Unitary approach to the quantum forced harmonic oscillator

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Abstract

In this paper we introduce an alternative approach to studying the evolution of a quantum harmonic oscillator subject to an arbitrary time dependent force. With the purpose of finding the evolution operator, certain unitary transformations are applied successively to Schrödinger’s equation reducing it to its simplest form. Therefore, instead of solving the original Schrödinger’s partial differential equation in time and space the problem is replaced by a system of ordinary differential equations. From the obtained evolution operator we workout the propagator. Even though we illustrate the use of unitary transformations on the solution of a forced harmonic oscillator, the method presented here might be used to solve more complex systems. The present work addresses many aspects regarding unitary transformations and the dynamics of a forced quantum harmonic oscillator that should be useful for students and tutors of the quantum mechanics courses at the senior undergraduate and graduate level.
I. INTRODUCTION

The analytic solution of the one-dimensional harmonic oscillator’s Schrödinger’s equation is one of the first triumphs of the undergraduate student in the quantum mechanics course. Usually, in a first approach, the harmonic oscillator’s (HO) energy eigenfunctions and eigenstates are obtained by solving Schrödinger’s differential equation. These are recalculated later by defining the ladder operators. Furthermore, the harmonic oscillator is a problem with a wide range of applications in modern physics. It is of enormous practical importance in quantum optics and solid state physics for example. Many of these problems involve the application of an external force to the harmonic oscillator. Of special interest is a bounded electrical charge subject to an oscillating electromagnetic field that under certain approximations can be reduced to a forced harmonic oscillator.

The quantum forced harmonic oscillator (QFHO) has been treated either by solving the corresponding differential Schrödinger’s equation or by computing Feynman’s path integrals. In the first case the solution to Schrödinger equation might be worked out through a series of standard variables changes\(^1\) by proposing Gaussian wave function\(^2\) or by defining a rather general form of the well known ladder operators\(^3\). Any of these procedures yields the QFHO’s evolution operator and propagator. Feynman’s path integral method, on the other hand\(^4–7\), involves the calculation of the classical action in order to obtain the quantum propagator. The QFHO quantum propagator posses a structure similar to the well known propagator for the simple quantum oscillator plus an interaction-dependent correction due to the forcing term in the Hamiltonian\(^7,8\).

In this paper we seek for an alternative procedure to obtain the QFHO’s evolution operator and propagator. Our approach exploits certain unitary transformations\(^9–12\) that reduce the Floquet operator to its trivial form i.e., the energy operator. This method may be applied to a wide variety of systems whose Hamiltonians are more involved and have more dimensions than the one presented in this work. We show that the evolution operator and propagator can be obtained from the set of unitary transformations mentioned above. Finally, from the evolution operator, we workout the position and momentum operators in the Heisenberg picture in terms of the corresponding operators in the Schrödinger picture. This transformation posses a symplectic form.

This work is organized as follows. In section II we introduce the philosophy behind the

\(^1\)\(^2\)\(^3\)\(^4\)\(^5\)\(^6\)\(^7\)\(^8\)\(^9\)\(^10\)\(^11\)\(^12\)
method to reduce Schrödinger’s equation and compute the evolution operator and propagator. The unitary transformations used to calculate the evolution operator are listed in section III. We also show how the Schrödinger’s equation is reduced through these transformations. Additionally, the Heisenberg picture position and momentum operators are explicitly calculated and it is demonstrated that their relation with the Schrödinger picture ones is symplectic. The propagator or Green’s function is calculated in section IV. Finally in section V we summarize.

II. UNITARY TRANSFORMATIONS

Unitary transformations are very useful tools in quantum mechanics. In particular, in the form of an evolution operator, they may describe the dynamics of a closed quantum system since they preserve the probability of quantum states.

Whereas, any time independent Hamiltonians $\hat{H}$ indistinctly yields the very well known evolution operator

$$ U = \exp \left( -i\hat{H}t/\hbar \right), $$

(1)
time dependent ones may lead to two radically different situations. On one hand, if the Hamiltonian commutes with itself at any two different times i.e. $[\hat{H}(t_1), H(t_2)] = 0$ for $t_1 \neq t_2$, the evolution operator is given by $U = \exp \left( -i/\hbar \int_{t_0}^{t_1} \hat{H}(s) ds \right)$. On the other, if $\hat{H}(t)$ does not commute with itself at two different times, the evolution operator is

$$ U = \mathcal{T} \exp \left[ -i/\hbar \int_{t_0}^{t_1} \hat{H}(s) ds \right] $$

(2)

where $\mathcal{T}$ is the time ordering operator. Although very often the evolution operator is hard to find, and for most physically interesting potentials only perturbative solutions can be obtained for Hamiltonians in this category, some systems allow analytic solutions to be found. Such is the case of the one dimensional harmonic oscillator with time dependent coefficients and nonlinear Hamiltonian systems. Although even in the case in which the system falls in the first category, the Hamiltonian is too complex to be treated through the evolution operator in (1). This is the case of a system of coupled harmonic oscillators, the Hamiltonian does not depend explicitly on time, however its structure is so involved that it has been studied via unitary transformation.
The QFHO’s Hamiltonian is given by

\[ \hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{q}^2 - f \hat{q}, \]  

(3)

where, \( \hat{p} \) is the energy operator \( i\hbar \partial_t \) and \( \hat{q}, \hat{p} \) are the space and momentum operators while \( f \equiv f(t) \) is the time dependent force. Clearly, the commutator \( [\hat{H}(t_1), \hat{H}(t_2)] = \hat{p} [f(t_2) - f(t_1)] / m \) does not vanish unless \( f \) is a constant, therefore the QFHO falls into the third category. Thus, the QFHO’s evolution operator takes the form (2), nevertheless here we propose the following alternative procedure to compute it. First, we consider Schrödinger’s equation

\[ \hat{H} |\psi(t)\rangle = \hat{p}_t |\psi(t)\rangle, \]  

(4)

and conveniently introduce the Floquet operator \( \hat{U} \) \( \hat{H} \) = \( \hat{H} - \hat{p}_t = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{q}^2 - f \hat{q} - \hat{p}_t \),

(5)

that allows to write Schrödinger’s equation in the compact form

\[ \hat{H} |\psi(t)\rangle = 0. \]  

(6)

Lets now assume that there is a set \( \{ \hat{U}_1, \hat{U}_2, \ldots, \hat{U}_n \} \) of unitary transformations such that if we apply \( \hat{U} = \hat{U}_n \ldots \hat{U}_2 \hat{U}_1 \) to the Floquet operator, it reduces to the energy operator as

\[ \hat{U} \hat{H} \hat{U}^\dagger = -\hat{p}_t, \]  

(7)

removing the Hamiltonian part. If such a transformation does exists, the Schrödinger’s equation takes the form

\[ \hat{U} \hat{H} \hat{U}^\dagger |\psi(t)\rangle = -\hat{p}_t \left( \hat{U} |\psi(t)\rangle \right) = 0. \]  

(8)

Reminding that \( \hat{p}_t \) is \( \hbar \) times a time derivative, it is easy to see that \( \hat{U} |\psi(t)\rangle \) is a constant ket, say

\[ \hat{U} |\psi(t)\rangle = |\psi(0)\rangle \]  

(9)

Therefore, if the set of unitary transformations is known, the evolution of a quantum state \( \psi \) can be easily calculated by multiplying the previous equation by the inverse of \( U \) \( (U^{-1} = U^\dagger) \)

\[ |\psi(t)\rangle = \hat{U}^\dagger |\psi(0)\rangle. \]  

(10)
This equation states that $\hat{U}^\dagger$ is the time evolution operator.

The Green’s function, or the propagator, is calculated as usual in terms of the evolution operator as

$$G(q, q'; t, 0) = \langle q | U^\dagger | q' \rangle.$$  \hspace{1cm} (11)

### III. EVOLUTION OPERATOR

In this section we aim to find a set of unitary operators that comply with equation (7) and an expression for the evolution operator. With this objective in mind it is necessary to consider the Hamiltonian symmetry in order to propose the unitary transformations that reduce the Floquet operator. Notice that the QFHO’s Hamiltonian is quadratic in position and momentum operators, thus our transformations may not contain terms of a superior order.

The purpose of the first transformation is to remove the linear term of the position operator $f \hat{q}$ from equation (5). We thus apply a translation in the position and momentum operators in the form of

$$\hat{U}_1 = \hat{U}_1^t \hat{U}_1^q \hat{U}_1^p,$$  \hspace{1cm} (12)

where

$$\hat{U}_1^t = \exp \left( \frac{i}{\hbar} S \right),$$  \hspace{1cm} (13)

$$\hat{U}_1^q = \exp \left( -i \frac{\pi}{\hbar} \hat{q} \right),$$  \hspace{1cm} (14)

$$\hat{U}_1^p = \exp \left( \frac{i}{\hbar} \lambda \hat{p} \right),$$  \hspace{1cm} (15)

where $S$, $\pi$ and $\lambda$ are time dependent transformation parameters. The above transformation yields the following transformation rules on the position, momentum and energy operators

$$\hat{U}_1 \hat{q} \hat{U}_1^\dagger = \hat{q} + \lambda,$$  \hspace{1cm} (16)

$$\hat{U}_1 \hat{p} \hat{U}_1^\dagger = \hat{p} + \pi,$$  \hspace{1cm} (17)

$$\hat{U}_1 \hat{p} \hat{U}_1^\dagger = \hat{p}_t + \hat{S} - \hat{\pi} \hat{q} + \hat{\lambda} \hat{p} + \hat{\lambda} \pi.$$  \hspace{1cm} (18)

Equation (16) shows that $\hat{U}_1$ performs a translation by $\lambda$ to the position operator $\hat{q}$. Similarly, in (17) and (18) reveal that $\hat{U}_1$ also performs shifts in momentum and energy. In the particular case where $\pi$ is a constant and $\pi = m \dot{\lambda}$, (16) and (17) characterize a Galilean
boost. Nevertheless, as we will show below, the most general case where \( f \) is an arbitrary function of time requires \( \lambda \) and \( \pi \) to be nontrivial functions of time.

Under the action of \( \hat{U}_1 \), the Floquet operator (5) becomes

\[
\hat{U}_1 \hat{H} \hat{U}_1^\dagger = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{q}^2 - \hat{p}_t + \left( \frac{\pi}{m} - \dot{\lambda} \right) \hat{p} + \left( m \omega^2 \lambda - f + \dot{\pi} \right) \hat{q} + \mathcal{L} - \hat{S},
\]

where we have introduced

\[
\mathcal{L} = \frac{1}{2m} \pi^2 + \frac{1}{2} m \omega^2 \dot{\lambda}^2 - f \lambda - \pi \dot{\lambda},
\]

As mentioned earlier, the purpose of this transformation is to reduce the terms proportional to \( \hat{q} \). Nonetheless, due to the action of \( U_1 \) new terms have appeared that also need to be canceled in order to simplify the Hamiltonian. Hence we now proceed to eliminate the terms proportional to \( \hat{q} \), \( \hat{p} \) and the independent ones by setting

\[
\frac{\pi}{m} - \dot{\lambda} = 0,
\]

\[
-\dot{\pi} + f - m \omega^2 \lambda = 0,
\]

\[
\dot{S} - \mathcal{L} = 0.
\]

The first of the previous equations is the standard relation between the classical velocity and momentum \( \dot{\lambda} \) and \( \pi \). The second is Newton’s second law of motion for the classical forced harmonic oscillator. The last equation states the relation between the action \( S \) and the Lagrangian function \( \mathcal{L} \). Furthermore, the somewhat astonishing and interesting point is that computing the Euler’s equations arising from the Lagrangian \( \mathcal{L} \) we obtain

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\pi}} - \frac{\partial \mathcal{L}}{\partial \pi} = \frac{\pi}{m} - \dot{\lambda} = 0,
\]

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} - \frac{\partial \mathcal{L}}{\partial \lambda} = - \left( m \omega^2 \lambda - f + \dot{\pi} \right) = 0,
\]

recovering the conditions (21) and (22) that make the linear terms vanish.

Here, it is important to establish the initial conditions \( S(t=0) = 0 \), \( \pi(t=0) = 0 \) and \( \lambda(t=0) = 0 \) on the transformation parameters in order to guarantee that \( \hat{U}_1 \) reduces to the identity operator when \( t \to 0 \) i.e.

\[
U^\dagger(t=0) = \mathbf{1}.
\]

Hence, the \( U_1 \) transforms the Floquet operator into

\[
\hat{U}_1 \hat{H} \hat{U}_1^\dagger = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{q}^2 - \hat{p}_t,
\]

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the Floquet operator of a simple harmonic oscillator. The first transformation has left a
time-independent system that falls in the first category of dynamics systems that yield the
simplest evolution operator (11) as
\[
\hat{U}_2 = \exp \left[ -\frac{i}{\hbar} \left( \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{q}^2 \right) t \right].
\] (28)
However, it is instructive to consider the last transformation in the general form of a shear
in the \( \hat{q} - \hat{p} \) space
\[
\hat{U}_2 = \exp \left[ i \theta \left( \frac{1}{\Delta} \hat{p}^2 + \Delta \hat{q}^2 \right) \right]
\] (29)
were \( \theta \) and \( \Delta \) are in general time dependent transformation parameters.

The unitary operator \( \hat{U}_2 \) acts on position, momentum and energy operators as follows
\[
\hat{U}_2 \hat{q} \hat{U}_2^\dagger = \hat{q} \cos \theta + \frac{1}{\Delta} \hat{p} \sin \theta,
\] (30)
\[
\hat{U}_2 \hat{p} \hat{U}_2^\dagger = \hat{p} \cos \theta - \Delta \hat{q} \sin \theta,
\] (31)
\[
\hat{U}_2 \hat{p} t \hat{U}_2^\dagger = \hat{p}_t + \frac{\dot{\theta}}{2} \left( \frac{1}{\Delta} \hat{p}^2 + \Delta \hat{q}^2 \right),
\] (32)
where, supposing beforehand that \( \Delta \) is a constant, suffices to eliminate all the Hamiltonian’s
terms.

Applying the last transformation (29) to the transformed Floquet operator (27) we have
\[
\hat{U}_2 \hat{H} \hat{U}_2^\dagger = \left( \frac{\Delta}{m} \cos^2 \theta + \frac{m\omega^2}{\Delta} \sin^2 \theta - \dot{\theta} \right) \frac{\hat{p}^2}{2\Delta}
\]
\[+ \left( \frac{\Delta}{m} \sin^2 \theta + \frac{m\omega^2}{\Delta} \cos^2 \theta - \dot{\theta} \right) \frac{\Delta \hat{q}^2}{2}
\]
\[+ \left( \frac{m\omega^2}{\Delta} - \frac{\Delta}{m} \right) \frac{1}{2} \cos \theta \sin \theta (\dot{\hat{p}} \hat{\hat{q}} + \dot{\hat{q}} \hat{\hat{p}}) - \hat{p}_t.
\] (33)
To reduce this last form of the Floquet operator to the energy operator \( \hat{p}_t \) we must eliminate
the coefficients of \( \hat{q}^2 \), \( \hat{p}^2 \) and \( \dot{\hat{p}} \hat{\hat{q}} + \dot{\hat{q}} \hat{\hat{p}} \) obtaining the system of differential equations
\[
\frac{\Delta}{m} \cos^2 \theta + \frac{m\omega^2}{\Delta} \sin^2 \theta - \dot{\theta} = 0,
\] (34)
\[
\frac{\Delta}{m} \sin^2 \theta + \frac{m\omega^2}{\Delta} \cos^2 \theta - \dot{\theta} = 0,
\] (35)
\[
\frac{m\omega^2}{\Delta} - \frac{\Delta}{m} = 0.
\] (36)
From (36) we obtain that \( \Delta = m\omega \) and consequently (34) and (35) reduce to
\[
\omega - \dot{\theta} = 0.
\] (37)
Once more, in order for $U_2$ to reduce to unity, the initial condition on the $\theta$ parameter must be set to $\theta(t = 0) = 0$ giving

$$\theta = \omega t.$$  \hfill (38)

Notice that the relations above turn the unitary transformation (29) into (28).

In this way all the Hamiltonian terms vanish to yield the simplest Floquet operator

$$\hat{U}_2\hat{U}_1\hat{H}\hat{U}_1^\dagger\hat{U}_2^\dagger = -\hat{p}_t.$$  \hfill (39)

As we mentioned above, this is precisely the evolution operator

$$\hat{U}^\dagger = \hat{U}_1^\dagger\hat{U}_2^\dagger.$$  \hfill (40)

Note that the time evolution operator’s time dependence enters through the parameters $S$, $\lambda$, $\pi$ and $\theta$. Once the explicit form of the force $f$ is given $\pi$ and $\lambda$ can be calculated from the system of differential equations formed by (21), (22) and the initial conditions $\lambda = \pi = 0$ with $t = 0$. With the expressions of $\lambda$ and $\pi$ at hand the Lagrangian $\mathcal{L}$ is a known function and $S$ can readily be worked out from (23)

$$S = \int_0^t \mathcal{L}(s) \, ds.$$  \hfill (41)

We have therefore reduced the problem to solving a system of on variable differential equations. Naturally the difficulty in finding solutions for $\lambda$ and $\pi$ hinges in the complexity of the function $f$.

In the following we obtain the Heisenberg picture position and momentum operators’ time evolution via the application of the above obtained evolution operator

$$\dot{\hat{q}}_H(t) = \hat{U}\hat{q}\hat{U}^\dagger = \hat{q}\cos \theta + \frac{1}{\Delta} \sin \theta + \lambda,$$

$$\dot{\hat{p}}_H(t) = \hat{U}\hat{p}\hat{U}^\dagger = \hat{p}\cos \theta - \hat{q}\Delta \sin \theta + \pi.$$  \hfill (42) \hfill (43)

Rearranging the previous results in in matricial form we get

$$\begin{bmatrix} \dot{\hat{q}}_H(t) \\ \dot{\hat{p}}_H(t) \end{bmatrix} = \begin{bmatrix} \cos \theta & \frac{1}{\Delta} \sin \theta \\ -\Delta \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \hat{q} \\ \hat{p} \end{bmatrix} + \begin{bmatrix} \lambda \\ \pi \end{bmatrix} = \mathcal{M} \begin{bmatrix} \hat{q} \\ \hat{p} \end{bmatrix} + \xi.$$  \hfill (44)

where

$$\mathcal{M} = \begin{bmatrix} \cos \theta & \frac{1}{\Delta} \sin \theta \\ -\Delta \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \omega t & \frac{1}{m}\sin \omega t \\ -m\omega \sin \omega t & \cos \omega t \end{bmatrix}$$  \hfill (45)

is clearly a symplectic matrix.
IV. GREEN’S FUNCTION

Now we focus on obtaining the Green’s function using the time evolution operator through the relation (11). The importance of obtaining the general form of this function lies in the fact that it provides us with a direct method to calculate the evolution of a given wave packet from \( t = 0 \) to \( t \).

At this point it is assumed that, given the explicit form of the force \( f \), the solution of equations (21), (22) and (23) for \( \lambda, \pi \) and consequently \( S \) are known. In order to find \( G \) we insert a complete basis of the position states so we can calculate separately the matrix elements of each unitary transformation \( \mathcal{U}_1^{\dagger} \) and \( \mathcal{U}_2^{\dagger} \) as follows

\[
G(q, q'; t, 0) = \int dq'' \left\langle q \left| \mathcal{U}_1^{\dagger} q'' \right\rangle \left\langle q'' \left| \mathcal{U}_2^{\dagger} q' \right\rangle \right. \tag{46}
\]

The first matrix element, corresponding to the unitary transformation related with a translation in position and momentum, is readily found as

\[
\left\langle q \left| \mathcal{U}_1^{\dagger} q'' \right\rangle = \exp \left[-\frac{i}{\hbar} S \right] \exp \left[\frac{i}{\hbar} \pi q''\right] \delta(q - q'' - \lambda). \tag{47}
\]

The second element corresponding to a shear is

\[
\left\langle q'' \left| \mathcal{U}_2^{\dagger} q' \right\rangle = \sqrt{\frac{\Delta}{2\pi \hbar \sin \theta}} \exp \left[\frac{i\Delta}{2\hbar \sin \theta} \left[(q'^2 + q''^2) \cos \theta - 2q'q''\right]\right] \tag{48}
\]

Note that this matrix element is in fact the propagator for a simple harmonic oscillator.

By integrating (46) we finally arrive to the explicit form of the Green’s function

\[
G(q, q'; t, 0) = \sqrt{\frac{\Delta}{2\pi \hbar \sin \theta}} \exp \left[-\frac{i}{\hbar} S \right] \exp \left[\frac{i}{\hbar} \left(\pi - \frac{\Delta}{\hbar \sin \theta} q'\right) (q - \lambda)\right] \times \exp \left[\frac{i\Delta}{2\hbar} \cot \theta ((q - \lambda)^2 + q'^2)\right] \\
= \sqrt{\frac{m\omega}{2\pi \hbar \sin \omega t}} \exp \left[-\frac{i}{\hbar} S \right] \exp \left[\frac{i}{\hbar} \left(\pi - \frac{m\omega}{\hbar \sin \omega t} q'\right) (q - \lambda)\right] \times \exp \left[\frac{im\omega}{2\hbar} \cot \omega t ((q - \lambda)^2 + q'^2)\right] \tag{49}
\]

This function is formed by linear and quadratic terms of the position eigenstates. It exhibits the correct structure predicted by Schwinger and others for Hamiltonians that are quadratic in the position and momentum operators.
V. CONCLUSIONS

We have obtained the evolution operator and the propagator of a forced harmonic oscillator by performing a series of unitary transformations. These transformations are chosen to eliminate certain terms of the Floquet operator to the point of reducing to its simplest form, the energy operator. In this particular case two transformations are needed: the first one shifts position, momentum and energy allowing to remove from the Hamiltonian the potential energy due to the force. The second and final transformation resembles a shear in the \( \hat{q} - \hat{p} \) space. The first transformation leaves the Floquet operator of a simple quantum harmonic oscillator. Finally the shear cancels all remaining terms to yield the quantum forced harmonic evolution operator and propagator.

The obtained result for the propagator is consistent with the one obtained by Feynman integration and with the general structure proposed in Ref[7] and Ref[8].

Despite the simplicity of the forced harmonic oscillator, it may serve to illustrate the use of unitary transformations to describe its dynamics. This method may be applied to the solution of more complex systems as the one of a charged particle subject to arbitrary electromagnetic fields. Unfortunately the choice on which unitary transformations should be used to reduce the Floquet operator is not systematic nor direct, and should be made intuitively for each case.

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1. Popov V S and Perelomov A M 1970 Parametric excitation of a quantum oscillator II Sov. Phys. JETP 30, 910-13
2. Husimi K 1953 Miscellanea in Elementary Quantum Mechanics II Prog. Theor. Phys. 9, 381-402
3. Kim H-C, Lee M-H, Ji J-Y, and Kim J K 1996 Heisenberg-picture approach to the exact quantum motion of a time-dependent forced harmonic oscillator Phys. Rev. A 53, 3767-72
4 Khandekar D C and Lawande S V 1978 Exact solution of a time-dependent quantal harmonic oscillator with damping and a perturbative force J. Math. Phys. 20, 1870-7
5 Feynman R P 1948 Space-Time Approach to Non-Relativistic Quantum Mechanics Rev. Mod. Phys. 20, 367-87
6 Feynman R P 1950 Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction Phys. Rev. 80, 440-57
7 Merzbarcher E 1998 Quantum mechanics 3rd edn (USA: John Wiley & Sons, Inc.) Chap. 15
8 Schwinger J 1951 On gauge invariance and vacuum polarization Phys. Rev. 82, 664-79
9 Maamache M 1996 Unitary transformation approach to the exact solution for the singular oscillator J. Phys. A: Math Gen. 29, 2833-7
10 Fan M 1990 Unitary transformation for four harmonically coupled identical oscillator Phys. Rev. A 42, 4377-80
11 Maamache M 1998 Unitary transformation approach to the exact solution for a class of time-dependent nonlinear Hamiltonian systems J. Math. Phys. 39, 161-9
12 Ibarra-Sierra V G, Anzaldo-Meneses A, Cardoso J L, Hernandez-Saldaña H, Kunold A, and Roa-Neri J A E 2013 Quantum and classical dissipation of charged particles Annals Phys. 335, 86-107
13 Breuer H and Petruccione F 2006 The theory of open quantum systems (New York: Oxford University Press) Chap. 8
14 Urrutia L F and Hernández E 1984 Calculation of the propagator for a time-dependent damped, forced harmonic oscillator using the Schwinger action principle Int. J. Theor. Phys. 23, 1105-27
15 Pepore S and Bodinchat S 2009 Schwinger method and path integral with generalized canonical transformation for a harmonic oscillator with time-dependent mass and frequency Chinese J. Phys. 47, 753-63