CHIRAL DIFFERENTIAL OPERATORS ON THE UPPER HALF PLANE AND MODULAR FORMS

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1. Introduction

Consider the Heisenberg Lie algebra with basis \( a_n, b_n \) \((n \in \mathbb{Z})\), the central element \( C \), and with commutation relations

\[
[a_m, b_n] = \delta_{m, -n} C. \tag{1.1}
\]

Its vacuum representation \( V = \mathbb{C}[a_{-1}, a_{-2}, \ldots, b_0, b_{-1}, \ldots] \) generated by the vacuum vector 1, with the relations

\[
a_m 1 = 0 \quad \text{if } m \geq 0; \quad b_n 1 = 0 \quad \text{if } n > 0; \quad C 1 = 1,
\]

has a structure of vertex operator algebra. Let \( H := \{ \tau \in \mathbb{C} \mid \text{Im } \tau > 0 \} \) be the upper half plane. By the result of Malikov, Schectman and Vaintrob [MSV],

\[
\mathcal{D}^{\text{ch}}(\mathbb{H}) := V \otimes_{\mathbb{C}[b_0]} \mathcal{O}(\mathbb{H}), \tag{1.2}
\]

where \( \mathbb{C}[b_0] \) is considered as a subring of the ring of holomorphic functions \( \mathcal{O}(\mathbb{H}) \) on \( \mathbb{H} \) by \( b_0 \mapsto \tau \), is also a vertex operator algebra, which is called the vertex algebra of chiral differential operators on \( \mathbb{H} \).

It can be proved that the \( SL(2, \mathbb{R}) \)-action on \( \mathbb{H} \) by the fractional linear transformation induces an action of \( SL(2, \mathbb{R}) \) on \( \mathcal{D}^{\text{ch}}(\mathbb{H}) \) as automorphisms of vertex algebras (see Section 2). Let \( \Gamma(1) := SL(2, \mathbb{Z}) \), and \( \Gamma \subset \Gamma(1) \) be an arbitrary congruence subgroup. In this work, we will study the fixed point vertex algebra \( \mathcal{D}^{\text{ch}}(\mathbb{H})^{\Gamma} \) under \( \Gamma \)-action. As in the theory of modular forms, we consider the subspace of \( \mathcal{D}^{\text{ch}}(\mathbb{H})^{\Gamma} \) consisting of elements that are holomorphic at the cusps (see Section 2 for definition), denoted by \( \mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma) \). Since the \( SL(2, \mathbb{R}) \)-action preserves the conformal weights of \( \mathcal{D}^{\text{ch}}(\mathbb{H}) \), \( \mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma) \) is naturally a \( \mathbb{Z}_{\geq 0} \) graded vertex operator algebra. One of the main purposes of this work is to understand the structure of \( \mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma) \) and compute its character.

We will show that the structure of \( \mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma) \) is closely related to the modular forms of level \( \Gamma \). Let \( M_k(\Gamma) \) be the space of modular forms of weight \( k \). For any \( f \in M_{2k}(\Gamma), k > 0 \), we introduce a certain subspace \( D_f \) of \( \mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma) \) which is obtained by applying invariant vertex operators to \( f \) (see Section 5 for precise definition). For \( f \equiv 1 \), \( D_1 \) is obtained by invariant vertex operators and a quasi-modular form \( E_2 \) (see also in Section 5).

Let \( \mathcal{B} \) be a homogeneous basis of \( \oplus_{k \geq 0} M_{2k}(\Gamma) \), then we will prove that \( \mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma) \) can be decomposed as a direct sum of \( D_f \) for \( f \in \mathcal{B} \), i.e.

\[
\mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma) = \bigoplus_{f \in \mathcal{B}} D_f.
\]

One of the main results of this work is the following.
Theorem 1.1. The character formula of \( \mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma) \) is given by
\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \dim M_{2m}(\Gamma) q^{2n+m} \prod_{i=1}^{n} \frac{1}{1-q^{i}} \prod_{j=1}^{m+n} \frac{1}{1-q^{j}}.
\]

For any partition \( \lambda = (\lambda_1, \cdots, \lambda_d) \), we define \(|\lambda| := \sum_{i=1}^{d} \lambda_i\), \(p(\lambda) := d\). Since \( SL(2, \mathbb{R}) \) preserves the conformal weight, we consider the conformal weight subspace by partition pairs in \( \mathcal{D}^{\text{ch}}(\mathbb{H})_N \). For each partition pair \((\lambda, \mu)\) such that \(|\lambda| + |\mu| = N\), we will introduce an \( SL(2, \mathbb{R}) \)-invariant subspace \( V_{\lambda, \mu} \subset \mathcal{D}^{\text{ch}}(\mathbb{H})_N \). And we introduce a total order on the partition pairs such that whenever \((\lambda', \mu') < (\lambda, \mu)\) and \(|\lambda'| + |\mu'| = |\lambda| + |\mu|\), we have \( V_{\lambda', \mu'} \subset V_{\lambda, \mu} \), thus we obtain a filtration labeled by partition pairs in \( \mathcal{D}^{\text{ch}}(\mathbb{H})_N \). Then we will consider a subspace \((V_{\lambda, \mu})_0^\Gamma \subset V_{\lambda, \mu}\), consisting of \( \Gamma \)-invariant elements that satisfy the cuspidal conditions (see Section 2). Let \((\lambda_1, \mu_1)\) be the largest partition pair under the condition that \((\lambda_1, \mu_1) < (\lambda, \mu)\) and \(|\lambda_1| + |\mu_1| = |\lambda| + |\mu|\), then another main result is (Theorem 2.5)
\[
(V_{\lambda, \mu})_0^\Gamma / (V_{\lambda', \mu'})_0^\Gamma \cong M_{2k}(\Gamma),
\]
where \(k = p(\mu) - p(\lambda)\).

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2. The Algebra of Chiral Differential Operators on the Upper Half Plane

In this section, we recall the construction of the vertex algebra \( \mathcal{D}^{\text{ch}}(\mathbb{H}) \) of chiral differential operators on \( \mathbb{H} \) and construct an \( SL(2, \mathbb{R}) \)-action as in [MSV]. And we introduce an \( SL(2, \mathbb{R}) \)-invariant filtration on \( \mathcal{D}^{\text{ch}}(\mathbb{H}) \), and a cuspidal condition on the \( \Gamma \)-fixed algebra \( \mathcal{D}^{\text{ch}}(\mathbb{H})^\Gamma \).

The vacuum representation \( V \) in Section 1 of the Heisenberg Lie algebra (1.1) is a polynomial algebra of variables \( b_0, b_{-1}, \cdots, a_{-1}, a_{-2}, \cdots \), and the Virasoro element is given by
\[
\omega = a_{-1} b_{-1}.
\]
Then \( L_0 = \omega_{(1)} \) gives \( V \) a gradation \( V = \bigoplus_{n=0}^{\infty} V_n \), where an element in \( V_n \) is said to have conformal weight \( n \). So \( V_0 = \mathbb{C}[b_0] \) and \( V_1 = \mathbb{C}[b_0] a_{-1} \mathbb{C}[b_0] b_{-1} \). We will write \( a = a_{-1} \cdot 1, b = b_0 \). The basic fields \( a(z) \) and \( b(z) \) are given by
\[
a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n}.
\]
According to [MSV], \( \mathcal{D}^{\text{ch}}(\mathbb{H}) \) as in (1.2) is also a vertex operator algebra generated by the basic fields, \( a(z), b(z) \) as above and \( Y(f, z), f \in O(\mathbb{H}) \), where
\[
Y(f, z) = \sum_{i=0}^{\infty} \frac{q^i}{i!} f(b) \left( \sum_{n \neq 0} b_n z^{-n} \right)^i.
\]
(2.1)
We write \( f(b)_{m+1} := f(b)_{(m)} \) for the coefficient of \( z^{-m-1} \) in the field \( Y(f, z) \).
Certain vertex operators on $\mathcal{D}^{ch}(\mathbb{H})$ generates representations of affine Kac-Moody algebra $\hat{\mathfrak{sl}}_2$. More precisely let
\[ E := -a_{-1}, \quad F := a_{-1}b_0 + 2b_{-1}, \quad H := -2a_{-1}b_0. \] We have the following theorem

**Theorem 2.1.** [W, FF, F] The coefficients of $E_{(n)}, F_{(n)}, H_{(n)}$ of fields $Y(E, z), Y(F, z), Y(H, z)$ satisfy the relations of affine Kac-Moody algebra $\hat{\mathfrak{sl}}_2$ of critical level $-2$, where $E, F, H$ corresponds to matrices
\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]
respectively.

This representation of $\hat{\mathfrak{sl}}_2$ on $V$ was first introduced by M. Wakimoto [W]. The general construction of Wakimoto modules was given by B. Feigin and E. Frenkel [FF]. Its connection with vertex algebras as in the above formulation can be found in [F].

It is well-known that there is a natural right $SL(2, \mathbb{R})$-action on the space
\[
\Omega(\mathbb{H}) = \Omega^0(\mathbb{H}) \oplus \Omega^1(\mathbb{H}) = \{ f(b) + g(b)db \}
\]
induced by the fractional linear transformation on $\mathbb{H}$:
\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \tau = \frac{\alpha \tau + \beta}{\gamma \tau + \delta}.
\]
So the Lie algebra $\mathfrak{sl}_2$ acts on $\Omega(\mathbb{H})$ as Lie derivatives, where
\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]
act as $-\frac{d}{d\tau}, \frac{1}{\tau} \frac{d}{d\tau}, -2b \frac{d}{d\tau}$ respectively. And we replace $\frac{d}{d\tau}$ by $a_{-1}$, we get the formula of $E$ and $H$ in (2.2). But for the formula of $F$, we need to add an extra term $2b_{-1}$.

For a vector $v$ of conformal weight $1$ in a vertex algebra with field $Y(v, z) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1}$, $v_{(0)}$ is a derivation. In our case $E_{(0)}, F_{(0)}$ and $H_{(0)}$ give an action of Lie algebra $\mathfrak{sl}_2$ on $\mathcal{D}^{ch}(\mathbb{H})$ as derivations. By the method in [MSV], we can show that this can be integrated to an $SL(2, \mathbb{R})$-action as automorphisms of vertex algebra. Because we will consider the action of a congruence subgroup $\Gamma \subset SL(2, \mathbb{R})$, and it will be related to the theory of modular forms, where the action of $SL(2, \mathbb{R})$ is always from the right (see, e.g., [B]), we will make our action of $SL(2, \mathbb{R})$ a right action. By definition, for $g = e^x \in SL(2, \mathbb{R})$, $x \in \mathfrak{sl}_2$, then
\[
\pi(g) = \sum_{n \geq 0} \frac{(-x_{(0)})^n}{n!}.
\]
And we have $\pi(g_1g_2) = \pi(g_2)\pi(g_1)$.

The $SL(2, \mathbb{R})$-action commutes with the translation operator $T = L_{-1} = \omega_{(0)}$ for the fact that
\[
[T, x_{(0)}] = (Tx)_{(0)} = 0, \quad \text{for } x \in \mathfrak{sl}_2 \subset \mathcal{D}^{ch}(\mathbb{H}).
\]
And it also commutes with the semisimple operator $L_0 = \omega_{(1)}$, so it preserves the gradation.

We now give the formula of the action of
\[
g = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \in SL(2, \mathbb{R})
\]
on generators $a, b, f(b) \in \mathcal{D}^{ch}(\mathbb{H})$
\[ \pi(g)a = a_{-1}(\gamma b + \delta)^2 + 2\gamma^2 b_{-1} \]
\[ \pi(g)b = \frac{ab + \beta}{\gamma b + \delta} \]
\[ \pi(g)f(b) = f(gb) = f\left(\frac{\alpha b + \beta}{\gamma b + \delta}\right) \]

For simplicity, we will introduce a notation \( a_{-\lambda} \) for a long expression
\[ a_{-\lambda_1} a_{-\lambda_2} \cdots a_{-\lambda_d}, \]
where \( \lambda = (\lambda_1, \cdots, \lambda_d) \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 1 \), i.e. \( \lambda \) is a partition. And we define \( p(\lambda) := d \), and \( |\lambda| := \sum_{i=1}^{d} \lambda(i) \). So \( \lambda \) is a partition of \( |\lambda| \) with \( p(\lambda) \) parts. Similarly we define \( b_{-\mu} \) for any partition \( \mu \). Every element in \( \mathcal{D}^\text{ch}(\mathbb{H}) \) can be written as a sum of elements of type \( a_{-\lambda} b_{-\mu} f(b) \) with \( f(b) \in \mathcal{O}(\mathbb{H}) \). Notice that \( a_{-\lambda} b_{-\mu} f(b) \) has conformal weight \( |\lambda| + |\mu| \). We also consider the empty set as a partition, and set \( a_{-\emptyset} = b_{-\emptyset} = 1 \), and \( |\emptyset| = p(\emptyset) = 0 \).

Notice that as a subgroup of \( SL(2, \mathbb{R}) \), the congruence subgroup \( \Gamma \subset SL(2, \mathbb{Z}) = \Gamma(1) \) also acts on \( \mathcal{D}^\text{ch}(\mathbb{H}) \). We denote by \( \mathcal{D}^\text{ch}(\mathbb{H})^\Gamma \) the \( \Gamma \)-fixed points of \( \mathcal{D}^\text{ch}(\mathbb{H}) \). \( \mathcal{D}^\text{ch}(\mathbb{H}) \) is not an interesting object as it is too big, so we consider the elements in \( \mathcal{D}^\text{ch}(\mathbb{H})^\Gamma \) satisfying the cuspidal conditions similar to the definition of modular forms of \( \Gamma \).

We consider \( \Gamma = \Gamma(1) \) first. Since \( \pi\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) a = a \), \( \pi\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) b = b + 1 \) by (2.4), \( \pi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \) preserves \( a_{-n} \) and \( b_{-n} \) for \( n \geq 1 \). And it acts as an automorphism on \( \mathcal{D}^\text{ch}(\mathbb{H}) \), so we have
\[ \pi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \sum a_{-\lambda} b_{-\mu} f_{\lambda, \mu}(b) = \sum a_{-\lambda} b_{-\mu} f_{\lambda, \mu}(b + 1). \]
Hence \( f_{\lambda, \mu}(b + 1) = f_{\lambda, \mu}(b) \), and \( f_{\lambda, \mu}(b) \) has a q-expansion at the cusp \( \infty \),
\[ f_{\lambda, \mu}(b) = \sum_{m=\infty}^{\infty} u_{\lambda, \mu}(m) q^m, \quad \text{where} \quad q = e^{2\pi ib}. \]

We call \( v = \sum_{\lambda, \mu} a_{-\lambda} b_{-\mu} f_{\lambda, \mu} \) is holomorphic at \( \infty \), if for arbitrary partitions \( \lambda, \mu \), we have \( u_{\lambda, \mu}(m) = 0 \) for \( m < 0 \). Since all the cusps \( \mathbb{Q} \cup \{\infty\} \) are \( SL(2, \mathbb{Z}) \)-equivalent, we call \( v \) is holomorphic at the cusps.

For a general congruence subgroup \( \Gamma \), the notion of holomorphicity at the cusp \( c \in \mathbb{Q} \cup \{\infty\} \) needs more discussions. Choose \( \rho \in SL(2, \mathbb{Z}) \) such that \( \rho(c) = \infty \). Then \( \pi(\rho)v = \sum a_{-\lambda} b_{-\mu} \tilde{f}_{\lambda', \mu'} \) is invariant under \( \rho^{-1} \Gamma \rho \) as the group action is a right action. And since \( \rho^{-1} \Gamma \rho \) contains the translation matrix \( \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} \), for some positive integer \( N \)(cf. [B] p.41-42), \( \pi(\rho)v \) is fixed by \( \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \), which implies that \( \tilde{f}_{\lambda', \mu'}(b_0 + N) = \tilde{f}_{\lambda', \mu'}(b_0) \). Hence \( \tilde{f}_{\lambda', \mu'} \) has a Fourier expansion \( \sum \tilde{u}_{\lambda', \mu'}(m) e^{2\pi imb/N} \).

We say that \( v \) is holomorphic at the cusp \( c \) if for arbitrary partitions \( \lambda', \mu' \), we have \( \tilde{u}_{\lambda', \mu'}(m) = 0 \) for \( m < 0 \). We use \( \mathcal{D}^\text{ch}(\mathbb{H}, \Gamma) \) to denote the \( \Gamma \)-invariant vectors in \( \mathcal{D}^\text{ch}(\mathbb{H}) \) that are holomorphic at all the cusps. Using (2.1), we can prove \( \mathcal{D}^\text{ch}(\mathbb{H}, \Gamma) \) is a vertex subalgebra.

**Proposition 2.2.** \( \mathcal{D}^\text{ch}(\mathbb{H}, \Gamma) \) is a vertex subalgebra of \( \mathcal{D}^\text{ch}(\mathbb{H}) \).
For $x \in \mathcal{D}^{ch}(\mathbb{H}, \Gamma)$, the adjoint action of $g \in SL(2, \mathbb{R})$ on the operator $x_{(n)}$ is defined to be

$$
\pi(g)x_{(n)}\pi(g)^{-1} = (\pi(g)x)_{(n)}. \tag{2.5}
$$

In particular, the formulas of the adjoint action on the operators $a_{-n}$ and $b_{-n}$ for $n \geq 1$ are given by

$$
\pi(g)a_{-n}\pi(g)^{-1} = \left( \pi(g)a \right)_{-n} = (a_{-1}(\gamma b + \delta)^2 - 2n\gamma^2 b_{-n}) - \sum_{k \geq 1} a_{-k}(\gamma b + \delta)^2 - n_{n+k} + \sum_{k \geq 0} (\gamma b + \delta)^2 - n_{n+k} a_k + 2n\gamma^2 b_{-n}
$$

$$
= a_{-n} \left( \gamma b + \delta \right)^2 + \gamma^2 \sum_{i \neq 0} b_{-i} \right) + \sum_{k \geq 1, k \neq n} a_{-k} \left( 2\gamma (\gamma b + \delta) b_{-n+k} + \gamma^2 \sum_{i, j \neq 0} b_{-i} b_{-j} \right) + \sum_{k \geq 0} \left( 2\gamma (\gamma b + \delta) b_{-n-k} + \gamma^2 \sum_{i, j \neq 0} b_{-i} b_{-j} \right) a_k + 2n\gamma^2 b_{-n}, \tag{2.6}
$$

$$
\pi(g)b_{-n}\pi(g)^{-1} = \left( \pi(g)b \right)_{-n} = c_0 \left( b \right)_{-n}
$$

$$
= \sum_{l \geq 1} \sum_{i_1, \cdots, i_l \in \mathbb{Z}, i_1 + \cdots + i_l = n} (-\gamma)^{l-1} (\gamma b + \delta)^{-l-1} b_{-i_1} \cdots b_{-i_l}, \tag{2.7}
$$

where the third equality in (2.6) is given by the Borcherds identity, and the last equalities in (2.6) and (2.7) are given by (2.1). Since $SL(2, \mathbb{R})$ acts on $\mathcal{D}^{ch}(\mathbb{H})$ as automorphisms, the action of $g$ is given by

$$
\pi(g)a_{-\lambda} b_{-\mu} f(b) = \left( \pi(g)a \right)_{-\lambda(1)} \cdots \left( \pi(g)a \right)_{-\lambda(n)} \left( \pi(g)b \right)_{-\mu(1)} \cdots \left( \pi(g)b \right)_{-\mu(m)} f(g b), \tag{2.8}
$$

where $\lambda = (\lambda(1), \cdots, \lambda(n)), \mu = (\mu(1), \cdots, \mu(m))$ are partitions.

If we replace $\left( \pi(g)a \right)_{-\lambda(1)}$ and $\left( \pi(g)b \right)_{-\mu(1)}$ in the right side of (2.8) by (2.6) and (2.7), we can prove that

**Lemma 2.3.** For $g$ as in (2.3), and holomorphic function $f$ on $\mathbb{H}$,

$$
\pi(g)a_{-\lambda} b_{-\mu} f(b) = a_{-\lambda} b_{-\mu} (\gamma b + \delta)^2(p(\lambda) - p(\mu)) f(g b) + \sum_{\lambda', \mu', p(\lambda'), p(\mu') \leq p(\lambda), p(\lambda') - p(\mu') < p(\lambda) - p(\mu)} a_{-\lambda} b_{-\mu} f_{\lambda', \mu'}(b), \tag{2.9}
$$

where $f_{\lambda', \mu'}$ is a holomorphic function on $\mathbb{H}$.

**Proof:** After replacing $\left( \pi(g)a \right)_{-\lambda(1)}$ and $\left( \pi(g)b \right)_{-\mu(1)}$ in the right side of (2.8) by (2.6) and (2.7), and moving the annihilation operators to the right, the result is a sum of elements of type $a_{-\lambda} b_{-\mu} f_{\lambda', \mu'}$. If $a_{-\lambda} b_{-\mu} f_{\lambda', \mu'}$ appears, then $|\lambda'| + |\mu'| = |\lambda| + |\mu|$, because the action preserves the conformal weight.

Since the formula of the adjoint action on $a_{-n}$ in (2.6) has at most one $a_m (m \in \mathbb{Z})$ and the adjoint action on $b_{-n}$ in (2.7) is free of $a_m (m \in \mathbb{Z})$, so the $\pi(g)$ action will not increase the number of $a_m (m \in \mathbb{Z})$, namely $p(\lambda') \leq p(\lambda)$. Notice that $p(\lambda') - p(\mu')$ means the difference of the number of $a_{-m} (m \geq 1)$ and the number of $b_{-m} (m \geq 1)$ in each expression $a_{-\lambda} b_{-\mu} f_{\lambda', \mu'}$. 

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From (2.6) and (2.7), we can see that all the terms except $a_{-n}(\gamma b+\delta)^2$ in (2.6) and $(\gamma b+\delta)^{-2}b_{-n}$, the case when $l = 1$ in (2.7), will decrease the difference of the number of $a_{-m}(m \geq 1)$ and the number of $b_{-m}(m \geq 1)$. For example the term $\gamma^2a_{-n}b_{-n}b_i(i \neq 0)$ in (2.6) will decrease the number of $a_{-m}(m \geq 1)$ by 1, and increase the number of $b_{-m}(m \geq 1)$ by 1; the term $2\gamma(\gamma b+\delta)b_{-n-k}a_k(k \geq 0)$ will decrease the number of $a_{-m}(m \geq 1)$ by 1, and it will preserve the number of $b_{-m}(m \geq 1)$ when $k > 0$, and increase the number of $b_{-m}(m \geq 1)$ by 1 when $k = 0$. Hence we have

$$p(\lambda') - p(\mu') \leq p(\lambda) - p(\mu),$$

where the equality holds only for the case $(\lambda', \mu') = (\lambda, \mu)$ and the corresponding term equals

$$a_{-\lambda}b_{-\mu}(\gamma b+\delta)^{2(p(\lambda) - p(\mu))}f(\gamma b).$$

Note that a conceptual explanation of the above lemma would be using the infinitesimal adjoint action on the operator $a_{-\lambda}b_{-\mu}$, which is a maximal vector (killed by $E(0)$) and the eigenvalue of which under the action of semisimple operator $H(0)$ equals $2(p(\lambda) - p(\mu))$. And the action of $F(0)$ will strictly lower the $H(0)$-weight by $\mathfrak{s}\mathfrak{l}_2$-theory, hence the infinitesimal action will not increase the $H(0)$-weight. The action will be discussed in detail in Section 3.

Now we will introduce a total order on the partitions and partition pairs to equip a filtration on $\mathcal{O}^{ch}(\mathbb{H})$.

For two partitions $\lambda, \lambda'$, we say $\lambda > \lambda'$ if either $\lambda_{(i)} = \lambda'_{(i)}$ for $1 \leq i \leq j - 1$ and $\lambda_{(j)} > \lambda'_{(j)}$, where $j \leq \min\{p(\lambda), p(\lambda')\}$; or $p(\lambda) > p(\lambda')$ and $\lambda_{(i)} = \lambda'_{(i)}$ for $1 \leq i \leq p(\lambda')$. Hence $\emptyset$ is strictly less than any partitions except itself.

And we say $(\lambda, \mu) > (\lambda', \mu')$, if one of the following conditions holds

(A1) $p(\lambda) - p(\mu) > p(\lambda') - p(\mu')$;

(A2) $p(\lambda) - p(\mu) = p(\lambda') - p(\mu')$, and $\lambda > \lambda'$;

(A3) $\lambda = \lambda', p(\mu) = p(\mu')$ and $\mu < \mu'$.

Obviously this gives a total order on the partition pairs.

Now we define a free $\mathcal{O}(\mathbb{H})$-module of finite rank for a partition pair $(\lambda, \mu)$:

$$V_{\lambda, \mu} := \text{Span}\{a_{-\lambda}b_{-\mu}f(b_0) \in \mathcal{O}^{ch}(\mathbb{H})|(\lambda', \mu') \leq (\lambda, \mu), |\lambda'| + |\mu'| = |\lambda| + |\mu|\}.$$ 

Then by Lemma 2.3, $V_{\lambda, \mu}$ is an $SL(2, \mathbb{R})$-submodule of $\mathcal{O}^{ch}(\mathbb{H})_N$ for $N = |\lambda| + |\mu|$, where $\mathcal{O}^{ch}(\mathbb{H})_N$ is the conformal weight $N$ subspace of $\mathcal{O}^{ch}(\mathbb{H})$. So we have a filtration of submodules $\{V_{\lambda, \mu}\}$, satisfying that

$$V_{\lambda', \mu'} \subset V_{\lambda, \mu} \quad \text{if} \quad (\lambda', \mu') < (\lambda, \mu), |\lambda'| + |\mu'| = |\lambda| + |\mu|.$$ 

The filtration of chiral differential operators of different types can be found in [MSV], [S].

Given partitions $\lambda_0, \mu_0$, there are only finitely many partition pairs $(\lambda, \mu)$ such that $(\lambda, \mu) < (\lambda_0, \mu_0)$ and $|\lambda| + |\mu| = |\lambda_0| + |\mu_0|$. Let $(\lambda_1, \mu_1)$ be the successive partition pair of $(\lambda_0, \mu_0)$ under the above two conditions, namely

$$\lambda_1 = \max\{\lambda, \mu < (\lambda_0, \mu_0) \mid |\lambda| + |\mu| = |\lambda_0| + |\mu_0|\}. \quad (2.10)$$
Since $V_{\lambda_0,\mu_0}$ and $V_{\lambda_1,\mu_1}$ are preserved under the group action, the quotient space $V_{\lambda_0,\mu_0}/V_{\lambda_1,\mu_1}$ is also an $SL(2,\mathbb{R})$-module under the induced group action. We have an exact sequence of $SL(2,\mathbb{R})$-modules

$$0 \to V_{\lambda_1,\mu_1} \to V_{\lambda_0,\mu_0} \to V_{\lambda_0,\mu_0}/V_{\lambda_1,\mu_1} \to 0.$$  

Taking the $\Gamma$-fixed points of the above sequence, we have the exact sequence

$$0 \to V_{\lambda_1,\mu_1}^\Gamma \to V_{\lambda_0,\mu_0}^\Gamma \to (V_{\lambda_0,\mu_0}/V_{\lambda_1,\mu_1})^\Gamma.$$  

By (2.9), all the terms of $\pi(g)\lambda b_{-\mu_0}f(b)$ are contained in $V_{\lambda_1,\mu_1}$ except for $a_{-\lambda_0}b_{-\mu_0}(\gamma b + \delta)^{2(p(\lambda_0) - p(\mu_0))}f(gb)$. Hence $a_{-\lambda_0}b_{-\mu_0}f(b) + V_{\lambda_1,\mu_1}$ is fixed by $\Gamma$, if and only if

$$f(b) = (\gamma b + \delta)^{2(p(\lambda_0) - p(\mu_0))}f(gb), \quad \text{for any } g \in \Gamma.$$  

(2.11)

Note that (2.11) is the main condition for modular forms of level $\Gamma$. We denote by $(V_{\lambda,\mu})_0^\Gamma$ the subspace of $V_{\lambda,\mu}^\Gamma$ consisting of elements holomorphic at all the cusps. And for any partition pair $(\lambda, \mu)$, we define

$$l(\lambda, \mu) := p(\lambda) - p(\mu).$$  

(2.12)

So we have shown that

**Lemma 2.4.** Given any partition pair $(\lambda_0, \mu_0)$ and take $(\lambda_1, \mu_1)$ as in (2.10), then

$$(V_{\lambda_0,\mu_0})_0^\Gamma/(V_{\lambda_1,\mu_1})_0^\Gamma \subset M_{-2l(\lambda_0,\mu_0)}(\Gamma) = M_{2(p(\mu_0) - p(\lambda_0))}(\Gamma).$$

Let $\alpha : (V_{\lambda_0,\mu_0})_0^\Gamma \to M_{-2l(\lambda_0,\mu_0)}(\Gamma)$ be the map defined by

$$\sum_{(\lambda, \mu) \leq (\lambda_0, \mu_0)} a_{-\lambda b_{-\mu}}f_{\lambda, \mu} \mapsto f_{\lambda_0, \mu_0}.$$  

Our first main result is

**Theorem 2.5.** For any two successive partition pairs $(\lambda_0, \mu_0) > (\lambda_1, \mu_1)$ as in (2.10), we have the short exact sequence:

$$0 \to (V_{\lambda_1,\mu_1})_0^\Gamma \to (V_{\lambda_0,\mu_0})_0^\Gamma \to M_{-2l(\lambda_0,\mu_0)}(\Gamma) \to 0.$$  

(2.13)

The proof of this theorem will be given in Section 3 and Section 4. As a direct corollary of Lemma 2.4, we have

**Proposition 2.6.** For any congruence subgroup $\Gamma$, we have

$$\dim \mathcal{D}^{ch}(\mathbb{H}, \Gamma)_N < \infty, \quad \text{for any } N \geq 0,$$

where $\mathcal{D}^{ch}(\mathbb{H}, \Gamma)_N$ denotes the conformal weight $N$ subspace of $\mathcal{D}^{ch}(\mathbb{H}, \Gamma)$.

**Proof:** By Lemma 2.4, the dimension of $(V_{\lambda_0,\mu_0})_0^\Gamma$ is bounded by $\dim(V_{\lambda_1,\mu_1})_0^\Gamma + \dim M_{-2l(\lambda_0,\mu_0)}(\Gamma)$. And notice that $(\emptyset, (1, 1, \cdots, 1)) \leq (\lambda, \mu)$ for any partition pair with $|\lambda| + |\mu| = N$. We claim that

$$(V_{\emptyset, (1, 1, \cdots, 1)})_0^N$$

is finite dimensional. Indeed, by (2.4) and (2.8), we have

$$\pi(g)b_{\lambda}^N f(b) = b_{\lambda}^{N - 1}(\gamma b + \delta)^{-2N} f(gb),$$
When $2.6$ in Section 2; \( a \) has the following basis conditions holds

\[ (V_0, (1,1,\cdots,1))_N^\infty \cong M_{2N}(\Gamma). \]

Hence we prove the result by induction. \( \square \)

3. Lifting of nonconstant modular forms

In this section, we will study the lifting under the map $\alpha$ in (2.13) of modular forms of positive even weight, to $\mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma)$. We will prove Theorem 2.5 when $l(\lambda, \mu) \leq -1$.

Let $\mathcal{U}$ be the quotient of universal enveloping algebra of the Heisenberg Lie algebra (1.1) by the ideal generated by $C - 1$, so $\mathcal{U}$ is a graded algebra with the gradation given by the conformal weight. Define a topology on $\mathcal{U}$ in which a fundamental system of neighborhoods of 0 consists of the left ideals $\mathcal{U}_n$ generated by the elements with conformal weight less or equal to $-n$. Then $\{\mathcal{U}_n\}_{n=0}^\infty$ is a decreasing series with the condition that $\cap_{n\geq 0} \mathcal{U}_n = \{0\}$. Let $\bar{\mathcal{U}}$ denote the completion of $\mathcal{U}$ with respect to the topology (see similar constructions in [FZ]). And $\bar{\mathcal{U}}$ has a fundamental system $\{\mathcal{U}_n\}_{n=0}^\infty$ of neighborhoods of 0. Note that $\bar{\mathcal{U}}$ acts on $\mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma)$.

For any $N \geq 0$, we can check that all but finitely many terms in (2.6) and (2.7) are contained in $\bar{\mathcal{U}}_N$. Thus the adjoint action of $g$ on $a_n$ and $b_n$ are contained in $\bar{\mathcal{U}}$ and hence the Lie group $SL(2, \mathbb{R})$ acts on $\bar{\mathcal{U}}$. Let $K$ be the left ideal in $\bar{\mathcal{U}}$ generated by elements $a_n$ and $b_n$ for $n \geq 1$. Then $K$ is preserved under the $SL(2, \mathbb{R})$-action by (2.6) and (2.7). Therefore $\bar{\mathcal{U}}/K$ has an $SL(2, \mathbb{R})$-module structure. For $f \in \mathcal{O}(\mathbb{H}) \subset \mathcal{D}^{\text{ch}}(\mathbb{H})$, because $a_nf = b_nf = 0$ for $n \geq 1$, so $Kf = 0$. Therefore we have a map

\[ \bar{\mathcal{U}}/K \times \mathcal{O}(\mathbb{H}) \rightarrow \mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma) \]

\[ (u + K)f \mapsto uf, \]

which is $SL(2, \mathbb{R})$-equivariant in the sense that

\[ \pi(g)Af = (\pi(g)A\pi(g)^{-1})\pi(g)f, \quad \text{for any } A \in \bar{\mathcal{U}}/K, g \in SL(2, \mathbb{R}). \]

According to PBW theorem, we may write a basis of $\mathcal{U}$ as

\[ a_{-\lambda}b_{-\mu}a_0^{k_0}b_0^{l_0}a_{\lambda'}b_{\mu'}, \quad \text{for all partitions } \lambda, \mu, \lambda', \mu', \text{ and } k, l \in \mathbb{Z}_{\geq 0}, \]

where $a_{\lambda'}$ denotes the expression $a_{\lambda'(1)} \cdots a_{\lambda'(d)}$, with $d = p(\lambda')$, and similarly for $b_{\mu'}$. Hence $\bar{\mathcal{U}}/K$ has the following basis

\[ S := \{a_{-\lambda}b_{-\mu}a_0^{k_0}b_0^{l_0} | \text{ for all partitions } \lambda, \mu, \text{ and } k, l \in \mathbb{Z}_{\geq 0}\}. \]

Now we will give an order on $S$. We say $a_{-\lambda_1}b_{-\mu_1}a_0^{k_1}b_0^{l_1} > a_{-\lambda_2}b_{-\mu_2}a_0^{k_2}b_0^{l_2}$ if one of the following conditions holds

\( (B_1) \ (\lambda_1, \mu_1) > (\lambda_2, \mu_2), \) where the order is defined as in (A1)-(A3) in Section 2;

\( (B_2) \ (\lambda_1, \mu_1) = (\lambda_2, \mu_2), k_1 > k_2; \)

\( (B_3) \ (\lambda_1, \mu_1) = (\lambda_2, \mu_2), k_1 = k_2, l_1 < l_2. \)
We will first study the lifting of a nonconstant modular form $f$ of weight $-2l(\lambda_0, \mu_0)$ with $l(\lambda_0, \mu_0) \leq -1$ in (2.13) to $(V_{\lambda_0, \mu_0})_0^\dagger$, the idea is to find an operator

$$A = a_{-\lambda_0}b_{-\mu_0} + l.o.t \in \bar{U}/K,$$

where l.o.t refers to terms which are strictly less than $a_{-\lambda_0}b_{-\mu_0}$, such that

$$\pi(g)A f = Af, \quad \text{for any } g \in \Gamma.$$

The left side of above equation equals

$$(\pi(g)A\pi(g)^{-1}) f(gb) = (\pi(g)A\pi(g)^{-1})(\gamma b + \delta)^{-2l(\lambda_0, \mu_0)} f(b).$$

So it suffices to find solutions of operator $A$ of the form (3.1), such that

$$\pi(g)A\pi(g)^{-1} = A(\gamma b + \delta)^{2l(\lambda_0, \mu_0)}, \quad \text{for any } g \in SL(2, \mathbb{R}).$$

Considering the infinitesimal action, (3.2) is equivalent to the following system

$$E_{(0)}.A = 0,$$

$$H_{(0)}.A = 2l(\lambda_0, \mu_0)A,$$

$$F_{(0)}.A = -2l(\lambda_0, \mu_0)Ab_0,$$

where for $x \in \mathfrak{sl}_2$, we denote by $x_{(0)}$, the infinitesimal adjoint action of $x$ on $\bar{U}/K$, which is given by:

$$x_{(0)}.B = x_{(0)}B - Bx_{(0)}, \quad \text{for any } B \in \bar{U}/K.$$

The formulas of $E_{(0)}$ and $H_{(0)}$ on an operator $a_{-\lambda}b_{-\mu}a_0^k b_0^l$ can be calculated easily,

$$E_{(0)}a_{-\lambda}b_{-\mu}a_0^k b_0^l = -l a_{-\lambda}b_{-\mu}a_0^k b_0^{l-1},$$

$$H_{(0)}a_{-\lambda}b_{-\mu}a_0^k b_0^l = 2(p(\lambda) - p(\mu) + k - l)a_{-\lambda}b_{-\mu}a_0^k b_0^l.$$

We will construct an $\mathfrak{sl}_2$-submodule of $\bar{U}/K$ for the partition pair $(\lambda_0, \mu_0)$. We first define a subset $S_{\lambda_0, \mu_0} \subset S$ as follows:

$$S_{\lambda_0, \mu_0} := \{ a_{-\lambda}b_{-\mu}a_0^k b_0^l \in S \mid a_{-\lambda}b_{-\mu}a_0^k b_0^l \leq a_{-\lambda_0}b_{-\mu_0}, |\lambda| + |\mu| = |\lambda_0| + |\mu_0|, l(\lambda, \mu) + k \leq l(\lambda_0, \mu_0)\},$$

where $l(\lambda, \mu)$ and $l(\lambda_0, \mu_0)$ as in (2.12). According to (3.8), the last condition $l(\lambda, \mu) + k \leq l(\lambda_0, \mu_0)$ in (3.9) implies the operators in $S_{\lambda_0, \mu_0}$ has $H_{(0)}$-eigenvalues less or equal to $2l(\lambda_0, \mu_0)$.

**Lemma 3.1.** If $a_{-\lambda}b_{-\mu}a_0^k b_0^l \in S_{\lambda_0, \mu_0}$ for some $l \geq 0$, then $a_{-\lambda}b_{-\mu}a_0^i b_0^j \in S_{\lambda_0, \mu_0}$, for all $0 \leq i \leq k$ and $j \geq 0$.

**Proof:** The case $(\lambda, \mu) < (\lambda_0, \mu_0)$ is trivial from the definition of $S_{\lambda_0, \mu_0}$. For the case $(\lambda, \mu) = (\lambda_0, \mu_0)$, we conclude that $k$ must be zero. And in this case $a_{-\lambda_0}b_{-\mu_0}b_0 \in S_{\lambda_0, \mu_0}$ for all $j \geq 0$. \hfill \Box

We let $S_{\lambda_0, \mu_0}^0 \subset S_{\lambda_0, \mu_0}$ consisting of elements which contain no $b_0$, for example, if $a_{-\lambda}b_{-\mu}a_0^k \in S_{\lambda_0, \mu_0}$ then it is also contained in $S_{\lambda_0, \mu_0}^0$. By (3.7), $E_{(0)}$ kills all elements in $S_{\lambda_0, \mu_0}^0$.

Let $M_{\lambda_0, \mu_0}$ be the space spanned by $S_{\lambda_0, \mu_0}$, and we view $M_{\lambda_0, \mu_0}$ as a subspace of $\bar{U}/K$. Then we have the following lemma:

**Lemma 3.2.** For any partition pair $(\lambda_0, \mu_0)$, $M_{\lambda_0, \mu_0}$ is stable under the action (3.6). And $H_{(0)}$ acts semisimply with maximal weight $2l(\lambda_0, \mu_0)$. 


Lemma 3.3. For any term $S$ of decreasing basis in $\mathfrak{sl}_2$, has weight 2. Then according to Borcherds identity, Using the above formulas, it is not hard to see that the following six types of terms appear in $S$.

(C1) $a_{-\lambda}b_{-\mu}a_0^kb_0^l$ with $p(\lambda') = p(\lambda), \lambda' < \lambda, p(\mu') = p(\mu) + 1, |\lambda'| + |\mu'| = |\lambda| + |\mu|$, 
(C2) $a_{-\lambda}b_{-\mu}a_0^kb_0^l$ with $p(\lambda') = p(\lambda) - 1, p(\mu') = p(\mu), |\lambda'| + |\mu'| = |\lambda| + |\mu|$, 
(C3) $a_{-\lambda}b_{-\mu}a_0^{k+1}b_0^l$ with $p(\lambda') = p(\lambda) - 1, p(\mu') = p(\mu) + 1, |\lambda'| + |\mu'| = |\lambda| + |\mu|$, 
(C4) $a_{-\lambda}b_{-\mu}a_0^kb_0^l$ with $p(\mu') = p(\mu) + 1, |\mu'| < |\mu|, |\mu'| = |\mu|$, 
(C5) $a_{-\lambda}b_{-\mu}a_0^{k-1}b_0^l$, 
(C6) $a_{-\lambda}b_{-\mu}a_0^{k+1}b_0^l$.

We can easily check that all of above types are contained in $S_{\lambda_0,\mu_0}^0$ if $a_{-\lambda}b_{-\mu}a_0^kb_0^l \in S_{\lambda_0,\mu_0}^0$. Hence $F(0)$ preserves $M_{\lambda_0,\mu_0}$. Therefore $M_{\lambda_0,\mu_0}$ has an $\mathfrak{sl}_2$-module structure.

Notice that the vectors in $S_{\lambda_0,\mu_0}^0$ form a basis of maximal vectors in $M_{\lambda_0,\mu_0}$. And the above proof also shows that $F(0)$ will decrease the order of $a_{-\lambda}b_{-\mu}a_0^kb_0^l$ and hence $\mathfrak{sl}_2$-action will not increase the order of elements in $S_{\lambda_0,\mu_0}^0$, which means every operator $X(0)$ for $X \in \mathfrak{sl}_2$ has lower triangular matrix with respect to decreasing basis in $S_{\lambda_0,\mu_0}^0$. This gives us a hint to find the solution of (3.3)-(3.5) under the decreasing basis. And since $F(0)$ will decrease the weight by 2, we will study the relations of decreasing basis in $S_{\lambda_0,\mu_0}^0$ of weight $2l(\lambda_0, \mu_0)$ and $2l(\lambda_0, \mu_0) - 2$.

Lemma 3.3. Let $S_{\lambda_0,\mu_0}^0(m) \subset S_{\lambda_0,\mu_0}^0$ be the weight $m$ subspace of $S_{\lambda_0,\mu_0}^0$ under $H(0)$-action, namely

$S_{\lambda_0,\mu_0}^0(m) := \{a_{-\lambda}b_{-\mu}a_0^k \in S_{\lambda_0,\mu_0}^0 \mid H(0)a_{-\lambda}b_{-\mu}a_0^k = ma_{-\lambda}b_{-\mu}a_0^k\}$.

Then we have

$$|S_{\lambda_0,\mu_0}^0(2l(\lambda_0, \mu_0))| > |S_{\lambda_0,\mu_0}^0(2l(\lambda_0, \mu_0) - 2)|.$$

Proof: For any term $a_{-\lambda}b_{-\mu}a_0^k \in S_{\lambda_0,\mu_0}^0(2l(\lambda_0, \mu_0) - 2)$, we will show that $a_{-\lambda}b_{-\mu}a_0^{k+1} \in S_{\lambda_0,\mu_0}^0(2l(\lambda_0, \mu_0))$. Indeed, if $(\lambda, \mu) < (\lambda_0, \mu_0)$, then $a_{-\lambda}b_{-\mu}a_0^{k+1} < a_{-\lambda}b_{-\mu}a_0^k$ by (B1). And the $H(0)$-eigenvalue of $a_{-\lambda}b_{-\mu}a_0^{k+1}$ is $2l(\lambda_0, \mu_0)$ by (3.8). So $a_{-\lambda}b_{-\mu}a_0^{k+1}$ is contained in $S_{\lambda_0,\mu_0}^0(2l(\lambda_0, \mu_0))$. If $(\lambda, \mu) = (\lambda_0, \mu_0)$, the condition $l(\lambda, \mu) + k \leq l(\lambda_0, \mu_0)$ implies that $k$ must be zero. But $a_{-\lambda}b_{-\mu} = a_{-\lambda_0}b_{-\mu_0}$ has weight $2l(\lambda_0, \mu_0)$, which contradicts to the assumption $a_{-\lambda}b_{-\mu} \in S_{\lambda_0,\mu_0}^0(2l(\lambda_0, \mu_0) - 2)$. So we have a one to one correspondence from $S_{\lambda_0,\mu_0}^0(2l(\lambda_0, \mu_0) - 2)$ to $S_{\lambda_0,\mu_0}^0(2l(\lambda_0, \mu_0))$. And notice that
Proof: Since $2l(\lambda_0, \mu_0)$ is the maximal weight by Lemma 3.2, $E(0)$ kills $M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0))$. So $M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0))$ is spanned by the maximal vectors of weight $2l(\lambda_0, \mu_0)$.

For $a_{-\lambda} b_{-\mu} a_0^k b_0^l \in M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0) - 2)$, we have
\[
p(\lambda) - p(\mu) + k - l = l(\lambda_0, \mu_0) - 1.
\]
And the last condition of (3.9) implies
\[
p(\lambda) - p(\mu) + k - l \leq p(\lambda_0) - p(\mu_0) = l(\lambda_0, \mu_0).
\]

From (3.12) and (3.13), we deduce that $l \leq 1$. If $l = 1$, the equality holds in (3.13). Hence $p(\lambda) - p(\mu) + k = l(\lambda_0, \mu_0)$. By Lemma 3.1, $a_{-\lambda} b_{-\mu} a_0^k b_0^l \in M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0))$. Hence $a_{-\lambda} b_{-\mu} a_0^k b_0^l \in M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0)) b_0$. If $l = 0$, $a_{-\lambda} b_{-\mu} a_0^k$ is already in $S_{\lambda_0, \mu_0}^0(2l(\lambda_0, \mu_0) - 2)$.

Consider the composition map
\[
\bar{F} = \pi' \circ F(0) : M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0)) \rightarrow M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0) - 2) \rightarrow M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0) - 2)/M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0)) b,
\]
where $F(0)$ stands for the restriction of $F(0)$ to $M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0)) = \text{Span}_{\mathbb{C}} S_{\lambda_0, \mu_0}^0(2l(\lambda_0, \mu_0))$, and $\pi'$ is the standard quotient map.

\textbf{Theorem 3.5.} When $l(\lambda_0, \mu_0) = p(\lambda_0) - p(\mu_0) \leq -1$, there exists an element of the type
\[
a_{-\lambda} b_{-\mu} + l.o.t
\]
where l.o.t means the terms strictly less than $a_{-\lambda_0} b_{-\mu_0}$, in the kernel of the map
\[
\bar{F} : M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0)) \rightarrow M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0) - 2)/M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0)) b.
\]

\textbf{Proof:} Suppose $a_{-\lambda} b_{-\mu} a_0^k$ is a maximal vector of weight $2l(\lambda_0, \mu_0)$, then $p(\lambda) - p(\mu) + k = l(\lambda_0, \mu_0)$ by (3.8). Let $a_{-\lambda} b_{-\mu} a_0^k b_0^l$ be another maximal vector of weight $2l(\lambda_0, \mu_0)$ with $k > k'$. Then $p(\lambda) - p(\mu) < p(\lambda') - p(\mu')$ and hence $a_{-\lambda} b_{-\mu} a_0^k < a_{-\lambda'} b_{-\mu'} a_0^k'$ by (A1).

We list all the elements in $S_{\lambda_0, \mu_0}^0(2l(\lambda_0, \mu_0))$ in the decreasing order
\[
a_{-\lambda_0} b_{-\mu_0} > a_{-\lambda_1} b_{-\mu_1} > \cdots > a_{-\lambda_t} b_{-\mu_t} > a_{-\lambda_{t+1}} b_{-\mu_{t+1}} a_0^t > \cdots
\]
\[
> a_{-\lambda_{t+1}} b_{-\mu_{t+1}} a_0^t \cdots > a_{-\lambda_{t+1}} b_{-\mu_{t+1}} a_0^t > \cdots > a_{-\lambda_{t+1}} b_{-\mu_{t+1}} a_0^{t-1},
\]
where $p(\lambda_{i+1}) - p(\mu_{i+1}) + i = l(\lambda_0, \mu_0)$, for $0 \leq i \leq t - 1$, $1 \leq j_i \leq l_{i+1} - l_i$, and $l_0 = 0$.

Then the maximal vectors below
\[
a_{-\lambda_{t+1}} b_{-\mu_{t+1}} > \cdots > a_{-\lambda_{i+1}} b_{-\mu_{i+1}} a_0^{i-1} > \cdots > a_{-\lambda_{i+1}} b_{-\mu_{i+1}} a_0^{i-1} > \cdots > a_{-\lambda_{i+1}} b_{-\mu_{i+1}} a_0^{i-2},
\]
forms a decreasing basis of $\text{Span}_{\mathbb{C}} S_{\lambda_0, \mu_0}^0(2l(\lambda_0, \mu_0) - 2)$, which is identified with the quotient space $M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0) - 2)/M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0)) b$ by Proposition 3.4.

Then the representative matrix of $\bar{F}$ under the two bases (3.14) and (3.15) is of the form
\[
(B, C)
\]
where $B$ is an $(l_t - l_1) \times (l_1 + 1)$ matrix and $C$ is lower triangular matrix of type $(l_t - l_1) \times (l_t - l_1)$, because by $(C_1)-(C_6)$,

$$F(0).a_{-\lambda_i+j_i}b_{-\mu_i+j_i}a_i^0 = c_{i,j_i}a_{-\lambda_i+j_i}b_{-\mu_i+j_i}a_i^0b_0 + d_{i,j_i}a_{-\lambda_i+j_i}b_{-\mu_i+j_i}a_i^{-1} + l.o.t$$

(3.16)

where $l.o.t$ refers to the terms less than $a_{-\lambda_i+j_i}b_{-\mu_i+j_i}a_i^{-1}$, and $c_{i,j_i}, d_{i,j_i}$ are constants. And now we will calculate the diagonal on $C$, in other word, the coefficients $d_{i,j_i}$ for $1 \leq i \leq t - 1$, and $1 \leq j_i \leq l_{i+1} - l_i$. Notice that

$$F(0).a_{-\lambda_i+j_i}b_{-\mu_i+j_i}a_i^0 = (F(0).a_{-\lambda_i+j_i})b_{-\mu_i+j_i}a_i^0 + a_{-\lambda_i+j_i}(F(0).b_{-\mu_i+j_i})a_i^0 + a_{-\lambda_i+j_i}b_{-\mu_i+j_i}(F(0).a_i^0)$$

(3.17)

The terms involving $b_0$ in (3.10) and (3.11) are $-2a_{-\mu}b_0$ and $2b_0b_{-m}$ respectively, and notice that $b_0a_i^0 = -ia_i^{-1} + a_i^0b_0$. So the contribution of the coefficients $d_{i,j_i}$ for the first two terms in the right side of (3.17) is $2ip(\lambda_{l+j_i}) - 2ip(\mu_{l+j_i})$.

And the action of $F(0)$ on the derivation operator $a_i^0$ equals

$$F(0).a_i^0 = -2\left(\sum_{j=1}^{i} a_i^{j-1}b_0a_i^0\right)$$

$$= -2 \sum_{j=1}^{i} a_i^{j-1}(-ja_i^{-1} + a_i^{j}b_0)$$

$$= i(i + 1)a_i^{-1} - 2i a_i^0b_0$$

(3.18)

Hence by (3.17) and (3.18),

$$d_{i,j_i} = 2ip(\lambda_{l+j_i}) - 2ip(\mu_{l+j_i}) + i(i + 1) = i(2l(\lambda_{l+j_i}, \mu_{l+j_i}) + i + 1).$$

This is always nonzero, because

$$2l(\lambda_{l+j_i}, \mu_{l+j_i}) + i + 1 \geq 2(l(\lambda_0, \mu_0) - i) + i + 1 = 2l(\lambda_0, \mu_0) - i + 1 < 0,$$

(3.19)

where the first inequality is because $a_{-\lambda_i+j_i}b_{-\mu_i+j_i}a_i^0 \in \mathcal{S}_{\lambda_0, \mu_0}$, and the second inequality is because $l(\lambda_0, \mu_0) \leq -1$. Thus $C$ is an invertible matrix.

Under the bases (3.14) and (3.15), consider the linear system $\bar{F}(x_0, x_1, \ldots, x_{l_t}) = 0$. Let $X = (x_0, \cdots, x_{l_t})^T \in \mathbb{C}^{l_t + 1}, Y = (x_{l_t+1}, \cdots, x_{l_t})^T \in \mathbb{C}^{l_t - l_1}$, then $\bar{F}(x_0, x_1, \cdots, x_{l_t}) = (B, C)(X^T, Y^T)^T = BX + CY$, where $M^T$ refers to the transpose of the matrix $M$. Hence for arbitrary $X \in \mathbb{C}^{l_t + 1}$, $Y = -C^{-1}BX$ is always solvable. We take $x_0 = 1$, and arbitrary $x_1, \ldots, x_{l_t}$, there exists $x_{l_t+1}, \ldots, x_{l_t}$, such that $(x_0, \cdots, x_{l_t})$ is a unique nonzero solution of $\bar{F}(x_0, x_1, \cdots, x_{l_t}) = 0$. \hfill \Box

Note that the proof of Theorem 3.5 doesn’t work for the case $l(\lambda_0, \mu_0) = 0$, because in this case the matrix $C$ is not invertible.

**Proof of Theorem 2.5** (For the case $l(\lambda_0, \mu_0) \leq -1$): We take a nonzero solution of $\bar{F}(x_0, x_1, \cdots, x_{l_t}) = 0$ with $x_0 = 1$ and let

$$A = \sum_{i=0}^{l_t} x_i v_i,$$

(3.20)

where we denote by $v_0, \cdots, v_{l_t}$ the decreasing basis in (3.14). And Theorem 3.5 shows that when $l(\lambda_0, \mu_0) \leq -1$, $F(0).A \in M_{\lambda_0, \mu_0}(2l(\lambda_0, \mu_0))b$.

The coefficient of $a_{-\lambda_i+j_i}b_{-\mu_i+j_i}a_i^0b_0$, however only comes from the term $F(0).a_{-\lambda_i+j_i}b_{-\mu_i+j_i}a_i^0$, so it equals the constant $c_{i,j_i}$ in (3.16). And $c_{i,j_i} = -2p(\lambda_{l+j_i}) + 2p(\mu_{l+j_i}) - 2i = -2l(\lambda_0, \mu_0)$. 



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Therefore we have the required operator $A$ satisfying (3.5). And conditions (3.3) and (3.4) are automatically satisfied.

Applying $A$ to $f$, we get the lifting of the modular form $f$ in $(V_{\lambda_0,\mu_0})_0^\Gamma$ with the leading term $a_{-\lambda_0}b_{-\mu_0}f$. \hfill $\square$

4. Lifting of the Constant Modular Forms

In this section, we will consider the lifting under $\alpha$ in (2.13) of the constant modular forms, namely the case when $l(\lambda_0,\mu_0) = 0$. We will prove Theorem 2.5 when $l(\lambda_0,\mu_0) = 0$.

To study the lifting of the constant function, we need

**Lemma 4.1.** For $g$ as in (2.3), the adjoint action of $g$ on an operator $a_{-\lambda}b_{-\mu} \in \bar{U}/K$ is given by

$$
\pi(g)a_{-\lambda}b_{-\mu}\pi(g^{-1}) = \sum_{(\lambda',\mu') \leq (\lambda,\mu)} \sum_{s=0}^{\lfloor l(\lambda,\mu) - l(\lambda',\mu') \rfloor} c_{\lambda,\mu,\lambda',\mu'}^{s} \gamma^{l(\lambda,\mu) - l(\lambda',\mu') - s} a_{-\lambda'}b_{-\mu'}(\gamma b + \delta)^{l(\lambda,\mu) + l(\lambda',\mu') + s} a_{0}^{s},
$$

where $l(\lambda,\mu), l(\lambda',\mu')$ are as in (2.12), $[m]$ denotes the greatest integer less than or equal to $m$, and $c_{\lambda,\mu,\lambda',\mu'}^{s}$ is a constant independent of $g$. Moreover $c_{\lambda,\mu,\lambda',\mu'}^{0} = 1$, and $c_{\lambda,\mu,\lambda',\mu'}^{s} = 0$ if either $|\lambda'| + |\mu'| \neq |\lambda| + |\mu|$ or $(\lambda',\mu') \neq (\lambda,\mu), l(\lambda',\mu') = l(\lambda,\mu)$.

We give an example of the result.

$$
\pi(g)a_{-1}b_{-1}\pi(g^{-1}) = a_{-1}b_{-1} + 2\gamma (\gamma b + \delta)^{-1}b_{-2} - \gamma^2 (\gamma b + \delta)^{-2}b_{-1}^2 + 2\gamma (\gamma b + \delta)^{-1}b_{-1}^2 h_0
$$

where in this case $(\lambda,\mu) = ((1), (1))$, and $c_{\lambda,\mu,\lambda',\mu'}^{s}$ is nonzero only for the cases: when $(\lambda',\mu') = ((1), (1)), (\emptyset, (2)), (\emptyset, (1,1))$ and $s = 0$; when $(\lambda',\mu') = (\emptyset, (1,1))$ and $s = 1$.

**Proof of Lemma 4.1:** Let $g = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right)$, and view $\alpha, \beta, \gamma, \delta$ as symbols. It is easy to see

$$
\pi(g)a_{-\lambda}b_{-\mu}\pi(g^{-1}) \in \mathbb{C}[(\gamma b + \delta), (\gamma b + \delta)^{-1}], \gamma] \otimes \bar{U}.
$$

We will do the symbolic computation below. We assign an additive degree (denoted by $h$) on the monomials in $\mathbb{C}[(\gamma b + \delta), (\gamma b + \delta)^{-1}], \gamma] \otimes \bar{U}$ given by:

$$
h(a_0) = -1, h(b_0) = 1, h(\gamma^n) = -n, h((\gamma b + \delta)^n) = 0, h(c) = 0, \text{ for } n \in \mathbb{Z},
$$

where $c$ represents the operator multiplied by a constant $c$, and the “additive” means for any monomials $X, Y \in \mathbb{C}[(\gamma b + \delta), (\gamma b + \delta)^{-1}], \gamma] \otimes \bar{U}$

$$
h(XY) = h(X) + h(Y).
$$

We also call a vector homogeneous if the $h$-degrees of its monomials are the same, and we may enlarge the definition of the $h$-degree to the homogeneous vectors in $\mathbb{C}[(\gamma b + \delta), (\gamma b + \delta)^{-1}], \gamma] \otimes \bar{U}$.
Obviously the commutation of the above operators will not change the $h$-degree. For example, the $h$-degrees of the left and right sides of the equations below are equal:

\[
[a_n, b_m] = \delta_{n+m,0},
\]
\[
[a_0, (\gamma b + \delta)^m] = m\gamma(\gamma b + \delta)^{m-1} \quad (4.2)
\]

We claim that the $h$-degrees of the vectors $a_{-n}$ and $b_{-n}$ are preserved under the adjoint action of $g$, namely $(\pi(g)a)_{-n}$ is homogeneous and $h((\pi(g)a)_{-n}) = h(a_{-n})$, similarly for $b_n$. Indeed, each term in the right side of (2.6) has $h$-degree $-1$ which equals the $h$-degree of $a_{-n}$. Similarly each term in the right side of (2.7) has $h$-degree $1 = h(b_{-n})$.

Suppose $\gamma' a_{-\lambda' b_{-\mu'}}(\gamma b + \delta)^k a_0^*$ is a term in $\pi(g)a_{-\lambda b_{-\mu}}\pi(g)^{-1}$, then

\[-p(\lambda) + p(\mu) = -l - p(\lambda') + p(\mu') - s.\]

Hence $l = p(\lambda) - p(\mu) - p(\lambda') + p(\mu') - s = l(\lambda, \mu) - l(\lambda', \mu') - s$.

Moreover we can show that if $\gamma' a_{-\lambda' b_{-\mu'}}(\gamma b + \delta)^k a_0^*$ is a term in $\pi(g)a_{-\lambda b_{-\mu}}\pi(g)^{-1}$, then $l + k$ must be $2p(\lambda) - 2p(\mu)$, and $l \geq s$. This also can be proved by (2.6) and (2.7), and the fact that whenever moving the annihilation operators to the right, the sum of the indexes of $(\gamma b + \delta)$ and $\gamma$ will not be changed, and the index of $\gamma$ will not be decreased, thanks to the relation (4.2). Hence $k = 2p(\lambda) - 2p(\mu) - l = p(\lambda) - p(\mu) + p(\lambda') - p(\mu') + s = l(\lambda, \mu) + l(\lambda', \mu') + s$, and $s \leq \frac{[l(\lambda, \mu) - l(\lambda', \mu')]}{2}$.

Hence $c_{\lambda, \mu, \lambda', \mu}'^s$ is a constant. And according to (2.6) and (2.7), powers of $\gamma$ and $(\gamma b + \delta)$ are the only information involved related to $g$ in the adjoint action, so the constant $c_{\lambda, \mu, \lambda', \mu}'$ has nothing to do with $g$.

The remaining properties are due to Lemma 2.3.

When (4.1) is applied to $f(b) \in \mathcal{O}(\mathbb{H})$, we get a refinement of (2.9).

Now we will prove the remaining part of Theorem 2.5, and we assume that $p(\lambda_0) = p(\mu_0)$. In this case it suffices to study the lifting of the constant function $f(b) \equiv 1$, since the only modular form of weight 0 is the constant function. We assume that there is a lifting of 1 in $(V_{\lambda_0, \mu_0})^0$, and say

\[
v = a_{-\lambda_0 b_{-\mu_0}} + \sum_{\lambda, \mu, p(\lambda) < p(\mu)} a_{-\lambda b_{-\mu}} h_{\lambda, \mu}
\]

is invariant under $g \in \Gamma$, for certain holomorphic functions $h_{\lambda, \mu} \in \mathcal{O}(\mathbb{H})$. Consider $\pi(g)v$, then the term corresponding to $a_{-\lambda b_{-\mu}}$ with $l(\lambda, \mu) = p(\lambda) - p(\mu) = -1$ only comes from $\pi(g)a_{-\lambda_0 b_{-\mu_0}}$ and $\pi(g)a_{-\lambda b_{-\mu}} h_{\lambda, \mu}$ by Lemma 4.1. So we have

\[(\gamma b + \delta)^{-2} h_{\lambda, \mu}(gb) - h_{\lambda, \mu}(b) + c_{\lambda_0, \mu_0, \lambda, \mu}^0 \gamma(\gamma b + \delta)^{-1} = 0. \quad (4.3)
\]

Whenever $c_{\lambda_0, \mu_0, \lambda, \mu}^0$ vanishes, the function $h_{\lambda, \mu}$ is a modular form of weight 2, and we may cancel the term $a_{-\lambda b_{-\mu}} h_{\lambda, \mu}$ with the leading term $a_{-\lambda_0 b_{-\mu_0}} h_{\lambda_0, \mu_0}$ as in Section 3. Hence we only need to consider the partition pairs with $c_{\lambda_0, \mu_0, \lambda, \mu}^0 \neq 0$.

Recall that the Eisenstein series

\[E_2(\tau) := 1 + \frac{3}{\pi^2} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2},\]

is holomorphic with the Fourier expansion

\[E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n\]
It is well-known that \( E_2(\tau) \) is a quasi-modular form of weight 2, with the transformation property (cf. [Z] p.19; [KZ])

\[
(g\tau + \delta)^{-2} E_2(g\tau) = E_2(\tau) - \frac{6i}{\pi} \gamma(\gamma \tau + \delta)^{-1}, \quad \text{for any } g \in SL(2, \mathbb{Z}).
\]  

(4.4)

Obviously a rescaling of \( E_2(b) \) by \( \frac{\pi}{6i} \lambda_0, \mu_0, \lambda, \mu \), satisfies (4.3), which is exactly the unique holomorphic (here the condition “holomorphic” means holomorphic on \( \mathbb{H} \) and all the cusps) solution up to modular forms.

Define \( E(b) := \frac{\pi}{6i} E_2(b) \), then (4.4) is equivalent to

\[
(g\tau + \delta)^{-2} E(g\tau) = E(b) - \gamma(\gamma \tau + \delta)^{-1} \quad \text{for any } g \in SL(2, \mathbb{Z}).
\]

Now we want to find an operator \( A \in \bar{U}/K \), such that

\[
\pi(g)(a - \lambda_0 b - \mu_0 + AE(b)) = a - \lambda_0 b - \mu_0 + AE(b), \quad \text{for any } g \in \Gamma.
\]

Notice that \( \pi(g)AE(b) = (\pi(g)A\pi(g)^{-1})E(gb) = (\pi(g)A\pi(g)^{-1})(\gamma b + \delta)^2 E(b) - \gamma(\gamma b + \delta)) \), it is natural to consider solutions of \( A \), such that for \( g \in SL(2, \mathbb{R}) \)

\[
\pi(g)A\pi(g)^{-1} = A(\gamma b_0 + \delta)^{-2},
\]

\[
A\gamma(\gamma b_0 + \delta)^{-1} = \pi(g)a - \lambda_0 b - \mu_0 - a - \lambda_0 b - \mu_0.
\]  

(4.5)

(4.6)

Similar to (3.2), the equation (4.5) can be replaced by the version of infinitesimal action, namely

\[
E_{(0)}.A = 0,
\]

(4.7)

\[
H_{(0)}.A = -2A,
\]

(4.8)

\[
F_{(0)}.A = 2Ab.
\]

(4.9)

Lemma 4.2. The condition (4.6) implies (4.9).

Proof: Notice that (4.9) is equivalent to

\[
F_{(0)}Ab^i - AF_{(0)}b^i = 2Ab^{i+1}, \quad \text{for all } i \geq 0,
\]

which is also equivalent to

\[
F_{(0)}A(-tb + 1)^{-1} - AF_{(0)}(-tb + 1)^{-1} = 2Ab(-tb + 1)^{-1},
\]  

(4.10)

where \( t \) is a sufficiently small parameter.

Let \( h(t) := \left( \begin{array}{cc} 1 & 0 \\ -t & 1 \end{array} \right) \). By (4.6), we have

\[
A(-tb + 1)^{-1} = -\frac{1}{t}(\pi(h(t))a - \lambda_0 b - \mu_0 - a - \lambda_0 b - \mu_0).
\]
Hence we have

\[
F_0 A(-tb + 1)^{-1} = -\frac{d}{ds}\bigg|_{s=0} \pi(h(s)) \left( \pi(h(t)) a_{-\lambda_0} b_{-\mu_0} - a_{-\lambda_0} b_{-\mu_0} \right)
\]

\[
= -\frac{d}{ds}\bigg|_{s=0} \left( \pi(h(s)) a_{-\lambda_0} b_{-\mu_0} - a_{-\lambda_0} b_{-\mu_0} \right)
\]

\[
= -\frac{d}{ds}\bigg|_{s=0} \left( A(-s-t)((-s-t)b + 1)^{-1} - A(-s)(-sb + 1)^{-1} \right)
\]

\[
= A(-tb^2 + 2b)(-tb + 1)^{-2}. \tag{4.11}
\]

For any holomorphic function \( f(b) \in \mathcal{O}({\mathbb{H}}) \),

\[
F_0 f(b) = \frac{d}{ds}\bigg|_{s=0} \pi(h(s)) f(b) = \frac{d}{ds}\bigg|_{s=0} f\left( \frac{b}{-sb + 1} \right) = f'(b)b^2. 
\]

Hence we have

\[
AF_0(-tb + 1)^{-1} = Atb^2(-tb + 1)^{-2}. \tag{4.12}
\]

Using (4.11) and (4.12), the left hand side of (4.10) equals

\[
2Ab(-tb + 1)^{-1},
\]

which is exactly the right hand side. \( \square \)

Now we may assume

\[
A = \sum_{(\lambda,\mu): p(\mu) > p(\lambda)} c_{\lambda,\mu} a_{-\lambda} b_{-\mu} a_0^{-l(\lambda,\mu)-1}, \tag{4.13}
\]

where for any partition pair \((\lambda, \mu)\), \( c_{\lambda,\mu} \) is a constant, and \( l(\lambda, \mu) \) as in (2.12). Then \( A \) satisfies (4.7) and (4.8). Applying (4.13) to (4.6), the left side becomes

\[
\sum_{(\lambda,\mu): p(\mu) > p(\lambda)} c_{\lambda,\mu} (-1)^{-l(\lambda,\mu)-1}(-l(\lambda,\mu) - 1)!\gamma^{-l(\lambda,\mu)} a_{-\lambda} b_{-\mu} (\gamma b + \delta)^{l(\lambda,\mu)}.
\]

which equals the right hand side of (4.6), namely

\[
\sum_{(\lambda,\mu): p(\mu) > p(\lambda)} c_{\lambda_0,\mu_0,\lambda,\mu}^0 \gamma^{-l(\lambda,\mu)} a_{-\lambda} b_{-\mu} (\gamma b + \delta)^{l(\lambda,\mu)}.
\]

Comparing the coefficients, we have

\[
c_{\lambda,\mu} = \frac{(-1)^{-l(\lambda,\mu)-1}}{(-l(\lambda,\mu) - 1)!} c_{\lambda_0,\mu_0,\lambda,\mu}^0.
\]

Substituting the expression of \( c_{\lambda,\mu} \) to (4.13), we have

\[
A = \sum_{(\lambda,\mu): p(\mu) > p(\lambda)} \frac{(-1)^{-l(\lambda,\mu)-1}}{(-l(\lambda,\mu) - 1)!} c_{\lambda_0,\mu_0,\lambda,\mu}^0 a_{-\lambda} b_{-\mu} a_0^{-l(\lambda,\mu)-1}, \tag{4.14}
\]

Thus, we obtain the following result.

**Theorem 4.3.** The operator \( A \) defined by (4.14) satisfies (4.6)-(4.9), hence \( a_{-\lambda_0} b_{-\mu_0} + AE(b) \) is invariant under \( \Gamma(1) = SL_2(\mathbb{Z}) \), and hence under arbitrary congruence subgroup \( \Gamma \).
Theorem 4.3 completes the proof of Theorem 2.5. Apply Theorem 4.3, we may calculate that the lifting of \( f \) in \( (V_{\lambda,\mu})^0 \), for \( \lambda = \mu = (1) \), equals
\[
a_{-1}b_{-1} + 2b_{-2}E(b) + b_{-1}^2E'(b) = \omega + L_{-1}b_{-1}E(b),
\]
which is a Virasoro element in \( \mathcal{D}^\text{ch}(\mathbb{H}, \Gamma) \) of central charge 2. Therefore \( \mathcal{D}^\text{ch}(\mathbb{H}, \Gamma) \) is a vertex operator algebra.

5. Properties of vertex operator algebras \( \mathcal{D}^\text{ch}(\mathbb{H}, \Gamma) \)

In this section, we study the structure of \( \mathcal{D}^\text{ch}(\mathbb{H}, \Gamma) \) and calculate the character formula of \( \mathcal{D}^\text{ch}(\mathbb{H}, \Gamma) \).

**Lemma 5.1.** Let \( 0 \neq h \in M_k(\Gamma) \) for \( k > 0 \), so \( h(z + N) = h(z) \) for some \( N \in \mathbb{Z}_{\geq 0} \). Suppose that \( \sum_{i=0}^l g_i(z)h^{(i)}(z) \) is periodic with period \( N \), where \( g_i(z) \in \mathbb{C}[z] \) and \( h^{(i)}(z) \) denotes \( i \)-th derivative of \( h(z) \), then \( g_i(z) \) is a constant function for each \( i \).

For simplicity, we denote by \( D_f \) the space spanned by the lifting of modular forms \( f \in M_{2k}(\Gamma), k \geq 1 \) to \( \mathcal{D}^\text{ch}(\mathbb{H}, \Gamma) \), namely
\[
D_f := \{ Af \in \mathcal{D}^\text{ch}(\mathbb{H}, \Gamma) \mid A \in \tilde{U}/K \}.
\]
And define
\[
D_1 := \text{Span}_\mathbb{C} \{ a_{-\lambda_0}b_{-\mu_0} + AE(b) \mid l(\lambda_0, \mu_0) = 0, A \text{ as in (4.14)} \}.
\]
The solution \( A \) of (3.2) (or equivalently (3.3)-(3.5)) has the form
\[
\sum_{(\lambda,\mu) \leq (\lambda_0,\mu_0) \atop i \in \mathbb{Z}_{\geq 0}} c^i_{\lambda,\mu}a_{-\lambda}b_{-\mu}a^i_0, \tag{5.1}
\]
where \( c^i_{\lambda,\mu} \in \mathbb{C} \), and \( c^i_{\lambda_0,\mu_0} \) vanishes unless \( i = 0 \). Now we will show that for any \( A \in \tilde{U}/K \), if \( Af \in \mathcal{D}^\text{ch}(\mathbb{H}, \Gamma) \), where \( f \in M_{2k}(\Gamma), k \geq 1 \), then \( A \) has the form (5.1). We assume that
\[
A = \sum_{(\lambda,\mu) \leq (\lambda_0,\mu_0) \atop i,j \in \mathbb{Z}_{\geq 0}} c^i_{\lambda,\mu}a_{-\lambda}b_{-\mu}a^i_0b^j_0,
\]
and our first step is to show that the expression of \( A \) is free of \( b_0 \). Indeed, since the congruence subgroup \( \Gamma \) contains the translation matrix \( \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \) for some positive integer \( N \), \( Af \) is fixed by
\[
\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix},
\]
which implies that
\[
\sum_{i,j} c^i_{\lambda,\mu}a^i_0( b + N )^j f(b) = \sum_{i,j} c^i_{\lambda,\mu}a^i_0b^j f(b), \quad \text{for any partition pair} \ (\lambda,\mu).
\]
Fix \( (\lambda,\mu) \) and let \( i_0 \) be the maximal \( i \), such that \( c^i_{\lambda,\mu} \neq 0 \), for some \( j \). Then \( \sum_j c^{i_0}_{\lambda,\mu}b^j f(i_0)(b) \) consists of the terms with highest derivation of \( f(b) \), therefore according to Lemma 5.1, all of \( c^{i_0}_{\lambda,\mu} \) must be zero except possibly for \( j = 0 \). By induction we can show that all of \( c^i_{\lambda,\mu} = 0 \) except for \( j = 0 \) and there would be no \( b_0 \) involved in the expression of \( A \). Hence
\[
A = a_{-\lambda_0}b_{-\mu_0} \sum_{i=0}^l c^i_{\lambda_0,\mu_0}a^i_0 + \sum_{(\lambda,\mu) < (\lambda_0,\mu_0) \atop i \in \mathbb{Z}_{\geq 0}} c^i_{\lambda,\mu}a_{-\lambda}b_{-\mu}a^i_0, \quad \text{with} \ c^l_{\lambda_0,\mu_0} \neq 0. \tag{5.2}
\]
Then we will show that \( l = 0 \) in (5.2). Comparing the terms corresponding to \( a_{-\lambda} b_{-\mu} \) in both sides of \( \pi(g)Af = Af \), we have

\[
\sum_{i=0}^{l} c^i_{\lambda_0, \mu_0} f^{(i)}(gb)(\gamma b + \delta)^{2l(\lambda_0, \mu_0)} = \sum_{i=0}^{l} c^i_{\lambda_0, \mu_0} f^{(i)}(b),
\]

which implies \( F(b) := \sum_{i=0}^{l} c^i_{\lambda_0, \mu_0} f^{(i)}(b) \) is a modular form of weight \(-2l(\lambda_0, \mu_0)\). By induction we can show that

\[
f^{(i)}(gb) = \sum_{j=0}^{i} c^j_{\lambda} \gamma^{i-j}(\gamma b + \delta)^{2k+i-j}f^{(j)}(b), \quad \text{for } g \in \Gamma,
\]

where \( c^j_{\lambda} \in \mathbb{Z}_{>0} \) for \( 0 \leq j \leq i \), \( c^i_{\lambda} = 1 \). Hence,

\[
\pi(g)F(b) = F(gb) = \sum_{i=0}^{l} \sum_{j=0}^{i} c^i_{\lambda_0, \mu_0} c^j_{\lambda} \gamma^{i-j}(\gamma b + \delta)^{2k+i-j} f^{(j)}(b).
\]

Viewing \( \pi(g)F(b) \) and \( (\gamma b + \delta)^{2l(\lambda_0, \mu_0)} F(b) \) as polynomials of \( \gamma \), the leading terms of the two polynomials should be equal. And the leading terms are

\[
\sum_{j=0}^{l} c^j_{\lambda_0, \mu_0} c^j_{\lambda} \gamma^{2k+l+2j} f^{(j)}(b),
\]

which is nonzero according to Lemma 5.1, and \( F(b) b^{-2l(\lambda_0, \mu_0)} \gamma^{2l(\lambda_0, \mu_0)} \) respectively. Hence \( l(\lambda_0, \mu_0) = -k - l \), and

\[
\sum_{j=0}^{l} c^j_{\lambda_0, \mu_0} c^j_{\lambda} \gamma^{2k+l+2j} f^{(j)}(b) = \sum_{i=0}^{l} c^i_{\lambda_0, \mu_0} b^{-2l(\lambda_0, \mu_0)} f^{(i)}(b),
\]

which is impossible unless \( l = 0 \), and \(-k = l(\lambda_0, \mu_0)\) by Lemma 5.1.

**Lemma 5.2.** Let \( A_i \), \( f_i \in D_{f_i} \) be a nontrivial lifting of \( f_i \) for \( i = 1, \ldots, n \), where \( f_1, \ldots, f_n \) are linearly independent modular forms in \( \oplus_{l \geq 1} M_{2l}(\Gamma) \). Then \( A_1 f_1, \ldots, A_n f_n \) are also linearly independent.

**Proof:** It suffices to prove the case for the modular forms of the same weight \( 2l \) with \( l \geq 1 \). Let \( N > 0 \) be a fixed integer. Suppose \( (\lambda_1, \mu_1) > \cdots > (\lambda_m, \mu_m) \) are all the partition pairs with the relation that \(|\lambda_i| + |\mu_i| = N\) and \( p(\lambda_i) - p(\mu_i) = -l\).

According to the discussion above Lemma 5.2, we may assume \( A_i \) has the leading term \( a_{-\lambda} b_{-\mu} \), and let

\[
A_i = \sum_{k_i \leq j \leq m} x_{ij} a_{-\lambda_j} b_{-\mu_j} + \text{l.o.t}, \quad \text{with } x_{ik_i} = 1
\]

where we omit the lower order terms less than \( a_{-\lambda} b_{-\mu} \) by l.o.t. Notice that the term \( a_{-\lambda} b_{-\mu} a_0^k \) for \( j > k_i \) and \( k > 0 \) may not appear in \( A_i \), otherwise a combination of \( A_i \) and the solutions \( A \) in (3.2) with \( (\lambda_0, \mu_0) = (\lambda_{k_1}, \mu_{k_1}), \ldots, (\lambda_{j+1}, \mu_{j+1}) \) respectively, will contradict the form (5.1).

Suppose \( \sum_{i=1}^{n} c_i A_i f_i = 0 \), for some constants \( c_i, i = 1, \ldots, n \). Then

\[
\sum_{i} c_i \sum_{k_i \leq j \leq m} x_{ij} a_{-\lambda_j} b_{-\mu_j} f_i = 0. \tag{5.3}
\]

But the left hand side of (5.3) equals \( \sum_{j=1}^{m} \sum_{i: k_i \leq j} c_i x_{ij} a_{-\lambda_j} b_{-\mu_j} f_i = 0 \), hence \( \sum_{i: k_i \leq j} c_i x_{ij} f_i = 0 \), for \( 1 \leq j \leq m \). Because \( f_1, \ldots, f_n \) are linear independent, \( c_i x_{ij} = 0 \) for any \( i \) such that \( k_i \leq j \).
So we have \( c_i = 0 \), since \( x_{ik} = 1 \). Therefore \( c_i = 0 \) for \( 1 \leq i \leq n \) and \( A_1 f_1, \ldots, A_n f_n \) are linearly independent.

**Theorem 5.3.** Let \( \mathcal{B} \) be a homogeneous linear basis of modular forms in \( \oplus_{k \geq 0} M_{2k}(\Gamma) \), then we have

\[
\mathcal{D}^{ch}(\mathbb{H}, \Gamma) = \oplus_{f \in \mathcal{B}} D_f.
\]

Now we will derive the character formula of \( \mathcal{D}^{ch}(\mathbb{H}, \Gamma) \). The character formula of \( \mathcal{D}^{ch}(\mathbb{H}, \Gamma) \) is the formal power series of variable \( q \) defined by \( \sum_{n=0}^{\infty} \dim \mathcal{D}^{ch}(\mathbb{H}, \Gamma)nq^n \), which is \( tr \, q^{L_0} \). We first consider that the trace \( tr \, t H(t) q^{L_0} \) of the vertex subalgebra \( \mathbb{C}[a_{-n}, b_{-n} | n \geq 1] \). According to \( (5.8) \), \( tr \, t H(t) q^{L_0} \) is

\[
\Pi_{n=1}^{\infty} \frac{1}{1 - t^2q^n} \Pi_{n=1}^{\infty} \frac{1}{1 - t^{-2}q^n} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} c(m, n)q^n t^m
\]

(5.4)

It is clear that \( c(m, n) \) is the number of partition pairs \( (\lambda, \mu) \) with \( |\lambda| + |\mu| = n \) and \( 2(p(\lambda) - p(\mu)) = m \).

As the character of \( \mathcal{D}^{ch}(\mathbb{H}, \Gamma) \) coincides with the character of the graded algebra \( \oplus_{\lambda \mu} gr \mathcal{D}^{ch}(\mathbb{H}, \Gamma)_{\lambda \mu} \), where \( gr \mathcal{D}^{ch}(\mathbb{H}, \Gamma)_{\lambda \mu} \) is the successive quotient \( (V_{\lambda, \mu})_0 / (V_{\lambda, \mu'})_0 \cong M_{-2l(\lambda, \mu)}(\Gamma) \) by Theorem 2.5, the character of \( \mathcal{D}^{ch}(\mathbb{H}, \Gamma) \) equals

\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} c(m, n) \dim M_m(\Gamma)q^n.
\]

(5.5)

**Proof of Theorem 1.1:** When \( m < 0 \) or \( m \) odd, \( \dim M_m(\Gamma) \) equals zero. So we let \( m = 2k \) for \( k \geq 0 \), and we will calculate \( \sum_{n=0}^{\infty} c(2k, n)q^n \).

The left side of (5.4) equals

\[
(1 + \sum_{n, s \geq 1} p_s(n)t^{2s}q^n)(1 + \sum_{m, l \geq 1} p_l(m)t^{-2l}q^m)
\]

\[
= 1 + \sum_{m, l \geq 1} p_l(m)(t^{-2l} + t^{2l})q^m + \sum_{m, n, l, s \geq 1} p_l(m)p_s(n)t^{-2l+2s}q^{m+n},
\]

where \( p_l(m) \) is the number of partitions of \( m \) into exactly \( l \) parts.

As \( c(2k, n) \) is the coefficient of \( q^k t^{2k} \) of the above formula, hence \( \sum_{n=0}^{\infty} c(2k, n)q^n \) equals

\[
\sum_{n \geq 1} p_k(n)q^n + \sum_{l \geq 1} \sum_{m, n \geq 1} p_l(m)p_{l+k}(n)q^{m+n}
\]

(5.6)

Recall that a partition \( \lambda \) has \( k \) parts if and only if its conjugate partition \( \lambda' \) has largest part \( k \), where the conjugate partition \( \lambda' \) is the partition whose Young diagram is obtained from interchanging rows and columns of \( \lambda \). So the generating function for partition with part \( k \), is

\[
\sum_{n \geq 0} p_k(n)x^n = x^k \prod_{i=1}^{k} \frac{1}{1 - x^i}.
\]

Hence (5.6) equals

\[
\sum_{l \geq 0} q^{2l+k} \prod_{i=1}^{l} \frac{1}{1 - q^i} \prod_{j=1}^{l+k} \frac{1}{1 - q^j}.
\]

\[ \square \]
6. Formulas for the lifting to a vertex subalgebra

In this section, we consider the vertex algebra \( \mathcal{B} := \mathbb{C}[b_0, b_{-1}, \cdots] \otimes_{\mathbb{C}[t]} \mathcal{O}(\mathbb{H}) \), generated by \( b_{-1} \) and \( f(b) \in \mathcal{O}(\mathbb{H}) \). And we will find an explicit formula of a lifting of any nonconstant modular form of even weight to \( \mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma(1)) \cap \mathcal{B} \).

As in Section 3, we prove that for any nonconstant modular form \( f \in M_2(\Gamma) \), there exists a lifting in \( (V_{\lambda_0, \mu_0})^\Gamma_0 \) for arbitrary partition pair \( (\lambda_0, \mu_0) \) with the condition that \( p(\mu_0) - p(\lambda_0) = l \). Now we will give an explicit formula for a lifting of \( f \in M_2(\Gamma(1)) \) to \( (V_{\lambda_0, \mu_0})^\Gamma_0 \subset \mathcal{D}^{\text{ch}}(\mathbb{H}, \Gamma(1)) \cap \mathcal{B} \) for the case \( \lambda_0 = \emptyset \) and \( p(\mu_0) = l \). Since the action preserve the conformal weight, we will focus on \( \mathcal{B}_n \) the weight \( n \) subspace of \( \mathcal{B} \). Define a subspace \( \mathcal{B}_n(l) \subset \mathcal{B}_n \) as follows
\[
\mathcal{B}_n(l) := \text{Span}_C \{ b_{-\mu} f(b) \in \mathcal{B}_n \mid p(\mu) \geq l \}.
\]

Then we have a filtration which is stable under \( \Gamma(1) \)-action:
\[
\mathcal{B}_n = \mathcal{B}_n(0) \supset \mathcal{B}_n(1) \supset \cdots \supset \mathcal{B}_n(n).
\]

Similar to the discussion in Section 2, \( b_{-\mu} f(b) + \mathcal{B}_n(l + 1) \), with \( p(\mu) = l \) is fixed by \( \Gamma(1) \) if and only if
\[
f(g b) = (\gamma b_0 + \delta)^{2l} f(b) \quad (6.1)
\]

Define \( t^\mu_\nu := (-1)^{p(\nu) - p(\mu)} c_\emptyset^{(0), \emptyset_{\mu}, \emptyset_{\nu}} \). Then (4.1) turns into
\[
\pi(g) b_{-\mu} f(b) = \sum_{\nu : p(\nu) \geq p(\mu)} t^\mu_\nu (-\gamma)^{-p(\mu) + p(\nu)} b_{-\nu} (\gamma b + \delta)^{-p(\mu) - p(\nu)} f(g b). \quad (6.2)
\]

Before we give the construction, we will explore some important properties of the integer coefficient \( t^\mu_\nu \).

**Lemma 6.1.** Given any two partitions \( \mu \) and \( \nu \) of \( n \) with the relation that \( p(\mu) \leq p(\nu) \), we have

1. for \( 0 \leq s \leq p(\nu) - p(\mu) \),
\[
\sum_{\mu' : p(\mu') = p(\mu) + s} t^\mu_{\mu'} t^\mu_{\nu'} = \left( \frac{p(\nu) - p(\mu)}{p(\mu') - p(\mu)} \right) t^\mu_{\nu'}.
\]

2. for \( s_i \geq 0 \), with \( i = 1, 2, \cdots, k - 1 \) and \( \sum_{i=1}^{k-1} s_i \leq p(\nu) - p(\mu) \),
\[
\sum_{\mu_1, \cdots, \mu_{k-1} : p(\mu_1) = p(\mu_1 - s) \cdots \mu_{k-1} = p(\mu_{k-1}) - p(\mu_{k-2})} t^\mu_{\mu_1} t^\mu_{\mu_2} \cdots t^\mu_{\nu_{k-1}} = \left( \frac{p(\nu) - p(\mu)}{p(\mu_1) - p(\mu_2) - p(\mu_1) - \cdots - p(\mu_{k-1}) - p(\mu_{k-2})} \right) t^\mu_{\nu},
\]

where \( \mu_0 \) is defined to be \( \mu \).

The above lemma is equivalent to the following one in a special form.

**Lemma 6.2.** Given any two partitions \( \mu \) and \( \nu \) of \( n \) with the relation that \( p(\nu) = p(\mu) + k \), for \( k \geq 2 \), we have
\[
\sum_{\mu_1, \cdots, \mu_{k-1} : p(\mu_1) = p(\mu_1 - s) \cdots p(\mu_{k-1}) = p(\mu_{k-1} + i)} t^\mu_{\mu_1} t^\mu_{\mu_2} \cdots t^\mu_{\nu_{k-1}} = k! t^\mu_{\nu}.
\]

**Proof:** Take \( S_3 \) to be an order three element in \( PSL(2, \mathbb{R}) \), for example, let
\[
S_3 := \begin{pmatrix}
\frac{1}{3} & \frac{\sqrt{3}}{2} \\
\frac{-\sqrt{3}}{2} & \frac{2}{3}
\end{pmatrix}
\]
We may compute the formula $\pi(S_3)^3 b_{\mu}$ by iterating \((6.2)\), and it equals

$$
\sum_{\mu_1, \mu_2, \mu_3: \, p(\mu) \leq p(\mu_1) \leq p(\mu_2) \leq p(\mu_3)} (-1)^{p(\mu_1) - p(\mu)} \left( \frac{\sqrt{3}}{2} \right)^{p(\mu) - p(\mu_1)} \mu_{\mu_1}^\mu \mu_{\mu_2}^\mu \mu_{\mu_3}^\mu \left( -\frac{\sqrt{3}}{2} b + \frac{1}{2} \right)^{p(\mu_1) - p(\mu_3)} \left( -\frac{\sqrt{3}}{2} b - \frac{1}{2} \right)^{p(\mu) - p(\mu_2)}.
$$

On the other hand, $\pi(S_3)^3 b_{\mu} = \pi(S_3)^3 b_{\mu} = b_{-\mu}$, since $S_3^3$ acts as identity operator. So comparing the terms corresponding to $b_{-\mu}$, with $p(\nu) > p(\mu)$ in the two expressions of $\pi(S_3)^3 b_{-\mu}$, we have for $b \in \mathbb{H}$ the following equation holds

$$
\sum_{\mu_1, \mu_2: \, p(\mu) \leq p(\mu_1) \leq p(\mu_2) \leq p(\nu)} (-1)^{p(\mu_1) - p(\mu)} \mu_{\mu_1}^\mu \mu_{\mu_2}^\nu \left( -\frac{\sqrt{3}}{2} b + \frac{1}{2} \right)^{p(\mu_1) - p(\mu)} \left( -\frac{\sqrt{3}}{2} b - \frac{1}{2} \right)^{p(\nu) - p(\mu_2)} = 0.
$$

Since the left side of the above equation is indeed a polynomial for $b$, it holds for arbitrary $b \in \mathbb{C}$. We take $b$ to be 0, the equation becomes

$$
\sum_{\mu_1, \mu_2: \, p(\mu) \leq p(\mu_1) \leq p(\mu_2) \leq p(\nu)} \mu_{\mu_1}^\mu \mu_{\mu_2}^\nu \left( -\frac{1}{2} \right)^{p(\mu_1) - p(\mu) + p(\nu) - p(\mu_2)} = 0, \quad (6.3)
$$

Suppose the lemma holds for $k < l$, we will use \((6.3)\) to show the case $k = l$ by induction. Assume that $p(\nu) = p(\mu) + l$ if \{\{p(\mu_1), p(\mu_2)\} \subset \{p(\mu), p(\nu)\}\}. Then \(t_{\mu_1}^\mu t_{\mu_2}^\nu\) is exactly \(t_{\mu}\) since whenever $p(\alpha) = p(\beta)$, then $t_{\beta} = \delta_{\alpha, \beta}$, where $\delta_{\alpha, \beta}$ is the Kronecker delta function. If either $p(\mu_1)$ or $p(\mu_2)$ is distinct from both $p(\mu)$ and $p(\nu)$, we assume that $p(\mu_1) = p(\mu) + i$ and $p(\mu_2) = p(\mu_1) + j$. Then running through all of $\mu_1$ and $\mu_2$ under the above condition, the summation of \(t_{\mu_1}^\mu t_{\mu_2}^\nu\) \((\frac{1}{2})^{p(\mu_1) - p(\mu) + p(\nu) - p(\mu_2)}\) equals

$$
\frac{1}{i! j! (l - i - j)!} \left( \frac{1}{2} \right)^{l - j} \sum_{\mu_1, \ldots, \mu_{l-1}: \, p(\mu) = p(\mu) + s} \mu_{\mu_1}^\mu \mu_{\mu_2}^\mu \cdots \mu_{\mu_{l-1}}^\mu,
$$

where we use the induction assumption.

Thanks to the combinatorial equation

$$
\sum_{i=0}^{l} \sum_{j=0}^{l-i} \frac{1}{i! j! (l - i - j)!} \left( \frac{1}{2} \right)^{l - j} = 0,
$$

the left side of \((6.3)\) equals

$$
(2(-\frac{1}{2})^l + 1) t_{\nu}^\mu - \frac{1}{l!} (2(-\frac{1}{2})^l + 1) \sum_{p(\mu_1), \ldots, p(\mu_{l-1}) = p(\mu) + s} \mu_{\mu_1}^\mu \mu_{\mu_2}^\mu \cdots \mu_{\mu_{l-1}}^\mu,
$$

hence we have proved the case $k = l$. \(\square\)

Now suppose $f$ is a modular form of weight $2l$, we will derive a formula for the lifting of $f$ in $\mathcal{D}^w(\mathbb{H}, \Gamma) \cap \mathcal{B}_n(l)$ with the leading term $b_{-\mu} f(b)$, where $p(\mu) = l$, and $|\mu| = n$.

We define

$$
F_k(b) = \sum_{s=0}^{k} \sum_{p(\mu') = p(\mu) + s} c_{\mu'} b_{-\mu'} f^{(s)}(b), \quad (6.4)
$$

where $f^{(s)}(b)$ denotes $s$-th derivatives of $f(b)$, $c_{\mu'} := t_{\mu'}^\mu \cdot \Gamma_{t=0}^{p(\mu') - l - 1} (2l + t)^{-1} \in \mathbb{R}$, for partition $\mu'$ such that $p(\mu') > l$ and $c_{\mu} = 1$. Oviously, $F_k(b) \in \mathcal{B}_n(l)$, for $k \geq 0$, and $F_0(b) = b_{-\mu} f(b)$.\[21\]
Hence we have for any \( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1) \), obviously \( \pi(g)F_0(b) - F_0(b) \in B_n(l) \). Suppose that \( \pi(g)F_{k-1}(b) - F_{k-1}(b) \in B_n(l + k) \), we will show that
\[
\pi(g)F_k(b) - F_k(b) \in B_n(l + k + 1). \tag{6.5}
\]
Note that \( \pi(g)F_{k-1}(b) - F_{k-1}(b) \) equals
\[
\sum_{s=0}^{k-1} \sum_{\mu':p(\mu')=l+s} c_\mu t_\nu^{\mu'} (-\gamma)^{p(\nu)-p(\mu')} (\gamma b + \delta)^{-p(\nu)-p(\mu')} b_{-\nu} f(s)(gb) - \sum_{s=0}^{k-1} \sum_{\mu':p(\mu')=p(\mu)+s} c_\mu b_{-\mu'} f(s)(b) \tag{6.6}
\]
As (6.6) is contained in \( B_n(l + k) \), the term \( b_{-\nu} h(b), 0 \neq h(b) \in \mathcal{O}(\mathbb{H}) \) will not appear for any partition \( \nu \) such that \( p(\nu) = l + k - 1 \). Therefore we have
\[
\sum_{s=0}^{k-1} \sum_{\mu':p(\mu')=l+s} c_\mu t_\nu^{\mu'} (2l + k + s - 1)(-\gamma)^{k-s}(\gamma b + \delta)^{-2l-k-s} f(s)(gb) + \sum_{s=0}^{k-1} \sum_{\mu':p(\mu')=l+s} c_\mu t_\nu^{\mu'} (-\gamma)^{k-s}(\gamma b + \delta)^{-2l-k-s} f(s+1)(gb) - c_\nu f^{(k)}(b) = 0
\]
Taking derivative towards \( b \) leads to the equation that
\[
(2l + k - 1)t_\nu^{\mu'}(-\gamma)^{k}(\gamma b + \delta)^{-2l-k} f(gb) + \sum_{s=1}^{k-1} \left( \sum_{\mu':p(\mu')=l+s} (2l + k + s - 1)c_\mu t_\nu^{\mu'} + \sum_{\mu':p(\mu')=l+s-1} c_\mu t_\nu^{\mu'} \right) (-\gamma)^{k-s}(\gamma b + \delta)^{-2l-k-s} f(s)(gb) + c_\nu (\gamma b + \delta)^{-2l-2k} f^{(k)}(gb) - c_\nu f^{(k)}(b) = 0
\]
We substitute \( s \) by \( s - 1 \) in the second summation above, the equation becomes
\[
(2l + k - 1)t_\nu^{\mu'}(-\gamma)^{k}(\gamma b + \delta)^{-2l-k} f(gb) + \sum_{s=1}^{k-1} \left( \sum_{\mu':p(\mu')=l+s} (2l + k + s - 1)c_\mu t_\nu^{\mu'} + \sum_{\mu':p(\mu')=l+s-1} c_\mu t_\nu^{\mu'} \right) (-\gamma)^{k-s}(\gamma b + \delta)^{-2l-k-s} f(s)(gb) + c_\nu (\gamma b + \delta)^{-2l-2k} f^{(k)}(gb) - c_\nu f^{(k)}(b) = 0
\]
Fix an arbitrary partition \( \chi \) of \( n \) with \( p(\chi) = l + k \). Then multiply \( t_\chi^{\nu} \) to the above equation and take sum of all partition \( \nu \) of \( n \) with \( p(\nu) = l + k - 1 \). We have the equation below
\[
\sum_{\nu:p(\nu)=l+k-1} (2l + k - 1)t_\chi^{\nu} t_\nu^{\mu'}(-\gamma)^{k}(\gamma b + \delta)^{-2l-k} f(gb) + \sum_{s=1}^{k-1} \left( \sum_{\mu':p(\mu')=l+s} (2l + k + s - 1)c_\mu t_\nu^{\mu'} + \sum_{\mu':p(\mu')=l+s-1} c_\mu t_\nu^{\mu'} \right) (-\gamma)^{k-s}(\gamma b + \delta)^{-2l-k-s} f(s)(gb) + c_\nu (\gamma b + \delta)^{-2l-2k} f^{(k)}(gb) - c_\nu f^{(k)}(b) = 0
\]
The left side of the above equation, according to Lemma 6.1 and definition of $c_{\mu}$, equals
\[(2l + k - 1)kt_{\chi}(-\gamma)^{k}(\gamma b + \delta)^{-2l-k}f(gb)\]
\[+ \sum_{s=1}^{k-1} \binom{k}{s} k(2l + k - 1)\Pi_{t=0}^{s-1}(2l + t)^{-1}t_{\chi}(-\gamma)^{k-s}(\gamma b + \delta)^{-2l-k-s}f(s)(gb)\]
\[+ k\Pi_{t=0}^{k-2}(2l + t)^{-1}t_{\chi}(\gamma b_0 + \delta)^{-2l-k}f(k)(gb) - k\Pi_{t=0}^{k-2}(2l + t)^{-1}t_{\chi}f(k)(b)\]
which is exactly the function corresponding to $b_{-\chi}$ in $gF_{k}(b) - F_{k}(b)$. Hence $gF_{n-l}(b) - F_{n-l}(b) \in \mathcal{B}_{n}(n + 1) = 0$, thus $F_{n-l}(b)$ defined in (6.5) is a lifting of $f$ to $\mathcal{O}^{\text{ch}}(\mathbb{H}, \Gamma) \cap \mathcal{B}_{0}^{(1)}$ with the leading term $b_{-\mu}f(b)$.

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