UNEXPECTED CURVES ARISING FROM SPECIAL LINE ARRANGEMENTS

MICHELA DI MARCA, GRZEGORZ MALARA, AND ALESSANDRO ONETO

ABSTRACT. In a recent paper [CMN17], Cook II, Harbourne, Migliore and Nagel related the splitting type of a line arrangement in the projective plane to the number of conditions imposed by a general fat point of multiplicity $j$ to the linear system of curves of degree $j+1$ passing through the configuration of points dual to the given arrangement. If the number of conditions is less than the expected, we say that the configuration of points admits unexpected curves. In this paper, we characterize supersolvable line arrangements whose dual configuration admits unexpected curves and we provide other infinite families of line arrangements with this property.

1. INTRODUCTION

Polynomial interpolation problems are among the most studied topics in algebraic geometry. A classical example deals with computing dimensions of linear systems of curves of given degree passing through a given set of points in the projective plane. In other words, on the space of coefficients of ternary homogeneous polynomials of degree $j$, we consider the system of linear equations given by imposing the vanishing at a set of $d$ points and we want to study the dimension of its solution. If the points are in general position, we may assume that this system of linear equations has maximal rank and the dimension of the solution is as small as possible, i.e., it is equal to $\binom{j+2}{2} - d$, unless this difference is negative, in which case it is zero [GO81].

If we consider points with some multiplicity, usually called fat points, where we require that the partial derivatives of the polynomial up to some order vanish at the points, the problem becomes much more complicated and only poorly understood. In other words, we consider a polynomial interpolation problem where we look at plane curves having singularities of certain order at a set of points. A complete answer is not known, even for points in general position.

Here is an example where the solution is not as expected. Consider the space of plane quartics, which has dimension $\binom{4+2}{2} = 15$, and consider a scheme of five double points in general position, i.e., we consider the linear system of plane quartics having five singularities at general points. Imposing a singularity at a point provides three linear equations, i.e., the vanishing of the three partial derivatives. Therefore, we have a system of 15 linear equations on the space of plane quartics and we expect to have no quartics with five general singularities. However, through five general points there exists always a conic and, therefore, the double conic is an unexpected quartic singular at every point and, in particular, at the set of five general points.

The case up to nine general points goes back to Castelnuovo and it can also be found in the work of Nagata [Nag60]. In the 1980s, Harbourne [Har86], Gimigliano [Gim87] and Hirschowitz [Hir89] independently gave conjectures on the dimension of a linear system of plane curves of given degree and with multiple general base points. In [CM01], these conjectures have been proved to be all equivalent to an older conjecture by B. Segre [Seg61] and, for this reason, we refer to them as the SHGH Conjecture.
In a recent paper [CHMN17], Cook II, Harbourne, Migliore and Nagel slightly changed the question. Instead of counting the number of linear conditions given by a set of general multiple points to the complete linear system of plane curves of given degree, as in the classical problem, they look at the conditions imposed by a general fat point to the linear system of plane curves of given degree and passing through some particular configuration of reduced points.

This new question was motivated by previous works. Faenzi and Vallès noticed the relation between the splitting type of a line arrangement and curves passing through the set of points dual to the line arrangement and a fat point of multiplicity one less than the degree of the curve [FV14]. Afterwards, Di Gennaro, Iarlori and Vallès gave an example of configuration of points admitting an unexpected curve [DIV14 Proposition 7.3]. We recall it in Example 1.2. Actually, in [DIV14], the authors were studying Lefschetz properties of power ideals, i.e., ideals generated by powers of linear forms. In [CHMN17], the authors formalize the relation between Lefschetz properties of power ideals and the existence of unexpected curves for the configuration of points dual to the linear forms that define the power ideal.

In particular, in [CHMN17], the authors gave a characterization of the existence of unexpected curves for a given set of point in terms of the splitting type of the dual line arrangement. It is worth mentioning, that they also relate this problem to the famous Terao’s Conjecture which claims that freeness of a hyperplane arrangement depends only on the incidence lattice of the arrangement. In particular, they show that, if the splitting types depend only on the combinatorics of the arrangement, or equivalently if the existence of unexpected curves depends only on the combinatorics of the configuration of the points, then Terao’s Conjecture holds [CHMN17 Corollary 7.11].

In this paper, we characterize supersolvable line arrangements whose dual configuration of points admits unexpected curves. We also present several infinite families of line arrangements having this unexpected behavior by computing their splitting types. These families generalize examples from [CHMN17, DIV14].

**Formulation of the problem.** Let $S = \mathbb{C}[x_0, x_1, x_2] = \bigoplus_{i \geq 0} S_i$ be the standard graded ring of polynomials with complex coefficients, i.e., $S_i$ is the $\mathbb{C}$-vector space of homogeneous polynomials of degree $i$. Any homogeneous ideal $I$ inherits the grading, i.e., $I = \bigoplus_{i \geq 0} I_i$, where $I_i = I \cap S_i$.

The fat point of multiplicity $j$ and support at $P \in \mathbb{P}^2$ is the 0-dimensional scheme defined by the $j$-th power $\rho^j$ of the ideal $\rho$ defining the point $P$. We denote it by $jP$. Observe that, a homogeneous polynomial $f \in S$ belongs to $\rho^j$ if and only if all partial derivatives of $f$ of order $j - 1$ vanish at $P$. This gives $\left(\begin{array}{c} j+1 \\ 2 \end{array}\right)$ linear equations, which justifies the following definition.

**Definition 1.1.** Let $Z = P_1 + \ldots + P_s$ be a set of reduced points in $\mathbb{P}^2$. We say that $Z$ admits **unexpected curves of degree** $j+1$ if, for a general point $Q \in \mathbb{P}^2$, we have that

$$\dim_{\mathbb{C}}[I(Z + jQ)]_{j+1} > \max \left\{ \dim_{\mathbb{C}}[I(Z)]_{j+1} - \left(\begin{array}{c} j+1 \\ 2 \end{array}\right), 0 \right\},$$

where $I(Z + jQ) = I(Z) \cap I(Q)^j$.

The general problem in this theory is the following.

**Problem A.** Classify all configurations of points $Z$ that admit unexpected curves.

If $Z$ has general support, then it is well known that there are no unexpected curves of any degree. The following is an example coming from [DIV14] of 9 points which admits an unexpected quartic (see also [Har17, Example 4.1.10]).

**Example 1.2.** The configuration is constructed, step by step, as follows (see Figure 7). Consider four general points in the projective plane (the black dots). Then, there are three pairs of lines that contain all four points.
Each pair has a singular point (the three dotted circles). Then, draw the line through two of these three points (the dotted line) and take the two points (the two white circles) where this line intersects the pair of lines whose singular point is the third point. This gives five additional points. The space of quartics passing through this configuration of points is 6-dimensional, therefore, we expect to have no quartics with an additional general triple point (denoted as the two concentric blue circles). However, there exists an unexpected quartic. See Example 3.1 for an explicit construction in projective coordinates.

In [FGST18], Farnik, Galuppi, Sodomaco and Trok show that this is, up to isomorphism, the only example of a configuration of points in the projective plane admitting an unexpected quartic.

The configuration described in the example has a very special combinatorics in relation to the $B_3$ arrangement (see [OT92, Example 1.7]). We describe it in more detail in the next section, see Example 3.1. In [CHMN17], the authors connect the existence of unexpected curves for a configuration of points to the computation of the splitting type of the dual line arrangement $\mathcal{A}_Z$ whose lines are defined by the linear equations having as coefficients the coordinates of the points in $Z$. We explain later in more detail these connection, but, in order to mention one of the main results in [CHMN17] for the reader already familiar with these combinatorial concepts, a necessary condition for a set of points $Z$ to admit an unexpected curve in degree $j+1$ is that $a_Z \leq j \leq b_Z - 2$, where $(a_Z, b_Z)$ is the splitting type of $\mathcal{A}_Z$ [CHMN17, Theorem 1.5].

In this paper, we generalize Example 1.2 to infinite families of configurations having unexpected curves. In particular, while studying the problem, we noticed that the configuration given in the Example 1.2 is the dual configuration of points to an arrangement of lines described in a paper of Grünbaum [Grü09] where the author explains particular families of (real) line arrangements. After some experiments with the algebra software Macaulay2 [GS] and Singular [DGPS], Grünbaum’s paper inspired us to find the examples we describe in this paper. The families of line arrangements that we consider here are simplicial, i.e., arrangements of lines where every cell is a triangle, or near-simplicial, i.e., sometimes we also have quadrilateral cells. As nicely explained in Grünbaum’s paper, these arrangements occur in the literature as
examples and counterexamples in many contexts of algebraic combinatorics and its applications. In this case, we related them to a new interesting question on polynomial interpolation for plane curves.

**Structure of the paper.** In Section 2 we recall the basic notions and constructions of algebraic geometry and combinatorics that we need to analyse the problem. In Section 3 we consider particular families of line arrangements that give unexpected curves. In Section 4 we provide more sporadic examples of line arrangements whose dual configurations have unexpected curves, but that we could not extend these to a general class of examples.

**Acknowledgements.** This project started during the “2017 Pragmatic Summer School: Powers of ideals and ideals of powers” which was held at the University of Catania, Italy (June 19th - July 7th, 2017). We are grateful to the organizers (Afio Ragusa, Elena Guardo, Francesco Russo and Giuseppe Zappalà) and the teachers (Brian Harbourne, Adam Van Tuyl, Enrico Carlini and Tài Hà) of the school. In particular, we want to thank Brian Harbourne for suggesting and supervising this project and for useful comments on an early version of this paper. We also want to thank Michael Cuntz for sharing with us a database of crystallographic simplicial arrangements. The first author was partially supported by the "National Group for Algebraic and Geometric Structure, and their Applications" (GNSAGA-INdAM). The second author was partially supported by National Science Centre, Poland, grant 2016/21/N/ST1/01491. The third author was partially supported by the Aromath team of INRIA Sophia Antipolis Méditerranée (France).

2. Basic notions and constructions

In this section, we describe the main combinatorial objects we want to consider. For more details, we refer to the classical textbook on hyperplane arrangements by Orlik and Terao [OT92].

**Dual line arrangement.** Given a configuration of reduced points \( Z = P_1 + \ldots + P_d \subset \mathbb{P}^2 \), we consider the arrangement \( \mathcal{A}_Z \) of dual lines \( L_1, \ldots, L_d \) in the dual space \((\mathbb{P}^2)^\vee\). More precisely, if \( P_i = (p_{i,0} : p_{i,1} : p_{i,2}) \), for any \( i = 1, \ldots, d \), then we define the line \( L_i := \{p_{i,0}y_0 + p_{i,1}y_1 + p_{i,2}y_2 = 0\} \), where \( T = \mathbb{C}[y_0, y_1, y_2] \) is the coordinate ring of the dual plane. Moreover, if \( \ell_i \in T_1 \) is the linear form defining the line \( L_i \), for any \( i = 1, \ldots, d \), the arrangement \( \mathcal{A}_Z \) is defined by the polynomial \( f_\mathcal{A} = \ell_1 \cdots \ell_d \in \mathcal{O}_T \).

**Remark 2.1.** When we say that a line arrangement admits unexpected curves we implicitly mean that the dual configuration of points admits unexpected curves, as defined in Definition 1.1.

**Splitting type of line arrangements.** Let \( \mathcal{A} \) be a line arrangement of \( d \) lines and let \( f_{\mathcal{A}} \in T_3 \) be the polynomial of degree \( d \) defining it. We consider the map defined by the gradient \( \nabla f_{\mathcal{A}} = \partial f_{\mathcal{A}} \) as \( \mathcal{O}_2^3 \to \mathcal{O}_2^2(d-1) \).

We call the kernel of such a map the **derivation bundle** of \( \mathcal{A} \), i.e., the rank 2 vector bundle \( \mathcal{D}_{\mathcal{A}} \) defined by

\[
0 \to \mathcal{D}_{\mathcal{A}} \to \mathcal{O}_2^3 \xrightarrow{J_{\mathcal{A}}} \mathcal{O}_2^2(d-1),
\]

Up to a twist, the derivation bundle is isomorphic to the **syzygy bundle** of the Jacobian ideal of \( f_{\mathcal{A}} \), i.e., the ideal \( J_{\mathcal{A}} = (\partial y_0 f_{\mathcal{A}}, \partial y_1 f_{\mathcal{A}}, \partial y_2 f_{\mathcal{A}}) \) generated by the first partial derivatives of the polynomial \( f_{\mathcal{A}} \).

**Definition 2.2.** A line arrangement \( \mathcal{A} \) is said to be **free with exponents** or splitting type, \((a_{\mathcal{A}}, b_{\mathcal{A}})\) if \( \mathcal{D}_{\mathcal{A}} \) is free, i.e., if it splits as \( \mathcal{D}_{\mathcal{A}} = \mathcal{O}_2^3(−a_{\mathcal{A}}) \oplus \mathcal{O}_2^2(−b_{\mathcal{A}}) \).
In general, the restriction of the derivation bundle on any line $\ell$ splits as $\mathcal{D}_{\mathcal{A}}|\ell = \mathcal{O}_P(-a) \oplus \mathcal{O}_P(-b)$. The splitting type $(a, b)$ is constant on a Zariski open subset of the dual projective plane, i.e., it is constant on a general line. We call this the splitting type of $\mathcal{A}$ when the arrangement is not free. For details on these facts, we refer to \cite{CHMN17} Appendix.

If $\mathcal{A}$ is a free line arrangement, then we have that the resolution of $S/J_{\mathcal{A}}$ has length 2. In particular, the resolution is

$$0 \rightarrow T(-(d-1)-a) \oplus T(-(d-1)-b) \rightarrow T(-(d-1))^3 \rightarrow T \rightarrow T/J_{\mathcal{A}} \rightarrow 0,$$

where $a, b \in \mathbb{N}$ satisfy $a + b = d - 1$.

Remark 2.3. The fact that the characteristic of the field does not divide $\deg(f_{\mathcal{A}})$ is crucial for this construction. The notion of a free line arrangement can be given more generally for any characteristic, but it is more complicated and it is not needed for the purposes of this paper. For this reason, in order to make the exposition clearer, we decided to give a definition which relies on the fact that we are in characteristic 0 and we refer to \cite{CHMN17} for the general case.

Remark 2.4. If the line arrangement $\mathcal{A}$ is actually the dual arrangement $\mathcal{A}_Z$ of a configuration of points $Z$, we denote its splitting type by $(a_Z, b_Z)$.

Conditions for unexpected curves. Finally, we give the connection between unexpected curves for a configuration of points and the splitting type of the dual line arrangement. These are the main results in \cite{CHMN17} that motivated this project.

In \cite{FV14}, the authors associate to a set of reduced points $Z$ a multiplicity index defined as

$$m_Z := \min\{j \mid \dim_C[I(Z + jQ)]_{j+1} > 0, \text{ for a general point } Q\}.$$

In \cite{CHMN17} Lemma 3.5(i), the authors associate directly the multiplicity index to the splitting type of the dual line arrangement $\mathcal{A}_Z$. In particular, they proved that

\begin{equation}
(1) \quad m_Z = \min\{a_Z, b_Z\}.
\end{equation}

Consequently, they obtain a characterization for configurations of points which admit unexpected curves. Here, another important numerical character is given by

$$t_Z := \min\{i \mid \dim_C[I(Z)]_{i+1} > \binom{i+1}{2}\}.$$ 

Theorem 2.5. \cite{CHMN17} Theorem 1.1] Let $Z$ be a configuration of points in $\mathbb{P}^2$ and let $\mathcal{A}_Z$ be its dual line arrangement with splitting type $(a_Z, b_Z)$, say $a_Z \leq b_Z$. Then, $Z$ admits unexpected curves if and only if $a_Z < t_Z$. In this case, $Z$ admits an unexpected curve of degree $j + 1$ if and only if $a_Z \leq j \leq b_Z - 2$.

Therefore, a solution to Problem A is given by the following theorem.

Theorem 2.6. \cite{CHMN17} Theorem 1.5] Let $Z$ be a configuration of points in $\mathbb{P}^2$ and let $\mathcal{A}_Z$ be its dual line arrangement with splitting type $(a_Z, b_Z)$. Then, $Z$ admits an unexpected curve of degree $j + 1$ if and only if:

\begin{enumerate}
  \item $a_Z \leq j \leq b_Z - 2$;
  \item $\dim_C[I(Z)]_{t_Z} = \binom{t_Z+1}{2} - |Z|$, where $|Z|$ denotes the cardinality of the set $Z$.
\end{enumerate}

From these results, it is clear that there is a close connection between the definition of unexpected curve for a set of points $Z$ and the splitting type of the dual line arrangement. Hence, our problem translates to a question about splitting types of line arrangements. By (1), the splitting type can be computed with any algebra software by finding the least $j$ such that $[I(Z + jQ)]_{j+1} \neq 0$, for a general point $Q$. Unfortunately,
this computation is very slow and inefficient because require to consider a field \( F \) which contains all the coordinates of the points in \( Z \) and then, if \( Q = (q_0 : q_1 : q_2) \), work over the field extension \( F(q_0, q_1, q_2) \).

In the next section, we focus on special line arrangements for which we can compute the splitting type and, consequently, deduce if the dual configuration of points admits an unexpected curve of some degree or not. Our computation mostly relies on the well-known Addition-Deletion Theorem and, as far as we know, this is the only theoretical tool to compute the splitting type without doing it by direct computation.

We now recall a different version of the results in [CHMN17] which is the precise way we use the aforementioned characterization of configurations of points having unexpected curves.

**Theorem 2.7.** [CHMN17] Theorem 1.2] Let \( Z \subset \mathbb{P}^2 \) be a finite set of points and let \((a_z, b_z)\) be the splitting type of the dual line arrangement, with \( a_z \leq b_z \). Then, \( Z \) admits an unexpected curve if and only if

i. \( 2a_z + 2 < |Z|; \)

ii. no subset of \( a_z + 2 \) (or more) of the points is collinear.

In this case, \( Z \) has an unexpected curve of degree \( j \) if and only if \( a_z < j \leq |Z| - a_z - 2 = b_z - 1 \).

**Proposition 2.8.** [CHMN17] Corollary 5.5] Let \( Z \subset \mathbb{P}^2 \) be a finite set of points admitting unexpected curves and let \((a_z, b_z)\) be the splitting type of the dual line arrangement, with \( a_z \leq b_z \). Then, \( Z \) has a unique unexpected curve \( C \) in degree \( a_z + 1 \). Moreover, for any \( a_z < j \leq b_z - 1 \) the unexpected curves of degree \( j \) are precisely the curves \( C + L_1 + \ldots + L_r \), with \( r = j - a_z - 1 \), where the \( L_i \)'s are arbitrary lines passing through the general point at which \( C \) is singular.

In particular, when \( Z \) admits unexpected curves, there is always a unique unexpected curve of degree \( a_z + 1 \).

3. **Line arrangements with expected and unexpected behavior**

While we were studying the problem and, in particular, **Example 1.2** we noticed that the dual line arrangement of the configuration of points appears under the name \( A(9, 1) \) in the list of simplicial line arrangements given by [Grü09] (see also [Cun11] for updated list of these arrangements). Here is the same example from this point of view.

**Example 3.1.** We construct a configuration of points in projective plane as described in **Example 1.2**. Consider the four vertices of a square: \((1 : 1 : 1), (1 : -1 : 1), (-1 : 1 : 1) \) and \((-1 : -1 : 1)\) and the intersection point of the diagonals of the square, i.e., the point \((0 : 0 : 1)\), and the intersections (at infinity) of the two pairs of parallel lines corresponding to the sides of the square, i.e., the points \((1 : 0 : 0)\) and \((0 : 1 : 0)\). Then, the line at infinity meets the two diagonals in two extra points \((1 : 1 : 0)\) and \((1 : -1 : 0)\). Thus, we have a set \( Z \) of nine points whose dual line arrangement \( \mathcal{A}_Z \) is defined by the polynomial \( f = xyz(x + y + z)(x - y + z)(-x + y + z)(-x - y + z)(x + y)(x - y) \) and is depicted in **Figure 2(b)**.

Now, we look at families of line arrangements generalising the one constructed in **Example 3.1**. In particular, we analyse their splitting type in order to establish for which arrangements the dual configuration of points admits unexpected curves of certain degrees.

3.1. **Supersolvable arrangements.** We consider now a special family of line arrangements.

**Definition 3.2.** A line arrangement \( \mathcal{A} \) is called supersolvable if there exists a modular point, i.e., a point \( P \) such that for every point \( Q \in \text{Sing}(\mathcal{A}) \), the line joining \( P \) and \( Q \) is an element of \( \mathcal{A} \).
Figure 2. The configuration of points in the projective plane and the dual line arrangement constructed in Example 3.1. The pictures represent the projective plane and we use the classical model of the projective plane where the line at infinity is represented by a circle on which opposite points are identified. For this reason, some straight lines are represented by circular curves.

We denote the multiplicity of a point \( P \) with respect to the arrangement \( \mathcal{A} \) as
\[
m(P, \mathcal{A}) = |\{ \ell \in \mathcal{A} | P \in \ell \}|.
\]
Moreover, we define
\[
\text{Sing}_k(\mathcal{A}) := \{ P \in \text{Sing}(\mathcal{A}) | m(P, \mathcal{A}) = k \} \quad \text{and} \quad \text{Sing}_{\geq k}(\mathcal{A}) := \bigcup_{i \geq k} \text{Sing}_i(\mathcal{A}).
\]

A useful property of supersolvable line arrangements is the following.

Lemma 3.3. [AT16, Lemma 2.1] Let \( \mathcal{A} \) be a supersolvable line arrangement. Let \( P, Q \in \text{Sing}(\mathcal{A}) \) such that \( P \) is modular and \( Q \) is not. Then, \( m(P, \mathcal{A}) > m(Q, \mathcal{A}) \). In particular, if a point has multiplicity
\[
m(\mathcal{A}) = \max \{ m(P, \mathcal{A}) | P \in \text{Sing}(\mathcal{A}) \},
\]
then it is modular.

Definition 3.4. Let \( \{\ell_1, \ldots, \ell_s\} \subset S_1 \) be the set of the linear polynomials defining the lines of a supersolvable line arrangement \( \mathcal{A} \). We say that \( \mathcal{A} \) has full rank if \( \dim_{\mathbb{C}} \text{Span}(\ell_1, \ldots, \ell_s) = 3 \).

In this section, we want to understand when supersolvable line arrangements admit unexpected curves. We give a necessary and sufficient condition to guarantee that a supersolvable line arrangement admits no unexpected curves and then we exhibit an infinite family of cases where we have unexpected curves. This family generalizes the configuration described in Example 3.1. Our main tool is Theorem 2.7 and, in order to use it, we need to compute the splitting type of supersolvable line arrangements. This is an easy application of the following well-known result which holds also in the more general setting of hyperplane arrangements.

Theorem 3.5. (Addition-Deletion Theorem; see [OT92, Theorem 4.51]) Let \( \mathcal{A} \) be a line arrangement in \( \mathbb{P}^2 \) and \( \ell \in \mathcal{A} \). Let \( \mathcal{A}' := \mathcal{A} \setminus \{\ell\} \). If the following conditions hold:

1. \( \mathcal{A}' \) is free and has splitting type \((a, b)\);
2. \( |\text{Sing}(\mathcal{A}) \cap \ell| = b + 1 \) (or \( a + 1 \), respectively);

then, \( \mathcal{A} \) is free with splitting type \((a + 1, b)\) (or \((a, b + 1)\), respectively).

Now, we can compute the splitting type for supersolvable line arrangements.

Lemma 3.6. Let \( \mathcal{A} \) be a supersolvable line arrangement where \( d := |\mathcal{A}| \) and \( m := m(\mathcal{A}) \). Then, the splitting type of \( \mathcal{A} \) is \((m - 1, d - m)\).
Proof. Let \( O \) be a modular point with maximal multiplicity and consider \( \mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \), where \( \mathcal{A}_0 \) is the subset of lines passing through the modular point \( O \) and \( \mathcal{A}_1 \) is the subset of lines not passing through \( O \). Then, \( m = |\mathcal{A}_0| \) and denote \( m' = |\mathcal{A}_1| \). Namely, \( d = m + m' \). We proceed by induction on \( m' \).

Let \( m' = 0 \). We have that \( \mathcal{A} = \mathcal{A}_0 \) is a central line arrangement given by \( m \) lines passing through the point \( O \). We compute the splitting type in this case by induction on \( m \). If \( m = 2 \), it is easy to check that by definition the splitting type is equal to \( (1, 0) \). If \( m > 2 \), by the Addition-Deletion Theorem and inductive hypothesis, we have that the splitting type of \( \mathcal{A} \) is \( (m - 1, 0) \).

If \( m' = 1 \), let \( \ell \in \mathcal{A}_1 \). Then,

\[
\left| \bigcup_{\ell' \in \mathcal{A}_0} \ell' \cap \ell \right| = m
\]

and, by the Addition-Deletion Theorem, we have that the splitting type of \( \mathcal{A} \) is \( (m - 1, 1) \). If \( m' > 1 \), let \( \mathcal{A}_1 = \{\ell_1, \ldots, \ell_{m'}\} \). Then, we notice that, since \( O \) is modular point, for every pair \( \ell_i, \ell_j \in \mathcal{A}_1 \), the intersection \( \ell_i \cap \ell_j \) lies on a line in \( \mathcal{A}_0 \). Therefore, for each \( i = 2, \ldots, m' \), if \( \mathcal{A}^{(i)} = \mathcal{A}_0 \cup \{\ell_1, \ldots, \ell_{i-1}\} \), we have

\[
\left| \bigcup_{\ell' \in \mathcal{A}^{(i)}} \ell' \cap \ell_i \right| = m,
\]

Therefore, by the Addition-Deletion Theorem and inductive hypothesis, we conclude that the splitting type of \( \mathcal{A} \) is \( (m - 1, m') = (m - 1, d - m) \).

We are now ready to give a necessary and sufficient condition for supersolvable arrangements to admit unexpected curves. Here, we denote \( d := |\mathcal{A}| \) and \( m := m(\mathcal{A}) \).

**Theorem 3.7.** A supersolvable line arrangement \( \mathcal{A} \) admits unexpected curves if and only if \( d > 2m \), where \( d \) is the number of lines and \( m \) is the maximum multiplicity of a point of intersection of the lines of \( \mathcal{A} \). Moreover, if \( d = 2m + 1 \), there is a unique unexpected curve and it has degree \( m \).

**Proof.** We use Theorem 2.7. We split the proof in two cases: (1) \( m - 1 \leq d - m \) and (2) \( m - 1 > d - m \). Observe that condition (ii) of Theorem 2.7, namely requiring to have no subset of \( m_Z + 2 \), or more, collinear points in the configuration of points is equivalent to requiring that the multiplicity of the intersection points in the dual line arrangement is at most \( m_Z + 1 \). Then:

1. If \( m - 1 \leq d - m \), by Lemma 3.6, \( m_Z = m - 1 \). Therefore, by Lemma 3.3, we may conclude that condition (ii) of Theorem 2.7 is always satisfied. Then, it is enough to observe that, since \( m_Z = m - 1 \), condition (i) is equivalent to having \( 2m < d \);

2. If \( d - m < m - 1 \), from condition (ii) of Theorem 2.7 we get that the multiplicity of each intersection point in the line arrangement is at most \( m_Z + 1 \). In particular, \( m < m_Z + 2 = d - m + 2 \), hence \( d > 2m - 2 \). From the condition (i), we have instead that \( 2(d - m) + 2 < d \), so \( d < 2m - 2 \). As these two conditions are incompatible, this situation cannot occur.

Note that, conversely, when \( 2m < d \) we are sure to be in the first case and then, the proof is concluded.

Uniqueness directly follows from Proposition 2.8.

**Remark 3.8.** In a recent paper, Dimca and Sticlaru introduced the notion of nearly supersolvable line arrangement [DS17]. They define a nearly modular point to be a point \( P \in \text{Sing}(\mathcal{A}) \) such that:

(i) for any point \( Q \in \text{Sing}(\mathcal{A}) \), with the exception of a unique point of multiplicity 2, say \( P' \), \( PQ \in \mathcal{A} \);

(ii) \( PP' \cap \text{Sing}(\mathcal{A}) = \{P, P'\} \).

Then, a line arrangement is nearly supersolvable if it has a nearly modular point.

Let \( \mathcal{A} \) be a nearly supersolvable line arrangement with \( m = m(\mathcal{A}) \) and \( d = |\mathcal{A}| \). In [DS17] Corollary 3.2, Dimca and Sticlaru prove that the splitting type of \( \mathcal{A} \) is \( (d - m, m - 1) \), if \( 2m \geq d \), and \( (\lfloor d/2 \rfloor, \lfloor d/2 \rfloor) \),
if \( 2m < d \). By using the same idea of the proof of Theorem 3.7, it follows that nearly supersolvable arrangements do not admit unexpected curves.

In the case of supersolvable real arrangements, we have the following property.

**Proposition 3.9.** [AT16, Corollary 2.3] Let \( \mathcal{A} \) be a full rank supersolvable real arrangement. Then,

\[
| \text{Sing}_2(\mathcal{A}) | + m(\mathcal{A}) \geq |\mathcal{A}|.
\]

As a direct consequence, we obtain the following.

**Proposition 3.10.** Let \( \mathcal{A} \) be a full rank supersolvable real line arrangement such that \( | \text{Sing}_2(\mathcal{A}) | = \frac{d}{2} \). Then, \( \mathcal{A} \) admits no unexpected curves.

**Proof.** By Proposition 3.9, we have \( 2m \geq d \). Hence the conclusion follows by Theorem 3.7. \( \square \)

**Remark 3.11.** Examples of full rank supersolvable real arrangements with \( | \text{Sing}_2(\mathcal{A}) | = \frac{d}{2} \) are given by the configurations of lines called Böröczky examples. These examples arise in the literature as sets of non-collinear points with the fewest number of ordinary lines, i.e. lines passing exactly through two points from the set. See [GT13, Proposition 2.1] for more details.

### 3.2. Polygonal arrangements.

Now, we consider a family of arrangements included in the list of simplicial line arrangements given by Grünbaum in [Grü09].

**Construction.** Consider a regular polygon with \( N \geq 3 \) edges. We construct the following arrangement:

1. \( e_i, \quad i = 1, \ldots, N \): the lines corresponding to the sides of the \( N \)-gone;
2. \( m_i, \quad i = 1, \ldots, N \): the lines corresponding to symmetry axes of the \( N \)-gone;
3. \( \ell_\infty \): the line at infinity.

**Definition 3.12.** The line arrangement \( \mathcal{P}_N = \{e_1, \ldots, e_N, m_1, \ldots, m_N\} \) is called the \( N \)-gonal arrangement. The line arrangement \( \overline{\mathcal{P}}_N = \mathcal{P}_N \cup \{\ell_\infty\} \) is called the complete \( N \)-gonal arrangement.

**Remark 3.13.** Note that in the literature (see e.g. [Grü09]) the arrangements \( \mathcal{P}_N \) are denoted by \( A(2N, 1) \), while \( \overline{\mathcal{P}}_N \) as \( A(2N + 1, 1) \). These special configurations of lines are simplicial arrangements, i.e., all cells are triangles, and appear often as examples or counterexamples to various combinatorial problems.

**Example 3.14.** In the next figures, we describe the construction of the arrangements \( \mathcal{P}_4 \) and \( \overline{\mathcal{P}}_4 \). Note that the latter is precisely the arrangement considered in Example 3.1.

**Theorem 3.15.** Let \( N > 2 \) be an integer. Then \( \mathcal{P}_N \) is always supersolvable, and \( \overline{\mathcal{P}}_N \) is supersolvable if and only if \( N \) is even. Moreover, \( \mathcal{P}_N \) never admits an unexpected curve, but if \( N \) is even, then \( \overline{\mathcal{P}}_N \) admits a unique unexpected curve and its degree is \( N \).

**Proof.** The line arrangements \( \mathcal{P}_N \) is always supersolvable because all the singular points lie on a symmetry line; hence, the barycenter is a modular point. In \( \overline{\mathcal{P}}_N \) we are adding the line at infinity; hence:

1. if \( N \) is even, every line corresponding to an edge is “parallel” to some symmetry line, i.e., they meet on the line at infinity; therefore, any singular point at infinity still lie on a symmetry line and the barycenter is still a modular point;
2. if \( N \) is odd, the line corresponding to the edges are not parallel to any symmetry line; therefore, the singular points obtained as intersection of the edge lines and the line at infinity are not connected to the barycenter of the polygon which is no longer a modular point.
The lines $e_i$'s corresponding to the sides of the square.

(b) The lines $m_i$'s corresponding to the symmetries of the square.

(c) The line $\ell_\infty$ at infinity.

**Figure 3.** Construction of $\mathcal{P}_4$ and $\overline{\mathcal{P}}_4$.

By construction, the number of lines is $2N$ for $\mathcal{P}_N$ and $2N + 1$ for $\overline{\mathcal{P}}_N$. Also, $m(\mathcal{P}_N) = m(\overline{\mathcal{P}}_N) = N$.

Then, our claim follows directly from Theorem 3.7. Moreover, for $N$ even, we have that the splitting type of $\mathcal{P}_N$ is $(N - 1, N + 1)$; therefore, by Theorem 2.7 and Proposition 2.8, there is a unique unexpected curve of degree $N$.

**Remark 3.16.** Theorem 3.15 generalizes the case described in [Har17, Example 4.1.10] which corresponds, in our notation, to the configuration dual to $\mathcal{P}_4$ for which we have an unexpected quartic.

**Remark 3.17.** Although $\overline{\mathcal{P}}_N$ is not supersolvable when $N > 2$ is odd, and thus Theorem 3.7 does not apply, computer experiments for low odd values of $N$ show that $\overline{\mathcal{P}}_N$ has no unexpected curves.

### 3.3. Tic-tac-toe arrangements

Here, we consider another family of line arrangements which generalizes [CHMN17, Example 6.14].

**Construction.** A tic-tac-toe arrangement of type $(k, j)$, denoted $\mathcal{T}_{k}^{j}$, is the arrangement defined by:

1. $v_i, i = -k, \ldots, k$: vertical lines $x = kz$;
2. $h_i, i = -k, \ldots, k$: horizontal lines $y = kz$;
3. $d_i, i = -j, \ldots, j$: the diagonals $x - y + jz = 0$;
4. $e_i, i = -j, \ldots, j$: the anti-diagonals $x + y + jz = 0$.

**Remark 3.18.** By symmetry, we may always assume that $k \geq j$. Indeed, thinking in the real projective plane, up to a 45°-rotation, we have that $\mathcal{T}^{j}_{k}$ coincides with $\mathcal{T}^{k}_{j}$. Moreover, we observe that the tic-tac-toe arrangement $\mathcal{T}^{0}_{1}$ coincides with the square arrangement $\mathcal{P}_4$ (see Figure 3(b)), while, for $k > 1$, tic-tac-toe arrangements cannot be viewed as polygonal arrangements.

Similarly as above, we denote by $\overline{\mathcal{T}}^{j}_{k}$ the complete tic-tac-toe arrangement of type $(k, j)$ obtained by adding also the line at infinity. In [CHMN17, Example 6.14], the authors observed that the splitting type of the complete tic-tac-toe arrangement of type $(k, 0)$ is $(2k + 1, 2k + 3)$ and they obtained the following.

**Proposition 3.19.** [CHMN17, Proposition 6.15] The tic-tac-toe arrangement $\overline{\mathcal{T}}^{0}_{k}$ of type $(k, 0)$ admits a unique and irreducible unexpected curve of degree $2k + 2$.

We may observe that $\overline{\mathcal{T}}^{0}_{k}$ is supersolvable and, in particular, free. By the Addition-Deletion Theorem, we can compute the splitting type of $\overline{\mathcal{T}}^{1}_{k}$ which, in particular, remains free. Therefore, we may inductively use the Addition-Deletion Theorem to compute the splitting type of $\overline{\mathcal{T}}^{j}_{k}$, as we show in the following.
LEMMA 3.20. Let $k, j$ be positive integers with $k \geq j$. Then, the tic-tac-toe arrangement $\mathcal{F}_k^j$ is free with splitting type equal to $(2k + 1 + 2j, 2k + 3 + 2j)$.

Proof. For any $k$ and $j = 0$, we know that the claim holds by [CHMN17, Example 6.14]. We proceed now by induction on $j$. Assume that $\mathcal{F}_k^j$ is free and has splitting type $(2k + 1 + 2j, 2k + 3 + 2j)$. We want to add the lines $d_{j+1}, d_{j-1}, e_{j+1}, e_{j-1}$ and use the Addition-Deletion Theorem four times to prove the claim for $\mathcal{F}_k^{j+1}$. First, we need to compute the intersection between the diagonal $d_{j+1}$ and $\mathcal{F}_k^j$. This is:

\[ |\mathcal{F}_k^j \cap d_{j+1}| = c_1 - c_2 + c_3 = [2(2k + 1) + 1] - [2k - j] + [j + 1] = 2k + 2j + 4; \]

where:

i. $c_1$: the number of vertical and horizontal lines in $\mathcal{F}_k^j$ plus the line at infinity;
ii. $c_2$: the number of points of the type $v_\alpha \cap h_\beta$ lying on $d_{j+1}$, i.e., the number of points of intersection in $\mathcal{F}_k^j \cap d_{j+1}$ that are counted twice by $c_1$;
iii. $c_3$: the number of diagonals $e_i$'s intersecting $d_{j+1}$ in points not of type $v_\alpha \cap h_\beta$, i.e., the remaining points in $\mathcal{F}_k^j \cap d_{j+1}$ not yet counted by $c_1$.

Then, by the Addition-Deletion Theorem, $\mathcal{F}' = \mathcal{F}_k^j \cup \{d_{j+1}\}$ is free and has splitting type $(2k + 2 + 2j, 2k + 3 + 2j)$. Now, since $d_{j-1} \cap d_{j+1} = d_{j-1} \cap d_0$, we have that the cardinality of the intersection $\mathcal{F}' \cap d_{j-1}$ is the same as counted in (2). Therefore, by the Addition-Deletion Theorem, $\mathcal{F}'' = \mathcal{F}' \cup \{d_{j-1}\}$ is free and has splitting type $(2k + 3 + 2j, 2k + 3 + 2j)$. Similarly, since for any $\alpha, \beta$, $e_\alpha \cap h_\beta = v_\alpha \cap h_\beta$, for some $\alpha', \beta'$, we have that also the intersections $\mathcal{F}'' \cap \{e_{j+1}\}$ and $(\mathcal{F}'' \cup \{e_{j+1}\}) \cap \{e_{j-1}\}$ have the same cardinality as counted in (2). Again, by the Addition-Deletion Theorem, we have that the line arrangement

\[ \mathcal{F}_k^j \cup \{d_{j+1}, d_{j-1}, e_{j+1}, e_{j-1}\} = \mathcal{F}_k^{j+1}, \]

is free and has splitting type $(2k + 3 + 2j, 2k + 5 + 2j) = (2k + 1 + 2(j + 1), 2k + 3 + 2(j + 1))$. \hfill \Box

THEOREM 3.21. The complete arrangement $\mathcal{F}_k^j$ admits a unique unexpected curve of degree $2(k + j + 1)$.

Proof. Observe that $m(\mathcal{F}_k^j) = 2k + 1$, that is the multiplicity of one of the points at infinity, e.g. the direction of the vertical lines. Hence in the dual configuration there are no more than $2k + 1$ collinear points. Then, we can use Lemma 3.20 and conclude by Theorem 2.7 and Proposition 2.8. \hfill \Box
4. OTHER EXAMPLES

In this section, we exhibit other examples of unexpected curves arising from special line arrangements.

4.1. Adding lines to polygonal arrangements. First, we construct them by using ideas from the previous sections. We may notice that tic-tac-toe arrangements are constructed from the square arrangement \( P_4 \) by adding lines parallel to the ones of \( P_4 \). Hence, we try to proceed in a similar way by starting from polygonal arrangements \( P_N \), with \( N \) even. Unfortunately, this procedure is not successful in the sense that we can use Addition-Deletion Theorem only in a very few cases, as we are going to explain, but in general we do not know how to efficiently compute the splitting type of these line arrangements, since they are not supersolvable (hence, we cannot use Lemma 3.6) and we cannot apply Addition-Deletion Theorem.

**Example 4.1.** Consider the set of lines from the arrangement \( \overline{P_6} \) and take the points \( P_1, \ldots, P_6 \) as in Figure 5. From the proof of Theorem 3.15 we know that the splitting type for \( P_6 \) is \((5, 7)\). Now, we construct a series of examples for which some of the dual configuration of the points give an unexpected curve.

**FIGURE 5.** Configuration \( \overline{P_6} \). The line at infinity is not shown.

We add, step by step, the lines \( \ell_1 := P_1P_2, \ell_2 := P_2P_3, \ldots, \ell_6 := P_6P_1 \) (blue dotted lines in Figure 6). Denote \( B_0 := \overline{P_6} \) and \( B_i := B_{i−1} \cup \{\ell_i\} \). By Theorem 3.5, since \( \left| \bigcup_{\ell \in B_{i−1}} \ell \cap \ell_i \right| = 8 \), for \( i = 1, \ldots, 6 \), the splitting types of the arrangements \( B_i \)'s are

\[
\begin{align*}
\overline{P_6} & \rightarrow B_1 = \overline{P_6} \cup \{\ell_1\} \\
(5, 7) & \rightarrow (6, 7) \\
B_2 & \rightarrow B_2 = \overline{P_6} \cup \{\ell_1, \ell_2\} \\
(7, 7) & \rightarrow (11, 7) \\
\ldots & \rightarrow \ldots \\
B_6 & \rightarrow B_6 = \overline{P_6} \cup \{\ell_1, \ldots, \ell_6\} \\
(11, 7) & \rightarrow (11, 13)
\end{align*}
\]

By Theorem 2.7 we have that the line arrangements \( B_4, B_5 \) and \( B_6 \) admit unexpected curves of degrees 8, 9 and 10. We continue by adding lines passing through the points \( P_1, \ldots, P_6 \), as indicated in Figure 6 (dashed-dotted red lines). Denote by \( \ell_i' \) the new line passing through \( P_i \), respectively, and denote the line arrangements \( B_i' := B_i \) and \( B_i' := B_i' \cup \{\ell_i'\} \). Since \( \left| \bigcup_{\ell \in B_{i−1}} \ell \cap \ell_i' \right| = 12 \), for \( i = 1, \ldots, 6 \), by Theorem 3.5 the splitting types of the arrangements \( B_i' \)'s are

\[
\begin{align*}
B_6 & \rightarrow B_1' = B_6 \cup \{\ell_1'\} \\
(11, 7) & \rightarrow (11, 8) \\
B_2' & \rightarrow B_2' = B_6 \cup \{\ell_1', \ell_2'\} \\
(11, 9) & \rightarrow (11, 13) \\
\ldots & \rightarrow \ldots \\
B_6' & \rightarrow B_6' = B_6 \cup \{\ell_1', \ldots, \ell_6'\} \\
(11, 13) & \rightarrow (11, 13)
\end{align*}
\]
We obtain three new arrangements $B_1', B_2'$ and $B_6'$ which admit unexpected curves of degree 9, 10 and 12. Moreover, we may check that the line arrangement $B_6'$ constructed in Figure 6 is dual to the configuration of points given by $\text{Sing}(\mathcal{P}_6)$. This procedure of adding lines can be repeated two more times. As indicated in Figure 7, we first add the 6 blue dashed lines, $m_1, \ldots, m_6$, and then 6 red dash-dotted lines $m'_1, \ldots, m'_6$.

In this process, by Theorem 3.5 we get the following series of exponents

\[
B_6' \rightarrow B_6' \cup \{m_1\} \rightarrow B_6' \cup \{m_1, m_2\} \rightarrow \ldots \rightarrow B_6'' := B_6' \cup \{m_1, \ldots, m_6\}
\]

\[
B_6' \rightarrow B_6'' \cup \{m'_1\} \rightarrow B_6'' \cup \{m'_1, m'_2\} \rightarrow \ldots \rightarrow B_6'' \cup \{m'_1, \ldots, m'_6\}
\]

from which, by Theorem 2.7 we find new examples of unexpected curves.
Figure 7. Bolded points indicate the original points $P_1, \ldots, P_6$; blue dotted lines $m_1, \ldots, m_6$ are added in the first step; red dash-dotted lines $m'_1, \ldots, m'_6$ are added in the second step.

Example 4.2. We can proceed in a similar way as in Example 4.1 but starting from configuration $\mathcal{F}_8$. As before, we denote by $P_1, P_2, \ldots, P_8$ vertices of the octagon as indicated in Figure 8.

Figure 8. Configuration $\mathcal{F}_8$. The line at infinity is not shown.

By Lemma 3.6, the splitting type of $\mathcal{F}_8$ is $(7,9)$. We add the lines $\ell_1 := P_1P_2$, $\ell_2 := P_2P_3$, $\ldots$, $\ell_8 := P_8P_1$ (blue dotted lines in Figure 9). By Theorem 3.5, the splitting type of $\mathcal{F}_8 \cup \{\ell_1, \ldots, \ell_i\} = (7 + i, 9)$, for all $i = 1, \ldots, 8$. Thus, the existence of unexpected curves for $i \in \{4, 5, \ldots, 8\}$ is guaranteed by Theorem 2.7.

This line arrangement can also be extended to new arrangements which admit unexpected curves. We add other 8 lines: $m_1 := P_1P_3$, $m_2 := P_2P_4$, $\ldots$, $m_8 := P_8P_2$ (red dash-dotted lines in Figure 9). By Theorem 3.5, the splitting type of $\mathcal{F}_8 \cup \{\ell_1, \ldots, \ell_6, m_1, \ldots, m_j\} = (15, 9 + j)$, for all $j = 1, \ldots, 8$. Thus, the existence of unexpected curves for $j \in \{1, 2, 3, 4, 8\}$ is guaranteed by Theorem 2.7.
Remark 4.3. Observe that the order of adding new lines in Example 4.2 is not relevant because the lines we are adding meet each other in points which are also intersections with the original lines of the polygonal arrangement.

4.2. Sporadic cases. It occurs that not only the dual configurations of points to the line arrangements presented in the previous sections are giving unexpected curves. There are other simplicial arrangements for which we have the same property. In Table 1 we present a list of arrangements, together with their splitting type, and with their original names coming as in [Grü09]. In this list, we consider line arrangements that are dual to the configurations $A(n, k)$ described in Grünbaum’s paper.

Definition 4.4. Given a line arrangement $\mathcal{A}$, we define its dual line arrangement, denoted by $\mathcal{A}^d$, as the line arrangement dual to the configuration of points $\text{Sing}(\mathcal{A})$.

Example 4.5. The next three figures illustrate examples of the previous definitions and notations.
Figure 10. Line arrangements $\overline{A}_4$.

Figure 11. Line arrangements $\overline{A}_4^d$.

Figure 12. Line arrangement $\overline{A}_4^d$ and the points in $\text{Sing}_{\geq 4}(\overline{A}_4^d)$ (red bolded), i.e., the singular points with multiplicity at least 3. In particular, $\text{Sing}_{\geq 4}(\overline{A}_4^d)$ consists only in the central point.

Table 1. Line arrangements duals to simplicial arrangements defined in [Grü09] and their splitting types. The computations have been made with the algebra software Singular. The code used can be found as additional file to the arXiv version of the paper and includes the full computation for the $A(31, 3)$ case. The coordinates of the points of all the configurations listed in the table have been kindly provided by M. Cuntz during a private communication.
In Table 2 we give a list of line arrangements such that, for some \( k \), the configuration of points \( \text{Sing}_{\geq k} \) admits unexpected curves. We also give the exponents which speak about the degrees of unexpected curves.

| \( A^d \) | \( k \) | \( |\text{Sing}_{\geq k}(A^d)| \) | \( a_{\text{Sing}_{\geq k}(A^d)} \) | \( b_{\text{Sing}_{\geq k}(A^d)} \) |
|-----------|-----|----------------|----------------|----------------|
| \( A^d(13, 2) \) | 4 | 9 | 3 | 5 |
| \( A^d(13, 2) \) | 3 | 13 | 5 | 7 |
| \( A^d(13, 2) \) | 1 | 25 | 11 | 13 |
| \( A^d(17, 2) \) | 4 | 9 | 3 | 5 |
| \( A^d(17, 4) \) | 4 | 9 | 3 | 5 |
| \( A^d(17, 4) \) | 3 | 25 | 11 | 13 |
| \( A^d(19, 3) \) | 4 | 13 | 5 | 7 |
| \( A^d(19, 3) \) | 3 | 25 | 11 | 13 |
| \( A^d(21, 3) \) | 4 | 13 | 5 | 7 |
| \( A^d(21, 3) \) | 3 | 37 | 17 | 19 |
| \( A^d(25, 2) \) | 4 | 21 | 9 | 11 |
| \( A^d(25, 4) \) | 4 | 19 | 7 | 11 |
| \( A^d(25, 7) \) | 4 | 18 | 7 | 10 |
| \( A^d(26, 3) \) | 4 | 19 | 7 | 11 |
| \( A^d(26, 4) \) | 4 | 18 | 7 | 10 |
| \( A^d(27, 2) \) | 4 | 20 | 8 | 11 |
| \( A^d(27, 3) \) | 4 | 21 | 8 | 11 |
| \( A^d(27, 4) \) | 4 | 19 | 7 | 11 |
| \( A^d(28, 4) \) | 4 | 21 | 9 | 11 |
| \( A^d(31, 3) \) | 4 | 31 | 13 | 17 |
| \( A^d(34, 2) \) | 4 | 13 | 5 | 7 |
| \( A^d(37, 3) \) | 4 | 37 | 17 | 19 |
| \( A^d(37, 3) \) | 6 | 13 | 5 | 7 |

Table 2. Configurations of points defined as high order points of some simplicial line arrangements.

4.3. Future directions. The question we considered so far (Problem A) is a special case of the following more general problem suggested by Cook II, Harbourne, Migliore and Nagel in [CHMN17].

**Problem B.** Let \( Z \) be a set of reduced points in \( \mathbb{P}^2 \) and let \( X = m_1Q_1 + \ldots + m_sQ_s \) be a scheme of fat points with general support.

For which \( (Z; m_1, \ldots, m_s; j) \) do we have that \( \dim\mathcal{C}[I(Z + X)] > \max\{\dim\mathcal{C}[I(Z)]_j - \deg(X), 0\} \)?

If so, we say that \( Z \) admits unexpected curves of degree \( j \) with respect to \( X \).

The examples from Table 1 and Table 2 give examples of unexpected curves according to Problem A (i.e., when \( X \) is just a fat point), but we can also extend them to get examples of unexpected curves according to Problem B. The idea is explained in the following fact.

**Proposition 4.6.** Let \( Z \) be a configuration of points \( \text{with splitting type} \ (a_Z, b_Z) \) \( \text{with} \ b_Z - a_Z \geq 2 \). Then, for any \( j \in \{0, \ldots, b_Z - a_Z - 2\} \), we have that \( Z \) admits a unique unexpected curve of degree \( a_Z + 1 + j \) with respect to \( X = (a_Z + j)P + A \), where \( A \) is a set of reduced points of cardinality \( |A| = j \).

**Proof.** Let \( P \) be a general point. By Proposition 2.8, we know that, for any \( j \in \{0, \ldots, b_Z - a_Z - 2\} \), we have that \( Z \) admits unexpected curves of degree \( a_Z + j + 1 \) with respect to \((a_Z + j)P \) and, in particular, that

\[
\dim\mathcal{C}[I(Z + (a_Z + j)P)]_{a_Z + j + 1} = j + 1.
\]

Moreover, we may observe that, by definition of unexpected curves, we have

\[
\max\left\{ \dim\mathcal{C}[I(Z)]_{a_Z + j + 1} - \left(\frac{a_Z + j + 1}{2}\right), 0\right\} \leq j.
\]

Now, since generic simple points always impose the expected number of conditions on a linear system of curves, we have that, for any \( j \in \{0, \ldots, b_Z - a_Z - 2\} \),

\[
\dim\mathcal{C}[I(Z + (a_Z + j)P + A)]_{a_Z + j + 1} = \dim\mathcal{C}[I(Z + (a_Z + j)P)]_{a_Z + j + 1} - j = 1;
\]
and, at the same time, by (3),
\[
\max \left\{ \dim C[I(Z)]_{a_Z+j+1} - \binom{a_Z + j + 1}{2} - j, 0 \right\} = 0.
\]
Therefore, we have that \( Z \) admits a unique unexpected curve of degree \( a_Z + j + 1 \) with respect to \( X = (a_Z + j)P + A \), where \( A \) is a set of generic simple points with \( |A| = j \).

Some of the examples provided in Table 1 and Table 2 satisfy the hypothesis of the latter proposition and give examples of unexpected curves with respect to Problem B.

Note that, by Proposition 2.8, the unexpected curves constructed in the latter proposition, whenever \( j \geq 1 \), are reducible. It would be interesting to construct an example of reducible unexpected curve with respect to a scheme of fat points \( X \) having support in more than one point.

Finding a characterization to answer Problem B for some particular non-connected scheme \( X \), e.g., the union of two fat points, or constructing additional interesting examples of reducible unexpected curve besides the ones constructed in Proposition 4.6 are problems worthy of further investigation.

**References**

[AF16] B. Anzis, and S. Tohâneanu. *On the geometry of real or complex supersolvable line arrangements*, Journal of Combinatorial Theory, Series A, 140 : 76–96 (2016).

[CM01] C. Ciliberto and R. Miranda. *The Segre and Harbourne-Hirschowitz conjectures*, Applications of algebraic geometry to coding theory, physics and computation, pp. 37–51 (2001).

[CHMN17] D. Cook II, B. Harbourne, J. Migliore, and U. Nagel. *Line arrangements and configurations of points with an unusual geometric property*. Preprint, arXiv:1602.02300.

[Cun11] M. Cuntz *Simplicial arrangements with up to 27 lines*. Preprint, arXiv:1108.3000v1.

[DGPS] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönenmann. *SINGULAR 4-1-0 — A computer algebra system for polynomial computations*. http://www.singular.uni-kl.de (2017).

[DIV14] R. Di Gennaro, G. Ilardi, and J. Vallès, *Singular hypersurfaces characterizing the Lefschetz properties*, J. London Math. Soc. 89(1) : 194 – 212 (2014).

[DS17] A. Dimca, and G. Sticlaru, *On supersolvable and nearly supersolvable line arrangements*, arXiv:1712.03885v1.

[FGST18] L. Farnik, F. Galuppi, L. Sodomaco, and B. Trok, *On the unique unexpected quartic in \( \mathbb{P}^4 \)*, in preparation.

[FV14] D. Faenzi, and J. Vallès, *Logarithmic bundles and line arrangements, an approach via the standard construction*, J. London Math. Soc. (2) 90 : 675–694 (2014).

[GO81] A.V. Geramita, and F. Orecchia. *On the Cohen-Macaulay type of s-lines in \( \mathbb{A}^n \)*, Journal of Algebra 70(1) : 116–140 (1981).

[Gim87] A. Gimigliano. *On Linear Systems of Plane Curves*, Ph. D. thesis, Queen’s University, Kingston, Ontario (1987).

[GT13] B. Green, and T. Tao. *On sets defining few ordinary lines*, Discrete Comput. Geom. 50 : 409–468 (2013).

[GS] D. Grayson, and M. Stillman. *Macaulay2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/

[Grü09] B. Grünbaum. *A catalogue of simplicial arrangements in the real projective plane*, Ars Mathematica Contemporanea 2 : 1–25 (2009).

[Har66] B. Harbourne. *The geometry of rational surfaces and Hilbert functions of points in the plane*, Can. Math. Soc. Conf. Proc. 6 : 95–111 (1986).

[Har17] B. Harbourne. *Asymptotics of linear systems, with connections to line arrangements*. To appear, 2016 miniPAGES Conf. Proc., Warsaw, Poland, (BCSim-2016-s02), arXiv:1705.09946.

[Hir89] A. Hirschowitz. *Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles génériques*, Journ. Reine Angew. Math. 397 : 208–213 (1989).

[Nag60] M. Nagata. *On rational surfaces, II*. Mem. Coll. Sci. Univ. Kyoto, Ser. A. Math. 33 : 271–293 (1960).

[OT92] P. Orlik, and H. Terao. *Arrangement of hyperplanes*, Grundlehren der Mathematischen Wissenschaften 300, Springer-Verlag, Berlin, (1992).

[Sch12] H. Schenck. *Hyperplane arrangements: computations and conjectures*, Adv. Stud. Pure Math., 62, Math. Soc. Japan, Tokyo (2012).
[Seg61] B. Segre. *Alcune questioni su insiemi finiti di punti in Geometria Algebrica*, Atti del Convegno Internaz. di Geom. Alg., Torino (1961).

(M. Di Marca) DIAPARTIMENTO DI MATHEMATICA, UNIVERSITÀ DEGLI STUDI DI GENOVA, GENOA, ITALY

_E-mail address:_ dimarca@dima.unige.it

(G. Malara) DEPARTMENT OF MATHEMATICS, PEDAGOGICAL UNIVERSITY OF CRACOW, KRAKÓW, POLAND

_E-mail address:_ grzegorzmalara@gmail.com

(A. Oneto) DEPARTMENT OF MATHEMATICS, UNIVERSITAT POLITÈCNICA DE CATALUNYA, BARCELONA, SPAIN

_E-mail address:_ alessandro.oneto@upc.edu