Optimal Artificial Boundary Condition for Random Elliptic Media

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Abstract
We are given a uniformly elliptic coefficient field that we regard as a realization of a stationary and finite-range ensemble of coefficient fields. Given a right-hand side supported in a ball of size $\ell \gg 1$ and of vanishing average, we are interested in an algorithm to compute the solution near the origin, just using the knowledge of the given realization of the coefficient field in some large box of size $L \gg \ell$. More precisely, we are interested in the most seamless artificial boundary condition on the boundary of the computational domain of size $L$. Motivated by the recently introduced multipole expansion in random media, we propose an algorithm. We rigorously establish an error estimate on the level of the gradient in terms of $L \gg \ell \gg 1$, using recent results in quantitative stochastic homogenization. More precisely, our error estimate has an a priori and an a posteriori aspect: with a priori overwhelming probability, the prefactor can be bounded by a constant that is computable without much further effort, on the basis of the given realization in the box of size $L$. We also rigorously establish that the order of the error estimate in both $L$ and $\ell$ is optimal, where in this paper we focus on the case of $d = 2$. This amounts to a lower bound on the variance of the quantity of interest when conditioned on the coefficients inside the computational domain, and relies on the deterministic insight that a sensitivity analysis with respect to a defect commutes with stochastic homogenization. Finally, we carry out numerical experiments that show that this optimal convergence rate already sets in at only moderately large $L$, and that more naive boundary conditions perform worse both in terms of rate and prefactor.

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1 Introduction and Main Results

Let the dimension $d \geq 2$ and the ellipticity ratio $\lambda > 0$ be fixed. We will be considering symmetric tensor fields $a$ on $d$-dimensional space $\mathbb{R}^d$ that are $\lambda$-uniformly elliptic:

$$|\xi|^2 \geq \xi \cdot a(x)\xi \geq \lambda |\xi|^2$$

for all points $x$ and vectors $\xi$, (1)

where without loss of generality we have set the upper bound to unity. Symmetry is notationally convenient at a few places, but by no means essential; while we use scalar notation and language, all results hold for systems unless specified. For some localized right-hand side $g$, say near the origin, we are interested in the decaying (i.e., Lax-Milgram) whole-space solution $u$ of

$$-\nabla \cdot a \nabla u = \nabla \cdot g.$$ (2)

More precisely, we are interested in $-\nabla u$ near the origin, say, at the origin: $-\nabla u(0)$. In the language of electrostatics, we are interested in the electric field generated by the neutral and localized charge distribution $\nabla \cdot g$. We pose the question to which precision $-\nabla u(0)$ can be inferred without solving a PDE in whole-space. Let us denote by $\ell$ the (linear) size of the support of $g$. In case of constant coefficients $a$, the explicit fundamental solution $G_h$ (of the constant coefficient elliptic operator) allows to obtain $-\nabla u(0)$ by the evaluation of an integral over $B_{\ell}$, the centered ball of radius $\ell$.

In our case of variable coefficients, we ask the question of whether one can do better than solving a boundary value problem with homogeneous boundary data, say, the Dirichlet problem

$$-\nabla \cdot a \nabla u_0 = \nabla \cdot g \text{ in } Q_L, \quad u_0 = 0 \text{ on } \partial Q_L$$

on the centered cube $Q_L := (-L, L)^d$ for some large scale $L \geq \ell$ (where we take cubes instead of balls for computational convenience). Under the assumption that $\ell$ is the only scale of $g$ in the sense that there exists a function $\hat{g}$ such that $^1$

$$g(x) = \hat{g}\left(\frac{x}{\ell}\right) \quad \text{with } \hat{g} \text{ supported in } B_1 \text{ and } \hat{V}\hat{g} \text{ Hölder continuous,}$$

one expects and we experimentally show in Sect. 2 that the approximation (3) is no better than what generically holds in the constant-coefficient case, namely $\nabla (u_0 -$

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$^1$ Control of some Hölder norm of $\hat{V}\hat{g}$ is necessary to control the supremum norm of $\nabla^2 \hat{u}_h$, see (9), which in turn is convenient when estimating the homogenization error, see the end of the proof of Proposition 1.
More precisely, we ask the question whether one can do better without knowing the coefficients outside $Q_{2L}$, which is hopeless when $a$ has no further structure. In this paper, we thus consider the case when $a$ comes from a stationary finite-range ensemble $\langle \cdot \rangle$ of uniformly elliptic coefficient fields; without loss of generality we assume the range to be unity. We recall that stationary means that $a$ and its shifted version $a(\cdot + z)$ for any shift vector $z \in \mathbb{R}^d$ have the same distribution under $\langle \cdot \rangle$; unit range means that for two subset $D, D' \subset \mathbb{R}^d$ with $\text{dist}(D, D') > 1$, the restrictions $a|_D$ and $a|_{D'}$ of the coefficient field $a$ are independent under $\langle \cdot \rangle$. We consider the finite-range ensembles because this is the framework in which we both have generic examples for lower bounds (Definition 1 and Theorem 2) and proofs of (almost) matching upper bounds (Theorem 1). Upper bounds could also, actually more easily, be obtained under the assumption of a suitable functional inequality on the ensemble.

### 1.1 Approximation Algorithm and Error Bound

Loosely speaking, our first main result, Theorem 1, states that there is an algorithm, the outcome of which is $u^{(L)}$, that

- only involves knowing the realization $a$ restricted to $Q_{2L}$,
- next to the solution of a Dirichlet problem on $Q_L$ only requires the solution of $2$ (in $d = 2$) respectively $8$ (in $d = 3$) further Dirichlet problems on $Q_{2L}$,
- improves upon (3) by (almost) a factor $L^{-\frac{d}{2}}$, albeit with a random prefactor,
- with overwhelming probability in $L \gg 1$, this prefactor is dominated by a constant that can be computed at the cost of further $2$ (for $d = 2$) respectively $24$ (for $d = 3$) (constant-coefficient) Dirichlet problems on $Q_{2L}$.

Note that Theorem 1 is a mixture of an a priori result, namely the probabilistic estimate on when the approach is successful at all, and an a posteriori result, namely the domination of the prefactor by a computable quantity. Loosely speaking, the second main result, Theorem 2, states that in terms of scaling (both in $L$ and $\ell$), there is no better algorithm since for a relevant class of ensembles $\langle \cdot \rangle$, the square root of the variance of $\nabla u(0)$ conditioned on $a|_{Q_{2L}}$ is of order $(\ell L)^d L^{-\frac{d}{2}}$. The argument shows that the factor $L^{-\frac{d}{2}}$ is of central limit theorem (CLT)-type and loosely speaking arises as the inverse of the square root of the volume of the neighboring “annulus” $Q_{4L} - Q_{2L}$.

This paper only discusses the algorithm that gives the (near) optimal result in case of $d = 2$; the optimal algorithm for the case of $d = 3$ (and higher $d$ in general) would require a refinement, namely the second-order corrector, but no new concepts. More precisely, the theory of dipoles developed in [4] would have to be replaced by its systematic generalization to multipoles (in particular quadrupoles) developed in [5], relying on second-order correctors. The detailed study will be left for future works.

Since we do not assume (Hölder-) continuity of the realization $a$, we do not have access to the point evaluation (at the origin) of the gradient. Hence in all statements, pointwise control is replaced by $L^2$-control over an order-one ball.

**Correctors $\phi_i$.** Not surprisingly, our algorithm makes use of the correctors $\phi_i$, which for every coordinate direction $i = 1, \ldots, d$ provide $a$-harmonic coordinates
\( x_i + \phi_i \) by satisfying

\[
-\nabla \cdot a(e_i + \nabla \phi_i) = 0,
\]

(5)

where \( e_i \) denotes the unit vector in direction \( i \). According to the classical qualitative theory of stochastic homogenization (by “qualitative” theory we mean the one only relying on ergodicity and stationarity of the ensemble \( \langle \cdot \rangle \)), for almost every realization \( a \), a corrector \( \phi_i \) of sublinear growth, that is,

\[
\lim_{r \to \infty} \frac{1}{r} \left( \int_{B_r} (\phi_i - \int_{B_r} \phi_i)^2 \right)^{\frac{1}{2}} = 0
\]

(6)

can be constructed. Here and in the sequel, \( \int_{B_r} \) denotes the average over the ball \( B_r \) (of radius \( r \) centered at the origin). Moreover, again almost surely, the homogenized coefficients \( a_h \) may be inferred from

\[
a_h e_i = \lim_{L \to \infty} \frac{1}{r} \int_{B_{2r}} q_i \quad \text{where} \quad q_i := a(e_i + \nabla \phi_i).
\]

(7)

Hence, a naive guess would be to replace the approximation \( u_0 \) defined through (3) by the approximation \( u_I \) defined through

\[
-\nabla \cdot a \nabla u_I = \nabla \cdot g \quad \text{in} \quad Q_L, \quad u_I = \tilde{u}_h \quad \text{on} \quad \partial Q_L,
\]

(8)

where \( \tilde{u}_h \) is the decaying solution of the homogenized problem in the whole space, that is,

\[
-\nabla \cdot a_h \nabla \tilde{u}_h = \nabla \cdot g.
\]

(9)

This can be seen to generically yield no improved scaling of the error in \( L \) (i.e., at fixed \( \ell \)); it only improves the scaling of the error when \( \ell \gg 1 \). Incidentally, it would not fall in the class of the algorithms we consider, since inferring the homogenized coefficient \( a_h \) requires solving the whole-space problem (5). In the context of multiscale methods, this is the approach taken in [16] (with additional steps to approximate \( a_h \)).

In both periodic and random homogenization, it is known that the so-called two-scale expansion

\[
(1 + \phi_i \partial_i)\tilde{u}_h,
\]

where we use Einstein’s convention of summation over repeated indices, provides a better approximation to \( u \) than \( \tilde{u}_h \) itself; in particular, this approximation is necessary to get closeness of the gradients (in the regime \( \ell \gg 1 \)). Hence, a second attempt would be to replace the approximation \( u_I \) defined through (8) by the approximation \( u_{II} \) defined through

\[
-\nabla \cdot a \nabla u_{II} = \nabla \cdot g \quad \text{in} \quad Q_L, \quad u_{II} = (1 + \phi_i \partial_i)\tilde{u}_h \quad \text{on} \quad \partial Q_L,
\]

(10)
Table 1 Approximations to solution $u$ of (2) using various boundary conditions

| Approximation | Description |
|---------------|-------------|
| $u_0$         | (3) Zero boundary condition |
| $u_I$         | (8) b.c. given by homogenized solution in whole space |
| $u_{III}$     | (10) b.c. with gradient correction |
| $u_{III}$     | (12) b.c. with gradient correction and dipole correction |
| $u^{(L)}$     | (25) Analogous to $u_{III}$ but using only $a_{|Q_{2L}}$ (see Theorem 1 below) |

As our numerical experiments in Sect. 2 show, this generically yields no improved scaling of the error in $L$.

**Dipoles**. The problem with all three approaches, (3), (8), and (10), is that as soon as $\ell \ll L$, the far field of $\tilde{u}_h$ generically has the wrong dipole behavior. This phenomenon was observed in [4], where the right-hand side $g$ in (9) was replaced by $g_i(e_i + \partial_i \phi)$ in order for the gradient of the two scale-expansion $\nabla (1 + \phi_i \partial_i)\tilde{u}_h$ to be $O((\frac{\ell}{L})^d (\frac{1}{L})^\beta)$-close to $\nabla u$, for some exponent $\beta$ arbitrarily close to 1. This is the right strategy for concentrated $g$, i.e., for $\ell \sim 1$. In order to also treat a more spread right-hand side, i.e., $\ell \gg 1$, we hold on to $\tilde{u}_h$ but correct it through an $a_{\ell}$-dipole $\delta u_h$ coming from the first moments of $g_i \partial_i \phi$. In formulas, we pass to

$$u_h := \tilde{u}_h + (\int \nabla \phi_i \cdot g) \partial_i G_h,$$

(11)

where $G_h$ is the fundamental solution for $-\nabla \cdot a_h \nabla$. Hence, our more educated ansatz is to replace the approximation $u_{III}$ defined through (10) by the approximation $u_{III}$ defined through

$$-\nabla \cdot a \nabla u_{III} = \nabla \cdot g \text{ in } Q_L, \quad u_{III} = (1 + \phi_i \partial_i)u_h \text{ on } \partial Q_L.$$

(12)

Corollary 1 shows that this approximation indeed reduces the (generic) error of (3) by a factor with the desired $L$-scaling $\frac{1}{L^\beta}$. Corollary 1 relies on Lemma 1, which is a minor modification of [4].

For readers’ convenience, we summarize in Table 1 the various approximations introduced so far and the approximation $u^{(L)}$ relying only on $a_{|Q_{2L}}$, which will be introduced later in (25).

**Flux correctors** $\sigma_i$. As can be seen from Lemma 1, the prefactor comes in form of $r_*^\beta$ for some length scale $r_*$. This length scale has the interpretation that for larger scales $r \geq r_*$, the quantified sublinear growth of the correctors $\phi_i$ sets in, cf (16). The sublinear growth is quantified through the exponent $\beta$ (with $\beta$ close to 0 meaning almost linear growth, and $\beta$ close to 1 meaning almost no growth). However, for quantitative results like Lemma 1, which closely follows [4], itself inspired by [7], it is not sufficient to monitor just $\phi_i$. In fact, the harmonic vector field $e_i + \nabla \phi_i$ (the electric field in the language of electrostatics) is not just a closed 1-form; but through the flux

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2 Here and in the sequel, we will adhere to the convention that in formulas as $\nabla (1 + \phi_i \partial_i)\tilde{u}_h$, it is understood that the gradient $\nabla$ acts on everything that follows.
\( q_i := a_i (e_i + \nabla \phi_i) \) (the electric current in the language of electrostatics) it provides a closed \((d - 1)\)-form. Hence, there is not just the 0-form (a scalar potential) \( x_i + \phi_i \), or rather its correction \( \phi_i \), but there naturally is also a \((d - 2)\)-form (a vector potential in the 3-d language, or a stream function in the 2-d language), or rather its correction \( \sigma_i \), which we can write as a skew-symmetric tensor field \( \sigma_i = \{\sigma_{ijk}\}_{j=1,...,d} \). In view of (7), this correction should satisfy

\[
q_i = a_i e_i + \nabla \cdot \sigma_i ,
\]

where \( (\nabla \cdot \sigma_i)_j := \partial_k \sigma_{ijk} \). Note that by skew symmetry of \( \sigma_i \) we have \( \nabla \cdot \nabla \cdot \sigma_i = 0 \) so that (13) contains the familiar (5), as it implies \( \nabla \cdot q_i = 0 \). Clearly, (13) determines \( \sigma_i \) only up to a \((d - 3)\)-form, i.e., the freedom of the choice of a gauge. A particularly simple choice of gauge is

\[
-\Delta \sigma_{ijk} = \partial_j q_{ik} - \partial_k q_{ij} .
\]

This skew-symmetric field is common in periodic homogenization (see [14, (3.4)] for a more recent application and [13, Section 1.4] for a textbook); in qualitative stochastic homogenization [9, Lemma 1, Corollary 1] it has been shown to almost surely exist with sublinear growth:

\[
\lim_{r \to \infty} \frac{1}{r} \left( \frac{1}{B_r} \left( \int_{B_r} |\sigma_i - \int_{B_r} \sigma_i|^2 \right)^{\frac{1}{2}} \right) = 0 .
\]

**Exponent \( \beta \) and radius \( r_* \).** Loosely speaking, starting from the length scales \( r \) at which the left-hand side expressions (6) and (15) drop below a threshold only depending on \( d \) and \( \lambda \), the operator \(-\nabla \cdot a \nabla \) inherits the regularity theory of \(-\nabla \cdot a_h \nabla \), both for Schauder theory on the \( C^{1,\alpha} \) level (then the threshold depends in addition on \( 0 < \alpha < 1 \)) [9, Corollary 3], and for the Calderon-Zygmund theory on the \( \dot{H}^{1,p} \) level [9, Corollary 4]. This type of theory had been developed by Avellaneda and Lin [3] for the periodic case; Armstrong and Smart [2] were first to extent this to the random case. We however need to quantify the decay of (6) and (15) as \( r \to \infty \). In terms of the exponent \( \beta \) in an \( r^{-\beta} \)-decay, \( \beta = 1 \) obviously is the best to be achieved, and is reached in the periodic case, when \( \phi_i \) and \( \sigma_i \) are actually bounded. In the random case, \( \phi_i \) and \( \sigma_i \), even if they are stationary as is the case in \( d \geq 2 \), will not be bounded and \( \beta = 1 \) is missed by an iterated logarithm. In the case of \( d = 2 \) included here, and for generic ensembles, \( \phi_i \) and \( \sigma_i \) will grow like the square root of a logarithm. For the purpose of this paper, we choose to ignore the logarithmic scale and will just fix a \( \beta < 1 \), which we think of being very close to 1. In this sense, we miss optimality by an exponent \( 1 - \beta \ll 1 \). Once \( \beta \) is fixed, the actual decay is then conveniently quantified by a radius \( r_* \): It is the smallest radius starting from which \( \beta \)-sublinear growth of the scalar and vector correctors kick in, that is,

\[
\frac{1}{r} \left( \int_{B_r} |(\phi, \sigma) - \int_{B_r} (\phi, \sigma)|^2 \right)^{\frac{1}{2}} \leq \left( \frac{r_*}{r} \right)^\beta \text{ for all } r \geq r_* .
\]
where \((\phi, \sigma)\) is the collection of all components \(\{\phi_i, \sigma_{ijk}\}_{i,j,k}\). While the form (16) is more natural, it is only seemingly weaker than

\[
\frac{1}{r} \left( \int_{B_r} |(\phi, \sigma) - \int_{B_{r_*}} (\phi, \sigma)|^2 \right)^{\frac{1}{2}} \leq C(\beta) \left( \frac{r_*}{r} \right)^\beta
\]

for all \(r \geq r_*\), (17)
as we shall show at the beginning of the proof of Lemma 1. In [8, Theorem 1 ii)] it is shown that \(r_*\) satisfies (optimal) stretched exponential bounds even under weak correlation decay of \(\langle \cdot \rangle\), and [9] gives estimates that are optimal in terms of \(\beta\) for any correlation decay exponent of the ensemble.

Clearly, this notion of \(r_*\) singles out the origin (it can and will be defined for other bounds \(r_*(y)\), making it a stationary random field). The origin plays a special role in our analysis in two ways: It is where we want to monitor the error (on the level of the gradient) and where the right-hand side \(g\) is concentrated (on scale \(\ell\)). In view of the above-mentioned \(C^{1,\alpha}\)-regularity theory that kicks in (only) from scales \(r_*\) onwards, it is not surprising that we can localize the error (on the level of the gradients) only to scales \(R \geq r_*\), see Proposition 1 (which in Theorem 1 is expressed in terms of the proxy \(r_*^{(L)}\)). For the same reason, we need the condition \(\ell \geq r_*\) in Proposition 1.

Algorithm. The “algorithm” (12) is not admissible, since it involves solving the \(d\) whole-space problems (5). The natural idea is to replace these whole-space problems by Dirichlet problems on \(Q_{2L}\); for reasons that will become clearer later, we do the same for the \(d \times \frac{d(d-1)}{2}\) whole-space problems (14) (while having constant coefficients they feature a non-decaying right-hand side). For any coordinate direction \(i = 1, \ldots, d\), let the function \(\phi_{i}^{(L)}\) and the skew-symmetric tensor field \(\sigma_{i}^{(L)} = [\sigma_{ijk}^{(L)}]_{j,k=1,\ldots,d}\) be determined through

\[
-\nabla \cdot (e_i + \nabla \phi_{i}^{(L)}) = 0 \text{ in } Q_{2L}, \quad \phi_{i}^{(L)} = 0 \text{ on } \partial Q_{2L},
\]

\[
-\Delta \sigma_{ijk}^{(L)} = \partial_j q_{ik}^{(L)} - \partial_k q_{ij}^{(L)} \text{ on } Q_{2L}, \quad \sigma_{ijk}^{(L)} = 0 \text{ on } \partial Q_{2L},
\]

where in line with (7) we have set for abbreviation \(q_{i}^{(L)} := a(e_i + \nabla \phi_{i}^{(L)})\). While an easy calculation shows that (18) and (19) imply \(\Delta (q_{i} - \nabla \cdot \sigma_{i}) = 0\), this does in general not yield \(\int_{Q_{2L}} q_{i}^{(L)} = \nabla \cdot \sigma_{i}^{(L)}\). The latter would be automatic in case of periodic boundary conditions, in which case we would replace the homogenized coefficient \(a_h\), cf (7), by \(a_{h}^{(L)} e_i = \int_{Q_{2L}} q_{i}^{(L)}\). In our (more ambitious) case of Dirichlet boundary conditions, we pick a mask \(\hat{\omega}\) of an averaging function with

\[
\omega(x) = \frac{1}{L^d} \hat{\omega}(\frac{x}{L}) \quad \text{with } \hat{\omega} \text{ supported in } Q_1, \quad \int \hat{\omega} = 1, \quad \text{and } \nabla \cdot \hat{\omega} \text{ bounded},
\]

and set

\[
a_{h}^{(L)} e_i := \int \omega q_{i}^{(L)};
\]

(21)
it is a consequence of (35) in Lemma 2 that $a_h^{(L)}$ is elliptic. Alternative efficient estimator for the homogenized coefficient was considered in the recent work [15], which is also based on quantitative theory of stochastic homogenization.

We now make the corresponding changes on the level of $\tilde{u}_h$ and $u_h$: We substitute $\tilde{u}_h$ defined in (9) by the decaying solution $\tilde{u}^{(L)}_h$ of

$$-\nabla \cdot a_h^{(L)} \nabla \tilde{u}^{(L)}_h = \nabla \cdot g$$  \hspace{1cm} (22)

and $u_h$ defined in (11) by $u_h^{(L)}$ defined through

$$u_h^{(L)} := \tilde{u}^{(L)}_h + \left( \int \nabla \phi_i^{(L)} \cdot g \right) \partial_i G_h^{(L)},$$ \hspace{1cm} (23)

where $G_h^{(L)}$ denotes the fundamental solution of $-\nabla \cdot a^{(L)}_h \nabla$ acting on $\mathbb{R}^d$. Now, only $a_{|Q_{2L}}$ enters the definition of $u^{(L)}$ in (25) below.

As mentioned above, Theorem 1 is a mixture of an a priori and an a posteriori result: Through the scale $L_0$, which only depends on the dimension $d \geq 2$, the ellipticity ratio $\lambda > 0$, the sublinear growth exponent $\beta < 1$, and the stretched exponential exponent $s < 2(1 - \beta)$, Theorem 1 provides an a priori estimate on the probability that the random pick of a realization $a$ is so bad that the algorithm fails. Through (24), which characterizes the scale $r_*(L)$ in a computable fashion, it provides an estimate on the constant $(r_*(L))^\beta$ in the $1/L^\beta$-improvement of the error estimate (26). We think of (26) as an a posteriori estimate, since it relies on an auxiliary computation based on the given realization $a$.

**Theorem 1** Let $\langle \cdot \rangle$ be a stationary ensemble of $\lambda$-uniformly elliptic coefficient fields $a$, cf (1), that is of unit range. Then for any exponents $\beta \in (0, 1)$ and $s \in (0, 2(1 - \beta))$, there exists a scale $L_0 = L_0(d, \lambda, \beta, s)$ so that for any scale $L \geq L_0$, with probability $1 - \exp(-\left(\frac{L}{L_0}\right)^s)$ a realization $a$ has the following property:

Let $\{\phi_i^{(L)}\}_{i=1,\ldots,d}$, $\sigma_i^{(L)} = [\sigma_{ijk}^{(L)}]_{i,j,k=1,\ldots,d}$, and $a_h^{(L)}$ be defined through (18), (19), and (21). Then there exists a scale $1 \leq r_*(L) \leq L$ such that $(\phi^{(L)}, \sigma^{(L)})$ is quantitatively sublinear in the sense of

$$\frac{1}{r} \left( \int_{Q_r} |(\phi^{(L)}, \sigma^{(L)})|^2 \right)^{1/2} \leq \left( \frac{r_*(L)}{r} \right)^\beta \text{ for } 2L \geq r \geq r_*(L).$$  \hspace{1cm} (24)

Let $g$ be of the form (4) for some scale $\ell \in [r_*(L), L]$ and mask $\hat{g}$, and let $u$ be the decaying solution of (2). Consider the solution $u^{(L)}$ of

$$-\nabla \cdot a \nabla u^{(L)} = \nabla \cdot g \text{ in } Q_L, \quad u^{(L)} = (1 + \phi_i^{(L)} \partial_i) u_h^{(L)} \text{ on } \partial Q_L.$$

\hspace{1cm} (25)

\[\text{Springer}\]
where \( u_h^{(L)} \) is defined through (22) and (23). Then we have

\[
\left( \frac{1}{2} \int_{B_R} |\nabla (u^{(L)} - u)|^2 \right)^{\frac{1}{2}} \leq C \left( \frac{\ell}{L} \right)^d \left( \frac{r_*}{L} \right) \beta \text{ for } L \geq R \geq r_*^{(L)}
\]

with a constant \( C \) that only depends on \( d, \lambda, \beta, \) and the Hölder norm of \( \hat{\nabla} \hat{g} \) as well as the supremum norm of \( \hat{\nabla} \hat{\omega} \).

Theorem 1 has three ingredients, the deterministic a priori error estimate provided by Proposition 1, the stochastic ingredient Lemma 4, and the deterministic Lemma 3, that allows to pass from an a priori to an a posteriori error estimate. We say that Proposition 1 provides an \textit{a priori} and deterministic error estimate since it is formulated in terms of \( r_* \) characterizing the sublinear growth of the extended corrector \((\phi, \sigma)\), cf (16). In addition, it starts from a given uniformly elliptic coefficient field \( a \), which might but does not have to be a realization under \( \langle \cdot \rangle \). The only assumption on the Dirichlet proxy \((\phi(L), \sigma(L))\) is that it is well-behaved on the large scale \( 2L \), cf (27), but not necessarily on smaller scales as in (24).

\textbf{Proposition 1} Consider a given \( \lambda \)-uniformly elliptic coefficient field \( a \) on \( \mathbb{R}^d \), cf (1). Suppose that there exists a tensor \( a_h \) and, for \( i = 1, \ldots, d \), a scalar field \( \phi_i \) and a skew-symmetric tensor field \( \sigma_i \) such that (13) holds. Suppose that for given \( \beta \in (0, 1) \) there exists a radius \( r_* \) such that (16) holds.

For given \( L \geq r_* \), let \( \phi_i^{(L)}, \sigma_i^{(L)} \), and \( a_h^{(L)} \) be defined through (18), (19), and (21). We assume that

\[
\frac{1}{2L} \int_{Q_{2L}} |(\phi^{(L)}, \sigma^{(L)})|^2 \leq \left( \frac{r_*}{2L} \right) \beta.
\]

Given \( \hat{g} \) and \( \ell \in [r_*, L] \) let \( u \) and \( u^{(L)} \) be defined as in Theorem 1. Then we have

\[
\left( \frac{1}{2} \int_{B_R} |\nabla (u^{(L)} - u)|^2 \right)^{\frac{1}{2}} \leq C \left( \frac{\ell}{R} \right)^d \left( \frac{r_*}{R} \right) \beta \text{ for } R \geq r_*.
\]

where the constant \( C \) is of the same type as in Theorem 1.

The following lemma is the key ingredient for Proposition 1; it shows that indeed the multipole has to be corrected in the sense of (11). Its proof essentially follows [4, Theorem 0.2].

\textbf{Lemma 1} Let \( a, a_h, \phi_i, \sigma_i, \beta, \) and \( r_* \) be as in Proposition 1. Let \( g \) be of the form (4) for some \( \ell \geq r_* \) and \( \hat{g} \), and let \( u \) be the decaying solution of (2). Let \( u_h \) be defined through (9) and (11). Then we have

\[
\left( \frac{1}{2} \int_{B_R^c} |\nabla (u - (1 + \phi_i \partial_i) u_h)|^2 \right)^{\frac{1}{2}} \leq C \left( \frac{\ell}{R} \right)^d \left( \frac{r_*}{R} \right) \beta \text{ for } R \geq r_*.
\]
where $\phi_i$ is normalized through
\[ \int_{B_{r_*}^c} \phi_i = 0. \]  

Here $C$ is a constant of the same type as in Theorem 1. Furthermore $\int_{B_R^c}$ is the abbreviation of $\frac{1}{K^d} \int_{B_R^c}$, where $B_{r_*}^c$ is the complement of $B_r$.

Equipped with Lemma 1, we may assess the effect of a computational domain $Q_L$ endowed with the Dirichlet conditions given by $(1 + \phi_i \partial_i)u_h$, cf (12).

**Corollary 1** Let $a, a_h, \phi_i, \sigma_i, \beta$, and $r_*$ be as in Proposition 1. Let $g$ be of the form (4) for some $\ell \geq r_*$ and $\hat{g}$, and let $u$ be the decaying solution of (2).

For $L \geq \ell$ let $u_{III}$ be the solution of the Dirichlet problem (12). Then we have
\[ \left( \int_{B_R} |\nabla (u_{III} - u)|^2 \right)^{\frac{1}{2}} \leq C \left( \frac{\ell}{L} \right)^d \left( \frac{r_*}{L} \right)^\beta \text{ for } L \geq R \geq r_*, \] where $C$ is of the same type as in Theorem 1.

In order to pass from Corollary 1 to Proposition 1, that is, from $u_{III}$ to $u^{(L)}$, we need to replace $(\phi, \sigma)$ by the computable $(\phi^{(L)}, \sigma^{(L)})$.

**Lemma 2** Let $a, a_h, \phi_i, \sigma_i$ be as in Proposition 1. Suppose that for given $\beta \in (0, 1)$ there exists a radius $r_*$ such that (16) holds in the weaker form of
\[ \frac{1}{2L} \left( \int_{Q_{2L}} |(\phi, \sigma) - \int_{Q_{2L}} (\phi, \sigma)|^2 \right)^{\frac{1}{2}} \leq \left( \frac{r_*}{2L} \right)^\beta \] for given $L \geq r_*$. Let $\phi_i^{(L)}, \sigma_i^{(L)}$, and $a_h^{(L)}$ be defined through (18), (19), and (21). Suppose that (27) holds. Then we have
\[ \left( \int_{Q_{2L}} |\nabla (\phi^{(L)} - \phi)|^2 \right)^{\frac{1}{2}} \leq C \left( \frac{r_*}{L} \right)^\beta, \] (33)
\[ \left( \int_{Q_{2L}} |\nabla (\sigma^{(L)} - \sigma)|^2 \right)^{\frac{1}{2}} \leq C \left( \frac{r_*}{L} \right)^\beta, \] (34)
and
\[ |a_h^{(L)} - a_h| \leq C \left( \frac{r_*}{L} \right)^\beta, \] (35)
where $C$ is of the same type as in Theorem 1.

As mentioned, the following Lemma 3 allows to pass from the deterministic *a priori* estimate of Proposition 1 to the deterministic *a posteriori* estimate of Theorem 1. It shows that sublinear growth of $(\phi, \sigma)$ on scales $R \geq L$ with (some) pre-factor $(r_*)^\beta$, cf (36), and sublinear growth of the Dirichlet proxy $(\phi^{(L)}, \sigma^{(L)})$ on scales $2L \geq r \geq r_*^{(L)}$, cf (37), implies sublinear growth of $(\phi, \sigma)$ on all scales $r \geq r_*^{(L)}$, that is, $r_* \lesssim r_*^{(L)}$.  

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Lemma 3 Let $a, a_h, \phi, \sigma$ be as in Proposition 1. For given $L \geq r_*$, let $a_h^{(L)}$, $\phi_i^{(L)}$, and $\sigma_i^{(L)}$ be defined through (21), (18), and (19). We assume that for a given $\beta \in (0, 1)$ there exists a scale $r_*^{(L)} \leq L$ such that we have on the one side

$$\frac{1}{R} \left( \int_{B_R} |(\phi, \sigma) - \int_{B_R} (\phi, \sigma)|^2 \right)^{\frac{1}{2}} \leq \left( \frac{r_*^{(L)}}{R} \right)^{\beta} \quad \text{for} \quad R \geq L$$

and on the other side

$$\frac{1}{r} \left( \int_{B_r} |(\phi^{(L)}, \sigma^{(L)})|^2 \right)^{\frac{1}{2}} \leq \left( \frac{r_*^{(L)}}{r} \right)^{\beta} \quad \text{for} \quad L \geq r \geq r_*^{(L)}.$$ 

Then we have

$$\frac{1}{r} \left( \int_{B_r} |(\phi, \sigma) - \int_{B_r} (\phi, \sigma)|^2 \right)^{\frac{1}{2}} \leq C \left( \frac{r_*^{(L)}}{r} \right)^{\beta} \quad \text{for} \quad r \geq r_*^{(L)}$$

with the constant $C$ being of the same type as in Theorem 1.

The only stochastic ingredient, which we “take from the shelf”, for Theorem 1 is part ii) of the following lemma; part i) is needed in the argument for Theorem 2 below. Lemma 4 provides stochastic bounds on the (extended) corrector $(\phi, \sigma)$, and essentially amounts to saying that it is bounded with overwhelming probability in $d > 2$ and almost so in $d = 2$ (the form of the statement of Lemma 4 is marginally weakened by not distinguishing the cases $d = 2$ and $d > 2$). By now, there are several approaches to such a result: The (historically) first result of this type is [10, Proposition 2.1], and is based on functional inequalities (at first in case of a discrete medium; see [12, Proposition 1] for an extension to the continuum case) and thus (indirectly) relies on an underlying product structure of the ensemble $(\cdot)$ (eg a Gaussian field or Poisson point process like in Definition 1 below). Here, we follow a more recent approach that is based on a finite, say unit, range assumption. There are two possible references for this second approach: [1] based on the variational approach of [2], and [11], based on a semi-group approach. For reasons of familiarity, we opt for the second reference, which also has the advantage of treating $\sigma$ next to $\phi$.

Lemma 4 Let $(\cdot)$ be a stationary ensemble of $\lambda$-uniformly elliptic coefficient fields, cf (1), that has unit range of dependence. Then for every $i = 1, \ldots, d$, there exist a (random) scalar field $\phi_i$ and a (random) skew symmetric tensor field $\sigma_i$ such that the gradient fields $\nabla \phi_i$ and $\nabla \sigma_i$ are stationary, of finite second moments, and of vanishing expectation, and such that (13) (and thus (5)) and (14) hold. Moreover,

(i) For every exponent $0 < \beta < 1$ there exists a (random) radius $r_* \in (0, \infty)$ such that (16) holds and which satisfies

$$\langle \exp(r_*^s) \rangle \leq C$$

for any exponent $0 < s < 2$ with $C = C(d, \lambda, \beta, s)$. 

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(ii) For every exponents $0 < \beta < 1$ and $0 < s < 2(1 - \beta)$, there exists a scale $L_0 = L_0(d, \lambda, \beta, s)$ such that for every $L \geq L_0$ the statement

$$
\frac{1}{R} \left( \int_{B_R} |(\phi, \sigma) - \int_{B_R} (\phi, \sigma)|^2 \right)^{\frac{1}{2}} \leq \left( \frac{1}{R} \right)^{\beta} \text{ for all } R \geq L
$$

fails with probability smaller than $\exp\left(-\left(\frac{L}{L_0}\right)^s\right)$.

### 1.2 Optimality and Lower Bound

Our second main result states that the fluctuations of $\nabla u(0)$, when conditioned on the coefficient field inside the ball $B_L$, are at least of the order of $(\ell L)^{d} (\ell L)^{-d}$. The first factor is a (deterministic) consequence of the fact that $\nabla u(0)$, in view of the right-hand side supported in $B_L$, is not very sensitive in the coefficients in $B_L^c$. The second factor scales as the inverse of the square root of the volume of the annulus $B_{2L} - B_L$, and thus has a CLT-flavor to it. More precisely, instead of $\nabla u(0)$ we monitor smooth averages of $\nabla u$ on a sufficiently large but order-one scale $R$ near the origin (which amounts to a strengthening of the lower bound result). Similarly to (20) we fix a (universal) averaging function on $B_R$ through

$$
\omega(x) = \frac{1}{R^d} \hat{\omega}\left(\frac{x}{R}\right) \text{ with } \hat{\omega} = 0 \text{ in } B_1^c, \quad \int \hat{\omega} = 1, \quad \hat{\nabla} \hat{\omega} \text{ bounded}
$$

and consider $\int \omega \nabla u$. We establish this lower bound on the fluctuations under convenient assumptions on the ensemble. The lower bound should also hold for general ensembles with finite-range dependence, the proof of which will be left for future works. We also restrict ourselves to elliptic equations (instead of elliptic systems) for the lower bound.

**Definition 1** Let $\langle \cdot \rangle$ denote the distribution of the Poisson point process $X$ in $\mathbb{R}^d$ of unit intensity. We assume that there exists a measurable map $A$ from the space of point configurations into the space of $\lambda$-uniformly elliptic coefficient fields; in other words, we consider the ensemble of such coefficient fields given by

$$
a(x) = A(x, \cdot)
$$

and thus make the following assumptions on $A$: For all points $x \in \mathbb{R}^d$, point configurations $X, X' \subset \mathbb{R}^d$, shift vectors $z \in \mathbb{R}^d$ we impose

- shift-invariance, that is,

$$
A(z + x, z + X) = A(x, X),
$$

- locality, that is,

$$
A(x, X) = A(x, X') \text{ provided } X \cap B_1(x) = X' \cap B_1(x),
$$
strict monotonicity, that is,
\[ X \subset X' \implies A(x, X) \leq A(x, X'), \]  
and the equality holds if and only if \( X = X' \). Here the inequality on the right-hand side of (44) is understood as inequality between symmetric matrices. Note that as a consequence, we have \( A(x, \emptyset) < a_h \).

In particular, \( \langle \cdot \rangle \) is stationary and of range unity, so that by Lemma 4 i), for every \( \beta < 1 \) and \( s < 2 \), there exists \( r_* \) with (16) and such that
\[
\langle \exp(r_*^2) \rangle \leq C(d, \lambda, \beta, s). \tag{45}
\]

Given the ensemble, we consider the conditional expectations: For a set \( D \subset \mathbb{R}^d \), we denote by \( \langle \cdot | D \rangle \) the expectation when conditioned on \( a|_D \). This allows us to measure fluctuations in terms of the variance of \( \int \omega \nabla u \) conditioned on \( a|_{B_L} \). The theorem below establishes a lower-bound for such fluctuations, on the order of \( (\ell L)^d \left( \frac{1}{L} \right)^d \). This means that, with only the knowledge of the elliptic coefficient inside \( B_L \) (such as considered in our algorithm), there is no hope to approximate \( \hat{\omega} \nabla u \) with better accuracy.

**Theorem 2** Let the ensemble be as in Definition 1. Consider the solution \( u \) of (2) with right-hand side of the form (4) for a given \( \hat{g} \). Then there exists a radius \( R \) such for all scales \( L, \ell \) with \( C \leq \ell \leq \frac{1}{C} L \) we have
\[
\langle \left| \int \omega \nabla u - \langle \int \omega \nabla u | B_L \rangle \right|^2 \rangle^{\frac{1}{2}} \geq \frac{1}{C} \left( \frac{\ell}{L} \right)^d \left( \frac{1}{L} \right)^d \int \hat{g}, \tag{46}
\]
where \( \omega \) is defined as in (41). Here the radius \( R \) and the constant \( C \) depend on the ensemble, on the sup norm of \( \nabla \omega \), and on the Hölder norm of \( \hat{\omega} \).

Like Theorems 1 and 2 relies on a purely deterministic result, namely Proposition 2 which is of independent interest. Proposition 2 monitors the effect of a “defect” in the medium \( a \); more precisely, we consider the medium \( a' \) given by
\[
a' = a_0 \text{ in } B_R(y) \quad \text{and} \quad a' = a \text{ outside of } B_R(y), \tag{47}
\]
where \( a_0 \) is some other \( \lambda \)-uniformly elliptic coefficient field, cf (1), and \( y \) is some point and \( R \) some radius. We are interested in the effect on the solution \( u \) of our whole-space problem (2) with localized right-hand side \( g \). Hence, we compare \( u \) to \( u' \) given by the decaying solution of
\[
-\nabla \cdot a' \nabla u' = \nabla \cdot g. \tag{48}
\]
Here, we think of \( B_R(y) \) as being far from the origin where \( g \) is localized, cf (4). Proposition 2 states that to leading order, the effect of the inclusion is captured by its
effect on the level of the homogenized coefficients. More precisely, we consider the coefficient field $a_h'$ given by

$$a_h' = a_0 \text{ in } B_R(y) \text{ and } a_h' = a_h \text{ outside of } B_R(y).$$

(49)

Recall $a_h$ denotes the homogenized coefficient matrix. Consider the decaying solution $u_h'$ of

$$-\nabla \cdot a_h' \nabla u_h' = -\nabla \cdot a_h \nabla u_h,$$

(50)

where $u_h$ is given by (11). Proposition 2 states that indeed $u' - u \approx u_h' - u_h$. As for the other results, Proposition 2 does so on the level of the gradient (and thus involves the two-scale expansion) and in a localized way. It is the positivity of the exponent $\alpha$ in (51) that ensures that indeed to leading order, $\nabla(u_h' - u_h)$ behaves like $\partial_i (u_h' - u_h)(e_i + \nabla \phi_i)$, since, as a classical argument shows, $\nabla(u_h' - u_h)(0)$ generically scales as $(\frac{\alpha}{|y|})^d (\frac{R}{|y|})^d$.

Loosely speaking, Proposition 2 states that sensitivity analysis (the dependence of a solution on the coefficient field) and homogenization commute.

**Proposition 2** Consider a given $\lambda$-uniformly elliptic coefficient field $a$ on $\mathbb{R}^d$, cf (1). Suppose that there exists a tensor $a_0$ and, for $i = 1, \ldots, d$, a scalar field $\phi_i$ and a skew-symmetric tensor field $\sigma_i$ such that (13) holds. Suppose that for given $\beta \in (0, 1)$ there exists a radius $r_*(y)$ such that (16) holds with the origin replaced by some point $y$.

We are given another $\lambda$-uniformly elliptic coefficient field $a_0$, cf (1), and consider $a'$ and $a_h'$ given through (47) and (49). Consider $u, u', u_h$ and $u_h'$ defined through (2), (48), (11), and (50).

Under the assumption that $\ell \geq r_*(0)$ and provided the radius $R$ satisfies $\frac{1}{2} |y| \geq R \geq \max\{r_*(0), r_*(y)\}$ we have

$$\left( \int_{B_R} |\nabla((u' - u) - (1 + \phi_i \partial_i)(u_h' - u_h))|^2 \right)^{\frac{1}{2}} \leq C \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d \left( \frac{r_*(y)}{R} \right)^\beta + \left( \frac{r_*(y)}{R} \right)^\alpha$$

(51)

with an exponent $\alpha > 0$ that only depends on $d, \lambda$ and $\beta$, and a constant $C < \infty$ that in addition depends on the Hölder norm of $\hat{g}$.

More general defects and their effect on the corrector have been investigated in [6, Section 3]. While that paper focusses on periodic media, part of the results hold for a general medium just endowed with sublinear correctors, which is the spirit this paper is written in. Proposition 2 relies on two ingredients: The first ingredient is Lemma 6 which to leading order characterizes $u' - u$ in terms of the homogenized solution $u_h$ and a tensor $\delta a_{ij}$, defined in (57), that captures how the corrector $\phi$ is affected by the defect. Since we are interested in $u' - u$ near the origin, the Green’s function $G_h$ of the homogenized operator is also involved. Since we characterize $u' - u$ on the level of gradients, the estimate involves the two-scale expansion and thus $\nabla \phi_k$ (see (59)).
The second ingredient is Lemma 5 that establishes a version of Proposition 2 with the general solution \( u \) of an equation involving \(-\nabla \cdot a \nabla\) replaced by a specific one, namely the corrector \( \phi_i \). With help of this lemma, we establish Corollary 2 that characterizes the tensor \( \delta a_{ij} \) in terms of its counterpart \( \delta a_h \).

How is the solution of the corrector equation (5) affected by replacing \( a \) by \( a' \), cf (47)? We denote by \( \phi'_i \) a solution of

\[-\nabla \cdot a'(e_i + \nabla \phi'_i) = 0 \tag{52}\]

that behaves as \( \phi_i \) at infinity. More precisely, let \( \delta \phi_i \) be a sublinear solution to

\[-\nabla \cdot a' \nabla \delta \phi_i = \nabla \cdot (a' - a)(e_i + \nabla \phi_i), \]

where the right-hand side is compactly supported. Then \( \phi'_i \) is defined by \( \phi'_i = \phi_i + \delta \phi_i \) (in particular, \( \nabla (\phi'_i - \phi_i) = \nabla \delta \phi_i \) is square integrable). Lemma 5 compares \( \phi'_i \) to \( \phi'_{ih} \), the decaying solution of

\[-\nabla \cdot a'_h(e_i + \nabla \phi'_{ih}) = 0. \tag{53}\]

More precisely, Lemma 5 compares the two solutions on the level of the potentials and of flux averages. Similarly to (20) we fix a (universal) averaging function on \( B_{2R}(y) \) through

\[
\omega(y)(x) = \frac{1}{R^d} \hat{\omega}(\frac{x - y}{R}) \quad \text{with} \quad \hat{\omega} = \frac{1}{2|B_1|} \quad \text{in} \quad B_1, \quad \hat{\omega} = 0 \quad \text{outside} \quad B_2, \quad \int \hat{\omega} = 1, \quad \nabla \hat{\omega} \text{ bounded.} \tag{54}\]

In terms of arguments, Lemma 5 takes inspiration from [9, Proposition 2].

**Lemma 5** We make the same assumptions on \( a, a_h, \phi_i, \sigma_i, r_*(y) \) as in Proposition 2. Let \( a_0, a', \) and \( a'_h \) be as in Proposition 2. For \( i = 1, \ldots, d \) we consider the solutions \( \phi'_i \) and \( \phi'_{ih} \) of (52) and (53). Then for \( R \geq r_*(y) \) and \( \omega_y, \) cf (54), we have

\[
\frac{1}{R} \left( \int_{B_{2R}(y)} (\phi'_i - \phi'_{ih})^2 \right)^{\frac{1}{2}} \leq C \left( \frac{r_*(y)}{R} \right)^{\alpha}, \tag{55}\]

\[
\left| \int \omega_y a'(e_i + \nabla \phi'_i) - \int \omega_y a'_h(e_i + \nabla \phi'_{ih}) \right| \leq C \left( \frac{r_*(y)}{R} \right)^{\alpha}, \tag{56}\]

where \( \alpha \) and \( C \) only depend on \( d, \lambda \) and \( \beta \).

As we shall see in Lemma 6, it is the tensor

\[
\delta a_{ij} := \int (e_j + \nabla \phi'_j) \cdot (a' - a)(e_i + \nabla \phi_i) \tag{57}\]
that to leading order governs the sensitivity of the solution operator of $-\nabla \cdot a \nabla$ under changing the coefficients from $a$ to $a'$. It is an easy consequence of Lemma 5 that to leading order, we may replace the medium described by $a$ (in both $a$ itself and in its perturbation $a'$, and the corresponding correctors) by its homogenized version $a_h$ in the expression (57).

**Corollary 2** Under the assumptions of Lemma 5 we have for $R \geq r_*(y)$

$$\left| \frac{1}{R^d} \int (e_j + \nabla \phi'_j) \cdot (a' - a)(e_i + \nabla \phi_i) \right| - \frac{1}{R^d} \int (e_j + \nabla \phi'_j) \cdot (a'_h - a_h)e_i \right| \leq C\left(\frac{r_*(y)}{R}\right)^\alpha,$$

where $\alpha$ and $C$ are as in Lemma 5.

The following lemma heavily relies on [4], indirectly through expanding on Lemma 1, but also more directly for localization.

**Lemma 6** We make the same assumptions on $a$, $a_h$, $\phi_i$, $\sigma_i$, $r_* = r_*(0)$, $r_*(y)$ as in Proposition 2. Let $a_0$, $a'$, $a'_h$, $u$, and $u'$ be as in Proposition 2. Then for $R \geq \max\{r_*, r_*(y)\}$ we have

$$\left(\int_{B_R} |\nabla (u' - u) - \partial_i u_h(y)\delta a_{ij} \partial_j G_h(-y)(e_k + \nabla \phi_k)|^2\right)^{\frac{1}{2}} \leq C\left(\frac{\ell}{|y|}\right)^d \left(\frac{R}{|y|}\right)^{d+\beta},$$

where $G_h$ is the fundamental solution for the homogenized operator $-\nabla \cdot a_h \nabla$ and $C$ depends on $d$, $\lambda$, $\beta$ and the Hölder norm of $\hat{g}$.

Equipped with Lemma 6 and Corollary 2, we now may show that $\nabla u(0)$, or rather a smooth average $\int \omega \nabla u$ of $\nabla u$ near the origin, substantially reacts to a change in the medium at $B_R(y)$. This reaction is characterized in terms of the tensor

$$\delta a_{hij} := \int (e_j + \nabla \phi'_{jh}) \cdot (a'_h - a_h)e_i$$

appearing in (58). In fact, Theorem 2 is not inferred from Proposition 2, but rather directly from Corollary 2 and Lemma 6, which we combine for that purpose to

**Corollary 3** We make the same assumptions on $a$, $a_h$, $\phi_i$, $\sigma_i$, $r_* = r_*(0)$, $r_*(y)$ as in Proposition 2. Let $a_0$, $a'$, $a'_h$, $u$, and $u'$ be as in Proposition 2. Then provided $R \leq \ell \leq |y|$ we have
where $\omega$ is defined in (41). Here $C$ depends on $d$, $\lambda$, $\beta$, and the $L^2$-norm of $\hat{\nabla} \hat{g}$.

The first right-hand side summand in (61) comes from Lemma 6, the second one from eliminating $\nabla \phi_k$ in (59) with help of $\omega$, the third one comes from Corollary 2, the fourth comes from the dipole expansion of $u_h$, and the fifth from simplifying the dipole moment. The last factor on the right-hand side of (61) arises because we avoid any smallness condition on $\max \{|r_s, r_s(y)|\}$ in terms of $R$ or $\ell$.

Under our assumptions on the ensemble, cf Definition 1, and equipped with the deterministic result of Corollary 3, we obtain a lower bound on the variance of the expectation of $\int \omega \nabla u$ conditioned on the Poisson process restricted to $B_R(y)$. Note that we first average over the Poisson process in the complement $B_{R}^c(y)$, and then consider the variance with respect to the Poisson process on $B_{R}(y)$. As opposed to the opposite order, which would amount to a weaker result, in particular too weak for the purpose of Theorem 2. This lower bound holds provided the order-one radius $R$, which also governs the average through $\omega$, cf (41), is sufficiently large. The lower bound is optimal in terms of the scaling in the ratios of the length scales 1 (the range $\ell$ of dependence of the random coefficient field $a$), $\ell$ (the scale of the source $g$), and $|y|$ (the distance between the source and the site at which the variance is probed).

**Lemma 7** Under the assumptions of Theorem 2 there exists an $R$ so that provided $C \leq \ell \leq \frac{1}{2} \frac{1}{\ell} y|$ we have

$$\left|\left\langle \int \omega \nabla u | B_R(y) \right\rangle - \left\langle \int \omega \nabla u \right\rangle \right|^2 \geq \frac{1}{C} \left( \frac{\ell}{|y|} \right)^d \left( \frac{1}{|y|} \right)^d | \int \hat{g} |,$$

where $R$ and $C$ are as in Theorem 2.

The following auxiliary lemma will be used in the proof of Lemma 6 and states that the random (scalar) field $r_s = r_s(a, x)$ is not very sensitive to local changes in the coefficient field $a$. Here we think of $r_s$ as being defined as the minimal radius with the property (16); we denote by $r_s'$ the radius for the medium $a'$, and by $r_s(y)$, $r_s'(y)$ the corresponding radii with the origin replaced by the point $y$ (in the neighborhood of which $a'$ differs from $a$). Here, and in analogy to $\phi_k'$, we think of $\sigma_{ij}'$ as the solution of $-\Delta \sigma_{ij}' = \partial_j q_{i}' - \partial_k q'_{ij}$, where $q_{i}' := a'(e_i + \nabla \phi_i')$, that behaves at $\infty$ as $\sigma_{ij}$ (i.e., $\int |\nabla (\sigma_{ij}' - \sigma_{ij})|^2 < \infty$). We note that by construction we have $\Delta (q_{i}' - \nabla \cdot \sigma_i') = 0$, so that by the decay of $q_{i}' - q_{i}$ and $\nabla \sigma_i' - \nabla \sigma_i$, the relation (13) is preserved, that is, $q_{i}' = a_h e_i + \nabla \cdot \sigma_{ij}$.

**Lemma 8** Under the assumptions of Lemma 6, we have

$$r_s'(y) + r_s' \leq C(d, \beta)(r_s(y) + r_s + R).$$

$$\Box$$
2 Numerical Results

In our numerical tests, we consider a random ensemble according to Definition 1. Let $X$ be the Poisson point process on $\mathbb{R}^2$ with unit intensity; the coefficient field $a$ is given by

$$a(x) = \max_{\xi_i \in X} \{ \varphi(x - \xi_i) \} + \frac{1}{2},$$  \hspace{1cm} (64)$$

where $\varphi$ is a bump function supported in $B_{1/2}$

$$\varphi(\hat{x}) = \exp\left( -\frac{1}{1 - 4|\hat{x}|^2} \right).$$  \hspace{1cm} (65)$$

The elliptic PDE is

$$- \nabla \cdot a \nabla u = f$$  \hspace{1cm} (66)$$

where the right-hand side is given by

$$f(x) = x_2 \exp\left( -\frac{5}{5 - |x|^2} \right)$$  \hspace{1cm} (67)$$

and thus $f$ is compactly supported inside the ball $B_{\sqrt{5}}$ with average 0. It is clear that we can rewrite $f = \nabla \cdot g$ with $g$ supported in $B_{\sqrt{5}}$.

To numerically approximate the solution of the Dirichlet problems (18), (19), and (25), we use a standard second-order centered finite-difference scheme with mesh size $\delta x = 0.1$.

In Fig. 1, we plot the numerically obtained $a_h^{(L)}$ as a function of $L$ for 4 independent realizations of the coefficient field $a$. It shows that the Dirichlet approximation $a_h^{(L)}$ converges as $L$ increases, qualitatively validating (35) in Lemma 2.

We now consider our algorithm to approximate $\nabla u(0)$ based on the $a|_{Q_2L}$. To validate the algorithm, we check the numerical convergence rate and compare it to two approximations with a slower convergence rate. Recall that our algorithm consists of the steps:

- Solve Eq. (18) for approximate correctors $\phi_i^{(L)}$;
- calculate the homogenized coefficient $a_h^{(L)}$ as in (21);
- obtain $u_h^{(L)}$ from the homogenized equation (22) and the dipole correction (23);
- solve Eq. (25) for $u^{(L)}$. 
In comparison, we consider two other algorithms:

1. Solving the equation with a homogeneous Dirichlet boundary condition, i.e., (3), referred as “Dirichlet algorithm”;
2. Dropping the dipole correction, i.e., instead of (25), one solves (10) using approximate homogenized coefficient and correctors

\[-\nabla \cdot a \nabla u^{(L)}_{II} = \nabla \cdot g \text{ in } Q_L, \quad u^{(L)}_{II} = (1 + \phi_i^{(L)} \partial_i) \tilde{u}^{(L)}_h \text{ on } \partial Q_L.\] (68)

Thus in the boundary condition, \(u^{(L)}_h\) is replaced by \(\tilde{u}^{(L)}_h\) so the dipole correction is dropped. This will be referred as the “no dipole algorithm”.

In Fig. 2, we compare the numerical convergence for the three algorithms with two different realization of the random media. The difference \(\nabla u^{(2L)}(0) - \nabla u^{(L)}(0)\) is plotted for various \(L\) for the three algorithms. As the plot is in loglog scale, a straight line shows algebraic convergence with the slope indicating the convergence rate. We observe from the numerical results that the proposed algorithm achieves almost \(O(L^{-3})\) convergence, which is consistent with the theoretical rate \(O((\ell/L)^2(1/L)^\beta)\) with \(\beta\) close to 1. The wiggled line is due to the randomness of the coefficients added in the annulus \(Q_{4L} - Q_{2L}\) when \(L\) is doubled. We also observe that the other two algorithms have the slower rate of \(O(L^{-2})\), the main reason being that they do not capture the correct dipole in the far field. It can be also seen that by using the information from homogenization, the “no dipole algorithm” indeed reduces the error compared with the simple minded Dirichlet approximation.

Finally, we compare the error of the proposed algorithm with the uncertainty of \(\nabla u(0)\) due to the unknown coefficient outside \(Q_{2L}\). In the second row of Table 2, we compare the approximation \(\nabla u^{(L)}(0)\) with the reference solution obtained by a
Fig. 2 Numerical convergence rate of $\nabla u(0)$ for the proposed algorithm (blue cross), an approximation without the dipole correction (light blue circle), and the Dirichlet approximation (pink diamond). The two figures correspond to two independent realizations of the random medium (Color figure online)

calculation with a large domain ($L = 768$ is chosen). In the remaining three rows, we show how $\nabla u(0)$ varies when we re-sample the random medium outside $Q_{2L}$ (where again, we take $\nabla u(768)(0)$ as a proxy for $\nabla u(0)$). These thus show the sensitivity of $\nabla u(0)$ with respect to the change of the media. See also Fig. 3. We observe from the numerical results that while the error made by our algorithm is larger than the typical fluctuations of $\nabla u(0)$, the decay rate matches and the error is also comparable to the sensitivity for various $L$. 
Table 2  The error of the proposed numerical scheme compared with sensitivity of \( \nabla u(0) \) when the media is re-sampled outside \( Q_{2L} \)

| \( L = 32 \) | \( L = 64 \) | \( L = 128 \) | \( L = 256 \) |
|---|---|---|---|
| \(| \nabla u^{(L)}(0) \) \( \nabla u^{(768)}(0) |\) | \( 2.7709 \times 10^{-6} \) | \( 1.2843 \times 10^{-6} \) | \( 8.3910 \times 10^{-8} \) | \( 5.8996 \times 10^{-9} \) |
| \( \nabla u(0) \) difference with re-sampled media in \( Q_{2L}^c \) | \( 4.8919 \times 10^{-8} \) | \( 7.7095 \times 10^{-9} \) | \( 4.9692 \times 10^{-9} \) | \( 7.8225 \times 10^{-10} \) |
| \( 5.3976 \times 10^{-8} \) | \( 1.9546 \times 10^{-8} \) | \( 8.6449 \times 10^{-9} \) | \( 6.7321 \times 10^{-10} \) |
| \( 1.8771 \times 10^{-7} \) | \( 3.1458 \times 10^{-8} \) | \( 1.0232 \times 10^{-8} \) | \( 5.9754 \times 10^{-10} \) |

Reference value \(| \nabla u^{(768)}(0) | = 3.5859 \times 10^{-1} \)

Fig. 3  The error of the proposed numerical scheme (blue cross) compared with sensitivity of \( \nabla u(0) \) when the media is re-sampled outside \( Q_{2L} \) (red circle), cf. Table 2 (Color figure online)

3 Proofs

3.1 Proof of Lemma 1

Proof  We start with the remark that (16) actually implies (17). Indeed, what separates (17) from (16) is

\[
\left| \int_{B_r} (\phi, \sigma) - \int_{B_{r'}} (\phi, \sigma) \right| \lesssim r \left( \frac{r_*}{r} \right)^\beta \quad \text{for all } r \geq r' \geq r_*.
\]

(69)

Since \( \beta < 1 \), we may apply dyadic decomposition to reduce this to

\[
\left| \int_{B_r} (\phi, \sigma) - \int_{B_{r'}} (\phi, \sigma) \right| \lesssim r \left( \frac{r_*}{r} \right)^\beta \quad \text{for all } 2r' \geq r \geq r' \geq r_*.
\]
which by the triangle inequality is a consequence of (16). In view of (17), the definition of \( r_* \) is oblivious to adding a constant to \((\phi, \sigma)\), and thus without affecting (13) to upgrade (16) to

\[
\frac{1}{r} \left( \int_{B_r} |(\phi, \sigma)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r_*}{r} \right)^{\beta} \quad \text{for all } r \geq r_*. \tag{70}
\]

This is in line with (30).

For later reference we also note that

\[
\left( \int_{B_R} |e_i + \nabla \phi_i|^2 \right)^{\frac{1}{2}} \lesssim 1 \quad \text{for } R \geq r_*, \tag{71}
\]

which is a consequence of Caccioppoli’s estimate on the \(a\)-harmonic function \(x_i + \phi_i\), cf (5), and (70) in form of \((\int_{B_R} (x_i + \phi_i)^2)^{\frac{1}{2}} \lesssim R (r/R)^{\beta}\).

We now follow the steps of the proof of [4, Theorem 0.2] and like there may without loss of generality assume that \( r_* = 1 \). In analogy to Step 2 in the proof of [4, Theorem 0.2] we claim

\[
\left( \int |\nabla u|^2 \right)^{\frac{1}{2}} \lesssim \ell^d, \tag{72}
\]

and

\[
|x|^d |\nabla u_h(x)| + |x|^{d+1} |\nabla^2 u_h(x)| \lesssim \ell^d \quad \text{for all } x \neq 0. \tag{73}
\]

We furthermore claim that the \( u \) and \( u_h \) have not only vanishing “constant invariant”

\[
\int \nabla \eta_r \cdot a \nabla u = \int \nabla \eta_r \cdot a_h \nabla u_h = 0 \quad \text{for } r \geq \ell, \tag{74}
\]

but also identical “linear invariants”, that is,

\[
\int \nabla \eta_r \cdot ((x_i + \phi_i)a \nabla u - u a(e_i + \nabla \phi)) = \int \nabla \eta_r \cdot (x_i a_h \nabla u_h - u_h a_h e_i) \quad \text{for } r \geq \ell, \tag{75}
\]

where \( \eta_r(x) = \eta(\frac{x}{r}) \), and \( \eta \) is a cut-off function for \( B_1 \) in \( B_2 \). Indeed, (72) follows from the energy estimate for (2) and the form (4) of the right-hand side. By the triangle inequality, we split (73) into the corresponding estimate for \( \tilde{u}_h \) and for \( \delta u_h := \xi_i \partial_i G_h \) with \( \xi_i := \int \nabla \phi_i \cdot g \). We first turn to \( \tilde{u}_h \) defined through (9) which in view of (4) is of the form \( \nabla \tilde{u}_h(x) = \nabla \tilde{u}_h(\frac{x}{\ell}) \) with \( -\hat{\nabla} a_h \hat{\nabla} \tilde{u}_h = \hat{\nabla} \cdot \hat{g} \), so that the desired estimate follows from

\[
|\nabla \tilde{u}_h| \lesssim \frac{1}{|x|^d + 1} \quad \text{and} \quad |\nabla^2 \tilde{u}_h| \lesssim \frac{1}{|x|^{d+1} + 1},
\]

\[ Springer \]
which in turn are a consequence of standard Schauder theory based on the Hölder continuity of \( \hat{\nabla} \cdot \hat{g} \) for the near-field, and the Green’s function representation for the far-field. We now turn to \( \delta u_h \) and note that the estimates

\[
|\nabla \delta u_h| \lesssim \frac{\ell^d}{|x|^d} \quad \text{and} \quad |\nabla^2 \delta u_h| \lesssim \frac{\ell^d}{|x|^{d+1}}
\]

follow from the homogeneity of \( \nabla^2 G_h \) of degree \(-d\) and the fact that \( |\xi_i| \lesssim \int_{B_{\ell}} |\nabla \phi_i| \) (as \( \ell \geq r_* \) is assumed), Jensen’s inequality together with (71). We now turn to (74). The identity for \( u \) follows directly from (2) and the fact that \( g \) is supported in \( B_{\ell}, \) cf (4). For the same reason, the contribution to the constant invariant of \( u_h \) coming from \( \tilde{u}_h \) vanishes; the one coming from \( \delta u_h, \) which is constant in \( r > 0 \) by \( a_h \)-harmonicity (thus the name “invariant”), must vanish by its homogeneity of order \( 1 - d. \) We finally turn to (75). By (2) and (5) we learn from an integration by parts and the support condition on \( g \) that for \( r \geq \ell, \)

\[
\int \nabla \eta_r \cdot ((x_i + \phi_i)a \nabla u - u a(e_i + \nabla \phi_i)) = \int (e_i + \nabla \phi_i) \cdot g. \tag{76}
\]

Likewise, we obtain from (9) that

\[
\int \nabla \eta_r \cdot (x_i a_h \nabla \tilde{u}_h - \tilde{u}_h a_h e_i) = \int e_i \cdot g.
\]

Finally, by definition of the Green’s function \( G_h \) we have

\[
\int \nabla \eta_r \cdot (x_i a_h \nabla \partial_j G_h - \partial_j G_h a_h e_i) = \delta_{ij}.
\]

By \( u_h = \tilde{u}_h + \xi_j \partial_j G_h \) and the definition of \( \xi, \) the last two identities combine to

\[
\int \nabla \eta_r \cdot (x_i a_h \nabla u_h - u_h a_h e_i) = \int (e_i + \nabla \phi_i) \cdot g.
\]

This and (76) yield the identity (75).

Following Step 3 in the proof of [4, Theorem 0.2] (with \( \eta = 0, \) the additional multiplicative factor \( \ell^d, \) and \( \beta \) playing the role of \( 1 - \alpha \)) we consider the error in the two-scale expansion \( w := u - (1 + \phi_i \partial_i) u_h \) we claim that

\[
\left( \int_{B_{\ell}} |\nabla w|^2 \right)^{\frac{1}{2}} \lesssim \ell^d \tag{77}
\]

and that

\[
- \nabla \cdot a \nabla w = \nabla \cdot h \quad \text{with} \quad h := (\phi_i a - \sigma_i) \nabla \partial_i u_h. \tag{78}
\]
where
\[
\left(\frac{1}{r^d} \int_{B_r} |h|^2\right)^{\frac{1}{2}} \lesssim \ell^d \left(\frac{1}{r}\right)^{d+\beta} \quad \text{for } r \geq 1.
\] (79)

We first turn to (77) and note
\[
\nabla w = \nabla u - \partial_i u_h (e_i + \nabla \phi_i) - \phi_i \nabla \partial_i u_h.
\]
Hence for the \(\nabla u\)-contribution the desired estimate follows from (72) (and \(\ell \geq 1\)). According to (73), the contribution from \(\partial_i u_h (e_i + \nabla \phi_i)\) is estimated by \(\ell^d \left(\int_{B_1} |x|^{-2d} |e_i + \nabla \phi_i|^2\right)^{\frac{1}{2}}\); dividing the integral into dyadic annuli and using (71), we see that the term is \(O(\ell^d)\). Still according to (73), the contribution from \(\phi_i \nabla \partial_i u_h\) is estimated by \(\ell^d \left(\int_{B_1} |x|^{-2(d+1)} \phi_i^2\right)^{\frac{1}{2}}\); dividing also this integral into dyadic annuli and using (70), we see that it is also \(O(\ell^d)\). As \(-\nabla \cdot a \nabla u = \nabla \cdot g = \nabla \cdot a_h \nabla u_h\) (in view of (2), (9), and the \(a_h\)-harmonicity of \(G_h\) and thus \(\delta u_h\) away from the origin), the identity (78) follows by a straightforward calculation as in Step 3 of [4]; it is the merit of \(\sigma_i\) that the right-hand side can be written in divergence-form \(\nabla \cdot h\), as is well-known from the periodic case, see, e.g., [14, Lemma 3.2]. We finally turn to the estimate (79); by (73) and (1) we have \(|h| \leq \ell^d |x|^{-(d+1)}|\phi, \sigma|\). Hence, the estimate follows once more from (70) and division into dyadic annuli.

Following Step 4 and Step 5 of [4, Theorem 0.2] (still with \(\alpha\) replaced by \(1 - \beta\) and the additional factor of \(\ell^d\)) we obtain from (77), (78), and (79) that
\[
\left(\frac{1}{r^d} \int_{B_r} |\nabla w|^2\right)^{\frac{1}{2}} \lesssim \ell^d \left(\frac{1}{r}\right)^{d+\beta} \quad \text{for } r \geq 2,
\]
which (in view of \(r_* = 1\)) turns into (29). Note that the outcome of Step 5 of [4, Theorem 0.2] is worse by a log \(r\); which however can be easily avoided, cf [5, Lemma 3] with \(k = 1\) and \(r = r_* = 1\).

3.2 Proof of Corollary 1

**Proof** Let us introduce a tool we will often use, namely the \(C^{0,1}\)-estimate for an \(a\)-harmonic function \(w\) in \(B_R\), the crucial role of which was recognized in [2]. As is obvious from its characterization (16), \(r_*\) dominates, up to a multiplicative constant \(C = C(d, \lambda, \beta)\), the “minimal radius” introduced in [9] and from which on the “mean-value property”, as an estimate, holds for \(|\nabla w|^2\), see [9, Theorem 1] for the proof. We record this
\[
\int_{B_r} |\nabla w|^2 \leq C(d, \lambda, \beta) \int_{B_R} |\nabla w|^2 \quad \text{for } r_* \leq r \leq R.
\] (80)

We note that the combination of (9), (11) and (12) may be rephrased in terms of \(w := u_{III} - u\) and \(w_D := (1 + \phi_i \partial_i) u_h - u\) as
\[
-\nabla \cdot a \nabla w = 0 \quad \text{in } Q_L \quad \text{and} \quad w = w_D \quad \text{on } \partial Q_L.
\] (81)
Once we show that this implies

\[
\left( \int_{Q_L} |\nabla w|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{Q_{2L}-Q_L} |\nabla w_D|^2 \right)^{\frac{1}{2}},
\]

(82)

we see that (31) follows from (29) for \( R = L \), and a subsequent application of the \( C^{0,1} \)-estimate for \( w \) to get from \( Q_L \) to \( B_R \) for \( R \geq r^* \).

We now turn to the (very standard) argument that (81) implies (82) under the mere assumption of uniform ellipticity, cf (1). Hence by rescaling, we may without loss of generality assume \( L = 1 \). Since (81) and (82) are oblivious to additive constant we may without loss of generality assume \( \hat{\mathbf{Q}}^2 - \mathbf{Q}^1 \mathbf{w}_D = 0 \), which allows to construct an extension \( \bar{w}_D \) of \( w_D \) on \( Q^1 \) such that

\[
\left( \int_{Q^2} |\nabla \bar{w}_D|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{Q^2 - Q_1} |\nabla w_D|^2 \right)^{\frac{1}{2}}.
\]

(83)

This extension allows us to reformulate (81) as

\[-\nabla \cdot a \nabla (w - \bar{w}_D) = \nabla \cdot a \nabla \bar{w}_D \text{ in } Q_1, \quad w - \bar{w}_D = 0 \text{ on } \partial Q_1.\]

Hence (82) follows by the energy estimate on the latter, followed by a triangle inequality in \( L^2 \) yielding \( \left( \int_{Q_1} |\nabla w|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{Q_1} |\nabla \bar{w}_D|^2 \right)^{\frac{1}{2}} \), into which we insert (83). \( \square \)

### 3.3 Proof of Lemma 2

**Proof** We first turn to (33) and, for notational simplicity, drop the index \( i = 1, \ldots, d \).

By (5) and (18) we have

\[-\nabla \cdot a \nabla (\phi(L) - \phi) = 0 \text{ in } Q_{2L}, \quad \text{so that by Caccioppoli’s estimate and the triangle inequality in } L^2 \text{ we have}\]

\[
\left( \int_{Q_{\frac{3}{2}L}} |\nabla (\phi(L) - \phi)|^2 \right)^{\frac{1}{2}} \lesssim \frac{1}{L} \left( \int_{Q_{2L}} ((\phi(L) - \phi) + \int_{Q_{2L}} \phi)^2 \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{L} \left( \int_{Q_{2L}} (\phi(L))^2 \right)^{\frac{1}{2}} + \frac{1}{L} \left( \int_{Q_{2L}} (\phi - \int_{Q_{2L}} \phi)^2 \right)^{\frac{1}{2}}.
\]

Hence (33) follows from the \( \phi \)-parts in (27) and (32).

For (34), we argue in a similar way: According to (14) and (19) we have

\[-\Delta (\sigma_{jk}^{(L)} - \sigma_{jk}) = \partial_j (q_k^{(L)} - q_k) - \partial_k (q_j^{(L)} - q_j) \text{ in } Q_{2L}, \]

where we recall \( q = a(e + \nabla \phi) \), \( q^{(L)} = a(e + \nabla \phi^{(L)}) \) and thus \( q^{(L)} - q = a \nabla (\phi^{(L)} - \phi) \).

By Caccioppoli’s estimate and the upper bound on \( a \) provided by (1) we therefore obtain

\[
\left( \int_{Q_L} |\nabla (\sigma^{(L)} - \sigma)|^2 \right)^{\frac{1}{2}}
\]
\[
\begin{align*}
&\lesssim \left( \int_{Q_{3L}} |\nabla (\phi^{(L)} - \phi)|^2 \right)^{\frac{1}{2}} + \frac{1}{L} \left( \int_{Q_{2L}} |\sigma^{(L)}|^2 \right)^{\frac{1}{2}} + \frac{1}{L} \left( \int_{Q_{2L}} |\sigma - \int_{Q_{2L}} \sigma|^2 \right)^{\frac{1}{2}}. \\
\text{Hence (34) follows from (33) and the } \sigma\text{-parts of both (27) and (32).}
\end{align*}
\]

We now turn to (35) and recall our averaging function \( \omega \), cf (20). We first claim that also for the whole-space case, the effective coefficient can approximately be recovered from averaging the flux with respect to \( \omega \):

\[
|a_h e - \int \omega a(e + \nabla \phi)| \lesssim \left( \frac{r_*}{L} \right)^\beta. \tag{84}
\]

The argument for this is based on (13), which yields the identity

\[
a_h e - \int \omega a(e + \nabla \phi) = \int (\sigma - \int_{B_L} \sigma) \nabla \omega,
\]

and thus by (20) the estimate \(|a_h e - \int \omega a(e + \nabla \phi)| \lesssim \frac{1}{L} \int_{Q_L} |\sigma - \int_{B_L} \sigma|\). Hence, we obtain (84) from (32) and Jensen’s inequality. We now may conclude: By the upper bound provided in (1), we obtain from (84) and the definition (21) by the triangle inequality

\[
|(a_h^{(L)} - a_h) e| \lesssim \int_{Q_L} |\nabla (\phi^{(L)} - \phi)| + \left( \frac{r_*}{L} \right)^\beta,
\]

so that (35) follows from (33) and Jensen’s inequality. \(\square\)

### 3.4 Proof of Proposition 1

**Proof** We first compare the two solutions \( \tilde{u}_h^{(L)} \) and \( \tilde{u}_h \) of (22) and (9), respectively, and claim that

\[
|\nabla (\tilde{u}_h^{(L)} - \tilde{u}_h)| + L|\nabla^2 (\tilde{u}_h^{(L)} - \tilde{u}_h)| \lesssim \left( \frac{r_*}{L} \right)^\beta (\frac{\ell}{L})^d \text{ on } Q_L^c. \tag{85}
\]

Indeed, the difference \( w := \tilde{u}_h^{(L)} - \tilde{u}_h \) satisfies

\[
-\nabla \cdot (a_h^{(L)} \nabla w) = \nabla \cdot (a_h^{(L)} - a_h) \nabla \tilde{u}_h.
\]

The form (4) of the right-hand side \( g \) in (9) transmits to the solution \( \tilde{u}_h \): \( \nabla \tilde{u}_h(x) = \hat{\nabla} \tilde{u}_h(\frac{x}{\ell}) \), and then also to \( w \): \( \nabla w(x) = \hat{\nabla} \tilde{w}(\frac{x}{\ell}) \). Recall that \( \hat{g} \) is compactly supported in \( B_1 \) with Hölder continuous derivatives; in particular, \( \hat{g} \) is in the class of vector fields decaying as \( O(|x|^{-d}) \) with derivatives decaying as \( O(|x|^{-d-1}) \) and differences of the derivatives decaying as \( O(|x|^{-d-1-\alpha}|x-y|^\alpha) \), in line with \( \alpha \)-Hölder continuity for some \( \alpha \in (0, 1) \). This class is preserved under the constant-coefficient Helmholtz
projection (recall that $a_{h}^{(L)}$ satisfies (1)). Hence, we obtain in particular

$$|\hat{\nabla} \hat{w}| \lesssim |a_{h}^{(L)} - a_{h}| \frac{1}{(|\hat{x}| + 1)^{d}}$$

$$|\hat{\nabla}^{2} \hat{w}| \lesssim |a_{h}^{(L)} - a_{h}| \frac{1}{(|\hat{x}| + 1)^{d+1}}.$$ Translating back to the microscopic variables and using (35) in Lemma 2 we obtain

$$|\nabla w| \lesssim (r^{\ast} L)^{\beta} (|x| + L)^{d}$$

$$|\nabla^{2} w| \lesssim (r^{\ast} L)^{\beta} (|x| + L)^{d+1}.$$ from which we extract (85).

We now compare $u_{h}^{(L)}$ and $u_{h}$ defined in (23) and (11) and claim that

$$|\nabla (u_{h}^{(L)} - u_{h})| + L|\nabla^{2} (u_{h}^{(L)} - u_{h})| \lesssim (r^{\ast} L)^{\beta} (\frac{\ell}{L})^{d}$$

on $Q_{c}^{L}.$ (86)

For later purpose, we also record

$$|\nabla u_{h}^{(L)}| + L|\nabla^{2} u_{h}^{(L)}| \lesssim (\frac{\ell}{L})^{d}$$

on $Q_{c}^{L}.$ (87)

In order to pass from (85) to (86), and from (73) and (86) to (87), it remains to control the dipole contributions:

$$|\nabla (\xi_{i}^{(L)} \partial_{i} G_{h}^{(L)} - \xi_{i} \partial_{i} G_{h})| + L|\nabla^{2} (\xi_{i}^{(L)} \partial_{i} G_{h}^{(L)} - \xi_{i} \partial_{i} G_{h})| \lesssim (r^{\ast} L)^{\beta} (\frac{\ell}{L})^{d}$$

and

$$|\nabla \xi_{i} \partial_{i} G_{h}^{(L)}| + L|\nabla^{2} \xi_{i} \partial_{i} G_{h}^{(L)}| \lesssim (\frac{\ell}{L})^{d}$$

both on $Q_{c}^{L},$

where we have set for abbreviation $\xi_{i}^{(L)} := \int \nabla \phi_{i}^{(L)} \cdot g$ and (as above) $\xi_{i} = \int \nabla \phi_{i} \cdot g.$ Because of the obvious estimates on the constant-coefficient (and thus homogeneous) fundamental solution

$$|\nabla \partial_{i} G_{h}^{(L)}| + L|\nabla^{2} \partial_{i} G_{h}^{(L)}| \lesssim \frac{1}{L^{d}}$$

on $Q_{c}^{L}$

and

$$|\nabla \partial_{i} (G_{h}^{(L)} - G_{h})| + L|\nabla^{2} \partial_{i} (G_{h}^{(L)} - G_{h})| \lesssim |a_{h}^{(L)} - a_{h}| \frac{1}{L^{d}}$$

$$\lesssim (r^{\ast} L)^{\beta} \frac{1}{L^{d}}$$

on $Q_{c}^{L},$

it suffices to show

$$|\xi| \lesssim \ell^{d}$$

and

$$|\xi^{(L)} - \xi| \lesssim (r^{\ast} L)^{\beta} \ell^{d}.$$.  

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By definition of $\xi^{(L)}$, $\xi$, the form (4) of $g$, and Jensen’s inequality the latter two estimates follow from
\[
\left( \int_{B_\ell} |\nabla \phi|^2 \right)^{\frac{1}{2}} \lesssim 1 \quad \text{and} \quad \left( \int_{B_\ell} |\nabla (\phi^{(L)} - \phi)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r_*}{L} \right)^{\beta}. \tag{88}
\]
The first estimate in (88) follows from (71) since $\ell \geq r_*$. The second estimate in (88) follows from (33) in Lemma 2 and the $C^{0.1}$-estimate applied to the $a$-harmonic function $w = \phi^{(L)} - \phi$ in $Q_L$, cf (80), which requires $L \geq \ell \geq r_*$. We finally compare $u^{(L)}$ defined through (25) with $u_{III}$ defined through (12) and claim that
\[
\left( \int_{B_R} |\nabla (u^{(L)} - u_{III})|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{L} \right)^{d} \left( \frac{r_*}{L} \right)^{\beta} \quad \text{for} \quad L \geq R \geq r_*.
\]
This allows us to pass from (31) in Corollary 1 to this proposition’s statement (28). We note that because $w := u^{(L)} - u_{III}$ satisfies
\[-\nabla \cdot a \nabla w = 0 \quad \text{in} \quad Q_L, \quad w = w_D \quad \text{on} \quad \partial Q_L,
\]
where $w_D := (1 + \phi_i^{(L)} \partial_i) u_h^{(L)} - (1 + \phi_i \partial_i) u_h$. Hence, we have by (a slight adaptation of) the argument at the beginning of the proof of Corollary 1 that
\[
\left( \int_{Q_L} |\nabla w|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{Q_{\frac{L}{2}} - Q_L} |\nabla w_D|^2 \right)^{\frac{1}{2}}.
\]
We combine this with the $C^{0.1}$-estimate for $a$-harmonic functions, cf (80), so that it remains to show
\[
\left( \int_{Q_{\frac{L}{2}} - Q_L} |\nabla w_D|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{L} \right)^{d} \left( \frac{r_*}{L} \right)^{\beta}. \tag{89}
\]
We appeal to the triangle inequality in $L^2$ to split $\nabla w_D$ into the four contributions
\[
\nabla w_D = \partial_i (u_h^{(L)} - u_h)(e_i + \nabla \phi_i) + \phi_i \nabla \partial_i (u_h^{(L)} - u_h) \\
+ \partial_i (u_h - u_h^{(L)}) \nabla (\phi_i^{(L)} - \phi_i) + (\phi_i^{(L)} - \phi_i) \nabla \partial_i u_h^{(L)}.
\]
For the first contribution, (89) follows from the first part of (86) and (71). For the second contribution, (89) follows from the second part of (86) and (70). For the third contribution, we appeal to the first part of (87) and (33). For the fourth contribution we appeal once more to (70), and now also to (27), which we combine to
\[
\left( \int_{Q_{\frac{L}{2}}} (\phi_i^{(L)} - \phi_i)^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r_*}{L} \right)^{\beta}.
\]
In connection with the second part of (87), we see that also that this last contribution is controlled as stated in (89). □
3.5 Proof of Lemma 3

**Proof** As mentioned in the introduction, (18) and (19) do not ensure the analogue of (13), that is,
\[ q_i^{(L)} = a_i^{(L)} e_i + \nabla \cdot \tilde{\sigma}_i \]
on Q_L, while retaining the \( \sigma \)-part of (24) in (the seemingly weaker) form of
\[ \frac{1}{r} \left( \int_{Q_r} |\tilde{\sigma}_i| - \int_{Q_r} \tilde{\sigma}_i^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r^{(L)}}{r} \right)^{\beta} \]for \( L \geq r \geq r^{(L)} \).

To this purpose, we first argue that the “defect” \( g_i^{(L)} := q_i^{(L)} - a_i^{(L)} e_i - \nabla \cdot \sigma_i^{(L)} \), which as mentioned in the introduction is component-wise harmonic on \( Q_{2L} \) by (18) and (19), satisfies
\[ \frac{1}{r} \left( \int_{Q_r} |g_i^{(L)}|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r^{(L)}}{r} \right)^{\beta} \]for \( q_i^{(L)} \) on \( Q_L \).

Indeed, this follows immediately from the identity (13) in conjunction with the three estimates (33), (34), and (35) in Lemma 2 (applied with \( r_* \) replaced by \( r^{(L)} \), which we may because of (36) for \( R \sim L \).

We now turn to (90) and (91). In view of the definition of the defect \( g_i^{(L)} \), Poincaré’s inequality on \( Q_r \), and (24), it is enough to construct \( \delta \tilde{\sigma}_i \) such that
\[ g_i^{(L)} = \nabla \cdot \delta \tilde{\sigma}_i \quad \text{on } Q_L \]
and
\[ \left( \int_{Q_r} |\nabla \delta \tilde{\sigma}_i|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r^{(L)}}{L} \right)^{\beta} \]for \( L \geq r \geq r^{(L)} \).

To this purpose, we extend the restriction of \( g_i^{(L)} \) to \( Q_L \) periodically (without changing the notation) and let \( \delta \sigma_i^{(L)} \) be the \( Q_L \)-periodic solution of
\[ -\Delta \delta \sigma_i^{(L)} = \partial_j g_{ik}^{(L)} - \partial_k g_{ij}^{(L)} \]with \( \int_{Q_L} \delta \sigma_i^{(L)} = 0 \).

As remarked in the introduction after (19), periodic boundary conditions ensure
\[ g_i^{(L)} - \int_{Q_L} g_i^{(L)} = \nabla \cdot \delta \sigma_i^{(L)} \].

\[ \text{☐ Springer} \]
We claim that \((94)\) holds for \(\delta \sigma_i^{(L)}\):

\[
\left(\int_{Q_r} |\nabla \delta \sigma_i^{(L)}|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r_\ast^{(L)}}{L} \right)^\beta \quad \text{for } L \geq r \geq r_\ast^{(L)}. \tag{97}
\]

Indeed, since by the energy estimate for the periodic problem \((95)\), that is,

\[
\left(\int_{Q_L} |\nabla \delta \sigma_i^{(L)}|^2 \right)^{\frac{1}{2}} \lesssim \left(\int_{Q_L} |g_i^{(L)}|^2 \right)^{\frac{1}{2}},
\]

\((97)\) holds for \(r \in [\frac{L}{4}, L]\). For \(r \leq \frac{L}{4}\), \((97)\) follows from the stronger estimate

\[
\sup_{Q_{\frac{L}{4}}} |\nabla \delta \sigma_i^{(L)}| \lesssim \left(\int_{Q_{\frac{L}{4}}} |\nabla^{n+1} \delta \sigma_i^{(L)}|^2 \right)^{\frac{1}{2}} + \left(\int_{Q_{\frac{L}{4}}} |\nabla \delta \sigma_i^{(L)}|^2 \right)^{\frac{1}{2}}, \tag{98}
\]

and to appeal to \((92)\). Estimate \((98)\) is a (standard) consequence of Sobolev’s estimate

\[
\sup_{Q_{\frac{L}{4}}} |\nabla \delta \sigma_i^{(L)}| \lesssim \left(\int_{Q_{\frac{L}{4}}} |\nabla^{n+1} \delta \sigma_i^{(L)}|^2 \right)^{\frac{1}{2}} + \left(\int_{Q_{\frac{L}{4}}} |\nabla \delta \sigma_i^{(L)}|^2 \right)^{\frac{1}{2}},
\]

where \(n\) is an integer larger than \(\frac{d}{2}\), of a localized energy estimate for the constant-coefficient equation \((95)\) in form of

\[
\left(\int_{Q_{\frac{L}{4}}} |\nabla^{n+1} \delta \sigma_i^{(L)}|^2 \right)^{\frac{1}{2}} + \left(\int_{Q_{\frac{L}{4}}} |\nabla \delta \sigma_i^{(L)}|^2 \right)^{\frac{1}{2}} \lesssim \left(\int_{Q_L} |\nabla^n g_i^{(L)}|^2 \right)^{\frac{1}{2}} + \left(\int_{Q_L} |\nabla \delta \sigma_i^{(L)}|^2 \right)^{\frac{1}{2}},
\]

and of a localized energy estimate for \(-\Delta g_i^{(L)} = 0\)

\[
\left(\int_{Q_{\frac{L}{2}}} |\nabla^n g_i^{(L)}|^2 \right)^{\frac{1}{2}} \lesssim \left(\int_{Q_L} |g_i^{(L)}|^2 \right)^{\frac{1}{2}}.
\]

In order to pass from \((96)\) to \((93)\), it is enough to (explicitly) construct an affine \(\delta \tilde{\sigma}_{ijk}\) with \(\int_{Q_L} g_i^{(L)} = \nabla \cdot \delta \tilde{\sigma}_{ijk}\) and such that \(|\nabla \delta \tilde{\sigma}_{ijk}| \lesssim |\int_{Q_L} g_i^{(L)}|\), which is elementary.

We now note that we control the gradient of the proxy \((\phi^{(L)}, \sigma^{(L)})\) down to scales \(r_\ast^{(L)}\):

\[
\left(\int_{B_r} |\nabla (\phi^{(L)}, \sigma^{(L)})|^2 \right)^{\frac{1}{2}} \lesssim 1 \quad \text{for } 2L \geq r \geq r_\ast^{(L)}. \tag{99}
\]

Indeed, for scales \(r \sim L\), \((99)\) follows from the energy estimate for \((18)\), inserted into the energy estimate for \((19)\). Hence, we may restrict to \(L \gg r_\ast^{(L)}\). Since by \((18)\),

\[\square\]
\( x_i + \phi_i^{(L)} \) is \( a \)-harmonic in \( B_L \), we have by Caccioppoli’s estimate

\[
\left( \int_{B_{\frac{r}{2}}} |e_i + \nabla \phi_i^{(L)}|^2 \right)^{\frac{1}{2}} \lesssim \frac{1}{L} \left( \int_{B_L} (x_i + \phi_i^{(L)}) - \int_{B_L} (x_i + \phi_i^{(L)}) \right)^{\frac{1}{2}} \lesssim 1 + \frac{1}{L} \left( \int_{B_L} (\phi_i^{(L)}) - \int_{B_L} (\phi_i^{(L)}) \right)^{\frac{1}{2}},
\]

so that by (37) for \( r = L \), we get (99) for \( r = \frac{L}{2} \). The remaining range of \( \frac{L}{2} \leq r \leq r_*^{(L)} \) follows since there, according to our hypotheses (37) for \( \phi^{(L)} \) and to (90) and (91) for \( \tilde{\sigma} \), the medium \( a \) is well-behaved and thus admits the \( C^{0.1} \)-estimate, cf (80), which applied to \( x_i + \phi_i^{(L)} \) yields

\[
\left( \int_{B_{\frac{r}{2}}} |e_i + \nabla \phi_i^{(L)}|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{B_{\frac{r}{2}}} |e_i + \nabla \phi_i^{(L)}|^2 \right)^{\frac{1}{2}} \quad \text{for} \quad \frac{L}{2} \geq r \geq r_*^{(L)}.
\]

This establishes the \( \phi^{(L)} \)-contribution to (99), which in particular implies for the flux \( q_i^{(L)} = a(e_i + \nabla \phi_i^{(L)}) \)

\[
\left( \int_{B_{\frac{r}{2}}} |q_i^{(L)}|^2 \right)^{\frac{1}{2}} \lesssim 1 \quad \text{for} \quad \frac{L}{2} \geq r \geq r_*^{(L)}.
\]

By the equation for \( \sigma_i^{(L)} \) with right-hand side given by the curl of \( q_i^{(L)} \), cf (19), we obtain from Caccioppoli’s estimate

\[
\left( \int_{B_r} |\nabla \sigma_{ijk}^{(L)}|^2 \right)^{\frac{1}{2}} \lesssim \frac{1}{r} \left( \int_{B_{2r}} (\sigma_{ijk}^{(L)})^2 \right)^{\frac{1}{2}} + \left( \int_{B_{2r}} |q_i^{(L)}|^2 \right)^{\frac{1}{2}},
\]

so that we obtain the \( \sigma^{(L)} \)-part of (99) from the \( \sigma^{(L)} \)-part of (37) and from (100).

We now come to the central piece, namely that the differences between proxy and true corrector are small down to scales \( r_*^{(L)} \):

\[
\left( \int_{Q_r} |\nabla (\phi^{(L)} - \hat{\phi}, \sigma^{(L)} - \sigma)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r_*^{(L)}}{L} \right)^{\beta} \quad \text{for} \quad L \geq r \geq r_*^{(L)}.
\]

By Lemma 2 (with as above \( r_*^{(L)} \) playing the role of \( r_* \)), it is enough to consider \( L \gg r \geq r_*^{(L)} \). As noticed above, \( \phi_i^{(L)} - \hat{\phi}_i \) is \( a \)-harmonic in \( B_L \) so that passing from \( r = L \) to \( L \geq r \geq r_*^{(L)} \) follows from the \( C^{0.1} \)-estimate already used earlier, (80). This settles the \( \phi \)-part of (101); in order to deal with the \( \sigma \)-part, we need the full \( C^{1.\alpha} \)-theory for \( a \)-harmonic functions from [9, Theorem 1] (for an \( \alpha \in (0, 1) \) fixed, say \( \alpha = \frac{1}{2} \)), which holds for radii \( L \geq r \geq r_*^{(L)} \) since, as already remarked above, the medium is well behaved there in the sense that there exist a tensor \( a_h^{(L)} \), scalar fields \( \phi_i^{(L)} \), and skew symmetric vector fields \( \tilde{\sigma}_i \), related by (90) and satisfying the estimates.
(37) and (91). Applied to the $a$-harmonic function $\phi_i - \phi_i^{(L)}$ in $B_L$ this yields a vector $\xi$ (which should carry an index $i$) such that

$$\left(\int_{B_r} |\nabla (\phi_i - \phi_i^{(L)} - \xi_j (x_j + \phi_j^{(L)}))|^2 \right)^{\frac{1}{2}} \lesssim \left(\frac{r}{L}\right)\alpha \left(\frac{r_*^{(L)}}{L}\right)^\beta$$

for all $L \geq r \geq r_*^{(L)}$ and $|\xi| \lesssim \left(\frac{r_*^{(L)}}{L}\right)^\beta$.

Recalling the definition of the fluxes $q_i = a(e_i + \nabla \phi_i)$ and $q_i^{(L)} = a(e_i + \nabla \phi_i^{(L)})$ and inserting (36), we obtain

$$\left(\int_{B_r} |q_i - q_i^{(L)} - \xi_j q_j^{(L)}|^2 \right)^{\frac{1}{2}} \lesssim \left(\frac{r}{L}\right)\alpha \left(\frac{r_*^{(L)}}{L}\right)^\beta$$

for $L \geq r \geq r_*^{(L)}$, $|\xi| \lesssim \left(\frac{r_*^{(L)}}{L}\right)^\beta$.

By (14) and (19) we have that $w := \sigma_i - \sigma_i^{(L)} - \xi_j \sigma_j^{(L)}$ solves a Poisson equation with the curl of $q_i - q_i^{(L)} - \xi_j q_j^{(L)}$ as right-hand side. Hence, the first part of (102) translates into

$$-\Delta w = \nabla \cdot h \quad \text{with} \quad \left(\int_{B_r} |h|^2 \right)^{\frac{1}{2}} \lesssim \left(\frac{r}{L}\right)\alpha \left(\frac{r_*^{(L)}}{L}\right)^\beta$$

for $L \geq r \geq r_*^{(L)}$.

In the next paragraph, we shall argue that thanks to $\alpha > 0$, this implies

$$\left(\int_{B_r} |\nabla w|^2 \right)^{\frac{1}{2}} \lesssim \left(\frac{r_*^{(L)}}{L}\right)^\beta + \left(\int_{B_{r/2}} |\nabla w|^2 \right)^{\frac{1}{2}}$$

for $L \geq r \geq r_*^{(L)}$.

By definition of $w$ and the triangle inequality, this yields

$$\left(\int_{B_r} |\nabla \sigma_i - \sigma_i^{(L)}|^2 \right)^{\frac{1}{2}} \lesssim \left(\frac{r_*^{(L)}}{L}\right)^\beta + \left(\int_{B_{r/2}} |\nabla \sigma_i - \sigma_i^{(L)}|^2 \right)^{\frac{1}{2}}$$

+ $|\xi_j| \left(\left(\int_{B_r} |\nabla \sigma_j^{(L)}|^2 \right)^{\frac{1}{2}} + \left(\int_{B_{r/2}} |\nabla \sigma_j^{(L)}|^2 \right)^{\frac{1}{2}} \right)$

for $L \geq r \geq r_*^{(L)}$.

Inserting the estimate on $|\xi|$ from (102), the localized estimates on $\sigma^{(L)}$ from (99), and the large-scale estimate on $\sigma - \sigma^{(L)}$ from (34), we get the localized estimate on $\sigma - \sigma^{(L)}$ stated in (101).

It remains to argue that (103) implies (104). Like in the proof of Lemma 1, we resort to a construction via a decomposition into dyadic annuli. For any dyadic multiple $r$ of $r_*^{(L)}$ with $r \leq \frac{L}{2}$ we consider the Lax-Milgram solution $w_r$ of

$$-\Delta w_r = \nabla \cdot (I (B_{2r} - B_r)h),$$
where \( I(B_{2r} - B_r) \) denotes the characteristic function of the annulus \( B_{2r} - B_r \). The solution of the Poisson equation with right-hand side \( \nabla \cdot (I(B_{r^*})(h) \) is denoted by \( w_{r^*}^{(L)} \) for notational consistency. By the energy estimate for the Poisson equation and (103) we have for all dyadic \( \frac{r^*}{2} \leq r \leq \frac{L}{2} \)

\[
\left( \frac{1}{r^d} \int |\nabla w_r|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r}{L} \right)^\alpha \left( \frac{r^*}{L} \right)^\beta.
\]

From this and the (standard) mean-value property (note that unless \( r = \frac{r^*}{2} \), \( w_r \) is harmonic in \( B_r \)) we obtain for every radius \( R \geq \frac{r^*}{2} \) (the lower bound arises because of \( w_{r^*}^{(L)} \))

\[
\left( \int_{B_R} |\nabla w_r|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r}{L} \right)^\alpha \left( \frac{r^*}{L} \right)^\beta.
\]

Because of \( \alpha > 0 \) we obtain for \( \tilde{w} := \sum_{\frac{r^*}{2} \leq r \leq \frac{L}{2}} w_r \) that

\[
\left( \int_{B_R} |\nabla \tilde{w}|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r^*}{L} \right)^\beta \text{ for } R \geq \frac{r^*}{2}.
\]

Since by construction, \( w - \tilde{w} \) is harmonic in \( B_{L/4} \), we obtain by the (standard) mean-value property

\[
\left( \int_{B_R} |\nabla (w - \tilde{w})|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{B_{L/4}} |\nabla (w - \tilde{w})|^2 \right)^{\frac{1}{2}} \text{ for } R \leq \frac{L}{4}.
\]

By the triangle inequality in \( L^2 \), the two last inequalities yield (104).

Equipped with the localized estimates on \( \nabla (\phi - \phi^{(L)}, \sigma - \sigma^{(L)}) \) from (101), it is easy to conclude: Because of (36), it is sufficient to establish (38) for \( L \geq r \geq r^* \). In fact, we shall show that (37) entails (38) in this range. This follows instantly from Poincaré’s inequality in form of

\[
\frac{1}{r} \left( \int_{B_r} |(\phi, \sigma) - \int_{B_r} (\phi, \sigma)|^2 \right)^{\frac{1}{2}} - \frac{1}{r} \left( \int_{B_r} |(\phi^{(L)}, \sigma^{(L)}) - \int_{B_r} (\phi^{(L)}, \sigma^{(L)})|^2 \right)^{\frac{1}{2}} \\
\lesssim \left( \int_{B_r} |\nabla (\phi^{(L)} - \phi, \sigma^{(L)} - \sigma)|^2 \right)^{\frac{1}{2}},
\]

into which we plug (101). \( \square \)
3.6 Proof of Lemma 4

Proof Since as a finite-range ensemble, $\langle \cdot \rangle$ is in particular ergodic, \[9, Lemma 1\] applies, and yields the existence of $\phi_i$ and $\sigma_i$ with the stated properties. We set for abbreviation $F_R := (f_{BR}(\phi) - f_{BR}(\phi, \sigma))^{\frac{1}{2}}$.

For given $1 \leq s' < 2$, we start by extracting the stochastic bound

$$\langle \exp \left( \frac{F_R}{C \log^{\frac{3}{2}} R} \right)^{s'} \rangle \leq C \quad \text{for all } R \geq 2 \quad (105)$$

from \[11, Corollary 2\]. Here $C = C(d, \lambda, s')$ denotes a generic constant the value of which may change from line to line. We focus on the case of $d = 2$ (the result is stronger for $d > 2$), in which case the statement of \[11, Corollary 2\] takes the form of

$$\log \langle \exp \left( \frac{1}{C \log (2 + |x|)} \int G_1(\cdot - x) |(\phi, \sigma) - \int G_1(\phi, \sigma) |^2 + 1 \right)^{\frac{1}{2}} \rangle \leq 1,$$

for all $x \in \mathbb{R}^d$, where the Gaussian $G_1(x) := (2\pi)^{-d/2} \exp\left(-\frac{|x|^2}{2}\right)$ plays the role of a spatial averaging function. We rewrite this

$$\langle \exp \left( \frac{1}{C \log (2 + |x|)} \int G_1(\cdot - x) |(\phi, \sigma) - \int G_1(\phi, \sigma) |^2 + 1 \right)^{\frac{1}{2}} \rangle \lesssim 1,$$

where $\lesssim$ stands for $\leq C(d, \lambda, s')$. Since $[0, \infty) \in F \mapsto \exp((F + 1)^{\frac{1}{2}})$ is convex for $s' \geq 1$, we obtain for arbitrary $R \geq 2$ from averaging over $x \in B_R$

$$\langle \exp \left( \frac{1}{C \log R} \int_{B_R} |(\phi, \sigma) - \int G_1(\phi, \sigma) |^2 + 1 \right)^{\frac{1}{2}} \rangle \lesssim 1,$$

where now $\int G_1(\phi, \sigma)$ may be replaced by $f_{BR}(\phi, \sigma)$, so that this turns into (105).

We now turn to part i) of the lemma, i.e., (105), and fix $\beta < 1$ and let the random radius $r_* \in [1, +\infty]$ be minimal with (16). Because of $\frac{1}{r'} \left( f_{BR} \cdot \right)^{\frac{1}{2}} \leq \left( f_{r'} \cdot \right)^{\frac{1}{2}} + \frac{1}{r} \left( f_{BR} \cdot \right)^{\frac{1}{2}}$ for $r' \leq r$, $r_*$ is dominated by the smallest dyadic radius, which for simplicity we call again $r_*$, with the property that

$$\frac{1}{R} F_R \leq 2^{-\frac{d}{2}-1} \left( \frac{r_*}{R} \right)^{\beta} \quad \text{for all dyadic } R > r_*.$$

Hence, we have for any (deterministic) threshold $r \geq 2$

$$\langle I(r_* \geq r) \rangle \leq \sum_{R \geq r \text{ dyadic}} \langle I(F_R \geq 2^{-\frac{d}{2}-1} r^{\beta} R^{1-\beta}) \rangle, \quad (106)$$
where \( I(\cdot) \) is again the indicator function. We obtain from (105) for \( s < s' < 2 \) (say \( s' = \frac{s + 2}{2} \)) by Chebyshev’s inequality

\[
\langle I(F_R \geq 2^{-\frac{d}{2}-1}r^\beta R^{1-\beta}) \rangle \leq \exp\left(-\left(\frac{r^\beta R^{1-\beta}}{C \log^{\frac{1}{2}} r}\right)^{s'} + 1\right),
\]

(107)

where \( C = C(d, \lambda, s, \beta) \) denotes a generic constant the value of which may change line by line. We now claim that \( \beta < 1 \) implies that the first term in the dyadic sum dominates, i.e.,

\[
\sum_{R > r \text{ dyadic}} \exp\left(-\left(\frac{r^\beta R^{1-\beta}}{C \log^{\frac{1}{2}} r}\right)^{s'}\right) \lesssim \exp\left(-\left(\frac{r}{C \log^{\frac{1}{2}} r}\right)^{s'}\right),
\]

(108)

where \( \lesssim \) stands short for \( \leq \) up to a generic constant \( C = C(d, \lambda, s, \beta) \).

Indeed, setting for abbreviation \( C(r) := \left(\frac{r}{C \log^{\frac{1}{2}} r}\right)^{s'} \), this amounts to showing

\[
\sum_{R > r} \exp\left(-C(r)\left(\frac{R}{2}\right)^{s'(1-\beta)}\right) \lesssim \exp(-C(r)),
\]

which because of \( C(r) \geq 1 \) (for \( r \gg 1 \), which means that \( r \) is larger than some constant only depending on \( d, \lambda, s, \beta \)) reduces to the elementary \( \sum_{n=1}^{\infty} \exp(-((2^n)^{s'(1-\beta)} - 1)) \lesssim 1 \), where we used \( \beta < 1 \) and \( s' > 0 \). The combination of (106), (107), and (108) yields

\[
\langle I(r \geq r) \rangle \lesssim \exp\left(-\left(\frac{r}{C \log^{\frac{1}{2}} r}\right)^{s'}\right) \text{ for } r \gg 1,
\]

which implies (39) thanks to \( s < s' \).

We finally turn to part ii) of the lemma and fix \( \beta < 1 \) and \( 0 < s < 2(1 - \beta) \). We have to show the existence of \( L_0 \) such that

\[
\langle I(\exists R \geq L, \text{ s.t. } F_R > R^{1-\beta}) \rangle \leq \exp\left(-\left(\frac{L}{L_0}\right)^{s'}\right) \text{ for all } L \geq L_0;
\]

by the same argument as for part i) this reduces to

\[
\sum_{R \geq L \text{ dyadic}} \langle I(F_R > 2^{-\frac{d}{2}-1} R^{1-\beta}) \rangle \leq \exp\left(-\left(\frac{L}{L_0}\right)^{s'}\right) \text{ for all dyadic } L \geq L_0.
\]

(109)

From (105) for \( s' = s' < 2(\frac{1}{2}(\frac{s}{1-\beta} + 2)) \) we obtain for each summand with \( R \geq 2 \)

\[
\langle I(F_R \geq 2^{-\frac{d}{2}-1} R^{1-\beta}) \rangle \leq \exp\left(-\left(\frac{R^{1-\beta}}{C \log^{\frac{1}{2}} R}\right)^{s'} + 1\right),
\]

(106)
and as for (108) we find for the sum

\[ \sum_{R>L \text{ dyadic}} \exp\left(-\frac{R^{1-\beta}C}{\log^2 R} s'\right) \lesssim \exp\left(-\frac{L^{1-\beta}}{C \log^2 L} s'\right), \]

Since \( s < s'(1 - \beta) \) the last two statements imply (109) for some \( L_0 \lesssim 1 \). \( \square \)

### Theorem 1

**Proof** According to Lemma 4 ii) and with probability \( 1 - \exp\left(-\frac{L}{L_0}\right)^s \), the hypothesis (36) of Lemma 3 is satisfied with \( r_s^{(L)} = 1 \); the second hypothesis (37) is satisfied by assumption (24) of the theorem, where we postpone the argument for the existence of such an \( r_s^{(L)} \leq L \) to the next step. Hence by (38) in Lemma 3, (16) holds with \( r_s = \max\{r_s^{(L)}, 1\} = r_s^{(L)} \), so that we may apply Proposition 1 with \( r_s^{(L)} \) playing the role of \( r_s \). Hence (28) turns into the desired (26).

We now turn to the claimed existence of \( r_s^{(L)} \leq L \) with (24) with probability \( 1 - \exp\left(-\frac{L}{L_0}\right)^s \). To this end we argue that the sublinearity of \( (\phi, \sigma) \) on scale \( L \) entails the sublinearity of \( (\phi^{(L)}, \sigma^{(L)}) \) on scale \( L \) in the sense that there exists an exponent \( \alpha = \alpha(d, \lambda) > 0 \) such that

\[ \frac{1}{L} \left( \int_{Q^{2L}} |(\phi^{(L)}, \sigma^{(L)})|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{1}{L} \left( \int_{Q^{4L}} |(\phi, \sigma) - \int_{Q^{4L}} (\phi, \sigma)|^2 \right)^{\frac{1}{2}} \right)^{\alpha} =: \Lambda^\alpha. \] (110)

We start with the \( \phi \)-part and to this purpose consider \( w := \phi^{(L)} - \zeta \phi \), where we dropped the index \( i = 1, \ldots, d \), and where \( \zeta \in [0, 1] \) is a cut-off function with

\[ \zeta = 1 \text{ in } Q_{2L-\rho}, \quad \zeta = 0 \text{ outside of } Q_{2L}, \quad |\nabla \zeta| \lesssim \frac{1}{\rho} \] (111)

for some boundary layer size \( \rho \in (0, L] \) to be optimized later. An elementary computation (which is a variation upon (78)) yields

\[-\nabla \cdot a \nabla w = \nabla \cdot \left( (a - \sigma) \nabla \zeta + (1 - \zeta)(a - a_h)e \right).\]

Appealing to \( w = 0 \) on \( \partial Q_{2L} \), the Poincaré estimate followed by the energy estimate yield

\[ \frac{1}{L} \left( \int_{Q^{2L}} w^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{Q^{2L}} |\nabla w|^2 \right)^{\frac{1}{2}} \]

(111), (1)

\[ \lesssim \frac{1}{\rho} \left( \int_{Q^{2L}} |(\phi, \sigma)|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{L^d} \int_{Q^{2L} - Q_{2L-\rho}} 1 \right)^{\frac{1}{2}}. \]
By definition of \( w \), Leibniz’ rule, and the triangle inequality in \( L^2 \), this entails, also for the flux difference \( q^{(L)} - q = a \nabla (\phi^{(L)} - \phi) \),

\[
\frac{1}{L} \left( \int_{Q_{2L}} (\phi^{(L)})^2 \right)^{\frac{1}{2}} + \left( \int_{Q_{2L}} |\nabla (\phi^{(L)} - \phi)|^2 \right)^{\frac{1}{2}} + \left( \int_{Q_{2L}} |q^{(L)} - q|^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \frac{1}{\rho} \left( \int_{Q_{2L}} |(\phi, \sigma)|^2 \right)^{\frac{1}{2}} + \left( \int_{Q_{2L} - Q_{2L - \rho}} |\nabla \phi|^2 + 1 \right)^{\frac{1}{2}}.
\]

(112)

We now turn to the \( \sigma \)-part of (110) and consider \( w := \sigma^{(L)} - \zeta \sigma \), which by (14) and (19) satisfies (in three-dimensional notation)

\[-\Delta w = \nabla \times (q^{(L)} - q) + \nabla \cdot (\sigma \nabla \zeta - (1 - \zeta) \nabla \sigma),\]

which, by the same ingredients as above, first gives

\[
\frac{1}{L} \left( \int_{Q_{2L}} |\nabla w|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{Q_{2L}} |\nabla \sigma|^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \left( \int_{Q_{2L}} |q^{(L)} - q|^2 \right)^{\frac{1}{2}} + \frac{1}{\rho} \left( \int_{Q_{2L}} |\sigma|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{L^d} \int_{Q_{2L} - Q_{2L - \rho}} |\nabla \sigma|^2 \right)^{\frac{1}{2}}
\]

and then

\[
\frac{1}{L} \left( \int_{Q_{2L}} |\sigma^{(L)}|^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \left( \int_{Q_{2L}} |q^{(L)} - q|^2 \right)^{\frac{1}{2}} + \frac{1}{\rho} \left( \int_{Q_{2L}} |\sigma|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{L^d} \int_{Q_{2L} - Q_{2L - \rho}} |\nabla \sigma|^2 \right)^{\frac{1}{2}}.
\]

The combination of this with (112) yields

\[
\frac{1}{L} \left( \int_{Q_{2L}} |(\phi^{(L)}, \sigma^{(L)})|^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \frac{1}{\rho} \left( \int_{Q_{2L}} |(\phi, \sigma)|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{L^d} \int_{Q_{2L} - Q_{2L - \rho}} (|\nabla (\phi, \sigma)|^2 + 1) \right)^{\frac{1}{2}}.
\]

(113)

We now appeal to (and below will argue for) the fact that there exists \( p = p(d, \lambda) > 2 \) such that

\[
\left( \int_{Q_{2L}} |\nabla (\phi, \sigma)|^p \right)^{\frac{1}{p}} \lesssim 1 + \frac{1}{L} \left( \int_{Q_{4L}} |(\phi, \sigma)|^2 \right)^{\frac{1}{2}},
\]

(114)

which by Hölder’s inequality allows to upgrade (113) to

\[
\frac{1}{L} \left( \int_{Q_{2L}} |(\phi^{(L)}, \sigma^{(L)})|^2 \right)^{\frac{1}{2}}
\]
\[
\begin{aligned}
\lesssim & \frac{1}{\rho} \left( \int_{Q_{4L}} |(\phi, \sigma)|^2 \right)^{1/2} + \left( \frac{\rho}{L} \right)^{1/2 - 1/p} (1 + \frac{1}{L} \left( \int_{Q_{4L}} |(\phi, \sigma)|^2 \right)^{1/2}).
\end{aligned}
\]

Since in the above argument we may without loss of generality assume \( \int_{Q_{4L}} (\phi, \sigma) = 0 \), the last estimate takes the form

\[
\frac{1}{L} \left( \int_{Q_{4L}} |(\phi^{(L)}, \sigma^{(L)})|^2 \right)^{1/2} \lesssim \frac{L}{\rho} \Lambda + \left( \frac{\rho}{L} \right)^{1/2 - 1/p} (1 + \Lambda).
\] (115)

By the energy estimate followed by Poincaré’s estimate, first for \( \phi^{(L)} \) and then for \( \sigma^{(L)} \), the left-hand side of (110) is seen to be \( \lesssim 1 \) so that we may without loss of generality assume \( \Lambda \leq 1 \). Optimizing in \( \rho \in [0, L] \) shows that the right-hand side of (115) can be replaced by \( \Lambda^\alpha \) with \( \alpha := \frac{1}{2} - \frac{1}{p} > 0 \). This yields (110).

We now turn to \( \sigma \) characterized by \( -\Delta \sigma = \nabla \times q \), so that by standard Calderon–Zygmund theory, localized as above, we have

\[
\left( \int_{Q_{4L}} |\nabla \sigma|^p \right)^{1/p} \lesssim \left( \int_{Q_{3L}} |q|^p \right)^{1/p} + \left( \int_{Q_{4L}} |\sigma|^2 \right)^{1/2}.
\]

and appeal to Caccioppoli’s estimate to obtain, also for the flux,

\[
\left( \int_{Q_{4L}} |\nabla \phi|^p \right)^{1/p} + \left( \int_{Q_{4L}} |q|^p \right)^{1/p} \lesssim 1 + \frac{1}{L} \left( \int_{Q_{4L}} \phi^2 \right)^{1/2}.
\] (116)

We now turn to \( \sigma \) characterized by \( -\Delta \sigma = \nabla \times q \), so that by standard Calderon–Zygmund theory, localized as above, we have

\[
\left( \int_{Q_{2L}} |\nabla \sigma|^p \right)^{1/p} \lesssim \left( \int_{Q_{3L}} |q|^p \right)^{1/p} + \left( \int_{Q_{2L}} |\sigma|^2 \right)^{1/2},
\]

and appeal to the localized energy estimate to obtain

\[
\left( \int_{Q_{2L}} |\nabla \sigma|^p \right)^{1/p} \lesssim \left( \int_{Q_{3L}} |q|^p \right)^{1/p} + \frac{1}{L} \left( \int_{Q_{4L}} |\sigma|^2 \right)^{1/2}.
\] (117)

The combination of (116) and (117) yields (114).

We are now ready to conclude on the existence of \( r_\ast^{(L)} \) with overwhelming probability. Setting momentarily \( \delta := \frac{1}{2L} \left( \int_{Q_{2L}} |(\phi^{(L)}, \sigma^{(L)})|^2 \right)^{1/2} \), and using the obvious

\[
\frac{1}{r} \left( \int_{Q_r} |(\phi^{(L)}, \sigma^{(L)})|^2 \right)^{1/2} \leq \left( \frac{2L}{r} \right)^{1/2} + \delta,
\]

we see that (24) holds with \( r_\ast^{(L)} = \delta^{1/2+1} 2L \). It then follows from (110) that \( r_\ast^{(L)} \leq L \) provided \( \Lambda \ll 1 \). The latter is a consequence

\[\Box\] Springer
of (40), which we are thus using once more, and \( L \geq L_0 \), provided \( L_0 \) is chosen large enough.

### 3.8 Proof of Lemma 5

**Proof** For notational simplicity, we drop the index \( i \). We start by collecting some estimates on \( \phi_h' \). Rewriting (53) as
\[
- \nabla \cdot a_h' \nabla \phi_h' = \nabla \cdot (a_h' - a_h) e
\]
and noting that the right-hand side \((a_h' - a_h) e\) is bounded and supported in \( B_R(y) \), cf (49), we obtain from the energy estimate
\[
\left( \frac{1}{R^d} \int |\nabla \phi_h'|^2 \right)^{\frac{1}{2}} \lesssim 1.
\] (118)

It is convenient to introduce
\[
v_h' := x \cdot e + \phi_h'
\]
so that
\[
\nabla \cdot a_h' \nabla v_h' \overset{(53)}{=} 0,
\] (119)
and to reformulate (118) as
\[
\left( \int_{B_3R(y)} |\nabla v_h'|^2 \right)^{\frac{1}{2}} \lesssim 1,
\] (120)
which in view of (119), by Meyer’s estimate, upgrades to
\[
\left( \int_{B_2R(y)} |\nabla v_h'|^p \right)^{\frac{1}{p}} \lesssim 1 \quad \text{for some } p = p(d, \lambda) > 2.
\] (121)

Rewriting (53) once more, this time as
\[
- \nabla \cdot a_h \nabla \phi_h' = \nabla \cdot (a_h - a_h') \nabla v_h',
\]
and noting that \( a_h \) is a constant coefficient and that \((a_h - a_h') \nabla v_h'\) is supported in \( B_R(y) \), cf (49), we see that \( \phi_h' \) decays like the gradient of the fundamental solution \( G_h \). Hence, from (118) we obtain the estimate
\[
R \sup_{x \in B_{2R}(y)} \left( \frac{|x - y|}{R} \right)^{d+1} |\nabla^2 v_h'| \overset{(119)}{=} R \sup_{x \in B_{2R}(y)} \left( \frac{|x - y|}{R} \right)^{d+1} |\nabla^2 \phi_h'| \lesssim 1.
\] (122)

We may even get closer to the boundary of \( B_{2R}(y) \) at the expense of a bad constant: For any boundary layer width \( 0 < \rho \leq R \) we obtain from (120) and the fact that \( v_h' \) is constant-coefficient harmonic in \( B_R(y) \), cf (119),
\[
\sup_{B_{2R}(y) - B_{R+\rho}(y)} (\rho |\nabla^2 v_h'| + |\nabla v_h'|) \lesssim \left( \frac{R}{\rho} \right)^{\frac{d}{2}}.
\] (123)
We now turn to the comparison of $\phi'$ and $\phi'_h$, at first in the strong topology on the level of gradients. We carry this out in terms of the harmonic functions

$$v' := e \cdot x + \phi'$$
so that

$$\nabla \cdot a' \nabla v' = 0$$ (52)

and $v'_h$, by monitoring the error in the two-scale expansion, that is,

$$w := v' - (1 + \zeta \phi_i \partial_i)v'_h.$$ (125)

Here $\zeta$ denotes a cut-off function with

$$\zeta = 0 \text{ in } B_{R+\rho}, \quad \zeta = 1 \text{ outside } B_{R+2\rho}, \quad |\nabla \zeta| \lesssim \rho^{-1}$$ (126)

for a boundary layer width $0 < \rho \leq R$ to be optimized at the end of the proof. Based on the equations (119), (124), and on (13), we obtain the following formula

$$- \nabla \cdot a' \nabla w = \nabla \cdot (h_{\text{far}} + h_{\text{near}}) \quad \text{with}$$

$$h_{\text{far}} := (\phi_i a - \sigma_i) \nabla (\zeta \partial_i v'_h) \quad \text{and} \quad h_{\text{near}} := (1 - \zeta)(a' - a'_h) \nabla v'_h;$$ (127)

which is the same (elementary) calculation as in Step 2 of the proof of [9, Proposition 2] and a slight variation of (78). For the proof of this lemma, we normalize $(\phi, \sigma)$ by

$$-\frac{1}{R} \left( \int_{B_r(y)} |(\phi, \sigma)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r_*(y)}{R} \right)^{\frac{1}{2} - \frac{1}{p}}$$ (128)

The near-field right-hand side $h_{\text{near}}$ is supported in the thin annulus $B_{R+2\rho} - B_R$ of thickness $2\rho$, as a consequence of (126), (47) and (49). With help of Meyer’s estimate (121), we may capitalize on this by Hölder’s inequality:

$$\left( \frac{1}{R} \int_{B_{2R}(y)} |h_{\text{near}}|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\rho}{R} \right)^{\frac{1}{2} - \frac{1}{p}}.$$ (129)

In the complement of $B_{2R}(y)$, in view of (126) the far-field term assumes the simpler form $h_{\text{far}} = (\phi_i a - \sigma_i) \nabla \partial_i v'_h$ and thus is easily estimated:

$$\left( \frac{1}{R} \int_{B_{2R}(y)} |h_{\text{far}}|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r_*(y)}{R} \right)^{\frac{1}{2} - \frac{1}{p}}.$$ (128)

the last estimate can be seen by a decomposition into dyadic annuli; here we also use our assumption that $R \geq r_*(y)$. In $B_{2R}(y)$ by (126) the far-field term is supported in
\[ B_{R+\rho}^c(y) \text{ and estimated by } |h_{far}| \lesssim |(\phi, \sigma)|(\|
abla^2 v'_{h}\| + \frac{1}{\rho} |\nabla v'_h|). \]

Hence, we obtain from (128) and (123)

\[ \left( \frac{1}{R^d} \int_{B_{2R}(y)} |h_{far}|^2 \right)^{1/2} \lesssim \left( \frac{R}{\rho} \right)^{1+\frac{d}{2}} \left( \frac{r_*(y)}{R} \right)^\beta. \]  

(131)

Using (129), (130), and (131) in the energy estimate for (127) we obtain

\[ \left( \frac{1}{R^d} \int |\nabla w|^2 \right)^{1/2} \lesssim \left( \frac{\rho}{R} \right)^{1+\frac{1}{p}} + \left( \frac{R}{\rho} \right)^{1+\frac{d}{2}} \left( \frac{r_*(y)}{R} \right)^\beta. \]  

(132)

We are interested in the difference of the potentials, cf (55), and in the difference of flux averages, cf (56), and thus need to post-process (132). We first turn to the difference of the potentials and note that by definitions (119), (124), and (125) of \( v'_h \), \( v' \) and \( w \) we have \( \phi' - \phi'_h = w + \xi \phi_i \partial_i v'_h \), so that by Poincaré’s inequality and the support condition on \( \zeta \), cf (126),

\[ \frac{1}{R} \int_{B_{2R}(y)} (\phi' - \phi'_h - \int_{B_{2R}(y)} (\phi' - \phi'_h))^2 \]  

\[ \lesssim \left( \frac{1}{R^d} \int |\nabla w|^2 \right)^{1/2} + \frac{1}{R} \int_{B_{2R}(y)} |\phi'|^2 \]  

\[ \sup_{B_{2R}(y) - B_{R+\rho}(y)} |\nabla v'_h|, \]  

so that from (132) for the first right-hand side term and (128) and (123) for the second right-hand side term we obtain

\[ \frac{1}{R} \int_{B_{2R}(y)} (\phi' - \phi'_h - \int_{B_R(y)} (\phi' - \phi'_h))^2 \]  

\[ \lesssim \left( \frac{\rho}{R} \right)^{1+\frac{1}{p}} + \left( \frac{R}{\rho} \right)^{1+\frac{d}{2}} \left( \frac{r_*(y)}{R} \right)^\beta. \]  

(133)

We now turn to the differences of flux averages. The first post-processing step is based on the formula

\[ a' \nabla v' - a'_h \nabla v'_h + \nabla \cdot (\xi \partial_i v'_h \sigma_i) = a' \nabla w + h_{far} + h_{near}, \]

which in fact is the basis for the formula in (127). Hence from (129), (130), (131), and (132) we obtain

\[ \left( \frac{1}{R^d} \int |a' \nabla v' - a'_h \nabla v'_h + \nabla \cdot (\xi \partial_i v'_h \sigma_i)|^2 \right)^{1/2} \]  

\[ \lesssim \left( \frac{\rho}{R} \right)^{1+\frac{1}{p}} + \left( \frac{R}{\rho} \right)^{1+\frac{d}{2}} \left( \frac{r_*(y)}{R} \right)^\beta. \]  

(134)
The second post-processing step consists in noting that the additional term \( \nabla \cdot (\zeta \partial_i v'_h \sigma_i) \) in the integrand has small average: From integration by parts,

\[
\int \omega_y \nabla \cdot (\zeta \partial_i v'_h \sigma_i) = - \int \nabla \omega_y \cdot (\zeta \partial_i v'_h \sigma_i),
\]

and since \( \omega_y \) is supported in \( B_{2R}(y) \), cf (54), this right-hand side is bounded by

\[
\sup |\nabla \omega_y| (\frac{1}{R} \int_{B_{2R}(y)} |\sigma|^2)^{\frac{1}{2}} \lesssim \frac{1}{R} (\frac{r^*(y)}{R})^\beta.
\]

Combining this with (134) yields

\[
|\int \omega_y (a' \nabla v' - a'_h \nabla v'_h)| \lesssim (\frac{\rho}{R})^{\frac{1}{2} - \frac{1}{p} + \frac{d}{2}} + (\frac{R}{\rho})^{1 + \frac{d}{2}} (\frac{r^*(y)}{R})^\beta.
\]

The last task is to upgrade (133) and (135) by optimizing in the boundary layer thickness \( \rho \). Indeed, choosing \( \frac{\rho}{R} = (\frac{r^*(y)}{R})^{\frac{1}{2} - \frac{1}{p} + \frac{d}{2}} \) we obtain (55) and (56) with the potentially small but positive exponent \( \alpha = \beta \frac{1}{2} - \frac{1}{p} + \frac{d}{2} \).

3.9 Proof of Corollary 2

**Proof** We start by noting that

\[
\left( \int_{B_{2R}(y)} |e_j + \nabla \phi'_j|^2 \right)^{\frac{1}{2}} \lesssim 1.
\]

Indeed, by (71) in form of

\[
\left( \int_{B_{2R}(y)} |e_j + \nabla \phi_j|^2 \right)^{\frac{1}{2}} \lesssim 1
\]

and the triangle inequality in \( L^2 \), it suffices to show

\[
\left( \int_{B_{2R}(y)} |\nabla (\phi'_j - \phi_j)|^2 \right)^{\frac{1}{2}} \lesssim 1.
\]

Since \( -\nabla \cdot a' \nabla (\phi'_j - \phi_j) = \nabla \cdot (a' - a)(e_j + \nabla \phi_j) \), cf (5) and (52), and since \( a' - a \) is supported in \( B_R(y) \), cf (47), we have by the energy estimate

\[
\left( \frac{1}{R^d} \int_{B_{2R}(y)} |\nabla (\phi'_j - \phi_j)|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{B_{2R}(y)} |e_j + \nabla \phi_j|^2 \right)^{\frac{1}{2}},
\]
so that (138) is a consequence of (137).

Since \( a' - a \) and \( a'_h - a_h \) are supported in \( B_R(y) \), cf (47) and (49), we may smuggle in the averaging function \( \omega_y \), cf (54), into the left-hand side of the desired (58) so that it is enough to show

\[
\left| \int \omega_y (e_j + \nabla \phi'_j) \cdot (a' - a) (e_i + \nabla \phi_i) \right| - \int \omega_y (e_j + \nabla \phi'_{jh}) \cdot (a'_h - a_h) e_i \right| \lesssim \left( \frac{r_*(y)}{R} \right)^\alpha. \quad (139)
\]

By integration by parts, and using (5) and (52), we obtain for the first integral in (139)

\[
\int \omega_y (e_j + \nabla \phi'_j) \cdot (a' - a) (e_i + \nabla \phi_i) = \int \omega_y (e_j + \nabla \phi'_j) \cdot a' e_i - \int \omega_y e_j \cdot a (e_i + \nabla \phi_i) - \int (\phi_i - \int_{B_{2R}(y)} \phi_i) (e_j + \nabla \phi'_j) \cdot a' \nabla \omega_y + \int (\phi'_j - \int_{B_{2R}(y)} \phi'_j) \nabla \omega_y \cdot a (e_i + \nabla \phi_i). \quad (140)
\]

On the first right-hand side term in (140) we apply (56) of Lemma 5:

\[
\left| \int \omega_y (e_j + \nabla \phi'_j) \cdot a' e_i - \int \omega_y (e_j + \nabla \phi'_{jh}) \cdot a'_h e_i \right| \lesssim \left( \frac{r_*(y)}{R} \right)^\alpha. \quad (141)
\]

Using (13), the second right-hand side term in (140) (without the minus sign) can be rewritten as

\[
\int \omega_y e_j \cdot a (e_i + \nabla \phi_i) = \int \omega_y e_j \cdot a_h e_i - \int e_j \cdot (\sigma_i - \int_{B_{2R}(y)} \sigma_i) \nabla \omega_y,
\]

where the second contribution is estimated as follows

\[
\left| \int e_j \cdot (\sigma_i - \int_{B_{2R}(y)} \sigma_i) \nabla \omega_y \right| \lesssim \frac{1}{R} \left( \int_{B_{2R}(y)} |\sigma_i - \int_{B_{2R}(y)} \sigma_i|^2 \right)^{1/2} \lesssim \left( \frac{r_*(y)}{R} \right)^\beta, \quad (16)
\]

so that we obtain

\[
\left| \int \omega_y e_j \cdot a (e_i + \nabla \phi_i) - \int \omega_y e_j \cdot a_h e_i \right| \lesssim \left( \frac{r_*(y)}{R} \right)^\beta. \quad (142)
\]

The third right-hand side term in (140) is estimated as follows

\[
\left| \int (\phi_i - \int_{B_{2R}(y)} \phi_i) (e_j + \nabla \phi'_j) \cdot a' \nabla \omega_y \right|
\]
\[
\begin{align*}
\lesssim & \frac{1}{R} \left( \int_{B_{2R}(y)} (\phi_i - \int_{B_{2R}(y)} \phi_i)^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}(y)} |e_j + \nabla \phi'_j|^2 \right)^{\frac{1}{2}} \quad (54), (136) \lesssim \left( \frac{r_*(y)}{R} \right)^{\beta}. \\
& (143)
\end{align*}
\]

We now turn to the last right-hand side term in (140). We first apply (55) in Lemma 5 to the effect of
\[
\begin{align*}
\lefthalfcup (\phi_j' - \int_{B_{2R}(y)} \phi_j') \nabla \omega_y \cdot a (e_i + \nabla \phi_i) & - \left( \int_{B_{2R}(y)} (\phi_{jh}' - \int_{B_{2R}(y)} (\phi_j' - \phi_{jh}')^2 \right) \left( \int_{B_{2R}(y)} |e_i + \nabla \phi_i|^2 \right)^{\frac{1}{2}} \\
\lesssim & \frac{1}{R} \left( \int_{B_{2R}(y)} (\phi_j' - \phi_{jh}') \nabla \omega_y \cdot a (e_i + \nabla \phi_i) \right) \\
& \lesssim \frac{1}{R} \left( \int_{B_{2R}(y)} (\phi_j' - \phi_{jh}') \nabla \omega_y \cdot a (e_i + \nabla \phi_i) \right) \lesssim \left( \frac{r_*(y)}{R} \right)^{\alpha}. \\
& (144)
\end{align*}
\]

We then note that by (13), the skew symmetry of \(\sigma_i\) and two integration by parts
\[
\begin{align*}
\int (\phi_{jh}' - \int_{B_{2R}(y)} \phi_{jh}') \nabla \omega_y \cdot a (e_i + \nabla \phi_i) & = - \int \omega_y \nabla \phi_{jh}' \cdot a_i e_i - \int \nabla \omega_y \cdot (\sigma_i - \int_{B_{2R}(y)} \sigma_i) \nabla \phi_{jh}',
\end{align*}
\]
so that
\[
\begin{align*}
\lefthalfcup (\phi_{jh}' - \int_{B_{2R}(y)} \phi_{jh}') \nabla \omega_y \cdot a (e_i + \nabla \phi_i) & - \left( \int_{B_{2R}(y)} (\phi_{jh}' - \phi_{jh}) \nabla \omega_y \cdot a (e_i + \nabla \phi_i) \right) \\
\lesssim & \frac{1}{R} \left( \int_{B_{2R}(y)} |\sigma_i - \int_{B_{2R}(y)} \sigma_i|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}(y)} |\nabla \phi_{jh}'|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r_*(y)}{R} \right)^{\beta}. \\
& (145)
\end{align*}
\]

The combination of (144) and (145) yields for the last term in (140):
\[
\begin{align*}
\lefthalfcup (\phi_j' - \int_{B_{2R}(y)} \phi_j') \nabla \omega_y \cdot a (e_i + \nabla \phi_i) & + \int \omega_y \nabla \phi_{jh}' \cdot a_i e_i \lesssim \left( \frac{r_*(y)}{R} \right)^{\alpha}. \\
& (146)
\end{align*}
\]

Inserting the four estimates (141), (142), (143) and (146) into (140), we obtain (139). \(\square\)

### 3.10 Proof of Lemma 6

**Proof** Starting point is Lemma 1, more precisely (29) for \(R := \frac{1}{2} |y|\) and (30) in form of
\[
\begin{align*}
\left( \int_{B_{2R}(y)} |\nabla (u - (1 + (\phi_i - \int_{B_{2R}(y)} \phi_i)) \partial_i) u_h)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{r_*(y)}{|y|} \right)^{\beta}. \\
& (147)
\end{align*}
\]
The first post-processing step is to replace $\int_{B_{r_*}} \phi_i$ by $\int_{B_{r_*}(y)} \phi_i$ in (147). Indeed, by (69) (for $r = L = \frac{1}{2}|y|$ and $r' = r_*$) we have

\[
\frac{1}{L} \left| \int_{B_L} (\phi, \sigma) - \int_{B_{r_*}(y)} (\phi, \sigma) \right| + \frac{1}{L} \left| \int_{B_{L}(y)} (\phi, \sigma) - \int_{B_{r_*}(y)} (\phi, \sigma) \right| \\
\lesssim \left( \frac{\max\{r_*, r_*(y)\}}{|y|} \right)^\beta. \tag{148}
\]

As a direct consequence of (16) (for $r = 3L$) and $B_L \cup B_L(y) \subset B_{3L}$ we have

\[
\frac{1}{L} \left| \int_{B_{r_*}} (\phi, \sigma) - \int_{B_{r_*}(y)} (\phi, \sigma) \right| \lesssim \left( \frac{r_*}{|y|} \right)^\beta,
\]

so that (148) implies

\[
\frac{1}{L} \left| \int_{B_{r_*}} (\phi, \sigma) - \int_{B_{r_*}(y)} (\phi, \sigma) \right| \lesssim \left( \frac{\max\{r_*, r_*(y)\}}{|y|} \right)^\beta.
\]

We combine this with (73) in form of

\[
\sup_{B_L^c} |\nabla u_h| + \sup_{B_L^c} L |\nabla^2 u_h| \lesssim \left( \frac{\ell}{|y|} \right)^d \tag{149}
\]

to the desired

\[
\left( \int_{B_L^c} |\nabla (u - (1 + \phi_i \partial_i)u_h)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{\max\{r_*, r_*(y)\}}{|y|} \right)^\beta,
\]

where we recall our notation $\int_{B_L^c} = \frac{1}{L^d} \int_{B_L^c}$. Hence without loss of generality we assume $\int_{B_{r_*}(y)} \phi_i = 0$ so that on the one hand, the above simplifies to

\[
\left( \int_{B_L^c} |\nabla (u - (1 + \phi_i \partial_i)u_h)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{\max\{r_*, r_*(y)\}}{|y|} \right)^\beta, \tag{150}
\]

and (17), with the origin replaced by $y$, assumes the form of

\[
\frac{1}{r} \left( \int_{B_{r}(y)} |(\phi, \sigma)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r_*(y)}{r} \right)^\beta \quad \text{for all } r \geq r_*(y). \tag{151}
\]

Following Step 6 in the proof of [4, Theorem 0.2], we now localize (150) around $y$, making use of (151). To this purpose, we appeal once more to the formula (78) for the error in the two-scale convergence

\[-\nabla \cdot a \nabla (u - (1 + \phi_i \partial_i)u_h) = \nabla \cdot h \quad \text{where} \quad h := (\phi_i a - \sigma_i) \nabla \partial_i u_h.\]
The combination of (149) (note $B_r(y) \subset B_L^c$ for $r \leq L = \frac{1}{2}|y|$) and (151) yields for the right-hand side

$$\left( \int_{B_r(y)} |h|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \frac{r}{L} \left( \frac{r_*}{r} \right)^\beta$$

for all $L \geq r \geq r_*(y)$. \hfill (152)

We now argue, starting from the large-scale anchoring of (150) in form of

$$\left( \int_{B_L(y)} |\nabla (u - (1 + \phi_\ell \partial_\ell u_h))|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{\max\{r_*, r_*(y)\}}{|y|} \right)^\beta,$$ \hfill (153)

that (152) allows for the desired localization to our scale of interest $R \geq r_*(y)$:

$$\left( \int_{B_R(y)} |\nabla (u - (1 + \phi_\ell \partial_\ell u_h))|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{\max\{r_*, r_*(y)\}}{|y|} \right)^\beta.$$ \hfill (154)

To this purpose, for every radius $r = R, 2R, \ldots$ with $r \leq L$ we consider the Lax-Milgram solution of

$$-\nabla \cdot a \nabla u_r = \nabla \cdot (I(B_r(y) - B_{2r}(y))h)$$

with the understanding that for $r = R$, the right-hand side is given by $I(B_R(y))h$. From the energy estimate and (152) we obtain

$$\left( \frac{1}{L^d} \int |\nabla u_r|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \frac{r}{L} \left( \frac{r_*}{r} \right)^\beta.$$ \hfill (155)

Since for $r \geq 2R$, $u_r$ is $a$-harmonic in $B_{2r}(y)$, we may apply the $C^{0,1}$-estimate, cf (80), to localize the above to

$$\left( \int_{B_R(y)} |\nabla u_r|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \frac{r}{L} \left( \frac{r_*}{r} \right)^\beta.$$ \hfill (156)

Since the exponent $1 - \beta$ of $r$ is positive, \( \tilde{u} := \sum_{L \geq r \geq R} u_r \) satisfies

$$\left( \int_{B_R(y)} |\nabla \tilde{u}|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{r_*}{|y|} \right)^\beta.$$ \hfill (156)

Likewise, we obtain directly from (155)

$$\left( \frac{1}{L^d} \int |\nabla \tilde{u}|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{r_*}{|y|} \right)^\beta.$$ \hfill (157)
Since by construction, \( u - (1 + \phi_i \partial_i)u_h - \tilde{u} \) is harmonic in \( B_{L_2}(y) \), we may apply the \( C^{0,1} \)-estimate once more to the effect of

\[
\left( \int_{B_{R}(y)} |\nabla (u - (1 + \phi_i \partial_i)u_h - \tilde{u})|^2 \right)^{\frac{1}{2}} \leq \left( \int_{B_{L}(y)} |\nabla (u - (1 + \phi_i \partial_i)u_h - \tilde{u})|^2 \right)^{\frac{1}{2}}. \tag{158}
\]

By the triangle inequality in \( L^2 \), we obtain (154) from (158), (156), (157), and (153).

We now post-process (154), making use of \( \max \{ \rho_\ast, \rho_\ast(y) \} \leq R \), to

\[
\left( \int_{B_{R}(y)} |\nabla u - \partial_i u_h(y)(e_i + \nabla \phi_i)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^\beta. \tag{159}
\]

and also note for later purpose

\[
\left( \int_{B_{R}(y)} |\nabla u|^2 \right)^{\frac{1}{2}} + \left( \int_{B_{R}(y)} |\partial_i u_h(y)(e_i + \nabla \phi_i)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d. \tag{160}
\]

In deriving (159) from (154), we first replace \( \nabla (1 + \phi_i \partial_i)u_h \) by \( \partial_i u_h(e_i + \nabla \phi_h) \), which means that by the triangle inequality in \( L^2 \), we need to estimate \( \phi_i \nabla \partial_i u_h \). We have

\[
\left( \int_{B_{R}(y)} |\phi_i \nabla \partial_i u_h|^2 \right)^{\frac{1}{2}} \leq \sup_{B_{R}(y)} |\nabla^2 u_h| \left( \int_{B_{R}(y)} |\phi|^2 \right)^{\frac{1}{2}};
\]

by \( B_{R}(y) \subset B_L \) and (149), the first factor is estimated by \( \frac{1}{|y|} \left( \frac{\ell}{|y|} \right)^d \). By (151), the second factor is estimated by \( R \left( \frac{r_\ast(y)}{|y|} \right)^\beta \). Hence because of \( R \leq |y| \), this contribution is contained in the right-hand side of (159). We now replace \( \partial_i u_h(e_i + \nabla \phi_h) \) by \( \partial_i u_h(y)(e_i + \nabla \phi_h) \). By the triangle inequality in \( L^2 \), we are lead to estimating

\[
\left( \int_{B_{R}(y)} |(\partial_i u_h - \partial_i u_h(y))(e_i + \nabla \phi_i)|^2 \right)^{\frac{1}{2}} \leq R \sup_{B_{R}(y)} |\nabla \partial_i u_h| \left( \int_{B_{R}(y)} |e_i + \nabla \phi_i|^2 \right)^{\frac{1}{2}}.
\]

As above, the first factor is estimated by \( \frac{R}{|y|} \left( \frac{\ell}{|y|} \right)^d \). According to (71) (with \( y \) playing the role of the origin and using \( R \geq r_\ast(y) \)), the second factor is estimated by 1. Because of \( \beta \leq 1 \), also this contribution is contained in the right-hand side of (159). The same argument leads to the second estimate in (160). The first estimate in (160) follows from the second one and (159) via the triangle inequality in \( L^2 \).

We now turn to the sensitivity estimate and consider

\[
w := u' - u, \quad st \quad - \nabla \cdot a' \nabla w = \nabla \cdot h' \quad \text{with} \quad h' := (a' - a) \nabla u. \tag{161}
\]
We want to apply the localized homogenization error estimate of [4, Theorem 0.2], for the medium given by $a'$. This means comparing $w$ to the solution of

$$-\nabla \cdot a_h \nabla w_h = \nabla \cdot (h'_i (e_i + \partial_i \phi')).$$  \hspace{1cm} (162)

Note that by assumption, and by Lemma 8 we have $r'_* (y), r'_* \lesssim R$. Since $h'$ is supported in $B_R(y)$, by [4, Theorem 0.2] (for the medium $a'$, with $R$ playing the role of $r_*$ there, the roles of $y$ and the origin exchanged, and $\beta$ replacing $1 - \alpha$ there) we have

$$\left( \int_{B_R} |\nabla w - \partial_k w_h (e_k + \nabla \phi'_k)|^2 \right)^{1/2} \lesssim \left( \frac{R}{|y|} \right)^{d+\beta} \left( \int_{B_R(y)} |h'|^2 \right)^{1/2}.$$  \hspace{1cm} (163)

(As explained in the proof of Lemma 1, the logarithm in [4, Theorem 0.2] can be avoided.) Among other ingredients, this estimate is based on the following estimate of $w_h$, cf Step 2 of the proof of [4, Theorem 0.2],

$$\sup_{B'_L (y)} (|\nabla w_h| + L |\nabla^2 w_h|) \lesssim \left( \frac{R}{L} \right)^d \left( \frac{R}{|y|} \right)^d.$$  \hspace{1cm} (164)

In view of the definition of $h'$, cf (161), and (160), these two estimates turn into

$$\left( \int_{B_R} |\nabla w - \partial_k w_h (e_k + \nabla \phi'_k)|^2 \right)^{1/2} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+\beta},$$  \hspace{1cm} (165)

$$\sup_{B'_L (y)} (|\nabla w_h| + L |\nabla^2 w_h|) \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{L} \right)^d.$$  \hspace{1cm} (166)

Like for the passage from (154) to (159), (163) may be post-processed with help of (164) to

$$\left( \int_{B_R} |\nabla w - \partial_k w_h (0) (e_k + \nabla \phi'_k)|^2 \right)^{1/2} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+\beta}.$$  \hspace{1cm} (167)

In this argument, we just have to replace the medium $a$ by the medium $a'$ and $y$ by the origin.

We continue with post-processing and argue that we may replace $\phi'_k$ by $\phi_k$ in (165):

$$\left( \int_{B_R} |\nabla w - \partial_k w_h (0) (e_k + \nabla \phi'_k)|^2 \right)^{1/2} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+\beta}.$$  \hspace{1cm} (168)

In fact, we claim that the error term is of (substantially) higher order:

$$|\partial_k w_h (0) | \left( \int_{B_R} |\nabla (\phi'_k - \phi_k)|^2 \right)^{1/2} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{2d}.$$  \hspace{1cm} (169)
which by (164) reduces to
\[
\left( \int_{B_R} |\nabla (\phi'_k - \phi_k)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{R}{|y|} \right)^d.
\] (167)

The latter can be seen noting that
\[-\nabla \cdot a \nabla (\phi'_k - \phi_k) = \nabla \cdot (a' - a)(e_k + \nabla \phi'_k)\]. (168)

From (168) we learn at first that \(\phi'_k - \phi_k\) is \(a\)-harmonic in \(B^c_{R}(y)\) so that by the \(C^{0,1}\)-estimate, cf (80), and since \(R \geq r^*_y\)
\[
\left( \int_{B_R} |\nabla (\phi'_k - \phi_k)|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{B^c_{R}(y)} |\nabla (\phi'_k - \phi_k)|^2 \right)^{\frac{1}{2}},
\]
where we recall our abbreviation \(L = \frac{1}{2}|y|\). Moreover, since the right-hand side of (168) is in divergence form, we have by the dualized \(C^{0,1}\)-estimate (see Step 5 in the proof of [4, Theorem 0.2] for such a duality argument) and since \(R \geq r^*_y\)
\[
\left( \frac{1}{L^d} \int_{B^c_{L}(y)} |\nabla (\phi'_k - \phi_k)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{R}{L^d} \int_{B^c_{R}(y)} |\nabla (\phi'_k - \phi_k)|^2 \right)^{\frac{1}{2}}.
\]

Finally, by the energy estimate for (168) we have
\[
\left( \frac{1}{R^d} \int_{B^c_{R}(y)} |\nabla (\phi'_k - \phi_k)|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{B^c_{R}(y)} |e_k + \nabla \phi'_k|^2 \right)^{\frac{1}{2}} \lesssim 1,
\]
where in the very last estimate, we have used (71) (with \((a, 0)\) replaced by \((a', y)\)), which we may since by Lemma 8 we have \(r^*_y(y) \lesssim R\). Since by definition of \(L\), \(B_L \subset B^c_{L}(y)\) the last three estimates combine to (167).

In the remainder of the proof we argue that we may replace \(w_h\) by a more explicit expression. More precisely, in order to pass from (166) to (59) we have to replace \(\partial_k w_h(0)\) by
\[
\partial_j \partial_k G_h(-y) \partial_i u_h(y) \int (e_j + \nabla \phi'_j) \cdot (a' - a)(e_i + \nabla \phi_i),
\] (169)
and then appeal to the definition (57) of \(\delta a_{ij}\). The basis for this is the representation
\[
\partial_k w_h(0) = \int \partial_j \partial_k G_h(-y)((e_j + \nabla \phi'_j) \cdot (a' - a) \nabla u)(x) dx,
\] (170)
which is a consequence of the definition (162) of \(w_h\), yielding the representation
\[
\partial_k w_h(0) = \int \nabla \partial_k G_h(-x) \cdot (h'_i(e_i + \partial_i \phi'))(x) dx
\]
= \int \partial_j \partial_k G_h(-x)(h' \cdot (e_j + \nabla \phi'_j))(x)dx,

into which we insert the definition (161) of \( h' \). We split the passage from (170) to (169) into two steps. In view of (184), which is a consequence of \( R \gtrsim r'_* \), cf (71), it is enough to show

\[
R^d \sup_{x \in BR(y)} |\partial_j \partial_k G_h(-x) - \partial_j \partial_k G_h(y)| \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+1},
\]

\[
R^d |\partial_j \partial_k G_h(-y)| \int_{BR} |e_j + \nabla \phi'_j||\nabla u - \partial_i u_h(y)(e_i + \nabla \phi_i)| \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+\beta},
\]

which by the obvious decay properties of \( G_h \) reduces to

\[
\int_{BR} |e_j + \nabla \phi'_j||\nabla u - \partial_i u_h(y)(e_i + \nabla \phi_i)| \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^\beta.
\]

By the Cauchy-Schwarz inequality, because of (176) below, we obtain

\[
(R^d |\partial_j \partial_k G_h(-y)| \int_{BR} |e_j + \nabla \phi'_j||\nabla u - \partial_i u_h(y)(e_i + \nabla \phi_i)| \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+\beta},
\]

which for the Cauchy-Schwarz inequality, because of (176) below, we obtain

\[
(\int_{BR} |\nabla (u'_h - u_h) - \partial_i u_h(y)\delta a_{hi} \partial_j \partial_k G_h(-y)e_k|^2)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+\beta}.
\]

We insert (58) into this estimate, in form of \( \frac{1}{R^d} |\delta a_{ij} - \delta a_{hij}|, \) cf definitions (57) and (60). Combined with (176) below, we obtain

\[
(\int_{BR} |\partial_k (u'_h - u_h) - \partial_i u_h(y)\delta a_{ij} \partial_j \partial_k G_h(-y)|^2)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+\beta}.
\]

3.11 Proof of Proposition 2

**Proof** We may apply Lemma 6 to the heterogeneous medium \( a \) replaced by the homogeneous \( a_h \); in which case we may choose \((\phi, \sigma) \equiv 0\) and thus in particular \( r_* = r_*(y) = 0 \), whereas in view of (52) and (53), \( \phi' \) is being replaced by \( \phi'_h \) and thus \( \delta a \) by \( \delta a_h \), cf (60). An inspection of the proof of Lemma 6 shows that me may replace \( w = u' - u \) by \( w_h := u'_h - u_h \), since the crucial property of that difference was, when translated to the homogeneous medium,

\[
w_h := u'_h - u_h,
\]

\[
\nabla \cdot a'_h \nabla w_h = \nabla \cdot h'_h \quad \text{with} \quad h'_h := (a'_h - a_h) \nabla u_h,
\]

cf (161) for \( w \), which for \( w_h \) follows immediately from the definition (50) of \( u'_h \). Hence, (59) assumes the form

\[
(\int_{BR} |\nabla (u'_h - u_h) - \partial_i u_h(y)\delta a_{hi} \partial_j \partial_k G_h(-y)e_k|^2)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+\beta}.
\]

We insert (58) into this estimate, in form of \( \frac{1}{R^d} |\delta a_{ij} - \delta a_{hij}|, \) cf definitions (57) and (60). Combined with (176) below, we obtain

\[
(\int_{BR} |\partial_k (u'_h - u_h) - \partial_i u_h(y)\delta a_{ij} \partial_j \partial_k G_h(-y)|^2)^{\frac{1}{2}} \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+\beta}.
\]
\[ \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d \left( \frac{R}{|y|} \right)^\beta + \left( \frac{r_*(y)}{R} \right)^\alpha. \]

In combination with \((71)\) in form of \((f_{B_R} |e_k + \nabla \phi_k|^2)^\frac{1}{2} \lesssim 1\), this assumes the form

\[
\left( \int_{B_R} |\partial_k (u' - u) (e_k + \nabla \phi_k) - \partial_i u (y) \delta_{aij} \partial_j \partial_k G_h (-y) (e_k + \nabla \phi_k)|^2 \right)^{\frac{1}{2}} 
\lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d \left( \frac{R}{|y|} \right)^\beta + \left( \frac{r_*(y)}{R} \right)^\alpha.
\]

By the triangle inequality in \(L^2\) with \((59)\) in its original form, we obtain \((51)\). \(\square\)

### 3.12 Proof of Lemma 8

**Proof** Clearly, it suffices to establish \((63)\) at the origin (which here plays the role of the general point), that is,

\[ r'_* \lesssim r_* + r_*(y) + R. \]

According to the definition of \(r_*\) as the minimal radius with the property \((16)\), by the triangle inequality in \(L^2\), it is enough to show

\[ \frac{1}{r} \left( \int_{B_r} |(\phi' - \phi, \sigma' - \sigma) - \int_{B_r} (\phi' - \phi, \sigma' - \sigma)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r_*(y) + R}{r} \right)^\beta \text{ for } r \geq r_*(y) + R, \]

which by Poincaré’s inequality follows from

\[ \left( \int_{B_r} |\nabla (\phi' - \phi, \sigma' - \sigma)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r_*(y) + R}{r} \right)^\beta \text{ for } r \geq r_*(y) + R. \]

In view of \(\beta < 1 \leq \frac{d}{2}\) and of \((f_{B_r} f^2)^{\frac{1}{2}} \lesssim \left( \frac{R}{r} \right)^\frac{d}{2} \left( \frac{1}{R^d} \int f^2 \right)^{\frac{1}{2}}\) this follow from

\[ \left( \frac{1}{R^d} \int |\nabla (\phi' - \phi, \sigma' - \sigma)|^2 \right)^{\frac{1}{2}} \lesssim \left( 1 + \frac{r_*(y)}{R} \right)^\frac{d}{2}, \tag{171} \]

which we shall establish now.

In order to tackle \((171)\), we fix a coordinate direction \(i = 1, \ldots, d\) and suppress the index \(i\) in \((\phi_i, \sigma_i)\) and \((\phi'_i, \sigma'_i)\). Since \(-\nabla \cdot a' \nabla (\phi' - \phi) = \nabla \cdot (a' - a) (e + \nabla \phi)\), we have by the energy inequality

\[ \left( \frac{1}{R^d} \int |\nabla (\phi' - \phi)|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{B_{Rr}(y)} |e + \nabla \phi|^2 \right)^{\frac{1}{2}} . \tag{172} \]
We now note that Caccioppoli’s estimate (71) implies
\[
\left( \int_{B_R(y)} |e + \nabla \phi|^2 \right)^{\frac{1}{2}} \lesssim \left( 1 + \frac{r_*(y)}{R} \right)^\frac{d}{2}; \tag{173}
\]
This is immediate in case of \( R \geq r_*(y) \) and follows from \( \left( \int_{B_{r_*(y)}} f^2 \right)^{\frac{1}{2}} \leq \left( \frac{R}{r_*(y)} \right)^\frac{d}{2} \) \( \left( \int_{B_{r_*(y)}} f^2 \right)^{\frac{1}{2}} \) in the other case. The combination of (172) and (173) yields (171) for the \( \phi \)-part.

We now turn to the \( \sigma \)-part of (171) and note that
\[
-\Delta(\sigma'_{jk} - \sigma_{jk}) = \partial_j(q' - q)_k - \partial_k(q' - q)_j \quad \text{where} \quad q' - q = a'(\phi' - \phi) + (a' - a)(e + \nabla \phi).
\]
Hence, we obtain by the energy estimate
\[
\left( \frac{1}{R^d} \int |\nabla(\sigma' - \sigma)|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{1}{R^d} \int |\nabla(\phi' - \phi)|^2 \right)^{\frac{1}{2}} + \left( \int_{B_R(y)} |e + \nabla \phi|^2 \right)^{\frac{1}{2}},
\]
so that it remains to appeal to (172) and (173) once more. \( \square \)

### 3.13 Proof of Corollary 3

**Proof** We start by post-processing Lemma 6 by getting rid of the constraint that \( R \geq r_*, r_*(y) \). Indeed, in case of \( R \leq \max\{r_*, r_*(y)\} \) we apply Lemma 6 with \( \max\{r_*, r_*(y)\} \) playing the role of \( R \) so that (59) turns into
\[
\left( \int_{B_{\max\{r_*, r_*(y)\}}} |\nabla(u' - u) - \partial_i u_h(y) \delta a_{ij} \partial_j \partial_k G_h(-y)(e_k + \nabla \phi_k)|^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \frac{\ell}{|y|} \right)^d \left( \max\{r_*, r_*(y)\} \right)^{d+\beta},
\]
which yields
\[
\left( \int_{B_R} |\nabla(u' - u) - \partial_i u_h(y) \delta a_{ij} \partial_j \partial_k G_h(-y)(e_k + \nabla \phi_k)|^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+\beta} \left( \max\{r_*, r_*(y)\} \right)^{d+\beta+\frac{d}{2}},
\]
since the integral over the smaller ball \( B_R \) can be controlled by that over the larger ball \( B_{\max\{r_*, r_*(y)\}} \), and hence the average up to a factor of volume ratio. Hence we obtain in either case
\[
\left( \int_{B_R} |\nabla(u' - u) - \partial_i u_h(y) \delta a_{ij} \partial_j \partial_k G_h(-y)(e_k + \nabla \phi_k)|^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^{d+\beta} \left( 1 + \frac{r_* + r_*(y)}{R} \right)^{d+\beta+\frac{d}{2}}. \tag{174}
\]
We now pass from a strong-$L^2$-estimate to an estimate of averages, which allows us to get rid of the corrector in (174):

$$\left| \int \omega \left( \nabla (u' - u) - \partial_i u_h(y) \delta \alpha_{ij} \partial_j \partial_k G_h(-y) e_k \right) \right| \leq (\ell |y|)^d \left( \frac{R}{|y|} \right)^d \left( \frac{R}{|y|} \right)^\beta \left( 1 + \frac{r_*(y)}{R} \right)^{3d+\beta}.$$  

(175)

Indeed, this follows from the obvious estimate $|\nabla g| \lesssim (\frac{R}{|y|})^d |\nabla u_h(y)|$ applied to the field $g := \nabla (u' - u) - \partial_i u_h(y) \delta \alpha_{ij} \partial_j \partial_k G_h(-y)(e_k + \nabla \phi_k)$, combined with an argument that the contribution from $\nabla \phi_k$ is negligible. For the latter we first note that

$$R^d |\nabla^2 G_h(-y)| \lesssim (\frac{R}{|y|})^d, \quad |\nabla u_h(y)| \lesssim (\frac{\ell}{|y|})^d$$  

(176)

and that

$$\frac{1}{R^d} |\delta \alpha_{ij}| \lesssim (1 + \frac{r_*(y)}{R})^{-\frac{d}{2}}.$$  

(177)

In case of $R \geq \max\{r_*(y), r'_*(y)\}$ (recall that $r'_*(y)$ for medium $a'$ is the counterpart of $r_*(y)$ for medium $a$), (177) follows immediately from the definition (57), the fact that the integral is restricted to $B_R(y)$ by (47), and the Caccioppoli estimate (71) with $y$ playing the role of the origin and also applied to the perturbed medium $a'$. In case of $R \leq \max\{r_*(y), r'_*(y)\}$ we argue as in the previous paragraph, that is, we apply the above estimate with $\max\{r_*(y), r'_*(y)\}$ playing the role of $R$, which is legitimate in view of the only constraint (47) on $R$. This yields (177) with a right-hand side given by $\left( 1 + \frac{\max\{r_*(y), r'_*(y)\}}{R} \right)^{\frac{d}{2}}$, so that it remains to appeal to Lemma 8. We now may turn to $\int \omega \nabla \phi_k$ itself: By integration by parts we obtain

$$\left| \int \omega \nabla \phi_k \right| = \left| \int (\phi_k - \int_{B_R} \phi_k) \nabla \omega \right| \lesssim \frac{1}{R} \left( \int_{B_R} (\phi_k - \int_{B_R} \phi_k)^2 \right)^{\frac{1}{2}}.$$  

(178)

In case of $R \geq r_*$, the right-hand side is controlled by $(\frac{r_*}{R})^\beta$ according to (16). In the other case, we argue as before to obtain an estimate by $(\frac{r_*}{R})^{\frac{d}{2}+1}$. Hence, we obtain in either case

$$\frac{1}{R} \left( \int_{B_R} (\phi_k - \int_{B_R} \phi_k)^2 \right)^{\frac{1}{2}} \lesssim \frac{r_*}{R} \beta (1 + \frac{\max\{r_*(y), r'_*(y)\}}{R})^{\frac{d}{2}+1-\beta}.$$  

(179)

Inserting (179) into (178) and combining with (176) and (177) yields

$$\left| \partial_i u_h(y) \delta \alpha_{ij} \partial_j \partial_k G_h(-y) \int \omega \nabla \phi_k \right|$$
\[ \leq \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d \left( 1 + \frac{r_*(y)}{R} \right)^d \left( \frac{r_*(y)}{R} \right)^{(d+1-\beta)} , \]

which completes the argument for (175) since \( (1 + \frac{r_*(y)}{R})^d (1 + \frac{r_*(y)}{R})^{d+1-\beta} \leq (1 + \frac{r_*(y)}{R})^{d+\beta}. \)

We now bring Corollary 2 into play, which in our abbreviations (60) and (57) reads  
\[ \frac{1}{R^d} |\delta a_{ij} - \delta a_{hij}| \lesssim (\frac{r_*(y)}{R})^\alpha \]  
in case of \( R \geq r_*(y) \). In the other case, we use the triangle inequality  \( \frac{1}{R^d} |\delta a_{ij} - \delta a_{hij}| \leq \frac{1}{R^d} |\delta a_{ij}| + \frac{1}{R^d} |\delta a_{hij}| \) recall the argument for (177) that gave \( \frac{1}{R^d} |\delta a_{ij}| \lesssim \begin{cases} \max\{r_*(y), r'_*(y)\} & \frac{d}{R} \lesssim (1 + \frac{r_*(y)}{R})^{d-\alpha}, \end{cases} \]  
while we easily get \( \frac{1}{R^d} |\delta a_{hij}| \leq 1 \) from (120). Hence in either case, we have

\[ \frac{1}{R^d} |\delta a_{ij} - \delta a_{hij}| \lesssim (\frac{r_*(y)}{R})^\alpha (1 + \frac{r_*(y)}{R})^{d-\alpha}. \]

This allows us to upgrade (175) to

\[ |\int \omega(\nabla(u' - u) - \partial_i u_h(y)\delta a_{hij}\partial_j \partial_k G_h(-y)e_k)| \]

\[ \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d \left( (\frac{R}{|y|})^\beta + (\frac{r_*(y)}{R})^\beta + (\frac{r_*(y)}{R})^\alpha \right) (1 + \frac{r_*(y)}{R})^{d+\beta}. \]  \hspace{1cm} (180)

Finally, we will substitute \( u_h \) by \( (\int g \cdot \nabla G_h \) in (180). Starting point is the following representation of \( u_h \) which follows from its definition in (9) and (11):

\[ u_h(x) = \int \nabla G_h(x - x') \cdot g(x')\,dx' + (\int \nabla \phi_m \cdot g) \partial_m G_h(x) \]  
and thus

\[ \partial_i u_h(y) = \int \partial_i \partial_m G_h(y - x')g_m(x')\,dx' + (\int \nabla \phi_m \cdot g) \partial_i \partial_m G_h(y). \]

Hence, we have

\[ |\partial_i u_h(y) - (\int (e_m + \nabla \phi_m) \cdot g) \partial_i \partial_m G_h(y)| \]

\[ \lesssim \ell^d \sup_{x' \in B_\ell} |\partial_i \partial_m G_h(y - x') - \partial_i \partial_m G_h(y)| \lesssim \left( \frac{\ell}{|y|} \right)^{(d+1)}. \]  \hspace{1cm} (181)

In order to argue that the contribution from \( \int \nabla \phi_m \cdot g \) is negligible we argue as for (178) and (179) to obtain

\[ |\int \nabla \phi_m \cdot g| \lesssim \ell^d \left( \frac{\ell}{\ell} \right)^\beta (1 + \frac{r_*(y)}{R})^{d+1-\beta}. \]
Hence with help of \(|\partial_i \partial_m G_h(y)| \lesssim \frac{1}{|y|^d}\) we may upgrade (181) to
\[
|\partial_i u_h(y) - (\int g_m) \partial_i \partial_m G_h(y)| \lesssim \left( \frac{\ell}{|y|} \right)^d \left( \frac{\ell}{|y|} + \left( \frac{r_*}{\ell} \right)^\beta (1 + \frac{r_*}{R})^{\frac{d+1-\beta}{2}} \right).
\] (182)
Since \(|\delta a_{hij}| \lesssim R^d\), see above, and \(|\partial_j \partial_k G_h(-y)| \lesssim \frac{1}{|y|^d}\) this allows to pass from (180) to (61).
\[
\square
\]

### 3.14 Proof of Lemma 7

**Proof** In view of the independence properties of the Poisson point process, it follows from elementary probability theory that for any random variable \(F\) (i.e., a function \(F = F(X)\) of the point configuration \(X\)) we have
\[
\left\langle \left( (F|_{B_{R-1}(y)} - (F))^2 \right) \right\rangle = \frac{1}{2} \left\langle \left( (F')_{out} - \langle F \rangle_{out} \right)^2 \right\rangle_{in|\{X'\}}',
\]
where \(\langle \cdot \rangle_{out} := \langle \cdot |_{B_{R-1}(y)} \rangle\), where \(\langle \cdot \rangle_{in}\) and \(\langle \cdot \rangle_{in}'\) denote two independent copies of the Poisson process restricted to \(B_{R-1}(y)\), and where \(F' := F(X')\) with \(X'\) denoting the realization of the Poisson point process that arises from concatenating \(X|_{B_{R-1}(y)}\) and \(X'|_{B_{R-1}(y)}\). We apply this to \(F = \int \omega \nabla u\) and note that \(F' = \int \omega \nabla u'\), provided \(a' := A(\cdot, X')\), cf Definition 1 and (48). By definition of \(X'\) and the locality assumption (43) on \(A\), this is consistent with the second condition in (47) where \(a := A(\cdot, X)\); the first condition in (47) comes for free by setting \(a_0 := a|_{B_{R}(y)}\). Hence in order to establish (62) (with \(R\) replaced by \(R - 1\)), we have to show for \(R \gg 1\) that
\[
\left\langle \left( (\int \omega \nabla u')_{out} - \langle \int \omega \nabla u \rangle_{out} \right)^2 \right\rangle_{in|\{X'\}} ' \gtrsim \left( \frac{\ell}{|y|} \right)^d \left( \frac{1}{|y|} \right)^d
\]
where by homogeneity and rotational invariance, we have assumed without loss of generality
\[
\int \hat{g} = e_1.
\] (184)

We apply \(\langle \cdot \rangle_{out}\) to (61) in Corollary 3 and obtain by Jensen’s inequality in probability
\[
\left| \langle \int \omega \nabla u' \rangle_{out} - \langle \int \omega \nabla u \rangle_{out} - (\int g_m) \partial_m \partial_i G_h(y) \langle \delta a_{hij} \rangle_{out} \partial_i \partial_j G_h(-y)e_k \right| \lesssim \langle \text{err} \rangle_{out},
\] (185)
where we have set for abbreviation
\[
\text{err} := \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d
\]
\[
\times \left( \frac{R}{|y|} \right)^{\beta} + \left( \frac{r_s}{R} \right)^{\beta} + \left( \frac{r_s(y)}{R} \right)^{\alpha} + \left( \frac{r_s}{\ell} \right)^{\beta} \right) \left( 1 + \frac{r_s + r_s(y)}{R} \right)^{3d + \beta}.
\]

Thanks to the normalization (184), which by (4) turns into \( \int g = \ell^d e_1 \), we obtain from just considering the first component, and using the symmetry and homogeneity of \( \nabla^2 G_h \)

\[
\left| \left( \int g_m \right) \partial_m \partial_i G_h(y) (\delta a_{ij})_{\text{out}} \partial_j \partial_k G_h(-y) e_k \right|
\geq \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d \left| \partial_i \partial_1 G_h(\hat{y}) (\frac{1}{R^d} \delta a_{ij})_{\text{out}} \partial_j \partial_1 G_h(\hat{y}) \right|.
\]

Hence under the proviso

\[-\xi \cdot \left( \frac{1}{R^d} \delta a_h \right)_{\text{out}} \xi \geq \lambda_0 |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d \quad (186)\]

for some \( \lambda_0 > 0 \) to be chosen later, we obtain

\[
\left| \left( \int g_m \right) \partial_m \partial_i G_h(y) (\delta a_{ij})_{\text{out}} \partial_j \partial_k G_h(-y) e_k \right| \geq \lambda_0 \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d |\nabla^2 G_h(\hat{y}) e_1|^2.
\]

which by the invertibility of the Hessian matrix \( \nabla^2 G_h(\hat{y}) \)\(^3\) and its continuity on \( \{|\hat{y}| = 1\} \) implies

\[
\left| \left( \int g_m \right) \partial_m \partial_i G_h(y) (\delta a_{ij})_{\text{out}} \partial_j \partial_k G_h(-y) e_k \right| \geq \lambda_0 \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d.
\]

Therefore, always under the proviso (186), we obtain the following lower bound from (185)

\[
\lambda_0 \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d \leq \left| \langle \int \omega \nabla u' \rangle_{\text{out}} - \langle \int \omega \nabla u \rangle_{\text{out}} \right| + \langle \text{err} \rangle_{\text{out}}. \quad (187)
\]

We note that by definition, cf (49), (53) and (60), \( \delta a_{ij} \) only depends on \( a_{0|B_{R}(\hat{y})} = a'_{B_{R}(\hat{y})} = A(\cdot, X')_{B_{R}(\hat{y})} \), which by the locality of A, cf (43), implies that \( \delta a_{ij} \) and \( \langle \delta a_{ij} \rangle_{\text{out}} \) are in particular independent of \( X|_{B_{R-1}(\hat{y})} \). Hence, the proviso (186) is not affected by (squaring and) applying \( \langle \cdot \rangle_{\text{in}} \) to (187): Under the proviso (186), we have

\[
\lambda_0 \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d \leq \left| \langle \int \omega \nabla u' \rangle_{\text{out}} - \langle \int \omega \nabla u \rangle_{\text{out}} \right| + \langle \text{err}^2 \rangle_{\text{in}},
\]

where we have used Jensen’s inequality in probability on the last term. In particular, we may multiply this estimate with the characteristic function \( I( -\frac{1}{R^d} (\delta a_h)_{\text{out}} \geq \lambda_0 \text{id}) \)

\(^3\) In the case of \( a_h = \text{id} \), the formula for \( \nabla^2 G_h(\hat{y}) \) is classical and easily seen to be invertible; for a general (symmetric and positive) \( d \times d \) matrix \( a_h \), a linear change of variables transforms \( -\nabla \cdot a_h \nabla \) into \( -\Delta \).
of the event (186) and then apply \( \langle \cdot \rangle'_{in} \), which yields
\[
\text{prob}(\lambda_0(\frac{\ell}{|y|})^d(\frac{R}{|y|})^d)^2 \lesssim \left( \left| \langle \int \omega \nabla u'|_{out} - \langle \int \omega \nabla u|_{out} \rangle'_{in} \right|_{in} \right)^2 + \text{prob}(\text{err}^2),
\]
where we have set for abbreviation
\[
\text{prob} := \langle I(-\frac{1}{R} \langle \delta a_h \rangle_{out} \geq \lambda_0 \text{id}) \rangle'_{in}.
\]
Since \( r_* \) has finite moments, cf (45), (and thus also \( r_*(y) \) in view of the shift-covariance (42), which translates into shift covariance of \( \nabla (\phi, \sigma) \) and thus of \( r_* \), in conjunction with the stationarity of \( \langle \cdot \rangle \) we have
\[
\langle \text{err}^2 \rangle \lesssim \left( \left( \frac{\ell}{|y|} \right)^d \left( \frac{R}{|y|} \right)^d \left( \frac{1}{R} \right)^{2\beta} + \left( \frac{1}{R} \right)^{\alpha} + \left( \frac{\ell}{|y|} \right)^2 \right)^2.
\]
Hence provided
\[
|y| \gg \lambda_0^{-1/\beta}, \quad R \gg \lambda_0^{-1/\alpha}, \quad |y| \gg \lambda_0^{-1/\ell}, \quad \ell \gg \lambda_0^{-1/\beta},
\]
we may absorb the second right-hand side term to obtain
\[
\text{prob}(\lambda_0(\frac{\ell}{|y|})^d(\frac{R}{|y|})^d)^2 \lesssim \left( \left| \langle \int \omega \nabla u'|_{out} - \langle \int \omega \nabla u|_{out} \rangle'_{in} \right|_{in} \right)^2.
\]
In order to derive (62) for a fixed, but sufficiently large \( R \), from (189), it remains to argue that there exists a \( \lambda_0 \in (0, 1] \) only depending on the ensemble such that
\[
\text{prob} = \langle I(-\frac{1}{R} \langle \delta a_h \rangle_{out} \geq \lambda_0 \text{id}) \rangle'_{in} \geq \exp(-CR^d)
\]
for some constant \( C = C(d) \), where it only matters that the right-hand side of (190) only depends on \( R \) and is positive for every finite \( R \). To this purpose, we first claim that
\[
\frac{1}{(R - 2)^d} \int_{B_{R-2}(y)} \langle a_h - a' \rangle \geq 2\lambda_0 \text{id} \quad \Longrightarrow \quad -\frac{1}{R^d} \delta a_h \geq \lambda_0 \text{id}.
\]
Here comes the argument: by definition (60) we have
\[
\xi \cdot \delta a_h \xi = \int (\xi + \nabla \phi'_{h\xi} \cdot (a'_h - a_h)\xi, \quad \phi'_{h\xi} := \xi \cdot \phi' f_{j}.
\]
Indeed, since by definition (192) of \( \phi'_{h\xi} \) and (53) we have \( -\nabla \cdot a'_h (\xi + \nabla \phi'_{h\xi}) = 0 \), which we rewrite as \( -\nabla \cdot a'_h \nabla \phi'_{h\xi} = \nabla \cdot (a'_h - a_h)\xi \), we obtain from (192) and (49)
the representation
\[ \tilde{\xi} \cdot \delta a_h \xi = \int_{B_R(y)} \xi \cdot (a_0 - a_h) \tilde{\xi} - \int \nabla \phi_h \cdot a_h' \nabla \phi_h'. \]

Hence for the quadratic part we have the inequality
\[ \tilde{\xi} \cdot \delta a_h \xi \leq \hat{BR}(y) \xi \cdot (a_0 - a_h) \xi, \]

which in view of \( a_0 = a'_{|BR(y)} \) implies
\[ \frac{1}{R^d} \int_{B_R(y)} (a_h - a') \geq \lambda_0 \text{id} \implies -\frac{1}{R^d} \delta a_h \geq \lambda_0 \text{id}. \]

This yields (191) since
\[ |\frac{1}{(R-2)^d} \int_{B_{R-2}(y)} (a_h - a') - \frac{1}{R^d} \int_{B_R(y)} (a_h - a')| \lesssim \frac{1}{R} \]

and since we are in the regime (188), which in view of \( \alpha \leq 1 \) includes \( \frac{R}{R} \ll \lambda_0 \).

Based on (191) we now argue that
\[ \langle I(-\frac{1}{R^d}(\delta a_h)_{\text{out}} \geq \lambda_0 \text{id}) \rangle'_{\text{in}} \geq \langle I(\frac{1}{(R-2)^d} \int_{B_{R-2}(y)} (a_h - a) \geq 2\lambda_0 \text{id}) \rangle. \]

This follows immediately from (191) and the observation that thanks to the locality assumption (43), the event \( \frac{1}{(R-2)^d} \int_{B_{R-2}(y)} (a_h - a') \geq 2\lambda_0 \text{id} \) only depends on \( X' \) via its restriction to \( B_{R-1}(y) \). Hence by the independence property of the Poisson point process, on the one hand, this event implies \( -\frac{1}{R^d}(\delta a_h)_{\text{out}} \geq \lambda_0 \text{id} \), and on the other hand, the conditional probability \( \langle \cdot \rangle'_{\text{in}} \) of this event agrees with the unconditional one \( \langle \cdot \rangle \).

In view of (193), in order to establish (190), it is enough to argue that there exists a \( \lambda_0 > 0 \) such that
\[ \langle I(\frac{1}{(R-2)^d} \int_{B_{R-2}(y)} (a_h - a) \geq 2\lambda_0 \text{id}) \rangle \geq \exp(-CR^d). \]

Here comes the argument for (194): Letting \( \lambda_0 > 0 \) be so small that \( a_h - A(0, \emptyset) \geq 2\lambda_0 \text{id} \), cf the monotonicity assumption (44), we have by the locality assumption (43) and shift-invariance assumption (42) that for any realization \( X \) of the Poisson point process:
\[ X \cap B_{R-1}(y) = \emptyset \implies a_h - a \geq \frac{2\lambda_0}{|B_1|} \text{id} \text{ in } B_{R-2}(y) \]

\[ \text{ Springer} \]
\[
\implies \frac{1}{(R - 2)^d} \int_{B_{R-2}(y)} (a_r - a) \geq 2\lambda_0 \text{id}.
\]

Hence (194) follows from the defining property of the Poisson point process:

\[
\langle I(X \cap B_{R-1}(y) = \emptyset) \rangle = \exp(-|B_{R-1}(y)|).
\]

\[\Box\]

3.15 Proof of Theorem 2

**Proof** As a consequence of the independence property of the ensemble \( \langle \cdot \rangle \) of the Poisson point process, we have for any random variable, in particular \( F := \int \omega \nabla u \), that

\[
\langle |F - \langle F|B_L\rangle|^2 \rangle \geq \sum_{n=1}^{N} \langle |\langle F|B_R(y_n)\rangle - \langle F \rangle|^2 \rangle,
\]

provided we have for the sets

\[
B_L, B_R(y_1), \ldots, B_R(y_N) \text{ are pairwise disjoint.}
\]

Hence Theorem 2 follows immediately from Lemma 7, since under the assumptions of the theorem there exists a family \( \{y_n\}_{n=1}^{N} \) of points such that (196) holds while

\[
|y_n| \leq 2L \quad \text{and} \quad N \gtrsim L^d,
\]

where \( R \) is the order-one radius given by the lemma. More precisely, we use (62) with \( y_n \) playing the role of \( y \) and which by the first property in (197) assumes the form of

\[
\langle |\langle F|B_R(y_n)\rangle - \langle F \rangle|^2 \rangle \gtrsim \left( \frac{\ell}{L} \right)^d \left( \frac{1}{L} \right)^d \int \hat{g}^2.
\]

By the second property in (197) we obtain for the sum

\[
\sum_{n=1}^{N} \langle |\langle F|B_R(y_n)\rangle - \langle F \rangle|^2 \rangle \gtrsim \frac{1}{L^d} \left( \frac{\ell}{L} \right)^d \int \hat{g}^2,
\]

so that (46) follows via (195).

\[\Box\]

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