Inference on Individual Treatment Effects in Nonseparable Triangular Models

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Abstract

In nonseparable triangular models with a binary endogenous treatment and a binary instrumental variable, Vuong and Xu (2017) and Feng, Vuong, and Xu (2019) respectively provide the identification results for the individual treatment effects (ITEs) under the rank invariance assumption and propose a uniformly consistent kernel density estimator using estimated ITEs for the density of the ITE. This paper establishes the asymptotic normality of the density estimator of Feng, Vuong, and Xu (2019) and shows that the estimation error of the ITEs that vanishes at the root \( n \) rate has a non-negligible effect on the asymptotic distribution. We propose asymptotically valid standard errors that account for estimated ITEs, as well as a bias correction. Furthermore, we develop uniform confidence bands for the density of the ITE using the jackknife multiplier or nonparametric bootstrap critical values.

Keywords: Individual treatment effects, nonparametric triangular models, two-step nonparametric estimation, bootstrap, uniform confidence bands, labor supply and family size

JEL classification: C12, C14, C31, C36

1 Introduction

Heterogeneous treatment effects have received increasing attention in the causal inference and policy evaluation literature (Angrist, 2004; Heckman et al., 1997, 2006). There is a vast literature studying the causal effect of ceteris paribus change of a treatment variable using triangular models (see, e.g., Chesher, 2003, 2005; D’Haultfoeuille and Février, 2015; Imbens and Newey, 2009; Jun et al., 2011; Newey et al., 1999; Torgovitsky, 2015; Vytucil and Yildiz, 2007 among others). In a triangular model, the outcome variable is generated by an outcome equation, and a selection equation determines the endogenous treatment variable. Recently, Vuong and Xu (2017, VX, hereafter) and

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Feng et al. (2019, FVX, hereafter) developed nonparametric identification and estimation methods for individual treatment effects (ITEs) in a triangular model with a nonseparable outcome equation, a selection equation which is a latent index model (Vytlačil, 2002), a binary endogenous treatment variable and a binary instrument under the rank invariance assumption. VX assumes that disturbances in both equations are scalar-valued and that the outcome is a strictly monotone function of the disturbance. The outcome equation in VX satisfies the rank invariance assumption (i.e., for given covariates, the ranks of the two potential outcomes are the same). See, e.g., Chernozhukov et al. (2020) for discussion of this assumption. The triangular model considered in VX and FVX is also closely related to the classical local average treatment effect (LATE) model (see, e.g., Abadie et al., 2002; Abadie, 2003; Frölich and Melly, 2013; Imbens and Angrist, 1994 among others) and the instrumental variable quantile regression (IVQR) model (see, e.g., Chernozhukov and Hansen, 2005 among others).\(^1\)

Since the ITE is defined and estimated for each individual, it is natural to focus on its probability density function (PDF) when assessing the heterogeneity of a treatment. For that purpose, FVX uses the conventional kernel density estimator applied to estimated ITEs. They show its uniform consistency as well as derive its rate of convergence. However, when it comes to inference for the density of the ITE, two theoretical problems still need to be solved. First, asymptotically valid standard errors for the density estimator should incorporate the uncertainty stemming from the estimation of ITEs. Second, as researchers are often interested in the shape of the distribution of the ITE, it is important to have asymptotically valid uniform confidence bands (UCBs) for the density of the ITE. Our paper contributes to the literature by providing easy-to-implement solutions to both problems.

We first provide a sharper bound for the uniform rate of convergence of the FVX density estimator and show that it attains the optimal rate under seemingly minimal conditions. We then show the asymptotic normality of the FVX estimator and derive an analytical formula for the standard error of the density estimator that incorporates the influence of ITEs’ estimation in the first step of the FVX procedure.

The asymptotic normality result in this paper is non-trivial as the asymptotic distribution of the FVX estimator is different from that of the infeasible estimator based on true unobserved ITEs. FVX uses nonparametrically estimated counterfactual mappings to generate pseudo (i.e., estimated) ITEs in the first step of their procedure. In the second step, they apply kernel density estimation to the pseudo ITEs to construct an estimator for the PDF of the ITE. While the estimated counterfactual mappings converge at the root \(n\) rate, we show that the first-step estimation errors’ contribution to the density estimator’s asymptotic variance is non-negligible and can substantially dominate that of the second stage. This phenomenon is due to discontinuities in the linearization of the first-step estimator. At the same time, the asymptotic bias is unaffected by the first-stage estimation errors and equal to that of the infeasible estimator.

\(^1\)A more detailed literature review about the triangular model with an endogenous treatment can be found in our online supplement available at: ruc-econ.github.io/ITE_Supp_Rev_V13.pdf.
The paper’s second contribution is to propose asymptotically valid UCBs for the density of the ITE. The proposed UCB captures the uncertainty about the entire estimated density function and, therefore, can be used for inference about the shape of the ITE’s distribution (e.g., the number and locations of the modes) and for comparisons between the distributions of the ITE in different sub-populations. UCBs can also be used for the specification of parametric models. Our bootstrap UCBs have the desirable property of polynomial coverage error decay rates. Following Calonico et al. (2014), we also propose bias-corrected UCBs using standard errors that incorporate additional variability from the estimated bias (Section 4.3). This approach is common in the recent literature, as it validates the use of conventional data-driven bandwidth selectors for inference. In addition, we consider an extension to inference on PDF conditional on a sub-vector of covariates (Section 4.4).

The FVX estimator and our UCBs require bandwidth selection. Our results explicitly allow for random data-dependent bandwidths. Following the literature (e.g., Hsiao et al., 2007; Li and Li, 2010) and the standard practice in applied work, we assume that the data-dependent bandwidth consistently estimates some deterministic bandwidth. Similarly to the existing literature, we verify that the uniform rate of convergence and asymptotic normality results are unaffected by the bandwidth estimation. However, we go a step further and provide an explicit estimate of the effect of bandwidth estimation on the coverage error of the UCBs. The result is new for this literature and adds to our understanding of the impact of bandwidth selection on inferential procedures.

In the empirical section of the paper, we use the FVX estimator with our UCBs to study the effect of having more than two children on their parents’ labor income using the instrument proposed in Angrist and Evans (1998). We show that the conditional distribution of the ITE is significantly different between households with high-school-only and college-educated mothers. In the latter case, the ITE’s distribution is more dispersed. However, it also places significantly more weight on positive effects.

From the perspective of nonparametric inference, our paper contributes to the literature on inference in the presence of nonparametrically generated variables. See, e.g., Mammen et al. (2012) and Ma et al. (2019). The asymptotic theory of the FVX estimator is different from the results obtained in Mammen et al. (2012) and Ma et al. (2019). E.g., among their other results, Mammen et al. (2012) show that the influence of variables’ estimation can be made asymptotically negligible using a proper choice of the bandwidth. However, in the case of the FVX estimator, the influence of ITEs’ estimation has a non-negligible effect regardless of the bandwidth choice. Ma et al. (2019) show that in the context of first-price auctions, the estimation of latent bidders’ valuations contributes to the asymptotic variance of the Guerre et al. (2000) estimator for the density of valuations. Moreover, the variance of the Guerre et al. (2000) estimator has a slower decay rate than that of the infeasible estimator constructed using the true latent valuations. Here, we show that while the variance of the FVX estimator has the same decay rate as that of the infeasible estimator, the estimation of ITEs cannot be ignored. Our paper also contributes to the literature on inference for nonparametrically estimated functions. In this paper, we take the intermediate Gaussian approximation approach to show the asymptotic validity of our UCBs by using tools developed by Chernozhukov et al. (2014b,a,
This approach was recently applied, e.g., in Chen and Christensen (2018); Cheng and Chen (2019); Kato and Sasaki (2019); Ma et al. (2019) among others, to show the asymptotic validity of bootstrap UCBs for various nonparametric curves in different contexts.

The rest of the paper is organized as follows. Section 2 reviews the model setup and identification of the distribution of the ITE and discusses the nonparametric estimation of the density of the ITE. Section 3 shows the density estimator’s uniform convergence rate and asymptotic normality. Section 4 provides standard errors for the density estimator that can be used for asymptotically valid inference. It also establishes the validity of the bootstrap UCBs. Section 4.2 describes the algorithm for our proposed confidence band. Section 5 presents the results from Monte Carlo experiments. Section 6 applies our inference method to study the effect of family size on labor income. The proofs of the theorems and statements of the technical lemmas are presented in the appendices. The proofs of the lemmas, auxiliary results, and additional simulation evidence are collected in the online supplement (ruc-econ.github.io/ITE_Supp_Rev_V13.pdf).

2 Model and the FVX estimator

For completeness, first, we describe the model setup of VX and FVX, and their estimator. Let \( \mathbb{1}(\cdot) \) denote the indicator function. The outcome and selection equations are given respectively by

\[
\begin{align*}
Y &= g(D, X, \epsilon) \\
D &= \mathbb{1}(\eta \leq s(Z, X)) , 
\end{align*}
\]

where \( Y \in \mathbb{R} \) is a continuously distributed outcome variable, \( D \in \{0, 1\} \) is an endogenous treatment variable, and \( X \in \mathcal{X} \) is a vector of observed explanatory variables (covariates) with \( \mathcal{X} \) denoting the support of the distribution of a random vector \( V \) (i.e., the smallest closed set \( C \) satisfying \( \Pr[V \in C] = 1 \)). \( Z \in \{0, 1\} \) is a binary instrumental variable that has no direct effect on \( Y \) and, therefore, are excluded from the outcome equation. \( (\epsilon, \eta) \) are unobserved scalar-valued disturbances conditionally independent of \( Z \) given \( X \). \( g \) and \( s \) are unknown functions.

The functions \( g(d, x, \cdot) \) and \( s(\cdot, x) \) are assumed to be strictly increasing. The selection equation in (1) has the form of a latent index selection model: treatment is assigned if some latent index or utility \( s(Z, X) \) crosses the threshold \( \eta \). The ITE is defined as

\[
\Delta := g(1, X, \epsilon) - g(0, X, \epsilon) ,
\]

where “\( a := b \)” is understood as “\( a \) is defined by \( b \)”. Note that \( \Delta \) is random conditionally on \( X \) due to the unobserved \( \epsilon \), i.e., the treatment effect varies among individuals with the same observed

\(^2\)Under certain conditions, it can often be shown that the suprema of estimation errors can be approximated by the suprema of tight Gaussian random elements using the theorems of Chen and Kato (2020); Chernozhukov et al. (2014b). Then theorems in Chen and Kato (2020); Chernozhukov et al. (2014a, 2016) show that the distributions of these Gaussian suprema can be approximated by bootstrapping.
One can now estimate the ITEs by replacing \( y \), where we write the leave-out nonparametric estimator of \( \phi \) defined below:

\[
Q_{dx}(t; y) := 
\left( E \left[ 1(D = d) | Y - t | \mid Z = d, X = x \right] - E \left[ 1(D = d') \operatorname{sgn}(Y - y) \mid Z = d, X = x \right] \times t \right)
- \left( E \left[ 1(D = d) | Y - t | \mid Z = d', X = x \right] - E \left[ 1(D = d') \operatorname{sgn}(Y - y) \mid Z = d', X = x \right] \times t \right),
\]

where \( \operatorname{sgn}(u) := 2 \times 1 (u > 0) - 1 \) denotes the left continuous sign function.

The econometrician observes \( \{(Y_i, D_i, X_i, Z_i) : i = 1, \ldots, n\} \), a sample of observations on \((Y, D, X^\top, Z)^\top\) generated by the model. Let \( \hat{Q}_{dx}^{(-i)}(t; y) \) denote the leave-i-out sample analogue of \( Q_{dx}(t; y) \) constructed under the FVX assumption that \( X \) is discretely distributed:

\[
\hat{Q}_{dx}^{(-i)}(t; y) := \frac{\sum_{j \neq i} \left\{ 1(D_j = d, Z_j = d, X_j = x) | Y_j - t | - 1(D_j = d', Z_j = d, X_j = x) \operatorname{sgn}(Y_j - y) t \right\}}{\sum_{j \neq i} 1(Z_j = d, X_j = x)}
- \frac{\sum_{j \neq i} \left\{ 1(D_j = d, Z_j = d', X_j = x) | Y_j - t | - 1(D_j = d', Z_j = d', X_j = x) \operatorname{sgn}(Y_j - y) t \right\}}{\sum_{j \neq i} 1(Z_j = d', X_j = x)}.
\]

The leave-i-out nonparametric estimator of \( \phi_{dx}(y), d \in \{0, 1\} \), can be constructed as

\[
\hat{\phi}_{dx}^{(-i)}(y) := \arg \min_{t \in [\underline{y}_{dx}, \overline{y}_{dx}]} \hat{Q}_{dx}^{(-i)}(t; y),
\]

where we write \( \mathcal{g}(d, x, \epsilon | X = x) = \left[ \underline{y}_{dx}, \overline{y}_{dx} \right] \).

\( ^{3} \)The model allows the ITEs to be “essentially heterogeneous” (Heckman et al., 2006) since whether or not individuals who have the same observed characteristics select into treatment can be correlated with the gain from treatment.

\( ^{4} \)As FVX, we assume that \( \underline{L}_{dx} \) and \( \overline{L}_{dx} \) are known. Lemma 1 of VX shows that the supports of the potential outcomes, \( \mathcal{g}(d, x, \epsilon | X = x) = \left[ \underline{y}_{dx}, \overline{y}_{dx} \right] \), are identified by \( \mathcal{g}(d, x, \epsilon | X = x) = \mathcal{g}(D = d, X = x) \). In practical implementation, \( \underline{y}_{dx} \) and \( \overline{y}_{dx} \) can be estimated. See Section 3 of FVX for discussion.
Theorem 4 in Section 4.2 is our main result. It establishes that the proposed UCB covers $f_{\Delta|X}(v \mid x)$ simultaneously over a range of $v$ values with a pre-specified confidence level in large samples.

3 Asymptotic properties of the FVX estimator

In this section, we establish two new asymptotic results for the FVX density estimator. Theorem 1 below shows that under seemingly minimal conditions (see Assumptions 1 and 2 ahead), the FVX estimator has the same uniform rate of convergence as that of the infeasible kernel density estimator that uses true ITEs, and attains the optimal uniform rate of convergence (see Stone, 1982). Theorem 2 shows that the FVX estimator is asymptotically normal. However, its asymptotic variance is larger than that of the infeasible estimator. We show that these results hold under either a deterministic bandwidth or a data-dependent bandwidth that satisfies Assumption 3 below.

3.1 Assumptions

The following assumption on the data generating process (DGP) is similar to those in VX and FVX.
Assumption 1 (DGP). (a) For all \((d, x) \in \mathcal{S}_{(D, X)}\), \(g(d, x, \cdot)\) is continuously differentiable and strictly increasing. (b) \(Z \in \{0, 1\}\) is independent of \((\epsilon, \eta)\) conditionally on \(X\). (c) For all \(x \in \mathcal{S}_X\), \(\Pr[D = 1 \mid Z = 1, X = x] \neq \Pr[D = 1 \mid Z = 0, X = x]\). (d) The conditional distribution of \((\epsilon, \eta)\) given \(X\) is absolutely continuous with respect to the Lebesgue measure, has a compact support, and its PDF is continuous and bounded. (e) The supports \(\mathcal{S}_{(D, X)}\) and \(\mathcal{S}_{(Z, X)}\) are equal to \([0, 1] \times \mathcal{S}_X\). (f) For all \(x \in \mathcal{S}_X\), \(s(0, x) < s(1, x)\). (g) For \(D_{xx} := 1(\eta \leq s(z, x))\), the complier group is given by \(D_{0x} < D_{1x}\). We assume that for all \((d, x) \in \mathcal{S}_{(D, X)}\), \(\mathcal{S}_{g(d, x, \cdot)|X=x, D_{0x}<D_{1x}} = \mathcal{S}_{g(d, x, \cdot)|X=x}\). (h) For all \(d \in \{0, 1\}\), the conditional distribution of \(g(d, x, \epsilon)\) given \(X = x\) and \(D_{0x} < D_{1x}\) has a bounded away from zero density \(f_{dx|C_x}\). (i) For all \(x \in \mathcal{S}_X\) and \(d \in \{0, 1\}\), the conditional distributions of \(g(d, x, \epsilon)\) have the support \(\mathcal{S}_{g(d, x, \cdot)|X=x} = \left[y_{dx}, \overline{y}_{dx}\right] \) with known boundaries \(-\infty < y_{dx} < \overline{y}_{dx} < +\infty\). (j) The data \(\left\{W_i := (Y_i, D_i, X_i^\top, Z_i)^\top : i = 1, \ldots, n\right\}\) are i.i.d. observations on \(W = (Y, D, X^\top, Z)^\top\). (k) \(X\) is discretely distributed and \(\mathcal{S}_X\) is finite.

In the above assumption, the continuity and monotonicity conditions in (a), the standard instrument exogeneity assumption in (b), the instrument relevance condition in (c), the absolute continuity condition in (d), (e,f), as well as the equality of the supports condition in (g) are imposed for identification. The assumption that \(\epsilon\) is scalar-valued and the condition in (a) impose rank invariance on the potential outcomes. See Section 2.1 of VX. Parts (d,e) of the assumption are mild regularity conditions. Parts (c,f) and the latent index assumption on the selection equation imply that \(\Pr[D = 1 \mid Z = 1, X = x] > \Pr[D = 1 \mid Z = 0, X = x]\). Under (c,f) and the latent index assumption, we have \(D_{0x} \leq D_{1x}\), for all \(x \in \mathcal{S}_X\). Clearly, the model satisfies the LATE independence and monotonicity assumptions (see, e.g., Vytlačil, 2002, Section 4). See Kitagawa (2015) for testable implications.

In part (g), it is assumed that conditionally on \(X = x\), the support of the conditional distribution of \(g(d, x, \epsilon)\) in the complier group \(D_{0x} < D_{1x}\) is the same as that of the conditional distribution of \(g(d, x, \epsilon)\) given \(X = x\). VX argues that (g) is satisfied if the conditional distribution of \((\epsilon, \eta)\) given \(X = x\) has a rectangular support, for all \(x \in \mathcal{S}_X\). Note that the identification result \(\mathcal{S}_{g(d, x, \cdot)|X=x} = \mathcal{S}_{Y|D=d, X=x}\) in VX, together with (a), implies that \(\mathcal{S}_{g|X=x} = \mathcal{S}_{g|D=d, X=x}\) for all \(d \in \{0, 1\}\). The rest of the conditions are imposed for estimation. As in VX, part (k) restricts the estimation framework to discretely distributed covariates \(X\). Under these assumptions, the conditional distribution of \(Y\) or \(\epsilon\) given \((D, X, Z)\) is absolutely continuous with respect to the Lebesgue measure and admits a continuous and bounded Lebesgue density. Under Assumption 1, \(f_{dx|C_x}\) is also continuous and bounded.\(^5\)

Theorem 1 of VX shows the asymptotic properties of the estimated counterfactual mappings \(\hat{\phi}_{dx}\) under Assumption 1. Let \(\Delta_x(\cdot) := g(1, x, \cdot) - g(0, x, \cdot)\), which is continuously differentiable under Assumption 1. VX assumes in their Assumption 5(i) that the conditional density of \(\Delta = \Delta_x(\epsilon)\) given \(X = x\) exists and is \(P\)-times continuously differentiable. Without imposing further restrictions,\(^5\) Let \(F_{dx|C_x}\) denote the conditional CDF of \(g(d, x, \epsilon)\) given \(X = x\) and \(D_{0x} < D_{1x}\). Then it is clear that \(F_{dx|C_x}(y) = \Pr[\epsilon \leq g^{-1}(d, x, y) \mid s(0, x) < \eta \leq s(1, x), X = x]\).
Assumptions 1 alone does not guarantee that the distribution of $\Delta = \Delta_x (\epsilon)$ is absolutely continuous with respect to the Lebesgue measure.\(^6\) The assumption below provides mild sufficient conditions for the existence and differentiability of the Lebesgue density of the ITE $\Delta$ given $X = x$ (see Lemma 1 in Appendix A). Let $F_{V|W} (\cdot \mid w)$ and $f_{V|W} (\cdot \mid w)$ denote the conditional cumulative distribution function (CDF) and PDF of $V$ given $W = w$, respectively.

**Assumption 2** (Existence and differentiability of the conditional PDF of the ITE). (a) For all $x \in \mathcal{X}$, the conditional CDF $F_{\epsilon|x} (\cdot \mid x)$ of $(\epsilon, \eta)$ given $X = x$ and $g (d, x, \cdot)$ are both $(P + 1)$-times continuously differentiable. (b) There is a partition of $\mathcal{S}_{\epsilon|x=x} = [\mathcal{S}_x]$ with $\epsilon_x = \epsilon_x, 0 < \epsilon_x, 1 < \cdots < \epsilon_x, m = \mathcal{T}_x$ such that $\Delta_x (\cdot)$ is piecewise monotone: for all $j = 1, ..., m$, the restriction of $\Delta_x (\cdot)$ on $[\epsilon_x, j-1, \epsilon_x, j]$, $\Delta_{x,j} (\cdot) := \Delta_x |_{[\epsilon_x, j-1, \epsilon_x, j]}$ is strictly monotone. (c) Let $\Delta_x ((\epsilon_x, j-1, \epsilon_x, j)) := \{ \Delta_x (e) : e \in (\epsilon_x, j-1, \epsilon_x, j) \}$ denote the image of $\Delta_{x,j}(\cdot)$. We assume that $\Delta_x ((\epsilon_x, 0, \epsilon_x, 1)) = \cdots = \Delta_x ((\epsilon_x, m-1, \epsilon_x, m))$.

The smoothness assumption imposed by (a) with $P \geq 1$ is stronger than that imposed by Assumption 1(a,d). Under (a), $f_{d|x|C_l}$ is $P$-times continuously differentiable. The piecewise monotonicity condition in (b) is easily satisfied if $\Delta_x$ has finitely many local extrema on $[\mathcal{S}_x]$.\(^7\) Parts (a,b) of the assumption guarantee the existence of the Lebesgue density $f_{\Delta|x} (\cdot \mid x)$. Note that the knowledge of the partition in (b) is not required for estimation or inference. Part (c) rules out discontinuities in the interior of $\mathcal{S}_{\Delta|x=x}$. See the proof of Lemma 1 in Appendix A for more details. We are unaware of any weaker conditions that could be imposed on $\Delta_x$ to guarantee the existence and differentiability of the conditional PDF of $\Delta = \Delta_x (\epsilon)$ given $X = x$.

Application of kernel-based nonparametric techniques is complicated by the bandwidth selection issue. A common practice in applied work is using a data-dependent bandwidth approximating some underlying deterministic bandwidth. We allow the bandwidth $h = \hat{h}_n$ used in the implementation to be data-dependent and, following Li and Li (2010), assume that $\hat{h}$ is a consistent estimator of some deterministic bandwidth sequence $h = h_n \downarrow 0$ in the sense that $\hat{h}_n / h_n \to_p 1$. To simplify the notation, we suppress the dependence of the bandwidths on $n$. Formally, we make the following assumption.

**Assumption 3.** $\Pr \left[ \left| \hat{h} / h - 1 \right| > \epsilon_n \right] \leq \delta_n$ for some deterministic bandwidth $h$ and positive sequences $\epsilon_n, \delta_n \downarrow 0$.\(^8\)

The deterministic bandwidth assumption ($\epsilon_n = \delta_n = 0$) is nested as a special case. Clearly, $\hat{h} / h - 1 = O_p (\epsilon_n)$, under Assumption 3. E.g., as in FVX, one can consider a feasible version of the Silverman rule-of-thumb (ROT) bandwidth by setting $\hat{h} = \hat{h}_n^{\text{rot}} := C_K \cdot \hat{\sigma}_{\Delta|x=x} \cdot n^{-1/5}$, where

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\(^6\)E.g., the distribution has a mass point if $\Delta_x (\cdot)$ is constant on some sub-interval of $\mathcal{S}_{\Delta|x=x}$.

\(^7\)Since it was assumed in Assumption 1 that $\Delta_x$ is continuously differentiable, this condition is satisfied if the set of zeros of the continuous derivative function $\Delta'_x$, $\{ e \in [\mathcal{S}_x] : \Delta'_x (e) = 0 \}$, contains only isolated points.

\(^8\)Assumption 3 is equivalent to requiring $\hat{h} / h \to_p 1$. It is clear that Assumption 3 implies $\hat{h} / h \to_p 1$. On the other hand, if $\hat{h} / h \to_p 1$, $\epsilon_n = \delta_n \downarrow 0$ can be taken to be the Ky Fan metric between $\hat{h} / h$ and 1, which converges to 0 as $n \uparrow \infty$. See Dudley (2002, Theorem 9.2.2).
\( \hat{\sigma}_{\Delta|X=x} \) is the sample analogue of the standard deviation \( \sigma_{\Delta|X=x} := \sqrt{\text{Var}[\Delta | X = x]} \) computed using the pseudo (estimated) ITEs, \( C_K \) is a known constant that depends only on the kernel function \( K \), and \( n_x := \sum_{i=1}^n 1(X_i = x) \).\(^9\) Let \( a_n \propto b_n \) denote \( a_n = C \cdot b_n \) for some constant \( C > 0 \). Let \( p_x := \Pr[X = x] \). By Lemma 2, one can show that such a choice satisfies Assumption 3 with \( h = C_K \cdot \left( \sigma_{\Delta|X=x} p_x^{-1/5} \right) \cdot n^{-1/5}, \varepsilon_n \propto \sqrt{\log(n)/n} \) and \( \delta_n \propto n^{-1} \). We also assume that the kernel function \( K \) is of order \( P \geq 2 \).

**Assumption 4.** (a) \( K \) is symmetric, compactly supported on \([-1,1]\) and twice continuously differentiable on \( \mathbb{R} \) with Lipschitz derivatives. (b) \( \int K(u) \, du = 1 \). (c) \( \int u^k K(u) \, du = 0 \), for all \( k = 1, 2, ..., P - 1 \) (\( P \geq 2 \)), and \( \int u^P K(u) \, du \neq 0 \).

### 3.2 Rate of convergence and asymptotic distribution

In this section, we first derive a linearization for the FVX estimator. The result is given by equation (18) below and serves as the basis for establishing the asymptotic properties of the FVX estimator. The latter are presented below as Theorems 1 and 2 (the rate of convergence and asymptotic distribution, respectively).

Let \( \tilde{f}_{\Delta|X}(v \mid x; b) \) denote the infeasible estimator of the density \( f_{\Delta|X}(v \mid x) \) that uses the true latent ITEs:

\[
\tilde{f}_{\Delta|X}(v \mid x; b) := \sum_{i=1}^n \frac{1}{n} K \left( \frac{\Delta_i - v}{h} \right) 1(X_i = x).
\]

(10)

Let \( I_x \) denote an inner closed sub-interval of \( \mathcal{I}_{\Delta|X=x} = [\underline{h}, \overline{h}] \). Under Assumption 3, \( \hat{\Delta}_i \in \frac{1}{h} \] with probability \( 1 - \delta_n \). For \( (v, b) \in I_x \times [\underline{h}, \overline{h}] \), we decompose the estimation error \( \tilde{f}_{\Delta|X}(v \mid x; b) - f_{\Delta|X}(v \mid x) \) into that of the infeasible estimator \( \tilde{f}_{\Delta|X}(v \mid x; b) - f_{\Delta|X}(v \mid x) \) and the difference between the feasible and infeasible estimators \( \tilde{f}_{\Delta|X}(v \mid x; b) - \hat{f}_{\Delta|X}(v \mid x; b) \). We show that the former satisfies

\[
\tilde{f}_{\Delta|X}(v \mid x; b) - f_{\Delta|X}(v \mid x) = p_x^{-1} \left( \tilde{f}_{\Delta X}(v, x; b) - m_{\Delta X}(v, x; b) \right) + p_x^{-1} \left( m_{\Delta X}(v, x; b) - f_{\Delta X}(v, x) \right) + O_p \left( \sqrt{\frac{\log(n)}{n}} \right),
\]

(11)

where \( \tilde{f}_{\Delta X}(v, x; b) := \tilde{f}_{\Delta|X}(v \mid x; b) \tilde{p}_x \) with \( \tilde{p}_x := n^{-1} \sum_{i=1}^n 1(X_i = x) \) is the infeasible estimator of the joint density function \( f_{\Delta X}(v, x) := \Delta X(v \mid x) p_x \), \( m_{\Delta X}(v, x; b) := E \left[ \tilde{f}_{\Delta X}(v, x; b) \right] \), and the remainder term is uniform in \( (v, b) \in I_x \times [\underline{h}, \overline{h}] \). Note that \( m_{\Delta X}(v, x; b) - f_{\Delta X}(v, x) \) is the bias term that depends on the bandwidth \( b \). Let \( \mu_{K,P} := \left( \int u^P K(u) \, du \right) / P! \), and let \( f^{(P)}_{\Delta|X}(v \mid x) := \partial_P f_{\Delta|X}(v \mid x) / \partial u^P \) denote the derivatives of the conditional PDF. It follows from

\(^9\)The ROT bandwidth is a parametric estimator of the asymptotic mean integrated squared error (AMISE) optimal bandwidth for \( \tilde{f}_{\Delta|X}(v \mid x; h) \) defined by (10) under \( P = 2 \). See Li and Racine (2007). Theorem 2 shows that the asymptotic mean squared error (AMSE) of \( \tilde{f}_{\Delta|X}(v \mid x; h) \) is given by AMSE \( (v \mid x) := f^{(P)}_{\Delta|X}(v \mid x)^2 \mu_{K,P}^2 h^{2P} + \gamma(v \mid x) / (nh) \). Hence, the bandwidth that minimizes the AMISE \( \int_{\mathcal{I}_x} \text{AMSE}(v \mid x) \, dv \) is also a multiple of \( n^{-1/(2P+1)} \).
standard arguments for kernel density estimators (see, e.g., Newey, 1994) that
\[ m_{\Delta X} (v, x; b) - f_{\Delta X} (v, x) = f_{\Delta X}^{(P)} (v \mid x) p_x \mu_{K, P b} + o (h^P) , \] (12)
uniformly in \((v, b) \in I_x \times [h, \overline{h}]\). For a deterministic bandwidth sequence \(h\) such that \(nh \uparrow \infty\), it follows from standard arguments that \(\sqrt{n h} \left( \hat{f}_{\Delta X} (v, x; h) - m_{\Delta X} (v, x; h) \right)\) is asymptotically normal.

Let \(K'\) denote the derivative of the kernel function \(K\). Denote \(\hat{f}_{\Delta X} (v, x; b) := \hat{f}_{\Delta X} (v \mid x; b) \hat{p}_x\). We approximate \(\hat{f}_{\Delta X} (v, x; b) - \hat{f}_{\Delta X} (v, x; b)\) by \((nb^2)^{-1} \sum_{i=1}^n K' ((\Delta_i - v) / b) \left( \hat{\Delta}_i - \Delta_i \right) \mathbb{1} (X_i = x)\).

The first-stage estimation errors \(\hat{\phi}^{(-i)}_{d X_i} (Y_i) - \phi_{d X_i} (Y_i)\) in \(\hat{\Delta}_i - \Delta_i\) can be approximated using its linear representation (see Theorem 1 of FVX and Lemma 2). After recalling that \(f_{dx|c_x}\) is the conditional PDF of \(g(d, x, \epsilon)\) given \(X = x\) in the complier group, we define:

\[
\begin{align*}
\zeta_{dx} (y) & := f_{dx|c_x} (y) \left( \Pr [D = d \mid Z = 1, X = x] - \Pr [D = d \mid Z = 0, X = x] \right), \\
R_{dx} (y) & := \Pr [Y \leq \phi_{dx} (y), D = d \mid X = x] + \Pr [Y \leq y, D = d' \mid X = x], \\
q_{dx} (W_i, W_j) & := \frac{\mathbb{1} (D_i = d', X_i = x)}{\zeta_{dx} (\phi_{dx} (Y_i))} \times \left\{ \mathbb{1} (Y_j \leq \phi_{dx} (Y_i), D_j = d) + \mathbb{1} (Y_j \leq Y_i, D_j = d') - R_{dx} (Y_i) \right\}, \\
q_x (W_i, W_j) & := q_{1x} (W_i, W_j) - q_{0x} (W_i, W_j), \\
p_{zx} & := \Pr [Z = z, X = x], \\
\pi_x (Z_i, X_i) & := \frac{\mathbb{1} (Z_i = 0, X_i = x)}{p_{0x}} - \frac{\mathbb{1} (Z_i = 1, X_i = x)}{p_{1x}}.
\end{align*}
\] (13)

Using the above definitions, we can write that the difference between the feasible and infeasible estimators as a \(U\)-statistic with a kernel that depends on the bandwidth:

\[ \hat{f}_{\Delta X} (v \mid x; b) - \hat{f}_{\Delta X} (v \mid x; b) = p_x^{-1} \frac{1}{n (n - 1)} \sum_{i=1}^n \sum_{j \neq i} G_x (W_i, W_j, v; b) + O_p \left( \frac{\log (n)}{nh^2} + \frac{\log (n)^{3/4}}{n^{3/4} h} \right), \] (15)

uniformly in \((v, b) \in I_x \times [h, \overline{h}]\), where

\[ G_x (W_i, W_j, v; b) := \frac{1}{b^2} K' \left( \frac{\Delta_i - v}{b} \right) q_x (W_i, W_j) \pi_x (Z_j, X_j) \] (16)

By Assumption 1(b) and (40) in Appendix A, \(G_x^{[2]} (w, v; b) := \mathbb{E}[G_x (w, W, v; b)] = 0\), for all \(w\) and \(\mathbb{E}[G_x (W_i, W_j, v; b)] = 0\), for all \(i \neq j\). The leading term (or the Hájek projection) in the Hoeffding decomposition of the \(U\)-statistic is given by \(G_x^{[1]} (w, v; b) := \mathbb{E}[G_x (W, w, v; b)]\). Therefore, the Hoeffding decomposition is given by

\[ \frac{1}{n (n - 1)} \sum_{i=1}^n \sum_{j \neq i} G_x (W_i, W_j, v; b) = \frac{1}{n} \sum_{i=1}^n G_x^{[1]} (W_i, v; b) \]
\[ + \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \left\{ G_x(W_i, W_j, v; b) - \varphi^{[1]}_x(W_j, v; b) \right\}. \tag{17} \]

By definition and since \( \varphi^{[2]}_x(w, v; b) = 0 \), the \( U \)-statistic \((n(n-1))^{-1} \sum_{i=1}^{n} \sum_{j \neq i} G_x(W_i, W_j, v; b) \) is non-degenerate (Chen and Kato, 2020) if \( \text{Var} \left[ \varphi^{[1]}_x(W, v; b) \right] > 0 \). In the proof of Lemma 4, we show that the condition holds for all \((v, b) \in I_x \times [\underline{h}, \overline{h}] \) when \( n \) is sufficiently large. We also show that the second term on the right-hand side of (17) is dominated by the first term. See Remark 3 below. Combining the result with (11) and (15), we can write the estimation error of the feasible estimator as

\[
\hat{f}_{\Delta|X}(v \mid x; b) = f_{\Delta|X}(v \mid x) + p_x^{-1} \left( \frac{\log (n)}{n} + \frac{\log (n)}{nh^2} + \frac{\log (n)^{3/4}}{n^{3/4}h} \right), \tag{18}
\]

where the remainder term is uniform in \((v, b) \in I_x \times [\underline{h}, \overline{h}] \).

The asymptotic variance of the FVX estimator \( \hat{f}_{\Delta|X}(v \mid x; h) \) under the deterministic bandwidth sequence is determined by the sum of \( \left( \frac{\hat{f}_{\Delta X}(v, x; h) - m_{\Delta X}(v, x; h)}{p_x} \right) \) and \( n^{-1} \sum_{i=1}^{n} G_x^{[1]}(W_i, v; h) / p_x \). The first term comes from the infeasible estimator \( \hat{f}_{\Delta X}(v, x; h) \) as in equation (11). The second term reflects the influence of the estimation of ITEs in the first stage. We show that these terms are both of order \( O_p \left( \sqrt{\log (n) / (nh)} \right) \) uniformly in \( v \in I_x \) and independent of each other. Consequently, the first-stage estimation errors unambiguously add to the asymptotic variance of \( \hat{f}_{\Delta|X}(v \mid x; h) \) and their contribution must be taken into account for valid inference.\(^{10}\) Equation (12) implies that \( m_{\Delta X}(v, x; h) - f_{\Delta X}(v, x) = O \left( h^p \right) \) uniformly in \( v \in I_x \). These results provide the uniform rate of convergence of \( \hat{f}_{\Delta|X}(\cdot \mid x; h) \). The bias expansion in (12) and the linearization in (18) are also valid for a continuum \([\underline{h}, \overline{h}]\) of bandwidths. Since \( \text{Pr} \left[ \hat{h} \in [\underline{h}, \overline{h}] \right] > 1 - \delta_n \) under Assumption 3, (12) and (18) with \( b \) replaced by \( \hat{h} \) still hold. We show that the first two terms on the right-hand side of the decomposition in (18) are of the same \( O_p \left( \sqrt{\log (n) / (nh)} \right) \) order uniformly in \((v, b) \in I_x \times [\underline{h}, \overline{h}] \). These results show that the uniform convergence rate remains the same if \( h \) is replaced by its estimator \( \hat{h} \).

We summarize the above results in Theorem 1 below, which is a refinement of Theorem 2 in FVX. In Appendix A, we prove a stronger version (Theorem A1) of Theorem 1. The latter establishes the non-asymptotic deviation bounds for the uniform estimation errors of \( \hat{f}_{\Delta|X}(\cdot \mid x; h) \)

\(^{10}\) Let \( \hat{\phi}_{ds}(y) \) be the leave-in version of \( \hat{\phi}_{ds}^{(d+1)}(y) \) (i.e., \( \hat{\phi}_{ds}(y) \) minimizes the sample analogue of \( Q_{ds}(\cdot; y) \)). The crucial observation is that the linearization of \( \hat{\phi}_{ds}(y) - \phi_{ds}(y) \) derived in FVX (also see Lemma 2) is discontinuous in both \( W_i \) and \( y \). As a result, the difference between the feasible and infeasible estimators \( \hat{f}_{\Delta|X}(v \mid x; h) - \hat{f}_{\Delta X}(v \mid x; h) \) converges at a rate slower than \( n^{-1/2} \). One can show that the difference would be of order \( O_p(n^{-1/2}) \) if the linearization were smooth. See the proof of Lemma 3 for more details on how the linearization is involved in the \( U \)-statistic representation given by \((n(n-1))^{-1} \sum_{i=1}^{n} \sum_{j \neq i} G_x(W_i, W_j, v; b) \).
implies that the FVX and infeasible estimators of Stone
Li and Li (1982) hold, and 

\[ \text{Theorem 1.} \]

Suppose that Assumptions 1-4 hold, and the deterministic bandwidth \( h \) is such that \( \log(n)/(nh^2) \downarrow 0 \). Then, for any \( x \in \mathcal{I}_x \) and compact \( I_x \subseteq \text{Int}(\mathcal{I}_{\Delta|x=x}) \),

\[
\left\| \hat{f}_{\Delta|x}(\cdot \mid x; \hat{h}) - f_{\Delta|x}(\cdot \mid x) \right\|_{I_x} = O_p\left( \frac{\sqrt{\log(n)}}{nh} + h^p \right)
\]

and

\[
\left\| \hat{f}_{\Delta|x}(\cdot \mid x; \hat{h}) - f_{\Delta|x}(\cdot \mid x) \right\|_{I_x} = O_p\left( \frac{\sqrt{\log(n)}}{nh} + h^p \right).
\]

\[ \text{Remark 1.} \]

In comparison, Theorem 2 of FVX has a slower \( O_p\left( \frac{\sqrt{\log(n)}}{(nh^2)} + h^p \right) \) convergence rate. Theorem 1 implies that the FVX and infeasible estimators of \( f_{\Delta|x}(\cdot \mid x) \) have the same uniform convergence rate. Moreover, the convergence rate is unaffected by the estimation of the bandwidth. The optimal bandwidth rate that leads to the fastest possible convergence rate is of order \( (\log(n)/n)^{1/(2P+1)} \). Hence, both the FVX and infeasible estimators attain the optimal uniform convergence rate \( (\log(n)/n)^{P/(2P+1)} \). Note that under our smoothness conditions, any uniformly consistent estimator cannot converge uniformly at a rate faster than \( (\log(n)/n)^{P/(2P+1)} \) (see Stone, 1982).

The next theorem establishes the asymptotic normality of the FVX estimator and quantifies the contribution of the first-stage estimation errors to the asymptotic variance. By using (12) and the linearization (18) for a single bandwidth \( h \), we show that for any fixed \( v \in I_x \), asymptotic normality holds for \( \sqrt{nh} \left( \hat{f}_{\Delta|x}(v \mid x; h) - f_{\Delta|x}(v \mid x) - f_{\Delta|x}^{(P)}(v \mid x) \mu_{K,P,h} \right) \). By using the uniform-in-bandwidth approximation ((11) - (15)) of \( \hat{f}_{\Delta|x}(v \mid x; \hat{h}) - f_{\Delta|x}(v \mid x) \) and an asymptotic equivalence result (Lemma 5), we show that the same normality result holds if \( h \) is replaced by its estimator \( \hat{h} \). The result is analogous to those in Li and Li (2010).

Let \( f_{\Delta|x}(e, d, x) := f_{\Delta|x}(e \mid d, x) \Pr[D = d, X = x] \) denote the joint density of \((\epsilon, D, X^\top)^\top\).

\[ \text{Theorem 2.} \]

Suppose that Assumptions 1-4 hold, and \( h \asymp n^{-\lambda} \) with \( 1/(2P+1) \leq \lambda < 1/3 \). Then, for any \( x \in \mathcal{I}_x \) and \( v \) in a compact sub-interval \( I_x \) of \( \mathcal{I}_{\Delta|x=x} \),

\[
\sqrt{nh} \left( \hat{f}_{\Delta|x}(v \mid x; h) - f_{\Delta|x}(v \mid x) - f_{\Delta|x}^{(P)}(v \mid x) \mu_{K,P,h} \right) \to_d N(0, \mathcal{Y}(v \mid x))
\]

and if \( \varepsilon_n = o\left( \log(n)^{-1/2} \right) \) in Assumption 3,

\[
\sqrt{nh} \left( \hat{f}_{\Delta|x}(v \mid x; \hat{h}) - f_{\Delta|x}(v \mid x) - f_{\Delta|x}^{(P)}(v \mid x) \mu_{K,P,\hat{h}} \right) \to_d N(0, \mathcal{Y}(v \mid x)),
\]
where \( \mathcal{Y}(v \mid x) := p_x^{-2}(\mathcal{Y}_1(v, x) + \mathcal{Y}_2(v, x)) \),

\[
\mathcal{Y}_1(v, x) := f_{\Delta X}(v, x) \int K(u)^2 \, du,
\]

\[
\mathcal{Y}_2(v, x) := \left\{ \sum_{j=1}^m \left( \frac{f_{\Delta X} \left( \Delta_{x,j}^{-1}(v), 0, x \right)}{\zeta_{1x} \left( g \left( 1, x, \Delta_{x,j}^{-1}(v) \right) \right)} - \frac{f_{\Delta X} \left( \Delta_{x,j}^{-1}(v), 1, x \right)}{\zeta_{0x} \left( g \left( 0, x, \Delta_{x,j}^{-1}(v) \right) \right)} \right)^2 \frac{f_{\Delta X} \left( \Delta_{x,j}^{-1}(v) \mid x \right)}{\Delta_{x,j}^{'} \left( \Delta_{x,j}^{-1}(v) \right)^3} \right\} \times (p_{1x}^{-1} + p_{0x}^{-1}) \int K(u)^2 \, du,
\]

and \( \Delta_{x,j}^{'} \) is the derivative of \( \Delta_{x,j} \) defined in Assumption 2.

**Remark 2.** Under the same assumptions, the infeasible kernel estimator that uses the true ITEs satisfies

\[
\sqrt{n\Delta} \left( \hat{f}_{\Delta \mid X}(v \mid x; h) - f_{\Delta \mid X}(v \mid x) - f_{\Delta \mid X}^{(P)}(v \mid x) \mu_K, p h^2 \right) \rightarrow_d N \left( 0, p_x^{-2} \mathcal{Y}_1(v, x) \right).
\]

Note that the estimation of ITEs does not affect the leading bias term.

**Remark 3.** In the proof of Lemma 4, we show that \( \text{Var} \left[ \mathcal{G}_x^{[1]}(W, v; b) \right] = b^{-1}(\mathcal{Y}_2(v, x) + o(1)) \) uniformly in \((v, b) \in I_x \times [\underline{h}, \overline{h}]\). The proof of this result and derivation of the form of \( \mathcal{Y}_2(v, x) \) crucially rely on Assumption 2(b,c). It is clear from the definition of \( \zeta_{dx} \), the fact that \( \mathcal{Y}_{1, X=x} = \mathcal{Y}_{1, D=d, X=x} \), and Assumption 1(g) that for all \( j \),

\[
\left( \frac{f_{\Delta X} \left( \Delta_{x,j}^{-1}(v), 0, x \right)}{\zeta_{1x} \left( g \left( 1, x, \Delta_{x,j}^{-1}(v) \right) \right)} - \frac{f_{\Delta X} \left( \Delta_{x,j}^{-1}(v), 1, x \right)}{\zeta_{0x} \left( g \left( 0, x, \Delta_{x,j}^{-1}(v) \right) \right)} \right)^2 \frac{f_{\Delta X} \left( \Delta_{x,j}^{-1}(v) \mid x \right)}{\Delta_{x,j}^{'} \left( \Delta_{x,j}^{-1}(v) \right)^3} > 0
\]

and therefore, \( \mathcal{Y}_2(v, x) \) > 0. Moreover, in the proof of Lemma 5, we show that \( \sqrt{n\Delta} \left( n^{-1} \sum_{i=1}^n \mathcal{G}_x^{[1]}(W_i, v; b) \right) = o_p(1) \), uniformly in \( b \in [\underline{h}, \overline{h}] \).

Further, in the proof of Theorem 2, we show that \( \sqrt{n\Delta} \left( n^{-1} \sum_{i=1}^n \mathcal{G}_x^{[1]}(W_i, v; h) \right) \rightarrow_d N \left( 0, \mathcal{Y}_2(v, x) \right) \) and the second term on the right-hand side of (17) is \( O_p \left( (nh^2)^{-1} \right) \). Therefore, in (17), \( n^{-1} \sum_{i=1}^n \mathcal{G}_x^{[1]}(W_i, v; b) \) dominates the second term under our assumption on the rate of \( h \).

The \( p_x^{-2} \mathcal{Y}_1(v, x) \) term in the asymptotic variance of the FVX estimator is the asymptotic variance of the infeasible estimator. The \( q_x^{-2} \mathcal{Y}_2(v, x) \) term is due to the estimation of the ITEs. Thus, the estimation of the ITEs increases the variance (but not the bias). To illustrate the effect of estimation of the ITEs numerically, consider the DGP used for the Monte Carlo simulations in Section 5 with no controls \( X \). The treatment status \( D \) is determined by the index model in (35) with coefficients \((\gamma_0, \gamma_1) = (-0.5, 0.5)\). The kernel function \( K \) is taken to be the triweight kernel. In this case for \( v = 2 \), \( \mathcal{Y}_1(v) = 0.16 \) and \( \mathcal{Y}_2(v) = 2.30 \). Hence, the contribution of the ITE estimation errors to the asymptotic variance of the FVX estimator can be substantial and even exceed the asymptotic variance of the infeasible estimator.
4 Inference

In this section, we discuss the construction of asymptotically valid standard errors as well as construction of asymptotically valid UCBs for \( \{ f_{\Delta|X} (v \mid x) : v \in I_x \} \). We maintain Assumptions 1, 2 and 4 with \( P = 2 \). We also maintain the assumption that, as in the practical implementation of many nonparametric econometric methods, the bandwidth is data-driven and satisfies Assumption 3.

4.1 Standard errors

Inference for \( f_{\Delta|X} (v \mid x) \) requires a consistent estimator of the asymptotic variance term \( \mathcal{V} (v \mid x) \) defined in Theorem 2. By the same arguments as those used to establish Theorems 1 and 2, one can show that

\[
\sqrt{n h} \left( \hat{f}_{\Delta|X} (v \mid x; \hat{h}) - f_{\Delta|X} (v \mid x) \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} p_{-1}^{x} \left( U_{x}^{[1]} (W_i, v; h) - \mu_{U_x} (v; h) \right) + O_p \left( \varepsilon_n \sqrt{\log (n)} + \frac{\log (n)}{\sqrt{nh^3}} + \left( \frac{\log (n)^3}{nh^2} \right)^{1/4} + \sqrt{\log (n)h} + \sqrt{nh^5} \right),
\]

(20)

where

\[
U_{x}^{[1]} (w, v; b) := E \left[ U_{x} (W, w, v; b) \right], \\
U_{x} (W_i, W_j, v; b) := \frac{1}{\sqrt{b}} K \left( \frac{\Delta_i - v}{b} \right) 1 (X_i = x) + \sqrt{b} \cdot G_x (W_i, W_j, v; b), \\
\mu_{U_x} (v; b) := E \left[ U_{x} (W_1, W_2, v; b) \right].
\]

Note that \( \Delta_j \) can be expressed as a function of \( W_j \) (see (3)). Also note that the second Hájek projection term \( U_{x}^{[2]} (w, v; b) := E \left[ U_{x} (w, W, v; b) \right] \) is constant and equal to \( \mu_{U_x} (v; b) = \sqrt{b} \cdot m_{\Delta X} (v, x; b) \).

Since \( \epsilon \) is conditionally independent of \( Z \) given \( X \), one can show that the finite-sample variance of the right-hand side term in (20) is given by \( V (v \mid x; h) := p_{-2}^x \text{Var} \left[ U_{x}^{[1]} (W, v; h) \right] = p_{-2}^x V (v, x; h) \) (see (43)), where \( V (v, x; b) := V_1 (v, x; b) + V_2 (v, x; b) \) and

\[
V_1 (v, x; b) := E \left[ \frac{1}{b} K \left( \frac{\Delta - v}{b} \right)^2 1 (X = x) \right] - b \cdot m_{\Delta X} (v, x; b)^2, \\
V_2 (v, x; b) := E \left[ \frac{1}{b^3} K' \left( \frac{\Delta_3 - v}{b} \right) q_x (W_3, W_1) K' \left( \frac{\Delta_2 - v}{b} \right) q_x (W_2, W_1) 1 (X_1 = x) \right] \\
\times p_{-1}^x \left( p_1^{-1} + p_0^{-1} \right).
\]
The plug-in estimator of the $V_1 (v, x; b)$ term is given by

$$
\hat{V}_1 (v, x; b) := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{b} K \left( \frac{\hat{\Delta}_i - v}{b} \right)^2 \mathbb{1} (X_i = x) - b \cdot \hat{f}_\Delta X (v, x; b)^2.
$$

(21)

Denote $\hat{p}_{z|x} := n^{-1} \sum_{i=1}^{n} \mathbb{1} (Z_i = z, X_i = x)$ and $\hat{p}_{z|z} := \hat{p}_{z|x} / \hat{p}_x$. Let

$$
\hat{\zeta}_{1x} (y; b_\zeta) := \frac{\sum_{i=1}^{n} \frac{1}{b_\xi} K_\zeta \left( \frac{Y_i - y}{b_\xi} \right) D_i \left( \frac{Z_i - \hat{p}_{1|x|x}}{\hat{p}_{1|x|x}} \right) \mathbb{1} (X_i = x)}{\sum_{i=1}^{n} \mathbb{1} (X_i = x)}
$$

$$
\hat{\zeta}_{0x} (y; b_\zeta) := \frac{\sum_{i=1}^{n} \frac{1}{b_\xi} K_\zeta \left( \frac{Y_i - y}{b_\xi} \right) (1 - D_i) \left( \frac{\hat{p}_{0|x|x} - (1 - Z_i)}{\hat{p}_{1|x|x}} \right) \mathbb{1} (X_i = x)}{\sum_{i=1}^{n} \mathbb{1} (X_i = x)}
$$

(22)

be the reweighted kernel estimator proposed by Abadie et al. (2002), where $b_\zeta > 0$ is the bandwidth and $K_\zeta (\cdot)$ is a second-order kernel. Let

$$
\hat{R}_{d'x} (y) := \frac{\sum_{i=1}^{n} \left\{ \mathbb{1} (Y_i \leq \hat{\phi}_{dx} (y), D_i = d, X_i = x) + \mathbb{1} (Y_i \leq y, D_i = d', X_i = x) \right\}}{\sum_{i=1}^{n} \mathbb{1} (X_i = x)}
$$

be the plug-in nonparametric estimator of $R_{d'x}$. The $V_2 (v, x; b)$ term can be estimated by a $U$-statistic with an estimated kernel:

$$
\hat{V}_2 (v, x; b, b_\zeta) := \frac{1}{n (n - 1) (n - 2)} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i, k \neq j} \frac{1}{b^3} K' \left( \frac{\hat{\Delta}_j - v}{b} \right) \hat{q}_x (W_j, W_i; b_\zeta)
$$

$$
\times \hat{q}_x (W_k, W_i; b_\zeta) \mathbb{1} (X_i = x) \hat{p}_x^{-1} (\hat{p}_{1x}^{-1} + \hat{p}_{0x}^{-1}),
$$

(23)

where $\hat{q}_x (W_i, W_j; b_\zeta) := \hat{q}_{1x} (W_i, W_j; b_\zeta) - \hat{q}_{0x} (W_i, W_j; b_\zeta)$, $\hat{q}_{dx} (W_i, W_j; b_\zeta)$ is the plug-in nonparametric estimator of $q_{dx} (W_i, W_j)$ defined in (13) constructed by replacing $\zeta_{dx}, \phi_{dx},$ and $R_{d'x}$ with their nonparametric estimators $\hat{\zeta}_{dx} (\cdot; b_\zeta), \hat{\phi}_{dx},$ and $\hat{R}_{d'x}$ respectively.\textsuperscript{11}

Let

$$
\hat{V} (v \mid x; b, b_\zeta) := \hat{p}_x^{-2} \left( \hat{V}_1 (v, x; b) + \hat{V}_2 (v, x; b, b_\zeta) \right).
$$

(24)

For estimating $V (v \mid x; \hat{h})$, we set $b = \hat{h}$ in $\hat{V} (v \mid x; b, b_\zeta)$, where $\hat{h}$ satisfies Assumption 3. Similarly, we set the second bandwidth $b_\zeta = \hat{h}_\zeta$, where $\hat{h}_\zeta$ is a random bandwidth that satisfies the following assumption similar to Assumption 3.

\textsuperscript{11}It is known that the kernel estimator $\hat{\zeta}_{dx} (y; b_\zeta)$ is asymptotically biased if $y$ is near the boundaries of the support $[\underline{Y}_{dx}, \overline{Y}_{dx}]$. As Guerre et al. (2000), we can trim off the estimated counterfactual outcomes $\hat{\phi}_{dx} (Y_i)$ that lie in the boundary region $[\underline{Y}_{dx}, \underline{Y}_{dx} + b_\zeta] \cup [\overline{Y}_{dx} - b_\zeta, \overline{Y}_{dx}]$ by multiplying $\hat{\zeta}_{dx} \left( \hat{\phi}_{dx} (Y_i); b_\zeta \right)^{-1}$ in $\hat{q}_{dx} (W_i, W_j; b_\zeta)$ by a trimming factor $\mathbb{1} (y_{dx} + b_\zeta \leq \hat{\phi}_{dx} (Y_i) \leq \overline{Y}_{dx} - b_\zeta)$. It can be shown that the effect of the trimming factor is asymptotically negligible. All of our asymptotic results remain true, and the finite-sample performances may improve when trimming is used.
Assumption 5.  \( \text{Pr} \left[ \left| \hat{h}_\zeta / h_\zeta - 1 \right| > \varepsilon_n^2 \right] \leq \delta_n^2 \) for some deterministic bandwidth \( h_\zeta \) and positive sequences \( \varepsilon_n^2, \delta_n \downarrow 0 \).

Suppose \( \hat{h}_\zeta \) is the Silverman ROT bandwidth of the form \( \hat{h}_\zeta = C_{K_\zeta} \cdot \hat{\sigma}_{Y|X=x} \cdot n^{-1/5} \), where \( \hat{\sigma}_{Y|X=x} \) is the sample analogue of \( \sigma_{Y|X=x} := \sqrt{\text{Var}[Y|X=x]} \) and \( C_{K_\zeta} \) is a constant that depends on \( K_\zeta \). In this case, Assumption 5 is satisfied with \( \varepsilon_n^2 \propto \sqrt{\log(n)/n} \) and \( \delta_n^2 \propto n^{-1} \).\(^{12}\) Theorem 3 below provides a uniform convergence rate for \( \hat{V} \left( v \mid x; \hat{h}, \hat{h}_\zeta \right) \). In Appendix B, Theorem B1 presents a non-asymptotic deviation bound for the uniform estimation error of \( \hat{V} \left( v \mid x; \hat{h}, \hat{h}_\zeta \right) \), which implies the result of Theorem 3. The stronger result of Theorem B1 is used in the proof of Theorem 4 below.

**Theorem 3.** Suppose that Assumptions 1-5 hold with \( P = 2 \), the third-order derivative functions in Assumption 2(a) are Lipschitz continuous, \( h \propto n^{-\lambda} \) with \( 0 < \lambda < 1/4 \), and \( h_\zeta \propto n^{-\lambda_\zeta} \) with \( 0 < \lambda_\zeta < 1 \). Then, for any \( x \in \mathcal{X} \) and compact \( I_x \subseteq \text{Int}(\mathcal{X}_{\Delta|X=x}) \),

\[
\left\| \hat{V} \left( \cdot \mid x; \hat{h}, \hat{h}_\zeta \right) - V \left( \cdot \mid x; h \right) \right\|_{I_x} = O_p \left( \sqrt{\frac{\log(n)}{n h_\zeta}} + h_\zeta^2 + \frac{\log(n)}{n h^4} + \sqrt{\frac{\log(n)}{n h^2}} + \varepsilon_n h \right).
\]

**Remark 4.** While the estimator \( \tilde{V}_2 \left( v, x; b, b_\zeta \right) \) may be negative in finite samples, its modification \( \tilde{V}_2 \left( v, x; b, b_\zeta \right) \) defined below is always non-negative:

\[
\tilde{V}_2 \left( v, x; b, b_\zeta \right) := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{b} \left\{ \frac{1}{b} \sum_{j=1}^{n} K_1 \left( \hat{D}_j - v \right) \hat{q}_x \left( W_j, W_i; b_\zeta \right) \right\}^2 \mathbb{1} \left( X_i = x \right) \right\} \tilde{p}_x^{-1} \left( \hat{p}_x^{-1} + \hat{p}_0 \right).
\]

One can show that the difference \( \left\| \tilde{V}_2 \left( v, x; \hat{h}, \hat{h}_\zeta \right) - \tilde{V}_2 \left( v, x; \hat{h}, \hat{h}_\zeta \right) \right\|_{I_x} \) is of a smaller order than \( \left\| \tilde{V}_2 \left( v, x; \hat{h}, \hat{h}_\zeta \right) - \tilde{V}_2 \left( v, x; h \right) \right\|_{I_x} \).

A pointwise \( 1 - \alpha \) asymptotic confidence interval for \( f_{\Delta|X} \left( v \mid x \right) \) can be constructed as

\[
\left[ \hat{f}_{\Delta|X} \left( v \mid x; \hat{h} \right) \pm z_{1-\alpha/2} \sqrt{\frac{\hat{V} \left( v \mid x; \hat{h}, \hat{h}_\zeta \right)}{n h}} \right],
\]

where \( z_{1-\alpha/2} \) is the \( 1 - \alpha/2 \) quantile of the standard normal distribution, and \( h \) satisfies \( n h^5 \downarrow 0 \). However, if one is interested in constructing valid confidence bands for the density function, the \( z_{1-\alpha/2} \) critical value must be replaced with a bigger one determined by the distribution of the

\(^{12}\)One may use estimators \( \left( \tilde{\zeta}_{o,a} \left( y; b_{o,a} \right), \tilde{\zeta}_{1a} \left( y; b_{1a} \right) \right) \) with different bandwidths \( \left( b_{o,a}, b_{1a} \right) \). By easily modifying the proofs, we get results similar to Theorems 3 and 4 under two data-dependent bandwidths \( \left( \hat{h}_{o,a}, \hat{h}_{1a} \right) \) that satisfy the same assumption for \( \hat{h}_\zeta \). The ROT bandwidths can be set as \( \hat{h}_{o,a} = C_{K_{o,a}} \cdot \hat{\sigma}_{Y|X=x,D=d} \cdot n_{d}^{-1/5} \), where \( n_{d} := \sum_{i=1}^{n} \mathbb{1} \left( D_i = d, X_i = x \right) \), for \( d = 0, 1 \), and \( \hat{\sigma}_{Y|X=x,D=d} \) denotes the sample analogue of \( \sigma_{Y|X=x,D=d} := \sqrt{\text{Var}[Y|X=x,D=d]} \).
supremum of the estimation errors along the domain, as interpolations of the pointwise confidence intervals (25) over the domain are invalid in the uniform sense. In the section below, we discuss the construction of valid UCBs.

4.2 Jackknife multiplier bootstrap UCB

Let \( S(v | x; b) \) and \( Z(v | x; b, b_\zeta) \) denote the non-studentized and studentized estimation errors, respectively:

\[
S(v | x; b) := \sqrt{n} b \left( \hat{f}_{\Delta|x}(v | x; b) - f_{\Delta|x}(v | x) \right) \quad \text{and} \quad Z(v | x; b, b_\zeta) := \frac{S(v | x; b)}{\sqrt{\hat{V}(v | x; b, b_\zeta)}}. 
\tag{26}
\]

Moreover, recall the expansion of the estimation error in (20). An asymptotically valid \( 1 - \alpha \) UCB simultaneously covers \( \{ f_{\Delta|x}(v | x) : v \in I_x \} \) with a pre-specified asymptotic coverage \( 1 - \alpha \). To construct a valid UCB, one has to replace the standard normal quantile \( z_{1-\alpha/2} \) in the pointwise confidence interval (25) with a critical value approximating the \( 1 - \alpha \) quantile of the distribution of \( \| Z(\cdot | x; \hat{h}, \hat{h}_\zeta) \|_{I_x} \). In this section, we discuss the validity of the computationally fast jackknife multiplier bootstrap (JMB). Appendix C provides the algorithm and theoretical results for the nonparametric bootstrap.

We consider the problem of estimating the distribution of \( \| S(\cdot | x; \hat{h}) \|_{I_x} \) or \( \| Z(\cdot | x; \hat{h}, \hat{h}_\zeta) \|_{I_x} \) using the linearization (20), under the “undersmoothing” assumption \( nh^5 \downarrow 0 \) to ensure that the bias is asymptotically negligible in comparison to the standard deviation. The conventional bandwidth selectors that estimate the AMISE-optimal bandwidth such as the Silverman ROT method or cross-validation (using the pseudo ITEs in our case) violate the undersmoothing assumption. The undersmoothing assumption requires that the selected bandwidth should vanish at a faster rate. In practical implementation of undersmoothing for many nonparametric econometric techniques, a commonly used strategy is to shrink a conventional approximately AMISE-optimal data-driven bandwidth by an ad hoc amount.

The JMB approach of Chen and Kato (2020, CK hereafter,) approximates the distribution of the supremum (with respect to \( v \)) of \( n^{-1/2} \sum_{i=1}^{n} \left( \mathcal{U}_x^{[1]}(W_i, v; h) - \mu_{\mathcal{U}_x}(v; h) \right) \) with that of the Gaussian multiplier process (e.g., Chernozhukov et al., 2014a) that uses the jackknife estimator of \( \mathcal{U}_x^{[1]}(W_i, v; h) \). However unlike in CK, in our case the kernel \( \mathcal{U}_x \) involves the unknown nonparametric objects \( (\phi_{dx}, \zeta_{dx}, R_{dx}) \), unknown probabilities \( (p_{0x} \text{ and } p_{1x}) \), and latent ITEs. Therefore, we use the estimated version of \( \mathcal{U}_x \) that replaces the unknown objects with their nonparametric estimators:

\[
\tilde{\mathcal{U}}_x(W_j, W_i, v; b, b_\zeta) := \frac{1}{\sqrt{b}} K \left( \frac{\hat{\Delta}_i - v}{b} \right) \mathbb{1}(X_i = x) + \frac{1}{b^{3/2}} K' \left( \frac{\hat{\Delta}_j - v}{b} \right) \tilde{\pi}_x(W_j, W_i; b_\zeta) \tilde{\pi}_x(Z_i, X_i), 
\tag{27}
\]

where \( \tilde{\pi}_x(Z_i, X_i) \) is constructed by replacing \( (p_{0x}, p_{1x}) \) with \( (\tilde{p}_{0x}, \tilde{p}_{1x}) \) in the definition of \( \pi_x(Z_i, X_i) \) in (14). Let \( (\nu_1, ..., \nu_n) \) denote i.i.d. standard normal random variables that are drawn independently.
from the data. Let \( \left\{ \hat{S}_{\text{jmb}}(\cdot | x; b, b_\zeta) : v \in I_x \right\} \) be the feasible JMB process, where

\[
\hat{U}^{[1]}_x (W_i, v; b, b_\zeta) := \frac{1}{n-1} \sum_{j \neq i} \hat{U}_j (W_j, W_i, v; b, b_\zeta),
\]

\[
\hat{S}_{\text{jmb}} (v | x; b, b_\zeta) := \frac{1}{\sqrt{n}} \sum_{i=1}^n \nu_i \hat{p}_x^{-1} \left\{ \hat{U}^{[1]}_x (W_i, v; b, b_\zeta) - \sqrt{\hat{V}} (v; x; b, b_\zeta) \right\}.
\]

We show in Appendix B (the proof of Theorem B2) that the distribution of \( \left\| Z (\cdot | x; \hat{h}, \hat{h}_\zeta) \right\|_{I_x} \) can be approximated by the conditional distribution of \( \left\| \hat{Z}_{\text{jmb}} (\cdot | x; \hat{h}, \hat{h}_\zeta) \right\|_{I_x} \) given the original sample \( W^n_1 := \left\{ W_1, ..., W_n \right\} \), where

\[
\hat{Z}_{\text{jmb}} (v | x; b, b_\zeta) := \frac{\hat{S}_{\text{jmb}} (v | x; b, b_\zeta)}{\hat{V} (v; x; b, b_\zeta)}.
\]

Let \( \text{Pr}_{W^n_1} [\cdot] \) and \( \text{E}_{W^n_1} [\cdot] \) denote the conditional probability and expectation respectively given \( W^n_1 \), and

\[
z_{1-\alpha}^{\text{jmb}} := \inf \left\{ t \in \mathbb{R} : \text{Pr}_{W^n_1} \left[ \left\| \hat{Z}_{\text{jmb}} (\cdot | x; \hat{h}, \hat{h}_\zeta) \right\|_{I_x} \leq t \right] \geq 1 - \alpha \right\}
\]

be the 1 – \( \alpha \) quantile of the conditional distribution of \( \left\| \hat{Z}_{\text{jmb}} (\cdot | x; \hat{h}, \hat{h}_\zeta) \right\|_{I_x} \) given \( W^n_1 \). Recall that \( CB_{\text{jmb}} (v | x; b, b_\zeta) \) is defined by (9). The JMB confidence band is given by the family of random intervals \( \left\{ CB_{\text{jmb}} (v | x; \hat{h}, \hat{h}_\zeta) : v \in I_x \right\} \). Note that one can approximate \( z_{1-\alpha}^{\text{jmb}} \) to any degree of accuracy by Monte Carlo simulations, and that the width of \( CB_{\text{jmb}} (v | x; \hat{h}, \hat{h}_\zeta) \) varies with \( \hat{V} (v | x; \hat{h}, \hat{h}_\zeta) \).\(^{13}\) The following algorithm summarizes the construction of the JMB confidence band for the density of the ITE.

**Algorithm 1** (JMB confidence band). **Step 1:** Compute the pseudo ITEs using (6)-(7). **Step 2:** Select the covariates’ value \( x \), the number of grid points \( G \), and a grid \( I_x^G := \left\{ v_1 < v_2 < \ldots < v_G \right\} \) over which the density is estimated. **Step 3:** Select a kernel \( K \), use the ROT bandwidth \( \hat{h}_{\text{rot}} \) with undersmoothing (e.g., \( \hat{h} = \hat{h}_{\text{rot}} n^{-1/40} \)), and for all \( v \in I_x^G \) compute the kernel density estimator \( \hat{f}_{\Delta|X} (v | x; \hat{h}) \) using (8) and the pseudo ITEs from Step 1. **Step 4:** Select a kernel \( K_\zeta \), use the ROT bandwidth \( \hat{h}_{\text{rot}} \), and for all \( v \in I_x^G \) compute the variance estimator \( \hat{V} (v | x; \hat{h}, \hat{h}_\zeta) \) using (21), (23), and (24). **Step 5:** Select the number of bootstrap repetitions \( B \), for \( r = 1, ..., B \) generate i.i.d. standard normal random variables \( (\nu_1^{(r)}, ..., \nu_n^{(r)}) \), and for all \( v \in I_x^G \) compute \( \hat{Z}_{\text{jmb}}^{(r)} (v | x; \hat{h}, \hat{h}_\zeta) = \)

\(^{13}\)Alternatively, a constant-width UCB (e.g., Cheng and Chen, 2019) \( \hat{f}_{\Delta|X} (v | x; \hat{h}) \pm s^{\text{jmb}}_{1-\alpha}/\sqrt{n} \hat{h} \) is based on the critical value \( s^{\text{jmb}}_{1-\alpha} := \inf \left\{ t \in \mathbb{R} : \text{Pr}_{W^n_1} \left[ \left\| \hat{S}_{\text{jmb}} (\cdot | x; \hat{h}, \hat{h}_\zeta) \right\|_{I_x} \leq t \right] \geq 1 - \alpha \right\} \) that approximates the 1 – \( \alpha \) quantile of the distribution of \( \left\| S (\cdot | x; \hat{h}) \right\|_{I_x} \). However, such a UCB cannot exploit the fact that the tails of a density function approach zero, and therefore the variable-width UCB is preferred in this context.
\( \hat{S}_{jmb}^{(r)}(v \mid x; \hat{h}, \hat{h}_\zeta) / \sqrt{V(v \mid x; \hat{h}, \hat{h}_\zeta)} \) using (28) with \((\nu_1, \ldots, \nu_n)\) replaced by \((\nu_1^{(r)}, \ldots, \nu_n^{(r)})\).

**Step 6:** Select the coverage level \(1 - \alpha\) and compute the critical value \(z_{1-\alpha}^{jmb}\):

\[
z_{1-\alpha}^{jmb} = \inf \left\{ t \in \mathbb{R} : \frac{1}{B} \sum_{r=1}^{B} \mathbb{1} \left( \max_{v \in I_x^{(r)}} \left| \hat{S}_{jmb}^{(r)}(v \mid x; \hat{h}, \hat{h}_\zeta) \right| \leq t \right) \geq 1 - \alpha \right\}.
\] (30)

**Step 7:** Compute the JMB confidence band \(CB_{jmb}\) using (9) over \(v \in I_x^{(r)}\).

Theorem 4 below shows that the proposed JMB confidence band is asymptotically valid and its coverage error decays at a polynomial rate. The result rules out coverage probability errors with logarithmic decay rates (see, e.g., Chernozhukov et al., 2014a for discussion).

**Theorem 4.** Suppose that Assumptions 1-5 hold with \(P = 2\), the third-order derivative functions in Assumption 2(a) are Lipschitz continuous, \(h \propto n^{-\lambda}\) with \(1/5 < \lambda < 1/4\), \(h_\zeta \propto n^{-\lambda_\zeta}\) with \(1/8 < \lambda_\zeta < 1/2\), \(\varepsilon_n = O(n^{-\theta_\varepsilon})\), \(\delta_n = O(n^{-\theta_\delta})\), and \(\delta_\zeta = O\left(n^{-\theta_\delta_\zeta}\right)\) for some \(\theta_\varepsilon, \theta_\delta, \theta_\delta_\zeta > 0\). Then, for all \(x \in \mathcal{S}_X\) and any compact \(I_x \subseteq \text{Int} (\mathcal{S}_X | X = x)\),

\[
\Pr \left[ \hat{f}_{\Delta | X}(v \mid x) \in CB_{jmb}(v \mid x; \hat{h}, \hat{h}_\zeta), \text{ for all } v \in I_x \right] = (1 - \alpha) + O\left(n^{-\varrho}\right)
\]

for some \(\varrho > 0\).

**Remark 5.** In the proof of Theorem 4, we explicitly derive an estimate of the coverage probability error, which is presented in Theorem B2 in Appendix B. We show that replacing the deterministic bandwidths \((h, h_\zeta)\) with estimators satisfying Assumptions 3 and 5 incurs an additional \(O\left(\log(n)\varepsilon_n + \delta_n + \delta_\zeta\right)\) coverage probability error term. This result shows how the noise in the estimated bandwidth translates into an error in the coverage probability. The JMB confidence band achieves a potentially higher coverage accuracy if \((\hat{h}, \hat{h}_\zeta)\) converge to their population counterparts at a fast rate, which is the case for the simple and feasible Silverman ROT approach we recommended.

**Remark 6.** Since density functions are non-negative, the lower bound of the UCB can be truncated to zero to avoid negative values. Thus, the lower bound of \(CB_{jmb}(v \mid x; \hat{h}, \hat{h}_\zeta)\) can be replaced with \(\max \left\{ 0, \hat{f}_{\Delta | X}(v \mid x; \hat{h}) - z_{1-\alpha}^{jmb} \sqrt{\hat{V}(v \mid x; \hat{h}, \hat{h}_\zeta) / (n\hat{h})} \right\}\) without affecting the coverage properties of the UCB. Hence, the result of Theorem 4 continues to hold with the modified lower bound.\(^{14}\)

Our main focus is on the multiplier bootstrap approach, as it is computationally fast even with large sample sizes. The more commonly used nonparametric bootstrap would require re-calculation of the bootstrap versions of the estimated ITEs at every bootstrap repetition, which can be computationally burdensome. However, the constant-width version of the nonparametric bootstrap confidence band has the advantage of fewer tuning parameters as it does not require estimation.

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\(^{14}\)We thank the associate editor for suggesting the modification to us.
of $\zeta_{dx}$ and therefore does not need the second bandwidth $b_\varsigma$. The validity of the nonparametric bootstrap approach is discussed in Appendix C.

4.3 Bias-corrected JMB UCB

In this section, we discuss the bias correction approach to inference that can accommodate conventional bandwidth selectors such as ROT bandwidths that decay at the $n^{-1/5}$ rate as in Calonico et al. (2014). We assume that the third-order derivatives in Assumption 2 with $P = 2$ are Lipschitz continuous. Theorem 2 implies that in large samples and when $X \sim \mathcal{N}(\mu, \Sigma)$, it is approximately distributed as $N(\tilde{f}_\Delta | x (v | x), \mathcal{V} (v | x) / (nh))$. We use an estimator of the density derivative $f^{(2)}_\Delta | x (v | x)$ to remove the bias. Let

$$f^{(2)}_\Delta | x (v | x) = \frac{\sum_{i=1}^n \frac{1}{b} K_{(b)} \left( \frac{X_i - v}{b} \right) \mathbb{1} (X_i = x)}{\sum_{i=1}^n \mathbb{1} (X_i = x)},$$

where $M(u; b, b) := K(u) - (b/b)^3 \mu_{K, 2} K_{(b)} ((b/b) u)$. Typical choices for the bandwidth $b_\varsigma$ used for bias correction include the estimation-optimal bandwidth for the second-order density derivative $h_{\varsigma}$ (denotes the second derivative of $b_\varsigma$). We assume that $K_{(b)}$ satisfies Assumption 4(a). The bias-corrected estimator of $f^{(2)}_\Delta | x (v | x)$ is given by

$$\hat{f}^{(2)}_{\Delta | x} (v | x) = \frac{\sum_{i=1}^n \hat{M} \left( \frac{X_i - v}{b}; b \right) \mathbb{1} (X_i = x)}{\sum_{i=1}^n \mathbb{1} (X_i = x)},$$

where $\hat{M}(u; b, b) := K(u) - (b/b)^3 \mu_{K, 2} K_{(b)} ((b/b) u)$. Typical choices for the bandwidth $b_\varsigma$ used for bias correction include the estimation-optimal bandwidth for the second-order density derivative (e.g., Xu, 2017) or the same bandwidth as that used for estimating the density (e.g., Cheng and Chen, 2019). We assume that $b_\varsigma$ is some random bandwidth $\hat{b}_\varsigma$ that consistently estimates some target bandwidth $h_\varsigma$ in the sense that $\hat{h}_\varsigma/h_\varsigma \to p_1$. We assume that $h/h_\varsigma \to \zeta \in [0, \infty)$ as in Calonico et al. (2014). In practice, we can take $\hat{h}_\varsigma$ to be the feasible Silverman ROT bandwidth for the second-order density derivative that uses the pseudo ITEs and aims to approximate the AMISE-optimal bandwidth (in this case, $h_\varsigma \propto n^{-1/9}$). Alternatively, one can set $\hat{h}_\varsigma = \hat{h}$.

One can show that the bias of $\hat{f}^{(2)}_{\Delta | x} (v | x)$ is of the order $O(h^2 b_\varsigma)$, and its standard deviation is of the order $(nh)^{-1/2}$.

15Hence, when the bandwidth $h$ is chosen AMISE-optimally so that $h \propto n^{-1/5}$, the bias of the bias-corrected estimator $\hat{f}^{(2)}_{\Delta | x} (v | x)$ is of smaller order than its standard deviation. Robust standard errors (Calonico et al., 2014) have to take into account the additional stochastic variability coming from the bias correction, i.e., estimation of $f^{(2)}_{\Delta | x} (v | x)$.

Let $\hat{V}^{(2)} (v | x; b_\varsigma, b)$ and $\hat{S}^{(b)}_{jmb} (v | x; b_\varsigma, b)$ be defined similarly to $\hat{V} (v | x; b_\varsigma, b)$ and $\hat{S}^{(b)}_{jmb} (v | x; b_\varsigma, b)$ respectively with $K(\cdot)$ replaced by the bias-correcting kernel $M(\cdot; b, b_\varsigma)$. Define

15Under Assumptions 1-4 with $P = 2$, the bias part of the bias-corrected estimator in (31) is $o(h^2)$. Lipschitz continuity ensures that the bias part is $O(h^2 h_\varsigma)$, if $h = O(h_\varsigma)$.
The bias-corrected UCB can be computed by replacing \( \hat{f}_{\Delta|x} (v | x; \hat{h}) \), \( \hat{V} (v | x; \hat{h}, \hat{h}_\zeta) \), and \( \hat{V}_b (v | x; \hat{h}, \hat{h}_\zeta, \hat{h}_b) \) in Algorithm 1 with their respective bias-corrected versions \( \hat{f}_{\Delta|x}^{bc} (v | x; \hat{h}) \), \( \hat{V}^{bc} (v | x; \hat{h}, \hat{h}_\zeta) \), and \( \hat{V}_b^{bc} (v | x; \hat{h}, \hat{h}_\zeta, \hat{h}_b) \). We show that the conclusion of Theorem 4 (asymptotic validity with a polynomial rate) holds for \( CB_{\Delta|x}^{bc}(v | x; \hat{h}, \hat{h}_\zeta, \hat{h}_b) \) for some \( \hat{h} \). The bias-corrected JMB UCB is given by

\[
CB_{\Delta|x}^{bc}(v | x; \hat{h}, \hat{h}_\zeta, \hat{h}_b) := \left[ \hat{f}_{\Delta|x}^{bc}(v | x; \hat{h}, \hat{h}_\zeta) \pm z_{1-\alpha} \frac{\hat{\sigma}}{\sqrt{n}} \right].
\]

4.4 Conditioning on sub-vectors of the covariates

In applications, researchers are often interested in the unconditional PDF of the ITE, or the conditional PDF of the ITE after conditioning only on some of the covariates. See, e.g., the application in Section 6. This section discusses how our results can be applied in such cases.

Partition the vector of covariates as \( X = (X_1^T, X_2^T) \) (similarly, \( X_i = (X_{1,i}^T, X_{2,i}^T) \)). Let \( f_{\Delta|x_1}(\cdot | x_1) \) denote the conditional density of the ITE for some fixed \( x_1 \in \mathcal{X}_1 \). The estimator of the conditional density is given by

\[
\hat{f}_{\Delta|x_1}(v | x_1; b) := \frac{\sum_{i=1}^n \mathbb{1}[X_{1,i} = x_1]}{\sum_{i=1}^n \mathbb{1}[X_{1,i} = x_1]} K \left( \frac{\Delta - v}{b} \right) \mathbb{1}[X_{1,i} = x_1].
\]

Let \( \hat{q}_{X_1}(W_i, W_j; b_\zeta) \) be defined similarly to \( \hat{q}_x(W_i, W_j; b_\zeta) \) by the same formula with \( x \) replaced by \( X_i \). Note that in the definition of \( \hat{q}_{X_1}(W_i, W_j; b_\zeta) \), \( b_\zeta \) may depend on \( X_i \). Next, let \( \hat{p}_{x_1}(W_i, W_j; b_\zeta) := \mathbb{1}[X_{1,i} = x_1] \hat{q}_{X_1}(W_i, W_j; b_\zeta) \), \( \hat{p}_{x_1} := n^{-1} \sum_{i=1}^n \mathbb{1}[X_{1,i} = x_1] \), \( \hat{f}_{\Delta|x_1}(v, x_1; b) := \hat{f}_{\Delta|x_1}(v | x_1; b) \hat{p}_{x_1} \), and

\[
\hat{U}_{x_1}(W_j, W_i, v; b, b_\zeta) := \frac{1}{\sqrt{b}} K \left( \frac{\Delta_i - v}{b} \right) \mathbb{1}[X_{1,i} = x_1]
\]

\[
+ \frac{1}{b^{3/2}} K' \left( \frac{\Delta_j - v}{b} \right) \hat{q}_{x_1}(W_j, W_i; b_\zeta) \mathbb{1}[X_{1,i} = x_1],
\]

\[
\hat{U}_{x_1}^{[j]}(W_i, v; b, b_\zeta) := \frac{1}{n-1} \sum_{j \neq i} \hat{U}_{x_1}(W_j, W_i, v; b, b_\zeta),
\]
Define also

\[\hat{S}_{\text{jmb}} (v \mid x_1; b, b_\zeta) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_i \hat{p}_{X_1}^{-1} \left\{ \hat{U}^{[1]}_{X_1} (W_i, v; b, b_\zeta) - \sqrt{b} \cdot \hat{f}_{\Delta X_1} (v, x_1; b) \right\}.\]

5 Monte Carlo experiments

This section evaluates the finite-sample performance of the UCBs proposed in Section 4 for the density \(f_\Delta (v)\) of the ITE. We consider the following experiment design based on FVX. The outcome
and treatment status variables are generated according to

\[
Y = (\epsilon + 1)^{2+D}
\]

\[
D = 1 (\gamma_0 + \gamma_1 \cdot Z + \eta \geq 0),
\]

where \((\epsilon, \eta) = (\Phi(U), \Phi(V))\), \((U, V)\) has a mean-zero bivariate normal distribution with \(\text{Var}[U] = \text{Var}[V] = 1\) and \(\text{Cov}[U, V] = 0.3\), and \(\Phi\) is the standard normal CDF. The instrument is generated according to \(Z = 1\) \((N > 0)\), where \(N\) is a standard normal random variable independent of \((\epsilon, \nu)\). In this design, the ITE satisfies \(\Delta = \epsilon (\epsilon + 1)^2\), where \(\epsilon\) is uniformly distributed on \([0, 1]\) and \(\Delta\) is supported on \([0, 4]\). We consider two sets of values for \((\gamma_0, \gamma_1) : (-0.5, 0.5)\) and \((-0.4, 0.6)\). We use the tricubic kernel for \(K, K_\zeta\) and \(K_b\), and the Silverman ROT bandwidths for \(\hat{h}, \hat{h}_b\) and \(\hat{h}_\zeta\). The number of Monte Carlo replications is set to 1,000.

Table 1 reports the simultaneous coverage rates of two types of UCBs: the bias-corrected JMB UCB defined in (32) and the bias-corrected nonparametric bootstrap (NPB) UCB described in Appendix C. FVX interpolates pointwise (nonparametric) bootstrap percentile confidence intervals (CIs) to construct a confidence band for the density of the ITE. It follows from our results that such intervals are valid in the pointwise sense. We also report the coverage probability of the confidence band constructed by interpolating the bootstrap percentile pointwise confidence intervals. The nominal coverage rates are 0.90, 0.95, and 0.99. We consider two ranges of \(\nu\): a longer interval \(I = [0.5, 3.5]\) and a shorter one \(I = [0.8, 3.2]\). We use grid search to solve the one-dimensional optimization problems in (6) when estimating the pseudo ITEs and calculating the supremum of the bootstrap process. The number of bootstrap replications is set to 5,000. Table 2 reports the average widths of the bias-corrected JMB and NPB UCBs relative to the interpolated pointwise CIs.

16To be specific, we take \(\hat{h} = 3.15 \cdot \hat{\sigma}_\Delta \cdot n^{-1/5}, \hat{h}_b = 2.7 \cdot \hat{\sigma}_\Delta \cdot n^{-1/9}\). When it comes to \(\hat{h}_\zeta\), we distinguish between the treated and control subsamples, i.e., \(\hat{h}_{\zeta,d} = 3.15 \cdot \hat{\sigma}_\Delta |D=d \cdot n_d^{-1/5}\) with \(n_d := \sum_{i=1}^n 1(D_i = d)\).

17Validity of pointwise bootstrap percentile confidence intervals follows from (20), Lemma ?? in the supplement and standard arguments (see the proof of Ma et al., 2019, Theorem 4.2).

18The average width of bias-corrected JMB (or NPB) UCB is computed by first averaging the widths of the confidence band over all grid points in the given range \(I\) and then averaging over all simulation replications. The
We make the following observations regarding the simulation results. First, as expected, interpolation of pointwise CIs exhibits substantial under-coverage, especially for the nominal coverage probabilities 0.90 and 0.95. Therefore, appealing as it looks to practitioners, interpolation of pointwise CIs fails to cover the true density curve with the desired coverage probability even in large samples. Second, the bias-corrected JMB and NPB UCBs yield reasonably good coverage rates across different setups and sample sizes. The JMB UCBs are slightly narrower than the NPB UCBs; however they are also less accurate. Third, we focus on the studentized UCB because it has variable width and thus is narrower than the non-studentized counterpart (Footnote 13). The additional simulation results in Section ?? of the online supplement confirm that the non-studentized UCBs are on average wider than the studentized ones. The computation of the JMB is faster than the NPB, as the former avoids the estimation of ITEs for each bootstrap sample. Therefore, we recommend the bias-corrected JMB UCB defined in (32) to practitioners for assessing the shape of the density of ITEs.

6 Empirical application: Childbearing and labor income

In this section, we apply the FVX estimator for the density of the ITE and our bias-corrected JMB UCB to investigate the effect of family size on labor income. Understanding the relationship between the two variables is important for policymakers; however, estimation of the effect can be complicated due to the simultaneity between the labor supply and fertility decisions (Angrist and Evans, 1998, AE hereafter).

We revisit the 1980 Census Public Use Micro Samples (PUMS) previously used by AE and other authors. Following AE, we focus on married women aged 21-35 with at least two children. The focus on households with at least two children is due to the identification strategy developed in AE, as explained below. Our outcome variable $Y$ is the sum of the mother’s and father’s 1979 labor incomes (in thousands of dollars). The binary treatment variable $D$ takes the value one if the mother has more than two children. The instrument proposed in AE is the “same-sex” dummy variable that reported number is the ratio of the average width of the UCBs to that of the interpolated pointwise CIs.
Table 3: Summary statistics for ITE estimates of the effect of having more than two children on parents’ labor income (in thousands of 1979 dollars).

|                          | Mean | Std.dev | 1st decile | 1st quartile | Median | 3rd quartile | 9th decile | Pr[Δ > 0] |
|--------------------------|------|---------|------------|--------------|--------|--------------|------------|----------|
| Full sample              | -2.67| 36.72   | -16.99     | -8.71        | -4.14  | 0.25         | 10.37      | 0.264    |
| (n = 224,692)            |      |         |            |              |        |              |            |          |
| High School, Age > 31    | -5.01| 30.75   | -10.93     | -7.59        | -5.45  | -3.00        | 1.03       | 0.122    |
| (n = 48,129)             |      |         |            |              |        |              |            |          |
| College, Age > 31        | 10.88| 64.20   | -39.99     | -14.00       | -1.42  | 21.69        | 53.22      | 0.462    |
| (n = 19,103)             |      |         |            |              |        |              |            |          |
| High School, Age ≤ 31    | -7.93| 23.56   | -12.89     | -8.96        | -4.70  | -2.06        | 0          | 0.097    |
| (n = 65,447)             |      |         |            |              |        |              |            |          |
| College, Age ≤ 31        | 25.93| 68.24   | -16.08     | -5.58        | 9.95   | 27.07        | 178.43     | 0.667    |
| (n = 8,547)              |      |         |            |              |        |              |            |          |

takes the value one when the first two children are of the same sex. This identification strategy relies on parental preferences for a mixed sibling-sex composition: parents whose first two children are of the same sex are more likely to have an additional child. AE shows that in 1980, the estimated probabilities of having a third child for women with same-sex and mixed-sex children were 0.432 and 0.372, respectively. Moreover, the difference between the two groups is highly significant.

Our covariates \( X \) include the mother’s education level (less than high school, high school, some college, college), the quartile in the age distribution, race (white, black, Hispanic, others), and the sex of the first birth.\(^{19}\) Following FVX, we drop observations in \( X \)-defined cells that contain less than 1% of the sample. Our remaining sample has 224,962 observations.

Using 2SLS with a linear IV regression model, the estimated effect of having more than two children on parents’ labor income is \(-3.54\) with a standard error of \(1.51\). The estimates suggest that in 1979, having more than two children reduced parents’ labor income by \$3,540\) (or \$12,624 in 2020 dollars).\(^{20}\) The effect is substantial and corresponds to 7.7% of the average household labor income in our sample.\(^{21}\)

Table 3 reports summary statistics for the ITE estimates. According to the results, estimated ITEs display substantial heterogeneity. For example, the median ITE in our sample is \(-4.14\) with an interquartile range of \(9.03\).\(^{22}\) Conditioning on the above median age and college-level education produces an even wide range of estimated ITEs: \(-1.42\) for the median effect with the interquartile range of \(35.69\). While for the below median age, college-educated mothers, the median effect is positive (9.95), the corresponding interquartile range is similarly wide (32.65). The table also shows that in the case of mothers with only high-school-level education, ITEs tend to be more negative. For example, conditional on the above median age and only high-school-level education, the median ITE is \(-5.45\), with an interquartile range of \(7.93\). Only 12.2% of the households in this group have

\(^{19}\)The three quartiles of the distribution of mother’s age in our sample are 28, 31, and 33.

\(^{20}\)We used the CPI series from FRED, Federal Reserve Bank of St. Louis, for the conversion.

\(^{21}\)The average household labor income in our sample is \$45,829 in 1979 dollars.

\(^{22}\)The results are consistent with the findings in Frölich and Melly (2013) who also report substantial heterogeneity using the 2000 PUMS data and quantile treatment effects.
Figure 1: The unconditional PDF of the ITE with the 95% pointwise and uniform confidence bands

positive estimated ITEs, compared to 46.2% of the households with college-educated mothers from the same age group. The group with the largest fraction of households with positive estimated ITEs is the below-median-age mothers with a college education: 66.7%.

Next, we use the FVX estimator with our bias-corrected JMB UCB to analyze the distribution of the ITE. Figure 1 shows the unconditional PDF of the ITE together with the 95% pointwise and uniform confidence bands for the density. Following Remark 6, the lower bounds of the confidence bands are truncated to zero. One can see that while the UCB developed in this paper is somewhat wider than the pointwise, it is still informative. The estimated mode of the unconditional distribution is -4.08, and according to the UCB the mode is located between -5.60 and -3.52.

Figure 2 shows the conditional PDFs of the ITE conditional on the mother’s age (above or below the median age in our sample) and education (high school only or college levels) with their 95% UCBs. Figure 2(a) displays the results conditional on the above median age for high school only and college education levels. As there are regions where the two UCBs do not intersect, we can conclude with at least 90.25% confidence that the two densities conditional on high school and college are different. In particular, while the distribution of the ITE conditional on college is much more

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23The two 95% UCBs are independent since they are computed on non-overlapping samples. Hence, the overall
Figure 2: The conditional PDFs of the ITE (solid lines) and their 95% UCBs (shaded areas) conditional on the mother’s age, high school (red), or college (black) education: the left panel is (a) Above the median age, and the right panel is (b) Below the median age.

dispersed, it also has more probability mass in the positive range. E.g., on the interval corresponding to ITEs between $20,000 and $30,000, the confidence band for the density conditional on high school is very narrow and close to zero. At the same time, the density conditional on college is significantly different from zero. Hence, a non-negligible fraction of households with college-educated mothers above the median age received a substantial positive effect of a magnitude between 43.6% and 65.5% of the average household labor income in our sample. There is no evidence that households with only high-school-educated mothers experienced ITEs of this magnitude.

In the case of high school only, the UCB does not rule out a bimodal density with the two modes at -8.80 and -2.56. According to these results, households with high-school-educated mothers above the median age are likely to experience either a strong negative effect around 19.2% of the average household labor income or a more moderate negative effect around 5.6% of the average household labor income.

Figure 2(b) shows similar results conditional on the below median age. We can again conclude with a 90.25% confidence that the conditional distributions by education level (high school or college) are different. The ITE distribution is more dispersed for households with college-educated mothers than for households with high-school-educated mothers. Similarly to the previous case, the distribution conditional on college has more mass in the positive range than the distribution conditional on high school. The results conditional on high school again cannot rule out bimodality; however, this time the first mode at -6.56 corresponds to a more moderate effect, and the second mode at 0.13 occurs in the positive range.

We conclude that there are significant differences in the distributions of the ITE across the education levels. For households with high-school-only-educated mothers, the distribution of the confidence level when comparing the two distributions is $0.95^2 = 0.9025$. 

ITE is heavily concentrated in the negative range and potentially bimodal. Households with college-educated mothers have a wider range of ITEs. However, such households can also experience positive effects of a large magnitude. Predicting the effect of having more than two children on labor income for such households is difficult as the distribution is thinly spread from large negative to large positive values.

Recently, Abrevaya and Xu (2021) studied the distributional effect of having a third child on female labor supply by applying a weakly nonseparable model (equipped with the mean-variance-effect structure) to the PUMS dataset in 2000. Similar to our findings, they also documented a large amount of heterogeneity in the ITE distributions (See their Figures 2 to 5). In terms of how the ITE distribution varies across mothers’ education levels, they found less variation with the 2000 data than we document in this paper with the 1980 data.

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Appendix A  Proofs of Theorems in Section 3

Lemma 1. Under Assumptions 1 and 2, the PDF $f_{\Delta|X}(\cdot | x)$ exists and is $P$-times continuously differentiable on any inner closed sub-interval $I_x$ of $\mathcal{J}_{\Delta|x=x}$.

Proof. Let $(\Delta_{x,j}^{-1})'$ denote the derivative of $\Delta_{x,j}^{-1}$. By Assumption 2(a,b) and Severini (2005, Theorem 7.3), the conditional distribution of $\Delta = \Delta_x(\epsilon)$ given $X = x$ admits a Lebesgue density $f_{\Delta|X}(\cdot | x)$:

$$f_{\Delta|X}(v | x) = \sum_{j=1}^{m} f_{\epsilon|x} \left( \frac{\Delta_{x,j}^{-1}(v) | x}{(\Delta_{x,j}^{-1})'(v)} \right) 1(v \in \Delta_x((\epsilon_{x,j-1}, \epsilon_{x,j})))$$

(36)

for $v \in \Delta_x\left( \bigcup_{j=1}^{m} (\epsilon_{x,j-1}, \epsilon_{x,j}) \right)$, and $\mathcal{J}_{\Delta|x=x}$ is the closure of $\Delta_x\left( \bigcup_{j=1}^{m} (\epsilon_{x,j-1}, \epsilon_{x,j}) \right)$. From (36), the density $f_{\Delta|X}(\cdot | x)$ has a jump at an interior point of $\mathcal{J}_{\Delta|x=x} (\Delta_x(\epsilon_{x,j-1})$ or $\Delta_x(\epsilon_{x,j+1})$ if $\Delta_x((\epsilon_{x,j-1}, \epsilon_{x,j})) \neq \Delta_x((\epsilon_{x,j}, \epsilon_{x,j+1}))$. Such cases are ruled out by Assumption 2(c).

A.1  Notations and mathematical definitions

Let $X_n$, $Y_n$ be (sequences of) random variables and $\alpha_n, \beta_n, \gamma_n, \alpha'_n, \beta'_n, \gamma'_n$ be sequences in $(0, \infty)$. We write $X_n = O_p^*(\alpha_n, \beta_n)$ if there exists positive constants $C_1, C_2$ such that $\Pr|X_n| > C_1 \alpha_n \leq C_2 \beta_n$. We write $X_n = O_p^*(\alpha_n)$ for simplicity if $\beta_n = n^{-1}$. It is straightforward to check that the following properties hold for the $O_p^*$ notations. If $X_n = O_p^*(\alpha_n, \beta_n)$ and $Y_n = O_p^*(\alpha'_n, \beta'_n)$, then $X_n + Y_n = O_p^*(\alpha_n + \alpha'_n, \beta_n + \beta'_n)$ and $X_n + Y_n = O_p^*(\alpha_n, \alpha'_n)$ if $\beta_n = \beta'_n$. It is easy to see that if $X_n = O_p^*(\alpha_n, \beta_n)$ and $\beta_n \downarrow 0$, then $X_n = O_p(\alpha_n)$. We write $\xi(v) = O_p^*(\alpha_n, \beta_n)$, uniformly in $v \in A$, if $\sup_{v \in A} |\xi(v)| = O_p^*(\alpha_n, \beta_n)$. We write $X_n = O_p^*(\alpha_n, \beta_n, \gamma_n)$ if there exists positive constants $C_1, C_2, C_3$ such that $\Pr\left[ \frac{|X_n - C_2 \beta_n|}{C_1 \alpha_n} \right] \leq C_3 \gamma_n$. For simplicity, we write $X_n = O_p^*(\alpha_n)$ if $\beta_n = \gamma_n = n^{-1}$. It is straightforward to verify that if $X_n = O_p^*(\alpha_n, \beta_n, \gamma_n)$ and $Y_n = O_p^*(\alpha'_n, \beta'_n, \gamma'_n)$, then $X_n + Y_n = O_p^*(\alpha_n + \alpha'_n, \beta_n + \beta'_n, \gamma_n + \gamma'_n)$ and $X_n Y_n = O_p^*(\alpha_n \alpha'_n, \beta_n + \beta'_n, \gamma_n + \gamma'_n)$. We say $\xi(v) = O_p^*(\alpha_n, \beta_n, \gamma_n)$, uniformly in $v \in A$, if $\sup_{v \in A} |\xi(v)| = O_p^*(\alpha_n, \beta_n, \gamma_n)$.

Let $a \wedge b$ and $a \vee b$ denote min $\{a, b\}$ and max $\{a, b\}$ respectively. Let $C, C_1, C_2, \ldots$ denote positive constants that are independent of the sample size and whose values may change in different places. $\preceq$ denotes an inequality up to a universal constant, i.e., $a \preceq b$ is understood as $a \leq C \cdot b$ for some $C > 0$ that does not depend on the distribution of the variables in the model, any unknown quantity.
related to the model or the sample size. “$\equiv_d$” is understood as being equal in distribution. For some set $A$, let $\ell^\infty (A)$ denote the space of all bounded functions $f : A \to \mathbb{R}$ endowed with the sup-norm $\|f\|_A := \sup_{x \in A} |f (x)|$ of $f$ on $A$. For $f : \mathbb{R} \to \mathbb{R}$, $\|f\|_\infty$ is understood as $\sup_{x \in \mathbb{R}} |f (x)|$. $\ell ([a, b])$ denotes the length $b - a$ of the interval $[a, b]$. “$a =: b$’” means “$b$ is defined by $a$”. $f^{(k)}$ denotes the $k$-th derivative of $f : \mathbb{R} \to \mathbb{R}$ and $(f', f'')$ are understood as $(f^{(1)}, f^{(2)})$.

Let $\mathcal{F}$ denote a class of $\mathbb{R}$-valued functions defined on a compact set in a finite-dimensional Euclidean space $\mathcal{S}$. Let $\mathcal{F}$ be equipped with a norm $\|\cdot\|$. We say that $\mathcal{F}\circ \subseteq \mathcal{F}$ is an $\varepsilon$-net if the union of the closed $\|\cdot\|$-balls of radius $\varepsilon$ centered at points in $\mathcal{F}\circ$ covers $\mathcal{F}$. Let the $\varepsilon$-covering number $N (\varepsilon, \mathcal{F}\circ, \|\cdot\|)$ be given by

$$N (\varepsilon, \mathcal{F}\circ, \|\cdot\|) := \inf \{\#A : \mathcal{F}\circ \text{ is an \varepsilon-net of } A\},$$

where $\#A$ denotes the cardinality of a set $A$. A function $F_{\mathcal{F}} : \mathcal{S} \to \mathbb{R}^+$ is an envelope of $\mathcal{F}$ if $\sup_{f \in \mathcal{F}} |f| \leq F_{\mathcal{F}}$. Some of the function classes that appear later in this paper depend on the sample size $n$. We suppress the dependence for notational simplicity. We say that $\mathcal{F}$ is a (uniform) Vapnik–Chervonenkis-type (VC-type) class with respect to the envelope $F_{\mathcal{F}}$ (see, e.g., Giné and Nickl, 2016, Definition 3.6.10) if there exist some positive constants (VC characteristics) $A_\mathcal{F} \geq \varepsilon$ and $V_\mathcal{F} > 1$ that are independent of the sample size $n$ such that

$$\sup_{Q \in \mathcal{Q}_{\mathcal{F}}} N \left( \varepsilon \|F_{\mathcal{F}}\|_{Q, 2}, \mathcal{F}, \|\cdot\|_{Q, 2} \right) \leq \left( \frac{A_\mathcal{F}}{\varepsilon} \right)^{V_\mathcal{F}}, \forall \varepsilon \in (0, 1],$$

where $\mathcal{Q}_{\mathcal{F}}$ denotes the collection of all finitely discrete probability measures on $\mathcal{S}$ and the symbol $\forall$ is understood as “for all”. We denote $P_n f := n^{-1} \sum_{i=1}^n f (W_i)$, $P^W f := \mathbb{E} [f (W)]$ and $G_n := \sqrt{n} (P_n^W - P^W)$. And also $\|G_n^W\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |G_n^W f|$, $\|P^W\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |P^W f|$ and $\|P_n^W - P^W\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |(P_n^W - P^W) f|$.

Denote $U_i := (\epsilon_i, D_i, Z_i, X_i)$ and $U := (\epsilon, D, Z, X)$. $(P_U, P^U, G_n^U)$ and $(\|G_n^U\|_{\mathcal{F}}, \|P^U\|_{\mathcal{F}}, \|P_n^U - P^U\|_{\mathcal{F}})$ are defined similarly. For some $f : \mathcal{S}^r \to \mathbb{R}$, let

$$U_n^{(r)} f := \sqrt{n} \left( \frac{1}{n^{(r)}} \sum_{(i_1, \ldots, i_r)} f (U_{i_1}, \ldots, U_{i_r}) - \mathbb{E} [f (U_1, \ldots, U_r)] \right),$$

where $n^{(r)}$ is understood as $n! / (n - r)!$ and $\sum_{(i_1, \ldots, i_r)}$ is understood as summation over indices $(i_1, \ldots, i_r) \in \{1, \ldots, n\}^r$ which are all distinct, and $\|U_n^{(r)}\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |U_n^{(r)} f|$.

### A.2 Proofs

Recall that $\hat{\phi}_{d+} (y)$ is the “leave-in” version of $\hat{\phi}_{d-} (y)$ (see Footnote 10). We refine the asymptotic expansion for $\hat{\phi}_{d+} (y)$ in FVX (see Theorem 1 therein)

$$\hat{\phi}_{d+} (y) - \phi_{d+} (y) = \frac{1}{n} \sum_{i=1}^n \mathcal{L}_{d+} (W_i, y) + o_p \left( n^{-1/2} \right),$$

(38)
uniformly in $y \in \mathcal{J}_{g(d', x, \epsilon)}|X=x = \left[ \frac{Y_{d', x}}{Y_{d', x}} \right]$, where

$$
\mathcal{L}_{d'}(W_i, y) := \zeta_{d'}(\phi_{d'}(y))^{-1} \left\{ \mathbf{1}(Y_i \leq \phi_{d'}(y), D_i = d') + \mathbf{1}(Y_i \leq y, D_i = d') - R_{d'}(y) \right\} \pi_x(Z_i, X_i).
$$

We provide a Bahadur-representation-type result for the estimated counterfactual mapping with an $O_p\left(\frac{n^{3/4}}{h^{3/4}}\right)$ (up to a logarithmic term) estimate for the order of magnitude of the remainder term in (38). The proof of our Theorem 1 (derivation of (18)) relies on the latter result. Proofs of all lemmas in the rest of the appendix can be found in the online supplement.

**Lemma 2.** Suppose that Assumption 1 holds. Then,

$$
\hat{\phi}_{d'}(y) - \phi_{d'}(y) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}_{d'}(W_i, y) + O_p^* \left( \frac{\log(n)}{n} \right)^{3/4},
$$

and the remainder term is uniform in $y \in I_{d'} := \mathcal{J}_{g(d', x, \epsilon)}|X=x = \left[ \frac{Y_{d', x}}{Y_{d', x}} \right]$.

For any $(v, b) \in I_x \times [\underline{h}, \overline{h}]$, decompose $\hat{f}_{\Delta X}(v, x; b) - f_{\Delta X}(v, x)$ into the sum of $\hat{f}_{\Delta X}(v, x; b) - \hat{f}_{\Delta X}(v, x; b)$ and $\hat{f}_{\Delta X}(v, x; b) - f_{\Delta X}(v, x)$. The following lemma shows that the leading term in the asymptotic expansion of $\hat{f}_{\Delta X}(v, x; b) - f_{\Delta X}(v, x; b)$ can be represented by a $U$-statistic, uniformly in $(v, b) \in I_x \times [\underline{h}, \overline{h}]$.

**Lemma 3.** Suppose that the assumptions of Theorem 1 hold. Then,

$$
\hat{f}_{\Delta X}(v, x; b) - \hat{f}_{\Delta X}(v, x; b) = \frac{1}{n(2)} \sum_{(i, j)} \mathcal{G}_x(W_i, W_j, v; b) + O_p^* \left( \frac{\log(n)}{nh^2} + \frac{\log(n)^{3/4}}{n^{3/4}h} \right),
$$

where the remainder term is uniform in $(v, b) \in I_x \times [\underline{h}, \overline{h}]$.

Note that if $X_i = x$ and $D_i = d'$, $\phi_{d'}(Y_i) = g(d, x, \epsilon_i)$ and

$$
\left\{ \mathbf{1}(Y_j \leq \phi_{d'}(Y_i), D_j = d) + \mathbf{1}(Y_j \leq Y_i, D_j = d') \right\} \mathbf{1}(X_j = x) = \mathbf{1}(\epsilon_j \leq \epsilon_i) \mathbf{1}(X_j = x).
$$

It is shown in the proof of Lemma 2 that $R_{d'}(y) = F_{g(d', x, \epsilon)|X}(y \mid x) = \Pr[g(d', x, \epsilon) \leq y \mid X = x]$. Therefore, if $X_i = x$ and $D_i = d'$, $R_{d'}(Y_i) = F_{\epsilon|X}(\epsilon_i \mid x)$. And therefore,

$$
q_{d'}(W_i, W_j) \mathbf{1}(X_j = x) = \mathbf{1}(D_i = d', X_i = x) \left\{ \mathbf{1}(\epsilon_j \leq \epsilon_i) - F_{\epsilon|X}(\epsilon_i \mid x) \right\} \mathbf{1}(X_j = x).
$$

Let

$$
\varpi_x(U_i) := \frac{\mathbf{1}(D_i = 0, X_i = x)}{\zeta^x(g(1, x, \epsilon_i))} - \frac{\mathbf{1}(D_i = 1, X_i = x)}{\zeta^x(g(0, x, \epsilon_i))} \\
C_x(U_i, U_j) := \varpi_x(U_i) \left\{ \mathbf{1}(\epsilon_j \leq \epsilon_i) - F_{\epsilon|X}(\epsilon_i \mid x) \right\} \pi_x(Z_j, X_j)
$$

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and

\[ \mathcal{H}_x (U_i, U_j, v; b) := \mathcal{G}_x ((g (D_i, X_i, \epsilon_i), D_i, Z_i, X_i), (g (D_j, X_j, \epsilon_j), D_j, Z_j, X_j), v; b) = \frac{1}{b^q} K' \left( \frac{\Delta_x (\epsilon_i) - v}{b} \right) c_x (U_i, U_j). \]

Denote \( \mathcal{H}_x^{[i]} (u, v; b) := E [\mathcal{H}_x (u, v; b)] \). Clearly, we have \( \mathcal{G}_x (W_i, W_j, v; b) = \mathcal{H}_x (U_i, U_j, v; b), \forall i \neq j \) and \( \mathcal{G}_x^{[i]} (W_i, v; b) = \mathcal{H}_x^{[i]} (U_i, v; b) \forall i \). Note that by conditional independence of \( \epsilon \) and \( Z \) given \( X \),

\( E [\mathcal{G}_x (w, W; v; b)] = E [\mathcal{H}_x (u, U; v; b)] = 0 \) and, \( E [\mathcal{G}_x (W_i, W_j; v; b)] = E [\mathcal{H}_x (U_i, U_j; v; b)] = 0, \forall i \neq j \). Then, we have the following result.

**Lemma 4.** Suppose that the assumptions of Theorem 1 hold. Then,

\[ \frac{1}{n(2)} \sum_{(i, j)} G_x (W_i, W_j, v; b) = \frac{1}{n(2)} \sum_{(i, j)} H_x (U_i, U_j, v; b) = O_p^* (\sqrt{\frac{\log (n)}{nh}} + \sqrt{\frac{\log (n)}{nh^3}}), \]

uniformly in \((v, b) \in I_x \times [h, \overline{h}]\).

The following theorem is a stronger version of Theorem 1. It is easy to check that the asymptotic results in Theorem 1 are straightforward implications of the non-asymptotic deviation bounds here.

**Theorem A1.** Under the assumptions of Theorem 1,

\[ \left\| \hat{f}_{\Delta |X} (\cdot | x; h) - f_{\Delta |X} (\cdot | x) \right\|_{I_x} = O_p^* \left( \sqrt{\frac{\log (n)}{nh}} + h^P, \sqrt{\frac{\log (n)}{nh^3}} \right) \]

and

\[ \left\| \hat{f}_{\Delta |X} (\cdot | x; h) - f_{\Delta |X} (\cdot | x) \right\|_{I_x} = O_p^* \left( \sqrt{\frac{\log (n)}{nh}} + h^P, \sqrt{\frac{\log (n)}{nh^3}} + \delta_n \right). \]

**Proof of Theorem A1.** Denote \( \mathcal{E}_x (U_i, v; b) := b^{-1} K ((\Delta_x (\epsilon_i) - v) / b) \mathbb{I} (X_i = x) \). Then we write

\[ \left\| \hat{f}_{\Delta X} (\cdot, x; \cdot) - m_{\Delta X} (\cdot, x; \cdot) \right\|_{I_x \times [\underline{h}, \overline{h}]} = \left\| \mathbb{P}^U_n - \mathbb{P}^U \right\|_{\mathcal{E}}, \text{ where } \mathcal{E} := \{ \mathcal{E}_x (\cdot, v; b) : (v, b) \in I_x \times [h, \overline{h}] \}. \]

By similar arguments used in the proof of Lemma 4 (decomposing \( K \) into the difference of two bounded monotone functions, using Kosorok (2007, Lemma 9.6), Kosorok (2007, Lemma 9.9(viii)), Giné and Nickl (2016, Theorem 3.6.9) and Chernozhukov et al. (2014a, Lemma B.2)), \( \mathcal{E} \) is uniformly VC-type with respect to a constant envelope \( F_{\mathcal{E}} = O (h^{-1}) \). By arguments used for proving \( \left\| \mathbb{P}^U_n - \mathbb{P}^U \right\|_{\mathcal{E}} = O_p^* \left( \sqrt{\log (n) / (nh)} \right) \) in the proof of Lemma 3, we have \( \left\| \mathbb{P}^U_n - \mathbb{P}^U \right\|_{\mathcal{E}} = O_p^* \left( \sqrt{\log (n) / (nh)} \right) \). By Lemmas 3 and 4, we have

\[ \left\| \hat{f}_{\Delta X} (\cdot, x; \cdot) - \hat{f}_{\Delta X} (\cdot, x; \cdot) \right\|_{I_x \times [\underline{h}, \overline{h}]} = O^*_p \left( \sqrt{\frac{\log (n)}{nh}}, \sqrt{\frac{\log (n)}{nh^3}} \right). \]

By Hoeffding’s inequality, \( \hat{p}_x - p_x = O_p^* \left( \sqrt{\log (n) / n} \right) \) (Pr \( \left| \hat{p}_x - p_x \right| > C_1 \sqrt{\log (n) / n} \leq C_2 n^{-1} \)).
Then, \( p_x/\hat{p}_x - 1 = O_p^* \left( \sqrt{\log(n)/n} \right) \) follows from \( \Pr \left[ |p_x/\hat{p}_x - 1| > (2C_1/p_x) \sqrt{\log(n)/n} \right] \leq \Pr \left[ |\hat{p}_x - p_x| > C_1 \sqrt{\log(n)/n} \right] + \Pr \left[ p_x < p_x/2 \right] \leq 2C_2n^{-1} \), where the last inequality holds when \( n \) is sufficiently large. Then, by these results, the equality \( a/b = a/c - a(b - c)/c^2 + a(b - c)^2/(bc^2) \) and Lemma 3,

\[
\tilde{f}_{\Delta|x} (v \mid x; b) - f_{\Delta|x} (v \mid x) = p_x^{-1} \left( \tilde{f}_{\Delta|x} (v, x; b) - m_{\Delta|x} (v, x; b) \right) + p_x^{-1} \frac{1}{n(2)} \sum_{(i,j)} \mathcal{H}_x (U_i, U_j; v, b) \\
+ p_x^{-1} \left( m_{\Delta|x} (v, x; b) - f_{\Delta|x} (v, x) \right) + O_p^* \left( \sqrt{\log(n)/n} + \log(n)/n^{3/4} + \sqrt{\log(n)/n^{3/4}h} \right),
\]

(41)

uniformly in \( (v, b) \in I_x \times [h, \overline{h}] \). By this result, Lemma 4, (12) and \( \|\mathbb{P}^U - \mathbb{P}^\hat{U}\|_\varepsilon = O_p^* \left( \sqrt{\log(n)/n} \right) \), we now have

\[
\sup_{(v, b) \in I_x \times [h, \overline{h}]} \left| \tilde{f}_{\Delta|x} (v \mid x; b) - f_{\Delta|x} (v \mid x) \right| = O_p^* \left( \sqrt{\log(n)/n^3} \right) + O(h^P).
\]

(42)

The first conclusion follows from this result. It follows from the assumption \( \Pr \left[ \hat{h} \notin [h, \overline{h}] \right] > 1 - \delta_n \) and (42) that

\[
\Pr \left[ \left\| \tilde{f}_{\Delta|x} (\cdot \mid x; \hat{h}) - f_{\Delta|x} (\cdot \mid x) \right\|_{I_x} > C_1 \left( \sqrt{\log(n)/n^3} + h^P \right) \right] \\
\leq \Pr \left[ \sup_{(v, b) \in I_x \times [h, \overline{h}]} \left| \tilde{f}_{\Delta|x} (v \mid x; b) - f_{\Delta|x} (v \mid x) \right| > C_1 \left( \sqrt{\log(n)/n^3} + h^P \right) \right] + \Pr \left[ \hat{h} \notin [h, \overline{h}] \right]
\leq \sqrt{\log(n)/n^3} + \delta_n.
\]

The second conclusion follows from this result.

The following lemma is a refinement of Lemma 4 and the result \( \left\| \tilde{f}_{\Delta X} (\cdot; \cdot) - m_{\Delta X} (\cdot; \cdot) \right\|_{I_x \times [h, \overline{h}]} = O_p^* \left( \sqrt{\log(n)/n} \right) \). It provides an estimate of the effect of a random bandwidth that satisfies Assumption 3.

Lemma 5. Suppose that the assumptions of Theorem 2 hold. Then, (a)

\[
\sqrt{n} \left( \frac{1}{n(2)} \sum_{(i,j)} \mathcal{H}_x (U_i, U_j; v; b, h) \right) = O_p^* \left( \varepsilon_n \sqrt{\log(n)} \sqrt{\frac{\log(n)}{n^3}} \right),
\]

uniformly in \( (v, b) \in I_x \times [h, \overline{h}] \), where \( \mathcal{H}_x^\Delta (U_i, U_j; v; b, h) := \sqrt{b} \cdot \mathcal{H}_x (U_i, U_j; v; b) - \sqrt{h} \).
\( \mathcal{H}_x (U_i, U_j, v; h) \).

\[
\sqrt{nh} \left( \tilde{f}_{\Delta X} (v, x; b) - m_{\Delta X} (v, x; b) \right) - \sqrt{nh} \left( \hat{f}_{\Delta X} (v, x; h) - m_{\Delta X} (v, x; h) \right) = O_p \left( \varepsilon_n \sqrt{\log (n)} \right),
\]

uniformly in \((v, b) \in I_x \times [h, \overline{h}]\).

Then by using these lemmas, we prove the asymptotic normality result with either a deterministic bandwidth or a random bandwidth that satisfies Assumption 3. For simplicity, denote

\[
\Psi (v \mid x; b) := \sqrt{nb} \left( \tilde{f}_{\Delta |X} (v \mid x; b) - f_{\Delta |X} (v \mid x) - \hat{f}_{\Delta |X} (v \mid x) \mu_{K, P} b^P \right).
\]

**Proof of Theorem 2.** The Hoeffding decomposition (17) can be equivalently written as

\[
\frac{1}{n(2)} \sum_{(i,j)} G_x (W_i, W_j, v; h) = \frac{1}{n(2)} \sum_{(i,j)} \mathcal{H}_x (U_i, U_j, v; h)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \mathcal{H}_x^{[1]} (U_i, v; h) + \frac{1}{n(2)} \sum_{(i,j)} \left\{ \mathcal{H}_x (U_i, U_j, v; h) - \mathcal{H}_x^{[1]} (U_j, v; h) \right\}.
\]

Then we show that the second term in the Hoeffding decomposition is negligible uniformly in \(v \in I_x\).

It is shown in the proof of Lemma 4 that \( \mathcal{F} := \{ \mathcal{H}_x (\cdot, v; b) : (v, b) \in I_x \times [h, \overline{h}] \} \) is uniformly VC-type with respect to a constant envelope \( F_\mathcal{F} = O (h^{-2}) \). Then, by Corollary 5.6 of CK,

\[
\mathbb{E} \left[ \sup_{(v, b) \in I_x \times [h, \overline{h}]} \left| \frac{1}{n(2)} \sum_{(i,j)} \left\{ \mathcal{H}_x (U_i, U_j, v; b) - \mathcal{H}_x^{[1]} (U_j, v; b) \right\} \right| \right] = O \left( (nh^2)^{-1} \right).
\]

Then, by Lemma 3, \( \hat{f}_{\Delta X} (v, x; h) - \tilde{f}_{\Delta X} (v, x; h) = n^{-1} \sum_{i=1}^{n} \mathcal{H}_x^{[1]} (U_i, v; h) + o_p \left( (nh)^{-1/2} \right) \). By this result, (12) and (41), \( \Psi (v \mid x; h) = p_x^{-1} \sum_{i=1}^{n} J_i + o_p (1) \), where

\[
J_i \approx n^{-1/2} \left\{ h^{1/2} \left( \frac{1}{h} K \left( \frac{\Delta - v}{h} \right) \mathbb{1} (X_i = x) - m_{\Delta X} (v, x; h) \right) + h^{1/2} \mathcal{H}_x^{[1]} (U_i, v; h) \right\}.
\]

Let \( \sigma_x^2 := \mathbb{E} \left[ \sum_{i=1}^{n} J_i \right] \). Denote

\[
\rho_x (e) := \frac{f_{\varepsilon DX} (e, 0, x)}{\zeta_{1x} (g (1, x, e))} - \frac{f_{\varepsilon DX} (e, 1, x)}{\zeta_{0x} (g (0, x, e))}
\]

\[
\Gamma_x (e, v; b) := \int_{\mathbb{R}} \frac{1}{b^2} K' \left( \frac{\Delta - v}{b} \right) \rho_x (e) \left\{ \mathbb{1} (e \leq e) - F_{\varepsilon|X} (e \mid x) \right\} \, de.
\]

Since \( \mathcal{H}_x^{[1]} (U, v; h) = \Gamma_x (e, v; h) \pi_x (Z, X) \), we have

\[
\mathbb{E} \left[ \mathcal{H}_x^{[1]} (U, v; h) \left( \frac{1}{h} K \left( \frac{\Delta - v}{h} \right) \mathbb{1} (X = x) - m_{\Delta X} (v, x; h) \right) \right] =
\]

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\[
E \left[ \Gamma_x (\epsilon, v; h) \left( \frac{1}{h} K \left( \frac{\Delta x (\epsilon) - v}{h} \right) \mathbb{1} (X = x) - m_{\Delta X} (v, x; h) \right) \pi_x (Z, X) \right] = 0, \tag{43}
\]
where the second equality follows from LIE, the fact that \( \epsilon \) is conditionally independent of \( Z \) given \( X \) and the fact that \( E [\pi_x (Z, X) | X] = 0 \). Therefore,
\[
\sigma_j^2 = E \left[ h \left( \frac{1}{h} K \left( \frac{\Delta - v}{h} \right) \mathbb{1} (X = x) - m_{\Delta X} (v, x; h) \right)^2 \right] + E \left[ h \cdot \mathcal{H}_x^{(1)} (U, v; h) \right] = \mathcal{V}_1 (v, x) + \mathcal{V}_2 (v, x) + o (1), \tag{44}
\]
where it follows from standard arguments for kernel density estimators (Newey, 1994) and (12) that the first term on the right hand side of the second equality is \( \mathcal{V}_1 (v, x) + o (1) \) and in the proof of Lemma 4, we show that \( E [h \cdot \mathcal{H}_x^{(1)} (U, v; h)] = \mathcal{V}_2 (v, x) + o (1) \). Then, we verify Lyapunov’s condition. By Loève’s \( c_r \) inequality,
\[
\sum_{i=1}^{n} E \left[ \left| \frac{J_i}{\sigma_j} \right|^3 \right] \leq \sigma_j^{-3} n^{-1/2} h^{3/2} \left\{ E \left[ \left| \mathcal{H}_x^{(1)} (U, v; h) \right|^3 \right] + E \left[ \left\{ \frac{1}{h} K \left( \frac{\Delta - v}{h} \right) \mathbb{1} (X = x) \right\}^3 \right] + \left| m_{\Delta X} (v, x; h) \right|^3 \right\}, \tag{45}
\]
where by change of variables, the second term in the bracket on the right hand side of (45) is \( O \left( h^{-2} \right) \) and the third term is \( O (1) \). Then by the \( c_r \) inequality,
\[
E \left[ \left| \mathcal{H}_x^{(1)} (U, v; h) \right|^3 \right] \leq E \left[ \int_{\mathbb{R}} \frac{1}{h^2} K' \left( \frac{\Delta x (e) - v}{h} \right) \rho_x (e) \mathbb{1} (\epsilon \leq e) \, de \right]^3
\]
\[
+ \left[ \int_{\mathbb{R}} \frac{1}{h^2} K' \left( \frac{\Delta x (e) - v}{h} \right) \rho_x (e) F_{\epsilon | X} (e | x) \, de \right]^3. \tag{46}
\]
By change of variables,
\[
E \left[ \left[ \int_{\mathbb{R}} \frac{1}{h^2} K' \left( \frac{\Delta x (e) - v}{h} \right) \rho_x (e) \mathbb{1} (\epsilon \leq e) \, de \right]^3 \right]
\]
\[
\leq \left( \int_{\mathbb{R}} \frac{1}{h^2} K' \left( \frac{\Delta x (e) - v}{h} \right) \rho_x (e) \, de \right)^3 = O \left( h^{-3} \right).
\]
It is shown in the proof of Lemma 4 that the second term on the right hand side of (46) is \( O (1) \). Therefore, \( E \left[ \left| \mathcal{H}_x^{(1)} (U, v; h) \right|^3 \right] = O \left( h^{-3} \right) \). Then, \( \sum_{i=1}^{n} E \left[ \left| J_i / \sigma_j \right|^3 \right] = O \left( (nh^3)^{-1/2} \right) \) follows from this result, (44) and (45). By Lyapunov’s central limit theorem, \( \sum_{i=1}^{n} J_i / \sigma_j \rightarrow_d N (0, 1) \). The first assertion \( \Psi (v | x; h) \rightarrow_d N (0, \mathcal{V} (v | x)) \) follows from this result, \( \Psi (v | x; h) = p_x^{-1} \sum_{i=1}^{n} J_i + o_p (1), \) (44) and Slutsky’s lemma.

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Recall that $S(v \mid x; b)$ is defined by (26). For the second part, by (41),

$$
S(v \mid x; b) - S(v \mid x; h) =
\begin{align*}
p_x^{-1} & \left\{ \sqrt{n b} \left( \bar{f}_{\Delta X}(v, x; b) - m_{\Delta X}(v, x; b) \right) - \sqrt{n h} \left( \bar{f}_{\Delta X}(v, x; h) - m_{\Delta X}(v, x; h) \right) \right\} \\
& + p_x^{-1} \left\{ \sqrt{n b} \left( m_{\Delta X}(v, x; b) - f_{\Delta X}(v, x) \right) - \sqrt{n h} \left( m_{\Delta X}(v, x; h) - f_{\Delta X}(v, x) \right) \right\} \\
& + p_x^{-1} \left\{ \sqrt{n} \left( \frac{1}{n(2)} \sum_{i,j} H_x^\Delta (U_i, U_j, v; b) \right) \right\} + O_p \left( v_n, \sqrt{\log(n) / n h^3} \right), \tag{47}
\end{align*}
$$

uniformly in $(v, b) \in I_x \times [\hat{b}, \bar{b}]$, where $v_n := \sqrt{\log(n) / \tilde{h} + \log(n) / \sqrt{n h^3}} + \left( \log(n)^3 / (n h^2) \right)^{1/4}$. It then follows from Lemma 5 and (12) that

$$
\Psi(v \mid x; b) - \Psi(v \mid x; h) = O_p \left( v_n + \varepsilon_n \sqrt{\log(n) / n h^3} \right) + o \left( \sqrt{n h^3} \right),
$$

uniformly in $(v, b) \in I_x \times [\hat{b}, \bar{b}]$. Then, by using $\Pr \left[ \hat{h} \in [\hat{h}, \bar{h}] \right] > 1 - \delta_n$,

$$
\left\| \Psi \left( \cdot \mid x; \hat{h} \right) - \Psi \left( \cdot \mid x; h \right) \right\|_{L^2_x} \leq \sup_{(v, b) \in I_x \times [\hat{b}, \bar{b}]} |\Psi(v \mid x; b) - \Psi(v \mid x; h)| = o_p(1),
$$

where the inequality holds with probability $1 - O(\delta_n)$, we have $\Psi(v \mid x; \hat{h}) = \Psi(v \mid x; h) + o_p(1)$. The second assertion follows from this result, $\Psi(v \mid x; h) \to_d N(0, \Psi'(v \mid x))$ and Slutsky’s lemma. 

### Appendix B Proofs of Theorems in Section 4

The following lemma gives rates of convergence of $\hat{R}_{d_x}$ and $\hat{\zeta}_{d_x}$.

**Lemma 6.** Suppose that the assumptions of Theorem 3 hold. Then, (a) $\left\| \hat{R}_{d_x} - R_{d_x} \right\|_{L^2_{d_x}} = O_p \left( \sqrt{\log(n) / n} \right)$. (b) $\hat{\zeta}_{d_x}(y; b) - \zeta_{d_x}(y) = O_p \left( \sqrt{\log(n) / (n h \zeta)} \right) + O \left( h^2 \zeta \right)$, uniformly in $(y, b) \in I_{d_x} \times [\hat{b}_{\zeta}, \bar{b}_{\zeta}]$, where $I_{d_x}$ is any inner closed sub-interval of $I_{d_x}$, $\hat{h}_{\zeta} := (1 - \varepsilon_{n \zeta}) h_{\zeta}$ and $\bar{h}_{\zeta} := (1 + \varepsilon_{n \zeta}) h_{\zeta}$.

Denote

$$
\kappa_1^V(\gamma) := \sqrt{\frac{\log(n)}{n h_{\zeta}}} + h_{\zeta}^2 + \frac{\log(n)^{2/3}}{(n h_{\zeta})^{2/3} h_{\gamma}^{1/3}} + \frac{\log(n)}{n h_{\zeta}^2} + \sqrt{\frac{\log(n)}{n h_{\zeta}^2}}\text{ and } \kappa_2^V(\gamma) := \gamma + \sqrt{\frac{\log(n)}{n h_{\zeta}^2}},
$$

for $\gamma \in (0, 1)$. The following lemma is useful in proving Theorem 3.

**Lemma 7.** Under the assumptions of Theorem 3, (a) for some constants $C_1, C_2 > 0$, when $n$ is
sufficiently large,

\[ \Pr \left[ \sup_{(v,b,c) \in I_x \times [h, \tilde{h}] \times [b, \tilde{b}, \tilde{c}]} \left| \hat{V} (v \mid x; b, b', c) - V (v \mid x; b) \right| > C_1 \kappa_1^V (\gamma) \right] \leq C_2 \kappa_2^V (\gamma), \forall \gamma \in (0, 1). \]

(b) \( V (v \mid x; b) - V (v \mid x; h) = O (\varepsilon_n h), \) uniformly in \((v, b) \in I_x \times [h, \tilde{h}].\)

Then, in the following theorem, we present a non-asymptotic deviation bound for \( \| \hat{V} (\cdot \mid x; \tilde{h}, \tilde{h}, c) - V (\cdot \mid x; \tilde{h}) \| \). It is easy to see that this result is stronger than Theorem 3 presented in the main text. Its proof is relegated to the online supplement.

**Theorem B1.** Under the assumptions of Theorem 3, for some constants \( C_1, C_2 > 0, \) when \( n \) is sufficiently large,

\[ \Pr \left[ \| \hat{V} (\cdot \mid x; \tilde{h}, \tilde{h}, c) - V (\cdot \mid x; \tilde{h}) \|_{I_x} > C_1 (\kappa_1^V (\gamma) + \varepsilon_n h) \right] \leq C_2 \left( \kappa_2^V (\gamma) + \delta_n + \delta_n^c \right), \forall \gamma \in (0, 1). \]

The following lemma states a useful property of the \( O_p^* \) notation.

**Lemma 8.** Let \( \alpha_n, \beta_n, \gamma_n, \delta_n \) be deterministic sequences in \((0, \infty)\). Suppose that \( Y_n \geq 0 \) depends only on the data \((W_i^n)\). Suppose that \( Y_n = O_p^*(\alpha_n, \beta_n) \) and \( \Pr_{W_i^n} [X_n > C_1 Y_n] = O_p^*(\gamma_n, \delta_n). \) Then, (a) \( X_n = O_p^*(\alpha_n, \gamma_n, \beta_n + \delta_n). \) (b) \( Y_n = O_p^*(\alpha_n, \varepsilon_n, \beta_n), \) where \( \varepsilon_n \) is an arbitrary deterministic sequence in \((0, \infty)\) that decays to zero as \( n \uparrow \infty.\)

Let the infeasible JMB process be given by

\[ S_{jmb} (v \mid x; b) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_{i}^{-1} \left\{ \left( \hat{U}_x^{[1]} (W_i, v; b) - \hat{\mu}_x (v; b) \right) + \left( \hat{U}_x^{[2]} (W_i, v; b) - \hat{\mu}_x (v; b) \right) \right\} \quad (48) \]

where \( \hat{U}_x^{[1]} (W_i, v; b) := (n - 1)^{-1} \sum_{j \neq i} U_x (W_j, W_i, v; b) \) is the jackknife estimator of \( U_x^{[1]} (W_i, v; b), \)
\( U_x^{[2]} (W_i, v; b) \) is defined similarly and \( \hat{\mu}_x (v; b) := n^{-1} \sum (i, j) U_x (W_i, W_j, v; b). \) The following lemma also shows that the difference of \( Z_{jmb} (v \mid x; \tilde{h}, \tilde{h}, c) \) and

\[ Z_{jmb} (v \mid x; \tilde{h}, \tilde{h}, c) = \frac{S_{jmb} (v \mid x; \tilde{h}, \tilde{h})}{\sqrt{V (v \mid x; \tilde{h})}} \]

is negligible.

**Lemma 9.** Suppose that the assumptions in the statement of Theorem 4 hold. Then,

\[ \tilde{Z}_{jmb} (v \mid x; \tilde{h}, \tilde{h}, c) - Z_{jmb} (v \mid x; \tilde{h}, \tilde{h}) = O_p^* \left( \kappa_1^V \sqrt{\log (n)} + \left( \frac{\log (n)}{n h^2} \right)^{1/4} + \varepsilon_n \sqrt{\log (n)} n^{-1}, \kappa_2^V + \delta_n + \delta_n^c \right), \]

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uniformly in \( v \in I_x \), where

\[
\kappa_{1,n}^V := \sqrt{\frac{\log(n)}{nh \zeta}} + h^2 + \sqrt{\frac{\log(n)}{nh^4}} \quad \text{and} \quad \kappa_{2,n}^V := \sqrt{\frac{\log(n)}{nh^4}}.
\]

Denote

\[
\mathcal{M}_x(U_i, U_j, v; b) := p_x^{-1} \mathcal{U}_x (\{g(D_i, X_i, \epsilon_i), D_i, Z_i, X_i\}, \{g(D_j, X_j, \epsilon_j), D_j, Z_j, X_j\}, v; b)
\]

\[
= p_x^{-1} \sqrt{\tilde{\xi}} \{ \mathcal{E}_x(U_i, v; b) + \mathcal{H}_x(U_i, U_j, v; b) \},
\]

(49)

\[
\tilde{\mathcal{M}}_x[1] (u, v; b) := E[\mathcal{M}_x(U, u, v; b)] \quad \text{and} \quad \mathcal{M}_x[2] (u, v; b) := E[\mathcal{M}_x(U, U, v; b)].
\]

Note that \( \tilde{\mathcal{M}}_x[2] (\cdot, v; b) \) is constant and equal to \( \mu_{\mathcal{M}_x} (v; b) := E[\mathcal{M}_x(U_1, U_2, v; b)] \) and \( V(v \mid x, b) = \text{Var} \left[ \tilde{\mathcal{M}}_x[1] (U, v; b) \right] \). By (41) and (49),

\[
S(v \mid x; h) = \sqrt{n} \left\{ \frac{1}{n(2)} \sum_{(i,j)} \mathcal{M}_x(U_i, U_j, v; h) - \mu_{\mathcal{M}_x} (v; h) \right\} + O_p \left( \sqrt{\log(n)} \left( \frac{n}{nh^3} \right) \right) + O \left( \sqrt{nh^3} \right),
\]

uniformly in \( v \in I_x \). Note that by (49),

\[
S_{jmb}(v \mid x; b) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \nu_i \left\{ \left( \tilde{\mathcal{M}}_x[1] (U_i, v; b) - \tilde{\mu}_{\mathcal{M}_x} (v; b) \right) + \left( \tilde{\mathcal{M}}_x[2] (U_i, v; b) - \tilde{\mu}_{\mathcal{M}_x} (v; b) \right) \right\},
\]

where \( \tilde{\mathcal{M}}_x[1] (U_i, v; b) := (n - 1)^{-1} \sum_{j \neq i} \mathcal{M}_x(U_j, U_i, v; b) \) is the jackknife estimator of \( \mathcal{M}_x[1] (U_i, v; b) \), \( \tilde{\mathcal{M}}_x[2] (U_i, v; b) \) is defined similarly and \( \tilde{\mu}_{\mathcal{M}_x} (v; b) := n(2)^{-1} \sum_{(i,j)} \mathcal{M}_x(U_i, U_j, v; b) \).

The proof of Theorem 4 hinges on the \( U \)-process representation (50), the coupling theorem for \( U \)-process suprema (Proposition 2.1 of CK) and the JMB coupling theorem (Theorem 3.1 of CK). Denote \( \mathfrak{M}[1] := \left\{ \mathcal{M}_x[1] (\cdot, v; h) / \sqrt{V(v \mid x; h)} : v \in I_x \right\} \). Let \( \{G^U(f) : f \in \mathfrak{M}[1]\} \) be a centered Gaussian process with the covariance structure \( E[G^U(f_1) G^U(f_2)] = \text{Cov}[f_1(U), f_2(U)] \), \( \forall (f_1, f_2) \in \mathfrak{M}[1] \times \mathfrak{M}[1] \). One can show that this Gaussian process admits a version that is a tight random element in \( C^\infty (\mathfrak{M}[1]) \), and we use \( \{G^U(f) : f \in \mathfrak{M}[1]\} \) to denote the tight version. By the coupling theorems of CK and the anti-concentration inequality of Chernozhukov et al. (2014a), the distribution of \( \|Z(\cdot \mid x; \tilde{h}, \tilde{h}_\zeta)\|_{I_x} \) and the conditional distribution of \( \|\mathcal{Z}_{jmb}(\cdot \mid x; \tilde{h}, \tilde{h}_\zeta)\|_{I_x} \) given the original sample can be approximated by that of \( \|G^U\|_{\mathfrak{M}[1]} \). I.e., there are positive constants \( C_1, C_2, C_3 \) such that

\[
\sup_{t \in \mathbb{R}} \left[ \text{Pr} \left[ \|Z(\cdot \mid x; \tilde{h}, \tilde{h}_\zeta)\|_{I_x} \leq t \right] - \text{Pr} \left[ \|G^U\|_{\mathfrak{M}[1]} \leq t \right] \right] \leq C_1 \kappa_{1,n}
\]

(51)

\[\text{The existence of } \{G^U(f) : f \in \mathfrak{M}[1]\} \text{ is guaranteed by the Kolmogorov extension theorem.}\]
and
\[
\Pr \left[ \sup_{t \in \mathbb{R}} \Pr_{\mathcal{W}^T} \left[ \| \bar{Z}_{\text{jmmb}} \left( \cdot \mid x; \hat{h}, \hat{h}_\zeta \right) \|_{L_x} \leq t \right] - \Pr \left[ \| G^U \|_{2[1]} \leq t \right] \mid C_2 \tilde{\kappa}_{2,n}^\sharp \right] > 1 - C_3 \tilde{\kappa}_{3,n}^\sharp, \tag{52}
\]
where
\[
\tilde{\kappa}_{1,n} := \log (n) \kappa_{1,n}^V + \sqrt{\log (n)} \sqrt{nh^5} + \log (n) \varepsilon_n + \left( \frac{\log (n)^7}{nh^3} \right)^{1/8} \delta_n + \delta^\zeta_n,
\]
\[
\tilde{\kappa}_{2,n}^\sharp := \left( \frac{\log (n)^5}{nh^3} \right)^{1/16} + \log (n) \kappa_{1,n}^V + \log (n) \varepsilon_n \quad \text{and} \quad \tilde{\kappa}_{3,n}^\sharp := \left( \frac{\log (n)^5}{nh^3} \right)^{1/16} + \delta_n + \delta^\zeta_n.
\]
Then we show that
\[
\left| \Pr \left[ \| Z \left( \cdot \mid x; \hat{h}, \hat{h}_\zeta \right) \|_{L_x} \leq z_{1-\alpha}^{\text{jmmb}} \right] - (1 - \alpha) \right| \quad \text{is bounded by} \quad C_1 \tilde{\kappa}_{1,n} + C_2 \tilde{\kappa}_{2,n}^\sharp + C_3 \tilde{\kappa}_{3,n}^\sharp. \quad \text{This result gives an estimate of the coverage error decay rate of the JMB UCB. We state it as the following theorem, which implies the conclusion of Theorem 4. It is clear that the coverage error decay rate is } O (n^{-\theta}) \text{ if the assumptions of Theorem 4 are satisfied.}
\]
\[\textbf{Theorem B2.} \text{ Suppose that the assumptions of Theorem 4 hold. Then,}
\]
\[
\Pr \left[ f_{\Delta | X} (v \mid x) \in \text{CB}_{\text{jmmb}} \left( v \mid x; \hat{h}, \hat{h}_\zeta \right), \forall v \in I_x \right] = (1 - \alpha) + O \left( \left( \frac{\log (n)^5}{nh^3} \right)^{1/16} + \log (n) \kappa_{1,n}^V + \sqrt{\log (n)} \sqrt{nh^5} + \log (n) \varepsilon_n + \delta_n + \delta^\zeta_n \right), \tag{53}
\]
\[\textbf{Proof of Theorem B2.} \text{ It follows from (12), (47) and Lemma 5 that}
\]
\[
S (v \mid x; b) - S (v \mid x; h) = O_p \left( \varepsilon_n \sqrt{\log (n)} + v_n, \sqrt{\frac{\log (n)}{nh^3}} \right) + O \left( \sqrt{nh \left( h^3 + \varepsilon_n h^2 \right)} \right)
\]
uniformly in \((v, b) \in I_x \times [\underline{h}, \bar{h}]\). Then, by this result and \(\Pr \left[ \hat{h} \in [\underline{h}, \bar{h}] \right] > 1 - \delta_n\), we have
\[
S \left( v \mid x; \hat{h} \right) - S (v \mid x; h) = O_p \left( \varepsilon_n \sqrt{\log (n)} + v_n + \sqrt{nh \left( h^3 + \varepsilon_n h^2 \right)}, \sqrt{\frac{\log (n)}{nh^3}} + \delta_n \right). \tag{54}
\]
Write
\[
Z \left( v \mid x; \hat{h}, \hat{h}_\zeta \right) - \frac{S (v \mid x; h)}{\sqrt{V (v \mid x; h)}} =
\]
\[
\frac{S \left( v \mid x; \hat{h} \right) - S (v \mid x; h)}{\sqrt{V (v \mid x; h)}} \left( \frac{\sqrt{V (v \mid x; h)}}{\sqrt{V (v \mid x; \hat{h}, \hat{h}_\zeta)}} - 1 \right) + \frac{S \left( v \mid x; \hat{h} \right) - S (v \mid x; h)}{\sqrt{V (v \mid x; h)}} \tag{55}
\]
Taking $\gamma = \sqrt{\log(n) / (n h^4)}$ in Theorem B1, we get

$$\left\| \hat{V} \left( \cdot \mid x; \hat{h}, \hat{h}_\zeta \right) - V \left( \cdot \mid x; h \right) \right\|_{L^q} = O_p \left( \kappa_{1,n}^V + \varepsilon_n h, \kappa_{2,n}^V + \delta_n + \delta_n^\zeta \right).$$

Note that the second equality of (44) holds uniformly in $v \in I_x$ so that we have $V \left( v \mid x; h \right) \rightarrow \mathcal{Y} \left( v \mid x \right)$ uniformly in $v \in I_x$, as $h \downarrow 0$ and therefore, $\mathcal{Y} := \inf_{v \in I_x} V \left( v \mid x; h \right) \rightarrow \inf_{v \in I_x} \mathcal{Y} \left( v \mid x \right)$ as $h \downarrow 0$. By (19), (36) and continuity of $f_{x|x}(\cdot \mid x)$, $\Delta_{x,j}^{-1}$, $\Delta'_{x,j}$, $f_{x\Delta}(\cdot, d, x)$, $g(d, x, \cdot)$ and $f_{dx|c}$, we have $\inf_{v \in I_x} \mathcal{Y} \left( v \mid x \right) > 0$. Therefore, when $h$ is sufficiently small, $\mathcal{Y} > \inf_{v \in I_x} \mathcal{Y} \left( v \mid x \right)/2 > 0$. By these results, we also have $\Pr \left[ \inf_{v \in I_x} \hat{V} \left( v \mid x; \hat{h}, \hat{h}_\zeta \right) > \inf_{v \in I_x} \mathcal{Y} \left( v \mid x \right)/2 \right] = 1 - O \left( \kappa_{2,n}^V + \delta_n + \delta_n^\zeta \right)$. By these results,

$$\left| \frac{\sqrt{V(v \mid x; h)}}{\hat{V}(v \mid x; \hat{h}, \hat{h}_\zeta)} - 1 \right| \leq \frac{\left| V(v \mid x; h) - \hat{V}(v \mid x; \hat{h}, \hat{h}_\zeta) \right|}{\hat{V}(v \mid x; \hat{h}, \hat{h}_\zeta)} = O_p^* \left( \kappa_{1,n}^V + \varepsilon_n h, \kappa_{2,n}^V + \delta_n + \delta_n^\zeta \right), \quad (56)$$

uniformly in $v \in I_x$. Then, by this result, (54), (55) and $S(v \mid x, h) = O_p^* \left( \sqrt{\log(n)}, \sqrt{\log(n) / (n h^4)} \right)$, we have

$$Z(v \mid x; \hat{h}, \hat{h}_\zeta) - \frac{S(v \mid x; h)}{\sqrt{V(v \mid x; h)}} = O_p^* \left( \varepsilon_n \sqrt{\log(n)} + \Delta, \varepsilon_n h \right) \sqrt{\Delta + \varepsilon_n h^2} + \kappa_{1,n}^V \sqrt{\log(n)}, \kappa_{2,n}^V + \delta_n + \delta_n^\zeta, \quad (56)$$

uniformly in $v \in I_x$. By (50) and $\mathcal{Y} > \inf_{v \in I_x} \mathcal{Y} \left( v \mid x \right)/2 > 0$ when $h$ is sufficiently small, we have the deviation bound:

$$\Pr \left[ \left\| Z(v \mid x; \hat{h}, \hat{h}_\zeta) \right\|_{L^q} - \left\| U_n^{(2)} \right\|_{L^q} \right] > C_1 \left( \kappa_{1,n}^V \sqrt{\log(n)} + \varepsilon_n \sqrt{\log(n)} + \sqrt{\Delta h^5} \right) \leq C_2 \left( \kappa_{2,n}^V + \delta_n + \delta_n^\zeta \right). \quad (57)$$

By similar arguments used in the proof of Lemma 4 and the fact that when $h$ is sufficiently small, $\mathcal{Y} > \inf_{v \in I_x} \mathcal{Y} \left( v \mid x \right)/2 > 0$, $\mathcal{M} := \left\{ \mathcal{M}(\cdot \mid v; h) / \sqrt{V(v \mid x; h)} : v \in I_x \right\}$ is uniformly VC-type with respect to a constant envelope $F_{3\mathcal{M}} = O \left( h^{-q/2} \right)$. By Lemma A.3 of CK, $\mathcal{M}^{[1]}$ is also uniformly VC-type with respect to a constant envelope $F_{3\mathcal{M}}^{[1]} = F_{3\mathcal{M}}$. Let $\sigma_{2\mathcal{M}}^2 := \sup_{f \in \mathcal{M}} E \left[ f(U_1, U_2)^2 \right]$ and $\sigma_{2\mathcal{M}}^2 := \sup_{f \in \mathcal{M}} \sup_{v \in I_x} E \left[ f(U_1, U_2)^2 \right]$. By $E \left[ \mathcal{M}(U_1, U_2)^2 / V(v \mid x; h) \right] = 1 + h \cdot \| m_{\Delta x} (\cdot, x; h) \|_{L^q}^2 / \left( \rho^2 \mathcal{V} \right) = O \left( 1 \right)$. By calculations in the proof of Lemma 4, $\sigma_{2\mathcal{M}}^2 = O \left( h^{-2} \right)$. Denote $\mathcal{M}_\pm := \mathcal{M} \cup (-\mathcal{M})$ and $\mathcal{M}_\pm^{[1]} := \mathcal{M}^{[1]} \cup (-\mathcal{M}^{[1]})$. The coupling theorem (CK Proposition 2.1 with $\mathcal{H} = \mathcal{M}_\pm$, $\sigma_\theta = \sigma_{2\mathcal{M}}^{[1]}$, $\sigma_h = \sigma_{2\mathcal{M}}$, $b_\theta = b_h = F_{3\mathcal{M}}$, $\chi_n = 0$ and $q = \infty$) implies that when $n$ is sufficiently large ($n \geq \left( V_{3\mathcal{M}} \log \left( A_{3\mathcal{M}} \cap \mathcal{M}_\pm \right) \right) \vee 3$), for each coupling error $\gamma \in (0, 1)$, one can construct a random variable $Z_{3\mathcal{M}, \gamma}$ that satisfies the
following conditions: $Z_{\mathfrak{M} \pm, \gamma} = \sup \{ G^U (f) = \| G^U \|_{\mathfrak{M}[1]} \}$, where $\{ G^U (f) : f \in \mathfrak{M}[1] \}$ is a centered separable Gaussian process that has the same covariance structure as the Hájek process $\{ G^U (f) : f \in \mathfrak{M}[1] \}$, for every $\mathfrak{M} \subseteq \mathfrak{M}[1]$, and the difference between $\sup_{f \in \mathfrak{M} \pm} \| G^U (f) \|_{\mathfrak{M}[1]}$ and $Z_{\mathfrak{M} \pm, \gamma}$ satisfies the deviation bound:

$$\Pr \left[ \left\| \| G^U \|_{\mathfrak{M}[1]} - Z_{\mathfrak{M} \pm, \gamma} \right\| > C_1 \kappa_{\mathfrak{M} \pm} (\gamma) \right] \leq C_2 \left( \gamma + n^{-1} \right),$$

(58)

where $\kappa_{\mathfrak{M} \pm} (\gamma) := \log (n)^{2/3} / \left( \gamma^{1/3} (n h^2)^{1/6} \right) + \log (n) / \left( \gamma \sqrt{nh^2} \right)$. By the Gaussian anti-concentration inequality (Chernozhukov et al., 2014a, Corollary 2.1), since $E \left[ \| G^U \|_{\mathfrak{M}[1]} \right] = 1 \forall f \in \mathfrak{M}[1]$, we have

$$\sup_{t \in \mathbb{R}} \Pr \left[ \| G^U \|_{\mathfrak{M}[1]} - t \leq \varepsilon \right] \leq \varepsilon \left( E \left[ \| G^U \|_{\mathfrak{M}[1]} \right] + 1 \right), \forall \varepsilon > 0.$$  

(59)

By Dudley’s metric entropy bound (Giné and Nickl, 2016, Theorem 2.3.7), Lemma A.2 of CK and calculations (see the proof of Lemma 4 for details),

$$E \left[ \| G^U \|_{\mathfrak{M}[1]} \right] \leq \left( \sigma_{\mathfrak{M}[1]} \vee n^{-1/2} \| F_{\mathfrak{M}[1]} \|_{\tau^U, 2} \right) \sqrt{\log (n)},$$

(60)

when $n$ is sufficiently large. Then, since $Z_{\mathfrak{M} \pm, \gamma} = \sup \{ G^U \|_{\mathfrak{M}[1]} \}$, by Chernozhukov et al. (2016, Lemma 2.1), (57) and (58), when $n$ is sufficiently large, $\forall \gamma \in (0, 1)$,

$$\sup_{t \in \mathbb{R}} \left| \Pr \left[ \left\| Z \right\|_{I_x} \leq t \right] - \Pr \left[ \| G^U \|_{\mathfrak{M}[1]} \leq t \right] \right| \leq \varepsilon \left( E \left[ \| G^U \|_{\mathfrak{M}[1]} \right] + 1 \right) \leq C_1 \left( \kappa_{\mathfrak{M} \pm} (\gamma) \sqrt{\log (n)} + \varepsilon \right) + C_2 \left( \gamma + n^{-1} \right).$$

(61)

By (59), (60) and optimally choosing $\gamma$ that gives the fastest rate of convergence of the right hand side of (61), which should balance $\gamma$ and $\kappa_{\mathfrak{M} \pm} (\gamma) \sqrt{\log (n)}$ and set $\gamma = \log (n)^{7/8} / (nh^2)^{1/8}$, we have (51).

We apply the JMB coupling theorem (Theorem 3.1 of CK) with $\mathcal{H} = \mathfrak{M}[1]$, $\sigma = \sigma_{\mathfrak{M}[1]}$, $b = b_0 = \nu = F_{\mathfrak{M}[1]}$, $\chi = 0$ and $q = \infty$. When $n$ is sufficiently large (so that the assumptions in Equation (9) of CK are satisfied and $n \geq 3$), for any coupling error $\gamma \in (0, 1)$, there exists a random variable $Z_{\mathfrak{M} \pm, \gamma}$ such that (1) $Z_{\mathfrak{M} \pm, \gamma}$ is independent of the data; (2) $Z_{\mathfrak{M} \pm, \gamma}$ has the same distribution as $\| G^U \|_{\mathfrak{M}[1]}$; (3) $Z_{\mathfrak{M} \pm, \gamma}$ and $Z_{\mathfrak{M} \pm, \gamma}$ satisfies the deviation bound: when $n$ is sufficiently large, $\forall \gamma \in (0, 1)$,

$$\Pr \left[ \left| \right\| Z_{\mathfrak{M} \pm, \gamma} \|_{I_x} - Z_{\mathfrak{M} \pm, \gamma} \right\| > C_1 \kappa_{\mathfrak{M} \pm} (\gamma) \right] \leq C_2 \left( \gamma + n^{-1} \right).$$
where \( \kappa_{2\nu \pm}^2(\gamma) := \log(n)^{3/4} / \left( \gamma^{3/2} \left( nh^3 \right)^{1/4} \right) \) and then, by Markov’s inequality,

\[
\Pr \left[ \Pr_{W_t^n} \left[ \left\| Z_{\text{jmb}} (\cdot | x; \bar{h}) \right\|_{L_2} - Z_{2\nu \pm}^2 \right] > C_1 \kappa_{2\nu \pm}^2(\gamma) \right] \leq \frac{\sqrt{C_2 (\gamma + n^{-1})}}{\sqrt{C_2 (\gamma + n^{-1})}}.
\]

By this result and Lemma 9, when \( n \) is sufficiently large, \( \forall \gamma \in (0, 1) \), with probability at least \( 1 - C_3 \left( \sqrt{\gamma} + \kappa_{2\nu \pm}^V + \delta_n + \delta_n^\epsilon \right) \),

\[
\Pr_{W_t^n} \left[ \left\| Z_{\text{jmb}} (\cdot | x; \hat{h}, \hat{h}_\zeta) \right\|_{L_2} - Z_{2\nu \pm}^2 \right] > C_1 \left( \kappa_{1,n}^V \sqrt{\log(n)} + \varepsilon_n \sqrt{\log(n)} + \kappa_{2\nu \pm}^2(\gamma) \right)
\leq C_2 \left( \sqrt{\gamma} + n^{-1/2} \right).
\]

Then, since \( Z_{2\nu \pm}^2 \) is independent of the data and \( Z_{2\nu \pm}^2 = \| G^U \|_{2\nu [1]} \), by the above deviation bound and Chernozhukov et al. (2016, Lemma 2.1), with probability at least \( 1 - C_3 \left( \sqrt{\gamma} + \kappa_{2\nu \pm}^V + \delta_n + \delta_n^\epsilon \right) \),

\[
\sup_{t \in \mathbb{R}} \Pr_{W_t^n} \left[ \left\| Z_{\text{jmb}} (\cdot | x; \hat{h}, \hat{h}_\zeta) \right\|_{L_2} \leq t \right] - \Pr \left[ \left\| G^U \right\|_{2\nu [1]} \leq t \right] \leq \sup_{t \in \mathbb{R}} \Pr \left[ \left\| G^U \right\|_{2\nu [1]} - t \right] \leq C_1 \left( \kappa_{1,n}^V \sqrt{\log(n)} + \varepsilon_n \sqrt{\log(n)} + \kappa_{2\nu \pm}^2(\gamma) \right) + C_2 \left( \sqrt{\gamma} + n^{-1/2} \right). \tag{62}
\]

Then (52) follows from (59), (60) and optimally choosing \( \gamma \) that gives the fastest rate of convergence of the upper bound in (62), which should balance \( \sqrt{\gamma} \) and \( \kappa_{2\nu \pm}^2(\gamma) \sqrt{\log(n)} \) and set \( \gamma = \log(n)^{5/8} / \left( nh^3 \right)^{1/8} \).

Let \( F_G(t) := \Pr \left[ \left\| G^U \right\|_{2\nu [1]} \leq t \right] \) and \( F_G^{-1}(1 - \alpha) := \inf \{ t \in \mathbb{R} : F_G(t) \geq 1 - \alpha \} \) denote the (unconditional) CDF and quantile of \( \left\| G^U \right\|_{2\nu [1]} \). It is easy to check that (59) implies that \( F_G \) is continuous everywhere. Suppose that \( \sup_{t \in \mathbb{R}} \Pr_{W_t^n} \left[ \left\| Z_{\text{jmb}} (\cdot | x; \hat{h}, \hat{h}_\zeta) \right\|_{L_2} \leq t \right] - F_G(t) \leq C_2 \kappa_{2\nu \pm}^2, \)

and (52) implies that this event happens with probability greater than \( 1 - C_3 \kappa_{3\nu \pm}^2 \). Then, \( F_G(z_{\text{jmb}}^{\alpha}) \geq 1 - \alpha - C_2 \kappa_{2\nu \pm}^2 \) and \( \Pr_{W_t^n} \left[ \left\| Z_{\text{jmb}} (\cdot | x; \hat{h}, \hat{h}_\zeta) \right\|_{L_2} \leq F_G^{-1}(1 - \alpha + C_2 \kappa_{2\nu \pm}^2) \right] \geq 1 - \alpha, \)

by Van der Vaart (2000, Lemma 21.1(ii)). Then we have \( F_G^{-1}(1 - \alpha - C_2 \kappa_{2\nu \pm}^2) \leq z_{\text{jmb}}^{\alpha} \leq F_G^{-1}(1 - \alpha + C_2 \kappa_{2\nu \pm}^2) \) and such an event happens with probability greater than \( 1 - C_3 \kappa_{3\nu \pm}^2 \). Now using \( \Pr \left[ z_{\text{jmb}}^{\alpha} \leq F_G^{-1}(1 - \alpha + C_2 \kappa_{2\nu \pm}^2) \right] > 1 - C_3 \kappa_{3\nu \pm}^2 \), we have

\[
\Pr \left[ \left\| Z (\cdot | x; \hat{h}, \hat{h}_\zeta) \right\|_{L_2} \leq z_{\text{jmb}}^{\alpha} \right] \leq \Pr \left[ \left\| Z (\cdot | x; \hat{h}, \hat{h}_\zeta) \right\|_{L_2} \leq F_G^{-1}(1 - \alpha + C_2 \kappa_{2\nu \pm}^2) \right] + C_3 \kappa_{3\nu \pm}^2
\leq F_G \left( F_G^{-1}(1 - \alpha + C_2 \kappa_{2\nu \pm}^2) \right) + C_1 \kappa_{1,n} + C_3 \kappa_{3\nu \pm}^2
\leq 1 - \alpha + C_1 \kappa_{1,n} + C_2 \kappa_{2\nu \pm}^2 + C_3 \kappa_{3\nu \pm}^2, \tag{63}
\]

where the second inequality follows from (51) and the equality follows from continuity of \( F_G \) and
Van der Vaart (2000, Lemma 21.1(ii)). By using \( \Pr \left[ F_G^{-1} \left( 1 - \alpha - C_2 \hat{\kappa}^z_{2,n} \right) \right] > z_{1-\alpha}^{\text{mb}} \geq 1 - C_3 \hat{\kappa}^z_{3,n} \) and similar arguments, we have \( \Pr \left[ \left\| Z \left( \cdot \mid x; \hat{h}, \hat{h}_\zeta \right) \right\|_{L_x} > z_{1-\alpha}^{\text{mb}} \right] \leq \alpha + C_1 \hat{\kappa}_{1,n} + C_2 \hat{\kappa}^z_{2,n} + C_3 \hat{\kappa}^z_{3,n} \). Then, the conclusion of the theorem follows.

Appendix C  Nonparametric bootstrap confidence band

A nonparametric bootstrap sample \( \{W_1^n, ..., W_n^n\} \) consists of \( n \) independent draws from the original sample \( W_1^n := \{W_1, ..., W_n\} \) (\( W_i := (Y_i, D_i, X_i, Z_i) \)) with replacement. Let \( \hat{\phi}^{(-i)*}_d(y) \) be the bootstrap analogue of \( \hat{\phi}^{(-i)}(y) \), i.e., \( \hat{\phi}^{(-i)*}_d(y) \) is the minimizer of the bootstrap analogue of (5) computed using the bootstrap sample. Similarly, we construct the bootstrap analogues \( \hat{\Delta}_i^* \) and \( \hat{f}_{\Delta|X}^* (v \mid x; b) \) of \( \Delta_i \) and \( f_{\Delta|X} (v \mid x; b) \) respectively by computing the corresponding terms in (7) and (8) using the bootstrap sample. Let

\[
S_{\text{npb}} (v \mid x; b) := \sqrt{n b} \left( \hat{f}_{\Delta|X}^* (v \mid x; b) - \hat{f}_{\Delta|X} (v \mid x; b) \right)
\]

be the nonparametric bootstrap analogues of \( S (v \mid x; b) \) and \( Z (v \mid x; b, b_\zeta) \) respectively. We define \( z_{1-\alpha}^{\text{npb}} \) as the \( 1 - \alpha \) quantile of the conditional distribution of \( \left\| Z_{\text{npb}} \left( \cdot \mid x; \hat{h}, \hat{h}_\zeta \right) \right\|_{L_x} \). A nonparametric bootstrap confidence band can be constructed similarly to (9):

\[
CB_{\text{npb}} (v \mid x; \hat{h}, \hat{h}_\zeta) := \left[ \hat{f}_{\Delta|X} (v \mid x; \hat{h}) \pm z_{1-\alpha}^{\text{npb}} \sqrt{\hat{V} (v \mid x; \hat{h}, \hat{h}_\zeta) / n \hat{h}} \right].
\]

The bootstrap critical value \( z_{1-\alpha}^{\text{npb}} \) can be estimated by Monte Carlo simulations. The procedure is summarized below. Undersmoothing in Step 3 is not needed if one takes the bias correction approach by replacing the kernel \( K \) with \( M (\cdot; b, b_\zeta) \) in (64). It is also straightforward to adapt the algorithm to construct nonparametric bootstrap confidence bands for the PDF conditioning on sub-vectors.

**Algorithm 2** (Nonparametric bootstrap). **Steps 1-4:** Same as those in Algorithm 1. **Step 5:** In each of the iterations \( r = 1, ..., B \), independently draw \( \{W_1^{(r)}, ..., W_n^{(r)}\} \) with replacement from the original sample; compute the pseudo ITEs \( \{\hat{\Delta}_1^{(r)}, ..., \hat{\Delta}_n^{(r)}\} \) by applying (5), (6), and (7) to the bootstrap sample \( \{W_1^{(r)}, ..., W_n^{(r)}\} \); compute \( \hat{f}_{\Delta|X}^{(r)} (v \mid x; \hat{h}) \) by applying (8) to \( \{\Delta_1^{(r)}, ..., \Delta_n^{(r)}\} \) and \( \{W_1^{(r)}, ..., W_n^{(r)}\} \); and compute \( Z_{\text{npb}}^{(r)} (v \mid x; \hat{h}, \hat{h}_\zeta) := S_{\text{npb}}^{(r)} (v \mid x; \hat{h}) / \sqrt{\hat{V} (v \mid x; \hat{h}, \hat{h}_\zeta)} \). **Step 6:** Compute the critical value:

\[
z_{1-\alpha}^{\text{npb}} = \inf \left\{ t \in \mathbb{R} : \frac{1}{B} \sum_{r=1}^{B} 1 \left( \max_{v \in I_x^r} \left| Z_{\text{npb}}^{(r)} (v \mid x; \hat{h}, \hat{h}_\zeta) \right| \leq t \right) \geq 1 - \alpha \right\}.
\]
Step 7: Compute the nonparametric bootstrap confidence band $CB_{npb}$ using (66) over $v \in I_x^G$.

Theorem C1 below extends the result of Theorem B2 to the nonparametric bootstrap. It shows that the coverage probability errors of the nonparametric bootstrap UCB in (65) decay at a polynomial rate. The proof is found in the online supplement. In the proof, we show a nonparametric bootstrap version of (20). By the nonparametric bootstrap coupling theorem of Chernozhukov et al. (2016), we have

\[
\Pr \left[ \sup_{t \in \mathbb{R}} \left\{ \Pr_{W^t} \left[ \left\| Z_{npb} \left( \cdot \mid \hat{h}, \hat{h}_\xi \right) \right\|_{I_x} \leq t \right] - \Pr \left[ \left\| G^U \right\|_{\mathbb{R}[1]} \leq t \right] \right\} \leq C_2 \kappa_{2,n}^* \right] > 1 - C_3 \kappa_{3,n}^*, \tag{67}
\]

where

\[
\kappa_{2,n}^* := \left( \frac{\log (n)^5}{nh^3} \right)^{1/12} + \log (n) \kappa_{1,n}^v + \log (n) \varepsilon_n \text{ and } \kappa_{3,n}^* := \left( \frac{\log (n)^5}{nh^3} \right)^{1/12} + \delta_n + \delta_n^\xi.
\]

The result is then implied by (51) and (67).

**Theorem C1.** Suppose that the assumptions of Theorem 4 hold. Then,

\[
\Pr \left[ f_{\Delta \mid X} (v \mid x) \in CB_{npb} \left( v \mid x; \hat{h}, \hat{h}_\xi \right), \forall v \in I_x \right] = (1 - \alpha) + O \left( \left( \frac{\log (n)^5}{nh^3} \right)^{1/12} + \kappa_{1,n}^v \sqrt{\log (n)} + \sqrt{\log (n)} \sqrt{nh^5} + \varepsilon_n \sqrt{\log (n)} + \delta_n + \delta_n^\xi \right). \tag{68}
\]

A non-studentized constant-width nonparametric bootstrap confidence band can be constructed as

\[
\tilde{CB}_{npb} \left( v \mid x; \hat{h} \right) := \left[ \tilde{f}_{\Delta \mid X} \left( v \mid x; \hat{h} \right) \pm s_{1-\alpha}^{npb} \right],
\]

where $s_{1-\alpha}^{npb}$ denotes the 1 − $\alpha$ quantile of the conditional distribution of $\left\| S_{npb} \left( \cdot \mid x; \hat{h} \right) \right\|_{I_x}$. Note that $\tilde{CB}_{npb} \left( v \mid x; \hat{h} \right)$ does not require estimation of $\zeta_{dx}$ and the additional tuning parameters $b_\xi$ and $K_\xi$. One can show deviation bounds similar to (51) and (52): there are positive constants $C_1, C_2, C_3$ such that

\[
\sup_{t \in \mathbb{R}} \left\{ \Pr \left[ S \left( \cdot \mid x; \hat{h} \right) \right\|_{I_x} \leq t \right] - \Pr \left[ \left\| G^U \right\|_{\mathbb{R}[1]} \leq t \right] \right\} \leq C_1 \kappa_{1,n}, \tag{69}
\]

and

\[
\Pr \left[ \sup_{t \in \mathbb{R}} \left\{ \Pr_{W^t} \left[ \left\| S_{npb} \left( \cdot \mid x; \hat{h} \right) \right\|_{I_x} \leq t \right] - \Pr \left[ \left\| G^U \right\|_{\mathbb{R}[1]} \leq t \right] \right\} \leq C_2 \kappa_{2,n}^* \right] > 1 - C_3 \kappa_{3,n}^*, \tag{70}
\]

where

\[
\kappa_{1,n} := \left( \frac{\log (n)^7}{nh^3} \right)^{1/8} + \sqrt{\log (n)} \sqrt{nh^5} + \log (n) \varepsilon_n + \delta_n.
\]
κ^*_{2,n} := \left( \frac{\log(n)^5}{nh^3} \right)^{1/12} + \log(n) \varepsilon_n \quad \text{and} \quad κ^*_{3,n} := \left( \frac{\log(n)^5}{nh^3} \right)^{1/12} + \delta_n.

Then by (69) and (70), we have the following result.

**Theorem C2.** Suppose that Assumptions 1-4 hold with \( P = 2 \), the third-order derivative functions in Assumption 2(a) are Lipschitz continuous and \( h \propto n^{-\lambda} \) with \( 1/5 < \lambda < 1/3 \),

\[
\Pr \left[ f_{\Delta | X}(v \mid x) \in \bar{CB}_{npb}(v \mid x; \hat{h}), \forall v \in I_x \right] = (1 - \alpha) + O \left( \left( \frac{\log(n)^5}{nh^3} \right)^{1/12} + \sqrt{\log(n)} \sqrt{nh^5} + \log(n) \varepsilon_n + \delta_n \right).
\]