Abstract
We study random series priors for estimating a functional parameter \( f \in L^2[0,1] \). We show that with a series prior with random truncation, Gaussian coefficients, and inverse gamma multiplicative scaling, it is possible to achieve posterior contraction at optimal rates and adaptation to arbitrary degrees of smoothness. We present general results that can be combined with existing rate of contraction results for various nonparametric estimation problems. We give concrete examples for signal estimation in white noise and drift estimation for a one-dimensional SDE.

1 Introduction
In Bayesian function estimation, a common approach to putting a prior distribution on a function \( f \) of interest, for instance a regression function in nonparametric regression models or a drift function in diffusion models, is to expand the function in a particular basis and to endow the coefficients in the expansion with prior weights. For computational
or other reasons the series is often truncated after finitely many terms, and the truncation level is endowed with a prior as well. The coefficients in the expansion are often chosen to be independent under the prior and distributed according to some given probability density.

It is of interest to understand whether, in addition to their attractive conceptual and computational aspects, nonparametric priors of this type enjoy favourable theoretical properties as well. Examples of papers in which this was studied for various families of series priors include Zhao (2000), Shen and Wasserman (2001), de Jonge and van Zanten (2012), Rivoirard and Rousseau (2012), Arbel et al. (2013), Shen and Ghosal (2015). The results in these papers show that when appropriately constructed, random series priors can yield posteriors that contract at optimal rates and that adapt automatically to the smoothness of the function that is being estimated.

To ensure that the nonparametric Bayes procedure not only adapts to smoothness, but is also flexible with respect to the multiplicative scale of the function of interest, a multiplicative hyperparameter with an independent prior distribution is often employed as well. Theoretically this is usually not needed for an optimal concentration rate of the posterior, but it can greatly improve performance in practice. See for instance van der Meulen et al. (2014), where it is explained why it is computationally attractive in certain settings to use Gaussian priors on the series coefficients in combination with a multiplicative (squared) scaling parameter with an inverse gamma prior. For a given truncation level, the prior is conjugate and allows for posterior computations using standard Gibbs sampling. The existing theoretical results do not cover this important case however. This is mainly due to the fact that essentially, the available rate of contraction theorems for series priors require that hyper priors have (sub-) exponential tails, which excludes the inverse gamma distribution. (For example the second part of condition (A2) of Shen and Ghosal (2015) is not satisfied in our setting.) The theoretical properties of random series priors with inverse gamma scaling have therefore remained unexplored. With this paper we intend to fill this gap.

Concretely, we consider statistical models in which the unknown object of interest is a square integrable function $f$ on $[0,1]$. We endow this function with a prior that is
hierarchically specified as follows:

\[ J \sim \text{Poisson or geometric}, \]
\[ s^2 \sim \text{inverse gamma}, \]
\[ f | s, J = \sum_{j \leq J} f_j \psi_j, \quad \text{with } (f_1, \ldots, f_J) \sim N(0, \text{diag}(s^2_j^{-1-2\alpha})), \quad j \leq J, \]

where \((\psi_j)\) is a fixed orthonormal basis of \(L^2[0,1]\) and \(\alpha > 0\) is a hyperparameter. (In fact, we will consider a somewhat broader class of hyper priors on \(J\) and \(s^2\), see Section 2.) In this paper we prove that this prior enjoys very favourable theoretical properties as well. We derive optimal posterior contraction rates and adaptation up to arbitrarily high degrees of smoothness.

In recent years, general rate of contraction theorems have been derived for a variety of nonparametric statistical problems. Roughly speaking, such theorems give sufficient conditions for having a certain rate of contraction in terms of (i) the amount of mass that the prior gives to neighbourhoods of the true function and (ii) the existence of growing subsets of the support of the prior, so-called sieves, that contain all but an exponentially small amount of the prior mass and whose metric entropy is sufficiently small. The statements of our main theorem match the conditions of these existing general results. This means that we automatically obtain results for different statistical settings, including for instance signal estimation in white noise and drift estimation for SDEs.

A simple but important observation that we make in this paper is that to obtain sharp rates for the priors we consider, it is necessary to use versions of the general contraction rate theorems that give entropy conditions on the intersection of the sieves with balls around the true function, as can be found for instance in Ghosal et al. (2000), van der Meulen et al. (2006) and Ghosal and van der Vaart (2007). As remarked in these papers, it is in many nonparametric problems sufficient to consider only the entropy of the sieves themselves, without intersecting them with a ball around the truth. For the priors we consider in this paper however, which in some sense are finite-dimensional in nature in certain regimes, this is not the case. It turns out that since the inverse gamma has polynomial tails, we need to make the sieves relatively large in order to ensure that they receive sufficient prior mass. Without intersecting them with a small ball around the truth, this would make their entropy too large, or even infinite.

The proof of our main results indicate that the good adaptation properties of series priors like (1.1) are really due to the fact that both the truncation level \(J\) and the scaling
constant $s$ are random. If the true function that is being estimated is relatively smooth, the prior can approximate it well by letting $J$ be small. If it is relatively rough however, the prior can adapt to it by letting $J$ be essentially infinite, or very large, to pick up all the fluctuations. The correct bias-variance trade-off is in that case achieved automatically by adapting the multiplicative scale. In some sense, priors like (1.1) can switch with sufficient probability between being essentially finite-dimensional, and being essentially infinite-dimensional. In combination with a random multiplicative scale, this gives them the ability to adapt to all levels of smoothness.

The remainder of the paper is organized as follows. In the next section we describe in detail the class of priors we consider. In Section 3 we present the main results of the paper, which give bounds on the amount of mass that the priors give to $L^2$-neighbourhoods of functions with a given degree of (Sobolev-type) smoothness, and the existence of appropriate sieves within the support of the prior. In Section 4 we link these general theorems to existing rate of contraction results for two different SDE models, to obtain concrete contraction results for signal estimation in white noise and drift estimation of a one-dimensional SDE with priors of the form (1.1). The proofs of the main results are given in Sections 5 and 6.

2 Prior model

We consider problems in which the unknown function of interest (e.g. a drift function of an SDE, a signal observed in noise, . . . ) is a square integrable function on $[0,1]$, i.e. an element of $L^2[0,1] = \{ f : [0,1] \to \mathbb{R} : \| f \|_2 < \infty \}$, where the $L^2$-norm is as usual defined by $\| f \|_2^2 = \int_0^1 f^2(x) \, dx$. We fix an arbitrary orthonormal basis $(\psi_j)$ of $L^2[0,1]$ (for instance the standard Fourier basis). Every element of $f \in L^2[0,1]$ can be represented as a series $f = \sum_j \langle f, \psi_j \rangle \psi_j$ where the convergence is in the $L^2$-norm and by the Plancherel formula $\| f \|_2^2 = \sum_j | \langle f, \psi_j \rangle |^2$. Finite series $\sum_{j \leq J} \langle f, \psi_j \rangle \psi_j$ approximate $f$ and the quality of this approximation depends on the decay of the coefficients $\langle f, \psi_j \rangle$, which also determines the “smoothness” of the function. The class of $\beta$-Sobolev smooth functions $H^\beta[0,1]$ is given by all $f \in L^2[0,1]$ for which the $\beta$-Sobolev norm $\| f \|_\beta := \sqrt{\sum_j k^{2\beta} | \langle f, \psi_j \rangle |^2}$ is finite. If $\psi_j$ is the classical Fourier series basis, these are the classical $\beta$-Sobolev spaces.

We define a series prior on a function $f \in L^2[0,1]$ through a hierarchical scheme which involves a prior on the point $J$ at which the series is truncated, a prior on the multiplicative scaling constant $s$ and conditionally on $s$ and $J$, a series prior with Gaussian coefficients.
Specifically, the prior on $J$ is defined through a probability mass function $p$ that is assumed to satisfy, for constants $C, C' > 0$,

$$p(j) \gtrsim e^{-Cj \log j}, \quad \sum_{i > j} p(i) \lesssim e^{-C'j}$$

for all $j \in \mathbb{N}$. (As usual, $a \lesssim b$ or $b \gtrsim a$ means that $a \leq cb$ for an irrelevant constant $c > 0$.) This includes for instance the cases of a Poisson or a geometric prior on $J$. For the scaling parameter we assume that the density $g$ of $s^2$ is positive and continuous and satisfies, for some $q < -1$ and $C'' > 0$,

$$g(x) \gtrsim e^{-C''/x} \quad \text{near 0,} \quad g(x) \gtrsim x^q \quad \text{near } \infty.$$

Hence in particular, the popular and computationally convenient choice of an inverse gamma prior on $s^2$ is included in our setup. The full prior $\Pi$ is then specified as follows:

$$J \sim p$$
$$s^2 \sim g$$
$$f \mid s, J \sim s \sum_{j=1}^{J} j^{-1/2-\alpha} Z_j \psi_j,$$

where $\alpha$ is a positive constant which determines the baseline smoothness of the prior, $p$ satisfies (2.1), $g$ satisfies (2.2) and the $Z_j$ are independent standard Gaussians.

### 3 Main results

Our main abstract result gives properties of the truncated series prior that link directly to the conditions of existing general theorems for posterior contraction in a variety of statistical settings. Combined with such existing results, we obtain concrete results for, for instance, signal estimation in white noise, drift estimation in diffusion models, etcetera. We give concrete examples in the next section.

As usual, if $\mathcal{F}$ is a subset of a normed vector space with norm $\| \cdot \|$, then we denote by $N(\varepsilon, \mathcal{F}, \| \cdot \|)$ the minimal number of balls of $\| \cdot \|$-radius $\varepsilon$ needed to cover the set $\mathcal{F}$.

**Theorem 3.1.** Let the prior $\Pi$ on $f$ be as defined in (2.3)–(2.5), with $\alpha > 0$ and $p$ and $g$ satisfying (2.1)–(2.2). Let $f_0 \in H^{\beta}[0,1]$ for $\beta > 0$. Then there exists a constant $c > 0$ such that for every $K > 1$, there exist $\mathcal{F}_n \subset L^2[0,1]$ such that with

$$\varepsilon_n = c \left( \frac{n}{\log n} \right)^{-\beta/(1+2\beta)},$$


we have
\[ \Pi(f : \|f - f_0\|_2 \leq \varepsilon_n) \geq e^{-n\varepsilon_n^2}, \]  
\[ \Pi(f \notin F_n) \leq e^{-K\varepsilon_n^2}, \]  
\[ \sup_{\varepsilon > \varepsilon_n} \log N(\varepsilon, \{f \in F_n : \|f - f_0\|_2 \leq \varepsilon\}, \| \cdot \|_2) \lesssim n\varepsilon_n^2, \]
for all \( a \in (0, 1) \).

The proof of the theorem is given in Section 5. The result matches with the sufficient conditions of existing posterior contraction theorems, provided that the relevant statistical distance-type quantities (e.g. Hellinger, Kullback-Leibler, ...) in the model can be appropriately linked to the \( L^2 \)-norm on the parameter \( f \). In the next section we give two concrete SDE-related examples, which motivated the present study.

Theorem 3.1 shows that with truncated series priors of the type (2.3)–(2.5) we can have adaption to arbitrary degrees of smoothness in certain function estimation problems, and achieve posterior contraction rates that are optimal up to a logarithmic factor. Inspection of the proof of Theorem 3.1 shows that in the range \( \beta \leq \alpha + 1/2 \), i.e. if the “baseline smoothness” \( \alpha \) of the prior happens to have been chosen large enough relative to the smoothness \( \beta \) of the true function, then we actually get the optimal rate \( n^{-\beta/(1+2\beta)} \) without additional logarithmic factors. This is true under a slightly stronger condition on the prior on the cut-off point \( J \). Instead of (2.1), we need to assume that for constants \( C, C' > 0 \) it holds that
\[ p(j) \gtrsim e^{-Cj}, \quad \sum_{i>j} p(i) \lesssim e^{-C'j} \]
for all \( j \in \mathbb{N} \). This means that the prior on \( J \) can still be geometric, but that the Poisson prior on \( J \) is excluded.

**Theorem 3.2.** Let the prior \( \Pi \) on \( f \) be as defined in (2.3)–(2.5), with \( \alpha > 0 \) and \( p \) and \( g \) satisfying (3.3) and (2.2). Let \( f_0 \in H^\beta[0,1] \) for \( 0 < \beta \leq \alpha + 1/2 \). Then there exists a constant \( c > 0 \) such that for every \( K > 1 \), there exist \( F_n \subset L^2[0,1] \) such that with
\[ \varepsilon_n = cn^{-\beta/(1+2\beta)}, \]
we have
\[ \Pi(f : \|f - f_0\|_2 \leq \varepsilon_n) \geq e^{-n\varepsilon_n^2}, \]  
\[ \Pi(f \notin F_n) \leq e^{-K\varepsilon_n^2}, \]  
\[ \sup_{\varepsilon > \varepsilon_n} \log N(\varepsilon, \{f \in F_n : \|f - f_0\|_2 \leq \varepsilon\}, \| \cdot \|_2) \lesssim n\varepsilon_n^2, \]  
for all \( a \in (0, 1) \).
for all \( a \in (0, 1) \).

The proof of this theorem is given in Section 6.

4 Specific statistical settings

4.1 Detecting a signal in Gaussian white noise

Suppose we observe a sample path \( X^{(n)} = (X_t^{(n)} : t \in [0, 1]) \) of stochastic process satisfying the SDE

\[
dX_t^{(n)} = f_0(t) \, dt + \frac{1}{\sqrt{n}} \, dW_t,
\]

where \( W \) is a standard Brownian motion and \( f_0 \in L^2[0, 1] \) is an unknown signal. To make inference about the signal we endow it with the truncated series prior \( \Pi \) described in Section 2 and we compute the corresponding posterior \( \Pi(\cdot \mid X^{(n)}) \). Theorem 3.1 of van der Meulen et al. (2006) or Theorem 6 of Ghosal and van der Vaart (2007), combined by our main result Theorem 3.1, imply that if \( f_0 \in H^\beta[0, 1] \) for \( \beta > 0 \), then we have the posterior contraction

\[
\Pi(f : \|f - f_0\|_2 > M_n(n/\log n)^{-\beta/(1+2\beta)} \mid X^{(n)}) \xrightarrow{P} 0
\]

for all \( M_n \to \infty \), where the convergence is in probability under the true model corresponding to the signal \( f_0 \).

4.2 Estimating the drift of an ergodic diffusion

Suppose we observe a sample path \( X^{(T)} = (X_t : t \in [0, T]) \) of an ergodic one-dimensional diffusion satisfying the SDE

\[
dx_t = b_0(X_t) \, dt + \sigma(X_t) \, dW_t, \quad X_0 = 0,
\]

where \( W \) is a standard Brownian motion, \( \sigma : \mathbb{R} \to \mathbb{R} \) is a know continuous function that is bounded away from 0, and \( b_0 : \mathbb{R} \to \mathbb{R} \) is a continuous function that satisfies the appropriate conditions to guarantee that the SDE indeed generates an ergodic diffusion (see for instance Kallenberg (2002)). The goal is to estimate the restriction \( b_0|_{[0,1]} \) of \( b_0 \) to the interval \([0, 1]\).

The likelihood for this model, given by Girsanov’s formula (e.g. Liptser and Shiryaev (2001)), factorizes into a factor involving only the drift on the interval \([0, 1]\) and a factor
involving only the restriction of the drift to the complement $\mathbb{R}\setminus [0, 1]$. As a result, since we are only interested in the drift on $[0, 1]$, we can effectively assume that it is known outside $[0, 1]$ and we only have to put a prior on the restriction of the drift to $[0, 1]$. We endow this with the truncated series prior $\Pi$ described in Section 2 and we compute the corresponding posterior $\Pi(\cdot | X(T))$. Theorem 3.3 of van der Meulen et al. (2006) and Theorem 5.1 then imply that if $b_0 | [0, 1] \in H^\beta [0, 1]$ for $\beta > 0$, then we have the posterior contraction

$$
\Pi(b : \|b - b_0\|_2 > M_T(T/\log T)^{-\beta/(1+2\beta)} | X(T)) \xrightarrow{P_{b_0}} 0
$$

as $T \to \infty$ for all $M_T \to \infty$, where the convergence is in probability under the true model corresponding to the drift function $b_0$.

5 Proof of Theorem 5.1

5.1 Prior mass

The following theorem implies that (5.2) holds with $\varepsilon_n$ as specified.

**Theorem 5.1.** Let the prior $\Pi$ on $f$ be defined according to (2.3) – (2.5), with $\alpha > 0$ and $p$ and $g$ satisfying (2.1) – (2.2), and let $f_0 \in H^\beta [0, 1]$ for $\beta > 0$. Then, for a constant $C > 0$, it holds that

$$
- \log \Pi(f : \|f - f_0\|_2 \leq 2\varepsilon) \leq C\varepsilon^{-1/\beta} \log 1/\varepsilon,
$$

for all $\varepsilon > 0$ small enough.

**Proof.** Recall that $s^2$ has density $g$ under the prior. Hence, by conditioning we see that the probability of interest is bounded from below by

$$
p\left(\frac{\varepsilon}{\|f_0\|_\beta}\right) \int_0^\infty \Pi\left(\left\|\sqrt{\eta} \sum_{j=1}^{(\varepsilon/\|f_0\|_\beta)^{-1/\beta}} j^{-1/2 - \alpha} Z_j \psi_j - f_0\right\|_2 \leq 2\varepsilon\right) g(\eta) d\eta,
$$

Now suppose first that $1 + 2\alpha - 2\beta \leq 0$. Then by Lemma 5.3 the preceding is further lower bounded by

$$
\exp\left(-C_1\varepsilon^{-1/\beta} \log 1/\varepsilon\right)p\left(\frac{\varepsilon}{\|f_0\|_\beta}\right) \int_{\varepsilon^{1/\beta}}^{2\varepsilon^{1/\beta}} g(\eta) d\eta
$$

for a constant $C_1 > 0$. By the assumptions on $p$ and $g$ this is bounded from below by a constant times $\exp(-C_2\varepsilon^{-1/\beta} \log 1/\varepsilon)$ for $\varepsilon$ small enough, for some constant $C_2 > 0$. 

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In the other case $1 + 2\alpha - 2\beta > 0$ we restrict the integral over $\eta$ to a different region to obtain instead the lower bound
\[
\exp \left( - C_1 \varepsilon^{-1/\beta} \log 1/\varepsilon \right) p\left( \left( \varepsilon/\|f_0\|_\beta \right)^{-1/\beta} \right) \int_{\varepsilon^{(2\beta-2\alpha)/\beta}}^{2\varepsilon^{(2\beta-2\alpha)/\beta}} g(\eta) \, d\eta
\]
for some $C_1 > 0$. For $\alpha < \beta \leq \alpha + 1/2$ the assumptions on $p$ and on the behaviour of $g$ near 0 ensure again that this is bounded from below by a constant times $\exp(-C_2\varepsilon^{-1/\beta} \log 1/\varepsilon)$ for $\varepsilon$ small enough. For the range $\beta < \alpha$ this holds as well, by the the assumptions on $p$ and on the behaviour of $g$ near $\infty$. When $\alpha = \beta$ we use the lower bound
\[
\exp \left( - C_1 \varepsilon^{-1/\beta} \log 1/\varepsilon \right) p\left( \left( \varepsilon/\|f_0\|_\beta \right)^{-1/\beta} \right) \int_{\varepsilon^{(2\beta-2\alpha)/\beta}}^{C_3} g(\eta) \, d\eta.
\]
Again by the behaviour of $g$ near infinity, the integral on the right is positive for $C_3$ big enough and the desired lower bound holds by the assumption on $p$.

**Lemma 5.2.** Let $Z_1, Z_2, \ldots$ be independent and standard normal. There exists a universal constant $K > 1$ such that for every $s > 0$, $\varepsilon > 0$, $J \in \mathbb{N}$ and $a \in \ell^2$,
\[
- \log \mathbb{P}\left( \| s \sum_{j=1}^J a_j Z_j \psi_j \|_2 \leq \varepsilon \right) \leq 2J \log \left( K \sqrt{\varepsilon^2/\|a\|^2} \right).
\]

**Proof.** Since the $\psi_j$ form an orthonormal basis, the probability we have to lower bound equals
\[
\mathbb{P}\left( s^2 \sum_{j=1}^J a_j^2 Z_j^2 \leq \varepsilon^2 \right) \geq \mathbb{P}\left( \max_{j \leq J} |Z_j| \leq \frac{\varepsilon}{s\|a\|_2} \right) = \left( \mathbb{P}\left( |Z_1| \leq \frac{\varepsilon}{s\|a\|_2} \right) \right)^J.
\]
If $\varepsilon/(s\|a\|_2) \geq \xi_{3/4}$, with $\xi_p$ the $p$-quantile of the standard normal distribution, $\mathbb{P}(|Z_1| \leq \varepsilon/(s\|a\|_2)) \geq 1/2$. In the other case, it is at least $\varphi(\xi_{3/4}) \times 2\varepsilon/(s\|a\|_2)$, with $\varphi$ the standard normal density. So in either case, it is at least a constant $C \in (0, 1)$ times $1 \wedge \varepsilon/(s\|a\|_2)$. It follows that
\[
\log \mathbb{P}\left( \| s \sum_{j=1}^J a_j Z_j \psi_j \|_2 \leq \varepsilon \right) \geq J \log C + J \log \left( 1 \wedge \frac{\varepsilon}{s\|a\|_2} \right)
\]
\[
\geq 2J \log \left( C \wedge \frac{\varepsilon}{s\|a\|_2} \right).
\]
This implies the statement of the lemma. \qed
Lemma 5.3. Let $Z_1, Z_2, \ldots$ be independent and standard normal. Let $\beta > 0$ and $f_0 \in H^\beta[0,1]$ be given. There exists a constant $K > 1$ such that for all $\varepsilon, s, \alpha > 0$ and $J \geq \varepsilon^{1/2 - \alpha} Z_j^\psi_j - f_0 \|^2 < 2s \varepsilon$.

Proof. For fixed $J, s$, the sum $s \sum_{j=1}^J j^{-1/2 - \alpha} Z_j^\psi_j$ is a centered Gaussian random element in $L^2[0,1]$ and has a reproducing kernel Hilbert space (RKHS), which is the space $H^{s,J}$ of all functions $h = \sum_{j \leq J} h_j^\psi_j$, with RKHS-norm $\| h \|_{H^{s,J}}^2 = J(1 + 2\alpha - 2\beta) s J^2$.

The function $f_0$ admits a series expansion $f_0 = \sum f_j^\psi_j$. For $J_0 \leq J$, consider the function $h_0 = \sum_{j \leq J_0} f_j^\psi_j$ in the RKHS. It holds that $\| f_0 - h_0 \|^2 = \sum_{j > J_0} f_j^2 \leq J_0^{-2\beta} \| f_0 \|_{\beta}^2$.

Hence for $J_0 = \lfloor \varepsilon / \| f_0 \|_{\beta} \rfloor^{1/\beta} - 1/\beta$, we have that $\| f_0 - h_0 \|^2 \leq \varepsilon$. The condition on $J$ ensures that $h_0$ is an element of the RKHS, and $\| h_0 \|^2_{H^{s,J}} = J(1 + 2\alpha - 2\beta) s J^2 f_j^2 \leq \| f_0 \|^2_{\beta} J_0(1 + 2\alpha - 2\beta) s J^2$.

It follows that

$$\inf_{h \in H^{s,J}} \| h \|^2_{H^{s,J}} \leq \frac{\| f_0 \|^2_{\beta}}{s^2} J(1 + 2\alpha - 2\beta) s J^2$$

Combining this with the preceding lemma and Lemma 5.3 of van der Vaart and van Zanten (2008) completes the proof.

5.2 Sieves, remaining mass and entropy

Let the sequence $\varepsilon_n \to 0$ and $\beta > 0$ be given. We consider sieves of growing dimension of the form $\mathcal{F}_n = \left\{ h = \sum_{j \leq J_n} h_j^\psi_j \right\}$, where $J_n = K_1 \varepsilon_n^{-1/\beta} \log 1/\varepsilon_n$.
for a constant $K_1 > 0$ specified below.

By assumption (2.1) we have

$$
\Pi(f \notin F_n) = \Pi(J > J_n) \lesssim e^{-C' K_1 \epsilon_n^{1/\beta} \log 1/\epsilon_n}.
$$

This implies that statement (3.3) of Theorem 3.1 holds if $K_1$ is chosen large enough.

As for the entropy condition (3.4), we note that if the function $f_0$ admits the series expansion $f_0 = \sum_{j} f_{0,j} \psi_j$, then a function $f \in F_n$ which satisfies $\|f - f_0\|_2 \leq \epsilon$ is of the form $f = \sum_{j \leq J_n} f_j \psi_j$, and $\sum_{j \leq J_n} (f_j - f_{0,j})^2 \leq \epsilon^2$. Hence, the covering number in (3.4) is bounded by the $a\epsilon$-covering number of a ball of radius $\epsilon$ in $\mathbb{R}^{J_n}$, which is bounded by $(3/a)^{J_n}$ (see, for instance, Pollard (1990)). In view of the choice (5.3) of $J_n$ it follows that (3.4) holds.

6 Proof of Theorem 3.2

Under the conditions of Theorem 3.2 we can replace the result of Theorem 5.1 by the following, which implies that (3.7) holds.

**Theorem 6.1.** Let the prior $\Pi$ on $f$ be defined according to (2.3) – (2.5), with $\alpha > 0$ and $p$ and $g$ satisfying (3.5) and (2.2), and let $f_0 \in H^{\beta}[0,1]$ for $0 < \beta \leq \alpha + 1/2$. Then, for a constant $C > 0$, it holds that

$$
- \log \Pi(f : \|f - f_0\|_2 \leq \epsilon) \leq C \epsilon^{-1/\beta},
$$

for all $\epsilon > 0$ small enough.

**Proof.** Instead of using Lemma 5.2 we simply note that for $s > 0$ and $J \in \mathbb{N}$, and $Z_1, Z_2, \ldots$ independent and standard normal,

$$
- \log \mathbb{P}\left( \| s \sum_{j=1}^{J} j^{-1/2-\alpha} Z_j \psi_j \|_2 \leq \epsilon \right) \leq - \log \mathbb{P}\left( \| \sum_{j=1}^{\infty} j^{-1/2-\alpha} Z_j \psi_j \|_2 \leq \epsilon/s \right).
$$

By Lemma 4.2 of van Waaij and van Zanten (2016) the right-hand side is bounded by a constant times $(\epsilon/s)^{-1/\alpha}$. Using (5.1) and Lemma 5.3 of van der Vaart and van Zanten (2008), we see that for $J \geq \left(\epsilon/\|f_0\|_\beta\right)^{-1/\beta}$,

$$
- \log \mathbb{P}\left( \| s \sum_{j=1}^{J} j^{-1/2-\alpha} Z_j \psi_j - f_0 \|_2 \leq 2\epsilon \right) \lesssim \left( \frac{s}{\epsilon} \right)^{1/\alpha} + 1 \sqrt{\epsilon^{-1/\beta} - 2^{-\beta}}.
$$

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For $\beta \leq \alpha + 1/2$ the two terms on the right are balanced for $s$ of the order $\varepsilon^{(\beta-\alpha)/\beta}$, in which case the right-hand side of bounded by a constant times $\varepsilon^{-1/\beta}$. In view of assumption (3.5) and (3.2), it follows by conditioning that

$$\Pi(f : \|f - f_0\|_2 \leq 2\varepsilon) \geq \exp\left(-c_1\varepsilon^{-1/\beta}\right)p\left([c_2\varepsilon^{-1/\beta}]\right) \int_{\varepsilon^{(2\beta-2\alpha)/\beta}}^{c_3\varepsilon^{(2\beta-2\alpha)/\beta}} g(\eta) \, d\eta.$$ 

The assumptions on $p$ and $g$ ensure that this is bounded from below by a constant times $\exp(-C\varepsilon^{-1/\beta})$ for some constants $C,c_1,c_2,c_3 > 1$.

To complete the proof of Theorem 3.2 we note that in this case we can use the same sieves $\mathcal{F}_n$ as defined in (5.2), but with a different choice for the dimension $J_n$, namely $J_n = \lceil K_1\varepsilon^{-1/\beta} \rceil$, for some $K_1 > 0$. The tail condition in (3.5) then ensures that (3.8) holds if $K_1$ is chosen large enough. The entropy bound (3.9) is obtained by the same argument as before, but now using the new choice of $J_n$.

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