^2}{H_{\beta}} \, df_{\beta} \right)^{-1} \left\{ 1 - \left( \int_0^{2\pi} \frac{r^2}{H_{\beta}} \, df_{\beta} \right)^{-1} \int_0^{2\pi} \frac{r^2}{H_{\beta}} \left( \frac{r^2 \, d\omega_{\beta}}{dt} \right) \, df_{\beta} \right\}. \quad (143)
\]

It can be easily verified that the last integral in Eq. (143) equals zero, within the required accuracy, and
\[
LHS = \left( \int_0^{2\pi} \frac{r^2}{H_{\beta}} \, df_{\beta} \right)^{-1} = \frac{\sqrt{\mu}(1 - \beta)}{2 \pi a_{\beta 0}^{3/2}} \left\{ 1 + \frac{\beta}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}1 - e_{\beta 0}^2}{\sqrt{a_{\beta 0}}LHS} \right\},
\]
\[
I_{LHS} = \frac{5}{2\pi} \int_0^{2\pi} \frac{x}{(1 + e_{\beta 0}\cos x)^2} \, dx - \frac{2}{2\pi} \int_0^{2\pi} \frac{x}{(1 + e_{\beta 0}\cos x)^2} \, dx. \quad (144)
\]

Finally, Eqs. (137) – (140), (142) – (144) yield
\[
\langle \frac{da_{\beta}}{dt} \rangle_I = -\beta \frac{\mu}{c} \frac{2 + 3e_{\beta 0}^2}{a_{\beta 0}^2 (1 - e_{\beta 0}^2)^{3/2}} \left\{ 1 + \frac{\beta}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}1 - e_{\beta 0}^2}{\sqrt{a_{\beta 0}}LHS} \times \left[ I_{LHS} - \frac{\pi + (10\pi + 3) e_{\beta 0}^2 + (9\pi + 2) e_{\beta 0}^4}{(1 + 3e_{\beta 0}^2/2 (1 - e_{\beta 0}^2)^{5/2}} \right] \right\}. \quad (145)
\]

Total secular change of semi-major axis is given as a sum of Eqs. (111) and (145).

Changing the index \(\beta 0\) into \(\beta\) and using Eq. (144), we can finally write
\[
\langle \frac{da_{\beta}}{dt} \rangle = -\beta \frac{\mu}{c} \frac{2 + 3e_{\beta}^2}{a_{\beta}^2 (1 - e_{\beta}^2)^{3/2}} \left\{ 1 + \frac{\beta}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}1 - e_{\beta}^2}{\sqrt{a_{\beta}}LHS} \times \left[ I_{n} - \frac{\pi + (10\pi + 3) e_{\beta}^2 + (9\pi + 2) e_{\beta}^4}{(1 + 3e_{\beta}^2/2) (1 - e_{\beta}^2)^{5/2}} \right] \right\},
\]
\[
I_{n} = \frac{5}{2\pi} \int_0^{2\pi} \frac{x}{(1 + e_{\beta}\cos x)^2} \, dx - \frac{2}{2\pi} \int_0^{2\pi} \frac{x}{(1 + e_{\beta}\cos x)^2} \, dx. \quad (146)
\]

Let us calculate secular change of \(e_{\beta}\) to the order \(1/c^2\), generated by osculating orbital change proportional to \(1/c\) – compare Eqs. (109) and (111). Putting Eq. (114) into Eq. (113), we can write to the required accuracy,
\[
\langle \frac{de_{\beta}}{dt} \rangle_I = \left\{ \int_0^{2\pi} \left( \frac{H_{\beta}}{r^2} - \frac{d\omega_{\beta}}{dt} \right)^{-1} \, df_{\beta} \right\}^{-1} \int_0^{2\pi} \frac{r^2}{H_{\beta}} \left( 1 + \frac{r^2 \, d\omega_{\beta}}{H_{\beta} \, dt} \right) \, df_{\beta} \, df_{\beta}. \quad (147)
\]
since time derivatives of \( e_\beta \) and \( \omega_\beta \) are proportional to \( 1/c \), and, thus, the third term on
the right-hand side of Eq. (114) would yield a higher order than \( 1/c^2 \) in the last integral.

The last integral in Eq. (147) consists of two terms:

\[
I_e = \int_0^{2\pi} \frac{r^2}{H_\beta} \left( 1 + \frac{r^2}{H_\beta} \frac{d\omega_\beta}{dt} \right) \frac{de_\beta}{dt} df_\beta \equiv I_{e1} + I_{e2}
\]

\[
I_{e1} = \int_0^{2\pi} \frac{r^2}{H_\beta} \frac{d\omega_\beta}{dt} \frac{de_\beta}{dt} df_\beta ,
\]

\[
I_{e2} = \int_0^{2\pi} \frac{2r^2}{H_\beta} \frac{de_\beta}{dt} df_\beta .
\]

(148)

Inserting relations for \( de_\beta/dt \) and \( d\omega_\beta/dt \) from Eq. (109), we can immediately write for
\( I_{e1} \), within the required accuracy:

\[
I_{e1} = \left( \beta \frac{\mu}{c} \right)^2 \int_0^{2\pi} \frac{2(1 - e_\beta \cos f_\beta)}{H_\beta e_\beta} \frac{2e_\beta + e_\beta \sin^2 f_\beta + 2\cos f_\beta}{H_\beta} \frac{e_\beta}{H_\beta} df_\beta
\]

\[
= \left( \beta \frac{\mu}{c} \right)^2 \int_0^{2\pi} \frac{2(1 - e_{\beta0} \cos f_\beta)}{H_{\beta0} e_{\beta0}} \frac{2e_{\beta0} + e_{\beta0} \sin^2 f_\beta + 2\cos f_\beta}{H_{\beta0}} \frac{e_{\beta0}}{H_{\beta0}} df_\beta
\]

\[
= 0 ,
\]

(149)

where results of Eqs. (121) and (122) were used.

\[
I_{e2} = -\beta \frac{\mu}{c} \int_0^{2\pi} \frac{2e_\beta + e_\beta \sin^2 f_\beta + 2\cos f_\beta}{H_\beta} \frac{e_\beta}{H_\beta} df_\beta .
\]

(150)

Considering Eq. (122) and the first relation of Eq. (141) for \( 1/H_\beta \), one finally obtains
from Eq. (150):

\[
I_{e2} = -\beta \frac{\mu}{c} \frac{5\pi e_{\beta0}}{H_{\beta0}} \left\{ 1 - \frac{\beta}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}}{c \sqrt{\beta_{\beta0}}} \pi \left( 2 + \frac{7}{10} e_{\beta0} \right) \right\} .
\]

(151)

Putting the results represented by Eqs. (144), (148), (149) and (151) into Eq. (147),
we receive

\[
\left\langle \frac{de_\beta}{dt} \right\rangle = -\beta \frac{\mu}{c} \frac{5e_{\beta0}/2}{a_{\beta0}^2 \sqrt{1 - e_{\beta0}^2}} \left\{ 1 + \frac{\beta}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}}{c} \frac{1 - e_{\beta0}^2}{\sqrt{a_{\beta0}}} \times \right.
\]

\[
\left[ I_{LHS} - \pi \frac{2 + 7e_{\beta0}/10}{(1 - e_{\beta0}^2)^{3/2}} \right] \right\} .
\]

(152)

Total secular change of eccentricity is given as a sum of Eqs. (111) and (152). Changing
the index \( \beta0 \) into \( \beta \) in Eq. (152), we can finally write

\[
\left\langle \frac{de_\beta}{dt} \right\rangle = -\beta \frac{\mu}{c} \frac{5e_\beta/2}{a_\beta \sqrt{1 - e_\beta^2}} \left\{ 1 + \frac{\beta}{\sqrt{1 - \beta}} \frac{\sqrt{\mu}}{c} \frac{1 - e_\beta^2}{\sqrt{a_\beta}} \times \right.
\]

\[
\left[ I_n - \pi \frac{2 + 7e_\beta/10}{(1 - e_\beta^2)^{3/2}} \right] \right\} ,
\]

(153)

\[
I_n = \frac{5}{2\pi} \int_0^{2\pi} \frac{x}{(1 + e_\beta \cos x)^3} dx - \frac{2}{2\pi} \int_0^{2\pi} \frac{x}{(1 + e_\beta \cos x)^2} dx .
\]
Let us calculate secular change of $p_\beta$ (see Eq. (116)) to the order $1/c^2$, generated by osculating orbital change proportional to $1/c$ — one can easily verify that $(dp_\beta/dt)_{II} = 0$.

Taking into account Eqs. (109) and (116), we can write to the required accuracy,

$$
\left( \frac{dp_\beta}{dt} \right)_I = \left\{ \int_0^{2\pi} \frac{H_\beta}{r^2} \left( 1 + \frac{r^2}{H_\beta} \frac{d\omega_\beta}{dt} \right) \frac{dp_\beta}{dt} \, df_\beta \right\}^{1}_{-1} \int_0^{2\pi} \frac{r^2}{H_\beta} \left( 1 + \frac{r^2}{H_\beta} \frac{d\omega_\beta}{dt} \right) \frac{dp_\beta}{dt} \, df_\beta ,
$$

(154)

since time derivatives of $p_\beta$ and $\omega_\beta$ are proportional to $1/c$, and, thus, the third term on the right-hand side of Eq. (114) would yield a higher order than $1/c^2$ in the last integral.

The last integral in Eq. (154) consists of two terms:

$$
I_p = \int_0^{2\pi} \frac{r^2}{H_\beta} \left( 1 + \frac{r^2}{H_\beta} \frac{d\omega_\beta}{dt} \right) \frac{dp_\beta}{dt} \, df_\beta \equiv I_{p1} + I_{p2} ,
$$

(155)

Inserting relations for $dp_\beta/dt$ and $d\omega_\beta/dt$ from Eqs. (116) and (109), we can immediately write for $I_{p1}$, within the required accuracy:

$$
I_{p1} = 0 ,
$$

$$
I_{p2} = -2 \beta \frac{\mu}{c} \left\{ \frac{1}{\sqrt{\mu(1-\beta)}} \int_0^{2\pi} \sqrt{p_{\beta0}} \, df_\beta = -4 \pi \beta \frac{\mu}{c} \left\{ \sqrt{\frac{p_{\beta0}}{\mu(1-\beta)}} - \frac{1}{\pi} \right\} ,
$$

(156)

where Eq. (121) was used. Putting results of Eqs. (143), (144), (155) and (156) into Eq. (154), one obtains

$$
\left( \frac{dp_\beta}{dt} \right)_I = -2 \beta \frac{\mu}{c} \left\{ \frac{1}{\sqrt{\mu(1-\beta)}} \int_0^{2\pi} \sqrt{p_{\beta0}} \, df_\beta = -4 \pi \beta \frac{\mu}{c} \left\{ \sqrt{\frac{p_{\beta0}}{\mu(1-\beta)}} - \frac{1}{\pi} \right\} \right\} \left\{ 1 + \frac{\beta}{\sqrt{1-\beta}} \frac{1-e_\beta^2/30}{\sqrt{p_{\beta0}}} I_{LHS} - \frac{\pi}{\sqrt{p_{\beta0}}} \right\} .
$$

(157)

Total secular change of $p_\beta$ is given as a sum of Eqs. (157) and $(dp_\beta/dt)_{II} = 0$. Changing the index $\beta_0$ into $\beta$, we can finally write

$$
\left( \frac{dp_\beta}{dt} \right)_I = -2 \beta \frac{\mu}{c} \left\{ \frac{1}{\sqrt{\mu(1-\beta)}} \int_0^{2\pi} \sqrt{p_{\beta0}} \, df_\beta = -4 \pi \beta \frac{\mu}{c} \left\{ \sqrt{\frac{p_{\beta0}}{\mu(1-\beta)}} - \frac{1}{\pi} \right\} \right\} \left\{ 1 + \frac{\beta}{\sqrt{1-\beta}} \frac{1-e_\beta^2/30}{\sqrt{p_{\beta0}}} \left[ I_n - \frac{\pi}{\sqrt{p_{\beta0}}} \right] \right\} ,
$$

(158)

$$
I_n = \frac{5}{2\pi} \int_0^{2\pi} \frac{x}{(1+e_\beta \cos x)^3} \, dx - \frac{2}{\pi} \int_0^{2\pi} \frac{x}{(1+e_\beta \cos x)^2} \, dx .
$$

Comparing Eqs. (153) and (158), one easily obtains

$$
\frac{de_\beta}{dp_\beta} = \frac{5}{4} \frac{e_\beta}{p_{\beta0}} \left\{ 1 - \frac{\beta}{\sqrt{1-\beta}} \frac{\mu}{c} \frac{\pi}{\sqrt{p_{\beta0}}} (1+7e_\beta/10) \right\} .
$$

(159)

This can be easily integrated and the results may be written as:

$$
e_\beta = e_{\beta in} \left( \frac{p_{\beta0}}{p_{\beta in}} \right)^{5/4} \left\{ 1 - \frac{\beta}{\sqrt{1-\beta}} \frac{\mu}{c} \frac{5\pi}{3} \frac{1}{\sqrt{p_{\beta0}}} \left[ 1 + \frac{21}{80} e_{\beta in} \left( \frac{p_{\beta0}}{p_{\beta in}} \right)^{5/4} \right] \right\} ,
$$

(160)

$$
p_{\beta0} = p_{\beta in} \left( \frac{e_{\beta}}{e_{\beta in}} \right)^{4/5} \left\{ 1 + \frac{\beta}{\sqrt{1-\beta}} \frac{\mu}{c} \frac{4\pi}{3} \frac{1}{\sqrt{p_{\beta in}}} \left( \frac{e_{\beta in}}{e_{\beta}} \right)^{2/5} \left[ 1 + \frac{21}{80} e_{\beta} \right] \right\} .
$$
8. Secular change of advancement of pericenter/perihelion – gravitation as a central acceleration

We will use $-\mu e R / r^2$ as a central acceleration determining osculating orbital elements. As we have seen in the previous subsection, corrections of the order $(v/c)^2$ represent very small corrections with respect to the order $v/c$ – only $d\omega/dt$ is important, as for secular changes. Thus, the most important results, for the case when $-\mu e R / r^2$ is used as a central acceleration, were presented in section 6.2. However, we have not calculated secular change of $d\omega/dt$ in the section 6.2. We will do this now, since results of the preceding section 7.1 have to be used.

We will use Eqs. (91), (93) and (94) in the form

$$
\sin (\Theta - \omega) = \{(1 - \beta) e_c \sin (\Theta - \omega_c)\} / e,
$$

$$
\cos (\Theta - \omega) = \{(1 - \beta) e_c \cos (\Theta - \omega_c) - \beta\} / e,
$$

$$
e = \sqrt{(1 - \beta)^2 e_c^2 + \beta^2 - 2 \beta (1 - \beta) e_c \cos (\Theta - \omega_c)}.
$$

(161)

Using the fact that $\Theta - \omega = \Theta - \omega_c + \omega_c - \omega \equiv f_c + \omega_c - \omega$, we can write

$$
\frac{d\cos (\Theta - \omega)}{dt} = - \left\{ \frac{df_c}{dt} + \frac{d\omega_c}{dt} - \frac{d\omega}{dt} \right\} \sin (\Theta - \omega).
$$

(162)

We admit that the subscript "c" may be changed into $\beta$. Using Eq. (162), one immediately obtains

$$
\frac{d\omega}{dt} = \frac{d\cos (\Theta - \omega)}{dt} \times \{\sin (\Theta - \omega)\}^{-1} + \frac{d\omega_c}{dt} + \frac{df_c}{dt}.
$$

(163)

Inserting the right-hand side of the second of equations of Eq. (161) into Eq. (163), one finally, after differentiation, obtains

$$
\frac{d\omega}{dt} = \frac{d\omega_c}{dt} + \frac{\cos f_c}{\sin f_c} \left( e_c^{-1} \frac{de_c}{dt} - e^{-1} \frac{de}{dt} \right) + \frac{\beta}{1 - \beta} \frac{1}{e_c \sin f_c} e^{-1} e \frac{de}{dt}.
$$

(164)

Differentiation of the third equation of Eq. (161) yields

$$
\frac{e^{-1} de}{dt} = \frac{1}{e^2} \left\{ (1 - \beta)^2 e_c \frac{de_c}{dt} + \beta (1 - \beta) \left[ -\frac{de_c}{dt} \cos f_c + \frac{df_c}{dt} e_c \sin f_c \right] \right\}.
$$

(165)

Inserting Eq. (165) into Eq. (164), we can write

$$
\frac{d\omega}{dt} = \frac{d\omega_c}{dt} + \frac{1}{2} \left[ 1 + \frac{\beta^2 - (1 - \beta)^2 e^2}{e^2} \right] \frac{df_c}{dt} + \beta (1 - \beta) \frac{1}{e^2} \frac{de_c}{dt} \sin f_c,
$$

$$
\frac{e}{dt} = (1 - \beta)^2 e_c^2 + \beta^2 - 2 \beta (1 - \beta) e_c \cos f_c.
$$

(166)

Eq. (166) is the decisive equation which enables us to find secular change of $\omega$. 


As a first approximation, let us consider Keplerian orbit characterized by conditions
\[ \frac{da_c}{dt} = \frac{dp_c}{dt} = \frac{dc_c}{dt} = \frac{d\omega_c}{dt} = 0, \quad p_c = a_c(1 - e_c^2). \]
Using averaging of the type of Eq. (98), or, Eq. (110), we can write for Eq. (166) (we use \( r^2 df_c/dt = \sqrt{\mu(1 - \beta)p_c} \))
\[
\frac{d\omega}{dt} = \frac{1}{2} \int_0^{2\pi} \frac{d\omega}{dt} = \frac{1}{a_c^2/\sqrt{1 - e_c^2}} \frac{1}{2\pi} \int_0^{2\pi} \frac{d\omega}{dt} (f_c) r^2 df_c = \frac{1}{a_c^2/\sqrt{1 - e_c^2}} \frac{1}{2\pi} \times
\[
\int_0^{2\pi} \left[ 1 + \frac{\beta^2 - (1 - \beta)^2 e_c^2}{(1 - \beta)^2 e_c^2 + \beta^2 - 2\beta (1 - \beta) e_c \cos f_c} \right] \sqrt{\mu(1 - \beta)p_c} df_c. \tag{167}
\]
If the result \( \int_0^{2\pi} dx/(1 + k \cos x) = 2\pi/\sqrt{1 - k^2} \) is used, one finally obtains
\[
\frac{d\omega}{dt} = \begin{cases} 
0 & \text{if } \beta < e_c/(1 + e_c) \\
\sqrt{\mu(1 - \beta)/a_c^{3/2}}/2 & \text{if } \beta = e_c/(1 + e_c) \\
\sqrt{\mu(1 - \beta)/a_c^{3/2}} & \text{if } \beta > e_c/(1 + e_c)
\end{cases} \tag{168}
\]
Let us remind that \( d\Theta_c/dt = \sqrt{\mu(1 - \beta)/a_c^{3/2}} \) (see Eq. (100)).

Let us consider P-R effect to the first order in \( v/c \). We will use Eq. (55) and also Eq. (122), which we summarize in the following Eq. (169):
\[
\begin{align*}
\frac{de_{\beta}}{dt} & = -\beta \frac{\mu}{r^2 c} \left( 2e_{\beta} + e_{\beta} \sin^2 f_{\beta} + 2 \cos f_{\beta} \right), \\
\frac{d\omega_{\beta}}{dt} & = -\beta \frac{\mu}{r^2 c} \left( 2 - e_{\beta} \cos f_{\beta} \right) \sin f_{\beta}, \\
\frac{df_{\beta}}{dt} & = H_{\beta} - \frac{d\omega_{\beta}}{dt}, \tag{169}
\end{align*}
\]
where \( H_{\beta} \equiv \sqrt{\mu(1 - \beta)p_{\beta}} \). Changing subscript “c” into “\( \beta \)” in Eq. (166) and inserting the third of Eq. (169) into Eq. (166), we can write
\[
\frac{d\omega}{dt} = \frac{1}{2} \left[ 1 + \frac{\beta^2 - (1 - \beta)^2 e_{\beta}^2}{e_{\beta}^2} \right] \frac{H_{\beta}}{r^2} + \frac{1}{2} \left[ 1 - \frac{\beta^2 - (1 - \beta)^2 e_{\beta}^2}{e_{\beta}^2} \right] \frac{d\omega_{\beta}}{dt} + \beta (1 - \beta) \frac{1}{e_{\beta}^2} \frac{de_{\beta}}{dt} \sin f_{\beta},
\]
\[
e^2 = (1 - \beta)^2 e_{\beta}^2 + \beta^2 - 2\beta (1 - \beta) e_{\beta} \cos f_{\beta}. \tag{170}
\]
On the basis of Eqs. (113)-(115), (143)-(144), (169)-(170), we can write within the required accuracy:
\[
\frac{d\omega}{dt} = A_{\omega} \int_0^{2\pi} \frac{r^2}{H_{\beta}} \left( 1 + \frac{r^2}{H_{\beta}} \frac{d\omega_{\beta}}{dt} \right) \frac{d\omega}{dt} df_{\beta} =
\]
\[
= \frac{A_{\omega}}{2} \int_0^{2\pi} \left[ 1 + \frac{\beta^2 - (1 - \beta)^2 e_{\beta}^2}{(1 - \beta)^2 e_{\beta}^2 + \beta^2 - 2\beta (1 - \beta) e_{\beta} \cos f_{\beta}} \right] df_{\beta},
\]
\[
A_{\omega} = \sqrt{\frac{\mu(1 - \beta)}{2\pi a_{\beta 0}^{3/2}}} \left\{ 1 + \frac{\beta}{\sqrt{1 - \beta}} \frac{1 - e_{\beta 0}^2}{\sqrt{a_{\beta 0} I_{\beta 0}}} \right\},
\]
\[
I_{\beta 0} = \frac{5}{2\pi} \int_0^{2\pi} \left( \frac{x}{(1 + e_{\beta 0} \cos x)^3} \right) dx - \frac{2}{2\pi} \int_0^{2\pi} \left( \frac{x}{(1 + e_{\beta 0} \cos x)^2} \right) dx. \tag{171}
\]
Using also Eq. (122) in the integral of the second line of Eq. (171), we finally receive
\[
\langle \frac{d\omega}{dt} \rangle = \sqrt{\mu} \left( \frac{1 - \beta}{1 - \theta} \right)^{3/2} \left\{ \vartheta_H \left( \beta - \frac{e_\beta}{1 + e_\beta} \right) + \frac{\beta}{\sqrt{1 - \beta} \ c} \ \sqrt{\frac{\beta}{\vartheta}} \ M_{\omega_1} + M_{\omega_2} \right\} ,
\]

\[
M_{\omega_1} = (1 - \beta^2)^{3/2} \left[ 5 I_3 (e_\beta) - 2 I_2 (e_\beta) \right] \ vartheta_H \left( \beta - \frac{e_\beta}{1 + e_\beta} \right) ,
\]

\[
M_{\omega_2} = \frac{5}{4} \left\{ I_1 (\xi) - \frac{\beta^2 - (1 - \beta^2)^2 e_\beta^3}{\beta^2 + (1 - \beta^2)^2 e_\beta^2} I_2 (\xi) \right\} ,
\]

\[
\xi = - \frac{2 \beta (1 - \beta) e_\beta}{(1 - \beta)^2 e_\beta^2 + \beta^2} ,
\]

\[
I_\alpha (e) = \frac{1}{2\pi} \int_0^{2\pi} \frac{x}{(1 + e \cos x)^\alpha} \ dx , \quad \alpha = 1, 2, 3 ,
\]

where \( \vartheta_H (x) = 1 \) if \( x > 0 \), \( \vartheta_H (x) = 0 \) if \( x < 0 \) (Heaviside’s step function); it is assumed that \( \beta \neq e_\beta/(1 + e_\beta) \). We see that advancement of pericenter/periheleon exists already in the first order in \( v/c \) if gravity alone is taken as a central acceleration.

9. P-R effect and near circular orbits

When the orbits are near circular (pseudo-circular) – central acceleration contains radiation pressure term – the orbit can not reduce in semi-major axis without increasing in eccentricity. Both types of orbital elements, defined by central accelerations, have been used in detail numerical calculations in papers by Klačka and Kaufmannová (1992, 1993). Due to the property \( \langle e \rangle \geq \beta \) (see section 6.2.2), one must be aware that also values \( \langle e \rangle > \beta \) may correspond to pseudo-circular orbits. The results for pseudo-circular orbits were analytically confirmed by Breiter and Jackson (1998). As an advantage, the analytical approach reproduces known results without detail numerical calculations. However, the analytical approach paralelly produces nonphysical results which may not be distinguishable from the correct results. The nonphysical analytical results are caused by use of the P-R effect in the first order in \( v/c \) – very special analytical solutions will diminish when higher orders in \( v/c \) are used. The nonphysical analytical results have been discussed in more detail by Klačka (2001; see also http://xxx.lanl.gov/abs/astro-ph/0004181).

10. Solar wind effect

We have calculated secular changes of orbital elements for the P-R effect up to the second order in \( v/c \). The effects of the second order in \( v/c \) seem to be small to play an important role in Solar System studies. In reality, similar effect coming from the Sun exists and this effect may play more important role. Although this effect is not connected with
electromagnetic radiation, its secular changes of orbital elements to the first order in \( v/c \) (more correctly, \( v/u \), where \( u \) is the speed of solar wind particles) correspond to the secular changes for P-R effect. This effect is caused by solar wind particles hitting an interplanetary dust particle. The aim of this section is to obtain secular changes of orbital elements of the interplanetary dust particle under the action of the solar wind, up to the second order in \( v/u \).

Let us consider equation of motion in the form (we neglect decrease of particle’s mass)

\[
\frac{dv_{sw}}{dt} = \eta \frac{\beta}{Q'_1} \frac{\mu}{r^2} \frac{u}{c} \left\{ \left[ 1 - \frac{v \cdot e_R}{c} \right] e_R - \frac{v}{c} \right\} + \\
\left( \frac{v \cdot e_R}{u} \right) \frac{v}{u} + \frac{1}{2} \left[ \left( \frac{v}{u} \right)^2 - \left( \frac{v \cdot e_R}{u} \right)^2 \right] e_T \right\},
\]

where, as standardly in this paper, \( \mu \equiv GM/\odot \) and non-dimensional parameter \( \beta \) is defined in Eq. (43) (“the ratio of radiation pressure force to the gravitational force”); \( \eta \approx 1/3 \). Adding the right-hand side of Eq. (49) to the right-hand side of Eq. (173), one obtains

\[
\frac{dv}{dt} = -\frac{\mu}{r^2} e_R + \beta \frac{\mu}{r^2} \left\{ \left[ 1 - \frac{v \cdot e_R}{c} \right] e_R - \frac{v}{c} \right\} + \\
\eta \frac{\beta}{Q'_1} \frac{\mu}{r^2} \frac{u}{c} \left\{ \left[ 1 - \frac{v \cdot e_R}{u} \right] e_R - \frac{v}{u} \right\} + \\
\left( \frac{v \cdot e_R}{u} \right) \frac{v}{u} + \frac{1}{2} \left[ \left( \frac{v}{u} \right)^2 - \left( \frac{v \cdot e_R}{u} \right)^2 \right] e_T \right\}.
\]

The first term in Eq. (174) is Newtonian gravity for two-body problem, the second term is the Poynting-Robertson effect in flat spacetime, the third term corresponds to the action of the solar wind up to the second order in \( v/u \).

Eq. (174) may be written in the following form:

\[
\frac{dv}{dt} = -\frac{\mu}{r^2} e_R + \beta \left( 1 + \eta \frac{u}{Q'_1} \frac{c}{r^2} \right) \frac{\mu}{r^2} e_R - \beta \left( 1 + \eta \frac{u}{Q'_1} \right) \frac{\mu}{r^2} \left( \frac{v \cdot e_R}{c} e_R + \frac{v}{c} \right) + \\
\eta \frac{\beta}{Q'_1} \frac{\mu}{r^2} \left\{ \frac{v \cdot e_R}{c} \frac{v}{u} + \frac{1}{2} \left[ \frac{v^2}{c u} - \frac{(v \cdot e_R)^2}{c u} \right] e_T \right\}.
\]

10.1. Secular changes of orbital elements – radiation pressure as a part of central acceleration

Neglecting solar wind pressure term \( \eta(\beta/Q'_1)(\mu/r^2)(u/c) \), we can rewrite Eq. (175) to the following form:

\[
\frac{dv}{dt} = -\frac{\mu (1 - \beta)}{r^2} e_R - \beta \frac{\mu}{r^2} \left( 1 + \eta \frac{u}{Q'_1} \right) \left( \frac{v \cdot e_R}{c} e_R + \frac{v}{c} \right) + \\
\eta \frac{\beta}{Q'_1} \frac{\mu}{r^2} \left\{ \frac{v \cdot e_R}{c} \frac{v}{u} + \frac{1}{2} \left[ \frac{v^2}{c u} - \frac{(v \cdot e_R)^2}{c u} \right] e_T \right\}.
\]
where the second term causes deceleration of the particle’s motion. We use \(-\mu (1 - \beta) e_R / r^2\) as a central acceleration determining osculating orbital elements; \(\beta\) is considered to be a constant during the motion. On the basis of Eq. (176), we can immediately write for components of disturbing acceleration to Keplerian motion

\[
F_{\beta R} = -2 \beta \left( 1 + \frac{\eta}{Q_1} \right) \frac{\mu}{r^2} \frac{v_{\beta R}}{c} + \eta \frac{\beta}{Q_1} \frac{\mu}{r^2} \frac{v_{\beta R}^2}{c u},
\]

\[
F_{\beta T} = -\beta \left( 1 + \frac{\eta}{Q_1} \right) \frac{\mu}{r^2} \frac{v_{\beta T}}{c} + \eta \frac{\beta}{Q_1} \frac{\mu}{r^2} \left\{ \frac{v_{\beta R} v_{\beta T}}{c u} + \frac{1}{2} \frac{v_{\beta T}^2}{c u} \right\},
\]

\[
F_{\beta N} = 0,
\]

(177)

where \(F_{\beta R}, F_{\beta T}\) and \(F_{\beta N}\) are radial, transversal and normal components of perturbation acceleration, and the two-body problem yields

\[
v_{\beta R} = \sqrt{\mu (1 - \beta) p_{\beta}} e_{\beta} \sin f_{\beta},
\]

\[
v_{\beta T} = \sqrt{\mu (1 - \beta) p_{\beta}} (1 + e_{\beta} \cos f_{\beta}),
\]

(178)

where \(e_{\beta}\) is osculating eccentricity of the orbit, \(f_{\beta}\) is true anomaly, \(p_{\beta} = a_{\beta}(1 - e_{\beta}^2)\), \(a_{\beta}\) is semi-major axis. The important fact that perturbation acceleration is proportional to \(v/c \ll 1\) ensures small changes of orbital elements during a time interval \(T\) (\(T\) is time interval between passages through two following pericenters / periheila). Thus, we may take a time average in an analytical way

\[
\langle g \rangle \equiv \frac{1}{T} \int_0^T g(t) dt = \frac{\sqrt{\mu (1 - \beta) a_{\beta}^{3/2}}}{2\pi} \frac{1}{2\pi} \int_0^{2\pi} g(f_{\beta}) \left( \frac{df_{\beta}}{dt} \right)^{-1} df_{\beta}
\]

\[
= \frac{1}{a_{\beta}^{3/2}} \frac{1}{2\pi} \int_0^{2\pi} g(f_{\beta}) \frac{r^2}{\sqrt{\mu (1 - \beta) p_{\beta}}} df_{\beta}
\]

\[
= \frac{1}{a_{\beta}^2 \sqrt{1 - e_{\beta}^2}} \frac{1}{2\pi} \int_0^{2\pi} g(f_{\beta}) r^2 df_{\beta},
\]

(179)

assuming non-pseudo-circular orbits and the fact that orbital elements exhibit only small changes during the time interval \(T\); the second and the third Kepler’s laws were used:

\[r^2 df_{\beta}/dt = \sqrt{\mu (1 - \beta) p_{\beta}} - \text{conservation of angular momentum}, a_{\beta}^3/T^2 = \mu (1 - \beta)/(4\pi^2).
\]

Relevant perturbation equations of celestial mechanics yield for osculating orbital elements \((\omega_{\beta} - \text{longitude of pericenter / periheilon}; \Theta_{\beta} \text{ is the position angle of the particle on the orbit, when measured from the ascending node in the direction of the particle’s motion}, \Theta_{\beta} = \omega_{\beta} + f_{\beta}):

\[
\frac{da_{\beta}}{dt} = \frac{2 a_{\beta}}{1 - e_{\beta}^2} \sqrt{\frac{p_{\beta}}{\mu (1 - \beta)}} \left\{ F_{\beta R} e_{\beta} \sin f_{\beta} + F_{\beta T} (1 + e_{\beta} \cos f_{\beta}) \right\},
\]

\[
\frac{de_{\beta}}{dt} = \sqrt{\frac{p_{\beta}}{\mu (1 - \beta)}} \left\{ F_{\beta R} \sin f_{\beta} + F_{\beta T} \left[ \cos f_{\beta} + \frac{e_{\beta} + \cos f_{\beta}}{1 + e_{\beta} \cos f_{\beta}} \right] \right\},
\]
\[ \frac{d\omega}{dt} = - \frac{1}{e^{2}} \sqrt{\frac{p_{\beta}}{\mu (1 - e^{2})}} \left\{ F_{R} \cos f_{\beta} - F_{\beta} \right\} \], \tag{180} \]

where \( r = p_{\beta}/(1 + e_{\beta} \cos f_{\beta}) \).

Putting Eq. (177) and (178) into Eq. (180), making procedure of a verification of the type given by Eq. (179), we finally receive

\[ \langle \frac{da_{\beta}}{dt} \rangle = - \frac{\beta}{c} \frac{2 + 3e_{\beta}^{2}}{a_{\beta} (1 - e_{\beta}^{2})} \left\{ 1 + \frac{\eta}{Q_{1}} \left[ 1 - \frac{\mu (1 - \beta)}{a_{\beta} (1 - e_{\beta}^{2})} \right] \right\}, \tag{181} \]

\[ \langle \frac{de_{\beta}}{dt} \rangle = - \frac{\beta}{c} \frac{5e_{\beta}/2}{a_{\beta}^{3/2} (1 - e_{\beta}^{2})} \left\{ 1 + \frac{\eta}{Q_{1}} \left[ 1 - \frac{\mu (1 - \beta)}{a_{\beta} (1 - e_{\beta}^{2})} \right] \right\}, \tag{182} \]

\[ \langle \frac{d\omega_{\beta}}{dt} \rangle = \frac{3\mu^{3/2}}{c^{2}a_{\beta}^{5/2}} \left( \frac{\beta (1 - \beta)}{1 - e_{\beta}^{2}} \right)^{1/2} \frac{1}{3} \frac{\eta}{Q_{1}} \frac{c}{u}. \tag{183} \]

Initial conditions are given by Eq. (59), or, Eqs. (60) – (62).

Similarly, for the quantity \( p_{\beta} = a_{\beta} (1 - e_{\beta}^{2}) \), one can easily obtain

\[ \langle \frac{dp_{\beta}}{dt} \rangle = - 2 \beta \frac{\beta}{c} \left( \frac{1 - e_{\beta}^{2}}{p_{\beta}} \right)^{3/2} \left\{ 1 + \frac{\eta}{Q_{1}} \left[ 1 - \frac{\mu (1 - \beta)/p_{\beta}}{2 u} \right] \right\}. \tag{184} \]

We come to the conclusion that the following relation results from Eqs. (181) and (182):

\[ p_{\beta}^{4/5} = p_{\beta} \sin e_{\beta}^{3/5}, \tag{185} \]

as for secular changes of \( p_{\beta} \) and \( e_{\beta} \), for the simple case of equation of motion of an interplanetary dust particle under the action of solar wind represented by Eq. (173) and for the P-R effect. Thus, Eq. (182) can be solved as a separate equation

\[ \langle \frac{dp_{\beta}}{dt} \rangle = - 2 \beta \frac{\mu}{c} \left[ \frac{1 - e_{\beta}^{2}}{p_{\beta} \sin (p_{\beta}/p_{\beta} \sin)^{5/2}} \right]^{3/2} \left\{ 1 + \eta \frac{1}{Q_{1}} \left[ 1 - \frac{\sqrt{\mu (1 - \beta)/p_{\beta}}}{2 u} \right] \right\}. \tag{186} \]

10.2. Secular changes of orbital elements -- gravitation as a central acceleration

We can immediately write, on the basis of Eqs. (103), (172) and (181):

\[ \frac{da_{\beta}}{dt} = - \beta \left( 1 + \frac{\eta}{Q_{1}} \right) \frac{\mu}{c} \frac{2 + 3e_{\beta}^{2}}{a_{\beta} (1 - e_{\beta}^{2})^{3/2}}, \]

\[ \frac{de_{\beta}}{dt} = - \beta \left( 1 + \frac{\eta}{Q_{1}} \right) \frac{\mu}{c} \frac{5}{2} \frac{e_{\beta}}{a_{\beta}^{3/2} \sqrt{1 - e_{\beta}^{2}}}. \tag{187} \]
\[ a = a_\beta \left(1 - e_\beta^2\right)^{3/2} \frac{1}{2\pi} \int_0^{2\pi} \frac{[1 + \beta \left(1 + e_\beta^2 + 2e_\beta \cos x\right) / \left(1 - e_\beta^2\right)]^{-1}}{(1 + e_\beta \cos x)^2} \, dx, \]
\[ e = \left(1 - e_\beta^2\right)^{3/2} \frac{1}{2\pi} \int_0^{2\pi} \frac{\sqrt{(1 - \beta^2) e_\beta^4 + \beta^2 - 2\beta (1 - \beta) e_\beta \cos x}}{(1 + e_\beta \cos x)^2} \, dx, \]  
(187)

and, for secular change of longitude of pericenter/perihelion we have

\[ \frac{d\omega}{dt} = \frac{\sqrt{\mu(1 - \beta)}}{a_\beta^{3/2}} \left\{ \vartheta_H \left(\beta - \frac{e_\beta}{1 - e_\beta}\right) + \frac{\beta \left(1 + \eta Q_i^0 \right)}{\sqrt{1 - \beta}} \frac{\sqrt{T}}{c} \left(\frac{M_{\omega_1}}{a_\beta (1 - e_\beta)}\right) \right\}, \]

\[ M_{\omega_1} = \left(1 - e_\beta^2\right)^{3/2} \left[5 I_3 (e_\beta) - 2 I_2 (e_\beta)\right] \vartheta_H \left(\beta - \frac{e_\beta}{1 - e_\beta}\right), \]

\[ M_{\omega_2} = \frac{5}{4} \left\{ I_3 \left(\xi\right) - \frac{\left(\beta^2 - (1 - \beta)^2 e_\beta^2\right)^2}{\left(\beta^2 + (1 - \beta)^2 e_\beta^2\right)} I_2 \left(\xi\right) \right\}, \]

\[ \xi = -\frac{2\beta (1 - \beta) e_\beta}{(1 - \beta)^2 e_\beta^2 + \beta^2}, \]

\[ I_\alpha (\varepsilon) = \frac{1}{2\pi} \int_0^{2\pi} \frac{x}{(1 + \varepsilon \cos x)^\alpha} \, dx, \quad \alpha = 1, 2, 3, \]  
(188)

where \( \vartheta_H(x) = 1 \) if \( x > 0 \), \( \vartheta_H(x) = 0 \) if \( x < 0 \) (Heaviside’s step function); it is assumed that \( \beta \neq e_\beta/(1 - e_\beta) \).

Initial conditions are given by:

i) Eq. (59), or, Eqs. (60) – (62) for the set of Eqs. (186) – (187), and,

ii) Eq. (59), or, Eqs. (60) – (62) and Eq. (72), or, Eq. (73) for the set of Eqs. (186) and (188): Eqs. (72) – (73) are required for initial value of \( \omega \).

10.3. Solar wind – discussion

The results presented in this section hold for the most simple approximation of the solar wind action – only radial component of the solar wind particles is considered (Eq. (173) is taken as an approximation to more general equation of motion presented in Klačka and Saniga 1993; see also Leinert and Grün 1990). Moreover, solar wind causes decrease of particle’s mass and the secular change of particle’s mass \( m \) (present in \( \beta \), see Eq. (43)) is given as \( dm/dt = -K A'_{eff} / (a_\beta^2 \sqrt{1 - e_\beta^2}) \), where \( K \) is a constant depending on the material properties of the particle and \( A'_{eff} \) is the proper effective cross sectional area of the particle. If \( A'_{eff} \) (area toward the Sun) is changing during the particle’s motion, one has to use \( dm/dt = -K A'_{eff} (1 - v_\beta R/u)/r^2 \) and no averaging is possible – considerations made in section 16 hold for the case \( A'_{eff} \equiv A' \), where \( A' \) was defined above Eq. (16), and, moreover, \( A'_{eff} \) does not change during the particle’s motion (e. g., spherical particle).

Real velocity vector of solar wind particles is nonradial and the nonradial component
increases with decreasing distance from the Sun (e. g., Stix 2002). As a consequence, real solar wind effect may cause acceleration of meteoroids in small distances from the Sun, instead of their deceleration.

Finally, solar wind causes also charging of meteoroids and Lorentz force has to be taken into account in the case of submicron grains.

11. Summary and conclusions

The paper derives and presents relativistically covariant equation of motion for dust particle under the action of electromagnetic radiation – see Eq. (40). As for most frequent applications to systems in the universe (e. g., meteoroids in the Solar System, dust particles in circumstellar disks), equation of motion in the form of Eqs. (41) and (42) are sufficient: application of Eq. (41) (for the case $F'_{e,j} = 0$, $j = 1, 2, 3$ and under some assumption about particle’s rotation) may be found in Kocifaj et al. (2000), Kláčka and Kocifaj (2001). Some other accelerations may be added to the right-hand side of Eq. (41) – e. g., gravitational perturbations of planets, solar wind effect (see Eq. (173) or some more precise form of equation of motion) or some other nongravitational accelerations.

Special attention was devoted to the Poynting-Robertson effect, since this effect is standardly used in Solar System studies. We have derived secular orbital evolution for the P-R effect up to the second order in $v/c$. General equation of motion for interaction between particle and incident electromagnetic radiation shows that radiation cannot be considered as a part of central acceleration. The central acceleration has to contain only gravity of the central body (star/Sun); moreover, this corresponds to the physical situation when radiation effect is considered as a disturbing effect. In order to compare secular changes of semi-major axis and eccentricity for real cosmic dust particle and the P-R effect, the paper derives and presents also secular changes of these orbital elements for the P-R effect: see Eq. (103) in section 6.2.3 and Eq. (172) in section 8 – advancement of pericenter/perihelion exists even in the first order of $v/c$.

Solar wind effect is also considered in section 10. Solar wind, in its simplest approximation, produces secular changes of the orbital elements analogous to the P-R effect, as for the first order in $v/c$ – see first two equations in Eq. (181) and Eq. (182). However, second order in solar wind effect produces more significant changes of orbits than it is in the case of the P-R effect, if radiation pressure is a part of central acceleration – compare Eqs. (136), (146), (153) and (181). Section 10.2 presents secular changes of orbital elements for the P-R effect and the most simple approximation of the solar wind effect when solar gravitation alone is considered to be a central acceleration. in its simplest produces
Application to larger bodies, e. g., asteroids, may be found in Klačka (2000c) – some kind of thermal emission has to be added (quantities $Q'_{ej}$, $j = 1, 2, 3$ present in Eqs. (40) – (42) have to be calculated).

**Appendix A: Another formulation of the equation of motion**

(Reference to equation of number (j) of this appendix is denoted as Eq. (A j). Reference to equation of number (i) of the main text is denoted as Eq. (i).)

**Proper reference frame of the particle – stationary particle**

The equation of motion of the particle in its proper frame of reference is taken in the form

$$\frac{d E'}{d \tau} = 0 ,$$

$$\frac{d p'}{d \tau} = \frac{1}{c} S' (C' e'_{1}) + \sum_{j=1}^{3} F'_{ej} e'_{j} , \quad (1)$$

where $E'$ is particle’s energy, $p'$ its momentum, $\tau$ is proper time, $S'$ is the flux density of radiation energy (energy flow through unit area perpendicular to the ray per unit time), $C'$ is the radiation pressure cross section $3 \times 3$ matrix, unit vector $e'_{1}$ is directed along the path of the incident radiation (it is supposed that beam of photons propagate in parallel lines) and its orientation corresponds to the orientation of light propagation; $F'_{ej}$ is emission component of the radiation force acting on the particle (see Eq. (6)).

**Stationary frame of reference**

Our aim is to derive equation of motion for the particle in the stationary frame of reference.

**Covariant equation of motion – first case**

Let the components of the pressure cross section $3 \times 3$ matrix $C'$ be an orthonormal basis $e'_{b1}$, $e'_{b2}$, $e'_{b3}$. We may then write

$$e'_{n} = \sum_{k=1}^{3} (e'_{bk} \cdot e'_{n}) e'_{bk} , \quad n = 1, 2, 3 , \quad (2)$$

and

$$e'_{bn} = \sum_{k=1}^{3} (e'_{bn} \cdot e'_{k}) e'_{k} , \quad n = 1, 2, 3 , \quad (3)$$

where $e'_{1}$, $e'_{2}$ and $e'_{3}$ form an orthonormal basis. Similarly
\[ C^\prime e_1^\prime = \sum_{k=1}^{3} \left( e_{bk}^\prime C^\prime e_1^\prime \right) e_{bk}^\prime. \]  

(4)

On the basis of Eqs. (A1), (A2), (A3) and (A4), one obtains

\[
\frac{d p^\prime}{d \tau} = \frac{S^\prime}{c} \sum_{k=1}^{3} \left\{ \left( e_{bk}^\prime C^\prime e_1^\prime \right) \sum_{j=1}^{3} (e_{bk}^\prime \cdot e_j^\prime) e_j^\prime \right\} + \sum_{j=1}^{3} F_{ej}^\prime e_j^\prime =
\]

\[
= \frac{S^\prime}{c} \sum_{j=1}^{3} \left( \sum_{k=1}^{3} \left( e_{bk}^\prime \cdot e_j^\prime \right) \left( e_{bk}^\prime C^\prime e_1^\prime \right) \right) e_j^\prime + \sum_{j=1}^{3} F_{ej}^\prime e_j^\prime =
\]

\[
= \frac{S^\prime}{c} \sum_{j=1}^{3} \{ \bar{Q}_j^\prime A^\prime \} e_j^\prime + \sum_{j=1}^{3} F_{ej}^\prime e_j^\prime ,
\]

(5)

which corresponds Eq. (37). Thus, the covariant form is represented by Eq. (38).

More straightforward consideration: Let the components of the matrix \( C^\prime \) be given in the orthonormal basis \( e_1^\prime, e_2^\prime, e_3^\prime \) and \( C_{kl}^\prime = e_{kl}^\prime e_1^\prime, k, l = 1 \text{ to } 3 \). Then

\[ C^\prime e_1^\prime = \sum_{j=1}^{3} \left( e_{j}^\prime T C^\prime e_1^\prime \right) e_j^\prime. \]  

(6)

Substitutions \( \bar{Q}_j^\prime A^\prime \equiv e_{j}^\prime T C^\prime e_1^\prime, j = 1, 2, 3 \) immediately yield Eq. (37) which has already been rewritten to the covariant form represented by Eq. (38).

**Covariant equation of motion – second case**

We want to derive an equation of motion for the particle in the frame of reference in which particle moves with actual velocity \( \mathbf{v} \). We will use the fact that we know this equation in the proper frame of reference – see Eq. (A1).

Let us have a four-vector \( A^\mu = (A^0, \mathbf{A}) \), where \( A^0 \) is its time component and \( \mathbf{A} \) is its spatial component. Since generalized special Lorentz transformations do not form a group (in general, composition of two generalized special Lorentz transformations is not a generalized special Lorentz transformation), we will consider more general Lorentz transformation. This Lorentz transformation can be written as

\[ A^\nu = \Lambda^\mu_{\nu} A^\nu, \]  

(7)

where summation over repeated indices is supposed (and also in all the following equations) – 0, 1, 2, 3 for Greek letters and 1, 2, 3 for Latin letters – and its inverse

\[ A^\mu = \Lambda_\alpha^\mu A^\alpha. \]  

(8)

Important property of the Lorentz transformation
\[ A'' = \Lambda A \]  
(9)

is, that it can be composed of the following two transformations:

\[ A' = L A , \]
\[ A'' = R A' , \]  
(10)

where \( L \) corresponds to generalized special Lorentz transformation and \( R \) represents rotation in 3-dimensional space:

\[ A'\mu = L'_{\nu} A''^\nu , \]  
(11)

and (see Eq. (13))

\[ L^0_0 = \gamma , \]
\[ L^0_i = L^i_0 = - \gamma \left( \frac{v}{c} \right)_i , \ i = 1, 2, 3 , \]
\[ L^i_j = \delta_{ij} + (\gamma - 1) \left( \frac{v}{c} \right)_i \left( \frac{v}{c} \right)_j / v^2 , \ i = 1, 2, 3 , \ j = 1, 2, 3 , \]  
(12)

where \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \),

\[ A^\mu = L_{\alpha}^\mu A'^\alpha , \]  
(13)

where (see Eq. (14))

\[ L^0_0 = \gamma , \]
\[ L^0_i = L^i_0 = \gamma \left( \frac{v}{c} \right)_i , \ i = 1, 2, 3 , \]
\[ L^i_j = \delta_{ij} + (\gamma - 1) \left( \frac{v}{c} \right)_i \left( \frac{v}{c} \right)_j / v^2 , \ i = 1, 2, 3 , \ j = 1, 2, 3 , \]  
(15)

\[ R^0_0 = 1 , \]
\[ R^0_i = R^i_0 = 0 , \ i = 1, 2, 3 , \]
\[ R^i_j = r_{ij} , \ i = 1, 2, 3 , \ j = 1, 2, 3 , \]  
(16)

where \( r_{ij} \) can be expressed in terms of Euler angles, and, moreover, orthogonality conditions are fulfilled:

\[ \sum_{i=1}^{3} r_{ij} r_{ik} = \delta_{jk} , \ j = 1, 2, 3 , \ k = 1, 2, 3 , \]
\[ \sum_{j=1}^{3} r_{ij} r_{kj} = \delta_{ik} , \ i = 1, 2, 3 , \ k = 1, 2, 3 . \]  
(17)

On the basis of Eqs. (A9) and (A10) we can write
\[ \Lambda^\mu_\nu = R^\mu_\nu \cdot L^\nu_\nu. \]  

(18)

Eqs. (A12), (A16) and (A18) yield

\[ \Lambda^0_0 = L^0_0 = \gamma, \]
\[ \Lambda^0_i = L^0_i = -\gamma (v/c)_i, \quad i = 1, 2, 3. \]  

(19)

Finally, requirement \( A^\mu A_\mu = A^{\prime \mu} A^{\prime \mu} \) yields,

\[ \Lambda^\mu_\nu \Lambda^{\nu_\mu} = \delta^\mu_\nu, \]  

(20)

where \( \delta^\mu_\nu = 1 \) if \( \mu = \nu \) and \( \delta^\mu_\nu = 0 \) if \( \mu \neq \nu \); Eq. (A7) was also used: \( A^{\prime \mu} = \Lambda^\mu_\nu A^\nu \), \( A^{\prime \mu} = \Lambda^\nu_\mu A^\nu \).

Incoming radiation

Applying Eqs. (A7) and (A19) to quantity \( (E_i/c, p_i) \) (four-momentum per unit time – proper time is a scalar quantity) and taking into account also the fact that \( p_i = E_i/c e_1 \), we can write

\[ E^{\prime}_i = E_i w_1, \]  

(21)

where

\[ w_1 \equiv \gamma \left( 1 - v \cdot e_1/c \right). \]  

(22)

Using the fact that \( p^\mu = (h \nu, h \nu e_1) \) for photons, we have

\[ \nu^{\prime} = \nu w_1. \]  

(23)

We have four-vector \( p^{\mu}_i = (E_i/c, p_i) = (1, e_1)E_i/c = (1/w_1, e_1/w_1) w_1 E_i/c \equiv b^{\mu}_1 w_1 E_i/c \). We have found a new four-vector \( b^{\mu}_1 \), which is given as \( b^{\prime \mu}_1 = \Lambda^\mu_\nu b^{\nu}_1 \) in the proper frame of reference of the particle: \( b^{\prime \mu}_1 = (1, e_1') \). The transformation of space components between \( b^{\mu}_1 \) and \( b^{\prime \mu}_1 \) corresponds to aberration of light.

For monochromatic radiation the flux density of radiation energy becomes

\[ S' = n' h \nu' c; \quad S = n h \nu c, \]  

(24)

where \( n \) and \( n' \) are concentrations of photons (photon number densities) in the corresponding reference frames. We also have continuity equation

\[ \partial_\mu j^\mu = 0, \quad j^\mu = (c n, c n e_1) , \]  

(25)

with current density \( j^\mu \). Application of Eqs. (A7) and (A19) then yields
\( n' = w_1 n \).  

(26)

Using Eqs. (A23), (A24) and (A26) we finally obtain

\[ S' = w_1^2 S. \]  

(27)

Eqs. (A21) and (A27) then together give \( E_i = w_1 S A' \), \( p_i = w_1 S A' e_1/c. \)

**Covariant equation of motion**

Inspiration comes from the fact that space components of four-momentum are written as a product with unit vector \( e'_1 \). We know that this unit vector can be generalized to a four-vector

\[ b_{1\mu} = \left( \frac{1}{w_1}, e_1 / w_1 \right) \]

\[ w_1 \equiv \gamma (1 - v \cdot e_1/c). \]  

(28)

Moreover, we know that \( w_1^2 S / c \) is a scalar quantity – invariant of the Lorentz transformation (see Eqs. (A27)).

The idea is to write covariant equation of motion in the form

\[
\frac{d p^\mu}{d \tau} = \frac{w_1^2 S}{c} G^{\mu \nu} b_{1\nu} + \frac{1}{c} \sum_{j=1}^{3} F'_{ej} \left( c b^\mu_j - w^\mu \right),
\]  

(29)

if the result for the emission component of the radiation force acting on the particle was added (see sections 2 and 3 in the main text). The only problem is to find components of the tensor of the second rank \( G^{\mu \nu}. \)

In order to find \( G^{\mu \nu} \), we will proceed in two steps. At first, we will rewrite Eq. (A29) in the proper frame of reference of the particle. Comparison with Eq. (A1) will yield components of \( G'^{\mu \nu}. \) The second step is transformation from \( G'^{\mu \nu} \) to \( G^{\mu \nu}. \)

In the proper frame of reference, Eq. (A29) yields

\[
\frac{d E'}{d \tau} = \frac{w_1^2 S}{c} \left\{ G'^{\ 0 \ 0} - \sum_{j=1}^{3} G'_{\ 0 \ j} \left( e'_1 \right)_j \right\},
\]

\[
\frac{d (p'_k)}{d \tau} = \frac{w_1^2 S}{c} \left\{ G'_{\ k \ 0} - \sum_{j=1}^{3} G'_{\ k \ j} \left( e'_1 \right)_j \right\} + \sum_{j=1}^{3} F'_{ej} \left( e'_j \right)_k,
\]  

(30)

where the term \( 1/w'_1 \) in brackets is omitted due to the simple fact that it equals 1 in the proper frame of reference. Comparison with Eq. (A1) yields

\[ G'_{\ 0 \ 0} = G'_{\ 0 \ j} = G'_{\ j \ 0} = 0, \ j = 1, 2, 3, \]

\[ G'_{\ k \ j} = -C_{k \ j}, \ j = 1, 2, 3, \ k = 1, 2, 3. \]  

(31)

In order to find \( G^{\mu \nu} \), we have to use the Lorentz transformation
\[ G^{\mu \nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta G^{\alpha \beta}, \]  

(32)

where

\[ \Lambda^\beta_\alpha = \eta^\beta_\rho \eta^\rho_\gamma \Lambda^\gamma_\alpha, \]

\[ \eta^\alpha_\beta = \text{diag}(+1, -1, -1, -1). \]  

(33)

Usage of the generalized special Lorentz transformation (Eqs. (A12) or (A15)) yields the results of Kimura et al. (2002) – the authors take the space tensor \( C' \) as a scalar quantity under Lorentz transformation. However, generalized special Lorentz transformation has not to be used. The reason is that any body in torque-free, accelerated motion undergoes rotation. This effect is known as the Thomas precession (see e.g., Robertson and Noonan 1968, pp. 66–69). Thus, the process of complete relativistic derivation and the corresponding result presented by Kimura et al. (2002) is incorrect. We can mention that the last statement is evident already at first glance: Kimura et al. (2002) take the radiation pressure cross section \( 3 \times 3 \) matrix – space tensor – as a relativistically invariant quantity.

**Consistency of the covariant formulations**

We have obtained equation of motion in the form of Eq. (A29). Another form of covariant equation is presented in Eq. (35) (Eq. (38)). Are these equations consistent?

Eq. (38) is covariant equation of motion in the form corresponding to \( \beta^\mu \equiv u^\mu/c \)

\[
\left( \frac{d p^\mu}{d \tau} \right)_I = \frac{w_2^2}{c} S \frac{A'}{c} \sum_{j=1}^{3} \bar{Q}^j_{\mu}(b^\mu_j - \beta^\mu) + \frac{1}{c} \sum_{j=1}^{3} F^\mu_{ej}(c b^\mu_j - u^\mu). 
\]  

(34)

This appendix has discussed an equation of motion of the form

\[
\left( \frac{d p^\mu}{d \tau} \right)_II = \frac{w_2^2}{c} S \frac{A'}{c} G^{\mu \nu} b_1 \nu + \frac{1}{c} \sum_{j=1}^{3} F^\nu_{ej}(c b^\nu_j - u^\nu). 
\]  

(35)

We want to show that Eqs. (A34) and (A35) are equivalent, i.e., that \( \left( \frac{d p^\mu}{d \tau} \right)_I = \left( \frac{d p^\mu}{d \tau} \right)_II \).

We will not write the terms \( F^\nu_{ej} \), in what follows.

Multiplication of Eq. (A34) by four-vector \( b_k \mu \) (and summation over \( \mu \)) yields

\[
\left( \frac{d p^\mu}{d \tau} \right)_I b_k \mu = \frac{w_2^2}{c} S \frac{A'}{c} \sum_{j=1}^{3} \bar{Q}^j_{\mu}(b^\mu_j b_k \mu - \beta^\mu b_k \mu), \quad k = 1, 2, 3. 
\]  

(36)

In calculations of \( \beta^\mu b_k \mu \) we will use the fact that it represents scalar product of two four-vectors. Thus, its value is independent on the frame of reference. For the proper frame of reference
$\beta^\mu b_{k\mu} = 1$, $k = 1, 2, 3$. \hfill (37)

It can be easily verified, that

$$ b_j^\mu b_{k\mu} = 1 - e_j' \cdot e_k' = 1 - \delta_j k, \quad j = 1, 2, 3, \quad k = 1, 2, 3, \hfill (38) $$

since in the optics of scattering processes it is assumed (defined) that unit vectors $e_1'$, $e_2'$ and $e_3'$ are orthogonal. Thus, we obtain (inserting results of Eqs. (A37) and (A38) into Eq. (A36))

$$ \left( \frac{d p^\mu}{d \tau} \right)_I b_{k\mu} = -\frac{w_1^2 S}{c} A' Q_k', \quad k = 1, 2, 3. \hfill (39) $$

Multiplication of Eq. (A35) by four-vector $b_{k\mu}$ (and summation over $\mu$) yields

$$ \left( \frac{d p^\mu}{d \tau} \right)_I b_{k\mu} = \frac{w_1^2 S}{c} G^{\mu^\nu} b_1^\nu b_{k\mu}, \quad k = 1, 2, 3. \hfill (40) $$

Again, the value is independent on the frame of reference. For the proper frame of reference Eq. (A31) yields

$$ \left( \frac{d p^\mu}{d \tau} \right)_I b_{k\mu} = -\frac{w_1^2 S}{c} C' j \cdot (e_1'), (e_j'), \quad \equiv -\frac{w_1^2 S}{c} (e_k')^T (C' e_1'), \quad k = 1, 2, 3. \hfill (41) $$

It was already shown (see Eq. (A6) and the text below it) that right-hand sides of Eqs. (A39) and (A41) are identical. Thus, also left-hand sides of Eqs. (A39) and (A40) are identical:

$$ \left( \frac{d p^\mu}{d \tau} \right)_I b_{k\mu} = \left( \frac{d p^\mu}{d \tau} \right)_I b_{k\mu}, \quad k = 1, 2, 3. \hfill (42) $$

If we take into account that four-vectors $b_k^\mu$ may be taken in various ways, Eq. (A42) yields

$$ \left( \frac{d p^\mu}{d \tau} \right)_I = \left( \frac{d p^\mu}{d \tau} \right)_I. \hfill (43) $$

Thus, Eqs. (A34) and (A35) are equivalent, Q. E. D.

**Appendix B: Einstein’s example**

(Reference to equation of number (j) of this appendix is denoted as Eq. (B j). Reference to equation of number (i) of the main text is denoted as Eq. (i).)

Let us consider a plane mirror moving (at a given moment) along x-axis (system S) with velocity $v = (v, 0, 0)$, $v > 0$; the mirror is perpendicular to the x-axis (the plane of the mirror is parallel to the yz-plane). A beam of incident (hitting) photons is
characterized by unit vector $\textbf{S}' = (\cos \theta', \sin \theta', 0)$ in the proper frame (primed quantities) of the mirror. Reflected beam is described by the unit vector $\textbf{e}' = (-\cos \theta', \sin \theta', 0)$ (in the proper frame $\textbf{S}'$).

The problem is: Find equation of motion of the mirror in the frame of reference $\textbf{S}$.

**Solution 1: trivial manner**

Consider one photon (frequency $f'$) in the proper frame of the mirror. Since the directions (and orientations) of the incident and outgoing photons are characterized by

$$S' = (\cos \theta', \sin \theta', 0),$$

$$e' = (-\cos \theta', \sin \theta', 0),$$

we can immediately write

$$p'_{i\mu} = \frac{h f'}{c} (1, \cos \theta', \sin \theta', 0),$$

$$p'_{o\mu} = \frac{h f'}{c} (1, -\cos \theta', \sin \theta', 0),$$

for the four-momentum of the photon before interaction with the mirror and after the interaction.

As a consequence, the mirror obtains four-momentum

$$p'^{\mu} = p'_{i\mu} - p'_{o\mu} = \frac{h f'}{c} (0, 2 \cos \theta', 0, 0).$$

Application of the special Lorentz transformation to Eq. (B3) yields

$$p^{\mu} = \frac{h f'}{c} 2 \gamma (\cos \theta') (\beta, 1, 0, 0),$$

where, as standardly abbreviated,

$$\gamma = 1 / \sqrt{1 - \beta^2}; \quad \beta = v / c.$$

On the basis of Eq. (B4), we can immediately write equation of motion of the mirror

$$\frac{dp^{\mu}}{d\tau} = \frac{E'_i}{c} 2 \gamma (\cos \theta') (\beta, 1, 0, 0),$$

where $E'_i$ is the total energy (per unit time) of the incident radiation measured in the proper frame of reference.
Solution 2: application of general theory presented in Sec. 3 of the main text

We have to choose orthonormal vectors in the systems \( S' \): we will use \( e_1' \equiv S' \) and \( e_2' \) and one can easily find

\[
e_1' = S' = (+ \cos \theta', \sin \theta', 0) ,
\]
\[
e_2' = (- \sin \theta', \cos \theta', 0) .
\] (7)

We have to write \( (Q'_3 = 0) \)

\[
p' = \frac{h f'}{c} (Q'_1 e'_1 + Q'_2 e'_2) .
\] (8)

On the basis of Eqs. (B3), (B7) and (B8) we have \( (Q'_3 = 0) \)

\[
Q'_1 = 2 (\cos \theta')^2 , \quad Q'_2 = - 2 (\sin \theta') (\cos \theta') .
\] (9)

Other prescription yields (see Eq. (34))

\[
b_1^0 = \gamma (1 + v \cdot e_1'/c) = \gamma (1 + \beta \cos \theta') ,
\]
\[
b_1 = e'_1 + [(\gamma - 1)v \cdot e'_1/v^2 + \gamma/c] \mathbf{v} = (\gamma \cos \theta' + \gamma \beta, \sin \theta', 0) ,
\] (10)

\[
b_2^0 = \gamma (1 + v \cdot e_2'/c) = \gamma (1 - \beta \sin \theta') ,
\]
\[
b_2 = e'_2 + [(\gamma - 1)v \cdot e'_2/v^2 + \gamma/c] \mathbf{v} = (-\gamma \sin \theta' + \gamma \beta, \cos \theta', 0) .
\] (11)

Inserting Eqs. (B9) – (B11) (and \( F'_{e_j} = 0 \) for \( j = 1, 2, 3 \)) into Eqs. (29) – (30), one obtains

\[
\frac{dt}{\mathbf{v}} = \frac{E'_i}{c} \left\{ [2(\cos \theta')^2] (b'_1 - \beta^0) + [-2(\sin \theta')(\cos \theta')] \ (b'_2 - \beta^0) \right\}
\]
\[
= \frac{E'_i}{c} 2\gamma (\cos \theta') (\beta, 1, 0, 0) .
\] (12)

Unit vectors \( e_1' \equiv S' \) and \( e_2' \) are used. They are orthonormal in the system \( S' \). However, corresponding vectors are not orthogonal in the system \( S \).

\[
e_1 = \frac{1}{w'} \left\{ e'_1 + [(\gamma - 1) v \cdot e'_1/v^2 + \gamma/c] \mathbf{v} \right\} ,
\]
\[
w' = \gamma (1 + v \cdot e_1'/c) ,
\] (13)

and analogous equation holds for vector \( e_2 \). Inserting Eqs. (B7), one obtains:

\[
e_1 = \left\{ \begin{array}{c} 
\cos \theta' + \beta \\
1 + \beta \cos \theta' \\
\end{array} \right\} \left\{ \begin{array}{c}
\sin \theta' \\
\gamma (1 + \beta \cos \theta') \\
\end{array} \right\} ,
\]
\[
e_2 = \left\{ \begin{array}{c} 
- \sin \theta' + \beta \\
1 - \beta \sin \theta' \\
\end{array} \right\} \left\{ \begin{array}{c}
\cos \theta' \\
\gamma (1 - \beta \sin \theta') \\
\end{array} \right\} .
\] (14)

It can be easily verified that scalar product of these two vectors is nonzero, in general.
Comparison with Einstein’s result

Inserting \( E_i' = w^2 S A'_\text{mirror} \cos \theta' \) into Eq. (B6) (or Eqs. (B12)) and using Eq. (B14) \((e_1 \equiv (\cos \theta, \sin \theta, 0))\) for the purpose of obtaining \( \cos \theta' = (\cos \theta - \beta)/(1 - \beta \cos \theta) \), one easily obtains:

i) \[ \frac{dE}{d\tau} = 2 \gamma^3 S A'_\text{mirror} (\cos \theta - \beta)^2 \beta; \text{ using definition of radiation pressure } dE/dt \equiv P v A'_\text{mirror}, \text{ we have } P = 2 (S/c) (\cos \theta - \beta)^2 / (1 - \beta^2), \] or,

ii) \[ \frac{dp}{d\tau} = 2 \gamma^3 (S A'_\text{mirror} / c) (\cos \theta - \beta)^2; \text{ using definition of radiation pressure } P \equiv (dp/dt) / A'_\text{mirror}, \text{ we have } P = 2 (S/c) (\cos \theta - \beta)^2 / (1 - \beta^2). \]

Result for \( P \) is consistent with the result presented in Einstein (1905).

Appendix C: Equation of motion presented by Kimura et al. (2002)

(Reference to equation of number (i) of this appendix is denoted as Eq. (C i). Reference to equation of number (j) of the appendix B is denoted as Eq. (B j). Reference to equation of number (k) of the main text is denoted as Eq. (k).)

It is said that the paper by Kimura et al. (2002) is relevant; irrelevant are papers by Klačka (2000a, 2000b, 2000c), or derivations and results presented in Sec. 3 of this paper. Thus, it is important to clarify the situation by presentation of detailed arguments.

Kimura et al. (2002) derive and present equation of motion of the form

\[
\frac{dv}{dt} = \frac{S A'}{mc} \times \zeta ,
\]

\[ \zeta = Q'_1 \left((1 - v \cdot k_1/c) \ k_1 - v/c\right) + Q'_2 \left((1 - 2 v \cdot k_1/c) \ k_2 + (v \cdot k_2/c) \ k_1\right) + Q'_3 \left(1 - 2 v \cdot k_1/c\right) \ k_3 ,
\]

\[ e'_1 = (1 + v \cdot k_1/c) \ k_1 - v/c ,
\]

\[ e'_2 = k_2 + (v \cdot k_2/c) \ k_1 ,
\]

\[ e'_3 = k_3 ,
\]

\[ e'_i \cdot e'_j = \delta_{ij} , \quad i, j \in \{1, 2, 3\} ,
\]

\[ k_i \cdot k_j = \delta_{ij} , \quad i, j \in \{1, 2, 3\} ,
\]

\[ v \cdot k_3 = 0 , \quad (1)
\]

(see Eqs. (1), (2), (3), (10), (14) in Kimura et al. 2002) and it is supposed that \( F'_e = 0 \).

There have appeared two independent suggestions:

i) Eq. (C1) is correct and Eq. (36) is incorrect.

ii) Eq. (C1) is equivalent to Eq. (36). As a proof, the following argument is presented.
Equations of motion presented by Klačka (2000a, 2000b, 2000c) and Eq. (36) (for the case $F'_e = 0$) and Kimura et al. (2002) are equivalent and all the difference comes from usage of different basic vectors. The set of non-orthogonal unit vectors $\{e_j; j = 1, 2, 3\}$ (see Eq. (36)) is replaced by new set of orthonormal vectors $\{k_j; j = 1, 2, 3\}$ in the way

\[
e_1 = k_1,
\]

\[
e_2 = k_2 + (v \cdot k_2/c) (k_1 - k_2) + v/c ,
\]

\[
e_3 = k_3 - (v \cdot k_3/c) k_3 + v/c ,
\]

\[
v \cdot k_3 = 0 .
\]

(2)

Eq. (C2) yields $e'_1 = (1 + v \cdot k_1/c)k_1 - v/c , e'_2 = k_2 + (v \cdot k_2/c)k_1$ and $e'_3 = k_3$, which is equivalent to Eq. (C1).

**Kimura et al. (2002) and Einstein’s example**

Let us consider a plane mirror moving (at a given moment) along x-axis (system S) with velocity $v = (v, 0, 0)$, $v > 0$; the mirror is perpendicular to the x-axis (the plane of the mirror is parallel to the yz-plane). A beam of incident (hitting) photons is characterized by unit vector $S' = (\cos \theta', \sin \theta', 0)$ in the proper frame (primed quantities) of the mirror. Reflected beam is described by the unit vector $e' = (- \cos \theta', \sin \theta', 0)$ (in the proper frame $S'$).

We are interested in application of Kimura et al. (2002) general equation of motion to this example.

On the basis of considerations presented in Appendix B, Eqs. (B7) and (B9) immediately yield

\[
e'_1 = (+ \cos \theta', \sin \theta', 0) ,
\]

\[
e'_2 = (- \sin \theta', \cos \theta', 0) .
\]

\[
Q'_1 = 2 (\cos \theta')^2 , \quad Q'_2 = - 2 (\sin \theta') (\cos \theta') , \quad Q'_3 = 0 .
\]

(3)

Eq. (C1) yields for orthonormal vectors

\[
k_1 = (1 - v \cdot e'_1/c) e'_1 + v/c ,
\]

\[
k_2 = e'_2 - (v \cdot e'_2/c) e'_1 .
\]

(4)

Using Eqs. (C3) and (C4),

\[
k_1 = \left\{ \cos \theta' + \frac{v}{c} (\sin \theta')^2 , \sin \theta' - \frac{v}{c} \cos \theta' \sin \theta' , 0 \right\} .
\]

\[
k_2 = \left\{ - \sin \theta' + \frac{v}{c} \cos \theta' \sin \theta' , \cos \theta' + \frac{v}{c} (\sin \theta')^2 , 0 \right\} .
\]

(5)
Inserting Eqs. (C3) and (C5) into Eqs. (C1), one easily obtains
\[
\frac{dv}{dt} = \frac{S}{m c} A'_\text{mirror} \cos \theta' \left\{ 2 \cos \theta' - \frac{v}{c} \left[ 1 + 2 (\cos \theta')^2 \right], 0, 0 \right\}.
\] (6)

Using the relation \( E'_i = S (1 - 2 v \cos \theta'/c) A'_\text{mirror} \cos \theta' \) in Eqs. (B6) or (B12), the correct result can be written in the form
\[
\frac{dv}{dt} = \frac{S}{m c} A'_\text{mirror} \cos \theta' \left\{ 2 \left( 1 - 2 \frac{v}{c} \cos \theta' \right) \cos \theta', 0, 0 \right\}.
\] (7)

Eq. (C6) was obtained on the basis of equation of motion presented by Kimura et al. (2002), while the correct result corresponds to Eq. (C7). Thus, Kimura et al. (2002) are incorrect – their general equation of motion yields result not consistent with Einstein’s result (Einstein 1905).

**Physics of transformation represented by Eq. (C2)**

Transformation \( \{e'_i; i = 1, 2, 3\} \rightarrow \{e_i; i = 1, 2, 3\} \) from frame of reference \( S' \) to frame of reference \( S \) corresponds to aberration of light (see Eq. (36) or heuristic derivation Sec. 3.4). In the given frame of reference \( S \) we may want to define two sets of unit vectors \( \{e_i; i = 1, 2, 3\} \) and \( \{l_i; i = 1, 2, 3\} \), where \( e_1 = l_1 \) and \( e_i \cdot e_j \neq \delta_{ij}, l_i \cdot l_j = \delta_{ij}, i, j \in \{1, 2, 3\} \). Relation between \( \{l_i; i = 1, 2, 3\} \) and \( \{e_i; i = 1, 2, 3\} \) corresponds to pure spatial rotation (geometry) and it can be described as
\[
e_1 = l_1,
\]
\[
e_2 = \xi_1 l_1 + \xi_2 l_2 + \xi_3 l_3,
\]
\[
e_3 = \eta_1 l_1 + \eta_2 l_2 + \eta_3 l_3.
\] (8)

The conditions
\[
l_i \cdot l_j = \delta_{ij}, \quad i, j \in \{1, 2, 3\}
\] (9)

lead to
\[
\sum_{i=1}^{3} \xi_i^2 = 1, \quad \sum_{i=1}^{3} \eta_i^2 = 1,
\]
\[
\xi_1 = e_1 \cdot e_2, \quad \eta_1 = e_1 \cdot e_3,
\]
\[
\sum_{i=1}^{3} \xi_i \eta_i = e_2 \cdot e_3.
\] (10)

The important property is that both sets of unit vectors \( \{l_i; i = 1, 2, 3\} \) and \( \{e_i; i = 1, 2, 3\} \) are defined in the same frame of reference \( S \). As for transformation defined by Eq. (C2), the sets of unit vectors \( \{e_i; i = 1, 2, 3\} \) and \( \{k_i; i = 1, 2, 3\} \) are also
defined in the system $S$. But what about translational velocity terms $+\mathbf{v}/c$ in Eq. (C2)?

Any velocity term $+\mathbf{v}/c$ in transformation corresponds to transformation from one (local) inertial frame of reference to another one. Thus, the conclusion is evident: Eq. (C2) is of no physical sense.

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