SPATIALITY OF DERIVATIONS ON THE ALGEBRA OF \( \tau \)-COMPACT OPERATORS

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ABSTRACT. This paper is devoted to derivations on the algebra \( S_0(M, \tau) \) of all \( \tau \)-compact operators affiliated with a von Neumann algebra \( M \) and a faithful normal semi-finite trace \( \tau \). The main result asserts that every \( \tau \)-continuous derivation \( D : S_0(M, \tau) \to S_0(M, \tau) \) is spatial and implemented by a \( \tau \)-measurable operator affiliated with \( M \), where \( \tau \) denotes the measure topology on \( S_0(M, \tau) \). We also show the automatic \( \tau \)-continuity of all derivations on \( S_0(M, \tau) \) for properly infinite von Neumann algebras \( M \). Thus in the properly infinite case the condition of \( \tau \)-continuity of the derivation is redundant for its spatiality.

1. INTRODUCTION

Given an algebra \( A \), a linear operator \( D : A \to A \) is called a derivation, if \( D(xy) = D(x)y + xD(y) \) for all \( x, y \in A \) (the Leibniz rule). Each element \( a \in A \) implements a derivation \( D_a \) on \( A \) defined as \( D_a(x) = [a, x] = ax - xa, x \in A \). Such derivations \( D_a \) are said to be inner derivations. If the element \( a \), implementing the derivation \( D_a \), belongs to a larger algebra \( B \) containing \( A \), then \( D_a \) is called a spatial derivation on \( A \).

One of the main problems in the theory of derivations is to prove the automatic continuity, “inner-ness” or “spatiality” of derivations, or to show the existence of non-inner, non-spatial and moreover discontinuous derivations on various topological algebras.

The theory of derivations in the framework of operator algebras is an important and well investigated part of this theory, with applications in mathematical physics. It is well known that every derivation of a \( C^* \)-algebra is bounded (i.e. is norm continuous), and that every derivation of a von Neumann algebra is inner. For a detailed exposition of the theory of bounded derivations we refer to the monographs of Sakai [19], [20].

Investigations of general unbounded derivations (and derivations on unbounded operator algebras) began much later and were motivated mainly by needs of mathematical physics, in particular by the problem of constructing the dynamics in quantum statistical mechanics (see, e.g. [15], [19], [20]).

The development of a non commutative integration theory was initiated by I. Segal [21], who considered new classes of (not necessarily Banach) algebras of unbounded operators, in particular the algebra \( S(M) \) of all measurable operators affiliated with a von Neumann algebra \( M \). Algebraic, order and topological properties of the algebra \( S(M) \) are somewhat similar to those of von Neumann algebras, therefore in [5] the first author have initiated the study of derivations on the algebra \( S(M) \).

If the von Neumann algebra \( M \) is abelian, then it is \(*\)-isomorphic to the algebra \( L^\infty(\Omega) = L^\infty(\Omega, \Sigma, \mu) \) of all (classes of equivalence of) complex essentially bounded measurable functions on a measure space...
(Ω, Σ, μ) and therefore $S(M) \cong L^0(Ω)$, where $L^0(Ω, Σ, μ)$ the algebra of all complex measurable functions on $(Ω, Σ, μ)$. In this case it is clear that every measure continuous derivation and, in particular, all inner derivations on $S(M)$ are identically zero, i.e. trivial.

Investigating the abelian case A. F. Ber, F. A. Sukochev, V. I. Chilin in [12] obtained necessary and sufficient conditions for the existence of non trivial derivations on commutative regular algebras. In particular they have proved that the algebra $L^0(0, 1)$ of all complex measurable functions on the $(0, 1)$-interval admits non trivial derivations. Independently A. G. Kusraev [16] by means of Boolean-valued analysis has established necessary and sufficient conditions for the existence of non trivial derivations and automorphisms on extended complete complex $f$-algebras. In particular he has also proved the existence of non trivial derivations and automorphisms on $L^0(0, 1)$. It is clear that these derivations are discontinuous in the measure topology, and hence they are not inner.

Therefore the properties of derivations on the unbounded operator algebra $S(M)$ turned to be very far from those on $C^*$- or von Neumann algebras. Nevertheless there was a conjecture that the existence of such “exotic” examples of derivations is deeply connected with the commutative nature of the underlying von Neumann algebra $M$. In view of this we have initiated investigations of the above problems in the non commutative case, namely on the algebra $LS(M)$ of all locally measurable operators affiliated with a von Neumann algebra $M$, and on its subalgebras, including the mentioned algebra $S(M)$, the algebra $S(M, τ)$ of all $τ$-measurable operators affiliated with $M$, non commutative Arens algebras, etc. [1,4,6,9].

In [3] and [8] derivations on various subalgebras of the algebra $LS(M)$ of locally measurable operators with respect to a von Neumann algebra $M$ has been considered. A complete description of derivations has been obtained in the cases when $M$ is of type I and III. Derivations on algebras of measurable and locally measurable operators, including rather non trivial commutative case, have been studied by many authors [1–14]. A comprehensive survey of results concerning derivations on various algebras of unbounded operators affiliated with von Neumann algebras can be found in [7].

If we consider the algebra $S(M)$ of all measurable operators affiliated with a type III von Neumann algebra $M$, then it is clear that $S(M) = M$. Therefore from the results of [3] it follows that for type $I_∞$ and type III von Neumann algebras $M$ every derivation on $S(M)$ is automatically inner and, in particular, is continuous in the local measure topology. The problem of description of the structure of derivations in the case of type II algebras has been open so far and seems to be rather difficult.

In this connection several open problems concerning innerness and automatic continuity of derivations on the algebras $S(M)$ and $LS(M)$ for type II von Neumann algebras have been posed in [7]. First positive results in this direction were recently obtained in [10], [11], where automatic continuity has been proved for derivations on algebras of $τ$-measurable and locally measurable operators affiliated with properly infinite von Neumann algebras.

Another problem [7, Problem 3] asks the following question:

Let $M$ be a type II von Neumann algebra with a faithful normal semi-finite trace $τ$. Consider the algebra $S(M)$ (respectively $LS(M)$) of all measurable (respectively locally measurable) operators affiliated with $M$ and equipped with the local measure topology $t$. Is every $t$-continuous derivation $D : S(M) \to S(M)$ (respectively, $D : LS(M) \to LS(M)$) necessarily inner?

The positive answer for the above question have been given independently in [9] and [10], [11]. It should be noted that the proofs of innerness of continuous derivations on algebras $S(M)$ and $LS(M)$
in the above mentioned papers are essentially based on the fact that the von Neumann algebra $M$ is a subalgebra in the considered algebras.

The present paper is devoted to derivations on the algebra $S_0(M, \tau)$ of all $\tau$-compact operators affiliated with a von Neumann algebra $M$ with a faithful normal semi-finite trace $\tau$. Since the algebra $S_0(M, \tau)$ does not contain the von Neumann algebra $M$, one cannot directly apply the methods of the papers [9], [11] in this case.

In Section 2 we give preliminaries from the theory of measurable operators affiliated with a von Neumann algebra $M$, and, in particular, we recall the notion of $\tau$-compact operator with respect to $M$.

In Section 3 we show that if $M$ is a properly infinite von Neumann algebra with a faithful normal semi-finite trace $\tau$, then any derivation $D : S_0(M, \tau) \to S_0(M, \tau)$ is automatically $t_\tau$-continuous.

In Section 4 we prove if $M$ is a von Neumann algebra with a faithful normal semi-finite trace $\tau$, then every $t_\tau$-continuous derivation $D : S_0(M, \tau) \to S_0(M, \tau)$ is spatial and implemented by an element from the algebra $S(M, \tau)$. As a corollary, we obtain that if $M$ is a properly infinite von Neumann algebra then arbitrary derivation $D : S_0(M, \tau) \to S_0(M, \tau)$ is spatial.

2. ALGEBRAS OF MEASURABLE OPERATORS

Let $B(H)$ be the $*$-algebra of all bounded linear operators on a Hilbert space $H$, and let $1$ be the identity operator on $H$. Consider a von Neumann algebra $M \subset B(H)$ with the operator norm $\| \cdot \|$ and with a faithful normal semi-finite trace $\tau$. Denote by $P(M) = \{p \in M : p = p^2 = p^*\}$ the lattice of all projections in $M$.

A linear subspace $D$ in $H$ is said to be affiliated with $M$ (denoted as $D\eta M$), if $u(D) \subset D$ for every unitary $u$ from the commutant$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$of the von Neumann algebra $M$.

A linear operator $x : D(x) \to H$, where the domain $D(x)$ of $x$ is a linear subspace of $H$, is said to be affiliated with $M$ (denoted as $x\eta M$) if $D(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in D(x)$ and for every unitary $u \in M'$.

A linear subspace $D$ in $H$ is said to be strongly dense in $H$ with respect to the von Neumann algebra $M$, if

1) $D\eta M$;
2) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow 1$, $p_n(H) \subset D$ and $p_n^* = 1 - p_n$ is finite in $M$ for all $n \in \mathbb{N}$.

A closed linear operator $x$ acting in the Hilbert space $H$ is said to be measurable with respect to the von Neumann algebra $M$, if $x\eta M$ and $D(x)$ is strongly dense in $H$.

Denote by $S(M)$ the set of all linear operators on $H$, measurable with respect to the von Neumann algebra $M$. If $x \in S(M)$, $\lambda \in \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, then $\lambda x \in S(M)$ and the operator $x^*$, adjoint to $x$, is also measurable with respect to $M$ (see [21]). Moreover, if $x, y \in S(M)$, then the operators $x + y$ and $xy$ are defined on dense subspaces and admit closures that are called, correspondingly, the strong sum and the strong product of the operators $x$ and $y$, and are denoted by $x + y$ and $x * y$. It was shown in [21] that $x + y$ and $x * y$ belong to $S(M)$ and these algebraic operations make $S(M)$ a $*$-algebra with the identity $1$ over the field $\mathbb{C}$. Here, $M$ is a $*$-subalgebra of $S(M)$. In what
follows, the strong sum and the strong product of operators \( x \) and \( y \) will be denoted in the same way as the usual operations, by \( x + y \) and \( xy \).

It is clear that if the von Neumann algebra \( M \) is finite then every linear operator affiliated with \( M \) is measurable and, in particular, a self-adjoint operator is measurable with respect to \( M \) if and only if all its spectral projections belong to \( M \).

Let \( \tau \) be a faithful normal semi-finite trace on \( M \). We recall that a closed linear operator \( x \) is said to be \( \tau \)-measurable with respect to the von Neumann algebra \( M \), if \( x\eta M \) and \( \mathcal{D}(x) \) is \( \tau \)-dense in \( H \), i.e. \( \mathcal{D}(x)\eta M \) and given \( \varepsilon > 0 \) there exists a projection \( p \in M \) such that \( p(H) \subset \mathcal{D}(x) \) and \( \tau(p^\perp) < \varepsilon \).

Denote by \( S(M, \tau) \) the set of all \( \tau \)-measurable operators affiliated with \( M \).

Consider the topology \( t_\tau \) of convergence in measure or measure topology on \( S(M, \tau) \), which is defined by the following neighborhoods of zero:

\[
U(\varepsilon, \delta) = \{ x \in S(M, \tau) : \exists e \in P(M), \tau(e^\perp) < \delta, xe \in M, \| xe \| < \varepsilon \},
\]

where \( \varepsilon, \delta \) are positive numbers.

It is well-known \([18]\) that \( M \) is \( t_\tau \)-dense in \( S(M, \tau) \) and \( S(M, \tau) \) equipped with the measure topology is a complete metrizable topological \(*\)-algebra.

In the algebra \( S(M, \tau) \) consider the subset \( S_0(M, \tau) \) of all operators \( x \) such that given any \( \varepsilon > 0 \) there is a projection \( p \in P(M) \) with \( \tau(p^\perp) < \infty, xp \in M \) and \( \| xp \| < \varepsilon \). Following \([22]\) let us call the elements of \( S_0(M, \tau) \) \( \tau \)-compact operators with respect to \( M \). It is known \([17]\) that \( S_0(M, \tau) \) is a *-subalgebra in \( S(M, \tau) \) and a bimodule over \( M \), i.e. \( ax, xa \in S_0(M, \tau) \) for all \( x \in S_0(M, \tau) \) and \( a \in M \).

The following properties of the algebra \( S_0(M, \tau) \) are known (see \([22]\)):

1) \( S(M, \tau) = M + S_0(M, \tau) \);
2) \( S_0(M, \tau) \) is an ideal in \( S(M, \tau) \).

Note that if the trace \( \tau \) is finite then

\[
S_0(M, \tau) = S(M, \tau) = S(M).
\]

It is well-known \([22]\) \( S_0(M, \tau) \) equipped with the measure topology is a complete metrizable topological \(*\)-algebra.

3. Continuity of derivations on \(*\)-algebras of \( \tau \)-compact operators in the properly infinite case

In this section we prove the automatic continuity of derivations in the measure topology on the algebra of \( \tau \)-compact operators affiliated with a properly infinite von Neumann algebra and a faithful normal semi-finite trace.

Denote by \( S_0(M, \tau)_b \) the bounded part of the \(*\)-algebra \( S_0(M, \tau) \), i.e.

\[
S_0(M, \tau)_b = M \cap S_0(M, \tau).
\]

Since \( S_0(M, \tau) \) is a complete metrizable topological \(*\)-algebra with respect to the measure topology, and the norm topology on \( M \) is stronger than the measure topology, we have that \( S_0(M, \tau)_b \) is a Banach \(*\)-algebra with respect to the norm topology.
Set
\[ P_\tau(M) = \{ p \in P(M) : \tau(p) < +\infty \}. \]

It is clear that
\[ P_\tau(M) = P(M) \cap S_0(M, \tau). \]

Let \( D \) be a derivation on \( S_0(M, \tau) \). Let us define a mapping \( D^* : S_0(M, \tau) \rightarrow S_0(M, \tau) \) by setting
\[ D^*(x) = (D(x^*))^*, \ x \in S_0(M, \tau). \]

A direct verification shows that \( D^* \) is also a derivation on \( S_0(M, \tau) \). A derivation \( D \) on \( S_0(M, \tau) \) is said to be
- **hermitian**, if \( D^* = D \),
- **skew-hermitian**, if \( D^* = -D \).

Every derivation \( D \) on \( S_0(M, \tau) \) can be represented in the form
\[ D = D_1 + iD_2, \]
where \( D_1 = (D + D^*)/2 \), \( D_2 = (D - D^*)/2i \) are hermitian derivations on \( S_0(M, \tau) \).

It is clear that a derivation \( D \) is continuous if and only if the hermitian derivations \( D_1 \) and \( D_2 \) are continuous.

Therefore further in this section we may assume that \( D \) is a hermitian derivation.

**Lemma 3.1.** Let \( M \) be a von Neumann algebra with a faithful normal semi-finite trace \( \tau \) and let \( D : S_0(M, \tau) \rightarrow S_0(M, \tau) \) be a derivation. Then the following assertions are equivalent:

i) \( D \) is \( t_\tau \)-continuous;

ii) \( D|_{S_0(M, \tau)_b} \) is \( \| \cdot \|_{t_\tau} \)-continuous.

**Proof.** Since the norm topology on \( S_0(M, \tau)_b \) is stronger than the measure topology, it follows that implication \( i \Rightarrow ii \) is true.

Let us show the converse implication. Note that \((S_0(M, \tau), t_\tau)\) is an \( F \)-space. Therefore by the "closed graph theorem" it is sufficient to show that the graph of the linear operator \( D \) is closed.

Let \( x_n \overset{t_\tau}{\rightarrow} 0 \) and \( D(x_n) \overset{t_\tau}{\rightarrow} y \). Take an arbitrary \( p \in P_\tau(M) \). Then \( x_n p \overset{t_\tau}{\rightarrow} 0 \) and
\[ D(x_n p) = D(x_n) p + x_n D(p) \overset{t_\tau}{\rightarrow} y p. \]

Therefore there exists a sequence \( n_1 < n_2 < \ldots < n_k < \ldots \) such that \( x_{n_k} p \in U \left( \frac{1}{k}, \frac{1}{2k+1} \right) \). For each \( k \in \mathbb{N} \) take a projection \( p_k \leq p \) such that \( \tau(p - p_k) \leq \frac{1}{2^{k+1}} \) and \( \| x_{n_k} p p_k \| \leq \frac{1}{k} \). Set
\[ q_i = \bigwedge_{k=i}^{\infty} p_k, \ i \in \mathbb{N}. \]

Then
\[ \tau(p - q_i) \leq \sum_{k=i}^{\infty} \tau(p - p_k) \leq \frac{1}{2^i} \]
and
\[ \| x_{n_k} p q_i \| \leq \frac{1}{k} \]
for all $k \geq i$. This means that $x_{n_k}pq_i \xrightarrow{\|\|} 0$ as $k \to \infty$ for all $i$. Therefore $D(x_{n_k}pq_i) \xrightarrow{t_\tau} 0$. Thus

$$D(x_{n_k}p)q_i = D(x_{n_k}pq_i) - x_{n_k}pD(q_i) \xrightarrow{t_\tau} 0.$$ 

On the other hand

$$D(x_{n_k}p)q_i \xrightarrow{t_\tau} ypq_i.$$

So $ypq_i = 0$ for all $i$. Since $q_i \uparrow p$ we have that $yp = 0$. Taking into account that $p \in P_\tau(M)$ is arbitrary and the trace $\tau$ is semi-finite, we obtain that $y = 0$. This means that $D$ is $t_\tau$-continuous. The proof is complete.

**Theorem 3.2.** Let $M$ be a properly infinite von Neumann algebra with a faithful normal semi-finite trace $\tau$. Then any derivation $D : S_0(M, \tau) \to S_0(M, \tau)$ is $t_\tau$-continuous.

**Proof.** By Lemma 3.1 it is suffices show that $D|_{S_0(M, \tau)_b}$ is $\| \cdot \|_{t_\tau}$-continuous. Since $(S_0(M, \tau)_b, \| \cdot \|)$ and $(S_0(M, \tau), t_\tau)$ are $F$-spaces, as above it is sufficient to show that the graph of the linear operator $D|_{S_0(M, \tau)_b}$ is closed. For convenience we denote $D|_{S_0(M, \tau)_b}$ also by $D$.

Let us suppose the converse, i.e. assume that the graph of $D$ is not closed. This means that there exists a sequence $\{x_n\} \subset S_0(M, \tau)_b$ such that $x_n \xrightarrow{\|\|} 0$ and $D(x_n) \xrightarrow{t_\tau} y \neq 0$. Taking into account that $S_0(M, \tau)$ is a $*$-algebra and that $D = D^*$, we may assume that $y = y^*$, $x_n = x_n^*$ for all $n \in \mathbb{N}$. In this case, $y = y_+ - y_-$, where $y_+, y_- \in S_0(M, \tau)$ are respectively the positive and the negative parts of $y$. Without loss of generality, we shall also assume that $y_+ \neq 0$, otherwise, instead of the sequence $\{x_n\}$ we may consider the sequence $\{-x_n\}$. Let us choose $\lambda_0 > 0$ such that the projection $p = 1 - e_{\lambda_0}(y) \neq 0$. We have that

$$0 < \lambda_0 p \leq yp.$$ 

Again replacing, if necessary, $x_n$ by $x_n = px_np/\lambda_0$, we may assume that

$$y \geq p,$$

because

$$D(px_n p) = D(p)x_n p + pD(x_n)p + px_n D(p) \to yrp.$$ 

Note that $y \in S_0(M, \tau)$ implies that $p \in P_\tau(M)$. By the assumption, $M$ is a properly infinite von Neumann algebra and therefore, there exist pairwise orthogonal projections $p_2, \ldots, p_k, \ldots$ in $p^\perp Mp^\perp$ such that $p_1 = p \sim p_k$ for all $k \geq 2$. Let $u_k$ be a partial isometry in $M$ such that $u_k^*u_k = p_1$, $u_ku_k^* = p_k$ for all $k$. Then

$$u_k x_n u_k^* \to 0 \text{ as } n \to \infty$$

and

$$D(u_k x_n u_k^*) = D(u_k) x_n u_k^* + u_k D(x_n) u_k^* + u_k x_n D(u_k^*) \to u_k y u_k^* \geq u_k p u_k^* = p_k$$

for all $k$.

Let $\tau(p) = 4 \varepsilon$. Since

$$u_k x_n u_k^* \xrightarrow{\|\|} 0$$

and

$$p_k D(u_k x_n u_k^*)p_k \xrightarrow{t_\tau} p_k u_k y u_k^* p_k$$

for all $k$.
there exist a projection $q_k \leq p_k$ and a number $n(k)$ such that
\[ \tau(p_k - q_k) < \varepsilon \]
and
\[ \|u_{k,x_n(k)}u_k^*\| < 1/k^3, \|p_k D(u_{k,x_n(k)}u_k^*)p_k - p_k u_k u_k^* p_k q_k\| < 1/2. \]
Taking into account these inequalities and since $D$ is hermitian we obtain that
\[ (3.1) \quad \tau(q_k) > 3\varepsilon \]
and
\[ (3.2) \quad q_k D(u_{k,x_n(k)}u_k^*)q_k \geq \frac{1}{2} q_k. \]
Now set
\[ x = \sum_{k=1}^{\infty} k u_{k,x_n(k)}u_k^* \in S_0(M, \tau)_b. \]
Then there exists $\lambda > 0$ such that
\[ D(x) \in U(\lambda, \varepsilon). \]
Therefore
\[ (3.3) \quad q_k D(x) q_k \in U(\lambda, 2\varepsilon). \]
On the other hand,
\[ p_k D(p_k x p_k) p_k = p_k D(p_k x p_k p_k + p_k p_k D(x) p_k p_k + p_k p_k x D(p_k) p_k). \]
Taking into account that
\[ p_k x = x p_k = k p_k u_{k,x_n(k)}u_k^* p_k = k u_{k,x_n(k)}u_k^* \]
and
\[ p_k D(p_k) p_k = 0 \]
we obtain
\[ p_k D(x) p_k = p_k D(k u_{k,x_n(k)}u_k^*) p_k. \]
Whence
\[ q_k D(x) q_k = q_k D(k u_{k,x_n(k)}u_k^*) q_k. \]
By \((3.2)\) we obtain that
\[ q_k D(x) q_k \geq \frac{k}{2} q_k. \]
Now using \((3.3)\) we have
\[ \frac{k}{2} q_k \in U(\lambda, 2\varepsilon) \]
for all $k$. Taking into account \((3.1)\) we have $k/2 \leq \lambda$ for all $k$. This contradiction shows that $x = 0$. Thus $D$ is continuous. The proof is complete.
4. THE MAIN RESULTS

In this section we prove the spatiality of measure continuous derivations on the algebra of \( \tau \)-compact operators with respect to a von Neumann algebra and a faithful normal semi-finite trace.

Denote by \( \mathcal{U}(M) \) and \( \mathcal{GN}(M) \) the set of all unitaries in \( M \) and the set of all partial isometries in \( M \), respectively.

A partial ordering on the set \( \mathcal{GN}(M) \) can be defined as follows:

\[
u \leq_1 v \Leftrightarrow uu^* \leq vv^*, \ u = uu^*v.
\]

It is clear that

\[
u \leq_1 v \Leftrightarrow u^*u \leq v^*v, \ u = vu^*u.
\]

Note that \( u^*u = r(u) \) is the right support of \( u \), and \( uu^* = l(u) \) is the left support of \( u \).

Similar to the previous section every derivation \( D \) on \( S_0(M, \tau) \) can be represented in the form \( D = D_1 + iD_2 \), where

\[
D_1 = (D - D^*)/2, \quad D_2 = (D + D^*)/2i
\]

are skew-hermitian derivations on \( S_0(M, \tau) \).

It is clear that a derivation \( D \) is inner or spatial if and only if the both skew-hermitian derivations \( D_1 \) and \( D_2 \) are inner or spatial respectively.

Therefore further in this section we may assume that \( D \) is a skew-hermitian derivation.

Since \( S_0(M, \tau) \) is an ideal in \( S(M, \tau) \) and algebraic operations on \( S(M, \tau) \) are \( t_\tau \)-continuous, each element \( a \in S(M, \tau) \) implements a \( t_\tau \)-continuous derivation on the algebra \( S_0(M, \tau) \) by the formula

\[
D(x) = ax - xa, \ x \in S_0(M, \tau).
\]

The main aim of the present paper is to prove the converse assertion. Namely, we shall prove the following

**Theorem 4.1.** Let \( M \) be a von Neumann algebra with a faithful normal semi-finite trace \( \tau \). Then every \( t_\tau \)-continuous derivation \( D : S_0(M, \tau) \to S_0(M, \tau) \) is spatial and implemented by an element \( a \in S(M, \tau) \).

For the proof of this theorem we need several lemmata.

Set

\[
\mathcal{GN}_\tau(M) = \{u \in \mathcal{GN}(M) : uu^* \in P_\tau(M)\}.
\]

It is clear that

\[
P_\tau(M) \subset \mathcal{GN}_\tau(M) \subset S_0(M, \tau)_b.
\]

The following three lemmata have been proved in [9] in the case of finite von Neumann algebras, but the proofs are similar in the semi-finite case.

**Lemma 4.2.** For every \( v \in \mathcal{GN}_\tau(M) \) the element \( vv^*D(v)v^* \) is hermitian.

**Lemma 4.3.** Let \( n \in \mathbb{N} \) be a fixed number and let \( v \in \mathcal{GN}_\tau(M) \) be a partially isometry. Then

\[
vv^*D(v)v^* \geq nvv^*
\]

if and only if

\[
v^*vD(v^*)v \leq -nv^*v.
\]
Lemma 4.4. Let \( v_1 \in \mathcal{GN}_\tau(M) \) be a partially isometry and let \( v_2 \in \mathcal{GN}_\tau(pMp) \), where 
\[
p = 1 - v_1 v_1^* \vee v_1^* v_1 \vee s(iD(v_1 v_1^*)) \vee s(iD(v_1^* v_1))
\] and \( s(x) \) denotes the support of a hermitian element \( x \). Then 
\[
(v_1 + v_2)(v_1 + v_2)^* D(v_1 + v_2)(v_1 + v_2)^* = v_1 v_1^* D(v_1) v_1^* + v_2 v_2^* D(v_2) v_2^*.
\]

For each \( n \in \mathbb{N} \) consider the set 
\[
\mathcal{F}_n = \{ v \in \mathcal{GN}_\tau(M) : vv^* D(v)v^* \geq n vv^* \}.
\]
Note that \( 0 \in \mathcal{F}_n \), so \( \mathcal{F}_n \) is not empty.

Lemma 4.5. Let \( \varepsilon_n = \sup \{ \tau(uu^*) : u \in \mathcal{F}_n \} \). Then \( \varepsilon_n \downarrow 0 \).

Proof. Since \( \mathcal{F}_n \supset \mathcal{F}_{n+1} \) we have that \( \varepsilon_n \downarrow \). Let us show that \( \varepsilon_n \downarrow 0 \). Let us suppose the opposite, e.g. there exists a number \( \varepsilon > 0 \) such that \( \varepsilon_n \geq n \varepsilon \) for all \( n \in \mathbb{N} \). There exists element \( v_n \in \mathcal{F}_n \) such that 
\[
\tau(v_n v_n^*) \geq \varepsilon
\]
for all \( n \geq 1 \). Since \( v_n \in \mathcal{F}_n \) we have
\[
(4.1) \quad v_n v_n^* D(v_n) v_n^* \geq n v_n v_n^*
\]
for all \( n \geq 1 \).

Now take an arbitrary number \( c > 0 \) and let \( n \) be a number such that \( n > c \delta \), where \( \delta = \frac{\varepsilon}{2} \). Suppose that 
\[
v_n v_n^* D(v_n) v_n^* \in c U (\delta, \delta) = U (c \delta, \delta).
\]
Then there exists a projection \( p \in M \) such that 
\[
(4.2) \quad ||v_n v_n^* D(v_n) v_n^* p|| < c \delta, \quad \tau(p^\perp) < \delta.
\]
Let \( v_n v_n^* D(v_n) v_n^* = \int_{-\infty}^{+\infty} \lambda d e_\lambda \) be the spectral resolution of \( v_n v_n^* D(v_n) v_n^* \). From (4.2) using [17, Lemma 2.2.4] we obtain that \( e_\delta^\perp \preceq p^\perp \). Taking into account (4.1) we have that \( v_n v_n^* \leq e_\delta^\perp \). Since \( n > c \delta \) it follows that \( e_n \leq e_\delta^\perp \). So 
\[
v_n v_n^* \leq e_n \leq e_\delta^\perp \leq p^\perp.
\]
Thus 
\[
\varepsilon \leq \tau(v_n v_n^*) \leq \tau(p^\perp) < \delta = \frac{\varepsilon}{2}.
\]
This contradiction implies that 
\[
v_n v_n^* D(v_n) v_n^* \notin c U (\delta, \delta)
\]
for all \( n > c \delta \). Since \( c > 0 \) is arbitrary it follows that the sequence \( \{ v_n v_n^* D(v_n) v_n^* \}_{n \geq 1} \) is unbounded in the measure topology. Therefore the set \( \{ vv^* D(v)v^* : v \in \mathcal{GN}_\tau(M) \} \) is also unbounded in the measure topology.

On the other hand, the continuity of the derivation \( D \) implies that the set \( \{ xx^* D(x)x^* : \|x\| \leq 1 \} \) is bounded in the measure topology. In particular, the set \( \{ uu^* D(u)u^* : u \in \mathcal{GN}_\tau(M) \} \) is also bounded in the measure topology. This contradiction implies that \( \varepsilon_n \downarrow 0 \). The proof is complete. \( \square \)

Lemma 4.5 implies that there exists a number \( k \in \mathbb{N} \) such that \( \varepsilon_n < +\infty \) for all \( n \geq k \).

Lemma 4.6. For \( n \geq k \) the set \( \mathcal{F}_n \) has a maximal element with respect to the partial ordering \( \leq_1 \).
Proof. Let \( \{v_\alpha\} \subset \mathcal{F}_n \) be a totally ordered net. Then

\[
v_\alpha v_\alpha^* \uparrow p, \quad v_\alpha^* v_\alpha \uparrow q, \tag{10}
\]

where \( p, q \in P(M) \). Since \( \tau(v_\alpha v_\alpha^*) \leq \varepsilon_n \) for all \( \alpha \), we have \( \tau(p), \tau(q) \leq \varepsilon_n \). Note that \( v_\alpha \in eMe \), where \( e = p \vee q \). Consider the \( L_2 \)-norm

\[
\|x\|_2 = \sqrt{\tau(x^*x)}, \quad x \in eMe.
\]

Let us show that \( v_\alpha \xrightarrow{t_\tau} v \) for some \( v \in \mathcal{F}_n \). For \( \alpha \leq \beta \) we have

\[
\|v_\beta - v_\alpha\|_2 = \|l(v_\beta) - l(v_\alpha)\|_2 = \|l(v_\beta) - l(v_\alpha)\|_2 = \sqrt{\tau(l(v_\beta) - l(v_\alpha))} \rightarrow 0,
\]

because \( \{l(v_\alpha)\} \) is an increasing net of projections. Thus \( \{v_\alpha\} \) is a \( \| \cdot \|_2 \)-fundamental, and hence there exists an element \( v \) in the unit ball \( eMe \) such that \( v_\alpha \xrightarrow{\| \|_2} v \). Therefore \( v_\alpha \xrightarrow{t_\tau} v \), and thus we have

\[
v_\alpha v_\alpha^* \xrightarrow{t_\tau} vv^*, \quad v_\alpha^* v_\alpha \xrightarrow{t_\tau} v^* v. \]

Therefore

\[
vv^*, \quad v^* v \in P_\tau(M).
\]

Thus \( v \in \mathcal{G}N_\tau(M) \).

Since \( v_\alpha = v_\alpha v_\alpha^* v_\alpha \) for all \( \beta \geq \alpha \) we have that \( v_\alpha = v_\alpha v_\alpha^* v_\alpha \). So \( v_\alpha \leq 1 \) for all \( \alpha \). Since \( v_\alpha \xrightarrow{t_\tau} v \) by \( t_\tau \)-continuity of \( D \) we have that \( D(v_\alpha) \xrightarrow{t_\tau} D(v) \). Taking into account that \( v_\alpha v_\alpha^* D(v_\alpha) v_\alpha^* \geq n v_\alpha v_\alpha^* \) we obtain \( vv^* D(v) v^* \geq n vv^* \), i.e. \( v \in \mathcal{F}_n \).

So, any totally ordered net in \( \mathcal{F}_n \) has the least upper bound. By Zorn’s Lemma \( \mathcal{F}_n \) has a maximal element, say \( v_n \). The proof is complete. \( \Box \)

The following lemma is one of the key steps in the proof of the main result.

**Lemma 4.7.** Let \( M \) be a von Neumann algebra with a faithful normal semi-finite trace \( \tau \). There exists a sequence of projections \( \{p_n\} \) in \( M \) with \( \tau(1 - p_n) \rightarrow 0 \) such that

\[
\|vv^* D(v) v^*\| \leq n
\]

for all \( v \in p_n \mathcal{G}N_\tau(M) p_n = \mathcal{G}N_\tau(p_n M p_n) \).

**Proof.** Let \( v_n \) be a maximal element of \( \mathcal{F}_n \). Put

\[
p_n = 1 - v_n v_n^* \vee v_n^* v_n \vee s(iD(v_n v_n^*)) \vee s(iD(v_n^* v_n)).
\]

Let us prove that

\[
\|vv^* D(v) v^*\| \leq n
\]

for all \( v \in \mathcal{G}N_\tau(p_n M p_n) \).

The case \( p_n = 0 \) is trivial.
Let us consider the case $p_n \neq 0$. Take $v \in \mathcal{S}N_\tau(p_n M p_n)$. Let $vv^* D(v)v^* = \int_{-\infty}^{+\infty} \lambda \, d\epsilon_\lambda$ be the spectral resolution of $vv^* D(v)v^*$. Assume that $p = e_n^\perp \neq 0$. Then

$$pvv^* D(v)v^* p \geq np.$$ 

Denote $u = pv$. Then since $p \leq vv^*$, we have

$$uu^* D(u)u^* = pvv^* pD(pv)v^* p = pvv^* pD(p)vv^* p + pvv^* ppD(v)v^* p = pvv^* pD(p)pvv^* + pvv^* D(v)v^* p = 0 + pvv^* D(v)v^* p \geq np,$$

i.e.

$$uu^* D(u)u^* \geq np.$$ 

Since $uu^*, u^* u \leq p_n = 1 - v_n v_n^* \vee v_n^* v_n \vee s(iD(v_n v_n^*)) \vee s(iD(v_n^* v_n))$ it follows that $u$ is orthogonal to $v_n$, i.e. $uv_n^* = v_n^* u = 0$. Therefore $w = v_n + u \in \mathcal{S}N_\tau(M)$. Using Lemma 4.4 we have

$$ww^* D(w)w^* = v_n v_n^* D(v_n) v_n^* + uu^* D(u)u^* \geq n(v_n v_n^* + p) = nww^*$$

because

$$ww^* = (v_n + u)(v_n + u)^* = v_n v_n^* + uu^* = v_n v_n^* + pvv^* p + v_n v_n^* + p.$$ 

So

$$ww^* D(w)w^* \geq nww^*.$$ 

This contradicts with the maximality $v_n$. From this contradiction it follows that $e_n^\perp = 0$. This means that

$$vv^* D(v)v^* \leq n vv^*$$

for all $v \in \mathcal{U}_\tau(p_n M p_n)$.

Set

$$\mathcal{S}_n = \{ v \in \mathcal{S}N_\tau(M) : vv^* D(v)v^* \leq -n vv^* \}.$$ 

By Lemma 4.3 it follows that $v \in \mathcal{F}_n$ is a maximal element of $\mathcal{F}_n$ with respect to the partial ordering $\leq_1$ if and only if $v^*$ is a maximal element of $\mathcal{S}_n$ with respect to this ordering.

Taking into account this observation in a similar way we can show that

$$vv^* D(v)v^* \geq -nvv^*$$

for all $v \in \mathcal{S}N_\tau(p_n M p_n)$. So

$$-nvv^* \leq vv^* D(v)v^* \leq n vv^*.$$ 

This implies that $vv^* D(v)v^* \in M$ and

$$\|vv^* D(v)v^*\| \leq n$$

for all $v \in \mathcal{S}N_\tau(p_n M p_n)$.

Finally let us show that

$$\tau(1 - p_n) \to 0.$$
It is clear that
\[ l(iD(v_n^*v_n^*v_n^*)) \leq v_nv_n^*, \]
\[ r(v_n^*iD(v_n^*)) \leq v_nv_n^*. \]

Since
\[ D(v_n^*) = D(v_n^*)v_nv_n^* + v_nv_n^*D(v_n^*) \]
we have
\[ \tau(s(iD(v_n^*))) = \tau(s(iD(v_n^*v_n^*v_n^*)) + v_nv_n^*iD(v_n^*)) \leq \]
\[ \leq \tau(v_nv_n^*) + \tau(v_nv_n^*) + 3\tau(v_nv_n^*) = 3\tau(v_nv_n^*), \]
i.e.
\[ \tau(s(iD(v_n^*))) \leq 3\tau(v_nv_n^*). \]

Similarly
\[ \tau(s(iD(v_n^*)) \leq 3\tau(v_n^*)v_n). \]

Now taking into account that
\[ v_nv_n^* \sim v_nv_n \]
we obtain
\[ \tau(1 - p_n) = \tau(v_nv_n^*v_nv_n^*v_n^*v_n^*v_n^*v_n^*) \leq \]
\[ \leq \tau(v_nv_n^*) + \tau(v_nv_n^*) + 3\tau(v_nv_n^*) + 3\tau(v_nv_n^*) = \]
\[ = 8\tau(v_nv_n^*) \to 0, \]
i.e.
\[ \tau(1 - p_n) \to 0. \]

The proof is complete. \(\square\)

Let \( p \in M \) be a projection. It is clear that the mapping
\[ (4.4) \quad D^{(p)} : x \to pD(x)p, \quad x \in pS_0(M, \tau)p \]
is a derivation on \( pS_0(M, \tau)p = S_0(pMp, \tau_p) \), where \( \tau_p \) is the restriction of \( \tau \) on \( pMp \).

**Lemma 4.8.** Let \( \{p_n\} \) be the sequence of projections from Lemma 4.7. Then for every \( n \in \mathbb{N} \) there exists an element \( a_n \in p_nMp_n \) such that
\[ (4.5) \quad D^{(p_n)}(x) = a_nx - xa_n \]
for all \( x \in S_0(p_nMp_n, \tau_{p_n}) \).

**Proof.** Let \( n \in \mathbb{N} \) be a fixed number. Since the trace \( \tau \) is semi-finite we have
\[ p_n = \sqrt{\{p \in P_r(p_nMp_n)\}}. \]

Let us show that the derivation \( D^{(p)} \) defined as in (4.4), maps \( pMp \) into itself for all \( p \in P_r(p_nMp_n) \).

Take \( v \in U(pMp) \). Then \( vv^* = v^*v = p \) and hence by Lemma 4.7 we have that
\[ D^{(p)}(v) = pD(pvp)p = vv^*D(v)v^*v \in pMp \]
and

$$\|D^{(p)}(v)\| \leq n.$$  

Since any element from the unit ball of $pMp$ can be represented of the form

$$x = \frac{1}{2}(v_1 + v_2 + v_3 + v_4),$$

where $v_i \in U(pMp), i \in \{1, 4\}$, it follows that

$$\|D^{(p)}(x)\| \leq 2n.$$  

for all $x \in pMp, \|x\| \leq 1$. So, for each $p \in P_\tau(p_nM_{p_n})$ the derivation $D^{(p)}$ maps $pMp$ into itself and $\|D^{(p)}\| \leq 2n$. By Sakai’s Theorem [19, Theorem 4.1.6] there is an element $a_p \in pMp$ such that $\|a_p\| \leq 2n$ and $D^{(p)}(x) = a_p x - xa_p$ for all $x \in pMp$.

If $\tau(p_n) < +\infty$ setting $p = p_n$, we can take $a_n = a_p$.

Now let us consider the case $\tau(p_n) = +\infty$. Since the net $\{a_p\}_{p \in P_\tau(p_nM_{p_n})}$ is norm bounded we have that it contains a subnet which $^*$-weakly converges in $p_nM_{p_n}$. Without loss of generality we may assume that $a_p \to a_n$ for some $a_n \in p_nM_{p_n}$.

Let $x \in M$ be an element with $s(x) \in P_\tau(p_nM_{p_n})$. Since $M$ is semi-finite there exist $f, e \in P_\tau(p_nM_{p_n})$ such that $s(x) \leq f \leq e$. Then

$$fD^{(p_n)}(x)f = fD(x)f = fD^{(e)}(x)f = fa_e x - xa_e f,$$

i.e.

$$fD^{(p_n)}(x) = fa_e x - xa_e f.$$  

Since $\{a_e\}_{e \in J}$ is a subnet of the net $\{a_p\}_{p \in P_\tau(p_nM_{p_n})}$, where $J = \{e \in P_\tau(p_nM_{p_n}) : f \leq e\}$, it follows that $a_e \to a_n$. Taking into account that the left side of the last equality is independent on $e$ we have

$$fD^{(p_n)}(x)f = f(a_n x - xa_n)f.$$  

Since $f$ is arbitrary it follows that

$$D^{(p_n)}(x) = a_n x - xa_n.$$  

Taking into account that the set $\{x \in S_0(p_nM_{p_n}, \tau_{p_n}) : \tau(s(x)) < +\infty\}$ is $t_\tau$-dense in $S_0(p_nM_{p_n}, \tau_{p_n})$ and that $D$ is $t_\tau$-continuous we obtain

$$D^{(p_n)}(x) = a_n x - xa_n$$

for all $x \in S_0(p_nM_{p_n}, \tau_{p_n})$. The proof is complete.

**Proof of Theorem 3.1.** Let $\{p_n\}$ be the sequence of projections from Lemma 4.7 and let $\{a_n\}$ be the sequence from Lemma 4.8. Since $\tau(1 - p_n) \to 0$, there exist a sequence $n_1 < n_2 < \ldots < n_k < \ldots$ such that $\tau(1 - p_{n_k}) < 1/2^{k+1}$. Set

$$q_k = \bigwedge_{i=k}^{\infty} p_{n_i}$$

and

$$b_k = q_k a_{n_k} q_k$$

for all $k \in \mathbb{N}$. Then

$$D^{(q_k)}(x) = b_k x - xb_k.$$
for all \( x \in S_0(q_k M q_k, \tau_{q_k}) \).

Now we will construct a sequence \( \{c_k\} \) such that \( c_k \in q_k M q_k \) and

\[
D^{(q_k)}(x) = c_k x - xc_k
\]

for all \( x \in S_0(q_k M q_k, \tau_{q_k}) \), and moreover \( q_i c_j q_i = c_i \) for all \( i < j \).

Set \( c_1 = b_1 \) and suppose that elements \( c_1, c_2, \ldots, c_k \) have been already constructed. Since

\[
D^{(q_k)}(q_k x q_k) = q_k D^{(q_{k+1})}(q_k x q_k) q_k
\]

we obtain that

\[
[c_k, q_k x q_k] = q_k [b_{k+1}, q_k x q_k] q_k = [q_k b_{k+1} q_k, q_k x q_k]
\]

for all \( x \in S_0(M, \tau) \). In particular, the element \( c_k - q_k b_{k+1} q_k \) commutes with any projection from \( P_\tau(q_k M q_k) \). Hence, the element \( c_k - q_k b_{k+1} q_k \) belongs to the center of the algebra \( q_k M q_k \). Then there exists a central element \( f_k \) from \( q_{k+1} M q_{k+1} \) such that \( c_k - q_k b_{k+1} q_k = f_k q_k \). Set \( c_{k+1} = b_{k+1} + f_k \). Then

\[
D^{(q_{k+1})}(x) = c_{k+1} x - xc_{k+1}
\]

for all \( x \in S_0(q_{k+1} M q_{k+1}, \tau_{q_{k+1}}) \). Further

\[
q_k c_{k+1} q_k = q_k b_{k+1} q_k + f_k q_k = c_k
\]

and

\[
q_i c_{k+1} q_i = q_i q_k c_{k+1} q_k q_i = q_i c_k q_i = c_i
\]

for \( i < k + 1 \).

Now let us show that \( \{c_k\} \) is a \( t_\tau \)-fundamental sequence. Let \( \varepsilon > 0 \). Take a number \( k_\varepsilon \) such that \( 1/2^{k_\varepsilon} < \varepsilon \).

Put \( p = q_{k_\varepsilon} \). For \( i, j > k_\varepsilon \) we have

\[
p(c_i - c_j) = q_{k_\varepsilon} c_i q_{k_\varepsilon} - q_{k_\varepsilon} c_j q_{k_\varepsilon} = c_{k_\varepsilon} - c_{k_\varepsilon} = 0.
\]

Since \( \tau(1 - q_{k_\varepsilon}) < 1/2^{k_\varepsilon} \) we have that \( \{c_k\} \) is fundamental in the so-called topology of two-side convergence in measure. But this topology coincides with the topology \( t_\tau \) (see [17, Proposition 3.4.11]), and therefore there exists an element \( c \in S(M, \tau) \) such that \( \{c_k\} \) converges to \( c \) in \( t_\tau \)-topology. Since

\[
q_k D(q_k x q_k) q_k = c_k (q_k x q_k) - (q_k x q_k) c_k
\]

due to the continuity of the derivation \( D \) we obtain that

\[
D(x) = cx - xc
\]

for all \( x \in S_0(M, \tau) \). The proof is complete.

From Theorems 3.2 and 4.1 we obtain the following result.

**Theorem 4.9.** If \( M \) is a properly infinite von Neumann algebra with a faithful normal semi-finite trace \( \tau \), then any derivation \( D : S_0(M, \tau) \to S_0(M, \tau) \) is spatial and implemented by an element \( a \in S(M, \tau) \).

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