1 Notations and definitions

Given a positive integer $n$ and an $r$-uniform hypergraph (or $r$-graph for short) $F$, the Turán number $\text{ex}(n, F)$ of $F$ is the maximum number of edges in an $r$-graph on $n$ vertices that does not contain $F$ as a subgraph. The extension $H^e$ of $F$ is obtained as follows: For each pair of vertices $v_i, v_j$ in $F$ not contained in an edge of $F$, we add a set $B_{ij}$ of $r-2$ new vertices and the edge $\{v_i, v_j\} \cup B_{ij}$, where the $B_{ij}$'s are pairwise disjoint over all such pairs $\{i, j\}$. Let $K_r^e$ denote the complete $r$-graph on $p$ vertices. For all sufficiently large $n$, we determine the Turán numbers of the extensions of a 3-uniform $t$-matching, a 3-uniform linear star of size $t$, and a 4-uniform linear star of size $t$, respectively. We also show that the unique extremal hypergraphs are balanced blowups of $K_{3t-1}^3, K_{2t}^3$, and $K_{3t}^4$, respectively. Our results generalize the recent result of Hefetz and Keevash [7].

Key Words: Turán number, Hypergraph Lagrangian

Abstract

Given a positive integer $n$ and an $r$-uniform hypergraph (or $r$-graph for short) $F$, the Turán number $\text{ex}(n, F)$ of $F$ is the maximum number of edges in an $r$-graph on $n$ vertices that does not contain $F$ as a subgraph. The extension $H^e$ of $F$ is obtained as follows: For each pair of vertices $v_i, v_j$ in $F$ not contained in an edge of $F$, we add a set $B_{ij}$ of $r-2$ new vertices and the edge $\{v_i, v_j\} \cup B_{ij}$, where the $B_{ij}$'s are pairwise disjoint over all such pairs $\{i, j\}$. Let $K_r^e$ denote the complete $r$-graph on $p$ vertices. For all sufficiently large $n$, we determine the Turán numbers of the extensions of a 3-uniform $t$-matching, a 3-uniform linear star of size $t$, and a 4-uniform linear star of size $t$, respectively. We also show that the unique extremal hypergraphs are balanced blowups of $K_{3t-1}^3, K_{2t}^3$, and $K_{3t}^4$, respectively. Our results generalize the recent result of Hefetz and Keevash [7].

Key Words: Turán number, Hypergraph Lagrangian


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partite Turán graph on \( n \) vertices. Let \( t_r^m(n) = |T_r^m(n)| \). For a positive integer \( n \), we let \([n]\) denote \( \{1, 2, 3, \ldots, n\} \). Given positive integers \( m \) and \( r \), let \([m]_r = m(m-1) \ldots (m-r+1)\).

Given an \( r \)-graph \( F \), an \( r \)-graph \( G \) is called \( F \)-free if it does not contain \( F \) as a subgraph. For a fixed positive integer \( n \) and an \( r \)-graph \( F \), the Turán number of \( F \), denoted by \( \text{ex}(n, F) \), is the maximum number of edges in an \( r \)-graph on \( n \) vertices that does not contain \( F \) as a subgraph. An averaging argument of Katona, Nemetz and Simonovits [10] shows that the sequence \( \text{ex}(n, F) \) is a non-increasing sequence of real numbers in \([0, 1]\). Hence, \( \lim_{n \to \infty} \frac{\text{ex}(n, F)}{\binom{n}{r}} \) exists. The Turán density of \( F \) is defined as

\[
\pi(F) = \lim_{n \to \infty} \frac{\text{ex}(n, F)}{\binom{n}{r}}.
\]

In this paper, we extend the work of Hefetz and Keevash in [7] and determine Turán numbers of several classes of \( r \)-graphs using so-called hypergraph Lagrangian method.

**Definition 1.1** Let \( G \) be an \( r \)-graph on \([n]\) and let \( \vec{x} = (x_1, \ldots, x_n) \in [0, \infty)^n \). For every subgraph \( H \subseteq G \), define

\[
\lambda(H, \vec{x}) = \sum_{e \in E(H)} \prod_{i \in e} x_i.
\]

The Lagrangian of \( G \), denoted by \( \lambda(G) \), is defined as

\[
\lambda(G) = \max \{ \lambda(G, \vec{x}) : \vec{x} \in \Delta \},
\]

where

\[
\Delta = \{ \vec{x} = (x_1, x_2, \ldots, x_n) \in [0, \infty)^n : \sum_{i=1}^n x_i = 1 \}.
\]

The value \( x_i \) is called the weight of the vertex \( i \) and a vector \( \vec{x} \in \Delta \) is called a feasible weight vector on \( G \). A feasible vector \( \vec{y} \in \Delta \) is called an optimum weight vector on \( G \) if \( \lambda(G, \vec{y}) = \lambda(G) \).

Given an \( r \)-graph \( F \), we define the Lagrangian density \( \pi_\lambda(F) \) of \( F \) to be

\[
\pi_\lambda(F) = \sup \{ r! \lambda(G) : F \not\subseteq G \}.
\]

**Proposition 1.2** \( \pi(F) \leq \pi_\lambda(F) \).

**Proof.** Let \( \varepsilon > 0 \) be arbitrary. Let \( n \) be large enough and let \( G_n \) be a maximum \( F \)-free \( r \)-graph on \( n \) vertices. We have

\[
\pi(F) \leq \frac{|G_n|}{\binom{n}{r}} + \varepsilon/2 \leq r! \sum_{e \in E(G_n)} \frac{1}{n^{|e|}} + \varepsilon = r! \lambda(G_n, (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})) + \varepsilon \leq r! \lambda(G_n) + \varepsilon \leq \pi_\lambda(F) + \varepsilon.
\]

The Lagrangian method for hypergraph Turán problems were developed independently by Sidorenko [18] and Frankl-Füredi [5], generalizing work of Motzkin and Straus [12] and Zykov [23]. More recent developments of the method were obtained by Pikhurko [16] and Norin and Yepremyan [14]. Based on these developments, Brandt, Irwin, and Jiang [2], and independently Norin and Yepremyan [15] were able to determine the Turán numbers of a large family of hypergraphs and thereby extending earlier
works in [1] [4] [8] [9] [17] [19]. The methods used by the two groups are quite different. The former group used Pikhurko’s stability method while the latter group used a refined stability method that they developed in [14]. In this paper, we extend a recent work on the topic by Hefetz and Keevash [7] on Lagrangians of intersecting 3-graphs to determine the maximum Lagrangian of a 3-graph not containing a matching of a given size. We also determine the maximum Lagrangian of a 3-graph not containing a linear star of a given size and the maximum Lagrangian of a 4-graph not containing a linear star of a given size. These results combined with the corresponding general theorems in [2] and [15] then allow us to determine the Turán numbers of some corresponding hypergraphs, which we now define as below.

We say that a pair of vertices \( \{i, j\} \) is covered in a hypergraph \( H \) if there exists \( e \in H \) such that \( \{i, j\} \subseteq e \). Let \( r \geq 3 \) and \( F \) be an \( r \)-graph. Let \( p \geq |V(F)| \). Let \( \mathcal{K}_p^F \) denote the family of \( r \)-graphs \( H \) that contains a set \( C \) of \( p \) vertices, called the core, such that the subgraph of \( H \) induced by \( C \) contains a copy of \( F \) and such that every pair in \( C \) that are not covered by \( F \) is covered by an edge of \( H \). We call \( \mathcal{K}_p^F \) the family of weak extensions of \( F \) for the given \( p \). If \( p = |V(F)| \), then we simply call \( \mathcal{K}_p^F \) the family of extensions of \( F \). Let \( H_p^F \) be a member of \( \mathcal{K}_p^F \) obtained as follows. Label the vertices of \( F \) as \( v_1, \ldots, v_{|V(F)|} \). Add new vertices \( v_{|V(F)|+1}, \ldots, v_p \). Let \( C = \{v_1, \ldots, v_p\} \). For each pair of vertices \( v_i, v_j \in C \) not covered in \( F \), we add a set \( B_{ij} \) of \( r - 2 \) new vertices and the edge \( \{v_i, v_j\} \cup B_{ij} \), where the \( B_{ij} \)'s are pairwise disjoint over all such pairs \( \{i, j\} \). We call \( H_p^F \) the extension of \( F \) for the given \( p \). If \( p = |V(F)| \), then we simply call \( H_p^F \) the extension of \( F \).

Let \( r, t \) be integers such that \( r \geq 3 \) and \( t \geq 2 \). The \( r \)-uniform \( t \)-matching, denoted by \( M_t^r \), is the \( r \)-graph with \( t \) pairwise disjoint edges. The \( r \)-uniform linear star of size \( t \), denoted by \( L_t^r \), is the \( r \)-graph with \( t \) edges such that these \( t \) edges contain a common vertex \( x \) but are pairwise disjoint outside \( \{x\} \).

In [7], Hefetz and Keevash determined the Lagrangian density of \( M_2^3 \) and the Turán number of the extension of \( M_2^3 \) for all sufficiently large \( n \). In this paper, we generalize their result to determine the Lagrangian density of \( M_t^3 \) for all \( t \geq 2 \). We also determine the Lagrangian densities of \( L_t^1 \) or \( L_t^4 \), for all \( t \geq 2 \). For each of the hypergraphs mentioned above, we determine the Turán numbers of their extensions for all sufficiently large \( n \). Our method differs from the one employed by Hefetz and Keevash [7]. For the matching problem, we use compression and induction. This allows us to obtain a short proof of the main result of [7] and solve the problem for general \( t \). We solve the linear star problem for \( r = 3, 4 \) by first studying a local version of the matching problem for \( r = 2, 3 \), respectively.

## 2 Preliminaries

In this section, we develop some useful properties of Lagrangian functions. The following fact follows immediately from the definition of the Lagrangian.

**Fact 2.1** Let \( G_1, G_2 \) be \( r \)-graphs and \( G_1 \subseteq G_2 \). Then \( \lambda(G_1) \leq \lambda(G_2) \).

Given an \( r \)-graph \( G \) and a set \( S \) of vertices, the link graph of \( S \) in \( G \), denoted by \( L_G(S) \), is the hypergraph with edge set \( \{e \in (V(G) \setminus S) : e \cup S \in E(G)\} \). When \( S \) has only one element, e.g. \( S = \{i\} \), we write \( L_G(i) \) for \( L_G(\{i\}) \). Furthermore, when there is no confusion, we will drop the subscript \( G \). Given \( i, j \in V(G) \), define

\[
L_G(j \setminus i) = \{f \in \left( V(G) \setminus \{i, j\} \right)^{r-1} : f \cup \{j\} \in E(G) \text{ and } f \cup \{i\} \notin E(G) \},
\]
and define
\[ \pi_{ij}(G) = (E(G) \setminus \{f \cup \{j\} : f \in L_G(j \setminus i)\}) \bigcup \{f \cup \{i\} : f \in L_G(j \setminus i)\} . \]

By the definition of \( \pi_{ij}(G) \), it’s straightforward to verify the following fact.

**Fact 2.2** Let \( G \) be an \( r \)-graph on the vertex set \([n]\). Let \( x = (x_1, x_2, \ldots, x_n) \) be a feasible weight vector on \( G \). If \( x_i \geq x_j \), then \( \lambda(\pi_{ij}(G), x) \geq \lambda(G, x) \).

Part (a) of the following lemma is well-known (see [3] for instance). We include a short proof of it for completeness.

**Lemma 2.3** Let \( r, t \geq 2 \) be integers. Let \( G \) be a \( M'_t \)-free \( r \)-graph on the vertex set \([n]\). Let \( i, j \) be a pair of vertices, then the following hold:
(a) \( \pi_{ij}(G) \) is \( M'_t \)-free.
(b) If \( G \) is also \( K_{tr-1} \)-free and \( \{i, j\} \) is contained in an edge of \( G \), then \( \pi_{ij}(G) \) is \( K_{tr-1} \)-free.

**Proof.** Suppose for contradiction that there exist \( i, j \) such that \( \pi_{ij}(G) \) contains a \( t \)-matching \( M \). Then there must be an edge \( e \) of \( M \) that is in \( \pi_{ij}(G) \) but not in \( G \). This implies that \( i \in e, j \notin e \) and \( e' = (e \setminus \{i\}) \cup \{j\} \in G \). If \( j \) is not covered by any edge of \( M \), then \((M \setminus \{e\}) \cup \{e'\}\) is a \( t \)-matching in \( G \), contradicting \( G \) being \( M'_t \)-free. Hence, \( \exists f \in M \) such that \( j \in f \). Let \( f' = (f \setminus \{j\}) \cup \{i\} \). By the definition of \( \pi_{ij}(G) \), \( f \) and \( f' \) must both exist in \( G \), or else \( f \) wouldn’t be in \( \pi_{ij}(G) \). But now, \((M \setminus \{e, f\}) \cup \{e', f'\}\) is a \( t \)-matching in \( G \), contradicting \( G \) being \( M'_t \)-free.

Next, suppose that \( G \) is \( K_{tr-1} \)-free and \( \{i, j\} \) is contained in some edge \( e \) of \( G \). Suppose for contradiction that \( \pi_{ij}(G) \) contains a \( t \)-matching \( M \). Clearly \( V(K) \) must contain \( i \). If \( V(K) \) also contains \( j \) then it is easy to see that \( K \) also exists in \( G \), contradicting \( G \) being \( K_{tr-1} \)-free. All the edges in \( K \) not containing \( i \) also exist in \( G \). By our assumption, \( V(K) \) contains at least \( tr - 1 - (r - 1) = (t - 1)r \) vertices outside \( e \). So \( K \) contains a \((t - 1)\)-matching \( M \) disjoint from \( e \), all of which lie in \( G \) by earliest discussion. Now, \( M \cup \{e\} \) is a \( t \)-matching in \( G \), a contradiction.

Next, we show that for \( r = 2 \), part (b) of Lemma 2.3 holds even without the assumption that \( \{i, j\} \) is contained in an edge.

**Lemma 2.4** Let \( t \geq 2 \). Let \( G \) be an \( M'_2 \)-free and \( K^2_{2t-1} \)-free graph on \([n]\) and \( i, j \in [n] \). Then \( \pi_{ij}(G) \) is also \( K^2_{2t-1} \)-free.

**Proof.** Suppose for contradiction that \( \pi_{ij}(G) \) contains a copy \( K \) of \( K^2_{2t-1} \). Then \( \pi_{ij}(G) \not\cong G \) and \( K \) contains \( i \) but not \( j \) (note that \( \pi_{ij} \) does not change the common link of \( i \) and \( j \)). Since \( \pi_{ij}(G) \not\cong G \), \( L_G(i \setminus j) \not\subseteq \emptyset \). Also, \( L_G(i \setminus j) \not\subseteq \emptyset \), since otherwise \( K \subseteq G \). Let \( a \in V(L_G(i \setminus j)) \), \( b \in V(L_G(j \setminus i)) \). Note that any edge in \( \pi_{ij}(G) \) not containing \( i \) also exist in \( G \). Hence, \( K - \{i, a, b\} \) is a complete graph on at least \( 2t - 4 \) vertices in \( G \), which contains a \((t - 2)\)-matching \( M \). Now, \( M \cup \{ia, jb\} \) is a \( t \)-matching in \( G \), a contradiction.

An \( r \)-graph \( G \) is *dense* if for every subgraph \( G' \) of \( G \) with \(|V(G')| < |V(G)|\) we have \( \lambda(G') < \lambda(G) \). This is equivalent to saying that all optimum weight vectors on \( G \) are in the interior of \( \Delta \), which means that no coordinate in an optimum weight vector is zero. We say that a hypergraph \( G \) *covers pairs* if every pair of its vertices is covered by an edge.
**Fact 2.5** ([3]) Let $G = (V, E)$ be a dense $r$-graph. Then $G$ covers pairs.

**Definition 2.6** Let $G$ be an $r$-graph on $[n]$ and a linear order $\mu$ on $[n]$. We say that $G$ is left-compressed (or simply compressed) relative to $\mu$ if for all $i, j \in [n]$ with $i <_\mu j$ we have $\pi_{ij}(G) = G$. Let $\bar{x}$ be a feasible weight vector on $G$. We say that $G$ is $\bar{x}$-compressed if there exists a linear order $\mu$ on $V(G)$ such that $\forall i, j \in V(G)$ whenever $i <_\mu j$ we have $x_i \geq x_j$ and that $G$ is left-compressed relative to $\mu$.

**Algorithm 2.7** Let $G$ be an $r$-graph on $[n]$. Let $\bar{x}$ be an optimum weight vector of $G$. If there exist vertices $i, j$, where $i < j$, such that $x_i > x_j$ and $L_G(j \setminus i) \neq \emptyset$, then replace $G$ by $\pi_{ij}(G)$, continue this process until no such pair exists.

In the above algorithm, by relabelling the vertices if necessary, we may assume that $x_1 \geq x_2 \cdots \geq x_n$. Note that $s(G) = \sum_{e \in G} \sum_{i \in e} i$ is a positive integer that decreases by at least 1 in each step. Hence the algorithm terminates after finite many steps.

**Algorithm 2.8** (Dense and compressed subgraph)

**Input:** An $r$-graph $G$.

**Output:** A dense subgraph $G' \subseteq G$ together with an optimum weight vector $\bar{y}$ such that $\lambda(G', \bar{y}) = \lambda(G)$ and that $G'$ is $\bar{y}$-compressed.

**Step 1.** If $G$ is not dense, then replace $G$ by a dense subgraph with the same Lagrangian. Otherwise, go to Step 2.

**Step 2.** Let $\bar{y}$ be an optimum weight vector of $G$. If $G$ is $\bar{y}$-compressed, then terminate. Otherwise, there exist vertices $i, j$, where $i < j$, such that $y_i > y_j$ and $L_G(j \setminus i) \neq \emptyset$, then replace $G$ by $\pi_{ij}(G)$ and go to step 1.

Note that the algorithm terminates after finite many steps since Step 1 reduces the number of vertices by at least 1 in each step and Step 2 reduces the parameter $s(G)$ (similarly defined as above) by at least 1 in each step.

**Lemma 2.9** Let $G$ be a $M^*_r$-free $r$-graph and $\bar{x}$ a feasible weight vector on $G$. Then the following hold:

(a) There exists a $M^*_r$-free $r$-graph $H$ with $V(H) = V(G)$ such that $\lambda(H, \bar{x}) \geq \lambda(G, \bar{x})$ and that $H$ is $\bar{x}$-compressed.

(b) There exists a dense $M^*_r$-free $r$-graph $G'$ with $|V(G')| \leq |V(G)|$ together with an optimum weight vector $\bar{y}$ such that $\lambda(G', \bar{y}) = \lambda(G') \geq \lambda(G)$ and that $G'$ is $\bar{y}$-compressed. Furthermore, if $G$ is $K^*_r[-1]$-free, then $G'$ is $K^*_r[-1]$-free.

**Proof.** For (a), we apply Algorithm 2.7 to $G$ and let $H$ be the final graph obtained. That $\lambda(H, \bar{x}) \geq \lambda(G, \bar{x})$ follows from Fact 2.2. That $H$ is $M^*_r$-free follows from Lemma 2.3. That $H$ is $\bar{x}$-compressed follows from the fact that algorithm terminates after finite steps and it only terminates when the $r$-graph becomes compressed.

For (b), we apply Algorithm 2.8 to $G$ and let $G'$ be the final graph and $\bar{y}$ the optimum weight vector on $G$ implied by the algorithm. Since Algorithm terminates after finite many steps, $G'$ and $\bar{y}$ are well-defined. By Fact 2.2 $\lambda(G') \geq \lambda(G)$. By Lemma 2.3 $G'$ is $M^*_r$-free. By the algorithm, $G'$ is $\bar{y}$-compressed. Assume that $G$ is $K^*_r[-1]$-free. In the process of obtaining $G'$ we always take a dense subgraph first before applying a compression $\pi_{ij}$. Taking a subgraph preserves $K^*_r[-1]$-free condition.
For a dense graph, by Lemma 2.3 part (b) performing $\pi_{ij}$ preserves $K_{t-1}^r$-free condition. So $G'$ is $K_{t-1}^r$-free.

In [12], Motzkin and Straus determined the Lagrangian of any given 2-graph.

**Theorem 2.10** (Motzkin and Straus [12]) If $G$ is a 2-graph in which a maximum complete subgraph has $t$ vertices, then $\lambda(G) = \lambda(K_t^2) = \frac{1}{2}(1 - \frac{1}{t})$.

Let $G$ be an $r$-graph on $[n]$ and $\vec{x} = (x_1, x_2, \ldots, x_n)$ be a weight vector on $G$. If we view $\lambda(G, \vec{x})$ as a function in variables $x_1, \ldots, x_n$, then

$$\frac{\partial \lambda(G, \vec{x})}{\partial x_i} = \sum_{j \in E(G)} \prod_{j \in \{i\}} x_j.$$  

We sometimes write $\frac{\partial \lambda}{\partial x_i}$ for $\frac{\partial \lambda(G, \vec{x})}{\partial x_i}$.

**Fact 2.11** ([6]) Let $G$ be an $r$-graph on $[n]$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimum weight vector on $G$. Then

$$\frac{\partial \lambda(G, \vec{x})}{\partial x_i} = r \lambda(G)$$

for every $i \in [n]$ with $x_i > 0$.

**Fact 2.12** Let $G$ be an $r$-graph on $[n]$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be a feasible weight vector on $G$. Let $i, j \in [n]$, where $i \neq j$. Suppose that $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$. Let $\vec{y} = (y_1, y_2, \ldots, y_n)$ be defined by letting $y_\ell = x_\ell$ for every $\ell \in [n] \setminus \{i, j\}$ and letting $y_i = y_j = \frac{1}{2}(x_i + x_j)$. Then $\lambda(G, \vec{y}) \geq \lambda(G, \vec{x})$.

Furthermore, if the pair $\{i, j\}$ is not covered by any edge of $G$ and $\lambda(G, \vec{y}) = \lambda(G, \vec{x})$, then $x_i = x_j$.

**Proof.** Since $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$, we have

$$\lambda(G, \vec{y}) - \lambda(G, \vec{x}) = \sum_{\{i,j\} \subseteq E(G)} \left[ \frac{(x_i + x_j)^2}{4} - x_i x_j \right] \prod_{k \in \{i,j\}} x_k \geq 0.$$  

If the pair $\{i, j\}$ is not covered by any edge of $G$ then equality holds only if $x_i = x_j$.

As usual, if $V_1, \ldots, V_s$ are disjoint sets of vertices then $\Pi_{i=1}^s V_i = V_1 \times V_2 \times \cdots \times V_s = \{(x_1, x_2, \ldots, x_s) : \forall i = 1, \ldots, s, x_i \in V_i\}$. We will use $\Pi_{i=1}^s V_i$ to also denote the set of the corresponding unordered $s$-sets.

If $L$ is a hypergraph on $[m]$, then a blowup of $L$ is a hypergraph $G$ whose vertex set can be partitioned into $V_1, \ldots, V_m$ such that $E(G) = \bigcup_{i \in L} \bigcap_{v \in V_i} V_i$. The following proposition follows immediately from the definition and is implicit in many papers (see [11] for instance).

**Proposition 2.13** Let $r \geq 2$. Let $L$ be an $r$-graph and $G$ a blowup of $L$. Suppose $|V(G)| = n$. Then $|G| \leq \lambda(L)n^r$.

### 3 Lagrangian of an $r$-graph not containing a $t$-matching and related Turán numbers

#### 3.1 Lagrangian density of $M^3_t$

**Lemma 3.1** Let $n, r, t$ be positive integers where $t \geq 2$ and $n \geq r \geq 2$. Let $F$ denote the family of all $r$-graphs $H$ with no isolated vertex on at most $n$ vertices such that $H$ is $M^r_t$-free and $H \neq K_{t-1}^r$. Then there
exists a dense \( r \)-graph \( G \in \mathcal{F} \) and an optimum vector \( \bar{x} \) on \( G \) such that \( \lambda(G, \bar{x}) = \max \{ \lambda(H) : H \in \mathcal{F} \} \) and that \( G \) is \( \bar{x} \)-compressed.

**Proof.** First note that if \( H \in \mathcal{F} \) then \( H \) is \( K_{10r-1} \)-free. Otherwise suppose \( H \) contains a copy \( K \) of \( K_{10r-1} \). Then since \( H \) has no isolated vertex and \( H \neq K_{10r-1} \), \( H \) contains some edge not in \( K \), in which case we can find a \( t \)-matching in \( H \), a contradiction. Let \( \lambda^* = \max \{ \lambda(H) : H \in \mathcal{F} \} \). Let \( G_1 \in \mathcal{F} \) be an \( r \)-graph with \( \lambda(G_1) = \lambda^* \). By Lemma 2.39 (b), there exists a \( M^r \)-free dense \( r \)-graph \( G' \) with \( |V(G'_1)| \leq |V(G_1)| \) such that \( \lambda(G'_1) \geq \lambda(G_1) \) and \( G'_1 \) is \( \bar{x} \)-compressed, where \( \bar{x} \) is an optimum vector of \( G'_1 \). Furthermore, \( G'_1 \) is \( K_{10r-1} \)-free. Hence \( G'_1 \in \mathcal{F} \). So \( \lambda(G'_1) = \lambda^* \). The claim thus holds by letting \( G = G'_1 \). \( \blacksquare \)

Hefetz and Keevash [7] established the Lagrangian density of \( M_3^3 \). We give a short new proof here.

**Theorem 3.2** ([7]) Let \( G \) be an \( M_2^3 \)-free 3-graph. Then \( \lambda(G) \leq \lambda(K_5^3) = \frac{2}{27} \). Furthermore, if \( G \neq K_5^3 \) and \( G \) has no isolated vertex, then \( \lambda(G) \leq \lambda(K_3^3) - 10^{-3} \).

**Proof.** (new proof) It suffices to prove that if \( G \) is an \( M_2^3 \)-free 3-graph with no isolated vertex and \( G \neq K_5^3 \) then \( \lambda(G) \leq \lambda(K_5^3) - 10^{-3} \). By Lemma 3.11 it suffices to assume that \( G \) is dense and has an optimum weight vector \( \bar{x} \) such that \( G \) is \( \bar{x} \)-compressed. Suppose \( V(G) = [n] \). If \( n \leq 5 \), then \( \lambda(G) \leq \lambda(K_5^3) - 10^{-3} \), where \( K_5^3 \) is the 3-graph obtained by removing one edge from \( K_5^3 \). Hence, we may assume that \( n \geq 6 \). By our assumption, there exists a linear order \( \mu \) on \( [n] \) such that \( \forall i, j \in [n] \) whenever \( i <_\mu j \) we have \( x_i \geq x_j \) and that \( G \) is compressed relative to \( \mu \) By relabelling if needed, we may assume that \( \mu \) is the natural order \( 1 < 2 < \cdots < n \). Then \( x_1 \geq x_2 \geq \cdots \geq x_n \). By Fact 2.34 \( G \) covers pairs. So \( i(n-1)n \in G \), for some \( i < n - 1 \). Since \( G \) is compressed relative to the natural order, we have \( 1(n-1)n \in G \). Again, since \( G \) is compressed relative to the natural order, this implies that \( \exists i, j, \text{ where } 2 \leq i < j \leq n, \text{ } 1ij \in G \). Suppose that \( G \{ \{2, \ldots, n\} \} \) contains an edge \( e \). Since \( n \geq 6 \), \( \exists i, j \in \{2, \ldots, n\} \), such that \( i, j \notin e \). Now, \( \{1ij, e\} \) forms a 2-matching in \( G \), contradicting \( G \) being \( M_2^3 \)-free. Hence \( G = \{1ij : 2 \leq i < j \leq n\} \). Assume that \( x_1 = a \). Since \( \vec{y} = (\frac{x_2}{1-a}, \ldots, \frac{x_n}{1-a}) \) is a feasible weight vector on \( L_G(1) \), by Theorem 2.10

\[
\lambda(G) = \lambda(G, \vec{x}) = a(1-a)^2 \lambda(L_G(1), \vec{y}) < \frac{1}{2} a(1-a)^2 \leq \frac{1}{4} \left[ \frac{2a+(1-a)+(1-a)}{3} \right] = \frac{2}{27} < \lambda(K_5^3)-10^{-3}. \]

\( \blacksquare \)

We now extend Theorem 3.2 to determine (with stability) the maximum Lagrangian of a 3-graph not containing a \( t \)-matching, for all \( t \geq 2 \). Given an \( r \)-graph \( G = (V, E) \) and \( i \in V \), let \( I_G(i) = \{ e \in G : i \in e \} \).

**Theorem 3.3** Let \( t \geq 2 \) be a positive integer. Let \( G \) be an \( M_t^3 \)-free 3-graph with no isolated vertex and \( G \neq K_3^{3t-1} \). Then there exists a positive real \( c_1 = c_1(t) \) such that \( \lambda(G) \leq \lambda(K_3^{3t-1}) - c_1 = \frac{1}{6} \left( \frac{3^t-1}{3^t-1} - 6c_1 \right) \).
Proof. By Lemma 3.1, it suffices to assume that $G$ is dense and has an optimum weight vector $\bar{x}$ such that $G$ is $\bar{x}$-compressed. Suppose $V(G) = [n]$. Let $K_{3t-1}^3$ be the 3-graph obtained by removing one edge from $K_{3t-1}^3$. If $n \leq 3t-1$, then since $G \neq K_{3t-1}^3$, $\lambda(G) \leq \lambda(K_{3t-1}^3)$. So, we may assume that $n \geq 3t$. We use induction on $t$, with Theorem 5.2 forming the basis step $t = 2$. For the induction step, let $t \geq 3$. By our assumption, there exists a linear order $\mu$ on $[n]$ such that $\forall i, j \in [n], x_i \geq x_j$ if $i <_\mu j$ and that $G$ is compressed relative to $\mu$. By relabelling if needed, we may assume that $\mu$ is the natural order $1 < 2 < \cdots < n$. Then $x_1 \geq x_2 \geq \cdots \geq x_n$. By Fact 250 $G$ covers pairs. So $i(n-1)n \in G$, for some $i < n-1$. Since $G$ is compressed relative to the natural order, this implies $1(n-1)n \in G$ and furthermore

$$I_G(1) = \{1ij : 2 \leq i < j \leq n\}. \tag{1}$$

Suppose $x_1 = a$. Then $0 < a < 1$. Since $\bar{x} = (\frac{x_2}{1-a}, \ldots, \frac{x_n}{1-a})$ is a feasible weight vector on $L_G(1) = K_{n-1}^2$. By Theorem 2.10 we have

$$\lambda(I_G(1), \bar{x}) = a \cdot \sum_{2 \leq i < j \leq n} x_i x_j = a(1-a)^2 \lambda(L_G(1), \bar{x}) < 1/2 a(1-a)^2.$$

Let $F = G(\{2, 3, \ldots, n\})$. Suppose $F$ contains a $(t-1)$-matching $M$. Since $n \geq 3t$, there exist distinct vertices $i, j \in [n] \setminus (V(M) \cup \{1\})$. By 1, $1ij \in G$. Now, $M \cup \{1ij\}$ is a $t$-matching in $G$, contradicting $G$ being $M_t^3$-free. Hence $F$ must be $M_{t-1}^3$-free. Note that $\bar{x}$ is a feasible weight vector on $F$. By the induction hypothesis (by considering $F = K_{3t-4}^3$ or not), we have $\lambda(F, \bar{x}) \leq \lambda(K_{3t-4}^3)$. Thus,

$$\lambda(F, \bar{x}) = (1-a)^3 \cdot \lambda(F, \bar{x}) \leq (1-a)^3 \lambda(F) \leq (1-a)^3 \lambda(K_{3t-4}^3) = \left(\frac{3t-4}{3}\right)^3 \left(\frac{1-a}{3t-4}\right)^3.$$

Let $s = 3t-4$ and $\mu = \frac{2s-3s+2}{6s^2}$. We have

$$\lambda(G) = \lambda(G, \bar{x}) \leq \lambda(I_G(1), \bar{x}) + \lambda(F, \bar{x})$$

$$< \frac{1}{2} a(1-a)^2 + \left(\frac{s}{3}\right)^3 \left(\frac{1-a}{s}\right)^3$$

$$= \frac{1}{2} a(1-a)^2 + \frac{s^2 - 3s + 2}{6s^2} (1-a)^3$$

$$= (1-a)^2 \left(\frac{1}{2} a + \mu(1-a)\right)$$

$$= (1-a)^2 \left(\frac{1}{2} - \mu\right) a + \mu$$

$$= (1-a)(1-a) \left(2a + \frac{\mu}{\frac{1}{2} - \mu}\right) \cdot \left(\frac{1}{4} - \frac{1}{2} \mu\right)$$

$$\leq \left[\frac{1}{3} \left(1-a + 1 - a + 2a + \frac{\mu}{\frac{1}{2} - \mu}\right)\right]^3 \cdot \left(\frac{1}{4} - \frac{1}{2} \mu\right) \quad \text{(by the AM-GM inequality)}$$

$$= \frac{1}{54} \left(\frac{1}{2} - \mu\right)^2$$

$$= \frac{2s^4}{3(2s^2 + 3s - 2)^2}.$$

Since $s = 3t-4$, we have

$$\lambda(K_{3t-1}^3) = \left(\frac{3t-1}{3}\right)^3 \cdot \left(\frac{1}{3t-1}\right)^3 = \left(\frac{s+3}{3}\right) \cdot \left(\frac{1}{s+3}\right)^3 = \frac{s^2 + 3s + 2}{6(s+3)^2}.$$
Hence,
\[
\lambda(G) - \lambda(K_{3m-1}^3) \leq \frac{2s^4}{3(2s^2 + 3s - 2)^2} - \frac{s^2 + 3s + 2}{6(s + 3)^2}
\]
\[
= \frac{4s^4(s + 3)^2 - (2s^2 + 3s - 2)(s^2 + 3s + 2)}{6(2s^2 + 3s - 2)(s + 3)^2}
\]
\[
= -\frac{9s^4 + 15s^3 - 30s^2 - 12s + 8}{6(2s^2 + 3s - 2)(s + 3)^2},
\]
which is negative for every \( s \geq 2 \). Let
\[
c_1 = \min \left\{ \lambda(K_{3m-1}^3) - \lambda(K_{3m+2}^3), \frac{9s^4 + 15s^3 - 30s^2 - 12s + 8}{6(2s^2 + 3s - 2)(s + 3)^2} \right\}.
\]
Then \( \lambda(G) \leq \lambda(K_{3m+2}^3) - c_1 \) and the proof is complete.

**Corollary 3.4** \( \pi_{\lambda}(M_t^3) = 3!\lambda(K_{3m-1}^3) = \frac{[3m-1]_3}{(3m-1)!} \).

**Proof.** Since \( K_{3m-1}^3 \) is \( M_t^3 \)-free, \( \pi_{\lambda}(M_t^3) \geq 3!\lambda(K_{3m-1}^3) \). On the other hand, by Theorem 3.3 \( \pi_{\lambda}(M_t^3) \leq 3!\lambda(K_{3m-1}^3) \). Therefore, \( \pi_{\lambda}(M_t^3) = 3!\lambda(K_{3m-1}^3) \).

### 3.2 Turán number of the extension of \( M_t^3 \)

The main result in this section is as follows.

**Theorem 3.5** Let \( t \geq 2 \) be an integer. Then \( ex(n, H_{3t}^{M_t^3}) = t_{3t-1}^t(n) \) for sufficiently large \( n \). Moreover, if \( n \) is sufficiently large and \( G \) is an \( H_{3t}^{M_t^3} \)-free \( 3 \)-graph on \( [n] \) with \( |G| = t_{3t-1}^t(n) \), then \( G = T_{3t-1}^3(n) \).

To prove the theorem, we need several results from [2]. Similar results are obtained independently in [15].

**Definition (2)** Let \( m, r \geq 2 \) be positive integers. Let \( F \) be an \( r \)-graph that has at most \( m + 1 \) vertices satisfying \( \pi_{\lambda}(F) \leq \frac{|m|}{m^r} \). We say that \( K_{m+1}^F \) is \( m \)-stable if for every real \( \varepsilon > 0 \) there are a real \( \delta > 0 \) and an integer \( n_1 \) such that if \( G \) is a \( K_{m+1}^F \)-free \( r \)-graph with at least \( n \geq n_1 \) vertices and more than \( \left( \frac{|m|}{m^r} - \delta \right) \binom{n}{r} \) edges, then \( G \) can be made \( m \)-partite by deleting at most \( \varepsilon n \) vertices.

**Theorem 3.7 (2)** Let \( m, r \geq 2 \) be positive integers. Let \( F \) be an \( r \)-graph that either has at most \( m \) vertices or has \( m + 1 \) vertices one of which has degree 1. Suppose either \( \pi_{\lambda}(F) < \frac{|m|}{m^r} \) or \( \pi_{\lambda}(F) = \frac{|m|}{m^r} \) and \( K_{m+1}^F \) is \( m \)-stable. Then there exists a positive integer \( n_2 \) such that for all \( n \geq n_2 \) we have \( ex(n, H_{m+1}^{F^3}) = t_{m+1}^n(n) \) and the unique extremal \( r \)-graph is \( T_{m+1}^3(n) \).

Given an \( r \)-graph \( G \) and a real \( \alpha \) with \( 0 < \alpha \leq 1 \), we say that \( G \) is \( \alpha \)-dense if \( G \) has minimum degree at least \( \alpha \binom{|V(G)|}{r-1} \). Let \( i, j \in V(G) \), we say \( i \) and \( j \) are nonadjacent if \( \{i, j\} \) is not contained in any edge of \( G \). Given a set \( U \subseteq V(G) \), we say \( U \) is an equivalence class of \( G \) if for every two vertices \( u, v \in U \), \( L_G(u) = L_G(v) \). Given two nonadjacent nonequivalent vertices \( u, v \in V(G) \), symmetrizing \( u \) to \( v \) refers to the operation of deleting all edges containing \( u \) of \( G \) and adding all the edges \( \{u\} \cup A, A \in L_G(v) \) to \( G \). We use the following algorithm from [2], which was originated in [16].
Algorithm 3.8 (Symmetrization and cleaning with threshold $\alpha$)

Input: An r-graph $G$.
Output: An r-graph $G^*$.

Initiation: Let $G_0 = H_0 = G$. Set $i = 0$.

Iteration: For each vertex $u$ in $H_i$, let $A_i(u)$ denote the equivalence class that $u$ is in. If either $H_i$ is empty or $H_i$ contains no two nonadjacent nonequivalent vertices, then let $G^*_i = H_i$ and terminate. Otherwise let $u, v$ be two nonadjacent nonequivalent vertices in $H_i$, where $d_{H_i}(u) \geq d_{H_i}(v)$. We symmetrize each vertex in $A_i(v)$ to $u$. Let $G_{i+1}$ denote the resulting graph. If $G_{i+1}$ is $\alpha$-dense, then let $H_{i+1} = G_{i+1}$. Otherwise we let $L = G_{i+1}$ and repeat the following: let $z$ be any vertex of minimum degree in $L$. We redefine $L = L - z$ unless in forming $G_{i+1}$ from $H_i$ we symmetrized the equivalence class of some vertex $v$ in $H_i$ to some vertex in the equivalence class of $z$ in $H_i$. In that case, we redefine $L = L - v$ instead. We repeat the process until $L$ becomes either $\alpha$-dense or empty. Let $H_{i+1} = L$. We call the process of forming $H_{i+1}$ from $G_{i+1}$ “cleaning”. Let $Z_{i+1}$ denote the set of vertices removed, so that $H_{i+1} = G_{i+1} - Z_{i+1}$. By our definition, if $H_{i+1}$ is nonempty then it is $\alpha$-dense.

Theorem 3.9 (2) Let $m, r \geq 2$ be positive integers. Let $F$ be an r-graph that has at most $m$ vertices or has $m + 1$ vertices one of which has degree 1. There exists a real $\gamma_0 = \gamma_0(m, r) > 0$ such that for every positive real $\gamma < \gamma_0$, there exist a real $\delta > 0$ and an integer $n_0$ such that the following is true for all $n \geq n_0$. Let $G$ be an $K_{m+1}^F$-free r-graph on $[n]$ with more than $(\frac{m}{m^r} - \delta \gamma)^\binom{n}{r}$ edges. Let $G^*$ be the final r-graph produced by Algorithm 3.8 with threshold $\frac{m}{r(m^r - 1)} - \gamma$. Then $|V(G^*)| \geq (1 - \gamma)n$ and $G^*$ is $(\frac{m}{m^r} - \gamma)$-dense. Furthermore, if there is a set $W \subseteq V(G^*)$ with $|W| \geq (1 - \gamma_0)|V(G^*)|$ such that $W$ is the union of a collection of at most $m$ equivalence classes of $G^*$, then $G[W]$ is $m$-partite.

The following corollary is implicit in [2] and [15].

Corollary 3.10 Let $m, r \geq 2$ be positive integers. Let $F$ be an r-graph that has at most $m + 1$ vertices with a vertex of degree 1 and $\pi_\lambda(F) \leq \frac{[m]}{m^r}$. Suppose there is a constant $c > 0$ such that for every $F$-free r-graph $L$ with no isolated vertex and $L \not\cong K_{m}^r$, $\lambda(L) \leq \lambda(K_{m}^r) - c$. Then $K_{m+1}^F$ is $m$-stable.

Proof. Let $\epsilon > 0$ be given. Let $\delta, n_0$ be the constants guaranteed by Theorem 3.9. We can assume that $\delta$ is small enough and $n_0$ is large enough. Let $\gamma > 0$ satisfy $\gamma < \epsilon$ and $\delta + \gamma r < c$. Let $G$ be a $K_{m+1}^F$-free r-graph on $n > n_0$ vertices with more than $(\frac{m}{m^r} - \delta\gamma)^\binom{n}{r}$ edges. Let $G^*$ be the final r-graph produced by applying Algorithm 3.8 to $G$ with threshold $\frac{m}{r(m^r - 1)} - \gamma$. By Algorithm 3.8, if $S$ consists of one vertex from each equivalence class of $G^*$, then $G^*[S]$ covers pairs and $G^*$ is a blowup of $G^*[S]$.

First, suppose that $|S| \geq m + 1$. If $F \subseteq G^*[S]$, then since $G^*[S]$ covers pairs we can find a member of $K_{m+1}^F$ in $G^*[S]$ by using any $(m + 1)$-set that contains a copy of $F$ as the core, contradicting $G^*$ being $K_{m+1}^F$-free. So $G^*[S]$ is $F$-free. Since $|S| \geq m + 1$ and $G^*[S]$ covers pairs, clearly $G^*[S] \not\cong K_{m}^r$. Also, $G^*[S]$ has no isolated vertex. Hence, by our assumption, $\lambda(G^*[S]) \leq \frac{1}{r} \frac{[m]}{m^r} - c$. By Proposition 2.13, we have

$$|G^*| \leq \lambda(G^*[S])n^r \leq \left(\frac{1}{r} \frac{[m]}{m^r} - c\right)n^r < \left(\frac{[m]}{m^r} - c\right) \frac{n^r}{r!}. \quad (2)$$

Now, during the process of obtaining $G^*$ from $G$, symmetrization never decreases the number of edges. Since at most $\gamma n$ vertices are deleted in the process (see Theorem 3.9),

$$|G^*| > |G| - \gamma n \left(\binom{n-1}{r-1}\right) \geq \left(\frac{[m]}{m^r} - \delta - \gamma r\right) \frac{n^r}{r} > \left(\frac{[m]}{m^r} - c\right) \frac{n^r}{r!}. \quad (3)$$
contradicting (2). So |S| ≤ m. Hence, W = V(G*) is the union of at most m equivalence classes of G*.

By Theorem 3.9, |W| ≥ (1 - γ)n and G[W] is m-partite. Hence, G can be made m-partite by deleting at most γn < εn vertices. Thus, K_{m+1}^F is m-stable.

**Proof of Theorem 3.5.** By Theorem 3.3 and Corollary 3.4, M_3^3 satisfies the conditions of Corollary 3.10. So, K_{3t-1}^F is (3t - 1)-stable. The theorem then follows from Theorem 3.7.

4 Local Lagrangians of M_t^r-free r-graphs and Lagrangians of L_t^r-free r-graphs and related Turán numbers

In this section we consider a local version of Lagrangians of M_t^r-free graphs for r = 2, 3. This will then be used to determine the Lagrangian density of a linear star L_t^r for r = 3, 4. Let 0 < b < 1 be a real. Given an r-graph G on [n], a feasible weight vector ⃗x = (x_1, . . . , x_n) is called a b-bounded feasible weight vector on G if ∀i ∈ [n], x_i ≤ b. If G has a b-bounded feasible weight vector, then we define the b-bounded Lagrangian of G as

\[
λ_0(G) = \max\{λ(G, ⃗x) : ⃗x is a b-bounded feasible weight vector on G\}.
\] (3)

If G does not have any b-bounded feasible weight vector, then we define λ_0(G) = 0. A feasible b-bounded weight vector ⃗x on G such that λ(G, ⃗x) = λ_0(G) is called an optimum b-bounded weight vector on G. We now consider λ_0(G) over M_t^r-free r-graphs for r = 2, 3 for appropriate values of b. For such a study, first we reduce the problem to the case where the r-graph in consideration is compressed and there exists an optimum b-bounded weight vector with some additional properties.

**Lemma 4.1** Let 0 < b < 1 be a real. Let r, t ≥ 2 be integers. Let F be the family of all M_t^r-free r-graphs. There exists G ∈ F and an optimum b-bounded weight vector ⃗x on G such that

1. λ(G, ⃗x) = λ_0(G) = \max\{λ_0(H) : H ∈ F\}.

2. G is ⃗x-compressed.

3. All vertices of G have positive weight under ⃗x.

4. If u, v are any two vertices in G with weight less than b under ⃗x then \{u, v\} is covered in G.

**Proof.** Clearly, F is closed under taking subgraphs. Let λ* = \max\{λ_0(H) : H ∈ F\}. Among all r-graphs H ∈ F with λ_0(H) = λ*, let G be the one with the fewest possible vertices. Let ⃗x be an optimum b-bounded weight vector on G that has the maximum number of b-components. By Lemma 2.9 (a), we may assume that G is ⃗x-compressed (or else we could replace G with one that is ⃗x-compressed). If some vertex in G has 0 weight under ⃗x then deleting that vertex would give us a graph G′ ∈ F with λ_0(G′) = λ* and having fewer vertices than G, contradicting our choice of G. Hence, all vertices in G have positive weights under ⃗x. Now, suppose u, v are two vertices with weight less than b under ⃗x. Suppose that no edge of G contains both u and v. Without loss of generality suppose that λ(L_G(u), ⃗x) ≥ λ(L_G(v), ⃗x).

If we decrease the weight of v and increase the weight of u by the same amount, the total weight does not decrease. Hence, we can obtain an optimum b-bounded weight vector on G that either has more
b-components than \( \bar{x} \) or has weight 0 on \( v \). In the former, we get a contradiction to our choice of \( \bar{x} \). In the latter case, we get a contradiction to our choice of \( G \). Hence there must be some edge in \( G \) containing both \( u \) and \( v \). ■

For the purpose of studying \( L^3_t \)-free graphs, we will also need the following short lemma.

**Lemma 4.2** Let \( r, t \geq 2 \). Let \( G \) be an \( L^3_t \)-free \( r \)-graph with at least \( t(r - 1) + 1 \) vertices and \( G \) covers pairs. Let \( x \in V(G) \). Then \( L(x) \) is \( K^{r-1}_{t(r-1)-1} \)-free. In particular, \( G \) is \( K^r_{t(r-1)-1} \)-free.

**Proof.** Suppose for contradiction that \( L(x) \) contains a copy \( K^{r-1}_{t(r-1)-1} \). By our assumption, \( \exists y \in V(G) \setminus (V(K) \cup \{x\}) \). Since \( G \) covers pairs, there exists \( e \in G \) that contains \( x \) and \( y \). Now we can find a copy of \( L^3_{t-1} \) using a \((t-1)\)-matching in \( K \) containing \( x \) that are disjoint from \( e \setminus \{x, y\} \), which together with \( e \) form a copy of \( L^3_t \) in \( G \), a contradiction. ■

### 4.1 Local Lagrangians of \( M^2_t \)-free graphs and Lagrangians of \( L^3_t \)-free 3-graphs and related Turán numbers

We start the subsection by developing some structural properties of \( M^2_t \)-free left-compressed graphs. Let \( n, t \) be positive integers, where \( t \geq 2 \) and \( n \geq 2t \). For each \( \ell \in [t-1] \cup \{0\} \), define

\[
F_{t,\ell}(n) = \left( \binom{2t-1-\ell}{2} \right) \cup \{ab : a \in \{1, \ldots, \ell\}, b \in \{2t-\ell, \ldots, n\}\}.
\]

Note that \( F_{t,\ell}(n) \) is \( M^2_t \)-free for each \( \ell \in [t-1] \cup \{0\} \).

**Lemma 4.3** Let \( n, t \) be positive integers, where \( t \geq 2 \) and \( n \geq 2t \). Let \( G \) be an \( M^2_t \)-free 2-graph on \([n]\) that is left-compressed relative to the natural order. Then \( G \subseteq F_{t,\ell}(n) \) for some \( \ell \in [t-1] \cup \{0\} \).

**Proof.** For each \( i \in [t] \), let \( N_i = \{j \in [n] : j > i, ij \in G\} \). Since \( G \) is left-compressed relative to the natural order on \([n]\), we have either \( N_i = \emptyset \) or \( N_i = \{i+1, i+2, \ldots, m_i\} \) for some \( m_i > i \). Furthermore, \( N_1 \supseteq N_2 \supseteq \cdots \supseteq N_t \). For convenience, we define \( m_i = 1 \) for those \( i \in [t] \) with \( N_i = \emptyset \). Then \( \{m_1, \ldots, m_t\} \) is non-increasing. Let \( h \) be the largest \( i \in [t] \) such that \( m_i \leq 2t - i \). Note that \( h \) exists; otherwise \( \{i(2t+1-i) : i \in [t]\} \) is a \( t \)-matching in \( G \), a contradiction. Let \( \ell = h - 1 \). Then \( \ell \in [t-1] \cup \{0\} \). By our assumption, there is no edge from \( [\ell+1, n] \) to \( [2t-\ell, n] \). So \( G \subseteq F_{t,\ell}(n) \). ■

**Lemma 4.4** Let \( n, t \) be positive integers, where \( t \geq 2 \) and \( n \geq 2t \). Let \( b \) be a real such that \( 0 < b \leq \frac{1}{2} \). For each \( \ell \in [t-1] \), we have \( \lambda_b(F_{t,\ell}(n)) \leq \left( 2^{t-1-2\ell} \right) b^2 + \ell b - \frac{\ell^2}{2} b^2 \).

**Proof.** Let \( \ell \in [t-1] \). Let \( \bar{x} = (x_1, \ldots, x_n) \) be a \( b \)-bounded feasible vector on \( F_{t,\ell}(n) \) such that \( \lambda(F_{t,\ell}(n), \bar{x}) = \lambda_0(F_{t,\ell}(n)) \). Using Fact 2.12 (note that any new weight vector produced by Fact 2.12 based on \( \bar{x} \) is also \( b \)-bounded), we may assume that \( x_1 = \cdots = x_\ell, x_{\ell+1} = \cdots = x_{2t-1-\ell} \text{ and } x_{2t-\ell} = \cdots = x_n \).
\[ \cdots = x_n. \text{ Let } a = x_1, c = x_{\ell+1}, \text{ and } d = x_{2t-\ell} + \cdots + x_n = 1 - \ell a - (2t - 1 - 2\ell)c. \text{ We have} \]
\[
\lambda(F_{\ell,t}(n), \bar{x}) = \left( \frac{\ell}{2} \right) a^2 + \left( \frac{2t - 1 - 2\ell}{2} \right) c^2 + (2t - 1 - 2\ell)\ell a + la[1 - \ell a - (2t - 1 - 2\ell)c] \\
= \left( \frac{\ell}{2} \right) a^2 + \left( \frac{2t - 1 - 2\ell}{2} \right) c^2 + \ell a(1 - \ell a) \\
\leq \left( \frac{2t - 1 - 2\ell}{2} \right) b^2 + \ell b - \frac{t^2 + \ell}{2} a^2,
\]
where we used the fact that \( f(x) = \ell x - \frac{t^2 + \ell}{2} x^2 \) is increasing on \((-\infty, \frac{1}{\ell+1})\) and that \( a, c \leq b \leq \frac{1}{\ell+1}. \]
\]

\[ \boxed{\text{Theorem 4.5}} \]
\[ \text{Let } t \geq 2 \text{ be an integer. If } G \text{ is an } L_t^1\text{-free 3-graph, then } \lambda(G) \leq \lambda(K_{2t}^3). \text{ Furthermore, there is } c_2 = c_2(t) > 0 \text{ such that if } G \text{ is an } L_t^1\text{-free 3-graph that covers pairs and } G \neq K_{2t}^3 \text{ then } \lambda(G) \leq \lambda(K_{2t}^3) - c_2 = \frac{(2t-1)(t-1)}{12t^2} - c_2. \]

\[ \text{Proof.} \quad \text{It suffices to assume that } G \text{ is dense (otherwise we consider an appropriate subgraph). So } G \text{ covers pairs. In this set up, it suffices to prove the second statement. So assume that } G \text{ covers pairs and } G \neq K_{2t}^3. \text{ Suppose } V(G) = [n] \cup \{0\}. \text{ If } n < 2t, \text{ then } \lambda(G) \leq \lambda(K_{2t}^3) \leq \lambda(K_{2t}^3) - c_2, \text{ by choosing } c_2 \text{ to be small enough, where } K_{2t}^3 \text{ denotes } K_{2t}^3 \text{ minus an edge. Hence, we may assume that } n \geq 2t. \text{ Let } \bar{x} = (x_0, x_1, \ldots, x_n) \text{ be an optimum weight vector on } G. \text{ Let } a = \max\{x_i : i \in V(G)\}. \text{ By relabeling if needed, we may assume that } x_0 = a. \text{ By Fact 2.11, } \lambda(L(0), \bar{x}) = \frac{\partial \lambda(G, \bar{x})}{\partial x_0} = 3\lambda(G), \text{ so it suffices to show that } \lambda(L(0), \bar{x}) \leq \frac{(2t-1)(t-1)}{12t^2} - 3c_2, \text{ for some sufficiently small positive real } c_2.
\]

Since \( G \) is \( L_t^1 \)-free, \( L(0) \) is \( M_t^2 \)-free. Since \( G \) covers pairs and \( n \geq 2t \), by Lemma 4.12, \( K_{2t-1}^3 \not\subset L(0). \)

We may view \( L(0) \) as a 2-graph on \([n] \). Let \( \tilde{y} = (\frac{x_0}{a}, \ldots, \frac{x_n}{a}) \).

Then \( \tilde{y} \) is a feasible weight vector on \( L(0) \). Furthermore, it is \( \frac{1}{1-a} \)-bounded. We consider two cases.

\[ \text{Case 1. } a \geq \frac{1}{2t}. \]

Since \( L(0) \) is \( K_{2t-1}^2 \)-free, by Theorem 2.10, \( \lambda(L(0)) \leq \frac{1}{2}(1 - \frac{1}{2t-1}) \). Hence, for sufficiently small \( c_2 > 0 \),
\[
\lambda(L(0), \bar{x}) = (1 - a)^2 \lambda(L(0), \tilde{y}) \leq (1 - a)^2 \lambda(L(0)) \leq \left( \frac{2t-1}{2t} \right)^2 \frac{1}{2t-2} \leq \frac{(2t-1)(t-1)}{4t^2} - 3c_2.
\]

\[ \text{Case 2. } a < \frac{1}{2t}. \]

Let \( b = \frac{a}{1-a} \). Then \( b < \frac{1}{2t-1} \leq \frac{1}{t} \). By Lemma 2.14(a), there exists a \( M_t^2 \)-free 2-graph \( H \) on \([n] \) such that \( \lambda(H, \tilde{y}) \geq \lambda(L(0), \tilde{y}) \) and such that \( H \) is \( \tilde{y} \)-compressed. Also, since \( L(0) \) is \( K_{2t-1}^2 \)-free, by Lemma 2.13 \( H \) is also \( K_{2t-1}^2 \)-free. By relabeling if needed, we may assume that \( y_1 \geq \cdots \geq y_n \) and that \( H \) is left-compressed relative to the natural order on \([n] \). By Lemma 4.3 \( H \subset F_{t,\ell}(n) \) for some \( \ell \in [t-1] \cup \{0\} \).

First, assume that \( \ell \in [t-1] \).

Since \( \tilde{y} \) is a \( b \)-bounded feasible weight vector on \([n] \), by Lemma 4.3, we have
Theorem 4.7

Since \( f \) if for every edge \( e \), we have

\[
\lambda(L(0), \vec{x}) = (1 - a)^2 \lambda(L(0), \vec{y}) \leq \lambda(H, \vec{y}) \leq \lambda(F_{t, \ell}(n), \vec{y})
\]

\[
\leq (1 - a)^2 \left( \frac{2t - 1 - 2\ell}{2} \right) \left( \frac{a}{1 - a} \right)^2 + \ell \frac{a}{1 - a} - \frac{\ell^2 + \ell}{2} \left( \frac{a}{1 - a} \right)^2
\]

\[
= \left( \frac{2t - 1 - 2\ell}{2} \right) a^2 + \ell a(1 - a) - \frac{\ell^2 + \ell}{2} a^2.
\]

Since \( f(x) = \ell x(1 - x) - \frac{\ell^2 + \ell}{2} x^2 \) increases on \( (-\infty, \frac{1}{\ell+3}) \) and \( a < \frac{1}{2t} \leq \frac{1}{\ell+3} \), we have

\[
\lambda(L(0), \vec{x}) \leq \left( \frac{2t - 1 - 2\ell}{2} \right) \left( \frac{1}{2t} \right)^2 + \ell \frac{1}{2t} \left( 1 - \frac{1}{2t} \right) - \frac{\ell^2 + \ell}{2} \left( \frac{1}{2t} \right)^2
\]

\[
= \left( \frac{2t - 1 - 2\ell}{2} \right) \left( \frac{1}{2t} \right)^2 + \ell \frac{1}{2t} \left( 1 - \frac{2}{2t} \right) - \left( \frac{\ell}{2} \right) \left( \frac{1}{2t} \right)^2
\]

\[
= \lambda(F_{t, \ell}(2t - 1), \vec{z}),
\]

where \( z \) is a weight vector on \( [2t - 1] \) with \( z = \left( \frac{1}{2t}, \ldots, \frac{1}{2t} \right) \). Since \( \ell \geq 1 \), \( F_{t, \ell}(2t - 1) \subseteq K_{2t-1}^{2t-1} \). Hence,

\[
\lambda(L(0), \vec{x}) \leq \lambda(K_{2t-1}^{2t-1}, \vec{z}) \leq \frac{(2t - 1)(t - 1)}{4t^2} - 3c_2,
\]

for sufficiently small \( c_2 > 0 \).

Finally, suppose \( \ell = 0 \). Note that \( F_{t,0}(n) \) consists of a copy of \( K_{2t-1}^{2t-1} \) and some isolated vertices. Since \( H \subseteq F_{t,0}(n) \) and \( H \) is \( K_{2t-1}^{2t-1} \)-free, we have \( \lambda(L(0), \vec{x}) \leq \lambda(H, \vec{x}) \leq \lambda(H) \leq \lambda(K_{2t-1}^{2t-1}) \). Hence (4) still holds for sufficiently small \( c_2 > 0 \). This completes our proof.

Corollary 4.6

\[
\pi_\lambda(L_t^3) = 3! \lambda(K_{2t}^3) = \frac{(2t)!}{(2t)^t}.
\]

Applying Theorem 4.5 Corollary 4.14 Corollary 3.11 and Theorem 3.7 we have

Theorem 4.7 Let \( t \geq 2 \) be an integer. Then \( \text{ex}(n, H_{2t+1}^3) = t_2^3(n) \) for sufficiently large \( n \). Moreover, if \( n \) is sufficiently large and \( G \) is an \( H_{2t+1}^3 \)-free 3-graph on \( n \) vertices with \( |G| = t_2^3(n) \) then \( G = T_{2t}^3(n) \). 

Theorem 4.7 is part of a more general theorem obtained in [2] and [15]. However, the method we used in this section is self-contained and is very different from those used in [2] and [15].

4.2 Local Lagrangians of \( M_t^3 \)-free 3-graphs and Lagrangians of \( L_t^4 \)-free 4-graphs and related Turán numbers

Next, we consider local Lagrangians of \( M_t^3 \)-free 3-graphs. First, we focus on the \( t = 2 \) case. As before, we first develop some structural properties of \( M_t^3 \)-free 3-graphs. Given a 3-graph \( G \) on \( [n] \), let \( L^+(1) \) and \( L^+(2) \) denote the links of 1, 2 of \( G \) in \([3, n]\) respectively, i.e.

\[
L^+(i) = \{ A \subseteq [3, n] : A \cup \{i\} \in G \}
\]

for \( i = 1, 2 \). We say a set \( S \subseteq V(G) \) is a vertex cover of \( G \) if for every edge \( e \) of \( G \), \( e \cap S \neq \emptyset \).
Lemma 4.8 Let $n \geq 6$ be an integer. Let $G$ be an $M_2^3$-free 3-graph on $[n]$ with no isolated vertex that is left-compressed relative to the natural order on $[n]$. Then

(a) $\forall i \in [3,n], 12i \in G$,
(b) $\{1,2\}$ is a vertex cover of $G$, and
(c) $L^+(2)$ is $M_2^3$-free. Thus, if $L^+(2) \neq \emptyset$ then $L^+(2)$ is either a triangle or a star.

Proof. By our assumption, for some $i < j < n$, $ijn \in G$. Since $G$ is left-compressed relative to the natural order on $[n]$, we have $12n \in G$. Since $G$ is left-compressed, this further implies that $12i \in G$ for every $i \in [3,n]$. If $G$ contains an edge $e$ not containing 1 or 2, then $\{12i,e\}$ would form a 2-matching in $G$, for some $i \in [n], i \notin e$ and $i \neq 1,2$, contradicting $G$ being $M_2^3$-free. Hence $\{1,2\}$ is a vertex cover of $G$. Finally, since $G$ is left-compressed, $L^+(2) \subseteq L^+(1)$. If $L^+(2)$ contains a 2-matching, then we would obtain a 2-matching in $G$, a contradiction. So $L^+(2)$ is intersecting and must be either a star or a triangle.

Lemma 4.8 allows us to describe all left-compressed $M_2^3$-free 3-graphs on $[n]$.

Definition 4.9 For all integers $n \geq 5$, let

\[ G_0(n) = \{1ij : 2 \leq i < j \leq n\}, \]

\[ G_1(n) = \{12i : 3 \leq i \leq n\} \cup \{134, 135, 145, 234, 235, 245\}, \]

\[ G_2(n) = \binom{[4]}{3} \cup \{12i, 13i, 14i : 5 \leq i \leq n\}, \]

\[ G_3(n) = \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{234, 235, 145\}, \]

\[ G_4(n) = \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{23i : 4 \leq i \leq n\}. \]

Lemma 4.10 Let $n \geq 6$ be an integer. Let $G$ be an $M_2^3$-free 3-graph on $[n]$ that is left-compressed relative to the natural order on $[n]$. Then $G$ is a subgraph of one of $G_0(n), G_1(n), G_2(n), G_3(n), G_4(n)$ given in Definition 4.9.

Proof. By Lemma 4.8 $\{1, 2\}$ is a vertex cover of $G$ and $L^+(2)$ is either empty, or a triangle or a star. We now consider three cases.

Case 1. $L^+(2) = \emptyset$.
Since $G$ is left-compressed, $L^+(i) = \emptyset$ for all $i \geq 2$. Hence $G \subseteq G_0(n) = \{1ij : 2 \leq i < j \leq n\}$.

Case 2. $L^+(2)$ is a triangle.
Since $G$ is left-compressed, we have $L^+(2) = \{34, 35, 45\}$ and $L^+(1) \supseteq L^+(2)$. Since $G$ contains no 2-matching, we must have $L^+(1) = L^+(2) = \{34, 35, 45\}$. Hence

\[ G \subseteq G_1(n) = \{12i : 3 \leq i \leq n\} \cup \{134, 135, 145, 234, 235, 245\}. \]

Case 3. $L^+(2)$ is a star.
Since $G$ is left-compressed, we have $L^+(2) = \{34, 35, \ldots, 3p\}$ for some $4 \leq p \leq n$. Since $G$ contains no 2-matching, every member of $L(1 \setminus 2)$ must contain either 3 or 4. Further, if $p \geq 6$ then every member of $L(1 \setminus 2)$ must contain 3.
If $p = 4$, then
\[ G \subseteq G_2(n) = \binom{[4]}{3} \cup \{12i, 13i, 14i : 5 \leq i \leq n\}. \]

If $p = 5$, then
\[ G \subseteq G_3(n) = \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{234, 235, 145\}. \]

If $p \geq 6$, then
\[ G \subseteq G_4(n) = \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{23i : 4 \leq i \leq n\}. \]

Let us recall the definition of the $b$-bounded Lagrangian $\lambda_b(G)$ of $G$, given in (3).

**Lemma 4.11** Let $b$ be a real with $0 < b \leq \frac{1}{3}$. Let $G$ be a 3-uniform star. Then $\lambda_b(G) \leq \frac{1}{2}b(1-b)^2$.

**Proof.** Suppose $V(G) = [n]$. Without loss of generality suppose vertex 1 is the center of the star. Let $\bar{x}$ be a $b$-bounded feasible vector on $G$ with $\lambda(G, \bar{x}) = \lambda_b(G)$. Let $a = x_1$. Then $a \leq b$. Note that $(\frac{1}{1-b}, \dots, \frac{1}{1-b})$ is a feasible weight vector on $L_G(1)$. By Theorem 2.10 $\lambda(G, \bar{x}) \leq a \frac{1}{2}(1-a)^2 \leq \frac{1}{2}b(1-b)^2$, where the last inequality follows from the fact that the function $\frac{1}{2}x(1-x)^2$ increases on $[0, \frac{1}{3}]$ and that $0 < a \leq b \leq \frac{1}{3}$.

**Lemma 4.12** Let $G$ be an $M^3_2$-free 3-graph. For $0 < b \leq \frac{1}{5}$, we have
\[ \lambda_b(G) \leq \max\{\frac{1}{2}b(1-b)^2, b^2 + 4b^3\}. \]

Furthermore, if $0 < b \leq \frac{1}{5}$ then $\lambda_b(G) \leq \frac{1}{4}b(1-b)^2$.

**Proof.** Suppose $V(G) = [n]$. By Lemma 4.11, we may assume that $G$ has an optimum $b$-bounded weight vector $\bar{x}$ such that $G$ is $\bar{x}$-compressed, all vertices of $G$ have positive weights under $\bar{x}$, and such that all pairs of vertices of weight less than $b$ are covered in $G$. By relabeling the vertices of $G$ if needed we may assume that $x_1 \geq \ldots \geq x_n$ and that $G$ is left-compressed relative to the natural order on $[n]$.

**Case 1.** $x_{n-1} < b$.

In this case we have $x_{n-1}, x_n < b$. By our assumption, $\{n-1, n\}$ is covered in $G$. Since $G$ is left-compressed, this implies that $\forall 2 \leq i < j \leq n, 1ij \in G$. If there is an edge of $G'$ in $\{2, \ldots, n\}$ then since $G$ is left-compressed, we have $234 \in G$. But then $234, 156$ forms a $M^3_2$ in $G'$, contradiction. Hence $G' \subseteq G_0(n) = \{1ij : 2 \leq i < j \leq n\}$. By Lemma 4.11 $\lambda_b(G') \leq \lambda_b(G_0(n)) \leq \frac{1}{2}b(1-b)^2$.

**Case 2.** $x_{n-1} = b$.

In this case we have $x_1 = x_2 = \cdots = x_{n-1} = b$, $x_n \leq b$. By Lemma 4.10 $G' \subseteq G_i$ for some $i = 0, 1, 2, 3, 4$. Since $\lambda_b(G_0(n)) \leq \frac{1}{2}b(1-b)^2$, we may assume that $G' \subseteq G_i(n)$ for some $i \in [4]$. Since $G_1(n) = \{12i : 3 \leq i \leq n\} \cup \{134, 135, 145, 234, 235, 245\}$, \[
\lambda(G_1(n), \bar{x}) \leq 6b^3 + b^2(1-2b) = b^2 + 4b^3.
\]
Since $G_2(n) = \left(\left\lfloor \frac{n}{3} \right\rfloor \right) \cup \{12i, 13i, 14i : 5 \leq i \leq n\}$,
\[\lambda(G_2(n), \bar{x}) \leq 4b^3 + 3b^2(1 - 4b) = 3b^2 - 8b^3.\]

Since $G_3(n) = \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{234, 235, 145\}$,
\[\lambda(G_3(n), \bar{x}) \leq b^2(1 - 2b) + b^2(1 - 3b) + 3b^3 = 2b^2 - 2b^3.\]

Since $G_4(n) = \{12i : 3 \leq i \leq n\} \cup \{13i : 4 \leq i \leq n\} \cup \{23i : 4 \leq i \leq n\}$,
\[\lambda(G_4(n), \bar{x}) \leq b^3 + 3b^2(1 - 3b) = 3b^2 - 8b^3.\]

So
\[\lambda(G', \bar{x}) \leq \max\left\{\frac{1}{2}b(1 - b)^2, b^2 + 4b^3, 3b^2 - 8b^3, 2b^2 - 2b^3\right\} = \max\left\{\frac{1}{2}b(1 - b)^2, b^2 + 4b^3, 3b^2 - 8b^3\right\}.
\]

Note that $\frac{1}{2}b(1 - b)^2 - (3b^2 - 8b^3) \geq 0$ on $[0, \infty)$. Also, $\frac{1}{2}b(1 - b)^2 - (b^2 + 4b^3) \geq 0$ on $[0, \frac{1}{3}]$. The conclusion follows.

Next, we establish an upper bound on $\lambda_0(G)$ for $M_3^t$-free graphs $G$, where $t \geq 3$. We need the following lemma of Frankl.

**Lemma 4.13** [3] If $G$ is an $n$-vertex $r$-graph with matching number $s$ then $|G| \leq s\left(\frac{n}{r-1}\right)^\frac{1}{2}$. ■

**Lemma 4.14** Let $n, r, t$ be positive integers, where $r, t \geq 2$, $n \geq tr$. Let $G$ be an $M_r^t$-free graph on $[n]$ that is left-compressed relative to the natural order. Then $L_G(n)$ is $M_r^{t-1}$-free. Furthermore, if $r = 3$ and $\{n - 1, n\}$ is covered then $G[\{2, \ldots, n\}]$ is $M_3^{t-1}$-free.

**Proof.** Suppose for contradiction that $M = \{f_1, \ldots, f_t\}$ is a $t$-matching in $L_G(n)$. Together they cover $t(r - 1)$ vertices in $[n - 1]$. Since $n \geq tr$, there exist distinct vertices $v_1, \ldots, v_{t-1} \in [n - 1]$ that are not covered by $M$. Since $G$ is left-compressed, $f_1 \cup \{v_1\}, \ldots, f_{t-1} \cup \{v_{t-1}\} \in G$, which together with $f_t \cup \{n\}$, form a $t$-matching in $G$, a contradiction.

Next, suppose $r = 3$ and $\{n - 1, n\}$ is covered. Since $G$ is left-compressed we have $\forall 2 \leq i < j \leq n, 1ij \in G$. Suppose $G[\{2, \ldots, n\}]$ contains $(t - 1)$-matching $M$. Then since $n \geq 3t$, $[n \setminus \{1\}$ contains two vertices $j, \ell$ not covered by $M$. Now, $M \cup \{1j\ell\}$ is a $t$-matching in $G$, a contradiction. ■

**Lemma 4.15** Let $t \geq 3$. Let $G$ be an $M_3^t$-free 3-graph. Let $0 < b < \frac{1}{3t-1}$. Let $\bar{x}$ be a $b$-bounded feasible weight vector on $G$ such that all but one of the components of $\bar{x}$ are $b$. Then
\[\lambda(G, \bar{x}) \leq \frac{t - 1}{2}b(1 - 3b + 4b^2).\]

**Proof.** Suppose $V(G) = [n]$. Note that $n \geq 3t$. By Lemma 2.9 (a), we may assume that $G$ is $\bar{x}$-compressed. By relabeling the vertices of $G$ if needed we may assume that $x_1 \geq \ldots \geq x_n$ and that $G$ is left-compressed relative to the natural order on $[n]$. By our assumption, $x_1 = \cdots = x_{n-1} = b$. Suppose $x_n = \alpha b$, where $0 < \alpha \leq 1$. By Lemma 4.14 $L(n)$ is $M_3^t$-free. Hence by Lemma 4.13 $|L(n)| \leq (t - 1)(n - 1)$. Let $G'$ denote the set of edges of $G$ not containing $n$. Since $G'$ is $M_3^t$-free, by Lemma 4.13 $|G'| \leq (t - 1)\left(\frac{n}{2}\right)$. Hence the contribution to $\lambda(G, \bar{x})$ of edges in $G$ containing $n$ or not
containing \(n\) are at most \((t - 1)(n - 1)b^2 \cdot \alpha b\) and \((t - 1)\left(\frac{n - 2}{2}\right)b^3\) respectively. Note that \((n - 1)b + \alpha b = 1\). Also, on \([0, 1]\) we have \(\alpha^2 - 3\alpha + \frac{1}{4} \geq -\frac{7}{4}\). Hence

\[
\lambda(G, \bar{x}) \leq (t - 1)\left(\frac{n - 2}{2}\right)b^3 + (t - 1)\alpha(n - 1)b^3
\]

\[
= \frac{t - 1}{2}(n^2 - 5n + 6 + 2\alpha n - 2\alpha)b^3
\]

\[
= \frac{t - 1}{2}\left(n - \frac{5}{2} + \alpha\right)^2 - \left(\frac{1}{4} - 3\alpha + \alpha^2\right)b^3
\]

\[
= \frac{t - 1}{2}\left(1 - \frac{3}{2}\right)b^2 - \left(\frac{1}{4} - 3\alpha + \alpha^2\right)b^3
\]

\[
\leq \frac{t - 1}{2}\left(1 - \frac{3}{2}\right)b^2 + \frac{7}{4}b^3
\]

\[
= \frac{t - 1}{2}b\left(1 - 3b + 4b^2\right).
\]

\[
\lambda(G, \bar{x}) \leq \frac{t - 1}{2}b(1 - 3b + 6b^2).
\]

**Lemma 4.16** Let \(t \geq 3\) be an integer and \(b\) a real with \(0 < b < \frac{1}{3t - 1}\). Let \(G\) be an \(M_1^3\)-free 3-graph with \(n \geq 3t\) vertices. Then

\[
\lambda_b(G) \leq \frac{t - 1}{2}b(1 - 3b + 6b^2).
\]

**Proof.** Suppose \(V(G) = [n]\). If no \(b\)-bounded feasible weight vector exists, then \(\lambda_b(G) = 0\) by definition and the claim holds trivially. So assume that there exist \(b\)-bounded feasible weight vectors. By Lemma 4.14 we may assume that \(G\) has an optimum \(b\)-bounded weight vector \(\bar{x}\) such that \(G\) is \(\bar{x}\)-compressed, all vertices of \(G\) have positive weights under \(\bar{x}\), and such that all pairs of vertices of weight less than \(b\) are covered in \(G\). By relabeling the vertices of \(G\) if needed we may assume that \(x_1 \geq \ldots \geq x_n > 0\) and that \(G\) is left-compressed relative to the natural order on \([n]\).

We use induction on \(t\). For the basis step, let \(t = 3\). If \(x_{n-1} = b\), then by Lemma 4.15,

\[
\lambda(G, \bar{x}) \leq b(1 - 3b + 4b^2) \leq b(1 - 3b + 6b^2).
\]

Hence, we may assume that \(x_{n-1}, x_n < b\). By our assumption, \([n - 1, n]\) is covered in \(G\). Since \(G\) is left-compressed, we have \(\forall 2 \leq i < j \leq n, \forall i < j \in G\). Let \(G' = G[\{2, \ldots, n\}]\). By Lemma 4.14 \(G'\) is \(M_2^3\)-free. Since \(x_2 + \ldots + x_n = 1 - x_1 = 1 - b\), \(\bar{y} = \frac{1}{1 - b}(x_2, \ldots, x_n)\) is a \((\frac{1}{1 - b})\)-bounded feasible weight vector on \(G'\). Let \(b' = \frac{b}{1 - b}\). Since \(b \leq \frac{1}{3t - 1}\), \(b' = \frac{b}{1 - b} \leq \frac{b}{3t - 1}\). Since \(G'\) is \(M_2^3\)-free, and \(\bar{y}\) is a \(b'\)-bounded feasible weight vector on \(G'\), by Lemma 4.12

\[
\lambda(G', \bar{x}) = (1 - b)^3\lambda(G', \bar{y}) \leq (1 - b)^3 \cdot \frac{1}{2}b'(1 - b')^2 = \frac{1}{2}(1 - b)^3 \frac{b}{1 - b} \left(\frac{1 - 2b}{1 - b}\right)^2 = \frac{1}{2}b(1 - 2b)^2.
\]

Since the total contribution to \(\lambda(G, \bar{x})\) from the edges containing 1 is at most \(\frac{1}{2}b(1 - b)^2\), we have

\[
\lambda(G, \bar{x}) \leq \frac{1}{2}b(1 - b)^2 + \frac{1}{2}b(1 - 2b)^2 = \frac{1}{2}b(2 - 6b + 5b^2) < b(1 - 3b + 6b^2).
\]

Hence the claim holds. For the induction step, let \(t \geq 4\). As before, if \(x_{n-1} = b\), then by Lemma 4.15

\[
\lambda(G, \bar{x}) \leq \frac{t - 1}{2}b(1 - 3b + 4b^2) \leq \frac{t - 1}{2}b(1 - 3b + 6b^2).
\]

18
Hence, we may assume that $x_{n-1}, x_n < b$. By our assumption, \( \{n-1, n\} \) is covered in $G$. Since $G$ is left-compressed we have $\forall 2 \leq i < j \leq n, 1ij \in G$. By Lemma 4.14 $G' = G'[\{2, \ldots, n\}]$ is $M_{t-1}^3$-free. Since $\bar{y} = \frac{1}{1-a}(x_2, \ldots, x_n)$ is a $(\frac{b}{1-a})$-bounded feasible weight vector on $G'$, by induction hypothesis,

$$\lambda(G', \bar{x}) = (1 - b)^3 \lambda(G', \bar{y}) \leq (1 - b)^3 \frac{t - 2}{2} \frac{b}{1 - b} \left(1 - 3 \frac{b}{1 - b} + 6 \left(\frac{b}{1 - b}\right)^2 \right) = \frac{t - 2}{2} b(1 - 5b + 10b^2).$$

Since the total contribution to $\lambda(G, \bar{x})$ from the edges containing 1 is at most $\frac{1}{2} b(1 - b)^2$, we have

$$\lambda(G, \bar{x}) \leq \frac{1}{2} b(1 - b)^2 + \frac{t - 2}{2} b(1 - 5b + 10b^2).$$

$$= \frac{1}{2} b[(t - 1) - (5t - 8)b + (10t - 19)b^2]$$

$$< \frac{t - 1}{2} b(1 - 3b + 6b^2),$$

where the last inequality can be verified using the condition that $0 < b \leq \frac{1}{3t - 1}$ and $t \geq 4$.

**Theorem 4.17** Let $t \geq 2$ be an integer. There exists a positive real $c_3 = c_3(t)$ such that the following holds. If $G$ is an $L_1^t$-free 4-graph then $\lambda(G) \leq \lambda(K_{3t}^4) = \frac{(3t - 1)(3t - 2)(3t - 3)}{24(3t - 1)^3}$. Furthermore, if $G$ also covers pairs and $G \neq K_{3t}^4$, then $\lambda(G) \leq \lambda(K_{3t}^4) - c_3$.

**Proof.** Since we may consider a dense subgraph covering pairs, it suffices to prove the second statement. Suppose that $G$ is on $[n]$. If $n \leq 3t$, then the result holds obviously since $G \neq K_{3t}^4$. Now suppose that $n \geq 3t + 1$. Let $\bar{x}$ be an optimum weight vector on $G$. Without loss of generality, suppose that $x_1 = \max\{x_i : i \in [n]\}$. Let $a = x_1$. By Fact 2.14 we have $\lambda(G) = \frac{1}{4} \frac{\partial \lambda}{\partial x_1}$. So it suffices to prove that

$$\frac{\partial \lambda}{\partial x_1} \leq \frac{(3t - 1)(3t - 2)(3t - 3)}{6(3t - 1)^3} - c_3$$

for some positive real $c_3 = c_3(t)$. Since $G$ is $L_1^t$-free, $L(1)$ is an $M_t^3$-free 3-graph. Since $G$ covers pairs, $L(1)$ is a 3-graph on $[n] \setminus \{1\}$ that contains no isolated vertex. Since $G$ covers pairs and $n \geq 3t + 1$, by Lemma 4.2 $K_{3t - 1}^3 \not\subseteq L(1)$. Let $\bar{y} = \frac{1}{1-a}(x_2, \ldots, x_n)$. Then $\bar{y}$ is an $(\frac{a}{1-a})$-bounded feasible weight vector on $L(1)$. We consider two cases.

**Case 1.** $a \geq \frac{1}{3t}$.

Since $L(1)$ is $M_t^3$-free, $L(1) \neq K_{3t-1}^3$ and has no isolated vertex, by Theorem 3.3

$$\lambda(L(1), \bar{y}) \leq \lambda(K_{3t-1}^3) - c_1 = \frac{(3t - 1)(3t - 2)(3t - 3)}{6(3t - 1)^3} - c_1.$$

Hence the claim holds by setting $c_3 = c_1$.

**Case 2.** $a < \frac{1}{3t}$.

Let $b = \frac{a}{1-a}$. Then $b < \frac{1}{3t - 1}$. By Lemma 4.10 we have

$$\frac{\partial \lambda}{\partial x_1} = (1 - a)^3 \lambda(L(1), \bar{y}) \leq (1 - a)^3 \frac{t - 1}{2} b(1 - 3b + 6b^2).$$

Substituting $b = \frac{a}{1-a}$ and simplifying we get

$$\frac{\partial \lambda}{\partial x_1} \leq (t - 1)a(5a^2 - \frac{5}{2}a + \frac{1}{2}).$$
Let \( f(a) = 5a^3 - \frac{1}{2}a^2 + \frac{1}{2}a \). Note that \( f'(a) > 0 \) always. So \( f(a) \) is increasing. Since \( a < \frac{1}{3t} \), we have
\[
\frac{\partial \lambda}{\partial x_1} \leq (t-1)f\left(\frac{1}{3t}\right) = \frac{(t-1)(9t^2 - 15t + 10)}{2(3t)^3} < \frac{(t-1)(3t-1)(3t-2)}{2 \cdot (3t)^3} - c_3 = \frac{(3t-1)(3t-2)(3t-3)}{6(3t)^3} - c_3,
\]
for \( t \geq 2 \) and sufficiently small positive real \( c_3 = c_3(t) \).

**Corollary 4.18** \( \pi_\lambda(L_4^4) = 3! \lambda(K_{3,3}^4) = \frac{303}{20} \).

By Theorem 4.17, Corollary 4.18, Corollary 3.10, and Theorem 3.7, we get the following result.

**Theorem 4.19** Let \( t \geq 2 \) be an integer. Then \( \text{ex}(n, H_{3t}^{4,4}) = t_{3t}^4(n) \) for sufficiently large \( n \). Moreover, if \( n \) is sufficiently large and \( G \) is an \( H_{3t}^{4,4} \)-free 4-graph on \( n \) vertices with \( |G| = t_{3t}^4(n) \) edges, then \( G = T_{3t}^4(n) \).

5 Concluding remarks

Another natural way to extend the Hefetz-Keevash result in [7] is to establish the maximum Lagrangian of an \( r \)-uniform intersecting family for \( r \geq 4 \), i.e. to determine the Lagrangian density of \( M_2^r \), for \( r \geq 4 \). The situation there is quite different from the \( r = 3 \) case. Hefetz and Keevash [7] conjectured that the maximum Lagrangian of an \( r \)-uniform intersecting family is achieved by a feasible weight vector on the star \( \{1ij : 2 \leq i < j \leq n \} \). This conjecture was recently confirmed for all \( r \geq 4 \) by Norin, Watts, and Yepremyan [13], who determined the Lagrangian density of \( M_2^r \) as well as the stability of the related Turán problem. For the stability part of their result, see also [22]. Independently, Wu, Peng, and Chen [21] had also confirmed the Hefetz-Keevash conjecture for \( r = 4 \).

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