Exotic superfluid of trapped Fermi gases with spin–orbit coupling in dimensional crossover

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Abstract
We investigate the ground state properties of Fermi gases in a planar array of one-dimensional potential tubes with spin–orbit coupling where the motion of atoms is free in the \( \hat{x} \)-direction and the tunneling between nearest tubes in the \( \hat{y} \)-direction is permitted. By using the mean-field method, the phase diagrams of the system at the dimensional crossover from quasi-one dimension to quasi-two dimensions is obtained. We find the existence of the topological state and Majorana mode in the weak tunneling case, and a rich phase diagram including two kinds of nodal superfluid phase and gapped superfluid phase, in the opposite case. The results show that topological pairing is favored in quasi-one dimension while nodal pairing state is favored in quasi-two dimensions.

Keywords: exotic superfluid, spin–orbit coupling, dimensional crossover, topological superfluid, nodal superfluid

(Some figures may appear in colour only in the online journal)

1. introduction
Spin–orbit coupling (SOC) has received tremendous attention due to its rich physics. For instance, it gives rise to novel transport properties in semiconductor materials [1–3], and plays a crucial role in the formation of many important phenomena including topological insulators [4], topological conductors [5], and the quantum spin Hall effect [6]. In recent years, with the help of Raman laser coupling, SOC has been implemented experimentally [7–9] in ultra-cold neutral atomic system and has attracted considerable attention both theoretically and experimentally [10–12]. Due to its highly flexibility, the cold atom system is considered as an ideal platform for the quantum simulation of interesting physics models, ranging from nuclear physics [13, 14] to condensed matter physics [15, 16]. This provides a great opportunity to investigate such novel SOC physics in a highly controllable way.

The Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) state was first proposed in the 1960s [17, 18] and is an exotic superfluid pairing state with finite momentum compared with the conventional superconductor with zero momentum pairing. In general, the FFLO state emerges with a mismatched Fermi surface of imbalanced Fermi gas. Therefore, it has been viewed as one of the most interesting superconduction phases, and is studied in various fields, ranging from heavy fermions [19–21], organic superconductors [22] to ultra-cold atomic gases [23–28]. However, due to the small pairing phase space
in 2D and 3D, only phase separation is observed in experiments rather than the FFLO state. This difficulty is avoided in an 1D system, where the Fermi surface is a point, and the pairing phase space is not affected by the spin imbalance. However, the large quantum fluctuations in 1D might suppress the long-range order. With spin–orbit coupling and a Zeeman field, the spatial inversion symmetry is broken and pairing on a single Fermi surface is possible. A theoretical investigation indicated that the FFLO state is favoured in this system with nontrivial topology [29–32].

In this paper, we consider an array of Fermi tubes with 1D SOC. The atoms can move freely along the $\hat{x}$-direction while nearest-tube tunneling is allowed in our model [33]. The tunneling can be adjusted over a wide range by tuning the optical lattice in this direction. The tunneling between nearest tubes on these novel phases is also studied. Detailed calculation shows that the presence of a transverse Zeeman field eradicates the zero center-of-mass momentum pairing, and stabilizes the nodal FFLO state with finite center-of-mass momentum pairing along the anisotropic SOC direction. We also obtain the phase diagrams with different interactions and tunneling, and show the interaction effect and how the crossover physics can be explored by tuning the tunneling.

The rest of this paper is organized as follows: in section 2, we introduce our model and detailed calculation method based on the standard Bogoliubov–de Gennes (BdG) theory. In section 3, we discuss the phase diagram of a quasi-1D system with small inter-tube tunneling. The two-coupled-tubes model is used to analyze the critical tunneling between a topological FFLO state and the normal FFLO state, we use the two-coupled-tubes model to analyze the phase transition and find the critical tunneling. It indicates that the topological FFLO state exists not only in 1D [34, 35], but also in quasi 1D. For the quasi-2D system, we find that the system supports novel gapless superfluid phases which can be classified by the number of gapless points in the energy spectra. The influence of the transverse Zeeman field, interaction strength, and tunneling between nearest tubes on these novel phases is also studied. Detailed calculation shows that the presence of a transverse Zeeman field eradicates the zero center-of-mass momentum pairing, and stabilizes the nodal FFLO state with finite center-of-mass momentum pairing along the anisotropic SOC direction. We also obtain the phase diagrams with different interactions and tunneling, and show the interaction effect and how the crossover physics can be explored by tuning the tunneling.

Figure 1. Illustration of the model in an array of coupled tubes in a 2D plane, which can realize dimensional crossover.

2. Model Hamiltonian and calculation method

We start by considering a two-component Fermi gas with SOC and a Zeeman field in an array of tubes. As shown in figure 1, the red tubes are parallel to the $\hat{x}$-direction. Two counter-propagating laser lights along the $\hat{y}$-direction construct a 1D optical lattice in this direction. The tunneling between nearest tubes is permitted in the model. The type of SOC is induced by another two counter-propagating Raman laser lights along the $\hat{y}$-direction. The Hamiltonian can be written as

$$H = -\sum_{\sigma} \mu_{\sigma} N_{\sigma} = H_0 + H_{SOC} + H_{Zee} + H_{int},$$

with

$$H_0 = \sum_{\sigma,k,j} \varepsilon_{k} C_{\sigma,k,j}^{\dagger} C_{\sigma,k,j} - t \sum_{\sigma,k_{i},(k_{f})} C_{\sigma,k_{i}}^{\dagger} C_{\sigma,k_{f}} + \text{h.c.},$$

$$H_{SOC} = \sum_{k,j} (\alpha k_{x} C_{\alpha,k,j}^{\dagger} C_{\alpha,k,j} - \alpha k_{x} C_{\alpha,k,j}^{\dagger} C_{\alpha,k,j}),$$

$$H_{Zee} = \sum_{k,j} \hbar g (C_{\sigma,k,j}^{\dagger} C_{\bar{\sigma},k,j} + C_{\bar{\sigma},k,j}^{\dagger} C_{\sigma,k,j}),$$

$$H_{int} = \sum_{k,k',Q} g \sum_{\sigma} C_{k+Q/2}^{\dagger} C_{k'}^{\dagger} C_{k'} + \text{c.c.},$$

where $\varepsilon_{k} = \frac{\hbar^{2} k^{2}_{x}}{2m} - \mu$, $C_{\sigma,k,j}^{\dagger}$ ($C_{\sigma,k,j}$) is the annihilation (creation) operator for the hyperfine spin state $\sigma$ in the $j$th tube. $m$ is the atom mass, $\alpha$ denotes the strength of the SOC, $t$ is the near-rest tunneling between different tubes, and $\text{h.c.}$ represents the Hermitian conjugate. The Zeeman field contains two terms: in-plane ($h_{\parallel}$) and out-of-plane ($h_{\perp}$) magnetic fields, which are included in the current experimental scheme. $g$ is the bare strength of the s-wave attractive interaction.

Throughout the paper, we only focus on the zero temperature properties of the system. The order parameter of the FF pairing state is defined as $\Delta = \Delta_{Q} = g \sum_{k} (C_{k+Q/2}^{\dagger} C_{k+Q/2})$. Within the mean-field approximation, we can rewrite the Hamiltonian based on the Nambu spinor base $\Psi_{k} = [C_{k+Q/2}^{\dagger}, C_{k+Q/2}^{\dagger}]^{T}$ as follows

$$H = -\sum_{\sigma} \mu_{\sigma} N_{\sigma} = \frac{1}{2} \sum_{k} \hat{H}_{BdG} \Psi_{k} + \sum_{k} \Psi_{k}^{\dagger} + \frac{\Delta_{Q}^{2}}{g},$$
with the BdG Hamiltonian
\[
H_{\text{BdG}} = \begin{pmatrix}
\lambda_{k+Q/2}^+ & h_x & 0 & \Delta_Q \\
h_x & -\lambda_{k+Q/2}^- & -\Delta_Q & 0 \\
0 & -\Delta_Q & -\lambda_{k+Q/2}^+ & -h_x \\
\Delta_Q & 0 & -h_x & -\lambda_{k+Q/2}^- \\
\end{pmatrix},
\]
where \( \lambda_{k+Q/2}^\pm = \hbar^2 k_{x}^2/2m - 2\hbar^2 (\cos(k_x) + 1) \pm (\omega_{k_x} - \hbar) \), and \( \omega_{k_x} = \hbar^2 k_{x}^2/2m - 2\hbar^2 (\cos(k_x) + 1) - \mu \). Since there is no constraint in the \( \bar{x} \)-direction, \( k_x \) still remains to be a good quantum number, while \( k_y \) is confined in the first Brillouin \( [k_y] \leq \pi \). The tunneling, \( t \), is highly controllable by changing the lattice height in the \( \bar{y} \)-direction, which allows us to test the crossover physics from 1D to 2D.

In order to get the zero temperature phase diagram, we straightforwardly diagonalize the BdG Hamiltonian and calculate the thermodynamic potential \( \Omega = -\frac{1}{\beta}\log\text{Tr}e^{-\beta H_{\text{BdG}}} \). For zero temperature, the thermodynamic potential is:
\[
\Omega = \frac{1}{2} \sum_{k} \Theta^{-1}(E_k) E_k^\dagger E_k^\nu + \frac{|\Delta_Q|^2}{g},
\]
where the quasi-particle (hole) dispersion \( E_k^\nu \) is the eigenvalues of the BdG Hamiltonian. And \( \Theta \) is the Heaviside step function. For the sake of simplicity, we can assume the order parameter \( \Delta_Q \) to be real throughout the whole work. In this model, due to the complex 4 \( \times \) 4 matrix \( H_{\text{BdG}} \), the analytical expression of quasi-particle dispersion \( E_k^\nu \) is too complicated to be presented here, so we will perform our calculation numerically. In general, the ground state can be obtained from the saddle point equations \( \partial \Omega / \partial Q = 0 \), \( \partial \Omega / \partial \mu = 0 \) and \( \partial \Omega / \partial \omega = -n \). However, Due to competition between pairing and the Zeeman field, the thermodynamic potential shows a double well structure. We may get a metastable state from the gap equation. In order to find the true ground state, we numerically search for the global minimum of the thermodynamic potential to obtain the order parameter \( \Delta_Q \) as well as the finite mass momentum \( Q \).

We note that in our work, the phase diagram is obtained by fixing the chemical potential throughout the whole system. For a trapped gas system, the local chemical potential is defined as \( \mu(\mathbf{r}) = \mu - V(\mathbf{r}) \), which is known as the local density approximation (LDA). In principle, based on our calculation, we can sweep the chemical potential to obtain the phase structure in a trapped system. The total particle number can be evaluated by integrating the local particle density from the trap center to the edge.

### 3. Quasi-1D result

#### 3.1. Phase diagram

For the fixed chemical potential, the pairing order parameter is obtained by searching for the global minimum of the thermodynamic potential. In the calculation, we find that with not large enough \( \hbar \), the FF state with finite pairing momentum \( Q = Q_0 \) is stable in a fairly large parameter region. We map out the phase diagram in \( \alpha - \mu \) plane, where a topological FF (tFF) phase is always sandwiched between a normal FF (gFF) state and a normal state (N) without pairing.

The tFF state can be understood from the symmetry of the BdG Hamiltonian. We note that this Hamiltonian possesses particle-hole symmetry as \( \Xi H_{\text{BdG}} \Xi^{-1} = \Delta h_{\text{BdG}} \Lambda \) with \( \Xi = \Lambda K, \Lambda = (\sigma, \tau_z) \) and \( K \) is the complex conjugation operator. The Pauli-spin matrices and \( \sigma_{x,y,z} \) are the Pauli particle-hole matrices. In this case, the topological character is verified by the topological number—Pfaffian invariant \( Q \) [39], which can in principle be computed in terms of the eigenvalues of \( H_{\text{BdG}} \). The topological phase corresponds to \( Q = -1 \), while the topological trivial state corresponds to \( Q = 1 \). In general, the topological nature is protected by the gap in the bulk quasi-particle excitation. The phase boundary between the tFF state and the gFF state can be determined by the gap condition. According to the symmetry, the gap closes at a single point \( k = (0, 0) \). With a simple matrix determinant calculation, we find the energy gap close and reopen condition is:
\[
h^2 = \Delta_Q^2 - h_x^2 + \alpha \hbar Q_x - \frac{1}{4} d^2 Q_x^2 + \left( \mu - \frac{Q_x^2}{4} \right)^2.
\]
In the special case that both the in-plane Zeeman field \( h_x \) and the pairing momentum \( Q_x \) are equal to zero, the above formula reduces to the standard condition for BCS topological superfluid \( h^2 = \Delta^2 + \mu^2 \) [40].

In figure 2, we show the phase diagrams of a quasi-1D system, where the first-order phase transition is shown with a black solid line while the topological phase transition is shown with a dashed burgundy curve. The red dotted line marks the threshold. To the left of the threshold curve, the pairing gap \( \Delta/h_x < 10^{-3} \), and it decreases exponentially as the spin–orbit coupling strength approaches zero. Considering the fluctuation in 2D system at finite temperature, experimental observation of the tFF state is only possible to the right of the threshold. So we define the normal state for the left of the threshold curve. The phase boundary between the gFF state and the normal state is determined from the double-well structure of the thermodynamic potential. The tFF state in the phase diagram is verified by the topological invariant Pfaffian and the topological condition above. Comparing the two plots in figure 2, it is obvious that the tFF state region is much larger with weaker tunneling \( t \). This means that the tFF state of our system is much more favored in quasi-1D geometry than in a quasi-2D one.

In order to capture the topological properties of different phases, we draw figure 3 to show the value of topological invariant Pfaffian. As shown in this figure, the topological state corresponds to \( -1 \), while the trivial state corresponds to 1. According to the value of the superfluid order parameter, the whole range of the chemical potential can be divided into five parts with different phases.

#### 3.2. Two coupled tubes model

To investigate the effect of tunneling on the phase transition, we further consider a simplified toy model with only two
coupling tubes. Within the mean-field approximation, we calculate the Bogliubov spectra using the standard real space Bogoliubov–de Gennes equation as

\[ H_{\text{BdG}}(x) \Psi(x) = E_n \Psi(x), \]  

where \( \Psi(x) = [u_{n,1}^i, u_{n,2}^i, u_{n,1}^j, u_{n,2}^j, v_{n,1}^i, v_{n,2}^i, v_{n,1}^j, v_{n,2}^j]^T \). Here, \( u_{n,1}^i, v_{n,1}^i \) and \( u_{n,1}^j, v_{n,1}^j \) are the wave functions of the \( n \)th quasiparticle and the \( i \)th tube. The pairing parameter of the \( i \)th tube is defined as

\[ D = \sum_{n} \left( v_{n,1}^i u_{n,1}^i + v_{n,1}^j u_{n,1}^j \right) \langle E_n \rangle \sin \left( \frac{m\pi x}{L} \right). \]  

The in-plane Zeeman field is taken as the energy unit, while the unit of momentum \( k_{x} \) is defined through \( \hbar^2 k_{x}^2 / 2m = \hbar_{x} \). The other parameters used in this plot are: \( g / \hbar_{x} = -0.5 \), \( h / \hbar_{x} = 0.2 \).

Figure 2. The phase diagram in the \( \alpha - \mu \) plane with different tunneling \( t \) (left: \( t = 0.02 \); right: \( t = 0.08 \)). The first-order phase transition is shown with a black solid line and the topological phase transition with a dashed burgundy curve. The red dotted line marks the \( \Delta / \hbar_{x} = 10^{-3} \) threshold. The in-plane Zeeman field is taken as the energy unit, while the unit of momentum \( k_{x} \) is defined through \( \hbar^2 k_{x}^2 / 2m = \hbar_{x} \). The other parameters used in this plot are: \( g / \hbar_{x} = -0.5 \), \( h / \hbar_{x} = 0.2 \).

Figure 3. The topological invariant Pfaffain is plotted for varying chemical potential. The spin–orbit coupling is fixed as \( \alpha k_{x} = 0.5 \). The other parameters of this figure are set as: \( t = 0.02 \), \( g = -0.5 \), \( h = 0.2 \), \( k_{x} = 1 \). According to the value of superfluid parameter, the whole range of the chemical potential is divided into five parts.

Figure 4. Phase diagram for a given spin–orbit coupling \( \alpha k_{x} / E_{F} = 0.6 \) of the two coupled tubes model, is determined from the real space BdG calculation. The Bogoliubov excitation spectra is calculated by diagonalizing a large matrix in the real space. As tunneling increases, a phase transition occurs, and the system evolves from a tFF state to a gFF state. The critical tunneling \( t = 0.1 \). The other parameter is set as: \( g = -0.3 \), \( h = 0.2 \), \( k_{x} = 0.2 \), \( \alpha k_{x} / E_{F} = 0.6 \). The energy unit \( E_{F} \) is determined by the total particle number.

The wave functions as

\[ u_{n,1}^i = \sum_{m} A_{n,m}^i \frac{2}{L} \sin \left( \frac{m\pi x}{L} \right), \]  

\[ u_{n,1}^j = \sum_{m} A_{n,m}^j \frac{2}{L} \sin \left( \frac{m\pi x}{L} \right), \]  

\[ v_{n,1}^i = \sum_{m} B_{n,m}^i \frac{2}{L} \sin \left( \frac{m\pi x}{L} \right), \]  

\[ v_{n,1}^j = \sum_{m} B_{n,m}^j \frac{2}{L} \sin \left( \frac{m\pi x}{L} \right). \]  

\( L \) is the length of the tube, which is fixed as \( 100\pi \). In our calculation, the cutoff number of the base function is set to be \( N = 300 \). An open boundary condition is used in the calculation. The emergence of a topological FF superfluid...
state can be clearly revealed by the behavior of the lowest eigenenergy of the quasi-particle energy spectrum. As shown in figure 4, when \( t \) is below 0.1, the absolute value of the minimum Bogoliubov energy is quite small (about \( 10^{-4} \) order). However, the minimum energy is not zero, due to the coupling between two tubes and coupling between two Majorana modes in the ends of each tube. The absolute value of the minimum energy changes to be \(-10^{-1} \) order with tunneling above 0.1. An obvious leap emerges at a critical value \( t = 0.1 \). This signifies that there must be a phase transition.

In order to get the topological properties of the phase transition at \( t = 0.1 \), we draw figure 5 to show the energy spectra and the wave functions of the states at two sides of the critical tunneling parameter. In the case with \( t = 0.05 \), there are two zero modes in the spectra, which are known as Majorana fermions. (b) The wave function of the zero mode. Obviously, it is localized at the edge of the tubes. (c) The energy spectra with \( t = 0.15 \), which is above the critical tunneling \( t = 0.1 \). In contrast with the previous case, the spectra are fully gapped. (d) The wave function of the fully gapped case, which is extensional in the whole tubes. The other parameters in this figure are set as: \( g = -0.3, \ h = 0.2, \ h_x = 0.2, \ \alpha k_e/E_F = 0.6 \). The Fermi energy \( E_F \) is determined by the total particle number.

Figure 5. Bogoliubov spectra \( E_n \) and wave functions with different tunnelings are shown in this figure. (a) The energy spectra with \( t = 0.05 \). There are two zero modes in the spectra, which are known as Majorana fermions. (b) The wave function of the zero mode. Obviously, it is localized at the edge of the tubes. (c) The energy spectra with \( t = 0.15 \), which is above the critical tunneling \( t = 0.1 \). In contrast with the previous case, the spectra are fully gapped. (d) The wave function of the fully gapped case, which is extensional in the whole tubes. The other parameters in this figure are set as: \( g = -0.3, \ h = 0.2, \ h_x = 0.2, \ \alpha k_e/E_F = 0.6 \). The Fermi energy \( E_F \) is determined by the total particle number.

In conclusion, we get the topological properties and phase transition of the ground state with different tunneling. As we know, the topological superfluid state is stable in a 1D system with spin–orbit coupling \([34, 35]\). We also show the topological phase transition for a quasi-1D system. The critical tunneling for the phase transition is obtained by using a real space BdG calculation method.

4. Quasi-2D result

4.1. Pairing state with zero center-of-mass momentum

In last section, we analyze the phase diagrams and the topological state with weak tunneling \( t \). To better understand the crossover physics, in this section, we focus on the quasi-2D geometry where the tunneling, \( t \), can be changed over a wide parameter range. We start by studying the pairing states with a zero out-of-plane Zeeman field. When \( h = 0, H_{\text{BdG}} \) can be diagonalized analytically. It can be verified numerically that in this case, pairing with zero center-of-mass momentum \( Q_z = 0 \) is stable against finite momentum pairing. The analytical expression of the quasi-particle and quasi-hole dispersions along \( k_z = 0 \) axis can be written as \( E_{\lambda}^{\lambda} = \lambda (\sqrt{\xi_{k_y}^2 + \Delta^2} \pm h_z) \) with \( \lambda = \pm \), and \( \xi_{k_y} = -2t (\cos(k_y) - 1) \). Apparently, only two of the branches \( E_{\lambda}^{\lambda} \) can cross zero.
And this leads to the exotic nodal superfluid (nSF) with gapless points in the energy spectra. It is interesting to classify nSF state with a different number of gapless points. Typically, there can be two kinds of nSF state with two or four isolated gapless points, as shown in Figure 6. A simple calculation shows that when \( \mu = \sqrt{h_1^2 - \Delta^2} \in [0, 4\pi] \), there will be four isolated gapless points in the energy spectra, while the energy spectra is fully gapped when \( \mu \leq \sqrt{h_1^2 - \Delta^2} \notin [0, 4\pi] \). Otherwise, we find that only two gapless points appear in the dispersion.

In Figure 7, we map out the phase diagram with different interaction \( g \) on the \( \alpha - \mu \) plane. Compared with phase diagrams with Rashba SOC in 2D [36], the topological phase
is now replaced by the nodal superfluid state. We focus on the nodal superfluid in our quasi-2D model. With anisotropic SOC, superfluid state with zero center-of mass momentum pairing is still stable against a normal state with a large in-plane Zeeman field $h_x$. As we have anticipated, there are two kinds of nodal superfluid states with either four isolated nodes or two isolated nodes in the energy spectra, as shown in figure 7(a). The four-node superfluid disappears in figure 7(b) as we increase the interaction strength to $g = -0.65$. It means that the nodal superfluid is more stable with weaker interaction. To study the phase transition in the case of dimensional crossover from 1D to 2D, we also plot the phase diagrams with different tunneling $t$ as shown in subgraphs (a, c, d) of figure 7. For fixed interaction strength $g = -0.35$, one can see that the four-nodal superfluid region shrinks rapidly and finally disappears from the phase diagram with $t = 0.7$ (figure 5(d)). Therefore, the four-nodal phase is favored with relatively larger ratio $|t/g|$. We note that in these cases, nodal

Figure 8. Phase diagrams with different tunneling $t$. The chemical potential $\mu$ in (a) and (b) is set as $\mu = 0.6$ and $\mu = 0.9$, respectively. The other parameters are set as follows: $g = -0.3$; $\alpha k_h/h_x = 1.2$, $h_x = 1$, $h = 0$.

Figure 9. Phase diagram of finite momentum pairing state in $\alpha - \mu$ plane. We still analyze quantum phase with different interaction and different tunneling. For (a) and (b), the interaction is set as: (a) $g = -0.35$; (b) $g = -0.65$, respectively. The other parameters for the two figures are set as: $t = 1$, $h_x = 1$, $h = 0.2$. For (c) and (d), the tunneling is set as: (c) $t = 0.9$; (d) $t = 0.5$, respectively. The other parameters of the two figures are set as: $g = 0.35$, $h_x = 1$; $h = 0.2$. In all the figures, the black solid lines represent the first-order phase transition boundaries while the dashed burgundy lines represent the continuous phase transition boundaries. The red dotted curve corresponds to a $\Delta = 10^{-3}$ threshold.
phases possess a relatively large region. This is very different from the quasi-1D case, where the gFF state dominates the superfluid region, and tFF state disappears very quickly as we increase the inter-tube tunneling. Therefore, we can claim that nodal-superfluid is more favored in quasi-2D geometry than the quasi-1D case.

To illustrate the crossover behaviors of the SF phases along with the tunneling, we also plot the evolution of pairing parameter $\Delta$ as a function of $t_{\perp}/h_{\perp}$ from 0.5 to 1.5 by fixing all other parameters. As shown in figure 8(a), one can see below a critical tunneling value $t_{\perp}/h_{\perp} = 1.04$, $\Delta$ changes smoothly. This corresponds to the normal pairing state and the energy spectra are gapped. However, the gap closes and the ground state changes to a four-node pairing phase (nSF2) when the tunneling is above the critical value. This transition is of first order which can be verified by the the discontinuity of $\partial \Delta / \partial t_{\perp}$ across the transition point. The ground state becomes more complicated in figure 8(b) where two critical values $t_{\perp}/h_{\perp} = 0.94$ and $t_{\perp}/h_{\perp} = 1.08$ are involved. Besides a first-order transition from the normal pairing state to the nSF2 phase, the ground state changes to nSF1 state when the tunneling is above $t_{\perp}/h_{\perp}$. We note that, in this case, the transition is continuous which is also consistent with the results shown in figure 7.

### 4.2. FF pairing state with an out-of-plane Zeeman field

When a transverse Zeeman field is introduced, the presence of SOC leads to an asymmetric deformed Fermi surface. In this case, s-wave pairing with zero center-of-mass momentum becomes unfavorable, which is replaced with FF states with finite $Q_x$. Qualitatively, this corresponds to a shift about the local minimum of the thermodynamic potential from $Q_x = 0$ to a finite $Q_x$ plane. When the shift is small, most of the properties of the FF state are similar with the pairing states with zero center-of-mass momentum.

In figure 9, we map out the phase diagrams in the $\alpha - \mu$ plane with different interaction and different tunneling. The calculation shows that these diagrams share similar patterns shown in figure 7, except that the nodal superfluid states are now replaced by nodal FF states. We stress that, in all these diagrams, the FF state is hosted in the $\alpha - \mu$ plane. This is very different from the usual 2D polarized Fermi gases without SOC, where FFLO states can only be stabilized within a small parameter region. Therefore, FF states becomes more stable due to the presence of a transverse Zeeman field and SOC, which greatly increases the possibility of finding such exotic pairing states in our system.

Figures 9(a) and (b) show the phase diagrams with different interactions. We find that the first-order phase transition line is longer with larger interaction. In addition, a nodal FF state is more favourable with smaller interaction. For instance, when interaction strength $g = -0.35$, there are two kinds of nodal-FF state, two-node FF state (nFF1) and four-node FF state (nFF2). As the interaction strength $g$ increases, nFF2 disappears gradually and the region for the nFF1 phase also becomes smaller. It is expected that when the interaction is
very strong, all the nodal FF states will be replaced by fully gapped FF states.

To illustrate the crossover physics of FF states, we also plot the phase diagrams for different tunnelings \( t = (1, 0.9, 0.5) \), as shown in figures 9(a), (c) and (d) respectively. When \( t = (1, 0.9) \), both nFF1 and nFF2 phase appear in the phase diagram, although the region of nFF2 shrinks rapidly as \( t \) decreases. When the tunneling decreases to 0.5, only nFF1 is left in the phase diagram. Therefore, we assert that nFF2 is favourable with larger tunneling. In this case, it is more appropriate to treat the system as a 2D one. When tunneling decreases further, the situation becomes more complicated, and the tFF state comes into play as shown in figure 2. Figure 10 shows the changes of the pairing order parameter and center-of-mass momentum as we increase the inter-tube tunneling \( t \). One can see that a fully gapped FF state always exists in the dimensional crossover region, while nodal an FF state emerges around \( t = 1 \). The center-of-mass momentum \( |Q_c| \) increases almost monotonically along with \( t \). The ground state becomes a normal state when \( t \) is large enough.

5. Conclusion

In summary, we have studied the pairing state in the dimensional crossover region in an array of coupling tubes. The interplay of SOC, Zeeman field, and tunneling leads to a rich phase structure with exotic pairing states. In particular, the ground state is an FFLO state with finite center-of-mass momentum pairing over a large parameter region, which facilitates the experimental verification of this kind of exotic pairing state. For a quasi-1D system, there is a tFF state in the phase diagram, which is characterized by the presence of a Majorana edge state in the model. In order to qualitatively show the transition from the tFF state to the normal SF state in the crossover region, a toy model composed of two-coupled tubes is studied to show the critical tunneling. Furthermore, phase diagrams with different interaction and inter-tube tunneling are also presented. Except for fully gapped states, we find that there is a large parameter region for which pairing states with isolated gapless excitations in the dispersion appears. nSF and nFF states exist when the system is a quasi-2D one, and finally they change to a normal state with further increasing the inter-tube tunneling.

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