Lorentz-violating extension of the spin-one Duffin-Kemmer-Petiau equation

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Abstract

We investigate the breaking of Lorentz symmetry caused by the inclusion of an external four-vector via a Chern-Simons-like term in the Duffin-Kemmer-Petiau Lagrangian for massless and massive spin-one fields. The resulting equations of motion lead to the appearance of birefringence, where the corresponding photons are split into two propagation modes. We discuss the gauge invariance of the extended Lagrangian. Throughout the paper, we utilize projection operators to reduce the wave-functions to their physical components, and we provide many new properties of these projection operators.

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1 Introduction

The brilliant success of the Standard Model (SM) of elementary particles is still hampered by some hurdles. For instance, the SM has not been successful in explaining the origin of electron’s electric dipole moment, $d_e$, and its experimental upper bounds \cite{1, 2}. This motivates investigations of physics beyond the SM. Along these lines, a possible way is to extend the mechanism of the spontaneous symmetry breaking through a background vector (or tensor) field such that the Lorentz symmetry is violated \cite{3}. In 1989, Kostelecký and Samuel \cite{4} discussed an interesting possibility of establishing the spontaneous violation of symmetry through non-scalar field (vacuum of fields that have a tensor nature) based on a string field theory environment. A general framework for testing the low-energy manifestations of CPT violation and Lorentz symmetry breaking is the Standard-Model Extension (SME) \cite{5}, where the effective Lagrangian corresponds to the usual SM Lagrangian, to which are added SM operators of any dimensionality contracted with Lorentz-violating tensorial background coefficients \cite{6, 7, 8}. With regard to the experimental searches for CPT- and Lorentz-violation, the generality of the SME has provided the basis for many investigations. In the flat spacetime limit, empirical studies include muons \cite{9}, mesons \cite{10, 11}, baryons \cite{12, 13}, photons \cite{14, 15, 16}, electrons \cite{17}, neutrinos \cite{18} and the Higgs sector \cite{19}. The gravity sector has also been explored in Refs. \cite{20, 21}. Current limits on the coefficients of the Lorentz symmetry violation are in Ref. \cite{22}. In recent years, Lorentz symmetry breaking effects have been investigated in the hydrogen atom \cite{23}, in the Rashba coupling \cite{24, 25}, in a quantum ring \cite{26}, in Weyl semi-metals \cite{27}, in tensor backgrounds \cite{28, 29}, in the quantum Hall effect \cite{30} and geometric quantum phases \cite{31, 32, 33}.

In this paper, we will apply a similar Lorentz-violation approach to the Duffin-Kemmer-Petiau (DKP) equation \cite{34, 35, 36}. Originally intended to describe mesons, and sometimes called the ‘meson algebra’, the DKP theory describes massive \cite{34} and massless \cite{37} scalar and vector bosons in a unified formalism based on a first-order wave equation analogous to the Dirac equation for spin-half fields. Hence, for spin-zero bosons, one replaces the second-order Klein-Gordon equation with the first-order DKP equation which involves matrices $\beta^\mu$, analogous to the Dirac gamma matrices, that satisfy a specific algebraic relation such that the DKP equation acquires a matrix form. Despite its similarity with the Dirac equation, the DKP formalism is more intricate; for instance, its field components are dependent, the use of specific representations can sometimes be replaced by component-projection operators, the treatment of massless fields requires more than simply setting the mass to zero and involves singular operators, etc. The DKP theory has been applied in different problems in quantum mechanics and field theory, such as the meson-nucleus elastic scattering \cite{38}, quantum chromodynamics \cite{39}, covariant Hamiltonian dynamics \cite{40}, studies on the $S$-matrix \cite{41}, calculations of the phase in Aharonov-Casher effect \cite{42}, on the causality of the DKP theory \cite{43}, Bose-Einstein condensation \cite{45, 46}, curved
spacetime [47], non-relativistic theories via Galilei covariant 5-dimensional formalism [48, 49, 50] and several other applications [51].

With a view to implementing Lorentz-symmetry breaking, the DKP formalism is interesting because it allows us to introduce various types of interactions through scalar, pseudo-scalar, vector or tensor couplings [52]. The richness of couplings introduced in the DKP theory allows us to examine Lorentz-symmetry breaking via non-minimal couplings of the massless DKP field with a background field. In this paper, we add a Chern-Simons-type term in the DKP Lagrangian for massless and massive spin-one fields, that causes the breaking of Lorentz symmetry. As studied in the literature, a sensitive phenomenological evidence of Lorentz-symmetry violation would be provided by the observation of vacuum birefringence. Carroll, Field and Jackiw observed that, in 3 + 1 dimensions, the Chern-Simons term, $n_\mu A_\nu \tilde{F}^{\mu\nu}$, which couples the dual electromagnetic tensor to an external (or background) four-vector (denoted $n_\mu$ hereafter) is gauge invariant but not Lorentz invariant [14]. One observes the birefringence of light in the vacuum when speeds which depend on polarization appear in the solutions of the modified Maxwell equations with Lorentz-violating terms (see also Kostelecky and Mewes (2009, 2013) in Ref. [15]). In this paper, we will point out that our model implies a connection between the photon dispersion relation and its polarization, which can therefore lead to a vacuum birefringence effect.

We examine a modified theory of electromagnetism with focus on the gauge sector of the SME. With a study based on the DKP formalism, we carry out the analysis of the odd sector [14]. In Sec. 2, we set up the model for spin-one massless DKP fields and we modify it by adding a Lorentz-violating background vector. In Sec. 3, we obtain and analyse the dispersion relations of the model. In Sec. 4, we study the massive DKP equation with a Lorentz-violating term and obtain the dispersion relation. Finally, we present concluding remarks in 5.

2 DKP equation for massless field in a background

As mentioned in the introduction, one aspect of the DKP theory which is less straightforward than the Dirac equation is the treatment of massless fields, described in this section. The Lagrangian for the massless DKP free field can be obtained in a manner similar to Harish-Chandra in Ref. [37]. We write it as follows:

$$\mathcal{L} = i\frac{1}{2} \overline{\Psi} \gamma^\mu \partial_\mu \Psi - i\frac{1}{2} \left( \partial_\mu \overline{\Psi} \right) \beta^\mu \Psi - \overline{\Psi} \gamma \Psi,$$

where $\mu = 0, 1, 2, 3$, and $\overline{\Psi} = \Psi^\dagger \eta$ is the adjoint DKP spinor, with $\eta = 2(\beta^0)^2 - 1$. We utilize the Minkowski metric $g^{\mu\nu}$ with signature $(+1, -1, -1, -1)$.

The corresponding free massless DKP equation obtained from this Lagrangian is

$$(i\beta^\mu \partial_\mu - \gamma) \Psi = 0.$$
Note the appearance here of a singular matrix $\gamma$ which takes the place of the mass term, in contrast with the Dirac equation, where one simply takes the mass equal to zero. For vector DKP field with spin one, we take $\gamma = 3 - \beta^\mu \beta_\mu$. The matrices $\beta^\mu$ and $\gamma$ satisfy the following algebra

$$
\beta^\mu \beta^\nu + \beta^\nu \beta^\mu = g^\mu_\nu \beta^\rho + g^\nu_\rho \beta^\mu, \quad \gamma^2 = \gamma.
$$

(3)

The DKP field, $\Psi$, has two different sectors: the scalar (or spin-zero) sector whose representation is by $5 \times 5$ beta-matrices, and the vector spin-one sector represented by $10 \times 10$ beta-matrices. The scalar DKP equation is equivalent to the Klein-Gordon equation, whereas the vector DKP equation with mass corresponds to the Proca field and to the Maxwell equation for massless fields. In both cases, the DKP equation provides rich possibilities to include interactions. Note that hereafter, we will not consider external sources, that is, we consider $j^\mu = 0$.

The Lorentz-symmetry breaking is produced by the addition of a background field to the Lagrangian, Eq. (1), analogous to the Chern-Simons term utilized in the literature. We write the DKP Lagrangian with these symmetry-breaking terms in the form

$$
L = \frac{i}{2} \overline{\Psi} \gamma^\mu \partial_\mu \Psi - \frac{i}{2} \left( \partial_\mu \overline{\Psi} \right) \beta^\mu \Psi - \overline{\Psi} \gamma \Psi - \frac{1}{4} \overline{\Psi} \epsilon_{\mu \nu \rho \sigma} [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma \Psi
$$

$$
+ \frac{1}{4} \overline{\Psi} \epsilon_{\mu \nu \rho \sigma} [\beta^\mu, \beta^\nu] n^\rho \left( - \partial^\sigma \Psi \right),
$$

(5)

where, for the vector DKP field with spin one, we have

$$
P = 1 - \gamma.
$$

(6)

The model is CPT-odd and predicts a rotation of the plane of polarization of radiation from distant galaxies such as in gamma-ray emission [14]. The other contribution to the pure-photon sector is a CPT-even Lorentz-violating term which does not have such properties [6]. For the scalar sector, the projection operator $P_S$ is defined in Appendix A. The four-vector $n^\mu$ is constant and acts as the background which breaks the Lorentz symmetry (see Ref. [14]). (Note that the commutator $[\beta^\mu, \beta^\nu]$, which is in the definition of the spin operator $S$,

$$
S_j = \frac{i}{2} \epsilon_{j k l} [\beta^k, \beta^l],
$$

(7)

therefore appears in the expression for the Pauli-Lubanski vector, which gives the field’s spin [44]. This suggests that the new interaction term in the Lagrangian should be trivial for spin-zero fields, as we will explain below Eq. (21).)

The interaction term between the DKP field $\Psi$ and the background field $n^\rho$ in Eq. (5) is related to the Chern-Simons term employed in Ref. [14]. In order to see this, let us consider the third and fourth terms of Eq. (5),

$$
L_{int} = -\frac{1}{4} \overline{\Psi} \epsilon_{\mu \nu \rho \sigma} [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma \Psi + \frac{1}{4} \overline{\Psi} \epsilon_{\mu \nu \rho \sigma} [\beta^\mu, \beta^\nu] n^\rho \left( - \partial^\sigma \Psi \right).
$$

(8)
If we consider the particular case where \( n^\rho = (n^0, \mathbf{0}) \), utilize \([\beta^i, \beta^j] = -i\epsilon^{ij}_l S^l\) (obtained from Eq. (7)), and use \( PS_i = S_i P \), we observe that, for a complex field \( \Psi \), the interaction term contains a spin-dependent structure,

\[
\mathcal{L}_{\text{int}} = i \frac{n_0}{2} \overline{\Psi} (S \cdot \nabla) P \Psi - i \frac{n_0}{2} \overline{P} (S \cdot \nabla) P \Psi.
\] (9)

For the spin-one DKP field considered here, if we use the identity in Eq. (B.7) together with Eqs. (B.2) and (B.3), we find

\[
\mathcal{L}_{\text{int}} = i \frac{n_0}{2} \overline{\omega} R^a (S \cdot \nabla) \Psi - i \frac{n_0}{2} \overline{P} R^a \Psi
\] (10)

and from Eq. (7) and the second identity in Eq. (B.6), we find

\[
\mathcal{L}_{\text{int}} = i \frac{n_0}{2} \overline{\omega} R^a (S \cdot \nabla) \Psi - i \frac{n_0}{2} \overline{P} R^a \Psi
\]

which, for a real field, reduces to

\[
\mathcal{L}_{\text{int}} = - \frac{n_0}{4} \overline{\omega} R \cdot (\nabla \times \Psi),
\] (11)

which has the form of a Chern-Simons term, where \( R\Psi \) plays the role of the usual electromagnetic potential \( A \).

Hereafter, we shall utilize the following spin-one representation of the beta matrices:

\[
\begin{align*}
\beta^0 &= \epsilon_{1,7} + \epsilon_{2,8} + \epsilon_{3,9} + \epsilon_{7,1} + \epsilon_{8,2} + \epsilon_{9,3}, \\
\beta^1 &= \epsilon_{1,10} + \epsilon_{5,9} - \epsilon_{6,8} + \epsilon_{8,6} - \epsilon_{9,5} - \epsilon_{10,1}, \\
\beta^2 &= \epsilon_{2,10} - \epsilon_{4,9} + \epsilon_{6,7} - \epsilon_{7,6} + \epsilon_{9,4} - \epsilon_{10,2}, \\
\beta^3 &= \epsilon_{3,10} + \epsilon_{4,8} - \epsilon_{5,7} + \epsilon_{7,5} - \epsilon_{8,4} - \epsilon_{10,3}.
\end{align*}
\] (12)

The (singular) gamma matrix for the massless term with a vector field is given by

\[
\gamma = 3 - \beta^\mu \beta_\mu = \epsilon_{1,1} + \epsilon_{2,2} + \epsilon_{3,3} + \epsilon_{4,4} + \epsilon_{5,5} + \epsilon_{6,6}.
\] (13)

The shorthand notation \( \epsilon_{ij} \) represents a \( 10 \times 10 \) matrix whose only non-zero entry is \( ij \), defined to be one, that is, \( (\epsilon_{ij})_{mn} = \delta_{im} \delta_{jn} \).

For the spin-one sector, we define the spinor \( \Psi \) by

\[
\Psi = \begin{pmatrix}
\Psi_1 \\
\vdots \\
\Psi_{10}
\end{pmatrix} = \begin{pmatrix}
-iE \\
-iB \\
A \\
\phi
\end{pmatrix}
\] (14)
where the electromagnetic fields are given by

\[\begin{align*}
-iE &= \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \\
-iB &= \begin{pmatrix} \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{pmatrix}, \\
A &= \begin{pmatrix} \Psi_7 \\ \Psi_8 \\ \Psi_9 \end{pmatrix}, \\
\phi &= \Psi_{10}.
\end{align*}\] (15)

We can see that by identifying \(E\) as the electric field, \(B\) as the magnetic field, and \(A_\mu = (\phi, A)\) as the usual electromagnetic gauge field, then the free massless DKP equation above reproduces the free Maxwell equations.

For many aspects, it is not necessary to recourse to a specific representation, by utilizing instead projection operators that select the wave-function for spin zero or spin one [35, 53]. Hereafter, we shall proceed as in Ref. [53] and construct operators of projection,

\[R^\mu = (\beta^1)^2(\beta^2)^2(\beta^3)^2[\beta^\mu \beta^0 - g^{\mu 0}]\] (16)

and

\[R^{\mu \nu} = R^\mu \beta^\nu,\] (17)

that select the spin-one sector when applied to the DKP field \(\Psi\). From the operator properties in Eq. (B.6) of the Appendix B, we see that when we apply these operators on Eq. (2), we find

\[\partial_\mu (G^{\mu \nu} \Psi) = 0, \quad \partial_\nu \partial^\mu (R^\nu \Psi) = 0,\] (18)

where

\[G^{\mu \nu} \Psi = \partial^\mu (R^\nu \Psi) - \partial^\nu (R^\mu \Psi).\] (19)

In other words, \(R^\mu \Psi\) can be interpreted as a massless vector field that satisfies the Klein-Gordon equation.

We can enforce a gauge symmetry which entails the interaction of the spin-one DKP field \(\Psi\) with an external gauge field \(A_\mu\). Clearly, we can render the Lagrangian in Eq. (1) invariant under some gauge group \(G\) in the usual manner: given a gauge transformation \(\Psi' = S \Psi\), where \(S\) belongs to \(G\), we replace the partial derivative \(\partial_\mu\) with the covariant derivative \(D_\mu = \partial_\mu - igA_\mu\), where \(A_\mu\) is a gauge field that transforms as \(A'_\mu = SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1}\). Note that \(A_\mu\) is different from the electromagnetic field \(A_\mu\) described by the DKP field.

Since the first three terms of the Lagrangian in Eq. (5) correspond to Eq. (1), we are left to verify the gauge invariance of the last two terms in Eq. (5). Let us modify \(\frac{1}{4} \epsilon_{\mu \nu \rho \sigma} [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma \Psi\) so that it becomes gauge-invariant under \(G\) (the last term of Eq. (5) being treated the same manner). This is done again by replacing the partial derivative \(\partial_\mu\) by the covariant derivative \(D_\mu = \partial_\mu - igA_\mu\). We find that

\[D^\sigma \Psi' = \left(\partial^\sigma - igSA^\sigma S^{-1} - (\partial^\sigma S)S^{-1}\right) S \Psi\]

\[= S(\partial^\sigma \Psi - igA^\sigma \Psi)\]

\[= SD^\sigma \Psi,\]
which shows the gauge invariance of \( \frac{1}{4} \nabla \epsilon_{\mu \nu \rho \sigma} [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma \Psi \), and hence of the Lagrangian in Eq. (5) when the gauge field is introduced with the covariant derivative.

The Lagrangian, Eq. (5), leads to the wave equation
\[
\left( i \beta^\mu \partial_\mu - \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} P [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma - \gamma \right) \Psi = 0
\]
and its adjoint equation,
\[
\overline{\Psi} \left( i \beta^\mu \overline{\partial}_\mu - \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} P [\beta^\mu, \beta^\nu] n^\rho \overline{\partial}^\sigma + \gamma \right) = 0.
\]

(Let us point out that if we wish to interpret Eq. (20) and its adjoint as describing a scalar field, then we must use projectors, \( P_S \) and \( P_\mu \), to select the scalar sector of the DKP field, in analogy with Eqs. (16) and (17) for the vector sector. However, when we apply these operators on Eq. (20), we can see that the Lorentz-breaking term disappears. As shown in the Appendix A, for the scalar DKP field, \( P_S [\beta^\mu, \beta^\nu] = 0 \) and \( P_\mu (1 - \gamma_S) = 0 \), so that the second term of Eq. (20) vanishes. This corroborates the fact, mentioned earlier, that since the commutator \([\beta^\mu, \beta^\nu]\) of the new interaction term is related to the field’s spin, then for spin-zero field, it will not contribute to the dynamics of the field.)

With the representations for \( \beta^\mu \) and \( \gamma \) given in Eqs. (12) and (13), we find that the DKP equation (20), modified by adding the background field \( n^\mu \), takes the form
\[
\begin{align*}
E &= -\nabla \phi - \partial_t A, \\
B &= \nabla \times A, \\
\nabla \times B &= \partial_t E + n^\mu B + n \times E, \\
\nabla \cdot E &= -n \cdot B.
\end{align*}
\]
One recognizes the Maxwell equations with the Coulomb and Ampère laws modified in terms of the background \( n^\mu \) (see in Ref. ([14])).

Note that the energy-momentum tensor for the Lagrangian in Eq. (5) is
\[
T^{\mu \nu} = \frac{i}{2} \overline{\Psi} \beta^\mu \partial^\nu \Psi - \frac{i}{2} \overline{\Psi} \left[ \partial^\mu \overline{\Psi} \right] \beta^\nu \Psi - \frac{1}{4} \overline{\Psi} \epsilon_{\lambda \rho \sigma \rho} P [\beta^\lambda, \beta^\rho] n^\rho g^{\sigma \mu} \partial^\nu \Psi + \frac{1}{4} \left[ \partial^\nu \overline{\Psi} \right] \epsilon_{\lambda \rho \sigma \rho} P [\beta^\lambda, \beta^\rho] n^\rho g^{\sigma \mu} \Psi - g^{\mu \nu} \mathcal{L}.
\]
Note that the extensions, which contributes the last terms to Eq. (5), renders the energy-momentum tensor nonsymmetric: \( T^{\mu \nu} \neq T^{\nu \mu} \) which again indicates the absence of Lorentz invariance. One can verify that \( \partial_\mu T^{\mu \nu} = 0 \) by using the equations of motion in Eqs. (20) and (21), thus showing that this tensor be conserved because the DKP theory is invariant under translations in the Minkowski space.

As shown in the Appendix C, we can write Eq. (23) as
\[
T^{\mu \nu} = -G^{\alpha \mu} G^{\nu \alpha} + \frac{1}{4} g^{\mu \nu} G^{\sigma \alpha} G_{\sigma \alpha} + \left( n^\nu \epsilon^{\mu \alpha \rho \sigma} G_{\rho \sigma} R_\alpha \Psi \right),
\]
so that, with the help of Eq. (15), the components are
\[ T^{00} = \frac{1}{2} (E^2 + B^2) + \frac{n^0}{2} (B \cdot A), \]
\[ T^{0i} = (E \times B)^i + \frac{n^i}{2} (B \cdot A). \]
\[ (25) \]
\[ (26) \]

As expected, if we take \( n^\mu = 0 \), these two expressions lead to the Maxwell field’s energy density and the Poynting vector, respectively. The \( n \)-dependent terms are similar to the ones obtained in Ref. [14].

We emphasize that the minimal coupling used with this formalism opens a window of possibilities for the investigation of contributions that can arise in this context. We therefore investigated the possibilities of a charged particle to bring information about the four-vector \( n^\rho \) which violates Lorentz symmetry via such minimal coupling \((n^\rho \partial^\sigma \rightarrow n^\rho D^\sigma)\), so that information can be obtained from experiments with interference phenomena and Berry phases (see in Ref. [54]). Another study would consist in investigating new ways to generate bound states (Landau levels [55]) and new Berry phases with non-minimal couplings in this formalism.

3 Birefringence for the massless DKP field

As mentioned earlier, vacuum birefringence may provide a sensitive phenomenological signature of Lorentz-symmetry breaking and it was amply studied (e.g. Carroll et all (1990) and Kostelécký-Mewes (2008) in Refs. [14, 15], and [6, 56]). This is analogous to the optical birefringence, associated to double-refraction, of anisotropic materials that have an index of refraction that depends on the polarization and the direction of propagation of light. Typically, the Maxwell equations modified with a Lorentz-breaking term will lead to dispersion relations that correspond to left-handed and right-handed modes. Hereafter we deduce a similar effect with our Lorentz-violating extension of the DKP equation.

If we multiply Eq. (20) by \( R^\alpha \) followed by \( R^{\alpha \delta} \), then we find
\[ \left( i \partial_\mu R^{\alpha \mu} - \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} R^\alpha P [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma - R^\alpha R^\beta \right) \Psi = 0. \] \[ (27) \]
\[ \left( i G^{\delta \alpha} - \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} R^{\delta \alpha} P [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma - R^{\delta \alpha} R^\beta \right) \Psi = 0. \] \[ (28) \]

If we use the identities (B.1)-(B.3) presented in the Appendix B, together with Eq. (27), we obtain
\[ \left( -\partial_\delta G^{\delta \alpha} - \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} R^\alpha [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma \right) \Psi = 0. \] \[ (29) \]

With the identity (B.4) in the Appendix B, the previous equation becomes
\[ \left( \partial_\delta G^{\delta \alpha} - \epsilon^{\alpha \nu \rho \sigma} n_\nu \partial_\rho R_\sigma \right) \Psi = 0. \] \[ (30) \]
In order to obtain a dispersion relation between the frequency $\omega$ and the wavevector $k$, we expand the field $\Psi$ in terms of plane waves, $\Psi = \frac{1}{(2\pi)^4} \int \tilde{\Psi}(k) e^{ik \cdot x} d^4k$, with the four-vector

$$k_\mu = (\omega, k),$$

and utilize Eq. (30), which leads to

$$\frac{1}{(2\pi)^4} \int [k_\mu k^{\mu} g^{\alpha \sigma} + k^\alpha k^\sigma - i \epsilon^{\alpha \sigma \rho \sigma} n_\nu k_\rho] R_\sigma \tilde{\Psi}(k) e^{ik_\mu x^\mu} d^4k = 0. \quad (32)$$

If we utilize the Lorentz gauge condition, $\partial_\mu R_\mu \tilde{\Psi}(k) = 0$, that is, $k_\mu R_\mu \tilde{\Psi}(k) = 0$, then we have

$$(k_\mu k^\mu g^{\alpha \sigma} - i \epsilon^{\alpha \sigma \rho \sigma} n_\nu k_\rho) R_\sigma \tilde{\Psi}(k) = 0. \quad (33)$$

When we multiply this expression by $n^\alpha$, we obtain $n^\alpha R_\sigma \tilde{\Psi}(k) = 0$. Then we multiply Eq. (33) by $k_\mu k^\mu g^{\lambda \alpha} + i \epsilon^{\lambda \alpha \beta \gamma} n^\beta k^\gamma$,

$$\left[ (k_\mu k^\mu)^2 g^{\lambda \sigma} - \epsilon^{\alpha \lambda \beta \gamma} \epsilon^{\alpha \sigma \nu \rho} n^\delta k^\gamma n_\nu k_\rho \right] R_\sigma \tilde{\Psi}(k) = 0. \quad (34)$$

From the properties of the tensor $\epsilon^{\alpha \lambda \beta \gamma}$, Eq. (34) leads to

$$\left[ (k_\mu k^\mu)^2 + (k_\mu k^\mu)(n_\nu n^\nu) - (n_\nu k^\nu)^2 \right] R_\lambda \tilde{\Psi}(k) = 0, \quad (35)$$

from which it follows that

$$(k_\mu k^\mu)^2 + (k_\mu k^\mu)(n_\nu n^\nu) - (n_\nu k^\nu)^2 = 0. \quad (36)$$

Let us choose $n_\mu = (n_0, 0)$ and use Eq. (31), so that Eq. (36) becomes

$$\omega^4 - 2|k|^2 \omega^2 + |k|^2 \left(|k|^2 - n_0^2\right) = 0, \quad (37)$$

which gives us the solution

$$\omega_\lambda = |k|(1 + \lambda n_0/|k|)^{1/2}, \quad (38)$$

where $\lambda = \pm 1$ implies that the background $n^\mu$ splits the photons into two modes of propagation. This dispersion relation is similar in Refs. [14, 15] and provides evidence for the violation of Lorentz invariance.

The dispersion relation, Eq. (38), leads to a modified group velocity,

$$v_g \equiv \frac{\partial \omega_\lambda}{\partial |k|} = \frac{(1 + \lambda n_0/2|k|)}{(1 + \lambda n_0/|k|)^{1/2}}. \quad (39)$$

This expression leads to rotations of the polarization of linearly polarized photons during their propagation (see, e.g. Ref. [57]). The group velocity, $v_g(\lambda = +1)$, can
exceed the speed of light, thereby introducing problems of causality (see also Ref. [58]). On the other hand, the phase velocity can be obtained with
\[ v_p \equiv \frac{\omega}{|k|}, \]
(40)
Notice that the phase and group velocities are related through Rayleigh’s formula:
\[ \frac{v_p - v_g}{v_g} = \frac{\lambda n_0}{|k| (1 + \lambda n_0/|k|)^{1/2}}, \]
(41)
for large momenta, |k|, such that \( n_0/|k| \ll 1 \). In the superluminal case, \( \lambda = +1 \), we have that \( v_p > v_g \), a normal dispersion medium. In the subluminal case, \( \lambda = -1 \), this implies at \( v_g > v_p \), an anomalous medium (from an influence of anisotropic effects). Therefore, we can conclude that a model truly isotropic, \( (n_\mu \equiv (n_0,0)) \), must be attributed only to superluminal case. This is important for phenomenological analyses.

4 The massive DKP field with a background

As is well known, the existence of a massive gauge field implies that the associated electromagnetic fields have short range, as we can see in a superconductor environment (e.g. Meissner effect). Another example is the electroweak theory where the vector bosons of the weak interaction acquire mass (which results in an interaction confined within the atomic nucleus), whereas the long-range photon remains massless. Hereafter, we observe that our model with the DKP equation reproduces previous results for the massive gauge fields with preferential spacetime directions which stem from the violating background of the Carroll-Field-Jackiw term. Our dispersion relations is very similar to results found in the literature.

The Lagrangian associated to the spin-one sector of the DKP field with mass \( m \) in a background \( n^\nu \) is given by
\[ \mathcal{L} = \frac{i}{2} \bar{\Psi} \beta^\mu \partial_\mu \Psi - \frac{i}{2} \left( \partial_\mu \bar{\Psi} \right) \beta^\mu \Psi - m \bar{\Psi} \Psi - \frac{1}{4m} \bar{\Psi} \epsilon_{\mu\nu\rho\sigma} P [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma \Psi + \frac{1}{4m} \bar{\Psi} \epsilon_{\mu\nu\rho\sigma} P [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma \Psi, \]
(42)
where \( P = 1 - \gamma = \beta^\mu \beta_\mu - 2 \).

As we did with the massless DKP field (in Eq. (11)), the interaction term
\[ \mathcal{L}_{\text{int}} = -\frac{1}{4m} \bar{\Psi} \epsilon_{\mu\nu\rho\sigma} P [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma \Psi + \frac{1}{4m} \bar{\Psi} \epsilon_{\mu\nu\rho\sigma} P [\beta^\mu, \beta^\nu] n^\rho \partial^\sigma \Psi \]
(43)
also leads to a Chern-Simons type expression,

\[ \mathcal{L}_{\text{int}} = -\frac{n_0}{4m} \overrightarrow{\Psi} \mathbf{R} \cdot (\nabla \times \mathbf{R}) \Psi + \frac{n_0}{4m} \overrightarrow{\nabla} \left( \overrightarrow{\nabla} \times \mathbf{R} \right) \cdot \mathbf{R} \Psi, \]  

which for a real DKP field becomes

\[ \mathcal{L}_{\text{int}} = -\frac{n_0}{2m} \overrightarrow{\Psi} \mathbf{R} \cdot (\nabla \times \mathbf{R}) \Psi. \]  

One should remember that the components of the DKP spinor \( \Psi \) are multiplied by \( m \), that is, \( \mathbf{R} \Psi = m \mathbf{A} \). This shows also that the analogous Eq. (11) for the massless DKP field cannot be obtained from Eq. (45) simply by setting \( m = 0 \).

From the Lagrangian in Eq. (42), we obtain the equation of motion,

\[ \left( i \beta^\mu \partial_\mu - \frac{1}{2m} \epsilon_{\mu\nu\rho\sigma} P [\beta^\nu, \beta^\rho] n^\sigma \partial^\sigma - m \right) \Psi = 0 \]  

(46)

and its adjoint

\[ \overrightarrow{\Psi} \left( i \beta^\mu \partial_\mu - \frac{1}{2m} \epsilon_{\mu\nu\rho\sigma} P [\beta^\nu, \beta^\rho] n^\sigma \partial^\sigma + m \right) = 0. \]  

(47)

By performing a calculation similar to the massless case, we use the representation in Eq. (12) to obtain the Proca field equations modified with the background field \( n^\mu \):

\[ \mathbf{E} = -\nabla \phi - \partial_t \mathbf{A}, \]
\[ \mathbf{B} = \nabla \times \mathbf{A}, \]
\[ \nabla \times \mathbf{B} = \partial_t \mathbf{E} - m^2 \mathbf{A} + n^0 \mathbf{B} + \mathbf{n} \times \mathbf{E} \]
\[ \nabla \cdot \mathbf{E} = -m^2 \phi - \mathbf{n} \cdot \mathbf{B}. \]  

(48)

The calculations related to the energy-momentum tensor are also similar to the massless case,

\[ T^{\mu\nu} = \frac{i}{2} \overrightarrow{\Psi} \beta^\mu \partial^\nu \Psi - \frac{i}{2} \left[ \partial^\nu \overrightarrow{\Psi} \beta^\mu \Psi - \frac{1}{4m} \overrightarrow{\Psi} \epsilon_{\alpha\lambda\rho\sigma} P [\beta^\lambda, \beta^\rho] n^\sigma g^{\sigma\mu} \partial^\nu \Psi \right. \]
\[ + \frac{1}{4m} \left[ \partial^\nu \overrightarrow{\Psi} \right] \epsilon_{\alpha\lambda\rho\sigma} P [\beta^\lambda, \beta^\rho] n^\sigma g^{\sigma\mu} \Psi - g^{\mu\nu} \mathcal{L}. \]  

(49)

With the help of the equations of motion (46) and (47), we find \( \partial_\mu T^{\mu\nu} = 0 \), as for the massless DKP field.

As shown in Appendix C, Eq. (49) can be expressed as Eq (C.18), and for \( R^\mu \Psi \) a real field, the tensor \( T^{\mu\nu} \) then becomes

\[ T^{\mu\nu} = -G^{\alpha\mu} G^\nu_\alpha + g^{\mu\nu} \left[ \frac{1}{4} G_{\mu\alpha} G^{\mu\alpha} + \frac{m^2}{2} \overrightarrow{\Psi} R R_\alpha \Psi \right] + \frac{n^\nu}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma} R_\alpha \Psi. \]  

(50)
Then we see that the components of this tensor are

\[ T^{00} = \frac{1}{2} \left( E^2 + B^2 + m^2 A^\alpha A_\alpha \right) + \frac{n^0}{2} (B \cdot A), \tag{51} \]

\[ T^{0i} = (E \times B)^i + \frac{n^i}{2} (B \cdot A). \tag{52} \]

Clearly, with \( m = 0 \) and \( n = 0 \), we recover the Maxwell field’s energy density and the Poynting vector, respectively.

The calculations related to the spatial anisotropy in this context are also similar to the massless case and result in the expression

\[ \left( k^\mu k^\nu - m^2 \right)^2 + \left( k^\mu k^\nu \right) (n_\nu n^\nu) - (n_\nu k^\nu)^2 = 0. \]

If we take \( n_\mu = (n_0, 0) \) and \( k_\mu = (\omega, k) \), we have

\[ \omega^4 - 2 \left( |k|^2 + m^2 \right) w^2 + \left( |k|^2 + m^2 \right)^2 - |k|^2 n_0^2 = 0, \]

which has solution

\[ \omega_\lambda = \left( |k|^2 + \lambda |k| n_0 + m^2 \right)^{\frac{1}{2}}, \tag{53} \]

with \( \lambda = \pm 1 \), that expresses a spatial anisotropy effect for massive vector bosons. The term \( |k| n_0 \) in Eq. \( (53) \) competes with the mass \( m \) and this implies that the range of electromagnetic interaction depends on linear momentum; in particular, some specific values of \( k \) are such that the model would be effectively massless. Eq. \( (53) \) is analogous to Eq. (3.5) of Ref. [59] and its physical implications are similar to those discussed for the massless DKP field, after Eq. (37). The dispersion relation \( (53) \) carries information about the propagation modes of the theory. Birefringence occurs when different polarization modes propagate at different (phase) velocities than the wave propagates, thus determining a rotation in the polarization plane. In our case, in the massless limit, we find that there are two modes which propagate at different velocities, that is, we have the wave propagating with the two (right and left) modes of polarization with different velocities. Thereby we conclude that the background field promotes the birefringence of the model.

### 5 Concluding remarks

In this work we have examined the spontaneous violation of Lorentz symmetry for spin-one massless and massive bosonic fields with the DKP formalism. This is an important question to verify what kind of contribution can emerge with this formalism. In analogy with previous analyses of the CPT- and Lorentz-symmetry breaking of the odd gauge sector of the Standard Model Extension, we added similar symmetry-breaking terms in the DKP Lagrangians for spin-one massless and massive fields.
These symmetry-breaking terms, analogous to the Chern-Simons term encountered in the literature, are defined in terms of the projection operators of the DKP theory. Motivated by similar results in previous studies, we verified that our model leads to dispersion and vacuum birefringence effects. In particular, our results are compatible with previous calculations for the massless DKP field. For both the massless and massive DKP theories, our dispersion relations clearly show that the birefringence and anisotropy effect disappear when the background field is equal to zero.

In this context, we can consider the appearance of Berry phases in the DKP equation and the study of anisotropies generated by a background. Potential applications to the quantum phase include the use of a non-null field $n_i$ to study the Aharonov-Casher effect, by extending the CPT-even, dimension-five, non-minimal coupling between the Dirac and gauge fields of Ref. [60] to a bosonic field described by the DKP equation. Also, along the lines of Ref. [61], we could use the DKP formalism described in this paper to keep extra dimensions hidden by adding Lorentz-violating tensor fields, or aether, with expectations values aligned along the extra dimensions. In relation with dimensional reduction, let us cite the reduction from $3+1$ spacetime by means of only one space-like component $n_k$ of the background field with the DKP formalism in order to study the Chern-Simons interaction in $2+1$ spacetime. Note also that the scenario described in this paper could be extended to use the DKP approach to describe scalar mesons, where the background field would be a tensor that violates Lorentz symmetry without violating CPT.

One can also investigate the analogous non-relativistic problems via the 5-dimensional Galilean DKP theory, whose results can be applied to phenomena in condensed matter physics. This is on particular interest, given, for instance, the recent study on the broken Galilean invariance at the quantum spin Hall edge [62] and in spin-orbit coupled Bose-Einstein condensates [63]. Clearly, a natural continuation of the present work would be the introduction of the background field interaction term within the Dirac Lagrangian.

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A Appendix: DKP scalar projection operators

Following Refs. [52] and [53], we define the operators $P^S$, $P^\mu$ and $\gamma^S$ as

\[
P^S = \frac{1}{3} (\beta^\mu \beta^\mu - 1),
\]

\[
P^\mu = P^S \beta^\mu,
\]

\[
\gamma^S = \frac{1}{3} (4 - \beta^\mu \beta^\mu).
\]

They satisfy many properties, among which we need $P^\mu \beta^\nu = P^S g^\mu\nu$ and $P^2 = P^S$. From the previous equations, we find $P^S(1 - \gamma^S) = P^S$, so that when we apply $P^S$ to Eq. (20), the second term which contains $(1 - \gamma^S)[\beta^\mu, \beta^\nu]$ becomes

\[
P^S(1 - \gamma^S)[\beta^\mu, \beta^\nu] = P^S[\beta^\mu, \beta^\nu] = 0.
\]

We proceed in a similar way with $P^\mu$. We can see that $P^\mu \gamma^S = P^\mu$, which implies $P^\mu(1 - \gamma^S)$, so that the second term of Eq. (20) is also annihilated by $P^\mu$.

Therefore, since both projection operators $P^S$ and $P^\mu$ annihilate the second term of Eq. (20), then when we apply both operators to the equation of motion and combine the results, we see that $P^S \Psi$ satisfies the Klein-Gordon equation: $\partial^\mu \partial^\mu P^S \Psi = 0$.

B Appendix: DKP vector projection operators

From the algebra in Eq. (4), we have

\[
\beta^\mu \beta^\nu \gamma = \gamma \beta^\mu \beta^\nu,
\]

which implies

\[
R^\mu \gamma = \gamma R^\mu = 0, \quad R^\mu (1 - \gamma) = R^\mu,
\]

\[
R^\mu \gamma = R^\mu, \quad R^\mu (1 - \gamma) = \gamma R^\mu = 0,
\]

We find also

\[
R^\alpha [\beta^\mu, \beta^\nu] = g^{\alpha\mu} R^\nu - g^{\alpha\nu} R^\mu.
\]

If we define

\[
S^\mu\nu = [\beta^\mu, \beta^\nu],
\]

then we see that the operators defined in Eqs. (16), (17) and (B.5) satisfy the following properties:

\[
R^\mu = - R^{\mu},
\]

\[
R^\mu \beta^\nu \beta^\alpha = g^{\mu\alpha} R^\nu - g^{\mu\nu} R^\alpha,
\]

\[
R^{\mu\nu} \beta^\alpha = g^{\nu\alpha} R^\mu - g^{\mu\alpha} R^\nu,
\]

\[
R^\mu S^\nu\alpha = g^{\mu\nu} R^\alpha - g^{\mu\alpha} R^\nu,
\]

\[
R^{\mu\nu} S^{\rho\alpha} = g^{\mu\rho} R^{\nu\alpha} - g^{\mu\alpha} R^{\nu\rho} + g^{\nu\rho} R^{\mu\alpha}.\]

13
Finally, let us note the following identity, which will be useful in Appendix C,

\[ \alpha RR_\alpha + \frac{1}{2} \alpha_\sigma RR_\sigma = 1, \]  

(B.7)

where

\[ \alpha R = \eta (R^\alpha)\dagger, \quad \alpha_\sigma R = \eta (R^{\alpha\sigma})\dagger. \]  

(B.8)

C Appendix: Energy-momentum tensors

In this appendix, we examine the energy-momentum tensors for the massive and massless DKP Lagrangians with a background field. First, we consider the complex massive DKP field in more details, and then briefly highlight the analogous derivations for the massless field.

Complex massive DKP field. The massive DKP Lagrangian in Eq. (42) leads to the wave equation

\[ \partial_\mu G^{\mu\nu} + \frac{1}{2} \varepsilon^{\mu
u\rho\sigma} n_\mu G_{\rho\sigma} + m^2 R^\nu \Psi = 0, \]  

(C.1)

and its adjoint

\[ \partial_\mu \overline{G}^{\mu\nu} + \frac{1}{2} \varepsilon^{\mu
u\rho\sigma} n_\mu \overline{G}_{\rho\sigma} + m^2 \overline{\Psi} R = 0. \]  

(C.2)

We wish to express the components \( T^{00} \) and \( T^{0i} \) of the tensor \( T^{\mu\nu} \) in terms of the components of \( \Psi \). Consider the expression for \( T^{\mu\nu} \):

\[
T^{\mu\nu} = \frac{i}{2} \overline{\Psi} \beta^\mu \partial^\nu \Psi - \frac{i}{2} \left[ \partial^\nu \overline{\Psi} \right] \beta^{\mu} \Psi - \frac{1}{4} \overline{\Psi} \varepsilon_{\lambda\rho\sigma} P \left[ \beta^\lambda, \beta^{\alpha} \right] n^\rho g^\sigma\mu \partial^\nu \Psi \\
+ \frac{1}{4} \left[ \partial^\nu \overline{\Psi} \right] \varepsilon_{\lambda\rho\sigma} P \left[ \beta^\lambda, \beta^{\alpha} \right] n^\rho g^\sigma\mu \Psi - g^{\mu\nu} \mathcal{L}. 
\]  

(C.3)

In order to find its components, we use its conservation (\( \partial_\mu T^{\mu\nu} = 0 \)) and the identity in Eq. (B.7). Note that

\[
m \overline{\Psi} \Psi = m \overline{\Psi} \left( \alpha RR_\alpha + \frac{1}{2} \alpha_\sigma RR_\sigma \right) \Psi = m^2 A^\alpha A_\alpha + \frac{1}{2} \overline{G}^{\alpha\sigma} G_{\alpha\sigma}. \]  

(C.4)

With Eq. (B.7), we see that the first term of the tensor \( T^{\mu\nu} \) is

\[
\frac{i}{2} \overline{\Psi} \beta^\mu \partial^\nu \Psi = \frac{i}{2} \overline{\Psi} \left( \alpha RR_\alpha + \frac{1}{2} \alpha_\sigma RR_\sigma \right) \beta^\mu \partial^\nu \Psi
\]
\[ = \frac{i}{2} \overline{\Psi} \left( \alpha R^{\nu \mu} + \frac{1}{2} \alpha R (g^{\sigma \mu} R^{\nu} - g^{\alpha \mu} R^{\sigma}) \right) \partial^{\nu} \Psi \]

\[ = \frac{i}{2} \overline{\Psi} (\alpha R^{\nu \mu}) \partial^{\nu} \Psi + \frac{i}{2} \overline{\Psi} (\alpha R^{\nu} \alpha) \partial^{\nu} \Psi \]

\[ = - \frac{1}{2} (\overline{\Psi} \alpha R(\mu) \partial^{\nu} \Psi + \frac{1}{2} \overline{\Psi} \alpha G^{\mu \alpha} (\partial^{\nu} R_{\alpha} \Psi). \quad (C.5) \]

Likewise, the second term of the tensor \( T^{\mu \nu} \) becomes:

\[ = - \frac{i}{2} (\partial^{\nu} \overline{\Psi}) \beta^{\mu} \Psi = \frac{1}{2} \left( \partial^{\nu} \overline{\Psi} \right) G^{\mu \sigma} - \frac{1}{2} \left( \partial^{\nu} \overline{\Psi} \right) \beta^{\mu} \Psi = - \frac{1}{2} \left( \overline{\Psi} \alpha R \Psi + \frac{1}{2} \overline{\Psi} \alpha G^{\mu \alpha} (\partial^{\nu} R_{\alpha} \Psi) \right) - \frac{1}{2} \left( \partial^{\nu} \overline{\Psi} \right) R_{\alpha} \Psi + \frac{1}{2} \left( \partial^{\nu} \overline{\Psi} \right) G^{\mu \sigma}. \quad (C.6) \]

Now it is crucial to add to the left-hand side of this expression (which is part of the tensor \( T^{\mu \nu} \)) the following zero-divergence term:

\[ = - \frac{1}{2} \left( \partial_{\sigma} \overline{\Psi} \right) G^{\mu \sigma} - \frac{1}{2} \overline{G}^{\mu \alpha} (\partial^{\nu} R_{\alpha} \Psi). \quad (C.8) \]

Indeed, its divergence takes the form

\[ \partial_{\mu} K^{\mu \nu} = - \frac{1}{2} \left( \partial_{\sigma} \overline{\Psi} \right) G^{\mu \sigma} - \frac{1}{2} \overline{G}^{\mu \alpha} (\partial^{\nu} R_{\alpha} \Psi) \]

and from the equation of motion and the definitions \( \tilde{G}^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} G_{\rho \sigma}, \quad \overline{G}^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} \overline{G}_{\rho \sigma} \), we have

\[ \partial_{\mu} K^{\mu \nu} = \frac{1}{2} \left( \partial_{\sigma} \overline{\Psi} \right) n_{\mu} \tilde{G}^{\mu \sigma} + \frac{m^{2}}{2} \left( \partial_{\sigma} \overline{\Psi} \right) R^{\sigma} \Psi + \frac{1}{2} n_{\mu} \tilde{G}^{\mu \alpha} (\partial_{\alpha} R^{\nu} \Psi) + \frac{m^{2}}{2} \overline{\Psi} R^{\sigma} \overline{\Psi} + \frac{m^{2}}{2} \left[ \left( \partial_{\sigma} \overline{\Psi} \right) \overline{\Psi} + \overline{\Psi} R^{\nu} \overline{\Psi} (\partial_{\alpha} \Psi) \right]. \quad (C.10) \]

If we use a representation, we can see that \( \nu R R_{\rho} \) is different from zero only when \( \nu = \rho \), which implies that the first term on the right-hand side becomes zero. Moreover, from this property, we can see that the term proportional to \( m^{2} \) becomes zero when we perform the Fourier transforming for the fields \( \Psi \) and \( \overline{\Psi} \). These two conclusions imply \( \partial_{\mu} K^{\mu \nu} = 0 \).
Then, we replace Eq. (C.7) with
\[
\frac{i}{2} \overline{\Psi} \beta^\mu \partial^\nu \Psi - \frac{i}{2} (\partial^\nu \overline{\Psi}) \beta^\mu \Psi = -\frac{1}{2} \left( \overline{\Psi}_\alpha R \right) \partial^\nu G^\mu \alpha + \frac{1}{2} \overline{G}^\alpha \Theta G^\nu \alpha \\
- \frac{1}{2} \left( \partial^\nu \overline{G}^\alpha \right) R_\alpha \Psi + \frac{1}{2} G^\nu \alpha .
\]
(C.11)

We note that the divergence of the first and the third terms on the right-hand side is
\[
\partial_\mu \left[ -\frac{1}{2} \left( \overline{\Psi}_\alpha R \right) \partial^\nu G^\mu \alpha - \frac{1}{2} \left( \partial^\nu \overline{G}^\alpha \right) R_\alpha \Psi \right] = -\frac{1}{2} \left( \partial_\mu \overline{\Psi}_\alpha R \right) \partial^\nu G^\mu \alpha - \frac{1}{2} \overline{\Psi}_\alpha R \partial_\mu G^\mu \alpha \\
- \frac{1}{2} \left( \partial^\nu \partial_\mu \overline{G}^\alpha \right) R_\alpha \Psi - \frac{1}{2} \left( \partial^\nu \overline{G}^\alpha \right) \partial_\mu R_\alpha \Psi \\
= \partial_\mu \left( \frac{g^{\mu \nu}}{4} \overline{G}^\alpha \mu G^\mu \alpha + \frac{m^2}{2} \overline{\Psi}^\alpha R R_\alpha \Psi \right) ,
\]
where we have used \( \epsilon^{\alpha \mu \rho \sigma} \sigma R R_\alpha = 0 \), since \( \sigma R R_\alpha \) is non-zero only when \( \sigma = \alpha \). Then we can substitute
\[
-\frac{1}{2} \left( \overline{\Psi}_\alpha R \right) \partial^\nu G^\mu \alpha - \frac{1}{2} \left( \partial^\nu \overline{G}^\alpha \right) R_\alpha \Psi = g^{\mu \nu} \left[ \frac{1}{4} \overline{G}^\alpha \mu G^\mu \alpha + \frac{m^2}{2} \overline{\Psi}^\alpha R R_\alpha \Psi \right] ,
\]
(C.13)
and
\[
\frac{i}{2} \overline{\Psi} \beta^\mu \partial^\nu \Psi - \frac{i}{2} (\partial^\nu \overline{\Psi}) \beta^\mu \Psi = -\frac{1}{2} \overline{G}^\alpha \theta G^\nu \alpha - \frac{1}{2} \overline{G}^\alpha \Theta G^\alpha \nu \\
+ g^{\mu \nu} \left[ \frac{1}{4} \overline{G}^\alpha \mu G^\mu \alpha + \frac{m^2}{2} \overline{\Psi}^\alpha R R_\alpha \Psi \right] .
\]
(C.14)

Now we consider the third term of \( T^\mu \nu \) in Eq. (49) and utilize the identity (B.7) to obtain
\[
-\frac{1}{4} \overline{\Psi} \epsilon_{\lambda \alpha \rho \sigma} P [\beta^\lambda, \beta^\alpha] \eta^\rho g^{\sigma \mu} \partial^\nu \Psi = -\frac{1}{4} \overline{\Psi} \epsilon_{\lambda \alpha \rho \sigma} P [\beta^\lambda, \beta^\alpha] \left( \delta R R_\delta + \frac{1}{2} \delta R \delta \Psi \right) \\
\times \eta^\rho g^{\sigma \mu} \partial^\nu \Psi \\
= -\frac{1}{2} \overline{\Psi} \epsilon_{\lambda \alpha \rho \sigma} P \beta^\lambda \beta^\alpha \left( \delta R R_\delta \right) \eta^\rho g^{\sigma \mu} \partial^\nu \Psi \\
= \frac{1}{2} \epsilon_{\lambda \rho \mu \nu} \eta^\rho \overline{\Psi}_\alpha R R_\lambda \left( \partial^\nu \Psi \right) \\
= 0.
\]
(C.15)
We show in a similar manner that the fourth term of \( T^\mu \nu \) on the right-hand side of Eq. (49) is also equal to zero.

Finally, let us add another divergenceless \( n^\mu \)-dependent term to the tensor \( T^\mu \nu \),
\[
M^\mu \nu = \frac{n^\nu}{4} \epsilon_{\mu \alpha \rho \sigma} \overline{G}^\rho \sigma R_\alpha \Psi + \frac{n^\nu}{4} \epsilon_{\mu \alpha \rho \sigma} \overline{\Psi}_\alpha R G^\rho \sigma .
\]
(C.16)
Its divergence is given by

\[
\partial_{\mu} M^{\mu \nu} = \frac{n^\nu}{4} \epsilon^{\mu \alpha \rho \sigma} (\partial_\mu G_{\rho \sigma}) R_\alpha \Psi + \frac{n^\nu}{4} \epsilon^{\mu \alpha \rho \sigma} G_{\rho \sigma} (\partial_\mu R_\alpha \Psi) \\
+ \frac{n^\nu}{4} \epsilon^{\mu \alpha \rho \sigma} (\partial_\mu \nabla_\nu) \alpha R_G \rho \sigma + \frac{n^\nu}{4} \epsilon^{\mu \alpha \rho \sigma} \nabla_\alpha R (\partial_\mu G_{\rho \sigma}) \\
= \frac{n^\nu}{4} \epsilon^{\mu \alpha \rho \sigma} (\partial_\rho \nabla_\sigma) R_{\alpha \beta} (\partial_\mu \Psi) + \frac{n^\nu}{4} \epsilon^{\mu \alpha \rho \sigma} (\partial_\mu \nabla_\nu) \alpha RR_\sigma (\partial_\rho \Psi) \\
= 0.
\]

(C.17)

Thus, we have for \( T^{\mu \nu} \):

\[
T^{\mu \nu} = \frac{1}{2} G^{\alpha \mu} G^{\nu \alpha} - \frac{1}{2} G^{\nu \alpha} G^{\alpha \mu} + g^{\mu \nu} \left[ \frac{1}{4} G_{\mu \alpha} G^{\mu \alpha} + \frac{m^2}{2} \nabla_\alpha RR_\sigma \Psi \right] \\
+ \frac{n^\nu}{4} \epsilon^{\mu \alpha \rho \sigma} G_{\rho \sigma} R_\alpha \Psi + \frac{n^\nu}{4} \epsilon^{\mu \alpha \rho \sigma} \nabla_\alpha R G_{\rho \sigma}
\]

(C.18)

with \( \partial_{\mu} T^{\mu \nu} = 0 \). If we take \( R^\mu \Psi \) to be a real field, then the tensor \( T^{\mu \nu} \) becomes (50).

**Complex massless DKP field.** The Lagrangian for the massless DKP field, given in Eq. (5), differs from the massive Lagrangian in Eq. (42) by the third term, with \( m \) replaced by the singular matrix \( \gamma \). Note that, in the end, it turns out that the components \( T^{00} \) and \( T^{0i} \) for the massless field are simply obtained from Eqs. (51) and (52) with \( m = 0 \). However, hereafter we highlight the independent approach, analogous to the massive field, starting with the Lagrangian in Eq. (5) to highlight a few subtleties. For instance, note that the wave equations for the field and its adjoint can be obtained by setting \( m = 0 \) in Eqs. (C.1) and (C.2), respectively. However, the tensor \( T^{\mu \nu} \) has the form in Eq. (C.3), the difference being hidden in the Lagrangian of the last term; this equation thus contains \( \gamma \), both the massive and the massless fields.

Let us point out that, for massless fields, Eq. (C.4) is replaced by

\[
\bar{\Psi} \gamma \Psi = \bar{\Psi} \left( a RR_\alpha + \frac{1}{2} \tilde{a} RR_{\alpha \sigma} \right) \gamma \Psi \\
= \frac{1}{2} \bar{\Psi} \left( b a RR_\sigma \right) \gamma \Psi \\
= -\frac{i}{2} \bar{\Psi} \left( \tilde{\partial}^{\alpha} R - \tilde{\partial}^{\sigma} R \right) \left( \tilde{\partial}^{\alpha} R_\alpha - \tilde{\partial}^{\sigma} R_\sigma \right) \Psi \\
= -\frac{1}{2} \bar{\Psi} \gamma \tilde{G}^{\alpha \sigma} G_{\alpha \sigma}
\]

(C.19)

where we use again Eq. (B.7). The expressions (C.7), (C.8), (C.15) and (C.16) are still valid for the massless field.
Thus, we obtain that for a complex massless DKP field, $T^{\mu\nu}$ is given by
\[
T^{\mu\nu} = -\frac{1}{2} G^{\alpha\mu} G^{\nu} + \frac{1}{2} G^{\alpha\nu} G^{\mu} + \frac{n^{\nu}}{4} \epsilon^{\mu\alpha\rho\sigma} G_{\rho\sigma} R_{\alpha} \Psi + \frac{n^{\nu}}{4} \epsilon^{\mu\alpha\rho\sigma} \overline{\Psi} G_{\rho\sigma} R_{\alpha},
\]  
and such that $\partial_{\mu} T^{\mu\nu} = 0$.

If we choose $R^{\mu}\Psi$ to be a real field, then the tensor $T^{\mu\nu}$ reduces to Eq. (24).

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