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Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials $R_\lambda(x; q, t) = R_\lambda(x_1, \ldots, x_n; q, t)$ form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in $n$ variables over the field $\mathbb{F} := \mathbb{Q}(q, t)$. They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most $n$ parts

$$\mathcal{P}_n := \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}.$$

For a partition $\mu \in \mathcal{P}_n$ we define $|\mu| = \mu_1 + \cdots + \mu_n$ and write

$$\overline{\mu} = (q^{\mu_1 \tau_1}, \ldots, q^{\mu_n \tau_n}) \text{ where } \tau := (\tau_1, \ldots, \tau_n) \text{ with } \tau_i := t^{1-i}.$$
Then \( R_\lambda(x) = R_\lambda(x; q, t) \) is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most \(|\lambda|\) satisfying the vanishing conditions

\[
R_\lambda(\mu) = 0 \quad \text{for} \quad \mu \in \mathcal{P}_n \quad \text{such that} \quad |\mu| \leq |\lambda|, \mu \neq \lambda.
\]

The normalization is fixed by requiring that the coefficient of \( x^{\lambda_1} \cdots x^{\lambda_n} \) in the monomial expansion of \( R_\lambda(x) \) is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of \( R_\lambda(x) \) is the Macdonald polynomial \( P_\lambda(x) \) [9] and \( R_\lambda(x) \) satisfies the extra vanishing property

\[
R_\lambda(\mu) = 0 \quad \text{unless} \quad \lambda \subseteq \mu \quad \text{as Ferrer diagrams.}
\]

Other key properties of \( R_\lambda(x) \), which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of \( R_\lambda(ax) = R_\lambda(ax_1, \ldots, ax_n; q, t) \) in terms of the \( R_\mu(x; q^{-1}, t^{-1})'s \) over the field \( K := \mathbb{Q}(q, t, a) \), and the duality or evaluation symmetry, which involves the evaluation points

\[
\tilde{\mu} = (q^{-\mu_1} \tau_1, \ldots, q^{-\mu_n} \tau_n), \quad \mu \in \mathcal{P}_n
\]

and takes the form

\[
\frac{R_\lambda(a\tilde{\mu})}{R_\lambda(\mu)} = \frac{R_\mu(a\tilde{\lambda})}{R_\mu(\mu)}.
\]

The interpolation polynomials have natural non-symmetric analogs \( G_\alpha(x) = G_\alpha(x; q, t) \), which were also defined in [4, 13]. These are indexed by the set of compositions with at most \( n \) parts, \( \mathcal{C}_n := (\mathbb{Z}_{\geq 0})^n \). For a composition \( \beta \in \mathcal{C}_n \) we define

\[
\overline{\beta} := w_\beta(\beta_+),
\]

where \( w_\beta \) is the shortest permutation such that \( \beta_+ = w_\beta^{-1}(\beta) \) is a partition. Then \( G_\alpha(x) \) is, up to normalization, characterized as the unique polynomial of degree at most \(|\alpha| := \alpha_1 + \cdots + \alpha_n\) satisfying the vanishing conditions

\[
G_\alpha(\overline{\beta}) = 0 \quad \text{for} \quad \beta \in \mathcal{C}_n \quad \text{such that} \quad |\beta| \leq |\alpha|, \beta \neq \alpha.
\]

The normalization is fixed by requiring that the coefficient of \( x^{\alpha_1} \cdots x^{\alpha_n} \) in the monomial expansion of \( G_\alpha(x) \) is 1.

Many properties of the symmetric interpolation polynomials \( R_\lambda(x) \) admit non-symmetric counterparts for the \( G_\alpha(x) \). For instance, the top homogeneous part of \( G_\alpha(x) \)
is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax; q, t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x; q, t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\tilde{\beta} := \langle -w_0\beta \rangle$, with $w_0$ the longest element of the symmetric group $S_n$:

$$G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.$$ 

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in $n$ variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators $H_w (w \in S_n)$ as described in the next section.

**Theorem A.** Write $I(\alpha) := \#\{i < j \mid \alpha_i \geq \alpha_j\}$. Then we have

$$G'_\alpha(t^{n-1}x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha|-I(\alpha)}w_0H_{w_0}G_\alpha(x; q, t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov’s duality result.

**Theorem B.** For all compositions $\alpha, \beta \in C_n$ we have

$$\frac{G_\alpha(a\tilde{\beta})}{G_\alpha(a\tau)} = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)}.$$ 

This is a special case of Theorem 17 below.

We now recall the interpolation $O$-polynomials introduced in [14, Thm. 1.1]. Write $x^{-1}$ for $(x_1^{-1}, \ldots, x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x; q, t; a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha(\bar{\beta}^{-1}) = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)} \text{ for all } \beta.$$ 

Our 3rd result is a simple expression for the $O$-polynomials in terms of the interpolation polynomials $G_\alpha(x)$. 
Theorem C. For all compositions $\alpha \in C_n$ we have

$$O_{\alpha}(x) = \frac{G_{\alpha}(t^{1-n}aw_0x)}{G_{\alpha}(ax)}.$$ 

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov’s [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials $G_{\alpha}'(x)$ in terms of the $G_{\beta}(ax)$'s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let $S_n$ be the symmetric group in $n$ letters and $s_i \in S_n$ the permutation that swaps $i$ and $i+1$. The $s_i$ ($1 \leq i < n$) are Coxeter generators for $S_n$. Let $\ell : S_n \to \mathbb{Z}_{\geq 0}$ be the associated length function. Let $S_n$ act on $\mathbb{Z}^n$ and $\mathbb{K}^n$ by $s_i \sigma := (\cdots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \cdots)$ for $\sigma = (v_1, \ldots, v_n)$. Write $w_0 \in S_n$ for the longest element, given explicitly by $i \to n+1-i$ for $i = 1, \ldots, n$.

For $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ define $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_n) \in \mathbb{F}^n$ by $\overline{v}_i := q^v_i t^{-k_i(v)}$ with

$$k_i(v) := \#\{k < i \mid v_k \geq v_i\} + \#\{k > i \mid v_k > v_i\}.$$ 

If $v \in \mathbb{Z}^n$ has non-increasing entries $v_1 \geq v_2 \geq \cdots \geq v_n$, then $\overline{v} = (q^{v_1} \tau_1, \ldots, q^{v_n} \tau_n)$. For arbitrary $v \in \mathbb{Z}^n$ we have $\overline{v} = w_\sigma(\overline{\sigma})$ with $w_\sigma \in S_n$ the shortest permutation such that $\overline{v}_+ := w_\sigma^{-1}(v)$ has non-increasing entries, see [4, Section 2]. We write $\overline{v} := -w_0 v$ for $v \in \mathbb{Z}^n$.

Note that $\overline{\alpha}_n = t^{1-n}$ if $\alpha \in C_n$ with $\alpha_n = 0$.

For a field $F$ we write $F[x] := F[x_1, \ldots, x_n]$, $F[x_{\pm 1}] := F[x_{\pm 1}, \ldots, x_{\pm n}]$ and $F(x)$ for the quotient field of $F[x]$. The symmetric group acts by algebra automorphisms on $F[x]$ and $F(x)$, with the action of $s_i$ by interchanging $x_i$ and $x_{i+1}$ for $1 \leq i < n$. Consider the $F$-linear operators

$$H_i = ts_i - \frac{(1-t)x_i}{x_i-x_{i+1}}(1-s_i) = t + \frac{x_i-tx_{i+1}}{x_i-x_{i+1}}(s_i - 1)$$.
on $\mathbb{F}(x)$ (1 ≤ $i$ < $n$) called Demazure-Lusztig operators, and the automorphism $\Delta$ of $\mathbb{F}(x)$ defined by

$$\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).$$

Note that $H_i$ (1 ≤ $i$ < $n$) and $\Delta$ preserve $\mathbb{F}[x^{\pm 1}]$ and $\mathbb{F}[x]$. Cherednik [1, 2] showed that the operators $H_i$ (1 ≤ $i$ < $n$) and $\Delta$ satisfy the defining relations of the type A extended affine Hecke algebra,

$$(H_i - t)(H_i + 1) = 0,$$

$$H_i H_j = H_j H_i, \quad |i - j| > 1,$$

$$H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1},$$

$$\Delta H_{i+1} = H_i \Delta,$$

$$\Delta^2 H_1 = H_{n-1} \Delta^2$$

for all the indices such that both sides of the equation make sense (see also [4, Section 3]).

For $w \in S_n$ we write $H_w := H_{i_1} H_{i_2} \cdots H_{i_\ell}$ with $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ a reduced expression for $w \in S_n$. It is well defined because of the braid relations for the $H_i$'s. Write $\overline{H}_i := H_i + 1 - t = t H_i^{-1}$ and set

$$\xi_i := t^{1-n} \overline{H}_{i-1} \cdots \overline{H}_1 \Delta^{-1} H_{n-1} \cdots H_i, \quad 1 \leq i \leq n. \quad (1)$$

The operators $\xi_i$'s are pairwise commuting invertible operators, with inverses

$$\xi_i^{-1} = \overline{H}_i \cdots \overline{H}_{n-1} \Delta H_1 \cdots H_{i-1}.$$

The $\xi_i^{-1}$ (1 ≤ $i$ ≤ $n$) are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ of degree $\alpha \in C_n$ is the unique polynomial satisfying

$$\xi_i^{-1} E_\alpha = \overline{u}_i E_\alpha, \quad i = 1, \ldots, n$$

and normalized such that the coefficient of $x^\alpha$ in $E_\alpha$ is 1.

Let $\iota$ be the field automorphism of $\mathbb{K}$ inverting $q$, $t$ and $a$. It restricts to a field automorphism of $\mathbb{F}$, inverting $q$ and $t$. We extend $\iota$ to a $\mathbb{Q}$-algebra automorphism of $\mathbb{K}[x]$
and $\mathbb{F}[x]$ by letting $\iota$ act on the coefficients of the polynomial. Write

$$G_\alpha^\circ := \iota(G_\alpha), \quad E_\alpha^\circ := \iota(E_\alpha)$$

for $\alpha \in C_n$. Note that $\overline{\nu}^{-1} = (\iota(\overline{v}_1), \ldots, \iota(\overline{v}_n))$.

Put $H_i^\circ, H_w^\circ, \overline{H}_i^\circ, \Delta^\circ$ and $\xi_i^\circ$ for the operators $H_i, H_w, \overline{H}_i, \Delta$ and $\xi_i$ with $q, t$ replaced by their inverses. For instance,

$$H_i^\circ = t^{-1}s_i - \frac{(1 - t^{-1})x_i}{x_i - x_{i+1}}(1 - s_i),$$

$$\Delta^\circ f(x_1, \ldots, x_n) = f(qx_n, x_1, \ldots, x_{n-1}).$$

We then have $\xi_i^\circ E_\alpha^\circ = \overline{\alpha}_i E_\alpha^\circ$ for $i = 1, \ldots, n$, which characterizes $E_\alpha^\circ$ up to a scalar factor.

**Theorem 1.** For $\alpha \in C_n$ we have

$$G'_\alpha(x) = t^{(1-n)|\alpha|+I(\alpha)}w_0H_w^\circ G_\alpha(t^{n-1}x)$$

(2)

with $I(\alpha) := \#\{i < j \mid \alpha_i \geq \alpha_j\}$.

**Remark.** Formally set $t = q^r$, replace $x$ by $1 + (q - 1)x$, divide both sides of (2) by $(q - 1)|\alpha|$ and take the limit $q \to 1$. Then

$$G'_\alpha(x; r) = (-1)^{|\alpha|} \sigma(w_0)w_0G_\alpha(-x - (n - 1)r; r)$$

(3)

for the non-symmetric interpolation Jack polynomial $G_\alpha(\cdot; r)$ and its primed version (see [14]). Here $\sigma$ denotes the action of the symmetric group with $\sigma(s_i)$ the rational degeneration of the Demazure-Lusztig operators $H_i$, given explicitly by

$$\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}}(1 - s_i),$$

see [14, Section 1]. To establish the formal limit (3) one uses that $\sigma(w_0)w_0 = w_0 \sigma^\circ(w_0)$ with $\sigma^\circ$ the action of the symmetric group defined in terms of the rational degeneration

$$\sigma^\circ(s_i) = s_i - \frac{r}{x_i - x_{i+1}}(1 - s_i)$$

of $H_i^\circ$. Formula (3) was obtained before in [14, Thm. 1.10].
Proof. We show that the right-hand side of (2) satisfies the defining properties of $G'_{\alpha}$.

For the vanishing property, note that

$$t^{n-1}w_0 \tilde{\beta} = \tilde{\beta}^{-1}$$

(this is the $q$-analog of [14, Lem. 6.1(2)]); hence,

$$(w_0 H_{w_0}^{\circ} G_{\alpha}^{\circ} (t^{n-1}x))|_{x=\tilde{\beta}} = (H_{w_0}^{\circ} G_{\alpha}^{\circ} (x))|_{x=\tilde{\beta}^{-1}}.$$

This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G_{\alpha}^{\circ} (w^\beta^{-1})$ ($w \in S_n$) by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that

$$E_{\alpha} = t^{1(\alpha)} w_0 H_{w_0}^{\circ} E_{\alpha}^{\circ}.$$

Note that $\Psi := w_0 H_{w_0}^{\circ}$ satisfies the intertwining properties

$$H_i \Psi = t \Psi H_i^{\circ},$$

$$\Delta \Psi = t^{n-1} \Psi H_{n-1}^{\circ} \cdots H_1^{\circ} (\Delta^{\circ})^{-1} H_{n-1}^{\circ} \cdots H_1^{\circ}$$

for $1 \leq i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi_i^{-1} \Psi = \Psi \xi_i^{\circ}$ for $i = 1, \ldots, n$. Therefore,

$$E_{\alpha} (x) = c_{\alpha} \Psi E_{\alpha}^{\circ} (x)$$

for some constant $c_{\alpha} \in \mathbb{F}$. But the coefficient of $x^\alpha$ in $\Psi x^\alpha$ is $t^{-1(\alpha)}$; hence, $c_{\alpha} = t^{1(\alpha)}$. ■

Consider the Demazure operators $H_i$ and the Cherednik operators $\xi_j^{-1}$ as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators $\xi_j^{-1}$ with eigenvalues $\bar{u}_j$ ($1 \leq j \leq n$), normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in $E_u$ is 1. For $u = \alpha \in C_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_{\alpha} \in \mathbb{F}[x]$ as defined before. Note that

$$E_{u+(1^n)} = x_1 \cdots x_n E_u (x).$$
It is now easy to check that formula (5) is valid with $\alpha$ replaced by an arbitrary integral vector $u$,

$$E_u = t^{l(u)}w_0H_{w_0}^\circ E_\circ$$

(7)

with $E_\circ := \iota(E_u)$. Furthermore, one can show in the same vein as the proof of (5) that

$$w_0E_{-w_0u}(x^{-1}) = E_u(x)$$

for an integral vector $u$, where $p(x^{-1})$ stands for inverting all the parameters $x_1, \ldots, x_n$ in the Laurent polynomial $p(x) \in \mathbb{F}[x^{\pm 1}]$. Combining this equality with (7) yields

$$E_{-w_0u}(x^{-1}) = t^{l(u)}H_{w_0}^\circ E_\circ(x),$$

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

$$G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n} - q^{a(s)} + t^{1-l(s)}}{1 - q^{a(s)+1}t^{l(s)+1}} \right) \prod_{s \in \alpha} (at^{l(s)} - q^{a'(s)})$$

(8)

was obtained, with $a(s), l(s), a'(s)$ and $l'(s)$ the arm, leg, coarm and coleg of $s = (i, j) \in \alpha$, defined by

$$a(s) := \alpha_i - j, \quad l(s) := \# \{ k > i \mid j \leq \alpha_k \leq \alpha_i \} + \# \{ k < i \mid j \leq \alpha_k + 1 \leq \alpha_i \},$$

$$a'(s) := j - 1, \quad l'(s) := \# \{ k > i \mid \alpha_k > \alpha_i \} + \# \{ k < i \mid \alpha_k \geq \alpha_i \}.$$ 

By (8) we have

$$E_\alpha(\tau) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n+l(s)} - q^{a'(s)+1}t^{l'(s)+1}}{1 - q^{a(s)+1}t^{l(s)+1}} \right),$$

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for $\alpha \in C_n$,

$$\ell(w_0) - I(\alpha) = \# \{ i < j \mid \alpha_i < \alpha_j \}. $$
Lemma 2. For $\alpha \in \mathcal{C}_n$ we have

$$G'_\alpha(\tau a) = t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)}G^0_\alpha(a\tau^{-1}).$$

Proof. Since $t^{n-1}w_0\tau = \tau^{-1} = \overline{\tau}^{-1}$ we have by Theorem 1,

$$G'_\alpha(\tau a) = t^{(1-n)|\alpha|+I(\alpha)}(H^0_{w_0}\circ G^0_\alpha)(a\overline{\tau}^{-1})$$

$$= t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)}G^0_\alpha(a\overline{\tau}^{-1}),$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality.

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G^0_\alpha(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^n \binom{\alpha_i}{2}$; hence, it only depends on the $S_n$-orbit of $\alpha$, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). \quad (9)$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in \mathcal{C}_n$ we have

$$G_\alpha(\tau a) = (-a)^{|\alpha|} t^{(1-n)|\alpha|-n(\alpha)} q^{n'(\alpha)} G^0_\alpha(a^{-1}\tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial $G_\alpha$.

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F} (\alpha \in \mathcal{C}_n)$ by

$$\tau_\alpha := (-1)^{|\alpha|} q^{n'(\alpha)} t^{-n(\alpha^+)}. \quad (10)$$

It only depends on the $S_n$-orbit of $\alpha$. 
Corollary 4. For $\alpha \in C_n$ we have

$$G'_\alpha(a^{-1}\tau) = \tau a^{-|\alpha|} G_\alpha(a\tau).$$

Proof. Use Lemmas 2 and 3 and (9).

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of $\mathbb{K}$-valued functions on $\mathbb{Z}^n$, which is constructed as follows.

For $v \in \mathbb{Z}^n$ and $y \in \mathbb{K}^n$ write $v^\flat := (v_2, \ldots, v_n, v_1 + 1)$ and $y^\flat := (y_2, \ldots, y_n, qy_1)$. Denote the inverse of $^\flat$ by $^\sharp$, so $v^\sharp = (v_n - 1, v_1, \ldots, v_1)$ and $y^\sharp = (y_n/q, y_1, \ldots, y_1)$. We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let $v \in \mathbb{Z}^n$ and $1 \leq i < n$. Then we have

1. $s_i(v) = s_i v$ if $v_i \neq v_{i+1}$.
2. $v_i = t v_{i+1}$ if $v_i = v_{i+1}$.
3. $v^\flat = v^\sharp$.

Let $\mathbb{H}$ be the double affine Hecke algebra over $\mathbb{K}$. It is isomorphic to the subalgebra of $\text{End}(\mathbb{K}[x^\pm 1])$ generated by the operators $H_i$ ($1 \leq i < n$), $\Delta^\pm 1$, and the multiplication operators $x_j^\pm 1$ ($1 \leq j \leq n$).

For a unital $\mathbb{K}$-algebra $A$ we write $\mathcal{F}_A$ for the space of $A$-valued functions $f : \mathbb{Z}^n \to A$ on $\mathbb{Z}^n$.

Corollary 6. Let $A$ be a unital $\mathbb{K}$-algebra. Consider the $A$-linear operators $\widehat{H}_i$ ($1 \leq i < n$), $\widehat{\Delta}$ and $\widehat{x}_j$ ($1 \leq j \leq n$) on $\mathcal{F}_A$ defined by

$$\begin{align*}
(\widehat{H}_i f)(v) := tf(v) + \frac{v_i - t v_{i+1}}{v_i - v_{i+1}} (f(s_i v) - f(v)), \\
(\widehat{\Delta} f)(v) := f(v^\sharp), \\
(\widehat{\Delta}^{-1} f)(v) := f(v^\sharp), \\
(\widehat{x}_j f)(v) := a v_j f(v)
\end{align*}$$

for $f \in \mathcal{F}_A$ and $v \in \mathbb{Z}^n$. Then $H_i \mapsto \widehat{H}_i$ ($1 \leq i < n$), $\Delta \mapsto \widehat{\Delta}$ and $x_j \mapsto \widehat{x}_j$ ($1 \leq j \leq n$) defines a representation $\mathbb{H} \to \text{End}_A(\mathcal{F}_A)$, $X \mapsto \widehat{X}$ ($X \in \mathbb{H}$) of the double affine Hecke algebra $\mathbb{H}$ on $\mathcal{F}_A$. 
Proof. Let $O \subset \mathbb{K}^n$ be the smallest $S_n$-invariant and $\mathfrak{z}$-invariant subset that contains \( \{a\mathfrak{v} \mid v \in \mathbb{Z}^n\} \). Note that $O$ is contained in \( \{y \in \mathbb{K}^n \mid y_i \neq y_j \text{ if } i \neq j\} \). The Demazure–Lusztig operators $H_i$ (1 \( \leq \) i \( \leq \) n), $\Delta^{\pm 1}$ and the coordinate multiplication operators $x_j$ (1 \( \leq \) j \( \leq \) n) act $A$-linearly on the space $F_A^O$ of $A$-valued functions on $O$, and hence turns $F_A^O$ into an $\mathbb{H}$-module. Define the surjective $A$-linear map

$$ \text{pr} : F_A^O \to F_A $$

by $\text{pr}(g)(v) := g(a\mathfrak{v})$ ($v \in \mathbb{Z}^n$).

We claim that $\text{Ker}(\text{pr})$ is an $\mathbb{H}$-submodule of $F_A^O$. Clearly $\text{Ker}(\text{pr})$ is $x_j$-invariant for $j = 1, \ldots, n$. Let $g \in \text{Ker}(\text{pr})$. Part 3 of Lemma 5 implies that $\Delta g \in \text{Ker}(\text{pr})$. To show that $H_j g \in \text{Ker}(\text{pr})$ we consider two cases. If $v_i \neq v_{i+1}$ then $s_i \mathfrak{v} = \overline{s_i \mathfrak{v}}$ by part 1 of Lemma 5.

Hence,

$$ (H_j g)(a\mathfrak{v}) = tg(a\mathfrak{v}) + \frac{\overline{v}_i - \overline{v}_{i+1}}{\overline{v}_i - \overline{v}_{i+1}} (g(as_i \mathfrak{v}) - g(a\mathfrak{v})) = 0. $$

If $v_i = v_{i+1}$ then $\overline{v}_i = t\overline{v}_{i+1}$ by part 2 of Lemma 5. Hence,

$$ (H_j g)(\mathfrak{v}) = tg(a\mathfrak{v}) + \frac{\overline{v}_i - \overline{v}_{i+1}}{\overline{v}_i - \overline{v}_{i+1}} (g(as_i \mathfrak{v}) - g(a\mathfrak{v})) = tg(a\mathfrak{v}) = 0. $$

Hence, $F_A$ inherits the $\mathbb{H}$-module structure of $F_A^O/\text{Ker}(\text{pr})$. It is a straightforward computation, using Lemma 5 again, to show that the resulting action of $H_i$ (1 \( \leq \) i \( \leq \) n), $\Delta$ and $x_j$ (1 \( \leq \) j \( \leq \) n) on $F_A$ is by the operators $\widehat{H}_i$ (1 \( \leq \) i \( \leq \) n), $\widehat{\Delta}$ and $\widehat{x}_j$ (1 \( \leq \) j \( \leq \) n).

Remark 7. With the notations from (the proof of) Corollary 6, let $\tilde{g} \in F_A^O$ and set $g := \text{pr}(\tilde{g}) \in F_A$. In other words, $g(v) := \tilde{g}(a\mathfrak{v})$ for all $v \in \mathbb{Z}^n$. Then

$$ (Xg)(v) = (X\tilde{g})(a\mathfrak{v}), \quad v \in \mathbb{Z}^n $$

for $X = H_i, \Delta^{\pm 1}, x_j$.

Remark 8. Let $F_A^+$ be the space of $A$-valued functions on $\mathcal{C}_n$. We sometimes will consider $\widehat{H}_i$ (1 \( \leq \) i \( \leq \) n), $\widehat{\Delta}^{-1}$ and $\widehat{x}_j$ (1 \( \leq \) j \( \leq \) n), defined by the formulas (11), as linear operators on $F_A^+$.

Definition 9. We call

$$ K_{\alpha}(x; q, t; a) := \frac{G_{\alpha}(x; q, t)}{G_{\alpha}(a\tau; q, t)} \in \mathbb{K}[x] $$

(12)

the normalized non-symmetric interpolation Macdonald polynomial of degree $\alpha$. 


We frequently use the shorthand notation $K_\alpha(x) := K_\alpha(x; q; t; a)$. We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that $a$ cannot be specialized to 1 in (12) since $G_\alpha(\tau) = G_\alpha(0) = 0$ if $\alpha \in \mathbb{C}_n$ is nonzero. Note furthermore that

$$\lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}$$

(13) since $\lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax) = E_\alpha(x)$.

Recall from [4] the operator $\phi_1 = (x_n - t^{1-n})\Delta \in \mathbb{H}$ and the inhomogeneous Cherednik operators

$$\Xi_j = \frac{1}{x_j} + \frac{1}{x_j} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{j-1} \in \mathbb{H}, \quad 1 \leq j \leq n.$$ 

The operators $H_i$, $\Xi_j$ and $\Phi$ preserve $\mathbb{K}[x]$ (see [4]); hence, they give rise to $\mathbb{K}$-linear operators on $F^+_{\mathbb{K}[x]}$ (e.g., $(H_i f)(\alpha) := H_i(f(\alpha))$ for $\alpha \in \mathbb{C}_n$). Note that the operators $H_i$, $\Xi_j$ and $\Phi$ on $F^+_{\mathbb{K}[x]}$ commute with the hat-operators $\hat{H}_i$, $\hat{\Xi}_j$ and $\hat{\Phi}$ on $F^+_{\mathbb{K}[x]}$ (cf. Remark 8). The same remarks hold true for the space $F^+_{\mathbb{K}(x)}$ of $\mathbb{K}(x)$-valued functions on $\mathbb{Z}^n$ (in fact, in this case the hat-operators define a $\mathbb{H}$-action on $F^+_{\mathbb{K}(x)}$).

Let $K \in F^+_{\mathbb{K}[x]}$ be the map $\alpha \mapsto K_\alpha(\cdot)$ ($\alpha \in \mathbb{C}_n$).

Lemma 10. For $1 < i < n$ and $1 \leq j \leq n$ we have in $F^+_{\mathbb{K}[x]}$,

1. $H_i K = \hat{H}_i K$.
2. $\Xi_j K = a \hat{\Xi}_j^{-1} K$.
3. $\Phi K = t^{1-n}(a^2 \hat{\Xi}_1^{-1} - 1)\hat{\Delta}^{-1} K$.

Proof. 1. To derive the formula we need to expand $H_i K_\alpha$ as a linear combination of the $K_\beta$'s. As a 1st step we expand $H_i G_\alpha$ as linear combination of the $G_\beta$'s.

If $\alpha \in \mathbb{C}_n$ satisfies $\alpha_i < \alpha_{i+1}$ then

$$H_i G_\alpha(x) = \frac{(t - 1)a_{i+1}}{a_i - a_{i+1}} G_\alpha(x) + G_{s_i \alpha}(x)$$

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that $H_i$ satisfies the quadratic relation $(H_i - t)(H_i + 1) = 0$, it follows that

$$H_i G_\alpha(x) = \frac{(t - 1)a_i}{a_i - a_{i+1}} G_\alpha(x) + \frac{t(\alpha_{i+1} - t\alpha_i)(\alpha_{i+1} - t \alpha_i - t^{-1} \alpha_i)}{(\alpha_{i+1} - \alpha_i)^2} G_{s_i \alpha}(x)$$
if \( \alpha \in C_n \) satisfies \( \alpha_i > \alpha_{i+1} \). Finally, \( H_iG_\alpha(x) = tG_\alpha(x) \) if \( \alpha \in C_n \) satisfies \( \alpha_i = \alpha_{i+1} \) by [4, Cor. 3.4].

An explicit expansion of \( H_iK_\alpha \) as linear combination of the \( K_\beta \)'s can now be obtained using the formula

\[
G_\alpha(\alpha \tau) = \frac{\alpha_{i+1} - \alpha_i}{\alpha_i - \alpha_i} G_{\alpha, \alpha}(\alpha \tau)
\]

for \( \alpha \in C_n \) satisfying \( \alpha_i > \alpha_{i+1} \), cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as \( H_iK = \hat{H}_iK \).

2. See [4, Thm. 2.6].

3. Let \( \alpha \in C_n \). By [14, Lem. 2.2 (1)],

\[
\Phi G_\alpha(x) = q^{-\alpha_1} G_{\alpha, z}(x).
\]

By the evaluation formula (8) we have

\[
\frac{G_{\alpha, z}(\alpha \tau)}{G_\alpha(\alpha \tau)} = at^{1-n+k_1(\alpha)} - q^{\alpha_1} t^{1-n}.
\]

Hence,

\[
\Phi K_\alpha(x) = t^{1-n}(a\alpha_{-1} - 1)K_{\alpha, z}(x).
\]

Remark 11. Note that

\[
\Phi K_\alpha(x) = (a\alpha_n - t^{1-n})K_{\alpha, z}(x)
\]

for \( \alpha \in C_n \) since \( \alpha^{-1} = t^{n-1}w_0\alpha \).

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials \( G_\alpha(x) \) and \( K_\alpha(x) \) to \( \alpha \in \mathbb{Z}^n \). It will be the unique extension of \( K \in \mathcal{F}^+(x) \) to a map \( K \in \mathcal{F}_{\mathbb{K}(x)} \) such that Lemma 10 remains valid.

Lemma 12. For \( \alpha \in C_n \) we have

\[
G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha + (1^n)}(qx)}{\prod_{i=1}^n (qx_i - t^{1-n})},
\]

\[
K_\alpha(x) = \left( \prod_{i=1}^n \frac{1 - a\alpha_i^{-1}}{1 - q t^{n-1} x_i} \right) K_{\alpha + (1^n)}(qx).
\]
Remarks on Interpolation Macdonald Polynomials

Proof. Note that for \( f \in \mathbb{K}[x] \),

\[
\Phi^n f(x) = \left( \prod_{i=1}^{n} (x_i - t^{1-n}) \right) f(q^{-1}x).
\]

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10. ■

For \( m \in \mathbb{Z}_{\geq 0} \) we define \( A_m(x; \nu) \in \mathbb{K}(x) \) by

\[
A_m(x; \nu) := \prod_{i=1}^{n} \left( \frac{q^{1-m} \nu^{-1} x_i; q}{q x_i^{-1} x_i; q} \right)_m \quad \forall \nu \in \mathbb{Z}^n,
\]

with \((y; q)_m := \prod_{j=0}^{m-1} (1 - q^j y)\) the \( q \)-shifted factorial.

Definition 13. Let \( \nu \in \mathbb{Z}^n \) and write \(|\nu| := \nu_1 + \cdots + \nu_n\). Define \( G_\nu(x) = G_\nu(x; q, t) \in \mathbb{F}(x) \) and \( K_\nu(x) = K_\nu(x; q, t; a) \in \mathbb{K}(x) \) by

\[
G_\nu(x) := q^{-m|\nu|-m^2} \frac{G_{\nu+(m^n)}(q^m x)}{\prod_{i=1}^{n} x_i^m (q^{-m} t^{1-n} x_i^{-1}; q)_m},
\]

\[
K_\nu(x) := A_m(x; \nu) K_{\nu+(m^n)}(q^m x),
\]

where \( m \) is a nonnegative integer such that \( \nu + (m^n) \in C_n \) (note that \( G_\nu \) and \( K_\nu \) are well defined by Lemma 12).

Example 14. If \( n = 1 \) then for \( m \in \mathbb{Z}_{\geq 0} \),

\[
K_{-m}(x) = \left( \frac{qa; q}{qx; q} \right)_m, \quad K_m(x) = \left( \frac{x}{a} \right)^m \frac{(x^{-1}; q)_m}{(a^{-1}; q)_m}.
\]

Lemma 15. For all \( \nu \in \mathbb{Z}^n \),

\[
K_\nu(x) = \frac{G_\nu(x)}{G_\nu(a \tau)}.
\]

Proof. Let \( \nu \in \mathbb{Z}^n \). Clearly \( G_\nu(x) \) and \( K_\nu(x) \) only differ by a multiplicative constant, so it suffices to show that \( K_\nu(\tau) = 1 \). Fix \( m \in \mathbb{Z}_{\geq 0} \) such that \( \nu + (m^n) \in C_n \). Then

\[
K_\nu(\tau) = A_m(\tau; \nu) K_{\nu+(m^n)}(q^m \tau a \tau) = A_m(\tau; \nu) \frac{G_{\nu+(m^n)}(q^m \tau a \tau)}{G_{\nu+(m^n)}(\tau a \tau)} = 1,
\]
where the last formula follows from a direct computation using the evaluation formula (8).

We extend the map $K : C_n \to \mathbb{K}[x]$ to a map

$$K : \mathbb{Z}^n \to \mathbb{K}(x)$$

by setting $v \mapsto K_v(x)$ for all $v \in \mathbb{Z}^n$. Lemma 10 now extends as follows.

**Proposition 16.** We have, as identities in $\mathcal{F}_\mathbb{K}(x)$,

1. $H_i K = \hat{H}_i K$.
2. $\Xi_j K = a\hat{x}_j^{-1} K$.
3. $\Phi K = t^{1-n}(a^2\hat{x}_1^{-1} - 1)\hat{\Delta}^{-1} K$.

**Proof.** Write $A_m \in \mathcal{F}_\mathbb{K}(x)$ for the map $v \mapsto A_m(x; v)$ for $v \in \mathbb{Z}^n$. Consider the linear operator on $\mathcal{F}_\mathbb{K}(x)$ defined by $(A_m f)(v) := A_m(x; v)f(v)$ for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_\mathbb{K}(x)$. For $1 \leq i < n$ we have $[H_i, A_m] = 0$ as linear operators on $\mathcal{F}_\mathbb{K}(x)$, since $A_m(x; v)$ is a symmetric rational function in $x_1, \ldots, x_n$. Furthermore, for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_\mathbb{K}(x)$,

$$(\widehat{H}_i \circ A_m f)(v) = (A_m \circ \widehat{H}_i f)(v) \quad \text{if} \quad v_i \neq v_{i+1}$$

(15)

by part 2 of Lemma 5 and the fact that $A_m(x; v)$ is symmetric in $\overline{v}_1, \ldots, \overline{v}_n$. Fix $v \in \mathbb{Z}^n$ and choose $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in C_n$. Since

$$K_v(x) = A_m(x; v)K_{v+(m^n)}(q^m x)$$

we obtain from $[H_i, A_m] = 0$ and (15) that $(H_i K)(v) = (\widehat{H}_i K)(v)$ if $v_i \neq v_{i+1}$. This also holds true if $v_i = v_{i+1}$ since then $(\widehat{H}_i K)(v) = t K_v$ and $H_i K_{v+(m^n)}(q^m x) = t K_{v+(m^n)}(q^m x)$. This proves part 1 of the proposition.

Note that $\Phi K_v(x) = t^{1-n}(a\bar{v}_1^{-1} - 1)K_v(x)$ for arbitrary $v \in \mathbb{Z}^n$ by Lemma 10 and the commutation relation

$$\Phi \circ A_m = A_m \circ \Phi(q^m),$$

(16)

where $\Phi(q^m) := (q^m x_n - t^{1-n})\Delta$. This proves part 3 of the proposition.

Finally we have $\Xi_j K_v(x) = \overline{v}_j^{-1} K_v(x)$ for all $v \in \mathbb{Z}^n$ by $[H_i, A_m] = 0$, (16) and Lemma 10. This proves part 2 of the proposition.
6 Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation \( \tilde{v} = -w_0 v \) for \( v \in \mathbb{Z}^n \).

**Theorem 17.** (Duality). For all \( u, v \in \mathbb{Z}^n \) we have

\[
K_u(a \tilde{v}) = K_v(a \tilde{u}).
\]  

(17)

**Example 18.** If \( n = 1 \) and \( m, r \in \mathbb{Z}_{\geq 0} \) then

\[
K_m(a q^{-r}) = q^{-mr} \frac{(a^{-1}; q)_{m+r}}{(a^{-1}; q)_{m}(a^{-1}; q)_r}.
\]  

(18)

by the explicit expression for \( K_m(x) \) from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of \( m \) and \( r \).

**Proof.** We divide the proof of the theorem in several steps.

**Step 1.** If \( K_u(a \tilde{v}) = K_v(a \tilde{u}) \) for all \( v \in \mathbb{Z}^n \) then \( K_{siu}(a \tilde{v}) = K_v(a \tilde{siu}) \) for \( v \in \mathbb{Z}^n \) and \( 1 \leq i < n \).

**Proof of Step 1.** Writing out the formula from part 1 of Proposition 16 gives

\[
\frac{(t - 1) \tilde{v}_i}{(\tilde{v}_i - \tilde{v}_{i+1})} K_u(a \tilde{v}) + \frac{(\tilde{v}_i - t \tilde{v}_{i+1})}{\tilde{v}_i - \tilde{v}_{i+1}} K_u(a \tilde{s}_n v) = \frac{(t - 1) \bar{u}_i}{(\bar{u}_i - \bar{u}_{i+1})} K_u(a \tilde{v}) + \frac{(\bar{u}_i - t \bar{u}_{i+1})}{\bar{u}_i - \bar{u}_{i+1}} K_{siu}(a \tilde{v}).
\]  

(19)

Replacing in (19) the role of \( u \) and \( v \) and replacing \( i \) by \( n - i \) we get

\[
\frac{(t - 1) \bar{u}_{n-i}}{(\bar{u}_{n-i} - \bar{u}_{n+1-i})} K_v(a \tilde{u}) + \frac{(\bar{u}_{n-i} - t \bar{u}_{n+1-i})}{\bar{u}_{n-i} - \bar{u}_{n+1-i}} K_v(a \tilde{s}_i \tilde{u}) = \frac{(t - 1) \tilde{v}_{n-i}}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})} K_v(a \tilde{u}) + \frac{(\tilde{v}_{n-i} - t \tilde{v}_{n+1-i})}{\tilde{v}_{n-i} - \tilde{v}_{n+1-i}} K_{s_{n-i}v}(a \tilde{u}).
\]  

(20)

Suppose that \( s_{n-i} v = v \). Then \( \tilde{v}_{n-i} = t \tilde{v}_{n+1-i} \) by the 2nd part of Lemma 5. Since \( \tilde{v} = t^{1-n} w_0 v^{-1} \), that is, \( \tilde{v}_i = t^{1-n} \bar{v}_{n+1-i} \), we then also have \( \tilde{v}_i = t \tilde{v}_{i+1} \). It then follows by a direct computation that (19) reduces to \( K_{siu}(a \tilde{v}) = K_u(a \tilde{v}) \) and (20) to \( K_v(a \tilde{s}_i \tilde{u}) = K_v(a \tilde{u}) \) if \( s_{n-i} v = v \).
We now use these observations to prove Step 1. Assume that $K_u(a \tilde{v}) = K_v(a \tilde{u})$ for all $v$. We have to show that $K_{s_i u}(a \tilde{v}) = K_v(a \tilde{s}_i \tilde{u})$ for all $v$. It is trivially true if $s_i u = u$, so we may assume that $s_i u \neq u$. Suppose that $v$ satisfies $s_{n-i} v = v$. Then it follows from the previous paragraph that

$$K_{s_i u}(a \tilde{v}) = K_u(a \tilde{v}) = K_v(a \tilde{u}) = K_v(a \tilde{s}_i \tilde{u}).$$

If $s_{n-i} v \neq v$ then (19) and the induction hypothesis can be used to write $K_{s_i u}(a \tilde{v})$ as an explicit linear combination of $K_v(a \tilde{u})$ and $K_{s_{n-i} v}(a \tilde{u})$. Then (20) can be used to rewrite the term involving $K_{s_{n-i} v}(a \tilde{u})$ as an explicit linear combination of $K_v(a \tilde{u})$ and $K_v(a \tilde{s}_i \tilde{u})$. Hence, we obtain an explicit expression of $K_{s_i u}(a \tilde{v})$ as linear combination of $K_v(a \tilde{u})$ and $K_v(a \tilde{s}_i \tilde{u})$, which turns out to reduce to $K_{s_i u}(a \tilde{v}) = K_v(a \tilde{s}_i \tilde{u})$ after a direct computation.

**Step 2.** $K_0(a \tilde{v}) = 1 = K_v(a \tilde{0})$ for all $v \in \mathbb{Z}_n$.

**Proof of Step 2.** Clearly $K_0(x) = 1$ and $K_v(a \tilde{0}) = K_v(a \tilde{v}) = 1$ for $v \in \mathbb{Z}_n$ by Lemma 15.

**Step 3.** $K_\alpha(a \tilde{v}) = K_v(a \tilde{\alpha})$ for $v \in \mathbb{Z}_n$ and $\alpha \in \mathcal{C}_n$.

**Proof of Step 3.** We prove it by induction. It is true for $\alpha = 0$ by Step 2. Let $m \in \mathbb{Z}_{>0}$ and suppose that $K_\gamma(a \tilde{v}) = K_\gamma(a \tilde{\gamma})$ for $v \in \mathbb{Z}_n$ and $\gamma \in \mathcal{C}_n$ with $|\gamma| < m$. Let $\alpha \in \mathcal{C}_n$ with $|\alpha| = m$.

We need to show that $K_\alpha(a \tilde{v}) = K_\gamma(a \tilde{\alpha})$ for all $v \in \mathbb{Z}_n$. By Step 1 we may assume without loss of generality that $\alpha_n > 0$. Then $\gamma := a \tilde{\gamma} \in \mathcal{C}_n$ satisfies $|\gamma| = m - 1$, and $\alpha = a \tilde{\gamma}$. Furthermore, note that we have the formula

$$K_\alpha(a \tilde{v}) = (a \tilde{v}_1^{-1} - 1)K_\gamma(a \tilde{v}^\gamma) = (a \tilde{u}_1^{-1} - 1)K_\gamma(a \tilde{v})$$

for all $u, v \in \mathbb{Z}_n$, which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain

$$K_\alpha(a \tilde{v}) = K_\gamma(a \tilde{v}) = (a \tilde{v}_1^{-1} - 1)K_\gamma(a \tilde{v})$$

$$= (a \tilde{u}_1^{-1} - 1)K_\gamma(a \tilde{v}) = K_\gamma(a \tilde{v}) = K_v(a \tilde{\alpha}),$$

where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step.
Step 4. \( K_u(a\tilde{v}) = K_v(a\tilde{u}) \) for all \( u, v \in \mathbb{Z}^n \).

Proof of Step 4. Fix \( u, v \in \mathbb{Z}^n \). Let \( m \in \mathbb{Z}_{\geq 0} \) such that \( u + (m^n) \in C_n \). Note that \( q^m\tilde{v} = v - (m^n) \) and \( q^{-m}\tilde{u} = u + (m^n) \). Then

\[
K_u(a\tilde{v}) = A_m(a\tilde{v}; u)K_{u+(m^n)}(q^m a\tilde{v})
\]

\[
= A_m(a\tilde{v}; u)K_{u+(m^n)}(a(v - (m^n)))
\]

\[
= A_m(a\tilde{v}; u)K_{v-(m^n)}(a(u + (m^n)))
\]

\[
= A_m(a\tilde{v}; u)K_{v-(m^n)}(q^{-m}a\tilde{u}) = A_m(a\tilde{v}; u)A_m(q^{-m}a\tilde{u}; v - (m^n))K_v(a\tilde{u}),
\]

where we used Step 3 in the 3rd equality. The result now follows from the fact that

\[
A_m(a\tilde{v}; u)A_m(q^{-m}a\tilde{u}; v - (m^n)) = 1,
\]

which follows by a straightforward computation using (4). \( \blacksquare \)

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial \( E_\alpha(x) \) of degree \( \alpha \) is the top homogeneous component of \( G_\alpha(x) \), i.e.,

\[
E_\alpha(x) = \lim_{a \to \infty} a^{-|\alpha|}G_\alpha(ax), \quad \alpha \in C_n.
\]

The normalized non-symmetric Macdonald polynomials are

\[
\overline{K}_\alpha(x) := \lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}, \quad \alpha \in C_n.
\]

We write \( \overline{K} \in \mathcal{F}^{\tau}_{\text{Fix}} \) for the resulting map \( \alpha \mapsto \overline{K}_\alpha \). Taking limits in Lemma 10 we get the following.

Lemma 19. We have for \( 1 \leq i < n \) and \( 1 \leq j \leq n \),

1. \( H_i\overline{K} = \overline{H}_i\overline{K} \).
2. \( \xi_j\overline{K} = \overline{\xi}_j^{-1}\overline{K} \).
3. \( x_n\Delta\overline{K} = t^{1-n}\overline{\Delta}^{-1}\overline{\Delta}^{-1}\overline{K} \).
Note that
\[(x_n \Delta)^n f(x) = \left( \prod_{i=1}^{n} x_i \right) f(q^{-1} x).\]

Then repeated application of part 3 of Lemma 19 shows that for \(\alpha \in C_n\),
\[
E_\alpha(x) = \frac{E_{\alpha+(1^n)}(x)}{x_1 \cdots x_n},
\]
\[
\bar{K}_\alpha(x) = q^{\lvert \alpha \rvert |t|^{(1-n)n}} \left( \prod_{i=1}^{n} (\bar{\alpha}_i x_i)^{-1} \right) \bar{K}_{\alpha+(1^n)}(x). \tag{22}
\]

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials \(E_\nu(x) := E_\nu(x; q, t) \in \mathbb{F}[x^{\pm 1}]\) for arbitrary \(\nu \in \mathbb{Z}^n\) to those labeled by compositions through the formula
\[
E_\nu(x) = \frac{E_{\nu+(m^n)}(x)}{(x_1 \cdots x_n)^m}.
\]

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees \(\nu \in \mathbb{Z}^n\).

**Definition 20.** Let \(\nu \in \mathbb{Z}^n\) and \(m \in \mathbb{Z}_{\geq 0}\) such that \(\nu + (m^n) \in C_n\). Then \(\bar{K}_\nu(x) := \bar{K}_\nu(x; q, t) \in \mathbb{F}[x^{\pm 1}]\) is defined by
\[
\bar{K}_\nu(x) := q^{m|\nu| |t|^{(1-n)n} m} \left( \prod_{i=1}^{n} (\bar{\nu}_i x_i)^{-m} \right) \bar{K}_{\nu+(m^n)}(x).
\]

Using
\[
\lim_{a \to \infty} A_m(ax; \nu) = q^{-m^2 n |t|^{(1-n)n} m} \prod_{i=1}^{n} (\bar{\nu}_i x_i)^{-m}
\]
and the definitions of \(G_\nu(x)\) and \(K_\nu(x)\) it follows that
\[
\lim_{a \to \infty} a^{-|\nu|} G_\nu(ax) = E_\nu(x),
\]
\[
\lim_{a \to \infty} K_\nu(ax) = \bar{K}_\nu(x)
\]
for all \(\nu \in \mathbb{Z}^n\), so in particular
\[
\bar{K}_\nu(x) = \frac{E_\nu(x)}{E_\nu(\tau)} \quad \forall \nu \in \mathbb{Z}^n.
\]
Lemma 19 holds true for the extension of $K$ to the map $K \in \mathcal{F}[x^{\pm 1}]$ defined by $v \mapsto K_v$ ($v \in \mathbb{Z}^n$). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

**Corollary 21.** For all $u, v \in \mathbb{Z}^n$,

$$K_u(\tilde{v}) = K_v(\tilde{u}).$$

### 7.2 $O$-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the $O$-polynomials $O_{\alpha}$ (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for $O_{\alpha}$ in terms of the non-symmetric interpolation Macdonald polynomial $K_{\alpha}$.

**Proposition 22.** For all $\alpha \in \mathcal{C}_n$ we have

$$O_{\alpha}(x) = K_{\alpha}(t^{1-n}aw_0x).$$

**Proof.** The polynomial $\tilde{O}_{\alpha}(x) := K_{\alpha}(t^{1-n}aw_0x)$ is of degree at most $|\alpha|$ and

$$\tilde{O}_{\alpha}(\beta) = K_{\alpha}(t^{1-n}aw_0\beta^{-1}) = K_{\alpha}(a\tilde{\beta}) = K_{\beta}(a\tilde{\alpha})$$

for all $\beta \in \mathcal{C}_n$ by (4) and Theorem 17. Hence, $\tilde{O}_{\alpha} = O_{\alpha}$. \[\blacksquare\]

### 7.3 Okounkov’s duality

Write $F[x]^S_n$ for the symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in a field $F$. Write $C_+: = \sum_{w \in S_n} H_w$. The symmetric interpolation Macdonald polynomial $R_{\lambda}(x) \in F[x]^S_n$ is the multiple of $C_+G_{\lambda}$ such that the coefficient of $x^\lambda$ is one (see, e.g., [13]). We write

$$K_+^{\lambda}(x) := \frac{R_{\lambda}(x)}{R_{\lambda}(at)} \in \mathbb{K}[x]^S_n$$

for the normalized symmetric interpolation Macdonald polynomial. Then

$$C_+K_{\alpha}(x) = \left( \sum_{w \in S_n} t^\ell(w) \right) K_+^{\alpha}(x) \tag{23}$$

for $\alpha \in \mathcal{C}_n$. Okounkov’s [10, Section 2] duality result now reads as follows.
Theorem 23. For partitions $\lambda, \mu \in P_n$ we have

$$K^+_{\lambda}(a\mu^{-1}) = K^+_{\mu}(a\lambda^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\hat{C}_+ = \sum_{w \in S_n} \hat{H}_w$, with $\hat{H}_w := \hat{H}_{i_1} \cdots \hat{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in F_K$ for the function $f_\mu(u) := K_u(a\mu)$ ($u \in \mathbb{Z}^n$). Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_{\lambda}(a\mu) = (C_+K_{\lambda})(a\mu) = (\hat{C}_+f_\mu)(\lambda)$$

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a\hat{\mu}) = (Jw_0K_\mu(t^{1-n}x))|_{x=a^{-1}\mu}$$

with $(Jf)(x) := f(x^{-1}, \ldots, x^{-1})$ for $f \in K(x)$. A direct computation shows that

$$JH_iJ = (H_i^\circ)^{-1}, \quad w_0H_iw_0 = (H_{n-i}^\circ)^{-1}$$

for $1 \leq i < n$. In particular, $Jw_0C_+ = C_+Jw_0$. Combined with Remark 7 we conclude that

$$(\hat{C}_+f_\mu)(\lambda) = (Jw_0C_+K_\mu(t^{1-n}x))|_{x=a^{-1}\lambda}. $$

By (23) and (4) this simplifies to

$$(\hat{C}_+f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_{\mu}(a\lambda).$$

Returning to (24) we conclude that $K^+_{\lambda}(a\mu) = K^+_{\mu}(a\lambda)$. Since $K^+_{\lambda}$ is symmetric we obtain from (4) that

$$K^+_{\lambda}(a\mu^{-1}) = K^+_{\mu}(a\lambda^{-1}),$$

which is Okounkov’s duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).
Lemma 24. For \( u, v \in \mathbb{Z}^n \) we have

\[
(H_{w_0} K_u)(a \tilde{v}) = (H_{w_0} K_v)(a \tilde{u}).
\]  

Proof. We proceed as in the previous subsection. Set \( f_v(u) := K_u(a \tilde{v}) \) for \( u, v \in \mathbb{Z}^n \). By part 1 of Proposition 16,

\[
(H_{w_0} K_u)(a \tilde{v}) = (\hat{H}_{w_0} f_v)(u).
\]

Since \( f_v(u) = (Iw_0 K_v)(a^{-1} t^{n-1} \tilde{u}) \) by (4), Remark 7 implies that

\[
(\hat{H}_{w_0} f_v)(u) = (H_{w_0} Jw_0 K_v)(a^{-1} t^{n-1} \tilde{u}).
\]

Now \( H_{w_0} Jw_0 = Jw_0 H_{w_0} \) by (26); hence,

\[
(\hat{H}_{w_0} f_v)(u) = (Jw_0 H_{w_0} K_v)(a^{-1} t^{n-1} \tilde{u}) = (H_{w_0} K_v)(a \tilde{u}),
\]

which completes the proof.

Recall from Theorem 1 that

\[
G'_\beta(x) = t^{(1-n)|\beta| + I(\beta)} \Psi G^\circ_\beta(t^{n-1} x)
\]

with \( \Psi := w_0 H_{w_0}^\circ \). We define normalized versions by

\[
K'_\beta(x) := \frac{G'_\beta(x)}{G'_\beta(a^{-1} \tau)} = t^{\ell(w_0)} \Psi K^\circ_\beta(t^{n-1} x), \quad \beta \in C_n,
\]

with \( K^\circ_\nu := (K_\nu) \) for \( \nu \in \mathbb{Z}^n \) (the 2nd formula follows from Lemma 2). More generally, we define for \( \nu \in \mathbb{Z}^n \),

\[
K'_\nu(x) := t^{\ell(w_0)} \Psi K^\circ_\nu(t^{n-1} x).
\]  

We write \( K' : \mathbb{Z}^n \rightarrow \mathbb{K}(x) \) for the map \( \nu \mapsto K'_\nu(\nu \in \mathbb{Z}^n) \). Since \( H_1 \Psi = \Psi H_1^\circ \), part 1 of Proposition 16 gives \( H_1 K' = \hat{H}_1^\circ K' \). Considering the action of \((x_n - 1) \Delta^\circ)^n\) on \( K'_\beta(x) \) we get, using the fact that \((x_n - 1) \Delta^\circ)^n\) commutes with \( \Psi \) and part 3 of Proposition 16,

\[
K'_\nu(x) = \left( \prod_{i=1}^n \frac{(1 - a^{-1} \nu_i)}{(1 - q^{-1} x_i)} \right) K'_\nu + (1^n)(q^{-1} x),
\]
in particular
\[ K'_v(x) = \left( \prod_{i=1}^{n} \frac{(a^{-1}v; q)_m}{(q^{-m}x_i; q)_m} \right) K'_{v+(m^n)}(q^{-m}x). \]

**Example 25.** For \( n = 1 \) we have \( K'_v(x) = K^0_v(x) \) for \( v \in \mathbb{Z} \); hence,
\[ K'_{-m}(x) = \frac{(q^{-1}a^{-1}; q^{-1})_m}{(q^{-1}x; q^{-1})_m} = (ax)^{-m} \frac{(qa; q)_m}{(q^{-1}x; q)_m}, \]
\[ K'_m(x) = (ax)^m \frac{(x^{-1}; q^{-1})_m}{(a; q^{-1})_m} = \frac{(x; q)_m}{(a^{-1}; q)_m} \]
for \( m \in \mathbb{Z}_{\geq 0} \) by Example 14.

**Proposition 26.** For all \( u, v \in \mathbb{Z}^n \) we have
\[ K'_v(a^{-1}u) = K'_u(a^{-1}v). \]

**Proof.** Note that
\[ K'_v(a^{-1}u) = t^{t^m(wv)} \Psi K^0_v(t^{m^{-1}}x)|_{x=a^{-1}u} = t^{t^m(wv)} (H^0_{wv} K^0_v)(a^{-1}u^{-1}) \]
by (4). By (27) the right-hand side is invariant under the interchange of \( u \) and \( v \).

### 7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of \( O_\alpha \) was used to prove the following binomial theorem [14, Thm. 1.3]. Define for \( \alpha, \beta \in C_n \) the generalized binomial coefficient by
\[ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} := \frac{G^\circ_\beta(\overline{\alpha})}{G^\circ_\beta(\overline{\beta})}. \tag{29} \]

Applying the automorphism \( t \) of \( F \) to (29) we get
\[ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1},t^{-1}} = \frac{G^\circ_\beta(\overline{\alpha}^{-1})}{G^\circ_\beta(\overline{\beta}^{-1})}. \]
Theorem 27. For $\alpha, \beta \in \mathcal{C}_n$ we have the binomial formula

$$K_\alpha(ax) = \sum_{\beta \in \mathcal{C}_n} a^{\beta} \binom{\alpha}{\beta}_{q^{-1},t^{-1}} \frac{G'_\beta(x)}{G_\beta(ax)}.$$  \hfill (30)

Remark 28. 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

$$K_\alpha(ax) = \sum_{\beta \in \mathcal{C}_n} \tau^{-1}_\beta \binom{\alpha}{\beta}_{q^{-1},t^{-1}} K'_\beta(x)$$

$$= \sum_{\beta \in \mathcal{C}_n} \frac{K^0_\beta(\alpha^{-1})K'_\beta(x)}{\tau^{-1}_\beta K^0_\beta(\beta^{-1})}$$

$$= t^{\ell(w_0)} \sum_{\beta \in \mathcal{C}_n} \frac{K^0_\beta(\alpha^{-1})\Psi K^0_\beta(t^{-1}w_0x)}{\tau^{-1}_\beta K^0_\beta(\beta^{-1})}$$ \hfill (31)

with $\Psi = w_0 H^0_{w_0}$ (note that the dependence on $a$ in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K^0_\beta(x)$ and $K'_\beta(x)$).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of $K_\alpha$ as follows. By the identity $Hw_0 \Psi = w_0$ the binomial formula (31) implies the finite expansion

$$(Hw_0 K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta} K^0_\beta(\alpha^{-1})K^0_\beta(t^{-1}w_0x).$$

Substituting $x = \tilde{\gamma}$ and using (4) we obtain

$$(Hw_0 K_\gamma)(a\tilde{\gamma}) = \sum_{\beta \in \mathcal{C}_n} K^0_\beta(\alpha^{-1})K^0_\beta(\gamma^{-1}) \frac{K^0_\beta(\beta^{-1})}{\tau^{-1}_\beta}.$$
The dual binomial formula \[8, \text{Thm. 4.4}\] in our notations reads as follows.

**Theorem 29.** For all \(\alpha \in C_n\) we have

\[
K'_{\alpha}(x) = \sum_{\beta \in C_n} \tau_\beta \left[\begin{array}{c} \alpha \\ \beta \end{array}\right]_{q,t} K_\beta(ax). \tag{32}
\]

The starting point of the alternative proof of (32) is the binomial formula in the form

\[
K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} G^\circ_\beta(\overline{\alpha}^{-1}) \psi K^\circ_\beta(t^{n-1}x) \frac{\tau_\beta G^\circ_\beta(\overline{\beta}^{-1})}{\tau_\beta G^\circ_\beta(\overline{\beta}^{-1})},
\]

see (31). Replace \((a, x, q, t)\) by \((a^{-1}, at^{n-1}x, q^{-1}, t^{-1})\) and act by \(w_0Hw_0\) on both sides. Since \(w_0Hw_0\psi = \text{Id}\) we obtain

\[
\psi K^\circ_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta \left[\begin{array}{c} \alpha \\ \beta \end{array}\right]_{q,t} K_\beta(ax).
\]

Now use (28) to complete the proof of (32).

**Remark 30.** It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

\[
\psi K^\circ_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta K_\beta(\overline{\alpha})K_\beta(ax), \tag{33}
\]

As observed in \[8, (4.11)\], the binomial and dual binomial formula directly imply the orthogonality relations

\[
\sum_{\beta \in C_n} \frac{\tau_\beta}{\tau_\alpha} \left[\begin{array}{c} \alpha \\ \beta \end{array}\right]_{q,t} \left[\begin{array}{c} \beta \\ \gamma \end{array}\right]_{q^{-1},t^{-1}} \delta_{\alpha,\gamma}.
\]

Since \(\left[\begin{array}{c} \delta \\ \epsilon \end{array}\right]_{q,t} = 0\) unless \(\delta \supseteq \epsilon\), the terms in the sum are zero unless \(\gamma \subseteq \beta \subseteq \alpha\).

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