CHEEGER-GROMOV COMPACTNESS FOR MANIFOLDS WITH BOUNDARY

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Abstract. We prove Cheeger-Gromov convergence for a subsequence of a given sequence of manifolds-with-boundary of bounded geometry. The method of the proof is to reduce, via height functions, the problem to the setting of Hamilton’s compactness theorem for manifolds without boundary.

1. Introduction and statement of the main result

1.1. Motivation. Let \( i \mapsto (M_i, g_i, x_i^0) \) be a sequence of pointed connected Riemannian \( n \)-dimensional manifolds \(^1\) and we ask whether it subconverges to a smooth pointed Riemannian manifold \((M_\infty, g_\infty, x_\infty^0)\), i.e., whether a subsequence converges in the Cheeger-Gromov sense to

\[ (M_\infty, g_\infty, x_\infty^0) = \lim_{i \to \infty} (M_i, g_i, x_i^0), \]

and \( M_\infty \) might be noncompact even if all \( M_i \) are compact, and have non-empty non-compact boundary. This is essentially the question which subsets of the set of Riemannian manifolds are precompact. The question has a long history, beginning with Cheeger’s famous result [6]. Subsequently, the use of harmonic coordinates (whose existence in uniformly large balls had been shown by Jost and Karcher [15] under the assumptions of bounds on the sectional curvature) replaced the one of geodesic coordinates, and allowed Peters [19] and Greene-Wu [9] to obtain results of better regularity. An interesting recent result in this context is the one by Portegies [20], who reproduced Jost’s and Karcher’s result under the mere assumption of Ricci bounds. For nice accounts on parts of the subject see [18], [22].

Most results in this context either fix the topology a priori (like the interesting approach taken by Allen and Perales in [1]) or conclude Gromov-Hausdorff subconvergence to a metric space (like in Wong’s results [24], [25], which allow concluding subconvergence to an Alexandrov space with convex boundary) or subconvergence in a space of currents instead of \( C^k \)-convergence to a Riemannian manifold as above. An important result [7], due to Cheeger and Colding, shows that the subset of regular points of such a limit, that is, of points around which the space looks like a manifold, has full \( n \)-Hausdorff measure. Among those results that conclude convergence of a subsequence to a Riemannian manifold, Gromov [10] assumes compactness, uniformly bounded diameter and nonnegative Ricci curvature resp. uniformly bounded sectional curvature and \( \text{vol}^{-1} \) and concludes Lipschitz resp. metric convergence to a \( C^{1,\alpha} \) resp. \( C^0 \) metric, Anderson [3] shows \( C^{1,\alpha} \)-precompactness of the class of compact connected Riemannian \( n \)-manifolds of bounded Ricci curvature and diameter and injectivity radius bounded from 0. The result of Hamilton [11], enlisted in this article as [2,5] for (compact or non-compact) manifolds without boundary states that under appropriate bounds there is \( C^k \)-precompactness in the class of open manifolds as well, in the sense above. In the context of manifolds with boundary, there are the results of Kodani [17], Knox [16] and Anderson-Katsuda-Kurrayev-Lassas-Taylor [4], the last one even without assuming a bound on sectional curvature but only on

\(^1\)Throughout this article, the term ‘manifold’ includes manifolds with boundary.
Ricci curvature. Those three results assume compactness of the manifolds and a uniform diameter bound and conclude subconvergence to a Riemannian $C^{1,\alpha}$ manifold. In comparison, Theorem A of this article assumes, for any desired $k \in \mathbb{N}$, a $C^{k+1}$ bound on sectional curvature, neither assumes any diameter bound nor keeps the topology fixed but still concludes $C^k$-subconvergence.

1.2. Bounded geometry and main result. For a Riemannian metric $h$, we denote by $\text{Rm}_h$ its Riemannian tensor, and by $\text{inj}_h$ its injectivity radius. Let $(M, g, x^0)$ be a pointed connected Riemannian manifold. If $\partial M \neq \emptyset$, we denote by $\partial g = g|_{\partial M}$ the induced metric. Denote by $d^g$ the distance function induced by the metric $g$, and in general for a metric space $M$ and $A \subset M$, we write $B(A, r) = \{ x \in M \mid d(x, A) < r \}$. In the following, we adopt the following definition of bounded geometry for manifolds with in general nonempty boundary (cf., e.g., [21]):

**Definition 1.1.** Fix $k \in \mathbb{N} \cup \{\infty\}$ and $c > 0$. A Riemannian manifold $(M, g)$ with non-empty boundary $\partial M$ has $(c, k)$-bounded geometry if:

1. for the inward normal vector field $\nu$, the normal exponential map $E : \partial M \times [0, c^{-1}] \to M$, $E(y, r) := \exp_y(r\nu)$, is a diffeomorphism onto its image;
2. $\text{inj}_{\partial g}(\partial M) \geq c^{-1}$;
3. $\text{inj}_g(M \setminus B(\partial M, r)) \geq r$ for all $r \leq c^{-1}$;
4. $|\nabla^l_g \text{Rm}_g|_g \leq c$ for all $l \leq k$;
5. $|\nabla^l_{\partial g} \text{II}_{\partial g}|_g \leq c$ for all $l \leq k$, where $\text{II}$ is the second fundamental form of the boundary.

For a pointed Riemannian manifold $(M, g, x^0)$ with basepoint $x^0$ we also require $d^g(x^0, \partial M) \geq 2c^{-1}$.

**Remark 1.2.** It is known that the above requirements guarantee that the boundary manifold $(\partial M, \partial g)$ also has $(c, k)$-bounded geometry, see [21]. In the case when $\partial M = \emptyset$, some of requirements are empty, and the condition (iii) is equivalent to $\text{inj}_g \geq c^{-1}$.

Our main result is:

**Theorem A** (Precompactness result). Let $i \mapsto (M_i, g_i, x^0_i)$ be a sequence of pointed Riemannian manifolds of dimension $n$ of $(c, k + 1)$-bounded geometry (if the boundary is empty, it is enough to assume $(c, k)$-bounded geometry). Then the sequence $\{(M_i, g_i, x^0_i)\}$ $C^k$-subconverges to a pointed manifold $(M_\infty, g_\infty, x^0_\infty)$ of bounded geometry to order $k$. Assume furthermore the sequence $\{d_i(x^0_i, \partial M_i)\}$ is bounded. Then $(M_\infty, g_\infty, x^0_\infty)$ has non-empty boundary.

**Remark 1.3.** The condition on the distances $d_i$ cannot be omitted: For the sequence $n \mapsto \{(B(0, n), ds^2, 0)\}$ of closed balls $B(0, n)$ around 0 of radius $n$ in Euclidean $\mathbb{R}^m$, the limit manifold has empty boundary.

1.3. Plan of the paper and acknowledgments. We review necessary results on smooth Cheeger-Gromov convergence in Section 2. Then we prove the Main Theorem in Section 3.

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From work of Ammann-Große-Nistor [2] we know that the second condition follows from the other ones.
2. Notation, Convergence for Manifolds with Boundary

2.1. Conventions, notation, definition of convergence. We first settle some conventions by 0 \in \mathbb{N}, and for n \in \mathbb{N} let \mathbb{N}_n := \{ m \in \mathbb{N} | m \leq n \}. Let (Z, d) be a metric space, and Y \subset Z. Let B(Y, r) be the d-ball of radius r around Y in Z, where r > 0. In the case when Y = \{ y \}, we define B(y, r) := B(\{ y \}, r). When it will be important to emphasize an ambient space Z or a metric d, we will use the notation B^Z(y, r) and B^d(y, r), respectively.

If Z_0, Z_1 \subset Z, then the Hausdorff distance d(Z_0, Z_1) is defined as

\[ d(Z_0, Z_1) = \inf \{ r > 0 \mid Z_0 \subset B(Z_1, r) \land Z_1 \subset B(Z_0, r) \}, \]

which of course generalizes d in the sense that d(\{ p \}, \{ q \}) = d(p, q). Balls and distances in a Riemannian manifold \((M, g)\) always refer to the geodesic distance \(d^g\). We say (see e.g. \cite{8,14,13}) that a sequence \( i \mapsto (M_i, g_i, x_i^0) \) of pointed Riemannian manifolds \(C^k\)-converges (in the sense of Cheeger-Gromov) to a Riemannian manifold \((M_\infty, g_\infty, x_\infty)\) if and only if for all integers \( m \geq 1 \) and for \( i \) sufficiently large

(a) there is an open subset \( U_m \supset B(x_\infty^0, m) \) of \( M_\infty \) and a diffeomorphism \( \phi_i^{(m)} : U_m \to M_i \) with \( B_m(x_i^0) \subset \phi_i^{(m)}(U_m) \) mapping \( x_\infty^0 \) to \( x_i^0 \);

(b) the metrics \( \phi_i^{(m)} \circ g_i \) converge to \( g_\infty|_{U_m} \) in the \(C^k\)-norm on \( U_m \).

2.2. Gromov-Hausdorff convergence. Let \((X, d)\) and \((X', d')\) be metric spaces. Following \cite{5} Chapter 3, we say that a continuous map \( \phi : X \to X' \) is an \( \epsilon \)-isometry if \( ||\phi^* d' - d||_\infty < \epsilon \).

Definition 2.1. A sequence \( i \mapsto (Y_i, d_i, y_i^0) \) of pointed proper complete metric spaces is said to \( GH\)-converge to a complete and proper pointed metric space \((Y_\infty, d_\infty, y_\infty^0)\) if one of the following equivalent conditions is satisfied (see \cite{5} Section 3.1.2):

(B') there are sequences \( i \mapsto r_i, i \mapsto \epsilon_i \) of positive real numbers, where \( r_i \to i \to \infty \infty, \epsilon_i \to i \to \infty 0 \), and \( \epsilon_i \)-isometries \( \phi_i : B_{r_i}^{Y_i}(y_i^0) \to B_{r_i}^{Y_i}(y_i^0) \) such that

\[ B_{\epsilon_i}(\text{Im} \phi_i) \supset B_{\epsilon_i}^{Y_i}(y_i^0) \quad \text{and} \quad d_i(\phi_i(y_i^0), y_i^0) < \epsilon_i. \]

(D') there is a metric space \((Z, d)\) and isometric embeddings \( \iota_i : Y_i \to Z, \iota_\infty : Y_\infty \to Z, \) such that

1. \( \lim_{i \to \infty} \iota_i(y_i^0) = \iota_\infty(y_\infty^0) \),
2. \( \lim_{i \to \infty} d(U \cap \iota_i(Y_i), U \cap \iota_\infty(Y_\infty)) = 0 \) for any open bounded set \( U \subset Z \).

We use the notation \( \lim_{i \to \infty} GH(Y_i, d_i, y_i^0) = (Y_\infty, d_\infty, y_\infty^0) \).

We need the following particular case of more general results, see, e.g. \cite{5} Prop. 3.1.2, Th. 3.1.3:

Theorem 2.2. Let \( A : i \mapsto (X_i, g_i, x_i^0) \) be a sequence of pointed complete \( n \)-dimensional Riemannian manifolds such that \( \text{Ric}_{g_i} \geq (n - 1)\kappa \) for some \( \kappa \in \mathbb{R} \) and all \( i = 1, 2, \ldots \). Then there exists a pointed proper complete metric space \( (Y_\infty, d_\infty, y_\infty^0) \) such that the sequence \( A \) \( GH\)-subconverges to \( (Y_\infty, d_\infty, y_\infty^0) \).
2.3. Smooth Cheeger-Gromov convergence. Let \( i \mapsto (X_i, g_i, x_i^0) \) be a sequence of pointed complete Riemannian manifolds of dimension \( n \) which GH-converges to a metric space \((Y_\infty, d_\infty, y_\infty)\) as in Definition 2.1. Assume that the metric space \((Y_\infty, d_\infty, y_\infty)\) is, in fact, a complete Riemannian manifold, and we use the notation: \((Y_\infty, d_\infty, y_\infty) = (X_\infty, g_\infty, x_\infty^0)\).

**Definition 2.3.** Assume that a sequence \( \{(X_i, g_i, x_i^0)\} \) GH-converges to a complete Riemannian manifold \((X_\infty, g_\infty, x_\infty^0)\). Then the sequence \( \{(X_i, g_i, x_i^0)\} \) \( C^k \)-converges to \((X_\infty, g_\infty, x_\infty^0)\) if there is an exhaustion of \( X_\infty \) by open sets \( U_j \), i.e.,

\[
U_1 \subset \cdots \subset U_j \subset \cdots \subset X_\infty, \quad X_\infty = \bigcup_j U_j,
\]

and there are diffeomorphisms onto their image \( \phi_j : U_j \to M_j \) such that \( \phi_j \to Id_{X_\infty} \) pointwise, and for the pullbacks of the metrics we have

\[
\phi_j^* g_j \to j \to \infty g_\infty
\]

in the pointwise \( C^k \) topology, i.e., there is a point-wise convergence \( \phi_j^* g_j \to g_\infty \) and \( \nabla^\ell \phi_j^* g_j \to \nabla^\ell g_\infty \) for all \( \ell = 1, \ldots, k \), where \( \nabla \) denotes the Levi-Civita connection of the metric \( g_\infty \) on \( X_\infty \).

**Remark 2.4.** Without loss of generality, we assume that \( U_j = B(x_\infty^0, j) \subset X_\infty \) for all \( j \in \mathbb{N} \setminus \{0\} \).

R. Bamler provides a detailed proof (see [5, Theorem 3.2.4]) of the following result:

**Theorem 2.5.** (cf. R. Hamilton [11]) Let \( k \in \mathbb{N} \cup \{\infty\}, k \geq 4 \). Let \( i \mapsto (X_i, g_i, x_i) \) be a sequence of pointed complete Riemannian manifolds of dimension \( n \). Assume that \( \text{inj}_{g_i} \geq c^{-1} \) and \( |\nabla^\ell \text{Rm}_{g_i}| \leq c \) for all \( \ell \in \mathbb{N}_k \). Then the sequence \( \{(X_i, g_i, x_i)\} \) \( C^k \)-subconverges to a pointed complete Riemannian manifold \((X_\infty, g_\infty, x_\infty^0)\) of dimension \( n \).

**Remark 2.6.** Strictly speaking, only the case \( k = \infty \) is treated in the Theorems of the references, but their proofs contain implicitly the statement for finite \( k \geq 4 \).

### 3. Proof of the main result (Theorem A)

To apply Hamilton’s result we need to make the boundary disappear. The main tool to do so are so-called *height functions*. The general strategy of the proof of Theorem A can be summarized as follows:

(i) First we extend in Sec. 3.1 the metrics beyond the boundary using height functions;
(ii) Then we apply in Sec. 3.2 Hamilton’s theorem Th. 2.5 to the extended metrics;
(iii) Finally we show in Sec. 3.3 that the convergence of the extended metric together with the convergence of the height functions entails the convergence of the original manifolds-with-boundary.
3.1. Step 1: An extension procedure.

3.1.1. Height functions. In order to reduce the problem of convergence for manifolds-with-boundary to the corresponding problem for manifolds without boundary, we now introduce height functions. To a (in general noncompact) manifold $M$ with non-empty boundary, we can attach a collar to get a complete manifold $X$ without boundary equipped with a height function $f: X \to (-\infty; 1]$ such that $M = f^{-1}([0; 1])$. Then a sequence $i \mapsto (M_i, g_i, x_i^0)$ of pointed compact manifolds with non-empty boundary gives a sequence $i \mapsto (X_i, \hat{g}_i, x_i^0)$ (where $\hat{g}_i$ extends $g_i$) of complete Riemannian manifolds with the additional data of the height functions.

**Definition 3.1.** Let $(X, g, x^0)$ be a pointed Riemannian manifold. A smooth function $f: X \to (-\infty; 1]$ is called a $(c, k)$-height function, where $k \in \mathbb{N}$, $c > 0$, if the following conditions are satisfied:

(i) $\delta^0(f) := \min\{ |\nabla_g f(x)|_g \mid x \in f^{-1}([-1/2; 1/2]) \} \geq c^{-1}$, $f^{-1}(\{0\}) \neq \emptyset$, in particular $0$ is a regular value for the function $f$;

(ii) $f(x^0) > 0$ and $d^0(x^0, f^{-1}(0)) \in (c^{-1}; c)$;

(iii) $|\nabla^\ell f| \leq c$ for all $\ell = 0, 1, \ldots, k$.

Of course, if $C \geq c$ and $\kappa \leq k$ then any $(c, k)$-height function is a $(C, \kappa)$-height function. Furthermore, a sequence $\{(M_i, g_i, x_i^0, f_i)\}$ is called of $(c, k)$-bounded geometry if $\{(M_i, g_i, x_i^0)\}$ is a sequence of $(c, k)$-bounded geometry and $f_i$ are $(c, k)$-height functions on $M_i$.

**Remark 3.2.** Let $(X, g, x^0)$ and $f: X \to (-\infty, 1]$ be as in Definition 3.1. Denote $X^f := f^{-1}([0; 1])$. Then by definition, $X^f$ is a smooth manifold with the boundary $\partial X^f = f^{-1}(\{0\}) \neq \emptyset$, i.e., the triple $(X^f, g, x^0)$ is a pointed manifold with non-empty boundary. Here we denote by $g$ the restriction $g|_{X^f}$ to avoid multiple subscripts in sequences.

Now it is easy to see that if $f$ is a $(c, k)$-height function on a manifold of $(c, k)$-bounded geometry, then $X^f = f^{-1}([0, 1])$ is a manifold with boundary of bounded geometry. It is a bit harder to see that actually also the converse is true:

**Theorem 3.3.** Let $c > 0$, then there exists $\bar{c} > 0$, depending on $c$, such that, for any compact pointed manifold $(M, g, x^0)$ of $(c, k)$-bounded geometry with $\partial M \neq \emptyset$, there exists a pointed isometric inclusion $\iota: (M, g, x^0) \to (X, \hat{g}, x^0)$ where $(X, \hat{g}, x^0)$ is a complete pointed manifold of $(\bar{c}, k)$-bounded geometry and $(\bar{c}, k)$-height function $f$ on $X$ with $\iota(M) = f^{-1}([0, 1])$.

We postpone a proof of this theorem to Subsection 3.1.3 as it requires the careful extension of tensors. In a first step, however, we want to consider a certain extension procedure for scalar functions instead of tensors.
3.1.2. Extending functions beyond a boundary. We will need the following technical result allowing us to extend functions beyond the boundary of a manifold in a way that respects infima. To that purpose, let us be given a pointed Riemannian manifold \((M, g, x^0)\) of \((c, k)\)-bounded geometry. We would like to construct a standard outer collar to \(\partial M\) of \((M, g, x^0)\) with the same favorable properties for metrics close to \(g\).

Let us be given a pointed Riemannian manifold \((M, g)\) with non-empty boundary \(\partial M\), respectively. From now on, we refer to these atlases as cylindrical atlases.

We would like to construct a standard outer collar to \(\partial M\) of \((M, g, x^0)\) with the same favorable properties for metrics close to \(g\). Let \((X, x^0)\) be a Riemannian manifold with non-empty boundary \(\partial M\) equipped with the metric \(\partial g = g|_{\partial M}\). We denote by \(\vec{\nu}\) the inward normal vector field along \(\partial M\). Then for a point \(x \in \partial M\) we fix an orthonormal basis on the tangent space \(T_x \partial M\) to identify it with \(\mathbb{R}^{n-1}\). Then for small enough \(r_1, r_2 > 0\) there is normal collar coordinates

\[
\kappa_x : B(0, r_1) \times [0, r_2) \to M, \quad \kappa_x : (v, t) \mapsto \exp^g_{\exp^g_{\partial g}(v)}(t\vec{\nu}),
\]

where the exponential maps of \(\partial M\) and of \(M\) are composed. By assumption, the manifold \((M, g)\) has \((c, k)\)-bounded geometry, in particular, the boundary \((\partial M, \partial g)\) also has \((c, k)\)-bounded geometry. Let us choose a collar \(\partial M \times [0, \delta)\) for a small enough \(\delta > 0\) such that it is covered by normal collar coordinates charts \(U_\ell\) with

\[
U_\ell = \kappa_\ell(V_\ell); \quad V_\ell := W_\ell \times [0, r_2), \quad W_\ell := B(0, r^{(\ell)}_1)
\]

where \(\kappa_\ell\) is the corresponding map from (3.1). Since the manifold \((M, g)\) has \((c, k)\)-bounded geometry, [21, Proposition 3.2] implies that there exist constants \(r_0 > 0\) and \(c_0\) and positive integer \(m_0\) depending only on \(c\) and \(k\), such that if \(r_1, r_2 \leq r_0\) the family of charts \(\{ \kappa_\ell \mid \ell \in \Lambda \}\) can be chosen locally finite (this finiteness is controlled by \(m_0\)), and there is a subordinate partition of unity \(\{ \psi_\ell \mid \ell \in \Lambda \}\) satisfying the bound

\[
|\psi_\ell|_{C^k} < c_0,
\]

where, again, \(c_0\) only depends on \(c\) and \(k\). We fix this atlas \(\{ U_\ell \mid \ell \in \Lambda \}\) of the interior neck neighborhood once and forever, as well as the subordinate partition of unity \(\{ \psi_\ell \mid \ell \in \Lambda \}\), as they have the same favorable properties for metrics close to \(g\) as well. Now the atlas \(\bigcup_i \kappa_i \cup \bigcup_j \kappa_{j}^{\text{int}}\) (where the \(\kappa_{j}^{\text{int}}\) are charts for the interior) can be extended to an atlas \(\hat{\kappa}_i \cup \bigcup_{j} \kappa_{j}^{\text{int}}\) of a manifold \(X\) without boundary diffeomorphic to the interior of \(M\) by extending the smooth chart transitions from

\[
V_{i_j} = W_{i_j} \times [0, r_2) \to V_{j_i} = W_{j_i} \times [0, r_2) \quad \text{to} \quad W_{j_i} \times (-\infty, r_2) \to W_{j_i} \times (-\infty, r_2)
\]

(where \(V_{i_j} := \kappa_{i_j}^{-1}(U_i \cap U_j)\)) providing gluing data for a manifold \(X\) preserving the bounds (3.2) for the chart transitions. From now on, we refer to these atlases as cylindrical atlas and extended cylindrical atlas, respectively.

Now let \((X, p)\) the extension of the manifold with boundary \(M\) as above and let \(h\) be a complete Riemannian metric on \(X\). Let, furthermore, \(0 < r \leq \infty\) be given and define \(B_r := B(x^0, r) \subset M\). Let \(\Lambda_r \subset \Lambda\) be the subset of boundary chart domains of the cylindrical atlas contained in \(B_r\). Then we define \(\partial_r M := \bigcup_{\ell \in \Lambda_r} U_\ell\) and let \(X_r\) be the union of \(M\) and of the images of the extended cylindrical charts belonging to \(\Lambda_r\).
Lemma 3.4. (Stable nonlinear extension operator) Let \((X, x^0)\) the extension of the manifold with boundary \(M\) as above and let \(h\) be a complete Riemannian metric on \(X\), which we can assume to satisfy \(\kappa^*_i h > m_0 \epsilon_i\) in every chart \(\kappa_i\) (where \(\epsilon_i\) is the Euclidean metric in the chart \(\kappa_i\)). Then there is a map \(F : C^0(M, (0; \infty)) \rightarrow C^0(X, (0; \infty))\) with the following properties:

(i) the map \(F\) is an extension operator, i.e., \(F(u)|_M = u\) for all \(u \in C^0(M, (0; \infty))\), and \(F_r(u) := F(u)|_{X_r}\) only depends on \(u|_{B_r}\);

(ii) for each \(k \geq 1\) and each \(b > 0\), \(F_r\) maps the space \(C^k(B_r, (b; \infty))\) to \(C^k(X_r, \mathbb{R})\) continuously with respect to the \(C^k(B_r)\)-norms for an entire open \(C^k(B_r)\)-neighborhood of metrics;

(iii) for each \(k \geq 1\), \(F_r\) maps \(C^k(B_r)\)-bounded sets uniformly to \(C^k\)-bounded sets, i.e., for every \(a > 0\) there is a constant \(c_1 > 0\) such that

\[
F_r(B^{C^k(B_r)}(0, a) \cap C^k(B_r, (0; \infty))) \subset B^{C^k}_{r, c_1}(0) \subset C^k(X_r).
\]

Finally, for every \(b > 0\) there is a constant \(\beta \in (0; b)\), \(\beta = \beta(b)\), such that the bound \(\inf(u|_{B_r}) \geq b\) implies the bound \(\inf(F_r(u)|_X) \geq \beta\), uniformly in a \(C^k(B_r)\)-neighborhood of metrics.

Proof. We use the extension operator \(E\) from [23], defined on the half-space \(\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, \infty)\). Namely, let \(C^\infty(\mathbb{R}^n_+)\) be the space smooth functions on \(\mathbb{R}^n_+\) with uniform convergence on its compact subsets of all derivatives. In [23], Seeley defines a continuous linear extension operator \(E : C^\infty(\mathbb{R}^n_+) \rightarrow C^\infty(\mathbb{R}^n)\). Denote by \(\Phi : \mathbb{R}^{n-1} \times [0; r_2) \rightarrow \mathbb{R}^{n-1} \times [0; \infty)\), defined by \(\Phi(x, r) := (x, \phi(r))\) for \(\phi : [0; r_2) \rightarrow [0; \infty)\), a stretching diffeomorphism. Define the operator \(E_2 := E \circ \Phi : C^\infty(\mathbb{R}^{n-1} \times [0; r_2)) \rightarrow C^\infty(\mathbb{R}^n)\).

We first extend every member \(\psi_\ell\) of the partition of unity fixed above by defining

\[
\mathring{\psi}_\ell \circ \hat{\kappa}_\ell := E_2(\psi_\ell \circ \kappa_\ell),
\]

which is well-defined as \(E_2\) is a combination of reflections at \(\{x_1 = 0\}\), which fits the cylindrical charts, and as \(\operatorname{supp}(\psi_\ell) \subset U_\ell\). \(^3\) We define an extension operator \(E_M : C^\infty(M) \rightarrow C^\infty(X)\) by

\[
E_M(u) := \sum \psi_i \cdot (E_2(u \circ \kappa_i) \circ \hat{\kappa}_i^{-1}).
\]

It is well-defined for the same reason as above, and by inspection, it is clear from [23] that \(E_M\) satisfies the above properties (i), (ii) and (iii) (here we use that the respective metrics in every boundary chart of \(M\) satisfy \(C^k\)-bounds with respect to the Euclidean metric on open subsets of half-spaces.). However, this construction of the extension map still does not imply a bound \(\beta > 0\) on the infimum of \(F(u)\) for all \(u\) with \(\inf(u) > b > 0\). Now, let \(u \geq b\) be a function on \(B_r\). We define:

\[
F(u)(x) := \exp(E_M(\ln u(x))).
\]

Indeed, the properties (i)–(iii) can be transferred from \(E_M\) to \(F\) by using additionally the uniform continuity of \(\ln |_{[\sigma; \infty)}\) for any \(\sigma > 0\). Since \(\ln |_{[\sigma; \infty)}\) is bounded away from \(-\infty\), there exists \(\beta = \beta(b)\) such that \(\inf(F(u)|_{X_r}) \geq \beta\) provided \(\inf(u|_{B_r}) \geq b\). \(\square\)

\(^3\)The family of the \(\mathring{\psi}_\ell\) is not a partition of unity beyond the boundary any more but still its sum nowhere vanishes and the family is locally finite, thus by the usual normalization procedure the family could be made a partition of unity. However, we do not need this property here.
3.1.3. Proof of Theorem 3.3. This is basically an extension of the proof of Lemma 3.4 to endomorphism-valued functions. Let \( \kappa_i : V_i \to U_i \) be a member of the cylindrical atlas. We denote by \( g_\ell \) the metric \( g \) restricted to \( U_\ell \), and by \( e_\ell = (\kappa_\ell^{-1})^* (ds^2) \), where \( ds^2 \) is the Euclidean metric on \( B(0,r_1^{(\ell)}) \times [0;r_2) \).

For each \( \ell \), we define the operator

\[
A_\ell := e_\ell^{-1} \circ g_\ell : TU_\ell \to TU_\ell
\]

(3.3)

where the metrics are understood as maps \( TU_\ell \to T^*U_\ell \). The operators \( A_\ell \) are positive-definite symmetric operators, their spectrum is therefore contained in \((0;\infty)\). In [21 Proposition 2.3], it is shown that there is constant \( a_0 > 0 \) such that \( |A_\ell(p)| \in (a_0^{-1};a_0) \) for all \( p \). This allows to define the maps \( a_\ell := \ln(A_\ell) \), which are smooth maps from \( U_\ell \) to the set of symmetric matrices \( \text{Mat}_s(\mathbb{R}^n,\mathbb{R}^n) \) whose operator norm is bounded by \( \ln a_0 \). We use the Seeley operator \( F \) from Lemma 3.4 to extend the coefficients of each matrix \( a_\ell \) to the members of the extended atlas

\[
\hat{U}_\ell := \kappa_\ell(V_\ell), \quad V_\ell := B(0,r_1^{(\ell)}) \times (-\infty;r_2)).
\]

This gives maps \( \hat{a}_\ell : \hat{U}_\ell \to \text{Mat}_s(\mathbb{R}^n,\mathbb{R}^n), \) such that \( \hat{a}_\ell|_{U_\ell} = a_\ell \).

Equally we define \( \hat{\psi}_\ell \) as the Seeley extensions of the partition of unity \( \psi_i \) fixed above. Then we define \( \hat{A}_\ell := \exp(\hat{a}_\ell) \), which is a positive-definite symmetric smooth extension of \( A_\ell \). Thus we can define the Riemannian metric \( \hat{g}_\ell := e_\ell \circ \hat{A}_\ell \). Finally, we put \( \hat{g} := \sum \hat{\psi}_\ell \circ \hat{g}_\ell \), which is a complete metric on the manifold \( X \). Now let \( e_\ell \) be the Euclidean metric in \( \hat{U}_\ell \), and, for \( r_1 \) and \( r_2 \) as in the definition of bounded geometry define the metric \( \tilde{g}_\ell := \kappa_\ell^{-1} \hat{g} \) on

\[
B(0,r_1^{(\ell)}) \times (-\infty;r_2) \subset \mathbb{R}^n.
\]

Denote \( \tilde{A}_\ell := \tilde{g}_\ell \circ e_\ell^{-1} \). By construction, each operator \( \tilde{A}_\ell \) is has norm bounded away from infinity. But also the norm of its inverse is bounded, since according to the definition of bounded geometry, for each point \( x \in \hat{M} \), there at most \( m_0 \) neighborhoods \( \tilde{U}_\ell \) such that \( x \in U_\ell \), and consequently, there is an index \( \ell_0 \) such that \( \psi_{\ell_0}(x) \geq m_0^{-1} \) and thus we have

\[
e(\tilde{A}_\ell v,v) = \sum \psi_{\ell_0} \tilde{A}_{\ell_0}(v,v) \geq m_0^{-1} \tilde{A}_{\ell_0}(v,v)
\]

(3.4)

for some \( \ell_0 \), as all summands are positive. Now \( \tilde{A}_{\ell_0}(v,v) \) in turn can be estimated by

\[
\tilde{A}_{\ell_0}(v,v) \geq \beta(a_0^{-1}) \cdot |v|.
\]

These and the more obvious \( C^k \)-bounds from \( \infty \) are exactly the estimates needed to show \( C^k \)-bounded geometry of \( (X,\hat{g}) \). As a height function we take, for \( \tau \in C^\infty((-\infty;r_2]) \) with \( \tau(r) = r \) for all \( r \in (-\infty;r_2/4] \) and \( \tau([r_2/2;r_2]) = r_2/2 \),

\[
f := \sum \psi_\ell \circ \tau \circ x_1 \circ \kappa_\ell^{-1},
\]

(3.6)

complemented by \( r_2/2 \) in the interior, which after a final rescaling (to have the required range and satisfy the bound from zero on \([-1/2;1/2]) \) is easily seen to satisfy all our requirements. \( \square \)
3.2. Step 2: Convergence with height functions.

Theorem 3.5. Let \( i \mapsto (X_i, g_i, x_i^0, f_i) \) be a sequence of complete pointed manifolds equipped with height functions of \((c, k + 1)\)-bounded geometry with \( c > 0, k \geq 4 \). Then the sequence \( i \mapsto (X_i, g_i, x_i^0, f_i) \) \( C^k \)-subconverges to \((X_\infty, g_\infty, x_\infty^0, f_\infty)\), where \((X_\infty, g_\infty, x_\infty^0)\) is a complete open manifold, and \( f_\infty : X_\infty \to (-\infty, 1) \) is a \((c, k)\)-height function.

Proof. Let \( i \mapsto (X_i, g_i, x_i^0, f_i) \) be a sequence from Theorem 3.5. By Theorem 2.5 we may assume that the sequence of manifolds \( i \mapsto (X_i, g_i, x_i^0) \) already \( C^k \)-converges to a pointed complete Riemannian manifold \((X_\infty, g_\infty, x_\infty^0)\) of dimension \( n \). Without loss of generality, we can also assume that the exhaustions of \( X_\infty \) is chosen as a systems of open balls \( \{B(x_\infty^0, i)|i \in \mathbb{N}\} \). Let \( \phi_i : B(x_\infty^0, i) \to X_i \) be the diffeomorphisms (on their image) from Definition 2.3.

We recall that \( f_i : X_i \to \mathbb{R} \) are \((c, k + 1)\)-height functions as in Definition 3.1. By definition, we have that \(|\nabla^\ell f_i| \leq c\) for all \( \ell = 0, 1, \ldots, k+1 \). Then by passing to a subsequence if necessary, the functions \( \tilde{f}_i := \phi_i^* f_i \) also \( C^k \)-converge to a function \( f_\infty : X_\infty \to \mathbb{R} \) on the ball \( B(x_\infty^0, n) \), and then we choose a diagonal sequence to get convergence on every ball. By assumptions on the sequence of functions \( i \mapsto f_i \), it is evident that the function \( f_\infty \) is also a \((c, k)\)-height function, and since \( \delta^\ell(f_i) \geq c^{-1} \) for all \( i = 1, 2, \ldots \) (where \( \delta^\ell \) is defined as in Definition 3.1), we obtain that \( \delta^\ell(f_\infty) \geq c^{-1} \). This completes the proof of Theorem 3.5. \( \square \)

3.3. Step 3: Coming back to manifolds-with-boundary.

Theorem 3.6. Let \( (X_i, g_i, x_i) \) be a \( C^k \)-convergent sequence and let \( f_i : X_i \to \mathbb{R} \) be \((c, k)\)-height functions, then the sequence \( i \mapsto (M_i := X_i^f_i, g_{M_i}, x_i^0) \) of manifolds with boundary \( C^{k-1} \)-subconverges to \((M_\infty := X_\infty^f, g_{M_\infty}, x_\infty^0)\).

Proof. By Theorem 3.5 the manifolds \((M_i := X_i^f_i, g_i, x_i^0)\) Hausdorff-converge (in the limit manifold) and thus Gromov-Hausdorff-converge to \((M_\infty := X_\infty^f, g_\infty, x_\infty^0)\) as pointed metric spaces. However, we need more: we need to prove Cheeger-Gromov convergence. Now, for any fixed radius \( r \) and \( i > r \), we construct diffeomorphisms \( D_i^r \) from the manifold with height function \( H := B(x_\infty^0, r) \cap f_\infty^{-1}((-1/2, 1/2)) \) to an open set in \( B(x_\infty^0, r) \cap \tilde{f}_i^{-1}((-1/2, 1/2)) \) (recall that \( x_\infty^0 = \Phi_i^{-1}(x_0^0) \)) and that the \( \tilde{f}_i \) are defined as in Section 3.2 by means of the gradient flows of \( \tilde{f}_i \):

\[
D_{i, \delta}^r(x) := \text{Fl}^{\delta(x)}_{\nabla \tilde{f}_i}(x) \forall x \in H,
\]

where \( t(x) \) is chosen such that

\[
\tilde{f}_i(\text{Fl}^{\delta(x)}_{\nabla \tilde{f}_i}(x)) = f_\infty(x).
\]

It is easy to see that \( t \) is a smooth function (using the product decomposition of a neighborhood \( U \) of \( \tilde{f}_i^{-1}(0) \) given by the gradient flow of \( \tilde{f}_i \) and the fact that \( B(x_\infty^0, r) \cap f_\infty^{-1}((-1/2, 1/2)) \subset U \), and \( D_{i, \delta}^r \) is a diffeomorphism onto its image. Standard integral estimates yield

\[
\varepsilon/2 > |\tilde{f}_i(\text{Fl}^{\delta(x)}_{\nabla \tilde{f}_i}(x)) - \tilde{f}_i(x)| \geq t(x) \cdot \delta^\ell(\tilde{f}_i) \forall x \in H,
\]
(δ_t^0 defined as in Definition 3.1) and with this uniform flow time estimate and the C^k-estimates on \(\text{grad} \tilde{f}_i\), we get C^k-bounds of the \(D^r_{i,\partial}\) tending to 0. Now \(D^r_{i,\partial}\) is a diffeomorphism from \(H\) to its image, which is in \(B(x^0_\infty, r + 1) \cap X^f_i\). For \(i\) large enough, we get \(d(D^r_{i,\partial}y, y)\) smaller than the convexity radius in \(B(x^0_\infty, r + 1)\). This allows us to interpolate \(D^r_{i,\partial}\) geodesically with the identity in \(\text{int}(M)\): We define

\[ D^r_i(y) := \exp_y(\phi(y) \cdot \exp_y^{-1}(D^r_{i,\partial}(y))) \]

for \(y \in B_r(x_\infty) \cap \tilde{f}^{-1}_\infty(-1/2, 1/2)\) and for a smooth function \(\phi\) supported in \(B(x^0_\infty, r) \cap \tilde{f}^{-1}_\infty(-1/2, 1/2)\) and identical to 1 in a neighborhood of \(\tilde{f}^{-1}_\infty(0)\), and extended by \(D^r_i(y) = y\) on the complement, getting a sequence of diffeomorphisms from \(M_i \cap B(x^0_\infty, r)\) as above that converges as well. The image of \(D^r_i\) is still contained in \(B(x^0_\infty, r + 1) \cap X^f_i\) and contains \(B_{r-1} \cap X^f_i\), which allows to show \(M_i \rightarrow M_\infty\). □

3.3.1. Proof of Theorem A. Now let us prove Theorem A. Assume that we are given a sequence of pointed manifold with boundary \((M_i, g_i, x^0_i)\) of \((c, k+1)\)-bounded geometry. Then we can extend every \((M_i, g_i, x^0_i)\) to a pointed boundaryless manifold \((X_i, \overline{g}_i, x^0_i, f_i)\) as in Theorem 3.3. Then Theorem 2.5 implies that there is a convergent subsequence for both manifolds and height functions, also denoted by \((X_i, \overline{g}_i, x^0_i, f_i)\). Finally, Proposition 3.6 implies that, in the C^k sense,

\[
\lim_{i \rightarrow \infty} (M_i, g_i, x^0_i) = \lim_{i \rightarrow \infty} (X^f_i, \overline{g}_i, x^0_i) = (X^f_\infty, \overline{g}_\infty, x^0_\infty).
\]

The last assertion follows from the existence of a zero locus of the height function, which in turn follows from the fact that the base point is mapped to a positive value and that there are also negative values in the image of the limiting function due to the definition of convergence. □
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