Malmsten’s integral and some related results: A different approach with Special functions

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Abstract

This paper is devoted to a case of the logarithmic integrals studied by Blagouchine in 2014. It was clarified in his work on the rediscovery of Malmsten’s integrals that the previously called Vardi integral was actually a particular case of a considered family of integrals, first evaluated by Malmsten and colleagues in 1842. The approach here is based completely on real methods. A new proof for the integral presented by Vardi is considered by an elementary method and other proofs for some related integrals presented by Blagouchine are considered by methods involving the use of some Special functions and their properties. In determining a closed form for one of the integrals presented by Blagouchine, several generalizations for integrals involving the combinations of some transcendental functions were discovered and examples were given relating them to known mathematical constants.

Keywords: gamma function, polygamma function, Barnes G-function, Riemann zeta function, Hurwitz zeta function

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1 Introduction

1.1 Introductory remarks and brief history

In an article which appeared in the American Mathematical Monthly in 1988 [1], Vardi treated several interesting logarithmic integrals found in Gradshteyn and Ryzhik’s tables [2]. His exposition began with the integrals

\[
\int_0^{\frac{\pi}{4}} \ln \ln x \, dx = \int_0^1 \frac{\ln \frac{1}{x}}{1 + x^2} \, dx = \int_1^\infty \frac{\ln x}{1 + x^2} \, dx = \frac{1}{2} \int_0^\infty \frac{\ln x}{\cosh x} \, dx
\]

\[
= \frac{\pi}{2} \ln \left( \frac{\Gamma \left( \frac{1}{4} \right) \sqrt{2\pi}}{\Gamma \left( \frac{1}{4} \right)} \right)
\] (1.1)
which can be deduced one from another by simple change of variable. Vardi proposed a method of proof based essentially on the use of Dirichlet \( L \)-function [1], Blagouchine showed that by substituting \( n = 1 \) and \( n = 2 \) into Malmsten’s result, changing the variable, taking into account that the definite integral of \( \text{sech}^n x \) over \( x \in [0, \infty) \) equals \( \frac{\Gamma(n+1)}{2^n} \) and further setting the constant in the latter integral to be 0, Vardi’s result in the hyperbolic form in (1.1) will immediately follow ([3], p. 32). He also provided generalizations to the definite integral of \( \ln \text{sech}^n x \) over \( x \in [0, \infty) \) by the contour integration method ([3], p. 80). Throughout this work, \( \Delta_n \) will represent the definite integral of \( \ln \text{sech}^n x \) over \( x \in [0, \infty) \) for brevity.

The aim of this work is to propose: new proofs for the closed forms of \( \Delta_n \) for \( n = 1, 2, 3, \ldots, 6 \), a reduction formula for the generalized definite integral of the logarithm of the Barnes \( G \)-function and the log-gamma function, multiplied, respectively, by \( t^n \) and \( t^{n+1} \) for positive integers \( n \), and a reduction formula for \( \Delta_n \) for integers \( n \geq 4 \). Blagouchine also noted a formula derived from Binet’s integral representation for the digamma function that yields \( \Delta_1 \) ([3], p. 83). It can be noted here that: the method used in evaluating \( \Delta_1 \) is different from the method of proof for Binet’s integral as found in [4], there will be no missing constant(s) needed to be derived for \( \Delta_1 \) and the methods for determining closed forms for \( \Delta_n \) in this note can be understood by a reader that has little or no knowledge of contour integration.

### 1.2 Notations

Throughout this manuscript, the following abbreviated notations are used:

- \( \gamma = 0.5772156649... \) for the Euler-Mascheroni constant
- \( e = 2.71828182845... \) for Euler’s number
- \( G = 0.9159655941... \) for Catalan’s constant
- \( A = 1.2824271291... \) for the Glaisher-Kinkelin constant
- \( \ln z \) for the natural logarithm of \( z \)
- \( \Re(z) \) for the real part of \( z \)
- \( H_n \) for the \( n \)-th harmonic number
- \( \cos z \) for the cosine of \( z \)
- \( \sin z \) for the sine of \( z \)
- \( \tan z \) for the tangent of \( z \)
- \( \tanh z \) for the hyperbolic tangent of \( z \)
- \( \cosh z \) for the hyperbolic cosine of \( z \)
- \( \arctan z \) for the compositional inverse of the tangent of \( z \)
- \( \text{sech} z \) for the hyperbolic secant of \( z \)
- \( H \) for the set of all natural numbers
- \( \mathbb{N}_0 \) for the set of all natural numbers including 0
- \( \mathbb{N} \) for the set of all elements of \( \mathbb{N}_0 \) multiplied by \( n \)
- \( \mathbb{R} \) for the set of all real numbers
- \( \mathbb{C} \) for the set of all complex numbers
- \( \mathbb{Z} \) for the set of all integers

When the symbol + (or −) is placed above the sets, it denotes the positive (or negative) members of the set. We denote, respectively, the gamma, digamma, trigamma, tetragamma and pentagamma functions of argument \( z \) with \( \Gamma(z) \), \( \psi_0(z), \psi_1(z), \psi_2(z), \psi_3(z) \), where \( \psi_n(z) := \frac{d^n \ln \Gamma(z)}{dz^n}, n \in \mathbb{N}_0 \). \( \psi_n(z) \) is generally called the polygamma function of argument \( z \). Letter \( i \) is never used as an index and is \( i = \sqrt{-1} \).

The Riemann zeta function and the Hurwitz zeta function are, respectively, defined as

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \zeta(s, z) = \sum_{n=1}^{\infty} \frac{1}{(n+z)^s}
\]

where \( z \not\in \mathbb{N}_0 \) and \( \Re(s) > 1 \) for both functions. Where necessary, the domains of these two functions may be extended to other domains by the principle of analytic continuation. The Hermite integral representation for Hurwitz zeta function is

\[
\zeta(s, z) = \frac{z^{-s}}{2} + \frac{z^{1-s}}{s-1} + 2 \int_0^{\infty} \frac{\sin \left( s \arctan \left( \frac{x}{z} \right) \right)}{\left( x^2 + z^2 \right)^{s/2}} \left( e^{2\pi x} - 1 \right) dx \quad (1.2)
\]
Integrating term-wise which provides analytic continuation of \( \zeta(s, z) \) to the domain \( s \in \mathbb{C} \setminus \{1\} \) [15, Eq. 1.10(7)]. We denote the \( n \)-th derivative of \( \zeta(s, z) \) wrt. \( s \) with \( \zeta^{(n)}(s, z) \).

The Barnes G-function\(^1\) is defined as

\[
G(z) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z + z^2(1 + \gamma)}{2}\right) \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} - z\right), \quad z \in \mathbb{C}. \tag{1.3}
\]

Other notations are standard and we remark that the references to the equations are given between parentheses “( )”; the bibliographic references are given in square brackets “[ ]”; p., Eq., Sect. within the box stands, respectively, for page, Equation, Section; wrt. is an abbreviation for with respect to and Prop. is an abbreviation for Proposition. To prevent confusion with the Barnes function, the G representing Catalan’s constant was boldened and for consistency, A for the Glaisher-Kinkelin constant was also boldened.

2 The proposed methods

Carl Malmsten and colleagues used the gamma function and some elementary (transcendental) functions for their derivations, such as the sine and logarithm functions; with the gamma function serving as the only special function [12]. The proposed methods will involve the use of some special functions and their properties. Before considering the proof for the closed form of \( \Delta \), we denote the \( n \)-th derivative of \( \zeta(s, z) \) with \( \zeta^{(n)}(s, z) \).

Proposition. Let \( a \in \mathbb{R} \), then

\[
\int_{0}^{\infty} \frac{\cos ax}{\cosh x} \, dx = \frac{\pi}{2} \frac{\text{sech} \left( \frac{\pi a}{2} \right)}{2}. \tag{2.1}
\]

Proof. We begin the proof of (2.1) by rewriting numerator and denominator in their exponential forms

\[
\int_{0}^{\infty} \frac{\cos ax}{\cosh x} \, dx = \int_{0}^{\infty} \frac{e^{iax} + e^{-iax}}{1 + e^{-2x}} \, dx.
\]

Integrating term-wise

\[
\int_{0}^{\infty} \frac{\cos ax}{\cosh x} \, dx = \sum_{k=0}^{\infty} (-1)^k \int_{0}^{\infty} \frac{e^{iax} + e^{-iax}}{1 + e^{-2x}} \, dx
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \int_{0}^{\infty} \frac{e^{-(2k+1)iax} + e^{-(2k+1+ia)x}}{1 + e^{-2x}} \, dx
\]

\[
= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k + \frac{1}{2}} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k + \frac{1}{2} \pm i \pi}. \tag{2.2}
\]

The relationship between an alternating series and the digamma function is:

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k + z} = \frac{1}{2} \left( \psi(\frac{z}{2}) - \psi\left(\frac{z+1}{2}\right) \right), \quad z \in \mathbb{C} \setminus \mathbb{N}_0. \tag{2.3}
\]

\(^1\)Equation (1.3) is the Weierstrass canonical product form defined by Barnes. Barnes introduced and studied in [6, 7, 8, 9], the multiple gamma function which has the recurrence functional equation \( \Gamma_{n+1}(z+1) = \Gamma_n(z) / \Gamma_n(z) \), \( z \in \mathbb{C}, n \in \mathbb{N} \). The relationship between \( \Gamma_2(z) \) and \( G(z) \) is \( G(z) = 1/\Gamma_2(z) \). Adamchik gave integral representations for \( G(z) \) in [10]. Srivastava and Choi in [11] and Adamchik in [10] gave some special cases of the Barnes function.
Replacing the series in (2.2) with the digamma representation in (2.3)

\[
\int_0^\infty \frac{\cos ax}{\cosh x} \, dx = \frac{1}{2} \sum_{k=0}^\infty \frac{(-1)^k}{k + \frac{1}{2} - ia} + \frac{1}{2} \sum_{k=0}^\infty \frac{(-1)^k}{k + \frac{1}{2} + ia} = \frac{1}{4} \left( \psi_0 \left( \frac{3 - ia}{4} \right) - \psi_0 \left( \frac{1 + ia}{4} \right) - \psi_0 \left( \frac{1 - ia}{4} \right) + \psi_0 \left( \frac{3 + ia}{4} \right) \right).
\]

(2.4)

We derive from Euler’s reflection formula

\[
\psi_0(z) - \psi_0(1 - z) = -\pi \cot(\pi z), \quad z \in \mathbb{C} \setminus \mathbb{Z}.
\]

(2.5)

Taking into account of the reflection formula in (2.5), Equation (2.4) can be further simplified.

We now have that

\[
\int_0^\infty \frac{\cos ax}{\cosh x} \, dx = \frac{\pi}{4} \cot \left( \frac{\pi + ia}{4} \right) + \frac{\pi}{4} \cot \left( \frac{\pi - ia}{4} \right).
\]

Recalling a property of the cot function

\[
\cot \left( \frac{\pi}{4} + z \right) = \sec(2z) + \tan(2z), \quad z \neq \frac{2n\pi + \pi}{4}, \ n \in \mathbb{Z}.
\]

Therefore

\[
\int_0^\infty \frac{\cos ax}{\cosh x} \, dx = \frac{\pi}{4} \left( \sec \left( \frac{ia}{2} \right) + \tan \left( \frac{ia}{2} \right) + \sec \left( \frac{ia}{2} \right) - \tan \left( \frac{ia}{2} \right) \right).
\]

Hence

\[
\int_0^\infty \frac{\cos ax}{\cosh x} \, dx = \frac{\pi}{2} \sech \left( \frac{\pi a}{2} \right).
\]

□

3 Derivations

3.1 Closed form for \( \Delta_1 \) and some other results

In this subsection, Vardi’s result in the hyperbolic form in (1.1) will be proved by a new real elementary method.

**Proposition 3.1.** Let \( \Re(b) > 0 \), then

1. \( \int_0^\infty \frac{\ln(x^2 + a^2)}{\cosh(\pi x)} \, dx = 2 \ln \left( \frac{\sqrt{2}\Gamma \left( \frac{|a|}{2} + \frac{1}{2} \right)}{\Gamma \left( \frac{|a|}{2} + \frac{1}{4} + \frac{1}{2} \right)} \right), \quad a \in \mathbb{R}. \)

2. \( \Delta_1 = \int_0^\infty \ln x \, \sech x \, dx = \pi \ln \left( \frac{\Gamma \left( \frac{1}{4} \right) \sqrt{2\pi}}{\Gamma \left( \frac{1}{4} \right)} \right). \)
3. \[ \int_0^\infty \ln \left( ax \right) \sech \left( bx \right) \, dx = \frac{\pi}{b} \ln \left( \frac{\sqrt{2a\pi}}{\sqrt{\Gamma \left( \frac{1}{2} \right)}} \right), \quad a \in \mathbb{R}^+. \]

**Proof of Proposition 3.1.**

Let \( \Delta_1 \left( a \right) := \int_0^\infty \frac{\ln \left( x^2 + a^2 \right)}{\cosh \left( \pi x \right)} \, dx. \)

Then

\[
\Delta_1 \left( a \right) = 2 \int_0^\infty \frac{\ln \left( |a| - ix \right)}{e^{-2\pi x} + 1} e^{-\pi x} \, dx + 2 \int_0^\infty \frac{\ln \left( |a| + ix \right)}{e^{-2\pi x} + 1} e^{-\pi x} \, dx \]

\[
= \frac{2}{\pi} \int_0^\infty \left( \ln \left( |a| - ix \right) + \ln \left( |a| + ix \right) \right) \, d\left( \arctan \left( e^{-\pi x} \right) \right) \]

\[
= -\frac{2i}{\pi} \int_0^\infty \frac{\arctan \left( e^{-\pi x} \right)}{|a| - ix} \, dx + 2 \int_0^\infty \frac{\arctan \left( e^{-\pi x} \right)}{|a| + ix} \, dx + \ln a. \quad (3.1)
\]

From the Laplace transform of 1 with \( s = |a| \pm ix \), we have that

\[
\mathcal{L} \left( 1 \right) \{ |a| \pm ix \} = \int_0^\infty e^{-t(|a|\pm ix)t} \, dt = \frac{1}{|a| \pm ix}, \quad a, x \in \mathbb{R}. \quad (3.2)
\]

Replacing the fractions in (3.1) with the integrals in (3.2), then interchanging the order of integration by Fubini’s theorem [13]

\[
\Delta_1 \left( a \right) - \ln a = -\frac{2i}{\pi} \int_0^\infty \frac{\arctan \left( e^{-\pi x} \right)}{|a| - ix} \int_0^\infty e^{-t(|a| - ix)t} \, dt \, dx + \frac{2i}{\pi} \int_0^\infty \arctan \left( e^{-\pi x} \right) \int_0^\infty e^{-t(|a| + ix)t} \, dt \, dx \]

\[
= -\frac{2i}{\pi} \int_0^\infty e^{-|a|t} \int_0^\infty e^{itx} \arctan \left( e^{-\pi x} \right) \, dx \, dt + \frac{2i}{\pi} \int_0^\infty e^{-|a|t} \int_0^\infty e^{-itx} \arctan \left( e^{-\pi x} \right) \, dx \, dt \]

\[
= -\frac{2i}{\pi} \int_0^\infty e^{-|a|t} \left( \int_0^\infty e^{itx} - e^{-itx} \right) \arctan \left( e^{-\pi x} \right) \, dx \, dt.
\]

By Euler’s complex exponential representation for the sine function

\[ e^{itx} - e^{-itx} = 2i \sin \left( tx \right), \quad t, x \in [0, \infty). \]

Therefore

\[
\Delta_1 \left( a \right) - \ln a = \frac{4}{\pi} \int_0^\infty e^{-|a|t} \int_0^\infty \sin \left( tx \right) \arctan \left( e^{-\pi x} \right) \, dx \, dt \]

\[
= \frac{4}{\pi} \int_0^\infty e^{-|a|t} \int_0^\infty d \left( \frac{-\cos \left( tx \right)}{t} \right) \arctan \left( e^{-\pi x} \right) \, dt \]

\[
= \frac{4}{\pi} \int_0^\infty e^{-|a|t} \left( \frac{\pi}{4t} - \frac{\pi}{2t} \int_0^\infty \frac{\cos \left( tx \right)}{\cosh \left( \pi x \right)} \, dx \right) \, dt.
\]
\[ \Delta_1(a) - \ln a = \frac{4}{\pi} \int_0^\infty e^{-|a|t} \left( \frac{\pi}{4t} - \frac{1}{2} \int_0^\infty \frac{\cos \left( \frac{t}{x} \right)}{\cosh \left( \frac{x}{a} \right)} \, dx \right) \, dt. \]

It follows from the integral representation in (2.1) that

\[ \Delta_1(a) - \ln a = \frac{4}{\pi} \int_0^\infty e^{-|a|t} \left( \frac{\pi}{4t} - \frac{1}{2} \frac{2e^{-\left(|a|+\frac{1}{2}\right)t}}{t(1+e^{-t})} \right) \, dt \]

\[ = \int_0^\infty e^{-|a|t} \left( \frac{1}{t} - \frac{1}{t} \text{sech} \left( \frac{t}{2} \right) \right) \, dt = \int_0^\infty \left( \frac{e^{-|a|t}}{t} - \frac{2e^{-\left(2|a|+1\right)t}}{t(1+e^{-2t})} \right) \, dt \]

\[ = \int_0^\infty \left( \frac{z^{2|a|}}{1+z^2} - \frac{2z^{2|a|+1}}{1+z^2} \right) \, dz = - \int_0^1 \frac{z^{2|a|}}{\ln z} \left( \frac{1}{\ln z} - \frac{2}{1+z^2} \right) \, dz = - \int_0^1 \frac{z^{2|a|}}{\ln z} \left( \frac{1-z^2}{z(1+z^2)} \right) \, dz. \quad (3.3) \]

By elementary integration

\[ \int_0^1 z^p \, dp = -\frac{1}{\ln z}, \quad z \in (0, 1). \quad (3.4) \]

Substituting the integral in (3.4) into (3.3)

\[ \Delta_1(a) - \ln a = \int_0^1 \frac{z^{2|a|-1}(1-z)}{1+z^2} \int_0^1 z^p \, dz = \int_0^1 \int_0^1 \frac{z^{2|a|+p-1}(1-z)}{1+z^2} \, dp \, dz \]

\[ = \int_0^1 \int_0^1 \sum_{k=0}^{\infty} (-1)^k \frac{z^{2|a|+p+2k-1}(1-z)}{1+z^2} \, dz \, dp \]

\[ = \int_0^1 \sum_{k=0}^{\infty} (-1)^k \int_0^1 z^{2|a|+p+2k-1}(1-z) \, dz \, dp \]

\[ = \int_0^1 \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{2|a|+p+2k} - \frac{1}{2|a|+p+2k+1} \right) \, dp \]

\[ = \frac{1}{2} \int_0^1 \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{k + \frac{2|a|+p}{2}} - \frac{1}{k + \frac{2|a|+p+1}{2}} \right) \, dp. \quad (3.5) \]

Replacing the series in (3.5) with the digamma representation in (2.3)

\[ \Delta_1(a) - \ln(a) = \frac{1}{4} \int_0^1 \left( \psi_0 \left( \frac{2|a|+p}{4} \right) - \psi_0 \left( \frac{2|a|+p+1}{4} \right) \right) \, dp \]

\[ = \ln \left( \frac{\Gamma \left( \frac{2|a|}{4} \right) \Gamma \left( \frac{2|a|+1}{4} + \frac{1}{2} \right)}{\Gamma \left( \frac{2|a|}{4} + \frac{1}{2} \right) \Gamma \left( \frac{2|a|+1}{4} \right)} \right) - \ln \left( \frac{\Gamma \left( \frac{2|a|+1}{4} + \frac{1}{2} \right) \Gamma \left( \frac{2|a|+2}{4} + \frac{1}{2} \right)}{\Gamma \left( \frac{2|a|+1}{4} + \frac{1}{2} \right) \Gamma \left( \frac{2|a|+2}{4} \right)} \right) \]

\[ = 2 \ln \left( \frac{\sqrt{2} \Gamma \left( \frac{2|a|+1}{4} + \frac{1}{2} \right)}{\sqrt{a} \Gamma \left( \frac{2|a|+1}{4} \right)} \right). \]
Hence
\[
\frac{1}{\sqrt{\pi}} \Gamma \left( \frac{2|a|+1}{4} \right) + \ln |a| = 2 \ln \left( \frac{\sqrt{\pi} \Gamma \left( \frac{|a|}{2} + \frac{3}{4} \right)}{\Gamma \left( \frac{|a|}{2} + \frac{1}{4} \right)} \right) \tag{3.6}
\]
concluding the proof of Prop. 3.1(1).

To prove Prop. 3.1(2), we take the limit of both sides of (3.6) as \( a \to 0 \).

Thus
\[
\int_0^\infty \ln x \sech (\pi x) \, dx = \ln \left( \frac{\sqrt{\pi} \Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{3}{4} \right)} \right)
\]
\[
\int_0^\infty \ln \left( \frac{x}{\pi} \right) \, \sech x \, dx = \pi \ln \left( \frac{\sqrt{\pi} \Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{3}{4} \right)} \right)
\]
\[
\int_0^\infty \ln x \sech x \, dx - \ln \pi \int_0^\infty \sech x \, dx = \pi \ln \left( \frac{\sqrt{\pi} \Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{3}{4} \right)} \right)
\]
\[
\int_0^\infty \ln x \sech x \, dx - \frac{\pi}{2} \ln \pi = \pi \ln \left( \frac{2 \pi \frac{3}{4}}{\Gamma \left( \frac{3}{4} \right)} \right). \tag{3.7}
\]
By Euler’s reflection formula
\[
\Gamma \left( \frac{1}{4} \right) \Gamma \left( \frac{3}{4} \right) = \sqrt{\pi}.
\]
Hence
\[
\int_0^\infty \ln x \sech x \, dx = \pi \ln \left( \frac{2 \pi \frac{3}{4}}{\Gamma \left( \frac{3}{4} \right)} \right). \tag{3.7}
\]
To prove Prop. 3.1(3), make the substitution \( x = \theta/b \) in Prop. 3.1(3), take into account that \( \ln(c/d) = \ln(c) + \ln(d) \) and \( \int_0^\infty \sech \theta \, d\theta = \pi/2 \), then apply the result in Prop. 3.1(2). \( \square \)

### 3.2 Closed form for \( \Delta_2 \) from certain integrals

Vălean presented two proofs for Prop. 3.2(2) ([14], Sect. 3.40), but we shall not be reconsidering those proofs. The integral in Prop. 3.2(2) shall be splitted into two convergent parts and both will prove to be very useful in evaluating \( \Delta_2 \). It shall be seen later that both integrals give rise to families of integrals not considered in [14].

**Proposition 3.2.**

1. (a) \[
\int_0^\infty \left( \frac{x}{x^2 + y^2} - \frac{x}{x^2 + y^2} \tanh \left( \frac{\pi x}{2} \right) \right) \, dx = \varphi_0 \left( \frac{1 - |y|}{2} \right) + \ln \left( \frac{2}{|y|} \right),
\]
   \( y \in \mathbb{R} \setminus \{0\} \).
Proposition 3.2.

Therefore

$\int_0^\infty \left( \frac{\tanh \left( \frac{\pi x}{2} \right)}{x} - \frac{x}{x^2 + y^2} \right) dx = -\gamma + \ln (2 |y|), \quad y \in \mathbb{R} \setminus \{0\}.$

2. \[ \int_0^\infty \tanh \left( \frac{\pi x}{2} \right) \left( \frac{1}{x} - \frac{x}{x^2 + y^2} \right) dx = -\psi_0 \left( \frac{1}{2} \right) + \psi_0 \left( \frac{1 + |y|}{2} \right), \quad y \in \mathbb{R}. \]

3. \[ \Delta_2 = \int_0^\infty \ln x \operatorname{sech}^2 x \, dx = -\gamma + \ln \left( \frac{\pi}{2} \right). \]

Proof of Proposition 3.2.

Let \( \kappa (y) = \int_0^\infty \tanh \left( \frac{\pi x}{2} \right) \left( \frac{1}{x} - \frac{x}{x^2 + y^2} \right) dx. \)

Then

\[ \kappa (y) = \int_0^\infty (1 - \cos (|y| s)) \int_0^\infty e^{-xs} \tanh \left( \frac{\pi x}{2} \right) dx \, ds. \]

\[ \nu = \int_0^\infty e^{-xs} \left( \frac{1 - e^{-\pi x}}{1 + e^{-\pi x}} \right) dx = \frac{1}{s} - 2 \sum_{k=0}^\infty \frac{(-1)^k}{s + (k + 1) \pi}. \]

\[ \Rightarrow \kappa (y) = \int_0^\infty (1 - \cos (|y| s)) \left( \frac{1}{s} - 2 \sum_{k=0}^\infty \frac{(-1)^k}{s + (k + 1) \pi} \right) ds \]

\[ = \int_0^\infty (1 - \cos (|y| s)) \left( \int_0^\infty e^{-st} dt - 2 \sum_{k=0}^\infty \int_0^\infty (-1)^k e^{-(s+(k+1)\pi)t} dt \right) ds \]

\[ = \int_0^\infty (1 - \cos (|y| s)) \left( \int_0^\infty e^{-st} dt - 2 \sum_{k=0}^\infty \int_0^\infty e^{-(s+\pi) t} \sum_{k=0}^\infty (-1)^k e^{-k \pi t} dt \right) ds \]

\[ = \int_0^\infty (1 - \cos (|y| s)) \left( \int_0^\infty e^{-st} dt - 2 \int_0^\infty \frac{e^{-(s+\pi) t}}{1 + e^{-\pi t}} dt \right) ds \]

\[ = \int_0^\infty \int_0^\infty (1 - \cos (|y| s)) \left( e^{-st} - \frac{2e^{-(s+\pi) t}}{1 + e^{-\pi t}} \right) dt \, ds \]

\[ = \int_0^\infty \int_0^\infty e^{-st} (1 - \cos (|y| s)) \left( 1 + \left( \tanh \left( \frac{\pi t}{2} \right) - 1 \right) \right) ds \, dt \]

\[ = \int_0^\infty \left( \frac{\tanh \left( \frac{\pi t}{2} \right)}{t} - \frac{1}{t^2 + y^2} \right) \, dt \left( \frac{t}{t^2 + y^2} - \frac{t}{t^2 + y^2} \tanh \left( \frac{\pi t}{2} \right) \right) + \frac{t}{t^2 + y^2} \, dt. \]

Therefore

\[ \kappa (y) = \int_0^\infty \left( \frac{\tanh \left( \frac{\pi t}{2} \right)}{t} - \frac{1}{t^2 + y^2} \right) \, dt \left( \int_0^\infty \left( \frac{t}{t^2 + y^2} - \frac{t}{t^2 + y^2} \tanh \left( \frac{\pi t}{2} \right) \right) \, dt \right) \]

\[ \kappa_1 (y) \]

\[ \kappa_2 (y) \]

\[ \kappa (y) \]

\[ \kappa (y) \] can be split into two integrals \( \kappa_1 (y) \) and \( \kappa_2 (y) \) since \( \kappa_1 (y) \) and \( \kappa_2 (y) \) are convergent.

\[ \kappa (y), \kappa_1 (y) \] and \( \kappa_2 (y) \) may be shortened, respectively, to \( \kappa, \kappa_1 \) and \( \kappa_2 \) for simplicity.
3.2.1 Closed form for $\kappa_1$ by a tanh-series derived from a product formula

Recalling a product formula for the hyperbolic cosine function ([15], p. 8)

$$\cosh x = \prod_{k=0}^{\infty} \left( 1 + \frac{4x^2}{\pi^2 (2k + 1)^2} \right), \quad x \in \mathbb{R}. \quad (3.8)$$

Taking the logarithmic derivative of both sides of (3.8), we obtain

$$\tanh x = 8x \sum_{k=0}^{\infty} \frac{1}{4k^2 + \pi^2 (2k + 1)^2}, \quad x \in \mathbb{R}. \quad (3.9)$$

Rewriting $\kappa_1$ as the limit of a definite integral, we have

$$\kappa_1 = \int_0^\infty \left( \frac{t}{t^2 + y^2} - \frac{t}{t^2 + y^2} \tanh \left( \frac{\pi t}{2} \right) \right) \, dt \quad (3.10)$$

$$= \lim_{N \to \infty} \int_0^{\epsilon_1 N} \left( \frac{t}{t^2 + y^2} - \frac{t}{t^2 + y^2} \tanh \left( \frac{\pi t}{2} \right) \right) \, dt$$

$$= \lim_{N \to \infty} \kappa_1 (N), \text{ where } \epsilon_1 \in \mathbb{R}^+. \quad (3.11)$$

Replacing $x$ in (3.9) with $\pi t/2$ and substituting the result into (3.11)

$$\kappa_1 = \lim_{N \to \infty} \int_0^{\epsilon_1 N} \left( \frac{t}{t^2 + y^2} - \frac{4t}{\pi} \sum_{k=0}^{N} \frac{t^2}{t^2 + (2k + 1)^2} \right) \, dt.$$ \quad (3.12)

$$\kappa_1 (N) = \frac{4}{\pi} \sum_{k=0}^{N} \left( \frac{1}{(2k+1)^2 - y^2} - \frac{2k + 1}{(2k + 1)^2 - y^2} \arctan \left( \frac{\epsilon_1 N}{2k + 1} \right) \right)$$

$$+ \frac{1}{2} \ln \left( \left( \frac{\epsilon_1 N}{y} \right)^2 + 1 \right). \quad (3.13)$$

It follows from (3.13) that

$$\kappa_1 (y) = \lim_{N \to \infty} \left( \frac{-2}{\pi} \sum_{k=0}^{N} \frac{1}{2k + 1 + |y|} + \frac{1}{2} \ln \left( \left( \frac{\epsilon_1 N}{y} \right)^2 + 1 \right) \right)$$

$$= \lim_{N \to \infty} \left( \sum_{k=0}^{N} \frac{1}{k + 1} - \frac{1}{k + 1 + |y|} \right) - \sum_{k=1}^{N+1} \frac{1}{k} + \frac{1}{2} \ln \left( \left( \frac{\epsilon_1 N}{y} \right)^2 + 1 \right)$$

$$= \lim_{N \to \infty} \left( \sum_{k=0}^{N} \frac{1}{k + 1} - \frac{1}{k + 1 + |y|} \right) - \sum_{k=1}^{N} \frac{1}{k} + \ln N + \ln \left( \frac{\epsilon_1}{|y|} \right)$$

$$= -\gamma + \sum_{k=0}^{\infty} \left( \frac{1}{k + 1} - \frac{1}{k + 1 + |y|} \right) - \ln |y| + \ln (\epsilon_1)$$

$$= \phi_0 \left( \frac{1 + |y|}{2} \right) - \ln |y| + \ln (\epsilon_1).$$
\[
\lim_{y \to \infty} \kappa_1(y) = \lim_{y \to \infty} \left( \psi_0 \left( \frac{1 + |y|}{2} \right) - \ln |y| + \ln (\epsilon_1) \right)
\]
\[
= \lim_{y \to \infty} \left( -y + H_{\frac{-1+|y|}{2}} - \ln \left( \frac{-1+|y|}{2} \right) + \ln \left( \frac{1+|y|}{2} \right) + \ln (\epsilon_1) \right)
\]
\[
= -y + y - \ln (2) + \ln (\epsilon_1), \quad \implies \epsilon_1 = 2.
\]

Therefore
\[
\kappa_1(y) = \int_0^\infty \left( \frac{t}{1+t^2+y^2} - \frac{t}{1+t^2+y^2} \tanh \left( \frac{\pi t}{y} \right) \right) dt = \psi_0 \left( \frac{1 + |y|}{2} \right) + \ln \left( \frac{2}{|y|} \right). \quad (3.14)
\]

### 3.2.2 Closed form for \( \kappa_2 \) by a similar approach used in 3.2.1

\[
\kappa_2(y) = \int_0^\infty \left( \frac{\tanh \left( \frac{\pi t}{y} \right)}{t} - \frac{t}{1+t^2+y^2} \right) dt
\]
\[
= \lim_{N \to \infty} \int_0^\infty \left( \sum_{k=0}^{N} \frac{1}{(2k+1)^2 + y^2} - \frac{t}{1+t^2+y^2} \right) dt, \quad \text{where } \epsilon_2 \in \mathbb{R}^+.
\]
\[
\kappa_2(y) = \lim_{N \to \infty} \left( \sum_{k=0}^{N} \frac{\ln \left( \frac{2}{|y|} \right)}{2k+1} - \ln \left( \frac{\epsilon_2 N}{2k+1} \right) \right) + \ln |y|
\]
\[
= \lim_{N \to \infty} \left( \sum_{k=0}^{N} \frac{1}{2k+1} - \ln \left( \frac{\epsilon_2 N}{2k+1} \right) \right) + \ln \left( \frac{|y|}{\epsilon_2} \right)
\]
\[
= \lim_{N \to \infty} \left( 2H_{2N+1} - H_N \ln N + \ln \left( \frac{|y|}{\epsilon_2} \right) \right)
\]
\[
= \lim_{N \to \infty} \left( 2H_{2N+1} - 2 \ln (2N+1) - H_N \ln + 2 \ln \left( \frac{2N+1}{N} \right) \right) + \ln \left( \frac{|y|}{\epsilon_2} \right)
\]
\[
= 2y - y + \ln \left( \frac{4|y|}{\epsilon_2} \right) = y + \ln \left( \frac{4|y|}{\epsilon_2} \right).
\]

To determine the value of \( \epsilon_2 \), we take limits as \( y \to 0 \) on both sides of \( \kappa(y) \).

\[
\lim_{y \to 0} \kappa(y) = \lim_{y \to 0} \int_0^\infty \tanh \left( \frac{\pi x}{y} \right) \left( \frac{1}{x} - \frac{x}{x^2+y^2} \right) dx
\]
\[
= \lim_{y \to 0} \left( y + \ln \left( \frac{4|y|}{\epsilon_2} \right) + \psi_0 \left( \frac{1 + |y|}{2} \right) + \ln \left( \frac{2}{|y|} \right) \right), \quad \implies \epsilon_2 = 2.
\]

Therefore
\[
\kappa_2(y) = \int_0^\infty \left( \frac{\tanh \left( \frac{\pi t}{y} \right)}{t} - \frac{t}{1+t^2+y^2} \right) dt = y + \ln (2 |y|). \quad (3.15)
\]

Adding (3.14) and (3.15), we conclude that
\[
\int_0^\infty \tanh \left( \frac{\pi t}{2} \right) \left( \frac{1}{t} - \frac{t}{1+t^2+y^2} \right) dt = -\psi_0 \left( \frac{1}{2} \right) + \psi_0 \left( \frac{1 + |y|}{2} \right). \quad (3.16)
\]
3.2.3 A more direct approach for the evaluation of $\kappa$

By a similar approach used in 3.2.1 and 3.2.2, we replace $x$ with $\pi t / 2$ in the tanh-series in (3.9) and substitute into $\kappa$.

$$\kappa(y) = \int_0^\infty \tanh \left( \frac{\pi t}{2} \right) \left( \frac{1}{t} - \frac{t}{t^2 + y^2} \right) dt$$

$$= \frac{4}{\pi} \sum_{k=0}^{\infty} \int_0^\infty \left( \frac{1}{t^2 + (2k+1)^2} - \frac{1}{t^2 + (2k+1)^2} \cdot \frac{t^2}{t^2 + y^2} \right) dt$$

$$= 2 \sum_{k=0}^{\infty} \left( \frac{1}{2k+1} + \frac{|y|}{(2k+1)^2 - y^2} - \frac{2k+1}{(2k+1)^2 - y^2} \right)$$

$$= -\left( -\gamma + \sum_{k=0}^{\infty} \left( \frac{1}{k+1} - \frac{1}{k+1/2} \right) \right) - \gamma + \sum_{k=0}^{\infty} \left( \frac{1}{k+1} - \frac{1}{k + \frac{|y|}{2}} \right)$$

$$= -\psi_0 \left( \frac{1}{2} \right) + \phi_0 \left( \frac{1 + |y|}{2} \right).$$

3.2.4 Deriving $\Delta_2$ from $\kappa_2$

We start by writing $\kappa_2(y)$ as a limit of a definite integral. By integration by parts, $\Delta_2$ is derived.

$$\kappa_2(y) = \lim_{N \to \infty} \int_0^N \left( \tanh \frac{t}{N} \frac{t}{t^2 + \frac{\pi^2 y^2}{4}} \right) dt = \gamma + \ln \left( 2 |y| \right)$$

$$= \lim_{N \to \infty} \int_0^N \left( \tanh t \ln t - \frac{\ln \left( t^2 + \frac{\pi^2 y^2}{4} \right)}{2} \right) dt$$

$$= \lim_{N \to \infty} \left( \ln t \tanh t - \frac{1}{2} \ln \left( t^2 + \frac{\pi^2 y^2}{4} \right) \right|_0^N - \int_0^N \ln t \sech^2 t \ dt$$

$$= \lim_{N \to \infty} \left( \ln N \tanh N - \frac{1}{2} \ln \left( N^2 + \frac{\pi^2 y^2}{4} \right) \right) - \int_0^\infty \ln t \sech^2 t \ dt + \frac{1}{2} \ln \left( \frac{\pi^2 y^2}{4} \right)$$

$$= \lim_{N \to \infty} \left( \ln N \tanh (N - 1) - \frac{1}{2} \ln \left( 1 + \frac{\pi^2 y^2}{4N^2} \right) \right) - \int_0^\infty \ln t \sech^2 t \ dt + \ln \left( \frac{\pi |y|}{2} \right)$$

$$= \ln \left( \frac{\pi |y|}{2} \right) - \int_0^\infty \ln t \sech^2 t \ dt.$$  

Hence

$$\int_0^\infty \ln t \sech^2 t \ dt = -\gamma + \ln \left( \frac{\pi}{4} \right).$$  \hspace{1cm} (3.17)

□
Proposition 3.3. Let $b \in \mathbb{R}^+$ and $\Re(a) > 0$, then

1. \[ \int_0^\infty \tanh(at) \left( \frac{1}{t} - \frac{t}{t^2 + y^2} \right) dt = -\psi_0 \left( \frac{1}{2} \right) + \psi_0 \left( \frac{1}{2} + \frac{a|y|}{\pi} \right), \quad y \in \mathbb{R}. \]

2. \[ \int_0^\infty \frac{t}{t^2 + y^2} (1 - \tanh(at)) dt = \psi_0 \left( \frac{1}{2} + \frac{a|y|}{\pi} \right) + \ln \left( \frac{\pi}{a |y|} \right), \quad y \in \mathbb{R} \setminus \{0\}. \]

3. \[ \int_0^\infty \ln(bt) \operatorname{sech}^2(at) dt = -\frac{y}{a} + \frac{1}{a} \ln \left( \frac{b \pi}{4a} \right). \]

4. \[ \int_0^\infty \left( \frac{\tanh(at)}{t} - \frac{t}{t^2 + y^2} \right) dt = \gamma + \ln \left( \frac{4a|y|}{\pi} \right), \quad y \in \mathbb{R} \setminus \{0\}. \]

Proof of Proposition 3.3. Proposition 3.3 is a generalization of Proposition 3.2. To obtain the results in Prop. 3.3(1), Prop. 3.3(2), and Prop. 3.3(4), replace $|y|$ with $2a|y|/\pi$ in Prop. 3.2(1) and Prop. 3.2(2) and make the substitution $x = 2at/\pi$. To obtain Prop. 3.3(3), make the substitution $t = x/a$ in Prop. 3.3(3) take into account that $\ln(cd) = \ln(c) + \ln(d)$ and $\int_0^\infty \operatorname{sech}^2 x dx = 1$, then apply the result in Prop. 3.2(3). \( \square \)

Proposition 3.4. Let $y \in \mathbb{R}$, then

1. \[ \int_0^\infty \ln \cosh(at) \left( \frac{1}{t^2 + y^2} \right) dt = -\frac{a}{y^2} \psi_0 \left( \frac{1}{2} \right) + \frac{\pi}{y^2 |y|} \ln \left( \frac{\Gamma \left( \frac{1}{2} + \frac{a|y|}{\pi} \right)}{\sqrt{\pi}} \right), \quad \Re(a) \geq 0. \]

2. \[ \int_0^\infty \tanh \left( \frac{a}{t} - \frac{a}{|y|} \right) \arctan \left( \frac{|y|}{t} \right) dt = -a \psi_0 \left( \frac{1}{2} \right) + \frac{\pi}{|y|} \ln \left( \frac{\Gamma \left( \frac{1}{2} + \frac{a|y|}{\pi} \right)}{\sqrt{\pi}} \right), \quad \Re(a) \geq 0. \]

3. \[ \int_0^\infty \arctan \left( \frac{|y|}{t} \right) (1 - \tanh(at)) dt = \frac{\pi}{a} \ln \left( \frac{\Gamma \left( \frac{1}{2} + \frac{a|y|}{\pi} \right)}{\sqrt{\pi}} \right) + |y| \ln \left( \frac{\pi}{a |y|} \right) + 1, \quad \Re(a) > 0. \]

4. \[ \int_0^\infty \ln \left( e^{at} \operatorname{sech}(at) \right) \left( \frac{1}{t^2 + y^2} \right) dt = \frac{\pi}{|y|} \ln \left( \frac{\Gamma \left( \frac{1}{2} + \frac{a|y|}{\pi} \right)}{\sqrt{\pi}} \right) + a \left( \ln \left( \frac{\pi}{a |y|} \right) + 1 \right), \quad \Re(a) > 0. \]

5. \[ \int_0^\infty \left( \frac{\ln \cosh(at)}{at^2} - \frac{t}{t^2 + y^2} \right) dt = \gamma - 1 + \ln \left( \frac{4a|y|}{\pi} \right), \quad \Re(a) > 0. \]

6. \[ \int_0^\infty \left( \frac{\ln \cosh(at)}{t^2} - \frac{a}{|y|} \arctan \left( \frac{|y|}{t} \right) \right) dt = a (\gamma - 2) + a \ln \left( \frac{4a|y|}{\pi} \right), \quad \Re(a) > 0. \]
7. \( \int_{0}^{\infty} \left( \frac{\tanh(at)}{t} - \frac{1}{|y|} \arctan \left( \frac{|y|}{t} \right) \right) \, dt = \gamma - 1 + \ln \left( \frac{4a|y|}{\pi} \right), \quad \Re(a) > 0. \)

**Proof of Proposition 3.4.** We prove each of the results in Prop. 3.4 by integrating, respectively, both sides of the results in Prop. 3.3 over \([0,a]\) wrt. \(a\) and \([0,|y|]\) wrt. \(|y|\). Only the result in Prop. 3.4(6) is obtained from the one in Prop. 3.4(5) by integrating both sides of Prop. 3.4(5) over \([0,|y|]\) wrt. \(|y|\). \(\square\)

**Remark 3.5.** None of 2, 3, 6 and 7 in Proposition 3.4 appear in Gradshteyn and Ryzhik’s tables ([2], 7th edition, 4.56, p. 605).

**Example.** We derive the following results from Proposition 3.4

1. \( \gamma = -\ln 2 + \int_{0}^{\infty} \frac{\tanh(t)}{t} \left( \frac{1}{t} - \frac{1}{\pi} \arctan \left( \frac{\pi}{t} \right) \right) \, dt. \)

2. \( \gamma = -\ln 2 + \int_{0}^{\infty} \frac{\pi^2 \ln \cosh t}{t^2 + \pi^2} \, dt. \)

3. \( \ln \left( \frac{e}{2} \right) = \int_{0}^{\infty} \frac{\ln (e \sech t)}{t^2 + \pi^2} \, dt. \)

4. \( \pi \ln \left( \frac{\pi}{2} \right) = \int_{0}^{\infty} \arctan \left( \frac{\pi}{t} \right) (1 - \tanh t) \, dt. \)

5. \( \gamma = 1 + \int_{0}^{\infty} \left( \frac{\ln \cosh (\pi t)}{\pi t^2} - \frac{16t}{16t^2 + 1} \right) \, dt. \)

6. \( \gamma = 1 + \int_{0}^{\infty} \left( \frac{\tanh (\pi t)}{t} - 4 \arctan \left( \frac{1}{4t} \right) \right) \, dt. \)

7. \( \gamma = 2 + \int_{0}^{\infty} \left( \frac{\ln \cosh (\pi t)}{\pi t^2} - 4 \arctan \left( \frac{1}{4t} \right) \right) \, dt. \)

8. \( 1 = \int_{0}^{\infty} \left( \frac{\tanh (\pi t)}{t} - \ln \cosh (\pi t) \right) \left( \frac{1}{t} - \frac{1}{\pi} \arctan \left( \frac{|y|}{t} \right) \right) \, dt. \)

**Remark 3.6.** Examples 1, 2, 5, 6 and 7 can be regarded as integral representations for the Euler-Mascheroni \(\gamma\) constant. Other integral representations for \(\gamma\) can also be written from Proposition 3.4, for \(y \in \mathbb{R} \setminus \{0\}\) and \(\Re(a) > 0\), as follows:

\[
\gamma = -\ln 4 + \frac{\pi}{a|y|} \left( \frac{1}{2} \ln \pi - \ln \Gamma \left( \frac{1}{2} + \frac{|y|}{\pi} \right) \right) + \int_{0}^{\infty} \frac{\tanh(at)}{t} \left( \frac{1}{t} - \frac{1}{|y|} \arctan \left( \frac{|y|}{t} \right) \right) \, dt
\]

\[
= -\ln 4 + \frac{\pi}{a|y|} \left( \frac{1}{2} \ln \pi - \ln \Gamma \left( \frac{1}{2} + \frac{|y|}{\pi} \right) \right) + \frac{y^2}{a} \int_{0}^{\infty} \frac{\ln \cosh (at)}{t^2 (t^2 + y^2)} \, dt
\]

\[
= 1 - \ln \left( \frac{4a|y|}{\pi} \right) + \int_{0}^{\infty} \frac{\ln \cosh (at)}{at^2} - \frac{t}{t^2 + y^2} \, dt
\]

\[
= 2 - \ln \left( \frac{4a|y|}{\pi} \right) + \int_{0}^{\infty} \frac{\ln \cosh (at)}{at^2} - \frac{1}{|y|} \arctan \left( \frac{|y|}{t} \right) \, dt
\]
\[= 1 - \ln\left(\frac{4a|y|}{\pi}\right) + \int_0^\infty \left(\frac{\tanh(at)}{t} - \frac{1}{|y|} \arctan\left(\frac{|y|}{t}\right)\right) dt.\] (3.18)

It is clear that Ex. 1, 2, 5, 6 and 7 are the simplest forms for \(\gamma\) in the families of integrals in (3.18). More integral representations can be written for \(\gamma\) from other propositions in this work and all the representations in (3.18) are symmetric, for \(y, a \in \mathbb{R}^n\).

**Remark 3.7.** It can be easily deduced from Proposition 3.3(2) that
\[\int_0^\infty \ln(t^2 + y^2) \, \text{sech}(at) \, dt = \frac{2}{a} \left(\psi_0\left(\frac{1}{2} + \frac{a|y|}{\pi}\right) + \ln\left(\frac{\pi}{a}\right)\right)\] (3.19)
where \(\Re(a) > 0, y \in \mathbb{R}\).

### 3.3 Closed forms for \(\Delta_n\) for \(3 \leq n \leq 6\)

Subsequently, \(\Delta_n\) for \(n \geq 3\) will be evaluated by integration by parts and each \(\Delta_n\) for \(n \geq 3\) will require the use of previous results obtained from evaluating \(\Delta_{n-2}\) for integers \(n \geq 3\), respectively. We hereby designate two important integrals in this work to \(\lambda\) and \(\delta\) as follows:
\[
\lambda_n = \int_0^\infty \frac{\tanh x \, \text{sech}^n x}{x} \, dx, \quad \delta_n = \int_0^\infty \frac{1 - \text{sech} x}{x^2} \, \text{sech}^n x \, dx.
\]

#### 3.3.1 Closed form for \(\Delta_3\) and some other results

We begin with integration by parts and continue by evaluating \(\lambda_1\) with an application of a property of the gamma function and the integral of the log-sine function over \((0, \pi/4]\) and \((0, \pi/2]\). Eventually, we have \(\lambda_1\) in terms of Catalan’s constant and \(\pi\). A closed form for \(\Delta_3\) then follows.

**Proposition 3.8.**

1. \(\lambda_1 = \delta_0 = \int_0^\infty \frac{\tanh x \, \text{sech} x}{x} \, dx = \int_0^\infty \frac{1 - \text{sech} x}{x^2} \, dx = \frac{4G}{\pi}\)

2. \(\Delta_3 = \int_0^\infty \ln x \, \text{sech}^3 x \, dx = -\frac{2G}{\pi} + \frac{\pi}{2} \ln\left(\frac{\sqrt{2\pi} \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}\right)\).

**Proof of Proposition 3.8.** Integrating by parts
\[
\Delta_3 = \frac{1}{2} (\lambda_1 - \lambda_1)
\] (3.20)

\[
\lambda_1 = \int_0^\infty \frac{\tanh x \, \text{sech} x}{x} \, dx = \int_0^\infty \frac{1}{x} \, d(1 - \text{sech} x) = \int_0^\infty \frac{1 - \text{sech} x}{x^2} \, dx
\]

\[
= \int_0^1 \int_0^1 \int_0^\infty \frac{e^{-(a+b)x}}{1 + e^{-2x}} \, dx \, da \, db = \int_0^1 \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + a + b} \, dx \, da \, db
\]

\[
= \frac{1}{4} \int_0^1 \int_0^1 \psi_0\left(\frac{a+b+2}{4}\right) - \psi_0\left(\frac{a+b}{4}\right) \, da \, db.
\]
Therefore
\[
\lambda_1 = \int_0^1 \left( \ln \Gamma \left( \frac{b+3}{4} \right) - \ln \Gamma \left( \frac{b+1}{4} \right) - \ln \Gamma \left( \frac{b+2}{4} \right) + \ln \Gamma \left( \frac{b}{4} \right) \right) \, db. \tag{3.21}
\]
Simplifying (3.21) with Euler’s reflection formula, we have that
\[
\lambda_1 = \frac{4}{\pi} \left( \int_0^1 \frac{\pi}{2} \ln \sin(b) \, db - \int_0^1 \frac{\pi}{2} \ln \sin(b) \, db \right).
\]
An integral representation for the beta function is given by
\[
B(a, b) = 2 \int_0^\pi \sin^{2a-1} \theta \cos^{2b-1} \theta \, d\theta. \tag{3.22}
\]
Differentiating (3.22) wrt. \( a \) at \( a = b = 1/2 \), we have that
\[
\int_0^1 \ln \sin(b) \, db = \frac{\pi}{2} \ln 2. \tag{3.23}
\]
Integrating the log-tangent function over \((0, \pi/4]\)
\[
\int_0^\pi \ln \tan(b) \, db = \int_0^1 \frac{\ln \frac{t}{1+t^2}}{1+t^2} \, dt = -\text{G} \tag{3.24}
\]
It follows from (3.23) and (3.24) that
\[
\int_0^1 \ln \sin(b) \, db = -\frac{\text{G}}{2} + \frac{\pi}{4} \ln 2.
\]
Hence
\[
\lambda_1 = \int_0^\infty \frac{\tanh x \sech x}{x} \, dx = \frac{4\text{G}}{\pi}. \tag{3.25}
\]
Recalling the closed form of \( \Delta_1 \) in (3.7) and using (3.25), we have from (3.20) that
\[
\int_0^\infty \ln x \, \sech^3 x \, dx = -\frac{2\text{G}}{\pi} + \frac{\pi}{2} \ln \left( \frac{\sqrt{2\pi} \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{4} \right)} \right). \tag{3.26}
\]

### 3.3.2 Closed form for \( \Delta_4 \) and some other results

We evaluate \( \Delta_4 \) by parts and then derive the definite integrals of \( x \ln \Gamma(x) \), \( \ln G(1 + x) \), \( \ln G(x) \) over \((0, z]\), where \( z \in (0, \infty) \). The derived integrals are later used to evaluate \( \delta_1 \) and determine a closed form for \( \Delta_4 \).

**Proposition 3.9.**

1. \( \delta_1 = \int_0^\infty \frac{1 - \sech x}{x^2} \sech x \, dx = -\frac{4\text{G}}{\pi} + \frac{14\zeta(3)}{\pi^2} \).
2. \( \lambda_2 = \int_0^\infty \frac{\tanh x \sech^2 x}{x} \, dx = \frac{7\zeta(3)}{\pi^2} \).
3. \( \Delta_4 = \int_0^\infty \ln x \sech^4 x \, dx = -\frac{2\gamma}{3} - \frac{7\zeta(3)}{3\pi^2} + \frac{2}{3} \ln \left( \frac{\pi}{4} \right) \).

Proof of Proposition 3.9. Integrating by parts

\[
\Delta_4 = \frac{1}{3} \lambda_2 + \frac{2}{3} \lambda_2, \quad \lambda_2 = \frac{1}{2} \delta_1 + \frac{1}{2} \Delta_1
\]

(3.27)

\[
\frac{1}{2} \delta_1 = \int_0^1 \int_0^1 \int_0^\infty \frac{e^{-(a+b)x}}{(1 + e^{-2x})^2} \, dx \, da \, db.
\]

For \( s \in \mathbb{C} \setminus -2\mathbb{N}_0 \)

\[
\int_0^\infty \frac{e^{-(s+2)x}}{(1 + e^{-2x})^2} \, dx = -\frac{1}{4} + \frac{s}{8} \left( \psi_0 \left( \frac{s}{4} + \frac{1}{2} \right) - \psi_0 \left( \frac{s}{4} \right) \right).
\]

It now follows that

\[
\frac{1}{2} \delta_1 = \int_0^1 \int_0^1 -\frac{1}{4} + \frac{a + b - 1}{8} \left( \psi_0 \left( \frac{a + b - 1}{4} + \frac{1}{2} \right) - \psi_0 \left( \frac{a + b - 1}{4} \right) \right) \, da \, db.
\]

(3.28)

\[
\varphi_1(b) := \int_0^1 \frac{a + b - 1}{8} \left( \psi_0 \left( \frac{a + b - 1}{4} + \frac{1}{2} \right) - \psi_0 \left( \frac{a + b - 1}{4} \right) \right) \, da
\]

\[
= 2 \int_{\frac{b}{4}}^{\frac{b}{2}} \left( \ln \Gamma(u) - \ln \Gamma(u + \frac{1}{2}) \right) \, du + \frac{b}{2} \left( \frac{\Gamma \left( \frac{b+2}{4} \right)}{\Gamma \left( \frac{b}{4} \right)} - \frac{b - 1}{2} \ln \left( \frac{\Gamma \left( \frac{b+1}{4} \right)}{\Gamma \left( \frac{b}{4} \right)} \right) \right).
\]

(3.29)

The integral representation of the Barnes \( G \)-function due to Alexejewsky [16] and Barnes [17] is:

\[
\int_0^1 \ln \Gamma(x) \, dx = \frac{1}{2} \ln 2\pi + \frac{t (1 - t)}{2} - \ln G(1 + t) + t \ln \Gamma(t), \quad \Re(t) > -1.
\]

(3.30)

In application of (3.30), we derive the following:

\[
\int_{\frac{b}{24}}^{\frac{b}{4}} \ln \Gamma \left( \frac{u + 1}{2} \right) - \ln \Gamma(u) \, du = \int_{\frac{b}{24}}^{\frac{b}{4}} \ln \Gamma(u + \frac{1}{2}) - \ln \Gamma(u) \, du
\]

\[
= \frac{b + 2}{4} \ln \left( \frac{b + 2}{4} \right) - \frac{b + 1}{4} \ln \left( \frac{b + 1}{4} \right) - \frac{b - 1}{4} \ln \left( \frac{b - 1}{4} \right) - \frac{b}{4} \ln \left( \frac{b}{4} \right) + \ln \left( \frac{b + 5}{4} \right) + \ln \left( \frac{b + 4}{4} \right) - \ln \left( \frac{b + 3}{4} \right) - \ln \left( \frac{b + 2}{4} \right).
\]

(3.31)

Substituting the result in (3.31) into (3.29)

\[
\varphi_1(b) = \frac{1}{4} - 2 \ln \left( \frac{G \left( \frac{b+5}{4} \right) G \left( \frac{b+4}{4} \right)}{G \left( \frac{b+6}{4} \right) G \left( \frac{b+3}{4} \right)} \right) + \ln \Gamma \left( \frac{b + 1}{4} \right) - \ln \Gamma \left( \frac{b + 2}{4} \right).
\]

(3.32)
Substituting (3.32) into (3.28)

\[
\frac{1}{2} \delta_1 = \int_0^1 -2 \ln \left( \frac{G \left( \frac{b+5}{4} \right) G \left( \frac{b+1}{4} \right)}{G \left( \frac{b+6}{4} \right) G \left( \frac{b+3}{4} \right)} \right) + \ln \Gamma \left( \frac{b+1}{4} \right) - \ln \Gamma \left( \frac{b+2}{4} \right) \, db. \tag{3.33}
\]

We define \( \varphi^* \) as follows:

\[
\frac{\varphi^*}{4} := \frac{1}{4} \int_0^1 \ln \left( \frac{G \left( \frac{b+5}{4} \right) G \left( \frac{b+1}{4} \right)}{G \left( \frac{b+6}{4} \right) G \left( \frac{b+3}{4} \right)} \right) \, db = 2 \int_0^{\frac{1}{4}} \ln G(b + 1) \, db - \int_{\frac{1}{4}}^1 \ln G(b) \, db. \tag{3.34}
\]

The functional relationship with the Hurwitz zeta function \( \zeta(s, t) \), due to Gosper [18] and Vardi [19] is:

\[
\ln G(1 + t) - t \ln \Gamma(t) = \zeta'(-1) - \zeta'(-1, t), \quad \Re(t) > 0. \tag{3.35}
\]

Substituting (3.35) into (3.30), we have that

\[
\int_0^t \ln \Gamma(x) \, dx = \frac{t}{2} \ln 2\pi + \frac{t(1-t)}{2} - \zeta'(-1) + \zeta'(-1, t). \tag{3.36}
\]

Integrating both sides of (3.36) from \( t = 0 \) to \( t = z \) and changing limits

\[
\int_0^z \int_x^z \ln \Gamma(x) \, dx \, dz = \frac{z^2}{4} \ln 2\pi + \frac{1}{12} \left( 3z^2 - 2z^3 \right) - z\zeta'(-1) + \int_0^z \zeta'(-1, t) \, dt
\]

\[
\int_0^z \ln \Gamma(x) \, dx = z \int_0^z \ln \Gamma(x) \, dx - \frac{z^2}{4} \ln 2\pi - \frac{1}{12} \left( 3z^2 - 2z^3 \right) + z\zeta'(-1)
\]

\[
- \frac{1}{4} \left( \zeta(-2, z) - \zeta(-2) \right) - \frac{1}{2} \left( \zeta'(-2, z) - \zeta'(-2) \right). \tag{3.37}
\]

The following are obtained from Hermite integral in (1.2)

\[
\zeta(-2, z) = \frac{1}{3} \left( -\zeta^3 + \frac{3\zeta^2}{2} - \frac{z}{2} \right), \quad \zeta(-2) = 0.
\]

The functional equation of the Riemann zeta function for all complex \( s \) is given by ([20], p. 27)

\[
\zeta(1 - s) = 2 (2\pi)^{-s} \cos \left( \frac{s\pi}{2} \right) \Gamma(s) \zeta(s). \tag{3.38}
\]

Differentiating (3.38) at \( s = 3 \), we have

\[
\zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}. \tag{3.39}
\]

Also, \( \zeta'(-1) = \frac{1}{12} - \ln A \).
Using the derived results and formula (3.30) in (3.37)

\[
\int_0^\gamma x \ln \Gamma(x) \, dx = \frac{-1}{8\pi^2} (2z + 1) (z - 1) + \frac{z^2}{4} \ln 2\pi + z^2 \ln \Gamma(z) - z \ln A - z \ln G(1 + z) - \frac{\zeta(3)}{8\pi^2} - \frac{1}{2} \zeta'(-2, z).
\]  

(3.40)

Integrating both sides (3.35) from \( x = 0 \) to \( x = \gamma \)

\[
\int_0^\gamma \ln G(1 + x) \, dx = \int_0^\gamma x \ln \Gamma(x) \, dx + z\zeta'(-1) - \int_0^\gamma \zeta'(-1, x) \, dx
\]

\[
= -\frac{1}{12}z(2z^2 - 3) + \frac{z^2 \ln 2\pi}{4} + z^2 \ln \Gamma(z) - 2z \ln A - z \ln G(1 + z)
\]

\[
- \frac{\zeta(3)}{4\pi^2} - \zeta'(-2, z).
\]  

(3.41)

Since \( G(1 + x) = G(x)\Gamma(x) \), we apply formula (3.30) and have that

\[
\int_0^\gamma \ln G(x) \, dx = -\frac{1}{12}z(2z^2 - 6z + 3) + \frac{z(z - 2)}{4} \ln 2\pi + z(z - 1) \ln \Gamma(z)
\]

\[
+ (1 - z) \ln G(1 + z) - 2z \ln A - \frac{\zeta(3)}{4\pi^2} - \zeta'(-2, z).
\]  

(3.42)

Setting \( z = \frac{3}{2} \) and \( z = \frac{7}{2} \) in (3.42), subtracting the first result obtained from the second, and using the property \( \zeta'(s, z + 1) = \zeta'(s, z) + z^{-s} \ln z \) for real \( s \) and \( \Re(z) > 0 \), the following result is obtained:

\[
\int_0^\frac{3}{2} \ln G(x) \, dx = \frac{17}{96} + \frac{1}{8} \ln 2\pi - \frac{1}{2} \ln A - \frac{1}{4} \ln \Gamma \left( \frac{3}{4} \right) - \ln G \left( \frac{3}{4} \right).
\]  

(3.43)

Setting \( z = \frac{1}{2} \) in (3.41)

\[
\int_0^\frac{1}{2} \ln G(b + 1) \, db = \frac{5}{48} + \frac{1}{16} \ln \left( \frac{2}{\pi} \right) - \frac{1}{2} \ln G \left( \frac{1}{2} \right) - \ln A - \frac{\zeta(3)}{4\pi^2} - \zeta'(-2, \frac{1}{2}).
\]  

(3.44)

Recall that

\[
\zeta \left( s, \frac{1}{2} \right) = (2^s - 1) \zeta(s).
\]  

(3.45)

Differentiating (3.45) at \( s = -2 \) and using (3.39), we have that

\[
\zeta'(s, -2, \frac{1}{2}) = -\frac{3}{4} \zeta'(-2) = \frac{3\zeta(3)}{16\pi^2}.
\]  

(3.46)

Thus

\[
\int_0^\frac{1}{2} \ln G(b + 1) \, db = \frac{5}{48} + \frac{1}{16} \ln \left( \frac{2}{\pi} \right) - \frac{1}{2} \ln G \left( \frac{1}{2} \right) - \ln A - \frac{7\zeta(3)}{16\pi^2}.
\]  

(3.47)

Substituting (3.34) into (3.33)

\[
\frac{1}{2} \delta_1 = -2\varphi^* + 4 \left( \int_0^\frac{1}{2} \ln \Gamma(b) \, db - \int_0^\frac{1}{2} \ln \Gamma'(b) \, db \right).
\]  

(3.48)
It follows from formula (3.30) that
\[
4 \left( \int_{\frac{1}{2}}^{\frac{3}{4}} \ln \Gamma(b) \, db - \int_{\frac{1}{4}}^{\frac{3}{4}} \ln \Gamma(b) \, db \right) = \frac{1}{4} + 2 \ln \left( \frac{\sqrt{\pi} G \left( \frac{1}{4} \right)^{3} \Gamma \left( \frac{1}{2} \right)}{G \left( \frac{1}{2} \right)^{4}} \right).
\]
(3.49)

Subtracting (3.43) from 2 times (3.47), we have that
\[
\frac{1}{4} \varphi^* = 2 \int_{0}^{\frac{1}{2}} \ln G(b + 1) \, db - \int_{\frac{1}{2}}^{\frac{1}{2}} \ln G(b) \, db
\Rightarrow \frac{1}{4} \varphi^* = \frac{1}{32} - \frac{1}{4} \ln \pi + \ln G \left( \frac{3}{4} \right) + \frac{1}{4} \ln \Gamma \left( \frac{3}{4} \right) - \ln G \left( \frac{1}{2} \right) - \frac{7 \zeta(3)}{8 \pi^{2}}.
\]
(3.50)

We have from (3.48), (3.49) and (3.50) that
\[
\frac{1}{2} \delta_1 = \ln \left( \sqrt{2} \pi \right) + 2 \ln \Gamma \left( \frac{1}{4} \right) + 4 \ln \left( \frac{G \left( \frac{1}{4} \right)}{G \left( \frac{1}{2} \right)} \right) - 2 \ln \Gamma \left( \frac{3}{4} \right) + \frac{7 \zeta(3)}{\pi^{2}}.
\]
(3.51)

Setting \( z = \frac{1}{4} \) in the reflection formula for \( G(z) \) [10, 11, 21]
\[
\ln \left( \frac{G \left( \frac{1}{4} \right)}{G \left( \frac{1}{2} \right)} \right) = \frac{1}{4} \ln \left( \sqrt{2} \pi \right) + \ln \Gamma \left( \frac{1}{4} \right) + \frac{1}{2} \ln \left( \frac{\pi}{2} \right) \text{ Cl}_2 \left( \frac{\pi}{2} \right)
\]
(3.52)

where \( \text{Cl}_2(z) \) is the Clausen function.

Considering (3.52), (3.51) becomes simplified as follows:
\[
\frac{1}{4} \delta_1 = - \frac{2}{\pi} \text{ Cl}_2 \left( \frac{\pi}{2} \right) + \frac{7 \zeta(3)}{\pi^{2}}.
\]
(3.53)

The relationship between the Clausen function and Catalan’s constant is \( \text{Cl}_2 \left( \pi/2 \right) = G \).

Therefore
\[
\frac{1}{4} \delta_1 = - \frac{2 G}{\pi} + \frac{7 \zeta(3)}{\pi^{2}}.
\]
(3.54)

Multiplying both sides of (3.53) by 2 completes the proof of Prop. 3.9(1).

It now follows from (3.27), (3.25) and (3.53) that
\[
\lambda_2 = - \frac{2 G}{\pi} + \frac{7 \zeta(3)}{\pi^{2}} + \frac{1}{2} \left( \frac{4 G}{\pi} \right) = \frac{7 \zeta(3)}{\pi^{2}}.
\]
(3.54)

Considering (3.27) and using the results in (3.17) and (3.54), we have that
\[
\int_{0}^{\infty} \ln x \, \text{sech}^{4} x \, dx = - \frac{1}{3} \lambda_2 + \frac{2}{3} \lambda_2 = \frac{2}{3} \left( - 2 \ln \left( \frac{\pi}{4} \right) \right).
\]
Hence
\[
\int_{0}^{\infty} \ln x \, \text{sech}^{4} x \, dx = - \frac{2 y}{3} - \frac{7 \zeta(3)}{3 \pi^{2}} + \frac{2}{3} \ln \left( \frac{\pi}{4} \right).
\]
(3.55)
Remark 3.10. It can be easily deduced from the closed forms of $\Delta_3$ and $\Delta_4$ that, for $a \in \mathbb{R}^+$, $\Re(b) > 0$

1. \[ \int_0^\infty \ln(ax) \sech^3(bx) \, dx = -\frac{2G}{b \pi} + \frac{\pi}{2b} \ln \left( \frac{\sqrt{2a\pi^3} \Gamma\left(\frac{1}{2}\right)}{\sqrt{b} \Gamma\left(\frac{1}{4}\right)} \right). \]

2. \[ \int_0^\infty \ln(ax) \sech^4(bx) \, dx = -\frac{2\gamma}{3b} - \frac{7\zeta(3)}{3b \pi^2} + \frac{2}{3b} \ln \left( \frac{\pi}{4b} \right). \]

3.3.3 Closed form for $\Delta_5$ and some other results

Two reduction formulas are put forward based on the results of Alexejewsky [16], Barnes [17], Gosper [18] and Vardi [19]. The derived formulas are subsequently used to determine a closed form for $\Delta_5$ and $\Delta_6$.

Proposition 3.11. Let $n$ be a positive integer, $s \in \mathbb{R} \setminus \{1, 2, \ldots, n+1\}$ and $\Re(z) > 0$, then

1. \[ \int_0^z x^{n+1} \ln \Gamma(x) \, dx = \frac{(-1)^{n+1}(n+1)(n(z-1)+4z-3)}{2(n+2)(n+3)} x^{n+2} \]
   \[ + \frac{(n+1)(-1)^n}{2(n+2)} x^{n+2} \ln(2\pi) + (-1)^n z^{n+1} \zeta'(-1,z) \]
   \[ - \frac{(-1)^n}{n+2} \left( \zeta'(-2-n,z) - \zeta'(-2-n) + (\zeta(-2-n,z) - \zeta(-2-n)) (H_{n+2}-1) \right) \]
   \[ + \sum_{k=1}^{n} \frac{\zeta^k \Gamma(n+2)(-1)^n}{k! \Gamma(n+3-k)} \left( \zeta(-2-n+k) (H_{n+2-k} - 1) + \zeta'(-2-n+k) \right) \]
   \[ + \sum_{k=1}^{n} \frac{(-1)^{k-1} \Gamma(n+2) z^k}{k! \Gamma(n+2-k)} \int_0^z x^{n-k+1} \ln \Gamma(x) \, dx. \]

2. \[ \int_0^z x^n \ln G(1+x) \, dx = \frac{(-1)^{n+1}(n+1)(n(z-1)+4z-3)}{2(n+2)(n+3)} x^{n+2} \]
   \[ + \frac{(n+1)(-1)^n}{2(n+2)} x^{n+2} \ln(2\pi) + (-1)^n z^{n+1} \zeta'(-1,z) + \frac{z^{n+1}}{n+1} \zeta'(-1) \]
   \[ - \frac{(-1)^n}{(n+2)(n+1)} \left( (H_{2+n} - 1) \left( \zeta(-2-n,z) - \zeta(-2-n) \right) + \zeta'(-2-n,z) - \zeta'(-2-n) \right) \]
   \[ - \frac{(-1)^n}{n+2} \left( \zeta'(-2-n,z) - \zeta'(-2-n) + (\zeta(-2-n,z) - \zeta(-2-n)) (H_{n+2}-1) \right) \]
   \[ + \sum_{k=1}^{n} \frac{\zeta^k \Gamma(n+2)(-1)^n}{k! \Gamma(n+3-k)} \left( \zeta(-2-n+k) (H_{n+2-k} - 1) + \zeta'(-2-n+k) \right) \]
   \[ + \sum_{k=1}^{n} \frac{(-1)^{k-1} \Gamma(n+2) z^k}{k! \Gamma(n+2-k)} \int_0^z x^{n-k+1} \ln \Gamma(x) \, dx. \]
Proof of Proposition 3.11. The following results are important for establishing the required proof:

\[
\begin{align*}
\int_0^z \cdots \int_0^z \frac{\zeta'(1, z) \, dz \cdots dz_{n-1}}{2} = \\
\frac{1}{\Gamma(n+2)} \left( \zeta'(-1, n, z) - \zeta'(-1 - n) + (\zeta(-1, n, z) - \zeta(-1 - n)) \right) (H_{n+1} - 1) \\
- \sum_{k=1}^{n-1} \frac{\zeta k}{k! \Gamma(n+2-k)} (\zeta(-1 - n + k) (H_{n+1-k} - 1) + \zeta'(-1 - n + k)).
\end{align*}
\]

\[
\int_0^z u^n \zeta(s, u) \, du = \sum_{k=1}^{n} \frac{(-1)^{k-1} u^{n-k+1} \Gamma(n+1) \Gamma(1-s)}{\Gamma(k+1-s) \Gamma(n-k+2)} \zeta(s-k, z) \\
+ \frac{(-1)^n \Gamma(n+1) \Gamma(1-s)}{\Gamma(n+2-s)} (\zeta(s-n+1, z) - \zeta(s-n)).
\]

\[
\int_0^z u^n \zeta'(-1, u) \, du \\
= \sum_{k=1}^{n} \frac{(-1)^{k-1} u^{n-k+1} \Gamma(n+1)}{\Gamma(n-k+2) \Gamma(2+k)} ((H_{k+1} - 1) (\zeta(-1 - k, z) + \zeta'(-1 - k, z))) \\
+ \frac{(-1)^n}{(n+2)(n+1)} ((H_{2n+1} - 1) (\zeta(-2 - n, z) - \zeta(-2 - n)) \\
+ \zeta'(-2 - n, z) - \zeta'(-2 - n)).
\]

Integrating both sides of (3.36) \(n\) times over \((0, z_1], (0, z_2], (0, z_3], \ldots, (0, z_n]\), each wrt. \(z, z_1, z_2, \ldots, z_{n-1}\), respectively, taking account of (3.56), (3.57), (3.58), (3.59) and (3.60), multiplying both sides by \((-1)^n\) and replacing \(z_n\) with \(z, n \text{ with } n+1\), Proposition 3.11(1) holds true.

Multiplying both sides (3.35) by \(t^n\), integrating over \((0, z]\) wrt. \(t\), taking account of Proposition 3.11(1) and Equation (3.62), Proposition 3.11(2) holds true. \(\Box\)
Proposition 3.12.

1. \[ \delta_2 = \int_0^\infty \frac{1 - \text{sech}^2 x}{x^2} \text{sech}^2 x \, dx = -\frac{14\zeta(3)}{\pi^2} + \frac{2G}{\pi} \left( \frac{\pi}{2} + \frac{1}{16\pi^3} \psi_3 \left( \frac{1}{4} \right) \right). \]

2. \[ \lambda_3 = \int_0^\infty \frac{\tanh x \, \text{sech}^3 x}{x} \, dx = \frac{2G}{3\pi} - \frac{\pi}{6} + \frac{1}{48\pi^3} \psi_3 \left( \frac{1}{4} \right). \]

3. \[ \Lambda_5 = \int_0^\infty \ln x \, \text{sech}^5 x \, dx = -\frac{5G}{3\pi} + \frac{\pi}{24} - \frac{1}{192\pi^3} \psi_3 \left( \frac{1}{4} \right) + \frac{3\pi}{8} \ln \left( \frac{\sqrt{2\pi} \Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{3}{4} \right)} \right). \]

Proof of Proposition 3.12. Integrating by parts, we have

\[ \Lambda_5 = -\frac{1}{4} \lambda_3 + \frac{3}{4} \lambda_3, \quad \lambda_3 = \frac{1}{3} \delta_2 + \frac{2}{3} \delta_2. \] (3.63)

For \( s \in \mathbb{C} \setminus -2\mathbb{N}_0, n \in \mathbb{Z}^+ \)

\[ \int_0^\infty \frac{e^{-sx}}{(1 + e^{-2x})^{n+1}} = \frac{-\Gamma \left( \frac{s}{2} \right)}{\Gamma(n+1)} \sum_{k=0}^{n-1} \frac{\Gamma(n-k)}{\Gamma \left( \frac{s}{2} - k \right)} 2^{n-k+1} + \frac{1}{4\Gamma \left( \frac{s}{2} - n \right)} \left( \psi_0 \left( s - \frac{2n}{4} \right) - \psi_0 \left( s - \frac{2n+2}{4} \right) \right). \] (3.64)

Rewriting \( \text{sech}^n x \) in its exponential form and \( 1 - e^{-x} \) as a definite integral

\[ \delta_n = 2^n \int_0^1 \int_0^1 \int_0^\infty \frac{e^{-(a+b+n)x}}{(1 + e^{-2x})^{n+1}} \, dx \, da \, db. \]

Replacing \( s \) in (3.64) with \( a + b + n \)

\[ \frac{\delta_n}{2^n} = \int_0^1 \int_0^1 \int_0^\infty \frac{-\Gamma \left( \frac{a+b+n}{2} \right)}{\Gamma(n+1)} \sum_{k=0}^{n-1} \frac{\Gamma(n-k)}{\Gamma \left( \frac{a+b+n}{2} - k \right) 2^{n-k+1}} \psi_0 \left( a + b - \frac{n}{4} \right) - \psi_0 \left( a + b - \frac{n+2}{4} \right) \, da \, db. \] (3.65)

At \( n = 2 \)

\[ \delta_2 = \int_0^1 \int_0^1 \frac{1}{4} + \frac{a+b}{4} + \frac{2(a+b) - (a+b)^2}{8} \psi_0 \left( \frac{a+b}{4} + \frac{1}{2} \right) - \psi_0 \left( \frac{a+b}{4} \right) \, da \, db \]

\[ = \int_0^1 2\varphi_1(b+1) + \int_0^1 \frac{1}{4} + \frac{a+b}{4} - \frac{(a+b)^2}{8} \psi_0 \left( \frac{a+b}{4} + \frac{1}{2} \right) - \psi_0 \left( \frac{a+b}{4} \right) \, da \, db. \] (3.66)
It follows from integration by parts then formula (3.40) that
\[
\int_0^1 \frac{(a + b)^2}{8} \left( \varphi_0 \left( \frac{a + b}{4} + \frac{1}{2} \right) - \varphi_0 \left( \frac{a + b}{4} \right) \right) da = \frac{2b + 3}{8} + \frac{b^2 + 4b}{2} \ln \Gamma \left( \frac{b + 2}{4} \right)
- \frac{b^2}{2} \ln \Gamma \left( \frac{b}{4} \right) - \frac{1}{2} (b + 1) (b + 5) \ln \Gamma \left( \frac{b + 3}{4} \right) + \frac{(b + 1)^2}{2} \ln \Gamma \left( \frac{b + 1}{4} \right)
- 4b \ln G \left( \frac{b + 6}{4} \right) + 4(b + 1) \ln G \left( \frac{b + 7}{4} \right) + 4b \ln G \left( \frac{b + 4}{4} \right) - 4(b + 1) \ln G \left( \frac{b + 5}{4} \right)
+ 8\zeta'(-2, b + 3/4) + 8\zeta'(-2, b/4) - 8\zeta'(-2, b + 2/4) - 8\zeta'(-2, b + 1/4).
\] (3.67)

Applying (3.32) in (3.66), we have that
\[
\begin{align*}
\int_0^1 \int_0^1 -\vartheta_1(a, b) \, da \, db & - \int_0^1 \vartheta_2(b) \, db + 2 \int_0^1 \varphi_1(b + 1) \, db = 32 \int_0^1 x^2 \ln \Gamma(x) \, dx \\
- 64 \int_{\frac{1}{4}}^1 x^2 \ln \Gamma(x) \, dx & - 96 \int_{\frac{1}{4}}^1 \ln G(1 + x) \, dx + 48 \int_0^1 \ln G(1 + x) \, dx \\
+ 128 \int_{\frac{1}{4}}^1 x \ln G(1 + x) \, dx & - 64 \int_0^1 x \ln G(1 + x) \, dx - 16 \int_{\frac{1}{4}}^1 \ln \Gamma(x) \, dx \\
+ 16 \int_{\frac{1}{4}}^1 \ln \Gamma(x) \, dx & + 32 \int_{\frac{1}{4}}^1 \ln G(1 + x) \, dx - 64 \int_{\frac{1}{4}}^1 \ln G(1 + x) \, dx.
\end{align*}
\] (3.68)

Setting \(n = 1\) and \(z = 1\) in formula 3.11(1), then \(z = 1\) in Equation (3.40)
\[
\int_0^1 x^2 \ln \Gamma(x) \, dx = -\frac{1}{3} \ln 2\pi + \ln A + \frac{\zeta(3)}{4\pi^2} + 2 \int_0^1 x \ln \Gamma(x) \, dx = \\
\ln A = -\frac{1}{3} \ln 2\pi + \frac{\zeta(3)}{4\pi^2} + 2 \left( \frac{1}{4} \ln 2\pi - \ln A \right) = \frac{1}{6} \ln 2\pi - \ln A + \frac{\zeta(3)}{4\pi^2}.
\] (3.69)

Setting \(n = 1\) in formula 3.11(1)
\[
\int_0^1 x^2 \ln \Gamma(x) \, dx = -\frac{z^3}{3} \ln 2\pi + \frac{z^3}{24} (10z - 8) - z^2 \zeta'(-1, z) + \frac{1}{3} (\zeta'(-3, z) - \zeta'(-3)) \\
+ \frac{5}{18} (\zeta(-3, z) - \zeta(-3)) - z\zeta'(-2) + 2z \int_0^z x \ln \Gamma(x) \, dx.
\] (3.70)

Setting \(z = 1/4\) and \(z = 3/4\) in (3.70), then subtracting the first result obtained from the second, we have that
\[
\int_{\frac{1}{4}}^1 x^2 \ln \Gamma(x) \, dx = -\frac{13}{96} \ln 2\pi - \frac{1}{192} + \frac{1}{16} \zeta'(-1, \frac{1}{4}) - \frac{9}{16} \zeta'(-1, \frac{3}{4}) - \frac{1}{2} \zeta'(-2) \\
+ \frac{1}{3} \left( \zeta'(-3, \frac{3}{4}) - \zeta'(-3, \frac{1}{4}) \right) + \frac{3}{2} \int_{\frac{1}{4}}^1 x \ln \Gamma(x) \, dx - \frac{1}{2} \int_{\frac{1}{4}}^1 x \ln \Gamma(x) \, dx.
\] (3.71)
It follows from formula (3.40) that
\[
\frac{3}{2} \int_0^\frac{1}{2} x \ln \Gamma(x) \, dx - \frac{1}{2} \int_0^\frac{3}{4} x \ln \Gamma(x) \, dx = -\frac{9}{128} + \frac{13}{64} \ln 2\pi + \frac{3}{32} \ln \Gamma\left(\frac{1}{4}\right) - \ln A
\]
\[
- \frac{9}{32} \ln \Gamma\left(\frac{3}{4}\right) + \frac{1}{8} \ln G\left(\frac{1}{4}\right) - \frac{9}{8} \ln G\left(\frac{3}{4}\right) - \frac{\zeta(3)}{8\pi^2} + \frac{1}{4} \zeta'\left(-\frac{1}{4}\right) - \frac{3}{4} \zeta'\left(-\frac{3}{4}\right).
\]  
(3.72)

The reflection formula below for the first derivative of the Hurwitz zeta function is due to Adamchik ([22], Eq. 9)
\[
\zeta'(-n, x) + (-1)^n \zeta'(-n, 1 - x) = \pi i \frac{B_{n+1}(x)}{n+1} + \frac{n! e^{-\pi in/2}}{(2\pi)^n} \text{Li}_{n+1}\left(e^{2\pi ix}\right),
\]  
(3.73)

where \( n \in \mathbb{Z}^+ \) and \( x \in (0, 1) \).

We easily derive the following closed forms from (3.73):
\[
\zeta'\left(-\frac{1}{4}\right) - \zeta'\left(-\frac{3}{4}\right) = \frac{G}{2\pi},
\]  
(3.74)
\[
\zeta'\left(-\frac{2}{4}\right) + \zeta'\left(-\frac{3}{4}\right) = \frac{3\zeta(3)}{64\pi^2},
\]  
(3.75)
\[
\zeta'\left(-\frac{3}{4}\right) - \zeta'\left(-\frac{5}{4}\right) = \frac{1}{2048\pi^3} \left(16\pi^2 - 2G\left(\frac{1}{4}\right)\right).
\]  
(3.76)

The first case in the following is due to Barnes [23], the second, Srivastava and Choi [24] and the third, Adamchik [25]:
\[
\ln G\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{\ln 2}{24} - \frac{\ln \pi}{4} - \frac{3 \ln A}{2}
\]  
(3.77)
\[
\ln G\left(\frac{1}{4}\right) = \frac{3}{32} - \frac{G}{4\pi} - \frac{3}{4} \ln G\left(\frac{1}{4}\right) - \frac{9}{8} \ln A
\]  
(3.78)
\[
\zeta'\left(-\frac{1}{4}\right) = \frac{G}{4\pi} - \frac{1}{96} + \frac{\ln A}{8}.
\]  
(3.79)

Using (3.52) and (3.78), (3.72) becomes simplified as follows:
\[
\frac{3}{2} \int_0^\frac{1}{2} x \ln \Gamma(x) \, dx - \frac{1}{2} \int_0^\frac{3}{4} x \ln \Gamma(x) \, dx = -\frac{3}{128} + \frac{13}{64} \ln 2\pi + \frac{\ln A}{8} - \frac{5G}{16\pi}
\]
\[
+ \zeta'\left(-\frac{1}{4}\right) - \frac{41}{256\pi^2} \zeta(3).
\]

Using (3.74), (3.76) and (3.79), (3.71) becomes simplified as follows:
\[
\int_0^\frac{3}{4} x^2 \ln \Gamma(x) \, dx = -\frac{3}{128} + \frac{13}{192} \ln 2\pi + \frac{\ln A}{16} + \frac{1}{3072\pi^3} \psi_3\left(\frac{1}{4}\right) - \frac{\pi}{384} - \frac{5G}{32\pi}
\]
\[
- \frac{9\zeta(3)}{256\pi^2} + \zeta'\left(-\frac{1}{4}\right).
\]  
(3.80)
Setting \( n = 1, \ z = 1/4 \) and \( n = 1, \ z = 3/4 \) in formula (3.11(2)), then subtracting the first result obtained from the second, we have that
\[
\int_{\frac{1}{4}}^{\frac{1}{2}} x \ln G(1 + x) \, dx = -\frac{5}{768} + \frac{13}{192} \ln 2\pi - \frac{3 \ln A}{16} + \frac{1}{2048\pi^3} \psi_3 \left( \frac{1}{4} \right) - \frac{\pi}{256} - \frac{5G}{32\pi} - \frac{27\zeta(3)}{512\pi^2} + \frac{3}{2} \zeta' \left( -2, \frac{1}{4} \right). \tag{3.81}
\]

Setting \( z = 1/4 \) and \( z = 3/4 \) in formula (3.41), subtracting the first result obtained from the second and using (3.75), (3.52) and (3.78), we have that
\[
\int_{\frac{1}{4}}^{\frac{1}{2}} \ln G(1 + x) \, dx = -\frac{7 \ln A}{16} + \frac{G}{4\pi} + \frac{1}{96} + \frac{\ln 2\pi}{8} - \frac{3\zeta(3)}{64\pi^2} + \frac{1}{2\pi} \ln 2 - \frac{G}{4\pi^2} - \frac{3\ln A}{4} + \frac{G}{4\pi} + \frac{7\zeta(3)}{16\pi^2}. \tag{3.82}
\]

Using formula (3.30) and taking account of (3.77) and (3.78), we have that
\[
\int_{0}^{1} \ln \Gamma(u) \, du = 2 \int_{0}^{1} \ln \Gamma(u) \, du - \int_{0}^{\frac{1}{4}} \ln \Gamma(u) \, du - \int_{0}^{1} \ln \Gamma(2a) \, du - \int_{0}^{1} \ln \Gamma(2b) \, du - \int_{0}^{1} \ln \Gamma(2d) \, du - \int_{0}^{1} \ln \Gamma(2e) \, du - \int_{0}^{1} \ln \Gamma(2f) \, du = \frac{\ln 2}{24} + \frac{3\ln A}{4} - \frac{G}{4\pi^2}. \tag{3.83}
\]

Using (3.77) to simplify (3.47), we obtain (3.84). Substituting \( z = 1/4 \) in (3.41) and using (3.78), (3.85) also follows.
\[
\int_{0}^{1} \ln G(1 + x) \, dx = \frac{1}{24} + \frac{\ln 4\pi^3}{48} - \frac{\ln A}{4} - \frac{7\zeta(3)}{16\pi^2}. \tag{3.84}
\]
\[
\int_{0}^{1} \ln G(1 + x) \, dx = \frac{7}{192} + \frac{\ln 2\pi}{64} - \frac{7 \ln A}{32} - \frac{\zeta(3)}{4\pi^2} + \frac{G}{16\pi} - \zeta' \left( -2, \frac{1}{4} \right). \tag{3.85}
\]

In view of (3.69), (3.80), (3.81), (3.82), (3.83), (3.84) and (3.85), we arrive at the following for Equation (3.68):
\[
\int_{0}^{1} \int_{0}^{1} -\vartheta_1(a, b) \, da \, db - \int_{0}^{1} \vartheta_2(b) \, db + 2 \int_{0}^{1} \varphi_1(b + 1) \, db = -\frac{14\zeta(3)}{\pi^2} + \frac{2G}{\pi} - \frac{\pi}{3} + \frac{1}{24\pi^3} \psi_3 \left( \frac{1}{4} \right). \tag{3.86}
\]
\[
\int_{0}^{1} \vartheta_2(b) \, db = 64 \int_{0}^{\frac{1}{4}} \zeta'(-2, b) \, db - 32 \int_{0}^{1} \zeta'(-2, b) \, db = \frac{64}{\pi} \left( \zeta' \left( -3, \frac{3}{4} \right) - \zeta' \left( -3, \frac{1}{4} \right) \right) = \frac{1}{48\pi^3} \psi_3 \left( \frac{1}{4} \right) - \frac{\pi}{6}. \tag{3.87}
\]

Substituting the derived results into (3.66), we have that
\[
\delta_2 = \int_{0}^{1} \int_{0}^{1} -\frac{1}{4} + \frac{a + b}{4} \, da \, db - \frac{14\zeta(3)}{\pi^2} + \frac{2G}{\pi} - \frac{\pi}{3} + \frac{1}{24\pi^3} \psi_3 \left( \frac{1}{4} \right) + \int_{0}^{1} \vartheta_2(b) \, db
\]

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\[ \Rightarrow \delta_2 = -\frac{14\zeta(3)}{\pi^2} + \frac{2G}{\pi} - \frac{\pi}{2} + \frac{1}{16\pi^3 \psi_3 \left( \frac{1}{4} \right)}. \] (3.88)

It follows from (3.63), (3.54) and (3.88) that

\[ \lambda_3 = \frac{2G}{3\pi} - \frac{\pi}{6} + \frac{1}{48\pi^3 \psi_3 \left( \frac{1}{4} \right)}. \] (3.89)

Taking into account of (3.63) and using the results in (3.26) and (3.89), a closed form for \( \Delta_5 \) is derived as follows:

\[ \int_0^\infty \ln x \, \text{sech}^5 x \, dx = -\frac{5G}{3\pi} + \frac{\pi}{24} - \frac{1}{192\pi^3 \psi_3 \left( \frac{1}{4} \right)} + \frac{3\pi}{8} \ln \left( \frac{\sqrt{2\pi} \Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{3}{4} \right)} \right). \] (3.90)

3.3.4 Closed form for \( \Delta_6 \) and some other results

**Proposition 3.13.**

1. \[ \delta_3 = \int_0^\infty \frac{1 - \text{sech} x}{x^2} \, \text{sech}^3 x \, dx = -\frac{2G}{\pi} + \frac{28\zeta(3)}{3\pi^2} + \frac{124\zeta(5)}{\pi^4} + \frac{\pi}{2} - \frac{1}{16\pi^3 \psi_3 \left( \frac{1}{4} \right)}. \]

2. \[ \lambda_4 = \int_0^\infty \frac{\tanh x \, \text{sech}^4 x}{x} \, dx = \frac{7\zeta(3)}{3\pi^2} + \frac{31\zeta(5)}{\pi^4}. \]

3. \[ \Delta_6 = \int_0^\infty \ln x \, \text{sech}^6 x \, dx = -\frac{8G}{15} + \frac{8}{15} \ln \left( \frac{\pi}{4} \right) - \frac{7\zeta(3)}{3\pi^2} - \frac{31\zeta(5)}{5\pi^4}. \]

*Proof of Proposition 3.13.* By virtue of Equation (3.65)

\[ \delta_3 = \frac{1}{72} - 32 \int_0^1 \int_{u=1}^{u=0} \frac{2}{3} \left( \psi_0 \left( \frac{u + 1}{2} \right) - \psi_0(u) \right) \, du \, db + \frac{2}{3} \delta_1. \] (3.91)

It follows from integration by parts, then the reduction formula in **Proposition 3.11(1)** that

\[
\begin{align*}
\mu &= -\frac{11}{512} + \frac{1}{4} \ln A + \int_{\frac{1}{2}}^1 \left( 4u^2 - 1 \right) \zeta'(-1, u) \, du + \int_{\frac{1}{2}}^1 4u^2 \zeta'(-1, u) \, du - \frac{10}{9} \int_{\frac{1}{2}}^1 (u - 1)^3 \, du \\
&- \int_{\frac{1}{2}}^1 \left( 4u^2 - 1 \right) \zeta'(-1, u) \, du - \int_{\frac{1}{2}}^1 4u^2 \zeta'(-1, u) \, du - \frac{10}{9} \int_{\frac{1}{2}}^1 (u - 1)^3 \, du \\
&+ \frac{8}{3} \int_{0}^1 \zeta'(-3, u) \, du + \frac{4}{3} \int_{\frac{1}{2}}^1 (u - 1)^3 \ln (u - 1) \, du + \int_{\frac{1}{2}}^1 (4u^2 - 2) \zeta'(-2, u) \, du \\
&- \int_{\frac{1}{2}}^1 (4u^2 - 2) \zeta'(-2, u) \, du + \int_{\frac{1}{2}}^1 4u^2 \zeta'(-2, u) \, du - \int_{0}^1 4u \zeta'(-2, u) \, du
\end{align*}
\]
We have the following closed forms:

\[ 2 \int_0^\frac{1}{4} \left( \frac{20u^3}{3} - 2u^2 - u + \frac{1}{6} \right) \ln \Gamma(u) \, du + \int_0^\frac{1}{4} \left( -18u^2 + 21u - \frac{41}{6} \right) \ln \Gamma(u) \, du \\
- \frac{20}{3} \int_0^1 (u - 1)^3 \ln \Gamma(u) \, du + \int_0^\frac{1}{4} \left( 2u^2 + u - \frac{1}{6} \right) \ln \Gamma(u) \, du + \frac{20}{3} \int_0^\frac{1}{4} u^3 \ln u \, du \\
+ \int_0^\frac{1}{4} (8u^2 - 4u) \ln G(u + 1) \, du - 2 \int_0^\frac{1}{4} (8u^2 - 4u) \ln G(u + 1) \, du \\
+ 2 \int_0^\frac{1}{4} (8u^2 - 4u) \ln G(u + 1) \, du + 8 \int_0^1 (u - 1)^2 \ln G(u) \, du - 8 \int_0^\frac{1}{4} u^2 \ln G(u + 1) \, du. \]

We have the following closed forms:

\[ 2 \int_0^\frac{1}{4} (8u^2 - 4u) \ln G(u + 1) \, du = -\frac{1961}{69120} + \frac{13}{96} \ln A + \zeta'(-2, \frac{1}{4}) + \frac{377}{96} \zeta'(-3) + \frac{\pi}{192} \\
- \frac{13}{96} \ln 2 - \frac{1}{1536\pi^3} \psi_3 \left( \frac{1}{4} \right) - \frac{16}{3} \zeta'(-4, \frac{1}{4}) - \frac{G}{16\pi} - \frac{5}{384} \ln \pi + \frac{4\zeta(5)}{\pi^4}. \] (3.92)

\[ \int_0^\frac{1}{4} (4u^2 - 1) \zeta'(-1, u) \, du + \int_0^\frac{1}{4} 4u^2 \zeta'(-1, u) \, du + \int_0^\frac{1}{4} (4u^2 - 1) \zeta'(-1, u) \, du \\
- \int_0^1 4u^2 \zeta'(-1, u) \, du - \frac{10}{9} \int_0^\frac{1}{4} (u - 1)^3 \, du + \frac{8}{3} \int_0^\frac{1}{4} \zeta'(-3, u) \, du - \int_0^\frac{1}{4} 4u \zeta'(-2, u) \, du \\
+ \frac{4}{3} \int_0^1 (u - 1)^3 \ln (u - 1) \, du - \int_0^\frac{1}{4} (4u - 2) \zeta'(-2, u) \, du + \int_0^1 4u \zeta'(-2, u) \, du \\
+ \int_0^\frac{1}{4} (4u - 2) \zeta'(-2, u) \, du = -\frac{7}{6} \zeta'(-3) - \frac{1}{3072} + \frac{67}{5760} \ln 2 - \frac{31}{32\pi^2} \zeta(5) + \frac{3\zeta(3)}{128\pi^2} \\
- \frac{\pi}{192} - \zeta'(-2, \frac{1}{4}) - \frac{4}{3} \zeta'(-3, \frac{1}{4}) + \frac{1}{1536\pi^3} \psi_3 \left( \frac{1}{4} \right) + \frac{5\pi i}{768}. \] (3.93)

\[ 2 \int_0^\frac{1}{4} \left( \frac{20u^3}{3} - 2u^2 - u + \frac{1}{6} \right) \ln \Gamma(u) \, du + \int_0^\frac{1}{4} \left( -18u^2 + 21u - \frac{41}{6} \right) \ln \Gamma(u) \, du \\
- \frac{20}{3} \int_0^1 (u - 1)^3 \ln \Gamma(u) \, du + \int_0^\frac{1}{4} \left( 2u^2 + u - \frac{1}{6} \right) \ln \Gamma(u) \, du + \frac{20}{3} \int_0^\frac{1}{4} u^3 \ln u \, du \\
= -\frac{1}{8} \ln A - \frac{35}{8} \zeta'(-3) + \frac{155}{32\pi^4} \zeta(5) + \frac{263}{9216} - \frac{241}{5760} \ln 2 - \frac{1}{16} \ln \pi - \frac{3}{8\pi^2} \zeta(3) - \frac{5}{1536\pi^3} \psi_3 \left( \frac{1}{4} \right) + 3 \zeta' \left( -2, \frac{3}{4} \right) - \zeta' \left( -2, \frac{1}{4} \right) + \frac{5\pi i}{192} + \frac{17G}{48\pi} - \frac{5\pi i}{768}. \] (3.94)

\[ \int_0^\frac{1}{4} (8u^2 - 4u) \ln G(u + 1) \, du = -\frac{9}{2} \zeta' \left( -2, \frac{3}{4} \right) \\
- \frac{1}{2} \zeta' \left( -2, \frac{1}{4} \right) + \frac{8}{3} \zeta' \left( -4, \frac{1}{4} \right) - \frac{8}{3} \zeta' \left( -4, \frac{3}{4} \right) + \frac{4}{3} \zeta' \left( -3, \frac{3}{4} \right) - \frac{G}{4\pi} + \frac{155}{96} \zeta'(-3) - \]
\[ \pi \frac{1}{48} + \frac{1}{12} \ln \pi + \frac{431}{5760} \ln 2 + \frac{1}{384\pi^3} \psi_3 \left( \frac{1}{4} \right) + \frac{763}{34560} - \frac{25}{96} \ln A - \frac{31 \zeta(5)}{4\pi^4} + \frac{3 \zeta(3)}{8\pi^2}. \]  
(3.95)

\[ 8 \int_{\frac{1}{2}}^{1} (u - 1)^2 \ln G(u) \, du = \frac{8}{3} \zeta' \left( -2, \frac{1}{4} \right) + 3 \zeta' \left( -2, \frac{3}{4} \right) + 2 \zeta' \left( -2, \frac{1}{4} \right) + 2 \zeta' \left( -3, \frac{3}{4} \right) \]

\[ - \frac{7}{64} \zeta'' \left( -3 \right) - \frac{3}{32\pi^2} \zeta(3) - \frac{193}{69120} - \frac{2 \zeta(5)}{\pi^4} - \frac{5}{192} \ln A - \frac{G}{32\pi} - \frac{\pi}{128} - \frac{1}{256} \ln \pi \]

\[ + \frac{1}{1024\pi^3} \psi_3 \left( \frac{1}{4} \right) - \frac{3}{640} \ln 2. \]  
(3.96)

\[ - 8 \int_{0}^{1} u^2 \ln G(u + 1) \, du = \frac{193}{69120} + \frac{5}{192} \ln A + \zeta' \left( -2, \frac{1}{4} \right) + \frac{7}{48} \zeta'' \left( -3 \right) - \frac{\pi}{96} \]

\[ - \frac{1}{3840} \ln 2 + \frac{1}{768\pi^3} \psi_3 \left( \frac{1}{4} \right) + \frac{8}{3} \zeta' \left( -4, \frac{1}{4} \right) - \frac{2 \zeta(5)}{\pi^4} - \frac{G}{32\pi} - \frac{1}{256} \ln \pi. \]  
(3.97)

Taking account of the closed forms in (3.92), (3.93), (3.94), (3.95), (3.96) and (3.97), \( \mu \) becomes simplified as follows:

\[ \mu = \frac{3}{2} \left( \zeta' \left( -2, \frac{1}{4} \right) + \zeta' \left( -2, \frac{3}{4} \right) \right) + \frac{1}{2304} + \frac{7}{192} \zeta'' \left( -3 \right) + \frac{7}{3} \zeta' \left( -3, \frac{3}{4} \right) \]

\[ - \frac{5\pi}{384} + \frac{\ln 2}{3840} - \frac{G}{48\pi} - \frac{9}{128\pi^2} \zeta(3) - \frac{5}{3072\pi^3} \psi_3 \left( \frac{1}{4} \right) - \frac{31}{8\pi^2} \zeta(5). \]  
(3.98)

Recalling the closed form of \( \delta_1 \) in (3.53) and using (3.98), (3.91) becomes

\[ \delta_3 = -\frac{2G}{\pi} + \frac{28\zeta(3)}{3\pi^2} + \frac{124\zeta(5)}{\pi^4} + \frac{\pi}{2} + \frac{1}{16\pi^3} \psi_3 \left( \frac{1}{4} \right). \]  
(3.99)

Integrating by parts

\[ \Delta_0 = \frac{4}{5} \Delta_4 - \frac{1}{5} \lambda_4, \quad \lambda_4 = \frac{1}{4} \delta_3 + \frac{3}{4} \lambda_3. \]  
(3.100)

It follows from (3.100), (3.89) and (3.99) that

\[ \lambda_4 = \frac{7\zeta(3)}{3\pi^2} + \frac{31 \zeta(5)}{4\pi^4}. \]  
(3.101)

Taking into account of (3.100) and using the results in (3.55) and (3.101), a closed form for \( \Delta_0 \) is derived as follows:

\[ \int_{0}^{\infty} \ln x \operatorname{sech}^6 x \, dx = \frac{8\gamma}{15} + \frac{8}{15} \ln \left( \frac{\pi}{4} \right) - \frac{7\zeta(3)}{3\pi^2} - \frac{31 \zeta(5)}{5\pi^4}. \]  
(3.102)
Remark 3.14. It can be easily deduced from the closed forms of $\Delta$ and $\Delta_0$ that, for $a \in \mathbb{R}^+$, $\Re(b) > 0$

1. $\int_0^\infty \ln(ax) \operatorname{sech}^b(bx) \, dx = -\frac{5G}{3b^3} + \frac{\pi}{24b} - \frac{1}{192b^3} \psi_3\left(\frac{1}{4}\right) + \frac{3\pi}{8b} \ln\left(\frac{\sqrt{2}a\pi\Gamma\left(\frac{1}{4}\right)}{\sqrt{b}\Gamma\left(\frac{1}{4}\right)}\right)$

2. $\int_0^\infty \ln(ax) \operatorname{sech}^6(bx) \, dx = \frac{8\gamma}{15b} + \frac{8}{15b} \ln\left(\frac{a\pi}{4b}\right) - \frac{7\zeta(3)}{3b^2} - \frac{31\zeta(5)}{5b^4}$.

3.4 Reduction formula for $\Delta_n$

Proposition 3.15. Let $n \geq 4$, where $n \in \mathbb{Z}$, then

$$\Delta_n = -\frac{4G}{\pi(n-2)(n-1)} + \frac{n-2}{n-1} \Delta_{n-2} = \sum_{k=1}^{n-3} \frac{2^{n-2-k}}{(n-2)(n-1)} \times\left[\Gamma\left(\frac{a+b+n-2-k}{2}\right) \Gamma(n-1-k) \sum_{j=0}^{n-3-k} \Gamma(\frac{a+b+n-2-k}{2} - j) \frac{\Gamma(n-2-k-j)}{\Gamma(n-1-k-j)} \left(\psi_0\left(\frac{a+b+n-2-k}{2}\right) - \psi_0\left(\frac{a+b-n+2+k}{4}\right)\right) \right] \, da \, db.$$  \hspace{1cm} (3.103)

Proof. Integrating $\lambda_n$ by parts, where $n \geq 2$, $n \in \mathbb{Z}$

$$\lambda_n = \frac{1}{n} \delta_{n-1} + \frac{n-1}{n} \lambda_{n-1}.$$  \hspace{1cm} (3.104)

After $n-2$ iterations, we obtain the following:

$$\lambda_n = \frac{1}{n} \sum_{k=1}^{n-1} \delta_{n-k} + \frac{1}{n} \lambda_1.$$  \hspace{1cm} (3.105)

Integrating $\Delta_n$ by parts, where $n \geq 3, n \in \mathbb{Z}$

$$\Delta_n = \frac{n-2}{n-1} \Delta_{n-2} - \frac{1}{n-1} \lambda_{n-2}.$$  \hspace{1cm} (3.106)

Replacing $n$ with $n-2$ in (3.105) and substituting into (3.106) (the domain of $n$ in (3.105) changes to $n \geq 4$ upon substitution).

$$\Delta_n = \frac{n-2}{n-1} \Delta_{n-2} - \frac{1}{n-2}(n-1) \sum_{k=1}^{n-3} \delta_{n-2-k} - \frac{1}{n-2}(n-1) \lambda_{n-2}.$$  \hspace{1cm} (3.107)

where $n \geq \max(3, 4) = 4, n \in \mathbb{Z}$.

Substituting $n$ for $n-2-k$ into (3.65), substituting back into (3.107) and recalling the closed form for $\lambda_1$ in (3.25), (3.103) follows. \hfill \Box

29
4 Main Results

1. \[ \int_{0}^{\infty} \ln(x^2 + a^2) \sech(bx) \, dx = \frac{2\pi}{b} \ln \left( \frac{2\pi}{b} \Gamma \left( \frac{b |a| + 3}{4} \right) \right) - \frac{2\pi}{b} \ln \Gamma \left( \frac{b |a| + 1}{4} \right) \]
   \[ a \in \mathbb{R}, \quad \Re(b) > 0. \]

2. \[ \int_{0}^{\infty} \ln(x^2 + a^2) \sech^2(bx) \, dx = \frac{2}{b} \left( \psi_0 \left( \frac{1}{2} + \frac{b |a|}{\pi} \right) + \ln \left( \frac{\pi}{b} \right) \right). \]
   \[ a \in \mathbb{R}, \quad \Re(b) > 0. \]

3. \[ \int_{0}^{\infty} \ln(ax) \sech(x) \, dx = \frac{\pi}{b} \ln \left( \frac{\Gamma \left( \frac{1}{4} \right)}{\sqrt{\pi} \Gamma \left( \frac{1}{4} \right)} \right), \quad a \in \mathbb{R}^+, \quad \Re(b) > 0. \]

4. \[ \int_{0}^{\infty} \ln(ax) \sech^2(x) \, dx = -\left( \frac{\gamma}{b} + \frac{1}{b} \ln \left( \frac{a \pi}{4b} \right) \right), \quad a \in \mathbb{R}^+, \quad \Re(b) > 0. \]

5. \[ \int_{0}^{\infty} \ln(ax) \sech^3(x) \, dx = -\frac{2G}{b \pi} + \frac{\pi}{2b} \ln \left( \frac{\sqrt{2a} \pi \Gamma \left( \frac{1}{4} \right)}{\sqrt{\pi} \Gamma \left( \frac{1}{4} \right)} \right), \]
   \[ a \in \mathbb{R}^+, \quad \Re(b) > 0. \]

6. \[ \int_{0}^{\infty} \ln(ax) \sech^4(x) \, dx = -\frac{2\gamma}{3b} + \frac{7\zeta(3)}{3b \pi^2} + \frac{2}{3b} \ln \left( \frac{a \pi}{4b} \right), \]
   \[ a \in \mathbb{R}^+, \quad \Re(b) > 0. \]

7. \[ \int_{0}^{\infty} \ln(ax) \sech^5(x) \, dx = -\frac{5G}{3b \pi} + \frac{\pi}{24b} \ln \left( \frac{1}{192b \pi^3} \psi_3 \left( \frac{1}{4} \right) \right) + \frac{3\pi}{8b} \ln \left( \frac{\sqrt{2a} \pi \Gamma \left( \frac{1}{4} \right)}{\sqrt{\pi} \Gamma \left( \frac{1}{4} \right)} \right), \]
   \[ a \in \mathbb{R}^+, \quad \Re(b) > 0. \]

8. \[ \int_{0}^{\infty} \ln(ax) \sech^6(x) \, dx = -\frac{8\gamma}{15b} + \frac{8}{15b} \ln \left( \frac{a \pi}{4b} \right) - \frac{7\zeta(3)}{3b \pi^2} - \frac{31\zeta(5)}{5b \pi^4}, \]
   \[ a \in \mathbb{R}^+, \quad \Re(b) > 0. \]

5 Conclusion

Closed forms for \( \Delta_n \) for \( 1 \leq n \leq 6 \) have been explicitly given via real methods, new integral representations for the Euler-Mascheroni \( \gamma \) constant and more closed forms for improper integrals involving combinations of the arctangent and a hyperbolic function have also been provided. A point of focus for other researchers is the simplification of the proposed reduction formula for \( \Delta_n \).

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References

[1] Vardi, I., Integrals, an introduction to analytic number theory, Am. Math. Mon. 95, 308–315 (1988).

[2] Gradshteyn, I. S., Ryzhik, I. M., Tables of Integrals, Series and Products, fourth edition, Academic Press, New York (1980).

[3] I. V. Blagouchine, Rediscovery of Malmsten’s integrals, their evaluation by contour integration methods and some related results, The Ramanujan Journal (2014).

[4] Whittaker, E. T., Watson, G. N., A Course of Modern Analysis. An Introduction to the General Theory of Infinite Processes and of Analytic Functions, with an Account of the Principal Transcendental Functions, fourth edition, Cambridge University Press, Cambridge (1927) 248–251.

[5] Bateman, H., Erdélyi, Higher Transcendental Functions, McGraw-Hill Book, New York vol. I ((1955) [in 3 volumes]) page 26.

[6] E. W. Barnes, The theory of the G-function, Quart. J. Math. 31(1899) 264–314.

[7] E. W. Barnes, Genesis of the double gamma function, Proc. London Math. Soc. 31(1900) 358–381.

[8] E. W. Barnes, The theory of the double gamma function, Philos. Trans. Roy. Soc. London, Ser. A 196(1901) 265–388.

[9] E. W. Barnes, On the theory of the multiple gamma function, Trans. Cambridge Philos. Soc. 19(1904) 374–425.

[10] V. S. Adamchik, Contributions to the Theory of the Barnes Function, International Journal of Mathematics and Computer Science, 9(2014), no. 1, 11–30.

[11] J. Choi, H. M. Srivastava, Certain classes of series involving the zeta function, J. Math. Anal. and Appl. 231(1999) 91–117.

[12] Malmsten, C. J., Almgren, T. A., Camitz, G., Danelius, D., Moder, D.H., Selander, E., Grenander, J. M. A., Themptander, S., Trozelli, L. M., Föräldrar, A., Ossbahr, G. E., Föräldrar, D. H., Ossbahr, C. O., Lindhagen, C. A., Moder, D. H., Syskon, Å., Lemke, O. V., Fries, C., Laurentius, L., Leijer, E., Gyllenberg, G., Morfader, M. V., Linderoth, A., Specimen analyticum, theoremeta quedam nova de integralibus definitis, summatione serierum earumque in alias series transformatione exhibens (Eng. trans.: “Some new theorems about the definite integral, summation of the series and their transformation into other series”) [Dissertation, in 8 parts], Upsaliæ, excudebant Regiae academiar typographi. Uppsala, Sweden (April–June 1842).
[13] Aarts, Ronald M., Weisstein, Eric W., Fubini Theorem, From Mathworld–A Wolfram Web Resource, https://mathworld.wolfram.com/FubiniTheorem.html.

[14] C. I. Vălean, (Almost) Impossible Integrals, Sums, and Series, Springer (2019).

[15] D. Salwinski, Euler’s Sine Product Formula: An Elementary Proof, Math. Assoc. Am., College Math. Journal (2018).

[16] W. P. Alexejewsky, Ueber eine Classe von Funktionen, die der Gamma-funktion analog sind, Leipzig, Weidmannsche Buchhandluns (1894) 268–275.

[17] E. W. Barnes, The theory of the G-function, Quart. J. Math. 31(1899) 264–314.

[18] R. W. Gosper, \( \int \frac{1}{z} \log \Gamma(z) \, dz \), in: M. Ismail, D. Masson, M. Rahman editors, Special functions, q-series and related topics, The Fields Institute Communications, Amer. Math. Soc., 14(1997) 71–76.

[19] I. Vardi, Determinants of Laplacians and multiple gamma functions, SIAM J. Math. Anal. 19(1998) 493–507.

[20] J. Spanier, K. B. Oldham, An Atlas of Functions, Hemisphere, New York (1987).

[21] J. Choi, H. M. Srivastava, V. S. Adamchik, Multiple Gamma and related functions, Appl. Math. and Comp. 134(2003) 515–533.

[22] V. S. Adamchik, Polygamma functions of negative order, J. Comput. Appl. Math. 100(1998) 191–199.

[23] E. W. Barnes, The theory of the G-function, Quart. J. Math. 31(1899) 261–314.

[24] J. Choi, H. M. Srivastava, Certain classes of series associated with the Zeta function and the multiple Gamma function, J. Comput. Appl. Math. 118(2000) 87–109.

[25] V. S. Adamchik, Contributions to the Theory of the Barnes Function, Int. J. Math. Comput. Sci. 9(2014) 11–30.