FREE FROBENIUS ALGEBRA ON THE DIFFERENTIAL FORMS OF A MANIFOLD

SCOTT O. WILSON

Abstract. We construct an action of a free resolution of the Frobenius properad on the differential forms of a closed oriented manifold. As a consequence, the forms of a manifold with values in a semi-simple Lie algebra have an additional structure given by an action of a free resolution of the properad describing Lie di-algebras with module compatibility.

Contents

1. Introduction 1
2. Frobenius properad and $H^*(M; \mathbb{R})$ 2
3. Resolutions and constructing algebras 3
4. Forms of a manifold 4
5. Lie algebra valued forms 6
6. Further remarks and questions 7
References 7

1. Introduction

In this paper we describe an action of a free resolution of the Frobenius properad on the differential forms of a closed oriented manifold. This is constructed using the cobar-bar resolution [3] and an inductive argument showing that an appropriate sequence of obstructions vanish. We expect this argument to be of use in other contexts.

With a view towards the algebraic structure of connections on a manifold, we show that the differential forms of a manifold with values in a semi-simple Lie algebra have additional structure. Namely, they admit an action of a free resolution of a Lie di-algebra with module compatibility.

We end with some questions and further problems on duality and the integral cochain version of our constructions.

I would like to thank Dennis Sullivan for many helpful discussions. Together we obtained a graphical description of a free resolution of the Frobenius properad and an argument for transferring such structures. We intend to use this in future work to give computable effective models for fluid dynamics.

I would also like to thank Kevin Costello, Thomas Tradler and Mahmoud Zeinalian for helpful correspondence. The action on forms constructed here may also be related to their recent work on the Hochschild complex (loopspace), and must

Date: February 2, 2008.
be compared with the already useful $V_\infty$-algebras developed by the Tradler and Zeinalian [6].

The results below on Lie-algebra valued forms resemble the recent work of Mnëv on simplicial BF-theory [1] and Costello on renormalization [1], since they all give some higher-homotopy Lie algebra with possibly non-zero higher genus operations. I am interested in understanding these connections.

2. Frobenius properad and $H^\ast(M;\mathbb{R})$

A Frobenius algebra is classically a unital commutative associative algebra with non-degenerate invariant inner product. In the finite dimensional case, such a structure can be “opened-up” (a term due to Sullivan) to give a compatible algebra and co-algebra, and is described by an algebra over a prop(eralad). We now give a definition of this Frobenius prop(eralad), and later take the homotopy invariant notion of a Frobenius algebra to be an algebra over a free resolution of this prop(eralad).

**Definition 1.** The Frobenius properad of degree $n$ is defined as follows. For each $j, k > 0$ the complex $\text{Frob}(j, k)$ is isomorphic to $\mathbb{Q}$ in degrees $(k - 1)n + gn$, for $g \in \{0, 1, 2, \ldots \}$, and equals zero in all other degrees. There is a degree zero map $\mathbb{Q} \to \text{Frob}(1, 1)$ giving the properadic unit. The space $\text{Frob}(0, k)$ is isomorphic to $\mathbb{Q}$ in degrees $0, n, \ldots, kn$ and zero otherwise. The space $\text{Frob}(j, 0)$ is isomorphic to $\mathbb{Q}$ in degrees $-jn, -(j - 1)n, \ldots, 0$ and zero otherwise. The differential is zero.

Let $\text{Frob}(j, k)_g$ denote degree $(k - 1)n + gn$ of $\text{Frob}(j, k)$. We call a generator of $\text{Frob}(j, k)_g$ a genus $g$ operation with $j$ inputs and $k$ outputs. Let $j = \sum_{i=1}^{r} j_i$ and $k = \sum_{i=1}^{r} k_i$. The non-zero composition maps,

$$\text{Frob}(j_1, k_1)_{g_1} \otimes \cdots \otimes \text{Frob}(j_r, k_r)_{g_r} \otimes \text{Frob}(k, l)_g \to \text{Frob}(j, l)_{g_1 + \cdots + g_r + g + k - r}$$

are all of degree zero and are given by the canonical isomorphism $\mathbb{Q}^\otimes \alpha \to \mathbb{Q}$.

The compositions maps may be understood as follows. For each $j, k, g$ regard a generator of $\text{Frob}(j, k)_g$ as a graph with $j$ inputs leaves, $k$ output leaves and genus $g$. Then composition is given by grafting output leaves of $r$ graphs to inputs leaves of one graph with $k$ input leaves, and the value of $g$ for the graph obtained is the glued graph’s genus.

The following is a presentation for the Frobenius properad. The generators are in $(2, 1)$ and $(0, 1)$ of degree 0, in $(2, 1)$ of degree $n$, and $(1, 0)$ of degree $-n$. Regarding these as a product, augmentation map, co-product and co-augmentation map, respectively, the relations are given by: the product is commutative and associative, the augmentation map is a unit for the product, the co-product is co-commutative and co-associative, the co-augmentation map is a co-unit for the co-product, and finally the Frobenius relation, the co-product is a bi-module map.

**Theorem 2.** The cohomology $H^\ast(M;\mathbb{R})$ of a closed oriented $n$-manifold is an algebra over the Frobenius properad of degree $n$. All operations of genus $g \geq 1$ vanish if and only if the Euler characteristic $\chi(M)$ is zero.

**Proof.** The generators of $\text{Frob}(j, k)$ act on $H^\ast(M;\mathbb{Q})$ by

- $\text{Frob}(2, 1)$ cohomological product
- $\text{Frob}(0, 1)$ cohomological unit
- $\text{Frob}(1, 0)$ capping on $[M]
- $\text{Frob}(1, 2)$ cohomological co-product
The latter exists, for example, by Poincaré Duality\(^1\). Together the relations: (co)units, (co)associativity, (co)commutativity, and the co-product is a map of \(H^*(M; \mathbb{R})\)-modules, imply there is a well defined action.

The last statement of the theorem holds since the genus one operation \(H^*(M; \mathbb{R}) \to H^*(M; \mathbb{R})\) is given by multiplication by \(\chi(M)\).

\section*{3. Resolutions and Constructing Algebras}

We begin with a review of the Bar-Cobar construction for properads, see \[3\] for more details. Let \(P\) be a properad. There is a free properad \(\Omega(B(P))\) and a morphisms \(\Omega(B(P)) \to P\) inducing an isomorphism on homology. The underlying space of \(\Omega(B(P))\) is given by

\[
\Omega(B(P)) = F(F_c(P))
\]

where \(F\) is the free properad functor and \(F_c\) is the co-free co-properad functor. (We have absorbed the usual (de)suspension operators into the notations \(F, F_c\).)

The differential on \(\Omega(B(P))\) is given by the sum of three terms, \(d_P + d + \partial\). Here \(d_P\) is induced by the internal differential of \(P\), \(d\) is induced by the bar-construction differential (i.e. the unique co-derivation that extends the partial product of \(P\)), and \(\partial\) is the differential of the cobar construction (i.e. the unique derivation that extends the partial co-product of the co-properad \(F_c(P)\)).

Recall \(F(P)\) has a graphical description in term of graphs whose vertices are labeled by elements of \(P\), and in this case \(d\) is given by contraction in the graph and composition of elements of \(P\). The composition in \(F(P)\) is given by grafting. Similarly, \(F_c(P)\) has the same underlying space of labeled graphs and the differential \(\partial\) is given by the splittings of a graph into two graphs.

\textbf{Theorem 3.} Let \(P\) be a dg-properad with zero differential. Let \(Q\) be any properad. There is an inductive sequence of obstructions to defining a morphism of properads \(\phi : \Omega(B(P)) \to Q\).

As is usually the case with obstruction theory, the vanishing of an obstruction up to some point allows the induction to proceed to the next stage. But, the construction depends on the choices made, and there is no canonical choice for realizing a null-homologous cycle as a boundary.

\textit{Proof.} Since \(\Omega(B(P))\) is a free properad, to define a morphism \(\phi : \Omega(B(P)) \to Q\), it suffices to define a chain map \(B(P) \to Q\) that is invariant with respect to the left and right symmetric group actions \((3\), Lemma 13).

There is a filtration of \(B(P)\) given by the weight, i.e. the number of vertices of the graph labeled by elements of \(P\). We induct on this. Given \(x \in B(P)\), the differential \(\tilde{d}\) of \(\Omega(B(P))\) applied to \(x\) is of the form \(dx + \partial x\) where \(dx\) is of lower weight and \(\partial x\) is a sum of terms which are given by splitting \(x\) into two parts, each of lower weight. Therefore, \(\phi(dx)\) is defined by induction and \(\phi(\partial x)\) is defined by freeness. Also, \(d_Q(\phi(dx)) = \phi(d^2 x) = 0\). The obstruction determined by \(x\) corresponds to filling in the cycle \(\phi(\tilde{d}\Omega(B(P)) \cdot x))\) of \(Q\). \(\square\)

\footnote{1}Below we define the co-product on differential forms using a representative for the Thom class.

\footnote{2}This property of the differential was pointed out by Sullivan, and given the name “triangular”. It generalizes minimality, in the sense that the differential respects a filtration, but with a linear term as well. Compare the structure of a “master-equation”.}
In the next section we use the previous theorem to construct algebra over \( \Omega(B(Frob)) \) by showing that the appropriate obstructions vanish.

4. Forms of a manifold

In this section we construct an action of the Cobar-Bar resolution of the Frobenius properad of degree \( n \) on the differential forms of a closed oriented \( n \)-manifold. This means a morphism of properads

\[
\Omega(B(Frob(j, k))) \to \text{End}\left(\Omega(M)^{\otimes j}, \Omega(M)^{\otimes k}\right)
\]

By the tensor product above, we mean the completed tensor product, so that we may identify \( \Omega(M)^{\otimes k} \) with \( \Omega(M)^{\otimes k} \).

Let \( M \) be a closed oriented \( d \)-manifold and let \( \Delta(M) \) be the diagonal of \( M \times M \). We begin with a description of the co-product on the forms of \( M \). Let \( T \) (for Thom) be a differential form on \( M \times M \) that is Poincaré Dual to \( \Delta(M) \) and is supported in a neighborhood \( U \) that contracts to \( \Delta(M) \). We further assume \( T \) is invariant under \( \tau \), which may be achieved by averaging.

Define a map

\[
\Delta : \Omega(M) \to \Omega(M \times M)
\]

by sending a form \( \omega \) to \( \Delta(\omega) = \frac{1}{2}(\pi_1^* + \pi_2^*)\omega \wedge T \). For all \( \omega \) this is a form on \( M \times M \) with support in \( U \).

Note \( \Delta \) is a degree +\( n \) chain map since \( T \) is a closed \( n \)-form and \( \Delta(\omega) \) is invariant under the change of factors on \( M \times M \).

Similarly, one can construct co-products \( \Omega(M) \to \Omega(M^k) \) for all \( k \) by choosing a Thom form dual to the diagonal of \( M^k \).

Theorem 4. There is a morphism of properads

\[
\Omega(B(Frob(j, k))) \to \text{End}\left(\Omega(M)^{\otimes j}, \Omega(M)^{\otimes k}\right)
\]

inducing on cohomology \( H^*(M) \) the structure given in theorem 2.

Proof. We apply theorem 3 using freeness of \( \Omega(B(Frob(j, k))) \) and inducting on the weight. For the initial weight zero case we must define a chain map

\[
Frob(j, k) \to \text{End}\left(\Omega(M)^{\otimes j}, \Omega(M)^{\otimes k}\right)
\]

that respects the symmetric group actions. We first consider the genus zero operations.

The single output part, \( Frob(j, 1) \), is defined by the usual unital DGA structure on differential forms. For \( j > 0, k = 0 \) the map is given by wedge product and integration, which is a chain map since \( M \) is closed. For \( j = 0 \) the maps \( \mathbb{Q} \to M^\times k \) are given by mapping a generator of \( \mathbb{Q} \) to a Thom form Poincaré dual to the diagonal of \( M^k \) and supported in a neighborhood that contracts to the diagonal. Finally, the co-product for \( j = 1, k > 1 \) are given by the symmetrized pull back and wedge product with the appropriate Thom form, as described before the theorem. This completes definition of the genus zero operations. The higher genus operations, in \( Frob(j, k) \), are defined by compositions of the lower genus operations.

We assume we have defined a chain map from the weight \( n \) and lower component of \( B(Frob(j, k)) \) to \( \text{End}\left(\Omega(M)^{\otimes j}, \Omega(M)^{\otimes k}\right) \). By induction we may also assume that the chain maps satisfy the following properties, considered in four cases. First, for \( k = 1, g = 0 \) these operations are part of a strict DGA on forms. Secondly, for

---

3Such a local Thom form exists for any homology manifold, but does not exist for any Poincaré Duality pseudomanifold that is not also a homology manifold \[\mathbb{Q}\].
\[ k > 1 \] and each genus \( g \) these operations satisfy the following locality condition:

There exists a neighborhood \( U \) and contracting homotopy to a submanifold \( N \subset U \) such that

1. The operations are supported in \( U \).
2. All the operations all agree when restricted to \( N \).

Third, for \( k = 1 \) and arbitrary \( g \) the operations are of the form \( m \circ f \) where \( f \) satisfies the locality condition and \( m \) is the restriction to some diagonal. And finally, forth, for \( k = 0 \) the operations are of the form \( f \circ f \) where \( f \) is as in case one or two.

These properties are satisfied for the weight zero case by construction, where \( N \) is the diagonal and \( U \) is a neighborhood supporting the Thom form.

Now let \( x \in B(Frob(j, k)) \). By induction \( \phi(dx) \) is defined and is a cycle in \( \text{End}(\Omega(M)^{\otimes j}, \Omega(M)^{\otimes k}) \). We show that in any of the four cases this cycle can be filled in, i.e. the obstruction vanishes.

In the case \( k = 1, g > 0 \) we may fill in this cycle trivially since the wedge product gives a strict DGA. For the second case, we build a homotopy using the locality condition. Namely, it is given by integrating out all operations with respect to \( t \), along the contracting homotopy to the diagonal. At \( t = 1 \) all of the agree, so this defines a homomorphisms \( h_x \in \text{End}(\Omega(M)^{\otimes j}, \Omega(M)^{\otimes k}) \) whose boundary is the cycle \( \phi(dx) \). Finally, \( h_x \) also satisfies the locality condition. For the third case, we use the locality condition as in the previous case, and the fact that any two Thom forms are cohomologous, to build a homotopy before restriction.

The forth and final case breaks into two parts. Each summand of \( dx \), in the canonical basis, is given by a by a graph labeled by elements of \( Frob(j, k) \). If for each summand, all graphs are labeled by degree zero elements (products) then all operations given by the summands of \( dx \) agree (since there is a unique genus zero \( j \) to 0 operation on forms). So, we fill in \( \phi(dx) \) with the zero homotopy as in the strict DGA case. Otherwise, there is at least one labeling operation of degree at least \( n \) (made of a co-product). Then we use the fact that the Thom form is homotopic to the diagonal to construct a homotopy from each summand and fill in the cycle \( \phi(dx) \).

This completes the induction and the proof. \( \square \)

The structure is independent (up to equivalence) of the initial choices of Thom forms since any two Thom forms are co-homologous. Likewise, the structure is independent of the choices of homotopies since the neighborhoods contract to the diagonals. We have not discussed functorialy yet since it is unclear to the author how best to consider and state this.

We would like to point out that, while every Poincaré Duality space \( X \) has the structure of a unital DGA with invariant inner product on its polynomial forms \( A(X) \), the general non-existence of a local Thom form \([2]\) prevents one from constructing on \( A(X) \) the structure considered here.

**Remark 5 (On a Co-unit).** It is worth noting that the natural candidate for a co-unit, given by the integration map \( \Omega(M^k) \to \mathbb{R} \), is not strictly a co-unit.

**The diagram for** \( i = 1, 2 \):
does not commute since, for example,

$$\int_M \omega \neq \int_{M \times M} \frac{1}{2}(\pi_1^* + \pi_2^*)\omega \wedge T$$

The diffusive nature of the Thom form leads to the failure of a strict co-unit, yet the maps above are homotopic, and integration provides a co-unit up-to-homotopy.

5. **Lie algebra valued forms**

In this section we describe an additional structure possessed by differential forms with values in a semi-simple Lie algebra. The following is generalization of a definition in [5].

**Definition 6.** A degree $n$ Lie di-algebra with module compatibility is a triple $(V, [\cdot, \cdot], \Delta)$ such that $(V, [\cdot, \cdot])$ is a differential graded Lie algebra, $(V, \Delta)$ is a differential graded Lie co-algebra, and $\delta : V \rightarrow V \otimes V$ is a degree $n$ map of Lie modules. Here $V$ acts on $V$ by the adjoint representation and $V$ acts on $V \otimes V$ by derivation. Let $\text{diLie}$ be the properad that describes degree $n$ Lie di-algebras with module compatibility.

**Lemma 7.** Any finite dimensional semi-simple Lie algebra $\mathfrak{g}$ induces a degree 0 Lie di-algebra with module compatibility.

**Proof.** Since $\mathfrak{g}$ is semi-simple, its Killing form is non-degenerate. We use this pairing and the finite dimensional hypothesis to define $\Delta$ by inverting the given Lie bracket. The co-Lie and module relations are a formal consequence. □

It is well known that a commutative algebra tensor a Lie algebra is a Lie algebra. The operadic viewpoint make this easy to see: the operad $\text{Comm}$ satisfies $\text{Comm}(k) \approx \mathbb{Q}$ for all $k$ and tensoring with $\mathbb{Q}$ is the identity operation. We use a similar statement for properads to show

**Theorem 8.** Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra and $M$ be a closed oriented n-manifold. There is an action of a free resolution of the properad describing degree $n$ Lie di-algebras with module compatibility on $\Omega^*(M) \otimes \mathfrak{g}$.

**Proof.** As in the proof of theorem [4] we construct an algebra on $\Omega(M)$ over a free resolution of the superoperad of $\text{Frob}(k, l)_{\mathfrak{g}}$ given by $g = 0$ and $k, l > 0$. (This may appropriately be called the Frobenius dioperad $\text{Frob}(k, l)_{\mathfrak{g}}$ of degree $n$). It is easy to check that $\text{Frob}(k, l)_{\mathfrak{g}}$ of degree $n$ tensored with diLie of degree 0 equals the diLie properad of degree $n$.

Using lemma [7] we then have a tensor action of diLie of degree $n$ on $\Omega^*(M) \otimes \mathfrak{g}$. Now one can pull back along any free resolution of diLie. □

There is also the obvious statement

$$\Omega(B(\text{Frob})) \otimes \text{diLie} \text{ acts on } \Omega^*(M) \otimes \mathfrak{g}$$

though it is not yet clear the the author what is the structure of an $(\Omega(B(\text{Frob})) \otimes \text{diLie})$-algebra.
6. Further remarks and questions

One can show that for any finite dimensional algebra $V$ over the properad $Frob$, the dual space $V^*$ is also an algebra over $Frob$. In fact, the generators and relations of $Frob$ are self dual, so one may Hom-dualize all of the operations, and the relations dualize as well. For a closed oriented manifold we obtain that $H_*(M)$ is a $Frob$-algebra. Moreover, Poincaré Duality induces an isomorphism between these two $Frob$-algebras. This suggest the following

**Problem 1.** Construct a equivalence between the $Frob_\infty$ algebra on $\Omega^*(M)$ and an induced $Frob_\infty$ algebra on the dual space $(\Omega^*(M))^*$ of currents of $M$.

**Problem 2.** Determine the deeper meaning of this structure as an invariant of the manifold $M$.

The constructions above have all been over $\mathbb{R}$, which was essential in allowing us to symmetrize operations. To define an action on the integral cochains of a manifold we need a doubly free resolution of $Frob$. A properad is doubly free if it is free as a properad and also the space of $(j, k)$ operations is a free $(\Sigma_j, \Sigma_k)$-bimodule.

**Problem 3.** Construct an action of a doubly free resolution of $Frob$ on the integral cochains of a manifold.

And now we are led back to the integral versions of the first two problems.

References

[1] Costello, K. “Renormalisation and the Batalin-Vilkovisky formalism,” preprint [arXiv:0706.1533]
[2] McCrory, C. “A characterization of homology manifolds,” J. London Math. Soc. (2) 16 (1977), no. 1, 149–159.
[3] Merkulov, S. and Vallette, B. “Deformation theory of representations of prop(erad)s,” preprint [arXiv:0710.0821]
[4] Mnëv, P. N. “On the simplicial super-$BF$ model,” (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 331 (2006), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody, 14, 84–90, 223; translation in J. Math. Sci. (N. Y.) 141 (2007), no. 4, 1429–1431.
[5] Sullivan, D. “Open and closed string field theory interpreted in classical algebraic topology,” Topology, geometry and quantum field theory, 344–357, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.
[6] Tradler, T. and Zeinalian, M. “Algebraic String Operations,” preprint arXiv:math/0605770

Scott O. Wilson, 127 Vincent Hall, 206 Church St. S.E., Minneapolis, MN 55455.

email: scottw@math.umn.edu