On projective billiards with open subsets of triangular orbits

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Abstract

Ivrii’s Conjecture states that in every billiard in Euclidean space the set of periodic orbits has measure zero. It implies that for every $k \geq 2$ there are no $k$-reflective billiards, i.e., billiards having an open set of $k$-periodic orbits. This conjecture is open in Euclidean spaces, with just few partial results. It is known that in the two-dimensional sphere there exist $3$-reflective billiards (Yu.M.Baryshnikov). All the $3$-reflective spherical billiards were classified in a paper by V.Blumen, K.Kim, J.Nance, V.Zharnitsky: the boundary of each of them lies in three orthogonal big circles. In the present paper we study the analogue of Ivrii’s Conjecture for projective billiards introduced by S.Tabachnikov. In two dimensions there exists a $3$-reflective projective billiard, the so-called right-spherical billiard, which is the projection of a spherical $3$-reflective billiard. We show that the only $3$-reflective planar projective billiard with piecewise analytic boundary is the above-mentioned right spherical billiard. We prove non-existence of $3$-reflective projective billiards with piecewise analytic boundary in higher dimensions.

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1 Introduction

Ivrii’s conjecture states that for any billiard with smooth boundary in a Euclidean space, the set of periodic orbits has measure zero. To prove this conjecture, it is enough to show that for any $k \in \mathbb{N}$, the set of periodic orbit of period $k$ has measure 0 (and in particular, has empty interior). We say that a billiard is $k$-reflective, if its set of $k$-periodic orbits has non-empty interior. For planar billiards, this means the existence of a two-dimensional family of $k$-periodic orbits. The proof of the conjecture was made in the case of a billiard with a regular analytic convex boundary, see [10]. The conjecture is also true for $3$-periodic orbits, and was proved in [11] [12] [17] [19]. For $4$-periodic orbits, it was proven in [8] [9] and a complete classification of $4$-reflective complex analytic billiards was presented in [7].
Ivrii's conjecture was also studied in manifolds of constant curvature: it was proven to be true for \( k = 3 \) in the hyperbolic plane \( \mathbb{H}^2 \), see [4]; the case of the sphere \( S^2 \) was apparently firstly studied in [3], as quoted in [4] but we were not able to find the corresponding paper. The sphere is in fact an example of space were Ivrii's conjecture is not true, see [4], which contains a classification of all the 3-reflective billiards on the sphere.

In this paper we study a generalization of billiards: the so-called projective billiards introduced in [13, 15]. A projective billiard is a domain \( \Omega \subset \mathbb{R}^n \) with boundary endowed with a transverse line field \( L \), called the projective field of lines (see Figure 1). The reflection in planar projective billiard with a boundary curve \( \partial \Omega \) acts on lines according to the following reflection law. Given a point \( A \in \gamma \), let \( L(A) \) denote the corresponding line of the field. Two lines \( \ell \) and \( \ell' \) through \( A \) are symmetric with respect to the pair \( (L(A), T_A\gamma) \), if the four lines \( L(A), T_A\gamma, \ell, \ell' \) (identified with the corresponding 1-subspaces in \( T_A\mathbb{R}^2 \)) form a harmonic tuple: their cross-ratio is equal to 1. See Section 2 for more details.

In higher dimensions the notion of symmetry with respect to \( L(A) \) is analogous. Namely, two lines \( \ell, \ell' \subset T_A\mathbb{R}^n \) are symmetric with respect to the pair \( (L(A), T_A\partial \Omega) \), if \( \ell, \ell' \), \( L(A) \) lie in the same 2-subspace \( \Pi \subset T_A\mathbb{R}^n \) and the four lines \( L(A), T(A) := \Pi \cap T_A\partial \Omega, \ell, \ell' \) in \( \Pi \simeq \mathbb{R}^2 \) form a harmonic tuple.

Remark 1.1. The above harmonicity condition is equivalent to the condition that there exists a linear isomorphism \( H : T_A\mathbb{R}^2 \to \mathbb{R}^2 \) (respectively, \( \Pi \to \mathbb{R}^2 \)) sending \( L(A) \) and \( T_A\gamma \) to orthogonal lines and \( \ell, \ell' \) to symmetric lines with respect to \( H(T_A\gamma) \) (or equivalently, with respect to \( H(L(A)) \)).

The definition of orbits for projective billiards is the same as in the usual billiard dynamic.

**Definition 1.2.** A \( k \)-orbit in a convex projective billiard \( \Omega \) with transversal line field \( L \) on \( \partial \Omega \) is a sequence of vertices \( A_1, \ldots, A_k \in \partial \Omega \) such that for every \( j = 2, \ldots, k - 1 \) one has \( A_j \neq A_{j+1} \), the lines \( A_{j-1}A_j \) are not tangent to \( \partial \Omega \) at \( A_j \) and are symmetric with respect to the pair \( (L(A_j), T_A\partial \Omega) \). A \( k \)-orbit is periodic, if the above statements also hold for \( j = 1 \), \( k - 1 \) with \( A_0 = A_k \), \( A_{k+1} = A_1 \). In the general case, when \( \Omega \) is not necessarily convex, a \( k \)-orbit (\( k \)-periodic orbit) is defined as above with the additional condition that each edge \( A_jA_{j+1} \) lies in \( \Omega \) (except for its endpoints).

**Example 1.3.** A billiard \( \Omega \subset \mathbb{R}^n \) in Euclidean space can be viewed as a projective billiard, with the boundary \( \partial \Omega \) being equipped with the normal line field. The reflection law and orbits in the Euclidean billiard and in the corresponding projective billiard are the same.

Furthermore, as described in [14], examples of projective billiards can be obtained from metrics which are projectively equivalent to the Euclidean one, meaning that their geodesics are straight lines. Indeed for a given metric \( g \) on \( \mathbb{R}^n \) projectively equivalent to the Euclidean one and any hypersurface \( \Gamma \subset \mathbb{R}^n \) one can define the field of \( g \)-orthogonal lines to \( \Gamma \). The projective billiard thus constructed is equivalent to the billiard on \( \Gamma \) with the reflection acting on geodesics as the reflection in the metric \( g \), (see [14], Example 7).

**Example 1.4.** Metrics of constant curvature on space forms are projectively equivalent to the Euclidean metric. Indeed each complete simply connected \( n \)-dimensional space \( \Sigma \) of non-zero constant curvature, i.e., a space form (sphere or hyperbolic space) is realized as the unit sphere (half-pseudosphere of radius 1) in a Euclidean (respectively, Minkowski) space \( \mathbb{R}^{n+1} \) (after rescaling the metric by constant factor). The tautological projection \( \Sigma \to \mathbb{R}^n \) sends the geodesics to straight lines and thus, sends the metric on \( \Sigma \) to a metric projective equivalent to the Euclidean one. Consider a billiard \( \Omega \subset \Sigma \) with reflection acting on geodesics and defined by the metric on \( \Sigma \). Its boundary equipped with the normal line field is projected to a projective billiard in \( \mathbb{R}^n \). The billiard orbits in \( \Omega \) are projected to orbits of the latter projective billiard.

Figure 1: Left: A piece of curve \( \gamma \) endowed with a projective field of lines. Right: A convex closed curve \( \gamma \) endowed with a projective field of lines.
Definition 1.5. A projective billiard is said to be \( k \)-reflective, if the set of its \( k \)-periodic orbits has a non-empty interior.

As was already mentioned above, the version of Ivrii’s conjecture for triangular orbits in billiards on 2-sphere is false. The following example represents a 3-reflective spherical billiard given in [4] and describes its tautological projection, which is a 3-reflective projective planar billiard.

Example 1.6. Cut the sphere into 8 equal parts by choosing 3 pairwise orthogonal great circles. Consider one of these parts, \( E \), which is a geodesic triangle with all angles being right. It is a 3-reflective spherical billiard, and moreover, all its orbits are 3-periodic, as was shown in [3, 4]. Now let us equip the great circles containing its sides with normal line fields. Their projections form a triple of lines intersecting at three non-collinear points \( P, Q, R \) and equipped with the following fields of lines (i.e., projective billiard structure).

Definition 1.7. Consider three non-collinear points \( P, Q, R \) in the projective plane. For every point \( M \) in the line \( PQ \) let \( L(M) \subset T_M \mathbb{R}^2 \) be the line \( MR \) through the opposite vertex \( R \). The definition of line fields on the other lines \( QR, RP \) is analogous. Let us now choose an affine chart \( \mathbb{R}^2 \) containing \( P, R, Q \). This yields a triangle \( PQR \subset \mathbb{R}^2 \) equipped with a projective billiard structure, which will be called the right-spherical billiard. In what follows the triple of lines \( PQ, QR, RP \) equipped with the above line fields will be also called the right-spherical billiard.

Remark 1.8. All right-spherical billiards are projectively isomorphic and 3-reflective. This holds by construction, 3-reflectivity of the above triangular billiard on the sphere, and since the tautological projection sends orbits to orbits.

Remark 1.9. We will give another proof in Proposition 3.3 found by Simon Allais, that the right-spherical billiard is 3-reflective. If the reader is interested, we give in [5] examples of \( 2n \)-reflective projective billiards constructed inside polygons.

Let \( \Omega \) be a \( k \)-reflective projective billiard, and let \( A_1 \ldots A_k \) be its \( k \)-periodic orbit lying in the interior of the set of \( k \)-periodic orbits. This means that for every \( A'_1 \in \partial \Omega \) and \( A'_2 \in \partial \Omega \) close to \( A_1 \) and \( A_2 \) respectively the edge \( A'_1 A'_2 \) extends to a \( k \)-periodic orbit \( A'_1 \ldots A'_k \) close to \( A_1 \ldots A_k \). The latter statement depends only on germs of the boundary \( \partial \Omega \) at \( A_1, \ldots, A_k \). This motivates the following definitions.

Consider the fiber bundle \( \mathbb{P}(T\mathbb{R}^n) \) which can be seen as the set of pairs \( (A, L) \) where \( A \) is a point of \( \mathbb{R}^n \) and \( L \) is a line through \( A \), with its natural projection \( \pi: (A, L) \mapsto A \in \mathbb{R}^n \).

Definition 1.10. A \textit{line-framed planar curve} is a regularly embedded connected curve \( \alpha \subset \mathbb{P}(T\mathbb{R}^2) \) with the following properties:

1) The projection \( \pi \) sends \( \alpha \) diffeomorphically to a regularly embedded curve \( a \subset \mathbb{R}^2 \), which will be identified with \( \alpha \) and called the \textit{classical boundary} of the curve \( \alpha \).

2) For every \((A, L) \in \alpha \) the line \( L \) is transversal to \( T_A a \).

A \textit{line-framed hypersurface} is a regularly embedded connected \((n-1)\)-dimensional surface \( \alpha \subset \mathbb{P}(T\mathbb{R}^n) \) satisfying the above properties 1) and 2), with \( a \) being a hypersurface.

A \textit{line-framed complex analytic planar curve (hypersurface)} is a regularly embedded connected holomorphic curve \((n-1)\)-dimensional surface in \( \mathbb{P}(T\mathbb{C}^2) \) (in \( \mathbb{P}(T\mathbb{C}^n) \)) satisfying the above conditions 1) and 2).

Definition 1.11. A \textit{local projective billiard} is a collection of germs of \( k \) line-framed hypersurfaces \( \alpha_1, \ldots, \alpha_k \subset \mathbb{P}(T\mathbb{R}^n) \); let \( a_j \) denote their classical boundaries. The billiard is called \textit{smooth (analytic)}, if so are \( \alpha_j \). The notion of complex-analytic local projective billiard is the same, with \( \alpha_j \) being line-framed complex planar curves (hypersurfaces) in \( \mathbb{P}(T\mathbb{C}^n) \). A \textit{k-periodic orbit} of a local projective billiard is a \( k \)-tuple \((A_1, L_1) \ldots (A_k, L_k), (A_j, L_j) \in \alpha_j \), such that for every \( j = 1, \ldots, k \) one has \( A_j \neq A_{j \pm 1} \) and the lines \( A_{j-1}A_j, A_jA_{j+1} \) are not tangent to \( a_j \) at \( A_j \) and are symmetric with respect to the pair \((L_j, T_{A_j}(a_j))\); here \( A_0 = A_k, A_{k+1} = A_1 \).

A local projective billiard is called \textit{k-reflective}, if the base points \( A_j \) of the classical boundaries \( a_j \) form a \( k \)-periodic orbit and every pair \((A'_1, A'_2) \in \alpha_1 \times \alpha_2 \) close to \((A_1, A_2)\) extends to a \( k \)-periodic orbit \( A'_1 \ldots A'_k \) close to \( A_1 \ldots A_k \).

Example 1.12. Let \( \Omega \) be a \( k \)-reflective projective billiard. Let \( A_1 \ldots A_k \) be its \( k \)-periodic orbit. Then the germs at \( A_j \) of its boundary equipped with the corresponding line fields form a \( k \)-reflective local projective billiard.
A version of Ivrii’s conjecture for projective billiards is to classify all the $k$-reflective local projective billiards. In the present paper we solve it for 3-reflective local (real and complex) analytic billiards in any dimension.

**Theorem 1.13.** The local 3-reflective real (complex) analytic planar billiards are the real (complex) right-spherical billiards.

**Theorem 1.14.** There are no 3-reflective local analytic real (complex) projective billiards in $\mathbb{R}^d$ ($\mathbb{C}^d$) with $d \geq 3$.

**Plan of the article.** We first give precise definitions of the projective reflection law (Section 2). Then we prove Theorem 1.13 in the case when the classical boundaries of the projective billiard are supported by lines (Propositions 3.4 and 3.3 in Section 3).

After that, we prove Theorem 1.14 in the general case in the following way: we suppose that we are given a 3-reflective analytic projective billiard $B = (\alpha, \beta, \gamma)$ such that one of the classical boundaries, say the classical boundary $a$ of $\alpha$, is not a line. We study a certain analytic distribution, called Birkhoff’s distribution: it is constructed so that open sets of triangular orbits of 3-reflective projective billiards yield its two-dimensional integral surfaces (see Section 4). We use methods of analytic geometry (like Remmert’s proper mapping theorem) to prove that this distribution is of dimension 2 on the analytic closure $M$ of the set of triangular orbits of $B$ (Proposition 4.13); and this allows us to conclude that it is integrable on $M$ (Corollary 4.14).

Finally, we exhibit a particular integral surface of Birkhoff’s distribution (Subsection 5.1) corresponding to a certain projective billiard $B_0 = (\alpha, \beta_0, \gamma_0)$ where the classical boundary $b_0$ of $\beta_0$ intersects a certain tangent line $L$ of $a$. But such projective billiard cannot be 3-reflective, as we show in Section 5.2 using asymptotic comparisons of complex angles (or azimuths) of orbits whose vertices converge to $L$.

We finally prove Theorem 1.14 in Section 6 using Theorem 1.13 and the fact that any triangular orbit has its vertices in the same plane.

## 2 Projective law of reflection

We first recall some definitions about the cross-ratio of four lines in $\mathbb{CP}^1$, we introduce harmonic quadruples of lines, and then give a formula for the projective symmetry of lines.

**Definition 2.1.** Fix any set of coordinates on $\mathbb{CP}^1$. Given three distinct points $X_2$, $X_3$, $X_4$ of $\mathbb{CP}^1$ and a fourth point $X_1$, the cross-ratio $(X_1, X_2; X_3, X_4)$ is defined to be the value $h(X_1)$ where $h : \mathbb{CP}^1 \to \mathbb{CP}^1$ is the only projective transformation sending $X_2$ to 1, $X_3$ to 0 and $X_4$ to $\infty$. 

![Figure 2: A projective billiard $B = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$: each $\alpha_i$ is in $\mathbb{P}(T\mathbb{CP}^2)$ and is projected on $a_i$ in $\mathbb{CP}^2$.](image)
Remark 2.2. The cross-ratio is a projective invariant in the sense that for any projective transformation $f : \mathbb{CP}^1 \to \mathbb{CP}^1$, it satisfies $(f(X_1), f(X_2); f(X_3), f(X_4)) = (X_1, X_2; X_3, X_4)$, and hence doesn’t depend on the initial choice of coordinates of $\mathbb{CP}^1$.

Definition 2.3. The cross-ratio of a quadruple of four distinct lines $(\ell_1, \ell_2, \ell_3, \ell_4)$ intersecting at a point $O$, is the cross-ratio of the quadruple formed by their intersection points $X_1, X_2, X_3, X_4$ with a fifth line $\ell$ not containing $O$, taken in the same order:

$$(\ell_1, \ell_2; \ell_3, \ell_4) = (X_1, X_2; X_3, X_4).$$

It does not depend on the choice of $\ell$.

![Figure 3](image)

Figure 3: $(\ell_1, \ell_2; \ell_3, \ell_4) = (X_1, X_2; X_3, X_4) = (Y_1, Y_2; Y_3, Y_4)$. The definition of the cross-ratio does not depend on the chosen line $\ell$.

We say that a quadruple $(\ell_1, \ell_2, \ell_3, \ell_4)$ of distinct lines is harmonic if its cross-ratio is $-1$:

$$(\ell_1, \ell_2; \ell_3, \ell_4) = -1.$$ 

If now we fix a set of coordinates on the line $\ell$ of Definition 2.3 such that each intersection point $X_i$ of $\ell_i$ with $L$ can be identified with $z_i \in \mathbb{CP}^1$, then $(\ell_1, \ell_2, \ell_3, \ell_4)$ is harmonic if and only if

$$z_4 = \frac{(z_1 + z_2)z_3 - 2z_1z_2}{2z_3 - (z_1 + z_2)}. \quad (1)$$

Using Equation (1), we define the symmetry of lines at a point $A \in \mathbb{CP}^2$:

Definition 2.4. Let $L, T$ be two distinct lines through $A$. Two lines $(\ell, \ell')$ are said to be symmetric with respect to $(L, T)$ if either $\ell = \ell' = T$, or $\ell = \ell' = L$, or $(L, T, \ell, \ell')$ is a harmonic quadruple of distinct lines.

Remark 2.5. Note that the three conditions are coherent with Formula (1).

Finally let us introduce the azimuth of a line. We consider a set of coordinates on $\mathbb{CP}^2$ such that $\mathbb{CP}^2$ can be seen as the disjoint union of $\mathbb{C}^2$ and a complex line, called infinity line and denoted by $\mathbb{C}_\infty$. A finite (respectively, an infinite) point of $\mathbb{CP}^2$ will be a point in $\mathbb{C}^2$ (respectively, in $\mathbb{C}_\infty$).

Definition 2.6. A complex line $d$ of $\mathbb{CP}^2$ distinct from the infinity line intersects $\mathbb{C}_\infty$ at a unique point called the azimuth of $d$, and written $\text{az}(d)$.

Now fix two lines $L$ and $T$, distinct from $\mathbb{C}_\infty$ and intersecting at a finite point $A$. The set $A^*$ of all lines passing through $A$ can be identified with $\mathbb{CP}^1$, via the map

$$d \in A^* \mapsto \text{az}(d). \quad (2)$$
Proposition 2.7. Via the identification \((2)\), the symmetry of lines with respect to \((L, T)\) is the unique nontrivial complex involution of \(A^{*}\), which has \(L\) and \(T\) as fixed points. It is of the form
\[
z \mapsto \frac{(\ell + t)z - 2\ell t}{2z - (\ell + t)} \tag{3}
\]
where \(\ell\) and \(t\) are the azimuths of \(L\) and \(T\) respectively.

Remark 2.8. Notice that when \(\ell = t\), the map of Equation \((3)\) is constant to \(\ell\). This shows that when both lines \(L\) and \(T\) are the same, the reflection law degenerates and we can say that any two lines \((d, d')\) are symmetric with respect to \((L, L)\) if and only if \(d = L\) or \(d' = L\).

3 Proof of Theorem 1.13 when \(a, b, c\) are supported by lines

In this section we prove Theorem 1.13 when \(a, b, c\) are supported by lines. Let us first define right-spherical billiards properly with the tools we introduced on projective billiards.

Take a line \(\ell \subset \mathbb{CP}^2\) and a point \(P \notin \ell\). One can define a line-framed curve \(\omega(\ell, P) \subset \mathbb{P}(T\mathbb{CP}^2)\) (which is in fact algebraic) as the set of \((A, L)\) where \(A\) is on \(\ell\) and \(L\) is the line \(AP:\)
\[
\omega(\ell, P) = \{(A, AP) \in \mathbb{P}(T\mathbb{CP}^2) \mid A \in \ell\}.
\]

Figure 4: The right-spherical billiard \(B(P, Q, R)\)

Definition 3.1. We call any projective billiard \(B\) right-spherical if there are three points \(P, Q, R\) not on the same line, such that
\[
B = (\omega(PQ, R), \omega(QR, P), \omega(RP, Q)) .
\]
see Figure 4. We can write \(B = B(P, Q, R)\).

Remark 3.2. As explained in the introduction, the name right-spherical billiard comes from the construction which let us understand billiards on \(S^2\) as projective billiards. The construction is the following: consider the projection
\[
\pi: S^2 \subset \mathbb{R}^3 \to \mathbb{RP}^2 .
\]
It sends the geodesics of \(S^2\) (which are its great circles) onto the lines of \(\mathbb{RP}^2\). If \(b\) is a smooth curve in \(S^2\) and \(M \in b\), the transverse line at \(\pi(M) \in \pi(b)\) is given by the projection of the orthogonal geodesics to \(T_M b\). Note then that when you project and then complexify the reflective billiards on \(S^2\) given in [4], you get right-spherical billiards.

Proposition 3.3. Any right-spherical billiard is 3-reflective.
Figure 5: The line $AB$ is reflected into the lines $L'$ and $L^*$ in $A$ and $B$ respectively. Their intersection point is $C$, which in fact lies on $QR$ by a simple computation.

**Proof.** This proof was found by Simon Allais in a talk we had about harmonicity in a projective space. Let $B = B(P, Q, R)$ be a right-spherical billiard. Let $A \in PQ$ and $B \in QR$ such that $A \neq B$.

Let $C \in RP$ be such that $AB$, $BC$, $QR$, $BP$ are harmonic lines. Define $C' \in RP$ similarly: $AB$, $AC''$, $PQ$, $AR$ are harmonic lines.

Let us first show that necessarily $C = C'$. Consider the line $RP$ and let $K$ be its point of intersection with $AB$. Let us consider harmonic quadruples of points on $RP$. By harmonicity of the previous defined lines passing through $B$, the quadruple of points $(K, C, R, P)$ is harmonic. Doing the same with the lines passing through $A$, the quadruple of points $(K, C', R, P)$ is harmonic. Hence $C = C'$ since the projective transformation defining the cross-ratio is one to one.

Now let us prove that the lines $BC$, $AC$, $PR$, $CQ$ through $C$ are harmonic lines. Consider the line $PQ$: $BC$ intersects it at a certain point denoted by $T$, $AC$ at $A$, $PR$ at $P$ and $CQ$ at $Q$. But the quadruple of points $(T, A, P, PQ)$ is harmonic since there is a reflection law at $B$ whose lines intersect $PQ$ exactly in those points.

**Proposition 3.4.** Suppose that $B = (\alpha, \beta, \gamma)$ is a 3-reflective complex-analytic local projective billiard such that its classical borders $a$, $b$, $c$ are supported by pairwise distinct lines. Then $B$ is right-spherical.

Figure 6: The local projective billiard $B = (\alpha, \beta, \gamma)$ is such that $a$, $b$, $c$ are supported by the lines $\ell_a$, $\ell_b$, $\ell_c$ respectively.
Proof. Write $\ell_a, \ell_b, \ell_c$ the lines which respectively support $a, b, c$. First note that two lines among $\ell_a, \ell_b, \ell_c$ cannot be the same (otherwise there cannot exists a 3-reflective set of billiard orbits). There are two cases to consider:

Case 1. The three lines $\ell_a, \ell_b, \ell_c$ intersect at the same point $R$.

Case 2. The three lines $\ell_a, \ell_b, \ell_c$ do not intersect at the same point.

**Case 1.** We suppose that the three lines $\ell_a, \ell_b, \ell_c$ intersect at the same point $R$ and we show a contradiction. Choose any $m_A = (A, L_A)$ on $\alpha$ such that $L_A \neq \ell_a$ and $A \neq R$. The line $L_A$ intersects $\ell_c$ at a certain point $R_{C} \neq R$ and $\ell_b$ at a certain point $R_B \neq R$. Fix $m_B = (B, L_B)$ in $\beta$ and consider $m_C = (C, L_C)$ in $\gamma$ such that $(m_A, m_B, m_C)$ is a triangular orbit. Since the quadruple of lines through $A$, $(\ell_a, L_A; AC, AB)$, is harmonic, so is the quadruple of lines through $B$, $(\ell_b, BR_C; BC, AB)$ (since the intersection points of these lines with $\ell_c$ are the same in each quadruple). We deduce that $L_B = BR_C$. The same arguments can be applied for $m_C$ to show that $L_C = CR_B$. Thus the curves $\beta$ and $\gamma$ can be extended to the curves $\{(B, BR_C) | B \in \ell_b\}$ and $\{(C, CR_B) | C \in \ell_c\}$ respectively. This implies that for $m'_A = (A', L'_A)$ in a neighborhood of $m_A$, the line $L'_A$ should also intersect $\ell_b$ at $R_B$ and $\ell_c$ at $R_C$ (by the same arguments). Hence $L'_A = R_B R_C$ and $A = A'$ which is impossible by choice of $m'_A$.

**Case 2.** Suppose that the three lines $\ell_a, \ell_b, \ell_c$ do not intersect at the same point. Choose any $m_A = (A, L_A)$ in $\alpha$ such that $A$ do not belong to $\ell_b$ nor $\ell_c$, and denote by $R$ the intersection point of $\ell_b$ with $\ell_c$. Let us show that $L_A = AR$.

First we extend the curves $\beta$ and $\gamma$ analytically. For any $B \in \ell_b$, the line $AB$ is reflected at point $m_A$ into a certain line intersecting $\ell_c$ at a point $C(B)$. In particular, when $B = R$, $C(B) \neq B$ otherwise $L_A$ would be the line $AR$. Hence $B \neq C(B)$ for any $B \in \ell_b$ and the map $B \mapsto BC(B)$ is well-defined and analytic on $\ell_b$ (by Equation (3)). Therefore there is an analytic field of lines $L_B : \ell_b \to \mathbb{CP}^2$ such that for any $B \in \ell_b$, $(\ell_b, L_B(B); AB, BC(B))$ is a harmonic quadruple of lines through $B$ (see Equation (3)), and $\beta$ extends to the curve $\{(B, L_B(B)) | B \in \ell_b\}$. We can apply the same arguments to $\gamma$ which extends to a curve $\{(C, L_C(C)) | C \in \ell_c\}$ where $L_C : \ell_c \to \mathbb{CP}^2$ is an analytic field of lines. Notice that the constructed extensions do not depend on the choice of $m_A$ since they are analytic and they contain the initial line-framed curves $\beta$ and $\gamma$ respectively.

Now take $m_B = (B, L_B) \in \beta$ such that $B = R$ and consider $C = C(B)$ and then $m_C = (C, L_C(C))$ in $\gamma$. As we noticed previously, $C \neq R$ and $BC = \ell_c$. Note that $(m_A, m_B, m_C)$ is a triangular orbit by analyticity of the projective reflection law at $B$ and $C$ (by Equation (3)). Therefore, the quadruple of lines through $B$, $(\ell_b, L_B; AR, \ell_c)$, is harmonic, and this is true for any $m_A$ in $\alpha$ which is impossible since the lines $AR$ cannot be the same (this follows from the fact that $\ell_a$ doesn’t go through $R$).

### 4 Study of Birkhoff’s distribution

#### 4.1 Singular analytic distributions

We recall some definitions and properties of singular analytic distributions, which can be found in [7].

**Definition 4.1** ([7], Lemma 2.27). Let $W$ be a complex manifold, $\Sigma \subset W$ a nowhere dense closed subset, $k \in \{0, \ldots, n\}$ and $D$ an analytic field of $k$-dimensional planes defined on $W \setminus \Sigma$. We say that $D$ is a singular analytic distribution of dimension $k$ and singular set $\text{Sing}(D) = \Sigma$ if $D$ extends analytically to no points in $\Sigma$ and if for all $x \in W$, one can find holomorphic 1-forms $\alpha_1, \ldots, \alpha_p$ defined on a neighborhood $U$ of $x$ and such that for all $y \in U \setminus \Sigma$,

$$D(y) = \bigcap_{i=1}^{p} \ker \alpha_i(y).$$

Singular analytic distribution can be restricted on analytic subsets:

**Proposition 4.2** ([7], Definition 2.32). Let $W$ be a complex manifold, $M$ an irreducible analytic subset of $W$ and $D$ a singular analytic distribution on $W$ with $M \subset \not \subset \text{Sing}(D)$. Then there exists an open dense subset $M'_{\text{reg}}$ of $x \in M_{\text{reg}}$ for which

$$D|_M(x) := D(x) \cap T_x M$$

has minimal dimension. We say that $D|_M$ is a singular analytic distribution on $M$ of singular set $\text{Sing}(D) := M \setminus M'_{\text{reg}}$. 

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Remark 4.3. When $M$ is not irreducible anymore, we still can restrict $\mathcal{D}$ to $M$ by looking at its restriction to each of the irreducible components of $M$.

As in the smooth case, we can look for integral surfaces defined by the following

**Definition 4.4** ([7], Definition 2.34). Let $\mathcal{D}$ be a $k$-dimensional analytic distribution on an irreducible analytic subset $M$ and $\ell \in \{0, \ldots, k\}$. An integral $\ell$-surface of $\mathcal{D}$ is a submanifold $S \subset M \setminus \text{Sing}(S)$ of dimension $\ell$ such that for all $x \in S$, we have the inclusion $T_x S \subset \mathcal{D}(x)$. The analytic distribution $\mathcal{D}$ is said to be integrable if each $x \in M \setminus \text{Sing}(S)$ is contained in an integral $k$-surface.

We can finally introduce the following lemma, which will be used in a key result (Corollary 4.14). We recall here that the analytic closure of a subset $A$ of a complex manifold $W$, is the smallest analytic subset of $W$ containing $A$. We denote it by $\overline{A}^\text{an}$.

**Lemma 4.5** ([7], Lemma 2.38). Let $\mathcal{D}$ be a $k$-dimensional singular analytic distribution on an analytic subset $N$ and $S$ be a $k$-dimensional integral surface of $\mathcal{D}$. Then the restriction of $\mathcal{D}$ to $\overline{S}^\text{an}$ is an integrable analytic distribution of dimension $k$.

The proof is the same as in [7]:

**Proof.** Write $M = \overline{S}^\text{an}$. First, let us prove that $\mathcal{D}_{|M}$ is $k$-dimensional. Consider the subset

$$A := \{ x \in M \setminus \text{Sing}(\mathcal{D}_{|M}) \mid \mathcal{D}(x) \subset T_x M \}.$$  

It contains $S \setminus \text{Sing}(\mathcal{D}_{|M})$, hence its closure, which is an analytic subset of $M$, contains $S$. By definition, $\overline{A}^\text{an} = M$ which implies that $\mathcal{D}_{|M}$ is $k$-dimensional.

Now let us show that $\mathcal{D}_{|M}$ is integrable. The argument is similar: define the subset $B$ of those $x \in M \setminus \text{Sing}(\mathcal{D}_{|M})$ such that the Frobenius integrability condition is satisfied. $B$ contains $S \setminus \text{Sing}(\mathcal{D}_{|M})$ and its closure is an analytic subset of $M$ containing $S$, hence it is the whole $M$. Thus Frobenius theorem can be applied on the manifold $M \setminus \text{Sing}(\mathcal{D}_{|M})$, which implies the result. $\square$

### 4.2 Birkhoff’s distribution and the 3-reflective billiard problem

Let us define Birkhoff’s distribution attached to a complex analytic line-framed curve $\alpha$, and establish its link with the local projective billiard (Proposition 4.7). We define the space $\mathcal{P}$ as the fiber bundle

$$\mathcal{P} = \mathbb{P}(T\mathbb{C}\mathbb{P}^2) \times \mathbb{P}(T\overline{\mathbb{C}\mathbb{P}^2})$$

that is the set of triples $(A, L, T)$ where $A \in \mathbb{C}\mathbb{P}^2$ and $L, T$ are lines in $T_A \mathbb{C}\mathbb{P}^2$. Consider the space $\alpha \times \mathcal{P}^2$ of triples $z = (m_A, m_B, m_C)$ where $m_A = (A, L_A) \in \alpha$, $m_B = (B, L_B, T_B) \in \mathcal{P}$ and $m_C = (C, L_C, T_C) \in \mathcal{P}$ (these notations will be used all along the paper). We define also a certain number of projections:

- $\pi_B, \pi_C : \alpha \times \mathcal{P}^2 \to \mathbb{C}\mathbb{P}^2$ such that $\pi_B(z) = B$ and $\pi_C(z) = C$;
- $\pi_\alpha : \alpha \times \mathcal{P}^2 \to \alpha$ the projection onto $\alpha$;
- $\pi_{\bar{B}}, \pi_C : \alpha \times \mathcal{P}^2 \to \mathbb{P}(T\overline{\mathbb{C}\mathbb{P}^2})$ such that $\pi_{\bar{B}}(z) = (B, L_B)$ and $\pi_C(z) = (C, L_C)$.

**Phase space.** In the space $\alpha \times \mathcal{P}^2$, we consider the subspace $M_0^\alpha$ of triangular billiard orbits having one reflection in $\alpha$, that is the set of triples

$$z = [(A, L_A), (B, L_B, T_B), (C, L_C, T_C)]$$

where $(A, L_A) \in \mathbb{C}\mathbb{P}^2$, $(B, L_B, T_B) \in \mathcal{P}$ and $(C, L_C, T_C) \in \mathcal{P}$ with further properties that $A, B, C$ do not lie on the same line, $L_B \neq T_B$, $L_C \neq T_C$, the lines $AB$ and $AC$ are symmetric with respect to $(L_A, T_A)$, the lines $AB$ and $BC$ are symmetric with respect to $(L_B, T_B)$, and the lines $AC$ and $CB$ are symmetric with respect to $(L_C, T_C)$. Take $M_\alpha$ to be the analytic closure of $M_0^\alpha$. The set $M_\alpha$ is a smooth analytic subset of $\alpha \times \mathcal{P}^2$ of dimension 6.

**Birkhoff’s distribution attached to $\alpha$.** We consider the distribution $D$ on $\alpha \times \mathcal{P}^2$ defined for all $z$ by

$$D(z) = d\pi_B^{-1}(T_B) \cap d\pi_C^{-1}(T_C).$$
Definition 4.6. We call Birkhoff’s distribution attached to $\alpha$ the restriction of $D$ to the phase space $M_\alpha$, and still denote it by $D$.

Proposition 4.7. Let $z_0 \in M_\alpha^0$ such that one can find a germ of 2-dimensional integral analytic surface $S$ of $D$ containing $z_0$. Suppose that $\pi_G$ has rank 1 on $S$ for $G \in \{\alpha, \beta, \gamma, B, C\}$. Then there exists a 3-reflective complex-analytic local projective billiard defined by $(\alpha, \pi_\beta(U), \pi_\gamma(U))$, with $U$ a sufficiently small neighborhood of $z_0$ in $S$.

Remark 4.8. The link between billiards and Birkhoff’s distribution is not new and was introduced in [2]. See [2] Proposition 4.1 for a similar result in classical billiards.

Proof. By the constant rank theorem, there is a neighborhood $U$ of $z_0$ in $S$ such that $\hat{b} := \pi_B(U)$ and $\hat{c} := \pi_C(U)$ are immersed curves of $\mathbb{C}P^2$, and such that $\pi_\beta(U)$ and $\pi_\gamma(U)$ are immersed curves of $\mathbb{P}(\mathbb{C}P^2)$. It follows from the assumptions that the restrictions $\pi : \pi_\beta(U) \to \hat{b}$ and $\pi : \pi_\gamma(U) \to \hat{c}$ are biholomorphisms. Therefore, the inverse maps of these restrictions, denoted by $\beta$ and $\gamma$ respectively, are line-framed curves.

Now since $S$ is an integral surface of $D$, for $z = (A, L_A, B, L_B, T_B, C, L_C, T_C) \in U$ we have $T_B \hat{b} = d\pi_B(T_zS) \subset T_B$ and $T_C \hat{c} = d\pi_C(T_zS) \subset T_C$. Yet these spaces have the same dimension 1, hence $T_B \hat{b} = T_B$ and $T_C \hat{c} = T_C$. Therefore any $z \in U$ corresponds to a periodic orbit $i(z) = ((A, L_A), (B, L_B), (C, L_C))$ of $(\alpha, \beta, \gamma)$.

The map $\pi_{\alpha, B, C} : U \to \alpha \times \hat{b} \times \hat{c}$ verifying $\pi_{\alpha, B, C}(z) = ((A, L_A), B, C)$ is an immersion, since there is a map $s : \alpha \times \hat{b} \times \hat{c} \to \alpha \times \mathbb{P}^2$ verifying $s \circ \pi_{\alpha, B, C}(z) = z$ for any $z \in U$. Hence $\pi_{\alpha, B, C}^\prime(U)$ is an immersed surface of $\alpha \times \hat{b} \times \hat{c}$ and projects into a non-empty open subset of the line at $C$ and at $A$. Therefore any $z \in U$ corresponds to a periodic orbit $i(z) = ((A, L_A), (B, L_B), (C, L_C))$ of $(\alpha, \beta, \gamma)$.

Lemma 4.9. Let $(\alpha_1, \alpha_2, \alpha_3)$ be a local projective billiard; let $\alpha_j$ denote their classical boundaries. Let $(A_1, L_1) \in \alpha_1$, $(A_2, L_2) \in \alpha_2$, $(A_3, L_3) \in \alpha_3$ such that $A_1$, $A_2$, $A_3$ are pairwise distinct, the line $A_1A_2$ is transverse to $A_1$, the line $A_2A_3$ is transverse to $A_3$ at $A_3$ and the reflection law holds at $A_3$. Then one can define a smooth map $T$ in a neighborhood $W \subset \alpha_1 \times \alpha_2$ of $((A_1, L_1), (A_2, L_2))$, $T : W \subset \alpha_1 \times \alpha_2 \to \alpha_3 \times \alpha_3$, such that $T$ is of rank 2 and such that $T((A_1', L_1'), (A_2', L_2')) = ((A_2', L_2'), (A_3', L_3'))$ where $A_3'$ is defined by the condition that the lines $A_1'A_2'$ and $A_2'A_3'$ are symmetric with respect to the pair $(L_2', T_{A_3'}a_2)$.

Proof. $T$ is well defined and smooth in a neighborhood of $((A_1, L_1), (A_2, L_2))$ by the implicit functions theorem and by the transversality conditions. $T$ is of rank 2 if and only if when $A_2' = A_2$ is fixed, the map $(A_1', L_1') \mapsto (A_3', L_3')$ is of rank 1. And this is true by a computation using Formula [3] which we omit (it uses the transversality condition at $A_1$).

4.3 Reduction of the space of orbits

Let $B = (\alpha, \beta, \gamma)$ be a 3-reflective complex-analytic local projective billiard, and suppose that $\alpha = \pi \circ \alpha$ is not supported by a line. We are interested in the problem of finding 2-dimensional integral surfaces of $D$ in $M_\alpha$ (see Subsection 4.22 for precise definitions). By hypothesis we already have one such surface $S$ given by the triangular orbits of the 3-reflective projective billiard $B$. Consider

$$M = \overline{S^{an}}$$

the analytic closure of $S$ in $M_\alpha$. In this subsection we want to prove that $\dim M \leq 4$.

Construction of two analytic subsets containing $M$. Consider the open subset $\Omega_1$ of $\alpha \times \mathbb{P}$ defined by those

$$(m_A, m_B), m_A = (A, L_A), m_B = (B, L_B, T_B)$$

for which $A$ is a regular point of $a$ with $L_A \not= T_Aa$, $B \not= L_A \cup T_Aa$, and $A \not= L_B \cup T_B$ (in particular $A \not= B$). Then for each $(m_A, m_B) \in \Omega_1$, set $AB^{m_B}$ the line obtained by the symmetry of the line $AB$ at $A$ with respect to $(L_A, T_Aa)$, and $AB^{m_B}$ the line obtained by the symmetry of the line $AB$ at $A$ with respect to

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(L_B, T_B). By construction of Ω_1, the lines AB^{m_A} and AB^{m_B} are distinct and intersect at a point C. This defines a map

\[ \hat{\gamma} : \Omega_1 \to \mathbb{CP}^2 \]

which is analytic by the implicit function theorem. Then, for each (m_A, m_B) ∈ Ω_1, the map \( \hat{\gamma}(\cdot, m_B) : \alpha \to \mathbb{CP}^2 \) is analytic and non constant (unless if at least α is a line through B). Define \( \Gamma(m_A, m_B) \) to be the tangent line to the germ in \( m_A \) of the analytic curve \( \hat{\gamma}(\cdot, m_B) \). The analytic map

\[ \Gamma : \Omega_1 \to \mathbb{P}(T\mathbb{CP}^2) \]

has the fortunate property that for all \( z = [m_A, m_B, (C, L_C, T_C)] \in S \) with \( (m_A, m_B) \in \Omega_1 \), we have \( T_C = \Gamma(m_A, m_B) \) by the 3-reflective property of the local projective billiard corresponding to S. Now, \( \Gamma \) extends analytically to \( \Gamma' : \Omega_1 \to \mathcal{P} \) by setting \( \Gamma'(m_A, m_B) = (\hat{\gamma}(m_A, m_B), L_C, \Gamma(m_A, m_B)) \) where \( L_C \) is chosen so that \( AB^{m_A} \) and \( AB^{m_B} \) are symmetric with respect to \((L_C, \Gamma(m_A, m_B))\) (see Equation (3)). Thus \( S \) is in the analytic set

\[ M_{a,\gamma} = \{(m_A, m_B, m_C) \in \Omega_1 \times \mathcal{P} \mid \Gamma'(m_A, m_B) = m_C\} \]

defined as the closure of the graph of \( \Gamma' \).

Now we can do the same constructions with \( m_C = (C, L_C, T_C) \) instead of \( m_B \), and define analogously \( \Omega_2 \subset \alpha \times \mathcal{P}, \beta : \Omega_2 \to \mathbb{CP}^2, B : \Omega_2 \to \mathbb{P}(T\mathbb{CP}^2), B' : \Omega_2 \to \mathcal{P} \) and \( M_{a,\beta} \) as the closure of the graph of \( B' : \Omega_2 \to \mathcal{P} \). We have obviously:

**Proposition 4.10.** \( M \subset M_a \cap M_{a,\beta} \cap M_{a,\gamma} \).

From this, we deduce:

**Proposition 4.11.** The natural projection

\[ \pi : M \to F = \{(m_A, B, C) \mid m_A \in \alpha, B, C \in \mathbb{CP}^2, C \in AB^{m_A}\} \]

where \( AB^{m_A} \) is the line symmetric to \( AB \) with respect to \((L_A, T_A)\), has generically finite fibers. In more details, the image \( \pi(M) \) is an analytic subset in \( F \) and there exists a dense subset \( U \subset \pi(M) \) (a complement to a proper analytic subset) such that \( \pi^{-1}(y) \) is finite for every \( y \in U \). Hence \( \dim M \leq 4 \).

**Proof.** Notice first that \( F \) is an analytic subset of \( \alpha \times (\mathbb{CP}^2)^2 \) of dimension 4 and \( \pi(M) \) is an analytic subset of \( F \). Consider the set \( U \subset F \) of \( (m_A, B, C) \in F \) for which \( A, B, C \) do not lie on the same line, \( A \in \alpha \) is regular, \( L_A \neq T_A(a) \), \( B \) and \( C \) do not lie in \( L_A \cup T_A(a) \) by analyticity of these conditions, \( U \) is an open set which is the complement of a proper analytic subset in \( F \).

Take \( (m_A, B, C) \in U \). The set \( \pi^{-1}(m_A, B, C) \) is an analytic set of \( \{m_A\} \times \mathbb{P}(T_B \mathbb{CP}^2)^2 \times \mathbb{P}(T_C \mathbb{CP}^2)^2 \) (which can be identified with \( (\mathbb{CP}^1)^4 \)) hence is algebraic by Chow’s theorem. Suppose \( \pi^{-1}(m_A, B, C) \) isn’t finite. Then at least one of the projections of \( \pi^{-1}(m_A, B, C) \) onto either \( L_B, T_B, L_C \) or \( T_C \) is infinite. We suppose that it is the projection onto \( T_B, \pi_{T_B} : \pi^{-1}(m_A, B, C) \to \mathbb{CP}^1 \) (the cases of the other projections are treated similarly). By Remmert proper mapping and Chow’s theorems, its image is \( \mathbb{CP}^1 \). Now take

\[ z_1 = (m_A, B, L_B, T_B, C, L_C, T_C) \in \pi^{-1}(m_A, B, C) \]

such that \( T_B \neq AB \) or \( BC \). Since \( \pi_{T_B} = \mathbb{CP}^1 \), one can also find

\[ z_2 = (m_A, B, L'_B, L_B, C, L'_C, T'_C) \in \pi^{-1}(m_A, B, C), \]

that is such that \( T_B \) has been replaced by the previous \( L_B \). Now \( L'_B \) is such that the lines \( AB \) and \( BC \) are symmetric with respect to \( L'_B \) and they are already symmetric with respect to \( (L_B, T_B) \), hence \( L'_B = T_B \) (by Equation (3)).

By Proposition 1.10 \((C, L_C, T_C) = \Gamma(m_A, B, L_B, T_B) \) and \( (C, L'_C, T'_C) = \Gamma'(m_A, B, T_B, L_B) \). But since the maps \( \hat{\gamma}(\cdot, (B, L_B, T_B)) \) and \( \hat{\gamma}(\cdot, (B, T_B, L_B)) \) are the same (because of the symmetry of lines through \( B \) with respect to \( (L_B, T_B) \)) is the same as the symmetry of lines through \( B \) with respect to \( (T_B, L_B) \), hence \( (C, L_C, T_C) = (C, L'_C, T'_C) \).

The same argument works with \( (C, L_C, T_C) \): one should have \( (B, L_B, T_B) = B'(C, L_C, T_C) \) and \( (B, T_B, L_B) = B'(C, L'_C, T'_C) = B'(C, L_C, T_C) \). It follows that \( (B, L_B, T_B) = (B, T_B, L_B) \) hence that \( L_B = T_B \). But this contradicts the harmonicity of \( \{AB, BC; T_B, L_B\} \) since the lines \( AB, BC, T_B \) are pairwise distinct.

Therefore \( \pi^{-1}(m_A, B, C) \) is finite as soon as \( (m_A, B, C) \in U \), which concludes the proof.
Now consider a point \( m_A \in \alpha \), and denote by \( W_{m_A} \) the set of \( z = (m_A, *, *) \in M \). It is an algebraic set by Chow’s theorem.

**Lemma 4.12.** If \( \dim M \geq 3 \), for a generic \( m_A \in \alpha \) we have either \( \pi_B(W_{m_A}) = \mathbb{CP}^2 \) or \( \pi_C(W_{m_A}) = \mathbb{CP}^2 \).

**Proof.** By Proposition 4.11 for a generic \( m_A \in \alpha \), \( \dim W_{m_A} \geq 2 \) and the map \( \pi \) of Proposition 4.11 restricts to a map with generically finite fibers on \( W_{m_A} \)

\[
\pi|_{W_{m_A}} : W_{m_A} \to F_{m_A} := \{(m_A, B, C) | B \in \mathbb{CP}^2, C \in AB^{m_A}\}.
\]

Fix such a \( m_A \) with further condition that \( A \notin b \) and \( A \) is a regular point of \( A \). Now \( \pi_B(W_{m_A}) \) contains \( b \), thus is of dimension at least one. Suppose it is of dimension 1. The map \( \pi|_{W_{m_A}} \) has its image in the algebraic set

\[
G_{m_A} := \{(m_A, B, C) | B \in \pi_B(W_{m_A}), C \in AB^{m_A}\} \subset F_{m_A}
\]

of dimension 2, thus is epimorphic. Hence \( \pi_C(W_{m_A}) = \pi_C(G_{m_A}) \) contains all lines \( AB^{m_A} \) with \( B \in \pi_B(W_{m_A}) \). Since \( A \notin b \), there is an uncountable number of distinct such lines (when \( B \) varies on \( b \) for example). Hence \( \pi_C(W_{m_A}) = \mathbb{CP}^2 \).

We have proven that for a generic \( m_A \in \alpha \), either \( \pi_B(W_{m_A}) = \mathbb{CP}^2 \) or \( \pi_C(W_{m_A}) = \mathbb{CP}^2 \).

**4.4 Integrability of Birkhoff’s distribution on \( M \)**

Consider the restriction of Birkhoff’s distribution \( D \) to \( M \), which is a singular analytic distribution on \( M \), and denote it by \( D_M \). Let us compute its dimension.

**Proposition 4.13.** The singular analytic distribution \( D_M \) is 2-dimensional.

**Proof.** We obviously have \( \dim D_M \geq 2 \) since \( T_zS \subset D_M(z) \) for \( z \in S \), \( S \) being two dimensional. By Proposition 4.11 \( 2 \leq \dim M \leq 4 \) and so is \( D_M \). Consider two cases: \( \dim M = 3 \) and \( \dim M = 4 \) (the case when \( \dim M = 2 \) being obvious). In both cases, take a regular \( z = (m_A, B, L_B, T_B, C, L_C, T_C) \in M \) such that \( \dim D_M(z) \) is minimal, the points \( A, B, C \) are not on the same line, \( A \) being a regular point of \( a \), \( L_A \neq T_Aa, \) and \( B, C \) do not lie in \( L_A \cap T_Aa \).

**Case when** \( \dim M = 3 \). We just have to find one \( U \in T_zM \) which is not in \( D_M(z) \). By Lemma 4.12 we can suppose that \( m_A \) is such that \( \pi_B(W_{m_A}) = \mathbb{CP}^2 \). Consider then a path \( u : ]-\epsilon, \epsilon[ \to M \) with \( \epsilon > 0 \), such that \( u(0) = z \) and \( \pi_B \circ u(t) \) is a path along the line \( AB \) with non-zero derivative at 0. Consider the vector \( U = u'(0) \in T_zM \). It has the property that \( d\pi_B(U) \) is a vector corresponding to the derivative of \( \pi_B \circ u(t) \) in 0 which is non zero and is directed along the line \( AB \). Hence \( d\pi_B(U) \notin T_B \), otherwise \( T_B = AB \) and \( C \) could not lie outside \( AB \) by the reflection law. We conclude that \( U \notin D_M(z) \).

**Case when** \( \dim M = 4 \). Let us find two independent \( U, V \in T_zM \) such that \( D_M(z) \) and the 2-plane spanned by \( (U, V) \) have 0-dimensional intersection. We can suppose that \( m_A \) is such that \( \dim W_{m_A} = 3 \) (generic condition), and hence by Proposition 4.11 that the projection

\[
\pi|_{W_{m_A}} : W_{m_A} \to F_{m_A} := \{(m_A, B, C) | B \in \mathbb{CP}^2, C \in AB^{m_A}\}
\]

is epimorphic (Remmert’s proper mapping theorem). Hence we can define \( U \in T_zM \) from a path \( u : ]-\epsilon, \epsilon[ \to M \) with \( \epsilon > 0 \), such that \( u(0) = z \), \( \pi_B \circ u(t) \) is a path along the line \( AB \) with nonzero derivative in 0, and \( \pi_C \circ u(t) \equiv C \) is constant. We change the roles of \( B \) and \( C \) and do the same construction to get a certain \( V \in T_zM \). We then check that

- \( U \) and \( V \) are independent since \( d\pi_B(U) \neq 0 \) and \( d\pi_C(V) \neq 0 \) while \( d\pi_C(U) = 0 \) and \( d\pi_B(V) = 0 \).
- If \( pU + qV \in D_M(z) \) for \( p, q \in \mathbb{C} \), then \( d\pi_B(pU + qV) \in T_B \) by definition of \( D_M \). Yet \( d\pi_B(pU + qV) = pdt_B(U) \). Thus \( p = 0 \) since otherwise \( AB = T_B \) and we get a contradiction with the fact that \( A, B, C \) do not lie on the same line. Similarly, we find that \( q = 0 \) by considering \( d\pi_C(pU + qV) = qdt_C(V) \).

This concludes the proof.

By Lemma 4.15 we have the

**Corollary 4.14.** The singular analytic distribution \( D_M \) is integrable.
5 Proof of Theorem 1.13

Let $B = (\alpha, \beta, \gamma)$ be a 3-reflective complex-analytic local projective billiard, whose classical boundaries are denoted by $a$, $b$, $c$. We say that a classical border $g$ is supported by a line if $\text{im} \ g$ is contained in a line of $\mathbb{CP}^2$. In this section we prove the

**Proposition 5.1.** The classical borders $a$, $b$, $c$ are supported by lines.

Proposition 5.1 combined with Proposition 3.3 will conclude the proof of Theorem 1.13. To prove Proposition 5.1 we suppose that one of the classical borders, say $a$, is not supported by a line, and show that a contradiction arises: the first important result is Proposition 5.7 giving the existence of a particular 3-reflective local projective billiard having $\alpha$ in its boundary. The contradiction comes from asymptotic comparisons of complex angles (or azimuths defined in Section 2) proved in subsection 5.2. The following remark will be useful:

**Remark 5.2.** Let $H : U \to \mathbb{P}(T \mathbb{CP}^2)$ be an analytic map on a connected riemann surface $U$ such that $h := \pi \circ H$ is non constant. Then $dh(x)$ is of rank 1 for all $x$ lying outside a discrete subset of $U$. A point $p \in \text{im} \ h$ is said to be regular if there exists $z \in h^{-1}(p)$ for which $dh(z) \neq 0$. By shrinking $U$ around $z$ if needed, one can suppose in this case that $h$ and $H$ are diffeomorphisms on their respective images, and therefore $H(U)$ is a complex-analytic line-framed curve and $h(U)$ is its classical boundary.

In this section, we will use the following classical statement, concerning duality of analytic curves (see [10]):

**Proposition 5.3.** Suppose $a$ is not supported by a line, and let $P \in \mathbb{CP}^2$. Then $P \in T_A a$ for at most a countable number of regular $A \in a$.

### 5.1 Existence of a particular 3-reflective local projective billiard

The main result of this subsection is Proposition 5.7, which shows (under the assumption that $a$ is not supported by a line) the existence of a particular 3-reflective local projective billiard having $\alpha$ in its boundary. We will prove then in next subsection that the existence of such billiard is impossible.

Given two analytic curves $h_1 : U_1 \to \mathbb{CP}^2$ and $h_2 : U_2 \to \mathbb{CP}^2$ defined on Riemann surfaces $U_1$, $U_2$ and two points $p_1 \in \text{im} \ h_1$, $p_2 \in \text{im} \ h_2$, we say that the germs $(h_1, p_1)$ and $(h_2, p_2)$ coincide if $p_1 = p_2$ and there is an open subset $V$ of $\mathbb{CP}^2$ containing the $p_i$ and for which $\text{im} \ h_1 \cap V = \text{im} \ h_2 \cap V$.

**Proposition 5.4.** Let $B = (\alpha, \beta, \gamma)$ be a 3-reflective complex-analytic local projective billiard such that $a$ is not supported by a line. Then there is a 3-reflective complex-analytic local projective billiard $B' = (\alpha', \beta', \gamma')$, $m_{A_0} = (A_0, L_{A_0}) \in \alpha$ and $m_{B_0} = (B_0, L_{B_0}) \in \beta'$, such that $a$ is regular at $A_0$, $b' := \pi(\beta')$ is regular at $B_0$, with $L_{A_0} \neq T_{A_0} a$, $L_{B_0} \neq T_{B_0} b'$, and at least one of the following cases holds:

1. $A_0 = B_0$ and the germs $(a, A_0)$ and $(b', B_0)$ coincide;
2. the points $A_0$ and $B_0$ are distinct, and $T_{A_0} a$ intersects $b'$ transversally at $B_0$.

Furthermore, if $F \subset \alpha$ is a discrete subset, we can choose $m_{A_0} \notin F$.

**Proof.** Consider the open subset of $M$ defined by

$$M^o = \{ z \in M_{\text{reg}} \cap M^0_{\alpha} \mid d\pi_G(z) \text{ has rank 1 on } D_M(z), G = \alpha, \beta, \gamma, B, C \}$$

where $\pi_\beta, \pi_\gamma : \alpha \times \mathbb{P}^2 \to \mathbb{P}$ are the projections onto respectively the second and the third factor (in $\mathbb{P}$). By definition, $M^o$ contains $S$. For $m_A \in \alpha$, the set $W_{m_A}^o := M^o \cap W_{m_A}$ is an open subset of $W_{m_A}$ such that $W_{m_A} \setminus W_{m_A}^o$ is an analytic subset of $W_{m_A}$, hence is algebraic by Chow’s theorem. Thus $W_{m_A}^o$ is a Zariski open subset of $W_{m_A}$. By Chevalley’s theorem, $\pi_B(W_{m_A}^o)$ is a constructible subset, hence is a dense Zariski-open subset of $\pi_B(W_{m_A})$, which is itself either $\mathbb{CP}^2$ or an algebraic curve.

**Lemma 5.5.** For $m_A \in \alpha$, we can choose $m_A' \in \alpha$ arbitrarily close to $m_A$, such that $T_{A_0} a \cap \pi_B(W_{m_A}^o)$ is nonempty.
Proof. Fix $m_A \in \alpha$. By previous discussion, $Z_{m_A} := \pi_B(W_{m_A}) \setminus \pi_B(W_{m_A}^o)$ is a strict algebraic subset of $\mathbb{CP}^2$.

If $Z_{m_A}$ has dimension 0, it is finite. Hence, since $a$ is not a line, the set $S$ of $m'_A \in \alpha$ such that $T_{A'}a$ does not intersect $Z_{m_A}$ is uncountable by Proposition 5.3. By Bezout’s theorem, for any $m_{A'}$ in $S$, $T_{A'}a$ intersects the algebraic subset $\pi_B(W_{m_A})$ of $\mathbb{CP}^2$ and the result is proved in this case.

If $Z_{m_A}$ has dimension 1, $\pi_B(W_{m_A})$ is of dimension 2, hence it equals $\mathbb{CP}^2$. Since $a$ is not a line, for any neighborhood $V$ of $m_A$ in $\alpha$, $Z_{m_A}$ does not contain all of the $T_{A'}a$ for $m_{A'} \in V$ with $A'$ a regular point of $a$. Hence we can choose a $m_{A'} \in \alpha$ arbitrarily close to $m_A$ such that $T_{A'}a \not\subset Z_{m_A}$. Thus $T_{A'}a \cap \pi_B(W_{m_A}^o)$ is nonempty.

Fix an $m_A = (A, L_A) \in \alpha$ such that $a$ is regular at $A$ and $L_A \neq T_{A_{\alpha}}a$. By Lemma 5.4 we can choose $m_{A_0} = (A_0, L_{A_0}) \in \alpha$ arbitrarily close to $m_A$, such that $T_{A_0}a \cap \pi_B(W_{m_{A_0}}^o)$ is a nonempty Zariski open subset of $T_{A_{\alpha}}a$. Thus we can also impose that $A_0$ is a regular point of $a$ with $T_{A_0}a \neq L_{A_0}$.

This means that there is a $z \in W_{m_A} \cap M^o$ such that $\pi_B(z) \in T_{A_0}a$. By Corollary 5.4 there is a 2-dimensional integral surface $S_z$ through $z$. Now we can apply the same construction as in the proof of Proposition 1.1 shrinking $S_z$ if necessary, $b' := \pi_B(S_z)$ and $c' := \pi_C(S_z)$ are immersed curves of $\mathbb{CP}^2$, $\pi_\beta(S_z)$ and $\pi_\gamma(S_z)$ are immersed curves of $\mathbb{CP}(T\mathbb{CP}^2)$, on which the restrictions $\pi_B : \pi_\beta(S_z) \to b'$ and $\pi_C : \pi_\gamma(U S_z) \to c'$ are biholomorphisms. Therefore, the inverse maps of these restrictions, denoted by $\beta'$ and $\gamma'$ respectively, are line-framed curves. Their classical borders, $b' = \pi_\beta(b')$ and $c' = \pi_\gamma(c')$, are curves in $\mathbb{CP}^2$ with $b'$ intersecting $T_{A_0}a$ (maybe tangentially) at a point $\neq A_0$. Furthermore, by definition of $M^o_{\alpha_{\ast}}$, generically on $\beta'$ we have $L_{B'} \neq T_{B'}b'$, and the same is true on $\gamma'$.

Case when the germs $(a, A_0)$ and $(b', \pi_B(z))$ coincide. By eventually moving $m_{A_0}$ a little, there is a point $m_{B_0} = (B_0, L_{B_0})$ of $\beta'$ close to $\pi_\beta(z)$, and such that $B_0 \in T_{A_0}a$ and $b'$ is regular at $B_0$ with $L_{B_0} \neq T_{B_0}b'$.

Case when the germs $(a, A_0)$ and $(b', \pi_B(z))$ do not coincide. We can move $m_{A_0}$ a little, so that there is a point $m_{B_0} = (B_0, L_{B_0})$ of $\beta'$ close to $\pi_\beta(z)$ for which $B_0 \neq A_0$, $T_{A_0}a$ is transverse to $b'$ at $B_0$, and $L_{B_0} \neq T_{B_0}b'$.

Finally, if needed we can move again $m_{A_0}$ a little so that $m_{A_0} \notin F$ and have the same situation.

We will investigate from now on what happens when $m_B$ is $\beta'$ close to $m_{B_0}$. We first can say that, under further conditions, the vertices will converge to a line. Thus we need the

**Definition 5.6.** Let $B_0 = (\alpha_0, \beta_0, \gamma_0)$ be any local projective billiard. We say that $B_0$ owns a flat orbit if there are $m_A \in \alpha_0$, $m_B = (B, L_B) \in \beta_0$, $m_C = (C, L_C) \in \gamma_0$ and a line $L$ such that $A, B, C$ lie on $L$ and there exists a sequence of usual triangular orbits $(m_A, m_B^0, m_C^0)$ of $B_0$ converging to $(m_A, m_B, m_C)$. The triple $(m_A, m_B, m_C)$ is called a flat orbit of $B_0$ on $L$. 

Figure 7: The local projective billiard in the second case of Proposition 5.4: $T_{A_0}a$ intersects $b'$ transversally at $B_0$. 

Proof. Fix $m_A \in \alpha$. By previous discussion, $Z_{m_A} := \pi_B(W_{m_A}) \setminus \pi_B(W_{m_A}^o)$ is a strict algebraic subset of $\mathbb{CP}^2$.
Proposition 5.7. Let \( \mathcal{B} = (\alpha, \beta, \gamma) \) be a 3-reflective complex-analytic local projective billiard such that \( a \) is not supported by a line. Then there is a 3-reflective complex-analytic local projective billiard \( \mathcal{B}_0 = (\alpha_0, \beta_0, \gamma_0) \) such that \( \mathcal{B}_0 \) owns a flat orbit.

The corresponding flat orbit \((m_{A_0}, m_{B_0}, m_{C_0})\) has the property that the points \( A_0 = \pi(m_{A_0}), B_0 = \pi(m_{B_0}), C_0 = \pi(m_{C_0}) \) are regular points of \( a, b_0 = \pi(\beta_0) \) and \( c_0 = \pi(\gamma_0) \) respectively, lying on \( T_{A_0a} \), with \( L_{A_0} \neq T_{A_0a}, L_{B_0} \neq T_{B_0b_0} \) and \( L_{C_0} \neq T_{C_0c_0} \).

Furthermore, if two points among \( \{A_0, B_0, C_0\} \) coincide, the corresponding classical borders coincide. And \( T_{A_0a} \) intersect \( b_0 \) or \( c_0 \) transversally if \( A_0 \neq B_0 \) or \( A_0 \neq C_0 \) respectively.

![Figure 8: The local projective billiard of Proposition 5.7](image)

Figure 8: The local projective billiard of Proposition 5.7 with a flat orbit \( (m_{A_0}, m_{B_0}, m_{C_0}) \) on \( T_{A_0a} \). Here the three points \( A_0, B_0, C_0 \) are pairwise distinct.

Proof. Let \( \mathcal{B}' = (\alpha', \beta', \gamma') \) be the local projective billiard of Proposition 5.4 such that if \( b_0 \) is the germ of a line \( \ell, A_0 \notin \ell \) (we can do it by choosing \( F \) in a convenient way). Choose any representatives \( \alpha \) of \( \alpha \) containing \( m_{A_0} \) and \( \beta_0 \) of \( \beta' \) containing \( m_{B_0} \).

First let us construct \( \gamma_0 \) in the following way: for any \( m_B = (B, L_B) \in \beta_0 \), denote by \( A_0B^{*m_A} \) the line reflected from \( A_0B \) at \( m_{A_0} \) with respect to \( (L_{A_0}, T_{A_0a}) \), and by \( A_0B^{*m_B} \) the line reflected from \( A_0B \) at \( m_B \) with respect to \( (L_B, T_{Bb_0}) \). Consider the map \( \tilde{\gamma} : \beta_0 \setminus K \to \mathbb{C}P^2 \), which associates to any \( m_B = (B, L_B) \in \beta_0 \) the point \( C \) of intersection of the lines \( A_0B^{*m_A} \) and \( A_0B^{*m_B} \). \( \tilde{\gamma} \) is defined and analytic on \( \beta_0 \setminus K \), where \( K \) is a discrete subset of \( \beta_0 \) corresponding to the points where the lines \( A_0B^{*m_A} \) and \( A_0B^{*m_B} \) are the same or are not well-defined. Note that \( \tilde{\gamma} \) can be extended analytically to \( \beta_0 \) by solving the corresponding equations of lines defining \( C \), we find that, around \( m_{B_0} \), the coordinates of \( \tilde{\gamma} \) are rational fractions in the coordinates of \( \beta_0 \) containing \( m_{B_0} \).

Now \( \tilde{\gamma} \) is non-constant since the line \( A_0B \) is not constant (otherwise \( b_0 \) would be the germ of a line through \( A_0 \)). Hence locally around each \( m_B \in \beta_0 \), \( \tilde{\gamma} \) parametrises an analytic curve inside an open subset of \( \mathbb{C}P^2 \) and we can define \( T(m_B) \) to be its tangent line at \( \tilde{\gamma}(m_B) \). Hence we can construct an analytic map \( \Gamma : \beta_0 \setminus K \to \mathbb{P}(TCP^2) \) by setting \( \Gamma(m_B) = (\tilde{\gamma}(m_B), L_C(m_B)) \) where \( L_C(m_B) \) is the unique line through \( C \) such that the lines \( A_0B^{*m_A}, A_0B^{*m_B}, T(m_B) \) and \( L_C(m_B) \) are harmonic (see Equation (3)). We set \( \gamma_0 = \Gamma \) which ends this first step.

Let us show that \( (\alpha, \beta_0, \gamma_0) \) is a 3-reflective local projective billiard. Indeed, \( \mathcal{B}' = (\alpha', \beta', \gamma') \) is a 3-reflective billiard, hence there are open subsets \( U \subset \alpha \) and \( V \subset \beta' \) verifying the following condition: for any \( (m_A, m_B) \in U \times V \) there is an \( m_C \in \gamma' \) such that \( (m_A, m_B, m_C) \) is a 3-periodic projective billiard orbit. As before, we can construct an analytic map \( \tilde{\gamma}' : \alpha \times \beta_0 \setminus K' \to \mathbb{C}P^2 \), where \( K' \) is a proper analytic subset of \( \alpha \times \beta_0 \), by setting \( \tilde{\gamma}'(m_A, m_B) \) to be the intersection point of \( AB^{*mA} \) and \( AB^{*mB} \) \( (K') \) corresponds to the set of \( (m_A, m_B) \) for which the lines \( AB^{*mA} \) and \( AB^{*mB} \) are the same or are not well-defined.

Now \( \tilde{\gamma}' \) is of rank one on \( U \times V \), hence on \( \alpha \times \beta_0 \setminus K' \), by connectedness of \( \alpha \times \beta_0 \setminus K' \) and by analyticity of the condition "being of rank at most one". As before, we can extend \( \tilde{\gamma}' \) into an analytic map \( \Gamma' : \alpha \times \beta_0 \setminus K' \to \mathbb{P}(TCP^2) \), where \( \Gamma'(m_A, m_B) = (\tilde{\gamma}(m_A, m_B), L_C(m_A, m_B)) \) and \( L_C(m_A, m_B) \) is defined by the same rule as
previously: it is the unique line through $C$ such that the lines $AB^{m_A}$, $AB^{m_B}$, $T(m_A, m_B)$ and $L_C(m_A, m_B)$ are harmonic, with $T(m_A, m_B)$ being the tangent line at $(m_A, m_B)$ of the germ of curve locally parametrized by $\gamma'$. Again $\Gamma'$ is of rank one on $U \times V$, hence on $\alpha \times \beta_0 \setminus K'$. In particular, for any $m_B \in \beta_0$ for which $(m_{A_0}, m_B) \not\in K'$, there is an open subset $U_0 \times V_0 \subset \alpha \times \beta_0$ containing $(m_{A_0}, m_B)$ such that $\Gamma'(U_0 \times V_0) \subset \text{im} \gamma_0$. Hence $(\alpha, \beta_0, \gamma_0)$ is a 3-reflective local projective billiard.

Finally, and by construction, when $m_B \to m_{B_0}$, $m_C := \Gamma(m_B)$ is such that $(m_{A_0}, m_B, m_C)$ is a triangular billiard orbit on $(\alpha, \beta_0, \gamma_0)$ accumulating on $T_{A_0}a$ (since $A_0B$ goes to $T_{A_0}a$ by definition, and thus also $A_0C = A_0B^{m_A}$). Write $m_{C_0} = \lim_{m_B \to m_{B_0}} m(m_B)$, $b_0 = \pi(\beta_0)$ and $c_0 = \pi(\gamma_0)$. If $A_0 = C_0$ and the germs $(c_0, C_0)$ and $(a, A_0)$ do not coincide, one can move $m_{A_0}$ a little such that $A_0 \not\in C_0$ and get the same conclusions with $A_0 \not\in C_0$. The same operation can be used to suppose that if $B_0 = C_0$, the germs $(c_0, C_0)$ and $(b_0, B_0)$ coincide. And again, by moving $m_{A_0}$ a little, one can suppose that the points $A_0 = \pi(m_{A_0})$, $B_0 = \pi(m_{B_0})$, $C_0 = \pi(m_{C_0})$ are regular points of $a$, $b_0 = \pi(\beta_0)$ and $c_0 = \pi(\gamma_0)$ respectively, with $L_{A_0} \neq T_{A_0}a$, $L_{B_0} \neq T_{B_0}b_0$ and $L_{C_0} \neq T_{C_0}c_0$.

5.2 The 3-reflective local projective billiard of Proposition [5.7] cannot exist

Let $B = (\alpha, \beta, \gamma)$ be a 3-reflective complex-analytic projective billiard such that $a$ is not supported by a line. Let $B_0 = (\alpha, \beta_0, \gamma_0)$ be the 3-reflective local projective billiard of Proposition [5.7] and $a, b_0, c_0$ be the classical borders of $\alpha, \beta_0, \gamma_0$ respectively. We want to show that $B_0$ cannot exist.

To prove that, define coordinates on $\mathbb{CP}^2$ such that the points $A_0, B_0, C_0$ are finite,
\[
\text{az}(T_{A_0}a) = 0 \quad \text{and} \quad \infty \not\in \{\text{az}(T_{B_0}b_0), \text{az}(T_{C_0}c_0), \text{az}(L_{A_0})\}.
\]

We will write until the end of this section
\[
z = \text{az}(A_0B), \quad z^* = \text{az}(BC), \quad z' = \text{az}(A_0C)
\]
for any orbit $(m_{A_0}, m_B, m_C)$ on $B_0$ (see Figure 9 and section 2 for further details on azimuths). To be more precise, we will show the:

Proposition 5.8. When $m_B \to m_{B_0}$, the following equivalences are satisfied:
\[
z' \sim (-z), \quad z^* \sim (2I_b - 1)z, \quad z^* \sim (2I_c - 1)z'
\]
where $I_b$ (respectively, $I_c$) is the intersection index of $b$ (respectively $c$) with the tangent line $T_{A_0}a$ at $B_0$ (respectively $C_0$).

From Proposition 5.8, we deduce that $2I_b - 1 = -(2I_b - 1)$ which is impossible since $2I_b - 1$ and $2I_c - 1$ are strictly positive integers. Hence $B_0$ cannot exist.

We will prove the three equivalences of Proposition 5.8 in what follows, separated in three propositions (Propositions 5.9, 5.10 and 5.12).

Proposition 5.9 ($z' \sim (-z)$). When $m_B = (B, L_B) \in \beta'$ goes to $m_{B_0}$, we have
\[
z' \sim (-z).
\]

Proof. Equation (5) of Proposition 5.4 implies that
\[
z' = \frac{(\ell + t)z - 2\ell t}{2z - (\ell + t)}
\]
where $t = \text{az}(T_{A_0}a)$, $\ell = \text{az}(L_{A_0})$. By choice of coordinates, when $m_B \to m_{B_0}$,
\[
z' = \frac{\ell z}{2z - \ell} \sim \frac{\ell}{-\ell} = -z.
\]
Proposition 5.10. Suppose \( a \) is not supported by a line. If \( A_0 = B_0 \) then when \( m_B = (B, L_B) \in \beta' \) goes to \( m_B_0 \), we have
\[
z^* \to (2I - 1)z
\]
where \( I \geq 2 \) is the index of intersection of \( a \) with the tangent line \( T_{A_0}a \) at \( A_0 \).

Proof. Suppose \( A_0 = B_0 \) : in this case \((b_0, B_0) = (a, A_0)\) by Proposition 5.7. Take an orbit of the form \((m_{A_0}, m_B, m_C)\) with \( m_B \) and \( m_C \) close to \( m_{B_0} \) and \( m_{C_0} \). Write \( t = az(T_Bb_0) \), \( \ell = az(L_B) \). Equation (3) of Proposition 2.7 implies that
\[
z^* = \frac{(\ell + t)z - 2\ell t}{z(2z - (\ell + t))}.
\]

Now, when \( m_B \to m_B_0 \), since \( a \) and \( b_0 \) are the same in a neighborhood of \( B_0 \), we can compute that \( t \sim I z \).
Thus
\[
z^* \sim \frac{1 - 2I}{-\ell z} = 2I - 1.
\]

Lemma 5.11. Suppose \( a \) is not supported by a line. If \( B_0 = C_0 \), or the germs \((a, A_0), (b_0, B_0), (c_0, C_0)\) coincide.

Proof. Suppose that the three germs do not coincide and that \( B_0 = C_0 \) : in this case \((c_0, C_0) = (b_0, B_0)\) but \( A_0 \neq B_0 \) with \( T_{A_0}a \) intersecting \( b_0 \) transversally at \( B_0 \) by Proposition 5.7. Take an orbit of the form \((m_{A_0}, m_B, m_C)\) with \( m_B \) and \( m_C \) close to \( m_{B_0} \) and \( m_{C_0} \). Then, write \( t = az(T_Bb_0) \) and \( \ell = az(L_B) \). Equation (3) of Proposition 2.7 implies that
\[
\ell = \frac{(z + z^*)t - 2z z^*}{2t - (z + z^*)}
\]

(\( L_B \) and \( T_Bb_0 \) are symmetric with respect to \((A_0B, BC)\)). Now, when \( m_B \to m_{B_0} \), we have
\[
z \to 0, \quad t \to t_0
\]
where \( t_0 := az(T_Bb_0) \notin \{0, \infty\} \) by transversality of \( b_0 \) with \( T_{A_0}a \) at \( B_0 \) and by choice of coordinates. But we also have
\[
z^* \to t_0
\]
because \( BC \to T_Bb_0 \) since \( B,C \) are distinct points (by definition of an orbit) of the same irreducible germ of curve \( b_0 = c_0 \) converging to the same point \( B_0 = C_0 \). Hence, when \( m_B \to m_{B_0} \),
\[
L \to \frac{t_0^2}{t_0} = t_0
\]
which means that $L_B = T_B b_0$. But this is not the case by Proposition 5.7, contradiction. 

\[ \text{Proposition 5.12. Suppose a is not supported by a line. Then when } m_B = (B, L_B) \in \beta \text{ goes to } m_B, \text{ we have} \]

\[ z^* \sim z \]

which allows to extend the formula of Proposition 5.10 by setting $I = 1$ in this case (transverse intersection).

\[ \text{Proof. First, let us prove the following lemma, which gives the form of the projective field of lines locally around } m_B:\]

\[ \text{Lemma 5.13. Suppose a is not supported by a line and } B_0 \neq A_0. \text{ Then for } m_B = (B, L_B) \in \beta_0 \text{ close to } m_B, \text{ there is a } m_A = (A, L_A) \in \alpha \text{ close to } m_A_0 \text{ for which } L_B \text{ is tangent to } a \text{ at } A. \]

\[ \text{Proof. Proposition 5.7 implies that } T_{A_0} a \text{ intersects } b_0 \text{ transversally at } B_0. \text{ By the implicit function theorem, there is an analytic map which associates to any } m_A = (A, L_A) \text{ close to } m_A_0 \text{ a point } m_B = (B, L_B) \text{ close to } m_B \text{ such that } B \text{ lies on } T_a a. \text{ Since } a \text{ is not a line, this map is not constant, hence is open and thus parametrizes the germ of } \beta_0 \text{ at } m_B. \text{ Then we can choose } m_A_1 \text{ in the neighborhood of } m_A_0, \text{ and denote } m_B_1 = (B_1, L_{B_1}) \text{ the corresponding point on } \beta_0 \text{ obtained via the previous described parametrization.} \]

\[ \text{We can suppose that } b_0 \text{ is regular at } B_1, \text{ that } T_{A_1} a \text{ is transverse to } b_0 \text{ at } B_1 \text{ and that } B_1 \notin c_0 \text{ (possible by Lemma 5.11). Let } m_B = (B, L_B) \in \beta_0 \text{ be a point converging to } m_B_1. \text{ The line } A_1 B \text{ converges to } T_{A_1} a \text{ and by the reflection law at } A_1 \text{ we get that the line } A_1 C \text{ also converges to } T_{A_1} a, \text{ hence the limit } C_1 \text{ of } C \text{ lies on } T_{A_1} a. \text{ We also have, by the same proof as for Proposition 5.11 that } C_1 \neq B_1. \text{ Hence } C_1 B_1 = T_{A_1} a = A_1 B_1: \text{ } T_{A_1} a \text{ is invariant by the reflection law in } B_1 \text{ with respect to } (L_{B_1}, T_{B_1} b_0). \text{ Since it is transverse to } b_0, \text{ we have } T_{A_1} a = L_{B_1}, \text{ and this concludes the proof for } m_B = (B, L_B) \in \beta_0 \text{ close to } m_B. \]

\[ \text{As in Lemma 5.13 when } m_B \text{ is close to } m_B, \text{ } L_B \text{ is tangent to } a \text{ at a point } A \text{ corresponding to a } m_A \text{ close to } m_A_0; \text{ when } m_B \text{ converges to } m_B, \text{ } m_A \text{ converges to } m_A_0. \text{ Write } t = az(T_B b_0), \ell = az(L_B). \text{ We have by Formula } 3 \text{ of Proposition 2.7} \]

\[ z^* = \frac{(\ell + t)z - 2t}{2z - (\ell + t)} \quad (4) \]

\[ \text{Now in this configuration, we easily compute using Lemma 5.13 that, when } m_B \rightarrow m_B, \]

\[ \ell \sim z. \]

But we have also $t \rightarrow t_0$ where $t_0 = az(T_B b_0) \notin \{0, \infty\}$ (by the transversality condition of the intersection with $T_{A_0} a$). Hence, Equation (4) implies, when $m_B \rightarrow m_B$, that

\[ \frac{z^*}{z} = \frac{(\ell + t)z - 2t\ell}{z(2z - (\ell + t))} \sim \frac{-t_0 z}{-t_0 z} = 1. \]

\[ \text{5.3 Conclusion} \]

Let $B = (\alpha, \beta, \gamma)$ be a 3-reflective complex-analytic local projective billiard, whose classical borders are $a, b, c$. Suppose a is not supported by a line. Let $B_0 = (\alpha, \beta_0, \gamma_0)$ be the 3-reflective local projective billiard of Proposition 5.7, and $a, b_0, c_0$ be the classical borders of $\alpha, \beta_0, \gamma_0$ respectively.

Now by Proposition 5.8 we deduce that $2I_a - 1 = -(2I_0 - 1)$ which is impossible since $2I_b - 1$ and $2I_0 - 1$ are strictly positive integers. Hence $B_0$ cannot exist, contradiction: a is supported by a line. By symmetry of previous argument, a, b, an c are supported by lines, which proves Proposition 5.4. Finally, by Proposition 5.4 $B$ is a right-spherical billiard as defined in Definition 5.1. Hence Theorem 1.1.3 is proved.
6 Proof of Theorem 1.14

In this section, we want to prove Theorem 1.14 using Theorem 1.13. Fix $d \geq 3$ to be the dimension in which the problem takes place. We will need this auxiliary lemma:

**Lemma 6.1.** Let $W \subset \mathbb{C}^d$ be an analytic hypersurface, $p \in W$ and $U$ a non-empty open subset of $\mathbb{P}(T_p W)$. Suppose that for any $\mathbb{P}(v) \in U$, $W$ contains the points $p + tv$ for $t$ sufficiently close to 0. Then $W$ is an hyperplane.

**Proof.** We can suppose that $p = 0$, $T_p W = z_d = 0$ and $W$ is locally the graph of an analytic map $f : V \to \mathbb{C}$ where $V \subset \mathbb{C}^{d-1}$ is an open subset containing 0. Let $v \in \mathbb{C}^{d-1}$ be a non-zero vector such that $\mathbb{P}(v) \in U$. By assumption, for $t$ close to 0 we have $g_v(t) := f(tv) = 0$. Since $g_v$ is analytic, it is 0 everywhere where it is defined. Yet the set $\{tv \mid t \in \mathbb{R}, \mathbb{P}(v) \in U\}$ contains a non-empty open subset of $V$, on which $f$ should vanish. By analyticity $f = 0$ and $W$ is the hyperplane defined by the equation $z_d = 0$.

**Proof of Theorem 1.14.** Suppose by contradiction that there is a 3-reflective complex-analytic local projective billiard $\mathcal{B} = (\alpha, \beta, \gamma)$ in $\mathbb{P}(TC\mathbb{P}^d)$. Let $a = \pi(\alpha)$, $b = \pi(\beta)$ and $c = \pi(\gamma)$. Let $U \times V \subset \alpha \times \beta$ be an open subset for which all $(m_A, m_B, m_C) \in U \times V$ can be completed in an 3-periodic orbit of $\mathcal{B}$. Let us state the following lemma, which isn’t worth of a proof:

**Lemma 6.2.** Let $(m_A, m_B, m_C)$ be a 3-periodic orbit of $\mathcal{B}$ where $m_A = (A, L_A)$, $m_B = (B, L_B)$, $m_C = (C, L_C)$. Then all lines $AB$, $BC$, $CA$, $L_A, L_B, L_C$ belong to the plane $ABC$, which is transverse to $a, b, c$ at $A, B, C$ respectively.

First let us show the

**Lemma 6.3.** The hypersurfaces $a$ and $b$ are supported by hyperplanes.

**Proof.** By symmetry, let us just show that $a$ is supported by a hyperplane. Fix $m_A \in U$. For $m_B \in V$, consider the plane $P_{m_B}$ containing the triangular orbit starting by $(m_A, m_B)$, as in Lemma 6.2. Consider $a_{m_B}$, $b_{m_B}$, $c_{m_B}$ to be the intersections of $P_{m_B}$ respectively with $a$, $b$, $c$: by transversality, and shrinking them if needed, we can suppose that they are immersed curve of $P_{m_B}$.

Now let $\alpha_{m_B} = \pi^{-1}(a_{m_B}) \cap \alpha$. Since $\alpha$ and $a$ have the same dimension, $\pi|_a : \alpha \to a$ is a diffeomorphism, and thus $\alpha_{m_B}$ is an immersed curve. Define $\beta_{m_B}$ and $\gamma_{m_B}$ similarly. Let us show that $(\alpha_{m_B}, \beta_{m_B}, \gamma_{m_B})$ is a planar 3-reflective projective billiard. Consider the open subsets $U' = U \cap \alpha_{m_B}$ of $\alpha_{m_B}$ and $V' = V \cap \beta_{m_B}$ of $\beta_{m_B}$. Any $m'_B \in V'$ is such that $(m_A, m'_B)$ can be completed in an orbit $(m_A, m'_B, m'_C)$ of $\mathcal{B}$ and by Lemma 6.2, $AB'C'$ defines a plane containing $L_A$ and $AB'$, which are intersecting lines inside $P_{m_B}$. Hence $AB'C' = P_{m_B}$ and thus $\beta_{m_B}$ is an analytic curve such that for all $m'_B = (B', L'_B) \in V'$, $B'$ and $L'_B$ are in $P_{m_B}$. The same argument work for $c_{m_B}$: hence also for $\gamma_{m_B}$ by completing any $(m'_A, m'_C) \in U' \times V'$ into a 3-periodic orbit. Finally we showed that $(\alpha_{m_B}, \beta_{m_B}, \gamma_{m_B})$ is a 3-reflective projective billiard inside $P_{m_B}$.

In particular, by Theorem 1.13 $a_{m_B}$ is supported by a line, $\ell_{m_B}$. Now $\ell_{m_B}$ is included in $T_A a$ (since the tangent space of $a_{m_B}$ is included in the tangent space of $a$) and in $P_{m_B}$. This result is true for any $m_B \in V$, implying the same result for lines in a neighborhood of $\ell_{m_B}$ in $T_A a$: hence by Lemma 6.1 $a$ is supported by an hyperplane, which concludes the proof.

Let $\hat{a}$ be the hyperplane supporting $a$ and $\hat{b}$ be the hyperplane supporting $b$.

**Lemma 6.4.** There is a point $B_0 \in \hat{b}$ such that for all $m_A = (A, L_A) \in a$ the line $L_A$ goes through $B_0$. Similarly, there is a point $A_0 \in \hat{a}$ such that for all $m_B = (B, L_B) \in \beta$ the line $L_B$ goes through $A_0$.

**Proof.** Let us show the existence of $B_0$, the existence of $A_0$ being analogous. Fix $m_A = (A, L_A) \in U$, and consider the point $B_0 \in \hat{b}$ of intersection of $L_A$ with $\hat{b}$. For $m_B \in V$, consider the plane $P_{m_B}$ containing the triangular orbit starting by $(m_A, m_B)$, as in Lemma 6.2 define $a_{m_B}$, $b_{m_B}$, $c_{m_B}$, $\alpha_{m_B}$, $\beta_{m_B}$, $\gamma_{m_B}$, $U'$, $V'$ as in the proof of Lemma 6.3.

We recall that $(\alpha_{m_B}, \beta_{m_B}, \gamma_{m_B})$ is a planar 3-reflective projective billiard. By Theorem 1.13 it is a right-spherical billiard, hence each $m'_A = (A', L'_A) \in U'$ is such that $L'_A$ and $L_A$ intersect $b_{m_B}$ at the same point. By construction, $L_A$ intersects $b$ hence $b_{m_B}$ at only one point: $B_0$, hence so does $L'_A$. Therefore, any
$m'_A = (A', L'_A) \in U'$ is such that $L'_A$ passes through $B_0$. Hence by analyticity, if $\ell_{m_B}$ is the line of intersection of $P_{m_B}$ with $a$, all $m'_A = (A', L'_A) \in \alpha \cap \pi^{-1}(\ell_{m_B})$ is such that $L'_A$ passes through $B_0$.

Now the union of all $\ell_{m_B}$ for $m_B \in V$ contains a non-empty open subset $\Omega$ of $a$, which by construction has the following property: all $m'_A = (A', L'_A) \in \alpha \cap \pi^{-1}(W)$ is such that $L'_A$ passes through $B_0$. By analyticity, all $m'_A = (A', L'_A) \in \alpha$ is such that $L'_A$ passes through $B_0$, and the proof is complete. 

Now we can finish the proof of Theorem 2. Indeed, any $z = (m_A, m_B) \in U \times V$, can be completed in an orbit, which lies in the plan $P_z$ of Lemma 6.2. $P_z$ contains $L_A$ and $L_B$, hence goes through $A_0$ and $B_0$. If $A_0 \neq B_0$, $P_z = AA_0B$, but this is impossible since it doesn’t depend on $m_B$ which can be chosen such that $B \notin AA_0B$. Hence $A_0 = B_0$ and $A_0 \in \hat{a}$, implying that all $m_A = (A, L_A) \in U$ are such that $L_A \subset T_Aa$. This contradicts the definition of $\alpha$, and the theorem is proved.
References

[1] Y.M. Baryshnikov, V. Zharnitsky, Billiards and nonholonomic distributions, *J. Math. Sci.* 128 (2005), 2706–2710.

[2] Y.M. Baryshnikov, V. Zharnitsky, Sub-riemannian geometry and periodic orbits in classical billiards, *Math. Res. Lett.* 13 (2006), no. 4, 587–598.

[3] Y.M. Baryshnikov, Spherical billiards with periodic orbits, *preprint*.

[4] V. Blumen, K. Kim, J. Nance, V. Zharnitsky, Three-Period Orbits in Billiards on the Surfaces of Constant Curvature, *International Mathematics Research Notices* (2012), 10.1093/imrn/rnr228.

[5] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, P. A. Griffiths, *Exterior Differential Systems*, Springer Science and Business Media, 2013.

[6] C. Fierobe, Examples of reflective projective billiards and outer ghost billiards, *https://arxiv.org/pdf/2002.09845.pdf*

[7] A. Glutsyuk, On 4-reflective complex analytic billiards, *Journal of Geometric Analysis* 27 (2017), 183–238.

[8] A.A. Glutsyuk, Yu.G. Kudryashov, On quadrilateral orbits in planar billiards, *Doklady Mathematics* 83 (2011), No. 3, 371–373.

[9] A. A. Glutsyuk, Yu. G. Kudryashov, No planar billiard possesses an open set of quadrilateral trajectories, *J. Modern Dynamics* 6 (2012), No. 3, 287–326.

[10] Ph. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley and Sons, 1978

[11] M. R. Rychlik, Periodic points of the billiard ball map in a convex domain, *Journal of Differential Geometry* 30 (1989), 191–205.

[12] L. Stojanov, Note on the periodic points of the billiard, *Journal of Differential Geometry* 34 (1991), 835–837.

[13] S. Tabachnikov, Introducing projective billiards, *Ergodic Theory and Dynamical Systems* 17 (1997), 957–976.

[14] S. Tabachnikov, Ellipsoids, complete integrability and hyperbolic geometry, Moscow Mathematical Journal 2 (2002), 185–198.

[15] S. Tabachnikov, Exact transverse line fields and projective billiards in a ball, GAFA Geom. funct. anal. 7 (1997), 594–608.

[16] D. Vasiliev, Two-term asymptotics of the spectrum of a boundary value problem in interior reflection of general form, *Funct Anal Appl.* 18 (1984), 267–277.

[17] Ya. B. Vorobets, On the measure of the set of periodic points of the billiard, *Math. Notes* 55 (1994), 455–460.

[18] H. Whitney, Differentiable manifolds, *Annals of Mathematics* 37 (1936), 645–680.

[19] M.P. Wojtkowski, Two applications of Jacobi fields to the billiard ball problem, *Journal of Differential Geometry* 40 (1994), 155–164.