Global hypoelliptic estimates for a linear model of non-cutoff Boltzmann equation

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Abstract. In this paper we study a linear model of spatially inhomogeneous Boltzmann equation without angular cutoff. Using the multiplier method introduced by F. Hérau and K. Pravda-Starov (2011), we establish the optimal global hypoelliptic estimate with weights for the linear model operator.

1. Introduction and main results

Inspired by the work of Hérau and Pravda-Starov [21] on the global hypoellipticity of Landau-type operator, we study in this paper the hypoellipticity of a linear model of spatially inhomogeneous Boltzmann equation without angular cutoff, which takes the following form:

\[ P = \partial_t + v \cdot \partial_x + a(v)(-\tilde{\Delta}_v)^s + b(v), \quad 0 < s < 1, \]

where the coefficients \( a, b \) are smooth real-valued functions of the velocity variable \( v \) with the properties subsequently listed below. There exist a number \( \gamma \in \mathbb{R} \) and a constant \( C \geq 1 \) such that for all \( v \in \mathbb{R}^n \) we have

\[ C^{-1} \langle v \rangle^\gamma \leq a(v) \leq C \langle v \rangle^{2s+\gamma}, \quad C^{-1} \langle v \rangle^{2s+\gamma} \leq b(v) \leq C \langle v \rangle^{2s+\gamma}, \]

and

\[ \forall \ |\alpha| \geq 0, \exists C_\alpha > 0, \quad |\partial^\alpha_v a(v)| + |\partial^\alpha_v b(v)| \leq C_\alpha \langle v \rangle^{2s+\gamma-|\alpha|}, \]

where and throughout the paper we use the notation \( \langle \cdot \rangle = (1 + |\cdot|^2)^{1/2} \). The notation \( (-\tilde{\Delta}_v)^s \) in (1) stands for the Fourier multiplier of symbol

\[ |\eta|^{2s} \omega(\eta) + |\eta|^2 (1 - \omega(\eta)), \]

with \( \omega(\eta) \in C^\infty(\mathbb{R}^n; [0, 1]) \), such that \( \omega = 1 \) if \( |\eta| \geq 2 \) and \( \omega = 0 \) if \( |\eta| \leq 1 \). Here \( \eta \) is the dual variable of \( v \).

Let’s first explain the motivation for studying such a kind of operator \( P \), which is closely linked with the spatially inhomogeneous Boltzmann equation which has singularity in both the kinetic part and the angular part. Precisely, non-cutoff Boltzmann equation in \( \mathbb{R}^n \) reads

\[ \partial_t f + v \cdot \partial_x f = Q(f, f), \]
where \( f(t, x, v) \) is a real-valued function, standing for the time-dependent probability density of particles with velocity \( v \) at position \( x \). The right hand side of (4) is the Boltzmann bilinear collision operator which acts only on the velocity variable \( v \) by

\[
Q(g, f)(v) = \int_{\mathbb{R}^n} \int_{S^{n-1}} B(|v - v_*|, \sigma) (g'_* f' - g f) \, dv_* d\sigma.
\]

Here we use the shorthand \( f = f(t, x, v), f_* = f(t, x, v_*), f' = f(t, x, v'), f'_* = f(t, x, v'_*), \) and for \( \sigma \in S^{n-1} \),

\[
v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2\sigma}, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2\sigma}.
\]

In the above relations, \( v', v'_* \) and \( v, v_* \) are the velocities of a pair of particles before and after collision. The collision cross-section \( B(|v - v_*|, \sigma) \) is a non-negative function which only depends on the relative velocity \( |v - v_*| \) and the deviation angle \( \theta \) through \( \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma \), and takes the following form

\[
B(|v - v_*|, \sigma) = \Phi(|v - v_*|) b(\cos \theta), \quad \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, \quad 0 \leq \theta \leq \frac{\pi}{2},
\]

where the kinetic part \( \Phi \) is given by

\[
\Phi(|v - v_*|) = |v - v_*|^\gamma, \quad \gamma > -3,
\]

and the angular part \( b \) satisfies, with \( 0 < s < 1 \),

\[
b(\cos \theta) \approx \theta^{-(n-1)-2s} \quad \text{as} \ \theta \to 0.
\]

We refer to [2, 3, 13, 20, 38] and the references therein for the physical background and derivation of the Boltzmann equation, as well as the mathematical theory on the Boltzmann equation. Note that the angular cross-section \( b \) is not integrable on the sphere due to the singularity \( \theta^{-(n-1)-2s} \), which leads to the conjecture that the nonlinear collision operator should behave like a fractional Laplacian; that is,

\[
Q(g, f) \approx -C_g (-\Delta)^s f + \text{lower order terms},
\]

with \( C_g > 0 \) a constant depending only on the physical properties of \( g \). Initiated by Desvillettes [16, 17], there have been extensive works which give partial support to the conjecture regarding the smoothness of solutions for the homogeneous Boltzmann equation without angular cutoff, c.f. [4, 10, 11, 14, 18, 19, 23, 31, 32, 34]. For the inhomogeneous case the study becomes more complicated, due to the coupling of the transport operator with the collision operator, and the commutator between pseudo-differential operators and the collision operator. Recent works [5, 6, 7, 8, 9, 24, 25, 31, 35, 36] indicate the linearized Boltzmann operator around a normalized Maxwellian distribution behaves essentially like the operator given in (4). To explain it more precisely, let’s first recall the linearization process. Denote by \( \mu \) the normalized Maxwellian distribution; that is

\[
\mu(v) = (2\pi)^{-n/2} e^{-|v|^2/2}.
\]

By setting \( f = \mu + \sqrt{\mu} g \), we see the perturbation \( g \) satisfies the equation

\[
\partial_t g + v \cdot \partial_x g - \mu^{-1/2} Q(\mu, \sqrt{\mu} g) - \mu^{-1/2} Q(\sqrt{\mu} g, \mu) = \mu^{-1/2} Q(\sqrt{\mu} g, \sqrt{\mu} g),
\]

since \( \partial_t f + v \cdot \partial_x f - Q(f, f) = 0 \) and \( Q(\mu, \mu) = 0 \). Using the notation

\[
\Gamma(g, h) = \mu^{-1/2} Q(\sqrt{\mu} g, \sqrt{\mu} h),
\]

then

\[
\partial_t g + v \cdot \partial_x g - \mu^{-1/2} Q(\mu, \sqrt{\mu} g) - \mu^{-1/2} Q(\sqrt{\mu} g, \mu) = \mu^{-1/2} \Gamma(g, g).
\]
we may rewrite the above equation as
\[
\partial_t g + v \cdot \partial_x g - \Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu}) = \Gamma(g, g).
\]
Due to the following coercivity and upper bound estimates established in [7], with \(H^m(\mathbb{R}^n_+), m \in \mathbb{R}\), the usual Sobolev space,
\[
C^{-1} \left( \| \langle v \rangle^{\frac{3}{2}} g \|_{H^m(\mathbb{R}^n_+)}^2 + \| \langle v \rangle^{\frac{s+2}{2}} g \|_{L^2(\mathbb{R}^n_+)}^2 \right) \leq \left( -\Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu}), g \right)_{L^2(\mathbb{R}^n_+)} + \| g \|_{L^2(\mathbb{R}^n_+)}^2
\]
and
\[
\left( -\Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu}), g \right)_{L^2(\mathbb{R}^n_+)} \leq C \| \langle v \rangle^{s+\frac{3}{2}} g \|_{H^m(\mathbb{R}^n_+)}^2,
\]
we see that the linear part \(-\Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu})\) of the Boltzmann collision operator behaves like a generalized Kolmogorov type operator
\[
\partial_t + v \cdot \partial_x + a(v)(-\Delta_v)^s + b(v),
\]
with \(a(v), b(v)\) satisfying the conditions (2) and (3). This motivates the present work on the global hypoellipticity of the operator \(P\) given in (1).

We remark that there have been some related works concerned with a linear model of spatially inhomogeneous Boltzmann equation, which takes the following form
\[
P = \partial_t + v \cdot \partial_x - a(t, x, v)(-\Delta_v)^s, \quad \inf_{t,x,v} a(t, x, v) > 0, \quad a \in C^\infty_b,
\]
where \(C^\infty_b\) stands for the space of smooth functions whose derivatives of any order are bounded. As far as we know, the model operator (5) was firstly studied by Morimoto and Xu [23] for \(1/3 < s \leq 1\), and then was improved by Chen et al. [15] by virtue of Kohn’s method. Recently Lerner et al. [29] established optimal results using the Wick quantization techniques [27, 28], and then a simpler proof was presented by Alexandre [1] following the ideas of Bouchut [12] and Perthame [37], completing established optimal results using the Wick quantization techniques [27, 28], and then a simpler proof was presented by Alexandre [1] following the ideas of Bouchut [12] and Perthame [37], completing the study of the operator \(P\) given in (5). However these works are mainly concerned with the local hypoelliptic estimates in the sense that the coefficient \(a\) in (5) has strictly positive lower bound and bounded derivatives. Compared with the operator in (5), our model operator \(P\) in (1) is closer to the linearized Boltzmann equation in view of the aforementioned coercivity estimate and upper bound estimate. Moreover we do not need the restrictions that \(\inf_{t,x,v} a(t, x, v) > 0\) and \(a \in C^\infty_b\), since the coefficients in (1) may tend to 0 or \(+\infty\) as \(|v| \to +\infty\), depending on the sign of \(\gamma\).

Now we state our main results as follows.

**Theorem 1.1.** Let \(P\) be given in (1) with \(a(v), b(v)\) satisfying the conditions (2) and (3). Then for all \(m \in \mathbb{R}\), there exists a constant \(C_m\) such that for all \(f \in \mathcal{S}(\mathbb{R}^{2n+1})\) we have
\[
\| \langle v \rangle^{\frac{3}{2}} |D_t|^{\frac{2}{1+2s}} |D_x|^{\frac{2}{1+2s}} f \|_{H^m} + \| \langle v \rangle^{\frac{s+2}{2}} |D_t|^{\frac{2}{1+2s}} |D_x|^{\frac{2}{1+2s}} f \|_{H^m} + \| \langle v \rangle^\gamma |D_v|^{2s} f \|_{H^m} + \| \langle v \rangle^{2s+\gamma} f \|_{H^m} \\
\leq C_m \left( \| Pf \|_{H^m} + \| f \|_{H^m} \right),
\]
where \(\| \cdot \|_{H^m}\) stands for \(\| \cdot \|_{H^m(\mathbb{R}^{2n+1}_{t,x,v})}\), and \(D_t = \frac{1}{t} \partial_t, D_x = \frac{1}{x} \partial_x\), etc.

**Remark 1.2.** It seems that the multiplier method used in the paper can also be applied to the linearized Boltzmann operator \(L\) given by
\[
Lg = \partial_t g + v \cdot \partial_x g - \Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu}),
\]
and gives the same hypoellipticity as above. But the situation is more complicated, and we should pay more attention to handling the commutators between \( L \) and pseudo-differential operators. We hope to study this issue in a future work.

We end up the introduction by a few comments on the exponents of derivative terms and weight terms in Theorem 1.1. These exponents seem to be optimal. When restricted to a fixed compact subset \( K \subset \mathbb{R}^{2n+1} \), instead of the whole space, the problems reduce to a local version, and the operator becomes the type given in (5), for which the exponent \( 2s/(2s+1) \) for the regularity in the time and space variables is indeed sharp by using a simple scaling argument (see [29] for more detail). In the particular case when \( s = 1 \), we have a type of differential operator, which seems simpler to handle than fractional derivatives, and our exponents in the regularity terms and weight terms coincide well with the ones in [21].

2. Notations and estimates on commutator with pseudo-differential operators

2.1. Notations and some basic facts on symbolic calculus

Notice that the diffusion term in (1) is an operator only with respect to the velocity variable \( v \). So it is convenient to take partial Fourier transform in the \( t, x \) variables, and then to study the operator on the Fourier side

\[
\hat{P} = i(\tau + v \cdot \xi) + a(v)(-\Delta_v)^s + b(v),
\]

where and throughout the paper, \((\tau, \xi)\) always stand for the dual variables of \((t, x)\) and are considered as parameters, while \( \eta \) will be used to denote the dual variable of \( v \). Since our analysis is on \( \mathbb{R}^n_v \), we will use \( \langle \cdot, \cdot \rangle_{L^2} \) and \( \| \cdot \|_{L^2}^2 \), instead of \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^n_\tau)} \) and \( \| \cdot \|_{L^2(\mathbb{R}^n_\tau)}^2 \) to denote the inner product and norm in \( L^2(\mathbb{R}^n_v) \), if no confusion occurs.

To simplify the notation, by \( A \lesssim B \) we mean there exists a positive harmless constant \( C > 0 \) such that \( A \leq C B \), and similarly for \( A \gtrsim B \). While the notation \( A \approx B \) means both \( A \lesssim B \) and \( B \lesssim A \) hold.

Now we recall some basic facts on symbolic calculus, and refer to Chapter 18 of [22] and [26] for detailed discussion on the pseudo-differential calculus. In the sequel discussion, let \( m(v, \eta) \) be an admissible weight with respect to the constant metric \(|dv|^2 + |d\eta|^2\). By admissible weight we mean that

\[
\exists C > 0, r > 0, \quad \forall (v, \eta), (\tilde{v}, \tilde{\eta}) \in \mathbb{R}^{2n}, \quad |(v, \eta) - (\tilde{v}, \tilde{\eta})| \leq r \implies C^{-1} \leq \frac{m(v, \eta)}{m(\tilde{v}, \tilde{\eta})} \leq C,
\]

and that

\[
\exists C > 0, N > 0, \quad \forall (v, \eta), (\tilde{v}, \tilde{\eta}) \in \mathbb{R}^{2n}, \quad \frac{m(v, \eta)}{m(\tilde{v}, \tilde{\eta})} \leq C \left( 1 + |(v, \eta) - (\tilde{v}, \tilde{\eta})| \right)^N.
\]

Consider a symbol \( p(\tau, \xi, v, \eta) \) as a function of \((v, \eta)\) with parameters \((\tau, \xi)\), and we say \( p \in S \left( m, |dv|^2 + |d\eta|^2 \right) \) uniformly with respect to \((\tau, \xi)\), if

\[
\forall \alpha, \beta \in \mathbb{Z}_+^n, \forall v, \eta \in \mathbb{R}^n, \quad |\partial_\tau^\alpha \partial_\xi^\beta p(\tau, \xi, v, \eta)| \leq C_{\alpha, \beta} m(v, \eta),
\]
with $C_{\alpha, \beta}$ a constant depending only on $\alpha$ and $\beta$, but independent of $(\tau, \xi)$. For simplicity of notations, we will omit the parameters $(\tau, \xi)$ in symbols, and by $p \in S \left( m, |dv|^2 + |d\eta|^2 \right)$ we always mean that $p$ satisfies the above inequality uniformly with respect to $(\tau, \xi)$. Denote by
\[ \text{Op} \left( S(m, |dv|^2 + |d\eta|^2) \right) \]
the class of pseudo-differential operators $p^w$ with $p \in S \left( m, |dv|^2 + |d\eta|^2 \right)$. Here $p^w$ stands for the Weyl quantization of symbol $p$, defined by
\[ p^w(u(v)) = \int_{\mathbb{R}^6} e^{2\pi (v-z) \cdot \eta} p \left( \frac{v+z}{2}, \eta \right) u(z) \, dz \, d\eta. \]
One of the elementary properties of the Weyl quantization is the boundedness in $L^2$ of the operator $p^w$ with $p \in S \left( 1, |dv|^2 + |d\eta|^2 \right)$. If $p_i \in S \left( m_i, |dv|^2 + |d\eta|^2 \right)$, $i = 1, 2$, then we have (see Theorem 2.3.8 of [26] for instance)
\[ p_1^w p_2^w \in \text{Op} \left( S \left( m_1 m_2, |dv|^2 + |d\eta|^2 \right) \right). \] (7)
In view of (3), symbolic calculus (Theorem 2.3.8 and Corollary 2.3.10 of [26]) shows that for any $m \in \mathbb{R}$ and any $\ell \in \mathbb{R}$ we have
\[ \left[ (D_\eta)^m, (v)^\ell \right] \in \text{Op} \left( S \left( (v)^{\ell-1} \langle \eta \rangle^{m-1}, |dv|^2 + |d\eta|^2 \right) \right), \] (8)
\[ \left[ (D_\eta)^m, a \right], \left[ (D_\eta)^m, b \right] \in \text{Op} \left( S \left( (v)^{2\gamma+\gamma-1} \langle \eta \rangle^{m-1}, |dv|^2 + |d\eta|^2 \right) \right), \] (9)
and
\[ \left[ p^w, a \right], \left[ p^w, b \right] \in \text{Op} \left( S \left( (v)^{2\gamma+\gamma-1}, |dv|^2 + |d\eta|^2 \right) \right), \] (10)
where $p \in S \left( 1, |dv|^2 + |d\eta|^2 \right)$ and $[A, B]$ stands for the commutator between $A$ and $B$ defined by $[A, B] = AB - BA$.

**Lemma 2.1.** Let $\tilde{\mathcal{P}}$ be given in (6) with $a, b$ satisfying the assumptions (2) and (3). Then for all $f \in S(\mathbb{R}^\nu)$ we have
\[ \left\| a^{\frac{1}{2}} (\Delta_v)^{s/2} f \right\|^2_{L^2} + \left\| (D_\eta)^s \langle v \rangle^{\gamma/2} f \right\|^2_{L^2} + \left\| \langle v \rangle^{s+\gamma/2} f \right\|^2_{L^2} \lesssim \left\| \tilde{\mathcal{P}} f, f \right\|_{L^2} + \left\| f \right\|^2_{L^2}. \] (11)

**Proof.** We only need to treat the first and third terms on the left hand side of (11), since by (2) and (3) one has
\[ \left\| (D_\eta)^s \langle v \rangle^{\gamma/2} f \right\|_{L^2} \lesssim \left\| a^{\frac{1}{2}} (\Delta_v)^{s/2} f \right\|_{L^2} + \left\| \langle v \rangle^{s+\gamma/2} f \right\|_{L^2} + \left\| f \right\|_{L^2}. \]
Let $f \in S(\mathbb{R}^\nu)$. Observe the operator $i (\tau + v \cdot \xi)$ is skew-adjoint. It then follows that
\[ \text{Re} \left( a(v)(-\Delta_v)^s f, f \right)_{L^2} + \text{Re} \left( b(v) f, f \right)_{L^2} = \text{Re} \left( \tilde{\mathcal{P}} f, f \right)_{L^2}. \]
Note that $a, b$ are real-valued functions. Then by virtue of the relation that
\[ \text{Re} \left( a(v)(-\Delta_v)^s f, f \right)_{L^2} = \left( (-\Delta_v)^{\frac{s}{2}} a(v)(-\Delta_v)^{\frac{s}{2}} f, f \right)_{L^2} - \text{Re} \left( [(-\Delta_v)^{\frac{s}{2}}, a](-\Delta_v)^{\frac{s}{2}} f, f \right)_{L^2}, \]
we have
\[
\|a^{1/2}(-\tilde{\Delta}_v)^{s/2} f\|_{L^2}^2 + \|b^{1/2} f\|_{L^2}^2 \lesssim \left| \left( \tilde{P} f, f \right)_{L^2} \right| + \left| \left[ (-\tilde{\Delta}_v)^{s/2}, a \right] (-\tilde{\Delta}_v)^{s/2} f, f \right|_{L^2}.
\] (12)

In view of (9), we see
\[
\left[ (-\tilde{\Delta}_v)^{s/2}, a \right] \in \text{Op} \left( S \left( \langle v \rangle^{2s+\gamma-1}, |dv|^2 + |d\eta|^2 \right) \right),
\]
and thus by (7)
\[
\langle v \rangle^{-(2s+\gamma-1)} \left[ (-\tilde{\Delta}_v)^{s/2}, a \right] \langle v \rangle^{-\gamma/2} \in \text{Op} \left( S \left( 1, |dv|^2 + |d\eta|^2 \right) \right).
\]

As a result, writing
\[
\left[ (-\tilde{\Delta}_v)^{s/2}, a \right] = \langle v \rangle^{2s+\gamma-1} \left[ (-\tilde{\Delta}_v)^{s/2}, a \right] \langle v \rangle^{-\gamma/2} \langle v \rangle^{\gamma/2},
\]
we have
\[
\left| \left[ (-\tilde{\Delta}_v)^{s/2}, a \right] (-\tilde{\Delta}_v)^{s/2} f, f \right|_{L^2} \lesssim \| \langle v \rangle^{\gamma/2} (-\tilde{\Delta}_v)^{s/2} f\|_{L^2} \| \langle v \rangle^{2s+\gamma-1} f\|_{L^2}.
\]
This along with the interpolation inequality
\[
\| \langle v \rangle^{2s+\gamma-1} f\|_{L^2} \lesssim \varepsilon \| \langle v \rangle^{s+\gamma} f\|_{L^2} + C_\varepsilon \| f\|_{L^2}
\]
due to the fact that \( s < 1 \), gives
\[
\left| \left[ (-\tilde{\Delta}_v)^{s/2}, a \right] (-\tilde{\Delta}_v)^{s/2} f, f \right|_{L^2} \lesssim \varepsilon \| \langle v \rangle^{\gamma/2} (-\tilde{\Delta}_v)^{s/2} f\|_{L^2}^2 + \varepsilon \| \langle v \rangle^{s+\gamma} f\|_{L^2}^2 + C_\varepsilon \| f\|_{L^2}^2
\]
\[
\lesssim \varepsilon \| a^{1/2}(-\tilde{\Delta}_v)^{s/2} f\|_{L^2}^2 + \varepsilon \| b^{1/2} f\|_{L^2}^2 + C_\varepsilon \| f\|_{L^2}^2,
\]
where the last inequality follows from (2). Combining (12) we conclude
\[
\| a^{1/2}(-\tilde{\Delta}_v)^{s/2} f\|_{L^2}^2 + \| b^{1/2} f\|_{L^2}^2 \lesssim \varepsilon \| a^{1/2}(-\tilde{\Delta}_v)^{s/2} f\|_{L^2}^2 + \varepsilon \| b^{1/2} f\|_{L^2}^2 + \left| \left( \tilde{P} f, f \right)_{L^2} \right| + C_\varepsilon \| f\|_{L^2}^2.
\]
Taking \( \varepsilon \) sufficiently small gives the desired estimate (11), completing the proof of Lemma 2.1□

**Corollary 2.2.** Let \( \tilde{P} \) be given in (6) with \( a, b \) satisfying the assumptions (2) and (3), and let \( p \in S(1, |dv|^2 + |d\eta|^2) \). Then
\[
\forall f \in S(\mathbb{R}^n_v), \quad \left| \left( a(v)(-\tilde{\Delta}_v)^{s} f + b f, p^{w} f \right)_{L^2} \right| \lesssim \left| \left( \tilde{P} f, f \right)_{L^2} \right| + \| f\|_{L^2}^2.
\] (13)

**Proof.** In view of (2), it is clear that
\[
\left| \left( b f, p^{w} f \right)_{L^2} \right| \lesssim \| \langle v \rangle^{s+\gamma/2} p^{w} f\|_{L^2} \| \langle v \rangle^{s+\gamma/2} f\|_{L^2} \lesssim \| \langle v \rangle^{s+\gamma/2} f\|_{L^2}^2,
\]
where the last inequality holds because
\[
\| \langle v \rangle^{s+\gamma/2} p^{w} f\|_{L^2} \lesssim \| p^{w} \langle v \rangle^{s+\gamma/2} f\|_{L^2} \lesssim \| p^{w} \langle v \rangle^{s+\gamma/2} f\|_{L^2} + \| \langle v \rangle^{s+\gamma/2} f\|_{L^2} \lesssim \| \langle v \rangle^{s+\gamma/2} f\|_{L^2},
\]
since \( p \in S(1, |dv|^2 + |d\eta|^2) \). By virtue of (11), the desired estimate (13) will follow if we could show that
\[
\left| \left( a(v)(-\tilde{\Delta}_v)^{s} f, p^{w} f \right)_{L^2} \right| \lesssim \| a^{1/2}(-\tilde{\Delta}_v)^{s/2} f\|_{L^2}^2 + \| \langle \Delta_{\eta} \rangle^{s} \langle v \rangle^{s/2} f\|_{L^2}^2 + \| \langle v \rangle^{s+\gamma/2} f\|_{L^2}^2.
\] (14)

Observing that the term \( \left| \left( a(v)(-\tilde{\Delta}_v)^{s} f, p^{w} f \right)_{L^2} \right| \) on the left hand side is bounded from above by
\[
\left| \left( (-\tilde{\Delta}_v)^{s/2} a(v)(-\tilde{\Delta}_v)^{s/2} f, p^{w} f \right)_{L^2} \right| + \left| \left( \left[ (-\tilde{\Delta}_v)^{s/2}, a(v) \right] (-\tilde{\Delta}_v)^{s/2} f, p^{w} f \right)_{L^2} \right|,
\]
and that
\[ \left| \left( \left[ (-\Delta_v)^{\gamma/2}, a(v) \right] (-\Delta_v)^{\gamma/2} f, p^w f \right) \right|_{L^2} \leq \left\| \langle v \rangle^{\gamma/2} (-\Delta_v)^{\gamma/2} f \right\|_{L^2} \left\| \langle v \rangle^{\gamma/2} f \right\|_{L^2} \]
due to (9) and the fact that \( p \in S(1, |dv|^2 + |d\eta|^2) \), we have
\[ \left| \left( a(v)(-\Delta_v)^{\gamma/2}, p^w f \right) \right|_{L^2} \leq \left\| a^{\frac{2}{\gamma}} (-\Delta_v)^{\gamma/2} p^w f \right\|_{L^2} + \left\| a^{\frac{2}{\gamma}} (-\Delta_v)^{\gamma/2} f \right\|_{L^2} \]

As for the second term on the right hand side, by virtue of (3) symbolic calculus (Theorem 2.3.8 and Corollary 2.3.10 of [26]) shows that
\[ a^{\frac{2}{\gamma}} (\tilde{\Delta}_v)^{\gamma/2}, p^w \] \( \in \text{Op} \left( S \left( \langle v \rangle^{\gamma/2} + \langle \tilde{\eta} \rangle^{\gamma/2}, |dv|^2 + |d\tilde{\eta}|^2 \right) \right), \]
and thus
\[ \left\| a^{\frac{2}{\gamma}} (\tilde{\Delta}_v)^{\gamma/2} p^w f \right\|_{L^2} \leq \left\| p^w a^{\frac{2}{\gamma}} (-\Delta_v)^{\gamma/2} f \right\|_{L^2} + \left\| a^{\frac{2}{\gamma}} (-\Delta_v)^{\gamma/2} f \right\|_{L^2} \]
\[ \leq \left\| a^{\frac{2}{\gamma}} (-\Delta_v)^{\gamma/2} f \right\|_{L^2} + \left\| D_\eta \langle v \rangle^{\gamma/2} f \right\|_{L^2} + \left\| \langle v \rangle^{\gamma/2} f \right\|_{L^2}. \]
Combining the above inequalities, we get (14), completing the proof.

2.2. Estimates of the commutators with pseudo-differential operators

The main result of this subsection is the following estimate on the commutator of \( \tilde{\mathcal{P}} \) with \( M_\epsilon^s \) which is defined by, with \( \epsilon > 0 \) and \( \xi \in \mathbb{R}^n \) arbitrary and fixed,
\[ M_\epsilon^s = (\varphi_\epsilon(v, \eta) \langle \eta \rangle^s), \]
with
\[ \varphi_\epsilon(v, \eta) \overset{\text{def}}{=} \chi \left( \frac{\langle \xi \rangle}{\epsilon \langle \eta \rangle^{\frac{s}{1+2s}}} \right), \]
where \( \chi \in C_0^\infty(\mathbb{R}; [0, 1]) \) such that \( \chi = 1 \) in \([-1, 1] \) and \( \text{supp} \chi \subset [-2, 2] \).

**Lemma 2.3.** Let \( \tilde{\mathcal{P}} \) be given in (4) with \( a, b \) satisfying the assumptions (2) and (3), and let \( M_\epsilon^s \) be defined in (15). Then for all \( f \in S(\mathbb{R}^n) \) we have
\[ \left| \left[ \left[ \tilde{\mathcal{P}}, M_\epsilon^s \right], \langle v \rangle^{\gamma} M_\epsilon^s f \right] \right|_{L^2} \leq \epsilon \left\| \langle v \rangle^{\gamma} D_\eta^{2s} f \right\|_{L^2}^2 + C_\epsilon \left( \left\| D_\eta \langle v \rangle^{s+\gamma} f \right\|_{L^2}^2 + \left\| f \right\|_{L^2}^2 \right). \]

In order to prove the above results we need some lemmas.

**Lemma 2.4.** Let \( \varphi_\epsilon \) and \( M_\epsilon^s \) be given in (16) and (15). Then \( \varphi_\epsilon \in S \left( 1, |dv|^2 + |d\eta|^2 \right) \) and \( M_\epsilon^s \in \text{Op} \left( S \left( \langle \eta \rangle^s, |dv|^2 + |d\eta|^2 \right) \right) \), uniformly with respect to \( \epsilon \) and \( \xi \). Moreover for any \( \alpha, \beta \in \mathbb{Z}_+^n \) there exists a constant \( C_{\alpha, \beta} \), depending only on \( \alpha \) and \( \beta \), such that
\[ \left| \partial_\alpha v \partial_\beta \eta \left( \varphi_\epsilon(v, \eta) \langle \eta \rangle^s \right) \right| \leq C_{\alpha, \beta} \langle v \rangle^{-|\alpha|} \langle \eta \rangle^{-|\beta|}, \]
and
\[ \left| \partial_\alpha v \partial_\beta \eta \left( \xi \cdot \partial_\eta \varphi_\epsilon \right) \right| \leq \epsilon C_{\alpha, \beta} \langle v \rangle^{\gamma} \langle \eta \rangle^{2s}. \]
Proof. It is just a straightforward verification, since
\[ \langle \xi \rangle \leq \varepsilon \langle v \rangle \langle \eta \rangle^{1+2s} \]
on the support of \( \varphi_\varepsilon \). The proof is completed. \( \square \)

**Lemma 2.5.** Let \( M_\varepsilon^s \) be given in (15). Then for all \( f \in S(\mathbb{R}^n) \) we have
\[ \left| \left( [i (\tau + v \cdot \xi), M_\varepsilon^s] f, \langle v \rangle^\gamma M_\varepsilon^s f \right) \right|_{L^2} \lesssim \varepsilon \| \langle v \rangle^\gamma \langle D_\eta \rangle^{2s} f \|_{L^2}^2. \] (20)

**Proof.** Observe
\[ [i (\tau + v \cdot \xi), M_\varepsilon^s] = \frac{1}{2\pi} \left\{ \tau + v \cdot \xi, \varphi_\varepsilon(v, \eta) \langle \eta \rangle^s \right\}^w = -\frac{1}{2\pi} \left( \xi \cdot \partial_\eta (\varphi_\varepsilon \langle \eta \rangle^s) \right)^w, \]
where \( \{, \} \) stands for the Poisson bracket defined by
\[ \{ p, q \} = \frac{\partial p}{\partial \eta} \cdot \frac{\partial q}{\partial v} - \frac{\partial p}{\partial v} \cdot \frac{\partial q}{\partial \eta}. \] (21)

Thus
\[ \left( [i (\tau + v \cdot \xi), M_\varepsilon^s] f, \langle v \rangle^\gamma M_\varepsilon^s f \right)_{L^2} = -\frac{1}{2\pi} \left( M_\varepsilon^s \langle v \rangle^\gamma \left( \xi \cdot \partial_\eta (\varphi_\varepsilon \langle \eta \rangle^s) \right)^w f, f \right)_{L^2}. \]

Moreover, in view of (19) and (7) we have
\[ M_\varepsilon^s \langle \eta \rangle^w \left( \xi \cdot \partial_\eta (\varphi_\varepsilon \langle \eta \rangle^s) \right)^w \in \text{Op} \left( S \left( \varepsilon \langle v \rangle^{2\gamma} \langle \eta \rangle^4, |dv|^2 + |d\eta|^2 \right) \right) \]
uniformly with respect to \( \varepsilon \) and \( \xi \). This implies
\[ \left| \left( [i (\tau + v \cdot \xi), M_\varepsilon^s] f, \langle v \rangle^\gamma M_\varepsilon^s f \right) \right|_{L^2} \lesssim \varepsilon \| \langle v \rangle^\gamma \langle D_\eta \rangle^{2s} f \|_{L^2}^2, \]
completing the proof of Lemma 2.5. \( \square \)

The rest of this subsection is occupied by

**Proof of Lemma 2.3** Write
\[ [\tilde{\Phi}, M_\varepsilon^s] = [i (t + v \cdot \xi), M_\varepsilon^s] + a(v) \left[ (-\tilde{\Delta}_\nu)^s, M_\varepsilon^s \right] + \left[ a, M_\varepsilon^s \right] (-\tilde{\Delta}_\nu)^s + \left[ b, M_\varepsilon^s \right]. \]

Then by (20) we have
\[ \left| \left( [\tilde{\Phi}, M_\varepsilon^s] f, \langle v \rangle^\gamma M_\varepsilon^s f \right) \right|_{L^2} \lesssim \varepsilon \| \langle v \rangle^\gamma \langle D_\eta \rangle^{2s} f \|_{L^2}^2 + \sum_{j=1}^3 A_j + A_2 + A_3, \] (22)
with
\[ A_1 = \left| \left( a(v) \left[ (-\tilde{\Delta}_\nu)^s, M_\varepsilon^s \right] f, \langle v \rangle^\gamma M_\varepsilon^s f \right) \right|_{L^2} ; \]
\[ A_2 = \left| \left[ a, M_\varepsilon^s \right] (-\tilde{\Delta}_\nu)^s f, \langle v \rangle^\gamma M_\varepsilon^s f \right) \right|_{L^2} ; \]
\[ A_3 = \left| \left[ b, M_\varepsilon^s \right] f, \langle v \rangle^\gamma M_\varepsilon^s f \right) \right|_{L^2} . \]

In view of (18) we see
\[ \left[ (-\tilde{\Delta}_\nu)^s, M_\varepsilon^s \right] \in \text{Op} \left( S \left( \langle v \rangle^{-1} \langle \eta \rangle^{3s-1}, |dv|^2 + |d\eta|^2 \right) \right) , \]
and thus
\[ a(v) [(-\Delta_v)^s, M_x^s] \in \text{Op} \left( S \left( \langle v \rangle^s \langle \eta \rangle^{2s}, |dv|^2 + |d\eta|^2 \right) \right) \]
due to (2) and the fact that \( s < 1 \). This implies
\[ A_1 \lesssim \| \langle v \rangle^\gamma \langle D_\eta \rangle^{2s} f \|_{L^2} \| \langle D_\eta \rangle^s \langle v \rangle^{s+\gamma} f \|_{L^2} \lesssim \| \langle v \rangle^\gamma \langle D_\eta \rangle^{2s} f \|_{L^2}^2 + C_\varepsilon \| \langle D_\eta \rangle^s \langle v \rangle^{s+\gamma} f \|_{L^2}^2. \]
Similarly, by (3) and (18), we conclude that \([b, M_x^s] \in \text{Op} \left( S \left( \langle v \rangle^{s+\gamma}, |dv|^2 + |d\eta|^2 \right) \right)\)
and
\[ [a, M_x^s] (-\Delta_v)^s \in \text{Op} \left( S \left( \langle v \rangle^{s+\gamma}, |dv|^2 + |d\eta|^2 \right) \right), \]
which implies
\[ A_2 \lesssim \| \langle v \rangle^\gamma \langle D_\eta \rangle^{2s} \|_{L^2} \| \langle D_\eta \rangle^s \langle v \rangle^{s+\gamma} f \|_{L^2} \lesssim \| \langle v \rangle^\gamma \langle D_\eta \rangle^{2s} f \|_{L^2}^2 + C_\varepsilon \| \langle D_\eta \rangle^s \langle v \rangle^{s+\gamma} f \|_{L^2}^2, \]
and
\[ A_3 \lesssim \| \langle D_\eta \rangle^s \langle v \rangle^{s+\gamma} f \|_{L^2} \| \langle v \rangle^\gamma f \|_{L^2} \lesssim \| \langle D_\eta \rangle^s \langle v \rangle^{s+\gamma} f \|_{L^2}^2. \]
These inequalities together with (22) give the desired estimate (17), completing the proof of Lemma 2.3.

3. Proof of the main results

In this section we will proceed to prove Theorem 1.1 by four steps. The first three subsections are devoted to proving the following proposition concerning the hypoellipticity of the operator with parameters, while in the last one we present the proof of Theorem 1.1. Since our main analysis is still on \( \mathbb{R}^n_\gamma \), we will use the same notation as in the previous section; that is, \((\cdot, \cdot)_{L^2}\) and \(\| \cdot \|_{L^2}\) stand for \((\cdot, \cdot)_{L^2(\mathbb{R}^n_\gamma)}\) and \(\| \cdot \|_{L^2(\mathbb{R}^n_\gamma)}\), respectively.

**Proposition 3.1.** Let \( \tilde{\mathcal{P}} \) be given in (8) with \( a, b \) satisfying the assumptions (2) and (3). Then for all \( f \in \mathcal{S}(\mathbb{R}^n_\gamma) \) we have
\[
\| (v)^{\frac{s-\gamma}{2+2\gamma}} (\tau)^{\frac{2s}{2+2\gamma}} f \|_{L^2} + \| (v)^{\frac{s-\gamma}{2+2\gamma}} (\xi)^{\frac{2s}{2+2\gamma}} f \|_{L^2} + \| (v)^{\gamma} (D_\eta)^{2s} f \|_{L^2} + \| (v)^{2s+\gamma} f \|_{L^2} \\
\lesssim \| \tilde{\mathcal{P}} f \|_{L^2} + \| f \|_{L^2}.
\]
Recall here \( \| \cdot \|_{L^2(\mathbb{R}^n_\gamma)} \) stands for \( \| \cdot \|_{L^2(\mathbb{R}^n_\gamma)} \).

3.1. The first part of the proof of Proposition 3.1

In this subsection we prove the weighted estimate; that is

**Lemma 3.2.** Let \( \tilde{\mathcal{P}} \) be given in (8) with \( a, b \) satisfying the assumptions (2) and (3). Then for all \( f \in \mathcal{S}(\mathbb{R}^n_\gamma) \) we have
\[
\| (D_\eta)^s \langle v \rangle^{s+\gamma} f \|_{L^2} + \| \langle v \rangle^{2s+\gamma} f \|_{L^2} \lesssim \| \tilde{\mathcal{P}} f \|_{L^2} + \| f \|_{L^2}. \] (23)
Proof. Let \( f \in S(\mathbb{R}^n) \). Using (11) to the function \( \langle v \rangle^{s+\frac{2}{p}} f \), we have
\[
\| (D_n)^s \langle v \rangle^{s+\gamma} f \|_{L^2}^2 + \| \langle v \rangle^{2s+\gamma} f \|_{L^2}^2 
\lesssim \left| \left( \tilde{P} \langle v \rangle^{s+\frac{2}{p}} f, \langle v \rangle^{s+\frac{2}{p}} f \right)_{L^2} \right| + \| \langle v \rangle^{s+\frac{2}{p}} f \|_{L^2}^2
\lesssim \left| \left( \tilde{P} f, \langle v \rangle^{2s+\gamma} f \right)_{L^2} \right| + \left| \left( [\tilde{P}, \langle v \rangle^{s+\frac{2}{p}}] f, \langle v \rangle^{s+\frac{2}{p}} f \right)_{L^2} \right| + \| \langle v \rangle^{2s+\gamma} f \|_{L^2} \| f \|_{L^2},
\]
which imply that
\[
\| (D_n)^s \langle v \rangle^{s+\gamma} f \|_{L^2}^2 + \| \langle v \rangle^{2s+\gamma} f \|_{L^2}^2 \lesssim \| \tilde{P} f \|_{L^2}^2 + \| f \|_{L^2}^2 + \left| \left( [\tilde{P}, \langle v \rangle^{s+\frac{2}{p}}] f, \langle v \rangle^{s+\frac{2}{p}} f \right)_{L^2} \right|. \tag{24}
\]
Moreover, note that \( [\tilde{P}, \langle v \rangle^{s+\frac{2}{p}}] = a(v) \left( -\tilde{\Delta}_v \right) + \langle v \rangle^{s+\frac{2}{p}} \), and thus by (7) and (8) we have
\[
[\tilde{P}, \langle v \rangle^{s+\frac{2}{p}}] \in \mathcal{Op} \left( S \left( \langle v \rangle^{3s+\frac{2}{p}-1} \langle \eta \rangle^s, |dv|^2 + |d\eta|^2 \right) \right).
\]
This implies, with \( \varepsilon \) sufficiently small,
\[
\left| \left( [\tilde{P}, \langle v \rangle^{s+\frac{2}{p}}] f, \langle v \rangle^{s+\frac{2}{p}} f \right)_{L^2} \right| \lesssim \| (D_n)^s \langle v \rangle^{s+\gamma} f \|_{L^2} \| \langle v \rangle^{3s+\gamma-1} f \|_{L^2}
\lesssim \varepsilon \| (D_n)^s \langle v \rangle^{s+\gamma} f \|_{L^2}^2 + \varepsilon \| \langle v \rangle^{2s+\gamma} f \|_{L^2}^2 + C_\varepsilon \| f \|_{L^2}^2,
\]
where in the last inequality we used the interpolation inequality
\[
\| \langle v \rangle^{3s+\gamma-1} f \|_{L^2} \leq \varepsilon \| \langle v \rangle^{2s+\gamma} f \|_{L^2} + C_\varepsilon \| f \|_{L^2},
\]
due to \( s < 1 \). Combining (24) we get
\[
\| (D_n)^s \langle v \rangle^{s+\gamma} f \|_{L^2}^2 + \| \langle v \rangle^{2s+\gamma} f \|_{L^2}^2 \lesssim \varepsilon \| (D_n)^s \langle v \rangle^{s+\gamma} f \|_{L^2}^2 + \varepsilon \| \langle v \rangle^{2s+\gamma} f \|_{L^2}^2 + C_\varepsilon \| f \|_{L^2}^2 + \| \tilde{P} f \|_{L^2}^2.
\]
Letting \( \varepsilon \) small enough gives the desired estimate (23). The proof is complete. \( \square \)

3.2. The second part of the proof of Proposition 3.1

The main result in this subsection is the following lemma.

Lemma 3.3. Let \( \tilde{P} \) be given in (6) with \( a, b \) satisfying the assumptions (2) and (3). Then for all \( f \in S(\mathbb{R}^n) \) we have
\[
\| \langle v \rangle^{\frac{s}{s+p}} \langle \xi \rangle^{\frac{p}{s+p}} f \|_{L^2} \lesssim \| \tilde{P} f \|_{L^2} + \| f \|_{L^2}. \tag{25}
\]

We would make use of the multiplier method used in [21, 30] to prove the above result. Firstly we need to find a suitable multiplier. In what follows let \( \xi \in \mathbb{R}^n \) be fixed, and define a symbol \( p \) by setting
\[
p = p_\xi (v, \eta) = \frac{\langle v \rangle^{\gamma/(1+2s)} \xi \cdot \eta}{\langle \xi \rangle^{2-\frac{s}{s+p}}}, \tag{26}
\]
with \( \psi \) given by
\[
\psi (v, \eta) = \chi \left( \frac{\langle v \rangle^{\gamma} \langle \eta \rangle^{1+2s}}{\langle \xi \rangle} \right), \tag{27}
\]
where \( \chi \in C_0^\infty(\mathbb{R}; [0, 1]) \) such that \( \chi = 1 \) in \([-1, 1]\) and \( \text{supp} \chi \subset [-2, 2] \).

**Lemma 3.4.** Let \( p, \psi \) be given above. Then one has \( p, \psi \in S(1, |dv|^2 + |d\eta|^2) \) uniformly with respect to \( \xi \).

**Proof.** It is just a straightforward verification. \( \square \)

**Lemma 3.5.** Let \( \psi \) be given in (27). Then for all \( |\alpha| + |\beta| \geq 0 \) the following inequality
\[
|\partial_v^\alpha \partial_\eta^\beta (\xi \cdot \partial_\eta \psi)| \lesssim \langle v \rangle^\gamma \langle \eta \rangle^{2s}
\] (28)
holds uniformly with respect to \( \xi \).

**Proof.** Note that
\[
\xi \cdot \partial_\eta \psi = \frac{(2s + 1) \langle v \rangle^\gamma \langle \eta \rangle^{2s-1} \xi \cdot \eta}{\langle \xi \rangle^2} \chi \left( \frac{\langle v \rangle^\gamma \langle \eta \rangle^{2s+1}}{\langle \xi \rangle^2} \right)
\]
Then by direct computation, (28) follows. The proof of Lemma 3.5 is thus complete. \( \square \)

The rest of this subsection is occupied by

**Proof of Lemma 3.3** Let \( f \in S(\mathbb{R}_n^\ast) \) and let \( p^w \) be the Weyl quantization of the symbol \( p \) given in (26). Then using (13) gives
\[
\left| \left( a(v)(-\tilde{\Delta}_v)^s f + b f, p^w f \right)_{L^2} \right| \lesssim \left| \left( \bar{p} f, f \right)_{L^2} \right| + \| f \|_{L^2}^2.
\]
This together with the relation
\[
\text{Re} \left( i (\tau + v \cdot \xi) f, p^w f \right)_{L^2} = \text{Re} \left( \bar{p} f, p^w f \right)_{L^2} - \text{Re} \left( a(v)(-\tilde{\Delta}_v)^s f + b f, p^w f \right)_{L^2}
\]
yields
\[
\text{Re} \left( i (\tau + v \cdot \xi) f, p^w f \right)_{L^2} \lesssim \left| \left( \bar{p} f, f \right)_{L^2} \right| + \left| \left( \bar{p} f, p^w f \right)_{L^2} \right| + \| f \|_{L^2}^2.
\] (29)
Next we will give a lower bound of the term on the left side. Observe that
\[
\text{Re} \left( i (\tau + v \cdot \xi) f, p^w f \right)_{L^2} = \frac{1}{2\pi} \left( \{ p, \tau + v \cdot \xi \}^w f, f \right)_{L^2},
\] (30)
where \( \{ \cdot, \cdot \} \) is the Poisson bracket defined in (21). Direct calculus shows
\[
\{ p, \tau + v \cdot \xi \} = \frac{\langle v \rangle^{\gamma/(1+2s)} \langle \xi \rangle^2}{\langle \xi \rangle^{2-\frac{1}{1+2s}}} \psi + \frac{\langle v \rangle^{\gamma/(1+2s)} \xi \cdot \eta}{\langle \xi \rangle^{2-\frac{1}{1+2s}}} \xi \cdot \partial_\eta \psi
\]
\[
= \frac{\langle v \rangle^{\gamma/(1+2s)} \langle \xi \rangle^{2s/(1+2s)}}{\langle \xi \rangle^{2-\frac{1}{1+2s}}} \psi - \frac{\langle v \rangle^{\gamma/(1+2s)} \langle \xi \rangle^{2s/(1+2s)}}{\langle \xi \rangle^{2-\frac{1}{1+2s}}} \psi + \frac{\langle v \rangle^{\gamma/(1+2s)} \xi \cdot \eta}{\langle \xi \rangle^{2-\frac{1}{1+2s}}} \xi \cdot \partial_\eta \psi
\]
\[
= \frac{\langle v \rangle^{\gamma/(1+2s)} \langle \xi \rangle^{2s/(1+2s)}}{\langle \xi \rangle^{2-\frac{1}{1+2s}}} \psi - \frac{\langle v \rangle^{\gamma/(1+2s)} \langle \xi \rangle^{2s/(1+2s)}}{\langle \xi \rangle^{2-\frac{1}{1+2s}}} (1 - \psi) \xi \cdot \partial_\eta \psi.
\]
The above equalities along with (29) and (30) yield

$$\| \langle v \rangle^{\frac{\gamma}{1+2s}} \langle \xi \rangle^{\frac{\gamma}{1+2s}} f \|_{L^2}^2 \lesssim \sum_{j=1}^{3} K_j + \left| \left( \bar{\partial} f, f \right)_{L^2} \right| + \left| \left( \bar{\partial} f, f^{\alpha} \right)_{L^2} \right| + \| f \|_{L^2}^2,$$

(31)

with

$$K_1 = \left( \left( \langle v \rangle^{\gamma/(1+2s)} \langle \xi \rangle^{2s/(1+2s)} (1 - \psi) \right)^w f, f \right)_{L^2},$$

$$K_2 = \left( \left( \langle v \rangle^{\gamma/(1+2s)} \langle \xi \rangle^{2s/(1+2s)} \psi \right)^w f, f \right)_{L^2},$$

$$K_3 = - \left( \left( \langle v \rangle^{\gamma/(1+2s)} \langle \xi \rangle^{2s/(1+2s)} \xi \cdot \partial_\eta \psi \right)^w f, f \right)_{L^2}.$$

Note that

$$\langle \xi \rangle^{2s/(1+2s)} \leq \langle \eta \rangle^{2s/(1+2s)}\langle \eta \rangle^{2s}$$

on the support of $\partial_\xi \partial_\eta (1 - \psi)$ with $|\alpha| + |\beta| \geq 0$. Then by virtue of the conclusion that $\psi \in S(1, |dv|^2 + |d\eta|^2)$ uniformly with respect to $\xi$ in Lemma 3.4, we have

$$\langle v \rangle^{\gamma/(1+2s)} \langle \xi \rangle^{2s/(1+2s)} (1 - \psi) \in S \left( \langle v \rangle^{\gamma} \langle \eta \rangle^{2s}, |dv|^2 + |d\eta|^2 \right)$$

uniformly with respect to $\xi$. This implies

$$\langle D_\eta \rangle^{-s} \langle v \rangle^{-\gamma/2} \left( \langle v \rangle^{\gamma/(1+2s)} \langle \xi \rangle^{2s/(1+2s)} (1 - \psi) \right)^w \langle v \rangle^{-\gamma/2} \langle D_\eta \rangle^{-s} \in \text{Op} \left( S \left( 1, |dv|^2 + |d\eta|^2 \right) \right),$$

and thus

$$K_1 \lesssim \| \langle D_\eta \rangle^{-s} \langle v \rangle^{\gamma/2} f \|_{L^2}^2 \lesssim \left| \left( \bar{\partial} f, f \right)_{L^2} \right| + \| f \|_{L^2}^2,$$

(32)

where the last inequality follows from (11). Furthermore since

$$\frac{\langle v \rangle^{\gamma/(1+2s)}}{\langle \xi \rangle^{2s/(1+2s)}} \leq \frac{\langle \xi \rangle^{1/(1+2s)}}{\langle \xi \rangle^{2s/(1+2s)}} \leq 1$$

on the support of $\psi$, then combining the fact that that $\psi \in S(1, |dv|^2 + |d\eta|^2)$ uniformly with respect to $\xi$ we conclude

$$\frac{\langle v \rangle^{\gamma/(1+2s)}}{\langle \xi \rangle^{2s/(1+2s)}} \psi \in S \left( 1, |dv|^2 + |d\eta|^2 \right),$$

which implies

$$K_2 \lesssim \| f \|_{L^2}^2.$$

(33)

It remains to treat $K_3$. Direct verification shows

$$\left| \partial_\xi \partial_\eta \left( \frac{\langle v \rangle^{\gamma/(1+2s)}}{\langle \xi \rangle^{2s/(1+2s)}} \xi \cdot \eta \right) \right| \leq 1$$

on the support of $\psi$. This along with (28) gives

$$\frac{\langle v \rangle^{\gamma/(1+2s)}}{\langle \xi \rangle^{2s/(1+2s)}} \xi \cdot \partial_\eta \psi \in S \left( \langle v \rangle^{\gamma} \langle \eta \rangle^{2s}, |dv|^2 + |d\eta|^2 \right).$$
As a result, repeating the arguments used in the treatment of $K_1$ yields

$$K_3 \lesssim \| (D_\eta)^\gamma <v>^{\gamma/2} f \|^4_{L^2} \lesssim \left( \| \tilde{\Phi} f, f \|^2_{L^2} + \| f \|^2_{L^2} \right).$$

This, together with (31), (32) and (33), gives

$$\| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} \langle \xi \rangle^{\frac{\gamma}{2} + \epsilon} f \|^2_{L^2} \lesssim \left( \| \tilde{\Phi} f, f \|^2_{L^2} + \| \tilde{\Phi} f, p^w <v>^\gamma f \|^2_{L^2} + \| f \|^2_{L^2} \right).$$

Now applying the above inequality to the function $\langle v \rangle^{\frac{\gamma}{2} + \epsilon} f$, we get

$$\| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} \langle \xi \rangle^{\frac{\gamma}{2} + \epsilon} f \|^2_{L^2} \lesssim \left( \| \tilde{\Phi} f, f \|^2_{L^2} + \| \tilde{\Phi} f, p^w <v>^\gamma f \|^2_{L^2} \right) + \| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} f \|_{L^2} \| f \|_{L^2}$$

$$\lesssim \| \tilde{\Phi} f \|_{L^2} \| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} f \|_{L^2} + \| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} f \|_{L^2} \| f \|_{L^2}$$

$$\lesssim \left( \| \tilde{\Phi} f, f \|^2_{L^2} + \| \tilde{\Phi} f, p^w <v>^\gamma f \|^2_{L^2} \right) + \| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} f \|_{L^2} \| f \|_{L^2}.$$

On the other hand, by (7) and (8) we have,

$$[\tilde{\Phi}, \langle v \rangle^{\frac{\gamma}{2} + \epsilon}] = a(v)\left( -\tilde{\Delta} v, \langle v \rangle^{\frac{\gamma}{2} + \epsilon} \right) \in \operatorname{Op} \left( S \left( <v>^{s+\gamma+\frac{\gamma}{2} + \epsilon} \langle \eta \rangle^s, |dv|^2 + |d\eta|^2 \right) \right).$$

Then symbolic calculus gives

$$\| \left( \tilde{\Phi}, \langle v \rangle^{\frac{\gamma}{2} + \epsilon} f, \langle v \rangle^{\frac{\gamma}{2} + \epsilon} f \right)_{L^2} \| \left( \tilde{\Phi}, \langle v \rangle^{\frac{\gamma}{2} + \epsilon} f, p^w <v>^\gamma f \right)_{L^2} \|$$

$$\lesssim \| (D_\eta)^\gamma <v>^{s+\gamma} f \|_{L^2} \| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} f \|_{L^2},$$

since $p \in S \left( 1, \|dv|^2 + |d\eta|^2 \right)$ uniformly with respect to $\xi$. Consequently combining the above inequalities, we have

$$\| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} \langle \xi \rangle^{\frac{\gamma}{2} + \epsilon} f \|^2_{L^2} \lesssim \| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} f \|^2_{L^2} \left( \| \tilde{\Phi} f \|_{L^2} + \| f \|_{L^2} + \| (D_\eta)^\gamma <v>^{s+\gamma} f \|_{L^2} \right).$$

Note that $\| \cdot \|_{L^2}$ stands for the norm in $L^2(\mathbb{R}^n)$. Then multiplying both sides the factor $\langle \xi \rangle^{2s/(1+2s)}$, we get

$$\| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} \langle \xi \rangle^{\frac{\gamma}{2} + \epsilon} f \|^2_{L^2} \lesssim \| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} \langle \xi \rangle^{\frac{\gamma}{2} + \epsilon} f \|^2_{L^2} \left( \| \tilde{\Phi} f \|_{L^2} + \| f \|_{L^2} + \| (D_\eta)^\gamma <v>^{s+\gamma} f \|_{L^2} \right),$$

and thus

$$\| \langle v \rangle^{\frac{\gamma}{2} + \epsilon} \langle \xi \rangle^{\frac{\gamma}{2} + \epsilon} f \|_{L^2} \lesssim \| \tilde{\Phi} f \|_{L^2} + \| f \|_{L^2} + \| (D_\eta)^\gamma <v>^{s+\gamma} f \|_{L^2} \lesssim \| \tilde{\Phi} f \|_{L^2} + \| f \|_{L^2},$$

where the last inequality follows from (23). This gives the desired estimate (25), completing the proof of Lemma 3.3.

\[ \square \]

3.3. End of the proof of Proposition 3.1

In view of (23) and (25), the proof of Proposition 3.1 will be complete if we could show the following lemma.
Lemma 3.6. Let $\tilde{\mathcal{P}}$ be given in (6) with $a, b$ satisfying the assumptions (2) and (3). Then for all $f \in \mathcal{S}(\mathbb{R}^n)$ we have
\begin{equation}
\| \langle \phi \rangle^{\frac{3}{2}} \left( \tau \right)^{\frac{3}{2}} f \|_{L^2} + \| \langle \phi \rangle \gamma \langle D_\phi \rangle^{2s} f \|_{L^2} \lesssim \| \tilde{\mathcal{P}} f \|_{L^2} + \| f \|_{L^2}.
\end{equation}

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$. We first treat the second term on the left hand side of (34). By (8) one has
\begin{align*}
\| \langle \phi \rangle \gamma \langle D_\phi \rangle^{2s} f \|_{L^2}^2 &\lesssim \| \langle D_\phi \rangle^{s} \langle \phi \rangle \gamma \langle D_\phi \rangle^{s} f \|_{L^2}^2 + \| \langle \langle D_\phi \rangle^{s} \langle \phi \rangle \gamma \rangle \langle D_\phi \rangle^{s} f \|_{L^2}^2 \\
&\lesssim \| \langle D_\phi \rangle^{s} \langle \phi \rangle \gamma \langle \langle \eta \rangle^{s} \rangle^{w} f \|_{L^2}^2 + \| \langle \phi \rangle \gamma \langle D_\phi \rangle^{s} f \|_{L^2}^2,
\end{align*}
Moreover for the last term in the above inequality we have
\begin{align*}
\| \langle \phi \rangle \gamma \langle D_\phi \rangle^{s} f \|_{L^2}^2 &\lesssim \| \langle \phi \rangle \gamma \langle D_\phi \rangle^{s} f \|_{L^2}^2 + \| \langle \langle D_\phi \rangle^{s} \langle \phi \rangle \gamma \rangle \langle D_\phi \rangle^{s} f \|_{L^2}^2 \\
&\lesssim \| \tilde{\mathcal{P}} f \|_{L^2}^2 + \| f \|_{L^2}^2,
\end{align*}
the last inequality using (23). As a result the desired upper bound for $\| \langle \phi \rangle \gamma \langle D_\phi \rangle^{2s} f \|_{L^2}$ will follow if we could prove that, with $\varepsilon > 0$ arbitrarily small,
\begin{equation}
\| \langle D_\phi \rangle^{s} \langle \phi \rangle \gamma \langle \langle \eta \rangle^{s} \rangle^{w} f \|_{L^2}^2 \lesssim \varepsilon \| \langle \phi \rangle \gamma \langle D_\phi \rangle^{2s} f \|_{L^2}^2 + \| f \|_{L^2}^2.
\end{equation}
In order to show the above inequality we write
\begin{equation}
\| \langle D_\phi \rangle^{s} \langle \phi \rangle \gamma \langle \langle \eta \rangle^{s} \rangle^{w} f \|_{L^2}^2 \lesssim J_1 + J_2,
\end{equation}
with
\begin{align*}
J_1 &= \| \langle D_\phi \rangle^{s} \langle \phi \rangle \gamma \langle \varphi \varphi \rangle^{s} \rangle^{w} f \|_{L^2}^2 = \| \langle D_\phi \rangle^{s} \langle \phi \rangle \gamma M_\xi f \|_{L^2}^2, \\
J_2 &= \| \langle D_\phi \rangle^{s} \langle \phi \rangle \gamma \langle 1 - \varphi \varphi \rangle^{s} \rangle^{w} f \|_{L^2}^2,
\end{align*}
where $M_\xi$ and $\varphi \varphi$ are defined in (15) and (16). Let’s first treat the term $J_2$. Writing
\begin{align*}
J_2 &= \left( \langle D_\phi \rangle^{2s} \langle \phi \rangle \gamma f, (1 - \varphi \varphi \rangle^{s} \rangle^{w} \langle \phi \rangle \gamma \langle 1 - \varphi \varphi \rangle^{s} \rangle^{w} f \right)_{L^2} \\
&+ \left( \langle D_\phi \rangle^{2s}, ((1 - \varphi \varphi \rangle^{s} \rangle^{w}) \langle \phi \rangle \gamma f, \langle \phi \rangle \gamma \langle 1 - \varphi \varphi \rangle^{s} \rangle^{w} f \right)_{L^2},
\end{align*}
we have by direct symbolic calculus
\begin{align*}
J_2 &\leq \| \langle \phi \rangle \gamma \langle D_\phi \rangle^{2s} f \|_{L^2} \| ((1 - \varphi \varphi \rangle^{s} \rangle^{w} \langle \phi \rangle \gamma \langle 1 - \varphi \varphi \rangle^{s} \rangle^{w} f \|_{L^2} \\
&+ \| \langle \phi \rangle \gamma \langle D_\phi \rangle^{2s} f \|_{L^2} \| \langle \phi \rangle \gamma \langle 1 - \varphi \varphi \rangle^{s} \rangle^{w} f \|_{L^2}.
\end{align*}
Moreover observe that the symbols of the operators
\begin{align*}
\langle 1 - \varphi \varphi \rangle^{s} \rangle^{w} \langle \phi \rangle \gamma \langle 1 - \varphi \varphi \rangle^{s} \rangle^{w} \chi \text{ and } \langle \phi \rangle \gamma \langle 1 - \varphi \varphi \rangle^{s} \rangle^{w} \chi
\end{align*}
belong to
\begin{align*}
S \left( e^{-\frac{1}{(\tau)\langle \eta \rangle^{w}}} (\tau)^{\frac{3}{2}}, (\xi)^{\frac{3}{2}}, |dv|^2 + |d\eta|^2 \right)
\end{align*}
uniformly with respect to $\varepsilon$ and $\xi$, due to the fact that
\begin{align*}
\langle \phi \rangle \gamma \langle \langle \eta \rangle^{s} \rangle^{w} \langle 1 + 2s \rangle \leq \varepsilon^{-1} \langle \xi \rangle
\end{align*}
on the support of \( \partial_{\xi}^\alpha \partial_{\eta}^\beta (1 - \varphi_\varepsilon) \) with \(|\alpha| + |\beta| \geq 0\). Then
\[
J_2 \leq \varepsilon^{-\frac{3}{2\beta_2}} \| \langle \eta \rangle \gamma \langle D_{\eta} \rangle^2 f \|_{L^2} \| \langle \xi \rangle \gamma \langle D_{\xi} \rangle^\gamma f \|_{L^2}^2
\leq \varepsilon \| \langle \eta \rangle \gamma \langle D_{\eta} \rangle^2 f \|_{L^2}^2 + C_\varepsilon \left( \| \tilde{\mathcal{P}} f \|_{L^2}^2 + \| f \|_{L^2}^2 \right),
\]
the last inequality using (23). Next we treat \( J_1 \). Applying (11) to the function \( \langle \eta \rangle^{\gamma/2} M_\varepsilon^s f \) gives
\[
J_1 \leq \left| \left( \tilde{\mathcal{P}}, \langle \eta \rangle^{\gamma/2} M_\varepsilon^s f, \langle \eta \rangle^{\gamma/2} M_\varepsilon^s f \right)_{L^2} \right| + \| \langle \eta \rangle^{\gamma/2} M_\varepsilon^s f \|_{L^2}^2
\leq J_{1,1} + J_{1,2} + J_{1,3},
\]
with
\[
J_{1,1} = \left| \left( \tilde{\mathcal{P}}, \langle \eta \rangle^{\gamma/2} M_\varepsilon^s f, \langle \eta \rangle^{\gamma/2} M_\varepsilon^s f \right)_{L^2} \right|,
J_{1,2} = \left| \left( \tilde{\mathcal{P}}, M_\varepsilon^s f, \langle \eta \rangle^{\gamma} M_\varepsilon^s f \right)_{L^2} \right|,
J_{1,3} = \| \tilde{\mathcal{P}} f \|_{L^2}^2 \| M_\varepsilon^s \langle \eta \rangle^{\gamma} M_\varepsilon^s f \|_{L^2}^2 + \| \langle \eta \rangle^{\gamma/2} M_\varepsilon^s f \|_{L^2}^2.
\]
Next we will proceed to handle the above three terms. It’s clear that
\[
J_{1,3} \leq \varepsilon \| \langle \eta \rangle^{\gamma} \langle D_{\eta} \rangle^2 f \|_{L^2}^2 + C_\varepsilon \left( \| \tilde{\mathcal{P}} f \|_{L^2}^2 + \| f \|_{L^2}^2 \right),
\]
since
\[
M_\varepsilon^s \langle \eta \rangle^{\gamma} M_\varepsilon^s \langle D_{\eta} \rangle^{-2\alpha} \langle \eta \rangle^{-\gamma} \in S \left( 1, |dv|^2 + |dn|^2 \right)
\]
uniformly with respect to \( \varepsilon \) and \( \xi \), by virtue of the conclusions in Lemma 2.4. Using (17) in Lemma 2.3 gives
\[
J_{1,2} \leq \varepsilon \| \langle \eta \rangle^{\gamma} \langle D_{\eta} \rangle^2 f \|_{L^2}^2 + C_\varepsilon \left( \| \langle D_{\eta} \rangle^s \langle \eta \rangle^{s+\gamma} f \|_{L^2}^2 + \| f \|_{L^2}^2 \right)
\]
\[
\leq \varepsilon \| \langle \eta \rangle^{\gamma} \langle D_{\eta} \rangle^2 f \|_{L^2}^2 + C_\varepsilon \left( \| \tilde{\mathcal{P}} f \|_{L^2}^2 + \| f \|_{L^2}^2 \right),
\]
the last inequality following from (23). Finally as for the term \( J_{1,1} \), by (7) and (8) we have
\[
[\tilde{\mathcal{P}}, \langle \eta \rangle^{\gamma/2}] = a(v)[(-\tilde{\Delta}_\varepsilon)^s, \langle \eta \rangle^{\gamma/2}] \in \text{Op} \left( S \left( \langle \eta \rangle^{s+3\gamma/2} \langle \eta \rangle^s, |dv|^2 + |dn|^2 \right) \right),
\]
due to \( 0 < s < 1 \). This implies
\[
J_{1,1} = \left| \left( \tilde{\mathcal{P}}, \langle \eta \rangle^{\gamma/2} M_\varepsilon^s f, \langle \eta \rangle^{\gamma/2} M_\varepsilon^s f \right)_{L^2} \right|
\leq \varepsilon \| \langle \eta \rangle^{\gamma} \langle D_{\eta} \rangle^2 f \|_{L^2}^2 + C_\varepsilon \left( \| \langle D_{\eta} \rangle^s \langle \eta \rangle^{s+\gamma} f \|_{L^2}^2 \right)
\leq \varepsilon \| \langle \eta \rangle^{\gamma} \langle D_{\eta} \rangle^2 f \|_{L^2}^2 + C_\varepsilon \left( \| \tilde{\mathcal{P}} f \|_{L^2}^2 + \| f \|_{L^2}^2 \right),
\]
the last inequality following from (23). This along with the estimates on the terms \( J_{1,2} \) and \( J_{1,3} \) gives
\[
J_1 \leq J_{1,1} + J_{1,2} + J_{1,3} \leq \varepsilon \| \langle \eta \rangle^{\gamma} \langle D_{\eta} \rangle^2 f \|_{L^2}^2 + C_\varepsilon \left( \| \tilde{\mathcal{P}} f \|_{L^2}^2 + \| f \|_{L^2}^2 \right).
\]
Then the desired estimate (35) follows from the combination of (36), (37) and the above inequality, giving the upper bound for the second term on the left hand side of (34), that is
\[
\| \langle \eta \rangle^{\gamma} \langle D_{\eta} \rangle^2 f \|_{L^2} \lesssim \| \tilde{\mathcal{P}} f \|_{L^2} + \| f \|_{L^2}.
\]
Now it remains to treat the first term. By computation, we have

\[ \langle v \rangle^{\frac{3+2s}{1+2s}} (\tau + v \cdot \xi) \leq \langle v \rangle^{\frac{3+2s}{1+2s}} (\tau \cdot v) + \langle v \rangle^{\frac{3+2s}{1+2s}} + \langle v \rangle^{\frac{3+2s}{1+2s}} \langle v \cdot \xi \rangle^{\frac{3+2s}{1+2s}} \]

where the last inequality follows from the Young’s inequality.

Proof.

By computation, we have

\[ \langle v \rangle^{\frac{3+2s}{1+2s}} \left( \langle v \rangle^{\frac{3+2s}{1+2s}} (\tau \cdot v) + \langle v \rangle^{\frac{3+2s}{1+2s}} \langle v \cdot \xi \rangle^{\frac{3+2s}{1+2s}} \right) \leq \frac{\langle v \rangle^{\frac{3+2s}{1+2s}}}{1+2s} + \frac{2s}{1+2s} \langle v \rangle^{\frac{3+2s}{1+2s}} (\tau \cdot v) \langle v \cdot \xi \rangle^{\frac{3+2s}{1+2s}} \]

As a result, using the relation \( i(\tau + v \cdot \xi) = \tilde{D}f - a(v)(-\Delta_v)^s f - b(v) f \), we compute

\[
\| \langle v \rangle^{\frac{3+2s}{1+2s}} (\tau) \|_{L^2} \leq \| \langle v \rangle^{-2} (\tau + v \cdot \xi) \|_{L^2} + \| \langle v \rangle^{2s+\gamma} f \|_{L^2} + \| \langle v \rangle^{-2} \tilde{D}f \|_{L^2} + \| \langle v \rangle^{-2} a(v)(-\Delta_v)^s f \|_{L^2} + \| \langle v \rangle^{-2} b(v) f \|_{L^2}
\]

where the last inequality follows from (2) and (3). Then using (23), (25) and (38) to control the last three terms, we get

\[
\| \langle v \rangle^{\frac{3+2s}{1+2s}} (\tau) \|_{L^2} \leq \| \tilde{D}f \|_{L^2} + \| f \|_{L^2},
\]

completing the proof of Lemma 3.6. \(\square\)

3.4. Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1 which can be deduced at once from the following lemma by taking the partial Fourier transform with respect to \( t, x \) variables.

Lemma 3.7. Given \( m \in \mathbb{R} \), there exist a constant \( C_m \) depending only on \( m \), such that for all \( \tau \in \mathbb{R} \) and all \( \xi \in \mathbb{R}^n \), and all \( f \in \mathcal{S}(\mathbb{R}^n) \) we have

\[
\| \Lambda^m \langle v \rangle^{\frac{3+2s}{1+2s}} (\tau) \|_{L^2} + \| \Lambda^m \langle v \rangle^{\frac{3+2s}{1+2s}} \langle \xi \rangle \|_{L^2} \leq C_m \left( \| \Lambda^m \tilde{D}f \|_{L^2} + \| \Lambda^m f \|_{L^2} \right),
\]

where \( \| \cdot \|_{L^2} \) stands for \( \| \cdot \|_{L^2(\mathbb{R}^n)} \) and \( \Lambda^m = \left( 1 + |\tau|^2 + |\xi|^2 + |\eta|^2 \right)^{\frac{m}{2}} \).

Proof. For any \( \tau \in \mathbb{R} \) and any \( \xi \in \mathbb{R}^n \), we denote

\[
\lambda(\eta) = \lambda_{\tau, \xi}(\eta) = \left( 1 + |\tau|^2 + |\xi|^2 + |\eta|^2 \right)^{\frac{m}{2}}.
\]
Then by direct verification we see $\Lambda^m \in \text{Op} \left( S \left( \lambda^m, |dv|^2 + |d\eta|^2 \right) \right)$ uniformly with respect to $\tau$ and $\xi$. Then symbolic calculus (Theorem 2.3.8 and Corollary 2.3.10 of [26]) shows that

$$\forall \ell \in \mathbb{R}, \quad [\Lambda^m, \langle v \rangle^\ell] \in \text{Op} \left( S \left( \langle v \rangle^{\ell-1} \lambda^{m-1}, |dv|^2 + |d\eta|^2 \right) \right)$$

and that

$$[\Lambda^m, a], \quad [\Lambda^m, b] \in \text{Op} \left( S \left( \langle v \rangle^{2s+\gamma-1} \lambda^{m-1}, |dv|^2 + |d\eta|^2 \right) \right),$$

uniformly with respect to $\tau$ and $\xi$. As a result, combining (7), (40) and the fact that $s < 1$, we have

$$[\Lambda^m, a] (-\tilde{\Delta}_v)^s \langle v \rangle^{-(s+\gamma)} \langle D_\eta \rangle^{-s} \Lambda^{-(m-1+s)} \in \text{Op} \left( S \left( 1, |dv|^2 + |d\eta|^2 \right) \right).$$

This along with the relation

$$[\Lambda^m, a] (-\tilde{\Delta}_v)^s = \left( [\Lambda^m, a] (-\tilde{\Delta}_v)^s \langle v \rangle^{-(s+\gamma)} \langle D_\eta \rangle^{-s} \Lambda^{-(m-1+s)} \right) \Lambda^{m-1+s} \langle D_\eta \rangle^s \langle v \rangle^{s+\gamma},$$

implies

$$\| [\Lambda^m, a] (-\tilde{\Delta}_v)^s f \|_{L^2} \lesssim \| \Lambda^{m-1+s} \langle D_\eta \rangle^s \langle v \rangle^{s+\gamma} f \|_{L^2} \lesssim \varepsilon \| \Lambda^m \langle D_\eta \rangle^s \langle v \rangle^{s+\gamma} f \|_{L^2} + C_\varepsilon \| \Lambda^m \langle v \rangle^{s+\gamma} f \|_{L^2},$$

the last inequality using the interpolation inequality that, with $\varepsilon$ arbitrarily small,

$$\| \Lambda^{m-1+s} \langle D_\eta \rangle^s \langle v \rangle^{s+\gamma} f \|_{L^2} \lesssim \varepsilon \| \Lambda^m \langle D_\eta \rangle^s \langle v \rangle^{s+\gamma} f \|_{L^2} + C_\varepsilon \| \Lambda^m \langle v \rangle^{s+\gamma} f \|_{L^2}.$$
The treatment of other commutators can be handled quite similarly. So we only state the conclusions without proof; that is
\[ \| [\Lambda^m, v \cdot \xi] f \|_{L^2} + \| [\Lambda^m, b] f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^{2s+\gamma}] f \|_{L^2} \lesssim \varepsilon \| \langle v \rangle^{2s+\gamma} \Lambda^m f \|_{L^2} + C_\varepsilon \| \Lambda^m f \|_{L^2}, \]
and
\[ \| [\Lambda^m, \langle v \rangle^{\frac{s}{2+\varepsilon}}] (\tau) \frac{v \cdot \xi}{\langle v \rangle^{2s+\gamma}} f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^{\frac{s}{2+\varepsilon}}} \langle \xi \rangle \frac{v \cdot \xi}{\langle v \rangle^{2s+\gamma}} f \|_{L^2} \lesssim \| \langle v \rangle^{2s+\gamma} \Lambda^m f \|_{L^2} + \| \Lambda^m f \|_{L^2}. \]

The above four inequalities yield that
\[ \| [\tilde{P}, \Lambda^m] f \|_{L^2} \lesssim \varepsilon \| \langle v \rangle^\gamma \langle D_\theta \rangle^{2s} \Lambda^m f \|_{L^2} + \varepsilon \| \langle v \rangle^{2s+\gamma} \Lambda^m f \|_{L^2} + C_\varepsilon \| \Lambda^m f \|_{L^2} \tag{42} \]

since \([\tilde{P}, \Lambda^m] = [v \cdot \xi, \Lambda^m] + [a, \Lambda^m] (\tilde{\Delta}_v) + [b, \Lambda^m],\) and that
\[ \| [\Lambda^m, \langle v \rangle^{\frac{s}{2+\varepsilon}}} (\tau) \frac{v \cdot \xi}{\langle v \rangle^{2s+\gamma}} f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^{\frac{s}{2+\varepsilon}}} \langle \xi \rangle \frac{v \cdot \xi}{\langle v \rangle^{2s+\gamma}} f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^\gamma] \langle D_\theta \rangle^{2s} \Lambda^m f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^{2s+\gamma}] \Lambda^m f \|_{L^2} \lesssim \| \langle v \rangle^{\frac{s}{2+\varepsilon}} \Lambda^m f \|_{L^2} + \| \langle v \rangle^{\frac{s}{2+\varepsilon}}} \langle \xi \rangle \frac{v \cdot \xi}{\langle v \rangle^{2s+\gamma}} f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^\gamma] \langle D_\theta \rangle^{2s} \Lambda^m f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^{2s+\gamma}] \Lambda^m f \|_{L^2} \]

As a result the conclusion in Lemma 3.7 will follow if we could show that
\[ \| \langle v \rangle^{\frac{s}{2+\varepsilon}}} (\tau) \frac{v \cdot \xi}{\langle v \rangle^{2s+\gamma}} \Lambda^m f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^{\frac{s}{2+\varepsilon}}} \langle \xi \rangle \frac{v \cdot \xi}{\langle v \rangle^{2s+\gamma}} \Lambda^m f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^\gamma] \langle D_\theta \rangle^{2s} \Lambda^m f \|_{L^2} \]
\[ \lesssim \| \Lambda^m \tilde{P} f \|_{L^2} + \| \Lambda^m f \|_{L^2}. \tag{43} \]

To prove the above inequality we use the estimate in Proposition 3.1 to the function \(\Lambda^m f;\) this gives that the terms on the left hand side is bounded from above by
\[ \| \tilde{P} \Lambda^m f \|_{L^2} + \| \Lambda^m f \|_{L^2}. \]

Then from (42), it follows that
\[ \| \langle v \rangle^{\frac{s}{2+\varepsilon}}} (\tau) \frac{v \cdot \xi}{\langle v \rangle^{2s+\gamma}} \Lambda^m f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^{\frac{s}{2+\varepsilon}}} \langle \xi \rangle \frac{v \cdot \xi}{\langle v \rangle^{2s+\gamma}} \Lambda^m f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^\gamma] \langle D_\theta \rangle^{2s} \Lambda^m f \|_{L^2} + \| [\Lambda^m, \langle v \rangle^{2s+\gamma}] \Lambda^m f \|_{L^2} \]
\[ \lesssim \| \Lambda^m \tilde{P} f \|_{L^2} + \varepsilon \| \langle v \rangle^\gamma \langle D_\theta \rangle^{2s} \Lambda^m f \|_{L^2} + \varepsilon \| [\Lambda^m, \langle v \rangle^{2s+\gamma}] \Lambda^m f \|_{L^2} + C_\varepsilon \| \Lambda^m f \|_{L^2}. \]

Letting \(\varepsilon\) small enough gives (43). The proof of Lemma 3.7 is thus complete. \(\square\)

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