H₂ Performance Analysis and Synthesis for Discrete-Time Linear Systems With Dynamics Determined by an i.i.d. Process

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Abstract—This letter is concerned with H₂ control of discrete-time linear systems whose dynamics is determined by an independent and identically distributed (i.i.d.) process. A definition of H₂ norm is first discussed for the class of systems. Then, a linear matrix inequality (LMI) condition is derived for the associated performance analysis, which is tractable in the sense of numerical computation. The results about analysis are also extended toward state-feedback controller synthesis.

Index Terms—Stochastic systems, discrete-time linear systems, H₂ control, LMIs, stochastic dynamics.

1. INTRODUCTION

THIS letter studies H₂ control of discrete-time linear stochastic systems whose dynamics is determined by an independent and identically distributed (i.i.d.) process. This class of systems are also known as discrete-time linear systems with white parameters [1] since “i.i.d.” implies “white” in discrete time. The systems with an i.i.d. process can be seen as a discrete-time linear case of random dynamical systems [2]; see also [3] for discrete-time linear systems with general stochastic dynamics, which properly include the present systems. The systems with an i.i.d. process are also closely related with switched linear systems [4], [5], and indeed, can be viewed as those with the switching signal given by an i.i.d. signal (whose support may be uncountable). This is a relationship similar to that between switched systems and Markov jump systems [5], [6], where the switching signal is given by a Markov chain; further information will be given in Section III-C (see also [3, Sec. II]).

In our earlier study [3], relations of several notions of second-moment stability (i.e., mean square stability) and associated Lyapunov inequalities were discussed for discrete-time linear systems with a general stochastic process. The Lyapunov inequalities about second-moment stability of any class of stochastic systems can be theoretically unified in the framework developed in this earlier study, provided that the systems are discrete-time and linear. Hence, the associated results could have a large impact in the related fields. For example, the Lyapunov inequality for the i.i.d. case in [7], [8] and that for the Markovian case in [6], [9] are naturally covered in the framework as special cases. Their extensions such as periodic versions are also not an exception. As stated in [3], however, the most general form of the Lyapunov inequality (i.e., that for the most general stochastic systems) involves conditional expectation operations that are difficult to numerically compute, and is not suitable for practical control problems. The systems with an i.i.d. process are one of the tractable classes of systems in the sense of numerical computations, as well as Markov jump systems. To pursue the usefulness of the proposed framework, we have been also studying practical linear matrix inequality (LMI) conditions for control of the systems with an i.i.d. process. LMI conditions for H₂ performance analysis and synthesis derived in this letter correspond to results in such a direction.

Our results are potentially useful for networked control systems (NCSs) with randomly time-varying communication delays. Although details are omitted here due to the limited space, our preliminary experiments using the Internet suggested that i.i.d. processes are prospective as a model of actual communication delays; this is also consistent with the classical queueing theory, where arrivals are supposed to occur independently at fixed rate. Motivated by this, the i.i.d. results in [8] about stabilization were exploited in [10] about NCSs with random communication delays. One of the advantages of using our i.i.d. results is that we can directly handle unbounded spaces for the values of delays in controller synthesis even when the plant in the NCS is unstable; most of the earlier studies assumed the existence of an upper bound for the delays even when they are random (e.g., in [11], [12]). Our stabilization results have been already applied to remote control of vehicles in [13], in which an experiment using an actual plug-in hybrid electric vehicle and the Internet is reported. Deriving LMI conditions not only for stabilization but also for H₂ control in this letter is expected to contribute to improving the associated control performance, as in the
cases with deterministic systems [14], [15] and Markov jump systems [16].

None of our earlier articles including those cited in this letter deals with control performance other than stability for stochastic systems. This letter provides first results of the authors about extending the stabilizing control approach toward $H_2$ control. The results in this letter also correspond to an extension of $H_2$ control for conventional deterministic systems [14], [15] and other possible special cases. The essential technical contributions are the derivations of associated inequality conditions with appropriate treatment of dependent random matrices in the expectation operation.

We use the following notation in this letter. The set of real numbers, that of positive real numbers, that of positive integers and that of non-negative integers are denoted by $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{N}_0$, respectively. The set of $n \times n$ real matrices are denoted by $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{m \times n}$, respectively. The set of $n \times n$ symmetric matrices and that of $n \times n$ positive definite matrices are denoted by $\mathbb{S}^{n \times n}$ and $\mathbb{S}^+_n$, respectively. The identity matrix of size $n$ is denoted by $I_n$; the subscript will be dropped when the size is obvious. The maximum singular value is denoted by $\sigma_{\text{max}}(\cdot)$. The Euclidean norm is denoted by $\| \cdot \|$. The trace of a matrix is denoted by $\text{tr}(\cdot)$. The vectorization of a matrix in the row direction is denoted by $\text{row}(\cdot)$, i.e., row$(\cdot) := [\text{row}_1(\cdot), \ldots, \text{row}_m(\cdot)]$, where $m$ is the number of rows of the matrix and row$(\cdot)$ denotes the $i$th row. The Kronecker product is denoted by $\otimes$. The expectation of a random variable is denoted by $E[\cdot]$; this notation is also used for the expectation of a random matrix. If $s$ is a random variable obeying the distribution $\mathcal{D}$, then we represent it as $s \sim \mathcal{D}$.

II. DISCRETE-TIME LINEAR SYSTEMS WITH DYNAMICS DETERMINED BY AN I.I.D. PROCESS AND $H_2$ NORM

A. System Class

Let us consider the $Z$-dimensional discrete-time stochastic process $\xi = (\xi_k)_{k \in \mathbb{N}_0}$ satisfying the following assumption.

Assumption 1: The random vector $\xi_k$ is independent and identically distributed (i.i.d.) with respect to the discrete time $k \in \mathbb{N}_0$, i.e., $\xi_0, \xi_1, \ldots$ are i.i.d.

The time-invariant support of $\xi_k$ satisfying this assumption is denoted by $\Xi$ ($\subseteq \mathbb{R}^Z$). The process $\xi$ satisfying the above assumption is obviously stationary and ergodic [17].

With such a process $\xi$, let us further consider the discrete-time linear system $G$ represented by

$$
\begin{align*}
\xi_{k+1} &= A(\xi_k)\xi_k + B(\xi_k)w_k, \\
\zeta_k &= C(\xi_k)\xi_k + D(\xi_k)w_k,
\end{align*}
$$

where $\xi_k$, $w_k$ and $\zeta_k$ are the state vector, the external input vector and the output vector, respectively. The initial state $\xi_0$ as well as the external input $w = (w_k)_{k \in \mathbb{N}_0}$ are supposed to be deterministic. In addition, $A : \Xi \rightarrow \mathbb{R}^{n \times n}$, $B : \Xi \rightarrow \mathbb{R}^{n \times p}$, $C : \Xi \rightarrow \mathbb{R}^{q \times n}$ and $D : \Xi \rightarrow \mathbb{R}^{q \times p}$ are matrix-valued Borel functions satisfying the following assumption.

Assumption 2: The squares of entries of $A(\xi_0)$ are all Lebesgue integrable, i.e.,

$$E[A_{ij}(\xi_0)^2] < \infty \quad (\forall i, j = 1, \ldots, n).$$

Similarly, the squares of entries of $B(\xi_0), C(\xi_0)$ and $D(\xi_0)$ are also all Lebesgue integrable.

The condition (2) is known as a minimal requirement for defining second-moment stability [3]. No other structural constraints are imposed on the functions $A$, $B$, $C$ and $D$ as well as the distribution of $\xi_k$ in this letter; our study views the distribution of $\xi_k$ as a part of the system model.

B. $H_2$ Norm

As an internal stability notion for the aforementioned class of systems, we use the following second-moment exponential stability, which is also called mean square exponential stability [18].

Definition 1: The system $G$ satisfying Assumptions 1 and 2 is said to be exponentially stable in the second moment if for $w \equiv 0$, there exist $a \in \mathbb{R}_+$ and $\lambda \in (0, 1)$ such that

$$\sqrt{E[|x_k|^2]} \leq a|x_0|^\lambda^k \quad (\forall k \in \mathbb{N}_0, \forall x_0 \in \mathbb{R}^n).$$

The assumption $w \equiv 0$ (i.e., $w_k = 0$ ($\forall k$)) is used in the above definition merely because internal stability is defined there. None of the results in this letter is discussed under such an assumption. The above internal stability notion is known to be characterized by a Lyapunov inequality as in the following theorem [8].

Theorem 1: Suppose that the system $G$ satisfies Assumptions 1 and 2. The following two conditions are equivalent.

1) The system $G$ is exponentially stable in the second moment.

2) There exists $P \in \mathbb{S}^{n \times n}_+$ such that

$$E[P - A(\xi_0)^T P A(\xi_0)] > 0.$$  

For internally stable systems, this letter defines an $H_2$ norm in the time domain. Let us consider

$$
\Phi_{k_1}^{k_2} := \left\{ \begin{array}{ll}
I & (k_2 = k_1) \\
A(\xi_{k_2-1})A(\xi_{k_2-2}) \ldots A(\xi_{k_1}) & (k_2 \geq k_1 + 1)
\end{array} \right. 
$$

for $k_1, k_2 \in \mathbb{N}_0$ satisfying $k_2 \geq k_1$. Then, the impulse response matrix of the system (1) is given by

$$
g_k = \left[ \begin{array}{c}
D(\xi_k) \\
C(\xi_k)\Phi_{1}^{k}(\xi_k)
\end{array} \right] \quad (k \geq 1).
$$

As in the case of multi-input deterministic systems [15], [19], we consider the series

$$s_K := \sum_{k=0}^{K} E[\text{tr}(g_k^T g_k)],$$

where expectation is taken for each term because $g_k$ is a random matrix in the present case. The convergence of this series is ensured by the following theorem.

Theorem 2: Suppose that the system $G$ satisfies Assumptions 1 and 2. If $G$ is exponentially stable in the second moment, then the corresponding $s_K$ in (7) converges to a constant as $K$ tends to infinity.

Proof: See the Appendix.
This theorem validates our defining the $H_2$ norm of the system $G$ as
\[ \|G\|_2 := \sqrt{\text{tr}(S)}. \tag{8} \]
This definition is consistent with that for deterministic systems [14], [15], [20]. Indeed, if we consider the case where $\xi$ is given by a deterministic time-invariant constant process, the $H_2$ norm defined in (8) of the corresponding deterministic time-invariant $G$ coincides with the usual $H_2$ norm for deterministic time-invariant systems. $H_2$ norm is a popular index for evaluating influence of disturbance on systems.

III. $H_2$ PERFORMANCE ANALYSIS

A. Expectation-Based Inequality Condition

This section first proves the following theorem about $H_2$ analysis of stochastic systems, which is one of the main results in this letter.

**Theorem 3:** Suppose that the system $G$ satisfies Assumptions 1 and 2. For given $\gamma \in \mathbb{R}_+$, the following two conditions are equivalent.

1) The system $G$ is exponentially stable in the second moment and satisfies $\|G\|_2 < \gamma$.

2) There exists $P \in \mathbb{S}^{n \times n}_+$ such that
\[ E[P - A(\xi_0)^T C(\xi_0)] > 0, \tag{9} \]
\[ E[\text{tr}(D(\xi_0)^T D(\xi_0) + B(\xi_0)^T PB(\xi_0))] < \gamma^2. \tag{10} \]

**Proof:** For notational simplicity, we use $A_k := A(\xi_k)$, $B_k := B(\xi_k)$, $C_k := C(\xi_k)$ and $D_k := D(\xi_k)$ in this proof.

1) $\Rightarrow$ 2): It follows from (9) and Theorem 1 that the system $G$ is exponentially stable in the second moment. Hence, it suffices to show $\|G\|_2 < \gamma$.

Since $\xi_k$ is i.i.d. with respect to $k$ by Assumption 1, the inequality (9) implies
\[ E[A_k^T P A_k + C_k^T C_k] < P \quad (\forall k \in \mathbb{N}_0). \tag{11} \]
Take $K \in \mathbb{N}$ and consider the case of $k = K$ in the above inequality. Then, by multiplying $A_{K-1} (k = K)$ and its transpose on the inequality, we have
\[ A_{K-1}^T E[A_k^T A_k + C_k^T C_k] A_{K-1} = A_{K-1}^T P A_{K-1}. \tag{12} \]
Since $A_{K-1}$ is a random matrix, taking expectation for both sides of this inequality, together with using the independence between $\xi_{K-1}$ and $\xi_k$, leads to
\[ E[(\Phi_{K-1}^{K+1})^T P \Phi_{K-1}^{K+1} + A_{K-1}^T C_k C_k A_{K-1}] \leq E[A_{K-1}^T P A_{K-1}]. \tag{13} \]
Adding $E[C_k^T C_k]$ to both sides of this inequality and using (11) further lead to
\[ E[(\Phi_{K-1}^{K+1})^T P \Phi_{K-1}^{K+1} + A_{K-1}^T C_k C_k A_{K-1}] \leq E[A_{K-1}^T P A_{K-1}] \leq P. \tag{14} \]
By repeating the operation from (12) to (14) for $k = K - 1, K - 2, \ldots$, we finally obtain
\[ E[(\Phi_{1}^{K+1})^T P \Phi_{1}^{K+1} + \sum_{k=1}^{K} (\Phi_{k}^{K+1})^T C_k C_k \Phi_{k}^{K+1}] \leq P. \tag{15} \]

It follows from this inequality and $E[(\Phi_{1}^{K+1})^T P \Phi_{1}^{K+1}] \geq 0$ that
\[ E\left[ \sum_{k=1}^{K} (\Phi_{k}^{K+1})^T C_k^T C_k \Phi_{k}^{K+1} \right] \leq P. \tag{16} \]

Multiplying $B_0$ and its transpose and taking expectation for this inequality lead to
\[ E\left[ \sum_{k=1}^{K} B_0^T (\Phi_{k}^{K+1})^T C_k^T C_k \Phi_{k}^{K+1} B_0 \right] \leq E[B_0^T P B_0]. \tag{17} \]
Adding $E[D_0^T D_0]$ and taking trace for this inequality further lead to
\[ E[\text{tr}(D_0^T D_0)] + \sum_{k=1}^{K} E[\text{tr}(B_0^T (\Phi_{k}^{K+1})^T C_k^T C_k \Phi_{k}^{K+1} B_0)] \leq E[\text{tr}(D_0^T D_0 + B_0^T P B_0)]. \tag{18} \]

The left-hand side of this inequality is nothing but $s_K$ defined in (7). Hence, by letting $K \to \infty$, the above inequality leads us to
\[ \|G\|_2^2 \leq E[\text{tr}(D_0^T D_0 + B_0^T P B_0)]. \tag{19} \]
The convergence of the associated infinite series is ensured by Theorem 2. This, together with (10), leads to $\|G\|_2 < \gamma$.

1) $\Rightarrow$ 2): We prove this assertion by four steps.

(Step 1) Since the system $G$ is exponentially stable in the second moment, there exists $\Pi \in \mathbb{S}^{n \times n}_+$ satisfying
\[ E[\Pi - A_0^T \Pi A_0] > 0 \tag{20} \]
by Theorem 1.

(Step 2) It follows from Assumption 2 and Definition 1 that there exist $a \in \mathbb{R}_+$ and $\lambda \in (0, 1)$ satisfying
\[ E[C_k^T C_k \xi_k] \leq a \lambda E[\sigma_{\max}(C_0)^2](\lambda_{\xi_k}^2 \lambda_k)^{2k} \] \[
\forall k \in \mathbb{N}_0, \forall x_0 \in \mathbb{R}^n \]
for $w \equiv 0$. Let $b = a \sqrt{E[\sigma_{\max}(C_0)^2]}$. Then, the above inequality implies
\[ E[(\Phi_{k}^{K+1})^T C_k \Phi_{k}^{K+1}] \leq b^2 \lambda^{2K} I \quad (\forall k \in \mathbb{N}_0). \tag{21} \]

Since $\xi_k$ is i.i.d. with respect to $k$, for $Q := E[C_0^T C_0]$, this implies
\[ E[(\Phi_{k}^{K+1})^T Q \Phi_{k}^{K+1}] \leq b^2 \lambda^{2(K+n)} I \quad (\forall k, k_2 \in \mathbb{N}_0 \text{ s.t. } k_2 \geq k_1 \geq 0). \tag{22} \]

(Step 3) Define
\[ p_{k_1}^{k_2} := \sum_{k=k_1}^{k_2} (\Phi_{k}^{K+1})^T Q \Phi_{k}^{K+1} \tag{23} \]
for $k_1, k_2 \in \mathbb{N}_0$ satisfying $k_2 \geq k_1 \geq 0$. This $p_{k_1}^{k_2}$ naturally satisfies
\[ p_{k_1}^{k_2} - A_{k_1}^T p_{k_1}^{k_2+1} A_{k_1} = Q. \tag{24} \]

The inequality (22) holds for general $w$ since it merely means a condition for coefficient matrices (recall the arguments about internal stability in Section II).

1The inequality (22) holds for general $w$ since it merely means a condition for coefficient matrices (recall the arguments about internal stability in Section II).
On the other hand, since \( Q \geq 0 \), the sequence of
\[
E[P^k_{k_1}] = \sum_{k=k_1}^{k_2} E[(\Phi^k_{k_1})^T Q \Phi^k_{k_1}]
\]
(26)
with respect to \( k_2 = k_1, k_1 + 1, \ldots \) for each fixed \( k_1 \) is monotonically nondecreasing under the semimatrix relation based on positive semidefiniteness, i.e., \( E[P^k_{k_1}] \leq E[P^{k+1}_{k_1}] \). In addition, it follows from (23) that
\[
E[P^k_{k_1}] \leq b^2 \left( \sum_{k=k_1}^{k_2} \lambda^{2(k-k_1)} \right) I.
\]
(27)
Since \( \lambda < 1 \), the right-hand side of this inequality converges to a constant matrix as \( k_2 \to \infty \). Hence, \( E[P^k_{k_1}] \) also converges to a constant matrix as \( k_2 \to \infty \) for each fixed \( k_1 \). Since \( \xi_k \) is i.i.d., this constant matrix is independent of \( k_1 \), and we denote it by \( P' \). Then, it follows from (25) that
\[
E[P' - A_0^T P' A_0 - C_0^T C_0] = 0.
\]
(28)
(Step 4) We have obtained \( \Pi > 0 \) satisfying (20) and \( P' \geq 0 \) satisfying (28). With those \( \Pi \) and \( P' \), we construct \( P \) satisfying (9) and (10).

Let us consider the case where \( E[\sigma_{\text{max}}(B_0)^2] = 0 \). Take \( P = P' + \Pi \). Then, this \( P \) is positive definite and satisfies (9). In addition, (10) becomes \( E[|\text{tr}(D_0^T B_0)|] < \gamma^2 \), which is automatically satisfied under condition 1 (recall the definition of \( H_2 \) norm).

Let us next consider the case where \( E[\sigma_{\text{max}}(B_0)^2] > 0 \). Take \( P = P' + \beta \Pi \) with \( \beta = \frac{1}{2E[|\text{tr}(B_0^T B_0)|]} \). This \( P \) is also positive definite and satisfies (9) as in the above case since \( \beta > 0 \). It follows from the definitions of \( H_2 \) norm and \( P' \) that
\[
\|G\|^2_2 = E[|\text{tr}(D_0^T B_0)| + \text{tr}(B_0^T P' B_0)]
\]
(29)
Hence, with the above \( P \),
\[
E[|\text{tr}(D_0^T B_0 + B_0^T P B_0)|] = \|G\|^2_2 + \beta E[|\text{tr}(B_0^T \Pi B_0)|]
\]
\[
= \frac{\gamma^2 + \|G\|^2_2}{2} < \gamma^2
\]
(30)
holds.

Hence, in any case, there exists \( \Pi > 0 \) satisfying (9) and (10). This completes the proof.

**B. Numerically Tractable Condition**

Theorem 3 gives an expectation-based inequality condition for \( H_2 \) analysis of stochastic systems. This form of condition, however, is generally not tractable from the aspect of numerical computation since products involving the decision variable \( P \) are in the expectation operation. This issue can be resolved through the following theorem.

**Theorem 4:** For given \( \gamma \in \mathbb{R}_+ \) and \( P \in S^{a \times n}_+ \), the following two conditions are equivalent.

1. The inequalities (9) and (10) hold.
2. The inequalities
\[
P - \widetilde{A}^T (P \otimes I_n) \widetilde{A} - E[C(\xi_0)^T C(\xi_0)] > 0,
\]
(31)

**C. Relationship With Switched Linear Systems**

In the field of switched systems [4], [5], a finite number of “switches” is often dealt with. To show the relationship of our results with such standard switched systems, this subsection considers confining the support \( \Xi \) of \( \xi_k \) to a finite set \( S = \{1, \ldots, N\} \subset \mathbb{R} \). We denote by \( p_i \) \((i \in S)\) the probability that \( \xi_k = i \). Obviously, \( \sum_{i \in S} p_i = 1 \). Then, the most important advantage of considering a finite set for \( \Xi \) is that we can write all the associated expectations by mere summations. For example, (9) and (10) in Theorem 3 can be immediately rewritten as
\[
\sum_{i \in S} p_i (P - A_i^T P A_i - C_i^T C_i) > 0,
\]
(36)
\[
\sum_{i \in S} p_i \left( \text{tr}(D_i^T D_i + B_i^T P B_i) \right) < \gamma^2,
\]
(37)
where \( A_i, B_i, C_i \) and \( D_i \) \((i \in S)\) are deterministic constant coefficient matrices for mode \( i \). The above inequalities give a necessary and sufficient condition for \( H_2 \) performance of the present stochastic switched linear system. As is obvious, the inequalities are already LMIs, and hence, the arguments in the preceding subsection about numerical computation are needless; the application of Theorem 4 to the above condition itself is possible, which may lead to another form of a necessary and sufficient condition.

**Remark 1:** The above simplification associated with the expectation operation may contribute to all the arguments in this letter. Since all the equations can be described with mere summations of deterministic terms by the simplification, even the derivations of inequality conditions can be tackled only with the standard LMI techniques. Although it is unclear whether the condition (36) and (37) is explicitly stated somewhere, at least a finite set counterpart to the Lyapunov inequality (4) is shown in [6, Corollary 3.26]. Many results have been already obtained for such a special case, and our results correspond to one of their extensions.
IV. \( H_2 \) State-Feedback Controller Synthesis

A. Synthesis Problem

This section extends the results about \( H_2 \) analysis in the preceding section toward state feedback systems. We first describe our synthesis problem. Let us consider the stochastic process \( \xi \) satisfying Assumption 1 and the associated generalized plant

\[
\begin{align*}
x_{k+1} &= A_0(\xi_k)x_k + B_{ow}(\xi_k)w_k + B_{oq}(\xi_k)u_k, \\
z_k &= C_0(\xi_k)x_k + D_{ow}(\xi_k)w_k + D_{oq}(\xi_k)u_k,
\end{align*}
\]

where

\[
\begin{align*}
A_0 &: \Xi \to \mathbb{R}^{n \times n}, B_{ow} &: \Xi \to \mathbb{R}_{+}^{n \times p_w}, B_{oq} &: \Xi \to \mathbb{R}_{+}^{n \times p_w}, \\
C_0 &: \Xi \to \mathbb{R}^{q_n \times n}, D_{ow} &: \Xi \to \mathbb{R}_{+}^{q_n \times p_w} \quad \text{and} \quad D_{oq} &: \Xi \to \mathbb{R}_{+}^{q_n \times p_w}
\end{align*}
\]

are matrix-valued Borel functions, and \( x_0 \) and \( w_0 \) are supposed to be deterministic. In this plant, \( u_0 \) is the control input. As is the case with \( G \) without the control input, we introduce the following assumption on the coefficient matrices of the plant.

Assumption 3: The squares of entries of \( A_0(\xi_0), B_{ow}(\xi_0), B_{oq}(\xi_0), C_0(\xi_0), D_{ow}(\xi_0) \) and \( D_{oq}(\xi_0) \) are all Lebesgue integrable.

Let us next consider the state-feedback controller

\[
u_k = Fx_k
\]

with the static time-invariant gain \( F \in \mathbb{R}_{+}^{p_w \times n} \). Then, the closed-loop system \( GF \) consisting of the plant (38) and this controller is described by (1) with

\[
\begin{align*}
A(\cdot) &= A_0(\cdot) + B_{ow}(\cdot)F, \\
B(\cdot) &= B_{ow}(\cdot), \\
C(\cdot) &= C_0(\cdot) + D_{ow}(\cdot)F, \\
D(\cdot) &= D_{ow}(\cdot).
\end{align*}
\]

Obviously, if the plant satisfies Assumption 3, then the corresponding \( GF \) satisfies Assumption 2 for each fixed gain \( F \). This section tackles the problem of designing a gain \( F \) minimizing the \( H_2 \) norm \( \|GF\|_2 \) of the corresponding closed-loop system \( GF \).

B. Synthesis-Oriented Inequality Condition

For given \( F \), the \( H_2 \) norm of the corresponding closed-loop system \( GF \) can be analyzed as an LMI optimization problem by Theorems 3 and 4. In the synthesis, however, the gain \( F \) is also viewed as a decision variable, and hence, the inequality condition (31) and (32) becomes nonlinear in the decision variables. To make matters worse, the direct linearization of the inequality condition is not straightforward since the decision variables are involved in the matrix \( \mathbf{A} \) in a complicated form. One of the ways to circumvent these issues is to return to the expectation-based inequality condition (9) and (10), and derive a numerically tractable synthesis-oriented inequality condition from it.

The following theorem is a main result about the LMI condition for \( H_2 \) performance synthesis.

Theorem 5: Suppose that the generalized plant (38) satisfies Assumptions 1 and 3. For given \( \gamma > 0 \), there exists a gain \( F \) such that the closed-loop system \( GF \) is exponentially stable in the second moment and satisfies \( \|GF\|_2 < \gamma \) if and only if there exist \( X \in \mathbb{S}_{+}^{n \times n}, Y \in \mathbb{R}_{+}^{q_n \times n} \) and \( R \in \mathbb{S}_{+}^{p_w \times p_w} \) satisfying

\[
\begin{align*}
&X > 0, \\
&\begin{bmatrix}
A_0X + B_{ow}Y & X \otimes I_{n_w} \\
C_0X + D_{ow}Y & 0
\end{bmatrix} > 0, \\
&\begin{bmatrix}
R - E_D & Y \otimes I_{p_w} \\
F - E_D & 0
\end{bmatrix} > 0, \\
&\text{tr}(R) < \gamma^2
\end{align*}
\]

(\( \text{tr}(R) \) denotes the transpose of an appropriate submatrix), where

\[
E_D := E[D_{ow}(\xi)(\cdot)^T]D_{ow}(\xi)\]

and \( \tilde{A}_0, \tilde{B}_{ow}, \tilde{C}_0, \tilde{D}_{ow} \) and \( \tilde{B}_{oq} \) are the matrices given by

\[
\begin{align*}
\tilde{A}_0 &= [\tilde{A}_{o1}, \ldots, \tilde{A}_{oq}]^T \in \mathbb{R}_{+}^{q_n \times n}, \\
\tilde{B}_{ow} &= [\tilde{B}_{ow1}, \ldots, \tilde{B}_{owq}]^T \in \mathbb{R}_{+}^{q_n \times p_w}, \\
\tilde{A}_{o1}, \ldots, \tilde{A}_{oq}, \tilde{B}_{ow1}, \ldots, \tilde{B}_{owq} &= \tilde{G}_{CD}, \\
\tilde{C}_{o1}, \ldots, \tilde{C}_{oq}, \tilde{D}_{ow1}, \ldots, \tilde{D}_{owq} &= \tilde{G}_{AB}, \\
\tilde{B}_{ow} &= [\tilde{B}_{ow1}, \ldots, \tilde{B}_{owq}]^T \in \mathbb{R}_{+}^{q_n \times p_w}, \\
&[\tilde{B}_{ow1}, \ldots, \tilde{B}_{owq}] \in \mathbb{R}_{+}^{q_n \times p_w}, \\
&[\tilde{B}_{ow1}, \ldots, \tilde{B}_{owq}] := \tilde{B}_{ow}
\end{align*}
\]

with matrices \( \tilde{G}_{CD} \in \mathbb{R}_{+}^{q_n \times n} \) and \( \tilde{G}_{AB} \in \mathbb{R}_{+}^{q_n \times n} \) satisfying

\[
\begin{align*}
&\tilde{G}_{CD} = E[\text{row}(\tilde{C}_{o1}(\xi_0)), \text{row}(\tilde{D}_{ow1}(\xi_0))]^T, \\
&\tilde{G}_{CD1} = E[\text{row}(\tilde{C}_{o1}(\xi_0)), \text{row}(\tilde{D}_{ow1}(\xi_0))]^T.
\end{align*}
\]

In particular, \( F = YX^{-1} \) is one such gain.

Proof: Although the details are omitted, this theorem can be proved by using Theorem 3 in this letter, [21, Lemma 2], and well-known LMI techniques such as the Schur complement technique, congruence transformation and change of variables.

V. Numerical Example

Let us consider the system (38) with

\[
A_0(\xi_k) = \begin{bmatrix}
1.3 + \xi_{2k} & 0.8 + \xi_{1k} & -0.5 \\
0.5 & 0.3 + \xi_{1k}\xi_{2k} & -1.2 + \xi_{1k}^2 \\
-0.2 & 0.8 & 0.6
\end{bmatrix}
\]

It is theoretically possible to consider a \( \xi \)-dependent gain \( F(\xi_k) \), as in the case with switched systems. To obtain such a gain, however, we have to search for a function \( F(\cdot) \) since the support \( \Xi \) is not finite. In addition, the dependency of \( F(\xi_k) \) on the coefficient matrices may be an obstacle in deriving LMIs. Resolving these issues would be a potential future work.
right-hand side of (7) can be equivalently rewritten as tr(\sum_{k=1}^{K} \xi_k (B(\xi_k)^T \Phi_1^{-T} C(\xi_k)^T C(\xi_k) + \Phi_1^{-1} B(\xi_k))) if the sum in this trace converges to a constant matrix as K → ∞, then s_K also converges to a constant. This convergence can be shown in a fashion similar to Steps 2 and 3 in the proof of Theorem 3; multiplying B matrices and taking expectation for (27) with appropriate k_1 and k_2 lead to proving the convergence.

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