Geometric Phase, Hannay’s Angle, and an Exact Action Variable

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Canonical structure of a generalized time-periodic harmonic oscillator is studied by finding the exact action variable (invariant). Hannay’s angle is defined if closed curves of constant action variables return to the same curves in phase space after a time evolution. The condition for the existence of Hannay’s angle turns out to be identical to that for the existence of a complete set of (quasi)periodic wave functions. Hannay’s angle is calculated, and it is shown that Berry’s relation of semiclassical origin on geometric phase and Hannay’s angle is exact for the cases considered.

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Berry’s phase, the geometric part of a change in the phase of a wave function under a cyclic evolution, has attracted great interest both theoretically and experimentally. A significant generalization of the Berry’s phase, relaxing the adiabatic approximation, has been given by Aharonov and Anandan. The price we have to pay for this generalization is that the quantum states do not necessarily return to the original states up to a phase after the evolution of Hamiltonian’s cycle, so that the geometric phase may not be defined through the method in [3]. This geometric phase may have a natural classical correspondent: Hannay’s angle. Hannay’s angle or geometric phase has long been studied for the (generalized) harmonic oscillator. In a recent study of the oscillators with time-periodic parameters, it has been proven that, if two linearly independent homogeneous classical solutions of the oscillator are bounded for all the time, there always exists a complete set of (quasi)periodic wave functions for an oscillator without driving force. For driven case, the ratio of the period of periodic particular solution to that of Hamiltonian should be rational for the existence of the complete set. This oscillator provides an ideal system for the study of dynamics of Gaussian wave packets which has been applied to many problems in atomic and molecular physics as well as in quantum optics.

Recently, the Hannay’s angle for generalized harmonic oscillators and its relation to geometric phase were studied by Ge and Child, and by Liu et al. Though Hannay’s angle was originally formulated relying on the existence of action variable in the canonical structure of a system, and the exact invariant (action variable) for the harmonic oscillator of time-dependent frequency has been given by Lewis more than thirty years ago, none of the works for Hannay’s angle of the oscillator has benefited from the exact action variable.
In this Letter, we will study Hannay’s angle of the oscillator by generalizing the exact action variable, and will compare the angle with the geometric phase. The model we will consider is described by the Hamiltonian:

\[ H(x, p, t) = \frac{p^2}{2M(t)} - a(t)(px + xp) + \frac{1}{2} M(t) c(t) x^2 - \frac{b(t)}{M(t)} p + d(t) x + \left( \frac{b^2(t)}{2M(t)} - f(t) \right), \quad (1) \]

where

\[ c(t) = w^2 + 4a^2 - 2\dot{a} - 2M \dot{a}, \quad d(t) = 2ab - \ddot{b} - F. \quad (2) \]

The smooth functions \( M(t), w(t) \) and \( F(t) \) denote the positive mass, real frequency and external force, respectively, while the overdots denote differentiations with respect to time. In the Lagrangian description, the terms of smooth functions \( a(t), b(t) \) and \( f(t) \) can be interpreted as the results of adding total derivative terms to Lagrangian which do not affect the classical equation of motion [11].

We require a periodicity for every coefficient, so that

\[ H(x, p, t + \tau) = H(x, p, t). \quad (3) \]

The classical equation of motion for the Hamiltonian is given as

\[ \frac{d}{dt}(M\dot{x}) + M(t)w^2(t)x = F(t). \quad (4) \]

The general solution of this equation is a linear combination of a particular solution \( x_p(t) \) and two linearly independent homogeneous solutions \( u(t), v(t) \). As in [11], \( u(t) \) and \( v(t) \) are dimensionless, while \( x_p(t) \) has the dimension of length. For later use, we define \( \rho(t) \) and a time-constant \( \Omega \) as

\[ \rho = \sqrt{u^2(t) + v^2(t)}, \quad \Omega = M(t)[\dot{v}(t)u(t) - \dot{u}(t)v(t)]. \quad (5) \]

The \( \rho \) satisfies the following differential equation

\[ \frac{d}{dt}(M\dot{\rho}) - \frac{\Omega^2}{M}\rho^3 + Mw^2\rho = 0. \quad (6) \]

For the case that \( M \) is constant and \( a, b, F, f \) are zero, the Hamiltonian describes the harmonic oscillator of time-dependent frequency \( w(t) \). In this case, the exact invariant

\[ I_0 = \frac{1}{2I_0} [\Omega_0 x^2 + (\rho_0 p - \rho_0 x)^2], \quad (7) \]

whose level surfaces form ellipses in the phase space of any time \( t \), has been found in [11] (For a derivation of \( I_0 \) from the Hamiltonian of a simple harmonic oscillator, see [12]). The subscript 0 is to denote that variables are defined when \( M \) is the unit mass and \( a, b, F, f \) are zero.

The area enclosed by the ellipse is given by \( 2\pi I_0 \), as can be easily proven by making use of Stokes’ theorem

\[ \int pdx = \int_{2\pi I_0} \int_{\xi_0}^{\frac{\Omega_0 x}{\rho_0}} [\rho_0 p - \rho_0 x]^2 dp \wedge dx \]

\[ = 4\pi I_0 \int_{1 \geq \xi_0^2 + \xi_0^2} d\xi_0 \wedge d\zeta_0 = 2\pi I_0 \quad (8) \]

with a parameterization \( \xi_0 = \sqrt{1 - \frac{2\pi}{\sqrt{2\pi I_0} \rho_0 x}} \), \( \zeta_0 = \frac{x}{\sqrt{2\pi I_0} \rho_0} \). Though the ellipse evolves as time passes due to the time dependence of \( \rho_0 \), the area enclosed by the curve or \( I_0 \) is a time-constant.

To find action variable for the Hamiltonian in (1), we make use of the unitary transformations given in [11]. By applying the unitary transformations to \( I_0 \), one can find that the transformed operator \( \tilde{I} \) is written as
\[ I = \frac{1}{2\Omega} \frac{\Omega^2}{\rho^2} (x-x_p)^2 \]
\[ + \{(M \dot{\rho} + 2Ma \rho)(x-x_p) - \rho(p-p_p)\}^2 \] \quad (9)
where \( p_p \) is defined as \( p_p = M \dot{x}_p + 2M ax_p + b \). The generalized harmonic oscillator is a system where the path integral for the kernel is Gaussian. Making use of this fact, the quantum theory of the model has been studied in Ref. [11]. In this paper, we will consider the \( I(x,p,t) \) in (3) as a classical object.

One can explicitly check that \( I \) is a constant of motion satisfying
\[ \frac{dI}{dt} = \frac{\partial I}{\partial t} + \{I,H\}_{PB} = 0, \] \quad (10)
where the subscript PB denotes that the term is a Poisson bracket. Eq. (10) shows that \( I \) is the action variable of the oscillator system described by the Hamiltonian in (1); The value of \( I \) for a point of phase space at a given time stays constant along the trajectory generated by \( H \).

Again, by making use of Stokes’ theorem as in (8), one can show that \( 2\pi I \) is the area of ellipse of constant \( I \) in the phase space. The term with coefficient \( a \) changes the shape of ellipse, while the terms with coefficients \( b, F \) just move the center of the ellipse and the purely time-dependent term in the Hamiltonian has no effect on the ellipse. We parameterize the ellipse as
\[ \cos Q = \sqrt{\frac{\Omega}{2I}} \frac{(x-x_p)}{\rho}, \]
\[ \sin Q = \frac{1}{\sqrt{2\Omega I}} [(M \dot{\rho} + 2Ma \rho)(x-x_p) - \rho(p-p_p)]. \] \quad (11)

Then the Poisson Bracket
\[ \{I,\tan Q\}_{PB} = -\frac{d}{dQ} \tan Q \] \quad (12)
shows that \( Q \) is the angle variable. For the evaluation of \( p \) as a function of \( I \) and \( x \), we should consider the two branches of an ellipse divided by the line
\[ p = (M\frac{\dot{\rho}}{\rho} + 2Ma)(x-x_p) + p_p. \] \quad (13)

In the branch above (below) the line, the momentum is written as
\[ p = p_p + (M\frac{\dot{\rho}}{\rho} + 2Ma)(x-x_p) \]
\[ \pm \frac{1}{\rho} \sqrt{2\Omega I} \frac{\Omega^2}{\rho^2} (x-x_p)^2, \] \quad (14)
with upper (lower) sign. A generating function of the canonical transformation from \( \{x,p\} \) to action-angle variables is given in the branch above (below) the line as
\[ F_2(x,I,t) = \delta + \int^t f(z) dz + M ax_p^2 + bx_p + M \dot{x}_p x_p \]
\[ + p_p(x-x_p) + (M\frac{\dot{\rho}}{\rho} + Ma)(x-x_p)^2 \]
\[ \pm \frac{1}{2} \sqrt{2\Omega I} \frac{\Omega^2}{\rho^2} (x-x_p)^2 - 1 \] \quad (14)
with upper (lower) sign. In (14), \( \delta \) is defined through the relation
\[ \dot{\delta} = \frac{1}{2} M \dot{w}^2 x_p^2 - \frac{1}{2} M \dot{x}_p^2. \] \quad (15)

One can find that \( p = \frac{\partial F_2}{\partial x} \) gives the relation (13) and \( Q = \frac{\partial F_2}{\partial t} \) is compatible with the parameterization (11).

The Hamiltonian for the same system in terms of action-angle variables is given as
\[ \tilde{H} = H(x(Q,I,t),p(Q,I,t),t) + \frac{\partial F_2}{\partial t} = \frac{\Omega}{M \rho^2} I, \] \quad (16)
showing again that \( I \) is the action variable. It is noteworthy that \( \frac{\partial P_2}{\partial t} \) is a single-valued function of \( Q \) as
\[ \frac{\partial \tilde{F}_2}{\partial t} |_{x=(Q,I,t)} = \frac{I}{\Omega}(-M\dot{\rho}^2 + \frac{\Omega^2}{\rho^2} - Mw^2\rho^2 + 2\frac{d(Ma)}{dt})\cos^2 Q + 2\frac{\dot{\rho}}{\rho} \cos Q \sin Q + \dot{x}_p \frac{\sqrt{2I\Omega}}{\rho} \sin Q + (\dot{p}_p - \dot{x}_p (M\dot{\rho} + 2Ma\rho)) \sqrt{\frac{2I}{\Omega}} \cos Q + x_p\dot{p}_p + \delta + f - \frac{d}{dt}(Max^2_p). \] \tag{17}

Hannay’s angle is defined when the closed curves of constant action variables return to the original curves after a time evolution. For the generalized harmonic oscillator, this can be satisfied if both of \( \rho(t) \) and \( x_p(t) \) are periodic with a period which is an integral multiple of \( \tau \). In the cases where such period exists, we shall denote \( \tau' \) as the period. The evolution generated by \( H(x,p,t) \) transports a family of ellipses through the phase space. If \( \tau' \) exists, then the given family of such ellipses get transported back to itself after a time \( \tau' \). Hannay’s angle is defined as the integral of angle-averaged value of \( \dot{Q} - \frac{\partial H}{\partial Q} \) for the period \( \tau' \):

\[ Q_H = \frac{1}{2\pi} \int_{t_0}^{t_0 + \tau'} \int_0^{2\pi} \frac{\partial \tilde{F}_2}{\partial t} |_{x=(Q,I,t)} dQ dt, \] \tag{18}

where \( t_0 \) is an arbitrary time. After some algebra, one can find, making use of (6), that the Hannay’s angle is written as:

\[ Q_H = -\frac{1}{\Omega} \int_0^{\tau'} (M\dot{\rho}^2 + 2Ma\rho\dot{\rho}) dt. \] \tag{19}

A general generating function \( \tilde{F}_2(x,I,t) \) for the action variable \( I(x,p,t) \) of (11) may be written as \( \tilde{F}_2(x,I,t) = F_2(x,I,t) + IQ_c(t) + \delta(t) \). \( Q_c(t) \) must be dimensionless, while \( \delta(t) \) has the physical dimension of \( \delta(t) \). With this general generating function, the angle variable \( \tilde{Q} \) is given through the parameterization of (11) where \( Q \) is replaced by \( \tilde{Q} - Q_c(t) \), so that

\[ x = \sqrt{\frac{2I}{\Omega}} \rho \cos(\tilde{Q} - Q_c) + x_p, \]
\[
p = \sqrt{\frac{2I}{\Omega}} \left( \frac{M\dot{\rho}}{\rho} + 2Ma\rho \right) \cos(\tilde{Q} - Q_c) - \frac{\sqrt{2\Omega I}}{\rho} \sin(\tilde{Q} - Q_c) + p_p. \] \tag{20}

This parameterization is compatible with \( \tilde{Q} = \frac{\partial \tilde{F}_2}{\partial I} \), and implies that \( Q_c(t) \) represents the rotational motion of the line in phase space from which the angle variable is measured. The Hamiltonian in terms of \( I \) and \( \dot{Q} \) has two additional terms to that of (16). Making use of the definition (18), the Hannay’s angle \( \tilde{Q}_H \) for the generating function \( \tilde{F}_2 \) is evaluated as \( \tilde{Q}_H = Q_H + \Delta Q(t_0) \), where \( \Delta Q(t_0) = Q_c(t_0 + \tau') - Q_c(t_0) \). Formally, \( Q_c(t) \) can be any function of \( t \), and thus \( \tilde{Q}_H \) depends on \( t_0 \) unless \( Q_c(t) \) is periodic with the period \( \tau' \). The condition for the existence of meaningful Hannay’s angle, therefore, is that the \( Q_c(t) \) is periodic with the period \( \tau' \). The condition is satisfied if the mapping from \( (x,p) \) to \( (\tilde{Q},I) \) is periodic with the period \( \tau' \); If the mapping has the periodicity, the value of \( \tilde{Q}(x,p,t_0 + \tau) \) is equal to that of \( \tilde{Q}(x,p,t_0) \) in addition to the time-periodicity of \( I(x,p,t) \) discussed above. From now on we only consider the case of the periodic \( Q_c(t) \), so that \( \tilde{Q}_H \) does not depend on \( Q_c(t) \) and is equal to \( Q_H \). In (19), one can find that there could be cases where the \( \rho(t) \) and thus the \( Q_H \) depend on the way of choosing \( u(t) \) and \( v(t) \), since the mapping is deter-
mined by the choice of classical solutions; However, for a
given mapping of the periodicity, the Hannay’s angle is
unique.

In general cases, it looks like that there is no reason of
Hannay’s angle being described by an integral of canoni-
cal variables (see Ref. [5]). For the generalized harmonic
oscillator considered here, the Hannay’s angle happens
to be written as
\[
Q_H = -\frac{1}{2\pi} \frac{\partial}{\partial I} \int_0^\tau \int_0^{2\pi} p(\tilde{Q}, I, t) \frac{\partial x}{\partial t} d\tilde{Q} dt. \quad (21)
\]
The expression of Hannay’s angle in (21) may be facil-
itated in easily finding the fact that the angle does not
depend on \( x_p \) and \( p_p \); The terms containing \( x_p \) and \( p_p \) in
the integral of (21) which comes from the linear terms in
the Hamiltonian are removed through the differentiation
with respect to \( I \) and the angle-average. The Hannay’s
angle does not depend on the motion of center of the
ellipses, reflecting the fact that the angle is a geometric
quantity. If there is no linear term in the Hamiltonian so
that \( x_p = p_p = 0 \), averaging over a half range of the an-
gle variable which corresponds to a branch of the ellipse
divided by any straight line passing through origin gives
the same Hannay’s angle.

In the quantum treatment of the Hamiltonian in (1),
a set of wave functions satisfying Schrödinger equation is
given as [11]
\[
\psi_m(x, t) = \frac{1}{\sqrt{2^m m!}} \left( \frac{\Omega}{\pi \hbar} \right)^{\frac{1}{4}} \exp \left[ \frac{i}{\hbar} \left( \delta(t) + \int_0^t f(z) dz \right) \right] \\
\times \exp \left[ \frac{i}{\hbar} \left( M(t) a(t) x^2 + (M(t) x_p(t) + b(t)) x \right) \right] \\
\times \exp \left[ \frac{(x - x_p(t))^2}{2\hbar} \left( -\frac{\Omega}{\rho^2(t)} + iM(t) \frac{\dot{\rho}(t)}{\rho(t)} \right) \right] \\
\times H_m \left( \sqrt{\frac{\Omega}{\hbar}} x - x_p(t) \rho(t) \right). \quad (22)
\]
For \( \Omega > 0 \), \( \psi_m \) is square-integrable. The set of wave
functions \( \{ \psi_m(x, t) | m = 0, 1, 2, \cdots \} \) are complete with
a choice of two linearly independent solutions \( u(t), v(t) \),
and a particular solution \( x_p(t) \) [11], while, with the same
choice of \( \Omega > 0 \), the mapping from \( (x, p) \) to \( (\tilde{Q}, I) \) at a
given time is one-to-one.

The condition for the existence of a complete set of
(quasi)periodic wave functions is satisfied, if \( \rho(t) \) and
\( x_p(t) \) are periodic with a period which is an integral mul-
tiple of \( \tau \). Therefore, the condition of the existence of
a family of time-periodic closed curves in phase space
needed for the definition of Hannay’s angle is exactly
same to that of the complete set of (quasi)periodicity.

In [6], it has been shown that, if a homogeneous solution
diverges then such complete set does not exist and thus
Hannay’s angle can not be defined. Making use of Flo-
quet’s theorem, it has also been proven that [3], if there
exist \( u(t), v(t) \) finite all over the time, which are the cases
we will consider from now on, there always exist two lin-
early independent homogeneous solutions which give pe-
riodic \( \rho(t) \) with period \( \tau \) or \( 2\tau \); For the case of \( x_p = 0 \),
there exists a complete set of (quasi)periodicity and thus
a family of closed curves of periodicity. For the case of
\[ x_p \neq 0, \text{ the ratio of the period of periodic } x_p \text{ to } \tau \text{ should be rational for the existence of Hannay’s angle and the complete set(s) of (quasi)periodicity.} \]

If \( \tau' \), an integral multiple of \( \tau \), is the common period of periodic \( \rho \) and \( x_p \), the overall phase change of the wave function \( \psi_m \) under the \( \tau' \)-evolution is given as [6]:

\[
\chi_m = -(m + \frac{1}{2}) \int_{0}^{\tau'} \frac{\Omega}{M \rho^2} dt + \frac{1}{\hbar} \int_{0}^{\tau'} (\dot{\delta} + f) dt.
\]

(23)

The expectation value of the Hamiltonian for a wave function \( \psi_m(x, t) \) is given as

\[
m < H >_m = \hbar (m + \frac{1}{2}) \left[ \frac{\Omega}{2M \rho^2} + \frac{M \dot{\rho}^2}{2\Omega} + \frac{M \rho^2}{2\Omega} - \frac{\rho^2}{\Omega} (\dot{\mathbf{a}} + \dot{\mathbf{Ma}}) \right] + \frac{M}{2} \dot{x}_p^2 + \frac{Mw^2}{2} x_p^2 - Fx_p
\]

\[-(\dot{\mathbf{Ma}} + \dot{\mathbf{Ma}}) x_p^2 - bx_p - f. \]

(24)

Geometric phase for the wave function \( \psi_m(x, t) \) is thus written as

\[
\gamma_m = \chi_m + \frac{1}{\hbar} \int_{0}^{\tau'} m < H >_m dt
\]

\[
= (m + \frac{1}{2}) \Omega \int_{0}^{\tau'} [\dot{\rho}^2 + 2M \rho \dot{\rho}] dt + \frac{1}{\hbar} \int_{0}^{\tau'} (M \dot{x}_p^2 + 2M \dot{x}_p x_p + bx_p) dt.
\]

(25)

In the Hannay’s angle, the effects of the linear terms in the Hamiltonian has been removed through the angle-average and differentiation with respect to \( I \). The calculation of geometric phase, the expectation values of the linear terms can not depend on the \( m \) due to the orthogonality of Hermitian polynomials and their recurrence relations. A simple relation is thus satisfied between the geometric phase and the Hannay’s angle

\[
Q_H = -\frac{\partial \gamma_m}{\partial m}.
\]

(26)

In Ref. [3], the relation (26) has been suggested for general models; However, the suggestion was made through a semiclassical treatment of geometric phase and using, in fact, the Eq.(21) as a defining relation for Hannay’s angle which may not be equal to the definition of Eq.(18) in general cases. A similar relation for the case without linear terms is given in Ref. [9].

In summary, we analyze the canonical structure of a generalized harmonic oscillator by finding the exact action variable. The analyses has then been used to find Hannay’s angle which is defined relying on the time-periodic closed curves of constant action variables in phase space. Hannay’s angle for the model considered can be defined if and only if there exist a complete set of (quasi)periodic wave functions whose geometric phases have been given in this paper. There could be cases where both of the Hannay’s angle and the geometric phase depend on the way of choosing classical homogeneous solutions, while the angle is unique for a given choice of classical solutions of the periodicity. It should be of interest if similar analyses would be possible for other models.

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