The static properties of nuclei can be deduced from the dynamics of a single quark

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Abstract

We show that the static properties of a nucleus could arise from a single quark moving in a mean field due to all other constituents of this nucleus. The resulting model provides a way for determining nuclear sizes characteristic of the liquid drop model and reasonably accurate values of magnetic moments of different nuclei.

Keywords: QCD phenomenology
1 Introduction

With the advent of quantum chromodynamics (QCD) a serious effort was mounted to understand nuclei in terms of quarks. Early in the development of this line of inquiry, a nucleus with mass number $A$ was conceived as a system of $3A$ quarks moving in a large bag and forming a shell structure [1], [2]. Were this model experimentally successful, one would regard it as an effective theory to low energy QCD which covers both free hadrons and nuclei. However, the predicted nuclear magnetic moments differ considerably from their observed values [3], [4]. Furthermore, the relationship between the number of quarks $N$ that are contained in a stable bag and its size $R$ is given by $R \sim N^{1/4}$ which implies that the link with the liquid drop model, exhibiting the relationship $R \sim A^{1/3}$, is lost in this approach. This seems to be the origin of the discrepancy stated by the authors of Refs. [3] and [4]. We thus should proceed from another paradigm.

A direct way for clarifying the properties of a bound many-quark system in the strong-coupling regime is to use the Feynman path integral machinery whereby all color degrees of freedom are integrated out, except for those of a single quark $Q$, so that this quark is affected by a mean field generated by all other constituents of the system. A systematic implementation of this calculation program is still a good distance in the future. And yet integrating out the remaining degrees of freedom can be neatly approximated if we make plausible assumptions about the color relief over which the quark $Q$ travels, and invoke the semiclassical approximation for exploring the behavior of the quark $Q$. This strategy, proposed in [6], enables the quarkonium spectra to be found in good agreement with experiment. Under a different boundary condition, the same procedure makes possible to describe some static properties of nuclei, in particular nuclear sizes characteristic of the liquid drop model. We will show in the present paper that reasonably accurate values of magnetic moments for a rich variety of nuclei are also attainable in this framework.

The dynamics of the quark $Q$, specified by the Dirac field $\psi$, is assumed to be encoded by the Lagrangian

$$\mathcal{L} = \bar{\psi} \left[ i \gamma^{\mu} \left( \partial_{\mu} + ig_{V}A_{\mu} \right) - m_{Q} \right] \psi + g_{S} \bar{\psi} \psi \Phi, \quad (1)$$

where $A_{\mu} = (A_{0}, \mathbf{A})$ and $\Phi$ are respectively the Lorentz vector- and scalar potential of the mean color field, $g_{V}$ and $g_{S}$ their associated couplings, and $m_{Q}$ the current-quark mass of the quark $Q$. The second interaction term in (1) is absent from the QCD Lagrangian because the scalar Yukawa coupling is contrary to asymptotic freedom. However, our concern is with the effective theory in the infrared region where the dynamics is anticipated to arrange itself into the form shown in Eq. (1). The pseudoscalar coupling $g_{P} \bar{\psi} \gamma_{5} \psi \chi$ is a further fascinating candidate for the color relief whose analysis will be given elsewhere.

The semiclassical treatment implies that the extremal path contribution dominates the Feynman path integral. This is another way of stating that the wave function $\psi(x)$ of the quark $Q$ is a solution to the Dirac equation in the background $A_{\mu}(x)$ and $\Phi(x)$ because this equation results from the requirement that the action with the Lagrangian (1) be

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1. See, e. g., Ref. [5], Eq. (18.20).
2. Setting $g_{V} = g_{S} = 0$, while keeping $g_{P}$ finite, we find that the Dirac equation diagonalizes, so that the system with a pseudoscalar relief enjoys the property of supersymmetry [7].
the Dirac eigenstates are the eigenstates of this operator with eigenvalues \( K = \ell \) coupled amplitude \( \sum_{\ell, j} \) total angular momentum quantum numbers, respectively, \( f \). Here, \( \ell \) is extremal. In fact, we restrict our attention to spherically symmetric static interactions, and assume that the contribution of the Lorentz vector potential to the mean field is given by \( A_0 \), that is, \( A = 0 \). What this means is a particle with reduced mass \( m \) orbits the center of mass being driven by central potentials \( A_0(r) \) and \( \Phi(r) \). We are thus dealing with the eigenvalue problem for the Dirac Hamiltonian,

\[
\{ -i \alpha \cdot \nabla + U_V(r) + \beta [m + U_S(r)] \} \psi(r) = \epsilon \psi(r),
\]

where \( U_V = g_V A_0 \), and \( U_S = g_S \Phi \).

The distinction between free hadrons and nuclei is controlled by the phenomenological spin- and pseudospin symmetry conditions (an extended discussion of spin and pseudospin symmetries can be found in Refs. 8 and 9). We ascribe the spin symmetry condition to free hadrons, and the pseudospin symmetry condition to nuclei.

We take the Cornell potential \([10]\)

\[
V_C(r) = -\frac{\alpha_s}{r} + \sigma r
\]

as a phenomenological realization of both \( U_V \) and \( U_S \), and put \( V_C = 2U_V \).

We separate variables in the two-row Dirac eigenstates,

\[
\psi_{n_r, \ell, j, M}(r) = \begin{pmatrix} f_{n_r, \ell}(r) \left[ Y^{(\ell)}(\theta, \phi) \chi_j^M \right] \\ ig_{n_r, \ell, j}(r) \left[ Y^{(\ell)}(\theta, \phi) \chi_j^M \right] \end{pmatrix}
\]

Here, \( f_{n_r, \ell} \) and \( g_{n_r, \ell, j} \) are the radial amplitudes, \( n_r, \ell, j \) are the radial, orbital, and total angular momentum quantum numbers, respectively, \( Y^{(\ell)}(\theta, \phi) \chi_j^M \) stands for the spherical harmonic of order \( \ell, j \) is given by \( \ell_{-1/2} = \ell - 1 \) and \( \ell_{1/2} = \ell + 1 \), \( \chi^M \) is the spin function. The operator \( K = -\beta (S \cdot L + 1) \) commutes with the spherically symmetric Dirac Hamiltonian. Thus, the Dirac eigenstates are the eigenstates of this operator with eigenvalues \( \kappa = \pm (j + \frac{1}{2}) \), with \( - \) for aligned spin \( (s_{1/2}, p_{3/2}, \text{etc.}) \), and \( + \) for unaligned spin \( (p_{1/2}, d_{3/2}, \text{etc.}) \), so that the quantum number \( \kappa \) is sufficient to label the orbitals \([3]\). The radial part of Eq. (2) is

\[
f' + \frac{1 + \kappa}{r} f - ag = 0, \tag{5}
\]

\[
g' + \frac{1 - \kappa}{r} g + bf = 0, \tag{6}
\]

\[
a(r) = \epsilon + m + U_S(r) - U_V(r), \tag{7}
\]

\[
b(r) = \epsilon - m - U_S(r) - U_V(r), \tag{8}
\]

where the prime stands for differentiation with respect to \( r \).

We use (5) for expressing \( g \) in terms of \( f \) and substitute the result in (6). Our concern is with \( f \) because it is \( f \) that survives in the nonrelativistic free-particle limit. We eliminate the first derivative of \( f \) from the resulting second-order differential equation to obtain

\[
F'' + k^2 F = 0, \tag{9}
\]

where
\[ k^2 = \varepsilon^2 - m^2 - 2U(r; \varepsilon) = -\frac{1}{2} A'(r) - \frac{1}{4} A^2(r) + B(r), \] (10)

\[ A = -a \frac{a'}{a} + \frac{2}{r}, \quad B = a (1 + \kappa) \left( \frac{1}{ra} \right)' + ab + \frac{1 - \kappa^2}{r^2}. \] (11)

Once all angular (orbital and spin) variables are integrated out, and the pertinent phenomenological condition is imposed on \( U_S \) and \( U_V \), the function \( U(r; \varepsilon) \) defined in (10) acts as the effective potential.

2 Quarkonia

Spin symmetry is inherent in free hadron states \([11, 8]\). This symmetry occurs when \( U_S = U_V \). With \( V_C = 2U_V \), Eqs. (10) and (11) can be solved to give the effective potential

\[ U(r; \varepsilon) = \frac{1}{2} \left[ \kappa(\kappa + 1) \frac{1}{r^2} + (\varepsilon + m) \left( -\frac{\alpha_s}{r} + \sigma r \right) \right]. \] (12)

The form of the effective potential (12) is shown in Figure 1.

\[ \begin{align*}
\text{Figure 1: The effective potential (12) with the parameters } & m = 0.34 \text{ GeV, } \varepsilon = 1 \text{ GeV, } \\
& \alpha_s = 0.73, \sigma = 0.14 \text{ GeV}^2, \kappa = 1. 
\end{align*} \]

In the nonrelativistic limit \( \varepsilon \to m \), \( U(r; \varepsilon) \) becomes the sum of the centrifugal term and the Cornell potential, and hence, the Dirac equation reproduces results for the spectrum of quarkonia which were obtained through the use of the Schrödinger equation \([12, 13]\).

We now outline the procedure of looking for numerical solutions to Eqs. (5)–(8) with imposing the conditions \( U_S = U_V = \frac{1}{2} V_C \). Since our interest is with solutions that are regular at \( r = 0 \), we use the ansätze \( f(r) = u(r) r^{\kappa - 1} \) and \( g(r) = v(r) r^{\kappa - 1} \). The functions \( u \) and \( v \) satisfy the following integral equations

\[ u(r) = \frac{\kappa - |\kappa|}{\alpha_s} + (\varepsilon + m) \int_0^r ds \left[ \theta(-\kappa) + \theta(\kappa) \left( \frac{s}{r} \right)^{2|\kappa|} \right] v(s), \] (13)
\[ v(r) = 1 + \int_{0}^{r} ds \left\{ (\sigma s - \varepsilon + m) \left[ \theta(\kappa) + \theta(-\kappa) \left( \frac{s}{r} \right)^{2|\kappa|} \right] u(s) + \frac{\alpha_s(\varepsilon + m)}{2|\kappa|} \left[ \left( \frac{s}{r} \right)^{2|\kappa|} - 1 \right] v(s) \right\}, \]

where \( \theta(\kappa) \) is the Heaviside step function. We write \( u \) and \( v \) as the Liouville–Neumann series,

\[ u = \sum_{k=0}^{\infty} u_k, \quad v = \sum_{k=0}^{\infty} v_k, \quad (15) \]

where \( u_0 = (\kappa - |\kappa|)/\alpha_s, \quad v_0 = 1, \]

\[ u_{k+1} = \Phi_{12} v_k, \quad v_{k+1} = \Phi_{21} u_k + \Phi_{22} v_k, \quad k \geq 0, \quad (16) \]

\[ \Phi_{12} v(r) = (\varepsilon + m) \int_{0}^{r} ds \left[ \theta(-\kappa) + \theta(\kappa) \left( \frac{s}{r} \right)^{2|\kappa|} \right] v(s), \quad (17) \]

\[ \Phi_{21} u(r) = \int_{0}^{r} ds \left( \sigma s - \varepsilon + m \right) \left[ \theta(\kappa) + \theta(-\kappa) \left( \frac{s}{r} \right)^{2|\kappa|} \right] u(s), \quad (18) \]

\[ \Phi_{22} v(r) = \frac{\alpha_s(\varepsilon + m)}{2|\kappa|} \int_{0}^{r} ds \left[ \left( \frac{s}{r} \right)^{2|\kappa|} - 1 \right] v(s). \quad (19) \]

Every term of these series is a polynomial. The coefficient of a given monomial can be found recursively taking into account that

\[ \Phi_{12}(r^k) = (\varepsilon + m) \left[ \frac{\theta(-\kappa)}{k+1} + \frac{\theta(\kappa)}{k+1+2|\kappa|} \right] r^{k+1}, \quad (20) \]

\[ \Phi_{21}(r^k) = \sigma \left[ \frac{\theta(\kappa)}{k+2} + \frac{\theta(-\kappa)}{k+2+2|\kappa|} \right] r^{k+2} - (\varepsilon - m) \left[ \frac{\theta(\kappa)}{k+1} + \frac{\theta(-\kappa)}{k+1+2|\kappa|} \right] r^{k+1}, \quad (21) \]

\[ \Phi_{22}(r^k) = -\frac{\alpha_s(\varepsilon + m)}{(k+1)(k+1+2|\kappa|)} r^{k+1}. \quad (22) \]

The agreement between the energy levels \( \varepsilon_n \) obtained by this means and the masses of well established \( c\bar{c} \) and \( b\bar{b} \) states is within \( \sim 1\% \) for \( \alpha_s = 0.7, \sigma = 0.14 \text{ GeV}^2, m_c = 1.45 \text{ GeV}, m_b = 4.92 \text{ GeV} \), see Table 1 which lists the experimental data from [14].

## 3 Nuclei

The wavefunction classification stemming from pseudospin symmetry is important for both light and very heavy nuclei whose superdeformation appears already at low spin [8, 9]. The pseudospin transformation \( \ell = j \pm \frac{1}{2} \rightarrow \ell = j \mp \frac{1}{2} \) acts on the angular-spin wave functions of the particle, without affecting the radial motion. Pseudospin degeneracy in heavy nuclei derives from the fact that nucleons in a nucleus move in an attractive scalar, \( -U_s \), and repulsive vector, \( U_V \), mean fields, which are nearly equal in magnitude, \( |U_s| \approx |U_V| \). This near equality of mean fields is likely a general feature of any relativistic model which fits nuclear binding energies [15].
Table 1: Quarkonium masses

| Quarkonium | Calculation | Experiment | Quarkonium | Calculation | Experiment |
|------------|-------------|------------|------------|-------------|------------|
| $J/\psi$ (1S) | 3.084 | 3.097 | $\gamma$ (1S) | 9.415 | 9.460 |
| $\eta_c$ (1S) | 3.084 | 2.980 | $\eta_b$ (1S) | 9.415 | 9.398 |
| $\psi$ (2S) | 3.635 | 3.686 | $\gamma$ (2S) | 10.04 | 10.023 |
| $\eta_c$ (2S) | 3.635 | 3.638 | $\eta_b$ (2S) | 10.04 | 9.999 |
| $\psi$ (3S) | 4 | 4.030 | $\gamma$ (3S) | 10.35 | 10.355 |
| $\psi$ (4S) | 4.3 | 4.421 | $\gamma$ (4S) | 10.582 | 10.579 |
| $\eta_c$ (1S) | 3.084 | 2.980 | $\eta_b$ (1S) | 9.415 | 9.398 |
| $\psi$ (2S) | 3.635 | 3.686 | $\gamma$ (2S) | 10.04 | 10.023 |
| $\eta_c$ (2S) | 3.635 | 3.638 | $\eta_b$ (2S) | 10.04 | 9.999 |
| $\psi$ (3S) | 4 | 4.030 | $\gamma$ (3S) | 10.35 | 10.355 |
| $\psi$ (4S) | 4.3 | 4.421 | $\gamma$ (4S) | 10.582 | 10.579 |

One can go a step further and consider this symmetry as that arising from a quark $Q$ moving in a mean field generated by other quarks in the nucleus [6]. We adopt the condition $U_S = -U_V + C_s$ where $C_s$ is a constant. The Hamiltonian governing the quark $Q$ becomes

$$H_s = \alpha \cdot p + U_V(r)(1 - \beta) + \beta (m + C_s).$$

We thus see that $m$ is shifted, $m \rightarrow m_s = m + C_s$. This shift may signal that the current-quark masses become the corresponding constituent-quark masses. In what follows $m$ will be regarded as constituent-quark mass of the quark $Q$, and $C_s$ will be omitted.

We put $V_C = 2U_V$, and solve Eqs. (10) and (11) to give

$$U(r; \varepsilon) = \frac{1}{2r^2} \left\{ \kappa (\kappa + 1) + (\varepsilon - m) \left( -\frac{\alpha_s}{r} + \sigma r \right) r^2 + \frac{3(\alpha_s + \sigma r^2)^2}{4[\sigma r^2 - (\varepsilon + m)r - \alpha_s]^2} + \frac{\alpha_s(\kappa + 1) + \kappa \sigma r^2}{\sigma r^2 - (\varepsilon + m)r - \alpha_s} \right\}. \quad (24)$$

The terms in the first line of (24) closely resembles the respective terms involved in (12), except for changing the overall factor of the Cornell potential, but the terms of the second line dramatically change the situation. They are singular at $r = r_{Sc}$ which is the positive root of the equation $\sigma r^2 - (\varepsilon + m)r - \alpha_s = 0$,

$$r_{Sc} = \frac{(\varepsilon + m) + \sqrt{(\varepsilon + m)^2 + 4\sigma \alpha_s}}{2\sigma}. \quad (25)$$
Figure 2: The effective potential (24) with the parameters $m = 0.33 \text{ GeV}$, $\varepsilon = 1 \text{ GeV}$, $\alpha_s = 0.7$, $\sigma = 0.14 \text{ GeV}^2$, $\kappa = 1$.

A plot of $U(r; \varepsilon)$ defined by (24) is depicted in Fig. 2.

It is thus seen that the condition $U_V = -U_S = \frac{1}{2} V_C$ vastly enhances the interaction between spin degrees of freedom and the mean field to yield a spherical shell of radius $r_{sc}$ on which $U(r; \varepsilon)$ is infinite. The boundary of the spherical cavity of radius $r_{sc}$ keeps colored objects in this cavity from escaping. It is well known [16] that the tunneling through a potential barrier of the form $\lambda(x - x_0)^{-2}$ with $\lambda \geq \frac{3}{4}$ is forbidden in one-dimensional quantum mechanics. This condition is fulfilled by Eq. (24). Therefore, the boundary of the cavity sets up an impenetrable quantum-mechanical barrier to every colored object. Specifically, this implies that the wavefunction of the quark $Q$ is zero outside the cavity.

A singular boundary arises whenever $U_V(r)$ grows indefinitely with $r$. This is because in going from Eqs. (5) and (6) to Eq. (9), we have to apply the factor $1/a$ which is infinite when $a = 0$. Such is not the case when we adopt the spin symmetry condition $U_S = U_V$ by which $a = \varepsilon + m$. In contrast, the pseudospin symmetry condition $U_S = -U_V$ implies that $a = \varepsilon + m - 2U_V$, and $a = 0$ has a positive root provided that $U_V$ increases monotonically with $r$ beginning at $r = 0$ where $U_V$ assumes a negative value. However, no singular boundary arises when $U_V \rightarrow C$ as $r \rightarrow \infty$, where $C$ is a constant which is less than $\frac{1}{2}(\varepsilon + m)$. This is the reason for the absence of confinement from systems with electromagnetic bindings.

To verify that the effective potential $U(r; \varepsilon)$ defined by Eq. (24) is indeed attributable to the description of nuclei, we solve numerically Eqs. (5) and (6) to Eq. (9), we have to apply the factor $1/a$ which is infinite when $a = 0$. Such is not the case when we adopt the spin symmetry condition $U_S = U_V$ by which $a = \varepsilon + m$. In contrast, the pseudospin symmetry condition $U_S = -U_V$ implies that $a = \varepsilon + m - 2U_V$, and $a = 0$ has a positive root provided that $U_V$ increases monotonically with $r$ beginning at $r = 0$ where $U_V$ assumes a negative value. However, no singular boundary arises when $U_V \rightarrow C$ as $r \rightarrow \infty$, where $C$ is a constant which is less than $\frac{1}{2}(\varepsilon + m)$. This is the reason for the absence of confinement from systems with electromagnetic bindings.

To verify that the effective potential $U(r; \varepsilon)$ defined by Eq. (24) is indeed attributable to the description of nuclei, we solve numerically Eqs. (5) and (6) using the parameters $\alpha_s = 0.7$ and $\sigma = 0.1 \text{ GeV}^2$ (borrowed from the description of quarkonia), and taking $m$ to be $0.33 \text{ GeV}$. The procedure closely parallels that outlined in the previous section. We seek solutions in the form $f(r) = u(r)r^{\kappa - 1}$ and $g(r) = v(r)r^{\kappa - 1}$ with $u$ and $v$ obeying...
the set of integral equations

\[ u(r) = 1 + \int_0^r ds \left\{ \frac{\alpha_s(\varepsilon - m)}{2|\kappa|} \left[ \left( \frac{s}{r} \right)^{2|\kappa|} - 1 \right] u(s) + (\varepsilon + m - \sigma s) \left[ \theta(-\kappa) + \theta(\kappa) \left( \frac{s}{r} \right)^{2|\kappa|} \right] v(s) \right\}, \]

\[ v(r) = \frac{\kappa + |\kappa|}{\alpha_s} - (\varepsilon - m) \int_0^r ds \left[ \theta(\kappa) + \theta(-\kappa) \left( \frac{s}{r} \right)^{2|\kappa|} \right] u(s). \]

(26)

(27)

The functions \( u \) and \( v \) can be expanded into the Liouville–Neumann series,

\[ u = \sum_{k=0}^{\infty} u_k, \quad v = \sum_{k=0}^{\infty} v_k \]

(28)

with \( u_0 = 1, \ v_0 = (\kappa + |\kappa|)/\alpha_s \), and

\[ u_{k+1} = \Phi_{11} u_k + \Phi_{12} v_k, \quad v_{k+1} = \Phi_{21} u_k, \quad k \geq 0, \]

(29)

\[ \Phi_{11} u(r) = \frac{\alpha_s(\varepsilon - m)}{2|\kappa|} \int_0^r ds \left[ \left( \frac{s}{r} \right)^{2|\kappa|} - 1 \right] u(s), \]

(30)

\[ \Phi_{12} v(r) = \int_0^r ds (\varepsilon + m - \sigma s) \left[ \theta(-\kappa) + \theta(\kappa) \left( \frac{s}{r} \right)^{2|\kappa|} \right] v(s), \]

(31)

\[ \Phi_{21} u(r) = (-\varepsilon + m) \int_0^r ds \left[ \theta(\kappa) + \theta(-\kappa) \left( \frac{s}{r} \right)^{2|\kappa|} \right] u(s). \]

(32)

Every term of these series is a polynomial. The coefficient of a given monomial can be found recursively taking into account that

\[ \Phi_{11}(r^k) = -\frac{\alpha_s(\varepsilon - m)}{(k+1)(k+1+2|\kappa|)} r^{k+1}. \]

(33)

\[ \Phi_{12}(r^k) = (\varepsilon + m) \left[ \frac{\theta(-\kappa)}{k+1} + \frac{\theta(\kappa)}{k+1+2|\kappa|} \right] r^{k+1} - \sigma \left[ \frac{\theta(-\kappa)}{k+2} + \frac{\theta(\kappa)}{k+2+2|\kappa|} \right] r^{k+2}, \]

(34)

\[ \Phi_{21}(r^k) = (-\varepsilon + m) \left[ \frac{\theta(\kappa)}{k+1} + \frac{\theta(-\kappa)}{k+1+2|\kappa|} \right] r^{k+1}. \]

(35)

We find the energy levels \( \varepsilon_{nr} \) for \( \kappa = -1 \) and the corresponding sizes of the cavities \( r_{Sc}(nr) \). To a good approximation the energy levels \( \varepsilon_{nr} \) turn out to be proportional to \( \sqrt{n_r} \). By assuming that \( n_r \) is proportional to \( A^{2/3} \), where \( A \) is the nucleon mass number, we come to the relationship \( r_{Sc} = R_0 A^{1/3} \) with \( R_0 \approx 1 \) fm for the chosen values of the \( \alpha_s \) and \( \sigma \), which is characteristic of the liquid drop model.

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3To offer a qualitative explanation of this relationship for large \( n_r \), we recast Eqs. 6 and 8 as \( G'' + [(\kappa - \kappa')/r^2 + \varepsilon' + m + \alpha_c/r - \sigma r] \) \( G = 0 \), where \( G = rg \). Since our concern is with large \( r_{Sc} \), we may safely omit the terms proportional to \( r^{-1} \) and \( r^{-2} \). This gives the Airy equation. Regular solutions to this equation behave asymptotically as \( \sin \left\{ (\varepsilon + m)/\sigma \right\} r^{1/2} \) \( (\varepsilon + m)/\sigma = r^{3/2} + \phi_0 \), which shows that the number of radial nodes \( n_r \) is estimated at \( \left[ \frac{2}{3}(\varepsilon r)^{1/2} \right] [(\varepsilon + m)/\sigma]^{3/2} \). Therefore, \( \varepsilon_{nr} \sim \sqrt{n_r} \).
The conjecture that \( n_r \) equals the integral part of \( A^{2/3} \) has far-reaching implications, sending us in search of evidence for or against this conjecture. With this in mind, we compare the magnetic dipole of the quark \( Q \) and that of the nucleus in which this quark is incorporated. Note that the magnetic moment of a nucleon in the states with isospin \( I \) and the total angular momentum \( j = I - \frac{1}{2} \) and \( j = I + \frac{1}{2} \) is given \([17]\), respectively, by

\[
\mu_{j,I} = -\frac{e\gamma_I}{2(j+1)} \int_0^\infty f_{\tilde{r},\tilde{\ell},j,I} g_{\tilde{r},\tilde{\ell},j,I} r^3 dr + \mu_{A,I} \left( 1 - \frac{2j+1}{j+1} \int_0^\infty g_{\tilde{r},\tilde{\ell},j,I}^2 r^2 dr \right), \quad (36)
\]

and

\[
\mu_{j,I} = -\frac{e\gamma_I}{2(j+1)} \int_0^\infty f_{\tilde{r},\tilde{\ell},j,I} g_{\tilde{r},\tilde{\ell},j,I} r^3 dr - \mu_{A,I} \frac{j}{j+1} \left[ j - (2j+1) \int_0^\infty g_{\tilde{r},\tilde{\ell},j,I}^2 r^2 dr \right]. \quad (37)
\]

Here, \( \gamma_I \) is the orbital gyromagnetic ratio, \( \mu_{A,I} \) is the anomalous magnetic moment, \( \mu_{A,\frac{1}{2}} = -1.913 \mu_B \), \( \mu_{A,\frac{3}{2}} = 1.793 \mu_B \), where \( \mu_B = \epsilon/2M \) is the nucleon magneton. We reiterate mutatis mutandis the arguments of Ref. \([17]\) to conclude that \((36)\) and \((37)\), in which the infinite limits of integration are replaced by \( r_{Sc} \), are well suited for the magnetic moment of neutron-odd nuclei with \( j = \frac{1}{2} \) and \( j = \frac{3}{2} \) to be represented by the magnetic moment of a constituent \( u \) quark contained in these nuclei, with the understanding that \( n_r \) is associated with \([A^{2/3}]\). We determine the quark wave functions \( f_{n_r,j} \) and \( g_{n_r,j} \) for \( \alpha_s = 0.7 \), \( \sigma = 0.14 \text{ GeV}^2 \), \( m = 0.33 \text{ GeV} \). As to the anomalous magnetic moments of quarks, the present notion of their values is far from complete. Among suggested values of \( \mu_A \) for \( u \) quarks \([18], [19]\), we adopt to test \( \mu_A = 0.15 \mu_B \), and 0.2 \( \mu_B \).

Fig. 3(a,b) shows that the observed magnetic moments of neutron-odd nuclei with \( j = \frac{1}{2} \) and \( j = \frac{3}{2} \) \([20]\) agree with the calculated magnetic moments of a \( u \) quark involved in those nuclei to an accuracy of \( \sim 20\% \). The exceptions are \(^{113}_{50}\text{Sn}\) and \(^{119}_{52}\text{Te}\) whose magnetic moments are consistent with the results of our calculation to within \( 50 \div 90\% \). \(^{113}_{50}\text{Sn}\) and \(^{119}_{52}\text{Te}\) have respectively 50 and 52 protons. 50 is a magic number whereas 52 is not. Note also that other neutron-odd nuclei with magic numbers of protons \((^{15}_8\text{O} \text{ and } ^{207}_{82}\text{Pb})\) in the state with \( j = \frac{1}{2} \), and \(^{30}_{16}\text{Ca}, ^{57}_{28}\text{Ni}\) and \(^{124}_{50}\text{Sn}\) in the state with \( j = \frac{3}{2} \) do not exhibit this discrepancy.

Similar calculations can be done for proton-odd nuclei with \( j = \frac{1}{2} \). It is seen from Fig. 3(c) that the accuracy of \( \sim 20\% \) between the results of our calculations and the data for the magnetic moments of \( 50 \div 90\% \) nuclei \([20]\) can be attained if we turn to the behavior of a \( u \) quark, but adopt \( \mu_A = -0.15 \mu_B \). The “wrong” sign of \( \mu_A \) suggests that we should speak of a hole instead of a real quark.

Since a single proton may be regarded as both a free hadron and the lightest nucleus, \(^1_1\text{H}\), that is, a system subject to both spin- and pseudospin symmetry conditions, which are in conflict with each other, our argument is unsuitable for it. In this connection, the idea of spherical singular cavity is to be applied to light nuclei (which fall within a transition region between the spin- and pseudospin symmetry coverages) with care.

A further justification for the rendition of a nucleus as a singular cavity resulting from the superstrong interaction between spin degrees of freedom of a single quark \( Q \) and the mean field generated by other constituents of this nucleus is the consistency of
Figure 3: Comparison of the calculated magnetic moments with their experimental values for (a) neutron-odd \((j = 1/2)\), (b) neutron-odd \((j = 3/2)\), (c) proton-odd \((j = 1/2)\) nuclei

the nodal wavefunction structure with the pseudospin symmetry condition. It is well known (see, e. g., Ref. \[8\], Sect. 15.2) that the relativistic harmonic oscillator in the pseudospin limit has positive energy bound states. Furthermore, the energy depends only on pseudo-oscillator quantum number, and the states with pseudo-angular momentum
\( \ell = \tilde{N}, \tilde{N} - 2, \ldots, 0 \) or 1 are all degenerate. However, the eigenstates of the relativistic harmonic oscillator have the nodal structure

\[
\kappa < 0 : \quad n_f = n_g + 1, \quad \kappa > 0 : \quad n_g = n_f
\]

which is characteristic of “antiparticles”, rather than the nodal structure

\[
\kappa < 0 : \quad n_f = n_g, \quad \kappa > 0 : \quad n_g = n_f + 1
\]

which is peculiar to conventional positive-energy particles. Here, \( n_f \) and \( n_g \) are the number of internal nodes of \( f_k(r) \) and \( g_k(r) \), respectively. By contrast, the eigenstates of a quark \( Q \) executing a periodic motion in the discussed singular cavity have the true nodal structure shown in Eq. (39).

This provides us with all essential prerequisites to a refined shell model which proceeds from quarks rather than nucleons. Two novel features of this model are the QCD inspired interquark potential \( U_S - U_V = V_C \) in combination with the Pauli exclusion principle for color-neutral three-quark clusters. This is a challenging task. Indeed, the singular cavity arises from the quark dynamics rather than the nucleon dynamics because a rising potential is peculiar to the former and alien to the latter. Therefore, the radial nodes \( n_r \) refer to energy levels of the quark \( Q \). The possibility to derive the static properties of a nucleus from the dynamics of a single quark suggests that the notion of nucleon structure of nuclei is somewhat smeared; when a quark enters a nucleus, it liberates from its parent nucleon. The evidence in support of this suggestion is that a neutron taken up by a nucleus loses its individuality which is apparent from the fact that the probability of \( \beta \)-decay of such neutrons may be arbitrarily small, and its life time arbitrarily long. On the other hand, the magic numbers 2, 8, 20, 28, 50, 82, 126 bear on nucleons rather than quarks. Why does the Pauli principle selects the three-quark cluster structures? Is the condition of color neutrality for the selected structures crucial? It is well known [21] that the spin-statistics theorem stems from the following conditions: (i) the vacuum is the lowest energy state, (ii) field variables either commute or anti-commute at space like separations, (iii) the norm of every physical state is positive definite. None of these conditions is dynamical. How can this theorem concerned with “elementary” entities (quarks) be adapted to systems composed of three such entities?

4 Conclusion

Let us briefly summarize the main features of our approach to understanding nuclei in terms of quarks. We integrate out the degrees of freedom of a single quark \( Q \) contained in some nucleus, based on the belief that all remaining degrees of freedom of this nucleus are already integrated out, and the result of this integration can be neatly approximated by using plausible assumptions about the color relief over which the quark \( Q \) moves. We anticipate that the quark \( Q \) is governed by the Lagrangian (11), and invoke the semiclassical treatment which implies that the extremal path contribution dominates the Feynman path integral, that is, the wave function of the quark \( Q \) is a solution to the Dirac equation in the color background \( A_\mu(x) \) and \( \Phi(x) \). We restrict our attention to spherically symmetric
static interactions between the quark $Q$ and the mean field due to all other constituents of the nucleus, and assume that the contribution of the Lorentz vector potential $A_\mu$ to the mean field is given by $A_0$, that is, $A = 0$. To control the fact that the quark $Q$ is a constituent of some nucleus, the pseudospin symmetry condition is imposed on $U_V = g_V A_0$ and $U_S = g_S \Phi$, namely $U_S = -U_V$. We take the Cornell potential (3) as a phenomenological realization of both $U_V$ and $U_S$, and put $V_C = 2U_V$. As this take place, the interaction between spin degrees of freedom of the quark $Q$ and the mean color field becomes superstrong to yield a spherical shell of radius $r_{Sc}$ on which the effective potential $U = U(r; \varepsilon)$ is infinite. The boundary of the spherical cavity of radius $r_{Sc}$ keeps colored objects in this cavity from escaping. We solve numerically the Dirac equation with the parameters of the Cornell potential borrowed from the description of quarkonia, and obtain the energy levels $\varepsilon_n$ for $s$-states and the corresponding sizes of the cavities $r_{Sc}$. The energy levels $\varepsilon_n$ turn out to be proportional to $\sqrt{n_r}$. By assuming that $n_r$ is proportional to $A^{2/3}$, where $A$ is the nucleon mass number, we come to the relationship $r_{Sc} = R_0 A^{1/3}$, which is characteristic of the liquid drop model. To verify that the assumption $n_r = [A^{2/3}]$ is consistent with the experimental data, we compare the magnetic dipole of the quark $Q$ and that of the nucleus in which this quark is incorporated. The agreement between the calculated and observed values of $\mu/\mu_B$ is for the most part within $\sim 20\%$ which is better than expected when taken into account that the picture in which a single quark moving in a static spherically symmetric mean field applies to a rich variety of nuclei whose dynamical contents are highly tangled.

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