THE MODULI SPACE OF CUBIC FOURFOLDS VIA THE PERIOD MAP

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Abstract. We characterize the image of the period map for cubic fourfolds with at worst simple singularities as the complement of an arrangement of hyperplanes in the period space. It follows then that the GIT compactification of the moduli space of cubic fourfolds is isomorphic to the Looijenga’s compactification associated to this arrangement. This work builds on and is a natural continuation of our previous paper on the GIT compactification of the moduli space of cubic fourfolds.

1. Introduction

An indispensable tool for the study of K3 surfaces is the period map. The period map for K3 surfaces is both injective (satisfies the global Torelli theorem) and surjective. It follows that the moduli space of algebraic K3 surfaces of a given degree is the quotient of a 19-dimensional bounded symmetric domain by an arithmetic group. This explicit description of the moduli space has numerous geometric applications, for example to the study of deformations of certain classes of surface singularities (e.g. the work of Pinkham [40], Looijenga [29], and others). The study of deformations of higher-dimensional singularities led us to consider the period map for cubic fourfolds, which is well known to behave similarly to the period map for K3 surfaces. Specifically, as in the case of K3 surfaces, the period domain $D$ for cubic fourfolds is a bounded symmetric domain of type IV of the right dimension, and the global Torelli theorem holds (Voisin [48]). In this paper we analyze the complementary question, the question of surjectivity (or more precisely of characterizing the image) of the period map for cubic fourfolds. A surjectivity type statement is needed in applications, especially the applications regarding the degenerations and singularities of cubic fourfolds. Our work builds on the results of Voisin [48] and Hassett [21], and was inspired by the recent work of Allcock-Carlson-Toledo [1, 2] and Looijenga-Swierstra [33] on the moduli space of cubic threefolds.

As announced, our main result is a positive answer to a conjecture of Hassett [21 §4.4].

Theorem 1.1. The image of the period map for cubic fourfolds $P_0 : M_0 \to D/\Gamma$ is the complement of the hyperplane arrangement $H_\infty \cup H_\Delta$ (see Def. 2.3). Furthermore, the period map extends to a regular morphism $P : M \to D/\Gamma$ over the simple singularities locus with image the complement of the arrangement $H_\infty$.

We note that the theorem is analogous to the corresponding statement for degree two K3 surfaces. Specifically, the hyperplane arrangement $H_\infty \cup H_\Delta$ is the analogue of the hyperplane arrangement corresponding to the $(-2)$-curves in the K3 case (see [22.2]). As in the case low degree K3 surfaces, the two components $H_\Delta$ and $H_\infty$ are distinguished by arithmetic properties and they parametrize mildly singular and respectively degenerate (in terms of polarization) varieties. Finally, again completely analogous to the K3 situation (see [30] for the degree 2 case), a stronger result holds: the GIT compactification $\overline{M}$ of the moduli of cubic fourfolds is an explicit birational modification of the Baily-Borel compactification $(D/\Gamma)^*$ of the period space.

Theorem 1.2. The period map for cubic fourfolds induces an isomorphism

$$\overline{M} \cong D/\Gamma,$$

1
where $\overline{M}$ is the GIT compactification of moduli space of cubic fourfolds, and $\overline{D}/\Gamma$ denotes the Looijenga’s compactification associated to the arrangement of hyperplanes $H_\infty$.

Theorem 1.2 follows immediately from Theorem 1.1 and the general results of Looijenga (esp. [31, Thm. 7.6]). Thus, we are mainly concerned here with establishing Theorem 1.1. For this we take an incremental approach, following the arguments of Shah [43] on the surjectivity of the period map for degree two K3 surface. We start by computing the GIT compactification $\overline{M}$ of the moduli space of cubic fourfolds. By studying the monodromy around cubic fourfolds with closed orbits, we conclude that the indeterminacy of the period map is a curve $\chi$. Next, we successively blow-up $\overline{M}$ first in a special point $\omega \in \chi$, and then in the strict transform of $\chi$. A new monodromy analysis for the blow-up, allows us to conclude that Theorem 1.1 holds. Further details on the organization of the paper and the main intermediary results are given below.

The GIT computation for the moduli space of cubic fourfolds was done in Laza [27] (for some partial results see also Yokoyama [50]). The relevant details from [27] are given in section 2. Basically, what we need from the GIT computation are the following results. First, a cubic fourfold having at worst simple isolated singularities, called of Type I, is GIT stable. In particular, we can talk about the moduli space $M$ of such cubic fourfolds; this is the natural space where the period map for cubic fourfolds extends. Next, the boundary of $M$ in the GIT compactification $\overline{M}$ is naturally stratified in 3 types, labeled II, III, and IV. This stratification is closely related to the stratification of Shah [43, Thm. 2.4]: all the singularities that occur for Type I–III fourfolds are double suspensions of Shah’s insignificant singularities ([42]). In particular, it follows that the singularities allowed for Type I–III are very mild and cause no problem for the period map. On the other hand, the Type IV fourfolds have “cohomologically significant” singularities. This produces singularities for the period map along the locus parametrizing Type IV fourfolds, a rational curve $\chi$ containing a special point $\omega$. We note that the Type IV fourfolds play the same role as the triple conic for plane sextics ([43]) or the secant threefold for cubic threefolds ([2, 33]).

Based on the GIT results mentioned above, the proof of the Theorem 1.1 follows in two main steps. The first step, completed in section 3, is to prove that we can control the monodromy of 1-parameter degenerations of cubic fourfolds with central fiber of Type I–III. There are two ingredients for the proof of this statement. First, in §3.1, we prove that the natural specialization morphism associated to the degeneration induces an isomorphism on certain pieces of the corresponding mixed Hodge structure. Thus, the question about the monodromy of the family is reduced to checking some statement about the mixed Hodge structure of the central fiber $X_0$. But then, the mixed Hodge structure of the singular cubic fourfold $X_0$ can be computed by using a projection from a singular point (see §3.2). As already hinted in the previous paragraph, the essential fact that makes the proof work is that the singularities of Type I-III fourfolds are suspensions of special surface singularities, the so-called insignificant limit singularities of Mumford and Shah.

In section 4 we analyze the degenerations to Type IV fourfolds, completing the second step of our proof. An analysis of the generic degenerations was done by Hassett [21, §4.4] (for $\omega$) and Allcock-Carlson-Toledo [2, §5] (for another special point on $\chi$). We need to see that this analysis works also in the non-generic case. For this we proceed as follows. From the monodromy computation of section 3, we know that the curve $\chi$ parametrizing the Type IV fourfolds constitutes the indeterminacy locus of the period map. Some blow-up of an ideal sheaf supported on $\chi$ resolves the indeterminacy of the period map. Thus, the essential question is to explicitly understand this blow-up. Using again the GIT results, we recall that $\overline{M}$ is a GIT quotient and $\chi \subset \overline{M}$ is characterized by the fact that it is the locus of semi-stable cubic fourfolds with the largest possible stabilizer. It is therefore natural to consider the canonical Kirwan partial desingularization ([24]) of $\overline{M}$ along $\chi$. This consists of blowing-up the special point $\omega \in \chi$, followed by the blow-up of the strict transform of $\chi$ in a natural way (see [41]). Over the resulting space $M \to \overline{M}$ the period
map essentially extends. Namely, we show that by doing these two blow-ups we replace the Type IV fourfolds by some fourfolds, that we call of Type I'–III', having the same type of singularities as the cubic fourfolds of Type I–III. The main point of this procedure being that due to the canonicity of the Kirwan’s procedure we can interpret geometrically the exceptional divisors of $\overline{M} \to \overline{M}$ (see §4.2). We conclude by controlling the monodromy for 1-parameter degenerations to Type I'–III' as in section 3. This is enough to complete the proof of Theorem 1.1.

In the last section, we put everything together and prove the two main theorems 1.1 and 1.2. As a simple application, we note that from Theorem 1.2 and the results of Looijenga [31] we can recover some information about the GIT compactification purely in arithmetic terms (see §5.3).

The Fano variety of lines on a cubic fourfold $X$ is an irreducible symplectic fourfold $F$ deformation equivalent to the Hilb$^2(S)$ for $S$ a K3 surface ([6]). The periods of $X$ are essentially the periods of $F$. For irreducible symplectic fourfolds the surjectivity of the period map is known. However, the linear systems on symplectic fourfolds are not well enough understood (see [22]) to make possible a characterization of the image of the period map for cubic fourfolds starting from the Fano variety. On the other hand we note that locus $H_\infty/\Gamma$ excluded from the image of the period map for cubic fourfolds corresponds precisely to irreducible symplectic fourfolds of type Hilb$^2(S)$ for $S$ a degree two K3 surface. These type of fourfolds are not Fano varieties of cubic fourfolds, but rather of certain quadric bundles associated to degree two K3 surfaces (see §4.2). The Kirwan blow-up enlarges the moduli space of cubic fourfolds $\overline{M}$ to include these quadric bundles. The first blow-up corresponds to generic degree two K3 surface, and the second to the special (i.e. unigonal or elliptic) ones (see §1.1). In conclusion, we essentially obtain surjectivity for the period map for cubic fourfolds compatible with the surjectivity for irreducible symplectic fourfolds.

Eduard Looijenga [28] obtains results similar to our main results by a different method based on the techniques developed in [31, 32]. We would like to thank him for informing us about his work.

1.1. Notations and Conventions. Our notations and terminology are based on Mumford [36] when we refer to GIT, on Arnold et. al. [3] when we refer to singularities, and on Griffiths et al. [17] when we refer to Hodge Theory. Additionally, we are using freely the notations of Laza [27] (esp. [27, §1.3]). In particular, $M_0$, $M$, and $\overline{M}$ denote the moduli space of smooth cubic fourfolds, of cubic fourfolds with simple (A-D-E) singularities, and the GIT compactification respectively.

2. Preliminary results

In this section we collect a series of results that we will need in the subsequent sections. Some are well known general results, others are specific to cubic fourfolds and are based mostly on Voisin [48], Hassett [21], and Laza [27].

2.1. The GIT compactification of the moduli space of cubic fourfolds. The computation of GIT compactification $\overline{M}$ of the moduli space of smooth cubic fourfolds $M_0$ was carried out in [27]. What we need from [27] are following three results:

(GIT 1) A cubic fourfold with at worst simple singularities is GIT stable (cf. [27, Thm. 1.1]). We call such a fourfold a Type I cubic fourfold, and denote by $M$ their moduli space.

(GIT 2) The boundary $\overline{M} \setminus M$ is naturally stratified in 3 types (II, III, and IV). The type II and III fourfolds have mild singularities. The stratification is explained below, and the allowed singularities are defined in §2.1.

(GIT 3) We have a good understanding of the closed orbits corresponding to Type IV. In particular, the fact that they have a large stabilizer plays an important role.

According to [27, Thm 1.2], the boundary of $M$ in $\overline{M}$ consists of 6 irreducible, closed components, that we labeled $\alpha$–$\phi$. The boundary $\overline{M} \setminus \overline{M}$ contains the following degeneraci loci (see table):
σ, τ, χ, ω, and ζ (by convention each stratum α, . . . , ω is considered closed by taking the closure in \( \overline{M} \)). We also note that inclusions ζ \( \in \) τ \( \subset \) σ, and ω \( \in \) χ \( \subset \) σ. With this notation, the stratification of the boundary \( \overline{M} \setminus M \) is given by:

(Type II) the open stratum \((α \cup \ldots \cup φ) \setminus σ\), which has the property that the corresponding minimal orbits are of Type II in the sense of definition 2.1 below;

(Type III) the surface \( σ \setminus χ \) (including the special cases \( ζ \in τ \subset σ \));

(Type IV) the curve \( χ \) (including the special case \( ω \)).

| Component | Excludes | Singularities | Dim. | Baily-Borel component |
|-----------|----------|---------------|------|-----------------------|
| α         | ζ        | enc of degree 4, and rnc of degree 1 | 1    | \( A_{11} \oplus D_7 \) |
| β         | σ        | 2 isolated \( E_8 \) | 3    | \( E_8^{\otimes 2} \oplus A_2 \) |
| γ         | τ        | \( E_7 \), and enc of degree 2 | 2    | \( E_7 \oplus D_{10} \) |
| δ         | ζ        | 3 isolated \( E_6 \) | 1    | \( E_6^{\otimes 3} \) |
| ε         | σ        | rnc of degree 4 | 3    | \( D_{16} \) |
| φ         | τ        | enc of degree 6 | 2    | \( A_{17} \) |

Table 1. The Type II boundary stratum of \( \overline{M} \)

Definition 2.1. We say that a cubic fourfold \( X \) is of type II if it is not of Type I, but all the singularities of \( X \) are of one of the following types:

(0) isolated singularities of type \( A_n, D_m \), or \( E_r \) (for \( r = 6, 7, 8 \));

(1) isolated singularities of type \( \widetilde{E}_r \) (for \( r = 6, 7, 8 \));

(2) non-isolated singularities of type \( A_\infty \);

(3) non-isolated singularities of type \( D_\infty \).

Remark 2.2. A Type II fourfold has naturally associated an elliptic curve, that we call the elliptic tail. Namely, the singular locus of a Type II fourfold can be as follows:

(1) The cubic fourfold has an isolated singularity of type \( \widetilde{E}_r \). The blow-up of this singularity produces a fourfold singular along an elliptic curve \( C \).

(2) The cubic fourfold is singular along an elliptic normal curve \( C \), and the singularities along \( C \) are of type \( A_\infty \).

(3) The cubic fourfold is singular along a rational normal curve \( C \) with \( A_\infty \) singularities at all but 4 distinct points on \( C \), where the singularity is of type \( D_\infty \).

The curve \( C \) (or its double cover) is called elliptic tail. A Type II fourfold can have several singularities as above, but the \( j \)-invariants of the corresponding elliptic curves are equal.

A type III fourfold is a degeneration of a Type II fourfold. Roughly, the elliptic tail introduced in the previous remark becomes singular, but only with ordinary double points singularities. For example, a generic Type III fourfold with closed orbit is singular along a rational normal curve of degree 4 with 2 special points. The most degenerate case of a cubic fourfold of type III is the case \( ζ \), corresponding to the closed orbit:

(2.3) \( ζ : g(x_0, \ldots, x_5) = x_0 x_4 x_5 + x_1 x_2 x_3 \).

The fourfold of type \( ζ \) is singular along 9 lines, meeting in triples. At the intersection point of 3 singular lines, the singularity is of type \( T_{\infty, \infty, \infty} \).
In the study of degenerations of K3 surfaces, Shah has observed that the following 6 types of singularities play a particular role both in terms of the GIT stability and of the topology of 1-parameter degenerations.

**Definition 2.4.** We say that a hypersurface \((X, 0) \subset (\mathbb{C}^3, 0)\) has a singularity of type \((t1\text{-}t6)\) at the origin if its defining equation is one of the following:

- \((t1)\): the equation of an isolated rational double point, i.e. \(A_n, D_m,\) or \(E_r;\)
- \((t2)\): \(x_2x_3,\) i.e. double line or \(A_{\infty};\)
- \((t3)\): \(x_2^2 + x_3^2,\) i.e. ordinary pinch point or \(D_{\infty};\)
- \((t4)\): \(x_2^2 + (x_1 + a_1x_2^2)(x_1 + a_2x_2^2)(x_1 + a_3x_2^2) + g(x_1, x_2)\) with ord\(g > 6\) with respect to the weights 2 and 1 for \(x_1\) and \(x_2\) respectively, and such that at least two of the \(a_i\) are distinct;
- \((t5)\): \(x_2^2 + f_4(x_1, x_2) + g(x_1, x_2)\) with \(f_4\) a homogeneous polynomial of degree 4 and ord\(g > 4,\) and such that \(f_4\) has no triple root;
- \((t6)\): \(f_3(x_1, x_2, x_3) + g(x_1, x_2, x_3)\) with \(f_3\) a homogeneous polynomial of degree 3 and ord\(g > 4,\) and such that \(f_3\) has at worst ordinary double points as singularities.

where \(g\) is a convergent power series in the appropriate variables.

**Remark 2.5.** The singularities occurring in Shah’s list are precisely the 2-dimensional semi-log-canonical hypersurface singularities (cf. \([3, \text{Ch. 15}]\) and \([25, \text{Thm. 4.21}]\)). The relevance of this fact is that for surfaces three different concepts of measuring the complexity of a singularity are almost equivalent: semi-log-canonical, cohomological insignificant \((11, 46)\), and GIT stable \((35, \S 3, 20, \S 10)\). This gives a conceptual explanation of the close relation between the GIT and Hodge theoretical construction of the moduli of low degree K3 surfaces.

As mentioned in the introduction, a key fact for us is that the singularities occurring for cubic fourfolds of Type I–III are precisely the suspensions of the singularities occurring in Shah’s list.

**Lemma 2.6.** Let \(X_0\) be a semi-stable cubic fourfold with minimal orbit of Type I–III, and \(p \in X_0\) a singular point. Then we can choose local analytic coordinates \(x_1, \ldots, x_5\) at \(p\) such that the equation of \(X_0\) is given by

\[
f(x_1, x_2, x_3) + x_4^2 + x_5^2
\]

with \(f\) defining a surface singularity of type \((t1\text{-}t6).\) \hfill \(\square\)

Finally, the Type IV cubic fourfolds are parameterized by a rational curve \(\chi\), containing a special point \(\omega\). A fourfold with closed orbit parametrized by \(\chi \setminus \omega\) is singular along a rational normal curve with \(A_2\) transversal singularity. The degenerate case \(\omega\) corresponds to the secant variety of the Veronese surface in \(\mathbb{P}^5\). In particular, the singular locus in this case is a surface. We note additionally that the fourfolds with closed orbit of type IV have large stabilizers: \(\text{SL}(2)\) for the generic case, and \(\text{SL}(3)\) for \(\omega\). The relevance of Type IV fourfolds comes from the fact that they lie in the indeterminacy locus of the period map. The degenerations to \(\omega\) are studied in Hassett \([21, \S 4.4]\), and the point of \(\chi\) with extra \(\mu_3\)-automorphism occurs in Alcock-Carlson-Toledo \([2]\).

### 2.2. The period map for cubic fourfolds.

The period map for cubic fourfolds is defined by sending a smooth cubic fourfold \(X\) to its periods:

\[
\mathcal{P}_0 : \mathcal{M}_0 \to \mathcal{D}/\Gamma.
\]

The space \(\mathcal{D}\) is the classifying space of polarized Hodge structures on the middle cohomology of a cubic fourfold, and \(\Gamma\) is the monodromy group.

**Notation 2.7.** We denote \(\Lambda := \langle 1 \rangle^\oplus 21 \oplus \langle -1 \rangle^\oplus 21\) the abstract lattice isometric to the integral cohomology of a cubic fourfold, by \(h \in \Lambda\) the polarization class (the square of the class of a hyperplane section), and by \(\Lambda_0 := \langle h \rangle^\perp \cong E_8^\oplus 2 \oplus U^\oplus 2 \oplus A_2\) the primitive cohomology.
The Hodge numbers of a cubic fourfold are $h^{4,0} = h^{0,4} = 0$, $h^{3,1} = h^{1,3} = 1$, and $h^{2,2} = 21$. Thus, the classifying space of the Hodge structures for cubic fourfolds is a 20-dimensional Type IV bounded symmetric domain $\mathcal{D} \cong \text{SO}_0(20,2)/\text{SO}(20) \times \text{SO}(2)$. We regard $\mathcal{D}$ as the space of lines in $\Lambda_0 \otimes \mathbb{C}$ satisfying the two Riemann-Hodge bilinear relations:

$$\mathcal{D} = \{ \omega \in \mathbb{P}(\Lambda_0 \otimes \mathbb{C}) \mid \omega^2 = 0, \ \omega \bar{\omega} > 0 \}.$$  

A result of Beauville [5 Thm. 2] identifies the monodromy group $\Gamma$ with the group of automorphisms $O^+_h(\Lambda)$ of the lattice $\Lambda$ which preserve the orientation of a negative definite 2-plane and the polarization class $h$. There is a natural morphism $O^+_h(\Lambda) \rightarrow O(\Lambda_0)$, whose image is $O^*(\Lambda_0)$, the automorphisms of the lattice $\Lambda$ preserving the orientation and discriminant group $A_{\Lambda_0} := (\Lambda_0)^*/\Lambda_0$. In what follows, we identify the $\Gamma$ with $O^*(\Lambda_0)$. In particular, the action of $\Gamma$ on $\mathcal{D}$ is induced by the action of $O^*(\Lambda_0)$ on $\Lambda_0$.

It is a simple general fact that the period map for cubic threefolds is a local isomorphism. In fact, the much stronger global Torelli theorem holds in this situation by a result of Voisin [48]. It follows that the period map $\mathcal{P}_0$ is a birational isomorphism between the quasi-projective varieties $\mathcal{M}_0$ and $\mathcal{D}/\Gamma$. For us it is more convenient to regard $\mathcal{P}_0$ as a birational isomorphism between the compactification $\overline{\mathcal{M}}$ of $\mathcal{M}_0$ and the Baily-Borel compactification $(\mathcal{D}/\Gamma)^*$ of $\mathcal{D}/\Gamma$.

For completeness, we note that the boundary of the Baily-Borel compactification $(\mathcal{D}/\Gamma)^*$ for cubic fourfolds consists of 6 Type II boundary components (labeled by root lattices), and a single Type III boundary component (see [27 Thm. 1.3]). In a weak sense (explained in [27 §4]) we can match the Type II GIT boundary components to the Type II Baily-Borel boundary components as given in table [1]. A more precise relation we will become clear in [5.3].

2.2.1. Special loci inside the period domain. Hassett [21] has studied the period map of cubic fourfolds in relation to the rationality question. He introduced the following notion:

**Definition 2.8.** Fix a lattice $\Lambda = (1) \oplus 21 \oplus (-1) \oplus 21$ and an element $h \in \Lambda$ of square 3. Let $\Lambda_0 = \langle h^2 \rangle \Lambda$, and $\mathcal{D}$ and $\Gamma$ as above. For a rank 2 lattice $M \looparrowleft \Lambda$ primitively embedded in $\Lambda$ with $h \in M$ define the hyperplane $\mathcal{D}_M$ by

$$\mathcal{D}_M = \{ \omega \in \mathcal{D} \mid \omega \perp M \}.$$  

We say $\mathcal{D}_M$ is a hyperplane of determinant $d = \det(M)$. The union of all the hyperplanes of a given determinant form an arithmetic arrangement of hyperplanes. We denote the arrangement of hyperplanes 2 and 6 by $\mathcal{H}_\infty$ and $\mathcal{H}_\Delta$ respectively.

The following result is contained in Hassett [21 Prop. 3.2.2, Prop. 3.2.4].

**Lemma 2.9.** With notations as in Def. 2.8, the following holds:

i) $\mathcal{D}_M$ is non-empty if and only if $\det(M) \equiv 0, 2 \mod 6$;

ii) if $M$ and $M'$ have the same determinant, then $\mathcal{D}_M$ and $\mathcal{D}_{M'}$ are conjugated by $\Gamma$.

Hassett [21 §4.4] (see also [48 pg. 596, Prop. 1]) noticed that the image of the period map is contained in the complement of the hyperplane arrangement $\mathcal{H}_\Delta \cup \mathcal{H}_\infty$.

**Proposition 2.10.** Let $X$ be a smooth cubic fourfold. Then, $\Pi^1_0(X, \mathbb{Z}) \cap H^{2,2}(X)$ can not contain a primitive class $\delta$ such that either

i) $\delta^2 = 2$,

ii) or $\delta^2 = 6$ and $\delta \cdot x \equiv 0 \mod 3$ for all $x \in \Pi^1_0(X, \mathbb{Z})$.

The hyperplanes orthogonal to $\delta$ are hyperplanes of determinant 6 and 2 respectively. Consequently, the image of the period map satisfies: $\text{Im}(\mathcal{P}_0) \subset (\mathcal{D} \setminus (\mathcal{H}_\infty \cup \mathcal{H}_\Delta))/\Gamma$. $\Box$
Remark 2.11. We call \( \delta \in \Lambda_0 \) with \( \delta^2 = 2 \) a root, and \( \delta \in \Lambda_0 \) with \( \delta^2 = 6 \) such that \( \delta \cdot x \equiv 0 \mod 3 \) for all \( x \in \Lambda_0 \) a generalized root. We note the following facts:

i) The monodromy group \( \Gamma \cong O^*(\Lambda_0) \) is generated by reflections \( s_\delta \) in the roots of \( \Lambda_0 \) (see \([5\, pg.9-10]\)) The reflection hyperplanes constitute the arrangement \( H_\Delta \).

ii) The reflection in a generalized root \( s_\delta(x) = x - 2\delta \langle x, \delta \rangle \) defines an automorphism of \( \Lambda_0 \). In fact, \( s_\delta \in O^+(\Lambda_0) \), but \( s_\delta \not\in \Gamma \). The group \( O^+(\Lambda_0) \) is generated by reflections in roots and generalized roots. We have

\[
O^+(\Lambda_0) \cong \Gamma \rtimes \mathbb{Z}/2\mathbb{Z},
\]

where \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \Gamma \) by a reflection in a generalized root. The reflection hyperplanes in generalized roots give the arrangement \( H_\infty \).

2.2.2. The discriminant automorphic form. We note that the moduli space of cubic fourfolds constructed via the period map behaves similarly to the moduli space of degree two K3 surfaces. The basic observation (used also in \([27, \S 4]\)) is that from arithmetic/combinatorial point of view the cubic fourfolds and degree two K3 surfaces correspond to the \( E_6 \) and \( E_7 \) lattices respectively.

Notation 2.12. Let \( \Lambda' := \Pi_{26,2} \) the unique even unimodular lattice of signature \((26, 2)\). We denote by \( \mathcal{D}' \) the type IV domains associated to \( \Lambda_0 \) and \( \Lambda' \) respectively.

We note that there exists a unique embedding of \( \Lambda_0 \) in \( \Lambda' \) and \( (\Lambda_0)_{\Lambda'}^{1} \cong E_6 \). We fix the mutual orthogonal embeddings \( \Lambda_0 \leftarrow \Lambda' \) and \( E_6 \leftarrow \Lambda \). In particular, it follows that we have a natural embedding \( \mathcal{D} \leftarrow \mathcal{D}' \). The same construction (modulo the sign) works also for degree two K3 surfaces, just replace \( E_6 \) by \( E_7 \). Following Borcherds et al. \([7]\) we obtain:

Proposition 2.13. There exists an automorphic form \( \Phi \) on \( \mathcal{D} \) such that

i) the weight of \( \Phi \) is 48;

ii) the vanishing locus of \( \Phi \) is \( H_\infty \cup H_\Delta \);

iii) \( \phi \) vanishes of order 1 along \( H_\Delta \) and of order 37 along \( H_\infty \).

We call \( \Phi \) the discriminant automorphic form.

Proof. The situation is formally the same for degree two K3 surface. We follow \([7\, Thm. 1.2, Ex. 2.1]\). On \( \mathcal{D}' \) there exists an automorphic form \( \Phi_{12} \) vanishing exactly along the hyperplanes \( H_\delta \), where \( \delta \) is a root of \( \Lambda' \). By restricting \( \Phi_{12} \) to \( \mathcal{D} \) and dividing by the product of the linear forms which vanish along the hyperplanes \( H_\delta \) containing \( \mathcal{D} \), we obtain an automorphic form \( \Phi \) on \( \mathcal{D} \). The weight of \( \Phi \) is the weight of \( \Phi_{12} \) plus half the number of roots of \( E_6 \).

The vanishing locus of \( \Phi \) is the union of hyperplanes \( \mathcal{H} \) which are the restrictions to \( \mathcal{D} \) of hyperplanes of type \( H_\delta \) for \( \delta \) a root of \( \Lambda' \). The condition that \( H_\delta \cap \mathcal{D} \neq \emptyset \) gives that \( (\Lambda_0)_{\Lambda'}^{1}, \delta \) is positive definite. Since \( (\Lambda_0)_{\Lambda'}^{1} \cong E_6 \), we obtain the following two possibilities:

1) either \( \delta \perp E_6 \), in which case \( (\Lambda_0)_{\Lambda'}^{1}, \delta \cong E_6 \oplus A_1 \),
2) or \( \delta \not\perp E_6 \), in which case \( (\Lambda_0)_{\Lambda'}^{1}, \delta \cong E_7 \).

In the first case \( H_\delta \) restricted to \( \mathcal{D} \) is a hyperplane of determinant 6. Similarly, the second case gives a hyperplane of determinant 2. Conversely, each hyperplane of determinant 2 or 6 is of type (1) or (2) respectively. The order of vanishing is computed as in \([7\, Ex. 2.1]\).
Definition 2.14. A 1-parameter degeneration \( f : \mathcal{X} \to \Delta \) is a proper analytic map with smooth generic fiber (where \( \mathcal{X} \) is an analytic variety and \( \Delta \) is the unit disk). We denote by \( X_0 \) the special fiber, by \( X_t \) the generic fiber, and by \( X_\infty = \mathcal{X}|_{\Delta^*} \times \Delta \cdot \mathfrak{h} \) the canonical fiber, where \( \mathfrak{h} \to \Delta^* \) is the universal cover of the punctured disk. We also say that \( f \) is a 1-parameter smoothing of \( X_0 \).

2.3.1. Limit Mixed Hodge Structures. To a 1-parameter degeneration there is associated a canonical limit mixed Hodge structure on \( H^n_{\lim} := H^n(X_\infty, \mathbb{Q}) \) by Schmid and Steenbrink. To fix the notation and terminology we recall a few general facts (for a survey see [34]). The monodromy \( T \) of the family over the punctured disk is quasi-unipotent. After a ramified base change of type \( t \to t' \) one can further assume that \( T \) is actually unipotent. We will assume this in what follows. We fix \( n \) to be the dimension of the fiber.

Let \( N = \log T \) be the logarithm of the monodromy. It is a nilpotent endomorphism acting on the cohomology \( H^n_{\lim} \) with index of nilpotence \( \nu \) (i.e. \( N^{\nu} = 0 \), but \( N^{\nu-1} \neq 0 \)). The weight filtration \( W_k \) on \( H^n_{\lim} \) is determined by \( N \) (see [34] pg. 106-107), and in particular we have

\[
\nu = \max \{k \mid \text{Gr}^{W}_{n-k}H^n_{\lim} \neq 0\} + 1.
\]

The limit mixed Hodge structure has built-in symmetry:

i) \( \text{Gr}^W_k(H^n_{\lim}) \) carries a pure Hodge structure of weight \( k \);

ii) \( N^k : \text{Gr}^W_{n+k}(H^n_{\lim}) \to \text{Gr}^W_{n-k}(H^n_{\lim}) \) is an isomorphism of Hodge structures of type \((-k, -k)\).

Additionally, the possibilities for the limit mixed Hodge structure are restricted by the Hodge structure of the general fiber \( X_t \). Namely, \( H^n_{\lim} \) is isomorphic as vector space to \( H^n(X_t) \) and

iii) \( \dim_{\mathbb{C}} F^pH^n_{\lim} = \dim_{\mathbb{C}} F^pH^n(X_t) \) ([39] Cor. 11.25)).

2.3.2. The case of cubic fourfolds. We specialize the above discussion to the case of degeneration of cubic fourfolds. First, since the Hodge structure on the middle cohomology of a smooth cubic fourfold is of level 2 (i.e. \( H^{p,q} = 0 \) if \( |p - q| > 2 \)), it follows that the index of nilpotence of \( N \) is at most 3. Similarly to the case of K3 surfaces, based on the nilpotence index we define 3 types of degenerations.

Definition 2.15. Let \( f : \mathcal{X} \to \Delta \) be a 1-parameter degeneration of cubic fourfolds. Assume that the monodromy \( T \) is unipotent. We say that \( f \) is a Type I (II, or III) degeneration if the index of nilpotence \( \nu \) of \( N = \log T \) is 1 (2, or 3 respectively).

A simple fact that we use is that the type of a degeneration is determined by the non-vanishing of a graded piece of \( H^4_{\lim} \).

Lemma 2.16. Let \( f : \mathcal{X} \to \Delta \) be a 1-parameter degeneration of cubic fourfolds. Then \( f \) is a Type II degeneration if and only if \( \text{Gr}^W_3H^4_{\lim} \neq 0 \). In this case \( \text{Gr}^W_3H^4_{\lim} \) is a Tate twist of the Hodge structure of an elliptic curve \( E \), i.e. \( \text{Gr}^W_3H^4_{\lim} \cong H^1(E)(-1) \). Similarly, \( f \) is a Type III degeneration if and only if \( \text{Gr}^W_2H^4_{\lim} \neq 0 \), in which case \( \text{Gr}^W_2H^4_{\lim} \) is a trivial 1-dimensional Hodge structure of weight 2.

Proof. Since \( X_t \) is a smooth cubic fourfold, we have \( \dim_{\mathbb{C}} F^3H^4(X_t) = 0 \) and \( \dim_{\mathbb{C}} F^3H^4(X_t) = 1 \). Thus, the only possibly non-zero Hodge numbers in weight at most 3 are \( h^{2,1} \), \( h^{1,2} \), and \( h^{1,1} \).

Additionally, they satisfy (by item iii above) \( h^{3,1} + h^{2,1} + h^{1,1} = 1 \). The claim follows. \( \square \)

We note that the type of a degeneration is closely related to the Baily-Borel compactification as follows. A 1-parameter degeneration of cubic fourfolds gives a variation of Hodge structures over the punctured disk. This induces a period map \( g : \Delta^* \to \mathcal{D}/\Gamma \), which then extends to a holomorphic map \( \Delta \to (\mathcal{D}/\Gamma)^* \). For Type I degenerations the limit point \( \lim_{z \to \infty} g(z) \) belongs to the interior \( \mathcal{D}/\Gamma \). In contrast, the limit point of a Type II (Type III) degeneration belongs to a Type II (Type III respectively) boundary component.
2.3.3. The specialization morphism. The formalism of vanishing cycles ([9, Exp. XIII]) relates the limit mixed Hodge structure of a smoothing to the mixed Hodge structure of the central fiber via the specialization morphism:

$$s_{p_n} : H^n(X_0) \rightarrow H^n_{\text{lim}}.$$  

We recall the basic construction as needed in our situation.

First, from the specialization diagram:

$$\begin{array}{ccc}
X_0 & \xrightarrow{i} & X \\
\downarrow f & & \downarrow \pi \\
\Delta & \xleftarrow{\Delta^*} & \downarrow h
\end{array}$$

we define the functor of nearby cycles \( \psi_f : D^b_c(X) \rightarrow D^b_c(X_0) \) by

$$\psi_f F^\bullet := i^* R\pi_* \pi^* F^\bullet.$$  

There exists a natural comparison map \( i^* F^\bullet \xrightarrow{c} \psi_f F^\bullet \), and the specialization morphism \( s_{p_n} \) is defined to be the cohomology map associated to \( c \). The functor of vanishing cycles \( \phi_f \) is the cone over the morphism \( c \). By definition there exists a distinguished triangle:

$$\begin{array}{ccc}
i^* F^\bullet & \xrightarrow{c} & \psi_f F^\bullet \\
\downarrow \text{can} & & \downarrow \phi_f F^\bullet
\end{array}$$

in derived category \( D^b_c(X_0) \) (see [10 §4.2]).

We are interested in the situation when \( F^\bullet \) is the constant sheaf \( \mathbb{C}_X \). Taking the hypercohomology associated to (2.18), we obtain a long exact sequence relating the cohomology of the central fiber with the cohomology of the canonical fiber \( X_\infty \):

$$\begin{array}{c}
\cdots \rightarrow H^n(X_0) \xrightarrow{s_{p_n}} H^n_{\text{lim}} \rightarrow H^n(\phi_f \mathbb{C}_X) \rightarrow H^{n+1}(X_0) \rightarrow \cdots
\end{array}$$

Furthermore, the vanishing cohomology \( H^n(\phi_f \mathbb{C}_X) \) can be endowed with a natural mixed Hodge structure making [2.19] an exact sequence of mixed Hodge structures. Since a morphism of mixed Hodge structures is strict with respect to both the weight and Hodge filtration, \( s_{p_n} \) maps \( H^{p,q}(X_0) \) to \( H^{p,q}_{\text{lim}} \) (where \( H^{p,q} = Gr^W_p Gr^V_{p+q} H \) for a mixed Hodge structure \( H \)). In particular, the statement \( s_{p_n} \) is an isomorphism on the \((p,q)\) components is well-defined.

In our situation, \( X_t \) is either a cubic fourfold or a K3 surface, and \( n \) represents the dimension of \( X_t \). In particular, there is no odd cohomology. Thus, we have:

$$H^{n-1}(X_t) = H^{n+1}(X_t) = 0,$$

It follows that (2.19) reduces to a five-term exact sequence:

$$0 \rightarrow H^{n-1}(\phi_f \mathbb{C}_X) \rightarrow H^n(X_0) \xrightarrow{s_{p_n}} H^n_{\text{lim}} \rightarrow H^n(\phi_f \mathbb{C}_X) \rightarrow H^{n+1}(X_0) \rightarrow 0.$$  

To compute the vanishing cohomology we note the following spectral sequence:

$$E_2^{p,q} = H^p(X_0, \mathcal{H}^q(\phi_f \mathbb{C}_X)) \Rightarrow H^{p+q}(\phi_f \mathbb{C}_X)$$

Furthermore, the stalk of the cohomology sheaf \( \mathcal{H}^q \) is the reduced cohomology of the Milnor fiber:

$$\mathcal{H}^q(\phi_f \mathbb{C}_X)_x = \overline{H}^q(F_x; \mathbb{C}),$$

where as usually \( F_x \) is the intersection of the generic nearby fiber with a small open ball centered at \( x \). In particular, \( \mathcal{H}^q(\phi_f \mathbb{C}_X) \) are supported on the singular locus of \( X_0 \) ([10 Prop. 4.2.8]). The total space \( \mathcal{X} \) of the degeneration might be singular, but in our situation (degeneration of hypersurfaces)
it has at worst local complete intersection singularities. Therefore, the range for which there exists non-vanishing cohomology for the Milnor fiber is the same as in the smooth case \([10\text{ Prop. } 6.1.2]\).

In particular, in the case that the special fiber \(X_0\) has at worst isolated singularities, one obtains that the specialization morphism \(sp_n\) is injective \((11\text{ Prop. } 2.7, 10\text{ 6.2.2, 6.2.4})\) and an exact sequence of mixed Hodge structures:

\[(2.24) \quad 0 \rightarrow H^n(X_0) \xrightarrow{sp_n} H^n_{\lim} \rightarrow \bigoplus_{x_i \in \text{Sing}(X_0)} H^n(x_i) \rightarrow H^{n+1}(X_0) \rightarrow 0\]

where \(H^n(x_i)\) is the cohomology of the Milnor fiber at \(x_i\) endowed with the mixed Hodge structure of Steenbrink [45]. Similarly, if the central fiber has 1-dimensional locus the first part of the exact sequence \((2.21)\) reads:

\[(2.25) \quad 0 \rightarrow H^0(X_0, \mathcal{H}^{n-1}(\phi_f \mathcal{L}_x)) \rightarrow H^n(X_0) \xrightarrow{sp_n} H^n_{\lim} \rightarrow \ldots\]

3. **The monodromy around semistable cubic fourfolds**

The main result of this section is the control of the monodromy for degenerations of cubic fourfolds which are not of Type IV.

**Theorem 3.1.** Let \(f : X \rightarrow \Delta\) be a 1-parameter smoothing of a semi-stable cubic fourfold \(X_0\) with closed orbit. The following holds:

i) if \(X_0\) has Type I then \(f\) is a Type I degeneration;

ii) if \(X_0\) has Type II then \(f\) is a Type II degeneration;

iii) if \(X_0\) has Type III then \(f\) is a Type III degeneration.

(see \([2,7\text{ and Def. } 2.75]\))

**Proof.** By Lemma \(2.16\), the monodromy of the family is determined by the non-vanishing of \(H^{p,1}_{\lim}\) for \(p = 3, 2,\) or 1. By Prop. \(3.4\), the non-vanishing of \(H^{p,1}(X_0)\) implies the non-vanishing of \(H^{p,1}_{\lim}\).

The claim now follows from the computation of the mixed Hodge structure of the central fiber (Prop. \(3.7\)). \(\square\)

Due to the finiteness of the monodromy, for Type I fourfolds a stronger result holds. Namely, the period map extends over the locus \(\mathcal{M}\) of such fourfolds.

**Proposition 3.2.** The period map for a cubic fourfolds \(\mathcal{P}_0 : \mathcal{M}_0 \rightarrow \mathcal{D}/\Gamma\) extends to a morphism \(\mathcal{P} : \mathcal{M} \rightarrow \mathcal{D}/\Gamma\) over the simple singularity locus \(\mathcal{M} \subset \mathcal{M}_0\). The image of \(\mathcal{M} \setminus \mathcal{M}_0\) under the extended period map \(\mathcal{P}\) is contained in \(\mathcal{H}_{\Delta}/\Gamma\).

**Proof.** Let \(o \in \mathcal{M} \setminus \mathcal{M}_0\) correspond to a cubic fourfold \(X_0\) with simple isolated singularities. The statement is analytically local at \(o\), and stable by finite base changes. Since \(\mathcal{M}\) is a geometric quotient, after shrinking and a possible finite cover, we can assume that a neighborhood of \(o\) in \(\mathcal{M}\) is a 20-dimensional ball \(S\). We can further assume that there exists a family of cubic fourfolds \(\mathcal{X} \rightarrow S\) with at worst simple isolated singularities and fiber \(X_0\) over \(o\). Let \(o \in \Sigma\) be discriminant hypersurface. Over \(S \setminus \Sigma\) the family \(\mathcal{X}\) gives a variation of Hodge structures defining the period map \(\mathcal{P}_0\). By the removable singularity theorem [18 pg. 41] (also [16 Thm. 9.5]), the extension statement is equivalent to the monodromy representation \(\pi_1(S \setminus \Sigma, t) \rightarrow \text{Aut}(H^4(X_t, \mathbb{Z}))\) (for \(t \in S \setminus \Sigma\)) having finite image.

The fourfold \(X_0\) is GIT stable, thus it has finite stabilizer. For cubic fourfolds we have the identity \(n(d-2)-2 = d\), where \(n-1 = 4\) is the dimension and \(d = 3\) the degree. It follows then from [12 Cor. 1.6] (also [13]) that the family \(\mathcal{X}\) gives a simultaneous versal deformation of the singularities of \(X_0\). Thus, the composition of the Kodaira- Spencer map \(T_oS \rightarrow \text{Ext}^1(\Omega^1_{X_0}, \mathcal{O}_{X_0})\) with the natural local-to-global map for \(Ext\) gives a surjection:

\[(3.3) \quad T_oS \rightarrow H^0(X_0, \mathcal{E}xt^1_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0})) \cong \bigoplus_{p \in \text{Sing}(X_0)} T^1_{\text{Sing}(X_0, p)}\]
where $T_{\mathcal{X},p_i}^1$ denotes the tangent space to the miniversal deformation of the hypersurface singularity $(X_0,p_i)$. Let $S_i$ be basis of the miniversal deformation of the singularity at $p_i$, and $\Sigma_i$ the corresponding discriminant. Since $S$ is smooth, after a possible shrinking, we obtain a natural submersive morphism $S \to \prod_i S_i$ such that $\Sigma$ is the pullback of the discriminant divisor in $\prod_i S_i$. Since all the singularities of $X_0$ are of type A-D-E, the structure of $(S_i, \Sigma_i)$ is well understood. In particular, it follows that the local monodromy group at $o$ is a product of (finite) Weyl groups of type A-D-E. Also, after a finite base change, we can assume that the discriminant $\Sigma$ is a normal crossing divisor. The extension statement now follows from the removable singularities theorem. Furthermore, the period point corresponding to $X_0$ is left invariant by the reflections in the vanishing cycles. Thus, it belongs to $\mathcal{H}_\Delta/\Gamma$ (Remark 2.11).

\[\square\]

3.1. Reduction to the central fiber.

**Proposition 3.4.** Let $X_0$ be a semistable cubic fourfold with closed orbit of Type I–III. Let $\mathcal{X} \to \Delta$ be any 1-parameter smoothing of $X_0$, and consider the associated specialization morphism $sp_4 : H^4(X_0) \to H^4_{\lim}$. If $X_0$ has isolated (non-isolated) singularities then $sp_4$ induces an isomorphism (resp. injection) on the $(p,q)$ components of corresponding mixed Hodge structures for all $p$ and $q$ with $p + q \leq 4$ and $(p,q) \neq (2,2)$.

**Proof.** We divide the proof in two cases: either $X_0$ has isolated singularities, or not.

**Case 1 (Isolated singularities):** Assume that $X_0$ has only isolated singularities which are suspensions of the types listed in definition 2.3. From the exact sequence (2.24):

$$0 \to H^4(X_0) \xrightarrow{sp_4} H^4_{\lim} \to \bigoplus_{x_i \in \text{Sing}(X_0)} H^4(X_i) \ldots$$

and the strictness of the morphisms of mixed Hodge structures, we see that it is enough to prove the following claim:

(*) If $Gr^p_F Gr^p_{p+q} H^4(X_i) \neq 0$ then $(p,q) \in \{(2,2),(3,2),(2,3),(3,3)\}$, where $H^4(X_i)$ is the vanishing cohomology of a smoothing of a simple, simple elliptic, or cusp singularity.

We note the following facts about the mixed Hodge structure on the vanishing cohomology:

i) (the local nature) the mixed Hodge structure on $H^4(X_i)$ depends only on the germ of the smoothing $(\mathcal{X},x_i)$ (cf. [17] pg. 560);

ii) (the semicontinuity property) $\dim Gr^p_F H^4(X_i)$ is independent of the smoothing (cf. [47] Cor. 2.6).

From the semicontinuity property, it follows immediately that if (*) holds for a smoothing, then it holds for any smoothing. We therefore check (*) for the standard Milnor fibrations of the singularities $A_n$, $D_m$, $E_r$, $E_r$, and $T_{p,q,r}$ respectively. For these singularities the computation of the mixed Hodge structure on the Milnor fiber is well known (see [26] II.8). In fact, (*) is equivalent to the statement that the spectrum of those singularities (in dimension 4) is included in the interval [1,2]. This settles the case of isolated singularities.

**Case 2 (Non-isolated singularities):** Let $C$ be the 1-dimensional singular locus of $X_0$, and $C_i$ its irreducible components. In our situation, the singularities of $C$ are of type $A_\infty$ at all but a finite number of points. At these special points the singularity is of type $D_\infty$ or degenerate cusp (suspensions of the singularities occurring in Def. 2.4). We denote by $C$ (and $\tilde{C}_i$) the non-special locus. We note two simple facts:

i) $\tilde{C} = \cup_i \tilde{C}_i$ is disconnected (i.e. any point of intersection is special);

ii) $\pi_1(\tilde{C}_i)$ is non-trivial (in fact, either $C_i$ is elliptic and $\tilde{C}_i = C_i$, or $C_i$ is rational and there are either 2 or 4 special points depending on the type of the degeneration).
We recall, the exact sequence \((2.25)\):
\[
0 \rightarrow H^0(X_0, \mathcal{H}^3(\phi_f \mathbb{C}_X)) \rightarrow H^2(X_0) \xrightarrow{\text{sp}_2} H^4_{\text{lim}} \rightarrow \ldots
\]
and that
\[
\mathcal{H}^3(\phi_f \mathbb{C}_X)_x \cong H^3(F_x, \mathbb{C})
\]
where \(F_x\) is the Milnor fiber at \(x\). In particular, the sheaf of vanishing cycles is supported on the 1-dimensional singular locus. The injectivity statement is equivalent to saying that the mixed Hodge structure on the vanishing cohomology satisfies \(\text{Gr}_p^2 H^0(X_0, \mathcal{H}^3(\phi_f \mathbb{C}_X)) = 0\) for \(p < 2\).

The analysis of the vanishing cycles in the case of 1-dimensional singular locus is relatively well understood (see \([44, 26, \text{II.8.10}]\)). Specifically, we stratify \(C = \Sigma_1 \cup \Sigma_0\) such that the vanishing cycles form a local system over \(\Sigma_1\) (for us \(\Sigma_1 = \bigcup \tilde{C}_i\), and \(\Sigma_0\) are the special points). Since, the transversal singularity, given by the natural action of \(\pi_1(\tilde{C}_i)\) (see \([44, \text{Ch. 3}]\)), is non-trivial on each component, there are no non-zero sections of the local system over \(\Sigma_1\). Thus, the sections of \(\mathcal{H}^3(\phi_f \mathbb{C}_X)\) are supported on the special points. In the type II case we only have special points of type \(D_\infty\), but then \(H^3(F_x) = 0\) (\([44, \text{pg. 5}]\)). Thus, \(H^0(X_0, \mathcal{H}^3(\phi_f \mathbb{C}_X)) = 0\) for the Type II case. For the Type III case, we note that the situation is local around the special points. Via the generalized Thom-Sebastiani theorem \([41, 19]\) the computation of the vanishing cohomology is reduced to the surface case, but since the singularities are restricted to the list given in \(2.3\) we obtain \(\text{Gr}_p^2 H^0(X_0, \mathcal{H}^3(\phi_f \mathbb{C}_X)) = 0\) for \(p < 2\) (see \([42, \text{Thm. 2}]\)).

\begin{remark}
We note that the analogue of \((*)\) in the surface case is equivalent to the condition that spectrum is included in \([0, 1]\). This in turn is equivalent to the singularities being log canonical. In conclusion, the isolated singularities for which Prop. \(3.4\) holds are precisely the simple, simple-elliptic, and cusp singularities (compare \([12, \text{Thm. 1.2}], [11, \text{Thm. 4.13}]\)).
\end{remark}

3.2. The Mixed Hodge Structure of Type I–III fourfolds. The middle cohomology of a smooth cubic fourfold carries a pure weight 4 Hodge structure. If the cubic fourfold is singular, it becomes rational; it is birational to \(\mathbb{P}^4\) via the projection map from any singular point. This allows us to reduce the computation of the mixed Hodge structure in the singular case to a standard computation for degenerations of K3 surfaces. Consequently, we obtain the following result, concluding the proof of Theorem \(3.1\).

\begin{proposition}
Let \(X_0\) be a semistable cubic fourfold with closed orbit. Then
\begin{enumerate}[i)]
\item if \(X_0\) is of Type I then \(H^4(X_0)\) is a pure Hodge structure of weight 4 with \(H^{3,1}(X_0) \neq 0\);
\item if \(X_0\) is of Type II then \(\text{Gr}_2^W H^4(X_0) \neq 0\);
\item if \(X_0\) is of Type III then \(\text{Gr}_2^W H^4(X_0) \neq 0\).
\end{enumerate}
\end{proposition}

\begin{proof}
If \(X_0\) is smooth there is nothing to prove. Assume that \(X_0\) is singular. Let \(p \in \text{Sing}(X_0)\) and \(\pi_p : X_0 \rightarrow \mathbb{P}^4\) be the projection map. In \(3.2.1\) (esp. Cor. \(3.13\)), we relate the Hodge structure of \(X_0\) to the Hodge structure of the surface \(S_p\), the base locus of \(\pi_p^{-1}\). The surface \(S_p\) is a K3 surface, or a degeneration of K3 surfaces. Thus, the proposition follows from standard facts on degenerations of K3s. The case by case analysis is done in \(3.14, 3.16\) and \(3.17\) for Type I-III respectively.
\end{proof}

3.2.1. The Hodge structure of a singular cubic fourfold. Let \(X_0\) be a singular cubic fourfold (irreducible and reduced). We choose a singular point \(p \in X_0\) and assume additionally:
\[(*)\ corank_p(X_0) \leq 3 \ and \ no \ line \ contained \ in \ \text{Sing}(X_0) \ passes \ through \ p.\]
The linear projection $\pi_p$ with center $p$ gives a birational isomorphism between $X_0$ and $\mathbb{P}^4$. The birational map $\pi_p$ can be resolved by blowing-up the point $p$. The result is the following commutative diagram:

\[
\begin{array}{c}
Q_p \xrightarrow{\pi_p} \tilde{X} \xrightarrow{g} E_p \\
\left\downarrow f \right. \quad \left\downarrow \quad \left. \right\downarrow \quad \left\downarrow \quad \left\downarrow \right. \\
P' \xrightarrow{\pi} X_0 \xrightarrow{\phi} \mathbb{P}^4 \xrightarrow{g} S_p
\end{array}
\]

where $Q_p$ is the projectivized tangent cone at $p$, $S_p \subset \mathbb{P}^4$ is the surface parametrizing the lines of $X_0$ through $p$, and $E_p = g^{-1}(S_p)$ is the exceptional divisor of $g$.

The following facts about the surface $S_p$ are essentially contained in O’Grady [38].

**Proposition 3.9.** With notations as above, assume that $p \in X_0$ is a singular point satisfying (*). Then the following hold:

i) $S_p$ is the scheme theoretical complete intersection of a quadric and cubic in $\mathbb{P}^4$; 
ii) $S_p$ is reduced (but possibly reducible); 
iii) $S_p$ has only hypersurface singularities; 
iv) the singularities of $S_p$ are in one-to-one correspondence, including the type, with the singularities of $\tilde{X}$.

Furthermore, the morphism $g$ of (3.8) is the blow-up of $\mathbb{P}^4$ along $S_p$. In particular, $E_p$ is a $\mathbb{P}^1$-bundle (locally trivial in the Zariski topology) over $S_p$.

**Proof.** The first statement is well-known (e.g. [38] Remark 5.11). Explicitly, we can assume that $X_0$ is given by:

\[X_0 : (x_0 Q(x_1, \ldots, x_5) + F(x_1, \ldots, x_5) = 0),\]

with $Q$ and $F$ non-vanishing homogeneous polynomials of degree 2 and 3 respectively. The surface $S_p$ is the complete intersection of $Q$ and $F$. The two assumptions from (*) are equivalent to saying that $Q$ is reduced, and that $Q$ and $F$ are not simultaneously singular. In particular, since at any point of $S_p$ we can choose either $Q$ or $F$ as a local parameter, we obtain iii). The relation between the singularities of $S_p$ and those of $X_0$ and $\tilde{X}$ was analyzed in O’Grady [38] Prop. 5.15] and Wall [49] §I.2]. We emphasize the fact that Arnold’s splitting lemma gives that the singularities of $\tilde{X}$ are double suspensions of the singularities of $S_p$, i.e. the singularities of $\tilde{X}$ and $S_p$ are of the same analytic type (see [49] pg. 7).

The statement about $g$ is [38] Prop. 5.14]. We note that since $S_p$ is a reduced complete intersection there is no ambiguity about the blow-up. Furthermore, the normal bundle $\mathcal{N}_{S_p/\mathbb{P}^4}$ is a rank 2 vector bundle over $S_p$, giving the statement about $E_p$. \hfill \square

Since both $f$ and $g$ are explicit blow-ups, we are able to compute the Hodge structures of $X_0$ from that of $S_p$. We recall for a proper birational modification there exists a Mayer-Vietoris relating the cohomologies of the spaces involved. Specifically, assume that we are given:

\[
\begin{array}{c}
E \xrightarrow{f} \tilde{X} \\
\left\downarrow \right. \quad \left\downarrow \right. \\
D \xrightarrow{\phi} X
\end{array}
\]

with $X$ and $\tilde{X}$ projective varieties, $f : \tilde{X} \to X$ a projective birational morphism, $D$ the discriminant of $f$ and $E = \pi^{-1}(D)$. Then there exists a long exact sequence of mixed Hodge structures, the Mayer-Vietoris for the discriminant square:

\[
\cdots \to H^{n-1}(E) \to H^n(X) \to H^n(\tilde{X}) \oplus H^n(D) \to H^n(E) \to H^{n+1}(X) \to \cdots
\]
worst du V al singularities.

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proof of [38, Prop. 5.9] (see [38, 3.15 Remark].

case they are simple singularities and the lemma follows. □

The singularities of \( S \) are in bijective correspondence, and of the same type, as the singularities of \( p \).

Proof. Assume \( p \) is singular, and choose any singular point \( 0 \). In conclusion, we obtain:

\[
\cdots \rightarrow H^3(Q_p) \rightarrow H^4(X_0) \rightarrow H^4(\tilde{X}) \rightarrow H^4(Q_p) \rightarrow \cdots
\]

It follows easily that for the reduced quadric \( Q_p \) we have \( H^3(Q_p) = 0 \) and \( H^4(Q_p) \) is of pure weight 4 and type \((2, 2)\).

Next, we relate the cohomology of the \( \tilde{X} \) with that of the surface \( S_p \).

Lemma 3.12. With notations and assumptions as above,

\[
\dim \text{Gr}_p^W \text{Gr}_{n+2} H^4(\tilde{X}) = \dim \text{Gr}_p^W \text{Gr}_{n}^W H^2(S_p)
\]

for \( n = 0, 1, 2 \).

Proof. We consider the exact sequence (3.10) for the birational morphism \( g \) of (3.8). We obtain:

\[
\cdots \rightarrow H^4(\mathbb{P}^4) \rightarrow H^4(\tilde{X}) \oplus H^4(S_p) \rightarrow H^4(E_p) \rightarrow H^5(\mathbb{P}^4) = 0
\]

Both \( H^4(\mathbb{P}^4) \cong \mathbb{C}, \) and \( H^4(S_p) \) carry a pure weight 4 Hodge structure of type \((2, 2)\) (for \( S_p \) this follows from [39, Thm. 6.32]). Thus, the restriction map \( H^4(\tilde{X}) \rightarrow H^4(E_p) \) is a \((p, q)\)-iso- morphism for all \((p, q) \neq (2, 2)\). The claim now follows from the fact that \( E_p \) is a projective \( \mathbb{P}^1 \)-bundle over \( S_p \) (cf. Prop. 3.9). □

In conclusion, we obtain:

Corollary 3.13. Let \( X_0 \) be an irreducible, reduced cubic fourfolds, and \( p \in X_0 \) a singular point satisfying (*). Then \( \text{Gr}_p^W \text{Gr}_{n+2} H^4(X_0) = 0 \) and

\[
\dim \text{Gr}_p^W \text{Gr}_{n+2} H^4(X_0) = \dim \text{Gr}_p^W \text{Gr}_{n}^W H^2(S_p)
\]

for \( n = 0, 1, 2 \).

3.2.2. The Type I case. The verification of the purity of the Hodge structure for a Type I fourfold is straightforward: either directly invoking a resolution or by reducing to K3 surfaces. We choose the latter approach as it works also for Type II and III fourfolds.

Lemma 3.14. Let \( X_0 \) be a cubic fourfold with at worst simple isolated singularities. Then \( H^4(X_0) \) carries a pure Hodge structure of weight 4. Furthermore, \( H^{3,1}(X_0) \neq 0 \).

Proof. Assume \( X_0 \) is singular, and choose any singular point \( p \). The claim follows from Cor. 3.13 provided that \( S_p \) has at worst simple (du Val) singularities. According to Prop. 3.9 the singularities of \( S_p \) are in bijective correspondence, and of the same type, as the singularities of \( \tilde{X} \). The singularities of \( \tilde{X} \) are of two types: either they come from \( X_0 \setminus \{p\} \), or they lie over \( p \). In either case they are simple singularities and the lemma follows.

Remark 3.15. A closely related statement is O'Grady [38 Prop. 5.9, 5.28]. We remark that the proof of [38 Prop. 5.9] (see [38 §5.4.4]) works under the assumption that \( S_p \) is a surface with at worst du Val singularities.

(see [39 Cor. 5.37]).

We apply the exact sequence (3.10) to the two birational modifications occurring in the diagram (3.8). First, we relate the cohomology of \( X_0 \) with that of the blow-up \( \tilde{X} \).

Lemma 3.11. Let \( X_0 \) be a singular cubic fourfold (irreducible, reduced), and \( p \) a singular point satisfying (*). Then, the pullback \( f^* : H^4(X_0) \rightarrow H^4(X) \) induces a \((p, q)\)-isomorphism for all \((p, q) \neq (2, 2)\).

Proof. The morphism \( f \) is the blow-up of the singular point \( p \). Thus, the relevant part of the sequence (3.10) reads:

\[
\cdots \rightarrow H^3(Q_p) \rightarrow H^4(X_0) \rightarrow H^4(\tilde{X}) \rightarrow H^4(Q_p) \rightarrow \cdots
\]

The verification of the purity of the Hodge structure for a Type I fourfold is straightforward: either directly invoking a resolution or by reducing to K3 surfaces. We choose the latter approach as it works also for Type II and III fourfolds. The claim now follows from the fact that \( E_p \) is a projective \( \mathbb{P}^1 \)-bundle over \( S_p \) (cf. Prop. 3.9). □
3.2.3. The Type II and III case.

**Lemma 3.16.** Let $X_0$ be a Type II cubic fourfold with closed orbit. Then we can choose a singular point $p \in X_0$ satisfying (*) (see [3.2.1]) such that the surface $S_p$, associated to the projection from $p$ is a Type II degeneration of K3 surfaces (i.e. $\text{Gr}^W_k H^2(S_p) \neq 0$). Thus, $\text{Gr}^W_3 H^4(X_0) \neq 0$.

**Proof.** All the singularities of a Type II fourfold $X_0$ have corank at most 3. Furthermore, for all the cases $\alpha - \phi$ we can find a singular point $p$ not lying on a singular line. Thus, we can choose a point $p$ satisfying (*). According to Prop. [3.9] the resulting surface $S_p$ is reduced having only hypersurface singularities. Since $X_0$ is of Type II, the surface $S_p$ has only simple, $E_r$, $A_\infty$, or $D_\infty$ singularities. Thus, the surface $S_p$ is a degeneration of degree 6 K3 surfaces with only insignificant singularities in the sense of Shah (see Def. [2.1]). From Shah [42, Thm. 2] it follows that $\dim \text{Gr}^W_k H^2(S_p) = 1$. Since by construction $S_p$ has at least one non-du Val singularity, we have $\text{Gr}^W_k H^2(S_p) \neq 0$ for either $k = 1$ or $k = 0$. A case by case analysis shows that $k = 1$ in our situation.

We exemplify the computation only in the case $\delta$. In this case, the cubic fourfold $X_0$ has 3 singularities of type $E_6$. We choose one of them as the projection center $p$. The resulting surface $S_p$ is the union of two surfaces $S_1$ and $S_2$. Both $S_1$ and $S_2$ are cones over the same elliptic curve $C = S_1 \cap S_2$. By the Mayer-Vietoris sequence, we have

$$\cdots \rightarrow H^1(S_1) \oplus H^1(S_2) \rightarrow H^1(C) \rightarrow H^2(S_p) \rightarrow H^2(S_1) \oplus H^2(S_2) \rightarrow \cdots$$

The resolution of $S_i$ (for $i = 1, 2$) is a ruled surface $S_i$ over $C$. Moreover, the exceptional divisor of $\tilde{S}_i \rightarrow S_i$ is isomorphic to $C$. It then follows that $H^1(S_i) = 0$ and $H^2(S_i)$ is 1-dimensional, carrying a Hodge structure of type (1, 1). In conclusion, we have $\text{Gr}^W_1 H^2(S_p) \cong H^1(C)$ and the claim follows.

The case of cubic fourfolds of Type III is similar. Only the case $\zeta$ is special (all the singularities are double lines), but this is handled by a direct computation. We get:

**Lemma 3.17.** Let $X_0$ be a Type III fourfold with closed orbit. Then $\text{Gr}^W_2 H^4(X_0) \neq 0$.  

**Remark 3.18.** The singularities of a Type II fourfold (see Def. [2.1]) can be resolved as follows:

1. $p \in X_0$ is a singularity of type $E_r$. Locally, the equation of $X_0$ at $p$ is given by $$(f(x_1, x_2, x_3) + x_4x_5 = 0) \in (C^5, 0),$$ where $f(x_1, x_2, x_3)$ is weighted homogeneous polynomial of degree 1 with respect to the weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for $r = 6, 7$ and 8 respectively. A weighted blow-up (ordinary for $r = 6$) of $p$ gives a birational morphism $X' \rightarrow X_0$ such that
   i) there are two exceptional divisors $E_1$ and $E_2$ isomorphic to weighted projective spaces;
   ii) the exceptional divisors meet transversally in a weighted projective space $P$;
   iii) the fourfold $X'$ is singular along an elliptic curve $C \subset P$, with only $A_\infty$ singularities. The singularities of $X'$ are now resolved by blowing $C$ (see step (2) below). We denote by $E$ the resulting exceptional divisor.

2. $X_0$ is a fourfold singular along an elliptic curve $C$, with only $A_\infty$ singularities. A blow-up along $C$ resolves the singularities of $X_0$. The resulting divisor $E$ has the structure of a non-degenerate quadric bundle over $C$ (i.e. every fiber is a smooth quadric surface).

3. $X_0$ is a fourfold singular along a rational curve $C$, with $A_\infty$ singularities at all but 4 distinct points along $C$, where the singularities are of type $D_\infty$. A blow-up along $C$ resolves the singularities of $X$. The resulting divisor $E$ has the structure of a quadric bundle over $C$ with 4 degenerate fibers.
We note that in all cases the threefold $E$ has the property that its intermediate Jacobian is the Jacobian of an elliptic curve. This follows from Deligne [9, pg. 75, Thm. 3.3] in the smooth case, or from Beauville [4] in the degenerate case.

The previous remark says that the computation of mixed Hodge structures for cubic fourfolds of Type II is formally the same as that of K3 surfaces of Type II: just replace an elliptic curve by a quadric bundle having the same Hodge structure. The only difficulty for such an approach is to understand the contribution of the smooth part $X^m_0$ to the computation of Hodge structures. In the K3 surface case this is easy because we are working with $S \setminus C$ where $S$ is a rational or elliptic ruled surface and $C$ is an elliptic curve.

4. Degenerations to Type IV fourfolds

Due to the presence of bad singularities, the monodromy arguments of the previous section fail in the case of cubic fourfolds of Type IV (see also [21, §4.4] and [2, §5]). This implies that the period map has singularities along the curve $\chi \subset \bar{\mathcal{M}}$. In this section we prove that a certain blow-up of $\bar{\mathcal{M}}$ resolves these singularities. More precisely we do the following. In 4.1 using that $\chi$ parametrizes fourfolds with large stabilizer we blow-up $\bar{\mathcal{M}}$ along $\chi$ according to the canonical Kirwan desingularization procedure (see [21]). Next, in §4.2 we prove that the resulting space $\tilde{\mathcal{M}}$ has a modular interpretation. Essentially, the exceptional locus of $\tilde{\mathcal{M}} \to \mathcal{M}$ parametrizes quadric bundles canonically associated to degree two K3 surfaces. Furthermore, the singularities allowed for these quadric bundles are of the same type as those of Type I–III fourfolds. Thus, via the blow-up procedure we replace the Type IV fourfolds by Type I–III fourfolds, for which we can control the monodromy as in section 3.

4.1. The Kirwan desingularization along $\chi$. We start by recalling the setup of Kirwan [24]. Let $G$ be a reductive group, acting on a smooth projective variety $P$. For $R$ a reductive subgroup of $G$ we denote by $Z^s_R$ the subset of semistable points stabilized by $R$. Then, to each conjugacy class of reductive subgroups with $Z^s_R \neq \emptyset$ there is associated a natural blow-up of the GIT quotient $P/G$. The center of the blow-up is the closed subset $(G \cdot Z^s_R)//G$. The resulting space is again a GIT quotient, but with simpler singularities. By performing this procedure a finite number of times starting with the highest dimensional subgroup $R$, one obtains a space with only quotient singularities.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The adjacencies of connected reductive subgroups with $Z^s_R \neq \emptyset$}
\end{figure}

In our situation $\bar{\mathcal{M}} = \mathbb{P}\text{Sym}^3 W//G$, where $G = \text{SL}(6)$ is the standard $G$-representation. The determination of the conjugacy classes of subgroups $R$ as above follows easily from the GIT computation of [27]. Specifically, the minimal subgroups are $R_\alpha, \ldots, R_\delta$ corresponding to the GIT boundary strata $\alpha-\delta$ ([27, §2.3]). These boundary strata further specialize to the strata $\tau, \chi, \zeta,$
and $\omega$. The resulting subgroups and the corresponding inclusions are given in figure [1] (for a similar situation see [23] pg. 66).

We now do the first two steps of Kirwan partial desingularization. Namely, we blow-up $\overline{M}$ with respect to $R_\omega$, followed by the blow-up with respect to $R_\chi$. We denote the resulting variety $\tilde{M}$, and by $\hat{M}$ the intermediary blow-up. Thus, $\tilde{M}$ is the blow-up of $\overline{M}$ in the point $\omega$, and $\hat{M}$ is the blow-up of $\tilde{M}$ along the strict transform of $\chi$. To understand the geometry associated to these two blow-ups we need to take a closer look at the local structure of $\overline{M}$ at $\omega$ and $\chi$.

4.1.1. The blow-up of the point $\omega$. The moduli of cubic fourfolds is obtained as the quotient of the action of $G = \text{SL}(6)$ on $P := \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))) = \mathbb{P}(\text{Sym}^3 W)$. Let $\omega \in M$ be the special boundary point. The preimage of $\omega$ in $P$ contains a unique closed orbit, say the orbit of $x \in P^{ss}$.

The local structure (in the étale topology) of the quotient $\overline{M}$ is described by Luna’s slice theorem [36, App. D]. Specifically, there exists a $G_x$-invariant slice $W$ to the orbit $G \cdot x$. The slice $W$ can be taken to be a smooth, affine, locally closed subvariety of $P$ such that $U = G \cdot W$ is open in $P$. We then have the following commutative diagram with Cartesian squares:

$$
\begin{array}{ccc}
G \times_{G_x} \mathcal{N}_x & \leftarrow & G \times_{G_x} W \\
\downarrow & & \downarrow \\
\mathcal{N}_x / G_x & \leftarrow & W / G_x
\end{array}
\quad
\begin{array}{ccc}
U & \subset & P \\
\downarrow & & \downarrow \\
U / G & \subset & P / G \\& & \simeq \overline{M}
\end{array}
$$

where $G_x$ denotes the stabilizer of $x$, and $\mathcal{N}_x$ the fiber of the normal bundle of the orbit $G \cdot x$. The Kirwan desingularization is compatible with Luna’s slice. In particular, the exceptional divisor of the blow-up of $\overline{M}$ at the point corresponding to $G \cdot x$ is the quotient $\mathbb{P}(\mathcal{N}_x) / G_x$ ([23] Rem. 6.4]).

We recall that the point $\omega$ corresponds to the secant variety $X$ of the Veronese surface $S$ in $\mathbb{P}^5$. It follows that $G_x \cong \text{SL}(3)$, acting in the standard way on the singular locus $S$ of $X$. The Veronese surface $S \subset \mathbb{P}^5$ is the image of $\mathbb{P}^2$ embedded by $\mathcal{O}_{\mathbb{P}^2}(2)$ in $\mathbb{P}^5$. Thus, there is a natural identification $W \cong \text{Sym}^2 V$ of $\text{SL}(3)$-representations such that $S \subset \mathbb{P}(W)$ is left invariant (where $V$ is the standard $\text{SL}(3)$ representation). We obtain the decomposition in irreducible summands

\[(4.1) \quad \text{Sym}^3 W \cong \text{Sym}^3 \text{Sym}^2 V \cong \text{Sym}^6 V \oplus \Gamma_{2,2} \oplus \mathbb{C}\] 

(cf. [15] (13.15)). It then follows easily that the natural representation of $\text{SL}(3)$ on the normal slice $\mathcal{N}_x$ is isomorphic to $\text{Sym}^6 V$. In conclusion, from Luna’s slice theorem and this computation we have obtained the following precise description of the blow-up of the point $\omega$:

**Corollary 4.2.** The point $\omega \in \overline{M}$ has an étale (or analytic) neighborhood isomorphic to the affine cone over the moduli space of plane sextic curves. The Kirwan blow-up of $\overline{M}$ at the point $\omega$ is isomorphic when restricted to this neighborhood to the natural blow-up of the vertex of the cone. In particular the exceptional divisor is isomorphic to the GIT quotient for plane sextics.

4.1.2. The blow-up of the curve $\chi$. We repeat the computation from the previous section for the strict transform of the curve $\chi$. We recall that the singular locus of a fourfold giving a minimal orbit of type $\chi$ is a rational normal curve of degree 4. It follows that (at least generically) $G_x \cong \text{SL}(2)$, for $x \in P$ with closed orbit and mapping to $\chi$. Similarly to the case $\omega$ we obtain:

**Lemma 4.3.** Let $x \in P$ be a semistable point with closed orbit, mapping to $\chi$. Then the natural representation of $G_x^0 \cong \text{SL}(2)$ on the normal $\mathcal{N}_x$ is isomorphic to

\[(4.4) \quad \text{Sym}^{12} V \oplus \text{Sym}^8 \oplus \mathbb{C},\]

where $V$ denotes the standard $\text{SL}(2)$ representation.
Proof. A fourfold of type $\chi$ is singular along a rational normal curve of degree 4 contained in a hyperplane of $\mathbb{P}^5$. It follows (as in \ref{4.1.1}) that we can naturally identify $W$ to the $\text{SL}(2)$ representation $\text{Sym}^4V \oplus \mathbb{C}$. We then have

\begin{align*}
\text{Sym}^3W & \cong \text{Sym}^3(\text{Sym}^4V \oplus \mathbb{C}) \cong \text{Sym}^3(\text{Sym}^4V) \oplus \text{Sym}^2(\text{Sym}^4V) \oplus \text{Sym}^4V \oplus \mathbb{C} \\
& \cong \text{Sym}^{12}V \oplus (\text{Sym}^8V)^{\oplus 2} \oplus \text{Sym}^6V \oplus (\text{Sym}^4V)^{\oplus 3} \oplus \mathbb{C}^{\oplus 3}
\end{align*}

The lemma now follows from the identification of the summands in the normal sequence:

\[0 \rightarrow T_{G,x,x} \rightarrow T_{P,x} \rightarrow N_x \rightarrow 0\]

(e.g. the representation on $T_{G,x,x}$ is computed by using $G \cdot x \cong G/G_x$).

To interpret geometrically the result of the previous lemma, we recall the situation of plane sextics studied by Shah \cite{43}. Namely, to resolve the rational map from GIT quotient of plane sextics to the period domain of degree two K3 surfaces, one needs to blow-up the point corresponding to the triple conic. The stabilizer is again $\text{SL}(2)$, and the representation on the normal slice is

\begin{equation}
\text{Sym}^{12}V \oplus \text{Sym}^8V
\end{equation}

The two summands of (4.6) correspond in Shah’s notation (\cite{43} pg. 498-500) to $\Phi$ and $\Xi$ respectively. In conclusion, by comparing (4.4) and (4.6), we can say that the blow-up of $\chi$ is locally the product of the affine line with the exceptional divisor $\mathcal{R}$ obtained by Shah \cite{43} §5. The divisor $\mathcal{R}$ corresponds to elliptic degree two K3 surfaces. These special degree two K3 surfaces are not double covers of $\mathbb{P}^2$, but instead are double covers of the Hirzebruch surface $F_4$ embedded in $\mathbb{P}^5$ as the cone $\Sigma^0_4$ over a rational normal curve of degree 4.

Remark 4.7. It is well known that a for degree two K3 surface $(S, H)$ the polarization $2H$ is base point free and defines a morphism to $\mathbb{P}^5$. In the generic case the image of $S$ is the Veronese surface, and in the special (unigonal) case the image is $\Sigma^0_4$. In both cases the branch curve is cut by a cubic fourfold. Thus, from our point of view the two cases behave similarly.

4.2. Modular interpretation of the blow-up $\mathcal{M} \rightarrow \overline{\mathcal{M}}$. In \ref{4.1.1} and \ref{4.1.2} we have noted the natural occurrence of the moduli of degree two K3 surfaces in the blow-up of $\overline{\mathcal{M}}$ along $\chi$. The statement there is purely representation theoretic. Here we give a geometric interpretation to this observation (e.g. we give a modular interpretation for Cor. \ref{4.2}).

To start, we recall the computation of the semi-stable reduction for generic pencils of cubic fourfolds degenerating to the secant variety of Veronese surface $V$ (Hassett \cite[§4.4]{21}). Let $\mathcal{X} \rightarrow \Delta$ be such a degeneration with $X_0$ the special fiber (the secant to the Veronese). The total space $\mathcal{X}$ is singular along the degree 12 curve $C = X_t \cap V$. After a base change of order 2, and the blow-up of the Veronese surface, we obtain a semi-stable model $\mathcal{X} \rightarrow \Delta$. The new central fiber $X'_0$ is the union of two normal crossing components $\mathcal{X}_0$ and $E$. The fourfold $X_0 \cong \text{Hilb}^2(\mathbb{P}^2)$ is the resolution of $X_0 \cong \text{Sym}^2(\mathbb{P}^2)$, and $E$ is a quadric bundle over $V \cong \mathbb{P}^2$ with discriminant the curve $C$. The two components meet in a $\mathbb{P}^1$-bundle $E_0$ over the surface $V$. The key point of this computation is that the Hodge structure of $E$ (at that of $X'_0$) is essentially the Hodge structure of the degree two K3 surface obtained as the double cover of $V \cong \mathbb{P}^2$ branched over $C$.

A closer look at Hasset’s computation and at \ref{4.1.1} shows that the entire construction is rather tautological. This means that we can do the semi-stable reduction simultaneously for all the degenerations to $X_0$. Namely, the exceptional divisor of the blow-up of $\omega$ can be interpreted as parametrizing fourfolds of type $X'_0 = \mathcal{X}_0 \cup_{E_0} E$, where $E$ is a quadric bundle over $\mathbb{P}^2$ depending only on the direction of approaching the point $\omega$. An equivalent formulation is that any 1-parameter family of cubic fourfolds degenerating to $X_0$, can be filled in (after a possible finite base change) with a fourfold of type $X'_0$. This follows from \ref{4.1.1} and the following lemma:
Lemma 4.8. The quadric bundles $E$ occurring in the semistable reduction of degenerations to the secant to Veronese are given by sections $q \in H^0(\mathbb{P}^2, \text{Sym}^2(N_{V/P_5}^* \otimes \mathcal{O}_{P_2}) \otimes \mathcal{O}_{P_2}(6)) =: H$. Furthermore, we have
\begin{equation}
H \cong H^0(\mathbb{P}^2, \text{Sym}^2(N_{V/P_5}^* \otimes \mathcal{O}_{P_2}) \otimes H^0(\mathbb{P}^2, \text{Sym}^2(N_{V/P_5}^* \otimes \mathcal{O}_{P_2}(6)) \otimes H^0(\mathbb{P}^2, \mathcal{O}_{P_2}(6))
\end{equation}
and the section $q$ can be taken of type $q = q_0 + f$ with $q_0 \in H^0(\mathbb{P}^2, \text{Sym}^2(N_{V/P_5}^* \otimes \mathcal{O}_{P_2}(6))$ a tautological $G$-invariant section and $f \in H^0(\mathbb{P}^2, \mathcal{O}_{P_2}(6))$ the equation of the discriminant curve $C$.

Proof. A quadric bundle $E$ over $\mathbb{P}^2$ of relative dimension $k$ can be embedded in a projective bundle $\mathbb{P}(N)$ over $\mathbb{P}^2$, and is defined by a section $q \in H^0(\mathbb{P}^2, \text{Sym}^2(N \otimes \mathcal{O}_{P_2}(\alpha))$ for some $\alpha \in \mathbb{Z}$ (cf. [4 Prop. 1.2]). The discriminant curve is then given by the induced section:
\[
\Delta \in H^0(\mathbb{P}^2, \text{Sym}^2(\det(E)) \otimes \mathcal{O}_{P_2}((k + 2)\alpha).
\]
We are interested in the quadric bundle $E$ over the Veronese surface $V \cong \mathbb{P}^2$, and its hyperplane section $E_0$ (viewed as a non-degenerate conic bundle). As the semi-stable reduction is obtained by blowing-up the Veronese surface, we have $E_0 \subset \mathbb{P}(N_{V/P_5}^*)$ and $E \subset \mathbb{P}(N_{V/P_5}^* \otimes \mathcal{O}_V)$ ([21 Lemma 4.4.3]). The following basic facts about the normal bundle $N_{V/P_5}$ are well known ([14 Lemma 2.7]):
\begin{enumerate}
  \item $N_{V/P_5}^* \cong \text{Sym}^2(\Omega_{\mathbb{P}^2}^1)$;
  \item $N_{V/P_5}^*(2) \cong N_{V/P_5}(-1)$ (the twist is taken with respect to $\mathcal{O}_{P_5}(1)$);
  \item $E_0$ corresponds to the section
        \[
        q_0 \in H^0(\text{Sym}^2(N_{V/P_5}^*(-1)) \otimes \mathcal{O}_V(-1)) \hookrightarrow \text{Hom}(N_{V/P_5}^*(1), N_{V/P_5}(-1) \otimes \mathcal{O}_V(-1))
        \]
        given by the duality isomorphism of ii). Furthermore, $q_0$ is SL(3)-invariant.
\end{enumerate}
Using ii), we rewrite iii) as $q_0 \in H^0(\text{Sym}^2(N_{V/P_5}^* \otimes \mathcal{O}_{P_2}(6))$. The lemma now follows from [21 Lemma 4.4.3] by noting the surjection $H^0(\mathbb{P}^2, \mathcal{O}_{P_5}(3)) \to H^0(V, \mathcal{O}_V(3)) \cong H^0(\mathbb{P}^2, \mathcal{O}_{P_2}(6))$. □

The computations of the previous lemma can be made explicit, in particular we get the following statement about the singularities of the quadric bundle $E$.

Lemma 4.10. Let $E$ be a quadric bundle as in lemma 4.8. Then the singularities of $E$ are in one-to-one correspondence, including the type, with the singularities of the discriminant curve $C$.

Proof. We choose $U \subset \mathbb{P}^2$ an open affine over which $N_{V/P_5}$ is trivialized. Then, locally over $U$, $E_0 \subset \mathbb{P}^2 \times U \to U$ is given by $x_1^2 + x_0x_2$ and $E \subset \mathbb{P}^3 \times U \to U$ by $f(x, y)t^2 + x_1^2 + x_0x_2$, where $(x, y)$ are affine coordinates on $U$, $f$ is the equation of the discriminant curve, and $(x_0 : x_1 : x_2 : t)$ are homogeneous coordinates on $\mathbb{P}^3$. Since $E_0$ is a smooth hyperplane section of $E$ we can assume $t = 1$. It follows that the singularities of $E$ are suspensions of the singularities of $C$. □

In conclusion, we can interpret the exceptional divisor of the blow-up of the point $\omega \in \overline{M}$ as the moduli space of the quadric bundles $E$ as described above. The stability condition on $E$ is equivalent to the stability condition on the discriminant curve $C$ as a plane sextic (i.e. lemma 4.8 and the decomposition (4.11) are compatible). Thus, the exceptional divisor of the blow-up $\overline{M} \to M$ is naturally stratified in 4 types:

(Type I) the open stratum parametrizing quadric bundles $E$ with at worst simple singularities;
(Type II) four strata parametrizing quadric bundles with $E_r$, $A_\infty$ or $D_\infty$ singularities (corresponding to the Type II in Shah [43 Thm. 2.4]);
(Type III) the locus of quadric bundles with cusp or degenerate cusp singularities (Type III in [43]);
(Type IV) a single point corresponding to the intersection of the exceptional divisor with the strict transform of the curve $\chi$ (corresponding to the triple conic).
We now blow-up $\tilde{M}$ along the curve $\chi$. The effect of this blow-up is to replace the Type IV’ case by introducing only fourfolds of Type I’–III’. The computation is based on [4.1.2] and is similar to the case of the blow-up of $\omega$. The stability condition is the stability condition of Shah [43, Thm. 4.3]. Therefore, using the blow-up $\tilde{M} \rightarrow \overline{M}$ any 1-parameter family of cubic fourfolds degenerating to Type IV can be filled in with a fourfold of type I’–III’. By definition the singularities of type I’–III’ are double suspensions of the singularities of the list [2.4] and are divided in types in the same way as the types I–III. Since the arguments of the proposition 3.4 are local, they apply in this situation as well. It follows that the type of the degeneration is determined by the mixed Hodge structure of the central fiber. By construction, except the $H^{2,2}$ part, this mixed Hodge structure is isomorphic to the Tate twist of the (mixed) Hodge structure of the associated (degeneration of) degree two K3 surface (see also [21, §4.4]). We conclude as in section 3.

**Proposition 4.11.** Let $f : \mathcal{X} \rightarrow \Delta$ be a 1-parameter degeneration (with unipotent monodromy) with the generic fiber a smooth cubic fourfold, and the special fiber $X'_0$ of Type I’, II’, or III’. Let $N$ be the logarithm of the monodromy, and $\nu$ its index of nilpotence. Then the following holds:

i) if $X'_0$ has Type I’ then $\nu = 1$;

ii) if $X'_0$ has Type II’ then $\nu = 2$;

iii) if $X'_0$ has Type III’ then $\nu = 3$.

Furthermore, in the case i) the limit mixed Hodge $H^4_{\text{lim}}$ is pure and the corresponding period point belongs to $H_{\infty}/\Gamma$. 

**Remark 4.12.** The occurrence of degree two K3 surfaces and of the secant to Veronese surface can be also explained as follows. The Beauville-Donagi construction ([6]) shows that the Fano variety $F$ of lines on a cubic fourfold $X$ is an irreducible symplectic fourfold with a degree 6 polarization. The condition that the period of $X$ lies in $H_{\infty}/\Gamma$ is equivalent (via the Abel-Jacobi map $\alpha : H^4(X, \mathbb{Z}) \rightarrow H^2(F, \mathbb{Z})$) to the condition that Pic($F$) contains two classes $g$ (the polarization class) and $f$ such that the intersection matrix (w.r.t. the Bogomolov-Beauville form) is: $\begin{pmatrix} 6 & 2 \\ 2 & 0 \end{pmatrix}$.

By considering the classes $g' = g - f$ and $\delta = g - 2f$ (of square 2 and $-2$ respectively, and mutually orthogonal), one recognizes $H_{\infty}/\Gamma$ as corresponding to the locus of symplectic fourfolds of type $F \cong \text{Hilb}^2(S)$ for $S$ a degree 2 K3 surface. The morphism given by $g'$ is the composition $\text{Hilb}^2(S) \rightarrow \text{Sym}^2(S) \rightarrow \text{Sym}^2\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, where $\text{Sym}^2\mathbb{P}^2 \cong X_0$ is the secant to the Veronese surface. It follows that the map defined by $g = 2g' - \delta$ is not birational onto the image, and $F$ is not the Fano variety on a cubic fourfold. Instead, $F$ can be viewed as parametrizing the lines on the quadric bundle $E$ considered in this section.

5. **Proof of the main results**

5.1. **Proof of Theorem 1.1** We consider the GIT compactification $\overline{M}$ of the moduli space of cubic fourfolds. We blow-up the variety $\overline{M}$ along the locus of Type IV fourfolds (the point $\omega$ and $\chi$) as described in [4.1.1]. The result is a projective variety $\widetilde{M}$ compactifying the moduli space of smooth cubic fourfolds. The statements about the boundary of $\overline{M} \setminus M$ from [27], and those about the exceptional locus of $\overline{M} \rightarrow \overline{M}$ from [4.1.2] give the following statement: Any 1-parameter family of smooth cubic fourfolds $f^* : \mathcal{X}^* \rightarrow \Delta^*$ can be filled in (after a possible finite base change) to a proper analytic family $f : \mathcal{X} \rightarrow \Delta$ with the central fiber of type I–III or I’–III’.

The period map $\mathcal{P}_0 : \mathcal{M}_0 \rightarrow \mathcal{D}/\Gamma$ is a birational morphism of quasi-projective varieties with its image contained in $(\mathcal{D} \setminus (\mathcal{H}_0 \cup \mathcal{H}_\infty))/\Gamma$ (see [4.2]). Furthermore, it extends to a regular morphism $\mathcal{P} : \mathcal{M} \rightarrow \mathcal{D}/\Gamma$ over the simple singularities locus $\mathcal{M} \subset \tilde{M}$. The statement about the image of the period now follows from the monodromy results. Namely, fix a point $o \in \mathcal{D}/\Gamma$. We choose a holomorphic arc $g : \Delta \rightarrow \mathcal{D}/\Gamma$ with special point $o$ such that the generic point belongs to
the image of the period map of smooth cubic fourfolds (an open and dense subset in $\mathcal{D}/\Gamma$). After pull-back, and a finite base change we can assume that $g_{\Delta^*}$ comes from a family of smooth cubic fourfolds $X^* \to \Delta^*$. Thus, we can fill-in to a family $\mathcal{X} \to \Delta$ with central fiber $X_0$ of Type I–III or I’–III’. Since by assumption the limit point $\lim_{z \to 0} g(z)$ belongs to $\mathcal{D}/\Gamma$, the monodromy of the family $\mathcal{X}^*$ is finite [8, Thm. 13.4.5]. From the monodromy computation of [8, 4.1] it follows that $X_0$ is either of Type I or I’. In the Type I’ case we point $o$ belongs to the arrangement $\mathcal{H}_\infty$, and in the singular case to $\mathcal{H}_\Delta$.

5.2. Proof of the Theorem [1.2]. The statement follows directly from Thm. [1.1] and the general result of Looijenga [31, Thm. 7.6]. The situation is formally the same as that of low degree K3 surfaces treated by Looijenga [31, 7.6] (esp. [31, Thm. 8.6]).

Let $G = \text{SL}(6)$, and $P = \mathbb{P}\text{Sym}^3 W$ be the linear system of cubics in $\mathbb{P}^5$. On $P$ we consider the natural polarization $\eta = \mathcal{O}_P(n)$ (for some appropriate $n$; here $n = 2$), and define $U \in Y$ to be the $G$-invariant open subset parametrizing cubic fourfolds with at worst simple isolated singularities. In order to apply [31, Thm. 8.6] we note two important facts:

i) $P \setminus U$ has high codimension (at least 2) in $P$;

ii) the points of $U$ are stable with respect to the action of $G$ on $P$ ([27, Thm. 1.1]).

Let $L$ be the automorphic line bundle over the domain $\mathcal{D}$, and $\mathcal{L}$ the induced line bundle on the quotient $\mathcal{D}/\Gamma$. We denote $\mathcal{D}^0 = \mathcal{D} \setminus \mathcal{H}_\infty$ and $X^0 = \mathcal{D}^0/\Gamma$. Since i) and ii) are satisfied, Theorem [31, 7.6] gives an isomorphism of graded algebras (and thus Theorem [1.2]):

$$\bigoplus_{k \geq 0} H^0(Y, \eta^\otimes k)^G \cong \bigoplus_{k \in \mathbb{Z}} H^0(\mathcal{D}^0, \mathcal{O}(L)^\otimes k)^\Gamma$$

provided an identification of $(U, \eta|_U)/G$ with $(X^0, \mathcal{L}|_{X^0})$. By definition $\mathcal{M}$ is the geometric quotient $(U, \eta|_U)/G$, and thus we have an identification $\mathcal{M} = U/G$ with $X^0$ (Thm. [1.1]), but without the polarization statement. In the case hypersurfaces the identification of polarizations is automatic from Griffiths’ residue theory (see [31, Thm. 8.6] for the similar case of degree 4 K3 surfaces).

5.3. Arithmetic properties of the arrangement of hyperplanes $\mathcal{H}_\infty$. We close by noting some arithmetic facts about the hyperplane arrangement $\mathcal{H}_\infty$, which, via the identification given by Theorem [1.2] recover a posteriori part of the information given by the GIT compactification $\mathcal{M}$ (see [33, Remark 3.2] for the similar situation of cubic threefolds).

We recall that the birational transformation between the Looijenga compactification $\widetilde{X}^0 \cong \mathcal{M}$ (Thm. [1.2]) associated to the hyperplane arrangement $\mathcal{H}_\infty$ and the Baily-Borel compactification $X^{bb} := (\mathcal{D}/\Gamma)^*$ fits in a diagram:

$$\begin{array}{ccc}
\widetilde{X}^0 & \longrightarrow & X^\Sigma \\
\downarrow & & \downarrow \\
X^0 & \sim & X^{bb}
\end{array}$$

(The notations are those of Looijenga [31]). The map $X^\Sigma \to X^{bb}$ is a small morphism that makes the Weil divisor defined by $\mathcal{H}_\infty$ $\mathbb{Q}$-Cartier. Then $\widetilde{X}^0 \to X^\Sigma$ blows-up the strict transform of the hyperplane arrangement $\mathcal{H}_\infty$ in the usual way (starting with the highest codimension self-intersection stratum). Finally, the morphism $\widetilde{X}^0 \to \widetilde{X}^0$ contracts the hyperplane arrangement. The effect of this birational morphism is to flip a codimension $k$ intersection of hyperplanes in $X^{bb}$ to a $(k - 1)$-dimensional stratum in $\widetilde{X}^0$.

The fact that the period map has indeterminacy locus the curve $\chi$ (with the special point $\omega \in \chi$) corresponds to the arithmetic statement that at most 2 hyperplanes from $\mathcal{H}_\infty$ intersect inside $\mathcal{D}$.
Lemma 5.2. Keep notations as in definition \([2.8]\) (in particular \(h \in \Lambda\) fixed). Let \(M_i\) be distinct rank 2 sublattices of determinant 2 of \(\Lambda\) with \(h \in M_i\) for \(i = 1, \ldots, n\) (with \(n \geq 2\)). Assume that the sublattice \(M = \text{Span}(M_1, \ldots, M_n)\) generated by the \(M_i\) is positive definite. Then the rank of \(M\) is 3. Furthermore, the isometry class of \(M\) and the embedding in \(M \hookrightarrow \Lambda\) are uniquely determined (modulo \(\Gamma\)), and \((M)_\Lambda^1 \cong E_8^{\oplus 2} \oplus U^{\oplus 2}\).

Proof. Each of the lattices \(M_i\) are spanned by \(h\) and an element \(x_i\) such that \(x_i^2 = h.x_i = 1\). Assuming rank\((M) = 3\), we have \(h, x_i, x_j \in M\) such that \(h^2 = 3, h.x_i = 1,\) and \(x_i.x_j = a\) for some \(a \in \mathbb{Z}\). The condition that \(M\) is positive definite gives \(a = 0\). Similarly, it follows that it is not possible to have rank\((M) \geq 4\). The embedding statement follows from Nikulin [37]. \(\square\)

Notation 5.3. In what follows we use the notation:

\[
M_2 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}
\]

for the positive definite lattices occurring in the previous lemma. The lattice \(M_2\) defines a hyperplane of determinant 2, and the lattice \(M_3\) corresponds to the intersection of two such hyperplanes.

As for the intersection of the hyperplanes from \(\mathcal{H}_\infty\) with the boundary of \(\mathcal{D}\), we have:

Lemma 5.4. Let \(M\) be a lattice spanned by linear independent elements \(\{h, x_1, \ldots, x_n\}\) with \(h.h = 3, h.x_i = 1,\) and \(x_i.x_i = 1\) for all \(i\). Assume that \(M\) is positive semidefinite with at most 2-dimensional null space. Then \(M\) is isometric with one of the following lattices:

i) the positive definite case: \(M_2, M_3\);

ii) rank 1 null space: \(M_2 \oplus 0, M_3 \oplus 0\);

iii) rank 2 null space: \(M_2 \oplus 0^{\oplus 2}, M_3 \oplus 0^{\oplus 2}\);

(where \(0\) denotes the rank 1 lattice with null intersection form). In each of these case \(M\) can be embedded primitively in the lattice \(\Lambda\) (with \(h \in M\)).

Proof. We start by making two basic observations. First, the condition of positive semidefinite lattice gives that \(x_i.x_j\) is either 0 or 1. We also observe that \(x_i.v = x_j.v\) for all \(v \in M\) implies that we can split off a null summand (by changing the base element \(x_j\) to \(x_j - x_i\)). The following are then easy to check:

- if \(x_i.x_j = 1\), then \(x_i.v = x_j.v\) for all \(v \in M\);
- at most 3 of the \(x_i\) are mutually orthogonal;
- if \(x_i.x_j = x_j.x_k = x_i.x_k = 0\) for \(i \neq j \neq k\), then the sublattice spanned by \(\{h^2, x_i, x_j, x_k\}\) is isometric to \(M_3 \oplus 0\).

The claim follows from these properties and the condition on the dimension of the null space. \(\square\)

To explain the effect of the birational transformation \(\tilde{X}_0 \rightarrow X^{bb}\) on the boundary of the Baily-Borel compactification, we recall that the Baily-Borel compactification is done by adding six 1-dimensional Type II boundary components and a single Type III boundary point. The Type II (Type III) boundary components correspond to rank 2 (rank 1 respectively) classes of isotropic sublattices \(E\) of \(\Lambda_0\) (mod \(\Gamma\)). These were classified in [27, Thm. 1.3], and are listed below:

- \(E_6\) type: \(E_6^{\oplus 3}, A_{11} \oplus D_7\);
- \(E_7\) type: \(E_7^{\oplus 2}, D_{10}, A_{17}\);
- \(E_8\) type: \(E_8^{\oplus 2}, A_2, D_{16} \oplus A_2\);

(the label corresponds to the root sublattice contained in \(E^\perp/E\); the “type” is explained by [27, Lemma 4.3]). From Lemma 5.3 it follows:
Corollary 5.5. Let \( h \in \Lambda \) and \( \Lambda_0 = \langle h \rangle \) be as above. Assume \( E \) is a primitive rank 2 isotropic sublattice of \( \Lambda_0 \). Then the following holds:

i) If \( E \) is of type \( E_6 \), then there is no extension of the embedding \( E \oplus \langle h \rangle \hookrightarrow \Lambda \) to an embedding \( E \oplus M_2 \hookrightarrow \Lambda \) (note that \( E \) and \( h \in M_2 \) are fixed);

ii) If \( E \) is of type \( E_7 \), then we can extend the embedding \( E \oplus \langle h \rangle \hookrightarrow \Lambda \) to an embedding \( E \oplus M_2 \hookrightarrow \Lambda \), but not to an embedding \( E \oplus M_3 \hookrightarrow \Lambda \);

iii) If \( E \) is of type \( E_8 \), then we can extend the embedding \( E \oplus \langle h \rangle \hookrightarrow \Lambda \) to an embedding \( E \oplus M_3 \hookrightarrow \Lambda \).

Furthermore, when the extension exists, it is unique mod \( \Gamma \).

In conclusion, we obtain the following geometric consequences:

Corollary 5.6. With notations as above, the following hold:

i) The GIT boundary components \( \alpha \) and \( \delta \) associated to the type \( E_6 \) Baily-Borel components are unaffected by the birational transformation from \( [5.1] \). In particular, these strata are 1-dimensional and the period map extends to a morphism over the stratum \( \alpha \) and \( \delta \).

ii) The strata \( \gamma \) and \( \phi \) associated to the \( E_7 \) type are 2-dimensional and are affected only by the blow-up of the point \( \omega \).

iii) The strata \( \beta \) and \( \epsilon \) are three dimensional, and they are affected by both blow-ups of \( [5.4] \).

Proof. From corollary [5.5] it follows that the Looijenga’s compactification \( \hat{X}^0 \) has six Type II boundary strata of dimensions 1, 2, 3 depending on the type. Theorem [1.2] identifies \( \hat{X}^0 \) with the GIT compactification \( \overline{M} \). The claim then follows from the generic identification of table [1].

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