Research Article

Fifteen Limit Cycles Bifurcating from a Perturbed Cubic Center

Amor Menaceur,¹ Mufda Alrawashdeh,² Sahar Ahmed Idris,³,⁴ and Hala Abd-Elmageed⁵

¹Laboratory of Analysis and Control of Differential Equations ACED, Department of Mathematics, University of Guelma, P.O. Box 401, Guelma 24000, Algeria
²Department of Mathematics, College of Sciences and Arts, ArRas, Qassim University, Buraydah, Saudi Arabia
³College of Industrial Engineering, King Khalid University, Abha, Saudi Arabia
⁴Department of Mathematics, College of Sciences, Juba University, Juba, Sudan
⁵Department of Mathematics, Faculty of Science, SVU, Qena 83523, Egypt

Correspondence should be addressed to Sahar Ahmed Idris; sa6044690@gmail.com

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In this work, we study the bifurcation of limit cycles from the period annulus surrounding the origin of a class of cubic polynomial differential systems; when they are perturbed inside the class of all polynomial differential systems of degree six, we obtain at most fifteenth limit cycles by using the averaging theory of first order.

1. Introduction and Statement of the Main Result

Hilbert in 1900 was interested in the maximum number of the limit cycles that a polynomial differential system of a given degree can have. This problem is the well-known 16th Hilbert problem, which together with the Riemann conjecture are the two problems of the famous list of 23 problems of Hilbert which remain open. See for more details [1, 2].

A classical way to produce limit cycles is by perturbing a system which has a center, in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits of the period annulus of the center of the unperturbed system [3–7].

In [8], the authors improved the result of the maximum number of limit cycles for a class of polynomial differential systems which bifurcate from the period annulus surrounding the origin of the system:

\[
\begin{align*}
\dot{u} &= v - v(u - y + a)(u + v + a), \\
\dot{v} &= -u + u(u - v + a)(u + v + a),
\end{align*}
\]

where \((u - y + a)(u + v + a) = 1\) is a conic, \(a^2 \neq 1\), and \(|a| \leq \sqrt{2}\) by using the first order of the averaging theory method.

In [9], the authors improved the result of the maximum number of limit cycles of sixth polynomial differential systems which bifurcate from the period annulus surrounding the origin of the system:

\[
\begin{align*}
\dot{u} &= -v(u - v^2 - a)^2, \\
\dot{v} &= u(u - v^2 - a)^2,
\end{align*}
\]

where \(u - v^2 - a = 0\) is a conic and \(a \neq 0\), by using the first order of the averaging theory method.

In this work, we perturb the cubic systems equation (1). Thus, we consider these classes of all polynomial differential systems of degree \(n\), i.e.,

\[
\begin{align*}
\dot{u} &= v - v(u - y + a)(u + v + a) + \varepsilon P(u, v), \\
\dot{v} &= -u + u(u - v + a)(u + v + a) + \varepsilon Q(u, v),
\end{align*}
\]

where \((u - v + a)(u + v + a) = 1\) is a conic, \(|a| > \sqrt{2}\), \(P(u, v)\) and \(Q(u, v)\) are the real polynomials of degree \(n \geq 3\), and \(\varepsilon\) is
Consider the following two initial value problems:
\[ \dot{x} = \varepsilon R(t, x) + \varepsilon \dot{G}(t, x, \varepsilon), \quad x(0) = x_0, \]  
and
\[ \dot{y} = \varepsilon f^0(y), \quad y(0) = y_0, \]
where \( x, y, \) and \( x_0, y_0 \) are elements of \( \mathbb{R} \), \( t \in [0, \infty) \), \( \varepsilon \in (0, \varepsilon_0] \), \( R \) and \( G \) are the periodic functions with their period \( T \) with respect to \( t \), and \( f^0(y) \) is the average function of \( R(t, y) \) with respect to \( t \), i.e.,
\[ f^0(y) = \frac{1}{T} \int_0^T R(t, y) dt. \]

Assume that
(i) \( R, \partial R/\partial x, \partial^2 R/\partial x^2, G \), and \( \partial G/\partial x \) are well-defined, continuous, and bounded by a constant independent by \( \varepsilon \in (0, \varepsilon_0] \) in \( [0, \infty) \times D \).
(ii) \( T \) is a constant independent of \( \varepsilon \).
(iii) \( y(t) \) belongs to \( D \) on the time scale \( 1/\varepsilon \). Then, the following statements hold.

(a) On the time scale \( 1/\varepsilon \), we have
\[ x(t) - y(t) = O(\varepsilon), \text{ as } \varepsilon \to 0. \]  
(b) If \( p \) is an equilibrium point of the averaged system equation (5), such that
\[ \frac{\partial f^0}{\partial y}|_{y=p} \neq 0. \]  
Then, system equation (4) has a \( T \)-periodic solution \( \phi(t, \varepsilon) \to p \) as \( \varepsilon \to 0. \)
(c) If equation (8) is a negative, the corresponding periodic solution \( \phi(t, \varepsilon) \) of equation (4) according to \( (t, x) \) is asymptotically stable for all \( \varepsilon \) sufficiently small, and if equation (8) is a positive, then it is unstable.

For more details on the averaging method, see [10, 11].

### 3. Proof of Theorem 1

For \( |a| > 2 \), the cubic system equation (1) has a unique period annulus:
\[ A = \left\{ (u, v): 0 < u^2 + v^2 < \frac{a^2 - 2}{2} \right\}. \]

According to Figures 1 and 2, this proof is based on the first order of the averaging theory method, in polar coordinates \( (r, \theta) \), where \( u = r \cos \theta, v = r \sin \theta, \) and \( r > 0 \). We take
\[ P(u, v) = \sum_{k=1}^{n} \sum_{i+j=k} p_{ij} u^i v^j, \quad Q(u, v) = \sum_{k=1}^{n} \sum_{i+j=k} q_{ij} u^i v^j. \]

Equation (3) can be written as follows:

\[
\begin{align*}
\dot{r} &= \varepsilon \sum_{k=1}^{n} \left( \cos \theta M_k (\cos \theta, \sin \theta) + \sin \theta N_k (\cos \theta, \sin \theta) \right) r^{k-1}, \\
\dot{\theta} &= S(r, \theta) + \varepsilon \sum_{k=1}^{n} \left( \cos \theta N_k (\cos \theta, \sin \theta) - \sin \theta M_k (\cos \theta, \sin \theta) \right) r^{k-1},
\end{align*}
\]

where
\[ M_k (\cos \theta, \sin \theta) = \sum_{i+j=k} p_{ij} \cos^i \theta \sin^j \theta, \]
\[ N_k (\cos x, \sin x) = \sum_{i+j=k} q_{ij} \cos^i \theta \sin^j \theta, \]
and
\[ S(r, \theta) = (r (\cos \theta - \sin \theta) + a) (r (\cos \theta + \sin \theta) + a) - 1. \]

Therefore, we have
\[ f^0(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^k \cos \theta M_k (\cos \theta, \sin \theta) + \sin \theta N_k (\cos \theta, \sin \theta)}{S(r, \theta)} d\theta. \]
For \( n = 6 \), we get

\[
f^0(r) = \sum_{k=1}^{6} \sum_{i+j=k} \left( p_{ij}A_{i+1,j}(r) + q_{ij}A_{i,j+1}(r) \right) r^k,
\]

where

\[
A_{\rho \eta} = \frac{1}{2\pi} \int_0^{2\pi} \cos^\rho \theta \sin^\eta \theta \frac{S(r, \theta)}{d\theta}.
\]

According to Theorem 2, every simple zero of the average function \( f^0(r) \) provides a limit cycle of system equation (3). Now, we prove Theorem 1; in the first step, we compute the integral \( f^0(r) \), and in the second step, the number of its simple zeros is studied.

**Lemma 1.** From the above, we have

\[
A_{0,0} = \frac{G_1 - G_2}{2SH_1H_2} \quad \text{and} \quad A_{1,0} = \frac{a(G_1 - G_2)}{4rSH_1H_2} + \frac{G_1 + G_2}{4rH_1H_2}.
\]
\[ H_1 = \sqrt{(a-1)^2 - r^2}, \]
\[ H_2 = \sqrt{(a+1)^2 - r^2}, \]
\[ G_1 = \sqrt{-2r^2 - 2aS + 2}, \]
\[ G_2 = \sqrt{-2r^2 + 2aS + 2}, \]

with
\[ S = \sqrt{2r^2 - a^2 + 2} \neq 0. \]

**Proof.** Assume that \( z = e^{i\theta} \) and \( C \) is the circle \( |z| = 1 \); we get
\[
2\pi \int_{0}^{2\pi} \frac{d\theta}{S(\theta)} = \frac{2}{ir^2c(z-z_1)(z-z_2)(z-z_3)(z-z_4)}dz,
\]
whose poles are
\[
z_{1,2} = \frac{-a - \sqrt{2r^2 - a^2 + 2 + \sqrt{-2r^2 + 2a\sqrt{2r^2 - a^2 + 2}}}}{2r},
\]
\[
z_{3,4} = \frac{-a + \sqrt{2r^2 - a^2 + 2 + \sqrt{-2r^2 + 2a\sqrt{2r^2 - a^2 + 2}}}}{2r}.
\]

By applying the residue theorem, for \( |a| > \sqrt{2} \), we obtain
\(|z_1| < 1, |z_3| < 1, C \) encloses the two singular points of the integrand, so

\[
A_{0,0} = \frac{1}{2\pi} \frac{z_1}{r^2(z-z_1)(z-z_2)(z-z_3)(z-z_4)}dz,
\]
\[
A_{1,0} = \frac{1}{2\pi} \cos \theta \frac{d\theta}{S(\theta)} = \frac{1}{2\pi} \frac{z_1^2 + 1}{r^2(z-z_1)(z-z_2)(z-z_3)(z-z_4)}dz,
\]
\[
A_{1,0} = \frac{1}{2\pi} \frac{z_2^2 + 1}{r^2(z-z_1)(z-z_2)(z-z_3)(z-z_4)}dz,
\]
\[
A_{1,0} = \frac{1}{2\pi} \frac{z_3^2 + 1}{r^2(z-z_1)(z-z_2)(z-z_3)(z-z_4)}dz,
\]
\[
A_{1,0} = \frac{1}{2\pi} \frac{z_4^2 + 1}{r^2(z-z_1)(z-z_2)(z-z_3)(z-z_4)}dz.
\]
Lemma 2. Under the previous notations, we have

\begin{align}
A_{k,0} &= -\frac{a}{4\pi r^2} \int_0^{2\pi} \cos^{n-1} \theta \cos^2 \theta d\theta + \frac{r^2 + 1 - a^2}{4\pi r^2} \int_0^{2\pi} \cos^{n-2} \theta d\theta, \\
A_{k,1,0} &= \frac{a}{r} A_{k,0} + \frac{r^2 + 1 - a^2}{2r^2} A_{k-1,0}, \\
A_{k,2,0} &= \frac{a}{r} A_{k,1,0} + \frac{r^2 + 1 - a^2}{2r^2} A_{k-2,0} + \frac{1}{4\pi r^2} \int_0^{2\pi} \cos^{n-2} \theta d\theta,
\end{align}

(28)

This completes the proof.

\[ \lambda_{k-1} = 3.5 \ldots (2k - 3), \lambda_k = (2k - 1)\lambda_{k-1} \] (12).

Thus,

\[ A_{2k,0} = -\frac{a}{r} A_{2k-1,0} + \frac{r^2 + 1 - a^2}{2r^2} A_{2k-2,0} + \frac{\lambda_{k-1}}{2^k (k-1)!} r^2. \]

(30)

(31)

Then

\[ A_{n-2m+2,0} = \sum_{i=0}^{m} (-1)^i C_i^m \frac{\cos^{n-2m+2i} \theta}{S(r, \theta)}, \]

(33)

\[ \sin^{2k} \theta = \cos^{2k} \theta S(r, \theta) \]

Thus,

\[ A_{n-2m+2,0} = \sum_{i=0}^{m} (-1)^i C_i^m \frac{\sin^{2m-2i} \theta}{S(r, \theta)} = \sum_{i=0}^{m} (-1)^i C_i^m \frac{\cos^{n-2m+2i} \theta}{S(r, \theta)}, \]

(34)

This completes the proof.

Remark 1. \( A_{p,2k+1} = 0. \)

By Lemmas 1 and 2, we have

\begin{align}
A_{2k,0} &= \frac{1}{4\pi r^2} \int_0^{2\pi} d\theta \left( -\frac{a}{r} A_{1,0} + \frac{r^2 + 1 - a^2}{2r^2} A_{0,0} \right), \\
&= \frac{1}{2r^2} \left( -\frac{4}{r^2} S(r, H_2) + aS(G_1 + G_2) + \frac{G_1 - G_2}{2SH_1 H_2} \right), \\
&= \frac{1}{4SH_1 H_2 r^2} (-2SH_1 H_2 + aS(G_1 + G_2) + (r^2 + 1)(G_1 - G_2)).
\end{align}

(35)
Then,

\[
A_{0,2} = A_{0,0} - A_{2,0} \\
= \frac{G_1 - G_2}{2SH_1H_2} \left( -\frac{1}{4SH_1H_2r^2} \left( -2SH_1H_2 + aS(G_1 + G_2) + (r^2 + 1)(G_1 - G_2) \right) \right) \\
= \left( -2SH_1H_2 + aS(G_1 + G_2) - (r^2 - 1)(G_1 - G_2) \right) \frac{1}{4SH_1H_2r^2} ,
\]

and we also have

\[
A_{5,0} = -\frac{4aSH_1H_2 + (r^2 + a^2 + 1)S(G_1 + G_2) + a(3r^2 - a^2 + 3)(G_1 - G_2)}{8SH_1H_2r^3} \\
A_{1,2} = -\frac{(-4aSH_1H_2 + (-r^2 + a^2 + 1)S(G_1 + G_2) + a(r^2 - a^2 + 3)(G_1 - G_2))}{8SH_1H_2r^3} \\
A_{4,0} = -\frac{1}{8SH_1H_2r^4} \left[ (-4r^2 - 2(a^2 + 1))SH_1H_2 + 2a(r^2 + 1)S(G_1 + G_2) \\
+ (r^2 + 2(a^2 + 1)r^2 + (-a^4 + 2a^2 + 1))(G_1 - G_2)) \right] , \\
A_{2,2} = \frac{1}{8SH_1H_2r^4} \left[ 2(-a^2 - 1)SH_1H_2 + 2aS(G_1 + G_2) \\
+ (-r^2 + 2a^2r^2 + (-a^4 + 2a^2 + 1))(G_1 - G_2)) \right] ,
\]

and

\[
A_{0,4} = \frac{1}{8SH_1H_2r^4} \left[ 2(2r^2 - a^2 - 1)SH_1H_2 + 2a(-r^2 + 1)S(G_1 + G_2) \\
+ (r^4 + 2(a^2 - 1)r^2 + (-a^4 + 2a^2 + 1))(G_1 - G_2)) \right] , \\
A_{5,0} = \frac{1}{16SH_1H_2r^5} \left[ 4a(-3r^2 - 2)SH_1H_2 + (r^4 + 2(2a^2 + 1)r^2 + (-a^4 + 4a^2 + 1))S(G_1 + G_2) \\
+ a(5r^4 + 10r^2 - a^4 + 5)(G_1 - G_2) \right] , \\
A_{3,2} = \frac{1}{16SH_1H_2r^5} \left[ 4a(-r^2 - 2)SH_1H_2 + (-r^4 + 2a^2r^2 + (-a^4 + 4a^2 + 1))S(G_1 + G_2) \\
+ (-a^4 + 2a(a^2 + 2)r^2 + a(-a^4 + 5))(G_1 - G_2)) \right] , \\
A_{1,4} = \frac{1}{16SH_1H_2r^5} \left[ 4a(r^2 - 2)SH_1H_2 + (r^4 - 2r^2 + (-a^4 + 4a^2 + 1))S(G_1 + G_2) \\
+ (-3ar^4 + 2a(2a^2 - 1)r^2 + a(5 - a^4))(G_1 - G_2)) \right] ,
\]
\[ A_{6,0} = -\frac{1}{16SH_1H_2r^6} \left[ (-7r^4 + 2(-5a^2 - 3)r^2 + 2(a^4 - 4a^2 - 1))SH_1H_2 ight. \\
\left. + a(3r^4 + 2(a^2 + 3)r^2 + (-a^4 + 2a^2 + 3))S(G_1 + G_2) \\
+ (r^6 + 3(2a^2 + 1)r^4 + 3(4a^2 - a^4 + 1)r^2 + (6a^2 - 3a^4 + 1))(G_1 - G_2) \right], \]
\[ A_{4,2} = \frac{1}{16SH_1H_2r^6} \left[ (r^4 + (-6a^2 - 2)r^2 + (2a^4 - 8a^2 - 2))SH_1H_2 ight. \\
\left. + ((-a)r^4 + (2a^3 + 2a)r^2 + (2a^3 - a^5 + 3a))S(G_1 + G_2) \\
+ (-r^6 + (2a^2 - 1)r^4 + (8a^2 - a^4 + 1)r^2 + (6a^2 - 3a^4 + 1))(G_1 - G_2) \right]. \]

In addition, we have

\[ A_{2,4} = \frac{1}{16SH_1H_2r^6} \left[ (r^4 + 2(1 - a^2))r^2 + 2(a^4 - 4a^2 - 1) \right)SH_1H_2 \\
\left. + (-ar^4 + 2a(a^2 - 1)r^2 + a(-a^4 + 2a^2 + 3))S(G_1 + G_2) \\
+ (r^6 + (-2a^2 - 1)r^4 + (a^4 + 4a^2 - 1)r^2 + (6a^2 - 3a^4 + 1))(G_1 - G_2) \right], \]
\[ A_{0,6} = \frac{1}{16SH_1H_2r^6} \left[ (-7r^4 + 2(a^2 + 3)r^2 + 2(a^4 - 4a^2 - 1))SH_1H_2 ight. \\
\left. + (3ar^4 + 2a(a^2 - 3)r^2 + a(-a^4 + 2a^2 + 3))S(G_1 + G_2) \\
+ (-r^6 + 3(-2a^2 + 1)r^4 + 3(a^4 - 1)r^2 + (-3a^4 + 6a^2 + 1))(G_1 - G_2) \right], \]
\[ A_{7,0} = \frac{1}{32SH_1H_2r^6} \left[ (4(-3a - 7))r^4 + 4(7a - 10a^2 - 8)r^2 + 8(a^4 - 4a^2 + 3a - 1) \right)SH_1H_2 \\
\left. + (r^6 + (4a^2 + 12a - 1)r^4 + (-a^4 + 8a^3 - 8a^2 + 24a - 5)r^2 \\
+ (-4a^5 + 3a^4 + 8a^3 - 12a^2 + 12a - 3))S(G_1 + G_2) \\
+ ((5a + 4)r^6 + (24a^2 - 5a + 12)r^4 + (-a^5 - 12a^4 + 48a^2 - 25a + 12)r^2 \\
+ (3a^5 - 12a^4 + 24a^2 - 15a + 4))(G_1 - G_2) \right], \]
\[ A_{5,2} = \frac{1}{32SH_1H_2r^6} \left[ (4(3a - 7))r^4 + 4(11a - 10a^2 - 8)r^2 + 8(a^4 - 4a^2 + 3a - 1) \right)SH_1H_2 \\
\left. + (-r^6 + (-4a^2 + 12a - 5)r^4 + (a^4 + 8a^3 - 16a^2 + 24a - 7)r^2 \\
+ (-4a^5 + 3a^4 + 8a^3 - 12a^2 + 12a - 3))S(G_1 + G_2) \\
+ ((-5a + 4)r^6 + (24a^2 - 25a + 12)r^4 + (a^5 - 12a^4 + 48a^2 - 35a + 12)r^2 \\
+ (3a^5 - 12a^4 + 24a^2 - 15a + 4))(G_1 - G_2) \right]. \]
\[ A_{3,4} = \frac{1}{32SH_1H_2r^4} \left[ (4(5a - 7)r^4 + 4(15a - 10a^2 - 7)r^2 + 8(a^4 - 4a^2 + 3a - 1)SH_1H_2 \right. \\
+ \left( r^6 + (-8a^2 + 12a - 5)r^4 + (3a^4 + 8a^3 - 24a^2 + 24a - 9)r^2 \\
+ (4a^5 + 3a^4 + 8a^3 - 12a^2 + 12a - 3) \right) S(G_1 + G_2) \\
+ \left( (-3a + 4)r^6 + (-4a^3 + 24a^2 - 33a + 12)r^4 \\
+ (3a^5 - 12a^4 + 48a^2 - 45a + 12)r^2 \\
+ (3a^5 - 12a^4 + 24a^2 - 15a + 4) \right) (G_1 - G_2) \], \\
\[ A_{1,6} = \frac{1}{32SH_1H_2r^4} \left[ (4(3a - 7)r^4 + 4(19a - 10a^2 - 7)r^2 + 8(a^4 - 4a^2 + 3a - 1))SH_1H_2 \right. \\
+ \left( r^6 + (-8a^3 + 12a - 1)r^4 + (5a^4 + 8a^3 - 32a^2 + 24a - 11)r^2 \\
+ (4a^5 + 3a^4 + 8a^3 - 12a^2 + 12a - 3) \right) S(G_1 + G_2) \\
+ \left( (3a + 4)r^6 + (-12a^3 + 24a^2 - 29a + 12)r^4 \\
+ (5a^5 - 12a^4 + 48a^2 - 55a + 12)r^2 \\
+ (3a^5 - 12a^4 + 24a^2 - 15a + 4) \right) (G_1 - G_2) \]. \\
\] 

Using equation (15), we get 
\[ f^0(r) = \frac{1}{rSH_1H_2} \left( XSH_1H_2 + YS(G_1 + G_2) + Z(G_1 - G_2) \right), \]
\[ (40) \]
where 
\[ X = (x_ir^4 + x_jr^2 + x_0), Y = (y_6r^6 + y_2r^4 + y_2r^2 + y_0), \]
\[ (41) \]
and 
\[ Z = \left( z_6r^6 + z_2r^4 + z_2r^2 + \frac{(a^2 - 1)x_0 + 2ay_0}{2} \right), \]
\[ (42) \]
with the coefficients \( x_i, y_j, \) and \( z_i \) the polynomials in the coefficients of \( a, p_{ij}, \) and \( q_{ij}. \)

In fact, there are only ten independent parameters between \( x_i, y_j, \) and \( z_i \) with respect to \( p_{ij}, q_{ij}, \) and \( a. \) In order to bound the zeros number of numerator of \( f^0(r), \) it is sufficient to bound the zeros number of 
\[ K(r) = X^2S^2H_1^2H_2^2 - [YS(G_1 + G_2) + Z(G_1 - G_2)]^2. \]
\[ (43) \]
Since 
\[ (G_1 + G_2)(G_1 - G_2) = -4aS, \]
\[ (G_1 + G_2)^2 = 4(1 - r^2) + 2G_1G_2, \]
\[ (44) \]
and 
\[ (G_1 - G_2)^2 = 4(1 - r^2) - 2G_1G_2, \]
\[ (45) \]
we have 
\[ K(r) = X^2S^2H_1^2H_2^2 - 4Y^2S^2(1 - r^2) - 4Z^2(1 - r^2) \\
+ 8aYZS^2 - 2(Y^2S^2 - Z^2)G_1G_2. \]
\[ (46) \]
Finally, in order to bound the zeros number of the above expression, we should bound the zeros of the following polynomial:
\[ H(r) = \left[ X^2S^2H_1^2H_2^2 - 4Y^2S^2(1 - r^2) - 4Z^2(1 - r^2) + 8aYZS^2 \right] \\
- 4(Y^2S^2 - Z^2)G_1G_2. \]
\[ (47) \]
We have
Therefore, we get
\[ H(r) = S^2 \left( d_0 r^{30} + \cdots + d_2 r^2 + d_0 \right), \tag{49} \]
where \( d_i \) are the polynomials in \( a, x_i, y_i, \) and \( z_i \). We conclude that \( f^0(r) \) has at most 15 simple zeros. Hence, Theorem 1 is proved.

4. Conclusion

As we know, the limit cycles and a polynomial differential system is the well-known 16th Hilbert problem, which together with the Riemann conjecture are the two problems of the famous list of 23 problems of Hilbert which remain open. In addition, a classical way to produce limit cycles is by perturbing a system which has a center, in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits of the period annulus of the center of the unperturbed system; in this work, by using the averaging theory of first order, we study the bifurcation of limit cycles from the period annulus surrounding the origin of a class of cubic polynomial differential systems; when they are perturbed inside the class of all polynomial differential systems of degree six, we have obtained at most 15th limit cycles for this kind of the problem; in the next study, we will try to extend the same tools but for higher degrees.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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