BOSONIC VERTEX REPRESENTATIONS OF THE
TORIODAL SUPERALGEBRAS IN TYPE $D(m, n)$

NAIHUAN JING AND CHONGBIN XU*

Abstract. In this paper, vertex representations of the 2-toroidal Lie superalgebras of type $D(m, n)$ are constructed using both bosonic fields and vertex operators based on their loop algebraic presentation.

1. Introduction

Let $\mathfrak{g} = \mathfrak{g}_\pi + \mathfrak{g}_\tau$ be a finite dimensional complex simple Lie superalgebra under the Lie superbracket, and let $R$ be the algebra of Laurent polynomials in $\nu$ commuting indeterminates. By definition, the $\nu$-toroidal Lie superalgebra associated to $\mathfrak{g}$ is the perfect universal central extension of the loop Lie-superalgebra $L(\mathfrak{g}) = \mathfrak{g} \otimes R$, equivalently, one can realize it as certain homomorphic image of the universal central extension $T(\mathfrak{g}) = \mathfrak{g} \otimes R + \Omega_R/dR$, where $\Omega_R/dR$ is the Kähler differential of $R$ modulo the exact forms.

Representations of toroidal Lie algebras have been actively studied and a lot of known constructions for classical affine Lie algebras \cite{3} have been extended to the toroidal setting. In \cite{16} Moody, Rao and Yokonuma gave the loop algebra realization of the 2-toroidal algebras and constructed vertex representation for the simply laced types. Vertex operator representations of toroidal Lie algebras in type $B$ were given in \cite{18}, and then generalized to multi-loop toroidal Lie algebras of the same type in \cite{7}. A uniformed fermionic construction of the 2-toroidal algebras of the classical types were given by Misra and the authors \cite{9} and subsequently a general bosonic construction was realized in \cite{10}. Moreover, representations of the universal toroidal Lie algebras have been studied in \cite{1}. A Wakimoto type realization was also given for the toroidal Lie algebra in type $A$ in \cite{2} using noncommutative differential operators (see \cite{8} for an earlier construction for type $A_1$) was also given.

\textit{2010 Mathematics Subject Classification.} Primary: 17B60, 17B67, 17B69; Secondary: 17A45, 81R10.

\textit{Key words and phrases.} toroidal Lie superalgebra, vertex operators, free fields.

*Corresponding author.
The study of toroidal super Lie algebras is more involved and requires new method to treat the odd subalgebra. Based on Kac-Wakimoto’s work on the affine algebras, Rao constructed vertex representations for the toroidal general linear superalgebra in [17]. The authors have showed a MRy-type presentation for the 2-toroidal Lie superalgebras, and constructed the unitary and orthosymplectic series by means of free fields in [10] based upon the well-known constructions of Lie superalgebras [3, 4] in level one. Moreover, a new vertex representation for 2-toroidal special linear superalgebra was also given in [11]. However, it is not known if other constructions of the toroidal Lie algebras such as the generalized Feingold-Frenkel construction given in [7] can be lifted to the super situation.

In this paper, we use vertex operators to realize the even part of the orthosymplectic toroidal Lie algebra and bosonic operators for the remaining portion and then combine these two types of operators to construct the whole algebra. In particular, we have generalized the level −1 construction of the orthogonal and symplectic affine Lie algebras [7] to the super case. Our method is a natural generalization of [11] given the close relationship between orthosymplectic superalgebras and classical Lie algebras.

The paper is organized as follows. In section 2 we recall the notion of 2-toroidal Lie superalgebras of type $D(m, n)$ and the loop-algebra presentation. In section 3 we use certain vertex operators and Weyl bosonic fields to give a level $−1$ representation of the Lie superalgebras.

2. Toroidal Lie Superalgebras of Type $D(m, n)$

For two fixed natural numbers $m, n \in \mathbb{N}$, let $V = V_T^\mathbf{0} \oplus V_T^\mathbf{1}$ be the super vector space with $\dim V_T^\mathbf{0} = 2m$, $\dim V_T^\mathbf{1} = 2n$. The super-endomorphisms of $V$ form the general linear superalgebra $\mathfrak{gl}(2m|2n)$ under the superbracket given by

$$[f, g] = fg - (-1)^{|f||g|}gf$$

for homogeneous linear operators $f, g$. Let $(\cdot | \cdot)$ be a non-degenerate bilinear form on $V$ such that $(V_T^\mathbf{0}|V_T^\mathbf{1}) = 0$, and the restriction of $(\cdot | \cdot)$ to $V_T^\mathbf{1}$ is symmetric and the restriction to $V_T^\mathbf{0}$ is skew-symmetric. For $\tau = 0, 1$, let

$$\mathfrak{osp}(2m|2n)_\tau$$

$$= \{ a \in \mathfrak{gl}(2m|2n)_\tau | (a(x)|y) + (-1)^{\mathbf{op}(x)}(x|a(y)) = 0, x, y \in V \}$$
and \( \mathfrak{osp}(2m|2n) = \mathfrak{osp}(2m|2n)_τ \oplus \mathfrak{osp}(2m|2n)_\overline{τ} \). Then \( \mathfrak{osp}(2m|2n) \) forms the Lie superalgebra of type \( D(m,n) \) if \( m > 1 \). Note that

\[
D(m|n)_τ \cong \mathfrak{so}(2m) \oplus \mathfrak{sp}(2n).
\]

Let us denote the Lie superalgebra \( D(m,n) \) by \( g \), and let \( R = \mathbb{C}[s^{\pm 1}, t^{\pm 1}] \) be the complex commutative ring of Laurent polynomials in \( s, t \). The loop Lie superalgebra \( L(g) := g \otimes R \) is defined under the Lie superbracket

\[
[x \otimes a, y \otimes b] = [x, y] \otimes ab.
\]

Let \( \Omega_R \) be the \( R \)-module of Kähler differentials \( \{ bda | a, b \in R \} \) and \( d\Omega_R \) be the space of exact forms. The quotient space \( \Omega_R/d\Omega_R \) has a basis consisting of \( s^{-1}kds, s^lt^{-1}dt \), where \( k, l \in \mathbb{Z} \). Here \( \overline{a} \) denotes the coset \( a + d\Omega_R \).

The toroidal superalgebra \( T(g) \) is defined to be the Lie superalgebra on the following vector space:

\[
T(g) = g \otimes R \oplus \Omega_R/d\Omega_R
\]

with the Lie superbracket \( (x, y \in g, a, b \in R) \):

\[
[x \otimes a, y \otimes b] = [x, y] \otimes ab + (x|y)(da)b, \quad \Omega_R/d\Omega_R \text{ is central}
\]

and the parity is specified by:

\[
p(x \otimes a) = p(x), \quad p(\Omega_R/d\Omega_R) = \overline{0}.
\]

Let \( A = (a_{ij}) \) be the extended distinguished Cartan matrix of the affine Lie superalgebra of type \( D(m,n)^{(1)} \), i.e.

\[
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-2 & 2 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \vdots & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & -1 & 0 & 1 & \ddots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \vdots & 0 & 0 & 0 & \ddots & 2 & -1 & -1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 & 0 & 2
\end{pmatrix}
\]

← \( (n+1) \)-th
and $Q = \mathbb{Z}\alpha_0 \oplus \cdots \oplus \mathbb{Z}\alpha_{m+n}$ be its root lattice. The odd simple root is $\alpha_n$. The standard invariant form is then given by $(\alpha_i, \alpha_j) = d_i a_{ij}$, where

$$(d_0, d_1, \cdots, d_n, d_{n+1}, \cdots, d_{n+m}) = (2, 1, \cdots, 1, -1, \cdots, -1).$$

Note that $d_i = \frac{1}{2}(\alpha_i, \alpha_i)$ for non-isotropic roots.

To organize the commutation relations for toroidal Lie algebras, we use formal series. The formal delta function is defined by

$$\delta(z - w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n,$$

which can be formally viewed as a sum of two power series expanded at opposite directions. For this purpose we denote that

$$i_{z,w} \frac{1}{(z - w)} = \sum_{n=0}^{\infty} z^{-n-1} w^n,$$

$$i_{w,z} \frac{1}{(z - w)} = -\sum_{n=0}^{\infty} w^{-n-1} z^n,$$

where $i_{z,w}$ means that the power series is expanded in the domain of $|z| > |w|$. Subsequently one has that $[12]$:

$$\partial^{(j)} w \delta(z - w) = i_{z,w} \frac{1}{(z - w)^{j+1}} - i_{w,z} \frac{1}{(-w + z)^{j+1}},$$

where $\partial^{(j)} w = \partial w / j!$. By convention if we write a rational function in the variable $z - w$ it is usually assumed that the power series is expanded in the region $|z| > |w|.$

The following loop algebra presentation of the 2-toroidal Lie superalgebras was proved in $[10]$.

**Theorem 2.1.** The toroidal Lie superalgebra $T(D(m,n))$ is isomorphic to the Lie superalgebra $\mathfrak{S}^1(A)$ generated by

$$\{\mathcal{K}, \alpha_i(k), x^+_i(k) | 0 \leq i \leq m + n, k \in \mathbb{Z}\}$$

with parities given as : $(0 \leq i \leq m + n, k \in \mathbb{Z})$

$$p(\mathcal{K}) = p(\alpha_i(k)) = 0, \quad p(x^+_i(k)) = p(\alpha_i).$$
subject to the following relations

1) \([K, \alpha_i(z)] = [K, x_i^±(z)] = 0;\)
2) \([\alpha_i(z), \alpha_j(w)] = (\alpha_i|\alpha_j)\partial_w \delta(z - w)K;\)
3) \([\alpha_i(z), x_j^±(w)] = \pm (\alpha_i|\alpha_j)x_j^±(w)\delta(z - w);\)
4) \([x_i^±(z), x_j^±(w)] = 0, \text{if } i \neq j;\)
\[x_i^±(z), x_i^−(w) = \left\{-\left((\alpha_i(w)\delta(z - w) + \partial_w \delta(z - w)K\right), \text{if } (\alpha_i|\alpha_i) = 0\right\}\]
\[x_i^±(z), x_i^−(w) = \frac{2}{(\alpha_i|\alpha_i)}\left\{(\alpha_i(w)\delta(z - w) + \partial_w \delta(z - w)K\right), \text{if } (\alpha_i|\alpha_i) \neq 0\right\}\]
5) \([x_i^±(z), x_i^±(w)] = 0;\)
\[x_i^±(z), x_j^±(w)] = 0, \text{if } a_{ii} = a_{ij} = 0, i \neq j;\)
\[x_i^±(z_1), [x_i^±(z_2), x_j^±(w)] = 0, \text{if } a_{ii} = 0, a_{ij} \neq 0, i \neq j;\]
\[x_i^±(z_1), \ldots, [x_i^±(z_{a_{ij}}), x_j^±(w)] \ldots] = 0, \text{if } a_{ii} \neq 0, i \neq j,\]
where we have used the generating series \(\alpha_i(z) = \sum_{k \in \mathbb{Z}} \alpha_i(k)z^{-k-1}, x_i^±(z) = \sum_{k \in \mathbb{Z}} x_i^±(k)z^{-k-1}.\)

Note that all brackets in the relations are understood as super-brackets.

3. Representations of toroidal Lie superalgebras

This section is devoted to realization of the toroidal Lie superalgebra of \(D(m, n)\) using both bosonic fields and vertex operators.

Let \(\varepsilon_i \ (0 \leq i \leq n + m + 1)\) be an orthonormal basis of the vector space \(\mathbb{C}^{n+m+2}\) and \(\delta_i = \sqrt{-1}\varepsilon_{n+i} \ (1 \leq i \leq m + 1)\), then the distinguished simple root systems, positive root systems and the longest distinguished root of the Lie superalgebra of type \(D(m, n)\) can be represented in terms of vectors \(\varepsilon_i\)'s and \(\delta_i\)'s as follows.

\[\Delta_+ = \{\varepsilon_i \pm \varepsilon_j, \delta_k \pm \delta_l | 1 \leq i < j \leq n, 1 \leq k < l \leq m\},\]
\[\cup \{2\delta_k, \varepsilon_i \pm \delta_i | 1 \leq i \leq n, 1 \leq k \leq n\},\]
\[\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_n - \delta_1,\]
\[\alpha_{n+1} = \delta_1 - \delta_2, \ldots, \alpha_{n+m-1} = \delta_{m-1} - \delta_m, \alpha_{n+m} = \delta_m - \delta_{m+1},\]
\[\theta = 2\alpha_1 + \cdots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} = 2\varepsilon_1.\]

Let \(\overline{\varepsilon} = \varepsilon_0 + \delta_{n+1}\) and define \(\alpha_0 = \overline{\varepsilon} - \theta, \beta = -\overline{\varepsilon} + \varepsilon_1\), then \(\alpha_0 = -\beta - \varepsilon_1.\)

Note that \((\beta|\beta) = 1, (\beta|\varepsilon_i) = \delta_1i.\) Let \(\mathcal{P}\) be the vector spaces spanned by the set \(\{\overline{\varepsilon}, \varepsilon_i | 1 \leq i \leq n + m\}\) and \(\mathcal{P}^*\) be its dual space. Let \(\mathcal{C} = \mathcal{P} \oplus \mathcal{P}^*\) and
define the bilinear form \( \langle \cdot, \cdot \rangle \) on \( C \) as follows: for \( a, b \in P \)
\[
\langle b^*, a \rangle = -\langle a, b^* \rangle = \langle a, b \rangle, \quad \langle b, a \rangle = \langle a^*, b^* \rangle = 0,
\]

Let \( A(\mathbb{Z}^{2n+2m+2}) \) be the Weyl algebra generated by \( \{u(k) \mid u \in C, k \in \mathbb{Z}\} \) with the defining relations:
\[
[u(k), v(l)] = \langle u, v \rangle \delta_k, -l
\]
for \( u, v \in C \) and \( k, l \in \mathbb{Z} \).

We define the representation space of \( A(\mathbb{Z}^{2(n+m+1)}) \) by
\[
\mathcal{F} = \bigotimes_{a_i} \left( \bigotimes_{k \in \mathbb{Z}_+^+} \mathbb{C}[a_i(-k)] \bigotimes_{k \in \mathbb{Z}^+} \mathbb{C}[a^*_i(-k)] \right),
\]
where \( a_i \) runs through a fixed basis in \( P \), consisting of, say \( \tau \) and \( \varepsilon_i \)'s. The algebra \( A(\mathbb{Z}^{2(n+m+1)}) \) acts on the space by the usual action: \( a(-k) \) acts as a creation operator and \( a(k) \) an annihilation operator.

For \( u \in C \), we define the formal power series with coefficients from the associative algebra \( A(\mathbb{Z}^{2(n+m+1)}) \):
\[
u(z) = \sum_{k \in \mathbb{Z}} u(k) z^{-k-1},
\]
which is a bosonic field acting on the Fock space \( \mathfrak{F} \).

**Proposition 3.1.** [9] The bosonic fields satisfy the following communication relation:
\[
[u(z), v(w)] = \langle u, v \rangle \delta(z - w).
\]

Let \( L = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_m \oplus \mathbb{Z}^n \) and \( \mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C} \) be its complex hull. We view \( \mathfrak{h} \) as an abelian Lie algebra and define its central extension
\[
\widehat{\mathfrak{h}} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{h} \otimes t^k \oplus \mathbb{C} \mathfrak{c}
\]
with the following Lie multiplication:
\[
[\alpha(k), \beta(l)] = k(\alpha, \beta) \delta_{k,l} \mathfrak{c}, \quad [\widehat{\mathfrak{h}}, \mathfrak{c}] = 0
\]
where \( \alpha(k) = \alpha \otimes t^k \) and \( \alpha, \beta \in L; k, l \in \mathbb{Z} \).

Let \( \widehat{\mathfrak{h}}_\pm = \bigoplus_{k \in \mathbb{Z}_\pm^+} \mathfrak{h} \otimes t^{\pm k} \) and \( S(\widehat{\mathfrak{h}}_-) \) the symmetric algebra of \( \widehat{\mathfrak{h}}_- \). We give \( S(\widehat{\mathfrak{h}}_-) \) an \( \widehat{\mathfrak{h}}_+ \oplus \mathfrak{h}_- \oplus \mathbb{C} \mathfrak{c} \)-module structure by letting \( \alpha(-k) \) act as the multiplication by \( \alpha(-k) \) for \( k > 0 \), \( \alpha(k) \) the derivation determined by \( \alpha(k) \cdot \beta(-l) = \delta_{k,l} k(\alpha, \beta) \) for \( k, l > 0 \) and \( \mathfrak{c} \) the identity operator.
For $i = 0, 1$, let $L_i = \{\alpha \in L \mid (\alpha, \alpha) \equiv i \pmod{2}\}$, then $L = L_\Pi \oplus L_T$. Let $F : L \times L \to \{\pm 1\}$ be the cocycle satisfying $F(0, \alpha) = F(\alpha, 0)$ and $F(\varepsilon_i, \varepsilon_j) = \begin{cases} 1, & \text{if } i \leq j; \\ -1, & \text{if } i > j. \end{cases}$

Let $\mathbb{C}[L]$ be the complex vector space spanned by the basis $\{e^\gamma \mid \gamma \in L\}$. We define a twisted group algebra structure on $\mathbb{C}[L]$ by

$$e^\alpha e^\beta = F(\alpha, \beta) e^{\alpha + \beta}.$$ 

We define the tensor space $V[L] = S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[L]$, and define the action of $\hat{\mathfrak{h}}$ on $V[\Gamma]$ as follows

$$\alpha(k) \cdot (v \otimes e^\beta) = \delta_{k,0}(\alpha, \beta)(v \otimes e^\beta) + (1 - \delta_{k,0})\alpha(k) \cdot v \otimes e^\beta.$$

Then the space $V[L]$ has a natural $\mathbb{Z}_2$-gradation:

$$V[L] = V[L]_0 \oplus V[L]_1$$

where $V[L]_0$ (resp. $V[L]_1$) is the vector space spanned by $e^\alpha \otimes \beta \otimes t^{-j}$ with $\alpha, \beta \in L; j \in \mathbb{Z}_+$ such that $(\alpha, \alpha) \in 2\mathbb{Z}$ (resp. $(\alpha, \alpha) \in 2\mathbb{Z} + 1$).

For $\alpha \in L$, we define

$$E^\pm(\alpha, z) = \exp\left(\sum_{k \in \mathbb{Z}_+} \frac{\alpha(k)}{k} z^{-k}\right) \in \text{End}S(\hat{\mathfrak{h}}^-)[[z^{\pm 1}]]$$

and introduce the operator $z^{\alpha(0)}$ as follows:

$$z^{\alpha(0)}(e^\beta \otimes \gamma \otimes t^{-j}) = z^{(\alpha, \beta)}(e^\beta \otimes \gamma \otimes t^{-j})$$

where $\beta, \gamma \in L$ and $j \in \mathbb{Z}_+$. Define the vertex operator $Y(\alpha, z)$:

$$Y(\alpha, z) = e^\alpha z^{\alpha(0)} E^-(\alpha, z) E^+(\alpha, z)$$

and denote by

$$X(\alpha, z) = \begin{cases} z^{(\alpha, \alpha)/2} Y(\alpha, z), & \text{if } \alpha \in L_\Pi; \\ Y(\alpha, z), & \text{if } \alpha \in L_T. \end{cases}$$

Expand $X(\alpha, z)$ in powers of $z$:

$$X(\alpha, z) = \sum_{j \in \mathbb{Z}} X(\alpha, j)z^{-j-1}.$$ 

Note the components $X(\alpha, j)$ are well-defined operators.
In addition, for $\alpha \in L$, we define
\[
\alpha(z) = \sum_{k \in \mathbb{Z}} \alpha(k) z^{-k-1}
\]
then we have
\[
[\alpha(z), \beta(w)] = (\alpha, \beta) \partial_w \delta(z - w), \quad \alpha, \beta \in L.
\]

**Proposition 3.2.** On the space $V[L]$ one has that
1) 
\[
[X(\varepsilon_i, z), X(\varepsilon_j - \varepsilon_k, w)] = \delta_{ik} F(\varepsilon_i, \varepsilon_j - \varepsilon_k) X(\varepsilon_j, w) \delta(z - w), \quad j \neq k,
\]
2) 
\[
[X(\varepsilon_i, z), X(-\varepsilon_j, w)] = \delta_{ij} F(\varepsilon_i, -\varepsilon_j) \partial_w \delta(z - w),
\]
3) 
\[
[X(\varepsilon_i - \varepsilon_j, z), X(\varepsilon_j - \varepsilon_i, w)] = F(\varepsilon_i - \varepsilon_j, \varepsilon_j - \varepsilon_i)((\varepsilon_i - \varepsilon_j)(z) \delta(z - w) + \partial_w \delta(z - w))
\]
4) 
\[
[\alpha(z), X(\beta, w)] = (\alpha, \beta) X(\beta, z) \delta(z - w), \quad \alpha, \beta \in L.
\]

**Proof.** 1), 2) and 4) are direct consequences of Lemma 1.8 in [17]. For 3), we refer the reader to [5]. □

In the following, we will give a representation of $\mathfrak{X}(A)$ on the tensor space $V[L] \otimes \mathfrak{F}$. It is easy to see that there is a $\mathbb{Z}_2$–gradation on this space with the parity given by $p(e^a \otimes x \otimes y) = p(\alpha)$ for $\alpha \in L, x \in S(\bigoplus_{j<0} (h \otimes t^j)), y \in \mathfrak{F}$. The vertex operators $X(\alpha, z), \alpha(z)$ act on the first component and the bosonic fields $u(z)$ act on the second component. It follows that
\[
p(X(\alpha, z)) = p(\alpha), \quad p(\alpha(z)) = p(u(z)) = \overline{\alpha}.
\]

For any two fields $a(z), b(w)$ with fixed parity, we define the normal ordered product by:
\[
: a(z)b(w) : = a(z)\pm b(w) - (-1)^{p(a)p(b)} b(w)a(z)_- = (-1)^{p(a)p(b)} : b(w)a(z) :
\]
where $a_\pm(z)$ is defined as usual.

Furthermore, we define the contraction of two fields $a(z), b(w)$ by
\[
\underbrace{a(z)b(w)} = a(z)b(w) - a(z)b(w) : : .
\]

We recall the general operator product expansion [12]. Suppose $a(z), b(w)$ are two fields such that
\[
[a(z), b(w)] = \sum_{j=0}^{N-1} c^j(w) \partial^j_w \delta(z - w),
\]
where $N$ is a positive integer and $c^j(w)$ are formal distributions in the indeterminate $z$, then we have that

$$a(z)b(w) = \sum_{j=0}^{N-1} c^j(w) \frac{1}{(z-w)^{j+1}}.$$  

**Corollary 3.3.** For $u, v \in \mathcal{C}; \alpha, \beta \in L$, one has

1) $u(z)v(w) =< u, v > \frac{1}{z-w}$, $u(z)X(\alpha, w) = 0$;

2) $X(\varepsilon_i, z)X(-\varepsilon_i, w) = \frac{1}{z-w}$

**Proof.** These are direct results of Proposition 3.1 and the OPE.  \hfill \Box

The following well-known Wick’s theorem is useful for calculating the operator product expansions of normally ordered products of free fields.

**Theorem 3.4.** (12) Let $A^1, A^2, \cdots, A^M$ and $B^1, B^2, \cdots, B^N$ be two collections of fields with definite parity. Suppose these fields satisfy the following properties:

1) $[A^1 B^j, Z^k] = 0$, for all $i, j, k$ and $Z = A$ or $B$;

2) $[A^i, B^j] = 0$, for all $i, j$.

then we have that

$$: A^1 \cdots A^M : : B^1 \cdots B^N : \bigg|_{\min\{M, N\}} = \sum_{s=0}^{\min\{M, N\}} \sum_{i_1 < \cdots < i_s \atop j_1 \neq \cdots \neq j_s} \pm \left( A^{i_1} B^{j_1} \cdots A^{i_s} B^{j_s} : A^1 \cdots A^M B^1 \cdots B^N : (i_1, \ldots, i_s, j_1, \ldots, j_s) \right)$$

where the subscript $(i_1, \cdots, i_s, j_1, \cdots, j_s)$ means the fields $A^{i_1}, \ldots, A^{i_s}, B^{j_1}, \ldots, B^{j_s}$ are removed and the sign $\pm$ is obtained by the rule: each permutation of the adjacent odd fields changes the sign.

Now we state the main result in this paper.

**Theorem 3.5.** The map defined below

$$x_i^+(z) \mapsto \begin{cases} \frac{1}{2} : \beta^*(z) \varepsilon_i^*(z) :, & i = 0; \\ \sqrt{-1} : \varepsilon_i(z) \varepsilon_{i+1}^*(z) :, & 1 \leq i \leq n - 1; \\ : X(\varepsilon_{n+1}, z) \varepsilon_i^*(z) :, & i = n; \\ X(\varepsilon_i - \varepsilon_{i+1}, z), & n + 1 \leq i \leq n + m - 1 \\ X(\varepsilon_{n+m-1} + \varepsilon_{n+m}, z), & i = n + m. \end{cases}$$
Proof. We prove the theorem by checking the field operators defined above satisfying relations 1) — 5) listed in Proposition 2.1.

First of all, we check 4) and 3) with the help of Wick’s theorem.

\[ [x_0^+, x_0^-] = \frac{1}{4} \left( (\varepsilon_1 \varepsilon_1^* + \varepsilon_1 \varepsilon_1^* + \beta \varepsilon_1^* + \beta \varepsilon_1^*) \delta(z-w) + 2\partial_w \delta(z-w) \right) \]

\[ = -\frac{2}{(\alpha_0, \alpha_0)} (\alpha_0 \delta(z-w) + \partial_w \delta(z-w) \cdot (-1)) \]

\[ [\alpha_0(z), x_0^\pm(w)] = 0 = \pm(\alpha_0, \alpha_0) x_0^\pm(w) \delta(z-w) \]

For \(1 \leq i \leq n - 1\), we have that

\[ [x_i^+, x_i^-] = -\left( (\varepsilon_i \varepsilon_i + \varepsilon_i \varepsilon_i + \beta \varepsilon_i^* + \beta \varepsilon_i^*) \delta(z-w) + \partial_w \delta(z-w) \right) \]

\[ = -\frac{2}{(\alpha_i, \alpha_i)} (\alpha_i \delta(z-w) + \partial_w \delta(z-w) \cdot (-1)) \]

and \([\alpha_i(z), x_i^\pm(w)] = \pm(\alpha_i, \alpha_i) x_i^\pm(w) \delta(z-w)\).

Next, one check that

\[ [x_n^+, x_n^-] = (\varepsilon_n \varepsilon_n + X(\varepsilon_{n+1}) X(\varepsilon_{n+1}, z) \delta(z-w) + \partial_w \delta(z-w) \]

\[ = -\alpha_n \delta(z-w) + \partial_w \delta(z-w) \cdot (-1) \]

gives rise to a level -1 representation on the space \( V[L] \otimes \mathcal{S} \) for the 2-toroidal Lie superalgebra of type \( D(m,n) \).
and
\[ [\alpha_n(z), x^+_n(w)] = 0 = \pm (\alpha_n, \alpha_n)x^+_n(w)\delta(z - w). \]

For \( n + 1 \leq i \leq m + n - 1 \), we have that
\[ [x^+_i(z), x^-_i(w)] = \pm ((\varepsilon_n + m + n)(z))\delta(z - w) + \partial_w\delta(z - w) \]
\[ = -\frac{2}{(\alpha_i, \alpha_i)}(\alpha_i(z)\delta(z - w) + \partial_w\delta(z - w) \cdot (-1)) \]
and \[ [\alpha_i(z), x^+_i(w)] = \pm (\alpha_i, \alpha_i)x^+_i(w)\delta(z - w). \]

For the \( n + m \)-th vertex, one has
\[ [x^+_{n+m}(z), x^-_{n+m}(w)] = -((\varepsilon_n + m + n)(z))\delta(z - w) + \partial_w\delta(z - w) \]
\[ = -\frac{2}{(\alpha_{n+m}, \alpha_{n+m})}(\alpha_{n+m}(z)\delta(z - w) + \partial_w\delta(z - w) \cdot (-1)) \]
and \[ [\alpha_{n+m}(z), x^+_{n+m}(w)] = \pm (\alpha_{n+m}, \alpha_{n+m})x^+_{n+m}(w)\delta(z - w). \]

For all \( i \neq j \), we have \([x^+_i(z), x^-_j(w)] = 0\) and for any unconnected vertices
\[ [\alpha_i(z), x^+_j(w)] = 0 = \pm (\alpha_i, \alpha_j)x^+_j(w)\delta(z - w) \]

All the rest can be checked by straightforward calculation, for examples
\[ [\alpha_0(z), x^+_1(w)] = -2\sqrt{-1} : \varepsilon_1(z)\varepsilon^*_2(z) : \delta(z - w) \]
\[ = (\alpha_0, \alpha_1)x^+_1(w)\delta(z - w), \]
where have use the property \( \beta_1(z) = -c(z) + \varepsilon_1(z) \) and \( : c(z)\varepsilon^*_2(z) := 0 \)
\[ [\alpha_{n-1}(z), x^+_n(w)] = : X(\varepsilon_{n-1}, z)\varepsilon^*_n(z) : \delta(z - w) \]
\[ = (\alpha_{n-1}, \alpha_n)x^+_n(w)\delta(z - w) \]

By proposition 3.2, we have
\[ [\alpha_{n+m-2}(z), x^+_{m+n-1}(w)] = (\alpha_{m+n-2}, \alpha_{m+n-1})x^+_{m+n-1}(w)\delta(z - w) \]
\[ [\alpha_{n+m-2}(z), x^-_{m+n}(w)] = (\alpha_{m+n-2}, \alpha_{m+n})x^-_{m+n}(w)\delta(z - w). \]

and others can be proved similarly.

Secondly, we can check 2) case by case by using Proposition 3.2 and we include the following examples
\[ [\alpha_0(z), \alpha_0(w)] = -4\partial_w\delta(z - w) = (\alpha_0, \alpha_0)\partial_w\delta(z - w) \cdot (-1) \]
\[ [\alpha_0(z), \alpha_1(w)] = 2\partial_w\delta(z - w) = (\alpha_0, \alpha_1)\partial_w\delta(z - w) \cdot (-1) \]
Finally, we proceed to check the Serre relations. It is easy to verify that 
\([x_i^+(z), x_i^+(w)] = 0\) for \(0 \leq i \leq m + n\) and \([x_i^+(z), x_j^+(w)] = 0\) for \(i \neq j, a_{ij} = 0\). The rest can be checked directly:

\[
[x_0^+(z_1), [x_0^+(z_2), x_1^+(w)]] = \frac{-1}{4} [:: \beta^*(z_1) \varepsilon_1^*(z_1) ; ; \beta^*(z_2) \varepsilon_1^*(z_2) ; ; \varepsilon_1(w) \varepsilon_2^*(w) ; ]
\]

\[
= \frac{-1}{4} [:: \beta^*(z_1) \varepsilon_1^*(z_1) ; ; \varepsilon_1(w) \varepsilon_2^*(w) ; ]\delta(z_2 - w)
\]

\[
= 0
\]

\[
[x_1^+(z_1), [x_1^+(z_2), [[x_1^+(z_3), x_0^+(w)]]]] = \frac{-1}{2} [:: \varepsilon_1(z_1) \varepsilon_2^*(z_1) ; ; \varepsilon_1(z_2) \varepsilon_2^*(z_2) ; ; \varepsilon_1(z_3) \varepsilon_2^*(z_3) ; ; \beta^*(w) \varepsilon_1^*(w) ; ]]
\]

\[
= \frac{-1}{2} [:: \varepsilon_1(z_1) \varepsilon_2^*(z_1) ; ; \varepsilon_2(z_2) \varepsilon_2^*(z_2) ; ; \varepsilon_2(z_3) \varepsilon_1^*(z_3) ; ]\delta(z_3 - w)
\]

\[
= \frac{-1}{2} [:: \varepsilon_2(z_2) \varepsilon_2^*(z_2) ; ]\delta(z_2 - z_3)\delta(z_3 - w)
\]

\[
= 0
\]

\[
[x_{m+1}^+(z_1), [x_{m+1}^+(z_2), x_m^+(w)]] = [:: X(\varepsilon_{m+1}, z_1) \delta_1^*(z_1) ; ; X(\varepsilon_{m+1}, z_2) \delta_1^*(z_2) ; ; X(\varepsilon_m - \varepsilon_{m+1}, w)]
\]

\[
= [:: X(\varepsilon_{m+1}, z_1) \delta_1^*(z_1) ; ; X(\varepsilon_m, w) \delta_1^*(w) ; ]\delta(z_2 - w)
\]

\[
= 0
\]

\[
[x_{m+1}^+(z_1), [x_{m+1}^+(z_2), x_{m+2}^+(w)]] = \sqrt{-1} [:: X(\varepsilon_{m+1}, z_1) \delta_1^*(z_1) ; ; X(\varepsilon_{m+1}, z_2) \delta_1^*(z_2) ; ; \delta_1(w) \delta_2^*(w) ; ]]
\]

\[
= -\sqrt{-1} [:: X(\varepsilon_{m+1}, z_1) \delta_1^*(z_1) ; ; X(\varepsilon_{m+1}, w) \delta_2^*(w) ; ]\delta(z_2 - w)
\]

\[
= 0
\]

The remaining relations follow similarly by Wick’s theorem. This completes the proof of the theorem.

\[\square\]

**Acknowledgments**

The research is supported by the National Natural Science Foundation of China (Nos. 11271138, 11531004, 11301393), Zhejiang Natural Science Foundation (grant No. LY16A010016), Project from Zhejiang province (grant No. FX2014099) and Simons Foundation (grant no. 198129).
BOSONIC REPRESENTATIONS OF THE TOROIDAL SUPERALGEBRAS OF TYPE $D$3


dReferences

[1] S. Berman, Y. Billig, *Irreducible representations for toroidal Lie algebras*, J. Algebra 221 (1999), 188–231.
[2] S. Buek, B. L. Cox, E. Jurisich, *A Wakimoto type realization of toroidal $\mathfrak{sl}_{n+1}$*, Algebra Colloq. 19 (2012), Special Issue no. 1, 841–866.
[3] A. J. Feingold, I. B. Frenkel, *Classical affine algebras*, Adv. Math. 56 (1985), 117–172.
[4] L. Frappat, *Vertex operator representation of $OSp(M|N)^{(1)}$*, Int. J. Mod. Phys. A 3 (1988), 2545–2566.
[5] I.B. Frenkel, V. G. Kac, *Basic representation of affine Lie algebra and dual resonance models*, Invent. Math. 62 (1980), 23–66.
[6] I. B. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the monster*, Academic Press, Boston, 1988.
[7] C. Jiang, D. Meng, *Vertex representations for the $\nu + 1$-toroidal Lie algebra of type $B_l$*, J. Algebra 246 (2001), 564–593.
[8] N. Jing, K. C. Misra, S. Tan, *Bosonic realizations of higher-level toroidal Lie algebras*, Pacific J. Math. 219 (2005), 285–301.
[9] N. Jing, K. C. Misra, C. Xu, *Bosonic realization of toroidal Lie algebras of classical types*, Proc. Amer. Math. Soc. 137 (2009), 3609–3618.
[10] N. Jing, C. Xu, *Toroidal Lie superalgebras and free field representations*, Contemp. Math. 623 (2014), 135–153.
[11] N. Jing, C. Xu, *Vertex representation of toroidal special linear superalgebras*, Chin. Ann. Math. Ser. B 36 (2015), 427–436.
[12] V. G. Kac, *Vertex algebras for beginners*, Univ. Lect. Ser. 10, Amer. Math. Soc., Providence, 1997.
[13] V. G. Kac, *Infinite-dimensional Lie Algebras*, 3rd ed., Cambridge Univ. Press, Cambridge, 1990.
[14] V. G. Kac, M. Wakimoto, *Integrable highest weight modules over affine superalgebras and Appell’s function*, Comm. Math. Phys. 215 (2001), 631–682.
[15] M. Lau, *Representations of multiloop algebras*, Pacific J. Math. 245 (2010), 167–184.
[16] R. E. Moody, S. E. Rao, T. Yokomama, *Toroidal Lie algebras and vertex representations*, Geom. Dedicata 35 (1990), 283–307.
[17] S. Eswara Rao, *Representation of toroidal general linear superalgebras*, Comm. Algebra 42 (2013), 2476–2507.
[18] S. Tan, *Vertex operator representations for toroidal Lie algebra of type $B_l$*, Comm. Algebra 27 (1999), 3593–3618.
[19] X. Xu, *Introduction to vertex operator superalgebras and their modules*, Kluwer Academic Publishers, Dordrecht, 1998.
