UNIQUENESS RESULTS FOR FULLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. In this paper we prove the comparison principle for viscosity solutions of second order, degenerate elliptic pdes with a discontinuous, inhomogeneous term having discontinuities on Lipschitz surfaces. It is shown that appropriate sub and supersolutions \( u, v \) of a Dirichlet type boundary value problem satisfy \( u \leq v \) in \( \Omega \). In particular, continuous viscosity solutions are unique. We also give examples of existence results and apply the comparison principle to prove convergence of approximations.

1. Introduction. In this paper, we consider the pde

\[
F(x, u(x), Du(x), D^2u(x)) = f(x), \quad \text{in } \Omega,
\]

where \( \Omega \subset \mathbb{R}^N \) is an open and bounded domain. Here \( F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N \to \mathbb{R} \), is a continuous function, while \( f : \mathbb{R}^N \to \mathbb{R} \) is piecewise continuous as described below in the main body of the paper. The notation \( \mathcal{S}^N \) indicates the set of \( N \times N \) symmetric matrices. Equation (1.1) is not the most general structure that we can deal with and extensions are described in Section 4 of the paper, see e.g. (5.1). The unknown function in the equation is \( u : \overline{\Omega} \to \mathbb{R} \) and \( Du \) indicates its gradient, while \( D^2u \) denotes the matrix of second partial derivatives. Since the solution \( u \) of the fully nonlinear pde above may be nonsmooth, in general, we adopt viscosity solutions as the concept of solution, as we recall below. We couple equation (1.1) with a Dirichlet boundary condition and we plan to investigate the uniqueness of solutions. We will discuss the comparison principle for viscosity solutions of the fully nonlinear, degenerate elliptic, partial differential equation (1.1). Due to the discontinuous coefficient \( f \), the class of equations that we consider in not included in the standard viscosity solutions theory, see Crandall, Ishii, Lions [11]. Indeed for degenerate equations with Borel measurable \( f \), uniqueness of solutions is not to be expected, in general. For first order equations, in [33] we found representation formulas for the minimal and maximal viscosity solutions of the Dirichlet problem and showed that they are different functions unless, ideally, characteristic curves of the equation are transversal to the set of discontinuities \( \Gamma \) of the discontinuous coefficient \( f \). This is impossible when \( \Gamma \) has nonempty interior and gives us a reason to limit ourselves to a piecewise continuous coefficient.

The systematic study of viscosity solutions for equations with discontinuous ingredients starts with the paper by Caffarelli, Crandall, Kocan and Swiech [7], which

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builds upon earlier work by Caffarelli on the maximum principle for Pucci operators, see also [12] and [20]. For more general uniqueness results see Swiech [35] and the references therein. The paper [7] is devoted to uniformly elliptic pdes and they prove existence and uniqueness results with $f \in L^p$, $p \geq N - \varepsilon$. Their theory also extends to fully parabolic equations. Even for linear, uniformly elliptic equations however, the uniqueness problem is subtle and it is not completely understood. Indeed the papers by Nadirashvili [28] and Safonov [30] prove that uniqueness of the Dirichlet problem fails, in general, if the coefficients of the second order derivatives are discontinuous. The more recent paper by Camilli, Siconolfi [8], on first order eikonal-type equations, exploits instead coercivity of the Hamiltonian and proves uniqueness for $f \in L^\infty$. Both papers [7] and [8] however use a notion of solution which is stronger than the usual one of viscosity solution, and its definition is heavily determined by the class of equations considered. The key point seems to be that, if we deal with classes of equations with some strong structure, by suitably reinforcing the notion of solution we can obtain uniqueness results for a more general coefficient $f$, other than piecewise continuous.

In order to allow general fully nonlinear degenerate elliptic equations, addressed here for the first time in the theory, we stick instead to the standard viscosity solution notion, see [11] and [17]. This however has some advantages: being the notion of solution less restrictive, it is easier to check and it is more natural when dealing with limits of approximations. The price we have to pay is the restriction on the type of discontinuity allowed on $f$.

When the set $\Gamma$ is the union of Lipschitz hypersurfaces, we can prove the comparison principle of viscosity solutions. The difficulty that we face is the fact that usually a byproduct of the comparison principle is the continuity of solutions. However equation (1.1) does have discontinuous solutions, in general, as we show in an example. We prove that discontinuous super and subsolutions can be compared provided at least one of the two functions satisfies suitable directional continuity in a transversal direction at the points of the boundary of the domain and on the discontinuity set $\Gamma$, see below for the precise statements. The former is sometimes known as ”the cone condition”, the latter is introduced here and is similar looking. However it is required also at points in $\Omega$ but it is one directional. Thus our comparison theorem is not a complete ”strong comparison principle”, i.e. it does not apply to any pair of discontinuous super and subsolutions satisfying the appropriate boundary condition.

We prove several applications of our comparison theorem related to the study of the Dirichlet problem for (1.1). On one hand we show that if a discontinuous solution has the appropriate directional continuity on $\partial \Omega \cup \Gamma$, it is necessarily continuous, making it an indirect and simple way to prove existence of continuous solutions, in addition to Perron’s method for instance. On the other hand continuous viscosity solutions turn out to be unique in the class of discontinuous solutions. Thirdly, whenever we can find a continuous solution of the Dirichlet problem, then it turns out to be the uniform limit of the vanishing viscosity approximations. The latter convergence result can be also extended to other kinds of approximations.

We leave largely open the question of existence of continuous solutions in a general framework, but we plan to come back to this issue in a future paper. Continuous solutions are however quite natural to arise, their existence depending only on the mentioned transversality conditions. As examples of existence of continuous solutions we quote from Caffarelli, Crandall, Kocan and Swiech [7] uniformly elliptic pdes, therefore putting their theory into the new context of discontinuous solutions,
at least in the case of piecewise continuous coefficients. Moreover we can also deal with linear subelliptic operators, by using uniform interior a-priori estimates on the modulus of continuity of the vanishing viscosity approximations proved by Krylov [24], and first order equations of optimal control, where the transversality conditions on $\Gamma \cup \partial \Omega$ can be well expressed in terms of the Lie algebra generated by the vector field of the underlying control system. For convex first order equations the existence and uniqueness results that we can obtain are rather complete, see also a detailed discussion in the other paper on the degenerate eikonal equation [34] whose structure is not directly contained in the framework of the present paper. The proof we propose for the comparison principle follows some ideas contained in our previous paper [32] and avoids the so called method of doubling the variables, although this technique is somewhat incorporated in the use of the nonlinear convolution regularization. It also couples key technical ingredients adopted from the classical paper on state constraints by Soner [31], as later developed for Dirichlet type problems by Ishii [15], see also [9], [4] and [23], and by Barles-Buredeau [6]. The proof of the comparison principle for (1.1) is as self-contained as possible.

We finally want to mention that previous existence and uniqueness results for piecewise continuous, uniformly elliptic operators were obtained by Kutev and Lions [25]. The connection between viscosity solutions and the notion of good solution introduced by Fabes was studied by Jensen [20], Crandall, Kocan, Soravia, Swiech [12] and by Jensen, Kocan, Swiech [22]. Uniqueness of viscosity solutions of second order, degenerate elliptic equations, of the form (1.1) but with continuous $f$, was first established by Lions [26] for equations of stochastic optimal control, by control theoretic arguments. Next Jensen [18] gave the first proof for general pdes. His approach was simplified and extended in Jensen, Lions, Souganidis [21] and Jensen [19], see also Ishii [16], Crandall, Ishii [10], Ishii, Lions [17]. Different arguments were proposed in Lions, Souganidis [27] and the author [32].

The plan of the paper in as follows. In Section 1 we discuss definitions, assumptions and some properties of nonlinear convolution. Section 2 proves the comparison principle and its main consequences to the Dirichlet problem for (1.1). Section 3 is devoted to vanishing viscosity approximations and to the question of existence of continuous solutions for (1.1). In Section 4 we extend the framework of our results. Finally in the Appendix we give some auxiliary results in particular on other notions of solution.

2. Preliminaries. In this section we discuss the notion of viscosity solution and present the main assumptions and some preliminary statements. We start with two general definitions. Our equation (1.1) is set in an open and bounded domain $\Omega \subset \mathbb{R}^N$. In some statements below the boundary $\partial \Omega$ will be required to be a Lipschitz hypersurface, which is equivalent to the following, as shown for instance in the appendix of Bardi-Soravia [3]. The notation $B(x,r)$ indicates the open ball of center $x \in \mathbb{R}^N$ and radius $r > 0$.

**Definition 2.1.** The set $\Gamma \subset \mathbb{R}^N$ is said to be a Lipschitz hypersurface if for all $\hat{x} \in \Gamma$ one of its neighborhoods is partitioned into two connected nonempty open sets $\Omega^+, \Omega^-$ and $\Gamma$ itself, and we can find a transversal unit vector $\eta \in \mathbb{R}^N$, $|\eta| = 1$, with the following property: there are $c, r > 0$ such that and if $x \in B(\hat{x}, r) \cap \Omega^+$ then $B(x \pm t\eta, ct) \subset \Omega^\pm$ for all $0 < t \leq c$, respectively.

We say that an open set $\Omega$ is a Lipschitz domain if $\partial \Omega$ is a Lipschitz hypersurface. In this case if for $\hat{x} \in \partial \Omega$ and transversal unit vector $\eta$ we have $\hat{\Omega}^+ \subset \Omega$, then we call $\eta = \eta_{\Omega}$ an inward unit vector.
The following regularity property for functions will be a key request in our statements.

**Definition 2.2.** Given a unit vector \( \eta \), we say that a function \( u : \overline{\Omega} \to \mathbb{R} \), is continuous at \( \hat{x} \in \overline{\Omega} \) in the direction of \( \eta \) if there are sequences \( t_n \to 0^+ \), and \( p_n \to 0 \), \( p_n \in \mathbb{R}^N \), such that

\[
\lim_{n \to +\infty} u(\hat{x} + t_n\eta + t_n p_n) = u(\hat{x}).
\]

Given a Lipschitz hypersurface \( \Gamma \subset \mathbb{R}^N \) and a transversal unit vector \( \eta \), it will be particularly relevant for us the case of nontangential continuity of a given function \( u \) at points of \( \Gamma \), namely the continuity in the direction of \( \eta \). We will also deal with the continuity of \( u \) at points of \( \partial \Omega \) in an inward direction.

We proceed by recalling the definition of viscosity solution for equations of the form (1.1). This definition follows the one proposed by Ishii [15]. In order to introduce it, we need the notion of upper and lower semicontinuous envelope of a locally bounded function \( v : \Omega \to \mathbb{R} \). They are, respectively,

\[
v^+(x) = \lim_{r \to 0^+} \sup_{\|y-x\| \leq r, y \in \Omega} v(y), \quad v_+(x) = \lim_{r \to 0^+} \inf_{\|y-x\| \leq r, y \in \Omega} v(y).
\]

The definition of viscosity solution for equation (1.1) is as follows.

**Definition 2.3.** A lower (resp. upper) semicontinuous function \( u : \Omega \to \mathbb{R} \) is a viscosity super- (resp. sub-) solution of (1.1) if for all \( \varphi \in C^2(\Omega) \) and \( x \in \arg\min_{x \in \Omega} (u - \varphi) \), (resp. \( x \in \arg\max_{x \in \Omega} (u - \varphi) \)), we have

\[
F(x, u(x), D\varphi(x), D^2\varphi(x)) \geq f_+(x), \quad (\text{resp.} \quad F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq f^+(x)).
\]

We also say in this case that \( (D\varphi(x), D^2\varphi(x)) \in D^2\hat{-} u(x) \) the subjet of \( u \) at \( x \) (resp. \( (D\varphi(x), D^2\varphi(x)) \in D^2\hat{+} u(x) \) the superjet). A locally bounded function \( u : \Omega \to \mathbb{R} \) is a discontinuous viscosity solution of (1.1) if \( u^+ \) is a subsolution and \( u^- \) is a supersolution.

We postpone to the appendix some discussion about the definition of solution, but we refer also to [15], [33] and the references therein.

We will couple equation (1.1) with a Dirichlet boundary condition, allowing enough generality so that solutions may not necessarily satisfy the boundary condition in a pointwise sense. Let \( g : \partial \Omega \to \mathbb{R} \) be a continuous function.

**Definition 2.4.** We say that an upper semicontinuous function \( u : \Omega \to \mathbb{R} \), subsolution of (1.1), satisfies the Dirichlet type boundary condition in the viscosity sense if for all \( \varphi \in C^2(\mathbb{R}^N) \) and \( x \in \partial \Omega \), \( x \in \arg\max_{x \in \overline{\Omega}} (u - \varphi) \) such that \( u(x) > g(x) \), then we have

\[
F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq f^+(x).
\]

Lower semicontinuous functions that satisfy

\[
u \geq g \quad \text{or} \quad F(x, u, Du, D^2u) \geq f_+(x), \quad \text{on} \ \partial \Omega,
\]
or discontinuous solutions of the Dirichlet-type boundary condition

\[ u = g \text{ or } F(x, u, Du, D^2u) = f(x), \quad \text{on } \partial \Omega, \quad \text{(2.0)} \]

are defined accordingly.

A locally bounded function \( u : \overline{\Omega} \to \mathbb{R} \) is a discontinuous solution of the Dirichlet type problem for (1.1) with boundary datum \( g \) if it is a discontinuous solution of (1.1) and of (2.0).

Remark 1.5. Dirichlet type boundary conditions in the viscosity sense are known to represent a weak version of the more classical Dirichlet boundary condition. They appear naturally when proving existence of solutions via deterministic or stochastic optimal control or when taking limits of solutions of approximate regularized problems, while for instance existence of solutions to classical Dirichlet problems may fail in general without appropriate assumptions. This fact is standard and appears in many references of the viscosity solutions literature, see e.g. [2], [6] and [11].

Example. We give an example of solvability of an equation with the notion of viscosity solution. The equation

\[-\varepsilon u'' + 2|u'| + u = f(x), \quad x \in (-1, 1),\]

where \( \varepsilon > 0 \) and \( f \) is the characteristic function of the positive real numbers, is solved in the viscosity sense by the function

\[ u_\varepsilon(x) = \begin{cases} \frac{\sqrt{1+\varepsilon} - 1}{2\sqrt{1+\varepsilon}} e^{\frac{1+\sqrt{1+\varepsilon}}{x}}, & x < 0; \\ 1 - \frac{\sqrt{1+\varepsilon} + 1}{2\sqrt{1+\varepsilon}} e^{\frac{1-\sqrt{1+\varepsilon}}{x}}, & x \geq 0. \end{cases} \]

The reason is that \( u_\varepsilon \) is a strong solution, as explained in the appendix. The function \( u_\varepsilon \) turns out to be the unique viscosity solution of the corresponding Dirichlet problem. Notice that \( u_\varepsilon(x) \to u(x) = (1 - e^{-\frac{1}{2}})f(x) \), uniformly for \( x \in [-1, 1] \), and that as we show below (but see also [33]) \( u \) is the unique solution of the first order problem

\[ u + 2|u'| = f(x), \quad x \in (-1, 1), \]

\[ u(-1) = 0, \quad u(1) = (1 - e^{-\frac{1}{2}}). \]

Therefore convergence of the vanishing viscosity approximations to the correct solution of the limiting problem holds in this example. This is a general fact and will be justified in Section 3.

We now explain the main regularity assumptions on the Hamiltonian \( F \) that appears in (1.1). In particular, in order to call (1.1) degenerate elliptic we suppose that for all compact subsets \( \mathcal{O} \subset \overline{\Omega} \) we can find modulus \( \omega_{\mathcal{O}} \), i.e. \( \omega_{\mathcal{O}} : \mathbb{R}_+ \to \mathbb{R}_+ \) nondecreasing, continuous and vanishing at the origin, such that \( F \) satisfies

\[ -\omega_{\mathcal{O}}(|p - q|) \leq F(x, r, p, X) - F(x, r, q, Y) \leq \omega_{\mathcal{O}}(|p - q| + \text{Tr}(Y - X)), \quad \text{(2.1)} \]

for \( Y \geq X, \ x \in \mathcal{O}, \ r \in \mathbb{R} p, \ q \in \mathbb{R}^N \), where we used the usual ordering on \( S^N \). Moreover we suppose that there is a constant \( \nu > 0 \) such that \( r \to F(x, r, p, X) - \nu r \) is nondecreasing, for all \( (x, p, X) \in \overline{\Omega} \times \mathbb{R}^N \times S^N \). We will refer below to these two properties together by saying that \( F \) is proper. Note that condition (2.1) is a
little stronger than the corresponding condition required in the standard theory for continuous Hamiltonians, see [11].

We now introduce the key assumption on the discontinuities allowed in the Hamiltonian. The following assumption has several motivations. From one side, the study of first order equations pointed out that, with our notion of solution, uniqueness cannot occur if the set of discontinuities for $f$ has nonempty interior. On the other side, in order to implement some ideas of the constrained viscosity solutions theory, we need some regularity of the boundary of the domain and of the set of discontinuities of $f$.

**Assumption (DC).** The set
\[ \Gamma = \{ x \in \mathbb{R}^N : f \text{ is discontinuous at } x \} \]
is the disjoint union of a finite family of connected Lipschitz hypersurfaces and $f$ is piecewise continuous across $\Gamma$. In particular, for $\hat{x} \in \Gamma$ we can find $c, r > 0$, open, nonempty, connected sets $\Omega^+, \Omega^-$ and inward unit vectors $\eta^+, \eta^- = -\eta^+$ to $\Omega^\pm$ respectively, such that if $x \in B(\hat{x}, r)$, then one and only one of the following occurs
\[ x \in \Omega^+, \quad x \in \Omega^-, \quad x \in \Gamma. \]
Moreover if $x \in B(\hat{x}, R) \cap (\Omega^\pm \cup \Gamma)$, respectively, then $B(x + t\eta^\pm, ct) \subset \Omega^\pm$ for all $0 < t \leq c$. We also suppose that $\Omega^\pm \subset \Omega$ if $\hat{x} \in \Omega_0$, assume that the discontinuous coefficient $f$ is continuous in each component $\Omega^\pm$ with a continuous extension in $\overline{\Omega^\pm}$, and that if $x \in \Gamma$
\[ f(x) \in \left[ \lim_{\Omega^- \ni y \to x} f(y), \lim_{\Omega^+ \ni y \to x} f(y) \right], \]
where it is assumed that the above limits exist and the notation $\pm$ is introduced in such a way that the interval is well defined.

If $\hat{x} \in \Gamma \cap \partial \Omega$ we assume that in the above we can choose $c, r, \eta^+, \eta^-$ in such a way that $\eta^+, \eta^-$ are also inward for $\Omega$, i.e. for instance
\[ B(x + t\eta^+, ct) \subset \Omega \cap \Omega^+, \quad B(x + t\eta^-, ct) \subset \Omega \cap \Omega^- \]
for all $x \in B(\hat{x}, R) \cap \overline{\Omega \cap \Omega^\pm}$ and $0 < t \leq c$, respectively. To this end we allow $\eta^+ \neq -\eta^-$, in general. However we suppose also that $-\eta^+, -\eta^-$ are inward vectors for $\Omega^+, \Omega^-$, respectively.

One of the technical difficulties of the problem comes from considering boundary points where the inhomogeneous term in the equation is discontinuous. In order to cope with this fact we require that $\partial \Omega$ and $\Gamma$ meet transversally, which is expressed in the above assumption for $\hat{x} \in \partial \Omega \cap \Gamma$.

In addition to properness, we require the Hamiltonian to satisfy the following regularity condition which refers to previous condition (DC). For all $\hat{x} \in \overline{\Omega}$ and sufficiently small $s > 0$, relative to $B(\hat{x}, s) \cap \overline{\Omega}$ there is a modulus $\omega_\delta$ such that
\[
\begin{align*}
F(x, r, \alpha(x - y) - \sqrt{\alpha} \eta, (I + \frac{1}{\alpha} X)^{-1} X) - F(y, r, \alpha(x - y) - \sqrt{\alpha} \eta, X) & \geq -\omega_\delta(|x - y| + \alpha|x - y - \sqrt{\alpha} \eta|^2), \\
& \quad \text{for all } x, y \in B(\hat{x}, s) \cap \overline{\Omega}, \quad (r, X) \in \mathbb{R} \times \mathcal{S}^N, \quad -\alpha I < X \leq 2\alpha I, \quad \alpha > 0, \\
& \quad \eta = 0 \text{ or } \eta = \delta \eta^- \text{ if } \hat{x} \in \Gamma, \quad \delta \in [0, 1].
\end{align*}
\]
(2.2)
\[
\begin{align*}
F(x, r, \alpha(x - y) + \sqrt{\alpha} \eta, X) - F(y, r, \alpha(x - y) + \sqrt{\alpha} \eta, (I + \frac{1}{\alpha} X)^{-1} X) & \geq -\omega_\delta(|x - y| + \alpha|x - y + \sqrt{\alpha} \eta|^2), \\
& \quad \text{for all } x, y \in B(\hat{x}, s) \cap \Omega, \quad (r, X) \in \mathbb{R} \times \mathcal{S}^N, \quad -2\alpha I \leq X < \alpha I, \quad \alpha > 0, \\
& \quad \eta = 0 \text{ or } \eta = \delta \eta^+ \text{ if } \hat{x} \in \Gamma, \quad \delta \in [0, 1].
\end{align*}
\]
Remark 2.6. Assumption (2.2) is a little stronger than the corresponding standard assumption in [11], but it is general enough to be satisfied by Hamiltonians having the structure of Isaacs equations of stochastic differential games, i.e.

\[ F(x, r, p, X) = \sup_{a \in A} \inf_{b \in B} \{- \text{Tr}A(x, a, b)X - f(x, a, b) \cdot p + c(x, a, b)r - l(x, a, b)\}, \]

with coefficients \( A, f, c, l \) continuous and uniformly bounded in \( W^{1,\infty}(\mathbb{R}^N) \) for all \((a, b) \in A \times B\), \( c \) strictly positive. Condition (2.2) and the more standard (4.14) in [11] look different, ours follows more closely Jensen [19]. About comparing the two different looking assumptions one may check e.g. Remark 1.1 in [32].

A regularization tool for functions, which we employ below is nonlinear convolution that we now recall. Part of its basic properties is collected in Proposition A.1 in the Appendix, here we anticipate some of them to point out the role of the directional continuity. If \( \Omega \subset \mathbb{R}^N \) is a closed set and \( w : \Omega \to \mathbb{R} \) is a lower semicontinuous function satisfying,

\[ \liminf_{x \to \Omega^+, x \to +\infty} \frac{w(x)}{|x|^2} > -\frac{1}{2\kappa^2}, \quad \text{for some } \kappa > 0, \tag{2.3} \]

then, for \( \varepsilon \in (0, \kappa) \) and \( \eta \in \mathbb{R}^N \), we define its inf-convolution as

\[ w_{\varepsilon}(x) = \inf_{y \in \Omega} \left\{ w(y) + \frac{1}{2} \frac{|x - y|}{\varepsilon} + \eta \right\}, \tag{2.4} \]

for \( x \in \mathbb{R}^N \). In a similar way, we can define the sup-convolution as well, namely if \( v : \Omega \to \mathbb{R} \) is an upper semicontinuous function and \(-v\) satisfies (2.3), we put \( v_{\varepsilon} = -(-v)_{\varepsilon} \). If \( \eta = 0 \) the above definition is well known. We extend it for \( \eta \neq 0 \) to implement ideas used in the theory of constrained viscosity solutions, see e.g. Soner [31].

We will always think below that a lower semicontinuous function \( w : \Omega \to \mathbb{R} \), \( \Omega \subset \mathbb{R}^N \) closed, is extended by \(+\infty\) off \( \Omega \) and write directly

\[ w_{\varepsilon}(x) = \inf_{y \in \mathbb{R}^N} \left\{ w(y) + \frac{1}{2} \frac{|x - y|}{\varepsilon} + \eta \right\}, \]

instead of (2.4). Notice that for nonempty \( \Omega w_{\varepsilon} : \mathbb{R}^N \to \mathbb{R} \). The following result illustrates the role of directional continuity of Definition 1.2. It also shows the meaning of the assumption on the regularity of the boundary of the domain.

Proposition 2.7. Let \( w : \overline{\Omega} \to \mathbb{R} \) be a lower semicontinuous, bounded function. Let \( \Omega \subset \subset \overline{\Omega} \) be such that for some \( \eta \in \mathbb{R}^N \), and \( c > 0 \), we have \( B(x + \varepsilon \eta, \varepsilon c) \subset \Omega \) for all \( x \in \Omega \), \( 0 < \varepsilon \leq c \). For \( x \in \mathbb{R}^N \), let \( T_{\varepsilon}x(\in \overline{\Omega}) \) be a point such that

\[ w_{\varepsilon}(x) = w(T_{\varepsilon}x) + \frac{1}{2} \frac{|x - T_{\varepsilon}x|}{\varepsilon} + \eta \]

For \( x \in \Omega \) given, and a sequence \( p_{\varepsilon_n} \), \( |p_{\varepsilon_n}| \leq c \), \( \varepsilon_n \to 0 \) thus

\[ x + \varepsilon_n(\eta + p_{\varepsilon_n}) \in \Omega. \]
Then, at least for the sequence \((\varepsilon_n)_n\),
\[
\liminf_{\varepsilon \to 0^+, \, O \ni y \to x} w_\varepsilon(y) \geq w(x). \tag{2.5}
\]
If moreover
\[
\lim_{\varepsilon_n \to 0^+} w(x + \varepsilon_n(\eta + p_\varepsilon)) = w(x),
\]
then
\[
\lim_{\varepsilon \to 0^+} w_\varepsilon(x) = w(x), \quad \lim_{\varepsilon \to 0^+} \frac{|x - T_\varepsilon x + \varepsilon \eta|}{\varepsilon} = 0, \quad \lim_{\varepsilon \to 0^+} w(T_\varepsilon x) = w(x). \tag{2.6}
\]
In particular \(T_\varepsilon x \in \Omega\) for \(\varepsilon\) sufficiently small (but depending on \(x \in \mathcal{O}\)).

**Proof.** We will outline this proof which is more or less standard for the reader’s convenience.

Let \(x \in \mathcal{O} \subset \subset \overline{\Omega}\). Observe that, by definition and the assumption,
\[
w(T_\varepsilon x) \leq w_\varepsilon(x) \leq w(x + \varepsilon \eta + \varepsilon p_\varepsilon) + \frac{1}{2} |p_\varepsilon|^2 < +\infty, \tag{2.7}
\]
hence if \(w\) is bounded in \(\overline{\Omega}\), then for \(\varepsilon\) small enough we obtain that \(w_\varepsilon\) are uniformly bounded in \(\mathcal{O}\) by the bounds of \(w\). In particular by definition of \(T_\varepsilon x\) we have that \(\frac{|x - T_\varepsilon x + \varepsilon \eta|}{\varepsilon}\) is bounded, uniformly for \(x \in \mathcal{O}\) and hence, for \(\varepsilon\) small enough
\[
|x - T_\varepsilon x| \leq C\varepsilon, \quad |x - T_\varepsilon x + \varepsilon \eta| \leq C\varepsilon, \quad x \in \mathcal{O}, \tag{2.8}
\]
where \(C\) does not depend on \(\varepsilon\) and \(\mathcal{O}\). Going back to (2.7) and taking limits, then we prove the first equation in (2.6) by lower semicontinuity of \(w\) and directional continuity.

From (2.7), (2.8) it follows that
\[
\inf_{|y - x| \leq C\varepsilon + r} w(y) \leq \inf_{O \ni y, \, |y - x| \leq r} \inf_{|y - x| \leq r} w_\varepsilon(y),
\]
and thus (2.5) follows from the lower semicontinuity of \(w\). By definition of \(T_\varepsilon x\), we also obtain
\[
\frac{|x - T_\varepsilon x + \varepsilon \eta|}{\varepsilon} \leq 2 \left( w(x + \varepsilon \eta + \varepsilon p_\varepsilon) - w(T_\varepsilon x) + \frac{|p_\varepsilon|^2}{2} \right)
\]
and therefore by lower semicontinuity of \(w\) and the assumption we obtain \(|x - T_\varepsilon x + \varepsilon \eta| = o_\varepsilon(\varepsilon)\) as \(\varepsilon \to 0\), at least for the sequence \((\varepsilon_n)_n\). The third equation in (2.6) is now a corollary.

**Remark 2.8.** The fact that the rate of the second limit in (2.6) may depend on \(x\) is quite inconvenient and will be adapted later in cases needed in the proof of Theorem 2.3. Notice that (2.6) is a consequence of the directional continuity assumption on the function \(w\), which is always satisfied if \(\eta = 0\), by choosing \(p_\varepsilon \equiv 0\). Of course Proposition 1.7 above can be restated for sup-convolutions as well, in an obvious way. Proposition 1.7 points out that the modified inf-convolution with \(\eta \neq 0\) is an approximation of \(w\) (for instance at boundary points), provided \(w\) is continuous in the direction of \(\eta\) (and \(\eta\) is an inward vector for \(\Omega\)). We will use the
modified inf-convolution with appropriate $\eta$ also at the points of $\Gamma$ to take care of the discontinuities in the equation.

3. Comparison Principle and Uniqueness. In this section we study the boundary value problem

$$\begin{cases}
F(x, u(x), Du(x), D^2u(x)) = f(x), & \text{in } \Omega, \\
u(x) = g(x), & \text{in } \partial \Omega,
\end{cases}$$

(3.1)

and discuss the main uniqueness results and some consequences. Below, by saying that $w$ is twice differentiable at a point $x$, we mean that it has a second order Taylor expansion

$$w(y) = w(x) + (p, y - x) + \frac{1}{2}(X(y - x), y - x) + o(|y - x|^2), \quad \text{as } y \to x,$$

for some $(p, X) \in \mathbb{R}^N \times S^N$. In this case, with a slight abuse of notation, we say that $(p, X) = (Du(x), D^2u(x))$. We recall that if $u$ is twice differentiable at $x$ then $(Du(x), D^2u(x)) \in D^{2,+}u(x) \cap D^{2,-}u(x)$.

We start by stating the following Lemma, which is a refinement of Lemma 2.1 in [32]. It is one of the crucial technical steps of the proof of the comparison theorem.

**Lemma 3.1.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $w : \overline{\Omega} \to \mathbb{R}$ be an upper semicontinuous function. We will set $w(x) = -\infty$ off $\Omega$. Let $w^\varepsilon : \mathbb{R}^N \to \mathbb{R}, \varepsilon > 0$, be a family of semiconvex functions such that

$$\limsup_{\varepsilon \to 0^+, y \to x} w^\varepsilon(y) \leq w(x), \quad \text{for } x \in \mathbb{R}^N.$$

Let $K \subset \mathbb{R}^N$ be a set of Lebesgue measure zero and suppose that $\hat{x} \in \arg \max_{\overline{\Omega}}(w)$ be such that for some $\eta \in \mathbb{R}^N, p_\varepsilon \to 0$

$$\lim_{\varepsilon \to 0^+} w^\varepsilon(\hat{x} + \varepsilon(\eta + p_\varepsilon)) = w(\hat{x}).$$

Then for $r > 0$ fixed and $\rho > 0$ there are points $z_{\varepsilon, \rho} \in \mathbb{R}^N \setminus K$ such that $w^\varepsilon$ is twice differentiable at $z_{\varepsilon, \rho}$ and

$$\lim_{\rho \to 0^+} z_{\varepsilon, \rho} = x_{\varepsilon}, \quad \lim_{\varepsilon \to 0^+} x_{\varepsilon} = \hat{x}, \quad D^2w^\varepsilon(z_{\varepsilon, \rho}) \leq (\rho + r)I,$$

$$\lim_{\varepsilon \to 0^+} \left( \lim_{\rho \to 0^+} w^\varepsilon(z_{\varepsilon, \rho}) \right) = w(\hat{x}), \quad \lim_{\rho \to 0^+} |Dw^\varepsilon(z_{\varepsilon, \rho}) - r(z_{\varepsilon, \rho} - \hat{x})| = 0. \quad (3.2)$$

**Proof.** Let $\hat{x} \in \arg \max_{\overline{\Omega}}(w)$, we are going to localize the argument around $\hat{x}$. For $r > 0$, let $x_{\varepsilon}$ be a maximum point of $w^\varepsilon(\cdot) - \frac{r}{2}|\cdot - \hat{x}|^2$ in $\overline{B(\hat{x}, 1)}$. At least along a subsequence $\varepsilon_n \to 0^+$ we may suppose that $x_{\varepsilon_n} \to \hat{y}$. Thus we get

$$w^\varepsilon(x_{\varepsilon}) \geq w^\varepsilon(\hat{x} + \varepsilon(\eta + p_\varepsilon)) + \frac{r}{2}|x_{\varepsilon} - \hat{x}|^2 - \frac{r}{2}|\eta + p_\varepsilon|^2,$$

and then by the assumptions

$$w(\hat{y}) \geq \limsup_{\varepsilon \to 0^+, y \to \hat{y}} w^\varepsilon(y) \geq \limsup_{\varepsilon \to 0^+} w^\varepsilon(x_{\varepsilon}) \geq \liminf_{\varepsilon \to 0^+} w^\varepsilon(x_{\varepsilon}) \geq w(\hat{x}) + \frac{r}{2}|\hat{y} - \hat{x}|^2.$$
Therefore we conclude that \( \hat{y} \in \overline{\Omega} \), \( \hat{y} = \hat{x} \) and then

\[
x_\varepsilon \to \hat{x}, \ w^\varepsilon(x_\varepsilon) \to w(\hat{x})..
\]

Given \( \varepsilon \) sufficiently small, for any \( \sigma > 0 \), the function

\[
w^\varepsilon(\cdot) - \frac{r}{2} |\cdot - \hat{x}|^2 - \frac{\sigma}{2} |\cdot - x_\varepsilon|^2
\]

then attains at \( x_\varepsilon \) a strict local maximum. Moreover, as this function is semiconvex, we can apply Jensen’s Lemma and Aleksandrov’s Theorem in the Appendix, and perturb such a point to gain in regularity. Given \( \alpha, s > 0 \) sufficiently small, we can find points \( z_\varepsilon, \rho \in \mathbb{R}^N \setminus K \), \( p \in \mathbb{R}^N \), \( |p| \leq \alpha \), such that the function

\[
\mu(\cdot) = w^\varepsilon(\cdot) - \frac{r}{2} |\cdot - \hat{x}|^2 - \frac{\sigma}{2} |\cdot - x_\varepsilon|^2 - (p, \cdot),
\]

attains a local maximum at \( z \), and moreover \( w^\varepsilon \) is twice differentiable at \( z \). In particular we can state that

\[
0 = D\mu(z) = Dw^\varepsilon(z) - r(z - \hat{x}) - \sigma(z - x_\varepsilon) - p,
\]

and

\[
0 \geq D^2\mu(z) = D^2w^\varepsilon(z) - (r + \sigma)I.
\]

Given \( \rho > 0 \), we can then choose the parameters \( \alpha, s, \sigma \) and points \( z_{\varepsilon, \rho} \in \mathbb{R}^N \setminus K \) in such a way that

\[
|z_{\varepsilon, \rho} - x_\varepsilon| \leq \rho, \quad |Du^\varepsilon(z_{\varepsilon, \rho}) - r(z_{\varepsilon, \rho} - \hat{x})| \leq \rho, \quad D^2w^\varepsilon(z_{\varepsilon, \rho}) \leq (r + \rho)I.
\]

The conclusions then follow. \( \square \)

**Remark 3.2.** Observe that if in Lemma 2.1 we know that \( \hat{x} \in \Omega \), then we may suppose \( x^\varepsilon, z_{\varepsilon, \rho} \in \Omega \) for \( \varepsilon \) and then \( \rho \) sufficiently small. It is important to notice that if \( \hat{x} \in \partial \Omega \), then we might need to choose the points \( x_\varepsilon, z_{\varepsilon, \rho} \) outside \( \overline{\Omega} \). However one can always avoid the fixed in advance null set \( K \).

The first statement of a comparison theorem relates a super and a subsolution of a Dirichlet type boundary value problem.

**Theorem 3.3.** Let \( \Omega \) be a bounded domain with Lipschitz boundary. Assume that the Hamiltonian \( F \) is proper and satisfies condition (2.2). Let us suppose that the assumption (DC) is satisfied. Let \( u, v : \overline{\Omega} \to \mathbb{R} \) be respectively an upper and a lower-semicontinuous function, respectively a subsolution and a supersolution of

\[
F(x, u, Du, D^2u) = f(x), \quad \text{in} \ \Omega.
\]

Let us assume that \( u, v \) satisfy the Dirichlet type boundary conditions in the viscosity sense

\[
u \leq g \ \text{or} \ F(x, u, Du, D^2u) \leq f^*(x), \quad \text{on} \ \partial \Omega,
\]

\[
v \geq g \ \text{or} \ F(x, v, Dv, D^2v) \geq f_*(x), \quad \text{on} \ \partial \Omega.
\]

Suppose that \( u, v \) are nontangentially continuous on \( \partial \Omega \setminus \Gamma \) in the inward direction \( \eta_\Omega \), and on \( \Gamma \cap \partial \Omega \) in the directions \( \eta^-_\Omega, \eta^+_\Omega \) respectively. Assume moreover that at
each point of $\Gamma \setminus \partial \Omega$, either $u$ is nontangentially continuous in the direction of $\eta^-$ or $v$ is nontangentially continuous in the direction of $\eta^+$. Then $u \leq v$ in $\overline{\Omega}$.

Remark 3.4. It is important to notice that, as the proof below will show, the non-tangential continuity of $u, v$ at $x \in \partial \Omega$ is only needed if we allow the Dirichlet type boundary condition, but can be avoided if the super and subsolutions satisfy the usual Dirichlet condition. The same goes for the Lipschitz regularity of the boundary $\partial \Omega$, which is not needed if $u \leq g$, $v \geq g$ on $\partial \Omega$. We notice that the boundary $\partial \Omega$ and the discontinuity set $\Gamma$ require similar assumptions on the super and subsolutions. However while on $\partial \Omega$ both functions $u, v$ need some tangential continuity, on $\Gamma \cap \Omega$ either $u$ or $v$ is required to satisfy it, provided the direction is appropriate, i.e. it is $\eta^+$ for $v$ or $\eta^-$ for $u$. We will need to be a little more careful in $\Gamma \cap \partial \Omega$ where the two conditions combine. The role of nontangential continuity in viscosity solutions theory is not new. It was pointed out by Ishii [15] for first order equations and used by Katsoulakis [23] in the study of constrained viscosity solutions, but see also the book by Barles [5] and the paper by Barles-Burdeau [6] for an extensive study of the condition.

Proof. In the argument below we always suppose that $u, v$ are extended as $u(x) = -\infty$ and $v(x) = +\infty$ for $x \in \mathbb{R}^N \setminus \overline{\Omega}$. We assume by contradiction that there is $\hat{x} \in \overline{\Omega}$ such that

$$u(\hat{x}) - v(\hat{x}) = \max_{x \in \Omega} u(x) - v(x) = 2\gamma > 0.$$  \hfill (3.3)

The proof below will be local in a neighborhood of the point $\hat{x}$, therefore we will omit the subscripts in the moduli $\omega$ that we will encounter. The localization argument is already incorporated in Lemma 2.1. We will leave undetermined at this point the position of $\hat{x}$, because the differences in the proof will come up in few places. The main new parts in the argument are of course when $\hat{x} \in \Gamma$.

We need to set the two following facts. If $\hat{x} \in \partial \Omega$, we start by checking the boundary condition and observe that it holds

$$\text{either } u(\hat{x}) > g(\hat{x}) \text{ or } v(\hat{x}) < g(\hat{x}).$$ \hfill (3.4)

We will suppose below, just to fix the ideas, that the former is attained. For the same reason, if instead $\hat{x} \in \Gamma \cap \Omega$ we will suppose that $v$ is nontangentially continuous in the direction $\eta^+$.

Step 1: smoothing up. We start regularizing and, for $\varepsilon > 0$, we introduce the nonlinear convolutions $u^\varepsilon$ and $v^\varepsilon$. More precisely, in both definitions we have to choose carefully the vector $\eta$ to be used. We will choose for $u, v$ respectively, by applying assumption (DC):

$$\eta_u = \begin{cases} 0, & \text{if } \hat{x} \in \overline{\Omega} \setminus \Gamma, \\ -\eta^+, & \text{if } \hat{x} \in \Gamma, \end{cases} \quad \eta_v = \begin{cases} 0, & \text{if } \hat{x} \in \Omega \setminus \Gamma, \\ \eta^-, & \text{if } \hat{x} \in \partial \Omega \setminus \Gamma, \\ \eta^+, & \text{if } \hat{x} \in \Gamma. \end{cases}$$

We recall that on $\Gamma \cap \Omega$ we have $\eta^- = -\eta^+$, while this fact may not be true on $\partial \Omega \cap \Gamma$. Observe that, by Proposition 1.7 and the regularity of $\partial \Omega \cup \Gamma$

$$\limsup_{\varepsilon \to 0^+, y \to x} u^\varepsilon(y) - v^\varepsilon(y) \leq u(x) - v(x), \quad \text{for } x \in \overline{\Omega}.$$
The same is also true in $\mathbb{R}^N \setminus \overline{\Omega}$, as easily checked. By the nontangential continuity in the direction of $\eta_t$, at least along a subsequence $\varepsilon_n \to 0$ we can find $\mathbb{R}^N \ni p_n \to 0$ such that
\[
\lim_{n \to +\infty} v(\hat{x} + 2\varepsilon_n(\eta_t + p_n)) = v(\hat{x}).
\]
Thus, by definition of nonlinear convolution:
- if $\hat{x} \in \Gamma$, so $\eta_t = -\eta_v$, we get
  \[
u_{\varepsilon}^n(\hat{x} - \varepsilon_n\eta_t) - v_{\varepsilon}^n(\hat{x} + \varepsilon_n\eta_t) \geq u(\hat{x}) - v(\hat{x} + 2\varepsilon_n(\eta_t + p_n)) - 2|p_n|^2;
\]
- if $\hat{x} \in \partial \Omega \setminus \Gamma$, so $\eta_t = 0$,
  \[
u_{\varepsilon}^{2\varepsilon}(\hat{x}) - v_{\varepsilon}^{2\varepsilon}(\hat{x}) \geq u(\hat{x}) - v(\hat{x} + 2\varepsilon_n(\eta_t + p_n)) - |p_n|^2/2;
\]
- if $\hat{x} \in \Omega \setminus \Gamma$ we have $u_{\varepsilon}(\hat{x}) - v_{\varepsilon}(\hat{x}) \geq u(\hat{x}) - v(\hat{x})$.

Taking limits, in all cases we can find $\eta \in \mathbb{R}^N$ such that
\[
\liminf_{n \to +\infty} \nu_{\varepsilon}^n(\hat{x} + \varepsilon_n\eta) - v_{\varepsilon}^n(\hat{x} + \varepsilon_n\eta) \geq u(\hat{x}) - v(\hat{x}),
\]

at least along a subsequence. Notice that no continuity of $v$ is needed to obtain the previous inequality, if $\eta_t = 0$, by choosing $p = 0$. We will avoid below the notation for the subsequence $\varepsilon_n$.

The assumptions of Lemma 2.1 are thus satisfied with $w = u - v$ and $u_{\varepsilon} = u_{\varepsilon} - v_{\varepsilon}$, and with the choice
\[
K = \Gamma \cup \{ x : \text{either } u_{\varepsilon}^n \text{ or } v_{\varepsilon}^n \text{ is not twice differentiable at } x \}.
\]

Therefore by that result, for a fixed $r > 0$ (that we need as small as required at the end of step 3) and given $\varepsilon, \rho > 0$, we construct points $\hat{x} \in \mathbb{R}^N$, $z_{\varepsilon,\rho} \in \mathbb{R}^N \setminus \Gamma$, such that both $u_{\varepsilon}$ and $v_{\varepsilon}$ are twice differentiable at $z_{\varepsilon,\rho}$ and satisfy (3.2). In particular, if we choose $\varepsilon$ and then $\rho$ sufficiently small, by the first equation of the second line of (3.2) and (3.3), we may suppose that
\[
u_{\varepsilon}(z_{\varepsilon,\rho}) - v_{\varepsilon}(z_{\varepsilon,\rho}) \geq \gamma, \quad u_{\varepsilon}(x) - v_{\varepsilon}(x) \geq \gamma.
\]

Step 2: nonlinear convolution. We will now use the properties of nonlinear convolution. Notice that for $\rho > 0$ we have
\[
(Du_{\varepsilon}(z_{\varepsilon,\rho}), D^2u_{\varepsilon}(z_{\varepsilon,\rho}) + \rho I) \in D^{2,+}u_{\varepsilon}(z_{\varepsilon,\rho}),
(Dv_{\varepsilon}(z_{\varepsilon,\rho}), D^2v_{\varepsilon}(z_{\varepsilon,\rho}) - \rho I) \in D^{2,-}v_{\varepsilon}(z_{\varepsilon,\rho}).
\]

We will denote below $X_{\rho} = D^2u_{\varepsilon}(z_{\varepsilon,\rho}) + \rho I$ and $X_\rho = D^2v_{\varepsilon}(z_{\varepsilon,\rho}) - \rho I$. By applying Propositions 1.7 and A.1, and Remark 1.8, for sufficiently small values of the parameters $\varepsilon, \rho$, we can find a unique point $T_{\varepsilon}z_{\varepsilon,\rho} \in \overline{\Omega}$ such that
\[
\begin{align*}
u_{\varepsilon}(z_{\varepsilon,\rho}) &= u(T_{\varepsilon}z_{\varepsilon,\rho}) - \frac{1}{2} \left| \frac{z_{\varepsilon,\rho} - T_{\varepsilon}z_{\varepsilon,\rho}}{\varepsilon} + \eta_t \right|^2, \\
Du_{\varepsilon}(z_{\varepsilon,\rho}) &= \frac{(T_{\varepsilon}z_{\varepsilon,\rho} - z_{\varepsilon,\rho} - \varepsilon \eta_t)}{\varepsilon^2}, \quad D^2u_{\varepsilon}(z_{\varepsilon,\rho}) \geq -\frac{1}{\varepsilon^2}I, \\
(Du_{\varepsilon}(z_{\varepsilon,\rho}), (I + \varepsilon^2X_{\rho})^{-1}X_{\rho}) &\in D^{2,+}u(T_{\varepsilon}z_{\varepsilon,\rho}).
\end{align*}
\]

\[\]
We can also find a unique point $T^\varepsilon z_{\varepsilon,\rho} \in \overline{\Omega}$ such that

$$v_\varepsilon(z_{\varepsilon,\rho}) = v(T^\varepsilon z_{\varepsilon,\rho}) + \frac{1}{2} \left| \frac{z_{\varepsilon,\rho} - T^\varepsilon z_{\varepsilon,\rho} + \eta_v}{\varepsilon} \right|^2,$$

$$Dv_\varepsilon(z_{\varepsilon,\rho}) = \frac{(z_{\varepsilon,\rho} - T^\varepsilon z_{\varepsilon,\rho} + \varepsilon \eta_v)}{\varepsilon^2}, \quad (D^2v_\varepsilon(z_{\varepsilon,\rho}) = \frac{1}{\varepsilon^2} I, \quad (Dv_\varepsilon(z_{\varepsilon,\rho}), (I - \varepsilon^2 X_\rho)^{-1} X_\rho) \in D^2v(T^\varepsilon z_{\varepsilon,\rho}).$$ (3.7)

As $\rho \to 0^+$, at least along a subsequence, by (3.2) of Lemma 2.1, (3.6) and (3.7), we may suppose below that, for $\varepsilon$ sufficiently small,

$$T^\varepsilon z_{\varepsilon,\rho} \to z^\varepsilon \in \overline{\Omega}, \quad T^\varepsilon z_{\varepsilon,\rho} \to z_\varepsilon \in \overline{\Omega}, \quad D^2v_\varepsilon(z_{\varepsilon,\rho}) \to Y_\varepsilon, \quad -(r + \frac{1}{\varepsilon^2}) I \leq Y_\varepsilon \leq \frac{1}{\varepsilon^2} I.$$

Observe also that from the second equation of the second line in (3.2) and (3.6), (3.7) it follows that

$$z^\varepsilon - x_\varepsilon - \varepsilon \eta_u = x_\varepsilon - z_\varepsilon + \varepsilon^2 r(x_\varepsilon - \hat{x}).$$ (3.8)

In order to improve the second equation in (2.6) we do the following. Notice that by the first equations in (3.6), (3.7) we obtain

$$\frac{1}{2} \left| \frac{x_\varepsilon - z_\varepsilon}{\varepsilon} + \eta_u \right|^2 + \frac{1}{2} \left| \frac{x_\varepsilon - z_\varepsilon}{\varepsilon} + \eta_v \right|^2 = \lim_{\rho \to 0^+} \frac{1}{2} \left| \frac{z_{\varepsilon,\rho} - T^\varepsilon z_{\varepsilon,\rho}}{\varepsilon} + \eta_u \right|^2 + \frac{1}{2} \left| \frac{z_{\varepsilon,\rho} - T^\varepsilon z_{\varepsilon,\rho}}{\varepsilon} + \eta_v \right|^2 \leq u(z^\varepsilon) - u^\varepsilon(x_\varepsilon) + v_\varepsilon(x_\varepsilon) - v(z_\varepsilon) = u(z^\varepsilon) - v(z_\varepsilon) - w^\varepsilon(x_\varepsilon).$$ (3.9)

Since the right hand side is bounded, as $\hat{x}, z^\varepsilon, z_\varepsilon \in \overline{\Omega}$, then we get that $z^\varepsilon \to \hat{x}$ and $z_\varepsilon \to \hat{x}$. Using this fact again in (3.9), the semicontinuity of $u, v$ and the first equation in the second line of (3.2) we conclude that

$$\limsup_{\varepsilon \to 0^+} \frac{1}{2} \left| \frac{z_{\varepsilon,\rho} - T^\varepsilon z_{\varepsilon,\rho}}{\varepsilon} + \eta_u \right|^2 + \frac{1}{2} \left| \frac{z_{\varepsilon,\rho} - T^\varepsilon z_{\varepsilon,\rho}}{\varepsilon} + \eta_v \right|^2 = 0.$$ (3.10)

An important consequence of this estimate is the fact that given $c > 0$ of Assumption (DC), for $\varepsilon$ and then $\rho$ sufficiently small we can suppose that

$$|z_{\varepsilon,\rho} - T^\varepsilon z_{\varepsilon,\rho} + \varepsilon \eta_u| \leq c\varepsilon/2, \quad |z_{\varepsilon,\rho} - T^\varepsilon z_{\varepsilon,\rho} + \varepsilon \eta_v| \leq c\varepsilon/2.$$ (3.11)

Notice that (3.11) also gives us

$$|T^\varepsilon z_{\varepsilon,\rho} + \varepsilon(\eta_v - \eta_u) - T^\varepsilon z_{\varepsilon,\rho}| \leq c\varepsilon,$$ (3.12)

which will be useful if $\hat{x} \in \partial\Omega$. 
Another consequence of (3.6) (3.7), Lemma 2.1 and the semicontinuity properties of \( u, v \) is that

\[
v(\hat{x}) \leq \limsup_{\varepsilon \to 0^+} \lim_{\rho \to 0^+} v(T^\varepsilon z_{\varepsilon, \rho}) = \limsup_{\varepsilon \to 0^+} v_\varepsilon(x_\varepsilon) = \limsup_{\varepsilon \to 0^+} u^\varepsilon(x_\varepsilon) - u^\varepsilon(x_\varepsilon) \leq v(\hat{x})
\]

and similarly for \( u \). Thus

\[
\lim_{\varepsilon \to 0^+} \lim_{\rho \to 0^+} v(T^\varepsilon z_{\varepsilon, \rho}) = v(\hat{x}), \quad \lim_{\varepsilon \to 0^+} \lim_{\rho \to 0^+} u(T^\varepsilon z_{\varepsilon, \rho}) = u(\hat{x}). \tag{3.13}
\]

Step 3: estimate of the left hand side of the equation. We now want to apply the definition of viscosity sub and supersolution at the points \( T^\varepsilon z_{\varepsilon, \rho}, T^\varepsilon z_{\varepsilon, \rho} \), via Proposition A.1. We have to consider a few separate cases.

- If \( \hat{x} \in \Omega \), by the first two equations in (3.2) and (3.11), we may suppose that, for \( \varepsilon, \rho \) sufficiently small \( T^\varepsilon z_{\varepsilon, \rho}, T^\varepsilon z_{\varepsilon, \rho} \in \Omega \);
- if \( \hat{x} \in \partial \Omega \setminus \Gamma \), then by our choice of \( \eta_\nu = \eta_\Omega \) and \( \eta_u = 0 \), and (3.12) we have that \( T^\varepsilon z_{\varepsilon, \rho} \in \Omega \) while \( T^\varepsilon z_{\varepsilon, \rho} \in \Omega \);
- if \( \hat{x} \in (\Gamma \cap \partial \Omega) \), then by (3.12), \( -\eta_\nu = \eta_\nu \) and \( \eta_u \) inward \( \Omega \), \( |T^\varepsilon z_{\varepsilon, \rho} + 2\varepsilon \eta_\nu - T^\varepsilon z_{\varepsilon, \rho}| \leq c\varepsilon \) and then \( T^\varepsilon z_{\varepsilon, \rho} \in \Omega \).

Thus we might need to use the boundary condition for \( u \) at \( T^\varepsilon z_{\varepsilon, \rho} \). To this end, observe that by (3.4), (3.13) and continuity of \( g \), we may suppose that, for \( \varepsilon \) and then \( \rho \) sufficiently small, if \( T^\varepsilon z_{\varepsilon, \rho} \in \partial \Omega \) then we have

\[
u u(T^\varepsilon z_{\varepsilon, \rho}) > g(T^\varepsilon z_{\varepsilon, \rho}).
\]

Our discussion then always allows us to use the equation at the points \( T^\varepsilon z_{\varepsilon, \rho}, T^\varepsilon z_{\varepsilon, \rho} \) for \( v, u \), respectively. By definition of viscosity solution, the properness of \( F \), assumption (2.2) and Lemma 2.1, we calculate

\[
f^\ast(T^\varepsilon z_{\varepsilon, \rho}) \geq F \left( T^\varepsilon z_{\varepsilon, \rho}, u^\varepsilon(z_{\varepsilon, \rho}) + \frac{1}{2} \left| \frac{z_{\varepsilon, \rho} - T^\varepsilon z_{\varepsilon, \rho}}{\varepsilon} + \eta_u \right|^2, \right.
\]

\[
\left. \frac{T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta_u}{\varepsilon^2}, (I + \varepsilon^2 X^\rho)^{-1} X^\rho \right)
\]

\[
\geq F \left( T^\varepsilon z_{\varepsilon, \rho}, v_\varepsilon(z_{\varepsilon, \rho}) + \gamma, \frac{T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta_u}{\varepsilon^2}, (I + \varepsilon^2 X^\rho)^{-1} X^\rho \right)
\]

\[
\geq F \left( z_{\varepsilon, \rho}, v_\varepsilon(z_{\varepsilon, \rho}) + \gamma, \frac{T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta_u}{\varepsilon^2}, X^\rho \right)
\]

\[
- \omega \left( |T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho}| + \frac{|T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho}|}{\varepsilon} - \eta_u \right)^2
\]

\[
\geq \nu \gamma + F \left( z_{\varepsilon, \rho}, v_\varepsilon(z_{\varepsilon, \rho}), \frac{T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta_u}{\varepsilon^2}, D^2 v_\varepsilon(z_{\varepsilon, \rho}) + (r + 2\rho) I \right)
\]

\[
- \omega \left( |T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho}| + \frac{|T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho}|}{\varepsilon} - \eta_u \right)^2.
\tag{3.14}
\]
Moreover

\[ f_s(T_\varepsilon z_{\varepsilon, \rho}) \leq F \left( T_\varepsilon z_{\varepsilon, \rho}, v_\varepsilon(z_{\varepsilon, \rho}) - \frac{1}{2} \frac{z_{\varepsilon, \rho} - T_\varepsilon z_{\varepsilon, \rho}}{\varepsilon} + \eta_\varepsilon \right)^2, \]

\[
\leq F \left( T_\varepsilon z_{\varepsilon, \rho}, v_\varepsilon(z_{\varepsilon, \rho}), \frac{z_{\varepsilon, \rho} - T_\varepsilon z_{\varepsilon, \rho}}{\varepsilon^2} + \frac{\varepsilon \eta_\varepsilon}{\varepsilon^2}, (I - \varepsilon^2 X_\rho)^{-1} X_\rho \right)
\]

\[
\leq F \left( z_{\varepsilon, \rho}, v_\varepsilon(z_{\varepsilon, \rho}), \frac{z_{\varepsilon, \rho} - T_\varepsilon z_{\varepsilon, \rho}}{\varepsilon^2} + \frac{\varepsilon \eta_\varepsilon}{\varepsilon^2}, D^2 v_\varepsilon(z_{\varepsilon, \rho}) - \rho I \right) + \omega \left( \frac{|z_{\varepsilon, \rho} - T_\varepsilon z_{\varepsilon, \rho}| + \frac{|z_{\varepsilon, \rho} - T_\varepsilon z_{\varepsilon, \rho}|}{\varepsilon} + \eta_\varepsilon \right)^2. \]  

Hence, subtracting (3.15) from (3.14), taking the limit as \( \rho \to 0^+ \) and then using (2.1), (3.10) and (3.8), we get

\[
\limsup_{\rho \to 0^+} f^*(T_\varepsilon z_{\varepsilon, \rho}) - f_s(T_\varepsilon z_{\varepsilon, \rho}) \
\geq \nu \gamma + F(x_\varepsilon, v_\varepsilon(x_\varepsilon), \frac{z_\varepsilon - x_\varepsilon - \varepsilon \eta_\varepsilon}{\varepsilon^2}, Y_\varepsilon + r I) - F(x_\varepsilon, v_\varepsilon(x_\varepsilon), \frac{x_\varepsilon - z_\varepsilon + \varepsilon \eta_\varepsilon}{\varepsilon^2}, Y_\varepsilon) \
- \omega \left( |z_\varepsilon - x_\varepsilon| + \frac{|z_\varepsilon - x_\varepsilon|}{\varepsilon} - \eta_\varepsilon \right)^2 - \omega \left( |z_\varepsilon - x_\varepsilon| + \frac{|z_\varepsilon - x_\varepsilon|}{\varepsilon} - \eta_\varepsilon \right)^2 \
\geq \nu \gamma - \omega(r |x_\varepsilon - \hat{x}| + r N) + o(1), \]  

as \( \varepsilon \to 0^+ \). Thus

\[
\limsup_{\varepsilon \to 0^+} \limsup_{\rho \to 0^+} f^*(T_\varepsilon z_{\varepsilon, \rho}) - f_s(T_\varepsilon z_{\varepsilon, \rho}) \geq \nu \gamma - \omega(r N) > 0, \]  

when \( r > 0 \) is fixed sufficiently small.

Step 4: the inhomogeneous term. In order to handle the discontinuous term, which is the final brick of the construction, we only need to discuss the case \( \hat{x} \in \Gamma \), where \( f \) is discontinuous. Notice that as we argued at the end of Step 1, \( z_{\varepsilon, \rho} \notin K \) as there defined. Hence in particular \( z_{\varepsilon, \rho} \notin \Gamma \). Thus for any \( \varepsilon \) fixed we may restrict ourselves to subsequences \( \rho_\varepsilon \to 0 \) such that it always happens either \( z_{\varepsilon, \rho} \in \Omega^- \) or \( z_{\varepsilon, \rho} \in \Omega^+ \) for \( \rho \) sufficiently small. We just pause to recall that \( z_{\varepsilon, \rho} \) may be not in \( \Omega \) but this will not affect the estimate below. At this point we may choose a subsequence \( \varepsilon_\rho \to 0 \) such that always \( z_{\varepsilon_\rho, \rho} \in \Omega^- \) or \( z_{\varepsilon_\rho, \rho} \in \Omega^+ \) for any \( \varepsilon, \rho \) sufficiently small. In order to simplify notations below, the choice of the subsequences will not appear explicitly. Therefore (3.11) implies that either \( T_\varepsilon z_{\varepsilon, \rho} \in \Omega^- \) or \( T_\varepsilon z_{\varepsilon, \rho} \in \Omega^+ \), respectively. Thus we obtain in the limit that \( z^\varepsilon \in \Omega^- \cup \Gamma \) or \( z^\varepsilon \in \Omega^+ \cup \Gamma \), respectively. In either case by assumption (DC) we can estimate the limit as follows

\[
\limsup_{\varepsilon \to 0^+} \limsup_{\rho \to 0^+} f^*(T_\varepsilon z_{\varepsilon, \rho}) - f_s(T_\varepsilon z_{\varepsilon, \rho}) = \limsup_{\varepsilon \to 0^+} \limsup_{\rho \to 0^+} \begin{cases} f(T_\varepsilon z_{\varepsilon, \rho}) - f_s(T_\varepsilon z_{\varepsilon, \rho}), & \text{if } z_{\varepsilon, \rho} \in \Omega^- \\ f^*(T_\varepsilon z_{\varepsilon, \rho}) - f(T_\varepsilon z_{\varepsilon, \rho}), & \text{if } z_{\varepsilon, \rho} \in \Omega^+ \end{cases} \\
\leq \limsup_{\varepsilon \to 0^+} \begin{cases} f_s(z^\varepsilon) - f_s(z), & \text{if } z_{\varepsilon, \rho} \in \Omega^- \\ f^*(z^\varepsilon) - f^*(z), & \text{if } z_{\varepsilon, \rho} \in \Omega^+ \end{cases} \\
\leq \begin{cases} f_s(\hat{x}) - f_s(\hat{x}), & \text{if } z_{\varepsilon, \rho} \in \Omega^- \\ f^*(\hat{x}) - f^*(\hat{x}), & \text{if } z_{\varepsilon, \rho} \in \Omega^+ \end{cases} = 0. \]  

(3.17)
Finally (3.17), and (3.16) provide the required contradiction.

Remark 3.5. The choice of the vectors $\eta_v, \eta_u$ in the proof above can be explained as follows. We only need them to be non null vectors on $\partial \Omega$ and on $\Gamma$ in order to "push" the variable inside $\Omega$ and/or inside $\Omega^\pm$ respectively. The main difficulty comes on $\partial \Omega \cap \Gamma$ where one needs to reach both these effects. Luckily enough, at such points by (3.4) only one of the two functions needs to have its variable pushed into $\Omega$.

A consequence of Theorem 2.3 is the following Corollary which gives us a set of sufficient conditions for the existence of a continuous viscosity solution of a boundary value problem.

**Corollary 3.6.** Let $\Omega$ be bounded and $u : \overline{\Omega} \to \mathbb{R}$ be a bounded viscosity solution of

$$F(x, u, Du, D^2u) = f(x), \quad \Omega,$$

where $F$ is proper and satisfies condition (2.2). Let us suppose that the assumption $(DC)$ is satisfied. If $u$ is continuous on $\partial \Omega$ and either $u^*$ or $u^*$ is nontangentially continuous on $\Gamma \setminus \partial \Omega$ in the direction of $\eta^+, \eta^-$, respectively, then $u \in C(\Omega)$.

**Proof.** Apply Theorem 3.3 to the supersolution $u^*$ and the subsolution $u^*$ and $g = u$ on $\partial \Omega$.

The following variant of the comparison theorem also holds. It does not refer to a Dirichlet-type problem, but compares directly the super and subsolutions. Notice that only the nontangential continuity of one of the two functions to be compared is now needed at the boundary.

**Theorem 3.7.** Let $\Omega$ be a bounded domain with Lipschitz boundary. Assume that the Hamiltonian $F$ is proper and satisfies condition (2.2). Let us suppose that the assumption $(DC)$ is satisfied. Let $u, v : \overline{\Omega} \to \mathbb{R}$ be respectively an upper and a lower-semicontinuous function, respectively a subsolution and a supersolution of

$$F(x, u, Du, D^2u) = f(x), \quad \text{in } \Omega.$$

Let us assume that $v$ restricted on $\partial \Omega$ is continuous and that $u$ satisfies the Dirichlet type boundary condition in the viscosity sense

$$u \leq v \text{ or } F(x, u, Du, D^2u) \leq f^*(x), \quad \text{on } \partial \Omega.$$

Suppose moreover that $v$ is nontangentially continuous on $\partial \Omega \setminus \Gamma$ in the inward direction $\eta_\Omega$ and on $\Gamma$ in the direction of $\eta^+$. Then $u \leq v$ in $\Omega$.

**Proof.** We can apply the same proof of Theorem 2.3 with the choice of $g = v$ on $\partial \Omega$. The reason why the continuity of $u$ is not needed on $\partial \Omega$ is that $v$ is dealt with as a boundary condition and thus for $\hat{x} \in \partial \Omega$ we may always use the equation for $u$ at $T_{\hat{x}z_{z,\rho}}$.

The following result is an obvious corollary of the previous statement.

**Corollary 3.8.** Let $\Omega$ be bounded and $u : \overline{\Omega} \to \mathbb{R}$ be a continuous viscosity solution of the Dirichlet boundary value problem

$$\begin{cases} F(x, u, Du, D^2u) = f(x), & \Omega, \\ u = g, & \partial \Omega, \end{cases}$$
where \( F \) is proper and satisfies condition (2.2). Let us assume condition (DC). Then \( u \) is unique in the class of discontinuous solutions of the corresponding Dirichlet type problem.

**Remark 3.9.** For degenerate elliptic equations the above comparison principle is new. Even when \( F \) is uniformly elliptic and condition (DC) holds, it somewhat strengthens the uniqueness results in Caffarelli, Crandall, Kocan, Swiech [7], Swiech [35] and the references therein where uniqueness is found in the class of continuous \( L^p \)-viscosity solutions.

The following example shows that discontinuous solutions may exist after all, thus uniqueness is not always ensured by Corollary 2.8. Let us consider the boundary value problem

\[
\begin{cases}
  u - xu' = f(x), & x \in (0, 1], \\
  u(0) = 0,
\end{cases}
\]

where \( f(x) = 1 \), for \( x > 0 \), and \( f(x) = 0 \) for \( x \leq 0 \). It is easy to verify that the function

\[
u(x) = \begin{cases}
  0, & x \leq 0, \\
  1 - x, & x > 0
\end{cases}
\]

is a viscosity solution of the problem. However we can still use Corollary 2.8 to state that the problem has no continuous solution. For a more detailed discussion about existence of discontinuous solutions to boundary value problems, we refer the reader to the papers by Garavello and the author [13], [33], showing for instance that \( u \) above is the unique lower semicontinuous solution.

**Remark 3.10.** Notice that the example above does not contradict the regularity Corollary 2.6. Indeed what follows from that statement is that, if \( \Omega \) is split into two regions \( \Omega^\pm \) separated by a Lipschitz hypersurface \( \Gamma \), a solution which is piecewise continuous across \( \Gamma \) of a Dirichlet boundary value problem, and which is continuous on \( \partial \Omega \), is required to jump in the same direction as \( f \), like in the example above. A jump in the opposite direction would force continuity in view of Corollary 2.6, see also the two dimensional example in [34].

### 4. Approximations and Existence

In this section we deal with two different applications of the comparison principle, namely the construction of approximations and the existence of continuous solutions for which the uniqueness Corollary 2.8 applies. Concerning existence, we will give here only a few examples for certain classes of equations, but we will outline a general method based on Corollary 2.6 which we plan to pursue for more general existence results in a future paper. We refer the reader also to the paper [34] where this idea is used in the case of the eikonal equation. The main examples that we describe below are the cases of uniformly elliptic pdes, linear subelliptic operators, and of first order equations.

As an example of approximations, we will consider the classical vanishing viscosity method, i.e. we study the equation

\[
-\varepsilon \Delta u(x) + F(x, u(x), Du(x), D^2u(x)) = f(x), \quad x \in \Omega
\]

which is uniformly elliptic. We couple it with a boundary condition of the form

\[
u(x) = g_\varepsilon(x), \quad x \in \partial \Omega,
\]

(we notice that in this section subscripts have nothing to do with inf-convolution).

The Dirichlet boundary value problem for a uniformly elliptic operator can be solved also with a discontinuous inhomogeneous term. The theory starts with [7], see also [12] and Swiech [35]. In particular a general existence result is the following.
Theorem 4.1. (Swiech [35]) Suppose that \( \Omega \) is open, bounded with Lipschitz continuous boundary and let \( g_\varepsilon \in C(\partial \Omega) \). Assume that the Hamiltonian \( F \) is proper and satisfies condition (2.2). Let \( f \in L^\infty(\Omega) \). Then there exists a unique \( L^\infty \)-viscosity solution \( u_\varepsilon \in C(\overline{\Omega}) \) of the Dirichlet problem (4.1-2).

We notice that Theorem 3.1 as formulated in [35] applies to more general uniformly elliptic operators than (4.1). When assumption (DC) applies to \( f \) our result Corollary 2.8 extends uniqueness to the Dirichlet-type boundary value problem and to discontinuous viscosity solutions (not necessarily \( L^\infty \)-solutions). For the notion of \( L^\infty \) viscosity solution that appears in Theorem 3.1 and which is a stronger notion than viscosity solution used in this paper, we refer the reader to the mentioned references and to the appendix.

The main question we are interested in concerns the convergence of the family of solutions \( \{u_\varepsilon\} \). In the presence of \( L^\infty \) estimates on the family \( \{u_\varepsilon\} \), one can construct bounded functions (half relaxed limits)

\[
v(x) = \liminf_{\varepsilon \to 0^+, y \to x} u_\varepsilon(y), \quad w(x) = \limsup_{\varepsilon \to 0^+, y \to x} u_\varepsilon(y).
\]

Their use is contained in the following statement.

Proposition 4.2. Let \( \Omega \) be open, bounded with Lipschitz continuous boundary, assume that the Hamiltonian \( F \) is proper and satisfies condition (2.2), and that (DC) holds. Let us suppose that the Dirichlet boundary value problem (3.1) has viscosity super and subsolutions \( \underline{u}, \bar{u} \in C(\overline{\Omega}) \) with \( \underline{u} = \bar{u} = g \) on \( \partial \Omega \) and that \( g_\varepsilon \to g \) uniformly in \( \partial \Omega \). If the family of solutions \( \{u_\varepsilon\} \) of the approximating problems (4.1-2) satisfies \( \|u_\varepsilon\| \leq C \) for all \( \varepsilon > 0 \), then, with the notation in (4.3),

\[
w \leq \underline{u}, \quad \bar{u} \leq v, \quad \text{in } \overline{\Omega}.
\]

Proof. We first notice that \( v, w \) are respectively viscosity super and subsolutions of the Dirichlet-type boundary value problem, see Definition 1.4. The proof of this fact is virtually identical to that of Theorem 1.7 and Proposition 4.7 of Chapter 5 in [2], see also [11]. At this point we apply Theorem 2.7 and conclude. \( \square \)

We have several ways to use Proposition 3.2. The first classical method seeks existence of solutions to (3.1) by using further estimates on the modulus of continuity for the family \( \{u_\varepsilon\} \) in order to apply Ascoli-Arzelà Theorem. When \( f \) is discontinuous and \( F \) is degenerate, finding such estimates is largely open, but they are proved for instance by Krylov [24] for linear subelliptic operators. Namely he shows that when \( g_\varepsilon(x) \equiv 0 \) and

\[
F(x, u(x), Du(x), D^2u(x)) = u(x) - \sigma^k(x)(\sigma^k(x)u_{x^j}(x))_{x^j} - b^i(x)u_{x^i}(x)
\]

(4.4)

(here we use the summation convention on the repeated indices), where \( \sigma^k = (\sigma^k)_{i} \), \( b = (b_i) \) are \( C^\infty \) vector fields, \( k = 1, \ldots, M \), \( (M \leq N) \) and the Lie algebra generated by \( \{\sigma^k, b : k = 1, \ldots, M\} \) has dimension \( N \) at every point in \( \mathbb{R}^N \), one can estimate for a given \( f \in L^\infty \), for some \( \alpha \in (0,1) \) small enough,

\[
\sup_{\Omega} u_\varepsilon \leq C, \quad \sup_{x,y \in \Omega_1} |u_\varepsilon(x) - u_\varepsilon(y)| \leq C|x - y|^\alpha
\]

for any \( \Omega_1 \subset \subset \Omega \), where \( \{u_\varepsilon\} \) is the family of solutions of (4.1-2). One knows that \( u_\varepsilon \in W^{1,p}(\Omega) \) for any \( p \in (1, \infty) \), hence \( u_\varepsilon \) is a strong solution of (4.1) and a viscosity
solution, see Proposition A.4 and Remark A.5. Thus, at least along a subsequence, one can find \( u = \overline{u} = \underline{u} \in C(\Omega) \) such that \( u_\varepsilon \rightarrow u \) locally uniformly in \( \Omega \) and in fact \( u \) is locally Hölder continuous. By using Proposition 3.2 we therefore obtain that \( u \in C(\overline{\Omega}) \) is a viscosity solution (thus unique in view of Corollary 2.8) of (3.1). We obtain the following.

**Proposition 4.3.** In the assumptions of Proposition 3.2, suppose moreover that \( F \) is the subelliptic operator (4.4). Then problem (3.1) has a unique continuous viscosity solution which is the uniform limit of the vanishing viscosity approximations.

We can use Proposition 3.2 in an alternative way if we know in advance by other methods that (3.1) has a unique continuous viscosity solution. For instance by applying Corollary 2.6 and 2.8 as we will do below for first order equations. Then we can apply Proposition 3.2 by choosing \( \overline{u} = \underline{u} = u \). The conclusion is that \( v = w \), that is the following holds.

**Proposition 4.4.** In the assumptions of Proposition 3.2, suppose moreover that problem (3.1) has a (unique) viscosity solution \( u \in C(\overline{\Omega}) \). Then \( u \) is the uniform limit of the family of vanishing viscosity approximations \( \{u_\varepsilon\} \).

We discuss now the case of first order convex equations of the form

\[
F(x, r, p, X) = r + \max_{a \in A} \{-c(x, a) \cdot p - h(x, a)\},
\]

where we assume throughout the rest of this section that \( A \) is a compact subset of a metric space, \( c : \overline{\Omega} \times A \to \mathbb{R}^N \) is continuous and \( c(\cdot, a) \) is Lipschitz continuous, uniformly in \( a \), \( h : \overline{\Omega} \times A \to \mathbb{R} \) is continuous and \( h(\cdot, a) \) is uniformly continuous with a uniform modulus of continuity with respect to \( a \). We also suppose that \( f, h \) are bounded. Thus our equation is the Bellman equation of an optimal control problem for the dynamical system

\[
\dot{y} = c(y, a), \quad y(0) = x,
\]

where \( a : [0, +\infty] \to A \) is measurable. The control problem has as payoff functional

\[
J(x, a) = \int_0^{\tau_x(a)} e^{-t}(h(y, a) + f(y)) \, dt + e^{-\tau_x(a)} g(y(\tau_x(a))) \rightarrow \min,
\]

where \( g : \partial \Omega \to \mathbb{R} \) is continuous and \( \tau_x(a) = \inf\{t \geq 0 : y(t) \notin \Omega\} \). We will indicate as \( y(\cdot, a) \) the trajectory of the dynamical system.

If we consider the Dirichlet-type problem

\[
\begin{cases}
  u(x) + \max_{a \in A} \{-c(x, a) \cdot Du(x) - h(x, a)\} = f(x), & x \in \Omega; \\
  u(x) = g(x) \text{ or } u(x) + \max_{a \in A} \{-c(x, a) \cdot Du(x) - h(x, a)\} = f(x), & x \in \partial \Omega,
\end{cases}
\]

then a standard application of viscosity solutions theory to optimal control, see [2], produces that the value functions

\[
V_M(x) = \inf_{a(\cdot)} \int_0^{\tau_x(a)} e^{-t}(h(y, a) + f^*(y)) \, dt + e^{\tau_x(a)} g(y(\tau_x(a))),
\]

\[
V_m(x) = \inf_{a^*(\cdot)} \int_0^{\tau_x(a^*)} e^{-t}(h^*(y, a^*) + f_*(y)) \, dt + e^{\tau_x(a^*)} g(y(\tau_x(a^*)))
\]
are both discontinuous solutions of such a boundary value problem, see also [33].
In the case of function \( V_n \), instead of measurable \( a(\cdot) \), we are using the so called
relaxed controls \( a^r : [0, +\infty] \to A^r \), \( a^r \in L^\infty \), where \( A^r \) is the set of Radon pro-
bability measures on \( A \) and \( h^r(x, \rho) = \int_A h(x, a)d\rho(a) \). In the considered assumptions
these two functions are respectively upper semicontinuous and maximal, lower semi-
continuous and minimal viscosity solutions of the boundary value problem, see e.g.
[33]. By using the comparison theorems of Section 2, we want to give new sufficient
conditions for existence of a unique continuous viscosity solution. Such conditions,
based on Corollary 2.6, widely extend our previous results in [33]. The first state-
ment is the following, where to simplify the arguments we will suppose that \( \Gamma \) is a
smooth hypersurface.

Proposition 4.5. Let us suppose that the assumption (DC) holds and that \( \Gamma \) is a
smooth hypersurface. At \( \hat{x} \in \Gamma \cap \Omega \) let \( n(\hat{x}) \) be the unit inward normal to \( \Omega^+ \)
and let us suppose that:

there is a sequence of control functions \( a_n(\cdot) \), positive numbers \( t_n \to 0^+ \),
\( k > 0 \) and vector \( v \in \mathbb{R}^N \) such that \( v \cdot n(\hat{x}) < 0 \) and
\( x_n = y(t_n, a_n) = \hat{x} + (t_n)^k v + o((t_n)^k) \), as \( n \to +\infty \).

Then the value function \( V_n \) is continuous at \( \hat{x} \) in the direction of \( v \) (thus from \( \Omega^- \)).

Proof. By the Dynamic Programming Principle, see [2], for \( n \) sufficiently large, we may write

\[
V_n(\hat{x}) \leq \int_0^{t_n} e^{-t}(h(y(t, a_n), a_n) + f^*(y(t, a_n))) \, dt + e^{-t_n} V_n(y(t_n, a_n))
\]
\[
\leq M t_n + e^{-t_n} V_n(x_n),
\]
where \( M \) is a bound for \( |h| \) and \( |f| \). As \( n \to +\infty \) we obtain

\[
V_n(\hat{x}) \leq \liminf_{n \to +\infty} V_n(x_n) \leq \limsup_{n \to +\infty} V_n(x_n) \leq V_n(\hat{x}),
\]

since \( V_n \) is upper semicontinuous.

The conclusion comes by construction of the sequence \( x_n \). \( \square \)

To discuss assumption (4.5), we notice that if at \( \hat{x} \) we can find a control \( \hat{a} \) such
that \( f(\hat{x}, \hat{a}) \cdot n(\hat{x}) < 0 \), then by choosing \( v = f(\hat{x}, \hat{a}) \), \( a_n(t) \equiv \hat{a} \), \( t_n = \frac{1}{n} \) we obtain

\[
x_n = \hat{x} + \frac{1}{n} v + o\left(\frac{1}{n}\right).
\]

Thus (4.5) is satisfied with \( k = 1 \).

To obtain a version of (4.5) with larger exponent \( k \), one can use the Lie algebra
generated by the vector field \( c \). Let us suppose for instance that \( c \) is symmetric,
i.e. \( A \subset \mathbb{R}^K \) is symmetric and \( c(x, -a) = -c(x, a) \) for all \( x, a \) and that there are
two controls \( a_1, a_2 \) such that the Lie bracket \( [c(\cdot, a_1), c(\cdot, a_2)](\cdot) \cdot n(\hat{x}) < 0 \). Then
by setting \( v = [c(\cdot, a_1), c(\cdot, a_2)](\hat{x}) \), \( k = 2 \), we can find a sequence of controls \( a_n(\cdot) \)
such that (4.5) is satisfied. This is well known by setting \( a_n(t) = a_1 \) for \( t \in [0, \frac{1}{n}] \),
\( a_n(t) = a_2 \) for \( t \in [\frac{1}{n}, \frac{2}{n}] \), \( a_n(t) = -a_1 \) for \( t \in [\frac{2}{n}, \frac{3}{n}] \), \( a_n(t) = -a_2 \) for \( t \in [\frac{3}{n}, \frac{4}{n}] \).

In general, assumption (4.5) is exactly what one can obtain by choosing as \( v \) as
above an appropriate Lie bracket of order \( k \). For this fact the reader can consult the
paper by Haynes-Hermes [14], or for a more up to date discussion and refinements
the work by Rampazzo-Sussmann [29]. We now prove the following statement.
**Proposition 4.6.** Let us suppose that the assumption (DC) holds and that $\Gamma$ is a smooth hypersurface. At $\dot{x} \in \Gamma \cap \Omega$ let $n(\dot{x})$ be the unit inward vector to $\Omega^+$ and let us suppose that:

there is an optimal (relaxed) control $\hat{a}^r(\cdot)$, for $V_m(\hat{x})$,

positive numbers $t_n \to 0^+$, and vector $v \in \mathbb{R}^N$ such that $v \cdot n(\dot{x}) > 0$ and (3.6)

$x_n = y(t_n, \hat{a}^r) = \dot{x} + t_nv + o(t_n), \quad as \ n \to +\infty.$

Then the value function $V_m$ is continuous at $\hat{x}$ in the direction of $v$ (thus from $\Omega^+$).

**Proof.** We use again the Dynamic Programming Principle for $V_m$ and deduce that, for $n$ sufficiently large

$$V_m(\hat{x}) = \int_0^{t_n} e^{-t}(h(y(t, \hat{a}^r), \hat{a}^r) + f_s(y(t, \hat{a}^r))) \, dt + e^{-t_n}V_m(x_n) \geq -Mt_n + V_m(x_n),$$

where $M$ is a bound for $h$ and $f$. As $n \to +\infty$ we obtain

$$V_m(\hat{x}) \geq \limsup_{n \to +\infty} V_m(x_n) \geq \liminf_{n \to +\infty} V_m(x_n) \geq V_m(\hat{x}),$$

since $V_m$ is lower semicontinuous.  

To comment on the assumptions of Proposition 3.6, observe that

$$\frac{x_n - \hat{x}}{t_n} = \frac{1}{t_n} \int_0^{t_n} c(y, \hat{a}^r) \, dt = \frac{1}{t_n} \int_0^{t_n} c(\hat{x}, \hat{a}^r) \, dt + o(1) = v_n + o(1),$$

as $n \to +\infty$, where $v_n \in \text{co } c(\hat{x}, A)$. This implies that $v_n \to v \in \text{co } c(\hat{x}, A)$

and therefore a necessary condition for (4.6) is that there exists $\hat{a} \in A$ such that $c(\hat{x}, \hat{a}) \cdot n(\hat{x}) > 0$. A clearly sufficient condition is instead the existence of $1 > c, d > 0$

such that

$$c(\hat{x}, A) \subset \{ x : |x - \hat{x} - \varepsilon n(\hat{x})| < c\varepsilon, \quad d \leq \varepsilon < c \}.$$

We conclude this example with the following existence Theorem.

**Theorem 4.7.** Let us suppose that the assumption (DC) holds. Assume that the Dirichlet boundary value problem

$$\begin{cases}
  u(x) + \max_{a \in A}\{-c(x, a) \cdot Du(x) - h(x, a)\} = f(x), & x \in \Omega; \\
  u(x) = g(x), & x \in \partial\Omega
\end{cases}$$

(3.7)

has continuous super and subsolutions, respectively $\underline{u}, \overline{u} : \Omega \to \mathbb{R}$ such that $\underline{u}(x) = u(x) = g(x)$ for $x \in \partial\Omega$. If at any $\dot{x} \in (\Gamma \cap \Omega)$ either (4.5) or (4.6) is satisfied, then (4.7) has a unique continuous solution (either $V_M$ or $V_m$) and it is the uniform limit of vanishing viscosity approximations.

**Proof.** It is known that $V_M, V_m$ are discontinuous solutions of the Dirichlet type problem version of (4.7), see [33]. We first use Theorem 2.7 to compare $\underline{u}$ with $(V_M)_c$ and $V_M$ with $\overline{u}$. This gives $\underline{u} \leq V_M \leq \overline{u}$, thus $V_M$ continuously extends to the points of $\partial\Omega$ taking up the boundary condition. Similarly this is also the case for $V_m$. At this point the other assumption allows us to use Theorem 2.3 and compare the subsolution $V_M$ with the supersolution $V_m$ which turn out to be equal,
and thus continuous solutions of (4.7). Finally Corollary 2.7 gives uniqueness in the class of discontinuous solutions and Proposition 3.4 the limit of vanishing viscosity approximations. □

The existence of a continuous subsolution of the Dirichlet problem (4.7), which is of course necessary to solve the problem, is a standard assumption and it is called compatibility of the boundary datum. The existence of a continuous supersolution is also well understood and it can be obtained for instance, when \( \partial \Omega \) is a smooth hypersurface, with the request that at each point \( x \in \partial \Omega \) there is an outward vector field, that is a control \( \hat{a} \in A \) such that \( n_{\Omega}(x) \cdot c(x, \hat{a}) > 0 \), where \( n_{\Omega} \) is the outward normal to \( \Omega \) at \( x \), or a similar condition on a Lie bracket. For this standard fact we refer the reader for instance to the book \([2]\). Here we prefer to concentrate on the new issues raised by the presence of the discontinuous coefficient.

5. Extensions to More General Hamiltonians. In this section we want to extend the comparison principles to a more general class of equations, namely

\[
F(x, u(x), Du(x), D^2u(x)) = f(x)H(x, Du(x)).
\]  

(5.1)

The statement requires more regularity on the discontinuous coefficient \( f \) and, according to that, it is divided into cases.

**Theorem 5.1.** Let \( \Omega \) be a bounded domain with Lipschitz boundary. Assume that the Hamiltonian \( F \) is proper and satisfies condition (2.2). Let us suppose that the assumption (DC) is satisfied by \( f \) and that \( H : \overline{\Omega} \times \mathbb{R}^N \to [0, +\infty] \) is continuous and satisfies assumptions (2.1), (2.2). Moreover we suppose that \( H(x, \cdot) \) is positively 1-homogeneous for all \( x \in \overline{\Omega} \). We also assume that \( f \) is \( \alpha \)-Hölder continuous in \( \Omega^+ \) and \( \Omega^- \) (following the notation of (DC)). Let \( u, v : \overline{\Omega} \to \mathbb{R} \) be respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution of

\[
F(x, u, Du, D^2u) = f(x)H(x, Du(x)), \quad \text{in } \Omega.
\]

Suppose moreover either one of the following:
1. \( \alpha = 1 \) and \( u, v \) satisfy boundary conditions and regularity on \( \Gamma \) as in the statements of Theorem 2.3 or 2.6;
2. \( \alpha \in (\frac{1}{2}, 1) \), \( u \) is \( \beta \)-Hölder continuous with

\[
\beta > 2(1 - \alpha)
\]

and \( v \) satisfies the Dirichlet type boundary condition in the viscosity sense

\[
v \geq u \text{ or } F(x, v, Dv, D^2v) \geq f_*(x)H(x, Du(x)), \quad \text{on } \partial \Omega.
\]

Then \( u \leq v \) in \( \overline{\Omega} \).

**Remark 5.2.** In case 2 of the previous statement, the roles of subsolution and supersolution can be exchanged as far as the required regularity is concerned. A slightly better looking assumption can be made if \( \beta = 1 \): in this case we do not need the homogeneity of \( H \) and we can allow any \( \alpha > 0 \). We will leave the straightforward adaptations to the reader.
Sketch of the proof. The proof goes along as before in Theorem 2.3. The only differences can be found in step 4 when we need to estimate, instead of (3.17), the term

\[ f^*(T^\varepsilon z_{\varepsilon, \rho}) - f_*(T^\varepsilon z_{\varepsilon, \rho}) \]

We proceed by estimating the two terms in the right hand side separately. The estimate of the second term in (5.3) goes as follows, using (2.1), (2.2) and the last equation in (3.2)

\[
\begin{align*}
  &f_*(T^\varepsilon z_{\varepsilon, \rho})[H(T^\varepsilon z_{\varepsilon, \rho}, \frac{T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta u}{\varepsilon^2}) - H(T^\varepsilon z_{\varepsilon, \rho}, \frac{z_{\varepsilon, \rho} - T^\varepsilon z_{\varepsilon, \rho} + \varepsilon \eta u}{\varepsilon^2})] \\
  \leq & ||f||_{\infty}[H(z_{\varepsilon, \rho}, \frac{T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta u}{\varepsilon^2}) - H(z_{\varepsilon, \rho}, \frac{z_{\varepsilon, \rho} - T^\varepsilon z_{\varepsilon, \rho} + \varepsilon \eta u}{\varepsilon^2})] \\
  &+ \omega(|T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho}|) \left\{ |\frac{T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta u}{\varepsilon^2}| \right\} \\
  &+ \omega(|T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho}|) \left\{ |\frac{z_{\varepsilon, \rho} - T^\varepsilon z_{\varepsilon, \rho} + \varepsilon \eta u}{\varepsilon^2}| \right\} \\
  \leq & ||f||_{\infty}[\omega(r(z_{\varepsilon, \rho} - \hat{x}) + o_\rho(1)) + \omega(|T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho}|) + |\frac{T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta u}{\varepsilon^2}|] \\
  &+ \omega(|T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho}|) + |\frac{z_{\varepsilon, \rho} - T^\varepsilon z_{\varepsilon, \rho} + \varepsilon \eta u}{\varepsilon^2}|. \\
\end{align*}
\]

(5.4)

where \( o_\rho(1) \to 0 \) as \( \rho \to 0^+ \). At this point, the right hand side of (5.4) goes to zero when we take the \( \text{lim sup} \) as \( \rho \to 0^+ \) first and \( \varepsilon \to 0^+ \) next.

To estimate the first term in (5.3) we need the assumptions. Following the discussion in step 4 of the proof of Theorem 2.3, we may suppose for instance that always \( z_{\varepsilon, \rho} \in \Omega^- \), thus \( T^\varepsilon z_{\varepsilon, \rho} \in \Omega^- \) along appropriate subsequences, the other case being similar. Having established this fact, we either find appropriate subsequences such that \( T^\varepsilon z_{\varepsilon, \rho} \in \Omega^+ \), or we may suppose that always \( T^\varepsilon z_{\varepsilon, \rho} \in \Omega^- \cap \Gamma. \) In the former case we have that

\[ f^*(T^\varepsilon z_{\varepsilon, \rho}) - f_*(T^\varepsilon z_{\varepsilon, \rho}) = f(T^\varepsilon z_{\varepsilon, \rho}) - f(T^\varepsilon z_{\varepsilon, \rho}) < 0 \]

for \( \varepsilon, \rho \) sufficiently small by assumption (DC), and then the first term in the right hand side of (5.3) is also negative. In the latter case by the assumption on Hölder continuity of \( f \) we can estimate

\[
\begin{align*}
  &|f^*(T^\varepsilon z_{\varepsilon, \rho}) - f_*(T^\varepsilon z_{\varepsilon, \rho})|H(T^\varepsilon z_{\varepsilon, \rho}, \frac{T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta u}{\varepsilon^2}) \\
  \leq & C_f |T^\varepsilon z_{\varepsilon, \rho} - T^\varepsilon z_{\varepsilon, \rho}|^\alpha H(T^\varepsilon z_{\varepsilon, \rho}, \frac{T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta u}{\varepsilon^2}) \\
  = & C_f H(T^\varepsilon z_{\varepsilon, \rho}, |T^\varepsilon z_{\varepsilon, \rho} - T^\varepsilon z_{\varepsilon, \rho}|^\alpha T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta u). \\
\end{align*}
\]
We succeed if we can show that $|T^\varepsilon z_{\varepsilon, \rho} - T_{\varepsilon z_{\varepsilon, \rho}}|^\alpha T^{\varepsilon z_{\varepsilon, \rho} - \varepsilon \eta_\varepsilon} \to 0$, as we let $\rho \to 0$ and then $\varepsilon \to 0$.

In case 1 with $\alpha = 1$ we know that $|T^\varepsilon z_{\varepsilon, \rho} - T_{\varepsilon z_{\varepsilon, \rho}}|/\varepsilon$ is bounded by (3.12) and that $\frac{T^\varepsilon z_{\varepsilon, \rho} - T_{\varepsilon z_{\varepsilon, \rho}}}{\varepsilon} = o(1)$ by (3.10) so we conclude.

In case 2 from one hand we have, by the last equation in (3.2)

$$\frac{|T^\varepsilon z_{\varepsilon, \rho} - T_{\varepsilon z_{\varepsilon, \rho}}|}{\varepsilon} \leq \frac{1}{\varepsilon} \sqrt{\frac{2C_u T^{\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho}}}{\varepsilon}} \leq \frac{1}{\varepsilon} \frac{\sqrt{2C_u}}{\varepsilon} \left( \frac{T^\varepsilon z_{\varepsilon, \rho} - z_{\varepsilon, \rho} - \varepsilon \eta_\varepsilon}{\varepsilon} \right)^\frac{1}{2} + 1,$$

where $C_u$ is the Hölder constant of $u$. Thus the right hand side is of the order $O(\varepsilon^{\frac{1}{2}})$. We win as soon as $\alpha > 1 - \frac{1}{2}$. □

**Appendix.** We collect here standard basic facts that we used in the paper, and some auxiliary results to compare viscosity solutions with other notions in the literature. We start with some known properties of semiconvex (or semiconcave) functions, that are helpful when dealing with nonlinear convolution. The first proposition exhibits a key property of semiconvex regularization to be used in the study of second order pdes. Initially due to Jensen [18], a simple proof of this fact can be found in [12].

**Proposition A.1.** Let $w : \Omega \to \mathbb{R}$ be upper semicontinuous, $-w$ satisfying (2.3). Then the function $w^\varepsilon$ is $\frac{1}{\varepsilon^2}$ semiconvex, hence, if at $x$ the function $w^\varepsilon$ is twice differentiable, then

$$D^2 w^\varepsilon(x) \geq \frac{1}{\varepsilon^2} I.$$

If moreover $\hat{x} \in \mathbb{R}^N$ and $(p, X) \in D^{2+} w^\varepsilon(\hat{x})$, then for every $n \times n$ real matrix $T$ we have

$$(p, \frac{1}{\varepsilon^2}(I - T^t)(I - T) + T^t X T) \in D^{2+} w(x),$$

where $x = \hat{x} - \varepsilon \eta + \varepsilon^2 p$ is the unique point at which $w^\varepsilon(\hat{x}) = w(x) - \frac{1}{2} |x - \hat{x}|^2$. If, in particular, $X > -\frac{1}{2} I$, then by choosing $T = (I + \varepsilon^2 X)^{-1}$ we get

$$(p, (I + \varepsilon^2 X)^{-1} X) \in D^{2+} w(x).$$

The next result is a classical regularity result, for its proof see also [11].

**Theorem A.2.** (Aleksandrov [1]) Let $w : \mathbb{R}^N \to \mathbb{R}$ be a convex (semiconvex) function. Then $w$ is twice differentiable almost everywhere.

The last result describes a perturbation property of local maxima of semiconvex functions.
Lemma A.3. (Jensen [18]) Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be semiconvex and let $\hat{x}$ be a strict local maximum point of $\varphi$. Then $\varphi$ satisfies the following upper perturbation property: for $r, k > 0$ the set

$$
\Gamma^+(\varphi, \hat{x}) = \{ x \in B(\hat{x}, r) : x \text{ is a local max for } \varphi(\cdot) - < p, \cdot > \text{ for some } |p| \leq k \}
$$

has positive measure.

We next relate the notion of viscosity solution with other notions in the literature for (1.1). We say that a function $u \in C^{1,1}(\Omega)$ is a strong solution of

$$
F(x, u, Du, D^2u) = f(x), \quad (A.1)
$$

if it satisfies the equation almost everywhere.

We also say that an upper semicontinuous function $u : \Omega \to \mathbb{R}$ is an $L^\infty$-viscosity subsolution of (A.1) (see [7]) if for any test function $\varphi \in C^{1,1}(\Omega)$ and a local maximum point $x_\sigma$ of $u - \varphi$, $(u - \varphi)(x_\sigma) = 0$, we obtain

$$
ess \liminf_{x \to x_\sigma} \left[ F(x, \varphi(x), D\varphi(x), D^2\varphi(x)) - f(x) \right] \leq 0,
$$

where we are using the essential lim inf in the sense of Lebesgue measure. There are some immediate facts. It is clear that when $F$ is continuous, an $L^\infty$-viscosity subsolution of (A.1) is also a viscosity subsolution in the sense of Section 2, since the former uses a bigger family of test functions and does not take into account sets of measure zero when constructing the envelopes of the discontinuous term $f$, i.e.

$$
\ess \limsup_{x \to x_\sigma} f(x) \geq f^*(x_\sigma).
$$

Moreover if $u \in C^{1,1}$ is a viscosity subsolution of (A.1), and the jump set $\Gamma$ of $f$ has measure zero, then $u$ is also a strong solution. If $\Gamma$ has positive measure, the previous property may not be necessarily true. The following result completes the picture.

Proposition A.4. If $F$ is proper, any strong subsolution of (A.1) is an $L^\infty$-viscosity subsolution, thus also a viscosity subsolution.

Proof. Suppose not. Then we may find $\varphi \in C^{1,1}(\Omega)$ and a strict local maximum point $x_\sigma$ of $u - \varphi$, $(u - \varphi)(x_\sigma) = 0$ such that

$$
ess \liminf_{x \to x_\sigma} \left[ F(x, \varphi(x), D\varphi(x), D^2\varphi(x)) - f(x) \right] > 0.
$$

Thus we may find $\varepsilon > 0$ such that

$$
F(x, \varphi(x), D\varphi(x), D^2\varphi(x)) \geq f(x) + \varepsilon, \quad \text{a.e. in } B(x_\sigma, \varepsilon) \subset \Omega. \quad (A.2)
$$

As $u - \varphi$ is locally semiconvex, we may apply Lemma A.3 and for all $\sigma > 0$ we find out that the set

$$
\Gamma^+(u - \varphi, x_\sigma)
$$

$$
= \{ x \in B(x_\sigma, \varepsilon) : u(\cdot) - \varphi(\cdot) - < p, \cdot > \text{ has a local max at } x \text{ for some } |p| \leq \sigma \}.
$$
has positive measure. We now pick any point \( x \in \Gamma^+(u - \varphi, x_o) \) where both \( u, \varphi \) are twice differentiable and the equation is satisfied pointwise by \( u \). Then we have that \( Du(x) = D\varphi(x) + p \), for some \( p, |p| \leq \sigma \), \( D^2u(x) \leq D^2\varphi(x) \), and by degenerate ellipticity

\[
F(x, u(x), D\varphi(x), D^2\varphi(x)) - \omega(\sigma) \leq F(x, u(x), D\varphi(x) + p, D^2\varphi(x)) \leq f(x)
\]
on a positive measure set of \( B(x_o, \varepsilon) \). This contradicts (A.2) by taking \( \sigma \) small enough. \( \square \)

**Remark A.5.** It is very important to notice that the definition of strong solution and the result of Proposition A.4 can be extended to functions in \( W^{2,p}(\Omega) \) for \( p > N \) because this is a class of continuous, almost everywhere twice differentiable functions which satisfy the upper and lower perturbation properties, similar to Lemma A.3. These are however deep facts, see [7] and the references therein.

Let us suppose that the discontinuous term \( f \) is piecewise continuous across a smooth hypersurface \( \Gamma \) in the sense of assumption (DC), that \( u \) is a strong solution of (A.1) and it is piecewise \( W^{2,1}(\Omega) \) in the sense of assumption (DC), that \( u \in C^1(\Omega) \cup C^2(\Omega^+ \cup \Gamma) \cup C^2(\Omega^- \cup \Gamma) \), where the notation \( \Omega^\pm \) comes from assumption (DC). We start observing the following.

**Lemma A.6.** Let \( u \) be piecewise \( C^2 \) across a hypersurface \( \Gamma \), then at \( x_o \in \Gamma \cap \Omega \), denoted by \( D^2u(x_o^+) \), \( D^2u(x_o^-) \) the Hessians of \( u \) from the two sides of the hypersurface, they are comparable in the order of symmetric matrices.

**Proof.** Let \( v \in T_{x_o} \Gamma \) the tangent space, and \( \gamma : (-1, 1) \to \Gamma \) be a smooth curve with \( \gamma(0) = x_o, \gamma'(0) = v \). Then

\[
\frac{d}{dt}Du(\gamma(t))|_{t=0} = v \cdot D^2u(x_o^+) = v \cdot D^2u(x_o^-).
\]

Thus if we choose as a base for \( \mathbb{R}^N \) the set \( \{v_1, \ldots, v_{n-1}, n(x_o)\} \) where the first \( n-1 \) vectors are a base of \( T_{x_o} \gamma \) and \( n(x_o) \) is a normal unit vector of \( \Gamma \) at \( x_o \), then the only possible unequal element of the two matrices \( D^2u(x_o^+), D^2u(x_o^-) \) is the element of place \( n \times n \), i.e. \( n(x_o) \cdot D^2u(x_o^+)n(x_o), \) or \( n(x_o) \cdot D^2u(x_o^-)n(x_o) \), respectively. Thus the two matrices are comparable. \( \square \)

Piecewise \( C^2 \) strong solutions satisfy the equation in a much stronger sense.

**Proposition A.7.** Let \( f \) be piecewise continuous across a smooth hypersurface \( \Gamma \) and let \( u \) be a strong solution which is piecewise \( C^2 \) across \( \Gamma \). Then \( u \) satisfies

\[
F(x, u(x), Du(x), D^2u(x)) \leq f_+(x),
\]

\[
F(x, u(x), Du(x), D^2u(x)) \geq f_-(x),
\]
in the viscosity sense.

**Proof.** Since \( u \) is a strong solution and it is piecewise \( C^2 \), then at \( x_o \in \Gamma \), by taking limits from the sides we get

\[
F(x_o, u(x_o), Du(x_o), D^2u(x_o^+)) = f_+(x),
\]

\[
F(x_o, u(x_o), Du(x_o), D^2u(x_o^-)) = f_-(x).
\]
By Lemma A.6, and by ellipticity of the operator $F$, we then get
\[ D^2 u(x_o^+) < D^2 u(x_o^-). \]
Then it is clear that super and superjets of $u$ are given by
\[ D^2 u(x_o^+) = \{(Du(x_o), X) : X \geq D^2 u(x_o^-)\}, \]
\[ D^2 u(x_o^-) = \{(Du(x_o), Y) : Y \leq D^2 u(x_o^+)\}. \]

from which the conclusion by ellipticity of the operator $F$. □

The situation of Proposition A.7 is that, for instance, of uniformly elliptic piecewise continuous equations as in Kutev-Lions [25]. Our framework is however simplified with respect to that paper. Notice that from the proof of Proposition A.7 it follows that, if at $x_o \in \Gamma$, $F$ is degenerate in the direction of $n(x_o)$, namely
\[ F(x_o, u(x_o), Du(x_o), X + cn(x_o) \otimes n(x_o)) = F(x_o, u(x_o), Du(x_o), X), \]
for all $X \in S^N$, $c \in \mathbb{R}$, then the solution of equation (A.1) cannot be piecewise $C^2$ across $\Gamma$.

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