Curie temperature for an Ising model on Archimedean lattices

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(Dated: April 16, 2019)

Critical temperatures for the ferro-paramagnetic transition in the Ising model are evaluated for five Archimedean lattices, basing on Monte Carlo simulations. The obtained Curie temperatures are 1.25, 1.40, 1.45, 2.15 and 2.80 [J/kB] for (3, 122), (4, 6, 12), (4, 82), (3, 6, 4) and (34, 6) lattices, respectively.

PACS numbers: 05.10.-a, 05.50.+q, 07.05.Tp, 68.35.Rh
Keywords: computer modelling and simulation; Ising model; phase transition; critical parameter

I. INTRODUCTION

Beauty of the Ising model (IM) manifests in its simplicity. The system considered is a network of \( N \) interacting spins \( S_i = \pm 1 \) which energy is

\[
E \equiv -J \sum_{(i,j)} S_i S_j , \quad (1)
\]

where \( J \) is an exchange integral. We assume homogeneous short range spin interactions, i.e. the summation in Eq. (1) goes over pairs \((i,j)\) of nearest neighbours. Positive sign of \( J > 0 \) gives ferromagnetic interaction among spins. The minimisation of energy (1) for temperature \( T = 0 \) produces spin dynamics which may be described by deterministic cellular automaton with rule

\[
S_i(t+1) = \text{sign} \left( J \sum_j S_j(t) \right) , \quad (2)
\]

where \( t \) denotes discrete time and summation goes over nearest neighbours of \( i \)-th spin.

For finite temperature \( T > 0 \) the deterministic rule (2) is replaced by a probabilistic cellular automaton with spin update rule \( S_i(t) \to S_i(t+1) \) described by Glauber or Metropolis dynamics. Then the phase transition may be observed: below critical temperature \( T < T_C \) spontaneous magnetisation \( m = \sum_i S_i/N \neq 0 \) is observed while \( m = 0 \) for \( T > T_C \). The IM was already investigated

- in many ways, including Monte Carlo simulation, series expansion, mean-field approach or partition function technique,
- and for many systems, for example: antiferromagnets, frustrated, disordered or diluted networks on complex or shuffled lattices, etc.

II. RESULTS OF THE SIMULATIONS

In this paper the critical temperature \( T_C \) is estimated for five two-dimensional lattices, basing on \( \langle m(T) \rangle \) dependence, where \( \langle \cdots \rangle \) denotes the time average. The Archimedean lattices are vertex transitive graphs that can be embedded in a plane such that every face is a regular polygon. Kepler showed that there is exactly eleven such graphs [15]. The names of the lattices are given according to the sizes of faces incident to a given vertex. The face sizes are listed in order, starting with a face such that the list is the smallest possible in the lexicographical order. In this way, the square lattice gets the name \((4, 4, 4, 4)\), abbreviated to \((4^4)\), honeycomb is called \((6^3)\) and Kagomé is \((3, 6, 3, 6)\). Some results concerning IM on AL were already presented in Refs. [20, 21, 22, 23, 24, 25]; however in a literature known to us the Curie temperatures of several AL are still missing.

Critical properties of these lattices were investigated in terms of site percolation [26] in Ref. [27]. Topologies of all eleven AL are given there as well.

| z | lattice | \( T_C \) [J/kB] | Ref. |
|---|---------|-----------------|------|
| 3 | \((3, 12^2)\) | 1.25 | — |
| 4 | \((3, 4, 6, 4)\) | 2.15 | — |
| 5 | \((3^4, 6)\) | 2.80 | — |
| 6 | \((3^6)\) | 3.64 | [22] |

In Table I the AL and associated critical temperatures \( T_C \) are presented.

### References

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In this paper the Curie temperatures for IM on all AL are collected. Among them, \( T_C \) for (3, 12\(^2\)), (4, 6, 12), (4, 8\(^2\)), (3, 4, 6, 4) and (3\(^4\), 6) AL are evaluated for the first time with the Monte Carlo simulation. In the latter case, an analytical expression \(^2\) is known \( \langle m \rangle \approx (3, 12\(^2\)), (4, 6, 12), (4, 8\(^2\)), (3, 4, 6, 4) and (3\(^4\), 6) AL. The simulations are carried out for \( N \approx 6 \cdot 10^4 \) spins during \( N_{\text{iter}} = 2 \cdot 10^5 \) [MCS]. The magnetisation \( \langle m \rangle \) is averaged over the last \( 10^5 \) [MCS].

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For all investigated AL, the shape of \( m(T/T_C) \) curve (see Fig. 2) is roughly the same as for the square lattice. In the latter case, an analytical expression \(^2\) is known

\[
|\langle m(\kappa) \rangle| = \sqrt{\frac{\cosh^2(2/\kappa)}{\sinh^2(2/\kappa)}} [\sinh^2(2/\kappa) - 1],
\]

where \( \kappa \equiv T/T_C \).

In contrast to Galam–Mauger \(^2\) semi-exact formula for \( T_C \) dependence on system dimensionality \( d \) and lattice coordination number \( z \), we show that critical temperature for IM differ slightly for several AL (where \( d = 2 \)) with the same values of \( z \). Similarly to the percolation phenomena \(^2\), also for IM the dimensionality \( d \) and the coordination number \( z \) are not sufficient \(^2\), \(^3\), \(^4\) for determining the critical point \( T_C \).

III. CONCLUSIONS

In this paper the Curie temperatures for IM on all AL are collected. Among them, \( T_C \) for \((3, 12\(^2\)), (4, 6, 12), (4, 8\(^2\)), (3, 4, 6, 4) and (3\(^4\), 6) AL are evaluated for the first time with the Monte Carlo simulation.

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Acknowledgments

K.M. is grateful to Krzysztof Kulakowski for many valuable and fruitful discussions. Calculations were carried out in ACK-CYFRONET-AGH. The machine time on HP Integrity Superdome is financed by the Polish Ministry of Science and Information Technology under Grant No. MNiI/HP_LD_AHG/047/2004.

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