When can the Fokker–Planck equation describe anomalous or chaotic transport? Intuitive aspects

D F Escande\textsuperscript{1,2} and F Sattin\textsuperscript{2}

\textsuperscript{1} UMR 6633 CNRS-Université de Provence, Marseille, France
\textsuperscript{2} Consorzio RFX, Associazione EURATOM-ENEA sulla fusione, Padova, Italy

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Abstract
The Fokker–Planck equation (FPE) is a quite flexible tool for experimental and numerical data analysis. However, advective–diffusive effects may compete with a curvature pinch or the impact of a localized source to determine profiles of transported quantities. Depending on the statistics of interest and/or on the radial extension of the transport process, the same dynamical system may be found diffusive or dominated by its Lévy flights. The Kubo number plays a leading role in determining whether diffusive transport is in the quasilinear regime. It would be advisable to systematically compute it from experimental or numerical data, in order to clarify the issue of its value in magnetic fusion turbulence.

1. Introduction

The Fokker–Planck equation (FPE) is a basic model for the description of transport processes in several scientific fields. It has been used a lot in plasma physics to model chaotic and/or collisional kinetic effects. Furthermore, the FPE backs up the advection–diffusion picture of anomalous transport in magnetized thermonuclear fusion plasmas. Though very popular, the advective–diffusive picture underlying the FPE breaks down in some cases. This was shown for electron dynamics due to Langmuir waves [1], for the transport of tracer particles suddenly released in pressure-gradient-driven turbulence [2], and for pollutant transport in fluid dynamics. These facts triggered a series of studies where the Brownian paradigm was abandoned, and transport was described in terms of Lévy jumps, and of fractional diffusion models [3–6]. This sets the issue: when is the FPE relevant for anomalous or chaotic transport and when is it not? Reference [7] provided a more precise answer to this issue by showing that (i) for particle transport ruled by chaotic Hamiltonian dynamics, the FPE can be justified for generic particle transport provided that there is enough randomness in the Hamiltonian describing the dynamics, (ii) except for one degree of freedom, the two transport coefficients of the FPE (diffusion coefficient and dynamic friction) are not correlated, (iii) depending on the kind of averaging performed on the orbits, a dynamical system with Lévy flights may
be found diffusive or dominated by these flights, (iv) the FPE may work even whenever
the dynamics of individual particles exhibit strong trapping motion, (v) diffusion is justified
by locality of trapping in phase space, or by locality in velocity of particle resonance with
fluctuating fields, then leading to a quasilinear (QL) estimate, (vi) for a system described by
a master equation (ME) correctly modeled by the FPE in the absence of a source, adding a
particle source that is narrower than the mean random step at work in the ME makes the FPE
to fail, while the density provided by the ME displays spatial features that are not related
at all to the transport coefficients. Reference [7] aimed at giving all this information in a
convincing way for theoreticians in a paper not longer than this one. Unfortunately it fell
short of being pedagogical. In contrast, this paper parallels the corresponding invited talk, and
aims at a more pedagogical presentation of the part of the previous paper that may be directly
relevant to numericists and experimentalists. While doing so, we point out several caveats
for experimental or numerical turbulence modeling. This work, though primarily meant for
magnetic fusion physicists is applicable to other plasmas and to tracer transport in fluids. We
use the word ‘density’ to qualify the transported quantity, but our reasoning applies to heat and
momentum as well.

2. What the FPE can do

In one dimension the FPE reads

\[ \frac{\partial}{\partial t} n(x, t) = - V(x) n + \frac{\partial^2}{\partial x^2} \left(D(x)n\right) + S(x) \]

\[ = - \partial_x \Gamma + S(x), \] (1)

where \( n \) is the density for a generic scalar quantity, \( D \) is the diffusion coefficient and \( V \)
is the dynamic friction, often referred to as a pinch velocity in the fusion community; we added \( S \)
for future discussion of the effect of a source term, but as yet \( S = 0 \). Finally, \( \Gamma \) is the flux
\[ \Gamma = Vn - \partial_x \left(Dn\right). \]

The FPE is endowed with a very high flexibility that must be pointed out. We now display
it in a series of steps, starting with trivial ones. If \( D = 0, \) and \( V = \) constant, the FPE describes
a ballistic motion. If \( V = 0, \) and \( D = \) constant, the FPE describes a purely diffusive motion.
When both \( V \) and \( D \) are non-vanishing constants, a cross-over scale length comes in naturally:
\( L_{\text{cross}} \) = \( |D/V| \). An initially Dirac-like perturbation travels with velocity \( V \), while diffusing
with coefficient \( D \). Let \( L \) be the size of the system we are considering. If \( L \gg L_{\text{cross}} \), over
scale \( L \) the perturbation travels almost ballistically. Thus FPE can model the non-diffusive
propagation of induced perturbations in a fusion machine [8]. The confinement time \( \tau_{\text{conf}} \)
scales like \( L^1 \). If \( L \ll L_{\text{cross}} \) over scale \( L \) the perturbation is almost purely diffusing, and
the confinement time \( \tau_{\text{conf}} \) scales like \( L^2 \). Thus anomalous scaling of confinement time with
system size [10] can be accounted for by the FPE. Even more so in a real plasma: (i) any other
\( L \) dependence may be found, according to the scaling of \( V \) or \( D \) with \( L \); in particular mixing
length estimates provide scalings going from Bohm to gyro-Bohm [9], (ii) \( V \) and \( D \) depend
on \( x \), which makes the cross-over scale length space dependent \( L_{\text{cross}}(x) = |L_s(x)| \), where
\( L_s(x) \) = \( D(x)/V(x) \).

If \( \Gamma = 0 \), for \( D(x) \neq 0 \), the stationary state of equation (1) with \( n(0) = 1 \) is

\[ n(x) = \frac{D(0)}{D(x)} \exp \left( \int_0^x \frac{dx'}{L_s(x')} \right). \] (2)

As a result, a given profile \( n(x) \) can be interpreted by a quite flexible choice of \((V(x), D(x))\)
since these two functions verify a single constraint. In particular solutions with \( V(x) = 0 \) or
\( D(x) = \) constant are possible. This flexibility holds in particular for ‘uphill transport’, i.e. in
the case of a density piling up toward the plasma center [11]. This is a first caveat for data analysis: a broad family of \((V, D)\) profiles can model the same experimental data. A second caveat is that nothing proves \textit{a priori} that transport may be described by the FPE: indeed Lévy flights might have a leading role in the underlying transport mechanism, at least in some finite part of the system. Then fractional diffusion models should be used (see [3, 4] and references therein).

3. Transport in presence of a source

When considering using the FPE, or equivalently the advection–diffusion model to interpret experimental data, it is important to remember that the FPE is justified only if some scale separation is present in the system. Indeed this equation is derived from the Chapman–Kolmogorov or master equation

\[
\frac{\partial}{\partial t} n(x, t) = \int P(x, x') \frac{n(x', t)}{\tau(x')} \, dx' - \frac{n(x, t)}{\tau(x)} + S(x),
\]

where \(S(x)\) is a possible localized source term and \(\tau(x)\) is a waiting time. This equation says that density at point \(x\) decays exponentially with a rate \(1/\tau(x)\) and is replenished by the transport process with particles coming from all over the available space, with a rate \(P(x, x')/\tau(x')\) for particles coming from point \(x'\). To illustrate the scale separation issue, we assume \(n(x)\) to be stationary, \(\tau(x) = 1\), \(P\) to be a function of \(x - x'\) only and \(P\) and \(S\) to be Gaussians of respective widths \(\sigma_P\) and \(\sigma_S\). Then equation (3) is nothing but a convolution equation in \(x\), which makes it natural to solve by a Fourier transform in \(x\). This yields

\[
n(k) = S_0 \frac{\exp(-k^2 \sigma_S^2)}{1 - \exp(-k^2 \sigma_P^2)}.
\]

First assume \(\sigma_P \gg \sigma_S\). Then for scales such that \(k \sigma_P \gg 1\) the denominator in (4) is almost 1, which makes \(n(x) \simeq S(x)\) in the source domain, a profile dictated by the source only, and not by equation (1). In contrast, if \(\sigma_P \ll \sigma_S\), then \(-k^2 \sigma_P^2 n(k) + S(k) \simeq 0\), which corresponds exactly to the Fourier transform of equation (1) with \(V = 0\) and \(D = \sigma_P^2\). Therefore \(\sigma_S\) must be much larger than \(\sigma_P\) for the FPE to describe correctly \(n(x, t)\) within the source domain. Note that the same caveat holds whenever \(P\) corresponds to Lévy flights: indeed in the previous argument \((k \sigma_P)^2\) is simply to be substituted with \((k \sigma_P)^a\) with \(1 < a < 2\). In either case a bump in a density profile may be due to a narrow source and not to the transport coefficients. This is a third caveat for data analysis.

4. Simple model for transport

In real magnetized plasmas several parameters influence particle transport such as geometry, finite Larmor radius and drifts. However, the leading role of statistics in eventually defining whether dynamics is diffusive or not can be explained by using a very simple model describing transport induced by electrostatic turbulence in the guiding center approximation. It is defined by

\[
\frac{dX}{dt} = -\nabla \Phi(X, t) \times \hat{b},
\]

where \(X = (x_1, x_2)\) is the guiding center position perpendicular to the magnetic field (normalized to 1 and parallel unit vector \(\hat{b}\)) and \(\Phi\) is the electrostatic potential. If on top of this potential there is a perturbing parallel vector potential \(A_{||}\), the guiding center dynamics
is determined by the same equation where now $\Phi$ is replaced by $\Phi' = \Phi - v_i A_i$, and $v_i$ is the parallel velocity [12]. Therefore model (5) is particularly interesting, because it describes transport due to pure magnetic chaos as well (case $\Phi = 0$). Finally we stress that this model describes as well tracer transport in two-dimensional (2D) incompressible turbulence. A simple inspection proves that the components of equation (5) are the canonical equations of Hamiltonian $\Phi(X,r)$ for the conjugate variables $x_1$ and $x_2$. This makes available the whole artillery of Hamiltonian mechanics to describe chaotic transport.

5. Derivation of the FPE from Hamiltonian dynamics

In real magnetized plasmas, the turbulence acting on particles may have quite intricate statistical properties. In order to show how the FPE can be derived for chaotic Hamiltonian dynamics, we focus on model (5), and we endow $\Phi$ with the simplest statistical properties: the stochastic potential $\Phi$ is statistically stationary, spatially homogeneous, isotropic, zero-mean-valued, with typical amplitude $\Phi_0$, and a given two-point, two-time correlation function, with correlation time $\tau_c$ and correlation length $\lambda_c$, as considered in [12, 13]. These references introduced simple ideas justifying the FPE in this case, which may be summarized as follows. At a given time $t$, potential $\Phi$ has troughs and peaks of typical amplitude $\Phi_0$ and typical width $\lambda_c$. If the potential is considered as frozen, these extrema trap particles, which bounce with a bouncing frequency $\omega_b = \Phi_0 / \lambda_c^2$. We now define the Kubo number

$$K = \omega_b \tau_c = \Phi_0 \tau_c / \lambda_c^2. \quad (6)$$

This number is the ratio of the correlation time of the stochastic potential as seen by the moving object to the (non-linear) time where the dynamics is strongly perturbed by this potential (trapping time, chaos separation time, etc), a definition suitable for more general systems. If $K \gg 1$, a quasi-adiabatic picture works: the particles make a lot of bounces before the potential changes its topography. The change in topography forces particles to jump to a nearby trough or peak. The successive jumps produce a random walk with step $\lambda_c$ and waiting time $\tau_c$, whose corresponding diffusion coefficient is $D \approx \lambda_c^2 / \tau_c$. This simple picture is almost correct for a Gaussian spatial correlation function of the potential [13]. More generally, the frozen potential displays as well ‘roads’ crossing the whole chaotic domain. This enables long flights in the dynamics that bring some dependence upon $K$ in the above estimate for $D$ [13]. As a result, for $K \gg 1$ diffusion is justified by locality of trapping in phase space.

When $K \ll 1$, the particles typically run only along a small arc of length $\Phi_0 \tau_c / \lambda_c$ of the trapped orbits of the instantaneous potential during a correlation time. During the next correlation time they perform a similar motion in a potential completely uncorrelated with the previous one. These uncorrelated random steps yield a 2D Brownian motion with a diffusion coefficient $D \approx \Phi_0^2 \tau_c / \lambda_c^2$ (QL estimate). A more rigorous picture for the QL diffusion process of the $K \ll 1$ regime can be given [7], which incorporates chaos as an essential ingredient, but exhibits the paramount importance of potential randomness. This picture is a translation for dynamics (5) of that described in [14, 15] for the dynamics of an electron in a set of Langmuir waves. It uses the locality of wave–particle resonance in phase velocity, and can be used to prove diffusion for a single typical outcome of the random phases, if the set of the initial particle velocities is spread enough in phase space. This is the rationale for the diffusive behavior found in [16] by averaging over initial particle positions.

The above two limit cases in $K$ show diffusion is a quite general behavior of particle transport, even whenever structures are visible in the electrostatic potential, such as trapping eddies or roads (streamers). However, the same dynamics may lead to non-diffusive behavior (i) over a finite radial extension for the above statistics [17], (ii) for a more limited statistics, as
in [2]; in the latter work the dynamics, except for isotropy, may be thought as one realization of that in [13]. This brings a fourth caveat: the statistics and spatial scale of a given transport calculation must agree with that of the experiment it aims at modeling.

As yet we have just discussed the diffusive part of the FPE. We now turn to the pinch or dynamic friction part. This can be easily done by introducing a spatial inhomogeneity in model (5), such that its r.h.s. is multiplied by a growing function of $x_1$ [18]. This was originally meant as a modeling of the increase in the magnetic field toward the main axis of a fusion machine. On top of the previous diffusive behavior (which becomes anisotropic), the new inhomogeneity brings a ‘radial’ drift velocity $V$ along $x_1$ due to the chaotic motion (ratchet pinch), which corresponds to the dynamic friction of the FPE. The sign of $V$ depends on $K$. If $K \ll 1$, since the velocity increases toward larger $x_1$’s, the displacement during a correlation time is larger toward the exterior than toward the interior, which brings an outgoing drift. If $K \gg 1$, the trapped particles are slower in the inner part of their orbit, which increases their probability to be there with respect to that to be in the outer part: this brings an ingoing drift. Since $D$ now grows with $x_1$, $V = dD/dx_1$ is impossible for $K \gg 1$.

In a magnetized toroidal plasma, the space dependence of the magnetic field makes the true density compressible. As a result density $n$ describing particle transport is the true particle density divided by a growing function $\tau(x)$ of the local magnetic field (see [19, 20] and references therein). This brings a so-called curvature pinch and a slanting of the density profile toward the outer part of the torus. This slanting has nothing to do with a turbulent transport phenomenon, in contrast to that in [18], and corresponds exactly to a waiting time $\tau(x)$ in equation (3). This is a fifth caveat for data analysis. The combined effect of curvature and ratchet pinches is described in [20].

6. $V$ must be equal to $dD/dx$ for one degree of freedom only

In a homogeneous system, Chapman–Kolmogorov equation (3) with a $P$ symmetrical in $x - x'$ yields a density flux $\Gamma = Vn - \partial_x (Dn)$ often written as

$$\Gamma = -D\partial_x n,$$

(7)

where $D$ is a constant (Fick’s law or Fourier law for heat). This expression combined with the second part of equation (1) has been an incentive in the literature to consider that the generalization of Fick’s law to inhomogeneous systems reads

$$\partial_t n = \partial_x [D(x) \partial_x n].$$

(8)

When translated in terms of the transport coefficients of FPE, this means

$$V = dD/dx.$$

(9)

This special generalization of Fick’s law was derived by Landau [22] for stochastic but non-chaotic Hamiltonian dynamics (indeed his derivation uses Taylor expansions of the orbit in time). Constraint (9) was proved to hold for the chaotic motion of particles in a prescribed set of Langmuir waves [14]. In the field of plasma turbulence, equation (8) was made popular by the famous 1962 papers introducing quasilinear theory [23, 24]. If (7–9) hold for a stationary magnetic fusion plasma, $\Gamma = 0$ at the position of the radial maximum of $n$. In the absence of sources in the plasma core, $\Gamma$ is a radial invariant, which requires $dn/dx = 0$ in this domain, and rules out uphill transport. In the past the implicit assumption (7–9) cast a shadow over FPE ability to describe uphill transport. Section 2 shows that this is not justified.

However for dynamics with more than two degrees of freedom, equations (8) and (9) are not mandatory by any means. For instance, they do not hold for the self-consistent motion
of particles in a set of Langmuir waves (here $x$ is the particle velocity): $V$ is equal to $dD/dx$ plus a drag force due to the spontaneous emission of waves by particles [14]. The analogy of Langmuir wave–electron interaction with the toroidal Alfven eigenmode–fast ion one, where $x$ is the radial position [25], shows that a similar effect may hold for a fusion machine. This is a particular instance revealing that the true Hamiltonian dynamics of particles in fusion machines has more than one degree of freedom. This complements the discussion of [21].

We now show that equations (8) and (9) are mandatory for one degree of freedom dynamics by focusing on model (5) where $\Phi$ has a bounded support. In this case motion is located within a bounded domain $R$ of phase space, for instance between two magnetic flux surfaces sandwiching a zone with a chaotic magnetic field. Because of conservation of the area of a phase space element during motion, an initially uniform guiding center density $n$ in $R$ must remain uniform for later times. Now, assume that the FPE describes the evolution of $n(x)$ ($x$ one of the two conjugate variables). Since $n$ is stationary, the particle flux must be a constant in $R$, and must vanish, since it does on the boundary of $R$. Since $\Gamma = Vn - \partial_x(Dn)$, condition (9) is required. This is the fundamental reason underlying Landau derivation [22].

When more degrees of freedom are present, (for instance by adding the parallel motion to the $E \times B$ drift), condition (9) breaks down. Indeed a Kolmogorov–Arnold–Moser torus is no longer able to separate the chaotic domain into disconnected sets (Arnold diffusion), and therefore there are no longer boundaries where $\Gamma$ must vanish. As a result, a description of transport less constrained than model (5) makes possible uphill transport and a central finite density gradient.

7. Transport in bounded systems

The FPE may be derived in two different ways. The traditional one consists of a Taylor expansion of $n(x', t)/\tau(x')$ in $x' - x$ in the Chapman–Kolmogorov equation (3) (referred to in the literature as Kramers–Moyal expansion). As indicated in section 3, it can also be obtained by taking the so-called hydrodynamic limit $k \to 0$ in the Fourier transformed Chapman–Kolmogorov equation. If transport occurs in a bounded system with size $a$ the first technique still works, but the second one runs into a problem since $k$’s with $ka \lesssim 1$ are meaningless. For the case of a transport dominated by Lévy flights, $\sigma_P$ is infinite, which rules out the Taylor expansion technique in an infinite system. Therefore fractional diffusion equations are derived from Chapman–Kolmogorov equation by using the hydrodynamic limit only. This questions their validity to describe transport over the full width of bounded systems. In contrast, the Taylor expansion technique involves integrals over the finite size $a$ of the system, which may kill the divergence of the moments occurring in infinite systems [26]. As a result, for density profiles with a width comparable to the system size, the FPE might work in the case of a transport dominated by Lévy flights. These issues are the subject of a work in progress.

8. Conclusion

The FPE turns out to be a quite flexible tool for experimental and numerical data analysis. However care should be exerted when using it, in order to avoid interpreting as advective–diffusive effects profile characteristics ruled by a localized source or by a curvature pinch.

Depending on the statistics of interest and/or on the radial extension of the transport process, the same dynamical system may be found diffusive or dominated by its Lévy flights. In particular, averaging over many random phases may lead to classical diffusion, while averaging only over a limited set of initial conditions for the particles may lead to fractional diffusion.
We found that for $K \ll 1$, the FPE with a QL diffusion coefficient is justified by chaotic Hamiltonian dynamics with random phases, even though structures may exist in phase space for one realization of the phases. This, and the fact that the QL diffusive modeling of transport is quite efficient [27], suggest that $K$ is small in magnetic fusion turbulence. This was proved to be the case for the gyrokinetic simulation of electron temperature gradient turbulence [28], questioning the relevance of streamers in turbulent transport. There are reasons for $K$ not to be large. First, the usual estimate in fluid mechanics of the correlation time as the eddy turn-over time yields $K = 1$ [29]. Furthermore strong turbulence theory predicts that $K$ is at most of order 1 (see equation (4.34) of [30]). However the issue of the typical value of $K$ is still unsettled, since an analysis of fluctuation data in the TEXT tokamak indicated $K$ of order 1 and a poor agreement of QL estimates for impurity transport [31]. Recently the gyrokinetic simulation of trapped electron mode turbulence provided $K \simeq 7$ [17] and $K \simeq 9$ [32]. To clarify the value of the Kubo number in magnetic fusion turbulence, it would be advisable to systematically compute it from experimental or numerical data. In rotating or stratified fluid turbulence a weak effect of structures holds as well: cigar-like or pancake-like structures are present, but turbulent diffusion is correctly modeled by assuming random phases for the Fourier components of the turbulent fluid [33].

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