A Phase-Fitted and Amplification-Fitted Explicit Runge–Kutta– Nyström Pair for Oscillating Systems

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Abstract: An optimized embedded 5(3) pair of explicit Runge–Kutta– Nyström methods with four stages using phase-fitted and amplification-fitted techniques is developed in this paper. The new adapted pair can exactly integrate (except round-off errors) the common test: \( y'' = -w^2y \). The local truncation error of the new method is derived, and we show that the order of convergence is maintained. The stability analysis is addressed, and we demonstrate that the developed method is absolutely stable, and thus appropriate for solving stiff problems. The numerical experiments show a better performance of the new embedded pair in comparison with other existing RKN pairs of similar characteristics.

Keywords: phase-fitted and amplification-fitted schemes; RKN pair; oscillatory systems; initial-value problems

1. Introduction

The aim of this paper is to efficiently solve special second-order initial-value systems of the form

\[
y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \tag{1}
\]

for which it is known that their solutions are oscillatory, where \( y \in \mathbb{R}^d \) and \( f: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \) is sufficiently differentiable. In recent and past years, the search for new numerical algorithms to efficiently solve (1) has attracted the attention of many researchers due to the great relevance of these problems in so many areas of applied sciences (as quantum chemistry, fluid mechanics, physical chemistry, astronomy and many others). To solve (1) directly, the class of Runge–Kutta– Nyström (RKN) methods has mostly been used. Regarding the efficient use of these methods, the embedded technique was firstly proposed by Fehlberg in [1] to provide an estimate of the error committed on each step. Since then there have been many researchers who have presented pairs of embedded RKN methods. Van de Vyver developed in [2] an explicit 5(3) embedded pair of RKN methods with four stages for solving (1). Franco developed a 5(3) embedded pair of explicit ARKN schemes with four stages in [3]. Simos [4], Kalogiratou and Simos [5], Van de Vyver [6] and Liu [7] derived...
different adapted RKN methods. Senu et al. constructed an explicit embedded pair of RKN methods in [8], Franco et al. [9] presented two embedded explicit RKN pairs for approximating the oscillatory solution of (1). Anastassi and Kosti developed a 6(4) embedded RKN optimized pair in [10]. Fang et al. developed a new pair of explicit ARKN methods in [11], for the numerical integration of general perturbed oscillators. Similarly, Fang et al. in [12] constructed an efficient energy-preserving method for general nonlinear oscillatory Hamiltonian systems. Also, Mei et al. in [13] derived an arbitrary order ERKN method based on group theory for solving oscillatory Hamiltonian systems, and Yang et al. in [14] developed an extended RKN-type method for the numerical integration of perturbed oscillators. Recently, Demba et al. [15,16] derived two new explicit RKN methods trigonometrically adapted for solving the kind of problems in (1). Most recently, Demba et al. [17] derived an exponentially-fitted explicit RKN pair for solving (1).

This work aims at the development of a new phase-fitted and amplification-fitted 5(3) embedded pair of explicit RKN methods based on the 5(3) pair presented by Van der Vyver [2] for solving the problem in (1). The derived method accurately solves the test equation $y'' = -w^2y$. The numerical experiments reveal the efficiency of the developed method when compared with other embedded RKN codes of orders 5(3) with four stages.

The remaining part of this paper is organized in this way: the description of a pair gives a conclusion.

2. Fundamental Concepts

2.1. Explicit Runge-Kutta-Nyström Methods

An explicit RKN method with $r$ stages for the problem (1) is generally expressed by the formulas:

\begin{align*}
y_{n+1} &= y_n + hy'_n + h^2 \sum_{i=1}^{r} b_i f(x_n + c_i h, Y_i), \\
y'_n &= y'_n + h \sum_{i=1}^{r} d_i f(x_n + c_i h, Y_i), \\
Y_l &= y_n + c_l h y'_n + h^2 \sum_{j=1}^{l-1} a_{lj} f(x_n + c_j h, Y_j), \quad l = 1, \ldots, r,
\end{align*}

where $y_{n+1}$ and $y'_{n+1}$ denote approximations for $y(x_{n+1})$ and $y'(x_{n+1})$ respectively, and the grid points on the integration interval $[x_0, x_N]$ are given by $x_j = x_0 + jh$, $j = 0, 1, \ldots, N$, with $h$ the fixed step-size considered. The above explicit method may be formulated compactly using the Butcher array in the form

$$
c \begin{array}{c}
A \\
h^T \\
d^T
\end{array}
$$

where $A = (a_{ij})_{r \times r}$ a lower triangular matrix of coefficients, $c = (c_1, c_2, \ldots, c_r)^T$ is the vector of stages, and $b = (b_1, b_2, \ldots, b_r)^T$, $d = (d_1, d_2, \ldots, d_r)^T$ are two vectors containing the remaining coefficients of the method. For short, this can be denoted as $(c, A, b, d)$.

An $m(n)$ embedded-type pair of RKN methods comprises two such methods, one given by $(c, A, b, d)$ with order $m$, and another one of order $n$ ($n < m$) given by $(c, A, \hat{b}, \hat{d})$ which shares the coefficients in $c$ and $A$. The higher order method provides for each step an approximate solution $y_{n+1}$, $y'_{n+1}$, while a second approximate solution $\hat{y}_{n+1}$, $\hat{y}'_{n+1}$ is provided by the method of lower order. The purpose of the second approximation is to
provide an estimate of the local truncation error. A RKN pair of embedded methods may be expressed using the Butcher array in the form

\[
\begin{array}{c|c}
 c & A \\
 \hline
 b^r & d^r \\
 \end{array}
\]

On the basis of the local error estimation provided by the embedding procedure, a variable step-size approach can be constructed. The local error estimate at \( x_{n+1} = x_n + h \) is obtained through the differences between the two approximations of the solution and of the derivative, that is, \( \eta_{n+1} = y_{n+1} - y_{n} \), and \( \eta'_{n+1} = y'_{n+1} - y'_{n} \).

Let \( \text{Est}_{n+1} = \max (\| \eta_{n+1} \|_{\infty}, \| \eta'_{n+1} \|_{\infty}) \) denote the local error estimate used to decide the step-length \( h_n \) on the \( n+1 \) iteration. In order to advance the solution of the problem in hand we use the step-length control strategy presented in [7]:

- if \( \text{Est}_{n+1} < \text{Tol} / 100 \), then \( h_{n+1} = 2h_n \),
- if \( \text{Tol} / 100 \leq \text{Est}_{n+1} < \text{Tol} \), then \( h_{n+1} = h_n \),
- if \( \text{Est}_{n+1} \geq \text{Tol} \), then take \( h_{n+1} = h_n / 2 \) and redo the computations of the current step being \( \text{Tol} \) the prescribed tolerance.

**Definition 1** ([2,18]). An explicit Runge-Kutta-Nyström method as given in the Equations (2)–(4) is said to have algebraic order \( k \) if it holds

\[
\begin{align*}
\{ y(x_0 + h) - y_1 = O(h^{k+1}), \\
y'(x_0 + h) - y'_1 = O(h^{k+1}).
\}
\end{align*}
\]

### 2.2. Analysis of Phase-Lag, Amplification Error and Stability

Applying the RKN method in (2)–(4) to the test equation \( y'' = -w^2 y \), the phase-lag, amplification error, and the linear stability are derived. In particular, letting \( \hat{h} = -\mu^2 \), \( \mu = wh \), the approximate solution provided by (2)–(4) verifies the recurrence equation:

\[
L_{n+1} = E(\hat{h})L_n,
\]

where

\[
L_{n+1} = \begin{bmatrix} y_{n+1} \\ hy'_{n+1} \end{bmatrix}, \quad L_n = \begin{bmatrix} y_n \\ hy'_n \end{bmatrix}, \quad E(\hat{h}) = \begin{bmatrix} 1 + \hat{h}b^T N^{-1}e & wh(1 + \hat{h}b^T N^{-1}e) \\ -whd^T N^{-1}e & 1 + \hat{h}d^T N^{-1}e \end{bmatrix},
\]

\[
N = I - \hat{h}A, \quad A = (a_{ij})_{4 \times 4}, \quad b, c, d \text{ are the corresponding matrix and vectors of coefficients, I is the identity matrix of order four, and } e = [1,1,1,1]^T.
\]

For sufficiently small values of \( \mu = wh \), it is assumed that the matrix \( E(\hat{h}) \) possesses complex conjugate eigenvalues [19]. Under this assumption, an oscillatory numerical solution is obtained, whose behavior depends on the eigenvalues of the stability matrix \( E(\hat{h}) \). The characteristic equation of this matrix can be expressed as:

\[
\lambda^2 - \text{tr}(E(\hat{h}))\lambda + \text{det}(E(\hat{h})) = 0.
\]

**Theorem 1** ([10]). If we apply to the common test equation \( y'' = -w^2 y \) the Runge–Kutta–Nyström scheme in (2)–(4), we get the formula for calculating the phase-lag directly (or dispersion error) \( \Psi(\mu) \) given by:

\[
\Psi(\mu) = \mu - \arccos \left( \frac{\text{tr}(E(\hat{h}))}{2 \sqrt{\text{det}(E(\hat{h}))}} \right).
\]
If $\Psi(\mu) = O(\mu^{l+1})$, then the method is said to have phase-lag order $l$. For an explicit RKN method, $\text{tr}(E(\tilde{h}))$ and $\text{det}(E(\tilde{h}))$ are polynomials in $\mu$ (in case of an implicit RKN method these would be rational functions).

**Definition 2.** An explicit Runge–Kutta–Nyström method as given in the Equations (2)–(4) is said to be phase-fitted, if the phase-lag is zero.

**Definition 3** ([10]). For the Runge–Kutta–Nyström method given in the Equations (2)–(4), the value $\beta(\mu) = 1 - \sqrt{\text{det}(E(\tilde{h}))}$ is called the amplification error (or dissipative error). If $\beta(\mu) = O(\mu^{s+1})$, then the method is said to have amplification error of order $s$.

**Definition 4.** An explicit Runge–Kutta–Nyström method as given in the Equations (2)–(4) is said to be amplification-fitted if the amplification-error is zero.

**Definition 5.** An interval $(0, \hat{b}_h), \hat{b}_h \in \mathbb{R}^+ \cup \{+\infty\}$ is called an absolute stability interval of the method in (2)–(4) if $\hat{h}$ is the highest value for which $|\lambda_{1,2}| < 1$, where $\lambda_{1,2}$ are the solutions of the Equation (6).

3. Derivation of the New Embedded Pair

In this section, we will obtain a new 5(3) pair of explicit phase-fitted and amplification-fitted embedded RKN methods based on the RKN5(3) embedded pair derived by Van de Vyver in [2]. The coefficients of the RKN pair in [2] are shown in Table 1.

**Table 1.** The RKN5(3) Method in [2].

| $0$ | $\frac{1}{5}$ | $\frac{1}{2}$ | $1$ |
|-----|---------------|---------------|-----|
|     | $\frac{1}{50}$ | $-\frac{1}{27}$ | $\frac{5}{17}$ |
|     | $\frac{2}{5}$ | $\frac{7}{27}$ | $-\frac{5}{54}$ |
|     | $\frac{1}{12}$ | $\frac{9}{55}$ |     |
|     | $\frac{1}{12}$ | $\frac{25}{84}$ | $\frac{9}{55}$ |
|     | $\frac{1}{12}$ | $\frac{6}{7}$ | $\frac{9}{55}$ |
|     | $\frac{1}{12}$ | $\frac{125}{168}$ | $\frac{9}{55}$ |
|     | $\frac{1}{12}$ | $\frac{25}{42}$ | $\frac{9}{55}$ |

In order to get the new adapted pair we first consider the coefficients of the third-order scheme in the RKN5(3) pair. Equating to zero the phase-lag $\Psi(\mu)$ and the amplification error $\beta(\mu)$ we get the system:

$$
\begin{cases}
\Psi(\mu) = 0 \\
\beta(\mu) = 0.
\end{cases}
$$

We solve this system considering the coefficients in Table 1 except two of them, which are taken as unknowns. Specifically, we take $\hat{b}_2$ and $\hat{b}_3$ as unknowns. We obtain the following values:

$$
\hat{b}_2 = \frac{P(\mu)}{840\mu^4(405,000 - 85,500 \mu^2 + 7455 \mu^4 - 288 \mu^6 + 4 \mu^8)},
\hat{b}_3 = \frac{3Q(\mu)}{280\mu^4(405,000 - 85,500 \mu^2 + 7455 \mu^4 - 288 \mu^6 + 4 \mu^8)}.
$$
where

\[
P(\mu) = \begin{pmatrix} 186 \mu^{14} - 9157 \mu^{12} + 11,175 \mu^{10} + 3,355,275 \mu^8 - 57,739,500 \mu^6 & -2,232,000 \mu^6 \cos(\mu) + 349,515,000 \mu^4 + 4,860,000 \mu^4 \cos(\mu) \\
-810,000,000 \mu^2 + 81,000,000 \mu^2 \cos(\mu) + 1,458,000,000 & -1,458,000,000 \mu^2 \cos(\mu) \end{pmatrix},
\]

\[
Q(\mu) = \begin{pmatrix} -162,000,000 + 132,840,000 \mu^2 - 41,985,000 \mu^4 + 5,890,500 \mu^6 \\
-511,245 \mu^8 + 27,540 \mu^{10} - 934 \mu^{12} + 12 \mu^{14} - 51,840,000 \mu^2 \cos(\mu) + 3,240,000 \mu^4 \cos(\mu) - 144,000 \mu^6 \cos(\mu) \end{pmatrix}.
\]

The corresponding Taylor series expansions in powers of \(\mu\) are given by

\[
\hat{b}_2 = \frac{125}{168} - \frac{11}{1050} \mu^2 - \frac{613}{235,200} \mu^4 - \frac{129,473}{1,587,600,000} \mu^6 - \frac{61,471}{39,293,100,000} \mu^8 - \frac{25,533,945,907}{128,724,195,600,000,000} \mu^{10} + \cdots,
\]

\[
\hat{b}_3 = -\frac{9}{56} + \frac{17}{1400} \mu^2 - \frac{9101}{17,640,000} \mu^4 - \frac{12,353}{396,900,000} \mu^6 - \frac{29,522,267}{7,858,620,000,000} \mu^8 - \frac{65,718,361,603}{128,724,195,600,000,000} \mu^{10} + \cdots.
\]

As \(\mu \to 0\), the newly obtained coefficients \(\hat{b}_2, \hat{b}_3\) become the coefficients of the counterpart scheme in the original pair.

Similarly, if we take the coefficients of the fifth-order scheme in the RKN5(3) pair, except \(b_1\) and \(b_2\), which are taken as unknowns in the Equation (8), the solution of this system results in

\[
b_1 = \frac{R(\mu)}{360 \mu^4(1200 \mu^2 - 60 \mu^4 - 7200 + \mu^6)},
\]

\[
b_2 = \frac{S(\mu)}{252 \mu^4(1200 \mu^2 - 60 \mu^4 - 7200 + \mu^6)}.
\]

where

\[
R(\mu) = \begin{pmatrix} 25,920,000 + 38,160 \mu^8 - 1161 \mu^{10} + 16 \mu^{12} + 5,810,400 \mu^4 \\
-475,200 \mu^4 \cos(\mu) + 8,294,400 \mu^2 \cos(\mu) + 14,400 \mu^6 \cos(\mu) \\
-651,600 \mu^6 - 25,920,000 \cos(\mu) - 21,254,400 \mu^2 \end{pmatrix},
\]

\[
S(\mu) = \begin{pmatrix} 5,508,000 \mu^4 - 18,144,000 \mu^2 - 716,400 \mu^6 + 50,310 \mu^8 \\
-1815 \mu^{10} + 28 \mu^{12} - 756,000 \mu^4 \cos(\mu) + 18,144,000 \\
+25,200 \mu^6 \cos(\mu) - 18,144,000 \cos(\mu) + 9,072,000 \mu^2 \cos(\mu) \end{pmatrix}.
\]
The corresponding Taylor series expansions in powers of \( \mu \) of the above coefficients are given by

\[
\begin{align*}
\frac{b_1}{24} &= \frac{1}{50,400} \mu^4 + \frac{17}{4,536,000} \mu^6 - \frac{1}{3,991,680} \mu^8 \\
&+ \frac{4393}{217,945,728,000} \mu^{10} + \frac{1949}{485,222,400,000} \mu^{12} + \cdots, \\
\frac{b_2}{84} &= \frac{25}{10,808} \mu^4 - \frac{1}{36,288} \mu^6 + \frac{1}{3,991,680} \mu^8 \\
&- \frac{4211}{217,945,728,000} \mu^{10} - \frac{45,823}{10,461,394,944,000} \mu^{12} + \cdots.
\end{align*}
\]

As \( \mu \to 0 \), the newly obtained coefficients \( b_1, b_2 \) in the fifth-order adapted scheme become those of the counterpart scheme in the original pair.

The new adapted RKN pair will be named as PFAFRKN5(3).

### 3.1. Order of Convergence

This section is devoted to presenting the local truncation errors of the proposed methods and to get the algebraic orders of convergence. This is accomplished by using the usual tool of Taylor expansions. The local truncation errors (LTE) at the point \( x_{n+1} \) of the solution and the first derivative are given respectively by:

\[
\begin{align*}
LTE &= y(x_0 + h) - y_1, \\
LTE_{der} &= y'(x_0 + h) - y'_1.
\end{align*}
\]

**Proposition 1.** For the lower order method, the corresponding LTEs are:

\[
\begin{align*}
LTE &= -\frac{h^4}{240}(w^2 y'') + O(h^5), \\
LTE_{der} &= \frac{h^4}{24}(f_{xxx} + 3y'f_{yx} + 3y''f_{xy} + 3(y')^2 f_{xyy} + 3y'y_{yy} + (y')^3 f_{yyy}) \\
&+ (y')^3 f_{yfy} + f_{fy} + (f_y)^2 y' + O(h^5),
\end{align*}
\]

where the functions in the right hand sides are evaluated at \( x_0 \).

**Proposition 2.** For the higher order method, we have:

\[
\begin{align*}
LTE &= \frac{h^6}{21600} (4(y')^3 f_{yy} + 3(y'')^2 f_{yy} + 6(y'')^2 f_{xyy} + 6(y')^2 f_{xyy} + (y')^4 f_{yfyy} \\
&+ 6y'f_{xy} + 12f_{fy} + 12(f_y)^2 y' + 6(y')^2 f_{yy} + 12(y')^2 f_{yy} + 12y'f_{yy} + 12y'f_{yy} \\
&+ 24f_{yy}f_{yy} + f_{xx} + 12y'f_{yy} + O(h^5), \\
LTE_{der} &= \frac{h^6}{720} (f_{xxx} + 18y'f_{yy}f_{yy} + 30(y'')^2 f_{yy} + 10y''f_{xy} + 10y'f_{xy} + (f_y)^2 f_{xx} \\
&+ 5f_{xx} + 5f_{y} + 10f_{xx} + 5(y')^4 f_{yy} + 5y'f_{xyy} + 10f_{xyy} + 10y''f_{xy} + 10y'f_{xy} \\
&+ 10(y')^2 f_{xyy} + 5(y')^4 f_{yfyy} + 15y'f_{yy} + 15y'f_{yy} + 15y'f_{yy} + 15(y')^2 f_{yy} \\
&+ 30y'f_{yy} + 30(y')^2 f_{yy} + 8y'f_{xy} + 10y''f_{yy} + 10y'f_{yy} + 10(y')^3 f_{yy} \\
&+ 10(y')^2 f_{yy} + 23(y')^2 f_{yy} + 15(y')^2 f_{yy} + 20y'f_{yy} + 13y'f_{yy} \\
&+ (f_y)^3 y' + 5y'f_{yy} + O(h^5),
\end{align*}
\]

where the functions in the right hand sides are evaluated at \( x_0 \).
To effectively determine the order of the proposed method, we have checked the order conditions as given in [2]. We obtained that the lower order method has algebraic order three and the higher order method has algebraic order five, thus resulting in a 5(3) RKN pair.

3.2. Absolute Stability Intervals of the New Adapted Pair

**Proposition 3.** The third-order method of the PFAFRKN5(3) pair has \((0, 23.83)\) as interval of absolute stability and the fifth-order scheme has the absolute stability interval \((0, 20.65)\).

Using the Maple package, from the definition in (5), the above results can be readily obtained.

4. Numerical Examples

To demonstrate the performance of the new pair, we have considered other 5(3) RKN pairs that appear in the literature to be used for the numerical comparisons:

- **PFAFRKN5(3):** The adapted RKN embedded pair developed in this paper,
- **RKN5(3):** An explicit 5(3) RKN pair presented by Van de Vyver in [2],
- **ARKN5(3):** An explicit 5(3) adapted RKN pair given by Franco in [3],
- **EFRKN5(3):** An embedded exponentially-fitted explicit RKN pair presented by Van de Vyver in [6],
- **EEERKN5(3):** An embedded exponentially-fitted explicit RKN method derived by Demba et al. in [17].

We will consider different oscillatory problems appeared in the literature to test the performance of the above methods:

**Problem 1.** (Almost Periodic Problem in [20])

\[
\begin{align*}
y_1'' & = -y_1 + \epsilon \cos(\Psi x), \quad y_1(0) = 1, y_1'(0) = 0, \\
y_2'' & = -y_2 + \epsilon \sin(\Psi x), \quad y_2(0) = 0, y_2'(0) = 1, \quad x \in [0, 100],
\end{align*}
\]

whose exact solution is

\[
\begin{align*}
y_1(x) & = \frac{(1-\epsilon - \Psi^2)}{(1-\Psi^2)} \cos(x) + \frac{\epsilon}{1-\Psi} \cos(\Psi x), \\
y_2(x) & = \frac{(1-\epsilon - \Psi^2)}{(1-\Psi^2)} \sin(x) + \frac{\epsilon}{1-\Psi} \sin(\Psi x).
\end{align*}
\]

For the numerical computations we have taken \(\epsilon = 0.001\) and \(\Psi = 0.1\). To use the adapted methods we have taken the parameter value \(w = 1\).

**Problem 2.** (Two Body Problem in [21])

\[
\begin{align*}
y_1'' & = -\frac{y_1}{(y_1^2 + y_2^2)^2}, \quad y_1(0) = 1, y_1'(0) = 0, \\
y_2'' & = -\frac{y_2}{(y_1^2 + y_2^2)^2}, \quad y_2(0) = 0, y_2'(0) = 1, \quad x \in [0, 100],
\end{align*}
\]

whose exact solution is

\[
\begin{align*}
y_1(x) & = \cos x, \\
y_2(x) & = \sin x.
\end{align*}
\]

To apply the adapted methods we have taken the value \(w = 1\).

**Problem 3.** (Nonlinear Problem in [22])

\[
y'' + y + y^3 = B \cos(\Omega x), \quad y(0) = 0.20042672806900, \quad y'(0) = 0, \quad x \in [0, 100].
\]
with $B = 0.002$ and $\Omega = 1.01$.

The reference solution is

$$y(x) = 0.200179477536 \cos(\Omega x) + 0.246946143 \times 10^{-3} \cos(3\Omega x) + 0.304016 \times 10^{-9} \cos(7\Omega x),$$

which is an accurate approximate solution given in [22].

Now we take $w = \Omega$ to apply our method and the ones in [3,6,17].

**Problem 4. (Non-homogeneous System) in [8]**

$$
\begin{align*}
y''_1 &= -v^2 y_1(x) + v^2 f(x) + f''(x), \\
y(0) &= a + f(0), \\
y'(0) &= f'(0), \\
y''_2 &= -v^2 y_2(x) + v^2 f(x) + f''(x), \\
y(0) &= f(0), \\
y'(0) &= av + f'(0),
\end{align*}
$$

whose exact solution is

$$
\begin{align*}
y_1(x) &= a \cos(vx) + f(x), \\
y_2(x) &= a \sin(vx) + f(x).
\end{align*}
$$

In the numerical computations we have taken the values $v = 4$, $a = 0.1$, and $f(x) = e^{-10x}$.

We have solved this problem on the interval $[0, 100]$ taking the value of the fitting parameter $w = 4$.

**Problem 5. (Linear Problem in [23])**

$$
\begin{align*}
y'' + y &= 2\Omega \cos(x), \\
y(0) &= 1, \\
y'(0) &= 0, \\
x \in [0, 100],
\end{align*}
$$

with $\Omega = 10^{-6}$.

The exact solution is

$$y(x) = \cos(x) + \Omega x \sin(x).$$

Now we take $w = 1$ to apply our method and the ones in [3,6,17].

**Discussion**

The numerical data are given in Tables 2–6, considering different tolerances. The tables contain the number of steps, NSTEP; the number of function evaluations, NFE; the number of rejected steps, RSTEP; the maximum absolute errors, MAXER, and the computational time in seconds.

**Table 2.** Numerical data corresponding to Problem 1.

| TOL   | METHOD     | NSTEP | NFE  | RSTEP | MAXER      | TIME(s) |
|-------|------------|-------|------|-------|------------|---------|
| $10^{-2}$ | PFARKN5(3) | 122   | 488  | 0     | 1.841691(-5) | 0.036   |
|       | RKN5(3)    | 122   | 488  | 0     | 1.078825(-2) | 0.054   |
|       | ARKN5(3)   | 242   | 968  | 0     | 9.806283(-1) | 0.066   |
|       | EFRKN5(3)  | 242   | 968  | 0     | 2.139264(-3) | 0.051   |
|       | EEERKN5(3)| 122   | 488  | 0     | 2.591319(-3) | 0.040   |
| $10^{-4}$ | PFARKN5(3) | 522   | 2088 | 0     | 2.989273(-9) | 0.037   |
|       | RKN5(3)    | 522   | 2088 | 0     | 7.172465(-6) | 0.046   |
|       | ARKN5(3)   | 1044  | 4179 | 1     | 6.403390(-2) | 0.049   |
|       | EFRKN5(3)  | 522   | 2088 | 0     | 3.930696(-4) | 0.047   |
|       | EEERKN5(3)| 522   | 2088 | 0     | 4.299671(-7) | 0.047   |
| $10^{-6}$ | PFARKN5(3) | 1123  | 4492 | 0     | 2.983191(-11)| 0.044   |
|       | RKN5(3)    | 1123  | 4492 | 0     | 1.542823(-7) | 0.051   |
|       | ARKN5(3)   | 4491  | 17,970 | 2 | 3.458063(-3) | 0.078   |
|       | EFRKN5(3)  | 2246  | 8987 | 1     | 2.054474(-5) | 0.062   |
|       | EEERKN5(3)| 1123  | 4492 | 0     | 4.355510(-9) | 0.047   |
Table 2. Cont.

| TOL     | METHOD         | NSTEP | NFE    | RSTEP | MAXER  | TIME(s) |
|---------|----------------|-------|--------|-------|--------|---------|
| $10^{-8}$ | PFAFRKN5(3)   | 2420  | 9680   | 0     | 3.727366($\times10^{-12}$) | 0.054   |
|         | RKN5(3)       | 2420  | 9680   | 0     | 3.324929($\times10^{-9}$)  | 0.062   |
|         | ARKN5(3)      | 19,347| 77,397 | 3     | 3.862032($\times10^{-4}$)  | 0.115   |
|         | EFRKN5(3)     | 19,348| 77,404 | 4     | 2.760600($\times10^{-7}$)  | 0.189   |
|         | EEERKN5(3)    | 2420  | 9680   | 0     | 4.516099($\times10^{-11}$) | 0.062   |

| $10^{-10}$ | PFAFRKN5(3)    | 10,421| 41,687 | 1     | 1.644109($\times10^{-11}$) | 0.084   |
|           | RKN5(3)        | 10,421| 41,687 | 1     | 1.658362($\times10^{-11}$) | 0.091   |
|           | ARKN5(3)       | 83,362| 333,460| 4     | 1.002430($\times10^{-5}$)  | 0.034   |
|           | EFRKN5(3)      | 83,364| 333,477| 7     | 1.501274($\times10^{-8}$)  | 0.518   |
|           | EEERKN5(3)     | 10,421| 41,694 | 2     | 1.643935($\times10^{-11}$) | 0.129   |

Table 3. Numerical data corresponding to Problem 2.

| TOL     | METHOD         | NSTEP | NFE    | RSTEP | MAXER  | TIME(s) |
|---------|----------------|-------|--------|-------|--------|---------|
| $10^{-2}$ | PFAFRKN5(3)   | 122   | 488    | 0     | 5.205025($\times10^{-2}$)  | 0.038   |
|         | RKN5(3)       | 122   | 488    | 0     | 8.478978($\times10^{-1}$)  | 0.071   |
|         | ARKN5(3)      | 270   | 1083   | 1     | 1.804551($\times10^{0}$)   | 0.046   |
|         | EFRKN5(3)     | 242   | 968    | 0     | 5.268747($\times10^{-2}$)  | 0.060   |
|         | EEERKN5(3)    | 122   | 488    | 0     | 1.227156($\times10^{-1}$)  | 0.047   |

| $10^{-4}$ | PFAFRKN5(3)   | 522   | 2088   | 0     | 3.035691($\times10^{-4}$)  | 0.046   |
|           | RKN5(3)       | 522   | 2088   | 0     | 6.990118($\times10^{-4}$)  | 0.041   |
|           | ARKN5(3)      | 1044  | 4179   | 1     | 1.480069($\times10^{-1}$)  | 0.041   |
|           | EFRKN5(3)     | 522   | 2088   | 0     | 4.558690($\times10^{-3}$)  | 0.046   |
|           | EEERKN5(3)    | 522   | 2088   | 0     | 3.621045($\times10^{-5}$)  | 0.041   |

| $10^{-6}$ | PFAFRKN5(3)   | 1123  | 4492   | 0     | 3.943697($\times10^{-6}$)  | 0.050   |
|           | RKN5(3)       | 1123  | 4492   | 0     | 1.520229($\times10^{-5}$)  | 0.055   |
|           | ARKN5(3)      | 4491  | 17,970 | 2     | 5.473843($\times10^{-3}$)  | 0.059   |
|           | EFRKN5(3)     | 2246  | 8987   | 1     | 8.448674($\times10^{-5}$)  | 0.064   |
|           | EEERKN5(3)    | 1123  | 4492   | 0     | 3.722093($\times10^{-7}$)  | 0.053   |

| $10^{-8}$ | PFAFRKN5(3)   | 2420  | 9680   | 0     | 5.728311($\times10^{-10}$) | 0.051   |
|           | RKN5(3)       | 2420  | 9680   | 0     | 3.282692($\times10^{-7}$)  | 0.056   |
|           | ARKN5(3)      | 19,347| 77,397 | 3     | 4.588825($\times10^{-4}$)  | 0.109   |
|           | EFRKN5(3)     | 19,348| 77,404 | 4     | 6.160820($\times10^{-7}$)  | 0.160   |
|           | EEERKN5(3)    | 2420  | 9680   | 0     | 3.718493($\times10^{-9}$)  | 0.067   |

| $10^{-10}$ | PFAFRKN5(3)   | 10,421| 41,687 | 1     | 1.672787($\times10^{-11}$) | 0.075   |
|            | RKN5(3)       | 10,421| 41,687 | 1     | 2.058225($\times10^{-10}$) | 0.075   |
|            | ARKN5(3)      | 83,362| 333,460| 4     | 2.680717($\times10^{-5}$)  | 0.265   |
|            | EFRKN5(3)     | 83,364| 333,477| 7     | 3.050323($\times10^{-8}$)  | 0.532   |
|            | EEERKN5(3)    | 10,421| 41,694 | 2     | 1.717850($\times10^{-11}$) | 0.103   |

Table 4. Numerical data corresponding to Problem 3.

| TOL     | METHOD         | NSTEP | NFE    | RSTEP | MAXER  | TIME(s) |
|---------|----------------|-------|--------|-------|--------|---------|
| $10^{-2}$ | PFAFRKN5(3)   | 122   | 488    | 0     | 4.384567($\times10^{-5}$)  | 0.039   |
|         | RKN5(3)       | 122   | 536    | 16    | 2.777502($\times10^{-3}$)  | 0.090   |
|         | ARKN5(3)      | 123   | 504    | 4     | 2.849535($\times10^{-1}$)  | 0.043   |
|         | EFRKN5(3)     | 122   | 512    | 8     | 3.964776($\times10^{-3}$)  | 0.050   |
|         | EEERKN5(3)    | 122   | 515    | 9     | 1.170545($\times10^{-3}$)  | 0.048   |

| $10^{-4}$ | PFAFRKN5(3)   | 262   | 1048   | 0     | 9.677039($\times10^{-7}$)  | 0.040   |
|           | RKN5(3)       | 262   | 1075   | 9     | 7.208088($\times10^{-5}$)  | 0.041   |
|           | ARKN5(3)      | 510   | 2076   | 12    | 4.321049($\times10^{-2}$)  | 0.058   |
|           | EFRKN5(3)     | 267   | 1077   | 3     | 2.048732($\times10^{-4}$)  | 0.049   |
|           | EEERKN5(3)    | 262   | 1072   | 8     | 1.356514($\times10^{-5}$)  | 0.073   |
Table 4. Cont.

| TOL   | METHOD          | NSTEP | NFE  | RSTEP | MAXER       | TIME(s) |
|-------|-----------------|-------|------|-------|-------------|---------|
| $10^{-6}$ | PFAFRKN5(3)     | 1123  | 4492 | 0     | 1.011154($-9$) | 0.047   |
|       | RKN5(3)         | 562   | 2260 | 4     | 1.659456($-6$) | 0.039   |
|       | ARKN5(3)        | 2085  | 8439 | 33    | 1.815951($-3$) | 0.058   |
|       | EFRKN5(3)       | 995   | 4076 | 32    | 1.381514($-3$) | 0.050   |
|       | EEERKN5(3)      | 573   | 2337 | 15    | 9.010356($-8$) | 0.045   |
| $10^{-8}$ | PFAFRKN5(3)     | 2420  | 9680 | 0     | 2.150155($-11$) | 0.046   |
|       | RKN5(3)         | 2324  | 9392 | 32    | 7.596427($-9$) | 0.046   |
|       | ARKN5(3)        | 9091  | 36,487 | 41  | 5.609382($-6$) | 0.180   |
|       | EFRKN5(3)       | 4485  | 18,048 | 36  | 5.225435($-7$) | 0.066   |
|       | EEERKN5(3)      | 1959  | 7932 | 32    | 9.751267($-10$) | 0.049   |

Table 5. Numerical data corresponding to Problem 4.

| TOL   | METHOD          | NSTEP | NFE  | RSTEP | MAXER       | TIME(s) |
|-------|-----------------|-------|------|-------|-------------|---------|
| $10^{-2}$ | PFAFRKN5(3)     | 491   | 1973 | 3     | 7.410682($-4$) | 0.045   |
|       | RKN5(3)         | 491   | 1973 | 3     | 4.286625($-3$) | 0.070   |
|       | ARKN5(3)        | 966   | 3891 | 9     | 1.563857($-1$) | 0.132   |
|       | EFRKN5(3)       | 493   | 1987 | 5     | 7.363927($-3$) | 0.059   |
|       | EEERKN5(3)      | 491   | 1973 | 3     | 7.410682($-4$) | 0.048   |
| $10^{-4}$ | PFAFRKN5(3)     | 1062  | 4257 | 3     | 9.654792($-6$) | 0.045   |
|       | RKN5(3)         | 1062  | 4257 | 3     | 9.105814($-5$) | 0.061   |
|       | ARKN5(3)        | 4202  | 16,838 | 10  | 2.511836($-2$) | 0.072   |
|       | EFRKN5(3)       | 2100  | 8424 | 8     | 1.548196($-4$) | 0.077   |
|       | EEERKN5(3)      | 1076  | 4325 | 7     | 1.094766($-5$) | 0.059   |
| $10^{-6}$ | PFAFRKN5(3)     | 4551  | 18,216 | 4   | 2.691087($-8$) | 0.067   |
|       | RKN5(3)         | 4551  | 18,216 | 4   | 6.186462($-8$) | 0.069   |
|       | ARKN5(3)        | 18,104 | 72,449 | 11  | 1.370302($-3$) | 0.173   |
|       | EFRKN5(3)       | 9036  | 36,177 | 11  | 8.133685($-6$) | 0.133   |
|       | EEERKN5(3)      | 4568  | 18,302 | 10  | 1.985082($-9$) | 0.088   |
| $10^{-8}$ | PFAFRKN5(3)     | 9947  | 39,803 | 5   | 7.650380($-11$) | 0.127   |
|       | RKN5(3)         | 9947  | 39,803 | 5   | 1.325328($-9$) | 0.174   |
|       | ARKN5(3)        | 78,087 | 312,384 | 12  | 7.383390($-5$) | 0.542   |
|       | EFRKN5(3)       | 38,992 | 156,013 | 15  | 4.394977($-7$) | 0.419   |
|       | EEERKN5(3)      | 19,535 | 78,176 | 12  | 1.616837($-11$) | 0.273   |
| $10^{-10}$ | PFAFRKN5(3)     | 42,417 | 169,686 | 6   | 1.017235($-11$) | 0.340   |
|       | RKN5(3)         | 42,271 | 169,099 | 5   | 1.003746($-11$) | 0.247   |
|       | ARKN5(3)        | 338,631 | 1,354,563 | 13  | 3.970905($-6$) | 2.409   |
|       | EFRKN5(3)       | 168,206 | 672,878 | 18  | 2.383359($-8$) | 1.731   |
|       | EEERKN5(3)      | 42,443 | 169,814 | 14  | 1.000001($-11$) | 0.500   |
Table 6. Numerical data corresponding to Problem 5.

| TOL | METHOD    | NSTEP | NFE  | RSTEP | MAXER     | TIME(s) |
|-----|-----------|-------|------|-------|-----------|---------|
| 10⁻² | PFAFRKN5(3) | 122   | 488  | 0     | 1.489108(−5) | 0.082   |
|     | RKN5(3)   | 122   | 506  | 6     | 1.054508(−2) | 0.099   |
|     | ARKN5(3)  | 233   | 974  | 14    | 1.390402(+0)  | 0.102   |
|     | EFRKN5(3) | 125   | 500  | 0     | 1.824478(−2)  | 0.104   |
|     | EEERKN5(3)| 122   | 488  | 0     | 2.993892(−3)  | 0.094   |
| 10⁻⁴ | PFAFRKN5(3) | 522   | 2088 | 0     | 2.390414(−10)| 0.047   |
|     | RKN5(3)   | 263   | 1061 | 3     | 2.255454(−4)  | 0.094   |
|     | ARKN5(3)  | 863   | 3548 | 32    | 1.248772(−1)  | 0.169   |
|     | EFRKN5(3) | 477   | 1974 | 22    | 7.272351(−4)  | 0.131   |
|     | EEERKN5(3)| 265   | 1066 | 2     | 2.292836(−5)  | 0.050   |
| 10⁻⁶ | PFAFRKN5(3) | 1123  | 4492 | 0     | 2.764413(−12)| 0.126   |
|     | RKN5(3)   | 1031  | 4217 | 31    | 1.686848(−6)  | 0.131   |
|     | ARKN5(3)  | 3900  | 15,735 | 45  | 4.686084(−3)  | 0.578   |
|     | EFRKN5(3) | 1833  | 7428 | 32    | 3.976141(−5)  | 0.426   |
|     | EEERKN5(3)| 1059  | 4332 | 10    | 5.131505(−8)  | 0.225   |
| 10⁻⁸ | PFAFRKN5(3) | 2420  | 9680 | 0     | 3.734436(−12)| 0.336   |
|     | RKN5(3)   | 2389  | 9649 | 31    | 8.609137(−9)  | 0.232   |
|     | ARKN5(3)  | 17,271 | 69,285 | 67  | 2.177597(−4)  | 2.561   |
|     | EFRKN5(3) | 8586  | 34,479 | 45  | 1.453551(−6)  | 2.477   |
|     | EEERKN5(3)| 2449  | 9994 | 66    | 9.596512(−11)| 0.617   |
| 10⁻¹⁰| PFAFRKN5(3)| 10,421 | 41,687 | 1   | 1.645062(−11)| 1.482   |
|      | RKN5(3)   | 9577  | 38,467| 53   | 2.599720(−11)| 0.628   |
|      | ARKN5(3)  | 76,283 | 305,345 | 71  | 1.081975(−5)  | 12.256  |
|      | EFRKN5(3) | 37,547 | 150,413 | 75  | 6.953047(−8)  | 11.711  |
|      | EEERKN5(3)| 9862  | 39,670 | 74  | 1.510088(−11)| 2.442   |

To better show the efficiency of the developed PFAFRKN5(3) pair, we present in Figures 1–5 the efficiency curves for the considered problems. It can be observed that the good behavior of the new pair for tolerances $\text{Tol} = 1/10^k$, with $k = 1, 2, 3, 4$. 

Figure 1. Efficiency curves corresponding to Problem 1.
Figure 2. Efficiency curves corresponding to Problem 2.

Figure 3. Efficiency curves corresponding to Problem 3.
The new pair PFAFRKN5(3) gives minimum error norm, minimum number of function evaluations and minimum computational cost. Tables 2–6 and Figures 1–5 present evidence that PFAFRKN5(3) is a very efficient scheme. Therefore, we can say that PFAFRKN5(3) is more appropriate for solving the type of problems in (1) than other existing embedded 5(3) pairs of RKN methods with four stages in the literature.
5. Conclusions

In this study, we have used the methodology for constructing phase-fitted and amplification-fitted methods to develop a new efficient explicit phase-fitted and amplification-fitted embedded RKN pair based on the 5(3) RKN pair of Van de Vyver in [2]. The newly developed pair has four variable coefficients depending on the parameter $\mu = wh$, which is usually known as the parameter frequency [24,25]. We computed the local truncation error for both the higher and lower order methods in the new pair PFAFRKN5(3), confirming that the algebraic orders of convergence of the underlying pair are maintained. In addition, the stability intervals for both the higher and lower order methods have been obtained. The numerical results obtained clearly show that PFAFRKN5(3) is more accurate and efficient than other 5(3) RKN pairs in the literature.

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