A FAMILY OF FUNCTIONAL EQUATIONS RELATED TO THE MONOMIAL FUNCTIONS AND ITS STABILITY

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ABSTRACT. Our aim of this paper is to study a family of functional equation in vector and Banach spaces with difference operators, where this family of functional equation is a general mixed additive-quadratic-cubic-quartic functional equations. We show that every function satisfies the our functional equation is a monomial function with a certain degree. Furthermore, we deal with the generalized Hyers-Ulam stability of this family of functional equations in Banach space.

1. Background results

Throughout this paper, assume that \( F = \mathbb{Q}, \mathbb{R} \) or \( \mathbb{C} \) and \( V \) and \( B \) are vector spaces over \( F \) and \( X \) is a Banach spaces over \( F \). We present some background results based of paper [1]. A function \( a : V \rightarrow B \) is said to be additive provided \( a(x+y) = a(x)+a(y) \) for all \( x, y \in V \); in this case it is easily seen that \( a(rx) = ra(x) \) for all \( x \in V \) and all \( r \in \mathbb{Q} \).

If \( k \in \mathbb{N} \) and \( a : V^k \rightarrow B \), then we say that \( a \) is \( k \)-additive provided it is additive in each variable; we say that \( a \) is symmetric provided \( a(x_1, x_2, \ldots, x_k) = a(y_1, y_2, \ldots, y_k) \) whenever \( x_1, x_2, \ldots, x_k \in V \) and \( (y_1, y_2, \ldots, y_k) \) is a permutation of \( (x_1, x_2, \ldots, x_k) \).

If \( k \in \mathbb{N} \) and \( a : V^k \rightarrow B \) is symmetric and \( k \)-additive, let \( a^*(x) = a(x, x, \ldots, x) \) for \( x \in V \) is and note that \( a^*(rx) = r^k a(x) \) whenever \( x \in V \) and \( r \in \mathbb{Q} \). Such a function \( a^* \) will be called a monomial function of degree \( k \) (assuming \( a^* \neq 0 \)).

A function \( p : V \rightarrow B \) is called a generalized polynomial (GP) function of degree \( m \in \mathbb{N} \) provided there exist \( a_0 \in B \) and symmetric \( k \)-additive functions \( a_k : V^k \rightarrow B \) (for \( 1 \leq k \leq m \)) such that

\[
p(x) = a_0 + \sum_{k=1}^{m} a_k^*(x) \quad \text{for all } x \in V,
\]

and \( a_m^* \neq 0 \). In this case

\[
p(rx) = a_0 + \sum_{k=1}^{m} r^k a_k^*(x) \quad \text{for all } x \in V \text{ and } r \in \mathbb{Q}.
\]

Let \( B^V \) denote the vector space (over \( F \)) consisting of all maps from \( V \) into \( B \). For \( h \in V \) define the linear difference operator \( \Delta_h \) on \( B^V \) by

\[
\Delta_h f(x) = f(x+h) - f(x) \quad \text{for all } f \in B^V \text{ and } x \in V.
\]

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Notice that these difference operators commute \((\Delta_{h_1} \Delta_{h_2} = \Delta_{h_2} \Delta_{h_1})\) for all \(h_1, h_2 \in V\) and if \(h \in V\) and \(n \in \mathbb{N}\), then \(\Delta_h^n\)–the \(n\)-th iterate of \(\Delta_h\)–satisfies
\[
\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh) \quad \text{for } f \in B^V \text{ and } x, h \in V.
\]

The following theorem were proved by Mazur and Orlicz [2] and [3], and in greater generality by Djoković [4].

**Theorem 1.1.** If \(n \in \mathbb{N}\) and \(f : V \to B\), then the following are equivalent.

1. \(\Delta_h^n f(x) = 0\) for all \(x, h \in V\).
2. \(\Delta_{h_1} \cdots \Delta_{h_n} f(x) = 0\) for all \(x, h_1, \ldots, h_n \in V\).
3. \(f\) is a GP function of degree at most \(n - 1\).

The starting point of the stability theory of functional equations was the problem formulated by S. M. Ulam in 1940 (see [5]), during a conference at Wisconsin University:

Let \((G,.)\) be a group \((B,.,d)\) be a metric group. Does for every \(\varepsilon > 0\), there exists a \(\delta > 0\) such that if a function \(f : G \to B\) satisfies the inequality
\[
d(f(xy), f(x)f(y)) \leq \delta, \quad x, y \in G,
\]
there exists a homomorphism \(g : G \to B\) such that
\[
d(f(x), g(x)) \leq \varepsilon, \quad x \in G?
\]

In 1941, D. H. Hyers [6] gave an affirmative partial answer to this problem. This is the reason for which today this type of stability is called Hyers-Ulam stability of functional equation. In 1950, Aoki [7] generalized Hyers’ theorem for approximately additive functions. In 1978, Th. M. Rassias [8] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. Taking this fact into account, the additive functional equation \(f(x + y) = f(x) + f(y)\) is said to have the Hyers-Ulam-Rassias stability on \((X,Y)\). This terminology is also applied to the case of other functional equations. For more detailed definitions of such terminology one can refer to [9] and [10]. Thereafter, the stability problem of functional equations has been extended in various directions and studied by several mathematicians [11-25].

The functional equation
\[
Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)
\]
is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic function. The Ulam problem for the quadratic functional equation was proved by Skof [26,27] for mappings \(f : X \to Y\), where \(X\) is a normed space and \(Y\) is a Banach space. Cholewa [28] noticed that the theorem of Skof is still true if the relevant domain \(X\) is replaced by an Abelian group. Several functional equations have been investigated by J. M. Rassias in [29-31], St. Czerwik in [32] and Th. M. Rassias in [33].

Jun and Kim [34] introduced the following functional equation
\[
f(2x + y) + f(2x - y) = 12f(x) + 2f(x + y) + 2f(x - y)
\]
and they established the general solution and the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.2). Also a quartic functional equation have been investigated by J. M. Rassias in [35].
Letting $x$, we find that
\[ F(2x + y) + F(2x - y) = 4F(x + y) + 4F(x - y) + 24F(x) - 6F(y) \]
and they established the general solution and the stability problem for the functional equation (1.3). Also the quartic functional equation have been investigated by J. M. Rassias in [37].

In this paper, we consider a family of functional equations as follows
\[ (1.4) \quad c_1f(ax + by) + c_2f(ax - by) = c_3f(x) + c_4f(y) + c_5f(x + y) + c_6f(x - y), \]
for a given mapping $f : V \to X$ and for given $a, b, c_1, \ldots, c_6$ members of $F$, where $a, b, c_1$ and $c_3$ are nonzero elements and $c_1 + c_2 \neq 0$. Our aim is to study functional equation (1.4), where its a general mixed additive-quadratic-cubic-quartic (1.1), (1.2) and (1.3) functional equations. We will establish the solution of the functional equation (1.4) with difference operators. In fact we proved that $f$ is a monomial function of degree $\log |\frac{c_3 + c_5 + c_6}{c_1 + c_2}|$, where its at most 4. Furthermore, by using fixed point methods, we are going to solve the generalized Hyers-Ulam stability problem for the functional equation (1.4) in Banach space.

2. Solution of equation (1.4)

For a given mapping $f : V \to B$, we define the function $f : V \times V \to B$ as follows
\[
Df(x, y) := c_1f(ax + by) + c_2f(ax - by) - c_3f(x) - c_4f(y) - c_5f(x + y) - c_6f(x - y)
\]
for all $x, y \in V$.

**Theorem 2.1.** Let $f : V \to B$ be a function such that $Df(x, y) = 0$ for all $x, y \in V$. Then $f$ is a GP function of degree at most 4.

**Proof.** Let $h_1, h_2, h_3, h_4, h_5 \in V$ be arbitrary fixed elements. Letting $x + h_1$ and $y - h_1$ instead of $x$ and $y$ in (1.4), we get
\[
c_1f(ax + by + (a - b)h_1) + c_2f(ax - by + (a + b)h_1) = c_3f(x + h_1) + c_4f(y - h_1) + c_5f(x + y) + c_6f(x - y + 2h_1)
\]
Using this equality and (1.4), we find that
\[
(2.1) \quad \Delta_{(a-b)h_1}c_1f(ax + by) + \Delta_{(a+b)h_1}c_2f(ax - by) = \Delta_{h_1}c_3f(x) + \Delta_{-h_1}c_4f(y) + \Delta_{2h_1}c_6f(x - y).
\]
Now, Letting $x + h_2$ and $y + h_2$ instead of $x$ and $y$ in (2.1), we get
\[
\Delta_{(a-b)h_1}c_1f(ax + by + (a + b)h_2) + \Delta_{(a+b)h_1}c_2f(ax - by + (a - b)h_2) = \Delta_{h_1}c_3f(x + h_2) + \Delta_{-h_1}c_4f(y + h_2) + \Delta_{2h_1}c_6f(x - y).
\]
Using this equality and (2.1), we find that
\[
(2.2) \quad \Delta_{(a-b)h_1}\Delta_{(a+b)h_2}c_1f(ax + by) + \Delta_{(a+b)h_1}\Delta_{(a-b)h_2}c_2f(ax - by) = \Delta_{h_1}\Delta_{h_2}c_3f(x) + \Delta_{-h_1}\Delta_{h_2}c_4f(y).
\]
Letting $x + bh_3$ and $y - ah_3$ instead of $x$ and $y$ in (2.1), we get
\[
\Delta_{(a-b)h_1}\Delta_{(a+b)h_2}c_1f(ax + by) + \Delta_{(a+b)h_1}\Delta_{(a-b)h_2}c_2f(ax - by + (2ab)h_3) = \Delta_{h_1}\Delta_{h_2}c_3f(x + bh_3) + \Delta_{-h_1}\Delta_{h_2}c_4f(y - ah_3).
\]
Using this equality and (2.2), we find that
\[(2.3)\quad \Delta_{(a+b)h_1} \Delta_{(a-b)h_2} \Delta_{(2ab)b_3} c_2 f(ax - by) = \Delta_{h_1} \Delta_{h_2} \Delta_{bh_3} c_3 f(x) + \Delta_{-h_1} \Delta_{h_2} \Delta_{-ah_3} c_4 f(y).\]

Letting \(x - bh_4\) and \(y - ah_4\) instead of \(x\) and \(y\) in (2.1), we get
\[
\Delta_{(a+b)h_1} \Delta_{(a-b)h_2} \Delta_{(2ab)b_3} c_2 f(ax - by) = \Delta_{h_1} \Delta_{h_2} \Delta_{bh_3} c_3 f(x - bh_4) + \Delta_{-h_1} \Delta_{h_2} \Delta_{-ah_3} c_4 f(y - ah_4).
\]

Using this equality and (2.3), we find that
\[(2.4)\quad \Delta_{h_1} \Delta_{h_2} \Delta_{bh_3} \Delta_{-bh_4} c_3 f(x) + \Delta_{-h_1} \Delta_{h_2} \Delta_{-ah_3} \Delta_{-ah_4} c_4 f(y) = 0.
\]

Letting \(x + h_5\) and instead of \(x\) and \(y\) in (2.1), we get
\[
\Delta_{h_1} \Delta_{h_2} \Delta_{bh_3} \Delta_{-bh_4} c_3 f(x + h_5) + \Delta_{-h_1} \Delta_{h_2} \Delta_{-ah_3} \Delta_{-ah_4} c_4 f(y) = 0.
\]

Finally, using this equality and (2.4) and since \(c_3 \neq 0\), then we obtain
\[
\Delta_{h_1} \Delta_{h_2} \Delta_{bh_3} \Delta_{-bh_4} \Delta_{h_5} f(x) = 0
\]
for all \(x \in V\). Therefore, by Theorem (1.1), \(f\) is a GP function of degree at most 4. So there exist \(a_0 \in B\) and symmetric \(k\)-additive functions \(a_k : V^k \to B\) (for \(1 \leq k \leq m\)) such that
\[
f(x) = a_0 + \sum_{k=1}^{m} a_k^*(x) \quad \text{for all } x \in V,
\]
and \(a_m^* \neq 0\), also
\[
f(rx) = a_0 + \sum_{k=1}^{m} r^k a_k^*(x) \quad \text{for all } x \in V \text{ and } r \in \mathbb{Q}.
\]

The proof is complete.

**Theorem 2.2.** Let \(f : V \to B\) be a function such that \(V\) and \(B\) is a vector space over \(\mathbb{Q}\), \(Df(x,y) = 0\) for all \(x, y \in V\) and \(f(0) = 0\). If \(\left|\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right| \neq 0, 1\), then \(f\) is a monomial function of degree \(\log_{|a|} \left(\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right)\).

**Proof.** By Theorem (2.1) \(f\) is a GP function of degree at most 4 and there exist \(a_0 \in B\) and symmetric \(k\)-additive functions \(a_k : V^k \to B\) (for \(1 \leq k \leq m\)) such that
\[
f(x) = a_0 + \sum_{k=1}^{m} a_k^*(x) \quad \text{for all } x \in V,
\]
and \(a_m^* \neq 0\), also
\[
f(rx) = a_0 + \sum_{k=1}^{m} r^k a_k^*(x) \quad \text{for all } x \in V \text{ and } r \in \mathbb{Q}.
\]

Letting \(y = 0\) in (1.4), then
\[
(c_1 + c_2)f(ax) = (c_3 + c_5 + c_6)f(x)
\]
for all \(x \in V\). Thus,
\[
f(ax) = \frac{c_3 + c_5 + c_6}{c_1 + c_2} f(x)
\]
for all $x \in V$. Since $|\frac{c_1+c_2+c_3}{c_1+c_2}| \neq 0,1$, so the GP function $f$ must be a monomial function of degree at most 4. Therefore there exists a $k \in \mathbb{N}$ such that $f = a_k^n$, where its a monomial function of degree $k$, so

$$|a|^k = |\frac{c_3 + c_5 + c_6}{c_1 + c_2}|,$$

where it implies that $f$ is a monomial function of degree $\log_{|a|}|\frac{c_3 + c_5 + c_6}{c_1 + c_2}|$ and the proof is complete.

3. The generalized Hyers-Ulam stability of equation (1.4)

In the following, for the readers convenience and explicit later use, we will recall some fundamental results in fixed point theory.

**Definition 3.1.** The pair $(X, d)$ is called a generalized complete metric space if $X$ is a nonempty set and $d : X^2 \to [0, \infty]$ satisfies the following conditions:

1. $d(x, y) \geq 0$ and the equality holds if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$;
4. every $d$-Cauchy sequence in $X$ is $d$-convergent.

for all $x, y \in X$.

Note that the distance between two points in a generalized metric space is permitted to be infinity.

**Definition 3.2.** Let $(X, d)$ be a metric space. A mapping $J : X \to X$ satisfies a Lipschitz condition with Lipschitz constant $L \geq 0$ if

$$d(J(x), J(y)) \leq Ld(x, y)$$

for all $x, y \in X$. If $L < 1$, then $J$ is called a strictly contractive map.

**Theorem 3.3.** (38). Let $(X, d)$ be a generalized complete metric space and $J : X \to X$ be strictly contractive mapping. Then for each given element $x \in X$, either

$$d(J^n(x), J^{n+1}(x)) = \infty$$

for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

1. $d(J^n(x), J^{n+1}(x)) < \infty$, for all $n \geq n_0$;
2. the sequence $\{J^n(x)\}$ converges to a fixed point $y^*$ of $J$;
3. $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X : d(J^{n_0}(x), y) < \infty\}$;
4. $d(y, y^*) \leq \frac{1}{1-L}d(J(y), y)$.

**Theorem 3.4.** Let $\lambda_1 = \frac{c_1+c_2+c_3}{c_1+c_2}$ and $a_1 = a$, when $|\frac{c_1+c_2+c_3}{c_1+c_2}| > 1$ and also $\lambda_2 = \frac{1}{\frac{c_1+c_2+c_3}{c_1+c_2}}$ and $a_2 = \frac{1}{a}$, when $0 < |\frac{c_1+c_2+c_3}{c_1+c_2}| < 1$. Suppose that the mapping $f : V \to X$ satisfies the conditions $f(0) = 0$ and

$$\|Df(x, y)\| \leq \psi(x, y)$$

for all $x, y \in V$, in which $\psi : V \times V \to [0, \infty)$ is a function. If there exists a positive real $L = L(i) < 1$ ($i = 1, 2$) such that

1. $\lim_{n \to \infty} \lambda_1^n \psi(a_1^n x, a_1^n y) = 0$,
2. $\lim_{n \to \infty} \lambda_2^n \psi(a_2^n x, a_2^n y) = 0$,
3. $\psi(a_i x, 0) \leq L_i |\lambda_i| \psi(x, 0)$
for all \( x, y \in V \), then there exists a unique \( T : V \to X \) such that \( DT(x, y) = 0 \) and
\[
\|f(x) - T(x)\| \leq \frac{L^{2-i}}{(1-L)|c_3 + c_5 + c_6|} \psi(x, 0)
\]
for all for \( x, y \in V \).

**Proof.** Let us consider the set \( A := \{ g : V \to X \} \) and introduce the generalized metric on \( A \):
\[
d(g, h) = \sup_{\{x \in X : \psi(x, 0) \neq 0\}} \frac{\|g(x) - h(x)\|}{\psi(x, 0)}.
\]
It is easy to show that \((A, d)\) is generalized complete metric space. Now we define the function \( J : A \to A \) with
\[
J(g(x)) = \frac{1}{\lambda_i} g(a_i x)
\]
for all \( g \in A \) and \( x \in V \). We can write,
\[
d(J(g), J(h)) = \sup_{\{x : \psi(x, 0) \neq 0\}} \frac{\|g(a_i x) - h(a_i x)\|}{|\lambda_i| \psi(x, 0)}
\leq L \sup_{\{x : \psi(x, 0) \neq 0\}} \frac{\|g(a_i x) - h(a_i x)\|}{\psi(a_i x, 0)} = Ld(g, h),
\]
that is \( J \) is a strictly contractive selfmapping of \( A \), with the Lipschitz constant \( L \). We set \( y = 0 \) in the hypothesis (3.1), then we obtain
\[
\|f(ax) - f(x)\| \leq \frac{1}{|c_3 + c_5 + c_6|} \psi(x, 0)
\]
for all \( x \in V \). Now if \( i = 1 \), then from (3.5), we get
\[
\|f(a_1 x) - f(x)\| \leq \frac{1}{|c_3 + c_5 + c_6|} \psi(x, 0)
\]
for all \( x \in V \) and it implies that \( d(J(f), f) < \frac{1}{|c_3 + c_5 + c_6|} < \infty \). If \( i = 2 \), then from (3.5) and (3.3), we get
\[
\|f(a_2 x) - f(x)\| \leq \frac{L}{|c_3 + c_5 + c_6|} \psi(x, 0)
\]
for all \( x \in V \) and it implies that \( d(J(f), f) < \frac{L}{|c_3 + c_5 + c_6|} < \infty \). By Theorem (3.3), there exists a mapping \( T : V \to X \) such that
\[
(1) \ T \text{ is a fixed point of } J, \ i.e.,
\]
\[
T(a_i x) = \lambda_i T(x)
\]
for all \( x \in V \). The mapping \( T \) is a unique fixed point of \( J \) in the set
\[
\bar{A} = \{ h \in A : d(f, h) < \infty \}.
\]
(2) \( d(J^n(f), T) \to 0 \) as \( n \to \infty \). This implies that
\[
T(x) = \lim_{n \to \infty} \frac{f(a^n_i x)}{\lambda^n_i}
\]
for all \( x \in V \) and also
\[
\lim_{n \to \infty} \frac{Df(a^n_i x, a^n_i y)}{\lambda^n_i} = DT(x, y)
\]
Theorem 3.5. Let $\gamma_1 = \frac{a_1 + c_4 + c_6}{c_1 + c_2}$ and $k_1 = a + b$, when $|\frac{a_1 + c_4 + c_6}{c_1 + c_2}| > 1$ and also $\gamma_2 = \frac{1}{\gamma_i}$ and $k_2 = \frac{1}{a + b}$, when $0 < |\gamma| < 1$. Suppose that the mapping $f : V \to X$ satisfies the conditions $f(0) = 0$ and

$$\frac{\|Df(x,y)\|}{\|x,y\|} \leq \frac{\psi(x,y)}{\lambda_i^n}$$

(3.7)

for all $x,y \in V$ and letting $n$ to infinity, we get $DT(x,y) = 0$ for all $x,y \in V$ and the proof is complete.

Proof. Let us consider the set $U := \{g : V \to X\}$ and introduce the generalized metric on $U$:

$$d(g,h) = \sup_{\{x \in X : \psi(x) \neq 0\}} \frac{\|g(x) - h(x)\|}{\psi(x,x)}.$$

It is easy to show that $(U,d)$ is generalized complete metric space. Now we define the function $J : A \to A$ with

$$J(g(x)) = \frac{1}{\gamma_i} g(k_i x)$$

(3.11)

for all $g \in U$ and $x \in V$. We can write,

$$d(J(g), J(h)) = \sup_{\{x : \psi(x,x) \neq 0\}} \frac{\|g(k_i x) - h(k_i x)\|}{\psi(k_i x, k_i x)} \leq L \sup_{\{x : \psi(x,x) \neq 0\}} \frac{\|g(k_i x) - h(k_i x)\|}{\psi(k_i x, k_i x)} = Ld(g,h),$$

that is $J$ is a strictly contractive selfmapping of $U$, with the Lipschitz constant $L$. We set $y = x$ in the hypothesis (3.8), then we obtain

$$\frac{f((a + b)x)}{|c_1 + c_4 + c_6|} - f(x) \leq \frac{1}{|c_3 + c_4 + c_6|} \psi(x,x)$$

(3.12)

for all $x \in V$. Now if $i = 1$, then from (3.12), we get

$$\frac{f(k_2 x)}{\gamma_1} - f(x) \leq \frac{1}{|c_3 + c_4 + c_6|} \psi(x,x).$$
for all \( x \in V \) and it implies that \( d(J(f), f) < \frac{1}{|c_3 + c_4 + c_6|} < \infty \). If \( i = 2 \), then from (3.12) and (3.10), we get
\[
\| \frac{f(k_2x)}{\gamma_2} - f(x) \| \leq \frac{L}{|c_3 + c_4 + c_6|} \psi(x, x)
\]
for all \( x \in V \) and it implies that \( d(J(f), f) < \frac{L}{|c_3 + c_4 + c_6|} < \infty \). By Theorem (3.3), there exists a mapping \( T : V \to X \) such that
(1) \( T \) is a fixed point of \( J \), i.e.,
\[
T(k_i x) = \gamma_i T(x)
\]
for all \( x \in V \). The mapping \( T \) is a unique fixed point of \( J \) in the set \( \tilde{U} = \{ h \in U : d(f, h) < \infty \} \).
(2) \( d(J^n(f), T) \to 0 \) as \( n \to \infty \). This implies that
\[
T(x) = \lim_{n \to \infty} \frac{f(k_n x)}{\gamma_i^n}
\]
for all \( x \in V \) and it implies that
\[
\lim_{n \to \infty} \frac{Df(k_n x, k_n y)}{\gamma_i^n} = DT(x, y)
\]
(3) \( d(f, T) \leq \frac{1}{1-L} d(J(f), f) \) and \( d(J(f), f) \leq \frac{L^{1-i}}{|c_3 + c_4 + c_6|} \), which implies,
\[
d(f, T) \leq \frac{L^{2-i}}{(1-L)|c_3 + c_4 + c_6|}.
\]
Now from (3.1), we get
\[
\| \frac{Df(k_n x, k_n y)}{\gamma_i^n} \| \leq \frac{\psi(k_n x, k_n y)}{|\gamma_i|^n}
\]
for all \( x, y \in V \) and letting \( n \) to infinity, we get \( DT(x, y) = 0 \) for all \( x, y \in V \) and the proof is complete.

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