Ori and Soen have proposed a spacetime which has closed causal curves on the boundary of a region of normal causality, all within a region where the weak energy condition (positive energy density) is satisfied. I analyze the causal structure of this spacetime in some simplified models, show that the Cauchy horizon is compactly generated, and argue that any attempt to build such a spacetime with normal matter might lead to singular behavior where the causality violation would otherwise take place.

I. INTRODUCTION

In the absence of any restrictions on the stress-energy tensor, general relativity permits an arbitrary spacetime. One simply writes down the desired metric, computes the curvature, and solves Einstein’s equations in reverse to find the required matter content. In particular, the spacetime may contain closed timelike curves (CTC’s), future-directed timelike paths which return to the same point in spacetime. Thus general relativity always permits time travel unless one restricts the matter content that one can use as a source.

To prove theorems about the properties of a spacetime, one uses energy conditions, i.e., restrictions on the stress-energy tensor $T_{\mu\nu}$. Perhaps the most important of these is the weak energy condition (WEC), which states that every timelike observer must see a nonnegative energy density, i.e., that $T_{\mu\nu}V^\mu V^\nu \geq 0$ for any timelike vector $V^\mu$. Tipler and Hawking have shown that if a spacetime obeys the weak energy condition, closed timelike curves cannot be produced in a compact region. They proved these theorems by considering the Cauchy horizon, which is the boundary of the region whose entire past intersects some initial surface $S$. The Cauchy horizon is composed of null geodesic generators which can have no past endpoints. If this horizon is to arise from a compact region, each generator must wind around indefinitely inside this region as one goes into the past. This in turn requires the generators to originate as a closed null geodesic called the fountain. This means they must be defocused, which is impossible if the weak energy condition is obeyed.

However, Ori and Soen exhibit a spacetime which casts some doubt on the effectiveness of these theorems at preventing the construction of time machines. This spacetime contains a closed null geodesic $N$; all points in the future of $N$ have causality violation, while all points in its chronological past have normal causality. The weak energy condition is obeyed in a region of finite size surrounding $N$. The Ori-Soen spacetime is sketched in Fig. 1.

How can this spacetime evade the theorems of Tipler and Hawking? The answer is that $N$ is not in fact the fountain where the causality violating region originates, but rather a place where the Cauchy horizon terminates. The origin of the causality violation lies outside the region where the weak energy condition is obeyed.

Ori and Soen suggest that one could perhaps produce causality violation without WEC violation, as follows: Let $U$ be the subset of the Ori-Soen spacetime which does not violate WEC and which is not in the future of any WEC-violating region. Now imagine that we specify the conditions on $U$, which we are presumably free to do, and then let the rest of the spacetime evolve according to some WEC-obeying equation of state. If the resulting spacetime is continuous, causality must still be violated, because the closed null geodesic $N$ lies on the boundary of $U$. Thus while the Ori-Soen spacetime does not violate...
the Tipler and Hawking theorems, it does seem to open a possibility that causality violation could nevertheless be produced using normal matter. (See [8] for another such possibility.)

In this paper, I will study a simplified model with the same properties as the Ori-Soen spacetime. I will analyze its causal structure and show how the unusual situation Ori and Soen describe can arise. I will then argue that the procedure described above may not in fact result in a continuous spacetime at the place where the closed null curve $N$ would otherwise appear.

II. SIMPLIFIED SPACETIME

The spacetime of [5–7] has causality violation in a toroidal region. We will simplify the situation as follows: first we will take the “cylindrical” metric of [6] in which the torus has been straightened out. We will take the $z$ axis to lie on the axis of the cylinder. This spacetime does not have CTC’s, but we can re-introduce them by making the $z$ direction periodic. We will then eliminate the azimuthal direction to produce a 2+1-dimensional spacetime with CTC’s lying in a strip. We will use the metric

$$ds^2 = f(t)[-(dt-h(x)t\,dz)^2 + dz^2 + (dx-h(x)bx\,dz)^2]$$

$$= f(t)[-dt^2 + 2h(x)t\,dt\,dz + (1-h(x)^2)dz^2 + b(x)bx\,dz\,dx + dx^2]$$

with $t > 0$, $z$ periodic, and $f(t) > 0$. The function $h(x)$ is a window function [5–7], with a parameter $d$,

$$h(x) = \begin{cases} 
1 - (x/d)^4 & \text{for } x < d \\
0 & \text{for } x > d.
\end{cases}$$

Outside the strip $x \in [-d,d]$, the spacetime is conformal to Minkowski space. The function $f(t)$ provides a conformal factor that does not affect the causal structure but allows energy conditions to be satisfied. We will take $f(1) = 1$ for simplicity, and adjust $f'(1)$ and $f''(1)$ as required. Since $-\det g = f(t)^3 > 0$ everywhere, the metric is never singular.

Since the $z$ coordinate is periodic, a path moving purely in the $z$ direction is closed. For points where $1 + h(x)^2[1 + b(x)^2] = 0$ this path is null. At $x = 0$, $t = 1$ it is a closed null geodesic which we call $N$, following [5–7]. For larger $t$, such paths are closed timelike curves.

However, the intrinsic metric of a surface given by $t = t_0 < 1$ is

$$dt^2 = f(t)[(1-h^2t_0^2)dz^2 + (dx-h(x)bx\,dz)^2],$$

which is positive definite. Thus the space with $t < 1$ can be foliated in constant-time surfaces and so has normal causality.

To study the causal structure, we first note that nothing depends on the $z$ coordinate. We will be concerned only with the set of motions in $t$ and $x$ that are possible within the light cone, when any motion in $z$ is permitted. Thus we project the light cone into the $t$-$x$ plane. If the $z$ direction is timelike, then motion in any direction in the $t$-$x$ plane is possible. If not, there will be maximum and minimum values of $dt/dx$ for a causal curve, which are given by

$$\left.\frac{dt}{dx}\right|_{\text{max}} = \frac{h^2bx\pm\sqrt{1 + h^2b^2x^2 - h^2t^2}}{1 + h^2b^2x^2}, \quad (4)$$

We will also define a function $t_n(x) = \sqrt{h^2 + b^2x^2}$ which gives the boundary of the region where the $z$ direction is timelike.

Now we can find the Cauchy horizon. Points with $t \geq t_n$ lie on closed causal curves with $t$ and $x$ constant, so any such point is in the causality violating region. It is clear that every point of the spacetime has causality violating points in its future, so a point is on a closed causal curve exactly if there is a point with $t \geq t_n$ in its past, and the Cauchy horizon is just the boundary of the causality violating region.

Now we integrate Eq. (4) to get the Cauchy horizon. There are two different situations depending on the magnitude of $dt/dx$. As $t \to t_n$, the projected light cones open out so that maximum and minimum values of $dt/dx$ become the same. If this value is smaller than $dt_n/dx$, then null rays leave the region where the $z$ direction is timelike and go outward, whereas if it is larger, then null rays leave the region where the $z$ direction is timelike and go inward.

This behavior can be understood by considering the case where $h(x) = 1$ everywhere, so that the entire spacetime, rather than just a strip, is modified. In this case we have $t_n(x) = \sqrt{1 + b^2x^2}$, and at $t = t_n(x)$,

$$\left.\frac{dt}{dx}\right|_{\text{max}} = \frac{bxt}{\sqrt{1 + b^2x^2}} = \frac{bx}{\sqrt{1 + b^2x^2}} \quad (5)$$

whereas

$$\frac{dt_n}{dx} = \frac{b^2x}{\sqrt{1 + b^2x^2}}. \quad (6)$$

Thus if $b > 1$ we have $dt/dx < dt_n/dx$, which leads to the situation shown in Fig. 2. Future-directed null curves cross $t = t_n$ in the direction of increasing $|x|$. The Cauchy horizon is the future-directed null curve from $N$ with $dt/dx$ minimal. In this case, the Cauchy horizon is compactly generated: it arises at the curve $N$.

In contrast, if $b < 1$, we have $dt/dx > dt_n/dx$, which leads to the situation shown in Fig. 3. Null curves cross $t = t_n$ in the direction of decreasing $|x|$. Any point with $t > 1$ is reachable by such a curve, so the Cauchy horizon is just the surface $t = 1$. It comes in from infinity and thus is not compactly generated.
It is straightforward to compute the stress-energy tensor for the metric of Eq. (1) with $h = 1$. First we look at $T_{\mu \nu}$ at $t = 1$, $x = 0$ projected on the tangent vector to $N$ given by $V^z = 1$, $V^t = V^x = 0$,

$$T_{\mu \nu} V^\mu V^\nu = T_{zz} = \frac{1}{8\pi G} h(1-b) .$$  

If $b > 1$, $T_{zz} = 0$, the weak energy condition cannot be obeyed, as one would expect since the Cauchy horizon is compactly generated in this case.

If $b < 1$, $T_{zz} > 0$, but we need also to look at other projections of $T^\mu_{\nu}$. The matrix of mixed-index components $T^\mu_{\nu}$ at $t = 1$ is

$$\frac{1}{32\pi G} \begin{pmatrix} 4b^2 + 2f'(1+b) & 4b(1-b) & 0 \\ 3f'^2 - 2f'' & 4b + 2f'(1+b) & 0 \\ bx(3f'^2 - 2f'') & -2b(1-b)x(f' + 2) & 4 + 4f'' \end{pmatrix} .$$

The eigenvalues of $T^\mu_{\nu}$ are $-\rho$, $p_1$, and $p_2$, where $\rho$ is the energy density and $p_i$ are the principal pressures. In terms of the density and pressures, the weak energy condition can be written $\rho \geq 0$ and $\rho + p_i \geq 0$. We can write also the dominant energy condition, $\rho \geq 0$ and $|p_i| \leq \rho$, and the strong energy condition, $\rho + p_i \geq 0$ and $\rho + \sum p_i \geq 0$. We can choose parameter values so that all these conditions will be satisfied. For example,

$$b = 0.02, f' = -1, f'' = 0.6$$

give $\rho = 0.59$, $p_1 = 0$, and $p_2 = -0.41$.

Now we include the full form of $h(x)$ from Eq. (2). At $t = t_a(x)$, we have

$$\frac{dt}{dx} \bigg|_{\text{max}} = \frac{hbx}{\sqrt{1 + h^2b^2x^2}}$$

$$\frac{dt}{dx} = \frac{-h^{-2}h' + hbx^2}{\sqrt{1 + h^2b^2x^2}} .$$

Null curves leaving the region where the $z$ direction is timelike go inward if

$$1 - b > \frac{-h'}{h^2bx} .$$

As $x \rightarrow 0$, $h' \sim x^3$, so the right hand side of Eq. (11) vanishes, and the condition is satisfied if $b < 1$. However, as $x \rightarrow d$, $h'/h \rightarrow -\infty$, so the condition is never satisfied. Thus null curves leaving the region where the $z$ direction is timelike go inward for small values of $x$, but outward for large values of $x$. There is some intermediate point where $-h'/h^2bx = 1 - b$, from which the two families of null curves diverge. Following [1] we will call this point (really a curve with fixed $x$ and $t$) $N'$. This situation is shown in Fig. (1).

In this case, the Cauchy horizon arises from $N'$ and spreads out in both directions. The generators going inward terminate at $x = 0$, but the generators going outward continue to infinity. By the theorems of Hawking...
FIG. 4. Causal structure of the full spacetime. In the region above the thick line, motion in the $z$ direction is timelike. The thin lines are null curves leaving this region. The dashed line is the Cauchy horizon, which is tangent to the thick line at $N'$. To the left of $N'$, the future-directed null curves go inward and backward in $t$; to the right, they go outward and forward in $t$.

FIG. 5. Density and pressures for the parameter values of Eq. (9) and $d = 40$. The solid line is $\rho$, the dotted lines are $|p|$, and the dashed line is $\rho + p_1 + p_2$.

and Tipler, the weak energy condition must be violated at $N'$, but it is possible to obey the weak, strong, and dominant energy conditions in a neighborhood of the slice $t = 1$. To see this, we can compute the eigenvalues of the full $T_{\mu\nu}$, e.g., using Mathematica. With the parameter values of Eq. (4) and $d = 40$, we find the results shown in Fig. 5. We see that for all values of $x$ we have $\rho > 0$, and $\rho > |p|$, so the dominant and weak energy conditions are satisfied. We also see that $\rho + p_1 + p_2 > 0$, so the strong energy condition is satisfied. Since all these conditions are satisfied strongly, i.e., with $>$ instead of $\ge$, they must be satisfied over some finite range of $t$ near 1.

III. DISCUSSION

From the point of view of the Tipler/Hawking theorems, the Ori-Soen spacetime is perfectly normal. It contains a region of closed timelike curves with a compactly-generated Cauchy horizon, and it violates the weak energy condition where the horizon arises. However, there still remains the original question of whether one can use this spacetime to generate causality violation with normal matter.

To make such an attempt, one removes from the spacetime all the areas where the weak energy condition is violated, and all areas that could be influenced by the WEC-violating regions. Thus one must remove at least a region around the point $N'$, and its causal future. This future contains $N$, but does not include any point with $x = 0$ and $t < 1$. We thus have the situation shown in Fig. 6. As discussed in [5,6], the causality violation occurs in the boundary of the region that we can control. In this case, we have just the single closed null curve $N$ that has this property; the entire rest of the causality-violating region has been removed. The system has no chronology violation, since $N$ is null. But it does have causality violation at $N$, which cannot be avoided unless there is a metric discontinuity there.

However, there is some reason to believe that such a discontinuity might be expected. As shown in Fig. 6, the past volume of $N$ is much larger than the past volume of points on the $t$ axis immediately before $N$. Past-volume discontinuities always exist when there is a CTC-containing region, because every point in such region is in the past of every other point. However, the present situation is different in that the discontinuity has appeared already on the supposed boundary of the CTC-containing region, and does not depend on the interior. As noted in [6], $N$ is also a curve where many null hypersurfaces intersect.

It seems reasonable to imagine that, in a realistic field theory, the past-volume discontinuity would lead to a piling up of modes at $N$, and consequently a discontinuity in the metric, which would prevent the curve $N$ from being causal. Similar dynamics have been discussed in [3], although there is some question about how general such
FIG. 7. Causal structure preceding the region of causality violation. Thin lines are the most left-going null paths. The dotted line near the origin is the boundary of the past of points immediately before $N$, and the dotted line at the right is the boundary of the past of $N$ in the unmodified spacetime. Removing some space around $N$ might reduce the volume of the past of $N$, but there will still be a discontinuity.

Another way to make this argument is to note that in a continuous spacetime, it should be possible to make small perturbations on the initial surface to produce another similar spacetime. But here, because of the divergence in past volume, small perturbations would not remain small at the point $N$.

The past-volume discontinuity is not accidental, but is required for causality violation to take place on the boundary of the controlled region. If the past volumes were continuous, then all points near the Cauchy horizon would be in the future of points near the fountain. When one excised the WEC-violating region and its future, one would remove a neighborhood around the entire Cauchy horizon, and so causality violation could not take place on the boundary of the controlled region.

In spite of the above arguments, I cannot rule out the alternative possibility that a naked singularity could form in place of the region surrounding $N'$ where the weak energy condition is violated. The generators of the Cauchy horizon could then arise from this singularity, and the theorems of Hawking and Tipler would be satisfied, because the singularity renders the region non-compact. In such a case, we would have succeeded in creating causality violation using normal matter, as described in the introduction. Although the details of the causality violation could not be predicted because of the singularity, we could predict that causality violation will occur, as long as there is no metric discontinuity at $N$ as above.

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\[ \text{\textsuperscript{1}}\text{This idea is due to L. H. Ford} \]