THE ALMOST UNCONDITIONAL CONVERGENCE OF THE EULER IMPLICIT/EXPLICIT SCHEME FOR THE THREE DIMENSIONAL NONSTATIONARY NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we present the Euler implicit/explicit scheme for the three dimensional nonstationary Navier-Stokes equations. The Galerkin mixed finite element satisfying inf-sup condition is used for the spatial discretization and the temporal treatment is implicit/explicit scheme, which is Euler implicit scheme for the linear terms and explicit scheme for the nonlinear term. We prove that this method is almost unconditionally convergent and obtain the optimal $H^1 - L^2$ error estimate of the numerical velocity-pressure under the hypothesis of $H^2$-regularity of the solution for the three dimensional nonstationary Navier-Stokes equations. Finally some numerical experiments are carried out to demonstrate the effectiveness of the method.

1. Introduction. Let $\Omega$ be a bounded convex polyhedral domain in the three dimensional space $\mathbb{R}^3$. We consider the nonstationary Navier-Stokes equations with the primitive variables in the $\Omega$:

$$
\begin{align*}
  u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f, \quad \text{div} \, u = 0, \quad (x, t) \in \Omega \times (0, T]; \\
  u(x, 0) &= u_0(x), \quad x \in \Omega; \quad u(x, t)|_{\partial \Omega} = 0, \quad t \in [0, T],
\end{align*}
$$

(1.1)

where $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ represents the velocity, $p = p(x, t)$ the pressure, $f = f(x, t)$ a given body force, $u_0(x)$ the initial velocity, $\nu > 0$ the viscosity, $T > 0$ the finite time.

In the past few decades, there were extensive study on the efficient schemes to solve the Navier-Stokes equations [2, 3, 4, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 27, 28, 29, 30, 31, 32, 34], which included fully implicit scheme, semi-implicit scheme, and implicit/explicit scheme. The convergence of these schemes has attracted considerable attention of many researchers. The explicit scheme has the advantage of easy calculation, but people rarely use it because of its harsh restriction of the time step in the convergence condition. The fully
implicit scheme is nearly unconditionally convergent, but it has the disadvantage of solving nonlinear problem in every time step. The semi-implicit scheme is almost unconditionally convergent too, but it has the defectiveness in the numerical calculation. In this scheme, the coefficient matrix of the linear system varies and needs recalculation in each time step, so it becomes inefficient. Relatively, the popular scheme to solve the Navier-Stokes equations is the implicit/explicit scheme which use the back Euler discretization in the linear terms to improve the stability and explicit scheme in the nonlinear term to keep the simplicity of the method. The convergence condition for the implicit/explicit scheme has been studied by many people. In the previous references\cite{1, 20, 29, 34}, the mesh size \( h(0 < h < 1) \) in the space discretization and time step \( \tau = \frac{1}{T} (0 < \tau < 1) \) need to satisfy the convergence condition(or CFL condition): i. e. there exist positive constants \( C_0 \) and \( r \) such that

\[
\tau \leq C_0 h^r. \tag{1.2}
\]

Recently for two dimensional nonstationary Navier-Stokes equations, He and Sun in reference\cite{19} \cite{22} improved the above convergence condition to be

\[
\tau \leq C_0. \tag{1.3}
\]

That is to say the implicit/explicit scheme is almost unconditionally convergent to solve the two dimensional Navier-Stokes equations.

The main purpose of this paper is to consider the convergence and the error estimate of the Euler implicit/explicit scheme to solve the three dimensional nonstationary Navier-Stokes equations. It is well known that there are the different Sobolev inequalities between the two and three dimensional spaces, refer to Lemma 2.1 in this paper, (3.16)-(3.17) in \cite{19} and Lemma 3.2 in \cite{22}. Due to the above fact, we can not establish the \( H^2 \)-regularity results of the fully discrete solution \( u^n_h \) as like in the references \cite{19} and \cite{22}. Hence, we can not use the same method as like in the reference \cite{19} \cite{22} to prove the the optimal \( H^1 - L^2 \) error estimate for the numerical velocity-pressure \((u^n_h, p^n_h)\). So we apply the mathematical induction method to prove the the optimal \( H^1 - L^2 \) error estimate of the numerical velocity-pressure \((u^n_h, p^n_h)\) which is the essential differences in arguments from the references \cite{19} \cite{22}.

The remainder of this paper is organized as follows. In Section 2, we discretize the equation with mixed finite element method in the space, and Euler implicit/explicit scheme in the time. Next the convergence and the error estimate of the numerical solution are discussed in Section 3 and 4. When the CFL condition \( \tau \leq C_0 \) is satisfied, the implicit/explicit scheme to solve the nonstationary Navier-Stokes equations in the three dimensional space is almost unconditionally convergent, which has the similar result with the two dimensional space. Furthermore, we obtain the optimal \( H^1 - L^2 \) error estimate for the velocity-pressure under the assumption of CFL condition. In Section 5 two examples are presented to verify the correctness of theoretical analysis and effectiveness of the implicit/explicit scheme.

2. Euler implicit/explicit scheme. We introduce the following Sobolev spaces\cite{19} \cite{23} \cite{33}:

\[
X = H^1_0(\Omega)^3, \quad Y = L^2(\Omega)^3, \quad M = L^2_0(\Omega) = \left\{ q \in L^2(\Omega); \int_\Omega q dx = 0 \right\}.
\]

The \( (\cdot, \cdot) \) and \( \| \cdot \|_0 \) denote the \( L^2 \) inner product and norm in the \( L^2(\Omega)^3 \) respectively. The \( X \) space is equipped with the usual inner product \( (\nabla u, \nabla v) \) and norm \( \|\nabla u\|_0 \).
We introduce the closed subspace $V$ of $X$ as follows:
$$V = \{ v \in X; \text{div} \, v = 0 \}$$
and the closed subspace $H$ of $Y$
$$H = \{ v \in Y; \text{div} \, v = 0, v \cdot n|_{\partial \Omega} = 0 \}.$$
Where $n$ is the unit normal vector directed towards the exterior of the $\Omega$. The Stokes operator is defined by $A = -P \Delta$, where $P$ is the $L^2$ orthogonal projection operator from $Y$ to $H$. We note the domain of the Stokes operator $A$ as $D(A) = H^2(\Omega)^3 \cap V$.

First, we assume the solution of (1.1) satisfies the following $H^2$ regularity, i.e. (A1).
$$\|Au(t)\|_0 + \|u_t(t)\|_0 + \|p(t)\|_1 \leq \kappa, \forall t \in [0, T],$$
where $\kappa > 0$ is a constant depending on the $v, \Omega, T, u_0, f$. The validity of the assumption (A1) can be found in reference [23].

Let $J_h$ be a regular and quasi uniform mesh generation which divide the $\bar{\Omega}$ into a series of tetrahedron element $K$, indexed by the parameter $h = \max_{K \in J_h} \{ h_K; h_K = \text{diam}(K) \}$. Based on the mesh generation $J_h$, the finite element space $X_h \times M_h$ can be constructed (see Ciarlet [6], Girault and Raviart [10]).

We define the subspace $V_h$ of $X_h$ given by
$$V_h = \{ v_h \in X_h; d(v_h, q_h) = 0, \forall q_h \in M_h \}.$$

Let $P_h : Y \rightarrow V_h$ represent $L^2$ orthogonal projection operator, which is defined as follows:
$$(P_h v, v_h) = (v, v_h), \forall v \in Y, \forall v_h \in V_h.$$ Furthermore, we assume $(X_h, M_h)$ satisfies the following approximation properties.

(A2) For all $v \in H^2(\Omega)^3 \cap X$ and $q \in H^1(\Omega) \cap M$, there exist $\pi_h v \in X_h$ and $\rho_h q \in M_h$ such that
$$\|\nabla(v - \pi_h v)\|_0 \leq c_h \|Av\|_0, \|q - \rho_h q\|_0 \leq c_h \|\nabla q\|_0.$$ For all $v_h \in X_h$, the following inverse inequality exists
$$\|\nabla v_h\|_0 \leq c_1 h^{-1} \|v_h\|_0, \forall v_h \in X_h;$$ and inf-sup condition: for all $q_h \in M_h$, there exists $v_h \in X_h, v_h \neq 0$ such that
$$d(v_h, q_h) \geq \beta \|q_h\|_0 \|\nabla v_h\|_0,$$
where $c_1$ and $\beta$ are some positive constants depending on $\Omega$.

For the non negative integer $l$, $P_l(K)$ presents the space of piecewise polynomial of degree $l$ on $K$. Next some examples which satisfy the assumption (A2) are given as follows.

**Example 1 (Girault-Raviart [10]).**

$$X_h = \{ v_h \in C^0(\Omega)^3 \cap X; v_h|_K \in P_2(K)^3 \forall K \in J_h \},$$
$$M_h = \{ q_h \in M; q_h|_K \in P_0(K) \forall K \in J_h \}.$$ **Example 2 (Bercovier-Pironneau [3]).** We discretize the element $K \in J_h$ into smaller element $K'$ to obtain a new mesh generation $J_{h/2}$. The diameter of the element $K' \in J_{h/2}$ is no more than $h/2$. Then we define the finite element spaces
$$X_h = \{ v_h \in C^0(\bar{\Omega})^3 \cap X; v_h|_K \in P_1(K)^3 \forall K \in J_{h/2} \},$$
$$M_h = \{ q_h \in C^0(\bar{\Omega}) \cap M; q_h|_K \in P_1(K) \forall K \in J_h \}.$$
Example 3 (Gerbeau [10]). Let $\tilde{b}$ be the bubble function which value equals to one at the barycenter of $K$ and zero on the boundary of $K$. We define

$$X_h = \{ v_h \in C^0(\Omega)^3 \cap X; v_h|_K \in (P_1(K) + \text{span}\{\tilde{b}\})^3, \quad \forall K \in J_h \},$$

$$M_h = \{ q_h \in C^0(\Omega) \cap M; q_h|_K \in P_1(K), \quad \forall K \in J_h \}.$$  

According to the above assumptions, the Galerkin mixed finite element approximation problem of (1.1) is given by: find $(u_h, p_h) \in (X_h, M_h)$ such that for all $0 < t \leq T$ and $(v_h, q_h) \in (X_h, M_h)$

$$(u_{ht}, v_h) + a(u_h, v_h) - d(v_h, p_h) + d(u_h, q_h) + b(u_h, u_h, v_h) = (f, v_h),$$

$$u_h(0) = u_{0h} = P_h u_0$$

(2.1)

(2.2)

here

$$a(u, v) = \nu(\nabla u, \nabla v), \quad u, v \in X, \quad d(v, q) = (\text{div} v, q),$$

$$b(u, v, w) = ((u \cdot \nabla)v + \frac{1}{2}(\text{div}u)v, w) = \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v).$$

Next we introduce the discrete Stokes operator $A_h = -P_h \Delta_h$, where $\Delta_h$ is defined as following

$$(-\Delta_h u_h, v_h) = (\nabla u_h, \nabla v_h) \quad \forall u_h, v_h \in X_h.$$

As the restriction of operator $A_h$ on $V_h$ is reversible and we denote its inverse operator as $A_h^{-1}$, we can define the discrete $r$ order Sobolev norm in the space $V_h$:

$$||v_h||_r = ||A_h^{r/2}v_h||_{L^2}, \quad v_h \in V_h, \quad r \in R.$$

Specially, we have

$$||v_h||_0 = ||v_h||_{L^2}, \quad ||v_h||_1 = ||\nabla v_h||_0, \quad ||v_h||_2 = ||A_h v_h||_0, \quad v_h \in V_h.$$

In addition, we need some discrete Poincaré and Sobolev inequalities (see Temam [33], Hill-Süli [25] and Heywood-Rannacher [23]).

Lemma 2.1. The following inequalities hold

$$||v_h||_0 \leq \gamma_0 ||\nabla v_h||_0, \quad ||v_h||_{L^2} \leq c_0 ||v_h||_0^{1/2} ||\nabla v_h||_0^{1/2}, \quad \forall v_h \in X_h, $$

$$||v_h||_{L^p} \leq c_0 ||\nabla v_h||_0, \quad \forall v_h \in X_h,$n

$$||v_h||_{L^\infty} + ||v_h||_{L^2} \leq c_0 ||\nabla v_h||_0^\frac{3}{2} ||A_h v_h||_0^\frac{1}{2}, \quad \forall v_h \in V_h,$$

(2.3)

where $\gamma_0$ and $c_0$ are some positive constants depending on $\Omega$.

Before analyzing the discrete time error, we recall the smooth properties of $(u_h, p_h)$ given by Heywood-Rannacher in the reference [24].

Lemma 2.2. Assume (A1)-(A2) hold and $f, f_t, f_{tt} \in L^\infty(0, T; Y)$, then the finite element solution $(u_h, p_h)$ satisfies

$$\|\nabla u_h(t)\|^2_0 + \|A_h u_h(t)\|^2_0 + \int_0^t \|A_h u_h(s)\|^2_0 ds \leq \kappa_0,$$

$$\sigma^r(t)\|u_{ht}(t)\|^r_0 \leq \kappa_0, \quad r = 0, 1, 2,$$

$$\int_0^t \{ \|u_h(t)\|^2_0 + \|u_{ht}(t)\|^2_0 + \sigma(s)\|A_h u_h(s)\|^2_0 \} ds \leq \kappa_0,$$

$$\sigma^2(t)\|u_{htt}(t)\|^2_0 + \int_0^t [\sigma(s)\|u_{htt}(s)\|^2_0 + \sigma^2(s)\|A_h^{-1/2} u_{htt}(s)\|^2_0] ds \leq \kappa_0,$$

where $0 \leq t \leq T, \quad \sigma(t) = \min\{1, t\}$.  

\[ \]
In this paper, we always use the $\kappa_0, \kappa_1, \cdots$ etc. to represent the positive constant depending on $\nu, \Omega, T, u_0, f$. Then the error $(u - u_h, p - p_h)$ satisfies the following estimates (see the reference of Heywood and Rannacher [23]).

**Lemma 2.3.** Under the assumption of Lemma 2.2, the finite element solution $(u_h, p_h)$ satisfies
\[
\|u(t) - u_h(t)\|_0^2 + \nu^2 \|\nabla (u(t) - u_h(t))\|_0^2 + \tau\|p(t) - p_h(t)\|_0^2 \leq \kappa h^4,
\]
where $t \in (0, T)$.

Next we consider the Euler implicit/explicit scheme of the finite element approximation problem (2.1)-(2.2). This approach adopts the implicit scheme in the linear terms and explicit scheme in the nonlinear term to discretize the temporal variable. That results the linear equations with the constant coefficient matrices which we have to solve in every time step.

We choose the initial value $u_0^h = u_0h = P_h u_0$. We find the $(u^n_h, p^n_h) \in (X_h, M_h)$ through the following Euler implicit/explicit scheme:
\[
\begin{aligned}
&\left(\begin{array}{c}
d_t u^n_h + v_h + a(u^n_h, v_h) - d(v_h, p^n_h) + d(u^n_h, q_h) + b(u^{n-1}_h, u^n_h, v_h) \\
&\quad = (f(t_n), v_h) \forall (v_h, q_h) \in (X_h, M_h),
\end{array}\right)
\end{aligned}
\]
(2.4)

here $d_t u^n_h = \frac{1}{\tau}(u^n_h - u^{n-1}_h)$.

According to the definition of $u^0_h$, the following estimate holds
\[
\|u^0_h\|_\alpha = \|P_h u_0\|_\alpha \leq c_\alpha \|A^{\alpha/2} u_0\|_0, \quad \alpha = 0, 1, 2.
\]
(2.5)

To analyze the convergence of the solution for the (2.4), we give the Gronwall lemma as follows.

**Lemma 2.4.** Suppose $C, \tau$, and $a_n, b_n, c_n, d_n (n \geq 0)$ are non negative numbers satisfying
\[
a_m + \tau \sum_{n=1}^m b_n \leq \tau \sum_{n=0}^{m-1} a_n d_n + \tau \sum_{n=0}^{m-1} c_n + C, \quad m \geq 1.
\]

Then
\[
a_m + \tau \sum_{n=1}^m b_n \leq (\tau \sum_{n=0}^{m-1} c_n + C) \exp \left(\tau \sum_{n=0}^{m-1} d_n\right), \quad m \geq 1.
\]

3. The error estimate of the velocity. In this section, we will derive the upper bound of the error estimate $e^n = u_h(t_n) - u^n_h$ in the $H^1$-$L^2$ norm, where $1 \leq n \leq N$.

First, we assume $t = t_n$ in (2.1) and use the equality
\[
u_t h(t_n) - du_h(t_n) = \tau^{-1} \int_{t_{n-1}}^{t_n} (u_{ht}(t_n) - u_{ht}(t)) dt = \tau^{-1} \int_{t_{n-1}}^{t_n} (t - t_{n-1})(u_{ht}(t), v_h) dt.
\]
We have
\[
\begin{aligned}
&\left(\begin{array}{c}
d_t u_h(t_n), v_h \\
&\quad + a(u_h(t_n), v_h) - d(v_h, p_h(t_n)) + d(u_h(t_n), q_h) \\
&\quad + b(u_h(t_n), u_h(t_n), v_h) \\
&\quad = (f(t_n), v_h) - \frac{1}{\tau} \int_{t_{n-1}}^{t_n} (t - t_{n-1})(u_{ht}(t), v_h) dt.
\end{array}\right)
\end{aligned}
\]
(3.1)

reduced by the subtraction of (2.4) gives the error equation
\[(d_t e^n, v_h) + a(e^n, v_h) - d(v_h, \eta^n) + d(e^n, q_h) + b(e^{n-1}, u_h(t_{n-1}), v_h) + b(u_h(t_{n-1}), e^{n-1}, v_h) - b(e^{n-1}, e^{n-1}, v_h) = (E_n, v_h) \quad \forall (v_h, q_h) \in (X_n, M_h), \quad (3.2)\]

where

\[(E_n, v_h) = -\frac{1}{\tau} \int_{t_{n-1}}^{t_n} (t - t_{n-1})(u_{htt}(t), v_h)dt - b(u_h(t_n) - u_h(t_{n-1}), u_h(t_n), v_h) - b(u_h(t_{n-1}), u_h(t_n) - u_h(t_{n-1}), v_h). \quad (3.3)\]

**Lemma 3.1.** Under the assumption of Lemma 2.1, \(E_n\) satisfies

\[
\tau \sum_{n=1}^{m} \|A_h^{-1/2} P_h E_n\|_0^2 \leq \kappa_1 \tau^2, \quad 1 \leq m \leq N, \quad (3.4)
\]

\[
\sigma^2(t_m)\|E_m\|_0^2 + \tau \sum_{n=2}^{m} \sigma(t_n)\|E_n\|_0^2 \leq \kappa_1 \tau^2, \quad 1 \leq m \leq N, \quad (3.5)
\]

\[
\tau \sum_{n=3}^{m} \sigma^2(t_n)\|A_h^{-1/2} P_h dt E_n\|_0^2 \leq \kappa_1 \tau^2, \quad 3 \leq m \leq N. \quad (3.6)
\]

**Proof.** First, using (3.3) and Lemma 2.1, we can obtain

\[
\|A_h^{-1/2} P_h E_n\|_0^2 \leq c \tau^2 \int_{t_{n-1}}^{t_n} \|A_h^{-1/2} u_{htt}\|_0^2 dt + c \tau^2 (\|u_h(t_n)\|_1^2 + \|u_h(t_{n-1})\|_1^2) \int_{t_{n-1}}^{t_n} \|u_{ht}(t)\|_1^2 dt, \quad (3.7)
\]

that holds for \(n = 1, \cdots, N\). Summing (3.7) over \(n\) from 1 to \(m\) and using the Lemma 2.2, we have (3.4).

Secondly, from (3.3) and Lemma 2.1, we can find

\[
\|E_n\|_0 \leq c \tau^{-1/2} \int_{t_{n-1}}^{t_n} (t - t_{n-1})^{-1/2} \|u_{htt}\|_0^2 dt \right)^{1/2} \quad (3.8)
\]

that holds for \(1 \leq n \leq N\). Hence

\[
\sigma(t_n)\|E_n\|_0^2 \leq c \tau^2 \int_{t_{n-1}}^{t_n} \sigma(t)\|u_{htt}\|_0^2 dt + c \tau^2 (\|A_h u_h(t_n)\|_0^2 + \|A_h u_h(t_{n-1})\|_0^2) \int_{t_{n-1}}^{t_n} \|u_{ht}(t)\|_1^2 dt, \quad (3.9)
\]

\[
\sigma^2(t_n)\|E_n\|_0^2 \leq c \tau \int_{t_{n-1}}^{t_n} \sigma^2(t)\|u_{htt}(t)\|_0^2 dt \quad + c \tau (\|A_h u_h(t_n)\|_0^2 + \|A_h u_h(t_{n-1})\|_0^2) \int_{t_{n-1}}^{t_n} \sigma(t)\|u_{ht}(t)\|_1^2 dt, \quad (3.10)
\]
hold for $2 \leq n \leq N$. In addition, using the integral by part and Lemma 2.1, we get
\[
\|E(t)\|_0 \leq \|u_h(t_1)\|_0 + \tau^{-1} \int_{t_1}^{t} \|u_h(t_1)\|_0 ds \\
+ c\|u_h(t_1) - u_h(t_0)\|_1 (\|A_h u_h(t_1)\|_0 + \|A_h u_h(0)\|_0).
\] (3.11)

Summing (3.9) over $n$ from 2 to $m$ and using (3.10)-(3.11) and Lemma 2.2, we derive (3.5).

Furthermore, from (3.3), we deduce
\[
(d_t E_n, v_h) = -\tau^{-2} \int_{t_{n-1}}^{t_n} (t - t_{n-1}) \int_{t_{n-1}}^{t} (u_{h\ttt}(s), v_h) ds dt \\
- \tau^{-1} b(u_h(t_n) - 2u_h(t_{n-1}) + u_h(t_{n-2}), u_h(t_n), v_h) \\
- \tau^{-1} b(u_h(t_{n-1}) - u_h(t_{n-2}), u_h(t_n) - u_h(t_{n-1}), v_h) \\
- \tau^{-1} b(u_h(t_{n-1}) - u_h(t_{n-2}), u_h(t_n) - u_h(t_{n-1}), v_h) \\
- \tau^{-1} b(u_h(t_{n-2}), u_h(t_n) - 2u_h(t_{n-1}) + u_h(t_{n-2}), v_h),
\] (3.12)

where $3 \leq n \leq N$. According to (3.12) and Lemma 2.1, we have
\[
\|A_h^{1/2} P_h d_t E_n\|_0 \leq c\tau^{-3/2} \int_{t_{n-1}}^{t_n} (t - t_{n-1}) ^2 \|\int_{t_{n-1}}^{t} A_h^{-1/2} u_{h\ttt}(s) ds\|_0 dt \|\int_{t_{n-1}}^{t} A_h^{-1/2} u_{h\ttt}(s) ds\|_0 dt \\
+ c\tau \int_{t_{n-1}}^{t_n} \|u_{h\ttt}(t)\|_0 dt \int_{t_{n-1}}^{t_n} \|u_{h\ttt}(t)\|_0 dt \\
+ c\tau \int_{t_{n-1}}^{t_n} \|A_h u_h(t_n)\|_0 + \|A_h u_h(t_{n-2})\|_0 \int_{t_{n-1}}^{t_n} \|u_{h\ttt}(t)\|_0 dt \\
+ c\tau \int_{t_{n-1}}^{t_n} \|u_{h\ttt}(t)\|_0 dt \int_{t_{n-1}}^{t_n} \|u_{h\ttt}(t)\|_0 dt,
\] (3.13)

where $3 \leq n \leq N$. Summing (3.13) over $n$ from 3 to $m$, and using Lemma 2.2, we obtain (3.6).

Lemma 3.2. Suppose that the assumption of the condition for Lemma 2.2 and the convergence condition $\tau < \min\{1, \frac{\nu}{\nu_0 \kappa_2} \left( \frac{\nu_0}{\kappa_2} \right)^{\frac{1}{4}} \}$ are valid. Then $e^m$ satisfies the following estimate:
\[
\|e^m\|_0^2 + \tau \sum_{n=1}^{m} (\|d_t e^n\|_0^2 + \nu \|e^n\|_1^2) \leq \kappa_2 \tau^2,
\] (3.14)

where $1 \leq m \leq N$ and $\kappa_2 = \frac{16 \kappa_1}{\nu} \left( \frac{\nu_0}{\kappa_2} \right)^{\frac{1}{4}} \tau$. 


Proof. Let \( v_h = 2e^n \tau \in V_h \) and \( q_h = 0 \) in (3.2). Then we have
\[
\begin{align*}
\|e^n\|^2_0 - \|e^{n-1}\|^2_0 + \|dte^n\|^2_0 \tau^2 + 2\nu\|e^n\|^2_0 \tau \\
+ 2b(e^{n-1}, u_h(t_{n-1}), e^n) \tau - 2b(e^{n-1}, e^{n-1}, e^n) \tau + b(u_h(t_{n-1}), e^{n-1}, e^n) \tau \\
\leq \frac{\nu}{8} \|e^n\|^2_1 \tau + 8\nu^{-1} ||A_h^{-1/2} P_h E_n||^2_0 \tau.
\end{align*}
\]
(3.15)
Using Lemma 2.1 yields
\[
2\|b(e^{n-1}, u_h(t_{n-1}), e^n)\|_\tau \\
\leq 2\|e^{n-1}\|_L^2 (\|\nabla u_h(t_{n-1})\|_L^2 \|e^n\|_L^0 + \|\nabla e^n\|_L^2 \|u_h(t_{n-1})\|_L^0) \tau \\
\leq 2\nu \nu \|e^n\|^2_1 + ||e^{n-1}||^2_1 \tau + 2(\frac{4}{\nu})^3 c_0^8 ||h(t_{n-1})||_1^4 ||e^{n-1}||^2_0 \tau,
\]
\[
2\|b(u_h(t_{n-1}), e^{n-1}, e^n)\|_\tau \\
\leq 2\|e^{n-1}\|_L^2 (\|\nabla u_h(t_{n-1})\|_L^2 \|e^n\|_L^0 + \|\nabla e^n\|_L^2 \|u_h(t_{n-1})\|_L^0) \tau \\
\leq 4\nu \|e^n\|^2_1 + 2||e^{n-1}||^2_1 \tau + 2(\frac{8}{\nu})^3 c_0^8 ||h(t_{n-1})||_1^4 ||e^{n-1}||^2_0 \tau,
\]
\[
2\|b(e^{n-1}, e^{n-1}, e^n)\|_\tau \leq \frac{\nu}{8} \|e^n\|^2_1 + ||e^{n-1}||^2_1 \tau + 2(\frac{4}{\nu})^3 c_0^8 ||e^{n-1}||^4_1 ||e^{n-1}||^2_0 \tau.
\]
Hence, combining the above estimates with (3.15), we have
\[
\begin{align*}
\|e^n\|^2_0 - \|e^{n-1}\|^2_0 + \|dte^n\|^2_0 \tau^2 + \nu\|e^n\|^2_0 \tau &+ \frac{\nu}{2} (\|e^n\|^2_1 - ||e^{n-1}||^2_1) \tau \\
&\leq d_{n-1} ||e^{n-1}||^2_0 \tau + 4\nu^{-1} ||A_h^{-1/2} P_h E_n||^2_0 \tau \\
&+ 2(\frac{4}{\nu})^3 c_0^8 ||e^{n-1}||^2_1 ||e^{n-1}||^2_0 \tau,
\end{align*}
\]
(3.16)
where \( 1 \leq n \leq N \)
\[
d_{n-1} = 3(\frac{8}{\nu})^3 c_0^8 ||h(t_{n-1})||_1^4.
\]
(3.17)
Furthermore, summing (3.16) over \( n \) from 1 to \( m \) leads to the following inequality:
\[
\|e^n\|^2_0 + \tau \sum_{n=1}^m (\|dte^n\|^2_0 \tau + \nu\|e^n\|^2_1) \\
\leq \tau \sum_{n=0}^{m-1} d_n ||e^n||^2_0 + 8\nu^{-1} \tau \sum_{n=1}^N ||A_h^{-1/2} P_h E_n||^2_0 \\
+ \tau \sum_{n=0}^{m-1} 2(\frac{4}{\nu})^3 c_0^8 ||e^n||^4_1 ||e^n||^2_0 \tau.
\]
(3.18)
Assuming
\[
a_n = \|e^n\|^2_0, \quad b_n = \|dte^n\|^2_0 \tau + \nu\|e^n\|^2_1, \\
c_n = 2(\frac{4}{\nu})^3 c_0^8 ||e^n||^4_1 ||e^n||^2_0, \quad C = 8\nu^{-1} \tau \sum_{n=1}^N ||A_h^{-1/2} P_h E_n||^2_0,
\]
and using the Lemma 2.4 in (3.18), we have
\[
\|e^m\|^2 + \tau \sum_{n=1}^{m} (\|d_t e^n\|^2_0 + \nu \|e^n\|^2_1) \\
\leq e \sum_{n=1}^{m} d_n (8 \nu^{-1} \tau \sum_{n=1}^{N} \|A_h^{-1/2} P_k E_n\|^2_0 + \tau \sum_{n=0}^{m} 2 (\frac{4}{\nu^3} c_0^8 \|e^n\|^2_1 \|e^n\|^2_0 \tau). \quad (3.19)
\]

According to the above discussion, we can prove (3.14) by the inductive method. From the definition of $w_0^0$, we know (3.14) holds for $m = 0$. In view of (3.16) and Lemma 3.1, (3.14) holds for $m = 1$. Assuming (3.14) holds for $m = 0, 1, \ldots, J$, then we need to prove it holds for $m = J + 1$.

Combining the induction hypothesis, Lemma 2.2 and convergence condition, we obtain
\[
\tau \sum_{n=0}^{J} d_n \leq 3(\frac{8}{\nu})^3 c_0^8 \kappa_2^2 \tau, \quad (3.20)
\]
\[
\tau \sum_{n=0}^{J} 2 (\frac{4}{\nu^3} c_0^8 \|e^n\|^2_1 \|e^n\|^2_0 \tau \leq 2 \nu^{-2} (\frac{4}{\nu})^3 c_0^8 \kappa_2^2 \tau^5 \leq \frac{8}{\nu} \kappa_1 \tau^2. \quad (3.21)
\]

Let $m = J + 1$ in (3.19). Then by substituting (3.20)–(3.21) into it, we obtain that (3.14) holds for $m = J + 1$.

**Lemma 3.3.** Suppose the assumption of Lemma 2.2 and convergence condition
\[
\tau < \min\{1, \frac{\nu}{4c_0^3 \kappa_2} \left(\frac{\nu \kappa_1}{c_0} \right)^{\frac{1}{2}}, \left(\frac{\nu}{4} \right)^2 \left(1 + c_0 \right)^2 \kappa_3 \}
\]
are valid. Then $e^m$ satisfies
\[
\sigma(t_m) \|e^m\|^2 + \tau \sum_{n=1}^{m} \sigma(t_n) (\|d_t e^n\|^2_1 \|e^n\|^2_0 + \nu \|A_h e^n\|^2_0) \leq \kappa_3 \tau^2, \quad (3.22)
\]
where $1 \leq m \leq N$, $\kappa_3 = \frac{12}{\nu} \kappa_1 + \nu^{-1} (1 + \frac{16}{\nu} c_0^2 (1 + c_0)^2 \kappa_2) \kappa_2$.

**Proof.** Let $v_h = 2 A_h e^n \tau \in V_h$ and $q_h = 0$ in (3.2), we have
\[
\|e^n\|^2_1 - \|e^{n-1}\|^2_1 + \|d_t e^n\|^2_1 \|e^n\|^2_0 \tau + 2 \nu \|A_h e^n\|^2_0 \tau + 2 b(u_h(t_{n-1}), A_h e^n) \tau \\
+ 2 b(u_h(t_{n-1}), e^{n-1}, A_h e^n) \tau - 2 b(e^{n-1}, e^{n-1}, A_h e^n) \tau \\
\leq \frac{\nu}{4} \|A_h e^n\|^2_0 \tau + 4 \nu^{-1} \|E_n\|^2_0 \tau. \quad (3.23)
\]

According to Lemma 2.1, we obtain
\[
2 b(e^{n-1}, u_h(t_{n-1}), A_h e^n) |\tau + 2 b(u_h(t_{n-1}), e^{n-1}, A_h e^n) |\tau \\
\leq 4 (\|e^{n-1}\|_{L^2} \left\| \nabla u_h(t_{n-1}) \right\|_{L^2} + \|e^{n-1}\|_{L^2} \left\| u_h(t_{n-1}) \right\|_{L^2}) \|A_h e^n\|_0 \tau \\
\leq 4 c_0 (1 + c_0) \|u_h(t_{n-1})\|^2_1 + \|e^{n-1}\|^2_1 \|A_h u_h(t_{n-1})\|^2_0 \|e^{n-1}\|^2_1 \|A_h e^n\|^2_0 \\
\leq \frac{\nu}{4} \|A_h e^n\|^2_0 \tau + \frac{16}{\nu} c_0^2 (1 + c_0)^2 \|u_h(t_{n-1})\|^2_1 + \|A_h u_h(t_{n-1})\|^2_0 \|e^{n-1}\|^2_1 \|A_h e^n\|^2_0 \tau \\
\leq \frac{\nu}{4} \|A_h e^n\|^2_0 \tau + \frac{\nu}{8} \|A_h e^{n-1}\|^2_0 \tau + \frac{1}{2} \left(\frac{4}{\nu^3} c_0^8 (1 + c_0)^4 \|e^{n-1}\|^2_1 \|A_h e^{n-1}\|^2_0 \right) \tau.
\]
Hence, taking the above estimate into (3.3) and using Lemma 2.2, we get
\begin{align*}
\|e^n\|_1^2 - \|e^{n-1}\|_1^2 + \|d_i e^n\|_1^2 \rho^2 + \frac{5\nu}{4} \|A_h e^n\|_0^2 \tau - \frac{\nu}{8} \|A_h e^{n-1}\|_0^2 \tau \\
\leq 4\nu^{-1}\|E_n\|_0^2 \tau + d_{n-1}\|e^{n-1}\|_1^2 \tau + \frac{4}{\nu} (\frac{\gamma}{\nu})^3 \sigma_0^2 (1 + \sigma_0)^4 \|e_h^{n-1}\|_1^6,
\end{align*}
where $1 \leq n \leq N$,
\begin{align*}
d_{n-1} = \frac{16}{\nu} \sigma_0^2 (1 + \sigma_0)^2 \|u_h(t_{n-1})\|_1 \|A_h u_h(t_{n-1})\|_0 \\
\leq \frac{16}{\nu} \sigma_0^2 (1 + \sigma_0)^2 \kappa_0.
\end{align*}
(3.25)

Using $\sigma(t_n)$ to multiply (3.24), we have
\begin{align*}
\sigma(t_n)\|e^n\|_1^2 - \sigma(t_{n-1})\|e^{n-1}\|_1^2 + \sigma(t_n)\|d_i e^n\|_1^2 \rho^2 + \frac{5\nu}{4} \sigma(t_n) \|A_h e^n\|_0^2 \tau \\
\leq \frac{\nu}{4} \sigma(t_{n-1}) \|A_h e^{n-1}\|_0^2 \tau + 4\nu^{-1} \sigma(t_n) \|E_n\|_0^2 \tau + (1 + d_{n-1})\|e^{n-1}\|_1^2 \tau \\
+ \frac{4}{\nu} (\frac{\gamma}{\nu})^3 \sigma_0^2 (1 + \sigma_0)^4 \sigma(t_{n-1})\|e_h^{n-1}\|_1^6,
\end{align*}
here $2 \leq n \leq N$. Summing (3.26) over $n$ from 2 to $m$ and using Lemma 3.2 and (3.25), we deduce
\begin{align*}
\sigma(t_m)\|e^m\|_1^2 + \tau \sum_{n=2}^m \sigma(t_n)\|A_h e^n\|_0^2 + \|d_i e^n\|_1^2 \tau \\
\leq \tau \|e^1\|_1^2 + \tau^2 \|A_h e^1\|_0^2 + \tau \sum_{n=1}^{m-1} (1 + \sigma_0)^2 \|e^n\|_1 \\
+ 4\nu^{-1} \tau \sum_{n=1}^N \sigma(t_n) \|E_n\|_0^2 + \tau \sum_{n=1}^{m-1} \frac{4}{\nu} (\frac{\gamma}{\nu})^3 \sigma_0^2 (1 + \sigma_0)^4 \sigma(t_{n-1})\|e_h^{n-1}\|_1^6,
\end{align*}
(3.27)
where $2 \leq m \leq N$. Owing to (3.24), we have
\begin{align*}
\|e^1\|_1^2 + \|d_i e^1\|_1^2 \tau^2 + \|A_h e^1\|_0^2 \leq \frac{4}{\nu} \|E_1\|_0^2 \tau.
\end{align*}
(3.28)

Taking (3.25) and (3.28) into (3.3), we obtain
\begin{align*}
\sigma(t_m)\|e^m\|_1^2 + \tau \sum_{n=2}^m \sigma(t_n)\|A_h e^n\|_0^2 + \|d_i e^n\|_1^2 \tau \\
\leq \nu^{-1} (1 + \frac{16}{\nu} \sigma_0^2 (1 + \sigma_0)^2 \kappa_0)^2 \frac{\gamma}{\nu} \tau^2 + 4\nu^{-1} \tau \sum_{n=1}^N \sigma(t_n) \|E_n\|_0^2 \\
+ \tau \sum_{n=1}^{m-1} \frac{4}{\nu} (\frac{\gamma}{\nu})^3 \sigma_0^2 (1 + \sigma_0)^4 \sigma(t_{n-1})\|e_h^{n-1}\|_1^6,
\end{align*}
(3.29)
here $2 \leq m \leq N$.

Because of the above discussion, we can prove (3.22) by the inductive method. Using the definition of $u_h^m$ and (3.28), (3.21) hold for $m = 0, 1$. Now we suppose (3.22) holds for $m = 0, 1, \cdots, J$, and we need to prove it holds for $m = J + 1$. 

Combining the inductive hypothesis, Lemma 2.2 with convergence condition leads to
\[
\tau \sum_{n=1}^{J} \left( \frac{4}{\nu} \right)^3 c_0^4(1 + c_0)^4 \sigma(t_n)\|e^n\|^6_1 \leq \left( \frac{4}{\nu} \right)^3 c_0^4(1 + c_0)^4 \nu^{-2} \kappa_3^2 \kappa_3^5 \tau^2 \leq \frac{4}{\nu} \kappa_1 \tau^2.
\]

(3.30)

Supposing \( m = J + 1 \) in (3.29), taking (3.30) into (3.29) and using Lemma 3.1 and (3.28), we obtain (3.22) holds for \( m = J + 1 \).

4. The error estimate of pressure. In this section, we will derive the upper bound of the pressure error \( p_h(t_n) - p_n^\epsilon \) in \( L^2 \) norm, where \( 1 \leq n \leq N \).

**Lemma 4.1.** Under the assumption of Lemma 3.3, \( e^m \) satisfies
\[
\tau \sum_{n=1}^{m} \sigma(t_n)\|d_t e^n\|^2_0 \leq \kappa_4 \tau^2,
\]
where \( 1 \leq m \leq N \).

**Proof.** According to (3.2), Lemma 2.1 and Lemma 2.2, we obtain
\[
\begin{align*}
\sigma(t_n)\|d_t e^n\|^2_0 \tau &\leq c\sigma(t_n)(1 + \|u_h(t_{n-1})\|_t^2)\|A_h e^n\|^2_0 \tau \\
&+ c\|e^{n-1}\|^2_1 \sigma(t_{n-1})\|A_h e^{n-1}\|^2_0 \tau + c\sigma(t_n)\|E_n\|^2_0 \tau.
\end{align*}
\]

(4.2)

where \( 1 \leq n \leq N \). Summing (4.2) over \( n \) from 1 to \( m \) and using Lemma 2.2, 3.1, 3.2 and 3.3 yields (4.1).

Next we will give the upper bound of the pressure error \( p_h(t_n) - p_n^\epsilon \) in \( L^2 \) norm. First, we need estimate the \( d_t e^n \) in \( L^2 \) norm. From (3.2), we have
\[
\begin{align*}
&\quad (d_t e^n, v_h) + a(d_t e^n, v_h) \\
&\quad + b(d_t e^{n-1}, u_h(t_{n-1}), v_h) + b(e^{n-2}, d_t u_h(t_{n-1}), v_h) \\
&\quad + b(d_t u_h(t_{n-1}), e^{n-1}, v_h) + b(u_h(t_{n-2}), d_t e^{n-1}, v_h) \\
&\quad - b(d_t e^{n-1}, e^{n-1}, v_h) - b(e^{n-2}, d_t e^{n-1}, v_h) \\
&\quad = (d_t E_n, v_h) \quad \forall v_h \in V_h,
\end{align*}
\]

(4.3)

where \( 2 \leq n \leq N \). We assume \( v_h = 2d_t e^n \tau \in V_h \), \( q_h = 0 \) in (4.3). Then we have
\[
\|d_t e^n\|^2_0 - \|d_t e^{n-1}\|^2_0 + 2\nu\|d_t e^n\|^2_1 \tau + 2b(d_t e^{n-1}, u_h(t_{n-1}), d_t e^n)\tau \\
+ 2b(d_t u_h(t_{n-1}), e^{n-1}, d_t e^n)\tau + 2b(u_h(t_{n-2}), d_t e^{n-1}, d_t e^n)\tau \\
- 2b(d_t e^{n-1}, e^{n-1}, d_t e^n)\tau - 2b(e^{n-2}, d_t e^{n-1}, d_t e^n)\tau \\
\leq \frac{\nu}{8}\|d_t e^n\|^2_1 \tau + 8\nu^{-1} \|A_h^{1/2} P_h d_t E_n\|^2_0 \tau.
\]

(4.4)

Using Lemma 2.1 leads to
\[
\begin{align*}
&\quad 2|b(d_t e^{n-1}, u_h(t_{n-1}), d_t e^n)|\tau + 2|b(u_h(t_{n-2}), d_t e^{n-1}, d_t e^n)|\tau \\
&\quad \leq 2\|d_t e^{n-1}\|_{L^2} (\|\nabla u_h(t_{n-1})\|_{L^3} + \|\nabla u_h(t_{n-2})\|_{L^3} \|d_t e^n\|_{L^6} \\
&\quad + 2\|d_t e^{n-1}\|_{L^2} (\|u_h(t_{n-1})\|_{L^\infty} + \|u_h(t_{n-2})\|_{L^\infty}) \|\nabla d_t e^n\|_{L^2}
\end{align*}
\]
Combining the above estimation formula, (4.4), Lemma 2.1 and Lemma 2.2, we have

\[
\begin{align*}
\|d_t e^n\|_0^2 + \|d_t e^{n-1}\|_0^2 + \nu \|d_t e^n\|_0^2 &\leq d_{n-1} \|d_t e^{n-1}\|_0^2 + \nu \|d_t e^n\|_0^2 + 4\nu^{-1} \|A_h^{-1/2} P_h d_t E_n\|_0^2 \tau \\
&\quad + 32\nu^{-1} \gamma_0 e_0^4 \int_{t_{n-2}}^{t_n} \|u_{ht}(t)\|_2^2 dt \left(\|e^{n-1}\|_0^2 + \|e^{n-2}\|_0^2\right) \\
&\quad + 8\nu^{-1} \gamma_0 e_0^2 (1 + c_0)^2 \|d_t e^{n-1}\|_0^2 \|A_h e^{n-1}\|_0^2 + \|A_h e^{n-2}\|_0^2 \|e^{n-2}\|_0^2 \|e^{n-2}\|_0^2 \tau,
\end{align*}
\]

where

\[
d_{n-1} = 8\nu^{-1} c_0^2 (1 + c_0)^2 \|u_h(t_{n-2})\|_1 \|A_h u_h(t_{n-2})\|_0 + \|u_h(t_{n-2})\|_1 \|A_h u_h(t_{n-2})\|_0
\]

\[
\leq 8\nu^{-1} c_0^2 (1 + c_0)^2 \kappa_0.
\]

Using \(\sigma^2(t_n)\) to multiply the above inequality and Lemma 2.1, we obtain

\[
\begin{align*}
\sigma^2(t_n) \|d_t e^n\|_0^2 - \sigma^2(t_{n-1}) \|d_t e^{n-1}\|_0^2 + \nu \sigma^2(t_n) \|d_t e^n\|_0^2 &\leq (3 + 2d_{n-1}) \sigma(t_{n-1}) \|d_t e^{n-1}\|_0^2 + 4\nu^{-1} \sigma^2(t_n) \|A_h^{-1/2} P_h d_t E_n\|_0^2 \tau \\
&\quad + 32\nu^{-1} \gamma_0 e_0^4 \int_{t_{n-2}}^{t_n} \|u_{ht}(t)\|_2^2 dt \left(2\sigma(t_{n-1}) \|e^{n-1}\|_0^2 + 3\sigma(t_{n-2}) \|e^{n-2}\|_0^2\right) \\
&\quad + 16\nu^{-1} \gamma_0 e_0^2 (1 + c_0)^2 \sigma(t_{n-1}) \|d_t e^{n-1}\|_0^2 \|A_h e^{n-1}\|_0^2 + \|A_h e^{n-2}\|_0^2 \|e^{n-2}\|_0^2 \|e^{n-2}\|_0^2 \tau,
\end{align*}
\]

holds for \(3 \leq n \leq N\). Summing (4.7) over \(n\) from 3 to \(m\), using (4.6), Lemma 2.2 and Lemma 3.1-3.3, we obtain

\[
\sigma^2(t_m) \|d_t e^m\|_0^2 \leq \sigma^2(t_2) \|d_t e^2\|_0^2 + \kappa e_0^2, \quad 3 \leq m \leq N.
\]

Supposing \(m = 1, 2\) in Lemma 3.2 and using (4.8), we have

\[
\sigma^2(t_m) \|d_t e^m\|_0^2 \leq \sigma^2(t_2) \|d_t e^2\|_0^2 + \kappa e_0^2, \quad 1 \leq m \leq N.
\]
Thanks to the assumption (A2), Lemma 2.1 and (3.2), we get
\[
\|\eta^m\|_0 \leq c(\|d_1e^m\|_0 + \|e^m\|_1) + c\|e^m\|_1(\|u_h(t_m)\|_1 + \|u_h^m\|_1)
\]
\[
+ c\|E_m\|_0,
\]
i.e.
\[
\sigma^2(t_m)\|\eta^m\|_0^2 \leq c\sigma^2(t_m)\|d_1e^m\|_0^2
\]
\[
+ c\sigma(t_m)\|e^m\|_1^2 + \sigma^2(t_m)\|E_m\|_0^2.
\] (4.10)
Because of (4.9)-(4.10) and Lemma 3.1, 3.2, we obtain
\[
\sigma^2(t_m)\|\eta^m\|_0^2 \leq \kappa\tau^2, \quad 1 \leq m \leq N.
\] (4.11)
Combining (4.11), Lemma 3.2-3.3 and Theory 2.3 yields the error estimate for the numerical solution \((u_h^m, p_h^m)\).

**Theorem 4.2.** Under the assumption of Lemma 3.3, \((u_h^m, p_h^m)\) satisfies
\[
\|u(t_m) - u_h^m\|_0 \leq \kappa(h^2 + \tau), \quad t_m \in [0, T],
\] (4.12)
\[
\|u(t_m) - u_h^m\|_1 \leq \kappa(h + \sigma^{-\frac{1}{2}}(t_m)\tau), \quad t_m \in (0, T],
\] (4.13)
\[
\|p(t_m) - p_h^m\|_0 \leq \kappa(\sigma^{-\frac{1}{2}}(t_m)h + \sigma^{-1}(t_m)\tau), \quad t_m \in (0, T],
\] (4.14)

5. **Numerical results.** In this section, we will present two numerical examples to verify the theoretical results. The domain \(\Omega\) in the two examples is the unit cube in \(R^3\). We divide the domain \(\Omega\) into the meshes of tetrahedron and use the \(P1b-P1\) finite element pair (see Gerbeau [9]) for the velocity and pressure.

The first example has an exact solution whose velocity \(u = (u_1, u_2, u_3)^T\) and pressure \(p\) are given by

\[
u_1 = 10x^2(x - 1)^2(y(y - 1)(2y - 1))z^2(z - 1)^2 - y^2(y - 1)^2z(z - 1)(2z - 1)e^{-t};
\]
\[
u_2 = 10y^2(y - 1)^2(-x(x - 1)(2x - 1)z^2(z - 1)^2 + x^2(x - 1)^2z(z - 1)(2z - 1))e^{-t};
\]
\[
u_3 = 10z^2(z - 1)^2(x(x - 1)(2x - 1)y^2(y - 1)^2 - x^2(x - 1)^2y(y - 1)(2y - 1))e^{-t};
\]
\[
p = 10(x^2 + y^2 + z^2 - 1)te^{-t}.
\] (5.1)

Obviously the velocity satisfies divergence free \(\text{div} u = 0\) and the boundary condition \(u|_{\partial \Omega} = 0\). Furthermore the body force \(f\) can be deduced by taking the exact solution into equation (1.1).

First, we compute the velocity and pressure at time \(T = 1.0\) with the viscosity \(\nu = 1.0\). The space mesh size is \(h = 1/24\) and the time step is \(\tau = 0.001, 0.01, 0.1\) respectively. Figure 1 shows the exact value and computational values of velocity \(u\) and pressure \(p\) at twenty five equidistant points in the vertical center line where \(x = 0.5, y = 0.5\). Seen from Figure 1 the values of the velocity component \(u_1, u_2\) and pressure \(p\) are very close to the exact value when we use the large time step. The component of velocity \(u_3\) has some tiny errors to the exact values, and the errors increase when the time steps increase, but the max error is still less than \(10^{-8}\).

Next we compute the velocity and pressure at time \(T = 6.0\) with different time steps, and compare the \(H^1\) and \(L^2\) norm with different space meshes under the same time step. When we fix a time step, the \(L^2\) and \(H^1\) norm in Tables 1 and 2 increase monotonically but the increased values reduce quickly. Furthermore the values of \(L^2\) norm of the pressure in the Table 3 are all 0.0128002 when the space meshes refine.
These results indicate the stability and convergence of the Euler implicit/explicit scheme. In addition, the errors of velocity and pressure with the different time steps \( \tau = 0.2, 0.4, 0.75, 1.0 \) are provided in Figure 2. Seen from Figure 2, the convergence rates of velocity and pressure keep well when we take the time step as \( \tau = 0.2, 0.4 \). The convergence rates of pressure error in \( L^2 \) norm reduce as the meshes refine when the time step increases to \( \tau = 0.75, 1.0 \). This is mainly because the large time step leads to more error coming from the temporal discretization, and this part of error is no longer small relative to the error of the space discretization.

Table 1. The norm \( \| u_h^m \|_0 \) of the Euler Explicit/Implicit scheme (\( T=6.0 \)).

| \( \tau \) | 0  | 0.2 | 0.3 | 0.4 | 0.5 | 0.75 | 1.0 |
|---|---|---|---|---|---|---|---|
| 16 | 0.641684e-5 | 0.642277e-5 | 0.642915e-5 | 0.643601e-5 | 0.645542e-5 | 0.64787e-5 |
| 24 | 0.655394e-5 | 0.65601e-5 | 0.656672e-5 | 0.65738e-5 | 0.659394e-5 | 0.661811e-5 |
| 32 | 0.660154e-5 | 0.660778e-5 | 0.661448e-5 | 0.662165e-5 | 0.664204e-5 | 0.66651e-5 |
| 40 | 0.662344e-5 | 0.662971e-5 | 0.663644e-5 | 0.664368e-5 | 0.666417e-5 | 0.66888e-5 |
| 48 | 0.663529e-5 | 0.664159e-5 | 0.664832e-5 | 0.665557e-5 | 0.667614e-5 | 0.670082e-5 |

Table 4 presents the error of the velocity under the \( L^2 \) and \( H^1 \) norm and the error of pressure under \( L^2 \) norm at \( T = 1.0 \) with time step \( \tau = 0.01 \) and different space mesh size. The convergence order of velocity is two in \( L^2 \) norm and one in \( H^1 \) norm. This phenomenon is consistent with the results of theoretical analysis (4.12) and (4.13). The convergence order of pressure in Table 4 is bigger than one which shows the super convergence property.
Table 2. The norm $\|\nabla u_m\|_0$ of the Euler Explicit/Implicit scheme ($T=6.0$).

| $1/h$ | $1/16$ | $1/24$ | $1/32$ | $1/40$ | $1/48$ |
|-------|-------|-------|-------|-------|-------|
| $\tau$ | 0.2   | 0.3   | 0.4   | 0.5   | 0.75  | 1.0   |
| 0.528852e-4 | 0.529345e-4 | 0.529924e-4 | 0.530582e-4 | 0.532305e-4 | 0.53425e-4 |
| 0.537892e-4 | 0.538435e-4 | 0.539056e-4 | 0.539624e-4 | 0.54138e-4 | 0.543468e-4 |
| 0.539084e-4 | 0.539595e-4 | 0.54019e-4 | 0.540862e-4 | 0.542633e-4 | 0.544733e-4 |
| 0.539725e-4 | 0.540276e-4 | 0.540809e-4 | 0.541456e-4 | 0.543187e-4 | 0.545238e-4 |

Table 3. The norm $\|p_m\|_0$ of the Euler Explicit/Implicit scheme ($T=6.0$).

| $1/h$ | $1/16$ | $1/24$ | $1/32$ | $1/40$ | $1/48$ |
|-------|-------|-------|-------|-------|-------|
| $\tau$ | 0.2   | 0.3   | 0.4   | 0.5   | 0.75  | 1.0   |
| 0.0128003 | 0.0128003 | 0.0128003 | 0.0128002 | 0.0128002 | 0.0128002 |
| 0.0128003 | 0.0128003 | 0.0128003 | 0.0128003 | 0.0128003 | 0.0128003 |
| 0.0128002 | 0.0128002 | 0.0128002 | 0.0128002 | 0.0128002 | 0.0128002 |
| 0.0128002 | 0.0128002 | 0.0128002 | 0.0128002 | 0.0128002 | 0.0128002 |

Figure 2. Comparison of the error for the velocity $u$ and pressure $p$ with different times steps $\tau$.

Table 4. The convergence of the Euler Explicit/Implicit scheme.

| $h$ | $1/16$ | $1/24$ | $1/32$ | $1/40$ | $1/48$ |
|-----|-------|-------|-------|-------|-------|
| rate | 0.402465e-1 | / | 0.187043 | 0.563091e-3 | / |
| $\|\nabla u_0\|_{L^2}$ | 0.402465e-1 | 2.0381 | 1.943 | 1.8235 |
| $\|\nabla u_0\|_{L^2}$ | 0.187043 | 1.1356 | 0.268825e-3 |
| $\|\nabla u_0\|_{L^2}$ | 0.563091e-3 | 1.8235 |
| $\|\nabla u_0\|_{L^2}$ | / | / |

The second example is the driven cavity flow which serves as a standard benchmark problem for the Navier-Stokes equations. We consider a unit cube $\Omega = [0,1] \times [0,1] \times [0,1]$. The velocity satisfies $u_1 = 1$, $u_2 = 0$, $u_3 = 0$ on the top surface of $\Omega$, and zero boundary condition on the other surface of $\Omega$. The distribution of pressure at time $T = 1.0$ with different time steps $\tau = 0.1, 0.05, 0.01$ is provided in Figure 3. Graphically, there is no observable difference between the three time steps for the pressure. Figure 4 presents the vector of the velocity at three surfaces ($y = 0.75, 0.5, 0.25$) under the different time steps. As can be seen, the numerical velocity at surface of $y = 0.75$ is almost as the same as $y = 0.25$ under the same time step. These results are in good agreement with the symmetry of physical phenomena.
Figure 3. Comparison of the pressure $p$ at $T = 1.0$ with different time steps ($\tau = 0.1, 0.05, 0.01$).

Figure 4. Comparison of the velocity $u$ at $T = 1.0$ in different $y$-plane ($y = 0.75, 0.5, 0.25$) with different time steps $\tau = 0.1$ (top), $0.05$ (middle), $0.01$ (bottom).

In conclusion, the Euler implicit/explicit scheme is almost unconditionally stable and convergent. Because of its advantage of excellent stability and convergence with the large time step and easy calculation, this scheme has even more potential in large-scale computation.

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