A Generalized Vaserstein Symbol

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Abstract
Let $R$ be a ring with $2 \in R^\times$. Then the usual Vaserstein symbol is a map from the orbit space of unimodular rows of length 3 under the action of the group $E_3(R)$ to the elementary symplectic Witt group. Now let $P_0$ be a projective module of rank 2 with trivial determinant. Then we provide a generalized symbol map which is defined on the orbit space of the set of epimorphisms $P_0 \oplus R \to R$ under the action of the group of elementary automorphisms of $P_0 \oplus R$. We also generalize results by Vaserstein and Suslin on the surjectivity and injectivity of the Vaserstein symbol. Finally, we use local-global principles for transvection groups in order to deduce that the generalized Vaserstein symbol is an isomorphism if $R$ is a regular Noetherian ring of dimension 2 or a regular affine algebra of dimension 3 over a field $k$ with $c.d.(k) \leq 1$ and $6 \in R^\times$.

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1 Introduction

Let $R$ be a ring and let $Um_n(R)$ denote the set of unimodular rows of length $n$, i.e. row vectors $(a_1, a_2, ..., a_n)$ such that $\langle a_1, a_2, ..., a_n \rangle = R$. Such row vectors obviously correspond to epimorphisms $R^n \to R$. Therefore the group $GL_n(R)$ of invertible $n \times n$-matrices acts on the right on $Um_n(R)$ (by precomposition); consequently the same holds for any subgroup of $GL_n(R)$, e.g. the group $SL_n(R)$ of invertible $n \times n$-matrices with determinant 1 or its subgroup $E_n(R)$ generated by elementary matrices. Note that the set $Um_n(R)$ has a canonical basepoint given by the row $e_1 = (1, 0, ..., 0)$.

Now assume that 2 is a unit in $R$. Let $n = 3$ and let $(a_1, a_2, a_3)$ be a unimodular row of length 3. By definition, there exist elements $b_1, b_2, b_3 \in R$ such that $\sum_{i=1}^3 a_i b_i = 1$. Therefore the skew-symmetric matrix

$$V(a, b) = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -b_3 & b_2 \\ a_2 & b_3 & 0 & -b_1 \\ a_3 & -b_2 & b_1 & 0 \end{pmatrix}$$

has Pfaffian 1 and represents an element of the so-called elementary symplectic Witt group $W_E(R)$. It was shown in [SV, Lemma 5.1] that this element is independent of the choice of the elements $b_1, b_2, b_3$. Furthermore, it follows from [SV, Theorem 5.2(a)] that this assignment is invariant under the action of $E_3(R)$ on $Um_3(R)$. Therefore one obtains a well-defined map

$$V : Um_3(R)/E_3(R) \to W_E(R)$$

called the Vaserstein symbol. Suslin and Vaserstein also found criteria for this map to be surjective or injective in terms of the right action of $E_n(R)$ on $Um_n(R)$ mentioned above. More precisely, they proved that the Vaserstein symbol is surjective if $Um_{2n+1}(R) = e_1 E_{2n+1}(R)$ for $n \geq 2$ (cp. [SV, Theorem 5.2(b)]) and injective if $E_{2n} = e_1 (E(R) \cap GL_{2n}(R))$ for $n \geq 3$ and $E(R) \cap GL_4(R) = E_4(R)$ (combine [SV, Theorem 5.2(c)] and the proof of [SV, Corollary 7.4]).

These criteria immediately enabled them to deduce that the Vaserstein symbol is a bijection for a Noetherian ring of Krull dimension 2 (cp. [SV, Corollary 7.4]). Using local-global principles, R. Rao and W. van der Kallen could prove in [RvdK, Corollary 3.5] that the Vaserstein symbol is also a bijection for a 3-dimensional regular affine algebra over a field $k$ with $c.d.(k) \leq 1$, which is supposed to be perfect if $\text{char}(k) = 3$. The elementary symplectic Witt group $W_E(R)$ is a subgroup of the group $W'_E(R)$ which is generated by skew-symmetric invertible matrices; $W_E(R)$ then
corresponds to matrices with Pfaffian 1. It is known that the group $W_2^e(R)$ is isomorphic to the higher Grothendieck-Witt group $GW^3_1(R)$ and also to the group $V(R)$ (cp. [FRS]). The latter group is generated by isometry classes of triples $(P, g, f)$, where $P$ is a finitely generated projective $R$-module and $g$ and $f$ are skew-symmetric isomorphisms on $P$ (or, equivalently, non-degenerate skew-symmetric forms on $P$). Under the isomorphism $W_2^e(R) \cong V(R)$ given in [FRS], the group $W_E^e(R)$ corresponds to a subgroup of $V(R)$. We denote this subgroup by $V_1(R)$.

Now let $P_0$ be a finitely generated projective $R$-module of rank 2 with a fixed trivialization $\theta_0 : R \xrightarrow{\cong} \det(P_0)$ of its determinant. We denote by $Um(P_0 \oplus R)$ the set of epimorphisms $P_0 \oplus R \to R$ and by $E(P_0 \oplus R)$ the group of elementary automorphisms of $P_0 \oplus R$. Any element $a : P_0 \oplus R \to R$ of $Um(P_0 \oplus R)$ has a section $s : R \to P_0 \oplus R$, which canonically induces an isomorphism $i : P_0 \oplus R \xrightarrow{\cong} P(a) \oplus R$, where $P(a) = \ker(a)$. We let $\chi_0$ be the skew-symmetric form on $P_0$ which sends a pair $(p, q)$ to the element $\theta_0^{-1}(p \wedge q)$ of $R$; similarly, there is an isomorphism $\theta : R \xrightarrow{\cong} \det(P(a))$ obtained as the composite of $\theta_0$ and the isomorphism $\det(P_0) \cong \det(P(a))$ induced by $a$ and $s$. We then denote by $\chi_a$ the skew-symmetric form on $P(a)$ which sends $(p, q)$ to the element $\theta^{-1}(p \wedge q)$ of $R$. We then consider the element

$$V(a) = [R_0 \oplus R^2, \chi_0 \bot \psi_2, (i \oplus 1)^t(\chi_a \bot \psi_2)(i \oplus 1)]$$

of $V(R)$. Our first result is the following (Theorem 4.1, Lemma 4.2 and Theorem 4.3 in the text):

**Theorem 1.** The element $V(a)$ is independent of the choice of a section $s$ of $a \in Um(P_0 \oplus R)$ and is an element of $V_1(R)$. Furthermore, we have $V(a) = V(\alpha \varphi)$ for all $a \in Um(P_0 \oplus R)$ and $\varphi \in E(P_0 \oplus R)$. Thus, the assignment above descends to a well-defined map $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to V_1(R)$, which we call the generalized Vaserstein symbol (associated to the trivialization $\theta_0$ of $\det(P_0)$).

The terminology is justified by the following observation: If we take $P_0 = R^2$ and let $e_1 = (1, 0)$ and $e_2 = (0, 1)$, then it is well-known that there is a canonical isomorphism $\theta_0 : R \xrightarrow{\cong} \det(R^2)$ given by $1 \mapsto e_1 \wedge e_2$. Then the generalized Vaserstein symbol associated to $-\theta_0$ coincides with the usual Vaserstein symbol via the identification $V_1 \cong W_2^e(R)$ mentioned above.

Of course, any two trivializations of $\det(P_0)$ are equal up to multiplication by a unit of $R$. We will actually make precise how the generalized Vaserstein symbol depends on the choice of a trivialization of $\det(P_0)$ by means of a canonical $R^\times$-action on $V_1(R)$.

We also generalize the criteria found by Suslin and Vaserstein on the injectivity and surjectivity of the Vaserstein symbol. For this, let $P_n = P_0 \oplus R^{n-2}$ for all $n \geq 3$ and let $E_\infty(P_0)$ be the direct limit of the groups $E(P_n)$ for $n \geq 3$. Note that $Um(P_n)$ has a canonical base-point given by the projection $\pi_{n,0}$ on the "last" free direct summand of rank 1. We then prove (Theorem 4.5 and
Theorem 4.14 in the text):

**Theorem 2.** The Vaserstein symbol $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \rightarrow V_1(R)$ is surjective if $Um(P_{2n+1}) = \pi_{2n+1,2n+1}E(P_{2n+1})$ for all $n \geq 2$. Furthermore, it is injective if $\pi_{2n,2n}E(P_{2n}) = \pi_{2n,2n}(E_\infty(P_0) \cap Aut(P_{2n}))$ for all $n \geq 3$ and $E_\infty(P_0) \cap Aut(P_4) = E(P_4)$.

Using local-global principles for transvection groups (cp. [BBR]), we may then prove the following result (Theorems 2.15, 2.16 and 4.15 in the text):

**Theorem 3.** The equality $E_\infty(P_0) \cap Aut(P_4) = E(P_4)$ holds if $R$ is a 2-dimensional regular Noetherian ring or if $R$ is a 3-dimensional regular affine algebra over a field $k$ such that $c.d.(k) \leq 1$ and $6 \in k^\times$. In particular, it follows that the generalized Vaserstein symbol $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \rightarrow V_1(R)$ is a bijection in these cases.

We remark that the corresponding result for the usual Vaserstein symbol in dimension 3 was crucially used in the proof of [FRS, Theorem 7.5] in order to deduce that stably free modules of rank $d-1$ are free over smooth affine $k$-algebras of dimension $d$ whenever $k$ is algebraically closed and $(d-1)! \in k^\times$. They also used Suslin’s result that unimodular rows of the form $(a_1,a_2,a_3^2)$ are completable to invertible matrices. An explicit completion of such a unimodular row is given e.g. in [Kr]. In fact, we can translate this result to our setting: Any epimorphism $a : P_0 \oplus R \rightarrow R$ can be written as $(a_0,a_R)$, where $a_0$ and $a_R$ are the restrictions of $a$ to $P_0$ and $R$ respectively. Then we can generalize Krusemeyer’s construction in order to give an explicit completion of an epimorphism of the form $a = (a_0,a_R^2)$ to an automorphism of $P_0 \oplus R$ (cp. Proposition 4.18).

The organization of the paper is as follows: In Section 2 we basically prove the technical ingredients for the proofs of the main results of this paper. In particular, we prove some lemmas on elementary automorphisms of projective modules and use local-global principles for transvection groups in order to derive stability results for automorphism groups of projective modules. Section 3 basically covers the definition of the elementary symplectic Witt group $W_E(R)$ and the identifications of $W'_E(R)$, $V(R)$ and $GW^1_3(R)$. In Section 4 we motivate and give the definition of the generalized Vaserstein symbol and begin to study its basic properties. We will then use all the technical lemmas proven in the previous sections in order to deduce the theorems stated above.

**Notation and Conventions**

In this paper, a ring $R$ will always be commutative with unit. If not otherwise stated, we will furthermore assume that $2 \in R^\times$. We will not use this assumption for most of the proofs in Section 2. Throughout Sections 3 and 4 however, we will assume that $2 \in R^\times$. This is mainly due to our use of the theory of higher Grothendieck-Witt groups and skew-symmetric forms on projective modules. If $k$ is a perfect field, we will denote by $\mathcal{H}(k)$ the $\mathbb{A}^1$-homotopy category as
defined by Morel and Voevodsky and by $\mathcal{H}_•(k)$ its pointed version. If $\mathcal{X}$ and $\mathcal{Y}$ are spaces (resp. pointed spaces), we will write $[\mathcal{X}, \mathcal{Y}]_{\mathcal{A}^1}$ (resp. $[\mathcal{X}, \mathcal{Y}]_{\mathcal{A}^1, •}$) for the set of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ in $\mathcal{H}(k)$ (resp. $\mathcal{H}_•(k)$).

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2 Preliminaries on projective modules

In this section, we recall some basic facts on projective modules over commutative rings and prove some technical lemmas on elementary automorphisms which will be largely used in the proofs of the main results of this paper. We also recall the local-global principle for transvection groups in [BBR] in order to prove a stability result on automorphisms of projective modules. At the end of this section, we briefly recall how projective modules stably isomorphic to a given projective module $P$ can be classified in terms of the orbit space of the set of epimorphisms $P \oplus R \to R$ under the action of the groups of automorphisms of $P \oplus R$.

2.1 Local trivializations and skew-symmetric isomorphisms on projective modules

Let $R$ be a commutative ring and $P$ be any projective $R$-module. For any prime ideal $\mathfrak{p}$ of $R$, the localized $R_\mathfrak{p}$-module $P_\mathfrak{p}$ is again projective and therefore free (because projective modules over local rings are free). In this weak sense, projective modules are locally free. If the rank of $P_\mathfrak{p}$ as an $R_\mathfrak{p}$-module is finite for every prime $\mathfrak{p}$, then we say that $P$ is a projective module of finite rank.

In this case, there is a well-defined map $\text{rank}_P : \text{Spec}(R) \to \mathbb{Z}$ which sends a prime ideal $\mathfrak{p}$ of $R$ to the rank of $P_\mathfrak{p}$ as an $R_\mathfrak{p}$-module. It is not true in general that projective modules of finite rank are finitely generated; nevertheless, this is true if $\text{rank}_P$ is a constant map (cp. [W] Chapter I, Ex. 2.14)). We will say that $P$ is locally free of finite rank (in the strong sense) if it admits elements $f_1, ..., f_n \in R$ generating the unit ideal such that the localizations $P_{f_k}$ are free $R_{f_k}$-modules of finite rank. In fact, it is well-known that this is true if and only
if $P$ is a finitely generated projective module. The following lemma follows from [W, Chapter I, Lemma 2.4] and [W, Chapter I, Ex. 2.11]:

**Lemma 2.1.** Let $R$ be a ring and $M$ be an $R$-module. Then the following statements are equivalent:

1. $M$ is a finitely generated projective $R$-module;
2. $M$ is locally free of finite rank (in the strong sense);
3. $M$ is a finitely presented $R$-module and $M_p$ is a free $R_p$-module for every prime ideal $p$ of $R$;
4. $M$ is a finitely generated $R$-module, $M_p$ is a free $R_p$-module for every prime ideal $p$ of $R$ and the induced map $\text{rank}_M : \text{Spec}(R) \to \mathbb{Z}$ is continuous.

Now let $R$ be a ring with $2 \in R^\times$. For any projective $R$-module $P$ of finite rank, there is a canonical isomorphism

$$\text{can} : P \to P^{\nu \nu}, \ p \mapsto (\text{ev}_p : P^{\nu} \to R, a \mapsto a(p))$$

induced by evaluation. A symmetric isomorphism on $P$ is an isomorphism $f : P \to P^{\nu}$ such that the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{f} & P^{\nu} \\
\downarrow{\text{can}} & & \downarrow{\text{id}} \\
P^{\nu \nu} & \xrightarrow{f^{\nu}} & P^{\nu}
\end{array}
$$

is commutative. Similarly, a skew-symmetric isomorphism on $P$ is an isomorphism $f : P \to P^{\nu}$ such that the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{f} & P^{\nu} \\
\downarrow{-\text{can}} & & \downarrow{\text{id}} \\
P^{\nu \nu} & \xrightarrow{f^{\nu}} & P^{\nu}
\end{array}
$$

is commutative.

A symmetric form on a projective $R$-module $P$ of finite rank is an $R$-bilinear map $\chi : P \times P \to R$ such that $\chi(p, q) = \chi(q, p)$ for all $p, q \in P$. Similarly, a skew-symmetric form on a projective $R$-module $P$ of finite rank is an $R$-bilinear map $\chi : P \times P \to R$ such that $\chi(p, q) = -\chi(q, p)$ for all $p, q \in P$. The form $\chi$ satisfies a strong non-degeneracy property if the map $P \to P^{\nu}, p \mapsto (q \mapsto \chi(p, q))$ is an isomorphism. We will refer to such forms as non-degenerate (skew-)symmetric forms or simply (skew-)symmetric inner products on $P$. Obviously, the data of a (skew-)symmetric isomorphism is equivalent to the data of a (skew-)symmetric
inner product.

Now let \( \chi : M \times M \to R \) be any \( R \)-bilinear form on \( M \). This form induces a homomorphism \( M \otimes_R M \to R \). For any prime \( p \) of \( R \), there is an induced homomorphism \( M_p \otimes_R M_p \cong (M \otimes_R M)_p \to R_p \). This gives an \( R \)-bilinear form \( \chi_p : M_p \times M_p \to R_p \) on \( M_p \). The following lemma shows that these localized forms completely determine \( \chi \):

**Lemma 2.2.** If \( \chi_1 \) and \( \chi_2 \) are \( R \)-bilinear forms on an \( R \)-module \( M \). Then \( \chi_1 = \chi_2 \) if and only if \( \chi_{1p} = \chi_{2p} \) for every prime ideal \( p \) of \( R \).

**Proof.** The forms \( \chi_1 \) and \( \chi_2 \) agree if and only if \( \chi_1(p, q) = \chi_2(p, q) = 0 \) for all \( p, q \in M \). Therefore the lemma follows immediately from the fact that being 0 is a local property for elements of any \( R \)-module. \( \square \)

### 2.2 Elementary automorphisms and unimodular elements

Again, let \( R \) be a ring and let \( M \cong \bigoplus_{i=1}^n M_i \) be an \( R \)-module which admits a decomposition into a direct sum of \( R \)-modules \( M_i, i = 1,...,n \). An elementary automorphism \( \varphi \) of \( M \) with respect to the given decomposition is an endomorphism of the form \( \varphi_{s_{ij}} = id_M + s_{ij} \), where \( s_{ij} : M_j \to M_i \) is an \( R \)-linear homomorphism for some \( i \neq j \) (cp. \[HB\] Chapter IV, §3]). Any such homomorphism automatically is an isomorphism with inverse given by \( \varphi^{-1}_{s_{ij}} = id_M - s_{ij} \).

For \( M = \bigoplus_{i=1}^n R \), one just obtains the automorphisms given by elementary matrices. We denote by \( Aut(M) \) the group of automorphisms of \( M \) and by \( E(M_1,...,M_n) \) (or simply \( E(M) \) if the decomposition is understood) the subgroup of \( Aut(M) \) generated by elementary automorphisms.

The following lemma gives a list of useful formulas which can be checked easily by direct computation:

**Lemma 2.3.** Let \( M = \bigoplus_{i=1}^n M_i \) be a direct sum of \( R \)-modules. Then we have

\( a) \varphi_{s_{ij}} \varphi_{t_{ij}} = \varphi(s_{ij} + t_{ij}) \) for all \( s_{ij} : M_j \to M_i, t_{ij} : M_j \to M_i \) and \( i \neq j \);

\( b) \varphi_{s_{ij}} \varphi_{s_{kl}} = \varphi_{s_{kl}} \varphi_{s_{ij}} \) for all \( s_{ij} : M_j \to M_i, s_{kl} : M_l \to M_k, i \neq j, k \neq l, j \neq k, i \neq l \);

\( c) \varphi_{s_{ij}} \varphi_{s_{jk}} \varphi_{s_{kj}} \varphi_{s_{ji}} = \varphi_{(s_{ij} s_{jk})} \) for all \( s_{ij} : M_j \to M_i, s_{jk} : M_k \to M_j \) and distinct \( i,j,k \);

\( d) \varphi_{s_{ij}} \varphi_{s_{ki}} \varphi_{s_{kj}} \varphi_{s_{ki}} = \varphi_{(s_{ki} s_{ji})} \) for all \( s_{ij} : M_j \to M_i, s_{ki} : M_i \to M_k \) and distinct \( i,j,k \).

If we restrict to the case \( M_i = R \) for \( i \geq 2 \), we obtain the following result on \( E(M) \):

**Corollary 2.4.** If \( M_i = M_n \) for \( i \geq 2 \), then the group \( E(M) \) is generated by the elementary automorphisms of the form \( \varphi_s = id_M + s \), where \( s \) is an \( R \)-linear map \( M_i \to M_n \) or \( M_n \to M_i \) for some \( i \neq n \). The same statement holds one replaces \( n \) by any other \( k \geq 2 \).
Proof. Since $M_i = M_n$ for all $i \geq 2$, we have identities $id_{M_i} : M_n \to M_i$ and $id_{M_i} : M_i \to M_n$ for all $i \geq 2$. Let $s_{ij} : M_j \to M_i$ be a morphism with $i \neq j$ and therefore either $i \geq 2$ or $j \geq 2$. We may assume that $i, j, n$ are distinct. If $i \geq 2$, then

$$\varphi_{s_{ij}} = \varphi_{id_n} \varphi_{id_{s_{ij}}} \varphi_{-id_n} \varphi(-id_{n} s_{ij})$$

by the third formula in Lemma 2.3. If $j \geq 2$, then

$$\varphi_{s_{ij}} = \varphi_{s_{ij} id_{n}} \varphi_{id_n} \varphi(-s_{ij} id_{n}) \varphi - id_n.$$

by the third formula in Lemma 2.3. This proves the first part of the corollary. The last part follows in the same way if $n$ is replaced by $k \geq 2$. □

The proof of Corollary 2.4 also shows:

**Corollary 2.5.** Let $M = \bigoplus_{i=1}^{n} M_i$ be a direct sum of $R$-modules and also let $s : M_j \to M_i$, $i \neq j$, an $R$-linear map. Assume that there is $k \neq i$ with $M_k = M_i$ or $k \neq j$ with $M_k = M_j$. Then the induced elementary automorphism $\varphi_s$ is a commutator.

The following lemma is a version of Whitehead’s lemma in our general setting:

**Lemma 2.6.** Let $M = M_1 \oplus M_2$ and let $f : M_1 \to M_2$, $g : M_2 \to M_1$ be morphisms. Assume that $id_{M_1} + gf$ is an automorphism of $M_1$. Then:

- $id_{M_2} + fg$ is an automorphism of $M_2$ and
- $(id_{M_1} + gf) \oplus (id_{M_2} + fg)^{-1}$ is an element of $E(M_1 \oplus M_2)$

Proof. We have $id_{M_1} \oplus (id_{M_2} + fg) = \varphi_f \varphi_{-g}((id_{M_1} + gf) \oplus id_{M_2}) \varphi_f \varphi_g$. This shows the first statement. For the second statement one checks that

$$(id_{M_1} + gf) \oplus (id_{M_2} + fg)^{-1} = \varphi_f \varphi_{-g} \varphi_{g} (id_{M_1} + gf)^{-1} \varphi_{g} \varphi_f + f.$$

So $(id_{M_1} + gf) \oplus (id_{M_2} + fg)^{-1}$ lies in $E(M_1 \oplus M_2)$. □

Now let $P$ be a finitely generated projective $R$-module. We denote by $Um(P)$ the set of epimorphisms $P \to R$. The group $Aut(P)$ of automorphisms of $P$ then acts on the right on $Um(P)$; consequently, the same holds for any subgroup of $Aut(P)$. In particular, it holds for the subgroup $SL(P)$ of automorphisms of determinant 1 and, if we fix a decomposition $P \cong \bigoplus_{i=1}^{n} P_i$, for the group $E(P) = E(P_1, ..., P_n)$ as well.

An element $p \in P$ is called unimodular if there is an $a \in Um(P)$ such that $a(p) = 1$; this means that the morphism $R \to P, 1 \mapsto p$ defines a section for the epimorphism $a$. We denote by $Unim.El.(P)$ the set of unimodular elements of $P$. Note that the group $Aut(P)$ and hence also $SL(P)$ and $E(P)$ act on the left on $P$; these actions restrict to actions on $Unim.El.(P)$.

The canonical isomorphism $can : P \to P^\nu$ identifies the set of unimodular elements $Unim.El.(P)$ of $P$ with the set $Um(P^\nu)$ of epimorphisms $P^\nu \to R$, i.e.
an element \( p \in P \) is unimodular if and only if \( ev_p : P^\nu \to R \) is an epimorphism. Furthermore, if \( p \) and \( q \) are unimodular elements of \( P \) and \( \varphi \in Aut(P) \) with \( \varphi(p) = q \), then \( ev_p \varphi^\nu = ev_q : P^\nu \to R \).

We therefore obtain a well-defined map

\[
Unim.El.(P)/Aut(P) \to Um(P^\nu)/Aut(P^\nu).
\]

Let us show that this map is actually a bijection. Since the map is automatically surjective, it only remains to show that it is injective. So let \( \psi \in Aut(P^\nu) \) such that \( ev_p \psi = ev_q \). One can easily check that the map \( Aut(P) \to Aut(P^\nu) \), \( \varphi \mapsto \varphi^\nu \), is bijective; hence \( \psi = \varphi^\nu \) for some \( \varphi \in Aut(P) \). Thus, we obtain \( ev_q = ev_p \varphi^\nu = ev_{\varphi(p)} \) and therefore \( \varphi(p) = q \), because \( \text{can} : P \to P^\nu \) is injective. Altogether, we obtain a bijection

\[
Unim.El.(P)/Aut(P) \cong \to Um(P^\nu)/Aut(P^\nu).
\]

In particular, if \( P \cong P^\nu \), then \( Unim.El.(P)/Aut(P) \cong Um(P)/Aut(P) \).

We introduce some notation. Let \( P_0 \) be a finitely generated projective \( R \)-module.

For any \( n \geq 3 \), let \( P_n = P_0 \oplus Re_3 \oplus \ldots \oplus Re_n \) be the direct sum of \( P_0 \) and free \( R \)-modules \( Re_i \), \( 3 \leq i \leq n \), of rank 1 with explicit generators \( e_i \). We denote by \( \pi_{k,n} : P_n \to R \) the projections onto the free direct summands of rank 1 with index \( k = 3, \ldots, n \). For any non-degenerate skew-symmetric form \( \chi \) on \( P_{2n} \), \( n \geq 2 \), we define \( Sp(\chi) = \{ \varphi \in Aut(P_{2n}) | \varphi^\chi \varphi = \chi \} \).

For \( n \geq 3 \), we have embeddings \( Aut(P_n) \to Aut(P_{n+1}) \) and \( E(P_n) \to E(P_{n+1}) \). We denote by \( Aut_\infty(P_0) \) (resp. \( E_\infty(P_0) \)) the direct limits of the groups \( Aut(P_n) \) (resp. \( E(P_n) \)) via these embeddings.

In the following lemmas, we denote by \( \psi_2 \) the non-degenerate skew-symmetric form on \( R^2 \) given by the matrix

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

Thus, for any non-degenerate skew-symmetric form \( \chi \) on \( P_{2n} \) for some \( n \geq 2 \), we obtain a non-degenerate skew-symmetric form on \( P_{2n+2} \) given by the orthogonal sum \( \chi \perp \psi_2 \).

With this notation in mind, we may now state and prove a few lemmas which provide the technical groundwork in the proofs of the main results in this paper:

**Lemma 2.7.** Let \( \chi \) be a non-degenerate skew-symmetric form on \( P_{2n} \) for some \( n \geq 2 \). Let \( p \in P_{2n-1} \) and \( a : P_{2n-1} \to R \). Then there are \( \varphi, \psi \in Aut(P_{2n-1}) \) such that

- the morphism \((\varphi \oplus 1)(id_{P_{2n}} + p\pi_{2n,2n})\) is an element of \( E(P_{2n}) \cap Sp(\chi) \)

- the morphism \((\psi \oplus 1)(id_{P_{2n}} + ae_{2n})\) is an element of \( E(P_{2n}) \cap Sp(\chi) \).
Proof. We let \( \Phi : P_{2n} \to P_{2n}' \) be the skew-symmetric isomorphism induced by \( \chi \) and \( \Phi^{-1} \) be its inverse.

For the first part, we introduce the following homomorphisms: Let \( d \) be the morphism \( R \to P_{2n-1} \) which sends 1 to \( \Phi^{-1}(\pi_{2n,2n}) \) (note that because of \( \pi_{2n,2n}(\Phi^{-1}(\pi_{2n,2n})) = \chi(\Phi^{-1}(\pi_{2n,2n}), \Phi^{-1}(\pi_{2n,2n})) = 0 \) it can be considered an element of \( P_{2n-1} \)). Furthermore, let \( \nu = \chi(p,-) : P_{2n-1} \to R \). We observe that \( \nu d = 0 \). By Lemma 2.8, the morphism \( \varphi = id_{P_{2n-1}} - d\nu \) is an automorphism and \( \varphi \oplus 1 \) is an elementary automorphism. In particular, \( (\varphi \oplus 1)(id_{P_{2n}} + p\pi_{2n,2n}) \) is an elementary automorphism. In light of the proof of [SV, Lemma 5.4] and Lemma 2.2, one can check locally that it also lies in \( Sp(\chi) \).

For the second part, we introduce the following homomorphisms: We will denote \( c = \chi(-,e_{2n}) : P_{2n-1} \to R \). Furthermore, we let \( a \oplus 0 : P_{2n} \to R \) be the extension of \( a \) to \( P_{2n} \) which sends \( e_{2n} \) to 0; then we denote by \( \vartheta \) the homomorphism \( R \to P_{2n-1} \) which sends 1 to \( \Phi^{-1}(a \oplus 0) \). Note that \( c\vartheta = 0 \). Again by Lemma 2.6, the morphism \( \psi = id_{P_{2n-1}} - c\vartheta \) is an automorphism and \( \psi \oplus 1 \) is an elementary automorphism. In particular, \( (\psi \oplus 1)(id_{P_{2n}} + ae_{2n}) \) is an elementary automorphism as well. Again, in light of the proof of [SV, Lemma 5.4] and Lemma 2.2, one can check locally that it also lies in \( Sp(\chi) \).

\[\text{Lemma 2.8.} \text{ Let } \chi \text{ be a non-degenerate skew-symmetric form on the module } P_{2n} \text{ for some } n \geq 2. \text{ Then } E(P_{2n})e_{2n} = (E(P_{2n}) \cap Sp(\chi))e_{2n}.\]

\[\text{Proof.} \text{ Let } p \in E(P_{2n})e_{2n}. \text{ We can write } (\alpha_1...\alpha_r)(p) = e_{2n}, \text{ where each } \alpha_i \text{ is one of the generators given in Corollary 2.4. We show by induction on } r \text{ that } p \in (E(P_{2n}) \cap Sp(\chi))e_{2n}. \text{ If } r = 0, \text{ there is nothing to show. So let } r \geq 1. \text{ Lemma 2.7 shows that there is } \gamma \in Aut(P_{2n-1}) \text{ such that } (\gamma \oplus 1)\alpha_r \text{ lies in } (E(P_{2n}) \cap Sp(\chi))e_{2n}. \text{ We set } \beta_i = (\gamma \oplus 1)\alpha_i(\gamma^{-1} \oplus 1) \text{ for each } i < r. \text{ Each of the } \beta_i \text{ lies in } E(P_{2n}) \text{ and } (\beta_1...\beta_{r-1}(\gamma \oplus 1)\alpha_r)(p) = e_{2n}. \text{ This enables us to conclude by induction.}\]

\[\text{Lemma 2.9.} \text{ Let } \chi_1 \text{ and } \chi_2 \text{ be non-degenerate skew-symmetric forms on } P_{2n} \text{ such that } \varphi'(\chi_1 \perp \psi_2)\varphi = \chi_2 \perp \psi_2 \text{ for some } \varphi \in E_{\infty}(P_0) \cap Aut(P_{2n+2}). \text{ Now let } \chi = \chi_1 \perp \psi_2. \text{ If } (E_{\infty}(P_0) \cap Aut(P_{2n+2}))e_{2n+2} = (E_{\infty}(P_0) \cap Sp(\chi))e_{2n+2} \text{ holds, then one has } \psi'\chi_2\psi = \chi_1 \text{ for some } \psi \in E_{\infty}(P_0) \cap Aut(P_{2n}).\]

\[\text{Proof.} \text{ Let } \psi''e_{2n+2} = \varphi e_{2n+2} \text{ for some } \psi'' \in E_{\infty}(P_0) \cap Sp(\chi). \text{ Then we set } \psi' = (\psi'')^{-1}\varphi. \text{ Since } (\psi')'(\chi_1 \perp \psi_2)\psi' = \chi_2 \perp \psi_2, \text{ the restriction of } \psi' \text{ to } P_{2n}, \text{ denoted } \psi, \text{ satisfies the following conditions:}\]

- \( \psi'(\chi_1 \perp \psi_2) = \chi_2; \)
- \( \psi'(e_{2n+2}) = e_{2n+2}; \)
- \( \pi_{2n+1,2n+2}\psi' = \pi_{2n+1,2n+2}; \)

The last condition implies that \( \psi \) equals \( \psi' \) up to elementary morphisms and \( \psi \in E_{\infty}(P_0) \cap Aut(P_{2n}), \text{ which finishes the proof.}\]
Lemma 2.10. Assume that $\pi_{2n+1,2n+1}(E_{\infty}(P_0) \cap Aut(P_{2n+1})) = Um(P_{2n+1})$ holds for some $n \in \mathbb{N}$. Then for any non-degenerate skew-symmetric form $\chi$ on $P_{2n+2}$ there exists an automorphism $\varphi \in E_{\infty}(P_0) \cap Aut(P_{2n+2})$ such that $\varphi^t \varphi = \psi \perp \psi_2$ for some non-degenerate skew-symmetric form $\psi$ on $P_{2n}$.

Proof. Let $d = \chi(-,e_{2n+2}) : P_{2n+1} \to R$. Since $d$ can be locally checked to be an epimorphism, there is an automorphism $\varphi' \in E_{\infty}(P_0) \cap Aut(P_{2n+1})$ such that $d\varphi' = \pi_{2n+1,2n+1}$. Then the skew-symmetric form $\chi' = (\varphi'^t \perp 1)\chi(\varphi' \perp 1)$ satisfies that $\chi'(-,e_{2n+2}) : P_{2n+1} \to R$ is just $\pi_{2n+1,2n+1}$. Now we simply define $c = \chi'(-,e_{2n+1}) : P_{2n+1} \to R$ and let $\varphi_c = id_{P_{2n+2}} + ce_{2n+2}$ be the elementary automorphism on $P_{2n+2}$ induced by $c$; then $\varphi_c^t \chi' \varphi_c = \psi \perp \psi_2$ for some non-degenerate skew-symmetric form $\psi$ on $P_{2n}$, as desired. \hfill $\Box$

Lemma 2.11. Let $P_0$ be a finitely generated projective $R$-module of rank 2. Then we have $E(P_0 \oplus R) \subset SL(P_0 \oplus R)$. Furthermore, if $\varphi \in SL(P_0 \oplus R)$, then the induced morphism $\varphi_* : det(P_0 \oplus R) \to det(P_0 \oplus R)$ is the identity on $det(P_0 \oplus R)$.

Proof. If $\varphi \in SL(P_0 \oplus R)$ and $p$ is any prime ideal of $R$, then $\varphi_p$ will obviously correspond to an elementary automorphism of $(P_0)_p \oplus R_p$. Choosing any isomorphism $(P_0)_p \cong R_p^2$, it will therefore correspond to an element of $E_3(R_p) \subset SL_3(R_p)$. Thus, $E(P_0 \oplus R) \subset SL(P_0 \oplus R)$, as desired. Since being 0 is a local property, the second statement can also be checked locally. Again, choosing any isomorphism $(P_0)_p \cong R_p^2$, the automorphism $\varphi_p$ will by assumption correspond to an element of $SL_3(R_p)$. But since for any automorphism of $R_p^2$ the induced automorphism on $det(R_p^2)$ is just multiplication by its determinant, the second statement follows immediately. \hfill $\Box$

2.3 The local-global principle for transvection groups

We will now briefly review the local-global principle for transvection groups proven in [BBR] and use it in order to deduce stability results for stably elementary automorphisms of $P_0 \oplus R^2$. For this, we only have to assume that $R$ is an arbitrary commutative ring with unit.

First of all, let $P$ be a finitely generated projective $R$-module and $q \in P$, $\varphi \in P^t$ such that $\varphi(q) = 0$. This data naturally determines a homomorphism $\varphi_q : P \to P$ by $\varphi_q(p) = \varphi(p)q$ for all $p \in P$. An automorphism of the form $id_P + \varphi_q$ is called a transvection if either $q \in Unim.El.(P)$ or $\varphi \in Um(P)$. We denote by $T(P)$ the subgroup of $Aut(P)$ generated by transvections.

Now let $Q = P \oplus R$ be a direct sum of a finitely generated projective $R$-module $P$ and the free $R$-module of rank 1. Then the elementary automorphisms of $P \oplus R$ are easily seen to be transvections and are also called elementary transvections. Consequently, we have $E(Q) \subset T(Q) \subset Aut(Q)$.

In the theorem stated below, we denote by $R[X]$ the polynomial ring in one variable over $R$ and let $Q[X] = Q \otimes_R R[X]$. The evaluation homomorphisms $ev_0, ev_1 : R[X] \to R$ induce maps $Aut(Q[X]) \to Aut(Q)$. If $\alpha(X) \in Aut(Q[X])$,
Theorem 2.14. Let $k$ be a fixed field.

Proof. We set $\tau$ for some $C$-dimensional ring in $\mathfrak{m}$, then we denote its images under these maps by $\alpha(0)$ and $\alpha(1)$ respectively. Similarly, the localization homomorphism $R \to R_{\mathfrak{m}}$ at any maximal ideal $\mathfrak{m}$ of $R$ induces a map $Aut(Q[X]) \to Aut(Q_{\mathfrak{m}}[X])$, where $Q_{\mathfrak{m}}[X] = Q[X] \otimes_{R[X]} R_{\mathfrak{m}}[X]$; if $\alpha(X) \in Aut(Q[X])$, we denote its image under this map by $\alpha_{\mathfrak{m}}(X)$.

We will use the following result proven by Bak, Basu and Rao (cp. $[BBR$, Theorems 3.6 and 3.10]):

**Theorem 2.12.** There is an equality $E(Q) = T(Q)$. If $\alpha(X) \in Aut(Q[X])$ satisfies $\alpha(0) = id_Q \in Aut(Q)$ and $\alpha_{\mathfrak{m}}(X) \in E(Q_{\mathfrak{m}}[X])$ for all maximal ideals $\mathfrak{m}$ of $R$, then $\alpha(X) \in E(Q[X])$.

In order to prove the desired stability results, we introduce the following property: Let $C$ be either the class of Noetherian rings or the class of affine $k$-algebras over a fixed field $k$. Furthermore, let $d \geq 1$ be an integer and $m \in \mathbb{N}$. We say that $C$ has the property $\mathcal{P}(d, m)$ if for $R$ in $C$ of dimension $d$ and for any finitely generated projective $R$-module $P$ of rank $m$ the group $SL(P \oplus R^n)$ acts transitively on $Um(P \oplus R^n)$ for all $n \geq 2$.

If $k$ is a field, we simply say that $k$ has the property $\mathcal{P}(d, m)$ if the class of affine $k$-algebras has the property $\mathcal{P}(d, m)$.

Of course, if the class of Noetherian rings has the property $\mathcal{P}(d, m)$, then the same holds for every field. The class of Noetherian rings has the properties $\mathcal{P}(d, 1)$ and $\mathcal{P}(d, m)$ for $m \geq d$. Furthermore, it follows from $[B]$ that any field $k$ of cohomological dimension $\leq 1$ satisfies property $\mathcal{P}(d, d - 1)$ if $(d - 1)! \in k^\times$.

In the remainder of this section, we will denote by $\pi$ the canonical projection $P \oplus R^n \to R$ onto the "last" free direct summand of $R^n$.

**Lemma 2.13.** Let $C$ be the class of Noetherian rings or affine $k$-algebras over a fixed field $k$. Assume that $C$ has the property $\mathcal{P}(d, m)$. Let $R$ be a $d$-dimensional ring in $C$, $P$ a projective $R$-module of rank $m$ and $a \in Um(P \oplus R^n)$ for some $n \geq 2$. Moreover, assume that there is an element $t \in R$ and a homomorphism $w : P \oplus R^n \to R$ such that $a - \pi = tw$. Then there is an automorphism $\varphi$ of $P \oplus R^n$ such that $a = \pi \varphi$ and $\varphi(x) \equiv id_{P \oplus R^n}(x)$ modulo $(t)$ for all $x$.

**Proof.** We set $B = R[X]/(X^2 - tX)$. By assumption, we have $a = \pi + tw$. We lift it to $\alpha(X) = \pi + Xw : (P \oplus R^n) \otimes_R B \to B$, which can be checked to be an epimorphism (as in the proof of $[RvdK$, Proposition 3.3]). Therefore we have $a(X) \in Um((P \oplus R^n) \otimes_R B)$. Since $B$ still is a ring in $C$ of dimension $d$, property $\mathcal{P}(d, m)$ now gives an element $\varphi(X) \in SL((P \oplus R^n) \otimes_R B)$ with $a(X) = \pi \varphi(X)$. Then $\varphi = \varphi(0)^{-1} \varphi(t)$ is the desired automorphism. \hfill $\square$

For any $n \geq 2$, we say that two automorphisms $\varphi, \psi$ of $P \oplus R^n$ are isotopic if there is an automorphism $\tau(X)$ of $(P \oplus R^n) \otimes_R R[X]$ such that $\tau(0) = \varphi$ and $\tau(1) = \psi$.

**Theorem 2.14.** Let $C$ be the class of Noetherian rings or affine $k$-algebras over a fixed field $k$. Assume that $C$ has the property $\mathcal{P}(d + 1, m + 1)$. Let $R$ be a $d$-dimensional ring in $C$, $P$ a projective $R$-module of rank $m$ and $\sigma \in Aut(P \oplus R^n)$ for some $n \geq 2$. Assume that $\sigma \oplus 1 \in E(P \oplus R^{n+1})$. Then $\sigma$ is isotopic to $id_{P \oplus R^{n+1}}$. Moreover, if $R$ is regular, then $\sigma \in E(P \oplus R^n)$. 


Proof. Since $\sigma + 1 \in E(P \oplus R^{n+1})$, it is clear that there is a natural isotopy $\tau(X) \in E((P \oplus R^{n+1}) \otimes_R R[X])$ with $\tau(0) = id_{P \oplus R^{n+1}}$ and $\tau(1) = \sigma + 1$. Now apply the previous lemma to $R[X]$, $X^2 - X$ and $a = \pi \tau(X)$ in order to obtain an automorphism $\chi \in SL((P \oplus R^{n+1}) \otimes_R R[X])$ with $\chi(X) = a$ such that $\chi(X)(x) \equiv x$ modulo $(X^2 - X)$. Thus, $\pi \tau(X) \chi(X)^{-1} = \pi$. Therefore $\tau(X) \chi(X)^{-1}$ equals $\rho(X) \oplus 1$ for some $\rho(X) \in SL((P \oplus R^n) \otimes_R R[X])$ up to elementary automorphisms. But then $\rho(X)$ is an isotopy from $id_{P \oplus R^n}$ to $\sigma$.

Now assume that $R$ is regular. By a famous theorem of Vorst (cp. [5]), we know that $SL_N(R_p[X]) = E_N(R_p[X])$ for any prime $p$ of $R$ and $N \in \mathbb{N}$. Thus, we can then apply Theorem 2.15 to obtain that $\rho(X) \in E((P \oplus R^n) \otimes_R R[X]))$. Hence also $\sigma = \rho(1) \in E(P \oplus R^n)$.

Theorem 2.14 immediately implies the following stability results:

**Theorem 2.15.** With the notation of Section 2.2, we further assume that $P_0$ has rank 2. If $R$ is a regular Noetherian ring of dimension 2, then there is an equality $E_\infty(P_0) \cap Aut(P_4) = E(P_4)$.

Proof. We apply Theorem 2.15 to $P = P_0$.

**Theorem 2.16.** With the notation of Section 2.2, we further assume that $P_0$ has rank 2. Let $k$ be a field with $P(4,3)$. If $R$ is a regular affine $k$-algebra of dimension 3, then $E_\infty(P_0) \cap Aut(P_4) = E(P_4)$.

Proof. We apply Theorem 2.15 to $P = P_0$.

**Theorem 2.17.** With the notation of Section 2.2, we further assume that $P_0$ has rank 2. Let $k$ be a field with $P(5,3)$. If $R$ is a regular affine $k$-algebra of dimension 4, then $E_\infty(P_0) \cap Aut(P_4) = E(P_4)$.

Proof. Again, we apply Theorem 2.15 to $P = P_0$.

### 2.4 Classification of stably isomorphic projective modules

We consider the map

$$\phi_n : V_n(R) \to V_{n+1}(R), [P] \mapsto [P \oplus R],$$

from isomorphism classes of rank $n$ projective modules to rank $n + 1$ projective modules and fix a projective module $P \oplus R$ representing an element of $V_n(R)$ in the image of this map. An element $[P']$ of $V_n(R)$ lies in the fiber over $[P \oplus R]$ if and only if there is an isomorphism $i : P' \oplus R \overset{\cong}{\to} P \oplus R$. Any such isomorphism yields an element of $Um(P \oplus R)$ given by the composite

$$a(i) : P \oplus R \overset{i^{-1}}{\to} P' \oplus R \overset{\pi_R}{\to} R.$$

Note that if one chooses another module $P''$ representing the isomorphism class of $P'$ and any isomorphism $j : P'' \oplus R \overset{\cong}{\to} P \oplus R$, the resulting element $a(j)$ of $Um(P \oplus R)$ still lies in the same orbit of $Um(P \oplus R)/Aut(P \oplus R)$: For if we choose an isomorphism $k : P' \overset{\cong}{\to} P''$, then we have an equality
Thus, we obtain a well-defined map
\[ \phi_n^{-1}([P \oplus R]) \to Um(P \oplus R)/\text{Aut}(P \oplus R). \]

Conversely, any element of \( a \in Um(P \oplus R) \) gives an element of \( V_n(R) \) lying over \([P \oplus R]\), namely \([P'] = [\ker(a)]\). Note that the kernels of two epimorphisms \( P \oplus R \to R \) are isomorphic if these epimorphisms are in the same orbit in \( Um(P \oplus R)/\text{Aut}(P \oplus R) \). Thus, we also obtain a well-defined map
\[ Um(P \oplus R)/\text{Aut}(P \oplus R) \to \phi_n^{-1}([P \oplus R]). \]

One can easily check that the maps \( \phi_n^{-1}([P \oplus R]) \to Um(P \oplus R)/\text{Aut}(P \oplus R) \) and \( Um(P \oplus R)/\text{Aut}(P \oplus R) \to \phi_n^{-1}([P \oplus R]) \) are inverse to each other. Note that \([P]\) corresponds to the class represented by the canonical projection \( \pi_R : P \oplus R \to R \) under these bijections. In conclusion, we have a pointed bijection between the sets \( Um(P \oplus R)/\text{Aut}(P \oplus R) \) and \( \phi_n^{-1}([P \oplus R]) \) equipped with \([\pi_R]\) and \([P]\) as basepoints respectively. Moreover, we also obtain a (pointed) surjection \( Um(P \oplus R)/E(P \oplus R) \to \phi_n^{-1}([P \oplus R]). \)

### 3 The elementary symplectic Witt group

In this section, we briefly recall the definition of the so-called elementary symplectic Witt group \( W'_E(R) \). Primarily, it appears as the kernel of a homomorphism \( W'_E(R) \to R^\times \) induced by the Pfaffian of invertible skew-symmetric matrices. As we will discuss, the group \( W'_E(R) \) itself can be identified with a group denoted \( V(R) \) and with \( GW_3^1(R) \), a higher Grothendieck-Witt group of \( R \). We will also prove some lemmas on the group \( V(R) \), which will be used to prove the main results of this paper. Furthermore, we introduce a canonical \( R^\times \)-action on \( V(R) \) and identify this action with an action of \( R^\times \) on \( GW_3^1(R) \) coming from the multiplicative structure of higher Grothendieck-Witt groups.

#### 3.1 The group \( W'_E(R) \)

Let \( R \) be a ring with \( 2 \in R^\times \). For any \( n \in \mathbb{N} \), we denote by \( A_{2n}(R) \) the group of skew-symmetric invertible matrices of rank \( 2n \). We inductively define an element \( \psi_{2n} \in A_{2n}(R) \) by setting
\[
\psi_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
and \( \psi_{2n+2} = \psi_{2n} \perp \psi_2 \). For any \( m < n \), there is an embedding of \( A_{2m}(R) \) into \( A_{2n}(R) \) given by \( M \mapsto M \perp \psi_{2n-2m} \). We denote by \( A(R) \) the direct limit of the sets \( A_{2n}(R) \) under these embeddings. Two skew-symmetric invertible matrices \( M \in A_{2m}(R) \) and \( N \in A_{2n}(R) \) are called equivalent, \( M \sim N \), if there is an integer \( s \in \mathbb{N} \) and a matrix \( E \in E_{2n+2m+2s} \) such that
\[ M \perp \psi_{2n+2s} = E_t(N \perp \psi_{2m+2s})E. \]

The set of equivalence classes \( A(R)/\sim \) is denoted \( W'_E(R) \). Since

\[
\begin{pmatrix}
0 & id_r \\
-id_r & 0
\end{pmatrix} \in E_{r+s}(R)
\]

for even \( rs \), it follows that the orthogonal sum equips \( W'_E(R) \) with the structure of an abelian monoid. As it is shown in [SV], this abelian monoid is actually an abelian group. An inverse for an element of \( W'_E(R) \) represented by a matrix \( N \in A_{2n}(R) \) is given by the element represented by the matrix \( \sigma_{2n}N^{-1}\sigma_{2n} \), where the matrices \( \sigma_{2n} \) are inductively defined by

\[ \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

and \( \sigma_{2n+2} = \sigma_{2n} \perp \sigma_2 \). Now recall that one can assign to any skew-symmetric invertible matrix \( M \) an element \( Pf(M) \) of \( R^\times \) called the Pfaffian of \( M \). The Pfaffian satisfies the following formulas:

- \( Pf(M \perp N) = Pf(M)Pf(N) \) for all \( M \in A_{2m}(R) \) and \( N \in A_{2n}(R) \)
- \( Pf(G^tNG) = det(G)Pf(N) \) for all \( G \in GL_{2n}(R) \) and \( N \in A_{2n}(R) \)
- \( Pf(N)^2 = det(N) \) for all \( N \in A_{2n}(R) \)
- \( Pf(\psi_{2n}) = 1 \) for all \( n \in \mathbb{N} \)

Therefore the Pfaffian determines a group homomorphism \( Pf : W'_E(R) \to R^\times \); its kernel is denoted \( W_E(R) \) and is called the elementary symplectic Witt group of \( R \).

### 3.2 The group \( V(R) \)

Again, let \( R \) be a ring with \( 2 \in R^\times \). Consider the set of triples \( (P,g,f) \), where \( P \) is a finitely generated projective \( R \)-module and \( f,g \) are skew-symmetric isomorphisms. Two such triples \( (P,f_0,f_1) \) and \( (P',f'_0,f'_1) \) are called isometric if there is an isomorphism \( h : P \to P' \) such that \( f_i = h^\nu f'_i h \) for \( i = 0, 1 \). We denote by \([P,g,f]\) the isometry class of the triple \( (P,g,f) \).

Let \( V(R) \) be the quotient of the free abelian group on isometry classes of triples as above by the subgroup generated by the relations

- \([P \oplus P',g \perp g',f \perp f'] = [P,g,f] + [P',g',f'] \) for skew-symmetric isomorphisms \( f,g \) on \( P \) and \( f',g' \) on \( P' \)
- \([P,f_0,f_1] + [P,f_1,f_2] = [P,f_0,f_2] \) for skew-symmetric isomorphisms \( f_0,f_1,f_2 \) on \( P \).

Note that these relations yield the useful identities
\[ [P, f] = 0 \text{ in } V(R) \text{ for any skew-symmetric isomorphism } f \text{ on } P, \]

\[ [P, g, f] = -[P, f, g] \text{ in } V(R) \text{ for skew-symmetric isomorphisms } f, g \text{ on } P, \]

\[ [P, g, \beta^\nu \alpha^\nu f \alpha \beta] = [P, f, \alpha^\nu] + [P, g, \beta^\nu f \beta] \text{ in } V(R) \text{ for all automorphisms } \alpha, \beta \text{ of } P \text{ and skew-symmetric isomorphisms } f, g \text{ on } P. \]

We may also restrict this construction to free \( R \)-modules of finite rank. The corresponding group will be denoted \( V_{\text{free}}(R) \). Note that there is an obvious group homomorphism \( V_{\text{free}}(R) \to V(R) \).

This homomorphism can be seen to be an isomorphism as follows: For any finitely generated projective \( R \)-module \( P \), we call

\[ H_P = \begin{pmatrix} 0 & id_P \\ -\text{can} & 0 \end{pmatrix} : P \oplus P^\nu \to P^\nu \oplus P^{\nu \nu} \]

the hyperbolic isomorphism on \( P \).

Now let \((P, g, f)\) be a triple as above. Since \( P \) is a finitely generated projective \( R \)-module, there is another \( R \)-module \( Q \) such that \( P \oplus Q \cong R^n \) for some \( n \in \mathbb{N} \).

In particular, \( P \oplus P^\nu \oplus Q \oplus Q^\nu \) is free of rank \( 2n \). Therefore the triple

\[ (P \oplus P^\nu \oplus Q \oplus Q^\nu, g \perp \text{can } g^{-1} \perp H_Q, f \perp \text{can } g^{-1} \perp H_Q) \]

represents an element of \( V_{\text{free}}(R) \); this element is independent of the choice of \( Q \). It follows that the assignment

\[ (P, g, f) \mapsto (P \oplus P^\nu \oplus Q \oplus Q^\nu, g \perp \text{can } g^{-1} \perp H_Q, f \perp \text{can } g^{-1} \perp H_Q) \]

induces a well-defined group homomorphism

\[ V(R) \to V_{\text{free}}(R). \]

Since

\[ [P, g, f] = [P \oplus P^\nu \oplus Q \oplus Q^\nu, g \perp \text{can } g^{-1} \perp H_Q, f \perp \text{can } g^{-1} \perp H_Q] \]

in \( V(R) \), this homomorphism is actually inverse to the canonical morphism \( V_{\text{free}}(R) \to V(R) \). Thus, \( V_{\text{free}}(R) \cong V(R) \).

In order to discuss the identification of \( V(R) \) with the group \( W'_E(R) \) described in the previous section, we first need to prove Lemma 3.1 and Corollaries 3.2 and 3.3 below. They will also be used in the proofs of the main results of this paper.

**Lemma 3.1.** Let \( P = \bigoplus_{i=1}^nP_i \) be a finitely generated projective module and \( f_i \) skew-symmetric isomorphisms on \( P_i \), \( i = 1, \ldots, n \). Let \( f = f_1 \perp \ldots \perp f_n \).

Then \([P, f, \varphi^\nu f \varphi] = 0 in V(R) for any element \varphi of the commutator subgroup of Aut(P).

In particular, the same holds for every element of \( E(P) \) with respect to the given decomposition.
Proof. By the third of the useful identities listed above, we have
\[
[P, f, \varphi_2 \varphi_1 f \varphi_2] = [P, f, \varphi_2 f \varphi_1] + [P, f, \varphi_2 f \varphi_2].
\]
Therefore we only have to prove the first statement for commutators. Now if \( \varphi = \varphi_1 \varphi_2 \varphi_1^{-1} \varphi_2^{-1} \) is a commutator, then the formula above yields
\[
[P, f, \varphi'^\prime f \varphi] = [P, f, \varphi'^\prime f \varphi_1] + [P, f, \varphi_2 f \varphi_2] + [P, f, (\varphi_1^{-1})' f \varphi_1^{-1}] + [P, f, (\varphi_2^{-1})' f \varphi_2^{-1}] = 0,
\]
which proves first part of the lemma.

For the second part, observe that by the formula above we only need to prove the statement for elementary automorphisms. So let \( \varphi_2 \) be the elementary automorphism induced by \( s : P_i \to P_i \). Since we can add the summand \([P, f, f_i] = 0\), we may assume that we are in the situation of Corollary \(2.3\). Therefore we may assume that \( \varphi_2 \) is a commutator and the second statement then follows from the first part of the lemma.

Corollary 3.2. Let \( P \) be a finitely generated projective \( R \)-module and \( \chi \) be a skew-symmetric isomorphism on \( P \). Then \( [P \oplus R^{2n}, \chi \perp \psi_{2n}, \varphi'^\prime (\chi \perp \psi_{2n}) \varphi] = 0 \) in \( V(R) \) for any elementary automorphism \( \varphi \) of \( P \oplus R^{2n} \). In particular, if \( f \) is any skew-symmetric isomorphism on \( P \oplus R^{2n} \), it follows that there is an equality \( [P \oplus R^{2n}, \chi \perp \psi_{2n}, \varphi'^\prime f \varphi] = [P \oplus R^{2n}, \chi \perp \psi_{2n}, f] \) in \( V(R) \).

Proof. The first part follows directly from the previous lemma. The second part is then a direct consequence of the second relation given in the definition of the group \( V(R) \).

Corollary 3.3. For an arbitrary elementary matrix \( E \in E_{2n}(R) \), we have \( [R^{2n}, \psi_{2n}, E^t \psi_{2n} E] = 0 \) in \( V(R) \). In particular, for any alternating matrix \( N \in GL_{2n}(R) \), we have \( [R^{2n}, \psi_{2n}, N] = [R^{2n}, \psi_{2n}, E^t N E] \) in \( V(R) \).

Using the previous corollary, the group \( V_{free}(R) \) can be identified with \( W_E'(R) \) as follows: If \( M \in A_{2n}(R) \) represents an element of \( W_E'(R) \), then we assign to it the class in \( V_{free}(R) \) represented by \([R^{2n}, \psi_{2n}, M]\). By Corollary 3.3 this assignment descends to a well-defined homomorphism \( \nu : W_E'(R) \to V_{free}(R) \).

Now let us describe the inverse \( \xi : V_{free}(R) \to W_E'(R) \) to this homomorphism. Let \( (L, g, f) \) be a triple with \( L \) free and \( g, f \) skew-symmetric isomorphisms on \( L \).

We can choose an isomorphism \( \alpha : R^{2n} \cong L \) and consider the skew-symmetric isomorphism
\[
(\alpha^t f \alpha) \perp \sigma_{2n}(\alpha^t g \alpha)^{-1} \sigma_{2n} : R^{2n} \oplus (R^{2n})' \to (R^{2n})' \oplus R^{2n}.
\]

With respect to the standard basis of \( R^{2n} \) and its dual basis of \((R^{2n})'\), we may interpret this skew-symmetric isomorphism as an element of \( A_{4n}(R) \) and consider its class \( \xi([L, g, f]) \) in \( W_E'(R) \). It is proven in [FRS] that this assignment induces a well-defined homomorphism \( \xi : V_{free}(R) \to W_E'(R) \). By construction, \( \nu \) and \( \xi \) are obviously inverse to each other and therefore identify \( W_E'(R) \) with \( V_{free}(R) \).
In order to conclude this section, let us now describe some group actions on $V(R)$. For any finitely generated projective $R$-module $P$, skew-symmetric isomorphism $\chi : P \to P^\nu$ and $u \in R^\times$, the morphism $u \cdot \chi : P \to P^\nu$ is again a skew-symmetric isomorphism on $P$. Note that $u \cdot \chi$ is canonically isometric to the skew-symmetric isomorphism $u \otimes \chi : R \otimes P \to R \otimes P^\nu \cong (R \otimes P)^\nu$ and we therefore have an equality

$$[P, u \cdot \chi_2, u \cdot \chi_1] = [R \otimes P, u \otimes \chi_2, u \otimes \chi_1] \text{ in } V(R)$$

for all $\chi_1, \chi_2$. One can check easily that the assignment

$$(u, (P, \chi_2, \chi_1)) \mapsto (P, u \cdot \chi_2, u \cdot \chi_1)$$

descends to a well-defined action of $R^\times$ on $V(R)$. More generally, let $\varphi : Q \to Q^\nu$ be a symmetric isomorphism on a finitely generated projective $R$-module $Q$. Then, for any skew-symmetric isomorphism $\chi$ on $P$ as above, the homomorphism $\varphi \otimes \chi : Q \otimes P \to Q^\nu \otimes P^\nu \cong (Q \otimes P)^\nu$ is again a skew-symmetric isomorphism on $Q \otimes P$. One can check easily that the assignment

$$(Q, \varphi, (P, \chi_2, \chi_1)) \mapsto (Q \otimes P, \varphi \otimes \chi_2, \varphi \otimes \chi_1)$$

induces a well-defined action of the Grothendieck-Witt group $GW(R) = GW^0_R(R)$ of $R$ on $V(R)$.

### 3.3 Grothendieck-Witt groups

In this section, we recall some basics about the theory of higher Grothendieck-Witt groups, which are a modern version of Hermitian $K$-theory. The general references of the modern theory are [MS1], [MS2] and [MS3]. We still assume $R$ to be a ring such that $2 \in R^\times$. Then we consider the category $P(R)$ of finitely generated projective $R$-modules and the category $C^b(R)$ of bounded complexes of objects in $P(R)$. The category $C^b(R)$ inherits a natural structure of an exact category from $P(R)$ by declaring $C^\cdot = C^\cdot \to C^\cdot \to C^\cdot$ exact if and only if $C^\cdot \to C^\cdot \to C^\cdot$ is exact for all $n$. The duality $\text{Hom}_R(-, L)$ associated to any line bundle $L$ induces a duality $\#_L$ on $C^b(R)$ and the identification of a finitely generated projective $R$-module with its double dual induces a natural isomorphism of functors $\varpi_L : \text{id} \cong \#_L$ on $C^b(R)$. Moreover, the translation functor $T : C^b(R) \to C^b(R)$ yields new dualities $\#^T_L = T \#_L$ and natural isomorphisms $\varpi^T_L = (-1)^{j(j+1)/2} \varpi_L$. We say that a morphism in $C^b(R)$ is a weak equivalence if and only if it is a quasi-isomorphism. For all $j$, the quadruple $(C^b(R), \text{qis}, \#^T_L, \varpi^T_L)$ is an exact category with weak equivalences and strong duality (cp. [MS2] §2.3). Following [MS2], one can associate a Grothendieck-Witt space $GW$ to any exact category with weak equivalences and strong duality. The (higher) Grothendieck-Witt groups are then defined to be its homotopy groups.
Definition 3.4. For any $i \geq 0$, let $\mathcal{GW}(C^b(R), q_{is}, \#^1_L, \varpi^1_L)$ be the Grothendieck-Witt space associated to the quadruple $(C^b(R), q_{is}, \#^1_L, \varpi^1_L)$ as above. Then we define $GW^j_i(R, L) = \pi_i \mathcal{GW}(C^b(R), q_{is}, \#^1_L, \varpi^1_L)$. If $L = R$, then we set $GW^j_i(R) = GW^j_i(R, L)$.

The groups $GW^j_i(R, L)$ are 4-periodic in $j$ and coincide with Hermitian $K$-theory and $U$-theory as defined by Karoubi (cp. [MK1] and [MK2]), at least if $2 \in R^\times$ (cp. [MS1] Remark 4.13) and [MS3] Theorems 6.1-2). In particular, we have isomorphisms $K_1O(R) = GW^0_1(R)$, $\pi_1U_1(R) = GW^1_1(R)$, $K_1Sp(R) = GW^2_1(R)$ and $U_1(R) = GW^3_1(R)$.

The group of our particular interest is $GW^3_1(R) = U_1(R)$. Indeed, it is argued in [ERS] that there is a natural isomorphism $GW^3_1(R) \cong V_{\text{free}}(R) \cong V(R)$. The Grothendieck-Witt groups defined as above carry a multiplicative structure. Indeed, the tensor product of complexes induces product maps

$$GW^j_i(R, L_1) \times GW^s(R, L_2) \to GW^j+s_{i+r}(R, L_1 \otimes L_2)$$

for all $i, j, r, s$ and lines bundles $L_1, L_2$ (cp. [MS3] §9.2)). In general, it is (probably) difficult to give explicit descriptions of this multiplicative structure; nevertheless, if we restrict ourselves to smooth algebras over a perfect field $k$ (with $\text{char}(k) \neq 2$), then it is known (cp. [MH] Theorem 3.1) that Grothendieck-Witt groups are representable in the (pointed) $\mathbb{A}^1$-homotopy category $\mathcal{H}_k$ as defined by Morel and Voevodsky. As a matter of fact, if we let $R$ be a smooth $k$-algebra over a perfect field $k$ and $X = \text{Spec}(R)$, it is shown that there are spaces $\mathcal{GW}^j$ such that

$$[S^1_+ \wedge X_+, \mathcal{GW}^j]_{\mathbb{A}^1, *_{\times}} = GW^j_i(R),$$

i.e. the spaces $\mathcal{GW}^j$ represent the higher Grothendieck-Witt groups. In order to make these spaces more explicit, we consider for all $n \in \mathbb{N}$ the closed embeddings $GL_n \to O_{2n}$ and $GL_n \to Sp_{2n}$ given by

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & (M^{-1})^t \end{pmatrix}.$$ 

These embeddings are compatible with the standard stabilization embeddings $GL_n \to GL_{n+1}$, $O_{2n} \to GL_{2n+2}$ and $Sp_{2n} \to Sp_{2n+2}$. Taking direct limits over all $n$ with respect to the induced maps $O_{2n}/GL_n \to O_{2n+2}/GL_{n+1}$ and $Sp_{2n}/GL_n \to Sp_{2n+2}/GL_{n+1}$, we obtain spaces $O/GL$ and $Sp/GL$. Similarly, the natural inclusions $Sp_{2n} \to GL_{2n}$ are compatible with the standard stabilization embeddings and we obtain a space $GL/Sp = \text{colim}_n GL_{2n}/Sp_{2n}$. As proven in [ST] Theorems 8.2 and 8.4, there are canonical $\mathbb{A}^1$-weak equivalences

$$\mathcal{GW}^j \cong \begin{cases} \mathbb{Z} \times OGr & \text{if } j \equiv 0 \text{ mod } 4 \\ O/GL & \text{if } j \equiv 1 \text{ mod } 4 \\ \mathbb{Z} \times HGr & \text{if } j \equiv 2 \text{ mod } 4 \\ Sp/GL & \text{if } j \equiv 3 \text{ mod } 4 \end{cases}$$
Following \cite{AF2}, we refer to this action as the conjugation action of $R$ for any ring $GW$.

Now let us examine the product map $a$ perfect field.

homotopy-theoretic interpretation of this action in case of a smooth algebra over $V$.

We describe an action of $A$ precisely $GL$.

Altogether, we obtain an isomorphism $[X,A]_\lambda = GW_1^3(R)$ and $[X,A]_\lambda$ is precisely $A(R)/ \sim = W'_E(R)$.

We describe an action of $\mathbb{G}_m$ on $GL/Sp$. For any ring $R$ and any unit $u \in R^\times$, we denote by $\gamma_{2n,u}$ the invertible $2n \times 2n$-matrix inductively defined by

$$
\gamma_{2,u} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}
$$

and $\gamma_{2n+2,u} = \gamma_{2n,u} \gamma_{2,u}$. Conjugation by $\gamma^{-1}_{2n,u}$ induces an action of $\mathbb{G}_m$ on $GL_{2n}$ for all $n$. As $Sp_{2n}$ is preserved by this action, there is an induced action on $GL_{2n}/Sp_{2n}$. Since all the morphisms $GL_{2n}/Sp_{2n} \to GL_{2n+2}/Sp_{2n+2}$ are equivariant for this action, we obtain an action of $\mathbb{G}_m$ on $GL/Sp$.

Using the isomorphism $GL/Sp \cong A$ described above, there is an induced action of $R^\times = \mathbb{G}_m(R)$ on $GW_1^3(R) \cong A(R)/ \sim = W'_E(R)$ for any smooth $k$-algebra $R$ by taking $A^1$-homotopy classes of morphisms. If $M \in A_{2n}(R)$ represents a class in $W'_E(R)$ and $u$ is a unit of $R$, this action is given by

$$(u, M) \mapsto \gamma^{-1}_{2n,u} M^t (u \cdot \psi_{2n}) M \gamma^{-1}_{2n,u}.$$ 

Note that the isometry induced by the matrix $\gamma_{2n,u}$ yields an equality

$$[R^{2n}, \psi_{2n}, \gamma^{-1}_{2n,u} M^t (u \cdot \psi_{2n}) M \gamma^{-1}_{2n,u}] = [R^{2n}, u \cdot \psi_{2n}, M^t (u \cdot \psi_{2n}) M]$$

in $V(R)$. As a consequence, the action of $R^\times$ on $GW_1^3(R)$ can be described via the isomorphism $GW_1^3(R) \cong V(R)$ as follows: If $(P,g,f)$ is a triple as in the definition of the group $V(R)$ and $u \in R^\times$, then the action is given by

$$(u,(P,g,f)) \mapsto (P, u \cdot g, u \cdot f).$$

Following \cite{AF2}, we refer to this action as the conjugation action of $R^\times$ on $GW_1^3(R) \cong V(R)$. Recall the we have already defined an action of $R^\times$ on $V(R)$ for any ring $R$ with $2 \in R^\times$ in section 3.2. The conjugation action is thus a homotopy-theoretic interpretation of this action in case of a smooth algebra over a perfect field.

Now let us examine the product map

$$GW_0^0(R) \times GW_1^3(R) \to GW_1^3(R)$$

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for smooth $k$-algebras. As described above, there is a canonical isomorphism $GW_3^0 = K_0 O(R) = GW(R)$, where $GW(R)$ is the Grothendieck completion of the abelian monoid of non-degenerate symmetric bilinear forms over $R$. Furthermore, there is a canonical map

$$R^\times \to GW(R), \ u \mapsto (R \times R \to R, (x, y) \mapsto uxy),$$

which induces an action of $R^\times = \mathbb{G}_m(R)$ on $GW_3^1(R)$ via the product map mentioned above. Again following [AF2], we refer to this action as the multiplicative action of $R^\times = \mathbb{G}_m(R)$ on $GW_3^1(R) \cong V(R)$. It follows from the proof of [AF1] Proposition 3.5.1 that the multiplicative action coincides with the conjugation action. Therefore we obtain another interpretation of the $R^\times$-action on $V(R)$ given in section 3.2 via the multiplicative structure of higher Grothendieck-Witt groups.

### 4 Main results

We finally give the definition of the generalized Vaserstein symbol in this section. As a first step, we recall the definition of the usual Vaserstein symbol introduced in [SV] and reinterpret it by means of the isomorphism $W_E^1(R) \cong V(R)$ discussed in the previous section. Then we define the generalized symbol and study its basic properties. In particular, we find criteria for the generalized Vaserstein symbol to be injective and surjective (onto the subgroup $V_1(R)$ of $V(R)$ corresponding to $W_E^1(R)$), which are the natural generalizations of the criteria found in [SV] Theorem 5.2. These criteria will enable us to prove that the generalized Vaserstein symbol is a bijection e.g. for 2-dimensional Noetherian rings and for 3-dimensional regular affine algebras over algebraically closed fields such that $6 \in k^\times$.

#### 4.1 The Vaserstein symbol for unimodular rows

For the rest of the section, let $R$ be a ring with $2 \in R^\times$. Let $Um_3(R)$ be its set of unimodular rows of length 3, i.e. triples $a = (a_1, a_2, a_3)$ of elements in $R$ such that there are elements $b_1, b_2, b_3 \in R$ with $\sum_{i=1}^3 a_i b_i = 1$. This data determines an exact sequence of the form

$$0 \to P(a) \to R^3 \xrightarrow{\alpha} R \to 0,$$

where $P(a) = ker(a)$. The triple $b = (b_1, b_2, b_3)$ gives a section to the epimorphism $a : R^3 \to R$ and induces a retraction $r : R^3 \to P(a), e_i \mapsto e_i - a_i b$, where $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. One then obtains an isomorphism $i = r + a : R^3 \to P(a) \oplus R$, which induces an isomorphism $det(R^3) \to det(P(a) \oplus R)$. Finally, by composing with the canonical isomorphisms $det(P(a) \oplus R) \cong det(P(a))$ and $R \to det(R^3), 1 \mapsto e_1 \wedge e_2 \wedge e_3$, one obtains an isomorphism $\theta : R \to det(P(a))$.

The matrix
therefore an isomorphism

\[ V(a, b) = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -b_3 & b_2 \\ a_2 & b_3 & 0 & -b_1 \\ a_3 & -b_2 & b_1 & 0 \end{pmatrix} \]

has Pfaffian 1 and its image in \( W_E(R) \) does not depend on the choice of the section \( b \). We therefore obtain a well-defined map \( V : Um_3(R) \to W_E(R) \) called the Vaserstein symbol.

Now let us reinterpret the Vaserstein symbol map in light of the isomorphism \( W'_E(R) \cong V(R) \) discussed in section \( \text{[12]} \). The symbol \( V(a) \) is sent to the element of \( V(R) \) represented by the isometry class \([R^4, \psi_4, V(a, b)]\). If we denote by \( \chi_a \) the skew-symmetric form \( P(a) \times P(a) \to R, (p, q) \to \theta^{-1}(p \wedge q) \), we obtain a skew-symmetric form on \( R^4 \) given by \((i \oplus 1)^t(\chi_a \perp \psi_2)(i \oplus 1)\). If we set

\[ \sigma = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in E_4(R), \]

then one can check that the form \((i \oplus 1)^t(\chi_a \perp \psi_2)(i \oplus 1)\) is given by the matrix \( \sigma^tV(a, b)^t\sigma \). In particular, if we let \( M : U m_3(R) \to U m_3(R) \) be the map which sends a unimodular row \( a = (a_1, a_2, a_3) \) to \( M(a) = (-a_1, -a_2, -a_3) \), then the map \( \nu \circ V \circ M \) is given by \( a \mapsto [R^4, \psi_4, (i \oplus 1)^t(\chi_a \perp \psi_2)(i \oplus 1)] \). Since both \( M \) and \( \nu \) are bijections, these considerations lead to a generalization of the Vaserstein symbol.

### 4.2 The generalized Vaserstein symbol

Now let \( P_0 \) be a projective \( R \)-module of rank 2. We will use the notation of section \( \text{[22]} \). For all \( n \geq 3 \), let \( P_n = P_0 \oplus R e_3 \oplus \ldots \oplus R e_n \) be the direct sum of \( P_0 \) and free \( R \)-modules \( R e_i \), \( 3 \leq i \leq n \), of rank 1 with explicit generators \( e_i \). We will sometimes omit these explicit generators in the notation. We denote by \( \pi_{k,n} : P_n \to R \) the projections onto the free direct summands of rank 1 with index \( k = 3, \ldots, n \).

We assume that \( P_0 \) admits a trivialization \( \theta_0 : R \to det(P_0) \) of its determinant. Then we denote by \( \chi_0 \) the non-degenerate skew-symmetric form on \( P_0 \) given by \( P_0 \times P_0 \to R, (p, q) \mapsto \theta_0^{-1}(p \wedge q) \).

Now let \( Um(P_0 \oplus R) \) be the set of epimorphism \( P_0 \oplus R \to R \). Any element \( a \) of \( Um(P_0 \oplus R) \) gives rise to an exact sequence of the form

\[ 0 \to P(a) \to P_0 \oplus R \xrightarrow{a} R \to 0, \]

where \( P(a) = ker(a) \). Any section \( s : R \to P_0 \oplus R \) of \( a \) determines a canonical retraction \( r : P_0 \oplus R \to P(a) \) given by \( r(p) = p - sa(p) \) and an isomorphism \( i : P_0 \oplus R \to P(a) \oplus R \) given by \( i(p) = a(p) + r(p) \).

The exact sequence above yields an isomorphism \( det(P_0) \cong det(P(a)) \) and therefore an isomorphism \( \theta : R \to det(P(a)) \) obtained by composing with \( \theta_0 \).
Now let us conduct the second step. By Lemma 2.2, we can check the desired equality locally: If we let
\[ V = V_{\theta_0} : Um(P_0 \oplus R) \to V(R) \]
associated to the fixed trivialization \( \theta_0 \) of \( det(P_0) \) by
\[ V(a) = V_{\theta_0}(a) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, (i \oplus 1)^t(\chi_a \perp \psi_2)(i \oplus 1)]. \]

In order to prove that this generalized symbol is well-defined, one has to show that our definition is independent of a section of \( a \):

**Theorem 4.1.** The generalized Vaserstein symbol is well-defined, i.e. the element \( V(a) \) defined as above is independent of the choice of a section of \( a \).

**Proof.** Let \( a \in Um(P_0 \oplus R) \) with two sections \( s, t : R \to P_0 \oplus R \). We denote by \( \iota_s \) and \( \iota_t \) the isomorphisms \( P_0 \oplus R \cong P(a) \oplus R \) induced by the sections \( s \) and \( t \) respectively. Since the isomorphism \( det(P(a)) \cong det(P_0) \) does not depend on the choice of a section (because the difference of two sections maps \( R \) into \( P(a) \)), the form \( \chi_a \) is independent of the choice of a section as well. Thus we have to show that the elements \( V(a, s) = [P_0 \oplus R^2, (\chi_0 \perp \psi_2), (i_s \oplus 1)^t(\chi_a \perp \psi_2)(i_s \oplus 1)] \) and \( V(a, t) = [P_0 \oplus R^2, (\chi_0 \perp \psi_2), (i_t \oplus 1)^t(\chi_a \perp \psi_2)(i_t \oplus 1)] \) are equal in \( V(R) \).

We do this in the following three steps:

- We define a map \( d : P_0 \oplus R \to R \). We get a corresponding automorphism \( \varphi \in E(P_0 \oplus R^2) \) defined by \( \varphi = id_{P_0 \oplus R^2} - de \);
- We show that \( \varphi^t(i_s \oplus 1)^t(\chi_a \perp \psi_2)(i_s \oplus 1)\varphi = (i_t \oplus 1)^t(\chi_a \perp \psi_2)(i_t \oplus 1)\);
- Using Corollary 3.2, we conclude that \( V(a, s) = V(a, t) \).

Let us carry out the first step: We first define a map \( d' : P_0 \oplus R \to det(P_0 \oplus R) \) by \( p \mapsto s(1) \wedge t(1) \wedge p \in det(P_0 \oplus R) \). Then \( d : P_0 \oplus R \to R \) is the map obtained from \( d' \) by composing with the isomorphisms \( det(P_0 \oplus R) \cong det(P_0) \cong R \). Let \( d_0 \) and \( d_R \) be its restrictions to \( P_0 \) and \( R \) respectively. Furthermore, let \( \varphi_0 = id_{P_0 \oplus R^2} - d_0 e \) and \( \varphi_R = id_{P_0 \oplus R^2} - d_R e \) be the elementary automorphisms of \( P_0 \oplus R^2 \) defined by \( -d_0 \) and \( -d_R \) respectively. Moreover, let \( \varphi = id_{P_0 \oplus R^2} - de \).

Note that \( \varphi = \varphi_0 \varphi_R = \varphi_R \varphi_0 \in E(P_0 \oplus R^2) \).

Now let us conduct the second step. By Lemma 2.3, we can check the desired equality locally. So let \( p \) be a prime ideal of \( R \) and \( (e_1^p, e_2^p) \) be a basis of the free \( R_p \)-module \( (P_0)_p \) of rank 2. We may further assume that \( (\theta_0^{-1})_p(e_1^p \wedge e_2^p) = 1 \).

With respect to the basis \( (e_1^p, e_2^p, e_3) \) of \( (P_0)_p \oplus R_p \), the epimorphism \( a_p \) can be represented by the unimodular row \( (a_1^p, a_2^p, a_3^p) \) and both sections \( s_p \) and \( t_p \) can be represented by the columns \( (s_1^p, s_2^p, s_3^p)^t \) and \( (t_1^p, t_2^p, t_3^p)^t \). Using the basis \( (e_1^p, e_2^p, e_3, e_4) \) of \( (P_0)_p \oplus R_0^2 \), we can check the desired equality locally: If we let \( d_1^p = t_3^p s_2^p - t_2^p s_3^p, \quad d_2^p = t_1^p s_3^p - t_3^p s_1^p \) and \( d_3^p = t_2^p s_1^p - t_1^p s_2^p \) and
By Corollary 3.2, we deduce that finally, we conclude by Corollary 3.2: Since this follows from the proof of \[SV\, \text{Lemma 5.1}\].

Proof. For this, we note that the Pfaffian of an element of \( P_0 \oplus R^2 \) is completely determined by the Pfaffians of all its images under the maps \( V \). But the localization \( (P_0)_p \) at any prime \( p \) is a free \( R_p \)-module of rank 2; choosing a basis \( (e_1^p, e_2^p) \) of \( (P_0)_p \) such that \((\theta_0^{-1})_p(e_1^p \wedge e_2^p) = 1\) as in the proof of 4.1, we may calculate the Pfaffian of any Vaserstein symbol by the usual formula for the Pfaffian of an alternating \( 4 \times 4 \)-matrix. The lemma then follows immediately.

Theorem 4.3. Let \( \varphi \) be an elementary automorphism of \( P_0 \oplus R \). Then we have \( V(a) = V(a\varphi) \) for any \( a \in Um(P_0 \oplus R) \). In particular, we obtain a well-defined map \( V : Um(P_0 \oplus R) / E(P_0 \oplus R) \rightarrow V_1(R) \).
Proof. Let \( \varphi \) be an elementary automorphism of \( P_0 \oplus R \), \( a \in Um(P_0 \oplus R) \) and \( s : R \to P_0 \oplus R \) a section of \( a \). Then \( \varphi^{-1}s \) is a section of \( a\varphi \). We let \( i : P_0 \oplus R \to P(a) \oplus R \) and \( j : P_0 \oplus R \to P(a\varphi) \oplus R \) be the isomorphisms induced by the sections \( s \) and \( \varphi^{-1}s \). We will show that

\[
(\varphi + 1)^{(i + 1)}(\chi_a \perp \psi_2)(i + 1)(\varphi + 1) = (j + 1)^{(\chi(a\varphi) \perp \psi_2)}(j + 1).
\]

The theorem then follows from Corollary 3.2.

So let us show the equality above. Directly from the definitions, one checks that \((i + 1)(\varphi + 1) = ((\varphi + 1) \oplus (j + 1)), \) where by abuse of notation we understand \( \varphi \) as the induced isomorphism \( P(a\varphi) \to P(a) \). Altogether, it only remains to show that \( \varphi^{\varphi} \chi_a \varphi = \chi_{a\varphi} \).

For this, let \((p, q)\) a pair of elements in \( P(a\varphi) \); by definition, \( \chi_{a\varphi} \) sends these elements to the image of \( p \wedge q \) under the isomorphism \( det(P(a\varphi)) \cong R \). This element can also be described as the image of \( p \wedge q \wedge \varphi^{-1}s(1) \) under the isomorphism \( det(P_0 \oplus R) \cong R \).

Analogously, the skew-symmetric form \( \varphi^{\varphi} \chi_a \varphi \) sends \((p, q)\) to the image of the element \( \varphi(p) \wedge \varphi(q) \wedge s(1) \) under the isomorphism \( det(P_0 \oplus R) \cong R \). Therefore Lemma 2.11 allows us to conclude as desired, which finishes the proof of the theorem.

Note that if we equip the set \( Um(P_0 \oplus R) \) with the projection \( \pi_R : P_0 \oplus R \to R \) onto \( R \) as a base-point, then the generalized Vaserstein symbol is a map of pointed sets, because \( V(\pi_R) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi_0 \perp \psi_2] = 0 \).

Let us briefly discuss how the generalized Vaserstein symbol depends on the choice of the trivialization \( \theta_0 \) of the determinant of \( P_0 \). For this, recall that we have defined an action of \( R^x \) on \( V(R) \) in section 3.2. In case of a smooth algebra over a perfect field, we saw in section 3.3 that this action can be identified with the multiplicative action induced by a product map in the theory of higher Grothendieck-Witt groups.

Now let \( P_0 \) be a projective \( R \)-module of rank 2 which admits a trivialization \( \theta_0 \) of its determinant. Furthermore, let \( a \in Um(P_0 \oplus R) \) with section \( s \) and let \( i, \chi_0, \chi_a \) as in the definition of the generalized Vaserstein symbol. We consider another trivialization \( \theta'_0 \) of \( det(P_0) \) and we let \( \chi'_0 \) and \( \chi'_a \) be the corresponding skew-symmetric morphisms on \( P_0 \) and \( P(a) \). Obviously, there is a unit \( u \in R^x \) such that \( u \cdot \theta_0 = \theta'_0 \); in particular, we have \( u \cdot \chi_0 = \chi'_0 \) and \( u \cdot \chi_a = \chi'_a \). Thus, if we denote the Vaserstein symbol associated to \( \theta'_0 \) by \( V_{\theta'_0} \), then

\[
V_{\theta'_0} = [P_0 \oplus R^2, (u \cdot \theta_0) \perp \psi_2, (i + 1)^{(u \cdot \chi_a) \perp \psi_2)(i + 1)}],
\]

Finally, the isometry given by \( P_0 \oplus R^2 \xrightarrow{id_{P_0} \oplus 1 \oplus u} P_0 \oplus R^2 \) yields an equality

\[
[P_0 \oplus R^2, (u \cdot \theta_0) \perp \psi_2, (i + 1)^{(u \cdot \chi_a) \perp \psi_2)(i + 1)}] = [P_0 \oplus R^2, u \cdot (\theta_0 \perp \psi_2), u \cdot (i + 1)^{(\chi_a \perp \psi_2)(i + 1)}].
\]

Thus, if we denote the Vaserstein symbol associated to \( \theta_0 \) by \( V_{s\theta_0} \), then
In particular, the property of the generalized Vaserstein symbol to be injective, surjective or bijective onto \( V_1(R) \) does not depend on the choice of \( \theta_0 \).

There is another immediate consequence of this: If we let \( P_0 = R^2 \) be the free \( R \)-module of rank 2 and let \( e_1 = (1,0), e_2 = (0,1) \in R^2 \) be the obvious elements, then there is a canonical isomorphism \( \theta_0 : R \xrightarrow{\cong} \text{det}(R^2) \) given by \( 1 \mapsto e_1 \wedge e_2 \). Then recall that the usual Vaserstein symbol can be described as \( V_{\theta_0} \circ M \) (up to the identification \( W_E(R) \cong V_1(R) \)). But by the formula above, it immediately follows that the generalized Vaserstein symbol associated to \(-\theta_0\) coincides with the usual Vaserstein symbol via the identification \( V_1 \cong W_E(R) \) mentioned above.

### 4.3 Criteria for the surjectivity and injectivity of the generalized Vaserstein symbol

The main purpose of this section is to find some criteria for the generalized Vaserstein symbol to be surjective onto \( V_1(R) \) or injective. We have already seen that these properties are independent of the choice of a trivialization of \( \text{det}(P_0) \). So let us fix such a trivialization \( \theta_0 : R \xrightarrow{\cong} \text{det}(P_0) \).

Recall that a unimodular row of length \( n \) is an \( n \)-tuple \( a = (a_1, ..., a_n) \) of elements in \( R \) such that there are elements \( b_1, ..., b_n \in R \) with \( \sum_{i=1}^{n} a_i b_i = 1 \). We denote by \( Um_n(R) \) the set of unimodular rows of length \( n \). For any \( n \geq 3 \), there are obvious maps \( U_n : Um_{n-2}(R) \to Um(P_n) \).

As a first step towards our criterion for the surjectivity of the generalized Vaserstein symbol (cp. Theorem 4.5 below), we prove the following statement:

**Lemma 4.4.** Any element of the form \( [P_4, \chi_0 \perp \psi_2, \chi] \) for a non-degenerate skew-symmetric form \( \chi \) on \( P_4 \) is in the image of the generalized Vaserstein symbol.

**Proof.** First of all, we set \( a = \chi(-, e_4) : P_1 \oplus Re_3 \to R \). Since \( P_0 \) is locally free of finite rank by Lemma 2.1 there are elements \( f_1, ..., f_n \in R \) which generate the unit ideal of \( R \) such that \((P_0)_{f_k}\) is a free \( R_{f_k} \)-module of rank 2 for all \( k = 1, ..., n \). We can therefore choose a basis \((e_1^{f_k}, e_2^{f_k})\) of \((P_0)_{f_k}\) such that \((\theta^{-1})_{f_k}(e_1^{f_k} \wedge e_2^{f_k}) = 1\). The epimorphism \( a_{f_k} : (P_0)_{f_k} \oplus R_{f_k} \to R_{f_k} \) is then represented by the unimodular row \((a_1^{f_k}, a_2^{f_k}, a_3^{f_k})\) (with respect to the basis \((e_1^{f_k}, e_2^{f_k}, e_3)\) of \((P_0)_{f_k} \oplus R_{f_k}\)). The localized non-degenerate skew-symmetric form \( \chi_{f_k} \) is then represented (with respect to the basis \((e_1^{f_k}, e_2^{f_k}, e_3, e_4)\) of the module \((P_0)_{f_k} \oplus R_{f_k}^2\)) by the matrix

\[
\begin{pmatrix}
0 & s_3^{f_k} & -s_2^{f_k} & a_1^{f_k} \\
-s_3^{f_k} & 0 & s_1^{f_k} & a_2^{f_k} \\
s_2^{f_k} & -s_1^{f_k} & 0 & a_3^{f_k} \\
-a_1^{f_k} & -a_2^{f_k} & -a_3^{f_k} & 0
\end{pmatrix}
\]
which has Pfaffian 1. By the formula for the Pfaffians of invertible alternating
$4 \times 4$-matrices, we therefore obtain a morphism $s_{fk} : R_{fk} \to (P_0)_{fk} \oplus R_{fk}$.
$1 \mapsto s_{fk}^1ek_1 + s_{fk}^2ek_2 + s_{fk}^3ek_3$ which is a section of $a_{fk}$. One can check that the
morphisms $s_{fk} : R_{fk} \to (P_0)_{fk} \oplus R_{fk}$ necessarily coincide over $R_{fj}$; so they
can be glued to a morphism $s : R \to P_0 \oplus R$, which is locally and hence globally
a section of $a$. The generalized Vaserstein symbol of $a$ may thus be computed by
means of this section: As in the definition of the generalized Vaserstein symbol,
we obtain an isomorphism $i : P_0 \oplus R \to P(a) \oplus R$ and a skew-symmetric form
$\chi_a$ on $P(a) = \ker(a)$ induced by $a$ and its section $s$. The generalized Vaserstein
symbol of $a$ is then given by $[P_0 \oplus R^2, \chi_0 \perp \psi_2, (i \oplus 1)^i(\chi_a \perp \psi_2)(i \oplus 1)]$. But
the form $(i \oplus 1)^i(\chi_a \perp \psi_2)(i \oplus 1)$ locally coincides with $\chi$ by construction. By
Lemma 2.2 it thus also coincides with $\chi$ globally. Therefore we obtain the
desired equality $V(a) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi]$. \hfill \qed

Using Lemma 4.4 and the technical lemmas proven in previous sections, we may
now prove the following criteria for the surjectivity of the generalized Vaserstein
symbol:

**Theorem 4.5.** Let $N \in \mathbb{N}$. Assume that an element $\beta$ of $V_1(R)$ is of the form
$[P_{2N+2}, \chi_0 \perp \psi_{2N}, \chi]$ for some non-degenerate skew-symmetric form on $P_{2N+2}$.
Moreover, assume that $\pi_{2n+1,2n+1}(E_{\infty}(P_0) \cap \text{Aut}(P_{2n+1})) = Um(P_{2n+1})$ for any
$n \in \mathbb{N}$ with $1 < n \leq N$. Then $\beta$ lies in the image of the generalized Vaserstein
symbol. Thus, the generalized Vaserstein symbol $V : Um(P_0 \oplus R) \to V_1(R)$ is
surjective if $\pi_{2n+1,2n+1}(E_{\infty}(P_0) \cap \text{Aut}(P_{2n+1})) = Um(P_{2n+1})$ for all $n \geq 2$.

**Proof.** By assumption, $\beta \in V_1(R)$ has the form $\beta = [P_{2N+2}, \chi_0 \perp \psi_{2N}, \chi]$ for
some non-degenerate skew-symmetric form on $P_{2N+2}$. Furthermore, we may
inductively apply Lemma 2.10 (because of the second assumption) in order
to deduce that there is an elementary automorphism $\varphi$ on $P_{2N+2}$ such that
$\varphi' \varphi = \varphi \perp \psi_{2N-2}$ for some non-degenerate skew-symmetric form $\psi$ on $P_4$. In
particular, $\beta = [P_4, \chi_0 \perp \psi_2, \psi]$ by Corollary 3.2. Finally, any element of this
form is in the image of the generalized Vaserstein symbol by Lemma 4.4. So $\beta$
is in the image of the generalized Vaserstein symbol.

For the last statement, note that any element of $V_1(R)$ is of the form $[R^{2n}, \psi_{2n}, \chi]$ for
some non-degenerate skew-symmetric form on $R^{2n}$ (because of the isomorphism
$V_1(R) \cong W_E(R)$). We may then artificially add a trivial summand
$[P_0, \chi_0, \chi_0]$; hence any element of $V_1(R)$ is of the form $[P_{2n+2}, \chi_0 \perp \psi_{2n}, \chi_0 \perp \chi]$ for
some non-degenerate skew-symmetric form on $R^{2n}$. We can then conclude
by the previous paragraph. \hfill \qed

**Theorem 4.6.** Let $N \in \mathbb{N}$. Assume that the following conditions are satisfied:

- Every element of $V_1(R)$ is of the form $[R^{2N}, \psi_{2N}, \chi]$ for some non-degenerate
  skew-symmetric form on $R^{2N}$

- One has $\pi_{2n+1,2n+1}(E_{\infty}(P_0) \cap \text{Aut}(P_{2n+1})) = Um(P_{2n+1})$ for any $n \in \mathbb{N}$
  with $1 < n \leq N$ and $U_{2n+1}(Um_{2n-1}(R)) \subset \pi_{2n+1,2n+1}E(P_{2n+1})$
Then the generalized Vaserstein symbol $V : Um(P_0 \oplus R) \to V_1(R)$ is surjective.

**Proof.** We proceed as in the proof of Theorem [1.3]. By the first assumption, any element of $V_1(R)$ is of the form $[R^{2N}, \psi_{2N}, \chi]$ for some non-degenerate skew-symmetric form on $R^{2N}$. Again adding a trivial summand $[P_0, \chi_0, \chi_0]$, we see that any element of $V_1(R)$ is of the form $[P_{2N+2}, \chi_0 \perp \psi_{2N}, \chi_0 \perp \chi]$ for some non-degenerate skew-symmetric form on $R^{2N}$. As in the proof of Theorem [1.3], it then follows inductively from Lemma [2.10] that any element of $V_1(R)$ is of the form $[P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi]$ for some non-degenerate skew-symmetric form $\chi$ on $P_0 \oplus R^2$. The generalized Vaserstein symbol is then surjective by Lemma [1.3]. Note that the condition $\pi_{2N+1,2N+1}E(P_{2N+1}) = Um(P_{2N+1})$ can be replaced by the weaker condition $U_{2N+1}(Um_{2N-1}) \subset \pi_{2N+1,2N+1}E(P_{2N+1})$ in our situation.

**Corollary 4.7.** Assume that the following conditions are satisfied:

- The usual Vaserstein symbol $V : Um_3(R) \to W_E(R)$ is surjective
- $U_5(Um_3(R)) \subset \pi_{5,5}(E_{\infty}(P_0) \cap Aut(P_5))$

Then the generalized Vaserstein symbol $V$ as $Um(P_0 \oplus R) \to V_1(R)$ is surjective.

**Proof.** The surjectivity of the usual Vaserstein symbol means that any element of $V_1(R)$ is of the form $[R^4, \psi_4, \chi]$ for some non-degenerate skew-symmetric form on $R^4$. Now the corollary follows from Theorem [1.3].

In order to prove our criterion for the injectivity of the generalized Vaserstein symbol, we introduce the following condition: We say that $P_0$ satisfies condition $(\ast)$ if $[P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi_1] = [P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi_2]$ for skew-symmetric forms $\chi_1, \chi_2$ on $P_0 \oplus R^2$ implies $\alpha'(\chi_1 \perp \psi_{2n}) = \chi_2 \perp \psi_{2n}$ for some automorphism $\alpha \in E_{\infty}(P_0) \cap Aut(P_{2n+4})$.

If $P_0$ is a free $R$-module, condition $(\ast)$ is satisfied, which basically follows from the isomorphism $V(R) \cong W^+_{E}(R)$. Furthermore, using the isomorphisms $V(R) \cong V_{free}(R) \cong W^+_{E}(R)$, we will see that it is possible to prove that condition $(\ast)$ is always satisfied (cp. Lemma [4.9]). First of all, we fix some notation. For any finitely generated projective $R$-module $P$, recall that we call

$$H_P = \begin{pmatrix} 0 & id_P^\nu \\ -can & 0 \end{pmatrix} : P \oplus P^\nu \to P^\nu \oplus P^{\nu\nu}$$

the hyperbolic isomorphism on $P$. Note that if we let

$$h_{2n} = \begin{pmatrix} 0 & id_n \\ -id_n & 0 \end{pmatrix} \in A_{2n}(R),$$

then $Pf(h_{2n}) = (-1)^{(n-1)/2}$. Thus $h_{2n} \sim \psi_{2n}$ for all $n \in 4\mathbb{Z} \cup 4\mathbb{Z} + 1$, because any even permutation matrix of rank $2n$ lies in $E_{2n}(R)$ (cp. [W] Chapter III, Example 1.2.1). As a first step towards Lemma [4.9] we observe:
Lemma 4.8. Let $\chi$ be a skew-symmetric isomorphism on a finitely generated projective $R$-module $P$ and furthermore $P \oplus Q \cong R^n$ for some $R$-module $Q$. Then there exists an automorphism $\alpha \in E(P \oplus P^\nu \oplus Q \oplus Q^\nu)$ such that the equality $\alpha^\nu(\chi \perp \text{can} \chi^{-1} \perp H_Q) = H_P \perp H_Q$ holds and furthermore such that $\alpha$ corresponds to an element of $E_{2n}(R)$ under the canonical identifications $P \oplus P^\nu \oplus Q \oplus Q^\nu \cong (P \oplus Q) \oplus (P^\nu \oplus Q^\nu) \cong R^n \oplus R^n \cong R^{2n}$.

Proof. First of all, we observe that there is an isometry $\nu : R^2 \to R^2$ that on the generators of $E$ for some automorphism $\beta$ isomorphism $P \oplus P^\nu \oplus Q \oplus Q^\nu$ and identifying both $R^n \cong P \oplus Q$ and $R^n \cong (P \oplus Q)^\nu \cong P^\nu \oplus Q^\nu$ yields elementary automorphisms of $R^{2n}$ with respect to the decomposition $R^{2n} \cong R^n \oplus R^n$. Since we have $E(R^n \oplus R^n) \subset E_{2n}(R)$, this proves the lemma.

Using Lemma 4.8, we may prove:

Lemma 4.9. Any $P_0$ satisfies condition $(*)$.

Proof. We use the explicit description of the inverse of $V_{\text{free}}(R) \to V(R)$ to prove Lemma 4.10 below, which obviously implies Lemma 4.9 for $P = P_0 \oplus R^2$ and $\chi = \chi_0 \perp \psi_2$:

Lemma 4.10. If $[P, \chi, \chi_1] = [P, \chi, \chi_2]$ for skew-symmetric isomorphisms $\chi, \chi_1$ and $\chi_2$ on a finitely generated projective $R$-module $P$, then we have an equality $\alpha^\nu(\chi_1 \perp \psi_2) \alpha = \chi_2 \perp \psi_2$ for some $n \in \mathbb{N}$ and some $\alpha \in E(R \oplus R^{2n})$.

Proof. For this, let $Q$ be a finitely generated projective $R$-module such that $P \oplus Q \cong R^n$ for some $n$. If $\psi$ is any skew-symmetric isomorphism on $P$, then

\[ [P, \chi, \psi] = [P, \chi, \psi] + [P^\nu, \text{can} \chi^{-1}, \text{can} \chi^{-1}] + [Q \oplus Q^\nu, H_Q, H_Q] = [P \oplus P^\nu \oplus Q \oplus Q^\nu, \chi \perp \text{can} \chi^{-1} \perp H_Q, \psi \perp \text{can} \chi^{-1} \perp H_Q], \]

where $H_Q$ denotes the hyperbolic isomorphism on $Q$. This gives the class of $[P, \chi, \psi]$ in $V_{\text{free}}(R)$, because $P \oplus P^\nu \oplus Q \oplus Q^\nu \cong R^{2n}$.

Now if $[P, \chi, \chi_1] = [P, \chi, \chi_2]$, we therefore directly deduce from the isomorphism $W'_C(R) \cong V_{\text{free}}(R)$ that

\[ \beta^\nu(\chi_1 \perp \text{can} \chi^{-1} \perp H_Q \perp \psi_2) \beta = \chi_2 \perp \text{can} \chi^{-1} \perp H_Q \perp \psi_2 \]

for some automorphism $\beta$ corresponding to an element of $E_{2n+2m}(R)$ under the isomorphism $P \oplus P^\nu \oplus Q \oplus Q^\nu \cong R^{2n} \oplus R^{2m}$. Note that any such $\beta$ also is an elementary automorphism of $P \oplus P^\nu \oplus Q \oplus Q^\nu \oplus R^{2m}$ (You can check that on the generators of $E_{2n+2m}(R)$ given in Corollary 2.4). Adding a direct summand $P^\nu = P$ with skew-symmetric isomorphism $\chi$, we therefore get that
and in particular

\[(\beta \oplus id_P)^{\nu}(\chi_1 \perp can\chi^{-1} \perp H_Q \perp \psi_{2m} \perp \chi)(\beta \oplus id_P) = \chi_2 \perp can\chi^{-1} \perp H_Q \perp \psi_{2m} \perp \chi\]

Again, note that under the isomorphism

\[P \oplus P^{\nu} \oplus Q \oplus Q^{\nu} \oplus R^{2m} \oplus P' \cong P \oplus P' \oplus Q \oplus P^{\nu} \oplus Q^{\nu} \oplus R^{2m} \cong P \oplus R^{2n+2m}\]

the automorphism corresponding to \(\beta \oplus id_P\) is an elementary automorphism of \(P \oplus R^{2n+2m}\). Thus, permuting the direct summands as above and using Lemma 4.8, we end up with an equality

\[\alpha^{\nu}(\chi_1 \perp H_{R^{2n+2m}})\alpha = \chi_2 \perp H_{R^{2n+2m}}\]

for an elementary automorphism \(\alpha\) of \(P \oplus R^{2n+2m}\). Since we may assume that \(2n + 2m\) is divisible by 4, this proves the lemma. \(\Box\)

Now that we have proven that condition (\(*\)) is always satisfied, we can find conditions which imply that to elements \(a, b \in Um(P_0 \oplus R)\) with the same Vaserstein symbol are equal up to a stably elementary automorphism of \(P_0 \oplus R\). More precisely:

**Theorem 4.11.** Assume that \(E(P_{2n})e_{2n} = (E_\infty(P_0) \cap Aut(P_{2n}))e_{2n}\) for \(n \geq 2\).

Then the equality \(V(a) = V(b)\) for \(a, b \in Um(P_0 \oplus R)\) implies that \(b = a\varphi\) for some \(\varphi \in E_\infty(P_0) \cap Aut(P_3)\).

**Proof.** Let \(a\) and \(b\) elements of \(Um(P_0 \oplus R)\) with sections \(s\) and \(t\) respectively and let \(i : P_0 \oplus R \rightarrow P(a) \oplus R\) and \(j : P_0 \oplus R \rightarrow P(a) \oplus R\) be the isomorphisms induced by these sections. Furthermore, we let \(V(a, s) = (i \oplus 1)^{\nu}(\chi_a \perp \psi_2)(i \oplus 1)\) and \(V(b, t) = (j \oplus 1)^{\nu}(\chi_b \perp \psi_2)(j \oplus 1)\) be the skew-symmetric forms on \(P_0 \oplus R^2\) appearing in the definition of the generalized Vaserstein symbols of \(a\) and \(b\) respectively. Now assume that \(V(a) = V(b)\). Since \(P_0\) satisfies condition (\(*\)), there exist \(n \in \mathbb{N}\) and an automorphism \(\alpha \in E_\infty(P_0) \cap Aut(P_{2n+4})\) such that \(\alpha^{\nu}(V(a, s) \perp \psi_{2n})\alpha = V(b, t) \perp \psi_{2n}\). Using Lemma 2.9 we may inductively deduce that \(\beta^{\nu}V(a, s)\beta = V(b, t)\) for some \(\beta \in E_\infty(P_0) \cap Aut(P_0 \oplus R^2)\). Now by Lemma 2.8 and the second assumption in the theorem, there exists an automorphism \(\gamma \in E(P_0 \oplus R^2) \cap Sp(V(a, s))\) such that \(\beta e_4 = \gamma e_4\).

We now define \(\delta : P_0 \oplus R \rightarrow P_0 \oplus R\) as the composite

\[P_0 \oplus Re_3 \rightarrow P_0 \oplus Re_3 \oplus Re_4 \xrightarrow{\gamma^{-1}\beta} P_0 \oplus Re_3 \oplus Re_4 \rightarrow P_0 \oplus Re_3.\]

One can then check that \(\delta\) is an element of \(E_\infty(P_0) \cap Aut(P_0 \oplus R)\). Moreover, we have

\[\beta^{\nu}(\gamma^{-1})V(a, s)\gamma^{-1}\beta = V(b, t)\]

and in particular \(a\delta = b\), as desired. \(\Box\)
Corollary 4.12. Under the hypotheses of Theorem 4.11, furthermore assume that \( a(E_\infty(P_0) \cap \text{Aut}(P_0 \oplus R)) = aE(P_0 \oplus R) \) for all \( a \in \text{Um}(P_0 \oplus R) \). Then the generalized Vaserstein symbol \( V : \text{Um}(P_0 \oplus R)/E(P_0 \oplus R) \to V_1(R) \) is injective.

Proof. By Theorem 4.11, we have that \( V(a) = V(b) \) implies \( b = a \varphi' \) for some \( \varphi' \in E_\infty(P_0) \cap \text{Aut}(P_0 \oplus R) \). Now by the additional assumption, there also exists an elementary automorphism \( \varphi \) of \( P_0 \oplus R \) such that \( b = a \varphi \). So the generalized Vaserstein symbol is injective. \( \square \)

Regarding the additional assumption in Corollary 4.12, it is possible to adapt the arguments in the proof of [SV, Corollary 7.4] to show that the desired equality \( a(E_\infty(P_0) \cap \text{Aut}(P_0 \oplus R)) = aE(P_0 \oplus R) \) holds for all \( a \in \text{Um}(P_0 \oplus R) \) if \( E_\infty(P_0) \cap \text{Aut}(P_0 \oplus R^2) = E(P_4) \):

Lemma 4.13. If \( E_\infty(P_0) \cap \text{Aut}(P_0 \oplus R^2) = E(P_4) \), then we have an equality \( a(E_\infty(P_0) \cap \text{Aut}(P_0 \oplus R)) = aE(P_0 \oplus R) \) for all \( a \in \text{Um}(P_0 \oplus R) \).

Proof. Let \( a \in \text{Um}(P_0 \oplus R) \) with section \( s \) and let \( \varphi \in E_\infty(P_0) \cap \text{Aut}(P_0 \oplus R) \). If we let \( V(a, s) \) be the skew-symmetric form from the definition of the generalized Vaserstein symbol, then it follows from the proof of Lemma 4.4 that

\[
(\varphi + 1)^t V(a, s)(\varphi + 1) = V(a', s')
\]

for some \( a' \in \text{Um}(P_0 \oplus R) \) with section \( s' \). By assumption, the automorphism \( \varphi + 1 \) of \( P_4 \) is an elementary automorphism. Moreover, by Corollary 2.4, the group \( E(P_4) \) is generated by elementary automorphisms \( \varphi_g = \text{id}_{P_4} + g \), where \( g \) is a homomorphism

1) \( g : Re_3 \to P_0 \),
2) \( g : P_0 \to Re_3 \),
3) \( g : Re_3 \to Re_4 \) or
4) \( g : Re_4 \to Re_3 \).

It therefore suffices to show the following: If \( \varphi_g^t V(a, s) \varphi_g = V(a', s') \) for some \( g \) as above, then \( a' = a \psi \) for some \( \psi \in E(P_0 \oplus R) \). The only non-trivial case is the last one, i.e. if \( g \) is a homomorphism \( Re_4 \to Re_3 \).

So let \( g : Re_4 \to Re_3 \) and let \( \varphi_g \) be the induced elementary automorphism of \( P_4 \). As explained above, we assume that

\[
\varphi_g^t V(a, s) \varphi_g = V(a', s')
\]

for some epimorphism \( a' : P_0 \oplus Re_3 \to R \) with section \( s' \). Now write \( a = (a_0, a_R) \), where \( a_0 \) and \( a_R \) are the restrictions of \( a \) to \( P_0 \) and \( Re_3 \) respectively. Furthermore, let \( p = \pi_{P_0}(s(1)) \). From now on, we interpret the skew-symmetric form \( \chi_0 \) in the definition of the generalized Vaserstein symbol as a skew-symmetric isomorphism \( \chi_0 : P \to P'' \). Then one can check locally that
$a' = (a_0 - g(1) \cdot \chi_0(p), a_R)$. 

Then let us define an elementary automorphism $\psi$ as follows: We first define an endomorphism of $P_0$ by

$$\psi_0 = id_{P_0} - g(1) \cdot \pi_0 \circ \chi_0(p) : P_0 \to P_0$$

and we also define a morphism $P_0 \to R_{\mathbb{E}3}$ by

$$\psi_R = -g(1) \cdot \pi_R \circ \chi_0(p) : P_0 \to R.$$ 

Then we consider the endomorphism of $P_0 \oplus R$ given by

$$\psi = \begin{pmatrix} \psi_0 & 0 \\ \psi_R & id_R \end{pmatrix}.$$ 

First of all, this endomorphism coincides up to an elementary automorphism with

$$\begin{pmatrix} \psi_0 & 0 \\ 0 & id_R \end{pmatrix}.$$ 

Since $\chi_0(p) \circ \pi_0 \circ s = 0$, this endomorphism is an element of $E(P_0 \oplus R)$ by Lemma 2.6. Hence the same holds for $\psi$. Finally, one can check easily that $a\psi = a'$ by construction.

As an immediate consequence, we can finally deduce our criterion for the injectivity of the generalized Vaserstein symbol:

**Theorem 4.14.** Assume that $E(P_{2n})e_{2n} = (E_{\infty}(P_0) \cap Aut(P_{2n}))e_{2n}$ for all $n \geq 3$ and $E_{\infty}(P_0) \cap Aut(P_4) = E(P_4)$. Then the generalized Vaserstein symbol $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to V_1(R)$ is injective.

**Proof.** Combine Corollary 4.12 and Lemma 4.13.

4.4 The bijectivity of the generalized Vaserstein symbol in dimension 2 and 3

Let us now study the criteria for the surjectivity and injectivity of the generalized Vaserstein symbol found in this section. In [HB] the conditions of Theorem 4.13 and Theorem 4.14 are studied in a very general framework. If $R$ is a Noetherian ring of Krull dimension $d$, it follows from [HB, Chapter IV, Theorem 3.4] that actually $\text{Unim.El.}(P_n) = E(P_n)e_n$ for all $n \geq d + 2$ (or $Um(P_n) = \pi_{n,n}E(P_n)$ for all $n \geq d + 2$). In particular, if $\dim(R) \leq 4$, then the generalized Vaserstein symbol is injective as soon as $E_{\infty}(P_0) \cap Aut(P_4) = E(P_4)$; if $\dim(R) \leq 3$, it is surjective. Hence the following results are immediate consequences of our stability results in Section 2.3.
Theorem 4.15. Let $R$ be a 2-dimensional regular Noetherian ring or a 3-dimensional regular affine algebra over a field $k$ with c.d.$(k) \leq 1$ and $6 \in k^\times$. Then the generalized Vaserstein symbol $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to V_1(R)$ is a bijection.

Theorem 4.16. Let $R$ be a 4-dimensional regular affine algebra over a field $k$ satisfying the property $P(5,3)$ (cp. Section 2.3). Then the generalized Vaserstein symbol $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to V_1(R)$ is injective.

Because of the pointed surjection $Um(P_0 \oplus R)/E(P_0 \oplus R) \to \phi^{-1}_2([P_0 \oplus R])$, the bijectivity of the generalized Vaserstein symbol always gives rise to a surjection $W_E(R) \to \phi^{-1}_2([P_0 \oplus R])$; in this case, it seems that the group structure of $W_E(R) \cong Um(P_0 \oplus R)/E(P_0 \oplus R)$ essentially governs the structure of the fiber $\phi^{-1}_2([P_0 \oplus R])$.

The following application follows - to some degree - the pattern of the proof of [FRS, Theorem 7.5] and illustrates the previous paragraph:

Theorem 4.17. Let $R$ be a ring and $P_0$ be a projective $R$-module of rank 2 which admits a trivialization $\theta_0$ of its determinant. Assume the following conditions are satisfied:

a) The generalized Vaserstein symbol $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to V_1(R)$ induced by $\theta_0$ is a bijection;

b) $2V(a_0, a_R) = V(a_0, a_R^2)$ for $(a_0, a_R) \in Um(P_0 \oplus R)$;

c) The group $W_E(R)$ is 2-divisible.

Then $\phi^{-1}_2([P_0 \oplus R])$ is trivial.

Proof. Assume $P' \oplus R \cong P_0 \oplus R$. As we have seen in section 2.3, $P'$ has an associated element of $Um(P_0 \oplus R)/Aut(P_0 \oplus R)$. We lift this element to an element $[b]$ of $Um(P_0 \oplus R)/E(P_0 \oplus R)$ ($[b]$ denotes the class of $b \in Um(P_0 \oplus R)$). Since the generalized Vaserstein symbol is a bijection and $W_E(R)$ is a 2-divisible group by assumption, we get that $[b] = 2[a]$, where $[a]$ denotes the class of an element $a = (a_0, a_R)$ of $Um(P_0 \oplus R)$ in the orbit space $Um(P_0 \oplus R)/E(P_0 \oplus R)$. But then the second assumption shows that $2[a] = [(a_0, a_R^2)]$. It follows from [B, Proposition 2.7] or [S1, Lemma 2] that any element of $Um(P_0 \oplus R)$ of the form $(a_0, a_R^2)$ is completable to an automorphism of $P_0 \oplus R$, i.e. $\pi_R = a\varphi$ for some automorphism $\varphi$ of $P_0 \oplus R$. Altogether, $\pi_R$ and $b$ therefore lie in the same orbit under the action of $Aut(P_0 \oplus R)$ and hence $P' \cong P$. Thus, $\phi^{-1}_2([P_0 \oplus R])$ is trivial. \hfill $\square$

As mentioned in the proof of Theorem 4.10, any element $a \in Um(P_0 \oplus R)$ of the form $a = (a_0, a_R^2)$ is completable to an automorphism of $P_0 \oplus R$. This follows directly from [B, Proposition 2.7] or [S1, Lemma 2], because $P_0$ has a trivial determinant. We now construct a more concrete completion of $a = (a_0, a_R^2)$. For this, let us first look at the case $P_0 \cong R^2$: If $(b, c, a^2)$ is a unimodular row and $qb + rc + ap = 1$, then it follows from [KT] that the matrix

\begin{center}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\end{center}
is a completion of \((b, c, a^2)\) with determinant 1. We observe that
\[
\begin{pmatrix}
-pqr & q^2 & -c + 2aq \\
-r^2 & -p + qr & b + 2ar \\
b & c & a^2
\end{pmatrix}
\]
is a completion of \((b, c, a^2)\) with determinant 1. We observe that
\[
\begin{pmatrix}
-qr & q^2 \\
r^2 & qr
\end{pmatrix} = \begin{pmatrix} q & -r \\
r & q \end{pmatrix}
\]
and also
\[
\begin{pmatrix}
-c \\
b
\end{pmatrix} = \begin{pmatrix} 0 & -1 \\
1 & 0 \end{pmatrix} \begin{pmatrix} b \\
c \end{pmatrix}.
\]
This shows how to generalize the construction of this explicit completion. We denote by \(\chi_0 : P_0 \to P_0^\nu\) the skew-symmetric isomorphism from the definition of the generalized Vaserstein symbol (we now interpret it as a skew-symmetric isomorphism and not as a non-degenerate skew-symmetric form). If \(a = (a_0, a_R)\) is an element of \(Um(P_0 \oplus R)\) with a section \(s\) uniquely given by the element \(s(1) = (q, p) \in P_0 \oplus R\), we consider the following morphisms: We define an endomorphism of \(P_0\) by
\[
\varphi_0 = -(\pi_{P_0}s) \circ \chi_0(q) - p \cdot id_{P_0} : P_0 \to P_0
\]
and we also define a morphism \(R \to P_0\) by
\[
\varphi_R : R \to P_0, \ 1 \mapsto 2a_R(1) \cdot q + \chi_0^{-1}(a_0).
\]
Then we consider the endomorphism of \(\varphi : P_0 \oplus R\) given by
\[
\begin{pmatrix}
\varphi_0 & \varphi_R \\
a_0 & a_R^2
\end{pmatrix}.
\]
Essentially by construction, \(\varphi\) is a completion of \((a_0, a_R)\):

**Proposition 4.18.** The endomorphism \(\varphi\) of \(P_0 \oplus R\) defined above is an automorphism of \(P_0 \oplus R\) of determinant 1 such that \(\pi_R \varphi = (a_0, a_R^2)\).

**Proof.** Choosing locally a free basis \((e_1^p, e_2^p) (P_0)_p\) at any prime \(p\) such that \((\theta_0^{-1})_p (e_1^p \wedge e_2^p) = 1\), we can check locally that this endomorphism is an automorphism of determinant 1 (because locally it coincides with the completion given in [Kr]); by definition, we also have \(\pi_R \varphi = (a_0, a_R^2)\). Thus, \(\varphi\) has the desired properties and generalizes the explicit completion given in [Kr]. \(\square\)
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