Transposition of a sphere's volume based on didactic phenomena

E Puspita*, Turmudi¹, and R Rosjanuardi¹

¹ Departemen Pendidikan Matematika, Universitas Pendidikan Indonesia, Jl. Dr. Setiabudi No. 229, Bandung 40154, Indonesia

*entipuspita@upi.edu

Abstract. Concerns about mathematics learning are rarely associated with life contexts, giving ideas to teachers and researchers developing didactic phenomenon-based learning. Didactic phenomenology is defined as a bridge to study mathematical concepts. Unfortunately, the use of didactic phenomena still leaves problems when formalizing the idea, including the concept of spherical volume. The qualitative method was chosen with a didactic phenomenological design, to explore the transposition of spherical volume construction, to strengthen the results of didactic phenomena based learning. By utilizing didactic phenomena and assuming students know the surface area of the sphere informally, the spherical volume can show. Formally with the concept of the volume of a rotating object and double integrals can be proven the volume of the sphere. With the idea of vector and limit the surface area and volume of the sphere can be determined as students know it. Further studies need to do regarding research collaboration mechanisms that involve teachers and academics so that the efforts of teachers in utilizing various didactic phenomena have a foundation of thinking that can be scientifically justified.

1. Introduction
Mathematics plays an essential role in building scientific attitudes and mindsets of students. Ideally, mathematics learning designed and directed to be able to meet the needs of students, both short-term interests and long-term interests. For the short-term investments of education, mathematics directed to understanding mathematical concepts and ideas. While for the long-term of learning mathematics needs to be designed in such a way that students have the opportunity to develop the ability to think critically, logically, systematically, creatively, carefully, foster a sense of confidence, have an objective, and open attitude. In line with this, Harel [1] views mathematics as ways of thinking and ways of understanding.

The National Council of Teachers of Mathematics [2], states that learning mathematics in schools must be student-centred. They must learn mathematics actively by building their new knowledge based on the prerequisite experience and expertise they have. On the other hand, so that teachers can carry out active mathematics learning requires an understanding of what students know and need to learn.

Learning mathematics is a complex and ongoing process; the interaction process in it involves students, teachers, and mathematics itself [3]. Based on these problems, how to package mathematics learning so that it can build meaning and a complete understanding of mathematical concepts. The above conditions encourage teachers to be creative in developing learning processes that involve students in building knowledge actively. Regarding the construction of accurate understanding, Hiebert and Carpenter [4], defines education as an attempt to make connections between ideas, facts, or procedures. The teacher acts as a mediator for the cognitive development of students [5]. The use of this learning
approach is expected that students can build their knowledge by observing, exploring a phenomenon, investigating the related components of the phenomenon, determining the relationship and determining the formula of the phenomenon.

Freudenthal [6] states that didactic phenomenology is how teachers can place students to step into the learning process. Didactic phenomena are understood as attempts to utilize a phenomenon as a bridge to study concepts. By presenting mathematical sources from real life, didactic phenomenology is strongly related to Realistic Mathematics Education (RME), which was developed in the Netherlands by Freudenthal. In RME, the rich and real situation has a prominent position in the learning process. This situation serves as a resource to begin the development of mathematical concepts, media, and procedures. The condition that serves as the context for developing ideas is the characteristics of didactic phenomenology.

The idea of didactic phenomenology from Freudenthal [6] inspired to explore mathematical content through phenomena that follow the environment in Indonesia. Several studies have utilized didactic phenomena to study mathematical concepts ([7], [8], [9], [10], [11]). These studies prove that many natural phenomena can be used as "bridges" to understand mathematical concepts. Other research has been conducted regarding the use of didactic phenomena in the learning process, namely Arifin [12] dan Hendroarto [13].

In their research, Turmudi and Julia [9] collaborated with junior high school mathematics teachers in learning activities, which were attended by mathematics teachers who were around the school, school principals, and school supervisors. There are three stages in the Lesson Study, namely Plan, Do, and See. During the Plan, the model teacher collaborates with the researchers to design a learning plan, including the selection of strategies, learning media, and class settings that will use. In the Do Stage, the teacher carries out a plan that has prepared in an open class attended by participants. At the See stage, the teacher and the participants reflect on the learning that has taken place. Furthermore, the professional community can help mathematics teachers so they can guide students to learn how to conduct investigations to learn mathematical concepts in the context of exploring didactic phenomena. Concepts related to the professional community were put forward by Wetbunpot and Inprasitha [14] and Rogoff [15].

Thomas et al. [16] say that mathematics teachers are part of the professional community involved in efforts to introduce their teaching practice processes, developing, implementing, modifying teaching programs, strengthen and condition the class so students can explore mathematics in depth. In line with this opinion, Stein [17], argues that we can view mathematics classes as an environment where students can make meaning actively about themselves by emphasizing the process of learning mathematics.

Selden and Selden [18], posed several questions: how can lecturers have a positive influence when students learn mathematics? Is the didactic obstacle caused by order of lecture material? Do lecturers in tertiary institutions ignore everyday concepts or materials at the school level? The question means that the lecturer has the responsibility in helping students understand mathematical concepts. One of the abilities that lecturers need to have is to interpret the didactic transposition concept of a mathematical theory (Chevallard and Bosch [19]).

Various attempts have been made by the teacher to create a situation through a phenomenon where students can construct mathematical knowledge, one of which is related to the construction of the sphere volume formula. Turmudi and Julia [9], in their research, succeeded in utilizing the didactic phenomenon in constructing sphere volumes (informally), using other concepts that were temporarily used (students considered to know already). We can interpret that, learning by utilizing didactic phenomena still leaves problems at the time of formalization of concepts. Bringing these problems into study material at the tertiary level is exciting and will provide useful insight for prospective mathematics teachers.

Regarding the idea of sphere volume, there has not been any research conducted to prove formally at the secondary school level. A teacher as a source of information in learning and spearheading the success of an educational process needs to know mathematical concepts formally so that what they do has a good foundation of thinking. Based on this, a theoretical study is required related to the
construction of sphere volume officially. Hence, the purpose of this study is "Transposition of sphere volume construction, to strengthen the learning outcomes based-on didactic phenomena."

2. Methods
The research method used is a qualitative method, with the design of phenomenology [20]. The phenomenology chosen was didactic phenomenology, which is a research method using the utilization of didactic phenomena around students in developing mathematical concepts [6]. Using the phenomenon of round watermelons in simple sphere volume construction. Next, we will explore various transpositions of the idea of spherical volume, using multiple concepts including rotating object volume, double integrals, vector concepts in determining the formal proof of spherical surface area, ending with the use of the idea of limits to show legal evidence of spherical volume. The strategy chosen was to conduct a theoretical study related to the concept of spherical volumes from various literature used as sourcebooks in lectures in the Mathematics Education Study Program, as a producer of mathematics teacher candidates.

3. Result and Discussion
3.1 The didactic phenomenon of the spherical surface area
Some researchers used the didactic phenomenon of orange peels to prove (informal) the surface area of the ball. The steps taken are: divide the oranges into two equal parts then trace a cross-section on the paper (two circles), then one part of the orange peel is cut into small pieces and then stick to the circle made. If done correctly, the orange peel flakes will cover the two circles that made, meaning that the orange peel can cover four circles. This condition shows that the surface area of the ball is four times the area of the circle. Assuming students already know the area of a circle that is \( \pi r^2 \), it is evident that the surface area of the ball is equal to \( 4\pi r^2 \).

Other didactic phenomena can be chosen for example, a tennis ball and mattress yarn. Activity steps: draw a rectangle with length (p) equal to the diameter of the circle of the sphere pieces (D), and width (l) equal to the circumference of the circle formed (K). Wrap the mattress yarn until it covers half the surface of the sphere. After that, wrap it around the rectangle that made. If done correctly, then the yarn wrapped around half the sphere will cover half of the rectangle, this means the surface area of the sphere = area of the rectangle = \( p.l = 4\pi r^2 \).

Can be concluded that, by utilizing the didactic phenomena in the environment around the students, the teacher can guide them in proving (informally) that the surface area of the sphere is equal to \( 4\pi r^2 \).

3.2 The didactic phenomenon of spherical volume
Consider the illustration of a watermelon, which cut into small pieces in such a way as to form a pyramid-rectangular model (can also be other shapes, for example, the triangular pyramid), while the right is an illustration of a spherical model constructed by pieces of a rectangular pyramid.

![Figure 1. A watermelon, pyramid-shaped watermelon piece, and a sphere model](https://cdn.intechopen.com/pdfs/55953.pdf)
From the pieces of the pyramid model in the middle part of Figure 1, then rearranged into a sphere as illustrated in the rightmost picture. The illustration indicates that the volume of the sphere will be equal to the volume of the pyramid that forms the sphere. Assuming students already know the volume of the pyramid, they will be able to determine the volume of the sphere in question. Some critical students might ask because the base of the pyramid is not flat. In this condition, the teacher’s role is crucial in helping students understand the problem. Some illustrations that can be given by the teacher include:

a) Assume the watermelon divided into very many pyramid models, then the base of the pyramid will eventually be as flat

b) Students can be invited to see the phenomenon of a football field if in the four corners of the field is embedded a nail and then from the nails drawn a line towards the centre of the earth, build space that is a quadrilateral pyramid. By assuming the sphere is identical to the globe, then the volume of the sphere is the sum of the volume of its forming pyramid.

3.3 Informal evidence of ball volume

Through the didactic phenomenon described in sections 3.1 and 3.2, informal evidence of sphere volume will be shown. Based on the illustration of the ball model in Figure 1, the sphere is formed from an infinite number of the pyramid with a rectangular pyramid bottom. For images, a pyramid base a rectangular shape. So visually, it can be seen that the surface area of the sphere is equal to the total area of the bottom of the pyramid that forms the sphere. Suppose the surface area of each pyramid denoted as $A_1, A_2, \ldots, A_n$, and the volume of the pyramid is denoted by $V_1, V_2, \ldots, V_n$. Relationship obtained:

$$V_{sphere} = \frac{1}{3}r, A_{sphere surface} = \frac{4}{3} \pi r^3$$  \hspace{1cm} (1)

Using the concept of the spherical surface area obtained by using didactic phenomena as shown in section 3.1, it can be proven that the volume of the sphere is $\frac{4}{3} \pi r^3$.

3.4 Formal proof of sphere volume with the concept of a rotary object volume

A sphere can be produced by rotating the area bounded by $y = (r^2 - x^2)^{1/2}$ to the x-axis, using the cylinder method on the volume of a rotating object, the following results obtained:

$$V = 2\pi \int_0^r r^2 - x^2 \, dx = \frac{4}{3} \pi r^3$$  \hspace{1cm} (2)

By using the sphere volume formula in (1), and by using $V_{sphere} = \frac{1}{3}r, L_{sphere surface}$ and indirect sphere surface area formula can derive, namely:

$$L_{sphere surface} = \frac{4}{3} \pi r^3 = 4\pi r^2$$  \hspace{1cm} (3)

From the description above, it can show that in teaching practice in class two different things can be done, namely: a) formal evidence of sphere volume can be used to prove the surface area of the sphere, b) the surface area of the sphere informally can be used to indicate the volume of the sphere. Of course, this is interesting when the problem becomes a study of students in a learning process.

3.5 Double integrals for formal proof of spherical volume

Consider the half-sphere equation with radius $a$ and equation $f (x, y) = (a^2 - x^2 - y^2)^{1/2}$. Based on the application of double integrals in determining the solid volume and using polar coordinate applications, the following results obtained:

$$V_{1/2sphere} = \int_0^{2\pi} \int_0^a (a^2 - r^2)^{1/2} r \, dr \, d\theta = \frac{2}{3} \pi a^3$$  \hspace{1cm} (4)

Thus, it can be proven that the volume of the sphere = $\frac{4}{3} \pi a^3$. If the radius of the sphere is $r$, then it is evident that the volume of the sphere is $\frac{4}{3} \pi r^3$.

3.6 Surface area concept for formal proof of spherical volume

By using the concept of the area of a parallelogram is the cross product of the vector forming it [21] and the idea of a limit, it can be shown that the surface area of a sphere is:
\[ A_{\text{sphere}} = 2 \int_S \left( f_x^2 + f_y^2 + 1 \right)^{1/2} dL \]  

From sections 3.2 and 3.3, it has been shown that based on the didactic phenomenon a sphere with radius \( a \), formed from the pyramid with \( V_i \) and \( A_i \), is the volume and surface area of the pyramid corresponding to index \( i \), with ignoring that the surface of the base of the pyramid is a curved plane. Note the shape \( V_{\text{sphere}} = \sum_{i=1}^{n} V_i \). If observed, the shape will be the same when the limas form so much that the base of the pyramid is no longer a curved area. Concepts like this in calculus are known as the concept of limits, so:

\[ V_{\text{sphere}} = \lim_{n \to \infty} \sum_{i=1}^{n} V_i = \frac{2}{3} a \int_0^r \frac{a}{\sqrt{a^2 - r^2}} r^2 \, dr = \frac{4}{3} \pi a^3 \]  

If the radius of the sphere is \( r \), then the volume of the sphere is \( \frac{4}{3} \pi r^3 \).

The results of this study are in line with the results of Turmudi & Julia [9], by utilizing the didactic phenomenon of a round watermelon it can be shown that the volume of the sphere is equal to the sum of the volume of the forming pyramid. But in this study produced a formal proof of the volume of the sphere using: the concept of the volume of a rotating object; the idea of double integrals; or by first determining the formal proof of surface area by using the concept of vector, the next with the idea of the limit proves the sphere volume formula. The results of this study are also in line with theories from Freire [22], which revealed two big ideas in learning, namely "There is no Teaching without Learning" and "Teaching is not Just Transferring knowledge", the statement has vast and profound implications. The first principle implies that the experience of an educator doing learning mathematics well will underlie the creation of useful mathematics learning for students. While the second principle means that teaching is far more profound than merely transferring knowledge, mathematical knowledge which is the knowledge that requires justification, cannot be done with transferring but must go through a transposition process.

4. Conclusion

The Conclusions of this study are as follows: (a). Utilization of the didactic phenomenon of a round watermelon gives the conclusion that the volume of the sphere is equal to the sum of the volume of the forming pyramid, then assuming students already know the surface area of the sphere it can be shown informally that the volume of the sphere is \( \frac{4}{3} \pi r^3 \); (b). Formally, using the concept of rotating object volume, it can be proven that the volume of the sphere is \( \frac{4}{3} \pi r^3 \); (c) Formally using the concept of double integrals can be proven that the volume of the sphere is \( \frac{4}{3} \pi r^3 \); (d). By using the vector concept it can be proven that the surface area of the sphere is \( \int_S \left( f_x^2 + f_y^2 + 1 \right)^{1/2} dL \); and (e).The concept of limits has strengthened the evidence that the volume of the sphere is \( \frac{4}{3} \pi r^3 \).

The results of this study provide recommendations that, for further studies related to the collaborative mechanism of research involving teachers and academics, so that the steps taken by teachers in utilizing various didactic phenomena, have a foundation of thinking that can be scientifically justified.

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