Quasi-periodic solutions of NLS with Liouvillean Frequency

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Abstract:

Quasi-periodic solutions with Liouvillean frequency of forced nonlinear Schrödinger equation are constructed. This is based on an infinite dimensional KAM theory for Liouvillean frequency.

Résumé:

Les solutions quasi-périodiques avec les fréquences de Liouville de l’équation de Schrödinger non linéaire forcée sont construites. C’est fondé sur la théorie KAM de dimension infinie pour la fréquence de Liouville.

1 Introduction

In 1989’s, Kuksin \cite{24} first constructed quasi-periodic solutions for 1d NLS equation with Dirichlet boundary conditions by infinite dimensional KAM theory. Following \cite{24}, mathematicians study Hamiltonian PDEs (such as wave equation, KDV and etc) with periodic boundary condition or in higher space dimension, many other methods are developed. They also consider Hamiltonian PDEs with derivative or finitely differentiable nonlinearities. For more details, one may refer to \cite{7, 9, 14, 17, 18, 20, 21, 22, 25, 28} and the references therein.
Note that all the quasi-periodic solution constructed above must satisfy some Diophantine condition. This is the key observation by Kolmogorov in 1954. We recall a vector \( \omega \in \mathbb{R}^d \) is said to be Diophantine if
\[
|\langle k, \omega \rangle| \geq \frac{\gamma}{|k| \tau}, \quad k \in \mathbb{Z}^d \setminus \{0\}.
\]
Later, people find results which works for Diophantine condition can be parallely generalized to Brjuno condition, which is
\[
\mathcal{B}(\omega) := \sum_{n \geq 0} 2^n \max_{0 < \|k\| \leq 2^n, k \in \mathbb{Z}^d} \ln \frac{1}{|\langle k, \omega \rangle|} < \infty.
\]
If \( \omega \) is not Brjuno, we call it is Liouvillean. The question is that whether it is possible to obtain some quasi-periodic solution with Liouvillian frequency? In this paper, we will establish the existence of quasi-periodic solution beyond Brjuno frequency. Before introducing the precise result, we need to give some necessary definitions.

For \( \bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2) \) with \( \bar{\omega}_1 = (\alpha, 1), \alpha \in \mathbb{R} \setminus \mathbb{Q} \), \( \bar{\omega}_2 \in \mathbb{R}^d \), we say that the frequency \( \bar{\omega} \) is weak Liouvillean, if there exist \( \gamma > 0 \) and \( \tau > d + 6 \), such that
\[
\begin{align*}
\beta(\alpha) := \limsup_{n \to 0} \frac{\ln \ln q_{n+1}}{\ln q_n} < \infty, \\
|\langle k, \bar{\omega}_1 \rangle + \langle l, \bar{\omega}_2 \rangle| \geq \frac{\gamma}{\|k\| + \|l\|} \tau, \quad \text{for } k \in \mathbb{Z}^2, l \in \mathbb{Z}^d \setminus \{0\}.
\end{align*}
\]
where \( \frac{p_n}{q_n} \) is the continued fraction approximates to \( \alpha \) (c.f. section 2.1). Denote by \( WL(\gamma, \tau, \beta) \) the set of such frequency and by \( WL \) the union
\[
WL = \bigcup_{\gamma > 0, \tau > d + 6, \beta < \infty} WL(\gamma, \tau, \beta).
\]
It is obvious that \( WL \) is of full Lebesgue measure, and if \( \bar{\omega} \in WL \), then it is not necessarily to be Brjuno. \(^1\)

While our method works for other Hamiltonian PDEs, as an example, we study the quasi-periodic solution of forced NLS:
\[
i u_t - u_{xx} + v(x)u + \epsilon f(\omega t, x, u, \bar{u}; \xi) = 0 \quad (1.2)
\]
on segment \([0, \pi]\) with Dirichlet boundary condition
\[
u(t, 0) = 0 = u(t, \pi), \quad -\infty < t < +\infty.
\]
Our main result is the following:

\(^1\)In the case \( d = 2 \), if \( \mathcal{B}(\bar{\omega}_1) < \infty \), then \( \beta(\alpha) = 0 \).
Theorem 1 Let \( \bar{\omega} \in WL, \xi \in O = (\frac{1}{2}, \frac{3}{2}) \). The function \( v(x) \) is real analytic with \( \int_0^\pi |v(x)|dx < 1 \), \( f(\Theta, x, u, \bar{u}; \xi) \) is assumed to be real analytic on \( \Theta, x, u, \bar{u} \) and Lipschitz on \( \xi \). Then for any small \( \gamma > 0 \), there exists \( \epsilon_0 > 0 \) and \( O_\gamma \subset O \) with \( |O \setminus O_\gamma| = O(\gamma) \), such that equation (1.2) has a \( C^\infty \) smooth quasi-periodic solution with frequency \( \omega = \xi \bar{\omega} \) for any \( \xi \in O_\gamma \) if \( \epsilon < \epsilon_0 \).

Before giving its proof, let us make some comments on the result.

We choose NLS as a model mainly because it is one of the most important equation in mathematical physics, many questions are still open. It has been a long time for people to construct quasi-periodic solution of NLS by KAM, real breakthrough was recently made by Eliasson-Kuksin [14], who established quasi-periodic solution of NLS with \( x \in \mathbb{T}^d \). We should mention that the existence of quasi-periodic with Diophantine frequency for NLS in higher dimension was first proved by Bourgain [9, 10] by CWB method. To ensure localization properties of the eigenfunctions, \( v(x) \) from the operator \( \partial_{xx} + v(x) \) is usually substituted by a “convolution potential” (see [14, 17, 27]), which will provide parameters required by KAM theorem. However, in our result, the potential \( v(x) \) serves as a multiplicative operator, the result holds for any fixed multiplicative operator \( v(x) \), we do not extract parameters from \( v(x) \), the role of parameter is being played by \( \xi \) from \( \omega = \xi (\bar{\omega}_1, \bar{\omega}_2) \). In fact, the existence of this kind of solution (quasi-periodic solution with frequency vary in a line) was first proposed by Bourgain [8] and Eliasson [12]. In the infinitely dimensional Hamiltonian setting, this has been first proved by Geng-Ren [16] for 1-dimensional wave equation and then Berti-Biasco [6] for 1-dimension NLS. Berti-Bolle [7] answered this question for the forced NLS like (1.2) with differentiable nonlinearity and \( x \in \mathbb{T}^d \). Comparing with [7], Berti-Bolle relax the perturbation to be finite differentiable and forced by Diophantine frequency, while the system we consider is forced by Liouvillian frequency and the perturbation is analytic.

As we should mentioned, our work is also motivated by Avila-Fayad-Krikorian, Hou-You’s recent work [3, 19], where they consider rotation reducibility result of quasi-periodic \( SL(2; \mathbb{R}) \) cocycles with Liouvillian frequency. Note quasi-periodic \( SL(2; \mathbb{R}) \) cocycles can be viewed as two dimensional linear Hamiltonian, reducibility of \( SL(2; \mathbb{R}) \) cocycles is equivalent to the quasi-periodic solution of the corresponding Hamiltonian systems. Readers can refer [1, 15, 31] for related results. We just emphasize that reducibility of quasi-periodic \( SL(2, \mathbb{R}) \) cocycles with Liouvillian frequency is quite meaningful, since the dynamics of quasi-periodic \( SL(2, \mathbb{R}) \) cocycles are closely related to the spectral theory of one-dimensional quasi-periodic
Schrödinger operators, the reducibility of $SL(2; \mathbb{R})$ cocycles with Liouvillean frequency plays a quite important role in recent advances of spectral theory of quasi-periodic Schrödinger operators, for example, Avila’s global theory of one-frequency quasi-periodic Schrödinger operators [1, 2], the solution of Aubry-André-Jitomirskaya’s conjecture [5]. We further mention that before our work, Wang-You-Zhou [29] already generalized Avila-Fayad-Krikorian’s result [3] to finite dimensional nonlinear Hamiltonian system, where they obtained response solution of harmonic oscillators, however, as we will discuss in section 5.1, the key techniques are quite different compared to this paper.

Finally, let’s comment on the innovations of our results. The proof of the theorem is based on infinite dimensional KAM theory, it is well-known that the key point of KAM theory is the solution of homological equation. The typical homological equation we meet can be written as

$$-i\partial_x u + \zeta u + b(x)u = f(x), \quad x \in \mathbb{T}^n. \quad (1.3)$$

In fact, this kind of equation was already met when Kuksin [25] studied KDV equations(also [21]), and also by Liu-Yuan [20] when they studied one-dimensional derivative NLS. We will provide a quite general method for solving this kind of equation, and is believed to have further applications. Compared to [20, 21, 25], the method is totally different and which even works for Liouvillean frequency (not merely Diophantine frequency as in [20, 21, 25]), this is one novelty of the paper. Readers are invited to consult section 5.1 for more discussions.

We emphasize that in all the results mentioned above [1, 2, 3, 5, 19, 29, 31], the frequency is one frequency (thus two frequencies in the continuous case), however, our method works for multifrequency. To the best knowledge of the authors, our result gives the first result regarding on the quasi-periodic solutions with Liouvillean frequency for Hamiltonian PDE, and it also gives the first positive result regarding on multifrequency Liouvillean frequency (even for the linear finite dimensional Hamiltonian case)!

One can not hope our result works for any Liouvillean frequency, since in the linear cocycle case, Avila and Jitomirskaya [4] already proved that there exists two dimensional frequency, such that for typical analytic potential, the corresponding Schrödinger cocycle has positive Lyapunov exponent for almost every energies. Thus the corresponding Hamiltonian system doesn’t exist quasi-periodic solution.
2 Preliminaries

2.1 Continued fraction expansion.

Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) be irrational. We first set
\[
a_0 = 0, \quad a_0 = \alpha
\]
and then we define inductively for \( n \geq 1 \):
\[
a_n = [\alpha_{n-1}], \quad \alpha_n = \alpha_{n-1} - a_n := G(\alpha_{n-1}) = \{1/\alpha_{n-1}\}.
\]

We define
\[
p_0 = 0, \quad q_1 = a_1, \quad q_0 = 1, \quad p_1 = 1
\]
and
\[
p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.
\]
Then \( (q_n) \) is the sequence of denominators of the best rational approximations for \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). To be more precise, for any \( 1 \leq k < q_n \), one has
\[
\|k\alpha\| \geq \|q_{n-1}\alpha\|, \quad \frac{1}{q_n + 1} \leq \|q_n\alpha\| \leq \frac{1}{q_n + 1}, \quad \text{(2.1)}
\]
where \( \|x\| = \inf_{p \in \mathbb{Z}} |x - p| \).

2.2 CD-Bridge

For any \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), let \( (q_n) \) be the sequence of denominators of best rational approximations. We choose two particular subsequences of \( (q_n) \), the first is \( (q_{n_k}) \) which we denote by \( (Q_k) \) for simple, the second is \( (q_{n_k+1}) \) which we denote by \( (\overline{Q}_k) \). The properties required from our choice of the subsequence \( (Q_k) \) are summarized below. The definition of CD-Bridge is required.

Definition 2.1 \([3]\) Let \( 0 \leq A \leq B \leq C \), We say that the pair of denominators \( (q_{\ell}, q_n) \) forms a CD-bridge if:

1. \( q_{i+1} \leq q^n_{\ell} \), \( i = \ell, \ldots, n - 1 \)
2. \( q^n_{\ell} \leq q_n \leq q^C_{\ell} \).
Lemma 2.1 [3] For any $A > 0$, there exists a subsequence $(Q_k)$, such that $Q_0 = 1$ and for each $k \geq 0$, $Q_{k+1} \leq Q_k^{A^4}$, and either $Q_k \geq Q_k^{A^4}$ or the pairs $(Q_{k-1}, Q_k)$ and $(Q_k, Q_{k+1})$ are both CD$(A, A^3)$ bridges.

In the sequel, we assume $A \geq 10$, and $(Q_n^{\ast})$ is the selected subsequence as in Lemma 2.1. Note if $\beta(\alpha) = \limsup_{n>0} \frac{\ln \ln q_{n+1}}{\ln q_n} < \infty$, then $\tilde{U}(\alpha) := \sup_{n>0} \frac{\ln \ln q_{n+1}}{\ln q_n} < \infty$. Then we have the following:

Lemma 2.2 [23] If $\tilde{U}(\alpha) < \infty$, then there is $Q_n \geq Q_{n-1}$ for any $n \geq 1$. Furthermore, one has

$$\sup_{n>0} \frac{\ln \ln Q_{n+1}}{\ln Q_n} \leq U(\alpha), \quad \ln Q_{n+1} \leq Q_n^U,$$

where $U(\alpha) = \tilde{U}(\alpha) + 4 \frac{\ln A}{\ln 2}$.

3 An Infinite Dimensional KAM Theorem

The main result will be proved by a generalized KAM theorem for Liouvillean frequency. In this section, we introduce this basic KAM result.

We start by introducing the notations. The Lipschitz norm of a function $f(\xi)$ with $\xi \in \mathcal{O} \subset \mathbb{R}$ is defined as

$$|f(\xi)|_{\mathcal{O}} = |f(\xi)|_0 + |f(\xi)|_C,$$

where $|f(\xi)|_0 = \sup_{\xi \in \mathcal{O}} |f(\xi)|$, $|f(\xi)|_C = \sup_{\xi, \xi \neq \eta \in \mathcal{O}} \frac{|f(\xi) - f(\eta)|}{|\xi - \eta|}$. Let $D_r = \{ (\theta, \varphi) \in \mathbb{T}^2 \times \mathbb{T}^d, |3\theta| + |3\varphi| < r \}$. For a bounded holomorphic (possibly with parameter) function $g(\theta, \varphi; \xi) = \sum_{(k,l) \in \mathbb{Z}^2 \times \mathbb{Z}^d} \hat{g}(k,l)(\xi)e^{i(k,l) \cdot (\theta, \varphi)}$ on $D_r$, we let

$$|g|^r_{\mathcal{O}} = \sum_{|k| + |l| \leq r} |\hat{g}(k,l)(\xi)|_{\mathcal{O}} e^{i(|k| + |l|)r}.$$

We denote by $\mathcal{B}(\mathcal{O})$ the set of these functions, and for any $K \in \mathbb{Z}^+$, we define the truncation operator $T_K$ as

$$T_K g(\theta, \varphi; \xi) = \sum_{|k| + |l| < K} \hat{g}(k,l)(\xi)e^{i(k,l) \cdot (\theta, \varphi)}, \quad (3.1)$$

also denote

$$[g(\theta, \varphi; \xi)]_\varphi = \int_{\mathbb{T}^d} g(\theta, \varphi) d\varphi, \quad [g(\theta, \varphi; \xi)] = \int_{\mathbb{T}^2 \times \mathbb{T}^d} g(\theta, \varphi) d\theta d\varphi. \quad (3.2)$$
Let $\ell^a_\rho$ be the Hilbert space of sequence $z = (z_1, z_2, \cdots)$ with
\[
|z|_{a,\rho}^2 = \sum_{p\geq 1} |z_p|^2 p^{2\rho} e^{2\rho |p|} < \infty,
\]
where $a > 0$ and $\rho > 0$. For $r, s > 0$, we then introduce the complex neighborhoods of $T^{2+d} \times \{0,0,0\}$ by
\[
D(r, s) = \{(\Theta, I, z, \bar{z}) : |\text{Im}\Theta| < r, |I| < s^2, |z|_{a,\rho} < s, |\bar{z}|_{a,\rho} < s\}
\subseteq C^{2+d} \times C^{2+d} \times \ell^a_\rho \times \ell^a_\rho \equiv \mathcal{P}^a_\rho,
\]
where $T^{2+d}$ is the usual $2+d$-torus, $|\cdot|$ denotes the sup-norm of complex vectors for $\Theta = (\theta, \varphi) \in T^2 \times T^d$, $I = (I, J) \in \mathbb{R}^2 \times \mathbb{R}^d$. (3.3)

For any $W = (X, Y, U, V) \in \mathcal{P}^a_\rho$, the weighted phase norm is defined to be
\[
|W|_s =: |W|_{s, a, \rho} = |X| + \frac{1}{s^2} |Y| + \frac{1}{s} |U|_{a, \rho} + \frac{1}{s} |V|_{a, \rho}.
\] (3.4)

For any map $W : D(r, s) \times O \to \mathcal{P}^a_\rho$, we define its norm as
\[
|W|_{s, D(r, s) \times O} = \sup_{D(r, s) \times O} |W|_s,
\]
\[
|W|_{s, D(r, s) \times O} =: |W|_{s, a, \rho} = \sup_{\xi, \eta \in O, \xi \neq \eta} \frac{\triangle_{\xi, \eta} W}{|\xi - \eta|},
\]
where $\triangle_{\xi, \eta} W = W(\cdot, \xi) - W(\cdot, \eta)$ and the supremum is taken over $O$.

We also need the operator norm $||\cdot||_{s, \tilde{s}}$ below,
\[
||A||_{s, \tilde{s}} = \sup_{W \neq 0} \frac{|AW|_s}{|W|_{\tilde{s}}},
\]
where $|\cdot|_s$ is the shorten of $|\cdot|_{s, a, \rho}$ defined in (3.4), and $|\cdot|_{\tilde{s}}$ defined similarly. For $s \geq \tilde{s}$, these norms satisfy $|AB|_{s, \tilde{s}} \leq |A|_{s, s} \cdot |B|_{s, \tilde{s}}$ since $|W|_s \leq |W|_{\tilde{s}}$.

If the function $F$ is analytic in space coordinate, we usually take Taylor–Fourier series as:
\[
F(\Theta, I, z, \bar{z}; \xi) = \sum_{\nu, \mu, \alpha, \beta} F_{\nu \mu \alpha \beta}(\xi) I^{\nu} e^{\nu(\mu, \Theta)} z^\alpha \bar{z}^\beta, \quad (3.5)
\]
where the coefficient functions $F_{\nu \mu \alpha \beta}(\xi)$ are Lipschitz on $\xi$, the vectors $\alpha \equiv (\cdots, \alpha_n, \cdots)_{n \geq 1}$, $\beta \equiv (\cdots, \beta_n, \cdots)_{n \geq 1}$ have finitely many non-zero
components $\alpha_n, \beta_n \in \mathbb{N}$, $z^\alpha z^\beta$ denotes $\prod_n z_n^\alpha \bar{z}_n^\beta$ and finally $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{C}^d$.

In this paper, we will consider the perturbed Hamiltonian on $D(r, s) \times \mathcal{O}$,

$$H = \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} \Omega_p(\xi)|z_p|^2 + P(\theta, \varphi, z, \bar{z}; \xi). \quad (3.6)$$

endowed with the symplectic structure

$$dI \wedge d\theta + dJ \wedge d\varphi + i \sum_{p \geq 1} dz_p \wedge d\bar{z}_p.$$

The perturbation $P(\theta, \varphi, z, \bar{z}; \xi)$ is real analytic in space coordinates $\theta, \varphi, z, \bar{z}$ and Lipschitz in parameters $\xi$. For each $\xi \in \mathcal{O}$, the Hamiltonian vector field $X_P = (-P(\theta, \varphi), 0, iP_z, -iP_{\bar{z}})$ defines a real analytic map $X_P : \mathcal{P}_{\mathbb{C}}^{a, \rho} \rightarrow \mathcal{P}_{\mathbb{C}}^{a, \rho}$ near $T^{2+d} \times \{0,0,0\}$. We denote the weighted norm of $X_P$ to be

$$|X_P|^s,D_{s,D}(r,s) \times \mathcal{O} = |X_P|^C_{s,D(r,s) \times \mathcal{O}} + |X_P|_{s,D(r,s) \times \mathcal{O}}.$$

Then we have the following infinite dimensional KAM theorem:

**Theorem 2** For any given $\beta < \infty, \tau > d + 6, s > 0, r > 0, \gamma > 0$, let $\omega = \xi \tilde{\omega}$ where $\tilde{\omega} \in WL(\gamma, \tau, \beta)$, $\xi \in \mathcal{O} = (\frac{1}{4}, \frac{3}{2})$. Suppose the Hamiltonian (3.6) satisfy

$$\sup_{p \geq 1} |\Omega_p - p^2| < \frac{1}{2}.$$

Then there exists $\epsilon_0(\tau, \beta, \gamma, s, r) > 0$, such that for any real analytic perturbation $P(\theta, \varphi, z, \bar{z}; \xi)$ with

$$\epsilon = |X_P|^s,D_{s,D(r,s) \times \mathcal{O}} \leq \epsilon_0,$$

there exists a Cantor set $\mathcal{O}_\gamma$ of $\mathcal{O}$ with $\text{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma)$ and a Lipschitz family symplectic map $\Phi : T^{2+d} \times \mathcal{O}_\gamma \rightarrow \mathcal{P}_{\mathbb{C}}^{a, \rho}$ which is $C^\infty$ smooth in $\theta, \varphi$, such that (3.6) is transformed to

$$H^* = e^*(\theta; \xi) + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} (\Omega_p^*(\xi) + B_p^*(\theta; \xi))|z_j|^2 + P^*(\theta, \varphi, z, \bar{z}; \xi),$$

where $P^*(\theta, \varphi, z, \bar{z}; \xi) = \sum_{|\alpha + \beta| \geq 3} P_{\alpha, \beta}^*(\theta, \varphi; \xi)z^\alpha \bar{z}^\beta$.

**Remark 3.1** We emphasize that the perturbation is independent of the action variable $I$ and $J$, this fact is crucial for our results.
3.1 Main ideas of the proof

Theorem 2 is proved by modified KAM theory which involves an infinite sequence of change of variables. The philosophy of KAM theory is to construct a series of symplectic transformation which makes the perturbation smaller and smaller at the cost of excluding a small set of parameters. Compared to the classical KAM scheme, due to the Liouvillean property of $\bar{\omega}_1$ by condition (1.1), some $\theta$ dependent terms have to be preserved as a normal form under KAM iteration. Thus we have a generalized Hamiltonian

$$H_n = e(\theta; \xi) + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} (\Omega_p^n(\xi) + B_p^n(\theta; \xi))|z_p|^2 + P_n(\theta, \varphi, z, \bar{z}; \xi).$$

(3.7)

where $B_p^n(\theta; \xi)$ is of size $\epsilon_0$, and the perturbation $P_n(\theta, \varphi, z, \bar{z}; \xi)$ is of size $\epsilon_n$. In the following, we will construct a symplectic transformation $\Phi_{n+1}$ which is close to the identity (Proposition 3.1), such that $\Phi_{n+1}$ transform (3.7) to

$$H_{n+1} = e(\theta; \xi) + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} (\Omega_p^{n+1}(\xi) + B_p^{n+1}(\theta; \xi))|z_p|^2 + P_{n+1}(\theta, \varphi, z, \bar{z}; \xi),$$

where $B_p^{n+1}(\theta; \xi)$ is still of size $\epsilon_0$, and the perturbation $P_{n+1}(\theta, \varphi, z, \bar{z}; \xi)$ is of size $\epsilon_{n+1}$. However, compared to classical KAM iteration, $\epsilon_n$ shrinks to $0$ much faster (other than $\epsilon_{n+1} = \epsilon_n^{3/2}$), and Proposition 3.1 is proved with finite KAM iteration steps. The reason is the following: to eliminate the effect taken by $B_p^n(\theta; \xi)$, when one solves the homological equation (Proposition 5.1), one has to shrink the analytical strip of $\theta$ very quickly (that’s reason why we can only obtain $C^\infty$ solution), as a consequence, $\epsilon_n$ has to shrink much faster otherwise the homological equation doesn’t admit any analytical solution. Finally, finite KAM iteration steps are needed to ensure the fast decay of $\epsilon_n$.

3.2 The infinite induction

To begin with iteration, we first fix $\epsilon_0, r_0 > 0, s_0 > 0, \tau > d + 6, A > 10$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $U(\alpha) < \infty$, let $(Q_n)$ be the selected subsequence as in
Lemma 2.1. We then define the iteration sequences for \( n \geq 1 \):

\[
\begin{align*}
    s_n &= \epsilon_{n-1} \cdot s_{n-1}, \\
    r_n &= \frac{r_0}{4Q_n^2}, \\
    \epsilon_n &= \epsilon_{n-1} \cdot Q_{n+1}^{-2n+1} U, \\
    E_n &= \sum_{m=0}^{n-1} \epsilon_m, \\
    \gamma_n &= \gamma_0 - 3 \sum_{m=0}^{n-1} \epsilon_m, \\
    K_n &= r_0^{-1} 40 Q_{n+1}^4 \ln \epsilon_n, \\
    D_n &= D(r_n, s_n).
\end{align*}
\]

where \( c \) is a global constant with \( c > \frac{18\tau + 27}{2\tau U} \).

For convenience, for \( r, s, M_1, M_2, m > 0 \) and the parameter set \( O \), we define the space \( F_{r,s,O}(M_1, M_2, m) \) to be the functions

\[
e(\theta; \xi) + \sum_{p=1}^{\infty} (\Omega_p(\xi) + B_p(\theta; \xi)) |z_p|^2 + P(\theta, \varphi, z, \bar{z}; \xi)
\]

which satisfy

\[
\begin{align*}
|e(\theta; \xi)|_{r,O}^* &\leq M_1, \\
|\Omega_p(\xi)|_{O}^* &\leq \frac{1}{2} + M_1, \\
|B_p(\theta; \xi)|_{r,O}^* &\leq M_2, \\
|X_p|_{s,D(r,s)\times O}^* &\leq m.
\end{align*}
\]

Now we have the following result:

**Proposition 3.1** Suppose that \( \epsilon_0 \) is small enough so that

\[
\begin{align*}
    \epsilon_0 &\leq \min \left\{ \left( \frac{r_0 s_0 \gamma_0}{2^{2\tau U}} \right)^{12\tau + 36} \frac{1}{Q_1^{2\tau U}}, e^{-2\tau U} \right\}, \\
    \ln \epsilon_0^{-1} &\leq \epsilon_0^{\frac{1}{12\tau + 18}}.
\end{align*}
\]

Then the following holds for all \( n > 0 \): Let

\[
H_n = e_n(\theta; \xi) + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p=1}^{\infty} (\Omega_p^n(\xi) + B_p^n(\theta; \xi)) |z_p|^2 + P_n(\theta, \varphi, z, \bar{z}; \xi)
\]

which satisfy

1. For parameter \( \xi \in O_n, (k, l) \in \mathbb{Z}^2 \times \mathbb{Z}^d \) with \( |k| + |l| \leq K_n \) and \( p, q \geq 1 \) there is

\[
\begin{align*}
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle \pm \Omega_p^n| &\geq \frac{\gamma_n}{(|k| + |l| + 1)^\tau}, \\
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle \pm (\Omega_p^n + \Omega_q^n)| &\geq \frac{\gamma_n}{(|k| + |l| + 1)^\tau}, \\
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle \pm (\Omega_p^n - \Omega_q^n)| &\geq \frac{\gamma_n}{(|k| + |l| + 1)^\tau}, \quad |l| + |p - q| \neq 0;
\end{align*}
\]
2. The functions $e_n(\theta; \xi), B_p^n(\theta; \xi) \in B_{r_n}(O_n)$ with $[B_p^n(\theta; \xi)] = 0$.

3. The functions $\Omega_p^n = \Omega_p^n(\xi) - |p|^2$ with

$$e_n + \sum_{p \geq 1} (\Omega_p^n + B_p^n)|z_p|^2 + P_n \in \mathcal{F}_{r_n, s_n, O_n}(\mathcal{E}_n, \mathcal{E}_n, \epsilon_n).$$

Then there exists a real analytic symplectic transformation

$$\Phi_{n+1}: D(r_{n+1}, s_{n+1}) \times O_{n+1} \to D(r_n, s_n)$$

with

$$\text{meas}(O_n \setminus O_{n+1}) \leq \frac{C \gamma_{n+1}}{K_{n+1}}$$

and

$$|\Phi_{n+1} - id|_{s_{n+1}, D_{n+1} \times O_{n+1}}^* \leq \epsilon_n^2,$$

$$|D(\Phi_{n+1} - id)|_{s_{n+1}, D_{n+1} \times O_{n+1}} \leq \epsilon_n^2,$$

such that $H_{n+1} = H_n \circ \Phi_{n+1}$ satisfies the assumptions of $H_n$ with $n + 1$ in place of $n$.

4 Proof of the main results

4.1 Proof of Theorem 2

We are now in position to prove Theorem 2. We start with the Hamiltonian

$$H_0 = e_0(\theta; \xi) + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} (\Omega_p^0 + B_p^0(\theta; \xi))|z_p|^2 + P_0(\theta, \varphi, z, \bar{z}; \xi)$$

(4.1)

which is defined on $D(r_0, s_0) \times O_0$, where

$$r_0 = r, s_0 = s, \gamma_0 = \gamma, K_0 = r_0^{-1} \ln \epsilon_0^{-1}, e_0 = 0, \Omega_p^0 = \Omega_p, B_p^0 = 0, P_0 = P.$$

$$O_0 = \left\{ \xi \in O : \begin{array}{c} |\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \Omega_p^0| \geq \frac{\gamma_0}{(|k| + |l| + 1)^r}, \\
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + (\Omega_p^0 + \Omega_q^0)| \geq \frac{\gamma_0}{(|k| + |l| + 1)^r}, \\
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + (\Omega_p^0 - \Omega_q^0)| \geq \frac{\gamma_0}{(|k| + |l| + 1)^r}, \ |l| + |p - q| \neq 0 \end{array} \right\}$$

Then the assumption of Proposition 3.1 are satisfied for $n = 0$ since $\epsilon < \epsilon_0$ and $\mathcal{E}_0 = 0$, we thus get the symplectic transformation $\Phi_1: D(r_1, s_1) \times O_1 \to D(r_0, s_0)$. Inductively we obtain a sequence:

$$\Phi_{n+1}: D(r_{n+1}, s_{n+1}) \times O_{n+1} \to D(r_n, s_n),$$

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such that

\[
\Phi^{n+1} = \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_{n+1} : D(r_{n+1}, s_{n+1}) \times O_{n+1} \to D(r_0, s_0), \quad n \geq 0,
\]

conjugate the Hamiltonian (4.1) to

\[
H_{n+1} = e^{n+1}_n(\theta; \xi) + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} (\Omega^{n+1}_p + B^{n+1}_p(\theta; \xi)) |z_p|^2 + P_{n+1}(\theta, \varphi, z, \bar{z}; \xi)
\]

with estimates:

\[
e_{n+1} + \sum_{p \geq 1} (\Omega^{n+1}_p + B^{n+1}_p) |z_p|^2 + P_{n+1} \in F_{r_{n+1}, s_{n+1}, O_{n+1}}(E_{n+1}, E_{n+1}, \epsilon_{n+1}).
\]

and the symplectic map satisfy

\[
|\Phi_{n+1} - id|^{s_{n+1}, D_{n+1} \times O_{n+1}} \leq \frac{\epsilon_n}{2},
\]

\[
|D(\Phi_{n+1} - id)|^{s_{n+1}, s_{n+1}, D_{n+1} \times O_{n+1}} \leq \frac{\epsilon_n}{2}.
\]

For \( n \geq 0 \), by the chain rule, we get

\[
|D\Phi^{n+1}|_{s_0, s_{n+1}, D_{n+1}} \leq \prod_{m=1}^{n+1} |D\Phi_m|_{s_{m-1}, s_m, D_m} \leq \prod_{m=1}^{n+1} (1 + \epsilon_{m-1}^2) \leq 2 \quad (4.2)
\]

\[
|D\Phi^{n+1}|_{s_0, s_{n+1}, D_{n+1}} \leq \sum_{m=1}^{n+1} (|D\Phi_m|_{s_{m-1}, s_m, D_m} \prod_{j=1, j \neq m}^{n+1} |D\Phi_j|_{s_j, s_j, D_j}) \leq \sum_{m=1}^{n+1} 2\epsilon_{m-1}^2 \leq 2. \quad (4.3)
\]

As a consequence, we have

\[
|\Phi^{n+1} - \Phi^n|_{s_0, D_{n+1}} \leq |D\Phi^n|_{s_0, s_n, D_n} |\Phi_{n+1} - id|_{s_n, D_{n+1}} \leq 2\epsilon_n^2,
\]

and

\[
|\Phi^{n+1} - \Phi^n|_{s_0, D_{n+1}} \leq |D\Phi^n|_{s_0, s_n, D_n} |\Phi_{n+1} - id|_{s_n, D_{n+1}} + |D\Phi^n|_{s_0, s_n, D_n} |\Phi_{n+1} - id|_{s_n, D_{n+1}} \leq 2|\Phi_{n+1} - id|_{s_n, D_{n+1}} \leq 2\epsilon_n^2.
\]
The remaining task is to prove $C^\infty$ smoothness of $\Phi^\infty$ on $\Theta \in \mathbb{T}^{2+d}$. From the choice of parameter $\epsilon_n$, and for any $b \in \mathbb{Z}^{2+d}$, there exists some $N \in \mathbb{N}$ such that $Q^{4|b|}_{n+1} < \epsilon_n^{-1/4}$ for any $n \geq N$, that is

$$Q^{4|b|}_{n+1} \epsilon_n^4 < \epsilon_n^{-1}$$

Then according to the Cauchy estimate, one has

$$|\partial^{|b|} \Theta (\Phi^{n+1} - \Phi^n)| \leq r^{-|b|} \epsilon_n^{n+1} \leq Q^{4|b|}_{n+1} \epsilon_n^{4} \leq \epsilon_n$$

and also

$$|\partial^{|b|} \Theta (\Phi^{n+1} - \Phi^n)| \leq r^{-|b|} \epsilon_n^{n+1} \leq Q^{4|b|}_{n+1} \epsilon_n^{4} \leq \epsilon_n.$$

Thus $\Phi^n$ converges uniformly on $\mathbb{T}^{2+d} \times \{0, 0, 0\} \times \mathcal{O}_\infty$, and the limit $\Phi^\infty = \lim_{n \to \infty} \Phi^{n+1}$ is $C^\infty$ smooth on $\Theta$. Let $\phi^t_H$ be the flow of $X_H$, since $H \circ \Phi^n = H_n$, there is

$$\phi^t_H \circ \Phi^n = \Phi^{n+1} \circ \phi^t_H.$$

The uniform convergence of $\Phi^n, D\Phi^n$ and $X_{H_n}$ implies that the limits can be taken on both sides of (4.6). Hence, on $D(0,0) \times \mathcal{O}$ we get

$$\phi^t_H \circ \Phi^\infty = \Phi^\infty \circ \phi^t_H$$

and

$$\Psi^\infty : D(0,0) \times \mathcal{O}_\infty \to D(r,s).$$

By (3.13), the total measure we excluded is

$$\text{meas}(\mathcal{O} \setminus \mathcal{O}^\infty) \leq \sum_{n=0}^\infty \frac{C \gamma_n}{K_n} \leq C \gamma_0.$$  

4.2 Proof of Theorem 1:

As an application of Theorem 2, we study the equation (1.2) on some suitable phase space. As it is well known, the operator $\partial_{xx} + v(x)$ has an orthonormal basis $\phi_p \in L^2[0, \pi], p \geq 1$, with corresponding eigenvalues $\Omega_p$ satisfying the asymptotics for large $p$,

$$\Omega_p = p^2 + \frac{1}{\pi} \int_0^\pi v(x) dx + o(p^{-1}) $$

(4.8)
To write (1.2) in infinitely many coordinates, we make the ansatz
\[ u(t, x) = \mathcal{S} z = \sum_{p \geq 1} z_p(t) \phi_p(x), \ p \geq 1. \]

Then (1.2) is written as a non-autonomous Hamiltonian
\[ H(u) = \sum_{p \geq 1} \Omega_p |z_p|^2 + \epsilon \int_0^\pi F(\omega t, x, \mathcal{S} z, \mathcal{S} \bar{z}; \xi) dx. \]

with symplectic structure \( i \sum_{p \geq 1} dz_p \wedge d\bar{z}_p \), where \( F \) is a function such that
\( F_u(\Theta, x, u, \bar{u}; \xi) = f(\Theta, x, u, \bar{u}; \xi) \). Then one has a modified system
\[
\begin{aligned}
\dot{\theta} &= \omega_1, \\
\dot{\varphi} &= \omega_2, \\
\dot{z}_p &= -i \Omega_p z_p - i \partial_{z_p} P(\theta, \varphi, z, \bar{z}; \xi), \quad p \geq 1, \\
\dot{\bar{z}}_p &= i \Omega_p \bar{z}_p + i \partial_{\bar{z}_p} P(\theta, \varphi, z, \bar{z}; \xi), \quad p \geq 1,
\end{aligned}
\]

We introduce auxiliary action variable \( I, J \) and rewrite (4.9) to an autonomous system for convenience
\[
\begin{aligned}
\dot{\theta} &= \omega_1, \\
\dot{\varphi} &= \omega_2, \\
\dot{I} &= -\partial_\theta P(\theta, \varphi, z, \bar{z}; \xi) \\
\dot{J} &= -\partial_\varphi P(\theta, \varphi, z, \bar{z}; \xi) \\
\dot{z}_p &= -i \Omega_p z_p - i \partial_{z_p} P(\theta, \varphi, z, \bar{z}; \xi), \quad p \geq 1, \\
\dot{\bar{z}}_p &= i \Omega_p \bar{z}_p + i \partial_{\bar{z}_p} P(\theta, \varphi, z, \bar{z}; \xi), \quad p \geq 1.
\end{aligned}
\]

That is we consider the Hamiltonian
\[
H = N + P(\theta, \varphi, z, \bar{z}; \xi) = \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} \Omega_p |z_p|^2 + \epsilon \int_0^\pi F(\theta, \varphi, x, \mathcal{S} z, \mathcal{S} \bar{z}; \xi) dx
\]

with symplectic structure \( dI \wedge d\theta + dJ \wedge d\varphi + i \sum_{p \geq 1} dz_p \wedge d\bar{z}_p \).

Next let us verify that \( H = N + P \) satisfies the assumptions of Theorem 2. Recall the eigenvalue \( \Omega_p \) satisfy (4.8), and \( v(x) \) is independent of \( \xi \), thus one has
\[
\sup_{p \geq 1} |\Omega_p - p^2|_\mathcal{O} = \sup_{p \geq 1} \frac{1}{\pi} \int_0^\pi v(x) dx + o(p^{-1})|_\mathcal{O} < \frac{1}{2}.
\]

The regularity of the perturbation is given by the following basic Lemma.
Lemma 4.1 Suppose that \( v(x) \) is real analytic in \( x \), then for small enough \( r, s, a, \rho > 0 \), \( X_P \) is real analytic as a map from some neighborhood of the origin in \( \ell^{a, \rho} \) to \( \ell^{a, \rho} \), in particularly
\[
|X_P|_{s, D(r, s) \times \mathcal{O}} \leq \epsilon.
\]

Proof: From the hypotheses that \( v(x) \) is real analytic, it follows that the eigenfunctions \( \phi_p, p \geq 1 \) are analytic. Let \( u(t, x) = \sum_{p \geq 1} z_p \phi_i \) and \( \bar{u}(t, x) = \sum_{p \geq 1} \bar{z}_p \bar{\phi}_i \) with \( z, \bar{z} \in \ell^{a, \rho} \). Since
\[
\partial_{z_p} P(\theta, \varphi, z, \bar{z}; \xi) = -i\epsilon \int_0^\pi f(x, \sum_{p \geq 1} z_p \phi_i, \sum_{p \geq 1} \bar{z}_p \bar{\phi}_i; \xi) \phi_p dx.
\]
It follows that \( |X_P|_{s, D(r, s) \times \mathcal{O}} \leq \epsilon \).

Thus Theorem 2 is applicable, and the system (4.9) is conjugate to
\[
\begin{align*}
\dot{\theta} &= \omega_1, \\
\dot{\varphi} &= \omega_2, \\
\dot{z}_p &= -i(\Omega_p^* + B_p^*(\theta; \xi)) z_p - i\partial_{\bar{z}_p} P^*(\theta, \varphi, z, \bar{z}; \xi), \ p \geq 1, \\
\dot{\bar{z}}_p &= i(\Omega_p^* + B_p^*(\theta; \xi)) \bar{z}_p + i\partial_{z_p} P^*(\theta, \varphi, z, \bar{z}; \xi), \ p \geq 1,
\end{align*}
\]
by \( \Phi : T^{d+1} \times \{0, 0\} \to D(r, s) \). Since \( \Theta = (\theta, \varphi), \omega = (\omega_1, \omega_2), (\Theta^*(0) + \omega t; 0, 0) \) is a solution of (4.11). Let \( (\Theta(t); z(t); \bar{z}(t)) = \Phi(\Theta^*(0) + \omega t; 0, 0) \), by Theorem 2, \( z(t) = g(\Theta^*(0) + \omega t) \) is \( C^\infty \) smooth in \( t \). Then \( (\Theta(t); z(t); \bar{z}(t)) = (\Theta^*(0) + \omega t; g(\Theta^*(0) + \omega t); \bar{g}(\Theta^*(0) + \omega t)) \) is a solution of (4.9) for any \( \xi \in \mathcal{O}_\gamma \), and the equation (1.2) has a quasi periodic solution
\[
u(t, x) = \mathcal{S} z = \sum_{j \geq 1} g_j(\Theta^*(0) + \omega t) \phi_j(x) = \sum_{j \geq 1} g_j(\Theta^*(0) + \xi \omega t) \phi_j(x)
\]
which is \( C^\infty \) smooth in \( t \). Thus we have our result of Theorem 1.

5 Proof of Proposition 3.1

This main proposition is proved by KAM iteration. As we mentioned before, finite many iterations are required. Since our homological equation depends on the angle \( \theta \), it will be hard for us to solve this equation. Thus in the following, we first introduce an abstract result on the homological equation, a finite iteration lemma will be given and then we complete the proof of Proposition 3.1.
5.1 Homological equation

During the KAM iteration, a more complicated homological equation come out, namely:

\[-i\langle \partial_\theta F_\ell(\theta, \varphi), \omega_1 \rangle - i\langle \partial_\varphi F_\ell(\theta, \varphi), \omega_2 \rangle + \langle \ell, \Omega + B(\theta) \rangle F_\ell(\theta, \varphi) = R_\ell(\theta, \varphi), \]

where \( \ell \in \mathbb{Z}^N \) with \(|\ell| = 1 \) or \( 2 \). As \( B(\theta) \) is of size \( \varepsilon_0 \) and \((\omega_1, \omega_2)\) is Liouvillean, \(5.1\) will have no analytic solution. Actually, Wang-You-Zhou \[29\] met similar problem when they consider response solutions of harmonic oscillators, then the first and second Melnikov conditions are required for \( \forall \ell \in \mathbb{Z}^d \) with \(|\ell| = 1 \) or \( 2 \),

\[|\langle k, \bar{\omega}_1 \rangle + \langle \ell, \Omega(\lambda) \rangle| \geq \frac{\gamma}{(|k| + |\ell|)^{\tau}}. \]

(5.2)

For small divisor as above, the key observation is the following: for a very large and specialized truncation \( K \), \(|\langle k, \bar{\omega}_1 \rangle + \langle \ell, \Omega(\lambda) \rangle| \) has an uniform relative large lower bound for any \( k \) such that \(|k| \leq K \) (Lemma 3.2 of \[29\]). With this observation, they construct \( C^\omega \) smooth response solution for any \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). However, this phenomena is not right for the problem we meet, since in our case, \( \ell \in \mathbb{Z}^N \) with \(|\ell| = 1 \) or \( 2 \), therefore there are infinitely many choices of \( l \). Similar problem was also met during the work of Krikorian-Wang-You-Zhou \[23\], where they solved the following homological equation

\[\partial_{\omega_1} h(\theta, \phi) + (\rho + b(\phi)) \frac{\partial h}{\partial \theta} = f(\theta, \phi)\]

with Melnikov condition

\[|\langle k, \bar{\omega}_1 \rangle + \rho l| \geq \frac{\gamma}{(|k| + |l|)^{\tau}}, \quad \forall \ell \in \mathbb{Z} \setminus \{0\}, k \in \mathbb{Z}^2.\]

by the method of diagonally dominant (Proposition 4.1 of \[23\]). In this paper, we will borrow some method developed in \[23\], but gave more concise argument and uniform ways to deal with this kind of equations, we believe it will have more applications. Also we stress that we can deal with multi-frequencies, while all the former results were restricted to one frequency (thus two frequency in the continuous case).

**Proposition 5.1** Let \( \gamma > 0, \lambda \geq 1, \tau > d+6 \) and \( \zeta \in \mathbb{R} \setminus \{0\}, 0 < \bar{\sigma} < \bar{\tau} < \tau, 0 < \eta_1, \eta_2, \bar{\eta} < 1 \). Consider the equation

\[-i\langle \partial_\theta F(\theta, \varphi), \bar{\omega}_1 \rangle - i\langle \partial_\varphi F(\theta, \varphi), \bar{\omega}_2 \rangle + (\zeta + B(\theta) + b(\theta))F(\theta, \varphi) = R(\theta, \varphi)\]

(5.3)
with $[B(\theta)] = [b(\theta)] = 0$. Suppose that $|B(\theta)|_r \leq \eta_1, |b(\theta)|_r \leq \eta_2, |R(\theta)|_r \leq \tilde{\eta}$ which furthermore satisfy the following condition:

1. $\eta_1 e^{-|r - \tilde{r}|Q_{n+1}} \leq \eta_2$
2. $2\eta_1 Q_{n+1} \tilde{r} \leq (r - \tilde{r})^4$,
3. $K = \frac{1}{\sigma} \ln \frac{1}{\eta} \leq (\frac{\gamma}{2\eta_2})^{\frac{1}{\gamma^2}}$
4. $|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \zeta| \geq \frac{\gamma^2}{(|k| + |l|) + 1}, \quad |k| + |l| \leq K$.

Then the equation (5.3) has an approximate solution $F(\theta, \varphi)$ with estimation

$$|F(\theta, \varphi)|_{\tilde{r} - \tilde{\sigma}} \leq \frac{c_{\tilde{\eta}}}{\lambda \gamma^{3+\tau}}.$$  \hspace{1cm} (5.4)

Moreover, the error term satisfies

$$|\tilde{R}|_{\tilde{r} - \tilde{\sigma}} = |e^{iB(\theta)}(I - T_K)(e^{-iB(\theta)}R(\theta, \varphi) - (b(\theta) + (I - T_K)B(\theta))F(\theta, \varphi))|_{\tilde{r} - \tilde{\sigma}} \leq \frac{\eta^2}{\lambda \gamma^{3+\tau}},$$ \hspace{1cm} (5.5)

where $B(\theta)$ is the solution of $\langle \partial_\theta B(\theta), \omega_1 \rangle = T_{Q_{n+1}}B(\theta)$.

**Remark 5.1** The above Proposition 5.1 holds irrespectively of any arithmetical property of $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2)$.

**Remark 5.2** The assumption 2 means there is a quick shrink from analytic radius $r$ to $\tilde{r}$. We have this assumption if $r = \frac{r_0}{Q_n}$ and $\tilde{r} \leq \frac{r_0}{Q_{n+1}}$.

**Proof:** Let $\tilde{F}(\theta, \varphi) = F(\theta, \varphi)e^{iB(\theta)}$, where $B(\theta)$ is the solution of

$$\langle \partial_\theta B(\theta), \bar{\omega}_1 \rangle = T_{Q_{n+1}}B(\theta).$$

Then we consider equation

$$T_K(-i\langle \partial_\theta \tilde{F}, \omega_1 \rangle - i\langle \partial_\varphi \tilde{F}, \omega_2 \rangle + (\zeta + \tilde{b}(\theta))\tilde{F}(\theta, \varphi)) = T_K\tilde{R}(\theta, \varphi)$$ \hspace{1cm} (5.6)

with $\tilde{b}(\theta) = (I - T_K)B(\theta) + b(\theta), \tilde{R}(\theta, \varphi) = e^{iB(\theta)}R(\theta, \varphi)$. By the assumption 1, one has

$$|\tilde{b}(\theta)|_{\tilde{r}} \leq e^{-|r - \tilde{r}|Q_{n+1}}\eta_1 + \eta_2 \leq 2\eta_2.$$ \hspace{1cm} (5.7)
Since we only seek approximation of (5.3), we set $T_{\tilde{F}} = \tilde{F}$ for convenience below.

In order to control the norm of $\tilde{R}$, which is a conjugation of $R$ by $e^{iB}$, it is sufficient to estimate $\text{Im } B$. As argued in Lemma 4.1 of [30], let $\theta = x + iy$ and recall $B(\theta) = \sum_{0 < |k| < Q_{n+1}} \frac{\hat{B}(k)}{i(k, \tilde{\omega}_1)} e^{i(k, \theta)}$ with $|\hat{B}(k)| \leq |B| e^{-|k|/\tau}$, we define

$$B^1 = \sum_{0 < |k| < Q_{n+1}} \frac{\hat{B}(k)}{i(k, \tilde{\omega}_1)} e^{i(k, x)} , \quad B^2 = B - B^1 . \quad (5.8)$$

Since $B(\theta)$ is real analytic, one has $\text{Im } B^1 = 0$ and $\text{Im } B = \text{Im } B^2$.

$$|\text{Im } B|_\tau = |\text{Im } B^2|_\tau \leq \sum_{1 \leq |k| < Q_{n+1}} \left| \frac{\hat{B}^n(k)}{i(k, \tilde{\omega}_1)} \right| |e^{-\langle k, y \rangle} - 1| \leq |B(\theta)|_\tau Q_{n+1} \sum_{1 \leq |k| < Q_{n+1}} 2e^{-|\tau - \tilde{\tau}|} |k| \tilde{\tau} \leq \frac{\eta Q_{n+1}^2 \tilde{\tau}}{(\tau - \tilde{\tau})^4} \leq \frac{1}{2} . \quad (5.9)$$

where the last inequality is given by assumption 2. As a consequence, we have

$$|\tilde{R}(\theta, \varphi)|_{\tilde{\tau}} \leq e^{|\text{Im } B|_\tau} \cdot |R(\theta, \varphi)|_{\tilde{\tau}} \leq 2\tilde{\eta} . \quad (5.10)$$

Now we start to solve the equation (5.6).

Let

$$\tilde{F}(\theta, \varphi) = \sum_l \tilde{F}_l(\theta)e^{i(l, \varphi)} , \quad \tilde{R}(\theta, \varphi) = \sum_l \tilde{R}_l(\theta)e^{i(l, \varphi)} ,$$

Then (5.6) is equivalent to the equations below for $|l| \leq K$,

$$(A_l + D_l) \tilde{F}_l = \tilde{R}_l , \quad (5.11)$$

where

$$A_l = \text{diag}(\cdots , \langle k, \tilde{\omega}_1 \rangle + (l, \tilde{\omega}_2) + \zeta, \cdots ))_{|k| < K - |l|} , \quad D_l = (\hat{b}(k_i - k_j))_{|k_i|, |k_j| \leq K - |l|}$$

$$\tilde{F}_l = (\cdots , \tilde{F}_l^k , \cdots )_{|k| \leq K - |l|} , \quad \tilde{R}_l = (\cdots , \tilde{R}_l^k , \cdots )_{|k| \leq K - |l|} .$$
Let $M_{l,r'} = \text{diag}(\cdots, e^{\|k^r\|}, \cdots)\|k\| \leq K - |l|$ for any $r' \leq \tilde{r}$, then (5.11) is equivalent to

$$M_{l,r'}(A_l + D_l)M_{l,r}^{-1}M_{l,r'}\tilde{F}_l = M_{l,r'}\tilde{R}_l.$$  

We rewrite it to be

$$(A_l + D_l)\tilde{F}_{l,r'} = \tilde{R}_{l,r'} \quad (5.12)$$

where $D_{l,r'} = M_{l,r'}D_l M_{l,r'}^{-1}$, $\tilde{F}_{l,r'} = M_{l,r'}\tilde{F}_l$, $\tilde{R}_{l,r'} = M_{l,r'}\tilde{R}_l$. A simple calculation show that $\|D_{l,r'}\| \leq (K - |l|)^{2/\eta_2}$.

By assumption 3 and 4, we have

$$|\langle k, \bar{\omega}_1 \rangle + \langle l, \bar{\omega}_2 \rangle + \zeta| \geq \frac{\gamma \lambda}{(\|k\| + |l| + 1)^{r'}} \geq \frac{\gamma \lambda}{K^{r'}} \geq (K - |l|)^{1/2} \eta_2.$$  

for all $|k| + |l| < K$. As a result, the diagonally dominant operators $A_l + D_{l,r'}$ has a bounded inverse and $\| (I + A_l^{-1}D_{l,r'})^{-1} \|_{\text{op}(l^1)} < 2$ for $r' = \tilde{r} - \tilde{\sigma}$, where $\| \cdot \|_{\text{op}(l^1)}$ denotes the operator norm associated to the $l^1$ norm $|u|_{l^1} = \sum_{|k| < K - |l|} |u^k|$. To see this, one can compute

$$\| A_l^{-1}D_{l,r'} \|_{\text{op}(l^1)} \leq \max_{|k| \leq K} \sum_{|k_j| \leq K} \frac{|\tilde{b}(k_i - k_j)| e^{\|k_i - |k_j|\|} r'}{(\langle k_i, \bar{\omega}_1 \rangle + \langle l, \bar{\omega}_2 \rangle + \zeta)} \quad (5.13)$$

$$\leq \max_{|k| \leq K} \sum_{|k_j| \leq K} \frac{e^{-|k_i - k_j| |\tilde{r} + (|k_i| - |k_j|)|} r'}{\gamma \lambda} \leq \frac{K^{\tau + 2/\eta_2}}{\gamma \lambda} \leq \frac{1}{2},$$

where the last inequality follows from the assumption 3.

Since

$$(I_l + A_l^{-1}D_{l,r'})^{-1} = \sum_{n=0}^{\infty} (-1)^n (A_l^{-1}D_{l,r'})^n,$$

one has $\| (I_l + A_l^{-1}D_{l,r'})^{-1} \|_{\text{op}(l^1)} \leq 2$. As a conclusion, the approximate
solution \( \mathcal{F}_{l,r'} = (I + A_{l}^{-1}D_{l,r'})^{-1}A_{l}^{-1}\mathcal{F}_{l,r'} \) is regular with
\[
|\tilde{F}(\theta, \varphi)|_{\tilde{\tau} - \tilde{\sigma}} \leq \sum_{|k| + |l| < K} |\tilde{F}_{l}^{k}| e^{(|k| + |l|)(\tilde{\tau} - \tilde{\sigma})} = \sum_{|l| \leq K} \sum_{|k| < K - |l|} |\tilde{F}_{l}^{k}| e^{(|k| + |l|)(\tilde{\tau} - \tilde{\sigma})} e^{l(\tilde{\tau} - \tilde{\sigma})} = \sum_{|l| \leq K} |\tilde{F}_{l}\tilde{\tau}\tilde{\sigma}|_{l} e^{l(\tilde{\tau} - \tilde{\sigma})} = \sum_{|l| \leq K} \| \tilde{R}_{l}\|_{l} e^{l(\tilde{\tau} - \tilde{\sigma})}
\]
\[
\leq 2 \sum_{|l| \leq K} \sum_{|k| < K - |l|} \frac{(|k| + |l| + 1)\gamma}{\lambda \gamma} |\tilde{R}_{l}^{k}| e^{(|k| + |l|)(\tilde{\tau} - \tilde{\sigma})} \leq \frac{c\gamma}{\lambda \gamma \tilde{\sigma}^{3+\tau}}.
\]

Let \( F = e^{-iB(\theta)} \tilde{F} \), like (5.10), one has
\[
|F(\theta, \varphi)|_{\tilde{\tau} - \tilde{\sigma}} \leq e^{\text{Im}B(\theta)\tilde{\tau}} |\tilde{F}(\theta, \varphi)|_{\tilde{\tau} - \tilde{\sigma}} \leq \frac{c\gamma}{\lambda \gamma \tilde{\sigma}^{3+\tau}}.
\]

Moreover, one can compute \( F \) will solve equation (5.3) with error term \( \tilde{R} \),
\[
\tilde{R} \equiv e^{iB(\theta)}(I - \mathcal{T}_{K})(e^{-iB(\theta)} R(\theta, \varphi) - (b(\theta) + (I - \mathcal{T}_{K})B(\theta))F(\theta, \varphi)).
\]

The estimation on \( \tilde{R} \) is a direct computation.

### 5.2 A finite KAM induction

We will prove Proposition 3.1 by induction. We start with the Hamiltonian
\[
\hat{H}_{0} = \tilde{c}_{0}(\theta; \xi) + \langle \omega_{1}, I \rangle + \langle \omega_{2}, J \rangle + \sum_{p \geq 1} (\tilde{\omega}_{p}(\xi) + B_{p}(\theta; \xi))|z_{p}|^{2} + \tilde{P}_{0}(\theta, \varphi, z, \tilde{z}; \xi)
\]
on \( D(\tilde{r}_{0}, \tilde{s}_{0}) \times \tilde{\mathcal{O}}_{0}, \tilde{c}_{0} = c_{n}, \tilde{\mathcal{H}}_{0} = \mathcal{E}_{n} \), where
\[
\tilde{r}_{0} = 2r_{n+1}, \tilde{s}_{0} = s_{n}, \tilde{\mathcal{O}}_{0} = \mathcal{O}_{n}, \quad \tilde{c}_{0} = \mathcal{E}_{n}, \quad \tilde{\mathcal{O}}_{0} = \mathcal{O}_{n},
\]
and \([B_{p}(\theta; \xi)] = 0, \]
\[
\tilde{c}_{0} + \sum_{p \geq 1} (\tilde{\omega}_{p}^{0} + B_{p})|z_{p}|^{2} + \tilde{P}_{0} \in \mathcal{F}_{\tilde{r}_{0}, \tilde{s}_{0}, \tilde{\mathcal{O}}_{0}}(\mathcal{E}_{n}, \mathcal{E}_{n}, \tilde{c}_{0}).
\]

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Let $N = 2 + \left[\frac{2^{n+1}rU\ln Q_{n+1}}{6r+9}\right]$ and $K^n = \left(\frac{2^n}{4\tau}\right)^{\frac{1}{1+\gamma}}$, where the former will control the steps of finite KAM iteration and the later is the upper bound control in assumption 4 of Proposition 5.1. Define the following iteration sequence for $j = 1,2,3,\cdots,N$,

\[
\tilde{e}_j = \left(\tilde{e}_{j-1}\right)^{\frac{1}{2}} = (\tilde{e}_0)^{\frac{1}{2}}, \quad \tilde{E}_j = \sum_{q=0}^{j-1} \tilde{e}_q,
\]

\[
\tilde{\gamma}_j = \gamma_n - 2 \sum_{q=0}^{j-1} \tilde{e}_q^2, \quad \tilde{\eta}_j = \left(\tilde{e}_{j-1}\right)^{\frac{1}{2}},
\]

\[
\tilde{\sigma}_j = \tilde{\sigma}_j^{-1} \ln \tilde{\epsilon}_j^{1}, \quad \tilde{\sigma}_j = \frac{1}{3} \tilde{\eta}_j, \quad \tilde{\epsilon}_j = \tilde{\epsilon}_j^{1}, \quad \tilde{\epsilon}_j = \tilde{\epsilon}_j^{1} - 5 \tilde{\sigma}_j,
\]

\[
\tilde{D}_j = D(\tilde{r}_j, \tilde{s}_j).
\]

We also set $\tilde{K}_0 = K_n$. Then we have the following iteration lemma:

**Lemma 5.1** The following holds for $0 \leq j \leq N - 1$. Suppose the Hamiltonian

\[
\tilde{H}_j = \tilde{e}_j(\theta; \xi) + \langle \omega_1, \ell \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} (\tilde{\Omega}_p^j(\xi) + B_p(\theta; \xi) + b_p^j(\theta; \xi)) |z_p|^2 + \tilde{P}_j(\theta, \varphi, z, \bar{z}; \xi)
\]

is defined on $\tilde{D}_j \times \tilde{O}_j$ with $B_p \in B^n(\mathcal{O}_n)$ and $[B_p(\theta; \xi)] = 0$, which furthermore satisfy

1. For any $\xi \in \tilde{O}_j$, $|k| + |l| \leq \tilde{K}_j$ and $p, q \geq 1$ there is

\[
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \tilde{\Omega}_p^j| \geq \frac{\tilde{\gamma}_j}{(|k| + |l| + 1)^{\tilde{\sigma}_j}},
\]

\[
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + (\tilde{\Omega}_p^j + \tilde{\Omega}_p^j)| \geq \frac{\tilde{\gamma}_j}{(|k| + |l| + 1)^{\tilde{\sigma}_j}},
\]

\[
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + (\tilde{\Omega}_p^j - \tilde{\Omega}_p^j)| \geq \frac{\tilde{\gamma}_j}{(|k| + |l| + 1)^{\tilde{\sigma}_j}}, |l| + |p - q| \neq 0.
\]

2. The functions $b_p^j \in B_{\tilde{r}_j}(\tilde{O}_j)$ have average zero: $[b_p^j(\theta; \xi)] = 0$, and there is

\[
\tilde{e}_j + \sum_{p \geq 1} (\tilde{\Omega}_p^j + b_p^j)|z_p|^2 + \tilde{P}_j \in F_{\tilde{r}_j, \tilde{s}_j, \tilde{O}_j}(\tilde{E}_n + \tilde{E}_j, \tilde{E}_j, \tilde{E}_j)
\]
Then there exists a subset \( \tilde{\mathcal{O}}_{j+1} = 0 \mathcal{O}_j \setminus \mathcal{R}^{j+1} \) with \( \operatorname{meas}(\mathcal{R}^{j+1}) \leq \frac{C_{r+1}}{R_{j+1}} \), and a symplectic transformation \( \tilde{\phi}_{F_j} : \tilde{D}_{j+1} \times \tilde{\mathcal{O}}_j \to \tilde{D}_j \) with estimate

\[
|\tilde{\phi}_{F_j}^t - id|_{\hat{D}_{j+1} \times \mathcal{O}_j}^e \leq \tilde{e}_j^3, \tag{5.18}
\]

\[
|D\tilde{\phi}_{F_j}^t - Id|_{\hat{D}_{j+1} \times \mathcal{O}_j}^e \leq \tilde{e}_j^3, \tag{5.19}
\]

such that \( \tilde{H}_{j+1} = \tilde{H}_j \circ \tilde{\phi}_{F_j}^1 \) satisfies the assumptions of \( \tilde{H}_j \) with \( j+1 \) in place of \( j \).

**Remark 5.3** The crucial point for us is that the functions \( B_p(\theta; \xi), p \geq 1 \) are fixed in the iteration.

Once we have Proposition 5.1, the proof of this Lemma is standard KAM, we leave it to the appendix.

### 5.3 The Construction of \( \Phi_{n+1} \) and \( H_{n+1} \)

Now we are going to finish the proof of Proposition 3.1. As a beginning, we fix \( \bar{H}_0 = H_n \) with

\[
\bar{H}_0 = \tilde{e}_0 + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} ((\tilde{\Omega}_p^0 + B_p + b_p^0)|z_p|^2 + \tilde{P}_0
\]

on \( D(\bar{r}_0, \bar{s}_0) \times \mathcal{O}_0 \), where

\[
\bar{e}_0 = e_n, \bar{r}_0 = 2r_{n+1}, \bar{s}_0 = s_n, \tilde{K}_0 = K_n, \mathcal{O}_n, \bar{e}_0 = e_n(\theta; \xi),
\]

\[
B_p = B_p^n(\theta; \xi), b_p^0 = 0, \tilde{\Omega}_p^0 = \Omega_p^n(\xi), \tilde{P}_0 = P_n(\theta, \varphi, z, \xi).
\]

By the assumption of Proposition 3.1, \( e_n = e_{n-1} \cdot Q^{-2n+1}_{n+1} r_n \), \( r_{n+1} = \frac{r_n}{4Q_{n+1}^4} \), the truncation parameter \( K_n = r_n^{-1}4Q_{n+1}^4 \ln e_n^{-1} \), and the Diophantine condition for \( \xi \in \mathcal{O}_n \) is given, thus the assumptions of iteration Lemma 5.1 are satisfied with \( j = 0 \). Inductively, we iterate Lemma 5.1 \( N \) times, we arrive at parameter set \( \tilde{\mathcal{O}}_N \) and the Hamiltonian

\[
\bar{H}_N = \tilde{e}_N + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} (\tilde{\Omega}_p^N + B_p + b_p^N)|z_p|^2 + \tilde{P}_N \quad \tag{5.20}
\]

on \( \tilde{D}_N \times \tilde{\mathcal{O}}_{N-1} \) with

\[
\tilde{e}_N + \sum_{p \geq 1} (\tilde{\Omega}_p^N + b_p^N)|z_p|^2 + \tilde{P}_N \in \mathcal{F}_{\bar{r}_N, \bar{s}_N, \mathcal{O}_{N-1}}(\bar{e}_N + \tilde{e}_N, \tilde{e}_N + \tilde{e}_N) \quad \tag{5.21}
\]

First we have the following observation:
Lemma 5.2 We can select

\[ O_{n+1} = \tilde{O}_N. \]  

(5.22)

Proof: By (5.15), we have

\[ \gamma_{n+1} = \gamma_0 - 3 \sum_{i=0}^{n} e_i^4 = \gamma_n - 3 \epsilon_n^4 \leq \gamma_n - 2 \sum_{i=0}^{N} e_i^4 = \tilde{\gamma}_N, \]

and

\[ K_{n+1} = \frac{40 \ln \epsilon_{n+1}}{r_{n+2}} \leq \frac{\ln \epsilon_0}{r_{n+1} \left( \frac{1}{2} \right)^{N-1}} = \tilde{K}_N. \]

Thus for any \(|k| + |l| \leq K_{n+1}\) we have

\[ |\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle \pm \tilde{\Omega}_p^N| \geq \frac{\tilde{\gamma}_N}{(|k| + |l| + 1)^r} \geq \frac{\gamma_{n+1}}{(|k| + |l| + 1)^r}, \]

\[ |\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle \pm (\tilde{\Omega}_p^N + \tilde{\Omega}_q^N)| \geq \frac{\gamma_N}{(|k| + |l| + 1)^r} \geq \frac{\gamma_{n+1}}{(|k| + |l| + 1)^r}, \]

\[ |\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle \pm (\tilde{\Omega}_p^N - \tilde{\Omega}_q^N)| \geq \frac{\gamma_N}{(|k| + |l| + 1)^r} \geq \frac{\gamma_{n+1}}{(|k| + |l| + 1)^r}, \]

which just ensures us to choose \( O_{n+1} = \tilde{O}_N. \)

Recall (5.15) and (3.8), one has

\[ \tilde{r}_N = \tilde{r}_0 - \sum_{j=0}^{N-1} \tilde{r}_0 \left( \frac{4}{3} \right)^j \geq \frac{\tilde{r}_0}{2} = r_n \geq r_{n+1}, \]

\[ \tilde{s}_N = \sum_{j=0}^{N-1} \left( \frac{\tilde{r}_0}{2} \right)^j \left( \frac{4}{3} \right)^j \tilde{s}_0 = \epsilon_n^2 \left( \frac{2^{n+1} c \ln q_{n+1}}{6r + 9} \right)^{1-1} \cdot s_n = s_{n+1}, \]

and then \( \tilde{H}_N \) is regular on \( D(r_{n+1}, s_{n+1}). \) Thus we can fix

\[ \Phi_{n+1} = \phi_{F_0} \circ \phi_{F_1} \circ \ldots \circ \phi_{F_{N-1}} : \tilde{D}_N \times \tilde{O}_{N-1} \rightarrow \tilde{D}_0, \]

which now is defined on \( D(r_{n+1}, s_{n+1}) \times O_{n+1} \rightarrow D(r_n, s_n). \) We also fix \( H_{n+1} = \tilde{H}_N \) with

\[ e_{n+1} = \tilde{e}_N, B_p^N = B_p^n + \hat{b}_p^N, \Omega_p^{n+1} = \tilde{\Omega}_p^N, P_{n+1} = \tilde{P}_N. \]

(5.23)

We have our Proposition once we have following estimate.
Lemma 5.3 We have the following estimate:

\[ e_{n+1} + \sum_{p\geq 1} (\Omega_p^{n+1} + B_p^{n+1}) |z_p|^2 + P_{n+1} \in F_{r_{n+1}, s_{n+1}, O_{n+1}} (\mathcal{E}_{n+1}, \mathcal{E}_{n+1}, e_{n+1}) \]

and furthermore we have

\[ |\Phi_{n+1} - id|_{s_{n+1}, D_{n+1} \times O_{n+1}} \leq \frac{\epsilon}{2}, \quad |D(\Phi_{n+1} - id)|_{s_{n+1}, s_{n+1}, D_{n+1} \times O_{n+1}} \leq \frac{\epsilon}{2}, \tag{5.24} \]

Proof: First we note

\[ \left( \frac{4}{3} \right)^N - 1 \geq \left( \frac{4}{3} \right)^{N-1} \geq \left( \frac{4}{3} \right)^{2n+1} e^{2n+1} \ln Q_{n+1} = Q_{n+1}^{2n+1} e^{2n+1} \ln Q_{n+1}, \]

and \( \tilde{\epsilon}_0 = \epsilon_0 \leq \epsilon_0 \leq e^{-2cU} \) by our selection, we have

\[ \frac{(4/3)^N}{\tilde{\epsilon}_0} = \tilde{\epsilon}_0 e^{(4/3)^N-1} \ln \tilde{\epsilon}_0 \leq \tilde{\epsilon}_0 e^{(4/3)^N-1} \ln Q_{n+1}^{2n+1} e^{2n+1} \ln Q_{n+1} \]

\[ \leq \tilde{\epsilon}_0 e^{-2cU} Q_{n+1}^{2n+1} e^{2n+1} \ln Q_{n+1} \leq \tilde{\epsilon}_0 Q_{n+1}^{2n+1} e^{2n+1} \ln Q_{n+1} = \epsilon_{n+1}, \]

therefore

\[ |X_{\Phi_{n+1}}|_{s_{n+1}, D_{n+1} \times O_{n+1}} \leq \tilde{\epsilon}_0 \leq \epsilon_{n+1}. \]

The estimation below is a direct calculous,

\[ |B_{n+1}^{n+1}|_{r_{n+1}, O_{n+1}} \leq \mathcal{E}_n + \tilde{\mathcal{E}}_N \leq \mathcal{E}_n + \sum_{i=0}^{j-1} \tilde{\epsilon}_i \leq \mathcal{E}_n + 2\epsilon_n \leq \mathcal{E}_{n+1}. \]

Similarly, one has \( |\epsilon_{n+1}|_{r_{n+1}, O_{n+1}} \leq \mathcal{E}_{n+1} \) and \( |\partial_{n+1}^{n+1}|_{O_{n+1}} \leq \mathcal{E}_{n+1}. \)

To prove (5.24) and (5.25), for any \( 0 \leq \nu \leq N - 1 \), we let

\[ \phi^\nu = \phi_{F_0} \circ \phi_{F_1} \circ \ldots \circ \phi_{F_\nu} : \tilde{D}_{\nu+1} \times \tilde{O}_{\nu} \to \tilde{D}_0. \]

By the chain rule, we have estimate,

\[ |D\phi^\nu|_{\tilde{s}_0, \tilde{s}_{\nu+1}, \tilde{D}_{\nu+1}} \leq \prod_{\mu=0}^{\nu} |D\phi_{F_{\mu}}|_{\tilde{s}_\mu+1, \tilde{s}_{\mu+1}, \tilde{D}_{\mu+1}} \leq \sum_{\mu=0}^{\nu} \tilde{\epsilon}_\mu \leq \epsilon, \]

\[ |D\phi^\nu|_{\tilde{s}_0, \tilde{s}_{\nu+1}, \tilde{D}_{\nu+1}} \leq \sum_{\mu=0}^{\nu} |D\phi_{F_{\mu}}|_{\tilde{s}_\mu, \tilde{s}_{\mu+1}, \tilde{D}_{\mu+1}} \prod_{\substack{j=0, j \neq \mu}}^{\nu} |D\phi_{\tilde{F}_j}|_{\tilde{s}_{j+1}, \tilde{s}_{j+1}, \tilde{D}_{j+1}} \]

\[ \leq e \sum_{\mu=0}^{\nu} |D\phi_{F_{\mu}}|_{\tilde{s}_\mu, \tilde{s}_{\mu+1}, \tilde{D}_{\mu+1}} \leq 2e \sum_{\mu=0}^{\nu} \tilde{\epsilon}_\mu \leq \frac{3}{3}. \]
With mean value theorem,
\[
|\phi^\nu - id|^*_{s_0, D_{\nu+1}} \leq \sum_{\mu=0}^{\nu-1} |\phi^{\nu+1}_\mu - \phi^{\mu+1}_\mu|^*_{s_\mu, s_{\mu+1}} |D\phi^{\mu+1}_\mu|_{s_\mu, s_{\mu+1}} |\phi^{\nu+1}_{F_{\mu+1}} - id|^*_{s_{\mu+1}, D_{\mu+1}}
\]
\[
\leq \frac{3}{2} \sum_{\mu=0}^{\nu-1} \epsilon^3_\mu \leq 2 \frac{N-1}{2} \sum_{\mu=0}^{\nu-1} \epsilon^3_\mu \leq c\epsilon^n_n,
\]
and then with generalized Cauchy estimate,
\[
|D\phi^\nu - I|^*_{s_0, s_{\nu+1}, D_{\nu+1}} \leq c\epsilon^n_n Q^3_n \leq \frac{2}{4}.\]

Therefore \( \Phi_{n+1} = \phi^{N-1} \) is the transformation we are searching.

5.4 Measure estimate

At the \( j \)-1-th finite KAM iteration of the \( n \)-th infinite iteration, we have to exclude the following resonant set:
\[
\tilde{R}^j = \bigcup_{|k| + |l| \leq \tilde{K}_j} (\bigcup_{p \geq 1} \mathcal{R}^j_{klp} \bigcup_{|l| + |p-q| \neq 0} \mathcal{R}^{j1}_{klpq} \bigcup_{p, q \geq 1} \mathcal{R}^{j2}_{klpq}),
\] (5.26)

where
\[
\mathcal{R}^j_{klp} = \{\xi \in \tilde{O}_{j-1} : |\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle \pm \tilde{\Omega}_p^j| < \frac{\tilde{\gamma}_j}{(|k| + |l| + 1)^{1/2}}\},
\]
\[
\mathcal{R}^{j1}_{klpq} = \{\xi \in \tilde{O}_{j-1} : |\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \tilde{\Omega}_p^j - \tilde{\Omega}_q^j| < \frac{\tilde{\gamma}_j}{(|k| + |l| + 1)^2}\},
\]
\[
\mathcal{R}^{j2}_{klpq} = \{\xi \in \tilde{O}_{j-1} : |\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle \pm (\tilde{\Omega}_p^j + \tilde{\Omega}_q^j)| < \frac{\tilde{\gamma}_j}{(|k| + |l| + 1)^2}\}.
\]

In order to estimate the measure of the resonant set \( \tilde{R}^j \), we first need the following observation:

**Lemma 5.4** For any \( |k| + |l| \leq \tilde{K}_{j-1} \) and \( p, q \geq 1 \), then the resonant set satisfy
\[
\mathcal{R}^j_{klp} = \mathcal{R}^{j1}_{klpq} = \mathcal{R}^{j2}_{klpq} = \emptyset.
\]

**Proof:** As an example, we prove that
\[
\mathcal{R}^{j1}_{klpq} = \emptyset, \text{ if } |k| + |l| \leq \tilde{K}_{j-1}.
\]

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By the regularity of $X_{\tilde{P}_{j-1}}$, one has
\[|\tilde{\Omega}_p^j - \tilde{\Omega}_q^j|_{C_{j-1}} \leq \tilde{\epsilon}_{j-1}, p \geq 1.\]

It follows that for any $|k| + |l| \leq \tilde{K}_{j-1}$, there is
\[
\begin{align*}
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \tilde{\Omega}_p^j - \tilde{\Omega}_q^j| & \geq |\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \tilde{\Omega}_p^j - \tilde{\Omega}_q^j| - |\tilde{\Omega}_p^j - \tilde{\Omega}_q^j| - |\tilde{\Omega}_p^j - \tilde{\Omega}_q^j| \\
& \geq \frac{\tilde{\gamma}_{j-1}}{|k| + |l| + 1} - 2\tilde{\epsilon}_{j-1} \geq \frac{\tilde{\gamma}_{j-1}}{|k| + |l| + 1}. 
\end{align*}
\]

The last inequality is possible since $\tilde{\epsilon}_{j-1}|\tilde{K}_{j-1}| \leq (\tilde{\gamma}_{j-1} - \tilde{\gamma}_j)$ by the iteration sequence (5.15).

Lemma 5.5 If $\max\{p, q\} > c(|k| + |l|), p \neq q$, then the resonant set satisfy
\[R_{klp}^j = R_{klpq}^{j1} = R_{klpq}^{j2} = \emptyset.\]

Proof: As an example, we only prove that
\[R_{klpq}^{j1} = \emptyset, \text{ if } \max\{p, q\} > c(|k| + |l|), p \neq q.\]

In fact, it follows from the following computations:
\[
\begin{align*}
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \tilde{\Omega}_p^j - \tilde{\Omega}_q^j| & \geq |p^2 - q^2| - |\tilde{\Omega}_p^j - p^2| - |\tilde{\Omega}_q^j - q^2| - |\tilde{\epsilon}(|k| + |l|)|\tilde{\omega}| \\
& \geq |p^2 - q^2| - 2\epsilon_0 - \frac{3}{2} |\tilde{\omega}||k| + |l| > \frac{|p^2 - q^2|}{2}.
\end{align*}
\]

The others can be handled in the same way.

Lemma 5.6 If $l \neq 0$ and $p = q$, then the resonant set satisfy $R_{klpq}^{j1} = \emptyset$.

Proof: In this case, since we assume $\tilde{\omega} \in W L(\gamma, \tau, \beta)$, then
\[
\begin{align*}
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \tilde{\Omega}_p^j - \tilde{\Omega}_q^j| & = \xi|\langle k, \tilde{\omega}_1 \rangle + \langle l, \tilde{\omega}_2 \rangle| \geq \frac{\tilde{\gamma}_0}{2(|k| + |l| + 1)^r} \geq \frac{\tilde{\gamma}_j}{2(|k| + |l| + 1)^r} 
\end{align*}
\]

which just means $R_{klpq}^{j1} = \emptyset.$
By Lemma 5.4, 5.5 and 5.6, one can reduce the resonant set (5.26) to be following set:

\[
\tilde{R}_j = \bigcup_{K_{j-1} < |k| + |l| \leq K_j} \left( \bigcup_{p \leq c(|k| + |l|)} \tilde{R}^{j}_{klp} \bigcup_{|l| + |p-q| \neq 0, p, q \leq c(|k| + |l|)} \tilde{R}^{j11}_{klpq} \bigcup_{p, q \leq c(|k| + |l|)} \tilde{R}^{j2}_{klpq} \right).
\]

(5.27)

Lemma 5.7 At \( n \)-th infinite iteration, we have to exclude the following resonant set \( \mathcal{O}_n \setminus \mathcal{O}_{n+1} = \bigcup_{j=1}^{N} \tilde{R}^j \), and there exist constant \( C \) such that

\[
\text{meas}(\tilde{R}^j) \leq \frac{C\tilde{\gamma}_j}{K_j^2},
\]

(5.28)

\[
\text{meas}(\mathcal{O}_n \setminus \mathcal{O}_{n+1}) \leq \frac{C\gamma_n}{K_n}.
\]

(5.29)

**Proof:** By the definition, we have \( \mathcal{O}_n = \bar{\mathcal{O}}_0 \). By Lemma 5.2, one has \( \mathcal{O}_{n+1} = \bar{\mathcal{O}}_N \). By (5.15), one has \( \mathcal{O}_n \setminus \mathcal{O}_{n+1} = \bar{\mathcal{O}}_0 \setminus \bar{\mathcal{O}}_N = \bigcup_{j=1}^{N} \tilde{R}^j \) at once.

Now, as an example, we will focus on the measure of resonant set

\[
\tilde{R}^{j11}_{klpq} = \{ \xi \in \bar{\mathcal{O}}_{j-1} : |\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \tilde{\Omega}_p^j - \tilde{\Omega}_q^j| < \frac{\tilde{\gamma}_j}{(|k| + |l| + 1)^\tau} \}
\]

with \( p \neq q \) and \( (k, l) \neq 0 \). Since \( \xi \in \bar{\mathcal{O}}_{j-1} \subseteq (\frac{1}{2}, \frac{3}{2}) \), it follows that

\[
\tilde{R}^{j11}_{klpq} \subseteq Q^{j11}_{klpq} = \{ \xi \in \bar{\mathcal{O}}_{j-1} : |\langle k, \bar{\omega}_1 \rangle + \langle l, \bar{\omega}_2 \rangle + \frac{\tilde{\Omega}_p^j - \tilde{\Omega}_q^j}{\xi}| < \frac{2\tilde{\gamma}_j}{(|k| + |l| + 1)^\tau} \}
\]

By a direct computation,

\[
\frac{d}{d\xi}(\langle k, \bar{\omega}_1 \rangle + \langle l, \bar{\omega}_2 \rangle + \frac{1}{\xi}(\tilde{\Omega}_p^j - \tilde{\Omega}_q^j))| \geq \frac{1}{9}|p^2 + q^2|,
\]

then

\[
\text{meas}(\tilde{R}^{j11}_{klpq}) \leq \text{meas}(Q^{j11}_{klpq}) \leq \frac{2\tilde{\gamma}_j}{9(|k| + |l| + 1)^\tau} \frac{1}{|p^2 - q^2|}.
\]
Thus the total measure can be estimates as:

\[
\text{meas}(\tilde{R}^j) \leq \sum_{K_j < |k| + |l| \leq K_{j+1}} 1 \leq \sum_{1 \leq p \leq c(|k| + |l|)} \frac{2\tilde{\gamma}_j}{9(|k| + |l| + 1)^{\tau}} \frac{1}{p^2} + \sum_{1 \leq p, q \leq c(|k| + |l|)} \frac{4\tilde{\gamma}_j}{9(|k| + |l| + 1)^{\tau}} \frac{1}{p^2} + \sum_{1 \leq p, q \leq c(|k| + |l|)} \frac{4\tilde{\gamma}_j}{9(|k| + |l| + 1)^{\tau}} \frac{1}{q^2} \leq \sum_{K_j < |k| + |l| \leq K_{j+1}} \frac{4\tilde{\gamma}_j}{(|k| + |l| + 1)^{\tau}} + \frac{2\tilde{\gamma}_j \ln(|k| + |l|)}{(|k| + |l| + 1)^{\tau-2}} + \frac{4\tilde{\gamma}_j}{(|k| + |l| + 1)^{\tau-2}} \leq \frac{C\tilde{\gamma}_j}{K_j}.
\]

The last inequality is possible since we choose \( \tau > d + 6 \). The measure of the other resonant set be estimates similarly. There is also

\[
\text{meas}(O_n \setminus O_{n+1}) \leq \sum_{i=1}^N \frac{C\tilde{\gamma}_j}{K_j} \leq \frac{C\tilde{\gamma}_1}{K_0} \leq \frac{C\gamma_n}{K_n}.
\]

\[\blacksquare\]

A Appendix: Proof of Lemma 5.1

Proof: At the \( j \)-th step of the finite iteration, the Hamiltonian \( \tilde{H}_j = \tilde{N}_j + \tilde{P}_j \) is studied as a small perturbation of some normal form \( \tilde{N}_j \). A transformation \( \phi_1 F_j \) is set up so that \( \tilde{H}_j \circ \phi_1 F_j = \tilde{N}_{j+1} + \tilde{P}_{j+1} \) with new normal form \( \tilde{N}_{j+1} \) and a much smaller perturbation \( \tilde{P}_{j+1} \). We drop the index \( j \) of \( \tilde{H}_j, \tilde{N}_j, \tilde{P}_j, \phi_1 F_j \) and shorten the index \( j + 1 \) to be +.

Let \( R \) to be 2-order Taylor polynomial truncation of \( \tilde{P} \), that is

\[
R = \sum_{|k| + |l| \leq \tilde{K}, |\alpha| + |\beta| \leq 2} \tilde{P}_{k\alpha\beta} e^{i(k,l), (\theta, \varphi)} z^\alpha \bar{\bar{z}}^\beta \quad (A.1)
\]

\[
= R^0 + \langle R^{01}, z \rangle + \langle R^{10}, z \rangle + \langle R^{02}, z \rangle + \langle R^{11}, z \rangle + \langle R^{20}, \tilde{z} \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is formal product for two column vectors, \( R^0, R^{01}, R^{10}, R^{02}, R^{11} \) and \( R^{20} \) depend on \( \theta, \varphi \) and \( \xi \).
By $[R]$ denote the part of $R$ in generalized average part as follows

$$[R] = [R^0]_\varphi + \langle \text{diag}(R^{11}) \rangle_\varphi z, \bar{z},$$

where $\text{diag}(R^{11})$ is the the diagonal of $R^{11}$. The transformation $\phi^1_F = X^1_F$ is constructed as the time-1-map of a Hamiltonian vector field $X_F$, where $F$ is of the same form as $R$,

$$F = F^0 + \langle F^{01}, z \rangle + \langle F^{10}, z \rangle + \langle F^{11}, z, \bar{z} \rangle + \langle F^{20}, \bar{z}, \bar{z} \rangle,$$

and $[F(\theta, \varphi)] = 0$. The function $F(\theta, \varphi)$ is also an approximate solution of the homological equation

$$\{\tilde{N}, F\} = R - \sum_{p \geq 1} [R_{pp}^{11}(\theta, \varphi)]_\varphi |z_p|^2 - [R] \quad (A.2)$$

with $\tilde{N} = \tilde{e}(\theta, \xi) + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \langle \Omega + B(\theta; \xi) + b(\theta; \xi) z, \bar{z} \rangle$. In the following, we denote $\partial_{(\omega_1, \omega_2)} = \langle \omega_1, \partial_\theta \rangle + \langle \omega_2, \partial_\varphi \rangle$. Then $F$ should satisfy the homological equations:

$$\begin{align*}
\partial_{(\omega_1, \omega_2)} F^0 &= R^0 - [R^0]_\varphi, \\
\partial_{(\omega_1, \omega_2)} F_{pq}^{01} - i(\tilde{\Omega}_p + B_p + b_p) F_{pq}^{01} &= R_{pq}^{11}, & p \geq 1, \\
\partial_{(\omega_1, \omega_2)} F_{pq}^{10} + i(\tilde{\Omega}_q + B_q + b_q) F_{pq}^{10} &= R_{pq}^{10}, & q \geq 1, \\
\partial_{(\omega_1, \omega_2)} F_{pq}^{20} - i(\tilde{\Omega}_q + B_q + \Omega_q + B_q + b_q) F_{pq}^{20} &= R_{pq}^{20}, & p, q \geq 1, \\
\partial_{(\omega_1, \omega_2)} F_{pq}^{11} + i(\tilde{\Omega}_p + B_p + b_p - \tilde{\Omega}_q - B_q - b_q) F_{pq}^{11} &= R_{pq}^{11}, & p \neq q, \\
\partial_{(\omega_1, \omega_2)} F_{pq}^{11} - R_{pq}^{11} - [R_{pq}^{11}]_\varphi &= p = q. \quad (A.3)
\end{align*}$$

For the first equation from (A.3),

$$\langle \partial_\theta F^0, \omega_1 \rangle + \langle \partial_\varphi F^0, \omega_2 \rangle = R^0 - [R^0(\theta, \varphi)]_\varphi.$$

The Fourier expansion is given, and we arrive at equation

$$i\langle (k, \omega_1) + (l, \omega_2) \rangle \tilde{R}_0^0(k, l) = \hat{R}_0^0(k, l)$$

for $(k, l) \in \mathbb{Z}^2 \times \mathbb{Z}^d$ with $|k| + |l| \leq K$ and $l \neq 0$. Recall $\omega = \xi \bar{\omega}$ and by weak Liouvillean condition (5.17), we have

$$|F^0(\theta, \varphi)|_{\tilde{p} - 3\tilde{q}} \leq \tilde{\gamma}^{-1} \tilde{\sigma}^{-3-\tau} |R^0|_{\tilde{p}}.$$

The last equation in (A.3) is considered in the same way and

$$|F_{pq}^{11}(\theta, \varphi)|_{\tilde{p} - 3\tilde{q}} \leq \tilde{\gamma}^{-1} \tilde{\sigma}^{-3-\tau} |R_{pq}^{11}|_{\tilde{p}}.$$
The left equations in (A.3) will be discussed in the same way, as an
example, we do this for
\[
\partial(\omega_1,\omega_2)F_{pq}^{11} + i(\bar{\Omega}_p + B_p + b_p - \bar{\Omega}_q - B_q - b_q)F_{pq}^{11} = R_{pq}^{11}, p \neq q. \tag{A.4}
\]
To obtain a solution of these equations with useful estimates we want to
apply Proposition 5.1. The assumptions of this proposition are now verified.

We set \((r, \tilde{r}, \bar{\sigma}, \gamma, K, (\frac{r}{2})^\frac{1}{1+\nu}, 1, \eta_2, \eta_2)\) to be \((r_n, \tilde{r}, \bar{\sigma}, \gamma, K, K^\nu, \mathcal{E}_n, \bar{\mathcal{E}}, \tilde{\mathcal{E}})\). The assumption \([B_p(\theta; \xi)] = |b_p(\theta; \xi)| = 0, |B_p|_r \leq \mathcal{E}_n, |b_p|_{\tilde{r}} \leq \mathcal{E}\) and
\(|R_{pq}^{11}|_{\tilde{r}} \leq \tilde{\mathcal{E}}\) are given by condition from Lemma 5.1.

**Verification of assumption 1:** In fact, by Lemma 2.2, one has \(Q_{n+1} \geq Q_n^A\) and \(\ln Q_{n+1} \leq \mathcal{Q}_n\) for any \(n \geq 1\). Then since \(r \leq 2r_{n+1} < r_n\), one has
\[
e^{-|r_n-r||Q_{n+1}|} \leq e^{-\frac{Q_{n+1}}{8Q_n^A}} \leq e^{-\frac{\frac{1}{8}Q_{n+1}}{Q_n^A}} \leq e^{-\ln Q_{n+1} Q_n^{A-3}} \leq e^{-\ln Q_{n+1} n^{2n+1} c U} = Q_{n+1}^{-n^{2n+1} c U} \leq \mathcal{E}_n \leq \bar{\mathcal{E}}.
\]
Thus we have our conclusion as \(\mathcal{E}_n < 1\).

**Verification of assumption 2:** This is a direct computation since \(r_1 = r_n, r_2 = \tilde{r} \leq r_{n+1}\) and \(|B_p|_r \leq \mathcal{E}_n\).

**Verification of assumption 3:** Since \(K = \left[(\frac{Q_n}{2\mathcal{Q}_n})^{\frac{1}{1+\nu}}\right]\) and by (5.15), one has
\[
\tilde{K} \leq \ln(\frac{1}{r_0})^N (\frac{2NQ_n^2}{3}) \leq \frac{1}{r_0} (\frac{2NQ_n^2}{3}) \geq 8r_0 n \ln \mathcal{E}_0^{-1} \tag{A.5}
\]
\[
= \frac{1}{r_0} (\frac{2NQ_n^2}{3}) \ln Q_{n+1}^{2n+1} c U \geq 2^{n+1} c U \ln Q_n^{n+3} \ln \mathcal{E}_0^{-1} \leq Q_n^{n+3} \ln \mathcal{E}_0^{-1} \leq \tilde{\mathcal{E}}_n^{-c(r)} \leq \tilde{\mathcal{E}}_n^{-c(r)} \leq \mathcal{E}_n^{-c(r)} \leq K^\nu.
\]

**Verification of assumption 4:** For any \(\xi \in \hat{\mathcal{O}},\) we consider any pair \((k, l, p, q)\) with \(|k| + |l| \leq \tilde{K}\) and \(p \neq q\):

If \(\max\{|p, q| \geq c(|k| + |l|)\},\) one has
\[
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \bar{\Omega}_p - \bar{\Omega}_q| \geq |p^2 - q^2| - |\bar{\Omega}_p - p^2| - |\bar{\Omega}_q - q^2| - (|k| + |l|)|\omega| \geq \frac{|p^2 - q^2|}{2}.
\]

If \(\max\{|p, q| \leq (1 + \alpha)(|k| + |l|)\},\) by (5.17), one has
\[
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \bar{\Omega}_p - \bar{\Omega}_q| \geq \frac{\tilde{\gamma}}{(|k| + |l| + 1)^{\tau - 2}} \geq \frac{\tilde{\gamma}}{(|k| + |l| + 1)^{\tau + 2}}|p^2 - q^2|.
\]
Recall $\omega = \xi \tilde{\omega}$, one has

$$|\langle k, \tilde{\omega}_1 \rangle + \langle l, \tilde{\omega}_2 \rangle + \frac{\tilde{\Omega}_p - \tilde{\Omega}_q}{\xi} \geq \frac{\tilde{\gamma}}{|k| + |l| + 1} |p^2 - q^2|.$$ \hfill (A.6)

We have our assumption once $\zeta$ and $\lambda$ are in places by $\frac{\tilde{\Omega}_p - \tilde{\Omega}_q}{\xi}$ and $p^2 - q^2$.

Thus, Proposition 5.1 applies, and the approximate solution $F_{pq}^{11}$ satisfies the estimate

$$|F_{pq}^{11}|_{D(\bar{r} - 3 \bar{\delta})} \leq \frac{\tilde{\gamma}^{-1} \sigma^{-3 - \tau}}{|p^2 - q^2|^{-1}} |R_{pq}^{11}|_{D(\bar{r} - \bar{\delta})}. \hfill (A.7)$$

To obtain the norm of $X_F$, we need following useful Lemma.

**Lemma A.1 (M.3 in [21])** Let $R = (R_{pq})_{p,q \geq 1}$ be a bounded operator on $\ell^2$ which depends on $x \in \mathbb{T}^n$ such that all elements $(F_{pq})$ are analytic on $D(r)$. Suppose $F = (F_{pq})_{p,q \geq 1}$ is another operator on $\ell^2$ depending on $x$ whose elements satisfy

$$\sup_{x \in D(r)} |F_{pq}(x)| \leq \frac{1}{|p - q|} \sup_{x \in D(r)} |R_{pq}(x)|, \quad p \neq q,$$

and $F_{pq} = 0$. Then $F$ is a bounded operator on $\ell^2$ for every $x \in D(r)$, and

$$\sup_{x \in D(r - \sigma)} \|F(x)\| \leq \frac{c}{\sigma^2} \sup_{x \in D(r)} \|R(x)\|.$$

Then recall (A.7), we have

$$|F_{pq}^{11}|_{a, \rho, D(\bar{r} - 3 \bar{\delta})} \leq \frac{256}{\tilde{\gamma} \sigma^{5 + \tau}} |R_{pq}^{11}|_{a, \rho, D(\bar{r} - \bar{\delta})} \leq \frac{256}{\tilde{\gamma} \sigma^{5 + \tau}} |X_R|_{D(\bar{r}, \bar{\delta})}.$$

Multiplying by $z, \tilde{z}$ we get

$$\frac{1}{\tilde{\gamma}^2} |\langle F_{pq}^{11} z, \tilde{z} \rangle|_{D(\bar{r} - 3 \bar{\delta}, \bar{\delta})} \leq |F_{pq}^{11}|_{a, \rho, D(\bar{r} - 3 \bar{\delta}, \bar{\delta})},$$

and finally by Cauchy’s estimate we have

$$|X_{\langle F_{pq}^{11} z, \tilde{z} \rangle}|_{\tilde{\delta}, D(\bar{r} - 3 \bar{\delta}, \bar{\delta})} \leq \frac{256}{\tilde{\gamma} \sigma^{6 + \tau}} |X_R|_{\tilde{\delta}, D(\bar{r}, \bar{\delta})}.$$

To obtain the estimate of the Lipschitz semi-norm, we proceed as follows. Shortening $\triangle \xi_\rho$ to be $\triangle$ and applying it to (A.4), one gets that

$$\partial_{(\omega, \omega)} \triangle F_{pq}^{11} + i T_K \left( (\tilde{\Omega}_p + B_p + b_p - \tilde{\Omega}_q - B_q - b_p) \triangle F_{pq}^{11} \right) \hfill (A.8)$$

$$= -i T_K \left( \triangle (\tilde{\Omega}_p + B_p + b_p - \tilde{\Omega}_q - B_q - b_p) F_{pq}^{11} \right) + i \triangle R_{pq}^{11} := Q_{pq}$$

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By (A.6) and (A.7),
\[ |Q_{pq}|_{D(\tilde{\vartheta} - 2\tilde{\sigma})} \leq |\triangle (\tilde{\Omega}_p + B_p + b_p - \tilde{\Omega}_q - B_q - b_q)| \cdot |F_{pq}^{11}|_{D(\tilde{\vartheta} - 2\tilde{\sigma})} + |\triangle R_{pq}^{11}|_{D(\tilde{\vartheta} - 2\tilde{\sigma})} \]
\[ \leq \frac{|\triangle (\tilde{\Omega}_p + B_p + b_p - \tilde{\Omega}_q - B_q - b_q)|}{|p^2 - q^2|^{\gamma \tilde{\sigma}^3}} |R_{pq}^{11}|_{D(\tilde{\vartheta} - \tilde{\sigma})} + |\triangle R_{pq}^{11}|_{D(\tilde{\vartheta} - \tilde{\sigma})}. \]

We again apply Proposition 5.1 to (A.8), one has
\[ |\triangle F_{pq}^{11}|_{D(\tilde{\vartheta} - 2\tilde{\sigma})} \leq \frac{|\triangle (\tilde{\Omega}_p + B_p + b_p - \tilde{\Omega}_q - B_q - b_q)|}{|p^2 - q^2|^{\gamma \tilde{\sigma}^3}} |R_{pq}^{11}|_{D(\tilde{\vartheta} - \tilde{\sigma})} + |\triangle R_{pq}^{11}|_{D(\tilde{\vartheta} - \tilde{\sigma})}. \]

Again by Lemma A.1, one has
\[ |\triangle F_{a,p,D(\tilde{\vartheta} - 3\tilde{\sigma})} \leq \frac{|\triangle (\tilde{\Omega}_p + B_p + b_p - \tilde{\Omega}_q - B_q - b_q)|}{|p^2 - q^2|^{\gamma \tilde{\sigma}^3}} |R_{a,p,D(\tilde{\vartheta} - \tilde{\sigma})} + |\triangle R_{a,p,D(\tilde{\vartheta} - \tilde{\sigma})}. \]

Dividing $|\triangle F_{pq}^{11}|_{D(\tilde{\vartheta} - 3\tilde{\sigma}, \tilde{\vartheta})}$ by $|\xi - \eta| \neq 0$ and take supreme over $\tilde{\vartheta}$, one gets
\[ |X_{\langle F_{11} z, \tilde{\vartheta} \rangle}|_{\tilde{\vartheta}} \leq \frac{|X_{\langle R^{11} z, \tilde{\vartheta} \rangle}|_{\tilde{\vartheta}}}{\gamma^{\frac{1}{2}} \tilde{\sigma}^{10 + 2\tau}} + |X_{\langle R^{11} z, \tilde{\vartheta} \rangle}|_{\tilde{\vartheta}} \times \tilde{\vartheta}. \]

Thus one has
\[ |\langle X_{\langle F_{11} z, \tilde{\vartheta} \rangle} \rangle|_{\tilde{\vartheta}} \leq \frac{|X_{\langle R^{11} z, \tilde{\vartheta} \rangle}|_{\tilde{\vartheta}}}{\gamma^{\frac{1}{2}} \tilde{\sigma}^{10 + 2\tau}} + |X_{\langle R^{11} z, \tilde{\vartheta} \rangle}|_{\tilde{\vartheta}} \times \tilde{\vartheta}. \]

For the left terms of $F$, that is $\langle F^{01} z, \rangle, \langle F^{10} z, \rangle, \langle F^{02} z, \rangle$ and $\langle F^{20} z, \rangle$, one can obtain the same results with similar technical.

The estimation on approximate solution $F$ is obvious,
\[ |X_F|_{\tilde{\vartheta}} \leq \frac{256}{\gamma^{\frac{1}{2}} \tilde{\sigma}^{10 + 2\tau}} |X_{\langle R^{11} z, \tilde{\vartheta} \rangle}|_{\tilde{\vartheta}} \times \tilde{\vartheta}. \]

Let $\phi_F^1$ to be the time-1 map of $X_F$, we have
\[ \tilde{H}_+ = \tilde{H} \circ \phi_F^1 = (\mathcal{N} + \mathcal{R}) \circ X_F^1 + (\tilde{P} - \mathcal{R}) \circ X_F^1 \]
\[ = \mathcal{N} + \{\mathcal{N}, F\} \circ \mathcal{R} \]
\[ + \int_0^1 \{(1 - t)\mathcal{N}, F\} + \mathcal{R}, F\} \circ \phi_F^1 dt + (\tilde{P} - \mathcal{R}) \circ \phi_F^1 \]
\[ = \mathcal{N} + [\mathcal{R}] + \mathcal{R}_e + \int_0^1 \{R(t), F\} \circ \phi_F^1 dt + (\tilde{P} - \mathcal{R}) \circ \phi_F^1 \]
\[ = \mathcal{N}_+ + \tilde{P}_e, \]
where $N_+ = N + [R]$. For the new normal form $N_+$, we set
\[
\tilde{e}_+ = \tilde{e} + [R^0(\theta, \varphi)]\varphi, \quad \tilde{\Omega}_+ = \tilde{\Omega}_p + [R^{11}_p(\theta, \varphi)], \quad \tilde{\theta}_+ = \tilde{\theta}_p + [R^{11}_{pp}(\theta)] \varphi - [R^{11}_{pp}(\theta, \varphi)].
\]

The perturbation
\[
\tilde{P}_+ = R_e + \int_0^1 \{ R(t), F \} \circ \phi_{\tilde{t}} dt + (\tilde{P} - R) \circ \phi_{\tilde{t}}
\]
with $R(t) = (1 - t)[R] + tR$ and
\[
R_e = \sum_{p \geq 1} e^{i \delta p} \Gamma_K (e^{-i \delta p} R_p - (b_p + \Gamma_K B_p) F^{01}_p)(\tilde{z}_p
\]
\[+ \sum_{p \geq 1} e^{-i \delta p} \Gamma_K (e^{i \delta p} R_p + (b_p + \Gamma_K B_p) F^{10}_p)(\tilde{z}_p
\]
\[+ \sum_{p, q \geq 1} e^{i \delta p + i \delta q} \Gamma_K (e^{i \delta p - i \delta q} R_p - (b_p - b_q + \Gamma_K (B_p - B_q)) F^{11}_{pq})(\tilde{z}_p \tilde{z}_q
\]
\[+ \sum_{p, q \geq 1} e^{-i \delta p + i \delta q} \Gamma_K (e^{-i \delta p - i \delta q} R_p - (b_p + b_q + \Gamma_K (B_p + B_q)) F^{20}_{pq})(\tilde{z}_p \tilde{z}_q
\]
\[+ \sum_{p, q \geq 1} e^{-i \delta p - i \delta q} \Gamma_K (e^{i \delta p + i \delta q} R_p + (b_p + b_q + \Gamma_K (B_p + B_q)) F^{02}_{pq})(\tilde{z}_p \tilde{z}_q,
\]
where operator $\Gamma_K = I - T_K$.

Following [27], one has
\[
|X_{\tilde{P}_+ - R_e}^*|_{\tilde{s}, D(\tilde{r}, \tilde{s})} \leq \tilde{\epsilon}^3.
\]

Recall (5.5),
\[
|X_{R_e}^*|_{\tilde{s}, D(\tilde{r} - 5\tilde{s}, \tilde{s})} \leq e^{-K\tilde{\sigma}} \cdot \sup_{p \geq 1} (|e^{i \delta p}| + |b_p|^2) \cdot |X_F|^*_{\tilde{s}, D(\tilde{r}, \tilde{s})} \leq \tilde{\epsilon}^3. \quad (A.12)
\]

Thus we have new bound on perturbation $\tilde{P}_+$ and Lemma 5.1 follows.

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