THE CONSTANT IN THE FUNCTIONAL EQUATION AND DERIVED EXTERIOR POWERS

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§0. Introduction

Let $X$ be a regular scheme, projective and flat over Spec $\mathbb{Z}$, of dimension $d$. Let $f : X \to \text{Spec } \mathbb{Z}$. The zeta-function $\zeta(X, s)$ of $X$ is defined to be $\prod_x 1/(1 - (Nx)^{-s})$, where the product runs over the closed points $x$ of $X$, and $N(x)$ is the order of the residue field $\kappa(x)$. This product is known to converge for $\text{Re}(s) > d$. It is conjectured that $\zeta(X, s)$ can be extended to a function meromorphic in the plane.

It is further conjectured [B1] that there is a product of Gamma-functions $\Gamma(X, s)$ and a positive rational number $A$ associated with $X$ such that if $\phi(X, s) = \zeta(X, s)\Gamma(X, s)a^{-s/2}$, then $\phi(X, s) = \pm \phi(X, d - s)$.

Let $\Omega$ be the sheaf of Kahler differentials $\Omega_{X/\mathbb{Z}}$ on $X$. We are going to give a series of conjectures relating $A$ to the étale cohomology of the derived exterior powers of $\Omega$, and a proof of these conjectures for $d = 1$ and $d = 2$.

We begin by reviewing the notion of derived exterior power. If $\mathcal{C}$ is any abelian category, there is an isomorphism of categories $N$ from the category of simplicial objects $\mathcal{S}(\mathcal{C})$ of $\mathcal{C}$ to the category $\mathcal{CH}(\mathcal{C})$ of chain complexes of objects of $\mathcal{C}$ whose degree is bounded below by 0, and an explicit inverse isomorphism $K$. Now let $\mathcal{C}$ be the category of coherent sheaves on $X$. Let $\Lambda^k$ be the usual $k$-th exterior power on coherent sheaves.

Let $E$ be any coherent sheaf on $X$. Let $P$ be a finite resolution of $E$ by locally free sheaves, and let the $k$-th derived exterior power $\check{\Lambda}^k(E)$ in the derived category of coherent sheaves.
sheaves on $X$ be $N\Lambda^k KP$.

This is independent (in the derived category) of the choice of resolution. (See the Appendix, Theorem A.1).

It is easy to see that there is a map $\rho^k$ in the derived category from $\tilde{\lambda}^k E$ to $\Lambda^k E$. Since $X$ is regular and projective over Spec $\mathbb{Z}$, we can embed $X$ as a locally complete intersection $i : X \to P$ in some projective space $P$ over $\mathbb{Z}$. If we let $I$ be the sheaf of ideals defining $X$, we have the exact sequence $0 \to I/I^2 \to i^*\Omega_P/\mathbb{Z} \to \Omega_{X/\mathbb{Z}} \to 0$, where $\Omega$ denotes as usual the Kahler differentials.

The sheaf $I/I^2$ is locally free of rank $m$, say, and the sheaf $i^*\Omega_P/\mathbb{Z}$ is locally free of rank $n$, with $n - m = d - 1$. The canonical class $\omega = \omega_{X/\mathbb{Z}}$ may be defined as $\text{Hom}(\Lambda^m I/I^2, \Lambda^n i^*\Omega_P/\mathbb{Z})$, from which we obtain a map from $\Lambda^{d-1}\Omega_{X/\mathbb{Z}}$ to $\omega$.

Composing this with $\rho^{d-1}$, we obtain a natural map $\psi$ from $\tilde{\lambda}^{d-1}\Omega$ to $\omega$, so we can compose the derived tensor product map from $\tilde{\lambda}^r \Omega \otimes_L \tilde{\lambda}^{d-1-r}\Omega$ to $\tilde{\lambda}^{d-1}\Omega$ with $\psi$ to obtain a map from $\tilde{\lambda}^r \Omega \otimes_L \tilde{\lambda}^{d-1-r}\Omega$ to $\omega$, and hence by adjointness, a map $\phi_{r,d}$ from $\tilde{\lambda}^r \Omega$ to $R\text{Hom}(\tilde{\lambda}^{d-1-r}\Omega, \omega)$. By taking the cone of $\phi_{r,d}$ we obtain an object $C_{r,d}$ of the derived category, defined up to a non-canonical isomorphism.

By Serre duality, $\phi_{r,d}$ is an isomorphism at smooth points of $X$, so the étale cohomology groups of $C_{r,d}$ have support in the bad fibres of $f$, hence are finite. So we may define the Euler characteristic $\chi_{r,d}$ of $C_{r,d}$ to be the alternating product of the orders of those étale cohomology groups.

In [L], I made a very general conjecture giving a formula for the value of the leading term of the Laurent series expansion of $\zeta(X,s)$ at a rational integer $r$. Roughly speaking, the compatibility of this conjecture with the functional equation would result from knowing the following conjecture.

**Conjecture 0.1.** : $\chi_{r,d}$ is equal to $A$ if $r$ is odd and equal to $A^{1}$ if $r$ is even.

In this paper we will give a proof of this result if $d$ is equal to 1 or 2. The proof for $d = 2$ relies heavily on deep results of Spencer Bloch [B]. I thank him for many extremely helpful discussions.

§1. Derived Tensor Product
Now assume that $X$ is an arithmetic surface, i. e. $d = 2$.

**Theorem 1.1.** The Grothendieck group $K(X_{\text{bad}})$ of the category of coherent sheaves on $X$ with support contained in the bad fibers of $f$ is generated by the sheaves $(i_P)_*\kappa(P)$ and $(i_Y)_*O_Y$, where $P$ is a closed point on a bad fiber, $\kappa(P)$ is the residue field at $P$, and $Y$ is an irreducible component of a bad fiber.

There are surjective maps $r_Y$ from $K(X_{\text{bad}})$ to $\mathbb{Z}$ which take the class of a sheaf $E$ to the length of $E$ where $Q$ is the generic point of $Y$.

**Lemma 1.2.** If an element $F$ of $K(X_{\text{bad}})$ is in the kernel $\text{Ker}(X)$ of all the maps $r_Y$, then $F$ is in the subgroup of $K(X_{\text{bad}})$ generated by the residue fields.

**Theorem 1.3.** (computation of Euler characteristics of derived tensor products) If $P_1$ and $P_2$ are closed points on $X$ the Euler characteristic of $\kappa(P_1) \otimes_L \kappa(P_2)$ is equal to 1, whether or not $P_1 = P_2$. The Euler characteristic of $O_Y \otimes_L \kappa(P)$ is also equal to 1, whether or not $P$ is on $Y$. The Euler characteristic of $O_{Y_1} \otimes_L O_{Y_2}$ is the intersection number $(Y_1, Y_2)$, whether or not $Y_1 = Y_2$.

Proof If $P \neq Q$ the support of $\kappa(P) \otimes_L \kappa(Q)$ is contained in $P \cap Q = \phi$. To compute $\kappa(P) \otimes_L \kappa(P)$ take the Koszul resolution $0 \to A \to A^2 \to A \to 0$ of $\kappa(P)$ and tensor it with $\kappa(P)$ to obtain $0 \to \kappa(P) \to \kappa(P)^2 \to \kappa(P) \to 0$ which clearly has Euler characteristic equal to 1.

**Corollary 1.4.** If $F_1$ and $F_2$ are in $\text{Ker}(X)$, $\chi(F_1 \otimes_L F_2) = 1$.

§2. Derived Exterior Powers.

We now give the basic lemma ([H], Chapter II, Exercise 5.16) about derived exterior powers:

**Lemma 2.1.** Let $0 \to E_1 \to E_2 \to E_3 \to 0$ be an exact sequence of coherent locally free sheaves on $X$. Let $r \geq 1$ be an integer. Then there is a filtration on $\Lambda^r E_2$:

$$0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_{r-1} \subseteq G_r = \Lambda^r E_2$$
and exact sequences \(0 \to G_{i-1} \to G_i \to \Lambda^{r-i}E_1 \otimes \Lambda^iE_3 \to 0\). The filtration and the associated maps are functorial in exact sequences.

If \(E\) is a coherent sheaf on \(X\), let \([E]\) denote the class of \(E\) in the Grothendieck group \(K(X)\) of \(X\).

**Corollary 2.2.** Let \(0 \to E_1 \to E_2 \to E_3 \to 0\) be an exact sequence of coherent sheaves on \(X\). Then \(\lambda^r(E_2) = \sum_{i=0}^r [\lambda^{r-i}(E_1) \otimes_L \lambda^i(E_3)]\) in the Grothendieck group \(K(X)\). If the cohomology groups of all the \(E_i\) are finite, then \(\chi(\tilde{\lambda}^r(E_2)) = \prod_0^r \chi(\tilde{\lambda}^{r-i}E_1)\chi(\tilde{\lambda}^iE_3)\)

**Proof.** Choose compatible coherent locally free finite resolutions \(P_1, P_2,\) and \(P_3\) of \(E_1, E_2,\) and \(E_3\). Use these resolutions to compute \(\tilde{\lambda}^r\) of \(E_1, E_2,\) and \(E_3\) then there exists a filtration on \(\Lambda^rKP_2\)

\[0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_{r-1} \subseteq G_r = \Lambda^rKP_2\]

and exact sequences \(0 \to G_{i-1} \to G_i \to \Lambda^{r-i}KP_1 \otimes_L \Lambda^iKP_3 \to 0\).

Recall that if \(F^\cdot\) and \(G^\cdot\) are simplicial sheaves, the simplicial tensor product of \(F^\cdot\) and \(G^\cdot\) is given by \((F^\cdot \otimes s G^\cdot)^n = F^n \otimes G^n\).

We now apply the inverse functor \(N\) to our filtration, getting

\[0 = NG_0 \subseteq NG_1 \subseteq \cdots \subseteq NG_{r-1} \subseteq NG_r = N\Lambda^rKP_2 = \tilde{\lambda}^rE_2\]

and exact sequences \(0 \to NG_{i-1} \to NG_i \to N(\Lambda^{r-i}KP_1 \otimes s \Lambda^iKP_3) \to 0\). The corollary then follows from the fact ([M], p. 129ff. May proves this for simplicial abelian groups, but the argument is valid for any abelian category with tensor products) that if \(F^\cdot\) and \(G^\cdot\) are simplicial sheaves, \(N(F^\cdot \otimes s G^\cdot)\) is isomorphic in the derived category to \(NF^\cdot \otimes_L NG^\cdot\).

**Definition 2.3.** Let \(\lambda^k\) be the usual \(k\)-th \(\lambda\) operation on \(K(X)\). Recall that \(\lambda\) is determined by the relations that \(\lambda^k([E]) = [\Lambda^k(E)]\) if \(E\) is locally free and if \(F_2 = F_1 + F_3\) in \(K(X)\), then \(\lambda^r(F_2) = \sum_{i=0}^r \lambda^{r-i}(F_1)\lambda^i(F_3)\).
**Theorem 2.4.** If $E$ is a coherent sheaf on $X$, $[\tilde{\lambda}^r(E)] = \lambda^r([E])$.

Proof. This is an easy double induction on $r$ and the length of a locally free resolution of $E$, using Definition 2.3 and Corollary 2.2.

**Theorem 2.5.** $\chi(\lambda^2(m)) = \chi(k)$, and $\chi(\lambda^2\kappa(P)) = (\chi(\kappa(P)))^{-2}$.

Proof. Let $P$ be a closed point of $X$, and let $A = O_{X,P}$ be the local ring of $P$ on $X$. Let $m$ be the maximal ideal of $A$, and $k$ the residue field of $A$. Since $A$ is regular, we have the Koszul resolution of $k$:

$$0 \to m \to A \to A^2 \to A \to k \to 0.$$

We start with the exact sequence $0 \to m \to A \to k \to 0$.

Since $\lambda^2(A) = 0$ Corollary 2.2 tells us that $\lambda^2(k) + \lambda^2(m) + m \otimes_L k = 0$. We have $\chi(m \otimes_L k) = \chi(A \otimes k) / \chi(k \otimes k)$ which implies because of Theorem 1.3 that $\chi(m \otimes_L k) = \chi(A \otimes_L k) = \chi(k)$, so $\chi(\lambda^2 k) = (\chi(\lambda^2(m)) \chi(k))^{-1}$.

From the exact sequence $0 \to A \to A^2 \to m \to 0$ and Corollary 2.2 we get the triangle in the derived category $0 \to m \to A \to \lambda^2(m) \to 0$, which implies that $\lambda^2(m) = k$, and hence that $\chi(\lambda^2(k)) = \chi(k)^{-2}$.

**Corollary 2.6.** If $F$ is in $\text{Ker}(X)$, then $\chi(\lambda^2(F)) = \chi(F)^{-2}$.

Proof. If we have the exact sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ with $F_i$ finite then $\chi(\lambda^2(F_2)) = \chi(\lambda^2(F_1)) \chi(\lambda^2(F_3))$. This is an immediate consequence by induction of Corollary 2.2 and Corollary 1.4. The Corollary then follows immediately from Lemma 1.2.

§ 3. The conjecture for $d = 1$.

If $d = 1$, $X = \text{Spec} \ O_F$, with $O_F$ the ring of integers in the number field $F$. The functional equation is well-known and $A$ is equal to $|d_F|$, where $d_F$ is the discriminant of $F$. If $r = 0$ our complex $C_{0,1}$ is $O_F \to D^{-1}$, namely the inclusion of the ring of integers in the inverse different. This complex has $H^0 = 0$ and $H^1$ isomorphic to $\Omega$, which has order equal to $|d_F|$. So $\chi(C_{0,1}) = A^{-1}$.

If $r = 1$, the complex $C_{1,1}$ is $\Omega \to 0$, so $\chi(C_{1,1}) = A$.

Now let $r \geq 2$. Applying Corollary 2.2 to the sequence $0 \to D \to A \to \Omega$ gives $0$, and calling that both $\Lambda^r A$ and $\Lambda^r D$ are $0$ for $r \geq 2$ yields $[\lambda^r \Omega] + [\lambda^{r-1} \Omega] = 0$, which of course implies
\( \chi(\lambda^r[\Omega])\chi(\lambda^{r-1}[\Omega]) = 1. \)

If \( r < 0 \), the complex \( C_{r,1} \) becomes \( 0 \to \text{RHom}(\tilde{\lambda}^{-r}, \omega) \) (since \( \tilde{\lambda}^r = 0 \) if \( r < 0 \)) and then the result follows by Serre duality from the cases where \( r \geq 1 \).

\( \S \) 4 The conjecture for \( d = 2 \)

From now on in this paper we will assume \( d = 2 \).

**Proposition 4.1.** Conjecture 0.1 is true for \( r = 0 \) and \( r = 1 \).

Proof. If \( r = 1 \), the complex \( C_{1,2} \) is \( \Omega \to \omega \), which is the complex called \( C \) in \([B]\). Bloch proves in \([B]\) that \( \chi(C) = A \)

If \( r = 0 \), \( C_{0,2} \) is \( O_X \to \text{RHom}(\Omega, \omega) \) and the conjecture follows from the conjecture for \( r = 1 \) and Serre duality.

If \( P \) is a closed point of \( X \) let \( B = O_{X,P}, m \) be the maximal ideal of \( B \), and \( k = \kappa(P) \) be the residue field of \( B \).

**Lemma 4.2.** Let \( r \geq 2 \). Then \( \lambda^r[(m)] = (-1)^r[k] \) in \( K(X) \), hence \( \chi([\lambda^r(m)]) = (\chi([k]))^{(-1)^r}. \)

Proof. Applying Corollary 2.2 to the exact sequence \( 0 \to B \to B^2 \to m \to 0 \), \( r \geq 3 \) implies that \( [\lambda^r(m)] + [\lambda^{r-1}(m)] = 0 \), so \( \chi([\lambda^r(m)])\chi([\lambda^{r-1}(m)]) = 1 \), and hence since Theorem 2.1 tells us that \( \chi([\lambda^2(m)]) = \chi([k]) \), Lemma 4.2 follows by induction.

**Lemma 4.3.** \( \chi([\lambda^r(k)]) = (\chi([k]))^r \) if \( r \) is odd and \( (\chi([k]))^{-r} \) if \( r \) is even.

Proof. Applying Corollary 2.2 to the exact sequence \( 0 \to m \to B \to k \to 0 \), we obtain \( \chi(\lambda^{r+1}(m))\chi(\lambda^r(k))\chi(\lambda^{r+1}(k)) = 1 \) (Note that Lemma 4.2 implies that \( \lambda^j(m) \) is in \( \text{Ker}(X) \) for \( j \geq 2 \), and hence Corollary 1.4 implies that \( \chi(\lambda^j(m) \otimes L \lambda^{r-j}(k)) = 1 \) for \( 2 \leq j \leq r-1 \.) Induction using Lemma 4.2 now completes the proof.

**Proposition 4.4.** If \( F \) is in \( \text{Ker}(X) \), then \( \chi(\lambda^r(F)) = (\chi(F))^r \) if \( r \) is odd and \( \chi(F)^{-r} \) if \( r \) is even.

Proof. The proof is the same as the proof of Corollary 2.6, starting from Lemma 4.3.

**Corollary 4.5.** \( \chi([\lambda^r([C])]) = A^r \) if \( r \) is odd and \( A^{-r} \) if \( r \) is even.
Proof. Since \( r_Y(\Omega) \) and \( r_Y(\omega) \) are both 1 for all \( Y \), \( [C] \) is in \( \text{Ker}(X) \). Then use Proposition 4.1.

**Theorem 4.6.** Conjecture 0.1 is true for \( r \geq 2 \) and \( r < 0 \).

Proof. Since \( \omega \) differs from \( B \) by something in \( \text{Ker}(X) \), \( \chi([\omega \otimes_L C]) = \chi([B \otimes_L C]) = \chi([C]) \). Then Corollary 2.2 applied to the triangle \( C \to \Omega \to \omega \to C[1] \) tells us that if \( r \geq 2 \), \( \chi([\tilde{\lambda}^r(\Omega)]) = \chi([\tilde{\lambda}^r(C)]) \chi([\tilde{\lambda}^{r-1}(C)]) \) and Conjecture 0.1 follows. The case when \( r < 0 \) follows from Serre duality.

**Appendix (Derived tensor products and derived exterior powers)**

The key to defining derived functors (both additive and non-additive) in the absence of projectives is contained in the classic paper \([BS]\) of Borel and Serre. The basic point is that if \( F \) is a coherent sheaf and \( P^\bullet \to F \) and \( Q^\bullet \to F \) are two finite resolutions of \( F \) by coherent locally free sheaves, there exists a finite resolution \( R^\bullet \to F \) by coherent locally free sheaves which dominates both \( P^\bullet \to F \) and \( Q^\bullet \to F \). (\( R^\bullet \to F \) dominates \( P^\bullet \to F \) if there is a surjective map of complexes from \( R^\bullet \to F \) to \( P^\bullet \to F \) which is the identity on \( F \).) This result follows immediately by induction from Lemma 14 of Borel-Serre.

Let \( L^\bullet \) be the kernel of the map from \( R^\bullet \to F \) to \( P^\bullet \to F \), Then \( L^\bullet \) is acyclic and locally free, so if \( G \) is any coherent sheaf \( L^\bullet \otimes G \) is acyclic. It immediately follows that the map from \( R^\bullet \otimes G \) to \( P^\bullet \otimes G \) is a quasi-isomorphism, and so \( P^\bullet \otimes G \) and \( Q^\bullet \otimes G \) are isomorphic in the derived category. So we may define the derived tensor product \( F \otimes^L G \) to be \( P^\bullet \otimes G \) and this is independent of the choice of the locally free resolution \( P^\bullet \).

We wish to define derived exterior powers in an analogous fashion. Let \( N \) be the functor which takes simplicial sheaves to bounded below complexes of sheaves, and \( K \) be its inverse functor. Let \( \Lambda^k \) denote the kth exterior power. We would like to define \( \tilde{\lambda}^k F \) to be \( N\Lambda^k K P^\bullet \), where \( P^\bullet \) is a resolution of \( F \), so we have to show that in the derived category this is independent of the choice of resolution.

**Theorem A,1.** \( N\Lambda^k K P^\bullet \) is independent (in the derived category) of the choice of locally free resolution \( P^\bullet \) of \( F \).
Lemma A.2. If $R^\bullet$ is a finite acyclic complex of locally free sheaves on a noetherian scheme $X$, $\Lambda^k K R^\bullet$ is acyclic for all $k \geq 1$.

Proof. Since exterior powers commute with restriction to open sets, we may assume $X$ is affine. But then the locally free sheaves are projective, and an acyclic projective complex is homotopically trivial. The functors $K$, and $\Lambda^k$ preserve homotopy, so the resulting complex is still homotopically trivial, so acyclic.

Proof of Theorem A.1 Again, by using Lemma 14 of [BS], we reduce to the case where we have a surjective map $f$ from $P^\bullet$ to $Q^\bullet$, where both $P^\bullet$ and $Q^\bullet$ are finite locally free resolutions of the coherent sheaf $F$. Let $R^\bullet$ be the kernel of $f$, $R^\bullet$ is clearly acyclic, so by the lemma $\Lambda^i K R^\bullet$ is acyclic, for all $i \geq 1$. We have the exact sequence of simplicial sheaves $0 \to K R^\bullet \to K P^\bullet \to K Q^\bullet \to 0$. Then $\Lambda^k K P^\bullet$ has a filtration whose associated graded pieces are $\Lambda^i K R^\bullet \otimes \Lambda^j K Q^\bullet$ for $i + j = k$. Since $\Lambda^i K R^\bullet$ is acyclic for $i \geq 1$ we get that $f$ induces a quasi-isomorphism from $\Lambda^k P^\bullet \to \Lambda^k Q^\bullet$. Since $N$ preserves quasi-isomorphisms we obtain the desired result.

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