Asymptotic behaviour of dynamical systems with plastic self-organising vector fields

N.B. Janson\textsuperscript{1} and P.E. Kloeden\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, Loughborough University, Loughborough LE11 3TU, UK
\textsuperscript{2}Mathematics Department, University of Tübingen, Tübingen 72074, Germany

Abstract

In [Janson & Marsden 2017] a dynamical system with a plastic self-organising velocity vector field was introduced, which was inspired by the architectural plasticity of the brain and proposed as a possible conceptual model of a cognitive system. Here we provide a more rigorous mathematical formulation of this problem, make several simplifying assumptions about the form of the model and of the applied stimulus, and perform its mathematical analysis. Namely, we explore the existence, uniqueness, continuity and smoothness of both the plastic velocity vector field controlling the observable behaviour of the system, and of the behaviour itself. We also analyse the existence of pullback attractors and of forward limit sets in such a non-autonomous system of a special form. Our results verify the consistency of the problem, which was only assumed in the previous work, and pave the way to constructing models with more sophisticated cognitive functions.

Keywords: non-autonomous dynamical system, plastic spontaneously evolving velocity field, pullback attractor, model of cognition

1. Introduction

It has been widely believed that replication in hardware or software of the brain’s physical architecture would automatically replicate the brain’s cognitive functions. To a certain extent, this assumption has been correct, since artificial neural networks have been highly successful in a range of applications requiring classification and pattern recognition \cite{1}. However, they are still far away from demonstrating human-level intelligence. There is a growing appreciation that to recreate the brain’s functions, one needs to reveal
and reproduce the brain’s working principles rather than the architecture per se. To achieve that, it is necessary to answer a number of fundamental questions posed by neuroscience, such as how memories are represented in the brain, how behaviour could be linked to the brain substance, and how cognitive processes could be described in rigorous terms [2, 3, 4, 5, 6, 7, 8].

Given that the neurons in the brain fire *spontaneously*, biologically relevant brain models take the form of dynamical systems with continuous time [9, 10], whose key element is the velocity vector field combining two features. Firstly, assuming that the model of a spontaneously evolving device is derived from the first principles and is accurate, its velocity field is a mathematical representation of the device’s physical architecture [11]. Secondly, this field is a mathematical expression of the force fully controlling the device’s behaviour, and is an embodiment of the full set of behavioural rules.

In [12] it was proposed that looking at the brain through the prism of its velocity vector field offers a solution to a number of fundamental questions asked by neuroscience. Firstly, the velocity field of the brain could represent the sought-after link between its physical properties and the resultant behaviour. Secondly, by hypothesising that memories could be imprints on the brain’s velocity field, one could unify several dominating memory theories. Thirdly, the principles of the brain cognitive function could amount to the ability to create a plastic self-organising velocity vector field evolving according to certain rules, which need to be of an appropriate form to enable cognition. Ultimately, it has been suggested that to be cognitive, the system does not necessarily need to be a neural network, but rather to be capable of spontaneous modifications of its velocity vector field according to some suitable rules.

Thus, a conceptual model of a cognitive system has been proposed, which represents a dynamical system with a plastic self-organising velocity vector field. The standard part of this model is given by the non-autonomous differential equation:

$$\frac{dx}{dt} = a(x, t). \quad (1)$$

Here, $x \in \mathbb{R}^d$ is the state of the cognitive system at any time $t$, which in the brain would be a collection of states of all the neurons, and $a \in \mathbb{R}^d$ is the velocity vector field governing evolution of the state. The unconventional part of the proposed model assumes that the vector field $a$ evolves with time *spontaneously* according to some pre-defined deterministic rules and is
affected by a stimulus \( \eta(t) \in \mathbb{R}^m \), \( m \leq d \), as expressed by an equation
\[
\frac{\partial a}{\partial t} = c(a, x, \eta(t), t),
\] (2)
with \( c \) taking values in \( \mathbb{R}^d \).

In [12] equation (2) was interpreted as a (degenerate) partial differential equation (PDE). For the mathematical analysis in this paper it is more appropriate to regard (2) as an ordinary differential equation (ODE) for the unknown variable \( a \) which depends on the parameter \( x \), and to write it as
\[
\frac{da}{dt} = c(a, x, \eta(t), t).
\]
For an additional clarity, (2) could be re-written as a parameterised ODE
\[
\frac{d}{dt}a(z, t) = c(a(z, t), z, \eta(t), t), \quad z \in \mathbb{R}^d,
\]
since the parameter \( z \) does not evolve according to the ODE (1).

Note, that although it has long been acknowledged that the brain can be regarded as a dynamical system [13], consideration of the brain within the classical framework of the dynamical systems theory did not fully explain its abilities for cognition and adaptation. As a possible reason for this, it has been suggested that the theory of dynamical systems has not been sufficiently developed and required extensions directly relevant to cognition [14]. In [12] the velocity vector field with an ability to self-organize has been suggested as an extension of the dynamical systems theory, which could explain adaptation of the behaviour to the environment as a spontaneous modification of the behavioural rules specified by this field, where this modification is induced by the self-organised architectural plasticity of the brain. Under the assumption that the conceptual model (1)–(2) was mathematically consistent and \( a(t) \) stayed smooth for all \( t \), a simple example of (2) was constructed, analysed numerically and shown to perform some basic cognition. One could potentially build more sophisticated examples of \( c \) leading to more advanced information-processing functions. However, before this can be done, one needs to address the consistency of (1)–(2) and the conditions on \( c \) and \( \eta(t) \) in (2), under which the solution of (1) exists and is unique, which has not been done to date and which is the purpose of the current paper.

In Section 2 we formulate a mathematical problem to be solved here. In Section 3 we establish the existence and uniqueness of solutions of the system (3)–(4). In Section 4 we show that the first equation (3) has a global non-autonomous attractor. In Section 5 we discuss the results obtained.
2. Problem statement

We can interpret (1)–(2) as a system of ODEs with an unconventional structure,
\[
\frac{dx(t)}{dt} = a(x(t), t)
\]
(3)
\[
\frac{d}{dt} a(z, t) = c(a(z, t), z, \eta(t), t), \quad z \in \mathbb{R}^d,
\]
(4)
with solutions \(x(t)\) and \(a(z, t)\) taking values in \(\mathbb{R}^d\). The solution \(a(t)\) of (4) depends on \(z \in \mathbb{R}^d\) as a fixed parameter. Note that the solution \(x(t)\) of the first equation (3) is not inserted into the second equation (4), i.e. equation (4) is decoupled from equation (3). Essentially, we need to solve the second equation (4) first, independently of equation (3), to obtain the vector field for the first equation (3).

The purpose of this paper is to provide a more precise mathematical formulation and an analysis of the modelling and numerical work in [12]. In particular, we show that the system (3)–(4) is well posed in the sense of the global existence and uniqueness of its solutions.

It is also important to consider a long-term behaviour of the newly introduced systems, which is usually described by attractors. The concept of an attractor has been successfully extended from the autonomous to the standard non-autonomous dynamical systems of the form \(\frac{dx}{dt} = f(x, t)\), where \(f\) is some fixed vector field function [15]. However, the existence of an attractor where the vector field itself evolves spontaneously according to (4) needs to be proved. We show that, under a mild dissipativity assumption, the non-autonomous system generated by (3) has a non-autonomous (or random) attractor.

3. Existence and uniqueness of solutions

In [12] it was proposed that in (1)–(2) the stimulus \(\eta(t)\) is used both to contribute to the modification of the vector field \(a\) according to (2), and to regularly reset the initial conditions of (1). Here, we consider a simplified case, in which we allow \(\eta(t)\) only to affect evolution of \(a\). Therefore, when considering system (3)–(4) we will handle equation (3) separately from (4), assuming that the vector field \(a\) is known. The existence and uniqueness of solutions of equations (3) and (4) require at least a local Lipschitz property.
of the right-hand sides $a$ and $c$ in the corresponding state variable, while the existence of an attractor in (3) requires a dissipativity property.

Since the equations (3)–(4) represent a non-autonomous or a random system, we need to consider them on the entire time axis $t \in \mathbb{R}$ (see the discussion in Section 5 for when this does not hold). In particular, the vector field $a$ should be defined for all values of time $t \in \mathbb{R}$. Below we formulate our assumption on the stimulus $\eta$.

**Assumption 1.** $\eta : \mathbb{R} \rightarrow \mathbb{R}^m$ is continuous.

This stimulus signal is considered as a given and fixed input in the model.

**3.1. Existence and uniqueness of the observable behaviour $x(t)$ of (3)**

**Assumption 2.** $a : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ and $\nabla_x a(x,t) : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ are continuous in both variables $(x,t)$.

This assumption ensures the vector field $a$ is locally Lipschitz in $x$. Hence, by standard theorems (see Walter [16, Chapter 2]), there exists a unique solution $x(t) = x(t; t_0, x_0)$ of the ODE (3) for each initial condition $x(t_0) = x_0$, at least for a short time interval.

**Assumption 3.** $a : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ satisfies the dissipativity condition $\langle a(x,t), x \rangle \leq -1$ for $\|x\| \geq R^*$ for some $R^*$.

(Here $\|a\| = \sqrt{\sum_{i=1}^d a_i^2}$ is the Euclidean norm on $\mathbb{R}^d$ and $\langle a, b \rangle = \sum_{i=1}^d a_i b_i$ is the corresponding inner product, for vectors $a, b \in \mathbb{R}^d$.)

This assumption (which may be stronger than we really need, but avoids assumptions about the specific structure of $a$) ensures that the ball $B^* := \{x \in \mathbb{R}^d : \|x\| \leq R^* + 1\}$ is positive invariant. This follows from the estimate

$$\frac{d}{dt} \|x(t)\|^2 = 2 \langle x(t), a(x(t), t) \rangle \leq -1 \quad \text{if } \|x(t)\| \geq R^*$$

and in turn ensures that the solution of the ODE (3) exists for all future time $t \geq t_0$. We thus formulate the following theorem.

**Theorem 1.** Suppose that Assumptions (1), (2) and (3) hold. Then for every initial condition $x(t_0) = x_0$, the ODE (3) has a unique solution $x(t) = x(t; t_0, x_0)$, which exists for all $t \geq t_0$. Moreover, these solutions are continuous in the initial conditions, i.e., the mapping $(t_0, x_0) \mapsto x(t; t_0, x_0)$ is continuous.
3.2. Existence and uniqueness of the vector field \( a(x, t) \) as a solution of (4)

The ODE (4) for the velocity field \( a(x, t) \) is independent of the solution \( x(t; t_0, x_0) \) of the ODE (3). We need the following assumption to provide the existence and uniqueness of \( a(x, t) \) for all future times \( t > t_0 \) and to ensure that this solution satisfies Assumptions (2) and (3).

**Assumption 4.** \( c : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^d \) and \( \nabla a c : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{d \times d} \) are continuous in all variables.

This assumption ensures the vector field \( c \) is locally Lipschitz in \( a \). Hence, by standard theorems (see Walter [16, Chapter 2]), there exists a unique solution \( a(t; t_0, a_0) \) of the ODE (4) for each initial condition \( a(t_0) = a_0 \), at least for a short time interval. This solution also depends continuously on the parameter \( x \in \mathbb{R}^d \). To ensure that the solutions can be extended for all future times \( t \), we need a growth bound such as in the following assumption.

**Assumption 5.** There exist constants \( \alpha \) and \( \beta \) (which need not be positive) such that
\[
\langle c(a(x, y, t)), x \rangle \leq \alpha \| a \|^2 + \beta \text{ for all } (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}.
\]

The next assumption ensures that the solution of the ODE (4), which we now write as \( a(x, t) \), is continuously differentiable and hence locally Lipschitz in \( x \), provided that the initial value \( a(x, t_0) = a_0(x) \) is continuously differentiable.

**Assumption 6.** \( \nabla x c : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{d \times d} \) is continuous in all variables.

The above statement then follows from the properties of the linear matrix-valued variational equation
\[
\frac{d}{dt} \nabla x a = \nabla a c \nabla x a + \nabla x c,
\]
which is obtained by taking the gradient \( \nabla x \) of both sides of the ODE (4).

Finally, we need to ensure that the solution \( a(x, t) \) satisfies the dissipativity property as in Assumption (3).

**Assumption 7.** There exist \( R^* \) such that
\[
\langle c(a(x, y, t)), x \rangle \leq 0 \text{ for } \| x \| \geq R^*, \quad (a, y, t) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}.
\]
To show this we write equation (4) in integral form
\[ a(x, t) = a_0(x) + \int_{t_0}^{t} c(a(x, s), x, \eta(s), s) \, ds \]
and then take the scalar product on both sides with a constant \( x \), which gives
\[ \langle a(x, t), x \rangle = \langle a_0(x), x \rangle + \left\langle \int_{t_0}^{t} c(a(x, s), x, \eta(s), s), x \right\rangle ds, x \]= \langle a_0(x), x \rangle + \int_{t_0}^{t} \langle c(a(x, s), x, \eta(s), s), x \rangle ds \leq -1 + 0 = -1 \quad \text{for} \quad \|x\| \geq R^*.

Summarising from the above, we can formulate the following theorem.

**Theorem 2.** Suppose that Assumptions (1) and (4)–(7) hold. Further, suppose that \( a_0(x) \) is continuously differentiable and satisfies the dissipativity condition in Assumption (3). Then the ODE (4) has a unique solution \( a(x, t) \) for the initial condition \( a(x, t_0) = a_0(x) \), which exists for all \( t \geq t_0 \) and satisfies Assumptions (2) and (3).

Thus, we have obtained a theorem for the existence, uniqueness, continuity and dissipativity of the velocity vector field \( a \) governing the behaviour of (3).

### 4. Asymptotic behaviour

Here we consider the conditions for the existence of two kinds of attractors in equation (3) describing the observable behaviour of a system with a plastic velocity field. The ODE (3) is non-autonomous and its solution mapping generates a non-autonomous dynamical system on the state space \( \mathbb{R}^d \) expressed in terms of a 2-parameter semi-group, which is often called a process (see Kloeden & Rasmussen [15]). Define
\[ \mathbb{R}^+_\geq = \{ (t, t_0) \in \mathbb{R} \times \mathbb{R} : t \geq t_0 \} . \]

**Definition 1.** A process is a mapping \( \phi : \mathbb{R}^+_\geq \times \mathbb{R}^d \to \mathbb{R}^d \) with the following properties:
(i) initial condition: \( \phi(t_0, t_0, x_0) = x_0 \) for all \( x_0 \in \mathbb{R}^d \) and \( t_0 \in \mathbb{R} \);

(ii) 2-parameter semi-group property: \( \phi(t_2, t_0, x_0) = \phi(t_2, t_1, \phi(t_1, t_0, x_0)) \) for all \( t_0 \leq t_1 \leq t_2 \) in \( \mathbb{R} \) and \( x_0 \in \mathbb{R}^d \);

(iii) continuity: the mapping \( (t, t_0, x_0) \mapsto \phi(t, t_0, x_0) \) is continuous.

The 2-parameter semi-group property is an immediate consequence of the existence and uniqueness of solutions of the non-autonomous ODE: the solution starting at \( (t_1, x_1) \), where \( x_1 = \phi(t_1, t_0, x_0) \), is unique so must be equal to \( \phi(t, t_0, x_0) \) for \( t \geq t_1 \).

4.1. Pullback attractors in equation (3)

Time in an autonomous dynamical systems is a relative concept since such systems depend on the elapsed time \( t - t_0 \) only and not separately on the current time \( t \) and initial time \( t_0 \), which means that limiting objects exist all the time and not just in the distant future. In contrast, non-autonomous systems depend explicitly on both \( t \) and \( t_0 \), which has a profound affect on the nature of limiting objects (see [15, 17]).

In particular, a non-autonomous attractor is a family \( \mathcal{A} = \{ A(t) : t \in \mathbb{R} \} \) of nonempty compact subsets \( A(t) \) of \( \mathbb{R}^d \) with the following properties:

1) invariance: \( A(t) = \phi(t, t_0, A(t_0)) \) for all \( t \geq t_0 \);
2) pullback attracting:

\[
\lim_{t_0 \to -\infty} \text{dist}_{\mathbb{R}^d} (\phi(t, t_0, B), A(t)) = 0 \quad \text{for all bounded subsets } B \text{ of } \mathbb{R}^d.
\]

It is called a pullback attractor since the starting time \( t_0 \) is pulled further and further back into the past. The dynamics then moves forwards in time from this starting time \( t_0 \) to the present time \( t \). Essentially, the pullback attractor takes into account the past history of the system, so we cannot expect it to say much about the future. In fact, a pullback attractor need not be forward attracting in the conventionally understood sense, i.e. as \( t \to +\infty \) for fixed \( t_0 \) (see the example in subsection in [5.3 below]).

The existence and uniqueness of a global pullback attractor for a non-autonomous dynamical system on \( \mathbb{R}^d \) is implied by the existence of a positive invariant absorbing set. The following theorem is adapted from [15, Theorem 3.18].
Theorem 3. Suppose that a non-autonomous dynamical system $\phi$ on $\mathbb{R}^d$ has a positive invariant absorbing set $B^*$. Then it has a unique pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ with component sets defined by

$$A(t) = \bigcap_{t_0 \leq t} \phi(t, t_0, B^*), \quad t \in \mathbb{R}.$$ 

An important characterization [15, Lemma 2.15] of a pullback attractor is that it consists of the entire bounded solutions of the system, i.e., $\chi : \mathbb{R} \to \mathbb{R}^d$ for which $\chi(t) = \phi(t, t_0, \chi(t_0)) \in A(t)$ for all $(t, t_0) \in \mathbb{R}^+$. 

In particular, under the above assumptions, the ODE (3) describing the observable behaviour of the model of a cognitive system generates a non-autonomous dynamical system, which has a global pullback attractor. Summarising, we formulate the following theorem.

Theorem 4. Suppose that Assumptions (1), (2) and (3) hold. Then the non-autonomous dynamical system generated by the ODE (3) describing the observable behaviour has a global pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$, which is contained in the absorbing set $B^*$.

Thus, Theorem 4 specifies the conditions under which the global pullback attractor exists in a dynamical system with plastic spontaneously evolving velocity vector field.

4.2. Forward limit sets in equation (3)

The concepts of pullback attraction and pullback attractors assume that the system exists for all time, in particular past time. This is obviously not true in many biological systems, though an artificial “past” can some times be usefully introduced (see the final section).

The above definition of a non-autonomous dynamical system can be easily modified to hold only for $(t, t_0) \in \mathbb{R}^+_\geq(T^*) = \{(t, t_0) \in \mathbb{R} \times \mathbb{R} : t \geq t_0 \geq T^*\}$ for some $T^* > -\infty$.

When the system has a nonempty positive invariant compact absorbing set $B^*$, as in the situation here, the forward omega limit set

$$\Omega(t_0) = \bigcap_{\tau \geq t_0} \bigcup_{t \geq \tau} \phi(t, t_0, B^*), \quad t_0 \in \mathbb{R},$$

exists for each $t_0 \geq T^*$, where the upper bar denotes the closure of the set under it. The set $\Omega(t_0)$ is thus a nonempty compact subset of the absorbing set $B^*$ for each $t_0 \in \mathbb{R}$. 

9
Moreover, these sets are increasing in $t_0$, i.e., $\Omega(t_0) \subset \Omega(t'_0)$ for $t_0 \leq t'_0$, and the closure of their union

$$\Omega^* := \bigcup_{t_0 \geq T^*} \Omega(t_0) \subset B^*$$

is a compact subset of $B^*$, which attracts all of the dynamics of the system in the forward sense, i.e.

$$\lim_{t \to \infty} \text{dist}_{\mathbb{R}^d} (\phi(t, t_0, B), \Omega^*) = 0$$

for all bounded subsets $B$ of $\mathbb{R}^d$, $t_0 \geq T^*$.

Vishik \[18\] called $\Omega^*$ the \emph{uniform attractor} \[^1\], although strictly speaking $\Omega^*$ do not form an attractor since it need not be invariant and the attraction need not be uniform in the starting time $t_0$. Nevertheless, $\Omega^*$ does indicate where the future asymptotic dynamics ends up. Moreover, Kloeden \[19\] showed that $\Omega^*$ is \emph{asymptotically positive invariant}, which means that the later the starting time $t_0$, the more and more it looks like an attractor as conventionally understood. In \[19\] $\Omega^*$ was called the \emph{forward attracting set}.

Summarising from the above, we formulate the following theorem.

**Theorem 5.** Suppose that Assumptions (1), (2) and (3) hold. Then the non-autonomous dynamical system generated by the ODE (3) describing the observable behaviour of the system has a forward attracting set $\Omega^*$, which is contained in the absorbing set $B^*$.

Theorem 5 expresses the conditions under which a forward attracting set exists in a dynamical system (3) with a plastic velocity vector field evolving according to (4).

5. Discussion

In the previous sections we explicated the mathematical formulation of, and analysed mathematically, the conceptual model of a cognitive system introduced in \[12\]. For clarity, we made some simplifying assumptions about the properties of the right-hand sides of this model and of the external stimulus.

\[^1\]He required the system to be defined in the whole past and the convergence to be uniform in $t_0 \in \mathbb{R}$.
If the model discussed here is to be used for the description of the cognitive function similar to that of a biological brain, one needs to take into account the different timescales at which different processes occur. It is known that the observable dynamics of neurons is much faster than the rate of change of the inter-neuron connections. Hence, the velocity vector field describing the dynamics of neurons in the brain should evolve at a much slower rate than the neural states. A realistic application of our model should take this into account.

Even the simplified cases studied here raise a number of questions, in particular about the relevance of pullback attractors for such models. These and some further issues will be briefly discussed here.

### 5.1. Use of pullback attractors

Pullback convergence requires the dynamical system to exist in the distant past, which is often not a realistic assumption in biological systems. Pullback attractors can nevertheless be used in such situations by inventing an artificial past. This and other aspects are discussed in [20, 15].

The simplest way to do this for this model is to set the vector field \(a(x, t) \equiv a_0(x)\) for \(t \leq t^*\) for some finite time \(t^*\), which could be the desired starting time \(t_0\). In this case \(a_0(x)\) would be the desired initial velocity vector field of the model of a cognitive system, which could be zero or contain some initial features representing previous memories. Then the ODE (1) should be replaced by the switching system

\[
\frac{dx}{dt} = \begin{cases} a_0(x) & : t \leq t_0 \\ a(x, t) & : t \geq t_0 \end{cases},
\]

where \(a(x, t)\) evolves according to the ODE (2) for \(t \geq t_0\) with the parameterised initial value \(a_0(x)\). If \(a_0(x)\) satisfies the dissipativity condition in Assumption (3), then the switching system (5) will also be dissipative and have a pullback attractor with component sets \(A(t) = A^*\) for \(t \leq t_0\) and \(A(t) = \phi(t, t_0, A^*)\) for \(t \geq t_0\), where \(A^*\) is the global attractor of the autonomous dynamical system generated by the autonomous ODE with the vector field \(a_0(x)\).

### 5.2. Random stimulus signals

The stimulus signal \(\eta(t)\) in Assumption (11) is a deterministic function. When this signal is random it would be a single sample path \(\eta(t, \omega)\) of a
stochastic process with $\omega \in \Omega$, where $\Omega$ is the sample space of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The above analysis holds, which is otherwise deterministic, for this fixed sample path. For emphasis, $\omega$ could be included in the system and the pullback attractor as an additional parameter, i.e., $\phi(t, t_0, x_0, \omega)$ and $\mathcal{A} = \{A(t, \omega) : t \in \mathbb{R}\}$. Cui et al. [21, 22] call these objects non-autonomous random dynamical systems and random pullback attractors, respectively.

This is the appropriate formulation for brain vector fields generated by the ODE (4), which has two sources of non-autonomy in its vector field $c$, i.e., indirectly through the stimulus signal $\eta(t)$ and directly through the independent variable $t$. The ODE (4) is then a random ODE (RODE), see [23]. Note, that without this additional independent variable $t$, the theory of random dynamical systems (RDS) in Arnold [24] could be used. It is also a pathwise theory with a random attractor defined through pullback convergence, but requires additional assumptions about the nature of the driving noise process, which is here represented in the stimulus signal.

Until now we considered the stimulus signal $\eta(t)$ with continuous sample paths. The above results remain valid when $\eta(t)$ has only measurable sample paths, such as for a Poisson or Lévy process, but the RODE must now be interpreted pathwise as a Carathéodory ODE, see [23].

5.3. Relevance of pullback attractors

Assuming, by nature or artifice, that the system does have a pullback attractor, what does this actually tell us about the asymptotic dynamics of the neural activity?

As mentioned above, a pullback attractor consists of the entire bounded solutions of the system, which is useful information. This characterisation is also true of attractors of autonomous systems, for which pullback and forward convergence are equivalent due to the fact that only the elapsed time is important in such systems.

In general, a pullback attractor need not be forward attracting. This is easily seen in the following switching system

$$\frac{dx}{dt} = \begin{cases} -x : t \leq 0 \\ x(1-x^2) : t > 0 \end{cases},$$

for which the set $B^* = [-2, 2]$ is positively invariant and absorbing. The pullback attractor $\mathcal{A}$ has identical component subsets $A_t \equiv \{0\}, t \in \mathbb{R}$, corresponding to the zero entire solution, which is the only bounded entire
solution of this switching system. This zero solution is obviously not forward asymptotically stable. The forward attracting set here is $\Omega^* = [-1, 1]$. It is not invariant (though it is positive invariant in this case), but contains all of the forward limit points of the system.

Nevertheless, a pullback attractor indicates where the system settles down to when more and more information of its past is taken into account. This is very useful in system which is itself evolving in time, as in the brain plasticity model under consideration, for which the future input stimulus is not yet known.

Interestingly, a random attractor for the RDS in the sense of [24] is pullback attracting in the pathwise sense and also forward attracting in probability, see [17].

5.4. Vector field from a potential function

In an example investigated numerically in [12], the vector field $a$ was generated from a potential function $U$, i.e. with $a = -\frac{1}{t} \nabla_x U$, and a differential equation was constructed for $U$, rather than for $a$. Componentwise $a_i = -\frac{1}{t} \frac{\partial U}{\partial x_i}$, so the existence of such a potential requires

$$\frac{\partial a_i}{\partial x_j} = -\frac{1}{t} \frac{\partial^2 U}{\partial x_j \partial x_i} = -\frac{1}{t} \frac{\partial^2 U}{\partial x_i \partial x_j} = \frac{\partial a_j}{\partial x_i}$$

From equation (4) this requires

$$\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}.$$ 

The example considered in [12] is a special case of the system (3)–(4). Namely, in [12] $U$ satisfies a scalar parameterised ordinary differential equation

$$\frac{d}{dt} U(x, t) = -k U(x, t) - g(x - \eta(t)), \quad (6)$$

where $k \geq 0$, $g$ is shaped like a Gaussian function

$$g(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}},$$

and $\eta(t)$ is the given input, which is assumed to be defined for all $t \in \mathbb{R}$ and is continuous.
The gradient $\nabla_x U$ of $U$ satisfies the scalar parameterised ordinary differential equation

$$\frac{d}{dt} \nabla_x U(x, t) = -k \nabla_x U(x, t) - G(x - \eta(t)),$$

where

$$G(x - \eta(t)) = \nabla_x g(x - \eta(t)) = \frac{2}{\sigma^2 \sqrt{2\pi \sigma^2}} (x - \eta(t)) e^{-\frac{(x - \eta(t))^2}{2\sigma^2}}.$$

The linear ODE (7) has an explicit solution

$$\nabla_x U(x, t) = \nabla_x U(x, t_0) e^{-k(t-t_0)} - \int_{t_0}^{t} e^{-k(t-s)} G(x - \eta(s)) ds.$$

Taking the pullback limit as $t_0 \to -\infty$ gives

$$\nabla_x U(x, t) = - \int_{-\infty}^{t} e^{-k(t-s)} G(x - \eta(s)) ds.$$

This solution is asymptotically stable and forward attracts all other solutions, since

$$|\nabla_x U(x, t) - \nabla_x \bar{U}(x, t)| \leq |\nabla_x U(x, t_0) - \nabla_x \bar{U}(x, t_0)| e^{-k(t-t_0)}$$

for every $x$ and any solution $\nabla_x U(x, t) \neq \nabla_x \bar{U}(x, t)$.

Finally, the asymptotic dynamics of this example system with a plastic vector field satisfies the scalar ODE

$$\frac{dx(t)}{dt} = -\frac{1}{t} \nabla_x \bar{U}(x(t), t) = \frac{1}{t} \int_{-\infty}^{t} e^{-k(t-s)} G(x(t) - \eta(s)) ds.$$

Since the integrand is uniformly bounded, it follows that $\left| \frac{dx(t)}{dt} \right| \leq \frac{C}{t} \to 0$ as $t \to \infty$. From numerical simulations, the system (3) with $a(x, t) = -\frac{1}{t} \nabla_x U(x, t)$ appears to have a forward attracting set.

From the argument presented above, equation (4) for the vector field $a$ has a pullback attractor consisting of singleton set, i.e. a single entire solutions, which is also Lyapunov forward attracting. This implies that starting from an arbitrary smooth initial vector field $a(x, t_0)$, the solution $a(x, t)$ of (4) converges to a time-varying function $\bar{a}(x, t) = -\frac{1}{t} \nabla_x \bar{U}(x, t)$.
Remark 1. The example considered in [12] actually involved a random forcing term $\eta(t)$, which was the stochastic stationary solution (essentially its random attractor) of the scalar Itô stochastic differential equation (SDE)

$$d\eta(t) = h(\eta(t))dt + 0.5dW(t),$$

where $W(t)$ was a two-sided Wiener process. For the function $h(u) = 3(u - u^3)/5$ used in [12], the representative potential function had two non-symmetric wells of different depths and widths. In such cases, the solutions of (3)–(4) depend on the sample path $\eta(t, \omega)$ of the noise process, and the convergences are pathwise, and random versions of the theorems formulated above apply. In particular, the random pullback attractor consists of singleton sets, i.e., it is essentially is a stochastic process. Moreover, it is Lyapunov asymptotically stable in probability.

6. Conclusion

To conclude, we reconsidered the problem from [12] from the perspective of recent developments in non-autonomous dynamical systems. In order to further develop modelling of information processing by means of dynamical systems with plastic self-organising vector fields, we needed to show that the problem is well-posed mathematically, which is one of the results of this paper obtained under some simplifying assumptions. At the same time, we have shown that asymptotic dynamics can be formulated in terms of non-autonomous and/or random attractors. This provides us with a firm foundation for a deeper understanding of the potential capabilities of systems with plastic adaptable rules of behaviour.

The model presented here offers many interesting mathematical challenges, such as the rigorous analysis of parameter-free bifurcations occurring as a result of spontaneous evolution of the velocity field of the dynamical system. The necessary background theory is yet to be developed.

Acknowledgements
The visit of PEK to Loughborough University was supported by London Mathematical Society.

Authors’ Contributions
NBJ and PEK jointly formulated the problem and interpreted the results. PEK obtained the mathematical results and NBJ put the problem into the context of applications to the brain and cognition.
References

[1] D. Hassabis, D. Kumaran, C. Summerfield, M. Botvinick, Neuroscience-inspired artificial intelligence, Neuron 95 (2) (2017) 245–258.

[2] A. Damasio, How the brain creates the mind, Sci Am 281 (6) (1999) 112–117.

[3] M. Carandini, From circuits to behavior: a bridge too far?, Nat Neurosci 15 (2012) 507–509.

[4] A. Abbott, Solving the brain., Nature 499 (2013) 272–274.

[5] R. Yuste, G. Church, The new century of the brain, Sci Am 310 (2014) 38–45.

[6] S. Grillner, Megascience efforts and the brain, Neuron 82 (6) (2014) 1209–1211.

[7] M. Katkov, S. Romani, M. Tsodyks, Memory retrieval from first principles, Neuron 94 (5) (2017) 1027 – 1032.

[8] T. Siegfried, There’s a long way to go in understanding the brain, Science News, July 25 (2017).

[9] A. Hodgkin, A. Huxley, A quantitative description of membrane current and its application to conduction and excitation in nerve, J Physiol (Lond) 117 (4) (1952) 500–544.

[10] P. Leon, S. Knock, M. Woodman, A. Spiegler, The virtual brain, Scholarpedia 8 (7) (2013) 30912.

[11] N. Janson, C. Marsden, Dynamical system with plastic self-organized velocity field as an alternative conceptual model of a cognitive system, Scientific Reports 7 (2017) 17007 (Supplementary Note).

[12] N. Janson, C. Marsden, Dynamical system with plastic self-organized velocity field as an alternative conceptual model of a cognitive system, Scientific Reports 7 (2017) 17007.

[13] T. van Gelder, The dynamical hypothesis in cognitive science, Behav Brain Sci 21 (1998) 615–665.
[14] J. Crutchfield, Dynamical embodiments of computation in cognitive processes, Behav. Brain Sci. 21 (1998) 635.

[15] P. Kloeden, M. Rasmussen, Nonautonomous Dynamical Systems, Amer. Math. Soc. Providence, 2011.

[16] W. Walter, Ordinary Differential Equations, Springer, New York, 1998.

[17] H. Crauel, P. Kloeden, Nonautonomous and random attractors, Jahresbericht der Deutschen Mathematiker-Vereinigung 117 (2015) 173–206.

[18] M. Vishik, Asymptotic Behaviour of Solutions of Evolutionary Equations, Cambridge University Press, Cambridge, 1992.

[19] P. Kloeden, Asymptotic invariance and the discretisation of nonautonomous forward attracting sets, J. Comput. Dynamics 3 (2016) 179–189.

[20] P. Kloeden, Pullback attractors of nonautonomous semidynamical systems, Stochastics & Dynamics 3 (1) (2003) 101–112.

[21] H. Cui, P. Kloeden, Invariant forward random attractors of non-autonomous random dynamical systems, J. Differential Eqns. 65 (2018) 6166–6186.

[22] H. Cui, J. Langa, Uniform attractors for non-autonomous random dynamical systems, J. Differential Equations 263 (2017) 1225–1268.

[23] X. Han, P. Kloeden, Random Ordinary Differential Equations and their Numerical Solution, Springer Nature, Singapore, 2017.

[24] L. Arnold, Random Dynamical Systems, Springer-Verlag, Berlin, 1998.