A Cheeger-Müller theorem for symmetric bilinear torsions on manifolds with boundary

Guangxiang Su

Abstract

In this paper, we extend Su-Zhang’s Cheeger-Müller type theorem for symmetric bilinear torsions to manifolds with boundary in the case that the Riemannian metric and the non-degenerate symmetric bilinear form are of product structure near the boundary. Our result also extends Brüning-Ma’s Cheeger-Müller type theorem for Ray-Singer metric on manifolds with boundary to symmetric bilinear torsions in product case. We also compare it with the Ray-Singer analytic torsion on manifolds with boundary.

1 Introduction

Let $F$ be a unitary flat vector bundle on a closed Riemannian manifold $X$. In [26], Ray and Singer defined an analytic torsion associated to $(X, F)$ and proved that it does not depend on the Riemannian metric on $X$. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on $X$ (cf. [21]). This conjecture was later proved in the celebrated papers of Cheeger [10] and Müller [23]. Müller generalized this result in [24] to the case where $F$ is a unimodular flat vector bundle on $X$. In [4], inspired by the considerations of Quillen [25], Bismut and Zhang reformulated the above Cheeger-Müller theorem as an equality between the Reidemeister and Ray-Singer metrics defined on the determinant of cohomology, and proved an extension of it to the case of general flat vector bundles over $X$. The method used in [4] is different from those of Cheeger and Müller in that it makes use of a deformation by Morse functions introduced by Witten [33] on the de Rham complex.

On the other hand, if there is a non-degenerate symmetric bilinear form $h_F$ on $F$, Burghelea and Haller [8], defined a complex valued analytic torsion in the spirit of Ray and Singer [26]. In [8], they also made an explicit conjecture between the complex valued analytic torsion and the Turaev torsion (cf. [14], [31]). In [30], Su and Zhang used the approach developed by Bismut-Zhang [4, 5], making use of the Witten deformation, proved the conjecture in full generality. In [9], Burghelea and Haller proved their conjecture, up to sign, in the case where $X$ is of odd dimensional.

Now consider $X$ with the boundary $Y \neq \emptyset$. In [19], [32] and [18], the authors studied the Ray-Singer analytic torsion under the assumption that the Hermitian metric $h_F$ on $F$ is flat and the Riemannian metric $g^{TX}$ has product structure near the boundary. Dai and Fang [12] studied the case that the Hermitian metric $h_F$ is flat but without assuming a product structure for $g^{TX}$ near $Y$. In [6], Brüning and Ma studied the general case without assumption on $h_F$ and $g^{TX}$ associated to the absolute boundary condition on $X$. In [7], Brüning and Ma studied the results for the relative boundary condition on $X$, proved a Cheeger-Müller type theorem for
the Ray-Singer metric on the manifolds with boundary and applied it derived the gluing formula for the analytic torsion in general setting.

For the complex-valued analytic torsion, Molina [22] extended the Burghelea-Haller analytic torsion to compact manifolds with boundary under the relative boundary condition and the absolute boundary condition.

In this paper, we extend the main result in [30] to manifolds with boundary for the couple \((g^T_X, b^F)\) are of product structure near the boundary. We use the method in [7]. We first double \(X\) along the boundary \(Y\) and get a closed Riemannian manifold \(\tilde{X} = X \cup Y\) \(X\) with Riemannian metric \(g^T_{\tilde{X}}\), then there is a \(\mathbb{Z}_2\)-action on \(\tilde{X}\) induced by the natural involution \(\phi\) on it. Also the flat bundle \(F\) extends to a flat bundle \(\tilde{F}\) on \(\tilde{X}\) which has a non-degenerate symmetric bilinear form \(b_{\tilde{F}}\). The metric \(g^T_{\tilde{X}}\) and the form \(b_{\tilde{F}}\) are all invariant under the action of \(\phi\). So we first extend the main result in [30] to \(\mathbb{Z}_2\)-equivariant case. Then we compare this \(\mathbb{Z}_2\)-equivariant symmetric bilinear torsions on \(\tilde{X}\) to the symmetric bilinear torsions on \(X\) with relative boundary condition and absolute boundary condition. Combining these and the result in [30], we get the main theorem in this paper. In a next paper, I will apply the techniques in this paper to deal with the Cappell-Miller analytic torsion [11].

The rest of the paper is organized as follows. In Section 2, we review the symmetric bilinear analytic torsion on manifolds with boundary and the anomaly formula from [22]. In Section 3, we define the symmetric bilinear Milnor torsion on manifolds with boundary. In Section 4, we study the doubling formulas for symmetric bilinear torsions and extend the result in [30] to \(\mathbb{Z}_2\)-equivariant case. In Section 5, we prove the Cheeger-M"uller type theorem in current case. In Section 6, we compare the symmetric bilinear analytic torsion with the Ray-Singer analytic torsion on manifolds with boundary.

## 2 Symmetric bilinear analytic torsion on manifolds with boundary

In this section, we will review the definition of the symmetric bilinear analytic torsion on manifolds with boundary and the anomaly formula of it.

Let \(X\) be a compact oriented Riemannian manifold with boundary \(Y\). Let \(F\) be a flat complex vector bundle over \(X\). We assume that there is a non-degenerate symmetric bilinear form \(b^F\) on \(F\). If the Riemannian metric \(g^T_X\) and the form \(b^F\) are of product structure near the boundary \(Y\). By the \((g^T_X, b^F)\), we can define a non-degenerate symmetric bilinear form on \(\Omega^*(X, F)\), i.e., for \(\omega_1, \omega_2 \in \Omega^*(X, F)\), then

\[
\langle \omega_1, \omega_2 \rangle_b = \int_X \text{Tr}(\omega_1 \wedge *_{b^F} \omega_2),
\]

where \(*_{b^F} = * \otimes b^F : \Omega^*(X, F) \to \Omega^*(X, F')\) and \(b^F : F \to F'\) is induced by the form \(b^F\), * is the Hodge star operator (cf. [34]). Let \(d_{b^F}^{\#}\) be the formal adjoint of \(d^F\) with respect to the form in (2.1). Then we have the Laplacian operator

\[
\Delta_b^F = (d^F + d_{b^F}^{\#})^2 = d^F d_{b^F}^{\#} + d_{b^F}^{\#} d^F.
\]

We can impose the relative boundary condition or the absolute boundary condition on \(Y\) for the operator \(\Delta_b\) (cf. [22]). That is, for \(\omega \in \Omega^*(X, F)\),

\[
i^* \omega = 0, \quad i^* d_{b^F}^{\#} \omega = 0
\]
We denote by $\Omega^*(X,F)_r$ (resp. $\Omega^*(X,F)_a$) the complex $(\Omega^*(X,F),d^F)$ with the relative (resp. absolute) boundary condition. Then we have

$$H^*(\Omega^*(X,F)_r,d^F) \cong H^*(X,Y,F), \quad H^*(\Omega^*(X,F)_a,d^F) \cong H^*(X,F).$$

For any $k \geq 0$, let $\Omega^*_{[0,k]}(X,F)_r$ (resp. $\Omega^*_{[0,k]}(X,F)_a$) denote the generalized eigenspace of $\Delta_b$ on $\Omega^*(X,F)_r$ (resp. $\Omega^*(X,F)_a$) with respect to the generalized eigenvalues with the absolute value in $[0,k]$. Let $b_{detH^*}(\Omega^*_{[0,k]}(X,F)_r)$ (resp. $b_{detH^*}(\Omega^*_{[0,k]}(X,F)_a)$) be the induced symmetric bilinear form on $detH^*(\Omega^*_{[0,k]}(X,F)_r)$ (resp. $detH^*(\Omega^*_{[0,k]}(X,F)_a)$) via the canonical isomorphisms

$$detH^*(\Omega^*_{[0,k]}(X,F)_r) \cong det(\Omega^*_{[0,k]}(X,F)_r),$$

$$detH^*(\Omega^*_{[0,k]}(X,F)_a) \cong det(\Omega^*_{[0,k]}(X,F)_a).$$

Then we can define the symmetric bilinear torsion $b_{detH^*(X,Y,F)}$ (resp. $b_{detH^*(X,F)}$) on $detH^*(X,Y,F)$ (resp. $detH^*(X,F)$) by

$$b_{detH^*(X,Y,F)} = b_{detH^*}(\Omega^*_{[0,k]}(X,F)_r) \cdot \prod_{i=0}^{\text{dim}X} \left( det(\Delta^F_{\Omega^*_{[k,\infty]}(X,F)_r}) \right)^{(-1)^i} \quad (2.3)$$

and

$$b_{detH^*(X,F)} = b_{detH^*}(\Omega^*_{[0,k]}(X,F)_a) \cdot \prod_{i=0}^{\text{dim}X} \left( det(\Delta^F_{\Omega^*_{[k,\infty]}(X,F)_a}) \right)^{(-1)^i}, \quad (2.4)$$

which are independent of $k \geq 0$.

Let $\theta(F,b^F) \in \Omega^1(M)$ be the Kamber-Tondeur form defined by (cf. [8] (4))

$$\theta(F,b^F) = \text{Tr} [(b^F)^{-1} \nabla^F b^F].$$

Now we state the anomaly formulas. If $(g^{TX},b_0^F)$ and $(g^{TY},b_0^F)$ are two couple of metric and symmetric bilinear form which are of product structure near the boundary (cf. [18], [1.2]) and in a same homotopy class. Then by [22] Theorem 3, we have

$$\log \left( \frac{b_{0,\text{det}^F}(X,Y,F)}{b_{1,\text{det}^F}(X,Y,F)} \right) = \int_X \log \left( \frac{b_{0,\text{det}^F}}{b_{1,\text{det}^F}} \right) e \left( TX, \nabla_1^{TX} \right)$$

$$- \int_X \theta(F,b_0^F)(\nabla f)^* \left( \psi(TX,\nabla_0^{TX}) - \psi(TX,\nabla_1^{TX}) \right)$$

$$- \frac{1}{2} \int_Y \log \left( \frac{b_{0,\text{det}^F}}{b_{1,\text{det}^F}} \right) e \left( TY, \nabla_1^{TY} \right) - \frac{1}{2} \int_Y \bar{e}(TY,\nabla_1^{TY},\nabla_0^{TY}) \theta(F,b_0^F), \quad (2.5)$$

and

$$\log \left( \frac{b_{0,\text{det}^F}(X,F)}{b_{1,\text{det}^F}(X,F)} \right) = \int_X \log \left( \frac{b_{0,\text{det}^F}}{b_{1,\text{det}^F}} \right) e \left( TX, \nabla_1^{TX} \right)$$

$$- \int_X \theta(F,b_0^F)(\nabla f)^* \left( \psi(TX,\nabla_0^{TX}) - \psi(TX,\nabla_1^{TX}) \right)$$

$$+ \frac{1}{2} \int_Y \log \left( \frac{b_{0,\text{det}^F}}{b_{1,\text{det}^F}} \right) e \left( TY, \nabla_1^{TY} \right) + \frac{1}{2} \int_Y \bar{e}(TY,\nabla_1^{TY},\nabla_0^{TY}) \theta(F,b_0^F), \quad (2.6)$$
where \( e(TX, \nabla^{T}X) \) is the Euler form, \( \tilde{e}(TY, \nabla_{0}^{T}Y, \nabla^{T}Y) \) is the Chern-Simons class (cf. [4, Chapter 4]) and \( \psi(TX, \nabla^{T}X) \) is the Mathai-Quillen current on \( TX \) constructed in [4, Chapter 3].

Remark 2.1. Throughout this paper, for complex numbers \( a, b \in \mathbb{C} \), \( \log a = b \) means that \( a = e^{b} \).

Remark 2.2. In [22], \( (g^{TX}, b^{F}) \) does not need the product structure near the boundary.

Remark 2.3. If \( \dim X = m \) is odd, then the first two terms in the right hand side of (2.5) and (2.6) vanish.

3 Symmetric bilinear Milnor torsion on manifolds with boundary

Let \( f \) be a Morse function on \( X \) and \( f|_{Y} \) be the restriction of \( f \) on \( Y \). Set

\[
B = \{ x \in X; df(x) = 0 \}, \quad B_{0} = \{ x \in Y; d(f|_{Y})(x) = 0 \}. \tag{3.1}
\]

For \( x \in B \), let \( \text{ind}(x) \) be the index of \( f \) at \( x \), i.e., the number of negative eigenvalues of the quadratic form \( d^{2}f(f)|_{n} \) on \( T_{x}X \).

Consider the differential equation

\[
\frac{\partial y}{\partial t} = -\nabla f(y), \tag{3.2}
\]

and denote by \( (\psi_{t}) \) the associated flow.

For \( x \in B \), the unstable cell \( W^{u}(x) \) and the stable cell \( W^{s}(x) \) of \( x \) are defined by

\[
W^{u}(x) = \{ y \in X; \lim_{t \to -\infty} \psi_{t}(y) = x \}, \tag{3.3}
\]

\[
W^{s}(x) = \{ y \in X; \lim_{t \to +\infty} \psi_{t}(y) = x \}.
\]

The Smale transversality condition means that

for \( x, y \in B, x \neq y, W^{u}(x) \) and \( W^{s}(y) \) intersect transversally. \tag{3.4}

Let \( n \) be the normal bundle to \( Y \) in \( X \).

Lemma 3.1. ([7, Lemma 1.5]) There exists a Morse function \( f \) on \( X \) such that \( f|_{Y} \) is a Morse function on \( Y \), \( B_{0} = B \cap Y \), and \( d^{2}f(x)|_{n} > 0 \) for \( x \in B_{0} \). Moreover, there exists a gradient vector field \( \nabla f \) of \( f \), verifying the Smale transversality condition \( \tag{3.4} \) and \( \nabla f|_{Y} \in TY \).

From now on, we choose a Morse function \( f \) on \( X \) fulfilling the condition of Lemma 3.1. For \( x \in B_{0} \), set

\[
W^{u}_{\partial}(x) = W^{u}(x) \cap Y, \quad W^{s}_{\partial}(x) = W^{s}(x) \cap Y. \tag{3.5}
\]

As \( d^{2}f(x)|_{n} > 0 \), for \( x \in B_{0} \), and \( \nabla f|_{\partial X} \in T\partial X \), we know that \( \nabla f|_{\partial X} \) verifies also the Smale transversality condition \( \tag{3.4} \).

For \( x \in B \), we denote by \([W^{u}(x)]\) the complex line generated by \( W^{u}(x) \), and by \([W^{u}(x)]^{*}\) the dual line. Set

\[
C_{j}(W^{u}, F^{*}) = \bigoplus_{x \in B, \text{ind}(x) = j} [W^{u}(x)] \otimes F_{x}^{*}, \tag{3.6}
\]
In this section, we will study the doubling formulas for symmetric bilinear tor-
sions.

Let \( b \) correspond to \( \text{Definition 3.2} \). Let \( b \) be the natural morphism of complexes \( \mathcal{C} \to \mathcal{C}/f \) such that \( \mathcal{C}/f \) is the symmetric bilinear form on \( \mathcal{C} \). Let \( \mathcal{C}/f \) be the symmetric bilinear form on \( \mathcal{C}/f \). Finally, let \( b_{\det C}(W^u/W^t, F) \) be the form on the complex line

\[
\text{det} C^*(W^u/W^t, F) = \bigotimes_{j=0}^{\dim X} (\text{det} C^j(W^u/W^t, F))^{(-1)^j}.
\]

**Definition 3.2.** Let \( b_{\det H^*}(X, Y, F) \) be the symmetric bilinear form on \( \text{det} H^*(X, Y, F) \) corresponding to \( b_{\det C^*}(W^u/W^t, F) \) via the canonical isomorphism

\[
\text{det} H^*(X, Y, F) \cong \text{det} C^*(W^u/W^t, F).
\]

The form \( b_{\det H^*}(X, Y, F) \) will be called the symmetric bilinear Milnor torsion.

**Remark 3.3.** We can also define the symmetric bilinear Milnor torsion \( b_{\det H^*}(X, Y, F) \) on \( \det H^*(X, Y, F) \).

4 Doubling formulas for symmetric bilinear torsions

In this section, we will study the doubling formulas for symmetric bilinear torsions and extend the main theorem in [30] to the \( \mathbb{Z}_2 \)-equivariant case.
4.1 Doubling formula for symmetric bilinear Ray-Singer torsion

We assume that the Riemannian metric $g^{TX}$ is product near the boundary $Y$, i.e., there exists a neighborhood $U_\varepsilon$ of $Y$ and an identification $Y \times [0, \varepsilon) \to U_\varepsilon$, such that for $(y, x_m) \in Y \times [0, \varepsilon)$,

$$g^{TX}|_{(y, x_m)} = dx_m^2 \oplus g^{TY}(y). \tag{4.1}$$

This condition insure that the manifold $\tilde{X} := X \cup_Y X$ has a canonical Riemannian metric $g^{\tilde{T}X} = g^{TX} \cup_Y g^{TY}$. The natural involution on $\tilde{X}$ will be denoted by $\phi$, it generates a $\mathbb{Z}_2$-action on $\tilde{X}$. Let $j_k : X \to \tilde{X}$ be the natural inclusion into the $k$-th factor, $k = 1, 2$, which identifies $X$ with $j_k(X)$.

Let $F$ be a complex flat vector bundle over $X$ with flat connection $\nabla^F$. Suppose that there exists a non-degenerate symmetric bilinear form $b^F$ on $F$. We trivialize $F$ on $U_\varepsilon$ using the parallel transport along the curve $u \in [0, 1) \to (y, u\varepsilon)$ defined by the connection $\nabla^F$, then we have $F|_{U_\varepsilon} = \pi_*^*F|_Y$, where $\pi_* : Y \times [0, \varepsilon) \to Y$ is the obvious projection on the first factor. We also assume that

$$b^F = \pi_*^*b^F|_Y \text{ on } U_\varepsilon. \tag{4.2}$$

Let $\tilde{F} = F \cup_Y F$ be the flat complex vector bundle with non-degenerate symmetric bilinear form $\tilde{b}^F$ on $\tilde{X}$ induced by $(F, b^F)$.

For the couple $(\tilde{X}, \tilde{F})$, with the Riemannian metric $g^{\tilde{T}X}$ and the non-degenerate symmetric bilinear form $\tilde{b}^F$, we denote by $D_0$ the operator defined as in [30] (2.20). Let $\langle \cdot, \cdot \rangle_b$ be the symmetric bilinear form on $\Omega^*(\tilde{X}, \tilde{F})$ defined as in (4.1).

For any $a \geq 0$, let $\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})$ be the generalized eigenspace corresponding to the generalized eigenvalue of $D_0^2$ with absolute value in $[0, a]$. Let $b_{\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})}$ be the induced symmetric bilinear form on $\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})$. As it in [30], we know that it is non-degenerate. Let $\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})^\pm$ be the $\pm 1$-eigenspace of $\phi$ on $\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})$. Let $b_{\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})^\pm}$ be the induced symmetric bilinear forms on $\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})^\pm$ from $b_{\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})}$. It is easily to see that $\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})^+$ and $\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})^-$ are orthogonal with respect to $b_{\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})}$, then $b_{\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})^\pm}$ are non-degenerate.

Let $\Omega^*_{(a, +\infty)}(\tilde{X}, \tilde{F})$ be the $\langle \cdot, \cdot \rangle_b$-orthogonal complement of $\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})$. For any $0 \leq i \leq m$, let $D_{b,i}^2$ be the restriction of $D_0^2$ on $\Omega^*_{(a, +\infty)}(\tilde{X}, \tilde{F})$. Then for $g \in \mathbb{Z}_2$, we can define the regularized equivariant zeta determinant

$$\det_g(D_{b,(a, +\infty), i}^2) = \exp \left( -\frac{\partial}{\partial s} \bigg|_{s=0} \Tr \left[ g \left( D_{b,i}^2|_{\Omega_{(a, +\infty)}^*}(\tilde{X}, \tilde{F}) \right)^{-s} \right] \right). \tag{4.3}$$

Let $b_{\det(H^*(\tilde{X}, \tilde{F})^\pm)}$ be the symmetric bilinear form on $\det(H^*(\tilde{X}, \tilde{F})^\pm)$ induced by the finite dimensional subcomplex of the de Rham complex according the $\pm 1$-eigenvalue. Then for $\mu = (\mu_1, \mu_2) \in \det(H^*(\tilde{X}, \tilde{F}), \mathbb{Z}_2)$, $g \in \mathbb{Z}_2$, the equivariant symmetric bilinear form Ray-Singer torsion is defined by

$$b^{RS}_{\det(H^*(\tilde{X}, \tilde{F}), \mathbb{Z}_2)}(\mu) = b_{\det(H^*(\Omega^*_{[0,a]}(\tilde{X}, \tilde{F})^\pm))}^{\mu_1} \cdot b_{\det(H^*(\Omega^*_{(a, +\infty)}(\tilde{X}, \tilde{F})))}^{\mu_2} \cdot \prod_{i=1}^{\dim X} \left( \det_g(D_{b,(a, +\infty), i}^2) \right)^{-1}, \tag{4.4}$$

which is independent of the choice of $a \geq 0$ (cf. [5] Theorem 1.4 and [8] Proposition 4.7)).
Denote by \((b_{(a,+\infty)}(\tilde{X}, g^{2r}x_{b}))^{-1}(g)\) the product
\[
\prod_{i=1}^{\dim X} \left( \det'(\tilde{D}_{b,(a,+\infty)}^{2r}) \right)^{-1} \tilde{\lambda}_i
\]
and similarly by \(b_{(a,+\infty)}(X,Y,g^{2r}x_{b}), b_{(a,+\infty)}(X,g^{2r}x_{b})\) the corresponding parts in \((2.3)\) and \((2.4)\). Let \(C^+, C^-\) be the trivial and nontrivial one dimension complex \(Z_2\)-representation, respectively. Then by the same argument in \([19\ \text{Proposition 2.1}]\), we have

**Proposition 4.1.** (Doubling formula for symmetric bilinear analytic torsion) For \(\lambda \geq 0\), let \(\Omega^\lambda(X, F)\) be the generalized eigenspace of \(D^\lambda_g\) corresponding to the generalized eigenvalue with absolute value equals \(\lambda\), we have a \(Z_2\)-equivariant map

\[
\tilde{\phi} : \Omega^\lambda(X, F) \to \Omega^\lambda(X, F) \otimes C^+ \oplus \Omega^\lambda(X,Y,F) \otimes C^-,
\]

where

\[
\tilde{\phi} (\sigma) = \frac{\sqrt{2}}{2} (\sigma + \phi^* \sigma)_{X} \otimes 1_{C^+} + \frac{\sqrt{2}}{2} (\sigma - \phi^* \sigma)_{X} \otimes 1_{C^-}.
\]

The map \(\tilde{\phi}\) preserves the non-degenerate symmetric bilinear forms. In particular, with \(\chi\) the nontrivial character of \(Z_2\), we have for \(g \in Z_2\),

\[
\left( b_{(0,+\infty)}(\tilde{X}, \tilde{F}) \right)(g) = b_{(0,+\infty)}(\tilde{X}, \tilde{F}) \cdot \left( b_{(0,+\infty)}(X, g^{r}x_{\tilde{b}}) \right)^{\chi(g)}.
\]

**Proof.** By the same argument in \([19\ \text{Proposition 1.27}]\), one finds that \(\tilde{\phi}\) is well-defined and injective. For the surjectivity, we need to show that for \(\omega \in \Omega^\lambda(X, F)\), \(\tilde{\omega} = \phi^* \omega\) on \(\phi(X)\) is a smooth form on \(\tilde{X}\) with coefficients in \(\tilde{F}\), and thus \(\tilde{\omega} \in \Omega^\lambda(\tilde{X}, \tilde{F})\). First, as the case in \([19\ \text{Proposition 1.27}]\), \(\tilde{\omega}\) is smooth on \(\tilde{X}\) and continuous on \(\tilde{X}\). Let \(P_{\lambda}\) be the spectral projection onto \(\Omega^\lambda(\tilde{X}, \tilde{F})\), then we have the decomposition

\[
L^2(\Omega^\lambda(\tilde{X}, \tilde{F})) = P_{\lambda} \left( L^2(\Omega^\lambda(\tilde{X}, \tilde{F})) \right) \oplus (I - P_{\lambda}) \left( L^2(\Omega^\lambda(\tilde{X}, \tilde{F})) \right).
\]

So that

\[
\tilde{\omega} = P_{\lambda}(\tilde{\omega}) + (I - P_{\lambda})(\tilde{\omega}).
\]

Let \(i : X \to \tilde{X} = X \cup Y X\) be the inclusion onto the first summand. Then similar as it in \([19\ \text{Proposition 1.27}]\), for any \(\eta \in \Omega^\lambda(\tilde{X}, \tilde{F})\), we have

\[
\langle \tilde{\omega}, \eta \rangle_b = \langle \omega, i^* \eta \rangle_b + \langle \omega, i^* \phi^* \eta \rangle_b.
\]

Then by \([19\ \text{Proposition 1.27}]\), the similar discussion in \([19\ \text{Proposition 1.27}]\) and the fact that for \(\lambda \neq \mu\), \(\Omega^\lambda(X, F)\) and \(\Omega^\mu(X, F)\) are \(\langle , \rangle_b\)-orthogonal (cf. \([22]\)), one finds

\[
\langle (I - P_{\lambda})\tilde{\omega}, \eta \rangle_b = \langle \tilde{\omega}, (I - P_{\lambda})\eta \rangle_b = 0,
\]

then \((I - P_{\lambda})\tilde{\omega} = 0\). So we have \(\tilde{\omega} = P_{\lambda}\tilde{\omega}\). Since \(\tilde{\omega}\) is continuous, then we get that \(\tilde{\omega}\) is smooth.

By Proposition 4.1 for \(\lambda = 0\), we have a natural isomorphism of \(Z_2\)-vector spaces

\[
\tilde{\phi} : \Omega^\lambda_{(0)}(\tilde{X}, \tilde{F}) \to \Omega^\lambda_{(0)}(X, F) \otimes C^+ \oplus \Omega^\lambda_{(0)}(X, F) \otimes C^-.
\]
\[ \tilde{\phi}(\sigma) = \frac{\sqrt{2}}{2} \cdot (\sigma + \phi^* \sigma)|_X + \frac{\sqrt{2}}{2} \cdot (\sigma - \phi^* \sigma)|_X. \] (4.11)

Then \( \tilde{\phi} \) preserves the non-degenerate symmetric bilinear form. From [22] Proposition 3, we have that the inclusion \( i: \Omega^*(X,F)_r \to \Omega^*(X,F)_r \) induces an isomorphism on cohomology. We also denote by \( \tilde{\phi} \) the induced map on the cohomology.

Let \( b_{detH^*\Omega^*(X,F)}(\tilde{X},\tilde{F}) \) be the induced non-degenerate symmetric bilinear form on \( \det H^*\Omega^*(X,F) \). Then for \( \mu = (\mu_1, \mu_2) \in \det(H^*(\tilde{X},\tilde{F}),Z_2), g \in Z_2, \) the equivariant Ray-Singer symmetric bilinear torsion is defined by

\[
\log \left( \frac{b_{detH^*\Omega^*(X,F)}(\mu)}{b_{detH^*\Omega^*(X,F)}(\mu)}(g) \right) = \log \left( \frac{b_{detH^*\Omega^*(X,F)}(\mu)}{b_{detH^*\Omega^*(X,F)}(\mu)}(g) \right) + \chi(g) \log \left( \frac{b_{detH^*\Omega^*(X,F)}(\mu)}{b_{detH^*\Omega^*(X,F)}(\mu)}(g) \right) + \log \left( \frac{b_{detH^*\Omega^*(X,F)}(\mu)}{b_{detH^*\Omega^*(X,F)}(\mu)}(g) \right). \] (4.12)

Note that \( \tilde{\phi} \) in (4.11) induces isomorphisms

\[ \tilde{\phi}_1: H^*(\tilde{X},\tilde{F}) \to H^*(X,F), \tilde{\phi}_2: H^*(\tilde{X},\tilde{F}) \to H^*(X,F). \]

By (4.11) and (4.12), for \( \mu = (\mu_1, \mu_2) \in \det(H^*(\tilde{X},\tilde{F}),Z_2), g \in Z_2, \)

\[
\log \left( \frac{b_{detH^*\Omega^*(X,F)}(\mu)}{b_{detH^*\Omega^*(X,F)}(\mu)}(g) \right) = \log \left( \frac{b_{detH^*\Omega^*(X,F)}(\mu)}{b_{detH^*\Omega^*(X,F)}(\mu)}(g) \right) + \chi(g) \log \left( \frac{b_{detH^*\Omega^*(X,F)}(\mu)}{b_{detH^*\Omega^*(X,F)}(\mu)}(g) \right) + \log \left( \frac{b_{detH^*\Omega^*(X,F)}(\mu)}{b_{detH^*\Omega^*(X,F)}(\mu)}(g) \right). \] (4.13)

From (4.13) and the anomaly formula [22] Theorem 3, one can get the following anomaly formula

**Theorem 4.2.** Let \( g^{TX}_u \) be a smooth one-parameter Riemannian metrics on \( X \) which is product near \( Y \) and \( b^F_u \) is a smooth one-parameter non-degenerate symmetric bilinear forms which is product near \( Y \) such that \( g^{TX}_u = g^{TX}, g^{TX}_1 = g^{TX} \) and \( b^F_0 = b^F, b^F_1 = b^F, \) then The following identity holds,

\[
\frac{b_{detH^*\Omega^*(X,F)}(\mu)}{b_{detH^*\Omega^*(X,F)}(\mu)}(\phi) = \exp \left( \int_Y \log \left( \det \left( \left(b^F_0 \right)^{-1} b^F_0 \right) \right) e(TY,\nabla^TY) \right) \cdot \exp \left( - \int_Y \theta (F,b^F) \bar{e}(TY,\nabla^TY,\nabla^TY) \right). \] (4.14)

### 4.2 Doubling formula for symmetric bilinear Milnor torsion

Let \( \tilde{F} = F \cup_Y F \) be the flat complex vector bundle with non-degenerate symmetric bilinear form \( b^F \) on \( \tilde{X} = X \cup_Y X \) induced by \( (F,b^F) \).

Let \( \tilde{f} \) be a Morse function on \( X \) satisfying Lemma 3.1 which is induced by a \( Z_2 \)-equivariant Morse function \( \tilde{f} \) on \( \tilde{X} \) with critical set \( B = \{ x \in \tilde{X}; \tilde{f}(x) = 0 \} \). Let \( \tilde{W}^u(x) \) be the unstable set of \( x \in B \subset \tilde{X} \).

Let \( C^*(\tilde{W}^u,F)^\pm \) and \( H^*(\tilde{X},\tilde{F})^\pm \) be the \( \pm 1 \)-eigenspaces of the \( Z_2 \)-action induced by \( \phi \) on \( C^*(\tilde{W}^u,F) \) and \( H^*(\tilde{X},\tilde{F}) \); then \( H^*(\tilde{X},\tilde{F})^\pm \) is the cohomology of the complex \( (C^*(\tilde{W}^u,F)^\pm,\phi) \).

Let \( C^+, C^- \) be the trivial and nontrivial one dimension complex \( Z_2 \)-representation, respectively, and let \( 1_{C^+}, 1_{C^-} \) be their unit elements.

Following [5] (1.10), we define

\[
\det \left( H^*(\tilde{X},\tilde{F}),Z_2 \right) = \det \left( H^*(\tilde{X},\tilde{F})^+ \right) \otimes C^+ \oplus \det \left( H^*(\tilde{X},\tilde{F})^- \right) \otimes C^- \cdot \] (4.15)
Let $b^{M,f}_{\det H^*(\tilde{X},\tilde{F})}$ be the symmetric bilinear form on $\det H^*(\tilde{X},\tilde{F})^\pm$ defined via the canonical isomorphism

$$\det H^*(\tilde{X},\tilde{F})^\pm \cong \det C^*(\tilde{W}^u,\tilde{F})^\pm.$$  

For $\mu = (\mu_1, \mu_2) \in \det (H^*(\tilde{X},\tilde{F}), \mathbb{Z}_2)$, $\phi \in \mathbb{Z}_2$, and $\chi$ the nontrivial character of $\mathbb{Z}_2$, we introduce the equivariant symmetric bilinear Milnor torsion by

$$b^{M,f}_{\det H^*(\tilde{X},\tilde{F}),\mathbb{Z}_2}(\mu)(\phi) = b^{M,f}_{\det H^*(\tilde{X},\tilde{F})^+}(\mu_1) \cdot b^{\chi(\phi)}_{\det H^*(\tilde{X},\tilde{F})^-}(\mu_2) = b^{M,f}_{\det H^*(\tilde{X},\tilde{F})^+}(\mu_1) \cdot b^{-1}_{\det H^*(\tilde{X},\tilde{F})^-}(\mu_2). \quad (4.16)$$

We have a $\mathbb{Z}_2$-equivariant isomorphism of complexes

$$\gamma : C^*(\tilde{W}^u, F) \otimes \mathbb{C}^+ \oplus C^*(\tilde{W}^u/\tilde{W}_Y^u, F) \otimes \mathbb{C}^- \to C^*(\tilde{W}^u, \tilde{F}), \quad (4.17)$$
given by

$$\gamma(a^* \otimes 1_{\mathbb{C}^+} \oplus b^* \otimes 1_{\mathbb{C}^-}) = \sqrt{2}((j_1^{-1})^* a^* + (j_2^{-1})^* a) + \sqrt{2}((j_1^{-1})^* b^* - (j_2^{-1})^* b^*), \quad (4.18)$$

which induces a $\mathbb{Z}_2$-isomorphism

$$\gamma : H^*(X,F) \otimes \mathbb{C}^+ \oplus H^*(X,Y,F) \otimes \mathbb{C}^- \to H^*(\tilde{X},\tilde{F}). \quad (4.19)$$

Note that as complex vector spaces, we have

$$C^j(\tilde{W}^u, F) = \bigoplus_{x \in B, \text{ind}(x) = j} [\tilde{W}^u(x)]^* \otimes F_x, \quad (4.20)$$

$$C^j(\tilde{W}^u/\tilde{W}_Y^u, F) = \bigoplus_{x \in B \setminus \text{ind}(x) = j} [\tilde{W}^u(x)]^* \otimes F_x. \quad (4.21)$$

Then $\gamma$ as a map from $C^j(\tilde{W}^u/\tilde{W}_Y^u, F) \otimes \mathbb{C}^+ \oplus C^j(\tilde{W}^u/\tilde{W}_Y^u, F) \otimes \mathbb{C}^- \to C^*(\tilde{W}^u, \tilde{F})$ preserves the symmetric bilinear form and for $a^* \in [\tilde{W}^u(x)]^* \otimes F_x$ with $x \in Y$, we have

$$\gamma(a^* \otimes 1_{\mathbb{C}^+}) = \sqrt{2}a^*. \quad (4.22)$$

Let $b^{M,f}_{\det C^*(\tilde{W}^u,\tilde{F})}$ be the symmetric bilinear form on $\det H^*(\tilde{X},\tilde{F})^\pm$. For $\mu = (\mu_1, \mu_2) \in \det (H^*(\tilde{X},\tilde{F}), \mathbb{Z}_2)$, $g \in \mathbb{Z}_2$, and $\chi$ the nontrivial character of $\mathbb{Z}_2$, we have defined the equivariant Milnor symmetric bilinear torsion by

$$b^{M,f}_{\det H^*(\tilde{X},\tilde{F}),\mathbb{Z}_2}(\mu)(g) = b^{C^*}(\tilde{W}^u,\tilde{F})^+ \cdot (b^{C^*}(\tilde{W}^u,\tilde{F})^-)(\mu_2))\chi(g). \quad (4.23)$$

Now $\gamma$ in (4.19) induces isomorphisms

$$\gamma_1^{-1} : H^*(\tilde{X},\tilde{F})^+ \to H^*(X,F),$$

$$\gamma_2^{-1} : H^*(\tilde{X},\tilde{F})^- \to H^*(X,Y,F). \quad (4.24)$$

From (4.21), (4.22) and (4.23), we get for $\mu = (\mu_1, \mu_2) \in \det (H^*(\tilde{X},\tilde{F}), \mathbb{Z}_2)$, $g \in \mathbb{Z}_2$,

$$\log\left(b^{M,f}_{\det H^*(\tilde{X},\tilde{F}),\mathbb{Z}_2}(\mu_1)\right)(g) = \log(2) \sum_{x \in B \setminus Y} (-1)^{\text{ind}(x)} \text{rk}(F_x) + \log\left(b^{M,f}_{\det H^*(X,F)}(\gamma_1^{-1}\mu_1)\right) + \chi(g) \log\left(b^{M,f}_{\det H^*(X,Y,F)}(\gamma_2^{-1}\mu_2)\right). \quad (4.24)$$

From (4.24) and the anomaly formulas for $b^{M,f}_{\det H^*(X,F)}$, $b^{M,f}_{\det H^*(X,Y,F)}$, we easily get the following anomaly formula

$$
Proposition 4.3. Let $b^F_u$, $0 \leq u \leq 1$, be a smooth one-parameter non-degenerate symmetric bilinear forms which are product near the boundary $Y$, then we have

\[
\left( \frac{b^M \nabla f}{\det(H^*(\tilde{X}, \tilde{F}), \mathbb{Z}_2)} \right) (\phi) = \prod_{x \in B_Y} \left( \frac{b^F_x}{\det(b^F_x, b^F_0)} \right)^{(-1)^{\operatorname{ind}_Y(x)}} . \tag{4.25}
\]

4.3 Comparison of $b^M \nabla f$ and $b^R \nabla f$

In this section, we will compare $b^M \nabla f$ and $b^R \nabla f$, we will use the method in [50].

Theorem 4.4. (Compare with [5, Theorem 5.1]) Let $\phi \in \mathbb{Z}_2$ be element induced by the involution on $\tilde{X}$, then the following identity holds,

\[
\frac{b^R}{b^M \nabla f} \left( \frac{b^M \nabla f}{\det(H^*(\tilde{X}, \tilde{F}), \mathbb{Z}_2)} \right) (\phi) = \\
\exp \left( - \int_Y \theta(\tilde{F}, b^F)(\nabla f)^* \psi(TY, \nabla^T Y) + \operatorname{rk}(F) \chi(Y) \log(2) \right) . \tag{4.26}
\]

Remark 4.5. By Theorem 4.2, Proposition 4.3 and [5, Section 4], we can assume that $g^TF$ and $\tilde{f}$ verifies the condition in [5, Section 5 b)]. Moreover, we may assume that $b^F$ is flat on an open neighborhood of the zero set $B$. By [8, Theorem 5.9], from $b^F$ we can get a Hermitian metric $g^F$, which can also be assumed to be flat on an open neighborhood of the zero set $B$.

For any $T \in \mathbb{R}$, let $b^F_T$ be the deformed symmetric bilinear form on $\tilde{F}$ defined by

\[
b^F_T(u, v) = e^{-2Tf} b^F(u, v) . \tag{4.27}
\]

Let $d b_T^{\#}$ be the associated formal adjoint in the sense of (2.1). Set

\[
D_{b_T} = d^F + d b_T^{\#}, \quad D_{b_T}^2 = \left( d^F + d b_T^{\#} \right)^2 = d^F d b_T^\# + d b_T^\# d^F . \tag{4.28}
\]

Let $\Omega^*_{[0,1],T}(\tilde{X}, \tilde{F})$ be defined as in Section 2 with respect to $D_{b_T}^2$, and let $\Omega^*_{[0,1],T}(\tilde{X}, \tilde{F})^\perp$ be the orthogonal complement with respect to the symmetric bilinear form. Let $P_T^{(3, +\infty)}$ be spectral projection onto $\Omega^*_{[0,1],T}(\tilde{X}, \tilde{F})^\perp$.

In the current case, comparing with the notations in [5, p. 168], we introduce

\[
\chi_\phi(\tilde{F}) = \operatorname{rk}(\tilde{F}) \chi(Y),
\]

\[
\chi_\phi'(\tilde{F}) = \operatorname{rk}(\tilde{F}) \sum_{x \in B_Y} (-1)^{\operatorname{ind}_Y(x)} \operatorname{ind}(x) = \operatorname{rk}(\tilde{F}) \sum_{x \in B_Y} (-1)^{\operatorname{ind}_Y(x)} \operatorname{ind}_Y(x)
\]

and

\[
\operatorname{Tr}_{B_Y}[f] = \operatorname{rk}(\tilde{F}) \sum_{x \in B_Y} (-1)^{\operatorname{ind}_Y(x)} f(x),
\]

where we used the fact that $d^2 f(x)|_{B_Y} > 0$ for $x \in B_Y$.

Let $N$ be the number operator on $\Omega^*(\tilde{X}, \tilde{F})$ acting on $\Omega^i(\tilde{X}, \tilde{F})$ by multiplication by $i$. Let $P_{T}^{[0,1],\det_H}$ be the restriction of the de Rham map $P_{\infty}$ (cf. [50] (3.1)-(3.6)) on $\Omega^*_{[0,1],T}(\tilde{X}, \tilde{F})$, and let $P_{T}^{[0,1],\det_H}$ be the induced isomorphism on cohomology.

We now state several intermediate results whose proofs will be given later. Note that $\phi$ acts as $\text{Id}$ on $\tilde{F}_x$ for $x \in Y$. 
Theorem 4.6. The following identity holds:

\[ \lim_{T \to +\infty} \frac{P_{T}^{[0,1],\det H}(b_{\det((H^*\cap(\Omega_{[0,1]}^{\text{int}})),Z_{2}))}}{\det H^*((X,F),Z_{2})} (\phi) \left( \frac{T}{T} \right)^{\frac{\chi_{\phi}(\bar{F})-\chi_{\phi}'(\bar{F})}{2}} \exp(2Tr_{s}B_{s}[f]T) = 1. \tag{4.29} \]

Theorem 4.7. For any \( t > 0 \),

\[ \lim_{T \to +\infty} Tr_{s} [\phi \exp(-tD_{b_{T}}^{2})P_{T}^{(1,+\infty)}] = 0. \tag{4.30} \]

Moreover, for any \( d > 0 \) there exist \( c > 0, C > 0 \) and \( T_{0} \geq 1 \) such that for any \( t \geq d \) and \( T \geq T_{0} \),

\[ \left| Tr_{s} [\phi \exp(-tD_{b_{T}}^{2})P_{T}^{(1,+\infty)}] \right| \leq c \exp(-Ct). \tag{4.31} \]

Theorem 4.8. The following identity holds

\[ \lim_{T \to +\infty} Tr_{s} [\phi P_{T}^{[0,1]}] = \chi_{\phi}(\bar{F}). \tag{4.32} \]

Also

\[ \lim_{T \to +\infty} Tr_{s} [\phi D_{b_{T}}^{2}P_{T}^{[0,1]}] = 0. \tag{4.33} \]

Theorem 4.9. As \( t \to 0 \), the following identity holds,

\[ Tr_{s} [\phi \exp(-tD_{b})] = \frac{m}{2} \chi_{\phi}(\bar{F}) + O(t) \text{ if } m \text{ is even}, \]

\[ = \text{rk}(F) \int_{Y} \int_{B} L \exp \left( -\frac{\tilde{R}^{TY}}{2} \right) \cdot \frac{1}{\sqrt{t}} + O(\sqrt{t}) \text{ if } m \text{ is odd}, \tag{4.34} \]

where \( L \) is defined similar as \( \text{(4, (3.52))} \) on \( Y \).

Theorem 4.10. There exist \( 0 \leq \alpha \leq 1, C > 0 \) such that for any \( 0 \leq t \leq \alpha, 0 \leq T \leq \frac{1}{t} \), then

\[ \left| Tr_{s} [\phi \exp(-(tD_{b}+\tilde{c} (\nabla f)^{2}))] - \frac{1}{t} \int_{Y} \int_{B} L \exp(-B_{T_{2}}) \text{rk}(\bar{F}) \right. \]

\[ \left. - \frac{T}{2} \int_{Y} \theta_{\phi}(\bar{F}, b_{T}) \int_{B} \tilde{d}f \exp(-B_{T_{2}}) - \frac{m}{2} \chi_{\phi}(\bar{F}) \right| \leq Ct. \tag{4.35} \]

Theorem 4.11. For any \( T > 0 \), the following identity holds

\[ \lim_{t \to 0} \left[ \phi \exp \left( - \left( tD_{b} + \frac{T}{t} \tilde{c}(\nabla f) \right)^{2} \right) \right] \]

\[ = \text{rk}(\bar{F}) \cdot \left( \frac{1}{1-e^{-2T}} \left( 1 + e^{-2T} \sum_{x \in Y \cap B} (-1)^{\text{ind}_{Y}(x)} \text{ind}_{Y}(x) - \dim Y e^{-2T} \chi(Y) \right) \right) \]

\[ - \frac{\text{rk}(\bar{F})^{1}}{2} \left( \frac{\sinh(2T)}{\cosh(2T) + 1} - 1 \right) \chi(Y). \tag{4.36} \]
Theorem 4.12. There exist \( \alpha \in (0, 1], c > 0, C > 0 \) such that for any \( t \in (0, \alpha) \), \( T \geq 1 \), then
\[
\left| \text{Tr}_a \left[ \phi N \exp \left( - \left( tD_b + \frac{T}{t} \tilde{c} \nabla f \right)^2 \right) \right] - \tilde{\chi}'(\tilde{F}) \right| \leq c \exp(-CT).
\] (4.37)

Now we give a proof of Theorem 4.4.

Theorem 4.13. The following identity holds,
\[
\frac{b^{RS}}{b^{Mf}} \left( \frac{\det(H^\ast, \tilde{X}, \tilde{F})}{\det(H^\ast, \tilde{X}, F)} \right)(\phi) = \exp \left( - \int_{\mathcal{Y}} \theta(\tilde{F}, b\tilde{F})(\nabla f)^*\psi(TY, \nabla^T Y) + \text{rk}(F)\chi(Y) \log(2) \right).
\] (4.38)

Proof. First of all, by the anomaly formula (4.14), for any \( T \geq 0 \), one has
\[
P^{[0, 1], \det H}_{\theta, T} \left( \frac{b^{RS}}{b^{Mf}} \left( \frac{\det(H^\ast, \tilde{X}, \tilde{F})}{\det(H^\ast, \tilde{X}, F)} \right)(\phi) \right) = \prod_{i=0}^{m} \left( \det(D^2_{b\|_{\Omega^\ast, T}}(\tilde{X}, \tilde{F}))(\phi) \right)^{(-1)^i}
\]
\[
= P^{\det H} \left( \frac{b^{RS}}{b^{Mf}} \left( \frac{\det(H^\ast, \tilde{X}, \tilde{F})}{\det(H^\ast, \tilde{X}, F)} \right)(\phi) \exp \left( -2\text{rk}(F) \int_{\mathcal{Y}} f e(TY, \nabla^T Y) \right) \right). \] (4.39)

From now on, we will write \( a \simeq b \) for \( a, b \in \mathbb{C} \) if \( e^a = e^b \). Thus, we can rewrite (4.39) as
\[
\log \left( \frac{P^{[0, 1], \det H}_{\theta, T} \left( \frac{b^{RS}}{b^{Mf}} \left( \frac{\det(H^\ast, \tilde{X}, \tilde{F})}{\det(H^\ast, \tilde{X}, F)} \right)(\phi) \right) \right)
\]
\[
\simeq \log \left( \frac{P^{[0, 1], \det H}_{\theta, T} \left( \frac{b^{RS}}{b^{Mf}} \left( \frac{\det(H^\ast, \tilde{X}, \tilde{F})}{\det(H^\ast, \tilde{X}, F)} \right)(\phi) \right) \right)
\]
\[
+ \sum_{i=0}^{m} (-1)^i \log \left( \det(D^2_{b\|_{\Omega^\ast, T}}(\tilde{X}, \tilde{F}))(\phi) \right)
\]
\[
+ 2\text{Trk}(F) \int_{\mathcal{Y}} f e(TY, \nabla^T Y). \] (4.40)

Let \( T_0 > 0 \) be as in Theorem 4.7. For any \( T \geq T_0 \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) \geq m+1 \), set
\[
\theta_{\phi, T}(s) = \frac{1}{\Gamma(s)} \int_{0}^{+\infty} t^{s-1} \text{Tr}_a \left[ \phi N \exp \left( -tD^2_{b\|_{\Omega^\ast, T}}(\tilde{X}, \tilde{F}) \right) P^{[1, +\infty]}_{T} \right] dt.
\] (4.41)

By (4.31), \( \theta_{\phi, T}(s) \) is well defined and can be extended to a meromorphic function which is holomorphic at \( s = 0 \). Moreover,
\[
\sum_{i=0}^{m} (-1)^i \log \left( \det(D^2_{b\|_{\Omega^\ast, T}}(\tilde{X}, \tilde{F}))(\phi) \right) \simeq - \frac{\partial \theta_{\phi, T}(s)}{\partial s} \bigg|_{s=0}. \] (4.42)
Let $d = \alpha^2$ with $\alpha$ being as in Theorem 4.12. From (4.41) and Theorems 4.7-4.9, one finds that

$$
\lim_{T \to +\infty} \left. \frac{\partial \theta_{\phi, T}(s)}{\partial s} \right|_{s=0}
= \lim_{T \to +\infty} \int_0^d \left( \text{Tr}_s \left[ \phi N \exp \left( -t D^2_{\theta r} \right) \right] - \frac{a-1}{\sqrt{t}} - \frac{m}{2} \chi_0(\bar{F}) \right) \frac{dt}{t}
- \frac{2a-1}{\sqrt{d}} - (T'(1) - \log d) \left( \frac{m}{2} \chi_0(\bar{F}) - \bar{\chi}_0(\bar{F}) \right),
$$

(4.43)

where we denote for simplicity that

$$
a_{-1} = \text{rk}(F) \int_Y \int B \exp \left( -\hat{R}^T Y \right).
$$

To study the first term in the right hand side of (4.43), we observe first that for any $T \geq 0$, one has

$$
e^{-T \tilde{f}} D^2_{\theta r} e^{T \tilde{f}} = \left( D_b + T \tilde{c}(\nabla \tilde{f}) \right)^2.
$$

(4.44)

Thus, one has

$$
\text{Tr}_s \left[ \phi N \exp \left( -t D^2_{\theta r} \right)^2 \right] = \text{Tr}_s \left[ \phi N \exp \left( -t \left( D_b + T \tilde{c}(\nabla \tilde{f}) \right) \right) \right].
$$

(4.45)

By (4.45), one writes

$$
\int_0^d \left( \text{Tr}_s \left[ \phi N \exp \left( -t D^2_{\theta r} \right) \right] - \frac{a-1}{\sqrt{t}} - \frac{m}{2} \chi_0(\bar{F}) \right) \frac{dt}{t}
= 2 \int_1^{\sqrt{d} T} \left( \text{Tr}_s \left[ \phi N \exp \left( - \left( \frac{t}{\sqrt{T}} D_b + t \sqrt{T} \tilde{c}(\nabla \tilde{f}) \right)^2 \right) \right] - \frac{\sqrt{T}}{t} a_{-1}
- \frac{m}{2} \chi_0(\bar{F}) \right) \frac{dt}{t}
+ 2 \int_0^{\sqrt{d} T} \left( \text{Tr}_s \left[ \phi N \exp \left( - \left( t D_b + t \sqrt{T} \tilde{c}(\nabla \tilde{f}) \right)^2 \right) \right] - \frac{a-1}{t} - \frac{m}{2} \chi_0(\bar{F}) \right) \frac{dt}{t}
$$

(4.46)

In view of Theorem 4.10, we write

$$
\int_0^{\sqrt{d} T} \left( \text{Tr}_s \left[ \phi N \exp \left( - \left( t D_b + t \sqrt{T} \tilde{c}(\nabla \tilde{f}) \right)^2 \right) \right] - \frac{a-1}{t} - \frac{m}{2} \chi_0(\bar{F}) \right) \frac{dt}{t}
= \int_0^{\sqrt{d} T} \left( \text{Tr}_s \left[ \phi N \exp \left( - \left( t D_b + t \sqrt{T} \tilde{c}(\nabla \tilde{f}) \right)^2 \right) \right] \right]
- \frac{1}{t} \text{rk}(F) \int_Y \int B \exp (\tilde{B}_{(T)})
- \frac{t T}{2} \int_Y \theta(F, b^F) \int B \tilde{d}\tilde{f} \exp (\tilde{B}_{(T)}) - \frac{m}{2} \chi_0(\bar{F}) \right) \frac{dt}{t}
+ \int_0^{\sqrt{d} T} \left( \frac{1}{t} \text{rk}(F) \int_Y \int B \exp (\tilde{B}_{(T)}) a_{-1} \right) \frac{dt}{t}
+ \int_0^{\sqrt{d} T} \frac{t T}{2} \int_Y \theta(F, b^F) \int B \tilde{d}\tilde{f} \exp (\tilde{B}_{(T)}) \frac{dt}{t}
$$

(4.47)
By Definitions 3.6, 3.12 and Theorem 3.18, one has that, as $T \to +\infty$,

$$
\int_0^T \frac{t}{2} \int_Y \theta(F, b^F) \int_0^B \frac{dt}{\chi} \exp\left( -B(t_T) \right) \frac{dt}{t} \to \\
\frac{1}{2} \int_Y \theta(F, b^F) \left( \nabla f \right)^* \psi \left( T_Y, \nabla T_Y \right) .
$$

From Theorems 4.10, 4.11, [4, Theorem 3.20], [4, (7.72) and (7.73)] and integration by parts, we have

$$
\int_0^T \left( \frac{1}{t} \text{rk}(F) \int_Y \int_0^B L \exp\left( -B(t_T) \right) - \frac{a-1}{t} \right) \frac{dt}{t} = -\sqrt{T} \text{rk}(F) \int_Y \int_0^B L \exp(-B_T) + \sqrt{T} a_{-1}
$$

$$
- \text{rk}(F) \int_Y \int_0^B f \left( \int_0^B \exp(-B_T) + \text{Tr}(F) \int_Y f \int_0^B \exp(-B_0) .
$$

From Theorems 4.10, 4.11 [4, Theorem 3.20], [4, (7.72) and (7.73)] and the dominate convergence, one finds that as $T \to +\infty$,

$$
\int_0^T \left( \text{Tr}_Y \left[ \phi N \exp\left( - \left( tD_y + tT \nabla \left( \sqrt{\nabla f} \right) \right)^2 \right) \right] \\
- \frac{1}{t} \text{rk}(F) \int_Y \int_0^B L \exp\left( -B(t_T) \right) \\
- \frac{1}{t} \int_Y \theta(F, b^F) \int_0^B \frac{dt}{\chi} \exp\left( -B(t_T) \right) - \frac{m}{2} \chi_\phi(F) \right) \frac{dt}{t} = \int_0^1 \left( \text{Tr}_Y \left[ \phi N \exp\left( - \left( \frac{t}{\sqrt{T}} D_y + t\sqrt{T} \nabla \left( \sqrt{\nabla f} \right) \right)^2 \right) \right] \\
- \frac{\sqrt{T}}{t} \text{rk}(F) \int_Y \int_0^B L \exp\left( -B(t_T) \right) \\
- \frac{t\sqrt{T}}{2} \int_Y \theta(F, b^F) \int_0^B \frac{dt}{\chi} \exp\left( -B(t_T) \right) - \frac{m}{2} \chi_\phi(F) \right) \frac{dt}{t} \\
\to \int_0^1 \left\{ \frac{1}{1 - e^{-2t^2}} \left( 1 + e^{-2t^2} \right) \chi_\phi(F) - \text{dim} Y e^{-2t^2} \chi_\phi(F) \\
- \text{rk}(F) \frac{1}{2} \left( \frac{\sinh(2t^2)}{\cosh(2t^2)} + 1 \right) \chi(Y) \\
+ \frac{1}{2t^2} \text{rk}(F) \sum_{x \in B_Y} \left( -1 \right)^{\text{ind}_Y(x)} \left( \text{dim} Y - 2\text{ind}_Y(x) \right) - \frac{m}{2} \chi_\phi(F) \right\} \frac{dt}{t}
$$

$$
= \frac{1}{2} \text{rk}(F) \left\{ \sum_{x \in B_Y} \left( -1 \right)^{\text{ind}_Y(x)} \text{ind}_Y(x) - \frac{1}{2} \chi(Y) \text{dim} Y \right\} \cdot \int_0^1 \left( \frac{1 + e^{-2t}}{1 - e^{-2t}} - \frac{1}{t} \right) \frac{dt}{t} - \text{rk}(F) \frac{1}{4} \chi(Y) \cdot \int_0^1 \frac{\sinh(2t)}{\cosh(2t) + 1} \frac{dt}{t} .
$$

On the other hand, by Theorems 4.11, 4.12 and the dominate convergence, we
have that as $T \to +\infty$,

$$
\int_1^{\sqrt{T d}} \left( \text{Tr}_s \left[ \phi N \exp \left( - \left( \frac{t}{\sqrt{T}} D_b + t \sqrt{T} \tilde{c}(\nabla \tilde{f}) \right)^2 \right) \right] - \frac{\sqrt{T}}{t} a - 1 - \frac{m}{2} \chi_{\phi}(\tilde{F}) \right) \frac{dt}{t}
$$

$$
= \int_1^{\sqrt{T d}} \left( \text{Tr}_s \left[ \phi N \exp \left( - \left( \frac{t}{\sqrt{T}} D_b + t \sqrt{T} \tilde{c}(\nabla \tilde{f}) \right)^2 \right) \right] - \tilde{\chi}'_{\phi}(\tilde{F}) \right) \frac{dt}{t}
$$

$$
+ \frac{1}{2} \tilde{\chi}'_{\phi}(\tilde{F}) \log(Td) + a - 1 \sqrt{T} \left( \frac{1}{\sqrt{T d}} - 1 \right) - \frac{m}{4} \chi_{\phi}(\tilde{F}) \log(Td)
$$

$$
= \int_1^{+\infty} \left\{ \frac{1}{1 - e^{-2t^2}} \left( 1 + e^{-2t^2} \right) \tilde{\chi}'_{\phi}(\tilde{F}) - \text{dim} Y e^{-2t^2} \chi_{\phi}(\tilde{F}) \right\}
$$

$$
- \text{rk}(\tilde{F}) \frac{1}{2} \left( \frac{\sinh(2t^2)}{\cosh(2t^2) + 1} - 1 \right) \chi(Y) - \tilde{\chi}_{\phi}(\tilde{F}) \right\} \frac{dt}{t}
$$

$$
+ \frac{1}{2} \chi_{\phi}(\tilde{F}) \log(Td) + a - 1 \sqrt{T} \left( \frac{1}{\sqrt{T d}} - 1 \right) - \frac{m}{4} \chi_{\phi}(\tilde{F}) \log(Td) + o(1)
$$

$$
= \text{rk}(\tilde{F}) \left\{ \sum_{x \in B_Y} (-1)^{\text{ind}_Y(x)} \text{ind}_Y(x) - \frac{1}{2} \chi(Y) \text{dim} Y \right\}
$$

$$
\cdot \int_1^{+\infty} \frac{e^{-2t^2}}{1 - e^{-2t^2}} \frac{dt}{t}
$$

$$
- \text{rk}(\tilde{F}) \frac{1}{4} \chi(Y) \int_1^{+\infty} \left( \frac{\sinh(2t^2)}{\cosh(2t^2) + 1} - 1 \right) \frac{dt}{t}
$$

$$
+ \frac{1}{2} \left( \tilde{\chi}'_{\phi}(\tilde{F}) - \frac{m}{2} \chi_{\phi}(\tilde{F}) \right) \log(Td) + \frac{a - 1}{\sqrt{d}} - \sqrt{T} a - 1 + o(1). \quad (4.51)
$$

Combining (4.29), (4.39) and (4.46)-(4.51), one deduces, by setting $T \to +\infty$,
that

\[
\log \left( \frac{P_{\text{det}H}^{\infty}}{b_{\text{det}K}^{M} \frac{\text{det}(H \chi_{F})_{Z_{2}}}{\text{det}(H \chi_{F})_{Z_{2}}}} \phi \right) 
\approx 
- 2T_{B}^{|s|}[f]T + \left( \chi_{\phi}(\bar{T}) - \frac{m}{2} \chi_{\phi}(\bar{T}) \right) \log T
\]

\[
- \int_{Y} \theta(F, b^F)(\nabla f)^{*} \psi (TY, \nabla TY)
\]

\[
+ 2\sqrt{rTk(F)} \int_{Y} f \int_{B} \exp(-B_{T})
\]

\[- 2T_{0} - 2T_{0} \exp(-B_{0})
\]

\[- \text{rk}(\bar{T}) \left\{ \sum_{x \in B_{Y}} (-1)^{\text{ind}_{Y}(x)} \text{ind}_{Y}(x) - \frac{1}{2} \chi(Y) \text{dim}Y \right\}
\]

\[
\cdot \left( \int_{0}^{1} \left( \frac{1 + e^{-2t}}{1 - e^{-2t}} - \frac{1}{t} \right) dt + \int_{1}^{+ \infty} \frac{2e^{-2t}}{1 - e^{-2t}} dt \right)
\]

\[+ 2\text{rk}(\bar{T}) \int_{Y} f \text{e}(TY, \nabla^{TY})
\]

\[+ \frac{2a_{1}}{\sqrt{d}} - (T'(1) - \log d) \left( \chi_{\phi}(\bar{T}) - \frac{m}{2} \chi_{\phi}(\bar{T}) \right) + o(1).
\]

(4.52)

By [4] Theorem 3.20] and [4] (7.72)], one has

\[
\lim_{T \to + \infty} 2T_{0} \exp(-B_{T}) - 2T_{0} \exp(-B_{T}) = - \text{rk}(\bar{T}) \left\{ \sum_{x \in B_{Y}} (-1)^{\text{ind}_{Y}(x)} \text{ind}_{Y}(x) - \frac{1}{2} \chi(Y) \text{dim}Y \right\},
\]

(4.53)

\[
\lim_{T \to + \infty} 2\sqrt{rTk(F)} \int_{Y} f \int_{B} \exp(-B_{T})
\]

\[= 2\text{rk}(\bar{T}) \left\{ \sum_{x \in B_{Y}} (-1)^{\text{ind}_{Y}(x)} \text{ind}_{Y}(x) - \frac{1}{2} \chi(Y) \text{dim}Y \right\}.
\]

(4.54)

On the other hand, by [4] (7.93)] and [5] (5.55)], one has

\[
\int_{0}^{1} \left( \frac{1 + e^{-2t}}{1 - e^{-2t}} - \frac{1}{t} \right) dt + \int_{1}^{+ \infty} \frac{2e^{-2t}}{1 - e^{-2t}} dt = 1 - \log \pi - \Gamma'(1),
\]

(4.55)

\[
\int_{0}^{1} \left( \frac{\sinh(2t)}{\cosh(2t) + 1} \right) dt + \int_{1}^{+ \infty} \left( \frac{\sinh(2t)}{\cosh(2t) + 1} - 1 \right) dt
\]

\[= - \log \pi - \Gamma' \left( \frac{1}{2} \right).
\]

(4.56)
From \((4.52)\) - \((4.56)\), we get
\[
\frac{b_{RS}^{H^*\Omega, \Omega_1}}{b_{RS}^{H^*\Omega, \Omega_1}} (\phi) = \exp \left( - \int_Y \theta(F, F) (\nabla f)^* \psi(TY, \nabla TY) \right. \\
\left. - \frac{1}{4} \sum_{x \in B_{\phi}} \text{rk}(F)(-1) \text{ind}_{Y}(x) \left( 2 \Gamma' \left( \frac{1}{2} \right) - 2\Gamma'(1) \right) \right). \tag{4.57}
\]

By \([5, (5.53)]\), we know
\[
\Gamma' \left( \frac{1}{2} \right) - \Gamma'(1) = -2 \log(2). \tag{4.58}
\]
Then from \((4.57)\), \((4.58)\) and Lemma 3.1 we get \((4.38)\).

\[\square\]

4.4 Proofs of the intermediate results

The purpose of this subsection is to prove the intermediate results. Since the methods of the proofs of these theorems are essentially the same as the corresponding theorems in \([30]\), so we refer to \([30]\) for related definitions and notations directly when there will be no confusion, such as \(B_{b,g}, A_{b,t,T}, A_{g,t,T}, C_{t,T}, \cdots\).

4.4.1 Proof of Theorem 4.6

First, as it in \([30, (4.45)]\), we have
\[
\frac{P_{T, \Omega_1}[\det H^{\Omega^* (\Omega_1)} (\bar{X}, \bar{F})]}{P_{T, \Omega_1}[\det H^{\Omega^* (\Omega_1)} (\bar{X}, \bar{F})]} (\phi) = \prod_{i=0}^{m} \det \left( P_{\Omega_1, T}^{\Omega_1, T} | \Omega_1^{(\Omega_1)} (\bar{X}, \bar{F}) \right)^{(1)} (\phi). \tag{4.59}
\]

From \([30]\) Propositions 4.4 and 4.5], one deduces that as \(T \to +\infty,
\[
\det \left( P_{\Omega_1, T}^{\Omega_1, T} | \Omega_1^{(\Omega_1)} (\bar{X}, \bar{F}) \right) (\phi) = \det \left( e_{T} P_{T}^{\Omega_1, T} | \Omega_1^{(\Omega_1)} (\bar{X}, \bar{F}) \right) (\phi) \cdot \det^{-1} \left( e_{T} P_{T}^{\Omega_1, T} | \Omega_1^{(\Omega_1)} (\bar{X}, \bar{F}) \right) (\phi)
\]
\[
= \det \left( (P_{\Omega_1, T} e_{T})^{\#} | C_{\Omega_{1}, T} (\bar{X}, \bar{F}) \right) (\phi) \cdot \det^{-1} \left( e_{T} C_{\Omega_{1}, T} (\bar{X}, \bar{F}) \right) (\phi)
\]
\[
= \det \left( (1 + O(e^{-cT}))^{\#} | C_{\Omega_{1}, T} (\bar{X}, \bar{F}) \right) e^{2T, T} (1 + O(e^{-cT})) (\phi). \tag{4.60}
\]

From \((4.59)\) and \((4.60)\), we get the result immediately.

4.4.2 Proof of Theorem 4.7

The proof of Theorem 4.7 is the same as the proof of \([30, \text{Theorem 3.4}]\) given in \([30, \text{Section 5}]\).

4.4.3 Proof of Theorem 4.8

Recall that the operator \(e_{T} : C^{*} (W^{u}, \bar{F}) \to \Omega^{*}_{1, T} (\bar{X}, \bar{F})\) has been defined in \([30, (4.38)]\), and in the current case, we also have that \(e_{T}\) commutes with \(Z_{2}\). So by \([30, \text{Proposition 4.4}]\), we have that for \(T \geq 0\) large enough, \(e_{T} : C^{*} (W^{u}, \bar{F}) \to \Omega^{*}_{1, T} (\bar{X}, \bar{F})\) is an identification of \(Z_{2}\)-spaces. So \((4.32)\) follows. Also \((4.33)\) is from \([30, \text{proposition 4.2}]\).
4.4.4 Proof of Theorem 4.9

In this section, we provide a proof of Theorem 4.9, which computes the asymptotic of \( \text{Tr}_s[gN\exp(-tD_g^2)] \) for fixed \( T \geq 0 \) as \( t \to 0 \). The method is essentially the same as it in [30].

By [30] (6.4), we have

\[
e^{-tD_g^2} = e^{-tD_g^2} + \sum_{k=1}^{m} (-1)^k t^k \int_{\Delta_k} e^{-t_1 t_2 D_g^2} B_{b,g} e^{-t_2 t_3 D_g^2} \cdots B_{b,g} e^{-t_{k+1} t D_g^2} dt_1 \cdots dt_k
\]

\[
+ (-1)^{m+1} t^{m+1} \int_{\Delta_{m+1}} e^{-t_1 t_2 D_g^2} B_{b,g} e^{-t_2 t_3 D_g^2} \cdots B_{b,g} e^{-t_{m+2} t D_g^2} dt_1 \cdots dt_{m+1},
\]

(4.61)

where \( \Delta_k, 1 \leq k \leq m+1, \) is the \( k \)-simplex defined by \( t_1 + \cdots + t_{k+1} = 1, t_1 \geq 0, \cdots, t_{k+1} \geq 0 \). Also, by the same proof of [30] Proposition 6.1], we have the following result.

Proposition 4.14. As \( t \to 0^+ \), one has

\[
t^{m+1} \int_{\Delta_{m+1}} \text{Tr}_s \left[ \phi N e^{-t_1 t_2 D_g^2} B_{b,g} e^{-t_2 t_3 D_g^2} \cdots B_{b,g} e^{-t_{m+2} t D_g^2} \right] dt_1 \cdots dt_{m+1} \to 0.
\]

(4.62)

Now for \( 1 \leq k \leq m, \) we want to prove

\[
\lim_{t \to 0^+} t^k \int_{\Delta_k} \text{Tr}_s \left[ \phi N e^{-t_1 t_2 D_g^2} B_{b,g} e^{-t_2 t_3 D_g^2} \cdots B_{b,g} e^{-t_{k+1} t D_g^2} \right] dt_1 \cdots dt_k = 0.
\]

(4.63)

First from [13] Theorem 3.3, in our case, we have, as \( t \to 0^+ \),

\[
t^k \int_{\Delta_k} \text{Tr}_s \left[ \phi N e^{-t_1 t_2 D_g^2} B_{b,g} e^{-t_2 t_3 D_g^2} \cdots B_{b,g} e^{-t_{k+1} t D_g^2} \right] dt_1 \cdots dt_k
\]

\[
= \sum_{|\lambda(k)| \leq m-k} \frac{(-1)^{|\lambda(k)|}}{\lambda(k)! \lambda(k)!} \text{Tr}_s \left[ \phi D_{b,g}^{|\lambda(k)|} \exp(-tD_g^2) \right] + O(\sqrt{t}),
\]

(4.64)

where

\[
D_{b,g}^{|\lambda(k)|} = t^{k+|\lambda(k)|} NB^{[\lambda_1]} B^{[\lambda_2]} \cdots B^{[\lambda_k]},
\]

\[
B^{[0]} = B_{b,g}, \quad B^{[k]} = \left[ D_g^2, B^{[k-1]} \right],
\]

and we refer to [13] (3.22)-(3.26) the other notations.

Let \( N_Y = TX/TY \) be the normal bundle to \( Y \) in \( X \). We identity \( N_Y \) with the orthogonal bundle to \( TY \) in \( TX \). By standard estimates of heat kernel, the problem in calculating \( t \to 0^+ \) in (4.64) can be localized to an open neighborhood \( U \) of \( Y \) in \( X \). Using normal geodesic coordinates to \( Y \) in \( X \), we will identify \( U \) to an \( \varepsilon \)-neighborhood of \( Y \) in \( N_Y \).

Since we have used normal geodesic coordinates to \( Y \) in \( X \), if \( (y, z) \in N_Y \),

\[
\phi^{-1}(y, z) = (y, \phi^{-1} z).
\]

(4.65)

Let \( dv_Y, dv_{N_Y} \) be the Riemannian volumes on \( TY, N_Y \) induced by \( g^{TX} \). Let \( k(y, z) \) \( (y \in Y, z \in N_Y, \left| z \right| < \varepsilon) \) be defined by

\[
dv_X = k(y, z) dv_Y(y) dv_{N_Y}(z).
\]

(4.66)

Then

\[
k(y, 0) = 1.
\]

(4.67)
Let $\rho(Z)$ be a smooth function which is equal to 1 if $|Z| \leq \frac{1}{8}t$ and equal to 0 if $|Z| \geq \frac{1}{2}t$. Take $x_0 \in Y$, let $F_{x_0}$ be the smooth sections of $(\Lambda^*(T^*X) \otimes \overline{F})_{x_0}$ over $T_{x_0}X$. Let $\Delta^*X$ be the standard Laplacian on $T_{x_0}X$ with respect to the metric $g^{T^*X}$.

Let $J^1_\ell$ be the operator acting on $F_{x_0}$

$$J^1_\ell = (1 - \rho^2(Z))(-t\Delta^{T_{x_0}X}) + \rho^2(Z)tD^2_g. \quad (4.68)$$

Let $H_t$ be the linear map

$$s(Z) \in F_{x_0} \rightarrow s\left(\frac{Z}{\sqrt{t}}\right) \in F_{x_0}. \quad (4.69)$$

Set

$$J^2_\ell = H^{-1}_t J^1_\ell H_t. \quad (4.70)$$

Let $e_1, \ldots, e_{m-1}$ be an oriented orthonormal base of $T_{x_0}Y$ and let $e_m$ be an orthonormal base of $N_Y$.

Let $J^3_\ell$ be the operator obtained from $J^2_\ell$ by replacing $c(e_i), \tilde{c}(e_i), 1 \leq i \leq m-1$ by

$$c_i(e_i) = \frac{e_i}{t^{\frac{1}{2}}} \wedge -t^\frac{1}{2} i_{e_i}, \tilde{c}_i(e_i) = \frac{\tilde{e}_i}{t^{\frac{1}{2}}} \wedge +t^\frac{1}{2} i_{\tilde{e}_i}, 1 \leq i \leq m-1. \quad (4.71)$$

Let $G_t$ be the process of (4.70) and (4.71) in the above. Let $P_t$ be the smooth kernel of $\exp(-tD^2_g)$ and let $P^1_t(z, z') (z, z' \in T_{x_0}X, i = 1, 2, 3)$ be the smooth kernel associated to $\exp(-J^1_\ell)$ with respect to $dv_{T_{x_0}X}(z')$. Then we have

$$\lim_{t \rightarrow 0^+} \int_{H_t/8} \text{Tr}_s \left[ \phi D^k_t \exp(-tD^2_g) \right] = \lim_{t \rightarrow 0^+} \int_{H_t/8} \text{Tr}_s \left[ \phi D^k_t P_t(\phi^{-1}x, x) \right] dv_X(x) =$$

$$\lim_{t \rightarrow 0^+} \int_{y \in Y} \int_{z \in N_Y, |z| \leq \varepsilon/8} \text{Tr}_s \left[ \phi D^k_t P_t(\phi^{-1}(y, z), (y, z)) \right] k(y, z) dv_Y(y) dv_{N_Y}(z).$$

(4.72)

By (4.68) and the finite propagation speed, there exist $c, C > 0$ such that for $z \in N_Y, |z| \leq \frac{1}{4}t, 0 < t \leq 1,$ we have

$$|P_t(\phi^{-1}(y, z), (y, z))k(y, z) - P^1_t(\phi^{-1}z, z)| \leq C \exp\left(-\frac{C}{t^2}\right). \quad (4.73)$$

Let $[G_t \left(D^k_t \right) P^3_t(\phi^{-1}z, z)]_{\text{max}} \in \text{End}(\Lambda^*(N_Y)) \otimes \text{End}\overline{F}$ be the coefficient of $\epsilon^1 \wedge \cdots \epsilon^{m-1} \wedge \tilde{c}^1 \cdots \wedge \tilde{c}^{m-1}$ in the expansion of it. Then by [4] Proposition 4.11, we have

$$\text{Tr}_s \left[ \phi D^k_t P^1_t(\phi^{-1}z, z) \right] = 2^{m-1}(-1)^{\frac{m(m+1)}{2}} \frac{1}{\sqrt{t}} \text{Tr}_s \left[ \phi \left[ G_t \left(D^k_t \right) P^3_t(\frac{\phi^{-1}z}{\sqrt{t}}, \frac{z}{\sqrt{t}}) \right] \right]_{\text{max}}. \quad (4.74)$$

Then

$$\lim_{t \rightarrow 0^+} \int_{z \in N_Y, |z| \leq \varepsilon/8} \text{Tr}_s \left[ \phi D^k_t P^1_t(\phi^{-1}z, z) \right] dv_{N_Y}(z) = \lim_{t \rightarrow 0^+} \int_{z \in N_Y, |z| \leq \varepsilon/8} 2^{m-1}(-1)^{\frac{m(m+1)}{2}} \frac{1}{\sqrt{t}} \text{Tr}_s \left[ \phi \left[ G_t \left(D^k_t \right) P^3_t(\frac{\phi^{-1}z}{\sqrt{t}}, \frac{z}{\sqrt{t}}) \right] \right]_{\text{max}} dv_{N_Y}(z). \quad (4.75)$$
Let \( a > 0 \) be the injectivity radius of \((\tilde{X}, g^{\tilde{X}})\). We identify the open ball 
\( B^{X}(0, \frac{a}{2}) \) with the open ball \( B^{\tilde{X}}(x, \frac{a}{2}) \) in \( \tilde{X} \) using geodesic coordinates. Then 
\( y \in T_x \tilde{X}, \|y\| \leq \frac{a}{2} \), represents an element of \( B^{\tilde{X}}(0, \frac{a}{2}) \). For \( y \in T_x \tilde{X}, \|y\| \leq \frac{a}{2} \), we identify \( T_y \tilde{X}, \tilde{F}_y \) to \( T_x \tilde{X}, \tilde{F}_x \) by parallel transport along the geodesic \( t \in [0, 1] \to ty \) with respect to the connections \( \nabla^{\tilde{X}}, \nabla^{\tilde{F}, u} \) respectively.

Let \( \Gamma^{X,x}, \Gamma^{\tilde{F}, u,x} \) be the connection forms for \( \nabla^{\tilde{X}}, \nabla^{\tilde{F}, u} \) in the considered trivialization of \( T \tilde{X} \). By \cite[Proposition 4.7]{1}, one has

\[
\Gamma^{X,x}(x) = \frac{1}{2} R^{\tilde{X}}_x (y, \cdot) + O(\|y\|^2),
\]

\[
\Gamma^{\tilde{F}, u,x} = O(\|y\|).
\]

Then by direct computation, we find that as \( t \to 0^+ \),

\[
G_t \left( \int^{t+1} B^l \right) = O(\sqrt{t}), \quad l \geq 0,
\]

and

\[
G_t(N) = \frac{1}{2\sqrt{t}} \sum_{i=1}^{m-1} \epsilon_i \wedge \hat{\epsilon}_i + O(1) = \frac{1}{\sqrt{t}} L_{\|N\|} + O(1),
\]

then

\[
\lim_{t \to 0^+} G_t \left( \mathcal{D}_t^k \right) \text{ exists and } \lim_{t \to 0^+} G_t \left( \mathcal{D}_t^k \right) = 0, \quad 1 < k \leq m.
\]

Using \cite[(4.29)]{1}, one can find that as \( t \to 0^+ \),

\[
J_t^3 \to J_0^3 = -\Delta^{\tilde{X}_0 \tilde{X}} + \frac{1}{2} \tilde{R}^{\tilde{Y}}.
\]

Then by \cite[(4.75), (4.77), (4.78) and (4.80)]{1}, we have

\[
\lim_{t \to 0^+} \int_{z \in N_Y \cdot [z] \leq e/8} \text{Tr} \left[ \phi \mathcal{D}_t^k \mathcal{P}_1^k (\phi^{-1} z, z) \right] dv_{N_Y}(z) = 2^{m-1} \int_{j \in N_{\tilde{Y}}} \text{Tr} \left[ \phi \left( \mathcal{D}_0^k \mathcal{P}_0^k (\phi^{-1} z, z) \right) \right] dv_{N_{\tilde{Y}}}(z).
\]

Then by \cite[(4.64), (4.72), (4.73), (4.79), (4.80) and (4.81)]{1}, we have that for any \( 1 < k \leq m \),

\[
\lim_{t \to 0^+} t^{k} \int_{\Delta_k} \text{Tr} \left[ \phi e^{-t_1 \mathcal{D}_1^k} B_{b,g} e^{-t_2 \mathcal{D}_2^k} \cdots B_{b,g} e^{-t_k \mathcal{D}_k^k} \right] dt_1 \cdots dt_k = 0,
\]

while for \( k = 1, 0 \leq t_1 \leq 1 \), using the standard heat kernel on \( \mathbb{R}^n, \phi^{-1} z = -z \), \cite[(6.16)]{6} and

\[
\frac{1}{2\sqrt{t} \pi} \int_{\mathbb{R}} \exp \left( -\frac{4|y|^2}{4t} \right) dy = \frac{1}{2} \quad \text{Tr} \left[ \mathcal{C}(e_m) \mathcal{C}(e_m) \right] = -2,
\]

we have

\[
\lim_{t \to 0^+} t \text{Tr} \left[ \phi e^{-t \mathcal{D}_1^k} B_{b,g} e^{-(1-t_1) \mathcal{D}_1^k} \right] = \lim_{t \to 0^+} t \text{Tr} \left[ \phi B_{b,g} e^{-t \mathcal{D}_1^k} \right] = \frac{1}{2} \int_{\mathbb{R}} \int_{Y} \text{Tr} \left[ \phi \left( \sum_{i,j=1}^{m-1} \epsilon_i \wedge \hat{\epsilon}_j (\nabla_{\epsilon_i} \omega F (\epsilon_j)) + \frac{1}{2} \left[ \omega F, \omega F - \hat{\omega F} \right] \right) \right] \cdot L \exp \left( -\frac{\tilde{R}^{\tilde{Y}}}{2} \right).
\]
So by [5] (2.13), and proceeding as in [30] (6.26)-(6.28), we have
\[
\lim_{t \to 0^+} t \text{Tr}_s \left[ \phi N e^{-tD_g^2} B_{b,g} e^{-(1-t_1)D_g^2} \right] = 0. \tag{4.84}
\]

From (4.61), (4.62), (4.82) and (4.84) and [3, Theorem 5.9], one can get the result.

**4.4.5 Proof of Theorem 4.10**

In order to prove (4.10), one needs only to prove that under the conditions of Theorem 4.10, there exists a constant \( C^n > 0 \) such that
\[
|\text{Tr}_s \left[ \phi N \exp \left( -(tD_b + T \tilde{c}(\nabla f))^2 \right) \right] - \text{Tr}_s \left[ \phi N \exp \left( -(tD_g + T \tilde{c}(\nabla f))^2 \right) \right] |
\leq C^n t. \tag{4.85}
\]

By [30] (7.8), we have
\[
e^{-A_{g,t}^2} = e^{-A_{g,t}^2} + \sum_{k=1}^{m} (-1)^k \int_{\Delta_k} e^{-t_1 A_{g,t}^2} C_{t_1} e^{-t_2 A_{g,t}^2} \cdots C_{t_k} e^{-t_{k+1} A_{g,t}^2} dt_1 \cdots dt_k + (-1)^{m+1} \int_{\Delta_{m+1}} e^{-t_1 A_{g,t}^2} C_{t_1} e^{-t_2 A_{g,t}^2} \cdots C_{t_m} e^{-t_{m+2} A_{g,t}^2} dt_1 \cdots dt_{m+1}. \tag{4.86}
\]

From the proof of [30] (7.21), we have that there exists \( C_1 > 0 \) such that for any \( t > 0 \) small enough and \( T \in [0, \frac{1}{t}] \),
\[
\left| \int_{\Delta_{m+1}} \text{Tr}_s \left[ \phi N e^{-t_1 A_{g,t}^2} C_{t_1} e^{-t_2 A_{g,t}^2} \cdots C_{t_m} e^{-t_{m+2} A_{g,t}^2} \right] dt_1 \cdots dt_{m+1} \right| \leq C_1 t. \tag{4.87}
\]

By the standard heat kernel expansion, we see that for \( 1 \leq k \leq m \), our problem can be localized near \( Y \).

Now for any \( x \in Y \), let \( e_1, \cdots, e_{m-1}, e_m \) be an orthonormal basis of \( T \tilde{X}|_Y \) such that \( e_1, \cdots, e_{m-1} \) is an orthonormal basis of \( TY \) and \( e_m \) is the normal vector field along \( Y \). Then we use the Getzler rescaling (cf. [2], [10], [17]) introduced in (4.71), with \( t \) there replaced by \( t^2 \) here. By using [30] (7.7), one has
\[
G_{t^2} (C_{t,T}) = G_{t^2} (t^2 B_{b,g}) + t \omega F (\nabla f). \tag{4.88}
\]

By (4.78), we have
\[
G_{t^2} (N) = \frac{1}{2t} \sum_{i=1}^{m-1} e_i \wedge \tilde{e}_i + O(1) = \frac{1}{t} L|_Y + O(1). \tag{4.89}
\]

By (4.88), (4.89), replacing (4.68) by [3] (13.8), that is,
\[
J^1_{t^2} = (1 - \rho^2 (Z)) \left( -t^2 \Delta_{T = \tilde{X}} + T^2 \right) + \rho^2 (Z) (t D_g + T \tilde{c}(\nabla f))^2, \tag{4.90}
\]

using \( \mathbb{Z}_2 \)-equivariant version of [3] Proposition 13.3 (cf. [3] Proposition 11.5) and applying the steps (4.64)-(4.84), we have that there exists \( C_2 > 0, 0 < d < 1 \) such
that for any $1 < k \leq m$, $0 < t \leq d$, $T \geq 0$ with $tT \leq 1$,
\[
\left| \int_{\Delta_k} \text{Tr}_s \left[ \phi N e^{-t_1 A^2_{p,t,T}} C_{t,T} e^{-t_2 A^2_{p,t,T}} \cdots C_{t,T} e^{-t_{k+1} A^2_{p,t,T}} \right] dt_1 \cdots dt_k \right| \leq C_2 t. \quad (4.91)
\]

For $k = 1$, we have
\[
\text{Tr}_s \left[ \phi N e^{-t_1 A^2_{p,t,T}} C_{t,T} e^{-(1-t_1) A^2_{p,t,T}} \right] = \text{Tr}_s \left[ \phi NC_{t,T} e^{-A^2_{p,t,T}} \right]. \quad (4.92)
\]

From (4.89), (4.90), (4.92) and proceeding as above, one has for any $0 < t \leq d$, $T \geq 0$ with $tT \leq 1$ and $0 \leq t_1 \leq 1$,
\[
\left| \text{Tr}_s \left[ \phi N e^{-t_1 A^2_{p,t,T}} C_{t,T} e^{-(1-t_1) A^2_{p,t,T}} \right] - T \int_Y \int_B \text{Tr} \left[ \phi \omega^F(\nabla f) \right] L \exp(-B T^2) \right| \leq C_2 t. \quad (4.93)
\]

Now similar as [30] (7.25), we have
\[
\int_Y \int_B \text{Tr} \left[ \phi \omega^F(\nabla f) \right] L \exp(-B T^2) = \frac{1}{2} \int_Y \left( \theta_\phi(\tilde{F}, \tilde{g}^e) - \theta_\phi(\tilde{F}, \tilde{b}^\nabla) \right) \int_B \nabla \phi \exp(-B T^2). \quad (4.94)
\]

From (4.86)-(4.91), we get (4.85), which completes the proof.

### 4.4.6 Proof of Theorem 4.11

In order to prove Theorem 4.11, we need only to prove that for any $T > 0$,
\[
\lim_{t \to 0^+} \left( \text{Tr}_s \left[ \phi N \exp \left( -A^2_{b,t,T} \right) \right] - \text{Tr}_s \left[ \phi N \exp \left( -A^2_{g,t,T} \right) \right] \right) = 0. \quad (4.95)
\]

By [30] (8.2) and (8.4), there exists $0 < C_0 \leq 1$, such that when $0 < t \leq C_0$, one has the absolute convergent expansion formula
\[
e^{-A^2_{b,t,T}} - e^{-A^2_{g,t,T}} = \sum_{k=1}^{+\infty} (-1)^k \int_{\Delta_k} e^{-t_1 A^2_{b,t,T} + t_2 A^2_{b,t,T} + \cdots + t_{k+1} A^2_{b,t,T}} dt_1 \cdots dt_k, \quad (4.96)
\]

and that
\[
= \sum_{k=m}^{+\infty} (-1)^k \int_{\Delta_k} e^{-t_1 A^2_{b,t,T} + t_2 A^2_{b,t,T} + \cdots + t_{k+1} A^2_{b,t,T}} dt_1 \cdots dt_k \quad (4.97)
\]
is uniformly absolute convergent for $0 < t \leq C_0$.

Proceeding as in [30] Section 8, for any $(t_1, \cdots, t_{k+1}) \in \Delta_k \setminus \{ t_1 \cdots t_{k+1} = 0 \}$, one has that
\[
\left| \text{Tr}_s \left[ \phi N e^{-t_1 A^2_{b,t,T}} C_{t,T} e^{-t_2 A^2_{b,t,T}} \cdots C_{t,T} e^{-t_{k+1} A^2_{b,t,T}} \right] \right| \leq C_3 t^k (t_1 \cdots t_k)^{-1} \left| \text{Tr} \left[ e^{-A^2_{b,t,T}} \right] \right| \left| \psi e^{-\frac{1}{2} t_{k+1} A^2_{b,t,T}} \right| \quad (4.98)
\]
for some positive constant $C_3 > 0$.

Also by \[30\] (8.4), \[4.95\] and the same assumption in \[30\] that $t_{k+1} \geq \frac{1}{k+1}$, one gets

$$
\left| \int_{\Delta_k} \text{Tr}_s \left[ \phi N e^{-t_1 A^2_{g,t}} + C_{1,t} e^{-t_2 A^2_{g,t}} + \cdots + C_{t-1,t} e^{-t_{k+1} A^2_{g,t}} \right] dt_1 \cdots dt_k \right| 
\leq C_4 t^{k-m} \left\| \psi e^{\frac{-2T t^{k+1}}{m+1}} \right\|^{A^2_{g,t}} \tag{4.99}
$$

for some constant $C_4 > 0$.

From \[4.36\], \[4.37\], \[4.99\], \[30\] (8.9) and (8.10) and the dominate convergence, we get \[4.95\].

**Remark 4.5.** The right hand side of \[4.36\] is not stated in \[3\], so we explain it in our case. First by direct computation, it equals

$$
\frac{1}{1 - e^{-2T}} \left( 1 + e^{-2T} \right) \chi_\phi(\Phi) - e^{-2T} (m + 1) \chi_\phi(\Phi) + \frac{e^{-T}}{e^T + e^{-T}} \chi_\phi(\Phi). \tag{4.100}
$$

Since near $x \in B_Y$, $\phi = 1$ in $T_x Y$ and $\phi = -1$ in $N_Y$. So the first term of \[4.100\] is just from \[5\] Theorem A.2] corresponding to $\phi = 1$. On $N_Y$, by \[5\] (8.15)], as the proof of \[5\] Theorem 5.12], we need to compute

$$
\int_{y \in N_Y, |y| \leq \varepsilon} \left( \frac{T e^{2T}}{2\pi t^2 \sinh(2T)} \right)^{\frac{1}{2}} \exp \left\{ -\frac{T}{t^2} \sinh(2T) \right\} dy 
\times \text{Tr}_s^{A(N_Y)} \left[ -N \exp[-2T(N^+ + \text{ind}_{N_Y}(x) - N^-)]] \chi_\phi(\Phi). \tag{4.101}
$$

Since $\text{ind}_{N_Y}(x) = 0$, $N^- = 0$, then as $t \to 0^+$, \[4.101\] equals

$$
\frac{e^T}{e^T + e^{-T}} \times e^{-2T} \chi_\phi(\Phi),
$$

which is equal to the second term in \[4.100\].

### 4.4.7 Proof of Theorem 4.12

In order to prove Theorem 4.12, we need only to prove that there exist $c > 0$, $C > 0$, $0 < C_0 \leq 1$ such that for any $0 < t \leq C_0$, $T \geq 1$,

$$
\left| \text{Tr}_s \left[ \phi N \exp \left( -A^2_{b,t} \right) \right] - \text{Tr}_s \left[ \phi N \exp \left( -A^2_{g,t} \right) \right] \right| \leq C \exp(-CT). \tag{4.102}
$$

First of all, one can choose $C_0 > 0$ small enough so that for any $0 < t \leq C_0$, $T > 0$, by \[4.96\], we have the absolute convergent expansion formula

$$
e^{-A^2_{b,t}} - e^{-A^2_{g,t}}
= \sum_{k=1}^{+\infty} (-1)^k \int_{\Delta_k} e^{-t_1 A^2_{b,t}} C_{1,t} e^{-t_2 A^2_{b,t}} \cdots C_{t-1,t} e^{-t_{k+1} A^2_{b,t}} dt_1 \cdots dt_k \tag{4.103}
$$

from which one has

$$
\sum_{k=1}^{+\infty} (-1)^k \int_{\Delta_k} \text{Tr}_s \left[ \phi N e^{-t_1 A^2_{b,t}} C_{1,t} e^{-t_2 A^2_{b,t}} \cdots C_{t-1,t} e^{-t_{k+1} A^2_{b,t}} \right] dt_1 \cdots dt_k.
$$

(4.104)
Thus, in order to prove \((4.102)\), we need only to prove
\[
\sum_{k=1}^{\infty} \int_{\Delta_k} \text{Tr} \left[ \phi N e^{-t_1 A_{g,t}^2} C_{t_1} e^{-t_2 A_{g,t}^2} \cdots C_{t_k} e^{-t_{k+1} A_{g,t}^2} \right] dt_1 \cdots dt_k
\]
\[
= \sum_{k=1}^{\infty} \int_{\Delta_k} \text{Tr} \left[ \phi N e^{-(t_1+t_{k+1}) A_{g,t}^2} C_{t_1} e^{-t_2 A_{g,t}^2} \cdots C_{t_k} e^{-t_{k+1} A_{g,t}^2} \right] dt_1 \cdots dt_k
\]
\[
\leq c \exp(-CT). \quad (4.105)
\]

By \([30] (8.6)\), we have for any \(t > 0, T \geq 1, (t_1, \ldots, t_{k+1}) \in \Delta_k \setminus \{t_1 \cdots t_{k+1} = 0\},
\]
\[
\text{Tr} \left[ \phi N e^{-(t_1+t_{k+1}) A_{g,t}^2} C_{t_1} e^{-t_2 A_{g,t}^2} \cdots C_{t_k} e^{-t_{k+1} A_{g,t}^2} \right]
\]
\[
= \text{Tr} \left[ \phi N \psi e^{-(t_1+t_{k+1}) A_{g,t}^2} C_{t_1} \psi e^{-t_2 A_{g,t}^2} \cdots \psi e^{-t_{k+1} A_{g,t}^2} C_{t_k} \right]. \quad (4.106)
\]

From \((4.106), \ [30] (9.18) \text{ and } (9.19)\), one sees that there exists \(C_5 > 0, C_6 > 0\) and \(C_7 > 0\) such that for any \(k \geq 1,
\]
\[
\left| \int_{\Delta_k} \text{Tr} \left[ \phi N e^{-t_1 A_{g,t}^2} C_{t_1} e^{-t_2 A_{g,t}^2} \cdots C_{t_k} e^{-t_{k+1} A_{g,t}^2} \right] dt_1 \cdots dt_k \right|
\]
\[
\leq C_5 (C_d t)^k \frac{T}{m} \exp \left( -\frac{C_7 T}{4} \right), \quad (4.107)
\]

from which one sees that there exists \(0 < c_1 \leq 1, C_8 > 0, C_9 > 0\) such that for any \(0 < t \leq c_1\) and \(T \geq 1\), one has
\[
\left\| \sum_{k=m}^{\infty} \int_{\Delta_k} \text{Tr} \left[ \phi N e^{-t_1 A_{g,t}^2} C_{t_1} e^{-t_2 A_{g,t}^2} \cdots C_{t_k} e^{-t_{k+1} A_{g,t}^2} \right] dt_1 \cdots dt_k \right\|
\]
\[
\leq C_8 \exp(-C_9 T). \quad (4.108)
\]

On the other hand, for any \(1 \leq k < m\), by proceeding as in \((4.99)\), one has that for any \(0 < t \leq c_1, T \geq 1,
\]
\[
\left| \int_{\Delta_k} \text{Tr} \left[ \phi N e^{-t_1 A_{g,t}^2} C_{t_1} e^{-t_2 A_{g,t}^2} \cdots C_{t_k} e^{-t_{k+1} A_{g,t}^2} \right] dt_1 \cdots dt_k \right|
\]
\[
\leq C_{10} t^{k-m} \left\| \psi e^{-\frac{1}{\pi e^{T/2}} A_{g,t}^2} \right\| \quad (4.109)
\]

for some constant \(C_{10} > 0\).

From \((4.109)\) and \([30] (9.23)\), one sees immediately that there exists \(C_{11} > 0, C_{12} > 0\) such that for any \(1 \leq k \leq m - 1, 0 < t \leq c_1\) and \(T \geq 1\), one has
\[
\left| \int_{\Delta_k} \text{Tr} \left[ \phi N e^{-t_1 A_{g,t}^2} C_{t_1} e^{-t_2 A_{g,t}^2} \cdots C_{t_k} e^{-t_{k+1} A_{g,t}^2} \right] dt_1 \cdots dt_k \right|
\]
\[
\leq C_{11} e^{-C_{12} T}. \quad (4.110)
\]

From \((4.106), (4.108)\) and \((4.110)\), one gets \((4.102)\).
Assume first that $f \in H_{\partial}^1(X, \nabla f)$. Then for the relative boundary condition, we have

\[
\left( \frac{b_{RS}(X, F)}{\det H^*(X, F)} \right)^2 = 2^{-rk(F) \chi(Y)} \times \exp \left( -2 \int_X \theta(F, b^F)(\nabla f)^* \psi(TX, \nabla^TX) \right.
\]
\[
+ \int_Y \theta(F, b^F)(\nabla f)^* \psi(TY, \nabla^TY) \right),
\]

and for the absolute boundary condition, we have

\[
\left( \frac{b_{RS}(X, F)}{\det H^*(X, F)} \right)^2 = 2^{-rk(F) \chi(Y)} \times \exp \left( -2 \int_X \theta(F, b^F)(\nabla f)^* \psi(TX, \nabla^TX) \right.
\]
\[
- \int_Y \theta(F, b^F)(\nabla f)^* \psi(TY, \nabla^TY) \right).
\]

Proof. Assume first that $f$ is a Morse function on $X$ induced by a $Z_2$-equivariant Morse function $\tilde{f}$ on $\tilde{X} = X \cup Y$ as in the proof of [7, Lemma 1.5].

We denote by $C_* (\tilde{W}^u/W_{\tilde{Y}}^u, \tilde{F}^*) = \oplus_{x \in \tilde{W}^u} [W_{\tilde{Y}}^u(x)] \otimes \tilde{F}^*$. Let $P_{\infty}$ be the isomorphism on the cohomology induced by the de Rham map $P_{\infty}$, then for $\sigma \in H^*(\tilde{X}, \tilde{F})$,

\[
(\gamma \circ P_{\infty} \circ \tilde{\phi}_1 \circ P_{\infty}^{-1})(\sigma) |_{C_* (\tilde{W}^u/W_{\tilde{Y}}^u, \tilde{F}^*)} = \frac{\sqrt{2}}{2} \gamma(\sigma + \phi^* \sigma) |_{C_* (\tilde{W}^u/W_{\tilde{Y}}^u, \tilde{F}^*)} = 2\sigma |_{C_* (\tilde{W}^u/W_{\tilde{Y}}^u, \tilde{F}^*)}.
\]
Set
\[ \tau_{\pm} = \gamma \circ P^H_\infty \circ \tilde{\phi} \circ P^{H,-1} : H^\bullet(\tilde{X}, \tilde{F})^\pm \to H^\bullet(\tilde{X}, \tilde{F})^\pm. \] (5.5)

By (5.4), we get (cf. [7 (2.22)])
\[ \prod_{j=0}^{m} \left( \det \tau_{j}|_{H^j(\tilde{X}, \tilde{F})^+} \right)^{(-1)^j} = 2^{\chi(Y)\text{rk}(F)} \prod_{j=0}^{m} \left( \det \tau_{-j}|_{H^j(\tilde{X}, \tilde{F})^-} \right)^{(-1)^j} = 1. \] (5.6)

By (4.24), (5.4), (5.6) and (4.13), for \( g \in \mathbb{Z}_2 \) and \( \chi \) the nontrivial character,
\[ \log \left( \frac{b_{RS}^{\det H^\bullet(\tilde{X}, \tilde{F}), z_2}}{b_{M,V,F}^{\det H^\bullet(\tilde{X}, \tilde{F}), z_2}} \right)(g) = \chi(Y)\text{rk}(F) \log 2 + \log \left( \frac{b_{RS}^{\det H^\bullet(X,F)}}{b_{M,V,F}^{\det H^\bullet(X,F)}} \right) + \chi(g) \log \left( \frac{b_{RS}^{\det H^\bullet(X,Y,F)}}{b_{M,V,F}^{\det H^\bullet(X,Y,F)}} \right). \] (5.7)

We denote by \( \psi(T\tilde{X}, \nabla^T\tilde{X}), \psi(TY, \nabla^{TY}) \) the Mathai-Quillen current on \( T\tilde{X}, TY \), respectively. Then by [30 Theorem 3.1] and Theorem 4.4, we get
\[ \log \left( \frac{b_{RS}^{\det H^\bullet(\tilde{X}, \tilde{F})}}{b_{M,V,F}^{\det H^\bullet(\tilde{X}, \tilde{F})}} \right) = -\int_{\tilde{X}} \theta(F, b^F)(\nabla f)^* \psi(T\tilde{X}, \nabla^T\tilde{X}), \]
\[ \log \left( \frac{b_{RS}^{\det H^\bullet(\tilde{X}, \tilde{F}), z_2}}{b_{M,V,F}^{\det H^\bullet(\tilde{X}, \tilde{F}), z_2}} \right)(g) = -\int_{\tilde{Y}} \theta(F, b^F)(\nabla f)^* \psi(TY, \nabla^{TY}) + \text{rk}(F)\chi(Y) \log 2. \] (5.8)

By (5.7) and (5.8), we get (5.1) and (5.2).

We established until now Theorem 5.1 for a special Morse function \( f \) on \( X \) induced by a \( \mathbb{Z}_2 \)-equivariant Morse function \( f \) on \( \tilde{X} \). By combining this with the argument in [4 Section 16], we know that Theorem 5.1 holds for any \( f \) verifying Lemma 3.1.

\[ \square \]

**Remark 5.2.** If \( \partial X = Y \cup V \), the metric \( g^{TX} \) and the symmetric bilinear form \( b^F \) are product near \( \partial X \). We impose the relative boundary condition on \( Y \) and absolute boundary condition on \( V \), then by the same proof of [7 Theorem 2.2], we have
\[ \frac{b_{RS}^{\det H^\bullet(X,Y,F)}}{b_{M,V,F}^{\det H^\bullet(X,Y,F)}}^2 = 2^{-\text{rk}(F)\chi(\partial X)} \exp \left( -\int_{V} \theta(F, b^F)\nabla f)^* \psi(TY, \nabla^{TY}) \right) \]
\[ \cdot \exp \left( -2 \int_{X} \theta(F, b^F)\nabla f)^* \psi(TX, \nabla^{TX}) \right) \]
\[ + \int_{Y} \theta(F, b^F)\nabla f)^* \psi(TY, \nabla^{TY}) \right). \] (5.9)

**Remark 5.3.** By the anomaly formula [22 Theorem 3] and the argument in [7 Proof of Theorem 0.1], we can easily extend Theorem 5.1 to the case that \( g^{TX} \) is not of product structure near the boundary.
6 Compare with the Ray-Singer analytic torsion

In this section, we assume that \( m \) is odd and \( \chi(Y) = 0 \). We will compare the symmetric bilinear analytic torsion to the Ray-Singer analytic torsion.

First from the anomaly formula of Ray-Singer metric on manifolds with boundary \([7, (3.25)]\) for the case that metrics \((g^{TX}, h^F)\) are product near boundary, we see that

\[
\log \left( \frac{\| \cdot \|_{\det H^* (X,Y,F), 1}}{\| \cdot \|_{\det H^* (X,Y,F), 0}} \right)^2 = \frac{1}{2} \int_Y \log \left( \frac{\| \cdot \|_{\det F, 1}}{\| \cdot \|_{\det F, 0}} \right)^2 e(TY, \nabla_0^{TY})
\]

\[
- \frac{1}{2} \int_Y \bar{e}(TY, \nabla_0^{TY}, \nabla_1^{TY}) \theta(F, h^F). \quad (6.1)
\]

By an observation due to Ma and Zhang \([20]\), we know that by combining \([4, \text{Theorem 6.1} ] \) and \((6.1)\), we have that

\[
\| \cdot \|_{\det H^* (X,Y,F), 1} : \| \cdot \|_{\det H^* (X,Y,F)} \cdot \| \cdot \|_{\det H^* (Y,F)} \quad (6.2)
\]

is independent of the metrics \((g^{TX}, h^F)\). In the same way, from the anomaly formulas \([8, \text{Theorem 4.2} ], [30, \text{Theorem 2.1} ] \) and \([22, \text{Theorem 3} ] \), we get that

\[
b^{RS}_R := b^{RS}_{\det H^* (X,Y,F)} \cdot b^{RS}_{\det H^* (Y,F)} \quad (6.3)
\]

does not depend on the choice of \( g^{TX} \) and the smooth deformations of the non-degenerate symmetric bilinear form \( b^F \).

Let \( h^F \) be a Hermitian metric on \( F \). Then one can construct the Ray-Singer analytic torsion as inner products on \( \det H^*(X,Y,F) \) and \( \det H^* (Y,F) \), we denote them by \( h^{RS}_{\det H^* (X,Y,F)} \) and \( h^{RS}_{\det H^* (Y,F)} \), respectively. Then by as it in \(6.3\) we have

\[
h^{RS} := \left( h^{RS}_{\det H^* (X,Y,F)} \right)^2 \cdot h^{RS}_{\det H^* (Y,F)}
\]

is independent of the choice of \((g^{TX}, h^F)\) and by \([4, \text{Theorem 0.2} ], [7, \text{Theorem 2.2} ] \), we have

\[
\left( \frac{h^{RS}_{\det H^* (X,Y,F)}}{h^{RS}_{\det H^* (Y,F)}} \right)^2 \cdot \left( \frac{h^{MS} \nabla f}{h^{MS} \nabla f} \right) = \exp \left( -2 \int_X \theta(F, h^F)(\nabla f)^* \psi(TX, \nabla^{TX}) \right), \quad (6.4)
\]

where we used the assumption \( \chi(Y) = 0 \).

On the other hand, if there exists a non-degenerate symmetric bilinear form \( b^F \) on \( F \) which is product near the boundary \( Y \), then by \([30, \text{Theorem 3.1} ] \) and \([5, \text{Theorem 5.1} ] \) we have

\[
\left( \frac{h^{RS}_{\det H^* (X,Y,F)}}{h^{RS}_{\det H^* (Y,F)}} \right)^2 \cdot \left( \frac{h^{MS} \nabla f}{h^{MS} \nabla f} \right) = \exp \left( -2 \int_X \theta(F, b^F)(\nabla f)^* \psi(TX, \nabla^{TX}) \right). \quad (6.5)
\]

Then by \((6.3) \) and \((6.3) \), we have

\[
\frac{b^{RS}}{h^{RS}} = \left( \frac{h^{RS}_{\det H^* (X,Y,F)}}{h^{RS}_{\det H^* (Y,F)}} \right)^2 \cdot \left( \frac{h^{MS} \nabla f}{h^{MS} \nabla f} \right) = \left( \frac{h^{MS} \nabla f}{h^{MS} \nabla f} \right)^2 \cdot \left( \frac{h^{MS} \nabla f}{h^{MS} \nabla f} \right) \cdot \exp \left( -2 \int_X \left( \theta(F, b^F) - \theta(F, h^F) \right)(\nabla f)^* \psi(TX, \nabla^{TX}) \right). \quad (6.6)
\]
Since \( \chi(Y) = 0 \), \( \theta(F, b^F) = \theta(F, h^F) \) in a neighbourhood of \( B \cap Y \), then by \([8\) (46)] and \([30\) (10.14)], we get
\[
\left| \frac{h_{\det H(Y,F)}^{M, \nabla f}}{b_{\det H(Y,F)}^{M, \nabla f}} \right| = 1. \tag{6.7}
\]
By \([7\) (3.18)] and \( \chi(Y) = 0 \), similarly we have
\[
\left| \frac{h_{\det H(X,Y,F)}^{M, \nabla f}}{b_{\det H(X,Y,F)}^{M, \nabla f}} \right| = 1. \tag{6.8}
\]
Since \( m \) is odd, \( \text{Re} \left[ \theta(F, b^F) \right] = \left[ \theta(F, g^F) \right] \) and \( \theta(F, b^F) = \theta(F, h^F) \) in a neighborhood of \( B \), then by an analogue formula of \([7\) (3.53)], we have
\[
\left| \exp \left( -2 \int_X (\theta(F, b^F) - \theta(F, h^F))(\nabla f)^* \psi(TX, \nabla TX) \right) \right| = 1. \tag{6.9}
\]
Then by \([6.6]-[6.9]\), we get
\[
\left| \frac{b_{RS}^{H, \det H(X,F)}}{h_{RS}^{H, \det H(Y,F)}} \right| = \left( \frac{b_{RS}^{H, \det H(X,Y,F)}}{h_{RS}^{H, \det H(X,Y,F)}} \right)^2 \cdot \left| \frac{b_{RS}^{H, \det H(Y,F)}}{h_{RS}^{H, \det H(Y,F)}} \right| = 1. \tag{6.10}
\]

Remark 6.1. For the absolute boundary condition, we have
\[
\left| \frac{b_{RS}^{H, \det H(X,F)}}{h_{RS}^{H, \det H(Y,F)}} \right| = \left( \frac{b_{RS}^{H, \det H(X,Y,F)}}{h_{RS}^{H, \det H(Y,F)}} \right)^2 \cdot \left( \frac{b_{RS}^{H, \det H(Y,F)}}{h_{RS}^{H, \det H(Y,F)}} \right)^{-1} = 1.
\]

References

[1] M. F. Atiyah, R. Bott and V. K. Patodi, On the heat equation and the index theorem. *Invent. Math.*, 19, (1973), 279-330.

[2] N. Berline, E. Getzler and M. Vergne, Heat Kernels and Dirac Operators. Springer, Berline-Heidelberg-New York, 1992.

[3] J.-M. Bismut, Equivariant immersions and Quillen metrics. *J. Diff. Geom.*, 41 (1995), 53-159.

[4] J.-M. Bismut and W. Zhang, An extension of a Theorem by Cheeger and Müller. *Asterisque*, 205, 1992, 236 pp.

[5] J.-M. Bismut and W. Zhang, Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle. *G.A.F.A.* 4 (1994), 136-212.

[6] J. Brüning and X. Ma, An anomaly formula for Ray-Singer metrics on manifolds with boundary. *G.A.F.A.* 16 (2006), 767-837.
[7] J. Brüning and X. Ma, On the gluing formula for the analytic torsion. *Math. Z.*, 273 (2013), 1085-1117.

[8] D. Burghelea and S. Haller, Complex valued Ray-Singer torsion. *J. Funct. Anal.*, 248 (2007), 27-78.

[9] D. Burghelea and S. Haller, Complex valued Ray-Singer torsion II. *Math. Nachr.*, 283 (2010), 1372-1402.

[10] J. Cheeger, Analytic torsion and the heat equation. *Ann. of Math.*, 109 (1979), 259-332.

[11] S. E. Cappell and E. Y. Miller, Complex-valued analytic torsion for flat bundles and for holomorphic bundles with (1,1) connections. *Comm. Pure and Applied Math.*, 133-202 (2010).

[12] X. Dai and H. Fang, Analytic torsion and R-torsion for manifolds with boundary. *Asian J. Math.*, 4 (2000), 695-714.

[13] X. Dai and W. Zhang, An index theorem for Toeplitz operators on odd-dimensional manifolds with boundary. arXiv:math/0103230v1. *J. Funct. Anal.*, 238 (2006), no. 1, 1-26.

[14] M. Farber and V. Turaev, Poincaré-Reidemeister metric, Euler structures and torsion. *J. Reine Angew. Math.*, 520 (2000), 195-225.

[15] G. Grubb, Functional calculus of pseudodifferential boundary problems. Second edition. Progress in Mathematics, 65. Birkhäuser Boston, Inc. Boston, MA, 1996.

[16] E. Getzler, Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem. *Commun. Math. Phys.*, 92, 1983, 163-178.

[17] E. Getzler, A short proof of the local Atiyah-Singer index theorem. *Topology*, 25, 1986, 111-117.

[18] A. Hassell, Analytic surgery and analytic torsion. *Comm. Anal. Geom.*, 6 (1998), 255-289.

[19] W. Lück, Analytic and topological torsion for manifolds with boundary and symmetry. *J. Diff. Geom.*, 37 (1993), 263-322.

[20] X. Ma and W. Zhang, Private discussion.

[21] J. Milnor, Whitehead torsion. *Bull. Amer. Math. Soc.*, 72 (1966), 358-426.

[22] O. Molina, Anomaly formulas for the complex-valued analytic torsion on compact bordims. *Diff. Geom. Appl.*, 31 (2013) 416-436.

[23] W. Müller, Analytic torsion and the R-torsion of Riemannian manifolds. *Adv. in Math.*, 28 (1978), 233-305.

[24] W. Müller, Analytic torsion and the R-torsion for unimodular representations. *J. Amer. Math. Soc.*, 6 (1993), 721-753.

[25] D. Quillen, Determinants of Cauchy-Riemann operators over a Riemann surface. *Funct. Anal. Appl.*, 14 (1985), 31-34.

[26] D. B. Ray and I. M. Singer, R-torsion and the Laplacian on Riemannian manifolds. *Adv. in Math.*, 7 (1971), 145-210.
[27] R. T. Seeley, Complex powers of an elliptic operator. Singular Integrals (Proc. Sympos. Pure Math., Chicago, 1996) Amer. Math. Soc., Providence, R.I. 1967, 99 288-307.

[28] R. T. Seeley, Analytic extension of the trace associated with elliptic boundary problems. Amer. J. Math. 91 (1969), 963-983.

[29] S. Smale, On gradient dynamical systems. Ann. of Math., 74 (1961), 199-206.

[30] G. Su and W. Zhang, A Cheeger-Müller theorem for symmetric bilinear torsions. Chin. Ann. Math. Ser. B vol. 29, no. 4, 385–424.

[31] V. Turaev, Euler structures, nonsingular vector fields, and Reidemeister-type torsion. Math. USSR-Izv., 34 (1990), 627-662.

[32] S. M. Vishik, Generalized Ray-Singer conjecture I: a manifold with a smooth boundary. Comm. Math. Phys. 167 (1995), 1-102.

[33] E. Witten, Supersymmetry and Morse theory. J. Diff. Geom., 17 (1982), 661-692.

[34] W. Zhang, Lectures on Chern-Weil Theory and Witten Deformations, Nankai Tracts in Mathematics, Vol. 4, World Scientific, Singapore, 2001.