Anomalous Transport in Complex Networks

Eduardo López¹, Sergey V. Buldyrev¹, Shlomo Havlin¹,², and H. Eugene Stanley¹

¹Center for Polymer Studies, Boston University, Boston, MA 02215, USA
²Minerva Center & Department of Physics, Bar-Ilan University, Ramat Gan, Israel

(Dated: last revised: 30 November 2004 3:30PM)

Abstract

To study transport properties of complex networks, we analyze the equivalent conductance \( G \) between two arbitrarily chosen nodes of random scale-free networks with degree distribution \( P(k) \sim k^{-\lambda} \) in which each link has the same unit resistance. We predict a broad range of values of \( G \), with a power-law tail distribution \( \Phi_{SF}(G) \sim G^{-g_{G}} \), where \( g_{G} = 2\lambda - 1 \), and confirm our predictions by simulations. The power-law tail in \( \Phi_{SF}(G) \) leads to large values of \( G \), thereby significantly improving the transport in scale-free networks, compared to Erdős-Rényi random graphs where the tail of the conductivity distribution decays exponentially. Based on a simple physical “transport backbone” picture we show that the conductances are well approximated by \( ck_Ak_B/(k_A + k_B) \) for any pair of nodes \( A \) and \( B \) with degrees \( k_A \) and \( k_B \). Thus, a single parameter \( c \) characterizes transport on scale-free networks.
Recent research on the topic of complex networks is leading to a better understanding of many real world social, technological, and natural systems ranging from the World Wide Web and the Internet to cellular networks and sexual-partner networks [1]. One type of network topology that appears in many real world systems is the scale-free network [2], characterized by a scale-free degree distribution:

\[ P(k) \sim k^{-\lambda}, \quad k_{\text{min}} \leq k \leq k_{\text{max}}, \]

where \( k \), the degree, is the number of links attached to a node. The distribution has two cutoff values for \( k \): \( k_{\text{min}} \), which represents the minimum allowed value of \( k \) on the graph \( (k_{\text{min}} = 2 \) here), and \( k_{\text{max}} \equiv k_{\text{min}}N^{1/(\lambda-1)} \), which is the typical maximum degree of a network with \( N \) nodes [3, 4]. The scale-free feature allows a network to have some nodes with a large number of links (“hubs”), unlike the case for the classic Erdős-Rényi model of random networks [3, 4].

Here we show that for scale-free networks with \( \lambda \geq 2 \), transport properties characterized by conductance display a power-law tail distribution that is related to the degree distribution \( P(k) \). We find that this power-law tail represents pairs of nodes of high degree which have high conductance. Thus, transport in scale-free networks is better than in Erdős-Rényi random networks. Also, we present a simple physical picture of transport in complex networks and test it with our data.

The classic random graphs of Erdős and Rényi [3, 6] have a Poisson degree distribution, in contrast to the power-law distribution of the scale-free case. Due to the exponential decay of the degree distribution, the Erdős-Rényi networks lack hubs and their properties, including transport, are controlled solely by the average degree \( \bar{k} \equiv \sum_{i=k_{\text{min}}}^{k_{\text{max}}} i P(i) \) [6, 7].

Most of the work done so far regarding complex networks has concentrated on static topological properties or on models for their growth [1, 3, 8, 9]. Transport features have not been extensively studied with the exception of random walks on complex networks [10, 11, 12], despite the fact that transport properties contain information about network function [13]. Here, we study the electrical conductance \( G \) between two nodes \( A \) and \( B \) of Erdős-Rényi and scale-free networks when a potential difference is imposed between them. We assume that all the links have equal resistances of unit value [14].

To construct an Erdős-Rényi network, we begin with a fully connected graph, and randomly remove \( 1 - \bar{k}/(N - 1) \) out of the \( N(N - 1)/2 \) links between the \( N \) nodes. To generate
a scale-free network with \( N \) nodes, we use the Molloy-Reed algorithm \[15\], which allows for the construction of random networks with arbitrary degree distribution. We generate \( k_i \) copies of each node \( i \), where the probability of having \( k_i \) satisfies Eq. \[1\]. These copies of the nodes are then randomly paired in order to construct the network, making sure that two previously-linked nodes are not connected again, and also excluding links of a node to itself \[16\].

The conductance \( G \) of the network between two nodes \( A \) and \( B \) is calculated using the Kirchhoff method \[17\], where entering and exiting potentials are fixed to \( V_A = 1 \) and \( V_B = 0 \). We solve a set of linear equations to determine the potentials \( V_i \) of all nodes of the network. Finally, the total current \( I \equiv G \) entering at node \( A \) and exiting at node \( B \) is computed by adding the outgoing currents from \( A \) to its nearest neighbors through \( \sum_j (V_A - V_j) \), where \( j \) runs over the neighbors of \( A \).

First, we analyze the pdf \( \Phi(G)dG \) that two nodes on the network have conductance between \( G \) and \( G + dG \). To this end, we introduce the cumulative distribution \( F(G) \equiv \int_G^\infty \Phi(G')dG' \), shown in Fig. \[1\](a) for the Erdős-Rényi and scale-free (\( \lambda = 2.5 \) and \( \lambda = 3.3 \), with \( k_{\text{min}} = 2 \)) cases. We use the notation \( \Phi_{\text{SF}}(G) \) and \( F_{\text{SF}}(G) \) for scale-free, and \( \Phi_{\text{ER}}(G) \) and \( F_{\text{ER}}(G) \) for Erdős-Rényi. The function \( F_{\text{SF}}(G) \) for both \( \lambda = 2.5 \) and 3.3 exhibits a tail region well fit by the power law

\[
F_{\text{SF}}(G) \sim G^{-(g_G - 1)},
\]

and the exponent \( (g_G - 1) \) increases with \( \lambda \). In contrast, \( F_{\text{ER}}(G) \) decreases exponentially with \( G \).

Increasing \( N \) does not significantly change \( F_{\text{SF}}(G) \) (Fig. \[1\](b)) except for an increase in the upper cutoff \( G_{\text{max}} \), where \( G_{\text{max}} \) is the typical maximum conductance, corresponding to the value of \( G \) at which \( \Phi_{\text{SF}}(G) \) crosses over from a power law to a faster decay. We observe no change of the exponent \( g_G \) with \( N \). The increase of \( G_{\text{max}} \) with \( N \) implies that the average conductance \( \bar{G} \equiv \int G\Phi(G)dG \) also increases slightly \[18\].

We next study the origin of the large values of \( G \) in scale-free networks and obtain an analytical relation between \( \lambda \) and \( g_G \). Larger values of \( G \) require the presence of many parallel paths, which we hypothesize arise from the high degree nodes. Thus, we expect that if either of the degrees \( k_A \) or \( k_B \) of the entering and exiting nodes is small, the conductance \( G \) between \( A \) and \( B \) is small since there are at most \( k \) different parallel branches coming
out of a node with degree $k$. Thus, a small value of $k$ implies a small number of possible parallel branches, and therefore a small value of $G$. To observe large $G$ values, it is therefore necessary that both $k_A$ and $k_B$ be large.

We test this hypothesis by large scale computer simulations of the conditional pdf $\Phi_{SF}(G|k_A,k_B)$ for specific values of the entering and exiting node degrees $k_A$ and $k_B$. Consider first the case $k_B \ll k_A$, and the effect of increasing $k_B$, with $k_A$ fixed. We find that $\Phi_{SF}(G|k_A,k_B)$ is narrowly peaked (Fig. 2(a)) so that it is well characterized by $G^*$, the value of $G$ when $\Phi_{SF}$ is a maximum. Further, for increasing values of $k_B$, we find [Fig. 2(b)] $G^*$ increases as $G^* \sim k_B^\alpha$, with $\alpha = 0.96 \pm 0.05$ consistent with the possibility that as $N \to \infty$, $\alpha = 1$ which we assume henceforth.

For the case of $k_B \gg k_A$, $G^*$ increases less fast than $k_B$, as can be seen in Fig. 2(c) where we plot $G^*/k_B$ against the scaled degree $x \equiv k_A/k_B$. The collapse of $G^*/k_B$ for different values of $k_A$ and $k_B$ indicates that $G^*$ scales as

$$G^* \sim k_B f\left(\frac{k_A}{k_B}\right).$$

(3)

The behavior of the scaling function $f(x)$ can be interpreted using the following simplified “transport backbone” picture [Fig. 2(c) inset], for which the effective conductance $G$ between nodes $A$ and $B$ satisfies

$$\frac{1}{G} = \frac{1}{G_A} + \frac{1}{G_{tb}} + \frac{1}{G_B},$$

(4)

where $1/G_{tb}$ is the resistance of the “transport backbone” while $1/G_A$ (and $1/G_B$) are the resistances of the set of bonds near node $A$ (and node $B$) not belonging to the “transport backbone”. It is plausible that $G_A$ is linear in $k_A$, so we can write $G_A = ck_A$. Since node $B$ is equivalent to node $A$, we expect $G_B = ck_B$. Hence

$$G = \frac{1}{1/ck_A + 1/ck_B + 1/G_{tb}} = k_B \frac{ck_A/k_B}{1 + k_A/k_B + c k_A/G_{tb}},$$

(5)

so the scaling function defined in Eq. (3) is

$$f(x) = \frac{cx}{1 + x + c k_A/G_{tb}} \approx \frac{cx}{1 + x}.$$

(6)

The second equality follows if there are many parallel paths on the “transport backbone” so that $1/G_{tb} \ll c k_A$ [19]. The prediction (6) is plotted in Fig. 2(c) and the agreement with the simulations supports the approximate validity of the transport backbone picture of conductance in complex networks.
The agreement of (6) with simulations has a striking implication: the conductance of a scale-free network depends on only one parameter \( c \). Further, since the distribution of Fig. 2(a) is sharply peaked, a single measurement of \( G \) for any values of the degrees \( k_A \) and \( k_B \) of the entrance and exit nodes suffices to determine \( G^* \), which then determines \( c \) and hence through Eq. (6) the conductance for all values of \( k_A \) and \( k_B \).

Within this “transport backbone” picture, we can analytically calculate \( F_{SF}(G) \). Using Eq. (3), and the fact that \( \Phi_{SF}(G|k_A, k_B) \) is narrow, yields

\[
\Phi_{SF}(G) \sim \int P(k_B) dk_B \int P(k_A) dk_A \delta \left[ k_B f \left( \frac{k_A}{k_B} \right) - G \right],
\]

(7)

where \( \delta(x) \) is the Dirac delta function. Performing the integration of Eq. (7) using (6), we obtain

\[
\Phi_{SF}(G) \sim G^{-g_G} \quad [G < G_{\text{max}}],
\]

(8)

where

\[
g_G = 2\lambda - 1.
\]

(9)

Hence, for \( F_{SF}(G) \), we have \( F_{SF}(G) \sim G^{-2\lambda-2} \). To test this prediction, we perform simulations for scale-free networks and calculate the values of \( g_G - 1 \) from the slope of a log-log plot of the cumulative distribution \( F_{SF}(G) \). From Fig. 3(b) we find that

\[
g_G - 1 = (1.97 \pm 0.04)\lambda - (2.01 \pm 0.13).
\]

(10)

Thus, the measured slopes are consistent with the theoretical value predicted by Eq. (9).

In summary, we find a power-law tail for the distribution of conductance for scale-free networks and relate the tail exponent \( g_G \) to the exponent \( \lambda \) of the distribution \( P(k) \). Our work is consistent with a simple physical picture of how transport takes place in complex networks.

We thank the Office of Naval Research, the Israel Science Foundation, and the Israeli Center for Complexity Science for financial support, and L. Braunstein, S. Carmi, R. Cohen, E. Perlsman, G. Paul, S. Sreenivasan, T. Tanizawa, and Z. Wu for discussions.

[1] R. Albert and A.-L. Barabási. Rev. Mod. Phys. 74, 47 (2002); R. Pastor-Satorras and A. Vespignani, \textit{Structure and Evolution of the Internet: A Statistical Physics Approach} (Cam-
bridge University Press, Cambridge, 2004); S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW* (Oxford University Press, Oxford, 2003).

[2] A.-L. Barabási and R. Albert, *Science* **286**, 509 (1999).

[3] R. Cohen et al., *Phys. Rev. Lett.* **85**, 4626 (2000).

[4] In principle, a node can have a degree up to $N - 1$, connecting to all other nodes of the network. The results presented here corresponds to networks with the upper cutoff $k_{\text{max}} = k_{\text{min}}N^{1/(\lambda-1)}$ imposed. We also studied networks for which this cutoff is not imposed, and found no significant differences in the pdf $\Phi_{\text{SF}}(G)$.

[5] P. Erdős and A. Rényi, *Publ. Math. (Debreccen)*, **6**, 290 (1959).

[6] B. Bollobás, *Random Graphs* (Academic Press, Orlando, 1985).

[7] G. R. Grimmett and H. Kesten, *J. Lond. Math. Soc.* **30**, 171 (1984); G. R. Grimmett and H. Kesten, [http://xxx.lanl.gov/list/math/0107068](http://xxx.lanl.gov/list/math/0107068).

[8] P. L. Krapivsky et al., *Phys. Rev. Lett.* **85**, 4629 (2000).

[9] Z. Toroczkai and K. Bassler, *Nature* **428**, 716 (2004).

[10] J. D. Noh and H. Rieger, *Phys. Rev. Lett.* **92**, 118701 (2004).

[11] V. Sood et al., [cond-mat/0410309](http://xxx.lanl.gov/abs/cond-mat/0410309).

[12] L. K. Gallos, *Phys. Rev. E* **70**, 046116 (2004).

[13] The dynamical properties that we study here are related to transport on networks and differ from those which treat the network topology itself as evolving in time.

[14] The study of community structure has led some authors (M. E. J. Newman and M. Girvan, *Phys. Rev. E* **69**, 026113 (2004), and F. Wu and B. A. Huberman, *Eur. Phys. J. B* **38**, 331 (2004)) to propose methods in which networks are considered as electrical networks in order to identify communities. In these studies, however, transport properties have not been addressed.

[15] M. Molloy and B. Reed, *Random Struct. Algorithms* **6**, 161 (1995).

[16] We performed simulations in which the random copies are generated and then randomly matched, and also simulated networks where the node copies were paired in order of degree from highest to lowest and obtained similar results.

[17] G. Kirchhoff, *Ann. Phys. Chem.* **72**, 497 (1847); N. Balabanian, *Electric Circuits* (McGraw-Hill, New York, 1994).

[18] Since $g_G > 2$ for $\lambda > 2$, it follows that although $\bar{G}$ increases with $N$, it finally converges to
FIG. 1: (a) Comparison for networks with $N = 8000$ nodes between the cumulative distribution functions for the Erdős-Rényi and the scale-free cases (with $\lambda = 2.5$ and 3.3). Each curve represents the cumulative distribution $F(G)$ vs. $G$. The simulations have at least $10^6$ realizations. (b) Effect of system size on $F_{SF}(G)$ vs. $G$ for the case $\lambda = 2.5$. The cutoff value of the maximum conductance $G_{\text{max}}$ progressively increases as $N$ increases.

Only for $g_G < 2$, the cutoff $G_{\text{max}}$ in the tail of $\Phi_{SF}(G)$ controls $\bar{G}$, making $\bar{G}$ an increasing unbounded function of $N$.

[19] Flux starts at node $A$, being controlled by the conductance of the bonds in the vicinity of node $A$. This flux passes into the “transport backbone”, which is comprised of many parallel paths and hence has a high conductance. Finally, flux ends at node $B$, being controlled by the conductance of the bonds in the vicinity of node $B$. This is similar to the traffic around a major freeway. Most of the limitations to transport occur in getting to the freeway (“node $A$”) and then after leaving it (“node $B$”), but flow occurs easily on the freeway (“transport backbone”).

[20] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).

[21] The $\lambda$ values explored here are limited by computer time considerations. In principle, we are unaware of any theoretical reason that would limit the validity of our results to any particular range of values of $\lambda$ (above 2).
FIG. 2: (a) Probability density function $\Phi_{SF}(G|k_A, k_B)$ vs. $G$ for $N = 8000$, $\lambda = 2.5$ and $k_A = 750$ ($k_A$ is close to the typical maximum degree $k_{\text{max}} = 800$ for $N = 8000$). (b) Most probable values of $G^*$, estimated from the maxima of the distributions in Fig. 2(a), as a function of the degree $k_B$. The data support a power law behavior $G^* \sim k_B^\alpha$ with exponent $\alpha = 0.96 \pm 0.05$. (c) Scaled most probable conductance $G^*/k_B$ vs. scaled degree $x \equiv k_A/k_B$ for system size $N = 8000$ and $\lambda = 2.5$, for several values of $k_A$ and $k_B$: □ ($k_A = 8, 8 < k_B < 750$), ◊ ($k_A = 16, 16 < k_B < 750$), △ ($k_A = 750, 4 < k_B < 128$), ○ ($k_B = 4, 4 < k_A < 750$), ◊ ($k_B = 256, 256 < k_A < 750$), and ▼ ($k_B = 500, 4 < k_A < 128$). The solid line is the predicted function $G^*/k_B = cx/(1 + x)$ obtained from Eq. (6). This plot shows the rapid approach of the scaling function $f(x)$ of Eq. (6) from a linear behavior to the constant $c$ (here $c = 0.87 \pm 0.02$, horizontal dashed line). The inset shows a schematic of the “transport backbone” picture, where the circles labeled $A$ and $B$ denote the nodes $A$ and $B$ and their associated links which do not belong to the “transport backbone”.
FIG. 3: (a) Simulation results for the cumulative distribution $F_{SF}(G)$ for $\lambda$ between 2.5 and 3.5, consistent with the power law $F_{SF} \sim G^{-(gG - 1)}$ (cf. Eq. (8)), showing the progressive change of the slope $gG - 1$. (b) The exponent $gG - 1$ from simulations (circles) with $2.5 < \lambda < 4.5$; shown also is a least square fit $gG - 1 = (1.97 \pm 0.04)\lambda - (2.01 \pm 0.13)$, consistent with the predicted expression $gG - 1 = 2\lambda - 2$ [cf. Eq. (9)].