PERIODIC POINTS FOR SPHERE MAPS PRESERVING MONOPOLE FoliATIONS

GRZEGORZ GRAFF, MICHAL MISIUREWICZ, AND PIOTR NOWAK-PRZYGODZKI

Abstract. Let $S^2$ be a two-dimensional sphere. We consider two types of its foliations with one singularity and smooth maps $f : S^2 \to S^2$ preserving these foliations, more and less regular. We prove that in both cases the lower growth rate of the number of fixed points of $f^n$ is at least $\log |\deg(f)|$, where $\deg(f)$ is a topological degree of $f$, confirming the Shub’s conjecture in these classes of maps.

1. Introduction

Estimating the growth rate of the number of fixed points of the $n$-th iterate of a smooth map of the $n$-dimensional sphere to itself is a challenging problem. It was conjectured by Michael Shub in 1974 that it must be (asymptotically) exponential (cf. [16, 17]):

\[
\limsup_{n \to \infty} \frac{\log \# \text{Fix}(f^n)}{n} \geq \log |\deg(f)|,
\]

where $\deg(f)$ denotes the degree of $f$.

On the other hand, (1.1) does not hold for every continues map $f$ (cf. an example of a map with only two periodic points and $\deg(f) = 2$ in [16]), while it is known that the smoothness implies that the growth rate of number of fixed points of $f^n$ is at least linear [1] and indeed could be linear up to any fixed period, cf. [4, 5]. Proving or disproving the Shub’s conjecture would substantially extend our knowledge about the role of the differentiability assumption in periodic point theory.

Even in the simplest case of two-dimensional sphere $S^2$ the conjecture still remains unsolved, although there has been a lot of progress in some particular cases. During the last few years the exponential growth was obtained under some topological conditions for $S^2$ and annulus ([2, 8, 9, 10, 11]).

Recently the problem for $S^2$ has been studied in [14, 15] (see also [6, 7] for higher dimensional spheres) under the assumption that the map preserves some “geographical” (singular) foliation. In this context a natural question arises, what happens when we change the “geography.”

In this paper we consider foliations with one singularity. In geographical terms this means that our planet, call it Monopole, has only one pole. After all, if physicists
can admit magnetic monopoles (see, e.g. [3]), there is no reason for not admitting geographical monopoles.

The natural coordinates on the surface of Monopole are easy to describe. Suppose that our sphere $S^2$ is given by the equation $x^2 + y^2 + (z - 1)^2 = 1$. The pole $P$ is the origin. Let us denote the $xy$-plane by $\pi$. A given point $Q = (x, y, z) \in S^2$ belongs to the half plane $\pi_x$ containing the $x$-axis and to the half-plane $\pi_y$ containing the $y$-axis. We denote by $\alpha$ ($\beta$) the angle between $\pi_y$ and $\pi$ ($\pi_x$ and $\pi$, respectively). We introduce $\alpha$ and $\beta$ as new coordinates of $Q$, both varying from 0 to 180 degrees. We will call them $x$-titude and $y$-titude.

Thus, the lines of constant $x$-titude (respectively, $y$-titude), which we call merallels (respectively, paridians), are circles that are the intersections of the sphere with the planes passing through the $y$-axis (respectively, the $x$-axis).

Locally, in a neighborhood of the pole, we can easily draw the foliation by the paridians. For this, we use the stereographic projection from the antipole $(0, 0, 2)$. The picture will be on the $xy$-plane, and the projections of the paridians will be the $x$-axis and the circles tangent to it at the origin (see Figure 1). We will consider paridional maps of the sphere, that is, $C^1$ maps preserving the foliation by the paridians.

We also propose a much less regular foliation, with two rabbit-like “ears.” The corresponding picture in the $xy$-plane will be the same as for the $y$-paridional case in the lower half-plane. In the upper half-pane the leaves will be graphs given by the polar equation $r = c|\sin(2\theta)|$, where $\theta \in (0, \pi]$, and then the smooth deformations of the circles, being more and more geometrically circle-like as the radius goes to the infinity (see Figure 2). We will call this foliation the rabbit foliation, and the smooth maps preserving this foliation the rabbit maps.

In our approach we consider some type of open subsets of the base of our foliation with non-zero Brouwer degree called preband and prove a “fixed point theorem” stating the existence of a fixed point of $f$ (or $f^n$ for iterations) in every preband. Now in case of homogeneous foliation (paridional maps) we get that exponential growth of global degree of iterations implies the same growth of the number of prebands, thus also the number of fixed points of $f^n$, so the Shub’s conjecture is valid (Theorem 2.12).

If the foliation is not homogeneous, i.e. we consider rabbit maps, the answer is the same. The Shub’s conjecture holds, but for a completely different reason: all maps preserving such foliation have degree 0 or $\pm 1$ (Theorem 3.5).

2. Paridional maps

We will use the general scheme from [14]. However, many elements of this scheme, in particular the discussion of the behavior of our map close to the pole, will be different. We use the classical definition of Brouwer degree, cf. [13], where the degree for a $C^1$ map $f$ is defined as a sum of signs of the Jacobian of $f$ at a finite set $f^{-1}(y)$ for $y$ being a regular value.

We consider the natural system of coordinates on the sphere $S^2$, defined in the introduction. We include the pole $P$, so the $y$-titude (after the affine rescaling) is an element of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. To denote specific points on this circle we will usually use the numbers from $[0, 1)$. We will denote the $y$-titude of a point $x \in S^2$ by $\ell(x)$. The function $\ell$ is well defined and continuous on $S^2 \setminus \{P\}$.
Observe that all paridians, except for the pole, are circles, so the pole belongs to all paridians. However, the singleton of $P$ is also a paridian. We will call paridians other than $\{P\}$ proper paridians. Note that if we used the convention that the pole belongs only to the paridian $\{P\}$, our class of maps preserving paridians would be much smaller.

**Definition 2.1.** A map $f : \mathbb{S}^2 \to \mathbb{S}^2$ of class $C^1$ will be called *paridinal* if the image of each paridian is contained in a paridian.

**Lemma 2.2.** Let $f$ be a paridinal map. If $\deg(f) \neq 0$ then $f(P) = P$.

**Proof.** If $f(P) \neq P$ then by the continuity of $f$, since every paridian contains $P$, the image of the whole sphere would be contained in one paridian. Therefore $f$ would not be injective and thus $\deg(f) = 0$. \hfill \Box

From now on, we will assume that $\deg(f) \neq 0$.

Fix a paridinal map $f$. Since $f$ maps paridians to paridians, there exists a map $\varphi : \mathbb{T} \to \mathbb{T}$ such that for $x, f(x) \neq P$

$$\varphi \circ \ell = \ell \circ f.$$ 

As we will consider only maps with non-zero degree, by Lemma 2.2 we may assume that $f(P) = P$. Since $P$ belongs to all paridians, $\varphi$ is not defined uniquely. To make
it unique, whenever the whole paridian $\ell^{-1}(y)$ is mapped by $f$ to $P$, we set $\varphi(y) = 0$ and define $\ell(P) = 0$. Then we get the following commutative diagram:

\[
\begin{array}{ccc}
S^2 & \xrightarrow{f} & S^2 \\
\downarrow\ell & & \downarrow\ell \\
T & \xrightarrow{\varphi} & T
\end{array}
\]

While we cannot claim that $\varphi$ is continuous everywhere, it is continuous where it is important from the point of view of our considerations (cf. Lemma 2.4 below).

Denote $A = \varphi^{-1}(0)$. Let us take $y \not\in A$. Then $S_y := \ell^{-1}(y)$ is homeomorphic to a circle. Observe that we have orientation of the proper paridians consistent throughout the sphere. The map $f$ restricted to the circle $S_y$ leads from the circle to the circle. Let us denote the degree of this map by $d_{S_y}(f)$. Now consider $B$, a connectivity component of the set $T \setminus A$.

**Lemma 2.3.** For $y \in B$, the degree $d_{S_y}(f)$ depends only on $B$, not on $y$.

*Proof.* Let us take a parametrization $\eta_y : S^1 \to S_y$, which depends continuously on $y$. Then, $d_{S_y}(f)$ is the degree of the self-map of the circle

\[(2.2) \quad f_y = \eta_{\varphi(y)}^{-1} \circ f|_{S_y} \circ \eta_y.
\]

As a consequence by changing $y$ we obtain a homotopy between $f_y'$ and $f_y''$ for all $y', y'' \in B$ and the result follows from homotopical invariance of the degree. \hfill $\square$

We will call the common value of $d_{S_y}(f)$ for $y \in B$ the *paridian degree* of $B$, and denote it $d(B)$.

**Lemma 2.4.** If $d(B) \ne 0$ then $\varphi$ is continuous on the closure of $B$.

*Proof.* Assume that $d(B) \ne 0$ and $B = (y^-, y^+)$. Continuity of $\varphi$ on $B$ follows from continuity of $f$. We will show that for $y > y^-$ which tends to $y^-$, $\varphi(y)$ tends to 0, i.e., we have right-continuity at $y^-$ (the argument for left-continuity at $y^+$ is the same). Notice that $f(S_y) = S_{\varphi(y)}$ and that $f(S_y)$ covers the whole $S_{\varphi(y)}$ because $d(B) \ne 0$. By continuity of $f$, $f(S_y)(= S_{\varphi(y)})$ tends to $f(S_{y^-}) = \{P\}$, which implies that $\varphi(y)$ tends to 0. \hfill $\square$

Thus, in case of $d(B) \ne 0$, we can define the *sign of $B$*, which we will denote $\Delta(B)$. If $B = (a, b)$, then it is $+1$ if $\varphi(a) = 0$ and $\varphi(b) = 1$; it is $-1$ if $\varphi(a) = 1$ and $\varphi(b) = 0$. Otherwise it is 0. Finally, we define the *degree of $B$* as $\deg(B) = \Delta(B) \cdot d(B)$ (and $\deg(B) = 0$ if $d(B) = 0$ and $\Delta(B)$ is not defined). We will call $B$ a *preband* \footnote{It would be more natural to use the name *band*, but in several papers this name is used for the sets like $\ell^{-1}(B)$.} if $\deg(B) \ne 0$. We will denote the set of all prebands of $f$ by $B(f)$.

**Lemma 2.5.** There are only finitely many prebands.

*Proof.* Suppose that the number of prebands is infinite. Then, there exists a sequence of prebands $(B_i)_i$ which converges to some $y_0 \in A$. We choose a point $Q \in S^2$ such that $Q \ne P$. As a consequence of the fact that $\deg(B_i) \ne 0$ we may find a sequence of points $(Q_i)_i$ such that

\[(2.3) \quad Q_i \in \ell^{-1}(B_i) \quad \text{and} \quad f(Q_i) = Q.
\]
On the other hand, \((Q_i)_i\) (or its subsequence) is convergent and we get: \(Q_i \to Q_0\)
and thus, by (2.3), \(f(Q_0) = Q\).

However, \(\ell(Q_0) = y_0\), which leads to the following contradiction:
\[0 = \varphi(y_0) = \varphi(\ell(Q_0)) = \ell(f(Q_0)) = \ell(Q) \neq 0,\]
where in the third equality we use the formula (2.1).

Note that outside the pole \(P\) the map \(f\) can be locally written down in the following
form (2.4)
\[f(\alpha, \beta) = (\varphi(\alpha), f|_{S_{\alpha}}(\alpha, \beta)),\]
where \(\alpha\) and \(\beta\) are the \(x\)- and \(y\)-titude, respectively.

Lemma 2.6. If \(f : S^2 \to S^2\) is a paridional map, then its degree is
(2.5) \(\deg(f) = \sum_{B \in B(f)} \deg(B)\).

Proof. We choose \(Q \neq P\), a regular value of \(f\). Then by the definition
(2.6) \(\deg(f) = \sum_{x_i \in f^{-1}(Q)} \text{sign } \det Df(x_i).\)

In the neighborhood of any point \(x_i \in f^{-1}(Q)\) the map \(f\) has the form (2.4) for
\(x_i = (\alpha_i, \beta_i) \in (0, \pi) \times (0, \pi)\). On the other hand, \(f\) is a paridional map and thus \(\varphi\)
is a function of only one variable \(y\). Thus

(2.7) \[Df_{x_i} = \begin{bmatrix} a_i & 0 \\ b_i & \end{bmatrix},\]

where \(a_i \in \mathbb{R}\) is the derivative of \(\varphi\) at \(\alpha_i\) and \(b_i = \frac{df|_{S_{\alpha_i}}}{dy}(\alpha_i, \beta_i)\).

For each fixed \(\alpha_i\) take the finite set of \(\beta_{ij}\) such \((\alpha_i, \beta_{ij}) \in f^{-1}(Q)\) and all \(\beta_{ij}\) belong
to the same paridian \(S_{\alpha_i} = \ell^{-1}(\alpha_i)\). Then, by the formula (2.6) and taking into
account that by Lemma 2.3 \(d_{S_{\alpha_i}}(f)\) is the same for all \(\alpha_i \in B\) and is equal to \(d(B)\)
we get:
\[
\deg(f) = \sum_B \sum_{x_i \in f^{-1}(B)} \text{sign } \det Df(x_i) = \sum_B \left( \sum_{\alpha_i \in B} \text{sign}(a_i) \cdot \sum_{\beta_{ij} \in S_{\alpha_i}} \text{sign}(b_{ij}) \right)
\]
\[
= \sum_B \left( \sum_{\alpha_i \in B} \text{sign}(a_i) d_{S_{\alpha_i}}(f) \right) = \sum_B \left( \sum_{\alpha_i \in B} \text{sign}(a_i) d(B) \right) = \sum_B \Delta(B) \cdot d(B),
\]
where the summation is taken over a finite number of bands \(B\), (for some of them
perhaps \(\deg(B) = 0\)). This gives us the formula (2.5).

Studying the derivative \(Df(P)\) we consider the local planar coordinate system near
\(P\) given by the stereographic projection from the antipole \((0, 0, 2)\). In this system \(0\)
is a fixed point representing \(P\) and circles represent paridians (see Figure 1). The \(x\)-axis
(with the point at infinity, which we will not mention later) is also a paridian. To
simplify the notation we use the same letter \(f\) for our map in this coordinate system.

Lemma 2.7. Assume that one of the following conditions hold:
(1) there is a preband $B \neq (0,1)$,
(2) $B = (0,1)$ is a preband and $|d(B)| > 1$.

Then $Df(P) = 0$.

Proof. We use the planar coordinates, so $P$ becomes 0, and proper paridians become the $x$-axis and circles tangent to it at 0 (see Figure 1).

We will start by proving that $Df(0)$ maps paridians that are circles to paridians. Let $L_r$ be a paridian of diameter $r$. Fix $\varepsilon > 0$. By the definition of the derivative, there exists $\delta_0$ such that for every vector (point) $v$ if $\|v\| \leq \delta_0r$ then
\[ \|f(v) - Df(0)(v)\| \leq \varepsilon\|v\|. \]

For $\delta \in (0,\delta_0)$, define the map $g_\delta$ by $g_\delta(v) = f(\delta v)/\delta$. Since $f$ maps paridians to paridians, the set $g_\delta(L_r)$ is contained in a paridian. For $v \in L_r$ we have $\|\delta v\| \leq \delta r$, and $Df(0)$ is linear, so
\[ \|g_\delta(v) - Df(0)(v)\| = \frac{1}{\delta}\|f(\delta v) - Df(0)(\delta v)\| \leq \frac{1}{\delta}\varepsilon\|\delta v\| \leq \varepsilon r. \]

Since $\varepsilon > 0$ was arbitrary, this means that $Df(0)$ is the uniform limit of maps $g_\delta$ as $\delta \to 0$. Therefore, $Df(0)(L_r)$ is contained in a paridian.

In case (1) there is always a proper paridian that is mapped by $f$ to $P$, but then the derivative $Df(0)$ in $x$-direction is 0. This direction is an eigendirection of $Df(0)$, so the only possibility for paridians to be mapped by $Df(0)$ to paridians is that $Df(0) = 0$.

Let us now consider case (2). Then sufficiently small proper paridians cannot be mapped to the $x$-axis, unless they are mapped to 0. Therefore, for sufficiently small $\delta$, the set $g_\delta(L_r)$ is either a proper paridian other than the $x$-axis, or $\{0\}$. Hence, the same is true for $Df(0)$ instead of $g_\delta$. This means that either $\det(Df(0)) \neq 0$ or $Df(0) = 0$.

If $\det(Df(0)) \neq 0$, then $f$ is one-to-one in a small neighborhood of 0. However, since $|d(B)| > 1$, in such neighborhood there are paridians, which are mapped by $f$ not in the one-to-one manner. This is a contradiction, so we must have $Df(0) = 0$. \( \square \)

Now we want to compare the number of fixed points of a paridional map $f$ with its degree if $|\deg(f)| > 1$.

We have to show that in each preband the map $\varphi$ has a fixed point (which must be different from 0 and 1 which are in the set $A$).

Lemma 2.8. Assume that $B \neq (0,1)$ is a preband with $d(B) > 0$, or $B = (0,1)$ with $|d(B)| > 1$. Then there is a fixed point of $\varphi$ in $B$.

Proof. We have to show that the graph of $\varphi$ on $B$ crosses the diagonal. This is clear if the closure of $B$ does not contain 0 and 1. Suppose now that one endpoint of $B$ is 0 (with 1 the situation is analogous).

If $\Delta(B) = -1$ then clearly $\varphi$ has a fixed point in $B$. Suppose that $\Delta(B) = 1$. By Lemma 2.7, $Df(P) = 0$. Therefore, $\varphi'(0) = 0$, so the point $(\varepsilon, \varphi(\varepsilon))$ lies below the diagonal for small $\varepsilon > 0$. thus, again the graph of $\varphi$ on $B$ crosses the diagonal. \( \square \)

Corollary 2.9. Let $B$ be a preband. Then either there is a fixed point of $\varphi$ in $B$, or $|\deg(f)| \leq 1$. 
A fixed point of \( \varphi \) in a preband gives us a proper paridian that is mapped by \( f \) to itself. The next step is to estimate the number of fixed points in this paridian, other than \( P \), compared to the degree of \( f \) restricted to this paridian. In general, any continuous map of a circle to itself of degree \( d \) has at least \( |d - 1| \) fixed points (cf. [12]). Thus, if \( d \leq 0 \), this number is at least \( |d| + 1 \). Taking into account that in our case one of those points is \( P \), we get for any preband \( B \) at least \( |d(B)| \) other fixed points. This leaves us with the case when \( d(B) > 0 \), when we have to take into account the local behavior of the map at \( P \).

**Lemma 2.10.** Assume that \( B \neq (0, 1) \) is a preband with \( d(B) > 0 \) or \( B = (0, 1) \) with \( d(B) > 1 \); and \( y \in B \) is a fixed point of \( \varphi \). Then there are at least \( d(B) \) fixed points of \( f \), other than \( P \), in \( \ell^{-1}(y) \).

**Proof.** By Lemma 2.7, \( Df(P) = 0 \). In our planar coordinate system this fact implies that the derivative of \( f \) at 0 in the \( x \)-direction is also 0. This means that \( f_y : S^1 \to S^1 \) (conjugate to \( f \) restricted to the paridian \( \ell^{-1}(y) \)) has the derivative 0 at 0 (here 0 \( \in \mathbb{T} \) corresponds to \( P \)).

Now consider \( \tilde{f}_y : [0, 1] \to \mathbb{R} \), the lift of \( f_y \). The fact that the derivative is 0 implies that the graph \( \tilde{f}_y \) intersects \( d(B) \) straight lines of the form \( y = x + k \), \( k = 0, \ldots, d(B) - 1 \), and thus there must be at least \( d(B) \) fixed points of \( \tilde{f}_y \) other than 0. This completes the proof. \( \square \)

By Lemma 2.2, \( P \) is always a fixed point of \( f \) in \( \ell^{-1}(y) \), where \( y \) is a fixed point of \( \varphi \). Thus, as a straightforward consequence of Lemma 2.10, we get the following corollary.

**Corollary 2.11.** Let \( |\deg(f)| > 1 \) and let \( B \) be a preband. Then

\[
\text{# Fix }|\ell^{-1}(B)| - 1 \geq |d(B)|.
\]

Now we get the estimate of the number of fixed points of a paridianal map \( f \) with \( |\deg(f)| > 1 \).

**Theorem 2.12.** For a paridianal map \( f \), let \( |\deg(f)| > 1 \). Then

\[
\text{# Fix } f \geq |\deg(f)| + 1.
\]

**Proof.** Using Lemma 2.6 and Corollary 2.11 we get

\[
|\deg(f)| = \left| \sum_{B \in \mathcal{B}(f)} \deg(B) \right| \leq \sum_{B \in \mathcal{B}(f)} |d(B)| \leq \sum_{B \in \mathcal{B}(f)} \text{# Fix }|\ell^{-1}(B)| - 1 \leq \text{# Fix } f - 1.
\]

\( \square \)

Taking into account Lemma 2.2 in case \( |\deg(f)| \leq 1 \), we get for arbitrary degree the following corollary, which is the main results of this section.

**Corollary 2.13.** If \( f \) is a paridianal map, then it has at least \( |\deg(f)| \) fixed points.

If \( f \) is a paridianal map then \( f^n \) is also a paridianal map. Moreover, \( \deg(f^n) = \deg(f)^n \). As a consequence, we obtain the conclusion, in which we obtain (1.1) in a stronger version with the upper limit replaced by the lower limit.

**Corollary 2.14.** If \( f \) is a paridianal map, then the lower growth rate of the number of fixed points of \( f^n \) is at least \( \log |\deg(f)| \).
3. Rabbit maps

Here we use the rabbit foliation, defined in the introduction, and consider smooth (of class $C^1$) sphere maps preserving this foliation, the rabbit maps. We denote the family of all rabbit maps by $\mathcal{R}$.

Let us be more precise. The pole is still $P$ and there are two ears, which are the sets given (in the plane, after the stereographic projection, see Figure 2) in the polar coordinates by $\{(r,\theta) : r = c\sin(2\theta), \; c \in [0,1]\}$ in the first quadrant and $\{(r,\theta) : r = -c\sin(2\theta),\; c \in [0,1]\}$ in the second quadrant. We denote them by $E_1$ and $E_2$, and their union by $E$. Each ear is foliated by the appropriate curves $r = \pm c\sin(2\theta)$; $P$ is a singular point and belongs to all leaves. We divide the rest of the sphere, $G = (S^2 \setminus E) \cup \{P\}$ into two areas $G_1$ which is the part of $G$ in the first and the second quadrant and $G_2$ which is the part in the third and fourth quadrant and define the foliation in the following way. In $G_1$ its leaves are given by $r = 2\sin\theta\sqrt{\cos^2\theta + c}$, where $c \geq 0$. In $G_2$ the leaves are circles $r = c\sin\theta$. Additionally, the $x$-axis (with the point at infinity) is also a leaf.

Two leaves are very special. They are boundaries of the ears, $\partial E_1$ and $\partial E_2$. The leaves other than $\partial E_1$, $\partial E_2$ and $\{P\}$ will be called regular.

Fix a rabbit map $f$.

**Lemma 3.1.** If $\deg(f) \neq 0$ then $f(P) = P$.

*Proof.* If $f(P) \neq P$ then, since every leaf contains $P$, the image of the whole sphere is contained in one leaf. Therefore $\deg(f) = 0$. \qed

In the rest of the paper we assume that $\deg(f) \neq 0$. In particular, $f(P) = P$. We will call the leaves of the rabbit foliation simply *leaves*.

As in the preceding section, we want to define the map $\ell$. However, now we need it only in a one-sided (from the side of $G_1$) neighborhood $U$ of $\partial E$. There we parametrize the set of leaves by an interval $[0,\alpha)$, for a small $\alpha > 0$, where $0$ corresponds to the union of two leaves $\partial E_1$ and $\partial E_2$. Thus, we have a projection $\ell : U \to [0,\alpha)$. We want to show that there exists $\varepsilon \in (0,\alpha)$ and $\varphi : [0,\varepsilon) \to [0,\alpha)$ such that $\varphi \circ \ell = \ell \circ f$.

We have the orientation of proper paridians consistent throughout the sphere. Therefore if $L, K$ are leaves other than $\{P\}$, and $f(L) \subset K$, the degree $d_L(f)$ of the map $f|_L : L \to K$ is well defined.

We start with three lemmas.

**Lemma 3.2.** Let $(L_n)$ and $(K_n)$ be sequences of leaves other than $\{P\}$, such that $f(L_n) \subset K_n$, $d_{L_n}(f) \neq 0$, and the sequence $(K_n)$ converges to $\partial E$ from the side of $G_1$. Then the sequence $(L_n)$ also converges to $\partial E$ from the side of $G_1$.

*Proof.* Observe that since $d_{L_n}(f) \neq 0$, we have $f(L_n) = K_n$. If there is a subsequence of $(L_n)$ convergent to a leaf $L$, then $f(L) = \partial E$, but $\partial E$ is not contained in any leaf, a contradiction. However, the only possibility that there is no subsequence of $(L_n)$ convergent to a leaf is that $(L_n)$ converges to $\partial E$ from the side of $G_1$. \qed

**Lemma 3.3.** We have $f(\partial E) = \partial E$. Moreover, either $f(\partial E_1) = \partial E_1$ and $f(\partial E_2) = \partial E_2$, or $f(\partial E_1) = \partial E_2$ and $f(\partial E_2) = \partial E_1$. If a leaf $L \subset G_1$ is sufficiently close to $\partial E$, then $|d_L(f)| = 1$.\[\]
Proof. Choose a point \( x \in \partial E \) and a sequence \((x_n)\) convergent to \( x \) from the side of \( G_1 \), which are regular values of \( f \). Since \( \deg(f) \neq 0 \), for each \( n \) there is a point \( y_n \in f^{-1}(x_n) \) such that if \( L_n \) is the leaf containing \( y_n \), then \( d_{L_n}(f) \neq 0 \) (if \( d_{L_n}(f) = 0 \) then the sum of the signs of the Jacobian of \( f \) over all elements of \( L_n \cap f^{-1}(x_n) \) is zero). Then, by Lemma 3.2, the sequence \((L_n)\) converges to \( \partial E \) from the side of \( G_1 \). The leaf \( K_n = f(L_n) \) contains \( x_n \), so the sequence \((K_n)\) also converges to \( \partial E \) from the side of \( G_1 \). Therefore, \( f(\partial E) = \partial E \).

The second statement of the lemma follows from the fact that \( f \) maps leaves to leaves. The third statement follows from the second one. \( \square \)

Lemma 3.4. There exists a one-sided (from the side of \( G_1 \)) neighborhood \( U \) of \( \partial E \), an interval \([0, \alpha)\), a projection \( \ell : U \to [0, \alpha) \), and a map \( \varphi : [0, \varepsilon) \to [0, \alpha) \) for some \( \varepsilon \in (0, \alpha) \), such that:

(a) \( \ell^{-1}(0) = \partial E \),
(b) \( \ell^{-1}(t) \) is a leaf for \( t > 0 \),
(c) \( \varphi \circ \ell = \ell \circ f \),
(d) \( \varphi(0) = 0 \),
(e) \( \varphi \) is continuous on \([0, \varepsilon)\),
(f) \( d_{\ell^{-1}(t)}(f) \) is independent of \( t \in (0, \varepsilon) \) and its modulus is 1.

Proof. Existence of \( U \), \( \alpha \) and \( \ell \) satisfying (a) and (b) is obvious. Since \( f \) maps leaves to leaves, it is clear how to define \( \varphi \) so that it satisfies (c). By Lemma 3.3, for a sufficiently small \( \varepsilon > 0 \), if a leaf \( L \) is contained in \( \ell^{-1}([0, \varepsilon)) \), then \( f(L) \subset U \). Therefore \( \varphi \) is well defined in some \([0, \varepsilon)\). Property (d) follows from Lemma 3.3. Properties (e) and (f) follow from Lemma 3.3 and continuity of \( f \). \( \square \)

Now we can prove the main theorem of this section.

Theorem 3.5. If \( f \) is a rabbit map, then \( |\deg(f)| \leq 1 \). Moreover, it has at least \( |\deg(f)| \) fixed points.

Proof. We may assume that \( \deg(f) \neq 0 \). Choose a point \( x \in G_1 \), which is a regular value of \( f \), and lies sufficiently close to \( \partial E \) (but far from \( P \)). By Lemma 3.2, all elements of \( f^{-1}(x) \) which belong to leaves \( L \) such that \( d_L(f) \neq 0 \), lie in \( \ell^{-1}([0, \varepsilon)) \). By Lemma 3.4 (f) and the same arguments as in the proof of Lemma 2.6, \( \deg(f) \) is equal to the common value of \( d_{\ell^{-1}(t)}(f) \) (which has modulus 1) multiplied by the sum of the signs of \( \varphi' \) at the points of \( \varphi^{-1}(\ell(x)) \). This sum has modulus not larger than 1, so \( |\deg(f)| \leq 1 \).

If \( \deg(f) \neq 0 \), then by Lemma 3.1 \( f \) has a fixed point. Therefore, \( f \) has at least \( |\deg(f)| \) fixed points. \( \square \)

From this theorem we get an obvious corollary.

Corollary 3.6. If \( f \) is a rabbit map, then the lower growth rate of the number of fixed points of \( f^n \) is at least \( \log |\deg(f)| \).

References

[1] I. K. Babenko, S. A. Bogatyi, The behavior of the index of periodic points under iterations of a mapping, Math. USSR Izv. 38 (1992), 1–26.
[2] J. P. Boroniński, A fixed point theorem for the pseudo-circle, Topology Appl. 158 (2011), 775–778.
[3] G. Brumfiel, ‘Overwhelming’ evidence for monopoles, Nature, 3 September 2009.
[4] G. Graff and J. Jezierski, On the growth of the number of periodic points for smooth self-maps of a compact manifold, Proc. Amer. Math. Soc. 135 (2007), no. 10, 3249–3254.
[5] G. Graff and J. Jezierski, Minimal number of periodic points for smooth self-maps of $S^3$, Fund. Math. 204 (2009), no. 2, 127–144.
[6] G. Graff, M. Misiurewicz and P. Nowak-Przygodzki, Periodic points of latitudinal maps of the $m$-dimensional sphere, Discrete Cont. Dynam. Sys. 36 (2016), no. 11, 6187–6199.
[7] G. Graff, M. Misiurewicz and P. Nowak-Przygodzki, Shub’s conjecture for smooth longitudinal maps of $S^m$, J. Difference Equ. Appl., DOI 10.1080/10236198.2018.1449840.
[8] L. Hernández-Corbato and F. R. Ruiz del Portal, Fixed point indices of planar continuous maps, Discrete Contin. Dyn. Syst. 35 (2015), 2979–2995.
[9] G. Honorato, J. Iglesias, A. Portela, A. Rovella, F. Valenzuela, J. Xavier, On the growth rate inequality for periodic points in the two sphere, 2017, preprint, (arXiv:1707.00144v1).
[10] J. Iglesias, A. Portela, A. Rovella and J. Xavier, Dynamics of annulus maps II: Periodic points for coverings, Fund. Math. 235 (2016), no. 3, 257–276.
[11] J. Iglesias, A. Portela, A. Rovella and J. Xavier, Dynamics of annulus maps III: completeness, Nonlinearity 29 (2016) 2641–2656.
[12] B. J. Jiang, Lectures on the Nielsen Fixed Point Theory, Contemp. Math. 14, Amer. Math. Soc., Providence 1983.
[13] N. G. Lloyd, Degree theory, Cambridge Tracts in Mathematics, no. 73. Cambridge University Press, Cambridge-New York-Melbourne, 1978.
[14] M. Misiurewicz, Periodic points of latitudinal maps, J. Fixed Point Theory Appl. 16 (2014), no. 1–2, 149–158.
[15] C. Pugh and M. Shub, Periodic points on the 2-sphere, Discrete Contin. Dynam. Sys. 34 (2014), 1171–1182.
[16] M. Shub, Dynamical systems, filtration and entropy, Bull. Amer. Math. Soc. 80 (1974), 27–41.
[17] M. Shub, All, most, some differentiable dynamical systems, Proceedings of the International Congress of Mathematicians, Madrid, Spain, (2006), European Math. Society, 99–120.
[18] M. Shub and F. Sullivan, A remark on the Lefschetz fixed point formula for differentiable maps, Topology 13 (1974), 189-191.

Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland
E-mail address: graff@mif.pg.gda.pl

Department of Mathematical Sciences, Indiana University–Purdue University Indianapolis, 402 N. Blackford Street, Indianapolis, IN 46202, USA
E-mail address: mmisiure@math.iupui.edu

E-mail address: piotrn@wp.pl