An Approach to Higher Dimensional Theories Based on Lattice Gauge Theory

M. Murata and H. So

Department of Physics, Niigata University, Ikarashi 2-8050, Niigata 950-2181, Japan

A higher dimensional lattice space can be decomposed into a number of four-dimensional lattices called as layers. The higher dimensional gauge theory on the lattice can be interpreted as four-dimensional gauge theories on the multi-layer with interactions between neighboring layers. We propose the new possibility to realize the continuum limit of a five-dimensional theory based on the property of the phase diagram.

1. MOTIVATION

In constructing a higher dimensional quantum field theory, the regularization and the continuum limit are two important keys. Particularly, the problem of the limit associates with hard difficulty in the higher dimensional case. Statistical mechanics usually insists that critical behaviors of the phase transition in the theory are equivalent to those of a mean field theory, which has only a trivial fixed point.

Many pioneering works on lattice gauge theories were trying to overcome the difficulty [1,2,3,4,5], although the continuum limit is not taken strictly. Where is the continuum limit? By using well-known 4-dimensional theories such as QED and QCD, is it possible to construct the continuum limit by the related critical behavior near the critical point? Our purpose is to construct a $D$-dimensional pure Yang-Mills theory by arranging a number of 4-dimensional Yang-Mills theories with appropriate couplings.

2. OUR APPROACH

A $D$-dimensional lattice space is decomposed into $N_D$ layers with 4 dimensions like as Fig.1. Originally, Fu and Nielsen investigated the system to show dynamical dimensional reduction by using the characteristic vacuum which confines for extra dimensional directions and deconfines for 4 dimensions called a layer phase [6]. We do not assume that the layer phase exists, but use only the decomposition.

![Layer decomposition of 5-dimensional lattice space.](image)

From now, we focus on a 5-dimensional theory to study the possibility of the construction for higher dimensional field theories explicitly but an extension to general dimensions shall be mentioned in the final section. Our starting action for SU(2) gauge group is written as

$$S_{\text{lat}} = \frac{\beta_4}{2} \sum_{P_4} |2 - \text{Tr} U_{P_4}| + \frac{\beta_5}{2} \sum_{P_5} |2 - \text{Tr} U_{P_5}|,$$ (1)
where $\beta_4$ is proportional to a usual 4-dimensional coupling constant inside a layer and $\beta_5$ is a coupling constant between neighboring layers. $P_4$ implies a plaquette inside a layer and $P_5$ does a plaquette between neighboring layers.

3. PHASE STRUCTURE

To display the phase diagram explicitly, two order parameters are introduced: (1) Creutz ratio ($\sigma_4$) for 4-dimensional Wilson loops and (2) 5-dimensional Polyakov loop ($\langle L_5 \rangle$). Since a theory with $\beta_5 = 0$ is equivalent to a pure 4-dimensional Yang-Mills theory which is one phase, we expect a phase structure such as Fig. 2.

![Figure 2](image)

Figure 2. Expected phase diagram. Layer phase cannot realize in this system.

We have calculated numerically these parameters for the system and obtained the numerical phase diagram in Fig. 3. In the figure, along line i we can see both 4-dimensional and 5-dimensional deconfinement transitions in the large $\beta_5$. By wider calculations along lines ii, we recognize the phase transition is of 1st order near $\beta_4 \approx \beta_5$. The results of lines iii and iv suggest that both 4-dimensional and 5-dimensional deconfinement transitions in the small $\beta_5$ are of 2nd or weakly 1st order, because we cannot see any hysteresis loop in the coupling region. This critical point in the $\beta_4 \to \infty$ is noted as $\beta_{5c}(\approx 0.6)$. Our surprising remark is that the diagram Fig. 3 is not quantitatively changed for various $N_5 = 2, 3, 5, 6, 8$ and its stable property may help us to take the limit $N_5 \to \infty$.

![Figure 3](image)

Figure 3. Numerical phase diagram. The vertical dashed line represents crossover, not phase transition one.

4. MULTI-LAYER WORLD AND 5-SPACE

4.1. Multi-Layer World

Our simple consideration finds our way to existence of a 4-dimensional continuum system with finite inter-layer coupling ($\beta_5$). Near the critical point ($\beta_5 \approx \beta_{5c}$), the inverse ($H$) of correlation length for Polyakov loop is written as

$$H = m_{h4} a_4 \propto |\beta_5 - \beta_{5c}|^{-\nu},$$

where the value of $\beta_{5c}$ is approximately 0.6 and $\nu$ implies the critical exponent which is 0.5 in a mean field theory. We call the picture that many gauge fields on different layers interacts each other with finite coupling as multi-layer world. It is noted that link variables with 5-th direction behave as bi-fundamental fields with a finite coupling.
4.2. 5-dimensional space

Can we construct a 5-dimensional space not an internal space? A straightforward way uses excitation masses $M_{ij}$ between layers corresponding to Kaluza-Klein (K-K) modes,

$$M_{ij}^2 \equiv \frac{\beta_5 g_i^2}{4 a_4^2} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & \ddots & -1 \\ & & & & 1 \end{bmatrix} + \langle L_5 \rangle^2 \begin{bmatrix} 1 & 0 & & & \\ 0 & 0 & & & \\ & & \ddots & & \\ & & & \ddots & 0 \\ & & & & 1 \end{bmatrix}. \quad (3)$$

For a lower excited mode with label $k$, the simple formula of the mass is obtained as

$$M_k \sim \sqrt{\frac{\beta_5}{N_5 \beta_4}} \left( \frac{2\pi k a_4}{1} \right). \quad (4)$$

In order to remain finite masses of K-K modes, we must keep $\sqrt{\lambda_5 a_4}$ finite for large $N_5$ and $\beta_4$ with small $a_4$ from Eq.(4). To go through a 5-space from our 4-dimensional multi-layer world, we need to balance inter-layer dynamics and inside-layer one,

$$R \equiv \frac{M_K^2}{m_h^2} \sim \frac{\beta_5 \lambda_2(N_5)}{\beta_4 H^2(\beta_4, \beta_5, N_5)} \propto \frac{\beta_5}{N_5^2 \beta_4 \lambda_5 \beta_5 - \beta_5 |\beta_5|^{2\nu}}. \quad (5)$$

From Eq.(5), three parameters ($\beta_4, \beta_5, N_5$) tunings are necessary (See Fig.4). Comparing $M_{K-K} \sim a_5^{-1}$ with Eq.(4), a lattice spacing $a_5$ along 5th dimension can be defined by $a_5 \equiv \sqrt{\frac{\lambda_5 a_4}{\nu}}$.

5. SUMMARY AND PROBLEMS

We summarize main three steps; a) To find a second order phase transition in the meaning of 4-dimensional statistical mechanics. b) To take a 4-dimensional continuum limit (Multi-Layer World). c) To compare a 4-dimensional scale with an extra dimensional scale, i.e. confinement and Kaluza-Klein modes. For detailed analysis, see Ref. [7].

Finally, the following problems remain: A) Further detailed study for large $N_5$ and $\beta_4$. B) To estimate contribution of bi-fundamental field for $\sigma_4$. C) To recover the rotational symmetry relating to anisotropy between 4-Scale and 5-Scale. D) 6 or higher-dimensional extension of these decomposition is straightforward but their phase diagram analysis is not so easy.

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