STRONG CONVEXITY IN FLIP-GRAPHS

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ABSTRACT

The triangulations of a surface $\Sigma$ with a prescribed set of vertices can be endowed with a graph structure $\mathcal{F}(\Sigma)$. Its edges connect two triangulations that differ by a single arc. It is known that, when $\Sigma$ is a convex polygon or a topological surface, the subgraph $\mathcal{F}_\epsilon(\Sigma)$ induced in $\mathcal{F}(\Sigma)$ by the triangulations that contain a given arc $\epsilon$ is strongly convex in the sense that all the geodesic paths between two such triangulations remain in that subgraph. Here, we provide a related result that involves a triangle instead of an arc, in the case when $\Sigma$ is a convex polygon. We show that, when the three edges of a triangle $\tau$ appear in (possibly distinct) triangulations along a geodesic path, $\tau$ must belong to a triangulation in that path. More generally, we prove that certain 3-dimensional triangulations related to the geodesics in $\mathcal{F}(\Sigma)$ are flag when $\Sigma$ is a convex polygon with flat vertices, and provide two consequences. The first is that $\mathcal{F}_\epsilon(\Sigma)$ is not always strongly convex when $\Sigma$ is a convex polygon with either two flat vertices or two punctures. The second is that the number of arc crossings between two triangulations of a topological surface $\Sigma$ does not allow to approximate their distance in $\mathcal{F}(\Sigma)$ by a factor of less than $3/2$.

1. Introduction

In order to study the properties of a surface, it is convenient to decompose it into elementary pieces such as triangles. This can be achieved by embedding a collection of pairwise non-crossing arcs in the surface, whose complement is a disjoint union of triangles. These arcs are geodesic when the surface is equipped...
with a metric and they are isotopy classes of paths otherwise. In both of these cases the two extremities of an arc are allowed to coincide. Such a collection of arcs is called a triangulation. Two triangulations that differ by a single arc can be thought of as related by a local operation called a \textit{flip} that removes an arc from one triangulation and replaces it with the only other arc such that the resulting set of arcs is a triangulation of the same surface. Observe that both of these triangulations share the same set of vertices as this operation does not change the set of the arcs’ endpoints. One can therefore associate a graph structure to the triangulations of a surface \( \Sigma \) with a prescribed set of vertices. The vertices of this graph are are the triangulations and its edges connect any two triangulations that are related by a flip. This graph, which we denote by \( \mathcal{F}(\Sigma) \) from now on, is called the \textit{flip-graph} of \( \Sigma \).

Flip-graphs appear in a number of contexts in geometry and topology. Their simplest manifestation, when \( \Sigma \) is a convex Euclidean polygon and the set of vertices of the triangulations coincide with that of the polygon, is a popular example. In this case, \( \mathcal{F}(\Sigma) \) is the graph of a convex polytope—the associahedron—whose dimension is the number of vertices of \( \Sigma \) minus three \([20]\). The genus of this graph \([23]\), its chromatic number \([10]\), and Hamiltonicity \([17]\) have been investigated. The geometry of \( \mathcal{F}(\Sigma) \) is specially interesting because of its relation with the rotation distance of binary trees \([34, 33]\). While its diameter is known exactly \([28]\), the time-complexity of computing distances in \( \mathcal{F}(\Sigma) \) remains an important unsolved problem \([1, 7, 21, 27]\). The flip-graph of a convex polygon coincides with that of a topological disk whose boundary contains the triangulations’ vertices. The generalization of this basic case to arbitrary topological surfaces has been studied in \([2, 9, 14, 24, 25]\). Its generalizations to higher dimensions have been considered both in the geometric \([8, 13, 31, 32]\) and topological \([3, 22]\) settings. Related flip-graphs, relevant to particular combinatorial or geometric objects, can be obtained with only a subset of all possible triangulations or a modified flip operation \([5, 6, 11, 12, 16, 26, 29]\).

It is shown in \([33]\) that, when \( \Sigma \) is a convex Euclidean polygon and the vertices of the triangulations are exactly those of \( \Sigma \), the subgraph \( \mathcal{F}_\epsilon(\Sigma) \) induced in \( \mathcal{F}(\Sigma) \) by the triangulations that contain a given arc \( \epsilon \) is \textit{strongly convex} in the sense that all the geodesic paths in \( \mathcal{F}(\Sigma) \) between two triangulations that contain \( \epsilon \) remain in \( \mathcal{F}_\epsilon(\Sigma) \). In other words, the arc \( \epsilon \) belongs to all the triangulations along any geodesic path in \( \mathcal{F}(\Sigma) \) between two triangulations that contain \( \epsilon \). This property is instrumental in a number of results on the geometry
of $\mathcal{F}(\Sigma)$ [7, 28, 33]. It is obtained by projecting the paths in $\mathcal{F}(\Sigma)$ between two triangulations that contain $\varepsilon$ to paths in $\mathcal{F}_\varepsilon(\Sigma)$ between the same two triangulations. It is proven in [9] that the same strong convexity property holds when $\Sigma$ is an oriented topological surface. It was used in this case to study the (coarse) geometry of the mapping class group of $\Sigma$ [9] and to establish sharp bounds on the diameter of the quotient of $\mathcal{F}(\Sigma)$ by the group of certain homeomorphisms of $\Sigma$ up to isotopy [24]. In a related combinatorial setting, the corresponding property was obtained [6, 35] for the graphs of all generalized associahedra that arise from the theory of cluster algebra [12]. Our first main result is a similar property, that involves a triangle instead of an arc.

**Theorem 1.1:** Consider a convex polygon $\Sigma$. If the three edges of a triangle $\tau$ appear in possibly distinct triangulations along a geodesic path in $\mathcal{F}(\Sigma)$, then $\tau$ must belong to some triangulation in that path.

This theorem is obtained by thinking of paths in $\mathcal{F}(\Sigma)$ as certain triangulations of a 3-dimensional space which we call blow-up triangulations of $\Sigma$. This space and these triangulations are geometric with respect to two dimensions and topological with respect to the third. The proof consists in showing that blow-up triangulations are necessarily flag complexes when they correspond to a geodesic path in $\mathcal{F}(\Sigma)$. The strategy is to project the paths in $\mathcal{F}(\Sigma)$ in the spirit of [33] except that the projection takes place with respect to a triangle within the blow-up triangulation that corresponds to a path instead of with respect to an arc within each of the triangulations of $\Sigma$ in that path.

Theorem 1.1 provides a powerful new tool to investigate the geometry of $\mathcal{F}(\Sigma)$. Consider a convex Euclidean polygon $\Sigma$ and assume that the vertices of the triangulations is any finite subset of $\Sigma$ that contains all the vertices of $\Sigma$. By analogy with the non-Euclidean case, we refer to the vertices that belong to the interior of $\Sigma$ as punctures and to the vertices that belong to the interior of an edge of $\Sigma$ as flat vertices of $\Sigma$. It is well-known that, when $\Sigma$ has six punctures or more, the subgraphs $\mathcal{F}_\varepsilon(\Sigma)$ are not always strongly convex [4, 6, 8, 18]. Our second main result, proven using a variant of Theorem 1.1, is that this remains the case down to two punctures or two flat vertices.

**Theorem 1.2:** Consider a number $i$ equal to either 0, 1, or 2. There exists a convex polygon $\Sigma$ with $i$ flat vertices and $2 - i$ punctures such that, for some arc $\varepsilon$, the subgraph $\mathcal{F}_\varepsilon(\Sigma)$ is not strongly convex within $\mathcal{F}(\Sigma)$. 
As we shall see, the subgraph $\mathcal{F}_e(\Sigma)$ is always strongly convex when $\Sigma$ is a convex polygon with no puncture and at most one flat vertex. In particular, Theorem 1.2 is sharp for convex polygons with flat vertices.

Our third main result and second application of Theorem 1.1 is on the problem of computing the distances within $\mathcal{F}(\Sigma)$ where $\Sigma$ is a topological surface. In the special case when $\Sigma$ is a convex polygon, a popular procedure to estimate the distance of two given triangulations in $\mathcal{F}(\Sigma)$ is based on the number of crossing arc pairs between them [7]. It is known that, in the more general cases of convex punctured polygons [15] and arbitrary topological surfaces [9], performing some flip in one of the triangulations makes the number of such crossings decrease. Hence, one can build a path in $\mathcal{F}(\Sigma)$ between the two triangulations, whose length can be taken as an estimation of the desired distance. It is shown in [7] that the resulting estimation is not always equal to the distance between the two considered triangulations. Here we show that, for pairs of triangulations of any oriented topological surface $\Sigma$ with sufficiently many vertices, this estimation is sometimes about $3/2$ times greater than their distance in $\mathcal{F}(\Sigma)$. This follows from determining the exact distance in $\mathcal{F}(\Sigma)$ of a family of triangulation pairs for which that computation was previously out of reach.

The outline of the article is the following.

In Section 2, we tell how paths in the flip-graph of a convex polygon with possibly flat vertices can be modeled as blow-up triangulations. We derive some of the geometric and combinatorial properties of these triangulations in Section 3. In particular, we translate in terms of blow-up triangulations the tools from [28] that allow to bound distances in flip-graphs. In Section 4, we recall how the projection from [33] with respect to an arc in the flip-graph of a convex polygon works, and show that it still works for a convex polygon with a single flat vertex. In Section 5, we introduce our projection with respect to triangles and prove a result which we call the decomposition lemma because it allows to split a blow-up triangulation into smaller ones. In Section 6, we give several variants of that lemma, one of whose is Theorem 1.1.

We establish Theorem 1.2 in Section 7 using a variant of the decomposition lemma from Section 6 that allows to treat convex polygons with flat vertices. Finally, in Section 8, we show that the number of arc crossings between two triangulations of any oriented topological surface $\Sigma$ does not allow to approximate their distance in $\mathcal{F}(\Sigma)$ by a factor of less than $3/2$. 
2. A topological model for sequences of flips

It is well known that a sequence of flips between two triangulations of a convex polygon can be thought of as a 3-dimensional triangulation \[33\]. Before we give a precise description of this correspondence in the more general context of convex polygons with flat vertices, let us illustrate it informally and give a quick word on how we will use these triangulations in the sequel. Consider the path in \(\mathcal{F}(\Sigma)\) shown in Fig. 1 between two triangulations \(T^-\) and \(T^+\) of a convex hexagon \(\Sigma\). Each of the flips in that path removes a diagonal of a convex quadrilateral from a triangulation and replaces it with the other diagonal of that quadrilateral. The tetrahedron depicted next to each flip is obtained from that quadrilateral by lifting one of the vertices of the arc introduced by the flip. Now consider the 3-dimensional triangulation \(K\) obtained by gluing these tetrahedra on top of one another in the order of the flips they correspond to. This triangulation, sketched at the center of Fig. 1, is an example of what we call a blow-up triangulation of \(\Sigma\) in this article. It should be noted here that

![Diagram of a path between two triangulations and the corresponding blow-up triangulation](image-url)

Figure 1. A path between two of triangulations \(T^-\) and \(T^+\) of a convex hexagon and the corresponding blow-up triangulation \(K\) of that hexagon (center). The tetrahedron in \(K\) that corresponds to each flip is depicted next to it.
this construction is topological with respect to the third dimension (so these
tetrahedra can be bent) and geometric with respect to the first two dimensions
(in particular, the orthogonal projection of each tetrahedron back on the plane
is the quadrilateral whose diagonals are exchanged by the corresponding flip).
Now observe that if a blow-up triangulation \( K \) of a convex polygon \( \Sigma \) contains a
triangle whose three edges are in the boundary of \( K \), then we can split \( K \) along
that triangle, as sketched in Fig. 2. This results into two blow-up triangulations
of smaller polygons. More generally, a blow-up triangulation can be cut along
a union of its triangles whose edges are not all contained in the boundary of
\( K \) and the corresponding path in \( \mathcal{F}(\Sigma) \) can be analyzed by looking at smaller
blow-up triangulations. Our main results deal with the existence of such cuts.
Under appropriate assumptions on the placement of the flat vertices of \( \Sigma \), the
decomposition results that we establish in Sections 5 and 6 tell that, when \( K \)
corresponds to a geodesic path in \( \mathcal{F}(\Sigma) \) and three of its arcs form a topological
circle, then these arcs necessarily bound a triangle in \( K \). This allows to cut
\( K \) along that triangle and, if the edges of that triangle do not belong to the
boundary of \( K \), along neighboring triangles.

We now give a detailed description blow-up triangulations. In the remainder
of this section and in the next four sections, \( \Sigma \) is a fixed convex Euclidean
polygon with a finite number of flat vertices. By a flat vertex of \( \Sigma \) we mean a
point in the interior of an edge of \( \Sigma \) that serves as a vertex of its triangulations.
It will be important to remember that for us, an edge of \( \Sigma \) is a Euclidean line
segment \( \varepsilon \) contained in the boundary of \( \Sigma \) between two (possibly flat) vertices
of \( \Sigma \), but that does not contain a flat vertex of \( \Sigma \) in its interior. In particular,
the vertices of \( \varepsilon \) are always adjacent in the boundary of \( \Sigma \). It will be assumed
that \( \Sigma \) is contained in the plane spanned by the first two coordinates of \( \mathbb{R}^3 \),

Figure 2. When a blow-up triangulation \( K \) of a convex polygon
(left) contains a triangle (colored yellow) whose three edges lie
in the boundary of \( K \), one can decompose \( K \) into two blow-up
triangulations of smaller polygons (right).
which we denote by $\mathbb{R}^2$ in the sequel. The orthogonal projection from $\mathbb{R}^3$ to $\mathbb{R}^2$ is denoted by $\pi$. We say that a point $x$ in $\mathbb{R}^3$ is below another point $y$ in $\mathbb{R}^3$ (and that $y$ is above $x$) when $\pi(x)$ coincides with $\pi(y)$ and the third coordinate of $x$ is less than the third coordinate of $y$.

Throughout the article, we identify any path in $\mathcal{F}(\Sigma)$ between two triangulations $T^-$ and $T^+$ of $\Sigma$ with the sequence $(T_i)_{0 \leq i \leq k}$ of the triangulations of $\Sigma$ in that path, ordered in such a way that $T^-$ is equal to $T_0$, $T^+$ to $T_k$, and $T_i$ is the triangulation that results from the $i$-th flip along the path (in particular, $T_{i-1}$ and $T_i$ are related by a flip). Now consider such a path $(T_i)_{0 \leq i \leq k}$ and let us explain how a blow-up triangulation of $\Sigma$ can be obtained from it.

First consider a topological disk $\Pi_0$ embedded in $\mathbb{R}^3$ in such a way that $\pi$ induces an homeomorphism $\Pi_0 \to \Sigma$ and the boundary of $\Pi_0$ coincides with that of $\Sigma$. We say that a point of $\mathbb{R}^3$ is below $\Pi_0$ when it is below some point in $\Pi_0$ and above $\Pi_0$ when it is above some point in $\Pi_0$. The set made up of the portions of $\Pi_0$ whose images by $\pi$ are a vertex, an arc, or a triangle of $T_0$ is a first example of a blow-up triangulation of $\Sigma$. Now if $k$ is positive, consider the portion $\Theta_1^-$ of $\Pi_0$ whose image by $\pi$ is the quadrilateral whose diagonals are exchanged by the flip between $T_0$ and $T_1$. One can embed a topological disk $\Theta_1^+$ in $\mathbb{R}^3$ in such a way that $\pi$ induces an homeomorphism from $\Theta_1^+ \to \Theta_1^-$ and the boundary of $\Theta_1^+$ coincides with that of $\Theta_1^-$. We also ask that all the points in the interior of $\Theta_1^+$ are above $\Pi_0$ (or, equivalently, above $\Theta_1^-$). Denote

$$\Pi_1 = [\Pi_0 \setminus \Theta_1^-] \cup \Theta_1^+$$

and observe that $\pi$ induces an homeomorphism $\Pi_1 \to \Sigma$. Further observe that $\Theta_1^+ \cup \Theta_1^-$ is a 2-dimensional topological sphere and denote by $\psi(1)$ the closed 3-dimensional topological ball bounded by that sphere. A second example of a blow-up triangulation of $\Sigma$ is the set made up of $\psi(1)$, of the portions of $\Pi_0$ whose images by $\pi$ are a vertex, an arc, or a triangle of $T_0$, and of the portions of $\Pi_1$ whose images by $\pi$ are a vertex, an arc, or a triangle of $T_1$. If $k$ is at least 2, one can repeat the procedure in order to build three families $(\Pi_i)_{0 \leq i \leq k}$, $(\Theta_i^-)_{1 \leq i \leq k}$, and $(\Theta_i^+)_{1 \leq i \leq k}$ of topological disks such that

$$\Pi_i = [\Pi_{i-1} \setminus \Theta_i^-] \cup \Theta_i^+$$

wherever $0 < i \leq k$ and $\pi$ induces homeomorphisms $\Pi_i \to \Sigma$. In addition, all the points in the interior of $\Theta_i^+$ are above $\Pi_i$ and the image by $\pi$ of both $\Theta_i^-$ and $\Theta_i^+$ is the quadrilateral whose diagonals are exchanged by the flip that
transforms $T_i$ into $T_{i+1}$. As above, $\Theta_i^- \cup \Theta_i^+$ is a 2-dimensional topological sphere that bounds a 3-dimensional closed topological ball $\psi(i)$. Just as for $\Theta_i^-$ and $\Theta_i^+$, the image by $\pi$ of $\psi(i)$ is the quadrilateral whose diagonals are exchanged by the flip that transforms $T_i$ into $T_{i+1}$.

Now consider the set $K$ made up of the portions of $\Pi_i$ whose image by $\pi$ is a vertex, an arc, or a triangle of $T_i$, when $i$ ranges from 0 to $k$ and of the tetrahedra $\psi(i)$ when $i$ ranges from 1 to $k$. We will refer to the 1-, 2-, and 3-dimensional objects contained in $K$ as arcs, triangles and tetrahedra.

**Remark 2.1:** If $\sigma$ is a tetrahedron contained in $K$, then by construction there exists an homeomorphism $\phi$ from $\sigma$ to a Euclidean tetrahedron that sends the set of the elements of $K$ contained in $\sigma$ to the face complex of $\phi(\sigma)$.

We will refer to the points contained in $K$ as vertices. Note that these points are exactly the vertices of $\Sigma$. Moreover, while $K$ contains all the edges of $\Sigma$, it can also contain arcs that are not edges of $\Sigma$.

**Definition 2.2:** A blow-up triangulation of $\Sigma$ is a set of vertices, arcs, triangles, and tetrahedra obtained from a path in $F(\Sigma)$ as $K$ is from $(T_i)_{0 \leq i \leq k}$.

We will call the elements of $K$ its faces. When $k$ is equal to zero, $K$ does not contain any tetrahedron and as mentioned above, its faces are precisely the portions of $\Sigma_0$ whose image by $\pi$ is a vertex, an arc, or a triangle of $T_0$. If $k$ is positive, then $K$ contains tetrahedra as well. For any face $\sigma$ of $K$, the faces of $K$ contained in $\sigma$ will also be referred to as the faces of $\sigma$. When a vertex of $K$ is a face of $\sigma$, it will be called a vertex of $\sigma$ and when an arc of $K$ is a face of $\sigma$ it will be called an edge of $\sigma$. A face of $\sigma$ whose dimension is less by one than the dimension of $\sigma$ will be referred to as a facet of $\sigma$.

**Remark 2.3:** Let us remark that blow-up triangulations differ from what is usually called a triangulation, though they share properties with them. For instance, two arcs in a blow-up triangulation can have the same pair of endpoints. Likewise, two tetrahedra can meet along the union of two facets. This happens when the two flips that correspond to these tetrahedra in the above construction perform opposite operations. Still, the intersection of two distinct faces of a blow-up triangulation $K$ is contained in the boundary of at least one of them and in turn, distinct faces of $K$ have disjoint interiors. In particular, a triangle of $K$ cannot be a face of more than two tetrahedra of $K$.
Given two distinct faces $\sigma$ and $\tau$ of $K$, we say that $\sigma$ is above $\tau$ (and $\tau$ below $\sigma$) when some point in $\sigma$ is above some point in $\tau$. Observe that $\sigma$ cannot be both below and above $\tau$. Indeed, by continuity, the interiors of $\sigma$ and $\tau$ would otherwise not be disjoint. Throughout the article, we will denote by $K^-$ the subset of the faces of $K$ that are not above any other face of $K$ and by $K^+$ the subset of its faces that are not below any other face of $K$. As mentioned above, a triangle $\tau$ in $K$ is a face of at most two tetrahedra of $K$. In fact, $\tau$ is not contained in any tetrahedron if and only if $\tau$ belongs to both $K^-$ and $K^+$. It is contained in a unique tetrahedron if and only if $\tau$ is contained in $K^-$ and not in $K^+$ or inversely. Otherwise, $\tau$ is a face of two tetrahedra in $K$.

While different blow-up triangulations can be obtained from $(T_i)_{0 \leq i \leq k}$, they are all identical up to continuous deformation. More precisely, consider the homeomorphisms $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\pi \circ h$ coincides with $\pi$, and, for any two points $x$ and $y$ from $\mathbb{R}^3$, if $x$ is below $y$ then $h(x)$ is below $h(y)$. In other words, these homeomorphisms fix the first two coordinates and preserve the order on the third coordinate. We consider the blow-up triangulations of $\Sigma$ up to the homeomorphisms among these that further fix boundary of $\Sigma$. Therefore, we will call $K$ the blow-up triangulation of $\Sigma$ associated with $(T_i)_{0 \leq i \leq k}$ throughout the article. We will often need to consider the family $(\Pi_i)_{0 \leq i \leq k}$ of topological disks that we used to build $K$, or the corresponding map $\psi$ from $\{1, \ldots, k\}$ to the set of the tetrahedra contained in $K$. In order to do that we will use the following statement, an immediate consequence of the above construction.

**Proposition 2.4:** Consider a path $(T_i)_{0 \leq i \leq k}$ in $\mathcal{F}(\Sigma)$ and the associated blow-up triangulation $K$ of $\Sigma$. There exists a bijection $\psi$ from $\{1, \ldots, k\}$ to the set of the tetrahedra in $K$ and a family $(\Pi_i)_{0 \leq i \leq k}$ of topological disks embedded within $\mathbb{R}^3$ in such a way that

1. $\pi$ induces homeomorphisms $\Pi_i \to \Sigma$,
2. $\psi(i)$ is below $\Pi_j$ if and only if $i \leq j$,
3. if $\sigma$ is a face of $T_i$, then $\pi^{-1}(\sigma) \cap \Pi_i$ is a face of $K$, and
4. the vertices of $\pi \circ \psi(i)$ are exactly the endpoints of the two arcs exchanged by the flip between $T_{i-1}$ and $T_i$.

As mentioned above there is a unique blow-up triangulation associated to a given path in $\mathcal{F}(\Sigma)$. It turns out that the inverse is not true. For instance, the two paths in the flip-graph of a convex hexagon shown in Fig. 3 are associated
3. The geometry and combinatorics of blow-up triangulations

In this section, we establish some of the properties of blow-up triangulations. We recall that, as in Section 2, Σ is a convex polygon, possibly with flat vertices. Throughout the section we also consider a blow-up triangulation $K$ of $\Sigma$.

First consider three arcs contained in $K$ and assume that the union of these arcs is a (topological) circle $C$. This circle may be bounding a triangle of $K$, if there is such a triangle. If there is no triangle in $K$ whose image by $\pi$ is bounded by $\pi(C)$ then, according to Lemma 3.2 below, $K$ must contain at least one arc that penetrates $C$ in the following sense.

Definition 3.1: Consider a topological circle $C$ obtained as the union of three arcs of a blow-up triangulation $K$. An arc of $K$ penetrates $C$ when it is above one of the arcs of $K$ contained in $C$ and below another such arc.
In order to prove the following lemma, the usual topological notions of the star and link of a face of a complex will be useful. More precisely, consider a face $\sigma$ of a blow-up triangulation $K$. The star of $\sigma$ in $K$ is the set $S_K(\sigma)$ of all the faces of $K$ that admit $\sigma$ as one of their faces and the link of $F$ in $K$ is the set $L_K(\sigma)$ made up of all the faces $\tau$ of $K$ such that $\tau$ is a face of some face of $K$ contained in $S_K(\sigma)$ but $\tau$ is disjoint from $\sigma$ itself.

**Lemma 3.2:** Consider a topological circle $C$ obtained as the union of three arcs of a blow-up triangulation $K$ and let $\beta$ be one of these arcs. If $S_K(\beta)$ does not contain a triangle whose image by $\pi$ is bounded by $\pi(C)$, then there exists a tetrahedron in $K$ incident to $\beta$ and to an arc in $K$ that penetrates $C$.

**Proof.** Denote by $\alpha$ and $\gamma$ the two arcs of $K$ other than $\beta$ that are contained in $C$. Assume that there is no triangle in $S_K(\beta)$ whose image by $\pi$ is bounded by $\pi(C)$. Observe that, under this assumption, the vertex $b$ shared by $\alpha$ and $\gamma$ cannot be contained in the link of $\beta$ in $K$.

If $\beta$ is contained in $K^-$, then denote by $\tau^-$ the triangle in $K^-$ that is incident to $\beta$ and that lies below $\alpha$ or below $\gamma$. If $\beta$ is not contained in $K^-$, consider the tetrahedron $\sigma^-$ in $K$ incident to and below $\beta$. Observe that, if the edge $\varepsilon^-$ of $\sigma^-$ that is disjoint from $\beta$ is above $\alpha$ or above $\gamma$, then that edge penetrates $C$ and the lemma is proven. Therefore, we assume that $\varepsilon^-$ is not above $\alpha$ and not above $\gamma$. Since $\varepsilon^-$ does not admit $b$ as a vertex, it must be either below $\alpha$ or below $\gamma$, and the same must be true of one of the facets of $\sigma^-$ incident to $\beta$. Let us denote by $\tau^-$ that facet of $\sigma^-$. Similarly, we can either find an arc in $L_K(\beta)$ that penetrates $C$ (in which case the lemma is proven) or a triangle $\tau^+$ in $S_K(\beta)$ that lies above $\alpha$ or above $\gamma$. Therefore, it suffices to show that, if some triangles $\tau^-$ and $\tau^+$ in $S_K(\beta)$ are such that $\tau^-$ is below $\alpha$ or below $\gamma$ and $\tau^+$ is above $\alpha$ or $\gamma$, then some arc in $L_K(\beta)$ penetrates $C$.

Consider the triangle in $S_K(\beta)$ below $\alpha$ or $\gamma$ that is above every other such triangle. Similarly, consider the triangle in $S_K(\beta)$ above $\alpha$ or $\gamma$ that is below every other such triangle. There exists a tetrahedron $\sigma$ in $S_K(\beta)$ that is incident to both of these triangles. This follows from the observation that, when two triangles in $S_K(\beta)$ are below or above $\alpha$ or $\gamma$, these triangles must be below or above one another. The tetrahedron $\sigma$ has an edge that belongs to $L_K(\beta)$ and that edge necessarily penetrates $C$, as desired. ■
Remark 3.3: In the statement of Lemma 3.2 it is not only assumed that $K$ does not contain a triangle bounded by $C$ but that it does not contain any triangle whose image by $\pi$ is bounded by $C$. It turns out that this stronger assumption is required. For instance, consider the path in $\mathcal{F}(\Sigma)$ depicted in Fig. 4 from a triangulation $T^-$ to a triangulation $T^+$ of a convex pentagon $\Sigma$ with no flat vertex and assume that $K$ denotes the blow-up triangulation of $\Sigma$ associated to that path. The three arcs in $K$ that correspond to the dotted arcs in Fig. 4 form a topological circle $C$. However, no triangle in $K$ is bounded by $C$ and no arc in $K$ penetrates $C$ either. Lemma 3.2 fails here because $K$ contains (three) triangles whose image by $\pi$ is bounded by $\pi(C)$.

Let us now translate in terms of blow-up triangulations some of the tools introduced in [28] to estimate distances in flip-graphs. In order to do that, we first need to define two operations. The first operation, already used in [28] consists in contracting an edge of $\Sigma$ toward one of its vertices within a triangulation of $\Sigma$. The second operation is related: it consists in pulling a tetrahedron (or one of its lower dimensional faces) to a vertex and may be carried out within a blow-up triangulation of $\Sigma$ in order to build another blow-up triangulation. Let us begin by describing edge contractions. Consider a triangulation $T$ of $\Sigma$, pick an edge $\varepsilon$ of $\Sigma$, and denote by $x$ and $y$ the vertices of $\varepsilon$. Now consider the convex polygon $\Sigma'$ whose vertices are the vertices of $\Sigma$ except for $y$ and assume that the vertex of $\Sigma$ adjacent to $x$ but distinct from $y$ is not flat. Under this assumption, once can build a triangulation $T'$ of $\Sigma'$ whose arcs connect any two vertices of $\Sigma'$ that are already connected by an arc of $T$, and further connect $x$ to a vertex $z$ whenever an arc of $T$ has vertices $y$ and $z$. We say that $T'$ is obtained by contracting $\varepsilon$ to $x$. This operation is illustrated on the left of Fig. 5.
and distinct from $y$ is a flat vertex of $\Sigma$ one cannot always build $T'$ from $T$, as shown on the right of Fig. 5. Indeed, in this case, one of the arcs of $T$ may be sent by the contraction to an arc that contains $v$ in its interior, as for instance the arc with vertices $y$ and $z$ on the right of Fig. 5.

We now tell how the pulling operation affects a tetrahedron.

**Definition 3.4:** Consider two tetrahedra $\sigma$ and $\sigma'$ and a vertex $x$ of $\sigma'$. We say that $\sigma'$ is obtained by pulling $\sigma$ to $x$ when the boundary of $\sigma'$ contains a triangle $\tau'$ and the boundary of $\sigma$ a triangle $\tau$ such that

1. $x$ is not a vertex of $\tau'$ or $\sigma$,
2. the images of $\tau$ and $\tau'$ by $\pi$ coincide, and
3. $\sigma'$ is below $\tau'$ if and only if $\sigma$ is below $\tau$.

In the case of an arc or a triangle, the pulling operation is similar but simpler to describe. Consider two such objects $\sigma$ and $\sigma'$ of the same dimension. We say that $\sigma'$ is obtained by pulling $\sigma$ to $x$ when $x$ is a vertex of $\sigma'$ but not a vertex of $\sigma$ and a facet $\tau$ of $\sigma$ has the same image by $\pi$ than the facet $\tau'$ of $\sigma'$ that does not admit $x$ as a vertex. This operation is illustrated in Fig. 6.

One may modify $K$ into another blow-up triangulation by removing a subset of its faces and then by pulling to a vertex $x$ of $\Sigma$ a subset of the remaining faces of $K$. When a face of $K$ is pulled to $x$, its lower-dimensional faces that admit $x$ as a vertex are pulled to $x$ along with the face itself. In addition, the faces of $K$ that are not removed or pulled to $x$ remain unaffected and belong to the resulting blow-up triangulation. The proof of the following lemma tells how the edge contraction and pulling operations are related.
Lemma 3.5: Consider an edge \( \varepsilon \) of \( \Sigma \) and denote by \( x \) and \( y \) its two vertices. Let \( \Sigma' \) be the polygon whose vertices are the vertices of \( \Sigma \) except for \( y \). If the vertex of \( \Sigma \) adjacent to \( x \) and distinct from \( y \) is not flat, then a blow-up triangulation of \( \Sigma' \) can be obtained from \( K \) by removing any face incident to \( \varepsilon \), and by pulling to \( x \) the remaining faces that are incident to \( y \).

Proof. Consider a path \( (T_i)_{0 \leq i \leq k} \) in \( \mathcal{F}(\Sigma) \) associated to \( K \) as well as the corresponding bijection \( \psi \) from \( \{1, \ldots, k\} \) to the set of the tetrahedra in \( K \) and the family of \( (\Pi_i)_{0 \leq i \leq k} \) of topological disks provided by Proposition 2.4. Under the assumption that the vertex of \( \Sigma \) adjacent to \( x \) but distinct from \( y \) is not a flat vertex, we can consider the triangulation \( T'_i \) of \( \Sigma' \) obtained by contracting \( \varepsilon \) to \( x \) within \( T_i \). Observe that \( T'_{i-1} \) and \( T'_i \) are identical when the quadrilateral \( \pi \circ \psi(i) \) admits \( \varepsilon \) as an edge. Moreover, when \( \pi \circ \psi(i) \) does not admits \( \varepsilon \) as an edge, then \( T'_{i-1} \) and \( T'_i \) are still related by a flip. Hence, one obtains a path in \( \mathcal{F}(\Sigma') \) by removing consecutive duplicates from the sequence \( (T'_i)_{0 \leq i \leq k} \). Let \( K' \) be the blow-up triangulation of \( \Sigma' \) associated to this path. By construction, \( K' \) is obtained by removing from \( K \) the faces that are incident to \( \varepsilon \) (these faces are precisely the ones whose image by \( \pi \) admits \( \varepsilon \) as an edge), and by pulling to \( x \) all the remaining faces incident to \( y \), as desired.

The following lemma corresponds to Theorem 5 from [28]. We state it here in terms of blow-up triangulations.

Lemma 3.6: Consider two edges \( \varepsilon \) and \( \varepsilon' \) of \( \Sigma \) that share a vertex \( z \). If \( z \) is incident to at least two arcs in \( K^- \setminus K^+ \) and not incident to any arc in \( K^+ \setminus K^- \), then \( \varepsilon \) and \( \varepsilon' \) cannot both be incident to only one tetrahedra of \( K \).
Proof. Assume that $z$ is not incident to any arc from $K^+ \setminus K^-$. In that case $K^+$ contains a triangle $\tau$ that admits $\varepsilon$ and $\varepsilon'$ as two of its edges. Let us assume that only one tetrahedron $\sigma_\varepsilon$ in $K$ is incident to $\varepsilon$. In this case, $\sigma_\varepsilon$ must admit $\tau$ as a facet. Therefore, $\sigma_\varepsilon$ is also incident to $\varepsilon'$. Denote by $a$ the vertex of $\sigma_x$ that is not a vertex of $\tau$. As $\sigma_\varepsilon$ is the only tetrahedron in $K$ incident to $\varepsilon$, its facet incident to $\varepsilon$ and to $a$ must be contained in $K^-$. Now assume that $z$ is incident to at least two arcs from $K^- \setminus K^+$. In this case the triangle in $K^-$ incident to $\varepsilon'$ cannot admit $a$ as a vertex or $\varepsilon$ as an edge. Hence, the tetrahedron $\sigma_{\varepsilon'}$ in $K$ incident to that triangle is distinct from $\sigma_\varepsilon$. Since both of the tetrahedra $\sigma_\varepsilon$ and $\sigma_{\varepsilon'}$ are incident to $\varepsilon'$, this proves the lemma. 

Combining Lemmas 3.5 and 3.6 one may show that certain blow-up triangulations of well-chosen convex polygons with one vertex less than $\Sigma$ contain at least two tetrahedra less than $K$. In Sections 7 and 8 we will need a more general result obtained from these two lemmas. Let us fix an arc $\delta$ in $K^+$. The two vertices of $\delta$ will be denoted by $x$ and $y$. If $\delta$ is not an edge of $\Sigma$, then we denote by $\Xi$ and $\Omega$ the two convex polygons obtained by cutting $\Sigma$ along $\pi(\delta)$. If $\delta$ is an edge of $\Sigma$, then we take $\Sigma$ for $\Omega$ and $\delta$ for $\Xi$.

In order to prove the announced consequence of Lemmas 3.5 and 3.6 we will need the following straightforward property.

**Proposition 3.7:** If $\Xi$ has at least three vertices, then it admits a vertex that is not incident to any arc from $K^+ \setminus K^-$. 

**Proof.** Assume that $\Xi$ has at least three vertices and consider the triangulation $U$ of $\Xi$ contained in $\pi(K^+)$. If $U$ contains a single triangle, then consider the vertex $z$ of that triangle other than $x$ and $y$. In that case, the only two arcs in $K^+$ that are incident to $z$ are the two edges of $\Xi$ incident to $z$. Therefore, $z$ cannot be incident to any arc from $K^+ \setminus K^-$, as desired.

Now assume that $U$ contains more than just one triangle, and let us recall the well-known property that in this case, at least two triangles of $U$ have two of their edges in the boundary of $\Xi$ (such triangles are often called *ears* [24, 30]). At least one of these triangles does not admit $\pi(\delta)$ as an edge. The vertex shared by the two edges of that triangle that are contained in the boundary of $\Xi$ cannot be incident to any arc from $K^+ \setminus K^-$. 

We can now state and prove the announced result, that can be thought of as a generalization of Corollary 2 from [28]. Under some conditions on $K$, it provides
a blow-up triangulation $L$ of the polygon $\Lambda$ whose vertices are the vertices of $\Omega$ except for $y$. A blow-up triangulation $K$ that satisfies the required conditions is sketched on the left of Fig. 7 and the resulting blow-up triangulation $L$ is sketched on the right of that figure (where the portion of $\Sigma$ outside of $\Lambda$ and the arc $\delta$ are drawn using dashed lines). In that figure, the arcs in $K^-\setminus K^+$ are colored blue as well as the links in $K^-$ of the edges of $\Omega$ contained in $\Xi$. Note that each of these links is made up of a single vertex. The arcs in $K^+\setminus K^-$ are colored red as well as the link of $\delta$ in $K^+$. On the left of the figure, the polygon $\Xi$ is striped. Note that, when $\Xi$ is a polygon (and not a line segment), the link of $\delta$ in $K^+$ is made up of two points, one in $\Xi$ and one in $\Omega$, but we only show the point contained in $\Omega$ in the figure because the required condition is always satisfied for the other point. The way the faces of $K$ are pulled to $x$ in order to build $L$ is sketched at the center of Fig. 7.

**Theorem 3.8:** Assume that $\Omega$ has at least four vertices and denote by $\Lambda$ the convex polygon whose vertices are the vertices of $\Omega$ except for $y$. If

(i) no flat vertex of $\Sigma$ is contained in $\Xi$ or adjacent to $x$ or $y$, and
(ii) the links in $K^-$ of the edges of $\Sigma$ contained in $\Xi$ are pairwise distinct, disjoint from $\Xi$, and disjoint from the link of $\delta$ in $K^+$,

then there exists a blow-up triangulation $L$ of $\Lambda$ such that

(iii) the number of tetrahedra in $L$ is less than the number of tetrahedra in $K$ by at least twice the number of vertices of $\Xi$ minus three, and
(iv) $L$ is obtained from $K$ by removing any face incident to more than one vertex of $\Xi$ and by pulling to $x$ all of the remaining faces that are incident to a vertex of $\Xi$ other than $x$. 

---

Figure 7. Partial sketch of a blow-up triangulation $K$ that satisfies the conditions of Theorem 3.8 (left) and of the blow-up triangulation $L$ provided by the theorem (right).
Proof. The theorem is proven by induction on the number of vertices of $\Sigma$ that are not contained in $\Omega$. If there is no such vertex, then $\delta$ is an edge of $\Sigma$. According to Lemma $3.5$, there exists a blow-up triangulation $L$ of $\Lambda$ obtained by removing from $K$ all the faces incident to $\delta$ and by pulling to $x$ all the remaining faces that are incident to $y$. In the case at hand, the condition that the links in $K^-$ of the edges of $\Sigma$ contained in $\Xi$ are disjoint from the link of $\delta$ in $K^+$ amounts to ask that the third vertex of the triangles incident to $\delta$ in $K^-$ and in $K^+$ are distinct. As a consequence, $\delta$ is incident to at least two triangles, and therefore to at least one tetrahedron of $K$. Hence, $L$ contains at least one tetrahedron less than $K$, as desired.

Now assume that some vertex of $\Sigma$ is not contained in $\Omega$. According to Proposition $3.7$, some vertex $c$ of $\Xi$ is not incident to any arc from $K^+ \setminus K^-$. Observe that, as $x$ and $y$ are incident to $\delta$, they are both distinct from $c$. Hence, the two edges $\varepsilon$ and $\varepsilon'$ of $\Sigma$ incident to $c$ are contained in $\Xi$. Under the assumption that the links in $K^-$ of $\varepsilon$ and $\varepsilon'$ are distinct and not contained in $\Xi$, at least two arcs of $K^- \setminus K^+$ must admit $c$ as a vertex. Therefore, by Lemma $3.6$, $\varepsilon$ and $\varepsilon'$ cannot both be incident to only one tetrahedron of $K$. We can assume that $\varepsilon$ is an edge of at least two tetrahedra of $K$, by exchanging the labels of $\varepsilon$ and $\varepsilon'$ if needed. Denote by $\Sigma'$ the convex polygon whose vertices are exactly the vertices of $\Sigma$ except for $c$ and by $a$ the vertex of $\varepsilon$ distinct from $c$. According to Lemma $3.5$, a blow-up triangulation $M$ of $\Sigma'$ can be obtained from $K$ by removing the faces incident to both $\varepsilon$ and by pulling to $a$ the remaining faces that are incident to $c$. Since $K$ contains at least two tetrahedra that admit $\varepsilon$ as an edge, the number of tetrahedra contained in $M$ is less by at least two than the number of tetrahedra contained in $K$.

Observe that $M$ satisfies assertions (i), (ii), (iii), and (iv) with respect to $\Sigma'$ as soon as $K$ does with respect to $\Sigma$. Note, in particular that the links in $M^-$ of the edges of $\Sigma'$ contained in $\Xi$ form a subset of the links in $K^-$ of the edges of $\Sigma$ contained in $\Xi$. Moreover, the number of vertices of $\Sigma'$ that do not belong to $\Omega$ is less by one than the number of vertices of $\Sigma$ that do not belong to $\Omega$. Hence, by induction, there exists a blow-up triangulation $L$ of $\Lambda$ whose number of tetrahedra is less than the number of tetrahedra in $M$ by at least twice the number of vertices of $\Xi$ minus five. It is obtained from $M$ by removing any face incident to more than one vertex of $\Xi$, and by pulling to $x$ the remaining faces that are incident to a vertex of $\Xi$ other than $x$. It immediately follows that the number of tetrahedra in $L$ is less than the number of tetrahedra in $K$ by at
least twice the number of vertices of $\Xi$ minus three. Moreover, by construction, the blow-up triangulation $L$ is obtained from $K$ by removing any face incident to more than one vertex of $\Xi$ and by pulling to $x$ all of the remaining faces that are incident to a vertex of $\Xi$ other than $x$, as desired.

4. Revisiting the projections related to arcs

Let us assume for a moment that $\Sigma$ does not have any flat vertex and recall how the projection from $F(\Sigma)$ to $F_\varepsilon(\Sigma)$ introduced in [33] works, where $\varepsilon$ is a Euclidean line segment between two vertices of $\Sigma$. Consider a triangulation $T$ of $\Sigma$. If $T$ admits $\varepsilon$ as an arc, then $T$ already belongs to $F_\varepsilon(\Sigma)$ and coincides with its projection $T^x_{\varepsilon}$. If $\varepsilon$ is not contained in $T$, then the union of the triangles in $T$ whose interior is non-disjoint from $\varepsilon$ is a convex polygon $\Sigma'$. In other words, $T$ admits as a subset a triangulation $T'$ of $\Sigma'$. The polygon $\Sigma'$ is striped on the left of Fig. 8 and the arc $\varepsilon$ is colored red.

Now recall that the comb triangulation of a convex polygon at one of its vertices $x$ is the triangulation that contains arcs incident to $x$ and to each of the other vertices of that polygon. One can build a new triangulation $T^x_{\varepsilon}$ of $\Sigma$ by replacing within $T$ the triangles and arcs contained in $T'$ by the comb triangulation of $\Sigma'$ at one of the vertices $x$ of $\varepsilon$, as illustrated in Fig. 8. The triangulation $T^x_{\varepsilon}$ is the projection of $T$ to $F_\varepsilon(\Sigma)$ described in [33].

It will be useful to remember that there are two different possible projections to $F_\varepsilon(\Sigma)$ depending on which vertex $x$ of $\varepsilon$ is chosen for the comb triangulations. An important property of these projections is that two triangulations related by a flip will be projected either to the same triangulation, or to two triangulations that are still related by a flip [33]. In fact the projections of two triangulations

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{The projection of a triangulation $T$ (left) to the triangulation $T^x_{\varepsilon}$ (right), where $\varepsilon$ is the arc colored red. The pulling operation is shown in the center of the figure.}
\end{figure}
related by a flip coincide exactly when that flip exchanges two arcs that cross $\varepsilon$ or an arc that crosses $\varepsilon$ and an arc incident to $x$.

The following is a consequence of these remarks.

**Lemma 4.1** (Lemma 3(b) in [33]): If $\Sigma$ does not have any flat vertex, then $\mathcal{F}_\varepsilon(\Sigma)$ is a strongly convex subgraph of $\mathcal{F}(\Sigma)$.

Informally, the projections to $\mathcal{F}_\varepsilon(\Sigma)$ can be thought of as pulling to $x$ the arcs that cross $\varepsilon$ in a triangulation of $\Sigma$—as if these arcs were rubber bands—as shown at the center of Fig. 8 and then inserting the arc $\varepsilon$ as a diagonal of the quadrilateral created by this pulling operation. If $K$ is the blow-up triangulation of $\Sigma$ associated to a path in $\mathcal{F}(\Sigma)$ and $\varepsilon$ an arc contained in $K^-$, then the projection of that path to $\mathcal{F}_{\pi(\varepsilon)}(\Sigma)$ can also be thought of as an operation within $K$. That operation will affect the tetrahedra of $K$ by removing a subset of them and by pulling to $x$ another subset in the sense of Definition 3.4. The tetrahedra that are not removed or pulled to $x$ are unaffected and belong to the remaining blow-up triangulation. For the sake of simplicity we alternatively denote $\mathcal{F}_{\pi(\varepsilon)}$ by $\mathcal{F}_\varepsilon$ and $T^x_{\pi(\varepsilon)}$ by $T^x_\varepsilon$ from now on.

**Theorem 4.2:** Consider an arc $\varepsilon$ of $K^-$ and a vertex $x$ of $\varepsilon$. If $\Sigma$ does not have any flat vertex, then there exists a blow-up triangulation $N$ of $\Sigma$ such that

(i) $\pi(N^-)$ is equal to $\pi(K^-)$ and $\pi(N^+)$ to $[\pi(K^+)]_\varepsilon$, and

(ii) the tetrahedra of $N$ are obtained from those of $K$ by removing the ones whose four facets are each incident to $x$ or above $\varepsilon$ and by pulling to $x$ all the remaining tetrahedra that are above $\varepsilon$.

**Proof.** Consider a path $(T_i)_{0 \leq i \leq k}$ in $\mathcal{F}(\Sigma)$ associated to $K$. Further denote by $\psi$ the bijection from $\{1, \ldots, k\}$ to the set of the tetrahedra in $K$ that corresponds to that path and that blow-up triangulation via Proposition 2.4.

As we mentioned above, $[T_{i-1}]^x_\varepsilon$ and $[T_i]^x_\varepsilon$ are either identical or related by a flip. Moreover, these triangulations are identical precisely when the four facets of $\psi(i)$ are each either incident to $x$ or above $\varepsilon$. When $\psi(i)$ is above $\varepsilon$ but one of its facets $\tau$ is not incident to $x$ and not above $\varepsilon$, observe that $\psi(i)$ cannot be incident to $x$. In particular, the quadrilateral whose diagonals are exchanged by the flip between $[T_{i-1}]^x_\varepsilon$ and $[T_i]^x_\varepsilon$ shares exactly the three vertices of $\tau$ with $\pi \circ \psi(i)$. The vertex of that quadrilateral that is not a vertex of $\pi \circ \psi(i)$ is necessarily equal to $x$. When $\psi(i)$ is not above $\varepsilon$, the flip that
changes $T_{i-1}$ into $T_i$ exchanges the same arcs than the one that changes $[T_{i-1}]_\varepsilon$ into $[T_i]_\varepsilon$. Now, consider the path from $\pi(K^-)$ to $[\pi(K^+)]_\varepsilon$ in $\mathcal{F}_\varepsilon(\Sigma)$ obtained by removing consecutive duplicates from the sequence $([T_i]_\varepsilon)_{0 \leq i \leq k}$ as well as the blow-up triangulation $N$ of $\Sigma$ associated to that path. By the above discussion, the tetrahedra of $N$ are obtained from those of $K$ by removing the tetrahedra whose four facets are each either incident to $x$ or above $\varepsilon$ and by pulling to $x$ all the remaining tetrahedra that still lie above $\varepsilon$. Moreover, by construction $\pi(N^-)$ is equal to $\pi(K^-)$ and $\pi(N^+)$ is equal to $[\pi(K^+)]_\varepsilon$, as desired.

The strategy in the proof of Theorem 4.2 is to consider a path in $\mathcal{F}(\Sigma)$ associated to a blow-up triangulation of $\Sigma$, to modify that path, and to consider the blow-up triangulation of $\Sigma$ associated to that modified path. We will use the same general strategy in the next section to prove the decomposition lemma. While the properties of the projection described above can be derived from only looking at paths in $\mathcal{F}(\Sigma)$, the interplay between the combinatorics of these paths and the geometry of the corresponding blow-up triangulations will become instrumental in our proofs of the decomposition lemma and Theorem 1.1.

Let us now allow $\Sigma$ to have flat vertices. Consider first a convex polygon $\Sigma$ with a single flat vertex $v$. Denote by $a$ and $b$ the vertices of $\Sigma$ adjacent to $v$. Pick a Euclidean line segment $\varepsilon$ that shares its two vertices with $\Sigma$ but that does not contain a flat vertex of $\Sigma$ in its interior. Further consider a vertex $x$ of $\varepsilon$, and a triangulation $T$ of $\Sigma$. A triangulation $T_x^\varepsilon$ of $\Sigma$ can be defined just as in the case when $\Sigma$ does not have a flat vertex, except when $x$ is equal to $a$ or $b$ and some arc $\delta$ in $T$ that crosses $\varepsilon$ is incident to the vertex distinct from $x$ among $a$ and $b$. This situation is depicted in Fig. 9, in the case when $x$ is equal to $a$ and $\delta$ is the dashed arc. The reason why the projection fails here, is that pulling $\delta$ to $a$ creates an arc that contains $v$ in its interior as shown in the

![Figure 9](image_url)
center of the figure. However, since $v$ is the only flat vertex of $\Sigma$, one can always construct the triangulation $T^c_\varepsilon$ shown on the right of Fig. 9 instead by using, for $x$ the vertex $c$ of $\varepsilon$ that is not adjacent to $v$. In other words, there is still a way to project any path in $\mathcal{F}(\Sigma)$ to $\mathcal{F}_\varepsilon(\Sigma)$. In this case, the same argument as for Lemma 4.1 still can be used and we obtain the following.

**Lemma 4.3:** If $\Sigma$ is a convex polygon with exactly one flat vertex, then $\mathcal{F}_\varepsilon(\Sigma)$ is a strongly convex subgraph of $\mathcal{F}(\Sigma)$.

Now assume that $\Sigma$ has two flat vertices $v$ and $w$. If $v$ and $w$ are adjacent in the boundary of $\Sigma$, then the situation is essentially similar to that of Lemma 4.3. Indeed, in this case, it is always possible to project a path to $\mathcal{F}_\varepsilon(\Sigma)$ by picking, for $x$, a vertex of $\varepsilon$ that is not adjacent to $v$ or $w$. However, if $v$ and $w$ are not adjacent, then one can always choose $\varepsilon$ in such a way that one of its ends is adjacent to $v$ and the other to $w$. If $w$ is not on the same side of $\varepsilon$ than $v$ then the modified projection illustrated in Fig. 10 is still possible.

Let us explain how that projection works. Denote by $a$ the vertex of $\varepsilon$ adjacent to $v$ and by $b$ the vertex of $\varepsilon$ adjacent to $w$. Consider a triangulation $T$ of $\Sigma$ and the convex polygon $\Sigma'$ (possibly with flat vertices) obtained as the union of the triangles of $T$ whose interior is non-disjoint from $\varepsilon$. Now replace the subset of $T$ that triangulates $\Sigma'$ with the triangulation of $\Sigma'$ that contains $\varepsilon$, an arc between $b$ and each of the vertices of $\Sigma'$ on the same side of $\varepsilon$ than $v$, and an arc between $a$ and each of the vertices of $\Sigma'$ on the same side of $\varepsilon$ than $w$. This projection is illustrated in Fig. 10. Again, the projections of two triangulations of $\Sigma$ that are related by a flip are still either identical or related by a flip. As a consequence, $\mathcal{F}_\varepsilon(\Sigma)$ is still a strongly convex subgraph of $\Sigma$.

![Figure 10](image)

**Figure 10.** A projection (right) of the triangulation $T$ (left) with respect to the arc $\varepsilon$ colored red by pulling (center) the arcs of $T$ crossing $\varepsilon$ to both of its extremities.
If, however, \( v \) and \( w \) are on the same side of \( \varepsilon \), then we shall see in Section 7 that \( F_{\varepsilon}(\Sigma) \) is no longer always a strongly convex subgraph of \( \Sigma \).

Finally note that the projection illustrated in Fig. 10 can still be performed when \( \Sigma \) has an arbitrary number of flat vertices, provided that \( \Sigma \) does not have two flat vertices on the same side of \( \varepsilon \) each adjacent to a vertex of \( \varepsilon \). We obtain the following as a consequence of this observation.

**Lemma 4.4:** Consider a convex polygon \( \Sigma \) with flat vertices and a line segment \( \varepsilon \) between two vertices of \( \Sigma \) that does not contain a flat vertex of \( \Sigma \) in its interior. If \( \Sigma \) does not have two flat vertices on the same side of \( \varepsilon \) each adjacent to one of the vertices of \( \varepsilon \), then \( F_{\varepsilon}(\Sigma) \) is a strongly convex subgraph of \( F(\Sigma) \).

It is noteworthy that, as the projection shown in Fig. 10, our proofs in the next section will consist in partly re-triangulating a blow-up triangulation of a convex polygon with flat vertices by pulling some of the tetrahedra it contains to one of its vertices and some tetrahedra to a different vertex.

### 5. A projection related to triangles

Throughout this section, \( K \) is a fixed blow-up triangulation of a convex polygon \( \Sigma \). We will first assume that \( \Sigma \) does not have flat vertices and then discuss the case when \( \Sigma \) has flat vertices at the end of the section. We also fix a topological circle \( C \) obtained as the union of three arcs \( \alpha, \beta, \) and \( \gamma \) of \( K \) such that \( \alpha \) is contained in \( K^- \) and \( \gamma \) in \( K^+ \). The vertices of these arcs will be denoted by \( a, b, \) and \( c \) in such a way that \( a \) does not belong to \( \alpha, \) \( b \) does not belong to \( \beta, \) and \( c \) does not belong to \( \gamma \). If \( \varepsilon \) is one of the arcs \( \alpha, \beta, \) or \( \gamma, \) we will denote by \( \Sigma^\varepsilon \) the portion of \( \Sigma \) sketched in Fig. 11 made up of the points of \( \Sigma \) that do not lie on the same side of \( \pi(\varepsilon) \) than the interior of the triangle bounded by \( \pi(C) \). Note that \( \Sigma^\varepsilon \) admits \( \pi(\varepsilon) \) as a subset. It is a convex polygon when \( \pi(\varepsilon) \) is not an edge of \( \Sigma \) and it shrinks to \( \pi(\varepsilon) \) otherwise.

We further consider a path \( (T_i)_{0\leq i\leq k} \) in \( F(\Sigma) \) associated to \( K \). In particular, there exist a bijection \( \psi \) from \( \{1,\ldots,k\} \) to the set of the tetrahedra in \( K \) and a family \( (\Pi_i)_{0\leq i\leq k} \) of topological disks satisfying assertions (i), (ii), (iii), and (iv) in the statement of Proposition 2.4. In the whole section, \( q \) is a fixed integer such that \( \beta \) is contained in \( \Pi_q \). Observe, in particular that most of the intermediate results established in the section depend on \( q \) and on other objects that we might need to define along the way, but these dependences will
disappear when we state the decomposition lemma. Further denote

\[
\begin{aligned}
\Pi^\alpha &= \pi^{-1}(\Sigma^\alpha) \cap \Pi_0, \\
\Pi^\beta &= \pi^{-1}(\Sigma^\beta) \cap \Pi_q, \text{ and} \\
\Pi^\gamma &= \pi^{-1}(\Sigma^\gamma) \cap \Pi_k.
\end{aligned}
\]

In other words, \(\Pi^\alpha, \Pi^\beta, \) and \(\Pi^\gamma\) are the portions of \(\Pi_0, \Pi_q,\) and \(\Pi_k\) whose orthogonal projections on \(\mathbb{R}^2\) are \(\Sigma^\alpha, \Sigma^\beta, \) and \(\Sigma^\gamma.\)

Now consider a triangulation \(T\) of \(\Sigma\) and pick an arc \(\varepsilon\) among \(\alpha, \beta,\) and \(\gamma.\) Let \(x\) be one of the vertices of \(\varepsilon.\) We can define a projection \(\langle T \rangle_\varepsilon^x\) of \(T\) to \(\mathcal{F}_\varepsilon(\Sigma)\) as follows. Consider the polygon \(\Sigma'\) obtained as the union of the triangles of \(T\) whose interior is non-disjoint from \(\Sigma\varepsilon.\) If \(\Sigma'\) coincides with \(\Sigma\varepsilon,\) then we build \(\langle T \rangle_\varepsilon^x\) by replacing the triangulation of \(\Sigma'\) contained in \(T\) with the images by \(\pi\) of the triangles and arcs of \(K\) contained in \(\Pi\varepsilon.\) If \(\Sigma'\) admits \(\Sigma\varepsilon\) as a strict subset then one obtains a convex polygon \([\Sigma'\backslash \Sigma\varepsilon] \cup \pi(\varepsilon)\) by cutting \(\Sigma\varepsilon \backslash \pi(\varepsilon)\) off from \(\Sigma'.\) In that case, we build a triangulation \(\langle T \rangle_\varepsilon^x\) from \(T\) by replacing the triangulation of \(\Sigma'\) contained in \(T\) with the images by \(\pi\) of the triangles and arcs of \(K\) contained in \(\Pi\varepsilon\) and with the triangles and arcs of the comb triangulation of \([\Sigma'\backslash \Sigma\varepsilon] \cup \pi(\varepsilon)\) at \(x.\) The projection \(T \mapsto \langle T \rangle_\varepsilon^x\) is illustrated on the left of Fig. 12 where the images by \(\pi\) of the triangles and arcs of \(K\) contained in \(\Pi\varepsilon\) is sketched at the top of the figure and \(\Sigma\varepsilon\) is striped.

A straightforward but important observation is that, just as for the projections we discussed in Section 4, the projection we just defined sends two triangulations related by a flip either to the same triangulation, or to two different
triangulations that are still related by a flip. By construction, we immediately obtain the following characterization of these two cases.

**Proposition 5.1:** If $U$ is obtained from $T$ by a flip, then the triangulations $\langle T \rangle^x_\varepsilon$ and $\langle U \rangle^x_\varepsilon$ coincide when the arcs exchanged by that flip are both non-disjoint from $\Sigma^x \setminus \{y\}$, where $y$ is the vertex of $\varepsilon$ other than $x$. Otherwise the triangulations $\langle T \rangle^x_\varepsilon$ and $\langle U \rangle^x_\varepsilon$ are still related by a flip.

We now consider two subsets of the tetrahedra contained in $K$ that will play an important role in the proof of the decomposition lemma.

**Definition 5.2:** We call a tetrahedron $\sigma$ contained in $K$ a lower tetrahedron with respect to $\Pi^\beta$ when there exists a triangle $\tau$ in $K$ such that $\sigma$ admits $\tau$ as a facet, $\tau$ does not admit $c$ as a vertex, and

1. $\tau$ is below $\Pi^\beta$ or $\Pi^\gamma$ and not above $\Pi^\alpha$ and $\Pi^\beta$, or
2. $\tau$ is above $\sigma$ and contained in either $\Pi^\beta$ or $\Pi^\gamma$,

and an upper tetrahedron with respect to $\Pi^\beta$ when $\sigma$ has a facet $\tau$ in $K$ that does not admit $a$ as a vertex and satisfies (i) or (ii), where the words “below” and “above” have been exchanged and well as $\Pi^\alpha$ and $\Pi^\gamma$.

In order to prove our decomposition lemma, we need to understand how the projections $T \rightarrow \langle T \rangle^c_\alpha$ and $T \rightarrow \langle \langle T \rangle^c_\alpha \rangle^c_\beta$ are related to the lower tetrahedra of $K$. We denote by $\mathcal{L}$ the set of the lower tetrahedra of $K$ with respect to $\Pi^\beta$.

**Lemma 5.3:** Consider an integer $i$ such that $1 \leq i \leq q$. The triangulations $\langle T_{i-1} \rangle^c_\alpha$ and $\langle T_i \rangle^c_\alpha$ are related by a flip if and only if $\psi(i) \in \mathcal{L}$.
Proof. Assume first that $\psi(i)$ belongs to $\mathcal{L}$. It follows from Definition 5.2 that $\psi(i)$ admits a facet $\tau$ that is not incident to $c$ and satisfies one of the assertions (i) and (ii) in the statement of the definition. In other words either $\tau$ lies below $\Pi^\beta$ or $\Pi^\gamma$ and not above $\Pi^\alpha$ and $\Pi^\beta$, or lies above $\psi(i)$ and is contained in $\Pi^\beta$ or $\Pi^\gamma$. In both of these cases, $\pi(\tau)$ is disjoint from $\Sigma^\alpha\setminus\{b\}$ and as a consequence, at least one of the arcs exchanged by the flip that transforms $T_{i-1}$ into $T_i$ is disjoint from $\Sigma^\alpha\setminus\{b\}$. It follows from Proposition 5.1 that the triangulations $\langle T_{i-1}\rangle^c_\alpha$ and $\langle T_i\rangle^c_\alpha$ must be related by a flip, as desired.

Now assume that the triangulations $\langle T_{i-1}\rangle^c_\alpha$ and $\langle T_i\rangle^c_\alpha$ are related by a flip. According to Proposition 5.1, $\psi(i)$ must then have a facet $\tau$ whose image by $\pi$ is disjoint from $\Sigma^\alpha\setminus\{b\}$. In particular, $\tau$ cannot admit $c$ as a vertex and it cannot be above $\Pi^\alpha$ either. In addition, some vertex of $\tau$ must be contained in $\Sigma^\beta\setminus\beta$ or in $\Sigma^\gamma\setminus\gamma$. It follows that $\tau$ must be below, above or contained in one of the disks $\Pi^\beta$ or $\Pi^\gamma$. As $\Pi^\gamma$ is a subset of $\Pi_k$, the tetrahedron $\psi(i)$ cannot be above the latter disk. It cannot be above the former either because $i$ is at most $q$. As a consequence, $\tau$ is either below or contained in $\Pi^\beta$ or $\Pi^\gamma$.

If $\tau$ is below $\Pi^\beta$ or $\Pi^\gamma$, then by Definition 5.2, $\psi(i)$ must be a lower tetrahedron of $K$ with respect to $\Pi^\beta$ because $\tau$ is not above $\Pi^\alpha$ or $\Pi^\beta$. If $\tau$ is contained in $\Pi^\beta$ or in $\Pi^\gamma$, then $\psi(i)$ it is also such a tetrahedron because it is not above either of these disks, and therefore, it must be below $\tau$.

An analogous lemma can be stated when $q < i \leq k$.

**Lemma 5.4:** Consider an integer $i$ such that $q < i \leq k$. The triangulations $\langle\langle T_{i-1}\rangle^c_\alpha\rangle^c_\beta$ and $\langle\langle T_i\rangle^c_\alpha\rangle^c_\beta$ are related by a flip if and only if $\psi(i) \in \mathcal{L}$.

**Proof.** The argument is similar to that in the proof of Lemma 5.3. Let us first assume that $\psi(i)$ belongs to $\mathcal{L}$. It follows from Definition 5.2 that $\psi(i)$ admits a facet $\tau$ that is not incident to $c$ and either lies below $\Pi^\beta$ or $\Pi^\gamma$ and not above $\Pi^\alpha$ and $\Pi^\beta$, or lies above $\psi(i)$ and is contained in $\Pi^\beta$ or $\Pi^\gamma$. As in the proof of Lemma 5.3, $\pi(\tau)$ must then be disjoint from $\Sigma^\alpha\setminus\{b\}$. By Proposition 5.1, the triangulations $\langle T_{i-1}\rangle^c_\alpha$ and $\langle T_i\rangle^c_\alpha$ are related by a flip. Observe that one of the diagonals exchanged by that flip is an edge of $\pi(\tau)$. Since $q$ is less than $i$, the triangle $\tau$ cannot be below or contained in $\Pi^\beta$. Hence, $\tau$ is either below $\Pi^\gamma$ and not above $\Pi^\beta$ or contained in $\Pi^\gamma$. In both of these cases, $\pi(\tau)$ is disjoint from $\Sigma^\beta\setminus\{a\}$ and according, again, to Proposition 5.1 the triangulations $\langle\langle T_{i-1}\rangle^c_\alpha\rangle^c_\beta$ and $\langle\langle T_i\rangle^c_\alpha\rangle^c_\beta$ are related by a flip.
Let us now assume that the triangulations $\langle \langle T_{i-1} \rangle \rangle_\alpha^c$ and $\langle \langle T_i \rangle \rangle_\beta^c$ are related by a flip. It turns out that $\psi(i)$ has a facet whose image by $\pi$ is disjoint from both $\Sigma^\alpha \setminus \{b\}$ and $\Sigma^\beta \setminus \{a\}$. Indeed, consider the quadrilateral $\Theta$ whose diagonals are exchanged by the flip that transforms $\langle T_{i-1} \rangle_\alpha^c$ into $\langle T_i \rangle_\alpha^c$. By Proposition 5.1, $\Theta$ and $\psi(i)$ share three or four vertices. If they share exactly three vertices, consider the facet $\tau$ of $\psi(i)$ incident to these vertices. The image by $\pi$ of that facet must be disjoint from $\Sigma^\alpha \setminus \{b\}$. Moreover the vertex of $\Theta$ that is not a vertex of $\tau$ must be $c$. Hence, according to Proposition 5.1, $\pi(\tau)$ must also be disjoint from $\Sigma^\beta \setminus \{a\}$ as $\langle \langle T_{i-1} \rangle \rangle_\alpha^c$ would otherwise coincide with $\langle \langle T_i \rangle \rangle_\alpha^c$. Now if $\Theta$ and $\psi(i)$ have the same four vertices, then $\Theta$ coincides with $\pi \circ \psi(i)$. In that case, $\pi \circ \psi(i)$ is disjoint from $\Sigma^\alpha \setminus \{b\}$ and so is the image by $\pi$ of every facet of $\psi(i)$. Therefore, by Proposition 5.1, $\psi(i)$ has, again, a facet $\tau$ that is disjoint from both $\Sigma^\alpha \setminus \{b\}$ and $\Sigma^\beta \setminus \{a\}$, as desired.

As $\pi(\tau)$ is disjoint from both $\Sigma^\alpha \setminus \{b\}$ and $\Sigma^\beta \setminus \{a\}$, the triangle $\tau$ cannot be incident to $c$. That triangle cannot be above, below, or contained in $\Pi^\alpha$ or $\Pi^\beta$ either. As a consequence, $\tau$ is necessarily below or contained in $\Pi^\gamma$. If $\tau$ is below $\Pi^\gamma$, then it immediately follows that $\psi(i)$ is a lower tetrahedron of $K$ with respect to $\Pi^\beta$ (because $\psi(i)$ is not above $\Pi^\alpha$ or $\Pi^\beta$). If $\tau$ is contained in $\Pi^\gamma$, then the tetrahedron $\psi(i)$ is also a lower tetrahedron of $K$ with respect to $\Pi^\beta$ because $\psi(i)$ is necessarily below $\Pi^\gamma$ in that case.

In the sequel, we denote by $\mathcal{P}$ the set of the tetrahedra in $K$ that are incident to at least one of the arcs of $K$ that penetrate $C$. Note that $\mathcal{P}$ can be empty if $K$ does not have an arc that penetrates $C$. Recall that following Remark 3.3, this can happen even when $K$ does not contain a triangle bounded by $C$.

**Lemma 5.5:** Consider an integer $i$ such that $1 \leq i \leq q$ and $\psi(i)$ belongs to $\mathcal{L}$. A diagonal of $\pi \circ \psi(i)$ crosses $\pi(\alpha)$ if and only if $\psi(i) \in \mathcal{P}$.

**Proof.** By Definition 5.2, we can consider a facet $\tau$ of $\psi(i)$ that does not admit $c$ as a vertex and such that $\tau$ is either below $\Pi^\beta$ or $\Pi^\gamma$ and not above $\Pi^\alpha$ and $\Pi^\beta$, or contained in $\Pi^\beta$ or in $\Pi^\gamma$ and above $\psi(i)$. Hence, $\pi(\tau)$ is disjoint from $\Sigma^\alpha \setminus \{b\}$ and some vertex $x$ of $\tau$ must be contained in $\Pi^\beta \setminus \beta$ or in $\Pi^\gamma \setminus \gamma$. Assume that a diagonal of the quadrilateral $\pi \circ \psi(i)$ crosses $\pi(\alpha)$. In this case, the vertex $y$ of $\psi(i)$ that is not incident to $\tau$ must be contained in $\Pi^\alpha \setminus \alpha$. The edge of $\psi(i)$ with vertices $x$ and $y$ is necessarily above $\alpha$ and, as $i$ is not greater than $q$, that edge is also below $\beta$ or below $\gamma$. This implies that $\psi(i)$ belongs to $\mathcal{P}$. 


Now assume that \( \psi(i) \) belongs to \( \mathcal{P} \) and consider an edge \( \varepsilon \) of \( \psi(i) \) that penetrates \( C \). As \( i \) is not greater than \( q \), that edge cannot be above \( \beta \) or \( \gamma \) and it must be above \( \alpha \). If \( \pi(\varepsilon) \) is a diagonal of \( \pi \circ \psi(i) \), the desired result immediately holds. If \( \pi(\varepsilon) \) is an edge of \( \pi \circ \psi(i) \), then the interior of the quadrilateral \( \pi \circ \psi(i) \) is non-disjoint from \( \pi(\alpha) \) and by convexity, some diagonal of that quadrilateral must cross \( \pi(\alpha) \), as desired.

A statement similar to that of Lemma 5.5 can be given when \( q < i \leq k \).

**Lemma 5.6:** Consider an integer \( i \) such that \( q < i \leq k \) and \( \psi(i) \) belongs to \( \mathcal{L} \). A diagonal of \( \pi \circ \psi(i) \) crosses \( \pi(\alpha) \) or \( \pi(\beta) \) if and only if \( \psi(i) \) belongs to \( \mathcal{P} \).

**Proof.** Consider a facet \( \tau \) of \( \psi(i) \) that does not admit \( c \) as a vertex and either lies below \( \Pi^\beta \) or \( \Pi^\gamma \) and not above \( \Pi^\alpha \) or \( \Pi^\beta \), or lies above \( \psi(i) \) and is contained in \( \Pi^\beta \) or in \( \Pi^\gamma \). As \( i \) is greater than \( q \), the tetrahedron \( \psi(i) \) cannot be below \( \Pi^\beta \) and the triangle \( \tau \) cannot be below or contained in \( \Pi^\gamma \). Therefore, as in the proof of Lemma 5.4, \( \pi(\tau) \) is disjoint from \( [\Sigma^\alpha \cup \Sigma^\beta] \setminus \{a, b\} \) and the three vertices of that triangle must be contained in \( \Pi^\gamma \). In particular, some vertex \( x \) of \( \tau \) belongs to \( \Pi^\gamma \setminus \gamma \). Assume that a diagonal of the quadrilateral \( \pi \circ \psi(i) \) crosses \( \pi(\alpha) \) or \( \pi(\beta) \). In this case, the vertex \( y \) of \( \psi(i) \) that is not incident to \( \tau \) must be contained in \( \Pi^\alpha \setminus \alpha \) or in \( \Pi^\beta \setminus \beta \). Since \( i \) is greater than \( q \), the edge of \( \psi(i) \) with vertices \( x \) and \( y \) is above \( \alpha \) or above \( \beta \). As \( x \) belongs to \( \Pi^\gamma \setminus \gamma \), that edge is also below \( \gamma \). Hence, \( \psi(i) \) belongs to \( \mathcal{P} \), as desired.

Now assume that \( \psi(i) \) belongs to \( \mathcal{P} \). In this case, some edge \( \varepsilon \) of \( \psi(i) \) penetrates \( C \). This edge must be must be below \( \gamma \) because \( i \) is greater than \( q \), and above \( \alpha \) or \( \beta \). By the same argument as in the proof of Lemma 5.5, some diagonal of \( \pi \circ \psi(i) \) must then cross \( \pi(\alpha) \) or \( \pi(\beta) \).

Denote by \( T^o \) the triangulation of \( \Sigma \) made up of the images by \( \pi \) of all the faces of \( K \) contained in \( \Pi^\alpha \cup \Pi^\beta \cup \Pi^\gamma \) together with the Euclidean triangle bounded by \( \pi(C) \). Equivalently, \( T^o \) shares its arcs contained in \( \Sigma^\alpha \) with \( \pi(K^-) \), its arcs contained in \( \Sigma^\beta \) with \( T_q \), and its arcs contained in \( \Sigma^\gamma \) with \( \pi(K^+) \).

**Lemma 5.7:** There exists a blow-up triangulation \( L \) of \( \Sigma \) such that

(i) \( \pi(L^-) \) is equal to \( \pi(K^-) \) and \( \pi(L^+) \) is equal to \( T^o \), and

(ii) the tetrahedra of \( L \) are obtained from the tetrahedra contained in \( \mathcal{L} \) by pulling to \( c \) the ones that belong to \( \mathcal{P} \cap \mathcal{L} \).
Proof. Consider an integer $i$ such that $0 \leq i \leq k$. If $i \leq q$, denote

$$T'_i = \langle T_i \rangle^c_\alpha,$$

and if $i > q$, denote

$$T'_i = \langle \langle T_i \rangle^c_\alpha \rangle^c_\beta.$$

By Proposition 5.1, the triangulations $T'_{i-1}$ and $T'_i$ are either identical or related by a flip. Moreover, $T'_k$ is precisely equal to $\pi(K^-)$. As a consequence, one obtains a path in $\mathcal{F}(\Sigma)$ from $\pi(K^-)$ to $T^\circ$ by removing consecutive duplicates from the sequence $(T'_i)_{0 \leq i \leq r}$. By construction, the set of the tetrahedra of the blow-up triangulation $L$ of $\Sigma$ associated to that path is obtained from the set of the tetrahedra of $K$ by removing a subset of them, by pulling to $c$ another subset, and by leaving unaffected any other tetrahedron of $K$.

According to Lemmas 5.3 and 5.4, $T'_{i-1}$ and $T'_i$ are related by a flip if and only if $\psi(i)$ belongs to $L$. Hence, the tetrahedra that are removed when $L$ is obtained from $K$ are exactly the ones that do not belong to $L$.

Now assume that $\psi(i)$ belongs to $L$. It follows from Lemmas 5.5 and 5.6 that the flip between $T'_{i-1}$ and $T'_i$ exchanges the same two arcs as the flip between $T'_{i-1}$ and $T'_i$ exactly when $\psi(i)$ does not belong to $P$. Hence, $\psi(i)$ belongs to $P \cap L$ exactly when the tetrahedron of $L$ that corresponds to the flip between $T'_{i-1}$ and $T'_i$ is obtained by pulling $\psi(i)$ to $c$, as desired.

Note that, by symmetry, a result similar to Lemma 5.7 can be stated for the set $U$ of the upper tetrahedra of $K$ with respect to $\Pi^\beta$. In this case, the tetrahedra contained in $P \cap U$ must be pulled to $a$ instead of $c$. Hence, we can prove the following theorem using Lemma 5.7. This theorem can be considered one of the main results of the section. In particular, both the decomposition lemma and Theorem 1.1 are obtained from it.

**Theorem 5.8:** There exists a blow-up triangulation $N$ of $\Sigma$ such that

(iii) $\pi(N^-)$ is equal to $\pi(K^-)$ and $\pi(N^+)$ is equal to $\pi(K^+)$, and

(iv) the tetrahedra of $N$ are obtained from those in $L \cup U$ by pulling to $c$ the tetrahedra in $P \cap L$ and to $a$ the tetrahedra in $P \cap U$.

**Proof.** Consider the blow-up triangulation $L$ provided by Lemma 5.7 whose set of tetrahedra is obtained from $L$ by pulling to $c$ the tetrahedra that also belong to $P$. That triangulation is such that $\pi(L^-)$ is equal to $\pi(K^-)$ and $\pi(L^+)$ to $T^\circ$. By symmetry, Lemma 5.7 also provides a triangulation $U$ whose set of
tetrahedra is obtained from $U$ by pulling to $a$ the tetrahedra that also belong to $P$. It further follows from the lemma that $\pi(U^-)$ is equal to $T^o$ and $\pi(U^+)$ to $\pi(K^+)$. In particular, $\pi(L^+)$ coincides with $\pi(U^-)$, and one can build the desired triangulation $N$ by gluing $U$ on top of $L$.

**Remark 5.9:** It is important to note here that the blow-up triangulation $N$ provided by Theorem 5.8 can be identical to $K$ even when $K$ does not contain a triangle bounded by $C$. For instance, consider the path in the flip-graph of a pentagon depicted in Fig. 4. If $K$ is the blow-up triangulation associated to that path, then $P$ is empty, $L = \{\psi(3), \psi(4)\}$, and $U = \{\psi(1), \psi(2)\}$.

We now turn our attention to the combinatorics of the blow-up triangulation $N$ provided by Theorem 5.8. By the following lemma, $L$ and $U$ to be disjoint. As a consequence, $N$ contains at most as many tetrahedra as $K$.

**Lemma 5.10:** The sets $L$ and $U$ are disjoint.

**Proof.** Consider a tetrahedron $\sigma$ contained in $K$ and assume for contradiction that $\sigma$ is a lower and an upper tetrahedron with respect to $\Pi^\beta$. According to Definition 5.2, $\sigma$ is incident to two triangles $\tau^-$ and $\tau^+$ in $K$ such $\tau^-$ satisfies assertions (i) or (ii) in the definition and $\tau^+$ satisfies one of these assertions wherein “below” and “above” have been exchanged as well as $\Pi^\alpha$ and $\Pi^\gamma$. In addition, $\tau^-$ is does not admit $c$ as a vertex and $\tau^+$ does not admit $a$ as a vertex. Note that $\tau^-$ and $\tau^+$ are necessarily distinct because otherwise, $\sigma$ would be both above and below $\Pi^\beta$. Denote by $\varepsilon$ be the edge shared by $\tau^-$ and $\tau^+$.

Since $\tau^-$ is not above and $\tau^+$ not below any of the disks $\Pi^\alpha$, $\Pi^\beta$, and $\Pi^\gamma$, their intersection cannot be below or above any of these disks. As a consequence, the arc $\varepsilon$ must be contained in $\Pi^\delta$ where $\delta$ is equal to either $\alpha, \beta$, or $\gamma$. As $\tau^-$ is not incident to $c$ and $\tau^+$ not incident to $a$, this arc is necessarily distinct from $\delta$. In this case, the only disk among $\Pi^\alpha$, $\Pi^\beta$, and $\Pi^\gamma$ that possibly contains the triangles $\tau^-$ or $\tau^+$ is $\Pi^\delta$. Moreover, if one of these triangles is not contained in $\Pi^\delta$, then it must either be above or below it. Let us reach a contradiction by showing that $\sigma$ is then necessarily both below and above $\Pi^\delta$. If the triangles $\tau^-$ and $\tau^+$ are both contained in $\Pi^\delta$, this immediately follows from assertion (ii) in the statement of Definition 5.2 and if neither of them are, then this follows from assertion (i). If exactly one of these triangles is contained in $\Pi^\delta$, this follows by combining the two assertions. ■
We now prove the decomposition lemma for polygons without flat vertices.

**Lemma 5.11:** If $\Sigma$ does not admit any flat vertex and the distance of $\pi(K^-)$ and $\pi(K^+)$ in $\mathcal{F}(\Sigma)$ is equal to the number of tetrahedra in $K$, then $S_K(\beta)$ contains a triangle whose image by $\pi$ is bounded by $\pi(C)$.

**Proof.** The proof is by contradiction and we assume that $K$ does not contain a triangle incident to $\beta$ whose image by $\pi$ is bounded by $\pi(C)$.

According to Theorem 5.8, there exists a blow-up triangulation $N$ of $\Sigma$ whose set of tetrahedra is obtained from $L \cup U$ by pulling to $c$ the tetrahedra that are contained in $P \cap L$ and to $a$ the tetrahedra that are contained in $P \cap U$. In addition, $\pi(N^-)$ coincides with $\pi(K^-)$ and $\pi(N^+)$ coincides with $\pi(K^+)$. By Lemma 5.10, the sets $L$ and $U$ are disjoint. Therefore, the number of tetrahedra in $N$ is not greater than the number of tetrahedra in $K$.

It now suffices to show that some tetrahedron of $K$ is not contained in $L \cup U$. In this case, any path from $\pi(K^-)$ to $\pi(K^+)$ in $\mathcal{F}(\Sigma)$ associated to the blow-up triangulation $N$ is shorter than $k$, resulting in a contradiction. Since $S_K(\beta)$ does not contain a triangle whose image by $\pi$ is bounded by $\pi(C)$, it follows from Lemma 3.2 that some tetrahedron $\sigma$ in $P$ is incident to $\beta$. The two facets of $\sigma$ that are not incident to both $a$ and $c$ are incident to an arc of $K$ that penetrates $C$. In particular, each of these facets is below $\Pi^\beta$ or $\Pi^\gamma$ and above $\Pi^\alpha$ or $\Pi^\gamma$. Hence, by Definition 5.2, $\sigma$ cannot belong to $L$ or $U$.

**Remark 5.12:** We assumed at the beginning of the section that $\alpha$ is contained in $K^-$ and $\gamma$ in $K^+$, an assumption that we do not recall in the statement of Lemma 5.11. While this allowed for simpler proofs, Lemma 5.11 still holds without that assumption. Indeed, consider two integers $p$ and $r$ such that $\alpha$ is contained in $\Pi_p$ and $\gamma$ in $\Pi_r$. We can assume without loss of generality that $p \leq q \leq r$ by permuting $\alpha$, $\beta$, and $\gamma$ if needed. With these notations, applying Lemma 5.11 to the blow-up triangulation whose tetrahedra are the tetrahedra of $K$ above $\Pi_p$ and below $\Pi_r$ provides the desired more general result.

Let us now turn our attention to the case when $\Sigma$ has flat vertices. In that case, the proofs in this section all work under the condition that the projections $\langle T_i \rangle^c_\alpha$, $\langle T_i \rangle^c_\beta$, $\langle T_i \rangle^a_\gamma$, and $\langle T_i \rangle^a_\beta$ we used along the way still exist. Indeed, as we discussed in Section 4, the presence of flat vertices may make it impossible to use the projection from 5.8 and the same holds for the projections we defined in this section: when pulling an arc of a triangulation of $\Sigma$ to a vertex, we
must make sure that the pulled arc will not contain a flat vertex in its interior. However, we can find conditions on the placement of the flat vertices of $\Sigma$ that make sure the desired projections can be built.

Consider a triangulation $T$ of $\Sigma$. When building $\langle T \rangle^c_\alpha$, we consider the polygon $\Sigma'$ obtained as the union of the arcs and triangles of $T$ whose interior is non-disjoint from $\Sigma^\alpha$, and we first remove these arcs and triangles. We replace them by the image by $\pi$ of the arcs and triangles of $K$ contained in $\Pi^\alpha$ as shown in Fig. 12. This can always be done, even when $\Sigma$ has flat vertices. If $[\Sigma \setminus \Sigma^\alpha] \cup \pi(\alpha)$ is not reduced to a line segment, we further add the arcs and triangles of the comb triangulation of $[\Sigma \setminus \Sigma^\alpha] \cup \pi(\alpha)$ at $c$. We need to make sure that these arcs do not contain any flat vertex of $[\Sigma \setminus \Sigma^\alpha] \cup \pi(\alpha)$ in their interior. For that, it suffices to require that no flat vertex of $[\Sigma \setminus \Sigma^\alpha] \cup \pi(\alpha)$ is adjacent to $c$ or, equivalently, that any flat vertex of $\Sigma$ adjacent to $c$ does not belong to $\Sigma^\beta$ and is distinct from $b$. If this holds, we can therefore build $\langle T \rangle^c_\alpha$. It is not difficult to see that the same assumption also allows to define $\langle \langle T_i \rangle^c_\alpha \rangle^c_\beta$. By symmetry, it suffices to assume that any flat vertex of $\Sigma$ adjacent to $a$ does not belong to $\Sigma^\beta$ and is distinct from $b$ to make sure that $\langle T_i \rangle^a_\gamma$ and $\langle \langle T_i \rangle^a_\gamma \rangle^a_\beta$ always exist. Hence, we obtain the following statement for our decomposition lemma, in the case when $\Sigma$ possibly has flat vertices.

**Lemma 5.13:** Assume that any flat vertex of $\Sigma$ adjacent to $a$ or $c$ is distinct from $b$ and not contained in $\Sigma^\beta$. If the distance between $\pi(K^-)$ and $\pi(K^+)$ in $\mathcal{F}(\Sigma)$ is equal to the number of tetrahedra contained in $K$, then $S_K(\beta)$ contains a triangle whose image by $\pi$ is bounded by $\pi(C)$.

**Remark 5.14:** According to Remark 5.12, Lemma 5.11 still holds without the assumption that $\alpha$ is contained in $K^-$ and $\gamma$ in $K^+$. Something similar is true for Lemma 5.13 but we need to be more careful. Let $p$ and $r$ be two integers such that $\alpha$ is contained in $\Pi_p$ and $\gamma$ is contained in $\Pi_r$. If $p \leq q \leq r$, then as in Remark 5.12 we can apply Lemma 5.13 to the blow-up triangulation of $\Sigma$ whose tetrahedra are the tetrahedra above $\Pi_p$ and below $\Pi_r$. However, permuting $\alpha$, $\beta$, and $\gamma$ is not always possible because the assumptions on the placement of the flat vertices of $\Sigma$ are not symmetric with respect to these three arcs.
6. Several variants of the decomposition lemma

In this section, we consider a blow-up triangulation $K$ of $\Sigma$ and three of its arcs $\alpha, \beta, \gamma$ whose union is a topological circle $C$. As in Section 5, we assume that $\alpha$ belongs to $\Sigma^-$ and $\gamma$ to $\Sigma^+$. The vertex $\beta$ shares with $\gamma$ is denoted by $a$ and the vertex it shares with $\alpha$ by $c$. The vertex common to $\alpha$ and $\gamma$ is denoted by $a$. The following theorem is a variant of Lemma 5.13. It states that there always exists a triangle in $K$ bounded by $C$ under the additional assumption that $b$ is not adjacent to a flat vertex of $\Sigma$. It should be noted here that Remark 5.14 also applies to the statement of that theorem.

**Theorem 6.1:** Assume that $b$ is not adjacent to a flat vertex of $\Sigma$ and that any flat vertex of $\Sigma$ adjacent to $a$ or $c$ is distinct from $b$ and is not contained in $\Sigma^\beta$. If the distance between $\pi(K^-)$ and $\pi(K^+)$ in $F(\Sigma)$ is equal to the number of tetrahedra in $K$, then $K$ contains a triangle bounded by $C$.

**Proof.** Under the assumptions that the distance between $\pi(K^-)$ and $\pi(K^+)$ in $F(\Sigma)$ is equal to the number of tetrahedra in $K$ and no flat vertex of $\Sigma$ adjacent to $a$ or to $c$ is contained in $\Sigma^\beta$, it follows from Lemma 5.13 that there must exist a triangle in $S_K(\beta)$ whose image by $\pi$ is bounded by $\pi(C)$. As a consequence, it suffices to show that $K$ does not contain any two arcs with the same pair of endpoints as $\alpha$ or as $\gamma$.

If $b$ is not adjacent to a flat vertex of $\Sigma$, then, by Lemma 4.4, $F_\alpha(\Sigma)$ and $F_\gamma(\Sigma)$ are strongly convex subgraphs of $F(\Sigma)$. As the number of tetrahedra in $K$ coincides with the distance between $\pi(K^-)$ and $\pi(K^+)$ in $F(\Sigma)$, there cannot be two arcs in $K$ with the same pair of endpoints as $\alpha$ or as $\gamma$.

Let us come back to the important special case when $\Sigma$ does not have any flat vertex. In this case, we have proved Lemma 5.13. By that lemma, there exists a triangle incident to $\beta$ whose image by $\pi$ is bounded by $\pi(C)$. Thanks to Theorem 6.1, we can now prove the stronger statement that there must be a single such triangle and that this triangle is in fact bounded by $C$. Moreover, we can also handle at once all the topological circles formed from three arcs of $K$. We recall that an abstract simplicial complex $K$ (and in particular any blow-up triangulation) is *flag* when a subset of the vertices of $K$ that induces a complete graph in the 1-skeleton of $K$ is always the vertex set of a face of $K$. Note that for a blow-up triangulation $K$ this amounts to require that any three arcs of $K$ whose union is a topological circle bound a triangle of $K$. 
Theorem 6.2: Consider a convex polygon $\Sigma$ without flat vertices and a blow-up triangulation $K$ of that polygon. If $K$ is associated to a geodesic path in $\mathcal{F}(\Sigma)$, then $K$ is a flag abstract simplicial complex.

Proof. Assume that $K$ is associated to a geodesic path $(T_i)_{0 \leq i \leq k}$ in $\mathcal{F}(\Sigma)$ and denote by $(\Pi_i)_{0 \leq i \leq k}$ the corresponding family of topological disks provided by Proposition 2.4. It suffices to show that any topological circle obtained as the union of three arcs of $K$ bounds a triangle of $K$. Consider such a topological circle $C$ and denote by $\alpha$, $\beta$, and $\gamma$ the three arcs of $K$ contained in $C$. Consider three integers $p$, $q$, and $r$ such that $\alpha$ is contained in $\Sigma_p$, $\beta$ in $\Sigma_q$, and $\gamma$ in $\Sigma_r$. We will assume that $p \leq q \leq r$. Note that this be required without loss of generality by permuting $\alpha$, $\beta$, and $\gamma$ if needed.

Applying Theorem 6.1 to the triangulation made up of the faces of $K$ that are not below $\Sigma_p$ and not above $\Sigma_r$ proves that this triangulation contains a triangle bounded by $C$, and in turn, so does $K$. 

We are finally ready to prove Theorem 1.1.

Proof of Theorem 1.1 Consider a geodesic path in the flip-graph of a convex polygon $\Sigma$ without flat vertices, and the blow-up triangulation $K$ of $\Sigma$ that is associated to that path. According to Theorem 6.2, $K$ is flag and as a consequence, if the three edges of a triangle appear in (possibly different) triangulations along the considered geodesic path, then that triangle must be contained in some triangulation along that path, as desired.

7. Strong convexity fails with two punctures

In this section, we consider two positive integers $n$ and $m$ and a convex polygon $\Sigma$ with $2n + 3m + 13$ vertices, exactly two of whose are flat. That polygon is sketched and the labels of its vertices shown on Fig. 13. It is obtained by gluing three triangles together with four convex polygons that we denote by $\Sigma_d$, $\Sigma_e$, $\Sigma_f$, and $\Sigma_h$. Observe that $\Sigma_d$ has a sequence of $n$ vertices labeled $d_1$ to $d_n$ counter-clockwise, $\Sigma_e$ a sequence of $m$ vertices labeled $e_1$ to $e_m$ counter-clockwise, and $\Sigma_h$ a sequence of $n$ vertices labeled $h_1$ to $h_n$ counter-clockwise. Further observe that $\Sigma_f$ has a sequence of $m+1$ vertices labeled $f_1$ to $f_{m+1}$ counter-clockwise, followed by a sequence of $m$ vertices labeled $g_1$ to $g_m$ counter-clockwise. The two flat vertices of $\Sigma$ are the ones denoted by $b$ and $r$. 
We now build a triangulation of $\Sigma$ as the union of the comb triangulation of $\Sigma_d$ at vertex $a$, the comb triangulation of $\Sigma_e$ at vertex $q$, the comb triangulation of $\Sigma_h$ at vertex $f$, and a zigzag triangulation of $\Sigma_f$. Recall that a zigzag triangulation is a triangulation whose arcs that are not contained in the boundary of the polygon form a simple path that alternates between left and right turns. Here, the zigzag triangulation of $\Pi_f$ we consider is the one where that simple path starts at vertex $f_{m+1}$ and does not contain the vertex $g$. In order to complete this triangulation, one needs to add the three triangles shown in Fig. 13 as well as their edges. We denote by $T^-$ the resulting triangulation of $\Sigma$. It is depicted on the left of Fig. 14 when $n = 4$ and $m = 2$. In this figure, some arcs have been slightly bent in order to make it easier to see the triangulation near vertices $o$, $c$, and $g$, but all of these arcs can be realized as straight line segments. We can define a triangulation $T^+$ of $\Sigma$ that is symmetric with $T^-$ with respect to the vertical axis in the representation of Fig. 14. That triangulation is sketched on the right of the figure where its arcs are colored red. Observe that the triangulations $T^-$ and $T^+$ share an arc $\eta$ with vertices $a$ and $s$, colored black on the right of Fig. 14. The triangulation $T^-$ is drawn below $T^+$ and its arcs (except for $\eta$) are colored blue.
Figure 14. The triangulations $T^-$ (left and blue) and $T^+$ (red).

The arc $\eta$ is precisely the kind of arcs for which Lemma 4.4 fails to establish the strong convexity of $F_\eta(\Sigma)$ as a subgraph of $F(\Sigma)$. We will prove that, in fact, $F_\eta(\Sigma)$ is not a strongly convex subgraph of $F(\Sigma)$. The strategy is to first establish an upper bound on the distance between $T^-$ and $T^+$ in $F(\Sigma)$ and then, to show that their distance in $F_\eta(\Sigma)$ is greater than this upper bound when $n$ and $m$ are large enough. In order to get a lower bound on the distance of $T^-$ and $T^+$ in $F_\eta(\Sigma)$, we will use one of the variants of our decomposition lemma, that allows to treat the case of a polygon with flat vertices. This will also illustrate why we call this result a decomposition lemma.

**Proposition 7.1:** $T^-$ and $T^+$ have distance at most $2n + 6m + 24$ in $F(\Sigma)$.

**Proof.** We only need to show that the distance between $T^-$ and the triangulation $T$ of $\Sigma$ sketched on the right of Fig. 15 is at most $n + m + 12$, and the result will follow by symmetry. We can transform $T^-$ into the triangulation shown on the left of Fig. 15 by performing a sequence of $m + 5$ flips that each introduce an arc incident to $o$. From that triangulation, we can first perform the flip that removes the arc with vertices $o$ and $q$, followed by a sequence of flips that each introduce one of the $n + 2$ arcs of $T^+$ incident to $s$. From there, the triangulation shown on the right of Fig. 15 can be reached with a sequence of flips that each introduce one of the remaining $2m + 4$ arcs incident to $o$.  

\[\square\]
In order to show that the distance of $T^-$ and $T^+$ in $\mathcal{F}_\eta(\Sigma)$ is greater than $2n+6m+24$ for some choice of $n$ and $m$, we consider from now on a geodesic path $(T_i)_{0 \leq i \leq k}$ from $T^-$ to $T^+$ in $\mathcal{F}_\eta(\Sigma)$ and the associated blow-up triangulation $K$ of $\Sigma$. We also consider the corresponding family $(\Pi)_{0 \leq i \leq k}$ of topological disks provided by Proposition 2.4. Note that here the union of the faces of $K$ is a 3-dimensional ball to which a triangle has been glued along one of its edges.

Consider the polygon $\tilde{\Sigma}$ obtained from $\Sigma$ by cutting off the triangle with vertices $o, a,$ and $s$. By construction, $k$ is, alternatively, the distance in the flip-graph of $\tilde{\Sigma}$ between the triangulations of $\tilde{\Sigma}$ contained in $T^-$ and $T^+$.

**Lemma 7.2:** If $k$ is less than $3n + 6$, then $K$ contains an arc with vertices $a$ and $g$ and, below this arc, an arc with vertices $f$ and $s$.

**Proof.** Denote by $S_x^-$ the set of the tetrahedra in $K$ incident to and above an arc of $K^-$ that admits $x$ as a vertex. Note that $S_a^-$ contains $n + 2$ different tetrahedra because there are $n + 2$ arcs in $K^-$ that are incident to $a$ and not contained in $K^+$ (whose other vertex is $d, e,$ or $d_i$ where $1 \leq i \leq n$). Likewise, denote by $S_x^+$ the set of the tetrahedra in $K$ incident to and below an arc of $K^+$ that admits $x$ as a vertex and observe that again, $S_g^+$ and $S_s^+$ each contain $n + 2$ different tetrahedra. Now observe that $S_s^+$ is disjoint from both $S_a^-$ and $S_g^+$. Indeed, a tetrahedron in $S_s^+$ cannot have two vertices on the same side of $\delta$ than $o$, where $\delta$ is the Euclidean line segment between $s$ and $h$.

Figure 15. The triangulations in the proof of Proposition 7.1.
As a consequence, if $S_a^-$ is disjoint from $S_g^+$ then the number $k$ of the tetrahedra in $K$ is at least $3n + 6$. Now assume that $k$ is less than $3n + 6$. In this case, some tetrahedron $\sigma^+$ belongs to $S_a^- \cap S_g^+$ and by symmetry, some tetrahedron $\sigma^-$ belongs to $S_s^+ \cap S_f^-$. Finally, observe that $\sigma^-$ is below $\sigma^+$ (otherwise these tetrahedra would not have disjoint interiors) and as can be seen in Fig. 14, this immediately implies that the edge of $\sigma^-$ with vertices $f$ and $s$ is below the edge of $\sigma^+$ with vertices $a$ and $g$, as desired.

Let us assume for a moment that $k$ is less than $3n + 6$. In this case, according to Lemma 7.2 $K$ must contain two arcs $\beta^-$ and $\beta^+$, the former with vertices $f$ and $s$, and the latter with vertices $a$ and $g$, such that $\beta^-$ is below $\beta^+$. Consider the four yellow triangles shown in Fig. 16. Two of these triangles, on the left of the figure, admit $\beta^-$ as an edge, while $\beta^+$ is an edge of the other two, on the right of the figure. The other two edges of each of these triangles are an arc contained in $K^-$ and an arc contained in $K^+$. We will denote by $C_d$, $C_e$, $C_h$, and $C_p$ the topological circles bounding these triangles, in such a way that $C_x$ contains $x$. It turns out that we can apply Theorem 6.1 for each of these circles within the blow-up triangulation $\tilde{K}$ obtained by cutting off from $K$ the triangles with vertices $a$, $o$, and $s$. In particular, recall that by assumption,
$k$ is precisely the distance between $\pi(\tilde{K}^-)$ and $\pi(\tilde{K}^+)$ in the flip-graph of the polygon $\tilde{\Sigma}$ obtained from $\Sigma$ by cutting the image by $\pi$ of that triangle. For instance, the circle $C_p$ is obtained by gluing the arc $\alpha$ of $K^-$ between vertices $f$ and $p$, the arc $\gamma$ of $K^+$ with vertices $s$ and $p$, and the arc $\beta^-$. Observe that $p$ is not adjacent to a flat vertex of $\tilde{\Sigma}$ and that $r$ is the only flat vertex of $\tilde{\Sigma}$ adjacent to $f$ or $s$. As $r$ does not lie in $\tilde{\Sigma}^{\beta^-}$ and is distinct from $p$, it follows from Theorem 6.1, that $K$ contains a triangle $\Delta_p$ bounded by $\alpha$, $\beta^-$, and $\gamma$, which we identify with the lower yellow triangle on the left of Fig. 16. We can use the same argument for the three other circles $C_d$, $C_e$, and $C_h$: by Theorem 6.1, these circles bound three triangles $\Delta_d$, $\Delta_e$, and $\Delta_h$ of $K$, respectively, that we identify with the yellow triangles in the figure.

Now recall that $(\Pi_i)_{0 \leq i \leq k}$ is the family of topological disks that corresponds to $K$ and to $(T_i)_{0 \leq i \leq k}$ via Proposition 2.4. Pick two integers $i$ and $j$ satisfying $0 < i < j < k$, such that the arcs $\beta^-$ and $\beta^+$ belong to $\Pi_i$ and $\Pi_j$, respectively (recall that, as $\beta^-$ is below $\beta^+$, one indeed has $i < j$). As $\Delta_e$ belongs to $K$, the arc $\delta$ in $K^-$ with vertices $a$ and $e$ is necessarily contained in $\Pi_i$. Similarly, the arc $\varepsilon$ in $K^+$ with vertices $s$ and $h$ is a subset of $\Pi_j$ because $\Delta_h$ belongs to $K$. Denote by $\Gamma^-$ the portion of $\Pi_i$ made up of the points $x$ such that $\pi(x)$ is not on the same side of $\pi(\beta^-)$ than $\pi(h)$ and not on the same side of $\pi(\delta)$ than $\pi(p)$. Likewise, let $\Gamma^+$ denote the subset of the points in $\Pi_j$ whose image by $\pi$ is not on the same side of $\pi(\beta^+)$ than $\pi(e)$ and not on the same side of $\pi(\varepsilon)$ than $\pi(p)$. According to Proposition 2.4, both $\Gamma^-$ and $\Gamma^+$ are the union of a subset of the triangles and arcs of $K$. As in addition the four triangles $\Delta_d$, $\Delta_e$, $\Delta_h$, and $\Delta_p$ belong to $K$, we can cut $K$ along $\Gamma^-$, $\Gamma^+$, and these four triangles. This results in the five blow-up triangulations $K_i$, $K_{ii}$, $K_{iii}$, $K_{iv}$, and $K_v$ of smaller polygons depicted in Fig. 17.

It is important to keep in mind that the decomposition of $K$ into $K_i$, $K_{ii}$, $K_{iii}$, $K_{iv}$, and $K_v$ exists thanks to the assumption that $k$ is less $3n + 6$ so that we know from Lemma 7.2 that the arcs $\beta^-$ and $\beta^+$ belong to $K$. Let us assume for a moment that this is the case in order to study $K_{iii}$. We can estimate the number of tetrahedra contained in $K_{iii}$ using Theorem 3.8. In fact, we will estimate the number of tetrahedra in a blow-up triangulation $\tilde{K}_{iii}$ sketched at the top of Fig. 18 on the left. It is obtained from $K_{iii}$ by removing all the faces incident to both $a$ and $e$ or to both $h$ and $s$, by pulling to $e$ the remaining faces incident to $a$, and by pulling to $h$ the remaining faces incident to $s$. In that process, the faces incident to both $a$ and $s$ (but not to $e$ or $h$) will first be
pulled to $e$ and then to $h$. Note that $\tilde{K}_{iii}$ indeed exists by Lemma 3.5, and that it contains at most as many tetrahedra than $K_{iii}$. This observation allows to bound the number of tetrahedra in $K_{iii}$ by that in $\tilde{K}_{iii}$. Let us denote by $l^-$ the number of arcs in $K_{iii}^- \setminus K_{iii}^+$ that are incident to $s$ and by $l^+$ the number of arcs in $K_{iii}^+ \setminus K_{iii}^-$ that are incident to $a$. By construction, $\tilde{K}_{iii}^- \setminus \tilde{K}_{iii}^+$ contains $l^- + 1$ arcs incident to $h$ and $\tilde{K}_{iii}^+ \setminus \tilde{K}_{iii}^-$ contains $l^+ + 1$ arcs incident to $e$.

**Lemma 7.3:** There are at least $5m + 4 - l^- - l^+$ tetrahedra in $\tilde{K}_{iii}$. 

Figure 17. A decomposition of $K$. 

Diagram showing a decomposition of $K$. There are several nodes labeled $a, b, c, d, e, f, g, h, s$, and $r, q, p$. The edges and face boundaries indicate the decomposition into various subgraphs such as $K_v, K_i, K_{iv}, K_{iii}$.
Proof. Consider the triangle in $\tilde{K}_{iii}^+$ incident to $e$ and $h$ and denote by $\delta$ the edge of that triangle that is not incident to $e$. If the vertices of $\delta$ are $g_m$ and $h$, then denote by $\Omega$ the convex polygon whose vertices are the vertices of $K_{iii}$ and by $\Xi$ the Euclidean line segment with vertices $g_m$ and $h$. Otherwise, consider the convex polygon whose vertices are exactly the vertices of $K_{iii}$, cut that polygon along $\pi(\delta)$, and denote by $\Omega$ and $\Xi$ the two polygons resulting from that cut with the convention that $\Omega$ contains $e$ and $\Xi$ contains $g_m$. Further denote by $x$ the vertex of $\delta$ other than $h$. By construction, the conditions of Theorem 3.8 are satisfied by $K_{iii}$ with respect to $\delta$, $\Xi$ and $\Omega$. Note in particular that the condition regarding flat vertices is void because $\Omega \cup \Xi$ does not have any flat vertex. Moreover, two edges of $\Omega \cup \Xi$ contained in $\Xi$ have distinct links in $\pi(K_{iii}^-)$ and these links are contained in $\Omega$. These links are also disjoint from the link of $\delta$ in $K_{iii}^+$ (because the point of $\Omega$ contained in the latter link is $e$).

Now consider the blow-up triangulation $L$ provided by Theorem 3.8 for this choice of $\delta$, $\Xi$ and $\Omega$: it is obtained from $K_{iii}$ by removing the faces incident to more than one vertex of $\Xi$ and by pulling to $x$ the remaining faces incident to a point in $\Xi \setminus \{x\}$. Moreover the number of tetrahedra in $L$ is less than the number of tetrahedra in $K_{iii}$ by at least twice the number of vertices of $\Xi$ minus three. This triangulation is sketched at the top of Fig. 18 on the right. It is

![Figure 18. The blow-up triangulations in the proof of Lemma 7.3](image-url)
important to note here that the number of arcs incident to $e$ in $L^+ \setminus L^-$ is now $l^+$ (so this number is down by one compared to $\tilde{K}_{iii}$). Now observe that, if $x$ is not equal to $g$, then one can invoke Theorem 3.8 again by considering the triangle of $L^+$ incident to $a$ and $s$, by using for $\delta$ the edge of that triangle that does not admit $a$ as a vertex, and for $\Omega$ and $\Xi$ the polygons (or, possibly a line segment for $\Xi$) defined by this arc with the convention that $\Omega$ contained $e$. In particular, repeating this procedure, one necessarily ends up with the blow-up triangulation $M$ shown at the bottom of Fig. 18 on the left.

Let us estimate the number of tetrahedra in $M$. By Theorem 3.8 that number is twice the number of the vertices of $\tilde{K}_{iii}$ that are no longer vertices of $M$ minus the number of times we invoked that theorem. However, each time we invoked Theorem 3.8 we have lost exactly one of the $l^+ + 1$ arcs incident to $e$ in $\tilde{K}_{iii}$ and there is only one such arc left in $M$. As $M$ has $m+1$ vertices less than $\tilde{K}_{iii}$, the number of tetrahedra in $M$ is less than the number of tetrahedra in $K_{iii}$ by at least $2m + 2 - l^+$. Now observe that the same procedure can be performed within $M$, but this time by considering the triangle in $M^-$ incident to $e$ and $g$ and its edge $\delta$ that is not incident to $g$. In this case, Theorem 3.8 can still be invoked with respect to the arc $\delta$ but one should take care to exchange $M^-$ and $M^+$ in the statement of the theorem. The conditions of this theorem remain satisfied during the process, as we invoke Theorem 3.8 for, possibly a succession of blow-up triangulations. This process ends up with the triangulation $N$ sketched at the bottom of Fig. 18 on the right. Similarly as between $\tilde{K}_{iii}$ and $M$, the number of tetrahedra in $N$ is less by at least $2m + 2 - l^+$ than the number of tetrahedra in $M$. Finally, the number of tetrahedra in $N$ is at least $m$ as there are $m$ arcs in $N^- \setminus N^+$ and the desired result follows. 

We now estimate the number of tetrahedra in $K_i$. By symmetry, this also allows to bound the number of tetrahedra in $K_v$.

**Lemma 7.4:** There are at least $m + l^-$ tetrahedra in $K_i$.

**Proof.** Consider the set of the $S^+$ of the tetrahedra from $K_i$ that are incident to and below an arc from $K_i^+$. Looking at the sketch of $K_i$ in Fig. 16 one can see that the only arc of $K_i^-$ that possibly also belongs to $K_i^+$ is the one incident to $a$ and $h$. Indeed, all the other arcs of $K_i^-$ must be below an arc of $K_i^+$. As $K_i^+ \setminus K_i^-$ contains $m + 4$ arcs, $S^+$ contains at least $m + 3$ tetrahedra. Now denote by $S_{a}^-$ the set of the tetrahedra from $K_i$ that are incident to and
above an arc incident to \(a\) and contained in \(K_{i}^{-}\). As at most one of these arcs can be contained in \(K_{i}^{+}\), there are at least \(l^{-} - 1\) tetrahedra in \(S_{a}^{-}\). Only two tetrahedra can belong to both \(S_{a}^{-}\) and \(S_{a}^{+}\). Indeed, if there were three or more, then one of the tetrahedra in \(S_{a}^{-}\cap S_{a}^{+}\) would be incident to and below an arc of \(K_{i}^{+}\) incident to \(c\). However, such a tetrahedron would be incident to both \(a\) and \(c\) and its edge with these vertices would contain \(b\) in its interior. Therefore the number of tetrahedra in \(K_{i}\) is at least \(|S_{a}^{+}| + |S_{a}^{-}| - 2\). Since \(|S_{a}^{+}|\) is at least \(m + 3\) and \(|S_{a}^{-}|\) at least \(l^{-} - 1\), the desired result holds.

Combining Lemmas 7.3 and 7.4 provides the following.

**Lemma 7.5:** The distance of \(T^{-}\) and \(T^{+}\) in \(F_{\eta}(\Sigma)\) is at least

\[
\min\{3n + 6, 2n + 7m + 6\}
\]

**Proof.** Assume that the distance of \(T^{-}\) and \(T^{+}\) in \(F_{\eta}(\Sigma)\) is less than \(3n + 6\) and recall that this distance is equal to \(k\). In this case, \(K\) can be decomposed into the blow-up triangulations \(K_{i}, K_{ii}, K_{iii}, K_{iv},\) and \(K_{v}\). Therefore, \(k\) is equal to the total number of tetrahedra in these five blow-up triangulations.

Now observe that the number of tetrahedra in \(K_{ii}\) is at least \(n + 1\) (because there are \(n + 1\) arcs in \(K_{ii}^{-}\backslash K_{ii}^{+}\)). Similarly, the number of tetrahedra in \(K_{iv}\) is also at least \(n + 1\). By Lemma 7.4, \(K_{i}\) contains at least \(m + l^{-}\) tetrahedra and by symmetry, \(K_{i}\) contains at least \(m + l^{+}\) tetrahedra. Finally, according to Lemma 7.3, there are at least \(5m + 4 - l^{-} - l^{+}\) tetrahedra in \(K_{iii}\), and the result is obtained by summing these numbers.

Now observe that, if \(n\) is at least \(7m\) and \(m\) at least 18, then according to Proposition 7.1 and Lemma 7.5, the distance of \(T^{-}\) and \(T^{+}\) in \(F(\Sigma)\) is less than their distance in \(F_{\eta}(\Sigma)\) and as a consequence, \(F_{\eta}(\Sigma)\) is not a strongly convex subgraph of \(F(\Sigma)\). It is noteworthy that, by choosing \(m\) and \(n\) large enough, we can make the gap between the distance of \(T^{-}\) and \(T^{+}\) in \(F_{\eta}(\Sigma)\) and their distance in \(F(\Sigma)\) arbitrarily large. This proves Theorem 1.2 in the case of a convex polygon with exactly two flat vertices and no puncture. Let us prove that theorem in the case of a convex polygon with two punctures and no flat vertex or one puncture and one flat vertex.

**Proof of Theorem 1.2** We have built a convex polygon \(\Sigma\) with two flat vertices such that \(F_{\eta}(\Sigma)\) is not a strongly convex subgraph of \(F(\Sigma)\) for some arc \(\eta\). The theorem follows from the observation that the flip-graph of \(\Sigma\) is not modified.
when one or both of the flat vertices of this convex polygon are moved into the interior of $\Sigma$ but remain close enough to their initial position.

8. Large gaps in distance estimates from arc crossings

It is well-known that, when $\Sigma$ is a convex polygon with $n$ vertices (none of whose is flat), the distance in $\mathcal{F}(\Sigma)$ between any two triangulations is at most $2n - 10$ when $n > 12$ [33]. It is also known that this bound is sharp: one can always pick two triangulations whose distance in $\mathcal{F}(\Sigma)$ is $2n - 10$ [28, 33]. However, there is no efficient algorithm for determining the distance in $\mathcal{F}(\Sigma)$ between any two triangulations [7, 19]. Interestingly, computing these distances is an important problem for a number of applications (see [7] and references therein). In order to estimate these distances anyways, a popular method is based on the number of arc crossings [7]. Indeed, when $T^-$ and $T^+$ do not coincide, one can always find an arc $\varepsilon$ in $T^-$ such that the arc introduced when $\varepsilon$ is flipped in $T^-$ crosses fewer arcs from $T^+$ than $\varepsilon$ does. This has been proven in [15] for triangulations of finite sets of points in the Euclidean plane and in [9] for triangulations of topological surfaces. Hence, by picking an arc from $T^-$ whose flip decreases as much as possible the number of crossings with $T^+$, by flipping that arc within $T^-$, and by repeating this procedure, one eventually reaches $T^+$. The length of the path in $\mathcal{F}(\Sigma)$ thus obtained between $T^-$ and $T^+$ can be used as an estimate of the distance of these triangulations in $\mathcal{F}(\Sigma)$.

However little is known on how accurate this estimation is. Only recently has it been observed the path in $\mathcal{F}(\Sigma)$ obtained from this method is not always a geodesic [7]. In other words, this distance estimate is sometimes one off from the distance in $\mathcal{F}(\Sigma)$ of the two considered triangulations. In this section, we prove that this method can overestimate the distances in $\mathcal{F}(\Sigma)$ by any factor arbitrarily close to $3/2$. In other words, the estimate is not always less than $3/2$ times the distance between the triangulations. We show this in the case when $\Sigma$ is a convex polygon and derive the same statement in the case when $\Sigma$ is an arbitrary oriented topological surface as a consequence.

Let us first describe a family of triangulation pairs of the convex polygon $\Sigma$ with $2n + 3m + 8$ vertices (none of whose is flat) obtained by gluing the four polygons $\Sigma_d$, $\Sigma_e$, $\Sigma_f$, and $\Sigma_h$ as shown in Fig. 19. Note that $\Sigma_f$ and $\Sigma_h$ coincide with the polygons already shown in Fig. 13. Further observe that $\Sigma_d$ has a sequence of $n$ vertices labeled $d_1$ to $d_n$ and $\Sigma_e$ a sequence of $m$ vertices
labeled \( e_1 \) to \( e_m \) counter-clockwise. Now consider a triangulation \( T^- \) of \( \Sigma \) that contains the arc with vertices \( f \) and \( p \), the comb triangulation of \( \Sigma_d \) at vertex \( o \) and the same zigzag triangulation of \( \Sigma_f \) as in Section 7. This triangulation is sketched on the left of Fig. 20. Note that this description does not tell all the arcs of \( T^- \). In particular \( T^- \) can admit, as a subset any triangulation of \( \Sigma_e \) and any triangulation of \( \Sigma_h \) (these polygons are striped on the left of Fig. 20). The special case when \( T^- \) admits as subsets the comb triangulation of \( \Sigma_e \) at vertex \( p \) and the comb triangulation of \( \Sigma_h \) at vertex \( f \) will be particularly interesting.

In fact, we will denote by \( T^+ \) the triangulation of \( \Sigma \) that is symmetric to this special case with respect to the vertical axis in the representation of Fig. 20. The two triangulations are shown together on the right of Fig. 20, \( T^- \) being drawn below \( T^+ \). In this figure, the arcs contained in \( T^- \setminus T^+ \) are colored blue and those contained in \( T^+ \setminus T^- \), red. Note that, as in Fig. 14 some of the arcs are bent in order to better show the triangulations near vertices \( d, f, g \) and \( p \), but all these arcs can be realized as straight line segments.

As a first step, let us look at how the number of arc crossings with \( T^+ \) varies when an arc of \( T^- \) is flipped. Our goal here is to identify which arc may be flipped first by the method we described above. Let us denote by \( A^-_f \) the set of the arcs in \( T^- \) whose two vertices are \( f \) and \( h_i \) where \( 1 \leq i \leq n \). Similarly we
Proposition 8.1: If \( n \) is at least \( m \) and \( A_f^- \cup A_p^- \) contains at least \( m + 3 \) arcs, then flipping any of the arcs of \( T^- \) that do not belong to \( A_f^- \cup A_p^- \) makes the number of arc crossings with \( T^+ \) decrease by at most \( n + m \).

Proof. Assume that \( A_f^- \cup A_p^- \) contains at least \( m + 3 \) arcs. By construction, \( n + m \) cannot be less than \( |A_f^- \cup A_p^-| \), and therefore, \( n \geq 3 \).

Note that the arc of \( T^- \) with vertices \( o \) and \( d_i \), where \( 0 \leq i \leq n \) crosses exactly \( m + i + 1 \) arcs of \( T^+ \) (see Fig. [20]). If that arc is flipped in \( T^- \), the arc that replaces it crosses a single arc from \( T^+ \) and therefore, this flip makes the number of arc crossings with \( T^+ \) decrease by at most \( n + m \). The arc of \( T^- \) with vertices \( e \) and \( o \) crosses \( n + m + 2 \) arcs of \( T^+ \) and flipping it introduces an arc that crosses \( 2n + m + 2 \) arcs of \( T^+ \). Hence, the desired result holds for this arc as well. Now consider an arc of \( T^- \setminus T^+ \) incident to \( f_i \), where \( 1 \leq i \leq m \) and observe that, when that arc is flipped in \( T^- \), the number of crossings with \( T^+ \) varies by at most 3. As \( n \geq 3 \), the desired result holds for any such arc. In

Figure 20. The triangulations \( T^- \) (left and blue) and \( T^+ \) (red).
addition, flipping the arc of $T^-$ with vertices $f_{m+1}$ and $g_1$ decreases the number of arc crossings with $T^+$ by exactly $n$ and the proposition also holds for it.

Let us now check the arcs of $T^-$ contained in $\Sigma_e$ or $\Sigma_h$. Observe that the arc with vertices $e$ and $p$ crosses $2n + m + 3$ arc of $T^+$. Flipping that arc introduces an arc incident to $o$ and whose other vertex is either $f$ or $e_i$ where $1 \leq i \leq m$. The introduced arc crosses at least $n + m + 4$ arcs of $T^+$. Thus that flip makes the number of arc crossings decrease by at most $n^2$, as desired. The arc of $T^-$ with vertices $f$ and $h$ crosses $n + 3m + 3$ arcs of $T^+$. When this arc is flipped, it is replaced by an arc $\delta$ with vertices $f_i$ and $h_j$ where $1 \leq i \leq n$ (note in particular that $\delta$ cannot be incident to $p$ because $A^-_f \cup A^-_p$ contains at least $m + 3$ arcs and, therefore, $A^-_f$ contains at least three arcs). The arc $\delta$ crosses at least $n + 3m + 4$ arcs of $T^+$ and the result also holds in this case.

Now consider the arc $\varepsilon$ of $T^-$ with vertices $f$ and $p$. For convenience, let us denote $e$ by $e_0$ and $h$ by $h_0$. Observe that $\varepsilon$ crosses $2n + 3m + 5$ arcs of $T^+$ (that is, all the arcs of $T^+$ that can possibly be crossed). When flipped, $\varepsilon$ is replaced by an arc $\delta$ incident to $e_i$ with $0 \leq i \leq m$ and $h_j$ with $0 \leq j \leq n$, that crosses exactly $n + m + 2 + 2i + j$ arcs of $T^+$. As $|A^-_f \cup A^-_p| \geq m + 3$, we must have $i + j \geq m + 3$. Therefore, $\delta$ crosses at least $n + 2m + 5$ arcs of $T^+$ and the number of arc crossings decreases by at most $n + m$ when $\varepsilon$ is flipped in $T^-$, as desired. Finally, observe that any of the arcs in $T^-$ that is contained in $\Sigma_e$ but not incident to $p$ crosses at most $2m$ arcs of $T^+$. Similarly, any arc of $T^-$ contained in $\Sigma_h$ but not incident to $f$ crosses at most $n$ arcs of $T^+$. Under the assumption that $m \leq n$, the number of crossing with $T^+$ cannot decrease by more than $n + m$ when such an arc is flipped in $T^-$.

Now let us look at the arcs contained in $A^-_f \cup A^-_p$.

**Proposition 8.2:** If $n$ is positive, at least $2m$ and at most $m|A^-_f \cup A^-_p|$, then there exists an arc in $A^-_f \cup A^-_p$ such that flipping that arc in $T^-$ makes the number of arc crossings with $T^+$ decrease by more than $n + m$.

**Proof.** Assume that $n$ is positive and that

$$2m \leq n \leq m|A^-_f \cup A^-_p|.$$  

First consider the case when $A^-_p$ is non-empty and pick an arc in that set. Such an arc crosses at least $2n + m + 5$ arcs of $T^+$. When the considered arc is flipped in $T^-$, the arc that replaces it crosses at most $2m$ arcs of $T^+$. Hence,
the number of arc crossings with $T^+$ decreases by at least $2n - m + 5$ which, by (1) is greater than $n + m$ and the result holds in this case.

Let us now review the case when $A^-_f$ is empty. As $n$ is positive, we obtain from (1) that there must be an arc in $A^-_f$. For the same reason $m$ must be positive as well. For each arc $\varepsilon$ contained in $A^-_f$, denote by $u(\varepsilon)$ the number of arcs of $T^+$ crossed by the arc $\delta$ introduced when $\varepsilon$ is flipped within $T^-$.

By construction,

\begin{equation}
\sum_{\varepsilon \in A^-_f} u(\varepsilon) \leq 2n. \tag{2}
\end{equation}

Moreover, as $A^-_p$ is empty, (1) yields

\begin{equation}
\frac{n}{m} \leq |A^-_f|. \tag{3}
\end{equation}

It follows from (2) and (3) that at least one arc $\varepsilon$ in $A^-_f$ satisfies

\begin{equation}
u(\varepsilon) \leq 2m. \tag{4}
\end{equation}

As any arc in $A^-_f$ crosses at least $n + 3m + 5$ arcs of $T^+$, the number of arc crossings with $T^+$ decreases by $n + 3m + 5 - u(\varepsilon)$ when $\varepsilon$ is flipped in $T^-$, and the desired result therefore follows from (4).

By Propositions 8.1 and 8.2, if $A^-_f \cup A^-_p$ contains at least $m + 3$ arcs and

\[2m \leq n \leq m(m + 3),\]

then the crossings-based method we described above that estimates distances in $\mathcal{F}(\Sigma)$ will always flip an arc from $A^-_f \cup A^-_p$ first. We will show that, for well-chosen values of $n$, $m$, and of the number of arcs in $A^-_f \cup A^-_p$, such a flip cannot begin a geodesic from $T^-$ to $T^+$. This will be done by using one of the variants of our decomposition lemma. From now on, we consider a geodesic path $(T_i)_{0 \leq i \leq k}$ from $T^-$ to $T^+$ in the flip-graph of $\Sigma$. We also consider the blow-up triangulation $K$ associated to that path and the corresponding family $(\Pi_i)_{0 \leq i \leq k}$ of topological disks provided by Proposition 2.4

**Lemma 8.3:** If $A^-_f$ contains more than $6m + 4$ arcs, then some arc in $K$ is incident to $o$ and $f$, and another to $o$ and $p$.

**Proof.** Let us recall that, given a vertex $x$ of $\Sigma$, the distance in $\mathcal{F}(\Sigma)$ between two triangulations is at most twice the number of the vertices of $\Sigma$ other than $x$ minus the total number of arcs incident to $x$ in the two triangulations.
Taking $o$ for $x$ in the case at hand shows that the distance between $T^-$ and $T^+$ is at most $2n + 6m + 8$. As a consequence,

\begin{equation}
    k \leq 2n + 6m + 8.
\end{equation}

The remainder of the proof proceeds like the proof of Lemma 7.2. Assume that $A_f^-$ contains more than $6m + 4$ arcs. For any vertex $x$ of $\Sigma$ denote by $S_x^-$ the set of the tetrahedra in $K$ incident to and above an arc of $K^-$ that admits $x$ as a vertex and by $S_x^+$ the set of the tetrahedra in $K$ incident to and below an arc of $K^+$ that admits $x$ as a vertex. There must be $n + 1$ tetrahedra in $S_o^-$ and in $S_o^+$ because $o$ is incident to exactly $n + 1$ arcs in $K^+ \setminus K^-$ and to exactly $n + 1$ arcs in $K^+ \setminus K^-$. Similarly, $S_g^+$ contains $n + 2$ tetrahedra. Observe that $S_o^+$ is disjoint from $S_g^+$ because all the tetrahedra in these sets are below exactly one arc from $K^+ \setminus K^-$. It is also disjoint from $S_o^-$ because two arcs of $K$ incident to the same vertex cannot be below or above one another. Hence, if $S_o^-$ and $S_g^+$ were also disjoint, then $k$ would be at least $3n + 4$. However, as $A_f^-$ contains more than $6m + 4$ arcs, $n$ is greater than $6m + 4$, and $3n + 4$ greater than $2n + 6m + 8$. Therefore, $k$ cannot be at least $3n + 4$ because this would contradict (5). Hence, some tetrahedron belongs to $S_o^- \cap S_g^+$. That tetrahedron has an edge with vertices $g$ and $o$ as desired.

Let us assume for a moment that $|A_f^-| > 6m + 4$. By Lemma 8.3, some arc $\beta^-$ in $K$ has vertices $f$ and $o$ and an arc $\beta^+$ has vertices $g$ and $o$. These arcs are colored gray in Fig. 21. Observe that the four triangles colored yellow in the figure are each bounded by three arcs of $K$, and by Theorem 6.2, they must belong to $K$. We will denote these four triangles by $\Delta_d$, $\Delta_e$, $\Delta_h$, and $\Delta_p$ in such a way that $\Delta_x$ admits $x$ as a vertex. As we did in Section 7, we will now decompose $K$ into five smaller triangulations $K_i$, $K_{ii}$, $K_{iii}$, $K_{iv}$, and $K_v$. 

By a similar argument as above, $S_o^-$ is disjoint from $S_f^-$ and as already mentioned, $S_o^-$ is also disjoint from $S_o^+$. Now recall that $S_o^-$ and $S_o^+$ each contain $n + 2$ tetrahedra. Hence, if $S_o^+$ and $S_f^-$ were disjoint, (6) would implies

\begin{equation}
    k \geq 2n + |A_f^-| + 4.
\end{equation}

However, as $A_f^-$ contains more than $6m + 4$ arcs, this would contradict (5) again. As a consequence, $S_o^+ \cap S_f^-$ contains a tetrahedron, and that tetrahedron has an edge with vertices $f$ and $o$ as desired.
Consider two integers $i$ and $j$ such that $\Delta_h$ is contained in $\Pi_i$ and $\Delta_e$ in $\Pi_j$. As can be seen in Fig. 21, $\Delta_e$ is below $\Delta_h$ and therefore, $i < j$. In this case, as can be seen in Fig. 21, $\Pi_i$ must contain the arc $\delta$ of $K^-$ with vertices $e$ and $o$. Denote by $\Gamma^-$ the portion of $\Pi_i$ made up of the points $x$ such that $\pi(x)$ is not on the same side of $\pi(\beta^-)$ than $h$ and not on the same side of $\pi(\delta)$ than $d$. By construction, $\Gamma^-$ is the union of a subset of the faces of $K$. Similarly, $\Pi_j$ must contain the arc $\varepsilon$ of $K^+$ with vertices $h$ and $o$. Let $\Gamma^+$ be the portion of $\Pi_j$ made up of the points $x$ such that $\pi(x)$ is not on the same side of $\pi(\beta^+)$ than $e$ and not on the same side of $\pi(\varepsilon)$ than $p$. Again, $\Gamma^+$ is the union of a subset of the faces of $K$. Therefore, cutting $K$ along $\Gamma^-$, $\Gamma^+$, and the four triangles $\Delta_d$, $\Delta_e$, $\Delta_h$, and $\Delta_p$ results in the five blow-up triangulations $K_i$, $K_{ii}$, $K_{iii}$, $K_{iv}$, and $K_v$ shown in Fig. 22. In that figure, the arcs contained in $\Gamma^-$, $\Gamma^+$ are not shown. In fact, we do not know what arcs of $K$ are contained in these disks. The arcs of $K^-$ incident to $e_i$ where $1 \leq i \leq m$ or to $h_i$ where $1 \leq i \leq n$ are not shown either because they depend on our choice for $T^-$. This decomposition of $K$ will allow us to determine the exact value of $k$. In order to do that it suffices to count the number of tetrahedra in each of the blow-up triangulations of that decomposition. We begin with $K_{iii}$. In the following, $l^-$ denotes the number of arcs in $K_{iii}^- \setminus K_{iii}^+$ incident to $o$. 

Figure 21. The four triangles $\Delta_d$, $\Delta_e$, $\Delta_h$ and $\Delta_p$ colored yellow and the arcs $\beta^-$ and $\beta^+$, colored gray. Two of these triangles are incident to $\beta^-$ (left) and two to $\beta^+$ (right).
Lemma 8.4: There are at least $5m + 5 - l^-$ tetrahedra in $K_{iii}$.

Proof. Consider the convex polygon whose vertices are the vertices of $K_{iii}$. Cutting that polygon along $\pi(\beta^+)$ results into two smaller polygons. Let us denote these smaller polygons by $\Omega$ and $\Xi$ with the convention that $\Omega$ contains $e$, and $\Xi$ contains $h$. One can see on the sketch of $K_{iii}$ at the center of Fig. 22 that $K_{iii}$ satisfies the conditions of Theorem 3.8 with respect to $\Xi$ and $\Omega$ when the arc $\beta^-$ is chosen for $\delta$. If in addition, we pick $g$ for the point $x$ in the statement of Theorem 3.8, then the blow-up triangulation $L$ provided by that
Theorem is the one we sketch on the left of Fig. 23. It is obtained by removing from $K_{iii}$ the faces that are incident to at least two vertices of $\Xi$ and by pulling to $g$ the remaining faces that are incident to a vertex of $\Xi$ other than $g$. According to Theorem 3.8, the number of tetrahedra in $L$ is less than the number of tetrahedra in $K_{iii}$ by at least twice the number of vertices of $\Xi$ minus three. In other words $L$ has at least $2m + 3$ tetrahedra less than $K_{iii}$.

The remainder of the proof proceeds just as that of Lemma 7.3. One should note here that the number of arcs in $L^- \setminus L^+$ incident to $g$ is equal to $l^- + m$. Consider the triangle in $L^-$ incident to $e$ and $g$ and denote by $\delta$ the edge of this triangle that does not admit $g$ as a vertex. Let $x$ be the vertex of $\delta$ distinct from $e$. If $x$ is equal to $e_1$, then $\pi(\delta)$ is an edge of the polygon $\Lambda$ whose vertices are the vertices of $L$. In that case, we denote by $\Omega$ that polygon and we denote by $\Xi$ the arc $\pi(\delta)$. Otherwise, we denote by $\Omega$ and $\Xi$ the two convex polygons obtained by cutting $\Lambda$ along $\pi(\delta)$ with the convention that $g$ belongs to $\Omega$ and $e_1$ to $\Xi$. One can see (for instance by looking at Fig. 23) that $L$ satisfies the conditions of Theorem 3.8 with respect to $\delta$, $\Xi$, and $\Omega$. Note in particular that, while $\delta$ belongs to $L^-$, the theorem can still be used, by symmetry. According to Theorem 3.8 one can build, from $L$ a blow-up triangulation $M$ whose number of tetrahedra in is less than the number of tetrahedra in $L$ by at least twice the number of vertices of $\Xi$ minus three. That blow-up triangulation is shown on the right of Fig. 23. It is obtained by removing from $L$ the faces that are incident to at least two points in $\Xi$ and by pulling to $x$ the remaining faces that are incident to a point in $\Xi \setminus \{x\}$. Now, observe that if $x$ is not equal to $g$, then can invoke Theorem 3.8 again, by considering the triangle of $M^-$ incident

![Figure 23. Two of the blow-up triangulations considered in the proof of Lemma 8.4. The blow-up triangulation $K_{iii}$ which they are obtained from, is shown using dashed lines.](image)
to $g$ and $x$ and its edge $\delta$ that does not admit $g$ as a vertex. Repeating this procedure will eventually result in the blow-up triangulation $N$ that we already considered in the proof of Lemma 7.3. Recall that this blow-up triangulation is shown at the bottom of Fig. 18 on the right.

According to Theorem 3.8, the number of tetrahedra in $N$ is less than the number of tetrahedra in $L$ by at least twice the number of vertices of $L$ that are no longer vertices of $N$ minus the number of times we invoked the theorem in order to change $L$ into $N$. Now recall that $L^− \setminus L^+$ contains $l^− + m$ arcs incident to $g$. As an arc of $L^− \setminus L^+$ incident to $g$ is removed each time we invoke Theorem 3.8 and exactly $m$ such arcs remains in $N$, the number of tetrahedra in $N$ is less than the number of tetrahedra in $M$ by at least $2m + 2 − l^−$. Finally, recall that $N$ contains at least $m$ tetrahedra (because there are $m$ arcs in $N^− \setminus N^+$). Summing $2m + 3$, $2m + 2 − l^−$, and $m$ results in the desired bound on the number of tetrahedra of $K$.  

When $A_f^−$ is large enough, we can now determine the exact value of $k$.

**Lemma 8.5:** If $A_f^−$ contains more than $6m + 4$ arcs, then $k = 2n + 6m + 8$.

**Proof.** Assume that $A_f^−$ contains more than $6m + 4$ arcs. We have already shown while proving Lemma 8.3 that $k$ is at most $2n + 6m + 8$. As a consequence, it suffices to establish the matching lower bound. As $A_f^−$ contains more than $6m + 4$ arcs, we know that $K$ must contain the arcs $\beta^−$, $\beta^+$ and the four triangles $\Delta_d$, $\Delta_e$, $\Delta_h$, and $\Delta_p$ shown in Fig. 21. Therefore, as discussed above, $K$ can be decomposed as shown in Fig. 22 into five blow-up triangulations $K_i$, $K_{iii}$, $K_{iii}$, $K_{iv}$, and $K_v$. Recall that this decomposition requires one to cut $K$ along the four mentioned triangles and two disks $\Gamma^−$ and $\Gamma^+$.

Now let us count the number of tetrahedra in each piece of the decomposition. By Lemma 8.4, $K_{iii}$ contains at least $5m + 5 − l^−$ tetrahedra, where $l^−$ is the numbers of arcs incident to $o$ contained in $K_{iii}^− \setminus K_{iii}^+$. Let us study the other four blow-up triangulations. Observe that $K_v$ contains at least $m + 1$ tetrahedra because $K_v^− \setminus K_v^+$ contains exactly $m + 1$ arcs. The situation is slightly different for $K_i$ because there may be less than $m + 1$ arcs in $K_i^− \setminus K_i^+$. However, we know that $o$ is incident to at least $l^−$ arcs from $K_i^+ \setminus K_i^−$ because $K_i$ and $K_{iii}$ are glued along $\Gamma^−$ (see Fig. 22). None of these arcs can be contained in both $K_i^−$ and $K_i^+$ because they are above the arc in $K_i^−$ with vertices $e$ and $p$. As a consequence, the number of tetrahedra in $K_i$ is at least $l^−$. Finally, observe that
$K_{ii}$ and $K_{iv}$ must each contain at least $n + 1$ tetrahedra. Summing $5m + 5 - l^-$, $m + 1$, $l^-$, and two times $n + 1$ provides the desired bound.

Interestingly, Lemma 8.5 shows that the number of tetrahedra in $K$, or equivalently the distance of $T^-$ and $T^+$ in $F(\Sigma)$ does not depend on $A^-_f$ or $A^-_p$ except maybe when $A^-_f$ contains few arcs. In particular, when $|A^-_f| > 6m + 4$ and

$$2m \leq n \leq m(7m + 5),$$

it follows from Lemma 8.5 and from Propositions 8.1 and 8.2 that the first flip in the crossings-based distance estimation method will never start a geodesic from $T^-$ to $T^+$. In fact, we have proven that the triangulation of $\Sigma$ resulting from that flip will be at the same distance of $T^+$ in $F(\Sigma)$ than $T^-$. 

**Theorem 8.6:** There exist triangulation pairs whose distance estimate via the crossings-based method is greater than their distance in $F(\Sigma)$ by a multiplicative factor that can be arbitrarily close to $3/2$.

**Proof.** Let us now assume that $T^-$ contains, as a subset, the comb triangulation of $\Sigma_h$ at vertex $f$ and the comb triangulation of $\Sigma_e$ at vertex $p$. Assume that $n$ is equal to $m(7m + 5)$ and that $m$ is positive. In this case, the assumptions of Propositions 8.1 and 8.2 are satisfied and it follows that the first $n - 7m - 5$ flips performed by the method will flip arcs contained in $A^-_f \cup A^-_p$. After these flips, $A^-_f \cup A^-_p$ still contains $7m + 5$ arcs and therefore $A^-_f$ contains at least $6m + 5$ arcs. By Lemma 8.5, the distance to $T^+$ of the last triangulation obtain in the process is still equal to $2n + 6m + 8$. Hence the method overestimates the distance of $T^-$ and $T^+$ by a factor of at least

$$1 + \frac{n - 7m - 5}{2n + 6m + 8}.$$

By our choice of $n$ as a quadratic function of $m$, it suffices to let $m$ go to infinity in this expression to prove the theorem.

Let us finally consider the case when $\Sigma$ is an arbitrary oriented topological surface with finite genus and with a finite number of boundary components. We will show that Theorem 8.6 still holds for $\Sigma$ but first, we briefly recall what we call a triangulation of $\Sigma$. Pick a finite number of marked points in $\Sigma$, making sure that there is at least one such point in each of the boundary components of $\Sigma$. A triangulation of $\Sigma$ is a set $T$ of pairwise non-crossing and pairwise non-isotopic arcs between these points that is maximal for the inclusion. In
fact, we consider the arcs of $T$ up to isotopy. In other words, the arcs in $T$ are isotopy classes of non-oriented paths that connect two vertices of $T$, and we ask that these classes can be realized disjointly. In such a triangulation of $\Sigma$, some arcs may be loops (when their ends coincide), and several arcs may connect the same pair of points, but they are still triangulations in the sense that the complement in $\Sigma$ of their union is a finite collection of triangles. As in the case of a polygon, a flip is the operation of removing an arc from a triangulation of $\Sigma$ and by replacing it with the only other arc such that the resulting set is a triangulation. The flip-graph $\mathcal{F}(\Sigma)$ of $\Sigma$ can then be defined just the same way as for polygons. This setup is considered for instance in [9].

Now consider a topological surface $\Sigma$ such that the set of marked points contains $2n + 6m + 8$ punctures (which, in this context means marked points that are not contained in a boundary component). Let $\gamma$ be a closed curve embedded in $\Sigma$ through all of these punctures. We can choose $\gamma$ in such a way that one of the surfaces obtained when $\Sigma$ is cut along $\gamma$ is a disk with no puncture in its interior. Denote by $\Sigma'$ and $\Sigma''$ the two surfaces we obtain this way, with the convention that $\Sigma'$ is a disk. By construction $\Sigma'$ has $2n + 6m + 8$ marked points in its boundary. Therefore, the flip-graph of this $\Sigma'$ is precisely the same as the flip-graph of a convex polygon with $2n + 6m + 8$ vertices. Moreover, if $T''$ is a triangulation of $\Sigma''$ it is known that the subgraph induced in $\mathcal{F}(\Sigma)$ by the triangulations of $\Sigma$ that admit $T''$ as a subset is strongly convex.

This is a consequence of the strong convexity in $\mathcal{F}(\Sigma)$ of the subgraph $\mathcal{F}_\varepsilon(\Sigma)$ induced by the triangulations that contain a given arc $\varepsilon$ [9]. Therefore, if we complete $T''$ by embedding into $\Sigma'$ each of the two triangulations $T^-$ and $T^+$ of a convex polygon with $2n + 6m + 8$, the distance of these triangulations in $\mathcal{F}(\Sigma)$ will be the same as in the flip-graph of the polygon, and the crossings-based distance estimation method will perform the same sequence of flips. Hence, as we announced, Theorem 8.6 still holds in that case.

Acknowledgement. Both authors are partially supported by the ANR project SoS (Structures on Surfaces), grant number ANR-17-CE40-0033.

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