The minimum color degree and a large rainbow cycle in an edge-colored graph

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Abstract

Let $G$ be an edge-colored graph with $n$ vertices. A subgraph $H$ of $G$ is called a rainbow subgraph of $G$ if the colors of each pair of the edges in $E(H)$ are distinct. We define the minimum color degree of $G$ to be the smallest number of the colors of the edges that are incident to a vertex $v$, for all $v \in V(G)$. Suppose that $G$ contains no rainbow-cycle subgraph of length four. We show that if the minimum color degree of $G$ is at least $\frac{2n+3k-2}{2}$, then $G$ contains a rainbow-cycle subgraph of length at least $k$, where $k \geq 5$. Moreover, if the condition of $G$ is restricted to a triangle-free graph that contains a rainbow path of length at least $\frac{3k}{2}$, then the lower bound of the minimum color degree of $G$ that guarantees an existence of a rainbow-cycle subgraph of length to at least $k$ can be reduced to $\frac{2n+3k-4}{4}$.

1 Introduction

For a finite simple undirected graph $G$, where $G = (V(G), E(G))$ and $|V(G)| = n$, we define an edge coloring $c$ of $G$ to be a function, where $c : E(G) \rightarrow \mathbb{Z}^+$. For a subgraph $H$ of $G$, the edge coloring of $H$ is the restriction of $c$ to $E(H)$. If the colors of each pair of the edges in $E(H)$ are distinct, then $H$ is called a rainbow subgraph or a heterochromatic subgraph of $G$. The works related to rainbow subgraphs in various types including paths, trees and cycles appear in the survey by M. Kano and X. Li [4].

In this paper, we are interested in a condition of an existence of a large rainbow-cycle subgraph in $G$. In 2005, J.J. Montellano-Ballesteros and V. Neumann-Lara [6] solved the conjecture of Erdős, Simonovits and Sós on the rainbow cycle. They gave a condition of the number of the colors in a complete graph $K_n$ that guarantees an existence of a rainbow-cycle subgraph. Let $c(E(G))$ be the set of the colors of the edges appearing in $G$. They showed that if $|c(E(K_n))| = n \left( \frac{k-2}{2} + \frac{1}{k-1} \right) + O(1)$, then $G$ contains a rainbow cycle of length at least $k$. Meanwhile, H. J. Broersma et. al. [1] showed that if $|c(E(G))| \geq n$, then $G$ contains a rainbow cycle of length at least $\frac{2|c(E(G))|}{n}$.

In 2012, H. Li and G. Wang [5] took a different approach and studied the existence of the rainbow-cycle subgraph of $G$ by considering its minimum color degree $\delta^c(G)$ of $G$, which is the smallest number of all distinct colors of the edges.
that are incident to a vertex \( v \), for all \( v \in V(G) \). In Theorem 1.1 H. Li and G. Wang showed that, for a triangle-free graph \( G \) with at least eight vertices, if \( \delta^c(G) \geq \frac{3}{5}n+1 \), then \( G \) contains a rainbow cycle of length at least \( \delta^c(G) - \frac{3}{5}n+2 \).

The largest lower bound of the guaranteed length of the rainbow-cycle subgraph is at least \( k \) in Theorem 1.2. We adopt the notations used by Čada et. al. [7]. For a pair of vertices \( u, v \) of \( G \), \( \delta^c \) is the set of the colors of the edges joining the vertex \( u \) and \( v \), \( |\delta^c(G)| \) is the set of the colors of the edges joining the vertex \( u \) and a vertex in \( H \). Let \( P \) be a path \( u_1u_2\ldots u_p \) and let \( u_iPu_j \) be a subpath of \( P \) that starts at \( u_i \) and ends at \( u_j \), where \( i, j \in \{1, \ldots, p\} \). Throughout this paper, we let \( G \) be an edge-colored graph with an edge-coloring \( c \) and \( |V(G)| = n \).

**Theorem 1.1.** [7] Let \( G \) be a triangle-free graph with \( n \) vertices, where \( n \geq 8 \). If \( \delta^c(G) \geq \frac{3}{5}n+1 \), then \( G \) contains a rainbow cycle of length at least \( \delta^c(G) - \frac{3}{5}n+2 \).

In 2016, R. Čada, A. Kaneko and Z. Ryjáček [7] gave a sufficient condition of the minimum color degree \( \delta^c(G) \) of \( G \) so that \( G \) contains a rainbow-cycle subgraph of length at least four in Theorem 1.2. We adopt the notations used by Čada et. al. [7]. For a pair of vertices \( u, v \in V(G) \), where \( uv \in E(G) \), let \( c(uv) \) be the color of the edge \( uv \). For a subgraph \( H \) of \( G \), the notation \( c(u, H) \) is the set of the colors of the edges joining the vertex \( u \) and a vertex in \( H \). Let \( P \) be a path \( u_1u_2\ldots u_p \) and let \( u_iPu_j \) be a subpath of \( P \) that starts at \( u_i \) and ends at \( u_j \), where \( i, j \in \{1, \ldots, p\} \). Throughout this paper, we let \( G \) be an edge-colored graph with an edge-coloring \( c \) and \( |V(G)| = n \).

**Theorem 1.2.** [7] If \( \delta^c(G) > \frac{3}{5}n+2 \), then \( G \) contains a rainbow cycle of length at least four.

Lemma 1.3 and 1.4 are used to prove Theorem 1.2. We will later use these lemmas to prove the main theorem.

**Lemma 1.3.** [7] For a graph \( G \), let \( P = u_1\ldots u_p \) be the longest rainbow path of \( G \). If \( G \) contains no rainbow cycle of length at least \( k \), where \( k \leq p \), then for any color \( a \in c(u_1, u_kPu_p) \) and vertex \( u_i \in V(u_kPu_p) \), where \( c(u_1u_i) = a \), there is an edge \( e \in E(u_1Pu_i) \) such that \( c(e) = a \).

**Lemma 1.4.** [7] For a graph \( G \), let \( P = u_1\ldots u_p \) be the longest rainbow path of \( G \). If \( G \) contains no rainbow cycle of length at least \( k \), where \( k \leq p \), then for any positive integers \( s, t \) such that \( s + t = k \),

\[
|c(u_1, u_kPu_p-(t-1)) \cap c(u_p, u_sPu_p-(k-1))| \leq 1.
\]

From Theorem 1.2 Čada et. gave the following conjecture.

**Conjecture 1.5.** If \( \delta^c(G) \geq \frac{n+k}{2} \), then \( G \) contains a rainbow-cycle subgraph of length at least \( k \).

In this work, by using the method appearing in Theorem 1.2, we give a progress toward Conjecture 1.5. In our main theorem, we showed that if \( G \) does not contain a rainbow cycle of length four and \( \delta^c(G) \geq \frac{n+3k-2}{2} \), then \( G \) contains a rainbow cycle of length at least \( k \). In Section 3 we will discuss the result in the main theorem in comparison with Theorem 1.4. We showed that by restricting the condition of Theorem 1.1 to be rainbow-\( C_4 \)-free, we can ignore the condition that the graph is triangle-free and the length of the guaranteed rainbow-cycle subgraph can be at least \( \frac{k}{2} - \frac{n+k}{4} \) larger than in Theorem 1.1.
In order to apply Theorem 2.1, we require an existence of a long rainbow path. In 2014, A. Das, S. V. Subrahmanya and P. Suresh [3] gave a lower bound of the length of the longest rainbow path in the term of the lower bound of the minimum color degree in Theorem 1.6.

**Theorem 1.6.** [3] Let $G$ be an edge-colored graph, where $\delta^c(G) \geq t$ and $t \geq 8$. The maximum length of the rainbow paths in $G$ is at least $\lceil \frac{3t}{5} \rceil$.

In 2016, H. Chen and X. Li gave a larger lower bound of the length of the longest rainbow path with respect to the lower bound of $\delta^c(G)$.

**Theorem 1.7.** [2] Let $G$ be an edge-colored graph. If $\delta^c(G) \geq t \geq 7$, then $G$ contains a rainbow path of length at least $\lceil \frac{2t}{3} \rceil + 1$.

## 2 Main Results

Theorem 2.1 is obtained by using the method in Theorem 1.2 with some generalization. As a result, Theorem 2.1 gives a lower bound of $\delta^c(G)$ guaranteeing a rainbow-cycle subgraph of length at least $k$, where $k \geq 5$, in a graph containing no rainbow-cycle subgraph of length four. The largest length of the guaranteed rainbow-cycle subgraph of $G$ in Theorem 1.1 is at most $\frac{n^2}{4} + 1$, while, it is $\frac{n^2}{4} + 4$ in Theorem 2.1.

**Theorem 2.1.** Let $G$ be a graph with no rainbow-cycle subgraph of length four. If $\delta^C(G) \geq \frac{n + 3k - 2}{2}$, then $G$ contains a rainbow-cycle subgraph of length at least $k$, where $k \geq 5$.

**Proof.** For a fixed $k \geq 5$. Suppose that $G$ contains no rainbow-cycle subgraph of length at least $k$. Since $|V(G)| \geq \delta^c + 1$, it follows that $n \geq 3k$. Let $P = u_1u_2\ldots u_p$ be the longest rainbow path in $G$. By Theorem 1.7 it follows that

$$p \geq \left\lceil \frac{(n + 3k - 4)}{3} \right\rceil + 1 \geq 2k.$$  

Let $s, t \in \mathbb{N}$ be such that $s = \lfloor \frac{k}{2} \rfloor$ and $t = \lceil \frac{k}{2} \rceil$. So $p - (t - 1) > k$ and $p - (k - 1) > s$. Let $A = c(u_1, u_kPu_{p-(t-1)})$ and $B = c(u_p, u_sPu_{p-(k-1)})$. By Lemma 1.4 we have $|A \cap B| \leq 1$. Let $P^C$ be the subgraph of $G$ induced by $V(G) - V(P)$ and let

$$C_0 = (c(u_1, P^C) \setminus c(u_1, P)) \cap (c(u_p, P^C) \setminus c(u_p, P)).$$

So $C_0 \cap (A \cup B) = \emptyset$. By Lemma 1.3 if $a \in A \cup B$, then $a \in c(E(P))$. If $a \in C_0$, then $a \in c(E(P))$; otherwise, there exists a longer rainbow path.

By Lemma 1.3 if $c(u_1u_2) \in B$, then there exists an edge $e \in E(u_kPu_p)$ such that $c(e) = c(u_1u_2)$; hence, the path $P$ is not rainbow which is a contradiction. So, $c(u_1u_2) \notin B$ and, similarly, $c(u_{p-1}u_p) \notin A$. Since $P$ is rainbow and $G$...
contains no rainbow-cycle subgraph of length at least \( k \), it follows that if \( u_1 u_p \in E(G) \), then \( c(u_1 u_p) \in c(E(P)) \). Let

\[
\epsilon_1 = \begin{cases} 
1, & \text{if } c(u_1 u_2) \notin A, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
\epsilon_2 = \begin{cases} 
1, & \text{if } c(u_{p-1} u_p) \notin B, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
\epsilon'_1 = \begin{cases} 
1, & \text{if } c(u_1 u_p) \notin A \cup c(u_1 u_2), \\
0, & \text{otherwise,}
\end{cases}
\]

\[
\epsilon'_2 = \begin{cases} 
1, & \text{if } c(u_1 u_p) \notin B \cup c(u_p u_{p-1}), \\
0, & \text{otherwise.}
\end{cases}
\]

So,

\[
|V(P)| = |E(P)| + 1 \\
\geq |c(E(P))| + 1 \\
\geq |A \cup B| + |C_0| + \epsilon_1 + \epsilon_2 + \epsilon'_1 + \epsilon'_2 + 1 \\
\geq |A| + |B| + |C_0| + \epsilon_1 + \epsilon_2 + \epsilon'_1 \epsilon_2. \quad (1)
\]

Let \( C_1 = c(u_1, P^C) \setminus (C_0 \cup c(u_1, P)) \) and \( C_2 = c(u_p, P^C) \setminus (C_0 \cup c(u_p, P)) \).

For each \( a \in C_1 \cap C_2 \), we have \( a \in c(E(P)) \); otherwise, there exists a longer rainbow path. By the construction of \( C_0, C_1 \) and \( C_2 \), it follows that

\[
C_1 \cap C_2 = ((c(u_1, P^C) \setminus c(u_1, P)) \cap (c(u_1, P^C) \setminus c(u_p, P))) \setminus C_0 = \emptyset.
\]

Suppose \( C_1 = \{c_1, \ldots, c_{|C_1|} \} \) and \( C_2 = \{c'_1, \ldots, c'_{|C_1|} \} \). Let \( X_{c_i} \) be a subset of \( N_{PC}(u_1) \), where \( X_{c_i} = \{v \in N_{PC}(u_1) : c(u_1 v) = c_i \} \), for all \( i \in \{1, \ldots, |C_1| \} \). Similarly, let \( X_{c'_j} \) be a subset of \( N_{PC}(u_p) \), where \( X_{c'_j} = \{v \in N_{PC}(u_p) : c(u_p v) = c'_j \} \), for all \( j \in \{1, \ldots, |C_2| \} \). Next, we choose one vertex \( x_i \) from each \( X_{c_i} \) and one vertex \( y_j \) from \( X_{c'_j} \), where \( i \in \{1, \ldots, |C_1| \} \) and \( j \in \{1, \ldots, |C_2| \} \). We have \( \{x_1, \ldots, x_{|C_1|} \} \) and \( \{y_1, \ldots, y_{|C_2|} \} \). Since \( C_1 \cap C_2 = \emptyset \) and \( G \) contains no rainbow-cycle subgraph of length four, it follows that

\[
|\{x_1, \ldots, x_{|C_1|} \} \cap \{y_1, \ldots, y_{|C_2|} \}| \leq 1.
\]

So,

\[
|V(P^C)| \geq |C_0| + |C_1| + |C_2| - 1. \quad (2)
\]

We note that \( |c(u_1, P)| \leq p - 1 \) and \( |A \cup \{c(u_1 u_2)\}| \leq p - t - k + 3 \). Suppose \( |A \cup \{c(u_1 u_2)\}| = l \) and \( l' = (p - t - k + 3) - l \). It follows that the number of the colors in \( c(u_1, P) \) is also less than \( |E(P)| \) by at least \( l' \). So,

\[
|c(u_1, P)| \leq (p - 1) - l' = t + k - 4 + l.
\]
Hence,

\[ |c(u_1, P) \setminus A \cup \{c(u_1u_2)\}| = |c(u_1, P)| - |A \cup \{c(u_1u_2)\}| \]
\[ \leq (t + k - 4 + l) - l \]
\[ = t + k - 4. \]  

(3)

Analogously, since \(|B \cup \{c(u_{p-1}u_p)\}| \leq p - s - k + 2\), it follows that

\[ |c(u_p, P) \setminus (B \cup \{c(u_{p-1}u_p)\})| \leq s + k - 3. \]  

(4)

Next, we consider the number of the colors of the edges that are incident to \(u_1\). Since, \(A, C_0, C_1\) and \(c(u_1, P) \setminus A \cup \{c(u_1u_2)\}\) are all disjoint, it follows that

\[ |A| + |C_0| + |C_1| + \epsilon_1 + \epsilon'_1 + (k + t - 4) \geq dc(u_1) \geq d^c(G). \]  

(5)

So, if we omit \(\epsilon'_1\), then

\[ |A| + |C_0| + |C_1| + \epsilon_1 \geq \delta^c(G) - k - t + 3. \]  

(6)

Similarly,

\[ |B| + |C_0| + |C_2| + \epsilon_2 + \epsilon'_2 + k + s - 3 \geq dc(u_p) \geq \delta^c(G). \]

Hence, if we omit \(\epsilon'_2\), then

\[ |B| + |C_0| + |C_2| + \epsilon_2 \geq \delta^c(G) - k - s + 2. \]  

(7)

By (1), (2), (6) and (7),

\[ |V(P)| + |V(PC)| \geq (|A| + |C_0| + |C_1| + \epsilon_1) + (|B| + |C_0| + |C_2| + \epsilon_2) - 1 + \epsilon'_1 \epsilon'_2 \]
\[ \geq (\delta^c - k - t + 3) + (\delta^c - k - s + 2) - 1 \]
\[ = 2\delta^c - 2k - t - s + 4 \]
\[ = 2\delta^c - 3k + 4 > n, \]

which is a contradiction. Therefore, \(G\) contains a rainbow-cycle subgraph of length at least \(k\). \(\square\)

If \(G\) is triangle-free, then the bound of sizes of \(c(u_1, P), A \cup \{c(u_1u_2)\}\) and \(B \cup \{c(u_{p-1}u_p)\}\) can be reduced to

\[ |c(u_1, P)| \leq \left\lfloor \frac{p-1}{2} \right\rfloor \]
\[ |A \cup \{c(u_1u_2)\}| \leq \left\lfloor \frac{p-t-k+3}{2} \right\rfloor \]
\[ |B \cup \{c(u_{p-1}u_p)\}| \leq \left\lfloor \frac{p-s-k+2}{2} \right\rfloor. \]
Hence, (3) and (4) can be reduced to
\[ |c(u_1, P) \setminus A \cup \{c(u_1u_2)\}| \leq \frac{t + k - 2}{2} \] (8)
and
\[ |c(u_p, P) \setminus (B \cup \{c(u_{p-1}u_p)\})| \leq \frac{s + k - 1}{2}. \] (9)

Thus, if \( G \) is triangle-free with no rainbow-cycle subgraph of length four, then the lower bound of \( \delta^c(G) \) can be reduced to \( \frac{3n+3k-1}{4} \); however, we need a condition of the existence of a rainbow path of length at least \( \frac{3k}{2} \) in \( G \).

**Theorem 2.2.** Let \( G \) be a triangle-free graph with no rainbow-cycle subgraph of length four. If \( G \) contains a rainbow path of length at least \( \frac{3k}{2} \) and \( \delta^c(G) \geq \frac{3n+3k-1}{4} \), then \( G \) contains a rainbow cycle of length at least \( k \), where \( k \geq 5 \).

We note that if we omit the condition of the length of the longest rainbow path in Theorem 2.2, then the condition of only the minimum degree is not able to guarantee the existence of the needed rainbow path. To omit such condition, we combine Theorem 1.7 and Theorem 2.1 which result to Corollary 2.3 as follows.

**Corollary 2.3.** If \( n \geq 3k+1 \) and \( \delta^c(G) \geq \frac{3n+3k-1}{4} \), then \( G \) contains a rainbow cycle of length at least \( k \).

**Proof.** By Theorem 1.7, there exists a path \( u_1 \ldots u_p \), where
\[ p \geq \left\lceil \frac{2n+3k-1}{6} \right\rceil + 1 > \frac{3k}{2}. \]

Hence, by Theorem 2.1, there exists a rainbow-cycle subgraph of length at least \( k \).

\[ \square \]

**3 Discussion**

In this section, we compare the length of the rainbow-cycle subgraphs guaranteed by Theorem 1.1 and Theorem 2.1. We consider a graph \( G \) satisfying the conditions in both theorems. The maximum length of the guaranteed rainbow-cycle subgraph in Theorem 1.1 is at most \( \frac{n}{4} + 1 \), whereas, the guaranteed length of the rainbow-cycle subgraph of \( G \) in Theorem 2.1 can be up to \( \frac{n+3k-2}{2} \). Let \( k \in \mathbb{N} \) be such that \( \delta^c(G) = \left\lceil \frac{n+3k-2}{2} \right\rceil \). Theorem 2.1 implies that \( G \) contains a rainbow-cycle subgraph of length at least \( k \). Next, we show that the guaranteed length of the rainbow-cycle subgraph obtained by Theorem 1.1 of such graph is less than \( k \). Since \( \delta^c(G) = \left\lceil \frac{n+3k-1}{2} \right\rceil \), it follows that
\[
\delta^c(G) = \frac{n + 3k - 1}{2} \quad \text{or} \quad \delta^c(G) = \frac{n + 3k - 2}{2}.
\]
So,

\[ \delta^c(G) = \left( \frac{3n}{4} + 1 \right) + \frac{6k - n - 6}{4} \text{ or } \delta^c(G) = \left( \frac{3n}{4} + 1 \right) + \frac{6k - n - 8}{4}. \]

We note that if \( k < \frac{n + 6}{4} \), then Theorem 1.1 is not applicable. We consider case \( k \geq \frac{n + 6}{4} \). By Theorem 1.1, \( G \) contains a rainbow cycle of length at least

\[ \delta^c(G) - \frac{3}{4}n + 2, \]

which is either

\[ k + \left( \frac{k}{2} - \frac{n + 6}{4} \right) \text{ or } k + \left( \frac{k}{2} - \frac{n + 4}{4} \right), \]

with respect to \( \delta^c(G) \). In order to guarantee a larger length of a rainbow cycle in \( G \), the value of \( k \) in Theorem 1.1 has to be larger than \( \frac{n}{2} \), and hence, \( \delta^c(G) - \frac{3}{4}n + 2 > \frac{k}{2} \) which is not possible, because the maximum guaranteed length of the rainbow cycle from Theorem 1.1 is at most \( \frac{k}{2} - 1 \). Therefore, Theorem 2.1 guarantees an existence of a larger length of a rainbow-cycle subgraph in \( G \) by at least \( \frac{n + 4}{4} - \frac{k}{2} \). However, in order to apply Theorem 2.1, the graph \( G \) cannot contain a rainbow cycle of length four, whereas, this condition is not necessary in Theorem 1.1.

The result in Theorem 2.1 is a progress toward the Conjecture 1.5 given by R. Čada, A. Kaneko and Z. Ryjáček [7]. However, the lower bound of \( \delta^c(G) \) is still larger than the conjectured bound. We also note that Theorem 2.1 is not a generalization of Theorem 1.2 because of the exclusion of the rainbow-cycle subgraph of length four.

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