Families of Painlevé VI equations having a common solution

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Abstract

We classify all functions satisfying non-trivial families of $PVI_\alpha$ equations. It turns out that, up to an Okamoto equivalence, there are exactly four families parameterized by affine planes or lines. Each affine space is generated by points of “geometric origin”, associated either to deformations of elliptic surfaces with four singular fibers, or to deformations of three-sheeted covers of $\mathbb{P}^1$ with branching locus consisting of four points.

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1 Introduction

Consider the Painlevé VI (PVI) equation

$$\frac{d^2\lambda}{dt^2} = \frac{1}{2}(\frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t})(\frac{d\lambda}{dt})^2 - (\frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t}) \frac{d\lambda}{dt}$$

$$+ \frac{\lambda(\lambda - 1)(\lambda - t)}{t^2(t - 1)^2} [\alpha_0 - \alpha_1 \frac{t}{\lambda^2} + \alpha_2 \frac{t - 1}{(\lambda - 1)^2} + \frac{1}{2} - \alpha_3 \frac{t(t - 1)}{(\lambda - t)^2}].$$

parameterized by $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^4$. Although any solution of $\text{PVI}_\alpha$, for generic $\alpha_i$, is transcendental (and even a ”new transcendental function”) there is a large amount of solutions which are algebraic in $t$. Their general classification is still an open problem (e.g. [15, Manin]), except in the particular case $\alpha_1 = \alpha_2 = \alpha_3 = 0$ [4, Dubrovin, Mazzocco]. The present paper addresses the question of classifying families of algebraic solutions. The simplest case occurs when a given algebraic solution satisfies each member of a non-trivial family of $\text{PVI}_\alpha$ equations. By a non-trivial family of $\text{PVI}_\alpha$ equations we mean a set $\{\text{PVI}_\alpha\}_\alpha$ containing at least two distinct elements corresponding to, say, $\alpha'$ and $\alpha''$. Then this solution satisfies the $\text{PVI}_\alpha$ equations corresponding to the affine line containing $\alpha'$ and $\alpha''$. It follows that each non-trivial family as above corresponds to an affine plane in the parameter space $\mathbb{C}^4\{\alpha\}$. We classify all such affine spaces, together with their associated algebraic solutions (Theorem 1). The proof of Theorem 1 does not use the notion of Picard-Fuchs equation. It turns out that the solutions which we obtain coincide surprisingly with the solutions obtained earlier by Doran, who used deformations of elliptic surfaces with four singular fibers and the related Picard-Fuchs equations, see Theorem 2.

The second purpose of the paper is to give a (partial) explanation of the above coincidence. It is well known that each solution $(\lambda(t), \alpha)$ of a given $\text{PVI}_\alpha$ equation governs the isomonodromy deformation of a $2 \times 2$ Fuchsian system with four singular points. We say that such a deformation is ”geometric”, if there is a fundamental matrix of solutions whose entries are Abelian integrals depending algebraically on the deformation parameter, see [3, Doran]. A geometric deformation of a Fuchsian system is isomonodromic and the associated solution of the corresponding Schlesinger system (or $\text{PVI}_\alpha$ equation) is an algebraic function in $t$. When this holds true, we say that the algebraic solution $(\lambda(t), \alpha)$ of $\text{PVI}_\alpha$ is of geometric origin. The solutions found by Doran are of geometric origin and they correspond to special values
of \(\alpha\) which belong to the affine spaces of \(PVI_\alpha\) equations described in Theorem 1. We shall prove that the same list of solutions can be obtained from deformations of ramified covers of \(\mathbb{P}^1\) with four ramification points. The corresponding values of the parameters \(\alpha\) are different and are shown on Table 5 see Theorem 3. Finally we note that all these points of geometric origin generate the affine planes of \(PVI_\alpha\) equations, described in Theorem 1.

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2 Families of Painlevé VI equations having a common solution

Let \(\lambda = \lambda(t)\) be a solution of the equations \(PVI_{\alpha'}, PVI_{\alpha''}, \alpha' \neq \alpha''\). Then \(\lambda(t)\) satisfies the implicit equation

\[
\beta_0 - \beta_1 \frac{t}{\lambda^2} + \beta_2 \frac{t - 1}{(\lambda - 1)^2} - \beta_3 \frac{t(t - 1)}{(\lambda - t)^2} = 0
\]

where \(\beta = \alpha' - \alpha'' = (\beta_0, \beta_1, \beta_2, \beta_3)\), and hence it is an algebraic function. The function \(\lambda(t)\) satisfies, moreover, the family \(\{PVI_\alpha\}_\alpha\), where \(\alpha\) belongs to the affine line \(\{\alpha' + s(\alpha' - \alpha''): s \in \mathbb{C}\} \subset \mathbb{C}^4\).

It is seen from this that the set of all \(\alpha\) such that \(PVI_\alpha\) is satisfied by the function \(\lambda(t)\) form an affine subspace of \(\mathbb{C}^4\). We refer to the set of these \(PVI_\alpha\) equations as to a family of Painlevé VI equations having a common solution.

**Theorem 1** The list of all families of Painlevé VI equations having a common solution, together with the corresponding solution, is shown in Table 1.

**Remark 1** Each solution \(\lambda(t)\) is defined by a relation \(P(\lambda(t), t) \equiv 0\) where \(P\) is an irreducible polynomial given in Table 1. The solutions in each of the 5 series of families on Table 1 are equivalent up to a \(S_4\)-symmetry of Painlevé VI equation (see section 2.1).
Moreover, the solutions \(1A, 1B, \ldots, 1F\) are Okamoto equivalent to the solutions \(2A, 2B, 2C\). More precisely, the solution \(1A\)
\[a\lambda^2 - bt = 0, \alpha = (a, b, \frac{1}{8}, \frac{1}{8})\]
is equivalent, after applying the transformation \(w_2\) of Okamoto [16, p.363], to the solution \(\lambda^2 - t = 0\) where the parameter \(\alpha\) equals to
\[
((\frac{\sqrt{2}a - \sqrt{2}b}{2\sqrt{2}})^2, (\frac{\sqrt{2}a - \sqrt{2}b}{2\sqrt{2}})^2, (1 - \frac{\sqrt{2}a - \sqrt{2}b}{2\sqrt{2}})^2, (1 - \frac{\sqrt{2}a - \sqrt{2}b}{2\sqrt{2}})^2).
\]
Thus, up to Okamoto equivalence, the families of Painlevé VI equations having a common solution are represented (for instance) by the four families \(2A, 3A, 4A, 5A\) on Table [1].

**Remark 2** The functions 0, 1, \(t\) are not considered as solutions of Painlevé VI equation, and therefore are excluded from Table [1].

**Outline of the Proof.** Denote by \(\Gamma_\beta\) the compactified and normalized algebraic curve defined by (1), with affine model
\[
\Gamma_{\beta}^{aff} = \{(\lambda, t) \in \mathbb{C}^2 : N(\lambda, t) = 0\}
\]
where
\[
N(\lambda, t) = \beta_0 \lambda^2 (\lambda - 1)^2 (\lambda - t)^2 - \beta_1 t (\lambda - 1)^2 (\lambda - t)^2 + \beta_2 (t - 1) \lambda^2 (\lambda - t)^2 - \beta_3 t (t - 1) \lambda^2 (\lambda - 1)^2 = 0.
\]
In the case when \(\Gamma_\beta\) is irreducible, the relation \(\{N(\lambda, t) = 0\}\) defines an algebraic function \(\lambda(t)\). If this function were a solution of some \(\text{PVI}_\alpha\) equation, then the only ramification points of \(\lambda(t)\) would be at \(t = 0, 1, \infty\) (because \(\text{PVI}_\alpha\) satisfies the so called Painlevé property [10]). Equivalently, the pair \((\Gamma_\beta, t)\) is a Belyi pair, which means that the only possible critical values of the map
\[
\pi : \Gamma_\beta \rightarrow \mathbb{CP}^1 : (\lambda, t) \rightarrow t
\]
are 0, 1 or \(\infty\). This means also that if \(\Delta(t)\) is the discriminant of \(N(\lambda, t)\) with respect to \(\lambda\), then it is a polynomial whose only roots are at \(t = 0\) and \(t = 1\). A direct computation shows that this is impossible. The more difficult case is when \(N(\lambda, t)\) is reducible over \(\mathbb{C}\). Then \(\Gamma_\beta\) defines several algebraic
| Name | Solution of $\text{PVI}_\alpha$ equation | $\text{PVI}_\alpha$ equation |
|------|----------------------------------------|--------------------------------|
| 1A   | $a\lambda^2 - bt$                     | $(a, b, \frac{1}{1_8}, \frac{1}{8})$ |
| 1B   | $a(\lambda - 1)^2 + b(t - 1)$         | $(a, \frac{1}{1_8}, b, \frac{1}{8})$ |
| 1C   | $a(\lambda - t)^2 - bt(t - 1)$        | $(a, \frac{1}{1_8}, \frac{1}{1_8}, b)$ |
| 1D   | $-at(\lambda - 1)^2 + b(t - 1)\lambda^2$ | $(\frac{1}{8}, a, b, \frac{1}{8})$ |
| 1E   | $a(\lambda - t)^2 + b(t - 1)\lambda^2$ | $(\frac{1}{8}, a, \frac{1}{1_8}, b)$ |
| 1F   | $a(\lambda - t)^2 - bt(\lambda - 1)^2$ | $(\frac{1}{8}, \frac{1}{1_8}, a, b)$ |
| 2A   | $\lambda^2 - t$                       | $(a, a, b, b)$ |
| 2B   | $\lambda^2 - 2\lambda + t$           | $(a, a, b, b)$ |
| 2C   | $\lambda^2 - 2\lambda t + t$         | $(b, a, a, b)$ |
| 3A   | $\lambda^4 - 6\lambda^2 t + 4\lambda t + 4\lambda^2 t - 3t^2$ | $(a, 9a, a, a)$ |
| 3B   | $3\lambda^4 - 4\lambda^3 - 4\lambda^2 t + 6\lambda^2 t - t^2$ | $(9a, a, a, a)$ |
| 3C   | $\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda^2 \lambda + t^2$ | $(a, a, 9a, a)$ |
| 3D   | $\lambda^4 - 4t\lambda^4 + 6\lambda^2 - 4t\lambda + t^2$ | $(a, a, a, 9a)$ |
| 4A   | $\lambda^4 - 2t\lambda^3 - 2\lambda^2 + 6\lambda^2$ | $(a, \frac{1}{8}, a, a)$ |
|      | $-2t^2\lambda - 2t\lambda + t^2 - t^2 + t$ |                                    |
| 4B   | $\lambda^4 - 2t\lambda^3 + 2t^2\lambda - t^3$ | $(a, a, \frac{1}{8}, a)$ |
| 4C   | $\lambda^4(t^2 - t + 1) - 2\lambda^2 t(t + 1) + 6t^2 \lambda^2$ | $(\frac{1}{8}, a, a, a)$ |
|      | $-2t^2(t + 1) + t^3$                   |                                  |
| 4D   | $\lambda^4 - 2\lambda^3 + 2t\lambda - t$ | $(a, a, a, \frac{1}{1_8})$ |
| 5A   | $-2\lambda^3 + 3t\lambda^2 + 3\lambda^2 - 6t\lambda + t^2 + t$ | $(4a, \frac{1}{8}, a, a)$ |
| 5B   | $\lambda^3 - 3\lambda^2 + 3t\lambda - 2t^2 + t$ | $(a, \frac{1}{1_8}, 4a, a)$ |
| 5C   | $\lambda^3 - 3t\lambda^2 + 3\lambda + t^2 - 2t$ | $(a, \frac{1}{1_8}, a, 4a)$ |
| 5D   | $2\lambda^3 - 3\lambda^2 + t^2$       | $(4a, a, \frac{1}{1_8}, a)$ |
| 5E   | $\lambda^3 - 3t\lambda + 2t^2$        | $(a, 4a, \frac{1}{1_8}, a)$ |
| 5F   | $\lambda^3 - 3t\lambda^2 + 3\lambda - t^2$ | $(a, a, \frac{1}{1_8}, 4a)$ |
| 5G   | $\lambda^3(2 - t) - 3\lambda^2 + 3t^2\lambda - t^2$ | $(\frac{1}{1_8}, a, 4a, a, a)$ |
| 5H   | $\lambda^4(t + 1) - 6t\lambda^4 + 3t(t + 1)\lambda - 2t^2$ | $(\frac{1}{1_8}, 4a, a, a, a)$ |
| 5I   | $(1 - 2t)\lambda^3 + 3t\lambda^2 - 3t\lambda + t^2$ | $(\frac{1}{1_8}, a, a, 4a)$ |
| 5J   | $\lambda^3 - 3\lambda^2 + 3t\lambda - t$ | $(a, a, 4a, \frac{1}{1_8})$ |
| 5K   | $\lambda^3 - 3t\lambda + 2t$          | $(a, 4a, a, \frac{1}{1_8})$ |
| 5L   | $\lambda^3 - 3\lambda^2 + t$          | $(4a, a, a, \frac{1}{1_8})$ |

Table 1: List of all algebraic solutions satisfying families of $\text{PVI}_\alpha$ equations
functions and we have to apply the above to each of them. Finally we have to check whether the obtained function is actually a solution of some $\text{PVI}_\alpha$ equation. To check whether a given polynomial $N(\lambda, t)$ is reducible over $\mathbb{C}$ is a difficult task in general. We shall make use of the action of the symmetric group $S_4$ (see section 2.1) on the set of curves $\Gamma_\beta$, parameterized by $\beta \in \mathbb{CP}^3$.

It turns out that, first, curves $\Gamma_\beta$ with a trivial stabilizer under the action of $S_4$ can not produce a solution of $\text{PVI}_\alpha$. The stabilizer of a curve acts on it as a group of automorphisms (symmetries) which imposes additional restrictions on $\beta$.

The second ingredient of the proof is the study of the Puiseux expansion of $\lambda(t)$ in a neighborhood of $t = 0, 1, \infty$ (section 2.2). These expansions depend on the stabilizer of $\Gamma_\beta$ only and imply the possible topological types of the solution $\lambda(t)$. Equivalently, to each solution $\lambda(t)$ we associate a Belyi pair and the Puiseux expansions determine their possible dessin d’enfant. The algebraic functions which we obtain in this way are, a posteriori, the solutions of the $\text{PVI}_\alpha$ presented in Table 1.

**Proof of Theorem 1**

### 2.1 The action of $S_4$

The set of automorphisms of the projective line $\mathbb{CP}^1$ which send four distinct points $(0, 1, t, \infty)$ to the points $(0, 1, \tilde{t}, \infty)$ ($\tilde{t} = \tilde{t}(t)$ is uniquely defined) form a group isomorphic to $S_4$ generated by the transpositions

$$
x_1^i: s \mapsto 1 - s, \quad x_2^i: s \mapsto \frac{1}{s}, \quad x_3^i: s \mapsto \frac{t - s}{t - 1}.
$$

Each $x^i$ sends an isomonodromic family of Fuchsian systems with singular points at $0, 1, t, \infty$ to an isomonodromic family of such systems with singular points at $0, 1, \tilde{t}, \infty$. Therefore $x^i$ induce an action of $S_4$ on the set of curves $\Gamma_\beta$. Explicitly we have

$$
x_1^i: \Gamma_\beta \to \Gamma_{x_1^i(\beta)}: (\lambda, t) \to (x^i(\lambda), x^i(t)), i = 1, 2
$$

and

$$
x_3^3: \Gamma_\beta \to \Gamma_{x_3^3(\beta)}: (\lambda, t) \to (x^3(\lambda), x^3(0)) = \left(\frac{t - \lambda}{t - 1}, \frac{t}{t - 1}\right)
$$

where

$$
\begin{align*}
x_1^1: (\beta_0, \beta_1, \beta_2, \beta_3) &\to (\beta_0, \beta_2, \beta_1, \beta_3) \\
x_1^2: (\beta_0, \beta_1, \beta_2, \beta_3) &\to (\beta_1, \beta_0, \beta_2, \beta_3) \\
x_1^3: (\beta_0, \beta_1, \beta_2, \beta_3) &\to (\beta_0, \beta_3, \beta_2, \beta_1)
\end{align*}
$$

\[ x_2^1: (\beta_0, \beta_1, \beta_2, \beta_3) \to (\beta_0, \beta_2, \beta_1, \beta_3) \]

\[ x_2^2: (\beta_0, \beta_1, \beta_2, \beta_3) \to (\beta_1, \beta_0, \beta_2, \beta_3) \]

\[ x_2^3: (\beta_0, \beta_1, \beta_2, \beta_3) \to (\beta_0, \beta_3, \beta_2, \beta_1) \]
which is the standard representation of $S_4$ on $\mathbb{C}^4$ (upon identifying $\infty, 0, 1, t$ to $\beta_0, \beta_1, \beta_2, \beta_3$ respectively). The proof of the above facts is straightforward, see [10] for details.

2.2 The topological type of the projection $\Gamma_\beta \to \mathbb{CP}^1$ in a neighborhood of the pre-image of $t = 0, 1, \infty$

Let $\Gamma_\beta$ be the compactified and normalized curve defined by (2) (it is a disjoint union of Riemann surfaces). In this section we determine the topological type of the projection (3)

$$\Gamma_\beta \to \mathbb{CP}^1$$

in a neighborhood of the pre-image of $t = 0, 1, \infty$ in $\Gamma_\beta$. In the projective space $\mathbb{CP}^3$ with coordinates $[\beta_0 : \beta_1 : \beta_2 : \beta_3]$ consider the complex polyhedron $W$ formed by the ten planes (2-faces)

$$W = \cup_{i \neq j} \{ \beta_i = \beta_j \} \cup_k \{ \beta_k = 0 \}. \tag{8}$$

It has also 45 1-faces (projective lines) and 120 0-faces (points). We shall see in the process of the proof that the topological type of the projection in a neighborhood of the pre-image of $t = 0, 1, \infty$ is one and the same when $\beta$ belongs to a given $i$-face, but does not belong to any other $j$-face with $j < i$. For this reason we shall use, until the end of this paper, the following convention. \textit{When we say that a point $\beta$ belongs to a given face (satisfies some set of relations \textbf{(8)}), then this will mean that it does not belong to any other face of smaller dimension (does not satisfy any other relation from the list \textbf{(8)})}. The topological type in a neighborhood of the pre-image of any point is determined by a partition of the degree of the map which is 6. Thus a partition $(1 + 1 + 1 + 1 + 2)$ means that we have 5 pre-images, and that the multiplicity of $\pi$ at each pre-image is $1, 1, 1, 1, 2$ respectively. Similarly, a partition $(1 + 1 + 2 + 2)$ means that have 4 pre-images with multiplicities $1, 1, 2, 2$ respectively etc. To formulate the result we note that the symmetric group $S_4$ acts on the polyhedron $W$ by its standard representation (7), as well on the set of curves $\Gamma_\beta$ by (5). The subgroup $S_3$ generated by $x^1, x^2$ permutes the ramification points $0, 1, \infty$ according to (4) without changing the topological type of the projection $\pi$ over each of these points.

\textbf{Proposition 1} \textit{The topological type of the projection (5) in a neighborhood of the pre-image of $t = 0, 1, \infty$ is one and the same when $\beta$ belongs to a given}
Figure 1: The Newton polygon of $N(\lambda, t)$ and $N^0(\lambda, t)$

The bi-rational transformations $x^1, x^2$ defined by (5) are compatible with the projection $\pi$ and permute the points $t = 0, 1, \infty$. Therefore it suffices to consider the pre-image of $0$. Let us consider in detail the "generic" case, when $\beta \notin W$. It follows from the Newton polygon of $N(\lambda, t)$, shown on fig. [1] that there are at least three Puiseux series in a neighborhood of $(0, 0)$ (for the terminology see for instance [12, Kirwan]). The first two correspond to the line segment $[(3, 0), (1, 2)]$ and have non-equivalent leading terms

$$\lambda = c_1 t + \ldots, \lambda = c_2 t + \ldots$$

where

$$(\beta_3 - \beta_1)c_{1,2}^2 + 2\beta_1 c_{1,2} - \beta_1 = 0$$

provided that

$$\beta_1 \neq \beta_3, \beta_1^2 + \beta_1 (\beta_3 - \beta_1) \neq 0.$$ 

The third one corresponds to the line segment $[(1, 2), (0, 4)]$ and has leading term

$$\lambda = c_3 t^{1/2} + \ldots$$
where
\[(\beta_0 - \beta_2)c_3^2 + \beta_3 - \beta_1 = 0,\]
provided that
\[\beta_0 \neq \beta_2, \beta_1 \neq \beta_3.\]
Taking into consideration that
\[N(\lambda, 0) = \lambda^4(\beta_0\lambda^2 - 2\beta_0\lambda + \beta_0 - \beta_2)\]
we conclude that we have at least five pre-images of multiplicities at least
1, 1, 2, 1, 1 respectively. As the degree of the map \(\pi\) is six, then its topological
type is exactly \((1 + 1 + 1 + 1 + 2)\). The topological type of the projection \(\pi\)
over 0 and 1 is obtained by acting with the group \(S_3\) generated by \(x^1, x^2\).
In a similar way one verifies that when \(\beta_0 = \beta_2\), or \(\beta_1 = \beta_3\) the multi-
plicities are \((1 + 1 + 1 + 3)\). If \(\beta_0 = \beta_2\) and \(\beta_1 = \beta_3\) the multiplicities are
\((1 + 1 + 2 + 2)\). The case \(\beta_0 = \beta_1 = \beta_3\) is the same as \(\beta_0 = \beta_2\) and the
multiplicity is \((1 + 1 + 1 + 3)\). The case \(\beta_0 = \beta_1 = \beta_2 = \beta_3\) is of the same type
as \(\beta_0 = \beta_2\) and \(\beta_1 = \beta_3\). The multiplicities of \(\pi\) over 1 and \(\infty\) are obtained
as before. This completes the study of faces of \(W\) for which \(\beta_i \neq 0\). In the
case \(\beta_3 = 0\) we consider the curve
\[\Gamma_\beta^0 = \{(\lambda, t) \in \mathbb{C}^2 : \beta_0 - \beta_1\frac{t}{\lambda^2} + \beta_2 \frac{t - 1}{(\lambda - 1)^2} = 0, \lambda \neq 0, 1\}.\]
The polynomial \(N(\lambda, t)\) is replaced by
\[N^0(\lambda, t) = \beta_0\lambda^2(\lambda - 1)^2 - \beta_1(\lambda - 1)^2 + \beta_2(t - 1)\lambda^2\]
whose Newton polygon is shown on fig. 1. It follows that there is at least
one Puiseux expansion with leading term
\[\lambda = ct^{1/2} + \ldots, \beta_1 + (\beta_2 - \beta_0)c^2 = 0\]
provided that \(\beta_1 \neq 0, \beta_2 \neq \beta_0\). As
\[N^0(\lambda, 0) = \lambda^2(\beta_0(\lambda - 1)^2 - \beta_2)\]
then \(t = 0\) has at least three pre-images, provided that \(\beta_0\beta_2 \neq 0\). We
conclude that \(t = 0\) has exactly three pre-images with multiplicities 2, 1, 1
respectively, provided that \(\beta\) belongs to the 2-face \(\beta_3 = 0\). The remaining
1-faces and 0-faces are studied in the same way. The result is summarized
on Table 2. It worth noting that in all cases the computing of the leading
term of the Puiseux expansion suffices to deduce the result.
| face of $W$ | stabilizer | $t = 0$ | $t = 1$ | $t = \infty$ |
|----------------|-------------|--------|--------|-------------|
| $\beta_i \neq \beta_j$ | $S_4$ | (1+1+1+1+2) | (1+1+1+1+2) | (1+1+1+1+2) |
| $\beta_0 = \beta_2$ | $S_2 \times S_2$ | (1+1+1+3) | (1+1+1+1+2) | (1+1+1+1+2) |
| $\beta_0 = \beta_2, \beta_1 = \beta_3$ | $D_4$ | (1+1+2+2) | (1+1+1+1+2) | (1+1+1+1+2) |
| $\beta_0 = \beta_1 = \beta_2$ | $S_3$ | (1+1+1+3) | (1+1+1+3) | (1+1+1+3) |
| $\beta_0 = \beta_1 = \beta_2 = \beta_3$ | $S_4$ | (1+1+2+2) | (1+1+2+2) | (1+1+2+2) |
| $\beta_3 = 0$ | $S_3$ | (1+1+2) | (1+1+2) | (1+1+2) |
| $\beta_0 = \beta_2, \beta_3 = 0$ | $S_2$ | (1+3) | (1+1+2) | (1+1+2) |
| $\beta_0 = \beta_1 = \beta_2, \beta_3 = 0$ | $S_3$ | (1+3) | (1+3) | (1+3) |
| $\beta_2 = \beta_3 = 0$ | $S_2 \times S_2$ | (2) | (1+1) | (2) |
| $\beta_2 = \beta_3 = 0, \beta_0 = \beta_1$ | $S_2 \times S_2$ | (2) | (1+1) | (2) |

Table 2: Multiplicity of $\pi$ at the pre-images of $t = 0, 1, \infty$.

We conclude this section by the following elementary claim which will be often useful in the computations.

**Proposition 2** Let $N_1(\lambda, t)$ be a polynomial of non-zero degree with respect to $\lambda$ and of non-zero degree with respect to $t$, which divides $N(\lambda, t)$, and $\beta_1 \beta_2 \beta_3 \neq 0$. Then

$$N_1(0, t) = c_0 t^{m_0}, c_0 \neq 0, 1 \leq m_0 \leq 3, N_1(1, t) = c_1 (t - 1)^{m_1}, c_1 \neq 0, 1 \leq m_1 \leq 3$$

and

$$N_1(t, t) = c_2 t^{m_0} (t - 1)^{m_1}, c_2 \neq 0, 1 \leq m_0, 1 \leq m_1, m_0 + m_1 \leq 3.$$ 

**Proof.** We have $N(0, t) = -\beta_1 t^3$. For a fixed $\lambda = c \sim 0$ the polynomial $N(c, t) \in \mathbb{C}[t]$ has exactly three roots which tend to zero when $c$ tends to zero. Therefore the polynomial $N_1(c, t) \in \mathbb{C}[t]$ has at least one and at most three roots which tends to zero when $c$ tends to zero, which proves the claim concerning $N_1(0, t)$. The claim concerning $N_1(1, t)$ is proved in the same way. As $N_1(0, 0) = N_1(1, 1) = 0$ then $N_1(t, t)$ is divided by $t(t - 1)$ but also divides $N(t, t) = -\beta_3 t^3 (t - 1)^3$. □

We are ready to compute the solutions of $\text{PVI}_\alpha$ corresponding to the faces of $W$. Let $\Gamma$ be the Riemann surface of an irreducible component of $\Gamma_\beta$, which defines a solution of some $\text{PVI}_\alpha$ equation. Then the only ramification points of the induced map

$$\pi : \Gamma \to \mathbb{CP}^1 : (\lambda, t) \to t$$

(9)
are at 0, 1, ∞, and Γ is connected. The pair (Γ, π) is called a Belyi pair, to which we associate a dessins d’enfant, which is the graph obtained as a pre-image of the segment [0, 1] under the map π. The degree of the dessin is the degree of π (see [18]). The dessin d’enfant will be useful when describing the topological type of the projection π.

2.3 The case β ∉ W

We suppose that λ(t) is an algebraic function, such that \( N(λ(t), t) \equiv 0 \), and consider the corresponding Belyi pair (Γ, π), (9). Let \( \{λ_1, \ldots, λ_d\} = π^{-1}(t_0) \) where \( t_0 \neq 0, 1, ∞ \). The loops originating from \( t_0 \) and going clockwise once around 0, 1 and ∞ induce permutations \( σ_0, σ_1, σ_∞ \) of the points \( λ_1, \ldots, λ_d \), such that \( σ_0σ_1σ_∞ = 1 \). According to Table 2, \( σ_0, σ_1, σ_∞ \) are transpositions, unless one of them is the identity permutation. In the former case \( (σ_0σ_1)^2 = 1 \) and hence \( σ_0σ_1 = σ_1σ_0 \). Thus \( σ_∞ \) is a product of two disjoint transpositions, which contradicts to Table 2. On the other, if one of the permutations \( σ_0, σ_1, σ_∞ \) is the identity, the group generated by them is either \( Z_2 \) or is trivial. This shows that the degree of π is either two (because the covering is connected), or one. The corresponding dessin d’enfants are shown on fig.2 (i) and (ii) (in particular \( N(λ, t) ∈ \mathbb{C}[λ, t] \) is reducible). If the dessin is of degree one, then the solution is defined as \( λ = P(t) \), where \( P \) is a polynomial (the coefficient of \( λ^6 \) in the polynomial \( N(λ, t) \) is \( β_0 \neq 0 \)). By Proposition 2 we conclude that either \( λ = t \), or \( λ = t^2 \) or \( λ = t^3 \). But \( λ - t \), \( λ - t^2 \), \( λ - t^3 \) can not divide \( N(λ, t) \), provided that \( β_i \neq 0 \). If the dessin is of degree two, then \( λ(t) \) is defined by \( λ^2 + 2p(t)λ + q(t) = 0 \). The functions \( p, q \) are polynomials in \( t \), because the coefficient of \( λ^6 \) in the polynomial \( N(λ, t) \) is \( β_0 \neq 0 \). Further, we may suppose (acting with an appropriate symmetry \( x^i \) on Γ, see section 2.1) that \( λ(t) \) is ramified over 0 and ∞ only. By Proposition
is a non-constant polynomial which divides $t^3$. As $p(t)^2 - q(t)$ is a non-constant monomial of odd degree, then $p(t) \equiv 0$. Proposition 2 implies that we have either $\lambda^2 = t$ or $\lambda^2 = t^3$. The polynomial $\lambda^2 - t$ divides $N(\lambda, t)$ if and only if $\beta_0 = \beta_1$ and $\beta_2 = \beta_3$ (this case is excluded, as $\beta \not\in W$). The curve $\Gamma_\beta$ does not define a solution.

2.4 The face $\beta_1 = \beta_2$

The possible dessins d’enfant are determined as above. Namely, when one of the permutations $\sigma_0, \sigma_1, \sigma_\infty$ is identity, the dessin is of degree one or two. Up to a symmetry it is equivalent to the one shown on fig. 2 (i) or (ii). Reasoning as in the case $\beta \not\in W$ we conclude that $\Gamma_\beta$ does not define a solution.

If, on the other hand, $\sigma_0, \sigma_1$ are non-trivial transpositions we have one more case compared to section 2.3: $\sigma_3$ is cyclic of order three, and $\sigma_0, \sigma_1$ are non-disjoined permutations. Taking into account that the covering is connected, we conclude that the dessin is of degree three. Up to a symmetry, it is shown on 2 (iii) and $\lambda(t)$ satisfies $N_1(\lambda(t), t) \equiv 0$ where $N_1$ is an irreducible polynomial of degree three in $\lambda$ dividing the polynomial $N(\lambda, t)$, defined after formula (2). We denote

$$N = N_1N_2, \Gamma_1^{aff} = \{N_1(\lambda, t) = 0\}, \Gamma_2^{aff} = \{N_2(\lambda, t) = 0\}, \Gamma_\beta^{aff} = \Gamma_1^{aff} \cup \Gamma_2^{aff}.$$ 

As before, let $\Gamma_1, \Gamma_2, \Gamma_\beta$ be the corresponding compactified and normalized curves. The symmetry $x^1$ is an automorphism of $\Gamma_\beta$ and hence it is either an automorphism of the curves $\Gamma_1$ and $\Gamma_2$, or it permutes these curves (we used that $\Gamma_1$ is irreducible). Suppose first that $x^1$ is an automorphism of $\Gamma_1$. Then the rational function

$$\frac{N_1(\lambda, t)}{\lambda(\lambda - 1)(\lambda - t)} = 1 + \frac{A_1}{\lambda} + \frac{B_1}{\lambda - 1} + \frac{C_1}{\lambda - t}$$

is invariant under the action of $x^1$ too. Here $A_1, B_1, C_1$ are polynomials in $t$ which divide $t, t - 1$ and $(t - \lambda)$ respectively (see 4), and hence we have

$$A_1(1 - t) = -B_1(t), B_1(1 - t) = -A_1(t), C_1(1 - t) = -C_1(t).$$

Similarly, if

$$\frac{N_2(\lambda, t)}{\lambda(\lambda - 1)(\lambda - t)} = 1 + \frac{A_2}{\lambda} + \frac{B_2}{\lambda - 1} + \frac{C_2}{\lambda - t}$$

with $A_2, B_2, C_2$ divisible by $\lambda - 1$ and $\lambda - \lambda$ respectively, then

$$(\lambda - 1)(\lambda - \lambda)$$

is a factor of $N_2(\lambda, t)$. Hence $\lambda(\lambda - 1)(\lambda - t)$ is a factor of $N_2(\lambda, t)$ and $\Gamma_\beta^{aff}$ is connected.

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then
\[ A_2(1 - t) = -B_2(t), B_2(1 - t) = -A_2(t), C_2(1 - t) = -C_2(t). \]

We conclude that \( C_1(t) = c_1(t - 1), C_2(t) = c_2(t - 1) \), which contradicts to \( C_1C_2 = -\beta_3(t - 1)/\beta_1, \beta_1, \beta_3 \neq 0 \). Suppose now that the map \( x^1 \) exchanges the curves \( \Gamma_1 \) and \( \Gamma_2 \). Then we have
\[ A_1(1 - t) = -B_2(t), B_1(1 - t) = -B_2(t), C_1(1 - t) = -C_2(t). \]

The polynomial \( N(\lambda, t) \) is of degree three with respect to \( t \) and
\[ A_1A_2 = -\frac{\beta_1}{\beta_0}t, B_1B_2 = -\frac{\beta_2}{\beta_0}(t - 1), C_1C_2 = -\frac{\beta_3}{\beta_0}t(t - 1). \]

Therefore without loss of generality we may suppose that \( C_1(t) = c_1t, C_2 = c_1(t - 1) \) and \( A_1(t) = a_1t, B_2 = b_2(t - 1) \) or \( A_2(t) = a_2t, B_1 = b_1(t - 1) \), where \( a_i, b_j \neq 0 \). In both cases the polynomials \( N_1(\lambda, t), N_2(\lambda, t) \) are of degree two in \( t \), in contradiction to the fact that the degree of \( N(\lambda, t) \) with respect to \( t \) is three. We conclude that the curve \( \Gamma_\beta \) does not define a solution.

### 2.5 The face \( \beta_0 = \beta_2, \beta_1 = \beta_3 \).

We have the identity
\[ N(\lambda, t) = (\lambda^2 - 2\lambda + t)(\beta_0\lambda^2(\lambda - t)^2 - \beta_1t^2(\lambda - 1)^2). \]

Indeed,
\[ \lambda^2 - 2\lambda + t = 0 \quad (10) \]
defines a solution of \( \text{PVI}_{\alpha} \), e.g. [3], Table 2, solution 2B. Its dessin is equivalent to the one on fig. 2 (ii). The function \( \lambda(t) \) defined by
\[ \lambda(\lambda - t) - ct(\lambda - 1) = 0, c = \pm \sqrt{\frac{\beta_1}{\beta_0}} \]
is ramified over \( 0, 1, \infty \) only provided that \( c = 0, \pm 1 \). This is, however, impossible as \( \beta_0 \neq \beta_1, \beta_i \neq 0 \). The curve \( \Gamma_\beta \) defines the solution (10).
2.6 The face $\beta_0 = \beta_1 = \beta_2$.

According to Table 2, each of the permutations $\sigma_0, \sigma_1, \sigma_\infty$ is either the identity, or is a cycle of length three.

If one of the permutations $\sigma_0, \sigma_1, \sigma_\infty$ is the identity then the group generated by $\sigma_0, \sigma_1, \sigma_\infty$ is either $\mathbb{Z}_3$ or the trivial one $\{1\}$, and hence the degree of the corresponding dessin is one or three. The case of degree one does not lead to a solution (see section 2.4). The case of degree three is studied as in section 2.4 and does not lead to a solution too (provided that $\beta_i \neq 0$).

If neither of the permutations $\sigma_0, \sigma_1, \sigma_\infty$ is the identity, then they are disjoint three-cycles. As the symmetric group $S_3$ contains only two three-cycles we conclude that the degree of the projection $\pi$ is at least four. Suppose that $\lambda(t)$ is defined by the polynomial $N_1(N_1, t) \equiv 0$, where $N_1 \in \mathbb{C}[\lambda, t]$ is irreducible of degree four in $\lambda$. Then $x^1, x^2$ are automorphisms of

$$\Gamma_1 = \{N_1(\lambda, t) = 0\}.$$ 

It follows that the curve

$$\Gamma_2 = \{N_2(\lambda, t) = 0\}$$

defined by the polynomial $N_2 = N/N_1$ is also invariant. We have

$$\frac{N_2(\lambda, t)}{\lambda(\lambda - 1)} = 1 + \frac{A}{\lambda} + \frac{B}{\lambda - 1}$$

where $A, B$ are polynomials in $t$ of degree at most three. The $x^{1,2}$ invariance of the above expression implies

$$A(t)A(1/t) = 1, A(1/t)B(t) = -B(1/t), B(t) = -A(1 - t).$$

with solutions

$$A(t) = t, B(t) = t - 1; A(t) = t^3, B(t) = (t - 1)^3; A(t) = -t^2, B(t) = (t - 1)^2.$$ 

The case $A(t) = t^3, B(t) = (t - 1)^3$ does not lead to a solution as $N_1(\lambda, t)$ does depend on $t$. The case $A(t) = -t^2, B(t) = (t - 1)^2$ implies $N_2(\lambda, t) = (\lambda - t)^2$, and hence $\beta_3 = 0$. Finally, in the case $A(t) = t, B(t) = t - 1$ we have $N_2(\lambda, t) = \lambda^2 - 2\lambda + 2\lambda t - t$ (which does not define a solution). The condition that $N_2(\lambda, t)$ divides $N(\lambda, t)$ leads to $\beta_3 = 9\beta_0$ and we get

$$N(\lambda, t) = (\lambda^2 - 2\lambda + 2\lambda t - t) \left( t^2 - 4\lambda^3t + 6\lambda^2t - 4\lambda t + \lambda^4 \right).$$
The function $\lambda(t)$ defined by
\[ t^2 - 4\lambda^3 t + 6\lambda^2 t - 4\lambda t + \lambda^4 = 0 \] (11)
is indeed a solution of $\text{PVI}_\alpha$, e.g. [3], Table 2, solution 3D. In the case when the dessin corresponding to $\lambda(t)$ is of degree five we conclude that the polynomial $N_2(\lambda, t)$ is linear in $\lambda$. By Proposition 2 we get $N_2(\lambda, t) = \lambda - t^2$ which implies $\beta_1 = 0$. To resume, the curve $\Gamma_\beta$ defines a solution, provided that $\beta_0 = \beta_1 = \beta_2 = \beta_3/9$.

2.7 The face $\beta_0 = \beta_1 = \beta_2 = \beta_3$.

We have
\[ N(\lambda, t) = \beta_0(\lambda^2 - 2\lambda + t)(\lambda^2 - 2\lambda t + t)(\lambda^2 - t) \]
and the three algebraic functions defined by $N(\lambda, t) = 0$ are solutions of suitable $\text{PVI}_\alpha$ equations, e.g. [3], Table 2, solutions 2B, 2C, 2A respectively. The curve $\Gamma_\beta$ defines three solutions.

2.8 The face $\beta_3 = 0$.

Recall that in this case $N(\lambda, t) = (\lambda - t)^2 N_0(\lambda, t)$ where
\[ N_0(\lambda, t) = \beta_0\lambda^2(\lambda - 1)^2 - \beta_1t(\lambda - 1)^2 + \beta_2(t - 1)\lambda^2. \]
The same arguments as in section 2.3 show that the corresponding dessin d’enfant is of degree one or two.

If the degree is one, then the solution is $\lambda = P(t)$ for some non-constant polynomial $P$. Therefore $N_0(P(t), t) \not\equiv 0$ and $P(t)$ can not be a solution.

If the degree is two, then $\lambda(t)$ has exactly two ramification points. Without loss of generality we suppose that these points a 0 and $\infty$, and as in section 2.3 we conclude that $\lambda(t)$ is defined by $\lambda^2 + 2p(t)\lambda + q(t) = 0$ for some $p, q \in \mathbb{C}[t]$. The polynomial $\lambda^2 + 2p(t)\lambda + q(t)$ divides
\[ N_0(\lambda, t) = t(\beta_2\lambda^2 - \beta_1(\lambda - 1)^2) + \beta_0\lambda^2(\lambda - 1)^2 - \beta_2\lambda^2 \]
and hence $p(t) = c_1$ and $q(t) = c_2t$ for some constants $c_1, c_2$. Without loss of generality we suppose that the ramification points of $\lambda(t)$ are 0 and $\infty$ and hence the discriminant $4(p^2 - q)$ is a power of $t$. This implies that $c_1 = 0$. Finally, a direct computation shows that the identity $N_0(\sqrt{-c_2t}, t) \equiv 0$ implies $\beta_2 = 0$ which is not true. The curve $\Gamma_\beta$ does not define a solution.
2.9 The face $\beta_3 = 0, \beta_0 = \beta_2$.

It is easier to analyze the face $\beta_3 = 0, \beta_1 = \beta_2$, which is equivalent to $\beta_3 = 0, \beta_0 = \beta_2$ after applying the transformation $x^2$. Suppose for a moment that $\beta_3 = 0, \beta_1 = \beta_2$. The dessin is of degree at most three, and hence $N^0(\lambda, t)$ is reducible. It follows that $\lambda - c$ divides $N^0(\lambda, t)$ for some constant $c$. As $N^0(\lambda, t)$ is linear in $t$, then $\lambda - c$ is deduced from the coefficient of $t$ which equals to $\beta_1(1 - 2\lambda)$. Thus $c = 1/2$ and the condition that $1 - 2\lambda$ divides $N^0(\lambda, t)$ leads to $\beta_0 = 4\beta_1 = 4\beta_2$, in which case

$$N^0(\lambda, t) = \beta_0 (2\lambda - 1) (2\lambda^3 - 3\lambda^2 + t).$$

The function $\lambda(t)$ defined by $(2\lambda^3 - 3\lambda^2 + t) = 0$ is indeed a solution, see [3], Table 2, solutions 5L. Applying the transformation $(x^2)^{-1} = x^2$ of section 2.1 we get the solution (see [3], Table 2, solution 5K

$$\lambda^3 - 3\lambda t + 2t = 0$$

defined by $\Gamma_\beta$ with $\beta_3 = 0, \beta_1 = 4\beta_0 = 4\beta_2$. The curve $\Gamma_\beta$ defines a solution provided that $\beta_3 = 0, \beta_1 = 4\beta_0 = 4\beta_2$.

2.10 The face $\beta_3 = 0, \beta_0 = \beta_1 = \beta_2$.

The polynomial $N^0(\lambda, t)$ is irreducible and defines a solution, see [3], Table 2, solution 4D.

2.11 The face $\beta_3 = 0, \beta_2 = 0$.

A direct computation shows that the relation $\beta_0 \lambda^2 - \beta_1 t = 0$ defines a solution.

2.12 The face $\beta_3 = 0, \beta_2 = 0, \beta_0 = \beta_1$.

The solution is $\lambda^2 = t$.

The results are summarized in Table 1. Theorem 1 is proved.

3 Algebraic solutions of PVI$_\alpha$ and Picard-Fuchs equations

It was noted in the Remark after Theorem 1 that the families 1A, 1B, \ldots, 1F are Okamoto equivalent to the families 2A, 2B, 2C. To this end we consider
the remaining 23 families $2A - 5L$, see Table 1. To each of them corresponds an affine plane or line in the parameter space $\mathbb{C}^4\{\alpha\}$ which, as we shall prove bel ow, is generated by special points $\alpha$ of geometric origin, see Tables 4 and 5. Indeed, observe that exactly the same 23 solutions $2A - 5L$ were already obtained by Doran, see Theorem 2 below, by making use of deformations of elliptic surfaces with four singular fibers. The corresponding special values of the parameter $\alpha$ are given in Table 4. The main result of this section is that exactly the same list of solutions can be obtained from deformations of ramified covers of $\mathbb{P}^1$ with four ramification points. The corresponding values of the parameters $\alpha$ are different and are shown on Table 5, see Theorem 3.

Recall that an elliptic surface is a complex compact analytic surface $S$ with a projection $S \rightarrow \mathbb{P}^1$, such that the general fiber $f^{-1}(z) = \Gamma_z$ is an elliptic curve. Two elliptic surfaces are equivalent, if there is a bi-analytic map compatible with the projections, see [14, Kodaira].

We may suppose that the fiber $\Gamma_z$ is written in the Weierstrass form

$$\Gamma_z = \{(x, y) \in \mathbb{C}^2 : y^2 = 4x^3 - g_2(z)x - g_3(z)\}$$

and consider the complete elliptic integrals of first and second kind

$$\eta_1 = \int_{\gamma(z)} \frac{dx}{y}, \quad \eta_2 = \int_{\gamma(z)} \frac{xdx}{y}$$

where $\gamma(z) \subset \Gamma_z$ is a continuous family of closed loops (representing a locally constant section $z \mapsto H_1(\Gamma_z, \mathbb{Z})$ of the associated homology bundle). Then $\eta_1, \eta_2$ satisfy the following Picard-Fuchs system (this goes back at least to [6, Griffiths], see [17, Sasai])

$$\Delta(z) \frac{d}{dz} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \frac{\Delta'}{12} & -\frac{3\delta}{2} \\ \frac{-g_2^3}{8} & \frac{\Delta}{12} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

(12)

where

$$\Delta(g_2, g_3) = g_3^2 - 27g_2^3,$$

and

$$\delta(z) = 3g_3 \frac{dg_2}{dz} - 2g_2 \frac{dg_3}{dz}.$$

The singular points of the system correspond to singular fibers of the surface. The elliptic surfaces with four singular fibers were classified by [9, Herfurtner] who obtained 50 distinct case, but only 5 of them contain an additional
parameter, see Table 3. They lead to non-trivial isomonodromic deformations of the above Picard-Fuchs system with four regular singular points. If we renormalize the singular points to be 0, 1, ∞, t then the zero λ of δ(z), considered as a function in t is a solution of an appropriate PVIα equation, see [16] for details. The result is summarized as follows

**Theorem 2** [3, Theorem 3.13] All algebraic solutions (λ(t), α) of PVIα equation coming from moduli of elliptic surfaces with four singular fibers are shown on Table 1. The corresponding values of α together with the stabilizer of the solution and the PVIα equation under the action of the symmetric group S4 are listed on Table 4.

The Picard-Fuchs system (12) has infinite monodromy group. We shall deduce similar families of Picard-Fuchs systems having a finite monodromy. Consider a ramified covering Γ → P1 of degree three with branching locus consisting of four points, where Γ is a Riemann surface. We choose an affine model Γaff = \{(x, z) ∈ C2 : f(x, z) = 0\} of Γ, where f(x, z) = 4x3 - g2x - g3, and g2 = g2(z), g3 = g3(z) are suitable polynomials. Moreover, without loss of generality, we suppose that the covering Γ → P1 is induced from the projection

\{(x, z) ∈ C2 : f(x, z) = 0\} → C : (x, z) ↦→ z. (13)

Let x1(z), x2(z) be two distinct roots of f. Then γ(z) = x1(z) - x2(z) is a 0-cycle of the fiber \{x : f(x, z) = 0\} and the Abelian integrals above are replaced by the algebraic functions

\[ η_1(z) = \int_{γ(z)} x = x_1(z) - x_2(z), \quad η_2(z) = \int_{γ(z)} x^2 = x_1^2(z) - x_2^2(z). \]

A straightforward computation shows that η1, η2 satisfy the following Picard-Fuchs system (see [5])

\[ \triangle(z) d \frac{d}{dz} \begin{pmatrix} η_1 \\ η_2 \end{pmatrix} = \begin{pmatrix} \frac{\Delta'}{6} & -3\delta \\ -\frac{g_2\delta}{2} & \frac{\Delta'}{3} \end{pmatrix} \begin{pmatrix} η_1 \\ η_2 \end{pmatrix}. \]

As before, if we renormalize the system (14) to have singular points at 0, 1, t, ∞, then the root λ(t) of δ(z) is a solution of a suitable PVIα equation, provided that the deformation is isomonodromic. The last property holds, strictly speaking, in the case when the system is non-resonant. In our case it holds too, because the deformation is isopincipal in the sense of [11]. This
can be also checked by a direct computation. Thus, if we consider the fibration (13) and take for \(g_2, g_3\) the expressions found by Herfutner, see Table 3, we get the same 23 algebraic solutions shown on Table 1. The corresponding values for \(\alpha\) are of course different and are computed in Table 5. In this way we prove

**Theorem 3** The algebraic solutions \((\lambda(t), \alpha)\) of \(\text{PVI}_\alpha\) equation coming from deformations of the covering (13) with \(g_2, g_3\) as on the Herfutner list, Table 3, are shown on Table 1. The corresponding values of \(\alpha\) together with the stabilizer of the solution and the \(\text{PVI}_\alpha\) equation under the action of the symmetric group \(S_4\) are listed on Table 5.

**Remark 3** As we already noted, the monodromy group of the Picard-Fuchs system (14) is finite. Particular cases of this system, in a more or less explicit way, were considered by many authors, e.g. [2, Boalch], [4, Dubrovin-Mazzocco], [7, 8, Hitchin], [13, Kitaev].

To this end we present, for convenience of the reader, one example of such computation. Using the Picard-Fuchs system (14), the Abelian integral of first kind \(\eta_1\) satisfies the following equation

\[
p_0(z, a)\eta_1'' + p_1(z, a)\eta_1' + p_2(z, a)\eta_1 = 0 \tag{15}
\]

where

\[
p_0(z, a) = 144\delta^2 \\
p_1(z, a) = 144\Delta(\delta \frac{d\Delta}{dz} - \Delta \frac{d\delta}{dz}) \\
p_2(z, a) = 12\delta \frac{d^2\Delta}{dz^2} - 216\delta^3g_2 - 12\Delta \frac{d\delta}{dz} \frac{d\Delta}{dz} - \delta(\frac{d\Delta}{dz})^2.
\]

Consider, for instance, the deformation 2 from the Herfurtner list (Table 3)

\[
g_2 = g_2(z, a) = 3z^3(z + a) \\
g_3 = g_3(z, a) = z^5(z + 1).
\]

We have

\[
\Delta = \Delta(g_2, g_3) = 27z^9((3a - 2)z^2 + (3a^2 - 1)z + a^3) \\
\delta = \delta(z, a) = -3z^7((3a - 2)z + a).
\]
Table 3: The Herfurtner list of "deformable" elliptic surfaces with four singular fibers

The Picard-Fuchs equation \([15]\) takes the form

\[
144z^2((3a - 2)z + a)((3a - 2)z^2 + (3a^2 - 1)z + a^3)^2 \eta''
+ 144z((3a - 2)z^2 + (3a^2 - 1)z + a^3)(3(3a - 2)^2z^3
+ 2(3a - 2)(3a - 1)(a + 1)z^2 + a(3a^3 + 7a^2 - 3)z + 2a^4) \eta'_1
+ [135(3a - 2)^3z^5 + (3a - 2)^2(468a^2 + 267a - 164)z^4
+ 2(3a - 2)(189a^4 + 522a^3 - 48a^2 - 208a + 10)z^3
- 2a(270a^5 - 1269a^4 + 252a^3 + 460a^2 - 70)z^2
- a^4(243a^3 - 666a^2 + 176)z + 27a^7] \eta_1 = 0
\]

and has four regular singular points at \(\infty\) and the roots of \((3a - 2)z^2 + (3a^2 - 1)z + a^3\) (the roots of \(\Delta\)), as well one apparent singularity at the root of \((3a - 2)z + a\) (which is a root of \(\delta\)). Re-normalizing the singular points to \(0, 1, t, \infty\) we get

\[
\lambda = \frac{a^2 - a + 1}{a^2(2 - a)}, \quad t = \frac{2a - 1}{a^3(2 - a)}, \quad a \in \mathbb{C}.
\]
The parameter $a$ defines an algebraic isomonodromic deformation of the Picard-Fuchs equation (15) with Riemann schema

$$
\begin{pmatrix}
0 & 1 & t & \lambda & \infty \\
-\frac{3}{4} & -\frac{1}{4} & 0 & 0 & \frac{3}{4} \\
-\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & 0 \\
\end{pmatrix}.
$$

Therefore the algebraic function $\lambda = \lambda(t)$ determined implicitly by (16) is an algebraic solution of \text{PVI}_\alpha equation with

$$
\alpha_0 = \frac{1}{8}, \quad \alpha_1 = \frac{1}{8}, \quad \alpha_2 = \frac{1}{8}, \quad \alpha_3 = \frac{1}{8}.
$$

(see [10] for details). Eliminating $a$ from (16) we get

$$
\lambda^4 - 2t\lambda^3 - 2\lambda^3 + 6t\lambda^2 - 2t^2\lambda - 2t\lambda + t^3 - t^2 + t = 0
$$

which is an equation for the solution 4A with $\alpha = (1/8, 1/8, 1/8, 1/8)$. Together with the Doran’s point $\alpha = (0, 1/8, 0, 0)$, this implies that the solution $\lambda(t)$ satisfies also the implicit equation (1)

$$
1 + \frac{t - 1}{(\lambda - 1)^2} - \frac{t(t - 1)}{(\lambda - t)^2} = 0
$$

corresponding to the affine line through $(1/8, 1/8, 1/8, 1/8)$ and $(0, 1/8, 0, 0)$ described in Table 1, 4A. The solution (16) with $\alpha = (1/8, 1/8, 1/8, 1/8)$ was found by Hitchin [7, section 6.1], [8, (34)].

**Remark 4** If we repeat the same computation, but making use of the Picard-Fuchs system (12) then of course we obtain the same algebraic solution but with $\alpha = (1/8, 0, 0, 0)$. This value has been erroneously computed by Doran [3] to be $(1/18, 0, 0, 0)$. This led him to the wrong conclusion that the solution 4C is equivalent by an Okamoto transformation to the "cubic" solution $B_3$ of Dubrovin-Mazzocco [4, p.140] with $\alpha = (25/18, 0, 0, 0)$, see [3, Remark 7]. As the Okamoto transformations of $\text{PVI}_\alpha$ act within the ring $\mathbb{Z}[1/2, \sqrt{2}\alpha_1, \sqrt{2}\alpha_1, \sqrt{2}\alpha_1, \sqrt{2}\alpha_1]$, then no solution of $\text{PVI}_{(25/18,0,0,0)}$ is equivalent to a solution of $\text{PVI}_{(1/8,0,0,0)}$. M. Mazzocco kindly informed us for a missprint in the formula for the $B_3$-solution, [4, p.140]. The corrected formula is reproduced in [3, formula (3.1)].
| Stabilizer of the solution | Name of the solution | $\text{PVI}_\alpha$ equation | Stabilizer of $\text{PVI}_\alpha$ equation |
|---------------------------|---------------------|-----------------------------|-------------------------------------------|
| $D_4$                     | $2A$                | $(0, 0, \frac{1}{18}, \frac{1}{18})$ | $S_2 \times S_2$                        |
|                           | $2B$                | $(\frac{1}{18}, 0, \frac{1}{18}, 0)$ |                                           |
|                           | $2C$                | $(\frac{1}{18}, 0, 0, \frac{1}{18})$ |                                           |
|                           |                     | $(0, \frac{1}{18}, 0, \frac{1}{18})$ |                                           |
| $D_4$                     | $2A$                | $(0, 0, 0, 0)$               | $S_4$                                     |
|                           | $2B$                |                               |                                           |
|                           | $2C$                |                               |                                           |
| $S_3$                     | $3A$                | $(0, 0, 0, 0)$               | $S_4$                                     |
|                           | $3B$                |                               |                                           |
|                           | $3C$                |                               |                                           |
|                           | $3D$                |                               |                                           |
| $S_3$                     | $4A$                | $(0, \frac{1}{5}, 0, 0)$     | $S_3$                                     |
|                           | $4B$                | $(0, 0, \frac{1}{5}, 0)$     |                                           |
|                           | $4C$                | $(\frac{1}{5}, 0, 0, 0)$     |                                           |
|                           | $4D$                | $(0, 0, 0, \frac{1}{5})$     |                                           |
| $S_2$                     | $5A$                | $(0, \frac{1}{18}, 0, 0)$    | $S_3$                                     |
|                           | $5B$                |                               |                                           |
|                           | $5C$                |                               |                                           |
|                           | $5D$                | $(0, 0, \frac{1}{18}, 0)$    |                                           |
|                           | $5E$                |                               |                                           |
|                           | $5F$                |                               |                                           |
|                           | $5G$                | $(\frac{1}{18}, 0, 0, 0)$    |                                           |
|                           | $5H$                |                               |                                           |
|                           | $5I$                |                               |                                           |
|                           | $5J$                | $(0, 0, 0, \frac{1}{18})$    |                                           |
|                           | $5K$                |                               |                                           |
|                           | $5L$                |                               |                                           |

Table 4: Solutions $(\lambda(t), \alpha)$ of $\text{PVI}_\alpha$ equations related to the Picard-Fuchs system [12].
| Stabilizer of the solution | Name of the solution | $\text{PVI}_\alpha$ equation | Stabilizer of $\text{PVI}_\alpha$ equation |
|---------------------------|---------------------|-------------------------------|------------------------------------------|
| $D_4$                     | 2A                  | $(\frac{1}{8}, \frac{1}{8}, \frac{1}{18}, \frac{1}{18})$ | $S_2 \times S_2$                       |
|                           | 2B                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{18}, \frac{1}{18})$ |                                          |
|                           | 2C                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{18}, \frac{1}{18})$ |                                          |
| $D_4$                     | 2A                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{18})$ | $S_2 \times S_2$                       |
|                           | 2B                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 2C                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{18})$ |                                          |
| $S_3$                     | 3A                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{18})$ | $S_3$                                   |
|                           | 3B                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 3C                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 3D                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{18})$ |                                          |
| $S_3$                     | 4A                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{18})$ | $S_4$                                   |
|                           | 4B                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 4C                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 4D                  | $(\frac{1}{8}, \frac{5}{8}, \frac{1}{8}, \frac{1}{18})$ |                                          |
| $S_2$                     | 5A                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ | $S_3$                                   |
|                           | 5B                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 5C                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 5D                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 5E                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 5F                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 5G                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 5H                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 5I                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 5J                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 5K                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ |                                          |
|                           | 5L                  | $(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18})$ |                                          |

Table 5: Solutions $(\lambda(t), \alpha)$ of $\text{PVI}_\alpha$ equations related to the Picard-Fuchs system (14).
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