Numerical investigation of magnetohydrodynamics Williamson nanofluid flow over an exponentially stretching surface

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Abstract
This research work describes the investigation of a magnetohydrodynamic flow of Williamson nanofluid over an exponentially porous stretching surface considering two cases of heat transfer i.e., prescribed exponential order surface temperature (PEST), and prescribed exponential order heat flux (PEHF). As a result of this infestation, a mathematical model of the problem based on conservation of linear momentum and law of conservation of mass and energy is developed. Whereas governing nonlinear partial differential equations (PDEs) are converted to nonlinear ordinary differential equations (ODEs). Subsequently, the velocity, concentration, and temperature profiles are developed by using the method of similarity transformation. Furthermore, the effects of various physical parameters of engineering interests are demonstrated graphically. It is highlighted that both the magnetic parameter ($M$) and Williamson parameter ($\lambda$) causes to reduce the boundary layer thickness.

Keywords
Williamson nanofluid, exponentially stretching sheet, similarity transformation, bvp4c

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Introduction
Numerical and experimental study of nanofluids has gained the attention of researchers considering the importance and practicality.\(^1\,^2\) The nanofluids, which are composed of nano-sized particles fairly homogeneously appended in the base fluid, have been demonstrated to improve fluid properties significantly.\(^3\,^4\) The nano-sized particles, suspended in the fluid, have the ability to augment the thermo-physical properties of the conventional base fluid.\(^6\,^9\) Conventionally, the size distribution of nanoparticles rests closely in the proximity of the size of the base fluid molecules. Therefore, the nanoparticles stay suspended in the base fluid homogeneously for a very long time period without settling or coagulation. Usually, the nanomaterials used for this purpose are carbon nanotubes, metal oxides, nanosized polymers, and nanosized clays.

The idea is principally based on enhancing the thermal properties of the base fluid by producing a nanofluid by enhancing overall thermal conductivity. Various nanofluids having the unique property of increasing the rate of heat transfer in a fluid are being used in many engineering processes like cooling of the vehicle engine, air conditioning cooling, cooling of electronic devices, cooling of power plants, etc. Dey et al.\(^11\) presented a review on nanofluids in which they elaborated one-step and two-step methods of preparation of
fluids. In his work, several methods of increasing the stability of nanofluids and discussed their thermophysical properties like thermal conductivity, the effect on viscosity, etc. were also reported. The idea was propagated by Choi and Eastman based on the fact that the metallic nanoparticles have higher thermal conductivity as compared to the liquid. For example, at room temperature, the thermal conductivity of copper is 700 times higher than water; therefore, the addition of conductive nanoparticles into fluid can enhance heat transfer rate as compared to the rate of heat transfer by conventional fluids alone. Masuda et al. in their research work described a rise in the thermal conductivity of base fluid by dispersal of ultra-fine particles. Wang et al. also presented his experimental work, based on the same principle, for the enhancement of thermal conductivity of nanofluid compared to the base fluid. They observed a reverse relationship exists between the thermal conductivity of nanofluid and the size of the particles. It was noted that on reducing the particle size, the thermal conductivity of a mixture of fluid is increased.

Magnetohydrodynamics plays a significant role in the domain of fluid dynamics where magnetic fields are considered vital. The term magnetohydrodynamics was coined by Swedish Physicist Noble Laureate Hannes Alfven. MHD flow of heat and mass transfer over a stretching surface has practical applications in the field of glass fiber production, polymer technology, and metallurgy. MHD demonstrates a dynamic role in nanofluid flow and heat transfer. Akbar et al. recently investigated the MHD flow of nanofluid due to stretching/shrinking surface with slip effect. Many other researchers have discussed the MHD flow of nanofluids and presented practical solutions.

MHD flow of nanofluid over an exponentially stretching surface and examined heat transfer. Waheed has discussed mixed convective heat transfer in rectangular enclosures driven by a continuously moving horizontal plate.

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The objective of this work is to elaborate the MHD flow of Williamson nanofluid over an exponentially stretching surface. This study particularly emphasizes the heat transfer analysis of Williamson nanofluid where the sheet stretches exponentially which is not addressed so far. Two instances of heat transfer, PEST and PEHF are also discussed. Since governing equations that describe the flow are complex in nature; therefore, analytical solutions are highly unlikely to be obtained satisfactorily. Considering this limitation, we attempted here to solve these equations numerically. Governing highly nonlinear PDEs are reduced into nonlinear ODEs by the assistance of a suitable similarity transformation and subsequently, solving it with the help of bvp4c code. There are seven parameters involved in resulting ODEs and their effects have been demonstrated graphically. The graphical results show that boundary layer thickness is decreasing with the increase of magnetic field \( M \) and Williamson parameter \( \lambda \). It is also seen that thermal boundary layer thickness is achieved little later than the momentum boundary layer.

**Problem description**

Here we have considered steady incompressible MHD Williamson nanofluid flow in two dimensions over a stretching plate. It is assumed that the plate is stretched along \( x \)-axis with the exponentially varying velocity \( U_w \) and \( y \) direction is taken perpendicular to the plate. The adjustable transverse magnetic field \( B = B_0 e^y \) is subjected in a direction perpendicular to the flow. The velocity, temperature, and nanoparticle concentration of the fluid near the surface is taken to be \( U_w \), \( T_w \) and \( C_w \) respectively. The governing equations for the model considered are given by Nadeem and Hussain.
The following similarity transformation is used to solve the governing equations

\[ u = U_0 e^{(\frac{x}{2l})f'(\eta)}, \quad v = -\sqrt{\frac{\nu U_0}{2l}} e^{\frac{x}{2l}}[f(\eta) + \eta f'(\eta)] \]

\[ \eta = \sqrt{\frac{U_0}{2l}} e^{\frac{x}{2l}}, \quad g = \frac{C - C_v}{C_w - C_v} \]

PEST Case:

\[ T = T_\infty + (T_w - T_\infty)e^{\frac{x}{2l}} \theta(\eta) \]

PEHF Case:

\[ T = T_\infty + \frac{(T_w - T_\infty)}{k} e^{\frac{2\nu l}{U_0}} \phi(\eta) \]

Using the above transformations, in equation (2) with the boundary equation given in equation (5) the governing equation takes the following form

\[ f'' - \frac{2f'^2}{f} + \frac{f''}{f} + \lambda f' f'' - Mf'' = 0 \]

\[ f(0) = -s, f'(0) = 1, f'(\infty) \to 0 \]

where, \( M = \frac{2\nu l \theta_i}{\rho U_0}, \lambda = \frac{\nu U_0}{\rho U_0}, s = V_0 \sqrt{\frac{2l}{U_0 \nu}}. \)

Subject to the boundary condition

\[ \theta(0) = 1, g(0) = 1, \theta(\infty) = 0, g(\infty) = 0 \]

where \( N_b = \frac{\rho_{\text{eff}}(C_v - C_w)}{\rho_{\text{eff}}(C_v - C_w)}D_B, N_{\text{f}} = \frac{\rho_{\text{eff}} D_B(T_w - T_\infty)}{\rho_{\text{eff}}}, \text{Sc} = \frac{D_B}{l}, \quad Pr = \frac{\nu}{\rho C_p} \)

Subject to the boundary condition

\[ \phi'' + Pr(\phi' - \phi'' + N_{\text{f}} g' \phi' + N_{\text{f}} \phi'^2) = 0 \]

\[ g'' + Sc g' + \frac{N_{\text{f}}}{N_{\text{b}}} \phi'' = 0 \]

Subject to the boundary condition

\[ \text{Schematic representation of boundary layer flow} \]
Table 1. Comparison of $\sqrt{2Re_c}c_f$ for different values of $\lambda$ and $s$, by fixing $Pr = 0.5$, $Nt = 0.5$, $Nb = 0.5$, $M = 0.0$, $Sc = 1.0$.

| $\lambda$ | $s = -0.1$ | $s = 0.1$ |
|-----------|-------------|------------|
|           | Nadeem and Hussain$^{22}$ | Present | Nadeem and Hussain$^{22}$ | Present |
| 0.0       | 1.33930     | 1.33930    | 1.23638     | 1.23638   |
| 0.1       | 1.29801     | 1.29801    | 1.20710     | 1.20710   |
| 0.2       | 1.26310     | 1.26310    | 1.17481     | 1.17481   |
| 0.3       | 1.22276     | 1.22276    | 1.13825     | 1.13825   |

Table 2. Values of $\sqrt{2Re_c}c_f$, for different values of $\lambda$, $s$, and $M$ by fixing of $Pr = 0.5$, $Nb = 0.5$, $Nt = 0.5$, and $Sc = 1.0$.

| $\lambda$ | $s$ | $M$ | $-(f'(0) + \frac{1}{2}f^3(0))$ |
|-----------|-----|-----|-------------------------------|
| 0.1       | 0.2 | 2.0 | 1.754213                      |
| 0.2       |     |     | 1.678675                      |
| 0.3       |     |     | 1.799249                      |
|           | 0.1 |     | 1.754213                      |
|           | 0.2 |     | 1.237223                      |
|           | 0.3 |     | 1.271816                      |

\[ \phi'(0) = -1, g(0) = 1, \phi(\infty) = 0, g(\infty) = 0, \]
where
\[ Nb = \frac{\mu_0 C_u - C_v}{\rho c} Db, \quad \frac{\rho_0 \sigma}{\kappa_0} \frac{D_T}{T_c - T_0} e^\gamma \sqrt{\nu_0} Sc = \frac{\nu}{D_b}, \]

Some important physical quantities are the local skin friction coefficient $c_f$, local Nusselt number $Nu_s$, and the local Sherwood number $Sh_s$, which are defined as
\[ c_f = \frac{1}{\rho U_w^2} \left( \mu_0 \left( \frac{\partial u}{\partial y} + \frac{\Gamma}{\sqrt{2}} \left( \frac{\partial u}{\partial y} \right)^2 \right) \right)_{y=0}, \]
\[ Nu_s = -\frac{\sqrt{2l}}{(T_w - T_0)} e^\gamma \left( \frac{\partial T}{\partial y} \right)_{y=0}, \]
\[ Nu_s = -\frac{lU_0 k}{(T_w - T_0)} \left( \frac{\partial T}{\partial y} \right)_{y=0}, \]
\[ Sh_s = -\frac{\sqrt{2l}}{(C_w - C_v)} \left( \frac{\partial C}{\partial y} \right)_{y=0}. \]

Using equation (7), we obtain
\[ \sqrt{2Re_c}c_f = \left( f''(0) + \frac{\lambda}{2} f''(0) \right) \eta = \frac{Nu_s}{\sqrt{Re_s}} = -\theta'(0), \]
\[ Nu_s = -\phi'(0), \quad Sh = \frac{Sh_s}{\sqrt{Re_s}} = -g'(0) \]
and
\[ \frac{Nu_s}{Re_s} = -\phi'(0), \quad Sh_s = \frac{Sh_s}{\sqrt{Re_s}} = -g'(0) \]
where $Re_s = \frac{U_0 l}{\nu}$.

Results and discussion

The numerical solution of magnetohydrodynamics Williamson nanofluid across an exponentially stretching surface is examined here. The impact of physical parameters that is Williamson fluid parameter $\lambda$, suction/injection parameter $s$, Magnetic parameter $M$, thermophoretic parameter $N_t$, and Brownian motion parameter $N_b$ on flow and Prandtl number $Pr$, Schmidt number $Sc$, heat and mass transfer characteristic has been investigated. The system of ODEs obtained in equations (10) to (13) are solved by using the MATLAB function bvp4c. In order to certify the code, developed in MATLAB, we obtain the results for the skin friction coefficient when $M = 0$ for distinct values of $s$ and $\lambda$, keeping other parameters fixed which are shown in Table 1. These outcomes are reliable and found in agreement with the results reported by Nadeem and Hussain.$^{22}$

In Tables 2 to 4, the effects on the $\sqrt{2Re_c}c_f$, $Nu_sRe_s^{-\frac{1}{2}}$, and $-g'(0)$ for several effective parameters are shown. equation (13) shows the dimensionless mathematical form of skin friction. As we increase the value of $\lambda$, the skin friction coefficient, local Nusselt number, and local Sherwood number decrease because greater values of $\lambda$ with more relaxation time offer more resistance to fluid motion. As we raise the value of suction/injection parameter $s$, the skin friction coefficient, local Nusselt number, and local Sherwood number decrease because the
fluid flow is caused only by the stretching sheet. When we increase $s$, it means an increase in porosity of the stretching sheet which produces the resistivity on the fluid flow. As we raise the value of $M$, the skin friction coefficient increases, whereas the local Nusselt number and local Sherwood number decrease because of Lorentz force which restrict fluid motion. Prandtl number $Pr$ is the ratio of momentum diffusivity to nanofluid thermal diffusivity. As we increase $Pr$, local Nusselt number increases whereas local Sherwood number decreases. An escalation in the value of $Nb$, causes a decrease in the local Nusselt number whereas an increase in the local Sherwood number. As we increase the value of $Nt$, both local Nusselt number and local

### Table 3. Variation in $-\theta'(0)$ for various values of $\lambda$, $s$, $M$, $Pr$, $Nb$, $Nt$, and $Sc$.

| $\lambda$ | $s$ | $M$ | $Pr$ | $Nb$ | $Nt$ | $Sc$ | $-\theta'(0)$ PESTCase |
|----------|-----|-----|------|------|------|------|------------------------|
| 0.1      | 0.2 | 2.0 | 0.5  | 0.5  | 0.5  | 1.0  | 0.355586               |
| 0.2      |     |     |      |      |      |      | 0.34313                |
| 0.3      |     |     |      |      |      |      | 0.34124                |
|          | 0.1 |     |      |      |      |      | 0.355786               |
|          | 0.2 |     |      |      |      |      | 0.355786               |
|          | 0.3 |     |      |      |      |      | 0.338681               |
|          | 0.1 | 0.2 |      |      |      |      | 0.471722               |
|          | 0.2 | 0.2 |      |      |      |      | 0.462801               |
|          | 0.3 | 0.2 |      |      |      |      | 0.454350               |
|          | 0.1 | 0.3 |      |      |      |      | 0.143461               |
|          | 0.2 | 0.3 |      |      |      |      | 0.206076               |
|          | 0.3 | 0.3 |      |      |      |      | 0.255045               |
|          | 1.0 | 0.3 |      |      |      |      | 0.365769               |
|          | 1.2 |     |      |      |      |      | 0.366728               |
|          | 1.3 |     |      |      |      |      | 0.363881               |
|          | 1.4 |     |      |      |      |      | 0.361076               |

### Table 4. Variation in $-g'(0)$ for various values of $\lambda$, $s$, $M$, $Pr$, $Nb$, $Nt$, and $Sc$.

| $\lambda$ | $s$ | $M$ | $Pr$ | $Nb$ | $Nt$ | $Sc$ | PESTCase | PEHFCase |
|----------|-----|-----|------|------|------|------|----------|----------|
| 0.1      | 0.2 | 2.0 | 0.5  | 0.5  | 0.5  | 1.0  | 0.015966 | -0.442629 |
| 0.2      |     |     |      |      |      |      | 0.013674 | -0.446536 |
| 0.3      |     |     |      |      |      |      | 0.011123 | -0.449414 |
|          | 0.1 |     |      |      |      |      | 0.059514 | -0.410602 |
|          | 0.2 |     |      |      |      |      | 0.015966 | -0.442629 |
|          | 0.3 |     |      |      |      |      | -0.02406 | -0.467926 |
|          |     | 0.1 |      |      |      |      | 0.079721 | -0.288410 |
|          |     | 0.2 |      |      |      |      | 0.072767 | -0.305662 |
|          |     | 0.3 |      |      |      |      | 0.066373 | -0.321260 |
|          |     | 0.1 |      |      |      |      | 0.221245 | -0.048631 |
|          |     | 0.2 |      |      |      |      | 0.165195 | -0.232101 |
|          |     | 0.3 |      |      |      |      | 0.112387 | -0.336034 |
|          |     |     | 0.1  |      |      |      | -0.416148 | -3.398190 |
|          |     |     | 0.2  |      |      |      | -0.175935 | -1.551180 |
|          |     |     | 0.3  |      |      |      | 0.044296 | -0.935144 |
|          |     |     |     | 0.1  |      |      | 0.225522 | 0.130956  |
|          |     |     |     | 0.2  |      |      | 0.171509 | -0.016000 |
|          |     |     |     | 0.3  |      |      | 0.118590 | -0.161674 |
|          |     |     |     |     | 0.1  |      | -0.187086 | -0.660461 |
|          |     |     |     |     | 0.2  |      | -0.168032 | -0.641514 |
|          |     |     |     |     | 0.3  |      | -0.147325 | -0.620342 |
Sherwood number decreases. Schmidt number $Sc$, is the ratio of momentum diffusivity to Brownian diffusivity. As we increase $Sc$, the local Nusselt number decreases whereas the local Sherwood increases.

Figures 1 and 2 displays the effect of velocity profile $f'(\eta)$ versus $\eta$ which depends on $M$, $l$, and $s$. Figure 1(a) illustrates that the increment in magnetic parameter $M$, causes velocity profile to decrease because of retarding force which is responsible for the decrease in velocity. Figure 1(b) shows the effect of $l$ showing the similar effect as before that is velocity profile decreases with an escalation in the values of $l$. Moreover, higher values of $M$ and $l$ reduce boundary layer thickness. Figure 2 shows the effect of $s$ on the velocity profile. It is evident from this figure that the velocity profile settles at higher values on raising $s$. Figure 3 shows temperature profile for various values of $Nb$. The PEST and PEHF cases represent the direct relationship between Brownian motion parameter $Nb$ and temperature profile as shown.
The thermal boundary layer for both the PEST and PEHF cases is also increased.

Figure 4(a) and (b) reveal the similar influence of the $N_t$ on $u(h)$ for PEST and PEHF cases, respectively. It happens because the strong temperature gradient forces particles in the fluid to move in the direction of decreasing temperature. For higher values of Prandtl number $Pr$, the temperature shows a decreasing behavior for both manifestations of PEST and PEHF as can be seen from Figure 5. It is due to the decrease in thermal diffusivity because the Prandtl number $Pr$ and thermal diffusivity $\alpha$ have inverse relationship with each other.

Figure 6 describes the impact of $M$ on a temperature profile for both instances of heat transform considered and increasing influence of $M$ on temperature profile is observed. Figure 7(a) and (b) depicts the impact of $\lambda$ on a temperature profile for PEST and PEHF case respectively. The escalation in temperature is observed as $\lambda$ increases. Moreover, momentum boundary layer thickness has increasing impact of both $M$ and $\lambda$. Figure 8 illustrate the effect of suction/injection parameter $s$ on temperature profile. An increase in $s$ results in rise in temperature for both PEST (Figure 8(a)) and PEHF (Figure 8(b)) cases. Consequently, thermal boundary layer thickness becomes large.

Figure 9(a) and (b) shows that concentration increases by increasing the magnetic parameter $M$ for both PEST and PEHF case. $Sc$ has adverse effect on concentration profile which is displayed by Figure 10(a) and (b). In Figure 11(a) and (b), we represent the behavior of Brownian motion parameter $Nb$ for PEST and PEHF cases respectively. Hypothetically, the enhanced thermal conductivity of nanofluid is primarily due to Brownian motion which produces macromixing. By increasing the $Nb$, it reduces the nanofluid concentration. Figures 12 and 13 depict the effects of $N_t$ and $s$ on concentration profile for PEST and PEHF cases, respectively. It is depicted that there is rise in...
Figure 6. Temperature profile versus $\eta$ by fixing $Nt = 0.5, Nb = 0.5, \lambda = 0.1, Pr = 0.5, s = 0.2, Sc = 1$: (a) for various values of $M$ (PEST Case) and (b) for various values of $M$ (PEHF Case).

Figure 7. Temperature profile versus $\eta$ by fixing $Pr = 0.5, Nt = 0.5, Nb = 0.5, Sc = 1, M = 2, s = 0.2$: (a) for various values of $\lambda$ (PEST Case) and (b) for various values of $\lambda$ (PEHF Case).

Figure 8. Temperature profile versus $\eta$ by fixing $Pr = 0.5, Nt = 0.5, \lambda = 0.1, M = 2, Nb = 0.5, Sc = 1$: (a) for various values of $s$ (PEST Case) and (b) for various values of $s$ (PEHF Case).
Figure 9. Concentration profile $g(h)$ versus $h$ by fixing $N_t = 0.5$, $N_b = 0.5$, $\lambda = 0.1$, $Pr = 0.5$, $s = 0.2$, $Sc = 1$: (a) for various values of $M$ (PEST Case) and (b) for various values of $M$ (PEHF Case).

Figure 10. Concentration profile $g(h)$ versus $h$ by fixing $Pr = 0.5$, $N_b = 0.5$, $\lambda = 0.1$, $M = 2$, $s = 0.2$, $N_t = 0.5$: (a) for various values of $Sc$ (PEST Case) and (b) for various values of $Sc$ (PEHF Case).

Figure 11. Concentration profile $g(h)$ versus $h$ by fixing $N_t = 0.5$, $Pr = 0.5$, $\lambda = 0.1$, $M = 2$, $s = 0.2$, $Sc = 1$: (a) for various values of $Nb$ (PEST Case) and (b) for various values of $Nb$ (PEHF Case).
concentration values by increasing the thermophoretic parameter $N_t$ and suction/injection parameter $s$ for both cases.

Conclusion

The following main remarks can be concluded from the results of the current investigation.

- Skin friction coefficient reduces by raising the value of Williamson parameter $\lambda$ and suction/injection parameter $s$, whereas it increases on increasing magnetic parameter $M$.
- Wall temperature gradient increases on an increasing Prandtl number $Pr$, whereas it decreases for an increase in Williamson parameter $\lambda$, suction/injection parameter $s$, magnetic parameter $M$, Brownian motion parameter $Nb$, thermophoresis parameter $N_t$, and Schmidt number $Sc$.
- $g(\eta)$ increases by raising the values of Brownian motion parameter $Nb$ and Schmidt number $Sc$, whereas it decreases by an increase in Williamson parameter $\lambda$, suction/injection parameter $s$, magnetic parameter $M$, Prandtl number $Pr$, and thermophoresis parameter $N_t$.
- Velocity profile settles at lower values for increasing Williamson parameter $\lambda$ and magnetic parameter $M$, whereas it settles at higher values for increasing suction/injection parameter $s$.
- $\theta(\eta)$ for both PEST and PEHF cases have similar behavior on $\lambda$, suction/injection parameter $s$, magnetic parameter $M$, Brownian motion parameter $Nb$, thermophoresis parameter $N_t$, and Prandtl number $Pr$.
- Concentration profile elevates on raising magnetic parameter $M$, suction/injection parameter $s$, thermophoresis parameter $N_t$, and Schmidt number $Sc$. 

Figure 12. Concentration profile $g(\eta)$ versus $\eta$ by fixing $Pr = 0.5, Nb = 0.5, \lambda = 0.1, M = 2, s = 0.2, Sc = 1$: (a) for various values of $N_t$ (PEST Case) and (b) for various values of $N_t$ (PEHF Case).

Figure 13. Concentration profile $g(\eta)$ versus $\eta$ by fixing $Nt = 0.5, Nb = 0.5, \lambda = 0.1, M = 2, Pr = 0.5, Sc = 1$: (a) for various values of $s$ (PEST Case) and (b) for various values of $s$ (PEHF Case).
...and \( N_f \) whereas it drops down for higher values of \( N_b \) and Schmidt number \( Sc \).

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Appendix

Notation

| Symbol | Description |
|--------|-------------|
| $U_0$ | rate of stretching surface |
| $B_0$ | magnetic field strength (N · m A$^{-1}$) |
| $C_f$ | skin friction coefficient |
| $Pr$ | Prandtl number |
| $M$ | magnetic parameter |
| $T$ | fluid temperature (K) |
| $C_w$ | concentration of nanoparticle at the surface |
| $u, v$ | velocity components (m s$^{-1}$) |
| $D_T$ | coefficient of thermophoresis diffusion (m$^{-2}$ s$^{-1}$) |
| $C$ | concentration of nanoparticle |
| $C_w'$ | ambient concentration of nanoparticle |
| $U_w$ | velocity at the wall |
| $Pr$ | Prandtl number |
| $s$ | suction/injection parameter |
| $\theta(\eta)$ | temperature profile for PEST case |
| $g(\eta)$ | concentration profile |
| $Nt$ | thermophoretic parameter |
| $Nu_s$ | represents local Nusselt |
| $Sh_x$ | local Sherwood number |

$Nb$ Brownian parameter

$T_w$ surface temperature (K)

$T_\infty$ ambient temperature (K)

$f$ dimensionless stream function

$x, y$ Cartesian coordinates (m)

$g$ nanoparticle volume fraction

$Sc$ Schmidt number

$D_B$ coefficient of Brownian diffusion (m$^{-2}$ s$^{-1}$)

$Re_x$ Reynolds number

$M$ magnetic parameter

$f'(\eta)$ velocity profile

$\phi(\eta)$ temperature profile for PEHF case

Greek letters

$\eta$ dimensionless Similarity variable

$\sigma$ electrical conductivity (S m$^{-1}$)

$\Gamma$ positive time constant

$\alpha$ thermal diffusivity (m$^{-2}$ s$^{-1}$)

$\theta$ dimensionless temperature

$\rho_f\epsilon_p$ heat capacity of nanoparticles (J m$^{-3}$ K$^{-1}$)

$\nu$ Kinematic viscosity (m$^{-2}$ s$^{-1}$)

$\rho$ density (kg m$^{-3}$)

$\lambda$ Williamson fluid parameter

$\mu$ dynamic viscosity (kg m$^{-1}$ s$^{-1}$)

$\rho_c$ heat capacity of the nanofluid (J m$^{-3}$ K$^{-1}$)

Superscripts

$w$ condition at the surface

$\infty$ condition at the free stream

Superscripts

$'$ derivative w.r.t $\eta$

Abbreviations

ODEs ordinary differential equations

PDEs partial differential equations