Exact Supersymmetric String Solutions in Curved Gravitational Backgrounds

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Abstract
We construct a new class of exact and stable superstring solutions based on $N = 4$ superconformal world-sheet symmetry. In a subclass of these, the full spectrum of string excitations is derived in a modular-invariant way.

In the weak curvature limit, our solutions describe a target space with non-trivial metric and topology, and generalize the previously known (semi) wormhole. The effective field theory limit is identified in certain cases, with solutions of the $N = 4$ and $N = 8$ extended gauged supergravities, in which the number of space-time supersymmetries is reduced by a factor of 2 because of the presence of non-trivial dilaton, gravitational and/or gauge backgrounds.

In the context of string theory, our solutions correspond to stable non-critical superstrings in the strong coupling region; the super-Liouville field couples to a unitary matter system with central charge $5 \leq \hat{c}_M \leq 9$. 

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1 Introduction

The study of string propagation in non-trivial gravitational backgrounds can provide a better understanding of quantum gravitational phenomena at short distances. Non-trivial classical string backgrounds can be obtained by two different methods. The first makes use of a two-dimensional $\sigma$-model where the space-time backgrounds correspond to field-dependent coupling constants [1]. The vanishing of the corresponding $\beta$-functions is identified with the background field equations of motion in the target space [1]-[3]. The second approach consists of replacing the free space-time coordinates by a non-trivial (super)conformal system, which, in the semiclassical limit, can be interpreted as describing a string propagation in non-trivial space-time [4]-[19].

The two methods are useful and complementary. The $\sigma$-model approach provides a clear geometric interpretation, but it has the disadvantage of the $\alpha'$-expansion, which is valid only when all curvatures and derivatives on space-time fields are small. In this way, one can easily obtain approximate solutions, but their possible extension to exact ones is in general difficult to prove. The conformal field theory approach takes into account all orders in $\alpha'$ automatically and has the main advantage of deriving exact string vacua. However, the background interpretation of a given exact string solution is, in general, an ill-defined notion [10]. Indeed, the notion of space-time dimensionality and topology breaks down for a solution that involves highly curved backgrounds, namely when the metric and/or gauge field curvatures are of the order of the string scale.

A typical example concerning the dimension and topology of space-time is that of the $SU(2)$ level $k$ group manifold compactification. For large $k$ (small curvature) the target space is a three-dimensional sphere $S^3$. For small $k$ (high curvature) this background interpretation fails. It is in fact well known that the $SU(2)_{k=1}$ WZW model is equivalent to a $c = 1$ conformal system defined by one free bosonic coordinate compactified on a cycle of radius $R = \frac{2}{k} = \sqrt{2}$ (self-dual point). Naively one may interpret this toroidal compactification as a one-dimensional space with $S^1$ topology, which is in contradiction with the three-dimensional interpretation with $S^3$.
topology of $SU(2)_{k=1}$. This shows that both the dimensionality and the topology of target space are not well-defined concepts in string theory. In general, a background interpretation of a given string solution exists only when the lower Kaluza-Klein excitations have masses much smaller than the typical string scale ($M_{st} = \alpha'^{-1/2}$).

In this work we present a special class of exact solutions of the heterotic and type-II superstrings, which are based on some $N = 4$ superconformal systems [10]. According to the realization of the underlying superconformal algebra, our solutions are classified into different subclasses. More explicitly, we arrange the degrees of freedom of the ten supercoordinates in three superconformal systems:

\[
\{\hat{c}\} = 10 = \{\hat{c} = 2\} + \{\hat{c} = 4\}_1 + \{\hat{c} = 4\}_2. \tag{1.1}
\]

The $\hat{c} = 2$ system is saturated by two free superfields. In one variation of our solutions, one of the two free superfields is chosen to be the time-like supercoordinate and the other to be one of the nine space-like supercoordinates. In other variations, both supercoordinates are Euclidean or even compactified on a one- or two-dimensional torus. The remaining eight supercoordinates appear in groups of four in $\{\hat{c} = 4\}_1$ and $\{\hat{c} = 4\}_2$. Both $\{\hat{c} = 4\}_A$ systems exhibit $N = 4$ superconformal symmetry of the Ademollo et al. type [20]. The non-triviality of our solutions follows from the fact that there exist realizations of the $\hat{c} = 4$, $N = 4$ superconformal systems that are based on geometrical and topological non-trivial spaces, other than the $T^4/Z_2$ orbifold and the $K_3$ compact manifold [11].

The first subclass is characterized by two integer parameters $k_1$, $k_2$, which are the levels of two $SU(2)$ group manifolds. For weakly curved backgrounds (large $k_A$) these solutions can be interpreted in terms of a ten-dimensional, but topologically non-trivial, target space of the form $R^4 \otimes S^3 \otimes S^3$. In the special limit $k_2 \to \infty$ one obtains the semi-wormhole solution of Callan, Harvey and Strominger [3], based on a six-dimensional flat background, combined with a four-dimensional space $W^{(4)}_{k_1} \equiv U(1) \otimes SU(2)_{k_1}$, which describes the semi-wormhole. The underlying superconformal field theory associated to $W^{(4)}_{k_1}$ includes a supersymmetric $SU(2)_{k_1}$ WZW model describing the three coordinates of $S^3$ as well as a non-compact dimension with a background
charge, describing the scale factor of the sphere. Furthermore, it was known that the five-brane background \[ M^6 \otimes W_{k_1}^{(4)} \] admits two covariantly constant spinors and, therefore, respects up to two space-time supersymmetries consistently with the \( N = 4 \) symmetry of the \( W_{k_1}^{(4)} \) superconformal system \[ 9 \]. The explicit representation of the desired \( N = 4 \) algebra is derived in \[ 22 \], \[ 10 \], while the target space interpretation as a four-dimensional semi-wormhole space is given in \[ 11 \]. In the context of this interpretation, the 10-d backgrounds of the first subclass of our solutions is that of a product of topologically non-trivial spaces, \( M^2 \otimes W_{k_1}^{(4)} \otimes W_{k_2}^{(4)} \) (\( M^2 \) is the flat \((1+1)\) space-time).

A second subclass of solutions is based on a different realization of the \( N = 4 \) superconformal system with \( \hat{c} = 4 \) \[ 10 \], \[ 3 \]. Here one replaces the \( W_{k}^{(4)} \) space by a new \( N = 4 \) system, \( \Delta_{k}^{(4)} \equiv \left\{ \left[ SU(2)\atop U(1) \right]_k \otimes \left[ SL(2,\mathbb{R})\atop U(1) \right]_{k+4} \right\}_{\text{SUSY}} \), i.e. a gauged supersymmetric WZW model with \( \hat{c}[\Delta_{k}^{(4)}] = 4 \) for any value of \( k \). The choice of the levels \( k \) and \( k+4 \) is necessary for the existence of an \( N = 4 \) symmetry with \( \hat{c} = 4 \). Using \( \Delta_{k}^{(4)} \) or \( W_{k}^{(4)} \) as four-dimensional subspaces, we can construct non-trivial 10-d solutions, which admit \( N = 2 \) target space supersymmetries in the heterotic string, or \( N = 2 + 2 \) target space supersymmetries in type-II strings.

Another subclass of solutions is obtained using the dual space of \( W_{k}^{(4)} \), \( C_{k}^{(4)} \) \[ 23 \], \[ 10 \], \[ 3 \]. It turns out that the \( C_{k}^{(4)} \) conformal system with \( \hat{c} = 4 \) shares with \( \Delta_{k}^{(4)} \) and \( W_{k}^{(4)} \) the same \( N = 4 \) superconformal properties. The explicit realization of the \( C_{k}^{(4)} \) space is given in \[ 10 \]. From the conformal theory viewpoint \( C_{k}^{(4)} \) is based on the supersymmetric gauged WZW model \( C_{k}^{(4)} \equiv \left[ SU(2)\atop U(1) \right]_k \otimes U(1)_R \otimes U(1)_Q \) with a background charge \( Q = \sqrt{\frac{2}{k+2}} \) in one of the two coordinate currents \( (U(1)_R) \). The other free coordinate \( (U(1)_Q) \) is compactified on a torus with radius \( R = \sqrt{2k} \).

Having at our disposal non-trivial \( N = 4 \), \( \hat{c} = 4 \) superconformal systems, we can use them as building blocks to obtain new classes of exact and stable string solutions around non-trivial backgrounds in both type-II and heterotic superstrings, as shown in \[ 11 \]. Some typical 10-d target spaces, obtained via the above-mentioned conformal
block construction, are:

\[ A) \quad i) F^{(2)} \otimes W_{k_1}^{(4)} \otimes W_{k_2}^{(4)} \]
\[ \quad ii) F^{(2)} \otimes F^{(4)} \otimes W_{k_1}^{(4)} \]

\[ B) \quad i) F^{(2)} \otimes C_{k_1}^{(4)} \otimes C_{k_2}^{(4)} \]
\[ \quad ii) F^{(2)} \otimes F^{(4)} \otimes C_{k_1}^{(4)} \]

\[ C) \quad i) F^{(2)} \otimes C_{k_1}^{(4)} \otimes W_{k_2}^{(4)} \]
\[ \quad ii) F^{(2)} \otimes C_{k_1}^{(4)} \otimes \Delta_{k_2}^{(4)} \]
\[ \quad iii) F^{(2)} \otimes \Delta_{k_1}^{(4)} \otimes W_{k_2}^{(4)} \]

\[ D) \quad i) F^{(2)} \otimes \Delta_{k_1}^{(4)} \otimes \Delta_{k_2}^{(4)} \]
\[ \quad ii) F^{(2)} \otimes F^{(4)} \otimes \Delta_{k_1}^{(4)} \]

(1.2)

In the above expressions, \( F^{(4)} \) stands for a four-dimensional Lorentzian flat space, compact or non-compact, as well as for a four-dimensional \( T^4/Z_2 \) orbifold; \( F^{(2)} \) denotes also a two-dimensional flat space, compact or non-compact, with Lorentzian or Euclidean signature. Note that the Euclidean version of the three cases (ii) (i.e. when \( F^{(2)} \otimes F^{(4)} \) is a compact six-dimensional flat space) can be identified with three different kinds of four-dimensional gravitational and/or dilatonic instanton solutions.

In this interpretation, the \( W_{k_1}^{(4)}, C_{k_1}^{(4)}, \) or \( \Delta_{k_1}^{(4)} \) subspace describes the Euclidean version of our space-time.

In Section 2, we discuss the connection of our constructions with stable solutions of extended gauge supergravities (type-A) and their relation with non-critical superstrings (type-A and -C). In Section 3, we describe the four different realizations of the \( N = 4 \hat{c} = 4 \) superconformal algebra, used to define the conformal blocks appearing in (1.2). In Section 4, we derive the spectrum of string excitations and give examples of one-loop partition functions for the semi-wormhole space (A-ii), as well as for the general (A-i) solution. In particular, we discuss the special values \( k_1 = 2, k_2 = \infty \) and \( k_1 = k_2 = 0 \) corresponding to non-critical superstrings with \( \hat{c}_M = 8 \) and \( \hat{c}_M = 5 \), respectively, where massless states appear in the twisted sector of the theory. Finally, Section 5 contains our conclusions.
2 Connection to gauged supergravities and non-critical strings

2.1 Gauged supergravity

The type-A constructions based on $W(4)$ conformal blocks describe, from the target space point of view, stable solutions of 4-d gauged supergravities [24], [25], which leave some of the space-time supersymmetries unbroken. In fact, consider the 10-d heterotic or type-II superstring compactified on a product of two three-dimensional spheres. The corresponding superconformal field theory is then given by a supersymmetric WZW model based on a $K(6) \equiv SU(2)_{k_1} \otimes SU(2)_{k_2}$ group manifold, where the Kac-Moody levels $k_A$ define the radii of the spheres $r_A$, $k_A = r_A^2$ [5]. In contrast with the toroidal compactification ($T^6 \equiv U(1)^6$), where the six graviphotons are Abelian, in $K(6)$ compactification they become non-Abelian. As expected from field theory Kaluza-Klein arguments, in the large radii limit the resulting effective theory is an $SU(2)_{k_1} \otimes SU(2)_{k_2}$ gauged supergravity [24]. This can be easily shown in the 2-d $\sigma$-model approach by means of the $\alpha'$-expansion. More precisely, the gauging is:

$$[SU(2)_{k_1} \otimes SU(2)_{k_2}]_{\text{left}} \otimes [SU(2)_{k_1} \otimes SU(2)_{k_2}]_{\text{right}}$$

in type-II construction ($N = 8$ gauged supergravity), and

$$[SU(2)_{k_1} \otimes SU(2)_{k_2}]_{\text{left}} \otimes [SU(2)_{k_1} \otimes SU(2)_{k_2}]_{\text{right}} \otimes G$$

in heterotic construction ($N = 4$ gauged supergravity). The gauge group $G$ depends on the particular embedding in the 10-d $SO(32)$ or $E_8 \otimes E_8$ gauge group, leading for instance to $G = E_7 \otimes E_8$ level-1, etc.

This connection is very important because it allows the derivation of the 4-d effective supergravity action, up to two space-time derivatives, which is induced by the $K(6)$ compactification. In the context of 2-d $\sigma$-model, it implies the knowledge of the corresponding $\beta$-functions of the background fields. For instance in the heterotic case, the induced $N = 4$ supergravity is uniquely fixed in terms of the heterotic gauge group mentioned above [25]. The strength of the gauge couplings is also fixed by the
levels of the affine algebras, $g_A = 1/\sqrt{k_A}$. The bosonic part of the $N = 4$ supergravity action, restricted in the supergravity multiplet, reads \cite{24,25}:

\[
S_{\text{eff \ bos}} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} R - e^{4\Phi} (\nabla a)^2 - \frac{1}{2g_A^2} (e^{-2\Phi} \text{Tr} F^A \cdot F^A + a \text{Tr} F^A \cdot \tilde{F}^A) - V(\Phi) \right\}
\]

(2.1)

where $\Phi$ is the dilaton field, $a$ is the pseudoscalar axion (dual to the two-index antisymmetric tensor), $F^A_{\mu\nu}$ are the $SU(2)_{k_A}$ field strengths, and $\tilde{F}^A_{\mu\nu}$ are their duals. The potential $V(\Phi)$ is non-vanishing, as expected from the $\sigma$-model evaluation of the dilaton $\beta$-function \cite{4}. In fact, the $K(6)$ compactification gives rise to a non-zero curvature contribution, inducing a non-trivial dilaton potential proportional to the central charge deficit, $\delta \hat{c}$ \cite{4}:

\[
V(\Phi) = \frac{1}{2} \delta \hat{c} e^{2\phi}
\]

\[
\delta \hat{c} = \frac{2}{3} \left( \frac{3k_1}{k_1 + 2} + \frac{3k_2}{k_2 + 2} - 6 \right) = -4 \left( \frac{1}{k_1 + 2} + \frac{1}{k_2 + 2} \right) \approx -4(g_1^2 + g_2^2) + O \left( \frac{1}{k_A^2} \right).
\]

(2.2)

It follows that the large $k_A$ limit of (2.2) reproduces the potential of the $N = 4$ gauged supergravity. The $O(1/k_A^2)$ corrections are due to curvature effects which are related to higher derivative terms neglected in the effective action.

### 2.2 Non-critical superstrings

The type-A, -B and -C constructions based on $W^{(4)}_{k_A}$ and $C^{(4)}_{k_A}$ superconformal systems are strongly connected to the non-critical superstrings in the so-called strong coupling regime (1 $\leq \hat{c}_{\text{matter}}$ $\leq$ 9) \cite{20,21}. In fact, the Liouville superfield of non-critical strings can be identified with the supercoordinate of the above spaces, which has a non-zero background charge. The central charge of the Liouville supercoordinate can be easily determined:

\[
\hat{c}_L = 1 + 2(Q_1^2 + Q_2^2) = 1 + 4 \left( \frac{1}{k_1 + 2} + \frac{1}{k_2 + 2} \right),
\]

(2.3)

where we have used the relation among the levels $k_A$ and the background charges $Q_A$, $Q_A^2 = 2/(k_A + 2)$. As we will see in the next Section, this relation follows from the
$N = 4$ superconformal symmetry in both $W$ and $C$ systems. The remaining matter part consists of tensor products of unitary $N = 1$ superconformal systems based on $SU(2)_{k_A}$ WZW, $[SU(2)/U(1)]_{k_A}$ GKO cosets, as well as $U(1)$ factors. The matter central charge is always

$$\hat{c}_M = 9 - 4\left(\frac{1}{k_1 + 2} + \frac{1}{k_2 + 2}\right),$$

and it varies in the region $5 \leq \hat{c}_M \leq 9$. Thus, our explicit constructions show the existence of super-Liouville theories coupled to $N = 1$ superconformal unitary matter systems in the strong coupling regime. The problematic complex conformal weights, usually present in this regime, are projected out by the $\hat{c}_M$-induced generalized GSO projection (see Section 4). This projection phenomenon is similar to the one observed in ref.\cite{27} for the $\hat{c}_M = 5$ case and in ref.\cite{30} for the case of the $N = 2$ globally defined superconformal symmetry.

The lower value $\hat{c}_M = 5$ corresponds to the type-A($i$) construction of (1.2), in the limiting case where $k_1 = k_2 = 0$. In this limit, the bosonic $SU(2)_{k_1} \otimes SU(2)_{k_2}$ currents decouple and only their free-fermionic superpartners remain which form a $\hat{c} = 2$, $N = 1$ superconformal system. This subsystem, combined with the two free supercoordinates $F^{(2)}$ in (1.2) and with the linear combination of the two $U(1)$’s with no background charge, define all together the $\hat{c}_M = 5$ unitary matter system. It was argued that this value of $\hat{c}_M$ is special in the sense that it is the only super-Liouville theory with massless excitation \cite{30}, and in that respect it has a behaviour similar to that in the $c_M = 1$ bosonic case. Furthermore, $\hat{c}_M = 5$ is one of the special dimensions of super-Liouville theories studied by Bilal and Gervais in ref.\cite{27}. The $\hat{c}_M = 5$ system will be further investigated in Section 4.

Another interesting value is $\hat{c}_M = 7$, obtained in particular when $k_1 = 0$ and $k_2 \to \infty$, or when $k_1 = k_2 = 2$. It turns out that this case corresponds to the high-temperature phase of the heterotic critical string \cite{31}.  


3 Exact realizations of $N = 4$, $\hat{c} = 4$ superconformal algebra

The desired $N = 4$ superconformal algebra involves operators of conformal weights 2, 3/2, and 1, namely the stress-energy tensor $T(z)$, four supercurrents $G_a(z)$, $a = 1, 2, 3, 4$, and three $SU(2)_n$ level-$n$ Kac-Moody currents $S_i(z)$, $i = 1, 2, 3$. The closure of the algebra implies the following OPE relations among these operators:

\begin{align*}
T(z)T(w) &\sim \frac{3\hat{c}}{4(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{(z - w)} \\
T(z)G_a(w) &\sim \frac{3G_a(w)}{2(z - w)^2} + \frac{\partial G_a(w)}{(z - w)} \\
T(z)S_i(w) &\sim \frac{S_i(w)}{(z - w)^2} + \frac{\partial S_i(w)}{(z - w)} \\
G_4(z)G_4(w) &\sim \frac{\hat{c}}{(z - w)^3} + \frac{2T(w)}{(z - w)} \\
G_i(z)G_j(w) &\sim \frac{\delta_{ij}}{(z - w)^3} - 4\epsilon_{ijl} \frac{S_l(w)}{(z - w)^2} + 2\delta_{ij} \frac{T(w)}{(z - w)} \\
G_4(z)G_i(w) &\sim \frac{4S_i(w)}{(z - w)} \\
S_i(z)G_4(w) &\sim -\frac{G_i(w)}{2(z - w)} \\
S_i(z)G_j(w) &\sim \frac{1}{2(z - w)} (\delta_{ij}G_4(w) + \epsilon_{ijl}G_l(w)) \\
S_i(z)S_j(w) &\sim -\delta_{ij} \frac{n}{2(z - w)^2} + \epsilon_{ijl} \frac{S_l(w)}{(z - w)}. \tag{3.1}
\end{align*}

The central charge $\hat{c}$ and the level $n$ of the $SU(2)_n$ currents are related by $\hat{c} = 4n$. The condition $\hat{c} = 4$ implies the existence of $SU(2)_1$ currents.

Below, we present the four different realizations of the algebra, used to define the $N = 4$ superconformal blocks that appear in (1.2).

3.1 The free-field realization

In this realization [20], the $N = 4$ basic operators are constructed in terms of four free supercoordinates $(\Phi_a, \Psi_a)$. The $U(1)^4$ bosonic currents $J_a = \partial \Phi_a$ and the Weyl-
Majorana 2-d fermions $\Psi_a$ are normalized as:

$$J_a(z)J_b(w) \sim -\frac{\delta_{ab}}{(z-w)^2}$$

$$\Psi_a(z)\Psi_b(w) \sim -\frac{\delta_{ab}}{(z-w)}.$$  \hfill (3.2)

The generators of the algebra are then given as:

$$T = -\frac{1}{2} \left( J_a^2 - \Psi_a \partial \Psi_a \right)$$

$$G_1 = +J_1 \Psi_1 + J_2 \Psi_2 + J_3 \Psi_3 + J_4 \Psi_4$$

$$G_2 = +J_1 \Psi_2 - J_2 \Psi_1 - J_3 \Psi_4 + J_4 \Psi_3$$

$$G_3 = -J_1 \Psi_4 + J_2 \Psi_3 - J_3 \Psi_2 + J_4 \Psi_1$$

$$G_4 = -J_1 \Psi_3 - J_2 \Psi_4 + J_3 \Psi_1 + J_4 \Psi_2$$

$$S_i = \frac{1}{2} \left( \Psi_4 \Psi_i + \frac{1}{2} \epsilon_{ijl} \Psi_j \Psi_l \right).$$  \hfill (3.3)

For later convenience, it is useful to complexify the coordinate currents as:

$$P = J_1 + iJ_2, \quad P^\dagger = -J_1 + iJ_2 ,$$

$$\Pi = J_4 + iJ_3, \quad \Pi^\dagger = -J_4 + iJ_3,$$  \hfill (3.4)

with

$$P(z)P^\dagger(w) \sim \frac{2}{(z-w)^2} + 2T_P$$

$$\Pi(z)\Pi^\dagger(w) \sim \frac{2}{(z-w)^2} + 2T_{\Pi},$$  \hfill (3.5)

where $T_P$ and $T_{\Pi}$ are the stress tensor of the $(P, P^\dagger)$ and $(\Pi, \Pi^\dagger)$ conformal subsystems.

It is also useful to bosonize the free fermions in terms of two bosons, $H^+$ and $H^-$. First, we decompose the $SO(4)_1$ level-1 fermionic currents, $\Psi_i \Psi_j$, in terms of two $SU(2)_1$ currents, $S_i$, $\tilde{S}_i$ using the $SO(4)$ self-dual and anti-self-dual decomposition:

$$S_i = \frac{1}{2} \left( +\Psi_4 \Psi_i + \frac{1}{2} \epsilon_{ijl} \Psi_j \Psi_l \right) \rightarrow \left( \frac{1}{2} \partial H^+, \ e^{\pm i\sqrt{2}H^+} \right)$$

$$\tilde{S}_i = \frac{1}{2} \left( -\Psi_4 \Psi_i + \frac{1}{2} \epsilon_{ijl} \Psi_j \Psi_l \right) \rightarrow \left( \frac{1}{2} \partial H^-, \ e^{\pm i\sqrt{2}H^-} \right).$$  \hfill (3.6)
In the above equation, $S_i$ and $\tilde{S}_i$ are parametrized in terms of the two free bosons, $H^+$ and $H^-$, which are both compactified on a cycle with radius $R_{H^+} = R_{H^-} = \sqrt{2}$ (the self-dual extended symmetry points).

In terms of $P, P^\dagger, \Pi, \Pi^\dagger, H^+$ and $H^-$, the $N = 4$ operators become:

\begin{align*}
T &= -\frac{1}{2} \left( (\partial H^+)^2 + (\partial H^-)^2 - PP^\dagger - \Pi\Pi^\dagger \right) \\
G &= -\left( \Pi^\dagger e^{-\sqrt{2}H^-} + P^\dagger e^{\sqrt{2}H^-} \right) e^{+\sqrt{2}H^+} \\
\tilde{G} &= \left( \Pi \ e^{+\sqrt{2}H^-} - P \ e^{-\sqrt{2}H^-} \right) e^{+\sqrt{2}H^+} \\
S_3 &= \frac{1}{\sqrt{2}} \partial H^+ , \quad S_\pm = e^{\pm i\sqrt{2}H^+} , \quad (3.7)
\end{align*}

where

\begin{align*}
G &= \frac{G_1 + iG_2}{\sqrt{2}} , \quad \tilde{G} = \frac{G_4 + iG_3}{\sqrt{2}} .
\end{align*}

The above expressions clearly show that the supercurrents $(G, G^\dagger)$ and $(\tilde{G}, \tilde{G}^\dagger)$ form two doublets under $SU(2)_{H^+}$. The $H^+$ factorization in the supercurrents is not a particular property of the free-field realization, but a generic property for any $c = 4$ system with $N = 4$ symmetry. Consequently, the supercurrents always have a factorized product form in terms of two conformal operators. The first one is not dependent on $H^+$ and has conformal dimension $\frac{5}{4}$, while the other is given only in terms of $H^+$ and has conformal dimension $\frac{1}{4}$. On the other hand, $G$ and $\tilde{G}$ do not transform covariantly under the action of $SU(2)_{H^-}$. They are odd, however, under a $Z_2$ transformation, defined by $(-)^{2\tilde{S}}$, which is the parity operator associated to the $SU(2)_{H^-}$ spin $\tilde{S}$ (integer spin representations are even, while half-integer representations are odd).

Another useful global quantum number is obtained by combining the bosonic oscillator number $N_P$, which counts the number of the $P$-oscillators minus the number of $P^\dagger$-ones, with the $SU(2)_{H^-}$ helicity $\tilde{S}_3$:

\[ \mathcal{N}^- \equiv N_P + \tilde{S}_3 . \quad (3.8) \]

$G$ and $\tilde{G}$ have $\mp 1/2$ $(\mathcal{N}^-)$-charges, respectively. As we will see in Section 4, the $\mathcal{N}^-$ charge, the $(-)^{2\tilde{S}}$ parity, as well as the $SU(2)_{H^+}$ spin $S$ play an important role in the definition of the induced $N = 4$ generalized GSO projections.
3.2 The semi-wormhole realization

In this case, the basic operators of the $N = 4$ algebra are constructed in terms of the $SU(2)_k \otimes U(1)_Q$ bosonic currents $J_a$ and their free-fermionic superpartners $\Psi^a$ [22],[10]:

$$T = -\frac{1}{2} \left[ \frac{2}{k+2} J_i^2 + J_4^2 - \Psi_a \partial \Psi_a + Q \partial J_4 \right]$$

$$G_4 = \sqrt{\frac{2}{k+2}} \left( J_i \Psi_i + \frac{1}{3} \epsilon_{ijl} \Psi_j \Psi_l \right) + J_4 \Psi_4 + Q \partial \Psi_4$$

$$G_i = \sqrt{\frac{2}{k+2}} \left( -J_i \Psi_4 + \epsilon_{ijl} J_j \Psi_l - \epsilon_{ijl} \Psi_4 \Psi_j \Psi_l \right) + J_4 \Psi_i + Q \partial \Psi_i$$

$$S_i = \frac{1}{2} \left( \Psi_4 \Psi_i + \frac{1}{2} \epsilon_{ijl} \Psi_j \Psi_l \right).$$

(3.9)

The closure of the $N = 4$ algebra can be easily verified using the OPE relations among $J_a$ and $\Psi_a$:

$$J_4(z)J_4(w) \sim -\frac{1}{(z-w)^2}$$

$$J_i(z)J_j(w) \sim -\frac{k}{2} \frac{\delta_{ij}}{(z-w)^2} + \epsilon_{ijl} J_l \frac{1}{(z-w)}$$

$$\Psi_a(z)\Psi_b(w) \sim -\frac{\delta_{ab}}{(z-w)}. \quad (3.10)$$

The relation $Q = \sqrt{\frac{2}{k+2}}$ between the background charge $Q$ and the level $k$ is necessary for the cancellation of the central charge deficit $\hat{c} = -4/(k+2)$, induced by the non-flat $S^3$ subspace, by the central charge benefit $2Q^2$, induced by the non-trivial dilaton. Indeed, the presence of a non-zero background charge for the $J_4$ coordinate current implies, in the $\sigma$-model representation, a dilaton background linear in the fourth coordinate.

The interesting observation here is that the $S_i \ N = 4$ currents are the same as in the free-field realization. The absence of any curvature correction $\mathcal{O}(1/k)$ is due to an exact cancellation among the contribution of the torsion terms $\Psi_a \Psi_b \Psi_c$ and the contribution of the background charge terms $Q \partial \Psi_a$ appearing in the $N = 4$ supercurrents. This cancellation is a consequence of the $N = 4$ algebra with $\hat{c} = 4$, which implies the existence of an $SU(2)_1$ level-1 current algebra. In fact, in any
$N = 4$ supersymmetric $\sigma$-model, the self-dual combination of the fermionic currents formed by the fermion bilinears $\Psi_a \Psi_b$ is always free (see (3.6)). It happens that in the $W_k^{(4)}$ space, the anti-self-dual combination remains also free. However, this is not true in general; the $N = 4$ algebra does not forbid non-trivial interactions of the second combination with the target space curvature.

As in the free-field realization, it is convenient to bosonize the free fermions by introducing the $H^+$ and $H^-$ scalar fields, as in (3.6), and thus to factorize the $SU(2)_{H^+}$ dependence of the supercurrents. After this bosonization, the $N = 4$ supercurrents $G$ and $\tilde{G}$ take the same form as in (3.7) in terms of the modified coordinate currents $P$ and $\Pi$:

\[
P \quad \rightarrow \quad P_k = Q(J_1 + iJ_2) \\
P^\dagger \quad \rightarrow \quad P^\dagger_k = Q(-J_1 + iJ_2) \\
\Pi \quad \rightarrow \quad \Pi_k = J_4 + iQ(J_3 + \sqrt{2}\partial H^-) \\
\Pi^\dagger \quad \rightarrow \quad \Pi_k^\dagger = -J_4 + iQ(J_3 + \sqrt{2}\partial H^-)
\]

(3.11)

while the energy-momentum tensor is shifted by the background charge:

\[
T = -\frac{1}{2} \left[ (\partial H^+)^2 + (\partial H^-)^2 + Q^2 (J_1^2 + J_2^2 + J_3^2) + J_4^2 + Q\partial J_4) \right].
\]

(3.12)

The modification $Q\sqrt{2}\partial H^-$ in the expression of $\Pi$ and $\Pi^\dagger$ gives rise at the same time to the standard fermionic torsion terms $\pm Q\Psi_a \Psi_b \Psi_c$, and to the fermionic background charge terms $\pm Q \partial \Psi_a$ in the expression of the supercurrents (3.9).

As in the free-field case, $(G, G^\dagger)$ and $(\tilde{G}, \tilde{G}^\dagger)$ form two doublets under $SU(2)_{H^+}$ while they are odd under the $\mathbb{Z}_2$ parity $(-)^{2\tilde{S}}$. Finally, the $\mathcal{N}^-$ charge (3.8) is now replaced by a global $SU(2)_{k+1}$ charge defined as the diagonal combination of $SU(2)_k$ and $SU(2)_{H^-}$:

\[
\mathcal{N}_i = J_i + \tilde{S}_i.
\]

(3.13)

Also $(G, G^\dagger)$ and $(\tilde{G}, \tilde{G}^\dagger)$ form two doublets under $SU(2)_\mathcal{N}$. Moreover $G$ and $\tilde{G}$ have $(\mathcal{N}_3, S_3)$ charges equal to $(-1/2, 1/2)$ and $(1/2, 1/2)$, respectively. As we will see in Section 4, the $SU(2)_\mathcal{N}$ and $SU(2)_{H^+}$ spins play an important role as classification charges of the string excitations around the semi-wormhole background.
The existence of non-trivial target spaces with \( \hat{c} = 4 \) for any value of \( k \) is interesting, since it allows the study of these models by means of the \( 1/k \) expansion, where the semiclassical approximation is valid. Indeed, the limit \( k \to \infty \) can be obtained after rescaling the \( SU(2)_k \) currents \( J_i \to \sqrt{k/2} J_i \), and it gives back the free-field case. As we mentioned in the introduction, the semi-wormhole interpretation of \( W^{(4)}_k \) is valid only in the large-\( k \) limit. This background interpretation fails for small \( k \) (large curvature). For instance at \( k = 2 \) the three \( SU(2)_2 \) currents are equivalent to three free world-sheet fermions, while at \( k = 1 \) they are equivalent to a free boson on a cycle of radius \( \sqrt{2} \).

In the \( k = 0 \) limit, the \( SU(2)_k \) currents decouple and the \( N = 4 \) operators are described in terms of the \( U(1)_Q \) current and the four free fermions \( \Psi_4 \) and \( \Psi_i \). From the point of view of \( N = 1 \) local world-sheet supersymmetry, this system is equivalent to a free super-coordinate \( (\partial X \equiv \Psi_1 \Psi_2, \Psi_3) \) coupled to super-Liouville \( (\partial \Phi \equiv J_4, \Psi_4) \). The coordinate \( X \) is compactified on a cycle of radius \( R_X = 1 \), which corresponds to the fermionization point. The heterotic superstring solution based on \( F^{(6)} \otimes W^{(4)}_{k=0} \) was identified in ref.\[31\] as the high-temperature phase of the ten-dimensional theory. There, the value of the radius of the \( X \) coordinate defines the temperature, while the special value \( R_X = 1 \) corresponds to a self dual thermal spectrum \[32\], and minimizes the free energy \[31\]. In the transition from the zero-temperature phase, described by the solution \( F^{(6)} \otimes W^{(4)}_{k=\infty} \), to the high-temperature phase \( (k = 0) \), the decoupling of the three \( SU(2) \) currents corresponds to a central charge deficit \( \delta \hat{c} = 2 \), which is balanced by the dilaton motion in the \( k = 0 \) phase.

### 3.3 The \( C^{(4)}_k \) torus-bell realization

In this case, the elementary fields are the \( \left( \frac{SU(2)}{U(1)} \right)_k \) parafermionic currents \( P_k \) and \( P_k^\dagger \) of conformal weight \( (1 - 1/k) \) \[33\], two free \( U(1) \) currents \( J_3 = \partial X_3 \) and \( J_4 \), and four free fermions, which are parametrized by the \( H^+ \) and \( H^- \) bosonic fields. The \( X_3 \) coordinate is compactified on a cycle of radius \( \sqrt{2k} \), while \( J_4 \) has a background charge \( Q = \sqrt{2/(k + 2)} \). In the large-\( k \) limit, \( C^{(4)}_k \) is factorized in two 2-d subspaces; the first subspace is described by the \( SU(2)/U(1) \) bell, while the second subspace
$U(1)_{X_3} \otimes U(1)_Q$ is a two-dimensional cylinder.

The $N = 4$ operators are given by [10]:

\begin{align*}
T &= - \frac{1}{2} \left( (\partial H^+)^2 + (\partial H^-)^2 + J_3^2 + J_4^2 + Q \partial J_4 \right) + T_{P_k} \\
G &= - \left( \Pi_k^+ e^{-\sqrt{2} H^-} + P_k^+ e^{\sqrt{2} H^- - i \sqrt{k} X_3} \right) e^{+ \sqrt{2} H^+} \\
\tilde{G} &= \left( \Pi_k e^{+ \sqrt{2} H^-} - P_k e^{-\sqrt{2} H^- + i \sqrt{k} X_3} \right) e^{+ \sqrt{2} H^+} \\
S_3 &= \frac{1}{\sqrt{2}} \partial H^+, \quad S_\pm = e^{\pm i \sqrt{2} H^+} ,
\end{align*}

(3.14)

where $\Pi_k$ and $\Pi_k^\dagger$ are defined in terms of $J_3$, $J_4$ and $H^-:

\begin{align*}
\Pi_k &= +J_4 + i \left( \sqrt{\frac{k}{k+2}} \partial X_3 + \sqrt{\frac{4}{k+2}} \partial H^- \right) \\
\Pi_k^\dagger &= -J_4 + i \left( \sqrt{\frac{k}{k+2}} \partial X_3 + \sqrt{\frac{4}{k+2}} \partial H^- \right) .
\end{align*}

(3.15)

In (3.14), $T_{P_k}$ is the energy-momentum tensor of the $(SU(2)_k)$ parafermions with central charge $c_P = 2 - 6/(k+2)$. It appears in the OPE of the (non-local) parafermionic currents $P_k$ and $P_k^\dagger$

\begin{equation}
P_k(z) P_k^\dagger(w) \sim \left[ \frac{k}{(k+2)} \frac{2}{(z-w)^2} + 2 T_{P_k}(w) \right] (z-w)^2 .
\end{equation}

(3.16)

In the supercurrent expression (3.14), the deviation from the free-field dimensionality $-1/k$ of the parafermionic currents $P_k$ is cancelled by the weight $+1/k$ of the $\exp(-i \sqrt{2/k} X_3)$ modification.

In contrast to the semi-wormhole case, there is no $SU(2)_N$ globally defined charge (3.13). Instead, there is only an abelian charge $N_3$ defined by:

\begin{equation}
N_3 = \sqrt{\frac{k}{2}} J_3 + \tilde{S}_3 .
\end{equation}

(3.17)

$G$ and $\tilde{G}$ have $U(1)_{N_3}$ charges $-1/2$ and $+1/2$, respectively.

### 3.4 The $\Delta_k^{(4)}$ cigar/trumpet-bell realization

In this case, the elementary fields are the $(SU(2)_{U(1)})_k$ (compact) parafermionic currents $P_k$ and $P_k^\dagger$, the $(SL(2,R)_{U(1)})_k$ non-compact parafermionic currents $\hat{\Pi}_k$ and $\hat{\Pi}_k^\dagger$. 

14
of conformal weight \((1 + 1/k')\), as well as the two free bosons \(H^+\) and \(H^-\). The level \(k' = k + 4\), so that the total central charge \(\hat{c}\) remains equal to 4 for any value of \(k\). In contrast with the three previous cases, the compactification radius of \(H^-\) deviates from its self-dual value due to \(1/k\) background curvature corrections. Its exact value is \(R_{H^-} = \sqrt{2}k'/k\) and, thus, breaks \(SU(2)_{H^-}\) to a \(U(1)\) subgroup. However, as we have already mentioned, \(H^+\) remains intact as a consequence of the \(N = 4\) algebra with \(\hat{c} = 4\). For large \(k\), \(\Delta_k^{(4)}\) is factorized in two 2-d subspaces; the first one is the \(SU(2)/U(1)\) bell, while the second one is described by the \(SL(2,R)/U(1)\) cigar (axial gauging) or trumpet (vector gauging) [35].

The \(N = 4\) operators are given by [10]:

\[
\begin{align*}
T &= -\frac{1}{2} \left( (\partial H^+)^2 + (\partial H^-)^2 \right) + T_{P_k} + T_{\tilde{H}_{k'}} \\
G &= -\left( \hat{\Pi}_{k'} e^{-i\sqrt{\frac{k}{2(k+4)}}H^-} + P_k e^{i\sqrt{\frac{k+4}{2k}}H^-} \right) e^{\pm \frac{1}{\sqrt{2}}H^+} \\
\tilde{G} &= \left( \hat{\Pi}_{k'} e^{+i\sqrt{\frac{k}{2(k+4)}}H^-} - P_k e^{-i\sqrt{\frac{k+4}{2k}}H^-} \right) e^{\pm \frac{1}{\sqrt{2}}H^+} \\
S_3 &= \frac{1}{\sqrt{2}} \partial H^+ , \quad S_\pm = e^{\pm i\sqrt{2}H^+} , \quad (3.18)
\end{align*}
\]

where \(T_{\tilde{H}_{k'}}\) is the energy-momentum tensor of the \(\left(\frac{SL(2,R)}{U(1)}\right)_{k'}\) non-compact parafermions with central charge \(c_{\tilde{H}} = 2 + 6/(k' - 2) = 2 + 6/(k + 2)\). It appears in the OPE of the non-compact parafermionic currents \(\hat{\Pi}_{k'}\) and \(\hat{\Pi}_{k'}^\dagger\):

\[
\hat{\Pi}_{k'}(z)\hat{\Pi}_{k'}^\dagger(w) \sim \left[ \frac{k'}{(k'-2)} \frac{2}{(z-w)^2} + 2T_{\tilde{H}_k}(w) \right] (z-w)^{-\frac{2}{k'}} . \quad (3.19)
\]

In the supercurrent expression (3.18), the deviation from the free-field dimensionality, \(1/(k + 4)\) of the non-compact parafermionic currents \(\hat{\Pi}_{k'}\) and \(-1/k\) of the compact ones \(P_k\), is cancelled by the deviation from the free-fermion dimensionality \(1/2\) of the exponential factors \(e^{\pm i\sqrt{\frac{k}{2k+4}}H^- + \frac{1}{\sqrt{2}}H^+}\) and \(e^{-i\sqrt{\frac{k+4}{2k}}H^- + \frac{1}{\sqrt{2}}H^+}\), respectively. The last deviation is due to the radius modification of \(H^-\) from its self-dual value [10].

In contrast with the three previous cases, here there is no globally defined charge analogous to \(N\) of (3.8), (3.13), (3.17). Furthermore the \(Z_2\) parity \((-)^{3S}\), under which
$G$ from $\bar{G}$ are odd, is replaced by

$$e^{2i\pi kQ_{H^-}} ; \quad Q_{H^-} = \sqrt{\frac{k+4}{2k}} \oint \partial H ,$$

(3.20)

when $k$ is odd, and $\exp \left( i\pi kQ_{H^-} \right)$ when $k = 2 \mod 4$. For $k = 0 \mod 4$, such a parity cannot be defined in terms of $H^-$ alone and it is necessary to use the $\mathbb{Z}_k$ discrete symmetry of compact parafermions. Under this symmetry $P_k$ picks up a phase $\exp \left( 2i\pi/k \right)$. The generalized $\mathbb{Z}_k$ operator for any $k$ even is a discrete analogue of the global charge (3.17), and it reads:

$$e^{2i\pi N} ; \quad N = \frac{k+2}{2}Q_{P_k} + Q_{H^-} ,$$

(3.21)

where $Q_{P_k}$ is the parafermionic charge (defined modulo integer) and equals $2/k$ for the $P_k$ currents. One can easily check that the $N = 4$ supercurrents change sign under the above $\mathbb{Z}_k$ transformation.

4 The spectrum of string excitations and partition functions

The explicit realizations of the $N = 4$ algebra that we presented in the previous section, in terms of known conformal field theories, allows us to compute the spectrum of string excitations around any of the background solutions given in (1.2). In this section, we present the method of constructing modular-invariant combinations respecting the $N = 4$ superconformal symmetry. The latter implies the existence of space-time supersymmetry [36] in the corresponding non-trivial target spaces, and thus guarantees the stability of these classical solutions in string perturbation theory. The origin of these new target space supersymmetries follows from the world-sheet $N = 4$ spectral flow relations that imply a spectrum degeneracy among space-time bosonic and fermionic string modes. This degeneracy guarantees the vanishing of the vacuum energy, and thus the background stability, at least at the one-loop level.
4.1 The required projections

In all constructions, the total number of space-time supersymmetries is reduced by a factor of 2 with respect to the flat (toroidal) compactifications. In the context of the $\sigma$-model approach, the $N = 4$ spectral flows are related to the number of covariantly constant spinors admitted by the corresponding non-trivial target spaces (1.2). Thus, from the point of view of four non-compact dimensions, there exist two covariantly constant spinors in heterotic and four in type-II backgrounds.

The reduction of space-time supersymmetries by a factor of 2 can be easily seen in the variation of our solutions (1.2), where the two-dimensional subspace $F^{(2)}$ is flat (non-compact), with Lorentzian signature. The supersymmetry generators are constructed by analytic (or antianalytic) dimension-1 currents whose transverse part is a spin-field of dimension $1/2$ constructed in terms of the $H_A^+$ and $H_A^-$ bosonized fermions (of the two $\hat{c}_A = 4$ systems $A = 1, 2$). In the toroidal case, there are four such spin-fields, which are even under the GSO parity:

$$e^{2i\pi(S_1+S_2)},$$

(4.1)

where $S_A$ are the two $SU(2)_{H_A^+}$ level-1 $N = 4$ spins:

$$\Theta_\pm = e^{i\sqrt{2}(H_1^\pm + H_2^\pm)}$$

and

$$\tilde{\Theta}_\pm = e^{i\sqrt{2}(H_1^- + H_2^-)}.$$  

(4.2)

In the case of the non-trivial spaces described in Section 3, only the two supersymmetry generators based on the operators $\Theta_\pm$, which are constructed with $H^+$'s, are BRS-invariant. Indeed, the other two operators $\tilde{\Theta}_\pm$ do not exist in the free-field ($\mathbb{Z}_2$ orbifold) and the $C^{(4)}$ realization; in the former case, this is because they are not $\mathbb{Z}_2$-invariant, while in the latter case, because $H^-$ is not compactified at the self-dual point. Moreover in $W^{(4)}$ and $C^{(4)}$ realizations, the $\tilde{\Theta}_\pm$ supersymmetry generators are not physical, due to the $\partial H^-$ modification, related to the torsion and/or background charge, in the supercurrent expressions (3.11) and (3.15).

The global existence of the (chiral) $N = 4$ superconformal algebra implies in all our constructions a universal GSO projection that generalizes the one of the $N = 2$
algebra [37], and it is responsible for the existence of space-time supersymmetry. This projection restricts the physical spectrum to being odd under the total $H_A^+$ parity (4.1). Thus, the supersymmetry generators based on $\Theta_{\pm}$, which are even under (4.1), when acting on physical states, create physical states with the same mass but with different statistics. The GSO projection restricts the (level-1) character combinations associated with the two $SU(2)^+_H$’s to appear in the form:

$$\frac{1}{2}(1 - (-)^{l_1+l_2})\chi_{H_1^+}^{l_1}\chi_{H_2^+}^{l_2} = \chi_{H_1^+}\chi_{H_2^+}^{1-l_1-l_2} \delta_{l_2,1-l_1},$$

with $l_A = 2S_A$ taking values 0 or 1 corresponding to the two possible characters, (spin-0 and spin-1/2) of the $SU(2)_{k=1}$ Kac-Moody algebras.

The above character combination is universal and is valid for all string solutions described in Section 3. For every particular solution, modular invariance and unitarity impose some restrictions to the remaining degrees of freedom. These extra restrictions are not universal but depend on the particular solution under consideration.

4.2 The required characters

The basic rules of our construction are similar to that of, the orbifold construction [38], the free 2-d fermionic constructions [39], and the Gepner construction [37] were, one combines in a modular invariant way the world-sheet degrees of freedom consistently with unitarity and spin-statistics of the string spectrum. We will choose at the beginning as first example the derivation of the string spectrum in a five-brane background $M^6 \otimes W^{(4)}_k$, where $M^6 = F^{(2)} \otimes F^{(4)}$ is a six-dimensional non-compact flat Minkowski space. The six non-compact coordinates, together with the reparametrization ghosts ($b, c$), provide a contribution to the (type-II) partition function:

$$Z_B[F^{(6)}; (b, c)] = \frac{\text{Im}^2}{\eta^4(\tau)\bar{\eta}^4(\bar{\tau})}.$$

The contribution of the $M^6$ world-sheet fermions together with the $\beta$ and $\gamma$ super-reparametrization ghosts is:

$$Z_F[M^6; (\beta, \gamma)] = (-)^{a+b} \frac{\theta^2(\beta)}{\eta^2(\tau)} (-)^{a+b} \frac{\bar{\theta}^2(\bar{\beta})}{\bar{\eta}^2(\bar{\tau})},$$

with $l_A = 2S_A$ taking values 0 or 1 corresponding to the two possible characters, (spin-0 and spin-1/2) of the $SU(2)_{k=1}$ Kac-Moody algebras.
where $\alpha$ and $\beta$ denote the spin structures. In (4.5), the spin-statistic factor $(-)^{\alpha+\beta}$ and $(-)^{\bar{\alpha}+\bar{\beta}}$ comes from the contribution of the (left- and right-moving) $F^{(2)}$ world-sheet fermions and the (left- and right-moving) $(\beta, \gamma)$-ghosts. The Neveu-Schwarz $(NS, \overline{NS})$ sectors correspond to $\alpha, \bar{\alpha} = 0$ and the Ramond $(R, \bar{R})$ sectors correspond to $\alpha, \bar{\alpha} = 1$. For later convenience we decompose the $O(4)$ level-1 characters, which are written with theta-functions, in terms of the $SU(2)_{H^+} \otimes SU(2)_{H^-}$ characters using the identity:

$$\frac{\theta^2(\alpha)}{\eta^2(\tau)} = \sum_{l=0}^{1} (-)^{\beta l} \chi_{H^+}^{l-\alpha} \chi_{H^-}^{-l+\alpha(1-2l)} , \quad (4.6)$$

and similarly for the right-movers. The modular transformations of $\theta$-functions are:

$$\tau \rightarrow \tau + 1 : \quad \frac{\theta^2(\alpha)}{\eta^2(\tau)} \longrightarrow e^{i\pi \alpha^2/2} e^{i\pi \alpha} \frac{\theta^2(\beta + \alpha + 1)}{\eta^2}$$

$$\tau \rightarrow -1/\tau : \quad \frac{\theta^2(\alpha)}{\eta^2(\tau)} \longrightarrow e^{i\pi \alpha \beta} \frac{\theta^2(\beta)}{\eta^2} . \quad (4.7)$$

Then, one must combine the above $M^6$ characters with those of $W^{(4)}_k$, namely: (i) the $SU(2)_k (\chi^L_k, \, L = 1, 2, \cdots, k)$, (ii) the $U(1)_Q$ Liouville-like ones, (iii) the $SU(2)_{H^+} (\chi^L_{H^+}, \, l = 0, 1)$, and (iv) the $SU(2)_{H^-} (\chi^L_{H^-}, \, l = 0, 1)$.

The $U(1)_Q$ Liouville-like characters can be classified in two categories. Those that correspond to the continuous representations generated by the lowest-weight operators:

$$e^{\beta X_L} ; \quad \beta = -\frac{1}{2} Q + ip , \quad (4.8)$$

having positive conformal weights $h_p = Q^2/8 + \frac{p^2}{2}$. The fixed imaginary part in the momentum $iQ/2$ of the plane waves, is due to the non-trivial dilaton motion. The second category of Liouville characters corresponds to lowest-weight operators (4.8) with $\beta = Q \tilde{\beta}$ real, leading to negative conformal dimensions $-1/2 \tilde{\beta}(\tilde{\beta}+1)Q^2 = -\tilde{\beta}(\tilde{\beta}+1)/k+2$.

Both categories of Liouville representations give rise to unitary representations of the $N = 4, \hat{c} = 4$ system $W^{(4)}_k$, once they are combined with the remaining degrees of freedom. The continuous representations (4.8) form long (massive) representations of $N = 4$ with conformal weights larger than the $SU(2)$ spin, $h > S$. On the other hand, the second category contains short representations of $N = 4$ (4.8) ($h = S$), while $\beta$ can take only a finite number of values, $-(k+2)/2 \leq \tilde{\beta} \leq k/2$. In fact, their
locality with respect to the $N = 4$ operators implies:

\[
S = \frac{1}{2}, \quad \tilde{S} = \frac{1}{2} : \quad \tilde{\beta} = -(j + 1) \\
S = 0, \quad \tilde{S} = 0 : \quad \tilde{\beta} = j.
\] (4.9)

In both cases of (4.9), the conformal weight $h = S$ is independent of $SU(2)_k$ and $SU(2)_{H^-}$ spins, due to the cancellation between the Liouville and $SU(2)_k$ contributions. The states associated to the short representations of $N = 4$ do not have the interpretation of plane waves, but they describe a discrete set of bound states. They are similar to the discrete states found in the $c = 1$ matter system coupled to the Liouville field and also to the two-dimensional coset models $[9]$. Although they play a crucial role in scattering amplitudes, they do not correspond to asymptotic states and nor do they contribute to the partition function. Indeed in our case they are not only discrete but also finite in number and thus, have zero measure compared to the contribution of continuous representations.

The presence of discrete representations with $\beta$ positive are necessary to define correlation functions. In fact, the balance of the background charge for an $N$-point amplitude at genus $g$ implies the relation

\[
N + 2(g - 1) + 2 \sum_I \tilde{\beta}_I = 0,
\] (4.10)

where the sum is extended over the vertices of the discrete representation states. Thus, these vertices define an appropriate set of screening operators, necessary to define amplitudes in the presence of non-vanishing background charge. In our case the screening procedure has an interesting physical interpretation similar to the scattering of asymptotic propagating states (continuous representations) in the presence of non-propagating bound states (discrete representations). The screening operation then describes the possible angular momentum excitations of the bound states. Below, we restrict ourselves to the one-loop partition function, where the discrete representations are not necessary (see eq.(4.10) with $g = 1$ and $N = 0$).

It is convenient to define appropriate character combinations of $SU(2)_k$, which
transform covariantly under modular transformations:

\[ Z_k[^{\alpha}\beta] = \sum_{L=0}^{k} e^{i\pi\beta L} X_k^L \bar{X}_k^{L+\alpha(k-2L)}, \tag{4.11} \]

where \( \alpha, \beta \) can be either 0 or 1. Under modular transformations, \( Z_k[^{\alpha}\beta] \) transforms as:

\[
\begin{align*}
\tau \to \tau + 1 &: Z_k[^{\alpha}\beta] \longrightarrow e^{-i\pi \frac{\alpha^2}{2}} Z_k[^{\alpha}\beta + \alpha] \\
\tau \to -1/\tau &: Z_k[^{\alpha}\beta] \longrightarrow e^{i\pi k \alpha \beta} Z_k[^{\beta}\alpha].
\end{align*}
\tag{4.12} \]

Because of the \( k \)-dependent phase in the \( \tau \to \tau + 1 \) transformation, we must distinguish four different cases corresponding to \( k = 0, 1, 2, 3 \) modulo 4.

The partition function must satisfy two basic constraints emerging from the \( N = 4 \) algebra. The first is associated to the two spectral flows of the \( N = 4 \) algebra which impose the universal GSO projection (4.1), (4.3) of the \( H^+ \) spin. The second constraint is associated to the reduction of space-time supersymmetries by a factor of 2. It imposes a second projection which eliminates half of the lower-lying states constructed with the \( H^{-2} \) field. Indeed, the vertex operators

\[ e^{ip\nu X^\mu + (ip - \frac{Q}{2}) X_L \epsilon^{\alpha\beta}(H_{1,2}^+ H_{1,2}^-)} = e^{ip\nu X^\mu + (ip - \frac{Q}{2}) X_L} [(\text{spin}\Psi^\mu)_+ \text{ or } (\text{spin}\Psi^I)_+] (\text{spin}\Psi^\mu)_- \tag{4.13} \]

create physical states (bosons and fermions correspond to \( H_{1,2}^+ \) and \( H_{1,2}^- \), respectively), while the vertex operators

\[ e^{ip\nu X^\mu + (ip - \frac{Q}{2}) X_L \epsilon^{\alpha\beta}(H_{1,2}^+ H_{1,2}^-)} = e^{ip\nu X^\mu + (ip - \frac{Q}{2}) X_L} [(\text{spin}\Psi^\mu)_+ \text{ or } (\text{spin}\Psi^I)_+] (\text{spin}\Psi^I)_- \tag{4.14} \]

are unphysical, since they are not local with respect to the \( N = 4 \) generators. These unphysical states should be eliminated from the spectrum by additional GSO projection(s). In (4.13) and (4.14), \( (\text{spin}\Psi^\mu)_\pm \) and \( (\text{spin}\Psi^I)_\pm \) are the spin-fields of \( SU(2)_{H_{1,2}^\pm} \) and \( SU(2)_{H_{1,2}^\pm} \), respectively.

### 4.3 Partition function for the wormhole solution

For \( k \) even, there is a \( Z_2 \) automorphism of \( SU(2)_k \) which leaves invariant the currents but acts non-trivially on the spinorial and odd spin representations. This allows to
correlate the $SU(2)_{H_2}$ and $SU(2)_k$ spins in a way which projects out of the spectrum the unphysical states (4.14). A modular-invariant partition function with this property is:

\[
Z_W = \frac{\text{Im } \tau^{-5/2}}{\eta^5 \bar{\eta}^5} \frac{1}{8} \sum_{\alpha,\beta,\bar{\alpha},\beta,\gamma,\delta} (-)^{\alpha+\beta} \frac{\theta^2(\alpha)}{\eta^2} \frac{\theta^2(\beta+\gamma)}{\eta^2} (-)^{\bar{\alpha}+\bar{\beta}} \frac{\bar{\theta}^2(\bar{\alpha}+\bar{\gamma})}{\bar{\eta}^2} \frac{\bar{\theta}^2(\beta+\delta)}{\bar{\eta}^2} (-)^{\delta(\alpha+\bar{\alpha}+\beta)} Z_k[\gamma],
\]

(4.15)

where the first factor in the r.h.s. represents the contribution of the non-compact coordinates (4.4) together with the contribution of the Liouville mode. Using the expressions (4.6) and (4.11), one can decompose the above partition function in terms of $SU(2)$ characters:

\[
Z_W = \frac{\text{Im } \tau^{-5/2}}{\eta^5 \bar{\eta}^5} \sum_{\alpha,\bar{\alpha},\gamma,l} (-)^{\alpha+\bar{\alpha}} \chi_{H_1}^l \chi_{H_2}^{1+l+\alpha+\gamma} \chi_{H_1}^{1\bar{l}+\alpha+\gamma} \chi_{H_2}^{1\bar{l}+\alpha+\gamma} \sum_{L} \frac{1}{2} [1 + (-)^{\alpha+l+\bar{l}+\bar{\alpha}+\gamma} L] \frac{\chi_k^L \chi_k^{L+\gamma(2L)}}{-\eta^2} Z_k[\gamma]
\]

(4.16)

where the summation indices of the $SU(2)$ level-1 characters $\chi_H$ take values modulo 2.

In the derivation of (4.16) from (4.15), the $\beta$ and $\bar{\beta}$ summations give rise to the universal (left- and right-moving) GSO projections, which imply the existence of space-time supersymmetry. The phase $(-)^{\alpha+\bar{\alpha}}$ guarantees the spin-statistics connection; it equals 1 for space-time bosons and $-1$ for space-time fermions. The summation over $\delta$ gives rise to an additional projection, which correlates the $SU(2)_{H_2}$ (left and right) spin together with the spin of $SU(2)_k$ and thus reduces the number of space-time supersymmetries by a factor of 2. This projection takes the form:

\[
2\bar{S}_2 + 2\bar{\bar{S}}_2 + L + \frac{k}{2} \gamma = \text{even },
\]

(4.17)

where $\bar{S}_2$ and $\bar{\bar{S}}_2$ are the left and right $SU(2)_{H_2}$ spin. Note that $L+k/2\gamma = J+(-)^{\gamma}J$, $J$ and $\bar{J}$ are the left and right $SU(2)_k$ spins.

In the $\gamma = 0$ sector, the lower-lying states have (left and right) mass-squared $Q^2/8$ and $L = 0$. Although there are seven non-compact dimensions, there is only six-dimensional Lorentz invariance, because of the non-trivial dilaton background.
It is then convenient to classify the states in the context of a six-dimensional theory. In fact, the lower-lying states form the gravitational supermultiplet of the six-dimensional $N = 2$ supergravity:

\[
(|\Psi^\mu > + |(\text{spin}\Psi^\mu)_- (\text{spin}\Psi^I)_+ > ) \otimes (|\bar{\Psi}^\mu > + |(\text{spin}\bar{\Psi}^\mu)_- (\text{spin}\bar{\Psi}^I)_+ > ) e^{ip_\mu X^\mu + (ip - Q)X_L} \quad (4.18)
\]

together with four vector multiplets:

\[
(|\Psi^I > + |(\text{spin}\Psi^\mu)_+ (\text{spin}\Psi^I)_- > ) \otimes (|\bar{\Psi}^I > + |(\text{spin}\bar{\Psi}^I)_+ (\text{spin}\bar{\Psi}^I)_- > ) e^{ip_\mu X^\mu + (ip - Q)X_L} \quad (4.19)
\]

As expected from the effective field theory point of view, their mass-squared $Q^2/8$ is due to the dilaton motion for bosons, and to the non-trivial torsion for fermions.

Note that the contribution of 2-d fermions in the partition function of the $\gamma = 0$ sector is identical to the fermionic part of the partition function of the ten-dimensional type II superstring with an additional $1/2$ factor:

\[
Z_{\gamma=0}^W = \frac{\text{Im } \tau^{-5/2} \frac{1}{8}|\theta_3^4 - \theta_4^4 - \theta_2^4|^2}{|\eta^5 \bar{\eta}^5|} \sum_L |\chi_L|^2 \quad (4.20)
\]

$Z_{\gamma=0}^W$ can be identified as the untwisted partition function of a $Z_2$ symmetric orbifold of the ten-dimensional theory compactified on a three-dimensional sphere $SU(2)_k$. The $Z_2$ acts on the four (left and right) world-sheet fermions associated to the wormhole $W^{(4)}_k$ space, as well as on the spinorial representations of the $SU(2)_k$. This $Z_2$ projection is dictated from the $N = 4$ superconformal algebra and eliminates from the untwisted sector the unphysical states. Modular invariance implies the presence of a twisted sector ($\gamma = 1$), which contains states with (left and right) mass-squared always larger than $(k - 2)/16$. The lower-lying twisted states are:

\[
(|(\text{spin}\Psi^\mu)_+ > |L = \frac{k}{2} > + |(\text{spin}\Psi^I)_+ > + |\bar{L} = \frac{k}{2} > ) \otimes \\
(|(\text{spin}\bar{\Psi}^\mu)_+ > |L = \frac{k}{2} > + |(\text{spin}\bar{\Psi}^I)_+ > + |\bar{L} = \frac{k}{2} > ) e^{ip_\mu X^\mu + (ip - Q)X_L} \quad (4.21)
\]

For any $k > 2$, the twisted states have masses larger than $Q^2/8$ and the lower mass spectrum comes always from the $L = 0$ states contained in the untwisted sector. In that sense $k = 2$ is an exceptional case, since the lower-lying twisted states are
massless with \( L = \bar{L} = k/2 = 1 \). These states form massless unitary representations of the \( N = 4 \) \( \hat{c} = 8 \) superconformal system which saturate the \( N = 4 \) unitarity bounds.

As we noted in Section 2.2, the above solution is connected to non-critical superstrings with \( \hat{c}_M = 9 - \left( \frac{4}{k+2} \right) \). Thus, for \( k = 2 \) the matter system has \( \hat{c}_M = 8 \) and, as we explicitly showed above, it contains massless states in the twisted sector. This seems to be in contradiction with the \( N = 2 \) super-Liouville analysis of Kutasov and Seiberg [30], where they claim the absence of massless states for all \( \hat{c}_M \neq 5 \). The reason they missed this possibility is because their analysis is valid only for \( (\hat{c}_M + 1) \) even integer, in order to bosonize by pairs the \( (\hat{c}_M + 1) \) two-dimensional fermions.

In our construction, the matter central charge is in general a fractional number with a lower value \( \hat{c}_M = 23/3 \) \( (k = 1) \). In Section 4.4 we extend our non-critical string constructions to the general \( K(6) \) space with \( \hat{c}_M = 9 - 4\left( \frac{1}{k_1+2} + \frac{1}{k_2+2} \right) \geq 5 \).

In the heterotic case, a modular-invariant partition function for \( k \) even can be easily obtained using the heterotic map [14], [37]. It consists of replacing in (4.15) the \( O(4) \) characters associated to the right-moving fermionic coordinates \( \bar{\Psi}^\mu \), with the characters of either \( O(12) \otimes E_8 \):

\[
(-)^{\alpha+\beta} \frac{\bar{\theta}^2(\frac{\alpha}{\beta})}{\eta^2} \rightarrow \frac{\bar{\theta}^8(\frac{\alpha}{\beta})}{\eta^8} + \frac{\bar{\theta}^8}{\eta^8} + \frac{\bar{\theta}^8}{\eta^8} + \frac{\bar{\theta}^8}{\eta^8},
\]

or \( O(28) \):

\[
(-)^{\alpha+\beta} \frac{\bar{\theta}^2(\frac{\alpha}{\beta})}{\eta^2} \rightarrow \frac{\bar{\theta}^{14}(\frac{\alpha}{\beta})}{\eta^{14}}.
\]

The lower-lying states having mass-squared \( Q^2/8 \) are those of a \((4,4)\) \( Z_2 \) symmetric orbifold and form the spectrum of an \( N = 2 \) six-dimensional supergravity with a gauge group either \( E_7 \otimes E_8 \), or \( SU(2) \otimes O(28) \).

For \( k \) odd, the above constructions are not valid because there is no \( Z_2 \) automorphism in \( SU(2)_k \). In fact, the phase in the \( \tau \rightarrow \tau + 1 \) transformation (4.12) cannot be cancelled in a modular-invariant way consistently with the \( N = (4,4) \) algebra, while keeping at the same time the \( SU(2)_k \) left and right symmetry. Therefore, in 6+1 non-compact dimensions, it is necessary to twist the \( SU(2)_k \) left and right currents and, thus, one must break \( SU(2)_L \otimes SU(2)_R \). However in lower non-compact
dimensions, it is possible to extend the above constructions keeping the left and right $SU(2)_k$ symmetry. For instance, compactifying one dimension $\phi$ on a cycle with radius $R_\phi = \sqrt{2}$ (the $SU(2)$ self-dual point), one can use the additional $SU(2)$ level-one characters $\chi_{\phi}^{L'}$ to define the $Z_2$ action consistently.

In the case $R_\phi = \sqrt{2}$, a type-II modular-invariant partition function, valid for any $k$, is:

$$ Z_W = \frac{\text{Im} \; \tau^{-2}}{\eta^4 \bar{\eta}^4} \sum_{\alpha, \beta, \gamma, \delta} \frac{1}{8} \left[ (-)^{\alpha+\beta} \frac{\theta^2(\bar{\alpha}) \theta^2(\alpha+\gamma)}{\eta^2} \right] \left[ (-)^{\alpha+\beta} \frac{\theta^2(\beta) \theta^2(\alpha+\gamma)}{\bar{\eta}^2} \right] \text{even} \left[ (-)^{\delta(\alpha+\bar{\alpha}+\frac{k(k+1)}{2})} Z_k(\bar{\beta}) Z_{\bar{\delta}}(\beta) \right] , \quad (4.24) $$

where $Z_{1[k\delta]}$ is defined, as in (4.11), at level 1 using the characters of the additional $SU(2)_\phi$ level-1 symmetry, associated to the compactified dimension $\phi$.

The partition function (4.24) can be decomposed as:

$$ Z_W = \frac{\text{Im} \; \tau^{-2}}{\eta^4 \bar{\eta}^4} \sum_{\alpha, \alpha, \gamma, \beta, l} \left[ (-)^{\alpha+\bar{\alpha}} \chi_{H^+}^l \chi_{H^-}^{l+\alpha} \chi_{H^+}^{l+\alpha+\gamma} \chi_{H^-}^{l+\alpha+\gamma} \chi_{H^+}^{l+\alpha+\gamma} \chi_{H^-}^{l+\alpha+\gamma} \right] \sum_{L, L'} \frac{1}{2} \left[ 1 + (-)^{\alpha+l+\bar{\alpha}+\gamma+L+kL'} L - \gamma + (k-2L) + L' - k' \right] \chi_k \chi_k \chi_k \chi_k \quad (4.25) $$

As already mentioned above, for $k$ odd, the $Z_2$ $\delta$-projection acts also in the spinorial representations of the additional $SU(2)_\phi$:

$$ 2\bar{S}_2 + 2\bar{S}_2 + L + kL' + \frac{k(k+1)}{2} \gamma = \text{even} . \quad (4.26) $$

Note that for $k$ even, this projection acts trivially on $SU(2)_\phi$ representations.

As in the case of $k$-even construction, the lower lying-states in the untwisted sector ($\gamma = 0$) have mass-squared $Q^2/8$ and $L = L' = 0$. Since one of the non-compact dimensions is compactified, we can classify these states in the context of a five-dimensional theory; they contain the states of the $N = 4$ supergravity multiplet, together with four extra vector multiplets. In the twisted sector ($\gamma = 1$) the states are always massive with lower left and right mass-squared equal to $\frac{(k+1)^2}{10(k+2)}$, corresponding to:
\[(|\text{spin} \Psi^\mu, \text{spin} \Psi^\phi\rangle_+ > + |\text{spin} \Psi^I\rangle_+ >) \otimes (|\text{spin} \bar{\Psi}^\mu, \text{spin} \bar{\Psi}^\phi\rangle_+ > + |\text{spin} \Psi^I\rangle_+ >)\]

\[\otimes (|J = \bar{J}' + \frac{1}{2} = \frac{k + 1}{4}, J' = 0, \bar{J}' = \frac{1}{2} > + |J = \bar{J}' - \frac{1}{2} = \frac{k - 1}{4}, J' = 1, \bar{J}' = 0 >)\]

\[e^{ip_\mu X^\mu + (ip - \frac{Q}{2})X_L}. \quad (4.27)\]

For \(k = 1\) the twisted and untwisted lower-mass states are degenerate; for all other values of \(k\), the twisted states are always heavier, with mass-squared larger than \(Q^2/8\).

In the heterotic case, a modular-invariant partition function can be easily obtained using the heterotic map described by (4.22) and (4.23). The corresponding gauge groups are now \(SU(2) \otimes E_7 \otimes E_8\) and \(SU(2) \otimes SU(2) \otimes O(28)\), respectively. For lower non-compact dimensions, several other constructions can be easily obtained.

Note that in the semiclassical limit \(k \to \infty\), the twisted states become infinitely heavy, since their mass-squared grows as \(k/16\) and thus decouple from the spectrum. The remaining states are those from the untwisted sector with lower mass-squared \(1/4k\). Moreover, the spectrum of lower modes with masses much less than the string scale \(L << \sqrt{k}\) forms the spectrum of the semi-wormhole, \([L(L + 2) + 1]/4k\). The first term \(L(L + 2)/4k\) is identical to the contribution of the angular-momentum excitations of an ordinary Kaluza-Klein field theory compactified on a three-dimensional sphere, while the second term \(1/4k\) is due to the dilaton motion (or torsion). This phenomenon is similar to the case of a toroidal compactification where the Kaluza-Klein states have mass-squared \(m^2/2R^2\) (quantized momenta) while the stringy-like states are superheavy (in the semiclassical limit) with mass-squared \(n^2R^2/4\). The twisted states in the wormhole space play the same role as the winding modes in toroidal compactifications.

The wormhole target space interpretation fails for \(k\) small, as mentioned in the introduction, since the field theory modes and string modes have comparable masses, of the order of the string scale. An interesting limit is when \(k = 0\), which corresponds to a non-critical superstring with \(\hat{c}_M = 7\). Furthermore, in the heterotic case it corresponds to the high-temperature phase of the critical heterotic superstring when one of the coordinates (associated to the temperature) is compactified on a cycle of
unit radius \([31]\). Unfortunately, this limit cannot be taken in the above partition functions \((4.13)\) and \((4.24)\), because the \(\delta\)-projection is inconsistent with the global existence of the 2-d supercurrent, when \(k = 0\). We were unable to find a consistent theory with \(k = 0\) and more than five non-compact dimensions. When two dimensions \(\phi_{1,2}\), in addition to that of the temperature, are also compactified on two tori of unit radius, an example of a consistent partition function is:

\[
Z[M^4 \otimes T^2 \otimes W^{(4)}_{k=0}] = \frac{\text{Im} \tau^{-3/2}}{\eta^3 3^3} \sum_{\alpha, \beta, \gamma, \delta} (-)^{\alpha+\beta} \frac{\theta^2(\alpha/\beta) \theta^2(\alpha+\gamma/\beta)}{\eta^2} \eta^2
\]

\[
(-)^{\alpha+\beta} \frac{\bar{\theta}^2(\bar{\alpha}/\bar{\beta}) \bar{\theta}^2(\bar{\alpha}+\gamma/\bar{\beta})}{\bar{\eta}^2} (-)^{\delta(\alpha+\bar{\alpha})} \frac{\theta^2(\bar{\delta}) \bar{\theta}^2(\bar{\delta})}{\bar{\eta}^2} ,
\]

where the \(\theta\)-functions with arguments \(\gamma\) and \(\delta\) denote the contribution of the two compactified coordinates \(\phi_{1,2}\) at radius 1.

The partition function \((4.28)\) can be decomposed as:

\[
Z[M^4 \otimes T^2 \otimes W^{(4)}_{k=0}] = \frac{\text{Im} \tau^{-3/2}}{\eta^3 3^3} \sum_{\alpha, \beta, \gamma, l, \bar{l}} (-)^{\alpha+\bar{\alpha}} \frac{\chi^l H^l + \chi^{l+\bar{l}} H^{l+\bar{l}} + \gamma}{\chi^l \chi^{l+\bar{l}} + \chi^{l+\bar{l}} \chi^l}
\]

\[
\sum_{l', \bar{l}'} \frac{1}{2} [1 + (-)^{\alpha+l+\bar{l}+l'+\bar{l}'}] \chi^{l' + \gamma} \chi^{l' + \bar{\gamma}} + \chi^{l' + \gamma} \chi^{l' + \bar{\gamma}} ,
\]

where \(\phi^\pm = (\phi_1 \pm \phi_2)/\sqrt{2}\) are both compactified at the \(SU(2)\) self-dual point. The \(\delta\)-projection is:

\[
2 \tilde{S}_2 + 2 \tilde{\bar{S}}_2 + l' + \bar{l}' = \text{even} .
\]

The lower-lying states from both the untwisted and twisted sectors are degenerate with mass-squared equal to \(Q^2/8 = 1/8\). The heterotic construction can be done using the maps \((4.22)\) and \((4.23)\).

### 4.4 The \(F^{(2)} \otimes W^{(4)}_{k_1} \otimes W^{(4)}_{k_2}\) partition function

The above constructions can be easily extended to the background solutions \(F^{(2)} \otimes W^{(4)}_{k_1} \otimes W^{(4)}_{k_2}\). In this case, the \(Z_2\) projection can act in different ways in the two spaces consistently with modular invariance. An interesting limit is when \(k_1 = k_2 = 0\), where the Kac-Moody currents decouple. Then, the \(N = 4\) algebra is realized only in terms of world-sheet fermions and the two \(U(1)\) currents with background charges.
\( Q_1 = Q_2 = 1 \). As mentioned in Section 2.2, this model can be seen as a non-critical superstring of \( \hat{c}_M = 5 \) coupled to the Liouville field, which is identified with the linear combination of the two \( U(1) \)'s carrying the background charge \( Q_L = \sqrt{Q_1^2 + Q_2^2} = \sqrt{2} \). For \( (k_1 + k_2) \) even, the partition function for the \( F^{(2)} \otimes W_{k_1}^{(4)} \otimes W_{k_2}^{(4)} \) model can be derived in a similar way as in (4.15) and (4.24): \[
Z_{W_{k_1} \otimes W_{k_2}} = \frac{\text{Im} \tau^{-1}}{\eta^2 \bar{\eta}^2} \sum_{\alpha, \beta, \alpha', \gamma, \delta} (-)^{\alpha + \beta} \frac{\theta^2(\alpha)}{\eta^2} \frac{\theta^2(\alpha + \gamma)}{\eta^2} (-)^{\alpha + \beta + \delta} \frac{\theta^2(\beta)}{\eta^2} \frac{\theta^2(\beta + \delta)}{\bar{\eta}^2} \]
\[
(-)^{\delta (\alpha + \bar{\alpha} + (k_1 + k_2)/2) \gamma} Z_{k_1}^{(3)} \; Z_{k_2}^{(3)} . \quad (4.31)
\]

After taking into account the \( \beta \) and \( \bar{\beta} \) projections, we can express \( Z_{W_{k_1} \otimes W_{k_2}} \) in terms of the various \( SU(2) \) characters:
\[
Z_{W_{k_1} \otimes W_{k_2}} = \frac{\text{Im} \tau^{-1}}{\eta^2 \bar{\eta}^2} \sum_{\alpha, \beta, \alpha', \gamma, \delta} (-)^{\alpha + \bar{\alpha}} \chi_{H_i}^{L_1} \chi_{H_i'}^{l+\alpha} \chi_{H_2}^{1+l+\alpha+\gamma} \chi_{H_2'}^{l+\bar{\alpha}+\gamma} \chi_{H_2}^{l+\bar{\alpha}+\gamma} \chi_{H_2'}^{l+\alpha+\gamma} \chi_{H_2}^{l+\alpha+\gamma}
\]
\[
\sum_{L, L'} \left[ 1 + (-)^{\alpha + l + \bar{\alpha} + (k_1 + k_2)/2 + L + L'} \right] \chi_{k_1}^{L} \chi_{k_1}^{L+\gamma(1-2L)} \chi_{k_2}^{L'} \chi_{k_2}^{L'+\gamma(k_2-2L')} \quad (4.32)
\]

The lower-lying states in the untwisted sector \( (\gamma = 0) \) have (left and right) mass-squared equal to \( \frac{Q_1^2 + Q_2^2}{8} \). In the general case, where \( k_1 \) and \( k_2 \) are non-zero, the \( \delta \)-projection becomes:
\[
2 S_2 + 2 \bar{S}_2 + L + L' + \frac{(k_1 + k_2)}{2} \gamma = \text{even} , \quad (4.33)
\]
correlating (for non-vanishing \( k_i \)) the \( SU(2)_{k_i} \) spins with those of \( SU(2)_{H_i} \). This correlation implies that the lower lying states in the twisted sector \( (\gamma = 1) \) are heavier than those of the untwisted sector, provided \( k_i \) are large. For \( k_i \) small, and in particular when both \( k_i \)'s are zero, the twisted states can have mass-squared lower than \( \frac{Q_1^2 + Q_2^2}{8} \). When \( k_1 \) and \( k_2 \) are even, the (left and right) mass-squared of the lower-lying twisted states are equal to \( (k_1 + k_2)/16 \), while those of the lower-lying untwisted states are equal to \( 1/4(k_1 + 2k_2) \). Thus, for \( k_i \) even, the twisted sector is heavier than the untwisted one for any value \( k_i \neq 0 \). In the limiting case \( k_i = 0 \), however, the twisted sector becomes massless as expected from the \( \hat{c}_M = 5 \) non-critical superstring theory [31]. When both \( k_i \)'s are odd, the mass-squared of the lower-lying twisted
states is \( \frac{1}{16} \left( \frac{(k_1+1)^2}{k_1+2} + \frac{(k_2+1)^2}{k_2+2} \right) \) and thus for \( k_1 = k_2 = 1 \) the twisted and untwisted lower mass states are degenerate.

As we mentioned in Section 2.2, a \( \hat{c}_M = 5 \) super-Liouville model exhibiting \( N = 4 \) superconformal symmetry was proposed in ref. [27]. This model has a five-dimensional Lorentz invariance while the Liouville zero-mode has a discrete spectrum. In our case, there are four non-compact dimensions but the Lorentz symmetry is reduced to three-dimensional due to the background charge in the Liouville coordinate; in addition, there are two compact dimensions described by four free fermions forming an \( SU(2) \otimes SU(2) \) level-1 group manifold. It is interesting to notice that the partition function of the Bilal-Gervais model, when one of the five dimensions is compactified at the \( SU(2) \) point \( (R = \sqrt{2}) \), is the same with our expression (4.31) in the limiting case \( k_i = 0 \). Despite this similarity, the two models are expected to be different since the spectra and their symmetries are not the same in the two models.

Note that when \( k_1 = k_2 = k \) the two \( SU(2)_k \)’s combine to form an \( SO(4)_k \) group manifold and the large-\( k \) limit of (4.31) corresponds to a classical solution of the \( SO(4)_k \) gauged supergravity theory [24], [25].

The heterotic construction can be done as before, using the maps given in (4.22) and (4.23). For \( k_1 = k_2 = 2 \), one obtains a different realization of the \( \hat{c}_M = 7 \) non-critical superstring (type-II or heterotic).

5 Conclusions

String solutions in the semiclassical limit define background solutions of some special effective field theories. This limit turns out to be very useful regarding the study of the string-induced low-energy theories, as well as the study of physics in weakly curved domains of space time. The field theory picture, however, completely fails when the involved curvatures are strong; it is then necessary to go beyond the semiclassical limit and work directly on the string level, using the powerful techniques of the underlying two-dimensional (super)conformal field theory. For a generic string background the stringy approach is at present non-accessible, because of some tech-
nical difficulties, which hopefully will be solved in the future. As we show in this work, it is possible to go further in the stringy direction for some special choices of the target-space backgrounds, namely when one chooses the world-sheet degrees of freedom to form non-trivial realizations of the $N = 4$ superconformal symmetry.

In the weak curvature limit, all our solutions have a ten-dimensional target space interpretation, and each one of them contains two curved four-dimensional subspaces; each 4-d subspace defines a $\hat{c} = 4$, $N = 4$ superconformal system and contains an integer parameter $k$ defining the strength of its curvature ($\mathcal{O}(\frac{1}{k+2})$). The semiclassical limit is when both 4-d curvatures are small: $(k_1, k_2 \to \infty)$. In our analysis we used as building blocks four topologically non-trivial 4-d subspaces found in ref. [10]: i) The $W_k^{(4)}$ space, which has the shape of a 4-d (semi)wormhole; ii) the $C_k^{(4)}$, with the shape of (2-d bell)$\otimes$(2-d cylinder); iii) and iv) the two versions of the $\Delta_k^{(4)}$ space, with the shape of (2-d bell)$\otimes$(2-d cigar) and (2-dbell)$\otimes$(2-d trumpet), respectively. In the large-$k_i$ limit, the constructions based on $W_{k_i}^{(4)}$ superconformal blocks describe some stable solutions of 4-d gauged supergravities, $N = 8$ in the type-II construction or $N = 4$ in the heterotic construction.

The constructions based on $W_k^{(4)}$ and/or $C_k^{(4)}$ blocks are connected to the non-critical strings and define super-Liouville theories in the strong coupling regime coupled to unitary matter systems. The Liouville and matter central charges, $\hat{c}_L = 1 + 4(\frac{1}{k_1+2} + \frac{1}{k_2+2})$ and $\hat{c}_M = 1 - 4(\frac{1}{k_1+2} + \frac{1}{k_2+2})$ are given in terms of the two-integer parameters $k_1$ and $k_2$. The lower value $\hat{c}_M = 5$ corresponds to the $W_{k_i}^{(4)}$ construction, in the limiting case where $k_1 = k_2 = 0$. Another interesting value is when $\hat{c}_M = 7$, obtained with $k_1 = 0$ and $k_2 \to \infty$, or when $k_1 = k_2 = 2$. It turns out that this case is in correspondence with the high-temperature phase of the heterotic critical superstring [31].

The full spectrum of excitations can be derived in all our constructions combining unitary representations of the $N = 4$ superconformal theory in a modular-invariant way. In the case of $W_k^{(4)}$ constructions, these representations are expressed in terms of the well-known $SU(2)$ characters, while in all other constructions one uses also the characters of some compact ($SU(2)/U(1)$) and/or non-compact ($SL(2, R)/U(1)$)
parafermions, as well as those of free boson compactified in a given special radius. For the $W_k^{(4)}$ constructions, we give the full spectrum of propagating states in terms of modular-invariant partition functions for all values of $k_i$’s. When $k_i$’s are large, all states in the $Z_2$ twisted sector have mass-squared which grows with $k_i$. The lower-lying states in this sector have mass-squared equal to $(k_1 + k_2)/16$ and, thus they decouple in the semiclassical limit. In this limit the remaining states are those of the $Z_2$ untwisted sector with masses lower than the string scale; these states are in one-to-one correspondence with those of a 10-d Kaluza-Klein field theory defined in a double wormhole space; their masses are just given in terms of the two $SU(2)_{k_i}$ spherical excitations with a shifting due to the non trivial dilaton and/or torsion background: $M_{l_1,l_2}^2 = \frac{1}{4} \left( \frac{(l_1+1)^2}{k_1+2} + \frac{(l_2+1)^2}{k_2+2} \right)$. The untwisted states are then similar to the quantized momenta of a toroidal compactification while the twisted states are in correspondence with the string-winding modes.

For small $k_i$, the lower-mass of the states, in both the twisted and untwisted sector, is of the same order as the string mass scale and thus, the wormhole target space-time interpretation fails. The untwisted states can be massless only for $k_i = \infty$. The twisted states are in general massive. We found, however, two special cases were some of the twisted states are massless. The existence of the first case was conjectured in the framework of super-Liouville theories in ref.\cite{30} and it corresponds to the limiting case where both $k_i$ are zero, with $\hat{c}_M = 5$. The second special case is new and it corresponds to a $\hat{c}_M = 8$ super-Liouville theory with $k_1 = 2$ and $k_2 = \infty$.

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