Analog gravity in nonisentropic fluids

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Abstract

The analog acoustic metric has been originally derived for adiabatic acoustic perturbations propagating in an isentropic irrotational ideal fluid. In the framework of a Lagrangian hydrodynamic description we demonstrate that under certain conditions the usual acoustic metric can be derived for nonisentropic fluids. In a special case when the pressure takes a special form and the nonadiabatic perturbations are neglected the adiabatic acoustic perturbations corresponding to massless phonons propagate in an analog metric of the usual type.

Keywords: analog gravity, nonisentropic flow, Lagrangian fluid dynamics

1. Introduction

The possibility that a pseudo-Riemannian geometry of spacetime can be mimicked by fluid dynamics in Minkowski spacetime has been exploited in various contexts including emergent gravity [1, 2], scalar theory of gravity [3], and acoustic geometry [4–7]. The basic idea is the emergence of an effective metric of the form

\[ G_{\mu\nu} = a [g_{\mu\nu} - (1 - c_s^2) u_\mu u_\nu], \]

which describes the effective geometry for acoustic perturbations propagating in a fluid potential flow with \( u_\mu \propto \partial_\mu \theta \). The quantity \( c_s \) is the adiabatic speed of sound, the conformal factor \( a \) is related to the equation of state of the fluid, and the background spacetime metric \( g_{\mu\nu} \) is usually assumed Minkowski. In an equivalent field-theoretical picture the fluid velocity \( u_\mu \) is derived from the scalar field as \( u_\mu = \partial_\mu \theta / \sqrt{X} \) and \( a \) and \( c_s \) are expressed in terms of the Lagrangian and its first and second derivatives with respect to the kinetic energy term \( X = g^{\alpha\beta} \dot{\theta}_\alpha \dot{\theta}_\beta \). The effective metric (1) has been originally derived for an isentropic irrotational perfect fluid. However, it has been recently demonstrated that the condition of vanishing vorticity can be relaxed for a Bose Einstein condensate coupled to the electromagnetic field [8].

In a slightly different context, the metric of the form (1) has been used to show that a pseudo-Riemann spacetime with Lorentz signature may be derived from a Riemann metric.
with Euclidean signature [9–11]. In that case, the vector $u_\mu$ represents the normalized gradient of a hypothetical scalar field which governs the dynamics and the signature of the effective spacetime.

In applications of analog geometry, in addition to the energy–momentum conservation or the Euler equation, the continuity equation is usually assumed. However, with this assumption some interesting geometries cannot be mimicked by analog geometry. For example, the Schwarzschild metric cannot be mimicked by the non-relativistic version of (1) unless the continuity equation is abandoned [7, 12] and the same holds true in the relativistic case. The fluid in which the particle number is not conserved is generally nonisentropic. Therefore, it is of considerable interest to study analog geometries in nonisentropic fluids.

The propagation of sound in an ideal isentropic fluid is automatically an adiabatic process. In a nonisentropic fluid flow the propagation of perturbations can be adiabatic (i.e. at fixed entropy) or nonadiabatic. In a field theoretical description Babichev, Mukhanov and Vikman have shown [1] that the scalar-field perturbations propagate in the analog metric of the form (1) and acquire an effective mass. In their approach the perturbations are generally nonadiabatic.

The work presented here is partially motivated by a recent article of Hossenfelder [13]. She has derived an analog metric that mimics the geometry of a planar black hole (BH) in asymptotic Anti-de Sitter (AdS) space. A careful analysis which will be provided in section 4 of the present paper reveals that the fluid in this model is essentially nonisentropic. In the present paper we will study adiabatic and nonadiabatic perturbations propagating in a general nonisentropic fluid. It turns out that under certain conditions, if nonadiabatic perturbations are ignored, the usual analog gravity description applies with the effective metric in the usual form (1). In this case the phonons corresponding to adiabatic acoustic perturbations remain massless.

We divide the remainder of the paper into four sections and an appendix. We start with section 2, in which we give a hydrodynamic and field-theoretic description of a nonisentropic fluid. In the following section, section 3, we study perturbations of the flow and derive conditions under which the equation of motion can be written in the form of a Klein–Gordon equation in an analog curved spacetime. In section 4 we apply our formalism to the model of an analog planar AdS5 BH. Concluding remarks are given in section 5. Finally, in appendix we provide a brief account of the second law of thermodynamics relevant for nonisentropic fluid flows.

2. Nonisentropic flow

The analog acoustic geometry has been derived under strict requirements of energy momentum conservations, particle number conservation, and vanishing vorticity. The first two restrictions are sufficient conditions for adiabaticity. If a stronger restriction of isentropy is assumed together with vanishing vorticity, the velocity field may be expressed as $w_\mu = \theta_\mu$, where $\theta$ is the velocity potential and $w$ is the specific enthalpy. The reverse of the above statement is not true: a potential flow alone implies only vanishing vorticity and implies neither particle number conservation nor isentropy.

Next we consider a nonisentropic fluid flow and derive general properties of relativistic nonisentropic fluids. To make it clear which properties are consequences of which assumptions, each subsection studies consequences of one additional assumption, from more fundamental assumptions towards less fundamental ones.
2.1. Energy–momentum conservation

The most fundamental assumption is the energy–momentum conservation

\[ T_{\mu\nu} = 0. \]  

The energy–momentum tensor of an ideal relativistic fluid can be expressed as

\[ T_{\mu\nu} = (p + \rho)u_\mu u_\nu - p g_{\mu\nu}, \]  

where \( p \) and \( \rho \) are the fluid pressure and energy density, respectively, and \( g_{\mu\nu} \) is the background metric with signature \( (+-\cdots) \). The contraction of (2) with \( u_\mu \) gives

\[ u_\mu (p + \rho) u_\mu ; \mu + (p + \rho) u_\mu u_\mu = 0. \]  

Inserting this into (2) with (3) gives the relativistic Euler equation

\[ (p + \rho) u_\nu u_\mu ; \nu - p , \mu + u_\nu p , \nu u_\mu = 0. \]  

2.2. The first law of thermodynamics

For a general thermodynamic system at nonzero temperature \( T \) the first law of thermodynamics may be written as

\[ dp = ndw - nTds, \]  

where \( n \) is the particle number density, \( s \) is the specific entropy, i.e. the entropy per particle, and \( w \) is the specific enthalpy defined as

\[ w = \frac{p + \rho}{n}. \]  

Since the pressure can be viewed as a function of two variables \( s \) and \( w \), a comparison of the total derivative

\[ dp = \frac{\partial p}{\partial w} dw + \frac{\partial p}{\partial s} ds, \]  

with (6) yields the thermodynamic relations

\[ n = \left. \frac{\partial p}{\partial w} \right|_s, \quad Tn = \left. \frac{\partial p}{\partial s} \right|_w. \]  

The second equation, which may be understood as a defining equation for the temperature, shows that in a realistic system the pressure is a non-increasing function of specific entropy at fixed specific enthalpy.

Using (7) and expressing (6) as \( p_{,\mu} = nw_{,\mu} - nT s_{,\mu} \), from equation (4) it follows

\[ w(nu^\mu)_{,\mu} + nTu^\mu s_{,\mu} = 0. \]  

Similarly, equation (5) with (6) gives

\[ u^\nu (wu_{\mu})_{,\nu} - w_{,\mu} = T(u^\nu s_{,\nu} u_\mu - s_{,\mu}). \]  

The sign of \( (nu^\mu)_{,\mu} \) determines whether we have local particle creation or destruction: the particles are locally created or destroyed if \( (nu^\mu)_{,\mu} \) is positive or negative, respectively. Hence, assuming that \( w \) is positive, Equation (10) states that the entropy per particle increases when the number of particles decreases.

In the following considerations the temperature will play no role. Therefore, we will use equation (10) in the form
Clearly, the particle number is generally not conserved. If the particle number were conserved, i.e. if the continuity equation \((nu^\mu)_{,\mu} = 0\) were true, equation (12) would imply the adiabatic condition \(u^\mu s_{,\mu} = 0\). It is worth mentioning that in most applications of thermodynamics and fluid dynamics in cosmology a conservation of particle number and entropy has been assumed (see, e.g. [15] and references therein).

### 2.3. Potential flow

Some of the equations will further simplify if we assume that the enthalpy flow \(wu_\mu\) is a gradient of a scalar potential, i.e. if there exist a scalar function \(\theta\) such that the velocity field satisfies [14]

\[ wu_\mu = \partial_\mu \theta. \]  

(13)

In this case the left-hand side of (11) vanishes identically, so equation (11) reduces to

\[ s_{,\mu} = u^\nu s_{,\nu} u_\mu. \]  

(14)

Hence, in a potential flow the entropy gradient is proportional to the gradient of the potential. The assumption (13) is automatically satisfied in the field-theory formalism, which will be discussed next.

### 2.4. Field-theory formalism

In order to give a more precise meaning to the quantity \(u^\mu s_{,\mu}\) which, generally, may be an arbitrary function of \(w\) and \(s\), it proves convenient to use the field-theoretical description of fluid dynamics [16, 17]. Consider a Lagrangian \(L(X, \theta)\) that depends on a dimensionless scalar field \(\theta\) and on the kinetic energy term

\[ X = g^{\mu\nu} \theta_\mu \theta_\nu. \]  

(15)

The corresponding energy–momentum tensor is given by

\[ T_{\mu\nu} = 2L_X \theta_\mu \theta_\nu - L g_{\mu\nu}. \]  

(16)

where the subscript \(X\) denotes a partial derivative with respect to \(X\). For \(X > 0\), the energy–momentum tensor takes the perfect fluid form (3) where the quantities

\[ p = L, \quad \rho = 2XL_X - L, \]  

(17)

and

\[ u_\mu = \frac{\partial_\mu \theta}{\sqrt{X}} \]  

(18)

are the pressure, energy density, and velocity of the fluid, respectively. Obviously, the field \(\theta\) serves as the velocity potential and comparing (18) with (13) we identify the specific enthalpy as

\[ w = \sqrt{X}. \]  

(19)

From (13) and (17) we find the expression for the particle number density

\[ n = 2\sqrt{X}L_X. \]  

(20)
It is understood that the quantities defined in equations (17)–(20) are derived from an on-shell Lagrangian, i.e. from the Lagrangian in which the field $\theta$ is a solution to the equation of motion
\[(2L_X g^{\mu\nu} \theta_{,\mu})_{,\nu} - \partial L / \partial \theta = 0, \tag{21}\]
which may be written as
\[(nu^\mu)_{,\mu} = \partial L / \partial \theta. \tag{22}\]
If the Lagrangian were a function of $X$ only, the right-hand side of (22) would vanish and this equation would expresses a conservation of the current $J_\mu = nu_\mu$. A comparison of (22) and (10) demonstrates that the conservation of $J_\mu$ is closely related to the isentropy of the fluid flow.

If the right-hand side of (22) were zero, i.e. if $\partial L / \partial \theta = 0$, equation (6) and the definitions (17)–(20) would imply $ds = 0$. In other words, if $L$ is a function of $X$ only the fluid flow is necessarily isentropic. The reverse is also true, although in somewhat weaker sense [18]: if $ds = 0$ there exists a field redefinition $\bar{\theta} = \bar{\theta}(\theta)$ such that the function $L = L(X, \theta)$ can be brought to the form $L = L(\bar{X})$, where $\bar{X} = g^{\mu\nu} \bar{\theta}_{,\mu} \bar{\theta}_{,\nu}$.

Next, we demonstrate that there exist a functional relationship between the specific entropy $s$ and $\theta$. Using the definitions (19) and (20) from (17) it follows
\[dp = L_X dX + \partial L / \partial \theta d\theta = ndw + \partial L / \partial \theta d\theta. \tag{23}\]
Comparing this with (6) we conclude that keeping $\theta$ fixed is equivalent to keeping the specific entropy $s$ fixed and hence
\[\left. \frac{\partial p}{\partial w} \right|_s = \left. \frac{\partial p}{\partial w} \right|_\theta = n. \tag{24}\]
Moreover, equations (6), (23) and (24) demonstrate that the the specific entropy $s$ is a function of the field $\theta$ such that
\[\frac{ds}{d\theta} = \frac{\partial L / \partial \theta}{\partial p / \partial s}. \tag{25}\]
Then, a comparison between (22) and (12) yields
\[u^\mu s_{,\mu} = wf(s), \tag{26}\]
where
\[f(s) = \frac{ds}{d\theta}. \tag{27}\]
Clearly, the functional relationship $s = s(\theta)$ depends on the Lagrangian. For example, if the Lagrangian is a function of the kinetic term $X$ only, i.e. if $\partial L / \partial \theta = 0$, the right-hand side of (25) will vanish yielding $s = \text{const}$, i.e. an isentropic fluid. It has been argued [18] that in most cases one can identify $\theta$ with $s$. However this identification cannot be generally correct since, as we have noted above, the flow could be isentropic even if $L(X, \theta)$ were a nontrivial function of $\theta$.

Hence, motivated by the Lagrangian description of fluid dynamics, in the following considerations we will use equation (12) in the form
\[(nu^\mu)_{,\mu} = f(s) \frac{\partial p}{\partial s}. \tag{28}\]
without specifying the function \( f(s) \). This is the key equation which will be used in the next section to derive a propagation equation for linear perturbations.

3. Acoustic metric

Acoustic metric is the effective metric perceived by acoustic perturbations propagating in a perfect fluid background. Under certain conditions the perturbations satisfy a Klein–Gordon equation in curved geometry with metric of the form (1).

We first derive a propagation equation for linear perturbations of a nonisentropic flow assuming a fixed background geometry. Given some average bulk motion represented by \( p, n, \) and \( u^\mu \), following the standard procedure [4, 5, 14], we make a replacement

\[
p \rightarrow p + \delta p, \quad n \rightarrow n + \delta n, \quad u^\mu \rightarrow u^\mu + \delta u^\mu,
\]

where the small disturbances \( \delta p, \delta n, \) and \( \delta u^\mu \) are induced by the perturbations \( \delta w \) and \( \delta s \) of two independent variables \( w \) and \( s \):

\[
w \rightarrow w + \delta w, \quad s \rightarrow s + \delta s.
\]

Then equation (28) at linear order yields

\[
\left[ \frac{\partial n}{\partial w} \delta w + \frac{\partial n}{\partial s} \delta s \right] u^\mu + n \delta u^\mu = f \left[ \frac{\partial n}{\partial s} \delta w + \frac{\partial}{\partial s} \left( f \frac{\partial p}{\partial s} \right) \right] \delta s,
\]

where we have employed (24) in the first term on the right-hand side. In contrast to the previous works, in this equation we have nonadiabatic terms related to the perturbation \( \delta s \), in addition to the adiabatic terms related to \( \delta w \). From (13) it follows

\[
\delta w = u^\mu \delta \theta_{\mu},
\]

\[
\delta u^\mu = (g^{\mu \nu} - u^\mu u^\nu) \delta \theta_{\nu}.
\]

To simplify the notation, in the following we introduce a perturbation \( \sigma \) such that \( \delta s = f \sigma \) and denote by \( \chi \) the perturbation \( \delta \theta \equiv \chi \) appearing in the variations \( \delta w \) and \( \delta u^\mu \). Hence, \( \chi \) and \( \sigma \) represent adiabatic and nonadiabatic perturbations, respectively. Then, combined with (32) and (33), equation (31) takes the form

\[
(f^{\mu \nu} \chi_{\nu},_{\mu} - f \frac{\partial n}{\partial s} u^\mu (\chi_{,\mu} - \sigma_{,\mu}) + \left[ \left( f \frac{\partial n}{\partial s} u^\mu \right),_{\mu} - f \frac{\partial}{\partial s} \left( f \frac{\partial p}{\partial s} \right) \right] \sigma = 0,
\]

where

\[
f^{\mu \nu} = \frac{n}{w} \left[ g^{\mu \nu} - \left( 1 - \frac{w}{n} \frac{\partial n}{\partial w} \right) u^\mu u^\nu \right].
\]

Next we derive conditions under which equation (34) can be written as the Klein–Gordon equation for the acoustic perturbation \( \chi \) in an effective curved geometry. Obviously, equation (34) with (35) in its most general form cannot be written as a Klein–Gordon equation because the term

\[
f \frac{\partial n}{\partial s} u^\mu \chi_{\mu}
\]

introduces an extra coupling between the velocity field and the derivative of \( \chi \). However, if we impose certain restrictions on perturbations, the term (36) could be eliminated. In that regard
we distinguish two cases: (a) nonadiabatic perturbations with $\sigma \equiv \chi$ and (b) purely adiabatic perturbations with $\sigma = 0$.

### 3.1. Nonadiabatic perturbations with $\sigma \equiv \chi$

In the field theoretical context, as in, e.g. [1] it seems natural to identify $\sigma \equiv \chi$. This is because in the variation of the Lagrangian one does not distinguish the perturbation $\delta \theta$ of the field $\theta$ in the explicit functions of $\theta$ from the perturbation $\delta \theta$ in the derivative $\partial_{\mu}$. Then the second term in (34) vanishes and applying the standard procedure [5] we can recast (34) into the form

$$
\frac{1}{\sqrt{-G}} \partial_{\mu} \left( \sqrt{-G} G^{\mu \nu} \partial_{\nu} \chi \right) + m_{\text{eff}}^2 \chi = 0.
$$

(37)

Here, the matrix

$$
G^{\mu \nu} = \frac{m^2 c_s w}{n} \left[ g^{\mu \nu} - (1 - \frac{1}{c_s^2}) u^\mu u^\nu \right],
$$

(38)
is the inverse of the effective metric tensor

$$
G_{\mu \nu} = \frac{n}{m^2 c_s w} \left[ g_{\mu \nu} - (1 - c_s^2) u_\mu u_\nu \right],
$$

(39)

with determinant

$$
G \equiv \det G_{\mu \nu} = \frac{n^4}{m^8 w^4 c_s^2} \det g_{\mu \nu}.
$$

(40)
The mass parameter $m$ in (37)–(40) is introduced to make $G_{\mu \nu}$ dimensionless. The effective mass squared is given by

$$
m_{\text{eff}}^2 = m^2 \frac{c_s w^2}{n^2} \left[ \left( f \frac{\partial n}{\partial s} u^\mu \right)_\mu - f \frac{\partial}{\partial s} \left( f \frac{\partial p}{\partial s} \right) \right],
$$

(41)

and the quantity $c_s$ is the so-called ‘adiabatic’ speed of sound defined as

$$
c_s^2 = \frac{\partial p}{\partial \rho} \bigg|_s = \frac{n}{w} \left( \frac{\partial n}{\partial w} \right)^{-1} = \frac{\mathcal{L}_X}{\mathcal{L}_X + 2X \mathcal{L}_{XX}}.
$$

(42)

Hence, the linear perturbations $\chi$ propagate in the effective metric (39) and acquire an effective mass. If we replace

$$
f \frac{\partial}{\partial s} \rightarrow \frac{\partial}{\partial \theta},
$$

(43)
equation (37) with (38)–(42) will coincide with that of [1] derived in a different way for a general Lagrangian of the form $\mathcal{L} = \mathcal{L}(X, \theta)$. Note that the particle number density $n$ and specific enthalpy $w$ in our notation differ from those of [1] by factors $\sqrt{2}$ and $1/\sqrt{2}$, respectively owing to a factor of $\sqrt{2}$ difference in the definition (13) of the velocity potential.

### 3.2. Purely adiabatic perturbations with $\sigma = 0$

To study the propagation of sound in an inhomogeneous medium one can consider only adiabatic perturbations and neglect the nonadiabatic ones. In this case by redefining the perturbation $\chi \rightarrow \tilde{\chi}$ so that
\( \chi_{,\mu} = h \tilde{\chi}_{,\mu}, \) \hspace{1cm} (44)

we will seek the function \( h = h(w, s) \) such that the unwanted term (36) is eliminated from (34). Substituting (44) into (34) with \( \sigma = 0 \) we find

\[
h \left( f^\mu{}_{;\nu} \right)_{,\mu} + \left( f^\mu{}_{;\nu} h_{,\mu} - fh \frac{\partial n}{\partial s} u^\nu \right) \tilde{\chi}_{,\nu} = 0.
\] \hspace{1cm} (45)

Now we demand that the second term in this equation vanishes identically. Since \( \tilde{\chi}_{,\mu} \) is basically arbitrary, this term will vanish if and only if the function \( h \) satisfies

\[
f^\mu{}_{;\nu} h_{,\mu} = fh \frac{\partial n}{\partial s} u^\nu.
\] \hspace{1cm} (46)

Noting that

\[
h_{,\mu} = \frac{\partial h}{\partial w} w_{,\mu} + \frac{\partial h}{\partial s} s_{,\mu} = \frac{\partial h}{\partial w} w_{,\mu} + f w \frac{\partial h}{\partial s} u_{,\mu},
\] \hspace{1cm} (47)

where the second equality follows from (14) and (26), equation (46) can be recast into the form

\[
\frac{\partial h}{\partial w} f^\mu{}_{;\nu} w_{,\mu} + f \left( \frac{\partial h}{\partial s} n - h \frac{\partial n}{\partial s} \right) u^\nu = 0.
\] \hspace{1cm} (48)

Since the vector \( f^\mu{}_{;\nu} w_{,\mu} \) is generally not parallel to \( u^\nu \), the above identity will hold true if and only if the function \( h \) does not depend on \( w \) and satisfies

\[
\frac{1}{h} \frac{\partial h}{\partial s} = \frac{c_s^2 \partial n}{n \partial s}.
\] \hspace{1cm} (49)

Clearly, this identity can hold true only if its right-hand side does not depend on \( w \). This together with the definition of the adiabatic speed of sound (42) yields a condition that the quantity

\[
\frac{1}{w} \frac{\partial n}{\partial s} \frac{\partial n}{\partial w}
\] \hspace{1cm} (50)

must be a function of \( s \) only. Applying a very general ansatz

\[
n(w, s) = \sum_{\alpha} w^\alpha \varphi_{,\alpha}(s),
\] \hspace{1cm} (51)

where \( \alpha \) can be integers or non-integers and \( \varphi_{,\alpha} \) are functions of \( s \), we find that \( n \) must be a function of the form

\[
n(w, s) = \sum_{\alpha} C_{\alpha} w^\alpha \varphi(s)^\alpha = n(x)
\] \hspace{1cm} (52)

where \( x = w\varphi(s) \), the quantities \( C_{\alpha} \) in the sum are real coefficients, and \( n(x) \) and \( \varphi(s) \) are arbitrary functions of single variables \( x \) and \( s \), respectively. From (49) and (42) it follows

\[
h(s) = \text{const} \varphi(s).
\] \hspace{1cm} (53)

From (6), (52) and (53) we deduce that the pressure must be of the form

\[
p = \frac{1}{h(s)} F(wh(s)) - V(s),
\] \hspace{1cm} (54)

where \( h \) and \( V \) are arbitrary functions of \( s \) and \( F(x) \) is an indefinite integral of \( n(x) \), i.e. \( n = dF/dx \). The energy density is then fixed by (7).
\[ \rho = wn - p. \]  

Thus, the second term in (45) can vanish identically if and only if the pressure is of the form (54). Then, we can write (45) in the form of a massless Klein–Gordon equation

\[ \frac{1}{\sqrt{-G}} \partial_\mu \left( \sqrt{-G} G^{\mu\nu} \partial_\nu \tilde{\chi} \right) = 0 \]  

in an effective curved background described by the metric (39).

In the language of field theory the pressure (54) corresponds to the Lagrangian of the form

\[ L = \frac{1}{h(\theta)} F(h(\theta) \sqrt{X}) - V(\theta), \]  

where \( h(\theta) = h(s(\theta)) \), \( V(\theta) = V(s(\theta)) \), and \( s(\theta) \) is a function that satisfies (27). This form includes the trivial case \( h = \text{const} \), i.e. \( \partial n / \partial \theta = 0 \), and the case of an isentropic fluid: \( V = 0 \) and \( F(x) \propto x^\alpha \).

4. Analog planar black hole

As an application of the formalism presented in sections 2 and 3, in this section we address the model of an analog planar BH in asymptotic AdS5 which may have interesting applications in condensed matter physics [19]. This model was discussed in detail by Hossenfelder [13, 20]. In her approach, a conservation of particle number is imposed so the fluid is required to be isentropic. However, in order to maintain the energy–momentum conservation, i.e. the Euler equation and a correct definition for the speed of sound, it is necessary to introduce an external pressure field. This in turn implies a violation of the Poincaré invariance of the Lagrangian in the field theoretical formulation.

In our approach we will consider a fluid with no external pressure field. We will demonstrate that this model then yields a nonisentropic fluid and derive a Poincaré invariant Lagrangian that reproduces the desired analog metric.

We start from a planar AdS5 BH with line element [19]

\[ ds^2 = \frac{\ell^2}{z^2} \left[ \gamma(z) dt^2 - \gamma(z)^{-1} dz^2 - \sum_{i=1}^{3} dx_i dx_i \right], \]  

where \( \ell \) is the curvature radius of AdS5,

\[ \gamma(z) = 1 - \left( \frac{z}{z_0} \right)^4, \]  

and \( z_0 \) is the location of the BH horizon. Following [13] we seek a fluid analog model on a 3 + 1 dimensional slice perpendicular to the BH horizon which would mimic the induced metric of the form (58) with the sum \( \sum_{i=1}^{3} \) replaced by \( \sum_{i=1}^{2} \). The basic idea is to find a suitable coordinate transformation \( t \rightarrow \tilde{t}, \ z \rightarrow \tilde{z} \) such that the new metric takes the form of the relativistic acoustic metric (39) with \( g_{\mu\nu} \) replaced by the Minkowski metric \( \eta_{\mu\nu} \). It has been shown [13] that this goal can be achieved by the coordinate transformation

\[ t = \tilde{t} + f(z), \ z = z(\tilde{z}) \]  

where the functions \( z(\tilde{z}) \) and \( f(z) \) are determined by the requirement that the transformed metric takes the form (39). Then, the speed of sound and the nonvanishing components of the velocity vector \( u_t \) and \( u_\tilde{z} \) in transformed coordinates are given by
\[ c_s \equiv \frac{dz}{d\tilde{z}}, \quad (61) \]

\[ u_t = \frac{(1 - \gamma)^{1/2}}{(1 - c_s^2)^{1/2}}, \quad u_z = -\frac{(e_s^2 - \gamma)^{1/2}}{(1 - c_s^2)^{1/2}}, \quad (62) \]

provided that the function \( f(z) \) satisfies

\[ \frac{df}{dz} = -\frac{1 - c_s^2}{\gamma c_s} u_t u_z. \quad (63) \]

Next, by applying the potential-flow equation (13) we derive closed expressions for \( w, n, \) and \( c_s \) in terms of the variable \( z \). Since the metric is stationary, the velocity potential must be of the form

\[ \theta = m \tilde{t} + g(z) \quad (64) \]

where \( m \) is an arbitrary mass and \( g(z) \) is a function of \( \tilde{z} \) through \( z \). Then from (13) it follows

\[ w = \frac{m}{u_t} = m \frac{1}{y} (1 - c_s^2)^{1/2}, \quad (65) \]

where we have introduced a dimensionless variable

\[ y = \frac{z^2}{z^2_0}. \quad (66) \]

Besides, it follows from (13) that the function \( g \) in (64) must satisfy

\[ \frac{dg}{dz} = \frac{w}{c_s} u_z = -\frac{m}{c_s} (e_s^2 - \gamma)^{1/2}. \quad (67) \]

Since the conformal factor in (58) must be equal to that of (39), i.e.

\[ \frac{n}{m^2 c_s w} = \frac{\ell^2}{z^2}, \quad (68) \]

using (65) one can also express \( n \) in terms of \( y \) and \( c_s \). In this way both \( w \) and \( n \) are expressed as functions of \( y \) and \( c_s \). However, \( c_s \) is not independent since by the definition (42)

\[ c_s^2 = \frac{n}{m^2 c_s w} = \frac{\ell^2}{z^2}, \quad (69) \]

where the subscript \( , y \) denotes a derivative with respect to \( y \).

At this point we depart from [13] in which the continuity equation \( (nu^\mu)_{\mu} = 0 \) was imposed. Instead, we require a strict validity of (69) and thus satisfying the Euler equation without introducing an external pressure field. As a consequence, the continuity equation in our model is not satisfied and the fluid is essentially nonisentropic.

The derivatives of \( w \) and \( n \) with respect to \( y \) may be easily calculated using (65) and (39) and using (69) a simple differential equation for \( c_s \) is obtained with solution

\[ c_s^2 = c_1 y^2 + 1/2 \quad (70) \]

where the integration ‘constant’ \( c_1 \) is generally a function of \( s \) and must satisfy the restriction \(-1/2 \leq c_1 \leq 1/2\). Plugging (70) into (65) and (68) one obtains \( w \) and \( n \) as functions of \( y \).
\[ w = m \left( \frac{1}{2y^2} - c_1 \right)^{1/2}, \]  
\[ n = m^3 \frac{c_2}{z_0^2} \left( \frac{1}{4y^4} - c_1^2 \right)^{1/2}. \] 

(71)

(72)

Now, one can easily verify that
\[ \partial_y (u \tilde{z} n) \neq 0, \]

(73)

hence, the particle number is not conserved. According to equation (12), a non-conservation of the particle number automatically implies a nonisentropic fluid.

Note that explicit functional forms of \( z(\tilde{z}), f(z), \) and \( g(z) \) can be obtained by making use of (70) and integrating respectively (61), (63) and (67). However, the precise forms of these functions are not really needed for obtaining the closed expression for the analog metric.

For the field theoretical description an important quantity is the pressure. The pressure may be derived from the equation
\[ p_y = nw_y, \]

(74)

which follows from the first thermodynamic relation in (9). With the help of (71) and (72) one finds a differential equation
\[ p_y = -\frac{c_2}{z_0^2} \left( \frac{1}{2y^2} + c_1 \right)^{1/2} \]

(75)

which may be easily integrated. It is convenient to express \( p \) as a function \( p = p(w, s) \). Using (75) and (71) one finds
\[ p = \frac{4}{3} \frac{c^2 m^4}{z_0^4} \left( \frac{w^2}{m^2} + 2c_1(s) \right)^{3/2} + c_2(s) \]

(76)

where the integration ‘constants’ \( c_1 \) and \( c_2 \) are arbitrary functions of \( s \) (up to the restriction on \( c_1 \) mentioned above). Note that with a particular choice \( c_1 = 0 \) the fluid would belong to the class described by (54) with \( h = \text{const} \).

Next we derive a Lagrangian that reproduces the desired fluid flow and analog metric. It is quite straightforward to apply the general considerations of section 2.4 to the model described above. First, using equations (19) and (74) one can determine a functional relationship between \( X \) and variable \( y \equiv z^2/z_0^2 \). Next, using this relation one can easily find field theoretical representations of all other fluid functions, such as \( c_s, n, p, \) and \( \rho \). In particular, the pressure \( p \) yields the desired Lagrangian \( \mathcal{L} \) if we replace \( w \) in (76) by \( \sqrt{X} \) and, assuming a functional relationship \( s = s(\theta) \), replace the functions \( c_1(s) \) and \( c_2(s) \) by functions of \( \theta \). In this way, we obtain
\[ \mathcal{L} = \frac{4}{3} \frac{c^2 m^4}{z_0^4} \left( \frac{X}{m^2} + V_1(\theta) \right)^{3/2} + V_2(\theta) \]

(77)

where we have identified
\[ V_1(\theta) = 2c_1(s), \quad V_2(\theta) = c_2(s). \]

(78)

Again, with the choice \( V_1 = 0 \) this field theory model would belong to the class of models described by the Lagrangian (57) with \( h(\theta) = \text{const} \). Finally, one can easily verify that the
acoustic metric (39) and hence the initial planar BH metric (58) are correctly reproduced by (77) provided the function $\theta(x)$ defined by (64) is a solution to the equation of motion (21).

Note that the $X$ dependence of the Lagrangian (77) is the same as that of the Lagrangian derived in [13]. However, in contrast to [13], our Lagrangian is Poincaré invariant as there is no explicit $z$-coordinate dependence.

5. Conclusions

We have demonstrated that the formalism of analog gravity under certain conditions can be extended to the case of nonisentropic fluids. First, if the flow is such that the fluid dynamics can be equivalently described by a scalar field theory the acoustic geometry can be fully applied but a phonon propagating in the fluid generally becomes effectively massive. In this case the nonadiabatic perturbations are identified with adiabatic ones. Second, if the nonadiabatic perturbations are neglected, the standard equations of analog acoustic geometry apply also for a nonisentropic flow with pressure of the form (54) in which case the phonons remain massless.

As a concrete example, we have applied our nonadiabatic formalism to the analog model of a planar BH in AdS$_5$.

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Appendix. The second law of thermodynamics

In general the second law of thermodynamics is a global law, which only tells that the entropy of the whole system cannot decrease. Different subsystems can exchange heat, so the entropy of a subsystem may decrease. In an ideal fluid, however, there in no exchange of heat between different parts of the fluid [14]. Noting that the entropy density equals $ns$, the second law of thermodynamics in an ideal fluid takes a local form

$$\left(nsu^\mu\right)_\mu \geq 0.$$  \hspace{1cm} (A.1)

Since $\left(nsu^\mu\right)_\mu = s(nu^\mu)_\mu + nu^\mu s_{\mu}$, equation (10) gives

$$\left(nsu^\mu\right)_\mu = -\frac{g}{T} (nu^\mu)_\mu,$$  \hspace{1cm} (A.2)

where

$$g = w - Ts$$  \hspace{1cm} (A.3)

is the specific Gibbs free energy. Thus the second law (A.1) is equivalent to the condition

$$g(nu^\mu)_\mu \leq 0.$$  \hspace{1cm} (A.4)

To clarify the physical meaning of (A.4), it is instructive to consider the case of a homogeneous fluid in the Minkowski background. In comoving coordinates, equation (10) reads

$$\omega \delta n + Tn \delta s = 0.$$  \hspace{1cm}

Assuming that the system has a fixed volume $V$, we can multiply this equation by $V$ and write it as
where $N = Vn$ is the total number of particles in the volume $V$. Equation (A.5) states that, in a system with conserved energy and positive $w$, the entropy per particle increases when the number of particles decreases. Using $S = Ns$ we have $dS = s dN + N ds$, so equation (A.5) becomes

$$dS = -\frac{g}{T} dN.$$  

(A.6)

Therefore the second law of thermodynamics $dS \geq 0$ is equivalent to

$$gdN \leq 0.$$  

(A.7)

Equations (A.2) and (A.4) are nothing but local covariant versions of (A.6) and (A.7), respectively.

Physically, equation (A.7) can be understood as a result of a competition between two effects. First, for a fixed entropy per particle the total number of particles increases with increasing total entropy. Second, owing to (A.5), the total number of particles decreases with increasing entropy per particle. Thus, depending on which effect prevails, the number of particles will increase or decrease. Equations (A.7) or (A.4) tell us that the sign of the Gibbs free energy is crucial: the number of particles will not increase (decrease) if the Gibbs free energy is positive (negative).

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