Microscopic theory of refractive index applied to metamaterials:
Effective current response tensor corresponding to standard relation \( n^2 = \varepsilon_{\text{eff}} \mu_{\text{eff}} \)

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In this article, we first derive the wavevector- and frequency-dependent, microscopic current response tensor which corresponds to the “macroscopic” ansatz \( D = \varepsilon_0 \varepsilon_{\text{eff}} E \) and \( B = \mu_0 \mu_{\text{eff}} H \) with wavevector- and frequency-independent, “effective” material constants \( \varepsilon_{\text{eff}} \) and \( \mu_{\text{eff}} \). We then deduce the electromagnetic and optical properties of this effective material model by employing exact, microscopic response relations. In particular, we argue that for recovering the standard relation \( n^2 = \varepsilon_{\text{eff}} \mu_{\text{eff}} \) between the refractive index and the effective material constants, it is imperative to start from the microscopic wave equation in terms of the transverse dielectric function, \( \varepsilon_T(k, \omega) = 0 \).

On the phenomenological side, our result is especially relevant for metamaterials research, which draws directly on the standard relation for the refractive index in terms of effective material constants. Since for a wide class of materials the current response tensor can be calculated from first principles and compared to the model expression derived here, this work also paves the way for a systematic search for new metamaterials.

INTRODUCTION

The traditional approach to electrodynamics in media is based on the division of electric charge and current densities into “free” and “bound” contributions, combined with the so-called “macroscopic” Maxwell equations, which are usually written in the form

\[
\nabla \cdot D = \rho, \quad \nabla \times E = -\partial_t B, \quad \nabla \cdot B = 0, \quad \nabla \times H = j + \partial_t D.
\]

These equations have to be complemented by the so-called “constitutive laws”, which are—more often than not—assumed to be simple linear relations between \( D \) and \( E \) as well as \( H \) and \( B \), i.e.,

\[
D = \varepsilon_0 \varepsilon_r E, \quad H = \mu_0^{-1} \mu_r^{-1} B,
\]

where \( \varepsilon_r \) and \( \mu_r \) are the relative permittivity and permeability, respectively. To lighten the notation, we will in the following suppress the subscript \( r \) and simply write \( \varepsilon \equiv \varepsilon_r \) and \( \mu \equiv \mu_r \) for these dimensionless quantities. (They should, however, not be confused with the corresponding absolute quantities given by \( \varepsilon_0 \varepsilon_r \) and \( \mu_0 \mu_r \).) In particular, the fundamental field equations determine the involved field quantities \( \{D, H, E, B\} \) only once the constitutive laws are specified. Without these the fields would remain underdetermined. The constitutive laws on their side are usually assumed to be given empirically. In the simplest case, one thinks of them as being formulated in terms of effective material constants, which are measurable in principle. In general, though, the constitutive laws could be much more complicated (involving retardation effects, non-linearities, etc.). Consequently, the field equation for, say, the divergence of the “magnetic field” \( H \) is material dependent in the Standard Approach described above. In principle, this divergence has to be determined by plugging the relation \( B = B(H) \) into the field equation.

However, along with the advent of ab initio materials physics, a new, microscopic approach to electrodynamics in media has been developed. Its basis is the division of both the electromagnetic source terms (i.e., charge and current densities) and the electromagnetic fields into their respective external and induced contributions, where the term “induced” means “generated under the influence of externally applied fields”. For convenience, one also considers “total” fields, which are defined as the sum of external and induced contributions. In this microscopic approach, all electric and magnetic fields are uniquely determined by the microscopic Maxwell equations

\[
\nabla \cdot E(x, t) = \rho(x, t)/\varepsilon_0, \quad \nabla \times E(x, t) = -\partial_t B(x, t), \quad \nabla \cdot B(x, t) = 0, \quad \nabla \times B(x, t) = \mu_0 j(x, t) + \varepsilon_0 \mu_0 \partial_t E(x, t).
\]
of an externally applied field via a corresponding response function. In principle, these microscopic response functions can be calculated from first-principles (many-body Schrödinger equation combined with Kubo formalism). Hence, they do not constitute freely adjustable material parameters. Instead, they can be predicted theoretically as well.

An important quantity in the *ab initio* context is given, for example, by the density response function \( \chi \), which is implicitly defined by

\[
\rho_{\text{ind}}(x, t) = \int d^3 x' \int c \, dt' \chi(x, x'; t - t') \varphi_{\text{ext}}(x', t'),
\]

where \( \varphi_{\text{ext}} \) denotes the externally applied scalar potential. In particular, in the microscopic approach, response functions are in general given in terms of non-local (possibly tensorial) integral kernels. Only for homogeneous systems (which, admittedly, constitute the most important practical application in theoretical materials science), the response functions depend only on the coordinate difference, such that the response laws have a purely multiplicative structure in Fourier space, i.e.,

\[
\rho_{\text{ind}}(k, \omega) = \chi(k, \omega) \varphi_{\text{ext}}(k, \omega).
\]

Once the microscopic density response function is given, a more traditional material property like the (relative) dielectric function can be calculated by means of the standard relation \( \varepsilon \equiv \frac{\chi(k, \omega)}{\varepsilon_0 |k|^2} \).

In principle, this quantity can be identified with the permittivity used in the Standard Approach (see Eq. 21). This has to be shown on the basis of the Fundamental Field Identifications given by

\[
D(x, t) = \varepsilon_0 E_{\text{ext}}(x, t),
\]

\[
P(x, t) = -\varepsilon_0 E_{\text{ind}}(x, t),
\]

\[
E(x, t) = E_{\text{tot}}(x, t),
\]

and by

\[
\mu_0 H(x, t) = B_{\text{ext}}(x, t),
\]

\[
\mu_0 M(x, t) = B_{\text{ind}}(x, t),
\]

\[
B(x, t) = B_{\text{tot}}(x, t).
\]

As a matter of principle, these identifications relate the microscopic fields used in *ab initio* materials science to their macroscopic counterparts used in the Standard Approach.

However, the Fundamental Field Identifications lead to the following problem which does not exist in the Standard Approach: if all electromagnetic fields (external, induced and total) are already completely determined by means of their respective Maxwell equations, while on the other hand the induced fields are also determined in terms of the external fields via the corresponding “direct” response functions (or in terms of the total fields via the “proper” response functions [24, § 2.3]), then apparently there exists an overdetermination in the theory which could in principle lead to contradictions. For example, in the traditional approach the expression \( \nabla \cdot H \) simply remains undetermined, while in the microscopic approach we necessarily obtain \( \nabla \cdot H = 0 \) on the basis of the Fundamental Field Identifications, although at the same time we have \( B = \mu H \) (within the limits of linear response theory, where \( \mu \) in general denotes a tensorial integral kernel). The resolution of this apparent paradox lies in another surprising feature of the microscopic approach, which distinguishes it sharply from its traditional macroscopic counterpart: the microscopic response functions cannot be prescribed arbitrarily. Instead, they are subject to constraints which guarantee the validity of the microscopic Maxwell equations. In particular, this also implies that the response functions are in general not independent of each other, but interrelated by the Universal Response Relations [22].

Conceptually, however, the somewhat shocking implication of this microscopic approach is that the standard formula for the refractive index in terms of the relative permittivity and permeability,

\[
n^2(\omega) = \frac{\varepsilon(\omega)}{\mu(\omega)},
\]

cannot be true in this form [21, 24, 25], *i.e.*, as a formula expressing the refractive index in terms of response functions. Instead, its allegedly approximate version,

\[
n^2(\omega) = \varepsilon(\omega),
\]

turns out to be the more correct formula, which can be justified microscopically [24]. Here, it is understood that the involved frequency-dependent quantities refer to the macroscopic limit \( (k \to 0) \) of microscopic response functions, which can be calculated from first principles.

Fortunately, in most cases the failure of the standard relation (20) does not pose any serious problems [26]. In fact, textbooks in condensed matter theory often even define [10, 21, 27] the refractive index by the allegedly approximate relation (21). Furthermore, a bulk material where the standard formula (20) would apply with independently obtained material parameters \( \varepsilon, \mu \) and \( n \) is not known. In the research domain of so-called metamaterials, however, one draws directly on the original Maxwell relation (20) if only with “effective” (i.e., not
calculated from first principles) material parameters (not response functions) \( \varepsilon_{\text{eff}}(\omega) \) and \( \mu_{\text{eff}}(\omega) \). Concretely, it has been argued by V. Veselago that \( n \) should be regarded as a negative number if both \( \varepsilon_{\text{eff}} < 0 \) and \( \mu_{\text{eff}} < 0 \). Such a negative effective permeability can occur in artificial materials by exploiting the concept of a splitting resonator. An anomalous light refraction at metamaterials has been observed experimentally. Therefore, metamaterials are regarded as promising candidates for technological applications such as superlenses and invisibility cloaks. For the longest time, this line of research has been pursued in complete independence from first-principles materials science.

However, in their recent groundbreaking work, D. Forcella et al. have—apparently for the first time—approached metamaterials theory from the \( ab \ initio \) framework of microscopic electrodynamics in media under suitable assumptions which then, admittedly, would have to be more specific than the general approach by D. Forcella et al.

With this state of affairs, we actually face two fundamental questions:

1. Although the refractive index is microscopically not determined by the standard formula, it is still possible to have a material whose microscopic response functions involve two (constant) material parameters, which have an interpretation as “effective” electric permittivity and permeability, such that the standard formula instead holds in terms of these “effective” material constants?

2. More generally, to which current response tensor does the simple macroscopic ansatz correspond, which is used in the traditional approach for the derivation of the standard formula?

To answer these questions is precisely the aim of the present article.

PHENOMENOLOGICAL DERIVATION

We start from the “macroscopic” Maxwell equations written in Fourier space as

\[
\begin{align*}
\mathbf{k} \cdot \mathbf{B}(k, \omega) &= 0, \\
\mathbf{k} \times \mathbf{E}(k, \omega) - \omega \mathbf{B}(k, \omega) &= 0, \\
\mathbf{i}k \cdot \mathbf{D}(k, \omega) &= \rho_{f}(k, \omega), \\
\mathbf{i}k \times \mathbf{H}(k, \omega) + \omega \mathbf{D}(k, \omega) &= \mathbf{j}_{f}(k, \omega).
\end{align*}
\]

By means of the first two, homogeneous equations, we can introduce the potentials

\[
\begin{align*}
\mathbf{B}(k, \omega) &= \mathbf{i}k \times \mathbf{A}(k, \omega), \\
\mathbf{E}(k, \omega) &= -\mathbf{i}k \varphi(k, \omega) + \omega \mathbf{A}(k, \omega).
\end{align*}
\]

Furthermore, in the last two, inhomogeneous Maxwell equations, we substitute

\[
\begin{align*}
\mathbf{D}(k, \omega) &= \varepsilon_{0} \varepsilon_{\text{eff}} \mathbf{E}(k, \omega), \\
\mathbf{H}(k, \omega) &= \mu_{0}^{-1} \mu_{\text{eff}}^{-1} \mathbf{B}(k, \omega),
\end{align*}
\]

with the effective permittivity and permeability, \( \varepsilon_{\text{eff}} \) and \( \mu_{\text{eff}} \), which are assumed to be (wavevector- and frequency-independent) constants. We note in passing that in principle, these material constants could \textit{ex post} be promoted to frequency-dependent functions, \( \varepsilon_{\text{eff}}(\omega) \) and \( \mu_{\text{eff}}(\omega) \), in order to include, for instance, the effects of dissipation without modifying the central results of the present article. In any case, we thus obtain the inhomogeneous equations for the potentials (suppressing \( k \) and \( \omega \) dependencies in the notation),

\[
\varepsilon_{0} \varepsilon_{\text{eff}} (|k|^2 \varphi - \omega k \cdot \mathbf{A}) = \rho_{f},
\]

and

\[
\frac{1}{\mu_{0} \mu_{\text{eff}}} (|k|^2 \mathbf{A} - k(k \cdot \mathbf{A})) + \varepsilon_{0} \varepsilon_{\text{eff}} (\omega k \varphi - \omega^2 \mathbf{A}) = \mathbf{j}_{f}.
\]

In matrix form, these equations can be rewritten in terms of Lorentz four-vectors as

\[
\begin{align*}
\mu_{0} \begin{pmatrix} \varepsilon \rho_{f} \end{pmatrix} &= \varepsilon_{\text{eff}} \begin{pmatrix} |k|^2 - \frac{\omega}{c} k^{T} \end{pmatrix} \begin{pmatrix} \varphi/c \end{pmatrix} \begin{pmatrix} \mathbf{A} \end{pmatrix} \\
&+ \frac{1}{\mu_{\text{eff}}} \begin{pmatrix} 0 & 0 \\
0 & |k|^2 - \omega k^{T} \end{pmatrix} \begin{pmatrix} \varphi/c \end{pmatrix} \begin{pmatrix} \mathbf{A} \end{pmatrix}.
\end{align*}
\]

Finally, defining the \((4 \times 4)\) matrices

\[
\begin{align*}
M_{e}(k, \omega) &= \begin{pmatrix} |k|^2 - \frac{\omega}{c} k^{T} \\
\frac{\omega}{c} k - \frac{\omega}{c} \end{pmatrix}, \\
M_{m}(k, \omega) &= \begin{pmatrix} 0 & 0 \\
0 & |k|^2 - \omega k^{T} \end{pmatrix},
\end{align*}
\]
we can rewrite Eq. (32) compactly as
\[ \mu_0 j = \left( \varepsilon_{\text{eff}} M_e + \frac{1}{\mu_{\text{eff}}} M_m \right) A, \]  
(35)
where \( j = j^\mu = (\epsilon \rho, j)^T \) and \( A^\nu = (\varphi/c, A) \) are the four-current and four-potential, respectively. In the following, this relation will form the basis for the identification of the fundamental response tensor \([22, \S 5.1]\) in this effective model for metamaterials.

PROPER FUNDAMENTAL RESPONSE TENSOR

In order to perform the transition from the traditional macroscopic approach to electrodynamics in media to its modern microscopic counterpart, we now identify \( j_{\text{ext}} \equiv j_\text{ext} \) and \( A = A_\text{tot} \) \([22, 24]\) such that Eq. (35) turns into
\[ \mu_0 j_{\text{ext}} = \left( \varepsilon_{\text{eff}} M_e + \frac{1}{\mu_{\text{eff}}} M_m \right) A_\text{tot}, \]  
(36)
Next, we use that \([22, \text{Eq. (3.30)}]\)
\[ \mu_0 j_{\text{tot}}(k) = k^2 P_T(k) A_\text{tot}(k), \]  
(37)
where \( k^\mu = (\omega/c, k)^T \) denotes the four-momentum, \( k^2 = k^\mu k_\mu = |k|^2 - \omega^2/c^2 \), and the Minkowskian transverse projector is given by \([38, \S 2.1 \text{ and } 2.2]\)
\[ P_T(k, \omega) = \frac{1}{|k|^2 - \omega^2/c^2} \left( \frac{|k|^2}{\omega} \frac{-\omega}{c^2} k^T \right), \]  
which is equivalent to
\[ P_T(k) = \frac{1}{k^2} (M_e(k) + M_m(k)). \]  
(38)
Together, these formulae imply the identity
\[ \mu_0 j_{\text{tot}} = (M_e + M_m) A_\text{tot}. \]  
(39)
Combining this with Eq. (38) yields
\[ \mu_0 j_{\text{ind}} = \mu_0 j_{\text{tot}} - \mu_0 j_{\text{ext}} \]  
\[ = \left( 1 - \varepsilon_{\text{eff}} \right) M_e + \left( 1 - \frac{1}{\mu_{\text{eff}}} \right) M_m A_\text{tot}. \]  
(40)
We now interpret the term in brackets as the proper fundamental response tensor \([22, 24, 37]\), which is hence given by
\[ \mu_0 \chi(k, \omega) = (1 - \varepsilon_{\text{eff}}) M_e(k, \omega) + \left( 1 - \frac{1}{\mu_{\text{eff}}} \right) M_m(k, \omega). \]  
(42)
One easily checks that \( M_e \) and \( M_m \) separately fulfill the constraints
\[ k_\mu M_{\mu \nu}^e(k) = M_{\mu \nu}^m(k) k^\nu = 0, \]  
(43)
and thus they are completely determined by their respective spatial parts,
\[ \tilde{M}_e(k, \omega) = -\frac{\omega^2}{c^2} \frac{1}{\varepsilon_{\text{eff}}}, \]  
(44)
\[ \tilde{M}_m(k, \omega) = |k|^2 \frac{1}{\varepsilon_{\text{eff}}} - k k^T = |k|^2 \tilde{P}_T(k), \]  
(45)
where \( \tilde{P}_T(k) \) denotes the Cartesian transverse projector \([21, \S 2.1]\). The proper fundamental response tensor \([22]\) therefore fulfills the same constraints, and it is completely determined by the proper current response tensor,
\[ \mu_0 \chi(k, \omega) = (\varepsilon_{\text{eff}} - 1) \frac{\omega^2}{c^2} \frac{1}{\varepsilon_{\text{eff}}} + \left( 1 - \frac{1}{\mu_{\text{eff}}} \right) |k|^2 \tilde{P}_T(k). \]  
(46)
We can write this equivalently as \([38, \text{Appendix D.1}]\)
\[ \tilde{\chi}(k, \omega) = \tilde{\chi}_L(k, \omega) P_L(k) + \tilde{\chi}_T(k, \omega) P_T(k), \]  
(47)
with the longitudinal and transverse proper current response functions
\[ \tilde{\chi}_L(k, \omega) = \varepsilon_0 (\varepsilon_{\text{eff}} - 1) \omega^2, \]  
(48)
\[ \tilde{\chi}_T(k, \omega) = \varepsilon_0 (\varepsilon_{\text{eff}} - 1) \omega^2 + \frac{1}{\mu_0} \left( 1 - \frac{1}{\mu_{\text{eff}}} \right) |k|^2. \]  
(49)
In particular, this shows that the phenomenological model defined by the fundamental response tensor \([22]\) describes a homogeneous and isotropic system.

Thus, Eq. (42) represents the first central result of this article. It gives the microscopic, wavevector- and frequency-dependent (proper) fundamental response tensor which corresponds to the traditional ansatz defined by Eqs. (5)–(6). In fact, this microscopic response tensor depends on only two “effective” material constants, \( \varepsilon_{\text{eff}} \) and \( \mu_{\text{eff}} \).

However, it remains to show that: (i) although these material constants do not coincide with the electric permittivity and the magnetic permeability (in the sense of response functions), they can still be interpreted as their “effective” versions; (ii) the microscopic wave equation leads to a refractive index which is simply given by the product of these effective material parameters, hence \( n^2 = \varepsilon_{\text{eff}} \mu_{\text{eff}} \).

EFFECTIVE PERMITTIVITY AND PERMEABILITY

For an isotropic system, the dielectric tensor has an analogous form as Eq. (47). The resulting longitudinal and transverse dielectric functions are related to the
corresponding proper current response functions by \[21\], § 5.1]
\[
\varepsilon_L(k, \omega) = 1 + \frac{1}{\varepsilon_0 \omega^2} \chi_L(k, \omega),
\]
\[
\varepsilon_T(k, \omega) = 1 + \frac{1}{\varepsilon_0 (\omega^2 - c^2|k|^2)} \chi_T(k, \omega).
\]
Using Eqs. (48)–(49), we therefore obtain
\[
\varepsilon_L(k, \omega) = \varepsilon_{\text{eff}},
\]
\[
\varepsilon_T(k, \omega) = \frac{\varepsilon_{\text{eff}} \omega^2 - \mu_{\text{eff}}^{-1} c^2|k|^2}{\omega^2 - c^2|k|^2}.
\]
Hence, both the longitudinal and the transverse dielectric function fulfill
\[
\lim_{|k| \to 0} \varepsilon_{L/T}(k, \omega) = \varepsilon_{\text{eff}},
\]
and thus \(\varepsilon_{\text{eff}}\) can indeed be interpreted as an “effective” permittivity. In particular, the longitudinal dielectric function is even constant and given by \(\varepsilon_{\text{eff}}\). We remark, however, that this does not imply a proportionality between the external and the total electric field. Instead, we have the following relations between the longitudinal and transverse components of the respective fields:
\[
E_{\text{ext},L}(k, \omega) = \varepsilon_{\text{eff}} E_{\text{tot},L}(k, \omega),
\]
\[
E_{\text{ext},T}(k, \omega) = \frac{\varepsilon_{\text{eff}} \omega^2 - \mu_{\text{eff}}^{-1} c^2|k|^2}{\omega^2 - c^2|k|^2} E_{\text{tot},T}(k, \omega).
\]
In particular, this shows that \(\varepsilon_0 E_{\text{ext}}\) does not coincide with the displacement field \(D\) used in the phenomenological derivation. Instead, the Fundamental Field Identification holds only for the respective longitudinal parts, such that the transverse displacement field \(D_T\) remains completely undetermined. In principle, it would then also be unclear how the corresponding transverse response function can actually be measured. In practice, however, this does not pose any problems since in the microscopic approach, all field quantities are uniquely determined by their respective Maxwell equations. Correspondingly, we here consider Eqs. (5)–(6) as a heuristic ansatz, whose sole purpose lies in the deduction of the proper fundamental response tensor defined by Eq. (42). The interpretation of the material parameters appearing there as “effective” permittivity and permeability can be justified ex post, i.e., independently of the originally macroscopic ansatz. For the electric case, this has already been shown by the above arguments. It remains to prove the analogous result for the magnetic material parameter.

Thus, let us next investigate the magnetic properties of the model defined by Eq. (42). We first note that the direct fundamental response tensor \[24\], § 2.3] has again the isotropic form \[17\], such that the longitudinal and transverse components can be calculated as \[21\], § 5.1]
\[
\chi_L(k, \omega) = \frac{\chi_L(k, \omega)}{\varepsilon_L(k, \omega)},
\]
\[
\chi_T(k, \omega) = \frac{\chi_T(k, \omega)}{\varepsilon_T(k, \omega)}.
\]
From our previous results, we obtain
\[
\chi_L(k, \omega) = \varepsilon_0 \omega^2 \left(1 - \frac{1}{\varepsilon_{\text{eff}}}\right),
\]
\[
\chi_T(k, \omega) = \varepsilon_0 (\omega^2 - c^2|k|^2)
\]
\[
\times \left(1 - \frac{\omega^2 - c^2|k|^2}{\varepsilon_{\text{eff}} \omega^2 - \mu_{\text{eff}}^{-1} c^2|k|^2}\right).
\]
Furthermore, with the Green function of the d’Alembert operator given by \[22\], Eq. (3.9)]
\[
\mathcal{D}_0(k, \omega) = \frac{1}{\varepsilon_0 (c^2|k|^2 - \omega^2)},
\]
we can write the transverse current response function as
\[
\chi_T(k, \omega) = \mathcal{D}_0^{-1}(k, \omega) \left(\omega^2 - c^2|k|^2\right)
\]
\[
\times \left(\varepsilon_{\text{eff}} \omega^2 - \mu_{\text{eff}}^{-1} c^2|k|^2\right) - 1).
\]
In particular, we note that the density response function (see Eq. (11)) is determined by the longitudinal current response function as \[22\], Eq. (7.11)]
\[
\chi(k, \omega) = -\frac{|k|^2}{\omega^2} \chi_L(k, \omega).
\]
From Eq. (59), we therefore obtain
\[
\chi(k, \omega) = \varepsilon_0 \left(\frac{1}{\varepsilon_{\text{eff}}} - 1\right) |k|^2.
\]
Finally, the magnetic susceptibility is determined by the transverse current response function as \[22\], Eq. (7.9)]
\[
\chi_{\text{m}}(k, \omega) = \mathcal{D}_0(k, \omega) \chi_T(k, \omega).
\]
From Eq. (62), we directly obtain
\[
\chi_{\text{m}}(k, \omega) = \frac{\omega^2 - c^2|k|^2}{\varepsilon_{\text{eff}} \omega^2 - \mu_{\text{eff}}^{-1} c^2|k|^2} - 1.
\]
In particular, the static susceptibility is given by
\[
\chi_{\text{m}}(k, \omega = 0) = \mu_{\text{eff}} - 1,
\]
and this shows that the material constant \(\mu_{\text{eff}}\) indeed has the interpretation of an effective permeability. Finally,
we remark that Eq. (68) can also be derived directly from
Eq. (69) by using the general identity [22, Eq. (6.48)]
\[
\chi_{\text{m}}(\mathbf{k}, \omega) = \mu(\mathbf{k}, \omega) - 1,
\] (68)

and this implies for the refractive index, \( n_k = c / u_k \), the standard relation
\[
 n^2 = \varepsilon_{\text{eff}} \mu_{\text{eff}}. \] (75)

In particular, this implies that the refractive index of our model is wavevector independent. We have thus shown that the standard formula for the refractive index is indeed recovered in this phenomenological model, but in terms of the effective permeability and permittivity. By contrast, the same does not hold true for the corresponding microscopic response functions.

**CONCLUSION**

We have derived a simple, phenomenological model for the microscopic current response tensor which corresponds to the macroscopic description of media in terms of “effective” permittivity and permeability constants. In particular, we have shown that the microscopic wave equation in media, \( \varepsilon \mathbf{\sigma}_T(\mathbf{k}, \omega) = 0 \), which is formulated in terms of the transverse, frequency- and wavevector-dependent dielectric function, yields back the standard equation \( n^2 = \varepsilon_{\text{eff}} \mu_{\text{eff}} \) for the refractive index in terms of these effective material constants, but not in terms of the corresponding microscopic response functions, i.e., \( n^2 \neq \varepsilon \mu \). Since the microscopic current response tensor is in principle accessible from \textit{ab initio} calculations, one could check whether for certain materials it reverts in some appropriate limit to the form \( n^2 = \varepsilon_{\text{eff}} \mu_{\text{eff}} \) with simultaneously negative material constants (as they are expected for metamaterials on grounds of V. Veselago’s argument [24]). Thus, our work also provides a criterion for the \textit{ab initio} screening, i.e., the systematic search for new metamaterials based on modern first-principles calculations.

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[1] J. D. Jackson, *Classical electrodynamics*, 3rd ed. (John Wiley & Sons, Inc., Hoboken, NJ, 1999).

[2] D. J. Griffiths, *Introduction to electrodynamics*, 3rd ed. (Prentice-Hall, Inc., Upper Saddle River, NJ, 1999).
[3] L. D. Landau and E. M. Lifshitz, Electrodynamics of continuous media, 2nd ed., Course of Theoretical Physics, Vol. 8 (Pergamon Press Ltd., Oxford, 1984).
[4] G. F. Giuliani and G. Vignale, Quantum theory of the electron liquid (Cambridge University Press, Cambridge, 2005).
[5] J. Kohanoff, Electronic structure calculations for solids and molecules: theory and computational methods (Cambridge University Press, Cambridge, 2006).
[6] R. M. Martin, Electronic structure: basic theory and practical methods (Cambridge University Press, Cambridge, 2008).
[7] P. Nozières and D. Pines, Phys. Rev. 109, 741 (1958).
[8] P. Nozières and D. Pines, Il Nuovo Cimento 9, 470 (1958).
[9] L. V. Keldysh, D. A. Kirzhnitz, and A. A. Maradudin, The dielectric function of condensed systems, Modern Problems in Condensed Matter Sciences (Elsevier Science Publishers B.V., Amsterdam, 1989).
[10] C. Kittel, Introduction to solid state physics, 7th ed. (John Wiley & Sons, Inc., New York, 1996).
[11] T. Fliessbach, Elektrodynamik: Lehrbuch zur Theoretischen Physik II, 6th ed. (Springer-Verlag, Berlin/Heidelberg, 2012).
[12] D. B. Melrose, Quantum plasmasdynamics: unmagnetized plasmas, Lecture Notes in Physics, Vol. 735 (Springer, New York, 2008).
[13] D. B. Melrose, Quantum plasmasdynamics: magnetized plasmas, Lecture Notes in Physics, Vol. 854 (Springer, New York, 2013).
[14] H. Bruus and K. Flensberg, Many-body quantum theory in condensed matter physics: an introduction (Oxford University Press, Oxford, 2004).
[15] P. A. Martin and F. Rothen, Many-body problems and quantum field theory: an introduction (Springer-Verlag, Berlin/Heidelberg, 2002).
[16] E. Kaxiras, Atomic and electronic structure of solids (Cambridge University Press, Cambridge, 2003).
[17] W. Schäfer and M. Wegener, Semiconductor optics and transport phenomena, Advanced Texts in Physics (Springer-Verlag, Berlin/Heidelberg, 2002).
[18] W. Hanke, Adv. Phys. 27, 287 (1978).
[19] G. Strinati, La Rivista del Nuovo Cimento 11, 1 (1988).
[20] A. Zangwill, Modern electrodynamics (Cambridge University Press, Cambridge, 2012).
[21] R. Starke and G. A. H. Schober, Phot. Nano. Fund. Appl. 26, 41 (2017), See also arXiv:1704.06615 [cond-mat.mtrl-sci].
[22] R. Starke and G. A. H. Schober, Phot. Nano. Fund. Appl. 14, 1 (2015), See also arXiv:1401.6800 [cond-mat.mtrl-sci].
[23] R. Starke and G. A. H. Schober, “Ab initio materials physics and microscopic electrodynamics of media,” arXiv:1606.00445 [cond-mat.mtrl-sci] (2016).
[24] R. Starke and G. A. H. Schober, Optik 140, 62 (2017), See also arXiv:1510.03404 [cond-mat.mtrl-sci].
[25] R. Starke and G. A. H. Schober, Optik 157, 275 (2018), See also arXiv:1705.11004 [physics.optics].
[26] N. W. Ashcroft and N. D. Mermin, Solid state physics (Harcourt, Inc., Orlando, FL, 1976).
[27] P. Y. Yu and M. Cardona, Fundamentals of semiconductors: physics and materials properties, 4th ed., Graduate Texts in Physics (Springer-Verlag, Berlin/Heidelberg, 2010).
[28] J. B. Pendry, A. J. Holden, D. J. Robbins, and W. J. Stewart, IEEE Trans. Microw. Theory Tech. 47, 2075 (1999).
[29] V. G. Veselago, Sov. Phys. Usp. 10, 509 (1968).
[30] D. R. Smith, W. J. Padilla, D. C. Vier, S. C. Nemat-Nasser, and S. Schultz, Phys. Rev. Lett. 84, 4184 (2000).
[31] R. A. Shelby, D. R. Smith, and S. Schultz, Science 292, 77 (2001).
[32] J. B. Pendry, Phys. Rev. Lett. 85, 3966 (2000).
[33] D. R. Smith, J. B. Pendry, and M. C. K. Wiltshire, Science 305, 788 (2004).
[34] T. Ergin, N. Stenger, P. Brenner, J. B. Pendry, and M. Wegener, Science 328, 337 (2010).
[35] D. Forcella, C. Prada, and R. Carminati, Phys. Rev. Lett. 118, 134301 (2017).
[36] G. A. H. Schober and R. Starke, “General form of the full electromagnetic Green function in materials physics,” arXiv:1704.07594 [physics.class-ph] (2017).
[37] R. Starke and G. A. H. Schober, Int. J. Mod. Phys. D 25, 1640010 (2016), See also arXiv:1409.3723 [math-ph].
[38] R. Starke and G. A. H. Schober, “Response Theory of the electron-phonon coupling,” arXiv:1606.00012 [cond-mat.mtrl-sci] (2016).
[39] R. Starke and G. A. H. Schober, Int. J. Mod. Phys. D 26, 1750163 (2017), See also arXiv:1702.06985 [physics.class-ph].
[40] O. V. Dolgov and E. G. Maksimov, in The dielectric function of condensed systems, Modern Problems in Condensed Matter Sciences, edited by L. V. Keldysh, D. A. Kirzhnitz, and A. A. Maradudin (Elsevier Science Publishers B.V., Amsterdam, 1989).