Quasitoric Totally Normally Split Representatives in the Unitary Cobordism Ring

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Abstract—A smooth stably complex manifold is said to be totally tangentially/normally split if its stably tangential/normal bundle is isomorphic to a sum of complex line bundles. It is proved that each class of degree greater than 2 in the graded unitary cobordism ring contains a quasitoric totally tangentially and normally split manifold.

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1. INTRODUCTION

In this paper, we construct quasitoric manifolds whose tangential and normal bundles split into a sum of complex line bundles. We show that the cobordism classes of these manifolds give a generating family of the ring \( \Omega^*_U \). In what follows, we refer to totally tangentially/normally split manifolds as TTS/TNS manifolds. We prove that each unitary cobordism class of degree greater than 2 has a representative which is a quasitoric TTS and TNS manifold. Recall that any quasitoric manifold is TTS [1].

A family of generators of the ring \( \Omega^*_U \) is called a minimal family of multiplicative generators if each (even) grading of \( \Omega^*_U \) contains a unique element of this family. In [2], Ray explicitly constructed a family of stably complex manifolds each of which is TTS and TNS. This family provides a set of multiplicative generators of the unitary cobordism ring \( \Omega^*_U \). Ray also produced stable complex TTS and TNS manifolds which represent the inverses of the classes of elements of the family constructed by him in \( \Omega^*_U \). The operation of taking the connected sum of stably complex manifolds agrees with the TTS and TNS properties. These observations imply the following theorem.

Theorem 1 ([2, Theorem 3.9]). Each element of degree greater than 2 in the unitary cobordism ring \( \Omega^*_U \) contains a representative which is simultaneously a TTS manifold and a TNS manifold.

In [3], Buchstaber and Ray constructed toric varieties multiplicatively generating the ring \( \Omega^*_U \). A normal algebraic variety \( X \) is called a toric variety if there exists an effective action on \( X \) of the algebraic torus \( (\mathbb{C}^*)^{\dim X} \) with dense open orbit (see [4]). Note that the classical Milnor hypersurfaces \( H_{i,j} \), which provide a family of generators of the ring \( \Omega^*_U \) [5], are not toric varieties for \( 2 \leq i \leq j \) (see, e.g., Proposition 4). The connected sum of any toric varieties is not (complex) cobordant to a toric variety; the obstruction is the Todd genus, which equals 1 for any toric variety.

In [1], Buchstaber, Panov, and Ray defined the diamond sum operation on the (larger) category of quasitoric manifolds. The diamond sum of any two quasitoric manifolds is cobordant to the connected sum of these manifolds. The “\(-1\)”-problem of finding a manifold with the opposite class of complex cobordisms was solved by taking the opposite orientation. (Orientation is not a part of the definition of a quasitoric manifold.) We do have the notion of omniorientation of a quasitoric manifold, which is related to a splitting of the natural stably tangent structure on a quasitoric manifold; see [4].) Using this, the following theorem was proved.

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Theorem 2 (1). Each element of degree at least 4 in the unitary cobordism ring \( \Omega_U^* \) contains a smooth quasitoric TTS manifold.

We also mention a result from [7] and [8] asserting the existence of nonsingular projective toric varieties which provide a generating family of the ring \( \Omega_U^* \).

The central result of this paper is the following theorem.

Theorem 3. Each element of the unitary cobordism ring \( \Omega_U^* \) of degree greater than 2 contains a representative which is a smooth quasitoric TTS and TNS manifold.

A brief plan of the paper is as follows. In Sec. 2, given a smooth compact stably complex manifold \( X \) and a set of complex line bundles

\[
\xi_i \rightarrow X, \quad i = 1, \ldots, k + 1,
\]

we define a smooth manifold \( BF(\xi_{k+1}, \ldots, \xi_1) \), called the bounded flag bundle over \( X \). The manifold thus obtained is a locally trivial bundle over \( X \), whose fiber is the bounded flag manifold \( BF_k \). The manifold \( BF(\xi_{k+1}, \ldots, \xi_1) \) is a tower of \( \mathbb{C}P^1 \) bundles over \( X \).

We prove that, given a TNS manifold \( X \) and any set of complex line bundles

\[
\xi_i \rightarrow X, \quad i = 1, \ldots, k + 1,
\]

\( BF(\xi_{k+1}, \ldots, \xi_1) \) is a TNS manifold. If \( X \) is a quasitoric manifold, then so is \( BF(\xi_{k+1}, \ldots, \xi_1) \). We use this construction to define quasitoric TNS manifolds in the proof of our main Theorem 3. The bounded flag bundle over a point is the bounded flag manifold \( BF_n \) of the corresponding dimension. We review properties of the manifolds \( BF_n \).

Section 3 considers known generators of the ring \( \Omega_U^* \), namely, the Milnor hypersurfaces \( H_{i,j} \) [5] and the Buchstaber–Ray varieties \( BR_{i,j} \) [3]. We show that, generally, these varieties are not totally normally split [3]. (For this reason, the proof of Theorem 3 uses other varieties.)

Section 4 presents other constructions of TNS manifolds, namely, the blow-up of a nonsingular complex projective TNS variety along a transversal intersection of codimension 2 of nonsingular hypersurfaces, the diamond sum of two quasitoric TNS manifolds, and the change of orientation of a quasitoric TNS manifold. Note that the first construction is well defined in the class of nonsingular projective toric TNS varieties (the submanifold being blown up must be invariant under the action of the torus \( (\mathbb{C}^*)^n \)).

In Sec. 5, we define nonsingular projective toric TNS varieties \( X_{i,j}, 0 < i \leq j \) (bounded flag bundles over \( BF_i \) and \( M_{i,j}, 0 < i \leq j \) (equivariant blow-ups of invariant subvarieties in \( X_{i,j} \) of codimension 2), which have complex dimension \( i + j \). Then, we introduce quasitoric TNS varieties \( N_{i,j}, 0 \leq i \leq j \), which coincide with \( M_{i,j} \) for \( 1 < i \) and are obtained from \( M_{1,j} \) and \( BF_{j+1} \) by applying the diamond sum operation and changing orientation for \( i = 0, 1 \) (see Definition 5). (The Milnor numbers of the varieties \( N_{i,j} \) turn out to be more convenient in considering congruences modulo a prime number.) A well-known result of Milnor and Novikov concerning generators of \( \Omega_U^* \) reduces the search for a generator among the integer linear combinations of the varieties \( N_{i,j} \) (in the sense of diamond summation and orientation change) to the study of integer linear combinations of the Milnor numbers of \( N_{i,j} \). Modulo finding the Milnor numbers of the varieties \( X_{i,j} \) and \( M_{i,j} \) (and, thereby, the numbers \( s_{i+j}(N_{i,j}) \)) (see Sec. 6) and certain number-theoretic computations (to which Sec. 7 is devoted), we solve this problem and prove the main Theorem 3.

In Sec. 8, we state the conjecture that certain explicitly specified projective algebraic varieties are toric. A proof of this conjecture would give a simpler proof of the main Theorem 3.
2. BOUNDED FLAG BUNDLES

This section contains a construction of a locally trivial fiber bundle over any smooth compact stably complex manifold $X^{2n}$. The fiber of this bundle is a bounded flag manifold (see [3], [4, Sec. 7.7]). The bounded flag bundle is the torus of $\mathbb{CP}^1$-bundles beginning with $X^{2n}$. We show that the bounded flag bundle over a TNS manifold is a TNS manifold as well. In the case where the base is a quasitoric manifold $X$, the bounded flag bundle over $X$ is endowed with the natural structure of a quasitoric manifold. (The complex bundles of rank 2 being projectivized split into sums of one-dimensional bundles.) (Here and in what follows, by complex bundles we always mean complex vector bundles.) In the proof of the main Theorem 3, we use only bounded flag bundles which are Bott towers, but the formalism of bounded flag bundles is convenient for computations.

2.1. Definitions and Properties

Suppose given a smooth compact stably complex manifold $X^{2n}$ and complex line bundles $\xi_i \to X$, $i = 1, \ldots, k + 1$, over it.

**Definition 1.** We set $BF(\xi_1) := X$ and denote the line bundle $\xi_1 \to X = BF(\xi_1)$ by $\zeta_1$. For $1 \leq i \leq k$, $BF(\xi_{i+1}, \ldots, \xi_1)$ is, by definition, a $\mathbb{CP}^1$-bundle over $BF(\xi_i, \ldots, \xi_1)$. Namely,

\[
BF(\xi_{i+1}, \ldots, \xi_1) = \mathbb{P}(\xi_i \oplus \xi_{i+1}) \to BF(\xi_i, \ldots, \xi_1). \tag{2.1}
\]

The tautological line bundle of this $\mathbb{CP}^1$-bundle is denoted by $\zeta_{i+1} \to BF(\xi_{i+1}, \ldots, \xi_1)$.

The projectivization of a Whitney sum of complex line bundles over a smooth complex (algebraic, toric) manifold has a natural complex (respectively, algebraic, toric) structure; see, e.g., [8]. Therefore, the bounded flag bundle $BF(\xi_{n+1}, \ldots, \xi_1)$ has a natural complex (respectively, algebraic, toric) structure under the corresponding assumption on the base $X$. For the natural complex structure on the bounded flag bundle $BF(\xi_{n+1}, \ldots, \xi_1)$, we have the isomorphism

\[
TBF(\xi_{k+1}, \ldots, \xi_1) \oplus \mathbb{C}^k \cong \bigoplus_{i=1}^{k} (\zeta_i \oplus \zeta_{i+1}) \mathbb{C} \oplus TX, \tag{2.2}
\]

where $\mathbb{C}$ is the trivial line bundle over $BF(\xi_{k+1}, \ldots, \xi_1)$ and $\overline{\zeta_{i+1}}$ denotes the complex conjugate of $\zeta_{i+1}$. (Here and in what follows, we omit the symbols of pullbacks and tensor products of vector bundles whenever possible.)

**Remark 1.** Generally, the stably complex structure of each bounded flag bundle $BF(\xi_{k+1}, \ldots, \xi_1)$ depends on the order of the elements of $\{\xi_{k+1}, \ldots, \xi_1\}$ (see (2.2)).

Let

\[
\zeta_{k+1}^* \to BF(\xi_{k+1}, \ldots, \xi_1) = \mathbb{P}(\zeta_k \oplus \zeta_{k+1}) \to BF(\xi_k, \ldots, \xi_1)
\]

denote the vector bundle in which the fiber over each point $l \subset (\zeta_k \oplus \zeta_{k+1})_p$, $x \in BF(\xi_k, \ldots, \xi_1)$, is the complex line $l_1^\perp$ orthogonal to $l$. (The bundle $\zeta_k \oplus \zeta_{k+1}$ is endowed with a fiberwise Hermitian metric, because this is a vector bundle over a compact stably complex manifold.) By

\[
\zeta_i^* \to BF(\xi_{k+1}, \ldots, \xi_1)
\]

we denote the pullback of the bundle $\zeta_i^* \to BF(\xi_i, \ldots, \xi_1)$ with respect to the composition

\[
BF(\xi_{k+1}, \ldots, \xi_1) \to BF(\xi_i, \ldots, \xi_1)
\]

of the projections (2.1) with $i = 1, \ldots, k + 1$. The vector bundle $\zeta_i^* \to BF(\xi_{k+1}, \ldots, \xi_1)$ is linear and satisfies the identities

\[
\zeta_{i+1} \oplus \zeta_{i+1}^* \cong \zeta_i \oplus \xi_{i+1}, \quad \zeta_{i+1} \oplus \bigoplus_{k=1}^{i} \zeta_{k+1}^* \cong \bigoplus_{k=1}^{i+1} \xi_i \tag{2.3}
\]
for \( i = 1, \ldots, k \) (see [2]).

Given any complex bundle \( \alpha \to X \), there exists a unique, up to a stable equivalence of vector bundles, “inverse” complex bundle, i.e., a complex bundle \( \theta \to X \) such that \( \alpha \oplus \theta \simeq \mathbb{C}^{r} \), where \( \mathbb{C}^{r} \) is the trivial complex vector bundle of rank \( r \) over \( X \). General properties of operations over vector bundles imply the following lemma.

**Lemma 1.** Let \( \alpha, \alpha' \to X \) be complex linear vector bundles whose stably inverse complex vector bundles are totally split, and let \( f : Y \to X \) be a continuous mapping \( (Y \text{ is any smooth compact manifold}) \). Then the stably inverse complex bundles of \( \mathfrak{g}, f^*\alpha, \alpha \oplus \alpha', \text{and } \alpha \alpha' \) are totally split.

Lemma 1 immediately implies the following proposition.

**Proposition 1.** If \( X \) is a TNS manifold and the bundles stably inverse to \( \xi_i \to X, i = 1, \ldots, k + 1 \), are totally split, then \( BF(\xi_{k+1}, \ldots, \xi_1) \to X \) is a TNS manifold.

The Milnor number \( s_n(X^{2n}) \) of a stably complex manifold \( X^{2n} \) is defined as
\[
s_n(X^{2n}) = (t_1^n + \cdots + t_k^n, [X^{2n}]) \in \mathbb{Z},
\]
where \( t_1, \ldots, t_k \) are the corresponding Wu generators.

We set
\[
x_i := c_1(\xi) \in H^2(X; \mathbb{Z}), \quad i = 1, \ldots, k + 1.
\]

**Proposition 2.** The following relation holds:
\[
s_{n+k}(BF(\xi_{k+1}, \ldots, \xi_1)) = (1 + (-1)^{n+k-1})(1 + x_{k+1})^{n+k-1}(1 + x_k)^{-1} \cdots (1 + x_1)^{-1}, [X]).
\]

### 2.2. Bounded Flag Varieties

Let \( X = pt. \) Then
\[
BF(\xi_{n+1}, \ldots, \xi_1) = BF(\mathbb{C}, \ldots, \mathbb{C}).
\]

This variety is called a *bounded flag variety* and denoted by \( BF_n \). Each \( BF_n \) is a toric variety of complex dimension \( n \). Let \( \beta_i \) and \( \beta^*_i \), \( i = 0, \ldots, n \), denote the vector bundles \( \zeta_{i+1}, \zeta^*_{i+1} \to BF_n \), respectively. (Here we use the notation of [2].) For \( BF_n \), identity (2.3) takes the form
\[
\beta_{i+1} \oplus \beta^*_{i+1} \simeq \beta_i \oplus \mathbb{C}, \quad \beta_i \oplus \bigoplus_{k=1}^{i} \beta_k^* \simeq \mathbb{C}^{i+1}, \quad (2.4)
\]
where \( i = 0, \ldots, n \) (see [2]).

**Corollary 1.** Each variety \( BF_n \) is TNS.

Let \( e_0, \ldots, e_n \) and \( z_0, \ldots, z_n \) be, respectively, a basis in \( \mathbb{C}^{n+1} \) and the basis dual to it. We set \( C_i := \mathbb{C}(e_i), i = 0, \ldots, n \), where each \( \mathbb{C}(e_i) \) is the linear span of the vector \( e_i \). The variety \( BF_n \) can be identified with the set of \( n \)-tuples \( (l_0, \ldots, l_n) \) of straight lines in \( \mathbb{C}^{n+1} \), where \( l_0 := \mathbb{C}_0 \), satisfying the inclusions
\[
l_{i+1} \subset l_i \oplus C_{i+1}, \quad i = 0, \ldots, n - 1. \quad (2.5)
\]

It follows that
\[
l_i \subset \mathbb{C}(e_0, \ldots, e_i) \quad \text{for } i = 0, \ldots, n. \quad (2.6)
\]

Therefore, \( BF_n \) is the subvariety in \( \prod_{i=1}^{n} \mathbb{C}P^n \) determined by the (algebraic) conditions (2.5). Let \([z_{i,0} : \cdots : z_{i,i}] \) be homogeneous coordinates in the \( i \)th factor of the product \( \prod_{i=1}^{n} \mathbb{C}P^n \). According to
a standard theorem of linear algebra, conditions (2.5) are equivalent to the vanishing of all $2 \times 2$ minors of the corresponding matrices, i.e., to the conditions

$$\text{rk} \begin{pmatrix} z_{i+1,0} & z_{i+1,1} & \cdots & z_{i+1,i} \\ z_{i,0} & z_{i,1} & \cdots & z_{i,i} \end{pmatrix} = 1,$$

(2.7)

where $i = 1, \ldots, n - 1$.

Let $f_n$ denote the map $f_n : BF_n \to \mathbb{C}P^n$ defined by $(l_0, \ldots, l_n) \mapsto l_n$. The map $f_n$ is the restriction of the projection morphism $\prod_{i=1}^n \mathbb{C}P^i \to \mathbb{C}P^n$ to $BF_n$. We have

$$f_n^* \eta_n = \beta_n,$$

(2.8)

where $\eta_n$ is the tautological line bundle over $\mathbb{C}P^n$. It can be shown that $f_n$ is the composition of successive blow-ups of strict transforms of the projective subspaces

$$\{z_0 = \cdots = z_{n-1} = 0\} \subset \cdots \subset \{z_0 = 0\} \subset \mathbb{C}P^n$$

determined by the corresponding equations in the homogeneous coordinates $[z_0 : \cdots : z_n]$ of $\mathbb{C}P^n$.

**Remark 2.** The bounded flag variety defined above differs from that defined in the standard way (see [3], [4, p. 292]), which we denote by $BF'_n$. However, these varieties are isomorphic. First, we change the basis of $\mathbb{C}^{n+1}$ for $\{v_1, \ldots, v_{n+1}\}$, $v_i = e_{n+1-i}$, $i = 0, \ldots, n$. Inclusion (2.5) takes the form

$$l_{i+1} \subset l_i \oplus \mathbb{C}\langle v_{n-i} \rangle \quad \text{for} \quad i = 0, \ldots, n.$$ 

Then, we change the order of lines, setting $L_i = l_{n-i+1}$, so that

$$L_{n-i} \subset L_{n-i+1} \oplus \mathbb{C}\langle v_{n-i} \rangle \quad \text{for} \quad i = 0, \ldots, n.$$

To obtain $BF'_n$, it remains to make the change $i = j - 1$.

### 3. REVIEW OF KNOWN MULTIPLICATIVE GENERATORS OF THE RING $\Omega_U^*$

In this section, we show that the known multiplicative generators of the unitary cobordism ring $\Omega_U^*$ are not generally quasitoric TNS manifolds. Therefore, they cannot be used in the proof of Theorem 3.

#### 3.1. Milnor Hypersurfaces

The Milnor hypersurfaces $H_{i,j}$, $0 \leq i \leq j$, are, by definition, the hypersurfaces in $\mathbb{C}P^i \times \mathbb{C}P^j$ of bidegree $(1, 1)$ (and complex dimension $i + j - 1$) determined by the equations

$$\sum_{k=0}^i z_k w_k = 0$$

(3.1)

in the homogeneous coordinates $[z_0 : \cdots : z_i]$ and $[w_0 : \cdots : w_j]$ of $\mathbb{C}P^i$ and $\mathbb{C}P^j$, respectively (by definition, $H_{0,0} = \emptyset$). The fiber of the tautological line bundle $\eta_i \to \mathbb{C}P^i$ over $[l] \in \mathbb{C}P^i$, $l \subset V$, is the line $l$. Let $\eta_i^j$ denote the vector bundle over $\mathbb{C}P^i$ whose fiber over $[l] \in \mathbb{C}P^i$, $l \subset V$, is the orthogonal complement $l^\perp$ in $\mathbb{C}^{i+1}$. Clearly,

$$\text{rk} \eta_i^* = i, \quad \eta_i \oplus \eta_i^* = \mathbb{C}^{i+1}.$$ 

**Proposition 3 ([9], [5]).** The restriction of the natural projection $\mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^i$ to $H_{i,j}$ induces the structure of a locally trivial $\mathbb{C}P^{j-1}$-bundle

$$H_{i,j} = \mathbb{P}(\eta_i^j \oplus \mathbb{C}^{j-i}) \to \mathbb{C}P^i.$$

The Milnor numbers of $H_{i,j}$ are easy to find. The following proposition holds.
Proposition 4 ([5]). The varieties $H_{i,j}$, $0 \leq i \leq j$, $i + j = n + 1$, are multiplicative generators of the ring $\Omega^n_U$ in degree $2n$.

Note that the varieties

$$H_{0,j} = \mathbb{C}P^{j-1}, \quad H_{1,j} = \mathbb{P}(\eta_1 \oplus \mathbb{C}^{j-1}) \to \mathbb{C}P^1$$

are toric. In fact, they exhaust all toric Milnor hypersurfaces. (The cohomology ring $H^*(H_{i,j}; \mathbb{Z})$ provides the corresponding obstruction.)

Proposition 5 ([4, Theorem 9.1.5]). For $2 \leq i \leq j$, $H_{i,j}$ is not a toric variety.

The dualization of the first Chern class $c_1(\xi) \in H^2(X; \mathbb{Z})$ of the line vector bundle $\xi \to X$ over a compact stably complex manifold gives a stably complex submanifold $D \subset X$ of real codimension 2 (see [10, p. 78 (Russian transl.)], [11]). The normal complex linear vector bundle of the inclusion $D \subset X$ coincides with the restriction of $\xi$ to $D$. Under Poincaré duality, the first Chern class $c_1(\xi) \in H^2(X; \mathbb{Z})$ corresponds to the class $[D] \in H_{2(n-1)}(X; \mathbb{Z})$.

Proposition 6 ([5], [4, Proposition D.6.3]). The variety $H_{i,j}$ is the dualization of the first Chern class of the complex line bundle $\overline{\mathbb{P}_i \eta_j}$ over $\mathbb{C}P^i \times \mathbb{C}P^j$.

3.2. Buchstaber–Ray Varieties

In [3], Buchstaber and Ray constructed nonsingular projective toric varieties $BR_{i,j}$, which provide multiplicative generators of the unitary cobordism ring $\Omega^n_U$ (in [3], they were denoted by $B_{i,j}$).

Definition 2 (see [3]). Let $0 \leq i \leq j$. Then

$$BR_{i,j} = \mathbb{P}(f_i^*\overline{\eta}_j \oplus \mathbb{C}^{j-i}) \to BF_i$$

is the pullback of

$$H_{i,j} = \mathbb{P}(\eta_i \oplus \mathbb{C}^{j-i}) \to \mathbb{C}P^i$$

under the map $f_i: BF_i \to \mathbb{C}P^i$. In particular, $BR_{0,j} = \mathbb{C}P^{j-1}$. (By definition, $BR_{0,0} = \emptyset$.)

Proposition 7. The variety $BR_{i,j}$ is a dualization of the first Chern class of the complex line bundle $\overline{\mathbb{P}_i \eta_j}$ over $BF_i \times \mathbb{C}P^j$.

The variety $BR_{i,j}$ is the preimage of the Milnor hypersurface $H_{i,j} \subset \mathbb{C}P^i \times \mathbb{C}P^j$ under the map $f_i \times \text{Id}_j: BF_i \times \mathbb{C}P^j \to \mathbb{C}P^i \times \mathbb{C}P^j$. Therefore, the hypersurface $BR_{i,j} \subset BF_i \times \mathbb{C}P^j$ is determined by the equation

$$\sum_{k=0}^i z_{i,k}w_k = 0, \quad (3.2)$$

where $[w_0: \cdots : w_j] \in \mathbb{C}P^j$ denotes homogeneous coordinates on $\mathbb{C}P^j$ and the $z_{i,k}$ are coordinates on $BF_i$ (see Sec. 2.2).

Proposition 8 ([4, Theorem 9.1.8]). The following relation holds:

$$s_{i+j-1}(BR_{i,j}) = s_{i+j-1}(H_{i,j}).$$

The varieties $BR_{i,j}$, $0 \leq i \leq j$, $i + j = n + 1$, are multiplicative generators of $\Omega^n_U$ in degree $2n$.

Just like the Milnor hypersurfaces $H_{i,j}$, the varieties $BR_{i,j}$ are projective bundles.
Lemma 2. For any $k = 1, \ldots, i$,

$$f_i^* \eta_k \simeq \bigoplus_{q=1}^{k} \beta_q$$

over $BF_i$.

**Proposition 9.** The following isomorphism holds:

$$BR_{i,j} = \mathbb{P} \left( \bigoplus_{k=1}^{i} \beta_k^j \oplus \mathbb{C}^{j-i} \right) \rightarrow BF_i.$$  

Unlike the Milnor hypersurfaces, the Buchstaber–Ray varieties are toric.

**Corollary 2 (see [3]).** Each $BR_{i,j}$ is a nonsingular projective toric variety.

**Remark 3.** The variety $BF_i \times \mathbb{C}P^j$, $0 \leq i \leq j$, is toric with respect to the torus action induced by the standard torus action on complex projective spaces and the equivariant embedding

$$BF_i \subset \prod_{k=1}^{n} \mathbb{C}P^k.$$  

Equation (3.2) is not invariant under this action of the torus (for $i \neq 0$). Therefore, the varieties $BR_{i,j}$, $1 \leq i \leq j$, are not invariant divisors of $BF_i \times \mathbb{C}P^j$. Nevertheless, using formula (2.4), we can identify the trivial $\mathbb{C}P^j$-bundle over $BF_i$ with a projectivization

$$\mathbb{P} \left( \beta_i \oplus \bigoplus_{k=1}^{i} \beta_k^j \oplus \mathbb{C}^{j-i} \right) \rightarrow BF_i.$$  

Now, we endow $BF_i \times \mathbb{C}P^j$ with the $(\mathbb{C}^*)^n$-action induced by the given projectivization structure of a totally split bundle. (This variety is equivariantly isomorphic to the preceding one. This follows, e.g., from the main result of [12].) The embedding

$$BR_{i,j} = \mathbb{P} \left( \bigoplus_{k=1}^{i} \beta_k^j \oplus \mathbb{C}^{j-i} \right) \subset \mathbb{P} \left( \beta_i \oplus \bigoplus_{k=1}^{i} \beta_k^j \oplus \mathbb{C}^{j-i} \right) = BF_i \times \mathbb{C}P^j$$

is equivariant with respect to the torus action on $BF_i \times \mathbb{C}P^j$ specified above.

As shown in what follows, the varieties $BR_{i,j}$ are not totally normal split, generally speaking.

Below we recall the definition of a quasitoric manifold.

**Definition 3.** A smooth manifold $M^{2n}$ is said to be quasitoric if there exists a smooth locally standard action of the $n$-torus $T^n$ on $M^{2n}$ whose orbit space is diffeomorphic to a simple polytope as a manifold with corners.

**Proposition 10 ([4]).** Any quasitoric manifold $M^{2n}$ over the $n$-simplex $\Delta^n$ is homeomorphic to $\mathbb{C}P^n$. The natural stably complex structure on $M^{2n}$ has the form

$$TM \oplus \mathbb{R} \simeq k\eta_n \oplus (n + 1 - k)\eta_n$$

for some $k = 0, \ldots, n + 1$.

The following proposition generalizes Theorem 1.5 of [13], which asserts that the $\mathbb{C}P^n$, $n > 1$, are not normally split.

**Proposition 11.** Any quasitoric manifold $M^{2n}$ over the $n$-simplex $\Delta^n$, $n > 1$, is not totally normally split.
Proof. Any complex line bundle over $\mathbb{C}P^n$ is topologically isomorphic to $\eta^a$, where $a \in \mathbb{Z}$. We set $x = c_1(\eta)$. For any totally split bundle $\alpha = \bigoplus_{i=1}^k \eta^{a_i}$, $a_i \in \mathbb{Z}$, the 4-component of the Chern character equals

$$\text{ch}_2(\alpha) = \frac{x^n}{2} \sum_{i=1}^k a_i^2.$$

Therefore, $(x^{n-2} \text{ch}_2(\alpha), [\mathbb{C}P^n]) \geq 0$.

According to Proposition 10, given a normal bundle $NM$, we have

$$(x^{n-2} \text{ch}_2(NM), [\mathbb{C}P^n]) = \frac{n+1}{2} < 0$$

for $n > 1$. It follows that $NM$ is not totally split. \hfill \Box

Remark 4. The projective line $\mathbb{C}P^1$, as well as any other Riemann surface $\Sigma_g$ of genus $g$, is a TNS variety, because any complex vector bundle over $\Sigma_g$ is (topologically) isomorphic to the Whitney sum of a complex line bundle and a trivial vector bundle.

Any quasitoric manifold can be endowed with the natural stably complex structure. Let $m$ be the number of facets of the moment polytope $P^m \subset \mathbb{R}^n$ of a quasitoric manifold $M^{2n}$. We number the facets of $P$ by integers from 1 to $m$ and denote the invariant submanifold of codimension 2 in $M$ corresponding to the $i$th facet of $P^m$ by $M^{2n-2}_i$ for $i = 1, \ldots, m$.

Theorem 4 (see, e.g., [4, Theorem 7.3.15]). There is an isomorphism

$$TM \oplus \mathbb{R}^{2n-2n} \simeq \rho_1 \oplus \cdots \oplus \rho_m$$

of real bundles over $M$, where the $\rho_i \to M$ are complex line bundles for $i = 1, \ldots, m$. The normal bundle of the embedding $M^{2n-2}_i \subset M^{2n}$ of oriented manifolds equals the realization of the bundle $\xi_i|_{M^i}$, $i = 1, \ldots, m$.

Proposition 12. Any invariant submanifold $Z$ of a quasitoric TNS manifold $M$ is totally normally split. In particular, $M$ contains no invariant submanifolds over the simplex $\Delta^k$, $k > 1$. Moreover, the moment polytope of $M$ has no triangular faces.

Proof. There exists a maximal chain

$$Z = Z_0 \subset \cdots \subset Z_k = M$$

of inclusions of invariant submanifolds of $M$ (each submanifold has codimension 2 in the succeeding one). The complete normal splitting of $Z$ follows from Theorem 4 and the transversality of the intersections of characteristic submanifolds of $M$ (see [4, p. 245]). The remaining assertions of the proposition follow from Propositions 10 and 11. \hfill \Box

In [14], Lû and Panov constructed another family of polynomial generators of $\Omega^*_{T^m}$ in relation to the problem of constructing generators of the ring $\Omega_{SU}/\text{Tors}$. Namely, these are the projective toric varieties $L(i, j) := \mathbb{P}(\eta \oplus \mathbb{C}^j) \to \mathbb{C}P^i$ of complex dimension $i + j$.

Corollary 3. The varieties $H_{i,j}$ and $BR_{i,j}$, $i \leq j, 2 < j$, are not totally normally split. The varieties $L(i, j)$, $1 < j$, are not totally normally split either.

4. CONSTRUCTIONS

In this section, we present two constructions of TNS manifolds (after the construction of the bounded flag bundle over a TNS manifold in Sec. 2): a blow-up of a complex codimension 2 complete intersection in a nonsingular complex projective TNS variety and the diamond sum of two quasitoric TNS manifolds. Note that the first construction has an equivariant analog, namely, the blow-up of a nonsingular complex projective toric TNS variety along an invariant subvariety of complex codimension 2. The second construction was introduced in [1]. We use these constructions in Sec. 5 to construct quasitoric TNS manifolds that provide generators of the ring $\Omega^*_{T^m}$.
4.1. Blow-ups of Nonsingular Complex Projective TNS Varieties along Complete Intersections of Complex Codimension 2

The following proposition holds.

**Proposition 13.** Let \( X \) be a nonsingular projective complex TNS variety of complex dimension \( n \), and let \( Z \subset X \) be the transversal intersection \( Z = D_1 \cap D_2 \) of two smooth hypersurfaces \( D_1, D_2 \subset X \). Then \( Bl_Z X \) is a TNS variety.

**Proof.** By definition (see [15, Sec. 6.2.1]), \( Bl_Z X \) is a hypersurface in the space of the projectivization

\[
P(L(D_1) \oplus L(D_2)) \to X,
\]

where \( L(D_1), L(D_2) \to X \) are the line bundles corresponding to the hypersurfaces \( D_1, D_2 \subset X \), respectively. We denote the corresponding embedding morphism by \( \iota \).

Let \( \nu \to Bl_Z X \) be the normal (line) bundle of the given embedding. The space \( P(L(D_1) \oplus L(D_2)) \) is the bounded flag bundle over the TNS variety \( X \). By Proposition 1, \( P(L(D_1) \oplus L(D_2)) \) is a TNS manifold. This means that there exists a totally split complex vector bundle

\[
\xi = \bigoplus_{i=1}^{k} \xi_i \to P(L(D_1) \oplus L(D_2)), \quad \text{rk}_C \xi_i = 1,
\]

so that

\[
T(P(L(D_1) \oplus L(D_2))) \oplus \xi \simeq \mathbb{C}^{n+k+1}.
\]

It remains to restrict this bundle to \( Bl_Z X \):

\[
T(Bl_Z X) \oplus (\nu \oplus \iota^* \xi) \simeq \mathbb{C}^{n+k+1}. \quad \square
\]

Note that the blow-up of a nonsingular projective toric variety \( X \) in its invariant subvariety provides a nonsingular projective toric variety (see, e.g., [4]).

4.2. The Diamond Sum and Orientation of Quasitoric TNS Manifolds

The existence of a canonical stably complex structure on a quasitoric manifold (see, e.g., [4, Sec. 7.3]) implies that the unitary cobordism class of a quasitoric manifold is well defined. (For an example of different stably complex structures on \( \mathbb{C}P^n \), see Proposition 10 above.)

In [2, Lemma 3.5], it was shown that the connected sum of two stable complex TTS/TNS manifolds is TTS/TNS, respectively. In [1], for any 2n-dimensional omnioriented quasitoric manifolds \( M \) and \( M' \), a 2n-dimensional omnioriented quasitoric manifold \( M \# M' \), called the diamond sum of \( M \) and \( M' \), was defined (see, e.g., [4, Sec. 9.1]). Let \( S(2n) \) denote the 2n-dimensional omnioriented quasitoric manifold \( M(I^n, (\mathrm{Id}_n \oplus \mathrm{Id}_n)) \), where \( I^n \) is the n-cube.

**Lemma 3.** The stably complex manifold \( S(2n) \) is the Cartesian product of \( n \) copies of the rational lines \( \mathbb{C}P^1 \) endowed with the nonstandard stably complex structure

\[
\mathbb{C}P^1 \oplus \mathbb{C} = \overline{\eta_1} \oplus \eta_1 \simeq \mathbb{C}^2.
\]

The manifold \( S(2n) \) is homeomorphic to \( (\mathbb{C}P^1)^n \). It has trivial unitary cobordism class and is a TNS manifold.

By definition, the diamond sum \( M \# M' \) of manifolds \( M \) and \( M' \) is oriented diffeomorphic to the connected sum \( M \# M' \# S(2n) \) of the manifolds \( M, M' \), and \( S(2n) \). The most important property of diamond sum is that this diffeomorphism is consistent with the canonical stably complex structures on the manifolds \( M \# M', M, M' \), and \( S(2n) \). Therefore, the unitary cobordism class of the diamond sum \( M \# M' \) equals the sum \([M] + [M']\). There is freedom in the definition of the diamond sum \( M \# M' \), namely, in the choice of the fixed points of \( M, M' \), and \( S(2n) \). Nevertheless, the unitary cobordism class of the diamond sum \( M \# M' \) does not depend on this choice and is well defined for omnioriented
quasitoric manifolds $M$ and $M'$. Moreover, the change of omni-orientation transforms the stably complex structure on any omni-oriented quasitoric manifold $M$ into the opposite one. Hence there exists a quasitoric manifold $\overline{M}$ representing the unitary cobordism class $-[M]$. Thus, we have proved the following proposition.

**Proposition 14.** Let $M_1$ and $M_2$ be quasitoric $2n$-manifolds. Then there exist quasitoric manifolds $\overline{M}_1$ and $M_1 \diamond M_2$ representing the unitary cobordism classes $-[M_1]$ and $[M_1] + [M_2]$, respectively. If the manifold $M_1$ is TNS, then so is $\overline{M}_1$. If both manifolds $M_1$ and $M_2$ are TNS, then so is $M_1 \diamond M_2$.

5. PROOF OF THE MAIN THEOREM 3

This section contains the proof of Theorem 3. We use the constructions of the auxiliary quasitoric TNS manifolds $X_{i,j}$, $M_{i,j}$, and $N_{i,j}$ presented in Secs. 2 and 4. To facilitate reading, we give the technically difficult parts of the proof of Theorem 3 (finding the Milnor numbers of $X_{i,j}$ and $M_{i,j}$ and the residues of binomial coefficients modulo $p$ and $p^2$) separately, in Secs. 6 and 7.

5.1. The Manifolds $X_{i,j}$, $M_{i,j}$, and $N_{i,j}$

**Definition 4.** For $1 \leq i \leq j$, we set

$$X_{i,j} = BF(\overline{\beta}_{i,1}, \cdots, \overline{\beta}_{i,j}, \overline{\zeta}_i, \cdots, \overline{\zeta}_j) \to BF_i, \quad X_{0,j} = BF_j.$$

For $i > 0$, by $Z_{i,j}$ we denote the invariant subvariety of $X_{i,j}$ of complex codimension 2 determined by the following conditions on the tautological bundle:

$$\beta_i = \beta_{i-1}, \quad \zeta_{j+1} = \zeta_j.$$

**Proposition 15.** For $i > 0$,

$$Z_{i,j} = BF(\overline{\zeta}_i, \overline{\beta}_{i-1}, \cdots, \overline{\beta}_1, \overline{\zeta}_j, \cdots, \overline{\zeta}_j) \to BF_{i-1}.$$

The normal bundle of the embedding $Z_{i,j} \subset X_{i,j}$ equals $\overline{\zeta}_j \beta_{i-1} \oplus \overline{\beta}_{i-1} \to Z_{i,j}$.

**Definition 5.** Let $M_{i,j} := Bl_{Z_{i,j}} X_{i,j}$. For even $n$, we set

$$N_{0,n} := M_{1,n-1}, \quad N_{1,n-1} = M_{1,n-1},$$

for odd $n$, we set

$$N_{0,n} := \overline{M}_{1,n-1} \diamond BF_n \diamond BF_n, \quad N_{1,n-1} = M_{1,n-1} \diamond BF_n,$$

and for $1 < i \leq j$, we set $N_{i,j} := M_{i,j}$.

**Proposition 16.** The varieties $X_{i,j}$, $M_{i,j}$, and $N_{i,j}$ are quasitoric TTS and TNS manifolds.

**Proof.** The manifold $BF_i$ is totally normally split by Corollary 1. The stably inverse complex bundles of $\overline{\beta}_k$ and $\overline{\beta}_k$, $k = 1, \ldots, i$, are totally split by (2.4). Therefore, $X_{i,j}$ is totally normally split by Proposition 1. Thus, $M_{i,j}$ is totally normally split by Proposition 13. The assertion concerning $N_{i,j}$ follows from properties of diamond sum (see Proposition 14).

**Proposition 17.** The relation $s_n([N_{0,n}]) = n + 1$ holds. Moreover, for $0 < i \leq j$ and $n = i + j$,

$$s_n([N_{i,j}]) = (-1)^{n+1} \binom{n}{j} - \sum_{k=j}^{n-1} \binom{k}{j}.$$

**Proof.** Recall that the Milnor number is linear with respect to taking diamond sum and reversing the orientation of a manifold. Now, to obtain the required relations, it remains to find the Milnor numbers of the varieties $BF_n$ (see Corollary 23) and $M_{i,j}$ (see Proposition 25).
5.2. Construction of Generators of the Ring $\Omega^*_U$

We need known facts of complex cobordism theory. According to the Milnor–Novikov theorem, the unitary cobordism ring is isomorphic to the countably generated graded polynomial ring:

$$\Omega^*_U \cong \mathbb{Z}[a_1, a_2, \ldots], \quad \text{deg}(a_i) = 2i$$

(see [10]).

**Theorem 5** (Milnor, Novikov; see [5], [10]). The cobordism class of a stably complex manifold $X^{2n}$ with $\dim \mathbb{R}X^{2n} = 2n$ can be taken for a multiplicative generator if and only if

$$s_n(X^{2n}) = \begin{cases} 
\pm 1 & \text{if } n \neq p^k - 1 \text{ for no prime } p, \\
\pm p & \text{if } n = p^k - 1 \text{ for some prime } p.
\end{cases}$$

Let us show that the greatest common divisor of the Milnor numbers of the varieties $BF_n$ and $N_{i,j}$, $i + j = n$, equals $p$ if $n = p^s - 1$, where $p$ is any prime and $s \in \mathbb{N}$, and equals 1 otherwise.

The proof of the following auxiliary proposition is given in Sec. 7.

**Proposition 18.** Let $s \geq 2$. Then, for any prime $p$,

$$\sum_{k=p^{s-1}-1}^{p^s-1} \binom{k}{p^s - p^{s-1} - 1} \equiv p \pmod{p^2}.$$

We introduce the notation

$$a_{i,j} := s_n([N_{i,j}]), \quad 0 \leq i \leq j, \quad i + j = n.$$

**Proposition 19.** Let $n = p^s - 1$ for $s \geq 1$ and prime $p$. Then

$$\text{GCD}\{s_n(BF_n), s_n(N_{i,j}) \mid i + j = n, 0 \leq i \leq j\} = p.$$

**Proof.** Consider the following cases. If $s = 1$, then $s_n(N_{0,0}) = n + 1 = p$. Otherwise, for $p = 2$, we have $s_n(BF_n) = 2$ (see (6.4)). Finally, if $s \geq 2$ and $p > 2$, then $p^{s-1} < p^s - p^{s-1} - 1$. Next, Proposition 17 implies $a_{0,n} = p^s$ and

$$a_{p^{s-1}, p^s - p^{s-1} - 1} = -\binom{n}{p^s - p^{s-1} - 1} - \sum_{k=p^{s-1}-1}^{p^s-2} \binom{k}{p^s - p^{s-1} - 1}$$

$$= -\sum_{k=p^{s-1}-1}^{p^s-1} \binom{k}{p^s - p^{s-1} - 1}.$$

Therefore, according to Proposition 18, we have $a_{p^{s-1}, p^s - p^{s-1} - 1} \equiv -p \pmod{p^2}$. Finally, we obtain

$$\text{GCD}(a_{0,n}, a_{p^{s-1}, p^s - p^{s-1} - 1}) = p.$$

We also need the following number-theoretic fact due to Lucas (its proof can be found in [5]).

**Theorem 6** (Lucas). Let $p$ be a prime, and let

$$n = n_0 + n_1p + \cdots + n_{r-1}p^{r-1} + n_rp^r, \quad (5.1)$$

$$m = m_0 + m_1p + \cdots + m_{r-1}p^{r-1} + m_rp^r \quad (5.2)$$

be the decompositions of positive integers $n$ and $m$ in base $p$. Then

$$\binom{n}{m} \equiv \binom{n_0}{m_0} \binom{n_1}{m_1} \cdots \binom{n_r}{m_r} \pmod{p}.$$
In what follows, we use the notation
\[ n = n_0 + n_1 p + \cdots + n_d p^d = [n_d, \ldots, n_0]_p \]
for the decomposition of a number in base \( p \), and we always omit the number of digits in decompositions in base \( p \).

**Proposition 20.** Let \( n \) be an integer such that \( n + 1 \) is not a prime power. Then
\[ \gcd\{ s_n(N_{i,j}) \mid i + j = n, 0 \leq i \leq j \} = 1. \]

**Proof.** It suffices to find a number \( a_{i,j} \) coprime to \( q \) for any prime divisor \( q \) of \( n + 1 \). Let us write \( n \) in the form \( [x_{s-1}, x_{s-2}, \ldots, x_0]_q \). Consider the following cases.

1. Suppose that
   \[ n = 2q^{s-1} - 1 = [1, q - 1, \ldots, q - 1]_q. \]
Then \( n + 1 = 2q^{s-1} \) is even, and the condition on \( n \) implies \( q > 2 \) and \( s > 1 \). We set
   \[ j = [1, q - 1, \ldots, q - 1, 0]_q. \]
Note that \( n - j = q - 1 < j \). By Lucas’s theorem, we have
   \[ a_{n-j,j} \equiv 1 - (q - 1) \equiv 2 \not\equiv 0 \pmod{q}. \]

2. Suppose that
   \[ n = (x_{s-1} + 1)q^{s-1} - 1 = [x_{s-1}, q - 1, \ldots, q - 1]_q, \quad x_{s-1} > 1. \]
Then \( 1 < x_{s-1} < q - 1, s > 1, \) and \( q > 3 \). We set
   \[ j = x_{s-1}q^{s-1} - 1 = [x_{s-1} - 1, q - 1, \ldots, q - 1]_q. \]
Note that \( n - j = q^{s-1} \). Since \( 1 < x_{s-1} \), we have \( n - j < j \). Lucas’s theorem implies
   \[ a_{n-j,j} \equiv \pm x_{s-1} - 1 \not\equiv 0 \pmod{q}. \]

3. Suppose that
   \[ n = [x_{s-1}, \ldots, x_a, \ldots, x_b, q - 1, \ldots, q - 1]_q, \]
where \( 0 < x_a, x_b < q - 1, b < a, \) and \( 0 < x_{s-1} (q^b \) is the maximum power of \( q \) dividing \( n + 1 \)). We set
   \[ j = [x_{s-1}, \ldots, x_a - 1, q - 1, \ldots, q - 1]_q, \]
where \( x_a - 1 \) is the \( a \)th digit. Then
   \[ n - j = [0, \ldots, x_{a-1}, \ldots, x_b + 1, 0, \ldots, 0]_q < j. \]
By Lucas’s theorem, we have \( \binom{k}{j} \equiv 0 \pmod{q} \) for \( j < k \leq n, \) and \( \binom{k}{j} = 1 \). Recall that
   \[ a_{n-j,j} = (-1)^{n+1} \binom{n}{j} - \sum_{k=j}^{n-1} \binom{k}{j}. \]
It follows that \( a_{n-j,j} \equiv -1 \pmod{q} \).

Thus, we have considered all possible values of \( n \) satisfying the assumption of the proposition. \( \square \)

**Proof of Theorem 3.** In Sec. 5.1, we constructed quasitoric TNS manifolds \( BF_n \) and \( N_{i,j}, 0 \leq i \leq j \). Their Milnor numbers are found in Proposition 17. According to Proposition 14, the integer combinations of quasitoric TNS manifolds (in the sense of diamond summation and orientation change) are quasitoric TNS manifolds as well. The greatest common divisor of the Milnor numbers of the manifolds \( BF_n \) and \( N_{i,j}, 0 \leq i \leq j, i + j = n, \) was calculated in Propositions 19 and 20. Therefore, for any \( n \in \mathbb{N} \), we can construct a quasitoric TNS manifold with Milnor number \( p \) if \( n = p^s - 1 \) for prime \( p \) and \( s \in \mathbb{N} \) or with Milnor number 1 otherwise. By Theorem 5, this manifold is a generator of the ring \( \Omega^*_U \) in degree \( 2n \), \( n \in \mathbb{N} \). According to the Milnor–Novikov theorem, any element of the ring \( \Omega^*_U \) of degree 2 can be represented as a linear combination of Cartesian products of manifolds constructed above, i.e., has a representative which is a quasitoric TNS manifold. \( \square \)
This section is devoted to the computation of the Milnor numbers $s_n$ of the toric varieties $X_{i,j}$ and their equivariant blow-ups $M_{i,j}$ defined in Sec. 5.1. A survey of the computation methods which we use can be found in [8].

**Proposition 21** (see [8, Sec. 4.5]). Let $Z$ and $X$, $Z \subset X$, be compact complex manifolds of dimensions $k$ and $n$, respectively. Consider the blow-up $\pi: Bl_Z X \to X$ along $Z$. The difference of the classes of $Bl_Z X$ and $X$ in the unitary cobordism ring equals

$$[Bl_Z X] - [X] = -[P(\nu(Z \subset X) \oplus \mathbb{C})];$$

(6.1)

where $\nu(Z \subset X)$ is the bundle normal to $Z$ and $P(\nu(Z \subset X) \oplus \mathbb{C})$ is the projective bundle endowed with the nonstandard stably complex structure

$$TP(\nu(Z \subset X) \oplus \mathbb{C}) \oplus \mathbb{C} \simeq (p^* \nu(Z \subset X) \otimes \gamma) \oplus \gamma^* \oplus p^*TB,$$

(6.2)

where $\gamma \to P(\nu(Z \subset X) \oplus \mathbb{C})$ is the corresponding tautological line bundle.

The cohomology ring $BF_i$ is easy to determine by using the Leray–Hirsch theorem.

**Proposition 22** (see [4, Theorem 7.8.2]). There is a graded ring isomorphism

$$H^*(BF_i; \mathbb{Z}) \simeq \mathbb{Z}[t_1,\ldots,t_i]/(t_a^2 - t_at_{a-1} | a = 1,\ldots,i),$$

where $t_0 := 0$.

The fundamental class of $BF_i$ is Poincaré dual to $t_i^i = t_i \cdots t_1 \in H^*(BF_i; \mathbb{Z})$. In the ring $H^*(BF_i; \mathbb{Z})$, the following identity holds:

$$(1 + t_i) \prod_{a=1}^{i} (1 - t_a + t_{a-1}) = 1.$$  

(6.3)

The Milnor number of the variety $X_{i,j}$ defined in Sec. 5.1 is found by using Proposition 2.

**Proposition 23.** The following relation holds:

$$s_{i+j}(X_{i,j}) = (1 + (-1)^{i+j+1}) \binom{i+j}{i}.$$  

**Corollary 4.** The following relation holds:

$$s_n(BF_n) = 1 + (-1)^{n+1}.$$  

(6.4)
We set $Z_{i+j-2} := Z_{i,j}$ and denote the “level” of complex dimension $k$ of the given Bott tower by $Z_k$.
By definition, we have $Z_{i-1} = BF_{i-1}$ for $i > 0$. For $i > 0$, we set

$$Y_{i,j} := \mathbb{P}(\zeta_j^{\beta_{i-1}} \oplus \beta_{i-1} \oplus \mathbb{C}) \to Z_{i,j}.$$ 

**Proposition 24.** The following identity holds:

$$s_{i+j}(Y_{i,j}) = \begin{cases} 
\sum_{k=j}^{i+j} \binom{k}{j}, & i \geq 2; \\
2^i & i = 1.
\end{cases}$$

Finally, we find the Milnor numbers of the varieties $M_{i,j}$.

**Proposition 25.** For $1 \leq i \leq j$ and $n := i + j$,

$$s_n(M_{i,j}) = \begin{cases} 
(-1)^{n+1} \binom{n}{j} - \sum_{k=j}^{n-1} \binom{k}{j}, & i \geq 2; \\
(-1)^{n+1} (n - 1) - 2, & i = 1.
\end{cases}$$

**Proof.** The required assertion follows from Propositions 21, 23, and 24 and the additivity of the Milnor number. 

### 7. NUMBER-THEORETIC RESULTS

In this section, we prove Proposition 18. Recall that, for any positive integer power $p^n$ of a prime $p$, the group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ of invertible elements of the residue ring $\mathbb{Z}/p^n\mathbb{Z}$ equals

$$\{k \in \mathbb{Z}/p^n\mathbb{Z} \mid \gcd(k, p) = 1\}.$$

In what follows, we understand expressions of the form $a/b = ab^{-1}$, where

$$a \in \mathbb{Z}/p^n\mathbb{Z} \quad \text{and} \quad b \in (\mathbb{Z}/p^n\mathbb{Z})^\times,$$

in the sense of inversion in this group.

It is easy to prove the following lemmas.

**Lemma 4.** For any nonnegative integers $0 \leq r < p$ (where $p$ is a prime),

$$\prod_{k=1}^{p-1} (pr + k) \equiv (p - 1)! \pmod{p^2}. \quad (7.1)$$

**Lemma 5.** For any positive integers $0 < a < p$ (where $p$ is a prime),

$$\prod_{k=1}^{a} \left( \frac{p(p-1) + k}{a!} \right) \equiv 1 - p \sum_{k=1}^{a} \frac{1}{k} \pmod{p^2}.$$
Lemma 6. For any positive integers $0 < a < p$ (where $p$ is a prime),
\[
\sum_{a=1}^{p-1} \frac{\prod_{k=1}^{a-1}(p(p-1)+k)}{a!} \equiv 0 \pmod{p^2}.
\] (7.2)

We also need yet another fact of number theory. Let $k!_p$ denote the product of positive integers between 1 and $k$ not divisible by $p$.

Theorem 7 (see [7, Theorem 1]). Consider a power $p^q$ of a prime $p$ and positive integers $m = n + r$. Let \( n = n_0 + n_1p + \cdots + n_dp^d \) be the decomposition of $n$ in base $p$. For each $j \geq 0$, let $N_j$ denote the number $\lfloor n/p^j \rfloor$ reduced modulo $p^q$, i.e.,
\[
N_j = n_j + n_{j+1}p + \cdots + n_{j+q-1}p^{q-1},
\]
and let $m_j$, $M_j$, $r_j$, and $R_j$ be the numbers defined in a similar way. Finally, let $e_j$ be the number of indices $i \geq j$ for which $n_i < m_i$ (i.e., the number of “carries” under the addition of $m$ and $r$ in base $p$ up to the $j$th digit). Then
\[
\frac{1}{p^e} \binom{n}{m} \equiv (-1)^{e-1} \left( \frac{(N_0)!_p}{(M_0)!_p (R_0)!_p} \right) \left( \frac{(N_1)!_p}{(M_1)!_p (R_1)!_p} \right) \cdots \left( \frac{(N_d)!_p}{(M_d)!_p (R_d)!_p} \right) \pmod{p^q},
\]
where $(-1)$ is $(-1)$, except in the case where $p = 2$ and $q \geq 3$.

Corollary 5 (Kummer; see [7]). For any positive integers $n$ and $m$ and any prime $p$, the maximum power of $p$ dividing the binomial coefficient $\binom{n}{m}$ equals the number of “carries” under the addition of $m$ and $n - m$ in base $p$.

Lemma 7 (see [7, Sec. 2, Lemma 1]). The following congruence holds:
\[
(p^2)!_p \equiv -1 \pmod{p^2}.
\] (7.3)

Theorem 6 implies the following lemma.

Lemma 8. The following congruence holds:
\[
\binom{p^s - 1}{p^s - p^{s-1} - 1} \equiv p - 1 \pmod{p^2}.
\]

Lemma 9. The following congruence holds:
\[
\sum_{k=p^s - p^{s-1}}^{p^{s-2}} \binom{k}{p^s - p^{s-1} - 1} \equiv 0 \pmod{p^2}.
\]

Proof. We have
\[
\sum_{k=p^s - p^{s-1}}^{p^{s-2}} \binom{k}{p^s - p^{s-1} - 1} = \sum_{k=0}^{s-2} \sum_{x_k=0}^{p-1} \sum_{x_k<p} \binom{p - 1, x_{s-2}, x_{s-3}, \ldots, x_k, p - 1, \ldots, p - 1}{p - 2, p - 1, p - 1, \ldots, p - 1, p - 1, \ldots, p - 1}.
\]
According to Kummer’s theorem (see Corollary 5), if at least two numbers among $x_k, \ldots, x_{s-2}$ are different from $p - 1$, then
\[
\binom{p - 1, x_{s-2}, x_{s-3}, \ldots, x_k, p - 1, \ldots, p - 1}{p - 2, p - 1, p - 1, \ldots, p - 1, p - 1, \ldots, p - 1} \equiv 0 \pmod{p^2}.
\]
Therefore,
\[
\sum_{k=0}^{s-2} \sum_{x_k,\ldots,x_{s-2} = 0}^{p-1} \left( \frac{[p - 1, x_{s-2}, x_{s-3}, \ldots, x_k, p - 1, \ldots, p - 1]}{p} \right) \equiv \sum_{k=0}^{s-2} \sum_{x_k = 0}^{p-2} \left( \frac{[p - 1, p - 1, \ldots, x_k, p - 1, \ldots, p - 1]}{p} \right) \pmod{p^2},
\]
where \( x_k \) is the \( k \)th digit. Let \( k = s - 2 \). Theorem 6 and Lemmas 6 and 7 imply
\[
\frac{1}{p} \cdot \sum_{x_{s-2} = 0}^{p-2} \left( \frac{[p - 1, x_{s-2}, p - 1, \ldots, p - 1]}{p} \right) \equiv \pm \sum_{x_{s-2} = 0}^{p-2} \frac{(x_{s-2} + p(p - 1))!}{(p - 1 + p(p - 2))!} \cdot \frac{1}{p(x_{s-2} + 1)!} \equiv \pm (p - 1) \frac{p - 1}{\prod_{a=1}^{p-1} (p(p - 1) + r) \pmod{p!}} \equiv 0 \pmod{p^2}.
\]
Suppose that \( 0 \leq k < s - 2 \). Then Theorem 6 and Lemmas 6 and 7 imply
\[
\frac{1}{p} \cdot \sum_{x_k = 0}^{p-2} \left( \frac{[p - 1, p - 1, \ldots, p - 1, x_k, p - 1, \ldots, p - 1]}{p} \right) \equiv \pm \sum_{x_k = 0}^{p-2} \frac{(x_k + p(p - 1))!}{(p - 1 + p(p - 1))!} \cdot \frac{1}{p(x_k + 1 + p(p - 1))!} \equiv \pm (p - 1) \frac{1}{\prod_{a=1}^{p-1} (p(p - 1) + r) \pmod{p!}} \equiv \pm (p - 1) \frac{(p - 1)^{p-1}}{p} \pmod{p^2}.
\]
Then
\[
\sum_{x_k = 0}^{p-2} \left( \frac{[p - 1, p - 1, \ldots, p - 1, x_k, p - 1, \ldots, p - 1]}{p} \right) \equiv 0 \pmod{p^2}.
\]
The required assertion follows from congruences (7.4), (7.4), (7.5), and (7.7).

Now Proposition 18 follows from Lemmas 8 and 9.

8. CONCLUDING REMARKS

It is natural to seek a shorter proof of Theorem 3. Let us introduce a family of nonsingular projective complex varieties.

Definition 6. By \( R_{i,j} \), \( 0 \leq i \leq j \), we denote the hypersurface in \( BF_i \times BF_j \) determined by the equation
\[
\sum_{k=0}^{i} z_{i,k,w_{j,k+j-i}} = 0.
\]
Note that \( R_{0,j} = BF_{j-1} \) and \( R_{0,0} = \emptyset \).
Remark 5. The order of \( w \)-variables in (8.1) is important. For example, consider the subvariety \( R_{1,2}' \subset BF_1 \times BF_2 \) given by
\[
z_{1,0}w_{2,0} + z_{1,1}w_{2,1} = 0.
\]
Recall that \( BF_1 \times BF_2 \subset CP^1 \times CP^1 \times CP^2 \) is determined by the single equation
\[
w_{2,0}w_{1,1} - w_{2,1}w_{1,0} = 0.
\]
It is easy to see that the variety \( R_{1,2}' \) is singular along the subvariety
\[
\{ w_{2,0} = w_{2,1} = 0, z_{1,0}w_{1,0} + z_{1,1}w_{1,1} = 0 \}
\]
isomorphic to \( CP^1 \).

These varieties are given explicitly. It is easy to prove the following proposition.

Proposition 26. (1) The variety \( R_{i,j} \) is the dualization of the first Chern class of the complex line bundle \( \mathcal{L}_i^{1/2} \) in \( BF_i \times BF_j \).
(2) For \( 0 < i \leq j \), \( s_{i+j-1}(R_{i,j}) = -(i+j) \).
(3) The varieties \( N_{0,n} \) and \( R_{i,j} \) with \( i + j = n + 1, n \in \mathbb{N} \), provide multiplicative generators of the ring \( \Omega_{T_i} \).
(4) The variety \( R_{i,j} \) is the strict transform of \( BR_{i,j} \) under a sequence of blow-ups of the strict transforms of the subvarieties \( \{ w_1 = \cdots = w_k = 0 \} \subset BF_i \times CP^j \), where \( k \) ranges over \( j-1, \ldots, 2 \).

The intersections of the subvarieties of \( BF_i \times CP^j \) specified in Proposition 26 with \( BR_{i,j} \) are not generally invariant under the torus action on \( BR_{i,j} \) introduced above. However, this does not prevent the existence of an effective torus action with dense orbit on \( R_{i,j} \). The following question arises.

Question. Is \( R_{i,j} \) a toric variety for \( 0 \leq i \leq j^2 \)?

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