Equivalent conditions for hyperbolicity on partially hyperbolic holomorphic map

Francisco Valenzuela-Henríquez*

fvalenzh@impa.br, pancho.valenzuela.math@gmail.com

May 14, 2010

Abstract

Let \( f : \mathbb{C}^n \to \mathbb{C}^n, \ n \geq 2, \) be a biholomorphism and let \( \Lambda \subseteq \mathbb{C}^n \) be a compact \( f \)-invariant set such that \( f|\Lambda \) is partially hyperbolic. We give equivalent conditions to hyperbolicity on \( \Lambda \). In the particular case of generalized Hénon map with dominated splitting in the Julia set \( J \), we characterize the hyperbolicity of \( J \).

Contents

1 Introduction 2
2 Preliminaries 5
3 Holomorphic Hadamard-Perron Theorem 7
   3.1 Sketch of proof of Hadamard-Perron Theorem 8
   3.2 Technical considerations 9
   3.3 Proof of Theorem 2.2 10
4 Dynamically defined and Overlapping property 11
5 Holomorphic center-unstable submanifolds 13
6 Forward Expansiveness in the center-unstable leaf 18
7 Proof of Theorem A 20

*Thanks for the financial support to CNPq-IMPA and PEC-PG and Universidad Católica de Valparaiso.
1 Introduction

In the theory of complex dynamical systems, a well known seminal area is the study of rational maps on the Riemann sphere. For complex dynamics in several variables, the study of polynomial automorphisms of $\mathbb{C}^2$ is the first step for a global understanding of holomorphics dynamics in higher dimension.

One of the first results in this direction, were given by Friedland and Milnor in [10]. They proved that for polynomial automorphism in $\mathbb{C}^2$, the only systems (module conjugation by a polynomial automorphism) that exhibit rich dynamics are the so called generalized Hénon maps (or by simplicity, complex Hénon maps).

Such a maps are obtained as a finite composition of maps of the form $(y, p(y) - bx)$, where $p$ is a polynomial of degree at least two, and $b \in \mathbb{C}^*$. Complex Hénon maps have been a subject of serious study, with foundational work done in the early 1990’s by Hubbard [11], Hubbard and Oberste-Vorth [12, 13], Bedford and Smillie [2, 3, 4], Bedford, Lyubich and Smillie [1] and Fornæss and Sibony [9].

As in the case of rational maps, complex Hénon map have well defined a Julia set $J$ (see [2]). This set is a compact invariant set, and it contains the supports of the unique measure of maximal entropy. Such measure exists due to the works of Bedford and Smillie [2]. Denote by $J^*$ the support of the measure of maximal entropy.

A significant open question in the study of complex Hénon maps is whether $J = J^*$. Bedford and Smillie [2] have shown that if $f$ is uniformly hyperbolic on $J$, then $J = J^*$. Moreover, Fornæss proved that if $f$ is uniformly hyperbolic on $J^*$ and $f$ is not volume preserving, then $J = J^*$ [8]. In the setting of complex Hénon maps, hyperbolicity is the natural generalization of the expansiveness on the Julia set for rational maps on $\mathbb{C}$.

Motivated by the results above, in this work we establish equivalent conditions to hyperbolicity for biholomorphisms of $\mathbb{C}^n$ with a compact invariant set $\Lambda \subset \mathbb{C}^n$, under the hypothesis of partial hyperbolicity. It is well known that for a partial hyperbolic map, we have strong stable manifolds and center-unstable leaves [15, 20]. So we can to characterize the uniform hyperbolicity in terms of the behavior along the center-unstable leaves.

More precisely, we say that the a local $cu$-leaf $W^u_{cu}(x)$ is dynamically defined, if for every $0 < \varepsilon \ll 1$, it is included in the local unstable set of $x$. We say that the $cu$-leaves are forward expansive if there exist a uniform constant $c > 0$ such...
that for every \( x \in \Lambda \), and any \( y \in W^c_{\varepsilon}(x) \), there exists \( n \in \mathbb{N} \), such that
\[
\text{dist}(f^n(y), f^n(x)) > c.
\]

The reader can find a more precise statement of the notions of partial hyperbolicity, forward expansiveness and dynamically defined in Section 2. Our main theorem is the following.

**Theorem A.** Let \( f : \mathbb{C}^n \to \mathbb{C}^n \), \( n \geq 2 \), be a biholomorphism which is partially hyperbolic on the compact invariant set \( \Lambda \). Then the following statements are equivalent:

1. The function \( f \) is uniformly hyperbolic on \( \Lambda \).
2. The \( \text{cu} \)-leaves are forward expansive.
3. The \( \text{cu} \)-leaves are dynamically defined.

One of the motivations of the previous Theorem appears in the study of complex Hénon map with dominated splitting. In the setting of dissipative Hénon maps, dominated splitting in \( J \) and partial hyperbolicity are equivalents (see Proposition 8.3). It is important to note that both, partially hyperbolic and dominated splitting are two ways to relax hyperbolicity.

Theorem A establishes that it is enough assume forward expansiveness or the dynamical definition of the \( \text{cu} \)-leaf to guarantee hyperbolicity.

We must remark that for real maps with dominated splitting in manifolds, the condition \( \text{cu} \)-leaf dynamically defined is not enough to conclude hyperbolicity. In order to conclude hyperbolicity it is required to add the hyphothesis of Kupka-Smale over the diffeomorphisms (see [22, 23] for surfaces and codimension one context respectively).

Nevertheless motivated by the works [22, 23], we conjecture that:

**Conjecture 1.** Generically (Kupka-Smale) complex Hénon maps with dominated splitting are hyperbolic.

We study the relation between uniform hyperbolicity and forward expansiveness because in the Hénon maps the saddle periodic points (which are dense in \( J^* \) [2]) satisfy a non-uniform forward expansivity condition. Moreover, for dissipative Hénon maps with dominated splitting, Proposition 8.3 states that in every center-unstable leaf there exist many points with (uniform) forward expansivity.

Another motivation of Theorem A is the notion of quasi-expanding due to Bedford and Smillie in [5]. Roughly speaking, quasi-expanding corresponds to an uniform forward expansivity in the periodic saddle point respect to the complex structure induced by the dynamics. The authors establish that a (topological) expansive quasi-hyperbolic map (quasi-expanding and quasi-contracting) is hyperbolic. Dissipative complex Hénon maps with dominated splitting are quasi-contracting. We obtain the same conclusion of Bedford and Smillie just assuming forward expansivity on the whole Julia set \( J \).
The sketch of the proof of the Theorem A, is essentially the following: firstly we establish the equivalence between forward expansivity and dynamically defined. Once the cu-leaves are dynamically defined, then they are holomorphics. In consequence, they are uniques and the center-unstable direction is in fact a strong unstable direction.

The existence of a cu-leaf, follows from a classical argument using the graph transform operator (see Theorem 3.1). It is possible to define the graph transform operator, in an appropriated (complete and metric) space of Lipschitz maps. In such case this operator is a contraction and the cu-leaf is the unique fixed point. The cu-leaf given by the graph transform operator is only $C^1$. To prove the holomorphy of the leaf it is necessary to prove that we can approximate the cu-leaf by holomorphic Lipschitz map (iterate of a holomorphic Lipschitz map by the graph transform operator). Hence, knowing that the convergence in the space of Lipschitz function is the convergence uniform on compact part, we conclude the holomorphy of the cu-leaf. The delicate step is guarantee the explained above, only using the dynamically defined property.

Among the dissipative Hénon maps, it is possible to obtain a more refined equivalence to the hyperbolicity of the Julia set. Follows from [1], that

$$J^* = \bigcup \{ \text{supp} \ (\nu) : \nu \text{ is } f \text{- invariant hyperbolic} \}.$$ 

So we can define the set

$$J_0 = \bigcup \{ \text{supp} \ (\nu) : \nu \text{ is } f \text{- invariant and has a zero exponent} \}.$$ 

Note that by definition, $J_0$ is a compact $f$-invariant set.

**Theorem B.** Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a dissipative complex Hénon map, with dominated splitting in $J$. The following statement are equivalents:

1. $J$ is uniformly hyperbolic,
2. $J_0 = \emptyset$,
3. The set of periodic (saddle) points is uniformly hyperbolic.
4. The set of periodic (saddle) points is uniformly expanding at the period.

Statements 3 and 4 in the theorem follows from $J^*$ be an homoclinic class. For a precise statement of uniform expansion at the period see Definition 7. An immediate Corollary from the Theorem B is the following result.

**Corollary C.** Let $f$ be a dissipative complex Hénon map, with dominated splitting in $J$. Then $J$ is hyperbolic if, and only if, every $f$-invariant measure supported in $J$ is hyperbolic.

The paper is organized as follows. In Section 2 we state some results and tools related with partially hyperbolicity, and we define the notions of forward expansivity and dynamically defined. Also we state the existence of
stable/center-unstable manifold for partially hyperbolic systems. In Section 3 we present the Theorem of existence of stable/center-unstable manifolds in the holomorphic context. From Section 4 until 7 we present the proof of the Theorem A. Finally in Section 8 we present some formalisms for complex Hénon maps and prove of Theorem B, and another.

Acknowledgements: This article is a part of my PhD Thesis under the direction of Enrique Pujals at IMPA. I would like to thank him for his guidance.

2 Preliminaries

In this section we recall several classic results of dynamical systems, and we write them in the context of complex and holomorphic dynamics in \( \mathbb{C}^n \) for any \( n \geq 2 \).

We define the open polydisc of center 0 and radio \( r > 0 \) in \( \mathbb{C}^k \) as the set
\[
\Delta_k(0,r) = \{ z \in \mathbb{C}^k : |z_i| < r, \text{ for every } i = 1, \ldots, k \}.
\]

We recall the notion of partially hyperbolic (see references [21] or [14]).

**Definition 1.** Let \( f : \mathbb{C}^n \to \mathbb{C}^n \) be a biholomorphism and \( \Lambda \subset \mathbb{C}^n \) denote be compact \( f \)-invariant set. We say that \( f \) is partially hyperbolic (in the broad sense) on \( \Lambda \), if there exist a \( Df \)-invariant splitting \( T\mathbb{C}^n = E \oplus F \), and constant \( 0 < \lambda < \mu \) and \( C > 0 \) such that,
1. \( ||Df^n(x)||_{E(x)}|| \leq C\lambda^n \) for all \( n \geq 0 \),
2. \( ||Df^{-n}(x)||_{F(x)}|| \leq C\mu^{-n} \) for all \( n \geq 0 \).

Clearly, either \( \lambda < 1 \) and/or \( \mu > 1 \) and without lost of generality in what follows we assume that \( \lambda < 1 \). In this case, the subspace \( E(x) \) is stable and it will denoted by \( E^s(x) \). Also we denote by \( l \) the complex dimension of the space \( E(x)^s \) and by \( k \) the complex dimension of the space \( F(x) \).

Denote by \( \text{Emb}^p(\Delta_k(0,1),\mathbb{C}^n) \) the set of \( C^p \)-embeddings of \( \Delta_k(0,1) \) on \( \mathbb{C}^n \). Two point \( x, y \in \mathbb{C}^n \) are **forward pseudo-asymptotic** under \( f \), if \( d(f^n(x), f^n(y)) \leq C\rho^n \) for all \( n \geq 0 \) and some constant \( C > 0 \). Similarly, we define **backward pseudo-asymptotic** as forward pseudo-asymptotic for \( f^{-1} \).

Recall by [15] (see also [20]) that a partially hyperbolic systems to admit the existence of stable/center-unstable manifolds.

**Theorem 2.1.** Let \( f \) be a biholomorphism in \( \mathbb{C}^n \), such that \( f \) is partially hyperbolic on \( \Lambda \) with splitting \( T\mathbb{C}^n = E^s \oplus F \). Then there exist two continuous functions \( \phi^s : \Lambda \to \text{Emb}^{\infty}(\Delta_k(0,1),\mathbb{C}^n) \) and \( \phi^{cu} : \Lambda \to \text{Emb}^{1}(\Delta_l(0,1),\mathbb{C}^n) \) such that, with \( W^s_\epsilon(x) = \phi^s(x)\Delta_k(0,\epsilon) \) and \( W^{cu}_\epsilon(x) = \phi^{cu}(x)\Delta_l(0,\epsilon) \), the following properties hold:

a) \( T_x W^s_\epsilon(x) = E^s(x) \) and \( T_x W^{cu}_\epsilon(x) = F(x) \),
b) for all $0 < \varepsilon_1 < 1$ there exist $\varepsilon_2$ such that

$$f(W^s_{\varepsilon_2}(x)) \subset W^s_{\varepsilon_1}(f(x))$$

and

$$f^{-1}(W^{cu}_{\varepsilon_2}(x)) \subset W^{cu}_{\varepsilon_1}(f^{-1}(x)).$$

Then sets $W^s_1(x)$ with $x \in \Lambda$ are submanifolds of $\mathbb{C}^n$, and are characterized as those points locally forward $\rho$-asymptotic with $x$, for some $\lambda \leq \rho < \mu$.

The sets $W^s_1(x)$ are the local stable manifolds in the point $x \in \Lambda$. We name the sets $W^{cu}_\varepsilon(x)$, the center-unstable leaf or cu-leaf. Clearly in the case $\mu > 1$, the subspace $F(x)$ is unstable and the cu-leaf are unstable manifolds.

In the holomorphic context we can say even more about the stable manifold.

**Theorem 2.2.** Let $f$ be as in the Theorem 2.1. Then the local stable manifolds \( \{W^s_1(x)\}_{x \in \Lambda} \) are holomorphic submanifolds of $\mathbb{C}^n$.

This Theorem is part of the folklore and we prove them in the following section. In his proof is introduced an important technique, that we use later in the proof of Proposition 5.1.

To end this section, we recall some basic definition. The *unstable set* of a point $x$ for $f$, is the set

$$W^u(x) = \{ y \in \mathbb{C}^n : d(f^{-n}(x), f^{-n}(y)) \to 0, \text{ when } n \to \infty \},$$

where $d$ is the Euclidean distance. Similarly, the *local unstable set* of size $\varepsilon$ is the set

$$W^u_\varepsilon(x) = \{ y \in W^u(x) : d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon, \text{ for every } n \geq 0 \}.$$

It is know that for every $0 < \varepsilon \leq 1$ there exist $\delta > 0$ such that for every $x \in \Lambda$, $W^u_\varepsilon(x) \subseteq W^{cu}_\varepsilon(x)$, however in general the opposite inclusion not hold if we not have good properties in the asymptotic behavior of $Df$.

We end this section, recalling the definition of cu-leaves dynamically defined and the notion cu-forward expansivity for a biholomorphism $f$ in $\mathbb{C}^n$, that is partially hyperbolic on $\Lambda$.

**Definition 2.** We say that the cu-leaves are dynamically defined, if for every $0 < \varepsilon \ll 1$, $W^{cu}_\varepsilon(x) \subset W^u_{loc}(x)$ for all $x \in \Lambda$.

**Definition 3.** We say that $f$ is forward expansive in the center-unstable leaves or cu-forward expansive, if there exist a uniform constant $c > 0$ such that for every $x \in \Lambda$, and any $y \in W^{cu}_\varepsilon(x)$, there exists $n \in \mathbb{N}$, such that

$$\text{dist} \left( f^n(y), f^n(x) \right) > c.$$

We say that the constant $c$ is the expansiveness constant.
3 Holomorphic Hadamard-Perron Theorem

A way to see the proof of the Theorem 2.1 is applying the classical Hadamard-Perron Theorem. We will use the notation and the “technique” of this Theorem, to prove many of the statement in the following sections, and use the version of this theorem stated in the book [17]. In this section we explain and present a sketch of the proof of this Theorem and prove the Theorem 2.2.

Theorem 3.1 (Hadamard-Perron Theorem). Let $\lambda < \mu, r \geq 1$ and for each $m \in \mathbb{Z}$ let $f_m : \mathbb{C}^n \to \mathbb{C}^n$ be a $C^r$ diffeomorphisms such that for $(x,y) \in \mathbb{C}^l \oplus \mathbb{C}^k$,

$$f_m(x,y) = (A_m x + \alpha_m(x,y), B_m y + \beta_m(x,y)),$$

for some linear maps $A_m : \mathbb{C}^l \to \mathbb{C}^l$ and $B_m : \mathbb{C}^k \to \mathbb{C}^k$ with $||A_m^{-1}|| \leq \mu^{-1}$, and $||B_m|| \leq \lambda$ and $\alpha_m(0) = 0, \beta_m(0) = 0$.

Then for $0 < \gamma < \min(1, \sqrt{\frac{\mu}{\lambda} - 1})$ and

$$0 < \delta < \min \left( \frac{\mu - \lambda}{\gamma + 2 + \gamma^{-1}}, \frac{\mu - (1 + \gamma)^2 \lambda}{(1 + \gamma)(\gamma^2 + 2\gamma + 2)} \right)$$

(1) we have the following property: If $||\alpha_m||_{C^1} < \delta$ and $||\beta_m||_{C^1} < \delta$ for all $m \in \mathbb{Z}$ then there is

(i) a unique family $\{W_m^+\}_{m \in \mathbb{Z}}$ of $l$-dimensional $C^1$ manifolds

$$W_m^+ = \{(x, \varphi^+_m(x)) : x \in \mathbb{C}^l\} = \text{graph } \varphi^+_m$$

and

(ii) a unique family $\{W_m^-\}_{m \in \mathbb{Z}}$ of $k$-dimensional $C^1$ manifolds

$$W_m^- = \{(\varphi^-_m(y), y) : y \in \mathbb{C}^n\} = \text{graph } \varphi^-_m,$$

where $\varphi^+_m : \mathbb{C}^l \to \mathbb{C}^k$, $\varphi^-_m : \mathbb{C}^k \to \mathbb{C}^l$, $\sup_{m \in \mathbb{Z}} ||D\varphi^+_m|| < \gamma$, and the following properties holds:

(i) $f_m(W_m^-) = W_{m+1}^-$, $f_m(W_m^+) = W_{m+1}^+$.

(ii) The inequalities

$$||f_m(z)|| < \lambda' ||z|| \text{ for } z \in W_m^-,$$

and

$$||f_m^{-1}(z)|| < \mu' ||z|| \text{ for } z \in W_m^+$$

hold, where $\lambda' = (1 + \gamma)(\lambda + \delta(1 + \gamma)) < \frac{\mu}{1+\gamma} - \delta = \mu'$.

(iii) Let $\lambda' < \nu < \mu'$. If $||f_{m+j-1} \circ \cdots \circ f_m(z)|| < C\nu^j$ for all $j \geq 0$ and some $C > 0$ then $z \in W_m^-$. Similarly, if $||f_{m-j}^{-1} \circ \cdots \circ f_m^{-1}(z)|| < C\nu^{-j}$ for all $j \geq 0$ and some $C > 0$ then $z \in W_m^+$. 

Finally, in the hyperbolic case \( \lambda < 1 < \mu \) the families \( \{ W^+_m \}_{m \in \mathbb{Z}} \) and \( \{ W^-_m \}_{m \in \mathbb{Z}} \) consist of \( C^r \) manifolds.

**Remark 1.** It is important to note that the axes \( C^l \) and \( C^k \) are not invariant by the action of \((Df)_0\). However, there exist a splitting \( \mathbb{C}^n = E^+_m \oplus E^-_m \) with \( \dim C E^+_m = l \) and \( \dim C E^-_m = k \), that are invariant by the action of \((Df)_0\), satisfy that

\[
|| (Df)_0^{-1} |_{E^+_m} || \leq (\mu')^{-1}, \quad \text{and} \quad || (Df)_0 |_{E^-_m} || \leq \lambda',
\]

and in this case \( T_0 W^+_m = E^+_m \). See [17] for details.

It is important also to note the following proposition, proved in [17].

**Proposition 3.1.** The invariant manifolds (with \( C^1 \) topology) obtained in the Hadamard-Perron Theorem, have continuous dependence with respect to the family \( f = \{ f_m \}_{m \in \mathbb{Z}} \), with the \( C^1 \) topology defined by

\[
d_1(f,g) = \sup_{m \in \mathbb{Z}} d_{C^1}(f_m,g_m).
\]

### 3.1 Sketch of proof of Hadamard-Perron Theorem

In the proof of Theorem 3.1 (see [17]), the functions \( \varphi^+_m \) are obtained as fixed point of a contractive operator in a space of Lipschitz maps. We enumerate the main fact:

1. Let \( C^0_\gamma \) the space of sequences as form \( \varphi_* = \{ \varphi_m \}_{m \in \mathbb{Z}} \) where each \( \varphi_m \) is in the set

   \[
   C^0_\gamma(C^l) = \{ \varphi : C^l \rightarrow \mathbb{C}^{n-l} : \text{Lip}(\varphi) < \gamma, \text{ and } \varphi(0) = 0 \}.
   \]

2. The set \( C^0_\gamma \) is a compact metric space with the metric defined by

   \[
d_*(\varphi_*,\phi_*) = \sup_{m \in \mathbb{Z}} d(\varphi_m,\phi_m);
\]

   where

   \[
d(\varphi,\phi) = \sup_{x \in C^l \setminus \{0\}} \frac{|| \varphi(x) - \phi(x) ||}{|| x ||}
\]

   is a metric in \( C^0_\gamma(C^l) \). Note that \( (C^0_\gamma(C^l),d) \) is also compact metric space.

3. The action of \( f = \{ f_m \}_{m \in \mathbb{Z}} \) in the space \( C^0_\gamma \) is the desired contraction; this action is defined as follows: denote by \( (f_m)_* \varphi \) the unique Lipschitz map that satisfy the equation

   \[
f_m(\text{graph } \varphi) = \text{graph } ((f_m)_* \varphi).
\]

   On the other hand, we have the bijection \( G^m_\varphi : C^l \rightarrow C^l \) defined by

   \[
   G^m_\varphi(x) = A_m x + \alpha_m(x,\varphi(x)),
   \]
and the map $F^m_{x_0} : \mathbb{C}^l \to \mathbb{C}^l$ given by
\[ F^m_{x_0}(x) = B_{x_0}\varphi(x) + \beta_{x_0}(x, \varphi(x)), \]

it follows that the function $(f_m)_\varphi$ is given by the expression
\[ (f_m)_\varphi(x) = F^m_{x_0} \circ (G^m_{x_0})^{-1}(x). \]

Finally if we define $f_{\varphi^*} = \{\psi_m\}_{m \in \mathbb{Z}}$, with $\psi_{m+1} = (f_m)_\varphi \psi_m$, we have that
\[ \lim_{n \to \infty} f^n_{\varphi^*} = \varphi^*_+, \tag{2} \]

where $\varphi^* \in C^0$ and $\varphi^*_+ = \{\varphi^*_m\}_{m \in \mathbb{Z}}$ is the sequences of function given by the Hadamard-Perron Theorem.

### 3.2 Technical considerations

To apply the previous Theorem and the subsequent results, it is necessary to construct the family $\{f_m\}_{m \in \mathbb{Z}}$ that carries the asymptotic information of the map $f$ along the whole orbit of some point $x \in \Lambda$. For this construction, we assume that $f$ is partially hyperbolic on $\Lambda$ (see Definition\[\text{B}\]) with $\|Df^n(x)\|_{E(x)} \leq C \lambda^n$ and $\|Df^{-n}(x)\|_{E^c(x)} \leq C \rho^{-n}$ for all $n \geq 0$.

First one, note that given $\delta > 0$ we can find $R > 0$ such that for every $x_0 \in \Lambda$ we can write
\[ f(x) = f(x_0) + Df(x_0)(x - x_0) + R_{x_0}(x - x_0) \]
on $\mathbb{C}^n$, and $\|R_{x_0}(x - x_0)\|_{C^1} < \delta$ for all $x \in \Delta_n(x_0, 2R)$.

Moreover, the following statement hold.

**Lemma 3.1.** For every $\delta > 0$, there exist $R > 0$ uniformly in $\Lambda$, and smooth diffeomorphisms $f_{x_0} : \mathbb{C}^n \to \mathbb{C}^n$ for $x_0 \in \mathbb{C}^n$, such that $f_{x_0}(0) = 0$,
\[ f_{x_0}(h) = f(x + h) - f(x_0) \quad \text{for all } h \in \Delta_n(0, R), \]

and $\|f_{x_0}(h) - Df(x_0)(h)\|_{C^1} < \delta$ for all $h \in \mathbb{C}^n$. Moreover, we can construct this family so that the functions $f_{x_0}$ depend continuously in the $C^1$ topology, of the point $x_0$.

**Proof.** Given $R > 0$ take $\rho : \mathbb{C}^n \to [0, 1]$ a smooth function such that $\rho = 1$ on $\Delta_n(x_0, R)$, $\rho = 0$ outside of $\Delta_n(x_0, 2R)$ and $\|D\rho\| \leq C/R$. So defining
\[ f_{x_0}(h) = \rho(h)(f(x + h) - f(x_0)) + (1 - \rho(h))Df(x_0) \cdot h \]
we conclude this proof, once we choose $R > 0$ small. \qed

Now taking $L_{x_0} : \mathbb{C}^n = \mathbb{C}^l \oplus \mathbb{C}^k \to \mathbb{C}^n$ a linear orthogonal complex map such that $L_{x_0}(\mathbb{C}^l) = F(x_0)$ and $L_{x_0}(\mathbb{C}^k) = F(x_0)^\perp$, and define the maps $\hat{f}_{x_0} = L_f^{-1}(x_0) \circ f_{x_0} \circ L_{x_0}$, then $\hat{f}_{x_0}$ has the form
\[ \hat{f}_{x_0}(x, y) = (A_{x_0}x + \alpha_{x_0}(x, y), B_{x_0}y + \beta_{x_0}(x, y)). \]

To finish, we denote $x_m = f^m(x_0)$ with $m \in \mathbb{Z}$ and $f_m = \hat{f}_{x_m}$, then:
1) \( f_m \) is holomorphic in \( \Delta_n(0, R') \) for every \( R' < R \).

2) since that the splitting \( T_A\mathbb{C}^n = E \oplus F \) varies continuously, and the angle between the subspaces \( F \) and \( E \) are uniformly away from zero (see [21] for instance), it follows that there exist \( \lambda < \tilde{\mu} \) such that are satisfied the hypothesis of Hadamard-Perron Theorem.

3) it follows from the previous construction that the correspondence \( x_0 \mapsto \{ f_m \}_{m \in \mathbb{Z}} \) is continuous in the \( C^1 \) topology.

### 3.3 Proof of Theorem 2.2

To proof the Theorem 2.2 is only necessary to observe the following Proposition.

**Proposition 3.2.** Under the hypothesis of Theorem 3.1, suppose that the following additional conditions hold:

1. \( \mu > 1 \).

2. There exists \( R > 0 \) such that, for each \( m \in \mathbb{Z} \), the map \( f_m \) is holomorphic in some neighborhood of the closed polydisc \( \Delta_n(0, R) \subset \mathbb{C}^n \).

Then there exists \( 0 < r < R \) such that each \( \varphi_m^+ \) is holomorphic in some neighborhood of \( \Delta_l(0, r) \subset \mathbb{C}^l \), where \( \varphi_m^+ \) is as in (1) in the Theorem 3.1.

So in the hypothesis of the Theorem 2.2 is only necessary to work with \( f^{-1} \) instead \( f \), and construct the family as in the previous subsection.

**Remark 2.** In several works (see for example [2] or [3]), is proved the holomorphy of the stable/unstable manifolds under the hypothesis of hyperbolicity in the compact invariant set. In our case, we only consider partially hyperbolic map with unstable direction.

**Proof of Proposition 3.2.** Denote by \( \mathcal{O}_r^0 \subset C_0^0 \), the set of sequences of functions that are holomorphic in some neighborhood of the closed polydisc \( \Delta_l(0, r) \) in each level \( m \in \mathbb{Z} \). To prove the Proposition, is only necessary to prove that there exists \( 0 < r < R \) such that:

(a) \( \mathcal{O}_r^0 \) is a closed space in \( C_0^0 \),

(b) \( \mathcal{O}_r^0 \) is invariant by the action \( f \).

If we assume that (a) and (b) holds, and since that equation (2) hold for every \( \varphi^* \in \mathcal{O}_r^0 \), the limit

\[
\lim_{n \to \infty} f^n \varphi^* = \varphi^+_r
\]

there exists and is an element of \( \mathcal{O}_r^0 \), so each function \( \varphi^+_m \) is holomorphic in some neighborhood of \( \Delta_l(0, r) \).

Observe that for proof the two previous assertions, is only necessary proof that:
(a') $O^0_\gamma(r, C')$ is a closed space in $C^0_\gamma(C')$,

(b') $O^0_\gamma(r, C')$ is invariant by the action $f_m$, for all $m \in \mathbb{Z}$.

where $O^0_\gamma(r, C')$ is the subset of $C^0_\gamma(C')$, whose elements are holomorphics function in some neighborhood of the polydisc $\Delta_l(0, r)$.

The first assertion (a'), follows after observing that the metric defined in the paragraph (2.) of the section 3.1, induce the uniformly convergence on compact topology in $O^0_\gamma(r, C')$, so if $\varphi_n \in O^0_\gamma(r, C')$ and $\varphi_n \to \varphi$ for some $\varphi \in C^0_\gamma(C')$ then, the limit map $\varphi$ is an element of the set $O^0_\gamma(r, C')$.

The proposition (b'), it follows from the following: in the proof of the Theorem 3.1 in [17], we can see that $\|G^m_\varphi(x)\| \geq \mu_0 \|x\|$. (3)

where the constant is $\mu_0 = (\mu - \delta(1 + \gamma))$. This constant is greater than 1 if and only if, $\mu > 1$ and $\delta$ and $\gamma$ are small enough. If we take $r = \mu_0^{-1} R$, the functions $F^m_\varphi$ and $G^m_\varphi$ are holomorphics in some neighborhood of $\Delta_l(0, r)$ when $\varphi \in O^0_\gamma(r, C')$. It follows by the equation (3) that $\Delta_l(0, R) \subset G^m_\varphi(\Delta_l(0, r))$, then the function $(G^m_\varphi)^{-1}$ is holomorphic in $\Delta_l(0, r)$, and also by equation (3), it follows that $(G^m_\varphi)^{-1}(\Delta_l(0, r)) \subset \Delta_l(0, \mu_0^{-1} r) \subset \Delta_l(0, r)$.

We obtain that $F^m_\varphi \circ (G^m_\varphi)^{-1}$ is holomorphic in some neighborhood of $\Delta_l(0, r)$, is as desired. \qed

4 Dynamically defined and Overlapping property

Now we return to the original context exposed in the Section 2. The map $f : \mathbb{C}^n \to \mathbb{C}^n$ is a biholomorphism partially hyperbolic on a compact $f$-invariant set $\Lambda$ with splitting $T_{\Lambda} \mathbb{C}^n = E^s \oplus F$. We recall the existence of the $cu$-leaf $W^c_{\varepsilon}(x)$ for every $x \in \Lambda$ and $0 < \varepsilon \leq 1$, that are locally $f$-invariant.

On the other hand, the notion of $cu$-leaves dynamically defined, say that the $cu$-leaf are locally, the local unstable set. Then is natural to expect that if the $cu$-leaf are dynamically defined, they have a similar asymptotic behavior than the unstable set. This is exemplified in the following Lemma.

**Lemma 4.1.** The $cu$-leaves are dynamically defined, if and only if, there exists $r \ll 1$ such that for all $x \in \Lambda$, the following statement holds:

1. For any $r_1 < r$, there exist $r_0 < r_1$ such that for every $n \geq 0$ and $x \in \Lambda$, $f^{-n}(W^{cu}_{r_0}(x)) \subset W^{cu}_{r_1}(f^{-n}(x))$.

2. For every $r_1 < r$ and $r_0 < r_1$, there exists $N = N(r_0, r_1)$ such that for all $x \in \Lambda$ and $n \geq N$ $f^{-n}(W^{cu}_{r_0}(x)) \subset W^{cu}_{r_0}(f^{-n}(x))$. 

11
**Proposition 4.1.** If the $r > x$ for every $n > 0$, where $dist$ is the induced distance in the center-unstable leaf. It follows by compactness of $\Lambda$ and continuity of the $W^u$-leaf become small after a fixed number of iterates to the past.

We can do a more detailed description of the asymptotic behavior of the $cu$-leaf. For this this we introduce the notion of overlapping property.

**Definition 4.** Given a number $r > 0$, we say that the $cu$-leaves has the overlapping property for $r$, if for every $0 < r < r$ there exist $0 < r_1 < r_0 < r_2$, a number $N = N(r_0, r_1)$ and closed topological balls $B^u(x)$ with $W^u_{r_1}(x) \subset B^u \subset (W^u_{r_0}(x))^\circ$ for every $x \in \Lambda$, such that the following statement holds:

1. For every $n \geq N$ we have the inclusion $f^{-n}(W^u_{r_1}(x)) \subset (W^u_{r_0}(f^{-n}(x)))^\circ$,
2. $W^u_{r_1}(f^{-N}(x)) \subset f^{-N}(W^u_{r_1}(x)) \subset (B^u(f^{-N}(x)))^\circ$,
3. For every $0 \leq k \leq N$, we have $f^k(B^u(f^{-N}(x))) \subset (W^u_{r_1}(f^{-N-k}(x)))^\circ$.

The overlapping is produced by the topological balls $B^u(x)$ after a finite number of iterations to the future. Moreover, the previous definition establish that the size of the topological balls increase because

$$B^u(f^N(x)) \subset W^u_{r_1}(f^N(x)) \subset f^N(B^u(x))^\circ,$$

but do not excessively (property 3 in the previous definition). Also note that we require that the size of the balls be in some sense uniform on $x$ (the condition $W^u_{r_2}(x) \subset B^u \subset (W^u_{r_0}(x))^\circ$).

The main result of this section is to proof the following Proposition.

**Proposition 4.1.** If the $cu$-leaves are dynamically defined, then there exists $r > 0$ such that such that the $cu$-leaves has the overlapping property for $r$.

**Proof.** Let $r > 0$ has in the Lemma 4.1. If let us take $r_2 < r$, then for every $x \in \Lambda$,

$$\text{dist} \left( \partial W^u_{r_2}(x), \partial W^u_{r_1}(x) \right) > 0,$$

where dist is the induced distance in the center-unstable leaf. It follows by compactness of $\Lambda$ and continuity of the $cu$-leaves, that there exist a positive number $\delta > 0$ such that

$$\text{dist} \left( \partial W^u_{r_2}(x), \partial W^u_{r_1}(x) \right) > \delta.$$

Now let us take $r_1 < r_2$ as in the item 1 in the previous Lemma. Since that for every $n \geq 0$, $f^{-n}(W^u_{r_1}(x)) \subset W^u_{r_2}(f^{-n}(x))$ we have in particular that

$$\text{dist} \left( \partial f^{-n}(W^u_{r_1}(x)), \partial W^u_{r_2}(f^{-n}(x)) \right) \geq \text{dist} \left( \partial W^u_{r_2}(f^{-n}(x)), \partial W^u_{r_1}(f^{-n}(x)) \right) > \delta.$$
If we take $r_0 < r_1$, and $\varepsilon$ small enough such that $r_0 - \varepsilon > 0$, we know from the item 2 in the previous Lemma, that there exist $N = N(r_0 - \varepsilon, r_1)$ such that $$f^{-n}(W^{cu}_{r_1}(x)) \subset W^{cu}_{r_0 - \varepsilon}(f^{-n}(x)) \subset (W^{cu}_{r_0}(f^{-n}(x)))^\circ$$ for every $n \geq N$, and this implies the first item.

On the other hand, we can define the function $$\rho(x) = \text{dist} \left( f^{-N}(x), \partial f^{-N}(W^{cu}_{r_1}(x)) \right) > 0$$ that is continuous in $\Lambda$. Let $\rho_0 = \inf_{x \in \Lambda} \rho(x)$. Then for every $x$ there exist a neighborhood $U_x$ and a radius $r_x$ such that for $y \in U_x$ $$\text{dist} \left( W^{cu}_{r_x}(f^{-N}(y)), \partial f^{-N}(W^{cu}_{r_1}(y)) \right) > \frac{\rho_0}{2}.$$ So by compactness, there exist an $r_{-1}$ such that $$\text{dist} \left( W^{cu}_{r_{-1}}(f^{-N}(x)), \partial f^{-N}(W^{cu}_{r_1}(x)) \right) > \frac{\rho_0}{2}$$ in $\Lambda$ and in particular, $W^{cu}_{r_{-1}}(f^{-N}(x)) \subset f^{-N}(W^{cu}_{r_1}(x))$, that is the first inclusion of the second item.

For the second inclusion of the item 2 and the item 3, we must first construct the sets $B^{cu}(x)$. For this, let us take $$B(x) = \{ z \in W^{cu}_{r}(x) : \text{dist} \left( z, \partial W^{cu}_{r}(x) \right) \geq \delta/2 \}.$$ Then it is clear that $f^{-n}(W^{cu}_{r_1}(x)) \subset (B(f^{-n}(x)))^\circ$ for all $n \geq 0$, thus we define $B^{cu}(f^{-N}(x))$ has the connected component that contain $f^{-N}(W^{cu}_{r_1}(x))$ of the intersection $$W^{cu}_{r_{0 - \varepsilon}}(f^{-N}(x))) \cap f^{-1}(B(f^{-(N-1)}(x))) \cap \ldots \cap f^{-N}(B(x)).$$ By construction the set $(B^{cu}(f^{-N}(x)))^\circ$ contain $f^{-N}(W^{cu}_{r_1}(x))$ and it follows the third item, that conclude the proof of this Lemma. \( \square \)

**Remark 3.** It is important to recall that the election of the constant $r_0 < r_1$ is arbitrary, once we take $r_1 < r_2$.

## 5 Holomorphic center-unstable submanifolds

This section is devote to prove the following Theorem.

**Theorem 5.1.** If the cu-leaf are dynamically defined, then they are holomorphic submanifolds of $\mathbb{C}^n$.

For this, we use the notation and the technique introduced in the Section 3. The principal key is the overlapping property in the cu-leaf. We rewrite the definition of overlapping property in Hadamard-Perron notation.

Let $f = \{ f_m \}_{m \in \mathbb{Z}}$ and $\varphi^+ = \{ \varphi^+_m \}_{m \in \mathbb{Z}}$ the families of function as in the Theorem 3.1. We recall that the graph of the functions $\varphi^+_m$ are, by some local change of chart, the center-unstable manifolds given in the Theorem 2.1.
Definition 5. We say that the family \( \varphi^+ \) has the overlapping property for \( r > 0 \) if: for every \( r_2 < r \) there exist \( r_{-1} < r_0 < r_1 < r_2 \), an integer \( N = N(r_0, r_1) \), and a family of closed topological balls \( U_m \) with \( \Delta_l(0, r_{-1}) \subset U_m \subset \Delta_l(0, r_0) \), such that if we denote \( D_m^+ = \text{graph}(\varphi_m^+|U_m) \) and \( W_m^+(r') = \text{graph}(\varphi_m^+|\Delta_l(0, r')) \) are satisfied the following properties:

a) For every \( n \geq N \) and \( m \in \mathbb{Z} \), hold that \( f_{m-n}^{-1} \circ \cdots \circ f_{m-1}^{-1}(W_{m}^+(r_1)) \subset (W_{m}^+(r_0))^o \),

b) \( W_{m-N}(r_{-1}) \subset f_{m-N}^{-1} \circ \cdots \circ f_{m-1}^{-1}(W_m^+(r_1)) \subset (D_{m-N}^+)^o \) for every \( m \in \mathbb{Z} \),

c) For every \( 0 \leq k \leq N - 1 \) we have

\[
 f_{m-(N-(k-1))}^{-1} \circ \cdots \circ f_{m-(N-1)}^{-1} \circ f_{m-N}(D_{m-N}^+) \subset W_{m-(N-k)}^+(r)^o
\]

for every \( m \in \mathbb{Z} \).

From the Proposition 4.1, cu-dynamically defined implies the overlapping property, hence the proof the Theorem 5.1 it follows directly from the following Proposition.

Proposition 5.1. Under the hypothesis of Theorem 3.1, suppose that the following additional conditions hold:

1. There exists \( R > 0 \) such that, for each \( m \in \mathbb{Z} \), the map \( f_m \) is holomorphic in some neighborhood of the closed polydisc \( \Delta(0, R) \subset \mathbb{C}^n \).

2. For every \( 0 < r < R \), the family \( \varphi^+ \) as the overlapping property for \( r > 0 \) then there exists \( R' < R \) such that \( \varphi_m^+ \) is holomorphic in \( \Delta_l(0, R') \).

Now, we want to highlight the main difference of the Proposition 3.2 with the Proposition 5.1. In the first of them, it is assumed that \( F \) is an unstable direction. Here we only assume that the center-unstable manifold are dynamically defined.

However, the states of the proof has many similarities. The goal is to show that using the graph transform operator it is possible to prove that the cu-leaves are limits of the graph of uniformly bounded holomorphic function, and therefore it is also holomorphic. The main difficulty is to show that only using the dynamically defined property, is arrange to recover, after some iterate, the overlapping property of the graph transform operator.

Proof of Proposition 5.1. We use the same notation of the proof of Proposition 3.2. Firstly let us take \( r_2 < r < R \) with

\[
 2\gamma r_2 < \frac{R - r}{2}. \tag{4}
\]

We recall that from the Remark 3, we can take \( r_0 < r_1 \) small enough such that

\[
 2\gamma r_0 < \frac{r_1 - r_0}{2}. \tag{5}
\]
The proof goes through a series of claims.

**Claim 1:** There exists $\lambda_0 < 1$, such that for every $\varphi$, $\phi \in C^\gamma_0(\mathbb{C}^l)$, $m \in \mathbb{Z}$ and $x \in \mathbb{C}^l$ we have the inequality

$$||f_m(x, \varphi(x)) - f_m(x, \phi(x))|| \leq \lambda_0||\varphi(x) - \phi(x)||.$$

**Proof of Claim 1.** We recall that $f_m(x, \varphi(x)) = (G^m_\varphi(x), F^m_\varphi(x))$, then we have

$$||f_m(x, \varphi(x)) - f_m(x, \phi(x))||$$

$$= ||(G^m_\varphi(x), F^m_\varphi(x)) - (G^m_\phi(x), F^m_\phi(x))||$$

$$\leq ||G^m_\varphi(x) - G^m_\phi(x)|| + ||F^m_\varphi(x) - F^m_\phi(x)||$$

$$\leq ||(A_m x + \alpha_m(x, \varphi(x))) - (A_m x + \alpha_m(x, \phi(x)))||$$

$$+ ||(B_m \varphi(x) + \beta_m(x, \varphi(x))) - (B_m \phi(x) + \beta_m(x, \phi(x)))||$$

$$\leq ||\alpha_m(x, \varphi(x)) - \alpha_m(x, \phi(x))|| + ||B_m(\varphi(x) - \phi(x))||$$

$$+ ||\beta_m(x, \varphi(x)) - \beta_m(x, \phi(x))||$$

$$\leq (\lambda + 2\delta)||\varphi(x) - \phi(x)||,$$

then let us take $\lambda_0 = (\lambda + 2\delta)$, and we will prove that $\lambda_0 < 1$. Firstly note that we can assume that $\mu \leq 1$, if not by Proposition 3.2 it follows that $\varphi_m^+$ is holomorphic in a polydisc $\Delta_l(0, R')$ for some $R' < R$, that is we want to prove. On other hand, by inequality (1) in the Theorem 3.1

$$\delta < \frac{\mu - \lambda}{\gamma + \gamma^{-1} + 2},$$

and this is less than $(1 - \lambda)/2 < 1$. This end the proof of the claim. \hfill \Box

In that follows, we fix $m \in \mathbb{Z}$ and define

$$g_k = f_{(m+kN)+(N-1)} \circ f_{(m+kN)+(N-2)} \circ \cdots \circ f_{(m+kN)+1} \circ f_{(m+kN)}.$$  \hfill (6)

Then we can write $g_k$ as the form

$$g_k(x, y) = (C_k x + c_k(x, y), D_k y + d_k(x, y)),$$

where

$$C_k = A_{(m+kN)+(N-1)} \cdot A_{(m+kN)+(N-2)} \cdot \cdots \cdot A_{(m+kN)+1} \cdot A_{(m+kN)}$$

and

$$D_k = B_{(m+kN)+(N-1)} \cdot B_{(m+kN)+(N-2)} \cdot \cdots \cdot B_{(m+kN)+1} \cdot B_{(m+kN)}.$$  

We recall that graph transform operator $(f_m)_*$ of a Lipschitz function $\varphi$ is defined by the equation

$$(x', (f_m)_*(x')) = f_m(x, \varphi(x)) = (A_m x + \alpha_m(x, \varphi(x)), B_m \varphi(x) + \beta_m(x, \varphi(x))).$$
It is possible to prove that the map $G_\varphi^m : \mathbb{C}^l \to \mathbb{C}^l$ given by

$$G_\varphi^m(x) = A_m x + \alpha_m(x, \varphi(x)),$$

is a bijection, and that if we define $F_\varphi^m : \mathbb{C}^l \to \mathbb{C}^l$ by

$$F_\varphi^m(x) = B_m \varphi(x) + \beta_m(x, \varphi(x)),$$

then the graph transform operator $(f_m)_* \varphi$, is given by the expression

$$(f_m)_* \varphi(x) = F_\varphi^m(\varphi^{-1}(x)).$$

Similarly, we denote by

$$\tilde{G}_k(x) = C_k x + c_k(x, \varphi(x)),$$

and

$$\tilde{F}_k(x) = D_k \varphi(x) + d_k(x, \varphi(x)),$$

the coordinates maps related with $g_k$ and $\varphi$.

For a fixed $k$ and $\varphi$, we denote:

1. $\varphi_1 = (f_{m+kN})_* \varphi$,
2. $\varphi_{j+1} = (f_{m+kN+j})_* \varphi_j$, for every $j = 1, \ldots, N - 2$,
3. $G_j = g_{j+kN+j}$, for every $j = 1, \ldots, N - 1$.

**Claim 2:** We have that

$$\tilde{G}_k = G_{N-1} \circ G_{N-2} \circ \ldots \circ G_1 \circ G_{m+kN},$$

and the graph transform operator of $g_k$, given by equality $\tilde{G}_k$, is equal to

$$(g_k)_* = (f_{(m+kN)+(N-1)})_*(f_{(m+kN)+(N-2)})_* \ldots (f_{(m+kN)+(2)})_*(f_{(m+kN)})_*.$$

**Proof.** This is elementary, and the proof is left to the reader.

As a consequence of the previous claim, we conclude that $\tilde{G}_k$ is a bijection of $\mathbb{C}^l$, and that the graph transform operator related with $g_k$ is given by the equality

$$(g_k)_* \varphi(x) = \tilde{F}_k(\varphi^{-1}(x)).$$

In that follows by simplicity, we will work with $m = k = \theta$, and the function $g_0 = f_{N-1} \circ \ldots \circ f_0$, but all the following results are true for any $m$ and $k$.

**Claim 3:** If $\varphi \in C_0^1(\mathbb{C}^l)$ is holomorphic in some neighborhood of $U_0^+$, then the function $\tilde{G}_k^0$ is holomorphic in some neighborhood of $U_0^+$.  

16
Proof. From the inequality \(1\), for any point \(x\) in the closed polydisc \(\Delta_t(0,r_2)\), we have that
\[
||\varphi^+_0(x) - \varphi(x)|| \leq 2\gamma||x|| \leq 2\gamma r_2 < \frac{R-r}{2}.
\]

We recall that each map \(f_j\) is holomorphic in the closed polydisc \(\Delta(0,R)\). From the item \((c)\), it follows that \(f_0(D^+_0) \subset (W^+_1(r))^o\) and we conclude that \(G^0_{\varphi^+_0}(U_0) \subset \Delta_t(0,r)^o\). It follows from the Claim 1 that for every \(x \in U_0\)
\[
||G^0_{\varphi^+_0}(x) - G^0_{\varphi^+_0}(x)|| \leq ||f_0(x,\varphi^+_0(x)) - f_0(x,\varphi(x))|| \\
\leq \lambda_0 ||\varphi^+_0(x) - \varphi(x)|| \\
< \lambda_0 \frac{R-r}{2},
\]
this implies that
\[
||G^0_{\varphi^+_0}(x)|| \leq ||G^0_{\varphi^+_0}(x) - G^0_{\varphi^+_0}(x)|| + ||G^0_{\varphi^+_0}(x)|| < \frac{R-r}{2} + r = \frac{R+r}{2} < R.
\]

As before, we denote \(\varphi_1 = (f_0)_*, \varphi_2 \varphi_2 = (f_j)_*, \varphi_j\), for every \(j = 1, \ldots, N-2\), and \(G_j = G^0_{\varphi_j}, \) for every \(j = 1, \ldots, N-1\). Again by item \((c)\), for every \(1 \leq k \leq N-1\) we have \(f_k \circ \cdots \circ f_0(D^+_0) \subset (W^+_k+1(r))^o\), it follows that \(G^k_{\varphi^+_k} \circ \cdots \circ G^0_{\varphi^+_0}(U_0) \subset \Delta_t(0,r)^o\). We use the following notation:
\[x_k^+ = G^k_{\varphi^+_k} \circ \cdots \circ G^0_{\varphi^+_0}(x) \quad \text{and} \quad x_k = G_k \circ \cdots \circ G^0_{\varphi^+_0}(x).\]

Then as before, we conclude that for every \(x \in U_0\)
\[
||x_k^+ - x_k|| \leq ||f_k(x^+_{k-1},\varphi^+_k(x^+_{k-1})) - f_k(x_{k-1},\varphi_k(x_{k-1}))|| \\
\leq \lambda_0 ||\varphi^+_k(x^+_{k-1}) - \varphi_k(x_{k-1})|| \\
\leq \lambda_0 ||f_{k-1}(x^+_{k-2},\varphi^+_k(x^+_{k-2})) - f_{k-1}(x_{k-2},\varphi_k(x_{k-2}))|| \\
\vdots \\
\leq \lambda_0^k ||\varphi^+_0(x) - \varphi(x)|| \\
< \lambda_0^k \frac{R-r}{2},
\]
then is follows that
\[
||G_k \circ \cdots \circ G^0_{\varphi^+_0}(x)|| \leq ||x_k^+ - x_k|| + ||x_k^+|| < \frac{R-r}{2} + r = \frac{R+r}{2} < R.
\]

To end, since that \(\varphi\) is holomorphic in some neighborhood of \(U_0\), \(f_0\) is holomorphic in \(\Delta_t(0,R)\) and \(\text{Im}(G^0_{\varphi^+_0}(U_0)) \subset \Delta_t(0,R)^o\) it follows that the map \(\varphi_1\) is holomorphic in some neighborhood of \(\text{Im}(G^0_{\varphi^+_0}(U_0))\). Similarly, since that \(G_{1}(\text{Im}(G^0_{\varphi^+_0}(U_0))) \subset \Delta_t(0,R)^o, \) and \(f_1\) is holomorphic in this domain, we conclude that \(\varphi_2\) is holomorphic in \(\text{Im}(G_1(\text{Im}(G^0_{\varphi^+_0}(U_0))))\), and so on. This implies that the map \(G^0_{\varphi^+_0} = G_{N-1} \circ \cdots \circ G_1 \circ G^0_{\varphi^+_0}\) is holomorphic in \(U_0\) and \(\text{Im}(G^0_{\varphi^+_0}(U_0)) \subset \Delta_t(0,R)^o\). \(\blacksquare\)
Claim 4: The image of $U_0$ from the map $\overline{G}_0$, contain the polydisc $\Delta_1(0, r_0)$.

Proof. From the item (b), we have that $W^+_N(r_1) \subset (g_0(D_0^+))^\circ$, and we recall that $g_0(D_0^+)$ is a topological ball that contain 0. Now for a point $x \in pr_1(g^{-1}_0(W^+_N(r_1))) \subset \Delta_1(0, r_0)$ we have that $||G_{\varphi_{N-1}}^{-1} \circ \cdots \circ G_{\varphi_0}^{-1}(x)|| \leq r_1$ and that

$$
||G_{\varphi_{N-1}}^{-1} \circ \cdots \circ G_{\varphi_0}^{-1}(x) - \overline{G}_0(x)|| \leq ||g_0(x, \varphi_{N}^{-1}(x)) - g_0(x, \varphi(x))||
$$

$$
< \lambda_0^N \gamma r_0
$$

$$
< \frac{r_1 - r_0}{2},
$$

and this last inequality comes from the inequality (5). This conclude the proof of the claim.

From the previous claim, in particular we have that $U_N \subset \text{Im}(g_0(D_0^+))$. Since $\overline{F}_0$ be a holomorphic map in some neighborhood of $U_0$, and $(\overline{G}_0)^{-1}$ is holomorphic in $\Delta_1(0, r_0) \supset U_N$ (and this because $\overline{G}_0$ is holomorphic and injective), it follows that the map $\varphi'(x) = (g_0)_* \varphi(x) = \overline{F}_0 \circ (\overline{G}_0)^{-1}(x)$ is holomorphic in $U_N$.

We conclude that for any $m$, the action of the graph transform operator associated with the family $g = \{g_k\}_{k \in \mathbb{Z}}$ defined as in the equation (6), leaves invariant the set of sequences of Lipschitz functions that in each level is holomorphic in some neighborhood of the sets $U_m$; and note that this set contain the linear maps. Passing to limit, we conclude that each $\varphi_{N}^{-1}$ is holomorphic in the set $U_m \supset \Delta_1(0, r_{-1})$. Thus taking $R' = r_{-1}$, we completed the proof of the Proposition.

6 Forward Expansiveness in the center-unstable leaf

In this section we will prove the following Theorem.

Theorem 6.1. If $f$ is cu-forward expansive then the cu-leaf are dynamically defined.

For this purpose, is only necessary to prove that to be satisfied the equivalents condition in the Lemma 4.1, which are proved in the following Propositions.

Proposition 6.1. Let $f$ be a forward expansive map in the cu-leaves, with constant of expansiveness $c$. Then for every $r_1 < c$ there exist $r_0 < r_1$ such that for all $x \in \Lambda$ and $n \geq 0$

$$
f^{-n}(W_{r_0}^{cu}(x)) \subset W_{r_1}^{cu}(f^{-n}(x)).
$$
Proof. We suppose that is not true, thus there exists $r_1$ such that the previous proposition not holds. Let $\rho \geq 1$ such that $\rho r_1 < c$ and let $(r_k)_{k}^{\infty}$ be a sequence of positive numbers such that $r_k \to 0$ and $r_k < r_1$. Thus there exist $x_k \in \Lambda$ and $(n_k)_k$ such that

$$f^{-n_k}(W^{cu}_{r_k}(x_k)) \not\subseteq W^{cu}_{r_1}(f^{-n_k}(x_k)) \subset W^{cu}_{\rho r_1}(f^{-n_k}(x_k)).$$

We take each $n_k$ minimal with this property. Let us take $y_k = f^{-n_k}(x_k)$ and take $z_k$ some point in the following intersection

$$f^{-n_k}(W^{cu}_{r_k}(x_k)) \cap W^{cu}_{\rho r_1}(y_k) \setminus W^{cu}_{r_1}(y_k).$$

Also we take $y_0$ and $z_0$ such that $z_k \to z_0$ and $y_k \to y_0$. By construction (and $C^1$ continuity of the $cu$-leaves) we have that $z_0 \in W^{cu}_{\rho r_1}(y_0) \setminus W^{cu}_{r_1}(y_0)$.

We assert that

$$\text{dist}(f^n(y_0), f^n(z_0)) \leq \rho r_1$$

for each $n \geq 1$, and since $\rho r_1 < c$ we have a contradiction with the expansiveness in the $cu$-leaves. Then to conclude the proof, is only necessary to prove the previous assertion.

By contradiction, we assume that there exist $n$ such that

$$\text{dist}(f^n(y_0), f^n(z_0)) = \gamma > \rho r_1.$$

By continuity of $f^n$, given $\varepsilon > 0$ we can take $k \gg 1$ such that $n_k > n$ and satisfied

$$\text{dist}(f^n(y_k), f^n(y_0)) < \varepsilon \quad \text{and} \quad \text{dist}(f^n(z_k), f^n(z_0)) < \varepsilon.$$

If we take $\varepsilon$ such that $\gamma - 2\varepsilon > \rho r_1$ we conclude that $\text{dist}(f^n(z_k), f^n(y_k)) > \gamma - 2\varepsilon > \rho r_1$. To end, taking $z_k \in W^{cu}_{r_1}(x_k)$ such that $f^{n_k}(z_k) = z_k$, the previous inequality implies that

$$\text{dist}(f^{-n_k}(z_k), f^{-n_k}(x_k)) > \rho r_1,$$

that is

$$f^{-n_k}(W^{cu}_{r_k}(x_k)) \not\subseteq W^{cu}_{r_1}(f^{-n_k}(x_k)).$$

that contradict the minimality of $n_k$. This ends the proof. \hfill $\Box$

**Proposition 6.2.** Let $f$ be a forward expansive map in the $cu$-leaves, and $r_0 < r_1$ such that $r_0 \in I(r_1)$. Then for every $0 < \varepsilon < r_1 < c$ there exists $N = N(\varepsilon, r_0)$ such that for all $x \in \Lambda$ and $n \geq N$

$$f^{-n}(W^{cu}_{r_0}(x)) \subset W^{cu}_{\varepsilon}(f^{-n}(x)).$$

**Proof.** We suppose that is not true. Thus there exist $\varepsilon$ such that for all $k \geq 0$ there exist $x_k \in \Lambda$ and $n_k > k$ such that

$$f^{-n_k}(W^{cu}_{r_0}(x_k)) \not\subseteq W^{cu}_{\varepsilon}(f^{-n_k}(x_k)) \subset W^{cu}_{r_1}(f^{-n_k}(x_k)).$$

19
We take each $n_k$ minimal with this property. Let us take $y_k = f^{-n_k}(x_k)$ and take $z_k$ some point in the following intersection

$$f^{-n_k}(W_{cu}^{\text{cu}}(x_k)) \cap W_{cu}^{\text{cu}}(y_k) \setminus W_{\varepsilon}^{cu}(y_k).$$

Note that in particular $\text{dist}(y_k, z_k) < c$.

Also we take $y_0$ and $z_0$ such that $z_k \to z_0$ and $y_k \to y_0$. By construction (and $C^1$ continuity of the $cu$-leaves) we have that $z_0 \in W_{cu}^{cu}(y_0) \setminus W_{\varepsilon}^{cu}(y_0)$ and $\text{dist}(y_0, z_0) \leq c$.

We assert that

$$\text{dist}(f^n(y_0), f^n(z_0)) \leq c$$

for each $n \geq 1$, and since $c$ the expansiveness constant, we have a contradiction with the hypothesis of expansiveness in the $cu$-leaves. Then to conclude the proof, is only necessary to prove the previous assertion.

By contradiction, and arguing as in the previous proposition, if we assume that there exist $n$ such that $\text{dist}(f^n(y_0), f^n(z_0)) > c > \varepsilon$, there exist $k \gg 1$ such that $n_k > n$ and satisfies $\text{dist}(f^n(z_k), f^n(y_k)) > \varepsilon$. Thus

$$f^{n - n_k}(W_{cu}^{cu}(x_k)) \not\subset W_{cu}^{cu}(f^{n - n_k}(x_k)),$$

that contradict the minimality of $n_k$. \hfill \Box

### 7 Proof of Theorem A

The proof of the Main Theorem, it follows after the following Theorem.

**Theorem 7.1.** If the $cu$-leaf are dynamically defined, then the center-unstable direction $F$, is an unstable direction

**Proof of Theorem A.** (1) $\Rightarrow$ (2) In the hyperbolic case, the $cu$-leaf is unique and equal to the unstable manifold. The forward expansiveness in the $cu$-leaf is a well know property of the unstable manifold (topological expansivity).

(2) $\Rightarrow$ (3) It follows from the Theorem 6.1.

(3) $\Rightarrow$ (1) It follows from the Theorem 7.1. \hfill \Box

To prove the Theorem A, it is only necessary to prove the Theorem 7.1 and for this, we use that the $cu$-leaf $W_1^{cu}(x)$ are holomorphic submanifolds of $\mathbb{C}^n$ (Theorem 6.1, biholomorphic to a polydisc.

Consider the infinitesimal Kobayashi metric on the polydisc that is the natural generalization of the Poincaré metric for the unitary disk in several variables (see [19] for instance). The Kobayashi metric on a polydisc $\Delta = \Delta_1(0, r_1) \times \cdots \times \Delta_1(0, r_n)$ is given by the equation

$$K_\Delta(x, \xi) = \max_i \frac{r_i |\xi_i|}{r_i^2 - |x_i|^2}.$$
Then if we consider $\Delta = \Delta_k(0, r)$ we have that $K_\Delta(0, \xi) = r^{-1}||\xi||$. The following proposition is an immediate consequence of the definition of $K_\Delta$.

**Proposition 7.1.** Let $f : \Delta \to \Delta'$ be a holomorphic map between two polydisc, then

$$K_{\Delta'}(f(x), Df_x(\xi)) \leq K_\Delta(x, \xi).$$

**Proof of Theorem 7.1.** Since that the $cu$-leaf are dynamically defined without loss of generality, we can assume that

$$f^{-1}(W^{cu}_1(x)) \subset W^{cu}_{1/2}(f^{-1}(x))$$

for every $x \in \Lambda$. Let $\phi^{cu}$ the continuous function given by the Theorem 2.1. From the Theorem 5.1, it follows that the function $\phi^{cu}(x) : \Delta_l(0, 1) \to \mathbb{C}^n$ is holomorphic, where $l$ is the complex dimension of $F(x)$.

We define the holomorphic map $f_x : \Delta_l(0, 1) \to \Delta_l(0, 1)$ given by

$$f_x(z) = ((\phi^{cu}(f^{-1}(x))))^{-1} \circ f^{-1} \circ \phi^{cu}(x)(z).$$

Applying the previous Proposition, it is follows that

$$2||Df_x(0)\xi|| = 2||D[(\phi^{cu}(f^{-1}(x))))^{-1}(f^{-1}(x)) \circ Df^{-1}(x) \circ D[\phi^{cu}(x)](0)\xi|| \leq ||\xi||,$$

then

$$||Df_{f^{-1}(x)}(0) \circ \cdots \circ Df_{f^{-1}(x)}(0) \circ Df_x(0)|| \leq \left(\frac{1}{2}\right)^n.$$

On the other hand, using the continuity of the function $x \mapsto \phi^{cu}(x)$ and the compactness of the set $\Lambda$ we conclude that there exist a constant $C > 0$ such that

$$C^{-1} \leq ||D\phi^{cu}(x)(0)|| \leq C.$$

To end, since that

$$Df^{-n}(x)|_{F(x)} = D[\phi^{cu}(f^{-(n-1)}(x))](0) \circ Df_{f^{-(n-1)}(x)}(0) \circ \cdots$$

$$\cdots \circ Df_{f^{-1}(x)}(0) \circ Df_x(0) \circ D[\phi^{cu}(x)](x),$$

it follows that for every $\xi \in F(x)$

$$||Df^{-n}(x)\xi|| \leq (1/2)^nC^2||\xi||,$$

as desired. \qed

### 8 Some remark for complex Hénon maps

This section is devote to prove the Theorem B. Also in the end of this section we prove the Propositions 8.3 and 8.5. For notations and definition of the Julia set $J$ and the support of the measure of maximal entropy $J^*$, see [2].
8.1 Zero Lyapunov exponent measure

In this subsection we introduce some definitions to enunciate the Theorem B. In what follows, we assume that $f$ is a dissipative generalized Hénon map in $\mathbb{C}^2$, with $|\det(Df)| = b < 1$.

Denote by $\nu$, to a $f$-invariant measure whose support is contained in $J$. Also, we denote by $R(\nu)$, the set of all regular point in $\text{supp}(\nu)$. By the classical Oseledets Theorem, we know that $\nu(R(\nu)) = 1$. Let $x \in J$ be a regular point and let $\lambda^-(x) \leq \lambda^+(x)$ its Lyapunov exponents, then they are related with a splitting $E_x^-$ and $E_x^+$ respectively. Since $J$ has no attracting periodic points, from the equation $\lambda^-(x) + \lambda^+(x) = \log(b)$ it follows that $\lambda^-(x) \leq \log(b) < 0 \leq \lambda^+(x)$.

Definition 6. We say that a $f$-invariant measure $\nu$:

1. is hyperbolic, if $\lambda^+(x) > 0$ for $\nu$-a.e.,
2. has a zero exponent, if $\lambda^+(x) = 0$ for $\nu$-a.e.,

Give $\nu$ a measure, we denote by $R^+(\nu)$ (resp. $R^0(\nu)$), the set of all regular points, that has the maximal exponent positive (resp. null). It is clear that $R(\nu) = R^+(\nu) \cup R^0(\nu)$, where $\cup$ is a disjoint union. It is easy to see from the definition that $\nu$ is hyperbolic (resp. be a zero exponent) if and only if $\nu(R^+(\nu)) = 1$ (resp. $\nu(R^0(\nu)) = 1$). A measure, is not of the above types if and only if $\nu(R^+(\nu)), \nu(R^0(\nu)) > 0$. We recall that supp $(\nu) = \overline{R(\nu)(\text{mod} 0)} = R(\nu)(\text{mod} 0)$.

We can write every measure $\nu$, as a direct sum of the form $\nu = \nu^+ \oplus \nu^0$, where $\nu^+ = \nu|_{R^+(\nu)}$ is hyperbolic and $\nu^0 = \nu|_{R^0(\nu)}$ is has a zero exponent. Naturally $\nu^0 \equiv 0$ when $\nu$ is hyperbolic, and $\nu^+ \equiv 0$ when $\nu$ has a zero exponent.

Remark 4. It is important to recall that, for a measure that is neither hyperbolic nor has zero exponent, the supports supp $(\nu^0) = R^0(\nu)(\text{mod} 0)$ and supp $(\nu^+) = R^+(\nu)(\text{mod} 0)$ can intersect, but this intersection has measure zero both for $\nu^0$ and for $\nu^+$.

We define the set support of $J$, as the set

$\text{supp} (J) = \bigcup \{\text{supp} (\nu) : \nu$ is $f$-invariant $\}$.

In the paper [1], the authors proof that the set $J^* = \text{supp} (\mu)$, where $\mu$ is the unique measure of maximal entropy $\log(\deg(f))$, and that any hyperbolic measure has support contained in $J^*$. Then we have that

$J^* = \bigcup \{\text{supp} (\nu) : \nu$ is hyperbolic $\}$.

Also we define the set

$J_0 = \bigcup \{\text{supp} (\nu) : \nu$ has a zero exponent $\}$.

Note that by definition, $J_0$ is a compact $f$-invariant set.
Proposition 8.1. The equality \( \text{supp}(J) = J^* \cup J_0 \) holds.

Proof. It is clear that \( J^* \cup J_0 \subset \text{supp}(J) \). On the other hand, Let \( x_n \to x \in \text{supp}(J) \) with \( x_n \in \text{supp}(\nu_n) \). Writing \( \nu_n = \nu_n^+ \oplus \nu_n^0 \), we have that there is an infinity times \( n \) such that either \( x_n \in \text{supp}(\nu_n^+) \) or \( x_n \in \text{supp}(\nu_n^0) \), and we can take a subsequence converging to \( x \). This conclude the proof. 

8.2 Proof of Theorem B

This subsection in devote to prove the Theorem B, and this proof will be supported essentially in the Fornæss Theorem (see \[8\]), and the Theorem 8.2.

Theorem 8.1 (Fornæss). Let \( f \) be a complex Hénon map which is hyperbolic in \( J^* \). If \( f \) is not volume preserving, then \( J^* = J \).

This implies that is sufficient to see hyperbolicity of the \( J^* \). This allows enunciate the following result.

Theorem 8.2. Let \( f \) be a complex Hénon map, dissipative with dominated splitting in \( J^* \). Then we have the following dichotomy:

i. The set \( J^* \) is hyperbolic.

ii. \( J^* \cap J_0 \neq \emptyset \).

In the next subsection, we shall prove this result, as a corollary of the Theorem 2.1 of the celebrated work of R. Mañe “A proof of the \( C^1 \) Stability Conjecture”. This Theorem can be also proved independently of the Mañe work. For this another proof see \[24\].

As a corollary of the previous Theorem, we have.

Corollary 8.1. The set \( J_0 \neq \emptyset \) if and only if \( J_0 \cap J^* \neq \emptyset \).

Proof. If \( J_0 \cap J^* = \emptyset \), then \( J^* \) is hyperbolic. Thus from Fornæss Theorem, \( J \) is hyperbolic and \( J_0 = \emptyset \). 

Let \( \text{Per} \) the set of all periodic point contained in \( J \). From \[2\] any periodic saddle point \( p \) of \( f \) is on \( \text{Per} \), and \( J^* = \text{Per} \). We recall that from Proposition 8.3 the dominated direction \( E \) in each periodic point is a stable direction. This justify the following definition.

Definition 7. 1. We say that \( \text{Per} \) is uniformly hyperbolic if there exist a \( C \geq 1 \) and \( 0 < \lambda_1 < 1 \) such that for every \( n \geq 1 \)

\[
\|Df^{-n}|_{F(p)}\| \leq C\lambda_1^n,
\]

for every \( p \in \text{Per} \).

2. We say that \( \text{Per} \) is uniformly expanding at the period, if there exist a \( C \geq 1 \) and \( 0 < \lambda_1 < 1 \) such that

\[
\|Df^{-\pi(p)}|_{F(p)}\| \leq C\lambda_1^{\pi(p)},
\]

for every \( p \in \text{Per} \), where \( \pi(p) \) is the period of \( p \).
Proof of Theorem B. (1) ⇔ (2) From the Theorem 8.2, it follows that if \( J \) is hyperbolic, then \( J_0 \cap J^* = \emptyset \). Thus, from Corollary 8.1 it follows that \( J_0 = \emptyset \).

The reciprocal direction, is essentially the same: Corollary 8.1 say that if \( J_0 = \emptyset \), then \( J_0 \cap J^* = \emptyset \). The hyperbolicity of \( J \), it follows from the Theorem 8.2 and the Fornæss Theorem.

It is clear that (1) ⇒ (3) ⇒ (4).

Then is only necessary to proof that (4) ⇒ (2) and we conclude the proof of Theorem B. This it follows directly from the following result.

Proposition 8.2. Let \( f \) be a complex Hénon map, dissipative with dominated splitting in \( J^* \). Then we have the following dichotomy:

i. The set \( \text{Per} \) is uniformly expanding at the period.

ii. \( J^* \cap J_0 \neq \emptyset \).

Proof. We assume that \( J^* \cap J_0 = \emptyset \), and that the set \( \text{Per} \) is not uniformly expanding at the period. In this case we can assume that for every \( n \geq 1 \), there exist a periodic point \( p_n \) such that

\[
||Df^{-k\pi(p_n)}||_{F(p_n)} < \left( \frac{n-1}{n} \right)^{k\pi(p_n)},
\]

for every \( k \geq 1 \). Thus we have

\[
\log \left( \frac{n}{n-1} \right) > \frac{1}{k\pi(p_n)} \log \left( ||Df^{k\pi(p_n)}||_{F(p_n)} \right).
\]

(7)

Since that \( \lambda^+(p_n) > 0 \), we can find \( k_n \) great enough such that

\[
\frac{1}{k_n \pi(p_n)} \log \left( ||Df^{k_n\pi(p_n)}||_{F(p_n)} \right) > \frac{1}{n}.
\]

(8)

Now we define

\[
\nu_n = \frac{1}{k_n \pi(p_n)} \sum_{j=1}^{k_n \pi(p_n)} \delta_{f^j(p_n)},
\]

be a sequence of \( f \)-invariant measures that, taking a subsequence if necessary, we can assume that \( \nu_n \rightarrow \nu \). It follows from the inequalities (7) and (8) that

\[
\int \log ||Df||_{F} \, d\nu = \lim_{n \to \infty} \int \log ||Df||_{F} \, d\nu_n = 0,
\]

that is a contradiction with \( J_0 = \emptyset \).

Recently Christian Bonatti, Shaobo Gan and Dawei Yang, have proven an more general case of the previous proposition and that contain this (see [6]). In the work of Bonatti Et al., an important hypothesis in the proof is that his compact invariant set is a homoclinic class, and these is the case of \( J^* \); but we don’t use this fact in the previous proof, however homoclinic class is a hypothesis used in the proof of Fornæss Theorem. We conclude this subsection with the statement of Theorem of Bonatti Et al.
Theorem 8.3 (Bonatti-Gan-Yang). Let $p$ be a hyperbolic periodic point of a diffeomorphism $f$ on a compact manifold $M$. Assume that its homoclinic class $H(p)$ admits a (homogeneous) dominated splitting $T_{H(p)}M = E \oplus F$ with $E$ contracting and $\dim(E) = \text{ind}(p)$.

If $f$ is uniformly $F$-expanding at the period on the set of periodic points $q$ homoclinically related to $p$, then $F$ is uniformly expanding on $H(p)$.

8.3 Proof of Theorem 8.2

First one, we present the Theorem 2.1 due to Mañé in [13]. Let $f$ be a diffeomorphisms of $\text{C}^1$ class in a Riemannian manifold $M$ of any dimension, and $\Lambda$ be a compact invariant by $f$. A dominated splitting $T\Lambda = E \oplus F$ is say homogeneous if the dimension of the subspace $E(x)$ is constant for every $x \in \Lambda$. We say that a compact neighborhood $U$ of $\Lambda$ is admissible if the set $M(f,U) = \cap_{n \in \mathbb{Z}} f^n(U)$ has one and exactly one homogeneous dominated splitting $TM(f,U) = \hat{E} \oplus \hat{F}$ extending the splitting $T\Lambda = E \oplus F$. It is known, that if $T\Lambda$ has a homogeneous dominated splitting, then $\Lambda$ has an admissible neighborhood $U$ (see [15] for instance).

Theorem 8.4. Let $\Lambda$ be a compact invariant set of $f \in \text{Diff}^1(M)$ such that $\Omega(f|_\Lambda) = \Lambda$, let $T\Lambda = E \oplus F$ be a homogeneous dominated splitting such that $E$ is contracting and suppose $c > 0$ is such that the inequality

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log \| (Df)^{-1} |_{f^j(x)} \| < -c$$

(9)

holds for a dense set of points $x \in \Lambda$. Then either $F$ is expanding (and therefore $\Lambda$ is hyperbolic) or for every admissible neighborhood $V$ of $\Lambda$ and every $0 < \gamma < 1$ there exists a periodic point $p \in M(f,V)$ with arbitrarily large period $N$ and satisfying

$$\gamma^N \leq \prod_{j=1}^{N} \| (Df)^{-1} |_{\hat{F}(f^j(p))} \| < 1$$

where $\hat{F}$ is given by the unique homogeneous dominated splitting $TM(f,V) = \hat{E} \oplus \hat{F}$ that extend $T\Lambda = E \oplus F$.

In terms of the hypothesis of the Mañé Theorem, is clear that are satisfied for a dissipative Hénon map: $f$ of $\text{C}^1$ class and homogeneous dominated splitting. The inequality (9) it is satisfies with $c = \log(d)$ where $d$ is the degree of the map $f$. In fact, in [1] the authors proof that

$$\lambda^-(p) = \log(b) - \lambda^+(p) \leq -\log(d) < 0 < \log(d) \leq \lambda^+(p)$$

for every regular point for maximal entropy measure $\mu$, every periodic saddle point is a regular point, and in [2] is proved that the saddle periodic point are dense in $J^*$.

Also we remark that any periodic point in $M(f,V)$ for some $V$ an admissible neighborhood of $J^*$, is in fact an element of $\text{Per} \subset J^*$.  

25
Proof of Theorem 8.2 By the Mañé Theorem, if $J^*$ is not hyperbolic then, in particular, for every $n > 0$ there exist a periodic point $p_n$ of period $N(n) \geq n$ such that
\[ \log \left( \frac{n-1}{n} \right) \leq \frac{1}{N(n)} \log \| (Df)^{-N(n)} \|_{F(p_n)} < 0. \]

To end the proof, proceed in the same way as in the Proposition 8.2.

8.4 Dominated splitting and partially hyperbolicity for Hénon maps

In this subsection, we prove that dominated splitting Hénon map, are in fact partially hyperbolic. The complete statement is the following.

Proposition 8.3. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ a complex Hénon map, with dominated splitting $T_J \mathbb{C}^2 = E \oplus F$ in $J$. Then

1. If $f$ is volume preserving, then $f$ is uniformly hyperbolic in $J$.

2. If $f$ is dissipative, then $f$ is partially hyperbolic in $J$ and the $E$ direction is a stable direction.

To the proof of this Proposition, we use a characterization of dominated splitting that is proved in [24].

Proposition 8.4. Let $f$ be a Hénon map with $b = |\det(Df)|$, and let $T_J \mathbb{C}^2 = E \oplus F$ be a splitting. The following statement are equivalents:

1. The splitting $T_J \mathbb{C}^2 = E \oplus F$ is dominated;

2. There exist $C > 0$ and $0 < \lambda < 1$ such that:
   a) For every unitary vector $v \in F$ and $n \geq 1$
      \[ \frac{b^n}{\|Df^n v\|^2} \leq C \lambda^n, \]
   b) For every unitary vector $v \in E$ and $n \geq 1$
      \[ \frac{b^{-n}}{\|Df^{-n} v\|^2} \leq C \lambda^n. \]

Proof of Proposition 8.3 From the previous Proposition, we can assume without loss of generality that there exist $0 < \lambda < 1$ such that

(a) \[ \frac{b^n}{\|Df^n_x u_x\|^2} < \lambda^n, \text{ for every } n \text{ and } x \in J, \]

(b) \[ \frac{b^{-n}}{\|Df^{-n}_x v_x\|^2} < \lambda^n, \text{ for every } n \text{ and } x \in J, \]
where \( u_x \in F(x) \) and \( v_x \in E(x) \) are unitary vectors, and every \( x \in J \).

Replacing the previous inequality for the direction \( E(x) \), it follows that
\[
\|Df^{-n}v_x\|^2 > \left( \frac{1}{b\lambda} \right)^n.
\]
Replacing the inverse function of \( Df^{-n} \) in the previous inequality, and taking \( \lambda_0 = \sqrt{b\lambda} \), we obtain that
\[
\|Df^n v_x\| \leq \lambda_0^n \implies \|Df^n|_{E(x)}\| \leq \lambda_0^n.
\]
Similarly for the direction \( F(x) \), let \( u_x \) a unitary vector in this direction we obtain
\[
\|Df^n u_x\|^2 > \left( \frac{b}{\lambda} \right)^n,
\]
and taking \( \mu_0 = \sqrt{b/\lambda} \) it follows that
\[
\|Df^n|_{F(x)}\| \geq \mu_0^n.
\]
Thus we have
\[
\lambda^2 < 1 \iff b\lambda < \frac{b}{\lambda} \iff \lambda_0 < \mu_0.
\]
This prove that any complex Hénon map with dominated splitting in \( J \) is partially hyperbolic, so this are equivalent notions in this context.

To prove the item 1, i.e. the volume preserving case \( b = 1 \), is only necessary to observe that
\[
\lambda_0 = \sqrt{\lambda} < 1 < \sqrt{1/\lambda} = \mu_0,
\]
and for the item 2, i.e. the dissipative case \( b < 1 \), we have that so \( \lambda_0 = \sqrt{\lambda b} < 1 \), then \( E \) is a stable direction, as is desired. \( \Box \)

### 8.5 Weak forward expansivity in \( J^* \)

Periodic saddle point in \( J^* \) have unstable manifold, and this can by characterized as the set of point in which the function \( f \) has asymptotically expansiveness, and the constant of expansivity is related with the rate of expansion of the derivative in the unstable direction. This implies, that the map is forward expansivity along the orbit of a periodic point. The problem appear because the constant of expansiveness in the unstable direction is not uniform in the set of periodic point, so the forward expansivity is not uniform in the set of periodic saddle point.

Notwithstanding the above fact, in each unstable manifold of a periodic saddle point, there are many point (an open set in each unstable manifold) that goes to infinity by positives iterates of \( f \). Then we can say that in this points, we have an uniform forward expansivity. This property over periodic saddle points, for a dissipative Hénon map with dominated splitting, can be recovered over each point in the support of the maximal entropy measure \( J^* \). This is stated in the following proposition.
For notations and definition of the Julia set $J$, the support of the measure of maximal entropy $J^*$ and the set $U^+$, see [2].

**Proposition 8.5.** Let $f$ be dissipative Hénon map, with dominated splitting in $J^*$. Then for every $x \in J^*$, holds that $W^u_{loc}(x) \cap U^+ \neq \emptyset$.

Proof. The statement of the Proposition is true for saddle periodic points. In fact, for a saddle periodic point $p$, we have that $W^u(p)$ is a copy of $C$ (see [2]), and is dense in $J^-$ (see [3]). Also, we have that $J^- \cap U^+ \neq \emptyset$ because $J^- = \partial K^-$ and $K^- \cap U^+ \neq \emptyset$.

Thus, to proof this Proposition, we assert that the stable manifold of $p$ intersect any local center-unstable disk. This it follows from the fact that $J^* = H(p)$ is a homoclinic class of any periodic saddle point, and that there is a uniformly contractive sub-bundle, i.e., the direction $E$ (see Proposition 8.3).

Let $p_k$ be a sequence of periodic saddle points, $p_k \to x \in J^*$. From the continuity of the splitting, it follows that for $k$ great enough $W^u_{loc}(p_k) \cap W^u_{loc}(x) \neq \emptyset$. Since that $p_k \in H(p)$, each $W^u_{loc}(p_k)$ is approximated by disc contained in $W^u(p)$. More precisely, there exist a disc $D_k \subset W^u(p)$ such that $\text{dist}_1(D_k, W^u_{loc}(p_k)) < 1/k$, where $\text{dist}_1$ is the metric of the $C^1$ topology. It follows that for $k$ great enough, $D_k \cap W^u_{loc}(x) \neq \emptyset$ thus $W^u(p)$ intersect to $W^u_{loc}(x)$.

Now since that $W^u(p) \cap U^+ \neq \emptyset$ and $U^+$ is open, backward iterates of $U^+$ accumulates on any compact part of $W^u(p)$, and imply that backward iterates of $U^+$ intersects $W^u_{loc}(x)$. \hfill \Box

**References**

[1] E. Bedford; M. Lyubich; J. Smillie. *Polynomial diffeomorphisms of $C^2$. IV. The measure of maximal entropy and laminar currents*. Invent. Math. 112 (1993), no. 1, 77–125.

[2] E. Bedford; J. Smillie. *Polynomial diffeomorphisms of $C^2$: currents, equilibrium measure and hyperbolicity*. Invent. Math. 103 (1991), no. 1, 69–99.

[3] E. Bedford; J. Smillie. *Polynomial diffeomorphisms of $C^2$. II. Stable manifolds and recurrence*. J. Amer. Math. Soc. 4 (1991), no. 4, 657–679.

[4] E. Bedford; J. Smillie. *Polynomial diffeomorphisms of $C^2$. III. Ergodicity, exponents and entropy of the equilibrium measure*. Math. Ann. 294 (1992), no. 3, 395–420.

[5] E. Bedford; J. Smillie. *Polynomial diffeomorphisms of $C^2$. VIII. Quasi-Expansion*. American Journal of Mathematics 124 (2002), 221271.

[6] C. Bonatti; S. Gan; D. Yang. *On the Hyperbolicity of Homoclinic Classes*. Discrete Contin. Dyn. Syst. 25 (2009), no. 4, 1143–1162.

[7] G. Buzzard; *Kupka-Smale theorem for automorphisms of $C^n$*. Duke Math. J. 93 (1998), no. 3, 487–503.
[8] J. E. Fornæss. The Julia set of Hénon maps. Math. Ann. 334 (2006), no. 2, 457–464.

[9] J. E. Fornæss; N. Sibony. Complex Hénon mappings in $C^2$ and Fatou-Bieberbach domains. Duke Math. J. 65 (1992), no. 2, 345–380.

[10] S. Friedland; J. Milnor. Dynamical properties of plane polynomial automorphisms. Ergodic Theory Dynam. Systems 9 (1989), no. 1, 67–99.

[11] Hubbard, John H. The Hénon mapping in the complex domain. Chaotic dynamics and fractals (Atlanta, Ga., 1985), 101–111, Notes Rep. Math. Sci. Engrg., 2, Academic Press, Orlando, FL, 1986.

[12] J. Hubbard; R. Oberste-Vorth. Hénon mappings in the complex domain. I. The global topology of dynamical space. Inst. Hautes Études Sci. Publ. Math. No. 79 (1994), 5–46.

[13] Hubbard, John H.; Oberste-Vorth, Ralph W. Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials. Real and complex dynamical systems (Hillerød, 1993), 89–132, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 464, Kluwer Acad. Publ., Dordrecht, 1995.

[14] B. Hasselblatt; Y. Pesin, Partially hyperbolic dynamical systems. Handbook of dynamical systems. Vol. 1B, 1–55, Elsevier B. V., Amsterdam, 2006.

[15] M. W. Hirsch; C. C. Pugh; M. Shub. Invariant manifolds. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977. ii+149 pp.

[16] M. Jonsson; D. Varolin. Stable manifolds of holomorphic diffeomorphisms. Invent. Math. 149 (2002), no. 2, 409–430.

[17] A. Katok; B. Hasselblatt. Introduction to the modern theory of dynamical systems. With a supplementary chapter by Katok and Leonardo Mendsoza. Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.

[18] R. Mañé. A proof of the $C^1$ stability conjecture. Inst. Hautes Études Sci. Publ. Math. No. 66 (1988), 161–210.

[19] H. L. Royden, Remarks on the Kobayashi metric. Several complex variables, II (Proc. Internat. Conf., Univ. Maryland, College Park, Md., 1970), pp. 125–137. Lecture Notes in Math., Vol. 185, Springer, Berlin, 1971.

[20] M. Shub, Global stability of dynamical systems. With the collaboration of Albert Fathi and Rémi Langevin. Translated from the French by Joseph Christy. Springer-Verlag, New York, 1987.
[21] Y. Pesin. *Lectures on partial hyperbolicity and stable ergodicity*. European Mathematical Society (EMS), Zürich, 2004. vi+122 pp.

[22] E. R. Pujals; M. Sambarino. *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms*. Annals of Mathematics, vol. 151, 2000, p. 961-1023.

[23] E. R. Pujals; M. Sambarino. *Density of hyperbolicity and tangencies in sectional dissipative regions*. Ann. I. H. Poincaré AN 26 (2009) 1971-2000.

[24] F. Valenzuela. *Dominated Splitting and Critical Sets for Polynomials Automorphisms on \( \mathbb{C}^2 \)*. Doctoral Thesis, IMPA (2009).