Some pricing tools for the Variance Gamma model

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We establish several closed pricing formula for various path-independent payoffs, under an exponential Lévy model driven by the Variance Gamma process. These formulas take the form of quickly convergent series and are obtained via tools from Mellin transform theory as well as from multidimensional complex analysis. Particular focus is made on the symmetric process, but extension to the asymmetric process is also provided. Speed of convergence and comparison with numerical methods are also discussed; notable feature is the accelerated convergence of the series for short term options, which constitutes an interesting improvement of numerical Fourier inversion techniques.

Keywords: Lévy process; Variance Gamma Process; Stochastic Volatility; Option Pricing.

1. Introduction

Lévy processes (Bertoin 1996) are well-known in Financial Modeling for their ability to reproduce realistic features of asset return distributions, such as skewness or leptokurtosis (Cont & Tankov 2004), that Gaussian models fail to describe. However, being stationary processes, the implied volatility surfaces they generate remain constant over time (as functions of moneyness and time horizon); this is in contradiction with the widely observed fact that volatility appears to be non constant, notably because large changes in asset prices tend to cluster together (Cont 2007), resulting in alternating periods of high and low variance.

In order to model this complex volatility behavior, two approaches have been introduced. The first one is to describe volatility by a positive stochastic process; such models can be either time-continuous (Heston 1993), or discrete (Heston & Nandi 2000). The second approach is to introduce a time change in the Lévy process describing asset prices (Barndorff-Nielsen et al. 2002, Carr et al. 2003); the stochastic process governing the evolution of time is called "stochastic clock" (Geman 2009) or "business time", while the Lévy process driving the asset’s returns is said to evolve in "operational time". A nice feature of this second category is its direct interpretation, the introduction of a random clock traducing the fact that trading activities are not uniform in time, but on the contrary display an alternation of peak and less busy periods; as a (drifted) Brownian motion whose time follows a Gamma process, the Variance Gamma process (Madan et al. 1998) belongs to this category.

For the purpose of derivatives pricing, the Variance Gamma process is typically implemented within the framework of exponential Lévy models (Tankov 2010), and we will therefore speak of the Variance Gamma model; such a model has been successfully tested on real market data and be shown to perform better than Black-Scholes or Jump-Diffusion models in multiple situations.
Pricing contingent claims in the Variance Gamma model, however, remains a complicated task, which can be a limitation to its popularity. Some exact formulas have been derived in particular cases (a Black-Scholes like formula for the European call involving products of Bessel and hypergeometric functions in (Madan et al. 1998), or generalized hyperbolic functions in (Ivanov 2018)) but, in practice, numerical methods are favored. Particularly popular are methods based on evaluation of Fourier integrals (see the classical references (Carr & Madan 1999, Lewis 2001)), because the characteristic function of the Variance Gamma process is known exactly, and admits a relatively simple form; other recent techniques include Fourier-related transforms for European options (Fang & Oosterlee 2008), local basis frame projection method for both vanilla (European and American) and exotic options (Kirkby 2015, 2018), or space-time discretization of the time process (Cantarutti & Guerra 2018).

In this article, we would like to show that it is possible to obtain tractable closed pricing formulas for European or digital prices, but also for more exotic payoffs (gap, power, log options) thanks to a remarkable factorization property in the Mellin space; option prices take the form of quickly convergent series whose terms resume to powers of the model’s parameters, and are straightforward to compute. We will particularly focus on the symmetric model (i.e., when the price process is a Brownian motion without drift), but will also show how to extend the technique to a drifted process. The technology, based on the Mellin factorization property and on residue calculation, has been previously implemented in exponential Lévy models driven by spectrally negative processes (Black-Scholes or Finite Moment Log Stable models) in (Aguilar & Korbel 2019, Aguilar 2019); in the present article, we will demonstrate that the technique is also well suited to two-sided processes. In particular, we will see that the obtained series converge extremely fast for very short maturities, which constitutes an advantage over numerical tools (as noticed in (Carr & Madan 1999), very short maturities create highly oscillatory integrands which considerably slow down the numerical Fourier inversion process).

The paper is organized as follows: in section 2 we start by recalling fundamental concepts on the Variance Gamma process, and on option pricing in exponential Lévy models; then, in section 3 we establish a generic pricing formula in the case of a symmetric process, which allows to determine closed-form series formulas for the price of several options. In section 4 we extend this pricing formula to the case of an asymmetric process, apply it to the case of a digital option, and compare our results with numerical techniques (in both the symmetric and asymmetric cases). Last, section 5 is dedicated to concluding remarks and discussion of future work.

2. Model definition

We start by recalling some basic facts about the Variance-Gamma process, and how it is used for option pricing within the framework of exponential Lévy models.

2.1. The Variance Gamma process

The Variance Gamma (VG) process is a pure jump Lévy process with finite variation and infinite activity; it can be defined either as a time-changed Brownian motion, a difference of two Gamma processes, or a particular case of a tempered stable Lévy process. In this subsection we briefly
recall, without proofs, the main features of the VG process as seen from these different point of views; detailed proofs and calculations can be found e.g. in the original paper by Madan, Carr and Chang [Madan et al. 1998].

**Gamma subordination.** In this approach, the VG process is obtained by evaluating a drifted Brownian motion at a random time following a Gamma process (see Bertoin 1999, Geman et al. 2001) for general results on Gamma subordination and time changes in Lévy processes. More precisely, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space, let $\gamma(t, \mu, \nu)$ be a Gamma process (i.e. a process whose increments $\gamma(t + h, \mu, \nu) - \gamma(t, \mu, \nu)$ follow a Gamma distribution with mean $\mu t$ and variance $\nu t$), and let $B(t)$ be a drifted Brownian motion:

$$B(t, \theta, \sigma) := \theta t + \sigma W(t) , \quad W(t) \sim N(0, t) \quad (1)$$

Then, the VG process is the stochastic process defined by:

$$X(t, \sigma, \nu, \theta) := B(\gamma(t, 1, \nu), \theta, \sigma) \quad (2)$$

$\sigma$ is the *scale parameter*, $\nu$ the *kurtosis parameter* and $\theta$ is the *asymmetry parameter*; the density function of the process is obtained by integrating the normal density conditionally to the Gamma time-change:

$$f_{\sigma, \nu, \theta}(x, t) = \int_0^\infty \frac{1}{\sigma \sqrt{2\pi y}} e^{-\frac{(x-\theta y)^2}{2\sigma^2 y}} \frac{y^{\nu-1}}{\nu^\nu \Gamma(\frac{\nu}{2})} e^{-\frac{y}{\nu}} dy \quad (3)$$

The density (3) can be expressed in terms of special functions as:

$$f_{\sigma, \nu, \theta}(x, t) = \frac{2e^{\theta x}}{\nu^{\frac{\nu}{2}} \sqrt{2\pi \sigma \Gamma(\frac{\nu}{2})}} \left( \frac{x^2}{2\sigma^2 + \theta^2} \right)^\frac{\nu}{2} \frac{1}{\nu} \frac{1}{\sqrt{\nu}} \left( \frac{1}{\sigma^2} \sqrt{\frac{2\sigma^2}{\nu} + \theta^2} \right) K_{\frac{\nu}{2}} \left( \frac{x}{\nu} \right) \quad (4)$$

where $K_{\alpha}(X)$ denotes the modified Bessel function of the second kind [Abramowitz & Stegun 1972]. The characteristic function admits a simple representation in the Fourier space:

$$\Phi_{\sigma, \nu, \theta}(u, t) := E^P \left[ e^{iuX(t, \sigma, \nu, \theta)} \right] = \left( \frac{1}{1 - i\theta \nu u + \frac{\sigma^2}{2} u^2} \right)^\frac{\nu}{2} \quad (5)$$

and, if we define the characteristic exponent by:

$$\Psi_{\sigma, \nu, \theta}(u) := -\log E^P \left[ e^{iuX(1, \sigma, \nu, \theta)} \right] = \frac{1}{\nu} \log \left( 1 - i\theta \nu u + \frac{\sigma^2}{2} u^2 \right) \quad (6)$$

then the following holds true:

$$\Phi_{\sigma, \nu, \theta}(u, t) = e^{-t\Psi_{\sigma, \nu, \theta}(u)} \quad (7)$$

**Decomposition.** As a consequence of the finite variations property, the VG process can be written down as a difference of two Gamma processes:

$$X(t, \sigma, \nu, \theta) = \gamma^+(t, \mu_+, \nu_) - \gamma^-(t, \mu_-, \nu_-) \quad (8)$$
where the parameters are defined by

\[
\begin{align*}
\mu_+ &:= \frac{1}{2} \sqrt{\theta^2 + 2\sigma^2} + \frac{\theta}{2} \\
\nu_+ &:= \mu_+^2 \\
\mu_- &:= \frac{1}{2} \sqrt{\theta^2 + 2\sigma^2} - \frac{\theta}{2} \\
\nu_- &:= \mu_-^2 
\end{align*}
\]  
(9)

**Lévy measure.** The VG process is a Lévy process without Brownian component nor additional drift, i.e., the Lévy-Khintchine representation for the characteristic exponent (6) is:

\[
\Psi_{\sigma,\nu,\theta}(k) = \int_{\mathbb{R}\setminus\{0\}} (1 - e^{ikx}) \Pi_{\sigma,\nu,\theta}(dx)
\]  
(10)

where the Lévy measure can be expressed as:

\[
\Pi_{\sigma,\nu,\theta}(dx) = \frac{1}{\nu} \left( e^{-\frac{1}{\nu^2} |x|} \mathbb{1}_{\{x<0\}} + e^{-\frac{1}{\nu^2} \frac{\theta}{\sigma^2} |x|} \mathbb{1}_{\{x>0\}} \right) dx
\]  
(11)

The representation (11) shows that the VG process is actually a particular case of a CGMY process (Carr et al. 2002), with overall activity level \(C = \frac{1}{\nu}\), skewness parameters \(G = \frac{1}{\mu - \nu}\) and \(M = \frac{1}{\mu + \nu}\), and fine structure parameter \(Y = 1\). Note that, using the transformation (9), the Lévy measure (11) can be re-written as:

\[
\Pi_{\sigma,\nu,\theta}(dx) = \frac{e^{\frac{\theta}{\sigma^2} |x|}}{\nu |x|} e^{-\sqrt{\frac{\theta^2}{\sigma^2} + 2} \frac{\nu}{\sigma^2} |x|} dx
\]  
(12)

and is symmetric around the origin when \(\theta = 0\).

### 2.2. Exponential VG model and option pricing

**Model specification.** Let \(T > 0\), and let \(S_t\) denote the value of a financial asset at time \(t \in [0, T]\); we assume that it can be modeled as the realization of a stochastic process \(\{S_t\}_{t \geq 0}\) on the canonical space \(\Omega = \mathbb{R}_+\) equipped with its natural filtration, and that, under the risk-neutral measure \(Q\), its instantaneous variations can be written down in local form as:

\[
\frac{dS_t}{S_t} = (r - q) dt + dX(t, \sigma, \nu, \theta)
\]  
(13)

In the stochastic differential equation (13), \(r \in \mathbb{R}\) is the continuously compounded risk-free interest rate and \(q \in \mathbb{R}\) is the dividend yield, both assumed to be deterministic, and \(\{X(t, \sigma, \nu, \theta)\}_{t \geq 0}\) is a VG process; for the simplicity of notations, we will assume that \(q = 0\), but all the results of the paper remain valid when replacing \(r\) by \(r - q\).

The solution to (13) is the exponential process defined by:

\[
S_T = S_t e^{(r + \omega_{\sigma,\nu,\theta}) \tau + X_{\tau}}
\]  
(14)

where \(\tau := T - t\) is the horizon (or time-to-maturity), and \(\omega\) is a convexity adjustment computed in a way that the discounted stock price is a \(Q\)-martingale, which resumes to the condition:

\[
\mathbb{E}^Q[e^{\omega_{\sigma,\nu,\theta} \tau + X_{\tau}}] = 1
\]  
(15)
or, equivalently, in terms of the characteristic exponent (16):

\[
\omega_{\sigma,\nu,\theta} = \Psi_{\sigma,\nu,\theta}(-i) = \frac{1}{\nu} \log \left( 1 - \theta \nu - \frac{\sigma^2 \nu}{2} \right)
\]  

(16)

**Option pricing.** Let \( N \in \mathbb{N} \) and \( \mathcal{P} : \mathbb{R}_{+}^{1+N} \to \mathbb{R} \) be a non time-dependent payoff function depending on the terminal price \( S_T \) and on some positive parameters \( K_n, n = 1 \ldots N \):

\[
\mathcal{P} : (S_T, K_1, \ldots, K_N) \to \mathcal{P}(S_T, K_1, \ldots, K_n) := \mathcal{P}(S_T, K)
\]  

(17)

The value at time \( t \) of a contingent claim \( C_{\sigma,\nu,\theta} \) with payoff \( \mathcal{P}(S_T, K) \) is equal to the risk-neutral conditional expectation of the discounted payoff:

\[
C(S_t, K, r, t, T, \sigma, \nu, \theta) = \mathbb{E}_t^Q \left[ e^{-r(T-t)} \mathcal{P}(S_T, K) \right]
\]  

(18)

As the VG process admits Q-density \( f_{\sigma,\nu,\theta}(x, t) \), then, using (14), we can re-write (18) by integrating all possible realizations for the terminal payoff over the martingale measure:

\[
C(S_t, K, r, \tau, \sigma, \nu, \theta) = e^{-r \tau} \int_{-\infty}^{+\infty} \mathcal{P} \left( S_t e^{(r+\omega_{\sigma,\nu,\theta}) \tau + x} K \right) f_{\sigma,\nu,\theta}(x, \tau) \, dx
\]  

(19)

In all the following and to simplify the notations, we will forget the \( t \) dependence in the stock price \( S_t \).

### 3. Symmetric Variance Gamma process

One speaks of a symmetric Variance Gamma process when \( \theta = 0 \), that is, when the process corresponds to a time-changed Brownian motion without drift; it was first considered in (Madan & Seneta 1990). In this section we start by showing that, under symmetric VG process, the price of a contingent claim admit a factorized representation in the Mellin space; then, we apply this pricing formula to the valuation of various payoffs.

#### 3.1. Pricing formula

Let \( \sigma, \nu > 0 \) and let us denote the density of the symmetric VG process by \( f_{\sigma,\nu}(x, t) := f_{\sigma,\nu,0}(x, t) \).

**Lemma 3.1** Let \( F_\nu^s \) be the meromorphic function on \( \mathbb{C} \) defined by

\[
F_\nu^s(s_1) = \Gamma \left( \frac{s_1 - \frac{t}{\nu} + \frac{1}{2}}{2} \right) \Gamma \left( \frac{s_1 + \frac{t}{\nu} - \frac{1}{2}}{2} \right)
\]  

Then, for any \( c_1 > |\frac{t}{\nu} - \frac{1}{2}| \), the following Mellin-Barnes representation holds true:

\[
f_{\sigma,\nu}(x, t) = \frac{1}{2\sqrt{\pi} \Gamma(\frac{1}{2})} \int_{c_1 - i\infty}^{c_1 + i\infty} F_\nu^s(s_1) (\sigma \sqrt{2\nu})^{s_1 - \frac{t}{2} - \frac{1}{2}} |x|^{-s_1 + \frac{t}{2} - \frac{1}{2}} \, ds_1
\]  

(21)
Proof. Taking $\theta = 0$ in (4) yields:

$$f_{\sigma,\nu}(x, t) = \frac{2}{\sqrt{2\pi}(\sigma^2\nu)^{\frac{1}{2}}} \left( \frac{|x|}{\frac{1}{\sigma}\sqrt{\nu}} \right)^{\frac{\nu}{2} - \frac{1}{2}} K_{\frac{\nu}{2}} \left( \frac{1}{\sigma}\sqrt{2\nu}|x| \right)$$

(22)

which, as expected, is symmetric around 0. The Mellin-Barnes representation (21) is easily obtained from the Mellin transform (Bateman 1954 p. 331):

$$K_{\alpha}(X) \rightarrow 2^{s_1-2}\Gamma\left(\frac{s_1-\alpha}{2}\right)\Gamma\left(\frac{s_1+\alpha}{2}\right), \quad X > 0$$

(23)

which converges for $\Re(s_1) > |\Re(\alpha)|$, and by applying the Mellin inversion formula (Flajolet et al. 1995).

Let us now introduce the Mellin-like transform of the payoff function:

$$P_{\sigma,\nu}^*(s_1) = \int_{-\infty}^{\infty} \mathcal{P}\left(Se^{(r+\omega_{\sigma,\nu})r+x,K} \right) |x|^{-s_1+\frac{1}{\nu}-\frac{1}{2}} dx$$

(24)

where the symmetric convexity adjustment resumes to:

$$\omega_{\sigma,\nu} := \omega_{\sigma,\nu,0} = \frac{1}{\nu} \log \left( 1 - \frac{\sigma^2\nu}{2} \right)$$

(25)

and let us assume that the integral (24) converges on some interval $s_1 \in (c_-,c_+)$, $c_- < c_+$. Then, as a direct consequence of the pricing formula (19) and of lemma 3.1, we have:

**Proposition 3.2 (Factorization in the Mellin space)** Let $c_1 \in (\bar{c}_-,\bar{c}_+)$ where $(\bar{c}_-,\bar{c}_+) := (c_-,c_+) \cap (-\infty,-1)$ is assumed to be nonempty. Then the value at time $t$ of a contingent claim $\mathcal{C}$ with maturity $T$ and payoff $\mathcal{P}(S_T,K)$ is equal to:

$$\mathcal{C}(S,K,r,\tau,\sigma,\nu) = \frac{e^{-rt}}{2\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)} \int_{c_1-i\infty}^{c_1+i\infty} P_{\sigma,\nu}^*(s_1) F_{\nu}^*(s_1) \left( \sigma\sqrt{2\nu} \right)^{s_1-\frac{1}{2}-\frac{1}{2}} \frac{ds_1}{2i\pi}$$

(26)

The factorized form (26) turns out to be a very practical tool for option pricing. Indeed, as an integral along a vertical line in the complex plane, it can be conveniently expressed as a sum of residues associated to the singularities of the integrand, i.e., schematically:

$$\frac{e^{-rt}}{2\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)} \times \sum \text{[residues of } P_{\sigma,\nu}^*(s_1) F_{\nu}^*(s_1) \times \text{powers of } \sigma\sqrt{2\nu}]$$

(27)

However, depending on the payoff’s complexity, it can be necessary to introduce a second Mellin variable in order to evaluate $P_{\sigma,\nu}^*(s_1)$. In this case, it is possible to express the arising multiple complex integral as a sum of multidimensional residues, in virtue of a multidimensional generalization of the Jordan lemma which goes as follow. Let the $\mathbf{a}_k$, $\mathbf{b}_j$ be vectors in $\mathbb{C}^n$, let the $b_k$, $\tilde{b}_j$ be complex numbers and let "," represent the euclidean scalar product. Assume that we are interested
in evaluating an integral of the type
\[ \int_{\mathbb{C} + i\mathbb{R}^n} \omega \] (28)
where the vector \( \xi \) belongs to the region of convergence \( P \subset \mathbb{C}^n \) of the integral (28), and where \( \omega \) is a complex differential \( n \)-form reading
\[ \omega = \frac{\Gamma(a_1, x + b_1) \cdots \Gamma(a_m, x + b_m)}{\Gamma(\tilde{a}_1, x + \tilde{b}_1) \cdots \Gamma(\tilde{a}_l, x + \tilde{b}_l)} x_1^{-s_1} \cdots x_n^{-s_n} \frac{ds_1 \cdots ds_n}{(2i\pi)^n} \quad x_1, \ldots, x_n \in \mathbb{R} \] (29)
The singular sets \( D_k \) induced by the singularities of the Gamma functions
\[ D_k := \{ \xi \in \mathbb{C}^n, a_k, b_k = -n_k, n_k \in \mathbb{N} \} \quad k = 1 \ldots m \] (30)
are called the divisors of \( \omega \). The characteristic vector of \( \omega \) is defined to be
\[ \Delta = \sum_{k=1}^{m} a_k - \sum_{j=1}^{l} \tilde{a}_j \] (31)
and, for \( \Delta \neq 0 \), its admissible region is:
\[ \Pi_{\Delta} := \{ \xi \in \mathbb{C}^n, \text{Re}(\Delta, \xi) < \text{Re}(\Delta, \xi) \} \] (32)
As a consequence of the Stirling approximation at infinity for the Gamma function (see e.g. (Abramowitz & Stegun 1972)), \( \omega \) decays exponentially fast in \( \Pi_{\Delta} \) and
\[ \int_{\mathbb{C} + i\mathbb{R}^n} \omega = \sum_{\Pi} \text{Res} \omega \] (33)
for any cone \( \Pi \subset \Pi_{\Delta} \) whose faces are interested by exactly one of the divisors of \( \omega \); such a cone is said to be compatible with \( \omega \), and generalizes the notion of left or right half plane in \( \mathbb{C} \). If \( \Delta = 0 \), then (33) holds in every compatible cone. Readers interested in this very rich theory will find full details in (Passare et al. 1997) (see also (Zhdanov & Tsikh 1998) for more details in the case \( n = 2 \)).

3.2. European and digital prices
In all the following, the forward strike price \( F \) and the log-forward moneyness \( k \) are defined to be:
\[ F := Ke^{-r\tau} \quad k := \log \frac{S}{F} = \log \frac{S}{K} + r\tau \] (34)
The re-scaled volatility \( \sigma_\nu \) and the (risk-neutral) moneyness \( k_{\nu} \) are:
\[ \sigma_\nu := \sigma \sqrt{\frac{\nu}{2}} \quad k_{\sigma,\nu} := k + \omega_{\sigma,\nu}\tau \] (35)
Instead otherwise stated, $k_{\sigma,\nu}$ will be our standard measure of moneyness, that is:

$$\begin{cases} 
  k_{\sigma,\nu} < 0 : & \text{Out-of-the-money (OTM) call} \\
  k_{\sigma,\nu} = 0 : & \text{At-the-money (ATM) call} \\
  k_{\sigma,\nu} > 0 : & \text{In-the-money (ITM) call} 
\end{cases}$$

(36)

By an abuse of notations, we will denote the price of contingent claims either by $C(S, K, r, \tau, \sigma, \nu)$ or $C(k_{\sigma,\nu}, \sigma\nu)$. We will also denote $\alpha := \frac{\tau}{\nu} - \frac{1}{2}$ and assume $\alpha \notin \mathbb{Z}$ to avoid the degenerate case for the Bessel function. Last, we will use the standard notation $X^+ := X\mathbb{1}_{\{X > 0\}}$.

**Digital option (asset-or-nothing).** The asset-or-nothing call option is a simple exotic option consisting in receiving a unit of the underlying asset $S_{\tau}$, on the condition that $S_{\tau}$ is greater than a predetermined strike price $K$; the payoff is therefore:

$$\mathcal{P}_{a/n}(S_{\tau}, K) := S_{\tau}\mathbb{1}_{\{S_{\tau} > K\}}$$

(37)

**Formula 1 (Asset-or-nothing call)**

The value at time $t$ of an asset-or-nothing call option is:

(i) (OTM price) If $k_{\sigma,\nu} < 0$,

$$
C_{a/n}^-(k_{\sigma,\nu}, \sigma\nu) = \frac{F}{2\Gamma\left(\frac{\tau}{\nu}\right)} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \frac{\Gamma\left(-\frac{n_1+n_2+1}{2}\alpha\nu\right) - \frac{k_{\sigma,\nu}}{\sigma\nu}}{\Gamma\left(-\frac{n_1+n_2}{2}+1\right) \Gamma\left(-\frac{n_1}{2}\right) \Gamma\left(-\frac{n_2}{2}\right)} 
\left[\frac{\Gamma\left(-n_1-n_2-1-2\alpha\nu\right)}{\Gamma\left(-n_1+\frac{1}{2}-\alpha\nu\right)} \left(-\frac{k_{\sigma,\nu}}{\sigma\nu}\right)^{2n_1+1+2\alpha} \left(-k_{\sigma,\nu}\right)^{n_2}\right]
$$

(38)

(ii) (ITM price) If $k_{\sigma,\nu} > 0$,

$$
C_{a/n}^+(k_{\sigma,\nu}, \sigma\nu) = S - C_{a/n}^-(k_{\sigma,\nu}, -\sigma\nu)
$$

(39)

(iii) (ATM price) If $k_{\sigma,\nu} = 0$,

$$
C_{a/n}^-(k_{\sigma,\nu}, \sigma\nu) = C_{a/n}^+(k_{\sigma,\nu}, \sigma\nu) = \frac{F}{2\Gamma\left(\frac{\tau}{\nu}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{2}+\alpha\nu\right)}{\Gamma\left(\frac{n+1}{2}+1\right)} \sigma\nu^n
$$

(40)

**Proof.** To prove (i), we first note that, using notations (34), we have:

$$
\mathcal{P}_{a/n}(Se^{(r+\omega_{\sigma,\nu})\tau+x}, K) = Ke^{k_{\sigma,\nu}+x}\mathbb{1}_{\{x>k_{\sigma,\nu}\}}
$$

(41)

Using a Mellin-Barnes representation for the exponential term (see e.g. [Bateman 1954] p. 312):

$$
e^{k_{\sigma,\nu}+x} = \int_{c_2-i\infty}^{c_2+i\infty} (-1)^{-s_2}\Gamma(s_2)(k_{\sigma,\nu} + x)^{-s_2}\frac{ds_2}{2i\pi} \quad (c_2 > 0)
$$

(42)
then the $P_{\sigma,\nu}^s(s_1)$ function \([24]\) becomes:

$$P_{\sigma,\nu}^s(s_1) = K \int_{c_2 - i\infty}^{c_2 + i\infty} (-1)^{-s_2} \Gamma(s_2) \int \frac{(k_{\sigma,\nu} + x)^{-s_2 x^{-s_1 + \alpha}} dx}{2i\pi}$$

$$= K \int_{c_2 - i\infty}^{c_2 + i\infty} (-1)^{-s_2} \frac{\Gamma(s_2)\Gamma(1 - s_2)\Gamma(s_1 + s_2 - 1 - \alpha)}{\Gamma(s_1 - \alpha)} (-k_{\sigma,\nu})^{-s_1 - s_2 + 1 + \alpha} \frac{ds_1 ds_2}{2i\pi}$$

(43)

(44)

where the $x$-integral is a particular case of a Bta integral, and converges because $-k_{\sigma,\nu} > 0$.

Inserting in the pricing formula \([26]\) and using the Legendre duplication formula, the asset-or-


nothing call thus reads:

$$C_{(a/n)}^-(-\sigma,\nu) = \frac{F}{2\Gamma(\frac{1}{2})} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} (-1)^{-s_2} \times$$

$$\frac{\Gamma(s_2 + \alpha)\Gamma(1 - s_2)\Gamma(s_1 + s_2 - 1 - \alpha)}{\Gamma(s_1 - \alpha)} (-k_{\sigma,\nu})^{-s_1 - s_2 + 1 + \alpha} \frac{ds_1 ds_2}{(2i\pi)^2}$$

(45)

where $(c_1, c_2)$ belongs to the convergence polyhedron of the integral $\Pi := \{(s_1, s_2) \in \mathbb{C}^2, 0 < Re(s_2) < 1, Re(s_1) > -\alpha, Re(s_1 + s_2) > 1 + \alpha\}$. The characteristic vector \([31]\) associated to \([45]\) is equal to:

$$\Delta = \left[ \frac{1}{2} \right]$$

(46)

resulting in the admissible region $\Pi_\Delta := \{(s_1, s_2) \in \mathbb{C}^2, Re(s_2) < \frac{1}{2}(c_1 - Re(s_1)) + c_2\}$. The cone $\Pi := \{(s_1, s_2) \in \mathbb{C}^2, Re(s_2) < 0, Re(s_1 + s_2) < 1 + \alpha\}$ satisfies $\Pi \subset \Pi_\Delta$ and, by definition, each divisor of \([45]\) intersects exactly one of its faces; as a consequence of the residue theorem \([33]\), the integral \([45]\) equals the sum of the residues located in $\Pi$; these residues are associated with two series of poles of the integrand:

- Series 1: when $\Gamma(s_1 + s_2 - 1 - \alpha)$ and $\Gamma(s_2)$ are singular;
- Series 2: when $\Gamma(s_1 + s_2 - 1 - \alpha)$ and $\Gamma(s_2)$ are singular.

The Gamma function being singular when its argument is a negative integer, poles of series 1 are located at $(s_1, s_2) = (-\alpha - 2n_1, -n_2)$, $n_1, n_2 \in \mathbb{N}$; a sequential application of the Cauchy formula yields the associated residues:

$$\frac{F}{2\Gamma(\frac{1}{2})} \frac{(-1)^{n_2}}{n_1! n_2!} \frac{2(-1)^{n_1 + n_2}}{\Gamma(1 + n_2)\Gamma(-2n_1 - n_2 - 1 - 2\alpha)} (-k_{\sigma,\nu})^{2n_1 + n_2 + 1 + 2\alpha}$$

(47)

which, after simplification, is the second term in the r.h.s. of \([38]\). Residues associated to the pole of series 2 can also be computed via the Cauchy formula (after making the change of variables $\tilde{s}_1 := s_1 + s_2 - 1 - \alpha$, $\tilde{s}_2 := s_2$), and give birth to the first term in the r.h.s. of \([38]\), which completes the proof of (i).

To prove (ii), it suffices to remark that, as $S$ is a Q-martingale, one has:

$$\mathbb{E}_t^Q[S_T 1_{\{S_T > K\}}] = Se^{r_T} - \mathbb{E}_t^Q[S_T 1_{\{S_T < K\}}]$$

(48)
The expectation in the right hand side can be computed with the same technique than for (i), resulting in the series (39).

Last, the proof of (iii) is straightforward: letting \( k_{\sigma,\nu} \to 0 \) in (38) or (39), we see that only the terms for \( n_1 = 0 \) survive; renaming \( n_2 := n \) yields formula (40) and completes the proof.

\[ P_{eur}(S_T, K) := [S_T - K]^+ \quad (49) \]

**European option.** The European call option pays \( S_T - K \) at maturity on the condition that the spot price is greater than the strike:

\[ P_{eur}(S_T, K) := \begin{cases} S_T - K & \text{if } S_T > K, \\ 0 & \text{otherwise} \end{cases} \quad (49) \]

**FORMULA 2 (European call)** The value at time \( t \) of an European call option is:

(i) (OTM price) If \( k_{\sigma,\nu} < 0 \),

\[ C^-_{eur}(k_{\sigma,\nu}, \sigma_{\nu}) = \frac{F}{2\Gamma(\frac{1}{2})} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \left[ \Gamma\left(\frac{n_1 + n_2 + 1}{2} + \alpha\right) \frac{n_1}{\sigma_{\nu}^{n_1 + 1}} \left(\frac{-k_{\sigma,\nu}}{\sigma_{\nu}}\right)^{-n_1 + n_2 + 1} \left(\frac{k_{\sigma,\nu}}{\sigma_{\nu}}\right)^{-n_1 + n_2 + 1} \right] \]

\[ + 2 \Gamma\left(\frac{-2n_1 - n_2 - 1 - 2\alpha}{\sigma_{\nu}}\right) \left(\frac{-k_{\sigma,\nu}}{\sigma_{\nu}}\right)^{2n_1 + 2n_2 + 1} \left(\frac{k_{\sigma,\nu}}{\sigma_{\nu}}\right)^{2n_1 + 1 + 2\alpha} \]

(ii) (ITM price) If \( k_{\sigma,\nu} > 0 \),

\[ C^+_{eur}(k_{\sigma,\nu}, \sigma_{\nu}) = S_T - Ke^{-rT} - C^-_{eur}(k_{\sigma,\nu}, -\sigma_{\nu}) \quad (51) \]

(iii) (ATM price) If \( k_{\sigma,\nu} = 0 \),

\[ C^-_{eur}(k_{\sigma,\nu}, \sigma_{\nu}) = C^+_{eur}(k_{\sigma,\nu}, \sigma_{\nu}) = \frac{F}{2\Gamma(\frac{1}{2})} \sum_{n_1=1}^{\infty} \Gamma\left(\frac{n_1 + 1}{2} + \alpha\right) \frac{n_1}{\sigma_{\nu}^{n_1 + 1}} \]

\[ (52) \]

**Proof.** To prove (i), we first note that, using notations (34), we have:

\[ P_{eur}(Se^{(r+\omega_{\sigma,\nu})T+x}, K) = K(e^{k_{\sigma,\nu}+x} - 1)1_{\{x>-k_{\sigma,\nu}\}} \quad (53) \]

Then, we use the Mellin-Barnes representation (see e.g. [Bateman 1954] p. 313):

\[ e^{k_{\sigma,\nu}+x} - 1 = \int_{c_2-i\infty}^{c_2+i\infty} (1-1)^{-s_2} \Gamma(s_2)(k_{\sigma,\nu} + x)^{-s_2} \frac{ds_2}{2\pi i} \quad (-1 < c_2 < 0) \quad (54) \]

and we proceed the same way than for proving (i) in Formula (1); note that the \( n_2 \)-summation in (50) now starts in \( n_2 = 1 \) instead of \( n_2 = 0 \), because the strip of convergence of (54) is reduced to \((-1, 0)\) instead of \((0, \infty)\) in (42).

To prove (ii), we write

\[ E_t^Q[(S_T - K)1_{\{S_T > K\}}] = Se^{rT} - K - E_t^Q[(S_T - K)1_{\{S_T < K\}}] \quad (55) \]

and compute the expectation in the r.h.s. following the same technique than for proving (i)

To prove (iii), we proceed like in the proof of formula (1) by letting \( k_{\sigma,\nu} \to 0 \) in both OTM and ITM prices (50) and (51).
An interesting particular case of formula (50) occurs when $S$ is at-the-money forward, i.e. when $S = F$ or, equivalently, $k = 0$. In this case, $k_{\sigma, \nu} = \omega_{\sigma, \tau}$ and, Taylor expanding the convexity adjustment (25) at first order for small $\sigma$:

$$\omega_{\sigma, \nu} \sim -\frac{\sigma^2}{2}$$  \hspace{1cm} (56)

we obtain:

**FORMULA 3 (European call - At-the-money forward price)** If $S = F$, then under the convexity adjustment approximation (25), the value at time $t$ of an European call option is:

$$C_{\text{eur}}^{\text{ATM}}(S, r, \tau, \sigma, \nu) = \frac{S}{2 \Gamma\left(\frac{1}{2}\right)} \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{\infty} \frac{(-1)^{n_1}}{n_1!} \left[ \frac{\Gamma\left(\frac{n_1+n_2+1}{2} + \alpha\right)}{\Gamma\left(\frac{n_1+n_2}{2} + 1\right)} \left(\frac{\sigma^2 \tau}{2} \nu^2\right)^{n_1} \right] (1-1)^{n_2} \left(\frac{\sigma \sqrt{2} \nu}{\sqrt{2} \nu}\right)^{n_2}$$  \hspace{1cm} (57)

A good estimate of (57) is given by the leading term of the series, i.e.:

$$C_{\text{eur}}^{\text{ATMF}}(S, r, \tau, \sigma, \nu) \simeq \frac{S}{\sqrt{2\pi} \Gamma\left(\frac{1}{2}\right)} \Gamma\left(\frac{1}{2} + \frac{\alpha}{\nu}\right) \sigma \sqrt{\nu}$$  \hspace{1cm} (58)

which allows for an estimate of the implied ATM-forward volatility:

$$\sigma_I \simeq \sqrt{\frac{2\pi}{\nu} \frac{\Gamma\left(\frac{\alpha}{\nu}\right)}{\Gamma\left(\frac{1}{2} + \frac{x}{\nu}\right)} \frac{C_0}{S}}$$  \hspace{1cm} (59)

where $C_0$ is the observed European ATM-forward market price. In the low $\gamma$-variance regime (small $\nu$), it follows from the Stirling approximation for the Gamma function that (59) becomes:

$$\sigma_I \xrightarrow{\nu \to 0} 2 \sqrt{\frac{\pi}{\tau}} \frac{C_0}{S}$$  \hspace{1cm} (60)

**Digital option (cash-or-nothing).** The payoff of the cash-or-nothing call option is

$$P_{c/n}(S_T, K) = \mathbb{1}_{\{S_T > K\}} \hspace{1cm} (61)$$

and therefore the option price itself is:

$$C_{c/n}(k_{\sigma, \nu}, \sigma_{\nu}) = \frac{1}{K} \left(C_{a/n}(k_{\sigma, \nu}, \sigma_{\nu}) - C_{\text{eur}}(k_{\sigma, \nu}, \sigma_{\nu})\right)$$  \hspace{1cm} (62)

Using formulas (1) and (2) it is immediate to see that:

**FORMULA 4 (Cash-or-nothing call)** The value at time $t$ of a cash-or-nothing call option is:
(i) (OTM price) If $k_{\sigma,\nu} < 0$,

$$C_{c/n}^-(k_{\sigma,\nu}, \sigma_\nu) = \frac{e^{-r\tau}}{2\Gamma\left(\frac{\alpha}{2}\right)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[ \frac{\Gamma\left(-n+\frac{\alpha}{2}+1\right)}{\Gamma\left(-n+\frac{1}{2}+1\right)} \left(\frac{-k_{\sigma,\nu}}{\sigma_\nu}\right)^n + 2 \frac{\Gamma(-2n-1-2\alpha)}{\Gamma(-n+\frac{1}{2}-\alpha)} \left(\frac{-k_{\sigma,\nu}}{\sigma_\nu}\right)^{2n+1+2\alpha} \right]$$



(63)

(ii) (ITM price) If $k_{\sigma,\nu} > 0$,

$$C_{c/n}^+(k_{\sigma,\nu}, \sigma_\nu) = e^{-r\tau} - C_{c/n}^-(k_{\sigma,\nu}, -\sigma_\nu)$$

(64)

(iii) (ATM price) If $k_{\sigma,\nu} = 0$,

$$C_{eur}^-(k_{\sigma,\nu}, \sigma_\nu) = C_{eur}^+(k_{\sigma,\nu}, \sigma_\nu) = \frac{e^{-r\tau}}{2}$$

(65)

3.3. Other payoffs

In this subsection, we show how the pricing formulas established in subsection 3.2 can be extended to other exotic options.

**Gap option.** A Gap (sometimes called Pay-Later) call has the following payoff:

$$P_{gap}(S_T, K_1, K_2) = \frac{S_T - K_1}{BD} \{S_T > K_2\}$$

(66)

and degenerates into the European call when trigger and strike prices coincide ($K_1 = K_2 = K$). From the definition (66), it is immediate to see that the value at time $t$ of the Gap call is:

$$C_{gap}(k_{\sigma,\nu}, \sigma_\nu) = C_{a/n}(k_{\sigma,\nu}, \sigma_\nu) - K_1 C_{c/n}(k_{\sigma,\nu}, -\sigma_\nu)$$

(67)

where the value of the asset-or-nothing and cash-or-nothing calls are given by formulas (1) and (4) for $K = K_2$.

**Power option.** Power options deliver a higher payoff than vanilla options, and are used to increase the leverage ratio of trading strategies. For instance, an asset-or-nothing power call has the non-linear payoff:

$$P_{pow,a/n}(S_T, K) = S_T^q \{S_T > K\}$$

(68)

for some $q \geq 1$. We can remark that

$$P_{pow,a/n}(S_T e^{(r+\omega_{\sigma,\nu})T+X}, K) = K e^{q(\tilde{k}_{\sigma,\nu}+X)} \{X > -\tilde{k}_{\sigma,\nu}\}$$

(69)

where

$$\tilde{k}_{\sigma,\nu} := \log \frac{S}{K^{\frac{\alpha}{2}}} + r\tau + \omega_{\sigma,\nu}$$

(70)

Introducing a Mellin-Barnes representation for the exponential term in (69) like we did in eq. (42) shows that the pricing formula for the OTM ($\tilde{k}_{\sigma,\nu} < 0$) power asset-or-nothing call is the same
than (38), when replacing \(k_{\sigma,\nu}\) by \(\tilde{k}_{\sigma,\nu}\) and multiplying out the series terms by \(q^{n_2}\), that is:

\[
C_{\text{pow.a/n}}^{-}(k_{\sigma,\nu},\sigma_{\nu}) = \frac{F}{2\Gamma(\frac{\tau}{\nu})} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}q}{n_1!} \left[ \frac{\Gamma\left(-\frac{n_1+n_2+1}{2}+\alpha\right)}{\Gamma\left(-\frac{n_1+n_2}{2}+1\right)} \left(-\tilde{k}_{\sigma,\nu}\right)^{n_1}\sigma_{\nu}^{n_2} \right. \\
+ 2 \frac{\Gamma(-n_1-n_2-1-2\alpha)}{\Gamma(-n_1+\frac{1}{2}-\alpha)} \left(-\tilde{k}_{\sigma,\nu}\right)^{2n_1+1+2\alpha} \left(-\tilde{k}_{\sigma,\nu}\right)^{n_2} \right] 
\]

(71)

Letting \(\tilde{k}_{\sigma,\nu} \to 0\) in (71) yields the ATM price:

\[
C_{\text{pow.a/n}}^{\text{atm}}(k_{\sigma,\nu},\sigma_{\nu}) = \frac{F}{2\Gamma(\frac{\tau}{\nu})} \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}+\alpha\right)}{\Gamma\left(n+\frac{1}{2}+1\right)} (q\sigma_{\nu})^n 
\]

(72)

The ITM price can be obtained from the parity relation:

\[
C_{\text{pow.a/n}}^{+}(k_{\sigma,\nu},\sigma_{\nu}) = S q e^{(q-1)(r-q\omega_{\sigma,\nu})\tau} - C_{\text{pow.a/n}}^{-}(k_{\sigma,\nu},-\sigma_{\nu}) 
\]

(73)

which follows from the risk-neutral expectation of \(S^q\). Extension of formulas (71), (72) and (73) to European and cash-or-nothing options are straightforward.

**Log option.** Log options were introduced in Wilmott (2000) and are basically options on the rate of return of the underlying asset. The call’s payoff is:

\[
P_{\text{log}}(S_T, K) := \left[ \log S_T - \log K \right]^+ 
\]

(74)

for \(K > 0\). Let us remark that we have:

\[
P_{\text{log}}(S e^{(r+\omega_{\sigma,\nu})\tau+x}, K) = [k_{\sigma,\nu} + x]^+ 
\]

(75)

and therefore, the \(P^{*}_{\sigma,\nu}\) function can be written down as:

\[
P^{*}_{\sigma,\nu}(s_1) = \frac{(-k_{\sigma,\nu})^{-s_1+2+\alpha}}{(s_1-2-\alpha)(s_1-1-\alpha)} 
\]

(76)

on the condition that \(k_{\sigma,\nu} < 0\). Inserting in the pricing formula (20) and summing up all arising residues yields the OTM log call price:

\[
C_{\text{log}}^{-}(k_{\sigma,\nu},\sigma_{\nu}) = \frac{e^{-r\tau}}{\sqrt{\pi}} \left[ \Gamma(1+\alpha)\sigma_{\nu} + \sqrt{\pi} \Gamma\left(\frac{1}{2}+\alpha\right)k_{\sigma,\nu} + 
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{\Gamma(\alpha-n)}{(2n+2)(2n+1)} (-k_{\sigma,\nu})^{2n+2} (2\sigma_{\nu})^{-2n-1} + \frac{\Gamma(-n)}{(2n+2\alpha+2)(2n+2\alpha+1)} \right. \\
\left. (-k_{\sigma,\nu})^{2n+2+\alpha+2} (2\sigma_{\nu})^{-2\alpha-2n-1} \right) \right] 
\]

(77)

Letting \(k_{\sigma,\nu} \to 0\) in (77) yields the ATM price:

\[
C_{\text{log}}^{\text{atm}}(k_{\sigma,\nu},\sigma_{\nu}) = \frac{e^{-r\tau}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2}+\frac{\tau}{\nu}\right)}{\Gamma\left(\frac{\tau}{\nu}\right)} \sigma_{\nu} 
\]

(78)
4. Extension to the asymmetric process, and numerical tests

In this section, we establish a factorized pricing formula for the asymmetric VG process \( \theta \neq 0 \) and apply the result to cash-or-nothing option. We also test the series formulas obtained in the paper for the European and digital options in both the symmetric and asymmetric case, by benchmarking the results with numerical evaluation of Fourier integrals.

4.1. Pricing formula

Let \( \sigma, \nu > 0 \), and let \( \theta \in \mathbb{R} \). Let us define the positive quantity

\[
q_{\sigma, \nu, \theta} := \frac{1}{2\sigma^2 \nu} + \left( \frac{\theta}{2\sigma^2} \right)^2 \tag{79}
\]

and let \( F^*_\nu \) be like in lemma (20). Introducing a Mellin representation for the Bessel function like in the proof of lemma 4.1, and a supplementary Mellin representation for the exponential term \( e^{\theta \frac{x}{\sigma}} \) in (4) yields:

**Lemma 4.1** Let \( P_1 := (|t - \frac{1}{2}|, \infty) \times (0, \infty) \); then for any \((c_1, c_2) \in P_1\), the following Mellin-Barnes representation holds true:

\[
f_{\sigma, \nu, \theta}(x, t) = \frac{(2\nu^2 + \theta^2)^{-\frac{\alpha}{2}}}{2 \sqrt{2\pi \sigma \nu^2} \Gamma(\frac{\alpha}{2})} \times \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} (-1)^{-s_2} \Gamma(s_2) F^*_\nu(s_1) q_{\sigma, \nu, \theta}^{-s_2} x^{-s_2} \frac{1}{x^{s_1 + \frac{1}{\nu} - \frac{1}{2}}} ds_1 ds_2 \tag{80}
\]

Let us now introduce the Mellin-like transform of the payoff function:

\[
P^*_{\sigma, \nu, \theta}(s_1, s_2) = \Gamma(s_2) \int_{-\infty}^{\infty} P \left( S e^{(r + \omega_{\nu, \theta}) \tau + x}, K \right) x^{-s_2} \frac{1}{x^{s_1 + \frac{1}{\nu} - \frac{1}{2}}} dx \tag{81}
\]

and assume that it converges for \( Re(s_1, s_2) \in P_2 \) for a certain subset \( P_2 \subset \mathbb{R}^2 \). As before, let us assume that \( \alpha := \frac{\nu}{2} - \frac{1}{2} \notin \mathbb{Z} \) and introduce the supplementary notation \( \theta_{\sigma} := \frac{\theta}{\sigma^2} \), so that we have:

\[
q_{\sigma, \nu, \theta} = \frac{1}{4} \left( \frac{1}{\sigma^2} + \theta^2_{\sigma} \right) \tag{82}
\]

where \( \sigma_{\nu} \) is the one defined in (35). As a consequence of lemma 3.1, and of the pricing formula (19), we have the following factorized formula:

**Proposition 4.2** (Factorization in the Mellin space) Let \((c_1, c_2) \in P\) where \( P := P_1 \cap P_2 \) is assumed to be nonempty. Then the value at time \( t \) of a contingent claim \( C \) with maturity \( T \) and
payoff $\mathcal{P}(S_T, K)$ is equal to:

$$C(S, K, r, \tau, \sigma, \nu, \theta) = \frac{e^{-r\tau}}{\sqrt{2\pi} \sigma^{1+2\alpha} \Gamma(\frac{\nu}{2})} \times$$

$$\int_{c_1-i\infty}^{c_1+i\infty} \int_{c_2-i\infty}^{c_2+i\infty} P^*_{\sigma, \nu, \theta}(s_1, s_2) F^*_\nu(s_1) (-\theta)^{-s_2} (q_{\sigma, \nu, \theta})^{-\frac{1+\alpha}{2}} \frac{ds_1 ds_2}{(2i\pi)^2}$$

(83)

Let us now demonstrate how to implement proposition 4.2 on the example of the cash-or-nothing option; we will denote the risk-neutral moneyness by

$$k_{\sigma, \nu, \theta} := k + \omega_{\sigma, \nu, \theta} \tau$$

(84)

**Example: Digital option (cash-or-nothing).**

**Formula 5 (Cash-or-nothing call)**  The value at time $t$ of a cash-or-nothing call option is:

(i) (OTM price) If $k_{\sigma, \nu} < 0$,

$$C^-_{c/n}(k_{\sigma, \nu, \theta}, q_{\sigma, \nu, \theta}, \sigma, \theta) = \frac{e^{-r\tau}}{2^{2+2\alpha} \sqrt{\pi} \sigma^{1+2\alpha} \Gamma(\frac{\nu}{2})} \times$$

$$\left[ \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{n+1}{2} + \alpha \right) q_{\sigma, \nu, \theta}^{-\frac{n+1}{2} - \alpha} \theta_n^\nu + \sum_{n=0}^{\infty} \frac{(-1)^{n_1}}{n_1! n_2!} \Gamma(\alpha - n) \frac{(-k_{\sigma, \nu, \theta})^{2n_1+n_2+1} q_{\sigma, \nu, \theta}^{-\alpha - n}}{-2n_1 - n_2 - 1 - 2\alpha} \theta_n^\nu \right]$$

(85)

(ii) (ITM price) If $k_{\sigma, \nu} > 0$,

$$C^+_{c/n}(k_{\sigma, \nu, \theta}, q_{\sigma, \nu, \theta}, \sigma, \theta) = e^{-r\tau} - C^-_{c/n}(-k_{\sigma, \nu, \theta}, q_{\sigma, \nu, \theta}, \sigma, -\theta)$$

(86)

(iii) (ATM price) If $k_{\sigma, \nu} = 0$,

$$C^{\text{atm}}_{c/n}(q_{\sigma, \nu, \theta}, \sigma, \theta) = \frac{e^{-r\tau}}{2^{2+2\alpha} \sqrt{\pi} \sigma^{1+2\alpha} \Gamma(\frac{\nu}{2})} \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{n+1}{2} + \alpha \right) q_{\sigma, \nu, \theta}^{-\frac{n+1}{2} - \alpha} \theta_n^\nu$$

(87)

**Proof.** To prove (i), we use notation (84) for the moneyness to write the cash-or-nothing payoff as:

$$\mathcal{P}_{c/n}(S e^{(r+\omega_{\sigma, \nu, \theta})\tau + x}, K) = 1_{\{x \geq -k_{\sigma, \nu, \theta}\}}$$

(88)

and therefore the $P^*_{\sigma, \nu, \theta}(s_1, s_2)$ function (81) is:

$$P^*_{\sigma, \nu, \theta}(s_1, s_2) = \Gamma(s_2) \int_{-k_{\sigma, \nu, \theta}}^{\infty} x^{-s_1-s_2+\alpha} dx = \Gamma(s_2) \frac{(-k_{\sigma, \nu, \theta})^{-s_1-s_2+1+\alpha}}{s_1 + s_2 - 1 - \alpha}$$

(89)

and exists as $k_{\sigma, \nu, \theta} < 0$. Inserting in the pricing formula (83), the OTM cash-or-nothing call thus
reads:

\[
\begin{align*}
C(S, K, r, \tau, \sigma, \nu, \theta) &= e^{-r\tau} \frac{1}{2^{2+2\alpha} \sqrt{\pi} \sigma^{1+2\alpha} \Gamma(\frac{1}{2})} \\
&\quad \times \int_{c_1-i\infty}^{c_1+i\infty} \int_{c_2-i\infty}^{c_2+i\infty} \frac{\Gamma(s_2) \Gamma(s_2)}{s_1 + s_2 - 1 - \alpha} (-k_{\sigma, \nu, \theta})^{-s_1-s_2+1+\alpha} (-\theta_{\sigma})^{-s_2} (q_{\sigma, \nu, \theta})^{-\frac{1}{2}} ds_1 ds_2
\end{align*}
\]

(90)

where \((c_1, c_2)\) belongs to the convergence polyhedron of \((90)\) \(P := \{(s_1, s_2) \in \mathbb{C}^2, \text{Re}(s_1) > |\alpha|, \text{Re}(s_2) > 0\}\). The associated characteristic vector is

\[
\Delta = [1, 1]
\]

(91)

Therefore, cone \(\Pi = \{(s_1, s_2) \in \mathbb{C}^2, \text{Re}(s_1) < |\alpha|, \text{Re}(s_2) < 0\}\) is located in the admissible region and is compatible with the divisors of \((90)\); thus, from the residue theorem \((83)\), the integral equals the sum of the residues of its integrand in \(\Pi\). There are three series of poles:

- Series 1: when \(\Gamma(s_2)\) is singular and \(s_1 + s_2 - 1 - \alpha = 0\), that is when \((s_1, s_2) = (n + 1 + \alpha, -n)\), \(n \in \mathbb{N}\), resulting in the first series in \((85)\);
- Series 2: when \(\Gamma(s_1 - \alpha)\) and \(\Gamma(s_2)\) are singular, that is when \((s_1, s_2) = (\alpha - 2n_1, -n_2)\), \(n_1, n_2 \in \mathbb{N}\);
- Series 3: when \(\Gamma(s_1 + \alpha)\) and \(\Gamma(s_2)\) are singular, that is when \((s_1, s_2) = (-\alpha - 2n_1, -n_2)\), \(n_1, n_2 \in \mathbb{N}\); series 2 and 3 result in the second series in \((85)\).

To prove (ii), we write

\[
E^Q[I_{\{S_T > K\}}] = 1 - E^Q[I_{\{S_T < K\}}]
\]

(92)

and apply the same technique than in (i) to the expectation in the right hand side.

Last, (iii) results from letting \(k_{\sigma, \nu, \theta} \to 0\) in \((83)\) or \((86)\). \(\square\)

4.2. **Comparison with numerical results**

In this section, we benchmark the various pricing formulas obtained in the paper with classical Fourier-related techniques. First, let us recall that, following \(\text{Lewis} [2001]\), digital option prices admit convenient representation involving the risk-neutral characteristic function and the log-forward moneyness; the asset-or-nothing call can be written as

\[
C_{a/n}(S, K, r, \tau, \sigma, \nu, \theta) = S \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{iuk\tilde{\Phi}_{\sigma, \nu, \theta}(u - i, \tau)}}{iu} \right] du \right)
\]

(93)

and the cash-or-nothing call as

\[
C_{c/n}(S, K, r, \tau, \sigma, \nu, \theta) = e^{-r\tau} \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{iuk\tilde{\Phi}_{\sigma, \nu, \theta}(u, \tau)}}{iu} \right] du \right)
\]

(94)
where \( k := \log \frac{S}{K} + r\tau \) as before, and where the characteristic function has been normalized by the convexity adjustment

\[
\tilde{\Phi}_{\sigma, \nu, \theta}(u, t) := e^{iu \omega_{\sigma, \nu, \theta} \Phi_{\sigma, \nu, \theta}(u, t)}
\]

(95)

so that the martingale condition \( \tilde{\Phi}_{\sigma, \nu, \theta}(-i, t) = 1 \) holds true. Second, regarding European options, we will use the representation given in (Carr & Madan 1999) based on the introduction of a damping factor \( a \) to avoid the divergence in \( u = 0 \); namely, let

\[
\varphi_{\sigma, \nu, \theta}(u, t) := e^{iu[\log S + (r + \omega_{\sigma, \nu, \theta}) \tau]} \tilde{\Phi}_{\sigma, \nu, \theta}(u, t)
\]

(96)

then the European option price admits the representation:

\[
C_{\text{eur}}(S, K, r, \tau, \sigma, \nu, \theta) = e^{-a \log K - r\tau} \int_0^\infty e^{-iu \log K} \text{Re} \left[ \frac{\varphi_{\sigma, \nu, \theta}(u - (a + 1)i, t)}{a^2 + a - u^2 + i(2a + 1)u} \right] du
\]

(97)

where \( a < 0 < a_{\text{max}} \), and \( a_{\text{max}} \) is determined by the square integrability condition \( \varphi_{\sigma, \nu, \theta}(0, t) < \infty \); in the symmetric VG model,

\[
a_{\text{max}} = \frac{1}{\sigma} \sqrt{\frac{2}{\nu}} - 1
\]

(98)

Integrals \( (93) \), \( (94) \) and \( (97) \) can be carried out very easily via a classical recursive algorithm after truncating the integration region (or, in the case of \( (97) \), by a Fast Fourier Transform algorithm); typically, restraining the Fourier variable to \( u \in [0, 10^4] \) is sufficient to obtain an excellent level of precision when the time to maturity is not too small; however, as we will see, in the limit \( \tau \to 0 \) the numerical evaluation of \( (97) \) has more difficulties to converge if the option is deep OTM, and in that case the series in formula \( (94) \) constitute a more efficient tool.

**Symmetric process.** In table 1 we compare the prices obtained by the series in formula \( (94) \) truncated at \( n = n_{\text{max}} \), with a numerical evaluation of \( (94) \); we investigate several market configurations, from ITM to OTM options, for long and short maturities.

| \( n_{\text{max}} \) | 3 | 5 | 10 | 15 |
|---------------------|---|---|----|----|
| **S** = 5000       | 0.8870 | 0.7885 | 0.7755 | 0.7754 |
| **S** = 4200       | 0.5372 | 0.5373 | 0.5373 | 0.5373 |
| **S** = 3800       | 0.3734 | 0.3739 | 0.3740 | 0.3740 |
| **S** = 3000       | -0.9399 | -0.1439 | 0.1159 | 0.1181 |
| **ATM**            | 0.4901 | 0.4901 | 0.4901 | 0.4901 |
| **S** = 5000       | 0.8415 | 0.9355 | 0.9410 | 0.9410 |
| **S** = 4200       | 0.7203 | 0.7104 | 0.7104 | 0.7104 |
| **S** = 3800       | 0.2487 | 0.2486 | 0.2486 | 0.2486 |
| **S** = 3000       | 0.4814 | 0.0731 | 0.0282 | 0.0281 |
We perform the same analysis for the asset-or-nothing call in Table 2; the series given by formula 1 are truncated at \( n = m = \max \), and compared to a numerical evaluation of (93).

Table 2. Asset-or-nothing prices given by formula 1 for various truncations and different market configurations, vs. numerical evaluation of the integral (93). Parameters: \( K = 4000, r = 1\%, \sigma = 0.2, \nu = 0.85 \).

| max = 3 | max = 5 | max = 10 | max = 15 | Lewis \([93]\) \([\theta = 0]\) |
|---------|--------|---------|---------|----------------|
| Long term options \((\tau = 2)\) |
| \( S = 5000 \) | 4758.18 | 4374.28 | 4307.02 | 4306.93 | 4306.93 |
| \( S = 4200 \) | 2739.14 | 2737.53 | 2737.49 | 2737.49 | 2737.49 |
| ATM | 2472.07 | 2474.66 | 2474.72 | 2474.72 | 2474.72 |
| \( S = 3800 \) | 1851.86 | 1855.51 | 1855.51 | 1855.51 | 1855.51 |
| \( S = 3000 \) | -3390.31 | -284.115 | 560.097 | 568.846 | 568.846 |

| Short term options \((\tau = 0.5)\) |
| \( S = 5000 \) | 4192.59 | 4769.19 | 4806.50 | 4806.52 | 4806.50 |
| \( S = 4200 \) | 3168.47 | 3168.75 | 3168.74 | 3168.74 | 3168.74 |
| ATM | 2196.77 | 2197.07 | 2197.07 | 2197.07 | 2797.07 |
| \( S = 3800 \) | 1113.19 | 1113.80 | 1113.80 | 1113.80 | 1113.80 |
| \( S = 3000 \) | 1265.39 | 222.654 | 127.749 | 127.292 | 127.293 |

We can remark that both in the cash-or-nothing and the asset-or-nothing cases, the convergence to the option price is very fast; in particular when the underlying is close to the money, it is sufficient to consider only terms up to \( n = 5 \) to get a precision of \( 10^{-3} \) in the asset-or-nothing price, independently of the maturity. We also note that, as a whole, the convergence goes faster in the ITM region, and is slightly slower when the option is very deep OTM; this is because, for fixed \( \tau \), the moneyness is logarithmically driven:

\[
|k_{\sigma,\nu}| \sim |\log S| \quad (99)
\]

and therefore becomes bigger when \( S \) decreases, which tends to slow down the convergence of the OTM series (38) and (63). On the contrary, as \( |k_{\sigma,\nu}| \) grows very slowly when \( S \) grows, the convergence of the ITM series (39) and (64) remains fast even for deep ITM situations. Last, a remarkable feature is the short maturity behavior of the series; as

\[
|k_{\sigma,\nu}| \sim \left| \frac{\log S}{K} \right| \quad (100)
\]

then if the ratio \( S/K \) is not too big or too small, the moneyness will be close to 0 and will trigger an accelerated convergence to the option price. This is a very interesting feature, as the short maturity situation is generally not favorable for numerical Fourier inversion (see Carr & Madan 1999): the numerical oscillation for \( \tau \to 0 \) leads the authors to introduce a "modified time" approach, involving a multiplication by a hyperbolic sine function instead of the usual Fourier kernel). This situation is displayed in Table 3. Observe in particular that, for deep OTM situations, the numerical inversion fails to converge even when integrating up to \( u = 10^4 \); on the contrary, the series (50) converges to the price with a precision of \( 10^{-3} \) when considering only terms up to \( n = m = 20 \) (or even 10 if the option is not to deep OTM).

Asymmetric process. In Table 4, we compare the prices obtained by the series in formula 5 truncated at \( n = m = \max \), with a numerical evaluation of (94), for negative or positive asymmetry parameter \( \theta \); again, the convergence is very fast in every situation. Note that the moneyness (84)
Table 3. Short maturity prices, obtained via the OTM series \((50)\) for the European prices truncated at \(n = m = \text{max}\) vs. numerical evaluations of \((97)\) for various upper bounds \(u_{\text{max}}\) of the integration region; the series \((50)\) converges extremely fast to the price, while numerical integration has more difficulty to converge, notably in the deep OTM region. Parameters: \(K = 4000\), \(r = 1\%\), \(\sigma = 0.2\), \(\nu = 0.85\), \(a = 1\).

| OTM (S=3000) | Series \((50)\) | \(\tau\) | \(\text{max} = 5\) | \(\text{max} = 10\) | \(\text{max} = 20\) |
|--------------|----------------|--------|----------------|----------------|----------------|
| \(\tau = \frac{1}{12}\) (1 month) | \(\tau = \frac{1}{52}\) (1 week) | \(\tau = \frac{1}{360}\) (1 day) | 3.953 | 1.802 | 1.802 |
| 0.804 | 0.388 | 0.388 |
| 0.113 | 0.055 | 0.055 |

| Carr-Madan \((97)\) | \(u_{\text{max}} = 10^2\) | \(u_{\text{max}} = 10^3\) | \(u_{\text{max}} = 10^4\) |
| \(\tau = \frac{1}{12}\) (1 month) | \(\tau = \frac{1}{52}\) (1 week) | \(\tau = \frac{1}{360}\) (1 day) | 1.870 | 1.803 | 1.802 |
| 0.476 | 0.390 | 0.388 |
| 0.149 | 0.057 | 0.055 |

| Deep OTM (S=2000) | Series \((50)\) | \(\tau\) | \(\text{max} = 10\) | \(\text{max} = 20\) | \(\text{max} = 30\) |
|-----------------|----------------|--------|----------------|----------------|----------------|
| \(\tau = \frac{1}{12}\) (1 month) | \(\tau = \frac{1}{52}\) (1 week) | \(\tau = \frac{1}{360}\) (1 day) | 4.9881 | 0.0470 | 0.0470 |
| 0.9093 | 0.0096 | 0.0096 |
| 0.1246 | 0.0013 | 0.0013 |

| Carr-Madan \((97)\) | \(u_{\text{max}} = 10^2\) | \(u_{\text{max}} = 10^3\) | \(u_{\text{max}} = 10^4\) |
| \(\tau = \frac{1}{12}\) (1 month) | \(\tau = \frac{1}{52}\) (1 week) | \(\tau = \frac{1}{360}\) (1 day) | 0.0410 | 0.0469 | 0.0467 |
| 0.0031 | 0.0093 | 0.0146 |
| -0.0051 | 0.0009 | 0.0115 |

is a function of \(\theta\); for the set of parameters of table \(1\) this results in two distinct ATM prices:

\[
\begin{align*}
S_{\text{ATM}}^+ &= Ke^{-(r+\omega_{\sigma,\nu,\theta})\tau} = 5050.24 \quad [\theta = +0.1] \\
S_{\text{ATM}}^- &= Ke^{-(r+\omega_{\sigma,\nu,\theta})\tau} = 3358.52 \quad [\theta = -0.1]
\end{align*}
\]

Table 4. Cash-or-nothing prices given by formula \((5)\) for various truncations and positive or negative asymmetry, vs. numerical evaluation of the integral \((94)\). Parameters: \(K = 4000\), \(r = 1\%\), \(\tau = 2\), \(\sigma = 0.2\), \(\nu = 0.85\).

| \(\text{max} = 3\) | \(\text{max} = 5\) | \(\text{max} = 10\) | \(\text{max} = 15\) | Lewis \((94)\) \([\theta = +0.1]\) |
|----------------|----------------|----------------|----------------|----------------|
| \(S = 6000\) | 0.9096 | 0.9008 | 0.8993 | 0.8993 | 0.8993 |
| ATM | 0.7064 | 0.7257 | 0.7288 | 0.7288 | 0.7288 |
| \(S = 3000\) | -15.545 | -0.3877 | 0.1364 | 0.1364 | 0.1364 |

| \(\text{max} = 3\) | \(\text{max} = 5\) | \(\text{max} = 10\) | \(\text{max} = 15\) | Lewis \((94)\) \([\theta = -0.1]\) |
|----------------|----------------|----------------|----------------|----------------|
| \(S = 5000\) | 1.7747 | 0.7749 | 0.7605 | 0.7605 | 0.7605 |
| ATM | 0.2412 | 0.2500 | 0.2514 | 0.2514 | 0.2514 |
| \(S = 2000\) | -1.1760 | -0.0400 | 0.0047 | 0.0047 | 0.0047 |

In table \(5\) we examine the case \(\tau \to 0\); like for the symmetric model, the convergence of formula \((5)\) is particularly fast; notably, for very short options (under one month), \(u_{\text{max}} = m_{\text{max}} = 3\) is sufficient to get a precision of \(10^{-2}\).
Table 5. Short maturity cash-or-nothing prices, obtained via the series in formula 5 truncated at $n = m = \text{max}$, positive and negative asymmetry; like in table 3 (symmetric model), the series converge extremely fast. Parameters: $S = 4200$, $K = 4000$, $r = 1\%$, $\sigma = 0.2$, $\nu = 0.85$.

| Positive asymmetry $[\theta = +0.1]$ | max = 3 | max = 5 | max = 10 | max = 15 |
|--------------------------------------|---------|---------|----------|---------|
| $\tau = \frac{1}{2}$ (6 months)     | 0.5370  | 0.5396  | 0.5398   | 0.5398  |
| $\tau = \frac{1}{12}$ (1 month)    | 0.9401  | 0.9399  | 0.9399   | 0.9399  |
| $\tau = \frac{1}{52}$ (1 week)     | 0.9873  | 0.9872  | 0.9872   | 0.9872  |
| $\tau = \frac{1}{360}$ (1 day)     | 0.9982  | 0.9982  | 0.9982   | 0.9982  |

| Negative asymmetry $[\theta = -0.1]$ | max = 3 | max = 5 | max = 10 | max = 15 |
|--------------------------------------|---------|---------|----------|---------|
| $\tau = \frac{1}{2}$ (6 months)     | 0.7315  | 0.7290  | 0.7287   | 0.7287  |
| $\tau = \frac{1}{12}$ (1 month)    | 0.9187  | 0.9184  | 0.9184   | 0.9184  |
| $\tau = \frac{1}{52}$ (1 week)     | 0.9786  | 0.9786  | 0.9786   | 0.9786  |
| $\tau = \frac{1}{360}$ (1 day)     | 0.9968  | 0.9968  | 0.9968   | 0.9968  |

5. Concluding remarks

In this paper, we have provided the reader with several ready-to-use formulas for pricing path independent options in the Variance Gamma model, with a particular focus on the symmetric model. These formulas, having the form of simple series expansions, can be easily implemented; the convergence is fast and can be made as precise as one wishes. Particularly remarkable is the short term behavior of the series: as function of powers of the moneyness, short maturities provide a very favorable situation and accelerate the convergence speed. This constitutes an interesting advantage over numerical evaluation of inverse Fourier integrals, which tend to oscillate and have a slower convergence when the time horizon is close to zero.

Future works should include the extension of the technology to other kind of payoffs, and to exponential Lévy models based on generalizations of the Variance Gamma process (e.g., multivariate processes) or on other Lévy processes. Models based on Normal Gaussian Inverse distributions (Barndorff-Nielsen 2002) are also particularly interesting, because the density function admits a convenient Mellin-Barnes representation; like in the Variance Gamma case, this opens the way to a factorized option pricing formula and to its evaluation via the residue technique.

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