Anomalous magnetohydrodynamics with longitudinal boost invariance and chiral magnetic effect

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Abstract

We study relativistic magnetohydrodynamics with longitudinal boost invariance in the presence of chiral magnetic effects and finite electric conductivity. With initial magnetic fields parallel or anti-parallel to electric fields, we derive the analytic solutions of electromagnetic fields and the chiral number and energy density in an expansion of several parameters determined by initial conditions. The numerical solutions show that such analytic solutions work well in weak fields or large chiral fluctuations. We also discuss the properties of electromagnetic fields in the laboratory frame.
I. INTRODUCTION

Recently some novel transport phenomena of chiral (massless) fermions in strong electromagnetic (EM) fields have been extensively studied in relativistic heavy ion collisions and condensed matter physics. One of them is the chiral magnetic effect (CME): an electric current can be induced by the strong magnetic field when the numbers of left and right handed fermions are not equal [1–3]. Similarly the strong magnetic field can also lead to the chiral separation effect (CSE) for the chiral charge current. These effects are associated with the chiral anomaly and can be described by chiral kinetic equations (CKE). The CKE are derived from various approaches, e.g. the path integral [4–6], the Hamiltonian approach [7, 8], the quantum kinetic theory via Wigner functions [9–17], and the world-line formalism [18, 19]. The chiral separation can also be induced by an electric field, which is called the chiral electric separation effect (CESE) [20–23]. If the electric field is perpendicular to the magnetic field, a Hall current for chiral fermions is expected, which is called chiral Hall separation effect (CHSE) [23]. The chiral particle production in strong EM fields are found to be directly connected to the Schwinger mechanism [24, 25], and similar calculation has been done analytically via the world-line formalism [26] and Wigner functions [27]. Recent reviews about chiral transport phenomena can be found in Ref. [28–31].

The chiral transport phenomena are expected to have observables in relativistic heavy ion collisions in which very strong magnetic fields of the order $B \sim 10^{18} \text{G}$ are produced [32–35]. At the very early stage of the quark-gluon plasma (QGP), the topological fluctuations in non-Abelian gauge fields give rise to the imbalance of chirality from event to event (event-by-event). Such an imbalance of chirality may lead to the charge separation with respect to the reaction plane in heavy ion collisions. The STAR collaboration have observed the charge separation in Au+Au collisions [36, 37]. However, due to the huge backgrounds from collective flows [38, 39] it is a challenge to extract the weak CME signal from the overwhelming backgroud. It is expected that the ongoing isobar collision experiment at STAR may shed light on the CME signal (see e.g. Ref. [40] for discussions on isobar collisions).

In order to extract the CME signal, we need the precise simulation of the QGP evolution in the time-evolving EM field. One approach is through the simulation of the CKE. Very recently, the boost invariant formulation of the CKE has been done with the chiral circular
displacement introduced [41]. The CKE has been solved numerically in heavy ion collisions [42, 43]. Another approach is the classical statistical simulation based on solving the coupled equations of Yang-Mills and Dirac applied to heavy ion collisions [44–46]. Besides the relativistic hydrodynamic is a widely-used model in relativistic heavy ion collisions.

The relativistic hydrodynamic model is one of the main approaches to the QGP evolution [47–53]. A natural extension of the hydrodynamic model in the presence of the magnetic field is the magento-hydrodynamics (MHD), which is hydrodynamics coupled with Maxwell’s equations. The ideal MHD equations with longitudinal boost invariance and a transverse magnetic field has been calculated [54, 55], where the magnetic field decays as $\sim 1/\tau$ with $\tau$ being the proper time, much slower than in vacuum [2]. The magnetization effect has also been systematically studied [54]. Later the calculation has been extended to 2+1 dimensions [56, 57]. There is an enhancement of the elliptic flow $v_2$ of $\pi^-$ from the external magnetic field [58]. Recently the MHD with the longitudinal boost invariance has been extended to include the finite conductivity in the Gubser flow [59]. Readers may look at Ref. [60] for recent numerical simulations of the ideal MHD.

In this work, we will consider the relativistic MHD in the presence of the CME and finite conductivity. Usually the numerical simulations of MHD with the CME could be very unstable because of chirality instability [61]. Therefore stable analytic solutions in some special cases are very important for providing a test of numerical simulations and a simple physical picture for such a complicated process. As a first attempt, we will consider the MHD with the longitudinal boost invariance. To avoid the acceleration of the fluid by the EM field, we will assume an electric charge neutral fluid. We then search for the EM fields that can keep the Bjorken fluid velocity unchanged. It is very similar to the case of the force-free magnetic field discussed in classical electrodynamics [62, 63]. To solve the coupled equations of the anomalous conservation equation and Maxwell’s equations, we assume that the terms proportional to the anomaly constant (proportional to the Planck constant $\hbar$) are perturbations, this is equivalent to an expansion in $\hbar$. We will compare our approximate analytic solutions with the numerical results. Finally we compute the EM field in the laboratory frame and discuss the coupling between the EM field and the chiral current.

The organization of the paper is as follows. In Sec. II, we give a brief review for the relativistic MHD with the CME. In Sec. III, we assume the form of the fluid velocity in longitudinal boost invariance. We choose a configuration of the EM field that is orthogonal

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to the fluid velocity. In Sec. IV A and IV B, we solve Maxwell’s equations coupled with
the anomalous conservation equation for the chiral charge. We obtain the approximate analytic
solutions for two different equations of state. We compare our approximate analytic solutions
with numerical ones. In Sec. IV C, we compute the EM field in the laboratory frame to show
the consistence with previous results. Finally we make a summary of our results in
Sec. V.

Throughout this work, we will use the metric \( g_{\mu\nu} = \text{diag}\{+,-,-,-\} \), thus, the fluid
velocity satisfies \( u^\mu u_\mu = 1 \), and the orthogonal projector to the fluid four-velocity is
\( \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u_\nu \). We also choose Levi-Civita tensor satisfying
\( \varepsilon_{0123} = -\varepsilon_{0123} = +1 \) and
\[ \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{\mu\nu\rho\sigma} = -2!(g^\alpha g^\beta - g^\alpha g^\beta). \]

II. ANOMALOUS MAGNETOHYDRODYNAMICS

In this section, we will give a brief preview to the relativistic MHD with CME which
is called anomalous magnetohydrodynamics. The MHD equations consist of conservation
equations and Maxwell’s equations (see, e.g., Ref. [54–57, 64–66] for details). The energy-
momentum conservation equation reads

\[ \partial_\mu T^{\mu\nu} = 0, \]  

where \( T^{\mu\nu} \) is the energy momentum tensor including the contributions from the fluid and the
EM fields

\[ T^{\mu\nu} = T^{\mu\nu}_F + T^{\mu\nu}_{EM}. \]

The fluid part has the usual form

\[ T^{\mu\nu}_F = \varepsilon u^\mu u^\nu - (p + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu}, \]

where \( \varepsilon \) and \( p \) are the energy density and pressure respectively, \( u^\mu = \gamma(1, v) \) is the fluid
velocity satisfying \( u^\mu u_\mu = 1 \), \( \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \) is the projector, and \( \Pi \) and \( \pi^{\mu\nu} \) are bulk
viscous pressure and shear viscous tensor respectively. For simplicity, we will neglect viscous
effects in this paper, i.e. \( \Pi = \pi^{\mu\nu} = 0 \). The EM field part of the energy-momentum tensor
reads

\[ T^{\mu\nu}_{EM} = -F^{\mu\lambda} F^{\nu}_\lambda + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}. \]
One can introduce the four-vector form of the electric and magnetic fields in terms of the fluid velocity

\[ E^\mu = F^\mu_\nu u_\nu, \quad B^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}, \]

which satisfy \( u^\mu E_\mu = 0 \) and \( u^\mu B_\mu = 0 \) meaning that both \( E^\mu \) and \( B^\mu \) are space-like. Then, the EM field strength tensor can be put into the form

\[ F^{\mu\nu} = E^\mu u^\nu - E^\nu u^\mu + \epsilon^{\mu\nu\alpha\beta} u_\alpha B_\beta, \]

Inserting the above formula into Eq. (4), we obtain the complete form of the energy-momentum tensor from Eq. (2)

\[ T^{\mu\nu} = (\varepsilon + p + E^2 + B^2) u^\mu u^\nu - (p + \frac{1}{2} E^2 + \frac{1}{2} B^2) g^{\mu\nu} - E^\mu E^\nu - B^\mu B^\nu - u^\mu \epsilon^{\nu\lambda\alpha\beta} E_\lambda B_\alpha u_\beta - u^\nu \epsilon^{\mu\lambda\alpha\beta} E_\lambda B_\alpha u_\beta, \]

where \( E \) and \( B \) are defined by

\[ E^\mu E_\mu = -E^2, \quad B^\mu B_\mu = -B^2. \]

The conservations equations are

\[ \partial_\mu j^\mu_e = 0, \]
\[ \partial_\mu j^\mu_5 = -e^2 C E \cdot B, \]

where \( j^\mu_e \) is the electric charge current and \( j^\mu_5 \) is the chiral (axial) charge current. Note that the chiral anomaly term appears in the second line of Eq. (9) with \( C = 1/(2\pi^2) \). These currents can be decomposed into three parts

\[ j^\mu_e = n_e u^\mu + \sigma E^\mu + \xi B^\mu, \]
\[ j^\mu_5 = n_5 u^\mu + \sigma_5 E^\mu + \xi_5 B^\mu, \]

where \( n_e \) and \( n_5 \) are the electric and chiral charge density respectively, \( \sigma \) and \( \sigma_5 \) are the electric and chiral electric conductivity respectively [20, 21, 23], and \( \xi \) and \( \xi_5 \) are associated with the CME and CESE [3, 9, 10] which are given by

\[ \xi = eC\mu_5, \quad \xi_5 = eC\mu_e. \]
For simplicity, we will neglect all other dissipative effects in $j^\mu_e$ and $j_5^\mu$ such as the heat conducting flow. The chiral electric conductivity $\sigma_5$ is usually parametrized as $\sigma_5 \propto \mu_e \mu_5$ in the small $\mu_e$ and $\mu_5$ limit [20, 21, 23].

Maxwell’s equations can be put into the following form

$$\partial_\mu F^{\mu \nu} = j_\nu^e,$$  \hspace{1cm} (12)
$$\partial_\mu (\epsilon^{\mu \nu \alpha \beta} F_{\alpha \beta}) = 0.$$  \hspace{1cm} (13)

To close the system of equations, we need to choose the equations of state (EoS) for the thermodynamic quantities. In the dense limit with high chemical potentials, we use

$$\epsilon = c_s^{-2} p,$$
$$n_e = a \mu_e (\mu_e^2 + 3\mu_5^2),$$
$$n_5 = a \mu_5 (\mu_5^2 + 3\mu_e^2),$$  \hspace{1cm} (14)

where $a$ is a dimensionless constant and $c_s$ is the speed of sound also taken as a constant. On the other hand, in the hot limit with high temperatures, we use

$$\epsilon = c_s^{-2} p,$$
$$n_e = a \mu_e T^2,$$
$$n_5 = a \mu_5 T^2,$$  \hspace{1cm} (15)

where $a$ is again a dimensionless constant. Note that the value of $a$ in Eq. (15) is different from that in Eq. (14). For the ideal fluid, we have $a = 1/(3\pi^2)$ and $a = 1/3$ for Eq. (14) and (15) respectively [9, 67].

Usually the electric field would accelerate charged particles and the charged fluid. To avoid such a problem, we simply set the chemical potential for electric charge vanishing, $\mu_e = 0$, which also leads to $n_e = \sigma_5 = \xi_5 = 0$. Such a condition means the fluid is neutral: the number of positively charged particles is the same as that of negatively charged particles. Actually we look for a special configuration of EM fields coupled with the media, very similar to the force-free case in classical electrodynamics. In Sec. IV C, we will discuss the details and check the consistence of this assumption.

Here are the whole system of equations we are going to solve: conservation equations (1, 9), Maxwell’s equations (12, 13), constitutive equations (7, 6, 10), and equations of state (14, 15).
III. EQUATIONS WITH LONGITUDINAL BOOST INVARIANCE

We assume that the fluid has longitudinal boost invariance. It is convenient to introduce the Milne coordinates \( z = \tau \sinh \eta \) and \( t = \tau \cosh \eta \), with \( \tau = (t^2 - z^2)^{1/2} \) being the proper time and \( \eta = \frac{1}{2} \ln[(t + z)/(t - z)] \) being the space-time rapidity. The fluid velocity with longitudinal boost invariance can be written as [68],

\[
\begin{align*}
    u^\mu &= (\cosh \eta, 0, 0, \sinh \eta) = \gamma (1, 0, 0, z/t),
\end{align*}
\]

where \( \gamma = \cosh \eta \) is the Lorentz contraction factor.

For simplicity we neglect the EM field in the longitudinal direction, so the general form of the EM field satisfying \( u \cdot E = u \cdot B = 0 \) is

\[
\begin{align*}
    E^\mu &= (0, E\cos\zeta, E\sin\zeta, 0), \\
    B^\mu &= (0, B\cos\varphi, B\sin\varphi, 0),
\end{align*}
\]

where \( \zeta \) and \( \varphi \) are the azimuthal angle of the electric and magnetic field in the transverse plane respectively. To search for possible analytic solutions, we assume that \( E^\mu \) and \( B^\mu \) will always be in the transverse plane and that \( E, B, \zeta, \varphi \) are only functions of \( \tau \). We can further simplify the problem by assuming that \( E^\mu \) and \( B^\mu \) are parallel or anti-parallel. Without loss of generality, the EM field can be put in the \( y \) direction

\[
E^\mu = (0, 0, \chi E(\tau), 0), \quad B^\mu = (0, 0, B(\tau), 0),
\]

where \( \chi = \pm 1 \). We will check the self-consistency of these assumptions after we find the solution in Sec. IV C. Note that the authors of Ref. [69] have found another possible configuration of the EM fields in the absence of the chiral magnetic effect, in which the direction of the electric and magnetic field depends on \( \eta \). As this configuration is irrelevant to the heavy ion collisions, we will not consider it in this paper.

By projecting the energy-momentum conservation equation (1) onto the spatial direction, \( \Delta_{\mu\alpha} \partial_\nu T^{\mu\nu} = 0 \), we obtain the acceleration of the fluid velocity

\[
\begin{align*}
    (u \cdot \partial) u_\alpha &= \frac{1}{(\varepsilon + p + E^2 + B^2)} \left[ \Delta^\nu_\alpha \partial_\nu(p + \frac{1}{2}E^2 + \frac{1}{2}B^2) + \Delta_{\mu\alpha} (E \cdot \partial) E^\mu + E_\alpha (\partial \cdot E) \ight. \\
    &+ \Delta_{\mu\alpha} (B \cdot \partial) B^\mu + B_\alpha (\partial \cdot B) + e^{\nu\lambda\rho\sigma} E_\lambda B_\rho u_\sigma (\partial_\nu u_\alpha) + (\partial \cdot u) \epsilon_{\alpha\lambda\rho\sigma} E^{\lambda} B^{\rho} u^{\sigma} \\
    &+ \Delta_{\mu\alpha} (u \cdot \partial) \epsilon^{\mu\lambda\rho\sigma} E_\lambda B_\rho u_\sigma \right].
\end{align*}
\]

(19)
According to our assumption that the electric and magnetic field are constant in transverse coordinates \((x, y)\), we have \((E \cdot \partial)E^\mu = (\partial \cdot E) = (B \cdot \partial)B^\mu = (\partial \cdot B) = 0\). Also, if \(p\), \(E^\mu\) and \(B^\mu\) are only the functions of \(\tau\), the first term inside the square brackets are vanishing. So we obtain the non-acceleration of the fluid velocity

\[
(u \cdot \partial)u_\alpha = 0, \quad (20)
\]

which means that the fluid velocity always takes the value in Eq. (16). This is consistent to the previous assumption that the fluid is charge neutral.

The energy conservation equation can be obtained by a contraction of \(u_\mu\) with Eq. (1) or \(u_\mu \partial_\nu T^{\mu\nu} = 0\),

\[
(u \cdot \partial)(\varepsilon + \frac{1}{2}E^2 + \frac{1}{2}B^2) + (\varepsilon + p + E^2 + B^2)(\partial \cdot u) = u_\mu (E \cdot \partial)E^\mu + u_\mu (B \cdot \partial)B^\mu + \epsilon^{\nu\lambda\alpha\beta} \partial_\nu (E_\lambda B_\alpha u_\beta) + u_\mu (u \cdot \partial)\epsilon^{\mu\lambda\alpha\beta} E_\lambda B_\alpha u_\beta. \quad (21)
\]

With Eq. (18), the above equation is reduced to

\[
(u \cdot \partial)(\varepsilon + \frac{1}{2}E^2 + \frac{1}{2}B^2) + (\varepsilon + p + E^2 + B^2)(\partial \cdot u) = 0. \quad (22)
\]

Now we look at Maxwell’s equations. Inserting Eq. (18) for the EM fields into Eq. (12) yields for \(\nu = y\)

\[
d \frac{d}{d\tau} E + \frac{1}{\tau} E + \sigma E + \chi \xi B = 0, \quad (23)
\]

where we have used \(d/d\tau \equiv (u \cdot \partial)\). For other indices \(\nu = t, x, z\), we obtain identities using \(\mu_\epsilon = 0\) and \(n_\epsilon = 0\). Similarly, from Eq. (13), we obtain for \(\nu = y\)

\[
d \frac{d}{d\tau} B + \frac{B}{\tau} = 0. \quad (24)
\]

For other indices \(\nu = t, x, z\), we obtain identities using \(\mu_\epsilon = 0\) and \(n_\epsilon = 0\).

Using the simplified Maxwell’s equations (23) and (24), we can rewrite Eq. (22) into a compact form

\[
\frac{d}{d\tau} \varepsilon + (\varepsilon + p) \frac{1}{\tau} - \sigma E^2 - \chi \xi E B = 0. \quad (25)
\]

This equation can also be derived by rewritten Eq. (1) as

\[
\partial_\mu T^{\mu\nu}_F = -\partial_\mu T^{\mu\nu}_{EM} = F^{\nu\lambda} j_\epsilon \ell_\lambda, \quad (26)
\]
Contracting the above equation with $u_\nu$ yields $u_\nu \partial_\mu T_{\mu\nu} = -E^\lambda j_{e\lambda}$, which is consistent with Eq. (25).

From Eq. (9) and using $\mu_e = 0$, the (anomalous) conservation equation of the chiral charge can be reduced to

$$\frac{d}{d\tau} n_5 + \frac{n_5}{\tau} = e^2 C \chi E B.$$  \hfill (27)

The conservation equation for $j^\mu_e$ is automatically satisfied with $\mu_e = 0$ and $E^\mu, B^\mu$ taking the form of Eq. (18).

Before we end this section, we make some remarks about the simplified equations with longitudinal boost invariance. To enforce the fluid velocity not accelerated, the EM field are assumed to take the form as Eq. (18). Using Maxwell’s equations the energy conservation equation $u_\mu \partial_\nu T_{\mu\nu} = 0$ is reduced to Eq. (25). The momentum conservation equation $\Delta_{\mu\alpha} \partial_\nu T_{\mu\nu} = 0$ is reduced to Eq. (20) meaning that the fluid velocity always takes value in (16). Maxwell’s equations (12, 13) are simplified to Eqs. (23, 24). The chiral charge conservation equation in Eq. (9) is simplified to Eq. (27).

IV. ANALYTIC SOLUTIONS

We will use the non-conserved charges method [69, 70] to solve Eqs. (23, 24, 25, 27) with the EoS (14) or (15).

The non-conserved charges method is to solve the equation for $f(\tau)$ in the following form

$$\frac{d}{d\tau} f(\tau) + m \frac{f(\tau)}{\tau} = f(\tau) \frac{d}{d\tau} \lambda(\tau),$$  \hfill (28)

where $m$ is a constant and $\lambda(\tau)$ is a known function. The general solution is

$$f(\tau) = f(\tau_0) \exp \left[ \lambda(\tau) - \lambda(\tau_0) \right] \left( \frac{\tau_0}{\tau} \right)^m,$$  \hfill (29)

where $\tau_0$ is an initial proper time and $f(\tau_0)$ is determined by an initial value at $\tau_0$. In this paper we will rewrite Eqs. (23, 24, 25, 27) into the form of Eq. (28) and obtain the solutions in the form of Eq. (29).

Note that generally $f$ can also be a function of rapidity $\eta$ [69, 70]. However, in this paper we focus on the central rapidity region in heavy ion collisions which implies $\eta \simeq 0$ with longitudinal boost invariance, therefore we will not consider the rapidity dependence.
From Eq. (24), we immediately obtain
\[ B(\tau) = B_0 \frac{\tau_0}{\tau}, \]  
(30)
where \( B_0 = B(\tau_0) \) is the initial value of the magnetic field. We see that the proper time behavior of the magnetic field seems to be the same as the case without CME [54–56]. But we will show in Sec. IV C the contribution from the CME and finite conductivity to the EM field appear in the Lab frame.

A. EoS (14)

For the EoS (14), we will solve Eq. (23) with Eq. (27) to obtain \( n_5(\tau) \) and \( E(\tau) \). Then we insert \( n_5(\tau) \) and \( E(\tau) \) into Eq. (25) to obtain the energy-density \( \varepsilon(\tau) \).

We need to put Eqs. (23, 27) into the form of Eq. (28)
\[ \frac{d}{d\tau} E + \frac{E}{\tau} = E \frac{d}{d\tau} \mathcal{E}, \]
\[ \frac{d}{d\tau} n_5 + \frac{n_5}{\tau} = n_5 \frac{d}{d\tau} \mathcal{N}, \]  
(31)
where
\[ \frac{d}{d\tau} \mathcal{E} = -\sigma - \chi \xi \frac{B}{E}, \]
\[ \frac{d}{d\tau} \mathcal{N} = \frac{e^2 C \chi E B}{n_5}, \]  
(32)
and \( \xi \) is given by Eq. (11) and depends on \( n_5 \) through the EoS (14). Following Eq. (29), the formal solutions are in the form
\[ n_5(\tau) = n_{5,0} \exp \left[ \mathcal{N}(\tau) - \mathcal{N}(\tau_0) \right] \frac{\tau_0}{\tau}, \]
\[ E(\tau) = E_0 \exp \left[ \mathcal{E}(\tau) - \mathcal{E}(\tau_0) \right] \frac{\tau_0}{\tau}, \]  
(33)
where \( n_{5,0} = n_5(\tau_0) \) and \( E_0 = E(\tau_0) \). Inserting the above \( n_5(\tau) \) and \( E(\tau) \) as well as \( B(\tau) \) in Eq. (30) into Eq. (32), we obtain
\[ \frac{d}{d\tau} x = -\sigma x - \frac{a_1}{\tau_0} \left( \frac{\tau_0}{\tau} \right)^{1/3} y^{1/3}, \]
\[ \frac{d}{d\tau} y = a_2 \frac{x}{\tau}, \]  
(34)
where we have introduced the new variables
\[
x(\tau) = \exp [\mathcal{E}(\tau) - \mathcal{E}(\tau_0)],
\]
\[
y(\tau) = \exp [\mathcal{N}(\tau) - \mathcal{N}(\tau_0)],
\]
(35)
with \(x(\tau_0) = y(\tau_0) = 1\), and \(a_1\) and \(a_2\) are dimensionless constants determined by the initial conditions
\[
a_1 = eC\chi \left( \frac{n_{5,0}}{a} \right)^{1/3} \frac{B_0}{E_0} \tau_0,
\]
\[
a_2 = \frac{e^2 C\chi E_0 B_0 \tau_0}{n_{5,0}}.
\]
(36)

Instead of solving Eqs. (23, 27) or Eq. (31), now we only need to solve Eq. (27) with the initial condition \(x(\tau_0) = y(\tau_0) = 1\). We see that both \(a_1\) and \(a_2\) are linearly proportional to the anomaly constant \(C\) which is linearly proportional to the Planck constant. This means \(a_1\) and \(a_2\) are of quantum nature.

Now we try to solve Eq. (34) under some approximations. We can rewrite Eq. (34) into an integral form
\[
x(\tau) = e^{-\sigma(\tau-\tau_0)} - \frac{a_1}{\tau_0} e^{-\sigma \tau} \int_{\tau_0}^{\tau} d\tau' e^{\sigma \tau'} \left( \frac{\tau_0}{\tau'} \right)^{1/3} y^{1/3}(\tau'),
\]
\[
y(\tau) = 1 + a_2 \int_{\tau_0}^{\tau} d\tau' \frac{x(\tau')}{\tau'},
\]
(37)
Since \(a_1\) and \(a_2\) terms are quantum corrections as \(a_1, a_2 \propto \hbar\), we can deal with these terms as perturbations to the classical terms, so Eq. (34) or (37) can be solved order by order in powers of \(\hbar\).

To the linear order in \(\hbar\), we have the solutions for \(x(\tau)\) and \(y(\tau)\)
\[
x(\tau) = e^{-\sigma(\tau-\tau_0)} - \frac{a_1}{\tau_0} e^{-\sigma \tau} \left[ \tau_0^{2/3} E_{1/3}(-\sigma \tau_0) - \tau^{2/3} E_{1/3}(-\sigma \tau) \right],
\]
\[
y(\tau) = 1 + a_2 \left[ e^{\sigma \tau_0} - a_1 E_{1/3}(-\sigma \tau_0) \right] \left[ E_1(\sigma \tau_0) - E_1(\sigma \tau) \right],
\]
where \(E_n(z) \equiv \int_1^\infty dt t^{-n} e^{-zt}\) is the generated exponential integral. Then we obtain the solutions for \(E(\tau)\) and \(n_5(\tau)\)
\[
E(\tau) = E_0 \frac{\tau_0}{\tau} \left\{ e^{-\sigma(\tau-\tau_0)} - \frac{a_1}{\tau_0^{2/3}} e^{-\sigma \tau} \left[ \tau_0^{2/3} E_{1/3}(-\sigma \tau_0) - \tau^{2/3} E_{1/3}(-\sigma \tau) \right] \right\},
\]
\[
n_5(\tau) = n_{5,0} \frac{\tau_0}{\tau} \left\{ 1 + a_2 e^{\sigma \tau_0} \left[ E_1(\sigma \tau_0) - E_1(\sigma \tau) \right] \right\}.
\]
(38)
At early proper time, \( \tau \to \tau_0 \), we can expand \( E_n(\tau) \) near \( \tau_0 \) and obtain

\[
E(\tau) \simeq E_0 \frac{\tau_0}{\tau} \left[ e^{-\sigma(\tau-\tau_0)} - \frac{a_1}{\tau_0}(\tau-\tau_0) + a_1 \frac{1 + 3\tau_0\sigma}{6\tau_0^2}(\tau - \tau_0)^2 \right],
\]

\[
n_5(\tau) \simeq n_{5,0} \frac{\tau_0}{\tau} \left\{ 1 + a_2 \frac{\tau-\tau_0}{\tau_0} - a_2 \frac{1 + \sigma\tau_0}{2\tau_0^2}(\tau - \tau_0) \right\}. \tag{39}
\]

Finally the energy density and the pressure can be solved by using the solutions for \( \mu_5, E, B \).

From Eq. (25), we obtain the energy density

\[
\varepsilon(\tau) = \varepsilon_0 \left( \frac{\tau_0}{\tau} \right)^{1+c_2^2} (1 + \Delta \varepsilon),
\]

\[
\Delta \varepsilon(\tau) = \frac{1}{\varepsilon_0} \int_{\tau_0}^{\tau} d\tau' \left( \frac{\tau'}{\tau_0} \right)^{1+c_2^2} \left[ \sigma E^2(\tau') + \chi\xi(\tau')E(\tau')B(\tau') \right]. \tag{40}
\]

We can also solve Eq. (34) numerically. We choose the initial proper time \( \tau_0 = 0.6 \, \text{fm/c.} \) The values of the electric conductivity vary in different situations. The lattice QCD calculations give \( \sigma \sim 5.8T/T_c \, \text{MeV} \) [71–73], while in holographic QCD models it takes the value \( \sigma \sim 20 - 30 \, \text{MeV} \) for \( T = 200 \, \text{MeV} \) [21, 23]. For \( \sigma \) in the weakly coupled QGP at finite temperature and chemical potential, see, e.g. Ref. [74]. In our numerical calculation, we choose \( \sigma \sim 5 - 30 \, \text{MeV} \simeq 0.04 - 0.25\tau_0. \)

In Fig. 1, we plot the normalized electric field \( E/E_0 \) and chiral charge density \( n_5/n_{5,0} \) as functions of the proper time \( \tau \). The solid lines are the numerical results from Eqs. (34), while the dashed lines are from the approximate analytic solution (38). Note that the approximate analytic solution for \( E(\tau) \) is independent of \( a_2 \) and \( n_5(\tau) \) independent of \( a_1 \) and \( \sigma \). From these results, we see that the approximation works very well for small \( a_1 \) and \( a_2 \). For positive \( a_1 \) and \( a_2 \), \( E \) decay faster as \( a_1 \) increases, while for negative \( a_1 \) and \( a_2 \), \( E \) decay slower as \( |a_1| \) increases. For positive \( a_1 \) and \( a_2 \), \( n_5 \) decays slower as \( a_2 \) grows, while for negative \( a_1 \) and \( a_2 \), \( n_5 \) decays faster as \( |a_2| \) grows. Such behaviors are obvious in the approximate analytic solution (38).

We observe that for large positive \( a_1 \) or large \( \sigma \) with positive \( a_1 \) and \( a_2 \), \( E/E_0 \) can be negative at late proper time. It means that the electric field flips its sign at the late time. From Eq. (38), one can see that a very large \( a_1 \) in the second term may dominate and make \( E/E_0 \) negative. Since \( a_1 \) is proportional to the initial chiral charge density, such a behavior may come from the competition between the anomalous conservation equation \( \partial_\mu J_5^\mu = -CE \cdot B \) and Maxwell’s equations.
Figure 1: The normalized electric field $E/E_0$ and chiral charge density $n_5/n_{5,0}$ as functions of the proper time $\tau$. We have chosen $\tau_0 = 0.6$ fm/c. The solid lines are obtained by solving Eq. (34) numerically and the dashed lines are from the approximate analytic solution (38). In the first row, we fix $\sigma/\tau_0 = 0.1$, $a_2 = \pm 0.2$ and change the values of $a_1$. In the second row, we fix $\sigma/\tau_0 = 0.1$, $a_1 = \pm 0.5$ and change the values of $a_2$. In the last row, we fix $(a_1, a_2) = \pm (0.05, 0.02)$ and change the values of $\sigma/\tau_0$. 

\[
\frac{\sigma}{\tau_0} = 0.1
\]

\[
\sigma/\tau_0 = 0.1
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\frac{\sigma}{\tau_0} = 0.1
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a_1 = 0.00, a_2 = 0.00
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a_1 = 0.05, a_2 = 0.02
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a_1 = 0.10, a_2 = 0.02
\]

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a_1 = 0.20, a_2 = 0.02
\]

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a_1 = -0.05, a_2 = -0.02
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a_1 = -0.10, a_2 = -0.02
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a_1 = -0.20, a_2 = -0.02
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\frac{\sigma}{\tau_0} = 0.1
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\sigma/\tau_0 = 0.1
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a_1 = 0.05, a_2 = 0.02
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a_1 = 0.10, a_2 = 0.02
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a_1 = 0.20, a_2 = 0.02
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a_1 = -0.05, a_2 = -0.02
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a_1 = -0.10, a_2 = -0.02
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a_1 = -0.20, a_2 = -0.02
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\frac{\sigma}{\tau_0} = 0.1
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\sigma/\tau_0 = 0.1
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\frac{\sigma}{\tau_0} = 0.1
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a_1 = 0.00, a_2 = 0.00
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a_1 = 0.05, a_2 = 0.02
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a_1 = 0.10, a_2 = 0.02
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a_1 = 0.20, a_2 = 0.02
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a_1 = -0.05, a_2 = -0.02
\]

\[
a_1 = -0.10, a_2 = -0.02
\]

\[
a_1 = -0.20, a_2 = -0.02
\]
Figure 2: The energy density correction $\Delta \varepsilon (\times 100)$ as functions the proper time $\tau$. The parameters are set to $\tau_0 = 0.6$ fm/c, $E_0/\epsilon_0 = 0.1$, $B_0/\epsilon_0 = 0.2$, $\mu_{5,0}/\tau_0 = 1$, and $c_s^2 = 1/3$. The solid lines are numerical solutions of Eq. (34) and the dashed lines are from the approximate analytic solution (38). In the left panel, we fix $\sigma/\tau_0 = 0.1$ and change the values of $a_1$ and $a_2$. In the right panel, we fix $(a_1, a_2) = \pm (0.05, 0.02)$ and change the values of $\sigma/\tau_0$.

One may expect that $n_5$ may have oscillation with time because it can be converted from the magnetic helicity and vice versa [61]. However, since the medium is expanding, the possible oscillation of $n_5$ is outperformed by its decay $n_5/n_{5,0} \sim \tau_0/\tau$.

In Fig. 2, we show the results of $\Delta \varepsilon$ in Eq. (40) which is amplified by a factor 100. The solid lines are numerical results from Eq. (34), while the dashed lines are given by the approximate analytic solution (38). Even with 100 times amplification of the difference, we see that the approximate analytic solution (38) still works well. For both positive and negative $a_1$ and $a_2$, $\Delta \varepsilon$ are positive because the first term dominates over the second one inside the square brackets in Eq. (40).

B. EoS (15)

For EoS (15), the equations for the energy density $\varepsilon(\tau)$, $E(\tau)$ and $n_5(\tau)$ are coupled together. We need to rewrite Eqs. (25, 23, 27) as

\[
\frac{d}{d\tau} \varepsilon + (1 + c_s^2)\varepsilon = \varepsilon \frac{d}{d\tau} \mathcal{L},
\]

\[
\frac{d}{d\tau} E + \frac{E}{\tau} = E \frac{d}{d\tau} \varepsilon,
\]
\[
\frac{d}{d\tau} n_5 + \frac{n_5}{\tau} = n_5 \frac{d}{d\tau} N, \tag{41}
\]
where
\[
\frac{d}{d\tau} \mathcal{L} = \frac{1}{\varepsilon} \sigma E^2 + \frac{1}{\varepsilon} e C \chi_5 E B,
\]
\[
\frac{d}{d\tau} \mathcal{E} = -\sigma - e C \chi_5 \frac{B}{E},
\]
\[
\frac{d}{d\tau} N = e^2 C \chi_5 E B n_5. \tag{42}
\]

With the help of Eq. (29), the solutions are,
\[
\varepsilon(\tau) = \varepsilon_0 \left(\frac{\tau_0}{\tau}\right)^{1+c_s^2} \exp \left[\mathcal{L}(\tau) - \mathcal{L}(\tau_0)\right], \tag{43}
\]
and \(n_5(\tau)\) and \(E(\tau)\) are similar to Eq. (33).

From the EoS (15), one can express all thermodynamic quantities as functions of \(T\) and \(\mu_5\). Since the critical temperature \(T_c \sim 200\) MeV is much larger than the chiral chemical potential in relativistic heavy ion collisions, i.e. \(\mu_5 \ll T\), all terms proportional to \(\mu_5\) in the thermodynamic relations are negligible. As a consequence, we obtain
\[
\varepsilon = \varepsilon_0 \left(\frac{T}{T_0}\right)^{1+e^{-2}} + \mathcal{O}(\mu_5^2/T^2), \tag{44}
\]
where \(\varepsilon_0 = \varepsilon(\tau_0)\) and \(T_0 = T(\tau_0)\). By introducing,
\[
x(\tau) = \exp[\mathcal{E}(\tau) - \mathcal{E}(\tau_0)], \quad y(\tau) = \exp[\mathcal{N}(\tau) - \mathcal{N}(\tau_0)], \quad z(\tau) = \exp[\mathcal{L}(\tau) - \mathcal{L}(\tau_0)], \tag{45}
\]
Equation (42) is reduced to
\[
\frac{d}{d\tau} x = -\sigma x - \frac{a_1}{\tau_0} y(\tau) \left(\frac{\tau}{\tau_0}\right)^{-1+2c_s^2} z^{-2c_s^2/(1+c_s^2)},
\]
\[
\frac{d}{d\tau} y = a_2 \frac{x(\tau)}{\tau},
\]
\[
\frac{d}{d\tau} z = \sigma x \frac{E_0^2}{\varepsilon_0} \left(\frac{\tau_0}{\tau}\right)^{1-c_s^2} x^2(\tau) + \frac{a_3}{\tau_0} \left(\frac{\tau}{\tau_0}\right)^{-2+3c_s^2} x(\tau) y(\tau) z^{-2c_s^2/(1+c_s^2)}, \tag{46}
\]
where \(x(\tau_0) = y(\tau_0) = z(\tau_0) = 1\), and \(a_1, a_2, a_3\) are dimensionless constants determined by the initial conditions
\[
a_1 = eC \chi_5 \frac{B_0 n_{5,0}}{\sigma T_0^2 E_0} \tau_0,
\]
Figure 3: The normalized electric field $E/E_0$ as functions of the proper time $\tau$. We have chosen $\tau_0 = 0.6 \text{ fm/c}$, $c_s^2 = 1/3$ and $E_0^2/\varepsilon_0 = 0.1$. The solid lines are obtained by solving Eq. (46) numerically and the dashed lines are from the approximate analytic solution (51). In the first row, we fix $\sigma/\tau_0 = 0.1$, $a_2 = \pm 0.2$, $a_3 = \pm 0.10$ and change the values of $a_1$. In the second row, we fix $\sigma/\tau_0 = 0.1$, $a_1 = \pm 0.5$ and change the values of $a_2$. In the last row, we fix $(a_1, a_2) = \pm (0.05, 0.02)$ and change the values of $\sigma/\tau_0$.

\[
\begin{align*}
a_2 &= \frac{e^2 C \chi E_0 B_0}{n_{5,0}} \tau_0, \\
a_3 &= \frac{e C \chi n_{5,0} E_0 B_0}{a} \frac{\tau_0}{\varepsilon_0 T_0^2} \tau_0.
\end{align*}
\] (47)

These dimensionless constants are all linearly proportional to $\hbar$ through the anomaly constant $C$, which means they are of quantum nature. So we can deal with the terms proportional to $a_1$, $a_2$ and $a_3$ in Eq.(46) as perturbations to the classical terms, and Eq. (46) can be solved order by order in powers of $\hbar$.

To the linear order in $\hbar$, we have the solutions for $x(\tau)$, $y(\tau)$ and $z(\tau)$.
\[ x(\tau) = e^{-\sigma(\tau - \tau_0)} - \frac{a_1}{\tau_0} e^{-\sigma \tau_0} \int_{\tau_0}^{\tau} d\tau' e^{\sigma \tau'} \left( \frac{\tau'}{\tau_0} \right)^{-1+2c_s^2} z_0(\tau')^{-2c_s^2/(1+c_s^2)}, \]

\[ y(\tau) = 1 + a_2 e^{\sigma \tau_0} [E_1(\sigma \tau_0) - E_1(\sigma \tau)], \]

\[ z(\tau) = z_0(\tau) + \frac{a_3}{\tau_0} \int_{\tau_0}^{\tau} d\tau' \left( \frac{\tau'}{\tau_0} \right)^{-2+3c_s^2} e^{-\sigma(\tau' - \tau_0)} [z_0(\tau')]^{-2c_s^2/(1+c_s^2)}, \]

where

\[ z_0(\tau) = 1 + \sigma \frac{E_0^2}{\varepsilon_0} e^{2\sigma \tau_0} \left[ \tau_0 E_{1-c_s^2}(2\sigma \tau_0) - \tau \left( \frac{\tau}{\tau_0} \right)^{c_s^2-1} E_{1-c_s^2}(2\sigma \tau) \right]. \]

We can further simplify the integration in \( x(\tau) \) and \( z(\tau) \). Since initial energy density \( \varepsilon_0 \) is much larger than the initial energy of the EM fields \( \varepsilon_0 \gg B_0^2, E_0^2, E_0B_0 \) (see, e.g., Ref. [34] for the values of \( B_0^2/\varepsilon_0 \) in the event-by-event simulation of relativistic heavy ion collisions), we can further simplify the integration in \( x(\tau) \) and \( z(\tau) \) in the linear order in \( E_0^2/\varepsilon_0 \) as

\[ x(\tau) = e^{-\sigma(\tau - \tau_0)} - \frac{a_1}{\tau_0} e^{-\sigma \tau_0} \left[ \tau_0 E_{1-c_s^2}(-\sigma \tau_0) - \tau \left( \frac{\tau}{\tau_0} \right)^{c_s^2-1} E_{1-c_s^2}(-\sigma \tau) \right] + O(a_2^2, a_4 E_0^2/\varepsilon_0), \]

\[ z(\tau) = 1 + \sigma \frac{E_0^2}{\varepsilon_0} e^{2\sigma \tau_0} \left[ \tau_0 E_{1-c_s^2}(2\sigma \tau_0) - \tau \left( \frac{\tau}{\tau_0} \right)^{c_s^2-1} E_{1-c_s^2}(2\sigma \tau) \right]
+ \frac{a_3}{\tau_0} e^{\sigma \tau_0} \left[ \tau_0 E_{2-3c_s^2}(\sigma \tau_0) - \tau \left( \frac{\tau}{\tau_0} \right)^{2-3c_s^2} E_{2-3c_s^2}(\sigma \tau) \right] + O(a_2^2, a_4 E_0^2/\varepsilon_0). \]

Then we obtain the solutions for \( E(\tau) \), \( n_5(\tau) \) and \( \varepsilon(\tau) \) in the linear order in \( h \) and \( E_0^2/\varepsilon_0 \)

\[ E(\tau) = E_0 \left( \frac{\tau}{\tau_0} \right) \left\{ e^{-\sigma(\tau - \tau_0)} - a_1 e^{-\sigma \tau_0} E_{1-c_s^2}(-\sigma \tau) - \left( \frac{\tau}{\tau_0} \right)^{2c_s^2} E_{1-c_s^2}(-\sigma \tau) \right\}, \]

\[ n_5(\tau) = n_{5,0} \left( \frac{\tau}{\tau_0} \right) \left[ 1 + a_2 e^{\sigma \tau_0} [E_1(\sigma \tau_0) - E_1(\sigma \tau)] \right], \]

\[ \varepsilon(\tau) = \varepsilon_0 \left( \frac{\tau}{\tau_0} \right)^{1+c_s^2} \left\{ 1 + \sigma \frac{E_0^2}{\varepsilon_0} e^{2\sigma \tau_0} \left[ \tau_0 E_{1-c_s^2}(2\sigma \tau_0) - \tau \left( \frac{\tau}{\tau_0} \right)^{c_s^2-1} E_{1-c_s^2}(2\sigma \tau) \right]
+ \frac{a_3}{\tau_0} e^{\sigma \tau_0} \left[ \tau_0 E_{2-3c_s^2}(\sigma \tau_0) - \tau \left( \frac{\tau}{\tau_0} \right)^{2-3c_s^2} E_{2-3c_s^2}(\sigma \tau) \right] \right\}. \]

In the leading order, we see \( E(\tau) \sim \frac{E_0}{\tau} x(\tau) \sim \frac{1}{\tau} e^{-\sigma \tau} \), i.e. the electric field decays in the conducting medium [69]. In the leading order, \( y(\tau) \sim 1 \) means \( n_5 \sim \frac{E_0}{\tau} \). We also see that when \( c_s^2 = 1/3 \), the analytic solutions of \( E(\tau) \) and \( n_5(\tau) \) have the same form as in Eq. (38) in previous subsection.

In Figs. 3, 4, 5, we plot the normalized \( E/E_0, n_5/n_{5,0} \) and \( \varepsilon/\varepsilon_0 \) as functions of the proper time \( \tau \). We choose the \( \tau_0 = 0.6 \text{ fm/c} \), the speed of sound \( c_s^2 = 1/3 \) and \( E_0^2/\varepsilon_0 = 0.1 \). The solid lines in those figures are the numerical results from Eqs. (46), while the dashed lines
Figure 4: The normalized electric field $n_5/n_{5,0}$ as functions of the proper time $\tau$. We have chosen $\tau_0 = 0.6$ fm/c, $c_s^2 = 1/3$ and $E_0^2/\varepsilon_0 = 0.1$. The solid lines are obtained by solving Eq. (46) numerically and the dashed lines are from the approximate analytic solution (51). In the first row, we fix $\sigma/\tau_0 = 0.1$, $a_2 = \pm 0.2$, $a_3 = \pm 0.10$ and change the values of $a_1$. In the second row, we fix $\sigma/\tau_0 = 0.1$, $a_1 = \pm 0.5$ and change the values of $a_2$.

are from approximate analytic solutions (51). We see that the approximation works very well for small $a_i$.

In Fig. 3, we find $E/E_0$ is almost independent of $a_2$ and $a_3$, as expected in Eq. (51). The $E/E_0$ decays rapidly as $a_1$ or $\sigma$ grows. Similar to the cases in Subsec. IV A, $E/E_0$ can be negative at the late proper time. Such a behavior may come from the competition between the anomalous conservation equation $\partial_j j_5^j = -CE \cdot B$ and Maxwell’s equations.

In Fig. 4, the numerical results show that $n_5$ is almost independent of $a_1$ and $a_3$ in small $a_i$ cases as expected in Eq. (51). The $n_5$ decays slowly as $a_2$ increases and the decay behavior of $n_5$ is also not sensitive to variation of $\sigma$.

In Fig. 5, we find that the time evolution of $\varepsilon(\tau)$ seems to be insensitive to $a_1$ and $a_2$. 

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Figure 5: The normalized electric field $\varepsilon/\varepsilon_0$ as functions of the proper time $\tau$. We have chosen $\tau_0 = 0.6$ fm/c, $c_s^2 = 1/3$ and $E_0^2/\varepsilon_0 = 0.1$. The solid lines are obtained by solving Eq. (46) numerically and the dashed lines are from the approximate analytic solution (51). In the first row, we fix $\sigma/\tau_0 = 0.1$, $a_2 = \pm 0.2$, $a_3 = \pm 0.10$ and change the values of $a_1$. In the second row, we fix $\sigma/\tau_0 = 0.1$, $a_1 = \pm 0.5$ and change the values of $a_2$.

Because $E_0^2/\varepsilon_0 \ll 1$, the contribution from the second term in Eq. (51) which is proportional to $\sigma E_0^2/\varepsilon_0$ is negligible. Interestingly, the energy density decays slower as $a_3$ grows. As shown in Fig. 5, for a large value of $a_3$, e.g. $a_3 = 3.0$, the energy density even increases at early time. That is because the fluid gain the energy from the EM fields, i.e. the $a_3$ term in Eq. (51) dominates. Similar behavior is also found in the ideal MHD with a background magnetic field [54, 55].

We make some remarks here. From analytic solutions (38) and (51), we conclude that the CME and chiral anomaly as quantum corrections play a role to the time evolution of the electric field $E(\tau)$, the chiral charge density $n_5(\tau)$ and the energy density $\varepsilon(\tau)$. With an initial magnetic field parallel to the electric field (with $\chi = 1$) and all $a_i$ ($i = 1, 2, 3$) are positive,
$E(\tau)/n_5(\tau)$ decay faster/slower than the cases without CME. If the initial magnetic field is anti-parallel to the electric field (with $\chi = -1$) and all $a_i$ are negative, $E(\tau)/n_5(\tau)$ decay slower/faster than the cases without CME. This behavior is consistent with the anomalous conservation equation $\partial_\mu j_\mu = -CE_\mu B_\mu = C\chi E(\tau)B(\tau)$ combined with Maxwell’s equations. For example, if $\chi = +1$, we have $\partial_\tau(n_5(\tau)) = C\gamma\chi E(\tau)B(\tau) > 0$, implying that $n_5(\tau)$ decays slower than the case $C = 0$. From Eq. (23), we have $\partial_\tau[E\exp(\sigma\tau)] = -\gamma\chi\xi B(\tau) < 0$, i.e. $E(\tau)$ decays faster than the case $C = 0$. Such a behavior is due to that the chiral charge density is converted from the magnetic helicity. For $\chi = -1$, the magnetic helicity will be converted from the chiral charge density so the behavior is opposite. The numerical results in Figs. 1, 2, 3, 4 and 5 are consistent with the above observation.

C. Discussions

In Subsec. IVA and IVB, we have obtained the approximate analytic solutions in two types of EoS. From Eq.(30), the proper time behavior of the magnetic field seems to be the same as the case without CME and finite conductivity, i.e. in an ideal MHD [54–56]. It seems to be counter-intuitive and inconsistent with the Maxwell’s equations. Our explanation is as follows. The $E^\mu$ and $B^\mu$ defined in the four vector form of EM fields in Eq. (5) are the fields in the co-moving frame of the fluid. The $B(\tau)$ in Eq.(30) is the length of the magnetic field three vector $B$. To show the explicit contribution from CME and finite conductivity to each component of $B$, we will compute EM fields three vector in the laboratory frame.

From Eq. (4), we observe that the EM field strength tensor $F^{\mu\nu}$ as well as the energy-momentum tensor $T^{\mu\nu}$ and fluid velocity $u^\mu$ is measured in the laboratory frame. According to the standard definitions of EM fields through the field strength tensor $F^{\mu\nu}$, i.e.

$$E^i_L = F^{i0}, \quad B^i_L = -\frac{1}{2}\epsilon^{ijk}F^{jk},$$

we can get the EM fields in the lab frame

$$E_L = (\gamma v^z B(\tau), \chi\gamma E(\tau), 0),$$
$$B_L = (-\gamma v^z\chi E(\tau), \gamma B(\tau), 0),$$

(52)

where in this subsection, we will use the lower index $L$ for the EM fields in the laboratory frame and $E(\tau)$ and $B(\tau)$ are the functions solved in previous Subsec. IV. We find that in
the lab frame $B_L^x$ and $E_L^y$ depend on the finite conductivity $\sigma$ and CME coefficient $\xi$ through $E(\tau)$.

Next, we will check the self-consistence of Maxwell’s equations. We will prove that the CME and finite conducting current will not generate the EM fields in the $z$ direction, i.e. $E_L^z$ and $B_L^z$ are always vanishing. From

$$\nabla \times E_L = -\partial_t B_L,$$  \hspace{1cm} (53)

we observe that with Eq. (52) the $\partial_t B_L^z = 0$ and $\partial_y E_L^z = -\partial_x B_L^x + \partial_z E_L^y = 0$ are automatically satisfied. With the solution (30), we can also obtain that $\partial_x E_L^z = \partial_t B_L^y + \partial_z E_L^x = 0$.

Similarly from

$$\nabla \cdot E_L = n_e, \quad \nabla \cdot B_L = 0,$$  \hspace{1cm} (54)

and Eq.(52), we can also obtain that $\partial_z E_L^z = -\partial_x E_L^x - \partial_y E_L^y = 0$ with $n_e = 0$, and $\partial_z B_L^x = -\partial_x B_L^x - \partial_y B_L^y = 0$.

We will focus on the last equation

$$\nabla \times B_L = j_e + \partial_t E_L.$$  \hspace{1cm} (55)

Different with the charge current in a static conductor, the charge current $j_e$ of a relativistic fluid includes two parts. The part parallel to the fluid velocity $u^\mu$ read

$$j_{e,\parallel} = \sigma E_L,\parallel + \xi B_L,\parallel,$$  \hspace{1cm} (56)

and the other part perpendicular to the fluid velocity is given by

$$j_{e,\perp} = \sigma(\gamma(E_L - v \times B_L),\perp + \xi \gamma(B_L - v \times E_L),\perp,$$  \hspace{1cm} (57)

with $v$ being the three vector of fluid velocity, i.e, $u^\mu = \gamma(1, v)$. In our case, since the fluid moves alone the $z$ direction, the charge current is given by

$$j_e = [\gamma(E_L^y + v^z B_L^x) + \xi \gamma(B_L^y - v^z E_L^x)] e_y.$$  \hspace{1cm} (58)

With Eq. (52), we find that $\partial_y B_L^z = \partial_x E_L^y + \partial_z B_L^y = 0$ and $\partial_t E_L^z = 0$. The space derivative of magnetic field in the $z$ direction is

$$\partial_z B_L^z = \partial_z B_L^x - \sigma \gamma(E_L^y + v^z B_L^x) - \xi \gamma(B_L^y - v^z E_L^x) - \partial_t E_L^y,$$  \hspace{1cm} (59)
where the left-handed-side of above equation equals to the right-handed-side of Eq. (23). Thus, inserting our solutions in Eqs. (38, 51) yields $\partial_z B^z_L = 0$.

Since both time and space derivatives of $E^x_L$ and $B^x_L$ vanish and initial $E^y_L$ or $B^y_L$ are chosen to be vanishing, we can conclude that in our setup the CME and conducting current will not generate EM fields in the $z$ direction in the lab frame. While only the space-time derivatives of EM fields in the transverse direction, e.g. $B^x_L$ and $E^y_L$, are non-vanishing. This is quite different with the case of a static media, in which the CME current can induce a circular magnetic field [61].

Thirdly, we will discuss the Bjorken fluid velocity. Usually, we can consider the right-handed-side of Eq. (26), as the covariant form of Lorentz force acting on the fluid. In the lab frame, we can rewrite it as

$$F^{\nu\lambda}j_{e\lambda} = (j_{e,0}E_L, j_e \times B_L).$$

(60)

Since we have chosen the $\mu_e = 0$, the electric field will not accelerate the fluid, i.e. the zeroth component $j_{e,0}E_L = 0$. The other component $j_e \times B_L$ is the Lorentz force driving by the magnetic field, where $j_e$ is given by Eqs. (56, 57). In our case, the EM fields with Lorentz force $j_e \times B_L$ is analogy to the so-called the force free fields (e.g. also see the discussion in the classical electrodynamics [62, 63] and recent studies in Ref. [75, 76]). Through Eqs. (19, 25), we have already shown that the EM fields in our setup will not modify the fluid velocity.

At last, we will check the consistence of (anomalous) current conservation equations. Since Eqs. (38, 51) are the solutions of anomalous current equation $\partial_\mu j^\mu_e = -CE \cdot B$, the anomalous current equation should be satisfied. Because EM fields are independent on $x, y$, the charge current conservation equation reduces to $\partial_\mu j^\mu_e = \partial_t j_{e,0} + \nabla \cdot j_e = \partial_z j_{e,z}$, with $j_{e,z} = j_{e,\parallel} = \sigma E_{L,z} + \xi B_{L,z} = 0$. We can conclude that the (anomalous) current conservation equations are satisfied.

Before we end this section, we make some remarks here. We have computed the EM fields in the lab frame and found our solutions satisfy the Maxwell’s equations. In our setup, the CME and electric conducting current will not generate the EM fields in $z$ direction in lab frame. It is quite different with the case in a static media. We have also shown the Lorentz force will not accelerate the fluid. At last, we have checked the self-consistence of (anomalous) current conservation equations.
V. SUMMARY AND CONCLUSIONS

We have solved MHD equations with longitudinal boost invariance and transverse EM fields in the presence of the CME and finite electric conductivity. The MHD equations involve the energy-momentum, the electric charge and chiral charge (anomalous) conservation equations coupled with Maxwell’s equations. We consider two types of EoS corresponding to the large chiral chemical potential and the high temperature cases respectively. For further simplification, we consider the electric charge neutral fluid and set the electric charge density \( n_e \) and its corresponding chemical potential \( \mu_e \) vanish.

We assume the Bjorken form of the fluid velocity in the longitudinal direction. To keep the fluid velocity unchanged, we obtain the four-vector form of the electric and magnetic field which are orthogonal to the fluid velocity. To solve the MHD equations, we treat the terms with the anomaly constant which is proportional to the Planck constant \( \hbar \) as perturbations. This is equivalent to an expansion in \( \hbar \). Then we apply the non-conserved charge method to obtain the approximate analytic solutions. The comparison of the analytic solutions with the exact numerical results shows good agreement.

Finally we compute the EM field in three-vector form in the lab frame and show the contributions from the electric conductivity and the CME. According to Maxwell’s equations, in our setup, the CME and electrically conducting current only modify the EM fields in the transverse direction in the lab frame. The electric and magnetic field in the z-direction does not grow with time and space. The Lorentz force only changes the time evolution of thermodynamic quantities and does not accelerate the fluid.

Our results can provide a future test of complete numerical simulations of the MHD with the CME. Since the polarization of chiral fermions in the strong magnetic field is different from the ordinary magnetization which is called chiral Barnett effect [77], the current method can be applied to study the magnetization effect in the future.

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