EXISTENCE AND ASYMPTOTIC STABILITY OF PERIODIC SOLUTIONS FOR NEUTRAL EVOLUTION EQUATIONS WITH DELAY

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Abstract. In this paper, we are devoted to consider the periodic problem for the neutral evolution equation with delay in Banach space. By using operator semigroups theory and fixed point theorem, we establish some new existence theorems of periodic mild solutions for the equation. In addition, with the aid of a new integral inequality with delay, we present essential conditions on the nonlinear function to guarantee that the equation has an asymptotically stable periodic mild solution.

1. Introduction. Let $X$ be a real Banach space with norm $\| \cdot \|$. The purpose of this paper is to discuss the existence and asymptotic stability of periodic mild solutions for the following abstract neutral evolution equation

$$(u(t) - G(t, u_t))' + Au(t) = F(t, u(t), u_t), \quad t \in \mathbb{R}, \quad (1.1)$$

where $A : D(A) \subset X \to X$ is a closed linear operator, and $-A$ generates a compact analytic operator semigroup $T(t)(t \geq 0)$ in $X$; $F : \mathbb{R} \times X \times \mathcal{B} \to X$ and $G : \mathbb{R} \times \mathcal{B} \to X$ are given appropriate continuous functions which are $\omega$-periodic in $t$. For $t \in \mathbb{R}$, we denote by $u_t$ the history function defined by $u_t(s) = u(t + s)$ for $s \in [-r, 0]$, where $u$ is a continuous function from $\mathbb{R}$ into $X$ and $r > 0$ is a constant. We denote also by $\mathcal{B} = C([-r, 0], X)$ the space of continuous functions from $[-r, 0]$ into $X$ provided with the uniform norm topology.

The theory of partial differential equations with delays has extensive physical background and realistic mathematical model, and it has undergone a rapid development in the last fifty years. The partial differential equations with delay are more realistic than the equations without delay in describing numerous phenomena observed in nature, hence the numerous properties of their solutions have been studied, see [14, 28] and references therein for more comments. During the last few decades, more researchers have given special attentions to the study of equations in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, so-called neutral differential equations.

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Partial neutral differential equations arise in many areas of applied mathematics. It can model a lot of problems arising from engineering, such as population dynamics, transmission line, immune response or distribution of albumin in the blood etc. For instance, in the theory of heat conduction in fading memory material, see [7, 25], the following partial neutral differential equation
\[
\frac{d}{dt} \left( u(t, x) + \int_{-\infty}^{t} k_1(t-s)u(s, x)ds \right) - c\Delta u(t, x) = \int_{-\infty}^{t} k_2(t-s)\Delta u(s, x)ds, \quad (1.2)
\]
has been frequently used to describe this phenomenon, which has better effects than partial differential equations without neutral item, where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with a sufficiently smooth boundary \( \partial \Omega \), \((t, x) \in [0, \infty) \times \Omega \) represents the temperature in \( x \) at the time \( t \), \( c \) is a physical constant, \( k_1, k_2 : \mathbb{R} \to \mathbb{R} \) are the internal energy and the heat flux relaxation respectively. If the solution \( u \) is known on \( (-\infty, 0] \), \( k_1 \equiv 0 \) on \( (r, \infty) \) and \( k_2 \equiv 0 \), then we can transform the system (1.2) into the abstract neutral evolution equation (1.1).

In fact, many of partial neutral differential equations can be written as first-order abstract neutral functional differential equations on an appropriate Banach space. There has been an increasing interest in the study of the abstract neutral evolution equations of form (1.1). The existence, uniqueness and stability of mild solutions to the abstract neutral evolution equations with delay have been considered by many authors in the literatures. Here we only mention [15, 2, 1, 17, 18, 10].

It is noteworthy at this point that the problem concerning periodic solutions of partial neutral functional differential equations has become an important area of investigation. The periodic problems can take into account seasonal fluctuations occurring in the phenomenon appearing in the models, and have been studied by some researchers in recent years. Specially, the existence of periodic solutions for the neutral evolution equations has been considered by several authors, see [8, 27, 16, 9, 3, 4, 12, 13, 5, 11] and references therein for more comments. We notice that, in many works, the key assumption of prior boundedness is employed and the most important ingredient to prove the existence of periodic solutions is to show that the Poincaré’s mapping
\[
P_\omega(\phi) = u_\omega(\cdot, \phi), \quad (1.3)
\]
is condensing, where \( \omega \) is a period of the system and \( u \) is the unique mild solution determined by \( \phi \). Thus, a fixed-point theorem can be used to derive periodic solutions.

For the delayed evolution equations without neutral item, the existence of periodic solutions has been discussed by more authors, see [6, 29, 24, 20, 21, 19, 22, 23] and references therein. It is worth noting that, in [20], Li discussed the existence and asymptotic stability of periodic solutions to the evolution equation with multiple delays in a Hilbert space \( H \)
\[
u'(t) + Au(t) = F(t, u(t), u(t-\tau_1), \cdots, u(t-\tau_n)), \quad t \in \mathbb{R}, \quad (1.4)
\]
where \( A : D(A) \subset H \to H \) is a positive definite selfadjoint operator, \( F : \mathbb{R} \times H^{n+1} \to H \) is a nonlinear mapping which is \( \omega \)-periodic in \( t \), and \( \tau_1, \tau_2, \cdots, \tau_n \) are positive constants which denote the time delays. By using periodic extension, Schauder fixed point theorem and the integral inequality with delays, the author obtained the essential conditions on the nonlinearity \( F \) to guarantee that the equation (1.4) has \( \omega \)-periodic solutions or an asymptotically stable \( \omega \)-periodic solution.
Although there have been many meaningful results on the neutral evolution equation periodic problem in Banach space, to our knowledge, these results have relatively large limitation. First of all, the most popular approach is the use of boundedness or ultimate boundedness of solutions and the compactness of Poincaré map realized through some compact embeddings. However, in some concrete applications, it is difficult to choose an appropriate initial condition to guarantee the boundedness of the solution. Secondly, we observe that the most popular condition imposed on the nonlinear term \( F \) is its Lipschitz-type condition. In fact, for equations arising in complicated reaction-diffusion processes, the nonlinear function \( F \) represents the source of material or population, which depends on time in diversified manners in many contexts. Thus, we may not hope to have the Lipschitz-type condition of \( F \). Finally, there are few papers to study the asymptotic stability of periodic solutions for the neutral evolution equation with delay.

Motivated by the papers mentioned above, we consider the periodic problem for the neutral evolution equation with delay (1.1) in Banach space. By using fixed point theorem, we study the existence of \( \omega \)-periodic mild solutions for Eq.(1.1). It is worth mentioning that the assumption of prior boundedness of solutions is not employed and the nonlinear term \( F \) satisfies some growth conditions, which are weaker than Lipschitz-type condition. On the other hand, by means of a new integral inequality with delay, we present the asymptotic stability result for Eq.(1.1), which will make up the research in this area blank.

The rest of this paper is organized as follows. In Section 2, we collect some known definitions and notions, and then provide preliminary results which will be used throughout this paper. In Section 3, we apply the operator semigroup theory to find the \( \omega \)-periodic mild solutions for Eq.(1.1) and in Section 4, by strengthening the condition, we obtain the global asymptotic stability theorems for Eq.(1.1). In the last section, we give an example to illustrate the applicability of abstract results obtained in Section 3 and Section 4.

2. Preliminaries. In this section, we introduce some notions, definitions, and preliminary facts which are used through this paper.

Throughout this paper, we assume that \( X \) is a Banach space with norm \( \| \cdot \| \), \( A : D(A) \subset X \to X \) is a closed linear operator and \( -A \) generates a compact analytic operator semigroup \( T(t)(t \geq 0) \) in Banach space \( X \). For the theory of semigroups of linear operators we refer to [26, 30] and references therein. We only recall here some notions and properties that are essential for us.

For a general \( C_0 \)-semigroup \( T(t)(t \geq 0) \), there exist \( M \geq 1 \) and \( \nu \in \mathbb{R} \) such that
\[
\| T(t) \| \leq Me^{\nu t}, \quad t \geq 0. \tag{2.1}
\]
Moreover, if there \( M \geq 1 \) and \( \delta > 0 \) such that \( \| T(t) \| \leq Me^{-\delta t} \) for all \( t \geq 0 \), the \( C_0 \)-semigroup \( T(t)(t \geq 0) \) is said to be exponentially stable. Let
\[
\nu_0 = \inf \{ \nu \in \mathbb{R} | \text{There exists } M \geq 1 \text{ such that } \| T(t) \| \leq Me^{\nu t}, \quad \forall t \geq 0 \},
\]
then \( \nu_0 \) is called the growth exponent of the semigroup \( T(t)(t \geq 0) \). Furthermore, \( \nu_0 \) can be also obtained by the following formula
\[
\nu_0 = \limsup_{t \to +\infty} \frac{\ln \| T(t) \|}{t}. \tag{2.2}
\]
Clearly, \( C_0 \)-semigroup \( T(t)(t \geq 0) \) is the exponentially stable if and only if \( \nu_0 < 0 \).
If $C_0$-semigroup $T(t)$ is continuous in the uniform operator topology for every $t > 0$ in $X$, it is well known that $\nu_0$ can also be determined by $\sigma(A)$ (the spectrum of $A$),

$$\nu_0 = -\inf\{\Re \lambda \mid \lambda \in \sigma(A)\},$$

(2.3)

where $-A$ is the infinitesimal generator of $C_0$-semigroup $T(t)(t \geq 0)$. We know that $T(t)(t \geq 0)$ is continuous in the uniform operator topology for $t > 0$ if $T(t)(t \geq 0)$ is compact semigroup.

In particular, if $T(t)(t \geq 0)$ is analytic semigroup with infinitesimal generator $-A$ satisfying $\nu_0 < 0$. Then for any $\alpha \in (0, 1)$, we can define $A^{-\alpha}$ by

$$A^{-\alpha}x := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}T(t)x\,dt, \quad x \in X,$$

(2.4)

where $\Gamma(\cdot)$ denotes the well known gamma function. Since $A^{-\alpha}$ is an injective continuous endomorphism of $X$, hence we can define $A^\alpha$ by $A^\alpha := (A^{-\alpha})^{-1}$, which is a closed bijective linear operator in $X$. Furthermore, the subspace $D(A^\alpha)$ is dense in $X$ and the expression

$$\|x\|_\alpha := \|A^\alpha x\|, \quad x \in D(A^\alpha),$$

(2.5)

defines a norm on $D(A^\alpha)$. Hereafter, we represent $X_\alpha$ as the space $D(A^\alpha)$ endowed with the norm $\| \cdot \|_\alpha$.

The following properties are well known([26]).

**Lemma 2.1.** If $T(t)(t \geq 0)$ is analytic semigroup with infinitesimal generator $-A$ satisfying $0 \in \rho(A)$, then

(i) $X_\alpha$ is a Banach space for $0 < \alpha < 1$;

(ii) $A^{-\alpha}$ is a bounded linear operator for $0 < \alpha < 1$ in $X$;

(iii) $T(t) : X \to X_\alpha$ for each $t > 0$;

(iv) $A^\alpha T(t)x = T(t)A^\alpha x$ for each $x \in D(A^\alpha)$ and $t \geq 0$;

(v) for every $t > 0$, $A^\alpha T(t)$ is bounded in $X$ and there exists $M_\alpha > 0$ such that

$$\|A^\alpha T(t)\| \leq M_\alpha t^{-\alpha}e^{\nu_0 t},$$

(2.6)

(vi) for $0 < \alpha \leq \beta < 1$, $X_\beta \hookrightarrow X_\alpha$, and the embedding $X_\beta \hookrightarrow X_\alpha$ is compact whenever the resolvent operator of $A$ is compact.

We denote by $C_B(\mathbb{R}, X)$ the Banach space of all bounded, continuous functions from $\mathbb{R}$ to $X$ equipped with the norm $\|u\|_B = \sup_{t \in \mathbb{R}}\|u(t)\|$. A function $u \in C_B(\mathbb{R}, X)$ is said to be $\omega$-periodic if $u(t + \omega) = u(t)$ for all $t \in \mathbb{R}$. In the rest of this paper, by $C_\omega(\mathbb{R}, X)$, we denote the set of all $\omega$-periodic functions from $\mathbb{R}$ to $X$. It is easy to see that $C_\omega(\mathbb{R}, X)$ is a Banach space equipped with the maximum norm $\|u\|_C = \max_{t \in [0, \omega]}\|u(t)\|$.

**Definition 2.2.** A function $u \in C_B(\mathbb{R}, X)$ is said to be a mild solution of the evolution equation (1.1) if the function $r \to AT(t-r)G(r, u_r)$ is integrable for $t > r$ and $u$ satisfies the following integral equation

$$u(t) = T(t-s)(u(s) - G(s, u_s)) + G(t, u_t) - \int_s^t AT(t-r)G(r, u_r)\,dr$$

$$+ \int_s^t T(t-r)F(r, u(r), u_r)\,dr$$

for all $t, s \in \mathbb{R}$ with $t > s$.

To prove our main results, we also need the following lemma.
Lemma 2.3. (Krasnoselskii’s Fixed Point Theorem). Let $X$ be a Banach space and $\Omega$ be a bounded closed and convex subset of $X$, and let $Q_1, Q_2$ be maps of $\Omega$ into $X$ such that $Q_1x + Q_2y \in \Omega$ for every pair $(x, y) \in \Omega$. If $Q_1$ is a contraction and $Q_2$ is compact, then the equation $Q_1z + Q_2z = z$ has a solution on $\Omega$.

Lemmas 2.3 is classical, which can be found in many books.

3. Existence of periodic solutions. Theorem 3.1. Let $X$ be a Banach space and $\alpha \in (0, 1)$, let $A : D(A) \subset X \to X$ be a closed linear operator, and $-A$ generate a compact and exponentially stable analytic operator semigroup $T(t)(t \geq 0)$ in $X$. We assume that $G : \mathbb{R} \times \mathcal{B} \to X_\alpha$ and $F : \mathbb{R} \times X \times \mathcal{B} \to X$ are continuous functions which are $\omega$-periodic functions in $t$. If the following conditions are satisfied

(H1) $G(t, 0) = 0$ and there exists $L > 0$ such that
$$\|A^\alpha G(t, \phi) - A^\alpha G(t, \psi)\| \leq L\|\phi - \psi\|_X, \quad t \in \mathbb{R}, \phi, \psi \in \mathcal{B},$$
(H2) for any $k > 0$, there exists a positive valued function $\Phi_k : \mathbb{R} \to \mathbb{R}^+$ such that
$$\sup_{\|\phi\|_X < k} \|F(t, x, \phi)\| \leq \Phi_k(t), \quad t \in \mathbb{R},$$
function $s \mapsto \Phi_k(s)$ belongs to $L^1_\text{loc}(\mathbb{R}, \mathbb{R}^+)$ and there is a positive constant $\gamma > 0$ such that
$$\liminf_{k \to \infty} \frac{1}{k} \int_{t-\omega}^t \Phi_k(s)ds = \gamma < \infty, \quad t \in \mathbb{R},$$
(H3) $C\alpha L + M_{1-\alpha}L\Gamma(\alpha)|\nu_0|^{-\alpha} + CM\gamma < 1$, where $\Gamma(\cdot)$ is the Gamma function, $C\alpha = \|A^{-\alpha}\|$ and $C = \frac{1}{1-e^{-\sigma}}$,
then Eq. (1.1) has at least one $\omega$-periodic mild solution $u \in C_\omega(\mathbb{R}, X)$.

Proof. Define an operator $Q$ on $C_\omega(\mathbb{R}, X)$ as follows
$$(Qu)(t) := G(t, u_t) - \int_{-\infty}^t AT(t-s)G(s, u_s)ds + \int_{-\infty}^t T(t-s)F(s, u(s), u_s)ds. \quad (3.1)$$
For any $u \in C_\omega(\mathbb{R}, X)$, it is easy to see that $u_t \in \mathcal{B}$ for $t \in \mathbb{R}$, and
$$u_{t+s}(s) = u(t + s) = u_t(s), \quad \text{for } s \in [-r, 0],$$
$$\|u_t\|_X = \sup_{s \in [-r, 0]} \|u(t + s)\| \leq \sup_{t \in [0, \omega]} \|u(t)\| = \|u\|_X.\quad (3.2)$$
Thus, by the assumptions of Theorem 3.1 and Lemma 2.1, it is not difficult to prove that $Q : C_\omega(\mathbb{R}, X) \to C_\omega(\mathbb{R}, X)$ is well defined.

Now, we prove that $u \in C_\omega(\mathbb{R}, X)$ is an $\omega$-periodic mild solution of the problem (1.1) if and only if it is a fixed point of $Q$. To this end, we first let $u \in C_\omega(\mathbb{R}, X)$ be an $\omega$-periodic mild solution of the problem (1.1), thus, the function $\tau \to AT(t - \tau)G(\tau, u_\tau)$ is integrable on $[\tau, t]$ for all $t > \tau$ and $u$ satisfies the integral equation
$$u(t) = T(t - \tau)(u(\tau) - G(\tau, u_\tau)) + G(t, u_t) - \int_\tau^t AT(t-s)G(s, u_s)ds$$
$$+ \int_\tau^t T(t-s)F(s, u(s), u_s)ds$$
for all $t, \tau \in \mathbb{R}$ with $\tau \leq t$. Let $t \in \mathbb{R}$ be fixed and $\tau \to -\infty$, from (2.1) and $\nu_0 < 0$, it follows that $\|T(t - \tau)\| \to 0$, hence, we have
$$u(t) = G(t, u_t) - \int_{-\infty}^t AT(t-s)G(s, u_s) + \int_{-\infty}^t T(t-s)F(s, u(s), u_s)ds, \quad (3.3)$$
which implies that $u$ is a fixed point of $Q$.

Conversely, if $u \in C_\omega(\mathbb{R}, X)$ is a fixed point of $Q$, then $u$ satisfies the integral equation (3.3), and for all $\tau < t$, we can obtain

$$T(t-\tau)(u(\tau) - G(\tau, u_\tau)) = - \int_{-\infty}^\tau AT(t-s)G(s, u_s) + \int_{-\infty}^\tau T(t-s)F(s, u(s), u_s)ds.$$  \hspace{1cm} (3.4)

Hence, it is easy to see, subtracting (3.3) and (3.4), that $u$ is a mild solution of the equation (1.1).

Next, by applying Krasnoselskii’s fixed point theorem, we will prove that $Q$ has at least one fixed point $u \in C_\omega(\mathbb{R}, X)$. To this end, we introduce the decomposition $Q = Q_1 + Q_2$ as

$$Q_1 u(t) := G(t, u_t) - \int_{-\infty}^t T(t-s)AG(s, u_s)ds, \quad t \in \mathbb{R},$$  \hspace{1cm} (3.5)
and

$$Q_2 u(t) := \int_{-\infty}^t T(t-s)F(s, u(s), u_s)ds, \quad t \in \mathbb{R}. \hspace{1cm} (3.6)$$

What follows is to show that $Q_1$ and $Q_2$ satisfy the conditions of Lemma 2.3. Choosing $R > 0$, let $\Omega_R = \{ u \in C_\omega(\mathbb{R}, X) \mid \|u\|_C \leq R \}$. Note that $\Omega_R$ is a closed ball in $C_\omega(\mathbb{R}, X)$ with centre 0 and radius $R$.

Firstly, we show that there is a positive constant $R_0$ such that $Q_1 u + Q_2 v \in \Omega_{R_0}$ for every pair $u, v \in \Omega_{R_0}$. If it is not true, then for any $R > 0$, there exist $u, v \in \Omega_R$ and $t \in \mathbb{R}$ such that $\|Q_1 u(t) + Q_2 v(t)\| > R$. Thus, we see by (H1) and (H2) that

$$R < \|Q_1 u(t) + Q_2 v(t)\|$$

$$\leq \|G(t, u_t)\| + \int_{-\infty}^t \|T(t-s)AG(s, u_s)ds\|$$

$$+ \int_{-\infty}^t \|T(t-s)F(s, v(s), v_s)ds\|$$

$$\leq \|A^{-\alpha} \| \cdot \|A^\alpha G(t, u_t)\| + \int_{-\infty}^t \|A^{1-\alpha}T(t-s)\| \cdot \|A^\alpha G(s, u_s)\|ds$$

$$+ \int_{-\infty}^t \|T(t-s)\| \cdot \|F(s, v(s), v_s)\|ds$$

$$\leq \|A^{-\alpha} \| \cdot \|u_t\|_B + \int_{-\infty}^t \frac{M_{1-\alpha}Le^{\nu_0(t-s)}}{(t-s)^{1-\alpha}} \|u_s\|_B ds$$

$$+ \int_{-\infty}^t M_1 e^{\nu_0(t-s)} \Phi_R(s) ds$$

$$\leq \frac{C_\alpha L \|u_t\|_B + M_1 \int_{-\infty}^t \frac{e^{\nu_0(t-s)}}{(t-s)^{1-\alpha}} \|u_s\|_B ds}{\alpha}$$

$$+ M \sum_{i=0}^{\infty} \int_{-\infty}^t e^{\nu_0(t-s)} \Phi_R(s) ds$$

$$\leq \frac{C_\alpha L \cdot \|u_t\|_B + M_1 \int_{-\infty}^t \frac{e^{\nu_0(t-s)}}{(t-s)^{1-\alpha}} \|u_s\|_B ds}{\alpha}$$

$$+ M \sum_{i=0}^{\infty} e^{\nu_0(t-s)} \Phi_R(s) ds$$

$$\leq \frac{C_\alpha L \cdot R + M_1 \Gamma(\alpha)(|\nu_0|^{-\alpha} R + CM \cdot \int_{-\infty}^t \Phi_R(s) ds}{\alpha}.$$
where \( C = \frac{1}{1-\omega}\). Dividing on both sides by \( R \) and taking the lower limit as \( R \to \infty \), we have

\[
C_{\alpha}L + M_{1-\alpha}L\Gamma(\alpha)|\nu_0|^{-\alpha} + CM\gamma \geq 1, \tag{3.7}
\]
which contradicts the condition (H3). Hence, there is a positive constant \( R_0 \) such that \( Q_1u + Q_2v \in \Omega_{R_0} \) for every pair \( u, v \in \Omega_{R_0} \).

Secondly, we show that \( Q_1 \) is a contractive mapping on \( \Omega_{R_0} \). Taking \( u, v \in \Omega_{R_0} \), thanks to (H1), one can find that

\[
\|Q_1u(t) - Q_1v(t)\|
\]
\[
\leq \|G(t, u) - G(t, v)\| + \left\| \int_{-\infty}^{t} T(t - s) \cdot (AG(s, u_s) - AG(s, v_s))ds \right\|
\]
\[
\leq \|A^{-\alpha}\| \cdot \|A^\alpha G(t, u) - A^\alpha G(t, v)\|
\]
\[
+ \int_{-\infty}^{t} \|A^{1-\alpha}T(t - s)\| \cdot \|A^\alpha G(s, u_s) - A^\alpha G(s, v_s)\|ds
\]
\[
\leq C_{\alpha}L \cdot \|u - v\|_C + M_{1-\alpha}L\Gamma(\alpha)|\nu_0|^{-\alpha} \cdot \|u - v\|_C
\]
\[
= \left( C_{\alpha}L + M_{1-\alpha}L\Gamma(\alpha)|\nu_0|^{-\alpha} \right) \cdot \|u - v\|_C.
\]
Combining this with the condition (H3), one can find that \( Q_1 \) is a contractive mapping on \( \Omega_{R_0} \).

Finally, we show that \( Q_2 \) is compact on \( \Omega_{R_0} \). Let \( \{u^{(n)}\} \subset \Omega_{R_0} \) with \( u^{(n)} \to u \) in \( \Omega_{R_0} \), then by the continuity of \( F \), we have

\[
F(t, u^{(n)}(t), u^{(n)}_s) \to F(t, u(t), u_s), \quad n \to \infty, \tag{3.8}
\]
for each \( t \in \mathbb{R} \). Since \( \|F(t, u^{(n)}(t), u^{(n)}_s) - F(t, u(t), u_s)\| \leq 2\Phi_{R_0}(t) \) for all \( t \in \mathbb{R} \), then the dominated convergence theorem ensure that

\[
\|Q_2u^{(n)}(t) - Q_2u(t)\|
\]
\[
= \left\| \int_{-\infty}^{t} T(t - s) \left( F(s, u^{(n)}(s), u^{(n)}_s) - F(s, u(s), u_s) \right)ds \right\|
\]
\[
\leq \int_{-\infty}^{t} \|T(t - s)\| \cdot \|F(s, u^{(n)}(s), u^{(n)}_s) - F(s, u(s), u_s)\|ds
\]
\[
\leq M \sum_{k=0}^{\infty} e^{k\nu_0} \int_{-\infty}^{t} e^{\nu_0(t-s)} \cdot \|F(s, u^{(n)}(s), u^{(n)}_s) - F(s, u(s), u_s)\|ds
\]
\[
\leq \frac{M}{1-e^{\nu_0}} \int_{-\infty}^{t} \|F(s, u^{(n)}(s), u^{(n)}_s) - F(s, u(s), u_s)\|ds
\]
\[
\to 0 \quad \text{as} \quad n \to \infty
\]
which implies that \( \|Q_2u^{(n)} - Q_2u\|_C \to 0 \) as \( n \to \infty \), i.e. \( Q_2 \) is continuous.

It is easy to see that \( Q_2 \) maps \( \Omega_{R_0} \) into a bounded set in \( C_{\omega}(\mathbb{R}, X) \). Now, we demonstrate that \( Q_2(\Omega_{R_0}) \) is equicontinuous. For every \( u \in \Omega_{R_0} \), and \( t_1, t_2 \in \mathbb{R} \)
with $t_1 < t_2$, we get that
\[
Q_2u(t_2) - Q_2u(t_1)
= \int_{-\infty}^{t_2} T(t_2 - s)F(s, u(s), u_s)ds - \int_{-\infty}^{t_1} T(t_1 - s)F(s, u(s), u_s)ds
= \int_{t_1}^{t_2} T(t_2 - s)F(s, u(s), u_s)ds
+ \int_{t_2 - \eta}^{t_1} (T(t_2 - s) - T(t_1 - s))F(s, u(s), u_s)ds
+ \int_{-\infty}^{t_2 - \eta} (T(t_2 - s) - T(t_1 - s))F(s, u(s), u_s)ds
:= I_1 + I_2 + I_3,
\]
where $\eta > 0$ is positive constant with $t_1 > t_2 - \eta$ yet to be determined.

For any $\varepsilon > 0$, we first note that there exists $\delta > 0$ small enough such that
\[
\|I_1\| \leq \int_{t_1}^{t_2} \|T(t_2 - s)\| \cdot \|F(s, u(s), u_s)\|ds
\leq M \int_{t_1}^{t_2} \Phi_{R_0}(s)ds < \frac{\varepsilon}{3}, \quad \text{as } t_2 - t_1 < \delta.
\]

For $I_3$, one can take a $\eta > 0$ big enough which is independent of $t_1$ and $t_2$ such that
\[
\|I_3\| \leq \int_{-\infty}^{t_2 - \eta} \|T(t_2 - s) - T(t_1 - s)\| \cdot \|F(s, u(s), u_s)\|ds
\leq M \int_{-\infty}^{t_2 - \eta} \left(e^{\rho_0(t_2 - s)} + e^{\rho_0(t_1 - s)}\right) \Phi_{R_0}(s)ds
= M e^{\rho_0 \eta} (1 + e^{-\rho_0(t_2 - t_1)}) \int_{-\infty}^{t_2 - \eta} e^{\rho_0(t_2 - s)} \Phi_{R_0}(s)ds
\leq M e^{\rho_0 \eta} (1 + e^{-\rho_0 \delta}) \int_{t_2 - \omega}^{t_2} \Phi_{R_0}(s)ds
\leq \frac{M e^{\rho_0 \eta} (1 + e^{-\rho_0 \delta})}{1 - e^{\rho_0 \omega}} \int_{t_2 - \omega}^{t_2} \Phi_{R_0}(s)ds < \frac{\varepsilon}{3}.
\]

For such $\eta$ fixed, taking $\tau > 0$ small enough which is independent of $t_1$ and $t_2$, from the continuity of $t \mapsto \|T(t)\|$ for $t > 0$, we have
\[
\|I_2\| \leq \int_{t_2 - \eta}^{t_1} \|T(t_2 - s) - T(t_1 - s)\| \cdot \|F(s, u(s), u_s)\|ds
= \int_{t_1 - \tau}^{t_1} \|T(t_2 - s) - T(t_1 - s)\| \cdot \|F(s, u(s), u_s)\|ds
+ \int_{t_2 - \eta}^{t_1 - \tau} \|T(t_2 - s) - T(t_1 - s)\| \cdot \|F(s, u(s), u_s)\|ds
\leq \int_{t_1 - \tau}^{t_1} (\|T(t_2 - s)\| + \|T(t_1 - s)\|) \cdot \Phi_{R_0}(s)ds
\]
relatively compact in $X$ where $Q$ which means that $t$

Since the operator $T$ $Q$ one can obtain that the set $(\epsilon R >$

Therefore, $\|Q_2u(t_2) - Q_2u(t_1)\|$ tends to 0 independently of $u \in \Omega R_0$ as $t_2 - t_1 \to 0$, which means that $Q_2(\Omega R_0)$ is equicontinuous.

It remains to show that $(Q_2(\Omega R_0))(t)$ is relatively compact in $X$ for all $t \in \mathbb{R}$. To do this, for fixed $t \in \mathbb{R}$, $\varepsilon > 0$, we define the set

$$(Q_\varepsilon(\Omega R_0))(t) := \{(Q_\varepsilon u)(t) \mid u \in \Omega R_0\},$$

where

$$(Q_\varepsilon u)(t) = \int_{-\infty}^{t-\varepsilon} T(t-s)F(s,u(s),u_s)ds = T(\varepsilon) \int_{-\infty}^{t} T(t-s)F(s,u(s),u_s)ds.$$

Since the operator $T(\varepsilon)$ is compact in $X$, it is follows that the set $(Q_\varepsilon(\Omega R_0))(t)$ is relatively compact in $X$. For any $u \in \Omega R_0$ and $t \in \mathbb{R}$, from the following inequality

$$\|Q_2u(t) - Q_\varepsilon u(t)\| \leq \int_{t-\varepsilon}^{t} \|T(t-s)F(s,u(s),u_s)\|ds$$

$$\leq \int_{t-\varepsilon}^{t} \|T(t-s)F(s,u(s),u_s)\|ds$$

$$\leq M \int_{t-\varepsilon}^{t} \Phi R_0(s)ds \to 0, \text{ as } \varepsilon \to 0,$$

one can obtain that the set $(Q_2(\Omega R_0))(t)$ is relatively compact in $X$ for all $t \in \mathbb{R}$.

Thus, the Arzela-Ascoli theorem guarantees that $Q_2$ is a compact operator on $\Omega R_0$.

Now, applying Lemma 2.3, we deduce that $Q$ has at least one fixed point $u \in C^0(\mathbb{R}, X)$, that is, Eq. (1.1) has at least one $\omega$-periodic mild solution. The proof is completed. $\square$

If the function $\Phi_k$ in (H2) is independent of $t$, situation becomes more simple. For example, we replace the condition (H2) with

(H2) there are positive constants $L_1$, $L_2$ and $L_0$ such that

$$\|F(t,x,\phi)\| \leq L_1 \|x\| + L_2 \|\phi\| + L_0, \quad t \in \mathbb{R}, \quad x \in X, \quad \phi \in B.$$

In this case, for any $R > 0$ and $x \in X, \phi \in B$ with $\|x\|, \|\phi\| \leq R$, we have

$$\|F(t,x_0,x_1)\| \leq R(L_1 + L_2) + L_0 := \Phi R(t), \quad t \in \mathbb{R},$$

which implies that

$$\liminf_{R \to \infty} \frac{1}{R} \int_{t-\omega}^{t} \Phi R(s)ds = (L_1 + L_2)\omega = \gamma > 0.$$
In fact, for any $u \in \Omega_R$, through careful calculation, one can see that for any $u \in \Omega_R$,

$$\lim inf_{R \to \infty} \left( \frac{1}{R} \int_{-\infty}^{t} \|T(t-s)\| \cdot \|F(s, u(s), u_s)\| ds \right) \leq \lim inf_{R \to \infty} \left( \frac{M}{R} \sum_{k=0}^{\infty} e^{k\nu_0} \int_{t-\omega}^{t} e^\nu_0 (t-s) \cdot \|F(s, u(s), u_s)\| ds \right) \leq \lim inf_{R \to \infty} \left( \frac{M}{R} \cdot \frac{1}{1-e^{\nu_0}} \int_{t-\omega}^{t} e^\nu_0 (t-s) \left( L_1 \|u(s)\| + L_2 \|u_s\|_B + L_0 \right) ds \right) \leq \frac{M(L_1 + L_2)}{|\nu_0|}.$$

Therefore, by the proof of Theorem 3.1, we have the following result.

**Theorem 3.2.** Let $X$ be a Banach space and $\alpha \in (0, 1)$, let $A : D(A) \subset X \to X$ be a closed linear operator, and $-A$ generate a compact and exponentially stable analytic operator semigroup $T(t)(t \geq 0)$ in $X$. We assume that $G : \mathbb{R} \times \mathcal{B} \to X_\alpha$ and $F : \mathbb{R} \times X \times \mathcal{B} \to X$ are continuous functions which are $\omega$-periodic functions in $t$. If the conditions $(H1),(H2')$ and $(H3') C_\alpha L + M_{1-\alpha} L_\Gamma(\alpha) |\nu_0|^{-\alpha} + \frac{M(L_1 + L_2)}{|\nu_0|} < 1$

hold, then Eq. (1.1) has at least one $\omega$-periodic mild solution $u \in C_\omega(\mathbb{R}, X)$.

Furthermore, we assume that $F$ satisfies Lipschitz condition, namely, $(H_4)$ there are positive constants $L_1, L_2$, such that

$$\|F(t, x, \phi) - F(t, y, \psi)\| \leq L_1 \|x - y\| + L_2 \|\phi - \psi\|_B,$$

for any $t \in \mathbb{R}, x, y \in X$ and $\phi, \psi \in \mathcal{B}$, then we can obtain the following result.

**Theorem 3.3.** Let $X$ be a Banach space and $\alpha \in (0, 1)$, let $A : D(A) \subset X \to X$ be a closed linear operator, and $-A$ generate a compact and exponentially stable analytic operator semigroup $T(t)(t \geq 0)$ in $X$. We assume that $G : \mathbb{R} \times \mathcal{B} \to X_\alpha$ and $F : \mathbb{R} \times X \times \mathcal{B} \to X$ are continuous functions which are $\omega$-periodic functions in $t$. If the condition $(H1), (H_4)$ and $(H3')$ hold, then Eq. (1.1) has unique $\omega$-periodic mild solution $u \in C_\omega(\mathbb{R}, X)$.

**Proof.** From $(H_4)$ we easily see that $(H2'$) holds. In fact, for any $t \in \mathbb{R}$ and $x \in X$, $\phi \in \mathcal{B}$, by the condition $(H4)$,

$$\|F(t, x, \phi)\| \leq \|F(t, x, \phi) - F(t, \theta, \theta)\| + \|F(t, \theta, \theta)\| \leq L_1 \|x\| + L_2 \|\phi\|_B + \|F(t, \theta, \theta)\|.$$

From the continuity and periodicity of $F$, we can choose $L_0 = \max_{t \in [0, \omega]} \|F(t, \theta, \theta)\|$, thus, the condition $(H2')$ holds. Hence by Theorem 3.2, Eq. (1.1) has $\omega$-periodic mild solutions. Let $u, v \in C_\omega(\mathbb{R}, X)$ be the $\omega$-periodic mild solutions of Eq. (1.1), then they are the fixed points of the operator $Q$ which is defined by (3.1). Hence,

$$\|Qu(t) - Qv(t)\| \leq \|G(t, u_t) - G(t, v_t)\|$$
and uniqueness of the following initial value problem

$$\begin{align*}
& + \left\| \int_{-\infty}^{t} T(t-s) \left( AG(s, u_s) - AG(s, v_s) \right) ds \right\| \\
& + \left\| \int_{-\infty}^{t} T(t-s) \left( F(s, u(s), u_s) - F(s, v(s), v_s) \right) ds \right\|
\end{align*}$$

$$\leq \left\| A^{-\alpha} \right\| \cdot \left\| A^\alpha G(t, u_t) - A^\alpha G(t, v_t) \right\|$$

$$\begin{align*}
& + \int_{-\infty}^{t} \left\| A^{1-\alpha} T(t-s) \right\| \cdot \left\| A^\alpha G(s, u_s) - AG(s, u_s) \right\| ds \\
& + \int_{-\infty}^{t} \left\| T(t-s) \right\| \cdot \left\| F(s, u(s), u_s) - F(s, v(s), v_s) \right\| ds
\end{align*}$$

$$\leq \left\| A^{-\alpha} \right\| \cdot \left\| L \cdot \left\| u_t - v_t \right\|_{B} + \int_{-\infty}^{t} \frac{M_{1-\alpha}}{(t-s)^{1-\alpha}} \cdot \left\| u_s - v_s \right\|_{B} ds \\
& + \int_{-\infty}^{t} M_{e^{\nu_0(t-s)}} \left( L_1 \left\| u_s - v_s \right\| \right) + L_2 \left\| u_s - v_s \right\|_{B} ds
\end{align*}$$

$$\leq \left( C_{\alpha} L + M_{1-\alpha} L \Gamma(\alpha) \right) \left\| v_0 \right\| - \alpha + \frac{M(L_1 + L_2)}{\left\| v_0 \right\|} \left\| u - v \right\|_{C},$$

which implies that

$$\left\| u - v \right\|_{C} \leq \left( C_{\alpha} L + M_{1-\alpha} L \Gamma(\alpha) \right) \left\| v_0 \right\| - \alpha + \frac{M(L_1 + L_2)}{\left\| v_0 \right\|} \left\| u - v \right\|_{C}. \quad (3.10)$$

From this and the condition (H3'), it follows that $u = v$. Thus, Eq.(1.1) has only one $\omega$-periodic mild solution. This completes the proof of theorem. \(\square\)

4. **Asymptotic stability of periodic solutions.** In order to prove the asymptotic stability of $\omega$-periodic mild solutions for Eq.(1.1), we need to discuss the existence and uniqueness of the following initial value problem

$$\begin{align*}
& \left\{ (u(t) - G(t, u_t))' + Au(t) = F(t, u(t), u_t), \quad t \geq 0, \\
& u_0 = \varphi, \quad (4.1)
\end{align*}$$

where $G : \mathbb{R}^+ \times \mathcal{B} \to X$, $\alpha \in (0, 1)$ and $F : \mathbb{R}^+ \times X \times \mathcal{B} \to X$ are continuous functions and $\varphi \in \mathcal{B}$.

A function $u : [-r, \infty) \to X$ is said to be a mild solution of the initial value problem (4.1) if $u(t) = \varphi(t)$ for $-\infty \leq t \leq 0$; for $t \in [0, \infty)$, the function $s \to AT(t-s)G(s, u_s)$ is integrable on $[0, t]$ and $u$ satisfies the following integral equation

$$\begin{align*}
u(t) &= T(t)(\varphi(0) - G(0, \varphi)) + G(t, u_t) - \int_{0}^{t} AT(t-s)G(s, u_s)ds \\
& + \int_{0}^{t} T(t-s)F(s, u(s), u_s)ds, \quad t \geq 0, \quad (4.2)
\end{align*}$$

In order to obtain the results about asymptotic stability, we need the following integral inequality of Gronwall-Bellman type with delay.
Lemma 4.1. Assume that \( \phi \in C([-r, \infty), \mathbb{R}^+) \). If there exist constants \( l_i \geq 0, i = 1, 2 \) and \( l_0 \in (0, 1) \) such that \( \phi \) satisfy the integral inequality

\[
\phi(t) \leq \phi(0) + l_0 \sup_{\tau \in [-r, 0]} \phi(t + \tau) + \int_0^t (l_1 \phi(s) + l_2 \sup_{\tau \in [-r, \tau]} \phi(s + \tau)) ds.
\]

for every \( t \geq 0 \), then

\[
\phi(t) \leq \frac{1}{1 - l_0} \| \phi \|_{C([-r, 0])} \exp \left( \frac{(l_1 + l_2)t}{1 - l_0} \right)
\]

for every \( t \geq 0 \), where \( \| \phi \|_{C([-r, 0])} = \max_{s \in [-r, 0]} |\phi(s)| \).

Proof. Define a function \( \psi \) on \([-r, \infty)\) as following

\[
\psi(t) = \sup_{s \in [-r, t]} \phi(s), \quad t \in [-r, \infty).
\]

Then \( \psi \in C([-r, \infty), \mathbb{R}^+) \) and \( \phi(t) \leq \psi(t) \) for \( t \in [-r, \infty) \). Similarly to the proof of [20, Lemma 4.1], we can get the following inequality

\[
\psi(t) \leq \frac{1}{1 - l_0} \| \phi \|_{C([-r, 0])} + \frac{l_1 + l_2}{1 - l_0} \int_0^t \psi(s) ds, \quad t > 0.
\]

By the well-known Bellman inequality, we have

\[
\psi(t) \leq \frac{1}{1 - l_0} \| \phi \|_{C([-r, 0])} \exp \left( \frac{(l_1 + l_2)t}{1 - l_0} \right)
\]

for every \( t \geq 0 \). Therefore,

\[
\phi(t) \leq \frac{1}{1 - l_0} \| \phi \|_{C([-r, 0])} \exp \left( \frac{(l_1 + l_2)t}{1 - l_0} \right)
\]

for every \( t \geq 0 \). This completes the proof of Lemma 4.1. \( \square \)

For the initial value problem (4.1), we have the following result.

Theorem 4.1. Let \( X \) be a Banach space and \( \alpha \in (0, 1) \). Assume that \( A : D(A) \subset X \to X \) is a closed linear operator, and \(-A\) generates an exponentially stable analytic operator semigroup \( T(t)(t \geq 0) \) in Banach space \( X \). If Assumptions (H1) and (H4) hold, then the initial value problem (4.1) has a unique mild solution \( u \in C([-r, \infty), X) \) provided \( C_\alpha L + M_{1-\alpha} L \Gamma(\alpha) |v_0|^{-\alpha} < 1 \).

Proof. From \( C_\alpha L + M_{1-\alpha} L \Gamma(\alpha) |v_0|^{-\alpha} < 1 \), it follows that there exist a constant \( \delta \in (0, 1) \) such that

\[
C_\alpha L + M_{1-\alpha} L \Gamma(\alpha) |v_0|^{-\alpha} + M(L_1 + L_2) \delta < 1. \tag{4.3}
\]

Now, we consider the initial value problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
(u(t) - G(t, u_t))' + Au(t) = F(t, u(t), u_t), \quad t \in [0, \delta], \\
u(t) = \varphi(t), \quad t \in [-r, 0].
\end{array} \right.
\end{aligned} \tag{4.4}
\]

Write

\[
C([-r, \delta], X) := \left\{ u : [-r, \delta] \to X \mid u|_{[0, \delta]} \in C([0, \delta], X) \text{ and } u|_{[-r, 0]} \in \mathcal{B} \right\},
\]

then \( C([-r, \delta], X) \) is a Banach space under the norm

\[
\|u\|_\delta = \sup_{t \in [0, \delta]} \|u(t)\| + \|u|_{[-r, 0]}\|_{\mathcal{B}}.
\]
For any $\varphi \in B$, let

$$C_{\varphi}([-r, \delta], X) := \{ u \in C([-r, \delta], X) \mid u_{[-r,0]} = \varphi \},$$

then $C_{\varphi}([-r, \delta], X)$ is a closed convex subset of $C([-r, \delta], X)$.

For each $\varphi \in B$ and $u \in C_{\varphi}([-r, \delta], X)$, define an operator as following

$$(Q u)(t) = \begin{cases}
T(t)(\varphi(0) - G(0, \varphi)) + G(t, u_t) - \int_0^t AT(t - s)G(s, u_s)ds \\
+ \int_0^t T(t - s)F(s, u(s), u_s)ds, & t \in [0, \delta], \\
\varphi(t), & t \in [-r, 0].
\end{cases}$$

It is easy to see that $Q : C_{\varphi}([-r, \delta], X) \to C_{\varphi}([-r, \delta], X)$ is well defined, and the mild solution of the initial value problem (4.4) is equivalent to the fixed point of $Q$ in $C_{\varphi}([-r, \delta], X)$.

Now, we prove that $Q$ has a fixed point in $C_{\varphi}([-r, \delta], X)$. For any $u, v \in C_{\varphi}([-r, \delta], X)$, by the definition of $Q$, we have

$$\|Q u(t) - Q v(t)\| \leq \|G(t, u_t) - G(t, v_t)\| + \left\| \int_0^t AT(t - s)(G(s, u_s) - G(s, v_s))ds \right\|$$

$$+ \left\| \int_0^t T(t - s)(F(s, u(s), u_s) - F(s, v(s), v_s))ds \right\|$$

$$\leq \|A^{-\alpha}\| \cdot \|A^\alpha G(t, u_t) - A^\alpha G(t, v_t)\|$$

$$+ \int_0^t \|A^{1-\alpha}T(t - s)\| \cdot \|A^\alpha G(s, u_s) - A^\alpha G(s, v_s)\| ds$$

$$+ \int_0^t \|T(t - s)\| \cdot \|F(s, u(s), u_s) - F(s, v(s), v_s)\| ds$$

$$\leq C_\alpha L \|u_t - v_t\|_B + \int_{-\infty}^t \frac{M_{1-\alpha}Le^{\nu_0(t-s)}}{(t-s)^{1-\alpha}} \cdot \|u_s - v_s\|_B ds$$

$$+ \int_0^t Me^{\nu_0(t-s)}(L_1\|u(s) - v(s)\| + L_2\|u_s - v_s\|_B)ds$$

$$\leq C_\alpha L \cdot \|u - v\|_\delta + M_{1-\alpha}LT(\alpha)|\nu_0|^{-\alpha} \cdot \|u - v\|_\delta + M(L_1 + L_2)t \cdot \|u - v\|_\delta$$

$$\leq \left(C_\alpha L + M_{1-\alpha}LT(\alpha)|\nu_0|^{-\alpha} + M(L_1 + L_2)\right) \cdot \|u - v\|_\delta,$$

thus, combining this with (4.3), one can find $\kappa \in (0, 1)$ such that

$$\|Q u - Q v\|_\delta < \kappa \|u - v\|_\delta,$$

which implies that $Q : C_{\varphi}([-r, \delta], X) \to C_{\varphi}([-r, \delta], X)$ is a contractive mapping. By the contraction principle, one shows that $Q$ has a unique fixed point $u_1 \in C_{\varphi}([-r, \delta], X)$, which is the mild solution of initial value problem (4.3), and

$$u_1(t) = \begin{cases}
T(t)(\varphi(0) - G(0, \varphi)) + G(t, u_t) - \int_0^t AT(t - s)G(s, u_s)ds \\
+ \int_0^t T(t - s)F(s, u(s), u_s)ds, & t \in [0, \delta], \\
\varphi(t), & t \in [-r, 0].
\end{cases}$$
Similarly to the process of the initial value problem (4.4), we can prove that the initial value problem (4.1) is in the following form:

\[
\begin{cases}
(u(t) - G(t, u_t))' + Au(t) = F(t, u(t), u_t), & t \in [\delta, 2\delta], \\
u(t) = u_1(t), & t \in [-r, \delta].
\end{cases}
\]

(4.5)

Doing this interval by interval, we obtain that there exists for \((t) = \phi(0)\) satisfying \(u(t) = \varphi(t)\) for \(-r \leq t \leq 0\) and

\[
u(t) = T(t)(\varphi(0) - G(0, \varphi)) + G(t, u_t) - \int_0^t T(t - s) AG(s, u_s)ds + \int_0^t T(t - s) F(s, u(s), u_s)ds
\]

(4.6)

for \(t \geq 0\), which is the mild solution of the initial value problem (4.1).

Next, we show the uniqueness. Let \(u, v \in C([-r, \infty), X)\) be the mild solutions of the initial value problem (4.1), hence they satisfy the initial value condition \(u(t) = v(t) = \varphi(t)\) for \(-r \leq t \leq 0\) and (4.2).

By the condition (H1) and (H2), for every \(t \geq 0\), one has

\[
\|u(t) - v(t)\| \\
\leq \|G(t, u_t) - G(t, v_t)\| + \int_0^t \|T(t - s)\| \|AG(s, u_s) - AG(s, v_s)\|ds \\
+ \int_0^t \|T(t - s)\| \|F(s, u(s), u_s) - F(s, v(s), v_s)\|ds \\
\leq \|A^{\alpha}\|L \cdot \|u_t - v_t\|_G + \int_0^t M_{1-a}L e^{\nu(t-s)} \frac{1}{(t-s)^{1-a}} \cdot \|u_s - v_s\|_G ds \\
+ \int_0^t M e^{\nu(t-s)} \cdot (L_1\|u(s) - v(s)\| + L_2\|u_s - v_s\|_G)ds \\
\leq C_0L \cdot \|u_t - v_t\|_G + \int_{-\infty}^t M_{1-a}L e^{\nu(t-s)} \frac{1}{(t-s)^{1-a}} \cdot \|u_s - v_s\|_G ds \\
+ \int_0^t M (L_1\|u(s) - v(s)\| + L_2\|u_s - v_s\|_G)ds.
\]

For \(t \in [-r, 2\delta]\), the initial value problem (4.1) is in the following form:

\[
\begin{cases}
(u(t) - G(t, u_t))' + Au(t) = F(t, u(t), u_t), & t \in [\delta, 2\delta], \\
u(t) = u_1(t), & t \in [-r, \delta].
\end{cases}
\]
alytic operator semigroup (H5) 

Let Theorem 4.2. From the condition (H5), it follows that (H3) is a closed linear operator, and \(-A\) generates an exponentially stable analytic operator semigroup \(T(t)(t \geq 0)\) in \(X\). If the conditions (H1), (H4) and the following condition 

\[(H5) \ (C_\alpha L + M_{1-\alpha} L \Gamma(\alpha)|v_0|^{-\alpha}) e^{-\nu_0 r} + \frac{M L_1 + M L_2 e^{-\nu_0 r}}{|\nu_0|} < 1,\]

hold, then the unique \(\omega\)-periodic mild solution of the periodic problem (1.1) is globally asymptotically stable.

Proof. From the condition (H5), it follows that (H3) holds, and 

\[C_\alpha L + M_{1-\alpha} L \Gamma(\alpha)|v_0|^{-\alpha} < 1. \tag{4.7}\]

By Theorem 3.3, the periodic problem (1.1) has a unique \(\omega\)-periodic mild solution \(u \in C_\omega(\mathbb{R}, X)\). For any \(\phi \in C([-r, 0], X)\), by Theorem 4.1, the initial value problem (4.1) has a unique global mild solution \(v = v(t, \phi) \in C([-r, \infty), X)\).

By the semigroup representation of the solutions, \(u\) and \(v\) satisfy the integral equation (4.2). Thus, by condition (H1), (H4), for any \(t \geq 0\), we have

\[
\|u(t) - v(t)\| \\leq \left\|T(t) \cdot \left((u(0) - v(0)) - (G(0, u_0) - G(0, v_0))\right)\right\| \\
+ \|G(t, u_t) - G(t, v_t)\| + \left\|\int_0^t T(t-s)(AG(s, u_s) - AG(s, v_s))ds\right\| \\
+ \left\|\int_0^t T(t-s)(F(s, u(s), u_s) - F(s, v(s), v_s))ds\right\| \\
\leq \|T(t)\| \cdot \left(\|u(0) - v(0)\| + \|A^{-\alpha}\| \cdot \|A^\alpha G(0, u_0) - A^\alpha G(0, v_0)\|\right) \\
+ \|A^{-\alpha}\| \cdot \|A^\alpha G(t, u_t) - A^\alpha G(t, v_t)\| \\
+ \int_0^t \|A^{1-\alpha}T(t-s)\| \cdot \|A^\alpha G(s, u_s) - A^\alpha G(s, v_s)\|ds \\
+ \int_0^t \|T(t-s)\| \cdot \|F(s, u(s), u_s) - F(s, v(s), v_s)\|ds \\
\leq Me^{\nu_0 t}(1 + \|A^{-\alpha}\| \cdot L)\|u_0 - v_0\|_B \\
+ \|A^{-\alpha}\| \cdot L\|u_t - v_t\|_B + \int_0^t \frac{M_{1-\alpha} L e^{\nu_0(t-s)}}{(t-s)^{1-\alpha}} \cdot \|u_s - v_s\|_B ds
\]
\[ + \int_0^t Me^{\nu t} (L_1 \|u(s) - v(s)\| + L_2 \|u_s - v_s\|) ds \]
\[ \leq Me^{\nu (t-s)} (1 + C_\alpha L) \|u_0 - v_0\| \]
\[ + (C_\alpha L + M_{1-\alpha} \Gamma(\alpha)|\nu_0|^{\alpha}) \sup_{\tau \in [-r, 0]} \|u(t + \tau) - v(t + \tau)\| \]
\[ + \int_0^t Me^{\nu_0 (t-s)} (L_1 \|u(s) - v(s)\| + L_2 \sup_{\tau \in [-r, 0]} \|u(s + \tau) - v(s + \tau)\|) ds, \]

let \( \phi(t) = e^{-\nu_0 t} \|u(t) - v(t)\| \) for \( t \in [-r, \infty) \), one can see
\[ \phi(t) \leq M (1 + C_\alpha L) \phi(0) \]
\[ \quad + (C_\alpha L + M_{1-\alpha} \Gamma(\alpha)|\nu_0|^{\alpha}) e^{-\nu_0 r} \sup_{\tau \in [-r, 0]} \phi(t + \tau) \]
\[ \quad + \int_0^t ML_1 \phi(s) + ML_2 e^{-\nu_0 r} \sup_{\tau \in [-r, 0]} \phi(s + \tau) ds, \quad (4.8) \]

let \( l_1 = ML_1, \quad l_2 = ML_2 e^{-\nu_0 r} \), and \( l_0 = (C_\alpha L + M_{1-\alpha} \Gamma(\alpha)|\nu_0|^{\alpha}) e^{-\nu_0 r} \), then
\[ e^{-\nu_0 t} \|u(t) - v(t)\| = \phi(t) \leq \frac{C(\varphi)}{1 - l_0} \exp \left( \frac{(l_1 + l_2)t}{1 - l_0} \right) \quad (4.9) \]

for \( t \geq 0 \), where \( C(\varphi) = M(1 + C_\alpha L) \sup_{s \in [-r, 0]} \{ e^{-\nu_0 s} \|u(s) - \varphi(s)\| \} \). By the assumption (H5), one can see
\[ \sigma := -\nu_0 - \frac{(l_1 + l_2)}{1 - l_0} \]
\[ = -\nu_0 - \frac{ML_1 + ML_2 e^{-\nu_0 r}}{1 - (C_\alpha L + M_{1-\alpha} \Gamma(\alpha)|\nu_0|^{\alpha}) e^{-\nu_0 r}} > 0, \]

Combining this with (4.9), we can obtain
\[ \|u(t) - v(t)\| \leq C(\varphi) e^{-\sigma t} \to 0, \quad (t \to \infty). \quad (4.10) \]

Thus, the \( \omega \)-periodic solution \( u \) is globally asymptotically stable and it exponentially attracts every mild solution \( v \) of the initial value problem. This completes the proof of Theorem 4.2. \( \square \)

5. Example. In this section, we present one example to illustrate our abstract results, which does not aim at generality but indicate how our theorems can be applied to a concrete problem.

Consider the partial neutral functional differential equation in the form
\[ \frac{\partial}{\partial t} (u(t, x) - \int_0^\pi g(t, x, y)u_t(y)dy) - \frac{\partial^2 u(t, x)}{\partial x^2} = f(t, x, u(t, x), u_t(x)), \quad (5.1) \]

for \( t \in \mathbb{R}, \quad x \in [0, \pi] \) under homogeneous Dirichlet boundary conditions
\[ u(t, 0) = u(t, \pi) = 0, \quad t \in \mathbb{R}, \]
where \( g : \mathbb{R} \times [0, \pi] \times [0, \pi] \to \mathbb{R} \) and \( f : \mathbb{R} \times [0, \pi] \times \mathbb{R} \times C([-r, 0], L^2[0, \pi]) \to \mathbb{R} \) are given functions to be specified below, \( r > 0 \) is a real number.

To treat this system in the abstract form (1.1), we choose the space \( X = L^2[0, \pi] \), equipped with the \( L^2 \)-norm \( \| \cdot \| \).

Define operator \( A : D(A) \subset X \to X \) by

\[
D(A) := \{ u \in X \mid u''(0, \pi) = u'(0) = 0 \}, \quad Au = -\frac{\partial^2 u}{\partial x^2},
\]

Then \(-A\) generates an exponentially stable compact analytic semigroup \( T(t)(t \geq 0) \) in \( X \). It is well known that \( 0 \in \rho(A) \) and so the fractional powers of \( A \) are well defined. Moreover, \( A \) has a discrete spectrum with eigenvalues of the form \( n^2 \), \( n \in \mathbb{N} \), and the associated normalized eigenfunctions are given by \( e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx) \) for \( x \in [0, \pi] \). Hence, the associated semigroup \( T(t)(t \geq 0) \) is explicitly given by

\[
T(t)u = \sum_{n=1}^{\infty} e^{-n^2t}(u, e_n)e_n, \quad t \geq 0, u \in X,
\]

where \((\cdot, \cdot)\) is the inner product on \( X \), and it is not difficult to verify that \( \|T(t)\| \leq e^{-t} \) for all \( t \geq 0 \). Hence, we take \( \kappa = 1 \) and \( M = 1 \) and \( M_2 = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \). The following results are also well known.

(e1) If \( u \in D(A) \) then

\[
Au = \sum_{n=1}^{\infty} n^2(u, e_n)e_n.
\]

(e2) For each \( u \in X \),

\[
A^{-\frac{1}{2}}u = \sum_{n=1}^{\infty} \frac{1}{n}(u, e_n)e_n,
\]

in particular, \( \|A^{-\frac{1}{2}}\| = 1 \).

(e3) For each \( u \in D(A^{\frac{1}{2}}) := \{ u \in X \mid \sum_{n=1}^{\infty} n(u, e_n)e_n \in X \} \),

\[
A^\frac{1}{2}u = \sum_{n=1}^{\infty} n(u, e_n)e_n.
\]

Define \( u(t)(x) = u(t, x) \) and \( u_t(s)(x) = u(t + s, x) \) for \( t \in \mathbb{R}, x \in [0, \pi] \) and \( s \in [-r, 0] \), then \( u \in X \) and \( u_t \in B = C([-r, 0], X) \). Thus, we can define \( F : \mathbb{R} \times X \times B \to X \) and \( G : \mathbb{R} \times X \to X \) as following

\[
F(t, u(t), u_t)(x) = f(t, x, u(t, x), u_t(x)),
\]

\[
G(t, u_t)(x) = \int_{\pi}^{\pi} g(t, x, y)u_t(y)dy.
\]

Hence, the partial neutral functional differential equation boundary value problem (5.1) can be reformulated as the abstract evolution equation (1.1) in \( X \).

Therefore, we can obtain the following results.

**Theorem 5.1.** If the following assumptions

(A1) \( g : \mathbb{R} \times [0, \pi] \times [0, \pi] \to \mathbb{R} \) is a continuous differentiable function, which is \( \omega \) periodic in \( t \), \( g(t, x, y) \) and \( \frac{\partial^i}{\partial x^i} g(t, x, y) \) are measurable and satisfies

\[
g(t, \pi, y) = g(t, 0, y) = 0, \quad t \in \mathbb{R}, y \in [0, \pi],
\]

\[
l := \max_{t \in [0, \omega]} \left( \max_{y \in [0, \pi]} \int_{0}^{\pi} \int_{0}^{\pi} \left( \frac{\partial^i}{\partial x^i} g(t, x, y) \right)^2 dy dx \right)^{\frac{1}{2}}, i = 0, 1 \leq +\infty,
\]
(A2) \( f : \mathbb{R} \times [0, \pi] \times \mathbb{R} \times C([-r, 0], L^2[0, \pi]) \to \mathbb{R} \) is a continuously differential function, which is \( \omega \) periodic in \( t \) and there exist positive constants \( l_1, l_2 \) such that

\[
|f(t, x, w, \phi) - f(t, x, v, \psi)| \leq l_1|w - v| + l_2\|\phi - \psi\|_{[-r, 0]}
\]

for \( t \in \mathbb{R}, x \in [0, 1] \) and \( w, v \in \mathbb{R}, \phi, \psi \in C([-r, 0], L^2[0, \pi]) \),

(A3) \( l(1 + \pi) + (l_1 + l_2) < 1 \),

hold, then the neutral partial differential equation with delays (5.1) has only one \( \omega \)-periodic mild solution.

**Proof.** From the definition of \( G \) and the assumption (A1), it follows that for all \( t \in \mathbb{R} \) and \( \psi \in \mathcal{B} \),

\[
(G(t, \psi), e_n) = \left(\frac{2}{\pi}\right)^\frac{1}{2} \int_0^\pi \int_0^\pi g(t, x, y)\psi(t, y)\sin(nx)dydx
\]

\[
= \frac{1}{n} \left(\frac{2}{\pi}\right)^\frac{1}{2} \int_0^\pi \int_0^\pi \frac{\partial g(t, x, y)}{\partial x}\psi(t, y)\cos(nx)dydx
\]

\[
= \frac{1}{n} \left(\frac{2}{\pi}\right)^\frac{1}{2} \int_0^\pi \frac{\partial g(t, x, y)}{\partial x}\psi(t, y)dy \left(\frac{2}{\pi}\right)^\frac{1}{2} \cos(nx)\right).
\]

Therefore, by (e3) and Bessel's inequality, one can obtain that for all \( t \in \mathbb{R} \) and \( \psi \in \mathcal{B} \),

\[
\|A^\frac{1}{2}G(t, \psi)\| = \left(\sum_{n=1}^{\infty} n^2(G(t, \psi), e_n)^2\right)^\frac{1}{2}
\]

\[
= \left(\sum_{n=0}^{\infty} \left(\int_0^\pi \frac{\partial g(t, x, y)}{\partial x}\psi(t, y)dy \left(\frac{2}{\pi}\right)^\frac{1}{2} \cos(nx)\right)^2\right)^\frac{1}{2}
\]

\[
\leq \left\| \int_0^\pi \frac{\partial g(t, x, y)}{\partial x}\psi(t, y)dy \right\| \leq l\|\psi\|_{\mathcal{B}},
\]

which implies that \( G : \mathbb{R} \times \mathcal{B} \to X_2^\frac{1}{2} \). Also, from the periodicity of \( g \), it is easy to see that \( G(t + 2\pi, \psi) = G(t, \psi) \) for \( t \in \mathbb{R} \) and \( \psi \in \mathcal{B} \). Moreover, for all \( t \in \mathbb{R} \) and \( \phi, \psi \in \mathcal{B} \), we find that

\[
\|A^\frac{1}{2}G(t, \phi) - A^\frac{1}{2}G(t, \psi)\| \leq l\left(\int_0^\pi (\phi(x) - \psi(s))^2dx\right)^\frac{1}{2} = l\|\phi - \psi\|_{\mathcal{B}},
\]

and \( G(t, 0) = 0 \) for \( t \in \mathbb{R} \). Therefore, the condition (H1) holds with \( L = l \).

On the other hand, by the definition of \( F \) and the assumption (A2), one can find that \( F \) satisfies the condition (H4) with \( L_1 = l_1, L_2 = l_2 \).

At last, combined Assumption (A3) with \( C_1^\frac{1}{2} = \|A^{-\frac{1}{2}}\| = 1, M = 1, M_1 = \Gamma(\frac{1}{2}) = \sqrt{\pi} \) and \( |v_0| = 1 \), one can easily get the condition (H3 ') holds.

Therefore, our conclusion follows from Theorem 3.3. This completes the proof of Theorem 5.1.

Furthermore, in Theorem 5.1, strengthening the condition (A3), we obtain the following asymptotic stability result of the periodic solution:

**Theorem 5.2.** Under the assumptions of Theorem 1.1, if the condition

(A4) \( (1 + \pi)e^r + (l_1 + l_2)e^r < 1 \),

then the unique \( \omega \)-periodic mild solution of problem (5.1) is globally asymptotically stable.
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Competing interests
Q. Li and M. Wei declare that they have no competing interests.

Consent for publication
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