Quantum critical effects on transition temperature of magnetically mediated p-wave superconductivity

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We determine the behavior of the critical temperature of magnetically mediated p-wave superconductivity near a ferromagnetic quantum critical point in three dimensions, distinguishing universal and non-universal aspects of the result. We find that the transition temperature is non-zero at the critical point, raising the possibility of superconductivity in the ferromagnetic phase.

Recent experimental work has shown that superconductivity in strongly correlated electron systems is closely associated with proximity to magnetic quantum critical points [1, 2], suggesting it is mediated by critical spin fluctuations [3], as indicated by theoretical calculations [4]. However, the interplay of superconductivity and criticality is not fully understood. In this paper we study the theoretically simplest case, namely p-wave superconductivity near a ferromagnetic quantum critical point in dimension $d = 3$. Our work is complimentary to that of Abanov, Chubukov and Schmalian [5] who studied pairing near a two dimensional antiferromagnetic quantum critical point.

We have two motivations. One is practical: one would like to know whether the superconducting $T_c$ critical point.

The parameter $r$ measures the distance from the quantum critical point (QCP), and alternatively b) $T_c$ is finite in QCP. The parameter $r$ measures the distance from the quantum critical point $r_c$. In experimental realizations $r$ corresponds to hydrostatic pressure [6].

Our second motivation is theoretical. Studies of magnetically mediated superconductivity have almost uniformly been based on the Eliashberg equations (defined below) [7] which are simple generalizations of the equations which describe conventional phonon-mediated superconductors [8]. While these equations are believed [9] to give the leading contributions to the low-energy behavior of systems near critical points, non-critical and high-frequency processes may also be important for the superconducting $T_c$ (These lead, e.g. to the $\mu^*$ familiar from conventional superconductivity) [10]. Our results show how to isolate the critical contributions and allow the magnetic analogue of $\mu^*$ to be estimated.

![FIG. 2. Eliashberg equations in diagrammatic form.](image)

$\chi(q, \nu) = \left[ \frac{|\nu|}{\Lambda} \cdot \frac{q}{p_F} \tan^{-1} \left( \frac{\Lambda}{|\nu|} \cdot \frac{p_F}{q} \right) + \left( \frac{q}{2p_F} \right)^2 + r \right]^{-1} + \ldots$ (1)

where $r$ is a parameter that measures the distance of the system from its quantum critical point (Fig. 1) and the ellipsis denotes less singular terms. Here $p_F$ is a momentum scale of the order of the Fermi momentum and $\Lambda \sim \nu_{FpF}$ is an energy scale of the order of the Fermi energy. We assume (following [10]) that the coupling of these to the electron system is given by the Eliashberg equations (shown diagrammatically in Fig. 2) for the electron self-energy $\Sigma(p, i\omega) = i\omega(1 - Z_p(\omega))$ and the anomalous self-energy $W(p, i\omega)$. $W(p, i\omega)$ vanishes at the...
superconducting critical temperature and grows continuously in the superconducting state. We find $T_c$ by solving the linearized Eliashberg equations, which are

$$i \omega (1 - Z_p(\omega)) = g^2 s_0 \int N(\Omega_p') d\Omega_p' \int_{-\infty}^{\infty} d\epsilon_p'$$

$$\pi T \sum_{\epsilon_p} \chi (p - p', i\omega - i\omega') \frac{1}{i\omega' Z_p(\omega') - \epsilon_p'}$$

$$W(p, i\omega) = g^2 s_1 \int N(\Omega_p') d\Omega_p' \int_{-\infty}^{\infty} d\epsilon_p'$$

$$\pi T \sum_{\epsilon_p} \chi (p - p', i\omega - i\omega') \frac{-W(p', i\omega')}{[i\omega' Z_p(\omega')]^2 - \epsilon_p'^2}.$$ 

Here the momentum integration has been separated into integration in a direction perpendicular to the Fermi surface ($\epsilon_p$ integration) and integration over angles $\Omega_p$ of the spherical Fermi surface; $N(\Omega_p)$ is the angle-dependent density of states of the quasiparticles on the Fermi surface. The numerical factors $s_1$ relate to the nature of the spin fluctuations and the symmetry of the pairing state.

For a system with a Heisenberg symmetry there are three independent soft spin components all of which contribute to $\omega Z$ so $s_0 = 3$. However, for spin triplet pairing only one combination can contribute to any given component of the gap function, so $s_1 = 1$. The importance of this factor was stressed by Monthoux and Lonzarich [6]. For a system with a strong Ising anisotropy, both $s_0$ and $s_1 = 1$. We will present results for the Heisenberg - Ising crossover elsewhere. $g$ is a constant vertex representing the interaction between spin fluctuations and low-energy quasiparticles. It may be experimentally defined from the singular (as $r \rightarrow 0$) behavior of the specific heat coefficient.

$$\gamma = \lim_{T \rightarrow 0} \frac{C}{T} = \frac{m_s p_F}{3h^3} k_B^2 Z(0).$$

Eqs. (3) and (4) apply only for frequencies much less than the electron bandwidth and only if the momentum dependence of $Z_p$ and $W$ is negligible relative to the frequency dependence, conditions which are satisfied for the leading singular behavior as $r \rightarrow 0$. We therefore employ the Migdal approximation [3] $Z_p(\omega) \rightarrow Z(\omega)$, $W(p, \omega) \rightarrow W(\Omega_p, i\omega)$ and perform the integral over the magnitude of the momentum. To perform the remaining integration over angles we note that $i\omega Z(\omega)$ has the full symmetry of the lattice, while for $p$-wave superconductivity $W$ corresponds to the $l = 1$ spherical harmonic.

The momentum transfer $q$ carried by the spin fluctuations in Eq. (4) is given by $q^2 = (p - p')^2 = 2p^2 (1 - (p \cdot p')/|p||p'|) + c_p^2/c_p^2$. The first term in $q^2$ is obtained by placing both momenta $p$ and $p'$ on the Fermi surface while the last term is a small correction $\delta p$ taking into account the fact that intermediate states can explore regions close to the Fermi surface (Fig. 3) and will be important as a cutoff. We perform the $\epsilon_p$ integral, use the angle dependences of $Z$ and $W$ and obtain

$$|\omega| (1 - Z(\omega)) = -\pi T s_0 \sum_{\omega} D_0(\omega - \omega') \text{sgn}(\omega')$$

$$W_1(\omega) = \pi T s_1 \sum_{\omega} D_1(\omega - \omega') \frac{W_1(\omega')}{|\omega' Z(\omega')|}.$$ 

where

$$D_1(\nu) = 16\pi^2 g^2 \int_0^1 N_0(x) \frac{x P_1(1 - 2x^2)dx}{U(U + |\omega' Z(\omega')|/\Lambda)}$$

with $U = [(\nu/\nu') \tan^{-1}(x/\nu) + x^2 + \nu]^{1/2}$, $\nu = \nu/\Lambda$, $P_1(x)$ is a Legendre polynomial and the $|\omega Z|/\Lambda$ comes from the $\epsilon_p^2$ term. It is numerically very small and is important only as a cutoff at $r < T^2$ and $\nu = 0$; except in these cases we drop it. In the following we combine all constant prefactors in $D_1$ into a single coupling constant: $16\pi^2 g^2 N_0 \rightarrow \lambda$.

FIG. 3. Fermi surface with momenta participating in the interaction. The dashed line is the momentum transfer $q$ when the correction $\delta p$ is neglected.

To solve Eqs. (3) - (6) we follow Bergmann and Rainer [4], defining a new order parameter $\Phi_1(\omega) = W_1(i\omega)/|\omega Z(\omega)|$ and casting Eqs. (3) and (4) into an eigenvalue problem for an eigenvalue $\rho$

$$\sum_{\omega} \left[ s_1 D_l(\omega - \omega') - s_0 \frac{|\omega' Z(\omega')|}{\pi T} \delta_{\omega,\omega'} \Phi_1(\omega') \right] \Phi_1(\omega) = \rho \Phi_1(\omega)$$

$$|\omega Z(\omega)| = |\omega| + \pi T \left( D_0(0) + 2 \sum_{\omega' = 0}^\omega D_0(\omega - \omega') \right).$$

At high temperatures the eigenvalues $\rho_n(T)$ are negative; at $T_c$ the leading eigenvalue crosses 0. We solve the matrix system numerically; the size of the kernel, $K_{nm} = s_1 D_l(\omega_n - \omega_m) - \delta_{nm} s_0 |\omega_m Z(\omega_m)|/\pi T \sim N/(2\pi T_c)$.

For $p$-wave pairing in systems of Heisenberg symmetry the critical temperatures are typically $\pi T_c \sim 10^{-5} \Lambda$ which translates into numerically unmanageable kernel sizes of $N \sim 50,000$. We therefore use a down-folding procedure: we separate $\Phi_1(\omega_n)$ in Eq. (8) into a low-frequency part $\Phi_1^{\text{LOW}}(\omega_n)$ with $0 \leq |n| \leq N_{\text{LOW}}$ and

$$\Sigma_{\omega} \left[ s_1 D_l(\omega - \omega') - s_0 \frac{|\omega' Z(\omega')|}{\pi T} \delta_{\omega,\omega'} \Phi_1^{\text{LOW}}(\omega') \right] \Phi_1^{\text{LOW}}(\omega) = \rho^{\text{LOW}} \Phi_1^{\text{LOW}}(\omega).$$

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$$|\omega Z(\omega)| = |\omega| + \pi T \left( D_0(0) + 2 \sum_{\omega' = 0}^\omega D_0(\omega - \omega') \right).$$
high-frequency part $\Phi^{\text{HIGH}}(\omega_n)$ with $N_{\text{LOW}} < |n| \leq N$. Then Eq. (8) can be written as a block linear system and formally solved for $\Phi^{\text{HIGH}}$, yielding $K^{\text{LOW}} \cdot \Phi^{\text{LOW}} = \rho \Phi^{\text{LOW}}$ with $K^{\text{LOW}}_{nn} = K_{nn} + \sum_{|i|,|j| > N_{\text{LOW}}} K_{ni}(\rho - K_{ij})^{-1} K_{jm}$. This transformation is exact. The simplification is that for large $N_{\text{LOW}}$ $K$ is nearly diagonal so $K^{-1}$ may easily be computed in the “high” subspace. In physical terms, this approximation retains only the diagonal scattering-dominated terms $K_{nn} \approx (s_1 - s_0)D_0(0) - (2n + 1)(1 + (2/3)\lambda s_0 + (2/3)\lambda s_0 \ln(\Lambda/2\pi Tn))$ and drops all off-diagonal pairing terms $s_1D_1(\nu_{nm}) \approx (\lambda s_1/3) \ln(\Lambda/2\pi T|n - m|)$ for $n, m > N_{\text{LOW}}$ in the high-frequency kernel. We have verified that this approximation reproduces faithfully the eigenvalues of Eq. (8) for large temperatures, and that our results are insensitive to the choice of $N_{\text{LOW}}$.

Our results for $T_c(r)$ are shown in Fig. 4. The top panel of the figure demonstrates the convergence of the scaling procedure with reduced kernel size $N_{\text{LOW}}$. Kernel sizes $N_{\text{LOW}} \geq 500$ show satisfactory convergence, so we have used sizes $N_{\text{LOW}} = 500$ in most of our work. That large kernels are needed shows that in this problem $T_c$ is not controlled by low-energy physics. As previously noted $T_c$ is very low in the $p$-wave Heisenberg case because of the factor of three between the pairing vertex and the self energy. The Ising case has not been previously studied; we see $T_c$ is much higher.

We find that in both Heisenberg and Ising cases, $T_c(r \to 0) > 0$, raising the interesting possibility of superconductivity extending into the magnetic phase. We confirm that $T_c(r = 0) > 0$ using a variational argument. The ansatz $W(\omega_m) = \Delta Z(\omega_m)\Theta(\omega^{*2} - \omega_m^2)$ allows the leading eigenvalue to be computed if $\omega^* \ll \Lambda$. At $r = 0$, the leading eigenvalue becomes positive for $T < T^\text{var}$ with

$$T^\text{var} = \frac{\omega^*}{\pi} \exp\left[-\frac{3/2\lambda + s_0}{s_1(1 + \ln(\Lambda/\omega^*))} - \frac{s_0}{s_1}\right] \quad (10)$$

which is thus a lower bound for $T_c$. Abanov, Chubukov and Schmalian [2], who studied a two dimensional antiferromagnetic problem, argued that the divergent mass enhancement associated with the critical fluctuation would drive the superfluid stiffness and thus $T_c$ to zero. In their case the divergent mass occurs only at one point on the Fermi surface, so it seems to us the considerations of Hulbina and Rice [13] should imply a non-zero superfluid stiffness. In any event, in the ferromagnetic problem of interest here the critical fluctuations are long-wavelength, and thus do not lead to divergences in the “transport mass” controlling the superfluid stiffness.

![FIG. 4. Results for $T_c(r)$ a) near $r = 0$ for a set of kernel sizes $N_{\text{LOW}}$ and b) in a broader range for two coupling constants $\lambda = 1.5$ and $\lambda = 10.0$; in the Ising case ($s_0 = 1$) we have plotted $T_c/100$ for better visual comparison with the Heisenberg ($s_0 = 3$) curves. $\Lambda \sim 2D_{\text{EFF}}$ is the characteristic spin-fluctuation frequency; the curve $2\pi T_c = r^{3/2}$ separates two regimes when $T_c < r^{3/2}$ and when $T_c > r^{3/2}$. For the Heisenberg case ($s_0 = 3$) $N_{\text{LOW}} = 500$ while for the Ising case ($s_0 = 1$) $N = \Lambda/2\pi T_c$. We see from Fig. 3 that the Heisenberg case $T_c(r)$ displays a maximum at a small non-zero $r$, whereas in the Ising case there is no maximum. We believe the small $r$ behavior is controlled by the interplay of pairing and scattering, as discussed by Bergmann and Rainer [14] for $s$-wave superconductivity and by Mills, Sachdev, Varma for $d$-wave superconductivity. To see this mathematically we compute $dT_c/dr$ using the Feynman-Hellman theorem [2]:

$$\frac{dT_c}{dr} = \left(\frac{d\rho}{dT_c}\right)^{-1} \frac{d\rho}{dr} = \left(\frac{d\rho}{dT_c}\right)^{-1} \langle \Phi | \frac{dK}{dr} | \Phi \rangle. \quad (11)$$

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The expectation value is infrared dominated numerically and we find that at small \( \omega \), \( \Phi'(\omega) \approx 1/|\omega| \). From Eqs. (6) and (7) we see there are two contributions to \( dK/dr \): one positive and proportional to \( s_l \), coming from the pairing term \( D_l(\nu) \) and one negative and proportional to \( s_0 \), from the depairing term \( |\omega Z| \) in Eq. (8). As \( r \to 0 \) the dominant term in \( D_0 \) becomes identical to the dominant term in \( D_1 \). It is convenient to isolate the contribution from zero-frequency spin fluctuations. For the leading singular behavior in \( r \) we find

\[
\frac{dD_0}{dr} \approx \sum_{n=0}^{\infty} \frac{s_0 - s_l}{(2n+1)^2} D_0(n) - 2 \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \left( \frac{-s_0}{(2n+1)^2} \right) D_0(n) - 2 \sum_{n=1}^{\infty} \frac{s_l}{(2n+1)^2} D_0(n) + \frac{s_l}{(2m+1)} D_0(n-m)
\]

(12)

Here

\[
D_0(n) = \frac{\lambda}{2} \sqrt{r/\pi} \frac{1}{|\omega_n Z|/\Lambda}
\]

(13)

comes from differentiating Eq. (6) at \( \bar{\nu} = 0 \) while

\[
D_\omega = \frac{\lambda}{r} F \left( \frac{2\pi T}{\sqrt{\pi r^2}} (n-m) \right)
\]

(14)

comes from differentiating Eq. (7) at \( \bar{\nu} \neq 0 \) and dropping the \( |\omega Z| \) term. The scaling function

\[
F(x) = \int_0^\infty \frac{y dy}{(x/y^2 + 1)^2};
\]

(15)

\( F(0) = 1/2 \) and as \( x \to \infty \), \( F(x) \to (2\pi/9\sqrt{3})x^{-2/3} \).

For \( s_0 = 1 \) (Ising case) the \( D_0 \) term vanishes and the \( D_\omega \) term is negative. The pairing and depairing effects of quasistatic (\( \omega < T \)) spin fluctuations exactly cancel (as in the s-wave case [14]) while at \( \omega > T \) the pairing effect wins. Thus \( T_c \) monotonically increases as \( r \to 0 \) because the spin fluctuations become stronger. At \( r = 0 \), \( dT_c/dr \sim -T^{-2/3} \), i.e. \( T_c(r) \) is linear; for \( r > T_c^{3/2} \) the derivative \( \sim (\ln 1/r)/r \), so we expect \( T_c \sim \ln^2 1/r \).

For \( s_0 = 3 \) (Heisenberg case) the \( D_0 \) term is non-vanishing, and indeed is dominant at small \( r \): quasistatic spin fluctuations are pairbreaking. At \( r = 0 \) \( T_c \) is set by the temperature at which the effect of these fluctuations becomes small enough to allow pairing. For \( r < \lambda^2(\pi T_c)^2 \ln^2 \Lambda/T_c \), the \( \omega Z \) term is important and \( dT_c/dr \sim -1/(\sqrt{\pi} \Lambda \pi T_c \ln \Lambda/T_c) \). For \( \lambda^2(\pi T_c)^2 \ln^2 \Lambda/T_c < r < (\pi T_c)^{3/2} \), \( dT_c/dr \sim 1/r \). In our calculations this intermediate regime is not wide enough to see. For larger \( r \), the variation of the pairing \( (D_\omega) \) term with \( r \) becomes most important.

To summarize, we have presented a theory of the variation of a p-wave superconducting \( T_c \) near a ferromagnetic quantum critical point. We have shown that the variation of \( T_c \) with distance from criticality is controlled by the low energy spin fluctuations which are theoretically tractable, and demonstrated the crucial role played by the symmetry of the magnetic fluctuations. We have found generically that \( T_c > 0 \) at the magnetic critical point, raising the interesting possibility of superconductivity within the ordered phase.

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