INJECTIVITY THEOREMS WITH MULTIPLIER IDEAL SHEAVES
FOR HIGHER DIRECT IMAGES UNDER KÄHLER MORPHISMS

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Abstract. The purpose of this paper is to establish injectivity theorems for higher
direct image sheaves of canonical bundles twisted by pseudo-effective line bundles and mul-
tiplier ideal sheaves. As applications, we generalize Kollár’s torsion freeness and Grauert-
Riemenschneider’s vanishing theorem. Moreover, we obtain a relative vanishing theorem
of Kawamata-Viehweg-Nadel type and an extension theorem for holomorphic sections from
fibers of morphisms to the ambient space. Our approach is based on transcendental meth-
ods and works for Kähler morphisms and singular hermitian metrics with non-algebraic
singularities.

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1. Introduction

The injectivity theorem, which has been studied in the last decades, is a very powerful tool to study higher dimensional algebraic geometry (in particular birational geometry) and complex geometry. After the pioneering work by Tankeev in [Tan71], Kollár established the celebrated injectivity theorem in [Kol86a] by using the Hodge theory. From the viewpoint of Hodge theory, we have already obtained many useful generalizations (for example, see [Amb03], [Amb14], [EV92], [Fuj09], [Fuj11], [Fuj13b], [Fuj14a], and [Kol86b]). Particularly, the following theorem is one of the most useful generalizations of Kollár’s injectivity theorem for deformations of projective varieties. On the other hand, also from the analytic viewpoint, we can approach to Kollár’s result (for example, see [Eno90], [Fuj12], [Fuj13a], [FM16], [Mat13], [Mat14], [Ohs04], [Take95], and [Take97]). This paper contributes to the study of the injectivity theorem and its applications from the analytic viewpoint.

**Theorem 1.1.** Let \( \pi: X \to \Delta \) be a surjective projective morphism from a smooth variety \( X \) to a quasi-projective variety \( \Delta \), and \( F \) be a \( \pi \)-semi-ample line bundle on \( X \).

Then, for a non-zero (holomorphic) section \( s \) of \( F^m \) \((m \geq 0)\), the multiplication map induced by the tensor product with \( s \)

\[
R^q \pi_* (K_X \otimes F) \xrightarrow{\otimes s} R^q \pi_* (K_X \otimes F^{m+1})
\]

is injective for every \( q \). Here \( K_X \) denotes the canonical bundle of \( X \) and \( R^q \pi_*(\bullet) \) denotes the \( q \)-th higher direct image sheaf.

In this paper, we consider a proper Kähler morphism \( \pi: X \to \Delta \) from a complex manifold \( X \) to an arbitrary analytic space \( \Delta \) and a (holomorphic) line bundle \( F \) on \( X \) equipped with a singular (hermitian) metric \( h \), and we study the direct image sheaves \( R^q \pi_* (K_X \otimes F \otimes \mathcal{I}(h)) \) of the canonical bundle \( K_X \) on \( X \) twisted by \( F \) and the multiplier ideal sheaf \( \mathcal{I}(h) \) of \( h \). As results, we establish two injectivity theorems formulated for singular metrics with arbitrary singularities, which can be seen as a generalization of Theorem 1.1 to pseudo-effective line bundles (see Theorem 1.2 and Theorem 1.3). As applications, we give a generalization of Kollár’s torsion freeness and Grauert-Riemenschneider’s vanishing theorem (see Corollary 1.5). Moreover, we obtain a relative vanishing theorem of Kawamata-Viehweg-Nadel type (see Theorem 1.7) and an extension theorem for (holomorphic) sections (see Corollary 1.9).

In [Eno90], Enoki obtained the special case (the absolute case) of Theorem 1.1 under the weaker assumption that \( F \) is semi-positive (namely, it admits a smooth (hermitian) metric with semi-positive curvature), as an application of the theory of harmonic integrals. Takegoshi proved the relative case of Enoki’s result in [Take95], which is a complete generalization of Theorem 1.1 from semi-ample line bundles to semi-positive line bundles. In this paper, we handle line bundles admitting a (possibly) singular metric with semi-positive curvature (that is, pseudo-effective line bundles). The study of pseudo-effective line bundles is one of the important subjects, and thus it is natural and of interest to study further generalizations of Theorem 1.1 and Takegoshi’s result from semi-positive line bundles to pseudo-effective line bundles.

The following theorems, which are the main results of this paper, can be seen as a generalization of Theorem 1.1 and Takegoshi’s result to pseudo-effective line bundles. Moreover
Theorem 1.2 and Theorem 1.3 include various injectivity theorems, for example, [Eno90], [FM16], [Fuj12], [Fuj13a], [GM13], [Kol86a], [Mat13], [Mat14], [Take95], [Take97], and so on.

**Theorem 1.2** (Main Result A). Let \( \pi: X \to \Delta \) be a surjective proper Kähler morphism from a complex manifold \( X \) to an analytic space \( \Delta \), and \( (F,h) \) be a (possibly) singular hermitian line bundle on \( X \) with semi-positive curvature.

Then, for a non-zero (holomorphic) section \( s \) of \( F^m \) \((m \geq 0)\) satisfying \( \sup_K |s|_{h^m} < \infty \) for every relatively compact set \( K \subset X \), the multiplication map induced by the tensor product with \( s \)

\[
R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h)) \overset{\otimes s}{\longrightarrow} R^q\pi_*(K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))
\]
is injective for every \( q \). Here \( \mathcal{I}(\bullet) \) denotes the multiplier ideal sheaf of \( \bullet \).

**Theorem 1.3** (Main Result B). Let \( \pi: X \to \Delta \) be a surjective proper Kähler morphism from a complex manifold \( X \) to an analytic space \( \Delta \). Let \( (F,h) \) be a (possibly) singular hermitian line bundle on \( X \) and \( (M,h_M) \) be a smooth hermitian line bundle on \( X \). Assume that

\[
\sqrt{-1}\Theta_{h_M}(M) \geq 0 \quad \text{and} \quad \sqrt{-1}(\Theta_h(F) - b\Theta_{h_M}(M)) \geq 0
\]

for some \( b > 0 \).

Then, for a non-zero (holomorphic) section \( s \) of \( M \), the multiplication map induced by the tensor product with \( s \)

\[
R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h)) \overset{\otimes s}{\longrightarrow} R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h) \otimes M)
\]
is injective for every \( q \).

**Remark 1.4.** (1) For every point \( t \in \Delta \) we can take an open neighborhood \( \Delta' \) of \( t \) such that \( \pi^{-1}(\Delta') \) is a Kähler manifold, when \( \pi: X \to \Delta \) is a Kähler morphism (for example see [Take95, Definition 6.1] for the definition of Kähler morphisms).

(2) The case \( m = 0 \) in Theorem 1.2 agrees with the case where \( (M,h_M) \) is trivial in Theorem 1.3. This case is important for applications.

(3) The assumption in Theorem 1.2 on the local sup-norm is a reasonable condition to make the multiplication map well-defined and it is always satisfied in the case \( m = 0 \).

In [Mat13] and [FM16], by combining the theory of harmonic integrals with the \( L^2 \)-method for the \( \overline{\partial} \)-equation, we succeeded to obtain the above results in the absolute case (see also [GM13] and [FM16] for applications). The proof of the main results is based on transcendental methods developed in [Mat13], [FM16], and [Take95]. One of the advantages of our method is that we can prove the main results for Kähler morphisms (not only projective morphisms) and singular metrics with non-algebraic singularities. We must sometimes handle the singular metric \( h \) obtained from taking the limit of suitable metrics \( \{h_m\}_{m=1}^\infty \). Under the regularity (smoothness) for singular metrics, a theorem similar to Theorem 1.3 was given in [Fuj13a]. However, it is quite hard to investigate the regularity of the limit \( h \), even if \( h_m \) has algebraic singularities. Therefore it is worth formulating Theorem 1.2 and Theorem 1.3 for singular metrics with arbitrary singularities.
As a direct corollary, we generalize Kollár’s torsion freeness ([Kol86a]) and Grauert-Riemenschneider’s vanishing theorem ([GR70]) for the higher direct images $R^q\pi_*(K_X \otimes F \otimes I(h))$.

**Corollary 1.5** (Kollár’s torsion freeness, Grauert-Riemenschneider’s vanishing theorem). Let $\pi: X \to \Delta$ be a surjective proper Kähler morphism from a complex manifold $X$ to an analytic space $\Delta$, and $(F, h)$ be a (possibly) singular hermitian line bundle on $X$ with semi-positive curvature.

Then the higher direct image sheaf $R^q\pi_*(K_X \otimes F \otimes I(h))$ is torsion free for every $q$. Moreover, we obtain

$$R^q\pi_*(K_X \otimes F \otimes I(h)) = 0 \text{ for every } q > \dim X - \dim \Delta.$$

As a further application, we obtain a vanishing theorem of Kawamata-Viehweg-Nadel type ([Kaw82], [Vie82], [Nad90]) for the higher direct images $R^q\pi_*(K_X \otimes F \otimes I(h))$ (see Theorem 1.7). For the proof of Theorem 1.7, we need the lower semi-continuity of the numerical Kodaira dimension of singular hermitian line bundles, which is of independent interest. If we can prove Proposition 1.6 for Kähler morphisms, we will be able to generalize Theorem 1.7 for them. See Definition 4.2 or [Cao14] for the definition of the numerical Kodaira dimension of singular hermitian line bundles.

**Proposition 1.6** (Quasi lower semi-continuity of the numerical Kodaira dimension). Let $\pi: X \to \Delta$ be a surjective projective morphism from a complex manifold $X$ to an analytic space $\Delta$, and $(F, h)$ be a (possibly) singular hermitian line bundle on $X$ with semi-positive curvature. Assume that $\pi$ is smooth at a point $t_0 \in \Delta$.

Then, there exist an open neighborhood $B$ of $t_0$ and a dense subset $Q \subset B$ with the following property:

For every $t \in Q$, we have $\text{nd}(F|_{X_t}, h|_{X_t}) \geq \text{nd}(F|_{X_{t_0}}, h|_{X_{t_0}})$.

Here $(F|_{X_t}, h|_{X_t})$ denotes the singular hermitian line bundle restricted to the fiber $X_t$ at $t$ and $\text{nd}(F|_{X_t}, h|_{X_t})$ denotes its numerical Kodaira dimension. (See Definition 4.2 for the precise definition).

By combining the celebrated vanishing theorem proved in [Cao14] and the (strong) openness theorem proved in [GZ15] with the proof of Proposition 1.6, we obtain a relative vanishing theorem of Kawamata-Viehweg-Nadel type.

**Theorem 1.7** (Relative vanishing theorem of Kawamata-Viehweg-Nadel type). Let $\pi: X \to \Delta$ be a surjective projective morphism from a complex manifold $X$ to an analytic space $\Delta$, and $(F, h)$ be a (possibly) singular hermitian line bundle on $X$ with semi-positive curvature.

Then we have

$$R^q\pi_*(K_X \otimes F \otimes I(h)) = 0 \text{ for every } q > f - \max_{\pi \text{ is smooth at } t \in \Delta} \text{nd}(F|_{X_t}, h|_{X_t}),$$
where $f$ is the dimension of general fibers. In particular, if $(F|_{X_t}, h|_{X_t})$ is big for some point $t$ in the smooth locus of $\pi$, then we have

$$R^q\pi_* (K_X \otimes F \otimes \mathcal{I}(h)) = 0 \quad \text{for every } q > 0.$$  

Moreover, we obtain an extension theorem (see Corollary 1.9), which is motivated by the following problem related to the invariance of plurigenera and the dlt extension conjecture in the minimal model program (see [Lev83], [Siu98], [Siu02], [Taka97], [Pău07] for the references therein).

**Problem 1.8.** Let $\pi: X \to \Delta$ be a surjective proper Kähler morphism from a complex manifold $X$ to an open disk $\Delta \subset \mathbb{C}$. Assume that $K_X$ is $\pi$-nef and the central fiber $X_0 := \pi^{-1}(0)$ is simple normal crossing. Then can we extend a section $u \in H^0(X_0, \mathcal{O}_{X_0}(K_X^{m}))$ to a section in $H^0(X, \mathcal{O}_X(K_X^{m}))$ (by replacing $\Delta$ with a smaller disk if necessary)?

The formulation of the above problem seems to be reasonable, since it can be seen as a relative version of the dlt extension conjecture and follows from the abundance conjecture. Under the same situation as in Problem 1.8, further let $(F, h)$ be a singular hermitian line bundle on $X$ with semi-positive curvature. Then every section in $H^0(X_0, \mathcal{O}_{X_0}(K_X \otimes F))$ that comes from $H^0(X_0, \mathcal{O}_{X_0}(K_X \otimes F) \otimes \mathcal{I}(h))$ can be extended to a section in $H^0(X, \mathcal{O}_X(K_X \otimes F) \otimes \mathcal{I}(h))$ by replacing $\Delta$ with a smaller disk. In particular, if $K_X$ admits a singular metric $h$ whose curvature is semi-positive and Lelong number is zero at every point in $X_0$, then Problem 1.8 is affirmatively solved.

**Corollary 1.9.** Under the same situation as in Problem 1.8, further let $(F, h)$ be a singular hermitian line bundle on $X$ with semi-positive curvature. Then every section in $H^0(X_0, \mathcal{O}_{X_0}(K_X \otimes F))$ that comes from $H^0(X_0, \mathcal{O}_{X_0}(K_X \otimes F) \otimes \mathcal{I}(h))$ can be extended to a section in $H^0(X, \mathcal{O}_X(K_X \otimes F) \otimes \mathcal{I}(h))$ by replacing $\Delta$ with a smaller disk. In particular, if $K_X$ admits a singular metric $h$ whose curvature is semi-positive and Lelong number is zero at every point in $X_0$, then Problem 1.8 is affirmatively solved.

At the end of this section, we briefly explain the sketch of the proof of Theorems 1.2, comparing with the absolute case established in [Mat13]. The proof of Theorem 1.3 is essentially the same as in Theorem 1.2. For the proof of the main results, we generalize methods in [Take95] and [Fuj13a], and combine them with techniques in [FM16] and [Mat13].

In Step 1, we approximate a given singular metric $h$ by singular metrics $\{h_\varepsilon\}_{\varepsilon > 0}$ that are smooth on a Zariski open set $Y_\varepsilon$, which enables us to use the theory of harmonic integrals on $Y_\varepsilon$. Note that we can not directly use the theory of harmonic integrals since $h$ may have non-algebraic singularities. The subvariety $Y_\varepsilon$ is independent of $\varepsilon$ in [Mat13], but $Y_\varepsilon$ may essentially depend on $\varepsilon$ in our case. For this reason, we construct a complete Kähler form $\omega_{\varepsilon, \delta}$ on $Y_\varepsilon$ such that $\omega_{\varepsilon, \delta}$ converges to a Kähler metric $\omega$ on $X$ as $\delta$ tends to zero. By the standard De Rham-Weil isomorphism, we can represent a given cohomology class $\{u\}$ by an $F$-valued differential form $u$. The $F$-valued form $u$ is locally $L^2$-integrable,
but unfortunately $u$ may not be $L^2$-integrable on $X$ due to the non-compactness of $X$. For this reason, in Step 2, we construct a new metric $H_\varepsilon$ on $F$ by suitably choosing an exhaustive plurisubharmonic function on $X$, which enables us to take harmonic $L^2$-forms $u_{\varepsilon,\delta}$ with respect to $H_\varepsilon$ and $\omega_{\varepsilon,\delta}$ representing the same cohomology class $\{u\}$. In Step 3, we reduce the proof to show that the $L^2$-norm $\|u_{\varepsilon,\delta}\|_{X_c,H_\varepsilon,\omega_{\varepsilon,\delta}}$ on a relatively compact set $X_c \Subset X$ converges to zero as $\varepsilon \to 0$ and $\delta \to 0$. For this step, we need that the quotient map from the $L^2$-space to the $\partial$-cohomology group is a compact operator (see Proposition 2.19). In Step 4, we construct a solution $v_{\varepsilon,\delta}$ of the $\partial$-equation $\partial v_{\varepsilon,\delta} = u_{\varepsilon,\delta}$ such that the $L^2$-norm $\|v_{\varepsilon,\delta}\|_{X_c,H_\varepsilon,\omega_{\varepsilon,\delta}}$ on $X_c$ is uniformly bounded, by using the Čech complex and the De Rham-Weil isomorphism. Finally we prove that

$$\|su_{\varepsilon,\delta}\|_{X_c,H_\varepsilon,\omega_{\varepsilon,\delta}}^2 = \langle \langle su_{\varepsilon,\delta}, \overline{\partial} v_{\varepsilon,\delta} \rangle \rangle_{X_c,H_\varepsilon,\omega_{\varepsilon,\delta}} + \langle \langle (d\Phi)^*su_{\varepsilon,\delta}, v_{\varepsilon,\delta} \rangle \rangle_{\partial X_c,H_\varepsilon,\omega_{\varepsilon,\delta}}$$

for almost all $X_c \Subset X$, by generalizing the formula in [FK] (see Proposition 2.5). Here $\bullet^*$ denotes the adjoint operator of $\bullet$ and $\langle \langle \bullet, \bullet \rangle \rangle_{\partial X_c}$ denotes the norm on the boundary $\partial X_c$. We remark that the norm on the boundary $\partial X_c$ appears due to the non-compactness of $X$. In Step 5, we show that the norm of $\overline{\partial}^*su_{\varepsilon,\delta}$ and $(d\Phi)^*su_{\varepsilon,\delta}$ converges to zero by using Ohsawa-Takegoshi’s twisted Bochner-Kodaira-Nakano identity.

This paper is organized as follows: In Section 2, we summarize the results needed in this paper. Moreover, in this section, we give a generalization of [FK, (1.3.2) Proposition] and recall the fundamental facts on the construction of the De Rham-Weil isomorphism in our situation. In Section 3, we prove Theorem 1.2 and Theorem 1.3. In Section 4, we prove Corollary 1.5, Proposition 1.6, Theorem 1.7, and Corollary 1.9.

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### 2. Preliminaries

In this section, we summarize the results used in this paper with our notation. Throughout this section, let $X$ be a complex manifold of dimension $n$ and $F$ be a (holomorphic) line bundle on $X$.

#### 2.1. $L^2$-spaces of differential forms.

In this subsection, we recall $L^2$-spaces of $F$-valued differential forms and operators defined on them. Let $\omega$ be a positive $(1,1)$-form on $X$ and $h$ be a smooth (hermitian) metric on $F$.

For $F$-valued $(p,q)$-forms $u$ and $v$, the (global) inner product $\langle \langle u, v \rangle \rangle_{h,\omega}$ is defined by

$$\langle \langle u, v \rangle \rangle_{h,\omega} := \int_X \langle u, v \rangle_{h,\omega} dV_\omega,$$

where $dV_\omega$ is the volume form defined by $dV_\omega := \omega^n/n!$ and $\langle u, v \rangle_{h,\omega}$ is the point-wise inner product with respect to $h$ and $\omega$. The $L^2$-space of $F$-valued $(p,q)$-forms with respect to $h$
and $\omega$ is defined by

$$L_{(2)}^{p,q}(X,F)_{h,\omega} := \{u \mid u \text{ is an } F\text{-valued } (p,q)\text{-form with } \|u\|_{h,\omega} < \infty.\}.$$ 

The Chern connection $D = D_{(F,h)}$ on $F$ is canonically determined by the holomorphic structure and the smooth metric $h$ on $F$, which can be written as $D = D'_h + D''_h$ with the $(1,0)$-connection $D'_h$ and the $(0,1)$-connection $D''_h$. We remark that $D''_h$ agrees with the $\bar{\partial}$-operator. The connections $\bar{\partial}$ and $D'_h$ (strictly speaking, their maximal extension) can be seen as a densely defined closed operator on $L_{(2)}^{p,q}(X,F)_{h,\omega}$ with the following domain:

$$\text{Dom } \bar{\partial} := \{u \in L_{(2)}^{p,q}(X,F)_{h,\omega} \mid \bar{\partial}u \in L_{(2)}^{p,q+1}(X,F)_{h,\omega}\}.$$ 

Strictly speaking, these operators depend on $h$ and $\omega$ since their domain and range depend on them, but we often omit the subscript (for example, we abbreviate $\bar{\partial}_{h,\omega}$ to $\bar{\partial}$).

We consider the Hodge star operator $\ast$ with respect to $\omega$

$$\ast = \ast_{\omega} : C^{p,q}_{\infty}(X,F) \rightarrow C^{n-q,n-p}_{\infty}(X,F),$$

where $C^{p,q}_{\infty}(X,F)$ is the set of smooth $F$-valued $(p,q)$-forms on $X$. By the definition, we have $\langle u,v \rangle_{h,\omega} dV_{\omega} = u \wedge H\bar{\partial}\bar{\partial}^*v$, where $H$ is a local function representing $h$. In this paper, the notation $A^*$ denotes the formal adjoint of an operator $A$. For example $D^*_{h,\omega}$ and $\bar{\partial}^*_{h,\omega}$ are respectively the formal adjoint operator of $D'_{h,\omega}$ and $\bar{\partial}$. We remark that

$$D^*_{h,\omega} = -* \bar{\partial}^* \quad \text{and} \quad \bar{\partial}^*_{h,\omega} = -* D'_{h,\omega} \ast.$$ 

Further, for a differential form $\theta$, the notation $\theta^*$ denotes the adjoint operator with respect to the point-wise inner product. When $\theta$ is of type $(s,t)$ and $\theta^*$ acts on $C^{p,q}_{\infty}(X,F)$, we have

$$\theta^* = (-1)^{(p+q)(s+t+1)} \bar{\partial}^*.$$ 

For operators $A$ and $B$ with pure degree, the graded bracket $[\cdot, \cdot]$ is defined by

$$[A,B] := AB - (-1)^{\deg A \deg B} BA.$$ 

In the proof of the main results, we often use the following lemmas, which are obtained from simple computations (for example, see [Take95] for Lemma 2.1).

**Lemma 2.1.** If $\omega$ is a Kähler form, then we have the following identities:

- $\theta^* = \sqrt{-1}[\bar{\partial}, \Lambda_\omega]$ for a $(1,0)$-form $\theta$.
- $\eta^* = \sqrt{-1}[\bar{\partial}, \Lambda_\omega]$ for a $(0,1)$-form $\eta$.
- $[\bar{\partial}, (\bar{\partial}\Phi)^*] + [D'_{h,\omega}, \partial\Phi] = [-1\bar{\partial}\bar{\partial}\Phi, \Lambda_\omega]$ for a smooth function $\Phi$.

Here $\Lambda_\omega$ denotes the adjoint operator defined by $\Lambda_\omega := \omega^*$.

**Lemma 2.2.** Let $\bar{\omega}$ and $\omega$ be positive $(1,1)$-forms such that $\bar{\omega} \geq \omega$. Then we have the following:

- There exists $C > 0$ such that $|\theta^* u|_{\omega} \leq C|\theta|_{\omega}|u|_{\omega}$ for differential forms $\theta, u$.
- The inequality $|\theta|_{\omega} \leq |\theta|_{\omega}$ holds for a differential form $\theta$.
- The inequality $|\theta|_{\bar{\omega}} dV_{\bar{\omega}} \leq |\theta|_{\omega} dV_{\omega}$ holds for an $(n,q)$-form $\theta$.
- The equality $|\theta|_{\bar{\omega}} dV_{\bar{\omega}} = |\theta|_{\omega} dV_{\omega}$ holds for an $(n,0)$-form $\theta$. 
Proof. For a given point \( x \in X \), we choose a local coordinate \((z_1, z_2, \ldots, z_n)\) such that
\[
\tilde{\omega} = \frac{-1}{2} \sum_{j=1}^{n} \lambda_j d\bar{z}_j \wedge dz_j \quad \text{and} \quad \omega = \frac{-1}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j \quad \text{at} \ x.
\]
When differential forms \( \theta \) and \( u \) are written as \( \theta = \sum_{I,J} \theta_{I,J} \ dz_I \wedge d\bar{z}_J \) and \( u = \sum_{K,L} u_{K,L} \ dz_K \wedge d\bar{z}_L \) in terms of this coordinate, we have
\[
|\theta|_\omega^2 = \sum_{I,J} |\theta_{I,J}|^2 \quad \text{and} \quad |\theta|_{\tilde{\omega}}^2 = \sum_{I,J} |\theta_{I,J}|^2 \prod_{(i,j) \in (I,J)} \lambda_i \lambda_j \quad \text{at} \ x,
\]
where \( I, J, K, L \) are ordered multi-indices. The second claim follows from \( \lambda_i \geq 1 \). Further we can easily check the third claim and the fourth claim from the above equalities. By \(|\theta|_{\omega} \leq |\theta|_{\omega} |u|_{\omega} \) and \(|u|_{K,L} \leq |u|_{\omega} \), we have
\[
|\theta \wedge u|_{\omega} = |\sum_{I,J,K,L} \theta_{I,J} u_{K,L} \ dz_I \wedge d\bar{z}_J \wedge dz_K \wedge d\bar{z}_L|_{\omega}
\leq \sum_{I,J,K,L} |\theta_{I,J} u_{K,L}| \leq \sum_{I,J,K,L} |\theta|_{\omega} |u|_{\omega} = C_1 |\theta|_{\omega} |u|_{\omega}.
\]
Here \( C_1 \) is a positive constant depending only on the degree of differential forms. On the other hand, since the Hodge star operator \( * \) preserves the point-wise norm, we have
\[
|\theta^* u|_{\omega} = |* \theta^* u|_{\omega} = |\theta^* u|_{\omega} \leq C_2 |\theta|_{\omega} |* u|_{\omega} = C_2 |\theta|_{\omega} |u|_{\omega}
\]
for some constant \( C_2 > 0 \). \( \square \)

2.2. Ohsawa-Takegoshi’s twisted Bochner-Kodaira-Nakano identity. The following proposition is obtained from the twisted Bochner-Kodaira-Nakano identity (cf. [DF83], [DX84], [OT87], [Ohs95]), which plays an important role in Step 5. For example, see [Take95, Theorem 2.2] for the precise proof.

Proposition 2.3 (Twisted Bochner-Kodaira-Nakano identity). Let \( \omega \) be a complete Kähler form on \( X \) such that \( \sqrt{-1} \Theta_h(F) \geq -C_1 \omega \) for some constant \( C_1 \). Further let \( \Phi \) be a bounded smooth function on \( X \) such that \( \sup_X |d\Phi|_{\omega} < \infty \) and \( \sqrt{-1} \partial \bar{\partial} \Phi \geq -C_2 \omega \) for some constant \( C_2 \).

Then, for every \( u \in \text{Dom} \ \bar{\partial}_{h,\omega} \cap \text{Dom} \ \bar{\partial} \subset L_{(2)}^{p,q}(X,F)_{h,\omega} \), we have
\[
\|\sqrt{\eta}(\bar{\partial} + \bar{\partial} \Phi)u\|_{h,\omega}^2 + \|\sqrt{\eta} \bar{\partial}^* u\|_{h,\omega}^2 = \|\sqrt{\eta}(D_{h,\omega}^* - (\partial \Phi)^*)u\|_{h,\omega}^2 + \langle \eta \sqrt{-1}(\Theta_h(F) + \partial \bar{\partial} \Phi) \Lambda_{\omega} u, u \rangle_{h,\omega},
\]
where \( \eta \) is the function defined by \( \eta := e^\Phi \).

We remark that the case \( \Phi \equiv 0 \) corresponds to the non-twisted version. In the proof of Proposition 2.5 (also Proposition 2.3), we use the following lemma due to Andreotti-Vesentini (see [AV65], [Dem82, LEMME 4.3], [Ves67]).
Lemma 2.4 (Density lemma). Let $\omega$ be a complete positive $(1,1)$-form on $X$.

- There exists a sequence of cut-off functions $\{\theta_k\}_{k=1}^{\infty}$ on $X$ such that $\text{Supp} \theta_k \subset X$, $|d\theta_k|_\omega \leq 1$, and that $\theta_k \to 1$ as $k \to \infty$.
- The set of smooth $F$-valued $(p,q)$-forms with compact support is dense in $\text{Dom} \overline{\partial}_{h,\omega}$, $\text{Dom} \overline{\partial}$, and $\text{Dom} \overline{\partial}_{h,\omega} \cap \text{Dom} \overline{\partial}$ respectively with respect to the following graph norms:
  $$||u||_{h,\omega} + ||\overline{\partial}_{h,\omega} u||_{h,\omega}, \quad ||u||_{h,\omega} + ||\overline{\partial} u||_{h,\omega}, \quad \text{and} \quad ||u||_{h,\omega} + ||\overline{\partial}_{h,\omega} u||_{h,\omega} + ||\overline{\partial} u||_{h,\omega}.$$  

2.3. Adjoint operators on domains with boundaries. Let $\Phi$ be a smooth function on $X$. In this subsection, we consider the level set $X_c$ defined by $X_c := \{x \in X \mid \Phi(x) < c\}$ such that $X_c \subset X$ and $d\Phi \equiv 0$ on the boundary $\partial X_c$ of $X_c$. The inner product on the boundary $\partial X_c$ is defined to be
  $$\langle (u,v) \rangle_{\partial X_c, h,\omega} := \int_{\partial X_c} \langle u,v \rangle_{h,\omega} dS_\omega$$  
for $F$-valued $(p,q)$-forms $u,v$ that are smooth on a neighborhood of $\partial X_c$. Here $dS_\omega$ denotes the volume form on $\partial X_c$ defined by $dS_\omega := *d\Phi/|d\Phi|^2$. Note that we have $dV_\omega = d\Phi \wedge dS_\omega$ by the definition. Then Stoke’s theorem yields
  $$\langle \overline{\partial} u, v \rangle_{X_c, h,\omega} = \langle u, \overline{\partial}^*_{h,\omega} v \rangle_{X_c, h,\omega} + \langle u, (\overline{\partial}\Phi)^* v \rangle_{\partial X_c, h,\omega}$$  
for a smooth $F$-valued $(p,q-1)$-form $u$ and a $(p,q)$-form $v$ on $X$ (see [FK, (1.3.2) Proposition]).

For our purposes, we need to generalize the above formula to a Zariski open set $Y \subset X$ equipped with a complete positive $(1,1)$-form $\tilde{\omega}$. In the following proposition, we consider the Hodge star operator $\ast := \ast_{\tilde{\omega}}$, the volume form $dS_{\tilde{\omega}} := \ast_{\tilde{\omega}} d\Phi/|d\Phi|^2$, the inner product $\langle \overline{\partial} u, v \rangle_{X_c, h,\tilde{\omega}} := \int_{\partial X_c \cap Y} \langle u,v \rangle_{h,\tilde{\omega}} dS_{\tilde{\omega}}$, and so on with respect to $\tilde{\omega}$ (not $\omega$).

Proposition 2.5. Let $\tilde{\omega}$ be a complete positive $(1,1)$-form on a Zariski open set $Y$ of a complex manifold $X$, and $u$ (resp. $v$) be a smooth $F$-valued $(p,q-1)$-form (resp. $(p,q)$-form) on $Y$ with the finite $L^2$-norms $||u||_{h,\tilde{\omega}}, ||v||_{h,\tilde{\omega}}, ||\overline{\partial} u||_{h,\tilde{\omega}}, ||\overline{\partial}^*_{h,\omega} v||_{h,\tilde{\omega}} < \infty$. Consider a smooth function $\Phi$ on $X$ and the level set $X_d$ defined by $X_d := \{x \in X \mid \Phi(x) < d\}$ for $d \in \mathbb{R}$. If $X_c \subset X$ and $d\Phi \equiv 0$ on the boundary $\partial X_c$ of $X_c$ for some $c \in \mathbb{R}$, then there exists a sufficiently small number $a > 0$ with the following properties:

- $d\Phi \equiv 0$ on $\partial X_d$ for every $d \in (c-a, c+a)$.
- $\langle \overline{\partial} u, v \rangle_{X_d, h,\tilde{\omega}} = \langle u, \overline{\partial}^*_{h,\omega} v \rangle_{X_d, h,\tilde{\omega}} + \langle u, (\overline{\partial}\Phi)^* v \rangle_{\partial X_d, h,\tilde{\omega}}$ for almost all $d \in (c-a, c+a)$.

Remark 2.6. (1) In the case of $Y = X$, the above equality in the second property holds for arbitrary $d \in (c-a, c+a)$ (see [FK, (1.3.2) Proposition]). Proposition 2.5 can be seen as a generalization of this result to Zariski open sets.

(2) By the proof, we see that $a$ depends only on $\Phi$, but does not depend on $u$, $v$, and $\tilde{\omega}$.

(3) In the proof of the main results, we apply Proposition 2.5 to a family of countably many differential forms. The subset $I$ defined by
  $$I := \{d \in (c-a, c+a) \mid \text{The above equality does "not" hold for } d. \}$$
depends on \( u, v, \bar{\omega} \). The Lebesgue measure of \( I \) is zero by the proposition, and a countable union of subsets of zero Lebesgue measure also has Lebesgue measure zero. Therefore, for given countably many differential forms, there exists a common \( I \) of zero Lebesgue measure such that the above equality in the second property holds for \( d \not\in I \).

**Proof.** For simplicity, we consider only the case where \((F, h)\) is trivial. For a sufficiently small \( a > 0 \), we have \( d\Phi \neq 0 \) on \( \partial X_d \) for every \( d \in (c - a, c + a) \) by the assumption \( d\Phi \neq 0 \) on \( \partial X_c \). For the proof of the second property, we take a sequence of cut-off functions \( \{\theta_k\}_{k=1}^{\infty} \) on \( Y \) such that \( \text{Supp} \theta_k \subseteq Y \) and \( \theta_k \to 1 \) as \( k \to \infty \). Since \( \bar{\omega} \) is complete on \( Y \), we can add the property \( |d\theta_k|_{\bar{\omega}} \leq 1 \) (see Lemma 2.4). Then Stoke’s theorem implies

\[
\langle \partial(\theta_k u), v \rangle_{X_d, \bar{\omega}} = \langle \theta_k u, \bar{\omega} v \rangle_{X_d, \bar{\omega}} + \langle \theta_k u, (\bar{\omega}^*)^* v \rangle_{\partial X_d, \bar{\omega}}.
\]

We remark that all integrals and adjoint operators are computed with respect to \( \bar{\omega} \) (not \( \omega \)). There is no difficulty in proving equality (2.1) since all integrands that appear in equality (2.1) are zero on a neighborhood of the subvariety \( X \setminus Y \). Indeed, we have

\[
\langle \partial(\theta_k u), v \rangle_{X_d, \bar{\omega}} = \langle \theta_k u, \bar{\omega} v \rangle_{X_d, \bar{\omega}} + \int_{\partial X_d} \theta_k u \wedge \bar{\omega} v
\]

by \( (d\theta_k u) \wedge \bar{\omega} v = -\theta_k u \wedge (\bar{\omega}^* d v) + d(\theta_k u \wedge \bar{\omega} v) \) and Stoke’s theorem. Further we have

\[
\{ u \wedge \bar{\omega} v \} \wedge d\Phi = -u \wedge \bar{\omega}^* d \Phi \wedge \bar{\omega} v = -\langle u, (d\Phi)^* v \rangle dV_{\bar{\omega}} = \{ \langle u, (d\Phi)^* v \rangle dS_{\bar{\omega}} \} \wedge d\Phi.
\]

Moreover it follows that \( d\Phi \) is non-zero in the normal direction of \( \partial X_d \) from \( d\Phi|_{\partial X_d} = 0 \) and \( d\Phi \neq 0 \) on \( \partial X_d \). Therefore we can conclude that \( \int_{\partial X_d} \langle u, (d\Phi)^* v \rangle dS_{\bar{\omega}} \), and thus we obtain equality (2.1).

Now we observe the limit of each term. By the bounded Lebesgue convergence theorem, \( \theta_k u \) converges to \( u \) as \( k \to \infty \) in the \( L^2 \)-topology with respect to \( \bar{\omega} \). Here we used the assumption \( ||u||_{\bar{\omega}} < \infty \). On the other hand, from \( \bar{\omega}(\theta_k u) = \bar{\omega}\theta_k \wedge u + \theta_k \bar{\omega} u \) and \( |d\theta_k|_{\bar{\omega}} \leq 1 \) and \( d\theta_k \to 0 \) in the point-wise sense, we can easily see that \( \bar{\omega}\theta_k \wedge u \to 0 \) and \( \theta_k \bar{\omega} u \to \bar{\omega} u \) in the \( L^2 \)-topology, by using the bounded Lebesgue convergence theorem again. Here we used the assumption \( ||\bar{\omega} u||_{\bar{\omega}} < \infty \). From the above argument, the left hand side (resp. the first term of the right hand side) in equality (2.1) converges to \( \langle \bar{\omega} u, v \rangle_{X_d, \bar{\omega}} \) (resp. \( \langle u, (\bar{\omega}^*)^* v \rangle_{X_d, \bar{\omega}} \)) for every \( d \).

It remains to show that the second term of the right hand side in equality (2.1) converges to \( \langle u, (\bar{\omega}^*)^* v \rangle_{\partial X_d, \bar{\omega}} \) for almost all \( d \). By Cauchy-Schwarz’s inequality and Lemma 2.2, the integrand of the second term can be estimated as follows:

\[
|\langle \theta_k u, (\bar{\omega}^*)^* v \rangle| \leq |\theta_k u|_{\bar{\omega}} |(\bar{\omega}^*)^* v|_{\bar{\omega}} \leq C \sup_{\partial X_d} (|\bar{\omega}^*|_{\bar{\omega}}) |u|_{\bar{\omega}} |v|_{\bar{\omega}}.
\]

We may assume \( \bar{\omega} \geq \bar{\sigma} \) for some positive \((1,1)\) form \( \bar{\sigma} \) on \( X \) since \( \bar{\omega} \) is complete on \( Y \). Then the inequality \( |\bar{\omega}^*|_{\bar{\omega}} \leq |\bar{\omega}^*|_{\bar{\sigma}} \) holds by Lemma 2.2. In particular, the sup-norm \( \sup_{\partial X_d} |\bar{\omega}^*|_{\bar{\omega}} \) is finite since the function \( \Phi \) is smooth on \( X \) (not \( Y \)). If the integral of \( |u|_{\bar{\omega}} |v|_{\bar{\omega}} \) on \( \partial X_d \) is finite, the integral \( \langle \theta_k u, (\bar{\omega}^*)^* v \rangle_{\partial X_d, \bar{\omega}} \) converges to \( \langle u, (\bar{\omega}^*)^* v \rangle_{\partial X_d, \bar{\omega}} \) by the bounded Lebesgue convergence theorem. In order to prove that the integral of \( |u|_{\bar{\omega}} |v|_{\bar{\omega}} \)
on $\partial X_d$ is finite for almost all $d$, we apply Fubini’s theorem and Hölder’s inequality. Then we obtain

$$
\int_{\partial X_d} |u| \, |v| \, dS \left( \int_{\partial X_d} |u| \, |v| \, dV \right) d\Phi = \int_{\{a < \Phi < b\}} |u| \, |v| \, dV
\leq \left( \int_{\{a < \Phi < b\}} |u|^2 \, dV \right)^{1/2} \left( \int_{\{a < \Phi < b\}} |v|^2 \, dV \right)^{1/2}.
$$

Here we used the equality $d\Phi \wedge dS \omega = dV \omega$. Since the right hand side is finite thanks to the assumptions $\|u\| \omega, \|v\| \omega < \infty$, the integral $\int_{\partial X_d} |u| \, |v| \, dS \omega$ should be finite for almost all $d$ in $(a, b)$. (Otherwise, the left hand side becomes infinity.) This completes the proof. \(\square\)

2.4. **Singular hermitian metrics and multiplier ideal sheaves.** We recall the definition of singular hermitian metrics and curvatures. Fix a smooth (hermitian) metric $g$ on $F$.

**Definition 2.7.** (Singular hermitian metrics and curvatures). (1) For an $L^1_{\text{loc}}$-function $\varphi$ on a complex manifold $X$, the hermitian metric $h$ defined by

$$
h := ge^{-2\varphi}
$$

is called a **singular hermitian metric** on $F$. Further $\varphi$ is called the **weight** of $F$ with respect to the fixed smooth metric $g$.

(2) The **curvature** $\sqrt{-1} \Theta_h(F)$ associated to $h$ is defined by

$$
\sqrt{-1} \Theta_h(F) := \sqrt{-1} \Theta_g(F) + 2 \sqrt{-1} \partial \bar{\partial} \varphi,
$$

where $\sqrt{-1} \Theta_g(F)$ is the Chern curvature of $g$.

For simplicity, we call the singular hermitian metric as the singular metric. In this paper, we consider only a singular metric $h$ such that $\sqrt{-1} \Theta_h(F) \geq \gamma$ holds for some smooth $(1, 1)$-form $\gamma$ on $X$. Then the weight function $\varphi$ becomes a quasi-plurisubharmonic (quasi-psh for short) function. In particular, the function $\varphi$ is upper semi-continuous. Moreover, then, the multiplier ideal sheaf defined below is coherent by a theorem of Nadel.

**Definition 2.8** (Multiplier ideal sheaves). Let $h$ be a singular metric on $F$ such that $\sqrt{-1} \Theta_h(F) \geq \gamma$ for some smooth $(1, 1)$-form $\gamma$ on $X$. The ideal sheaf $I(h)$ defined to be

$$
I(h)(B) := I(\varphi)(B) := \{ f \in \mathcal{O}_X(B) \mid |f| e^{-\varphi} \in L^2_{\text{loc}}(B) \}
$$

for every open set $B \subset X$, is called the **multiplier ideal sheaf** associated to $h$.

In Step 1, we approximate a given singular metric by singular metrics that are smooth on a Zariski open set. The following theorem is a reformulation of the equisingular approximation, which is proved by a slight revision of the proof of [DPS01, Theorem 2.3].

**Theorem 2.9** ([DPS01, Theorem 2.3]). Let $\omega$ be a positive $(1, 1)$-form on a complex manifold $X$ and $(F, h)$ be a singular hermitian line bundle on $X$. Assume that $\sqrt{-1} \Theta_h(F) \geq \gamma$ holds for a smooth $(1, 1)$-form $\gamma$ on $X$. Then, for a relatively compact set $K \subset X$, there exist singular metrics $\{ h_\varepsilon \}_{1 \gg \varepsilon > 0}$ on $F|_K$ with the following properties:
(a) $h_\varepsilon$ is smooth on $K \setminus Z_\varepsilon$, where $Z_\varepsilon$ is a proper subvariety of $K$.
(b) $h_\varepsilon'' \leq h_\varepsilon \leq h$ holds on $K$ for any $0 < \varepsilon' < \varepsilon''$.
(c) $I(h) = I(h_\varepsilon)$ on $K$.
(d) $\sqrt{1 - \Theta_{h_\varepsilon}(F)} \geq \gamma - \varepsilon \omega$ on $K$.

Remark 2.10. For a complete positive $(1, 1)$-form $\omega_K$ on $K$, we may assume that property (d) holds for $\omega_K$, that it, $\sqrt{1 - \Theta_{h_\varepsilon}(F)} \geq \gamma - \varepsilon \omega_K$ holds on $K$. Indeed we have $\omega_K \geq a \omega$ for some small number $a > 0$ since $\omega_K$ is complete, $\omega$ is defined on $X$, and $K$ is relatively compact in $X$. By applying Theorem 2.9 to $a \omega$, we can easily see that $\sqrt{1 - \Theta_{h_\varepsilon}(F)} \geq \gamma - \varepsilon a \omega \geq \gamma - \varepsilon \omega_K$.

Proof. In [DPS01], the theorem has been proved in the case where $K = X$ and $X$ is compact. In their proof, we essentially use the assumption that $K$ is compact only when we take a finite cover of $X$ with inequality [DPS01, (2.1)]. Since $K$ is a relatively compact in $X$, for a given $\varepsilon > 0$, we can take a finite open cover of $K$ by open balls $B_i := \{|z^{(i)}| < r_i\}$ with a local coordinate $z^{(i)}$ satisfying inequality [DPS01, (2.1)]. Then we can easily see that the same argument works by suitably replacing $X$ in [DPS01] with $K$. Indeed, uniform estimates [DPS01, (2.4), (2.5), (2.6)] can be checked in the same way since it is sufficient to consider only $B_i$ in this step. Further the function $\psi_{\varepsilon, \nu}$ can be defined on a neighborhood of $K$ by the fourth line in [DPS01, page 700]. Therefore we can repeat the same argument. □

2.5. Fréchet spaces. In this subsection, we summarize fundamental facts on Hilbert spaces and Fréchet spaces. We give a proof of them for the reader’s convenience.

Lemma 2.11. Let $L$ be a closed subspace in a Hilbert space $\mathcal{H}$. Then $L$ is closed with respect to the weak topology of $\mathcal{H}$, that is, if a sequence $\{w_k\}_{k=1}^\infty$ in $L$ weakly converges to $w$, then the weak limit $w$ belongs to $L$.

Proof. By the orthogonal decomposition, there exists a closed subspace $M$ satisfying $L = M^\perp$. Then we obtain that $0 = \langle w_k, v \rangle_{\mathcal{H}} \to \langle w, v \rangle_{\mathcal{H}}$ as $k \to \infty$ for every $v \in M$. Therefore we have $w \in M^\perp = L$. □

Lemma 2.12. Let $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator (continuous linear map) between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. If $\{w_k\}_{k=1}^\infty$ weakly converges to $w$ in $\mathcal{H}_1$, then $\{\varphi(w_k)\}_{k=1}^\infty$ weakly converges to $\varphi(w)$ in $\mathcal{H}_2$.

Proof. By taking the adjoint operator $\varphi^*$, we obtain

$$\langle \varphi(w_k), v \rangle_{\mathcal{H}_2} = \langle w_k, \varphi^*(v) \rangle_{\mathcal{H}_1} \to \langle w, \varphi^*(v) \rangle_{\mathcal{H}_1} = \langle \varphi(w), v \rangle_{\mathcal{H}_2}$$

for every $v \in \mathcal{H}_2$. This completes the proof. □

We apply the following lemma for the compact operator in Proposition 2.19.

Lemma 2.13. Let $\varphi : \mathcal{H} \to \mathcal{F}$ be a compact operator from a Hilbert space $\mathcal{H}$ to a Fréchet space $\mathcal{F}$. Assume $\mathcal{F}$ is the inverse limit of Hilbert spaces. If $\{w_k\}_{k=1}^\infty$ weakly converges to $w$ in $\mathcal{H}$, then $\{\varphi(w_k)\}_{k=1}^\infty$ converges to $\varphi(w)$ in $\mathcal{F}$. 

Proof. By the assumption, there exist Hilbert spaces $F_m$ such that $F = \varprojlim F_m$. The operator $\varphi_m : H \to F \to F_m$ induced by $\varphi$ is a compact operator. In particular, the vector $\varphi_m(w_k)$ converges to $\varphi_m(w)$ in the Hilbert space $F_m$. This means that $\varphi(w_k)$ converges to $\varphi(w)$ in $F$. \hfill \box

2.6. De Rham-Weil isomorphisms. In this subsection, we observe the De Rham-Weil isomorphism from the $\Omega$-cohomology to the Čech cohomology and give a refinement of [Mat13, Section 5] for our purpose. The contents in this subsection (including [Mat13, Section 5]) may be known for specialists, but we will explain them in detail for the reader’s convenience.

Throughout this subsection, let $\omega$ be a Kähler form on a complex manifold $X$ and $(F, h)$ be a singular hermitian line bundle on $X$ such that $\sqrt{-1}\Theta_h(F) \geq -\omega$. Fix a locally finite open cover $U := \{B_i\}_{i \in I}$ of $X$ by sufficiently small Stein open sets $B_i \subset X$. We consider the set of $q$-cochains $C^q(U, K_X \otimes F \otimes \mathcal{I}(h))$ with coefficients in $K_X \otimes F \otimes \mathcal{I}(h)$ calculated by $U$ and the coboundary operator $\mu$ defined to be

$$\mu\{\alpha_{i_0...i_q}\}_{i_0...i_q} := \{\sum_{\ell=0}^{q+1} (-1)^\ell \alpha_{i_0...i_{\ell-1}i_\ell...i_q+1} |_{B_{i_0...i_q+1}}\}_{i_0...i_q+1}$$

for every $q$-cochain $\{\alpha_{i_0...i_q}\}_{i_0...i_q}$, where $B_{i_0...i_q+1} := B_{i_0} \cap B_{i_1} \cap \cdots B_{i_q+1}$. In this paper, we will omit the notation of the restriction, the subscript “$i_0...i_q$”, and so on. The semi-norm $p_{K_{i_0...i_q}}(\bullet)$ is defined to be

$$(2.2) \quad p_{K_{i_0...i_q}}(\{\alpha_{i_0...i_q}\})^2 := \int_{K_{i_0...i_q}} |\alpha_{i_0...i_q}|^2_{h, \omega} dV_\omega$$

for a $q$-cochain $\{\alpha_{i_0...i_q}\}$ and a relatively compact set $K_{i_0...i_q} \subset B_{i_0...i_q}$. Note that the semi-norm $p_{K_{i_0...i_q}}(\bullet)$ is independent of the choice of $\omega$ (see Lemma 2.2). The set of $q$-cochains can be regarded as a topological vector space by the family of the semi-norms $\{p_{K_{i_0...i_q}}(\bullet)\}_{K_{i_0...i_q} \subset B_{i_0...i_q}}$. Then we have the following lemma.

Lemma 2.14. The set of $q$-cochains $C^q(U, K_X \otimes F \otimes \mathcal{I}(h))$ and the set of $q$-cocycles $Z^q(U, K_X \otimes F \otimes \mathcal{I}(h)) := \ker \mu$ are Fréchet spaces with respect to the semi-norms $\{p_{K_{i_0...i_q}}(\bullet)\}_{K_{i_0...i_q} \subset B_{i_0...i_q}}$. Moreover, if $X$ is holomorphically convex, then the set of $q$-coboundaries $B^q(U, K_X \otimes F \otimes \mathcal{I}(h)) := \im \mu$ is also a Fréchet space.

Proof. We first remark that the topology of coherent ideal sheaves induced by the local sup-norms $\sup_{K_{i_0...i_q}}(\bullet)$ is equivalent to the topology induced by the local $L^2$-norms $p_{K_{i_0...i_q}}(\bullet)$ (for example see [GR65, Theorem 2, Section D, Chapter II] or [Mat13, Lemma 5.2, Theorem 5.3, Lemma 5.7]). We can easily see that the metric induced by $p_{K_{i_0...i_q}}(\bullet)$ is complete, and thus $C^q(U, K_X \otimes F \otimes \mathcal{I}(h))$ is a Fréchet space. It follows that $Z^q(U, K_X \otimes F \otimes \mathcal{I}(h)) = \ker \mu$ is a closed subspace (in particular a Fréchet space) since the coboundary operator $\mu$ is continuous. For the latter conclusion, we consider the Čech cohomology group

$$\check{H}^q(U, K_X \otimes F \otimes \mathcal{I}(h)) := \frac{\ker \mu}{\im \mu} \text{ of } C^q(U, K_X \otimes F \otimes \mathcal{I}(h)).$$
Since $X$ is a holomorphically convex, there exists a proper holomorphic map $\pi : X \to \Delta$ to a Stein space $\Delta$. Then, for a coherent sheaf $\mathcal{F}$ on $X$, the natural morphism
\[
(2.3) \quad \pi_* : H^q(X, \mathcal{F}) \to H^0(\Delta, R^q\pi_*\mathcal{F})
\]
is an isomorphism of topological vector spaces (for example, see [Pri71, Lemma II.1]). Hence we have the isomorphism between topological vector spaces
\[
\pi_* : H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \to H^0(\Delta, R^q\pi_* (K_X \otimes F \otimes \mathcal{I}(h))).
\]
In particular, the Čech cohomology group is a separated topological vector space when $X$ is holomorphically convex. Therefore $B^q(U, K_X \otimes F \otimes \mathcal{I}(h)) = \text{Im} \mu$ must be closed. □

Now we observe the $\overline{\partial}$-cohomology group of $L^2$-spaces defined on Zariski open sets equipped with suitable Kähler forms. Let $Z$ be a proper subvariety on $X$ and $\tilde{\omega}$ be a Kähler form on the Zariski open set $Y := X \setminus Z$ with the following properties:

(B) $\tilde{\omega} \geq \omega$ on $Y = X \setminus Z$.

(C) For every point $p \in X$, there exists a bounded function $\Psi$ on an open neighborhood of $p$ such that $\tilde{\omega} = \sqrt{-1} \partial\overline{\partial} \Psi$.

The important point here is that $\tilde{\omega}$ locally admits a bounded potential function on $X$ (not $Y$). The above situation seems to be rather technical, but naturally appears in the proof of the main results. We define the local $L^2$-space of $F$-valued $(p, q)$-forms as follows:
\[
(2.4) \quad L^{p,q}_{(2,\text{loc})}(F)_{h,\tilde{\omega}} := \{u \mid u \text{ is an } F\text{-valued } (p, q)\text{-form with } \|u\|_{K,h,\tilde{\omega}} < \infty \text{ for every } K \in X\},
\]
where $\|u\|_{K,h,\tilde{\omega}}$ is the $L^2$-norm on a relatively compact set $K \subset X$, that is,
\[
\|u\|_{K,h,\tilde{\omega}}^2 := \int_{K \setminus Z} |u|^2_{h,\tilde{\omega}} dV_{\tilde{\omega}}.
\]

For the proof of Proposition 2.19, we concretely construct the De Rham-Weil isomorphism from the $\overline{\partial}$-cohomology to the Čech cohomology in Proposition 2.16. This construction plays an important role in the proof of the main results. To construct the De Rham-Weil isomorphism, we locally solve the $\overline{\partial}$-equation by Lemma 2.15, which is obtained from the standard technique of the theory of Kodaira-Andreotti-Vesentini-Hörmander ([AV61], [AV65], [Hor65], [Kod53]). The reader can check the proof of Lemma 2.15 in [Mat13, Lemma 5.4] with the same notation. We need to generalize Lemma 2.15 to Lemma 3.8 for the proof of the main results.

Lemma 2.15. Under the same situation as above, we assume that $B$ is a sufficiently small Stein open set in $X$. Then, for an arbitrary $U \in \text{Ker} \overline{\partial} \subset L^{n,q}_{(2)}(B \setminus Z, F)_{h,\tilde{\omega}}$, there exist $V \in L^{n,q-1}_{(2)}(B \setminus Z, F)_{h,\tilde{\omega}}$ and a positive constant $C$ (depending only on $\Psi$, $q$) such that
\[
\overline{\partial}V = U \quad \text{and} \quad \|V\|_{h,\tilde{\omega}} \leq C\|U\|_{h,\tilde{\omega}}.
\]
Proposition 2.16. We consider the same situation as above. That is, we consider a singular hermitian line bundle $(F, h)$ on a Kähler manifold $(X, \omega)$ such that $\sqrt{-1} \Theta_h(F) \geq -\omega$ and a Kähler form $\tilde{\omega}$ on a Zariski open set $Y$ satisfying properties (B), (C). Then there exist continuous maps

$$f : \text{Ker } \overline{\partial} \in L^{n,q}_{(2,\text{loc})}(F)_{h,\tilde{\omega}} \to \text{Ker } \mu \in C^q(U, K_X \otimes F \otimes \mathcal{I}(h))$$

$$g : \text{Ker } \mu \in C^q(U, K_X \otimes F \otimes \mathcal{I}(h)) \to \text{Ker } \overline{\partial} \in L^{n,q}_{(2,\text{loc})}(F)_{h,\tilde{\omega}}$$

satisfying the following properties:

- $f$ induces the isomorphism

$$\overline{\partial} : \frac{\text{Ker } \overline{\partial} \in L^{n,q}_{(2,\text{loc})}(F)_{h,\tilde{\omega}}}{\text{Im } \overline{\partial}} \cong \frac{\text{Ker } \mu \in C^q(U, K_X \otimes F \otimes \mathcal{I}(h))}{\text{Im } \mu} \quad \text{of } C^q(U, K_X \otimes F \otimes \mathcal{I}(h)).$$

- $g$ induces the isomorphism

$$\overline{\partial} \circ g : \frac{\text{Ker } \mu \in C^q(U, K_X \otimes F \otimes \mathcal{I}(h))}{\text{Im } \mu} \cong \frac{\text{Ker } \overline{\partial} \in L^{n,q}_{(2,\text{loc})}(F)_{h,\tilde{\omega}}}{\text{Im } \overline{\partial}} \quad \text{of } L^{n,q}_{(2,\text{loc})}(F)_{h,\tilde{\omega}}.$$

- $\overline{\partial} f$ is the inverse map of $\overline{\partial} g$.

Proof. The construction is essentially the same as the standard De Rham-Weil isomorphism. We briefly review only the construction of $f$ and $g$. See [Mat13, Proposition 5.5] and [Fuj13a, Lemma 3.20] for more details.

We first define $f(U)$ for $U \in \text{Ker } \overline{\partial} \subset L^{n,q}_{(2,\text{loc})}(F)_{h,\tilde{\omega}}$ by using the local solution of the $\overline{\partial}$-equation with minimum $L^2$-norm. We consider the $\overline{\partial}$-equation $\overline{\partial} \beta_{i_0} = U|_{B_{i_0}\setminus Z}$ on $B_{i_0}\setminus Z$. Then we can take the solution $\beta_{i_0}$ whose $L^2$-norm is minimum among all solutions. Further, we can see that $\|\beta_{i_0}\|_{h,\tilde{\omega}} \leq C\|U\|_{B_{i_0} \setminus h,\tilde{\omega}} \leq C\|U\|_{h,\tilde{\omega}}$ for some constant $C$ (independent of $U$) by Lemma 2.15. Next we take the solution of the $\overline{\partial}$-equation $\overline{\partial} \beta_{i_{01}} = \beta_{i_1} - \beta_{i_0}$ on $B_{i_{01}} \setminus Z$ with minimum $L^2$-norm. By Lemma 2.15, we have

$$\|\beta_{i_{01}}\|_{h,\tilde{\omega}} \leq D\|\beta_{i_1} - \beta_{i_0}\|_{B_{i_{01}} \setminus h,\tilde{\omega}} \leq D(\|\beta_{i_1}\|_{h,\tilde{\omega}} + \|\beta_{i_0}\|_{h,\tilde{\omega}}) \leq 2CD\|U\|_{h,\tilde{\omega}}$$

for some constant $D$. By repeating this process, we can obtain the $F$-valued $(n, q - k - 1)$-forms $\beta_{i_{0}...i_{k}}$ on $B_{i_{0}...i_{k}} \setminus Z$ satisfying

\[
\begin{cases}
\overline{\partial}\{\beta_{i_0}\} = \{U|_{B_{i_0}\setminus Z}\}; \\
\overline{\partial}\{\beta_{i_{01}}\} = \mu\{\beta_{i_0}\}; \\
\overline{\partial}\{\beta_{i_{01}i_1}\} = \mu\{\beta_{i_{01}}\}; \\
\vdots \\
\overline{\partial}\{\beta_{i_{0}...i_{q-1}}\} = \mu\{\beta_{i_{0}...i_{q-2}}\};
\end{cases}
\]

Then $\mu\{\beta_{i_{0}...i_{q-1}}\} = \{\beta_{i_{0}...i_{q}}\}$ is a $q$-cocycle of $\overline{\partial}$-closed $F$-valued $(n, 0)$-forms on $B_{i_{0}...i_{q}} \setminus Z$ such that $\|\beta_{i_{0}...i_{q}}\|_{h,\tilde{\omega}} < \infty$. We remark that $\|\beta_{i_{0}...i_{q}}\|_{h,\tilde{\omega}} = \|\beta_{i_{0}...i_{q}}\|_{h,\omega}$ by Lemma 2.2. We locally regard $\beta_{i_{0}...i_{q}}$ as a holomorphic function. Then, by the Riemann extension theorem, $\beta_{i_{0}...i_{q}}$ can be extend from $B_{i_{0}...i_{q}} \setminus Z$ to $B_{i_{0}...i_{q}}$. Therefore $\mu\{\beta_{i_{0}...i_{q-1}}\} = \{\beta_{i_{0}...i_{q-1}}\}$ determines a $q$-cocycle in $C^q(U, K_X \otimes F \otimes \mathcal{I}(h))$. We define $f$ by $f(U) := \mu\{\beta_{i_{0}...i_{q-1}}\}$. 

Remark 2.17. By the construction, we can obtain the $L^2$-estimate $\|\beta_{i_0\ldots i_k}\|_{\omega} \leq C \|U\|_{\omega}$ for some constant $C$. Here $C$ essentially depends on a constant that appears when we solve the $\overline{\partial}$-equation by the $L^2$-method, such as Lemma 2.15. We remark that the constant $C$ does not depend on $U$.

In order to define $g$, we fix a partition of unity $\{\rho_i\}_{i \in I}$ of $U = \{B_i\}_{i \in I}$. For a $q$-cocycle $\alpha = \{\alpha_{i_0\ldots i_q}\} \in C^q(U, K_X \otimes F \otimes \mathcal{I}(h))$, we define $g(\alpha)$ by

$$g(\alpha) := \overline{\partial} \left( \sum_{k_q \in I} \rho_{k_q} \overline{\partial} \left( \sum_{k_{q-1} \in I} \rho_{k_{q-1}} \ldots \overline{\partial} \left( \sum_{k_1 \in I} \rho_{k_1} \overline{\partial} (\sum_{k_{i_0} \in I} \rho_{k_{i_0}} \alpha_{k_1\ldots k_q}) ) \right) \right)$$

$$= \sum_{k_q \in I} \overline{\partial} \rho_{k_q} \wedge \sum_{k_{q-1} \in I} \overline{\partial} \rho_{k_{q-1}} \wedge \ldots \wedge \sum_{k_1 \in I} \overline{\partial} \rho_{k_1} \wedge \overline{\partial} (\sum_{k_{i_0} \in I} \rho_{k_{i_0}} \alpha_{k_1\ldots k_q}).$$

This determines the $\overline{\partial}$-closed $F$-valued $(n,q)$-form with locally bounded $L^2$-norm.

From Proposition 2.16, we obtain the following proposition.

**Proposition 2.18.** Under the same situation as in Proposition 2.16, we assume that $X$ is holomorphically convex. Then $\text{Im} \overline{\partial}$ is a closed subspace in $L^{n,q}_{(2,\omega)}(F)_{\omega}$. In particular the $\overline{\partial}$-cohomology group $\text{Ker} \overline{\partial} / \text{Im} \overline{\partial}$ of $L^{n,q}_{(2,\omega)}(F)_{\omega}$ is a Fréchet space.

**Proof.** Since $X$ is holomorphically convex, the Čech cohomology group is a separated topological vector space (see Lemma 2.14). Therefore $\text{Im} \overline{\partial}$ is a closed subspace by Proposition 2.16.

We close this subsection with the following proposition:

**Proposition 2.19.** Under the same situation as in Proposition 2.16, we assume that $X$ is holomorphically convex. Then the following composite map is a compact operator:

$$\text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2,Y,F)_{\omega}} \longrightarrow \text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2,\omega)}(F)_{\omega} \longrightarrow \text{Ker} \overline{\partial} / \text{Im} \overline{\partial} \text{ of } L^{n,q}_{(2,\omega)}(F)_{\omega}.$$

**Remark 2.20.** By Proposition 2.18, the $\overline{\partial}$-cohomology group of $L^{n,q}_{(2,\omega)}(F)_{\omega}$ is a Fréchet space. The natural quotient map

$$\text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2,\omega)}(F)_{\omega} \rightarrow \text{Ker} \overline{\partial} / \text{Im} \overline{\partial} \text{ of } L^{n,q}_{(2,\omega)}(F)_{\omega}$$

is not a compact operator, and thus we need to consider the map defined on $\text{Ker} \overline{\partial} \subset L^{n,q}_{(2,Y,F)_{\omega}}$.

In the proof of Proposition 2.19, we use the construction of $f$ and $g$ in Proposition 2.16. For the reader’s convenience, we first give a proof for the case $q = 1$. 
Proof of Proposition 2.19 for the case $q = 1$. We take a bounded sequence $\{U_\ell\}_{\ell=1}^\infty$ in $\text{Ker} \overline{\partial} \subset L^{n,1}_{(2)}(Y,F)_{h,\bar{\omega}}$, that is, $F$-valued $(n,1)$-forms $\{U_\ell\}_{\ell=1}^\infty \subset \text{Ker} \overline{\partial} \subset L^{n,1}_{(2)}(Y,F)_{h,\bar{\omega}}$ such that $\|U_\ell\|_{h,\bar{\omega}} \leq C_1$ for some constant $C_1$ (independent of $\ell$). For the restriction $U_{\ell,i} := U_\ell|_{B_i \setminus Z}$, by solving the $\overline{\partial}$-equation $\overline{\partial} \beta_{\ell,i} = U_{\ell,i}$ on $B_i \setminus Z$, we obtain $\beta_{\ell,i}$ and a constant $C$ independent of $\ell$ with the following properties:

$$\overline{\partial} \beta_{\ell,i} = U_{\ell,i} \quad \text{on } B_i \setminus Z \quad \text{and} \quad \|\beta_{\ell,i}\|_{h,\bar{\omega}} \leq C\|U_{\ell,i}\|_{B_i,h,\bar{\omega}} \leq C\|U_\ell\|_{h,\bar{\omega}}.$$ 

Here we used the assumption that $\bar{\omega}$ locally admits a bounded potential function $\Psi$ (see Lemma 2.15). In particular, the $F$-valued $(n,0)$-form $\beta_{\ell,j} - \beta_{\ell,i}$ is $\overline{\partial}$-closed on $B_{ij} \setminus Z$ and it is (uniformly) $L^2$-bounded. Therefore it can be extended to the $\overline{\partial}$-closed $F$-valued $(n,0)$-form on $B_i \cap B_j$ by the Riemann extension theorem.

From now on, we construct the $F$-valued $(n,1)$-form $V_\ell$ such that $V_\ell$ determines the same cohomology class as $U_\ell$ and $V_\ell$ converges to some $F$-valued $(n,1)$-form. (This completes the proof.) By the construction, the sup-norm $\sup_K |\beta_{\ell,j} - \beta_{\ell,i}|$ is uniformly bounded for every $K \Subset B_{ij}$. (Recall the local sup-norm of holomorphic functions can be bounded by the $L^2$-norm). Therefore, by Montel's theorem, there exists a subsequence of $\{\beta_{\ell,j} - \beta_{\ell,i}\}_{\ell=1}^\infty$ such that it uniformly converges to some $\alpha_{ij}$ as $\ell \to \infty$ on every relatively compact set in $B_{ij}$. We use the same notation for this subsequence. This subsequence also converges to $\alpha_{ij}$ with respect to the local $L^2$-norms (for example see [Mat13, Lemma 5.2]), that is, $p_{K,ij}(\beta_{\ell,j} - \beta_{\ell,i} - \alpha_{ij}) \to 0$.

For a fixed partition of unity $\{\rho_i\}_{i \in I}$ of $U$, we define the $F$-valued $(n,1)$-form $V_{\ell,i}$ by

$$V_{\ell,i} := \overline{\partial} \left( \sum_{k \in I} \rho_k(\beta_{\ell,i} - \beta_{\ell,k}) \right) = \overline{\partial}(\beta_{\ell,i} - \sum_{k \in I} \rho_k \beta_{\ell,k}) = U_{\ell,i} - \overline{\partial}(\sum_{k \in I} \rho_k \beta_{\ell,k}) \quad \text{on } B_i \setminus Z.$$ 

Since $\sum_{k \in I} \rho_k \beta_{\ell,k}$ is independent of $i$, the family $\{V_{\ell,i}\}_{i \in I}$ determines the $F$-valued $(n,1)$-form $V_\ell$ globally defined on $Y$. Further the $F$-valued $(n,1)$-form $V_\ell$ determines the same cohomology class as $U_\ell$. It is sufficient for the proof to show that $V_\ell$ converges to $\sum_{k \in I} \overline{\partial}(\rho_k \alpha_{ki}) = \sum_{k \in I} \overline{\partial} \rho_k \wedge \alpha_{ki}$ in $L^{n,1}_{(2,\text{loc})}(F)_{h,\bar{\omega}}$. For a given $K \Subset X$, the cardinality of $I_K$ defined by

$$I_K := \{i \in I \mid B_i \cap K \neq \emptyset\}$$

is finite. Hence we obtain

$$\|V_\ell - \sum_{k \in I} \overline{\partial}(\rho_k \alpha_{ki})\|_{K,h,\bar{\omega}} \leq \sum_{i \in I_K} \|\sum_{k \in I} \overline{\partial} \rho_k \wedge (\beta_{\ell,i} - \beta_{\ell,k} - \alpha_{ki})\|_{B_i,h,\bar{\omega}} \leq C_2\|\beta_{\ell,i} - \beta_{\ell,k} - \alpha_{ki}\|_{B_i \cap \text{Supp} \rho_k,h,\bar{\omega}}$$

for some constant $C_2 > 0$. Here we used $|\overline{\partial} \rho_k|_{\bar{\omega}} \leq |\overline{\partial} \rho_k|_{\omega}$ (see Lemma 2.2 and property (B)). By $K_i \cap \text{Supp} \rho_k \Subset B_{ik}$, the right hand side converges to zero as $\ell$ tends to $\infty$. This completes the proof.

Proof of Proposition 2.19 for the general case. We take a bounded sequence $\{U_\ell\}_{\ell=1}^\infty$ in $\text{Ker} \overline{\partial} \subset L^{n,q}_{(2)}(Y,F)_{h,\bar{\omega}}$. Then, by the construction of $f$, we can obtain the $F$-valued
Therefore it is sufficient to show that the multiplication map
\[ f(U) = \mu(\{\beta_{\ell,i_0...i_q-1}\}) =: \{\beta_{\ell,i_0...i_q}\} \quad \text{and} \quad \|\beta_{\ell,i_0...i_q}\|_{B_{i_0...i_q},h,\tilde{\omega}} \leq C\|U\|_{h,\tilde{\omega}}. \]

Here \( C \) is a positive constant independent of \( \ell \) (see Remark 2.17). Note that \( \beta_{\ell,i_0...i_q} \) can be regarded as a holomorphic function on \( B_{i_0...i_q} \) since it is a \( \overline{\partial} \)-closed \((n,0)\)-form. The local sup-norm \( \sup_K |\beta_{\ell,i_0...i_q}| \) is uniformly bounded for every relatively compact set \( K \subset B_{i_0...i_q} \). Therefore, by Montel’s theorem, we can take a subsequence of \( \{f(U)\}_\ell \) converging some \( q \)-cocycle \( \alpha \) with respect to the local sup-norms. This subsequence converges to \( \alpha \) also with respect to the local \( L^2 \)-norms. We use the same notation \( \{f(U)\}_\ell \) for this subsequence.

When we fix a partition of unity \( \{\rho_1\}_I \) of \( U \), we can define the map \( g \) in Proposition 2.16. It follows that \( g(f(U)) \) converges to \( g(\alpha) \) in \( L_{u,loc}(F)_{h,\tilde{\omega}} \) since the map \( g \) is continuous. Further we can see that \( g(f(U)) \) determines the same cohomology class as \( U_\ell \) by Proposition 2.16. This completes the proof. \( \square \)

3. Proof of the Main Results

3.1. Proof of Theorem 1.2. In this subsection, we prove Theorem 1.2. The proof can be divided into five steps. The proof of Theorem 1.3 will be obtained from a slight revision of Step 5 (see subsection 3.2). In the following proof, we often write (possibly different) positive constants as \( C \) and use the same notation for suitably chosen subsequences.

Step 1 (Reduction of the proof to the local problem). In this step, we fix the notation used in this subsection and reduce the proof to a local problem on \( \Delta \). At the end of this step, the rough strategy of the proof will be given, which helps us to understand the complicated and technical arguments.

The problem is local on \( \Delta \), and thus we may assume that \( \pi : X \rightarrow \Delta \) is a surjective proper holomorphic map from a Kähler manifold \( X \) to a Stein subvariety \( \Delta \) in \( \mathbb{C}^N \). Then the manifold \( X \) is holomorphically convex since \( X \) admits a proper holomorphic map to a Stein space. In particular, for a coherent sheaf \( F \) on \( X \), the natural morphism
\[
\pi_* : H^q(X,F) \rightarrow H^0(\Delta, R^q\pi_*F)
\]
is an isomorphism of topological vector spaces (for example, see [Pri71, Lemma II.1]). Therefore it is sufficient to show that the multiplication map
\[
H^q(X, K_X \otimes F \otimes I(h)) \stackrel{\otimes s}{\rightarrow} H^q(X, K_X \otimes F^{m+1} \otimes I(h^{m+1}))
\]
is injective. In [Mat13], we have already proved that the above multiplication map is injective when \( X \) is compact. One of the difficulties of the proof is to deal with the non-compact manifold \( X \).

By replacing \( \Delta \) with smaller one (if necessary), we may assume that \( X \) is a relatively compact set in the initial ambient space. In particular, the point-wise norm \( |s|_{h^m} \) can be assumed to be bounded on \( X \) by the assumption. Note that \( X \) admits a complete Kähler form since \( X \) is a weakly pseudoconvex Kähler manifold. For a fixed complete Kähler form \( \omega \) on \( X \), by applying Theorem 2.9 and Remark 2.10 for \( \gamma = 0 \), we can take a family of singular metrics \( \{h_\varepsilon\}_{\varepsilon > 0} \) on \( F \) with the following properties:
(a) $h_\varepsilon$ is smooth on $X \setminus Z_\varepsilon$ for some proper subvariety $Z_\varepsilon$.
(b) $h_{\varepsilon'} \leq h_{\varepsilon''} \leq h$ holds on $X$ for any $0 < \varepsilon' < \varepsilon''$.
(c) $\mathcal{I}(h) = \mathcal{I}(h_\varepsilon)$ on $X$.
(d) $\sqrt{-1} \Theta h_\varepsilon(F) \geq -\varepsilon \omega$ on $X$.

Remark 3.1. In the case $m > 0$, the set $\{ x \in X \mid \nu(h, x) > 0 \}$ is contained in the zero set $s^{-1}(0)$ of $s$ by $\sup |s|_{h^m} < \infty$, where $\nu(h, x)$ is the Lelong number of the weight of $h$ at $x$. Then we can assume that $Z_\varepsilon$ is independent of $\varepsilon$. However, in the case $m = 0$, the subvariety $Z_\varepsilon$ may essentially depend on $\varepsilon$, which is different from [Mat13]. For this reason, we need to consider a complete Kähler form $\omega_{\varepsilon, \delta}$ on $Y_\varepsilon := X \setminus Z_\varepsilon$ such that $\omega_{\varepsilon, \delta}$ converges to $\omega$ as $\delta$ goes to zero.

To use the theory of harmonic integrals on the Zariski open set $Y_\varepsilon$, we first take a complete Kähler form $\omega_\varepsilon$ on $Y_\varepsilon$ with the following properties:

- $\omega_\varepsilon$ is a complete Kähler form on $Y_\varepsilon$.
- $\omega_\varepsilon \geq \omega$ on $Y_\varepsilon$.
- $\omega_\varepsilon = \sqrt{-1} \partial \overline{\partial} \Psi_\varepsilon$ for some bounded function $\Psi_\varepsilon$ on a neighborhood of every $p \in X$.

See [Fuj13a, 3.11] for the construction of $\omega_\varepsilon$. The key point here is the third property on the bounded potential function, which enables us to construct the De Rham-Weil isomorphism from the $\mathcal{O}$-cohomology group on $Y_\varepsilon$ to the Čech cohomology group on $X$ (see Proposition 2.16). We define the Kähler form $\omega_{\varepsilon, \delta}$ on $Y_\varepsilon$ by

$$\omega_{\varepsilon, \delta} := \omega + \delta \omega_\varepsilon$$

for $0 < \delta \ll \varepsilon$.

It is easy to see the following properties:

(A) $\omega_{\varepsilon, \delta}$ is a complete Kähler form on $Y_\varepsilon = X \setminus Z_\varepsilon$ for every $\delta > 0$.
(B) $\omega_{\varepsilon, \delta} \geq \omega$ on $Y_\varepsilon$ for every $\delta > 0$.
(C) For every point $p \in X$, there exists a bounded function $\Psi_{\varepsilon, \delta} = \Psi + \delta \Psi_\varepsilon$ on an open neighborhood of $p$ in $X$ such that $\sqrt{-1} \partial \overline{\partial} \Psi_{\varepsilon, \delta} = \omega_{\varepsilon, \delta}$ and $\lim_{\delta \to 0} \Psi_{\varepsilon, \delta} = \Psi$. Here $\Psi$ is a local potential function of $\omega$.

Remark 3.2. Strictly speaking, by Theorem 2.9 ([DPS01, Theorem 2.3]), we obtain a countable family $\{h_{\varepsilon_k}\}_{k=1}^{\infty}$ of singular metrics satisfying the above properties and $\varepsilon_k \to 0$. In our proof, we actually consider only countable sequences $\{\varepsilon_k\}_{k=1}^{\infty}$ and $\{\delta_k\}_{k=1}^{\infty}$ converging to zero since we need to use Cantor’s diagonal argument, but we often use the notations $\varepsilon$ and $\delta$ for simplicity.

We define the function $\Phi$ on $X$ by

$$\Phi := \pi^* i^* (|z_1|^2 + |z_2|^2 + \cdots + |z_N|^2),$$

where $i : \Delta \to \mathbb{C}^N$ is a local embedding of the Stein subvariety $\Delta$ and $(z_1, z_2, \ldots, z_N)$ is a coordinate of $\mathbb{C}^N$. By the construction, the function $\Phi$ is a psh function on $X$. Since $\pi$ is a proper morphism, the function $\Phi$ is an exhaustive function on $X$ (that is, the level set $X_c := \{ x \in X \mid \Phi(x) < c \}$ is relatively compact in $X$ for every $c$ with $c < \sup_X \Phi$).
Moreover, we may assume that
\[ \sup_{X} \Phi < \infty \quad \text{and} \quad \sup_{X} |d\Phi|_{\omega_{\varepsilon, \delta}} < C \]
by replacing \( \Delta \) with smaller one. Indeed, we can assume that \( \Phi \) is defined on a neighborhood of \( \partial X \) in the initial ambient space by taking smaller \( \Delta \). This implies that \( \sup_{X} \Phi < \infty \) and \( \sup_{X} |d\Phi|_{\omega} < \infty \) for some positive \((1, 1)\)-form \( \omega \) defined on the neighborhood of \( \partial X \).

We may assume that \( \omega_{\varepsilon, \delta} \geq \omega \geq \omega_{\omega} \) since \( \omega \) is a complete form on \( X \). By Lemma 2.2, we have the inequality \( \sup_{X} |d\Phi|_{\omega_{\varepsilon, \delta}} \leq \sup_{X} |d\Phi|_{\omega} < \infty \). In particular, it was shown that the function \( \Phi \) and the complete Kähler form \( \omega_{\varepsilon, \delta} \) satisfy the assumptions in Proposition 2.3.

Let \( A \) be a cohomology class in \( H^{q}(X, K_{X} \otimes F \otimes \mathcal{I}(h)) \) such that \( sA = 0 \in H^{q}(X, K_{X} \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1})) \). Our goal is to prove that \( A \) is actually the zero cohomology class.

We briefly explain the strategy of the proof with the above notation. In Step 2, by suitably choosing an increasing convex function \( \chi : \mathbb{R} \to \mathbb{R} \), we represent \( A \) by harmonic \( L^{2} \)-forms \( u_{\varepsilon, \delta} \) with respect to \( \omega_{\varepsilon, \delta} \) and the new metric \( H_{\varepsilon} \) defined by \( H_{\varepsilon} := h_{\varepsilon} e^{-\chi_{0}\Phi} \). In Step 3, we consider the level set \( X_{c} := \{ x \in X | \Phi(x) < c \} \), and show that if the \( L^{2} \)-norm \( \|su_{\varepsilon, \delta}\|_{X_{c}, H_{\varepsilon} h^{m}_{\varepsilon, \omega_{\varepsilon, \delta}}} \) on \( X_{c} \) converges to zero for almost all \( c \), then \( A \) is zero.

To prove this convergence, in Step 4, we construct a solution \( v_{\varepsilon, \delta} \) of the \( \partial \)-equation \( \partial v_{\varepsilon, \delta} = su_{\varepsilon, \delta} \) such that the \( L^{2} \)-norm \( \|v_{\varepsilon, \delta}\|_{X_{c}, h, \omega_{\varepsilon, \delta}} \) on \( X_{c} \) is uniformly bounded. In Step 5, we show that

\[
\|su_{\varepsilon, \delta}\|_{X_{c}, H_{\varepsilon} h^{m}_{\varepsilon, \omega_{\varepsilon, \delta}}}^{2} = \langle su_{\varepsilon, \delta}, \partial v_{\varepsilon, \delta} \rangle_{X_{c}, H_{\varepsilon} h^{m}_{\varepsilon, \omega_{\varepsilon, \delta}}} + \langle (d\Phi)^{*} su_{\varepsilon, \delta}, v_{\varepsilon, \delta} \rangle_{\partial X_{c}, H_{\varepsilon} h^{m}_{\varepsilon, \omega_{\varepsilon, \delta}}} \to 0
\]

by using the twisted Bochner-Kodaira-Nakano identity.

**Step 2** (\( L^{2} \)-spaces and representations by harmonic forms). In this step, we construct harmonic \( L^{2} \)-forms \( u_{\varepsilon, \delta} \) representing the cohomology class \( A \in H^{q}(X, K_{X} \otimes F \otimes \mathcal{I}(h)) \), and prove Proposition 3.6, which says that if \( u_{\varepsilon, \delta} \) converges to zero in a suitable sense, then \( A \) is zero (that is, the proof is completed).

We first consider the standard De Rham-Weil isomorphism:

\[
H^{q}(X, K_{X} \otimes F \otimes \mathcal{I}(h)) \cong \frac{\text{Ker} \left( \partial^{*} : L^{n,q}_{(2, \text{loc})}(F)_{h, \omega} \to L^{n,q+1}_{(2, \text{loc})}(F)_{h, \omega} \right)}{\text{Im} \left( \partial^{*} : L^{n,q-1}_{(2, \text{loc})}(F)_{h, \omega} \to L^{n,q}_{(2, \text{loc})}(F)_{h, \omega} \right)},
\]

where \( L^{n,q}_{(2, \text{loc})}(F)_{h, \omega} \) is the set of \( F \)-valued \((n, \bullet)\)-forms on \( X \) with locally bounded \( L^{2} \)-norm (that is, the \( L^{2} \)-norm \( \|f\|_{K_{h, \omega}} \) on \( K \) with respect to \( h \) and \( \omega \) is finite for every relatively compact set \( K \subset X \)). By the above isomorphism, the cohomology class \( A \) can be represented by a \( \partial^{*} \)-closed \( F \)-valued \((n, q)\)-form \( u \) on \( X \) with locally bounded \( L^{2} \)-norm.

Our goal is to prove that \( u \in \text{Im} \left( \partial^{*} \right) \subset L^{n,q}_{(2, \text{loc})}(F)_{h, \omega} \).

We want to represent the cohomology class \( A \) by harmonic forms in \( L^{2} \)-spaces, but unfortunately, \( u \) may not be globally \( L^{2} \)-integrable on \( X \). For this reason, we construct a new metric on \( F \) that makes \( u \) globally \( L^{2} \)-integrable. Since \( u \) is locally \( L^{2} \)-integrable and \( \Phi \) is exhaustive, there exists an increasing convex function \( \chi : \mathbb{R} \to \mathbb{R} \) such that the
$L^2$-norm $\|u\|_{h^\epsilon e^{-\chi(\Phi)}, \omega}$ on $X$ is finite. For simplicity we put
\[ H := h e^{-\chi(\Phi)}, \quad H_\epsilon := h_\epsilon e^{-\chi(\Phi)}, \quad \text{and} \quad \|u\|_{\epsilon, \delta} := \|u\|_{H_\epsilon, \omega_\epsilon, \delta}. \]

Then we obtain the following inequality:
\[
\|u\|_{\epsilon, \delta} \leq \|u\|_{H, \omega_{\epsilon, \delta}} \leq \|u\|_{H_\epsilon, \omega_{\epsilon, \delta}} < \infty. \tag{3.4}
\]

Strictly speaking, the left hand side should be $\|u|_{Y_\epsilon}\|_{\epsilon, \delta}$, but we often omit the symbol of restriction. The first inequality follows from property (b) of $h_\epsilon$, and the second inequality follows from Lemma 2.2 and property (B) of $\omega_{\epsilon, \delta}$. Here we used a special characteristic of the canonical bundle $K_X$ since the second inequality holds only for $(n,q)$-forms. The norm $\|u\|_{\epsilon, \delta}$ is uniformly bounded since the right hand side is independent of $\epsilon$, $\delta$. These inequalities play an important role in the proof.

We consider the $L^2$-space
\[
L^{n,q}_{(2)}(Y_\epsilon, F) := L^{n,q}_{(2)}(Y_\epsilon, F)_{H_\epsilon, \omega_{\epsilon, \delta}}
\]
on $Y_\epsilon$ with respect to $H_\epsilon$ and $\omega_{\epsilon, \delta}$ (not $H$ and $\omega$). In general, we have the following orthogonal decomposition:
\[
L^{n,q}_{(2)}(F)_{\epsilon, \delta} = \overline{\text{Im } \partial} \oplus \mathcal{H}^{n,q}_{\epsilon, \delta}(F) \oplus \overline{\text{Im } \overline{\partial}}^\perp,
\]
where $\overline{\partial}$ denotes the closure of $\partial$ with respect to the $L^2$-topology and $\mathcal{H}^{n,q}_{\epsilon, \delta}(F)$ denotes the set of harmonic $F$-valued $(n,q)$-forms on $Y_\epsilon$, namely
\[
\mathcal{H}^{n,q}_{\epsilon, \delta}(F) := \{ v \in L^{n,q}_{(2)}(F)_{\epsilon, \delta} \mid \overline{\partial} v = \overline{\partial} v = 0 \}.
\]

We remark that (the maximal extension of) the formal adjoint $\overline{\partial}_{\epsilon, \delta}$ agrees with the Hilbert space adjoint since $\omega_{\epsilon, \delta}$ is complete for $\delta > 0$ (see Lemma 2.4). Strictly speaking, $\overline{\partial}$ also depends on $H_\epsilon$ and $\omega_{\epsilon, \delta}$ since the domain and range of the closed operator $\overline{\partial}$ depend on them, but we abbreviate $\overline{\partial}_{\epsilon, \delta}$ to $\overline{\partial}_{\epsilon, \delta}$.

The $F$-valued $(n,q)$-form $u$ belongs to $L^{n,q}_{(2)}(F)_{\epsilon, \delta}$ by inequality (3.4). By the above orthogonal decomposition, the $F$-valued $(n,q)$-form $u$ can be decomposed as follows:
\[
u = w_{\epsilon, \delta} + u_{\epsilon, \delta} \quad \text{for some} \quad w_{\epsilon, \delta} \in \overline{\text{Im } \partial} \quad \text{and} \quad u_{\epsilon, \delta} \in \mathcal{H}^{n,q}_{\epsilon, \delta}(F) \text{ in } L^{n,q}_{(2)}(F)_{\epsilon, \delta}.
\]

Note that the orthogonal projection of $u$ to $\overline{\text{Im } \overline{\partial}_{\epsilon, \delta}}$ is zero by the following lemma.

**Lemma 3.3.** If $u$ belongs to $\text{Ker } \overline{\partial}$, then the orthogonal projection of $u$ to $\overline{\text{Im } \overline{\partial}_{\epsilon, \delta}}$ is zero.

**Proof.** For an arbitrary element $\lim_{k \to \infty} \overline{\partial}_{\epsilon, \delta} c_k \in \overline{\text{Im } \overline{\partial}_{\epsilon, \delta}}$, we have
\[
\langle u, \lim_{k \to \infty} \overline{\partial}_{\epsilon, \delta} c_k \rangle_{\epsilon, \delta} = \lim_{k \to \infty} \langle u, \overline{\partial}_{\epsilon, \delta} c_k \rangle_{\epsilon, \delta} = \lim_{k \to \infty} \langle \overline{\partial} u, c_k \rangle_{\epsilon, \delta} = 0.
\]
This leads to the conclusion. \[\square\]
From now on, we take a suitable limit of \( u_{\varepsilon,\delta} \). We need to carefully choose the \( L^2 \)-space, since the \( L^2 \)-space \( L^{n,q}_{(2)}(F) \varepsilon,\delta \) depends on \( \varepsilon,\delta \) although we have property (c). We remark that \( \{ \varepsilon \}_{\varepsilon > 0} \) and \( \{ \delta \}_{\delta > 0} \) denote countable sequences converging to zero (see Remark 3.2). Let \( \{ \delta_0 \}_{\delta_0 > 0} \) be another countable sequence converging to zero.

**Proposition 3.4.** There exist a subsequence \( \{ \delta_\nu \}_{\nu = 1}^\infty \) of \( \{ \delta \}_{\delta > 0} \) and \( \alpha_\varepsilon \in L^{n,q}_{(2)}(F)_{H,\omega} \) with the following properties:

- For any \( \varepsilon,\delta_0 > 0 \), as \( \delta_\nu \) tends to 0,
  
  \( u_{\varepsilon,\delta_\nu} \) converges to \( \alpha_\varepsilon \), with respect to the weak \( L^2 \)-topology in \( L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0} \).

- For any \( \varepsilon > 0 \), we have
  
  \[
  \| \alpha_\varepsilon \|_{H,\omega} \leq \lim_{\delta_0 \to 0} \| \alpha_\varepsilon \|_{\varepsilon,\delta_0} \leq \lim_{\delta_\nu \to 0} \| u_{\varepsilon,\delta_\nu} \|_{\varepsilon,\delta_\nu} \leq \| u \|_{H,\omega}.
  \]

**Remark 3.5.** The subsequence \( \{ \delta_\nu \}_{\nu = 1}^\infty \) does not depend on \( \varepsilon,\delta_0 \). The \( F \)-valued form \( \alpha_\varepsilon \) is independent of \( \delta_0 \) and \( L^2 \)-integrable with respect to \( H,\omega \) (not \( \omega_{\varepsilon,\delta} \)).

**Proof.** For any \( \varepsilon,\delta_0 > 0 \), by taking \( \delta \) with \( \delta < \delta_0 \), we have

\[
(3.6) \quad \| u_{\varepsilon,\delta} \|_{\varepsilon,\delta_0} \leq \| u_{\varepsilon,\delta} \|_{\varepsilon,\delta} \leq \| u \|_{\varepsilon,\delta} \leq \| u \|_{H,\omega}.
\]

The first inequality follows from \( \omega_{\varepsilon,\delta} \leq \omega_{\varepsilon,\delta_0} \) and Lemma 2.2, the second inequality follows since \( u_{\varepsilon,\delta} \) is the orthogonal projection of \( u \) with respect to \( \varepsilon,\delta \), and the last inequality follows from inequality (3.4). From this estimate, we know that \( \{ u_{\varepsilon,\delta} \}_{\delta > 0} \) is uniformly bounded in \( L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0} \). Therefore, there exist a subsequence \( \{ \delta_\nu \}_{\nu = 1}^\infty \) of \( \{ \delta \}_{\delta > 0} \) and \( \alpha_{\varepsilon,\delta_0} \in L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0} \) such that \( u_{\varepsilon,\delta_\nu} \) converges to \( \alpha_{\varepsilon,\delta_0} \) with respect to the weak \( L^2 \)-topology in \( L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0} \). The choice of this subsequence \( \{ \delta_\nu \}_{\nu = 1}^\infty \) may depend on \( \varepsilon,\delta \), but by extracting a suitable subsequence, we can easily choose a subsequence independent of \( \varepsilon,\delta_0 \) by Cantor’s diagonal argument.

Now we prove that \( \alpha_{\varepsilon,\delta_0} \) does not depend on \( \delta_0 \). For arbitrary \( \delta',\delta'' \) with \( 0 < \delta' \leq \delta'' \), the natural inclusion \( L^{n,q}_{(2)}(F)_{\varepsilon,\delta'} \to L^{n,q}_{(2)}(F)_{\varepsilon,\delta''} \) is a bounded operator (continuous linear map) by \( \| \cdot \|_{\varepsilon,\delta''} \leq \| \cdot \|_{\varepsilon,\delta'} \) (see Lemma 2.2). By Lemma 2.12, the \( F \)-valued form \( u_{\varepsilon,\delta_\nu} \) weakly converges to \( \alpha_{\varepsilon,\delta} \) in not only \( L^{n,q}_{(2)}(F)_{\varepsilon,\delta'} \) but also \( L^{n,q}_{(2)}(F)_{\varepsilon,\delta''} \). Hence we have \( \alpha_{\varepsilon,\delta'} = \alpha_{\varepsilon,\delta''} \) since the weak limit is uniquely determined.

Finally we prove the estimate in the proposition. It is easy to see that

\[
\| \alpha_\varepsilon \|_{\varepsilon,\delta_0} \leq \lim_{\delta_\nu \to 0} \| u_{\varepsilon,\delta_\nu} \|_{\varepsilon,\delta_0} \leq \lim_{\delta_\nu \to 0} \| u_{\varepsilon,\delta_\nu} \|_{\varepsilon,\delta_\nu} \leq \| u \|_{H,\omega}.
\]

The first inequality follows since the norm is lower semi-continuous with respect to the weak convergence, the second inequality follows from \( \omega_{\varepsilon,\delta_0} \geq \omega_{\varepsilon,\delta_\nu} \), and the last inequality follows from inequality (3.6). Fatou’s lemma yields

\[
(3.7) \quad \| \alpha_\varepsilon \|^2_{H,\omega} = \int_{Y_\varepsilon} |\alpha_\varepsilon|^2_{H,\omega} dV_\omega \leq \lim_{\delta_0 \to 0} \int_{Y_\varepsilon} |\alpha_\varepsilon|^2_{H,\omega,\varepsilon,\delta_0} dV_{\omega,\varepsilon,\delta_0} = \lim_{\delta_0 \to 0} \| \alpha_\varepsilon \|^2_{\varepsilon,\delta_0}.
\]

These inequalities lead to the estimate in the proposition. \( \Box \)
For simplicity, we use the same notation \( u_{\varepsilon,\delta} \) for the subsequence \( u_{\varepsilon,\delta_0} \) in Proposition 3.4. Next we take a suitable limit of \( \alpha_{\varepsilon} \). For a fixed positive number \( \varepsilon_0 > 0 \), by taking a sufficiently small \( \varepsilon \), we have

\[
\| \alpha_{\varepsilon} \|_{H_{\varepsilon_0,\omega}} \leq \| \alpha_{\varepsilon} \|_{H_{\varepsilon,\omega}} \leq \| u \|_{H_{\omega}}
\]

by property (b) and Proposition 3.4. By taking a subsequence of \( \{ \alpha_{\varepsilon} \}_{\varepsilon > 0} \), we may assume that \( \{ \alpha_{\varepsilon} \}_{\varepsilon > 0} \) weakly converges to some \( \alpha \) in \( L^{n,q}_{(2)}(F)_{H_{\varepsilon_0,\omega}} \). The following proposition says that the proof of Theorem 1.2 is completed if the weak limit \( \alpha \) is shown to be zero.

**Proposition 3.6.** If the weak limit \( \alpha \) is zero in \( L^{n,q}_{(2)}(F)_{H_{\varepsilon_0,\omega}} \), then the cohomology class \( A \) is zero in \( H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \).

**Proof.** First we consider the De Rham-Weil isomorphism constructed in Proposition 2.16.

\[
\begin{array}{c}
\text{Ker} \overline{\partial} \\
\text{Im} \partial
\end{array}
\text{of } L^{n,q}_{(2,\text{loc})}(F)_{\varepsilon,\delta_0} \xrightarrow{\cong} \tilde{H}^q(X, K_X \otimes F \otimes \mathcal{I}(h_\varepsilon)) = \tilde{H}^q(X, K_X \otimes F \otimes \mathcal{I}(h)).
\]

We remark that the \( \check{\text{C}}ech \) cohomology group does not depend on \( \varepsilon \) by property (c). By Proposition 2.18, the subspace \( \text{Im} \overline{\partial} \) is closed in \( L^{n,q}_{(2,\text{loc})}(F)_{\varepsilon,\delta_0} \). Hence, for every \( \delta \) with \( 0 < \delta \leq \delta_0 \), we can easily see that

\[
(3.9) \quad u - u_{\varepsilon,\delta} \in \text{Im} \overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{\varepsilon,\delta} \subset \text{Im} \overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0}
\]

by the construction of \( u_{\varepsilon,\delta} \) and Proposition 2.18. As \( \delta \) tends to zero, we obtain

\[
u - \alpha_{\varepsilon} \in \text{Im} \overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0} \subset \text{Im} \overline{\partial} \text{ in } L^{n,q}_{(2,\text{loc})}(F)_{\varepsilon,\delta_0} = \text{Im} \overline{\partial} \text{ in } L^{n,q}_{(2,\text{loc})}(F)_{\varepsilon,\delta_0}.
\]

by Lemma 2.11 and Proposition 3.4.

On the other hand, we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0} & \xrightarrow{q_1} & \text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2,\text{loc})}(F)_{\varepsilon,\delta_0} \xrightarrow{\cong} \tilde{H}^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \\
& \downarrow j_1 & \downarrow \phi_1 \text{ in } \tilde{H}^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \\
\text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{H_{\varepsilon,\omega}} & \xrightarrow{j_2} & \text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2,\text{loc})}(F)_{H_{\varepsilon,\omega}} \xrightarrow{q_2} \text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2,\text{loc})}(F)_{H_{\varepsilon,\omega}}.
\end{array}
\]

Here \( j_1, j_2 \) are the natural inclusions, \( q_1, q_2 \) are the natural quotient maps via the local \( L^2 \)-spaces, and \( \phi_1, \phi_2 \) are the De Rham-Weil isomorphisms. We remark that \( j_2 \) is well-defined. Indeed, by the \( L^2 \)-integrability and [Dem82, LEMME 6.9], the equality \( \overline{\partial} U = 0 \) can be extended from \( Y_\varepsilon \) to \( X \) (in particular to \( Y_{\varepsilon_0} \)) for \( U \in \text{Ker} \overline{\partial} \subset L^{n,q}_{(2)}(F)_{H_{\varepsilon,\omega}} \). The key point here is the \( L^2 \)-integrability with respect to \( \omega \) (not \( \omega_{\varepsilon,\delta} \)). By Proposition 2.19, the map \( q_2 \) is a compact operator, and thus we obtain

\[
\lim_{\varepsilon \to 0} q_2(u - \alpha_{\varepsilon}) = q_2(u - \alpha) = q_2(u)
\]
by Lemma 2.13 and the assumption $\alpha = 0$. On the other hand, we can see that $q_1(u - \alpha_\varepsilon) = 0$ by the first half argument. Therefore we obtain $q_2(u) = 0$ by the above commutative diagram. Then we can conclude that $u$ belongs to $\text{Im} \overline{\partial}$ in $L^{n,q}_{(2,\text{loc})} (F)_{H,\varepsilon,\omega}$. Indeed, we can easily see that $q_3(u) = 0$ by the following commutative diagram:

\[
\begin{array}{cccc}
\text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2)} (F)_{H,\varepsilon,\omega} & \xrightarrow{q_2} & \text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2,\text{loc})} (F)_{H,\varepsilon,\omega} \\
\downarrow j_3 & & \downarrow j_3 \\
\text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2)} (F)_{H,\varepsilon,\omega} & \xrightarrow{q_3} & \text{Ker} \overline{\partial} \text{ in } L^{n,q}_{(2,\text{loc})} (F)_{H,\varepsilon,\omega} \\
\end{array}
\]

Step 3 (Relations between weak limits and $L^2$-norms). In this step, we consider the norm

\[
\|su_{\varepsilon,\delta}\|_{\varepsilon,\delta} := \|su_{\varepsilon,\delta}\|_{H,\varepsilon,\omega_{\varepsilon,\delta}}
\]

and prove Proposition 3.7, which says that it is sufficient for the proof to show that

\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \|su_{\varepsilon,\delta}\|_{K,\varepsilon,\delta} = 0
\]

for every relatively compact set $K \subseteq X$.

In order to clarify a relation between the weak limit $\alpha$ and the asymptotic behavior of the norm of $su_{\varepsilon,\delta}$, we compare the norm of $u_{\varepsilon,\delta}$ with the norm of $su_{\varepsilon,\delta}$. We define $Y_{\varepsilon_0}^k$ and $X_c$ by

\[
Y_{\varepsilon_0}^k := \{ y \in Y_{\varepsilon_0} \mid |s|_{H_{\varepsilon_0}}(y) > 1/k \} \quad \text{and} \quad X_c := \{ x \in X \mid \Phi(x) < c \}
\]

for $k \gg 0$ and $c$. The subset $X_c$ is a relatively compact set in $X$ for every $c$ with $c < \sup_X \Phi$ by the construction of $\Phi$. Further $Y_{\varepsilon_0}^k$ is an open set in $Y_{\varepsilon_0}$ since $|s|_{H_{\varepsilon_0}}$ is lower semi-continuous. Then we prove the following proposition:

Proposition 3.7. Under the above situation, if we have

\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \|su_{\varepsilon,\delta}\|_{X_c,\varepsilon,\delta} = 0
\]

for every $c$ with $c < \sup_X \Phi$, then the weak limit $\alpha$ is zero. In particular, the cohomology class $A$ is zero by Proposition 3.6.

Proof. By the argument on inequality (3.8), we are assuming that $\alpha_\varepsilon$ weakly converges to $\alpha$ in $L^{n,q}_{(2)} (F)_{H,\varepsilon,\omega}$. The restriction $\alpha_\varepsilon|_{X_c \cap Y_{\varepsilon_0}^k}$ also weakly converges to $\alpha|_{X_c \cap Y_{\varepsilon_0}^k}$ in $L^{n,q}_{(2)} (X_c \cap Y_{\varepsilon_0}^k, F)_{H,\varepsilon,\omega}$ by Lemma 2.12, since the restriction map

\[
L^{n,q}_{(2)} (F)_{H,\varepsilon,\omega} \longrightarrow L^{n,q}_{(2)} (X_c \cap Y_{\varepsilon_0}^k, F)_{H,\varepsilon,\omega}
\]

is a bounded operator (continuous linear map). Therefore we obtain

\[
\|\alpha\|_{X_c \cap Y_{\varepsilon_0}^k, H,\varepsilon,\omega} \leq \lim_{\varepsilon \to 0} \|\alpha_\varepsilon\|_{X_c \cap Y_{\varepsilon_0}^k, H,\varepsilon,\omega} \leq \lim_{\varepsilon \to 0} \|\alpha_\varepsilon\|_{X_c \cap Y_{\varepsilon_0}^k, H,\varepsilon,\omega}.
\]
The first inequality follows since the norm is lower semi-continuous with respect to the weak convergence, and the second inequality follows from property (b). By the same argument, the restriction of $u_{\varepsilon, \delta}$ weakly converges to $\alpha_{\varepsilon}$ in $L^{n,q}_{(2)}(X_c \cap Y^k_{\varepsilon, \delta}, F)_{\varepsilon, \delta}$, and thus we obtain

$$\|\alpha\|_{X_c \cap Y^k_{\varepsilon, \delta}, H_{\varepsilon, \delta}, \omega} \leq \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \|u_{\varepsilon, \delta}\|_{X_c \cap Y^k_{\varepsilon, \delta}, \omega, \varepsilon, \delta},$$

Moreover, we can obtain

$$\|\alpha\|_{X_c \cap Y^k_{\varepsilon, \delta}, \omega} \leq \lim_{\delta \to 0} \|\alpha\|_{X_c \cap Y^k_{\varepsilon, \delta}, \omega, \varepsilon, \delta},$$

by the above inequality and Fatou's lemma (see the argument for inequality (3.7)). These inequalities yield

$$\|\alpha\|_{X_c \cap Y^k_{\varepsilon, \delta}, \omega} \leq \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \|u_{\varepsilon, \delta}\|_{X_c \cap Y^k_{\varepsilon, \delta}, \omega}. $$

On the other hand, from $1/k < |s|_{h^m_{\varepsilon, \delta}} \leq |s|_{h^m_{\varepsilon, \delta}}$ on $Y^k_{\varepsilon, \delta}$, we have

$$\|u_{\varepsilon, \delta}\|_{X_c \cap Y^k_{\varepsilon, \delta}, \omega} \leq k \|s_{u_{\varepsilon, \delta}}\|_{X_c \cap Y^k_{\varepsilon, \delta}, \omega} \leq k \|s\|_{X_c \cap Y^k_{\varepsilon, \delta}, \omega}.$$ 

By the assumption, we can conclude that $\alpha = 0$ on $X_c \cap Y^k_{\varepsilon, \delta}$ for arbitrary $c < \sup \Phi$ and $k \gg 0$. From $\cup_{\varepsilon > c, k \gg 0}(X_c \cap Y^k_{\varepsilon, \delta}) = Y_0$, we obtain the conclusion. \hfill \Box

**Step 4** (Construction of solutions of the $\overline{\partial}$-equation). In this step, by using the construction of the De Rham-Weil isomorphism in subsection 2.6, we prove Proposition 3.9, which gives a solution $w_{\varepsilon, \delta}$ of the $\overline{\partial}$-equation $\overline{\partial}w_{\varepsilon, \delta} = u - u_{\varepsilon, \delta}$ with uniformly bounded (local) $L^2$-norm.

Fix a locally finite open cover $U := \{B_i\}_{i \in I}$ of $X$ by sufficiently small Stein open sets $B_i \subseteq X$. Since $h_{\varepsilon}, \omega_{\varepsilon, \delta}$, and $Y_\varepsilon$ satisfy the assumptions in Proposition 2.16, we have the continuous maps

$$f_{\varepsilon, \delta} : \text{Ker} \overline{\partial} \in L^{n,q}_{(2,\text{loc})}(F)_{\varepsilon, \delta} \to \text{Ker} \mu \in C^q(U, K_X \otimes F \otimes \mathcal{I}(h_{\varepsilon})),$$

$$g_{\varepsilon, \delta} : \text{Ker} \mu \in C^q(U, K_X \otimes F \otimes \mathcal{I}(h_{\varepsilon})) \to \text{Ker} \overline{\partial} \in L^{n,q}_{(2,\text{loc})}(F)_{\varepsilon, \delta}$$

such that they determine the isomorphism between the $\overline{\partial}$-cohomology and the Čech cohomology. For the construction of $f_{\varepsilon, \delta}$ in Proposition 2.16, we locally solved the $\overline{\partial}$-equation and estimated the $L^2$-norm of the solution by Lemma 2.15. In this subsection, for the $L^2$-estimate of the solution, we use the following lemma instead of Lemma 2.15

**Lemma 3.8.** Let $B \subseteq X$ be a sufficiently small Stein open set. Then, for an arbitrary $U \in \text{Ker} \overline{\partial} \subset L^{n,q}_{(2)}(B \setminus Z_\varepsilon, F)_{\varepsilon, \delta}$, there exist $V \in L^{n,q-1}_{(2)}(B \setminus Z_\varepsilon, F)_{\varepsilon, \delta}$ and a positive constant $C_{\varepsilon, \delta}$ (depending only on $\Psi_{\varepsilon, \delta}, q$) such that

- $\overline{\partial}V = U$ and $\|V\|_{\varepsilon, \delta} \leq C_{\varepsilon, \delta}\|U\|_{\varepsilon, \delta}$.
- $\lim_{\delta \to 0} C_{\varepsilon, \delta}$ is independent of $\varepsilon$.

**Proof.** We may assume that $\varepsilon < 1/2$. Further, by property (C), we may assume that there exists a bounded function $\Psi_{\varepsilon, \delta}$ on $B$ such that $\omega_{\varepsilon, \delta} = \sqrt{-1}\overline{\partial}\overline{\partial}^*\Psi_{\varepsilon, \delta}$ and $\Psi_{\varepsilon, \delta} \to \Psi$ as $\delta \to 0.$
The function $\Psi$ is independent of $\varepsilon$ since it is the local weight function of $\omega$. The curvature of $G_{\varepsilon,\delta}$ defined by $G_{\varepsilon,\delta} := H_{\varepsilon}e^{-\Psi_{\varepsilon,\delta}}$ satisfies
\[
\sqrt{-1}\Theta_{G_{\varepsilon,\delta}}(F) = \sqrt{-1}\Theta_{H_{\varepsilon}}(F) + \sqrt{-1}\partial\bar{\partial}\chi(\Phi) + \sqrt{-1}\partial\bar{\partial}\Psi_{\varepsilon,\delta}
\geq -\varepsilon\omega + \omega_{\varepsilon,\delta}
\geq (1 - \varepsilon)\omega_{\varepsilon,\delta}
\]
by property (a) and property (B). Here we used the inequality $\sqrt{-1}\partial\bar{\partial}\chi(\Phi) \geq 0$, which follows since $\Phi$ is a psh function and $\chi$ is an increasing convex function. It follows that $\|U\|_{G_{\varepsilon,\delta},\omega_{\varepsilon,\delta}}$ is finite since $\Psi_{\varepsilon,\delta}$ is a bounded function. Hence, by the standard result for the $\bar{\partial}$-equation, there exist a solution $V$ such that $\bar{\partial}V = U$ and $\|V\|_{G_{\varepsilon,\delta},\omega_{\varepsilon,\delta}} \leq (1/q(1 - \varepsilon))\|U\|_{G_{\varepsilon,\delta},\omega_{\varepsilon,\delta}}$. By $(1 - \varepsilon) > 1/2$ and the definition of $G_{\varepsilon,\delta}$, we can easily see that
\[
\|V\|_{\varepsilon,\delta} \leq \sqrt{\frac{2}{q}} \sup_B e^{-\Psi_{\varepsilon,\delta}} \|U\|_{\varepsilon,\delta}.
\]

The above constant converges to $(2/q)^{1/2}$ as $\delta$ tends to zero.

**Proposition 3.9.** For every $c$ with $c < \sup_X \Phi$, there exists $w_{\varepsilon,\delta} \in L_{(2,\text{loc})}^{n,q-1}(F)_{\varepsilon,\delta}$ with the following properties:
- $\bar{\partial}w_{\varepsilon,\delta} = u - u_{\varepsilon,\delta}$.
- $\lim_{\delta \to 0} \|w_{\varepsilon,\delta}\|_{X_{\varepsilon,\delta}}$ can be bounded by a constant independent of $\varepsilon$.

**Remark 3.10.** We have already known that there exists a solution $w_{\varepsilon,\delta}$ of the $\bar{\partial}$-equation $\bar{\partial}w_{\varepsilon,\delta} = u - u_{\varepsilon,\delta}$ by $u - u_{\varepsilon,\delta} \in \text{Im} \bar{\partial} \subset L_{(2,\text{loc})}^{n,q}(F)_{\varepsilon,\delta}$ (see (3.9) in the proof of Proposition 3.6). The important point here is the second property on the local $L^2$-norm of solutions.

The strategy of the proof is the same as in the proof of [FM16, Proposition 5.9] and [Mat13, Theorem 5.9]. The main idea is to change the $\bar{\partial}$-equation $\bar{\partial}w_{\varepsilon,\delta} = u - u_{\varepsilon,\delta}$ to the equation $\mu\gamma_{\varepsilon,\delta} = f_{\varepsilon,\delta}(u - u_{\varepsilon,\delta})$ of the coboundary operator $\mu$ in the set of cochains $C^\bullet(K_X \otimes F \otimes \mathcal{I}(h_\varepsilon))$, by using the Čech complex and pursuing the De Rham-Weil isomorphism. (A similar argument can be found in [Ohs84].) Here $f_{\varepsilon,\delta}$ is the continuous map constructed in Proposition 2.16. The $L^2$-space $L_{(2)}^{n,q}(F)_{\varepsilon,\delta}$ depends on $\varepsilon,\delta$, but $C^\bullet(K_X \otimes F \otimes \mathcal{I}(h_\varepsilon))$ does not depend on them thanks to property (c). This is one of the important points. In the proof, we will show that $f_{\varepsilon,\delta}(u - u_{\varepsilon,\delta})$ converges to some $q$-coboundary $\alpha_{0,0}$ in $C^q(K_X \otimes F \otimes \mathcal{I}(h_\varepsilon))$ with respect to the topology induced by the local $L^2$-norms $\{p_{K_{i_0\ldots i_q}}(\bullet)\}_{K_{i_0\ldots i_q} \in B_{i_0\ldots i_q}}$ (see (2.2) for the definition). Further we will show that the coboundary operator $\mu$ is an open map. Then, by these observations, we will construct a solution $\gamma_{\varepsilon,\delta}$ of the equation $\mu\gamma_{\varepsilon,\delta} = f_{\varepsilon,\delta}(u - u_{\varepsilon,\delta})$ with suitable local $L^2$-norm. Finally, by the continuous map $g_{\varepsilon,\delta}$ constructed by a partition of unity, we conversely construct $w_{\varepsilon,\delta}$ satisfying the properties in Proposition 3.9.

For the reader’s convenience, we first give a proof for the case $q = 1$. This case helps us to follow the essential arguments.
Proof of Proposition 3.9 for the case $q = 1$. We may assume that the cardinality of $I_c$ defined by

$$I_c := \{ i \in I \mid B_i \cap X_c \neq \emptyset \}$$

is finite by $X_c \subseteq X$. For simplicity we put $U_{\varepsilon,\delta} := u - u_{\varepsilon,\delta}$. By Lemma 3.8, we can take a solution $\beta_{\varepsilon,\delta,i}$ of the $F$-equation $\overline{\partial} \beta_{\varepsilon,\delta,i} = U_{\varepsilon,\delta}$ on $B_i \setminus Z_c$ such that

$$\| \beta_{\varepsilon,\delta,i} \|_{B_i,\varepsilon,\delta} \leq C_{\varepsilon,\delta} \| U_{\varepsilon,\delta} \|_{B_i,\varepsilon,\delta} \leq C_{\varepsilon,\delta} \| U_{\varepsilon,\delta} \|_{\varepsilon,\delta}$$

for some constant $C_{\varepsilon,\delta}$. In the proof, the notation $C_{\varepsilon,\delta}$ denotes a (possibly different) positive constant with the property in Lemma 3.8 (that is, $\lim_{\delta \to 0} C_{\varepsilon,\delta}$ is independent of $\varepsilon$). Inequality (3.6) yields

$$\| U_{\varepsilon,\delta} \|_{\varepsilon,\delta} \leq \| u \|_{\varepsilon,\delta} \leq \sum_{i,j} \| u_{\varepsilon,\delta} \|_{\varepsilon,\delta} \leq 2 \| u \|_{H,\omega}.$$ 

In particular, the norm $\| \beta_{\varepsilon,\delta,i} \|_{B_i,\varepsilon,\delta}$ can be bounded by a constant $C_{\varepsilon,\delta}$.

Now we consider the $F$-valued $(n,0)$-form $(\beta_{\varepsilon,\delta,j} - \beta_{\varepsilon,\delta,i})$ on $B_{ij} \setminus Z_c$, where $B_{ij} := B_i \cap B_j$. Then $(\beta_{\varepsilon,\delta,j} - \beta_{\varepsilon,\delta,i})$ can be seen as a holomorphic function with bounded $L^2$-norm, since it is a $\overline{\partial}$-closed $F$-valued $(n,0)$-form satisfying $\| \beta_{\varepsilon,\delta,j} - \beta_{\varepsilon,\delta,i} \|_{H,\omega} = \| \beta_{\varepsilon,\delta,j} - \beta_{\varepsilon,\delta,i} \|_{\varepsilon,\delta} < \infty$ (see Lemma 2.2). By the Riemann extension theorem, it can be extended to the $F$-valued $(n,0)$-form on $B_{ij}$ (for which we use same the notation). Further it belongs to $H^0(B_{ij}; K_X \otimes F \otimes \mathcal{I}(h))$ by property (c). Note that we can use property (c) thanks to a special property of $(n,0)$-forms (holomorphic functions).

We define the 1-cocycle $\alpha_{\varepsilon,\delta}$ by

$$\alpha_{\varepsilon,\delta} := \mu \{ \beta_{\varepsilon,\delta,i} \} = \{ \beta_{\varepsilon,\delta,j} - \beta_{\varepsilon,\delta,i} \},$$

where $\mu$ is the coboundary operator. The topology of $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ is induced by the semi-norms $\{ p_K(\bullet) \}_{K \subseteq B_{i_0 \ldots i_q}}$ defined to be

$$p_K^2(\{ \alpha_{i_0 \ldots i_q} \}) := \int_K |\alpha_{i_0 \ldots i_q}|^2_{H,\omega} dV_\omega$$

for every $\{ \alpha_{i_0 \ldots i_q} \} \in C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ and $K \subseteq U_{i_0 \ldots i_q}$. The above integrand is independent of $\omega$ since $\alpha_{i_0 \ldots i_q}$ is an $F$-valued $(n,0)$-form (see Lemma 2.2). Then $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ becomes a Fréchet space with respect to these semi-norms by Lemma 2.14. Then we prove the following claim :

Claim 3.11. There exist subsequences $\{ \varepsilon_k \}_{k=1}^\infty$ and $\{ \delta_\ell \}_{\ell=1}^\infty$ with the following properties:

- $\alpha_{\varepsilon_k,\delta_\ell} \to \alpha_{\varepsilon_0,0}$ in $C^1(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ as $\delta_\ell \to 0$.
- $\alpha_{\varepsilon_k,0} \to \alpha_{0,0}$ in $C^1(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ as $\varepsilon_k \to 0$.

Proof of Claim 3.11. We regard $\alpha_{\varepsilon,\delta,i} := \beta_{\varepsilon,\delta,j} - \beta_{\varepsilon,\delta,i}$ as a holomorphic function on $B_{ij}$. By the construction of $\beta_{\varepsilon,\delta,i}$, the norm $\| \alpha_{\varepsilon,\delta,i} \|_{B_{ij},\varepsilon,\delta}$ can be bounded by a constant $C_{\varepsilon,\delta}$. This implies that the sup-norm $\sup_K |\alpha_{\varepsilon,\delta,i}|$ is also uniformly bounded with respect to $\delta$ for every $K \subseteq B_{ij}$. (Recall the local sup-norm of holomorphic functions can be estimated by the $L^2$-norm). By Montel’s theorem, we can take a subsequence $\{ \delta_\ell \}_{\ell=1}^\infty$ with the first property. Then the norm of the limit $\alpha_{\varepsilon,0}$ can be bounded by a positive constant independent of $\varepsilon$ since $\lim_{\delta \to 0} C_{\varepsilon,\delta}$ is independent of $\varepsilon$. Thus we can take a subsequence $\{ \varepsilon_k \}_{k=1}^\infty$ with the
second properties. The convergence with respect to the local sup-norms implies the convergence with respect to the local $L^2$-norms $\{p_K(\bullet)\}_{K \in \mathcal{B}_{0,q}}$ (for example see [Mat13, Lemma 5.2]). This completes the proof. \hfill \Box

For simplicity, we continue to use the same notation for the subsequence in Claim 3.11.

**Claim 3.12.** The cocycle $\alpha_{\varepsilon,\delta}$ is a coboundary. In particular the limit $\alpha_{0,0}$ is also a coboundary.

**Proof of Claim 3.12.** By Remark 3.10, we can see that $U_{\varepsilon,\delta} = u - u_{\varepsilon,\delta}$ belongs to $\operatorname{Im} \overline{\partial}$ in $L^{n,q}_{(2,\text{loc})}(F)_{\varepsilon,\delta}$. Further, by the isomorphism in Proposition 2.16, we can see that $\alpha_{\varepsilon,\delta}$ is a coboundary. Since we are assuming that $X$ is holomorphically convex (see Step 1), the set of $q$-coboundaries $B^q(U, K_X \otimes F \otimes I(h)) = \operatorname{Im} \mu$ is a Fréchet space by Lemma 2.14. Therefore we obtain the latter conclusion. \hfill \Box

We will construct a solution $\gamma_{\varepsilon,\delta}$ of the $\mu$-equation $\mu \gamma_{\varepsilon,\delta} = \alpha_{\varepsilon,\delta}$ with suitable local $L^2$-norm. The coboundary operator

$$
\mu : C^{q-1}(U, K_X \otimes F \otimes I(h)) \to B^q(U, K_X \otimes F \otimes I(h))
$$

is continuous and surjective between Fréchet spaces, and thus it is an open map by the open mapping theorem. From the latter conclusion of Claim 3.12, there exists $\gamma_{0,0} \in C^0(U, K_X \otimes F \otimes I(h))$ such that $\mu \gamma_{0,0} = \alpha_{0,0}$. For an arbitrary family $K := \{K_i\}_{i \in I_c}$ of relative compact sets $K_i \subset B_i$, the image $\mu(\Delta_K)$ of $\Delta_K$ is an open neighborhood of $\alpha_{0,0}$, where $\Delta_K$ is an open neighborhood of $\gamma_{0,0}$ defined by

$$
\Delta_K := \{\gamma \in C^0(U, K_X \otimes F \otimes I(h)) \mid p_{K_i}(\gamma - \gamma_{0,0}) < 1 \text{ for every } i \in I_c\}.
$$

Since the image $\mu(\Delta_K)$ is an open neighborhood of $\alpha_{0,0}$ and $\alpha_{\varepsilon,\delta}$ converges to $\alpha_{0,0}$, we can take $\gamma_{\varepsilon,\delta} = \{\gamma_{\varepsilon,\delta,i}\} \in \Delta_K$ such that

$$
\{\gamma_{\varepsilon,\delta,j} - \gamma_{\varepsilon,\delta,i}\} = \mu \gamma_{\varepsilon,\delta} = \alpha_{\varepsilon,\delta} = \{\beta_{\varepsilon,\delta,j} - \beta_{\varepsilon,\delta,i}\},
$$

$$
p_{K_i}^2(\gamma_{\varepsilon,\delta}) = \int_{K_i} |\gamma_{\varepsilon,\delta,i}|^2_{H,\omega} dV_{\omega} \leq C_K \text{ for every } i \in I_c
$$

for some positive constant $C_K$ (depending on $K$, $\gamma$ but does not depend on $\varepsilon, \delta$).

Let us construct a solution $w_{\varepsilon,\delta}$ with the properties in Proposition 3.9. We fix a partition of unity $\{\rho_i\}_{i \in I}$. Then, by $\overline{\partial} \gamma_{\varepsilon,\delta,i} = 0$ and $\overline{\partial} \beta_{\varepsilon,\delta,i} = U_{\varepsilon,\delta}$, we have

$$
\overline{\partial} \sum_{k \in I} \rho_k (\gamma_{\varepsilon,\delta,i} - \gamma_{\varepsilon,\delta,k}) = \overline{\partial} \sum_{k \in I} \rho_k \gamma_{\varepsilon,\delta,k},
$$

$$
\overline{\partial} \sum_{k \in I} \rho_k (\beta_{\varepsilon,\delta,i} - \beta_{\varepsilon,\delta,k}) = U_{\varepsilon,\delta} - \overline{\partial} \sum_{k \in I} \rho_k \beta_{\varepsilon,\delta,k}.
$$

When we define $w_{\varepsilon,\delta}$ by

$$
w_{\varepsilon,\delta} := \sum_{k \in I} \rho_k \beta_{\varepsilon,\delta,k} + \sum_{k \in I} \rho_k \gamma_{\varepsilon,\delta,k},
$$
it is easy to check \( U_{\varepsilon,\delta} = \overline{\partial} w_{\varepsilon,\delta} \) by equality (3.11). It remains to estimate the \( L^2 \)-norm of \( w_{\varepsilon,\delta} \). By putting \( K_i \) by \( K_i := \text{Supp} \rho_i \), we may assume that the inequality \( p_{K_i}^2(\gamma_{\varepsilon,\delta}) = \int_{\text{Supp} \rho_i} \left| \gamma_{\varepsilon,\delta} \right|^2_{H,\omega} d\omega \leq C_K \) holds for every \( i \in I_c \) by inequality (3.12). Therefore we obtain

\[
\| \sum_{k \in I} \rho_k \gamma_{\varepsilon,\delta,k} \|^2_{X,\varepsilon,\delta} = \int_{X_c} \sum_{k \in I} \rho_k \gamma_{\varepsilon,\delta,k}^2_{H,\omega} d\omega \leq \sum_{k \in I_c} \int_{B_k \cap \text{Supp} \rho_k} \left| \gamma_{\varepsilon,\delta,k} \right|^2_{H,\omega} d\omega \leq C_K \# I_c.
\]

Note that the cardinality of \( I_c \) is finite by the choice of \( U \). Further, we obtain

\[
\| \sum_{k \in I} \rho_k \beta_{\varepsilon,\delta,k} \|^2_{X,\varepsilon,\delta} = \int_{X_c} \sum_{k \in I} \rho_k \beta_{\varepsilon,\delta,k}^2_{\varepsilon,\delta} d\omega_{\varepsilon,\delta} \leq \sum_{k \in I_c} \int_{B_k} \left| \beta_{\varepsilon,\delta,k} \right|^2_{\varepsilon,\delta} d\omega_{\varepsilon,\delta} \leq C_{\varepsilon,\delta} \# I_c \| u \|_{H,\omega}
\]

for some \( C_{\varepsilon,\delta} > 0 \) by the construction of \( \beta_{\varepsilon,\delta,i} \). These inequalities lead to the desired estimate of \( w_{\varepsilon,\delta} \).

**Proof of Proposition 3.9 for the general case.** For simplicity, we put \( U_{\varepsilon,\delta} := u - u_{\varepsilon,\delta} \in \text{Im} \overline{\partial} \subset L^q_{(2,\text{loc})}(F)_{\varepsilon,\delta} \). Then there exist the \( F \)-valued \( (n,q-k-1) \)-forms \( \beta_{i_0...i_k}^\varepsilon \) on \( B_{i_0...i_k} \setminus Z_\varepsilon \) satisfying

\[
\begin{align*}
\overline{\partial} \{ \beta_{i_0}^\varepsilon \} &= \{ U_{\varepsilon,\delta} |_{B_{i_0} \setminus Z_\varepsilon} \}, \\
\overline{\partial} \{ \beta_{i_0i_1}^\varepsilon \} &= \mu \{ \beta_{i_0i_1}^\varepsilon \}, \\
\overline{\partial} \{ \beta_{i_0i_1i_2}^\varepsilon \} &= \mu \{ \beta_{i_0i_1i_2}^\varepsilon \}, \\
&\vdots \\
\overline{\partial} \{ \beta_{i_0...i_{q-1}}^\varepsilon \} &= \mu \{ \beta_{i_0...i_{q-1}}^\varepsilon \}, \\
\beta_{\varepsilon,\delta}^{i_0...i_k}(U_{\varepsilon,\delta}) &= \mu \{ \beta_{i_0...i_{q-1}}^\varepsilon \},
\end{align*}
\]

Here \( \beta_{i_0...i_k}^\varepsilon \) is the solution of the above equation whose norm is minimum among all the solutions (see the construction of \( f \) in Proposition 2.16). For example, \( \beta_{i_0}^\varepsilon \) is the solution of \( \overline{\partial} \beta_{i_0}^\varepsilon = U_{\varepsilon,\delta} \) on \( B_{i_0} \setminus Z_\varepsilon \) whose norm \( \| \beta_{i_0}^\varepsilon \|_{\varepsilon,\delta} \) is minimum among all the solutions. In particular, we have \( \| \beta_{i_0}^\varepsilon \|^2_{\varepsilon,\delta} \leq C_{\varepsilon,\delta} \| U_{\varepsilon,\delta} \|_{B_{i_0} \setminus Z_\varepsilon} \leq C_{\varepsilon,\delta} \| U_{\varepsilon,\delta} \|_{\varepsilon,\delta}^2 \) for some constant \( C_{\varepsilon,\delta} \) by Lemma 3.8, where \( C_{\varepsilon,\delta} \) is a constant such that \( \lim_{\delta \to 0} C_{\varepsilon,\delta} \) is independent of \( \varepsilon \). Similarly, \( \beta_{i_0i_1}^\varepsilon \) is the solution of \( \overline{\partial} \beta_{i_0i_1}^\varepsilon = (\beta_{i_1}^\varepsilon - \beta_{i_0}^\varepsilon) \) on \( B_{i_0i_1} \setminus Z_\varepsilon \) and the norm \( \| \beta_{i_0i_1}^\varepsilon \|_{\varepsilon,\delta} \) is minimum among all the solutions. In particular, we have \( \| \beta_{i_0i_1}^\varepsilon \|_{\varepsilon,\delta} \leq D_{\varepsilon,\delta} \| (\beta_{i_1}^\varepsilon - \beta_{i_0}^\varepsilon) \|_{\varepsilon,\delta} \) for some constant \( D_{\varepsilon,\delta} \). It is easy to see that

\[
\| \beta_{i_0i_1}^\varepsilon \|_{\varepsilon,\delta} \leq D_{\varepsilon,\delta} \| (\beta_{i_1}^\varepsilon - \beta_{i_0}^\varepsilon) \|_{\varepsilon,\delta} \leq 2C_{\varepsilon,\delta} D_{\varepsilon,\delta} \| U_{\varepsilon,\delta} \|_{\varepsilon,\delta} \leq 2C_{\varepsilon,\delta} D_{\varepsilon,\delta} \| u \|_{H,\omega}.
\]

By repeating this process, we obtain

\[
\| \beta_{i_0...i_k}^\varepsilon \|^2_{i_0...i_k,\varepsilon,\delta} \leq C_{\varepsilon,\delta} \| u \|^2_{H,\omega}
\]

for a constant \( C_{\varepsilon,\delta} \) such that \( \lim_{\delta \to 0} C_{\varepsilon,\delta} \) is independent of \( \varepsilon \). Moreover, by property (c), we obtain

\[
\alpha_{\varepsilon,\delta} := f_{\varepsilon,\delta}(U_{\varepsilon,\delta}) = \mu \{ \beta_{i_0...i_{q-1}}^\varepsilon \} \in C^q(U, K_X \otimes F \otimes I(h_\varepsilon)) = C^q(U, K_X \otimes F \otimes I(h)).
\]

By the same arguments as in Claim 3.11 and Claim 3.12, we obtain the following:
**Claim 3.13.** There exist subsequences \( \{ \varepsilon_k \}_{k=1}^{\infty} \) and \( \{ \delta_k \}_{k=1}^{\infty} \) with the following properties:

- \( \alpha_{\varepsilon_k, \delta_k} \to \alpha_{\varepsilon_k, 0} \) in \( C^q(\mathcal{U}, K_{X} \otimes F \otimes \mathcal{I}(h)) \) as \( \delta_k \to 0 \).
- \( \alpha_{\varepsilon_k, 0} \to \alpha_{0, 0} \) in \( C^q(\mathcal{U}, K_{X} \otimes F \otimes \mathcal{I}(h)) \) as \( \varepsilon_k \to 0 \).

Moreover, the limit \( \alpha_{0, 0} \) belongs to \( B^q(\mathcal{U}, K_{X} \otimes F \otimes \mathcal{I}(h)) \) if \( \mu \).

By the latter conclusion of the claim, there exists \( \gamma_{0, 0} \in C^{q-1}(\mathcal{U}, K_{X} \otimes F \otimes \mathcal{I}(h)) \) such that \( \mu \gamma_{0, 0} = \alpha_{0, 0} \). The coboundary operator

\[
\mu : C^{q-1}(\mathcal{U}, K_{X} \otimes F \otimes \mathcal{I}(h)) \to B^q(\mathcal{U}, K_{X} \otimes F \otimes \mathcal{I}(h)) = \text{Im } \mu
\]

is an open map by the open mapping theorem. For an arbitrary family \( K := \{ K_i \}_{i \in I_c} \) of relative compact sets \( K_i \in B_i \), we define \( \Delta_K \) by (3.10). Then since \( \mu(\Delta_K) \) is an open neighborhood of the limit \( \alpha_{0, 0} \) in \( \text{Im } \mu \), we can obtain \( \gamma_{\varepsilon, \delta} \in C^{q-1}(\mathcal{U}, K_{X} \otimes F \otimes \mathcal{I}(h)) \) such that

\[
\mu \gamma_{\varepsilon, \delta} = \alpha_{\varepsilon, \delta} \quad \text{and} \quad p_{K_{i_0 \ldots i_k}}(\gamma_{\varepsilon, \delta})^2 \leq C_K
\]

for some positive constant \( C_K \). The above constant \( C_K \) depends on the choice of \( K, \gamma \), but does not depend on \( \varepsilon, \delta \).

By the same argument as in [Mat13, Claim 5.11 and Claim 5.13], we can obtain \( F \)-valued \( (n, q-1) \)-forms \( w_{\varepsilon, \delta} \) with the desired properties. The strategy is as follows: The inverse map \( \overline{\frac{\mu}{\mu}} \) of \( \frac{\mu}{\mu} \) is explicitly constructed by using a partition of unity (see Proposition 2.16). It is easy to see that \( g_{\varepsilon, \delta}(\mu \gamma_{\varepsilon, \delta}) = \overline{\partial}v_{\varepsilon, \delta} \) and \( g_{\varepsilon, \delta}(\alpha_{\varepsilon, \delta}) = \partial \overline{v}_{\varepsilon, \delta} \) hold for some \( v_{\varepsilon, \delta} \) and \( \overline{v}_{\varepsilon, \delta} \) by the De Rham-Weil isomorphism. In particular, we have \( U_{\varepsilon, \delta} = \overline{\partial}(v_{\varepsilon, \delta} - \overline{v}_{\varepsilon, \delta}) \) by \( \mu \gamma_{\varepsilon, \delta} = \alpha_{\varepsilon, \delta} \). The important point here is that we can explicitly compute \( v_{\varepsilon, \delta} \) and \( \overline{v}_{\varepsilon, \delta} \) by using the partition of unity \( \beta_{i_0 \ldots i_k}^{\varepsilon, \delta} \) and \( \gamma_{\varepsilon, \delta} \). From this explicit expression, we obtain the \( L^2 \)-estimate for \( v_{\varepsilon, \delta} \) and \( \overline{v}_{\varepsilon, \delta} \). (In the case \( q = 1 \), we have already obtained the \( L^2 \)-estimate.)

See [Mat13, Claim 5.11 and 5.13] for the precise argument.

We close this subsection with the following corollary:

**Corollary 3.14.** For every \( c \) with \( c < \sup_{X} \Phi \), there exist \( v_{\varepsilon, \delta} \in L_{(2)}^{n,q-1}(F^{m+1})_{\varepsilon, \delta} \) with the following property:

- \( \overline{\partial}v_{\varepsilon, \delta} = su_{\varepsilon, \delta} \).
- \( \lim_{\delta \to 0} \| v_{\varepsilon, \delta} \|_{X, \varepsilon, \delta} \) can be bounded by a constant independent of \( \varepsilon \).

**Proof.** Take \( w_{\varepsilon, \delta} \) with the properties in Proposition 3.9. On the other hand, we are assuming that the cohomology class \( sA = \{ su \} \) is zero, and thus there exists \( w \) such that \( \overline{\partial}w = su \) and \( \| w \|_{X, H^{m}} < \infty \). Then \( F \)-valued \( (n, q-1) \)-form \( v_{\varepsilon, \delta} \) defined by \( v_{\varepsilon, \delta} := w - sw_{\varepsilon, \delta} \) satisfies the desired properties by \( \sup_{X} \| s \|_{H^{0}} \leq \sup_{X} | s |_{H^{m}} < \infty \).}

**Step 5** (Asymptotics of norms of differential forms). In this step, for every \( b \) with \( b < \sup_{X} \Phi \), we show that

\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \| su_{\varepsilon, \delta} \|_{X, b, \varepsilon, \delta} = 0.
\]

This completes the proof by Proposition 3.7. For every \( b \) with \( b < \sup_{X} \Phi \), there exists \( c \) such that \( b < c < \sup_{X} \Phi \) and \( d\Phi \neq 0 \) on \( \partial X_c \) since the set of the critical values of \( \Phi \)
has Lebesgue measure zero by Sard’s theorem. Fix such $c$ in this step. We want to apply Proposition 2.5 to $su_{\varepsilon, \delta}$ and $v_{\varepsilon, \delta}$, but we do not know whether $v_{\varepsilon, \delta}$ is smooth on $Y_{\varepsilon}$. For this reason, for given $\varepsilon, \delta > 0$, we take smooth $F$-valued $(n, q - 1)$-forms $\{v_{\varepsilon, \delta, k}\}_{k=1}^{\infty}$ such that $v_{\varepsilon, \delta, k}$ (resp. $\partial v_{\varepsilon, \delta, k}$) converges to $v_{\varepsilon, \delta}$ (resp. $\partial v_{\varepsilon, \delta} = su_{\varepsilon, \delta}$) in the $L^2$-space $L^2((\mathbb{F}^{m+1})_{\varepsilon, \delta})$ (see Lemma 2.4). From now on, we consider only $d(>c)$ satisfying the properties in Proposition 2.5 for countably many differential forms (see Remark 2.6). Then Proposition 2.5 yields

\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \|su_{\varepsilon, \delta}\|_{X_{\varepsilon, \delta}}^2 \leq \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \|su_{\varepsilon, \delta}\|_{X_{\varepsilon, \delta}}^2
\]

\[
= \lim_{\varepsilon \to 0} \lim_{\delta \to 0} k \to \infty \langle su_{\varepsilon, \delta}, \partial \bar{v}_{\varepsilon, \delta, k}\rangle_{X_{\varepsilon, \delta}}
\]

\[
= \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \lim_{k \to \infty} \{ \lim_{k \to \infty} \langle \partial \bar{v}_{\varepsilon, \delta} su_{\varepsilon, \delta}, v_{\varepsilon, \delta, k}\rangle_{X_{\varepsilon, \delta}} + \langle (\bar{\partial} \Phi)^*su_{\varepsilon, \delta}, v_{\varepsilon, \delta, k}\rangle_{\partial X_{\varepsilon, \delta}} \}.
\]

Note that $(\bar{\partial} \Phi)^*$ is the adjoint operator of the wedge product $\bar{\partial} \Phi \wedge \bullet$ with respect to $\omega_{\varepsilon, \delta}$. We will show that the first term (resp. the second term) is zero in Proposition 3.16 (resp. Proposition 2.5). For this purpose, we first prove the following proposition.

**Proposition 3.15.** Under the above situation, we have

\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \|D^*_{\varepsilon, \delta} u_{\varepsilon, \delta}\|_{\varepsilon, \delta} = 0.
\]

Moreover we have

\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \|D^*_{\varepsilon, \delta} su_{\varepsilon, \delta}\|_{\varepsilon, \delta} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \|\partial \bar{v}_{\varepsilon, \delta} su_{\varepsilon, \delta}\|_{\varepsilon, \delta} = 0.
\]

**Proof.** By applying the Bochner-Kodaira-Nakano identity (Proposition 2.3 of the case $\Phi \equiv 0$) to $u_{\varepsilon, \delta}$ and $su_{\varepsilon, \delta}$, we obtain

\[
0 = \|D^*_{\varepsilon, \delta} u_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 + \int_{Y_{\varepsilon}} g_{\varepsilon, \delta} dV_{\varepsilon, \delta},
\]

(3.13) \[
\|\partial \bar{v}_{\varepsilon, \delta} su_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 = \|D^*_{\varepsilon, \delta} su_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 + \int_{Y_{\varepsilon}} |s_{\varepsilon, \delta}|^2 g_{\varepsilon, \delta} dV_{\varepsilon, \delta}.
\]

(3.14) Here we used the equality $\partial \bar{v}_{\varepsilon, \delta} = s_{\varepsilon, \delta} u_{\varepsilon, \delta} = 0$ and the fact that $u_{\varepsilon, \delta}$ is harmonic with respect to $H_{\varepsilon, \omega_{\varepsilon, \delta}}$. The integrand $g_{\varepsilon, \delta}$ is the function defined by $g_{\varepsilon} := \langle \sqrt{-1} \Theta_{H_{\varepsilon, \omega_{\varepsilon, \delta}}}, u_{\varepsilon, \delta}\rangle_{\varepsilon, \delta}$. By property (d) and property (B), we have

\[
\sqrt{-1} \Theta_{H_{\varepsilon}}(F) = \sqrt{-1} \Theta_{H_{\varepsilon}}(F) + \sqrt{-1} \partial \bar{\partial} \chi(\Phi) \geq -\varepsilon \omega \geq -\varepsilon \omega_{\varepsilon, \delta}.
\]

From the above inequalities, we can obtain

\[
\|\partial \bar{v}_{\varepsilon, \delta} su_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 \geq -\varepsilon q \|u_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2.
\]

(For example, see [Mat13, Step 2]). Therefore we obtain

\[
0 \geq \int_{\{g_{\varepsilon, \delta} \leq 0\}} g_{\varepsilon, \delta} dV_{\varepsilon, \delta} \geq -\varepsilon q \int_{\{u_{\varepsilon, \delta} \leq 0\}} |u_{\varepsilon, \delta}|_{\varepsilon, \delta}^2 dV_{\varepsilon, \delta} \geq -\varepsilon q \|u_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 \geq -\varepsilon q \|u\|_{H, \omega}^2.
\]
from inequality (3.6). By equality (3.13), we obtain

$$\|D_{\varepsilon,\delta}^* u_{\varepsilon,\delta}\|_{L^2}^2 + \int_{\{g_{\varepsilon,\delta} \geq 0\}} g_{\varepsilon,\delta} dV_{\varepsilon,\delta} \leq - \int_{\{g_{\varepsilon,\delta} \leq 0\}} g_{\varepsilon,\delta} dV_{\varepsilon,\delta} \leq \varepsilon q \|u\|_{H,\omega}^2.$$  

Hence we can see that

$$\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int_{\{g_{\varepsilon,\delta} \geq 0\}} g_{\varepsilon} dV_{\varepsilon,\delta} = 0$$

and

$$\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \|D_{\varepsilon,\delta}^* u_{\varepsilon,\delta}\|_{L^2} = 0.$$  

On the other hand, by sup\(_X|s|_{h^p} \leq \sup\_X|s|_{h^m} < \infty\), we have

$$\int_{Y_{\varepsilon}} |s|_{h^p}^2 g_{\varepsilon} dV_{\varepsilon,\delta} \leq \int_{\{g_{\varepsilon,\delta} \geq 0\}} |s|_{h^p}^2 g_{\varepsilon,\delta} dV_{\varepsilon,\delta} \leq \sup\_X|s|_{h^m}^2 \int_{\{g_{\varepsilon,\delta} \geq 0\}} g_{\varepsilon,\delta} dV_{\varepsilon,\delta},$$

$$\|D_{\varepsilon,\delta}^* u_{\varepsilon,\delta}\|_{L^2} = \|\partial_{\varepsilon,\delta} u_{\varepsilon,\delta}\|_{L^2} \leq \sup\_X|s|_{h^m} \|D_{\varepsilon,\delta}^* u_{\varepsilon,\delta}\|_{L^2}.$$  

These inequalities and equality (3.14) lead to the conclusion.

**Proposition 3.16.** Under the above situation, we have

$$\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \lim_{k \to \infty} \langle(\overline{\partial}_{\varepsilon,\delta}^* u_{\varepsilon,\delta}, v_{\varepsilon,\delta,k}\rangle_{X_{d,\varepsilon,\delta}} = 0.$$  

**Proof.** Cauchy-Schwarz’s inequality yields

$$\langle(\overline{\partial}_{\varepsilon,\delta}^* u_{\varepsilon,\delta}, v_{\varepsilon,\delta,k}\rangle_{X_{d,\varepsilon,\delta}} \leq \|\overline{\partial}_{\varepsilon,\delta}^* u_{\varepsilon,\delta}\|_{X_{d,\varepsilon,\delta}} \|v_{\varepsilon,\delta,k}\|_{X_{d,\varepsilon,\delta}}.$$  

By the construction of \(v_{\varepsilon,\delta,k}\), we may assume that

$$\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \lim_{k \to \infty} \|v_{\varepsilon,\delta,k}\|_{X_{d,\varepsilon,\delta}} = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \|v_{\varepsilon,\delta}\|_{X_{d,\varepsilon,\delta}}$$

is finite (see Corollary 3.14). On the other hand, the \(L^2\)-norm \(\|\overline{\partial}_{\varepsilon,\delta}^* u_{\varepsilon,\delta}\|_{L^2}\) converges to zero by Proposition 3.15.

We prove the following proposition by using the twisted Bochner-Kodaira-Nakano identity, which completes the proof of Theorem 1.2.

**Proposition 3.17.** Under the above situation, we have

$$\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \lim_{k \to \infty} \langle(\overline{\partial}_{\varepsilon,\delta}^* u_{\varepsilon,\delta}, v_{\varepsilon,\delta,k}\rangle_{\partial X_{d,\varepsilon,\delta}} = 0$$

for almost all \(d\).

**Proof.** Cauchy-Schwarz’s inequality and Hölder’s inequality yield

$$\langle(\overline{\partial}_{\varepsilon,\delta}^* u_{\varepsilon,\delta}, v_{\varepsilon,\delta,k}\rangle_{\partial X_{d,\varepsilon,\delta}} = \int_{\partial X_{d,\varepsilon,\delta}} \langle(\overline{\partial}_{\varepsilon,\delta}^* u_{\varepsilon,\delta}, v_{\varepsilon,\delta,k}\rangle_{\varepsilon,\delta} dS_{\omega_{\varepsilon,\delta}}$$

$$\leq \int_{\partial X_{d,\varepsilon,\delta}} |(\overline{\partial}_{\varepsilon,\delta}^* u_{\varepsilon,\delta})_{\varepsilon,\delta}|_{\varepsilon,\delta} \|v_{\varepsilon,\delta,k}\|_{\varepsilon,\delta} dS_{\omega_{\varepsilon,\delta}}$$

$$\leq \langle(\overline{\partial}_{\varepsilon,\delta}^* u_{\varepsilon,\delta}, (\overline{\partial}_{\varepsilon,\delta}^* u_{\varepsilon,\delta})_{\partial X_{d,\varepsilon,\delta}} (v_{\varepsilon,\delta,k}, v_{\varepsilon,\delta,k})_{\partial X_{d,\varepsilon,\delta}}.$$  


We first show that the limit of \( \langle v_{\epsilon,\delta,k}, v_{\epsilon,\delta,k} \rangle_{D_{X\delta}} \) is finite for almost all \( d \). By Fubini’s theorem and \( dV_{\epsilon,\delta} = d\Phi \wedge dS_{\omega_{\epsilon,\delta}} \), we have

\[
\int_{d \in (c-a,c+a)} \langle v_{\epsilon,\delta,k}, v_{\epsilon,\delta,k} \rangle_{D_{X\delta}} d\Phi = \int_{\{c-a < \Phi < c+a\}} |v_{\epsilon,\delta,k}|^2_{\epsilon,\delta} dV_{\epsilon,\delta}.
\]

Further, by Fatou’s lemma, we have

\[
\int_{d \in (c-a,c+a)} \lim_{\epsilon \to 0} \lim_{\delta \to 0} \int_{k \to \infty} \langle v_{\epsilon,\delta,k}, v_{\epsilon,\delta,k} \rangle_{D_{X\delta}} d\Phi \leq \lim_{\epsilon \to 0} \lim_{\delta \to 0} \int_{d \in (c-a,c+a)} \|v_{\epsilon,\delta}\|^2_{X\delta}\epsilon,\delta}.
\]

We are assuming that the right hand side is finite by Corollary 3.14. Therefore the integrand of the left hand side must be finite for almost all \( d \in (c-a,c+a) \).

Finally we will show that the norm of \( (\nabla\Phi)^* su_{\epsilon,\delta} \) on \( \partial X_d \) converges to zero for almost all \( d \). By \( (\nabla\Phi)^* su_{\epsilon,\delta} = s(\nabla\Phi)^* u_{\epsilon,\delta} \) and sup \( |s|_{H_{\epsilon,\delta}} \leq \sup |s|_{H_{\epsilon,\delta}} < \infty \), we have

\[
\langle (\nabla\Phi)^* su_{\epsilon,\delta}, (\nabla\Phi)^* su_{\epsilon,\delta} \rangle_{D_{X\delta}} \leq \sup_X |s|^2_{H_{\epsilon,\delta}} \langle (\nabla\Phi)^* u_{\epsilon,\delta}, (\nabla\Phi)^* u_{\epsilon,\delta} \rangle_{D_{X\delta}}
\]

Hence it is sufficient to show the norm of \( (\nabla\Phi)^* u_{\epsilon,\delta} \) converges to zero. By applying Proposition 2.5 to \( (\nabla\Phi)^* u_{\epsilon,\delta} \) and \( u_{\epsilon,\delta} \), we obtain

\[
\langle (\nabla\Phi)^* u_{\epsilon,\delta}, u_{\epsilon,\delta} \rangle_{D_{X\delta}} = \langle (\nabla\Phi)^* u_{\epsilon,\delta}, (\nabla\Phi)^* u_{\epsilon,\delta} \rangle_{D_{X\delta}} + \langle (\nabla\Phi)^* u_{\epsilon,\delta}, (\nabla\Phi)^* u_{\epsilon,\delta} \rangle_{D_{X\delta}}
\]

Here we used the equality \( \partial u_{\epsilon,\delta} = 0 \). For the proof, we will compute the left hand side. Note that we have the equalities \( \partial u_{\epsilon,\delta} = 0 \), \( \partial \Phi \wedge u_{\epsilon,\delta} = 0 \) and \( \sqrt{-1} \partial \nabla\Phi \wedge u_{\epsilon,\delta} = 0 \) since \( u_{\epsilon,\delta} \) is a \( \nabla\Phi \)-closed \( F \)-valued \( (n,q) \)-form. Therefore, by Lemma 2.1, we obtain

\[
\langle (\nabla\Phi)^* u_{\epsilon,\delta}, u_{\epsilon,\delta} \rangle_{D_{X\delta}} = -\langle \partial \Phi \wedge (D_{\epsilon,\delta}^* u_{\epsilon,\delta}), u_{\epsilon,\delta} \rangle_{D_{X\delta}} + \langle \sqrt{-1} \partial \nabla\Phi A u_{\epsilon,\delta}, u_{\epsilon,\delta} \rangle_{D_{X\delta}}.
\]

From Lemma 2.2, inequality (3.6), and Cauchy-Schwartz’s inequality, we can estimate the first term of equality (3.6) as follows:

\[
|\langle \partial \Phi \wedge (D_{\epsilon,\delta}^* u_{\epsilon,\delta}), u_{\epsilon,\delta} \rangle_{D_{X\delta}}| \leq \sup |\partial \Phi|_{\omega,\delta} \|D_{\epsilon,\delta}^* u_{\epsilon,\delta} \|_{\omega,\delta} \|u_{\epsilon,\delta}\|_{\epsilon,\delta} \\leq \sup |\partial \Phi|_{\omega} \|D_{\epsilon,\delta}^* u_{\epsilon,\delta} \|_{\epsilon,\delta} \|u\|_{H,\omega}.
\]

The right hand side converges to zero by Proposition 3.15.

To estimate the second term of equality (3.6), by applying Ohsawa-Takegoshi’s twisted Bochner-Kodaira-Nakano identity (Proposition 2.3), we obtain

\[
\|\sqrt{\eta} (\nabla\Phi) u_{\epsilon,\delta} \|_{\epsilon,\delta}^2 = \|\sqrt{\eta} (D_{\epsilon,\delta}^* - (\nabla\Phi)^* ) u_{\epsilon,\delta} \|_{\epsilon,\delta}^2 + \langle \eta \sqrt{-1} (\Theta_{H_{\epsilon}} + \partial \Phi \Lambda) u_{\epsilon,\delta}, u_{\epsilon,\delta} \rangle_{\epsilon,\delta}
\]

\[
\geq \|\sqrt{\eta} (D_{\epsilon,\delta}^* - (\nabla\Phi)^* ) u_{\epsilon,\delta} \|_{\epsilon,\delta}^2 - \varepsilon C \sup_X \eta \|u_{\epsilon,\delta}\|_{\epsilon,\delta}^2 + \langle \sqrt{-1} \partial \Phi \Lambda u_{\epsilon,\delta}, u_{\epsilon,\delta} \rangle_{\epsilon,\delta},
\]

where \( \eta \) is the bounded function defined by \( \eta := e^{\Phi} \). The above inequality follows from inequality (3.15). We compute the first term in the right hand side by using Lemma 2.1
and Cauchy-Schwarz’s inequality. It is easy to check that
\[
\|\sqrt{\eta}(D^*_e u_{e,\delta} - (\partial \Phi)^* u_{e,\delta})\|^2_{e,\delta} \\
\geq \|\sqrt{\eta}D^*_e u_{e,\delta}\|^2_{e,\delta} - 2\|\sqrt{\eta}D^*_e u_{e,\delta}\|_{e,\delta}\|\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}\|_{e,\delta} + \|\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}\|^2_{e,\delta} \\
\geq -2\|\sqrt{\eta}D^*_e u_{e,\delta}\|_{e,\delta}\|\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}\|_{e,\delta} + \|\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}\|^2_{e,\delta}.
\]

On the other hand, Lemma 2.1 implies \(\|\partial \Phi)^* u_{e,\delta}\|^2 = |\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}|^2\), and thus we obtain
\[
\|\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}\|^2_{e,\delta} = \|\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}\|^2_{e,\delta} + \|\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}\|^2_{e,\delta} \\
\geq \|\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}\|^2_{e,\delta}
\]
From these inequalities, we have
\[
\epsilon C \sup_X \eta \|u_{e,\delta}\|^2_{e,\delta} + 2\|\sqrt{\eta}D^*_e u_{e,\delta}\|_{e,\delta}\|\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}\|_{e,\delta} \geq \|\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}\|^2_{e,\delta}
\]

The norm \(\|\sqrt{\eta}D^*_e u_{e,\delta}\|^2_{e,\delta}\) converges to zero by Proposition 3.15 and the norm \(\|\sqrt{\eta}(\partial \Phi)^* u_{e,\delta}\|^2_{e,\delta}\) is uniformly bounded by Lemma 2.2. This completes the proof.

3.2. Proof of Theorem 1.3. In this subsection, we explain how to modify the proof of Theorem 1.2 to obtain Theorem 1.3.

**Theorem 3.18 (Theorem 1.3).** Let \(\pi: X \to \Delta\) be a surjective proper Kähler morphism from a complex manifold \(X\) to an analytic space \(\Delta\). Let \((F, h)\) be a (possibly) singular hermitian line bundle on \(X\) and \((M, h_M)\) be a smooth hermitian line bundle on \(X\). Assume that
\[
\sqrt{-1}\Theta_{h_M}(M) \geq 0 \quad \text{and} \quad \sqrt{-1}(\Theta_h(F) - b\Theta_{h_M}(M)) \geq 0
\]
for some \(b > 0\). Then, for a non-zero (holomorphic) section \(s\) of \(M\), the multiplication map induced by the tensor product with \(s\)
\[
R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h)) \xrightarrow{\otimes s} R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h) \otimes M)
\]
is injective for every \(q\).

**Proof.** The proof is a slight revision of Theorem 1.2. We give only several differences with the proof of Theorem 1.2.

In Step 1, by applying Theorem 2.9 for \(\gamma = b\sqrt{-1}\Theta_{h_M}(M)\), we take a family of singular metrics \(\{h_\epsilon\}_{1\geq \epsilon > 0}\) on \(F\) with the following properties:

(a) \(h_\epsilon\) is smooth on \(X \setminus Z_\epsilon\) for some proper subvariety \(Z_\epsilon\).
(b) \(h_{\epsilon'} \leq h_{\epsilon''} \leq h\) holds on \(X\) for any \(0 < \epsilon' < \epsilon''\).
(c) \(\mathcal{I}(h) = \mathcal{I}(h_\epsilon)\) on \(X\).
(d) \(\sqrt{-1}\Theta_{h_\epsilon}(F) \geq b\sqrt{-1}\Theta_{h_M}(M) - \epsilon\omega\) on \(X\).

Note that property (d) is obtained from the assumption \(\sqrt{-1}\Theta_h(F) \geq b\sqrt{-1}\Theta_{h_M}(M)\). We can see that property (e) is stronger than property (d) in the proof of Theorem 1.2. Indeed, by the assumption \(\sqrt{-1}\Theta_{h_M}(M) \geq 0\), we obtain property (d)
\[
(d) \sqrt{-1}\Theta_{h_\epsilon}(F) \geq -\epsilon\omega \text{ on } X.
\]
By property (d), we can see that the same argument as in Step 2 works.

In Step 3, by considering the norm \( \| su_{\varepsilon, \delta} \|_{H^{\varepsilon}_{h, h_M}, H^{m}_{\omega_{\varepsilon, \delta}}} \) instead of \( \| su_{\varepsilon, \delta} \|_{H^{h_{\varepsilon}}_{h, h_M}, H^{m}_{\omega_{\varepsilon, \delta}}} \), we can easily prove the same conclusion as in Proposition 3.7.

In Step 4, we can obtain \( v_{\varepsilon} \in L^{n,q-1}_{(2)}(F \otimes M)_{\varepsilon, \delta} \) with the properties Corollary 3.14, since we do not use the line bundle \( M \) when we prove Proposition 3.9.

In Step 5, we need to prove the following proposition (see Proposition 3.15). Recall that Proposition 3.16 and Proposition 3.17 finish the proof of Theorem 1.2 and they are obtained from Proposition 3.15.

**Proposition 3.19** (cf. Proposition 3.15). We have

\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \| D^{\ast}_{\varepsilon, \delta} u_{\varepsilon, \delta} \|_{\varepsilon, \delta} = 0.
\]

Moreover we have

\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \| D^{\ast}_{\varepsilon, \delta} su_{\varepsilon, \delta} \|_{\varepsilon, \delta} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \| \partial^{\ast}_{\varepsilon, \delta} su_{\varepsilon, \delta} \|_{\varepsilon, \delta} = 0.
\]

**Proof of Proposition 3.19.** By applying the Bochner-Kodaira-Nakano identity to \( u_{\varepsilon, \delta} \) and \( su_{\varepsilon, \delta} \), we obtain the following equalities:

\[
0 = \| D^{\ast}_{\varepsilon, \delta} u_{\varepsilon, \delta} \|_{\varepsilon, \delta}^2 + \int_{Y_{\varepsilon}} g_{\varepsilon, \delta} dV_{\varepsilon, \delta},
\]

\[
\| \partial^{\ast}_{\varepsilon, \delta} su_{\varepsilon, \delta} \|_{\varepsilon, \delta}^2 = \| D^{\ast}_{\varepsilon, \delta} su_{\varepsilon, \delta} \|_{\varepsilon, \delta}^2 + \int_{Y_{\varepsilon}} \left| s \right|^2_{h_M} (f_{\varepsilon, \delta} + g_{\varepsilon, \delta}) dV_{\varepsilon, \delta},
\]

where the integrands \( g_{\varepsilon, \delta} \) and \( f_{\varepsilon, \delta} \) are the functions defined by

\[
g_{\varepsilon, \delta} := \langle \sqrt{-1} \Theta_{H_{\varepsilon}}(F) \Lambda_{\varepsilon, \delta} u_{\varepsilon, \delta}, u_{\varepsilon, \delta} \rangle_{\varepsilon, \delta},
\]

\[
f_{\varepsilon, \delta} := \langle \sqrt{-1} \Theta_{h_M}(M) \Lambda_{\varepsilon, \delta} u_{\varepsilon, \delta}, u_{\varepsilon, \delta} \rangle_{\varepsilon, \delta}.
\]

Since we have property (d), we obtain

\[
(3.17) \quad g_{\varepsilon} \geq -\varepsilon q |u_{\varepsilon, \delta}|^2_{\varepsilon, \delta}.
\]

By the same argument as in Proposition 3.15, we can see that

\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \| D^{\ast}_{\varepsilon, \delta} u_{\varepsilon, \delta} \|_{\varepsilon, \delta}^2 = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int_{\{ g_{\varepsilon, \delta} \geq 0 \}} g_{\varepsilon, \delta} dV_{\varepsilon, \delta} = 0.
\]

Therefore we can see that \( \| D^{\ast}_{\varepsilon, \delta} su_{\varepsilon, \delta} \|_{\varepsilon, \delta} = \| s D^{\ast}_{\varepsilon, \delta} u_{\varepsilon, \delta} \|_{\varepsilon, \delta} \) converges to zero from \( \sup_X |s|_{h_M} < \infty \). On the other hand, from property (e), we can easily check

\[
f_{\varepsilon, \delta} \leq \frac{1}{b} (g_{\varepsilon, \delta} + \varepsilon q |u_{\varepsilon, \delta}|^2_{\varepsilon, \delta}).
\]
This implies that
\[ \int_{Y} |s|_{h_{M}}^{2}(f_{\varepsilon, \delta} + g_{\varepsilon, \delta}) \, dV_{\varepsilon, \delta} \leq \int_{Y} |s|_{h_{M}}^{2} \left\{ (1 + \frac{1}{b})g_{\varepsilon, \delta} + \frac{\varepsilon q}{b} |u_{\varepsilon, \delta}|_{\varepsilon, \delta}^{2} \right\} \, dV_{\varepsilon, \delta} \]
\[ \leq \int_{\{g_{\varepsilon, \delta} \geq 0\}} |s|_{h_{M}}^{2} \left\{ (1 + \frac{1}{b})g_{\varepsilon, \delta} + \frac{\varepsilon q}{b} |u_{\varepsilon, \delta}|_{\varepsilon, \delta}^{2} \right\} \, dV_{\varepsilon, \delta} \]
\[ \leq \sup_{X} |s|_{h_{M}}^{2} (1 + \frac{1}{b}) \int_{\{g_{\varepsilon, \delta} \geq 0\}} g_{\varepsilon, \delta} \, dV_{\varepsilon, \delta} + \frac{\varepsilon q}{b} \|u\|_{H, \omega}^{2}. \]

This completes the proof. \(\square\)

By this proposition, we can prove the same conclusion as in Proposition 3.16 and Proposition 3.17. Therefore we obtain the conclusion. \(\square\)

4. Applications

4.1. Proof of Corollary 1.5. In this subsection, we prove Corollary 1.5.

**Corollary 4.1** (Corollary 1.5). Let \( \pi : X \to \Delta \) be a surjective proper Kähler morphism from a complex manifold \( X \) to an analytic space \( \Delta \), and \( (F, h) \) be a (possibly) singular hermitian line bundle on \( X \) with semi-positive curvature. Then the higher direct image sheaf \( R^{q}\pi_{*}(K_{X} \otimes F \otimes \mathcal{I}(h)) \) is torsion free for every \( q \). Moreover, we obtain
\[ R^{q}\pi_{*}(K_{X} \otimes F \otimes \mathcal{I}(h)) = 0 \quad \text{for every} \quad q > \dim X - \dim \Delta. \]

**Proof.** We apply Theorem 1.2 in the case of \( m = 0 \) to a holomorphic function \( s \). For an open set \( B \subset \Delta \) and a holomorphic function \( s \) on \( \pi^{-1}(B) \), the multiplication map
\[ \Phi_{s} : R^{q}\pi_{*}(K_{X} \otimes F \otimes \mathcal{I}(h)) \otimes_{\mathcal{O}} R^{q}\pi_{*}(K_{X} \otimes F \otimes \mathcal{I}(h)) \]
is injective for every \( q \). This implies that \( R^{q}\pi_{*}(K_{X} \otimes F \otimes \mathcal{I}(h)) \) is torsion free. \(\square\)

4.2. Proof of Theorem 1.7. In this subsection, we prove Theorem 1.7. We first recall the definition of the numerical Kodaira dimension of singular hermitian line bundles on projective varieties (see [Cao14] for Kähler manifolds).

**Definition 4.2** (Numerical Kodaira dimension, [Cao14]). Let \( (F, h) \) be a singular hermitian line bundle on a smooth projective variety \( X \) such that \( \sqrt{-1} \Theta_{h}(F) \geq 0 \). Then the numerical Kodaira dimension \( \text{nd}(F, h) \) is defined to be \( \text{nd}(F, h) := -\infty \) if \( h \equiv \infty \), otherwise
\[ \text{nd}(F, h) := \dim X - \lim_{\varepsilon \to 0} \frac{\log \text{vol}_{X}(A^{\varepsilon} \otimes F, h)}{\log \varepsilon} \]
where \( \text{vol}_{X}(A^{\varepsilon} \otimes F, h) \) is defined by
\[ \text{vol}_{X}(A^{\varepsilon} \otimes F, h) := \lim_{m \to \infty} \frac{h^{0}(X, A^{m\varepsilon} \otimes F^{m} \otimes \mathcal{I}(h^{m}))}{m^{\dim X}}. \]

By combining Cao’s result in [Cao14] with the openness theorem proved by Guan-Zhou in [GZ15], we have the following vanishing theorem. (See [Hie14] and [Lem14] for another proof for the openness theorem.)
**Theorem 4.3** ([Cao14, Theorem 1.3], [GZ15, Theorem 1.1]). Let $(F, h)$ be a singular hermitian line bundle on a compact Kähler manifold $X$ such that $\sqrt{-1}\Theta_h(F) \geq 0$. Then we have

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) = 0 \quad \text{for every } q > \dim X - \text{nd}(F, h).$$

We first prove Proposition 1.6.

**Proposition 4.4** (Proposition 1.6). Let $\pi : X \to \Delta$ be a surjective projective morphism from a complex manifold $X$ to an analytic space $\Delta$, and $(F, h)$ be a (possibly) singular hermitian line bundle on $X$ with semi-positive curvature. Assume that $\pi$ is smooth at a point $t_0 \in \Delta$. Then, there exists a dense subset $Q \subset B$ in some open neighborhood $B$ of $t_0$ with the following properties:

For every $t \in Q$, we have $\text{nd}(F|_{X_t}, h|_{X_t}) \geq \text{nd}(F|_{X_{t_0}}, h|_{X_{t_0}})$.

**Remark 4.5.** By the proof of Proposition 1.6, we can add the property that $I(h^m|_{X_t}) = I(h^m)|_{X_t}$ for every $t \in Q$.

**Proof.** For a positive integer $m$, we define $Q_m$ by

$$Q_m := \{ t \in \Delta | I(h^m|_{X_t}) = I(h^m)|_{X_t} \}.$$

Note that we have $I(h^m|_{X_t}) \subset I(h^m)|_{X_t}$ by the Ohsawa-Takegoshi $L^2$-extension theorem. By Fubini’s theorem, we can see that $\Delta \setminus Q_m$ has zero Lebesgue measure. We put $Q := \cap_{m=1}^{\infty} Q_m$. Then $\Delta \setminus Q$ also has zero Lebesgue measure. Let $A$ be a relatively ample line bundle $A$ on $X$. By the definition of the numerical dimension, it is sufficient to show that

$$h^0(X_t, \mathcal{O}_{X_t}(A^{\mathrm{me}} \otimes F^m) \otimes I(h^m|_{X_t})) \geq h^0(X_{t_0}, \mathcal{O}_{X_{t_0}}(A^{\mathrm{me}} \otimes F^m) \otimes I(h^m|_{X_{t_0}}))$$

for every $t \in Q$ and $m \gg 0$.

For the canonical bundle $K_X$ on $X$, we have

$$A^{\mathrm{me}} \otimes F^m = K_X \otimes (A^{\mathrm{me}} \otimes K_X^{-1}) \otimes F^m.$$

$A^{\mathrm{me}} \otimes K_X^{-1}$ admits a smooth (hermitian) metric $g_m$ with positive curvature for a sufficiently large $m \gg 0$. We can extend a basis $\{ s_i \}_{i \in I}$ in $H^0(X_{t_0}, \mathcal{O}_{X_{t_0}}(A^{\mathrm{me}} \otimes F^m) \otimes I(h^m|_{X_{t_0}})$ to sections $\{ \tilde{s}_i \}_{i \in I}$ in $H^0(X_t, \mathcal{O}_X(A^{\mathrm{me}} \otimes F^m) \otimes I(h^m))$, by applying the Ohsawa-Takegoshi $L^2$ extension theorem to $(A^{\mathrm{me}} \otimes K_X^{-1} \otimes F^m, g_m h^m)$ (see [OT87] and [Man93]).

We can easily see that $\{ \tilde{s}_i|_{X_t} \}_{i \in I}$ is linearly independent in $H^0(X_t, \mathcal{O}_{X_t}(A^{\mathrm{me}} \otimes F^m) \otimes I(h^m)|_{X_t})$ for every $t$ in some open neighborhood of $B_{t_0}$. Indeed, if there exist a point $t$ converging to $t_0$ and $a_{t,i} \in \mathbb{C}$ such that $\sum_{i \in I} a_{t,i} \tilde{s}_i|_{X_t} = 0$, then we may assume that $a_{t,i}$ converges to some $a_i$ as $t \to t_0$. As $t$ tends to $t_0$, we obtain $\sum_{i \in I} a_i \tilde{s}_i|_{X_{t_0}} = 0$ from $\sum_{i \in I} a_{t,i} \tilde{s}_i|_{X_{t_0}} = 0$. Therefore $\{ \tilde{s}_i|_{X_t} \}_{i \in I}$ is linearly independent. If $t \in Q$, the restriction $\tilde{s}_i|_{X_t}$ to $X_t$ is a section in $H^0(X_t, \mathcal{O}_{X_t}(A^{\mathrm{me}} \otimes F^m) \otimes I(h^m|_{X_t}))$. This completes the proof. □

**Theorem 4.6** (Theorem 1.7). Let $\pi : X \to \Delta$ be a surjective projective morphism from a complex manifold $X$ to an analytic space $\Delta$, and $(F, h)$ be a (possibly) singular hermitian line bundle on $X$ with semi-positive curvature. Then we have

$$R^p \pi_* (K_X \otimes F \otimes \mathcal{I}(h)) = 0 \quad \text{for every } q > \max_{t \in \Delta} \text{nd}(F|_{X_t}, h|_{X_t}).$$
where \( f \) is the dimension of general fibers. In particular, if \((F|_{x_t}, h|_{x_t})\) is big for some point \( t \) in the smooth locus of \( \pi \), then we have

\[
R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h)) = 0 \quad \text{for every } q > 0.
\]

Proof. We take a point \( t_0 \) with

\[
\text{nd}(F|_{x_{t_0}}, h|_{x_{t_0}}) = \max_{\pi \text{ is smooth at } t \in \Delta} \text{nd}(F|_{x_t}, h|_{x_t}).
\]

By Proposition 1.6 and Remark 4.5, we can take a dense subset \( Q \) in some neighborhood \( B \) of \( t_0 \) such that \( \text{nd}(F|_{x_t}, h|_{x_t}) \geq \text{nd}(F|_{x_{t_0}}, h|_{x_{t_0}}) \) and \( \mathcal{I}(h|_{X|_{t_0}}) = \mathcal{I}(h^m)|_{X_t} \) for every \( t \in Q \). Therefore, By Cao’s result and the openness theorem (see Theorem 4.3), we obtain

\[
H^q(X_t, \mathcal{O}_{X_t}(K_X \otimes F) \otimes \mathcal{I}(h|_{X_t})) = 0
\]

for \( q > f - \text{nd}(F|_{x_{t_0}}, h|_{x_{t_0}}) \geq f - \text{nd}(F|_{x_t}, h|_{x_t}) \) and for every \( t \in Q \cap \Delta' \). Here \( \Delta' \) is the Zariski open set in \( \Delta \) defined by

\[
\Delta' := \{ t \in \Delta | \pi \text{ is smooth at } t \text{ and } R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h)) \text{ is locally free at } t \}.
\]

By the flat base change theorem, we obtain \( R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h))_t = 0 \) for every \( t \in Q \cap \Delta' \). This implies that \( R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h))_t = 0 \) on \( \Delta' \). We obtain the conclusion since \( R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h)) \) is torsion free.

\[\square\]

4.3. Proof of Corollary 1.9. Finally, we prove Corollary 1.9.

Corollary 4.7 (Corollary 1.9). Let \( \pi: X \rightarrow \Delta \) be a surjective proper Kähler morphism from a complex manifold \( X \) to an open disk \( \Delta \subset \mathbb{C} \) and \((F, h)\) be a singular hermitian line bundle with semi-positive curvature. Then every section in \( H^0(X_0, \mathcal{O}_{X_0}(K_X \otimes F)) \) that comes from \( H^0(X_0, \mathcal{O}_{X_0}(K_X \otimes F) \otimes \mathcal{I}(h)) \) can be extended to a section in \( H^0(X, \mathcal{O}_X(K_X \otimes F) \otimes \mathcal{I}(h)) \) by replacing \( \Delta \) with a smaller disk. In particular, if \( K_X \) admits a singular metric \( h \) whose curvature is semi-positive and Lelong number is zero at every point in \( X_0 \), then Problem 1.8 is affirmatively solved.

Proof. Let \( s \) be a holomorphic function on \( X \) with \( X_0 = s^{-1}(0) \). By Theorem 1.2, we can conclude

\[
H^1(X, K_X \otimes F \otimes \mathcal{I}(h)) \xrightarrow{\cong} H^1(X, K_X \otimes F \otimes \mathcal{I}(h))
\]

is injective for a sufficiently small \( \Delta \). On the other hand, since \( X_0 \) is a subvariety of codimension one and \( R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h)) \) is torsion free, the following sequence is exact:

\[
0 \rightarrow \mathcal{O}_X(K_X \otimes F \otimes \mathcal{I}(h)) \otimes \mathcal{I}_{X_0} \rightarrow \mathcal{O}_X(K_X \otimes F \otimes \mathcal{I}(h)) \rightarrow \mathcal{O}_{X_0}(K_X \otimes F \otimes \mathcal{I}(h)) \rightarrow 0.
\]

The induced long exact sequence implies that for every section \( t \) in \( H^0(X_0, \mathcal{O}_{X_0}(K_X \otimes F \otimes \mathcal{I}(h))) \), there exists a section \( T \) in \( H^0(X, \mathcal{O}_X(K_X \otimes F \otimes \mathcal{I}(h))) \) such that \( T|_{X_0} = t \). Further
we have the following commutative diagram:

\[
\begin{array}{ccc}
H^0(X, \mathcal{O}_X(K_X \otimes F \otimes \mathcal{I}(h))) & \longrightarrow & H^0(X_0, \mathcal{O}_{X_0}(K_X \otimes F \otimes \mathcal{I}(h))) \\
\downarrow & & \downarrow \\
H^0(X, \mathcal{O}_X(K_X \otimes F)) & \longrightarrow & H^0(X_0, \mathcal{O}_{X_0}(K_X \otimes F)).
\end{array}
\]

Therefore we can extend a section in \(H^0(X, \mathcal{O}_X(K_X \otimes F))\) that comes from \(H^0(X_0, \mathcal{O}_{X_0}(K_X \otimes F \otimes I(h)))\) to \(X\).

\[\square\]

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