Product Representations and the Quantization of Constrained Systems

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Abstract

We study special systems with infinitely many degrees of freedom with regard to dynamical evolution and fulfillment of constraint conditions. Attention is focused on establishing a meaningful functional framework, and for that purpose, coherent states and reproducing kernel techniques are heavily exploited. Several examples are given.

1 Introduction

Generally speaking, the quantum theory of infinitely many degrees of freedom (i.e., quantum field theory) exhibits a number of complications. However, the quantum theory of “product systems”, also involving infinitely many degrees of freedom, is especially simple, and such examples can serve as training models for more complicated cases. Initially, one starts with a basic system composed of a finite number of degrees of freedom. To be specific, let us say standard canonical degrees of freedom, which is the case we study. Subsequently, one adjoins an infinite number of identical and independent basic systems to build a model with an infinite number of degrees of freedom. The quantum theory of such systems involves (tensor) product representations of
the basic operators, and generally needs only an energy scale renormalization. (Some aspects of product representations may be found in [1].) On the other hand, such models—just like far more complicated examples—require that the field-operator representation be carefully chosen with the dynamics in mind. In the present paper we extend the discussion of such models to include constraints of a rather general nature and do so in such a way that the original product representation is maintained. We start with a discussion of basic classical models for finitely many degrees of freedom and then illustrate the extension of these classical models to infinitely many degrees of freedom in a manner that preserves the equality and the independence of each of the basic units that make up the infinite system.

1.1 Classical formulation

From a classical point of view, let us start with a $J$ degree of freedom model, $1 \leq J < \infty$, and a classical action given by

$$I = \int \left[ \frac{1}{2}(p \cdot \dot{q} - q \cdot \dot{p}) - H(p, q) - \lambda^\alpha(t)\phi_\alpha(p, q) \right] dt,$$  \hspace{1cm} (1)

where $p = \{p^j\}_{j=1}^J$ and $q = \{q^j\}_{j=1}^J$ are dynamical variables, $p \cdot \dot{q} \equiv \Sigma_j p^j \dot{q}^j$, etc., and $\{\lambda^\alpha\}_{\alpha=1}^A$ denote Lagrange multipliers. We next extend this model to $N$ identical and independent copies, $N < \infty$, leading to $NJ < \infty$ degrees of freedom. This procedure gives rise to the classical action

$$I_N = \Sigma_{n=1}^N \int \left[ \frac{1}{2}(p_n \cdot \dot{q}_n - q_n \cdot \dot{p}_n) - H(p_n, q_n) - \lambda^\alpha_n(t)\phi_\alpha(p_n, q_n) \right] dt,$$  \hspace{1cm} (2)

an expression which exhibits an interchange symmetry $(p_n, q_n) \leftrightarrow (p_m, q_m)$, $1 \leq n, m \leq N$, for any pairs $m$ and $n$. Here $H(p, q)$ denotes the classical Hamiltonian and $\{\phi_\alpha(p, q)\}_{\alpha=1}^A$ the constraints. So long as $N < \infty$ this generalization is straightforward. However, things become much more interesting when $N \to \infty$. Our ultimate interest lies in studying the quantum theory of the classical theory characterized by the classical action

$$I_\infty = \Sigma_{n=1}^\infty \int \left[ \frac{1}{2}(p_n \cdot \dot{q}_n - q_n \cdot \dot{p}_n) - H(p_n, q_n) - \lambda^\alpha_n(t)\phi_\alpha(p_n, q_n) \right] dt.,$$  \hspace{1cm} (3)

Already at the classical level, in order for this expression to make sense, it is necessary that

$$I_n \equiv \int \left[ \frac{1}{2}(p_n \cdot \dot{q}_n - q_n \cdot \dot{p}_n) - H(p_n, q_n) - \lambda^\alpha_n(t)\phi_\alpha(p_n, q_n) \right] dt.$$  \hspace{1cm} (4)
vanish as $n \to \infty$. Without loss of generality, we may assume this will occur provided $p_n \to 0$ and $q_n \to 0$ combined with the condition that $H$ and all $\phi_\alpha$ are continuous functions and that $H(0,0) = 0$ as well as $\phi_\alpha(0,0) = 0$. In that case, as $p_n \to 0$ and $q_n \to 0$, then $I_n \to 0$. However, that behavior is not quite enough since it does not automatically imply convergence of the series in (3). We do not pursue the classical story further but simply assume that $I_n \to 0$ sufficiently rapidly so that (3) converges absolutely. The resultant sequences characterize the domain of the classical theory. For instance, some examples may satisfy the criterion $\sum_{n=1}^{\infty} \left[ \sum_{j=1}^{J} (|p_j| + |q_j|) \right] < \infty$.

Observe that we can also recover $I_{(N)}$ from $I_{(\infty)}$ merely by setting $p_n \equiv 0$ and $q_n \equiv 0$ for all $n > N$. In this sense we also have the rule that

$$I_{(\infty)} = \lim_{N \to \infty} I_{(N)}$$

provided $I_n \to 0$ in a suitable fashion, which, in turn, will hold if $p_n \to 0$ and $q_n \to 0$ in an appropriate manner.

Our goal is to discuss the quantum theory of the models classically described by (3). Several simple examples are discussed in Section 3.

## 2 Quantum Theory

### 2.1 Basic systems

Our goal here is to find a meaningful functional formalism for the quantum theories involved, including dynamics and constraints. In our quantum analysis, we shall exploit canonical coherent states and for that purpose we choose (with $\hbar = 1$)

$$|p,q\rangle \equiv \exp(i p \cdot Q - i q \cdot P) |\eta\rangle$$

expressed in conventional terms and where the fiducial vector $|\eta\rangle$ is, for the present, a general unit vector. For any $|\eta\rangle$, such coherent states admit a resolution of unity in the form

$$1 = \int |p,q\rangle \langle p,q| d\mu(p,q), \quad d\mu(p,q) = \Pi_{j=1}^{J} dp^j dq^j / 2\pi,$$

with integration over the entire phase space $\mathbb{R}^{2J}$. If $\mathcal{H}$ denotes the quantum Hamiltonian operator, then the propagator in the coherent-state representation is determined by

$$\langle p'', q''| e^{-iHT} |p', q'\rangle,$$

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and this expression may be given a coherent-state path-integral representation with no difficulty. Following conventional notation \[2\],

\[
\langle p'', q'' | e^{-i\mathcal{H}T} | p', q' \rangle = \lim_{\epsilon \to 0} \int \prod_{i=1}^{N} \langle p_{(i+1)} , q_{(i+1)} | (\mathbb{1} - i\epsilon \mathcal{H}) | p_{(i)} , q_{(i)} \rangle \prod_{i=1}^{N} d\mu(p_{(i)} , q_{(i)}) = \mathcal{M} \int e^{i \int \frac{1}{2} (p\cdot\dot{q} - q\cdot\dot{p}) - H(p,q)\] \[dt} \mathcal{Dp} \mathcal{Dq}, (9)
\]

the last relation being formal but standard. In making this identification we have set \(H(p,q) = \langle p,q | H(P,Q) | p,q \rangle\).

Next, let us temporarily set \(\mathcal{H} = 0\) and focus on the constraints. To introduce quantum constraints, we adopt the projection-operator approach \[3\] in which one focuses on the projection operator \(E\) onto the physical Hilbert space \(H_{\text{phys}} = E \mathbb{H}\) composed of vectors \(|\psi\rangle_{\text{phys}} = E |\psi\rangle\) for arbitrary \(|\psi\rangle \in \mathbb{H}\). It is possible to construct a general \(E\) by a linear operation on the set of unitary operations generated by the constraints. In particular, if the several self-adjoint operators \(\Phi_\alpha(P,Q)\) denote the quantum constraint operators with the property that \(\Sigma \Phi_\alpha(P,Q)^2\) is essentially self adjoint, then there exists \[4\] a linear operation that is independent of the specific constraint operators themselves, denoted by an integral with measure \(R(\lambda)\), and such that

\[
\int \mathcal{T} \exp[-i \int^{t+\epsilon}_{t} \lambda^\alpha(s) \Phi_\alpha(P,Q) ds] DR(\lambda) = E (\Sigma \Phi_\alpha(P,Q)^2 \leq \delta(h)^2), (10)
\]

Here \(\mathcal{T}\) denotes time ordering, \(\epsilon > 0\), and \(\delta(h)^2 > 0\) denotes a suitable, and possibly provisional, precision with which the constraints are enforced.

A few examples will illustrate how this concept may be used. If \(\{\Phi_\alpha\}\) denotes operators with discrete spectra, say angular momentum operators \(J_k, k \in \{1, 2, 3\}\), then \(\delta(h)^2 \leq h^2/10\) ensures that \(E = E (\Sigma J_k^2 = 0)\). If \(\{\Phi_\alpha\}\) denotes second-class constraints, say \(\Phi_1 = P\) and \(\Phi_2 = Q\), then \(\delta(h)^2 = h\) ensures that \(E = E (P^2 + Q^2 \leq h) = |0\rangle\langle 0|\), the projection operator onto the oscillator ground state. If \(\{\Phi_\alpha\}\) denotes an operator with zero in its continuous spectrum, say \(\Phi_1 = P\), then \(E = E (P^2 \leq \delta^2)\) and \(\delta^2 > 0\) can be chosen arbitrarily small, e.g., \(\delta^2 = 10^{-100}\). For all practical purposes it is not necessary that \(\delta \to 0\); however, that limit can also be incorporated with a possible change of the Hilbert space involved.
The mechanism for a possible change of Hilbert space arises by a reduction of the reproducing kernel \[3\]. In particular, if \(K(p'', q''; p', q') \equiv \langle p'', q'' | p', q' \rangle\) denotes the reproducing kernel \[3\] for the full Hilbert space of the unconstrained system, then \(K_E(p'', q''; p', q') \equiv \langle p'', q'' | E | p', q' \rangle\) denotes the reproducing kernel for the (provisional) physical Hilbert space appropriate to the constrained system. To illustrate a reduction of such expressions, we set \(J = 1\) and focus on the example

\[
K_E(p'', q''; p', q') = \langle p'', q'' | E (P^2 \leq \delta^2) | p', q' \rangle,
\]

As \(\delta \to 0\) this expression vanishes, but if we first divide by \(\delta\) before taking the limit, we can generate a positive-definite function which, if continuous, characterizes a new Hilbert space, the true \(H_{\text{phys}}\). In particular, let us assume that \(\eta(k)\) is a continuous function, multiply by \(1/2\), and take the limit \(\delta \to 0\), leading to the result

\[
e^{-i\frac{1}{2}(p''q'' - p'q')} \int e^{ik(q'' - q')} \eta(k - p') dk.
\]

(11)

In more abstract terms, and in cases where the dependence of \(K_E\) on \(\delta\) is less clear, we can proceed as follows. Let

\[
W \equiv \limsup_{(p,q) \in \mathbb{R}^2} \langle p, q | E | p, q \rangle,
\]

for which, provided \(E \neq 0\), \(W > 0\). To show that \(W\) is positive, we observe that

\[
0 \leq |\langle p, q | E | r, s \rangle|^2 \leq \langle p, q | E | p, q \rangle \langle r, s | E | r, s \rangle \leq W^2.
\]

(13)

If \(W = 0\), then it would follow that \(\langle p, q | E | r, s \rangle = 0\) for all arguments, which can only hold if \(E = 0\), contrary to our assumption. Armed with \(W\) we next define

\[
K_W(p'', q''; p', q') \equiv W^{-1} \langle p'', q'' | E | p', q' \rangle.
\]

(14)

Note that |\(K_W| \leq 1\). We first observe that \(K_W\) corresponds to a new (simply rescaled) reproducing kernel for which every element of the associated Hilbert space is already a member of the space determined by \(K_E\). To reduce this expression we simply take the limit \(\delta \to 0\), namely,

\[
K_R(p'', q''; p', q') \equiv \lim_{\delta \to 0} K_W(p'', q''; p', q').
\]

(15)
If the result of this δ-limiting procedure exists and is continuous, then the result is a reproducing kernel for the ultimate physical Hilbert space. As we have already seen, the dimensionality of the Hilbert space can change dramatically in this limit and, moreover, some of the variables may no longer be relevant. Such a procedure may also change the measure (if any) by which the inner product in the new space may be evaluated.

We can combine constraints with a nonvanishing Hamiltonian by the observation that

\[
\langle p'', q'' | e^{-i(EH\delta_\lambda)T} E | p', q' \rangle = \lim_{\epsilon \to 0} \langle p'', q'' | e^{-i\mathcal{H}\epsilon E} e^{-i\mathcal{H}\epsilon E} \cdots E e^{-i\mathcal{H}\epsilon E} | p', q' \rangle = \mathcal{M} \int e^{i \int \left[ \frac{1}{2} (p \cdot \dot{q} - q \cdot \dot{p}) - H(p, q) - \lambda^\alpha \phi_\alpha(p, q) \right] dt} \mathcal{D}p \mathcal{D}q \mathcal{D}E(\lambda),
\]

where \( \phi_\alpha(p, q) = \langle p, q | \Phi_\alpha(P, Q) | p, q \rangle \), and \( E(\lambda) \) is a measure, based on \( R(\lambda) \), that is designed to introduce the projection operator \( E \) at every time slice. When \( E \mathcal{H} = \mathcal{H} E \), then a significant simplification occurs. In that case we may make use of the relation

\[
E e^{-i(EH\delta_\lambda)T} E = e^{-i\mathcal{H}T} E
\]

which holds as an identity. Thus, in this case, it is only necessary to put \textit{one} projection operator \( E \) inside the matrix elements to achieve the same result. Although it is possible to use \( E(\lambda) \) in this latter case as well, it may be easier to use a measure \( C(\lambda) \) designed to insert \textit{one} projection operator \( E \). In (16), observe how the evolution operator in \( \mathcal{H}_{\text{phys}} \), namely \( \exp[-i(EH\delta_\lambda)T] \), is evaluated in terms of matrix elements of vectors in the physical Hilbert space, namely \( E | p, q \rangle \). Such an expression is fully consistent with the constraints. For instance, in the case of closed first-class constraints, the propagator within the physical Hilbert space (16) is \textit{manifestly gauge invariant}, provided one has also used a δ-limiting procedure if necessary.

As (16) shows, the propagator within the physical Hilbert space is obtained by means of a formal path integral (with a meaningful lattice formulation and lattice limit) involving just the original dynamical variables and the Lagrange multipliers. No other variables are needed. How this quantization procedure for constrained systems relates to other, better known procedures is briefly discussed elsewhere [3].

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2.2 Finitely many basic systems

In this subsection we take up the quantization of a classical system described by (2) based on our discussion of the quantization of (1). Due to the independence of the separate basic systems, this extension is straightforward. Let us extend our notation so that now \( p = \{p_n\}_{n=1}^N \) and \( p_n = \{p_n^j\}_{j=1}^J \), etc., and therefore as a consequence

\[
\mathcal{K}_N(p''', q'''; p', q') \equiv \prod_{n=1}^N \mathcal{K}(p''_n, q''_n; p'_n, q'_n) \tag{18}
\]
denotes the reproducing kernel for the \( NJ \) degree of freedom system. Since the separate reproducing kernels in (18) do not depend on \( n \), it is evident that \( \mathcal{K}_N \) is invariant under the interchange of variables for any pair of independent basic systems just as is the case for the classical theory.

Dynamics (without constraints) takes the form

\[
\langle p''', q''' | e^{-i\mathcal{H}_N} T | p', q' \rangle = \prod_{n=1}^N \langle p''_n, q'''_n | e^{-i\mathcal{H}_n} T | p'_n, q'_n \rangle, \tag{19}
\]
while the imposition of constraints (without dynamics) leads to

\[
\langle p''', q''' | \mathcal{E}_N | p', q' \rangle = \prod_{n=1}^N \langle p''_n, q'''_n | \mathcal{E}_n | p'_n, q'_n \rangle. \tag{20}
\]
Reduction of such reproducing kernels follows the pattern described in the previous subsection. Finally, combining dynamics and constraints generally leads to

\[
\langle p''', q''' | e^{-i(\mathcal{E}_N \mathcal{H}_N \mathcal{E}_N)T} \mathcal{E}_N | p', q' \rangle = \prod_{n=1}^N \langle p''_n, q'''_n | e^{-i(\mathcal{E}_n \mathcal{H}_n \mathcal{E}_n)T} \mathcal{E}_n | p'_n, q'_n \rangle, \tag{21}
\]
or, in the special case that \( \mathcal{E}_n \mathcal{H}_n = \mathcal{H}_n \mathcal{E}_n \), to

\[
\langle p''', q''' | e^{-i\mathcal{H}_N} T \mathcal{E}_N | p', q' \rangle = \prod_{n=1}^N \langle p''_n, q'''_n | e^{-i\mathcal{H}_n} T \mathcal{E}_n | p'_n, q'_n \rangle. \tag{22}
\]
All these expressions exhibit the interchange symmetry inherit in the classical system.

2.3 Infinitely many basic systems

Due to the elementary structure of product representations, the analysis of infinitely many independent basic systems largely involves only a study of
the limit $N \to \infty$ in several formulas of the preceding section. Naturally, convergence of such limits will be a critical issue.

First, just for kinematics, with neither dynamics nor constraints, we require that

$$\langle p''', q''' | p', q' \rangle = \prod_{n=1}^{\infty} \langle p''_n, q''_n | p'_n, q'_n \rangle,$$

and convergence of the right-hand side dictates what elements $p = \{p_n\}_{n=1}^{\infty}$ and $q = \{q_n\}_{n=1}^{\infty}$ may enter on the left-hand side. The value of zero for this product may be arrived at in two different ways: (i) either one (or more) of the factors vanishes, or (ii) every factor is nonzero, but the infinite product leads to zero. This latter situation is called "divergence to zero", and in discussions regarding this subject it is not considered convergence. To have convergence, and to exclude divergence to zero, we need that

$$\sum_{n=1}^{\infty} |1 - \langle p''_n, q''_n | p'_n, q'_n \rangle| < \infty.$$  

Since each coherent state is a unit vector, convergence only occurs provided that $q''_n - q'_n \to 0$ and $p''_n - p'_n \to 0$. To preserve interchange symmetry, we require, in turn, that $q''_n \to q$, $q'_n \to q$ and $p''_n \to \overline{p}$, $p'_n \to \overline{p}$, where $(\overline{p}, \overline{q}) \in \mathbb{R}^{2J}$ is arbitrary. Observe that the variables $(\overline{p}, \overline{q})$ which label the asymptotic dependence actually label orthogonal Hilbert spaces. This statement holds because if $(\overline{p}, \overline{q}) \neq (\overline{r}, \overline{s})$, then $|\langle \overline{p}, \overline{q} | \overline{r}, \overline{s} \rangle| < 1$ and the infinite power of this factor yields zero; this is an example of divergence to zero. No change of a finite number of labels in either the bra or the ket, nor finite linear superpositions and arbitrary Cauchy sequences thereafter, can ever change the vanishing result. We deal here with an uncountable number of disjoint reproducing kernel Hilbert spaces (save for the zero element). (Stated alternatively, if one were to realize the underlying field operators in a common Hilbert space, then $(\overline{p}, \overline{q})$ would label unitarily inequivalent irreducible representations.) At this stage, there is no distinguished property that would help us choose which $(\overline{p}, \overline{q})$ set or which fiducial vector $|\eta\rangle$ is correct. In fact, that is as it should be since we have not specified any particular dynamics. In summary, labels for the coherent states are given by the set

$$T(\overline{p}, \overline{q}) \equiv \{(p_n, q_n) : p_n \to \overline{p}, q_n \to \overline{q}\}$$

where convergence means that

$$\lim_{N \to \infty} \prod_{n=N}^{\infty} \langle p_n, q_n | \overline{p}, \overline{q} \rangle = 1.$$  


or equivalently
\[
\lim_{N \to \infty} \sum_{n=N}^{\infty} |1 - \langle p_n, q_n | \vec{p}, \vec{q} \rangle| = 0. 
\] (26)

This criterion applies for any choice of \( |\eta\rangle \), and leads to acceptable (possibly \( |\eta\rangle \)-dependent) momentum and coordinate variable sets \( (p, q) = \{p_n, q_n\}_{n=1}^{\infty} \).

If one adds a modest domain requirement on the fiducial vector, such as \( \langle \eta | (P^2 + Q^2) | \eta \rangle < \infty \), then the convergence criterion in (26) is equivalent to
\[
\sum_{n=1}^{\infty} \left[ \sum_{j=1}^{\infty} \left( |p_j^n - \vec{p}| + |q_j^n - \vec{q}| \right) \right] < \infty , 
\] (27)
a relation that captures the allowed sequences \( \{p_n, q_n\}_{n=1}^{\infty} \) for a wide class of examples.

Next, let us consider the case of dynamics without constraints. Thus we initially study
\[
\langle p'', q'' | e^{-iH(\infty)T} | p', q' \rangle = \Pi_{n=1}^{\infty} \langle p''_n, q''_n | e^{-iHT} | p'_n, q'_n \rangle . 
\] (28)

In the case of dynamics, proper convergence of (28) requires, for acceptable sets \( (p, q) \in T(\vec{p}, \vec{q}) \), for some \( (\vec{p}, \vec{q}) \), that
\[
\sum_{n=1}^{\infty} |1 - \langle p''_n, q''_n | e^{-iHT} | p'_n, q'_n \rangle| < \infty . 
\] (29)

Without loss of generality, we shall assume that \( (\vec{p}, \vec{q}) = (0, 0) \). In that case, the criterion (29) requires that \( H|\eta\rangle = 0 \). If \( H \) has a (partially) discrete spectrum, then \( |\eta\rangle \) may be taken as an eigenvector whose eigenvalue has been adjusted to vanish. If \( H \) has only a continuous spectrum, then it is not possible to satisfy (29) as it stands unless we allow for \( n \)-dependent fiducial vectors, a modification that would destroy interchange symmetry. Since we wish to preserve interchange symmetry, we must confine attention to \( |\eta\rangle \) being a fixed, normalized eigenvector of \( H \). Observe that this condition links kinematics and dynamics, a condition generally regarded as a hallmark of infinitely many degrees of freedom (c.f., Haag’s Theorem [7]). In other words, in order for \( |\eta\rangle = \Pi_{n=1}^{\infty} (|\eta\rangle) \) to be a unit vector in the full Hilbert space in which the Hamiltonian \( H(\infty) \) is a well-defined (and self-adjoint) operator requires that \( H|\eta\rangle = 0 \). Hence, far from being chosen arbitrarily, \( |\eta\rangle \) is now determined to be an eigenvector of \( H \); when \( H \geq 0 \), we may even choose \( |\eta\rangle \) to be one of the ground states. If, as is often the case, the
ground state is unique, then $|\eta\rangle$ is fixed. Thus we see that the introduction of dynamics has effectively selected the fiducial vector $|\eta\rangle$, as well as the parameters $(\bar{p}, \bar{q}) = (0, 0)$, in order that $\mathcal{H}(\infty)$ is a self-adjoint operator. For convenience in what follows, we generally restrict attention to Hamiltonian operators $\mathcal{H}$ with a purely discrete spectrum.

Next, we consider constraints but no dynamics, a situation which—for the moment—restores general values of $(\bar{p}, \bar{q})$ and general $|\eta\rangle$ to consideration. As a first approach to this problem, consider

$$\langle p''', q''' | \mathcal{E}(\infty) | p', q' \rangle = \Pi_{n=1}^\infty \langle p''_n, q''_n | \mathcal{E} | p'_n, q'_n \rangle,$$  

(30)

where each argument set is a member of $T(p, q)$. By Schwarz’s inequality the right-hand side of (30) is a product of factors each of which is at most unity in magnitude. Therefore, as it stands, in order for this product to converge (and not diverge to zero), it is necessary, for some $(p, q)$ and $|\eta\rangle$, that $\mathcal{E} | p, q \rangle = | p, q \rangle$, namely that the vector $| p, q \rangle$ already belongs entirely to the physical Hilbert space. If this condition is fulfilled, then (30) defines a valid reproducing kernel on the physical Hilbert space for infinitely many degrees of freedom. On the other hand, this condition is a very strong restriction. We shall next see how we can significantly relax this requirement.

Suppose, as dictated by the future dynamics, that $(p, q) = (0, 0)$ and that $|\eta\rangle$ satisfies $\mathcal{E} | \eta \rangle \neq | \eta \rangle$. Several situations are then possible. If $\mathcal{E} | \eta \rangle = 0$, then the vector $| \eta \rangle$ lies entirely in the unphysical Hilbert space and $S = \langle \eta | \mathcal{E} | \eta \rangle = 0$. This property simply means that the chosen eigenvector of $\mathcal{H}$ is incompatible with $\mathcal{E}$. We cannot change $\mathcal{H}$ or $\mathcal{E}$, but we can change the fiducial vector. Hence, we introduce a new and distinct fiducial vector $|\bar{\eta}\rangle$ such that $\mathcal{E} | \bar{\eta} \rangle \neq 0$ and thus $\mathcal{S} = \langle \bar{\eta} | \mathcal{E} | \bar{\eta} \rangle > 0$. It may even be appropriate to choose $|\bar{\eta}\rangle = | p, q \rangle$ for some $(p, q) = (0, 0)$. Further conditions on $|\bar{\eta}\rangle$ will appear below.

Armed with $\mathcal{S}$ we introduce the rescaled reproducing kernel

$$\mathcal{K}_R(p'', q''; p', q') = \Pi_{n=1}^\infty [ \mathcal{S}^{-1} \langle p''_n, q''_n | \mathcal{E} | p'_n, q'_n \rangle ],$$  

(31)

where in this expression $|p, q\rangle$ is defined as in (6) with $|\bar{\eta}\rangle$ used in place of $|\eta\rangle$, and also such that $(p_n, q_n) \rightarrow (0, 0)$. In this language, $\mathcal{S} = 1$ corresponds to the case of (30) where $\mathcal{E} | \bar{\eta} \rangle = | \bar{\eta} \rangle$.

Finally, we turn to the case of dynamics plus constraints which will lead to additional conditions on $|\bar{\eta}\rangle$. The putative propagator reads

$$K(p'', q''; p', q', 0) = \Pi_{n=1}^\infty [ \mathcal{S}^{-1} \langle p''_n, q''_n | \mathcal{E} e^{-i(E \mathcal{E} T)} \mathcal{E} | p'_n, q'_n \rangle ],$$  

(32)
where \( \mathcal{H} = \mathcal{H} - \mathcal{E} \), with \( \mathcal{E} \) to be fixed. In order for this product to converge it is necessary that

\[
\mathcal{E} e^{-i(\mathcal{H} - \mathcal{E}) T} \mathcal{E} |\eta\rangle = \mathcal{E} |\eta\rangle .
\]  

(33)

Assume that \( \mathcal{H} \mathcal{E} \equiv \mathcal{E} \mathcal{H} \) is self adjoint, with a discrete spectrum, and let \( |\xi_l\rangle \), \( l = 0, 1, 2, \ldots \), be a complete orthonormal set of eigenvectors that satisfy \( \mathcal{H} \mathcal{E} |\xi_l\rangle = \mathcal{E} \mathcal{H} \mathcal{E} |\xi_l\rangle = \sigma_l |\xi_l\rangle \), with \( \sigma_0 \leq \sigma_1 \leq \sigma_2 \cdots \). Choose \( |\eta\rangle \) such that \( \mathcal{E} |\eta\rangle = c |\xi_p\rangle \), \( c \neq 0 \), for the least \( p \) value, and then choose \( \mathcal{E} \) so that \( \sigma_p = 0 \). If \( \mathcal{E} \neq 0 \), then this last condition has involved an infinite renormalization of the energy. With all conditions satisfied if follows that (33) holds and (32) determines the dynamics and constraints together.

The situation is simpler if \( \mathcal{H} \mathcal{E} = \mathcal{E} \mathcal{H} \), and in that case it is sufficient to consider

\[
K(p'', q'', T; p', q', 0) = \prod_{n=1}^{\infty} \left[ S^{-1} \langle p''_n, q''_n | e^{-i\mathcal{H}T} \mathcal{E} | p'_n, q'_n \rangle \right] ,
\]  

(34)

where each \( (p_n, q_n) \to (0, 0) \), \( S = \langle \eta | \mathcal{E} | \eta \rangle > 0 \), and \( \mathcal{H} = \mathcal{H} - \mathcal{E} \). Convergence of this expression requires, along with \( \mathcal{E} |\eta\rangle \neq 0 \), that

\[
\mathcal{E} |\eta\rangle = e^{-i\mathcal{H}T} \mathcal{E} |\eta\rangle = \mathcal{E} e^{-i\mathcal{H}T} |\eta\rangle ,
\]  

(35)

which implies that \( \mathcal{H} |\eta\rangle = 0 \). Let \( \mathcal{H} |\xi_l\rangle = \mu_l |\xi_l\rangle \), \( l = 0, 1, 2, \ldots \), \( \mu_0 \leq \mu_1 \leq \mu_2 \cdots \), and set \( |\eta\rangle = |\zeta_r\rangle \) for the least \( r \) such that \( \langle \zeta_r | \mathcal{E} | \zeta_r \rangle > 0 \), adjusting \( \mathcal{E} \) to ensure that \( \mu_r = 0 \). In this case the product in (34) converges to an acceptable propagator.

### 3 Examples

We illustrate some of the concepts in the previous sections with several examples. In order to do so we shall give the classical action for the basic system and then present the propagator on the physical Hilbert space in the form of a suitable coherent-state functional. In our examples we shall exclusively use the harmonic oscillator ground state (for unit angular frequency) as the fiducial vector. Thus we deal with the canonical coherent states in the so-called holomorphic representation; for notation consult, e.g., [2]. For clarity, we only present simple and explicitly soluble examples.
Example 1. Choose $J = A = 1$ and the classical action
\[
I = \int \left[ \frac{1}{2} (p\dot{q} - q\dot{p}) - \lambda (p^2 + q^2) \right] dt , \tag{36}
\]
which has a vanishing Hamiltonian. In this case
\[
E = \mathbb{E} \left( (P^2 + Q^2) \leq \hbar^2 \right) = \mathbb{E} \left( (P^2 + Q^2) \leq \hbar \right) = |0\rangle \langle 0| , \tag{37}
\]
namely the projection operator onto the harmonic oscillator ground state. In terms of the complex variable $z \equiv (q + ip)/\sqrt{2}$ and coherent states $|z\rangle \equiv |p, q\rangle$ and $\langle z| \equiv \langle p, q|$, we determine that
\[
\langle z'|z''\rangle = \exp \left( -\frac{1}{2} |z''|^2 + z''^* z' - \frac{1}{2} |z'|^2 \right) , \tag{38}
\]
and that
\[
\langle z''|E|z'\rangle = \exp \left( -\frac{1}{2} (|z''|^2 + |z'|^2) \right) . \tag{39}
\]
Observe that $\langle 0|E|0\rangle = 1$ so that the propagator for an infinite product system satisfying the constraints is given by
\[
K(z'', T; z', 0) = \Pi_{n=1}^{\infty} e^{-\frac{1}{2} \left( |z''|^2 + |z'|^2 \right)} . \tag{40}
\]
Convergence of this expression requires for each argument that $\Sigma_{n=1}^{\infty} |z_n|^2 < \infty$. Observe that the physical Hilbert space is one dimensional for all $N$, $1 \leq N \leq \infty$.

Example 2. Let $J = 1$, $A = 2$ and choose
\[
I = \int \left[ \frac{1}{2} (p\dot{q} - q\dot{p}) - \frac{1}{2} (p^2 + q^2) - \lambda_1 p - \lambda_2 q \right] dt . \tag{41}
\]
Here
\[
E = \mathbb{E} \left( P^2 + Q^2 \leq \hbar \right) = |0\rangle \langle 0| \tag{42}
\]
again. With $\mathcal{H} = \frac{1}{2} (P^2 + Q^2)$, and after adjusting for the zero-point energy, the solution is identical to Example 1 and given by (40).

Example 3. Again $J = A = 1$, and consider
\[
\int \left[ \frac{1}{2} (p\dot{q} - q\dot{p}) - \frac{1}{2} (p^2 + q^2) - \lambda (p^2 + q^2 - 2) \right] dt . \tag{43}
\]
This expression fails to satisfy the conditions following (4), so we must already make an energy renormalization and instead choose

$$I = \int \left[ \frac{1}{2} (p \dot{q} - q \dot{p}) - \frac{1}{2} (p^2 + q^2 - 2) - \lambda (p^2 + q^2 - 2) \right] dt ,$$

where, e.g., $\langle p_n, q_n \rangle \to (1, 1)$ classically, i.e., $z_n \to \sqrt{i}$. In this case,

$$E = E (P^2 + Q^2 = 3\hbar)$$

$$= E (P^2 + Q^2 \leq 3\hbar) - E (P^2 + Q^2 \leq \hbar)$$

$$= |1 \rangle \langle 1| ,$$

where $|1 \rangle$ denotes the first excited state of the harmonic oscillator. It follows, therefore, that

$$\langle z'' | z' \rangle = \exp \left( -\frac{1}{2} |z''|^2 + z''^{*} z' - \frac{1}{2} |z'|^2 \right) ,$$

$$\langle z'' | E | z' \rangle = \exp \left[ -\frac{1}{2} (|z''|^2 + |z'|^2) \right] z''^{*} z' .$$

In this case the chosen fiducial vector—the harmonic oscillator ground state—is incompatible with the constraint condition, namely $E |0 \rangle = 0$. Thus we need to change the fiducial vector, and for that purpose we choose $|\eta\rangle = |z = \sqrt{i} \rangle \equiv \sqrt{i}$. Next, we set $\mathcal{H} = \mathcal{H} - E = \mathcal{H} - 1$, then $\mathcal{H}|1 \rangle = 0$ and $E |1 \rangle = |1 \rangle$. Since $S = \langle \sqrt{i} | | \sqrt{i} \rangle = 1/e$, the propagator coupled with the constraints, following (34), is given by

$$K (z'', T; z', 0) = \Pi_{n=1}^{\infty} \exp \left[ -\frac{1}{2} (|z''|^2 + |z'|^2 - 2) \right] z''^{*} z' ,$$

an expression which describes a valid propagator on the one-dimensional physical Hilbert space. In the present case convergence means that $\sum_{n=1}^{\infty} |z_n - \sqrt{i}| < \infty$.

Example 4. Here $J = A = 3$ and

$$I = \int \left[ \frac{1}{2} (p \dot{q} - q \dot{p}) - \frac{1}{2} (p^2 + q^2) - \lambda (p \wedge q) \right] dt .$$

In this case the three constraints are the angular momentum generators and $E = E (\sum_{j=1}^{3} J_j^2 = 0)$, i.e., a projection operator onto the spherically symmetric subspace. For convenience, let us introduce the notation $z = (z^1, z^2, z^3)$ for the three components, and for two such vectors let $z \cdot w \equiv \sum_{j=1}^{3} z^j w^j$ and
\[(z)^2 \equiv \sum_{j=1}^{3} (z^j)^2\], etc. Then the reproducing kernel for the full Hilbert space is given by
\[
\langle z'' | z' \rangle = \exp[\sum_{n=1}^{\infty} \left(-\frac{1}{2} |z''_n|^2 + z''_n \cdot z'_n - \frac{1}{2} |z'_n|^2\right)] . \tag{49}
\]
After a modest computation the reproducing kernel for the physical Hilbert space is given by
\[
\langle z'' | \mathcal{E} | z' \rangle = e^{-\frac{1}{2} \sum_{n=1}^{\infty} \left(|z''_n|^2 + |z'_n|^2\right)} \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{[(z''_n)^2(z'_n)^2]^m}{(2m + 1)!} . \tag{50}
\]
Inclusion of the Hamiltonian follows simply be the change \(z'_n \rightarrow e^{-iT} z'_n\), and leads to
\[
\langle z'' | e^{-iHT} \mathcal{E} | z' \rangle = e^{-\frac{1}{2} \sum_{n=1}^{\infty} \left(|z''_n|^2 + |z'_n|^2\right)} \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{[(z''_n)^2(z'_n)^2]^m}{(2m + 1)!} e^{-i2mT} . \tag{51}
\]
We observe that with the constraints in force, the energy spectrum is \(E_m = 2m\) rather than \(E_m = m\) which applies to the unconstrained oscillators. Convergence of this expression requires that each sequence \(\{z_n\}\) satisfy
\[
\sum_{n=1}^{\infty} |z_n|^2 < \infty . \tag{52}
\]

\section*{Dedication}

It is a pleasure to dedicate this article to the 65th birthday of Ludwig Faddeev. His contributions, in a wide range of scientific fields, already place him in the Pantheon of Truly Great Scientists. May he continue to enlighten us all for many years to come.

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