Galois covers of the open \( p \)-adic disc

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Abstract

Motivated by the local lifting problem for Galois covers of curves, this paper investigates Galois branched covers of the open \( p \)-adic disc. Using the field of norms functor of Fontaine and Wintenberger, we show that the special fiber of a Galois cover is determined by arithmetic and geometric properties of the generic fiber and its characteristic zero specializations. As applications, we derive a characteristic zero reformulation of the abelian local lifting problem, and give a new proof of the liftability of \( p \)-cyclic covers.

1. Introduction

The main result of this paper (Theorem 4.1) says that the special fiber of a Galois branched cover of the open \( p \)-adic disc is determined by characteristic zero fibers. The motivation for such a theorem comes from the global lifting problem for Galois covers of curves: if \( G \) is a finite group, \( k \) is an algebraically closed field of characteristic \( p > 0 \), and \( f : C \to C' \) is a finite \( G \)-Galois branched cover of smooth projective \( k \)-curves, does there exist a lifting of \( f \) to a \( G \)-Galois branched cover of smooth projective \( R \)-curves, where \( R \) is a discrete valuation ring of mixed characteristic with residue field \( k \)? This problem has been much studied; see e.g. [Gro71], [OSS89], [GM98], [Pag02], [BW06], [CGH].

As a brief survey of the subject, we mention the following results:

- (Grothendieck, [Gro71]) If \( f \) is at most tamely ramified, then a lifting exists over \( R \), where \( R \) is any complete DVR with residue field \( k \). Moreover, the lifting is unique once we fix a lifting of \( C' \) and the branch locus of \( f \).

- Not all wildly ramified covers are liftable: there exist curves of genus \( g \geq 2 \) in positive characteristic whose automorphism groups are too large to be automorphism groups of genus \( g \) curves in characteristic zero, by the Hurwitz genus bounds. If \( C \) is such a curve, it is then clear that \( C' = C/\text{Aut}(C) \) is not liftable to mixed characteristic.

- There are examples of non-liftable \( p \)-elementary abelian covers. See [GM98] for an example with \( G = (\mathbb{Z}/p\mathbb{Z})^2 \) for \( p > 2 \).

- Oort, Sekiguchi, and Suwa showed in [OSS89] that cyclic covers lift if \( p \) divides \( |G| \) at most once. Their method was global in nature, involving a group scheme degeneration of Kummer Theory to Artin-Schreier Theory.

- Using local methods (see below), Green and Matignon proved in [GM98] that cyclic covers lift if \( p \) divides \( |G| \) at most twice.

- Pagot has shown in [Pag02] that Klein-four covers always lift.

- \( D_p \)-covers are liftable by [BW06], where the method of differential data is used.

- Green has shown in [Gre04] that the \( p^n \)-cyclic covers that occur as automorphisms of Lubin-Tate formal groups are liftable.

2000 Mathematics Subject Classification 11S15, 12F10, 12F15 (primary), 13B05, 14D15, 14H30 (secondary).
Keywords: open \( p \)-adic disc, field of norms, Galois groups, characteristic \( p \), lifting, ramification, Oort Conjecture.
The guiding conjecture in the subject was provided by F. Oort \([\text{Oor87}, \text{I.7}]\) who suggested the **Oort Conjecture**. *Cyclic covers always lift.*

In section 5, we derive an arithmetic reformulation of a stronger form of this conjecture, which specifies the ring \(R\) over which the lifting should occur:

**Ring Specific Oort Conjecture.** \[^1\]**If \(f : C \to C'\) is cyclic of order \(p^e n\) with \((p, n) = 1\), then \(f\) lifts over \(R = W(k)[\overline{q_p}]\), where \(W(k)\) denotes the Witt vectors of \(k\).*

A major breakthrough in the subject came when Green and Matignon discovered that the obstructions to lifting are not global in nature, but rather local. Indeed, using either rigid patching \([\text{GM98}]\) or deformation theory \([\text{BM00}]\), the global lifting problem reduces to a local one, due to the **Local-to-Global Principle.** \([\text{GM98}}\) section III, \([\text{BM00}]\) Corollaire 3.3.5) Let \(y \in C\) be a ramification point for the \(G\)-Galois cover \(f : C \to C'\), and consider the \(G_y\)-Galois cover \(f_y : \text{Spec}(O_{C,y}) \to \text{Spec}(O_{C',f(y)})\) obtained by completion, where \(G_y\) is the inertia subgroup at \(y\). Suppose that for each such ramified \(y \in C\), the map \(f_y\) can be lifted to a \(G_y\)-Galois cover of open \(p\)-adic discs, \(F_y : D \to D\), where \(D = \text{Spec}(R[[Z]])\). Then the local liftings, \(F_y\), can be patched together to yield a lifting of \(f\) to a \(G\)-Galois cover of smooth projective \(R\)-curves.

Since \(O_{C,y} \cong k[[t]]\), we are led to consider the following **local lifting problem for Galois covers of curves**; given a finite \(G\)-Galois extension of power series rings \(k[[t]][k[[z]]\), does there exist a lifting to a \(G\)-Galois extension \(R[[T]][R[[Z]]\), where \(R\) is a mixed characteristic DVR with residue field \(k\)? One could also consider the (weaker) *birational local lifting problem for Galois covers of curves*; given a finite \(G\)-Galois extension of Laurent series fields \(k((t))|k((z))\), does there exist a lifting to a \(G\)-Galois extension of normal rings \(A|R[[Z]]\)? By a lifting in this case, we mean that \(A_s := A/\overline{\omega}A\) is an integral domain (where \(\overline{\omega}\) is a uniformizer for \(R\)), and the fields \(\text{Frac}(A_s)\) and \(k((t))\) are isomorphic as \(G\)-Galois extensions of \(k((z))\). In particular, in the birational version of the local problem, we do not require \(\text{Spec}(A)\) to be smooth: in terms of geometry, this corresponds to allowing the curve \(C\) to acquire singularities.

As mentioned above, the local lifting problem does not always have a positive solution. On the other hand, Garuti has shown in \([\text{Gar96}]\) that the birational local lifting problem *does* always have a positive solution, so the birational problem is indeed weaker than the local lifting problem. It is clearly advantageous to work with Laurent series fields rather than power series rings, however, and the following criterion for good reduction ensures that we may do so without sacrificing the smoothness of our liftings.

**Local Criterion for Good Reduction.** \([\text{Kat87}\) section 5, \([\text{GM98}\) 3.4) Let \(A\) be a normal integral local ring, which is also a finite \(R[[Z]]\)-module. Assume moreover that \(A_s := A/\overline{\omega}A\) is reduced and \(\text{Frac}(A_s)|k((z))\) is separable. Let \(\tilde{A}_s\) be the integral closure of \(A_s\), and define \(\delta_k := \dim_k(\tilde{A}_s/A_s)\). Also, setting \(K = \text{Frac}(R)\), denote by \(d_\eta\) the degree of the different of \((A \otimes K)|R[[Z]]|K\), and by \(d_s\) the degree of the different of \(\text{Frac}(A_s)|k((z))\). Then \(d_\eta = d_s + 2\delta_k\), and if \(d_\eta = d_s\), then \(A \cong R[[T]]\).

Using this criterion, we obtain the **Birational Criterion for Local Lifting.** Suppose that \(k[[t]][k[[z]]\) is a \(G\)-Galois extension of power series rings. Then a \(G\)-Galois extension of normal integral local rings, \(A|R[[Z]]\), is a lifting of \(k[[t]][k[[z]]\) if and only if it is a birational lifting of \(k((t))|k((z))\), and \(d_s = d_\eta\).

[^1]: We had originally intended to use the name Strong Oort Conjecture for this statement, but that name has recently been used in \([\text{CGH}]\) for a different strengthening (and generalization) of the Oort Conjecture.
Hence, the local lifting problem can be reformulated as follows: given a $G$-Galois extension $k((t))|k((z))$, does there exist a $G$-Galois birational lifting $\mathcal{A}|R[[Z]]$ which preserves the different, i.e., such that $d_s = d_\eta$?

It is this last formulation of the local lifting problem that provides our motivation for studying Galois covers of the open $p$-adic disc. In particular, given such a $G$-Galois branched cover $Y = \text{Spec}(A) \rightarrow D$, we are interested in determining geometric and arithmetic properties of the special fiber $Y_k \rightarrow D_k$ (such as irreducibility, separability, and the degree of the different $d_s$) from the corresponding properties of the generic fiber $Y_K \rightarrow D_K$ and its specializations at various points $x \in D_K$. Our main result (Theorem 4.1) provides precisely such a characterization of the special fiber in terms of characteristic zero data. Roughly speaking, our result says that the special fiber of a Galois cover of the open $p$-adic disc “wants” to be the field of norms of the characteristic zero fibers, and the degree to which this fails is the phenomenon of inseparability. Our work can be regarded as a concrete investigation of the class field theory of the open $p$-adic disc, and our main result suggests that the local lifting problem would be answered by a Grunwald-Wang type theorem for the open disc, with control over the generic different.

In section 2 of this paper we review the basic structure of the open $p$-adic disc, and then in section 3 we describe the theory of the field of norms due to Fontaine and Wintenberger, which plays a major role in our main result. Section 4 contains the proof of our main theorem characterizing the special fiber of a Galois branched cover of the open $p$-adic disc in terms of the characteristic zero fibers of the cover. An arithmetic reformulation of the Oort Conjecture is deduced in section 5, together with a new proof of this conjecture in the $p$-cyclic case.

1.1 Notation
Let $K$ be a complete discretely valued field with residue characteristic $p > 0$. We make the following notational conventions:

- $R_K$ denotes the valuation ring of $K$;  
- $\mathfrak{m}_K$ denotes the maximal ideal of $R_K$;  
- $k_K$ denotes the residue field of $R_K$;  
- $\nu_K$ denotes the normalized discrete valuation on $K$, so that $\nu_K(K^\times) = \mathbb{Z}$;  
- $|\cdot|_K$ is the absolute value on $K$ induced by $\nu_K$, normalized so that $|\alpha|_K = p^{-\nu_K(\alpha)}$.  
- if $L$ is the completion of an algebraic extension of $K$, then we also denote by $\nu_K$ (resp. $|\cdot|_K$) the unique prolongation of $\nu_K$ (resp. $|\cdot|_K$) to $L$.

2. The Open $p$-adic Disc
Let $K$ be a complete discretely valued $p$-adic field, with valuation ring $R = R_K$. Then the open $p$-adic disc (over $K$) is defined to be $D_K := \text{Spec}(R[[Z]] \otimes_R K)$, and its smooth integral model is denoted by $D = \text{Spec}(R[[Z]])$. The key result for understanding the structure of the open $p$-adic disc is the

**Weierstrass Preparation Theorem.** ([Bou89] VII.3.8, Prop. 6) Suppose that $g(Z) \in R[[Z]]$ has a nonzero reduced series $\mathcal{F}(z) \in k[[z]]$, of valuation $\nu_{k((z))}(\mathcal{F}(z)) = d \geq 0$. Then $g(Z)$ can be written uniquely as

$$g(Z) = (Z^d + a_{d-1}Z^{d-1} + \cdots + a_0)U(Z),$$

where all $a_i \in \mathfrak{m}$ and $U(Z)$ is a unit in $R[[Z]]$. The degree $d$ is called the Weierstrass degree of $g(Z)$.

Polynomials $Z^d + a_{d-1}Z^{d-1} + \cdots + a_0$ with coefficients from $\mathfrak{m}$ as in the proposition are called
2.2 The Ramification Argument

Weierstrass Argument

In what follows, we will refer to this argument (which allows us to specialize polynomials almost one-to-one correspondence with the irreducible distinguished polynomials over \(R\). Finally, the geometric points of \(D_K\) can be described as:

\[
D_K(\overline{K}) = \{ \alpha \in \overline{K} \mid f(\alpha) = 0 \text{ for some } f \in R[Z] \text{ irred. distinguished} \}
\]

\[
= \{ \alpha \in \overline{K} \mid |\alpha|_K < 1 \},
\]

which explains the name of \(D_K\).

2.1 The Weierstrass Argument

As a consequence of the Weierstrass Preparation Theorem, we see that an arbitrary nonzero power series \(g(Z) \in R[[Z]]\) can be written in the form \(g(Z) = \varpi^e f(Z) U(Z)\), where \(c \geq 0\), the polynomial \(f(Z)\) is distinguished of degree \(d \geq 0\), and \(U(Z)\) is a unit. In the course of our investigation, we will often have to work with the ring \(R[[Z]]_m\), the local ring of \(D\) at the generic point of the special fiber \(D_{\alpha} = \text{Spec}(k[[z]]) \subset D\). The previous remarks imply that an arbitrary nonzero element \(A(Z) \in R[[Z]]_m\) has the form

\[
A(Z) = \frac{\varpi^e f_1(Z) U(Z)}{f_2(Z)},
\]

where the \(f_i(Z)\) are distinguished polynomials. In particular, the denominator \(f_2(Z)\) will be relatively prime to almost all height one primes of \(R[[Z]]\), so if \(P = (h(Z))\) is one of these primes, we will have \(A(Z) \in R[[Z]]_P\), and it will make sense to look at the image of \(A(Z)\) in \(R[[Z]]_p/P \cong K(\alpha)\), where \(\alpha\) is a root of \(h(Z)\) in \(\overline{K}\). When we have chosen a particular root \(\alpha\), we will refer to the image of \(A(Z)\) in \(K(\alpha)\) as the specialization of \(A\) at the point \(Z = \alpha\), and denote it by \(A(\alpha)\). More generally, if

\[
S(T) = T^N + A_{N-1}(Z)T^{N-1} + \cdots + A_0(Z)
\]

is a polynomial with coefficients in \(R[[Z]]_m\), then we can apply the previous reasoning to each of the finitely many coefficients \(A_i(Z)\). We conclude that for almost all points \(Z = \alpha\), we can specialize to obtain the polynomial

\[
S(T)|_{Z=\alpha} = T^n + A_{N-1}(\alpha)T^{N-1} + \cdots + A_0(\alpha) \in K(\alpha)[T].
\]

In what follows, we will refer to this argument (which allows us to specialize polynomials almost everywhere) as the Weierstrass Argument.

2.2 The Ramification Argument

Suppose that \(\{x_m\}_m \subset D_K\) is a sequence of points corresponding to a sequence \(\{\alpha_m\}_m \in \overline{K}\) with each \(\alpha_m\) being a uniformizer for the discrete valuation field \(K(\alpha_m)\). Moreover, suppose that \(|\alpha_m|_K \to 1\) as \(m \to \infty\), so that the points \(x_m\) are approaching the boundary of \(D_K\). Equivalently, we are assuming that the ramification index \(e_m := e(K(\alpha_m)|K)\) goes to \(\infty\) with \(m\). Given \(A(Z) = \varpi^e f_1(Z) U(Z) \in R[[Z]]_m\), we can consider the specialization of \(A\) at \(Z = \alpha_m\) for \(m >> 0\) (by the Weierstrass Argument). We find that

\[
\nu_{K(\alpha_m)}(A(\alpha_m)) = e\nu_{K(\alpha_m)}(\varpi) + \nu_{K(\alpha_m)}\left(\frac{f_1(\alpha_m)}{f_2(\alpha_m)}\right).
\]

Note that for any \(a \in \mathfrak{m}_K\) we have \(\nu_{K(\alpha_m)}(a) \geq \nu_{K(\alpha_m)}(\varpi) = e_m\). Hence, if \(d_i\) is the Weierstrass degree of \(f_i(Z)\), then for \(m >> 0\) we have \(\nu_{K(\alpha_m)}(a) \geq d_1 + d_2\) for all \(a \in \mathfrak{m}_K\), since \(e_m \to \infty\).

It follows that \(\nu_{K(\alpha_m)}(f_i(\alpha_m)) = \nu_{K(\alpha_m)}(\alpha_m^{d_i}) = d_i\), so \(\nu_{K(\alpha_m)}(A(\alpha_m)) = ce_m + (d_1 - d_2) \geq -d_2\).
Thus, we see that the normalized valuations of the specializations $A(\alpha_m) \in K(\alpha_m)$ are bounded below by $-d_2 = -(\text{degree of the denominator of } A(Z))$. Moreover, if $c > 0$ (i.e. if $\mathbb{T}(z) = 0$), then $\nu_{K(\alpha_m)}(A(\alpha_m)) \to \infty$ as $m \to \infty$. Finally, if $d_1 \geq d_2$, then $A(\alpha_m) \in R_{K(\alpha_m)}$ for $m > 0$, even if $c = 0$.

The preceding remarks can also be applied to the finitely many coefficients of a polynomial $S(T) \in R[[Z]]_m$ as in (2.1). We thereby obtain a uniform lower bound on the normalized valuations of the coefficients of $S(T)|_{Z=\alpha_m}$, independently of $m$. Moreover, it is easy to check whether these specialized coefficients are integral, and whether their valuations remain bounded as $m \to \infty$. We will refer to this argument (which yields information on the valuations of specializations) as the Ramification Argument in the sequel.

### 3. The Field of Norms

#### 3.1 Review of ramification theory

In this section we briefly recall the definitions and important properties of the upper and lower ramification filtrations. See [Ser62], Chapter IV for a complete treatment.

Let $K$ be a complete discretely valued field, and $L|K$ a finite Galois extension. Then we define a function $i_L : \text{Gal}(L|K) \to \mathbb{Z} \cup \{\infty\}$ by

$$i_L(\sigma) := \min_{x \in K} (\nu_L(\sigma(x) - x) - 1).$$

**Remark 3.1.** The function $i_L$ just defined differs slightly from the function $i_G$ defined by Serre in [Ser62]: $i_L = i_G - 1$ for $G = \text{Gal}(L|K)$. We prefer to use the function $i_L$ in order to match the notations of [Win83].

**Definition 3.2.** Let $L|K$ be a finite Galois extension with group $G$. Then for an integer $i \geq -1$,

$$G_i := \{\sigma \in G \mid i_L(\sigma) \geq i\}$$

is called the $i$th ramification subgroup in the lower numbering. We extend the indexing to the set of real numbers $\geq -1$ by setting $G_t := G_{[t]}$ for $t \in \mathbb{R}_{\geq -1}$.

The normal subgroups $G_i$ form a decreasing and separated filtration of $G$, with $G_{-1} = G$ and $G_0 = \text{the inertia subgroup of } G$.

**Definition 3.3.** The Herbrand function of $L|K$, $\varphi_{L|K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$, is given by

$$\varphi_{L|K}(t) := \int_0^t \frac{\#G_s}{\#G_0} ds.$$  

The function $\varphi_{L|K}$ is an increasing, continuous, piecewise-linear bijection, and hence has an increasing, continuous, and piecewise-linear inverse $\psi_{L|K}$. Using the inverse function $\psi_{L|K}$, we define a new ramification filtration on the Galois group $G$.

**Definition 3.4.** For a real number $s \geq -1$, define $G^s := G_{\psi_{L|K}(s)}$. The decreasing and separated filtration $\{G^s\}_s$ is called the ramification filtration in the upper numbering. Note that $G^{-1} = G_{-1} = G$ and $G^s = G_0$ for $-1 < s \leq 0$.

The importance of the upper ramification filtration stems from the fact that it behaves well under the formation of quotient groups:

**Proposition 3.5** ([Ser62], Chapter IV, Prop. 14). Let $H$ be a normal subgroup of $G = \text{Gal}(L|K)$, with fixed field $K'$. Then the upper ramification filtration on $G/H = \text{Gal}(K'|K)$ is induced by the upper ramification filtration of $G$:

$$(G/H)^s = G^s H/H \quad \forall s \in \mathbb{R}_{\geq -1}.$$
Now suppose that \( L|K \) is an infinite Galois extension. Then Proposition 3.3 allows us to define an upper ramification filtration on the profinite group \( G = \text{Gal}(L|K) \):

**Definition 3.6.** For a real number \( s \geq -1 \), define
\[
G^s := \lim_{\rightarrow} \text{Gal}(K'|K)^s
\]
where the limit is over all finite Galois subextensions \( K'|K \) of \( L|K \). The groups \( \{G^s\}_s \) form a decreasing, exhaustive, and separated filtration of \( G \) by closed normal subgroups, called the upper ramification filtration. We say that a real number \( r \geq -1 \) is a jump for the upper ramification filtration if \( G^{s+\varepsilon} \neq G^s \) for all \( \varepsilon > 0 \).

In particular, if \( K^{sep}|K \) is a separable closure of \( K \), then the absolute Galois group \( G_K := \text{Gal}(K^{sep}|K) \) is equipped with its upper ramification filtration \( \{G^s_K\}_s \).

**3.2 Arithmetically profinite extensions**

The field of norms construction applies to a certain type of field extension, which we now describe. The basic reference for this material is [Win83].

**Definition 3.7.** Let \( K \) be a complete discrete valuation field with perfect residue field \( k_K \) of characteristic \( p > 0 \), and \( K^{sep} \) a fixed separable closure. Then an extension \( L|K \) contained in \( K^{sep}|K \) is called arithmetically profinite (APF) if for all \( u \geq -1 \), the group \( G^u_K \cap G_L \) is open in \( G_K \).

If we set \( K^u := \text{Fix}(G^u_K) \subset K^{sep} \), then this definition means simply that \( L^u := K^u \cap L \) is a finite extension of \( K \) for all \( u \geq -1 \). Since the upper ramification filtration is separated, it follows that \( K^{sep} = \cup_u K^u \), which implies that \( L = \cup_u L^u \). A concrete example of an infinite APF extension is \( \mathbb{Q}_p(\zeta_{p^\infty})|\mathbb{Q}_p \), and in this case \( L^m = \mathbb{Q}_p(\zeta_{p^m}) \).

The APF property allows us to define an (inverse) Herbrand function: indeed, if \( L|K \) is a (possibly infinite) APF extension, then we set \( G^0_L := G^0_K \cap G_L \) and define
\[
\psi_{L|K}(u) := \begin{cases} 
\int_0^u (G^0_K : G^0_L G^t_K)dt & \text{if } u \geq 0 \\
0 & \text{if } -1 \leq u < 0.
\end{cases}
\]
Then \( \psi_{L|K} \) is increasing, continuous, and piecewise-linear, with inverse \( \varphi_{L|K} \) which is also increasing, continuous, and piecewise-linear. Of course, when \( L|K \) is finite, \( \varphi_{L|K} \) coincides with our previous definition of the Herbrand function.

An important quantity attached to an APF extension \( L|K \) is
\[
i(L|K) := \sup\{u \geq -1 \mid G^u_K G_L = G_K \}.
\]
In terms of the ramification subextensions \( L^u|K \), the quantity \( i(L|K) \) is the supremum of the indices \( u \) such that \( L^u = K \). In the case where \( L|K \) is Galois with group \( G = G_K|G_L \), we have \( G^u = G^u_K G_L/G_L \), and \( i(L|K) \) is the first jump in the upper ramification filtration on \( G \). Note that \( i(L|K) \geq 0 \) if and only if \( L|K \) is totally ramified, and \( i(L|K) > 0 \) if and only if \( L|K \) is totally wildly ramified. Because the inverse Herbrand function \( \psi_{L|K} \) and the quantity \( i(L|K) \) will be essential for our later work, we include here the

**Proposition 3.8** ([Win83], Proposition 1.2.3). Let \( M \) and \( N \) be two extensions of \( K \) contained in \( K^{sep} \) with \( M \subset N \). Then
\begin{itemize}
  \item[i)] if \( M|K \) is finite, then \( N|K \) is APF if and only if \( N|M \) is APF;
  \item[ii)] if \( N|M \) is finite, then \( N|K \) is APF if and only if \( M|K \) is APF;
  \item[iii)] if \( N|K \) is APF then \( M|K \) is APF;
\end{itemize}
if $N|K$ is APF, then $i(M|K) \geq i(N|K)$, and if in addition $M|K$ is finite, then $i(N|M) \geq \psi_{M|K}(i(N|K)) \geq i(N|K)$.

Parts i) and ii) of this proposition say that the APF property is insensitive to finite extensions of the top or bottom, while part iii) says that the APF property is inherited by subextensions. Part iv) says that the quantity $i(\cdot)$ can only increase in subextensions or under a finite extension of the base $M|K$. In the latter case, we get a lower bound on the increase of $i(\cdot)$ in terms of the inverse Herbrand function $\psi_{M|K}$.

Given an infinite APF extension $L|K$, let $\mathcal{E}_{L|K}$ denote the set of finite subextensions of $L|K$, partially ordered by inclusion. The key technical fact about the extension $L|K$ is the following property of the quantity $i(\cdot)$, which generalizes part iv) of the previous proposition:

**Proposition 3.11** ([Win83], Lemme 2.2.3.1). The numbers $i(L|E)$ for $E \in \mathcal{E}_{L|K}$ tend to $\infty$ with respect to the directed set $\mathcal{E}_{L|K}$.

### 3.3 The field of norms

Having discussed some general properties of infinite APF extensions, we are now ready to describe the field of norms construction, following [Win83]: given an infinite APF extension $L|K$, set

$$X_K(L^*) = \lim_{\mathcal{E}_{L|K}} E^*,$$

the transition maps being given by the norm $N_{E'|E} : E'^* \to E^*$ for $E \subset E'$. Then define $X_K(L) = X_K(L^*) \cup \{0\}$. Thus, a nonzero element $\alpha$ of $X_K(L)$ is given by a norm-compatible sequence $\alpha = (\alpha_E)_{E \in \mathcal{E}_{L|K}}$. We wish to endow this set with an additive structure in such a way that $X_K(L)$ becomes a field, called the field of norms of $L|K$. This is accomplished by the following

**Proposition 3.10** ([Win83], Théorème 2.1.3 (i)). If $\alpha, \beta \in X_K(L)$, then for all $E \in \mathcal{E}_{L|K}$, the elements $\{N_E(\alpha_E + \beta_E)\}_{E'}$ converge (with respect to the directed set $\mathcal{E}_{L|E}$) to an element $\gamma_E \in E$. Moreover, $\alpha + \beta := (\gamma_E)_{E \in \mathcal{E}_{L|K}}$ is an element of $X_K(L)$.

With this definition of addition, the set $X_K(L)$ becomes a field, with multiplicative group $X_K(L)^*$. Moreover, there is a natural discrete valuation on $X_K(L)$. Indeed, if $L^0$ denotes the maximal unramified subextension of $L|K$ (which is finite over $K$ by APF), then $\nu_{X_K(L)}(\alpha) := \nu_E(\alpha_E) \in \mathbb{Z}$ does not depend on $E \in \mathcal{E}_{L|L^0}$. In fact ([Win83], Théorème 2.1.3 (ii)), $X_K(L)$ is a complete discrete valuation field with residue field isomorphic to $k_L$ (which is a finite extension of $k_K$). The isomorphism of residue fields $k_{X_K(L)} \cong k_L$ comes about as follows. For $x \in k_L$, let $[x] \in L^0$ denote the Teichmüller lifting. That is, $[\cdot] : k_L \to L^0$ is the unique multiplicative section of the canonical map $R_{L^0} \to k_{L^0} = k_L$. Note that $E|L^1$ is of $p$-power degree for all $E \in \mathcal{E}_{L|L^1}$, so $x^{\frac{1}{[E:L^1]}} \in k_L$ for all such $E$, since $k_L$ is perfect. The element $([x^{\frac{1}{[E:L^1]}}])_{E \in \mathcal{E}_{L|L^1}}$ is clearly a coherent system of norms, hence (by cofinality) defines an element $f_{L|K}(x) \in X_K(L)$. The map $f_{L|K} : k_L \to X_K(L)$ is a field embedding which induces the isomorphism $k_L \cong k_{X_K(L)}$ mentioned above.

The following result will be used several times in the proof of our main result, Theorem 4.1.

**Definition 3.11.** For any subfield $F \in \mathcal{E}_{L|K}$, define $r(F) := \left[\frac{1}{p}i(L|F)\right]$.

**Proposition 3.12** ([Win83], Proposition 2.3.1 & Remarque 2.3.3.1). Let $L|K$ be an infinite APF extension and $F \in \mathcal{E}_{L|L^1}$ be any finite extension of $L^1$ contained in $L$. Then

i) for any $x \in R_F$, there exists $\hat{x} = (\hat{x}_E)_{E \in \mathcal{E}_{L|K}} \in X_K(L)$ such that $\nu_F(\hat{x}_F - x) \geq r(F)$.
ii) for any $\alpha, \beta \in R_{X_K(L)}$, we have

$$ (\alpha + \beta)_F \equiv \alpha_F + \beta_F \mod m^r_F. $$

The construction just described, which produces a complete discrete valuation field of characteristic $p = \text{char}(k_K)$ from an infinite APF extension $L/K$ is actually functorial in $L$. Precisely, $X_K(-)$ can be viewed as a functor from the category of infinite APF extensions of $K$ contained in $K^{sep}$ (where the morphisms are $K$-embeddings of finite degree) to the category of complete discretely valued fields of characteristic $p$ (where the morphisms are separable embeddings of finite degree). Moreover, this functor preserves Galois extensions and Galois groups.

Fixing an infinite APF extension $L/K$, the functorial nature of $X_K(-)$ allows us to define a field of norms for any separable algebraic extension $M|L$. Namely, given such an $M$, define the directed set $\mathcal{M} := \{L' \subset M \mid [L' : L] < \infty\}$, and note that

$$ M = \lim_{\mathcal{M}} L'. $$

Then we define the field of norms

$$ X_{L|K}(M) := \lim_{\mathcal{M}} X_K(L'). $$

With this definition, we can consider $X_{L|K}(-)$ as a functor from the category of separable algebraic extensions of $L$ to the category of separable algebraic extensions of $X_K(L)$.

**Proposition 3.13** ([Win83], Théorème 3.2.2). The field of norms functor $X_{L|K}(-)$ is an equivalence of categories.

In particular, $X_{L|K}(K^{sep})$ is a separable closure of $X_K(L)$, and we have an isomorphism $G_{X_K(L)} \cong G_L$.

Since $X_K(L)$ is a complete discrete valuation field with residue field $k_L$, it follows that any choice of uniformizer $\pi = (\pi_E)_E$ for $X_K(L)$ yields an isomorphism $k_L((z)) \cong X_K(L)$, defined by sending $z$ to $\pi$. Via this isomorphism, an element $\alpha = (\alpha_E)_E \in R_{X_K(L)}$ corresponds to a power series $g_\alpha(z) \in k_L[[z]]$. The following lemma describes the relationship between $g_\alpha(z)$ and the coherent system of norms $\alpha = (\alpha_E)_E$ in terms of the chosen uniformizer $\pi = (\pi_E)_E$. First we need to introduce some notation. Given a power series

$$ g(z) = \sum_{i=0}^{\infty} a_i z^i \in k_L[[z]], $$

define for each $E \in \mathcal{E}_{L|L^1}$ a new power series

$$ g_E(z) := \sum_{i=0}^{\infty} a_i [E:L^1] z^i = \sum_{i=0}^{\infty} (f_{L|K}(a_i))_E z^i \in R_E[[z]]. $$

**Lemma 3.14.** For all $\alpha = (\alpha_E)_E \in X_K(L)$, we have $\alpha_E \equiv g_\alpha, E(\pi_E) \mod m^r_E$ for all $E \in \mathcal{E}_{L|L^1}$.

**Proof.** By definition of the isomorphism $k_L((z)) \cong X_K(L)$, if $g_\alpha(z) = \sum_{i=0}^{\infty} a_i z^i$, then

$$ \alpha = \sum_{i=0}^{\infty} f_{L|K}(a_i) \pi^i = \lim_{n \to \infty} \sum_{i=0}^{n} f_{L|K}(a_i) \pi^i. $$

Now by Proposition 3.12, for any $E \in \mathcal{E}_{L|L^1}$ we have

$$ \left( \sum_{i=0}^{n} f_{L|K}(a_i) \pi^i \right)_E \equiv \sum_{i=0}^{n} (f_{L|K}(a_i))_E \pi^i \mod m^r_E. $$
Thus we see that

\[ \alpha_E \equiv \lim_{n \to \infty} \sum_{i=0}^{n} (f_{L|K}(a_i)) \pi_E^i = g_{\alpha,E}(\pi_E) \mod \mathfrak{m}_{E}^{\nu(E)}. \]

The congruence of the previous lemma can be replaced by an equality if one is willing to restrict attention to Lubin-Tate extensions of local fields. This is a theorem of Coleman ([Col79], Theorem A), to which we now turn.

### 3.4 Lubin-Tate extensions and Coleman’s Theorem

A special class of infinite APF extensions are the Lubin-Tate extensions, which we now briefly recall (see [LT65] and [Neu99] Chapter V). Let \( H \) be a finite extension of \( \mathbb{Q}_p \), and \( \Gamma \) a Lubin-Tate formal group associated to a uniformizer \( \varpi \) of \( H \). Then \( \Gamma \) is a formal \( R_H \)-module, and interpreting the group \( \Gamma \) in \( \mathfrak{m}_{\text{sep}} \) makes this ideal into an \( R_H \)-module (here \( \mathfrak{m}_{\text{sep}} \) is the maximal ideal of the valuation ring of \( H_{\text{sep}} \)). Let \( \Gamma_m \subset \mathfrak{m}_{\text{sep}} \) be the \( \varpi^m \)-torsion of this \( R_H \)-module. Then \( R_H/\varpi^m R_H \cong \Gamma_m \) for all \( m \), and in particular \( \Gamma_m \) is finite. Now define \( L_0 := \mathcal{O}_m H(\Gamma_m) \), which is an infinite totally ramified abelian extension of \( H \), the Lubin-Tate extension of \( H \) associated to \( \varpi \). Let \( K \) be a complete unramified extension of \( H \) with Frobenius element \( \phi \in \text{Gal}(K|H) \), and define \( L := L_0 K \). Then \( L|K \) is an infinite abelian APF extension, with ramification subfields \( L^m := \text{Fix}(G(L|K)^m) = K(\Gamma_m) \) ([Neu99], Corollary V.5.6). Moreover, we have \([L^m : K] = q^{m-1}(q-1)\), where \( q := \#(k_H) \). We will refer to extensions of this type as Lubin-Tate extensions, despite the fact that they are really the composition of a Lubin-Tate extension with an unramified extension.

Now fix a primitive element \((\omega_m)_m\) for the Tate module \( T_{\varpi}(\Gamma) := \lim_{\leftarrow} \Gamma_m \). That is, \( \omega_m \) is a generator for \( \Gamma_m \) as an \( R_H \)-module, and if \([\varpi]_\Gamma\) denotes the endomorphism of \( \Gamma \) corresponding to \( \varpi \), then \([\varpi]_\Gamma(\omega_{m+1}) = \omega_m \) for all \( m \geq 1 \). Note that the Frobenius \( \phi \in \text{Gal}(K|H) \) acts coefficient-wise on the ring \( R_K[[Z]][\frac{1}{Z}] \).

**Theorem 3.15 (Col79, Theorem A).** For all \( \alpha = (\alpha_m)_m \in X_K(L)^* \), there exists a unique \( f_\alpha(Z) \in R_K[[Z]][\frac{1}{Z}]^* \) such that for all \( m \geq 1 \),

\[ (\phi^{(m-1)} f_\alpha)(\omega_m) = \alpha_m. \]

Coleman’s Theorem has the following consequence in our situation (we omit the proof, since we will not make use of this result in what follows):

**Lemma 3.16.** Let \( L|K \) be a Lubin-Tate extension as described above, and suppose that \( \pi = (\pi_E)_E \) is a uniformizer for \( X_K(L) \) which is also a primitive element for the Tate module \( T_{\varpi}(\Gamma) \) (this is always possible if \( p \neq 2 \)). As described previously, \( \pi \) determines an isomorphism \( k_K[[z]] \to R_{X_K(L)} \) by sending \( z \) to \( \pi \). Consider a power series \( g_\alpha(z) \in \mathbb{F}_q[[z]] \) corresponding to \( \alpha \in R_{X_K(L)}^* \) under the isomorphism above, where \( q = \#(k_H) \). Then

\[ f_\alpha(Z) \equiv g_\alpha(z) \mod (\varpi) \]

where \( f_\alpha(Z) \in R_K[[Z]] \) is the Coleman power series for \( \alpha \) coming from Theorem 3.15.

Hence a choice of uniformizer (that is also a primitive element) defines a lifting of the multiplicative monoid \((\cup_{l|K,H} \mathbb{F}_q[[z]]) - \{0\}\) to \( R_K[[Z]]^* \). Of course, if \( K|H \) is finite then the first monoid above is simply \( k_K[[z]] - \{0\} \). Note that the multiplicity of this lifting is guaranteed by the uniqueness in Coleman’s Theorem. We extend this to a lifting \( C : (\cup_{l|K,H} \mathbb{F}_q[[z]]) \to R_K[[Z]] \) by setting \( C(0) := 0 \), and we note that this Coleman lifting provides an alternative to the more obvious Teichmüller lifting (coefficient-wise) of power series. The use of \( C \) allows for some simplification in
the proof of Theorem 4.1 in the case $p \neq 2$ and $\pi$ a primitive element. However, Theorem 4.1 holds for $p = 2$ and also for an arbitrary choice of uniformizer $\pi$, and the proof in the general case given below has recourse to the usual Teichmüller lifting, $\tau$.

3.5 Connection with the open $p$-adic disc

Given a totally ramified infinite APF extension $L|K$, we have seen how any choice of a uniformizer $\pi = (\pi_E)_E \in X_K(L)$ determines an isomorphism $k((z)) \cong X_K(L)$ defined by sending $z$ to $\pi$ (here we set $k := k_K = k_L$). We would now like to explicitly describe a connection between the field of norms $X_K(L)$ and the open $p$-adic disc $D_K := \text{Spec}(R[[Z]] \otimes K)$ that will underly the rest of our investigation (here $R := R_K$). Namely, the special fiber of the smooth integral model $D := \text{Spec}(R[[Z]])$ is $D_k = \text{Spec}(k[[z]])$, with generic point $D_{k,\eta} = \text{Spec}(k((z)))$. Via the isomorphism above coming from the choice of uniformizer $\pi$, we can thus identify $D_{k,\eta}$ with $\text{Spec}(X_K(L))$. On the other hand, each component $\pi_E$ of $\pi$ is a uniformizer in $E$, and in particular has absolute value $|\pi_E|_K < 1$. Hence, each $\pi_E$ corresponds to a point $x^E \in D_K$ with residue field $E$. In terms of the Dedekind domain $R[[Z]] \otimes K$, the point $x^E$ corresponds to the maximal ideal $\mathcal{P}_E$ generated by the minimal polynomial of $\pi_E$ over $R$. Thus, the uniformizer $\pi$ defines a net of points $\{x^E\}_E \subset D_K$ which approaches the boundary. In summary, we have the following picture:

\[
\begin{array}{ccc}
D_{k,\eta} & \longrightarrow & \text{Spec}(X_K(L)) & \longrightarrow & D_K \\
((z)) & \longrightarrow & X_K(L) & \longrightarrow & R[[Z]] \otimes K \\
z & \longrightarrow & \pi = (\pi_E)_E & \sim & \{x^E\}_E
\end{array}
\]

4. The Main Theorem

Let $L|K$ be a Lubin-Tate extension as described in section 3.3 with residue field $k := k_K = k_L$. Hence, there exists a $p$-adic local field $H$ such that $K|H$ is unramified and $L = KL_0$, where $L_0|H$ is an honest Lubin-Tate extension, associated to a formal group $\Gamma$. As usual, we let $L^m := \text{Fix}(G(L/K)^m)$, and we recall that $[L^m:L^1] = \#(k_H)^m - 1 = q^m - 1$. Choose a uniformizer $\pi = (\pi_E)_E \in X_K(L)$, which yields the identification $D_{k,\eta} = \text{Spec}(X_K(L))$ as well as the net of points $\{x^E\}_E \subset D_K$ as described in the last section.

Consider a $G$-Galois regular branched cover $Y \rightarrow D$, with $Y$ normal. We consider this cover to be a family over $\text{Spec}(R_K)$, and we introduce the following notations:

- $Y_k \rightarrow D_k$ denotes the special fiber of the cover, obtained by taking the fibered product with $\text{Spec}(k)$;
- $Y_K \rightarrow D_K$ denotes the generic fiber, obtained by taking the fibered product with $\text{Spec}(K)$;
- for each $E \in \mathcal{E}_{L|K}$, we denote by $Y_E$ the fiber of $Y_K$ at $x^E \in D_K$;
- if $X$ is an affine scheme, then $F(X)$ denotes the total ring of fractions of $X$, obtained from the ring of global sections, $\Gamma(X)$, by inverting all non-zero-divisors;
- if the special fiber $Y_k$ is reduced, then $F(Y_k) \cong \prod_{j=1}^{n_k} K$ is a product of $n_k$ copies of a field $K$, which is a finite normal extension of $k((z)) = X_K(L)$;
- since only finitely many of the points $x^E$ are ramified in the cover $Y_K \rightarrow D_K$, for $E$ large the fiber $Y_E$ is reduced and we have an isomorphism $F(Y_E) \cong \prod_{j=1}^{n_E} E'$, where $E'|E$ is a finite Galois extension of fields;
- $d_E := \nu_E(D(E'|E))$ denotes the degree of the different of $E'|E$;
- $L_E := LE'$ denotes the compositum of the fields $L$ and $E'$ in $K^{\text{sep}}.$
Galois covers of the open p-adic disc

Theorem 4.1. Let $Y \rightarrow D$ be a $G$-Galois regular branched cover of the open p-adic disc, with $Y$ normal and $Y_k$ reduced. Then

i) If $Y_k \rightarrow D_k$ is generically separable, then there exists a cofinal set $\mathcal{C}_Y \subset \mathcal{E}_{L|K}$ such that for $E \in \mathcal{C}_Y$ large, we have $n_E = n_s$ and $\mathcal{K} = X_{L|K}(L_E)$ as subfields of $X_K(L^{sep}) = X_{L|K}(K^{sep})$. Moreover, for these $E$, the functor $X_{L|K}(-)$ induces an isomorphism $\text{Gal}(\mathcal{K}|X_K(L)) \cong \text{Gal}(E'|E)$ which respects the ramification filtrations. In particular, if $d_s$ is the degree of the different of $\mathcal{K}|X_K(L)$, then $d_s = d_E$.

ii) If $Y_k$ is irreducible, then $Y_k \rightarrow D_k$ is generically inseparable if and only if $d_E \rightarrow \infty$.

iii) If $G$ is abelian, then $n_s \leq n_E$ for $E$ large, independently of any separability assumption. In particular, $Y_k$ is irreducible if $Y_E$ is irreducible for $E$ large.

Remark 4.2. If $k$ is a finite field, say $\#(k) = q^t$, then we can take the cofinal set $\mathcal{C}_Y$ in i) of the Theorem to be $\{L^m \mid m \equiv 1 \mod t\}$. That is, in the case of a finite residue field, $\mathcal{C}_Y$ is independent of the particular cover $Y \rightarrow D$.

Remark 4.3. The knowledgeable reader will note that much of our proof of i) is inspired by the proof in [Win83] of the essential surjectivity statement in Proposition 3.13. The main difficulty is to spread the construction of [Win83] over the open p-adic disc.

Proof of Theorem 4.1. Let $Y = \text{Spec}(\mathcal{A})$, so that $\mathcal{A}|R[[Z]]$ is a $G$-Galois extension of normal rings (here $R = R_K$). We are assuming that $\mathcal{A}_s := \mathcal{A}/\varpi \mathcal{A}$ is reduced, where $\varpi$ is a uniformizer of $R$. Moreover, we have $(\mathcal{A} \otimes K)/\mathcal{P}_E(\mathcal{A} \otimes K) = \prod_{j=1}^{n_E} E'$ for $E$ large (here $\mathcal{P}_E$ is the maximal ideal of $R[[Z]] \otimes K$ corresponding to $x^E$).

i): Suppose that $Y_k \rightarrow D_k$ is generically separable, which means that the field extension $\mathcal{K}|k((z))$ is separable, hence Galois. By the Primitive Element Theorem, there exists $\chi \in \mathcal{K}$ such that $\mathcal{K} = k((z))[\chi]$. Moreover, we can choose $\chi$ to be integral over $k[[z]]$, say with minimal polynomial $f(T) \in k[[z]][T]$. Further, since $k((z))$ is infinite, we can choose $n_s$ different primitive elements $\chi_j \in \mathcal{K}$ such that the corresponding minimal polynomials $f_j(T) \in k[[z]][T]$ are distinct. Even more, by Krasner’s Lemma, we may assume that each $f_j(T) \in k[z][T]$, so that in fact $f_j(T) \in \mathbb{F}_q[T]$ for some $l > 0$.

Having fixed this $l$, we take $\mathcal{C}_Y = \{L^m \mid m \equiv 1 \mod l\}$.

Setting $f(T) := \prod_{j=1}^{n_s} f_j(T)$, the Chinese Remainder Theorem implies that we have an isomorphism

$$
k((z))[T]/(f(T)) \cong \prod_{j=1}^{n_s} k((z))[\chi_j] = \prod_{j=1}^{n_s} \mathcal{K} \cong F(Y_k)
$$

Let $\overline{\chi}$ be the element of $F(Y_k)$ corresponding to $\overline{T}$ under this isomorphism, and choose a lifting, $\xi$, of $\overline{\chi}$ to $\mathcal{A}_{(\varpi)}$. Denote the minimal polynomial of $\xi$ over $R[[Z]](\varpi)$ by $F(T)$, so that $\overline{F}(T) = f(T)$. Then $F(T)$ has the form

$$
F(T) = T^N + A_{N-1}(Z)T^{N-1} + \cdots + A_0(Z) \in R[[Z]](\varpi)[T].
$$

Now by the Weierstrass Preparation Theorem, the coefficients of $F(T)$ have the form

$$
A_i(Z) = \frac{g(Z)}{Z^n + a_{n-1}Z^{n-1} + \cdots + a_0},
$$

where $g(Z) \in R[[Z]]$ and the denominator is a distinguished polynomial. Moreover, because $\overline{F}(T) = f(T) \in k[[z]][T]$, it follows that each $\overline{A}_i(z) \in k[[z]]$, which implies that either $\varpi | g(Z)$ in $R[[Z]]$ (in which case $\overline{A}_i(z) = 0$), or the Weierstrass degree of $g(Z)$ is greater than $n$ (the degree of the denominator).

Now by the Weierstrass Argument described in section 2.1 for $E$ large we can specialize the polynomial $F(T)$ at the point $Z = \pi_E$ to obtain the polynomial $F_E(T) \in E[T]$. 

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Lemma 4.4. For \( E \) large, the specialized polynomial \( F_E(T) \) lies in \( R_E[T] \), where \( R_E \) is the valuation ring of \( E \).

Proof. This follows immediately from the previous remarks and the Ramification Argument applied to \( S(T) = F(T) \) in the notation of section 2.2.

Since \( f(T) \in k[[z]][T] = R_{X_k(L)}[T] \) is separable, we have \( \text{disc}(f) = (\text{disc}(f))_E \neq 0 \) in \( X_k(L) \). Since \( r(E) := \left[ \frac{p-1}{p} i(L/E) \right] \), we know by Proposition 3.9 that \( \lim_{E \in E_k} r(E) = \infty \), so there exists \( E_0 \) such that for \( E \geq E_0 \) we have

\[
r(E) \geq r(E_0) > \nu_{X_k(L)}(\text{disc}(f)) := \nu_E(\text{disc}(f)).
\]

Now \( f(T) = T^N + \alpha_{N-1} T^{N-1} + \cdots + \alpha_0 \in R_{X_k(L)}[T] \).

Lemma 4.5. For \( E \in C_Y \) large, we have \( \nu_{X_k(L)}(\text{disc}(f)) = \nu_E(\text{disc}(F_E)). \)

Proof. Let \( G(T) \in R[Z][T] \) be the Teichmüller lifting of \( f(T) \):

\[
G(T) := \tau(f)(T) = T^N + \tau(\alpha_{N-1})(Z) T^{N-1} + \cdots + \tau(\alpha_0)(Z).
\]

Then \( G \) and \( F \) both reduce mod \( \varpi \) to \( f \), hence \( F(T) = G(T) + \varpi g(Z, T) \) for some \( g(Z, T) \in R[[Z]][\varpi][T] \). Specializing at \( Z = \pi_E \) for \( E \in C_Y \) now yields the equation

\[
F_E(T) = f_{E}(T) + \pi_E r(E) h_E(T) + \varpi g(\pi_E, T)
\]

for some \( h_E(T) \in R_E[T] \). Indeed, by Lemma 3.10 we have \( \overline{A}_i(E)(\pi_E) \equiv \alpha_i E \mod m^{r(E)}_E \). But \( [E : L] = q^{r(L)} = (q^i)^{r(E)} \) for \( E \in C_Y \). The operation of raising to the \( q^i \)-th power on \( \overline{F}_{q^i} \) is the identity, and since the coefficients of \( A_i(z) \) lie in \( \mathbb{F}_{q^i} \), it follows that \( \overline{A}_i(E)(Z) = \tau(\overline{A}_i)(Z) \). Hence \( \tau(\overline{A}_i)(\pi_E) \equiv \alpha_i E \mod m^{r(E)}_E \), from which equation (4.1) follows immediately.

Consider

\[
\nu_{E}(\text{disc}(f_E) - \text{disc}(F_E)) = \nu_E(\text{disc}(f_E) - \text{disc}(f_E) + \text{disc}(f_E) - \text{disc}(F_E)).
\]

Now by Proposition 3.12 \( \nu_E(\text{disc}(f_E) - \text{disc}(f_E)) \geq r(E) \). Moreover, for \( E \) large we have

\[
\nu_{E}(\text{disc}(f_E) - \text{disc}(f_E) + \text{disc}(f_E, \pi_E, T)) \geq r(E_0).
\]

Indeed, the Ramification Argument (section 2.2) applied to \( S(T) = \varpi g(Z, T) \) shows that the valuations of the coefficients of \( \varpi g(\pi_E, T) \) go to infinity as \( E \) gets large, and we know that \( r(E) \to \infty \).

It follows that \( \nu_{E}(\text{disc}(f_E) - \text{disc}(F_E)) \geq \min\{r(E), r(E_0) = r(E_0) \} \) for \( E \in C_Y \) large, which implies by (4.1) that \( \nu_{X_k(L)}(\text{disc}(f)) := \nu_{E}(\text{disc}(f_E) = \nu_E(\text{disc}(F_E)) \) for \( E \in C_Y \) large.

Now let \( y^E \) be a root of \( F_E(T) \) in \( K^{\text{sep}} \), and define \( \tilde{E} := E(y^E), \tilde{L}_E := L \tilde{E} \). Then \( \tilde{L}_E[\tilde{E}] \) is APF by Proposition 3.8 and we set \( r(\tilde{E}) := \left[ \frac{p-1}{p} i(\tilde{L}_E/\tilde{E}) \right] \). Moreover, note that for \( E \) large we have \( E' = \tilde{E} \) and thus \( L_E := L' \tilde{E} \) (recall that \( F(Y_E) \cong \prod_{i=1}^{n_E} E' \)). This follows from the fact that there exists \( g \in R[[Z]] \) such that \( \xi \in (\mathcal{A} \otimes K)g \) (for example, take \( g \) to be the product of the
denominators of the coefficients $A_i(Z)$ of $F(T) \in R[[Z]](\varpi)[T])$. Then the conductor of the subring $(R[[Z]] \otimes K)_{\varpi} \subset (A \otimes K)_{\varpi}$ defines a closed subset of $Y_K$, and if $x^E$ lies outside the image of this set in $D_K$, then the splitting of $F(T) \mod \mathcal{P}_E$ determines the fiber $Y_E$ (see [Neu99, Prop. I.8.3]). In particular, $L_E = L_T$ is Galois over $L$.

At this point we introduce the following lemma from [Win83], which it is easy to check holds in our situation, since it is a direct consequence of Lemma [4.5].

**Lemma 4.6** ([Win83], Lemme 3.2.5.4). For $E \subset C_Y$ large, the extensions $L|E$ and $\tilde{E}|E$ are linearly disjoint. Moreover, we have

$$i(\tilde{L}_E|\tilde{E}) = \psi_{E|L}(i(L|E)) \geq i(L|E).$$

Since for $E \subset C_Y$ large, $L|E$ is totally wildly ramified, it follows from this lemma that $i(\tilde{L}_E|\tilde{E}) \geq i(L|E) > 0$, so $\tilde{L}_E|\tilde{E}$ is totally wildly ramified. Hence, Proposition 3.12 says that there exists $\tilde{y}^E = (\tilde{y}^E)_{E} \in X_K(\tilde{L}_E)$ such that $\nu_E(\tilde{y}^E - y^E) \geq r(\tilde{E})$. Our immediate goal is to prove the following lemma about the polynomial $f(T) \in R_{X_K(L)}[T]$.

**Lemma 4.7.** $\lim_{E \in C_Y} f(\tilde{y}^E) = 0$.

**Proof.** Note that $\nu_{X_K(L)}(f(\tilde{y}^E)) \geq \frac{1}{\deg(f)} \nu_{X_K(\tilde{L}_E)}(f(\tilde{y}^E))$. Moreover, as mentioned above, $\tilde{L}_E|\tilde{E}$ is totally ramified, hence

$$\nu_{X_K(\tilde{L}_E)}(f(\tilde{y}^E)) = \nu_{\tilde{E}}(f(\tilde{y}^E)).$$

Denote by $f_{\tilde{E}} \in \tilde{E}[T]$ the polynomial obtained by replacing each coefficient of $f \in X_K(L)[T] \subset X_K(\tilde{L}_E)[T]$ by its component in $\tilde{E}$. Then by the linear disjointness of $L|E$ and $\tilde{E}|E$, it follows that $f_E = f_{\tilde{E}}$ and we have

$$\nu_{\tilde{E}}(f(\tilde{y}^E) - F_E(\tilde{y}^E)) = \nu_{\tilde{E}}(f(\tilde{y}^E) - f_{\tilde{E}}(\tilde{y}^E) + f_{\tilde{E}}(\tilde{y}^E)) = \nu_{\tilde{E}}(f(\tilde{y}^E) - f_{\tilde{E}}(\tilde{y}^E) + f_{\tilde{E}}(\tilde{y}^E)).$$

By Proposition 3.12 we have

$$\nu_{\tilde{E}}(f(\tilde{y}^E) - f_{\tilde{E}}(\tilde{y}^E)) \geq r(\tilde{E}).$$

On the other hand, by the Ramification Argument (again applied to $S(T) = \varpi g(Z,T)$) and equation 4.1,

$$\nu_{\tilde{E}}(f_{\tilde{E}}(\tilde{y}^E) - F_E(\tilde{y}^E)) = \nu_{\tilde{E}}(f_{\tilde{E}}(\tilde{y}^E) - f_{\tilde{E}}(\tilde{y}^E) + \pi_{\tilde{E}} h_E(\tilde{y}^E) + \varpi g(\pi_{\tilde{E}}, \tilde{y}^E))$$

$$\geq \min\{r(E), \nu_{\tilde{E}}(\varpi) - B\} = \min\{r(E), e(E|K) - B\},$$

where $B$ is the maximal degree of the denominators in the coefficients of $g \in R[[Z]]_{m}[T]$. Thus, we see that $\nu_{\tilde{E}}(f(\tilde{y}^E) - F_E(\tilde{y}^E)) \geq \min\{r(E), e(E|K) - B\}$, since $r(\tilde{E}) \geq r(E)$. Together with the fact that $\nu_{\tilde{E}}(\tilde{y}^E - y^E) \geq r(\tilde{E})$, this implies that

$$\nu_{\tilde{E}}(f(\tilde{y}^E)) = \nu_{\tilde{E}}(f(\tilde{y}^E) - F_E(\tilde{y}^E)) \geq \min\{r(E), e(E|K) - B\}.$$ 

Thus we have shown that

$$\nu_{X_K(L)}(f(\tilde{y}^E)) \geq \frac{1}{\deg(f)} \nu_{X_K(\tilde{L}_E)}(f(\tilde{y}^E)) \geq \frac{1}{\deg(f)} \min\{r(E), e(E|K) - B\}.$$ 

But $r(E) \to \infty$ for $E$ large, and since $B$ is a constant, we also have $e(E|K) - B \to \infty$. It follows that $\nu_{X_K(L)}(f(\tilde{y}^E)) \to \infty$ so that $\lim_{E \in C_Y} f(\tilde{y}^E) = 0$ as claimed. \qed
Proposition 3.12, choose an element 

\[ \tilde{\chi} \]

Proof. Given \( \sigma \in \text{Gal}(\tilde{L}_E|L) \), suppose that \( y \in \tilde{E} \) is such that \( \nu_{\tilde{E}}(\sigma(y) - y) < r(\tilde{E}) \). Using Proposition 3.12, choose an element \( \tilde{y} \in X_K(\tilde{L}_E) \) such that \( \nu_{\tilde{E}}(\tilde{y}_E - y) \geq r(\tilde{E}) \). Then

\[ \nu_{X_K(\tilde{L}_E)}(X_{L|K}(\sigma)(\tilde{y}) - \tilde{y}) = \nu_{E}(\sigma(y) - y). \]

Lemma 4.8. Given \( \sigma \in \text{Gal}(\tilde{L}_E|L) \), we wish to apply this lemma with \( \tilde{y} \), and \( \tilde{y} \) are conjugate to one of the roots \( \chi_j \) from the beginning of this proof, and since \( K|k((z)) \) is Galois, we have that \( K = k((z))(\chi_j) = k((z))(\tilde{\chi}) \). Moreover, by Krasner’s Lemma, \( \tilde{\chi} \in X_K(L)(\tilde{y}^E) \subset X_K(L_E) \) for \( E \in C_Y \) large. This implies that \( K \subset X_K(L_E) \) for \( E \in C_Y \) large, and I claim that this inclusion is actually an equality. For this we need a simple preliminary lemma.

Note that if \( \sigma \in \text{Gal}(\tilde{L}_E|L) \), then \( X_{L|K}(\sigma) \in \text{Gal}(X_K(\tilde{L}_E)|X_K(L)) \) and we have by definition

\[ X_{L|K}(\sigma) \tilde{y} = (\sigma(y_B))_{B \in E_{L|K}} \quad \forall \tilde{y} \in X_K(\tilde{L}_E). \]

\[ \text{Lemma 4.8. } \]

Replacing the net \( \{ \tilde{y}^E \}_E \) by a subnet, we may assume that it converges to a root \( \tilde{\chi} \) of \( f \). But then \( \tilde{\chi} \) is conjugate to one of the roots \( \chi_j \) from the beginning of this proof, and since \( K|k((z)) \) is Galois, we have that \( K = k((z))(\chi_j) = k((z))(\tilde{\chi}) \). Moreover, by Krasner’s Lemma, \( \tilde{\chi} \in X_K(L)(\tilde{y}^E) \subset X_K(L_E) \) for \( E \in C_Y \) large. This implies that \( K \subset X_K(L_E) \) for \( E \in C_Y \) large, and I claim that this inclusion is actually an equality. For this we need a simple preliminary lemma.

We wish to apply this lemma with \( y = y^E \) and \( \tilde{y} = \tilde{y}^E \), so we compute

\[ \nu_{E}(\sigma(y^E) - y^E) \leq (\deg F)\nu_{E}(\text{disc}(F_E)) \leq (\deg F)\nu_{X_K(L)}(\text{disc}(f)) \quad (\dagger) \]

for \( E \) large by Lemma 4.5. Since \( r(\tilde{E}) \to \infty \), it follows that \( y^E \) satisfies the hypothesis of Lemma 4.8 for \( E \) large, and we conclude that

\[ \nu_{X_K(\tilde{L}_E)}(X_{L|K}(\sigma)(\tilde{y}^E) - \tilde{y}^E) = \nu_{E}(\sigma(y^E) - y^E). \]

This immediately implies that \( X_K(L)(\tilde{y}^E) = X_K(L_E) \), because if the inclusion were proper, then there would exist \( \sigma \neq 1 \) in \( \text{Gal}(\tilde{L}_E|L) \) such that \( X_{L|K}(\sigma)(\tilde{y}^E) = \tilde{y}^E \), which is a contradiction since \( \sigma(y^E) \neq y^E \).

Thus, in order to show that \( K = X_K(L_E) \), we just need to show that \( X_K(L)(\tilde{y}^E) \subset X_K(L)(\tilde{\chi}) \). But the net \( \{ \tilde{y}^E \} \) converges to \( \tilde{\chi} \), and \( (\dagger) \) shows that the Krasner radii

\[ \max\{\nu_{X_K(L)}(X_{L|K}(\sigma)(\tilde{y}^E) - \tilde{y}^E) \mid \sigma \in G(\tilde{L}_E|L), \sigma \neq 1\} < C \]

for some constant \( C \) independent of \( E \). Hence for \( E \) sufficiently large so that \( \nu_{X_K(L)}(\tilde{x} - \tilde{y}^E) > C \), Krasner’s lemma tells us that \( X_K(L)(\tilde{x} - \tilde{y}^E) \subset X_K(L)(\tilde{\chi}) \) as required.

Thus, we have shown that \( K = X_{L|K}(\tilde{L}_E) = X_{L|K}(L_E) \) for \( E \in C_Y \) large. It now follows from the fundamental equality that \( n_s = n_E \):

\[ n_s = \frac{\deg f}{[K : k((z))]}, \quad \frac{\deg F}{[L_E : L]} = \frac{\deg F}{[E' : E]} = n_E. \]

It remains to prove the statement about the Galois groups. By the general theory of the field of norms, we have

\[ \text{Gal}(L_E|L) \cong \text{Gal}(X_K(L_E)|X_K(L)) = \text{Gal}(K|X_K(L)). \]

Moreover, since \( L_E = L_E' \) and \( L_E \) and \( E'|E \) are linearly disjoint, it follows that

\[ \text{Gal}(L_E|L) = \text{Gal}(L_E'|L) \cong \text{Gal}(E'|E' \cap L) = \text{Gal}(E'|E). \]

Thus, we just need to show that the ramification filtrations are preserved under these isomorphisms.

First note that for all \( E, B \in C_Y \) sufficiently large, we have \( L_E = L_B \), since by the preceding proof we have that \( X_K(L_E) = K = X_K(L_B) \) and \( X_{L|K}(-) \) is an isomorphism. Denote this common field by \( L' \).

Lemma 4.9 (compare [Win83], Proposition 3.3.2). For \( \sigma \in \text{Gal}(L'|L) \) and \( E \) large, we have \( i_{E'}(\sigma) = i_{X_K(L')}(X_{L|K}(\sigma)). \)

Since the lower ramification filtration is determined by the function \( i \), it follows that the isomorphism \( \text{Gal}(E'|E) \cong \text{Gal}(L'|L) \cong \text{Gal}(K|X_K(L)) \) induced by \( X_{L|K}(-) \) preserves the ramification filtrations. Since the degree of the different depends only on the ramification filtration, it follows
that $d_s = d_E$ for $E$ large. This completes the proof of i).

ii): Note that by part i), if $d_E \to \infty$, then the special fiber must be generically inseparable, without any irreducibility hypothesis.

Now suppose that $Y_k$ is irreducible and $Y_k \to D_k$ is generically inseparable. Let $V$ be the first ramification group at the unique prime of $\mathcal{A}$ lying over $(\varpi)$. Taking $V$-invariants, we obtain the tower $Y \to Y^V \to D$. Since $V$ is a nontrivial $p$-group, it has a $p$-cyclic quotient. Hence $Y \to Y^V$ has a $p$-cyclic subcover $W \to Y^V$, and we have the tower $Y \to W \to Y^V \to D$. Now consider the associated tower of special fibers $Y_k \to W_k \to Y^V_k \to D_k$, which corresponds (by considering the generic points) to a chain of field extensions $k((z)) \subset k((s)) \subset k((x)) \subset K$. Here the extension $k((x))|k((s))$ is purely inseparable of degree $p$, defined by $x^p = s$. Note that there is no extension of constants in this tower because the cover $Y \to D$ was assumed to be regular.

The extension $k((s))|k((z))$ is separable and totally ramified, so the minimal polynomial of $s$ over $k((z))$ is Eisenstein:

$$g(T) = T^c + za_d-1(z)T^{c-1} + \cdots + za_1(z)T + zu(z) \in k[[z]][T],$$

where $u(z)$ is a unit. It follows that $g(T_p)$ is the minimal polynomial of $x$ over $k((z))$. Now let $\xi$ be a lifting of $x$ to the localized ring of global sections $\Gamma(W)(\varpi)$. Then $\xi$ is integral over $\mathcal{A}(\varpi)$ and we let $G(T) \in \mathcal{A}(\varpi)[T]$ be its minimal polynomial. Since $\deg(W|D) = \deg(k((x))|k((z))) = pc$ and $\xi$ is a lifting of $x$, it follows that the degree of $G(T)$ is also $pc$, and $G(T) \equiv g(T_p)$ modulo $\varpi$. Using the Teichmüller lifting $\tau : k[[z]] \to R[[Z]]$ we find that:

$$G(T) = \tau(g)(Z,T_p) + \varpi P(Z,T) = T^{pc} + Z\tau(a_d-1)(Z)T^{pc-1} + \cdots + Z\tau(u)(Z) + \varpi P(Z,T),$$

for some polynomial $P(Z,T) \in R[[Z]][\varpi][T]$ of degree at most $pc - 1$ in $T$.

Setting $Z = \pi_E$, we get the specialized polynomial

$$G_E(T) = T^{pc} + \pi_E\tau(a_d-1)(\pi_E)T^{pc-1} + \cdots + \pi_E\tau(u)(\pi_E) + \varpi P(\pi_E,T) \in E[T],$$

which for $E$ sufficiently large is Eisenstein by the Ramification Argument applied to $S(T) = \varpi P(Z,T)$. Letting $\xi_E$ denote the image of $\xi$ in $F(W_E)$, it follows by degree considerations that $W_E$ is irreducible for $E$ large and $\xi_E$ is a uniformizer for the field $F(W_E)$.

We obtain the chain of field extensions $E \subset E(\xi_E) = F(W_E) \subset E'$, and can compute the different as follows:

$$\mathcal{D}(E(\xi_E)|E) = (G'_E(\xi_E)) = (p\xi_E^{p-1}\tau(g)'(\pi_E,\xi_E^p) + \varpi P'(\pi_E,\xi_E)).$$

But

$$\nu_E(\xi_E)(p\xi_E^{p-1}\tau(g)'(\pi_E,\xi_E^p) + \varpi P'(\pi_E,\xi_E)) \geq \min\{\nu_E(\xi_E)(p\xi_E^{p-1}\tau(g)'(\pi_E,\xi_E^p)),\nu_E(\xi_E)(\varpi P'(\pi_E,\xi_E))\},$$

and the latter quantity goes to $\infty$ with $E$. By multiplicativity of the different in towers we conclude that

$$d_E \geq \nu_E(\xi_E)(\mathcal{D}(E(\xi_E)|E)),$$

so $d_E$ goes to $\infty$ with $E$ as claimed.

iii): We now assume that $G$ is abelian, but make no separability assumption on the special fiber $Y_k \to D_k$. Since $G$ is abelian, the decomposition groups at the $n_s$ primes of $\mathcal{A}(\varpi)$ lying over $(\varpi) \in \operatorname{Spec}(R[[Z]])$ all coincide. Call this decomposition group $Z$. Taking $Z$-invariants, we observe that
Proof. First suppose that $Y \to D$ is a finite cyclic extension of $L$. Using Theorem 4.1, we deduce a local lifting criterion for finite abelian extensions of $L$. Since $Y \to Z$ is surjective, it follows that $n_E \geq n_E^Z = n_s$ as claimed. This completes the proof of part iii), and hence of Theorem 4.1.

\section{5. Arithmetic Form of the Ring Specific Oort Conjecture}

Using Theorem 4.1 we deduce a local lifting criterion for finite abelian extensions of $\mathbb{F}_p((z))$. For this, consider a Lubin-Tate extension $L/K$ as described in section 3.3, with $K = H\mathbb{Q}^\text{un}_p$ for some finite extension $H/\mathbb{Q}_p$. Then choose a uniformizer $\pi = (\pi_E) \in X_K(L)$, which defines an isomorphism $\mathbb{F}_p((z)) \cong X_K(L)$ as well as a net of points $\{x_E\} \subset D_K$ (see section 3.5). We will be interested in the cofinal sequence of ramification subfields $\{L^n\} \subset D_{L|K}$, and will use the simplified notation $\pi_m := \pi_{L^n}, x_m = x^{L^n}, L_m = L_{L^n}$, etc.

\begin{proposition}
Suppose that $G$ is a finite abelian group, and let $M|L$ be a $G$-Galois extension, corresponding to the $G$-Galois extension $X_K(M)|X_K(L)$ via the field of norms functor. Suppose that $Y \to D$ is a $G$-Galois regular branched cover with $Y$ normal and $Y_k$ reduced. Then $Y \to D$ is a smooth lifting of $X_K(M)|X_K(L)$ if and only if there exists $l > 0$ such that for $m >> 0$ and $m \equiv 1$ mod $l$, we have $L_m = M$ as $G$-Galois extensions of $L$, and $d_m = d_\eta$.
\end{proposition}

\begin{proof}
First suppose that $Y \to D$ is a smooth lifting of $X_K(M)|X_K(L)$. Then $Y_k \to D_k$ is generically separable, so by the proof of part i) of Theorem 4.1 there exists $l > 0$ such that for $m >> 0$ and $m \equiv 1$ mod $l$, we have $X_K(M) = X_K(L_m)$ and $d_m = d_s$. Since $X_K(-)$ is an equivalence of categories, we conclude that $M = L_m$ for these values of $m$. Moreover, by the Local Criterion For Good Reduction (see Introduction), we have $d_s = d_\eta$, which implies that $d_m = d_\eta$ for $m >> 0$ and $m \equiv 1$ mod $l$, as claimed.

Now suppose that there exists $l > 0$ so that $L_m = M$ and $d_m = d_\eta$ for $m >> 0$ and $m \equiv 1$ mod $l$. Then by part iii) of Theorem 4.1 $Y_k$ is irreducible, and then by part ii), $Y_k \to D_k$ is generically separable. Hence we may apply part i) to conclude that there exists $l_1 > 0$ such that $F(Y_k) = X_K(L_{m1})$ and $d_s = d_m$ for $m >> 0$ and $m \equiv 1$ mod $l_1$. But the two arithmetic progressions $\{tl + 1\}$ and $\{tl_1 + 1\}$ have a common subsequence. It follows that $F(Y_k) = X_K(M)$ and $d_s = d_\eta$, so $Y \to D$ is a birational lifting of $X_K(M)|X_K(L)$ which preserves the different. By the Local Criterion For Good Reduction, it follows that $Y \to D$ is actually a smooth lifting.
\end{proof}

In particular, we obtain an arithmetic reformulation of the Ring Specific Oort Conjecture concerning the liftability of cyclic covers over $\mathbb{F}_p$. Set $K = \mathbb{Q}^\text{un}_p$, and let $L = K(\zeta_{p^n})$. Then $L|K$ is Lubin-Tate for $H = \mathbb{Q}_p$ and $\Gamma = \mathbb{G}^\text{un}_m$. Moreover, if $C$ is a finite cyclic group, define $R_C := R_K(\zeta_{|C|}) \subset R_L$. Then we have the following arithmetic form of the Ring Specific Oort Conjecture from the Introduction:

\textbf{Arithmetic Form of the Ring Specific Oort Conjecture (over $\mathbb{F}_p$). Suppose that $M|L$ is a finite cyclic extension of $L$, with group $C$. Then there exists $l > 0$ and a normal, $C$-Galois, regular branched cover $Y \to D := \text{Spec}(R_C[[Z]])$ such that}

\begin{itemize}
  \item[i)] $Y_k$ is reduced;
\end{itemize}
ii) $L_m = M$ for $m >> 0$ and $m \equiv 1 \mod l$;
iii) $d_\eta = d_m$ for $m >> 0$ and $m \equiv 1 \mod l$.

**Proposition 5.2.** The Arithmetic Form of the Ring Specific Oort Conjecture over $\mathbb{F}_p$ is equivalent to the Ring Specific Oort Conjecture over $\mathbb{F}_p$.

**Proof.** This follows immediately from Proposition 5.1.

**Remark 5.3.** It can be shown by standard techniques of model theory that the Ring Specific Oort Conjecture over $\mathbb{F}_p$ implies the Ring Specific Oort Conjecture over $k$, where $k$ is an arbitrary algebraically closed field of characteristic $p$.

**5.1 An application**

We conclude this paper by giving a direct proof of the Arithmetic Form of the Ring Specific Oort Conjecture for $p$-cyclic covers over $\mathbb{F}_p$. We begin by recalling some facts about $p$-cyclic extensions for which we omit the straightforward proofs. Recall that $K = \mathbb{Q}_p^m$, and $L = K(\zeta_p^\infty)$.

**Lemma 5.4.** Suppose that $\pi = (\pi_m)$ is a uniformizer for $X_K(L)$, so that $N_{L^{m+1}|L^m}(\pi_{m+1}) = \pi_m$. Let $p_m(T) = T^p + a_{m}T^{p-1} + \cdots + a_{p-1,m}T + (-1)^p \pi_m$ be the minimal polynomial of $\pi_{m+1}$ over $L^m$. Then for any $B > 0$, there exists $m_B >> 0$ such that if $m \geq m_B$, the coefficients of $p_m(T) \in R_m[T]$ satisfy $\nu_m(a_{i,m}) \geq B$ for $i = 1, \ldots, p - 1$.

We will also need a result about the stability of the ramification filtration under base change by subfields of an infinite APF extension $L/K$. This result is actually an immediate consequence of Lemma 4.9 which we used in the proof of Theorem 4.1.

**Lemma 5.5 (compare [Win83], Proposition 3.3.2).** Suppose that $\tilde{L}/L$ is a finite $G$-Galois extension, obtained by base change to $L$ from a $G$-Galois extension $\tilde{L}^{m_0}|L^{m_0}$. Let $\sigma \in G$ be an element of the Galois group, and for $m \geq m_0$ define $\tilde{L}^m := L^m\tilde{L}^{m_0}$. Then there exists $m_1 \geq m_0$ such that for $m \geq m_1$ we have $i_{\tilde{L}^m}(\sigma) = i_{X_K(L)}(X_K(\sigma))$. Consequently, the canonical isomorphism $G(\tilde{L}^m|L^m) \cong G(\tilde{L}^{m_1}|L^{m_1})$ preserves the ramification filtrations on these groups. In particular, the conductor of $\tilde{L}^m|L^m$ is equal to the conductor of $\tilde{L}^{m_1}|L^{m_1}$ for $m \geq m_1$.

Finally, we state the following lemma which gives a standard form for the Kummer equation defining a $p$-cyclic extension of $L$.

**Lemma 5.6.** Let $M|L$ be a $p$-cyclic field extension. Then there exists $m_0 > 0$ and a principal unit $u \in L^{m_0}$ such that $M = L\tilde{L}^{m_0}$, where $\tilde{L}^{m_0} = L^{m_0}(u^{\frac{1}{p}})$. Moreover, $m_0$ may be chosen so that the conductor of $\tilde{L}^{m_0}|L^{m_0}$ is stable under base change by $L^{m}|L^{m_0}$ (Lemma 5.5). Modifying $u$ by a $p$-power in $L^{m_0}$, we may assume that $u = 1 + \frac{\lambda}{\pi^{m_0}_{m_0}} v$, where $v$ is a unit in $L^{m_0}$, $\lambda = \zeta_p - 1$, and $c + 1$ is the Artin conductor of $\tilde{L}^{m_0}|L^{m_0}$.

**Proof.** See [Gra03] Proposition 1.6.3.

**Theorem 5.7.** The Arithmetic Form of the Ring Specific Oort Conjecture over $\mathbb{F}_p$ holds for $p$-cyclic covers.

**Proof.** Let $M|L$ be a $p$-cyclic extension, and choose $m_0$ as described in Lemma 5.6 so that $M = L\tilde{L}^{m_0} = LL^{m_0}(u^{\frac{1}{p}})$, where $u = 1 + \frac{\lambda}{\pi^{m_0}_{m_0}} v$. Now set $N := 2c$ and increase $m_0$ if necessary so that $\min\{d_m, \frac{e_m}{p}\} > N$. 

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By Proposition \[3.12\] there exists a unit $\hat{v} \in X_K(L)$ such that $\nu_{m_0}(\hat{v}L^{m_0} - v) \geq r_{m_0} > N$. Hence we can write

$$u = 1 + \frac{\lambda^p}{\pi_{m_0}^c}((\hat{v}L^{m_0} - (\hat{v}L^{m_0} - v)) = 1 + \frac{\lambda^p\hat{v}L^{m_0}}{\pi_{m_0}^c}w$$

where $w \in L^{m_0}$ is a unit of index at least $N$. Now $(c, p) = 1$, so by Hensel’s Lemma $\hat{v}$ has a $c$th root in $X_K(L)$. Then replace the uniformizer $\pi = (\pi_m)$ by $\pi\hat{v}^{-\frac{1}{c}}$. Again calling this uniformizer $\pi$, we find that $u = 1 + \frac{\lambda^p}{\pi_{m_0}}w$ where $w$ is a unit of index at least $N$, say $w = 1 + b_1\pi_{m_0}^N + b_2\pi_{m_0}^{N+1} + \cdots$, where the $b_i \in K$ are Teichmüller representatives. Now set $W = 1 + b_1Z^N + b_2Z^{N+1} + \cdots \in R[[Z]]^\times$ and consider the extension of normal rings $A | R_1[[Z]]$ defined generically by the Kummer equation $T^p = 1 + \frac{\lambda^p}{\pi_{m_0}^c}W$. By \[Epp73\], Proposition 1.4, the special fiber of this extension, $A_s = A/\lambda A$, is reduced. Hence, we can apply Theorem \[4.1\] to the $p$-cyclic cover $\text{Spec}(A) \to D$.

Setting $\tilde{T} = T Z^c$ yields the integral equation $\tilde{T}^p = Z^{cp} + \lambda^p Z^{(p-1)c}W = Z^{(p-1)c}(Z^c + W\lambda^p)$. The right hand side has $c + 1$ zeros, yielding $c + 1$ ramified points, each with ramification index $p$. It follows that the degree of the generic different is $d_{\eta} = (c + 1)(p - 1)$.

Specializing the equation at $Z = \pi_m$ yields $T^p = 1 + \frac{\lambda^p}{\pi_m^c}W(\pi_m)$, and so all of the specializations $L^{m_1}/L^m$ are field extensions with conductor $c + 1$, hence degree of different $d_m = (c + 1)(p - 1) = d_\eta$. Theorem \[4.1\] says the special fiber is separable and irreducible, and $d_\eta = d_s$. Hence, the lifting is smooth, and it remains to verify that $M = L^{m_1}L$ for $m > 0$. But at level $m_0$ we have $L^{m_0} = \tilde{L}^{m_0}$, hence $M = \tilde{L}^{m_0}L = L^{m_0}L$. For $m > m_0$, the extension $L^{m_1}/L^m$ is defined by adjoining a $p$th root of $1 + W(\pi_m)^{\lambda p}$, so by Kummer Theory it suffices to show that this element differs from $1 + W(\pi_m)^{\lambda p}$ by a $p$th power in $L$. The following Lemma thus completes the proof of the $p$-cyclic case of the Arithmetic Form of the Ring Specific Oort Conjecture over $\overline{\mathbb{F}}_p$. \[\square\]

**Lemma 5.8.** The following quantity is a $p$th power in $L$:

$$(1 + u_m\frac{\lambda^p}{\pi_m})(1 + u_{m+1}\frac{\lambda^p}{\pi_{m+1}})^{-1},$$

where $u_m := W(\pi_m)$ is a unit of index at least $N$ in $L^m$.

**Proof.** In fact, I claim that

$$(1 + u_m\frac{\lambda^p}{\pi_m})(1 + u_{m+1}\frac{\lambda^p}{\pi_{m+1}})^{-1}(1 - \frac{\lambda}{\pi_{m+1}})^p (\ast)$$

is a $p$th power in $L^{m+1}$. For simplicity, we will assume that $p \neq 2$ in the computation that follows (the case $p = 2$ is similar). Moreover, since any principal unit in $L^{m+1}$ of index greater than $D := \nu_{m+1}(\lambda^p)$ is a $p$th power in $L^{m+1}$, we will work in the quotient ring $\mathcal{R}_{m+1} := R_{m+1}/u_{m+1}D_{m+1}$ to simplify the computation.

By Lemma \[5.4\] we have $\pi_m = \pi_{m+1} + \sum_{i=1}^{p-1}a_i\pi_{m+1} = \pi_{m+1}(1 + \sum_{i=1}^{p-1}a_i\pi_{m+1})$, and we may assume $\nu_{m+1}(\frac{a_i}{\pi_{m+1}}) > pN$ for each $i$. Then since $u_m$ has index at least $N > c$ in $L^m$, we have the following congruence in $\mathcal{R}_{m+1}$:

$$(1 + u_m\frac{\lambda^p}{\pi_m}) \equiv 1 + \frac{\lambda^p}{\pi_{m+1}}.$$ 

Multiplying by a $p$th power and remembering that $\frac{p\nu_{m+1}}{p-1} > pN = 2pc$ yields

$$(1 + u_m\frac{\lambda^p}{\pi_m})(1 - \frac{\lambda}{\pi_{m+1}})^p \equiv (1 + \frac{\lambda^p}{\pi_{m+1}})^p(1 - \frac{\lambda^p}{\pi_{m+1}} - \frac{p\lambda}{\pi_{m+1}} + \sum_{i=2}^{p-1}(-1)^i \binom{p}{i}(\frac{p\lambda}{\pi_{m+1}})^i) \equiv 1 - \frac{p\lambda}{\pi_{m+1}}.$$
Finally, we compute
\[
(1 + u_{m+1} \frac{\lambda^p}{\pi_{m+1}})^{-1}(1 + u_m \frac{\lambda^p}{\pi_m})(1 - \frac{\lambda}{\pi_m})\equiv (1 - \frac{\lambda^p}{\pi_{m+1}})(1 - \frac{p\lambda}{\pi_{m+1}})
\equiv 1 - \frac{\lambda^p + p\lambda}{\pi_{m+1}}
\equiv 1.
\]

Indeed, \(0 = (\lambda + 1)^p - 1 = \lambda^p + p\lambda + p\lambda^2\alpha\) for some \(\alpha \in R_1\), and so
\[
\nu_{m+1}(\frac{\lambda^p + p\lambda}{\pi_{m+1}}) \geq \nu_{m+1}(p\lambda^2) - c = e_{m+1} + 2\frac{e_{m+1}}{p-1} - c = \nu_{m+1}(\lambda^p) + \frac{e_{m+1}}{p-1} - c > \nu_{m+1}(\lambda^p).
\]
Hence, the quantity (\(\ast\)) is a \(p\)th power in \(I^{m+1}\) as required.

\[\square\]

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