Multi-scale Representation of High Frequency Market Liquidity

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February 10, 2014

Abstract

We introduce an event based framework of directional changes and overshoots to map continuous financial data into the so-called Intrinsic Network - a state based discretisation of intrinsically dissected time series. Defining a method for state contraction of Intrinsic Network, we show that it has a consistent hierarchical structure that allows for multi-scale analysis of financial data. We define an information theoretic measurement termed Liquidity that characterises the unlikeliness of price trajectories and argue that the new metric has the ability to detect and predict stress in financial markets. We show empirical examples within the Foreign Exchange market where the new measure not only quantifies liquidity but also acts as an early warning signal.

Keywords: Liquidity, Information Theory, Multi-Scale, Foreign Exchange, High Frequency Trading

1 Introduction

The notion of market liquidity is nowadays almost ubiquitous. It quantifies the ability of a financial market to match buyers and sellers in an efficient way, without causing a significant movement in the price, thus delivering low transaction costs. It is the lifeblood of financial markets (Fernandez 1999) without which market dislocations can show as in the recent well documented crisis: 2007 Yen carry trade unwind (Brunnermeier et. al. 2008), 2008 Credit Crunch (Brunnermeier 2009), May 6th 2010 Flash Crash (Kirilenko et. al. 2011, SEC 2011) or the numerous Mini Flash Crashes (Golub et. al. 2012, Johnson et. al. 2013) occurring in US equity markets, but also in many others cases that go unnoticed but are potenit candidate to become more important. While omnipresent, liquidity is an elusive concept. Several reasons may account for this ambiguity; some markets, such as the foreign exchange (FX) market with the daily turnover of $5.3 trillion (BIS 2013), are mistakenly assumed to be extremely liquid, whereas the generated volume is equated with liquidity. Secondly, the structure of modern markets with its high degree of decentralization generates fragmentation and low transparency.

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of transactions which complicates the way to define market liquidity as a whole. For instance, the implementation of Regulation NMS in US equity markets has created a fragmented ecosystem where trading is split between 13 public exchanges, more than 30 dark pools and over 200 internalizing broker-dealers (Shapiro 2010). Aggregating liquidity from all trading sources can be quite daunting and even with all of the market fragmentation, as new venues with different market structure continue to be launched. Furthermore, the landscape is continuously changing as new players emerge, such as high frequency traders that have taken over the role of liquidity intermediation in many markets, accounting between 50% and 70% of all trading (Chaboud et al. 2012). The shift in liquidity provision was the result of legislative changes (in US Regulation NMS, 2005 and in Europe MiFID, 2007) which was fostered greater competition and prompted by substantial technological advances in computation and communication that made high-speed trading possible between different trading venues. Last, but not least, important participants influencing the markets are the central banks with their myriad of market interventions, whereas it is indirectly through monetization of substantial amount of sovereign and mortgage debt with various quantitative easing programs, or in a direct manner as with Swiss National Bank setting the floor on EUR/CHF exchange rate, providing plenty of arguments they have overstepped their role of last resort liquidity providers and at this stage they hamper market liquidity1,2,3, potentially exposing themselves to massive losses in the near future4,5,6.

Despite the obvious importance of liquidity there is little agreement on the best way to measure and define market liquidity (von Wyss 2004, Sarr and Lybek 2002, Kavajecz and Odders-White 2008, Gabrielsen et al. 2011). Liquidity measures can be classified into different categories. Volume-based measures: liquidity ratio, Martin index, Hui and Heubel ratio, turnover ratio, market adjusted liquidity index (see Gabrielsen et al. 2011 for details) where, over a fixed period of time, the exchanged volume is compared to price changes. This class implies that non-trivial assumptions are made about the relation between volume and price movements. Other classes of measures include price based measures: Marsh and Rock ratio, variance ratio, vector autoregressive models; transaction costs based measures: spread, implied spread, absolute spread or relative spread see; or time based measures: number of transactions or orders per time unit. There exists plenty of studies that analyse these measures in various contexts (see von Wyss 2004, Gabrielsen et al. 2011 and references therein) without reaching a consensus. The aforementioned approaches suffer from many drawbacks. They provide a top-down approach of analysing a complex system, where the impact of the variation of liquidity is analysed rather than providing a bottom-up approach where liquidity lacking times are identified and quantified. These approaches also suffer from a specific choice of physical time, that does not reflect the correct and multi-scale nature of any financial market. Finally, we argue that some of the data could be hard to get or even not available as it is the case for the full limit order book, or trade direction, in the FX market. To circumvent these issues and to move forward into modelling the market dynamics, we propose an event based framework of directional changes and overshoots to map continuous financial data into the so-called intrinsic network - a state based discretisation of intrinsically dissected time series, whereas the resulting structure is modelled as a multi-scale Markov chain. We define liquidity, an information theoretic measurement that characterises the unlikeliness of price trajectories and argue that this new metric has the ability to detect and predict stress in financial markets and show examples within the FX market. Finally, the optimal choice of scales is derived using the Maximum

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1 “Minutes of the Monetary Policy Meeting on April 26, 2013” Bank of Japan, 2013.  
2 “Here Comes the Great Bond Liquidity Crisis”, Lee, P., September 26, 2013, Euromoney  
3 “The Fed Now Owns One Third Of The Entire US Bond Market”, Darden, T., 2013, Zero Hedge  
4 “SNB Losses 1.85 Billion Francs in Just One Day, 231 Francs per Inhabitant”, Dorgan, G., 2012, SNBCHF.com  
5 “SNB Losses in October and November: 8.4 Billion Francs, 1.5% of GDP”, Dorgan, G., 2012, SNBCHF.com  
6 “The Fed’s Impersonation Of The Hunt Brothers Continues”, Tchir, P., 2013, TF Market Advisors
Entropy Principle.

The rest of this paper is organised as follows; Section 2 describes the event based framework of directional changes and overshoots. Section 3 defines the state based discretisation of price trajectory movement termed intrinsic network. In section 4 we demonstrate the multi-scale property of intrinsic network. In section 5 we derive the probability matrix of the Markov chain modelling the transitions on the intrinsic network. Section 6 describes the information-theoretic concept termed Liquidity that characterises the unlikeliness of price trajectories. In section 7 we discuss the optimal choice of scales applying Maximum Entropy Principle. Finally, in section 8 we demonstrate the application of Liquidity on 2007 Yen carry trade unwind and on Swiss National Bank (SNB) August 2011 intervention setting the floor of 1.20 on EUR/CHF exchange rate.

2 The event based framework

Traditional high frequency finance models use equidistantly spaced data for their inputs, yet markets are known to not operate in a uniform fashion: during the weekend the markets come to a standstill, while unexpected news can trigger a spur of market activity. The idea of modelling financial series using a different time clock can be traced back to the seminal work of Mandelbrot and Taylor (1967) and Clark (1973), advocating the use of transaction and volume based clock, moving away from chronological time. One other area of research that analyses high-frequency time series from the perspective of fractal theory was initiated by Mandelbrot (1963). This seminal work has inspired others to search for empirical patterns in market data – namely scaling laws that would enhance the understanding of how markets work, providing fixed points of reference and blurring out the irrelevant details. One of the most reported scaling laws in capital markets (Müller et al. 1990, Galluccio et al. 1997, Dacorogna et al. 2001, Di Matteo, Aste and Dacorogna 2005), relating the average absolute price change $\langle \Delta x \rangle$ and the time interval of its occurrence $\Delta t$:

$$\langle \Delta x \rangle \sim \Delta t^{1/2}$$

sparked an attempt to move beyond the constraints of physical time devising a time-scale named "θ-time" to account for seasonal patterns correlated with the changing presence of main market places in the FX market (Guillaume et al. 1995). This approach was not flawless, since aggregating and interpolating tick data amongst fixed or predetermined time intervals, important information about the market microstructure and trader behaviour is lost (Bauwens and Hautsch 2009).

A major breakthrough occurred with the discovery of a scaling law that relates the number of rising and falling price moves of a certain size (threshold) with that respective threshold, which produces an event-based time scale named “intrinsic time” that ticks according to an evolution of a price move (Guillaume et al. 1997). The intrinsic time dissects the time series based on market events where the direction of the trend alternates (Figure 1). These directional change events are identified by price reversals of a given threshold value set ex-ante. Once a directional change event has been confirmed an overshoot event has begun and they continue the trend identified by the directional changes. So, if an upward event has been confirmed, an upward overshoot follows and vice versa. An overshoot event is confirmed when the opposite directional change occurs. With each directional change event, the “intrinsic time” ticks one unit. Details of the algorithm are provided in appendix A. The benefits of this approach in the analysis of high-frequency data are threefold; firstly, it can be applied to non-homogeneous

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7A scaling law establishes a mathematical relationship between two variables that holds true over multiple orders of magnitude.
Figure 1: Directional change events (squares) act as natural dissection points, decomposing a total-price move between two extremal price levels (bullets) into so-called directional-change (solid lines) and overshoot (dashed lines) sections. Time scales depict physical time ticking evenly across different price curve activity regimes, whereas intrinsic time triggers only at directional change events, independent of the notion of physical time.

time series without the need for further data transformations. Secondly, multiple directional change thresholds can be applied at the same time for the same tick-by-tick data. And thirdly, it captures the level of market activity at any one time.

Using the aforementioned framework Glattfelder et al. (2011) have discovered 12 independent scaling laws that hold true across 13 currency pairs across multiple orders of magnitude, linking together concepts like the length of the overshoot, the length of the directional change, the number of directional changes and the number of overshoot ticks with the threshold value. Although the scaling law discoveries were acknowledged, there was no consensus on what drives this behaviour. This concept was predominantly used to design trading models. For instance, in (Dupuis and Olsen 2011) and (Voicu 2012) a basic counter-trending trading model is described which exploits profit opportunities contained in the long coastline of prices.

Let us formalise the considerations of the framework. With $\delta > 0$ we will denote the directional change threshold, as well as the directional change itself, while the corresponding overshoot is denoted with $\omega$. We argue that the length of the overshoot $\omega$ is a proxy for liquidity as a long overshoot seems to imply that the market had to further progress in the direction of the overshoot to find the necessary liquidity to eventually retrace and exhibit the next directional change.

Claiming overshoot length is too long or too short is an abstract notion without a reference providing an “equilibrium” length. The following theorem establishes that given a stylized model of Brownian motion driving the price movement, the expected length of the overshoot $\omega$ equals the directional change threshold $\delta$;
Theorem 2.1. (Fundamental intrinsic theorem) Let the price \( (P_t, t \in \mathbb{R}_+) \) be modelled as Brownian motion \( (W_t, t \in \mathbb{R}_+) \) with volatility \( \sigma > 0 \)

\[
dP_t = \sigma dW_t. \tag{1}
\]

Let \( \omega(\delta; \sigma) \) denote the length of the overshoot for directional change threshold \( \delta \). Then

\[
E[\omega(\delta; \sigma)] = \delta. \tag{2}
\]

The proof is provided in appendix B. Unexpectedly, the average length of the overshoot \( E[\omega(\delta; \sigma)] \) is completely insensitive to volatility; regardless of the volatility level the average length of the overshoot will be proportional to the directional change threshold. Finally, we note that the theorem establishes a scaling law relationship between overshoot length and directional change threshold.

3 The intrinsic network

This section proposes a novel way to describe the evolution of a time series of prices by discretising the price movement over various scales and modelling it as transitions on a structure termed intrinsic network.

We consider \( n \in \mathbb{N} \) ordered thresholds \( \delta_1 < \delta_2 < \cdots < \delta_n \) that dissect the price curve into directional changes of fixed length \( \delta_i \) and overshoots \( \omega_i \) of varying length, assigning the states of the market for a given directional change threshold \( \delta_i \) either to be 1 or 0, depending whether the overshoot related to the corresponding threshold \( \delta_i \) is moving upwards or downwards. With this procedure at each physical time we can assign a binary vector \( b = (b_1, \ldots, b_n) \) consisting of 1 or 0, describing the market over various scales. The binary encoding \( b = (b_1, \ldots, b_n) \) can express the state of the market \( s \) in numeric terms as follows

\[
s = b_1 \cdot 2^0 + b_2 \cdot 2^1 + \cdots + b_n \cdot 2^{n-1}.
\]

The set of binary vectors will be denoted by \( B \), set of states in numeric representation with \( S \) and due to 1-1 correspondence between \( S \) and \( B \) we will interchangeably use both notations, whereas it is straightforward to notice that both sets have a total of \( 2^n \) possible states.

The ascending sorted thresholds and the underlying intrinsic time dynamics imply a simple rule describing the transition between states. Indeed in any state a move up would turn the smallest \( i \) state \( s_i \) showing a previous move down (i.e. \( s_i = 0 \)) into an upward state \( s_i = 1 \) state and similarly a move down would flip the first \( i \) state \( s_i = 1 \) state into an \( s_i = 0 \) state. The precise transition rules are stated bellow;

state \( b = (b_1, \ldots, b_n) \in B \) can transition to state

\[
b' = (\overline{b}_1, b_2, \ldots, b_n)
\]

or to state

\[
b'' = (b_1, b_2, \ldots, \overline{b}_i, \ldots, b_n), \ i = \min\{k : b_k \neq b_1\}
\]

where \( \overline{1} = 0 \) and \( \overline{0} = 1 \).

For a given \( n \in \mathbb{N} \) and directional change thresholds \( \delta_1 < \cdots < \delta_n \), we obtain a set of states \( S \) with the transitions prescribed by the above rule. In other words, the procedure creates a network which we call intrinsic network, denoted with \( \mathcal{IN}(n; \{\delta_1, \ldots, \delta_n\}; W) \) where \( W \) denotes the transition probability matrix of the stochastic process modelling the transitions on the network.
Figure 2: From left to right: 2-, 3- and 4-dimensional intrinsic networks $\mathcal{I}\mathcal{N}$, with states denoted in numeric presentation.

The intrinsic network exhibits a couple of peculiar states where the network is non-reactive: the downward blind-spot $(0, \ldots, 0)$ when the market can keep on moving down and the upward blind-spot $(1, \ldots, 1)$ when the market can keep on moving up without being traced. From the aforementioned states the intrinsic network has only one possible transition, state $(1, 1, \ldots, 1)$ transitions with certainty to state $(0, 1, \ldots, 1)$,

$$P((1, 1, \ldots, 1) \rightarrow (0, 1, \ldots, 1)) = 1,$$

while state $(0, 0, \ldots, 0)$ transitions with certainty to state $(1, 0, \ldots, 0)$,

$$P((0, 0, \ldots, 0) \rightarrow (1, 0, \ldots, 0)) = 1.$$

Figure 2 shows examples of respectively 2-, 3- and 4-dimensional intrinsic networks $\mathcal{I}\mathcal{N}$.

4 Implicit hierarchy

In this section we demonstrate that intrinsic networks have a convenient multi-scale property where one can dismiss the smallest threshold from the framework and still preserve the structure.
In order words, if we remove the directional change threshold $\delta_1$ of an $n$-dimensional intrinsic network $\mathcal{I}N(n; \{\delta_1, \ldots, \delta_n\}; W)$, the resulting structure is an $n-1$-dimensional intrinsic network $\mathcal{I}N(n-1; \{\delta_2, \ldots, \delta_n\}; \hat{W})$, whereas there is an explicit connection between transition matrices $W$ and $\hat{W}$, assuming the transitions on the network are modelled as first order Markov chain process.

Firstly, we introduce the concept of islands which are subsets of all possible states $S = \{0, 1, \ldots, 2^n - 1\}$ the set of states of an $n$-dimensional intrinsic network $\mathcal{I}N(n; \{\delta_1, \ldots, \delta_n\}; W)$. We define the $k$-th island as the following subset of states

$$\mathcal{I}_k = \{s \in S: \lfloor \frac{s}{2} \rfloor = k\}, \tag{3}$$

where $\lfloor \cdot \rfloor$ denotes the floor function, i.e. $\lfloor x \rfloor = \max\{k \in \mathbb{N}: k \leq x\}$. In our cases, the $k$-th island equals $\mathcal{I}_k = \{2k, 2k + 1\}$. For instance, island $\mathcal{I}_0$ equals to subset $\{0, 1\}$ or island $\mathcal{I}_7$ equals to subset $\{14, 15\}$. It is easy to notice that for an $n$-dimensional intrinsic network with $2^n$ states, there are $2^{n-1}$ islands. Note that we can again label the islands in numeric terms,

$$\mathcal{I}_k = k,$$

creating a new set of states with numeric labels $S^{(1)} = \{0, \ldots, 2^{n-1} - 1\}$. Let us remark on the transitions among islands $\mathcal{I}_k$. Given the numeric notation $s = b_1 \cdot 2^0 + \cdots + b_n \cdot 2^{n-1}$ for states of $S$, each state $s^{(1)} \in S^{(1)}$ can be written in numeric notation as

$$s^{(1)} = b_2 \cdot 2^0 + \cdots + b_n \cdot 2^{n-2},$$

hence the transitions among islands are equivalent to occurrences of directional changes for thresholds $\delta_2, \ldots, \delta_n$. In other words, transitions among islands are insensitive to changes in the market state related to first directional change thresholds $\delta_1$, hence the resulting structure is an $n-1$ dimensional intrinsic network $\mathcal{I}N(n-1; \{\delta_2, \ldots, \delta_n\}, \hat{W})$. The probabilities of transition matrix $\hat{W}$ and its connection to transition matrix $W$ is established later in this section. We refer to the presented method as state contraction.

Let us extend the concept of state contraction by defining islands of level 0, $\mathcal{I}^{(0)}_k$ as the states of an $n$-dimensional intrinsic network

$$\mathcal{I}^{(0)}_k = \{k\}, \quad k = 0, 1, \ldots, 2^n - 1.$$

If $S^{(0)} = \{0, 1, \ldots, 2^n - 1\} = \{\mathcal{I}^{(0)}_0, \ldots, \mathcal{I}^{(0)}_{2^n-1}\}$ denotes the states of $n$ dimensional intrinsic network then we define islands of level 1, in notation $\mathcal{I}^{(1)}_k$ as subsets

$$\mathcal{I}^{(1)}_k = \{\mathcal{I}^{(0)} \in S^{(0)}: \lfloor \frac{\mathcal{I}^{(0)}}{2} \rfloor = k\} \tag{4}$$

for $k = 0, 1, \ldots, 2^{n-1} - 1$. Using the aforementioned iterative process we can obtain islands of level $j$, in notation $\mathcal{I}^{(j)}$, by applying the process of state contraction $j$ times, hence we can then define an intrinsic network of level $j + 1$, $\mathcal{I}N^{(j+1)}$ whereas the states space $S^{(j+1)}$ are defined as islands $\mathcal{I}^{(j+1)}$, in other words $S^{(j+1)} = \{\mathcal{I}^{(j+1)}_0, \ldots, \mathcal{I}^{(j+1)}_{2^n-j-1}\}$. In more general sense, we can define islands of level $i$ as a subset of islands of level $i - 1$, i.e.

$$\mathcal{I}^{(i)}_l = \{\mathcal{I}^{(i-1)} \in S^{(i-1)}: \lfloor \frac{\mathcal{I}^{(i-1)}}{2} \rfloor = l\}.$$

Having started with an intrinsic network with a total of $2^n$, the intrinsic network of level $j + 1$, $\mathcal{I}N^{(j+1)}$ will have a total of $2^{n-j}$ states.
Figure 3: Illustration of the contraction procedure of a 4-dimensional intrinsic network $\mathcal{IN}(4; \{\delta_1, \ldots, \delta_4\}; W)$ to a 3-dimensional intrinsic network $\mathcal{IN}(3; \{\delta_2, \delta_3, \delta_4\}; \hat{W})$, whereas coloured shading graphs islands, contracted states and the resulting new states.

Figure 3 illustrates the process of state contraction, of 4-dimensional intrinsic network $\mathcal{IN}(4; \{\delta_1, \ldots, \delta_4\}; W)$ whereas the islands $\mathcal{I}_0^{(0)}, \ldots, \mathcal{I}_{15}^{(0)}$ are contracted in the following manner

$$\mathcal{I}_i^{(1)} = \{\mathcal{I}_2^{(0)}, \mathcal{I}_{2i+1}^{(0)}\}.$$ 

The coloured shading graphs islands, contracted states and the resulting new states of the three dimensional intrinsic network $\mathcal{IN}(3; \{\delta_2, \delta_3, \delta_4\}; \hat{W})$.

Assuming the transitions on the intrinsic network are modelled as a first order Markov chain, there is an explicit connection between the transition matrix $W$ of the $n$-dimensional intrinsic network and transition matrix $\hat{W}$ of the $n-1$ dimensional intrinsic network obtained through the contraction process described above.

Let us assume that the current state of the network is $\mathcal{I}_k^{(i)}$ and we are interested in finding the probability of transitioning to state $\mathcal{I}_j^{(i)}$. Note that observing the process from viewpoint of islands of level $(i - 1)$, the system can oscillate among states $\mathcal{I}_{2k}^{(i-1)}$ and $\mathcal{I}_{2k+1}^{(i-1)}$ arbitrary many times before taking the transition that links the islands $\mathcal{I}_k^{(i)}$ and $\mathcal{I}_j^{(i)}$. Hence, the desired probability can be easily derived using the closed form of geometric series. The three distinct cases are presented below, with the proof in appendix C,
for mod$(k + j, 2) = 0$

$$
P(I_k^{(i)} \rightarrow I_j^{(i)}) = \frac{P(I_{2k+1}^{(i-1)} \rightarrow I_{2j+1}^{(i-1)})}{1 - P(I_{2k+1}^{(i-1)} \rightarrow I_{2j}^{(i-1)}) \cdot P(I_{2k}^{(i-1)} \rightarrow I_{2j+1}^{(i-1)})} \tag{5}
$$

for mod$(k + j, 2) = 1$ and $k > j$

$$
P(I_k^{(i)} \rightarrow I_j^{(i)}) = \frac{P(I_{2k-1}^{(i-1)} \rightarrow I_{2j}^{(i-1)}) \cdot P(I_{2k+1}^{(i-1)} \rightarrow I_{2j}^{(i-1)})}{1 - P(I_{2k+1}^{(i-1)} \rightarrow I_{2j+1}^{(i-1)}) \cdot P(I_{2k}^{(i-1)} \rightarrow I_{2j+1}^{(i-1)})} \tag{6}
$$

for mod$(k + j, 2) = 1$ and $k < j$

$$
P(I_k^{(i)} \rightarrow I_j^{(i)}) = \frac{P(I_{2k+1}^{(i-1)} \rightarrow I_{2j}^{(i-1)}) \cdot P(I_{2k}^{(i-1)} \rightarrow I_{2j+1}^{(i-1)})}{1 - P(I_{2k}^{(i-1)} \rightarrow I_{2j+1}^{(i-1)}) \cdot P(I_{2k+1}^{(i-1)} \rightarrow I_{2j}^{(i-1)})} \tag{7}
$$

5 Transition probabilities

In this section we present the transition probabilities of an intrinsic network modelling the transitions as a first order Markov chain process.

Firstly, we stress that given a Brownian motion $(\sigma W_t, t \in \mathbb{R}_+)$ as the process modelling the price movement $(P_t, t \in \mathbb{R}_+)$

$$
dP_t = \sigma dW_t, \tag{8}
$$

the resulting stochastic process of transitions on intrinsic networks, in notation $(X_{\tau_{\alpha}}, \alpha \in \mathcal{A})$ is the set of intrinsic times when a transition occurred is in fact a non-Markovian process, i.e.

$$
P(X_{\tau_{\alpha}} = s_i|X_{\tau_{\alpha-1}} = s_j, \ldots, X_{\tau_{m-k}} = s_r) \neq P(X_{\tau_{\alpha}} = s_i|X_{\tau_{m-k}} = s_j, \ldots, X_{\tau_{m}} = s_l), \quad \forall k < m.
$$

In other words, the process requires the full history to derive the transition probabilities. The non-Markovian property stems from the fact that directional change thresholds can have different reference points in their memory. In what follows, for the sake of simplicity, we will nevertheless adopt a Markovian description. Properly quantifying the impact of such a choice is an interesting and ambitious question that we will address in the future. However, we have noticed that, in our case, transitions probabilities between two states happen to be close whatever the length of the memory considered in the description, and also that, for the application we have in mind -namely quantifying market liquidity- our liquidity measure seems rather insensitive to whether or not we take memory effects into account.

In order to simplify the framework we model the transitions on the intrinsic network as a first order Markov chain process, assuming that the transitions for threshold $\delta_1$ correspond to a length of the overshoot of $\delta_2 - \delta_1$, i.e. let

$$
i = \min\{k: b_k \neq b_1\},
$$

for $i = 2$

$$
P(b_1, b_2, \ldots, b_n) \rightarrow (b_1, b_2, \ldots, b_n) = P(\omega(\delta_1; \sigma) \geq \delta_2 - \delta_1) \tag{9}
$$

$$
P(b_1, b_2, \ldots, b_n) \rightarrow (\bar{b_1}, b_2, \ldots, b_n) = P(\omega(\delta_1; \sigma) < \delta_2 - \delta_1) \tag{10}
$$

where $\bar{I} = 0$ and $\bar{0} = 1$. Now we present the analytic expression for transition probabilities assuming (9)-(10):
Theorem 5.1. Let \( n \in \mathbb{N} \) and \( \delta_1 < \cdots < \delta_n \) directional change thresholds of intrinsic network \( \mathcal{I}N(n; \{\delta_1, \ldots, \delta_n\}; W) \) and \((b_1, \ldots, b_n) \in \mathcal{B}\) the current state of the market. Let
\[
i = \min\{k: b_k \neq b_1\},
\]
for \( i = 2 \)
\[
\mathbb{P}((b_1, b_2, \ldots, b_n) \rightarrow (b_1, \overline{b_2}, \ldots, b_n)) = e^{-\frac{\delta_2 - \delta_1}{\delta_1}}
\]
(11)
\[
\mathbb{P}((b_1, b_2, \ldots, b_n) \rightarrow (\overline{b_1}, b_2, \ldots, b_n)) = 1 - e^{-\frac{\delta_2 - \delta_1}{\delta_1}}
\]
(12)
while for \( i > 2 \)
\[
\mathbb{P}((b_1, b_2, \ldots, b_n) \rightarrow (b_1, \overline{b_2}, \ldots, b_n)) = \frac{\prod_{k=2}^{i} e^{-\frac{\delta_k - \delta_{k-1}}{\sigma_{k-1}}}}{1 - \sum_{k=2}^{i-1} (1 - e^{-\frac{\delta_k - \delta_{k-1}}{\sigma_{k-1}}}) \prod_{j=k+1}^{i} e^{-\frac{\delta_j - \delta_{j-1}}{\sigma_{j-1}}}}
\]
(13)
\[
\mathbb{P}((b_1, b_2, \ldots, b_n) \rightarrow (\overline{b_1}, \overline{b_2}, \ldots, b_n)) = 1 - \frac{\prod_{k=2}^{i} e^{-\frac{\delta_k - \delta_{k-1}}{\sigma_{k-1}}}}{1 - \sum_{k=2}^{i-1} (1 - e^{-\frac{\delta_k - \delta_{k-1}}{\sigma_{k-1}}}) \prod_{j=k+1}^{i} e^{-\frac{\delta_j - \delta_{j-1}}{\sigma_{j-1}}}}
\]
(14)
where \( \mathbb{I} = 0 \) and \( \overline{0} = 1 \).

The proof for the analytical expressions in theorem 5.1 can be found in appendix C, and it boils down to deriving the expressions (11) and (12) and then applying the explicit formulas for transition probabilities of contracted intrinsic network presented in section 4.

6 Price trajectory unlikeliness

Here we introduce the Liquidity, an information theoretic value that measures the unlikeliness of price trajectories mapped to the intrinsic network. Before proceeding let us simplify the notation used so far. The stochastic process modelling the transition on the intrinsic network \( \mathcal{I}N \) was denoted with \( (X_{\tau_n}) \) where \( (\tau_\alpha, \alpha \in \mathcal{A}) \) are the times of occurrences of transitions on the network. In other words, \( (\tau_n) \) is the waiting times process when any of the directional changes related to thresholds \( \delta_i, i = 1, \ldots, n \) has occurred. Hence, modelling the transitions on the network as a first order Markov chain, the corresponding probabilities are denoted with
\[
\mathbb{P}(X_{\tau_n} = s_i | X_{\tau_{n-1}} = s_j), s_i, s_j \in \mathcal{S}.
\]
We will shorten this notation by writing only that a transition is made from state \( s_i \) to \( s_j \), neglecting the intrinsic times when the transition occurred, i.e. we will simplify the notation by writing
\[
\mathbb{P}(s_i \rightarrow s_j).
\]

Firstly, let \( \mathcal{I}N(n; \{\delta_1, \ldots, \delta_n\}; W) \) denote the \( n \) dimensional intrinsic network with ordered directional change thresholds \( \delta_1 < \cdots < \delta_n \), state space \( \mathcal{S} = \{0, 1, \ldots, 2^n - 1\} \) and let \( W \) denote the corresponding transition probability matrix
\[
W = [\mathbb{P}(s_i \rightarrow s_j)]_{i,j=0}^{2^n-1}.
\]
We define the surprise of transition from state \( s_i \) to state \( s_j \), in notation \( \gamma_{s_i, s_j} \), by the following expression
\[
\gamma_{s_i, s_j} = -\log \mathbb{P}(s_i \rightarrow s_j).
\]
(15)
Let us explain the intuition behind the surprise $\gamma_{s_i,s_j}$, if the transition from state $s_i$ to $s_j$ is very unlikely, i.e. $\mathbb{P}(s_i \to s_j) \approx 0$, the corresponding surprise $\gamma_{s_i,s_j}$ will be very large, i.e. $\gamma_{s_i,s_j} \gg 0$, implying that the price trajectory experienced an unlikely movement within the context of its intrinsic network. Note that the above-introduced surprise of transition should be familiar to anyone working in information theory, since entropy is obtained by simply averaging the surprise. In other words, entropy is the average uncertainty of the process, while surprise is this same quantity evaluated for a particular realization of the process, i.e. the surprise is a point-wise entropy.

Let us now assume that we are observing the price trajectory $(P_t)$ within a time interval $[0, T]$, $T > 0$ and within this time interval the price trajectory experienced $K$ transitions

$$s_{i_1} \to s_{i_2} \to \cdots \to s_{i_K} \to s_{i_{K+1}}$$

hence residing in $K + 1$ states during time interval $[0, T]$. We define the surprise of price trajectory within time interval $[0, T]$, in notation $\gamma^{[0,T]}_{s_{i_1},\ldots,s_{i_{K+1}}}$ as

$$\gamma^{[0,T]}_{s_{i_1},\ldots,s_{i_{K+1}}} = -\log \mathbb{P}(s_{i_1} \to s_{i_2} \to s_{i_3} \to \cdots s_{i_K} \to s_{i_{K+1}})$$

$$= \sum_{k=1}^{K} -\log \mathbb{P}(s_{i_k} \to s_{i_{k+1}})$$

$$= \sum_{k=1}^{K} \gamma_{s_{i_k},s_{i_{k+1}}}$$

If the $K$ transitions taken on the intrinsic network are clear from the context, we will denote the surprise with $\gamma^{[0,T]}_{K}$. The defined value, as sum of unlikeliness of individual transitions, measures the unlikeliness of price trajectory within a given time interval. Should the value be very large, it would indicate that the trajectory experienced an unlikely movement. Furthermore, it is important to stress that the surprise is a price trajectory dependent measurement: two price trajectories of same volatility $\sigma$ can have widely different surprise values.

Suppose one wants to compute the surprise of a price trajectory within a certain time interval $[0, T]$ using empirical data, applying the formula listed above. An easily noticeable caveat arises as the price movement can result in a few transitions on the intrinsic network in some time periods, while others time periods can be marked by higher activity. Therefore, by construction the hectic time periods will have higher surprise purely due to higher number of transitions. In order to remove the market activity from the analysis, we need to remove the component related to the number of transitions from the surprise within the mentioned time interval. From Shannon-McMillan-Brieman theorem on convergence of sample entropy and the corresponding Central Limit Theorem (Pfister et al. 2001), we notice that the centring value in the expression is in fact equal to $K \cdot H^{(1)}$, while the rescaling value equals $\sqrt{K \cdot H^{(2)}}$, involving the first order informativeness (i.e. entropy)

$$H^{(1)} = \sum_{i=0}^{2^n-1} \mu_i \mathbb{E}[-\log \mathbb{P}(s_i \to \cdot)],$$

and second orders of informativeness

$$H^{(2)} = \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^n-1} \mu_i \mu_j \text{Cov} (-\log \mathbb{P}(s_i \to \cdot), -\log \mathbb{P}(s_j \to \cdot)).$$

where $\mu$ is the stationary distribution of the corresponding Markov chain, i.e. left normalised eigenvector of transition matrix $W$. The centred and rescaled expression converges to normal
distribution (Pfister et al. 2001)

\[
\frac{\gamma_{K}^{[0,T]} - K \cdot H^{(1)}}{\sqrt{K \cdot H^{(2)}}} \to N(0,1) \quad \text{as} \quad K \to \infty,
\]

(19)

hence we call Liquidity\(^8\) the corresponding quantile

\[
\mathcal{L} = 1 - \Phi\left(\frac{\gamma_{K}^{[0,T]} - K \cdot H^{(1)}}{\sqrt{K \cdot H^{(2)}}}\right), \quad K \gg 0
\]

(20)

where \(\Phi\) is the cumulative distribution function of normal distributions. The Liquidity \(\mathcal{L}\) provides an activity-free measurement of price trajectory unlikelines where value close to zero indicates illiquid market conditions as overshoots, observed in a multi-scale framework, are long; while values close to one indicates ample liquidity in the market and overshoots are short. Note that \(K\), the number of transitions within time interval \([0,T]\) on the intrinsic network, which is explicitly expressed in the Liquidity formula, in fact equals to the sum of all directional changes related to thresholds \(\delta_1, \ldots, \delta_n\) that occurred within time interval \([0,T]\), representing the measurement of activity across multiple scales. Hence by subtracting \(K \cdot H^{(1)}\) from surprise \(\gamma_{K}^{[0,T]}\) and rescaling it by \(\sqrt{K \cdot H^{(2)}}\) we remove the contribution of market activity from the analysis of price trajectory, allowing us to purely observe the lengths of the overshoots in a multi-scale framework.

### 7 Preferred Scales

Throughout the paper we have been elusive about optimal setting of the thresholds that map the continuous financial data into the intrinsic network. In this section we discuss the optimal choice of directional change thresholds, exploring three possibilities;

- setting all achievable transition probabilities to be equally likely, i.e. \(P(s_i \to s_j) = 0.5, \{s_i \to s_j\} \neq \emptyset, s_i, s_j \in \mathcal{S}\),
- maximizing the entropy, i.e. the first order informativeness \(H^{(1)}\), of the Markov chain process modelling the transitions on the intrinsic network,
- maximizing the second order informativeness, \(H^{(2)}\) of the Markov Chain process and relating the process to optimal setting of thresholds for Liquidity measurement \(\mathcal{L}\).

Notice that all of the mentioned optimization methods can be viewed as the consequence of application of maximum entropy principle on the selected probability distributions in the framework. In a nutshell, the maximum entropy principle states (Jaynes 1957a, Jaynes 1957b) that

"...subject to precisely stated prior data (such as a proposition that expresses testable information), the probability distribution which best represents the current state of knowledge is the one with largest entropy."

First we discuss the resulting thresholds when setting all of the achievable transition probabilities to be equally likely. Since the transition from each state \(s_i \in \mathcal{S}\) can be viewed as

---

\(^8\)When Shannon had invented his quantity and consulted von Neumann how to call it, von Neumann replied: “Call it entropy. It is already in use under that name and besides, it will give you a great edge in debates because nobody knows what entropy is anyway”, (Denbigh 1981). Using the same rational we decided to call our measurement Liquidity.
Bernoulli random variable, its maximal entropy is achieved by setting the probability of all non-trivial transitions to be equally likely, i.e.

\[ P(s_i \to s_1^i) = P(s_i \to s_2^i) = 0.5 \Rightarrow E[-\log P(s_i \to \cdot)] = \log 2, \ \forall i = 1, \ldots, 2^n - 2. \]

In that case we can directly apply the analytical formulas for probabilities of transitions presented in section 4, and given the first thresholds \( \delta_1 \), the \( k \)-th threshold equals

\[
\delta_k = \delta_1 \prod_{i=1}^{k} \left( 1 + \log(1 + \frac{1}{k}) \right). \quad (21)
\]

We observe that such a choice of transition probabilities induces approximately linear setting of thresholds as the limit of the product approaches fast its limit

\[
\lim_{k \to \infty} \prod_{i=1}^{k} \left( 1 + \log(1 + \frac{1}{k}) \right) = 0.8625576, \ldots,
\]

in other words, for \( k \gg 0 \), the \( k \)-th directional change thresholds \( \delta_k \) is approximately

\[
\delta_k \approx \delta_1 \cdot k \cdot 0.8625576.
\]

Unfortunately, setting the thresholds as in (21) disables the use of surprise \( \gamma_{\mathcal{K} \{0,T\}}^K \) in studying Liquidity as it becomes completely deterministic and proportional to number of transitions \( K \)

\[ \gamma_{\mathcal{K} \{0,T\}}^K \sim K \cdot \log 2. \]

Secondly, the potential method for setting the thresholds would be to maximise the entropy of the corresponding Markov chain modelling the transitions on the intrinsic network,

\[
(\delta_1^*, \ldots, \delta_n^*) = \arg \max_{(\delta_1, \ldots, \delta_n)} H^{(1)} = \arg \max_{(\delta_1, \ldots, \delta_n)} \sum_{i=0}^{2^n-1} \mu_i E[-\log P(s_i \to \cdot)],
\]

where \( \mu \) is the corresponding stationary distribution, i.e. left normalized eigenvector of \( W \).

For a two dimensional intrinsic network \( \mathcal{I} \mathcal{N}(2; \{ \delta_1, \delta_2 \}; W) \) setting all transition probabilities to be equally likely and maximising the entropy \( H^{(1)} \) from formula (21) yield equivalent optimal thresholds \( (\delta_1^*, \delta_2^*) \) where

\[
\delta_2^* = (1 + \log 2) \cdot \delta_1^*
\]

for a given \( \delta_1^* \). For \( n > 2 \) the claim does not hold.

The final proposal to set the thresholds in an optimal manner steams from the central measurement in our analysis, the surprise \( \gamma_{\mathcal{K} \{0,T\}}^K \) which is known, when properly adjusted, to converge to normal distribution for \( K \gg 0 \). Reshuffling the expression of surprise, for large but fixed \( K \) the distribution is approximately normal

\[ \gamma_{\mathcal{K} \{0,T\}}^K \sim N(K \cdot H^{(1)}, K \cdot H^{(2)}). \]

Normally distributed random variable \( X \) with mean \( \mu \) and standard deviation \( \sigma \), i.e. \( X \sim N(\mu, \sigma^2) \), has entropy given by

\[
H(X) = \frac{1}{2} \log(2\pi e \sigma^2)
\]

therefore entropy of surprise equals

\[
H(\gamma_{\mathcal{K} \{0,T\}}^K) = \frac{1}{2} \log \left( 2\pi e (K \cdot H^{(2)}) \right)
\]
and applying the maximum entropy principle on the distribution of surprise $\gamma_K^{[0,T]}$ we conclude that the optimal choice of thresholds $(\delta_1^*, \ldots, \delta_n^*)$ is the one that maximizes the second order informativeness $H^{(2)}$, i.e.

$$(\delta_1^*, \ldots, \delta_n^*) = \arg \max_{(\delta_1, \ldots, \delta_n)} H^{(2)}$$

$$= \arg \max_{(\delta_1, \ldots, \delta_n)} \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^n-1} \mu_i \mu_j \Cov \left( -\log P(s_i \rightarrow \cdot), -\log P(s_j \rightarrow \cdot) \right).$$

We note that the aforementioned optimisation process is a complex mathematical problem, which is intended to be solved using numerical procedures. For small $n$, one can explore the space of solutions of (23) and estimate the necessary joint probabilities. For larger $n$ an exploration becomes computationally undoable and we suggest to set $\delta_1$ to a fixed value and let $\delta_i = \lambda \delta_{i-1}, \lambda > 0$ for $i = 2 \ldots n$.

Choosing to maximise the first or second level of informativeness, we stress that there exists a solution for every $n \in \mathbb{N}$ as we note that the entropy of an $n$-dimensional intrinsic network, in notation $H_n^{(1)}$, is bounded

$$H_n^{(1)} = \frac{H_n^{(1)}}{\sum_{i=0}^{2^n-1} \mu_i} = \frac{\sum_{i=0}^{2^n-1} \mu_i \mathbb{E}[-\log P(s_i \rightarrow \cdot)]}{\sum_{i=0}^{2^n-1} \mu_i} \leq \max_{s_i \in S} \{\mathbb{E}[-\log P(s_i \rightarrow \cdot)]\} \leq \log 2$$

and it follows from the fact that weighted arithmetic average is always less than the maximum value of the components, as is also the second order informativeness $H_n^{(2)}$

$$H_n^{(2)} \leq \max_{s_i \in S} \{\text{Var} \left( -\log P(s_i \rightarrow \cdot) \right)^2 \}. \quad (25)$$

8 Liquidity Shocks

In this section we illustrate the application of the proposed Liquidity measurement $\mathcal{L}$ on well-known FX market crisis and argue that the Liquidity measurement can be used as an early warning signal for stress in financial markets, focusing on the August 2007 Yen carry trade collapse and the Swiss National Bank implementation of 1.20 floor for EUR/CHF exchange rate.

Before we proceed, let us explain the details of the intrinsic network used in the examples presented below. Firstly, as we are concerned with high frequency market conditions we choose the first threshold $\delta_1$ to be 0.025% (≈ 2.5 pips) and taking each next thresholds as the double of its predecessor. We use a total of twelve thresholds, i.e.

$$\delta_i = 2 \cdot \delta_{i-1}, \delta_1 = 2^{-1} \cdot \cdot \cdot 0.025\%, \quad i = 2, \ldots, 12.$$
the first $H^{(1)}_{12} = 0.4604$ and second order informativeness $H^{(2)}_{12} = 0.70818$, yielding the Liquidity measurement

$$\mathcal{L} = 1 - \Phi\left(\gamma_K [0,T] - K \cdot H^{(1)}_{12} / \sqrt{K \cdot H^{(2)}_{12}}\right).$$

whereas the sliding window is set to one day, i.e. $T = 1$ day, $K$ is the number of transitions within the sliding window. The Liquidity $\mathcal{L}$ is graphed every minute, dismissing the inactive periods during the week-ends.

First we present the Liquidity measurement $\mathcal{L}$ during the 2007 Yen carry trade unwind. The term carry trade in context of FX markets refers to strategy of shorting low-yielding currencies and buying high-yielding currencies, earning the interest differential. Carry trades are usually done with a lot of leverage\(^{10}\), so a small movement in exchange rates can result in huge losses, causing massive price reversals (Brunnermeier et al. 2008). The August 2007 USD/JPY price drop was the result of the unwinding of large Yen carry-trade positions; many hedge funds and banks with proprietary trading desks had large positions at risk and decided to buy back yen to pay back low-interest loans (Chaboud et al. 2012).

Figure 4 shows the time evolution of the tick-by-tick USD/JPY exchange rate and minute-by-minute Liquidity $\mathcal{L}$. Notable shocks to market liquidity occurred in mid July with almost 2% drop in matter of hours. From there on, relatively illiquid conditions is showed by the measurement, as traders started unwinding carry trade positions. Complete loss of market liquidity is demonstrated in the three weeks preceding the spectacular 6% drop, which occurred on August

\(^{10}\)By early 2007, it was estimated that some US$1 trillion have been staked on the yen carry trade (Lee 2008).
Next we focus on the Swiss National Bank (SNB) setting the floor rate after the Franc appreciated by a quarter of its value within a few months as the debt crisis caused money to flee from the Euro zone. Since then the SNB has systematically prevented the Euro from falling below the 1.20 level, buying Euros and selling Francs, resulting in a massive increase of FX reserves, which as of November 2013 stand at approximately 434 billion Francs\textsuperscript{11,12}.

Figure 5 shows the time evolution of the tick-by-tick EUR/CHF exchange rate and minute-by-minute Liquidity $\mathcal{L}$ in the months of the SNB intervention. Our measurement shows slow but steady deterioration of liquidity conditions during the time of Franc appreciation. In fact, as the graph demonstrates the Liquidity $\mathcal{L}$ measurement indicate a complete loss of liquidity during the week proceeding spectacular near 1000pip (approximately 10%) gain in Franc against Euro, reaching near parity on August 9th 2011. Our measurement shows that illiquid market conditions continue in the next weeks following a almost 20% reversal, with the liquidity conditions finally being restored after the SNB intervention on September 6th 2011 (Schmidt 2011).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Time evolution of (left axis) tick-by-tick EUR/CHF exchange rate and (right axis) the corresponding one minute Liquidity measurement $\mathcal{L}$ in the months of Swiss National Bank intervention, setting the floor of 1.20 on the EUR/CHF exchange rate.}
\end{figure}

\textsuperscript{11}Source: SNB (http://www.snb.ch/ext/stats/imfsdds/pdf/deenfr/IMF.pdf)
\textsuperscript{12}“Swiss to Join Euro: 73% of Every Swiss Yearly Income Invested in Euro via SNB”, Dorgan, G., 2012 SNBCHF.com
9 Conclusion

In this article we use an event based framework to map continuous financial data into the so-called intrinsic network. We define a method for state contraction of intrinsic networks, and show that the network has a consistent hierarchical structure, allowing multi-scale analysis of financial data. We define the concept of Liquidity that characterises the unlikeliness of price trajectories. Focusing on well know currency crisis, we argue and show that the new metric has the ability to detect and predict stress in financial markets.

Acknowledgements

The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement no 317534 (the Sophocles project).
A Intrinsic time framework

Formally, we map the time series of prices into sequences of directional changes and price overshoots as follows. Let $\Delta = \{\delta_0, \ldots, \delta_{n_\delta-1}\}$ be the set of $n_\delta$ directional change thresholds onto which time series is mapped. The initial condition of $x_0$ the initial price; $t_0$, the initial physical time; and $m_0$ the mode that switches between up and down indicating in which direction the directional change is expected. An initial condition affects at most the first two pairs (directional change, overshoot), and let the subsequent pairs in the sequence to synchronise with any other sequence obtained with a different initialization the sequence obtained with a different initialisation.

A given $\delta_i$ discretises the time series into a set of prices $X_i(t) = \{x_0^i(t_0), \ldots, x_{n_i-1}^i(t_{n_i-1}), x(t)\}$ occurring at times $T_i(t) = \{t_0^i, \ldots, t_{n_i-1}^i, t\}$ where $x(t) = (\text{bid}(t) + \text{ask}(t))/2$ is the midprice at time $t$. We highlight that the last elements of the set $(x(t), t)$ are temporary, as they do not correspond to a turning point yet but represent the state of the process at time $t$. We compute the number of turning points (i.e., the occurrence of a directional change) as $n_t^i = \lfloor \frac{n_i}{T} \rfloor$. The series of amplitude of directional changes $\Delta_i$ is defined as

$$\Delta_i(t) = \{\delta_0^i, \ldots, \delta_{n_i^e-1}^i, \delta_{n_i^e}^i(t)\} = \{x_{2j+1}^i - x_{2j}^i\}$$

(26)

where $0 \leq j \leq n_i^e$. The discreteness of the time series of prices prevents $|\delta_i^e| = \delta_i$. The discrepancy is, however, small and is on average within the spread. The series of amplitudes of overshoots $\Omega_i$ is written as

$$\Omega_i(t) = \{\omega_0^i, \ldots, \omega_{n_i^e-1}^i, \omega_{n_i^e}^i(t)\} = \{x_{2j+1}^i - x_{2j}^i\}.$$  

(27)

Algorithm 1. Dissect the price curve from time $t_0$ and measure overshoots with a $\delta_i$ price threshold

Require: initialise variables ($x^{ext} = x(t_0)$, mode is arbitrarily set to up, $X_i = x_0$, $T_i = t_0$)

1: update latest $X_i$ with $x(t)$
2: update latest $T_i$ with $t$
3: if mode is down then
4: if $x(t) > x^{ext}$ then
5: $x^{ext} \leftarrow x(t)$
6: else if $x(t) - x^{ext} \leq -\delta_i$ then
7: $x^{ext} \leftarrow x(t)$
8: mode $\leftarrow$ up
9: $X_i \leftarrow x(t)$
10: $T_i \leftarrow t$
11: end if
12: else if mode is up then
13: if $x(t) < x^{ext}$ then
14: $x^{ext} \leftarrow x(t)$
15: else if $x(t) - x^{ext} \geq \delta_i$ then
16: $x^{ext} \leftarrow x(t)$
17: mode $\leftarrow$ down
18: $X_i \leftarrow x(t)$

18
The analytical Gaussian benchmark

In the special case where the price follows a Brownian motion, the transition matrix can be derived analytically. Since the hierarchical nature of the intrinsic network allows deducing the transition matrix for any number of thresholds by a contracting process, the problem actually boils down to solving the two-thresholds case, which we will do now. In case the transitions on the Intrinsic Network are modelled as first order Markov Chain, the matrix has the following form

$$W = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 - \alpha & 0 & 0 & \alpha \\
\beta & 0 & 0 & 1 - \beta \\
0 & 0 & 1 & 0
\end{pmatrix}$$

with the convention that the states are numbered \((0, 0) = 0, (1, 0) = 1, (0, 1) = 2\) and \((1, 1) = 3\). We thus have only to calculate two probabilities which will be taken as

$$P((1, 0) \rightarrow (1, 1)) = \alpha$$ and $$P((0, 1) \rightarrow (0, 0)) = \beta$$.

For reasons of convenience and without loss of generality, we will simplify the calculations by considering the case where the thresholds are fixed in terms of absolute value instead of a percentage, which makes little difference -if any- as long the the volatility is not too hectic.

Let us now focus on the situation where the system just turned to the \((1, 0)\) state. As we can read it from \(W\), it was previously in \((0, 0)\) and just bounced back from some minimum by an amount \(\delta_1\). Two events may occur now

- either the upward move goes further by an amount \(\delta_2 - \delta_1\) and the system turns to the \((1, 1)\) state,
- either after having reached some maximum \(M < \delta_2 - \delta_1\) the walk goes downward by an amount \(\delta_1\) and the system turns back to \((0, 0)\).

The question is therefore to determine the probability of each of these two scenarios to occur. This is somewhat reminiscent of the famous gambler’s ruin, but the situation is more involved here due to the presence of two absorbing barriers, one of which is moving in time.

Let us denote \(A \equiv x_0 + \Delta\) the upper fixed barrier and \(B \equiv M - \delta\) the moving lower one (for the sake of notation we put \(\Delta \equiv \delta_2 - \delta_1\) and \(\delta \equiv \delta_1\)). We now dissect the interval \((x_0, A)\) in small intervals \((x_0, x_0 + \epsilon), (x_0 + \epsilon, x_0 + 2\epsilon), \ldots, (A - \epsilon, A)\) with \(\epsilon \equiv \Delta/n\) for some \(n\). In order for the walk to reach \(A\) before \(B\), it has, as a very first step, to reach \(x_0 + \epsilon\) before \(x_0 - \delta\) and then to reach \(x_0 + 2\epsilon\) before \(x_0 + \epsilon - \delta\), and so on. We are thus led to rewrite the probability (let us denote it \(P(A\backslash B)\)) to reach the fixed threshold \(A\) before the moving one \(B\) as

$$P(A\backslash B) = \prod_{k=1}^{\Delta/\epsilon} P(x_0 + k\epsilon\backslash x_0 + (k - 1)\epsilon - \delta\backslash x_0 + (k - 1)\epsilon\backslash x_0 + (k - 2)\epsilon - \delta)$$

But then invariance properties (Markovianity and translation invariance) of Brownian motion allow us to simplify this expression as

$$P(A\backslash B) = (P(x_0 + \epsilon\backslash x_0 - \delta))^{\Delta/\epsilon}$$
and it remains to take the limit \( \epsilon \to 0 \).

Let us now simplify a bit further the notation and assume we have a Brownian motion with mean \( \mu \) and variance \( \sigma^2 \) starting at a position \( x_0 \) somewhere between two absorbing barriers \( U \) (upper) and \( L \) (lower). The probability density of finding the walk at position \( x \) at time \( t \) will obey the backward diffusion equation

\[
\partial_t p(x_0, x, t) = \mu \partial_{x_0} p(x_0, x, t) + \frac{\sigma^2}{2} \partial_{xx} p(x_0, x, t)
\]  

(31)

with boundary conditions \( p(x_0, x, 0) = \delta(x_0) \) and \( p(x_0, U, t) = p(x_0, L, t) = 0 \). The best way to proceed is now to define

\[
g(x_0, t) \equiv -\partial_t \int_U^L p(x_0, x, t) dx,
\]  

(32)

which denotes the probability to be absorbed around time \( t \) by any of the barriers, and then take the Laplace transform

\[
G(x_0, s) \equiv \int_0^\infty e^{-st} g(x_0, t) dt
\]  

(33)

which has the very interesting property that evaluating it at \( s = 0 \) yields exactly the probability to be caught by any of the barriers. Some standard manipulations allow us to transfer the backward equation to the Laplace domain so as to obtain

\[
sG(x_0, s) = \mu \partial_{x_0} G(x_0, s) + \frac{\sigma^2}{2} \partial^2_{xx} G(x_0, s)
\]

(34)

with boundary conditions \( G(U, s) = G(L, s) = 1 \) (which means nothing but immediate absorption if the walk starts on either barrier).

We then split the total probability of absorption as \( g_-(x_0, t) + g_+(x_0, t) \), where \( g_\pm(x_0, t) \) denotes the probability of absorption by the upper, respectively lower, barrier. The transform is split accordingly as \( G_-(x_0, s) + G_+(x_0, s) \) with boundary conditions \( G_+(U, s) = G_-(L, s) = 1 \) and \( G_+(L, s) = G_-(U, s) = 0 \). Equation (34) can thus be solved separately for \( G_+ \) and \( G_- \). We use the standard ansatz \( G_\pm = \exp(\theta x_0) \) which boils down the differential equation to a quadratic algebraic equation for \( \theta \) easily solved to yield

\[
\theta_{1,2} = -\mu \mp \sqrt{\mu^2 + 2s\sigma^2}
\]  

(35)

(34) is then solved by

\[
G_\pm(x_0, s) = K_1 e^{\theta_1 x_0} + K_2 e^{\theta_2 x_0}
\]

(36)

for constants chosen to match the boundary conditions. We skip the details to quote the expression for \( G_+ \), taking according to our previous notations \( U = x_0 + \epsilon \) and \( L = x_0 - \delta \), and putting \( s = 0 \),

\[
G_+(x_0, 0) = \frac{1 - \exp \left( -\frac{2\delta|\mu|}{\sigma^2} \right)}{1 - \exp \left( -\frac{2(\delta + \epsilon)|\mu|}{\sigma^2} \right)} \exp \left( \frac{\epsilon(\mu - |\mu|)}{\sigma^2} \right)
\]  

(37)

This quantity is therefore the probability to get caught by the upper barrier without having ever met the lower one, which is \( P(x_0 + \epsilon \setminus x_0 - \delta) \) we introduced at the beginning. It thus remains to calculate

\[
\lim_{\epsilon \to 0} G_+(x_0, 0)^{\Delta/\epsilon}
\]  

(38)

which is easily found to be

\[
P(A \setminus B) = \exp \left( -\frac{\Delta}{\sigma^2} \cdot \frac{(|\mu| - \mu) + (|\mu| + \mu) \exp \left( -\frac{2\delta|\mu|}{\sigma^2} \right)}{1 - \exp \left( -\frac{2\delta|\mu|}{\sigma^2} \right)} \right)
\]  

(39)
This expression happens to simplify in the driftless case to the harmless formula

\[
\mathbb{P}(A\backslash B) = \exp \left( -\frac{A}{\delta} \right). \quad (40)
\]

This is the expression we were searching for the probability of transitioning from \((1, 0)\) to \((1, 1)\). The very same reasoning applies using \(G_-\) for the transition from \((0, 1)\) to \((0, 0)\), while other transitions are now trivial. \(W\) for a two-thresholds system can therefore be written as

\[
W = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 - \exp \left( -\frac{\delta_2 - \delta_1}{\delta_1} \right) & 0 & 0 & \exp \left( -\frac{\delta_2 - \delta_1}{\delta_1} \right) \\
\exp \left( -\frac{\delta_2 - \delta_1}{\delta_1} \right) & 0 & 0 & 1 - \exp \left( -\frac{\delta_2 - \delta_1}{\delta_1} \right) \\
0 & 0 & 1 & 0
\end{pmatrix} \quad (41)
\]

Obviously in that case the ratio of the thresholds only matters, and not the thresholds themselves.

The derivation established that the probability of overshoot \(\omega(\delta_1; \sigma)\) reaching the length \(\delta_2 - \delta_1 = \exp \left( -\frac{\delta_2 - \delta_1}{\delta_1} \right)\), i.e.

\[
\mathbb{P}(\omega(\delta_1; \sigma) \geq \delta_2 - \delta_1) = \exp \left( -\frac{\delta_2 - \delta_1}{\delta_1} \right) \quad (42)
\]

hence we conclude that the overshoot lengths are exponentially distributed. From there it straightforwardly follows that the average length of the overshoot equals the directional change threshold

\[
\mathbb{E}[\omega(\delta; \sigma)] = \delta \quad (43)
\]
thus proving the fundamental intrinsic theorem 2.1.

### C Contraction of probabilities

We demonstrate that assuming the transitions on the Intrinsic Network are modelled as a first order Markov Chain, there is an explicit connection between the transition matrix \(W\) of the \(n\)-dimensional Intrinsic Network and transition matrix \(\tilde{W}\) of the \(n - 1\) dimensional Intrinsic network obtained through the contraction process.

Let us assume that the current state of the network is \(I_k^{(i)}\) and we are interested in finding the probability of transitioning to state \(I_j^{(i)}\). Note that Island \(I_k^{(i)}\) consists of states \(\{I_{2k}^{(i-1)}, I_{2k+1}^{(i)}\}\), while Island \(I_j^{(i)}\) consists of state \(\{I_{2j}^{(i-1)}, I_{2j+1}^{(i)}\}\). For \(\text{mod}(k + j, 2) = 0\), from \(I_{2k+1}^{(i-1)}\) the system can directly transition to \(I_{2j+1}^{(i-1)}\) with probability \(\mathbb{P}(I_{2k}^{(i-1)} \to I_{2j+1}^{(i-1)})\). On the other hand, the system can oscillate once within Island \(I_k^{(i)}\) before proceeding to Island \(I_j^{(i)}\), i.e. transition from \(I_{2k+1}^{(i-1)}\) to \(I_{2k}^{(i-1)}\) and back to \(I_{2k+1}^{(i-1)}\), before transitioning to \(I_{2j+1}^{(i-1)}\), with probability

\[
\mathbb{P}(I_{2k}^{(i-1)} \to I_{2k}^{(i-1)}) \cdot \mathbb{P}(I_{2k}^{(i-1)} \to I_{2k+1}^{(i-1)}) \cdot \mathbb{P}(I_{2k+1}^{(i-1)} \to I_{2j+1}^{(i-1)}).
\]

Likewise, oscillation within Island \(I_k^{(i)}\) can occur \(k\) times, before proceeding to Island \(I_j^{(i)}\), with probability,

\[
(\mathbb{P}(I_{2k}^{(i-1)} \to I_{2k+1}^{(i-1)}) \cdot \mathbb{P}(I_{2k+1}^{(i-1)} \to I_{2k}^{(i-1)}))^{k} \cdot \mathbb{P}(I_{2k}^{(i-1)} \to I_{2j+1}^{(i-1)}),
\]

21
hence the probability to transition from Island $I^{(i)}_k$ before to Island $I^{(i)}_j$ equals
\[
\mathbb{P}(I^{(i)}_k \to I^{(i)}_j) = \sum_{k=0}^{\infty} (\mathbb{P}(I^{(i-1)}_{2k} \to I^{(i-1)}_{2k+1}) \cdot \mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_{2k+2}))^k \cdot \mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_{2j+1})
\]
\[
= \frac{\mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_{2j+1})}{1 - \mathbb{P}(I^{(i-1)}_{2k} \to I^{(i-1)}_{2k+1}) \cdot \mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_{2k})}.
\]

For $\text{mod}(k + j, 2) = 1$ and $k > j$, the system has to make an interim transition from $I^{(i-1)}_{2k+1}$ to $I^{(i-1)}_2$, before transitioning from Island $I^{(i)}_k$ to Island $I^{(i)}_j$, with probability
\[
\mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_2) \cdot \mathbb{P}(I^{(i-1)}_2 \to I^{(i-1)}_j).
\]

Likewise, as before the system can oscillate $k$ times within Island $I^{(i)}_k$ before proceeding to Island $I^{(i)}_j$, with probability
\[
\mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_2) \cdot (\mathbb{P}(I^{(i-1)}_2 \to I^{(i-1)}_{2k+1}) \cdot \mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_2))^k \cdot \mathbb{P}(I^{(i-1)}_2 \to I^{(i-1)}_{2j})
\]

hence the probability to transition from Island $I^{(i)}_k$ before to Island $I^{(i)}_j$ equals
\[
\mathbb{P}(I^{(i)}_k \to I^{(i)}_j) = \sum_{k=0}^{\infty} (\mathbb{P}(I^{(i-1)}_{2k} \to I^{(i-1)}_{2k+1}) \cdot (\mathbb{P}(I^{(i-1)}_2 \to I^{(i-1)}_{2k+1}) \cdot \mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_2))^k \cdot \mathbb{P}(I^{(i-1)}_2 \to I^{(i-1)}_{2j})
\]
\[
= \frac{\mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_2) \cdot \mathbb{P}(I^{(i-1)}_2 \to I^{(i-1)}_{2j})}{1 - \mathbb{P}(I^{(i-1)}_{2k} \to I^{(i-1)}_{2k+1}) \cdot \mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_2)}.
\]

Similarly, it can be shown that for $\text{mod}(k + j, 2) = 1$ and $k < j$, it can be shown
\[
\mathbb{P}(I^{(i)}_k \to I^{(i)}_j) = \sum_{k=0}^{\infty} \mathbb{P}(I^{(i-1)}_{2k} \to I^{(i-1)}_{2k+1}) \cdot (\mathbb{P}(I^{(i-1)}_2 \to I^{(i-1)}_{2k+1}) \cdot \mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_2))^k \cdot \mathbb{P}(I^{(i-1)}_2 \to I^{(i-1)}_{2j+1})
\]
\[
= \frac{\mathbb{P}(I^{(i-1)}_{2k} \to I^{(i-1)}_{2k+1}) \cdot \mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_{2j+1})}{1 - \mathbb{P}(I^{(i-1)}_{2k} \to I^{(i-1)}_{2k+1}) \cdot \mathbb{P}(I^{(i-1)}_{2k+1} \to I^{(i-1)}_2)}.
\]

### D Transition probability derivation

We will demonstrate the derivation of the analytic expressions of transition probabilities presented in Section 5. Firstly, we prove the claim holds for 3-dimensional intrinsic network, let $\delta_1 < \delta_2 < \delta_3$ denote the ordered directional change thresholds, $\mathcal{IN}(3; \{\delta_1, \delta_2, \delta_3\}; W)$ 3-dimensional intrinsic network and $\mathcal{IN}(2; \{\delta_2, \delta_3\}; \tilde{W})$ the contracted intrinsic network. Since the contracted intrinsic network is 2-dimensional, it is known that
\[
\mathbb{P}((1, 0) \to (1, 1)) = e^{-\frac{\delta_3 - \delta_2}{\delta_2}},
\]
while the explicit analytic expression of transition probabilities between 3- and contracted 2-dimensional intrinsic network presented in Section 4 states that
\[
e^{-\frac{\delta_3 - \delta_2}{\delta_2}} = \frac{\mathbb{P}((1, 1, 0) \to (1, 1, 1))}{1 - (1 - \mathbb{P}((1, 1, 0) \to (1, 1, 1))) \cdot \left(1 - e^{-\frac{\delta_3 - \delta_1}{\delta_1}}\right)}
\]

22
and untangling the formula we find
\[
\mathbb{P}((1, 1, 0) \to (1, 1, 1)) = \frac{e^{-\frac{\delta_3 - \delta_2}{\gamma_2}} e^{-\frac{\delta_2 - \delta_1}{\gamma_1}}}{1 - e^{-\frac{\delta_3 - \delta_2}{\gamma_2}} \left(1 - e^{-\frac{\delta_2 - \delta_1}{\gamma_1}}\right)}
\]
yielding the desired expression. Let us assume that the claim holds for \(n\)-dimensional intrinsic network, and we will prove that the claim holds for \(n+1\). Let \(\delta_1 < \cdots < \delta_{n+1}\) denote the ordered directional change thresholds, \(\mathcal{L}\mathcal{N}(n+1; \{\delta_1, \ldots, \delta_{n+1}\}; \mathcal{W})\) \(n+1\)-dimensional intrinsic network and \(\mathcal{L}\mathcal{N}(n; \{\delta_2, \ldots, \delta_{n+1}\}; \tilde{\mathcal{W}})\) the contracted intrinsic network. Since the contracted intrinsic network is \(n\)-dimensional, together with explicit analytic expression of transition probabilities between \(n+1\)- and contracted \(n\)-dimensional intrinsic network presented in Section 4 states that
\[
\frac{\prod_{k=3}^{n+1} e^{-\frac{\delta_k - \delta_{k-1}}{\gamma_k}}}{1 - \sum_{k=3}^{n} \left(1 - e^{-\frac{\delta_k - \delta_{k-1}}{\gamma_k}}\right)} = \frac{\mathbb{P}((1, \ldots, 1, 0) \to (1, \ldots, 1, 1))}{1 - \left(1 - \mathbb{P}((1, \ldots, 1, 0) \to (1, \ldots, 1, 1))\right) \left(1 - e^{-\frac{\delta_2 - \delta_1}{\gamma_1}}\right)}
\]
we find
\[
\mathbb{P}((1, \ldots, 1, 0) \to (1, \ldots, 1, 1)) = \frac{\prod_{k=3}^{n+1} e^{-\frac{\delta_k - \delta_{k-1}}{\gamma_k}}}{1 - \sum_{k=3}^{n} \left(1 - e^{-\frac{\delta_k - \delta_{k-1}}{\gamma_k}}\right)} \prod_{j=k+1}^{n+1} e^{-\frac{\delta_j - \delta_{j-1}}{\gamma_j}} - \left(1 - e^{-\frac{\delta_2 - \delta_1}{\gamma_1}}\right) \cdot \prod_{k=3}^{n+1} e^{-\frac{\delta_k - \delta_{k-1}}{\gamma_k}}
\]
obtaining the desired expression.

## E Convergence theorems

In this section we present Shannon-McMillan-Brieman theorem on convergence of sample entropy and the corresponding central limit theorem, in a more general setting following (Pfister et al. 2001); let \(\{X_t\}_{t \geq 1}\) denote an ergodic finite state process which gives rise to the conditional probability sequence \(\{A_t\}_{t \geq 1}\) where \(A_t = \mathbb{P}(X_t|X_{t-1}^{t-1})\) and \(H(X)\) denotes the entropy of the process, while \(\hat{H}_n(X)\) denotes the sample entropy rate, i.e.
\[
\hat{H}_n(X) = -\frac{1}{n} \mathbb{E}(X_1^n).
\]

**Theorem E.1.** *(Shannon-McMillan-Breiman theorem)*
\[
\hat{H}_n(X) = -\frac{1}{n} \sum_{t=1}^{n} \log \mathbb{P}(X_t|X_{t-1}^{t-1}) \to H(X) \text{ (a.s.)}
\]

**Theorem E.2.** *(Central limit theorem)* If
\[
\lim_{t \to \infty} \mathbb{E}[(\log A_t)^{2+\varepsilon}] < \infty,
\]
then the sample entropy rate obeys a central limit theorem of the form
\[
\sqrt{n}[\hat{H}_n(X) - H(X)] \to N(0, \sigma^2).
\]
The variance, \( \sigma^2 \), of the estimate is given by

\[
\sigma^2 = R(0) + 2 \sum_{\tau=1}^{\infty} R(\tau) \tag{56}
\]

where \( R(\tau) = \lim_{t \to \infty} \mathbb{E}[-\log A_t - H(X)(-\log(A_{t-\tau}) - H(X))] \). If we also have that

\[
\lim_{t \to \infty} \mathbb{E}[-\log A_t]^{4+\varepsilon} < \infty
\]

then we can estimate the variance using finite truncations of (22) with \( R(\tau) \) set to the sample autocorrelation

\[
\hat{R}_n(\tau) = \frac{1}{n-\tau} \sum_{t=\tau+1}^{n} (-\log A_t - \hat{H}_n(X))(-\log A_{t-\tau} - \hat{H}_n(X)). \tag{57}
\]
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