Aiming to explore physical limits of wind turbines, we develop a model for determining the work extractable from a compressible fluid flow. The model employs conservation of mass, energy and entropy and leads to a universal bound for the efficiency of the work extractable from kinetic energy. The bound is reached for a sufficiently slow, weakly-forced quasi-one-dimensional, dissipationless flow. In several respects the bound is similar to the Carnot limit for the efficiency of heat-engines. More generally, we show that the maximum work-extraction demands a contribution from the enthalpy, and is reached for sonic output velocities and strong forcing.

How much work can be extracted from the kinetic energy of a fluid flow? The question is old [1–3], but it is still of obvious practical importance for wind energy usage [4]; e.g. it is relevant for shaping renewable energy policies [5]. Wind turbines cannot extract the whole kinetic energy, otherwise the flow will stall. The question is of fundamental importance, since it asks about the operational meaning of energy stored in a continuous medium.

No satisfactory answer to the above question is known. A popular model developed by Betz [2] (and independently by Lanchester [1] and Joukowsky [3]) studies a quantity $\zeta_B$, which is smaller than the efficiency of work extracted from kinetic energy and proposes for it an upper bound by Lanchester [1] and Joukowsky [3]) studies a quantity $\zeta_B$, which is smaller than the efficiency of work extracted from kinetic energy and proposes for it an upper bound $\zeta_B \leq \frac{16}{27}$; see [4–8] for reviews. Betz’s model makes an unwarranted assumption about the pressure distribution [9, 10]. The proper efficiency in the model is attained for a weakly forced, subsonic, quasi-1d flow, where the fluid undergoes a cyclic process: its density and pressure after action of the force are equal to their initial values. This resembles Carnot’s bound for heat engines that is also reached for cyclic, slow and dissipationless processes [14]. We also determine the maximal work extracted from flow, without demanding that it necessarily comes from the kinetic energy. The maximum is reached for sonic output velocities and strong forcing. In this regime the work comes from enthalpy and can relate to increasing kinetic energy.

The model. The filled domain in Fig. 1 shows the stationary flow model. Here are our assumptions about it.

1. The fluid is dissipationless and compressible.
2. The work-extracting part of the turbine is modeled by a stationary space-dependent force $\vec{F}(x)$, which is zero out of a finite domain $\Omega$; see Fig. 1.
3. Homogeneous input flow: at the input $\vec{r}_1 \equiv (x_1, y, z)$, which is far from $\Omega$ (to the left in Fig. 1) the pressure $p$, velocity $\vec{v}$ and density $\rho$ do not depend on $(y, z)$, and transverse velocities are absent:

   $\vec{v}(\vec{r}_1) = (v_1, 0, 0), \quad p(\vec{r}_1) = p_1, \quad \rho(\vec{r}_1) = \rho_1$. \hspace{1cm} (1)

The control volume $B$ in Fig. 1 is defined along the flow lines via 2 conditions: $(i)$ it can be used to calculate the total work (volume integral) $\int_{\Omega} dV (\vec{v} \cdot \vec{F}) = \int_B dV (\vec{v} \cdot \vec{F})$ done by $\vec{F}$. $(ii)$ The area $a(x_1)$ of the input surface $A_1 = A(x_1)$ is possibly small, as needed for ensuring assumption 5 below, and for calculating the efficiency; see (15, 34) below. Hence $B$ encircles $\Omega$; cf. Fig. 1.
4. The cross-section $A(x)$ of $B$ grows with $x$ from input $A(x_1)$ to output $A(x_2)$. This assumption is needed for achieving work-extraction. The general bounds (19, 35) on the efficiency of work-extraction demand a weaker condition $a(x_2) > a(x_1)$, where $a(x)$ is the area of $A(x)$.

![FIG. 1: The model. The flow (denoted by blue) goes from $x_1$ (input) to $x_2$ (output). $\vec{F}$ is the external force. The control volume $B$ is filled. $A(x)$ (dashed line) is the cross-section. $A_1 = A(x_1)$ and $A_2 = A(x_2)$ are (resp.) input and output surfaces. Red curves bound the domain $\Omega$, where $\vec{F}$ is localized. Arrows denote stationary flow velocities.](image)

5. $v_x$ is constant along the output surface $A_2$:

   $\vec{v}(\vec{r}_2) = (v_2, v_y(\vec{r}_2), v_z(\vec{r}_2)), \quad \vec{r}_2 \equiv (x_2, y, z)$. \hspace{1cm} (2)
At the output $\vec{r}_2$ we apply the following notation
\[ p(x_2, y, z) = p_2 \tilde{p}(y, z), \quad \rho(x_2, y, z) = \rho_2 \tilde{\rho}(y, z), \tag{3} \]
where $\tilde{p}(y, z)$ and $\tilde{\rho}(y, z)$ are defined so as to hold
\[ \langle p \rangle = \int_{A_2} \frac{dy \, dz \, \tilde{p}(y, z)}{a_2} = 1, \quad \langle \tilde{\rho} \rangle = 1. \tag{4} \]
Eq. (2) is a weak form of the plug-flow assumption done in hydraulics and quasi-1d motion [12, 13]; see [15–17] for reviews that explore limits of plug-flows.

6. The fluid is an ideal gas with constant heat-capacities $c_v$ and $c_p$. This implies for the entropy density $s$ and internal energy density $\varepsilon$ [12]:
\[ \frac{s}{c_v} = \ln p - \gamma \ln \rho, \quad \varepsilon = \frac{1}{\gamma - 1} \frac{p}{\rho}, \quad \gamma \equiv \frac{c_p}{c_v} > 1, \tag{5} \]
where the integration constant in $s$ was fixed as in [12]. For air $\gamma = 1.4$ in agreement with the thermodynamic bound $\gamma > 1$ [12]. The local speed of sound reads [12]
\[ v_s^2 = (\partial p/\partial \rho)|_s = \gamma \rho p/\rho. \tag{6} \]
The set-up is a generalization of Betz’s model [1–10], because we do not assume that flow is incompressible, and we do not restrict $\vec{F}$ to be localized in a thin surface. Limitations of the set-up are discussed in §3 of [11].

Conservation laws of mass, entropy and energy read for stationary flow [12] $[\nabla \cdot (\rho \vec{v}) = 0, \quad \nabla \cdot (\rho \tilde{\rho} \vec{s}) = 0, \quad \nabla \cdot (\rho \tilde{\rho} \vec{v}/2 + \rho (\tilde{\rho} + \partial \vec{p}/\partial \rho) \vec{v}) = \vec{v} \cdot \nabla \vec{F}, \tag{7} \]
where $\varepsilon + \frac{\rho \tilde{\rho} v^2}{2}$ is the enthalpy density, and where the external force $\vec{F}$ enters into stationary Euler’s equation as:
\[ \rho \vec{v} \partial \vec{v}/\partial t = \rho (\vec{v} \cdot \nabla) \vec{v} = -\nabla \tilde{p} + \vec{F}. \tag{8} \]
The momentum conservation is not employed, since it is useless without restrictive assumptions; see §1 of [11].

We apply (7, 8) to the control domain $B$ in Fig. 1. Integrate $\nabla \cdot (\rho \vec{v})$ in (7) over the volume $B$ [cf. Fig. 1], and employ Gauss theorem to get 3 integrals over the surface of $B$: $\langle f_{A_1} + f_{A_2} + f_B \rangle d\vec{n} \cdot \tilde{\vec{v}} \rho = 0$, where $d\vec{n}$ points outward. Boundary conditions for a dissipative fluid imply $d\vec{n} \cdot \tilde{\vec{v}} |_{\partial B} = 0$ [12]. Then employ (1–4) in $\langle f_{A_1} + f_{A_2} \rangle d\vec{n} \cdot \tilde{\vec{v}}$. Two other relations in (7, 8) are treated in the same way, also using (5):
\[ a_1 p_1 v_1 = a_2 p_2 v_2, \quad (p_2/p_1) = (\rho_2/\rho_1)^\gamma e^{\sigma}, \tag{10} \]
\[ -\int dV \vec{v} \cdot \vec{F} = \frac{v_1^2 - v_2^2 - \tilde{v}_1^2}{2} + \frac{\gamma}{\gamma - 1} \left( \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} \right), \tag{11} \]
where $\int dV$ goes over volume $B$ (colored blue in Fig. 1), $a_k$ is the area of $A_k$, and where [cf. (2–4)]
\[ \tilde{v}_1^2 = \langle \tilde{\rho} [v_2^2(r_2) + v_2^2(r_2)] \rangle, \tag{12} \]
\[ \gamma \equiv (\tilde{\rho} \ln (\tilde{\rho}/\tilde{\rho})) + (\gamma - 1) (\tilde{\rho} \ln \tilde{\rho}) \geq 0. \tag{13} \]
The LHS of (11) is the extracted work that amounts to the kinetic energy+enthalpy difference between input and output. Here $\tilde{v}_1^2$ is the output transverse velocity contribution including vorticity. Both terms in (13) are non-negative [20] due to spatial inhomogeneities at the final surface $A_2$.

Now $\sigma$ corresponds to an effective entropy production [cf. (4)]. If we include dissipative effects by introducing in (7) a non-zero entropy production $\nabla \cdot (\rho \tilde{\rho} s) = s_{\text{prod}}$, then above formulas will hold upon $\sigma \rightarrow \sigma + \frac{1}{\rho_2 a_2 \rho_1} \int dV s_{\text{prod}}$. Thus even for a dissipationless fluid, the inhomogeneity of the output plays the role of an effective entropy production $\sigma > 0$.

To simplify (10, 11), employ dimensionless parameters:
\[ a_2 = \frac{a_2}{a_1}, \quad \tilde{v}_2 = \frac{v_2}{v_1}, \quad \tilde{\rho}_2 = \frac{\rho_2}{\rho_1}, \quad \tilde{v}_1 = \frac{v_1}{v_1}, \tag{14} \]
\[ M_1^2 = \frac{\rho_1 v_1^2}{\gamma p_1}, \quad \bar{w} = \frac{\int dV \vec{v} \cdot \vec{F}}{a_1 p_1 v_1^2}, \tag{15} \]
where $M_1$ (Mach number) is ratio of the input velocity to the speed of sound (6) at the input, and $\bar{w}$ is the dimensionless work defined as the ratio of the work to the inflow $\frac{1}{2} a_1 p_1 v_1^3$ of kinetic energy. Eqs. (10) lead to $\tilde{v}_2^2 \tilde{\rho}_2^2 = e^\sigma$ to be used together with (14, 15) in (11):
\[ \bar{w} = 1 - \tilde{v}_2^2 - \tilde{v}_1^2 + \frac{2}{M_1^2 (\gamma - 1)} (1 - e^{\sigma} \tilde{v}_2^{1+\gamma} \tilde{v}_1^{1-\gamma}). \tag{16} \]

Our purpose is to extract work, hence to achieve $\bar{w} > 0$.

Work-extraction from kinetic energy. We demand in (16) that the work is extracted from kinetic energy only:
\[ 1 = e^{\sigma} \tilde{v}_2^{1-\gamma} \tilde{v}_1^{1+\gamma}. \tag{17} \]

Due to $\sigma > 0$ and $\gamma > 1$, condition (17) can be achieved for $\nu_2 < 1$ (smaller kinetic energy) only for $a_2 > 1$ [cf. 4]. Using (17) and $\tilde{a}_2 > 1$ we get from (16, 14)
\[ \bar{w} = 1 - \tilde{a}_2^{-2} e^{\frac{2\bar{w}}{\tilde{a}_2}} - \tilde{v}_1^2 \leq 1 - \tilde{a}_2^{-2} = 1 - (a_1/a_2)^2, \tag{18} \]
where in deriving (19) we employed $\sigma \geq 0$, $\gamma > 1$ and $\tilde{v}_1^2 \geq 0$. Once the work is extracted from the kinetic energy only, the latter is the resource and then (18) is also the efficiency, i.e. the result over resource. Two hindrances for reaching (19) from (18) are $\tilde{v}_2^2 > 0$ and $\sigma > 0$.

The bound (19) is attained for $\tilde{v}_2^2 = 0$ (no tangential velocity) and $\sigma = 0$ (no effective entropy production). The latter relation means $\tilde{\rho} = \tilde{\rho} = 1$; cf. (13, 3). §4 of [11] shows that (19) holds for non-ideal gases.

Below we demonstrate that bound (19) is attained for quasi-1d motion, where $\bar{w} = \tilde{v}_1 = 0$ and $\tilde{\rho} = \tilde{\rho} = 1$ take place naturally. Then as (17, 10) show, work-extraction from kinetic energy demands cyclicity:
\[ \rho_1 = \rho_2, \quad p_1 = p_2. \tag{20} \]
Note that only requiring \( \rho_1 = \rho_2 \) we establish the bound (19) from (10) and \( \sigma \geq 0 \). The shape of (19), cyclicity condition (20) and no entropy production \( \sigma = 0 \) (needed for attaining (19)) make analogy between (19) and Carnot’s bound for heat-engines.

Work maximization over the final velocity. The work (16) is formally maximized over \( \bar{v}_2 \) for fixed values of other parameters—via \( \frac{d\bar{w}}{d\bar{v}_2} = 0 \) and \( \frac{d^2\bar{w}}{d\bar{v}_2^2} < 0 \). The second relation holds always, while the first one produces:

\[
\bar{v}_2 = \bar{v}_m = \left( e^{\sigma_1^2 v_1^2 M_1^2} \right)^{1/2} \tag{21}
\]

\[
\bar{w}_m = \bar{w}(\bar{v}_m) = 1 - \bar{v}_m^2 - \bar{v}_m^4 + \frac{2(1 - M_1^2 \bar{v}_m^2)}{(\gamma - 1)M_1^2} \tag{22}
\]

The output velocity that corresponds to \( \bar{v}_m \) equals to the speed of sound, as seen by starting from (6, 10, 14):

\[
\frac{u_s^2(x_2)}{v_s^2} = \frac{1 - (p_{2}/p_1)}{M_1^2 (p_{2}/p_1)} = \frac{e^{\sigma_1^2 v_1^2 M_1^2} - 1}{M_1^2} = \bar{v}_m^2 \tag{23}
\]

and noting that the last equality amounts to (21). The maximal work \( \bar{w}_m \) can be attained, as seen below.

Work-extraction in quasi-1d flow. Eqs. (10, 11) are useful for bounding the work, but they do not determine it, since \( v_2, \sigma, v_r \) are unknown. A more specific and informative approach is needed that allows to address the attainability of bounds. Since the flow (shown in Fig. 1) has a smooth and slowly varying cross-section \( A(x) \), we apply the quasi-1d approach [12, 13]. It assumes a stationary flow with the axial flow velocity \( \bar{v} = (v, 0, 0) \), pressure \( p \), density \( \rho \), and the external force \( \tilde{F} = (F, 0, 0) \) depending only on the axial variable \( x \). Hence transverse velocities and effective entropy production nullify: \( v_r = \sigma = 0 \); cf. (12, 13). I.e. two hindrances for reaching (19) from (18) are absent for the quasi-1d model.

We use scaled functions of \( x \) [cf. (14)]:

\[
\bar{v} = \frac{v}{v_1}, \quad \bar{\rho} = \frac{\rho}{\rho_1}, \quad \bar{p} = \frac{p}{p_1}, \quad \bar{a} = \frac{a}{a_1}, \quad \bar{F} = \frac{F}{p_1} \tag{24}
\]

Conservation laws of mass and entropy [12, 13] are to be taken from volume integrals of (7) [cf. (10, 24)]

\[
\bar{p}(x)\bar{a}(x)\bar{v}(x) = 1, \quad \bar{\rho}(x) = \bar{\rho}(x). \tag{25}
\]

Eqs. (25) go together with the stationary Euler equation (9) written with the 1d assumption [cf. (1, 15)]:

\[
\rho u v' = -p' + F, \quad \gamma M_1^2 \bar{\rho} \bar{v} \bar{v}' = -p' + \bar{F}, \tag{26}
\]

where \( \frac{dX}{dx} \equiv X' \) for any \( X \). Eqs. (26, 25) lead to

\[
\left[ \frac{\gamma M_1^2 \bar{v}^2}{2} + \frac{\bar{\rho} - 1}{\gamma - 1} \right]' = \frac{\bar{F}}{\bar{\rho}}. \tag{27}
\]

The work will be directly calculated from its definition (11, 15) by employing (25, 27):

\[
\bar{w} \gamma M_1^2 \frac{\bar{v}^2}{2} = - \frac{\int_0^{x_2} dV \bar{v} \vec{F}}{\bar{a}_1 p_1 v_1} = - \int_{x_1}^{x_2} dx \bar{a}(x) \bar{v}(x) \bar{F}(x) \tag{28}
\]

\[
= - \int_{x_1}^{x_2} dx \frac{\bar{F}(x)}{\bar{\rho}(x)} = \int_{x_2}^{x_1} dx \left[ \frac{\gamma M_1^2 \bar{v}^2}{2} + \frac{\gamma \bar{\rho} - 1}{\gamma - 1} \right]' \tag{29}
\]

Eq. (29) recovers the general formula (16) with \( \sigma = v_r = 0 \), as a consequence of the quasi-1d approach.

To understand the physics of this problem, let us note that (25) can be written as, respectively,

\[
\frac{\bar{p}'}{\bar{\rho}} + \frac{v'}{\bar{v}} + \frac{a'}{a} = 0, \quad \frac{\gamma \bar{p}}{\bar{\rho}} = \frac{\bar{p}}{\bar{\rho}} \tag{30}
\]

We take the derivative in (27) and work it out in 2 different ways using (30) and \( \bar{\rho}(x) = \bar{\rho}(x) \):

\[
\frac{p'}{p} \left[ 1 - \frac{v^2}{v_s^2} \right] = \frac{F + \gamma v^2 a'}{p v_s^2} \tag{31}
\]

\[
\frac{v'}{v} \left[ \frac{v^2}{v_s^2} - 1 \right] = \frac{a'}{a} + \frac{F}{\gamma p} \tag{32}
\]

where \( v_s = v_s(x) \) is the speed of sound defined in (6). In the subsonic case \( v^2 < v_s^2 \) consider firstly (31, 32) for \( F = 0 \) [12, 13]. Now \( a'(x) > 0 \) implies expected trends: \( p'(x) > 0 \) and \( v'(x) < 0 \). Eqs. (31, 32) show that a \( F > 0 \) can reverse those trends for \( a'(x) < 0 \). This reversing will be seen to be the mechanism of work-extraction.

Fig. 2 exemplifies the first scenario of work extraction, where \( \tilde{F} \) is weak. The velocity \( \bar{v}(x) \) decays with \( x \); its behavior is close to the case \( \tilde{F} = 0 \) in (26). But the density \( \bar{\rho}(x) \) does feel the weak force, since it changes cyclically returning to the initial value once the force ceases to act. We define \( x_2 \) such that \( \bar{\rho}(x_1) = \bar{\rho}(x_2) \); see (20) and Fig. 2. Hence the work is extracted from the kinetic energy only, and the efficiency equals its maximal value (19).

Eqs. (31, 32) explain why the weak force changes qualitatively the behavior of \( \bar{\rho}(x) = \bar{\rho}(x) \), and does not change the behavior of \( \bar{v}(x) \): the geometric factor \( \frac{\bar{\rho}}{\bar{\rho}} \) in (31) is multiplied by a factor \( \frac{\gamma v^2}{v_s^2} \), which is small for the subsonic flow, and which is lacking in (32).

Fig. 2 shows that the change of density \( \bar{\rho}(x) \) is small. Hence we can put \( \bar{\rho}(x) \approx 1 \) in (28, 25) obtaining

\[
\bar{w} \approx - \frac{2 \gamma M_1^2}{\int_{x_1}^{x_2} dx \bar{F}(x)} \tag{33}
\]

For parameters of Fig. 2 both work and efficiency can be maximized simultaneously. But generally there is a conflict between these two maximizations; see §5 of [11].

Maximal work-extraction. Eqs. (21, 22) show that in the quasi-1d case (\( \sigma = v_r = 0 \)) the maximal work-extraction \( \bar{w}_m > 0 \) demands a positive contribution \( x - M_1^2 \bar{v}_m^2 \) from enthalpy due to \( M_1^2 < 1 \) (subsonic
The dimensionless work $\bar{w}$ (black curve) in (21) versus $x$ for $x > 0.45$ reaching the sonic value (21) at the end point $x = x_2 = 0.6714$, where the work attains its maximum (22). Cyclic value (20), where bound (19) is attained: $\bar{p}(x_2) = 0.3547 = 1$.

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Supplementary material.

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Indeed, $\langle \tilde{\rho} \ln \tilde{\rho} \rangle \geq 0$, because it is a relative entropy between two distributions. To show $L \equiv \langle \tilde{\rho} \ln \tilde{\rho} \rangle \geq 0$ from (4), minimize $L$ over $\tilde{\rho}$ as independent functional variable keeping track of (4) via a Lagrange multiplier: $L$ is minimized for $\tilde{\rho} = 1$, and $L = 0$ at the minimum.
SUPPLEMENTARY MATERIAL

Supplementary Material consists of 9 chapters (referred to via §). References to equations and figures of the main text are marked bold. §1 discusses momentum conservation. §2 studies Betz’s model in great detail and explains why specifically it is inapplicable. §3 discusses limitations of the model. §4 shows that the bound (19) applies to non-ideal gases. §5 and §6 discuss the work versus efficiency in two scenarios of work-extraction for (respectively) weak and strong force. §7 explains details of the sonic limit. §8 considers implications of the Bernoulli equation. §9 studies work-extraction in a cylindric tube and makes relation with physics of d’Alembert’s paradox.

1. Conservation laws of momentum

![Diagram of conservation laws](image)

FIG. 4: Schematic representation of the integration domain in (38). B is the control volume and A(x) is its cross-section. Magenta lines divide the output surface A(x) = A2 over A1 and A3: A2 = A1 ∪ A3. h(y,z) relates to a point (y,z) ∈ A1; see (39).

The conservation of momentum reads [1]

$$\nabla \cdot (\rho v \vec{v}) + \partial_t \rho = F_\ell, \quad \ell = x, y, z,$$

where $v_\ell$ and $F_\ell$ are components of (resp.) $\vec{v}$ and $\vec{F}$. Eq. (36) is a combination of the mass conservation and Euler’s equation. Eq. (36) is considered separately from other conservation laws, since within the general approach it does not lead to useful constraints. Let us see why.

Integrating (36) with $\ell = x$ over the volume B, and using the same arguments as before (10), we get

$$\rho_2 a_2 v_2^2 - \rho_1 a_1 v_1^2 + \int dV \partial_x p(x, y, z) = \int dV F_x. \quad (37)$$

We need to treat the volume integral $\int dV \partial_x p(x, y, z)$ from (37); see Fig. 4. Given $(y, z) ∈ A(x_2)$, let $h(y, z)$ be the minimal possible value of $x$; see Fig. 4. Note that $h(y, z) = x_1$, if $(y, z) ∈ A_1 \cap A(x_2)$. Let also $A_3$ be that part of $A_2 = A(x_2)$, which does not project to $A_1$; see Fig. 4. The integral is taken as follows

$$\int dV \partial_x p(x, y, z) = \int dA_{x_2} \int_{h(y,z)}^{x_2} dx \partial_x p(x, y, z)$$

$$= \int dA_{x_2} \int_{h(y,z)}^{x_1} dy \int_{h(y,z)}^{x_2} dz \partial_x p(x, y, z)$$

$$- \int_{A_3} dy \int_{h(y,z)}^{x_1} dz \partial_x p(h(y,z), y, z)$$

$$= a_2 p_2 - a_1 p_1 - \zeta \quad (38)$$

where in (39) we employed (1-4) and denoted

$$\zeta = \int_{A_3} dy \int_{h(y,z)}^{x_1} dz \partial_x p(h(y,z), y, z) > 0. \quad (40)$$

Here $\zeta > 0$ is due to $p > 0$.

We now deduce from (37, 40) and from the mass conservation $\rho_1 v_1 a_1 = \rho_2 v_2 a_2$:

$$v_1 + \frac{p_1}{\rho_1 v_1} + \frac{\zeta + \int dV F_x}{a_1 \rho_1 v_1} = v_2 + \frac{p_2}{\rho_2 v_2}. \quad (41)$$

Eq. (41) is not useful, because it contains an unknown factor $\zeta$. More specifically, note that even the sign of the effective parameter $\zeta + \int dV F_x$ is not fixed, since $\zeta > 0$, while $\int dV F_x < 0$, as needed for work-extraction.

Likewise, we can consider (36) with $\ell = y$:

$$a_2 \rho_2 v_2 \left( \bar{p}(y,z) v_y(x_2, y, z) \right)$$

$$+ \int dV \partial_y p(x, y, z) = \int dV F_y, \quad (42)$$

where $\bar{p}(y,z)$ and $\langle ... \rangle$ are defined in (resp.) (3) and (4).

Here as well, there is an unknown factor $\int dV \partial_y p(x, y, z)$.

2. Betz’s model

Here we study in detail the Betz’s model that was reviewed in literature several times [2–5, 8, 9]. Similar approaches were developed by Lanchester and Joukowsky; see [10] for details.

Our conclusion will be that the model makes irrelevant assumptions and that anyhow its conclusions do not concern the efficiency of work-extraction from kinetic energy. Not all of our negative conclusions are new. The objection P1 (below in §2.3) was in fact formulated in [8] though in a less explicit way. Objection P2 was legitimately raised in [9].

2.1 Assumptions of the model

We now spell out assumptions of the model in great detail and put them in the context of conservation laws.

B0. The flow model is shown in Fig. 5.
Due to (43, 46, 48) the work is extracted from the kinetic energy only, as confirmed below.

**B5.** Assumptions expressed by Eq. (1, 2) on the homogeneous axial velocity \( v_x \) at input and output are naturally done also for Betz’s model. Moreover, it is assumed that the analogue of (1, 2) hold as well at \( A(x_0) \), i.e. altogether we have [cf. Fig. (5)]

\[
\begin{align*}
  v_x(x_1, y, z) &= v_1, \\
  v_x(x_2, y, z) &= v_2, \\
  v_x(x_0, y, z) &= v(x_0).
\end{align*}
\]

Eqs. (43, 49, 50) allow us to write the mass conservation as [cf. (10)]:

\[
a_1 v_1 = a(x_0) v(x_0) = a_2 v_2.
\]

**B6.** We employ (43, 46, 48) in momentum conservation relation (41), and assume additionally that the following relation takes place in (41):

\[
\frac{p}{v_1} + \frac{\zeta}{a_1 v_1} = \frac{p}{v_2},
\]

which via the mass conservation \( a_1 v_1 = a_2 v_2 \) amounts to:

\[
\zeta = p(a_2 - a_1).
\]

Assumption (52) allows to fix the unknown \( \zeta \).

Looking at definition (40) of \( \zeta \), we can replace (53) by an assumption that pressure is constant over the whole surface of the volume \( B \) between \( A(x_1) \) and \( A(x_2) \); cf. (48). Assumption (53) is normally made implicitly [2–5, 8]. It was spelled out explicitly in [9].

### 2.2 Derivation of Betz’s limit

Recall the energy conservation law (8), where in view of incompressibility assumption (43) we should skip the internal energy \( \varepsilon \) [1]:

\[
\nabla \cdot \left[ \frac{\rho \mathbf{v}^2 \mathbf{v}}{2} + p \mathbf{v} \right] = \mathbf{\dot{v}} \cdot \mathbf{\dot{F}}.
\]

Then instead of (11) we get from (48, 51, 54):

\[
- \int \frac{d\mathbf{V}}{a_1 v_1} \cdot \mathbf{\dot{F}} = \frac{v_1^2 - v_2^2}{2},
\]

where the transverse velocity contribution in (55) is already skipped due to (47), and we already employed (48). Eq. (55) makes it clear that the work is extracted from kinetic energy only. Using (44, 51) in (55) we get

\[
\frac{v_1^2}{2} - \frac{v_2^2}{2} = \frac{a(x_0) v(x_0) f}{a_1 v_1 \rho} = \frac{f}{\rho}.
\]
Likewise, (41) leads together with (53), (44) and (43, 48):

\[ v_1 - v_2 = \frac{a(x_0) f}{a_1 v_1 \rho}. \]  

(57)

Eqs. (51, 56, 57) imply Drude’s relation for (50) [13]:

\[ v(x_0) = (v_1 + v_2)/2. \]  

(58)

Using (44, 56, 58) we write for the work:

\[ - \int dV \bar{v} \cdot \bar{F} = f v(x_0) a(x_0) \]

\[ = \rho a(x_0) \frac{(v_1 + v_2)(v_1^2 - v_2^2)}{4} \]

\[ = \frac{v_1^2}{4} \rho a(x_0)(1 + \bar{v}_2)(1 - \bar{v}_2^2), \]  

(59)

where \(\bar{v}_2 \equiv v_2/v_1\). Now consider the ratio

\[ - \int \frac{dV \bar{v} \cdot \bar{F}}{\frac{1}{2} \rho a(x_0) v_1^3} = \frac{1}{2}(1 + \bar{v}_2)(1 - \bar{v}_2^2). \]  

(60)

The RHS of (60) maximizes for

\[ \bar{v}_2 = v_2/v_1 = 1/3, \]  

(61)

leading from (59) to the Betz’s (upper) limit for the ratio in the LHS of (60):

\[ - \int \frac{dV \bar{v} \cdot \bar{F}}{\frac{1}{2} \rho a(x_0) v_1^3} \leq \frac{16}{27}. \]  

(62)

2.3 Problems of Betz’s model

**P1.** In (62), \(16/27\) is now interpreted as an upper limit of the efficiency for the work-extraction from kinetic energy [2–5, 8]. This is not correct. The correct efficiency \(\eta\) of the work extraction from kinetic energy is defined as the work divided over the influx of kinetic energy:

\[ \eta = - \int \frac{dV \bar{v} \cdot \bar{F}}{\frac{1}{2} a_1 \rho v_1^3}. \]  

(63)

We remind that the LHS of (62) cannot be an efficiency, also because it came out of idealization (44). If the force were not artificially localized around \(x_0\) [as (44) does], the surface \(A(x_0)\) would not have any specific meaning [12]. Once the correct quantity (63) is employed the above derivation becomes pointless. Indeed, we return to (59) and note that upon using (51, 58), \(a(x_0) = \frac{2a_1}{v_1 + v_2}\) can also be presented as a function of \(\bar{v}_2\). Hence for the correct efficiency \(\eta\) we get from (59): \(\eta = 1 - \bar{v}_2^2\), whose upper limit is just 1.

One can try to apply (62) to \(\eta\) assuming \(a(x_1) \simeq a(x_0)\). This assumption is untenable, because then the mass conservation law (51) implies \(v(x_1) \simeq v(x_0)\), and then (58) leads to \(v_1 \simeq v_2\).

**P2.** Motivations for assuming (53) are unclear. A rationale for (53) can relate to boundary conditions \(\bar{v} = 0\) on the surface \(B\) of the volume \(B\) (see Fig. 5), which from the Euler equation leads to \(\nabla p|_B = 0\) and hence to \(p(\bar{r}) = \text{const} \in B\). However, \(B\) is defined via flow lines, i.e. the boundary condition \(\bar{v}_B = 0\) does not apply. Moreover, when Betz’s model is presented graphically one notes that the pressure is non-monotonic and depends only on \(x\). Hence making it constant for the whole surface between \(A(x_1)\) and \(A(x_2)\) is an arbitrary assumption.

**P3.** In the main text we implemented some assumptions of the Betz’s model and got different results. In particular, assumptions (46, 47) of the Betz’s model lead to \(\sigma = v_{tr} = 0\). Using these in (17)—which is necessary and sufficient for the work-extraction from kinetic energy—we get

\[ v_2 = v_1 a_1/a_2. \]  

(64)

The work is then extracted from the kinetic energy only and amounts to (18, 19)

\[ - \int dV \bar{v} \cdot \bar{F} = \frac{v_1^2 a_1}{2} (1 - \frac{a_2^2}{a_1^2}). \]  

(65)

It is seen that (64) and (65) are different from (respectively) (61) and (62). In particular, (65)—in contrast to (62)—refers to the efficiency of work-extraction. Hence assumptions of the Betz’s model are not consistent with each other, since we implemented some of them into the conservation laws and got different results.

2.4 Limits on Betz’s efficiency

Let us call the LHS of (62) Betz’s efficiency:

\[ \zeta_B = - \int \frac{dV \bar{v} \cdot \bar{F}}{\frac{1}{2} \rho a(x_0) v_1^3}. \]  

(66)

Since \(a(x_0) \geq a(x_1)\) (\(a(x_1) = a_1\) is the input area), \(\zeta_B\) is smaller than the actual efficiency:

\[ \zeta_B = \frac{a(x_1)}{a(x_0)} \eta \leq \eta. \]  

(67)

Hence when the efficiency \(\eta\) holds the bound (19), \(\zeta_B\) will hold:

\[ \zeta_B \leq \frac{a(x_1)}{a(x_0)} \left(1 - \frac{a_2^2(x_1)}{a_1^2(x_2)}\right). \]  

(68)

2.5 Conclusion on Betz’s model

The model together with its assumptions is problematic for several reasons. But it did motivate the development of the model in the main text.
3. Limitations of the model and open problems

3.1 Remarks on the general structure of the model

The most general approach for solving the model pictured in Fig. 1 is to give the shape of the force $\vec{F}$, provide boundary conditions at the input (which are homogeneous for the present model), and to determine the 3d flow in the full space by solving the (compressible) Euler equation together with continuity equations for mass and entropy. In particular, such a solution will determine the input and output cross-section areas $a_1 = A(x_1)$ and $a_2 = A(x_2)$ of the control volume $B$; see Fig. 1. Recall that the choice of $B$ has to hold the following conditions (see after 1 in the main text): (i) integration over $B$ suffices for calculating the total work done by $\vec{F}$:

$$\int_{\Omega} d^3r (-\vec{v} \cdot \vec{F}) = \int d^3r (-\vec{v} \cdot \vec{F}) = \int_B d^3r (-\vec{v} \cdot \vec{F}).$$ (69)

(ii) The input surface $A_1 = A(x_1)$ is possibly small. This is needed for ensuring that the output surface $A(x_2)$ is small as well (hence $v_z$ can be homogeneous on it, see (2)), and also for maximizing the efficiency of work-extraction, where the work is divided on the area $a_1$ of $A(x_1)$.

The general approach is not practical, since the 3d solution is certainly not available for any sufficiently non-trivial $\vec{F}$. Instead, the main text introduced the control volume $B$, applied to $B$ conservation laws of mass, energy and entropy. This led to upper bounds for the efficiency (19) of work-extraction from kinetic energy, and for the efficiency at the maximal work (extracted from enthalpy); see (35).

These expressions are universal in the sense that they do not depend on details of $\vec{F}$. Eq. (19) also does not depend on the input characteristics of the flow, and on the assumed ideal gas feature of the fluid. Eq. (35) depends on the adiabatic index of the ideal gas and on the initial Mach number of the flow.

Before applying these bounds in practice, one needs to estimate the input and output areas $a(x_1)$ and $a(x_2)$. The situation here is similar to applying the Carnot bound $1 - (T_{\text{cold}}/T_{\text{hot}})$ to realistic heat-engines, e.g. to internal combustion engines. Here $T_{\text{cold}}$ is given as the atmosphere temperature, but $T_{\text{hot}}$ depends on the very functioning of the engine, since this is the temperature that is created by the combustion process. Hence estimating $T_{\text{hot}}$ demands a knowledge of the heat-engine functioning.

3.2 Limitations of concrete assumptions on the flow

– Assumption 2 on a stationary force $\vec{F}(\vec{x})$ is restrictive, because wind turbines have blades that move faster than wind [5], and do not just exert a stationary force on the flow.

– Assumptions 3 and 5 in the main text are limited by turbulence [6, 7], since the turbulence makes the velocity time-dependent and space-dependent (i.e. inhomogeneous). However, after taking time-averages, the homogeneity is frequently recovered [1, 3, 20], and together with it assumptions 4 and 5 are supported.

– Assumption 3 is also limited by vorticity of the input flow. Excluding input vorticity seems legitimate if we want to understand work-extraction from the simplest form of kinetic energy. Clearly, vorticity is a separate resource for work-extraction and should be studied in future for its own sake.

– Assumption 5 is standard for quasi-1d and/or hydraulic flows; see [14–16] for recent expositions and reviews, earlier literature on the subject is reviewed in [17, 18, 20, 21]. (Note that assumption 5 and (2) is not limited to the quasi-1d situation, since it only concerns the longitudinal component of the flow; i.e. tangential components need not nullify; cf. (2).) The assumption is feasible for ideal (dissipationless) fluid, since it is consistent with the corresponding boundary conditions, i.e. the normal components to surface nullifies; see Fig. 1. It is less suitable for viscous fluid in relatively thin pipes and ducts (as well as in open flow), but even for such cases the deviations from it are well-controllable and frequently small, as experiments show [19]. Thus assumption 5 does have both empirical and theoretical support. Moreover, it is known what one can do when it does not hold, e.g. introduce additional variables or improve the velocity behavior next to boundaries. Unfortunately, all (improving) works rely on the incompressibility assumption [14–18] and hence do not apply directly to the considered situation, where the compressibility can be small, but instrumental. At least some of them ought to be generalizable to the compressible case, i.e. it should be possible to improve on assumption 5 and explore situations, where it does not hold.

– Assumption 6 on the ideal gas feature of the fluid is partially relaxed in §4 below.

– Note that (16) assumes that for the purpose of maximizing the (dimensionless) work, the final dimensionless velocity $\bar{v}_2$ can be varied independently from other parameters involved in (16): $\bar{v}_1^0$, $\bar{a}_2$ and $M_1^2$. This assumption does make sense for the following reasons. First, if there are relations between these parameters, then the maximal work will be smaller. I.e. the expression obtained in (21, 22) via the unconditional maximization still provides an upper bound on the work. Eqs. (21, 22) allow to conclude on the sonic character of optimal output velocities. Second, this assumption is confirmed in the quasi-1d approach.
4. Efficiency of work-extraction from kinetic energy holds (18) for non-ideal gases

Eqs. (18, 19) show that the efficiency of work extraction from kinetic energy of an ideal-gas flow holds an upper bound (19). The ideal-gas is understood in terms of (5). Eq. (18) shows that the real efficiency is always smaller due to inhomogeneous output pressure and density; see (4).

Here we relax the assumption (5) on the ideal gas, but are able to prove a more restrictive statement: if we assume that the output pressure and density are homogeneous, then the efficiency of work extraction from kinetic energy is given by (19). Put differently, we were not able to show that for non-ideal gases inhomogeneous output pressure or density decreases the efficiency.

Now conservation of entropy and mass amount to

\[ s(p_1, \rho_1) = s(p_2, \rho_2), \quad (70) \]

while the fact that no work is extracted from enthalpy reads

\[ \psi(p_1, s(p_1, \rho_1)) = \psi(p_2, s(p_2, \rho_2)), \quad (71) \]

where \( \psi = \varepsilon + \frac{\rho}{p} \) is the enthalpy density, and where in (71) we recalled that natural variables of \( \psi \) are \( p \) and \( s \). Now (70, 71) imply \( p_1 = p_2 \) from \( \psi(p_1, s) = \psi(p_2, s) \), because \( \partial \psi / \partial p |_s = 1/\rho > 0 \). It remains to show that \( s(p, \rho_1) = s(p, \rho_2) \) has the only solution \( \rho_1 = \rho_2 \). This will be shown via demonstrating that \( \partial s / \partial \rho |_p \) has a constant sign. Employing thermodynamic inequalities we show below that

\[ \text{sign} [\partial s / \partial \rho |_p] = -\text{sign} [\partial p / \partial T |_p]. \quad (72) \]

Now for many cases of practical interest one can demonstrate

\[ \text{sign} [\partial p / \partial T |_p] > 0, \quad (73) \]

directly from the equation of state. Though (73) is not among standard thermodynamic inequalities, we are not aware of any realistic example, where it is violated. Here is the example of the van der Waals gas, where it holds:

\[ p = \frac{\partial \rho}{1 - \rho b} - \rho^2 a, \quad \varrho \equiv RT/\mu, \quad (74) \]

where \( a > 0 \) and \( b > 0 \) are the van derWaals parameters, \( T \) is temperature, \( R \) is the gas constant and \( \mu \) is the molar mass \([1]\). Recall that \( 1 > \rho b \) is a strict constraint for the van der Waals gas \([11]\).

Once (70, 71) are solved only for \( p_1 = p_2 \) and \( \rho_1 = \rho_2 \) (cyclicality conditions), we employ the conservation of mass \( a_1 \rho_1 \varrho_1 = a_2 \rho_2 \varrho_2 \) to show that the efficiency of work-extraction from kinetic energy equals (19).

To show (72), we focus on \( \partial s / \partial V |_p \) (derivative of entropy over volume at fixed pressure), and write in natural thermodynamic variables \( (V, T) \):

\[ \frac{\partial s}{\partial V} \bigg|_T = \frac{\partial V}{\partial T} \bigg|_V \frac{\partial s}{\partial T} \bigg|_V \frac{\partial T}{\partial V} \bigg|_V \cdot (75) \]

Likewise the equation of state \( p = p(V, T) \) implies

\[ \frac{\partial p}{\partial V} \bigg|_T = \frac{\partial V}{\partial T} \bigg|_V \frac{\partial p}{\partial T} \bigg|_V \frac{\partial T}{\partial V} \bigg|_V \cdot (76) \]

Now a constant pressure implies \( dp = 0 \) in (76). Employing this in (75) we get

\[ \frac{\partial s}{\partial V} \bigg|_p = \frac{\partial s}{\partial V} \bigg|_T - \frac{\partial s}{\partial T} \bigg|_V \times \frac{\partial p}{\partial V} \bigg|_V. \quad (77) \]

Employing Maxwell’s relation \( \frac{\partial p}{\partial V} \bigg|_V = \frac{\partial T}{\partial V} \bigg|_T \) \([11]\), the fact of \( \frac{\partial s}{\partial T} \bigg|_V = c_v / T > 0 \) (the constant volume heat-capacity is positive due to a thermodynamic inequality), and \( \frac{\partial p}{\partial V} \bigg|_V < 0 \) (another known thermodynamic inequality) \([11]\) we conclude from (77) that \( \frac{\partial s}{\partial V} \bigg|_p \) and \( \frac{\partial p}{\partial V} \bigg|_V \) have the same sign. Eq. (72) follows from here upon noting \( V = 1/\rho \) and hence

\[ \frac{\partial s}{\partial V} \bigg|_p = -\rho^2 \frac{\partial s}{\partial p} \bigg|_p. \quad (78) \]

FIG. 6: This figure demonstrates the conflict between maximization of the extracted work and maximization of efficiency. The main figure shows dimensionless density \( \rho \) (black curve) and pressure \( \rho \) (blue curve) versus \( x \). The inset shows the dimensionless work \( \dot{\psi} \) (red curve), see (16), and the enthalpic part \( \dot{\psi}_{\text{enat}} \) given by (79). All these quantities are obtained from solving (25, 26).

Parameters are those of Fig. 2 (of the main text). But now \( f = 0.2 \), i.e. the force is two times stronger than in Fig. 2.
5. Weak force: conflict between maximization of the extracted work and maximization of efficiency

Eq. (19) deduces an upper bound for the efficiency of work-extraction assuming that the work is extracted from kinetic energy and the enthalpic contribution in (16):

\[ \bar{w}_{\text{ent}} = \frac{2}{M_1^2(\gamma - 1)} \left(1 - e^{\frac{1}{2}a_2^{-\gamma}v_2^{-\gamma}}\right), \]  

(79)
is precisely zero: \( \bar{w}_{\text{ent}} = 0 \). Eq. (19) does not refer to maximizing the work and it is interesting to see how the efficiency changes when the work is optimized over the choice of the end-point \( x_2 \) for a given force \( F \) in (25, 26).

Fig. 6 studies the same situation as Fig. 2, but now \( f = 0.2 \), i.e. the force is two times stronger. It is seen from Fig. 6 that if we choose \( x_2 = 0.49 \), then the process is cyclic, i.e. \( \bar{p}_1 = \bar{p}_2 \) and hence \( \bar{v}_1 = \bar{v}_2 \); cf. (20). Then the enthalpy does not contribute to the work, and since for the considered quasi-1d situation we have \( \sigma = \bar{v}_r = 0 \), then the efficiency of work-extraction from the kinetic energy is given by its value (19). With parameters of Fig. 6 this value amounts to 0.8897.

But with this choice of \( x_2 \) we loose nearly the half of the available work, as shown the red curve (for \( \bar{w} \)) and the green curve (for \( \bar{w}_{\text{ent}} \)) in Fig. 6. Choosing a larger value of \( x_2 \), i.e. \( x_2 > 0.7 \), we shall increase the extracted work, but now the work comes both from the enthalpy and kinetic energy, i.e. \( \bar{w}_{\text{ent}} > 0 \) in (79). This fact implies that the efficiency of work-extraction should be redefined, i.e. it is now given as the ratio of the work to the full input of energy [cf. (8, 33)]:

\[ \frac{-\int dV \bar{v} \cdot \vec{F}}{a_1(\rho_1v_1 + p_1)v_1 + \frac{2}{3}a_1\rho_1 v_1^3} = \frac{-\int dV \bar{v} \cdot \vec{F}}{a_1(\rho_1v_1 + p_1)v_1 + \frac{2}{3}a_1\rho_1 v_1^3} = \frac{2}{(\gamma - 1)M_1^2 + 1} \bar{w}. \]  

(80)

For parameters of Fig. 6 the value of this redefined efficiency (80) is 0.0555, which is expected (much) smaller than the efficiency 0.8897 obtained above.

We conclude that there is generally a conflict between maximizing the efficiency and maximizing the work. The core of this conflict is that the work can be increased due to a contribution from enthalpy. This fact leads to redefining—and thereby decreasing—the efficiency. Such a conflict need not be present always, i.e. it is absent for parameters employed in Fig. 2.

6. Strong force: the maximal work extraction

6.1 Upper bound on the efficiency of the maximal work-extraction

Let us now turn to discussing the work-extraction in the strong-force regime, which in the main text is exemplified by Fig. 3. For convenience this figure is reproduced as Fig. 7. Let us repeat the maximal work expressions (21, 22) as

\[ \bar{v}_2 = \bar{v}_m = \left( e^{\frac{1}{2}a_2^{-\gamma}M_1^{-2}} \right)^{\frac{1}{\gamma - 1}}, \]  

(81)

\[ \bar{w}_m = \bar{w}(\bar{v}_m) = 1 - \bar{v}_m^2 - \bar{v}_r^2 + \frac{2(1 - M_1^2\bar{v}_m^2)}{(\gamma - 1)M_1^2}, \]  

(82)

where the last equality can be also rewritten as

\[ \bar{w}_m = 1 - \bar{v}_m^2 - \bar{v}_r^2 + \frac{2}{(\gamma - 1)M_1^2} \cdot \left[ 1 - e^{\frac{2}{\gamma - 1}(M_1^2a_2^2 \bar{v}_m ^{\gamma - 2})} \right]. \]  

(83)

If the maximal work-extraction \( \bar{w}_m > 0 \) takes place from enthalpy only (not from kinetic energy), i.e. if in (83)

\[ 1 - \bar{v}_m^2 - \bar{v}_r^2 \leq 0, \]  

(84)

then the efficiency of work-extraction is obtained by analogy to (80), where only the influx of enthalpy is to be retained, since it is now the only resource [cf. (33)]:

\[ \eta = \frac{-\int dV \bar{v} \cdot \vec{F}}{a_1(\rho_1v_1 + p_1)v_1} = \frac{-\int dV \bar{v} \cdot \vec{F}}{a_1(\rho_1v_1 + p_1)v_1} \bar{w}(\gamma - 1)M_1^2 \frac{2}{\gamma - 1} \]  

(85)

Using (84), \( \sigma > 0 \) and \( \gamma > 1 \) we obtain from (83, 85) the following upper bound for the efficiency \( \eta \) at the maximal work-extraction from enthalpy:

\[ \eta \leq 1 - \left[ M_1^2a_2^2(x_2) \right]^{\frac{\gamma - 1}{\gamma - 2}}. \]  

(86)

Let us compare features of (86) with bound (19) for the efficiency of work-extraction from kinetic energy.

![FIG. 7: This figure demonstrates the reachability of the maximum work for a strong force. Dimensionless density \( \bar{p} \) (black curve), pressure \( \bar{p} \) (blue curve), velocity \( \bar{v} \) (green curve), and work \( w = \bar{w}/10 \) (red curve) versus \( x \) obtained from solving (25, 26) (of the main text) for \( x \in [0, 0.6714] \) under the same parameters as in Fig. 2 and Fig. 6), but now \( f = 1 \), i.e. the force is stronger. The dimensionless velocity \( \bar{v}(x) \) increases for \( x > 0.45 \) reaching the sonic value (21) at the end point \( x = x_2 = 0.6714 \), where the work attains its maximum (22). Cyclic values: \( \bar{p}(x_2) = 0.3547 \) = 1, \( \bar{v}(x_2) = 0.6331 \) = 1.}
The meaning of (86) is that a positive work is extracted from enthalpy in the maximum work regime. The contribution of the kinetic energy to work is non-positive. The meaning of (19) is that work is extracted from the kinetic energy, when the enthalpic contribution is zero. Bound (19) cannot be derived if we demand that the enthalpic contribution is non-positive, and the positivity of work is due to kinetic energy. This is the reason why (19)—in contrast to (86)—relates to cyclic processes.

Eq. (86) is non-trivial (i.e. smaller than one) under $M_f^2 \bar{a}_2^{-2} < 1$. This inequality is ensured by our consideration, since we consider the initially subsonic regime $M_f^2 < 1$ ($\gamma > 1$ for thermodynamic reasons), and because $\bar{a}_2 > 1$. Eq. (19) is non-trivial under $\bar{a}_2 > 1$ only.

Bound (86) is similar to bound (19) for the efficiency of work-extraction from kinetic energy, because both (86) and (19) do not depend on details of the force $\vec{F}$. But in contrast to (19), bound (86) depends also on the initial Mach number $M_f^2$ and on the adiabatic index $\gamma$ of the fluid ideal gas. The latter fact is natural, since (86) refers to work-extraction from enthalpy.

Yet another difference between (86) and (19) is that (86)—in contrast to (19)—is attainable under a rather restrictive condition $1 - \bar{v}_m^2 - \bar{v}_t^2 = 0$, which for the quasi-1d situation transforms to $1 - \bar{v}_m^2 = 0$. This condition does not generally hold.

When comparing (86) with (19) within the same setup, we should recall that $\bar{a}_2$ in (19) and $\bar{a}_2(x_2)$ in (86) refer to different choices of $x_2$, as determined from (17) and (84), respectively.

Another general conclusion follows from (83) upon noting that in the quasi-1d situation we get $\bar{v}_t = \sigma = 0$, i.e. in the maximal work-extraction regime some (positive) work should come also from enthalpy.

### 6.2 Numerical studies

Fig. 7 shows that the maximal work (82) is reached for a subsonic, quasi-1d flow; hence we should put $\sigma = \bar{v}_t = 0$ in (81–83). The dimensionless velocity $\bar{v}(x)$ is a non-monotonic function of $x$: first it decays, as expected due to expanding domain $\bar{u}(x)$, and then it starts to increase and reaches for $x_2 = 0.6714$ the sonic value (81) [cf. (23)]; see Fig. 7. Such a non-monotonic behavior is in accord with (31, 32) in the strong-force situation. The work grows and reaches the maximal value (82) at the interval-end $x_2 = 0.6714$; see Fig. 7. We remind that this is the maximal possible value of work for a fixed input Mach number $M_1$, $\gamma$ and $\bar{a}(x_2)$. The solution of (25, 26) shown in Fig. 7 cannot be continued for $x > x_2$, because the sonic value of the velocity is a singularity point [1]; see §7 for details.

Note from Fig. 7 that there is another choice of $x_2$: $x_2 = x_2^* = 0.3547$, where the cyclic condition holds: $\hat{p}(x_2^* = 0.3547) = 1$, and hence $\hat{p}(x_2^* = 0.3547) = 1$; cf. (20). Under this choice we return to the work-extracted from kinetic energy. The efficiency is given by (19) (i.e. the bound in (19) is reached), which for parameters of Fig. 7 reads

$$\hat{w}(x_2^* = 0.3547) = 0.8185. \quad (87)$$

But it is clear from Fig. 7 that at $x_2 = x_2^* = 0.3547$ the work is far from its maximal value. Taking $x_2 \in (0.3547, 0.6333)$ will lead to efficiencies sizably lower than (87), because the work is now extracted both from kinetic energy and enthalpy; cf. (80). However for

$$\hat{v}_2(x) > \hat{v}_2(x_2^* = 0.6333) = 1 \quad (88)$$

the extracted work comes from enthalpy only, and the efficiency is given by (85). Since the maximal work is reached for $\hat{v}(x_2 = 0.6714) > 1$, Eq. (88) means that the kinetic energy increases due to work-extraction.

The efficiency $\eta(x_2 = 0.6714)$ at the maximal work in Fig. 7 can be calculated from (82, 85). For parameters of Fig. 7, we get

$$\eta(x_2 = 0.6714) = 0.4833. \quad (89)$$

It is seen that the efficiency (89) at the maximal work is smaller than the efficiency (87) extracted from the kinetic energy only.

We emphasize that the efficiency at the maximal work (82) can be close to 1. Indeed, recalling in (82) that in the quasi-1d situation we have $\sigma = \bar{v}_t = 0$ we get:

$$\eta = 1 + \frac{(\gamma - 1)M_f^2}{2} - \frac{(\gamma + 1)}{2} (M_f^2 \bar{a}_2^{-2})^{\frac{\gamma - 1}{\gamma + 1}}. \quad (90)$$

Hence $\eta \to 1$ for initially vanishing Mach number $M_f^2 \to 0$. Fig. 8 illustrates this situation: the maximal work (22) is reached at $x_2 = 0.8011$, where $\hat{v}(x_2 = 0.8011) > 1$. The efficiency at the maximal work equals $\eta(x_2 = 0.8011) = 0.8377$; see Fig. 8.

For parameters of Fig. 8 the density $\hat{\rho}(x)$ (and pressure $\hat{p}(x)$) monotonously decay. Hence the scenario where work is extracted from the kinetic energy only is absent: there is always a contribution to work coming from enthalpy. Moreover, this is the main contribution into the work. Thus no comparison with the efficiency (19) of the work-extraction from kinetic energy can be carried out.

It remains to stress that $\bar{v}_m > 1$ holds for many reasonable values of parameters, but not always. E.g. if in parameters of Fig. 8 we increase the initial Mach number to $M_f^2 = \frac{95}{19}$ (which is still subsonic, but it already close to the sonic threshold), the velocity (81) at the maximal work-extraction equals $\bar{v}_m = 0.9727 \leq 1$. Since $0.9727 \approx 1$, this example illustrates the attainability of bound (86).

### 7. Sonic output velocity

In Fig. 3 (of the main text) we saw that under a sufficiently strong negative force $F(x) < 0$ the output axial
FIG. 8: Dimensionless density \( \bar{\rho} \) (black curve), pressure \( \bar{p} \) (blue curve), velocity \( \bar{v} \) (green curve), and the efficiency \( \eta \) (red curve, defined by \( \text{(85)} \)) versus \( x \) obtained from solving \( \text{(25, 26)} \) for \( x \in [0, 0.8011] \) under the same parameters as in Fig. 7, but now \( M_1^2 = 10^{-3} \times 4 \), i.e. the initial Mach number is very low.

The dimensionless velocity \( \bar{v}(x) \) reaches the sonic value \( \text{(21)} \) at the end point \( x = x_2 = 0.8011 \), where the work attains its maximum \( \text{(22)} \). The density and pressure decay monotonously, i.e. the work-extraction always takes place also from enthapy.

velocity \( v(x) \) can reach the sonic value. This reachability is studied here in more detail. Let us integrate \( \text{(27)} \) from \( x_1 \) to \( x \), and write it using \( \text{(25)} \)

\[
F(x) = \frac{2}{\gamma M_1^2} \int_{x_1}^{x} \bar{F}(y) \frac{d\bar{\rho}}{\bar{\rho}(y)}.
\]

Eq. \( \text{(91)} \) is a quadratic equation for \( \bar{\rho}^2(x) \) for \( \gamma = 3 \), which is not close to the air value \( \gamma = 1.4 \), but otherwise is physically sensible. It is solved as

\[
2\bar{\rho}^2(x) = 1 + M_1^2(1 - F(x)) \pm \sqrt{\left[1 + M_1^2(1 - F(x))\right]^2 - 4M_1^2\bar{\rho}^2(x)}.
\]

The choice of signs in \( \text{(93)} \) is regulated by \( \bar{\rho}(x_1) = 1 \), which implies the + sign in \( \text{(93)} \) for initially subsonic velocities \( M_1^2 < 1 \). If \( F(x) < 0 \) and \( |F(x)| \) is sufficiently large at least for some \( x \), then it is possible to nullify the square-root in \( \text{(93)} \):

\[
2\bar{\rho}^2(x) = 1 + M_1^2(1 - F(x)) = 2M_1/\bar{a}(x). \tag{94}
\]

This expression is equivalent to the local speed of sound \( v_s^2(x) = 3p(x)/\rho(x) \), as seen using \( \text{(25, 6)} \).

8. Implications of the Bernoulli equation for the incompressible situation

Here we shall work out some implications of the Bernoulli equation for the incompressible situation. The equation applies to the flow shown in Fig. 5 (after the singularity \( \text{(44)} \) of the force on \( x_0 \) is removed).

The incompressible situation reads:

\[
\rho = \text{const.} \tag{95}
\]

Now any potential force can be written as

\[
\vec{F} = -\rho \bar{\nabla} U(x, y, z) = -\nabla \left[\rho U(x, y, z)\right], \tag{96}
\]

where \( U(x, y, z) \) is the suitable potential. In particular, any force that depends only on \( x \) can be written as in \( \text{(96)} \). We shall assume that no potential is present initially, while its final value is independent from \( (y, z) \):

\[
U(x_1, y, z) = 0 \quad U(x_2, y, z) = U_2. \tag{97}
\]

Conditions \( \text{(97)} \) are consistent with having localized force inside of the flow volume; see Fig. 5.

Recall that under \( \text{(95)} \) the internal energy is constant and hence drops out from conservation laws \([1]\); only the term \( \frac{\bar{p}}{\rho} \) is relevant \([1]\). Also, the entropy is not involved. The potential \( U(x, y, z) \) can be incorporated into the Bernoulli equation \([1]\):

\[
\frac{\bar{v}^2(x, y, z)}{2} + \frac{p(x, y, z)}{\rho} + U(x, y, z) = \text{const}. \tag{98}
\]

Assuming that the flow lines are continuous and that for any point on \( A(x_2) \) there is a unique point on \( A(x_1) \) related by a flow line, we get from \( \text{(98, 97)} \):

\[
\frac{\bar{v}^2(x, y, z)}{2} + p_2 \frac{\bar{P}(y, z)}{\rho} + U_2 = \frac{v_1^2}{2} + \frac{p_1}{\rho}, \tag{99}
\]

where we recall \( \text{2 definitions:} \)

\[
p(x_1, y, z) = p_1, \quad p(x_2, y, z) = p_2 \bar{P}(y, z), \tag{100}
\]

\[
\langle \bar{p} \rangle \equiv \int_{A_2} \frac{dy\,dz}{a_2} \frac{\bar{p}(y, z)}{\bar{a}(y, z)} = 1. \tag{101}
\]

The energy conservation law reads \([\text{see (55)}]\):

\[
\frac{v_1^2}{2} + \frac{p_1}{\rho} = \frac{v_2^2 + \bar{v}^2}{2} + \frac{p_2}{\rho} - \int_a^b \bar{\psi} \cdot \vec{F} \, da. \tag{102}
\]

\[
\bar{v}^2 = \int_{A_2} \frac{dy\,dz}{a_2} \left[ v_y^2(x_2, y, z) + v_z^2(x_2, y, z) \right]. \tag{103}
\]

Denoting

\[
\bar{v}_{tr}(y, z) \equiv (0, v_y(x_2, y, z), v_z(x_2, y, z)), \tag{104}
\]

we conclude from \( \text{(102, 99)} \):

\[
\frac{\bar{v}_{tr}^2(y, z) - \bar{v}^2}{2} + \frac{p_2}{\rho} (\bar{p}(y, z) - 1) = -U_2 - \int_a^b \bar{\psi} \cdot \vec{F} \, da. \tag{105}
\]
due to \( v_x(x_1, y, z) = v_y(x_1, y, z) = 0 \) and (100). Integrating (105) by \( \int_{A_2} \frac{dV}{a_2^2} \) we conclude that

\[
U_2 + \int \frac{dV}{a_1 \rho v_1} \bar{F} = 0, 
\]

(106)

\[
\bar{v}_2^2(y, z) - \bar{v}^2 + \frac{\rho_2}{\rho_1} (\bar{p}(y, z) - 1) = 0. 
\]

(107)

Eq. (107) shows that homogeneous pressure \( \bar{p}(y, z) = 1 \) leads to zero transverse velocities: \( \bar{v}_2^2(y, z) \) not depending on \( (y, z) \) means \( \bar{v}_2^2(y, z) = 0 \), since \( \bar{v}_2^2(y, z) \) has to nullify on the boundaries of \( A(x_2) \).

Note that (106) automatically holds within the quasi-1d approach, where

\[
\int \frac{dV}{a_1 \rho v_1} \bar{F} = \frac{\int x^2 \, dx \, a(x) v(x) F(x)}{a_1 \rho v_1} = \frac{1}{\rho} \int_{x_1}^{x_2} dx \, F(x) 
\]

(108)

![Graph](image)

**FIG. 9:** Dimensionless work \( \bar{w} \) (red curve) and force \( \bar{f} \) (blue curve) versus \( x \) for \( M_f^2 = 0.3 \). Here \( \bar{w} \) and \( \bar{f} \) are obtained from (resp.) (111) and (112) under \( \sigma = \bar{v}_x = 0 \).

### 9. Work-extraction in a cylindrical tube: relations with d’Alembert paradox

#### 9.1 No work-extraction from kinetic energy

Recall that the choice of the control volume \( B \) in **Fig. 1** is conventional (i.e., other choices are also possible) and is subject to 2 conditions discussed after (1). Here we choose the control volume differently. We take it so large as it includes the domain \( \Omega \) of work-extraction and it is cylindrical, i.e. the cross-section \( A(x) \) and its area \( a(x) \) are constants. Since \( B \) is defined along the flow lines, this choice of a large \( B \) will first of all violate assumption 5 [see (2)], since now the output velocity \( v_x(x_2, y, z) \) will essentially depend on \( (y, z) \): for \( (y, z) \) close to the boundary of \( A(x_2) \) the flow will be practically unperturbed, \( v_x(x_2, y, z) \sim v_x(x_1) \) but closer to the center of \( A(x_2) \) we do expect serious differences between \( v_x(x_2, y, z) \) and \( v_x(x_1) \).

However, for methodological reasons it is still interesting to assume a cylindrical shape of \( B \) and implement all assumptions including 5. To avoid the above inconsistency with assumption 5, \( B \) can be regarded as a real cylindrical tube in which the fluid flows in the stationary regime. Now (41) is useful, since \( \zeta = 0 \) due to \( A_3 = 0 \). Once \( B \) is a cylinder, we have \( \bar{a}_2 = \bar{a}_2/\bar{a}_1 = 1 \), and the conservations of mass, entropy and momentum read in the dimensionless form [cf. (14, 15)]:

\[
\bar{p} \bar{v} = 1, \quad \bar{p}_2 = \bar{p}_2 \bar{e}^\sigma, \quad (109)
\]

\[
\gamma M_f^2 (1 - \bar{v}_2) + 1 - \bar{p}_2 = \bar{f}, \quad \bar{f} \equiv \frac{-\int dV F_x}{a_1 \rho v_1}. \quad (110)
\]

while the energy conservation leads to the definition of work (16) that we copy below:

\[
\bar{w} = 1 - \bar{v}_2^2 - \bar{v}_x^2 + \frac{2}{M_f^2 (\gamma - 1)} (1 - \bar{e}^\sigma \bar{v}_2^{1-\gamma}). \quad (111)
\]

Recalling that \( \sigma > 0 \) and \( \gamma > 1 \) (which have a thermodynamic origin) and the subsonic condition \( M_f^2 < 1 \), we see from (111) that no work-extraction from kinetic energy is possible, since \( \bar{w} < 0 \) for all \( \bar{v}_2 < 1 \). This fact is easily seen from differentiating (111) over \( \bar{v}_2 \).

#### 9.2 Relations with d’Alembert paradox

The above conclusion relates to d’Alembert’s paradox [21]. Recall the set-up of this paradox [21]. One considers a smooth body immersed into a cylindrical tube and by-passed by a dissipationless fluid. Formally, no volume force is present here, but the effective force appears due to integration of the momentum conservation relation over the volume of the body that is excluded from the control volume (i.e. the cylindrical tube). Due to boundary conditions no contribution from the body enters into the energy equation. Hence \( \bar{w} = 0 \). One also assumes that both input and output flows are homogeneous, i.e. \( \sigma = \bar{v}_x = 0 \). Then \( \bar{w} = 0 \) from (111) leads to \( \bar{v}_2 = 1 \), which together with (109) implies from (110): \( \int dV F_x = 0 \), i.e. the \( x \)-component of the force acting on the body nullifies [21].

Note that generally \( \int dV F_y \neq 0 \) [21]. This is seen from (42) even if we put there \( \langle \bar{p}(y, z) v_y(x_2, y, z) \rangle = 0 \) assuming a homogeneous output.

#### 9.3 Work-extraction from enthalpy

Let us now return to (109–111) and continue to assume there that \( \sigma = \bar{v}_x = 0 \). Then we get from (110):

\[
\gamma M_f^2 (1 - \bar{v}_2) + 1 - \bar{v}_2^{-\gamma} = \bar{f}, \quad (112)
\]

i.e. given the external force (given \( \bar{f} \)) we can determine \( \bar{v}_2 \) from (112) and find out the work \( \bar{w} \). **Fig. 9** shows
that for a given $M_1^2 < 1$ there are forces $\vec{f} > 0$ that can lead to work-extraction $\vec{w} > 0$ under $\vec{v}_2 > 1$. Thus the only possible scenario here is the work-extraction from enthalpy with increasing kinetic energy.

\[ \frac{m}{2}(v_1^2 - v_2^2) = f \equiv w, \quad m(v_1 - v_2) = f/v(0), \quad (113) \]

where $w$ is the work, while $v_1 = v(x < 0)$ and $v_1 = v(x > 0)$ are, respectively, the initial and final velocities. The second relation in (113) is got after integrating the Newton equation of motion in the vicinity of $x = 0$. Two equations in (113) imply Drude’s relation $v(0) = \frac{f}{mv}$ for the velocity at the work-extraction point; hence $v(x)$ is continuous in the vicinity of $x = 0$, though changes there fastly. It is seen that the conservation laws allow the extraction of the full kinetic energy, i.e. $v_2 \approx 0$ and $v \approx \frac{mv_1^2}{2}$, provided that $mv_1^2/2 \leq f$.

Not to mention that the force localization in (44) is in potential conflict with incompressibility (43). We shall not expand on drawback further, since the set-up does have more straightforward problems.

Drude’s relation (58) can be rigorously deduced from classical mechanics of a point particle. While this derivation is interesting by itself and hence is reproduced below, it by no means implies that this relation also holds in hydrodynamics. Note that “particles” of a compressible ideal fluid are not independent mechanical points, rather they are fluid parcels that move with conserving their mass (but not the shape), and interact via their thermally isolated (adiabatic) boundaries. A point particle with mass $m$ that moves in the positive $x$-direction and is subject to a localized (delta-function) force $-f\delta(x)$ located at $x = 0$ and directed against the motion: $f > 0$. This force may be generated by a step-function potential $U(x) = f\delta(x)$. Now energy and momentum conservation imply, respectively:

\[ \frac{m}{2}(v_1^2 - v_2^2) = f \equiv w, \quad m(v_1 - v_2) = f/v(0), \quad (113) \]

\[ \frac{m}{2}(v_1^2 - v_2^2) = f \equiv w, \quad m(v_1 - v_2) = f/v(0), \quad (113) \]