Some monoids of Pisot matrices

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Abstract

A matrix norm gives an upper bound on the spectral radius of a matrix. Knowledge on the location of the dominant eigenvector also leads to upper bound of the second eigenvalue. We show how this technique can be used to prove that certain semi-group of matrices arising from continued fractions have a Pisot spectrum: namely for all matrices in this semi-group all eigenvalues except the dominant one is smaller than one in absolute value.

1 Introduction

A dominant eigenvalue of a real square matrix is an eigenvalue of maximum modulus. We call a square matrix Pisot if it has non-negative integer entries, its dominant eigenvalue is simple and all eigenvalues different from the dominant one have absolute values less than one. We prove that several monoids of non-negative matrices enjoy the property of all being Pisot.

Our first family of matrices is related to the so called fully subtractive (multidimensional) continued fraction algorithm. For an integer \( d \geq 2 \) we define for each \( k = 1, \ldots, d \) the matrix \( A^{(k)}_{FS,d} \)

\[
(A^{(k)}_{FS,d})_{ij} = \begin{cases} 
1 & \text{if } j = k \text{ or } i = j, \\
0 & \text{otherwise}
\end{cases}
\]

For \( d = 3 \) this boils down to the three matrices

\[
A^{(1)}_{FS,3} = \begin{pmatrix} 1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1 \end{pmatrix}, \quad A^{(2)}_{FS,3} = \begin{pmatrix} 1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1 \end{pmatrix}, \quad A^{(3)}_{FS,3} = \begin{pmatrix} 1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \end{pmatrix}.
\]

All non-degenerate products of the matrices \( A^{(k)}_{FS,d} \) satisfy the Pisot property.

Theorem 1. Let \( A = A^{(i_1)}_{FS,d} A^{(i_2)}_{FS,d} \cdots A^{(i_n)}_{FS,d} \) be a product of the fully subtractive matrices in dimension \( d \). Then the matrix \( A \) is primitive if and only if all letters \( \{1, \ldots, d\} \) appear in the sequence \( (i_1, i_2, \ldots, i_n) \). Moreover, if the matrix \( A \) is primitive then it is Pisot.

Recall that a non-negative square matrix \( A \) is primitive if there exists a positive integer \( n \) so that \( A^n \) has all its entries positive. The case \( d = 3 \) of Theorem 1 was proved in [ArIt01]. The authors used an induction on characteristic polynomials and our approach is radically different.

The same result holds for another set of \( 3 \times 3 \) matrices related to the Brun multidimensional continued fractions. Let

\[
A^{(1)}_{Br} = \begin{pmatrix} 1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}, \quad A^{(2)}_{Br} = \begin{pmatrix} 1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1 \end{pmatrix}, \quad A^{(3)}_{Br} = \begin{pmatrix} 1 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \end{pmatrix}.
\]

Theorem 2. Let \( A = A^{(i_1)}_{Br} A^{(i_2)}_{Br} \cdots A^{(i_n)}_{Br} \) be a product of the \( 3 \times 3 \) Brun matrices. Then, \( B \) is primitive if and only if the matrix \( B^{(3)} \) appears in the product. Moreover, if \( B \) is primitive then it is Pisot.

This result was already known since the work of Brun [Br57].

The proofs of Theorem 1 and 2 are elementary and uses the following inequality. Given a non-negative primitive \( d \times d \) matrix \( A \) and its Perron-Frobenius eigenvector \( v \in \mathbb{R}_d^+ \) the absolute value \( \lambda_2 \) of its second largest eigenvalue satisfies

\[
\lambda_2 \leq \sup_{x \in v^+ \setminus \{0\}} \frac{\|Ax\|}{\|x\|}.
\]
where \( \| \cdot \| \) is any norm on \( \mathbb{R}^d \). In our proof, the information we have on the localization of the Perron-Frobenius eigenvector comes from the dynamical systems induced by the matrices; in other words the fully subtractive and Brun continued fraction algorithms.

From a diophantine approximation point of view, the Pisot property is particularly interesting because it provides the so called exponential convergence of the continued fraction expansion for almost every vectors (see [La93]). We show that the above results naturally extends to this situation in Section 3.

Beyond continued fractions, Pisot matrices are of special interest in substitutive dynamical systems. More precisely, replacing matrices with so called substitutions, the associated dynamical systems admit \( d - 1 \) eigenvalues and in many cases the dynamical system can be proved to have purely discrete spectrum (see [PV02] Chapter 7). This was the main motivation for the study of the fully subtractive matrices in [Ar101].

2 Fully subtractive and Brun continued fractions

Let us consider a finite or countable set \( \mathcal{A} \) that we call alphabet and for each \( i \in \mathcal{A} \) a matrix \( A(i) \in \text{SL}(d, \mathbb{Z}) \). We already saw two examples of this with the fully subtractive algorithm where \( A_{FS,d} = \{1, 2, \ldots, d\} \) and for Brun algorithm where \( A_{Br} = \{1, 2, 3\} \).

To the data \((\mathcal{A}, (A(i))_{i \in \mathcal{A}})\) we associate the set of infinite words \( \Delta = \mathbb{A}^\infty \), the shift map \( T : \Delta \to \Delta \) and a cocycle
\[
\forall x \in \Delta, \forall n \geq 0, \quad A_n(x) = A(x_0)A(x_1)\ldots A(x_{n-1}).
\]
The maps \( A_n : \Delta \to \text{SL}(d, \mathbb{Z}) \) satisfy the so called cocycle property: \( A_{m+n}(x) = A_m(x)A_n(T^m x) \).

**Definition 3.** Let \((A(i))_{i \in \mathcal{A}}\) be a set of matrices in \( \text{SL}(d, \mathbb{Z}) \) where \( \mathcal{A} \) is a finite or countable alphabet. We say that a set \( D \subset \mathbb{P}(\mathbb{R}^d) \) is adapted to these matrices if it is non-empty, it is the closure of its interior and for all \( i \in \mathcal{A} \) we have \( A(i)D \subset D \).

For example \( PP(\mathbb{R}^d) \) is always adapted. But we will be interested in the somewhat smallest adapted set in order to localize the dominant eigenvector.

Given \((A(i))_{i \in \mathcal{A}}\) and \( D \subset \mathbb{P}(\mathbb{R}^d) \) adapted, we define \( D(i) = A(i)D \) and more generally for a finite word \( w = i_0 i_1 \ldots i_{n-1} \) we define \( D(w) = A(i_0)A(i_1)\ldots A(i_{n-1})D \). Note that for any \( w \) the set \( D(w) \) is not empty. Given an infinite word \( x = x_0x_1\ldots \in \Delta \) we also set \( D_n(x) = A_n(x)D \) and \( D_\infty(x) = \bigcap_{n \geq 0} D_n(x) \).

Let \( A \in \text{SL}(d, \mathbb{Z}) \) and let \( \lambda \) be its spectral radius. Consider its Jordan decomposition over \( \mathbb{C} \) and the Jordan blocks associated with an eigenvalue of modulus \( \lambda \) and being of maximal dimension. To each of these maximal Jordan block is associated exactly one eigenvector \( v_i \). The dominant eigenspace of \( A \) is \( \mathbb{R}^d \cap (Cv_1 \oplus Cv_2 \oplus \ldots \oplus Cv_k) \). We have the following elementary result.

**Lemma 4.** Let \((A(i))_{i \in \mathcal{A}}\) be a finite or countable set of matrices in \( \text{SL}(d, \mathbb{Z}) \) and let \( D \) be adapted. Let \( x = x_0x_1\ldots \in \Delta \) be an infinite word over \( \mathcal{A} \). Then for any \( n \), the set \( D_n(x) \) contains a basis of the dominant eigenspace of \( A_n(x) \).

We omit the proof that only uses the fact that the maximum growth of \( \| A^n u \| \) is \( \lambda^n n^k \) where \( k \) is the maximal dimension of a Jordan block associated with an eigenvalue of maximal modulus of \( A \).

Let
\[
D_{FS,d} = \{(x_1, \ldots, x_d) \in \mathbb{P}(\mathbb{R}^d_+) : \forall i, j, k \quad x_i < x_j + x_k \}\]
and
\[
D_{Br} = \{(x, y, z) \in \mathbb{P}(\mathbb{R}^3_+) : x > y > z \}.
\]

Then it is easily seen that \( D_{FS,d} \) is adapted for the fully subtractive matrices in dimension \( d \) and \( D_{Br} \) is adapted for the Brun matrices. In figures 4 and 5 one can see the projective picture of the domains \( D^{(1)} \), \( D^{(2)} \) and \( D^{(3)} \). Note that in these cases, the domains \( D^{(i)} \) are disjoint but that it is not a requirement in our definition. Moreover, one can see that in the Brun case the \( D^{(i)} \) form a partition while it is not the case for the fully subtractive.

If the \( D^{(i)} \) are disjoint one can define a continued fraction algorithm as follows. One defines a partial map \( f : D \to D \) by setting \( f(x) = (A(i))^{-1}x \) on \( D^{(i)} \).

One can compute that for Brun one has
\[
f_{Br}(x) = \text{sort}(x - y, y, z)
\]
A_{FS,3}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_{FS,3}^{(2)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_{FS,3}^{(3)} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.

Figure 1: Fully subtractive partition of the domains with $d = 3$.

A_{Br}^{(1)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_{Br}^{(2)} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_{Br}^{(3)} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.

Figure 2: The matrices and domains for the Brun algorithm.

where sort $: \mathbb{P}(\mathbb{R}_+^d) \to D$ is the map which permutes the coordinates in order to sort them. While for the fully subtractive one has

$f_{FS,d}(x) = (x_1 - x_i, x_2 - x_i, \ldots, x_{i-1} - x_i, x_i, x_{i+1} - x_i, \ldots, x_d - x_i)$ if $x_i = \min(x_1, \ldots, x_d)$.

3 Strategy

The proofs of Theorems 1 and 2 follow a general strategy that we describe now. We let $\|\cdot\|$ be the $L_\infty$ norm on $\mathbb{R}^d$ and the associated operator norm on matrices. That is for a vector $v$ and a matrix $A$

$$\|v\| = \max(|v_1|, \ldots, |v_d|) \quad \text{and} \quad \|A\| = \max_{i=1,\ldots,d, j=1,\ldots,d} |a_{ij}|.$$ 

In this section the norm used on $\mathbb{R}^d$ has no importance. But it turns out that, to apply the results to continued fraction algorithms, the most convenient one was always the $L_\infty$ norm.

To a non-zero vector $v$ in $\mathbb{R}^d$, we associate its dual hyperplane $H_v = \{ z \in \mathbb{R}^d; (v, z) = 0 \}$. Given a non-zero vector $v$ in $\mathbb{R}^d$ we define the following semi-norm on $d \times d$ matrices

$$\|B\|_v = \sup_{z \in H_v \setminus \{0\}} \frac{\|Bz\|}{\|z\|} = \max_{z \in H_v} \|Bz\|.$$ 

More generally, if $\Lambda \subset \mathbb{R}^d$ is a cone, we define

$$\|B\|_{\Lambda} = \sup_{v \in \mathbb{P}(\Lambda)} \|B\|_v.$$ 

Let $(A^{(i)})_{i \in \Lambda}$ be a finite or countable set of matrices as in Section 2. Let also $\Delta = A^N$ and $D \subset \mathbb{P}(\mathbb{R}^d)$ be adapted. Recall that $D_n(x) = A_n(x)D = D(xa; \ldots, x_0)\Delta$ (in particular, $D_0(x) = D$ and $D_1(x) = D^{(x_0)}$) and that $D_\infty(x) = \bigcap_{n \geq 0} D_n(x)$.

**Lemma 5.** Let $(A^{(i)})_{i \in \Lambda}$ be a finite or countable set of matrices in $\text{SL}(d, \mathbb{Z})$ and let $D \subset \mathbb{P}(\mathbb{R}^d)$ be adapted. Let $B^{(i)}$ (respectively $B_n(x)$) denote the transposed of $A^{(i)}$ (resp. $A_n(x)$). If for all $i \in \Lambda$ we have

$$\|B^{(i)}\|_{D^{(i)}} \leq 1.$$ 

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Then for any point $x = x_0x_1 \ldots \in \Delta$ we have

$$\|B_n(x)\|_{D_n(x)} \leq 1.$$  

In particular, if $x = (x_0x_1 \ldots x_{p-1})^\infty$ is periodic the matrix $A_p(x) = A^{(x_0)}A^{(x_1)} \ldots A^{(x_{p-1})}$ has at most one eigenvalue greater than one in absolute value.

Proof. Let $B$ be the transposed cocycle. The hypothesis is just the case $n = 1$. Assume that this inequality holds for $n$. By definition $A_{n+1}(x) = A_1(x)A_n(Tx)$ and $D_{n+1}(x) = A_1(x)D_n(Tx)$. Hence $v \in D_n(Tx)$ if and only if $A_1(x)v \in D_{n+1}(x)$. Let us choose $v \in D_n(Tx)$, then

$$\|B_{n+1}(x)\|_{A_1(x)v} = \|B_n(Tx)B_1(x)\|_{A_1(x)v} \leq \|B_n(Tx)\|v \cdot \|B_1(x)\|_{A_1(x)v}.$$  

Now, let $x = (x_0x_1 \ldots x_{p-1})^\infty$ be a periodic point. Because $D_{p-1}(x) \supset D_\infty(x)$, the matrix $B_p(x)$ satisfies

$$\|B_p(x)\|_{D_\infty(x)} \leq 1.$$  

By Lemma 7 a basis of the dominant eigenspace of $B_p$ belongs to $D_\infty(x)$. Consequently, the union of the orthogonal of $D_\infty(x)$ contains all eigenspaces corresponding to the non-dominant eigenvalues. From the above inequality, we deduce that the absolute value of all eigenvalues different from the first one are bounded by $1$. \hfill \Box

4 Pisot property for Arnoux-Rauzy matrices

In this section we prove Theorem 1. Let $d \geq 2$ be an integer and let $e_1, e_2, \ldots e_d$ be the canonical basis of $\mathbb{R}^d$. Let $e = e_1 + e_2 + \ldots + e_d$ and for $i = 1, \ldots, k$ let $f_i = e - e_i$. The domain $D_{FS,d}$ is the convex hull of the rays vectors $\mathbb{R}_+ f_i$.

Let as usual $B^{(w)} = (A^{(w)})^*$ and $B_n(x) = (A_n(x))^*$. We claim that we even have a stronger property than what is required in Lemma 5

$$\forall i = 1, \ldots, d, \qquad \|B(i)\|_D \leq 1.$$  

Let us prove this claim. Let $v \in D_\infty$, then we may write $v = \mu_1 f_1 + \mu_2 f_2 + \ldots + \mu_d f_d$ for some non-negative numbers $\mu_i$ that satisfy $\mu_1 + \mu_2 + \ldots + \mu_d = 1$. We hence have $v = e - \sum \mu_i e_i$ and

$$H_v = \{z \in \mathbb{R}^d ; \langle z, v \rangle = 0 \} = \{z \in \mathbb{R}^d ; \sum z_j = 0 \}.$$  

Given $z \in H_v$ we have $B^{(i)} z = (z_1, \ldots, z_{i-1}, \sum \mu_z z_j, z_{i+1}, \ldots, z_d)$. In other words, $B^{(i)}$ acts on $H_v$ as a stochastic matrix $P(i, v)$ which is the identity except its $i$-th row which is $(\mu_1, \mu_2, \ldots, \mu_d)$. In particular, $\|B^{(i)}\|_v \leq 1$.

Now for a given finite product $A = A^{(i_0)}A^{(i_1)} \ldots A^{(i_{p-1})}$ if one of the letter $\{1, \ldots, d\}$ is missing in the sequence $(i_0, i_1, \ldots, i_{p-1})$ then $A e_i = e_i$ and so the matrix is not primitive. On the other hand, if all letters appear it is easy to see that all entries in $A$ are positive.

Now let $x = (x_0x_1 \ldots x_{p-1})^\infty$ be a periodic point that contains all letters from $A$. Because of positivity, all the orbits is contained in the interior of $D$ and the dominant eigenvalue is simple. Let $v_0 \in D$ be a dominant eigenvector and let $v_n = A_n(x)v_0$. Because all $(v_n)$ belongs to the interior of $D$ the coefficients $\mu_1, \mu_2, \ldots, \mu_d$ that appear in the stochastic matrices $P(x_i, v_i)$ are all positive. Now, the product $P(x_{p-1}, v_{p-1}) \ldots P(x_1, v_1)P(x_0, v_0)$ is a stochastic matrix with all its entries positive. Hence, its second eigenvalue, which is also the second eigenvalue of $A_p(x)$, is less than 1 in absolute value.

5 Pisot property for Brun algorithm (in dimension 3)

We now turn to the proof of Theorem 2. Let $A = \{1, 2, 3\}$ and $A^{(1)}, A^{(2)}, A^{(3)}$ be the matrix of the Brun algorithm. We let $\Delta = \mathbb{A}^{3}$ and denote by $A$ and $B$ respectively the cocyle and the transposed cocyle. We claim that, as in the case of the fully subtractive, we have the stronger property that

$$\forall i \in \{1, 2, 3\}, \quad \|B^{(i)}\|_{D^{(i)}} \leq 1.$$  


We only need to consider the matrix \( B^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) since the other two are obtained by multiplying by a permutation matrix which will not change the \( L^\infty \)-norm.

Let \( v = \mu_1(1 : 0 : 0) + \mu_2(1 : 1 : 0) + \mu_3(1 : 1 : 1) \in D \) for some \( \mu_1, \mu_2, \mu_3 \) such that \( \mu_1 + \mu_2 + \mu_3 = 1 \) and \( H_v = \{ z \in \mathbb{R}^3; z_1 + z_2 = \mu_1 z_2 - \mu_3 z_3 \} \). Now, for any \( z \in H_v \) we have \( B^{(1)}(z_1, z_2, z_3) = (z_1, z_1 + z_2, z_3) = (z_1, \mu_1 z_2 - \mu_3 z_3, z_3) \). In other words \( \| B^{(1)}(z_1, z_2, z_3) \|_1 \leq \| (z_1, z_2, z_3) \|_1 \).

Now, given a product \( A = A_{Br} r^{(i_1)} \ldots A_{Br} r^{(i_n)} \) it is easy to see that if 3 does not appear in the sequence \((i_1, i_2, \ldots, i_n)\) then \( A_{e3} = e_3 \) and hence the matrix \( A \) can not be irreducible. Conversely, if 3 appears then \( A^3 \) is easily seen to be positive.

Now consider the matrix \( P \) built from the beginning of the proof. As in the case of the fully subtractive algorithm for a primitive product we got that the \( \mu_i \) are all positive. Given a product \( A \) where the matrix \( A_{Br}^{(3)} \) appears, the matrix \( A^3 \) is then such that all rows are such that sum of their absolute values is strictly less than one. In other words \( \| A^3 \| < 1 \).

## 6 Lyapunov exponents

Let \((A^{(i)})_{i \in A}\) be a finite or countable set of matrices. Let \( \Delta, T, A, B \) denote as before the infinite words, the shift map the cocycle and the transposed cocycle. Let also \( D \) be adapted to these matrices.

The asymptotic of the cocycle (or the transposed cocycle) are studied through Lyapunov exponents. Given a \( T \)-invariant ergodic probability measure \( \mu \) on \( \Delta \), we associate the real numbers \( \gamma^\mu_1 \geq \gamma^\mu_2 \geq \ldots \geq \gamma^\mu_d \) defined by

\[
\forall k \in \{1, 2, \ldots, d\}, \quad \gamma_1 + \gamma_2 + \ldots + \gamma_k = \lim_{n \to \infty} \int_{\Delta} \log \| A_n(x) \| d\mu(x).
\]

In order to be well defined we assume that

\[
\int_{\Delta} \max \left( \log \| A_1(x) \|, \log \| A_1(x)^{-1} \| \right) d\mu(x) < \infty \tag{1}
\]

and we refer to this condition as the log-integrability of the cocycle. If the alphabet \( A \) is finite the cocycle is automatically log-integrable. If \( x \) is a periodic point of \( T \) and \( \mu = (\delta_x + \delta_{Tx} + \ldots + \delta_{T^{n-1}x})/n \) is the sum of Dirac masses distributed along its orbit, then the associated Lyapunov exponents are the logarithms of the average values of eigenvalues of \( A_n(x) \) where \( n \) is the period of \( x \). In that sense, Lyapunov exponents generalize eigenvalues.

Given a measure \( \mu \) for which the cocycle is log-integrable, we say that \((\Delta, T, A, \mu)\) has Pisot spectrum if the associated Lyapunov exponents satisfy \( \gamma^\mu_1 > 0 > \gamma^\mu_2 \). This property is related to the strong convergence of higher dimensional continued fraction algorithm [La93].

Now we restate Lemma 5 in a more dynamical context.

**Lemma 6.** Let \((A^{(i)})_{i \in A}\) be a finite or countable set of non-negative matrices in \( SL(d, \mathbb{Z}) \). Let \((\Delta, T, A, B)\) be the associated full shift with its cocycle and its transposed cocycle. Let also \( D \) be adapted. Assume that

\[
\forall i \in A, \quad \left\| \begin{pmatrix} A^{(i)} \end{pmatrix} \right\|_{D_{\mu}(i)} \leq 1.
\]

Let \( \mu \) be a \( T \)-invariant and ergodic measure on \( D \) so that

- the cocycle \( A_n \) is log-integrable,
- there exists a cylinder \([w]\) such that \( \mu([w]) > 0 \), \( A^{(w)} \) is positive and \( \| A^{(w)} \|_{D_{\mu}(\cdot)} < 1 \).

Then two first Lyapunov exponents of the cocycle \( A_n \) for the measure \( \mu \) satisfies \( \gamma^\mu_1 > 0 > \gamma^\mu_2 \).

**Proof.** Let us first prove that \( \gamma_1 > 0 \).

Now, by definition, for \( \mu \)-almost every \( x \)

\[
\gamma_1 = \lim_{n \to \infty} \frac{\log \| A_n(x) \|}{n}
\]

Let \( m = |w| \) be the length of \( w \) and consider the position which are multiple of \( m \). For a \( \mu \)-generic \( x \) we have by Birkhoff theorem that

\[
\lim_{n \to \infty} \frac{\# \{ i \leq n: T^i(x) \in [w] \}}{n} = \mu([w])
\]
In other words, given sequence of length \( n \) large enough we can find a linear number of disjoint occurrences of \( w \) (up to a sublinear error). Let \( k_n \) be the number of these occurrences, then necessarily each entry of \( A_n(x) \) is larger than the corresponding one in \( C^{k_n} \) where \( C \) is the matrix which contains a 1 in every position. In particular \( \gamma_1 > 0 \).

From the existence of \( w \) it also follows that for a \( \mu \)-generic \( x \) the cone \( D_\infty(x) \) is reduced to a line contained in the interior of \( \mathbb{R}^d_+ \). We can hence define \( \mu \)-almost everywhere a function \( v : \Delta \rightarrow \mathbb{R}^d_+ \) by \( D_\infty(x) = \mathbb{R}_+ v(x) \) and \( \|v(x)\| = 1 \). We then have the following formulas which holds for \( \mu \)-almost every \( x \)

\[
\gamma_1 = \lim_{n \to \infty} -\frac{\log \|A_n(x)^{-1}v(x)\|}{n} \quad \text{and} \quad \gamma_2 = \lim_{n \to \infty} \frac{\log \|B_n(x)\|}{n} \; d\mu(x).
\]

It is then easy to derive the estimate for \( \gamma_2 \). The map \( x \mapsto v(x) \) and the dual hyperplanes \( H_v(x) \) satisfy the following covariance properties

\[
D_\infty(Tx) = A(x)^{-1}D_\infty(x) \quad \text{and} \quad H_{A^{-1}v} = A^*H_v.
\]

Hence as in the proof of Lemma 5 we deduce that for all \( v \in D_\infty(x) \)

\[
\|B_{m+n}(x)\|_{v(x)} \leq \|B_m(x)\|_{v(x)}\|B_n(T^m x)\|_{v(T^m x)}
\]

In particular, for \( \mu \)-almost every \( x \in [w] \), any \( n \geq |w| \) we get that \( \|B_n(x)\|_{v(x)} < 1 \). Let \( \delta = \|B^{(w)}\|_{D^{(w)}} < 1 \). Using the same argument as in the estimation of \( \gamma_1 \) we get that

\[
\gamma_2 \leq \lim_{n \to \infty} \frac{k_n \log \delta}{n}.
\]

And the above limit is strictly negative. \( \square \)

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