Numerical model of elastic laminated glass beams under finite strain

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Abstract

Laminated glass structures are formed by stiff layers of glass connected with a compliant plastic interlayer. Due to their slenderness and heterogeneity, they exhibit a complex mechanical response that is difficult to capture by single-layer models even in the elastic range. The purpose of this paper is to introduce an efficient and reliable finite element approach to the simulation of the immediate response of laminated glass beams. It proceeds from a refined plate theory due to Mau (1973), as we treat each layer independently and enforce the compatibility by the Lagrange multipliers. At the layer level, we adopt the finite-strain shear deformable formulation of Reissner (1972) and the numerical framework by Ibrahimbegović and Frey (1993). The resulting system is solved by the Newton method with consistent linearization. By comparing the model predictions against available experimental data, analytical methods and two-dimensional finite element simulations, we demonstrate that the proposed formulation is reliable and provides accuracy comparable to the detailed two-dimensional finite element analyzes. As such, it offers a convenient basis to incorporate more refined constitutive description of the interlayer.

Keywords laminated glass; finite-strain Reissner beam theory; finite element method; Lagrange multipliers

1 Introduction

Due to present trends in architecture and photovoltaics, the use of structural glass is expanding from traditional window panes to large-area surfaces, roof and floor systems, columns or staircases. This interest leads to an increased emphasis on the safety and the mechanical performance of structural members made of glass. Perhaps the most popular material system meeting these criteria is laminated glass. It is a composite structure produced by bonding multiple layers of glass together with a transparent plastic interlayer, typically made of polyvinyl butyral (PVB) foil [10, 27]. The interlayer absorbs the energy impact, thereby resisting the glass penetration, and keeps the layers of glass bonded when fractured.

Laminated glass units exhibit a complex mechanical response, as a consequence of their heterogeneity. Namely, the contrast in elastic properties between glass and the interlayer typically exceeds three orders of magnitude, which renders classical laminate theories inapplicable since the glass layers deform mainly due to bending, whereas the interlayer experiences a pure shear. Moreover, PVB is a viscoelastic material exhibiting a high sensitivity to temperature changes,
Finally, glass structures are very slender and must be analyzed using geometrically non-linear theories.

In practical applications, the behavior of laminated glass units is often approximated by an equivalent, geometrically linear, single-layer elastic system. According to the performance of the interlayer, we distinguish the layered case, in which the structure responds as an assembly of independent layers, and the monolithic model with thickness equal to the combined thickness of glass layers and interlayer. This approach was pioneered by experimental studies of Behr et al. [3], who demonstrated that the degree of coupling due to the interlayer is significant at room temperatures and ceases at elevated temperatures. This was extended later in [4] to quantify the validity of the monolithic approximation in terms of temperature range and load duration. Norville et al. [23] further refined these results by studying the shear coupling around the transition temperature of the interlayer and demonstrated that the performance of laminated units exceeds the layered limit even above the transition point. Vallabhan et al. [28] introduced a notion of the strength factor as a ratio between the maximal principal strength in the equivalent monolithic unit and the laminated system and demonstrated that it can reach values smaller than one for cases of practical interest. They attributed this to the effects of geometrical non-linearity which become significant earlier for the laminated units than for the monolithic ones. These developments have been recently put on a rigorous basis by Galuppi and Royer-Carfagni [8], who derived explicit variationally-based formulas for deflection- and stress-equivalent thicknesses of monolithic systems.

Apart from experimental results, the validity of simplified models was assessed by several analytical studies. When restricting our attention to planar glass beams, the first study is due to Hooper [12], who derived a three-layer model under small deflections and presented the solution for the four point bending setup with different duration of loading and ambient temperatures. Later, Aşık and Tescan [2] extended this model to account for large deflections and demonstrated that they significantly contribute to the overall response when the normal forces start to develop, see also Section 5 for a concrete example. In the linear setting, Ivanov [17] proposed a procedure for thickness optimization of triplex glasses and Foraboschi [7] demonstrated that the empirical rules proposed by Behr et al. [4] may lead to unsafe designs. Recently, Schultze et al. [26] performed an experimental-analytical study into the response of the laminated structures used in photovoltaic applications under three-point bending.

Even though the analytical approaches give valuable insight into the behavior of laminated glass structures, they still suffer from two major limitations. First, the only closed-form solutions we are aware of hold only in the absence of membrane effects. In the opposite case, the governing differential equations have to be discretized anyhow and the discrete problem is often solved using the problem-specific procedures that experience convergence problems, e.g. [1]. Second, the analytical approaches require the interlayer to be replaced by an equivalent elastic material, with properties adjusted to the loading duration and ambient temperature. As shown by Galuppi and Royer-Carfagni [8], this approach leads to significant errors in local stresses and strains.

With these limitations in mind, we proposed in [30] a numerical approach to the analysis of laminated glass beams based on a refined laminate theory due to Mau [22]. In this framework, the structure is seen as an assembly of shear-deformable layers with independent kinematics, and the inter-layer interaction is enforced by the Lagrange multipliers. Such an approach may appear costly, as it increases the number of unknowns and tends to produce badly conditioned systems of equations, but it offers several advantages due to the modular format. Namely, each layer can be discretized by an appropriate type of finite elements, and different constitutive models can be used at the layer level (even in the multi-scale setting [29]). From the computational perspective, the resulting system is ideally suited for the deployment of efficient duality-based iterative solvers [20] that can also be extended to account for inter-layer delamination, e.g. [19, 23].

In this paper, we extend our previous results [30], valid in small strains, to the finite-strain
regime. For this purpose, each layer is modeled by the Reissner beam theory \cite{24}, briefly summarized in Section \ref{sec:2} in a variational format. Discretization in Section \ref{sec:3} is based on a robust finite element formulation proposed by Ibrahimbegović and Frey in \cite{14} and later applied, e.g., to discrete materials modeling \cite{13}, or optimal control and design of structures \cite{15}. In Section \ref{sec:4} we derive the discretized system arising from the optimality conditions of the associated constrained optimization problem, and perform its solution by the Newton method. Accuracy of the implementation is examined in Section \ref{sec:5} by verifying our results against data presented by Aşık and Tescan \cite{2}. To make the paper self-contained, in Appendix A we collect the details on tangent operators needed when implementing the Newton method. Note that this paper is focused on the immediate elastic response of laminated structures. The effect of temperature-dependent viscous properties of the interlayer will be discussed independently and in more details in a forthcoming publication.

The following nomenclature is used in the text. Scalar quantities are denoted by lightface letters, e.g. $a$, and the bold letters are reserved for matrices, e.g. $A$. $A^T$ standardly stands for the matrix transpose and $A^{-1}$ for the matrix inverse. The subscript in parentheses, e.g. $a^{(i)}$, is used to emphasize that the variable $a$ is associated with the $i$-th layer.

\section{Model formulation}

In our setting, a glass beam is considered as an assembly of three beams of identical length $L$, with the cross-section dimensions $b \times h^{(i)}$. Each layer is considered to behave according to the Reissner finite-strain beam theory \cite{24}, i.e. we assume that cross-sections remain planar, but not necessarily perpendicular to the deformed beam curve, and that the distance of a point at the cross-section from the centerline remains unchanged. For the $i$-th layer, the coordinates of a point in the deformed configuration can be determined as, Figure \ref{fig:1}

\begin{align}
  x^{(i)}(X^{(i)},Z^{(i)}) &= O_X^{(i)} + X^{(i)} + u_0^{(i)}(X^{(i)}) + \sin(\varphi^{(i)}(X^{(i)}))Z^{(i)}, \quad (1a) \\
  z^{(i)}(X^{(i)},Z^{(i)}) &= O_Z^{(i)} + w_0^{(i)}(X^{(i)}) + \cos(\varphi^{(i)}(X^{(i)}))Z^{(i)}, \quad (1b)
\end{align}

where $O_X^{(i)}$ and $O_Z^{(i)}$ stand for the coordinates of the beam origin, $X^{(i)}$ is the ordinate of a cross-section, $u_0^{(i)}$ and $w_0^{(i)}$ are centerline displacements measured in the global coordinate system, $\varphi^{(i)}$ is the rotation of the cross-section, and $Z^{(i)}$ is the coordinate measured along the cross-section. The inter-layer interaction is ensured via the geometric continuity conditions at the interfaces between the layers ($i = 1, 2$)

\begin{align}
  x^{(i)}(X^{(i)},\frac{1}{2}h^{(i)}) - x^{(i+1)}(X^{(i)},\frac{1}{2}h^{(i+1)}) &= 0, \quad (2a) \\
  z^{(i)}(X^{(i)},\frac{1}{2}h^{(i)}) - z^{(i+1)}(X^{(i)},\frac{1}{2}h^{(i+1)}) &= 0. \quad (2b)
\end{align}

As demonstrated, e.g. by Ibrahimbegović and Frey \cite{14} and Iscrich and Gerstmayr \cite{16}, the Reissner beam theory can be consistently derived from the continuum framework by the Biot-type strain tensors for the kinematics parametrization \cite{11}. To this purpose, we use the deformation gradient in the form

\begin{equation}
  F^{(i)}(X^{(i)},Z^{(i)}) = \begin{bmatrix}
    \frac{\partial x^{(i)}}{\partial X^{(i)}} & \frac{\partial x^{(i)}}{\partial Z^{(i)}} \\
    \frac{\partial z^{(i)}}{\partial X^{(i)}} & \frac{\partial z^{(i)}}{\partial Z^{(i)}}
  \end{bmatrix}(X^{(i)},Z^{(i)}),
\end{equation}
with individual entries provided by

\[ F_{11}^{(i)} = 1 + \frac{d w_0^{(i)}(x^{(i)})}{dX^{(i)}} + \cos(\varphi^{(i)}(x^{(i)})) \frac{d\varphi^{(i)}(x^{(i)})}{dX^{(i)}} Z^{(i)}, \quad (4a) \]
\[ F_{12}^{(i)} = \sin(\varphi^{(i)}(x^{(i)})), \quad (4b) \]
\[ F_{21}^{(i)} = \frac{d w_0^{(i)}(x^{(i)})}{dX^{(i)}} - \sin(\varphi^{(i)}(x^{(i)})) \frac{d\varphi^{(i)}(x^{(i)})}{dX^{(i)}} Z^{(i)}, \quad (4c) \]
\[ F_{22}^{(i)} = \cos(\varphi^{(i)}(x^{(i)})). \quad (4d) \]

The deformation gradient \( F^{(i)} \) admits a multiplicative decomposition

\[ F^{(i)}(X^{(i)}, Z^{(i)}) = R^{(i)}(X^{(i)})U^{(i)}(X^{(i)}, Z^{(i)}), \quad (5) \]

where the rotation matrix of the \( i \)-th layer is given by, recall Figure 1

\[ R^{(i)} = \begin{bmatrix}
\cos(\varphi^{(i)}(x^{(i)})) & \sin(\varphi^{(i)}(x^{(i)})) \\
-\sin(\varphi^{(i)}(x^{(i)})) & \cos(\varphi^{(i)}(x^{(i)}))
\end{bmatrix}, \quad (6) \]

and \( U^{(i)} \) follows from the inverse relation

\[ U^{(i)}(X^{(i)}, Z^{(i)}) = [R^{(i)}(X^{(i)})]^{-1} F^{(i)}(X^{(i)}, Z^{(i)}), \quad (7) \]

that provides

\[ U_{11}^{(i)} = \cos(\varphi^{(i)}(x^{(i)})) \left( 1 + \frac{d w_0^{(i)}(x^{(i)})}{dX^{(i)}} \right) - \sin(\varphi^{(i)}(x^{(i)})) \frac{d w_0^{(i)}(x^{(i)})}{dX^{(i)}} + \frac{d \varphi^{(i)}(x^{(i)})}{dX^{(i)}} Z^{(i)}, \quad (8a) \]
\[ U_{12}^{(i)} = 0, \quad (8b) \]
\[ U_{21}^{(i)} = \sin(\varphi^{(i)}(x^{(i)})) \left( 1 + \frac{d w_0^{(i)}(x^{(i)})}{dX^{(i)}} \right) + \cos(\varphi^{(i)}(x^{(i)})) \frac{d w_0^{(i)}(x^{(i)})}{dX^{(i)}}, \quad (8c) \]
\[ U_{22}^{(i)} = 1. \quad (8d) \]

Employing the definition of the Biot-type strain tensor, e.g. [18, Eq. (24.27)],

\[ H^{(i)}(X^{(i)}, Z^{(i)}) = U^{(i)}(X^{(i)}, Z^{(i)}) - I, \quad (9) \]
the non-zero strain components are given by
\[ H_{11}^{(i)}(X^{(i)}, Z^{(i)}) = E^{(i)}(X^{(i)}) + K^{(i)}(X^{(i)})Z^{(i)}, \quad H_{21}^{(i)}(X^{(i)}) = \Gamma^{(i)}(X^{(i)}), \]  
(10)

where \( E^{(i)}, K^{(i)} \) and \( \Gamma^{(i)} \) denote the generalized normal strain, pseudo-curvature and shear strain introduced by Reissner [23]:

\[
E^{(i)} = E(u_0^{(i)}, w_0^{(i)}, \varphi^{(i)}) = \cos(\varphi^{(i)}(X^{(i)}))(1 + \frac{d\varphi_0^{(i)}(X^{(i)})}{dX^{(i)}}) - \sin(\varphi^{(i)}(X^{(i)})) \frac{dW_0^{(i)}(X^{(i)})}{dX^{(i)}} - 1, \quad (11a)
\]

\[
\Gamma^{(i)} = \Gamma(u_0^{(i)}, w_0^{(i)}, \varphi^{(i)}) = \sin(\varphi^{(i)}(X^{(i)}))(1 + \frac{d\varphi_0^{(i)}(X^{(i)})}{dX^{(i)}}) + \cos(\varphi^{(i)}(X^{(i)})) \frac{dW_0^{(i)}(X^{(i)})}{dX^{(i)}}, \quad (11b)
\]

\[
K^{(i)} = K(u_0^{(i)}, w_0^{(i)}, \varphi^{(i)}) = \frac{d\varphi^{(i)}(X^{(i)})}{dX^{(i)}}. \quad (11c)
\]

It is useful to express these relations in a compact form

\[
\mathbf{u}^{(i)}(X^{(i)}) = \begin{bmatrix} u_0^{(i)} \\ w_0^{(i)} \\ \varphi^{(i)} \end{bmatrix}, \quad \mathbf{E}^{(i)}(X^{(i)}) = \begin{bmatrix} E^{(i)} \\ \Gamma^{(i)} \\ K^{(i)} \end{bmatrix}(X^{(i)}), \quad (12)
\]

and denote by \( \mathbf{E}(\mathbf{u}) \) a mapping assigning the generalized strain measures to the generalized centerline displacements \( \mathbf{u} \) according to Eq. (11).

The model is completed by specifying the energy functionals associated with the deformation of the laminated beam. In particular, the internal energy of the \( i \)-th layer is provided by [14]

\[
\Pi_{\text{int}}^{(i)}(\mathbf{E}^{(i)}) = \frac{1}{2} \int_{0}^{L} E^{(i)} A^{(i)}(E^{(i)}(X^{(i)}))^2 + G^{(i)} A_s^{(i)}(\Gamma^{(i)}(X^{(i)}))^2 + E^{(i)} I^{(i)} (K^{(i)}(X^{(i)}))^2 \, dX^{(i)}, \quad (13)
\]

where \( E^{(i)} \) and \( G^{(i)} \) denote the Young and shear moduli, and \( A^{(i)} = bh^{(i)}, A_s^{(i)} = \frac{5}{6}A^{(i)} \) and \( I^{(i)} = \frac{1}{12}bh^{(i)^2} \) stand for the the cross-section area, effective shear area, and the second moment of area, respectively. The external energy due to loading acting at the \( i \)-th layer assumes the form

\[
\Pi_{\text{ext}}^{(i)}(\mathbf{u}^{(i)}) = - \int_{0}^{L} u_0^{(i)}(X^{(i)}) \bar{F}^{(i)}(X^{(i)}) \, dX^{(i)} - \sum_{p=1}^{n_{p}^{(i)}} w_0^{(i)}(X^{(i)}) \bar{F}^{(i)}(X^{(i)})^{(i)}, \quad (14)
\]

where, for simplicity, we assume that the structure is subjected to the distributed loading with intensity \( \bar{F}^{(i)} \) and to \( n_{p}^{(i)} \) concentrated forces \( \bar{F}_p^{(i)} \) acting at \( X_p^{(i)} \). The total energy of the \( i \)-th layer is then given by

\[
\Pi^{(i)}(\mathbf{u}^{(i)}) = \Pi_{\text{int}}^{(i)}(\mathbf{E}(\mathbf{u}^{(i)})) + \Pi_{\text{ext}}^{(i)}(\mathbf{u}^{(i)}), \quad (15)
\]

and at the level of the whole structure it reads

\[
\Pi(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}) = \sum_{i=1}^{3} \Pi^{(i)}(\mathbf{u}^{(i)}). \quad (16)
\]
3 Finite element discretization

It is convenient for the numerical treatment to discretize all layers identically with $n^e$ two-node elements per each layer; by $\Omega^{(i)}_e$ we denote the $i$-th element of the $e$-th layer. The displacement fields at the element level are approximated as

$$u_0^{(i)}(x) \approx N_e(x)d_c^{(i)}, \quad w_0^{(i)}(x) \approx N_e(x)\varphi_e^{(i)}, \quad \varphi^{(i)}(x) \approx N_e(x)d_e^{(i)} \quad \text{for } x \in \Omega^{(i)}_e,$$

where $N_e$ is the matrix of piecewise linear basis functions, and e.g. $d_e^{(i)} = [\varphi^{(i)}_{e,1}, \varphi^{(i)}_{e,2}]^T$ collects the nodal rotations $\varphi^{(i)}$, cf. Figure 2.

![Figure 2: Scheme of discretization and element degrees of freedom.](image)

Inserting the approximate fields (17) into the energy functionals (15) results in

$$\Pi^{(i)}_{\text{int}}(u^{(i)}) \approx \sum_{e=1}^{n^e} \Pi^{(i)}_{\text{int},e}(d_e^{(i)}), \quad \Pi^{(i)}_{\text{ext}}(u^{(i)}) \approx \sum_{e=1}^{n^e} \Pi^{(i)}_{\text{ext},e}(d_e^{(i)}) = -\sum_{e=1}^{n^e} d_e^{(i)T}f_{\text{ext},e}^{(i)}$$

where the matrix $d_e^{(i)}$ collects the nodal unknowns,

$$d_e^{(i)} = \begin{bmatrix} u_{0,e,1}^{(i)} & w_{0,e,1}^{(i)} & \varphi_{e,1}^{(i)} & u_{0,e,2}^{(i)} & w_{0,e,2}^{(i)} & \varphi_{e,2}^{(i)} \end{bmatrix}^T,$$

recall Figure 2 and $f_{\text{ext},e}^{(i)}$ stores the corresponding generalized nodal forces. As demonstrated in detail by Ibrahimbegović and Frey [14], the normal and shear locking even for the non-linear kinematics can be suppressed by adopting the selective one-point integration of the corresponding terms in (13). Thus, the approximate generalized strain measures (11) are taken as element-wise constants:

$$E_e^{(i)}(d_e^{(i)}) = \frac{1}{L_e^{(i)}} \left( L_e^{(i)} + \Delta u_{0,e}^{(i)} \right) \cos \beta_e^{(i)} - \frac{1}{L_e^{(i)}} \Delta w_{0,e}^{(i)} \sin \beta_e^{(i)} - 1,$$

$$\Gamma_e^{(i)}(d_e^{(i)}) = \frac{1}{L_e^{(i)}} \left( L_e^{(i)} + \Delta u_{0,e}^{(i)} \right) \sin \beta_e^{(i)} + \frac{1}{L_e^{(i)}} \Delta w_{0,e}^{(i)} \cos \beta_e^{(i)},$$

$$K_e^{(i)}(d_e^{(i)}) = \frac{\Delta \varphi_e^{(i)}}{L_e^{(i)}},$$

with, e.g., $\Delta \varphi_e^{(i)} = \varphi_{e,2}^{(i)} - \varphi_{e,1}^{(i)}$, $\beta_e^{(i)} = \frac{1}{2}(\varphi_{e,1}^{(i)} + \varphi_{e,2}^{(i)})$, and $L_e^{(i)}$ denoting the element length. The contribution of the $e$-th element to the internal energy simplifies to

$$\Pi^{(i)}_{\text{int},e}(d_e^{(i)}) = \frac{1}{2} \left( E_e^{(i)} A_e^{(i)} E_e^{(i)}(d_e^{(i)})^2 + \Gamma_e^{(i)} A_s^{(i)} \Gamma_e^{(i)}(d_e^{(i)})^2 + E_e^{(i)} F_e^{(i)} K_e^{(i)}(d_e^{(i)})^2 \right) L_e^{(i)}.$$

The discretization is completed by enforcing the inter-layer compatibility conditions (2a) at the element nodes,

$$c_{X,j}^{(i,i+1)} = 0, \quad c_{Z,j}^{(i,i+1)} = 0,$$

indexed at the level of a single layer by $j = 1, 2, \ldots, n^e + 1$, cf. Figure 1 and with

$$c_{X,j}^{(i,i+1)} = u_0^{(i)} - u_0^{(i+1)} + \frac{1}{2}(h^{(i)} \sin \varphi_j^{(i)} + h^{(i+1)} \sin \varphi_j^{(i+1)}),$$

$$c_{Z,j}^{(i,i+1)} = -\frac{1}{2}(h^{(i)} + h^{(i+1)}) + w_0^{(i)} - w_0^{(i+1)} + \frac{1}{2}(h^{(i)} \cos \varphi_j^{(i)} + h^{(i+1)} \cos \varphi_j^{(i+1)}).$$
For later reference, these are introduced in a compact form as
\[ c(d) = 0, \] (24)
where \( d = [d^{(1)}, d^{(2)}, d^{(3)}] \) is a \( 9(n^e + 1) \times 1 \) column matrix of nodal degrees of freedom and \( c \) collects the \( 4(n^e + 1) \) compatibility conditions \(^{22}\).

4 Governing equations

The true nodal displacements \( d^* \) follow from the minimization of the discretized energy functional
\[
\Pi(d) = \sum_{i=1}^{3} \sum_{e=1}^{n^e} \Pi_{\text{int},e}^{(i)}(d_e^{(i)}) + \Pi_{\text{ext},e}^{(i)}(d_e^{(i)}),
\] (25)
subject to both kinematic constraints and compatibility conditions \(^{22}\). While the kinematic constrains are dealt with by matrix reduction techniques, e.g. \(^{18, \text{Appendix A}}\), the compatibility is enforced via the Lagrange multipliers \(^{22, 29}\). This procedure yields the Lagrangian function in the form
\[
\mathcal{L}(d, \lambda) = \Pi(d) + \lambda^T c(d) = \Pi(d) + \sum_{m=1}^{4(n^e + 1)} \lambda_m c_m(d) \] (26)
where \( \lambda \) is a \( 4(n^e + 1) \times 1 \) matrix storing the corresponding Lagrange multipliers.

The corresponding Karush-Kuhn-Tucker optimality conditions read, e.g. \(^{6, \text{Chapter 14}}\),
\[
\nabla_d \mathcal{L}(d^*, \lambda^*) = \nabla \Pi(d^*) + \nabla c(d^*)^T \lambda^* = 0, \quad (27a)
\n\nabla \lambda \mathcal{L}(d^*, \lambda^*) = c(d^*) = 0, \quad (27b)
\]
with \( \nabla \) denoting the gradient operator and \( \nabla_\bullet \) designating the partial derivative with respect to variable \( \bullet \). These relations represent a system of non-linear equations, to be solved using the Newton iterative scheme.

To that end, we assume that displacements at iteration \( (k_d, k_\lambda) \) are known and search for the iterative correction in the form
\[
k^{+1}_d = k_d + k^{+1} \delta d. \quad (28)
\]
The values of \( k^{+1} \delta d \) are obtained by means of the linearized expressions
\[
\nabla \Pi^{(k+1)}(d) \approx \nabla \Pi^{(k)}(d) + \nabla^2 \Pi^{(k)}(d)k^{+1} \delta d, \quad (29a)
\c^{(k+1)}(d) \approx c^{(k)}(d) + \nabla c^{(k)}(d)k^{+1} \delta d, \quad (29b)
\nabla c^{(k+1)}(d) \approx \nabla c^{(k)}(d) + \nabla^2 c^{(k)}(d)k^{+1} \delta d, \quad (29c)
\]
which, when introduced into the optimality conditions \(^{27a} \) and \(^{27b} \), result in a linear system of equations, cf. \(^{6} \) and \(^{30} \),
\[
\begin{bmatrix} kK & kC^T \\ kC & 0 \end{bmatrix} \begin{bmatrix} k^{+1} \delta d \\ k^{+1} \lambda \end{bmatrix} = - \begin{bmatrix} k f_{\text{int}} - f_{\text{ext}} \\ k c \end{bmatrix}. \quad (30)
\]
Here, we employ the short-hand notation
\[
kK = \nabla^2 \Pi^{(k)}(d) + \sum_{m=1}^{4(n^e + 1)} k \lambda_m \nabla^2 c_m^{(k)}(d) = K_k(d) + K_\lambda^{(k)}(d, k_\lambda), \quad (31a)
\]
\[
kC = \nabla c^{(k)}(d), \quad (31b)
\]
\[
k f_{\text{int}} - f_{\text{ext}} = \nabla \Pi^{(k)}(d), \quad (31c)
\]
\[
k c = c^{(k)}(d). \quad (31d)
\]
It follows from the specific form of the energy function \[25\] that the stiffness matrix \( K_t \) and the matrices of internal \( f_{int} \) and external forces \( f_{ext} \) exhibit the block structure

\[
K_t(d) = \begin{bmatrix}
K_t^{(1)}(d^{(1)}) & & \\
& K_t^{(2)}(d^{(2)}) & \\
& & K_t^{(3)}(d^{(3)})
\end{bmatrix},
\quad f_{int} = \begin{bmatrix}
f_{int}^{(1)}(1) \\
f_{int}^{(2)}(2) \\
f_{int}^{(3)}(3)
\end{bmatrix},
\quad f_{ext} = \begin{bmatrix}
f_{ext}^{(1)}(3) \\
f_{ext}^{(2)}(2) \\
f_{ext}^{(3)}(1)
\end{bmatrix}.
\]

(32)

For the \( i \)-th layer, these are obtained by the assembly of contributions of the \( e \)-th element \[18\], provided by

\[
f_{int,e}^{(i)} = \frac{\partial \Pi_{int,e}^{(i)}}{\partial d_{int,e}^{(i)}},
\quad K_{ke}^{(i)} = \frac{\partial^2 \Pi_{int,e}^{(i)}}{\partial d_{int,e}^{(i)}},
\quad \frac{\partial f_{int,e}^{(i)}}{\partial d_{int,e}^{(i)}}.
\]

(33)

In order to keep the paper self-contained, explicit expressions for the matrices needed to set up the linearized system \[30\] are summarized in Appendix \[A\]. Termination of the iterative process \[30\] is driven by two residuals \[6, \text{Section 14.1}\]

\[
k_1 \eta_1 = \frac{\|k f_{int} - f_{ext} + k e C T_k \lambda\|_2}{\|f_{ext}\|_2},
\quad k_2 \eta_2 = \frac{\|k c\|_2}{\min_i h^{(i)}},
\]

(34)

quantifying the validity of the first-order optimality conditions \[27\] that are related to equilibrium conditions for all three layers and the displacement compatibility at inter-layer interfaces, respectively. This provides the last component of the non-linear iterative solver, the implementation of which is outlined in Algorithm \[1\]

\begin{algorithm}
\caption{Conceptual implementation of the Newton method.}
\begin{algorithmic}
\Data: initial displacement \( 0d \), tolerances \( \epsilon_1 \) and \( \epsilon_2 \)
\Result: \( d^*, \lambda^* \)
\begin{algorithmic}
\State \( k \leftarrow 0, 0 \lambda \leftarrow 0 \), assemble \( k f_{int}, k c \) and \( k C \)
\While {\( k \eta_1 > \epsilon_1 \) or \( k \eta_2 > \epsilon_2 \)}
\State assemble \( k K \)
\State solve for \( k+1 \delta d, k+1 \lambda \) from Eq. \[30\]
\State \( k+1 d \leftarrow k d + k+1 \delta d \)
\State assemble \( k f_{int}, k c \) and \( k C \)
\State \( k \leftarrow k + 1 \)
\EndWhile
\State \( d^* \leftarrow k d, \lambda^* \leftarrow k \lambda \)
\end{algorithmic}
\end{algorithmic}
\end{algorithm}

5 Examples

In this section, the proposed finite element formulation is verified and partially validated against two benchmarks after Aşık and Tescan \[2\], involving simply supported, Section \[5.1\] and fixed-end, Section \[5.2\] three-layer beams subjected to three-point bending. The comparison is based on the centerline deflections and the extreme normal (at top and bottom surfaces) and shear stresses at the \( i \)-th layer. These quantities are obtained from the element-wise constant strain measures in the form

\[
S_{e,\text{top}}^{(i)} = E^{(i)} \left( E^{(i)}_c - \frac{1}{2} K^{(i)}_c h^{(i)} \right),
\quad S_{e,\text{bot}}^{(i)} = E^{(i)} \left( E^{(i)}_c + \frac{1}{2} K^{(i)}_c h^{(i)} \right),
\quad T^{(i)} = G^{(i)} \Gamma^{(i)}_c.
\]

(35)

cf. Eq. \[10\] and Eq. \[20\], and extrapolated to nodes by the least-square method assuming the piecewise linear distribution of stresses, e.g. \[11\]. Accuracy of our results against reference experimental data is quantified by

\[
\eta_{exp} = \frac{(\bullet) - (\bullet)_{\text{exp}}}{(\bullet)_{\text{exp}}},
\]

(36)

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and a similar approach is adopted for reference data obtained by analytical model (an) or two-dimensional finite element simulations (num), both from [2].

Results of the finite-strain formulation are complemented with the small-strain version [30] that corresponds to the first iteration of Algorithm 1 and the linearized strain measures given by

\[
E_e^{(i)}(d_e^{(i)}) = \frac{\Delta u_e^{(i)}}{L_e^{(i)}}, \quad \Gamma_e^{(i)}(d_e^{(i)}) = \beta_e^{(i)} + \frac{\Delta u_0_e}{L_e^{(i)}}, \quad K_e^{(i)}(d_e^{(i)}) = \frac{\Delta \varphi_e}{L_e^{(i)}},
\]

recall Eq. (20). The corresponding stresses follow from the same relation as for the non-linear model, (35). For completeness, we also provide the response of the equivalent monolithic beam of total thickness \((h^{(1)} + h^{(2)} + h^{(3)})\) and the layered beam corresponding to two independent layers of thicknesses \(h^{(1)}\) and \(h^{(3)}\), under the assumption of geometric linearity. Note that to avoid confusion with the terminology used for composite structures [22], the layered approximation will be referred to as two-layer and the present formulation is denoted as refined in what follows.

5.1 Simply supported beam

The first example concerns a simply supported beam with a span of 0.8 m and the total length of 1 m, with a symmetric layer setup of 5–0.38–5 mm in thickness. The structure is subject to a concentrated force at the mid-span ranging from 50 to 200 N and the quantities of interest are registered at the bottom layer, cf. Figure 3(a). The material data for individual layers appear in Table 1.

Table 1: Material data for simply supported beam, after [2].

| Property                        | Value   |
|---------------------------------|---------|
| Young’s modulus of glass, \(E^{(1)} = E^{(3)}\) | 64.5 GPa |
| Shear modulus of glass, \(G^{(1)} = G^{(3)}\) | 26.2 GPa |
| Young’s modulus of PVB, \(E^{(2)}\) | 3.61 MPa |
| Shear modulus of PVB, \(G^{(2)}\) | 1.28 MPa |

The centerline deflections as predicted by the considered models appear in Figure 3(b). In this case, the deflection of the refined formulation remains bounded by the two-layer and monolithic cases. The response of the laminated structure is in fact closer to the monolithic beam than to the layered approximation, thereby demonstrating that the interlayer provides sufficient coupling to achieve the composite action. We also observe, in agreement with assumptions of many analytical models discussed in Section 1, the absence of non-linear effects due to the statically-determinate character of the test.

These findings are further supported by Table 2 collecting the numerical values of mid-span deflections and experimental data. The results confirm that predictions of the numerical scheme are in a perfect agreement with the analytical model and correlate well with the experimental data. The limit cases, on the other hand, are too far apart to be of practical use and exhibit errors that far exceed the difference between the refined models and experiments.

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1. All results in the present section are reproducible with MATLAB® scripts available at [http://arxiv.org/abs/1303.6314](http://arxiv.org/abs/1303.6314).

2. Since PVB layer exhibits viscoelastic and temperature-dependent behavior, the shear modulus \(G^{(2)}\) represents an effective secant value related to a given temperature and load duration. For example, for data presented in [7] and the temperature of 20°C, the value in Section 5.1 corresponds to loads with duration of \(\sim 100\) s, whereas the value in Section 5.2 corresponds to loads with duration of \(\sim 1,000\) s.
Table 2: Mid-span deflections \( w^{(3)}_0 \) at \( X^{(3)} = L/2 \) for the simply supported beam subjected to \( F = 50 \) N.

| Model                      | Deflection [mm] | \( \eta_{\text{exp}} \) [%] | \( \eta_{\text{an}} \) [%] |
|----------------------------|-----------------|-------------------------------|----------------------------|
| Experiment\(^2\)           | 1.27 \( \times \) | -5.2                          |                            |
| Analytical model\(^2\)     | 1.34 5.5 \( \times \) | 0.1                           |                            |
| Refined linear             | 1.34 5.6       | 0.1                           |                            |
| Refined non-linear         | 1.34 5.6       | 0.1                           |                            |
| Monolithic linear          | 0.89 -30.1     | -33.8                         |                            |
| Two-layer linear           | 3.97 213       | 196                           |                            |

The response of the structure to an increasing force expressed in terms of the deflections and stresses appears in Tables 3 and 4, respectively. The data clearly demonstrate that the response of the structure remains linear even for larger loading; the differences appear only for \( F = 200 \) N and are indeed negligible. The deflections remain sufficiently accurate with respect to both the analytical model and experiments, while stress values display larger discrepancies. We attribute the \( \sim 1.3\% \) difference between the analytical and numerical models to the fact that Asik and Tescan in \(^2\) assume glass to deform exclusively in bending and the PVB layer in shear only, while the present approach accounts for both effects simultaneously in all layers. This error is still significantly smaller compared to the one measured against the experimental data, which might be attributed to inaccuracies in the experiment as explained in detail in \(^2\).

The negligible difference between the linear and non-linear models is finally confirmed by the identical values of the normal and shear stresses, recall (35), shown in Figure 4. Their distribution, however, differs significantly from any single-layered approximation, as manifested by, e.g. the presence of non-zero shear and normal stresses at free ends of the beams and the non-uniform distribution of shear stresses along the beam axis. We also observe, in agreement with \(^2\), that the distribution of extreme normal stresses is anti-symmetric in the vertical direction, i.e. \( S^{(1)}_{\text{top}} = -S^{(3)}_{\text{bot}} \), and that shear strains in the interlayer (\( \sim 3\% \)) are significantly larger than normal strains in the glass (\( \approx 10^{-4} \)).

### 5.2 Fixed-end beam

Effects of geometric non-linearity are illustrated by means of a thin 1.5 m long three-layer beam of thicknesses 2.12–0.76–2.12 mm subjected to a concentrated force at the mid-span with intensity ranging from 15 to 150 N, Figure 5(a). The tolerances for the Newton method were set to the same value as in the previous example, and so were the material constants for the glass layers. For the PVB interlayer, we used \( E^{(2)} = 2.8 \) MPa and \( G^{(2)} = 1 \) MPa.\(^2\) The reference data from \(^2\) include the results of the analytical model, obtained by a finite-difference iterative solver.\(^1\)
Table 3: Comparison of mid-span deflections $w_0^{(3)}$ at $X^{(3)} = L/2$ for the simply supported beam.

| Load [N] | Reference data | Linear model | Non-linear model |
|----------|----------------|--------------|------------------|
|          | $w_{\text{exp}}$ [mm] | $w_{\text{an}}$ [mm] | $w$ [mm] | $\eta_{\text{exp}}$ [%] | $\eta_{\text{an}}$ [%] | $w$ [mm] | $\eta_{\text{exp}}$ [%] | $\eta_{\text{an}}$ [%] |
| 50       | 1.27            | 1.34         | 1.34 | 5.6 | 0.1 | 1.34 | 5.6 | 0.1 |
| 100      | 2.55            | 2.69         | 2.68 | 5.2 | -0.3 | 2.68 | 5.1 | -0.3 |
| 150      | 4.12            | 4.03         | 4.02 | -2.3 | -0.1 | 4.02 | -2.5 | -0.3 |
| 200      | 5.57            | 5.38         | 5.37 | -3.7 | -0.3 | 5.35 | -4.0 | -0.6 |

Table 4: Comparison of extreme stresses $S_{\text{bot}}^{(3)}$ at $X^{(3)} = L/2$ for the simply supported beam.

| Load [N] | Reference data | Linear model | Non-linear model |
|----------|----------------|--------------|------------------|
|          | $S_{\text{exp}}$ [MPa] | $S_{\text{an}}$ [MPa] | $S$ [MPa] | $\eta_{\text{exp}}$ [%] | $\eta_{\text{an}}$ [%] | $S$ [MPa] | $\eta_{\text{exp}}$ [%] | $\eta_{\text{an}}$ [%] |
| 50       | 9.55           | 7.23         | 7.14 | -25.3 | -1.3 | 7.14 | -25.2 | -1.3 |
| 100      | 12.34          | 14.45        | 14.27 | 15.7 | -1.2 | 14.28 | 15.7 | -1.2 |
| 150      | 21.89          | 21.68        | 21.41 | -2.2 | -1.2 | 21.42 | -2.2 | -1.2 |
| 200      | 26.27          | 28.90        | 28.55 | 8.6 | -1.2 | 28.55 | 8.7 | -1.2 |

Figure 4: Stress distributions in simply supported three-layer beam; (a) normal and (b) shear stresses for $F = 50$ N.

and of detailed two-dimensional large-deformation finite element simulations. To make a direct comparison to these values, we employ the same number of elements per layer, $n^e = 150$, but analogous results are obtained for coarser discretizations.

The centerline deflections for geometrically linear and non-linear refined beam formulations appear in Figure 5(b) and are compared with the response of equivalent monolithic and two-layer formulations. While the response of the linear model still falls within the monolithic–two-layer bounds (with a closer proximity to the monolithic beam), the deflection of the fully non-linear model is considerably smaller that for the monolithic case. Moreover, the gap between the monolithic and two-layer cases becomes even more pronounced than in the previous example.

The detailed numerical values presented in Table 5 further support our findings. We see that the layered assumption as well as small-strain hypothesis are too conservative and lead to highly inefficient designs, as their errors exceed 100%. The accuracy of the monolithic approximation is comparable to the simply supported setup.

Basically the same conclusions follow from the response of the structure to an increasing load, Tables 6 and 7. For the largest load of $F = 150$ N, the error of the linear laminated model reaches $\sim 800\%$ for the deflections and $\sim 250\%$ for the extreme stresses. The finite-strain formulation remains accurate in the whole range of loading, and the errors with respect to the detailed two-dimensional finite element model do not exceed $\sim 1\%$ for the deflections and $\sim 2\%$ for stresses. In fact, it slightly outperforms the analytical model [2] in terms of the stress values, probably due to the more refined representation of the deformation in individual layers.
Figure 5: Fixed-end three-layer beam; (a) experiment setup and (b) centerline deflection for $F = 15$ N.

Table 5: Mid-span deflections $w^{(3)}_0$ at $X^{(3)} = L/2$ for the fixed beam subjected to $F = 15$ N.

| Model                        | Deflection [mm] | $\eta_{num}$ [%] | $\eta_{an}$ [%] |
|------------------------------|-----------------|-------------------|-----------------|
| 2D finite element model [2]  | 5.92            | ×                  | 0.0             |
| Analytical model [2]         | 5.92            | 0.0               | ×               |
| Refined linear               | 14.44           | 144               | 144             |
| Refined non-linear           | 6.00            | 1.3               | 1.3             |
| Monolithic linear            | 7.85            | 32.6              | 32.6            |
| Two-layer linear             | 51.48           | 770               | 770             |

as discussed above.

Unlike the previous example, the stress distributions for the linear and non-linear models differ to a significant amount. As for the extreme normal stresses, Figure 6(a), in the non-linear model their magnitude reaches $\sim 65\%$ of the value obtained by the linear one and their distribution is no longer antisymmetric in the thickness direction, i.e. $S^{(1)}_\text{top} \neq -S^{(3)}_\text{bot}$. These effects are due to additional membrane stresses acting in the glass layers. The same mechanism reduces the magnitude of the shear stresses in the interlayer to $\sim 40\%$ of value determined for the linear analysis, Figure 6(b). We note again that these observations are in an agreement with the results reported by Askik and Tescan [2].

Figure 6: Stress distributions in fixed-end three-layer beam; (a) normal and (b) shear stresses for $F = 15$ N.

As the final check of our implementation, in Table 8 we present the convergence progress for the load level $F = 150$ N. In order to investigate reliability of the method, we expose the structure to the full load, instead of applying it incrementally as in [2]. The results confirm significant
Table 6: Comparison of mid-span deflections \( w^{(3)}_0 \) at \( X^{(3)} = L/2 \) for the fixed-end beam.

| Load [N] | Reference data | Linear model | Non-linear model |
|----------|----------------|--------------|-----------------|
|          | \( w_{an} [\text{mm}] \) | \( w_{num} [\text{mm}] \) | \( w [\text{mm}] \) | \( \eta_{an} [%] \) | \( \eta_{num} [%] \) | \( w [\text{mm}] \) | \( \eta_{an} [%] \) | \( \eta_{num} [%] \) |
| 15       | 5.92           | 5.92         | 14.44          | 143.9          | 143.9          | 6.00          | 1.3          | 1.3          |
| 30       | 8.10           | 8.10         | 28.88          | 256.6          | 256.6          | 8.17          | 0.8          | 0.8          |
| 45       | 9.60           | 9.60         | 43.32          | 351.3          | 351.3          | 9.66          | 0.6          | 0.6          |
| 60       | 10.78          | 10.78        | 57.76          | 435.9          | 435.9          | 10.83         | 0.5          | 0.5          |
| 90       | 12.64          | 12.63        | 86.65          | 585.5          | 586.0          | 12.68         | 0.3          | 0.4          |
| 120      | 14.10          | 14.09        | 115.53         | 719.4          | 719.9          | 14.14         | 0.3          | 0.3          |
| 150      | 15.34          | 15.32        | 144.41         | 841.4          | 841.6          | 15.36         | 0.1          | 0.3          |

Table 7: Comparison of extreme stresses \( S^{(3)}_{bot} \) at \( X^{(3)} = L/2 \) for the fixed-end beam.

| Load [N] | Reference data | Linear model | Non-linear model |
|----------|----------------|--------------|-----------------|
|          | \( S_{an} [\text{MPa}] \) | \( S_{num} [\text{MPa}] \) | \( S [\text{MPa}] \) | \( \eta_{an} [%] \) | \( \eta_{num} [%] \) | \( S [\text{MPa}] \) | \( \eta_{an} [%] \) | \( \eta_{num} [%] \) |
| 15       | 12.87          | 12.46        | 19.51          | 51.6           | 56.6           | 12.60         | -2.1         | 1.1          |
| 30       | 20.69          | 19.89        | 39.02          | 88.6           | 96.2           | 20.12         | -2.7         | 1.2          |
| 45       | 27.13          | 25.94        | 58.53          | 115.7          | 125.6          | 26.28         | -3.2         | 1.3          |
| 60       | 32.82          | 31.25        | 78.03          | 137.8          | 149.7          | 31.69         | -3.4         | 1.4          |
| 90       | 42.82          | 40.51        | 117.05         | 173.4          | 188.9          | 41.18         | -3.8         | 1.6          |
| 120      | 51.68          | 48.64        | 156.07         | 202.0          | 220.9          | 49.53         | -4.2         | 1.8          |
| 150      | 59.76          | 56.00        | 195.09         | 226.4          | 248.4          | 57.13         | -4.4         | 2.0          |

degree of non-linearity in the structural response. After the first iteration, corresponding to the linear model, the structure is in an out-of-equilibrium state and the layer compatibility is violated. In the following iterations, the residuals are gradually reduced in a non-monotonic way until the ninth iteration, after which the method exhibits quadratic convergence in the equilibrium residual \( \eta_1 \) and super-linear convergence in the compatibility residual \( \eta_2 \). This behavior is in full agreement with available results for the Newton method, e.g. [6, Theorem 13.6], and also explains the convergence difficulties of the heuristic finite-difference solver reported in [2].

Table 8: Convergence of residuals, defined by Eq. (34), for \( F = 150 \text{ N} \).

| Iteration \( k \) | \( \eta_1 \) | \( \eta_2 \) |
|-------------------|--------------|--------------|
| 1                 | \( 8.49 \times 10^2 \) | \( 7.94 \times 10^{-1} \) |
| 2                 | \( 1.50 \times 10^3 \) | \( 4.65 \times 10^{-1} \) |
| 3                 | \( 1.02 \times 10^2 \) | \( 6.12 \times 10^{-2} \) |
| 4                 | \( 2.07 \times 10^2 \) | \( 5.61 \times 10^{-2} \) |
| 5                 | \( 2.31 \times 10^1 \) | \( 1.11 \times 10^{-2} \) |
| 6                 | \( 2.43 \times 10^1 \) | \( 7.53 \times 10^{-3} \) |
| 7                 | \( 4.93 \times 10^0 \) | \( 2.58 \times 10^{-3} \) |
| 8                 | \( 1.41 \times 10^0 \) | \( 8.17 \times 10^{-4} \) |
| 9                 | \( 1.38 \times 10^{-1} \) | \( 8.23 \times 10^{-5} \) |
| 10                | \( 1.58 \times 10^{-3} \) | \( 8.33 \times 10^{-7} \) |
| 11                | \( 2.53 \times 10^{-7} \) | \( 1.18 \times 10^{-10} \) |
6 Conclusions

In this paper, we have presented an efficient approach to the analysis of the laminated glass beams. Based on the Mau theory for layered structures, it treats each layer separately and enforces the inter-layer compatibility by the Lagrange multipliers. In our implementation, we utilize a reliable finite element formulation of the Reissner finite-strain beam theory due to Ibrahimbegović and Frey to discretize individual layers, and solve the resulting system of equations iteratively by the Newton method with consistent linearization. On the basis of the performed simulations, we conjecture that

- in the absence of membrane effects, the formulation reduces exactly to the small-strain model introduced in our previous work,
- although the discretization is based on the lowest-order polynomial basis functions, the method provides results with accuracy comparable to the detailed two-dimensional large-strain finite element simulations,
- the Newton method exhibits a reliable super-linear convergence even for high degrees of non-linearity.

Extension of the current framework to include temperature- and time-dependent properties of the interlayer will be reported independently.

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By an analogous procedure one obtains from Eq. (33)

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### A Sensitivity analysis

The expression for the internal nodal forces follows directly from Eq. (33). After certain manipulations we arrive at, cf. \[ 21 \]

\[
\begin{align*}
\mathbf{f}^{(i)}_{\text{int, e}} &= \begin{bmatrix} f_{\text{int, e, 1}}^{(i)} \\ f_{\text{int, e, 2}}^{(i)} \\ f_{\text{int, e, 3}}^{(i)} \\ f_{\text{int, e, 4}}^{(i)} \\ f_{\text{int, e, 5}}^{(i)} \\ f_{\text{int, e, 6}}^{(i)} \end{bmatrix} \\
&= \begin{bmatrix}
-E^{(i)} A^{(i)} E_{e}^{(i)} \cos \beta_{e}^{(i)} - G^{(i)} A^{(i)} \Gamma_{e}^{(i)} \sin \beta_{e}^{(i)} \\
E^{(i)} A^{(i)} E_{e}^{(i)} \sin \beta_{e}^{(i)} - G^{(i)} A^{(i)} \Gamma_{e}^{(i)} \cos \beta_{e}^{(i)} \\
-\frac{1}{2} (F_{e}^{(i)} + \Delta u_{0, e}^{(i)}) f_{\text{int, e, 2}}^{(i)} + \frac{1}{2} \Delta u_{0, e}^{(i)} f_{\text{int, e, 1}}^{(i)} - E^{(i)} f^{(i)} K_{e}^{(i)} \\
-\frac{1}{2} (F_{e}^{(i)} + \Delta u_{0, e}^{(i)}) f_{\text{int, e, 2}}^{(i)} + \frac{1}{2} \Delta u_{0, e}^{(i)} f_{\text{int, e, 1}}^{(i)} + E^{(i)} f^{(i)} K_{e}^{(i)} \\
\end{bmatrix}. \\
\end{align*}
\]

By an analogous procedure one obtains from Eq. (33)

\[
K_{t, e}^{(i)} = \begin{bmatrix}
K_{t, e, 11}^{(i)} & K_{t, e, 12}^{(i)} & K_{t, e, 13}^{(i)} & -K_{t, e, 11}^{(i)} & -K_{t, e, 12}^{(i)} & K_{t, e, 13}^{(i)} \\
K_{t, e, 11}^{(i)} & K_{t, e, 22}^{(i)} & K_{t, e, 23}^{(i)} & -K_{t, e, 12}^{(i)} & -K_{t, e, 22}^{(i)} & -K_{t, e, 23}^{(i)} \\
K_{t, e, 11}^{(i)} & K_{t, e, 23}^{(i)} & K_{t, e, 33}^{(i)} & -K_{t, e, 13}^{(i)} & -K_{t, e, 23}^{(i)} & -K_{t, e, 33}^{(i)} \\
-K_{t, e, 11}^{(i)} & -K_{t, e, 12}^{(i)} & -K_{t, e, 13}^{(i)} & K_{t, e, 11}^{(i)} & K_{t, e, 12}^{(i)} & K_{t, e, 13}^{(i)} \\
-K_{t, e, 12}^{(i)} & K_{t, e, 22}^{(i)} & K_{t, e, 23}^{(i)} & -K_{t, e, 13}^{(i)} & -K_{t, e, 22}^{(i)} & -K_{t, e, 23}^{(i)} \\
-K_{t, e, 13}^{(i)} & K_{t, e, 23}^{(i)} & K_{t, e, 33}^{(i)} & -K_{t, e, 12}^{(i)} & -K_{t, e, 23}^{(i)} & -K_{t, e, 33}^{(i)} \\
\end{bmatrix}, \quad \text{(38)}
\]
where the individual entries read

\[
K_{t,e,11}^{(i)} = \frac{\partial f_{i,11}}{\partial u_{0,11}} = \frac{1}{L_e} \left( E_e A_e \cos^2 \beta_e + G_e A_s \sin^2 \beta_e \right),
\]

\[
K_{t,e,12}^{(i)} = \frac{\partial f_{i,12}}{\partial u_{0,12}} = \frac{1}{2L_e} \left( -E_e A_e + G_e A_s \right) \sin 2\beta_e,
\]

\[
K_{t,e,13}^{(i)} = \frac{\partial f_{i,13}}{\partial \varphi_{e,13}} = \frac{1}{2} \left[ \left( E_e A_e - G_e A_s \right) \left( E_e \cos \beta_e + \Gamma_e \cos \beta_e \right) - G_e A_s \sin \beta_e \right],
\]

\[
K_{t,e,22}^{(i)} = \frac{\partial f_{i,22}}{\partial u_{0,22}} = \frac{1}{L_e} \left( E_e A_e \sin^2 \beta_e + G_e A_s \cos^2 \beta_e \right),
\]

\[
K_{t,e,23}^{(i)} = \frac{\partial f_{i,23}}{\partial \varphi_{e,23}} = \frac{1}{2} \left[ \left( E_e A_e - G_e A_s \right) \left( E_e \cos \beta_e - \Gamma_e \sin \beta_e \right) - G_e A_s \cos \beta_e \right],
\]

\[
K_{t,e,33}^{(i)} = \frac{\partial f_{i,33}}{\partial \varphi_{e,33}} = \frac{1}{2} \left[ - \left( E_e + \Delta u_{0,e} \right) K_{t,e,23}^{(i)} + \Delta u_{0,e} K_{t,e,13}^{(i)} \right] + \frac{E_e I_e}{L_e},
\]

\[
K_{t,e,36}^{(i)} = \frac{\partial f_{i,36}}{\partial \varphi_{e,36}} = K_{t,e,33}^{(i)} - 2E_e I_e L_e.
\]

The remaining terms in the system \([30]\) originate from the compatibility conditions \([22]\). In particular, the matrix \(C\) is analogous to the small-strain tying condition \([30, \text{Section 4}]\). The block of \(C\), associated with a node \(j\) and layers \(i\) and \((i+1)\) attains the form

\[
C_j^{(i,i+1)} = \begin{bmatrix}
1 & 0 & \frac{1}{2} h^{(i)} \cos \varphi_j^{(i)} & \cdots & -1 & 0 & \frac{1}{2} h^{(i+1)} \cos \varphi_j^{(i+1)} \\
0 & 1 & -\frac{1}{2} h^{(i)} \sin \varphi_j^{(i)} & \cdots & 0 & -1 & -\frac{1}{2} h^{(i+1)} \sin \varphi_j^{(i+1)}
\end{bmatrix}.
\]

The second derivatives of the compatibility conditions quantify their contributions to the tangent stiffness

\[
K_{\lambda_{j}}^{(i,i+1)} = \frac{\partial^2 F_{X}^{(i,i+1)}}{\partial d_{j}^{(i,i+1)} d_{j}^{(i,i+1)}} \lambda_{X,j}^{(i,i+1)} + \frac{\partial^2 F_{Z}^{(i,i+1)}}{\partial d_{j}^{(i,i+1)} d_{j}^{(i,i+1)}} \lambda_{Z,j}^{(i,i+1)}.
\]

This additional term is expressed as

\[
K_{\lambda_{j}}^{(i,i+1)} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & K_{\lambda_{j}}^{(i,i+1)} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & K_{\lambda_{j}}^{(i,i+1)}
\end{bmatrix},
\]

with non-zero entries provided by

\[
K_{\lambda_{j}}^{(i,i+1)} = -\frac{1}{2} h^{(i)} (\sin \varphi_j^{(i)} \lambda_{X,j}^{(i,i+1)} + \cos \varphi_j^{(i)} \lambda_{Z,j}^{(i,i+1)}),
\]

\[
K_{\lambda_{j}}^{(i,i+1)} = -\frac{1}{2} h^{(i)} (\sin \varphi_j^{(i+1)} \lambda_{X,j}^{(i,i+1)} + \cos \varphi_j^{(i+1)} \lambda_{Z,j}^{(i,i+1)}).
\]