Toward Scalable Risk Analysis for Stochastic Systems Using Extreme Value Theory

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Abstract—We aim to analyze the behaviour of a finite-time stochastic system, whose model is not available, in the context of more rare and harmful outcomes. Standard estimators are not effective in making predictions about such outcomes due to their rarity. Instead, we use Extreme Value Theory (EVT), the theory of the long-term behaviour of normalized maxima of random variables. We quantify risk using the upper-semideviation \( \rho(Y) := E(\max\{Y - \mu, 0\}) \) of an integrable random variable \( Y \) with mean \( \mu := E(Y) \). \( \rho(Y) \) is the risk-aware part of the common mean-upper-semideviation functional \( \varphi_\lambda(Y) := \mu + \lambda \rho(Y) \) with \( \lambda \in [0,1] \). To assess more rare and harmful outcomes, we propose an EVT-based estimator for \( \rho(Y) \) in a given fraction of the worst cases. We show that our estimator enjoys a closed-form representation in terms of the popular conditional value-at-risk (CVaR), which represents the expectation of \( \max\{0,Y - \eta\} \) conditioned on \( \max\{0,Y - \eta\} \). An expected exceedance above a threshold \( \eta \in \mathbb{R} \), i.e., \( E(\max\{Y - \eta, 0\}) \).

A popular criterion is the conditional value-at-risk (CVaR), which represents the expectation of \( Y \) conditioned on \( Y \) being larger than a particular quantile [10, Th. 6.2]. CVaR has been useful for defining performance or safety objectives [5], [8] and constraints [2], [4] for control systems. Another standard criterion is the mean-upper-semideviation (MUSD) \( \varphi_\lambda(Y) := \mu + \lambda \rho(Y) \) with \( \lambda \in [0,1] \), which is a weighted sum of the mean \( \mu := E(Y) \) and the upper-semideviation \( \rho(Y) \) defined by

\[
\rho(Y) := E(\max\{Y - \mu, 0\}).
\]

The above criteria are examples of risk functionals. A risk functional is a map from a space of random variables to the extended real line. CVaR and MUSD are real-valued coherent risk functionals on the space of integrable random variables [10, Ex. 6.16, Ex. 6.20]. A real-valued coherent risk functional can be expressed as a robust expectation with respect to a particular family of distributions [10, Th. 6.6], connecting risk analysis with distributionally robust optimization. Recent empirical evidence suggests that MUSD can provide protection against value function approximation errors [9, Sec. 7], reflecting its coherence. The expressive nature of risk functionals and the distributionally robust attributes of coherent ones suggest broad applicability to systems operating under uncertainties in practice.

However, current approaches to analyze and optimize risk-aware non-linear systems suffer from scalability challenges. Most approaches are based on dynamic programming (DP) [1], [6], [8] or Q-learning [3], [7]. These algorithms were developed originally to minimize an expected cumulative cost for a Markov decision process (MDP). When reformulating one of the above algorithms for a risk-aware setting, the new algorithm inherits the scalability issues of the original algorithm. It is well-known that DP cannot apply to high-dimensional state spaces without function approximations. Q-learning typically involves extensive exploration, finite state spaces, and infinite time horizons [3], [7]. These conditions need not apply when sampling is moderately expensive, the state space is continuous, or analysis on a finite time horizon is needed. A recent scalable approach for risk analysis combines temporal difference learning with value function approximation and quantifies risk using a composition of coherent risk functionals [9]. The approach does not require extensive exploration but involves a finite-state infinite-time MDP [9]. Similar to [9], we focus on the analysis of risk rather than its optimization. However, we adopt statistical tools from Extreme Value Theory (EVT).
EVT is the study of the long-term behaviour of normalized maxima of random variables. It has been useful for examining extreme events in hydrology [11], seismology [12], and disease transmission [13]. This theory offers tools to extrapolate beyond the available data to estimate the upper tail of a distribution [14], [15]. Theoretical connections between EVT and hitting time statistics for discrete-time systems have been established [16]. EVT has been applied to compute properties of chaotic systems, for example, the dimensions of an invariant measure [17]. Statistical applications of EVT include estimating extreme quantiles [18], CVaR [19]–[21], and extreme probabilities [18, Ch. 4.4]. A distributionally robust approach for extreme quantile estimation with sensitivity to modeling errors has been proposed [18]. An EVT-based formula for CVaR [19] has been applied to a multi-armed bandit problem [20]. Deo and Murthy have estimated CVaR while risk can be quantified in different ways, we consider a stochastic system, whose dynamical model is not available, operating on a discrete finite time horizon [49]. An element is not available, operating on a discrete finite time horizon [49, Fig. 1]. We also showcase our approach using data of total overflow volumes from combined sewer systems throughout Canada [23]. This letter initiates a new avenue for scalable risk analysis in data-sparse applications.

II. Preliminaries

Notation. \( \mathbb{N} := \{1, 2, \ldots \} \) is the set of natural numbers, \( \mathbb{R} \) is the set of real numbers, and \( \mathbb{R}_+ := (0, +\infty) \). \( Y \in L^1(\Omega, \mathcal{F}, P) \) or \( Y \in L^1 \) means that \( Y \) is an integrable random variable on \( (\Omega, \mathcal{F}, P) \), i.e., \( E(|Y|) \) is finite. The function \( 1_A : \Omega \to \{0, 1\} \) is the indicator on \( A \in \mathcal{F} \). If \( F \) is a distribution function, \( z^* := \sup\{z \in \mathbb{R} : F(z) < 1\} \) is its right endpoint. For an \( \mathbb{R} \)-valued function \( f \), we define \( \bar{f} := 1 - f \).

If \( S \) is a metric space, \( B_S \) is the Borel \( \sigma \)-algebra on \( S \), and \( \text{int}(S) \) is the interior of \( S \). Abbreviations: i.i.d. = independently and identically distributed; a.e. = almost everywhere or almost every; w.r.t. = with respect to; d.o.f. = degrees of freedom.

Since EVT is not well-known in control theory, it is necessary to summarize some fundamentals, which we adopt from [14] and [15]. Let \( (Z_i)_{i\in\mathbb{N}} \) be an i.i.d. sequence of random variables defined on a probability space \( (\Omega, \mathcal{F}, P) \) with distribution function \( F \). We do not know \( F \) or \( P \).

The partial maximum for \( m \in \mathbb{N} \) is defined by \( M_m := \max\{Z_1, Z_2, \ldots, Z_m\} \). Since \( M_m \) converges in probability to the right endpoint \( z^* \) of \( F \), a normalization of \( M_m \) can be useful for revealing characteristics of \( F \).

Definition 1 \((F \in D(G_\gamma))\): Suppose that there exist sequences \( (a_m)_{m \in \mathbb{N}} \subseteq \mathbb{R}_+ \) and \( (b_m)_{m \in \mathbb{N}} \subseteq \mathbb{R} \) such that

\[
\lim_{m \to +\infty} P \left( \left\{ \omega \in \Omega : \frac{M_m(\omega) - b_m}{a_m} \leq z \right\} \right) = G_\gamma(z)
\] (2)

for every continuity point \( z \) of \( G_\gamma \), where \( G_\gamma \) is a non-degenerate distribution function. Then, we say that \( F \) belongs to the maximum domain of attraction of \( G_\gamma \), i.e., \( F \in D(G_\gamma) \).

\( G_\gamma \) being non-degenerate means that it does not correspond to a point mass. There are many examples of \( F \) that satisfy Definition 1 including: Pareto, Burr, Fréchet, t-Student, Cauchy, Log-gamma, Uniform on \((0, 1)\), Beta, Exponential, Logistic, Gumbel, Normal, Lognormal, and Gamma [15]. The extreme value index \( \gamma \) is a qualitative measure for tail heaviness, i.e., how fast the tail of \( F \), provided that \( F \in D(G_\gamma) \), decays to zero [15, p. 63].

The next theorem provides an equivalent characterization for \( F \in D(G_\gamma) \), offering an approximation for the upper tail of \( F \). For convenience, we define the interval \( \mathcal{J}_\gamma := [0, +\infty) \) if \( \gamma \geq 0 \); \( \mathcal{J}_\gamma := [0, -1/\gamma) \) if \( \gamma < 0 \). Moreover, we define the function \( \phi_{\gamma} : \mathcal{J}_\gamma \to [0, 1] \) by

\[
\phi_{\gamma}(z) := \begin{cases} 
(1 + \gamma z)^{-1/\gamma}, & \text{if } \gamma \neq 0, \\
\exp(-z), & \text{if } \gamma = 0.
\end{cases}
\] (3)

\( \tilde{\phi}_{\gamma} := 1 - \phi_{\gamma} \) corresponds to the Generalized Pareto distribution [15, Eq. (4.6)].
Theorem 1: [14, Th. 1.1.6, Part 4]: \( F \in \mathcal{D}(G_\gamma) \) for some \( \gamma \in \mathbb{R} \) if and only if there is an \( \mathbb{R}_+ \)-valued function \( g \) s.t.

\[
\lim_{s \to z^+} \frac{1 - F(s + z g(s))}{1 - F(s)} = \phi_\gamma(x), \quad x \in J_\gamma,
\]

(4)

where \( s \to z^+ \) means that \( s \) approaches \( z^+ \) from below.

For brevity, we use the notations \( g_k := g(s) \) and \( \tilde{F} := 1 - F \). Motivated by (4), the following heuristic is commonly used, e.g., see [14, pp. 65–66], to approximate the upper tail of \( F \) above some sufficiently large threshold \( s \in \mathbb{R} \):

\[
\tilde{F}(z) \approx \tilde{F}(s) \cdot \phi_\gamma((z - s)/g_k), \quad z \geq s.
\]

(5)

The subsequent lemma formalizes the tail approximation (5) and uses the following definitions. We define a parameter vector \( \theta \) by

\[
\theta := \{k, m, \gamma, s, g_k\},
\]

(6)

where \( k \in \mathbb{N} \) and \( m \in \mathbb{N} \) with \( k < m \), \( \gamma \in \mathbb{R} \), \( s \in \mathbb{R} \), and \( g_k \in \mathbb{R}_+ \). If \( F \in \mathcal{D}(G_\gamma) \), then \( \gamma \) is an extreme value index, \( s \) is a threshold, and \( g_k = g(s) \), where \( g \) comes from Theorem 1.

In addition, we define the interval \( I_\theta \) by

\[
I_{\theta} := \begin{cases} (s, +\infty), & \text{if } \gamma \geq 0, \\ (s, s - g_k/\gamma), & \text{if } \gamma < 0. \end{cases}
\]

(7)

Lemma I (Formalized tail approximation): Let \( F \in \mathcal{D}(G_\gamma) \) for some \( \gamma \in \mathbb{R} \), and let \( \epsilon \in \mathbb{R}_+ \) and \( z \in I_\theta \) be given. If \( z^* \in \mathbb{R} \), then there exists a \( \delta_{z^*} \in \mathbb{R}_+ \) such that

\[
|\tilde{F}(z) - \tilde{F}(s) \cdot \phi_\gamma((z - s)/g_k)| < \epsilon \tilde{F}(s)
\]

(8)

for every \( s \in (-\delta_{z^*}, z^* + \epsilon) \). Otherwise, if \( z^* = +\infty \), then \( \exists \epsilon_{z^*} \in \mathbb{R}_+ \) such that (8) holds for every \( s \in (\epsilon_{z^*}, z^*) \).

Proof: First, note that \( \tilde{F}(z) < 1 \) for every \( z \in (-\delta_{z^*}, z^*) \) as a consequence of \( z^* \) being the right endpoint of the distribution function \( F \). Then, the result follows from applying the definition of the limit [24, Def. 4.33, p. 98] to (3). \( \square \)

Lemma I formalizes the tail approximation (5) by providing a pointwise bound (8) on the magnitude of the approximation error. Next, we will apply the tail approximation to estimate the upper-semideviation in a fraction \( \alpha \) of the worst cases, to be denoted by \( \rho_\alpha(Y) \). We will derive a closed-form expression for an EVT-based estimator for \( \rho_\alpha(Y) \) using first principles from probability theory.

III. ESTIMATING EXTREMAL UPPER-SEMIDEVIATION

Let \( Y \in L^1(\Omega, \mathcal{F}, P) \) be a random cost, whose distribution function, \( F_Y(y) := \Pr(\omega \in \Omega : Y(\omega) \leq y) \) with \( y \in \mathbb{R} \), is not known. As we have motivated in Sec. I-A, our focus is estimating the upper-semideviation \( \rho(Y) \) when considering a given fraction \( \alpha \in (0, 1) \) of the largest realizations of \( Y \). We call this quantity the upper-semideviation of \( Y \) at level \( \alpha \), which we define by

\[
\rho_\alpha(Y) := \int_{\omega \in \Omega: Y(\omega) \geq v_\alpha(Y)} \max\{Y - \mu, 0\} \, dP,
\]

(9)

where \( v_\alpha(Y) \in \mathbb{R} \) is a threshold (to be specified). The meaning of \( \rho_\alpha(Y) \) is the expected exceedance of \( Y \) above the mean in a fraction \( \alpha \) of the worst cases. In the definition of \( \rho_\alpha(Y) \) (9), the integral is over a subset of the sample space \( \Omega \) corresponding to a fraction of the largest realizations of \( Y \). In contrast, in the definition of \( \rho(Y) \) (1), the integral is over the entire sample space \( \Omega \). The threshold \( v_\alpha(Y) \) is the value-at-risk of \( Y \) at level \( \alpha \) [10, Sec. 6.2.3], i.e.,

\[
v_\alpha(Y) := \inf\{z \in \mathbb{R} : F_Y(z) \geq 1 - \alpha\}, \quad \alpha \in (0, 1).
\]

(10)

To estimate \( \rho_\alpha(Y) \) (9), we will apply properties of CVaR. The CVaR of \( Y \in L^1 \) at level \( \alpha \in (0, 1) \) is defined by

\[
c_\alpha(Y) := \inf_{\tau \in \mathbb{R}} \left( \tau + \frac{1}{\alpha} E\{\max\{Y - \tau, 0\} \} \right)
\]

(11)

as per [10, Eq. (6.22)]. A useful fact is that a minimizer of the right side of (11) is \( v_\alpha(Y) \) [10, p. 258], and therefore,

\[
c_\alpha(Y) = v_\alpha(Y) + \frac{1}{\alpha} E\{\max\{Y - v_\alpha(Y), 0\} \}.
\]

(12)

We will use a data set \( \{y_{i,m} : i = 1, 2, \ldots, m\} \subset \mathbb{R} \) of size \( m \in \mathbb{N} \) that is sampled independently from \( Y \) and satisfies

\[
y_{1,m} \leq \cdots \leq y_{m-k,m} \leq \cdots \leq y_{m-1,m} \leq y_{m,m}
\]

(13)

with \( k \in \mathbb{N} \) and \( k < m \). The notation \( \mu_m := \frac{1}{m} \sum_{i=1}^{m} y_{i,m} \) denotes the sample mean of the data (13). To describe the data formally, let \( (Y_i)_{i \in \mathbb{N}} \) be i.i.d. random variables defined on \((\Omega, \mathcal{F}, P)\) with distribution function \( F_Y \). For a given \( m \in \mathbb{N} \), the notation \( y_{i,m} \) denotes a realization of \( Y_i \) for a particular \( j \in \{1, 2, \ldots, m\} \), i.e., \( y_{i,m} = Y_j(\omega) \) for some \( \omega \in \Omega \). A data set \( \{y_{i,m} : i = 1, 2, \ldots, m\} \) corresponds to an \( \omega \in \Omega \), where we choose each index \( i \) to satisfy the inequalities in (13).

A. Typical empirical estimator for \( \rho_\alpha(Y) \)

A typical estimator for \( \rho_\alpha(Y) \) (9) is computed using the \( k + 1 \) largest samples for some large enough \( k \in \mathbb{N} \):

\[
\hat{\rho}_{\alpha,k,m} := \frac{1}{m} \sum_{i=m-k}^{m} \max\{y_{i,m} - \mu_m, 0\},
\]

(14)

where \( y_{m-k,m} \) is an approximation for \( v_\alpha(Y) \) (10). \( \hat{\rho}_{\alpha,k,m} \) is not designed to represent the upper tail of \( \tilde{F}_Y \) when the number \( m \) of samples is limited; we will illustrate limitations of \( \hat{\rho}_{\alpha,k,m} \) numerically in Sec. IV. We aim to estimate \( \rho_\alpha(Y) \) (9) using a data set (13) in a manner that is sensitive to the burden of collecting numerous samples and the challenge of observing large realizations of \( Y \) in practice.

B. EVT-based estimator for \( \rho_\alpha(Y) \)

Here, we propose an EVT-based estimator for \( \rho_\alpha(Y) \) (9) that enjoys a closed-form representation in terms of CVaR. Our tactic is to construct a random variable \( Y_\theta \), whose extremal distribution approximates the extremal distribution of \( Y \).

We suppose that an experiment has been conducted, providing a data set (13). We let \( \theta \) (6) be a parameter vector, which has been estimated using this data set. There are different ways to estimate \( \theta \) [14], and we will illustrate one way in Sec. IV. Motivated by the analysis of Lemma I we define for \( z \in \mathbb{R} \)

\[
F_\theta(z) := \begin{cases} 0, & \text{if } z < s, \\ 1 - \frac{k}{m} \phi_\gamma \frac{z - s}{g_k}, & \text{if } z \in I_\theta, \\ 1, & \text{if } z \geq s - g_k/\gamma, \quad \gamma < 0, \end{cases}
\]

(15)

with \( s := y_{m-k,m} \). The factor \( \frac{k}{m} \) is an estimate for \( \tilde{F}_Y(s) = 1 - F_Y(s) \), where the empirical distribution \( \tilde{F}_m(x) := \)
Theorem 2 (Closed-form expression for $\rho_{\alpha,\theta}$): Assume the conditions of Lemma 3 and let the data $\{y_i : i = 1, 2, \ldots, m\}$ be given. Suppose that $\theta \in \Theta$ is estimated from the data with $s := y_{m-k_m}$, and let $0 < \alpha < \frac{k_m}{m}$. If $v_{\alpha}(Y_0) \geq \mu_m$, then $\hat{\rho}_{\alpha,\theta} = \alpha (c_n(Y_0) - \mu_m)$. 

Proof: Lemma 3 guarantees that $Y_0 \in L^1(\mathbb{R}, \mathcal{B}_R, P_0)$. Thus, the CVar of $Y_0$ at level $\alpha$ satisfies 

$$c_{\alpha}(Y_0) = v_{\alpha}(Y_0) + \frac{1}{\alpha} \int \max\{z - v_{\alpha}(Y_0), 0\} \, dP_0(z)$$  

(21)

by utilizing the argument underlying (12). For brevity, we use the notation $(x)^+ := \max\{x, 0\}$ for every $x \in \mathbb{R}$. By applying the definition of $A_{\alpha,\theta}$ (19) and the assumption $v_{\alpha}(Y_0) \geq \mu_m$, it follows that 

$$1_{A_{\alpha,\theta}}(z) (z - \mu_m)^+ = 1_{A_{\alpha,\theta}}(z) (z - v_{\alpha}(Y_0))^+ + 1_{A_{\alpha,\theta}}(z) (v_{\alpha}(Y_0) - \mu_m)$$  

(22)

for every $z \in \mathbb{R}$. By re-expressing $\hat{\rho}_{\alpha,\theta}$ (18) and noting that a sum of non-negative Borel-measurable functions can be integrated term by term [25, Cor. 1.6.4], it holds that 

$$\hat{\rho}_{\alpha,\theta} = \int 1_{A_{\alpha,\theta}}(z) (z - \mu_m)^+ \, dP_0(z) = \psi_{1,\theta} + \psi_{2,\theta},$$

where $\psi_{1,\theta}$ and $\psi_{2,\theta}$ are defined by 

$$\psi_{1,\theta} := \int 1_{A_{\alpha,\theta}}(z) (z - v_{\alpha}(Y_0))^+ \, dP_0(z),$$  

(23)

$$\psi_{2,\theta} := (v_{\alpha}(Y_0) - \mu_m) \cdot P_0(A_{\alpha,\theta}),$$  

(24)

respectively. Since $A_{\alpha,\theta} = [v_{\alpha}(Y_0), +\infty)$, we have that 

$$1_{A_{\alpha,\theta}}(z) (z - v_{\alpha}(Y_0))^+ = (z - v_{\alpha}(Y_0))^+, \quad z \in \mathbb{R},$$  

(25)

and therefore, $\psi_{1,\theta} = \int (z - v_{\alpha}(Y_0))^+ \, dP_0(z)$. Next, we will simplify $\psi_{2,\theta}$ (24). Using $A_{\alpha,\theta} = [v_{\alpha}(Y_0), +\infty)$, $F_0$ being a distribution function (Lemma 2), and $P_0$ being the corresponding Lebesgue-Stieljes measure, we have that 

$$P_0(A_{\alpha,\theta}) = 1 - \lim_{z \uparrow v_{\alpha}(Y_0)} F_0(z)$$  

by [25, 1.4.5 (9), p. 25]. Since $F_0$ is continuous on $(\int(I_0), v_{\alpha}(Y_0) \in \int(I_0)$, and $F_0(v_{\alpha}(Y_0)) = 1 - \alpha$ (Lemma 9), we find that 

$$\lim_{z \uparrow v_{\alpha}(Y_0)} F_0(z) = F_0(v_{\alpha}(Y_0)) = 1 - \alpha.$$

(27)

Using (26)–(27), we simplify $\psi_{2,\theta}$ (24) as follows: 

$$\psi_{2,\theta} = (v_{\alpha}(Y_0) - \mu_m)(1 - (1 - \alpha)) = \alpha (v_{\alpha}(Y_0) - \mu_m).$$

(28)
Using our simplifications for $\psi_{1,\theta}$ and $\psi_{2,\theta}$, we conclude that

$$
\hat{\rho}_{\alpha,\theta} = \alpha \left( \frac{1}{\alpha} \int_\mathcal{R} (z - v_{\alpha}(Y_\theta))^+ \, dP_\theta(z) + v_{\alpha}(Y_\theta) - \mu_m \right) \\
= \alpha (c_\alpha(Y_\theta) - \mu_m),
$$

where we use (21) in the final line.

Remark 2 (Theorem 2 assumptions): Typically, $\alpha$ is small, e.g., less than 0.05, to emphasize rare high-consequence outcomes. Thus, we anticipate $v_{\alpha}(Y_\theta) \geq \mu_m$ to hold in applications. We will present an example in Sec. IV.2.

Subsequently, we use techniques from [10], [19] to provide expressions for $v_{\alpha}(Y_\theta)$ (20) and $c_\alpha(Y_\theta)$ (21).

Remark 3 (Expressions for $v_{\alpha}(Y_\theta)$ and $c_\alpha(Y_\theta)$): Assume the conditions of Theorem 2. Since $0 < \alpha < \frac{k}{m} < 1$, it holds that $v_{\alpha}(Y_\theta) \in \text{int}({\mathcal{I}}_\theta)$ by Lemma 4. Since $F_\theta$ (15) is continuous and strictly increasing on $\text{int}({\mathcal{I}}_\theta)$, we invert $F_\theta$ on $\text{int}({\mathcal{I}}_\theta)$ to derive the following expression for $v_{\alpha}(Y_\theta)$:

$$
v_{\alpha}(Y_\theta) = \left\{ \begin{array}{ll}
\left( -g_s \frac{v_{\alpha}(Y_\theta)}{m} \right)^{\frac{1}{\gamma}} - 1, & \text{if } \gamma \neq 0, \\
-\frac{g_s}{\log(m)} \cdot \log\left( -\frac{v_{\alpha}(Y_\theta)}{m} \right), & \text{if } \gamma = 0.
\end{array} \right.
$$

As well as $\alpha \in (0, 1)$, we have that $Y_\theta \in L^1(\mathbb{R}, B_\theta, P_\theta)$ by Lemma 5. Further, CVaR can be expressed as an average of the value-at-risk, $c_\alpha(Y_\theta) = \frac{1}{\alpha} \int_{1-\alpha}^1 v_{1-\tau}(Y_\theta) \, d\tau$ [10, Th. 6.2]. We use this expression with $v_{1-\tau}(Y_\theta)$ from (20) and the assumption $\gamma < 1$ from Theorem 2 to derive

$$
c_\alpha(Y_\theta) = \left( v_{\alpha}(Y_\theta) + g_s - \gamma \cdot s \right) (1 - \gamma)^{-1}.
$$

Theorem 2 and Remark 3 together provide a closed-form expression for $\hat{\rho}_{\alpha,\theta}$ (18), which we use for computation.

IV. NUMERICAL EXPERIMENTS

We conduct two experiments regarding the estimation of $\rho_{\alpha,Y}(Y)$ (9). First, we compare the performance of a typical estimator $\hat{\rho}_{\alpha,k,m}$ (14) to our EVT-based estimator $\hat{\rho}_{\alpha,\theta}$ (18) using six benchmark distributions. The second experiment uses data of combined sewer overflows in Canada [23]. Our code is available from https://github.com/eArsenault/evt-control.

Given a data set (13), we estimate a parameter vector $\theta = \{k, m, \gamma, s, g_s\}$ (6) using the probability-weighted moment estimator (PWME). The PWME is consistent under appropriate conditions [14, Th. 3.6.1], simple to implement, and produces a value for $\gamma$ that is strictly less than 1. The last characteristic is useful in light of Lemma 3. For a given data set, we use the following procedure with $\alpha = 0.01$ and $\frac{k}{m} \approx 0.10$:

1) We choose $s$ to be an estimate for $v_{0.10}(Y)$ (10). We use $s := \gamma_{[0.90-m,m]}$, where $[\cdot]$ is the ceiling function [20].

2) We assign $k$ to be the cardinality of $\{y_{i,m} : y_{i,m} > s, i = 1, 2, \ldots, n\}$ so that $y_{m-k,m} = s$ [14, p. 66].

3) We set $\gamma$ and $g_s$ as per the PWME expressions, which are closed-form and provided by [14, Eqs. (3.6.9), (3.6.10)].

Remark 4 (Consistency discussion): The PWME is consistent when there is a family $(\mathcal{Z}_i)_{i \in \mathbb{N}}$ of i.i.d. random variables with distribution function $F \in \mathcal{D}(G_\gamma)$ (2) with $\gamma < 1$ such that the $m$th order statistics $Z_{1,m} \leq \cdots \leq Z_{m-k,m} \leq \cdots \leq Z_{m,m}$ satisfy $k \to +\infty$ and $\frac{k}{m} \to 0$ as $m \to +\infty$ [14, Th. 3.6.1]. Our numerical procedure uses $m < 100$ samples because we are concerned with data-sparse applications. Further experiments could be designed for consistency in particular by considering significantly larger data sets, whose order statistics satisfy the limiting conditions in principle (e.g., choose $k$ to grow logarithmically with respect to $m$). Each benchmark distribution $F_Y$ satisfies $F_Y \in \mathcal{D}(G_\gamma)$ with $\gamma < 1$ (see below).

1) Benchmark distributions: We compare the performance of the estimators $\hat{\rho}_{0.01,k,m}$ (14) and $\hat{\rho}_{0.01,\theta}$ (18) for $m \in \{20, 21, 22, \ldots, 99\}$ samples. For each $m$, we have conducted 10,000 runs of the following: 1) draw $m$ i.i.d. samples from a random variable $Y$; 2) estimate $\theta$ as described previously; and 3) compute the errors $\hat{\rho}_{0.01,k,m} - \rho_{0.01}(Y)$ and $\hat{\rho}_{0.01,\theta} - \rho_{0.01}(Y)$. We approximate the ground-truth value of $\rho_{0.01}(Y)$ using a Monte Carlo simulation with over 4 million samples.

We consider six distributions for $Y$. The Pareto(2) and t-Student (5 d.o.f.) distributions have $\gamma = 0.5$ and $\gamma = 0.2$, respectively; the Exponential(1) and Gumbel distributions have $\gamma = 0$; the Uniform(0,1) and Beta(1,2) distributions have $\gamma < 0$ [15]. Fig. 1 presents the results. The EVT-based estimator $\hat{\rho}_{0.01,\theta}$ has lower average error versus the typical estimator $\hat{\rho}_{0.01,k,m}$ for smaller values of $m$. The error decreases as $m$ increases, and for sufficiently large $m$, the typical estimator is superior. These results support using the EVT-based estimator for risk analysis when a small number of samples is available (which is our focus). We illustrate one application next.

2) System example: Monthly combined sewer overflow volumes in Canada during the years of 2013–2017 are available from [23]. Each data point is the total volume of water (a mixture of stormwater and untreated wastewater) that was released into the environment during a particular month. (Combined sewers are present in older cities throughout North America.) The combined sewer network that underlies this data set is a vast hard-to-model dynamical system with safety requirements to protect environmental and public health. For the sake of illustration, we assume that the data during the spring and early summer months (March–June) is i.i.d., providing $m = 20$. We acknowledge that even data from a single month over consecutive years need not be i.i.d. due to climate trends. We compute $\hat{\rho}_{0.01,k,m} = 0.59 \cdot 10^6$ $m^3$ and $\hat{\rho}_{0.01,\theta} = 0.23 \cdot 10^6$ $m^3$ with $k = 2$. We verify some assumptions from Theorem 2, $\rho_{0.01}(Y_\theta) = 26 \cdot 10^6$ $m^3 > \mu_m = 15 \cdot 10^6$ $m^3$ and $\alpha = 0.01 < \frac{20}{20} = \frac{k}{m}$. We compute $\gamma = 0.87$, suggesting a heavy-tailed distribution. Interestingly, $\rho_{0.01,k,m} > \rho_{0.01,\theta}$, which also occurs for smaller values of $m$ in Fig. 1. Hence, the typical estimator may over-approximate the risk compared to the EVT-based estimator.

V. CONCLUSIONS

We have shown the ability of a new EVT-based estimator for a risk functional to perform well when data is limited. Our estimator was developed from measure-theoretic first principles, initiating a pathway for broader EVT-based risk analysis. We are in the process of deriving a consistency proof by applying theory from extreme quantile estimation [14, Th. 3.4.1]; a key aspect is to provide interpretable conditions under which the EVT-based CVaR estimator (31).
Fig. 1. The plots depict the errors, $\hat{\rho}_{0.01,k,m} - \rho_{0.01}(Y)$ (blue) and $\hat{\rho}_{0.01,\theta} - \rho_{0.01}(Y)$ (orange), versus the number of samples $m$. $\hat{\rho}_{0.01,k,m}$ is a typical estimator, and $\hat{\rho}_{0.01,\theta}$ is our EVT-based estimator. The average error across 10,000 trials is plotted using the solid lines, and the coloured bands represent 50% confidence intervals. The EVT-based estimator outperforms the typical estimator for smaller values of $m$.

is consistent using the facts that convergence in probability is preserved under continuous mappings and sums [26], [27]. The derivation of a convergence rate for $\rho_{\alpha,\theta}$ is preserved under continuous mappings and sums [26], [27]. We hypothesize faster rates when $F_Y$ is a Generalized Pareto distribution, based on (4) and an analogous result about convergence rates for $\alpha$ [28]. A study about convergence rates for the tail approximation $\rho_{\alpha,\theta}$ for twice-differentiable invertible distribution functions [29] may also be useful. Ultimately, we hope to develop controllers that are sensitive to more rare and harmful outcomes for high-dimensional systems when data is limited.

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REFERENCES

[1] N. Bauèrle and U. Rieder, “More risk-sensitive Markov decision processes,” Math. Oper. Res., vol. 39, no. 1, pp. 105–120, 2014.

[2] V. Borkar and R. Jain, “Risk-constrained Markov decision processes,” IEEE Trans. Autom. Control, vol. 59, no. 9, pp. 2574–2579, 2014.

[3] Y. Shen, M. J. Tobia, T. Sommer, and K. Obermayer, “Risk-sensitive reinforcement learning,” Neural Comput., vol. 26, no. 7, pp. 1298–1328, 2014.

[4] B. P. van Parys, D. Kuhn, P. J. Goulart, and M. Morari, “Distributionally robust control of constrained stochastic systems,” IEEE Trans. Autom. Control, vol. 61, no. 2, pp. 430–442, 2015.

[5] C. W. Miller and I. Yang, “Optimal control of Conditional Value-at-Risk in continuous time,” SIAM J. Control Optim., vol. 55, no. 2, pp. 856–884, 2017.

[6] N. Baurerle and A. Glauner, “Minimizing spectral risk measures applied to Markov decision processes,” Math. Methods Oper. Res., pp. 1–35, 2021.

[7] W. Huang and W. B. Haskell, “Stochastic approximation for risk-aware Markov decision processes,” IEEE Trans. Autom. Control, vol. 66, no. 3, pp. 1314–1320, 2021.

[8] M. P. Chapman, R. Bonalli, K. M. Smith, I. Yang, M. Pavone, and C. J. Tomlin, “Risk-sensitive safety analysis using Conditional Value-at-Risk,” IEEE Trans. Autom. Control, in press, 2022.

[9] U. Köse and A. Ruszczyński, “Risk-averse learning by temporal difference methods with Markov risk measures,” J. Mach. Learn. Res., vol. 22, pp. 1–34, 2021.

[10] A. Shapiro, D. Dentcheva, and A. Ruszczyński, Lectures on Stochastic Programming: Modeling and Theory. Philadelphia, PA: SIAM, 2009.

[11] E. Towler, B. Rajagopalan, E. Gilleland, R. S. Summers, D. Yates, and R. W. Katz, “Modeling hydrologic and water quality extremes in a changing climate: A statistical approach based on extreme value theory,” Water Resour. Res., vol. 46, no. 11, 2010.

[12] R. Shcherbakov, J. Zhuang, G. Zoller, and Y. Ogata, “Forecasting the magnitude of the largest expected earthquake,” Nature Commun., vol. 10, no. 1, pp. 1–11, 2019.

[13] J. T. Lim, Y. Han, B. Sue Lee Dickens, L. C. Ng, and A. R. Cook, “Time varying methods to infer extremes in dengue transmission dynamics,” PLoS Comput. Biol., vol. 16, no. 10, p. e1008279, 2020.

[14] L. de Haan and A. Ferreira, Extreme Value Theory: An Introduction. New York, NY: Springer, 2006, vol. 21.

[15] I. F. Alves and C. Neves, Extreme Events in Finance: A Handbook of Extreme Value Theory and its Applications, Chapter 4, F. Longin, Ed. Hoboken, NJ: John Wiley & Sons, Inc., 2017.

[16] A. C. M. Freitas, J. M. Freitas, and M. Todd, “Hitting time statistics and extreme value theory,” Probab. Theory Relat. Fields, vol. 147, no. 3, pp. 675–710, 2010.

[17] T. Caby, “Extreme value theory for dynamical systems, with applications in climate and neuroscience,” Ph.D. dissertation, Université de Toulon; Università degli studi dell’Insubria (Como, Italie), 2020.

[18] J. Blanchet, F. He, and K. Murthy, “On distributionally robust extreme value analysis,” Extremes, vol. 23, no. 2, pp. 317–347, 2020.

[19] A. J. McNeil, “Extreme value theory for risk managers,” Departement Mathematik ETH Zentrum, vol. 12, no. 5, pp. 1–22, 1999.

[20] D. Troop, F. Godin, and J. Y. Yu, “Risk-averse action selection using Extreme Value Theory estimates of the CVaR,” arXiv preprint arXiv:1912.01718, 2019.

[21] A. Deo and K. Murthy, “Optimizing tail risks using an importance sampling based extrapolation for heavy-tailed objectives,” in Proc. Conf. Decis. Control. IEEE, 2020, pp. 1070–1077.

[22] D. B. Bertsekas and S. E. Shreve, Stochastic Optimal Control: The Discrete-Time Case. Athena Scientific, 1996.

[23] Monthly combined sewer overflow volumes, Canada, https://www150.statcan.gc.ca/n1/daily-quotidien/190625/cg-c003-eng.htm, Statistics Canada, 2019, accessed: March 10, 2022.

[24] W. Rudin, Principles of Mathematical Analysis. New York, NY: McGraw Hill, 1964, vol. 3.

[25] R. Ash, Real Analysis and Probability. New York, NY: Academic Press, Inc., 1972.

[26] G. B. Folland, Real Analysis: Modern Techniques and Their Applications, 2nd ed. New York, NY: John Wiley & Sons, 1999, vol. 40.

[27] A. W. van der Vaart, Asymptotic Statistics. Cambridge, U.K.: Cambridge University Press, 1998.

[28] J. Rootzen, “Attainable rates of convergence of maxima,” Stat. Probab. Lett., vol. 2, no. 4, pp. 219–221, 1984.

[29] J.-P. Raoult and R. Worms, “Rate of convergence for the generalized Pareto approximation of the excesses,” Adv. Appl. Probability, vol. 35, no. 4, pp. 1007–1027, 2003.