Quiver mutation loops and partition $q$-series

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Joint work with Yuji Terashima (TIT)
arXiv:1403.6569 & 1408.0444
Ubiquity of quiver mutations

Quiver mutations appear in many fields in different guise: cluster algebras, 3-dimensional topology, gauge theory, Donaldson-Thomas theory, stability conditions, wall-crossing, WKB analysis, ...

Our Strategy

- Define a key mathematical object, à la partition function of statistical mechanics, by using *only combinatorial data* of quiver mutations.
- Every property of such object would be shared by various “realizations” of mutations.

In this talk, we introduce *partition $q$-series* for mutations and explain some nice properties of them.

Based on papers with Y. Terashima:
arXiv:1403.6569 & 1408.0444 (published online in Comm. Math. Phys.)
Mutation of a quiver $Q$ at a vertex $k$:

1. For each path $i \rightarrow k \rightarrow j$, add a new arrow $i \rightarrow j$; 
2. Reverse all arrows with source or target $k$. 
3. Remove newly created 2-cycle, if any.

Example

![Diagram of quiver mutation]
A **mutation loop** is a triple $\gamma = (Q, m, \varphi)$, where

- an initial quiver $Q$
- a mutation sequence $m = (m_1, m_2, \ldots, m_T)$
- boundary condition $\varphi$ is an isomorphism of the initial quiver $Q_0$ and the final quiver $Q_T$.

**Example**

- $Q = (a \to b \leftarrow c)$
- Mutation sequence $m = (b, a, c)$
- Boundary condition $\varphi : a' = a$, $b' = b$, $c' = c$
Motivation: quiver $\iff$ surface triangulation
Motivation: mutation $\iff$ flip

Mutation sequence $\iff$ surface diffeomorphism
Motivation: flip $\iff$ tetrahedron

tetrahedron

mutation sequence $\iff$ mapping cylinder
Motivation: Triangulation of 3-manifolds

Surface bundle with a mapping class $\varphi$ of a surface $\Sigma$

$$M := (\Sigma \times [0, 1])/(x, 0) \sim (\varphi(x), 1)$$
Motivation: Mutations $\iff$ 3-dim. topology

| Combinatorial       | Geometric                                      |
|---------------------|------------------------------------------------|
| Quiver              | Triangulation of a surface                    |
| Mutation            | Tetrahedron                                   |
| Mutation sequence   | Mapping class                                 |
| Mutation network    | Triangulation of a 3-manifold                 |
| Mutation loop       | Surface bundle over $S^1$                     |

Cluster transformations $\iff$ Hyperbolic geometry
Gekhtman-Shapiro-Vainshtein, Fock-Goncharov, Fomin-Shapiro-Thurston, Nagao-Terashima-Yamazaki, Hikami-Inoue, ...

??? $\iff$ Quantum field theories
Statistical model from quiver mutation sequence

- Statistical model on lattice $\Lambda$ with local variables $\phi$ taking their values in $S$.
- Partition function
  \[ Z = \sum_{\phi: \Lambda \to S} \prod_{\sigma \in \Lambda} W(\phi(\sigma)) \]
  $\phi: \Lambda \to S$: field
  $W = $ Boltzmann weight (defined locally in $\Lambda$)
- Quantum machanics = Path integral
- Space direction = quiver
- Matsubara (imaginary time) direction = mutation sequence
- Periodic boundary condition in time direction = mutation loop
- N.B. mutation sequence is an *inhomogeneous* time-evolution of space (=quiver). The space-time graph $\Lambda$ is dynamically generated from the initial quiver $Q$ and the mutation sequence $m$. 
Weights of mutations

- Attach a variable $s_j$ at each vertex $j$.
- Weight of a mutation at vertex $i$:

$$q^{\frac{1}{2}}(s_i + s_i' - \sum_{j:j\to i} s_j)(s_i + s_i' - \sum_{l:i\to l} s_l)$$

$$\frac{(q)^{s_i + s_i' - \sum_{j:j\to i} s_j}}{(q)}$$

- $k$-variable with a mutation $m_i$ at a vertex $i$:

$$k_i := s_i + s_i' - \sum_{j:j\to i} s_j$$

$q$-factorial

$$(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)$$
Weights of mutations — Example

mutation at vertex $a$:

$$W = q^{\frac{1}{2}}(a+a'-b_1-b_2-2b_3)(a+a'-c_1-c_2)$$

Weight of the mutation

$$W = \frac{q^{\frac{1}{2}}(a+a'-b_1-b_2-2b_3)(a+a'-c_1-c_2)}{(q)^{a+a'-b_1-b_2-2b_3}}$$

$k$-variable:

$$k = a + a' - b_1 - b_2 - 2b_3$$
For a mutation loop $\gamma = (Q, m, \varphi)$ with $m = (m_1, \cdots, m_T)$, we define partition $q$-series by

\[
Z(\gamma) := \sum_{k \in \mathbb{N}^T} \prod_{t=1}^{T} W(m_t)
\]

The relation between $k$- and $s$- variables depends on the global topology, especially on the boundary condition $\varphi$. 
The alternating quiver $Q$ of type $A_3$:

- Mutation sequence $m = (b, a, c)$
- Boundary condition $\varphi = \text{id}: a' = a, \ b' = b, \ c' = c$

![Diagram of the alternating quiver $A_3$ with mutation sequence $m = (b, a, c)$ and boundary conditions $a' = a, b' = b, c' = c$.]
Typical example

- Mutation sequence $m = (b, a, c)$
- Boundary condition $\varphi : a' = a$, $b' = b$, $c' = c$
Typical example

Product of weights:

\[
q^\frac{1}{2}((2b-a-c) \cdot 2b+(2a-b) \cdot 2a+(2c-b) \cdot 2c)\\(q)2b-a-c(q)2a-b(q)2c-b
\]

\[
k_1 = 2b - a - c,
\]
\[
k_2 = 2a - b,
\]
\[
k_3 = 2c - b.
\]
Product of weights:

\[
q^\frac{1}{2} \left( (2b-a-c) \cdot 2b + (2a-b) \cdot 2a + (2c-b) \cdot 2c \right)
\]

\[
\frac{1}{(q)2b-a-c(q)2a-b(q)2c-b}
\]

\(s\)-variables from \(k\)-variables:

\[
a = \frac{1}{4} \left( 3k_1 + 2k_2 + k_3 \right),
\]

\[
b = \frac{1}{2} \left( k_1 + 2k_2 + k_3 \right),
\]

\[
c = \frac{1}{4} \left( k_1 + 2k_2 + 3k_3 \right).
\]
Typical example

Product of weights:

\[
\frac{3k_1^2}{4} + k_1 k_2 + k_2^2 + k_2 k_3 + \frac{3k_3^2}{4} + \frac{k_3 k_1}{2}
\]

\[
(q)_k (q)_k (q)_k (q)_k
\]

s-variables from k-variables:

\[
a = \frac{1}{4} (3k_1 + 2k_2 + k_3),
\]

\[
b = \frac{1}{2} (k_1 + 2k_2 + k_3),
\]

\[
c = \frac{1}{4} (k_1 + 2k_2 + 3k_3).
\]

\[
Q(0) \quad a \rightarrow b \leftarrow c
\]

\[
\mu_2 \quad \mu_1 \quad \mu_3
\]

\[
Q(1) \quad a \leftarrow b' \rightarrow c
\]

\[
Q(2) \quad a' \rightarrow b' \rightarrow c
\]

\[
Q(3) \quad a' \rightarrow b' \leftarrow c'
\]
Partition $q$-series:

$$Z(\gamma) = \sum_{k \in \mathbb{N}^3} \frac{q^{k^\top D k}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}}$$

$$D = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

The inverse matrix of $D$ is the Cartan matrix of type $A_3$!

$$C := D^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$
Typical example

Partition $q$-series:

$$Z(\gamma) = \sum_{k \in \mathbb{N}^3} \frac{q^{k^\top} D k}{(q)_{k_1}(q)_{k_2}(q)_{k_3}} = \frac{1}{(q)_{\infty}} \sum_{n \in \mathbb{Z}} q^{3n^2}$$

$$D = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

The inverse matrix of $D$ is the Cartan matrix of type $A_3$!

$$C := D^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

In particular, $Z(\gamma)$ is a modular function — a character of a module associated with an affine Lie algebras. (Lepowsky-Primc, Terhoeven)
Alternating Quivers

A quiver is **alternating** if each vertex is source or sink.

Alternating quiver of type $ADE$:
Product Quivers

Product of quivers:

\[ \begin{array}{c}
\bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \\
\bigcirc \rightarrow \bigcirc \leftarrow \bigcirc \rightarrow \bigcirc \leftarrow \bigcirc \\
\uparrow \uparrow \uparrow \uparrow \uparrow \\
\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc 
\end{array} \quad Q \hspace{1cm} \begin{array}{c}
\bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \\
\bigcirc \rightarrow \bigcirc \leftarrow \bigcirc \rightarrow \bigcirc \leftarrow \bigcirc \\
\uparrow \uparrow \uparrow \uparrow \uparrow \\
\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc 
\end{array} \quad Q
\]

\[ \begin{array}{c}
\bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \\
\bigcirc \rightarrow \bigcirc \leftarrow \bigcirc \rightarrow \bigcirc \leftarrow \bigcirc \\
\uparrow \uparrow \uparrow \uparrow \uparrow \\
\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc 
\end{array} \quad Q' \hspace{1cm} \begin{array}{c}
\bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \\
\bigcirc \rightarrow \bigcirc \leftarrow \bigcirc \rightarrow \bigcirc \leftarrow \bigcirc \\
\uparrow \uparrow \uparrow \uparrow \uparrow \\
\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc 
\end{array} \quad Q \square Q'
\]

A special mutation sequence \( m = m_+ m_- \) of \( Q \square Q' \).
Fermionic formula

Theorem (K.-Terashima)

Consider the product $Q \square Q'$ of alternating quivers of ADE type. Let $\gamma = (Q \square Q', m, id)$ be the mutation loop with $m = m_+ m_-$. Then we have

$$Z(\gamma) = \sum_{k \in \mathbb{N}^n} \frac{q^{\frac{1}{2}k(C_Q \otimes C_{Q'}^{-1})}k}{(q)_k}$$

In particular, when $Q' = A_n$ type, $Z(\gamma)$ coincides with a fermionic character formula in the Kuniba-Nakanishi-Suzuki conjecture!

“Parafermionic” system associated with affine Lie algebra of type $Q$. 
Pentagon move (Pachner 2-3 move)

2 tetrahedra $\leftrightarrow$ 3 tetrahedra
Generalized pentagon move

\[ \gamma = (\cdots, x, y, \cdots) \leftrightarrow \gamma' = (\cdots, y, x, y, (xy), \cdots) \]
Theorem (K.-Terashima)

The partition $q$-series are invariant under generalized pentagon moves:

$$Z(\gamma) = Z(\gamma')$$

- $Z(\gamma)$ is invariant under local deformation of the loop $\gamma$ in the quiver exchange graph $\Gamma$.
  $(\Gamma : \text{vertex} = \text{quiver}, \text{edge} = \text{mutation})$.

- $Z(\gamma) =$ "holonomy" or "Wilson loop" along $\gamma \subset \Gamma$
Quantum Dilogarithm

Definition

\[ E(y) = \prod_{n=0}^{\infty} (1 + q^{n+\frac{1}{2}}y)^{-1} \in \mathbb{Q}(q^{1/2})[[y]] \]

\[ = 1 + \frac{q^{1/2}}{q - 1}y + \cdots + \frac{q^{n^2/2}y^n}{(q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})} + \cdots \]

Theorem (Fadeev-Kashaev-Volkov 1993)

\[ y_1y_2 = qy_2y_1 \implies E(y_1)E(y_2) = E(y_2)E(q^{-1/2}y_1y_2)E(y_1) \]
Ice-quivers, $c$-vectors, and reddening sequence
Non-commutative algebra

**Definition**

The non-commutative algebra $\mathbb{A}_Q$

- generators: $y_1, y_2, \cdots, y_n$
- relations: $y^\alpha y^\beta = q^{\frac{1}{2}\langle \alpha, \beta \rangle} y^{\alpha + \beta}$, $\quad (\alpha, \beta \in (\mathbb{Z}_{\geq 0})^n)$

where $\langle e_i, e_j \rangle = \#(i \to j) - \#(j \to i)$.

**Example**

\[
\mathbb{A}_Q = \langle y_1, y_2, y_3 \mid y_1 y_2 = q y_2 y_1, \quad y_2 y_3 = q y_3 y_2, \quad y_1 y_3 = q^{-1} y_3 y_1 \rangle
\]

\[
y^{(i_1, i_2, i_3)} = q^{-\frac{1}{2}(i_1 i_2 + i_2 i_3 - i_1 i_3)} y_{i_1} y_{i_2} y_{i_3}
\]
Dilogarithm product along reddening sequence

**Definition (Keller)**

Reddning sequence

\[ m = (m_1, m_2, \cdots, m_T) \]

c-vectors at the mutated vertices are

\[ \varepsilon_1 \alpha_1, \varepsilon_2 \alpha_2, \cdots, \varepsilon_T \alpha_T \in \mathbb{Z}^n, \quad \varepsilon_t = \pm 1, \ \alpha_t \in \mathbb{N}^n \]

\[ E(m) := E(y^{\alpha_1})^{\varepsilon_1} E(y^{\alpha_2})^{\varepsilon_2} \cdots E(y^{\alpha_T})^{\varepsilon_T} \]

**Theorem (Keller; Nagao, Reineke)**

*If m and m' are reddening sequence starting from Q. Then \( E(m) = E(m') \).*
combinatorial DT-invariant

Donaldson-Thomas (DT) invariant (Kontsevich-Soibelman) is the generating function which counts the number of “stable objects”.

**Definition**

\[ E_Q := E(m) \]

is well-defined as a formal power series intrinsically associated with \( Q \). This is called the combinatorial DT-invariant of \( Q \).

**Example**

\( Q = (1) : A_1 \)-quiver

\[ E_Q = E(y_1) \]

**Example**

\( Q = (1 \to 2) : A_2 \)-quiver

\[ y_1y_2 = qy_2y_1 \quad \implies \quad E_Q = E(y_1)E(y_2) = E(y_2)E(q^{-1/2}y_1y_2)E(y_1) \]
Theorem (K.-Terashima)

Let $\gamma$ be a mutation loop associated with a reddening mutation sequence starting from $Q$.
Then the non-commutative partition $q$-series $Z(\gamma; y)$ coincides with the combinatorial Donaldson-Thomas invariant $E_Q(y)$:

$$Z(\gamma; y) = E_Q(y)|_{q \to q^{-1}} \in \mathbb{A}_Q$$
Example

\[
Q = \begin{array}{c}
1 \\
\downarrow \quad \downarrow \\
2 & \rightarrow & 3
\end{array}
\]

The mutation sequence \( m = (1, 2, 3, 1) \) is reddening.

The non-commutative partition \( q \)-series

\[
Z(\gamma) = \sum_{k \geq 0} q^{\frac{1}{2}(k_1^2+k_2^2+k_3^2+k_4^2-k_1k_2+k_1k_3+k_1k_4-k_2k_4+k_3k_4)} \frac{(q)_{k_1}(q)_{k_2}(q)_{k_3}(q)_{k_4}}{y^{(k_1+k_3,k_2,k_3+k_4)}}
\]

Combinatorial Donaldson-Thomas invariant of \( Q \):

\[
E_Q(y) = E(y^{(1,0,0)})E(y^{(0,1,0)})E(y^{(1,0,1)})E(y^{(0,0,1)})
\]

quantum dilogarithm

\[
E(x) = \prod_{n=0}^{\infty} \left(1 + q^{n+\frac{1}{2}}y\right)^{-1}
\]
Reddening mutation sequence

The mutation sequence $m = (1, 2, 3, 1)$ applied to “ice-quiver”

- Only one out-going arrow on every frozen vertex in the final quiver.
- The associated boundary condition $\varphi : 1 \leftrightarrow 3, 2 \leftrightarrow 2, 3 \leftrightarrow 1$
Non-commutative grading

The mutation sequence \( m = (1,2,3,1) \) applied to “ice-quiver”

- Non-commutative grading \( y^{(k_1+k_3,k_2,k_3+k_4)} \in A_Q \)
- “orientation” \((+,+,+,+)\) (green/red mutation)
Non-commutative partition $q$-series

1. The associated boundary condition $\varphi$: $1 \leftrightarrow 3$, $2 \leftrightarrow 2$, $3 \leftrightarrow 1$
2. The non-commutative grading: $y^{(k_1+k_3,k_2,k_3+k_4)} \in \mathbb{A}_Q$
3. The orientation: $(+,+,+,+,+)$. 

$\tilde{Q}(0)$

$\begin{array}{ccc}
1' & \downarrow \mu_2 & 1' \\
1 & \leftarrow & 3 \\
2 & \rightarrow & 3' \\
2' & \rightarrow & 3' \\
\end{array}$

$\varphi = (13)$

$\begin{array}{ccc}
2' & \rightarrow & 1 \\
2 & \rightarrow & 1 \\
\end{array}$

$\tilde{Q}(1)$

$\begin{array}{ccc}
1' & \downarrow \mu_1 & 1' \\
1 & \leftarrow & 3 \\
2 & \rightarrow & 3' \\
2' & \rightarrow & 3' \\
\end{array}$

$\tilde{Q}(2)$

$\begin{array}{ccc}
1' & \downarrow \mu_2 & 1' \\
1 & \leftarrow & 3 \\
2 & \rightarrow & 3' \\
2' & \rightarrow & 3' \\
\end{array}$

$\tilde{Q}(3)$

$\begin{array}{ccc}
1' & \downarrow \mu_3 & 1' \\
1 & \leftarrow & 3 \\
2 & \rightarrow & 3' \\
2' & \rightarrow & 3' \\
\end{array}$

$\tilde{Q}(4)$

$\begin{array}{ccc}
1' & \downarrow \mu_1 & 1' \\
1 & \leftarrow & 3 \\
2 & \rightarrow & 3' \\
2' & \rightarrow & 3' \\
\end{array}$
The combinatorial Donaldson-Thomas invariant $E_Q(y)$ is defined as an ordered product of quantum dilogarithms along a maximal green sequence. (Keller; following Kontsevich-Soibelman, Nagao, Reineke)

$$E_Q(y) = E(y^{(1,0,0)})E(y^{(0,1,0)})E(y^{(1,0,1)})E(y^{(0,0,1)}) \in \mathbb{A}_Q$$

where

$$E(y) = \prod_{n=0}^{\infty} (1 + q^{n+\frac{1}{2}}y)^{-1}$$

**Theorem (K.-Terashima)**

For any reddening mutation sequence $m$,

$$Z(\gamma; y) = E_Q(y)|_{q \rightarrow q^{-1}} \in \mathbb{A}_Q$$
Summary

Partition $q$-series enjoy the following nice properties:

- Invariant under generalized pentagon moves.
- For (product of) Dynkin quivers and special mutation sequences, they coincide with characters of appropriate conformal field theories. Kuniba-Nakanishi-Suzuki formula
- For reddening mutation sequences, they coincide with the combinatorial Donaldson-Thomas invariants.
- Path integral formalism v.s. operator formalism

Various “realizations”

- 3-dimensional quantum topology
- Gauge theories Cecotti-Neitzke-Vafa, Terashima-Yamazaki, Dimofte-Gaiotto-Gukov, ...
- Categorification of cluster transformations Nagao, Reineke, Iyama, ...