ADDENDUM

Generating new perfect-fluid solutions from known ones: II

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Received 30 December 1996

Abstract. The properties of a transformation previously considered for generating new perfect-fluid solutions from known ones are further investigated. It is assumed that the 4-velocity of the fluid is parallel to the stationary Killing field, and also that the norm and the twist potential of the stationary Killing field are functionally related. This case is complementary to the case studied in our previous paper. The transformation can be applied to generate possibly new perfect-fluid solutions from known ones only for the case of the barotropic equation of state $\rho + 3P = 0$ or, alternatively, for the case of a static spacetime. For static spacetimes our method recovers the Buchdahl transformation. It is demonstrated, moreover, that Herlt’s technique for constructing stationary perfect-fluid solutions from static ones is, actually, a special case of the method considered in this paper.

PACS numbers: 0420J, 0440N

1. Introduction

A method has been developed in [1–3] for generating stationary perfect-fluid solutions from known ones was studied. This method was introduced as a generalization of the Geroch transformation [4] (see also [5]) given for vacuum spacetimes. It was assumed there that the 4-velocity of the fluid is parallel to the stationary Killing field, and also that the norm and the twist potential of the stationary Killing field are functionally independent. With these assumptions it was shown in [3] that the generalized transformation can be applied to create possibly new perfect-fluid solutions from known ones only for the case of the barotropic equation of state $\rho + 3P = 0$. In the present work we study the complementary problem, that is, here we assume that the norm and the twist potential of the stationary Killing field are functionally related and investigate the applicability of the generalized transformation. It turned out that either the equation of state of the fluid has to be the above restricted one for both the initial and the resultant configurations or the spacetime has to be static. In the latter case our transformation reduces to the Buchdahl transformation of perfect-fluid spacetimes [7, 8].

We would like to emphasize that most of the analysis given in section 2 of the present paper is also valid in the functionally independent case of [3]. In [3] we recalled that the functional independence of the twist potential and the norm of the Killing field ensures that

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§ We have just learnt that the transformation described in [1–3] is a rediscovery of a general Lie–Bäcklund transformation studied earlier by Stephani [6]. Nevertheless, due to the different approach we used to get our transformation, there are particular results (e.g. the relation of the generalized Geroch transformation to the Buchdahl transformation) described in the present paper which were out of the scope of [6] and deserve further attention.
any solution to the reduced set of equations (2.7)–(2.9) of [3] satisfies the complete system of Einstein’s equations. Using this fact it was possible to generalize the Geroch transformation to the selected perfect-fluid configurations. In the present paper, however, we use the full set of Einstein’s equations in order to introduce the generalized transformation and to study the restrictions on its applicability. This analysis is, in turn, actually independent of any assumptions about the functional dependence of the twist potential and norm of the Killing field.

This paper is organized as follows. We recall the field equations for the selected perfect-fluid configurations and study their symmetry properties. Then we determine the restrictions on the applicability of the generalized transformation for generating new perfect-fluid solutions for the case under consideration. Next, we examine the particular case of a static spacetime and determine the transformation of the equation of state. Finally, we study the relationship between Herlt’s technique [9] developed for generating stationary perfect-fluid solutions from static ones and the method considered in this paper. It is shown that the method introduced by Herlt can be considered as a special case of the one presented in this paper. In particular, it follows that the equation of state for both the initial and resulting perfect-fluid configurations has to be \( \rho + 3P = 0 \), whenever Herlt’s transformation is applied in order to get stationary solutions from static ones.

2. The symmetry properties of the field equations

Start with a stationary perfect-fluid spacetime with the 4-velocity vector, \( u^a \), parallel to the timelike Killing field, \( \xi^a \). Then

\[
u^a = (−v)^{1/2}\xi^a, \quad \text{(1)}
\]

where \( v = \xi_a\xi^a \) is the norm of the timelike Killing field. Consequently, the flow of the fluid is ‘rigid’ in the sense that it is shear- and expansion-free. It was shown in [10] that these perfect-fluid spacetimes can be considered as faithful representations of equilibrium states of real, dissipative relativistic fluids.

The existence of a Killing field allows one to use the three-dimensional formulation (projection formalism) of general relativity [4, 11]. Moreover, for the perfect-fluid configurations under consideration the twist \( \omega_a := \epsilon_{abcd}\xi^b \nabla_c \xi^d \) of the Killing field can always be expressed as a gradient of a function, \( \omega \), called the twist potential of the Killing field. The field equations for stationary perfect-fluid spacetimes having the 4-velocity of the fluid parallel to the timelike Killing field read

\[
H_{ab} := R_{ab} - 16\pi v^{-1}P h_{ab} = \frac{1}{4}v^{-2}[(D_a v)(D_b v) + (D_a \omega)(D_b \omega)], \quad \text{(2)}
\]

\[
D_m D^n v = v^{-1}[(D_m v)(D^n v) - (D_m \omega)(D^n \omega)] - 8\pi(\rho + 3P), \quad \text{(3)}
\]

\[
D_m D^n \omega = 2v^{-1}(D_m \omega)(D^n v), \quad \text{(4)}
\]

where \( \rho \) is the mass density and \( P \) is the pressure of the fluid, while \( R_{ab} \) and \( D_a \) are the Ricci tensor and covariant derivative associated with the three-dimensional Riemannian metric \( h_{ab} := −g_{ab} + \xi_a \xi_b \) (where \( g_{ab} \) denotes the spacetime metric). In addition to the above equations the Euler–Lagrange equation

\[
\partial_a P + \frac{1}{2}(\rho + P)\frac{\partial_a v}{v} = 0, \quad \text{(5)}
\]

which follows from the above ones, has to be satisfied.

In [3] it was shown that whenever the twist and the gradient of the norm of the Killing field are linearly independent then one can choose geometrically preferred local coordinates

† Note, however, that our conclusion applies merely to one of the methods introduced by Herlt in [9].
in which the functions \( v \) and \( \omega \) depend merely on two coordinates, say \( x^1 \) and \( x^2 \). In such coordinates it is straightforward to realize the analogy between the field equation (2) and the corresponding vacuum equation (for the study of the vacuum case see [4] and also [5]). Both equations have a symmetry property, which allows one to introduce a transformation for generating new solutions of Einstein’s equations from known ones. In this way, it was possible to generalize the vacuum transformation, given by Geroch [4], to the selected perfect-fluid spacetimes. However, in the perfect-fluid case there are additional field equations which have to be satisfied when applying the generalized transformation. This, in turn, means that the generalized transformation of [1–3] can be applied only if the equation of state of the fluid is \( \rho + 3P = 0 \), for both the initial and the final configurations.

In this paper we consider the complementary case, when the twist, \( \omega_a = D_a \omega \), and the gradient of the norm, \( D_a v \), of the Killing field are linearly dependent. This is possible whenever at least one of the functions \( v \) and \( \omega \) is constant throughout or there exists a function \( \omega = \omega(v) \) (or, alternatively, \( v = v(\omega) \)). Provided that at least one of the functions \( v \) and \( \omega \) is non-constant†, there exists at any point \( p \) of the three-dimensional space of Killing orbits, a two-dimensional subspace of the tangent space spanned by geometrically preferred vectors satisfying all the equations (3.6)–(3.9) of [1]. These geometrically preferred vectors are perpendicular to the one-dimensional subspace of the tangent space at \( p \) spanned by \( D_a v \) or \( D_a \omega \). Then, in all these cases one can choose, at least in an appropriately small neighbourhood of \( p \), two non-vanishing linearly independent preferred vector fields, \( k^a_{(2)} \) and \( k^a_{(3)} \), and introduce a local coordinate system \((x^1, x^2, x^3)\) adapted to these vector fields, so that \( k^a_{(2)} = (\partial/\partial x^2) \) and \( k^a_{(3)} = (\partial/\partial x^3) \). Then it follows immediately that

\[
v = v(x^1), \quad \omega = \omega(x^1). \tag{6}
\]

Furthermore, in such a local coordinate system \( h_{22} \) and \( h_{33} \) do not vanish and we get from the field equation (2) with the choice of indices \( a, b = 2 \) and \( a, b = 3 \)

\[
\frac{R_{22}}{h_{22}} = \frac{R_{33}}{h_{33}} = 16\pi v^{-1} P. \tag{7}
\]

Therefore the quantity \( P v^{-1} \), and thereby the left-hand side of (2), depends merely on the three-dimensional metric \( h_{ab} \), just as in the linearly independent case of [1–3]. Following [4] one may look for a transformation which would generate from a known solution \((h_{ab}, v, \omega)\) a new solution \((h'_{ab}, v', \omega')\) by leaving the three-dimensional metric, \( h_{ab} \), intact. Since \( h_{ab} \) (and therefore \( R_{ab} \)) is required to remain unchanged during the transformation, we get from (7)

\[
P' v'^{-1} = P v^{-1}, \tag{8}
\]

where \( P \) and \( v \) are the original pressure and norm, while \( P' \) and \( v' \) denote the corresponding transformed quantities. This is one of the necessary conditions for the applicability of the generalized transformation.

Next, in order to determine the exact form of the transformation we recall the complex notation of [4]

\[
\tau = \omega + iv. \tag{9}
\]

With this notation the field equations (2)–(4) can be written as

\[
H_{ab} = -2(\tau - \bar{\tau})^{-2}(D_a \tau)(D_b \bar{\tau}), \tag{10}
\]

\[
D_a D^m \tau = 2(\tau - \bar{\tau})^{-1}(D_a \tau)(D^m \tau) - 8i\pi(\rho + 3P), \tag{11}
\]

† If both \( v \) and \( \omega \) are constant, then we have from the field equations that the equation of state is \( \rho + 3P = 0 \), where both \( \rho \) and \( P \) are constant. Clearly, the transformation introduced below (see equations (12)–(14)) yields only a gauge transformation of these spacetimes.
where a bar denotes complex conjugation. Furthermore, in this complex notation the transformation we are looking for is expected to have the form \( h'_{ab} = h_{ab} \), \( \tau' = \tau'(\tau) \).

It can be shown that, just like in the vacuum case [4], the only solution to (10) and (11) of this form is

\[
\tau' = \frac{a\tau + b}{c\tau + d},
\]

(12)

where \( a, b, c \) and \( d \) are real constants which can be chosen, without loss of generality, to satisfy \( ad - bc = 1 \). From equation (12) it follows immediately that repeated applications of the transformation do not yield new (i.e. not gauge-related) solutions. It was shown for the vacuum case in [4] that by factoring out with respect to gauge transformations from the three parameters of an \( \text{SL}(2, \mathbb{R}) \) group there remains only one independent parameter.

Clearly the same argument applies for the examined perfect-fluid case as well. One of the possible parametrizations is the choice \( a = \cos \theta, b = \sin \theta, c = -\sin \theta \) and \( d = \cos \theta \), where now \( \theta \) is the only independent parameter of the transformation. Then, the real and imaginary parts of (12) give that

\[
v' = \frac{v}{(\cos \theta - \omega \sin \theta)^2 + v^2 \sin^2 \theta},
\]

(13)

\[
\omega' = \frac{(\omega \cos \theta + \sin \theta)(-\omega \sin \theta + \cos \theta) - v^2 \sin \theta \cos \theta}{(\cos \theta - \omega \sin \theta)^2 + v^2 \sin^2 \theta},
\]

(14)

where \( v' \) and \( \omega' \) are the transformed norm and twist potential of the Killing field.

The transformation of the equation of state can be determined as follows. The analogue of (11),

\[
D_m D_m \tau' = 2(\tau' - \bar{\tau}')^{-1}(D_m \tau')(D^m \tau') - 8\pi(\rho' + 3P'),
\]

(15)

must hold, where \( \rho' \) and \( P' \) are the transformed mass-density and pressure. Calculating \( D_m \tau' \) and \( D_m D_m \tau' \) from (12) and substituting into (15), using (9) we get a complex equation with the real part

\[
\rho + 3P = (\rho' + 3P')[c^2 \omega^2 - c^2 v^2 + 2cd\omega + d^2],
\]

(16)

and imaginary part

\[
0 = (\rho' + 3P')cv(c\omega + d).
\]

(17)

Obviously, the norm of the Killing field, \( v \), is non-zero and, also, it is reasonable to assume that the parameter of the transformation \( \theta \neq 0 \) (i.e. \( c \neq 0 \)). Therefore, there remain two possibilities so that (17) is satisfied. First, we may assume that the equation of state of the transformed perfect-fluid configuration is \( \rho' + 3P' = 0 \), which, in turn, gives along with (16) that \( \rho + 3P = 0 \). Unfortunately, this is a highly unphysical equation of state. One would expect that a stationary rigidly rotating perfect-fluid solution of Einstein’s equations which gives a faithful representation of a finite rigidly rotating body (e.g. a star) has to possess everywhere positive mass-density and pressure in the interior. Obviously, with \( \rho + 3P = 0 \) this cannot be the case.

There is, however, another possibility for equation (17) to be satisfied. Namely, take \( \omega = -d/c = \cos \theta / \sin \theta = \text{constant} \), i.e. suppose that the initial spacetime is static. The transformation of the equation of state for this latter possibility can be determined as follows.

Without loss of generality, in the static case we may assume that \( \omega = 0 \) which corresponds to \( \theta = \pi/2 \) (i.e. \( c = -1 \) and \( d = 0 \)) and \( \rho' + 3P' \neq 0 \). Then the transformation (13), (14) reduces to

\[
v' = \frac{1}{v}, \quad \omega' = 0,
\]

(18)
yielding a new static solution. This particular form of the transformation exactly reproduces the Buchdahl transformation of perfect-fluid spacetimes [7, 8] which has already been known for a long time. The transformed mass-density, $\rho'$, and pressure, $P'$, can be determined as follows. Substituting $\omega = 0$, $c = -1$ and $d = 0$ into (16) we get

$$\rho' + 3P' = -\frac{\rho + 3P}{v^2}. \tag{19}$$

Moreover, from (8) and (18) we get

$$P' = \frac{P}{v^2}, \tag{20}$$

which along with (19) gives

$$\rho' = -\frac{\rho + 6P}{v^2}. \tag{21}$$

The last two equations exactly recover the corresponding equations of [7, 8]. The transformed pressure and mass-density will both be positive only if the initial pressure, $P$, is positive and the initial mass-density, $\rho$, satisfies $\rho < -6P$ (compare with [8]). In most cases these conditions can be easily satisfied for an appropriate choice of parameters in the initial solution$^\dagger$. However, for a physically meaningful spherical star model in addition to the positivity of the pressure and mass-density, these quantities are expected to be monotonically decreasing functions of the radial coordinate. It is also required that the solution be regular at the origin$^\ddagger$. Unfortunately, none of the metrics which we generated by applying this form of the Buchdahl transformation to known solutions$^\S$ satisfy all criteria for physical acceptability listed in [12].

3. Herlt’s transformation

In this section we demonstrate that a technique given previously by Herlt in [9] for generating stationary perfect-fluid solutions from static ones is a special case of the above described transformation. In particular, we show that the fundamental equations of Herlt’s transformation, equations (4.4a) and (4.4b) of [9], follow from the above considerations corresponding to a special setting of the examined transformation.

Since in [9] the initial spacetime is taken to be static, we write $\omega = 0$ into (13) and (14). In this way we obtain

$$v' = \frac{(1 + \tan^2 \theta)v}{1 + v^2 \tan^2 \theta}, \tag{22}$$

$$\omega' = \frac{(1 - v^2) \tan^2 \theta}{1 + v^2 \tan^2 \theta}. \tag{23}$$

In the following we show that equations (4.4a) and (4.4b) of [9], i.e.

$$D_\alpha \dot{U} = \sqrt{1 + e^{-4U} \left( \frac{db}{dU} \right)^2} D_\alpha U, \tag{24}$$

$$\bar{p}e^{-2U} = pe^{-2U}, \tag{25}$$

$^\dagger$ For example, with a suitable choice of parameters, the transformed pressure and density can be made positive when one takes the interior (perfect-fluid) Schwarzschild solution as the initial spacetime [11] (see the discussion given by Stewart in [8]). Further simple examples are the configurations obtainable from the metrics denoted as Kuch71 II and R-R IV in [12].

$^\ddagger$ For the regularity conditions see e.g. [12].

$^\S$ These are the ones denoted as Tolman I, II, V, VI, Kuch2 VII, Kuch71 II, Whittaker, B-L, Heint IIIe, R-R I, III, IV, V, VI, Bayin II and III in [12].
follow as a special case from (8), (22) and (23) of the present paper, where our notation is related to the quantities $U$, $\bar{U}$, $p$ and $\bar{p}$ as
\begin{align}
v &= -e^{2\bar{U}}, \\
v' &= -e^{2U}, \\
\omega &= 0, \\
\omega' &= 2b, \\
P &= \bar{p}, \\
P' &= p.
\end{align}

(26) (27) (28)

Now equation (25) (equation (4.4b) of [9]) follows immediately by substituting (26)–(28) into (8).

To get (24) (equation (4.4a) of [9]), first we substitute expressions (26) and (27) into (22), (23) and to the derivative of these equations. Then, using the resulted system of equations, we eliminate $e^{\bar{U}}$ and $\tan \theta$ from the equation extracted by the substitution of (26) and (27) into (22). In this way we obtain
\begin{align}
(d\bar{U})^2 &= \left[1 + e^{-4U} \left(\frac{db}{dU}\right)^2\right](dU)^2,
\end{align}

(29)

from which (24) follows. Note, however, that there is a sign ambiguity arising from (29), which is missing from (24) (the original form of (4.4a) in [9]). Due to this, for instance in the particular case of static spacetimes, $\omega' = 2b = 0$, the special transformation $d\bar{U} = -d\bar{U}$ ($v' = 1/v$ in our notation) which recovers the Buchdahl transformation, was left out from the analysis in [9].

Let us, finally, mention that the equations used in [9] (see equations (24) and (25)) impose a strong restriction on the possible form of the equation of state. Since Herlt's transformation turned out to be a special case of the one examined throughout the previous section, it is obvious that when the resulting spacetime is non-static the relevant perfect-fluid configurations have to possess the equation of state $\rho + 3P = 0$.

Acknowledgments

We wish to thank T Dolinszky for carefully reading the manuscript. This research was supported in parts by the OTKA grants F14196 and T016246.

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