BALANCING STRAIGHT-LINE PROGRAMS

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Abstract. It is shown that a context-free grammar of size \( m \) that produces a single string \( w \) (such a grammar is also called a string straight-line program) can be transformed in linear time into a context-free grammar for \( w \) of size \( \mathcal{O}(m) \), whose unique derivation tree has depth \( \mathcal{O}(\log |w|) \). This solves an open problem in the area of grammar-based compression. Similar results are shown for two formalisms for grammar-based tree compression: top daggs and forest straight-line programs. These balancing results are all deduced from a single meta theorem stating that the depth of an algebraic circuit over an algebra with a certain finite base property can be reduced to \( \mathcal{O}(\log n) \) with the cost of a constant multiplicative size increase. Here, \( n \) refers to the size of the unfolding (or unravelling) of the circuit. In particular, this results applies to standard arithmetic circuits over (noncommutative) semirings.

1. Introduction

Grammar-based string compression. In grammar-based compression a combinatorial object is compactly represented using a grammar of an appropriate type. In many grammar-based compression formalisms such a grammar can be exponentially smaller than the object itself. A well-studied example of this general idea is grammar-based string compression using context-free grammars that produce only one string. This can be syntactically enforced by allowing for each variable \( X \) only one production with left-hand side \( X \) and excluding cycles in the derivation. Such context-free grammars are also known as straight-line programs. Since the term “straight-line programs” is used in the literature for different kinds of objects (e.g. arithmetic straight-line programs) and we will also deal with different types of straight-line programs, we use the term string straight-line program, SSLP for short.

Grammar-based string compression is tightly related to dictionary based compression: the famous LZ78 algorithm can be viewed as a particular grammar-based compressor, and the number of phrases in the LZ77-factorization is a lower bound for the smallest SSLP for a string \([34]\). For various other aspects of grammar-based string compression see \([11, 28]\).

Balancing string straight-line programs. The two important measures for an SSLP are size and depth. To define these measures, it is convenient to assume that all right-hand sides of the grammar have length two (as in Chomsky normal form). Then, the size \( |G| \) of an SSLP \( G \) is the number of variables (nonterminals) of \( G \) and the depth of \( G \) (depth(\( G \)) for short) is the depth of the unique derivation tree of \( G \). It is straightforward to show that any string \( s \) of length \( n \) can be produced by an SSLP of size \( \mathcal{O}(n) \) and depth \( \mathcal{O}(\log n) \). A more difficult problem is to balance a given...
SSLP: assume that the SSLP $G$ produces a string of length $n$. Several authors have shown that one can restructure $G$ in time $O(|G| \cdot \log n)$ into an equivalent SSLP $H$ of size $O(|G| \cdot \log n)$ and depth $O(\log n)$ [11, 22, 34].

Finding SSLPs of small size and small depth is important in many algorithmic applications. A prominent example is the random access problem for grammar-compressed strings. For a given SSLP $G$ that produces the string $s$ of length $n$ and a given position $p \in [1, n]$ one wants to access the $p$-th symbol in $s$. As observed in [8] one can solve this problem in time $O(\text{depth}(G))$. Using several levels of sophisticated data structures, it is shown in [8] that one can compute from $G$ a data structure of size $O(|G|)$ (measured in words of bit length $\log n$) which allows to access every position in time $O(\log n)$. As remarked in [8], one can obtain $O(\log n)$ access time using one of the known SSLP balancing procedures [11, 34], but this increases the size to $O(|G| \cdot \log n)$. Our main result for string straight-line programs states that SSLP balancing is in fact possible with a constant blow-up in size: a given SSLP of size $m$ that produces a string of length $n$ can be transformed in time $O(m)$ into an equivalent SSLP of size $O(m)$ and depth $O(\log n)$ (Theorem 10.3). As a corollary we obtain a very simple and clean algorithm for the random access problem with access time $O(\log n)$ that uses a data structure of size $O(m)$ (in words of bit length $\log n$). We can also obtain an algorithm for the random access problem with running time $O(\log n / \log \log n)$ using $O(m \cdot \log^2 n)$ words, previously this bound was only shown for balanced SSLPs [2]. Section 11 contains a list of further applications of Theorem 10.3, which include the following problems on SSLP-compressed strings: rank and select queries [2], subsequence matching [3], computing Karp-Rabin fingerprints [5], computing runs, squares, and palindromes [22], and real-time traversal [13, 31]. In all these applications we either improve existing results or significantly simplify existing proofs by replacing $\text{depth}(G)$ by $O(\log n)$ in time/space bounds.

We will derive our balancing result for string straight-line programs from a more general result that applies to circuits over algebras with a certain finite base property (string straight-line programs are circuits over free monoids), and that will be explained in the next paragraph. For those readers who are only interested in the balancing result for string straight-line programs we offer a direct proof in Appendix A.

Balancing circuits over algebras. Our balancing result for string straight-line programs is an instance of a more general result. A string straight-line program is the same thing as a bounded fan-in circuit over a free monoid. The circuit gates compute the concatenation of their inputs and correspond to the variables of the string straight-line program. We prove a general balancing result that applies to a large class of algebras (that contains free monoids). The definition of this class of algebras uses unary linear term functions. Fix an algebra $A$ (a set together with finitely many operations of possibly different arities). A unary linear term function is a unary function on $A$ that is computed by a term (or algebraic expression) that contains a single variable $x$ (which stands for the function argument) and, moreover, $x$ occurs exactly once in the term. For instance, a unary linear term function over a commutative ring is of the form $x \mapsto ax + b$ for ring elements $a, b$. A subsumption base for an algebra $A$ is, roughly speaking, a finite set $C(A)$ of unary linear term function that are described by terms with parameters such that every unary linear term function can be obtained from one of the terms in $C(A)$ by instantiating the parameters. In the above example for a commutative ring the set $C(A)$ consists of the single term $ax + b$, where $a$ and $b$ are the parameters. The main result

\[^1\]This uses the RAM model where arithmetic operations on numbers from the interval $[0, n]$ need constant time.
of this paper states that for every algebra \( A \) that has a finite subsumption base there is a linear time algorithm that transforms a given circuit \( G \) over \( A \) of size \( m \) into an equivalent circuit (i.e., one that computes the same element of \( A \)) of size \( \mathcal{O}(m) \) and depth \( \mathcal{O}(\log n) \). Here, \( n \) is the size of the algebraic expression obtained by unfolding the circuit \( G \) into a tree (Theorem 9.2). Our balancing result for string straight-line programs is an immediate corollary of this result, since every monoid has a finite subsumption base. Semirings (not necessarily commutative) have finite subsumption bases as well. Hence, for every semiring circuit one can reduce with a linear size blow-up the depth to \( \mathcal{O}(\log n) \), where \( n \) is the size of the circuit unfolding.

**Balancing forest straight-line programs and top dags.** Another application of our general balancing result concerns grammar-based tree compression. Grammar-based compression has been generalized from strings to ordered ranked node-labelled trees. In fact, the representation of a tree \( t \) by its smallest directed acyclic graph (DAG) is a form of grammar-based tree compression. This DAG is obtained by merging nodes where the same subtree of \( t \) is rooted. It can be seen as a regular tree grammar that produces only \( t \). A drawback of DAG-compression is that the size of the DAG is lower-bounded by the height of the tree \( t \). Hence, for deep shallow trees (like for instance caterpillar trees), the DAG-representation cannot achieve good compression. This can be overcome by representing a tree \( t \) by a linear context-free tree grammars that produces only \( t \). Such grammars are also known as tree straight-line programs in the case of ranked trees \([10, 29, 30]\) and forest straight-line programs in the case of unranked trees \([17]\). The latter are tightly related to top dags \([7, 4, 13, 21]\), which are another tree compression formalism, also akin to grammars. Forest straight-line programs and top dags can be defined as circuits over certain algebras, called forest algebras \([9, 17]\) and cluster algebras \([17]\). Both types of algebras turn out to have finite subsumption bases. With Theorem 9.2 it follows that from a forest straight-line program (resp., top dag) of size \( m \) that defines a tree of size \( n \), one can compute in linear time an equivalent forest straight-line program (resp., top dag) of size \( \mathcal{O}(m) \) and depth \( \mathcal{O}(\log n) \). This solves an open problem from \([7]\), where the authors proved that from a tree \( t \) of size \( n \), whose minimal DAG has size \( m \) (measured in number of edges in the DAG), one can construct in linear time a top dag for \( t \) of size \( \mathcal{O}(m \cdot \log n) \) and depth \( \mathcal{O}(\log n) \). It remained open whether one can get rid of the additional factor \( \log n \) in the size bound. For the specific top dag constructed in \([7]\), it was shown in \([4]\) that the factor \( \log n \) in the size bound \( \mathcal{O}(m \cdot \log n) \) cannot be avoided. On the other hand, our results yield another top dag of size \( \mathcal{O}(m) \) and depth \( \mathcal{O}(\log n) \). To see this note that one can easily convert the minimal DAG of \( t \) into a top dag of roughly the same size, which can then be balanced.

**Proof strategy of Theorem 9.2.** Our proof of Theorem 9.2 consists of two main steps. We start with a circuit \( G_0 \) over an algebra \( A \). Unfolding this circuit yields an expression tree \( t \) over the algebra \( A \). Let \( n \) be the size of \( t \) and \( m \) be the size of the circuit \( G_0 \).

In the previous paragraph we mentioned tree straight-line programs that have been used for the succinct representation of ranked trees. A tree straight-line program is a particular context-free tree grammar that produces exactly one tree. In a first step we compute from the circuit \( G_0 \) in linear time a tree straight-line program \( G_1 \) for the expression tree \( t \). The size of \( G_1 \) is \( \mathcal{O}(m) \), whereas the depth of \( G_1 \) is \( \mathcal{O}(\log n) \). This first step is purely syntactic and does not depend on the algebra

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2Instead of circuits, we prefer to work with the equivalent formalism of straight-line programs in the main part of the paper.
Our algorithm decomposes $G_0$ (viewed as a directed acyclic graph) into disjoint paths such that each path from the root to a leaf only intersects $O(\log m)$ paths from the decomposition (Section 13). Each path from the decomposition is then viewed as a string of integer-weighted symbols. These weights are the sizes of the trees obtained by unfolding the circuit nodes that branch off from the path. For this weighted string we construct an SSLP of linear size that produces all suffixes of the path in a weight-balanced way (Section 14). Plugging these SSLPs together yields the tree straight-line program $G_1$.

The second step in our proof of Theorem 9.2 assumes that we interpret the expression $t$ in an algebra $A$ with a finite subsumption base. We show that for every such algebra $A$ one can compute from a tree straight-line program $G_1$ that defines the expression tree $t$ in linear time a circuit $G_2$ over $A$ that defines the same element of $A$ as $t$ (Lemma 8.7). Moreover, the size and depth of $G_2$ are linearly bounded in the size and depth, respectively, of $G_1$. This construction was used before for the special cases of semirings and regular expressions [15, 16].

Related work. Algebras with a finite subsumption base have been implicitly used in our recent papers [15, 16]. There, we proved that a ranked tree $t$ of size $n$ can be transformed in linear time into a tree straight-line program of size $O(n/\log_\sigma n)$ and depth $O(\log n)$, where $\sigma$ is the the number of different node labels that appear in $t$. With Lemma 8.7 it follows that for every algebra $A$ having a finite subsumption base one can compute in linear time from a given expression tree of size $n$ an equivalent circuit of size $O(n/\log_\sigma n)$ and depth $O(\log n)$ ($\sigma$ is a constant here, namely the number of operations of the algebra $A$). Applying this to top dags gives an alternative proof for a result from [13], according to which one can construct in linear time a top dag of size $O(n/\log_\sigma n)$ and depth $O(\log n)$ for a given tree of size $n$ containing $\sigma$ many different node labels.

Note that in the depth bound $O(\log n)$ in our balancing result for string straight-line programs (Theorem 10.3), $n$ refers to the length of the produced string. A string straight-line program can be viewed as a circuit for a non-commutative semiring circuit that produces a single monomial (the symbols in the string correspond to the non-commuting variables). If one considers arbitrary circuits over non-commutative semirings (that produce a sum of more than one monomial), depth reduction is not possible in general by a result of Kosaraju [26]. For circuits over commutative semirings depth reduction is possible by a seminal result of Valiant, Skyum, Berkowitz and Rackoff [30]: for any commutative semiring, every circuit of size $m$ and formal degree $d$ can be transformed into an equivalent circuit of depth $O(\log m \log d)$ and size polynomial in $m$ and $d$. This result led to many further investigations on depth reduction for bounded degree circuits over various classes of commutative as well as noncommutative semirings; see [1] for an excellent survey. If one drops the restriction to bounded degree circuits, then depth reduction gets even harder. For general Boolean circuits, the best known result states that every Boolean circuit of size $m$ is equivalent to a Boolean circuit of depth $O(m/\log m)$ [33].

Some of the concepts in this paper can be traced back to the area of parallel algorithms. Linear term function were also used in [32] in the context of efficient parallel evaluation of expression trees. Our path decomposition for DAGs from Section 13 is related to the centroid path decomposition of trees [12], where it is the key technique in several parallel algorithms on trees. Moreover, the SSLP of linear size that produces all suffixes of a string with weights (Section 14) can be seen as a weight-balanced version of the optimal prefix sum algorithm.
2. General notations

Given an alphabet of symbols Σ, Σ+ denotes the set of all finite words over
the alphabet Σ, including the empty word ε. The set of non-empty words is
denoted by Σ+ = Σ+ \ {ε}. The length of a word w is denoted with |w|.

The composition of two functions f : A → B and g : B → C is denoted by g ∘ f.
Note that we first apply f followed by g.

3. Ranked trees

Let us fix a finite set S of sorts. Later, we will assign to each sort i ∈ S a
set A_i (of elements of sort i). An S-sorted signature is a set of symbols Π and a
mapping type : Π → S^+ that assigns to each symbol from Π a non-empty word over
the alphabet S. The number |type(f)| − 1 ≥ 0 is also called the rank of f. Let
Γ_i ⊆ Γ (i ≥ 0) be the set of all symbols in Γ of rank i.

Let us also fix a second (infinite) S-sorted signature X, where every x ∈ X has
rank zero. Elements of X are called variables. Since x ∈ X has rank zero, type(x)
is an element of S. For p ∈ S let X_p = {x | type(x) = p}. We assume that every set
X_p is infinite. We will always work with a finite subset Y of X. Take such a Y.
For each sort p ∈ S we define the set of terms T_p(Γ, Y) of sort p by simultaneous
induction as the smallest set such that the following holds:

- Every x ∈ X_p ∩ Y belongs to T_p(Γ, Y).
- If f ∈ Γ_n with type(f) = p_1 ⋯ p_n and t_1 ∈ T_{p_1}(Γ, Y), . . . , t_n ∈ T_{p_n}(Γ, Y),
  then f(t_1, t_2, . . . , t_n) ∈ T_p(Γ, Y).

We write T_p(Γ) for T_p(Γ, Y), and call its elements ground terms (of sort p). Note
that if a ∈ Γ_0 and type(a) = p ∈ S then a() ∈ T_p(Γ). In this case, we write a for
a() and call a a constant of sort p. Let T(Γ, Y) = ∪_{p ∈ S} T_p(Γ, Y).

Elements of T(Γ, Y) can be viewed as node labeled trees, where leaves are labeled
with symbols form Γ_0 ∪ Y and every internal node is labeled with a symbol from
some Γ_n with n ≥ 1: The root of the tree corresponding to the term f(t_1, t_2, . . . , t_n)
is labeled with f and its direct subtrees are the trees corresponding to t_1, . . . , t_n.

For a term t we define the size |t| of t as the number of edges of the corresponding
tree. Equivalently, |t| is inductively defined as follows: If t = x is a variable, then
|t| = 0. If t = f(t_1, t_2, . . . , t_n) for f ∈ Γ, then |t| = n + ∑_{i=1}^n |t_i|. The depth
of a term t is denoted by depth(t) and defined inductively as usual: If t = x is a
variable, then depth(t) = 0. If t = f(t_1, t_2, . . . , t_n) for f ∈ Γ, then depth(t) =
max{1 + depth(t_i) | 1 ≤ i ≤ n} with max{0} = 0.

Definition 3.1 (substitutions). A substitution is a mapping η : Y → T(Γ, Z) for
finite (not necessarily disjoint) subsets Y, Z ⊆ X such that y ∈ Y \ Y_0 implies η(y) ∈
T_p(Γ, Z). If Z = ∅, we speak of a ground substitution. For t ∈ T(Γ, Y) we define
the term η(t) by replacing simultaneously all occurrences of variables in t by their
images under η. Formally we extend η : Y → T(Γ, Z) to a mapping η : T(Γ, Y) →
T(Γ, Z) by η(f(t_1, . . . , t_n)) = f(η(t_1), . . . , η(t_n)) (in particular, η(a) = a for a ∈ Γ_0).
A variable renaming is a bijective substitution η : Y → Z for finite variable sets Y
and Z of the same size.

Definition 3.2 (contexts). Let p, q ∈ S. We define the set of contexts C_{pq}(Γ, Y) as
the set of all terms t ∈ T_p(Γ, Y \ {x}), where x ∈ X_p \ Y is a fresh variable such that
(i) t ̸= x, (ii) x occurs exactly once in t. We call x the main variable of t and Y
the set of auxiliary variables of t.\(^3\) We write C_{pq}(Γ) for C_{pq}(Γ, Y). Elements of C_{pq}(Γ)

\(^3\)Since also Y may contain a variable y that occurs exactly once in t, we explicitly have to
declare a variable as the main variable. Most of the times, the main variable will be denoted with
x.
are called ground contexts. Let \( \mathcal{C}(\Gamma, \mathcal{Y}) = \bigcup_{p,q \in \mathcal{S}} \mathcal{C}_{pq}(\Gamma, \mathcal{Y}) \) and \( \mathcal{C}(\Gamma) = \mathcal{C}(\Gamma, \emptyset) \). For \( s \in \mathcal{C}_{qp}(\Gamma, \mathcal{Y}) \) and \( t \in \mathcal{T}_p(\Gamma, \mathcal{Y} \cup \mathcal{Z}) \) (or \( t \in \mathcal{C}_{pq}(\Gamma, \mathcal{Z}) \)) we define \( s[t] \in \mathcal{T}_p(\Gamma, \mathcal{Y} \cup \mathcal{Z}) \) \((s[t] \in \mathcal{C}_{pq}(\Gamma, \mathcal{Y} \cup \mathcal{Z}))\) as the result of replacing the unique occurrence of the main variable in \( s \) by \( t \). Formally, we can define \( s[t] \) as \( \eta(s) \) where \( \eta \) is the substitution with domain \( \{ x \} \) and \( \eta(x) = t \), where \( x \) is the main variable of \( s \). An atomic context is a context of the form \( f(y_1, \ldots, y_k, x, y_{k+1}, \ldots, y_k) \) where \( x \) is the main variable and the \( y_i \) are the auxiliary variables (we can have \( y_i = y_j \) for \( i \neq j \)). Note that there are only finitely many atomic contexts up to renaming of variables.

4. Algebras

We will produce strings, trees and forests by ground terms (also called algebraic expressions in this context) over certain algebras. These expressions will be compressed by directed acyclic graphs. In this section, we introduce the generic framework, which will be instantiated several times in this paper.

Fix a finite \( \mathcal{S} \)-sorted signature \( \Gamma \). A \( \Gamma \)-algebra is a tuple \( \mathcal{A} = ((A_p)_{p \in \mathcal{S}}, (f^\mathcal{A})_{f \in \Gamma}) \) where every \( A_p \) is a non-empty set (the universe of sort \( p \)) and every \( f \in \Gamma \), with \( \text{type}(f) = p_1p_2 \cdots p_nq \), \( f^\mathcal{A} : \prod_{1 \leq j \leq n} A_{p_j} \rightarrow A_q \) is an \( n \)-ary function. We also say that \( \Gamma \) is the signature of \( \mathcal{A} \). In our settings, the sets \( A_p \) will always be pairwise disjoint, but formally we do not need this. Quite often, we will identify the function \( f^\mathcal{A} \) with the symbol \( f \). Functions of arity zero are elements of some \( A_p \). A ground term \( t \in \mathcal{T}_p(\Gamma) \) can be viewed as algebraic expressions over \( \mathcal{A} \) that evaluate to an element \( t^\mathcal{A} \in A_p \) in the natural way. For \( x \in \bigcup_{p \in \mathcal{S}} A_p \), we also write \( x \in \mathcal{A} \) and for \( A_p \), we also write \( A_p \).

When we define a \( \Gamma \)-algebra, we usually will not specify the types of the symbols in \( \Gamma \). Instead, we just list the sets \( A_p \) (\( p \in \mathcal{S} \)) and the functions \( f^\mathcal{A} \) (\( f \in \Gamma \)) including their domains. The latter implicitly determine the types of the symbols in \( \Gamma \).

From the sets \( \mathcal{T}_p(\Gamma) \) one can construct the free term algebra
\[
\mathcal{T}(\Gamma) = ((\mathcal{T}_p(\Gamma))_{p \in \mathcal{S}}, (f^\mathcal{A})_{f \in \Gamma}),
\]
where every ground term evaluates to itself. For every \( \Gamma \)-algebra \( \mathcal{A} \), the mapping \( t \mapsto t^\mathcal{A} \) \((t \in \mathcal{T}(\Gamma))\) is a homomorphism from the free term algebra to \( \mathcal{A} \). We need the technical assumption that this homomorphism is surjective, i.e., for every \( a \in \mathcal{A} \) there exists a ground term \( t \in \mathcal{T}(\Gamma) \) with \( a = t^\mathcal{A} \). In our concrete applications this assumption will be satisfied. Moreover, one can always replace \( \mathcal{A} \) by the subalgebra induced by the elements \( t^\mathcal{A} \) (we will say more about this later).

For a \( \Gamma \)-algebra \( \mathcal{A} = ((A_p)_{p \in \mathcal{S}}, (f^\mathcal{A})_{f \in \Gamma}) \), a variable \( x \in A_p \), and \( a \in A_q \), we define the \((\Gamma \cup \{ x \})\)-algebra \( \mathcal{A}[x/a] = ((A_p)_{p \in \mathcal{S}}, (f^\mathcal{A}[x/a])_{f \in \Gamma \cup \{ x \}}) \) by \( f^\mathcal{A}[x/a] = f^\mathcal{A} \) for \( f \in \Gamma \) and \( x^\mathcal{A}[x/a] = a \).

**Definition 4.1** (unary linear term functions). Given a \( \Gamma \)-algebra \( \mathcal{A} \) and a ground context \( t \in \mathcal{C}_{pq}(\Gamma) \) with main variable \( x \), we define the function \( t^\mathcal{A} : A_p \rightarrow A_q \) by \( t^\mathcal{A}(a) = t^\mathcal{A}[x/a] \) for all \( a \in A_p \). We call \( t^\mathcal{A} \) a unary linear term function, ULTF for short. We write \( \text{lin}_{pq}(\mathcal{A}) \) for the set of all ULTFs \( t^\mathcal{A} \) with \( t \in \mathcal{C}_{pq}(\Gamma) \).

5. Straight-line programs

A straight-line program over the \( \mathcal{S} \)-sorted signature \( \Gamma \) (\( \Gamma \)-SLP for short) is a tuple \( \mathcal{G} = (\mathcal{V}, \mathcal{P}, S) \), where \( \mathcal{V} \subseteq \mathcal{A} \) is a finite set of variables, \( S \in \mathcal{V} \) is the start variable and \( \rho : \mathcal{V} \rightarrow \mathcal{T}(\Gamma, \mathcal{V}) \) is a substitution (the so called right-hand side mapping) such that the edge relation \( E(\mathcal{G}) = \{(y, z) \in \mathcal{V} \times \mathcal{V} \mid z \text{ occurs in } \rho(y) \} \) is acyclic. This implies that there exists an \( n \geq 1 \) such that \( \rho^n : \mathcal{T}(\Gamma, \mathcal{V}) \rightarrow \mathcal{T}(\Gamma) \) (the \( n \)-fold composition of \( \rho \)) is a ground substitution (we can choose \( n = |\mathcal{V}| \)). For this \( n \), we write \( \rho^n \) for...
\[ \hat{\Gamma} \text{-algebra} \hat{\Gamma} \]

The operations \[ \llbracket X \rrbracket_\hat{\Gamma} \] (or \[ \llbracket X \rrbracket_G \] if \( G \) is clear from the context) for the ground term \( \rho^*(X) \).

Let \( A \) be a \( \Gamma \)-algebra. A \( \Gamma \)-SLP \( \hat{G} = (V, \rho, S) \) is also called an SLP over the algebra \( A \). We can evaluate every variable \( X \in V \) to its value \( \rho^*(X)^A = \llbracket X \rrbracket^A \in A \) in \( A \). It is important to distinguish this value from the syntactically computed ground term \( \rho^*(X) \) (which is the evaluation of \( X \) in the free term algebra).

The term \( \rho(X) \) is also called the right-hand side of the variable \( X \in V \). By adding fresh variables, we can transform every \( \Gamma \)-SLP in linear time into a so-called standard \( \Gamma \)-SLP, where all right-hand sides have the form \( f(X_1, \ldots, X_n) \) for variables \( X_1, \ldots, X_n \) (we can have \( X_i = X_j \) for \( i \neq j \)). A standard \( \Gamma \)-SLP \( \hat{G} \) is the same object as a DAG (directed acyclic graph) with \( \Gamma \)-labelled nodes: the DAG is \(( V, E(\hat{G}) ) \) and if \( \rho(X) = f(X_1, \ldots, X_n) \) then node \( X \) is labelled with \( f \). Since the order of the edges \( (X_i, X_j) \) \((1 \leq i \leq n) \) is important and we may have \( X_i = X_j \) for \( i \neq j \) we formally replace the edge \( (X_i, X_j) \) by the triple \( (X_i, i, X_j) \). A \( \Gamma \)-SLP interpreted over a \( \Gamma \)-algebra \( A \) is also called an algebraic circuit over \( A \).

Consider a (possibly non-standard) \( \Gamma \)-SLP \( \hat{G} = (V, \rho, S) \). We define the size of \( |\hat{G}| \) as \( \sum_{X \in V} |\rho(X)| \). For a standard \( \Gamma \)-SLP this is the number of edges of the corresponding DAG \(( V, E(\hat{G}) ) \). The \textit{depth} of \( \hat{G} \) is defined as \( \text{depth}(\hat{G}) = \text{depth}(\llbracket \hat{G} \rrbracket) \), i.e. the depth of the derivation tree of \( \hat{G} \). For a standard \( \Gamma \)-SLP \( \hat{G} \) this is the maximum length of a directed path in the DAG \(( V, E(\hat{G}) ) \). Our definitions of size and depth ensure that both measures does not increase when one transforms a given \( \Gamma \)-SLP into a standard \( \Gamma \)-SLP. In this paper, the sizes of the right-hand sides will be always bounded by a constant that only depends on the underlying algebra \( A \).

6. Functional extensions

An important concept in this paper is a functional extension \( \hat{T}(\Gamma) \) of the free term algebra \( T(\Gamma) \). We define an algebra \( \hat{T}(\Gamma) \) over an \( S \cup S^2 \)-sorted signature \( \hat{\Gamma} \).

Def. 6.1 \( (\hat{\Gamma} \text{-algebra} \hat{T}(\Gamma)) \). The \( S \cup S^2 \)-sorted signature \( \hat{\Gamma} \) is

\[ \hat{\Gamma} = \Gamma \cup \{ \hat{f}_i \mid f \in \Gamma, 1 \leq i \leq \ell \} \cup \{ \gamma_{pqr} \mid p, q, r \in S \} \cup \{ \alpha_{pq} \mid p, q \in S \}. \]

We define the \( \hat{\Gamma} \)-algebra \( \hat{T}(\Gamma) = ((A_s)_{s \in S \cup S^2}, (f^{\hat{T}(\Gamma)})_{f \in \Gamma}) \) as follows: the sets \( A_p \) and \( A_{pq} \) for \( p, q \in S \) are defined as

- \( A_p = T_p(\Gamma) \)
- \( A_{pq} = C_{pq}(\Gamma) \).

The operations \( g^{\hat{T}(\Gamma)} \) \((g \in \hat{\Gamma}) \) are defined as follows, where we write \( g \) instead of \( g^{\hat{T}(\Gamma)} \):

- For every symbol \( f \in \Gamma_n \) the algebra \( \hat{T}(\Gamma) \) inherits the function \( f^{\hat{T}(\Gamma)} \) from \( T(\Gamma) \).
- For every symbol \( f \in \Gamma_n \) with type \( (f) = p_1 \cdots p_n q \) \((n \geq 1) \) and every \( 1 \leq k \leq n \) we define the \((n - 1)\)-ary operation \( \hat{f}_k : \prod_{1 \leq i \leq k} T_{p_i}(\Gamma) \to C_{pq}(\Gamma) \) by \( \hat{f}_k(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n) = f(t_1, \ldots, t_{k-1}, x, t_{k+1}, \ldots, t_n) \) for all \( t_i \in T_{p_i}(\Gamma) \) \((1 \leq i \leq n, i \neq k) \).
• For all \( p, q, r \in S \) the binary operation \( \gamma_{pq} : C_{pq}(\Gamma) \times C_{qr}(\Gamma) \to C_{pr}(\Gamma) \) is defined by \( \gamma_{pq}(t, s) = s[t] \).
• For all \( p, q \in S \) the binary operation \( \alpha_{pq} : T_p(\Gamma) \times C_{pq}(\Gamma) \to T_q(\Gamma) \) is defined by \( \alpha_{pq}(t, s) = s[t] \).

The definition of the operations \( \alpha_{pq} \) and \( \gamma_{pq} \) suggests to write \( s[t] \) instead of \( \alpha_{pq}(s, t) \) or \( \gamma_{pq}(s, t) \), which we will do most of the times.

Recall the definition of unary linear term functions (ULTFs) from Definition 6.1. An atomic ULTF is of the form \( \gamma \) or \( \alpha \) with \( \gamma(t) = \gamma(p, q, r) \) and \( \alpha(a) = \alpha(A) \) for \( a \in A_p \) and \( A \in \mathcal{S} \) for \( (1 \leq i \leq n, i \neq k) \). We now extend this function with \( f^A(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \) in the following. At this point, we use the assumption that every element of \( \mathcal{A} \) can be written as \( t^A \) for a ground term \( t \). Hence, the elements \( a_i \) are defined by terms, which ensures that \( f^A(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \) is indeed a ULTF. It is easy to see that every ULTF is the composition of finitely many atomic ULTFs.

**Definition 6.2** (\( \hat{\Gamma} \)-algebra \( \hat{\mathcal{A}} \)). Given a \( \Gamma \)-algebra \( \mathcal{A} = ((A_p)_{p \in S}, (f^A)_{f \in \Gamma}) \) we define the \( \hat{\Gamma} \)-algebra \( \hat{\mathcal{A}} = ((B_s)_{s \in S \cup S^2}, (f^A)_{f \in \Gamma}) \) as follows: The sets \( B_s \) and \( B_{pq} \) for \( p, q \in S \) are defined as:

- \( B_p = A_p \)
- \( B_{pq} = \text{lin}_{pq}(A) \)

The operations \( g^A (g \in \hat{\Gamma}) \) are defined as follows, where we write \( g \) instead of \( g^A \).

- Every \( f \in \Gamma \) is interpreted as \( f^A = f^A \).
- For every symbol \( f \in \Gamma \) with \( \text{type}(f) = p_1 \cdots p_n q \) \((n \geq 1)\) and every \( 1 \leq k \leq n \) we define the \((n - 1)\)-ary operation \( \hat{f}_k : \prod_{1 \leq i \leq n, i \neq k} A_{p_i} \to \text{lin}_{pq}(A) \)

  by \( \hat{f}_k(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) = f^A(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \) for all \( a_i \in A_{p_i} \) \((1 \leq i \leq n, i \neq k)\).
- For all \( p, q, r \in S \) the binary operation \( \gamma_{pq} : \text{lin}_{pq}(A) \times \text{lin}_{qr}(A) \to \text{lin}_{pr}(A) \) is defined as function composition: \( \gamma_{pq}(g, h) = h \circ g \).
- For all \( p, q \in S \) the binary operation \( \alpha_{pq} : A_p \times \text{lin}_{pq}(A) \to A_q \) is defined as function application: \( \alpha_{pq}(a, g) = g(a) \).

Note that Definitions 6.1 and 6.2 are consistent in the following sense: If we apply the construction from Definition 6.2 for \( \mathcal{A} = T(\Gamma) \) (the free term algebra) then we obtain an isomorphic copy of the algebra \( \hat{T}(\Gamma) \) from Definition 6.1 i.e., \( \hat{T}(\Gamma) \cong \hat{T}(\Gamma) \). Moreover, the mappings \( t \mapsto t^A \) (for ground terms \( t \)) and \( c \mapsto c^A \) (for ground contexts \( c \)) yield a canonical surjective morphism from \( \hat{T}(\Gamma) \) to \( \hat{\mathcal{A}} \) that extends the canonical morphism from the free term algebra \( T(\Gamma) \) to \( \mathcal{A} \).

7. TREE STRAIGHT-LINE PROGRAMS

Recall the definition of the \( \mathcal{S} \cup \mathcal{S}^2 \)-sorted signature \( \hat{\Gamma} \) in [1]. A \( \hat{\Gamma} \)-SLP \( G \) which evaluates in the \( \hat{\Gamma} \)-algebra \( \hat{T}(\Gamma) \) to a ground term (i.e., \( [G]^{\hat{T}(\Gamma)} \in T(\Gamma)) \) is also called a tree straight-line program over \( \Gamma \) (T-TSLP for short) [15 16 29].

Recall that \( \hat{\Gamma} \) contains for every \( f \in \Gamma_n \) with \( n \geq 1 \) the unary symbols \( \hat{f}_k \) \((1 \leq k \leq n)\). Right-hand sides of the form \( \hat{f}_k(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n) \) in a \( \Gamma \)-TSLP are written as \( f(X_1, \ldots, X_{k-1}, x, X_{k+1}, \ldots, X_n) \) for better readability. This is also the notation used in [15 16 29]. For right-hand sides of the form \( \alpha_{pq}(X, Y) \) or \( \gamma_{pq}(X, Y) \) we write \( X[Y] \).
Example 7.1. Let us assume that $S$ consists of a single sort. Consider the $\Gamma$-
TSLP $G = \{\{S, X_1, \ldots, X_7\}, \rho, S\}$ with $\Gamma_2 = \{f, g\}, \Gamma_0 = \{a, b\}$ and $\rho(S) = X_1[X_2], \rho(X_1) = X_3[X_5], \rho(X_2) = X_4[X_5], \rho(X_3) = f(x, X_7), \rho(X_4) = X_6[X_6], \rho(X_5) = a$, $\rho(X_6) = g(X_7, x), \rho(X_7) = b$. We get

- $[X_6]^{\Gamma} = \rho^*(X_6)^{\Gamma} = g(b, x),$
- $[X_4]^{\Gamma} = \rho^*(X_4)^{\Gamma} = g(b, x)[g(b, x)] = g(b, g(b, x)),$
- $[X_3]^{\Gamma} = \rho^*(X_3)^{\Gamma} = f(x, b),$
- $[X_2]^{\Gamma} = \rho^*(X_2)^{\Gamma} = g(b, g(b, x))[a] = g(b, g(b, a)),$
- $[X_1]^{\Gamma} = \rho^*(X_1)^{\Gamma} = f(x, b)[f(x, b)] = f(f(x, b), b),$ and
- $[g]^{\Gamma} = \rho^*(S)^{\Gamma} = f(f(x, b), b)[g(b, g(b, a))] = f(f(g(b, g(b, a)), b), b).

8. FROM TSLPS TO SLPs

Fix a $\Gamma$-algebra $A$. Our first goal is to transform a $\Gamma$-TSLP $G$ into a $\Gamma$-SLP $H$ of

size $O(|G|)$ and depth $O(\text{depth}(G))$ such that $[H]^A = [G]^A$. For this, we have to restrict the class of $\Gamma$-algebras. For instance, for the free term algebra the above transformation cannot be achieved in general: the chain tree $t_n = f(f(f(\cdots f(a)\cdots)))$ with $2^n$ occurrences of $f$ can be easily produced by a $\{a, f\}$-TSLP of size $O(n)$ but the only DAG (= SLP over the free term algebra $T(\{a, f\})$) for $t_n$ is $t_n$ itself. The following concepts turn out to be useful.

Definition 8.1 (equivalence and subsumption preorder in $A$). For contexts $s, t \in C_{pq}(\Gamma, \mathcal{Y})$ we say that $s$ and $t$ are equivalent in $A$ if for every ground substitution $\eta: \mathcal{Y} \rightarrow T(\Gamma)$ we have $\eta(s)^A = \eta(t)^A$ (which is an ULTF).

For contexts $s \in C_{pq}(\Gamma, \mathcal{Y})$ and $t \in C_{pq}(\Gamma, \mathcal{Z})$ we say that $t$ subsumes $s$ in $A$ or that $s$ is subsumed by $t$ in $A$ ($t \preceq A s$ for short) if there exists a substitution $\zeta: \mathcal{Z} \rightarrow T(\Gamma, \mathcal{Y})$ such that $s$ and $\zeta(t)$ are equivalent in $A$.

A subsumption base of $A$ is a set of (not necessarily ground) contexts $C$ such that for every context $s$ there exists a context $t \in C$ with $t \preceq A s$.

It is easy to see that $\preceq A$ is reflexive and transitive but in general not asymmetric. Moreover, the relation $\preceq A$ satisfies the following monotonicity property:

Lemma 8.2. Let $s \in C_{pq}(\Gamma, \mathcal{Y}), t_1 \in C_{pq}(\Gamma, \mathcal{Z}_1)$ and $t_2 \in C_{pq}(\Gamma, \mathcal{Z}_2)$ be contexts such that $\mathcal{Y} \cap \mathcal{Z}_1 = \emptyset$ and $\mathcal{Y} \cup \mathcal{Z}_1 \cup \mathcal{Z}_2$ contains none of the main variables of $s, t_1, t_2$. If $t_1 \preceq A t_2$ then $s[t_1] \preceq A s[t_2]$.

Proof. Since $t_1$ subsumes $t_2$ in $A$ there exists a substitution $\zeta: \mathcal{Z}_1 \rightarrow T(\Gamma, \mathcal{Z}_2)$ such that for every ground substitution $\eta: \mathcal{Z}_2 \rightarrow T(\Gamma)$ we have

$$\eta(t_2)^A = \eta(\zeta(t_1))^A.$$

Define the substitution $\zeta': \mathcal{Y} \cup \mathcal{Z}_1 \rightarrow T(\Gamma, \mathcal{Y} \cup \mathcal{Z}_2)$ by

$$\zeta'(y) = \left\{ \begin{array}{ll}
\zeta(y) & \text{if } y \in \mathcal{Z}_1, \\
y & \text{if } y \in \mathcal{Y}.
\end{array} \right.$$

It satisfies $\zeta'(t_1) = \zeta(t_1)$ and $\zeta'(s) = s$. For any ground substitution $\eta: \mathcal{Y} \cup \mathcal{Z}_2 \rightarrow T(\Gamma)$ we have:

$$\eta(s[t_2])^A = (\eta(s)[\eta(t_2)])^A = \eta(s)^A \circ \eta(t_2)^A = \eta(s)^A \circ \eta(\zeta(t_1))^A = \eta(\zeta'(s))^A \circ \eta(\zeta'(t_1))^A = (\eta(\zeta'(s))[\eta(\zeta'(t_1))])^A = \eta(\zeta'(s[t_1]))^A.$$
This implies \( s[t_1] \leq^A s[t_2] \). □

We will be interested in algebras that have a finite subsumption base. In order to show that a set \( C \) is a finite subsumption base we will use the following lemma.

**Lemma 8.3.** Let \( A \) be a \( \Gamma \)-algebra and let \( C \) be a finite set of contexts with the following properties:

- For every atomic context \( s \) there exists \( t \in C \) with \( t \leq^A s \).
- For every atomic context \( s \) and every \( t \in C \) such that \( s[t] \) is defined and \( s \) and \( t \) do not share auxiliary variables, there exists \( t' \in C \) with \( t' \leq^A s[t] \).

Then \( C \) is a subsumption base.

**Proof.** Assume that the two conditions from the lemma hold. We show by induction on \( s \) that for every context \( s \) there exists a context \( t \in C \) with \( t \leq^A s \). If \( s = f(s_1, \ldots, s_i, x, s_{i+1}, \ldots, s_n) \) for some terms \( s_1, \ldots, s_i, s_{i+1}, \ldots, s_n \) then \( s \) is subsumed in \( A \) by the atomic context \( f(y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_n) \), which in turn is subsumed in \( A \) by some \( t \in C \). If \( s = f(s_1, \ldots, s_i, s', s_{i+1}, \ldots, s_n) \) for some terms \( s_1, \ldots, s_n \) and some context \( s' \) then \( f(y_1, \ldots, y_{i-1}, s', y_{i+1}, \ldots, y_n) \leq^A s \) for fresh auxiliary variables \( y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n \) (that neither occur in \( s' \) nor any context from \( C \)). By induction there exists \( t' \in C \) with \( t' \leq^A s \). By Lemma 8.2 we have \( f(y_1, \ldots, y_{i-1}, t', y_{i+1}, \ldots, y_n) \leq^A f(y_1, \ldots, y_{i-1}, s', y_{i+1}, \ldots, y_n) \). By the second assumption from the lemma, we have \( t'' \leq^A f(y_1, \ldots, y_{i-1}, t', y_{i+1}, \ldots, y_n) \) for some \( t'' \in C \). We get \( t'' \leq^A s \) by transitivity of \( \leq^A \).

**Remark 8.4.** Recall that we made the technical assumption that every element \( a \) of \( A \) can be written as \( t^A \) for a ground term \( A \). Let \( B \) be the subalgebra of \( A \) that is induced by all elements \( t^A \) for \( t \in \mathcal{T}(A) \). It is obvious that every subsumption base of \( A \) is also a subsumption base of \( B \).

**Example 8.5.** Every semiring \( A = (A, +, \times, a_1, \ldots, a_n) \), where \( a_1, \ldots, a_n \in A \) are arbitrary constants, has a finite subsumption base. Here we do not assume that \( \times \) is commutative, nor do we assume that identity elements with respect to \( + \) or \( \times \) exist. In other words: \((A, +)\) is a commutative semigroup, \((A, \times)\) is a semigroup and the left and right distributive law holds. The finite subsumption base \( C(A) \) consists of the following contexts \( axb + c, ax + c, xb + c, x + c, axb, ax, xb \), and \( x \), where \( x \) is the main variable and \( a, b, c \) are auxiliary variables. We write \( ab \) instead of \( a \times b \) and omit in \( axb \) brackets that are not needed due to the associativity of multiplication. To see that every context \( s \) is subsumed in \( A \) by one of the contexts from \( C(A) \), observe that a context defines a linear polynomial in the main variable \( x \). Hence, every context is equivalent in \( A \) to a context of the form \( sxt + u, sx + u, xt + u, x + u, sxt, sx, xt \) or \( x \), where \( s, t, u \) are terms that contain the auxiliary parameters. Each of these contexts is subsumed by a context from \( C(A) \) by the substitution \( \zeta \) with \( \zeta(a) = s \), \( \zeta(b) = t \), and \( \zeta(c) = u \).

Let us remark that the above proof can be adapted to the situation that also \( + \) is not commutative. In that case, we have include the terms \( c' + axb + c, c' + ax + c, c' + xb + c, c' + x + c, c' + axb, c' + ax, c' + xb \) and \( c' + x + c \) to the set \( C(A) \).

**Example 8.6.** If \( \Gamma \) contains a symbol of rank at least one, then the free term algebra \( T(\Gamma) \) has no finite subsumption base. If \( C \) is a finite subsumption base of \( T(\Gamma) \), then every ground context could be obtained from some \( t \in C \) by replacing the auxiliary parameters in \( t \) by ground terms. But this replacement does not change the length of the path from the root of the context to its main variable. Hence, we would obtain a bound for the length of the path from the root to the main variable in a ground context, which clearly does not exist.
Lemma 8.7. Assume that the \( \Gamma \)-algebra \( A \) has a finite subsumption base. Then from a given \( \Gamma \)–TSLP \( \mathcal{G} \) one can compute in time \( O(|\mathcal{G}|) \) a \( \Gamma \)–SLP \( \mathcal{H} \) of size \( O(|\mathcal{G}|) \) and depth \( O(depth(\mathcal{G})) \) such that \( [\mathcal{G}]A = [\mathcal{H}]A \).

Proof. Let \( C(\mathcal{A}) \) be a finite subsumption base for \( \mathcal{A} \). We say that a context \( s \in C(\Gamma, \mathcal{Y}) \) belongs to \( C(\mathcal{A}) \) up to variable renaming if there is a variable renaming \( \theta : \mathcal{Y} \to \mathcal{Z} \) such that \( \theta(s) \in C(\Gamma) \). Since the algebra \( \mathcal{A} \) is fixed, the set \( C(\mathcal{A}) \) has size \( O(1) \). Assume that \( s \) and \( t \) are contexts with the following properties: (i) \( s[t] \) is defined, (ii) \( s \) and \( t \) have no common auxiliary variable, and (iii) \( s \) and \( t \) belong to \( C(\mathcal{A}) \) up to variable renaming. We denote with \( s \cdot t \) a context from \( C(\mathcal{A}) \) with \( s \cdot t \leq_A s[t] \). Since \( s \) and \( t \) have size \( O(1) \) \((C(\mathcal{A}) \) is a fixed set of contexts), we can compute from \( s \) and \( t \) in constant time the context \( s \cdot t \) and a substitution \( \zeta \) such that \( s[t] \) and \( \zeta(s \cdot t) \) are equivalent in \( A \). Similarly, one can compute from a given atomic context \( s \) in constant time a context \( t \in C(\mathcal{A}) \) and a ground substitution \( \zeta \) such that \( s \) and \( \zeta(t) \) are equivalent in \( A \).

Let \( \mathcal{G} = (V, \rho, S) \). We define \( V_0 = \{ X \in V \mid \rho^*(X)^{\hat{G}}(\Gamma) \in T(\Gamma) \} \) and \( V_1 = \{ X \in V \mid \rho^*(X)^{\hat{G}}(\Gamma) \in C(\Gamma) \} = V \setminus V_0 \). The \( \Gamma \)-SLP \( \mathcal{H} \) to be constructed will be denoted with \( \mathcal{H} = (V', \tau, S) \). We will have \( V_0 \subseteq V' \). A variable \( X \in V_1 \) is replaced in \( \mathcal{H} \) by a finite set \( \mathcal{Y}_X \) of variables. Moreover, we will compute a context \( t_X \in C(\Gamma, \mathcal{Y}_X) \) that belongs to \( C(\mathcal{A}) \) up to variable renaming. We can assume that \( \mathcal{Y}_X \cap \mathcal{Y}_{X'} = \emptyset = \mathcal{Y}_X \cap V_0 \) for all \( X, X' \in V_1 \) with \( X \neq X' \). The set of variables of \( \mathcal{H} \) is then \( V' = V_0 \cup \bigcup_{X \in V_1} \mathcal{Y}_X \). Moreover, \( \mathcal{H} \) will satisfy the following conditions:

(a) If \( X \in V_0 \) then \( \rho^*(X)^A = \tau^*(X)^A \) (which is an element of \( A \)).

(b) If \( X \in V_1 \) then \( \rho^*(X)^A = \tau^*(t_X)^A \) (which is a ULTF on \( A \)).

We construct \( \mathcal{H} \) bottom-up. That means that we process all variables in \( V \) in a single pass over \( \mathcal{G} \). When we process a variable \( X \in V \) we have already processed all variables \( X' \) that appear in \( \rho(X) \). In particular, the set \( \mathcal{Y}_{X'} \) and the context \( t_{X'} \in C(\Gamma, \mathcal{Y}_{X'}) \) (in case \( X' \in V_1 \)) are defined. In addition, \( X' \) satisfies the above conditions (a) and (b).

We proceed by a case distinction according to the right-hand side \( \rho(X) \) of \( X \in V \).

This right-hand side has one of the following four forms:

Case 1. \( X \in V_0 \) and \( \rho(X) = f(X_1, \ldots, X_n) \) for \( f \in \Gamma_n, n \geq 0 \) and \( X_1, \ldots, X_n \in V_0 \). Then we set \( \tau(X) = \rho(X) \). Clearly, the above condition (a) holds.

Case 2. \( X \in V_0 \) and \( \rho(X) = X'[X''] \) with \( X' \in V_1 \), \( X'' \in V_0 \). By induction we have \( \rho^*(X'')^A = \tau^*(X'')^A \). Moreover, we have computed a context \( t_{X'} \in C(\Gamma, \mathcal{Y}_{X'}) \) that belongs to \( C(\mathcal{A}) \) up to variable renaming and such that \( \rho^*(X')^A = \tau^*(t_{X'})^A \). We define \( \tau(X) = t_{X'}[X''] \in T(\Gamma, \mathcal{Y}_{X'} \cup \{ X'' \}) \) (that is, we replace the main variable in \( t_{X'} \) by \( X'' \)) and get

\[
\rho^*(X)^A = \rho^*(X')^A(\rho^*(X'')^A) = \tau^*(t_{X'})^A(\tau^*(X'')^A) = \tau^*(t_{X'}[X''])^A = \tau^*(\tau(X))^A = \tau^*(X)^A.
\]

Case 3. \( X \in V_1 \) and \( \rho(X) = f(X_1, \ldots, X_{k-1}, x, X_{k+1}, \ldots, X_n) \) for \( f \in \Gamma_n, n \geq 1 \) and \( X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n \in V_0 \). By induction we have \( \rho^*(X_i)^A = \tau^*(X_i)^A \) for \( 1 \leq i \leq n, i \neq k \). We can view \( \rho(X) \) as an atomic context with main variable \( x \) and auxiliary variables \( X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n \). Hence, we can compute \( t_X \in C(\mathcal{A}) \) with \( t_X \leq_A \rho(X) \). We rename the auxiliary variables of \( t_X \) such that they do not already belong to \( \mathcal{H} \). Let \( \mathcal{Y}_X \) be the set of auxiliary variables of \( t_X \). We then add all variables in \( \mathcal{Y}_X \) to \( \mathcal{H} \). By the definition of \( \leq_A \) there is a substitution \( \zeta : \mathcal{Y}_X \to T(\Gamma, \{X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n\}) \) such that

\[
\rho^*(X)^A = \rho^*(\rho(X))^A = \tau^*(\rho(X))^A = \tau^*(\zeta(t_X))^A.
\]
We define the right-hand side for every new variable \( Y \in \mathcal{Y}_X \) by \( \tau(Y) = \zeta(Y) \) and get \( \rho^\ast(X)^\mathcal{A} = \tau^\ast(\zeta(t_X))^\mathcal{A} = \tau^\ast(\tau(t_X))^\mathcal{A} = \tau^\ast(t_X)^\mathcal{A} \), which is point (b).

**Case 4.** \( X = \mathcal{V}_1 \) with \( \rho_0(X) = X' \{ X'' \} \) and \( X', X'' \in \mathcal{V}_1 \). We have already defined the terms \( t_X', t_X'' \) that belong to \( C(\mathcal{A}) \) up to variable renaming. The set of auxiliary variables of \( t_X' \) (resp., \( t_X'' \)) is \( \mathcal{Y}_{X'} \) (resp., \( \mathcal{Y}_{X''} \)) and we have \( \mathcal{Y}_X \cap \mathcal{Y}_{X'} = \emptyset \).

Moreover, by the induction hypothesis for \( X' \) and \( X'' \) we have \( \rho^\ast(X')^\mathcal{A} = \tau^\ast(t_X')^\mathcal{A} \) and \( \rho^\ast(X'')^\mathcal{A} = \tau^\ast(t_X'')^\mathcal{A} \). We set \( t_X := t_X' \cdot t_X'' \in C(\mathcal{A}) \). We rename the auxiliary variables of \( t_X \) such that they do not already belong to \( \mathcal{H} \). Let \( \mathcal{Y}_X \) be the set of auxiliary variables of \( t_X \). We then add every \( Y \in \mathcal{Y}_X \) to \( \mathcal{H} \). By definition of \( t_X \) we have \( t_X \leq^\mathcal{A} t_X' \{ t_X'' \} \), which implies that there is a substitution \( \zeta : \mathcal{Y}_X \rightarrow \mathcal{T} \mathcal{H}(\mathcal{V}, \mathcal{Y}_X \cup \mathcal{Y}_{X''}) \) with
\[
\rho^\ast(X)^\mathcal{A} = \rho^\ast(X')^\mathcal{A} \cup \rho^\ast(X'')^\mathcal{A} = \tau^\ast(t_X')^\mathcal{A} \cup \tau^\ast(t_X'')^\mathcal{A} = \tau^\ast(t_X)^\mathcal{A}.
\]

We define the right-hand side for every new variable \( Y \in \mathcal{Y}_X \) by \( \tau(Y) = \zeta(Y) \) and get \( \rho^\ast(X)^\mathcal{A} = \tau^\ast(\zeta(t_X))^\mathcal{A} = \tau^\ast(\tau(t_X))^\mathcal{A} = \tau^\ast(t_X)^\mathcal{A} \), which is point (b).

The running time for the construction of \( \mathcal{H} \) is \( O(|\mathcal{G}|) \), since for each variable \( X \in \mathcal{V} \) we only spend constant time (see the remark from the first paragraph of the proof). In each step we have to take a constant number of fresh auxiliary variables. We can take them from a list \( Y_1, Y_1, Y_3, \ldots \) and store a pointer to the next free variable. \( \square \)

9. Main result of the paper

Let us now state the main technical result of this paper. For some applications we need a signature \( \Gamma \) that is part of the input.

**Theorem 9.1.** From a given signature \( \Gamma \) and a \( \Gamma \)-SLP \( \mathcal{G} \), which defines the tree \( t = [\mathcal{G}] \in \mathcal{T}_0(\Gamma) \), one can compute in time \( O(|\mathcal{G}|) \) a \( \Gamma \)-TSLP \( \mathcal{H} \) such that \( [\mathcal{H}]^{\mathcal{T}(\Gamma)} = t \), \( |\mathcal{H}| \in O(|\mathcal{G}|) \) and \( depth(\mathcal{H}) \in O(\log |t|) \).

We will prove Theorem 9.1 in Section 15. Together with Lemma 8.7, Theorem 9.1 yields the following result:

**Theorem 9.2.** Take a fixed signature \( \Gamma \) and a fixed \( \Gamma \)-algebra \( \mathcal{A} \) that has a finite subsumption base. From a given \( \Gamma \)-SLP \( \mathcal{G} \), which defines the derivation tree \( t = [\mathcal{G}] \in \mathcal{T}_0(\Gamma) \), one can compute in time \( O(|\mathcal{G}|) \) a \( \Gamma \)-TSLP \( \mathcal{H} \) such that \( [\mathcal{H}]^\mathcal{A} = [\mathcal{G}]^\mathcal{A} \), \( |\mathcal{H}| \in O(|\mathcal{G}|) \) and \( depth(\mathcal{H}) \in O(\log |t|) \).

**Proof.** Using Theorem 9.1 we obtain from \( \mathcal{G} \) in time \( O(|\mathcal{G}|) \) a \( \Gamma \)-TSLP \( \mathcal{G}' \) such that \( [\mathcal{G}']^{\mathcal{T}(\Gamma)} = t \), \( [\mathcal{G}'] \in O(|\mathcal{G}|) \) and \( depth(\mathcal{G}') \in O(\log |t|) \). From \( [\mathcal{G}']^{\mathcal{T}(\Gamma)} = t \) we get \( [\mathcal{G}']^\mathcal{A} = [\mathcal{G}]^\mathcal{A} \). By Lemma 8.7 we can compute from \( \mathcal{G}' \) in time \( O(|\mathcal{G}'|) = O(|\mathcal{G}|) \) a \( \Gamma \)-SLP \( \mathcal{H} \) of size \( O(|\mathcal{G}'|) = O(|\mathcal{G}|) \) and depth \( O(depth(\mathcal{G}')) = O(\log |t|) \) such that \( [\mathcal{H}]^\mathcal{A} = [\mathcal{G}']^\mathcal{A} = [\mathcal{G}]^\mathcal{A} \). \( \square \)

**Remark 9.3.** Recall that we made the technical assumption that every element \( a \) of \( \mathcal{A} \) can be written as \( t^A \) for a ground term \( A \). We can still prove Corollary 9.2 in case \( \mathcal{A} \) does not satisfy this assumption: let \( \mathcal{B} \) be the subalgebra of \( \mathcal{A} \) that is induced by all elements \( t^A \) for \( t \in \mathcal{T}(\mathcal{A}) \). By Remark 8.4, \( \mathcal{B} \) has a finite subsumption base as well. Moreover, for every \( \Gamma \)-SLP \( \mathcal{G} \) we obviously have \( \mathcal{G}^\mathcal{A} = \mathcal{G}^\mathcal{B} \). Hence, Corollary 9.2 applied to the algebra \( \mathcal{B} \) yields the statement for \( \mathcal{A} \).

**Remark 9.4.** Theorem 9.2 only holds for a fixed \( \Gamma \)-algebra because Lemma 8.7 assumes a fixed \( \Gamma \)-algebra. Nevertheless there are settings, where we consider a family of algebras \( \{ \mathcal{A}_i \mid i \in I \} \) with \( \mathcal{A}_i \) is a \( \Gamma \)-algebra. An example is the family of
all free monoids $\Sigma^*$ for a finite alphabet $\Sigma$ that is part of the input. Under certain assumptions, the statement of Theorem 9.2 can be extended to the uniform setting, where the signature $\Gamma_i$ ($i \in I$) is part of the input and SLPs are evaluated in the algebra $A_i$. First of all we have to assume that every symbol $f \in \Gamma_i$ fits into a machine word of the underlying RAM model, which is a natural assumption if the signature $\Gamma_i$ is part of the input. For the $\Gamma_i$-algebras $A_i$ we need the following assumptions:

(i) There is a constant $r$ such that the rank of every symbol $f \in \bigcup_{i \in I} \Gamma_i$ is bounded by $r$.

(ii) There is a constant $c$ and a finite subsumption base $C(A_i)$ for every $i \in I$ such that the size of every context $s \in \bigcup_{i \in I} C(A_i)$ is bounded by $c$. With the above assumption on the word size of the RAM this ensures that a context $s \in \bigcup_{i \in I} C(A_i)$ fits into $O(1)$ many machine words.

(iii) There is a constant time algorithm that computes from a given atomic context $s$ over the signature $\Gamma_i$ a context $t \in C(A_i)$ and a substitution $\zeta$ such that $\zeta(t)$ and $s$ are equivalent in $A_i$.

(iv) There is a constant time algorithm that takes two contexts $s$ and $t$ over the signature $\Gamma_i$ such that $s[t]$ is defined, $s$ and $t$ have no common auxiliary variable, and $s$ and $t$ belong to $C(A_i)$ up to variable renaming, and computes the context $s \cdot t$ (see the first paragraph in the proof of Lemma 8.7) and a substitution $\zeta$ such that $\zeta(t)$ and $s[t]$ are equivalent in $A_i$.

Under these assumptions the construction from the proof of Lemma 8.7 can still be carried out in linear time. Since the statement of Theorem 9.1 holds for a signature $\Gamma_i$ that is part of the input, this allows to extend Theorem 9.2 to the setting where the signature $\Gamma_i$ ($i \in I$) is part of the input. This situation will be encountered for the class of free monoids (Section 10), forest algebras (Section 11.1) and top dags (Section 12).

Before we go into the proof of Theorem 9.1 we first discuss some applications of Theorem 9.2. A first application concerns straight-line programs over a semiring $A$. Such circuits are also known as arithmetic circuits in the literature. We view addition and multiplication in $A$ as binary operations. In other words, we consider bounded fan-in arithmetic circuits. We also include arbitrary constants in the algebra $A$ (this is necessary in order to build expressions). The following result follows directly from Theorem 9.1 and the fact that every semiring has a finite subsumption base; see Example 8.7.

**Corollary 9.5.** Let $A$ be an arbitrary semiring with constants (we neither assume that $A$ is commutative nor that identity elements with respect to $+$ or $\times$ exist). Given an arithmetic circuit $G$ over $A$ such that the corresponding derivation tree $t$ has $n$ nodes, one can compute in time $O(|G|)$ an arithmetic circuit $H$ over $A$ such that $|H|^A = |G|^A$, $|H| \in O(|G|)$ and $\text{depth}(H) \in O(\log n)$.

### 10. String straight-line programs

Recall that for a finite alphabet $\Sigma$, $\Sigma^*$ denotes the set of all finite words over $\Sigma$. Together with the associative binary concatenation operator $\cdot$, $\Sigma^*$ forms the free monoid. As usual, we just write $uw$ for $u \cdot v$. We consider the algebra $(\Sigma^*, \cdot, \varepsilon, (\alpha)_{\alpha \in \Sigma})$ which extends the free monoid $(\Sigma^*, \cdot)$ by the constants $\alpha \in \Sigma$ and the empty word $\varepsilon$. We denote this algebra by $\Sigma^*$ as well. SLPs over $\Sigma^*$ are widely studied in the area of stringology and data compression; see [28] for a survey. In many papers, the term “straight-line program” refers to SLPs over $\Sigma^*$. Recall that since we deal with SLPs over various algebras, we use the term “SSLP” (string straight-line program) for
an SLP over $\Sigma^*$. We will also speak of an SSLP over $\Sigma$. Occasionally we consider SSLPs without a start variable $S$.

**Example 10.1.** Consider the SSLP $G = \{S, X_1, X_2, X_3, X_4\}, \rho, S\}$ over $\Sigma^* = \{a, b\}$ with $\rho(S) = X_1 X_4$, $\rho(X_1) = X_2 X_3$, $\rho(X_2) = X_3 X_4$, $\rho(X_3) = X_4 X_5$, $\rho(X_4) = a$, and $\rho(X_5) = b$. We have $[X_1]^{\Sigma^*} = a b$, $[X_2]^{\Sigma^*} = a b a b$, $\rho^{*}(X_1) = a b a b$, and $\rho^{*}(S) = a b a b a b a b$. The size of $G$ is 5 and the depth of $G$ is 4 due to the path $S \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4$ in the derivation tree of $G$.

**Lemma 10.2.** For every finite alphabet $\Sigma$, the algebra $\Sigma^*$ has a finite subsumption base.

**Proof.** The proof follows the arguments for semirings in Example 8.5. A finite subsumption base is $\{u x v\}$ where $u$ and $v$ are the auxiliary variables. □

A direct proof (that does not use Theorem 9.2) of the following result is given in Appendix A.

**Theorem 10.3.** Given a finite alphabet $\Sigma$ and an SSLP $G$ over $\Sigma^*$ defining the string $s = [G]^{\Sigma^*}$, one can compute in time $O(|G|)$ an SSLP $H$ such that $[H]^{\Sigma^*} = s$, $|H| \in O(|G|)$ and depth$(H) \in O(\log |s|)$.

**Proof.** We first eliminate all occurrences of the empty word on right-hand sides. This is possible in a single pass over $G$ (unless $s = \varepsilon$ in which case the statement is trivial): if $\rho(X) = \varepsilon$ then we eliminate $X$ and replace all occurrences of $X$ in right-hand sides by $\varepsilon$. This may lead to right-hand sides consisting of a single variable. If $\rho(X) = Y$ then we eliminate $X$ and replace all occurrences of $X$ in right-hand sides by $Y$. The resulting SSLP (which we denote with $G$ as well) has the property that its derivation tree $t = [G]$ has exactly $|s|$ leaves and $|s| - 1$ internal nodes (since concatenation is a binary operator). Hence, we have $O(\log |t|) = O(\log |s|)$.

We can then apply Theorem 9.2 if the alphabet $\Sigma$ has size $O(1)$. The statement for the case where the alphabet $\Sigma$ is part of the input follows from Remark 9.4 where $\{A_i \mid i \in I\}$ is the class of all free monoids $\Sigma^*$. The assumption that symbols $f \in \Gamma_i$ fit into a single machine words means that alphabet symbols $a \in \Sigma$ can be stored in a single machine word. Moreover, all assumptions on the class $\{A_i \mid i \in I\}$ are satisfied: All operations in a free monoid have rank zero or two and the subsumption base $\{u x v\}$ of a free monoid does not depend on the alphabet $\Sigma$ at all. From this it follows easily that the constant time algorithms in points (iii) and (iv) from Remark 9.4 exist. □

Let us discuss several algorithmic applications of Theorem 10.3. The idea is always the same: let $G$ be an SSLP of size $m$ for a string $s$ of length $n$. In many algorithms for SSLP-compressed strings the running time or space consumption depends on depth$(G)$, which can be $m$ in the worst case. Theorem 10.3 shows that we can replace depth$(G)$ by $O(\log n)$. This is the best we can hope for since depth$(G) \geq \Omega(\log n)$ for every SSLP $G$. Moreover, SSLPs that are produced by practical grammar-based compressors (e.g., LZ78 or RePair) are in general unbalanced in the sense that depth$(G) \geq \omega(\log n)$.

The time bounds in the following results refer to the RAM model, where arithmetic operations on numbers from the interval $[0, n]$ need time $O(1)$. The size of a data structure is measured in the number of words of bit length $\log_2 n$.

A random access query for a string $s$ takes a position $1 \leq i \leq |s|$ and returns the letter at position $i$ in $s$. The following result was shown in [8] using several quite sophisticated data structures.

**Corollary 10.4** (random access to grammar-compressed strings, cf. [8]). From a given SSLP $G$ of size $m$ such that the string $s = [G]^{\Sigma^*}$ has length $n$, one can
construct in time $O(m)$ a data structure of size $O(m)$ that allows to answer random access queries in time $O(\log n)$.

**Proof.** Using Theorem 10.3 we compute in time $O(m)$ an equivalent SSLP $\mathcal{H}$ for $s$ of size $O(m)$ and depth $O(\log n)$. By a single pass over $\mathcal{H}$ we compute for every variable $X$ of $\mathcal{H}$ the length of the word $[X]^{\Sigma^*}$. Using these lengths one can descend in the derivation tree $[\mathcal{H}]$ from the root to the $i$-th leaf node (which is labelled with the $i$-th symbol of $s$) in time $O(\text{depth}(\mathcal{H})) \leq O(\log n)$.

**Remark 10.5.** It is easy to see that our balancing algorithm from Theorem 10.3 can be implemented on a pointer machine. Thus, also the random access data structure from Corollary 10.4 can be implemented on a pointer machine. In contrast, the random access data structure from [8] needs the RAM model (for the pointer machine model only preprocessing time and size $O(m \cdot \alpha_k(m))$ for any fixed $k$, where $\alpha_k$ is the $k$-th inverse Ackermann function, is shown in [8]). On the other hand, recently, in [6], the $O(m)$-space data structure from [8] has been modified so that it can be implemented on a pointer machine as well.

Using fusion trees [14] one can improve the time bound $O(\log n)$ in Corollary 10.4 to $O(\log n / \log \log n)$ at the cost of an additional factor of $O(\log^\epsilon n)$ in the size bound.

**Corollary 10.6.** Fix an arbitrary constant $\epsilon > 0$. From a given SSLP $\mathcal{G}$ of size $m$ such that the string $s = [\mathcal{G}]^{\Sigma^*}$ has length $n$, one can construct in time $O(m \cdot \log^\epsilon n)$ a data structure of size $O(m \cdot \log^\epsilon n)$ that allows to answer random access queries in time $O(\log n / \log \log n)$.

**Proof.** The proof is exactly the same as for [2] Theorem 2]. There, the author have to assume that the input SSLP has depth $O(\log n)$, which we can enforce by Theorem 10.3. Roughly speaking, the idea in [2] is to reduce the depth of the SSLP to $O(\log n / \log \log n)$ by expanding right-hand sides to length $O(\log^\epsilon n)$. Then for each right-hand side a fusion tree is constructed, which allows to spend constant time at each variable during the navigation to the $i$-th symbol.

Let us also remark that the size bound for the computed data structure in [2] is given in bits, which yields $O(m \cdot \log^{1+\epsilon} n)$ bits since numbers from $[0, n]$ have to be encoded with $\log_2 n$ bits.

Given a string $s \in \Sigma^*$, a rank query gets a position $1 \leq i \leq |s|$ and a symbol $a \in \Sigma$ and returns the number of $a$’s in the prefix of $s$ of length $i$. A select query gets a symbol $a \in \Sigma$ and returns the position of the $i$-th $a$ in $s$ (if it exists).

**Corollary 10.7.** Fix an arbitrary constant $\epsilon > 0$. From a given SSLP $\mathcal{G}$ of size $m$ such that the string $s = [\mathcal{G}]^{\Sigma^*}$ has length $n$, one can construct in time $O(m \cdot |\Sigma| \cdot \log^\epsilon n)$ a data structure of size $O(m \cdot |\Sigma| \cdot \log^\epsilon n)$ that allows to answer rank and select queries in time $O(\log n / \log \log n)$.

**Proof.** Again we follow the proof [2] Theorem 2] but first apply Theorem 10.3 in order to reduce the depth of the SSLP to $O(\log n)$.

**Corollary 10.7** improves [2] Theorem 2], where the query time is $O(\log n)$ and the space is $O(m \cdot |\Sigma| \cdot \log n)$.

Our balancing result also yields an improvement for the *compressed subsequence problem* [3]. Bille et al. [3] present an algorithm based on a labelled successor data structure. Given a string $s = a_1 \cdots a_n \in \Sigma^*$, a labelled successor query gets a position $1 \leq i \leq n$ and a symbol $a \in \Sigma$ and returns the minimal position $j > i$ with $a_j = a$ (or rejects if it does not exist). The following result is an improvement.

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4See [35] for a discussion of the pointer machine model.
over \[3\], where the authors present two algorithms for the compressed subsequence problem: one with \(O(m + m \cdot |\Sigma|/w)\) preprocessing time and \(O(\log n \cdot \log w)\) query time, and another algorithm with \(O(m + m \cdot |\Sigma| \cdot \log w/w)\) preprocessing time and \(O(\log n)\) query time.

**Corollary 10.8.** There is a data structure supporting labelled successor (and predecessor) queries on a string \(s \in \Sigma^*\) of length \(n\) compressed by an SSLP of size \(m\) in the word RAM model with word size \(w \geq \log_2 n\) using \(O(m + m \cdot |\Sigma|/w)\) space, \(O(m + m \cdot |\Sigma|/w)\) preprocessing time, and \(O(\log n)\) query time.

**Proof.** In the preprocessing phase we first reduce the depth of the given SSLP to \(O(\log n)\) using Theorem 10.3 We compute for every variable \(X\) the length of \([X]\Sigma^*\) in time and space \(O(m)\) as in the proof of Corollary 10.3. Additionally for every variable \(X\) we compute a bitvector of length \(|\Sigma|\) which encodes the set of symbols \(a \in \Sigma\) that occur in \([X]\Sigma^*\). Notice that this information takes \(O(m \cdot |\Sigma|)\) bits and fits into \(O(m \cdot |\Sigma|/w)\) memory words. If \(p(X) = YZ\) then the bitvector of \(X\) can be computed from the bitvectors of \(Y\) and \(Z\) by \(O(|\Sigma|/w)\) many bitwise OR operations. Hence in total all bitvectors can be computed in time \(O(m \cdot |\Sigma|/w)\).

A labelled successor query (for position \(i\) and symbol \(a\)) can now be answered in \(O(\log n)\) time in a straightforward way: First we compute the path \((X_0, X_1, \ldots, X_l)\) in the derivation tree from the root \(X_0\) to the symbol at the \(i\)-th position. Then we follow the path starting from the leaf upwards to find the maximal \(k\) such that \(p(X_k) = X_{k+1}Y\) and \([Y]\Sigma^*\) contains the symbol \(a\), or reject if no such \(k\) exists. Finally, starting from \(Y\) we navigate in time \(O(\log n)\) to the leftmost leaf in the derivation tree which produces the symbol \(a\).

A minimal subsequence occurrence of a string \(p = a_1a_2\cdots a_k\) in a string \(s = b_1b_2\cdots b_l\) is given by two positions \(i, j\) with \(1 \leq i \leq j \leq l\) such that \(p\) is a subsequence of \(b_ib_{i+1}\cdots b_j\) (i.e., \(b_ib_{i+1}\cdots b_j\) belongs to the language \(\Sigma^*a_1\Sigma^*a_2\cdots \Sigma^*a_k\Sigma^*\)) but \(p\) is neither a subsequence of \(b_{i+1}\cdots b_j\) nor of \(b_i\cdots b_{j-1}\). Following the proof of \(3\) Theorem 1 we obtain:

**Corollary 10.9.** Given an SSLP \(G\) of size \(m\) producing a string \(s \in \Sigma^*\) of length \(n\) and a pattern \(p \in \Sigma^*\) one can compute all minimal subsequence occurrences of \(p\) in \(s\) in space \(O(m + m \cdot |\Sigma|/w)\) and time \(O(m + m \cdot |\Sigma|/w + |p| \cdot \log n \cdot \text{occ})\) where \(w \geq \log n\) is the word size and \(\text{occ}\) is the number of minimal subsequence occurrences of \(p\) in \(s\).

**Corollary 10.9** improves \(3\) Theorem 1, which states the existence of two algorithms for the computation of all minimal subsequence occurrences with the following running times (the space bounds are the same as in Corollary 10.9):

- \(O(m + m \cdot |\Sigma|/w + |p| \cdot \log n \cdot \log w \cdot \text{occ})\),
- \(O(m + m \cdot |\Sigma| \cdot \log w/w + |p| \cdot \log n \cdot \text{occ})\).

Let us list further applications of Theorem 10.3 (recall that \(G\) is an SSLP of size \(m\) for a string \(s\) of length \(n\)):

**Computing fingerprints of SSLP-compressed texts.** Given two positions \(i \leq j\) in \(s\) one wants to compute the Karp-Rabin fingerprint of the factor of \(s\) that starts at position \(i\) and ends at position \(j\). In \(5\) it was shown that one can compute from \(G\) a data structure of size \(O(m)\) that allows to compute fingerprints in time \(O(\log n)\). First, the authors of \(5\) present a very simple data structure of size \(O(m)\) that allows to compute fingerprints in time \(O(\text{depth}(G))\). With Theorem 10.3 we can use this data structure to obtain a \(O(\log n)\)-time solution. This simplifies the proof in Theorem 10.3 considerably.
Computing runs, squares, and palindromes in SSLP-compressed strings.

It is shown in [22] that certain compact representations of the set of all runs, squares and palindromes in $s$ (see [22] for precise definitions) can be computed in time $O(|P| \cdot \text{depth}(G))$. We can improve the time bound to $O(m^3 \cdot \log n)$.

**Real time traversal for SSLP-compressed strings.** One wants to output the symbols of $s$ from left to right and thereby spend constant time per symbol. A solution can be found in [18]: a two-way version (where one can navigate in each step to the left or right neighboring position in $s$) can be found in [21]. The drawback of these solutions is that they need space $O(\text{depth}(G))$. With Theorem 10.3, we can reduce this to space $O(\log n)$.

**Compressed range minimum queries.** Range minimum data structure preprocesses a given string $s$ of integers so that the following queries can be efficiently answered: given $i \leq j$, what is the minimum element in $s_i, \ldots, s_j$? The substring of $s$ from position $i$ to $j$. We are interested in the variant of the problem, in which the input is given as an SSLP $G$. It is known, that after a preprocessing taking $O(|G|)$ time, one can answer range minimum queries in time $O(\log n)$ [19] Theorem 1.1]. This implementation extends the data structure for random access for SSLP [8] with some additional information, which includes in particular adding standard range minimum data structures for subtrees leaving the heavy path and extending the original analysis. Using the balanced SSLP the same running time can be easily obtained, without the need of hacking into the construction of the balanced SSLP.

To this end for each variable $X$ we store the length $\ell_X$ of the derived word $[X]$ as well the minimum value in $[X]$. In the following, let RMQ($X, i, j$) be the range minimum query called on $[X]$ for interval $[i, j]$. Given RMQ($X, i, j$), with the rule for $X$ being $X \rightarrow YZ$ we proceed as follows:

- If the query asks about the minimum in the whole $[X]$, i.e., $i = 1$ and $j = \ell_X$, then we return the minimum of $[X]$; we call this case trivial in the following.
- If the whole range is within the substring generated by the first variable in the rule, i.e., $j \leq \ell_Y$, then we call RMQ($Y, i, j$).
- If the whole range is within the substring generated by the second nonterminal in a rule, i.e., $i > \ell_Y$, then we call RMQ($Z, i - \ell_Y, j - \ell_Y$).
- Otherwise, i.e., when $i \leq \ell_Y$ and $j > \ell_Y$ and $(i, j) \neq (1, \ell_X)$, the range spans over the substrings generated by both terminals. Thus we compute the queries for two substrings and take their minimum, i.e., we return the minimum of RMQ($Y, i, \ell_Y$) and RMQ($Z, 1, j - \ell_Y$).

To see that the running time is $O(\text{depth}(G)) = O(\log n)$ observe first that the cost of trivial cases can be charged to the function that called them. Thus it is enough to estimate the number of nontrivial recursive calls. In the second and third case there is only one recursive call for a variable that is deeper in the derivation tree of the SSLP. In the fourth case there are two calls, but two nontrivial calls are made at most once during the whole computation: if two nontrivial calls are made in the fourth case then one of them asks for the RMQ of a suffix of $[Y]$ and the other call asks for the RMQ of a prefix of $[Z]$. Moreover, every recursive call on a prefix of some string $[X']$ leads to at most one nontrivial call, which is again on a prefix of some string $[X'']$; and analogously for suffixes.

**Lifshits’ algorithm for compressed pattern matching [27].** The input consists of an SSLP $P$ for a pattern $p$ and an SSLP $T$ for a text $t$ and the question is whether $p$ occurs in $t$. Lifshits’ algorithm has a running time of $O(|P| \cdot |T|^2)$. It was conjectured by the author that the running time could be improved to $O(|P| \cdot |T| \cdot \log |t|)$. This follows easily from Theorem 10.3: the algorithm fills a table of
size \(|P| \cdot |T|\) and on each entry it calls a recursive subprocedure, whose running time is at most depth(\(T\)). By Theorem 10.83 we can bound the running time by \(O(\log |T|)\), which proves Lifshits’ conjecture. Note, that in the meantime a faster algorithm with running time \(O(|T| \cdot \log |p|)\) [24] was found.

Remark 10.10 (smallest grammar problem). In the “smallest grammar problem” (for strings) for a given string \(w\) we want to construct a smallest SSLP defining \(w\). The decision variant of this problem is \(NP\)-hard, the best known approximation lower bound is \(\frac{8}{3}\) [11], and the best known approximation algorithms have an approximation ratio of \(O(\log n)\), where \(n\) is the length of the input string [11] [14] [23] [25]. Except for [23], all these algorithms produce SSLPs of depth \(O(\log n)\). It was discussed in [23] that the reason for the lack of constant-factor approximation algorithms might be the fact that smallest SSLPs can have larger than logarithmic depth. Theorem 10.83 refutes this approach.

11. Forest algebras and forest straight-line programs

11.1. Forest algebra. Let us fix a finite set \(\Sigma\) of node labels. In this section, we consider \(\Sigma\)-labelled rooted ordered trees, where “ordered” means that the children of a node are totally ordered. Every node has a label from \(\Sigma\). In contrast to the context. Most of the time, we simply write \(i, j\) omit the subscripts \(i, j\) in \(\circ_{ij}\) and \(\oplus\), since they will be always clear from the context. Most of the time, we simply write \(uv\) instead of \(u \oplus v\), \(a(u)\) instead of \(a(*) \oplus u\), and \(u\) instead of \(a(e)\). With these abbreviations, a forest \(u \in F(\Sigma)\) can be also viewed as an algebraic expression over the algebra \(F(\Sigma)\), which evaluates to \(u\) itself (analogously to the free term algebra).

Lemma 11.1. Every forest algebra \(F(\Sigma)\) has a finite subsumption base.
Proof. In the following we denote by \( x \) and \( y \) the main variables of sorts \( F_0(\Sigma) \) and \( F_1(\Sigma) \), respectively, and by \( \tau, \tau_1, \tau_2, \ldots \) (resp., \( \sigma, \sigma_1, \sigma_2, \ldots \)) auxiliary variables of sorts \( F_0(\Sigma) \) (resp., \( F_1(\Sigma) \)). In the following, subsumption and equivalence of contexts are always meant with respect to the forest algebra \( F(\Sigma) \).

Let \( C \) be the set of containing the following contexts:

(a) \( \sigma_1 \otimes x \),
(b) \( \sigma_1 \otimes y \otimes \tau_1 \) and \( \sigma_1 \otimes y \otimes \sigma_2 \),
(c) \( \sigma_1 \otimes (\sigma_2 \otimes (\sigma_3 \otimes x)) \) and \( \sigma_1 \otimes ((\sigma_2 \otimes x) \otimes \sigma_3) \),
(d) \( \sigma_1 \otimes (\sigma_2 \otimes (\sigma_3 \otimes y \otimes \tau_1)) \) and \( \sigma_1 \otimes ((\sigma_2 \otimes y \otimes \tau_1) \otimes \sigma_3) \).

A context from point (x) (for \( x = a, b, c, d \)) will be also called a \( (x) \)-context below.

First notice that every atomic context is of the form \( \sigma \otimes x, \sigma \otimes y, y \otimes \tau, y \otimes \sigma, \tau \otimes x, x \otimes \tau, \tau \otimes y, y \otimes \tau, \sigma \otimes x, \tau \otimes \sigma \) (up to variable renaming). Each of these contexts is subsumed by a context in \( C \). For the atomic contexts \( \sigma \otimes x, \sigma \otimes y, y \otimes \tau, y \otimes \sigma, \sigma \otimes x, \tau \otimes \sigma \) this is obvious. For \( \tau \otimes x \) note that \( \tau \otimes x \) is equivalent to the context \( (\tau \otimes \ast) \otimes x \), which is subsumed by \( \sigma_1 \otimes x \). A similar argument also applies to \( x \otimes \tau, \tau \otimes y \) and \( y \otimes \tau \).

Now consider any context \( s \in C \). We prove that for any atomic context \( s' \) from above, \( s'[s] \) is subsumed by some context from \( C \).

**Case** \( \sigma \otimes s \): Since \( s \) is of the form \( s = \sigma_1 \otimes s' \) for some \( s' \), the context \( \sigma \otimes s = \sigma \otimes (\sigma_1 \otimes s') \) is subsumed by \( s \in C \) itself.

**Case** \( s \otimes \tau \) and \( s \otimes \sigma \): In this case \( s \) must be either the (b)-context \( s = \sigma_1 \otimes y \otimes \sigma_2 \), a (c)-context or a (d)-context.

(1) If \( s = \sigma_1 \otimes y \otimes \sigma_2 \), then \( s \otimes \tau \) and \( s \otimes \sigma \) are subsumed by a (b)-context.

(2) Assume that \( s \) is a (c)-context, say \( s = \sigma_1 \otimes (\sigma_2 \otimes (\sigma_3 \otimes x)) \). Then \( s \otimes \tau \) is equivalent to \( (\sigma_1 \otimes ((\sigma_2 \otimes \tau) \otimes \sigma_3)) \otimes x \) which is subsumed by \( \sigma_1 \otimes x \). Moreover, \( s \otimes \sigma \) is equivalent to \( \sigma_1 \otimes ((\sigma_2 \otimes \sigma) \otimes (\sigma_3 \otimes x)) \), which is subsumed by \( s \) itself.

(3) Assume that \( s \) is a (d)-context, say \( s = \sigma_1 \otimes (\sigma_2 \otimes (\sigma_3 \otimes y \otimes \tau_1)) \). Then \( s \otimes \tau \) is equivalent to \( (\sigma_1 \otimes (\sigma_2 \otimes \tau) \otimes \sigma_3) \otimes y \otimes \tau_1 \), which is subsumed by the serial context \( \sigma_1 \otimes x \otimes \tau_1 \). Secondly, \( s \otimes \sigma \) is equivalent to \( \sigma_1 \otimes ((\sigma_2 \otimes \sigma) \otimes (\sigma_3 \otimes y \otimes \tau_1)) \), which is subsumed by \( s \) itself.

**Case** \( \tau \otimes s \) and \( \sigma \otimes \tau_1 \). Since \( s \) is of the form \( s = \sigma_1 \otimes s' \) for some \( s' \) the context \( \tau \otimes s \) is equivalent to \( (\tau \otimes \sigma_1) \otimes s' \), which is subsumed by \( s \) itself. The case \( s \otimes \tau \) is similar.

**Case** \( \sigma \otimes s \) and \( s \otimes \sigma \): In this case \( s \) must be either the (a)-context or the (b)-context \( \sigma_1 \otimes y \otimes \sigma_1 \). If \( s \) is the (a)-context \( \sigma_1 \otimes x \) then \( \sigma \otimes s \) is subsumed by the (c)-context \( \sigma_1 \otimes (\sigma_2 \otimes (\sigma_3 \otimes x)) \). If \( s \) is the (b)-context \( \sigma_1 \otimes y \otimes \sigma_1 \) then \( \sigma \otimes s \) is subsumed by the (d)-context \( \sigma_1 \otimes (\sigma_2 \otimes (\sigma_3 \otimes y \otimes \tau_1)) \). The case \( s \otimes \sigma \) is similar.

By Lemma \[ \ref{lemma:finite-subsumption-base} \] \( C \) is a finite subsumption base. \( \square \)

Remark 11.2. Similarly to the proof of Lemma \[ \ref{lemma:finite-subsumption-base} \] one can also show that for every signature \( \Gamma \) the functional extension \( \hat{T}(\Gamma) \) of the free term algebra \( T(\Gamma) \) has a finite subsumption base as well. Recall from Example \[ \ref{example:finite-free-algebra} \] that the free term algebra \( T(\Gamma) \) has no finite subsumption base if \( \Gamma \) contains a symbol of rank at least one.

11.2. **Forest straight-line programs.** A forest straight-line program over \( \Sigma \), FSLP for short, is a straight-line program \( \mathcal{G} \) over the algebra \( F(\Sigma) \) such that \( \mathcal{G} \in F_0(\Sigma) \). Iterated vertical and horizontal concatenations allow to generate forests, whose depth and width is exponential in the size of the FSLP. For an FSLP \( \mathcal{G} = (V, \rho, S) \) and \( i \in \{0, 1\} \) we define \( V_i = \{ X \in V \mid [X]^{F(\Sigma)} \in F_i(\Sigma) \} \). Every right-hand side of a standard FSLP \( \mathcal{G} \) must have one of the following forms: (i)
Example 11.3. Let $n \in \mathbb{N}$. Consider the (non-standard) FSLP
\[ \mathcal{G} = (\{ S, X_0, \ldots, X_n, Y_0, \ldots, Y_n \}, \rho, S) \]
over $\{ a, b, c \}$ with $\rho$ defined by $\rho(X_0) = a$, $\rho(X_i) = X_{i-1}X_{i-1}$ for $1 \leq i \leq n$, $\rho(Y_0) = b(X_n \ast X_0)$, $\rho(Y_i) = Y_{i-1} \ast Y_{i-1}$ for $1 \leq i \leq n$, and $\rho(S) = Y_n \ast c$. We have $|\mathcal{G}|^F(\Sigma) = b(a^n b(a^{2^n} \cdots b(a^2 c a^2) \cdots a^n a^n))$, where $b$ occurs $2^n$ many times, see Figure 1 for $n = 2$.

Let us first show that most occurrences of $\varepsilon$ and $\ast$ can be eliminated in an FSLP.

**Lemma 11.4.** From a given FSLP $\mathcal{G}$ with $|\mathcal{G}|^F(\Sigma) \neq \varepsilon$ one can compute in linear time an FSLP $\mathcal{H}$ such that $|\mathcal{G}|^F(\Sigma) = |\mathcal{H}|^F(\Sigma)$, $|\mathcal{H}| \in O(|\mathcal{G}|)$, depth($\mathcal{H}$) $\in O(\text{depth}(\mathcal{G}))$, and $\mathcal{H}$ does not contain occurrences of the constants $\varepsilon$ and $\ast$, except for right-hand sides of the form $a(\varepsilon)$.

**Proof.** Let $\mathcal{G} = (\mathcal{V}, \rho, S)$. We first construct an equivalent FSLP which does not contain the constant $\ast$. Let us denote with $\mathcal{V}_s \subseteq V_1$ the set of all variables $X \in V_1$ such that $|X|^F(\Sigma)$ is of the form $u_t \ast u_r$ for forests $u_t, u_r \in F_0(\Sigma)$. In other words: $\ast$ occurs at a root position in the forest $|X|^F(\Sigma)$. The set $\mathcal{V}_s$ can be easily computed in linear time by a single pass over $\mathcal{G}$. Every variable $X \in \mathcal{V}_s$ with $|X|^F(\Sigma) = u_t \ast u_r$ is replaced in $\mathcal{H}$ by two variables $X_t$ and $X_r$ that produce in $\mathcal{H}$ the forests $u_t$ and $u_r$, respectively. Every variable $X \in V_1 \setminus \mathcal{V}_s$ is replaced in $\mathcal{H}$ by three variables $X_t, X_r, X_s$. Since $X \in V_1 \setminus \mathcal{V}_s$, $|X|^F(\Sigma)$ contains a unique subtree of the form $a(u_t \ast u_r)$. Let us denote with $u_t$ (the top part of $u$) the forest that is obtained from $u$ by replacing the subtree $a(u_t \ast u_r)$ by $a(\ast)$. We then will have $|X_t|^F(\Sigma) = u_t$, $|X_r|^F(\Sigma) = u_r$, and $|X_s|^F(\Sigma) = u_r$. Finally, all variables from $V_0$ also belong to $\mathcal{H}$ and produce in $\mathcal{H}$ the same forests as in $\mathcal{G}$.

It is straightforward to define the right-hand sides of $\mathcal{H}$ such that the variables indeed produce the desired forests ($\tau$ denotes the right-hand side mapping of $\mathcal{H}$):

- If $\rho(X) = \varepsilon$ then $\tau(X_t) = \tau(X_r) = \varepsilon$.
- If $\rho(X) = a(\ast)$ then $\tau(X_t) = a(\ast)$ and $\tau(X_r) = \tau(X_r) = \varepsilon$.
- If $\rho(X) = \varepsilon$ or $\rho(X) = YZ$ with $X, Y, Z \in V_0$ then $\tau(X) = \rho(X)$.
- If $\rho(X) = YZ$ with $X, Y \in \mathcal{V}_s$ and $Z \in V_0$ then $\tau(X_t) = Y_t$ and $\tau(X_r) = Y_r$, and $\tau(X_s) = Y_tZ$, and analogously for $X, Z \in \mathcal{V}_s$ and $Y \in V_0$.
- If $\rho(X) = YZ$ with $X, Y \in V_1 \setminus \mathcal{V}_s$ and $Z \in V_0$ then $\tau(X_t) = Y_t$, $\tau(X_r) = Y_r$, and $\tau(X_s) = Y_tZ$, and analogously for $X, Z \in V_1 \setminus \mathcal{V}_s$ and $Y \in V_0$.
- If $\rho(X) = Y \ast Z$ with $X, Z \in V_0$ and $Y \in \mathcal{V}_s$, then $\tau(X) = Y_tZY_t$.
- If $\rho(X) = Y \ast Z$ with $X, Z \in V_0$ and $Y \in V_1 \setminus \mathcal{V}_s$ then $\tau(X) = Y_t \ast (Y_tZY_t)$.

\footnote{Constants $a(\ast)$ are allowed as well. Formally, $a(\ast)$ is a constant symbol that is interpreted by the forest context $a(\ast)$.}
Remark 11.6. From $V$ is defined as follows: Note that the construction does not introduce new occurrences of Corollary 11.5. Finally note that both constructions increase the size and depth of the FSLP only $G$.

Lemma 11.1 we can apply Theorem 9.2 in order to get the FSLP also contained in $H$. We construct an equivalent FSLP $\mathcal{H}$ which neither contains $*$ nor $\varepsilon$, except for right-hand sides of the form $a(*)$ and $a(\varepsilon)$. All variables from $G$ are also contained in $\mathcal{H}$, except for variables in $V_\varepsilon$. For every variable $X \in V_1$, $\mathcal{H}$ also contains a copy $X_\varepsilon$ that produces $[X]^{(\Sigma)} \cup \varepsilon$. The right-hand side mapping $\tau$ of $\mathcal{H}$ is defined as follows:

- If $\rho(X) = a(*)$ then $\tau(X) = a(*)$ and $\tau(X_\varepsilon) = a(\varepsilon)$.
- If $\rho(X) = \varepsilon$ then $X$ does not belong to $\mathcal{H}$.
- If $\rho(X) = YZ$ with $Y, Z \in V_\varepsilon$ then $X \notin V_\varepsilon$ does not belong to $\mathcal{H}$.
- If $\rho(X) = YZ$ or $\rho(X) = ZX$ with $Y \in V_\varepsilon$ and $Z \in V_0 \setminus V_\varepsilon$ then $\tau(X) = Z$.
- If $\rho(X) = YZ$ with $Y, Z \in V_0 \setminus V_\varepsilon$ then $\tau(X) = \varepsilon$.
- If $\rho(X) = YZ$ or $\rho(X) = ZX$ with $Y \in V_\varepsilon$ and $X, Z \in V_1$ then $\tau(X) = Z$ and $\tau(X_\varepsilon) = Z_\varepsilon$.
- If $\rho(X) = YZ$ with $Y \in V_0 \setminus V_\varepsilon$ and $Z \in V_1$ then $\tau(X) = YZ_\varepsilon$.
- If $\rho(X) = YZ$ with $Y \in V_0 \setminus V_\varepsilon$ and $Z \in V_1$ then $\tau(X_\varepsilon) = YZ_\varepsilon$.
- If $\rho(X) = Y \cdot Z$ with $X, Y, Z \in V_1$ and $X, Y \in V_1$ then $\tau(X) = YZ_\varepsilon$.
- If $\rho(X) = Y \cdot Z$ with $X \in V_1$ and $Y \in V_\varepsilon$ then $\tau(X) = \varepsilon$.
- If $\rho(X) = YZ$ with $X \in V_1$ and $Z \in V_0 \setminus V_\varepsilon$ then $\tau(X_\varepsilon) = Y \cdot Z$.

Note that the construction does not introduce new occurrences of $*$. All variables from $V \setminus V_\varepsilon$ produce the same forest in $G$ and $\mathcal{H}$, which implies $[G]^{(\Sigma)} = [\mathcal{H}]^{(\Sigma)}$. Finally note that both constructions increase the size and depth of the FSLP only by a constant factor. \hfill \square

Corollary 11.5. Given a finite alphabet $\Sigma$ and an FSLP $G$ over the forest algebra $F(\Sigma)$ defining the forest $u = [G]^{(\Sigma)}$, one can compute in time $O(|G|)$ an FSLP $H$ such that $[H]^{(\Sigma)} = u$, $|H| \in O(|G|)$ and $\text{depth}(H) \in O(\log |u|)$.

Proof. The case $u = \varepsilon$ is trivial. Let us now assume that $u \neq \varepsilon$. We first apply Lemma 11.1 and construct from $G$ in linear time an equivalent FSLP $G'$ which does not contain occurrences of the constants $*$ and $\varepsilon$, except for right-hand sides of the form $a(\varepsilon)$. This ensures that the derivation tree $t = [G']$ has size $O(|u|)$. The size and depth of $G'$ are linearly bounded in the size and depth, respectively, of $G$. By Lemma 11.1 we can apply Theorem 10.3 in order to get the FSLP $H$ with the desired properties for the situation where the alphabet $\Sigma$ is fixed. For the situation where $\Sigma$ is part of the input one has to use Remark 11.3. The arguments are analogous to the proof of Theorem 10.3. Note in particular that the subsumption base from the proof of Lemma 11.1 does not depend on the alphabet $\Sigma$ of the forest algebra $F(\Sigma)$.

Remark 11.6. Using Remark 11.2 one can show the following variant of Corollary 11.5. Take a fixed signature $\Gamma$. From a given $\Gamma$-TSLP $G$ defining the tree
of rank one. The root node of a cluster boundary node of top dag of bottom boundary node is of the form $a$ with a linear size increase. Note that this is a much stronger statement than Theorem 11.4, which states that a $\Gamma$-SLP can be balanced into an equivalent $\Gamma$-TSLP with a linear size increase. On the other hand, the above balancing result for $\Gamma$-TSLPs finally uses the weaker Theorem 9.1 in its proof. We have to assume a fixed signature $\Gamma$ in the above argument since the size of the contexts in a finite subsumption $T(\Gamma)$ depends on the maximal rank of the symbols in $\Gamma$.

Alternatively, the balancing result for $\Gamma$-TSLPs can be deduced from the corresponding balancing result for FSLPs (Corollary 11.5): a given $\Gamma$-TSLP $\mathcal{G}$ can be directly translated into an FSLP $\mathcal{G}_1$ for the tree $[\mathcal{G}]^T(\Gamma)$. The size of $\mathcal{G}_1$ is $O(|\mathcal{G}|)$. Using Corollary 11.5 one can compute from $\mathcal{G}_1$ a balanced FSLP $\mathcal{G}_2$ of size $O(|\mathcal{G}_1|)$. Finally, the FSLP $\mathcal{G}_2$ can be easily transformed back into a $\Gamma$-TSLP of size $O(r \cdot |\mathcal{G}|)$, where $r$ is the maximal rank of a symbol in $\Gamma$. For this one has to eliminate horizontal concatenations in the FSLP. Since we assumed $\Gamma$ to be a fixed signature, $r$ is a constant.

12. CLUSTER ALGEBRAS AND TOP DAGS

FSLPs are very similar to top daggs that were introduced in [7] and further studied in [4, 13, 21]. In fact, top dags can be defined in the same way as FSLPs, one only has to slightly change the two concatenation operations $\otimes$ and $\hat{\otimes}$, which yields the so-called cluster algebra defined below.

Let us fix an alphabet $\Sigma$ of node labels and define for $a \in \Sigma$ the set $\mathcal{K}_a(\Sigma) = \{ a(u) \mid u \in F_0(\Sigma) \setminus \{ z \} \}$. Note that $\mathcal{K}_a(\Sigma)$ consists of unranked $\Sigma$-labelled trees of size at least two, where the root is labeled with $a$. Elements of $\mathcal{K}_a(\Sigma)$ (for any $a$) are also called clusters of rank 0. For $a, b \in \Sigma$ let $\mathcal{K}_{ab}(\Sigma)$ be the set of all trees $t \in \mathcal{K}_a(\Sigma)$ together with a distinguished $b$-labelled leaf of $t$, which is called the bottom boundary node of $t$. Elements of $\mathcal{K}_{ab}(\Sigma)$ (for any $a, b$) are called clusters of rank one. The root node of a cluster $t$ (of rank zero or one) is called the top boundary node of $t$. When writing a cluster of rank one, we underline the bottom boundary node. For instance $a(b \cdot (bc))$ is an element of $\mathcal{K}_{ab}(\Sigma)$. An atomic cluster is of the form $a(b)$ or $a(\hat{b})$ for $a, b \in \Sigma$.

We define the cluster algebra $\mathcal{K}(\Sigma)$ as an algebra over a $(\Sigma \cup \Sigma^2)$-sorted signature. The universe of sort $a \in \Sigma$ is $\mathcal{K}_a(\Sigma)$ and the universe of sort $ab \in \Sigma^2$ is $\mathcal{K}_{ab}(\Sigma)$. The operations of $\mathcal{K}(\Sigma)$ are the following:

- There are $|\Sigma| + 2|\Sigma|^2$ many horizontal merge operators; we denote all of them with the same symbol $\otimes$. Their domains and ranges are specified by: $\otimes: \mathcal{K}_a(\Sigma) \times \mathcal{K}_a(\Sigma) \to \mathcal{K}_a(\Sigma)$, $\otimes: \mathcal{K}_a(\Sigma) \times \mathcal{K}_{ab}(\Sigma) \to \mathcal{K}_{ab}(\Sigma)$, and $\otimes: \mathcal{K}_{ab}(\Sigma) \times \mathcal{K}_a(\Sigma) \to \mathcal{K}_{ab}(\Sigma)$, where $a, b \in \Sigma$. All of these merge operators are defined by $a(u) \otimes a(v) = a(uv)$, where sorts of the clusters $u, v$ must match the input sorts for one of the merge operators.

- There are $|\Sigma|^2 + |\Sigma|^3$ many vertical merge operators; we denote all of them with the same symbol $\hat{\otimes}$. Their domains and ranges are specified by: $\hat{\otimes}: \mathcal{K}_{ab}(\Sigma) \times \mathcal{K}_b(\Sigma) \to \mathcal{K}_b(\Sigma)$ and $\hat{\otimes}: \mathcal{K}_{ab}(\Sigma) \times \mathcal{K}_{bc}(\Sigma) \to \mathcal{K}_{ac}(\Sigma)$ for $a, b, c \in \Sigma$. For clusters $s \in \mathcal{K}_{ab}(\Sigma)$ and $t \in \mathcal{K}_b(\Sigma) \cup \mathcal{K}_{bc}(\Sigma)$ we obtain $s \hat{\otimes} t$ by replacing in $s$ the bottom boundary node by $t$. For instance, $a(bc(\hat{b}a)) \hat{\otimes} b(ac) = a(bc(b(ac)a))$.

- The atomic clusters $a(b)$ and $a(\hat{b})$ are constants of the cluster algebra.

In the following, we just write $\mathcal{K}_a$ and $\mathcal{K}_{ab}$ for $\mathcal{K}_a(\Sigma)$ and $\mathcal{K}_{ab}(\Sigma)$, respectively. A top dag over $\Sigma$ is an SLP $\mathcal{G}$ over the algebra $\mathcal{K}(\Sigma)$ such that $[\mathcal{G}]^{K(\Sigma)}$ is a cluster.
Figure 2. The shapes of the contexts in $C$ (proof of Lemma 12.2).
Bullet nodes represent boundary nodes. Symmetric shapes where $\sigma_3$ is to the right of $\sigma_1$ are omitted.

of rank zero. In our terminology, cluster straight-line program would be a more appropriate name, but we prefer to use the original term “top dag”.

**Example 12.1.** Consider the top dag $G = (\{S, X_0, \ldots, X_n, Y_0, \ldots, Y_n\}, \rho, S)$ with $\rho(X_0) = b(a^{2^n} b(a^{2^n} b(c)) \cdots a^{2^n})), where $b$ occurs $2^n + 1$ many times.

In [17] it was shown that from a top dag $G$ one can compute in linear time an equivalent FSLP of size $O(|G|)$. Vice versa, from an FSLP $H$ for a tree $t \in C_a$ (for some $a \in \Sigma$) one can compute in time $O(|\Sigma| \cdot |H|)$ an equivalent top dag of size $O(|\Sigma| \cdot |H|)$. The additional factor $|\Sigma|$ in the transformation from FSLPs to top dag s is unavoidable; see [17] for an example.

**Lemma 12.2.** Every cluster algebra $K(\Sigma)$ has a finite subsumption base.

**Proof.** The proof is similar to the proof of Lemma 11.1. Let the set $C$ contain the following contexts, where in each context, each of the auxiliary variables $\tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2, \sigma_3, \sigma_4$ can be also missing (this is necessary since in the cluster algebra, the merge operations have no neutral elements). The main variable $x$ and the auxiliary variables $\tau_1, \tau_2, \tau_3$ must have sorts from $\Sigma$ (rank zero), whereas the main variable $y$ and the auxiliary variables $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ must have sorts from $\Sigma \Sigma$ (rank one). The concrete sorts must be chosen such that all horizontal and vertical merge operations are defined.

(a) $\sigma_1 \uplus (\tau_1 \ominus x \ominus \tau_2)$
(b) $\sigma_1 \uplus (\tau_1 \ominus y \ominus \tau_2) \ominus \tau_3$
(c) $\sigma_1 \uplus (\tau_1 \ominus y \ominus \tau_2) \ominus \sigma_2$
(d) $\sigma_1 \uplus (\tau \ominus (\sigma_1 \ominus (\tau_1 \ominus x \ominus \tau_2)))$

Note that the definition of a top dag in [1] refers to the outcome of a particular top dag construction. In other words: for every tree $t$ a very specific SLP over the cluster algebra is constructed and this SLP is called the top dag of $t$. Here, as in [17], we call any SLP over the cluster algebra a top dag.
Note that these forms are very similar to the forms (a)–(g) for forest algebras from the proof of Lemma 11.1. Only the variables \( \tau_1 \) and \( \tau_2 \) that are horizontally merged with \( x \) (resp., \( y \)) are new.

Figure 3 shows the shapes of the above contexts. Let us explain the intuition behind these shapes. Take a cluster \( s \) (of rank zero or one) and cut out from \( s \) a subcluster \( x \) of rank zero or a subcluster \( y \) of rank one. We do not give a formal definition of subclusters (see [7]), but roughly speaking this means that \( x \) (resp., \( y \)) is a cluster that occurs somewhere in \( s \). In Figure 2 these subclusters are the red triangles. The part of \( s \) that does not belong to the subcluster \( x \) (resp., \( y \)) can be partitioned into finitely many subclusters, and these are the white triangles in Figure 2.

Using Lemma 11.3, we can show that \( C \) is a finite subsumption base for the cluster algebra \( K(\Sigma) \). The atomic clusters are \( \sigma \otimes x, \tau \otimes x, \tau \otimes y, x \otimes \sigma, x \otimes \tau, y \otimes \tau, \sigma \otimes y, x \otimes \sigma, x \otimes \tau \) (where \( x \) and \( \tau \) have sorts from \( \Sigma \) and \( y \) and \( \sigma \) have sorts from \( \Sigma \)). Each of these atomic contexts belongs to \( C \) up to renaming of auxiliary variables. For this it is important that every context from the above list (a)–(g), where some of the auxiliary variables are omitted, belongs to \( C \) as well.

Let us now consider a context \( s'[s] \), where \( s \in C \) and \( s' \) is atomic. We have to show that \( s'[s] \) is subsumed in \( K(\Sigma) \) by a context from \( C \). The case distinction is very similar to the proof of Lemma 11.1. Two examples are shown in Figure 3. The left figure the case \( s = \sigma_1 \otimes (\tau_1 \otimes x \otimes \tau_2) \otimes \tau_3 \) and \( s' = \sigma \otimes x \). In this case \( s'[s] = \sigma \otimes (\sigma_1 \otimes (\tau_1 \otimes x \otimes \tau_2)) \otimes \tau_3 \) is subsumed in \( K(\Sigma) \) by \( \sigma_3 \otimes (\sigma_1 \otimes (\tau_1 \otimes x \otimes \tau_2)) \otimes \tau_3 \) (the latter is obtained from the context in (f) by removing \( \sigma_4 \)).

Figure 3 on the right shows the case \( s = \sigma_1 \otimes (\sigma_3 \otimes (\tau_1 \otimes y \otimes \tau_2) \otimes \tau_3) \) and \( s' = y \otimes \tau \). We have \( s'[s] = \sigma_4 \otimes ((\sigma_3 \otimes \tau) \otimes (\sigma_1 \otimes (\tau_1 \otimes y \otimes \tau_2) \otimes \tau_3)) \) which is equivalent in \( K(\Sigma) \) to \( (\sigma_3 \otimes (\tau_1 \otimes y \otimes \tau_2)) \otimes (\tau_1 \otimes y \otimes \tau_2) \otimes \tau_3 \). The latter context is subsumed in \( K(\Sigma) \) by \( \sigma_1 \otimes (\tau_1 \otimes y \otimes \tau_2) \otimes \tau_3 \in C \).

We can now show the main result for top dags:

**Corollary 12.3.** Given a finite alphabet \( \Sigma \) and a top dag \( \mathcal{G} \) over the cluster algebra \( K(\Sigma) \) producing the tree \( t = \mathcal{G}^{|K(\Sigma)|} \), one can compute in time \( O(|\mathcal{G}|) \) a top dag \( \mathcal{H} \) for \( t \) of size \( O(|\mathcal{G}|) \) and depth \( O(\log |t|) \).

**Proof.** Note that in the derivation tree \( [\mathcal{G}] \) of a top dag \( \mathcal{G} \), all leaves are labelled with atomic clusters and all internal nodes have rank two. Hence, the size of the derivation tree \( [\mathcal{G}] \) is linearly bounded in the size of the generated tree \( [\mathcal{G}]^{K(\Sigma)} \) (in the forest algebra, we needed Lemma 11.3 to enforce this property). For the case of a fixed alphabet \( \Sigma \), the statement of the corollary follows from Lemma 12.2 and Theorem 9.2 analogously to Corollary 11.5 for FSLPs. For the general case of a
variable-size alphabet $\Sigma$ we have to use again Remark 9.4. As for SSLPs and FSLPs we need the natural assumption that symbols from the input alphabet fit into a single machine word of the RAM. All operations from a cluster algebra have rank zero and two, and the subsumption base $C$ from the proof of Lemma 12.3 has the property that every context $s \in \Sigma$ has constant size. In contrast to free monoids and forest algebras, the subsumption base depends on the alphabet $\Sigma$. Basically, we need to choose the sorts of the variables $x, y, \tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2, \sigma_3, \sigma_4$ in each of the contexts from $C$. This implies that every context $s \in C$ can be represented by a constant number of symbols from $\Sigma$ and hence can be stored in a constant number of machine words. The constant time algorithms from point (iii) and (iv) from Remark 9.4 make a constant number of comparisons between the $\Sigma$-symbols representing the input contexts.

In [19] top dags have been used for compressed range minimum queries (RMQs). It is well known that for a string $s$ of integers one can reduce RMQs to lowest common ancestor queries on the Cartesian tree corresponding to $s$. Two compressed data structures for answering RMQs for $s$ are proposed in [19]: one is based on an SSLP for $s$, we commented on it already in Section 10, the other one uses a top dag for the Cartesian tree corresponding to $s$. The following result has been shown, see [19, Corollary 1.4]:

Given a string $s$ of length $n$ over an alphabet of $\sigma$ many integers, let $m_{\text{opt}}$ denote the size of a smallest SSLP for $s$. There is a top dag $G$ for the Cartesian tree corresponding to $s$ of size $|G| \leq \min(\mathcal{O}(n/\log n), \mathcal{O}(m_{\text{opt}} \cdot \log n \cdot \sigma))$, and there is a data structure of size $\mathcal{O}(|G|)$ that answers range minimum queries on $s$ in time $\mathcal{O}(\log \sigma \cdot \log n)$.

As the time bound $\mathcal{O}(\log \sigma \cdot \log n)$ comes from the height of the constructed top dag, using Corollary 12.3 we can enforce the bound $\mathcal{O}(\log n)$ on the height of the constructed top dag and ensure that the transformation can be applied to any input SSLP. This yields the following improvement of the result of [19]:

**Theorem 12.4.** Given an SSLP of size $m$ generating a string $s$ of length $n$ over an alphabet of $\sigma$ many integers one can compute a top dag $G$ for the Cartesian tree corresponding to $s$ of size $|G| \leq \min(\mathcal{O}(n/\log n), \mathcal{O}(m_{\text{opt}} \cdot \sigma))$, and there is a data structure of size $\mathcal{O}(|G|)$ that answers RMQs on $s$ in time $\mathcal{O}(\log n)$. If $m_{\text{opt}}$ denotes the size of a smallest SSLP generating $s$ then, using Rytter’s algorithm, we can assume that $m \leq \mathcal{O}(m_{\text{opt}} \cdot \log n)$.

The rest of the paper is devoted to the proof of Theorem 12.4.

13. **The symmetric centroid decomposition of a DAG**

We start with a new decomposition of a DAG (directed acyclic graph) into disjoint paths. We believe that this decomposition might have further applications. For trees, several decompositions into disjoint paths with the additional property that every path from the root to a leaf only intersects a logarithmic number of paths from the decomposition exist. Examples are the heavy path decomposition [20] and centroid decomposition [12]. These decompositions can be also defined for

Note that [19, Corollary 1.4] states the size bound $|G| \leq \min(\mathcal{O}(n/\log n), \mathcal{O}(m_{\text{opt}} \cdot \sigma))$, i.e., without the additional $\log n$ factor. However, this seems to be a typo: the proof follows by an algorithmic transformation of an SSLP $G_0$ for $s$ to a top dag for the Cartesian tree of size $\mathcal{O}(|G_0| \cdot \sigma)$ and depth $\mathcal{O}(\log \sigma \cdot \text{depth}(G_0))$. In order to get query time $\mathcal{O}(\log \sigma \cdot \log n)$ the authors have to apply this transformation to an SSLP of depth $\mathcal{O}(\log n)$. They obtain such an SSLP by Rytter’s algorithm [33], which computes from the input string $s$ an SSLP of size $\mathcal{O}(m_{\text{opt}} \cdot \log n)$ and depth $\mathcal{O}(\log n)$. This yields the additional $\log n$ factor in the size bound.
DAGs but a technical problem is that the resulting paths are no longer disjoint and form, in general, a subforest of the DAG, see e.g. [8].

Our new path decomposition can be seen as a symmetric form of the centroid decomposition of [12]. Consider a DAG \( D = (V, E) \) with node set \( V \). In our application this DAG will come from a straight-line program. Hence we have to allow multi-edges. We therefore define the edge relation of \( \pi \) paths of multi-edges, this inequality also holds for \( \pi(u, v) \) is the set of sinks, we get

\[ \lambda(D) = (\lfloor \log_2 \pi(r, v) \rfloor, \lfloor \log_2 \pi(v, W) \rfloor). \]

If \( \lambda(D) = (k, \ell) \), then \( k, \ell \leq \lfloor \log_2 n(D) \rfloor \) because \( \pi(r, v) \) and \( \pi(v, W) \) are both bounded by \( n(D) \). Let \( u \) be a node whose path is of length \( 8 \) (number of edges); all other nodes form symmetric centroid paths of length zero.

**Lemma 13.2.** Let \( D = (V, E) \) be a DAG with \( n = n(D) \). The following hold:

1. Every node has at most one outgoing edge from \( E_{\text{scd}}(D) \).
2. Every node has at most one incoming edge from \( E_{\text{scd}}(D) \).
3. Every path from the root \( r \) to a sink node contains at most \( 2 \log_2 n \) edges that do not belong to \( E_{\text{scd}}(D) \).

**Proof.** Consider a node \( v \in V \) and assume that \( v \) has two different outgoing edges \((u, v), (u, w) \in E_{\text{scd}}(D)\). Hence, \( \lambda(u) = \lambda(v) = \lambda(w) \). Let \( \lambda(u) = (k, \ell) \). If \( W \) is the set of sinks, we get \( \pi(u, W) \geq \pi(v, W) + \pi(v, W) \) since we consider paths of multi-edges, this inequality also holds for \( v = W \). W.l.o.g. assume that \( \pi(u, W) \geq \pi(v, W) \) and thus \( \pi(u, W) \geq 2 \pi(v, W) \). We get

\[
\lfloor \log_2 \pi(u, W) \rfloor \geq 1 + \lfloor \log_2 \pi(v, W) \rfloor = 1 + \lfloor \log_2 \pi(u, W) \rfloor,
\]

where the last equality follows from \( \lambda(u) = \lambda(v) \). This is a contradiction and proves statement (1). Statement (2) can be shown in the same way, this time using \( \pi(v, W) \).

For statement (3) consider a path \((v_0, d_1, v_1), (v_1, d_2, v_2), \ldots, (v_{p-1}, d_p, v_p)\), where \( v_0 \) is the root and \( v_p \) is a sink. Let \( \lambda(v_i) = (k_i, \ell_i) \). We must have \( k_i \leq k_{i+1} + 1 \) and \( \ell_i \geq \ell_{i+1} \) for all \( 0 \leq i \leq p - 1 \). Moreover, \( k_0 = \ell_p = 0 \) and \( \ell_0, k_p \leq \lfloor \log_2 n \rfloor \).

Consider now an edge \((v_i, d_i, v_{i+1}) \in E \setminus E_{\text{scd}}(D)\). Since \( \lambda(v_i) \neq \lambda(v_{i+1}) \), we have \( k_i < k_{i+1} \) or \( \ell_i > \ell_{i+1} \). Hence, there can be at most \( 2 \lfloor \log_2 n \rfloor \leq 2 \log_2 n \) edges from \( E \setminus E_{\text{scd}}(D) \) on the path. \( \square \)
Figure 4. A symmetric centroid path decomposition.

Points (1) and (2) from Lemma 13.2 imply that the subgraph $\langle V, E_{scd}(D) \rangle$ is a disjoint union of possibly empty paths, which we call the symmetric centroid paths of $D$. It is straight-forward to compute the edge set $E_{scd}(D)$ in time $O(|D|)$, where $|D|$ is defined as the number of edges of the DAG: By traversing $D$ in both directions (from the root to the sinks and from the sinks to the root) one can compute all pairs $\lambda(v)$ for $v \in V$ in linear time.

One can use Lemma 13.2 in order to simplify the original proof of Corollary 10.4 from [8]: in [8], the authors use the heavy-path decomposition of the derivation tree of an SSLP. In the SSLP (viewed as a DAG that defines the derivation tree), these heavy paths lead to a forest, called the heavy path forest in [8]. The important property used in [8] is the fact that any path from the root of the DAG to a sink node contains only $O(\log n)$ edges that do not belong to a heavy path, where $n$ is the length of string produced by the SSLP. Using point (3) from Lemma 13.2 one can replace this heavy path forest by the decomposition into symmetric centroid paths. The fact that the latter is a disjoint union of paths in the DAG simplifies the technical details in [8] a lot. On the other hand, as shown in Section 10, Corollary 10.4 also directly follows from Theorem 10.3.
14. Suffixes of weighted strings

A weighted string is a string $s \in \Sigma^*$ equipped with a weight function $\| \cdot \| : \Sigma \to \mathbb{N} \setminus \{0\}$, which is extended to a homomorphism $\| \cdot \| : \Sigma^* \to \mathbb{N}$ by $\|a_1a_2\cdots a_n\| = \sum_{i=1}^{n} \|a_i\|$. If $X$ is a variable in an SSLP $\mathcal{G}$, we also write $\|X\|$ for the weight of the string $[X]_{\mathcal{G}}^*$ derived from $X$. In the rest of the section we will omit the superscript $\Sigma^*$ in the notation $[[X]]_{\mathcal{G}}$. Moreover, when we speak of suffixes of a string, we always mean non-empty suffixes.

The SSLPs in this section do not have a distinguished start variable. Moreover, it is convenient to allow the empty SSLP, where the set of variables is the empty set.

Lemma 14.1. For every weighted string $s$ of length $n \geq 0$ one can construct in linear time an SSLP $\mathcal{G}$ with the following properties:

- $\mathcal{G}$ contains at most $3n$ variables,
- all right-hand sides of $\mathcal{G}$ have length at most 4,
- $\mathcal{G}$ contains variables $S_1, \ldots, S_n$ (called the suffix variables of $\mathcal{G}$) producing all suffixes of $s$, and
- every path from $S_i$ to some terminal symbol $a$ in the derivation tree of $\mathcal{G}$ has length at most $3 + 2(\log_2 \|S_i\| - \log_2 \|a\|)$.

Proof. First, the presented algorithm never uses the fact that some letters of $s$ may be equal. Thus it is more convenient to assume that letters in $s$ are pairwise different—in this way the path from a variable $S_i$ to a terminal symbol $a$ in the last condition is defined uniquely.

For the sake of the inductive proof, the constructed SSLP will satisfy a slightly stronger and more technical variant of the last condition: every path from $S_i$ to some terminal symbol $a$ in the derivation tree of $\mathcal{G}$ has length at most $1 + 2(\log_2 \|S_i\| - \log_2 \|a\|)$. The trivial estimation $\|S_i\| \leq 1 + \log_2 \|S_i\|$ then yields the announced variant.

We first show how to construct $\mathcal{G}$ with the desired properties and then prove that the construction can be done in linear time.

The case $n \leq 1$ is trivial (note that for $n = 0$ we can take the empty SSLP since we only want to produce all non-empty suffixes of $s$). Now assume that $n \geq 2$ and let

$$s = a_1 \cdots a_k c b_1 \cdots b_m$$

where $c b_1 \cdots b_m$ is the shortest suffix of $s$ such that $\log_2 \|c b_1 \cdots b_m\| = \log_2 \|s\|$. Clearly such a suffix exists (in the extreme cases it is the whole $s$ or a single letter).

Note that

$$\log_2 \|c b_1 \cdots b_m\| = \log_2 \|a_1 \cdots a_k c b_1 \cdots b_m\|$$

for $1 \leq i \leq k + 1$. Moreover, the following inequalities hold:

\[ \log_2 \|c b_1 \cdots b_m\| \geq \log_2 \|b_1 \cdots b_m\| + 1 \]
\[ \log_2 \|c b_1 \cdots b_m\| \geq \log_2 \|a_1 \cdots a_k\| + 1 \]

(here, we define $\log_2(0) = -\infty$). The former is clear from the definition of $c b_1 \cdots b_m$, as $b_1 \cdots b_m$ satisfies $\log_2 \|b_1 \cdots b_m\| < \log_2 \|s\| = \log_2 \|c b_1 \cdots b_m\|$. If (3) does not hold then both $a_1 \cdots a_k$ and $c b_1 \cdots b_m$ have weights strictly more than $2^{\log_2 \|s\| - 1}$ and so their concatenation $s$ has weight strictly more than $2^{\log_2 \|s\|} \geq \|s\|$, which is a contradiction.

Recall that by the convention from the first paragraph of the proof, the symbols $a_1, \ldots, a_k, c, b_1, \ldots, b_m$ are pairwise different.
For $b_1 \cdots b_m$ we make a recursive call and include the produced SSLP in the output SSLP $G$. Let $V_1, V_2, \ldots, V_m$ be the variables such that
\[
[V_i]_G = b_1 \cdots b_m.
\]
By the inductive assumption, every path $V_i \rightarrow^* b_j$ in the derivation tree has length at most
\[
1 + 2 \log_2 ||V_i|| - 2 \log_2 ||a_j||.
\]
Add a variable $V_0$ with right-hand side $cV_1$ (or $c$ if $m = 0$) which derives the suffix $cb_1 \cdots b_m$. The path from $V_0$ to $c$ in the derivation tree has length 1, which is fine, and the path $V_0 \rightarrow^* a_j$ is one larger than the path $V_i \rightarrow^* a_j$ and hence has length at most
\[
1 + 1 + 2 \log_2 ||V_1|| - 2 \log_2 ||a_j|| \leq 2 \log_2 ||V_0|| - 2 \log_2 ||a_j||,
\]
as $1 + \log_2 ||V_1|| \leq \log_2 ||V_0||$ by (3).

Next we decompose the prefix $a_1 \cdots a_k$ into $\lfloor k/2 \rfloor$ many blocks of length two and, when $k$ is odd, one block of length 1. We add to the output SSLP $G$ new variables $X_1, \ldots, X_{\lfloor k/2 \rfloor}$ and define their right-hand sides by
\[
\rho(X_i) = a_{2i-1}a_{2i}.
\]
The number of variables in $G$ is $\lceil \frac{k}{2} \rceil$. For ease of presentation, when $k$ is odd, define $X_{\lfloor k/2 \rfloor} = a_k$, this is not a new variable, rather just a notational convention to streamline the presentation. Note that for $k$ even $\lfloor k/2 \rfloor = \lfloor k/2 \rfloor$ and in this case $X_{\lfloor k/2 \rfloor}$ is already defined. Viewing $X_1 \cdots X_{\lfloor k/2 \rfloor}$ as a weighted string of length $\lfloor k/2 \rfloor$ over the alphabet $\{X_1, \ldots, X_{\lfloor k/2 \rfloor}\}$, we obtain inductively an SSLP $G_X$ with at most $3\lfloor k/2 \rfloor$ variables and right-hand sides of length at most 4. Moreover, $G_X$ contains variables $U_1, U_2, \ldots, U_{\lfloor k/2 \rfloor}$ with
\[
[U_i]_{G_X} = X_iX_{i+1} \cdots X_{\lfloor k/2 \rfloor}
\]
such that any path of the form $U_i \rightarrow X_j$ in the derivation tree of $G_X$ has length at most
\[
1 + 2 \log_2 ||U_i|| - 2 \log_2 ||X_j||.
\]
By adding all variables and right-hand side definitions from $G_X$ to $G$ (where all symbols $X_i$ are variables, except $X_{\lfloor k/2 \rfloor}$ when $k$ is odd, in which case $X_{\lfloor k/2 \rfloor} = a_k$) we obtain
\[
[U_i]_G = a_{2i-1}a_{2i} \cdots a_k
\]
for all $1 \leq i \leq \lfloor k/2 \rfloor$. Any path $U_i \rightarrow a_j$ in the derivation tree of $G$ has length at most
\[
(5) \quad 2 + 2 \log_2 ||U_i|| - 2 \log_2 ||a_j||.
\]
Now, every suffix of $s$ that includes some letter of $a_1 \cdots a_k$ (note that we already have variables for all other suffixes) can be defined by a right-hand side of the form $U_i cV_1$ or $a_{2i-2} U_i cV_1$ ($U_i c$ or $a_{2i-2} U_i c$ if $m = 0$). As in the statement of the lemma, denote those variables by $S_1, \ldots, S_k$. Let us next verify the condition on the path lengths for derivations from those variables. All paths $S_i \rightarrow c$ have length one. Now consider a path $S_i \rightarrow a_j$. If the path has length one then we are done. Otherwise, the path must be of the form $S_i \rightarrow U_i \rightarrow a_j$. Therefore, by (5) the path length is at most
\[
3 + 2 \log_2 ||U_i|| - 2 \log_2 ||a_j|| \leq 3 + 2 \log_2 ||U_i|| - 2 \log_2 ||a_j||
\]
\[
\leq 1 + 2 \log_2 ||c b_1 \cdots b_m|| - 2 \log_2 ||a_j||
\]
\[
= 1 + 2 \log_2 ||S_i|| - 2 \log_2 ||a_j||,
\]
where the second inequality follows from (4) and the third inequality follows from (2).
Paths of the form $S_i \to^* b_j$ can be treated similarly: they are of the form $S_i \to V_i \to^* b_j$, where the path $V_i \to b_j$ is of length at most $1 + 2\log_2 ||V_i|| - 2\log_2 ||a_j||$ by the inductive assumption. Thus, the whole path is of length at most

$$2 + 2\log_2 ||V_i|| - 2\log_2 ||b_j|| \leq 2\log_2 ||cb_1 \cdots b_m|| - 2\log_2 ||b_j||$$

$$= 2\log_2 ||S_i|| - 2\log_2 ||b_j||,$$

where the first inequality follows from (3) and the second inequality follows from (2).

This concludes the proof of the lemma. $\square$

The SSLP $G$ consists of $[k/2]$ variables $X_1$, $3([k/2])$ variables from the recursive call for $X_1 \cdots X_{[k/2]}$, $3m = 3(n-k-1)$ variables from the recursive call for $b_1 \cdots b_m$, and $1 + k$ new suffix variables for suffixes beginning at $a_1 \cdots a_k c$ (note that those beginning at $b_1 \cdots b_m$ are taken care of by the recursive call). Therefore $G$ contains at most $[k/2] + 3[k/2] + 3(n-k-1) + 1 = 3n + 2[k/2] - k - 2 < 3n$

variables. Also note that all right-hand sides of $G$ have length at most four.

It remains to show that the construction works in linear time. To this end we need a small trick: we assume that when the algorithm is called on $s$, we supply the algorithm with the value $||s||$. More formally, the main algorithm applied to a string $s$ computes $||s||$ in linear time by going through $s$ and adding weights. Then it calls a subprocedure main$'(s, ||s||)$, which performs the actions described above. To find the appropriate symbol $c$, main$'$ computes the weights of consecutive prefixes $s_1 s_2 \cdots s_i$, until it finds the first such that $\log_2 ||s_i|| > \log_2 ||s_1 \cdots s_i||$. Then $k = i - 1$ and so $a_1 \cdots a_k = s_1 \cdots s_{i-1}$, $c = s_i$, $b_1 \cdots b_m = s_{i+1} \cdots s_{||s||}$. Moreover, we can compute $||a_1 \cdots a_k||$ and $||b_1 \cdots b_m||$ for the recursive calls of main$'$ in constant time.

Let $T(n)$ be the running time of main$'$ on a word of length $n$. Then all operations of main$'$, except the recursive calls, take at most $\alpha(k + 1)$ time for some constant $\alpha \geq 1$, where $s$ is represented as $a_1 \cdots a_k cb_1 \cdots b_m$. Thus $T(n)$ satisfies $T(1) = 1$ and

$$T(n) = T([k/2]) + T(n - k - 1) + \alpha(k + 1).$$

We claim that $T(n) \leq 2\alpha n$. This is true for $n = 1$ and inductively for $n \geq 2$ we get

$$T(n) \leq 2\alpha([k/2]) + 2\alpha(n - k - 1) + \alpha(k + 1)$$

$$\leq 2\alpha\frac{k + 1}{2} + 2\alpha n - \alpha(k + 1)$$

$$= 2\alpha n.$$

This concludes the proof of the lemma. $\square$

15. Proof of Theorem 9.1

For the proof of Theorem 9.1 let us take a signature $\Gamma$ and a standard $\Gamma$-SLP $G = (\mathcal{V}, \rho, S)$. Let $t = ||G||$ be its derivation tree and $n = ||t||$. We view $G$ also as a DAG $D := (\mathcal{V}, E)$ with node labels from $\Gamma$. The edge relation $E$ contains all edges $(X, i, X_i)$ where $\rho(X)$ is of the form $f(X_1, \ldots, X_n)$ and $1 \leq i \leq n$. We can assume that all nodes of the DAG are reachable from the start variable $S$. All variables from $\mathcal{V}$ also belong to the TSLP $H$ and produce the same trees in $G$ and $H$. The right-hand side mapping of $H$ will be denoted by $\tau$.

We start with the symmetric centroid decomposition of the DAG $D$, which can be computed in linear time as remarked in Section 9.3. Note that the number $n(D)$ defined in Section 9.3 is the number of leaves of $t$. Hence, we have $n(D) \leq n$.

Consider a symmetric centroid path

$$(X_0, d_0, X_1), (X_1, d_1, X_2), \ldots, (X_{p-1}, d_{p-1}, X_p)$$
in \( \mathcal{D} \), where all \( X_i \) belong to \( \mathcal{V} \) and \( d_i \geq 1 \). Thus, for all \( 0 \leq i \leq p-1 \), the right-hand side of \( X_i \) in \( \mathcal{G} \) has the form
\[
\rho(X_i) = f_i(X_{i,1}, \ldots, X_{i,d_i-1}, X_{i+1,1}, X_{i,d_i+1}, \ldots, X_{i,n_i})
\]
for \( f_i \in \Gamma_{n_i}, X_{i,j} \in \mathcal{V} \) for \( 1 \leq j \leq n_i, j \neq d_i \). Figure 5 shows such a path. Note that the variables \( X_{i,j} \) do not have to be pairwise different (as Figure 5 might suggest). Also note that the variables \( X_{i,j} \) from (7) and all variables in \( \rho(X_p) \) belong to other symmetric centroid paths.

We will introduce \( \mathcal{O}(p) \) many variables in the TSLP \( \mathcal{H} \) to be constructed and the sizes of the corresponding right-hand sides will sum up to \( \sum_{i=0}^{p} |\rho(X_i)| \). By summing over all symmetric centroid paths of \( \mathcal{D} \), this yields the size bound \( \mathcal{O}(|\mathcal{G}|) \) for \( \mathcal{H} \).

Define the ground terms \( t_i = \llbracket X_i \rrbracket_\mathcal{G} \) for \( 0 \leq i \leq p \) and \( t_{i,j} = \llbracket X_{i,j} \rrbracket_\mathcal{G} \) for \( 0 \leq i \leq p - 1 \) and \( 1 \leq j \leq n_i, j \neq d_i \). Recall that every variable \( X_i \) (\( 0 \leq i \leq p \)) of \( \mathcal{G} \) also belongs to \( \mathcal{H} \). For every \( 0 \leq i \leq p - 1 \) we introduce a fresh variable \( Y_i \) which will evaluate in \( \mathcal{H} \) to the context obtained by taking the tree \( t_i \) and cutting out the occurrence of the subtree \( t_k \) that is reached via the directions \( d_i, d_{i+1}, \ldots, d_{p-1} \) from the root of \( t_i \). In Figure 5 this context is visualized for \( i = 4 \) by the red part. Hence, we set
\[
\tau(X_i) = Y_i[X_p]
\]
for \( 0 \leq i \leq p \). For \( X_p \) we define
\[
\tau(X_p) = \rho(X_p).
\]
It remains to come up with right-hand sides such that every \( Y_i \) derives to the intended context. For this, we introduce variables \( Z_i \) (\( 0 \leq i \leq p - 1 \)) and define
\[
\tau(Z_i) = f_i(X_{i,1}, \ldots, X_{i,d_i-1}, x, X_{i,d_i+1}, \ldots, X_{i,n_i})
\]
for \( 0 \leq i \leq p - 1 \). It remains to add variables and right-hand sides such that every \( Y_i \) derives in \( \mathcal{H} \) to \( Z_i[Z_{i+1} \cdots Z_{p-1}] \). This is basically a string problem: we want to produce an SSLP for all suffixes of \( Z_0 Z_1 \cdots Z_{p-1} \). This SSLP should have small depth in order to keep the total depth of the final TSLP bounded by \( \mathcal{O}(\log n) \). Here we use Lemma 14.1. For this we have to define the weights of the variables \( Z_i \). We set \( \|Z_i\| = |t_i| - |t_{i+1}| \). We additively extend the weight function to strings over the symbols \( Z_0, \ldots, Z_{p-1} \).

Using Proposition 14.1 we can construct in time \( \mathcal{O}(p) \) a single SSLP \( \mathcal{I} \) with the following properties:

- \( \mathcal{I} \) has \( \mathcal{O}(p) \) many variables and all right-hand sides have length at most four,
- \( \mathcal{I} \) contains the variables \( Y_0, \ldots, Y_{p-1} \), where \( Y_i \) produces \( Z_i[Z_{i+1} \cdots Z_{p-1}] \) for \( 0 \leq i \leq p - 1 \) and
- every path from a variable \( Y_i \) to a variable \( Z_k \) in the derivation tree of \( \mathcal{I} \) has length at most \( 3 + 2 \log_2 |Y_i| - 2 \log_2 |Z_k| \) for \( 0 \leq i \leq p - 1 \).

Note that \( |Y_i| = |t_i| - |t_p| \). We finally add to the TSLP \( \mathcal{H} \) all right-hand side definitions (8), (9), (10), and all right-hand side definitions from the SSLP \( \mathcal{I} \). Here, we have to replace a concatenation \( YZ \) in a right-hand side of \( \mathcal{I} \) by \( Y[Z] \).

Using the size bound for \( \mathcal{I} \), it follows that the number of variables that are added to \( \mathcal{G} \) in the above construction is bounded \( \mathcal{O}(p) \) and that the sizes of the corresponding right-hand sides sum up to \( \sum_{i=0}^{p} |\rho(X_i)| \). We make the above construction for every symmetric centroid path of \( \mathcal{G} \). Hence, the total size of the TSLP \( \mathcal{H} \) is indeed \( \mathcal{O}(|\mathcal{G}|) \). Moreover, the construction of \( \mathcal{H} \) needs linear time. It remains to show that the depth of \( \mathcal{H} \) is \( \mathcal{O}(\log n) \).
where all variables $X_i \ (0 \leq i \leq p)$ to a variable $X_{j,k} \ (i \leq j \leq p - 1, 1 \leq k \leq n_j, k \neq d_j)$ or a variable from $\rho(X_p)$. Let us define the weight $\|X\|$ for a variable $X \in V$ of $\mathcal{G}$ as the size of the tree $[X]_{\mathcal{G}}$. A path from $X_i$ to a variable $Y$ in $\rho(X_p)$ has the form $X_i \rightarrow Y$ or $X_i \rightarrow X_p \rightarrow Y$ (since $\tau(X_p) = \rho(X_p)$) and hence has length at most two. Now consider a path from $X_i$ to a variable $X_{j,k}$ with $i \leq j \leq p - 1$. We claim that the length of this path is bounded by $5 + 2 \log_2 \|X_i\| - 2 \log_2 \|X_{j,k}\|$. The path $X_i \rightarrow X_{j,k}$ has the form

$$X_i \rightarrow Y_i \rightarrow Z_j \rightarrow X_{j,k},$$

where $Y_i \rightarrow Z_j$ is a path in $\mathcal{I}$ and hence has length at most $3 + 2 \log_2 \|Y_i\| - 2 \log_2 \|Z_j\|$. Hence, the length of the path is bounded by

$$5 + 2 \log_2 \|Y_i\| - 2 \log_2 \|Z_j\| \leq 5 + 2 \log_2 \|X_i\| - 2 \log_2 \|X_{j,k}\|$$

since $\|Y_i\| = |t_i| - |\rho| \leq |t_i| = \|X_i\|$ and $\|Z_j\| = |t_j| - |t_{j+1}| \geq |t_{j,k}| = \|X_{j,k}\|$.

Finally, we consider a maximal path in the derivation tree of $\mathcal{H}$ that starts in the root $S$ and ends in a leaf. We can factorize this path as

$$S = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_k$$

(11)

where all variables $X_i$ belong to the original $\Gamma$-SLP $\mathcal{G}$, and every subpath $X_i \rightarrow X_{i+1}$ has the form considered in the last paragraph. The right-hand side of $X_k$ is a single symbol from $\Gamma_0$ (such a right-hand side can appear in (11)). In the $\Gamma$-SLP $\mathcal{G}$ we have a corresponding path $X_i \rightarrow X_{i+1}$ that is contained in a single symmetric centroid path except for the last edge leading to $X_{i+1}$. By the above consideration, the length of the path (11) is bounded by

$$\sum_{i=0}^{k-1} (5 + 2 \log_2 \|X_i\| - 2 \log_2 \|X_{i+1}\|) \leq 5k + 2 \log_2 \|S\| = 5k + 2 \log_2 n.$$

By the third point from Lemma 13.2 we have $k \leq 2 \log_2 n$ which shows that the length of the path (11) is bounded by $7 \log_2 n$. This concludes the proof of Theorem 9.1. \qed
16. Open problems

For SSLPs one may require a strong notion of balancing. Let us say that an SSLP \( G \) is \( c \)-balanced if (i) the length of every right-hand side is at most \( c \) and (ii) if a variable \( Y \) occurs in \( \rho(X) \) then \( |\llbracket Y \rrbracket^*_G| \leq |\llbracket X \rrbracket^*_G|/2 \). It is open, whether there is a constant \( c \) such that for every SSLP of size \( m \) there exists an equivalent \( c \)-balanced SSLP of size \( O(m) \).

Another important open problem is whether the query time bound in Corollary 10.4 (random access to grammar-compressed strings) can be improved from \( O(\log n) \) to \( O(\log n/\log \log n) \). If we allow space \( O(m \cdot \log^\epsilon n) \) (for any small \( \epsilon > 0 \)) then such an improvement is possible by Corollary 10.6, but it is open whether query time \( O(\log n/\log \log n) \) can be achieved with space \( O(m) \). By the lower bound from [37] this would be an optimal random-access data structure for grammar-compressed strings.

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This appendix is written for readers that are only interested in our balancing result for string straight-line programs (Theorem 10). It allows to skip Sections 13 and 14. Only the material from Sections 13 and 14 is needed.

Let \( \Sigma \) be a finite alphabet of terminal symbols. A string straight-line program (SSLP for short) over the alphabet \( \Sigma \) is a triple \( G = (V, \rho, S) \), where \( V \) is a finite set of variables, \( S \in V \) is the start variable, and \( \rho : V \to (\Sigma \cup V)^* \) (the right-hand side mapping) has the property that the binary relation \( E(G) = \{(X, Y) \in V \times V ; Y \text{ occurs in } \rho(X)\} \) is acyclic. This allows to define for every variable \( X \in V \) a string \( [X]_G \) as follows: if \( \rho(X) = u_0 X_1 u_1 X_2 \cdots u_{n-1} X_n u_n \) with \( u_0, u_1, \ldots, u_n \in \Sigma^* \) and \( X_1, \ldots, X_n \in V \) then \( [X]_G = u_0 [X_1]_G u_1 [X_2]_G \cdots u_{n-1} [X_n]_G u_n \). We omit the subscript \( G \) if \( G \) is clear from the context. Finally, we define \( [G] = [S] \).

An SSLP \( G \) can be seen as a context-free grammar that produces the single string \( [G] \). Quite often, one assumes that all right-hand sides \( \rho(X) \) are from \( \Sigma \cup V \). This corresponds to the Chomsky normal form. For every SSLP \( G \) with \( [G] \neq \varepsilon \) one can construct in linear time an equivalent SSLP in Chomsky normal form by replacing every right-hand side by a balanced binary derivation tree.

Fix an SSLP \( G = (V, \rho, S) \). We define the size \( |G| \) of \( G \) as \( \sum_{X \in V} |\rho(X)| \). Let \( d \) be the length of a longest path in the graph \( (V, E(G)) \) and \( r = \max \{|\rho(X)| : X \in V\} \). We define the depth of \( G \) as \( \text{depth}(G) = d \cdot \lceil \log_2 r \rceil \). These definitions ensure that depth and size only increase by fixed constants when an SSLP is transformed into Chomsky normal form. Note that for an SSLP in Chomsky normal form, the definition of the depth simplifies to \( \text{depth}(G) = d \).

**Theorem A.1.** From a given SSLP \( G \) such that \( [G] \) has length \( n \) one can construct in linear time an SSLP \( H \) with the following properties: \( |H| = |G| \), \( |H| \leq O(|G|) \) and \( \text{depth}(H) \leq O(\log n) \).

**Proof.** Let \( G = (V, \rho_G, S) \). W.l.o.g. we can assume that \( G \) is in Chomsky normal form (the case that \( [G] = \varepsilon \) is of course trivial). Note that the graph \( (V, E(G)) \) is a directed acyclic graph (DAG). We can assume that every variable is reachable from the start variable \( S \). Consider a variable \( X \) with \( \rho_G(X) = Y Z \). Then \( X \) has the two outgoing edges \( (X, Y) \) and \( (X, Z) \) in \( (V, E(G)) \). We replace these two edges by the triples \( (X, 1, Y) \) and \( (X, 2, Y) \). Hence, \( D := (V, E(G)) \) becomes a DAG with multi-edges (triples from \( V \times \{1, 2\} \times V \)). Figure 1 shows the DAG \( D \) for an example SSLP. The right-hand sides for the two sink variables \( X_{13} \) and \( X_{14} \) are terminal symbols. The start variable is \( X_0 \).

We define for every \( X \in V \) the weight \( \|X\| \) as the length of the string \( [X]_G \). Moreover, for a string \( w = X_1 X_2 \cdots X_n \) we define the weight \( \|w\| = \sum_{i=1}^n \|X_i\| \). Note that \( \|S\| = n \) is the length of the derived string.

At this point, we use the material from Section 13. We start with the symmetric centroid decomposition of the DAG \( D \), which can be computed in linear time as remarked in Section 13. The red edges in Figure 4 are those edges that belong to the symmetric centroid path decomposition. Note that the DAG in Figure 4 is the same DAG as in Figure 1. The second components of the node labels in Figure 4 are the weights of the corresponding variables in Figure 1. Hence, we have \( \|X_0\| = 62, \|X_1\| = 61, \|X_2\| = 60, \|X_3\| = 58 \), etc.

Note that the variable \( n(D) \) defined in Section 13 is exactly the length of \( [G] \), i.e., \( n = n(D) \). Consider a symmetric centroid path

\[(12) \quad (X_0, d_0, X_1), (X_1, d_1, X_2), \ldots, (X_{p-1}, d_{p-1}, X_p)\]

Note that in the main part of the paper, \( [X]_G \) denotes the derivation tree rooted in \( X \), whereas the string derived from \( X \) is denoted with \( [X]_G^\tau \).
in $\mathcal{D}$, where all $X_i$ belong to $V$ and $d_i \in \{1, 2\}$. Thus, for all $0 \leq i \leq p - 1$, the right-hand side of $X_i$ in $\mathcal{G}$ has the form

- $\rho_{\mathcal{G}}(X_i) = X_{i+1}X'_{i+1}$ (if $d_i = 1$)
- $\rho_{\mathcal{G}}(X_i) = X'_{i+1}X_{i+1}$ (if $d_i = 2$)

for some $X'_{i+1} \in V$. Note that we can have $X'_{i} = X'_j$ for $i \neq j$. The right-hand side $\rho_{\mathcal{G}}(X_p)$ belongs to $\Sigma \cup VV$. Note that the variables $X'_i$ ($1 \leq i \leq p$) and the variables in $\rho_{\mathcal{G}}(X_p)$ (if they exist) belong to other symmetric centroid paths. We will introduce $O(p)$ many variables in the SSLP $\mathcal{H}$ to be constructed. Moreover, all right-hand sides of $\mathcal{H}$ have length at most four. By summing over all symmetric centroid paths, this yields the size bound $O(|\mathcal{G}|)$ for $\mathcal{H}$.

We now define the right-hand sides of the variables $X_0, \ldots, X_p$ in $\mathcal{H}$. We write $\rho_{\mathcal{H}}$ for the right-hand side mapping of $\mathcal{H}$. For $X_p$ we set $\rho_{\mathcal{H}}(X_p) = \rho_{\mathcal{G}}(X_p)$. For the variables $X_0, \ldots, X_{p-1}$ we have to “accelerate” the derivation somehow in order to get the depth bound $O(\log n)$ at the end. For this, we apply Proposition 14.1 from Section 14. Let $L_1 \cdots L_s$ be the subsequence obtained from $X'_1X'_2 \cdots X'_p$ by keeping only those $X'_i$ with $d_i = 2$ and let $R_1 \cdots R_t$ be the subsequence obtained from the reversed sequence $X'_pX'_{p-1} \cdots X'_1$ by keeping only those $X'_i$ with $d_i = 1$. Take for instance the symmetric centroid path consisting of the nodes $X_0, X_1, \ldots, X_8$ (hence, $p = 8$) from our running example in Figure 6. We have $L_1 \cdots L_s = X_{13}X_{12}X_{11}X_{10}$.
Note that every string $[X_i]$ ($0 \leq i \leq p-1$) can be derived in $G$ from a word $w_0X_pw_t$, where $w_t$ is a suffix of $[L_1 \cdots L_s]$ and $w_r$ is a prefix of $[R_1 \cdots R_s]$. For instance, $[X_2]$ can be derived from $(X_{12}X_{11}X_{10})X_8(X_{10}X_{11}X_{12})$ in our running example. We now apply Proposition 14.1 to the sequence $\{L_1 \cdots L_s\}$ in order to get an SSLP $G_r$ of size $O(s) \leq O(p)$ that contains variables $S_1, \ldots, S_s$ for the empty suffixes of $L_1 \cdots L_s$. Moreover, every path from a variable $S_i$ to some $L_j$ in the derivation tree has length at most $3 + 2 \log_n |S_i| - 2 \log_n |L_j|$, where $|S_i|$ is the weight of $[S_i]_{G_r}$. Analogously, we obtain an SSLP $G_t$ of size $O(t) \leq O(p)$ that contains variables $P_1, \ldots, P_t$ for the non-empty prefixes of $R_1 \cdots R_t$. Moreover, every path from a variable $P_i$ to some $R_j$ in the derivation tree has length at most $3 + 2 \log_n |P_i| - 2 \log_n |R_j|$. We can then define every right-hand side $\rho_H(X_i)$ as $S_jX_pP_k$, $X_pP_k$, $S_jX_p$, or $X_p$ for suitable $j$ and $k$. Moreover, we add all variables and right-hand side definitions of $G_r$ and $G_t$ to $\mathcal{H}$.

We make the above construction for all symmetric centroid paths of the DAG $\mathcal{D}$. This concludes the construction of $\mathcal{H}$. In our running example we set $\rho_H(X_i) = \rho_G(X_i)$ for $8 \leq i \leq 14$. Since we introduce $O(p)$ many variables for every symmetric centroid path of length $p$ and all right-hand sides of $\mathcal{H}$ have length at most four, we obtain the size bound $O(|G|)$ for $\mathcal{H}$. It remains to show that the depth of $\mathcal{H}$ is $O(\log n)$. Let us first consider the path (12) and a path in the derivation tree of $\mathcal{H}$ from a variable $X_i$ ($0 \leq i \leq p$) to a variable $Y$, where $Y$ is

(a) a variable in $\rho_G(X_p)$ or

(b) a variable $X_j^i$ for some $i < j \leq p$.

In case (a), the path $X_i \rightarrow Y$ has length at most two. In case (b) the path $X_i \rightarrow Y$ is of the form $X_i \rightarrow S_k \rightarrow X_j^i = Y$ or $X_i \rightarrow P_k \rightarrow X_j^i = Y$. Here, $S_k \rightarrow X_j^i$ (resp., $P_k \rightarrow X_j^i$) is a path in $G_t$ (resp., $G_r$) and therefore has length $3 + 2 \log_n |S_k| - 2 \log_n |Y|$ (resp., $3 + 2 \log_n |P_k| - 2 \log_n |Y|$). In both cases, we can bound the length of the path $X_i \rightarrow Y$ by $4 + 2 \log_n |X_i| - 2 \log_n |Y|$.

Finally, we consider a maximal path in the derivation tree of $\mathcal{H}$ that starts in the root $S$ and ends in a leaf. We can factorize this path as

$$S = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_k$$

where all variables $X_i$ belong to the original SSLP and every subpath $X_i \rightarrow X_{i+1}$ is of the form $X_i \rightarrow Y$ considered in the previous paragraph. The right-hand side of $X_k$ is a single symbol from $\Sigma$. In the DAG $\mathcal{D}$ we have a corresponding path $X_i \rightarrow X_{i+1}$ that is contained in a single symmetric centroid path except for the last edge leading to $X_{i+1}$. By the above consideration, the length of the path (13) is bounded by

$$\sum_{i=0}^{k-1} (4 + 2 \log_n |X_i| - 2 \log_n |X_{i+1}|) \leq 4k + 2 \log_n |S| = 4k + 2 \log_n n.$$

By the third point from Lemma 13.2 we have $k \leq 2 \log_n n$ which shows that the length of the path (13) is bounded by $6 \log_n n$. This concludes the proof of the theorem. \qed