Non-relativistic limit of multidimensional gravity: exact solutions and applications

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Received 20 October 2009, in final form 6 January 2010
Published 11 February 2010
Online at stacks.iop.org/CQG/27/055002

Abstract
We found the exact solution of the Poisson equation for the multidimensional space with topology $M_{3+d} = \mathbb{R}^3 \times T^d$. This solution describes the smooth transition from the newtonian behavior $1/r^3$ for distances bigger than periods of tori (the extra dimension sizes) to multidimensional behavior $1/r^{1+2/d}$ in the opposite limit. In the case of one extra dimension $d = 1$, the gravitational potential is expressed via compact and elegant formulae. These exact solutions are applied to some practical problems to get the gravitational potentials for considered configurations. Found potentials are used to calculate the acceleration for point masses and gravitational self-energy. Models with test masses smeared over some (or all) extra dimensions are proposed. In ten-dimensional spacetime with three smeared extra dimensions, it is shown that the size of three rest extra dimensions can be enlarged up to a submillimeter for the case of 1 TeV fundamental Planck scale $M_{Pl}(\sim 10^{19})$. In the models where all extra dimensions are smeared, the gravitational potential exactly coincides with the newtonian one regardless of the size of the extra dimensions. Nevertheless, the hierarchy problem can be solved in these models.

PACS numbers: 04.50.-h, 11.25.Mj, 98.80.-k

1. Introduction

There are two well-known problems which are related to each other. They are the discrepancies in gravitational constant experimental data and the hierarchy problem. Discrepancies (see e.g. figure 2 in the ‘CODATA Recommended Values of the Fundamental Constants: 2006’) are usually explained by extreme weakness of gravity. It is very difficult to measure Newton’s gravitational constant $G_N$. Certainly, for this reason geometry of an experimental setup can affect data. However, it may well be that the discrepancies can also be explained (at least partly) by underlying fundamental theory. Formulae for an effective gravitational constant...
following from such theory can be sensitive to the geometry of experiments. For example, if correction to Newton’s gravitational potential has the form of a Yukawa potential, then the force due to this potential at a given minimum separation per unit test-body mass is at its least for two spheres and its greatest for two planes (see e.g. [1]). Therefore, an effective gravitational constant obtained from these formulas acquires different values for different experimental setup.

The hierarchy problem—the huge gap between the electroweak scale \( M_{\text{EW}} \sim 10^3 \text{ GeV} \) and the Planck scale \( M_{\text{Pl}}(4) = 1.2 \times 10^{19} \text{ GeV} \)—can be also reformulated in the following manner: why is gravity so weak? The smallness of \( G_N \) is the result of the relation \( G_N = M_{\text{Pl}}^{-2} \), and huge value of \( M_{\text{Pl}}(4) \). The natural explanation was proposed in [2, 3]: the gravity is strong: \( G_D = M_{\text{Pl}}^{(2d)} \sim M_{\text{EW}}^{(2d)} \) and it happens in \((D = 4 + d)\)-dimensional spacetime. It becomes weak when gravity is ‘smeared’ over large extra dimensions: \( G_N \sim G_D / V_d \) where \( V_d \) is a volume of internal space.

To shed light on both of these problems from a new standpoint we intend to investigate multidimensional gravity in non-relativistic limit. To do that, first, we are going to obtain a solution of the \((D = 3 + d)\)-dimensional Poisson equation in the case of toroidal extra dimensions. From previous years we know that the newtonian gravitational potential of a body with mass \( m \) has the form \( \varphi(r_3) = -G_N m / r_3 \) where \( G_N \) is Newton’s gravitational constant and \( r_3 = |r_3| \) is magnitude of a radius vector in three-dimensional space. This expression is the solution of three-dimensional Poisson equation with a point-mass source (and corresponding boundary condition \( \varphi(r_3) \to 0 \) for \( r_3 \to +\infty \)) or it can be derived from Gauss’s flux theorem in three-dimensional space. To investigate the effects of extra dimensions, it is necessary to generalize Newton’s formula to the case of extra dimensions. Clearly, the result depends on the topology of investigated models. We consider models where the \( D \)-dimensional spatial part of factorizable geometry is defined on a product manifold \( M_D = \mathbb{R}^3 \times T^d \). \( \mathbb{R}^3 \) describes three-dimensional flat external (our) space and \( T^d \) is a torus which corresponds to \( d \)-dimensional internal space. Let \( V_d \) be a volume of the internal space and \( a \sim V_d^{1/d} \) is a characteristic size of extra dimensions. Then, Gauss’s flux theorem leads to the following asymptotes for the gravitational potential (see e.g. [3] and our appendix for alternative derivation): \( \varphi \sim 1/r_3 \) for \( r_3 \gg a \) and \( \varphi \sim 1/r_3^{3+d} \) for \( r_3 \ll a \) where \( r_3 \) and \( r_3^{3+d} \) are magnitudes of radius vectors in three-dimensional and \((3 + d)\)-dimensional spaces, respectively. Obviously, an exact solution of the \( D \)-dimensional Poisson equation should show smooth transition between both of these asymptotes. This formula gives the possibility of investigating the characteristic features of multidimensional gravity in the non-relativistic limit which enables us to reveal extra dimensions (or to establish experimental limitations on extra dimensions).

In our paper we obtain such an exact solution for an arbitrary number of extra dimensions. In the case of one extra dimension, this expression acquires a very compact and elegant form. The found exact solution is applied to a number of practical problems, e.g. to calculate the gravitational force between two spherical shells (or balls). It gives a possibility of calculating an effective four-dimensional gravitational constant for given configurations. For example, in the case of two balls we show that, in principle, the inverse-square law experiments enable us to detect a deviation from Newton’s gravitational constant. Then, we generalize our model to the case of smeared extra dimensions. It means that we suppose that test bodies are uniformly smeared/spreaded over all or part of the extra dimensions. We prove that the gravitational potential does not feel the smeared extra dimensions. In particular, if all extra dimensions are smeared, then the inverse-square law experiment does not show any deviation from the ordinary Newton’s formula, and this conclusion does not depend on the sizes of the smeared extra dimensions. Nevertheless, the hierarchy problem can be solved in these models.
The paper is structured as follows. In section 2 we get the exact solution of the multidimensional Poisson equation in the case of spacial topology $\mathbb{R}^3 \times T^d$. This formula is applied to some practical problems in section 3 to get the gravitational potential, gravitational acceleration of a point mass and gravitational self-energy for these problems. In section 4 we investigate the gravitational interaction of two spherical shells. Then, in section 5 we generalize our model to the case of smeared extra dimensions. Here, we prove that the gravitational potential does not ‘feel’ the smeared extra dimensions and demonstrate that the hierarchy problem can be solved in this case. A brief discussion of the obtained results is presented in the concluding section 6. In the appendix, we get the expressions for the gravitational force law at small and large separation between point masses on the multidimensional manifold with the topology $\mathbb{R}^3 \times T^d$.

2. Multidimensional gravitational potentials

In the $D$-dimensional space, the Poisson equation reads

$$\Delta_D \phi_D = S_D G_D \rho_D (\mathbf{r}_D),$$

(2.1)

where $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is a total solid angle (square of $(D-1)$-dimensional sphere of a unit radius), $G_D$ is a gravitational constant in $(D = D+1)$-dimensional spacetime and $\rho_D(\mathbf{r}_D) = m \delta(x_1) \delta(x_2) \ldots \delta(x_D)$.

2.1. Spatial topology $\mathbb{R}^D$

In the case of topology $\mathbb{R}^D$, (2.1) has the following solution:

$$\phi_D (\mathbf{r}_D) = -\frac{G_D m}{r_D^{D-2}}, \quad D \geq 3.$$  

(2.2)

This is the unique solution of (2.1) which satisfies the boundary condition: $\lim_{r_D \to \infty} \phi_D (\mathbf{r}_D) = 0$. Gravitational constant $G_D$ in (2.1) is normalized in such a way that the strength of gravitational field (acceleration of a test body) takes the form:

$$-d \phi_D / dr_D = -G_D m / r_D^{D-1}.$$  

(2.3)

To get this result we, first, use the formula

$$\delta(\xi_i) = \frac{1}{a_i} \sum_{k_i=-\infty}^{+\infty} \cos \left( \frac{2\pi k_i}{a_i} \xi_i \right),$$

and, second, put the following relation between the gravitational constants in four- and $D$-dimensional spacetimes:

$$S_D S_3 \cdot G_D \prod_{i=1}^{d} a_i = G_N.$$  

(2.4)
The latter relation provides the correct limit when all $a_i \to 0$. In this limit zero modes $k_i = 0$ give the main contribution and we obtain $\psi_D(r_3, \xi_1, \ldots, \xi_d) \to -G_N m/r_3$. (2.4) was widely used in the concept of large extra dimensions which gives the possibility of solving the hierarchy problem [2, 3]. It is also convenient to rewrite (2.4) via fundamental Planck scales:

$$\frac{S_D}{S_3} M_{\text{Pl}(a)}^2 = M_{\text{Pl}(D)}^2 \prod_{i=1}^{d} a_i,$$

(2.5)

where $M_{\text{Pl}(a)} = G_N^{-1/2} = 1.2 \times 10^{19}$ GeV and $M_{\text{Pl}(D)} \equiv G_{D}^{-1/(2d)}$ are fundamental Planck scales in four and $D$ spacetime dimensions, respectively.

In the opposite limit when all $a_i \to +\infty$ the sums in (2.3) can be replaced by integrals. Using the standard integrals (e.g. from [5]) and relation (2.4), we can easily show that, for example, in particular cases $d = 1, 2$ we get the desired result: $\psi_D(r_3, \xi_1, \ldots, \xi_d) \to -G_N m \left[ (D - 2) r_3^{D-1} \right]$. From (2.3), it follows that the potential energy of gravitational interaction between two point masses $m^{(a)}$ and $m^{(b)}$ with radius vectors $r_D^{(a)}$ and $r_D^{(b)}$ reads as

$$U_D(r_D^{(a)}, r_D^{(b)}) = -\frac{G_N m^{(a)} m^{(b)}}{|r_D^{(a)} - r_D^{(b)}|} \times \sum_{k_1=+\infty}^{+\infty} \cdots \sum_{k_{D-1}=+\infty}^{+\infty} \exp \left[ -2\pi \left( \sum_{i=1}^{D-3} \frac{k_i}{a_i} \right)^2 \right] \exp \left( \frac{2\pi k_D}{a_D} \right) \cdots \exp \left( \frac{2\pi k_{D-1}}{a_{D-1}} \right) \left( \xi_1^{(a)} - \xi_1^{(b)} \right) \cdots \left( \xi_{D-1}^{(a)} - \xi_{D-1}^{(b)} \right).$$

(2.6)

2.3. Spatial topology $\mathbb{R}^3 \times T^1$

In the case of one extra dimension $d = 1$, we can perform summation of series in (2.3). To do it, we can apply the Abel–Plana formula or simply use the tables of series [5]. As a result, we arrive at a compact and nice expression:

$$\psi_D(r_3, \xi) = -\frac{G_N m}{r_3} \frac{\sinh \left( \frac{2\pi r_3}{a} \right)}{\cosh \left( \frac{2\pi r_3}{a} \right) - \cos \left( \frac{2\pi \xi}{a} \right)},$$

(2.7)

where $r_3 \in [0, +\infty)$ and $\xi \in [0, a]$. It is not difficult to verify that this formula has the correct asymptotes when $r_3 \gg a$ and $r_3 \ll a$. Figure 1 demonstrates the shape of this potential. Dimensionless variables $\eta_1 \equiv r_3/a \in [0, +\infty)$ and $\eta_2 \equiv \xi/a \in [0, 1]$. With respect to variable $\eta_2$, this potential has two minima at $\eta_2 = 0, 1$ and one maximum at $\eta_2 = 1/2$. We continue the graph to negative values of $\eta_2 \in [-1, 1]$ to show in more detail the form of minimum at $\eta_2 = 0$. The potential (2.7) is finite for any value of $r_3$ if $\xi \neq 0, a$ and goes to $-\infty$ as $-1/r_3^2$ if simultaneously $r_3 \to 0$ and $\xi \to 0, a$ (see figure 2). We would like to mention that in the particular case $\xi = 0$, formula (2.7) was also found in [6].

2.4. Yukawa approximation

Having at hand formulae (2.3) and (2.7), we can apply them for calculation of some elementary physical problems and compare the obtained results with known newtonian expressions. For a working approximation, it is usually sufficient to summarize in (2.3) up to the first Kaluza–Klein modes $|k_i| = 1 (i = 1, \ldots, d)$:

$$\psi_D(r_3, \xi_1, \xi_2, \ldots, \xi_d) \approx -\frac{G_N m}{r_3} \left[ 1 + 2 \sum_{i=1}^{d} \exp \left( \frac{2\pi}{a_i} \right) \cos \left( \frac{2\pi}{a_i} \xi_i \right) \right].$$

(2.8)
Then, the terms with the biggest periods \(a_i\) give the main contributions. If all test bodies are on the same brane \((\xi_i = 0)\) we obtain:

\[
\psi_D(r_3, \xi_1 = 0, \xi_2 = 0, \ldots, \xi_d = 0) = -\psi_\alpha(r_3) \approx -\frac{G_N m}{r_3} \left[ 1 + \alpha \exp \left( -\frac{r_3}{\lambda} \right) \right],
\]

(2.9)

where \(\alpha = 2s\) \((1 \leq s \leq d)\), \(\lambda = a/(2\pi)\) and \(s\) is the number of extra dimensions with periods of tori \(a_i\) which are equal (or approximately equal) to \(\alpha = \max a_i\). If \(a_1 = a_2 = \ldots = a_d = a\), then \(s = d\). Thus, the correction to Newton’s potential has the form of a Yukawa potential. It is now customary to interpret tests of gravitational inverse-square law (ISL) as setting limits on an additional Yukawa contribution. The overall diagram of the experimental constraints can be found in [7] (see figure 6 therein) and we shall use these data for limitation \(a\) for given \(\alpha\).
3. Application ($\xi_i = 0$)

Now, we apply formulae (2.3) and (2.7) to some particular geometrical configurations. For our calculations we shall use the case of $\xi_1 = \xi_2 = \cdots = \xi_d = 0$. It means that test bodies have the same coordinates in the extra dimensions. It takes place e.g. when test bodies are on the same brane. Also, to get numerical results we should define the sizes $a_i$ of the extra dimensions. The $e^+e^-$ leptonic interaction experiments at high-energy colliders show that there is no deviations from Coulomb’s law for separations down to $10^{-16}$ cm [1]. Therefore, if the standard model fields are not localized on the brane, this value can be used for the upper bound of $a_i$: $a_i \lesssim 10^{-17}$ cm (in section 5 we demonstrate how to avoid this argument). These values $a_i$ can be greatly increased if we suppose that the standard model fields are localized on the brane. In this case, we can obtain the upper bound for $a = \max a_i$ from the gravitational inverse-square law experimental data depicted in figure 6 of paper [7]. For example, the Yukawa approximation (2.9) shows that $\alpha = 2$ for $d = 1$. For this value $\alpha$, the figure gives $\lambda \leq 4.7 \times 10^{-3}$ cm $\Rightarrow a \lesssim 3.0 \times 10^{-2}$ cm.

3.1. Infinitesimally thin shell

Let us consider first an infinitesimally thin shell of mass $m = 4\pi R^2 \sigma$, where $R$ and $\sigma$ are the radius and surface density of the shell. Then, the gravitational potential of this shell in a point with radius vector $r_3$ (from the center of the shell) is

$$
\phi_D(r_3) = -\frac{2\pi G_N \sigma R}{r_3} \int_{r_3-R}^{r_3+R} \sum_k \exp[-2\pi \chi_k r'] dr' 
$$

$$
= -\frac{G_N m}{r_3} - \frac{2G_N \sigma R}{r_3} \sum_k \frac{1}{\chi_k} e^{-2\pi \chi_k r_3} \sinh(2\pi \chi_k R), \quad r_3 > R (3.1)
$$

and

$$
\phi_D(r_3) = -\frac{2\pi G_N \sigma R}{r_3} \int_{R-r_3}^{R+r_3} \sum_k \exp[-2\pi \chi_k r'] dr' 
$$

$$
= -\frac{G_N m}{R} - \frac{2G_N \sigma R}{r_3} \sum_k \frac{1}{\chi_k} e^{-2\pi \chi_k R} \sinh(2\pi \chi_k r_3), \quad r_3 < R, (3.2)
$$

where

$$
\chi_k \equiv \left[ \sum_{i=1}^d \left( \frac{k_i}{a_i} \right)^2 \right]^{1/2}, \quad \sum_k \equiv \sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_d=-\infty}^{+\infty} (3.3)
$$

and the prime in sums denotes that the zero mode $k_1 = \cdots = k_d = 0$ is absent in summation. It means that Newton’s expressions (which correspond to the zero mode) are singled out from sums. In the case $d = 1$, these expressions can be written in the compact form:

$$
\phi_d(r_3) = -\frac{G_N \sigma Ra}{r_3} \ln \left\{ \frac{\cosh \left( \frac{2\pi (r_3+R)}{a} \right) - 1}{\cosh \left( \frac{2\pi (r_3-R)}{a} \right) - 1} \right\}, \quad r_3 \geq R. (3.4)
$$

These formulas demonstrate two features of the considered models. Firstly, we see that inside ($r_3 < R$) of the shell, the gravitational potential is not a constant. Thus, a test body undergoes an acceleration in contrast to the newtonian case (and to Birkhoff’s theorem of general relativity in four-dimensional spacetime which states that the metric inside an empty spherical cavity in the center of a spherically symmetric system is the Minkowski metric).
Secondly, if \( r_3 \to R \), these potentials have a logarithmic divergency of the type: \( \sum_{k=1}^{+\infty} 1/k \). For example, in this limit \( r_3 \to R \) (3.4) has the following asymptotic behavior:

\[
\varphi_D(r_3) \approx -\frac{G_N m}{R} \left[ 1 - \frac{a}{2\pi R} \ln \left( \frac{2\pi |R - r_3|}{a} \right) \right] = -\frac{G_N m}{R} \left[ 1 + \delta \right],
\]

where we took into account \( R \gg a \) and \( |R - r_3| \ll a \). In the particular case \( 2\pi R = 10 \text{ cm} \) and \( 2\pi |R - r_3| = 10^{-3} a \), the deviation \( \delta \) constitutes \( 2.3 \times 10^{-3} \) and \( 2.3 \times 10^{-15} \) parts of the newtonian value \(-G_N m/R\) for \( a = 10^{-2} \text{ cm} \) and \( a = 10^{-17} \text{ cm} \), respectively. In principle, the former estimate is not very small. However, it is very difficult to set an experiment which satisfies the condition \(|R - r_3| \ll a\). If the shell has a finite thickness, then the divergence disappears.

### 3.2. Spherical shell

Here, we consider a spherical shell of inner radius \( R_1 \) and outer radius \( R_2 \) and mass \( m = 4\pi \int_{R_1}^{R_2} \rho(R)R^2\,dR \) where \( \rho(R) \) is a volume density of the shell (in the case of constant volume density \( \rho = m/\left[ \frac{4\pi}{3} (R_2^3 - R_1^3) \right] \)). Then, the potential outside of the shell is

\[
\varphi_D(r_3) = -\frac{2 G_N m}{r_3} \left\{ \sum_k \frac{1}{\chi_k} e^{-2\pi \chi_k r_3} \int_{R_1}^{R_2} \rho(R)R \sinh(2\pi \chi_k R) \, dR \right\} = \varphi_D^{(1)}(r_3), \quad r_3 > R_2,
\]

which for the constant \( \rho \) reads

\[
\varphi_D(r_3) = -\frac{G_N m}{r_3} - \frac{G_N \rho}{2\pi^2 r_3} \sum_k \frac{1}{\chi_k^3} e^{2\pi \chi_k R} \left[ \rho(R) \left\{ \frac{R_3}{R_2} - \frac{1}{2\pi \chi_k} e^{-2\pi \chi_k R} \right\} dR \right] = \varphi_D^{(2)}(r_3), \quad r_3 < R_1
\]

Inside of the shell we obtain:

\[
\varphi_D(r_3) = -\frac{2 G_N m}{r_3} \left\{ \sum_k \frac{1}{\chi_k} \sinh(2\pi \chi_k r_3) \int_{R_1}^{R_2} \rho(R)R e^{-2\pi \chi_k R} \, dR \right\} = \varphi_D^{(2)}(r_3), \quad r_3 < R_1
\]

and for the constant \( \rho \):

\[
\varphi_D(r_3) = -2\pi G_N \rho \left( R_2^3 - R_1^3 \right) + \frac{G_N \rho}{2\pi^2 r_3} \sum_k \frac{1}{\chi_k^3} \sinh(2\pi \chi_k r_3) \left[ \left( R + \frac{1}{2\pi \chi_k} \right) e^{-2\pi \chi_k R} \right]_{R_1}^{R_2}.
\]

To get potential within the shell, we can use the following relation:

\[
\varphi_D(r_3) = \varphi_D^{(1)}(r_3|R_2 = r_3) + \varphi_D^{(2)}(r_3|R_1 = r_3), \quad R_1 \leq r_3 \leq R_2.
\]

Thus, in the case of constant \( \rho \) the gravitational potential within the shell reads

\[
\varphi_D(r_3) = 2\pi G_N \rho \left[ \frac{r_3^2}{3} - R_2^2 + \frac{2 R_1^3}{3} \right] + \frac{G_N \rho}{2\pi^2 r_3} \sum_k \frac{1}{\chi_k^3} \sinh(2\pi \chi_k r_3) \left[ 2\pi \chi_k R_2 + 1 \right] e^{-2\pi \chi_k R_2} - e^{-2\pi \chi_k r_3} h_1(R_1).
\]
For one extra dimension $d = 1$ we obtain a compact expression which is valid for the full range of variable $r_3 \geq 0$:

$$\varphi_4(r_3) = -\frac{G_N \rho a}{r_3^3} \int_{R_1}^{R_2} R \ln \left( \frac{\cosh \left( \frac{2\pi(R-r_3)}{a} \right)}{\cosh \left( \frac{2\pi(R-R_3)}{a} \right)} - 1 \right) dR,$$  \hspace{1cm} (3.13)

where $\rho$ is taken to be constant.

It can be easily seen that all these potentials are finite in the limit $r_3 \to R_1, R_2$ and $R_1 \neq R_2$, i.e. divergency of the potential is absent for finite thickness of the shell. However, divergency takes place for acceleration of a test body. We can see it from exact formulas for acceleration outside of the shell:

$$\frac{-d\varphi_D}{dr_3} = -\frac{G_N m}{r_3^2}$$

$$-\frac{G_N \rho}{2\pi^2 r_3^2} \sum_k \frac{1}{\chi_k} (2\pi \chi_k r_3 + 1) e^{-2\pi \chi_k R_2} \{ h_k(R_2) - h_k(R_1) \} < 0, \hspace{1cm} r_3 > R_2,$$  \hspace{1cm} (3.14)

within the shell:

$$\frac{-d\varphi_D}{dr_3} = -\frac{4}{3} \pi G_N \rho \frac{r_3^3 - R_1^3}{r_3^2}$$

$$-\frac{G_N \rho}{2\pi r_3^2} \sum_k \frac{1}{\chi_k} \{ h_k(r_3) (2\pi \chi_k R_2 + 1) e^{-2\pi \chi_k R_2}$$

$$- (2\pi \chi_k r_3 + 1) e^{-2\pi \chi_k R_1} \} , \hspace{1cm} R_1 < r_3 < R_2$$  \hspace{1cm} (3.15)

and inside of the shell:

$$\frac{-d\varphi_D}{dr_3} = -\frac{G_N \rho}{\pi r_3^2} \sum_k \frac{1}{\chi_k} h_k(r_3) \left[ \left( R + \frac{1}{2\pi \chi_k} \right) e^{-2\pi \chi_k R} \right]_{R_1}^{R_2} \geq 0, \hspace{1cm} r_3 < R_1.$$  \hspace{1cm} (3.16)

All of these equations (3.14)–(3.16) have logarithmic divergency in the limit $r_3 \to R_1, R_2$, e.g. in $d = 1$ case:

$$\frac{-d\varphi_4}{dr_3} \to \pm 2G_N \rho a \ln \frac{2\pi |R_1,2 - r_3|}{a},$$  \hspace{1cm} (3.17)

where the sign ‘−’ corresponds to $r_3 \to R_1$ and the sign ‘+’ corresponds to $r_3 \to R_2$.

Putting the limit $R_1 \equiv 0, R_2 \equiv R$ in expressions obtained above in this subsection, we can get the corresponding equations for a sphere. For example, using the same conditions as for (3.5), we can get for a sphere ($d = 1$):

$$\frac{-d\varphi_4}{dr_3} \approx -\frac{G_N m}{R^2} \left[ 1 - \frac{3a}{2\pi R} \ln \left( \frac{2\pi |R - r_3|}{a} \right) \right] = -\frac{G_N m}{R} [1 + \frac{3}{2}],$$  \hspace{1cm} (3.18)

Therefore, deviation from the newtonian acceleration $\vartheta = 3\delta$ and we can conclude that this deviation is also difficult to observe in experiments for considered parameters. (3.14) and (3.16) show that acceleration changes the sign from negative outside of the shell to positive inside of the shell (see figure 3). This change happens within the shell.
Figure 3. Graph of acceleration \( w = -\frac{d\phi}{dr^3} \) in dimensionless units (see (3.14)-(3.16)). Here, \( a = 2.5 \), \( R_1 = 5 \) and \( R_2 = 10 \). The dashed lines correspond to radii of the shell. The rightmost line goes to the newtonian asymptote \(-1/r^2_3\) when \( r_3 \to +\infty \). Point \( r_3 = 0 \) corresponds to unstable equilibrium.

3.3. Gravitational self-energy

Gravitational self-energy of the spherical shell with potential \( \phi_D(r_3) \) given by formulas (3.12) and (3.13) reads

\[
U = 2\pi \rho \int_{R_1}^{R_2} r_3^2 \phi_D(r_3) \, dr_3 = \frac{3G_N m^2}{10 (R_2^3 - R_1^3)^2} \left(-2R_2^3 - 3R_1^3 + 5R_1^3 R_2^2\right)
\]

\[- \frac{9G_N m^2}{2 (R_2^3 - R_1^3)^2} \sum'_{k} \left( \frac{1}{(2\pi \chi_k)^3} \left[ \frac{1}{3} (R_2^3 - R_1^3) + \frac{1}{(2\pi \chi_k)^3} E_k \right] \right),
\]

(3.19)

where

\[
E_k = (1 + 2\pi \chi_k R_2) e^{-2\pi \chi_k R_1} [h_k(R_2) + 2h_k(R_1)] - (1 + 2\pi \chi_k R_1) e^{-2\pi \chi_k R_1} h_k(R_1).
\]

(3.20)

In the case of sphere \( R_1 = 0, R_2 \equiv R \) and this expression is reduced to the following formula:

\[
U = -\frac{3G_N m^2}{5R} \left\{ 1 + \frac{15}{2(2\pi)^3} \right. \left. \sum'_{k} \frac{1}{(2\pi \chi_k)^3} \left[ \frac{2\pi \chi_k R}{3} - \left( \frac{1}{2\pi \chi_k R} \right)^2 \left( 1 + 2\pi \chi_k R \right) e^{-2\pi \chi_k R} h_k(R) \right] \right\}
\]

\[\equiv U_N + \delta U.\]

(3.21)

Here, \( U_N \) and \( \delta U \) are the newtonian self-energy of the sphere and the deviation from it, respectively. In the case of one extra dimension \( d = 1 \) and for condition \( a \ll R \) we obtain:

\[
\delta U \approx U_N \frac{5}{4\pi^2} \left( \frac{a}{R} \right)^2 \sum_{k=1}^{\infty} \frac{1}{k^2} = U_N \frac{5}{24} \left( \frac{a}{R} \right)^2,
\]

(3.22)

where we used that \( \sum_{k=1}^{\infty} 1/k^2 = \pi^2/6 \). It is worth noting that the correction to the newtonian formula has a power law suppression instead of an exponential one as is usually expected in Kaluza–Klein models. Nevertheless, for astrophysical objects this correction term is negligibly small because of a big difference between \( a \) and \( R \). For example, if \( a = 10^{-2} \) cm, we obtain

\[\delta U \approx 4 \times 10^{-27} U_N \text{ and } \delta U \approx 4 \times 10^{-19} U_N \text{ for Sun (} R \approx 7 \times 10^{10} \text{ cm) and neutron star (} R \approx 7 \times 10^{6} \text{ cm), respectively.}\]
4. Gravitational interaction of two spherical shells

Let us consider two spherical shells with radii $R_1 < R_2$ and a constant density $\rho$ for the first shell and radii $R'_1 < R'_2$ and a constant density $\rho'$ for the second shell. Then, the potential energy of gravitational interaction between these shells reads

$$U(r_3) = \int \int \int \varphi_D(r'_3) \rho' \, dV'.$$

(4.1)

Here, $r_3 \geq R_2 + R'_2$ is the magnitude of the three-dimensional vector between centers of shells and $r'_3$ is the magnitude of the three-dimensional vector between the center of the first shell and an arbitrary elementary mass $dm' = \rho' \, dV'$ within the second shell. The potential $\varphi_D(r'_3)$ is given by equation (3.7). After integration, we obtain

$$U(r_3) = -\frac{G_N m m'}{r_3} - \frac{G_N \rho \rho'}{4\pi r_3} \sum_k^' \frac{1}{\lambda_k^6} e^{-2\pi \lambda_k r_3} [h_k(R_2) - h_k(R_1)] [h_k(R'_2) - h_k(R'_1)].$$

(4.2)

In the case of two spheres ($R_1 \equiv 0$, $R_2 \equiv R$ and $R'_1 \equiv 0$, $R'_2 \equiv R'$), this formula is reduced to

$$U(r_3) = -\frac{G_N m m'}{r_3} - \frac{G_N \rho \rho'}{4\pi r_3} \sum_k^' \frac{1}{\lambda_k^6} e^{-2\pi \lambda_k r_3} h_k(R) h_k(R').$$

(4.3)

4.1. The Yukawa approximation

For a working approximation, it is often sufficient to keep in sums the first Kaluza–Klein modes, i.e. to use the Yukawa approximation. In this approximation, (4.2) and (4.3) can be rewritten correspondingly:

$$U(r_3) \approx -\frac{G_N m m'}{r_3} - \frac{16\pi^2 G_N \rho \rho' \alpha \lambda^6}{r_3} e^{-r_3 / \lambda} \tilde{h}(R) \tilde{h}(R').$$

(4.4)

and

$$U(r_3) \approx -\frac{G_N m m'}{r_3} \left[1 + 9\alpha \left(\frac{\lambda}{R}\right)^3 \left(\frac{\lambda}{R'}\right)^3 e^{-r_3 / \lambda}\right] \tilde{h}(R) \tilde{h}(R').$$

(4.5)

where

$$\tilde{h}(R) = \frac{R}{\lambda} \cosh \left(\frac{R}{\lambda}\right) - \sinh \left(\frac{R}{\lambda}\right).$$

(4.6)

Here, $\lambda = a/(2\pi)$ and parameter $\alpha = 2s$ is described in (2.9).

4.2. Gravitational force between two spheres

In the Yukawa approximation, the gravitational force between two spheres is

$$F(r_3) = -\frac{dU}{dr_3} \approx -\frac{G_N m m'}{r_3^2} \left[1 + 9\alpha \left(\frac{\lambda}{R}\right)^3 \left(\frac{\lambda}{R'}\right)^3 \frac{r_3}{\lambda} e^{-r_3 / \lambda}\right] \tilde{h}(R) \tilde{h}(R').$$

$$\approx -\frac{G_N m m'}{r_3^2} \left[1 + \frac{9}{4} \alpha \left(\frac{\lambda}{R}\right)^2 \left(\frac{\lambda}{R'}\right)^2 \frac{r_3}{\lambda} e^{-r_3 / (R+R') / \lambda}\right].$$

(4.7)
where in the last expression we use conditions $R, R' \gg \lambda$. If the surfaces of the spheres are on the distances of the order of the maximal period: $r_3 = R - R' \sim a$, then we obtain

$$F(r_3) \approx -\frac{G_N m m'}{r_3^2} \left[1 + 0.0084s \left(\frac{\lambda}{R}\right)^2 \left(\frac{\lambda}{R}\right)^2 \frac{r_3}{\lambda}\right] = -\frac{G_N m m'}{r_3^2} (1 + \tilde{s}). \quad (4.8)$$

For example, in the case of one extra dimension and $R = R' = 1$ cm, $\lambda = 4.7 \times 10^{-3}$ cm, we obtain for the deviation from the newtonian formula the estimate $\tilde{s} = 1.8 \times 10^{-9}$. To obtain it, it is necessary to remember that for $d = 1$ parameter $s = 1$.

5. Smeared extra dimensions

Now, we move onto the asymmetrical extra dimension models (cf with [8]) with the topology

$$M_D = \mathbb{R}^3 \times T^{d-p} \times T^p, \quad p \leq d,$n

where we suppose that $(d - p)$ tori have the same ‘large’ period $a$ and $p$ tori have ‘small’ equal periods $b$. In this case, the fundamental Planck scale relation (2.5) reads

$$S_D \frac{\tilde{G}_D^{2d}}{3^2} = \frac{M_D^{2d}}{\tilde{G}_D^{2d+1}} d^{d-p} b^p. \quad (5.2)$$

Additionally, we assume that test bodies are uniformly smeared/spread over small extra dimensions. Thus, test bodies have a finite thickness in small extra dimensions (thick brane approximation). For short, we shall call such small extra dimensions ‘smeared’ extra dimensions. If $p = d$ then all extra dimensions are smeared.

It is not difficult to show that the gravitational potential does not feel smeared extra dimensions. We can prove this statement by three different methods. First, we can directly solve the D-dimensional Poisson equation (2.1) with the periodic boundary conditions for the extra dimensions $\xi_{p+1}, \ldots, \xi_d$ and the mass density $\rho = (m / \prod_{i=p+1}^d a_i) \delta(r_3) \delta(\xi_{p+1}) \cdots \delta(\xi_d)$. As a result, we obtain the unique solution $\varphi_D(r_3, \xi_{p+1}, \ldots, \xi_d)$ of the form (2.3) which does not depend on $\xi_1, \ldots, \xi_p$ and satisfies the limit $\lim_{r_3 \to +\infty} \varphi_D(r_3, \xi_{p+1}, \ldots, \xi_d) = 0$.

Second, we can average solutions (2.3) (where the point mass $m$ is replaced by $m \prod_{i=p+1}^d (d\xi_i/a_i)$) over smeared dimensions $\xi_1, \ldots, \xi_p$ and take into account that $\int_0^a \cos(2\pi k \xi/a) \, d\xi = 0$, $a$ for $k \neq 0$ and $k = 0$, respectively. Thus, the terms with these variables disappear from (2.3). For example, direct integration of the compact expression (2.7) results in the pure newtonian formula:

$$-\frac{G_N m}{ar_3} \sinh \left(\frac{2\pi r_3}{a}\right) \int_0^a \left[\cosh \left(\frac{2\pi r_3}{a}\right) - \cos \left(\frac{2\pi \xi}{a}\right) \right]^{-1} \, d\xi = -\frac{G_N m}{\pi r_3} \arctan \left(\frac{\cos \left(\frac{2\pi r_3}{a}\right) + 1}{\cos \left(\frac{2\pi \xi}{a}\right) - 1}\right)^{-1} = -\frac{G_N m}{r_3}. \quad (5.3)$$

Finally, from the symmetry of the model, it is clear that in the case of a test mass smeared over extra dimensions, the gravitational field vector $\mathbf{E}_D = -\nabla_D \varphi_D$ does not have components with respect to smeared extra dimensions. For example, if we consider the model with the topology of a cylinder $S^1 \times \mathbb{R}$ where $S^1$ is the smeared extra dimension and $\mathbb{R}$ is the external dimension, then $\mathbf{E}_z$ will be parallel to an element of the cylinder. For simplicity, let us consider a model where all extra dimensions are smeared. Then, the Poisson equation (2.1) can be rewritten in the form

$$\nabla_D \mathbf{E}_D = -S_D G_D \rho_D(r_3) = -4\pi G_N m \delta(r_3), \quad (5.4)$$
where we use relation \((2.4)\) between gravitational constants and the formula \(\rho_D(r_0) = m \delta(r_0) / \prod_{i=1}^d \alpha_i\). After integrating both parts of this formula over the multidimensional volume \(V = V_3 \cup V_{\text{internal}}\) we obtain

\[
\int_V E^{\alpha_i}_{D,\alpha j} d^D V = -4 \pi G_N m V_{\text{internal}} = \text{const.} \quad (5.5)
\]

Here, \(d^D V = \delta_{\{a_1,...,a_D\}} dx^{a_1} \wedge ... \wedge dx^{a_D} = dx dy dz dx^{i_1} \ldots dx^{i_D}\) and \(\alpha_i = 1, 2, 3, i_1, \ldots, i_D\). Indices \(i_1, \ldots, i_D\) enumerate extra dimensions. Applying the Gauss theorem, we get

\[
\int_V E^{\alpha_i}_{D,\alpha j} d^D V = \int_{\partial V} E^{\alpha_i}_{D} d^{D-1} \Sigma_{\alpha_i} = 4 \pi R_3^2 E_D V_{\text{internal}}, \quad (5.6)
\]

where

\[
d^{D-1} \Sigma_{\alpha_i} = \delta_{\{a_1,...,a_D\}} dx^{a_1} \wedge ... \wedge dx^{a_D} \prod_{i=1}^{D-1} \]

and in three-dimensional spherical coordinates \(d^{D-1} \Sigma_{r_i} = R_3^2 \sin \theta d \theta d \phi dx^i \ldots dx^{i_D-1}\). To get (5.6), we use the fact that \(E^D_D\) is the only non-zero component of the gravitational field vector. Therefore, comparing (5.5) and (5.6), we obtain

\[
E_D(r_3) = -\frac{G_N m}{r_3^2} \implies \varphi_D(r_3) = -\frac{G_N m}{r_3}. \quad (5.8)
\]

Thus, all the three approaches show that the gravitational potential does not feel smeared extra dimensions. It means that in the case of \(p\) smeared extra dimensions, the wave numbers \(k_1, \ldots, k_p\) disappear from (2.3) and we should perform summation only with respect to \(k_{p+1}, \ldots, k_d\).

### 5.1. Effective gravitational constant

As it follows from (4.7), in the Yukawa approximation, the gravitational force between two spheres with masses \(m_1, m_2\), radii \(R_1, R_2\) and distance \(r_3\) between the centers of the spheres reads

\[
F(r_3) = -\frac{G_N(\text{eff})(r_3)m_1 m_2}{r_3^2}, \quad (5.9)
\]

where

\[
G_N(\text{eff})(r_3) \approx G_N \left[ 1 + \frac{9}{2} (d - p) \left( \frac{\lambda}{R_1} \right)^2 \left( \frac{\lambda}{R_2} \right)^2 e^{-r_3/(r_1-r_2)/\lambda} \right]
\]

\[\equiv G_N (1 + \delta_G) \quad (5.10)\]

is an effective gravitational constant. Here, we took into account that in the case of \(p\) smeared dimensions, the prefactor \(a\) should be replaced by \(2(d - p)\). Now, we want to evaluate the corrections \(\delta_G\) to Newton’s gravitational constant and to estimate their possible influence on the experimental data. As it follows from figure 2 in the CODATA 2006, the most precise values of \(G_N\) were obtained in the University Washington and the University Zürich experiments [9, 10]. They are \(G_N / 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2} = 6.674215 \pm 0.000092\), and \(6.674252 \pm 0.000124\), respectively. Let us consider two particular examples: the \((D = 5)\)-dimensional model with \(d = 1, p = 0 \rightarrow \alpha = 2\) and the \((D = 10)\)-dimensional model with \(d = 6, p = 3 \rightarrow \alpha = 6\). For these values of \(\alpha\), figure 6 in [7] gives the upper limits for \(\lambda = a/(2\pi)\) correspondingly \(\lambda \approx 4.7 \times 10^{-3}\) cm and \(\lambda \approx 3.4 \times 10^{-3}\) cm. To calculate \(\delta_G\), we take parameters of the Moscow experiment [11]: \(R_1 \approx 0.087\) cm for a platinum.
ball with the mass \( m_1 = 59.25 \times 10^{-3} \) g, \( R_2 \approx 0.206 \) cm for a tungsten ball with the mass \( m_2 = 706 \times 10^{-3} \) g and \( r_3 = 0.3773 \) cm. For both of these models we obtain \( \delta_G \approx 8.91 \times 10^{-12} \) and \( \delta_G \approx 1.06 \times 10^{-14} \), respectively. These values are rather far from the measurement accuracy of \( G_N \) in [9, 10]. In future, if we can achieve in the Moscow-type experiments the accuracy within these values, then changing radii \( R_{1,2} \) and distance \( r_3 \), we can reveal extra dimensions or obtain experimental limitations on considered models. We should note that small changes in experimental bounds for \( \lambda \) result in drastic changes of \( \delta_G \). For example, if for \( \lambda \) we take the upper limits \( \lambda \approx 2 \times 10^{-2} \) cm and \( \lambda \approx 1.3 \times 10^{-2} \) cm which follow from previous experiments (see figure 5 in [1] for \( \alpha = 2 \) and 6, respectively), then \( \delta_G \approx 0.0006247 \) and \( \delta_G \approx 0.0000532 \) and these figures are very close to the measurement accuracy of \( G_N \) in [9, 10].

5.2. Model: \( D = 10 \) with \( d = 6, p = 3 \)

Let us consider in more detail the (\( D = 10 \))-dimensional model with three smeared dimensions. Here, it is very symmetrical with respect to a number of spacial dimensions structure: three our external dimensions, three large extra dimensions with periods \( a \) and three small smeared extra dimensions with periods \( b \). For \( b \) we put a limitation: \( b \leq h_{\text{max}} = 10^{-17} \) cm which is usually taken for the thick brane approximation. This limitation follows from the electrical inverse-square law experiments. Although, in the next subsection we show that such an approach can be significantly relaxed for models with smeared extra dimension, here we still use this bound.

As we mentioned above, in the case of \( \alpha = 6 \), for \( a \) we should take a limitation \( a \leq a_{\text{max}} = 2.1 \times 10^{-2} \) cm. To solve the hierarchy problem, the multidimensional Planck scale is usually considered from 1 TeV up to approximately 130 TeV (see e.g. [8, 12]). To make some estimates, we take \( M_{\text{min}} \approx 1 \) TeV \( \lesssim M_{\text{Pl}(10)} \lesssim M_{\text{max}} = 50 \) TeV. Thus, as it follows from (5.2), the allowed values of \( a \) and \( b \) should satisfy inequalities:

\[
\frac{S_0}{S_3} M_{\text{Pl}(4)}^2 \lesssim a^3 b^3 \lesssim \frac{S_0}{S_3} M_{\text{Pl}(4)}^8.
\]

Counting all limitations, we find the allowed region for \( a \) and \( b \) (shadow area in figure 4). In this trapezium, the upper-horizontal and right-vertical lines are the decimal logarithms of \( a_{\text{max}} \) and \( h_{\text{max}} \), respectively. The right and left inclined lines correspond to \( M_{\text{Pl}(10)} = 1 \) TeV and \( M_{\text{Pl}(10)} = 50 \) TeV, respectively. To illustrate this picture, we consider two points A and B on the line \( M_{\text{Pl}(10)} = 1 \) TeV. Here, we have \( a = 2.1 \times 10^{-2} \) cm, \( b = 10^{-20.9} \) cm for A and \( a = 10^{-4} \) cm, \( b = 10^{-18.6} \) cm for B. These values of large extra dimensions \( a \) are much bigger than in the standard approach \( a \sim 10^{30/6} \approx 10^{-17} \) cm [2, 3].

5.3. Model: \( D \)-arbitrary and \( d = p \)

In this model, the test masses are smeared over all extra dimensions. Therefore, in the non-relativistic limit, there is no deviation from Newton’s law at all. Surprisingly, this result does not depend on the size of the smeared extra dimensions. The ISL experiments will not show any deviation from Newton’s law without regard to the size \( b \) (see also (5.10) where \( d = p = 0 \)). Similar reasoning is also applicable to Coulomb’s law. It is necessary to suggest other experiments which can reveal the multidimensionality of our spacetime. Nevertheless, we can solve the hierarchy problem in this model because (5.2) (where \( d = p \)) still works. For example, in the case of bosonic string dimension \( D = 26 \) we find \( M_{\text{Pl}(26)} \approx 31 \) TeV for \( b = 10^{-17} \) cm. In the case \( D = 10 \) we get \( M_{\text{Pl}(10)} \approx 30 \) TeV for \( b = 5.59 \times 10^{-14} \) cm. In
general, if we suppose that $1 \text{ TeV} \lesssim M_{pl(D)} \lesssim 30 \text{ TeV}$, then we obtain

\begin{align*}
p = 1, & \quad 1.65 \times 10^{11} \text{ cm} \lesssim b \lesssim 4.46 \times 10^{15} \text{ cm}, \\
p = 2, & \quad 3.81 \times 10^{-4} \text{ cm} \lesssim b \lesssim 3.43 \times 10^{-1} \text{ cm}, \\
p = 3, & \quad 4.83 \times 10^{-9} \text{ cm} \lesssim b \lesssim 1.40 \times 10^{-6} \text{ cm}, \\
& \quad \vdots \\
p = 6, & \quad 5.59 \times 10^{-14} \text{ cm} \lesssim b \lesssim 5.21 \times 10^{-12} \text{ cm},
\end{align*}

where the lower limit corresponds to 30 TeV and the upper limit agrees with 1 TeV.

6. Conclusions

We have considered generalization of Newton’s potential to the case of extra dimensions where multidimensional space has the topology $M_D = \mathbb{R}^3 \times T^d$. We obtained the exact solution (2.3) which describes the smooth transition from the newtonian behavior $1/r^3$ for distances bigger than periods of tori (the extra dimension sizes) to multidimensional behavior $1/r^{1+d}_n$ in the opposite limit. In the case of one extra dimension, the gravitational potential is expressed via compact and elegant formula (2.7). We applied these exact solutions to some practical problems to obtain the gravitational potentials for considered configurations. Found potentials were used to calculate the acceleration for point masses and gravitational self-energy.

To estimate corrections to the newtonian expressions, it is sufficient to keep only the first Kaluza–Klein modes. Then, we found that the correction term has the form of the Yukawa potential with parameters defined by multidimensional models. Such representation gave us a possibility of using the results of the inverse-square law experiments to get limitations on periods of tori. As a Yukawa potential approximation, it was shown that in the Cavendish-type experiments the corrections (due to the extra dimensions) to the newtonian gravitational constant are still far from the measurement accuracy of experiments for the determination of $G_N$.

Then, we proposed models where test masses can be smeared over the extra dimensions. The number of the smeared dimensions can be equal or less than the total number of the extra dimensions. We proved that the gravitational potential does not feel the smeared dimensions and this conclusion does not depend on the size of these dimensions. Such an approach opens new remarkable possibilities. For example in the case $D = 10$ with three large and three
smeared extra dimensions and $M_{Pl(10)} = 1$ TeV, the large extra dimensions can be as big as the upper limit established by the ISL experiments for $\alpha = 6$, i.e. $a \approx 2.1 \times 10^{-2}$ cm. This value of $a$ is in many orders of magnitude bigger than the rough estimate $a \approx 10^{-11.7}$ cm obtained from the fundamental Planck scale relation of the usual form (2.5). The limiting case where all extra dimensions are smeared is another interesting example. Here, there is no deviation from Newton’s law at all. Thus, these models can explain negative results for the detection of the extra dimensions in the ISL experiments irrespective of the size of the extra dimensions. Nevertheless, these models can still be used to solve the hierarchy problem.

Acknowledgments

We thank Uwe Günther for his stimulating discussions. AZh acknowledges the hospitality of the Theory Division of CERN and the High Energy, Cosmology and Astroparticle Physics Section of the ICTP during preparation of this work. This work was supported in part by the ‘Cosmomicrophysics’ programme of the Physics and Astronomy Division of the National Academy of Sciences of Ukraine.

Appendix. The gravitational force law under dimensional reduction

Let us consider a product space $M_D$ consisting of two components $M_D = M_3 \times M_d$, where for simplicity of the subsequent calculations we assume $M_3 = \mathbb{R}^3$ as a three-dimensional flat external space and a $d$-torus $M_d = T^d$ (with the same characteristic length $a$ along each of the $d$ dimensions) as compact internal space. The volume of $M_d$ is hence given as $\text{vol}(M_d) = V_d = a^d$.

The aim of this appendix is to demonstrate that the force laws for the full theory on $M_D$ and the effective theory on $M_3$ have correspondingly the form

$$F(r_{3+d}) = G_D \frac{m_1 m_2}{r_{3+d}^3},$$

(A.1)

$$F(r_3) = G_N \frac{m_1 m_2}{r_3^2},$$

(A.2)

where gravitational constants $G_D$ and $G_N$ are related to each other in accordance with (2.4). To demonstrate, we consider separately the regimes of small distances $r_{3+d} \ll a$ compared to the size of the internal space and of large distances $r_3 \gg a$. For $r_{3+d} \ll a$ the field lines penetrate all $D$-spatial dimensions, whereas at length scales $r_3 \gg a$ the internal dimensions are not accessible and effectively the field lines are spreading only along the three spatial dimensions of the external space $M_3$.

As in [3], we pass for simplicity of the calculations from the compact $d$-torus to its covering space $\mathbb{R}^d$. The single mass $m_1$ in $T^d$ is then mapped to a mass lattice in $\mathbb{R}^d$, which is build from $m_1$ and its mirror images in the cover.

For a test mass $m_2$ at small distances $r_{3+d} \ll a$ from $m_1$ the mirror masses give a negligible contribution to the gravitational force and in first-order approximation the Gauss law in $D$-space dimensions holds in accordance with (A.1). In this case the infinite set of mirror masses can in rough approximation be considered as symmetrically distributed around $m_1, m_2$ so that their contributions will compensate each other.

For the analysis of the opposite case of a large separation $r_3 \gg a$ between the masses $m_1$ and $m_2$, we assume the mass $m_1$ placed in the origin of the $D$-dimensional space. The lattice of the mirror masses spans then a $d$-dimensional subspace $\mathbb{R}^d$ (‘wire’) in the space
with a ‘surface’ density \( m_1/a^d \). Let us choose coordinates
\[
(z, r_3) \in \mathbb{R}^d \times \mathbb{R}^3
\]
and a test mass \( m_2 \) located at the point \((0, r_3)\). The force orthogonal to the lattice produced on \( m_3 \) by a small volume element \( dV_d \) at a point \((z, 0)\) of the (lattice) covering space \( \mathbb{R}^d \) is given by
\[
dF(z, r_3) = G_D \frac{r_3}{(z^2 + r_3^2)^{1/2}} \frac{m_1 dV_d}{a^d} \frac{m_2}{(z^2 + r_3^2)^{(3d)/2}} = G_D \frac{m_1 m_2}{a^d} \frac{r_3 dV_d}{(z^2 + r_3^2)^{(3d)/2}}.
\]
(A.4)
The factor \( r_3/(z^2 + r_3^2)^{1/2} \) is the cosine between the force direction and the lattice normal. Due to the lattice symmetry, the force component parallel to the lattice is compensated by an opposite force component from the point \((-z, 0)\). The total effective force \( F_{\text{eff}}(r_3) \) of the mass lattice on the test particle can now be easily obtained by integrating over the volume of \( \mathbb{R}^d \). Choosing spherical coordinates on \( \mathbb{R}^d \), the volume of a thin shell with radius \( z \) is
\[
dV_d = S_d z^{d-1} dz,
\]
(A.5)
where \( S_d \) is square of \((d-1)\)-dimensional sphere of a unit radius (see (2.1)) and the total effective force can be calculated as
\[
F_{\text{eff}}(r_3) = G_D \frac{m_1 m_2}{a^d} r_3 S_d \int_0^\infty \frac{z^{d-1} dz}{(z^2 + r_3^2)^{(3d)/2}}.
\]
(A.6)
With the substitution \( t = z^2 \) the integral can be transformed to the standard integral [5]
\[
\int_0^\infty x^{a-1} dx (x+b)^\rho = B(a, \rho - a), \quad 0 < \Re a < \Re \rho
\]
(A.7)
so that for the corresponding term in (A.6) we find
\[
\int_0^\infty \frac{z^{d-1} dz}{(z^2 + r_3^2)^{(3d)/2}} = \frac{1}{2} \int_0^\infty \frac{t^{d/2-1} dt}{(t + r_3^2)^{(3d)/2}} = \frac{1}{2r_3^d} \frac{3}{2} \frac{\Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{3}{2} \right)}{\Gamma \left( \frac{3d}{2} \right)}.
\]
(A.8)
Expressing the \( \Gamma \)-functions in (A.8) in terms of surface areas: \( \Gamma(D/2) = 2\pi^{D/2}/S_D \) gives
\[
\int_0^\infty \frac{z^{d-1} dz}{(z^2 + r_3^2)^{(3d)/2}} = \frac{1}{2r_3^d} \frac{3}{2} \frac{S_{3d} S_d}{S_d^3}.
\]
(A.9)
Hence, the total effective force takes the simple form
\[
F_{\text{eff}}(r_3) = \frac{G_D S_{3d} m_1 m_2}{S_d a^d r_3^3},
\]
(A.10)
and equating it with the gravitational force law (A.2) in three-dimensional space, we reproduce the result (2.4) for the relation between three-dimensional and multidimensional gravitational constants.
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