On spacetimes with constant scalar invariants

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Received 13 September 2005, in final form 16 February 2006
Published 5 April 2006
Online at stacks.iop.org/CQG/23/3053

Abstract

We study Lorentzian spacetimes for which all scalar invariants constructed from the Riemann tensor and its covariant derivatives are constant (CSI spacetimes). We obtain a number of general results in arbitrary dimensions. We study and construct warped product CSI spacetimes and higher-dimensional Kundt CSI spacetimes. We show how these spacetimes can be constructed from locally homogeneous spaces and VSI spacetimes. The results suggest a number of conjectures. In particular, it is plausible that for CSI spacetimes that are not locally homogeneous the Weyl type is II, III, N or O, with any boost weight zero components being constant. We then consider the four-dimensional CSI spacetimes in more detail. We show that there are severe constraints on these spacetimes, and we argue that it is plausible that they are either locally homogeneous or that the spacetime necessarily belongs to the Kundt class of CSI spacetimes, all of which are constructed. The four-dimensional results lend support to the conjectures in higher dimensions.

PACS numbers: 04.20.−q, 04.20.Jb, 02.40.−k

1. Introduction

Let \((M, g)\) denote a differentiable manifold, either of Riemannian or of Lorentzian signature. We are interested in all spacetimes for which all scalar invariants constructed from the Riemann tensor and its covariant derivatives are constant (denoted as CSI spacetimes). In the case of a Riemannian manifold, if it is CSI then it is locally homogeneous [1]. This theorem is false for Lorentzian manifolds (there are a number of counterexamples; e.g., the non-homogeneous VSI spacetimes [2]).

There are a number of CSI spacetimes that arise from homogeneous spacetimes and simple warped products (see section 3). Some examples of CSI spacetimes that arise as simple tensor sum generalizations of VSI spacetimes with a cosmological constant are given in [3] (some higher-dimensional versions are given in [4]). Simple examples of fibred products were given...
in [5, 6] (see also [7]). It is also straightforward to construct CSI spacetimes as warped products. All of these spacetimes belong to the class $\text{CSI}_R$ described below.

We begin with some definitions and a brief discussion of the four-dimensional case.

1.1. Notation

Let us denote by $\mathcal{I}_k$ the set of all scalar invariants constructed from the curvature tensor and its covariant derivatives up to order $k$.

**Definition 1.1** ($\text{VSI}_k$ spacetimes). $\mathcal{M}$ is called $\text{VSI}_k$ if for any invariant $I \in \mathcal{I}_k$, $I = 0$ over $\mathcal{M}$.

**Definition 1.2** ($\text{CSI}_k$ spacetimes). $\mathcal{M}$ is called $\text{CSI}_k$ if for any invariant $I \in \mathcal{I}_k$, $\partial \mu I = 0$ over $\mathcal{M}$.

Moreover, if a spacetime is $\text{VSI}_k$ or $\text{CSI}_k$ for all $k$, we will simply call the spacetime $\text{VSI}$ or $\text{CSI}$, respectively. The set of all locally homogeneous spacetimes will be denoted by $\mathcal{H}$. Clearly, $\text{VSI} \subset \text{CSI}$ and $\mathcal{H} \subset \text{CSI}$.

**Definition 1.3** ($\text{CSI}_R$ spacetimes). Let us denote by $\text{CSI}_R$ all reducible CSI spacetimes that can be built from $\text{VSI}$ and $\mathcal{H}$ by (i) warped products, (ii) fibred products and (iii) tensor sums (defined more precisely later).

**Definition 1.4** ($\text{CSI}_F$ spacetimes). Let us denote by $\text{CSI}_F$ those spacetimes for which there exists a frame with a null vector $\ell$ such that all components of the Riemann tensor and its covariant derivatives in this frame have the property that (i) all positive boost weight components (with respect to $\ell$) are zero and (ii) all zero boost weight components are constant.

Note that $\text{CSI}_R \subset \text{CSI}$ and $\text{CSI}_F \subset \text{CSI}$. (There are similar definitions for $\text{CSI}_{F,k}$, etc [8].)

**Definition 1.5** ($\text{CSI}_K$ spacetimes). Finally, let us denote by $\text{CSI}_K$ those CSI spacetimes that belong to the (higher-dimensional) Kundt class (defined later), the so-called Kundt CSI spacetimes.

In particular, we shall study the relationship between $\text{CSI}_R$, $\text{CSI}_F$, $\text{CSI}_K$ and especially with $\mathcal{H}$. We note that by construction $\text{CSI}_K$ is at least of Weyl type II (i.e., of type II, III, N or O [9]) and by definition $\text{CSI}_F$ and $\text{CSI}_K$ are at least of Weyl type II (more precisely, at least of Riemann type II). In 4D, $\text{CSI}_R$, $\text{CSI}_F$ and $\text{CSI}_K$ are closely related, and it is plausible that $\text{CSI} \setminus \mathcal{H}$ is at least of Weyl type II (see section 7).

In four dimensions, the Weyl classification and the Petrov classification are closely related [9]. However, due to the fact that the Weyl classification contains many more possibilities in dimensions higher than 4, we will restrict the term Petrov classification to four dimensions only.

1.2. 4D CSI

We are particularly interested in the four-dimensional CSI spacetimes. For a Riemannian manifold, every CSI is homogeneous ($\text{CSI} \equiv \mathcal{H}$). This is not true for Lorentzian manifolds. However, for every CSI with particular constant invariants there is a homogeneous spacetime (not necessarily unique) with precisely the same constant invariants. This suggests that CSI can be ‘built’ from $\mathcal{H}$ and $\text{VSI}$ (e.g., $\text{CSI}_R$).
In addition, there is a relationship between the CSI conditions and (i) the rank of the Riemann tensor and its holonomy class [12], (ii) the existence of curvature collineations [12], (iii) the condition of non-complete backsolvability\(^1\) (NCB) [11], in addition to (iv) curvature homogeneity\(^2\) and (v) sectional curvature [12]. The relationship between CSI and curvature homogeneity in 4D is studied in [13]. In locally homogeneous spacetimes, all of the sectional curvatures (Gaussian curvatures) are constant. With the exception of certain special plane wave and constant curvature spacetimes (and all vacuum spacetimes), the sectional curvature uniquely determines the spacetime metric [12].

It is clear that CSI consists of CSI\(_R\) and, possibly, some other very special spacetimes. Let us present a heuristic argument for this in 4D. Let us suppose that the spacetime is CSI. For spacetimes that are completely backsolvable (CB), there is a special invariant frame such that all components of the Riemann tensor are constant. With respect to this invariant frame, there exist smooth vector fields \(\zeta_i (i = 1, 2, 3, 4)\) that act transitively on the manifold whose directional derivatives leave the Riemann tensor invariant (i.e., \(\zeta_i\) form a Lie algebra of curvature collineations that span \(\mathcal{M}\)). In general, every curvature collineation is a homothetic vector, so as a result there is a homothetic group acting transitively on \(\mathcal{M}\). Therefore, this then implies that in general the CSI spacetime is locally homogeneous. The exceptions are those spacetimes that are not CB (NCB [11]) and those spacetimes for which a curvature collineation is not a homothetic vector [12], and these spacetimes are related to the VSI spacetimes.

### 1.3. Overview

In this paper, we shall study CSI spacetimes. In the next section, we shall begin by summarizing some important results. In section 3, we construct a subclass of CSI\(_R\) spacetimes that arise as warped products of a homogeneous space and a VSI spacetime. In section 4, we consider the higher-dimensional Kundt class. In section 5, examples of CSI\(_K\) spacetimes in higher dimensions are given. We discuss the results in section 6 and present a number of conjectures. In section 7, we summarize the results in 4D in detail.

There are three appendices. In appendix A, we present all of the three-dimensional VSI metrics, which is necessary for the explicit determination of the metrics in the four-dimensional CSI\(_R\) spacetimes. In appendix B, four-dimensional CSI spacetimes are given explicitly and the relationship between CSI\(_R\), CSI\(_F\) and CSI\(_K\) in 4D is discussed. Finally, in appendix C, we write the metric for higher-dimensional VSI spacetimes in a canonical form.

## 2. Spacetimes with constant scalar invariants

### 2.1. Riemannian case

Let us first consider the Riemannian case where a great deal about these spacetimes is known.

The reason the Riemannian case is easier to deal with is because the orthogonal group \(O(d)\) is compact, and hence, the orbits of its group action are also compact.

**Theorem 2.1.** If \(\mathcal{M}\) is a Riemannian spacetime, then

1. \(\text{VSI}_0 \Rightarrow \text{VSI}\);  
2. there exists a \(k \in \mathbb{N}\) such that \(\text{CSI}_k \Rightarrow \text{CSI}\).

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\(^1\) Complete backsolvability (CB) refers to when all of the components of the Riemann tensor can be determined from the CZ set [11] of zeroth-order curvature invariants, once all the remaining frame freedom has been used to fix components of the Riemann tensor.

\(^2\) \(\mathcal{M}\) is curvature homogeneous (of order zero) if there exists a frame with respect to which the Riemann tensor has constant components.
Proof. (1) See proof below. (2) See [1].

Theorem 2.2 (Riemannian VSI). A Riemannian space is VSI if and only if it is flat.

Proof. Consider the invariant $R_{ABCD}R^{ABCD}$. Using an orthonormal frame, this invariant is a sum of squares; hence, $R_{ABCD}R^{ABCD} = 0 \iff R_{ABCD} = 0$.

Theorem 2.3 (Riemannian CSI). A Riemannian space is CSI if and only if it is locally homogeneous. Moreover, the set of curvature invariants, $I$, uniquely determines the metric up to isometries.

Proof. See [1].

Theorem 2.4 (Singer). If $\mathcal{M}$ is curvature homogeneous of order $k \geq \frac{1}{2}n(n-1)$, then $\mathcal{M}$ is locally homogeneous.

Proof. See [14].

2.2. Lorentzian case

The Lorentzian case is more difficult to deal with and only partial results are known. However, in the VSI case, it was shown in [15] that VSI$_2$ implies VSI. A similar result in the CSI case is not known; however, in this case we believe that an $n$ exists such that CSI$_n$ implies CSI.

In the Riemannian case, $\mathcal{M}$ is locally homogeneous if and only if $\mathcal{M}$ is curvature homogeneous to all orders (theorem 2.4). In the Lorentzian case, we have that

Theorem 2.5 (Podesta and Spiro). There is an integer $K_{p,q}$ such that if $(\mathcal{M}, g)$ is a complete, simply connected pseudo-Riemannian manifold of type $(p, q)$ which is $K_{p,q}$-curvature homogeneous, then $\mathcal{M}$ is locally homogeneous.

Proof. See [16].

We argued above that in 4D Lorentzian manifolds for every CSI with particular constant invariants there is a locally homogeneous spacetime with the same constant invariants. It is plausible that this is true in higher dimensions.

Conjecture 2.6. Assume that a Lorentzian spacetime $\mathcal{M}$ is a CSI spacetime with curvature invariants $\tilde{I}$. Then there exists a locally homogeneous space $\tilde{\mathcal{M}}$ with curvature invariants $\tilde{\tilde{I}} = \tilde{I}$.

2.3. Boost weight decomposition

Consider an arbitrary covariant tensor $T$ and a null frame $\{\ell_A, n_A, m^\hat{A}\}$, i.e.,

$\ell_A n^A = 1, \quad m^\hat{A} m^\hat{B} = \delta^{\hat{A}\hat{B}}, \quad \ell_A \ell^A = n_A n^A = \ell_A m^\hat{A} = n_A m^\hat{A} = 0$.

Consider now a boost in the $\ell^A-n^A$-plane:

$\{\tilde{\ell}_A, \tilde{n}_A, \tilde{m}^\hat{A}\} = \{e^\lambda \ell_A, e^{-\lambda} n_A, m^\hat{A}\}$.

We can consider the vector-space decomposition of the tensor $T$ in terms of the boost weight with respect to the above boost (see [10]):

$T = \sum_b (T)_b$, \quad (2)$
where \((T)_b\) denotes the projection of the tensor \(T\) onto the vector space of boost weight \(b\). The components of the tensor \((T)_b\) with respect to the frame will transform according to
\[
(T)_b^{AB...} = e^{-b\lambda} (T)_{\tilde{\lambda}\tilde{\mu}...}.
\]
Furthermore, we note that
\[
(T \otimes S)_b = \sum_{b' + b'' = b} (T)_{b'} \otimes (S)_{b''}.
\]
Moreover, the connection \(\nabla\) can similarly be decomposed according to whether it raises, lowers or preserves the boost weight:
\[
\nabla = (\nabla)_{-1} + (\nabla)_0 + (\nabla)_1,
\]
where \((\nabla)_{\lambda}\) is defined by
\[
(\nabla)_{\lambda}X \equiv \nabla (X)_{\lambda},
\]
for all vectors \(X\). We note that for the metric, \(g = (g)_0\); hence, raising or lowering tensor indices preserves the boost weight.

An invariant of a tensor, \(T\), is necessarily \(SO(1, n)\)-invariant; in particular, for any full contraction of \(T\):
\[
\text{Cont}[T] = \text{Cont}[(T)_0].
\]
This property can now be utilized to construct spaces with constant curvature invariants. In the VSI case, there exists a null frame such that the Riemann tensor, \(R\), has the property
\[
(R)_b = 0, \quad b \geq 0.
\]
This property alone implies that \(I_0 = 0\), regardless of \((R)_b (b < 0)\). If the curvature tensor is of the form
\[
(R)_b = 0, \quad b > 0, \quad \text{and} \quad (R)_{ABCD,E} = 0, \quad b = 0,
\]
in a frame for which \(g_{AB,E} = 0\), it is necessarily a CSI\(_0\) spacetime. In general, further restrictions must be put on \(\nabla\) in order to make it CSI. We note that spacetimes can be classified according to their boost weight components \((R)_b\). In particular, a classification using the Weyl tensor \(C\), which generalizes the Petrov classification in 4D, has been employed based on the components \((C)_b\) [9, 10].

2.4. Kundt metrics

In the 4D VSI spacetimes, all of the positive boost weight spin coefficients (e.g., \(\kappa, \rho\) and \(\sigma\)) are zero, and hence it follows that \(\ell\) is geodesic, non-expanding, shear-free and non-twisting and the spacetime is Kundt \(K\) [17]. This is also true for CSI spacetimes, in the sense that if the CSI spacetime is not locally homogeneous, then in general it belongs to \(K\). That is, it is plausible that in 4D, CSI \(\equiv H \cup \text{CSI}_K\).

In higher dimensions, it was also shown that \(\ell\) is geodesic, non-expanding, shear-free and non-twisting (i.e., \(L_{ij} = 0\)) in VSI spacetimes [15]. In locally homogeneous spacetimes, in general there exists a null frame in which the \(L_{ij}\) are constants. We therefore anticipate that in CSI spacetimes that are not locally homogeneous, \(L_{ij} = 0\). For higher-dimensional CSI spacetimes with \(L_{ij} = 0\), the Ricci and Bianchi identities appear to be identically satisfied [19]. A higher-dimensional spacetime which admits a null vector \(\ell\) which is geodesic, non-expanding, shear-free and non-twisting will be denoted as a higher-dimensional Kundt spacetime.
Let us assume that the spacetime admits a geodesic, non-twisting, non-expanding, shear-
free null vector $\ell$. It can then be shown that the existence of a twist-free and geodesic null vector implies that there exists a local coordinate system $(u, v, x^k)$ such that
\[
d s^2 = 2\, du\, (dv + H\, du + W_i\, dx^i) + \tilde{g}_{ij}(u, v, x^k)\, dx^i\, dx^j. \tag{9}
\]
In essence, the twist-free condition implies that the vector field is ‘surface forming’ so that there exists locally an exact null 1-form $du$. Using a coordinate transformation and a null rotation, we can now bring the metric into the above form. In this coordinate system, $\ell = \frac{\partial}{\partial v}$. In addition, requiring that this vector field is non-expanding and shear-free implies that $\tilde{g}_{ij,v} = 0$.

We will therefore consider metrics of the from
\[
d s^2 = 2\, du\, (dv + H\, du + W_i\, dx^i) + \tilde{g}_{ij}(u, x^k)\, dx^i\, dx^j, \tag{10}
\]
where $H = H(v, u, x^k)$ and $W_i = W_i(v, u, x^k)$. The metric (10) possesses a null vector field $\ell$ obeying
\[
\epsilon^A\ell_{;A} = \epsilon^A;\ell_{;A} = 0, \tag{11}
\]
i.e., it is geodesic, non-expanding, shear-free and non-twisting. We will refer to the metrics (10) as higher-dimensional Kundt metrics (or simply Kundt metrics), since they generalize the four-dimensional Kundt metrics.

2.5. Lorentzian CSI

All examples to the authors’ knowledge of Lorentzian CSI spacetimes are of the following two forms:

(1) homogeneous spaces;
(2) a subclass of the Kundt spacetimes.

For the homogeneous spaces, their decomposition can be of the most general type,
\[
\nabla^{(k)} R = \sum_{b=-2k}^{2k} (\nabla^{(k)} R)_b,
\]
since a frame can be chosen such that all components are constants. However, for all known examples of non-homogeneous CSI metrics, a frame can be found such that we have the decomposition
\[
\nabla^{(k)} R = \sum_{b=-2k}^{0} (\nabla^{(k)} R)_b,
\]
with constant boost weight zero components.

Since a homogeneous space is automatically a CSI space, we will in this paper consider metrics which are not necessarily homogeneous. Since all known examples of non-homogeneous metrics are of Kundt form, we will henceforth only consider Kundt metrics.

3. Warped product CSI

It is known that CSI R spacetimes can be constructed in all dimensions via warped products. Let us first determine necessary and sufficient conditions so that the warped product of a homogeneous space and a VSI is CSI.

We consider the warped product metric $g_{AB} = g_{ab} \oplus e^{2\tau} g_{ab}$, where $\frac{1}{2} g$ is a Riemannian homogenous space and $\tilde{g}$ is a $k$-dimensional VSI Lorentzian manifold, the warping function $\tau$ only depends on points of the homogenous space. Using the left-invariant frame of the
homogeneous space\(^3\),

\[ e_0 = \{ m_1, \ldots, m_{N-1} \}, \tag{12} \]

we complete it to a basis of the warped product by appending the null frame,

\[ e_\hat{0} = \{ \ell = e^{-\tau'}, n = e^{-\tau}n', m_2 = e^{-\tau}m'_2, \ldots, m_{N-1} = e^{-\tau}m'_{N-1} \}, \tag{13} \]

where the primed vectors represent the canonical VSI frame in which \( \ell' \) (and hence \( \ell \)) is the aligned geodesic null congruence that is expansion, shear and twist free. The non-vanishing inner products are such that

\[ g_{\hat{A}\hat{B}} = 2\ell (\ell B) + \delta_{\hat{A}\hat{B}} A^\mu A_\mu B + \delta_{\hat{A}\hat{B}} m^\mu A_\mu B. \tag{14} \]

Defining \( T_{\hat{A}\hat{B}} = \tau_{\hat{A}\hat{B}} \equiv \tau_{\hat{A}} \tau_{\hat{B}} \), we shall show that if \( \tau_{\mu} \tau^\mu \) and \( T_{\hat{A}\hat{B}} \) are constant then this implies CSI. These two conditions imply that \( \tau_{\mu} \) is an eigenvector of \( T_{\hat{A}\hat{B}} \) with constant eigenvalue \( \tau_\mu \tau^\mu \), and thus \( \tau_{\hat{A}} \) must be constant.

Following [15], we perform a decomposition of the Riemann tensor for the warped product to obtain

\[ R_{\hat{A}\hat{B}\hat{C}\hat{D}} = 4R^{\hat{A}\hat{B}\hat{C}\hat{D}}_{\hat{0}\hat{1}\hat{0}\hat{1}} + 8R^{\hat{A}\hat{B}\hat{C}\hat{D}}_{\hat{1}\hat{0}\hat{0}\hat{1}} + 8R^{\hat{A}\hat{B}\hat{C}\hat{D}}_{\hat{0}\hat{0}\hat{1}\hat{1}} + 8R^{\hat{A}\hat{B}\hat{C}\hat{D}}_{\hat{0}\hat{1}\hat{1}\hat{0}} \]

\[ + R^{\hat{A}\hat{B}\hat{C}\hat{D}}_{\hat{0}\hat{0}\hat{0}\hat{1}} + 8R^{\hat{A}\hat{B}\hat{C}\hat{D}}_{\hat{0}\hat{1}\hat{0}\hat{1}} + 8R^{\hat{A}\hat{B}\hat{C}\hat{D}}_{\hat{1}\hat{0}\hat{1}\hat{0}} + 8R^{\hat{A}\hat{B}\hat{C}\hat{D}}_{\hat{1}\hat{1}\hat{0}\hat{0}} \]

\[ + R_{\hat{A}\hat{B}\hat{C}\hat{D}} + R_{\hat{A}\hat{B}\hat{C}\hat{D}} \]

\[ + R_{\hat{A}\hat{B}\hat{C}\hat{D}} \]

\[ + R_{\hat{A}\hat{B}\hat{C}\hat{D}}. \tag{15} \]

Evidently, the Riemann tensor is of boost order zero with boost weight zero components:

\[ R_{\hat{0}\hat{1}\hat{0}\hat{1}} = \tau_{\mu} \tau^{\mu}, \tag{16} \]

\[ R_{\hat{0}\hat{1}\hat{1}\hat{1}} = -\tau_{\mu} \tau^{\mu} 2 \delta_{\mu \hat{A}}, \tag{17} \]

\[ R_{\hat{1}\hat{0}\hat{1}\hat{1}} = -T_{\hat{0}\hat{1}}, \tag{18} \]

\[ R_{\hat{1}\hat{0}\hat{0}\hat{2}} = \tau^{\mu} \tau_{\mu} \left[ \frac{2}{\delta_{\mu \hat{A}}} \delta_{\mu \hat{A}} - \frac{2}{\delta_{\mu \hat{A}}} \delta_{\mu \hat{A}} \right], \tag{19} \]

\[ R_{\hat{0}\hat{0}\hat{1}\hat{1}} = -T_{\hat{0}\hat{1}} \delta_{\mu \hat{A}}, \tag{20} \]

\[ R_{\hat{1}\hat{0}\hat{1}\hat{2}} = R_{\hat{0}\hat{1}\hat{1}\hat{1}}. \tag{21} \]

The negative boost weight components, arising from the VSI spacetime, are given by

\[ R_{\hat{1}\hat{1}\hat{0}\hat{1}} = e^{-2\tau} \frac{2}{\delta_{\mu \hat{A}}} \hat{R}_{\hat{1}\hat{1}\hat{0}\hat{1}}, \tag{22} \]

\[ R_{\hat{1}\hat{0}\hat{0}\hat{2}} = e^{-2\tau} \frac{2}{\delta_{\mu \hat{A}}} \hat{R}_{\hat{1}\hat{0}\hat{0}\hat{1}}, \tag{23} \]

\[ R_{\hat{1}\hat{0}\hat{1}\hat{1}} = e^{-2\tau} \frac{2}{\delta_{\mu \hat{A}}} \hat{R}_{\hat{1}\hat{0}\hat{1}\hat{1}}. \tag{24} \]

The condition that \( \tau_{\hat{A}} \) is constant (and hence \( \tau_{\mu} \tau^{\mu} \) and \( T_{\hat{0}\hat{1}} \) are constant) implies that the boost weight zero components of the Riemann tensor are constant; the absence of positive boost weight components then gives CSI\(_0\).

The covariant derivative of the null frame has the form

\[ \ell_{K:E} = -\tau_{\hat{0}} m^k_E \ell_K + \gamma_{\hat{0} \hat{0}} m^\hat{E}_K \ell_E + \gamma_{\hat{0} \hat{1}} \ell_K m^\hat{E}_E + \cdots, \tag{25} \]

\[ n_{K:E} = -\tau_{\hat{0}} m^k_E n_K - \gamma_{\hat{0} \hat{0}} n_{\hat{K}} m^\hat{E}_E + \cdots. \tag{26} \]

\(^3\) Hatted indices are the preferred null-frame indices.
As expected, all components in (30)–(33) are constant (as a result of requiring
\[ m_{j,k,E} = -\gamma_{j}^b m_k^b k_{j,E} + \gamma_{j}^b m_\ell^\ell k_{j,E} - \gamma_{j}^b n_{K,E} + \cdots, \]  
(27)
\[ m_{a,K,E} = \tau_{a} \ell_{K,E} + \tau_{a} n_{K,E} + \tau_{a} \delta_{a} \delta_{a} m_j^j + \gamma_{a}^j m_j^j E + \gamma_{a}^j m_j^j E, \]  
(28)
where \('\cdots\') denote terms of lower-order boost weight with respect to previous terms in an
expression and the unspecified \(\gamma^{s}\) correspond to VSI Ricci-rotation coefficients [15] (for
example, \(\gamma_{0}^1 = e^{-\tau L_{1,1}}\), except for \(\gamma^{j}\) which are the rotation coefficients associated with the
homogeneous space. Equations (25)–(28) have leading order boost weight terms of +1, \(-1\), 0
and 0, respectively, which is the same boost weight as the corresponding null-frame vector
before taking a covariant derivative. It follows that the covariant derivative of the \{-\} quantities
appearing in (15) will produce terms of equal or lesser boost weight. Furthermore, in [15] it
was proven that the negative boost weight curvature components appearing in (15) will remain
negative upon covariant differentiation. Therefore, if the boost weight zero components of the
Riemann tensor and its derivatives are constant then any number of covariant derivatives of the
Riemann tensor will be of boost order zero, which implies that the warped product will be
CSI.

Supposing that the boost weight zero components of \(\nabla^{(n)} R\) are constant, then for
\(\nabla^{(n+1)} R\) the boost weight zero components will have contributions from \(\tau_{a},\) which is constant and the
non-constant \(\gamma^{s}\) arising from the VSI spacetime; however, these boost weight zero VSI \(\gamma^{s}\) do not occur as a result of theorem 5.1. Therefore, we have that if \(\tau_{b}\) is constant, then the warped
product is CSI.

For reference, we include the covariant derivative of the Riemann tensor. By using the
Ricci identity, we have
\[ \tau_{,a}^1 R_{b,c,dE} + 2 T_{b[a} \tau_{c,d]} = T_{b,c,d} - T_{b,c,d}, \]  
(29)
which can be used to express the non-vanishing boost weight zero components of the covariant
derivative of the Riemann tensor as
\[ R_{b,c,d} = T_{b,c,d} - T_{b,c,d} = R_{b,c,d}, \]  
(30)
\[ R_{b,c,d,k} = [T_{b,c,d} - T_{b,c,d}^\delta_{b}^\delta_{c}^\delta_{d}], \]  
(31)
\[ R_{b,c,d,k} = -T_{b,c,d}, \]  
(32)
\[ R_{b,c,d,k} = -T_{b,c,d}^\delta_{b}^\delta_{c}^\delta_{d}. \]  
(33)
As expected, all components in (30)–(33) are constant (as a result of requiring \(\tau_{b}\) to be constant).

We shall now show the converse; that is, if the Ricci invariants are constant then \(\tau_{b}\) is constant. In local coordinates, the Ricci tensor and the Ricci scalar are
\[ R_{\mu}^\rho = e^{-2\tau} \frac{1}{2} R_{\mu}^\rho - [T + (k - 1)\tau_{m} \tau_{m}^m] \delta_{\mu}^\rho, \]  
(34)
\[ R_{m}^b = R_{m}^p - k T_{m}^p, \]  
(35)
\[ R = -k [2T + (k - 1)\tau_{m} \tau_{m}^m]. \]  
(36)
where in (36) we have used that \(\frac{2}{2} = 0\) and set \(T = \text{Tr}(T_{m}^p)\). Choosing new coordinates
\(\tilde{x}^{1} = \tau(x^{m}), \tilde{x}^{2} = x^{2}, \ldots\) so that \(\tau_{m} \tau_{m}^m = \tilde{g}^{11}\), then
\[ \tilde{T}_{mn} = -\tilde{\Gamma}_{11,m} + \tilde{g}_{m}^1 \tilde{g}_{n}^1, \]  
(37)
Now, with normal coordinates along \( \check{x}^1 \) we have \( \check{g}^{11} = 1 \). Therefore, from (36), since \( R \) is constant we have that \( \check{T} \), and hence \( \check{T} \), is constant. In this coordinate system, the constraint \( \check{T} \) constant implies that \( \check{g}^{1m} \check{g}_{m1} \) is constant. Constant Ricci scalar implies that \( T \) and \( \tau_m \tau^m \) are constant. Further conditions are then provided by the constancy of higher-degree Ricci invariants.

From (34) and (35), the Ricci tensor of a warped product is \( R_M^N = R_m^n \oplus R^v_\mu v \); thus, a degree \( d \) Ricci invariant always has the form

\[
R_{N_1}^{N_2} R_{N_3}^{N_4} \cdots R_{N_d}^{N_1} = R_m^{n_2} R_m^{n_1} \cdots R_m^{n_d} + R_v^{y_2} R_v^{y_1} \cdots R_v^{y_1}.
\]

The second term of (38) is constant since \( T \) and \( \tau_m \tau^m \) are constant; hence, invariants constructed from (35) must be constant. Assuming that the determinant of \( R_m^p \) or \( R_\mu^m \) is nonzero, then there exists a frame in which \( R_m^n \) and \( R^v_\mu \) can simultaneously be expressed in block diagonal form with constant eigenvalues [18]. Therefore, (35) implies that in this frame \( T_m^p \) must also have constant components and thus constant invariants. In this case, the constancy of the Ricci invariants completely determines the conditions on \( T_m^p \) and specifies a frame in which the components of each term in (35) are constant. The frame in which \( \check{R}_\mu^m \) and \( T_m^p \) are constant is determined by the left-invariant 1-forms. As before, the conditions that \( T_{\check{g}b} \) and \( \tau_m \tau^m \) are constant imply \( \tau_{\check{d}} \) is constant, and consequently constant Ricci invariants imply CSI.

Many examples of CSI spacetimes can be found among the cases where the homogeneous space is a solvable Lie group\(^4\).

**Theorem 3.1.** Assume that \( \mathcal{M}_H \) is a solvable Lie group equipped with a left-invariant metric. Assume also that \( \mathcal{M}_H \) is connected and simply connected (i.e., \( H^1(\mathcal{M}_H) = 0 \)). Then there exists a non-constant \( \tau \) such that the warped product of \( \mathcal{M}_H \) and a VSI, as constructed above, is a CSI.

**Proof.** We identify \( \mathcal{M}_H = G \), where \( G \) is a solvable Lie group, with the Lie algebra \( \mathfrak{g} = T_e G \). Since \( \mathfrak{g} \) is solvable, the set \([\mathfrak{g}, \mathfrak{g}]\) will be a proper vector subspace of \( \mathfrak{g} \). Thus, for a solvable Lie algebra, there exists a nonzero \( X \in \mathfrak{g} \cap [\mathfrak{g}, \mathfrak{g}]^\perp \), where \([\mathfrak{g}, \mathfrak{g}]^\perp \) is the complement of the derived algebra \([\mathfrak{g}, \mathfrak{g}]\). This further implies that there is a left-invariant 1-form \( \omega \) such that \( d\omega = 0 \). Since \( H^1(\mathcal{M}_H) = 0 \), there exists a non-constant function \( \phi \) such that \( \omega = d\phi \). We can now choose \( \tau = p\phi \) for a constant \( p \).

In two dimensions, flat space is the only VSI metric, and in appendix A we list all of the three-dimensional VSI metrics. Examples of CSI warped products where the homogeneous space is either \( \mathbb{R}^n \) or \( \mathbb{H}^n \) yield a linear function for \( \tau \) (since both of these can be considered as solvmanifolds); however, for \( \mathbb{R}^n \), since the coordinates are cyclic, we find that \( \tau \) is a constant. Another example, in five dimensions, is where the homogeneous space is a Bianchi type VIII Lie group (\( \cong SL(2, \mathbb{R}) \)) warped with a two-dimensional VSI; with coordinates \( u, v, x, y, \) and \( z \), the metric is

\[
ds^2 = e^{2\tau} [2 du(dv + H du)] + \frac{a^2}{y^2} (\cos z \, dx + \sin z \, dy)^2 \\
+ \frac{b^2}{y^2} (\cos z \, dy - \sin z \, dx)^2 + c^2 \left( \frac{dz}{y} + \frac{dx}{y} \right)^2.
\]  

\(^4\) Recall that a Lie group can be equipped with a left-invariant metric (i.e., invariant under the left action) in the standard way, see e.g. [20].
In this case, $H(u, v)$ must be linear in $v$ to give CSI, and it turns out that for the warped product to be CSI we have to consider two separate cases.

1. The maximal isometry group of the homogeneous space is three dimensional ($a \neq b$): $\tau$ is a constant.
2. The maximal isometry group of the homogeneous space is four dimensional ($a = b$): $\tau$ can be non-constant, $\tau = \alpha \ln y + \beta z$.

The reason why we have to distinguish these two cases is that in the first case the isometry group is the semi-simple group, $\text{SL}(2, \mathbb{R})$. However, in the second case, the isometry group allows for a transitive group which is solvable (Bianchi type III).

4. Kundt CSI metrics

Let us next consider the higher-dimensional Kundt metrics (10) in the form

$$ds^2 = 2 \, du(du + H\, du + W^j \, m^j) + \delta_{ij} \, m^i \, m^j.$$  \hspace{1cm} (40)

It is convenient to introduce the null frame

$$\ell = du,$$ \hspace{1cm} (41)

$$n = dv + H\, du + W^j \, m^j,$$ \hspace{1cm} (42)

$$m^i = \theta^i_j (u; x^k) \, dx^l,$$ \hspace{1cm} (43)

$$(\eta_{AB}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \delta_{ij} \end{bmatrix},$$ \hspace{1cm} (44)

such that

$$dS^2_H = \delta_{ij} \, m^i \, m^j = \bar{g}_{ij} \, dx^i \, dx^j, \quad \bar{g}_{ij,u} = 0.$$ \hspace{1cm} (45)

There is a class of coordinate transformations that preserve the form of the metric (40). In particular, we can define new variables,

$$(v', u', x') = (v, u, f^j (u; x^k)).$$ \hspace{1cm} (46)

For our purposes, we can always use this coordinate transformation to simplify the spatial metric $\bar{g}_{ij}$.

**Theorem 4.1.** Consider the metric (10) and assume that there exists a null frame $\{\ell_A, n_A, m^i_A\}$ such that all scalars $R_{ijkl} = R_{ABDC} m^A_i m^B_j m^C_k m^D_l$ and $R_{i...j...k} = R_{A...C...D...} m^A_i m^B_j m^C_k m^D_l$ are constants. Then there exists (locally) a coordinate transformation $(v', u', x') = (v, u, f^j (u; x^k))$ such that

$$\bar{g}_{ij} = \tilde{g}_{ij} \, \frac{\partial f^j}{\partial x^i} \, \frac{\partial f^i}{\partial x^j}, \quad \bar{g}_{ij,u} = 0.$$ \hspace{1cm} (47)

Moreover, $dS^2_H = \tilde{g}_{ij} \, dx^i \, dx^j$ is a locally homogeneous space.

**Proof.** By calculating the curvature tensor of the metric (10), and its covariant derivatives, we note that

$$R_{ijkl} = \tilde{R}_{ijkl}, \quad R_{i...j...k} = \tilde{R}_{i...j...k},$$ \hspace{1cm} (48)
where the tilde refers to curvature tensors with respect to the spatial metric \( \tilde{g}_{ij}(u, x^k) \). Since this metric is Riemannian with all its components constant, we can use the results of [1] which state that the metrics \( \tilde{g}_{ij}(u, x^k) \), for different \( u \), are equivalent up to isometries. Thus, given \( u_0 \), there exists for every \( u \) sufficiently close to \( u_0 \) an isometry \( \psi_u(x^k) \) such that 
\[
(\psi_u)_* \tilde{g}_u = \tilde{g}_{u_0}.
\]

Since we assume that \( \tilde{g} \) is sufficiently smooth in a neighbourhood of \( u_0 \), we can find a sufficiently smooth \( \psi_u \) in \( u \). The map \( \psi_u(x^k) \) provides us with the functions \( f_i \) in the theorem by composition with the coordinate charts. Hence, we have \( \tilde{g}' = \tilde{g}_{u_0} |_{u_0} \). Moreover, \( \tilde{g}' \) is a locally homogeneous space [1].

Henceforth, we shall assume that we consider solutions in the set \( \text{CSI}_F \cap \text{CSI}_K \).

Consequently, from theorem 4.1, there is no loss of generality in assuming that the metric (10) has \( \tilde{g}_{ij,u} = 0 \). The remaining coordinate freedom preserving the Kundt form is then
\[
(1) \quad (v', u', x^i) = (v, u, f^i(x^k)) \quad \text{and} \quad J^i = \frac{\delta^i}{\delta x^i},
\]
\[
H' = H, \quad W_i' = W_i(J^{-1})^j_i, \quad \tilde{g}_{ij}' = \tilde{g}_{kl}(J^{-1})^k_i(J^{-1})^l_j.
\]
\[
(2) \quad (v', u', x^i) = (v + h(u, x^k), u, x^i).
\]
\[
H' = H - h_u, \quad W_i' = W_i - h_j, \quad \tilde{g}_{ij}' = \tilde{g}_{ij}.
\]
\[
(3) \quad (v, u, x^i) = (v/g(u), u, g(u), x^i).
\]
\[
H' = \frac{1}{g_{,uv}} \left( H + v \frac{g_{,uv}}{g_{,u}} \right), \quad W_i' = \frac{1}{g_{,u}} W_i, \quad \tilde{g}_{ij}' = \tilde{g}_{ij}.
\]

4.1. CSI₀ spacetimes

The linearly independent components of the Riemann tensor with boost weights 1 and 0 are
\[
R_{0i0i} = -\frac{1}{2} W_{i,vv},
\]
\[
R_{0i0i} = -H_{vv} + \frac{1}{4}(W_{i,v})(W^{i,v}),
\]
\[
R_{0i0j} = W_{[i} W_{j,v] + W_{[i,j],v},
\]
\[
R_{0i0j} = \frac{1}{2} \left[ -W_{j} W_{i,v} + W_{i,v} - \frac{1}{2}(W_{i,v})(W_{j,v}) \right],
\]
\[
R_{ij00} = \tilde{R}_{ij00}.
\]

Hence, the spacetime is a CSI₀ spacetime if there exists a frame \( \{ \ell, n, m^i \} \), a constant \( \sigma \), an anti-symmetric matrix \( a_{ij} \) and a symmetric matrix \( s_{ij} \) such that
\[
W_{i,vv} = 0,
\]
\[
H_{vv} - \frac{1}{4}(W_{i,v})(W^{i,v}) = \sigma,
\]
\[
W_{[i,j],v} = a_{ij},
\]
\[
W_{[i,j],v} - \frac{1}{2}(W_{i,v})(W_{j,v}) = s_{ij},
\]
and the components \( \tilde{R}_{ij00} \) are all constants (i.e., \( dS^2_H \) is curvature homogeneous).

We note that the first equation implies that
\[
W_i(v, u, x^k) = v W^{(1)}_i(u, x^k) + W^{(0)}_i(u, x^k),
\]
while the second implies

$$H(v, u, x^k) = \frac{v^2}{8} \left[ 4\sigma + (W^{(1)}_i(W^{(1)}_j)) \right] + vH^{(1)}(u, x^k) + H^{(0)}(u, x^k). \quad (59)$$

If $dS^2_H = \delta_{ij} \hat{m}^i \hat{m}^j$ is a (locally) homogeneous Riemannian space, then there exists a frame where $\hat{R}_{ij\hat{k}\hat{l}}$ are all constants. In general, curvature homogeneous does not imply homogeneous; however, since we are mostly interested in the CSI case, in light of theorem 4.1, we will henceforth take $dS^2_H$ as a (locally) homogeneous metric.

### 4.2. CSI$_1$ spacetimes

For a CSI$_0$ spacetime, we have

$$R_{i\hat{j}\hat{k}\hat{l}} = -\frac{1}{2}[\sigma W_{i\hat{u}} - \frac{1}{2}(s_{\hat{i}\hat{j}} + a_{\hat{i}\hat{j}})W_{j\hat{k}}], \quad (60)$$

$$R_{i\hat{j}\hat{k}\hat{l}} = -\frac{1}{2}[W_{i\hat{k}} \hat{R}_{i\hat{j}\hat{k}\hat{l}} - W_{i\hat{j}} a_{\hat{j}\hat{k}} + (s_{\hat{i}\hat{j}} + a_{\hat{i}\hat{j}})W_{\hat{j}\hat{i}\hat{k}}]. \quad (61)$$

For the spacetime to be CSI$_1$, it is sufficient to require that the above components are constants, i.e.

$$\sigma W_{i\hat{u}} - \frac{1}{2}(s_{\hat{i}\hat{j}} + a_{\hat{i}\hat{j}})W_{j\hat{k}} = \alpha_i, \quad (62)$$

$$(s_{\hat{i}\hat{j}} + a_{\hat{i}\hat{j}})_{\hat{k}} - (s_{\hat{i}\hat{k}} + a_{\hat{i}\hat{k}})_{\hat{j}} = \beta_{\hat{i}\hat{j}\hat{k}}, \quad (63)$$

where the Ricci identity has been used to rewrite the latter condition.

### 5. Examples of CSI spacetimes of Kundt form

Various classes of CSI spacetimes that arise as members of $\text{CSI}_F \cap \text{CSI}_K$ can now be found. The solutions come in classes according to the properties of $W^{(1)}_i(u, x^k)$. In the following, $\mathcal{M}_H$ is a locally homogeneous Riemannian space.

#### 5.1. $W^{(1)}_i(u, x^k) = 0$

Assuming $W^{(1)}_i(u, x^k) = 0$ immediately makes the metric (40) a CSI space.

A special subcase of this class which is worth mentioning is the Brinkmann metrics for which $H$ and $W_i$ are all independent of $v$; thus, $\sigma = s_{ij} = a_{ij} = 0$. In this case, the expressions for the curvature tensors simplify drastically. In particular, if $F_{\hat{j}\hat{i}} = 2W_{\hat{i}\hat{j}}$, the Ricci tensor is

$$R_{\hat{i}\hat{j}} = \Box H - \frac{1}{4} F^2, \quad R_{\hat{i}\hat{l}} = \nabla^j F_{\hat{j}\hat{i}}, \quad R_{\hat{i}\hat{j}} = \hat{R}_{\hat{i}\hat{j}}. \quad (64)$$

#### 5.2. $W^{(1)}_i(u, x^k) = \text{constant}$

This case is CSI if and only if the simpler metric, $\tilde{\mathcal{M}}$:

$$\tilde{d}s^2 = 2 du \left( dv + \frac{v^2}{2} \tilde{\sigma} du + vW^{(1)}_i \hat{m}^i + \delta_{ij} \hat{m}^i \hat{m}^j \right), \quad (65)$$

is a homogeneous space. The curvature invariants of the Kundt metrics in this class will have the same invariants as $\mathcal{M}$, i.e. $\mathcal{I} = \mathcal{I}(\mathcal{M})$. 
The homogeneous space $\mathcal{M}_H$ can be considered as a quotient space $\mathcal{M}_H = G/H$, where $G$ is the isometry group and $H$ is the isotropy group. The corresponding Lie algebra decomposition is $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and we let this correspond to the algebra of left-invariant vectors. We note that if there exists a subalgebra $\mathfrak{k} \subset \mathfrak{m}$ such that $[\mathfrak{k}, \mathfrak{k}] = 0$, where $\mathfrak{k}$ is the complement of $\mathfrak{t}$ in $\mathfrak{g}$, then we can choose an $\text{Ad}(H)$-invariant $W$:

$$W \in \mathfrak{t}.$$ 

By letting $W_i^{(1)} m^i$ be its corresponding left-invariant 1-form, the associated Kundt spacetime will be a CSI.

Note that some homogeneous spaces may be represented by inequivalent Lie algebra decompositions. For example, hyperbolic space, $\mathbb{H}^n$, can be considered both as the quotient $SO(1, n)/SO(n)$ and as a solvable Lie group $G$ [7]. In the latter case, the decomposition gives $\mathfrak{g} = \mathfrak{m}$ and $\mathfrak{h} = 0$; hence, $[\mathfrak{h}, \mathfrak{m}] = 0$.

5.3. $W_i^{(1)}(u, x^k) = \phi_i = \text{non-constant}$

5.3.1. $I = \mathcal{I} (dS \times \mathcal{M}_H), \sigma > 0$:

$$ds^2 = \cos^2(\sqrt{\sigma} x)[2 du(dv + \bar{H} du + \bar{W}_i m^i)] + dx^2 + \frac{1}{\sigma} \sin^2(\sqrt{\sigma} x) dS^2 + dS_H^2,$$

where

$$W_i^{(1)} = 2\sqrt{\sigma} \tan(\sqrt{\sigma} x), \quad \bar{W}_i = \bar{W}_i^{(0)}(u, x^j), \quad \bar{H} = \frac{v^2}{2} \sigma + v \bar{H}^{(1)}(u, x^j) + \bar{H}^{(0)}(u, x^j).$$

The Riemann tensor is of the type

$$R = (R)_0 + (R)_{-1} + (R)_{-2}.$$ 

Special cases.

1. If $\bar{W}_i^{(0)} = \frac{1}{\sigma} \bar{H}^{(1)},$ then $(R)_{-1} = 0.$
2. If $\bar{W}_i^{(0)} = \bar{H}^{(1)} = \bar{H}^{(0)} = 0,$ then $(R)_{-1} = (R)_{-2} = 0$ and $dS \times \mathcal{M}_H.$

5.3.2. $I = \mathcal{I} (\text{AdS} \times \mathcal{M}_H), \sigma < 0$:

$$ds^2 = \cosh^2(\sqrt{\sigma} x)[2 du(dv + \bar{H} du + \bar{W}_i m^i)] + dx^2 + \frac{1}{|\sigma|} \sinh^2(\sqrt{\sigma} x) dS^2 + dS_H^2,$$

where

$$W_i^{(1)} = 2\sqrt{|\sigma|} \tanh(\sqrt{|\sigma|} x), \quad \bar{W}_i = \bar{W}_i^{(0)}(u, x^j), \quad \bar{H} = -\frac{v^2}{2} |\sigma| + v \bar{H}^{(1)}(u, x^j) + \bar{H}^{(0)}(u, x^j).$$

The Riemann tensor is of the type

$$R = (R)_0 + (R)_{-1} + (R)_{-2}.$$ 

Special cases.

1. If $\bar{W}_i^{(0)} = -\frac{1}{\sqrt{|\sigma|}} \bar{H}^{(1)},$ then $(R)_{-1} = 0.$
2. If $\bar{W}_i^{(0)} = \bar{H}^{(1)} = \bar{H}^{(0)} = 0,$ then $(R)_{-1} = (R)_{-2} = 0$ and $\text{AdS} \times \mathcal{M}_H.$
Let us present another example:

\[
\text{d}s^2 = \sinh^2(\sqrt{|\sigma|}x)[2 \text{d}u(\text{d}v + \tilde{H} \text{d}u + \tilde{W}_a \text{d}x^a)] + \text{d}x^2 + \frac{1}{|\sigma|} \cosh^2(\sqrt{|\sigma|}x) \text{d}S^2_{\mathcal{I}} + \text{d}S^2_{\mathcal{H}},
\]

where

\[
\tilde{W}_i = \tilde{W}_i^{(0)}(u, x^j), \quad \tilde{H} = \frac{v^2}{2} |\sigma| + v \tilde{H}^{(1)}(u, x^j) + \tilde{H}^{(0)}(u, x^j).
\]

The Riemann tensor is of the type

\[
R = (R)_{0} + (R)_{-1} + (R)_{-2}.
\]

**Special cases.**

1. If \(\tilde{W}_j^{(0)} = \frac{1}{\tilde{m}} \tilde{H}^{(1)}_j\), then \((R)_{-1} = 0\).
2. If \(\tilde{W}_j^{(0)} = \tilde{H}^{(1)} = \tilde{H}^{(0)} = 0\), then \((R)_{-1} = (R)_{-2} = 0\).

There are more CSI spacetimes with \(I = I((A)\text{d}S \times \mathcal{M}_H)\). They are constructed similarly to those above. They are all related to the fact that de Sitter space or anti-de Sitter space can be written in many ways as fibred spaces over the spheres, \(S^n\), or the hyperbolic space, \(\mathbb{H}^n\), respectively (see appendix B for the 4D case).

### 5.4. \(W_a^{(1)}(u, x^k) = \text{non-constant}, W_a^{(1)}(u, x^k) = \text{constant}\)

This is a generalization of the warped CSI spacetimes considered earlier. The metric can be written as

\[
\text{d}s^2 = e^{-2\phi(x^\sigma)}[2 \text{d}u(\text{d}v + \tilde{H} \text{d}u + \tilde{W}_a \text{d}x^a + \tilde{W}_a \omega^a) + \delta_{ab} \text{d}x^a \text{d}x^b] + \delta_{ab} \omega^a \omega^b,
\]

where

\[
\begin{align*}
\tilde{W}_a &= v \tilde{W}_a^{(1)}(u, x^a) + \tilde{W}_a^{(0)}(u, x^a, x^a), \\
\tilde{W}_a &= \tilde{W}_a^{(0)}(u, x^a, x^a), \\
\phi &= \phi(x^a), \quad \phi_{,a} = \text{constant}, \\
\tilde{H} &= \frac{v^2}{8} (\tilde{W}_a^{(1)} \tilde{W}_a^{(1)} + v \tilde{H}^{(1)}(u, x^a, x^a) + \tilde{H}^{(0)}(u, x^a, x^a)).
\end{align*}
\]

Furthermore, we require that \(\delta_{ab} \omega^a \omega^b = \delta_{ab}(x^\sigma) \text{d}x^a \text{d}x^b\) is a Riemannian homogeneous space, \(\mathcal{M}_H\). The allowed forms of the function \(\phi\) depend on the homogeneous space \(\mathcal{M}_H\). Topologically, this space is a fibred manifold with base manifold \(\mathcal{M}_H\) and we will require that the fibre is a VSI space; in particular, this means that \(\tilde{W}_a\) satisfy the VSI equations (54)–(57) with \(\sigma = a_{ij} = s_{ij} = 0\).

**Special cases.**

1. If \(\tilde{W}_j^{(0)} = \Psi_i, \tilde{W}_j^{(0)} = \tilde{H}^{(1)} = 0\), then \((R)_{-1} = 0\).
2. If \(\tilde{W}_j^{(0)} = \tilde{H}^{(1)} = \tilde{H}^{(0)} = 0\), then \((R)_{-1} = (R)_{-2} = 0\).

The connection \(\nabla\) for the above metric can be decomposed as

\[
\tilde{\nabla} = \tilde{\nabla} - \tau,
\]

where \(\tau\) is a (2, 1) tensor (or operator acting in the obvious way) with only non-positive boost weight components, while \(\tilde{\nabla}\) has the following property. There exists a null frame such that the connection coefficients of \(\tilde{\nabla}\) have the following properties.
(1) Positive boost weight connection coefficients are all zero.
(2) Zero boost weight connection coefficients are either all constants or connection coefficients of $\mathcal{M}_H$.

Henceforth, we will assume that this null frame is chosen. Moreover, $\tau$ and $\tilde{\nabla}$ can be chosen such that
\begin{align}
\tau e_\alpha X &= \tau X e_\alpha = 0, \\
\tilde{\nabla} g &= 0, \\
g(\tau X, Z) + g(Y, \tau X Z) = 0, & \forall X, Y, Z \in TM.
\end{align}
(78)

Note that the torsion of $\tilde{\nabla}$ is $S_{XY} = \tau Y X - \tau X Y$.

Let $R$ and $\tilde{R}$ be the curvature tensors to the connections $\nabla$ and $\tilde{\nabla}$, respectively. Then
$$R = \hat{R} + \tilde{R},$$
(79)
where $(\hat{R})_b = 0$ for $b \geq 0$ while $(\tilde{R})_b = 0$ for $b > 0$. Moreover, all boost weight 0 components are constants.$^5$ We also have

**Theorem 5.1.** For all $k$, $\nabla^{(k)} R$ has maximal boost order 0 where all boost weight 0 components are constants.

**Proof.** We will prove the theorem by induction. Let us in the following denote the projection of a tensor $T$ onto the vector space spanned by the components of boost weight $b$ by $(T)_b$. The key observation is that $(R)_0 = (\tilde{R})_0$ is of constant curvature over the VSI fibres. As a result, $\tau (R)_0 = 0$. Thus, the theorem is true for $\nabla (R)_0$. Moreover, by applying the Bianchi identity for $\nabla R$, we can show that it is true for $\nabla (R)_{-1}$. Hence, the theorem is true for $k = 1$.

Assume the theorem is true for $k$. Then consider $\nabla \nabla^{(k)} R$. Immediately, we have that it is true for $\nabla (\nabla^{(k)} R)_0$. Moreover, the curvature tensor is of constant curvature over the VSI fibres, so $\tau (\nabla^{(k)} R)_0 = 0$; hence, the theorem is true for $\nabla (\nabla^{(k)} R)_0$.

For $\nabla (\nabla^{(k)} R)_{-1}$, the critical part is the one that raises the boost weight by 1. However, by using the identity
\begin{equation}
[\nabla_A, \nabla_B]T_{C_1...C_p} = \sum_{i=1}^p R^E_{CAB} T_{C_1...E...C_p}
\end{equation}
(80)
recursively, and also the Bianchi identity, we see that it is also true for $\nabla (\nabla^{(k)} R)_{-1}$. Hence, by induction, the theorem follows. \hfill \Box

These results consequently imply that the ‘fibred VSI’ spacetime given by equation (72) is a CSI spacetime. A crucial part of the proof is that $(R)_0$ is of constant curvature over the VSI fibres, which implies that $\tau (\nabla^{(k)} R)_0 = 0$.

**Theorem 5.2.** Consider the metric (72). If for all $k$ $\nabla^{(k)} R$ has maximal boost order 0, where all boost weight 0 components are constants, then there exists a homogenous space $(\tilde{\mathcal{M}}, \tilde{g})$ with the same curvature invariants as the metric (72).

**Proof.** The proof essentially follows along the same lines as above by noting that the functions $\tilde{W}_i$ and $\tilde{H}$ do not contribute to the curvature invariants. The homogeneous space can be obtained by setting all of these to zero. \hfill \Box

$^5$ This can be seen by explicit calculation of the curvature tensors.
6. Discussion

We have obtained a number of general results. We have studied warped product CSI spacetimes (theorem 3.1). We have presented the canonical form for the Kundt CSI metric (theorem 4.1) and studied higher-dimensional Kundt spacetimes in some detail. In sections 3 and 5, we have constructed higher-dimensional examples of CSI spacetimes that arise as warped products and that belong to the Kundt class. In the appendices, we shall present all of the three-dimensional VSI metrics, explicitly construct the metrics in the four-dimensional CSI spacetimes, and we shall establish the canonical higher-dimensional VSI metric.

Motivated by our results and examples we propose the following conjectures.

**Conjecture 6.1** (CSI conjecture). A spacetime is CSI iff there exists a null frame in which the Riemann tensor and its derivatives can be brought into one of the following forms: either

1. the Riemann tensor and its derivatives are constant, in which case we have a locally homogeneous space, or
2. the Riemann tensor and its derivatives are of boost order zero with constant boost weight zero components at each order. This implies that Riemann tensor is of type II or less.

Whereas the proof of the ‘if’ direction is trivial, the ‘only if’ part is significantly more difficult and would require considering second-order curvature invariants. We also point out that if we relax CSI to CSI$_I$ then in (1) we must now include all 1-curvature homogeneous spacetimes as well, since these naturally imply CSI$_I$ and are not necessarily homogeneous spaces.

Assuming that the above conjecture is correct and that there exists such a preferred null frame, then

**Conjecture 6.2** (CSI$_I$ conjecture). If a spacetime is CSI, then the spacetime is either locally homogeneous or belongs to the higher-dimensional Kundt CSI class.

Finally, we have that

**Conjecture 6.3** (CSI$_K$ conjecture). If a spacetime is CSI, then it can be constructed from locally homogeneous spaces and VSI spacetimes.

This construction can be done by means of fibering, warping and tensor sums.

From the above results and these conjectures, it is plausible that for CSI spacetimes that are not locally homogeneous, the Weyl type is II, III, N or O, and that all boost weight zero terms are constant.

We intend to study the general validity of these conjectures in future work. The relationship to curvature homogeneous spacetimes is addressed in [13]. Support for these conjectures in the 4D case is discussed in the next section.

7. Four-dimensional CSI

Let us now consider the 4D CSI spacetimes in more detail. We assume that a CSI spacetime is either of Petrov (P)-type I or Plebanski–Petrov (PP)-type I, or of P-type II and PP-type II (branch A or B of figure 1, respectively). In branch A, we explicitly assume that the spacetime is not of P-type II and PP-type II, otherwise we would be in branch B. It is plausible that in branch A the spacetimes must be of P-type I and PP-type I (but we have not established this here). Since the spacetime is not of P-type II and PP-type II, it is necessarily completely backsolvable (CB) [11]. Since all of the zeroth-order scalar curvature invariants are constant, there exists a frame in which the components of the Riemann tensor are all constant, and
the spacetime is curvature homogeneous CH₀ (i.e., CSI₀ ≡ CH₀). If the spacetime is also CH₁, then it is necessarily locally homogeneous H [13]. Therefore, if it is not locally homogeneous, it cannot be CH₁. Finally, by considering differential scalar invariants, we can obtain information on the spin coefficients in the CSI spacetime. Although a comprehensive analysis of the differential invariants is necessary, a preliminary investigation indicates that τ, ρ, and σ must all be constant and are likely zero. Therefore, in branch A, there are severe constraints on those spacetimes that are not locally homogeneous, and it is plausible that there are no such spacetimes.

In branch B of figure 1, we have that the spacetime is necessarily of P-type II and PP-type II. It then follows from theorem 7.1 that all boost weight zero terms are necessarily constant (i.e., the spacetime is CSIₐ₀). The spacetime is then either CB, in which case all of the results in branch A apply, or they are NCB and a number of further conditions apply (these conditions are very severe [11]). In either case, there are a number of different classes characterized by their P-type and PP-type (in each case at least of type II), and in each class there are a number of further restrictions. By investigating the differential scalar invariants we then find conditions on the spin coefficients. By considering each class separately, we can then establish that τ = ρ = σ = 0, and that the spacetime is necessarily CSIₖ. All of these spacetimes are constructed in appendix B.

The results are summarized in figure 1. In branch A of figure 1, we have established that if the CSI spacetime is not locally homogeneous, then necessarily it is of P-type I or PP-type I (and does not belong to CSIₖ, CSI₀ or CSI₁), it belongs to CSI₀ ≡ CH₀, but not CH₁ and there are a number of further constraints arising from the non-CB conditions and the differential constraints (i.e., constraints on the spin coefficients). This exceptional set is very sparse, and
Proof. We have explicitly constructed the 4D CSI_F, CSI_K and CSI_R, and established the relationship between these CSI subsets themselves and with CSI_H, thereby lending support to the conjectures in the previous section. The metrics of these CSI spacetimes are described in appendix B. From this calculation of all of the members of the set CSI_F \ CSI_K, we see that they all belong to CSI_K. That is, we have established the uniqueness of CSI_K within this particular context. In particular, we have found all CSI in branch B of figure 1.

Finally, let us consider Petrov (P)-type II and Plebanski–Petrov (PP)-type II Lorentzian CSI spacetimes. We shall show that if all zeroth-order curvature invariants are constant then the boost weight zero curvature scalars, \( \Psi_5, \Phi_{11} \) and \( \Phi_{02} \), are constant. For these algebraically special spacetimes, the converse follows straightforwardly.

**Theorem 7.1.** If all zeroth-order curvature invariants are constant in Petrov-type II and Plebanski–Petrov-type II Lorentzian CSI spacetimes, then the boost weight zero curvature scalars, \( \Psi_2, \Phi_{11} \) and \( \Phi_{02} \), are constant.

**Proof.** We make use of the following curvature invariants:

\[
\begin{align*}
  w_1 &= \Psi_{ABCD} \Psi^{ABCD}, & r_1 &= \Phi_{ABAB} \Phi^{ABAB}, & r_2 &= \Phi_{ABAB} \Phi^B_C \Phi^C_{ABA},
\end{align*}
\]

It is worth noting that in general there is an additional independent degree 4 Ricci invariant \( r_3 \); however, in PP-types II or less there always exists a syzygy for \( r_3 \) in terms of \( r_1 \) and \( r_2 \) (or else it vanishes). This is analogous to the well-known syzygy for the Weyl invariants, \( f^3 = 27 f^2 \), signifying an algebraically special spacetime.

Assuming first that the Weyl and Ricci canonical frames are aligned, then the non-vanishing curvature scalars are \( \Psi_2, \Psi_4 = 1, \Phi_{11}, \Phi_{20} = \Phi_{02}, \Phi_{22} = 1 \) and we have \( w_1 = 6 \Psi_2^2, r_1 = 2 \Phi_{02}^2 + 4 \Phi_{11}^2, r_2 = -6 \Phi_{02} \Phi_{11} \). Clearly \( w_1 \) constant implies \( \Psi_2 \) constant. If \( \Phi_{02}(2 \Phi_{11} - \Phi_{02})(2 \Phi_{11} + \Phi_{02}) \neq 0 \), then \( \Phi_{02} \) and \( \Phi_{11} \) can be expressed in terms of \( r_1 \) and \( r_2 \); therefore, constant \( r_1 \) and \( r_2 \) give constant \( \Phi_{11} \) and \( \Phi_{02} \). If \( \Phi_{02} = 0 \) or \( \Phi_{02} = \pm 2 \Phi_{11} \), then \( r_1 \) can be used to give the same result. In this aligned case, we find that constant \( R, w_1, r_1 \) and \( r_2 \) imply that all curvature scalars are constant (including the boost weight 0 scalars); therefore, we have a curvature homogeneous spacetime.

In the non-aligned case, we refer to [11] where it was shown that for this PP-type, complete back-solving (CB) of the Carminati–Zakhary (CZ) invariants can be achieved and all curvature scalars can be expressed in terms of zeroth-order invariants; thus, constant CZ invariants imply constant curvature scalars. In the exceptional case in which \( \Psi_0 = \Psi_1 = 0 \) in the Ricci canonical frame, complete back-solvability is not possible since \( \Psi_3 \) and \( \Psi_4 \) cannot be determined from any zeroth-order curvature invariant [11]. However, this exceptional case occurs when the Weyl and Ricci canonical frames are aligned, for P-type II \( \Psi_1 = 0 \) and \( \Psi_4 = 1 \) and the remaining curvature scalars were shown above to be constant (this is a particular instance of a not completely backsolvable (NCB) case becoming completely backsolvable; see comment in reference [3] of [11]).

We have consequently also shown that for P-type II and PP-type II spacetimes, if the zeroth-order invariants are constant then all curvature scalars are constant. These results support the higher-dimensional conjectures discussed in the previous section.

**Acknowledgments**

We would like to thank R Milson, V Pravda and A Pravdová for helpful comments. This work was supported by NSERC (AC and NP) and the Killam Foundation (SH).
On spacetimes with constant scalar invariants

Table 1. Segre type (3), i.e. $R_{22} \neq 0$ and $R_{33} \neq 0$.

| $\gamma_{12}$ | Constraints on $W_1(u, v, x), W_2(u, v, x)$ and $H(u, v, x)$ |
|---------------|-------------------------------------------------------------|
| $=0$          | A1 (P-III; $\tau = 0$; PP-N)                               |
|               | $W_1 = W_0(u, x), W_2 = W_0(u, x), H = vh_1(u, x) + h_0(u, x)$ |
| $\neq 0$      | B1 (P-III; $\tau \neq 0$; PP-N)                           |
|               | $W_1 = -\frac{2}{5} v + W_0(u, x), W_2 = W_0(u, x), H = -\frac{3}{5} v^2 + vh_1(u, x) + h_0(u, x)$ |

Table 2. Segre type (211), i.e. $R_{22} \neq 0$ and $R_{33} = 0$.

| $\gamma_{12}$ | Constraints on $W_1(u, v, x), W_2(u, v, x)$ and $H(u, v, x)$ |
|---------------|-------------------------------------------------------------|
| $=0$          | D1 (P-N; $\tau = 0$; PP-O) (special case of A1)             |
|               | $W_1 = x f(u) + f_0(u), W_2 = x g(u) + g_0(u), H = h_0(u, x)$ |
| $\neq 0$      | F1 (P-III; $\tau \neq 0$; PP-O) (special case of B1)       |
|               | $W_1 = -\frac{2}{5} v + W_0(u, x), W_2 = W_0(u, x), H = -\frac{3}{5} v^2 + \frac{3}{5} W_0(u) + h_0(u, x)$ |

Appendix A. Three-dimensional VSI metrics

By analogy with the four-dimensional VSI spacetimes, we begin by considering a real null frame $e_i = \{l, n, m\}$ such that the only non-vanishing inner products are $l^a n_a = 1 = -m^a m_a$.

In three dimensions, the Riemann tensor is equivalent to the Ricci tensor; therefore, all zeroth-order invariants vanish if

$$ R_{ab} = R_{22} l_a l_b + 2 R_{23} l_a m_b. \quad (A.1) $$

We identify $R_{22} \sim \phi_{22}$ and $R_{23} \sim \phi_{12} + \phi_{21}$ and we use the $n$-dimensional version of the VSI theorem to conclude that the analogues of $\kappa, \sigma$ and $\rho$ must vanish. In three dimensions, we find that $\kappa \sim \gamma_{111}, \sigma = \rho \sim \gamma_{313}$ and $\tau \sim \gamma_{012}$, where $\gamma_{ijk}$ are the Ricci-rotation coefficients. Therefore, we have that a three-dimensional spacetime is VSI if and only if $\gamma_{311} = \gamma_{313} = 0$ and has Segre type [3], [(211)] or [(111)] with the possibility that the (non-)vanishing of $\gamma_{112}$ may lead to distinct subclasses.

The method we use to explicitly obtain the three-dimensional VSI spacetimes involves writing the four-dimensional VSI spacetimes in terms of real coordinates $(u, v, x, y)$ then considering the resulting frame when either $x = \text{const}$ or $y = \text{const}$. We set $\xi = (x + iy)/\sqrt{2}$ and $W = (W_1 + iW_2)/\sqrt{2}$ throughout, and in every case the restriction to $y = \text{const}$ yields nothing new; hence, the relevant null frame, in coordinates $(u, v, x)$, is

$$ l = \partial_u, \quad n = \partial_v - \left[H + \frac{1}{2} (W_1^2 + W_2^2)\right] \partial_x + W_1 \partial_y, \quad m = \partial_y. \quad (A.2) $$

Upon restriction to $x = \text{const}$, most of the four-dimensional VSI spacetimes become either flat or a special case of a null frame given in the following tables. In addition, whenever $\gamma_{112}$ is nonzero then it is always equal to $-1/x$.

Tables 1–3 provide a list of the three-dimensional VSI spacetimes. Here, we have chosen to characterize the 3D cases according to Segre type. In this regard, the classification in [22] of three-dimensional spacetimes is worth noting. In 4D, the Segre types are related to the PP types [17]. We have also indicated the corresponding class of four-dimensional VSI spacetimes (in parentheses) from which the 3D solutions were obtained. It is also worth noting that some of the spacetimes given here may be equivalent and so these tables may reduce further. For example, in table 3 the vanishing of the Ricci tensor implies that all spacetimes listed there are flat; therefore, using the full three-parameter Lorentz group, $SO(1, 2)$, it should be possible to
Table 3. Segre type \((111)\), i.e. \(R_{ij} = 0\).

| \(\gamma_{12}\) | Constraints on \(W_1(u, v, x), W_2(u, v, x)\) and \(H(u, v, x)\) |
|-----------------|-------------------------------------------------------------|
| \(\neq 0\)     | L1 (P-III; \(\tau = 0\); vacuum) \(W_1 = W_{01}(u), W_2 = W_{02}(u), H = xh_{01}(u) + h_{02}(u)\) |
| \(= 0\)        | L1 (P-N; \(\tau \neq 0\); vacuum) \(W_1 = -\frac{\sqrt{2}}{\tau}, W_2 = 0, H = -\frac{x^2}{2\tau} + \sqrt{2}x[xh_{01}(u) + h_{02}(u)]\) |

Set \(\gamma_{312} = 0\) (i.e., the flat metric should appear in this table). In tables 2 and 3, it is necessary to consider the frame freedom left after the Ricci tensor has been brought to its canonical form. This can then be used to determine if \(\gamma_{312}\) can be made to vanish; alternatively, it may be an invariant of this left-over frame freedom and the metrics would be inequivalent. If \(\gamma_{312}\) can be set to zero, then it still remains to be shown if a coordinate transformation can be found relating the two metrics.

Appendix B. 4D CSI

We know that (Lorentzian) homogeneous spaces and VSI spacetimes are necessarily CSI spacetimes. Let us display all 4D spacetimes that are CSI\(_F\) and CSI\(_K\). We shall find that all such spacetimes are necessarily inCSI\(_R\).

The 4D spacetimes in CSI\(_F\) \(\cap\) CSI\(_K\) are as follows.

**B.1. \(I(AdS_4)\)**

\[
ds^2 = e^{-2py} [2 du (dv + \tilde{H} du + \tilde{W}_x dx + \tilde{W}_y dy) + dx^2] + dy^2, \tag{B.1}
\]

where

\[
\tilde{W}_x = v \tilde{W}^{(1)}(u, x) + \tilde{W}^{(0)}(u, x, y),
\]

\[
\tilde{W}_y = \tilde{W}^{(0)}(u, x, y),
\]

\[
\tilde{H} = \frac{v^2}{8} (\tilde{W}^{(1)}_x)^2 + v \tilde{H}^{(1)}(u, x, y) + \tilde{H}^{(0)}(u, x, y),
\]

and \(\tilde{W}^{(1)}_x\) satisfy the 3D VSI equations (see appendix A).

Since \(\tilde{W}^{(0)}(u, x, y)\) depends on \(y\), in addition to \(u\) and \(x\), this metric is fibred. Due to the terms \(e^{-2py}\) and \(dy^2\) in the metric, the spacetime is a warped product. By redefining \(\tilde{W}_x\) in the above metric by omitting the term \(v \tilde{W}^{(1)}(u, x)\) (the ‘\(vW\)’ term) and in \(\tilde{H}\) omitting the term \(\frac{v^2}{8}(\tilde{W}^{(1)}_x)^2\) (the ‘\(v^2H\)’ term), we can rewrite the metric as the tensor sum of the redefined metric (B.1) and the metric

\[
ds^2 = e^{-2py} \left[ 2 du \left( du + \frac{v^2}{8} (\tilde{W}^{(1)}_x)^2 du + v \tilde{W}^{(1)}_x dx \right) + dx^2 \right] + dy^2. \tag{B.2}
\]

Clearly, the metric can be written in terms of the combined operations of fibering, warping and tensor summing, and hence belongs to CSI\(_R\).

**B.2. \(I(Sol)\)**

\[
ds^2 = e^{-2py} [2 du (dv + \tilde{H} du + \tilde{W}_x dx + \tilde{W}_y dy)] + e^{-2py} dx^2 + dy^2, \tag{B.3}
\]
where
\[ \tilde{W}_x = \tilde{W}_x^{(0)}(u, x, y), \quad \tilde{W}_y = \tilde{W}_y^{(0)}(u, x, y), \]
\[ \tilde{H} = v\tilde{H}^{(1)}(u, x, y) + \tilde{H}^{(0)}(u, x, y). \]
This metric is fibred and warped.

B.3. \( \mathcal{I}(M_1 \times M_2) \)
\[
\begin{align*}
\text{ds}^2 &= 2 du (dv + H du + W_i \mathbf{m}^i) + \delta_{ij} \mathbf{m}^i \mathbf{m}^j, \\
\end{align*}
\]
where
\[ W_i = v W_i^{(1)} + \tilde{W}_i^{(0)}(u, x, y), \]
\[ H = v^2 H^{(2)} + v \tilde{H}^{(1)}(u, x, y) + \tilde{H}^{(0)}(u, x, y), \]
and \( W_i^{(1)} \) and \( H^{(2)} \) are given in the table below. This metric is fibred and warped, and due to the ‘\( vW \)’ and ‘\( v^2H \)’ terms, can be written as a tensor sum.

\[
\begin{array}{cccc}
\delta_{ij} \mathbf{m}^i \mathbf{m}^j & dx^2 + dy^2 & \sigma^2(dx^2 + \sin^2 x dy^2) & dx^2 + e^{-2y} dy \\
W_i^{(1)} \mathbf{m}^i & \alpha dx + \beta dy & 0 & \alpha dx + \beta e^{-y} dy \\
H^{(2)} & \frac{1}{2} (4\sigma + \alpha^2 + \beta^2) & \frac{1}{2} & \frac{1}{2} (4\sigma + \alpha^2 + \beta^2) \\
\end{array}
\]

B.4. \( \mathcal{I}((A) dS_3 \times \mathbb{R}) \)

Metrics of the form (66), (68) and (70). These metrics are fibred, warped and tensor summed.

B.5. \( \mathcal{I}((A) dS_4) \)

Metrics of the form (66), (68) and (70). In addition, for \( \mathcal{I}(dS_4) \)
\[
\begin{align*}
\text{ds}^2 &= \cos^2(\sqrt{\sigma} x) \cos^2(\sqrt{\sigma} y) [2 du (dv + H du + \tilde{W} \mathbf{m}^i)] + \cos^2(\sqrt{\sigma} y) dx^2 + dy, \\
\end{align*}
\]
where
\[ \tilde{W}_i = \tilde{W}_i^{(0)}(u, x, y), \quad \tilde{H} = \frac{v^2}{2} \sigma + v \tilde{H}^{(1)}(u, x, y) + \tilde{H}^{(0)}(u, x, y), \]
and in the case of \( \mathcal{I}(AdS_4) \), similar versions of the metrics (68) and (70) are obtained. These metrics are fibred, warped and tensor summed.

All of the metrics in sections B.1–B.5 can be written in terms of the combined operations of fibering, warping and tensor summing. Therefore, all of the spacetimes constructed belong to CSI\(_R\). We consequently have the result that in 4D CSI\(_F\), CSI\(_K\) and CSI\(_R\) are equivalent. It is plausible that a similar result applies in higher dimensions.

Finally, let us consider the tensor sum in a little more detail. Let us consider the CSI\(_R\) metric (66). Redefining the metric by omitting the ‘\( v^2H \)’ term, it can be rewritten as the ‘tensor sum’ of the redefined metric and
\[
\begin{align*}
\text{ds}^2 &= \left[ 2 du \left( dv + \frac{v^2}{2} \sigma du + 2 \sqrt{\sigma} \tan(\sqrt{\sigma} x) dx \right) \right] + dx^2. \\
\end{align*}
\]
The term \( 2 du (dv + \frac{v^2}{2} \sigma du) \) can be rewritten as \( (1 + \frac{v}{2} UV)^{-2} dU dV \), which is a metric of constant Ricci scalar curvature.

It is plausible that we have constructed all of the 4D CSI spacetimes here. To prove this, we must prove the conjectures outlined above. This is perhaps best done by considering higher-order differential scalar invariants in 4D [2], which we hope to do in a future paper.
Appendix C. VSI metrics

We can now write the metric for higher-dimensional VSI spacetimes in a canonical form. An immediate corollary of theorem 4.1 is

**Corollary C.1.** Any VSI metric can be written as
\[
ds^2 = 2 du[dv + H(v, u, x^k) du + W_i(v, u, x^k) dx^i] + \delta_{ij} dx^i dx^j. \tag{C.1}
\]

**Proof.** In [15], it was proven that all VSI metrics have \( L_{ij} = 0 \) and \( \hat{R}_{ijkl} = 0 \). The first condition implies that the metric is Kundt, hence of the form (10), while the second implies that the spatial metric, \( \tilde{g}_{ij} \), is flat. Using theorem 4.1, the corollary now follows. □

We are now in a position to determine the explicit metric forms for higher-dimensional VSI spacetimes [21], which we shall return to in future work.

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