Gravitomagnetic draconitic clock effect for inclined and quasi-circular orbits around a spinning body arbitrarily oriented in space

L. Iorio
Ministero dell’Istruzione, dell’Università e della Ricerca (M.I.U.R.)-Istruzione Fellow of the Royal Astronomical Society (F.R.A.S.)
Viale Unità di Italia 68, 70125, Bari (BA), Italy
email: lorenzo.iorio@libero.it

July 21, 2014

Abstract

In the weak-field and slow-motion approximation of general relativity, the rotation of a body discriminates between the opposite directions of motion of a pair of counter-revolving tests particles orbiting it along geometrically identical trajectories: it is the so-called gravitomagnetic clock effect. In this paper, we analytically calculate the gravitomagnetic corrections to both the draconitic and to the anomalistic periods of arbitrarily inclined, quasi-circular orbits for a generic orientation of the spin axis of the primary. While the anomalistic period is left unchanged, the draconitic one experiences a generally non-vanishing correction which, to zero order in the eccentricity, gains a minus sign if the velocity of the test particle is reversed. As a result, a gravitomagnetic draconitic clock effect arises since a generally non-zero difference of the draconitic periods of a pair of counter-orbiting test particles arises. Remarkably, it is independent of their initial conditions, with some advantages from an experimental point of view because, in principle, the data could be collected independently of each other and at different epochs. Numerically integrations of the equations of motion confirm our analytical results. The relativistic time difference turns out to be up to about 0.5 ms and 0.1 ms around Jupiter and the Sun, respectively, while it is as little as 0.6 µs around the Earth.

PACS: 04.20.-q; 04.80.-y; 04.80.Cc; 91.10.Sp
Keywords: Classical general relativity; Experimental studies of gravity; Experimental tests of gravitational theories; Satellite orbits

1 Introduction

According to the linearized weak-field and slow-motion approximation of the General Theory of Relativity (GTR), the gravitational field outside a rotating body is characterized by a peculiar term proportional to its proper angular momentum $S$ [1–3]. It has no classical analogue since, in the Newtonian theory, the gravitational field of an object depends only on its mass $M$. Such an extra-field is dubbed “gravitomagnetic” [2, 4] because of its formal resemblance with the magnetic field generated by a magnetic dipole moment. For a historical overview on general relativistic gravitomagnetism, see [5–8]. The gravitomagnetic field induces several effects [9–12] on the orbital motion of test particles [13], orbiting gyroscopes [14], electromagnetic waves [15], etc. Some of them have been subjected to intense experimental/observational scrutiny so far; for recent reviews, see, e.g., [16–18] and references therein.

At present, the empirical situation should not be considered, perhaps, as fully satisfying as it might be argued at first sight. Indeed, if, on the one hand, the outcome of the GP-B mission aimed to measure the Schiff effect on orbiting gyroscopes was successful [19], on the other hand, it will likely be doomed to remain an unique experiment, without practical possibilities of being repeated; only independent analyses of the existing data record will be possible\(^{1}\). Moreover, the obtained accuracy turned out to be worse than expected. As far as the satellite-based Lense-Thirring tests are concerned, there is still a lingering debate on several aspects of their reliability and overall accuracy [16–18].

In view of such considerations and by recalling that a fundamental pillar of our knowledge of the natural world as GTR requires to be put on the test in as many independent ways as possible, we will deal with another peculiar gravitomagnetic orbital effect. In [20], it was noted that the gravitomagnetic acceleration experienced by a test particle moving in the equatorial plane of a rotating mass at a constant distance from it becomes purely radial, acting as a supplement to the centripetal Newtonian monopole. Such an additional term may be directed outward if the particle’s motion coincides with the direction of rotation of the central body or inward if the motion is opposite to the primary’s rotation. As such, the overall gravitational attraction felt

\(^{1}\)To the best of our knowledge, none has been announced so far.
by the particle is weakened in the first case, while it is enhanced in the second case. Thus, the orbital period gets longer in the first case and shorter in the second by a quantity proportional to the angular momentum of the primary; notably, such a time interval is independent both of the orbit radius and of the Newtonian gravitational constant $G$. See also [21–23]. In [24], it was suggested to look at the difference between the orbital periods of two counter-revolving clocks orbiting a rotating astronomical body along circular and equatorial paths since they cancel the common Keplerian terms enhancing the gravitomagnetic ones which, instead, sum up. This is the so-called gravitomagnetic clock effect [3, 25]. Several studies focusing on various aspects of both theoretical and experimental/observational nature aimed to investigate different versions of the gravitomagnetic clock effect and the possibility of detecting it have appeared in the literature so far [3, 26–41].

In several practical astronomical scenarios, the tracking of spacecraft reaches its highest level of accuracy usually at their closest approaches to their target bodies thanks to a variety of techniques such as Doppler ranging rate, laser ranging, etc [42, 43]. Thus, we will look at the influence of the gravitomagnetic field on the anomalistic period $T_{an}$ of a test particle, i.e. the time elapsed between two consecutive passages at the pericenter. Another characteristic orbital temporal interval which can be measured from observations [44–46] is the draconitic period $T_{dr}$, which refers to two successive passages at the ascending node. The gravitomagnetic effect on it will be calculated as well. Our analytical results, which will be obtained with a first-order perturbative approach, are valid for an arbitrary orientation of the primary’s spin axis; moreover, they are not restricted to idealized circular orbits lying in the equatorial plane of the primary. The removal of such limitations, common to almost all the existing papers in the literature, is important because the spatial orientation of the spin axes of astronomical bodies of potential interest is often either imperfectly known or unknown at all. This fact may become important in a foreseeable future when suitably exoplanetary systems could be looked for: indeed, the possibility of using transit timing variations (TTVs) to distinguish between a prograde and a retrograde planetary motion with respect to the stellar rotation is currently being investigated [47]. Also when the situation is more favorable, as in our Solar System, the direction of the planetary spin axes is known with a necessarily limited accuracy. Finally, it must be remarked that, when data analyses spanning several decades are involved, the fact that the spins of many non-isolated bodies undergo slow precessional motions must be taken into account as well.

The paper is organized as follows. In Section 2, the perturbative treat-
ment of the gravitomagnetic Lense-Thirring acceleration for an arbitrary orientation of $S$ is performed. It is subsequently applied in Section 3 to calculate the effect of the gravitomagnetic field on the draconitic period, while Section 4 is devoted to the anomalistic period. In Section 5 our conclusions are offered.

**Notations**

Here, basic notations and definitions used in the text are presented [48, 49].

- $G$: Newtonian constant of gravitation
- $c$: speed of light in vacuum
- $T$: interval in Julian centuries (of 36525 days) from the standard epoch JD 2451545.0, i.e. 2000 January 1 12 hours TDB
- $M$: mass of the primary
- $\mu = GM$: gravitational parameter of the primary
- $S$: angular momentum of the primary
- $\hat{S}$: unit vector of the spin axis of the primary
- $r$: radius vector of the test particle
- $\hat{r} = r/r$: unit vector of the radius vector of the test particle
- $v$: velocity vector of the test particle
- $h = r \times v$: orbital angular momentum per unit mass of the test particle
- $\hat{h} = h/h$: unit vector of the orbital angular momentum per unit mass of the test particle
- $a$: semimajor axis
- $n_b = \sqrt{\mu a^{-3}}$: Keplerian mean motion
- $T_0 = 2\pi n_b^{-1}$: Keplerian orbital period
- $e$: eccentricity
- $p = a(1 - e^2)$: semilatus rectum
\(I\) : inclination of the orbital plane
\(\Omega\) : longitude of the ascending node
\(\omega\) : argument of pericenter
\(f\) : true anomaly
\(u = \omega + f\) : argument of latitude
\(q = e \cos \omega\) : nonsingular orbital element \(q\)
\(k = e \sin \omega\) : nonsingular orbital element \(k\)
\(\hat{l}\) : unit vector directed along the line of the nodes toward the ascending node
\(\hat{m}\) : unit vector directed transversely to the line of the nodes in the orbital plane
\(\hat{P}\) : unit vector directed along the line of the apsides toward the pericenter
\(\hat{Q}\) : unit vector directed transversely to the line of the apsides in the orbital plane
\(A\) : disturbing acceleration
\(A_R, A_T, A_N\) : radial, transverse and normal components of \(A\)
\(T_{dr}\) : draconitic period
\(T_{an}\) : anomalistic period

2 The gravitomagnetic Lense-Thirring acceleration

The Lense-Thirring acceleration experienced by a test particle orbiting a rotating primary is [50]

\[
A^{(LT)} = \frac{2GS}{c^2 r^3} \left[ \frac{3(\hat{S} \cdot \mathbf{r}) \mathbf{r} \times \mathbf{v}}{r^2} + \mathbf{v} \times \hat{S} \right].
\]  

(1)

It can be considered as a small perturbation of the Newtonian monopole.

In perturbative calculations, it is customarily required to project the disturbing acceleration \(A\) onto three mutually orthogonal directions, and to
evaluate the resulting components onto the unperturbed Keplerian ellipse; this can be done as follows. The components of the unit vectors $\hat{l}, \hat{m}, \hat{k}$ are [48]

\begin{align*}
\hat{l}_x &= \cos \Omega, \\
\hat{l}_y &= \sin \Omega, \\
\hat{l}_z &= 0, \\
\hat{m}_x &= -\cos I \sin \Omega, \\
\hat{m}_y &= \cos I \cos \Omega, \\
\hat{m}_z &= \sin I, \\
\hat{h}_x &= \sin I \sin \Omega, \\
\hat{h}_y &= -\sin I \cos \Omega, \\
\hat{h}_z &= \cos I,
\end{align*}

so that it is [48]

\begin{align*}
\hat{P} &= \hat{l} \cos \omega + \hat{m} \sin \omega, \\
\hat{Q} &= -\hat{l} \sin \omega + \hat{m} \cos \omega.
\end{align*}

Thus, the position and the velocity vectors can be expressed as [48]

\begin{align*}
r &= r \left( \hat{P} \cos f + \hat{Q} \sin f \right), \\
\mathbf{v} &= \sqrt{\frac{\mu}{r}} \left[ -\hat{P} \sin f + \hat{Q} (\cos f + e) \right].
\end{align*}
The radial, transverse and normal components of $A$ can be finally calculated as 

\[
A_R = A \cdot \hat{r},
\]

\[
A_T = A \cdot (\hat{h} \times \hat{r}),
\]

\[
A_N = A \cdot \hat{h}.
\]

In the case of eq. (1), they turn out to be

\[
A_R^{(LT)} = 2GSn (1 + e \cos f)^4 \frac{c^2a^2(1 - e^2)^{7/2}}{4} \left[ \hat{S}_z \cos I + \sin I \left( \hat{S}_x \sin \Omega - \hat{S}_y \cos \Omega \right) \right],
\]

\[
A_T^{(LT)} = -2eGSn (1 + e \cos f)^3 \sin f \frac{c^2a^2(1 - e^2)^{7/2}}{4} \left[ \hat{S}_z \cos I + \sin I \left( \hat{S}_x \sin \Omega - \hat{S}_y \cos \Omega \right) \right],
\]

\[
A_N^{(LT)} = -GSn (1 + e \cos f)^3 \frac{c^2a^2(1 - e^2)^{7/2}}{4} \left\{ - \hat{S}_z \sin I [e \sin \omega + 4 \sin (f + \omega) + 3e \sin (2f + \omega)] + 
\right. 
\]

\[
+ \cos I [e \sin \omega + 4 \sin (f + \omega) + 3e \sin (2f + \omega)] \left( \hat{S}_x \sin \Omega - \hat{S}_y \cos \Omega \right) - 
\]

\[
- [e \cos \omega + 4 \cos (f + \omega) + 3e \cos (2f + \omega)] \left( \hat{S}_x \cos \Omega + \hat{S}_y \sin \Omega \right)
\right\}.
\]

In the particular case $\hat{S}_x = \hat{S}_y = 0, \hat{S}_z = 1$, eq. (19)-eq. (21) agree with Eq. (4.2.18a)- Eq. (4.2.18c) in [51].
3 The draconitic period in a post-Keplerian orbit

The draconitic\(^2\) period \(T_{\text{dr}}\), defined for a perturbed trajectory as the time interval between two successive instants when the real position of the test particle coincides with the ascending node position on the corresponding osculating orbit, can be calculated as \([53]\)

\[
T_{\text{dr}} = \int_{0}^{2\pi} \left( \frac{dt}{du} \right) du. \tag{22}
\]

When a disturbing acceleration \(A\) is present, the derivative of \(t\) with respect to \(u\) becomes\(^3\) \([53, 54]\)

\[
\frac{dt}{du} = \frac{r^2\gamma}{\sqrt{\mu p}}, \tag{23}
\]

\[
\gamma = \frac{1}{1 - \left( \frac{r^2 \cos I}{\sqrt{\mu p}} \right) \frac{d\Omega}{dt}}, \tag{24}
\]

and the draconitic period \(T_{\text{dr}}\) generally differs from \(T_0\) by an additive term \(T_{\text{dr}}^{(A)}\). One of the causes of it is the shift of the node itself due to \([53, 54]\)

\[
\frac{d\Omega}{du} = \frac{r^3\gamma \sin u A_N}{\mu p \sin I}, \tag{25}
\]

it yields the direct contribution of the disturbing acceleration to \(T_{\text{dr}}^{(A)}\). An indirect contribution arises from \(r^2/\sqrt{\mu p}\) when the instantaneous shifts of the orbital elements entering it are considered as well \([53]\). All in all, the draconitic time lapse can be analytically computed as \([53]\)

\[
T_{\text{dr}}^{(A)} = T_{\text{dr}} - T_0 = I_1^{\text{dr}} + I_2^{\text{dr}} + I_3^{\text{dr}} + I_4^{\text{dr}}, \tag{26}
\]

\(^2\)This adjective was originally referred to the Moon’s passage at its ascending node, when an eclipse occurs. Indeed, the ancient Greeks thought that, during an eclipse, our natural satellite was swallowed up by a dragon (“δράκων”, meaning literally “which stares”) hiding near the nodes of the lunar orbit \([52]\).

\(^3\)A mistake is present in the expression for \(dt/du\) in Eq.(1) of \([53]\): actually, the correct multiplicative factor in front of \(r^2/\sqrt{\mu p}\) is the inverse of the square bracket in \([53]\).
with

\[ I_{1}^{dr} = \frac{3}{2} \sqrt{\frac{p}{\mu}} \int_{0}^{2\pi} \frac{\Delta p}{(1 + q \cos u + k \sin u)^{2}} du, \]  
\[ I_{2}^{dr} = -2 \sqrt{\frac{p^{3}}{\mu}} \int_{0}^{2\pi} \frac{\Delta q \cos u}{(1 + q \cos u + k \sin u)^{2}} du, \]  
\[ I_{3}^{dr} = -2 \sqrt{\frac{p^{3}}{\mu}} \int_{0}^{2\pi} \frac{\Delta k \sin u}{(1 + q \cos u + k \sin u)^{2}} du, \]  
\[ I_{4}^{dr} = \int_{0}^{2\pi} \frac{r^{4} \cos I d\Omega}{\mu p} \frac{dt}{du} du. \]  

In eq. (27)-eq. (30), the instantaneous shifts of \( p, q, k \) are to be calculated as [53]

\[ \Delta p = \int_{u_{0}}^{u} \left( \frac{dp}{du'} \right) du', \]  
\[ \Delta q = \int_{u_{0}}^{u} \left( \frac{dq}{du'} \right) du', \]  
\[ \Delta k = \int_{u_{0}}^{u} \left( \frac{dk}{du'} \right) du', \]  

by using the analytical expressions [53, 54]

\[ \frac{dp}{du} = \frac{2r^{3}T_{A}}{\mu}, \]  
\[ \frac{dq}{du} = \frac{r^{3}k \sin u \cot IA_{N}}{\mu p} + \frac{r^{2}T_{A} \left[ \frac{r}{T} (q + \cos u) + \cos u \right]}{\mu} + \frac{r^{2} \gamma \sin u A_{R}}{\mu}. \]
\[ \frac{dk}{du} = - \frac{r^3 \gamma q \sin u \cot IA}{\mu p} + \frac{r^2 \gamma \left[ \frac{\gamma}{\mu} (k + \sin u) + \sin u \right] A_T}{\mu} - \frac{r^2 \gamma \cos u A_R}{\mu}. \] (36)

of the derivatives of \( p, q, k \) with respect to \( u \) to the first order in the disturbing acceleration, i.e. by setting \( \gamma = 1 \). Moreover, it is intended that the right-hand-sides of eq. (27)-eq. (30) are evaluated onto the unperturbed Keplerian ellipse. In particular, the time derivative of the node entering eq. (30) has to be calculated, to the first order in the perturbation, from eq. (25) and eq. (23) with \( \gamma = 1 \).

The difference \( T_{dr}^{(LT)} \) between draconitic and osculating periods caused by the disturbing gravitomagnetic acceleration can be worked out by using eq. (19)-eq. (21) in eq. (27)-eq. (30). To the first order in \( e \), one obtains

\[ T_{dr}^{(LT)} = \frac{4\pi S}{c^2 M} \left\{ 2 \sin I \left( \hat{S}_x \sin \Omega - \hat{S}_y \cos \Omega \right) + \right. \]

\[ + \cos I \left[ 3 \hat{S}_z + \cot I \left( \hat{S}_y \cos \Omega - \hat{S}_x \sin \Omega \right) \right] - \]

\[ - e \cos \omega \left[ 6 \hat{S}_z \cos I + \csc I (3 \cos 2I - 1) \left( \hat{S}_y \cos \Omega - \hat{S}_x \sin \Omega \right) \right] \}

\[ + O \left( e^2 \right). \] (37)

It is worthwhile noticing that eq. (37) is independent of both \( G \) and of \( a \). Another important feature of eq. (37) is that it does not depend on the initial value of the argument of latitude \( u_0 \).

By reversing the direction of motion of the test particle, the following changes in its orbital parameters occur. The set of the osculating Keplerian orbital elements corresponding to the reversed motion are conventionally denoted with the superscript \((-)\), while \((+)\) is for the original direction of motion. The ascending node for \((-)\) corresponds to the descending node for \((+)\), so that \( \hat{l}^{(-)} = -\hat{l}^{(+) \ \text{and}} \)

\[ \cos \Omega^{(-)} = - \cos \Omega^{(+)}, \] (38)

\[ \sin \Omega^{(-)} = - \sin \Omega^{(+)}. \] (39)
Also the orbital angular momentum gets reversed: \( \hat{h}(-) = -\hat{h}(+) \). As such, and in view of eq. (38)-eq. (39), it is

\[
\cos I(-) = -\cos I(+), \\
\sin I(-) = \sin I(+).
\]  

As a consequence of eq. (38)-eq. (41), the unit vector \( \hat{m} \) remains unchanged. Since the unit vector \( \hat{P} \) directed along the line of the apsides toward the pericenter is the same for both the directions of motion, it turns out

\[
\cos \omega(-) = -\cos \omega(+), \\
\sin \omega(-) = \sin \omega(+),
\]  

which implies \( \hat{Q}(-) = -\hat{Q}(+) \) as well. Thus, from eq. (13), it turns out

\[
\cos f(-) = \cos f(+), \\
\sin f(-) = -\sin f(+).
\]  

As far as the argument of latitude \( u \) is concerned, eq. (42)-eq. (45) yield

\[
\cos u(-) = -\cos u(+), \\
\sin u(-) = \sin u(+),
\]  

implying

\[
u(-) = \pi - u(+), \quad \text{and} \quad du(-) = -du(+).
\]

By repeating the same calculation for \( T_{\text{dr}}^{(LT)} \) with eq. (38)-eq. (49), one obtains the term of zero order in \( e \) of eq. (37) with an overall minus sign, while the term of order \( O(e) \) has the same sign as in eq. (37). As a consequence, there is a non-vanishing difference between the draconitic periods of two counter-orbiting test particles following the same trajectories
in the gravitomagnetic field of a rotating primary. Such a time lapse amounts to

\[ \Delta T_{\text{dr}}^{(\text{LT})} = \frac{8\pi S}{c^2 M} \left\{ 2 \sin I \left( \hat{S}_x \sin \Omega - \hat{S}_y \cos \Omega \right) + \right. \]

\[ + \cos I \left[ 3 \hat{S}_z + \cot I \left( \hat{S}_y \cos \Omega - \hat{S}_x \sin \Omega \right) \right] \} + O\left(e^2\right). \quad (50) \]

It can be noted that the terms of order \( O(e) \) mutually cancel out. It is important to remark how eq. (50) is independent of the initial value of the argument of latitude \( u_0 \). It is a relevant feature in view of possible experimental tests: indeed, it would allow, in principle, to collect the data of the spacecrafts independently of each other and even at different epochs.

The analytical results of eq. (37) can be confirmed also by a numerical integration of the equations of motion of a fictitious satellite affected by the Lense-Thirring acceleration imparted by the gravitomagnetic field of a rotating body with the mass of the Earth. For illustrative purposes, we tune the magnitude of the angular momentum of the primary and the orbital parameters of the satellite in such a way that the gravitomagnetic acceleration felt by it amounts to about 5\% of the Newtonian monopole; this assures that a comparison with our analytical results, obtained perturbatively, is actually meaningful. The orientation of \( \hat{S} \) is taken from Table 1 of [55] with \( T = 0.14 \). We adopt a 2-hr circular orbit inclined by \( I = 50 \) deg to the Earth’s equator and starting at the ascending node, located at \( \Omega = 30 \) deg. We plot the time series of \( z/r \), so that the draconitic period can be easily obtained by reading the instant when the curve crosses the horizontal axis of the graph for the second\(^4 \) time. It turns out that the comeback of to the ascending node occurs with a delay with respect to the Newtonian case which is in agreement with the prediction of eq. (37).

In Figure 1, the dimensionless multiplicative factor of \( \Delta T_{\text{dr}}^{(\text{LT})} \) in eq. (50) is plotted as function of \( I \) and \( \Omega \) for the Earth, Sun and Jupiter. It turns out that the most favorable situation occurs for Jupiter, with a time shift which can be as large as \( \Delta T_{\text{dr}}^{(\text{LT})} \lesssim 0.5 \) ms. In the case of a Sun-like star, it can reach up to 0.1 ms, while for the Earth it is at 0.6 \( \mu \)s level.

\(^4\)The first crossings corresponds to the passage at the descending node.
Figure 1: Geometric multiplicative factor of $8\pi S c^{-2} M^{-1}$ in eq. (50) as a function of $I$ and $\Omega$ for the Earth ($\frac{8\pi S}{c^2 M} = 3 \times 10^{-7}$ s), Sun ($\frac{8\pi S}{c^2 M} = 2.67 \times 10^{-5}$ s), Jupiter ($\frac{8\pi S}{c^2 M} = 1.016 \times 10^{-4}$ s). The orientations of $\hat{S}$ with respect to the Celestial Equator at the epoch J2000.0, assumed as reference $\{x, y\}$ plane, were retrieved from Table 1 of [55].
4 The anomalistic period

The anomalistic period $T_{\text{an}}$, defined as the time interval between two successive instants when the real position of the test particle coincides with the pericenter position on the corresponding orbit, can be calculated as \[56, 57\]

$$T_{\text{an}} = \int_{0}^{2\pi} \left( \frac{dt}{df} \right) df. \quad (51)$$

In presence of a disturbing acceleration $A$, one has \[57, 58\]

$$\frac{dt}{df} = \frac{r^2 \beta}{\sqrt{\mu p}}, \quad (52)$$

$$\beta = \frac{1}{1 - \frac{r^2}{\sqrt{\mu p}} \left( \frac{d\omega}{dt} + \cos I \frac{d\Omega}{dt} \right)}. \quad (53)$$

The anomalistic period $T_{\text{an}}$ generally differs from $T_0$ by an additive term $T_{\text{an}}^{(A)}$ which can be analytically worked out as \[57\]

$$T_{\text{an}}^{(A)} = T_{\text{an}} - T_0 = I_{1\text{an}} + I_{2\text{an}} + I_{3\text{an}}, \quad (54)$$

with \[56, 57\]

$$I_{1\text{an}} = \frac{3}{2} \sqrt{\frac{p}{\mu}} \int_{0}^{2\pi} \frac{\Delta p}{(1 + e \cos f)^2} df, \quad (55)$$

$$I_{2\text{an}} = -2 \sqrt{\frac{p^3}{\mu}} \int_{0}^{2\pi} \frac{\Delta e \cos f}{(1 + e \cos f)^3} df, \quad (56)$$

$$I_{3\text{an}} = \int_{0}^{2\pi} \frac{r^4}{\mu p} \left( \frac{d\omega}{dt} + \cos I \frac{d\Omega}{dt} \right) df. \quad (57)$$

\[5\] This adjective refers to the fact that the three anomalies are all zero (modulo $2\pi$) at the pericenter \[52\].
In eq. (55)-eq. (56), the instantaneous shifts of $p, e$ are to be calculated as

$$\Delta p = \int_{f_0}^{f} \left( \frac{dp}{df'} \right) df', \quad (58)$$

$$\Delta e = \int_{f_0}^{f} \left( \frac{de}{df'} \right) df', \quad (59)$$

by using the analytical expressions [57,58]

$$\frac{dp}{df} = \frac{2r^3\beta A_T}{\mu}, \quad (60)$$

$$\frac{de}{df} = \frac{r^2\beta}{\mu} \left[ \sin fA_R + \left(1 + \frac{r}{p}\right) \cos fA_T + e \left(\frac{r}{p}\right) A_T \right], \quad (61)$$

of the derivatives of $p, e$ with respect to $f$ to the first order in the disturbing acceleration, i.e. by setting $\beta = 1$. Moreover, it is intended that the right-hand-sides of eq. (55)-eq. (57) are evaluated onto the unperturbed Keplerian ellipse. In particular, the time derivatives entering eq. (57) has to be calculated, to the first order in the perturbation, from [57,58]

$$\frac{d\Omega}{df} = \frac{r^3\beta \sin (\omega + f) A_N}{\mu p \sin I}, \quad (62)$$

$$\frac{d\omega}{df} = \frac{r^2\beta}{\mu} \left[ -\frac{\cos fA_R}{e} + \left(1 + \frac{r}{p}\right) \frac{\sin fA_T}{e} - \left(\frac{r}{p}\right) \cot I \sin (\omega + f) A_N \right], \quad (63)$$

and eq. (52) with $\beta = 1$.

The difference $T_{\text{an}}^{(\text{LT})}$ between anomalous and osculating periods caused by the disturbing gravitomagnetic acceleration can be worked out by using eq. (19)-eq. (21) in eq. (54)-eq. (57). It turns out

$$T_{\text{an}}^{(\text{LT})} = 0 \quad (64)$$

to all orders in $e$. The same result can be obtained by following the method in [59], which turns out to be equivalent\(^6\) to the one in [57].

\(^6\)The expression for $dt_a^{(1)}/d\theta$ in Eq. (3) of [59] is incorrect: an overall multiplicative factor $\sin \theta$ is missing in the second term of the right-hand-side. In the notation of [59], $\theta$ is the true anomaly.
A numerical analysis analogous to that performed in Section ?? confirms the analytical result of eq. (64). In this case, by adopting \( e = 0.1 \), it is possible to notice that the numerically integrated time series of the distance from the primary reaches its minimum value at the same instants of the Newtonian one.

In view of the high accuracy with which the pericenter passages can usually be recorded with modern tracking data, it is disappointing that the apsidal period is not affected by the gravitomagnetic field.

5 Summary and conclusions

In this paper, we analytically calculated the effect of the gravitomagnetic field of an arbitrarily oriented spinning body on both the draconitic and on the anomalistic periods of a test particle orbiting it along an inclined and quasi-circular orbit.

While the anomalistic period turns out to be unaffected, the draconitic one is changed by an additional term which is independent of the Newtonian gravitational constant, of the semimajor axis of the satellite, and of its initial position as well. To the first order in the eccentricity, such a gravitomagnetic correction depends on the direction of motion along the orbit in such a way that a generally non-vanishing difference between the draconitic periods of two counter-rotating test particles moving along otherwise identical trajectories occurs.

The largest effect occur around Jupiter and the Sun, with maximum values of the time shift of about 0.5 ms and 0.1 ms, respectively. As far as the Earth is concerned, the gravitomagnetic draconitic clock effect can reach up to 0.6 \( \mu \)s.

Our results have a general validity since they are not restricted to any a-priori direction for the angular momentum of the primary nor the equatorial alignment of the test particle’s orbital plane. As such, they can be applied, in principle, to a variety of different scenarios.

Further studies need to be focussed on the sources of systematic bias arising by the necessarily imperfect cancellation of the classical components of the draconitic periods.

References

[1] K. S. Thorne, D. A. MacDonald, and R. H. Price, eds., Black Holes: The Membrane Paradigm. Yale University Press, Yale, 1986.
[2] K. S. Thorne, “Gravitomagnetism, jets in quasars, and the Stanford Gyroscope Experiment,” in Near Zero: New Frontiers of Physics, J. D. Fairbank, B. S. Deaver, Jr., C. W. F. Everitt, and P. F. Michelson, eds., pp. 573–586. 1988.

[3] B. Mashhoon, F. Gronwald, and H. I. M. Lichtenegger, “Gravitomagnetism and the Clock Effect,” in Gyros, Clocks, Interferometers ...: Testing Relativistic Gravity in Space, C. Lämmerzahl, C. W. F. Everitt, and F. W. Hehl, eds., vol. 562 of Lecture Notes in Physics, pp. 83–108. Springer Verlag, Berlin/Heidelberg, 2001. gr-qc/9912027.

[4] W. Rindler, Relativity: special, general, and cosmological. Oxford University Press, Oxford, 2001.

[5] H. Pfister, “On the history of the so-called Lense-Thirring effect,” General Relativity and Gravitation 39 no. 11, (Nov., 2007) 1735–1748.

[6] H. Pfister, “The History of the So-Called Lense-Thirring Effect,” in The Eleventh Marcel Grossmann Meeting On Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories, H. Kleinert, R. T. Jantzen, and R. Ruffini, eds., pp. 2456–2458. World Scientific, Singapore, Sept., 2008.

[7] H. Pfister, “Editorial note to: Hans Thirring, On the formal analogy between the basic electromagnetic equations and Einsteins gravity equations in first approximation,” General Relativity and Gravitation 44 no. 12, (December, 2012) 3217–3224.

[8] H. Pfister, “Gravitomagnetism: From Einstein’s 1912 Paper to the Satellites LAGEOS and Gravity Probe B,” in Relativity and Gravitation, J. Bičák and T. Ledvinka, eds., vol. 157 of Springer Proceedings in Physics, pp. 191–197. Springer Verlag, Berlin/Heidelberg, 2014.

[9] I. G. Dymnikova, “REVIEWS OF TOPICAL PROBLEMS: Motion of particles and photons in the gravitational field of a rotating body (In memory of Vladimir Afanas’evich Ruban),” Soviet Physics Uspekhi 29 no. 3, (Mar., 1986) 215–237.

[10] M. L. Ruggiero and A. Tartaglia, “Gravitomagnetic effects,” Nuovo Cimento B Serie 117 no. 7, (July, 2002) 743–768, gr-qc/0207065.
[11] G. Schäfer, “Gravitomagnetic Effects,” General Relativity and Gravitation 36 no. 10, (Oct., 2004) 2223–2235, gr-qc/0407116.

[12] G. Schäfer, “Gravitomagnetism in Physics and Astrophysics,” Space Science Reviews 148 no. 1-4, (Dec., 2009) 37–52.

[13] J. Lense and H. Thirring, “Über den Einfluß der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie,” Physikalische Zeitschrift 19 (1918) 156–163.

[14] L. Schiff, “Possible new experimental test of general relativity theory,” Physical Review Letters 4 no. 5, (March, 1960) 215–217.

[15] G. W. Richter and R. A. Matzner, “Second-order contributions to gravitational deflection of light in the parametrized post-Newtonian formalism,” Physical Review D 26 no. 6, (Sept., 1982) 1219–1224.

[16] L. Iorio, H. I. M. Lichtenegger, M. L. Ruggiero, and C. Corda, “Phenomenology of the Lense-Thirring effect in the solar system,” Astrophysics and Space Science 331 no. 2, (Feb., 2011) 351–395, arXiv:1009.3225 [gr-qc].

[17] G. Renzetti, “History of the attempts to measure orbital frame-dragging with artificial satellites,” Central European Journal of Physics 11 no. 5, (May, 2013) 531–544.

[18] I. Ciufolini, A. Paolozzi, R. Koenig, E. C. Pavlis, J. Ries, R. Matzner, V. Gurzadyan, R. Penrose, G. Sindoni, and C. Paris, “Fundamental Physics and General Relativity with the LARES and LAGEOS satellites,” Nuclear Physics B Proceedings Supplements 243-244 (Oct., 2013) 180–193, arXiv:1309.1699 [gr-qc].

[19] C. W. F. Everitt, D. B. Debra, B. W. Parkinson, J. P. Turneauer, J. W. Conklin, M. I. Heifetz, G. M. Keiser, A. S. Silbergleit, T. Holmes, J. Kolodziejczak, M. Al-Meshari, J. C. Mester, B. Muhlfelder, V. G. Solomonik, K. Stahl, P. W. Worden, Jr., W. Bencze, S. Buchman, B. Clarke, A. Al-Jadaan, H. Al-Jibreen, J. Li, J. A. Lipa, J. M. Lockhart, B. Al-Suwaidan, M. Taber, and S. Wang, “Gravity Probe B: Final Results of a Space Experiment to Test General Relativity,” Physical Review Letters 106 no. 22, (June, 2011) 221101, arXiv:1105.3456 [gr-qc].
[20] Y. Vladimirov, N. Mitskiévic, and J. Horský, *Space Time Gravitation*. Mir, Moscow, 1987.

[21] A. Rosenblum, “Clock transport synchronisation and the dragging of inertial frames,” *Classical and Quantum Gravity* **4** no. 6, (Nov., 1987) L215–L216.

[22] A. Rosenblum and M. Treber, “Clock Transport Synchronization and the Dragging of Inertial Frames for Elliptical Orbits,” *International Journal of Theoretical Physics* **27** no. 7, (July, 1988) 921–924.

[23] A. Rosenblum and M. Treber, “Satellite Test for Dragging of Inertial Frames,” *International Journal of Theoretical Physics* **27** no. 3, (Mar., 1988) 365–368.

[24] J. M. Cohen and B. Mashhoon, “Standard clocks, interferometry, and gravitomagnetism,” *Physics Letters A* **181** no. 5, (Oct., 1993) 353–358.

[25] B. Mashhoon, F. Gronwald, and D. S. Theiss, “On measuring gravitomagnetism via spaceborne clocks: a gravitomagnetic clock effect,” *Annalen der Physik* **511** no. 2, (1999) 135–152, gr-qc/9804008.

[26] F. Gronwald, E. Gruber, H. Lichtenegger, and R. A. Puntigam, “Gravity Probe C(lock) - Probing the gravitomagnetic field of the Earth by means of a clock experiment,” *ArXiv e-prints* (Dec., 1997), gr-qc/9712054.

[27] R. J. You, “The gravitational Larmor precession of the Earths artificial satellite orbital motion,” *Bollettino di Geodesia e Scienze Affini* **57** no. 4, (1998) 453–460.

[28] W. B. Bonnor and B. R. Steadman, “The gravitomagnetic clock effect,” *Classical and Quantum Gravity* **16** no. 6, (June, 1999) 1853–1861.

[29] O. Semerák, “COMMENT: Gravitomagnetic clock effect and extremely accelerated observers,” *Classical and Quantum Gravity* **16** no. 11, (Nov., 1999) 3769–3770.

[30] A. Tartaglia, “Detection of the gravitomagnetic clock effect,” *Classical and Quantum Gravity* **17** no. 4, (Feb., 2000) 783–792, gr-qc/9909006.
[31] A. Tartaglia, “Geometric Treatment of the Gravitomagnetic Clock Effect,” *General Relativity and Gravitation* **32** no. 9, (Sept., 2000) 1745–1756, gr-qc/0001080.

[32] A. Tartaglia, “Influence of the angular momentum of astrophysical objects on light and clocks and related measurements,” *Classical and Quantum Gravity* **17** no. 12, (June, 2000) 2381–2384.

[33] H. I. M. Lichtenegger, F. Gronwald, and B. Mashhoon, “On Detecting the Gravitomagnetic Field of the Earth by Means of Orbiting Clocks,” *Advances in Space Research* **25** no. 6, (2000) 1255–1258.

[34] D. Bini, R. T. Jantzen, and B. Mashhoon, “Gravitomagnetism and relative observer clock effects,” *Classical and Quantum Gravity* **18** no. 4, (Feb., 2001) 653–670, gr-qc/0012065.

[35] L. Iorio, “Satellite Gravitational Orbital Perturbations and the Gravitomagnetic Clock Effect,” *International Journal of Modern Physics D* **10** no. 4, (2001) 465–476, gr-qc/0007014.

[36] L. Iorio, “Satellite non-gravitational orbital perturbations and the detection of the gravitomagnetic clock effect,” *Classical and Quantum Gravity* **18** no. 20, (Oct., 2001) 4303–4310, gr-qc/0007057.

[37] B. Mashhoon, L. Iorio, and H. Lichtenegger, “On the gravitomagnetic clock effect,” *Physics Letters A* **292** no. 1-2, (Dec., 2001) 49–57, gr-qc/0110055.

[38] L. Iorio, H. Lichtenegger, and B. Mashhoon, “An alternative derivation of the gravitomagnetic clock effect,” *Classical and Quantum Gravity* **19** no. 1, (Jan., 2002) 39–49, gr-qc/0107002.

[39] L. Iorio and H. I. M. Lichtenegger, “On the possibility of measuring the gravitomagnetic clock effect in an Earth space-based experiment,” *Annalen der Physik* **15** no. 12, (Dec., 2006) 868–876, gr-qc/0211108.

[40] H. Lichtenegger, L. Iorio, and B. Mashhoon, “The gravitomagnetic clock effect and its possible observation,” *Annalen der Physik* **15** no. 1, (Jan., 2005) 119–132, gr-qc/0210030.

[41] E. Hackmann and C. Lämmerzahl, “A generalized gravitomagnetic clock effect,” *ArXiv e-prints* (June, 2014), arXiv:1406.6232 [gr-qc].
[42] L. Iess and S. Asmar, “Probing Space-Time in the Solar System: from Cassini to Bepicolombo,” *International Journal of Modern Physics D* **16** no. 12a, (2007) 2117–2126.

[43] L. Iess, S. Asmar, and P. Tortora, “MORE: An advanced tracking experiment for the exploration of Mercury with the mission BepiColombo,” *Acta Astronautica* **65** no. 5-6, (Sept., 2009) 666–675.

[44] I. D. Zhongolovich, “On the Use of the Results Obtained from Synchronous Observations of the Artificial Satellites of the Earth from the INTEROBS Programme for Scientific Purposes,” in *Trajectories of Artificial Celestial Bodies as Determined from Observations / Trajectoires des Corps Celestes Artificiels Détéminées D’après les Observations*, J. Kovalevsky, ed., pp. 1–5. Springer Verlag, Berlin/Heidelberg, 1966.

[45] V. M. Amelin, “Determination of the Quasi-Nodal Period of the Satellite 1960 ε 3 from Simultaneous Visual Tracking Data,” in *Trajectories of Artificial Celestial Bodies as Determined from Observations / Trajectoires des Corps Celestes Artificiels Détéminées D’après les Observations*, J. Kovalevsky, ed., pp. 15–18. Springer Verlag, Berlin/Heidelberg, 1966.

[46] T. V. Kassimenko, “Evaluation of the Satellite Period on the Base of Simultaneous Visual Tracking from Two Given Stations,” in *Trajectories of Artificial Celestial Bodies as Determined from Observations / Trajectoires des Corps Celestes Artificiels Détéminées D’après les Observations*, J. Kovalevsky, ed., pp. 19–22. Springer Verlag, Berlin/Heidelberg, 1966.

[47] T. Mazeh, T. Holczer, and A. Shporer, “Time variation of Kepler transits induced by stellar rotating spots - a way to distinguish between prograde and retrograde motion I. Theory,” *ArXiv e-prints* (July, 2014), arXiv:1407.1979 [astro-ph.SR].

[48] V. A. Brumberg, *Essential Relativistic Celestial Mechanics*. Adam Hilger, Bristol, 1991.

[49] O. Montenbruck and E. Gill, *Satellite Orbits*. Spinger-Verlag, Berlin Heidelberg, 2000.

[50] G. Petit, B. Luzum, and et al., “IERS Conventions (2010),” *IERS Technical Note* **36** (2010) 1.
[51] M. H. Soffel, *Relativity in Astrometry, Celestial Mechanics and Geodesy*. Astronomy and Astrophysics Library. Springer-Verlag, Berlin Heidelberg, 1989.

[52] M. Capderou, *Satellites. Orbits and Missions*. Springer-Verlag France, Paris, 2005.

[53] V. Mioc and E. Radu, “The influence of direct solar radiation pressure on the nodal period of artificial earth satellites,” *Astronomische Nachrichten* 298 no. 2, (1977) 107–110.

[54] D. E. Ochocimskij, T. M. Eneev, and G. P. Taratynova, “Bestimmung der Lebensdauer eines künstlichen Erdsatelliten und Untersuchung der säkularen Störungen seiner Bahn,” *Fortschritte der Physik* 7 no. S2, (1959) 34–54.

[55] P. K. Seidelmann, B. A. Archinal, M. F. A’Hearn, A. Conrad, G. J. Consolmagno, D. Hestroffer, J. L. Hilton, G. A. Krasinsky, G. Neumann, J. Oberst, P. Stooke, E. F. Tedesco, D. J. Tholen, P. C. Thomas, and I. P. Williams, “Report of the IAU/IAG Working Group on cartographic coordinates and rotational elements: 2006,” *Celestial Mechanics and Dynamical Astronomy* 98 no. 3, (July, 2007) 155–180.

[56] I. D. Zhongolovich, “Nekotoryye formuly otnosyashchiyesya k dvizheniyu materiyal’noy tochki v pole tyagoteniya urovennogo ellipsoida vrashcheniya,” *Byul. Inst. Teor. Astron.* 7 no. 90, (1960) 521–536.

[57] V. Mioc and E. Radu, “Perturbations in the anomalistic period of artificial satellites caused by the direct solar radiation pressure,” *Astronomische Nachrichten* 300 no. 6, (1979) 313–315.

[58] G. P. Taratynova, “Über die Bewegung von künstlichen Satelliten im nicht-zentralen Schwerefeld der Erde unter Berücksichtigung des Luftwiderstandes,” *Fortschritte der Physik* 7 no. S2, (1959) 55–64.

[59] E. A. Roth, “On the perturbation of the anomalistic period of a highly eccentric orbit satellite due to the zonal harmonics,” *Celestial Mechanics* 23 no. 1, (Jan., 1981) 83–87.