A Fedosov Star Product of Wick Type for Kähler Manifolds

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Abstract

In this letter we compute some elementary properties of the Fedosov star product of Weyl type, such as symmetry and order of differentiation. Moreover, we define the notion of a star product of Wick type on every Kähler manifold by a straightforward generalization of the corresponding star product in \( \mathbb{C}^n \): the corresponding sequence of bidifferential operators differentiates its first argument in holomorphic directions and its second argument in antiholomorphic directions. By a Fedosov type procedure we give an existence proof of such star products for any Kähler manifold.

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1 Introduction

The concept of deformation quantization has been defined by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in 1978 [2]: A formal star product on a symplectic manifold \((M, \omega)\) is defined as a formal local associative deformation \(\ast\) of the commutative algebra of complex-valued smooth functions \(C^\infty(M)\) on \(M\) where the formal parameter is identified with \(\hbar\). These data are equivalent to giving a sequence \(M_t, t \geq 0\), of locally bidifferential operators on \(M\) which satisfy the following axioms for any three \(f, g, h \in C^\infty(M)\):

\[
f \ast g = \sum_{t=0}^{\infty} (i\hbar/2)^t M_t(f, g) \quad (1)
\]

\[
M_0(f, g) = fg \quad (2)
\]

\[
M_1(f, g) - M_1(g, f) = \{f, g\} \quad (3)
\]

\[
0 = \sum_{t+u=s} (M_t(M_u(f, g), h) - M_t(f, M_u(g, h))) \quad \text{for all } s \geq 0 \quad (4)
\]

\[
M_t(1, f) = M_t(f, 1) = 0 \quad \text{for all } t \geq 1 \quad (5)
\]

\[
M_t(f, g) = (-1)^t M_t(g, f) \quad \text{for all } t \geq 0 \quad (6)
\]

In most of the literature the axiom \(M_t(g, f) = (-1)^t M_t(f, g)\) is added, leading to what we shall call a star product of Weyl type because in flat \(\mathbb{R}^{2n}\) with the standard Poisson structure it exactly corresponds to the Weyl symmetrization rule in canonical quantization. The existence proofs of DeWilde-Lecomte [7] and Fedosov [8] refer to this kind of star products. However, in Berezin’s 1974 paper [3] on quantization via coherent states on Kähler manifolds another type of star products appears where the operators \(M_t\) are complex and have the property that one of their arguments is differentiated in holomorphic directions whereas the other arguments is differentiated in antiholomorphic directions only (which one does strongly depend on the sign conventions applied). See also [6], [5], and [4] for more details and examples of such star products which we shall call star products of Wick type because in flat \(\mathbb{C}^{n}\) they correspond to the Wick ordering (or normal ordering) rule.

The aim of this Letter is twofold: firstly we are going to prove some properties of the operators \(M_t\) occurring in the star products constructed by Fedosov [9]: in particular we prove the (no doubt folklore) statement that these star products are of Weyl type and that the operators \(M_t\) are bidifferential of order \(t\) (Lemma 3.3, Theorem 3.4).

Secondly, motivated by a discussion with Fedosov we transfer his procedure of constructing star products to Kähler manifolds by which we get a relatively simple direct proof of existence of star products of Wick type on each such manifold (Theorem 4.7). Our proof does not make use of analytic techniques usually employed in the literature to obtain star products as an asymptotic expansion of, say, a geometric quantization scheme.
2 Some Notation

We will use Einstein’s summation convention (i.e. summation over repeated indices is understood) and make use of the notation of Fedosov as in \[2\] with the following exceptions: We denote a connection by the symbol $\nabla$ instead of Fedosov’s $\partial$ and use symmetric tensor fields and insertion maps $i_s$ instead of Fedosov’s $y$’s and derivations with respect to $y$.

Let $(M, \omega)$ be a $2n$-dim. symplectic manifold with symplectic form $\omega = \frac{1}{2} \omega_{ij} dx^i \wedge dx^j$ where $(x^1, \ldots, x^{2n})$ are local coordinates. The Poisson tensor corresponding to $\omega$ is defined by $\Lambda = \frac{1}{2} \omega_{ij} \partial_i \wedge \partial_j$ where $\partial_i = \partial/\partial x^i$ and $\Lambda^{ij} \omega_{kj} = \delta^i_k$ according to the sign convention of \[1\]. Then the Poisson bracket of two functions $f, g$ is given by $\Lambda(df, dg)$.

Next one defines the Fedosov algebra $\mathcal{W} \otimes \Lambda$ by

$$\mathcal{W} \otimes \Lambda := (\mathcal{X}_s, \omega^\infty \mathcal{C}(T^*M)) [[\hbar]] (7)$$

and hence elements $a \in \mathcal{W} \otimes \Lambda$ are of the form $a = \sum_{r,s=0}^\infty \hbar^r a_{rs}$ where $a_{rs} \in \mathcal{C}(\mathcal{V}^*T^*M \otimes \wedge T^*M)$ are smooth sections. Then in $\mathcal{W} \otimes \Lambda$ a fibrewise product is declared by the $\vee$-product of symmetric forms in the first factor and the $\wedge$-product of antisymmetric forms in the second factor. For factorizing elements $f \otimes \alpha$ and $g \otimes \beta$ this reads $(f \otimes \alpha)(g \otimes \beta) = f \vee g \otimes \alpha \wedge \beta$. Clearly $\mathcal{W} \otimes \Lambda$ is a formally $\mathbb{Z} \times \mathbb{Z}$-graded algebra with respect to this product in the sense that the symmetric degree map $\deg_s$ and the antisymmetric degree map $\deg_a$ of forms are derivations. We denote by $i_s(X)a$ the insertion (symmetric substitution) of a vector field $X$ in the symmetric part of $a$ and by $i_a(X)a$ the usual insertion (inner product) in the antisymmetric part. Then the fibrewise Weyl product is defined by

$$a \circ b := \sum_{r=0}^\infty \left( \frac{i\hbar}{2} \right)^r \Lambda^{(r)}(a, b)$$

and

$$\Lambda^{(r)}(a, b) := \frac{1}{r!} \Lambda^{i_1 j_1} \cdots \Lambda^{i_r j_r} i_s(\partial_{i_1}) \cdots i_s(\partial_{i_r}) a i_a(\partial_{j_1}) \cdots i_a(\partial_{j_r}) b \quad (8)$$

where $a, b \in \mathcal{W} \otimes \Lambda$. This product is known to be an associative deformation of the fibrewise product. Denote by $\deg_h$ the degree map with respect to the order of the formal parameter $\hbar$ (i.e. $\deg_h(h^k a) = k \hbar^k a$) and define the total degree by $\Deg := 2\deg_h + \deg_s$. Then $(\mathcal{W} \otimes \Lambda, \circ)$ is a formally graded algebra with respect to $\Deg$ and $\deg_a$ again in the sense that $\Deg$ and $\deg_a$ are derivations with respect to the fibrewise Weyl product. The notation $a^{(k)}$ stands for an element in the Fedosov algebra with total degree $k$ and $a^{(k)}_s$ stands for an element with total degree $k$ and symmetric degree $s$. We define the graded $\circ$-commutator by $[a, b]_{\text{Weyl}} := a \circ b - (-1)^{kl} b \circ a$ where $k, l$ are the antisymmetric degrees of $a$ and $b$. Moreover we define $\text{ad}_{\text{Weyl}}(a) := [a, \cdot]_{\text{Weyl}}$ which is a graded derivation of antisymmetric degree $k$ if $\deg_a a = ka$. We denote by $\mathcal{W}$ the elements in $\mathcal{W} \otimes \Lambda$ with vanishing antisymmetric degree and by $\mathcal{W} \otimes \Lambda^a$ the elements with antisymmetric degree $a$. Denote by $C$ the complex conjugation $Ca := \overline{a}$ and by $P_h := (-1)^{\deg_h}$ the $\hbar$-parity. Then clearly $C$ and $P_h$ are $\deg_a$-graded involutions of the algebra $\mathcal{W} \otimes \Lambda$ with respect to $\circ$, i.e. for $a, b \in \mathcal{W} \otimes \Lambda$ with $\deg_a a = ka$, $\deg_a b = lb$ we have $C(a \circ b) = (-1)^{kl} b \circ a$ and analogously for $P_h$.

Now let $M$ be a Kähler manifold. In a holomorphic chart $(z^1, \ldots, z^n)$ the symplectic form can be written as $\omega = \frac{1}{2} \omega_{kj} dz^k \wedge d\overline{z}^j$ with a positive definite Hermitian matrix $\omega_{kj}$. The Poisson tensor is given by $\Lambda = \frac{1}{2} \omega^{kl} Z_k \wedge \overline{Z}_l$ where $\omega^{kl} \omega_{kl} = \delta^l_l$ is the inverse matrix and
$Z_k = \partial/\partial z^k$ and $\overline{Z}_l = \partial/\partial \overline{z}^l$ are local base fields of type $(1, 0)$ and $(0, 1)$. Since $M$ is Kähler we can characterize the forms by their type and we will denote by $\pi_{s}^{(k,l)}$ the projection on symmetric forms of type $(k,l)$ (see [11, II chap. 9, 10] for details). Moreover, in addition to the fibrewise Weyl product we can define a fibrewise Wick product for $a, b \in \mathcal{W} \otimes \Lambda$ by

$$
a \circ' b := \sum_{r=0}^{\infty} \left( \frac{i\hbar}{2} \right)^r \Lambda^{(r)}(a, b)
$$

$\Lambda^{(r)}(a, b) := \frac{1}{r!} \left( \frac{4}{i} \right)^r \omega^{k_1 l_1} \cdots \omega^{k_r l_r} i_s(Z_{k_1}) \cdots i_s(Z_{k_r}) a i_s(\overline{Z}_{l_1}) \cdots i_s(\overline{Z}_{l_r}) b.$

Then $(\mathcal{W} \otimes \Lambda, \circ')$ is again formally graded with respect to $\deg$ and we define graded $\circ'$-commutators analogously as the graded $\circ$-commutators and set $\text{ad}_{\text{Wick}}(a) := [a, \cdot]_{\text{Wick}}$ which is again a graded derivation with respect to the fibrewise Wick product. On a Kähler manifold the fibrewise Weyl and Wick product are fibrewise equivalent in the following sense:

**Lemma 2.1** For $a, b \in \mathcal{W} \otimes \Lambda$ we have

$$a \circ b = S^{-1} (S a \circ' S b)$$

where $S := e^{\hbar \Delta}$ and $\Delta := \omega^l i_s(Z_k) i_s(\overline{Z}_l)$.

If $\ast$ is a star product for the symplectic manifold $M$ then

$$f \ast g = \sum_{r=0}^{\infty} \left( \frac{i\hbar}{2} \right)^r M_r(f, g)$$

for $f, g \in C^\infty(M)$ and the $M_r$ are some bidifferential operators. The star product is said to be of **Weyl type** iff $M_r(f, g) = (-1)^r M_r(g, f)$ and the $M_r$ are real. If $M$ is a Kähler manifold a differential operator $L : f \to L(f)$ is called of type $(1, 0)$ resp. of type $(0, 1)$ iff in a holomorphic chart the function $f$ is only differentiated in holomorphic resp. antiholomorphic directions. This characterization is clearly independent of the holomorphic chart. Then a star product on a Kähler manifold is said to be of **Wick type** iff $M_r$ is of type $(1, 0)$ in the first argument and of type $(0, 1)$ in the second argument for all $r \geq 1$.

### 3 The Fedosov star product of Weyl type

We shall now briefly recall Fedosov’s construction of a star product for an arbitrary symplectic manifold (see [9]). First we have to define some maps. Let $a \in \mathcal{W} \otimes \Lambda$ with $\deg_s a = ka$, $\deg_a a = la$ and let $\nabla$ be an (always existing) symplectic torsion-free connection with curvature tensor $R$.

$$\delta a := (1 \otimes dx^i) i_s(\partial_i) a \quad \delta^{-1} a := \left\{ \begin{array}{ll} 0 & \text{if } k + l = 0 \\ \frac{1}{k+l} (dx^i \otimes 1) i_a(\partial_i) a & \text{else} \end{array} \right.$$  

$$\nabla a := (1 \otimes dx^i) \nabla_{\partial_i} a \quad R := \frac{1}{i} \omega_{ij} R^k_{jkl} dx^i \wedge dx^j \otimes dx^k \wedge dx^l \in \mathcal{W} \otimes \Lambda$$
Then $\delta$ and $\nabla$ turn out to be graded derivations with respect to the fibrewise Weyl product and we have $\delta^2 = (\delta^{-1})^2 = 0$, $\nabla^2 = \frac{i}{\hbar} \text{ad}_{\text{Weyl}}(R)$, $\nabla \delta + \delta \nabla = 0$ and $\delta \delta^{-1} a + \delta^{-1} \delta a = a$ if $\deg a + \deg a \neq 0$. Moreover Fedosov has proved the following two theorems:

**Theorem 3.1** There exists a unique section $r \in \mathcal{W} \otimes \Lambda^1$ with $\delta^{-1} r = 0$ and $\delta r = R + \nabla r + \frac{i}{\hbar} \text{ad}_{\text{Weyl}}(r)$. It can be calculated recursively with respect to the total degree $\text{Deg}$ by

$$
\begin{align*}
    r^{(3)} &= \delta^{-1} R \\
    r^{(k+3)} &= \delta^{-1} \left( \nabla r^{(k+2)} + i \sum_{l=1}^{k-1} r^{(l+2)} \circ r^{(k-l+2)} \right).
\end{align*}
$$

(14)

Then the Fedosov derivation $D := -\delta + \nabla + \frac{i}{\hbar} \text{ad}_{\text{Weyl}}(r)$ has square zero: $D^2 = 0$.

Since $D$ is a graded derivation the kernel of $D$ is a $\circ$-subalgebra and one defines

$$
\mathcal{W}_D := \ker D \cap \mathcal{W}.
$$

(15)

**Theorem 3.2** Let $D$ be constructed as in the last theorem. Then $\mathcal{W}_D$ is in bijection with $C^\infty(M)[[\hbar]]$ and the projection $\sigma : \mathcal{W}_D \to C^\infty(M)[[\hbar]]$ onto the part with symmetric degree zero is a $\mathbb{C}[[\hbar]]$-linear isomorphism. The inverse $\tau = \sigma^{-1}$ for a function $f \in C^\infty(M)$ can be constructed recursively with respect to the total degree $\text{Deg}$ by

$$
\begin{align*}
    \tau(f)^{(0)} &= f \\
    \tau(f)^{(s+1)} &= \delta^{-1} \left( \nabla \tau(f)^{(s)} + i \sum_{t=1}^{s-1} \text{ad}_{\text{Weyl}} \left( r^{(t+2)} \circ \tau(f)^{(s-t)} \right) \right).
\end{align*}
$$

(16)

Then a star product for $M$ is given by $f \ast g := \sigma(\tau(f) \circ \tau(g))$.

Since $f \ast g = \sigma(\tau(f) \circ \tau(g))$ is a star product it can be written in the form (11) with some bidifferential operators $M_r$. Although the following results may generally be believed to be true they do nevertheless not seem to have appeared in the literature:

**Lemma 3.3**

i.) $Cr = r$ and $P_h r = r$ hence $r$ is real and depends only on $\hbar^2$.

ii.) $C \tau = \tau C$ and $P_h \tau = \tau P_h$ and hence $\tau(f)$ depends only on $\hbar^2$ for $f \in C^\infty(M)$.

iii.) $M_s(f, g) = (-1)^s M_s(g, f)$ and hence $\ast$ is of the Weyl type.

iv.) $C(M_s(f, g)) = M_s(Cf, Cg)$ and hence the $M_r$ are real.

v.) Complex conjugation is an antilinear $\ast$-involution: $\overline{f \ast g} = \overline{f} \ast \overline{g}$.

**Theorem 3.4**

i.) For all nonnegative integers $k, l$ with $l \leq \lfloor k/4 \rfloor$ and $k - 4l \geq 1$ if $k \geq 1$ the map $f \mapsto \tau(f)^{(k)}_{k-4l}$ is a differential operator of order $k - 2l$. 


ii.) The Fedosov star product is a Vey star product, i.e. the bidifferential operator $M_s$ is of order $s$ in both arguments and we have the formula

$$M_s(f, g) = \sum_{k=0}^{\frac{s-1}{2}} \sum_{l=0}^{k} (-4)^k A_s 2^{s-2k} \left( \tau(f)_s^{-2k+4l}, \tau(g)_s^{-2k-4l} \right).$$  

(17)

**Proof:** This theorem is proved by lengthy but straightforward induction using the preceding lemma in particular the facts that $r$ and $\tau(f)$ depend only on $\hbar^2$ for $f \in C^\infty(M)$ as well as the recursion formulas for $r$ and $\tau$. □

4 The Fedosov star product of Wick type

Now we consider a Kähler manifold $M$ and investigate the fibrewise Wick product. For the map $\nabla$ we will always use the Kähler connection. Then we notice the following properties of the fibrewise Wick product using lemma 2.1:

**Proposition 4.1** For $a, b \in \mathcal{W} \otimes \Lambda$ we have:

i.) $\text{ad}_{\text{Wick}}(a) = S \circ \text{ad}_{\text{Weyl}}(S^{-1}a) \circ S^{-1}$.

ii.) $\text{ad}_{\text{Wick}}(a) = 0 \iff \text{ad}_{\text{Weyl}}(a) = 0 \iff \text{deg}_s a = 0$

iii.) In the first order of $\hbar$ the graded fibrewise Weyl and the graded fibrewise Wick commutators are the same.

iv.) $[\delta, \Delta] = [\delta, S] = [\delta, S^{-1}] = 0$.

v.) $\delta = -\frac{\iota}{\hbar} \text{ad}_{\text{Wick}}(\tilde{\delta}) = -\frac{\iota}{\hbar} \text{ad}_{\text{Weyl}}(\tilde{\delta})$ with $\tilde{\delta} = \omega_{k^l} d\tilde{z}^k \otimes d\tilde{z}^l + \omega_{k^l} d\tilde{z}^l \otimes d\tilde{z}^k$ and hence $\delta$ is a graded derivation of antisymmetric degree 1 with respect to the fibrewise Wick product.

vi.) $[\nabla, \Delta] = [\nabla, S] = [\nabla, S^{-1}] = 0$. Furthermore $\nabla$ is a graded derivation of antisymmetric degree 1 with respect to the fibrewise Wick product and we have

$$\nabla^2 = \frac{\iota}{\hbar} \text{ad}_{\text{Wick}}(SR) = \frac{\iota}{\hbar} \text{ad}_{\text{Wick}}(R)$$

with $R = \frac{1}{2} i \omega_{k^l} R_{k^l} dz^k \vee d\bar{z}^l \otimes d\bar{z}^l \wedge d\bar{z}^l$ being the same element in $\mathcal{W} \otimes \Lambda$ as in (13).

The proofs can be done easily in a holomorphic chart using the properties of the Kähler connection. Note that $SR = R + \hbar \Delta R$ with $\text{deg}_s \Delta R = 0$ and hence $\text{ad}_{\text{Wick}}(\Delta R) = 0$.

The following two theorems can be proved in a manner completely analogous to the original theorems of Fedosov (Theorem 3.1 and 3.2) if we use the properties of the maps $\delta$, $\nabla$ and the preceding proposition [11].

**Theorem 4.2** There exists a unique section $r' \in \mathcal{W} \otimes \Lambda^1$ with $\delta^{-1} r' = 0$ and $\delta r' = R + \nabla r' + \frac{\iota}{\hbar} r' \circ' r'$. It can be calculated recursively with respect to the total degree $\text{Deg}$ by

$$r'^{(k+3)} = \delta^{-1} R$$

$$r'^{(k)} = \delta^{-1} \left( \nabla r'^{(k+2)} + \frac{\iota}{\hbar} \sum_{l=1}^{k-1} r'^{(l+2)} \circ' r'^{(k-l+2)} \right).$$  

(18)
Then the Fedosov derivation \( D' := -\delta + \nabla + \frac{i}{\hbar} \text{ad}_{\text{Wick}}(r') \) has square zero: \( D'^2 = 0 \).

**Theorem 4.3** Let \( D' \) be constructed as in theorem 4.2. Then \( \mathcal{W}_{D'} \) is again in bijection to \( C^\infty(M)[[\hbar]] \) and the projection \( \sigma : \mathcal{W}_{D'} \to C^\infty(M)[[\hbar]] \) is again an isomorphism. The inverse \( \tau' = \sigma^{-1} \) for a function \( f \in C^\infty(M) \) can be constructed recursively with respect to the total degree \( \text{Deg} \) by

\[
\tau'(f)(0) = f, \\
\tau'(f)(s+1) = \delta^{-1} \left( \nabla\tau'(f)(s) + \frac{i}{\hbar} \sum_{t=1}^{s-1} \text{ad}_{\text{Wick}}(r'(t+2)) \tau'(f)(s-t) \right). \tag{19}
\]

Then a star product for \( M \) is given by \( f \ast' g := \sigma(\tau'(f) \circ' \tau'(g)) \). The maps \( \tau' \cdot(k) \) are differential operators of order \( k \).

We shall call this star product the **Fedosov star product of Wick type** on \( M \). The first two bidifferential operators of this star product are easily seen to take the following form:

\[
f \ast' g = fg + i\hbar \Lambda(\partial f, \overline{g}) + \cdots.
\]

We will now prove that this star product is indeed a star product of Wick type as defined in section 2: to this end we will examine the section \( r' \) and the corresponding maps \( \tau' \) in some more detail:

**Lemma 4.4** With the notations from above we have:

i.) \( Cr' = r' \) and hence \( CD'a = D'Ca \) for \( a \in \mathcal{W} \otimes \Lambda \).

ii.) \( C\tau'(f) = \tau'(Cf) \) for \( f \in C^\infty(M)[[\hbar]] \).

iii.) \( \overline{f \ast' g} = \overline{g} \ast' \overline{f} \) and hence the complex conjugation is a \( \ast' \)-involution too.

iv.) For all nonnegative integers \( k, l \) with \( l \leq \lfloor k/2 \rfloor \) and \( k - 2l \geq 1 \) for all \( k \geq 1 \) the linear map \( f \mapsto \tau'(f)_{k-2l} \) is a differential operator of order \( k - l \).

**PROOF:** Using the recursion formulas for \( r' \) and \( \tau' \) this lemma is proved by a straightforward induction. \( \square \)

**Lemma 4.5** With the notations form above we have for the section \( r' \) and any \( p \geq 0 \)

\[
\pi^{(0,p)}_{r'} = \pi^{(p,0)}_{r'} = 0. \tag{20}
\]

**PROOF:** The crucial point for this lemma is the choice of \( R \) as starting point in the recursion formula (18) and not \( SR = R + \hbar \Delta R \) since \( \text{deg}_s \Delta R = 0 \). Then it is proved again by induction. \( \square \)
Lemma 4.6 Let \( f \in C^\infty(M) \) be a holomorphic function in an open set \( U \subseteq M \) and smooth outside of \( U \). Then we have for \( 0 < p \in \mathbb{N} \)

\[
\pi_s^{(0,p)} \tau'(f) |U = 0. \tag{21}
\]

If on the other hand \( f \) is antiholomorphic in \( U \) then we have

\[
\pi_s^{(p,0)} \tau'(f) |U = 0. \tag{22}
\]

Proof: Again we prove this lemma by induction using the preceding lemma and the recursion formula for the \( \tau'(f) \).

Now we can prove that the star product constructed in theorem 4.3 is indeed a star product of Wick type. Note that in the proof the associativity of \( \ast' \) is crucial.

Theorem 4.7 Let \( \ast' \) be the star product constructed as in theorem 4.3. Then the bidifferential operators \( M'_r \) are of order \( r \) and hence \( \ast' \) is a Vey star product. Moreover we have

\[
M'_r(f, g) = \sum_{k=0}^{s} \sum_{l=0}^{k} (-2i)^k \Lambda^{k \ell(s-k)} \left( \tau'(f)^{(s-k+2l)}, \tau'(g)^{(s-k-2l)} \right)
\]

and the operators \( M'_r \) are of type \((1,0)\) in the first and of type \((0,1)\) in the second argument for all \( r \). Hence the star product is of Wick type. For a function \( f \) holomorphic on the open set \( U \subseteq M \) and for \( g \) antiholomorphic on \( U \) and \( h \) arbitrary we have

\[
(h \ast' f) |U = hf |U \quad (g \ast' h) |U = gh |U. \tag{24}
\]

Proof: The first part is again proved by an analogous induction as in theorem 3.4. Secondly we prove equation (23). Since this statement is local we can work in a holomorphic chart \((z^1, \ldots, z^n)\). Let \( f \) be a holomorphic function in the domain \( U \) of this chart. Then for \( p > 0 \) we have \( \pi_s^{(0,p)} \tau'(f) |U = 0 \) according to the last lemma. This implies that

\[
\pi_s^{(0,0)} \omega^{k_1 \tau_1} \cdots \omega^{k_r \tau_r} i_{s}(Z_{k_1}) \cdots i_{s}(Z_{k_r}) \tau'(h) i_{s}(Z_{l_1}) \cdots i_{s}(Z_{l_r}) \tau'(f) |U = 0
\]

for \( r > 0 \). Hence in \( h \ast' f = \pi_s^{(0,0)} (\tau'(h) o' \tau'(f)) \) only the lowest order \( r = 0 \) of the fibrewise Wick product contributes and this implies that \( h \ast' f = hf \). The antiholomorphic case is proved analogously.

In order to prove that all \( M'_r \) are of type \((1,0)\) in the first argument we use induction on \( r \). For \( M'_0 \) and \( M'_1 \) this is obviously true so let us assume that \( M'_0, \ldots, M'_{r-1} \) are all of type \((1,0)\) in the first argument. Since \( M'_r \) is of order \( r \) it can be written as

\[
M'_r(f, g) = \sum_{|I| + |J| \leq r} M'_{r, IJKL} \frac{\partial^{I+J} f}{\partial z^I \partial \bar{z}^J} \frac{\partial^{|K|+|L|} g}{\partial z^K \partial \bar{z}^L}
\]

with some smooth locally defined functions \( M'_{r, IJKL} \). Now let \( p \) be a point such that for some multi-indices \( I_0, J_0, K_0, L_0 \) we have \( M'_{r, I_0J_0K_0L_0}(p) \neq 0 \). Then we can adjust our holomorphic chart such that \( p = 0 \) in this chart. Now we consider polynomials in the coordinates \( z^{K_0} \bar{z}^{L_0} \).
and \( \overline{z}^{i_0} *' z^{i_0} = \overline{z}^{i_0} z^{i_0} \) since \( z^{i_0} \) is locally holomorphic and we use equation (24). By the associativity of \( *' \) we get the equation

\[
(\overline{z}^{i_0} z^{i_0}) *' (z^{k_0} \overline{z}^{l_0}) = \overline{z}^{i_0} *' (z^{i_0} *' (z^{k_0} \overline{z}^{l_0})) = \overline{z}^{i_0} (z^{i_0} *' (z^{k_0} \overline{z}^{l_0}))
\]

(25)
since \( \overline{z}^{i_0} \) is locally antiholomorphic. Then at \( z = 0 \) the term proportional to \( \hbar^r \) of \( (\overline{z}^{i_0} z^{i_0}) *' (z^{k_0} \overline{z}^{l_0}) \) is equal to \( c M'_{i_0,j_0,k_0,l_0}(0) \neq 0 \) with some positive combinatorical factor \( c \). But this leads to a contradiction if \( |J_0| > 0 \) since then the right hand side in (24) vanishes at 0 but the term proportional to \( \hbar^r \) of the left hand side is equal to \( c M'_{i_0,j_0,k_0,l_0}(0) \neq 0 \) at \( z = 0 \). Hence only holomorphic derivatives can occur in the first argument of \( M'_{r} \). The statement about its second argument is proved analogously. \( \square \)

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References

[1] R. Abraham, J. E. Marsden: Foundations of Mechanics, second edition. (Addison Wesley Publishing Company, Inc., Reading Mass. 1985)

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer: Deformation Theory and Quantization. Annals of Physics 111 (1978), part I: 61-110, part II: 111-151.

[3] F. Berezin: Quantization. Izv.Mat.NAUK 38 (1974), 1109-1165.

[4] M. Bordemann, M. Brischle, C. Emmrich, S. Waldmann: Phase Space Reduction for Star-Products: An Explicit Construction for \( \mathbb{C}P^n \) Lett. Math. Phys. 36 (1996), 357-371.

[5] M. Bordemann, E. Meinrenken, M. Schlichenmaier: Toeplitz Quantization of Kähler Manifolds and \( gl(N), N \to \infty \)-Limits, Comm. Math. Phys. 165 (1994), 281-296.

[6] M. Cahen, S. Gutt, J. Rawnsley: Quantization of Kähler Manifolds. II. Trans.Am.Math.Soc 337 (1993),73-98.

[7] M. DeWilde, P.B.A. Lecomte: Existence of star-products and of formal deformations of the Poisson Lie Algebra of arbitrary symplectic manifolds. Lett. Math. Phys. 7 (1983), 487-496.

[8] B. Fedosov: Quantization and the index. Sov. Phys. Dokl. 31(11) (1986) 877-878.

[9] B. Fedosov: A Simple Geometrical Construction of Deformation Quantization. J. of Diff. Geom. 40 (1994), 213-238.
[10] S. Kobayashi, K. Nomizu: *Foundation of Differential Geometry I, II*. John Wiley & Sons, New York, London 1963, 1969.

[11] S. Waldmann: *Ein Sternprodukt für den komplex projektiven Raum und die Fedosov Konstruktion für Kähler-Mannigfaltigkeiten* (in German), Diploma Thesis, 97 pages, Univ. Freiburg (1995).