Construction of higher order digital nets for numerical integration in weighted Sobolev spaces of high smoothness∗

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May 8, 2013

Abstract

Higher order digital nets have recently been recognized as one of the most promising branches of quasi-Monte Carlo methods. The notable feature of higher order digital nets is that they can exploit the smoothness of a function for numerical integration and achieve the improved convergence rate of the integration error for smooth functions. One prominent construction of such nets is based on digitally interlacing the components of the classical digital net whose number of components is a multiple of the dimension. In this study we consider the weighted unanchored Sobolev spaces of high smoothness and derive an upper bound on the mean square worst-case error for digitally shifted higher order digital nets. Employing our obtained bound as a quality criterion, we prove that the component-by-component construction can be made efficient use of to obtain good polynomial lattice rules that are used for interlaced components. Through this approach we are able to get some tractability results under certain conditions on the weights. Numerical experiments confirm that the performance of our constructed point sets often outperforms in terms of our introduced quality criterion the performances of higher order digital nets with Sobol’ sequences and Niederreiter-Xing sequences used as interlaced components, indicating the usefulness of our algorithm.

1 Introduction

In this paper we study the multivariate integration of smooth functions defined over the s-dimensional unit cube \([0, 1)^s\),

\[
I(f) = \int_{[0, 1)^s} f(x) dx.
\]

Quasi-Monte Carlo (QMC) methods approximate \(I(f)\) by averaging function evaluations at uniformly distributed \(N\) points \(x_0, \ldots, x_{N-1}\),

\[
\hat{I}(f) = \frac{1}{N} \sum_{n=0}^{N-1} f(x_n).
\]

∗The support of Grant-in-Aid for JSPS Fellows No.24-4020 is gratefully acknowledged.
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We refer to [10, 18] for general information on QMC methods. The Koksma-Hlawka inequality ensures that QMC rules typically achieve the worst-case error of order $N^{-1+\delta}$ for any $\delta > 0$ when the integrand has bounded variation in the sense of Hardy and Krause. Two prominent ways to generate a uniformly distributed point set are lattice rules, see for example [20, 27], and digital $(t, m, s)$-nets, see for example [13, 20]. It has long been known that it is possible to achieve the improved convergence rate for periodic smooth functions by using lattice rules, whereas neither rule can exploit the smoothness of a non-periodic function so as to achieve the improved convergence rate.

Recently, higher order digital nets have been proposed in [7, 8] and shown to achieve the improved convergence rate for numerical integration of both periodic and non-periodic smooth functions. Numerical integration using higher order digital nets in a reproducing kernel Hilbert space consisting of smooth periodic functions was studied in [7], and that in a normed Walsh space consisting of all smooth non-periodic functions was analyzed in [8]. Later it was shown in [2] that the result from [8] can be achieved in a reproducing kernel Hilbert space setting that is different from what is studied in [7]. More specifically, a weighted unanchored Sobolev space of smoothness $\alpha \in \mathbb{N}$, where $\mathbb{N}$ is a set of positive integers, was considered as a reproducing kernel Hilbert space and it was proved that higher order digital nets achieve the optimal convergence rate not only of the worst-case error but also of the mean square worst-case error with respect to random digital shift. We note that the case with $\alpha = 1$ has been extensively studied in the literature already, see for example [13, Chapter 12]. More recently, higher order scrambling was studied in [9, 15] to achieve the (nearly) optimal rate of the root mean square error for smooth functions.

There are two construction algorithms of higher order digital nets so far as known. One has been studied in several papers such as [3, 4, 12], which makes use of polynomial lattice rules introduced in [21] by generalizing the original definition. These rules are referred to as higher order polynomial lattice rules. Regarding explicit constructions of higher order polynomial lattice rules which achieve the nearly optimal rate of convergence, the component-by-component (CBC) construction has been studied in [3, 4]. Even with efficient calculation of the worst-case error implemented in [4], however, a computational cost of $O(sdN^d \log N)$ operations using $O(N^d)$ memory is required for $d \in \mathbb{N}$. Whereas higher order polynomial lattice rules thus constructed can achieve the worst-case error of order $N^{-\min(\alpha, d)+\delta}$ for functions in a normed Walsh space of smoothness $\alpha$, the exponential dependence on $d$ of the number of operations degrades the availability of these rules as $d$ increases.

The other algorithm for constructing higher order digital nets has been proposed in [7, 8], which applies a digit interlacing function to the components of the classical digital net whose number of components is a multiple of the dimension. Hence the major advantage of this algorithm is that one can make use of the existing digital $(t, m, s)$-nets including those of Sobol’ [29], Faure [14], Niederreiter [19] and Niederreiter-Xing [22]. Suppose we have digital $(t', m, ds)$-net in base $b$ for $d \in \mathbb{N}$. We denote by $y_n = (y_{n,1}, \ldots, y_{n,ds})$ the $n$-th point of this net and by $y_{n,j} = y_{n,j,1}b^{-1} + y_{n,j,2}b^{-2} + \cdots$ the $b$-adic expansion of $y_{n,j}$ for $1 \leq j \leq ds$. Then the $n$-th point $x_n = (x_{n,1}, \ldots, x_{n,s}) \in [0, 1)^s$ of higher order
digital nets consisting of $b^m$ points is given by

$$x_{n,j} = D_d(y_{n,d(j-1)+1}, \ldots, y_{n,dj}) := \sum_{a=1}^{\infty} \sum_{r=1}^{d} y_{n,d(j-1)+r,a} b^{-r-(a-1)d},$$

for $1 \leq j \leq s$. Here we refer to $D_d$ as a digit interlacing function of factor $d$. The disadvantage of this construction algorithm is that the $t$-value of digital $(t, m, s)$-nets increases at least linearly in $s$, so that it becomes hard to obtain a bound on the worst-case error independent of the dimension. This observation motivates us to replace the existing digital $(t, m, ds)$-nets by suitably chosen polynomial lattice rules in dimension $ds$ that are used for interlaced components. In the similar context, there exists a successful result in [15] where scrambled polynomial lattice rules are used as interlaced components to construct higher order scrambled digital nets. It was shown that we are able to obtain a better dependence on the dimension of the root mean square error. Moreover, as compared to deterministic higher order polynomial lattice rules, the computational cost for the CBC construction could be significantly reduced to $O(sdN \log N)$ operations using $O(N)$ memory. Thus, it is quite natural to think of using polynomial lattice rules for interlaced components to achieve the nearly optimal rate either of the worst-case error or the mean square worst-case error with respect to some randomization while obtaining a better dependence on the dimension.

In this study, we consider the weighted unanchored Sobolev spaces of high smoothness as studied in [2]. We derive a computable upper bound of the mean square worst-case error in this space setting for digitally shifted higher order digital nets. Employing this upper bound as a quality criterion, we prove that the CBC construction can be made efficient use of to obtain good polynomial lattice rules that are used for interlaced components. In the next section, we introduce the necessary background and notation including Walsh functions, polynomial lattice rules, higher order digital nets and their randomization, and the weighted unanchored Sobolev spaces of high smoothness $\alpha \in \mathbb{N}$. In Section 3, we study the mean square worst-case error in this space setting for digitally shifted higher order digital nets with an aim to introduce a computable upper bound on the error. We show in Section 4 that the CBC construction is applicable to obtain good polynomial lattice rules as interlaced components such that digitally shifted higher order digital nets thus constructed achieve the nearly optimal rate of convergence and that we are able to get some tractability results under certain conditions on the weights. We remark that the fast CBC construction using the fast Fourier transform as introduced in [23, 24] is available in our current setting. This enables us to proceed the construction with $O(sdN \log N)$ operations using $O(N)$ memory. Finally, we conclude this paper with numerical experiments in Section 5.

2 Preliminaries

Throughout this paper, we use the following notation. Let $\mathbb{N}$ be a set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Given a prime $b$, let $\mathbb{F}_b := \{0, \ldots, b-1\}$ be the finite field consisting of $b$ elements. The operators $\oplus$ and $\ominus$ denote the digitwise addition and subtraction modulo $b$, respectively. That is, for $x, x' \in [0, 1)$ with
b-adic expansions $x = \sum_{i=1}^{\infty} x_i b^{-i}$ and $x' = \sum_{i=1}^{\infty} x'_i b^{-i}$ where $x_i, x'_i \in \mathbb{F}_b$. ⊕ and ⊖ are defined as

$$x \oplus x' = \sum_{i=1}^{\infty} y_i b^{-i}$$
and

$$x \ominus x' = \sum_{i=1}^{\infty} y'_i b^{-i},$$

where $y_i = x_i + x'_i \mod b$ and $y'_i = x_i - x'_i \mod b$, respectively. Similarly, we define digitwise addition and subtraction for non-negative integers based on those $b$-adic expansions. In case of vectors in $[0, 1)^s$ or $\mathbb{N}_0^s$, the operators ⊕ and ⊖ are carried out componentwise. We call $x \in [0, 1)$ a $b$-adic rational if it can be represented by a finite $b$-adic expansion. Further we shall use the notation $I_a := \{1, \ldots, a\}$ for simplicity.

2.1 Walsh functions

Walsh functions were first introduced in [30] for the case with base 2 and were generalized later, see for example [5]. We refer to [13, Appendix A] for general information on Walsh functions. We first give the definition for the one-dimensional case.

**Definition 1** Let $b \geq 2$ be an integer and $\omega_b = e^{2\pi i / b}$. Let us denote the $b$-adic expansion of $k \in \mathbb{N}_0$ by $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^{-a-1}$ with $\kappa_i \in \{0, \ldots, b-1\}$. Then the $k$-th $b$-adic Walsh function $b_{\text{wal}}k : [0, 1) \to \{1, \omega_b, \ldots, \omega_b^{b-1}\}$ is defined as

$$b_{\text{wal}}k(x) = \omega_b^{x_1 \kappa_0 + \cdots + x_a \kappa_a - 1},$$

for $x \in [0, 1)$ with its $b$-adic expansion $x = x_1 b^{-1} + x_2 b^{-2} + \cdots$, that is unique in the sense that infinitely many of the $x_i$ are different from $b-1$.

This definition can be generalized to higher-dimensional case.

**Definition 2** For dimension $s \geq 2$, let $x = (x_1, \ldots, x_s) \in [0, 1)^s$ and $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$. Then the $k$-th $b$-adic Walsh function $b_{\text{wal}}k : [0, 1)^s \to \{1, \omega_b, \ldots, \omega_b^{b-1}\}$ is defined as

$$b_{\text{wal}}k(x) = \prod_{j=1}^{s} b_{\text{wal}}k_j(x_j).$$

Since we shall always use Walsh functions in a fixed base $b$, we omit the subscript and simply write wal_k or wal_k in the remaining of this paper.

2.2 Polynomial lattice rules

Given a prime $b$, let us denote by $\mathbb{F}_b((x^{-1}))$ the field of formal Laurent series over $\mathbb{F}_b$. Every element of $\mathbb{F}_b((x^{-1}))$ has a form

$$L = \sum_{l=w}^{\infty} l x^{-l},$$

where $w \in \mathbb{N}$. We call $L$ a polynomial lattice rule. For any $L$, we denote by $\mathbb{E}_L(\mathbb{F}_b)$ the corresponding function space, which consists of all formal Laurent series with rational coefficients whose coefficients are $b$-adic rationals.
where $w$ is an arbitrary integer and all $t_i \in \mathbb{F}_b$. Further, we denote by $\mathbb{F}_b[x]$ the set of all polynomials over $\mathbb{F}_b$. For a given $m \in \mathbb{N}$, we define the mapping $v_m$ from $\mathbb{F}_b((x^{-1}))$ to the interval $[0, 1)$ by

$$v_m \left( \sum_{l=w}^{\infty} t_l x^{-l} \right) = \sum_{l=\max(1,w)}^{m} t_l b^{-l}.$$ 

We often identify a non-negative integer $k$ whose $b$-adic expansion is given by $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_{a-1} b^{a-1}$ with the polynomial over $\mathbb{F}_b[x]$ as $k(x) = \kappa_0 + \kappa_1 x + \cdots + \kappa_{a-1} x^{a-1}$. For $k = (k_1, \ldots, k_a) \in (\mathbb{F}_b[x])^a$ and $q = (q_1, \ldots, q_s) \in (\mathbb{F}_b[x])^s$, we define the inner product as

$$k \cdot q = \sum_{j=1}^{s} k_j q_j \in \mathbb{F}_b[x],$$

and we write $q \equiv 0 \mod p$ if $p$ divides $q$ in $\mathbb{F}_b[x]$. Using these notations, a polynomial lattice point set is constructed as follows.

**Definition 3** Let $m, s \in \mathbb{N}$. Let $p \in \mathbb{F}_b[x]$ be an irreducible polynomial with $\deg(p) = m$ and let $q = (q_1, \ldots, q_s) \in (\mathbb{F}_b[x])^s$ such that $q_j \neq 0$ and $\deg(q_j) < m$ for $1 \leq j \leq s$. A polynomial lattice point set is a point set consisting of $b^m$ points such that

$$x_n := \left( v_m \left( \frac{n(x) q_1(x)}{p(x)} \right), \ldots, v_m \left( \frac{n(x) q_s(x)}{p(x)} \right) \right) \in [0, 1)^s,$$

for $0 \leq n < b^m$. A QMC rule using this point set is called a polynomial lattice rule with generating vector $q$ and modulus $p$.

In the remaining of this paper, we denote by $Q_{b^m,s}(q,p)$ a polynomial lattice point set, implicitly meaning that $\deg(p) = m$ and the number of components in a vector $q$ is $s$. The concept of the so-called dual net plays an important role in the subsequent analysis.

**Definition 4** For $k \in \mathbb{N}$ with its $b$-adic expansion $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_{a-1} b^{a-1}$, we denote the truncation of the associated polynomial $k(x) \in \mathbb{F}_b[x]$ by

$$\text{tr}_m(k) = \kappa_0 + \kappa_1 x + \cdots + \kappa_{\min(a,m)-1} x^{\min(a,m)-1}.$$ 

For a polynomial lattice point set $Q_{b^m,s}(q,p)$, the dual net is defined as

$$D^\perp(q,p) = \{ k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s : \text{tr}_m(k) \cdot q \equiv 0 \mod p \},$$

where $\text{tr}_m(k) = (\text{tr}_m(k_1), \ldots, \text{tr}_m(k_s))$.

Furthermore, we shall use the following two lemmas in this paper. The first lemma bridges between the dual net $D^\perp(q,p)$ and Walsh functions. The proof is straightforward from the above definition of $D^\perp(q,p)$, [13, Lemma 10.6] and [13, Lemma 4.75]. The second lemma states about the group structure of a polynomial lattice point set $Q_{b^m,s}(q,p)$. The proof is also straightforward from [13, Lemma 4.72].
Lemma 1 Let \( Q_{b^m,s}(q,p) \) be a polynomial lattice point set and \( D^\perp(q,p) \) be its dual net. Then we have
\[
\frac{1}{b^m} \sum_{n=0}^{b^m-1} \text{walk}(x_n) = \begin{cases} 1 & \text{if } k \in D^\perp(q,p), \\ 0 & \text{otherwise.} \end{cases}
\]

Lemma 2 For a prime \( b \), any polynomial lattice point set \( Q_{b^m,s}(q,p) \) is a subgroup of \((\mathbb{F}_b^{m,s}, \oplus)\).

2.3 Higher order digital nets

QMC rules based on higher order digital nets can exploit the smoothness of an integrand so that they achieve the optimal convergence rate of the integration error for functions with smoothness \( \alpha \in \mathbb{N} \). This result is based on a bound on the decay of the Walsh coefficients of smooth functions [2, 8]. Explicit construction of higher order digital nets by means of a digit interlacing function was given in [7, 8]. Although we have already mentioned about that function in the previous section, we describe the interlacing algorithm in more detail in the following.

Since the interlacing is applied to each point separately, we use just one point to describe the procedure. Let \( y \in [0,1)^d \) with \( y = (y_1, \ldots, y_d) \) and denote the \( b \)-adic expansion of each coordinate by \( y_j = y_j, b^{-1} + y_j, b^{-2} + \cdots \), that is unique in the sense that infinitely many digits are different from \( b - 1 \).

We then obtain a point \( x \in [0,1)^s \) by interlacing the digits of non-overlapping \( d \) components of \( y \) in the following way: Let \( x = (x_1, \ldots, x_s) \) where
\[
x_j = \sum_{a=1}^{\infty} \sum_{r=1}^{d} y_d(j-1)+r.a. b^{-r-(a-1)d},
\]
for \( 1 \leq j \leq s \). We denote this mapping by \( D_d : [0,1)^d \to [0,1) \) and we simply write \( x_j = D_d(y_{d(j-1)+1}, \ldots, y_d) \). Further we write \( x = D_d(y) := (D_d(y_1), \ldots, y_d), \ldots, D_d(y_{d(s-1)+1}, \ldots, y_{ds})) \) when \( x \) is obtained by interlacing the components of \( y \). Thus it is obvious that in order to construct a good higher order digital net consisting of \( N \) points in \([0,1)^s\) we need to carefully choose \( N \) points in \([0,1)^{ds}\).

In this paper, we are interested in applying polynomial lattice rules to generate a point set in \([0,1)^{ds}\) that serves as interleaved components for higher order digital nets. For clarity, we give the definition of higher order digital nets based on polynomial lattice rules.

Definition 5 Let \( b \) be prime and \( m, s, d \in \mathbb{N} \). Let \( p \in \mathbb{F}_b[x] \) be an irreducible polynomial with \( \deg(p) = m \) and let \( q = (q_1, \ldots, q_{ds}) \in (\mathbb{F}_b[x])^{ds} \) such that \( q_j \neq 0 \) and \( \deg(q_j) < m \) for \( 1 \leq j \leq ds \). We construct a higher order digital net consisting of \( b^m \) points \( \{x_0, \ldots, x_{b^m-1}\} \subseteq [0,1)^s \) as follows. The \( n \)-th point \( x_n \) is obtained by
\[
x_n = D_d(y_n),
\]
where the point \( y_n \in [0,1)^{ds} \) is the \( n \)-th point of a polynomial lattice point set \( Q_{b^m,s}(q,p) \) which is given as
\[
y_n = \left( v_m \left( \frac{n(x)q_1(x)}{p(x)} \right), \ldots, v_m \left( \frac{n(x)q_{ds}(x)}{p(x)} \right) \right),
\]
for \( n = 0, 1, \ldots, b^m-1 \).
for $0 \leq n < b^m$.

In this construction algorithm, the search for good $b^m$ points in $[0,1)^d$ to be interlaced has now been reduced to finding good polynomials $q = (q_1, \ldots, q_{ds})$, which is the particular interest of this paper.

Randomization of QMC point sets is useful to obtain some statistical estimate on the integration error. Especially for randomization of higher order digital nets, two algorithms have been discussed in the literature. One is digital shift, see [2, 7], and the other is higher order scrambling that is a generalization of Owen’s scrambling introduced in [25], see [9, 15]. Since we are concerned with the former in this paper, we follow [7, Section 6] to introduce some basic concepts of digital shift here.

Let $Q_{b^m,s} = \{x_0, \ldots, x_{b^m-1}\} \subseteq [0,1)^s$ with $x_n = (x_{n,1}, \ldots, x_{n,s})$. We denote the $b$-adic expansion of $x_{n,j}$ by $x_{n,j} = x_{n,j,0}b^{-1} + x_{n,j,1}b^{-2} + \cdots$ for $1 \leq j \leq s$. Also, let $\sigma = (\sigma_1, \ldots, \sigma_s)$ where $\sigma_j$ is independently and uniformly distributed in $[0,1)$ for $1 \leq j \leq s$. We also denote the $b$-adic expansion of $\sigma_j$ by $\sigma_j = \sigma_{j,0}b^{-1} + \sigma_{j,1}b^{-2} + \cdots$ for $1 \leq j \leq s$. Then the randomly digitally shifted point set $Q_{b^m,s,\sigma} = \{z_0, \ldots, z_{b^m-1}\}$ is given by

$$z_n = x_n \oplus \sigma = (z_{n,1}, \ldots, z_{n,s}),$$

where

$$z_{n,j} = (x_{n,j,0} + \sigma_{j,0})b^{-1} + (x_{n,j,1} + \sigma_{j,1})b^{-2} + \cdots,$$

for $1 \leq j \leq s$. In order to compute the mean square worst-case error, the next lemma is required. We refer to [13, Lemma 16.37] for the proof.

**Lemma 3** Let $x_1, x_2 \in [0,1)$ and let $z_1, z_2 \in [0,1)$ be the points obtained by applying an i.i.d. random digital shift to $x_1$ and $x_2$, respectively. Then we have

$$E[\text{wal}(z_1, z_2)] = \begin{cases} \text{wal}(x_1 \oplus x_2) & \text{if } k = l, \\ 0 & \text{otherwise}. \end{cases}$$

**2.4 Weighted Sobolev spaces of high smoothness**

Here we follow [2] and introduce the weighted reproducing kernel Hilbert space considered in this paper. The concept of weighted spaces was introduced in [28], which provides an essential insight into tractability properties of multivariate problems from the viewpoint of information-based complexity. Let us start with the one-dimensional unweighted case. The inner product of the Hilbert space is defined for $\alpha \in \mathbb{N}$ by

$$\langle f, g \rangle_{H_{1,\alpha}(1)} = \sum_{r=0}^{\alpha-1} \int_0^1 f^{(r)}(x)dx \int_0^1 g^{(r)}(x)dx + \int_0^1 f^{(\alpha)}(x)g^{(\alpha)}(x)dx,$$

where we denote by $f^{(r)}$ the $r$-th derivative of $f$ and set $f^{(0)} = f$. Let $\|f\|_{H_{1,\alpha}(1)} = \sqrt{\langle f, f \rangle_{H_{1,\alpha}(1)}}$ be the norm of $f$ associated with $H_{1,\alpha}(1)$. We now define the function

$$K_{1,\alpha}(1)(x, y) = \sum_{r=1}^{\alpha} \frac{B_r(x)B_r(y)}{r!} + (-1)^{\alpha+1} \frac{B_{2\alpha}(|x-y|)}{(2\alpha)!},$$
where $B_r$ denotes the Bernoulli polynomial of degree $r$. We refer to [1, Chapter 23] for information on Bernoulli polynomials. The reproducing kernel for this one-dimensional case is given by $1 + K_{1,\alpha}^{(1)}(x, y)$. That is, for any $f \in H_{1,\alpha}^{(1)}$, we have

$$f(x) = \langle f, 1 + K_{1,\alpha}^{(1)}(\cdot, x) \rangle_{H_{1,\alpha}^{(1)}}.$$  

We now consider the higher-dimensional weighted case. The inner product for the $s$-dimensional weighted unanchored Sobolev space $H_{s,\alpha,\gamma}$ of smoothness $\alpha \in \mathbb{N}$ and non-negative weights $\gamma = (\gamma_u)_{u \subseteq I}$ is defined by

$$\langle f, g \rangle_{H_{s,\alpha,\gamma}} = \sum_{u \subseteq I} \gamma_u^{-1} \sum_{r_u \setminus v \subseteq \{1, \ldots, \alpha-1\}} \sum_{u \subseteq v} \int_{[0,1)^{|v|}} f(r_u \setminus v, \alpha_u, 0)(x) dx \cdot \int_{[0,1)^{|v|}} g(r_u \setminus v, \alpha_u, 0)(x) dx,$$

where we have used the following notation. For $r_u \setminus v = (r_j)_{j \in u \setminus v}$, we denote by $(r_u \setminus v, \alpha_u, 0)$ the vector in which the $j$-th component is $r_j$ for $j \in u \setminus v$, $\alpha$ for $j \in v$, and 0 for $I \setminus u$. For $u \subseteq I$, such that $\gamma_u = 0$, we assume that the corresponding inner double sum equals 0 and we set $0/0 = 0$. Let $\|f\|_{H_{s,\alpha,\gamma}} = \sqrt{\langle f, f \rangle_{H_{s,\alpha,\gamma}}}$ be again the norm of $f$ associated with $H_{s,\alpha,\gamma}$.

The reproducing kernel for $H_{s,\alpha,\gamma}$ becomes

$$K_{s,\alpha,\gamma}(x, y) = \sum_{u \subseteq I \setminus y} \prod_{j \in u} K_{1,\alpha}^{(1)}(x_j, y_j)$$

$$= \sum_{u \subseteq I} \gamma_u \prod_{j \in u} \left( \sum_{r=1}^{\alpha} B_r(x_j) B_r(y_j) + (-1)^{\alpha+1} \frac{B_{2\alpha}(|x_j - y_j|)}{(2\alpha)!} \right),$$

where we set

$$\prod_{j \in \emptyset} K_{1,\alpha}^{(1)}(x_j, y_j) = 1.$$

That is, we have for any $f \in H_{s,\alpha,\gamma}$

$$f(x) = \langle f, K_{s,\alpha,\gamma}(\cdot, x) \rangle_{H_{s,\alpha,\gamma}}.$$

### 3 Mean square worst-case error

In this section, we derive an upper bound on the mean square worst-case error in the space $H_{s,\alpha,\gamma}$ for digitally shifted higher order digital nets. To begin with, the worst-case error in the space $H_{s,\alpha,\gamma}$ is defined by

$$e(Q_{b}, H_{s,\alpha,\gamma}) = \sup_{f \in H_{s,\alpha,\gamma}, \|f\|_{H_{s,\alpha,\gamma}} \leq 1} |I(f) - \hat{I}(f)|,$$

where an arbitrary point set $Q_{b}$ consisting of $b^n$ points is used for quadrature points. The initial error is given as

$$e(Q_{0}, H_{s,\alpha,\gamma}) = \sup_{f \in H_{s,\alpha,\gamma}, \|f\|_{H_{s,\alpha,\gamma}} \leq 1} |I(f)|.$$
From [2, Theorem 13], we have the following simple expressions for the squared worst-case error and the squared initial error in the space $H_{s, \alpha, \gamma}$

$$
\hat{e}^2(Q_{b^m, s}, H_{s, \alpha, \gamma}) = \int_{[0,1]^2} K_{s, \alpha, \gamma}(x, y) dxy - \frac{2}{b^m} \sum_{n=0}^{b^m-1} \int_{[0,1]^2} K_{s, \alpha, \gamma}(x_n, x) dx + \frac{1}{b^{2m}} \sum_{n, n'=0}^{b^m-1} K_{s, \alpha, \gamma}(x_n, x_n')
$$

$$
= - \gamma_0 + \frac{1}{b^{2m}} \sum_{n, n'=0}^{b^m-1} K_{s, \alpha, \gamma}(x_n, x_n'),
$$

and

$$
\hat{e}^2(Q_{0, s}, H_{s, \alpha, \gamma}) = \gamma_0.
$$

respectively. Let us now consider a randomization of $Q_{b^m, s}$ by using digital shift. The mean square worst-case error for digitally shifted point set $Q_{b^m, s, \sigma}$, denoted by $\hat{e}^2(Q_{b^m, s}, H_{s, \alpha, \gamma})$, is defined and calculated as

$$
\hat{e}^2(Q_{b^m, s}, H_{s, \alpha, \gamma}) := \mathbb{E}[\hat{e}^2(Q_{b^m, s, \sigma}, H_{s, \alpha, \gamma})]
$$

$$
= - \gamma_0 + \frac{1}{b^{2m}} \sum_{n, n'=0}^{b^m-1} \mathbb{E}[K_{s, \alpha, \gamma}(x_n \oplus \sigma, x_n' \oplus \sigma)]
$$

$$
= \frac{1}{b^{2m}} \sum_{n, n'=0}^{b^m-1} \sum_{\emptyset \neq u \subseteq I_s} \gamma_u \prod_{j \in u} \mathbb{E}[K_{1, \alpha, (1)}(x_{n, j} \oplus \sigma_j, x_{n', j} \oplus \sigma_j)].
$$

On the other hand, the mean square initial error is still given as

$$
\hat{e}^2(Q_{0, s}, H_{s, \alpha, \gamma}) = \gamma_0.
$$

We now consider the Walsh expansion of $K_{1, \alpha, (1)}(x, y)$. We write

$$
K_{1, \alpha, (1)}(x, y) = \sum_{k, l=1}^{\infty} \hat{K}_{1, \alpha, (1)}(k, l) \text{wal}_k(x) \text{wal}_l(y),
$$

where the $(k, l)$-th Walsh coefficient $\hat{K}_{1, \alpha, (1)}(k, l)$ is defined by

$$
\hat{K}_{1, \alpha, (1)}(k, l) = \int_0^1 \int_0^1 K_{1, \alpha, (1)}(x, y) \text{wal}_k(x) \text{wal}_l(y) dx dy.
$$

Using the above Walsh expansion of $K_{1, \alpha, (1)}(x, y)$ and Lemma 3, we further have on the mean square worst-case error

$$
\hat{e}^2(Q_{b^m, s}, H_{s, \alpha, \gamma})
$$

$$
= \frac{1}{b^{2m}} \sum_{n, n'=0}^{b^m-1} \sum_{\emptyset \neq u \subseteq I_s} \gamma_u \prod_{j \in u} \hat{K}_{1, \alpha, (1)}(k, l) \text{wal}_k(x_{n, j} \oplus x_{n', j})
$$

$$
= \sum_{\emptyset \neq u \subseteq I_s} \gamma_u \sum_{k_u \in \mathbb{N}} \hat{K}_{1, \alpha, (1)}(k_u) \frac{1}{b^{2m}} \sum_{n, n'=0}^{b^m-1} \text{wal}_{k_u, 0}(x_n \oplus x_{n'}),
$$

(1)

where we write $\hat{K}_{1, \alpha, (1)}(k) = \hat{K}_{1, \alpha, (1)}(k, k)$ for short and $\hat{K}_{1, \alpha, (1)}(k_u) = \prod_{j \in u} \hat{K}_{1, \alpha, (1)}(k_j)$. 9
3.1 A bound on Walsh coefficients

From [2, (13) and Proposition 20], we have a bound on Walsh coefficients

$$|\hat{K}_{1,\alpha,1}(k)| \leq D_{\alpha,b}b^{-2\mu_\alpha(k)},$$

where $\mu_\alpha(k)$ is the weight first introduced in [8], which is defined as

$$\mu_\alpha(k) := a_1 + \cdots + a_{\min(\nu,\alpha)},$$

for $k \in \mathbb{N}$ with its $b$-adic expansion $k = \kappa_1 b^{\kappa_1 - 1} + \cdots + \kappa_\nu b^{\kappa_\nu - 1}$ such that $0 < \kappa_1, \ldots, \kappa_\nu < b$ and $a_1 > \cdots > a_\nu > 0$. We set $\mu_\alpha(0) = 0$. Moreover, $D_{\alpha,b}$ is given by

$$D_{\alpha,b} = \max_{1 \leq \mu \leq \alpha} (C'_{\alpha,b,\nu} + \tilde{C}_{2\alpha,b}b^{-2(\alpha-\mu)}),$$

in which $C'_{\alpha,b,\nu}$ and $\tilde{C}_{2\alpha,b}$ are respectively given by

$$C'_{\alpha,b,\nu} = \sum_{\tau=\nu}^{\alpha} C_{\tau,b} b^{-2(\tau-\nu)},$$

where $C_{1,b} = (2 \sin(\pi/b))^{-1}$ and $C_{\tau,b} = (1 + 1/b + 1/(b(b+1)))^{\tau-2}/(2 \sin(\pi/b))^{\tau-2}$ for $\tau \geq 2$, and

$$\tilde{C}_{2\alpha,b} = \frac{2(1+b+1/(b(b+1)))^{2\alpha-2}}{(2 \sin(\pi/b))^{2\alpha}}.$$

Furthermore, we write $\mu_\alpha(k_u) = \sum_{j \in u} \mu_\alpha(k_j)$. We then have

$$|\hat{K}_{1,\alpha,1}(k_u)| \leq \prod_{j \in u} |\hat{K}_{1,\alpha,1}(k_j)| \leq D_{\alpha,b}^{[u]} b^{-2\mu_\alpha(k_u)}.$$

Inserting this bound on the Walsh coefficient into (1), we have thus far

$$\hat{c}^2(Q_{b^{(u,s)}, H_{s,a,\gamma}}) \leq \sum_{\emptyset \neq u \subseteq I_a} \gamma_u D_{\alpha,b}^{[u]} \sum_{k_u \in \mathbb{N}^{[u]}} b^{-2\mu_\alpha(k_u)} \frac{1}{b^{2m}} \sum_{n,n' \in \mathbb{N}} \text{wal}(k_u,0)(x_n \ominus x_{n'}).$$

3.2 A bound on mean square worst-case error

We continue the study of the mean square worst-case error in the space $H_{s,a,\gamma}$ to derive a computable upper bound. Following [9], we first define a digit interlacing function of factor $d$ for non-negative integers. For $d \in \mathbb{N}$ and $k_1, \ldots, k_d \in \mathbb{N}_0^d$, we denote the $b$-adic expansion of $k_j$ by $k_j = \kappa_{j,0} + \kappa_{j,1} b + \cdots$ for $1 \leq j \leq d$. Then a digit interlacing function $\mathcal{E}_d$ gives one non-negative integer from $k_1, \ldots, k_d$ by

$$\mathcal{E}_d(k_1, \ldots, k_d) = \sum_{r=1}^{d} \sum_{a=0}^{\infty} \kappa_{(j-1)d+r,a} b^{ad+r-1}. \quad (3)$$

We extend this function to vectors as

$$\mathcal{E}_d(k_1, \ldots, k_d) = (\mathcal{E}_d(k_1, \ldots, k_d), \ldots, \mathcal{E}_d(k_{(d-1)+1}, \ldots, k_{d+1})).$$
From the definition of Walsh functions, we have
\[ \text{wal}_{\mathcal{E}_d(k_1, \ldots, k_d)}(\mathcal{D}_d(y_1, \ldots, y_d)) = \prod_{j=1}^d \text{wal}_{k_j}(y_j), \]
for any \( k_1, \ldots, k_d \in \mathbb{N}_0^d \) and \((y_1, \ldots, y_d) \in \{0, 1\}^d\).

We now recall that the point set \( Q_{b^m,s} = \{x_0, \ldots, x_{b^m-1}\} \) is constructed according to Definition 5. Using the above property on the digit interlacing functions \( \mathcal{D}_d \) and \( \mathcal{E}_d \), and the group structure of polynomial lattice rules as described in Lemma 2, we have
\[ \bar{e}^2(Q_{b^m,s}, H_{s,\alpha,\gamma}) \]
\[ \leq \sum_{\emptyset \neq u \subseteq I_{ds}} \gamma(v(u))D_{\alpha,b}^{[v(u)]} \sum_{k_u \in \mathbb{N}^{[u]}} b^{-2\mu_u(\mathcal{E}_d(k_u, 0))} \frac{1}{b^m} \sum_{n,m=0}^{b^m-1} \text{wal}_{\mathcal{E}_d(k_u, 0)}(\mathcal{D}_d(y_n \oplus y_{n'})) \]
\[ = \sum_{\emptyset \neq u \subseteq I_{ds}} \gamma(v(u))D_{\alpha,b}^{[v(u)]} \sum_{k_u \in \mathbb{N}^{[u]}} b^{-2\mu_u(\mathcal{E}_d(k_u, 0))}, \quad (4) \]
where \((k_u, 0)\) is here a vector in \( \mathbb{N}^{d_u}_0 \) such that \( k_j \in \mathbb{N} \) for \( j \in u \) and \( k_j = 0 \) for \( j \notin u \). Furthermore, we have denoted by \( v(u) \) the set of \( 1 \leq j \leq s \) such that \( u \cap \{d(j-1) + 1, \ldots, dj\} \neq \emptyset \).

The next lemma gives a lower bound of \( \mu_\alpha(\mathcal{E}_d(k_u, 0)) \), from which we can derive a computable upper bound on \( \bar{e}^2(Q_{b^m,s}(q,p), H_{s,\alpha,\gamma}) \).

**Lemma 4** For \( \emptyset \neq u \subseteq I_{ds} \) and \( k_u \in \mathbb{N}^{[u]} \), we have
\[ \mu_\alpha(\mathcal{E}_d(k_u, 0)) \geq \min(\alpha, d) \sum_{j\in u} \mu_1(k_j) + \frac{1}{2} \alpha|u| - \frac{1}{2} \alpha(2d-1)|v(u)|. \]

**Proof.** We denote by \( u_j \) the set of \( u \cap \{d(j-1) + 1, \ldots, dj\} \) for \( j \in v(u) \). Then obviously we have
\[ \mu_\alpha(\mathcal{E}_d(k_u, 0)) = \sum_{j \in v(u)} \mu_\alpha(\mathcal{E}_d(k_{u_j}, 0)_d), \]
where \((k_{u_j}, 0)_d\) is a vector in \( \mathbb{N}^d_0 \) such that \( k_l \in \mathbb{N} \) for \( l \in u_j \) and \( k_l = 0 \) for \( l \in \{d(j-1) + 1, \ldots, dj\} \setminus u_j \). Thus it suffices to prove that
\[ \mu_\alpha(\mathcal{E}_d(k_{u_j}, 0)_d) \geq \min(\alpha, d) \sum_{l \in u_j} \mu_1(k_l) + \frac{1}{2} \alpha|u_j| - \frac{1}{2} \alpha(2d-1), \]
for \( j \in v(u) \). Considering the definition of the weight \( \mu_\alpha(k) \) as in (2), in order to precisely evaluate the weight \( \mu_\alpha(\mathcal{E}_d(k_1, \ldots, k_d)) \) in (3) for a given \( k_1, \ldots, k_d \in \mathbb{N}^d_0 \), we need to reorder the summand according to the value of \( ad + r \). Instead we give a lower bound by only looking at the most significant digit in the inner sum of (3). We denote by \( \{\pi_j(1), \ldots, \pi_j(d)\} \) the permutation of \( \{1, \ldots, d\} \) for a fixed \( j \) such that
\[ (\mu_1(k_{d(j-1)+\pi_j(1)}) - 1) d + \pi_j(1) > \cdots > (\mu_1(k_{d(j-1)+\pi_j(d)}) - 1) d + \pi_j(d). \]
Here we note that only the first \(|u_j|\) terms are positive since \(k_{d(j-1) + π_j(0)} = 0\) for \(|u_j| < j \leq d\). If \(α > |u_j|\), it is obvious from the definition of \(E_d(k_{u_j}, 0)_d\) to have

\[
\mu_α(E_d(k_{u_j}, 0)_d) \geq \sum_{i=1}^{\lfloor u_j \rfloor} \left[ (μ_1(k_{d(j-1) + π_j(0)}) - 1) d + π_j(i) \right] 
\]

\[
\geq d \sum_{i \in u_j} μ_1(k_i) - d|u_j| + \frac{1}{2}|u_j|(|u_j| + 1) 
\]

\[
\geq \min(α, d) \sum_{i \in u_j} μ_1(k_i) - αd + \frac{1}{2}α(|u_j| + 1). 
\]

Otherwise if \(α \leq |u_j|\), owing to the averaging argument of the permutated terms, we obtain

\[
\mu_α(E_d(k_{u_j}, 0)_d) \geq \sum_{i=1}^{\lfloor u_j \rfloor} \left[ (μ_1(k_{d(j-1) + π_j(0)}) - 1) d + π_j(i) \right] 
\]

\[
\geq \frac{α}{|u_j|} \sum_{i=1}^{\lfloor u_j \rfloor} \left[ (μ_1(k_{d(j-1) + π_j(0)}) - 1) d + π_j(i) \right] 
\]

\[
\geq α \sum_{i \in u_j} μ_1(k_i) - αd + \frac{1}{2}α(|u_j| + 1). 
\]

Putting the two cases above together, the result follows.

Now we arrive at the following theorem on an upper bound on the mean square worst-case error for digitally shifted higher order digital nets that are constructed using the digitally interlacing of polynomial lattice rules. Since the proof is straightforward by applying Lemma 4 to (4), we omit the proof.

**Theorem 1** For higher order digital nets constructed according to Definition 5, the mean square worst-case error in the space \(H_{s, α, γ}\) with respect to digital shift is bounded from above by

\[
\varepsilon^2(Q_{b^m, s}, H_{s, α, γ}) \leq \sum_{\emptyset \neq u \subseteq I_d} \gamma_{v(u)} \tilde{D}_{α, d}^{\nu(u)} \sum_{k_u \in N^{|u|}} \sum_{s, α, γ} r_{α, d}(k_u), 
\]

where we define \(\tilde{D}_{α, d} = b^{(2d-1)α}D_{α, d}\),

\[
r_{α, d}(k) = \begin{cases} 
1 & \text{if } k = 0, \\
b^{-2 \min(α, d)μ_1(k) - α} & \text{otherwise},
\end{cases}
\]

and \(r_{α, d}(k_u) = \prod_{j \in u} r_{α, d}(k_j)\).

In the remaining of this paper, we employ this upper bound as a quality criterion for construction of polynomial lattice rules. For simplicity, we denote the bound by

\[
B_{α, d, γ}(q, p) = \sum_{\emptyset \neq u \subseteq I_d} \gamma_{v(u)} \tilde{D}_{α, d}^{\nu(u)} \sum_{k_u \in N^{|u|}} \sum_{s, α, γ} r_{α, d}(k_u). 
\]
The next corollary provides us with a computable formula of $B_{\alpha,d,\gamma}(q,p)$. We write $\log_b$ for the logarithm in base $b$ and set $b(2\min(\alpha,d)-1)\log_b 0 = 0$.

**Corollary 1** Let $B_{\alpha,d,\gamma}(q,p)$ be given as above. We have

$$B_{\alpha,d,\gamma}(q,p) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \sum_{\emptyset \neq u \subseteq I_s} \gamma_u \tilde{D}^{[u]}_{\alpha,d} \prod_{j \in u} \left[ -1 + \frac{d}{l} \left( 1 + \phi_{\alpha,d}(y_{n,d(j-1)+1}) \right) \right],$$

where for $y \in [0,1)$

$$\phi_{\alpha,d}(y) = \frac{b - 1 - b(2\min(\alpha,d)-1)\log_b y (b^{2\min(\alpha,d)} - 1)}{b^n (b^{2\min(\alpha,d)} - b)}.$$

Particularly in case of product weights, that is $\gamma_u = \prod_{j \in u} \gamma_j$, we reduce to

$$B_{\alpha,d,\gamma}(q,p) = -1 + \frac{1}{b^m} \sum_{n=0}^{b^m-1} \prod_{j=1}^{s} \left[ 1 - \gamma_j \tilde{D}_{\alpha,d} + \gamma_j \tilde{D}_{\alpha,d} \prod_{l=1}^{d} \left( 1 + \phi_{\alpha,d}(y_{n,d(j-1)+1}) \right) \right].$$

**Proof.** Because of the property of the dual net $D^{\perp}(q,p)$, we have

$$B_{\alpha,d,\gamma}(q,p) = \sum_{\emptyset \neq u \subseteq I_s} \gamma_v(u) \tilde{D}^{[v(u)]}_{\alpha,d} \sum_{k_u \in \mathbb{N}^{|u|}} r_{\alpha,d}(k_u) \frac{1}{b^m} \sum_{n=0}^{b^m-1} \operatorname{val}(k_u,0)(y_n)$$

$$= \frac{1}{b^m} \sum_{n=0}^{b^m-1} \sum_{\emptyset \neq u \subseteq I_d} \gamma_v(u) \tilde{D}^{[v(u)]}_{\alpha,d} \prod_{j \in u} \sum_{k_j=1}^{\infty} r_{\alpha,d}(k_j) \operatorname{val}(y_{n,j}). \tag{5}$$

We follow the similar lines as in [11, Section 2.2] to obtain for $y \in [0,1)$

$$\sum_{k=1}^{\infty} r_{\alpha,d}(k) \operatorname{val}(y) = \phi_{\alpha,d}(y).$$

We arrange (5) by collecting the terms associated with a given $w \subseteq I_s$ such that $v(u) = w$. We then have

$$B_{\alpha,d,\gamma}(q,p) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \sum_{\emptyset \neq u \subseteq I_s} \gamma_w \tilde{D}^{[w]}_{\alpha,d} \prod_{j \in w} \left[ -1 + \frac{d}{l} \left( 1 + \phi_{\alpha,d}(y_{n,d(j-1)+1}) \right) \right].$$

Hence the result for the first part follows. For product weights $\gamma_w = \prod_{j \in w} \gamma_j$, we have more simplified expression for $B_{\alpha,d,\gamma}(q,p)$ as

$$B_{\alpha,d,\gamma}(q,p) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \prod_{\emptyset \neq u \subseteq I_s} \gamma_j \tilde{D}_{\alpha,d} \left[ -1 + \frac{d}{l} \left( 1 + \phi_{\alpha,d}(y_{n,d(j-1)+1}) \right) \right]$$

$$= -1 + \frac{1}{b^m} \sum_{n=0}^{b^m-1} \prod_{j=1}^{s} \left[ 1 + \gamma_j \tilde{D}_{\alpha,d} \left[ -1 + \frac{d}{l} \left( 1 + \phi_{\alpha,d}(y_{n,d(j-1)+1}) \right) \right] \right].$$

Hence the result for the second part follows. \qed
4 Component-by-component construction of polynomial lattice rules

4.1 Construction algorithm

In order to efficiently find good polynomials \( q = (q_1, \ldots, q_{ds}) \) such that \( B_{\alpha,d,\gamma}(q, p) \) becomes small, we investigate the CBC construction. We denote by \( R_{b,m} \) the set of all non-zero polynomials over \( \mathbb{F}_b \) with degree less than \( m \), that is,

\[
R_{b,m} = \{ q \in \mathbb{F}_b[x] : \deg(q) < m \text{ and } q \neq 0 \}.
\]

We search \( q \) from \( R_{b,m} \) component by component. Without loss of generality we can restrict \( q_1 = 1 \). Thus, the CBC construction proceeds as follows.

**Algorithm 1** For a prime base \( b \), a dimension \( s \), an integer \( m \), an interlacing factor \( d \), and a smoothness \( \alpha \) and weights \( \gamma = (\gamma_u)_{u \subseteq I_s} \):

1. Choose an irreducible polynomial \( p \in \mathbb{F}_b[x] \) such that \( \deg(p) = m \).
2. Set \( q_1^* = 1 \).
3. For \( r = 2, \ldots, ds \), find \( q_r^* \) by minimizing \( B_{\alpha,d,\gamma}(q_{r-1}^*, q_r, p) \) as a function of \( q_r \in R_{b,m} \) where \( q_{r-1}^* = (q_1^*, \ldots, q_{r-1}^*) \).

The following theorem gives a bound on \( B_{\alpha,d,\gamma}(q_r^*, p) \) for \( 1 \leq r \leq ds \), which justifies the CBC construction. In the proof of the theorem, we shall use the following inequality that is sometimes referred to as Jensen’s inequality. For a sequence \( (a_n)_{n \in \mathbb{N}} \) of non-negative real numbers

\[
\left( \sum a_n \right)^\lambda \leq \sum a_n^\lambda,
\]

where \( 0 < \lambda \leq 1 \).

**Theorem 2** Suppose \( q_{ds}^* = (q_1^*, \ldots, q_{ds}^*) \) is found by Algorithm 1 for a prime base \( b \), a dimension \( s \), an integer \( m \), an interlacing factor \( d \), and a smoothness \( \alpha \) and weights \( \gamma = (\gamma_u)_{u \subseteq I_s} \). Then for any \( r = 1, \ldots, ds \) we have

\[
B_{\alpha,d,\gamma}(q_r^*, p) \leq \frac{1}{(b^m - 1)^{1/\lambda}} \left[ \sum_{u \neq u \subseteq I_{j_0-1}} \gamma_u^* C_{\alpha,d,\gamma}^{\left| u \right|} + G_{\alpha,d,\lambda,d_0} \sum_{u \subseteq I_{j_0-1}} \gamma_u^{*\left| u \right|} C_{\alpha,d,\lambda,d}^{\left| u \right|} \right]^{1/\lambda},
\]

(6)

for \( 1/(2 \min(\alpha, d)) < \lambda \leq 1 \), where we write \( r = d(j_0 - 1) + d_0 \) such that \( j_0, d_0 \in \mathbb{N} \) and \( d_0 \in \{1, \ldots, d\} \) and

\[
G_{\alpha,d,\lambda,a} = \tilde{D}^{\lambda}_{\alpha,d} (-1 + (1 + C_{\alpha,d,\lambda})^\alpha),
\]

where

\[
C_{\alpha,d,\lambda} = \frac{1}{b^{\alpha \lambda}} \max \left\{ \left( \frac{b - 1}{b^{2 \min(\alpha,d)} - b} \right)^\lambda, \frac{b - 1}{b^{2 \lambda \min(\alpha,d)} - b} \right\}
\]
We now consider

$$\gamma(q_{r+1}) = \gamma(q_{r+1})$$

where we have defined

$$B_{\alpha,d,\gamma}(q^*, q_{r+1}, p)$$

and

$$\theta(q_{r+1}) = \sum_{u \subseteq I_r} \gamma(u \cup \{r+1\}) \bar{D}_{\alpha,d}^{[s(u \cup \{r+1\})]}(k_u, k_{r+1} + 1)$$

where we have defined

$$\theta(q_{r+1}) = \sum_{u \subseteq I_r} \gamma(u \cup \{r+1\}) \bar{D}_{\alpha,d}^{[s(u \cup \{r+1\})]}(k_u, k_{r+1} + 1)$$

In order to minimize $B_{\alpha,d,\gamma}(q^*, q_{r+1}, p)$ as a function of $q_{r+1}$, we only need to consider $\theta(q_{r+1})$. Due to the averaging argument, the minimum value of $\theta(q_{r+1})$ has to be less than or equal to the average value of $\theta(q_{r+1})$ over $R_{b,m}$. Applying Jensen’s inequality, we have for $1/(2 \min(\alpha, d)) < \lambda < 1$

$$\theta^\lambda(q^*) \leq \frac{1}{b^m - 1} \sum_{q_{r+1} \in R_{b,m}} \theta^\lambda(q_{r+1})$$

$$\leq \frac{1}{b^m - 1} \sum_{q_{r+1} \in R_{b,m}} \sum_{u \subseteq I_r} \gamma^\lambda(u \cup \{r+1\}) \bar{D}_{\alpha,d}^{[s(u \cup \{r+1\})]}(k_u, k_{r+1} + 1)$$

Proof. We prove the theorem by induction. For $r = 1$, we have $q^*_1 = 1$ and $B_{\alpha,d,\gamma}(1, p)$ can be calculated as

$$B_{\alpha,d,\gamma}(1, p) = \gamma(q_{r+1}) \bar{D}_{\alpha,d}^{[s(u \cup \{r+1\})]}(k_u, k_{r+1} + 1)$$

$$= \gamma(q_{r+1}) \bar{D}_{\alpha,d}^{[s(u \cup \{r+1\})]}(k_u, k_{r+1} + 1)$$

$$= \frac{1}{b^m - 1} \sum_{q_{r+1} \in R_{b,m}} \sum_{u \subseteq I_r} \gamma^\lambda(u \cup \{r+1\}) \bar{D}_{\alpha,d}^{[s(u \cup \{r+1\})]}(k_u, k_{r+1} + 1)$$

Hence the result follows. Next we suppose that (6) has been established for $r = d(j_0 - 1) + d_0$. We then write $r + 1 = d(j_1 - 1) + d_1$ such that $j_1, d_1 \in \mathbb{N}$ and $d_1 \in \{1, \ldots, d\}$. It is obvious that

$$(j_1, d_1) = \begin{cases} (j_0 + 1, 1) & \text{if } d_0 = d, \\ (j_0, d_0 + 1) & \text{otherwise.} \end{cases}$$

We now consider

$$B_{\alpha,d,\gamma}(q^*, q_{r+1}, p)$$

and

$$\theta(q_{r+1}) = \sum_{u \subseteq I_r} \gamma(u \cup \{r+1\}) \bar{D}_{\alpha,d}^{[s(u \cup \{r+1\})]} (k_u, k_{r+1} + 1)$$

where we have defined

$$\theta(q_{r+1}) = \sum_{u \subseteq I_r} \gamma(u \cup \{r+1\}) \bar{D}_{\alpha,d}^{[s(u \cup \{r+1\})]} (k_u, k_{r+1} + 1)$$

In order to minimize $B_{\alpha,d,\gamma}(q^*, q_{r+1}, p)$ as a function of $q_{r+1}$, we only need to consider $\theta(q_{r+1})$. Due to the averaging argument, the minimum value of $\theta(q_{r+1})$ has to be less than or equal to the average value of $\theta(q_{r+1})$ over $R_{b,m}$. Applying Jensen’s inequality, we have for $1/(2 \min(\alpha, d)) < \lambda < 1$

$$\theta^\lambda(q^*) \leq \frac{1}{b^m - 1} \sum_{q_{r+1} \in R_{b,m}} \theta^\lambda(q_{r+1})$$

$$\leq \frac{1}{b^m - 1} \sum_{q_{r+1} \in R_{b,m}} \sum_{u \subseteq I_r} \gamma^\lambda(u \cup \{r+1\}) \bar{D}_{\alpha,d}^{[s(u \cup \{r+1\})]} (k_u, k_{r+1} + 1)$$

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Here in the first term of the right-hand side we have
\[ r^\lambda_{\alpha,d}(k_u, k_{r+1}) \]
and in the second term
\[ \frac{1}{b^m - 1} \sum_{q_{r+1} \in R_{h,m}} \sum_{(k_u, k_{r+1}) \in \mathbb{N}_{[\lambda]}^{n+1}} r^\lambda_{\alpha,d}(k_u, k_{r+1}) \]
\[ = \sum_{u \subseteq I_r} \gamma_u \sum_{(k_u, k_{r+1}) \in \mathbb{N}_{[\lambda]}^{n+1}} r^\lambda_{\alpha,d}(k_u, k_{r+1}) \]
\[ \times \frac{1}{b^m - 1} \sum_{q_{r+1} \in R_{h,m}} \sum_{(k_u, k_{r+1}) \in \mathbb{N}_{[\lambda]}^{n+1}} r^\lambda_{\alpha,d}(k_u, k_{r+1}). \]

For a fixed \( u \subseteq I_r \) we have the following for the inner double sum in the last expression. If \( k_{r+1} \) is a multiple of \( b^m \), we always have \( \text{tr}_m(k_{r+1}) = 0 \) and the corresponding term becomes independent of \( q_{r+1} \), or otherwise we have \( \text{tr}_m(k_{r+1}) = 0 \) and \( \text{tr}_m(k_{r+1})q_{r+1} \) cannot be a multiple of \( p \) by considering that \( p \) is irreducible. Hence we have
\[ \sum_{k_{r+1}=1}^{\infty} \frac{1}{b^m |k_{r+1}|} \sum_{k_u \in \mathbb{N}_{[\lambda]}} r^\lambda_{\alpha,d}(k_u) \]
\[ = \frac{1}{b^m - 1} \sum_{k_{r+1}=1}^{\infty} \sum_{k_u \in \mathbb{N}_{[\lambda]}} r^\lambda_{\alpha,d}(k_u). \]

Here in the first term of the right-hand side we have
\[ \sum_{k_{r+1}=1}^{\infty} \frac{1}{b^m |k_{r+1}|} \sum_{k_u \in \mathbb{N}_{[\lambda]}} r^\lambda_{\alpha,d}(k_u) = \frac{b - 1}{b^{2\lambda \lambda (\min(\alpha,d)m + n)}} \leq \frac{C_{\alpha,d,\lambda}}{b^{2\lambda \lambda (\min(\alpha,d)m)}} \]
and in the second term
\[ \sum_{k_{r+1}=1}^{\infty} \frac{1}{b^m |k_{r+1}|} \sum_{k_u \in \mathbb{N}_{[\lambda]}} r^\lambda_{\alpha,d}(k_u) = \frac{b - 1}{b^{2\lambda \lambda (\min(\alpha,d)m)}} \leq C_{\alpha,d,\lambda}. \]

By inserting these inequalities into the above expression, we have
\[ \sum_{k_{r+1}=1}^{\infty} \frac{1}{b^m - 1} \sum_{q_{r+1} \in R_{h,m}} \sum_{(k_u, k_{r+1}) \in \mathbb{N}_{[\lambda]}^{n+1}} r^\lambda_{\alpha,d}(k_u, k_{r+1}) \]
\[ \leq \frac{C_{\alpha,d,\lambda}}{b^{2\lambda \lambda (\min(\alpha,d)m)}} \sum_{k_u \in \mathbb{N}_{[\lambda]}} r^\lambda_{\alpha,d}(k_u) \]
\[ + \frac{C_{\alpha,d,\lambda}}{b^m} \sum_{q_{r+1} \in \mathbb{N}_{[\lambda]}} \sum_{(k_u, k_{r+1}) \in \mathbb{N}_{[\lambda]}^{n+1}} r^\lambda_{\alpha,d}(k_u, k_{r+1}) \]
\[ \leq \frac{C_{\alpha,d,\lambda}}{b^{2\lambda \lambda (\min(\alpha,d)m)}} \sum_{k_u \in \mathbb{N}_{[\lambda]}} r^\lambda_{\alpha,d}(k_u) \]
\[ + \frac{C_{\alpha,d,\lambda}}{b^m} \sum_{q_{r+1} \in \mathbb{N}_{[\lambda]}} r^\lambda_{\alpha,d}(k_u). \]

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Thus, we have a bound on $\theta(q_{r+1}^*)$ as

$$\theta^\lambda(q_{r+1}^*) \leq \frac{1}{b^m - 1} \sum_{u \leq I_{r+1}} \gamma_{v(u \cup \{r+1\})}^\lambda I_{\alpha,d}^\lambda(u \cup \{r+1\}) C^{[u]+1}_{\alpha,d,\lambda}. \tag{8}$$

We arrange the sum of the right-hand side of (8). We recall that $r = (j_0 - 1)d + d_0 = (j_1 - 1)d_1 + d_1 - 1$. Let us define two disjoint subsets $I_1 := \{1, \ldots, d(j_1 - 1)\}$ and $J_2 := \{d(j_1 - 1) + 1, \ldots, (j_1 - 1) + d_1 - 1\}$. In case of $d_1 = 1$, $J_2$ is empty. Every subset $u \subseteq I_r$ can be classified by whether it includes at least one element of $J_1$ and by whether it includes at least one element of $J_2$. Since $\{r+1\}$ is one of $d$ components for the $j_i$-th dimension, whether or not $u$ includes some element of $J_2$ does not affect $v(u \cup \{r+1\})$. Through this argument, we have

$$\sum_{u \leq I_r} \gamma_{v(u \cup \{r+1\})}^\lambda I_{\alpha,d}^\lambda(u \cup \{r+1\}) C^{[u]+1}_{\alpha,d,\lambda} = \sum_{u_2 \subseteq J_2} C^{[u_2]+1}_{\alpha,d,\lambda} \sum_{u_1 \subseteq I_{j_1}} \gamma_{v(u_1) \cup \{j_1\}}^\lambda I_{\alpha,d}^\lambda(u_1 \cup \{j_1\}) C^{[u_1]}_{\alpha,d,\lambda}. \tag{9}$$

We further arrange the inner sum of (9) by collecting the terms such that $v(u_1) = u$ for $u \subseteq I_{j_1-1}$. For such terms, at least one component from $\{d(j_1 - 1) + 1, \ldots, dj\}$ for every $j \in \mathbb{D}$ must be included. Thus we have

$$\sum_{u_1 \subseteq I_{j_1-1}} \gamma_{v(u_1) \cup \{j_1\}}^\lambda I_{\alpha,d}^\lambda(u_1 \cup \{j_1\}) C^{[u_1]}_{\alpha,d,\lambda} = \sum_{u_1 \subseteq I_{j_1-1}} \gamma_{u \cup \{j_1\}}^\lambda I_{\alpha,d}^\lambda(u \cup \{j_1\}) \left[-1 + (1 + C_{\alpha,d,\lambda})d \right]^{[u]}$$

$$= \frac{\lambda}{b^m - 1} \sum_{u_2 \subseteq J_2} (G_{\alpha,d,\lambda,d_1} - G_{\alpha,d,\lambda,d_1-1}) \sum_{u \subseteq I_{j_1-1}} \gamma_{u \cup \{j_1\}}^\lambda I_{\alpha,d}^\lambda(u \cup \{j_1\}) G^{[u]}_{\alpha,d,\lambda,d}$$

From (8) we obtain a bound on $\theta^\lambda(q_{r+1}^*)$. By using Jensen’s inequality for (7), we finally have

$$B_{\alpha,d,\lambda}(q_{r+1}^*, p) = (B_{\alpha,d,\gamma}(q_r^*, p) + \theta(q_{r+1}^*))^\lambda$$
\[ B_{\alpha,d,q}^\lambda(q_r^*; p) + \theta^\lambda(q_{r+1}^*) \leq \frac{1}{b^m-1} \left[ \sum_{0 \neq u \subseteq I_{j_0-1}} \gamma^\lambda G_{\alpha,d,\lambda,d}^u + G_{\alpha,d,\lambda,d} \sum_{u \subseteq I_{j_0-1}} \gamma^\lambda (u \cup \{j_0\}) G_{\alpha,d,\lambda,d}^u \right] \]
\[ + \frac{1}{b^m-1} (G_{\alpha,d,\lambda,d_1} - G_{\alpha,d,\lambda,d_1-1}) \sum_{u \subseteq I_{j_1-1}} \gamma^\lambda (u \cup \{j_1\}) G_{\alpha,d,\lambda,d}^u \]
\[ = \frac{1}{b^m-1} \left[ \sum_{0 \neq u \subseteq I_{j_1-1}} \gamma^\lambda G_{\alpha,d,\lambda,d}^u + G_{\alpha,d,\lambda,d_1} \sum_{u \subseteq I_{j_1-1}} \gamma^\lambda (u \cup \{j_1\}) G_{\alpha,d,\lambda,d}^u \right]. \]

Hence the result follows. \( \square \)

4.2 Fast component-by-component construction

Here we assume product weights for sake of simplicity and show how one can apply the fast CBC construction using the fast Fourier transform. The cost of the CBC construction by naive implementation of Algorithm 1 is at least of order \( O(dsb^m) \) operations, which can be reduced to order \( O(dsmb^m) \) operations for the fast CBC construction using the fast Fourier transform.

According to Algorithm 1, we set \( q_1^* = 1 \) and construct the polynomials \( q_2^*, \ldots, q_d^* \) inductively in the following way. Assume that \( q_i^* = (q_1^*, \ldots, q_i^*) \) are already found. Let \( r = d(j_0 - 1) + d_0 \) and \( r + 1 = d(j_1 - 1) + d_1 \) such that \( j_0, d_0, j_1, d_1 \in \mathbb{N} \) and \( 0 < d_0, d_1 \leq d \). As in the proof of Theorem 2, \((j_1, d_1) = (j_0 + 1, 1)\) if \( d_0 = d \), or otherwise \((j_1, d_1) = (j_0, d_0 + 1)\). Here we introduce the following notation

\[ \eta^{(1)}_{n,r} := \prod_{j=1}^{j_1-1} \left[ 1 - \gamma_j \bar{D}_{\alpha,d} + \gamma_j \bar{D}_{\alpha,d} \prod_{l=1}^{d} (1 + \phi_{\alpha,d}(y_{n,d(j-1)+l})) \right], \]
\[ \eta^{(2)}_{n,r} := \prod_{i=1}^{d_1-1} (1 + \phi_{\alpha,d}(y_{n,d(j_1-1)+i})), \]

and

\[ \eta_{n,r} := \eta^{(1)}_{n,r} \eta^{(2)}_{n,r}, \]

for \( 0 \leq n < b^m \). We set \( \eta^{(2)}_{0,r} = 1 \) for \( 0 \leq n < b^m \) when \( d_1 = 1 \), or equivalently when \( d_0 = d \). It is straightforward to confirm that we only need to minimize

\[ \sum_{n=1}^{b^m-1} \eta_{n,r} \phi_{\alpha,d}(y_{n,r+1}), \tag{10} \]

for \( q_{r+1} \in R_{b,m} \). Since \( p \) is an irreducible polynomial over \( \mathbb{F}_b[x] \), there exists a primitive polynomial \( g \) in \( \mathbb{F}_b[x]/p \). When \( q_r+1 = g^i \), (10) can be rewritten into

\[ c_i = \sum_{n=1}^{b^m-1} \eta_{n,r} \phi_{\alpha,d} \left( \nu_{\alpha,d} \left( \frac{g^{i-n}(x)}{p(x)} \right) \right), \]

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for $1 \leq i < b^m$.

We now define the following circulant matrix

$$
\Omega_p := \begin{bmatrix}
\phi_{\alpha,d}(v_m\left(\frac{g^{i-n}(x)}{p(x)}\right))
\end{bmatrix}_{1 \leq i,n < b^m}.
$$

Let us denote $c = (c_1, \ldots, c_{b^m-1})^\top$ and $\eta_r = (\eta_{1,r}, \ldots, \eta_{b^m-1,r})^\top$, and let $c = \Omega_p \eta_r$.

Then for an integer $i_0$ $(1 \leq i_0 < b^m)$ such that $c_{i_0} \leq c_i$ for $1 \leq i < b^m$, we set $g_{i_0+1} = g^{c_{i_0}}$. After finding $g_{i_0+1}$, we need to update $\eta_{n,i}^{(1)}$ and $\eta_{n,i}^{(2)}$ as follows.

- If $d_1 = d$, then
\[
\begin{align*}
\eta_{n,i+1}^{(1)} &= \eta_{n,i}^{(1)}(1 - \gamma_{i_0} D_{\alpha,d} + \gamma_{i_0} D_{\alpha,d} \eta_{n,i}^{(2)}(1 + \phi_{\alpha,d}(v_m\left(\frac{n(x)g_{i_0}^{*+1}(x)}{p(x)}\right)))), \\
\eta_{n,i+1}^{(2)} &= 1.
\end{align*}
\]

- Otherwise if $d_1 = 1, \ldots, d-1$, then
\[
\begin{align*}
\eta_{n,i+1}^{(1)} &= \eta_{n,i}^{(1)}, \\
\eta_{n,i+1}^{(2)} &= \eta_{n,i}^{(2)}(1 + \phi_{\alpha,d}(v_m\left(\frac{n(x)g_{i_0}^{*+1}(x)}{p(x)}\right)))).
\end{align*}
\]

Since the matrix $\Omega_p$ is circulant, the matrix vector multiplication $\Omega_p \eta_r$ can be efficiently done by using the fast Fourier transform as shown in [23, 24], which significantly reduces the computational cost as compared to the naive matrix vector multiplication. Whereas we focus only on product weights here, it is possible to apply the fast CBC construction to the case with another form of weights by minor modifications of the above procedure. We refer to [6, 17] for the fast CBC construction of lattice rules for order-dependent weights and POD (product and order-dependent) weights.

### 4.3 Tractability properties

Finally in this section, we briefly mention about the tractability properties of our algorithm. In the concept of tractability of multivariate integration, we study the dependence of the minimum number of points $N(\epsilon, s)$ on $\epsilon$ and the dimension such that $\tilde{e}^2(Q_{N(\epsilon,s,s), H_{s,\alpha,\gamma}}) \leq \tilde{c}^2(Q_{0,s,s}, H_{s,\alpha,\gamma})$. Given that the number of points is $N = b^m$ and that $B_{\alpha,d,\gamma}(q,p)$ is a bound on $\tilde{c}^2(Q_{b^m,s,s}, H_{s,\alpha,\gamma})$, we have from Theorem 2

$$
N(\epsilon, s) \leq \gamma_0 \left[1 + \epsilon^{-\lambda} \sum_{\emptyset \neq u \subseteq I_s} \gamma_{\alpha,d,\lambda,d}^{l(u)}\right],
$$

for $1/(2 \min(\alpha, d)) < \lambda \leq 1$. Hence, it is already obvious that $N(\epsilon, s)$ depends polynomially on $\epsilon^{-1}$ with its degree at most one. As for the dependence on the dimension, we have the following corollary. Since the results immediately follow, we omit the proof.

**Corollary 2** Let $b$ a prime, $p \in \mathbb{F}_b[x]$ be an irreducible polynomial with $\deg(p) = m$. Suppose that $q = (q_1, \ldots, q_s)$ is constructed according to Algorithm 1. Then we have the following
1. Assume that
\[ \sum_{0 \neq u \subseteq I_\omega} \gamma_d^u G_{0, d, \lambda, d}^u < \infty. \]
for some \( 1/(2 \min(\alpha, d)) < \lambda \leq 1 \). Then \( N(\epsilon, s) \) is bounded above independently of the dimension.

2. Assume that
\[ \limsup_{s \to \infty} \left[ \frac{1}{s^q} \sum_{0 \neq u \subseteq I_s} \gamma_d^u G_{0, d, \lambda, d}^u \right] < \infty, \]
for some \( 1/(2 \min(\alpha, d)) < \lambda \leq 1 \) and \( q > 0 \). Then \( N(\epsilon, s) \) depends polynomially on the dimension with its degree at most \( q \).

5 Numerical experiments

We conclude this paper with numerical experiments. We present the values of \( B_{\alpha, d, \gamma}(q, p) \) for different choices of \( s, \alpha, d \) and \( \gamma \), where \( q \) is found by Algorithm 1.

In our computation, the base \( b \) is fixed at 2 and product weights are considered. As a competitor, we use Sobol’ sequences and Niederreiter-Xing sequences as interlaced components in Definition 5, instead of polynomial lattice rules. We use Sobol’ sequences as constructed in [16] for any \( ds \) and Niederreiter-Xing sequences as implemented in [26] as long as \( 4 \leq ds \leq 16 \). For these rules, we denote a bound on the mean square worst-case error by
\[ B_{\alpha, d, \gamma}(C_1, \ldots, C_{ds}) = \sum_{\emptyset \neq u \subseteq I_{ds}} \gamma_{\alpha, d}(u) D_{\alpha, d}^{[u]} \sum_{k_u \in N^{[u]}} r_{\alpha, d}(k_u), \]
where \( C_1, \ldots, C_{ds} \) are generating matrices of the digital net and \( D^+(C_1, \ldots, C_{ds}) \) is its dual space.

We compare the values of \( B_{\alpha, d, \gamma}(q, p) \) with the values of \( B_{\alpha, d, \gamma}(C_1, \ldots, C_{ds}) \) in Tables 1-4. In these tables, our constructed point sets based on polynomial lattice rules are denoted by PLR for short, and similarly, point sets based on Sobol’ sequences and Niederreiter-Xing sequences are respectively denoted by Sobol’ and N-X for short. In Tables 1 and 2, we consider \( \gamma_j = 1 \) for \( 1 \leq j \leq s \), that is the so-called unweighted case. In Table 1, we fix \( \alpha = d = 2 \) and change the dimension from \( s = 1 \) to \( s = 5 \). In Table 2, we fix \( s = 3 \) and change \( \alpha \) and \( d \) simultaneously. In Tables 3 and 4, we do the similar comparisons for the case with \( \gamma_j = j^{-2} \) for \( 1 \leq j \leq s \).

In most cases, PLR outperforms both Sobol’ and N-X. A few exception can be seen for \( s = 3 \) and \( (\alpha, d) = (1, 1) \) where Sobol’ slightly outperforms PLR. We have to emphasize that what we compare here is not the mean square worst-case error itself, but a bound on the mean square worst-case error. Thus, it is possible that Sobol’ and N-X outperform PLR in terms of the mean square worst-case error. However, our result is a good indicator of the usefulness of our algorithm in the sense that the actual performance of our constructed point sets is no worse than the result here.
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Table 1: Values of $B_{a,d}(q,p)$ and $B_{a,d}(C_1,\ldots,C_{d_5})$ for $\gamma_j = 1, 1 \leq j \leq s$ and $(\alpha,d) = (2,2)$, with various choices of $s = 1, 2, 5$.

| $m$ | $s = 1$ | $s = 2$ | $s = 5$ |
|-----|---------|---------|---------|
| Solob PLR | Sobol N-X PLR | Sobol N-X PLR | Sobol N-X PLR |
| 4 | 2.13e-5 | 2.11e-5 | 2.10e-5 | 2.70e-3 | 1.76e+0 | 9.81e-1 |
| 5 | 1.51e-6 | 1.42e-6 | 1.42e-6 | 3.05e-4 | 8.53e-1 | 2.91e-1 |
| 6 | 1.18e-7 | 9.56e-8 | 2.40e-8 | 7.58e-5 | 1.50e-1 | 7.42e-2 |
| 7 | 2.77e-8 | 6.38e-9 | 6.94e-6 | 4.37e-2 | 6.13e-2 | 2.59e-2 |
| 8 | 1.11e-8 | 4.24e-10 | 1.74e-7 | 4.82e-7 | 1.84e-2 | 5.56e-3 |
| 9 | 5.26e-9 | 2.81e-11 | 3.08e-8 | 8.09e-8 | 5.97e-3 | 1.94e-3 |
| 10 | 2.62e-9 | 1.86e-12 | 5.78e-9 | 1.50e-3 | 4.32e-2 | 3.97e-4 |
| 11 | 1.31e-9 | 1.24e-13 | 5.39e-10 | 8.74e-10 | 1.74e-3 | 7.42e-5 |
| 12 | 6.55e-10 | 6.44e-15 | 6.44e-11 | 1.35e-5 | 9.45e-5 | 1.82e-5 |
| 13 | 3.28e-10 | 4.44e-16 | 4.85e-12 | 1.36e-5 | 5.36e-5 | 3.65e-6 |
| 14 | 1.64e-10 | $< 10^{-16}$ | 6.12e-13 | 1.38e-5 | 6.12e-6 | 7.18e-7 |
| 15 | 8.19e-11 | $< 10^{-16}$ | 3.49e-14 | 7.52e-7 | 2.93e-7 | 1.35e-7 |

Table 2: Values of $B_{a,d}(q,p)$ and $B_{a,d}(C_1,\ldots,C_{d_5})$ for $\gamma_j = 1, 1 \leq j \leq s$ and $s = 3$ with various choices of $(\alpha,d) = (1,1),(2,2),(3,3)$.

| $m$ | $(\alpha,d) = (1,1)$ | $(\alpha,d) = (2,2)$ | $(\alpha,d) = (3,3)$ |
|-----|----------------|----------------|----------------|
| Solob PLR | Sobol N-X PLR | Sobol N-X PLR | Sobol N-X PLR |
| 4 | 2.12e-2 | 4.52e-3 | 2.06e-1 | 3.17e-2 | 4.86e+2 | 1.34e-2 |
| 5 | 6.68e-3 | 6.94e-3 | 3.06e-2 | 1.33e-2 | 8.05e-3 | 2.72e+2 | 1.37e-1 |
| 6 | 2.02e-3 | 2.17e-3 | 6.18e-3 | 2.92e-3 | 1.90e-3 | 4.06e+2 | 1.14e+1 |
| 7 | 6.01e-4 | 6.54e-4 | 9.08e-4 | 1.30e-3 | 2.79e-4 | 1.01e+2 | 1.35e+0 |
| 8 | 2.77e-5 | 3.74e-5 | 9.02e-5 | 1.16e-4 | 1.65e-5 | 1.22e+2 | 1.34e+1 |
| 9 | 5.18e-5 | 6.23e-5 | 8.86e-6 | 4.40e-6 | 7.53e-6 | 1.65e+0 | 1.74e+2 |
| 10 | 1.50e-5 | 1.86e-5 | 1.58e-6 | 7.80e-7 | 9.00e-7 | 1.59e+0 | 2.29e-3 |
| 11 | 4.29e-6 | 5.51e-6 | 1.29e-6 | 1.66e-7 | 1.45e-7 | 1.77e-2 | 8.70e-3 | 1.34e-4 |
| 12 | 1.21e-6 | 1.61e-6 | 6.41e-8 | 1.69e-8 | 1.61e-8 | 3.92e-3 | 8.24e-2 |
| 13 | 3.39e-7 | 4.43e-7 | 7.57e-9 | 4.92e-9 | 3.08e-9 | 1.97e-3 | 8.32e-3 |
| 14 | 9.53e-8 | 1.34e-7 | 2.43e-9 | 2.96e-9 | 2.37e-10 | 9.82e-4 | 5.14e-3 |
| 15 | 2.63e-8 | 3.73e-8 | 1.21e-9 | 9.80e-10 | 3.18e-11 | 4.91e-4 | 4.79e-4 | 2.75e-9 |

Table 3: Values of $B_{a,d}(q,p)$ and $B_{a,d}(C_1,\ldots,C_{d_5})$ for $\gamma_j = j^{-2}, 1 \leq j \leq s$ and $(\alpha,d) = (2,2)$ with various choices of $s = 1, 2, 5$.

| $m$ | $(\alpha,d) = (2,2)$ |
|-----|----------------|
| Solob PLR | Sobol N-X PLR | Sobol N-X PLR | Sobol N-X PLR |
| 4 | 2.12e-3 | 2.11e-3 | 6.89e-4 | 7.34e-4 | 6.91e-4 | 2.78e-2 | 1.31e-1 | 6.67e-3 |
| 5 | 1.51e-4 | 1.42e-4 | 2.66e-4 | 6.76e-5 | 7.72e-5 | 3.33e-3 | 1.04e-3 | 1.38e-3 |
| 6 | 1.38e-5 | 9.56e-8 | 2.39e-5 | 5.26e-5 | 1.90e-5 | 6.07e-4 | 5.56e-4 | 3.16e-4 |
| 7 | 2.77e-6 | 6.38e-9 | 8.38e-7 | 4.30e-7 | 1.74e-6 | 1.61e-4 | 1.32e-4 | 6.41e-5 |
| 8 | 1.18e-7 | 4.24e-10 | 1.17e-7 | 5.71e-8 | 1.21e-7 | 7.98e-5 | 4.98e-5 | 1.46e-5 |
| 9 | 5.26e-9 | 2.81e-11 | 1.62e-8 | 1.37e-8 | 2.92e-8 | 1.94e-5 | 1.41e-5 | 2.35e-6 |
| 10 | 1.31e-9 | 1.24e-13 | 1.28e-9 | 1.30e-9 | 1.53e-10 | 1.44e-6 | 2.30e-6 | 5.98e-7 |
| 11 | 3.28e-10 | 6.44e-15 | 1.13e-9 | 1.14e-9 | 1.16e-11 | 5.38e-8 | 1.06e-7 | 1.70e-8 |
| 12 | 1.64e-10 | $< 10^{-16}$ | 1.41e-10 | 1.41e-10 | 1.09e-14 | 4.33e-9 | 1.57e-9 | 3.92e-10 |
| 13 | 8.19e-11 | $< 10^{-16}$ | 1.41e-10 | 1.41e-10 | 1.09e-14 | 1.06e-9 | 3.91e-10 | 7.29e-11 |

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Table 4: Values of $B_{\alpha,d,\gamma}(q,p)$ and $B_{\alpha,d,\gamma}(C_1,\ldots,C_{ds})$ for $\gamma_j = j^{-2}$, $1 \leq j \leq s$ and $s = 3$ with various choices of $(\alpha, d) = (1, 1), (2, 2), (3, 3)$.

| $m$ | $(\alpha, d) = (1, 1)$ | $(\alpha, d) = (2, 2)$ | $(\alpha, d) = (3, 3)$ | Sobol\PLR | Sobol\N-X | PLR | Sobol\N-X | PLR |
|-----|-------------------------|-------------------------|-------------------------|-----------|-----------|-----|-----------|-----|
| 4   | 3.74e-3                 | 3.07e-3                 | 5.49e-3                 | 2.21e-2   | 2.38e-3   | 1.12e+2 | 1.31e+2   | 6.13e+0 |
| 5   | 1.04e-3                 | 1.04e-3                 | 2.03e-3                 | 9.75e-4   | 4.25e-4   | 1.86e+1 | 2.42e+1   | 6.05e-1 |
| 6   | 2.85e-4                 | 2.85e-4                 | 2.00e-4                 | 2.84e-4   | 9.00e-5   | 7.21e+0 | 1.13e+1   | 3.72e-1 |
| 7   | 7.77e-5                 | 8.04e-5                 | 2.84e-5                 | 3.93e-5   | 1.37e-5   | 5.82e-1 | 2.81e+0   | 5.32e-2 |
| 8   | 2.16e-5                 | 2.23e-5                 | 6.95e-6                 | 1.14e-5   | 2.21e-6   | 1.03e-1 | 2.73e+0   | 4.58e-3 |
| 9   | 5.74e-6                 | 6.04e-6                 | 2.87e-7                 | 2.64e-7   | 2.53e-7   | 5.10e-2 | 4.60e-2   | 5.02e-4 |
| 10  | 1.58e-6                 | 1.68e-6                 | 5.04e-8                 | 7.85e-8   | 3.22e-8   | 6.09e-3 | 4.41e-2   | 7.55e-5 |
| 11  | 4.25e-7                 | 4.71e-7                 | 3.61e-8                 | 8.29e-9   | 4.35e-9   | 5.20e-4 | 2.67e-4   | 3.96e-6 |
| 12  | 1.14e-7                 | 1.27e-7                 | 3.01e-9                 | 1.93e-9   | 5.93e-10  | 1.26e-4 | 1.16e-4   | 2.38e-7 |
| 13  | 3.05e-8                 | 3.37e-8                 | 8.19e-10                | 8.93e-10  | 9.78e-11  | 6.17e-5 | 6.10e-5   | 2.32e-8 |
| 14  | 8.25e-9                 | 9.52e-9                 | 3.72e-10                | 3.79e-10  | 7.46e-12  | 3.08e-5 | 3.39e-5   | 2.02e-9 |
| 15  | 2.17e-9                 | 2.52e-9                 | 1.85e-10                | 2.27e-10  | 1.14e-12  | 1.54e-5 | 1.52e-5   | 1.05e-10 |