Violation of the QNEC in a holographic wormhole and IR effects

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We examine the quantum null energy condition (QNEC) for a 2 + 1-dimensional conformal field theory (CFT) at strong coupling in the background of a wormhole spacetime by employing the AdS/CFT correspondence. First, we numerically construct a novel 3 + 1-dimensional vacuum AdS black hole solution with non-trivial topology, which is dual to a wormhole geometry connecting two flat universes. Although the bulk null energy condition (NEC) is not violated, the NEC for the holographic stress-energy tensor is violated near the wormhole throat. We focus on the von Neumann entropy for a half-space anchored to the boundary wormhole throat. We propose a natural prescription for regularizing the IR divergent part of the entanglement entropy and show that the QNEC is violated at the throat. This is the first counterexample to the QNEC, indicating that IR effects are crucial.

Introduction.—The null energy condition (NEC) is key to understanding the basic properties of spacetime structure. It holds for most physically reasonable classical fields and plays a crucial role in various theorems concerning singularities [1] and black hole (BH) mechanics [2]. However, as a local condition, the NEC can be violated [3] when quantum effects are considered. As an improved condition, an (achronal) averaged null energy condition (ANEC) that integrates the NEC along a null geodesic was proposed and used in improved versions of singularity theorems [4, 5] and topological censorship [6].

The quantum null energy condition (QNEC) [7, 8] is a new alternative condition to the NEC which is nonlocal, as it involves the von Neumann entropy or entanglement entropy (EE) S of quantum fields in some subregion A of the spacetime considered. More concretely, the QNEC gives a lower bound for the null-null component of the stress-energy tensor Tkk as

\[
2\pi \int_{\partial A} \gamma T_{kk} \geq \frac{D^2 S}{D\lambda^2},
\]

where \(D^2 S/D\lambda^2\) is the second variation under null deformations of the von Neumann entropy for A, and \(\gamma\) is the determinant of the boundary metric on the subregion boundary \(\partial A\). The QNEC was originally derived from the quantum focussing conjecture [7], where a “quantum expansion” of the null geodesic congruence never increases toward the future. The QNEC was first shown in Minkowski spacetime for free bosonic field theories [8] and later for the cases of holographic CFTs in Minkowski space [9] or in a class of curved spacetimes [10].

In this paper, we study the QNEC for quantum field theories at strong coupling on a wormhole geometry via the AdS/CFT correspondence [11] and show that it can be violated. Recently, bulk wormholes have gained some attention in the context of the AdS/CFT duality due to puzzles they raise when a bulk geometry connects multiple boundaries that each allow a well-defined QFT [12, 13]. Here, we numerically construct a novel 3 + 1-dimensional static-vacuum-AdS BH solution with non-trivial topology, where the AdS boundary metric is conformal to a wormhole geometry that connects two flat universes. According to the AdS/CFT dictionary [11], this corresponds to a thermal state in the boundary field theory at strong coupling. As in the case of Ref. [12], \(T_{kk}\) is negative near the wormhole throat. We focus on the von Neumann entropy for a half-space subregion whose boundary is the wormhole throat. According to the HRT formula [14, 15], the corresponding minimal surface is anchored to the AdS boundary at the throat and extends to the bulk BH at spatial infinity. Because the half-space minimal surface asymptotically approaches the IR region of the BH horizon, it shares the same IR divergence, and we propose a novel definition of the IR-regularized entropy \(S_{reg}\) for the half-space:

\[
S_{reg} := S_{UV} - \frac{1}{2} S_{BH}.
\]

Here, \(S_{UV}\) denotes the UV-regularized entropy for the given half-space and \(S_{BH}\) denotes the Bekenstein-Hawking entropy of the bulk BH. By this definition, we can subtract the thermal part of the entropy, and thereby manifest a purely entanglement part of the entropy for the boundary wormhole. We find that although Eq. (1) is satisfied in a UV expansion near the AdS boundary, it can be violated when IR-effects near spatial infinity are taken into account. As far as we know, this is the first counterexample to the QNEC.

The bulk geometry.—We first recall the 3 + 1-dimensional static-vacuum-AdS BH metrics for different horizon topologies, in units where \(L_{AdS} = 1\) and the conformal boundary is at \(z = 0\):

\[
ds^2 = \frac{1}{z^2} \left[ -f_k(z) dt^2 + \frac{dz^2}{f_k(z)} + d\Sigma_k^2 \right],
\]

\[
f_k(z) = 1 + k z^2 - \mu z^3,
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\[f_k(z)\] is the second variation under null deformations of the von Neumann entropy for A, and \(\gamma\) is the determinant of the boundary metric on the subregion boundary \(\partial A\). The QNEC was originally derived from the quantum focussing conjecture [7], where a “quantum expansion” of the null geodesic congruence never increases toward the future. The QNEC was first shown in Minkowski spacetime for free bosonic field theories [8] and later for the cases of holographic CFTs in Minkowski space [9] or in a class of curved spacetimes [10].

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\]

\[
f_k(z) = 1 + k z^2 - \mu z^3,
\]
with $\mu$ determining the BH mass and $k$ the horizon topology. For $k = 1$, $d\Sigma_k^2$ is the metric on the unit sphere. For the $k = 0$ solution, $d\Sigma_0^2$ is a planar metric. For $k = -1$, $d\Sigma_1^2$ is the unit hyperboloid metric $dr^2 + \cosh^2 r \, d\theta^2$ and near the throat at $r = 0$, the boundary metric is conformal to a cylinder. A static spacetime interpolating between a $k = -1$ BH near $r = 0$ and a $k = 0$ BH as $r \to \infty$ will have, on the boundary, a wormhole connecting two flat universes.

To find such a spacetime, we numerically solve the Einstein-DeTurck equation,

$$R_{\mu \nu} + 3 g_{\mu \nu} - \nabla_\mu \xi_\nu = 0. \quad (4)$$

The DeTurck vector is defined in terms of a reference metric $\bar{g}$ as $\xi^\mu = g^{\alpha \beta} \left( \nabla^\mu \bar{g}_{\alpha \beta} - \bar{g}_{\alpha \beta} \right)$. The final term in Eq. (4) fixes a gauge for the vacuum Einstein equation resulting in a set of elliptic rather than hyperbolic equations better suited to numerics [18, 19]. Solutions to Eq. (4) with $\xi = 0$ also solve the Einstein equation. We use $\zeta_2$ as a test of numerical accuracy, as shown in Fig. 1.

An ansatz for $g$ (modified from Ref. [20]) is

$$ds^2 = \frac{1}{g(x)^2 y^2} \left[ -(1 - y)f(x,y)T \, dt^2 + \frac{g(x)^2 A}{1 - y} f(x,y) \, dy^2 + 4B(dx + x(1 - x)^2 F \, dy)^2 + \frac{\ell(x) S}{(1 - x)^4} \, d\phi^2 \right]. \quad (5)$$

where $0 \leq \phi \leq 2\pi$, $X = \{T, S, A, B, F\}$ are functions of $x$ and $y$ and

$$f(x,y) = 1 + y + y^2 x^2(3 - 2x^2), \quad g(x) = 2 + x^2(3 - 2x^2), \quad \ell(x) = (1 - x^2)^4 + [1 + x^2(1 - x^2)^2]^{-1}. \quad (6)$$

Here, $\zeta$ is a parameter that controls curvature on the throat, as shown below. We choose $\{T, S, A, B\} = 1$ and $F = 0$ for $\bar{g}$ and use $\bar{g}$ as a seed for a Newton-Raphson pseudospectral numerical method over an $N \times N$ Chebyshev grid. The numerical domain is $x \in [0, 1]$ and $y \in [0, 1]$ with the $-1 \leq x \leq 0$ region being given by reflection across $x = 0$. We impose the boundary conditions

$$x = 0: \partial_x X = 0, \quad x = 1: \{T, S, A, B\} = 1, \quad F = 0,$$

$$y = 0: \{T, S, A, B\} = 1, \quad F = 0, \quad y = 1: \frac{T}{A} = 1. \quad (7)$$

At $x = 0$, the choice $\{T, S, A, B\} = 1, F = 0, t \to 2t$ would give the $k = -1$ metric in Eq. (3), but all we require is a reflection symmetry. At $x = 1$, the redefinitions $x = \sqrt{1 - \frac{1}{3\pi} t}, \quad t \to 3t$ give the $k = 0$ metric in Eq. (3). At $y = 0$, with $R = \int^x \frac{2dx}{(1-x^2)^2}$, the metric is

$$ds^2 = \frac{dy^2}{y^2} + \frac{1}{y^2 g(x)} ds^2_\theta$$

$$ds^2_\theta = -dt^2 + dR^2 + \ell(R) d\phi^2. \quad (8)$$

Near $x = 0$, $\ell(R) = 1 + \zeta + (1 - \frac{1}{2}) R^2 + O(R^4)$ and near $x = 1$, $\ell(R) = R^2 + O(R)$. When $-1 < \zeta < 2$, the $S^1$ is minimized at $R = 0 \ (x = 0)$ corresponding to the throat of a wormhole connecting two flat universes. A smooth, constant temperature horizon requires $\frac{T}{\ell_{v=1}} = 1$. Further boundary conditions come from inserting the near horizon expansions $X(x,y) \approx X(x,1) + (y - 1) \partial_y X(x,y)|_{y=1} + ...$ into Eq. (4) [21].

For all $\zeta$, the BH has temperature $T_{BH} = \frac{1}{2\pi}$. The spacetime is asymptotically locally AdS with Fefferman-Graham (FG) expansion near $z = 0$.

$$ds^2 = \frac{1}{z^2} \left[ dz^2 + (h_0^{(0)} + z^2 h_0^{(2)}) dx^a dx^b \right]. \quad (9)$$

The following expansions, inserted into Eq. (4) and solved order by order in $y$, can be used to find $h_{\alpha \beta}^{(0)}$.

$$X(z, r) \approx X^{(0)}(z) + X^{(2)}(z) \frac{y^2}{2} + X^{(3)}(z) \frac{y^3}{6} + O(y^4)$$

$$y(z, r) \approx z \left[ \frac{1}{g(r)} + z^2 g^{(4)}(r) + O(z^4) \right]$$

$$x(z, r) \approx r + z^2 x^{(2)}(r) + z^4 x^{(4)}(r) + O(z^5). \quad (10)$$

Analytic expressions exist for $X^{(2)}$ but $X^{(3)}$ require numerics [23]. The expressions are not illuminating and are omitted. An example of this procedure is in Ref. [24].

The regularized holographic stress-energy tensor is [23]:

$$\langle T_{\alpha \beta} \rangle = \frac{3 h_{\alpha \beta}^{(3)}}{16 \pi G_D}. \quad (11)$$

The off-diagonal terms vanish and the diagonal terms are $h_{\alpha \beta}^{(3)} = \frac{1}{6 g(r)^2} \times$

$$\left\{ h_{t t}^{(0)} (T^{(3)} - 2j), h_{r t}^{(0)} (B^{(3)} + j), h_{\phi \phi}^{(0)} (S^{(3)} + j) \right\} \quad (12)$$

where $j = 2 g(r) - 4$. Importantly, we find $T^{(3)} + S^{(3)} + B^{(3)} = 0$ implying a vanishing trace of the stress-energy tensor, as required for a CFT$_3$.

The null-null component of the stress-energy tensor, plotted in Fig. 2(a), can be defined in terms of the future
directed null vector \( k^a = \partial_t + \partial_R \) with \( R \) defined as before:
\[
R \equiv \int dr \sqrt{h_{rr}^{(0)}}.
\]
For \(-1 < \zeta < 2\), \( T_{kk} \) \( < 0 \) near the throat and near spatial infinity, \( T_{kk} = \frac{2}{\pi r_{\text{ct}}} (\frac{1}{2}) > 0\), which is the value for a \( T_{BH} = \frac{1}{2c}, k = 0 \) BH divided by \( 2\) due to the choice of conformal frame [see Eq. (10)].

Half-space entanglement entropies. — We are interested in the EE of the subregion \( A \) defined by \( r_b \leq r < \infty \). We call this a “half-space” EE since \( \partial A \) at \( r = r_b \) splits the fixed time \( t \) Cauchy surface into two pieces (if \( r_b = 0 \), this is the wormhole throat). The holographic EE of the subregion \( A \) is given by the HRT formula [14, 15].
\[
S(A) = \frac{\mathcal{A}(\Sigma_A)}{4G_4}, \tag{13}
\]
where \( \mathcal{A}_A \) is the codimension-2 minimal area surface in the bulk anchored to the boundary at \( \partial A \) and homologous to \( A \) with \( \mathcal{A}(\Sigma_A) \) denoting its area [25]. As shown in Fig. 2(b), there are two competing minimal surfaces: \( \Sigma_{A,1} \), which touches the BH horizon as \( x \to 1 \), and \( \Sigma_{A,2} \), which touches the BH horizon as \( x \to -1 \) (the reflection of \( \Sigma_{A,1} \) for \( r \geq -r_b \)) and also includes the BH horizon, as required by the homology constraint. The divergent BH area results in \( \Sigma_{A,1} < \Sigma_{A,2} \).

To find \( \Sigma_A \), we minimize the area functional
\[
A = 2\pi \int_0^1 ds \sqrt{g_{\mu\nu}} \partial_s Y^\mu \partial_s Y^\nu \tag{14}
\]
where \( Y^\mu(s) = \{x(s), y(s)\}^\mu\). The minimal surfaces have boundary conditions \( \{x, y\}|_{\int = 0} = \{r_b, 0\} \) and \( \{x, \partial_s y\}|_{\int = 1} = \{1, 0\} \) and solutions have \( y|_{\int = 1} = 1 \).

Eq. (14) is both UV and IR divergent, the former due to short distance correlations across \( \partial A \) and the latter to thermal correlations extending to spatial infinity. To eliminate the UV divergence, we introduce a bulk UV cutoff for the integral Eq. (14), \( z(x, y) = \epsilon \). As usual, the UV-part can be regularized by subtracting a counter-term proportional to the area of \( \partial A \) as \( \mathcal{A}_{UV} := A - 2\pi \sqrt{\frac{(r_{\text{ct}})}{g(r_b)}} \frac{\epsilon^3}{\epsilon} \). In the bulk, the IR divergence comes from the \( \{x, y\} \to \{1, 1\} \) region and has the same divergence as half the BH area. As a concrete realization of Eq. 2, we define a regularized area, \( A_{reg} \) as
\[
A_{reg} := \mathcal{A}_{UV} - \frac{1}{2} \mathcal{A}_{BH}. \tag{15}
\]
A plot of \( A_{reg} \) is shown in Fig. 2(c).

For future reference, we note the half-space minimal surface for a static cylindrical BH where \( d\Sigma_2^2 = dz_2^2 + dx_2^2 \) in Eq. (3) with \( x_2 = x_2 + L_2 \). With fixed \( x_1 \) subregion boundaries, the minimal surfaces solve the equation
\[
x_1' (z) = \frac{(c_1 z)^2}{\sqrt{(1 - (c_1 z)^2) (1 - \mu z^3)}}. \tag{16}
\]
For \( |c_1| > \mu^{1/3} \), the minimal surface has a turning point and gives the EE for a strip-shaped subregion. Half-space subregions have \( |c_1| \leq \mu^{1/3} \) and the surface with minimal area that touches the horizon at \( x_1 \to \infty \) has \( |c_1| = \mu^{1/3} \). Near \( z = 0 \), \( x_1(z) = x_{\text{min}} + c_1^2 z^3 + \mathcal{O}(z^4) \).

Violation of the QNEC. — The QNEC relates the null energy density of a QFT to the second null variation of the EE at a point \( p \) on \( \partial A \). Though the individual pieces of the QNEC are UV divergent, given certain conditions at \( p \), the combination in Eq. (1) is finite. The condition for a \( 2 + 1 \)-dimensional curved spacetime is that the expansion \( \theta|_p \) vanishes [10]. In our spacetime, this criteria is satisfied at the wormhole throat, \( r_b = 0 \). IR divergences do not contribute to the QNEC as they are the same for all half-space EE’s [see Eq. (13)].

To simplify the numerics and exploit the isometries along \( \partial_t \) and \( \partial_\phi \), we investigate an integrated form of the QNEC where all points of \( \partial A \) are moved equally in the \( k^a \) direction, as in Eq. (1). Importantly, because our spacetimes are static and asymptotically flat (AF), the half-space minimal surface lies on a single \( t \) slice. Hence, null and spatial variations of the EE are proportional. This is shown in Fig. 3(a). For example, in the case of the static cylindrical BH, the null variation of the half-space EE vanishes, due to translation invariance in the \( x_1 \)-direction. This is not the case for strip-shaped subregions whose minimal surface will be time-dependent when the two subregion endpoints lie on different \( t \) slices.

In terms of regularized quantities and our wormhole metric, Eq. (1) becomes [26],
\[
Q \equiv 2\pi \langle T_{kk} \rangle - \frac{1}{32\pi G_4 \sqrt{1 + \zeta}} \frac{\delta^2 A_{reg}}{\delta r^2} \bigg|_{r = 0} \geq 0. \tag{17}
\]
In Fig. 3(b), this inequality is violated at the throat.

The QNEC violation is surprising because many proofs exist [8–10, 27]. However, these proofs do not consider thermal states, where IR degrees of freedom play an important role. Proofs in Refs. [9] [10] rely on entanglement wedge nesting (EWN), a statement that for subregions \( A \) and \( B \), if the domains of dependence satisfy \( D(B) \subseteq D(A) \), then \( \Sigma_A \) and \( \Sigma_B \) are achronally separated [28]. The minimal surface embedding satisfies a UV expansion, \( x(z) = r_b + \frac{K}{2} z^2 + \frac{3}{4} z^3 + \mathcal{O}(z^4) \), where \( K \) is the trace of the extrinsic curvature at \( r_b \) [29]. The main assumption of Refs. [9] [10] is that, combining this expansion with EWN, when \( K = 0 \) at a point \( p \),
\[
W \equiv \frac{1}{\sqrt{7}} \bigg| \frac{\delta A(\Sigma_A)}{\delta r} + h_\text{(0)}^\text{(0) c_1} \bigg| = 0, \tag{18}
\]
resulting in the QNEC. However, thermal states do not generically obey this equation. For a strongly-coupled thermal CFT on a cylinder (dual to the cylindrical BH), \( \delta A(\Sigma_A)/\delta r = 0 \); but, \( c_1 \) vanishes only when \( T_{BH} = 0 \) [\( \mu \to 0 \) in Eq. (16)]. Likewise, in Fig. 3(c), we show that in the wormhole, Eq. (18) does not hold. Also shown in Fig. 3(c), the statement of EWN [30], which agrees with
Eq. (17) in the vacuum, is

\[ E \equiv 2\pi G_A \langle T_{kk} \rangle + \frac{h_{(3)}}{4} k^r \delta_{\epsilon_1} |_{r=0} \geq 0 \]

and is nearly saturated [31]. Hence, the entanglement variation in Eq. (17) crucially includes an IR contribution. For the cylindrical BH, the everywhere positive energy density realizes the QNEC, but in the wormhole, a negative energy at the throat leads to its violation.

Summary and discussion.—The QNEC illustrates new and beautiful connections between interacting QFTs and gravity. Intriguingly, it relates the variation of a non-local observable, the EE of a subregion, to a local observable, \( T_{kk} \), and is believed to hold even in curved spacetime at points where \( \theta \) vanishes [where Eq. (1) and Eq. (17) are equal]. However, for thermal states, which have finite energy density that extends to spatial infinity, less is known. In this Letter, we have demonstrated that for these states on a particular wormhole background, the QNEC is violated. On the other hand, a purely UV expression, Eq. (19), is obeyed and nearly saturated. This is a hint that the QNEC in Eq. (1) may govern states perturbatively close to the vacuum with no flux at spatial infinity. One could investigate this in zero temperature analogues of our spacetime, though such solutions have yet to be found. Thermal states, dual to bulk black holes, may instead involve a coarse-grained version of Eq. (1) [32, 33]. Our work emphasizes that some such modification of Eq. (1) to account for IR effects is necessary.

Acknowledgments.—We would like to thank G. Horowitz,
D. Marolf, A. Shahbazi-Moghaddam, and T. Takayanagi for useful discussions. This work was supported in part by JSPS KAKENHI Grant Number 17K05451 (KM), 15K05092 (AI), and NSF Grant Number PHY-1504541 (EM).

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[26] At the wormhole throat, the translation symmetry along $\phi$ says $2\pi \int_0^{\Delta A} \sqrt{g(T_{kk})} = (2\pi)^2 \sqrt{1 + \zeta(T_{kk})}|_{\tau = 0}$, and for affine parameter, $d\lambda = dR \approx 2dr$. In other words $\frac{\partial^2 S_{reg}}{\partial r^2} \equiv k^a \delta_a (k^b S_{reg})|_{r = 0} = \frac{1}{2} \delta^a_S S_{reg}|_{r = 0}$. These combine to give Eq. (17). Notably, Q involves only UV and IR regular quantities and agrees with Eq. (1) only at the throat where $\theta = 0$.

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