Fermions and Condensates on the Light-Front

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Abstract. Light-Front quantization is one of the most promising and physical tools towards studying deep inelastic scattering on the basis of quark gluon degrees of freedom. The simplified vacuum structure (nontrivial vacuum effects can only appear in zero-mode degrees of freedom) and the physical basis allows for a description of hadrons that stays close to intuition. I am reviewing recent progress in understanding the deep connection between renormalization of light-front Hamiltonians, effective light-front Hamiltonians and nontrivial vacuum condensates.

1 Advantages of Light-Front Coordinates

Deep inelastic lepton-nucleon scattering (DIS) provides access to quark and gluon degrees of freedom in nucleons and nuclei. In these experiments one shoots high energy leptons (e.g. electrons) at a hadronic target (usually protons or nuclei) and measures the energy and momentum transfer to the target by detecting the final state lepton (Fig. 1). In the most simple version of DIS, the hadronic final state $X$ is not measured (usually the nucleon is destroyed in these reactions and the hadronic final state consists of many particles). Because of the extremely large momentum transfer to the target (typical momentum transfers in DIS experiments are several GeV/c or more), the inclusive cross sections are dominated by single particle response functions along the light-cone. To illustrate this let us use the optical theorem which relates the differential lepton nucleon cross section to the imaginary part of the forward Compton amplitude [1] (Fig. 2). One finds

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{q^4} \left(\frac{E'}{E}\right) l_{\mu\nu} \Im T_{\mu\nu} \frac{2\pi}{2\pi}$$

where $E, E'$ are the energies of the initial and final lepton. Furthermore, $l_{\mu\nu} = 2k_\mu k'_\nu + 2k_\nu k'_\mu + q^2 g_{\mu\nu}$ is the leptonic tensor and $q = k - k'$ is the four momentum transfer of the lepton on the target. The hadronic tensor

$$T_{\mu\nu}(P, q) = \frac{i}{2M_N} \sum_S \int \frac{d^4x}{2\pi} e^{iqz} \langle P, S | T (J_{\mu}(x)J_{\nu}(0)) | P, S \rangle$$

1 Lecture Notes, based on three lectures given at the “School on Light-Front Quantization and Non-Perturbative Dynamics - Theory of Hadrons and Light-Front QCD”, IITAP, Ames, IA, May 1996.
Fig. 1. Inclusive process $e^- + N \rightarrow e'^- + X$, where $X$ is an unidentified hadronic state.

$(S$ is the spin of the target proton$)$ contains all the information about the parton substructure of the target proton.

In the Bjorken limit ($Q^2 \equiv -q^2 \rightarrow \infty$, $P \cdot q \rightarrow \infty$, $x_{Bj} = Q^2/2P \cdot q$ fixed), deep inelastic structure functions exhibit Bjorken scaling: up to kinematic coefficients, the hadronic tensor (2) depends only on $x_{Bj}$ but no longer on $Q^2$ (within perturbatively calculable logarithmic corrections). In order to understand this result, it is convenient to introduce light-front variables $a_\pm = (a^0 \pm a^3)/\sqrt{2}$ so that the scalar product reads $a \cdot b = a_+ b^+ + a_- b^- - a_1 b_1 = a_+ b_+ + a_- b_- - a_1 b_1$. Furthermore let us choose a frame where $q_\perp = 0$. The Bjorken limit corresponds to $p^0$ and $q_-$ fixed, while $q_+ \rightarrow \infty$. Bjorken scaling is equivalent to the statement that the structure functions become independent of $q_+$ in this limit (again up to trivial kinematic coefficients). In this limit, the integrand in Eq.(2) contains the rapidly oscillating factor $\exp(i q_+ x^+)$, which kills all contributions to the integral except those where the integrand is singular [2]. Due to causality, the integrand must vanish for $x^2 = 2x^+ x^- - x_\perp^2 < 0$ and the current product is singular at $x^+ = 0, x_\perp = 0$. The leading singularity can be obtained from the operator product expansion by contracting two fermion operators in the product $T(J_\mu(x)J_\nu(0)) \equiv T(\bar{\psi}(x)\gamma_\mu(\mathcal{D}_\mu\psi(0))\gamma_\nu(\mathcal{D}_\nu\psi(0)))$, yielding a nonlocal term bilinear in the fermion field multiplying a free (asymptotic freedom!) fermion propagator from 0 to $x$ which gives rise to the abovementioned singularity structure [3]. The $x^+ = x_\perp = 0$ dominance in the integral has two consequences. First it explains Bjorken scaling, because $q_+$ enters the hadronic tensor only via the term $x^+ q_+$ in the exponent and for $x^+ = 0$ the $q_+$ dependence drops out. Second, and this is very important for practical calculations,
\[
\sum_x \sigma (e p \rightarrow e' X) = \Im \nonumber
\]

\[
= \Im \nonumber
\]

\[
+ \Im \nonumber
\]

The physical origin of this result can be understood as follows. Consider again the virtual forward Compton amplitude (Fig. 2). In principle, the photons in the first and second interaction in Fig. 2 can couple to the same as well as to different quarks in the target. However, the hadronic wavefunction can only absorb momenta which are of the order of the QCD-scale \(\Lambda_{QCD} \approx 200 \text{ MeV}\). Therefore, in the limit of large momentum transfer, only such diagrams survive where the two photons in Fig. 2 couple to the same quark. All other diagrams have large momenta flowing through the wavefunction or they involve extra hard gluon exchanges which results in their suppression at large \(Q^2\). The large momentum transfer is also important because of asymptotic freedom. Since \(\alpha_s(Q^2) \sim 1/\log (Q^2/\Lambda_{QCD}^2)\), the running coupling constant of QCD, goes to zero for large \(Q^2\), all interactions of the struck quark can be neglected and it propagates essentially without interaction between the two photon-vertices. Furthermore, since the momentum transfer is much larger than the masses of the quarks in the target, the struck quark’s propagation between becomes ultra-relativistic, i.e. it moves
exceedingly close to the light cone $x^2 = 0$. Due to the high-energy nature of the scattering, the relativistic structure function is a LF correlation [4], [5]. Already at this point it should be clear that LF-coordinates play a distinguished role in the analysis of DIS experiments — a point which will become much more obvious after we have introduced some of the formal ideas of LF quantization.

LF quantization is very similar to canonical equal time (ET) quantization [6] (here we closely follow Ref. [7]). Both are Hamiltonian formulations of field theory, where one specifies the fields on a particular initial surface. The evolution of the fields off the initial surface is determined by the Lagrangian equations of motion. The main difference is the choice of the initial surface, $x^0 = 0$ for ET and $x^+ = 0$ for the LF respectively. In both frameworks states are expanded in terms of fields (and their derivatives) on this surface. Therefore, the same physical state may have very different wavefunctions$^2$ in the ET and LF approaches because fields at $x^0 = 0$ provide a different basis for expanding a state than fields at $x^+ = 0$. The reason is that the microscopic degrees of freedom — field amplitudes at $x^0 = 0$ versus field amplitudes at $x^+ = 0$ — are in general quite different from each other in the two formalisms.

This has important consequences for the practical calculation of parton distributions (3) which are real time response functions in the equal time formalism. $^3$ In order to evaluate Eq.(3) one needs to know not only the ground state wavefunction of the target, but also matrix elements to excited states. In contrast, in the framework of LF quantization, parton distributions are correlation functions at equal LF-time $x^+$, i.e. within the initial surface $x^+ = 0$ and can thus be expressed directly in terms of ground state wavefunctions (As a reminder: ET wavefunctions and LF wavefunctions are in general different objects). In the LF framework, parton distributions $f(x_{Bj})$ can be easily calculated and have a very simple physical interpretation as single particle momentum densities, where $x_{Bj}$ measures the fraction of the hadron’s momentum that is carried by the parton $^4$

$$x_{Bj} = \frac{p^\text{parton}_{-}}{P^\text{hadron}_{-}}. \quad (4)$$

Although DIS is probably the most prominent example for practical applications of LF coordinates, they prove useful in many other places as well.

$^2$ By “wavefunction” we mean here the collection of all Fock space amplitudes.

$^3$ The arguments of $\tilde{\psi}$ and $\psi$ in Eq.(3) have different time components!

$^4$ In DIS with non-relativistic kinematics (e.g. thermal neutron scattering off liquid $^4$He) one also observes scaling and the structure functions can be expressed in terms of single particle response functions. However, due to the different kinematics, non-relativistic structure functions at large momentum transfer are dominated by Fourier transforms of equal time response functions, i.e. ordinary momentum distributions.
For example, LF coordinates have been used in the context current algebra sum rules in particle physics [8]. Another prominent example is form factors, where moments of the wave function along the LF determine the asymptotic falloff at large momentum transfer [9]. More recently, LF quantization found applications in inclusive decays of heavy quarks [10], [11], [12].

From the purely theoretical point of view, various advantages of LF quantization derive from properties of the ten generators of the Poincaré group (translations $P^\mu$, rotations $L$ and boosts $K$) [6], [7]. Those generators which leave the initial surface invariant ($P$ and $L$ for ET and $P_-, P_\perp, L_3$ and $K$ for LF) are “simple” in the sense that they have very simple representations in terms of the fields (typically just sums of single particle operators). The other generators, which include the “Hamiltonians” ($P_0$, which is conjugate to $x^0$ in ET and $P_+$, which is conjugate to the LF-time $x^+$ in LF quantization) contain interactions among the fields and are typically very complicated. Generators which leave the initial surface invariant are also called kinematic generators, while the others are called dynamic generators. Obviously it is advantageous to have as many of the ten generators kinematic as possible. There are seven kinematic generators on the LF but only six in ET quantization.

The fact that $P_-$, the generator of $x^-$ translations, is kinematic (obviously it leaves $x^+ = 0$ invariant!) and positive has striking consequences for the LF vacuum[7]. For free fields $p^2 = m^2$ implies for the LF energy $p_+ = (m^2 + p_\perp) / 2p_-$. Hence positive energy excitations have positive $p_-$. After the usual reinterpretation of the negative energy states this implies that $p_-$ for a single particle is positive (which makes sense, considering that $p_- = (p_0 - p_3) / \sqrt{2}$). $P_-$ being kinematic means that it is given by the sum of single particle $p_-$. Combined with the positivity of $p_-$ this implies that the Fock vacuum (no particle excitations) is the unique state with $P_- = 0$. All other states have positive $P_-$. Hence, even in the presence of interactions, the LF Fock vacuum does not mix with any other state and is therefore an exact eigenstate of the LF Hamiltonian $P_+$ (which commutes with $P_-$). If one further assumes parity invariance of the ground state this implies that the Fock vacuum must be the exact ground state of the fully interacting LF quantum field theory. In sharp contrast to other formulations of field theory, the LF vacuum is trivial! This implies a tremendous technical advantage but also raises the question whether non-perturbative LF-field theory is equivalent to conventional field theory, where non-perturbative effects usually result in a highly nontrivial vacuum structure. This very deep issue will be the main topic of these lecture notes.

Practical calculations show that typical LF Hamiltonians are either unbounded from below or their ground state is indeed the Fock vacuum.
2 A First Look at the Light-Front Vacuum

In the Fock space expansion one starts from the vacuum as the ground state and constructs physical hadrons by successive application of creation operators. In an interacting theory the vacuum is in general an extremely complicated state and not known a priori. Thus, in general, a Fock space expansion is not practical because one does not know the physical vacuum (i.e. the ground state of the Hamiltonian). In normal coordinates, particularly in the Hamiltonian formulation, this is a serious obstacle for numerical calculations. As is illustrated in Table 1, the LF formulation provides a dramatic simplification at this point. While all components of the momentum in normal

| normal coordinates | light-front |
|--------------------|-------------|
| free theory | |
| \[ P^0 = \sqrt{m^2 + \mathbf{P}^2} \] | \[ P_+ = \frac{m^2 + \mathbf{P}^2}{2P^0} \] |
| \[ P_z \] | \[ P_- \] |
| vacuum (free theory) | |
| \[ P^0 = \sum_k a_k^\dagger a_k \sqrt{m^2 + \mathbf{k}^2} \] | \[ P_+ = \sum_{k_- k_\perp} a_{k_- k_\perp}^\dagger a_{k_- k_\perp} \frac{m^2 + \mathbf{k}_\perp^2}{2k_-} \] |
| \[ a_k |0\rangle = 0 \] | \[ a_{k_- k_\perp} |0\rangle = 0 \] |
| vacuum (interacting theory) | |
| many states with \( \mathbf{P} = 0 \) (e.g. \( a_k^\dagger a_k |0\rangle \)) \[ \leftrightarrow |0\rangle \text{ very complex} \] | \[ k_- \geq 0 \] \[ \leftrightarrow \text{only pure zero-mode} \] \[ \text{excitations have } P_- = 0 \] |

**Table 1. Zero Modes and the Vacuum**
coordinates can be positive as well as negative, the longitudinal LF momentum $P_-$ is always positive. In free field theory (in normal coordinates as well as on the LF) the vacuum is the state which is annihilated by all annihilation operators $a_k$. In general, in an interacting theory, excited states (excited with respect to the free Hamiltonian) mix with the trivial vacuum (i.e. the free field theory vacuum) state resulting in a complicated physical vacuum. Of course, there are certain selection rules and only states with the same quantum numbers as the trivial vacuum can mix with this state: for example, states with the same momentum as the free vacuum ($P = 0$ in normal coordinates, $P_- = 0$, $P_\perp = 0$ on the LF). In normal coordinates this has no deep consequences because there are many excited states which have zero momentum. On the LF the situation is completely different. Except for pure zero-mode excitations, i.e. states where only the zero-mode (the mode with $k_- = 0$) is excited, all excited states have positive longitudinal momentum $P_-$. Thus only these pure zero-mode excitations can mix with the trivial LF vacuum. Thus with the exception of the zero-modes the physical LF vacuum (i.e. the ground state) of an interacting field theory must be trivial (the only exceptions are pathological cases, where the LF Hamiltonian is unbounded from below).

Of course, this cannot mean that the vacuum is entirely trivial. Otherwise it seems impossible to describe many interesting problems which are related to spontaneous symmetry breaking within the LF formalism. For example one knows that chiral symmetry is spontaneously broken in QCD and that this is responsible for the relatively small mass of the pions — which play an important role in strong interaction phenomena at low energies. What it means is that one has reduced the problem of finding the LF vacuum to the problem of understanding the dynamics of these zero-modes.

First this sounds just like merely shifting the problem about the structure of the vacuum from nonzero-modes to zero-modes. However, as the free dispersion relation on the LF,

$$k_+ = \frac{m^2 + k_\perp^2}{2k_-},$$

(5)

indicates, zero-modes are high energy modes! Hence it should, at least in principle, be possible to eliminate these zero-modes systematically giving rise to an effective LF field theory [14].

3 Light-Front Hamiltonians without Zero-Modes

In the following Sections I will discuss several models that were solved using LF quantization. The solutions to these models have been obtained by unashamedly omitting explicit zero-mode degrees of freedom. Nevertheless, physical spectra and condensates (obtained using current algebra techniques) agree with results obtained using conventional (non-LF) frameworks. The models are ordered with increasing complexity.
3.1 The ’t Hooft Model

From the discussion in the previous section, it first seems that LF Hamiltonians that do not include zero-mode degrees of freedom simply give wrong results. A first indication that this is not necessarily the case was found in the ’t Hooft model: QCD\(_{1+1}(N_c \to \infty)\) [15]. Despite being 1+1 dimensional, this model has a nonzero fermion condensate in the chiral limit, since \(N_c \to \infty\). It is thus interesting to ask what LF quantization predicts for this case.

The original solution presented by ’t Hooft [15] was obtained in the LF formulation (quantization & gauge) did not involve any zero modes. Furthermore, only a simple principal value prescription was used to regulate the \(q_- = 0\) singularity in the gluon propagator. Nevertheless, the spectrum obtained by ’t Hooft agreed with the spectrum that was obtained later in an ET approach [16]. This is even more surprising if one considers that the vacuum state in the ET calculation had to be determined by solving coupled, nonlinear integral equations, whereas the LF vacuum is just empty space. Already at this point it was clear that the LF calculation cannot be complete nonsense.

A direct evaluation of the quark condensate in the ET case, gave a nonzero result in the chiral limit. Since the LF vacuum is the Fock vacuum, a direct evaluation gave of course zero on the LF. However, application of current algebra techniques to meson masses and coupling constants determined by solving the (zero-mode free) LF equations gives a nonzero result for the condensate — even in the zero quark mass limit:

\[
0 = \lim_{q \to 0} \frac{i q^\mu}{q^2} \int d^2 x e^{iqx} \langle 0| T \left( \bar{\psi} \gamma_\mu \gamma_5 \psi(x) \bar{\psi} i \gamma_5 \psi(0) \right) |0\rangle = -\langle 0| \bar{\psi} \psi|0\rangle - 2m_q \int d^2 x \langle 0| T \left[ \bar{\psi} i \gamma_5 \psi(x) \bar{\psi} i \gamma_5 \psi(0) \right] |0\rangle.
\]  

(6)

Upon inserting a complete set of meson states\(^6\) one thus obtains

\[
\langle 0| \bar{\psi} \psi|0\rangle = -m_q \sum_n \frac{f_P^n(n)}{M_n^2},
\]  

(7)

where

\[
f_P(n) \equiv \langle 0| \bar{\psi} i \gamma_5 \psi|n\rangle = \sqrt{\frac{N_C}{\pi}} \frac{m_q}{2} \int_0^1 dx \frac{1}{x(1-x)} \phi_n(x)
\]  

(8)

and the wave functions \(\phi_n\) and invariant masses \(M_n^2\) are obtained from solving ’t Hooft’s bound state equation for mesons in QCD\(_{1+1}\)

\[
M_n^2 \phi_n(x) = \frac{m_q^2}{x(1-x)} \phi_n(x) + G^2 \int_0^1 dy \frac{\phi_n(x) - \phi_n(y)}{(x-y)^2}.
\]  

(9)

\(^6\) Because we are working at leading order in \(1/N_C\), the sum over one meson states saturates the operator product in Eq.(6).
The result for \( \langle 0 | \bar{\psi} \psi | 0 \rangle \) obtained this way agrees with the ET calculation [17].

\[
m_q = \frac{m}{G/2} \frac{h}{J/0} / \frac{h}{J/0}
\]

Fig. 3. Chiral condensate obtained by evaluating Eq. (7) as a function of the quark mass. For nonzero quark mass, the (infinite) free part has been subtracted. The result agrees for all quark mass with the calculation done using equal time quantization.

This seemingly paradoxical result (peaceful coexistence of a Fock vacuum and a nonzero fermion condensate) can be understood by defining LF quantization through a limiting procedure [14], where the quantization surface is kept space-like, but being carefully “rotated” to the LF. Not all physical quantities behave continuously under this procedure as the LF is approached. For example, the chiral condensate \( \langle 0 | \bar{\psi} \psi | 0 \rangle \) has a discontinuous LF limit. On the other hand, the equation of motion for mesons in QCD_{1+1} does have a smooth LF limit. This result explains why the current algebra relation gives the right result for the condensate, even though \( \langle 0 | \bar{\psi} \psi | 0 \rangle \) vanishes when evaluated directly on the LF: Since the bound state equation for mesons has a smooth LF limit, both meson masses and coupling constant can be evaluated directly on the LF. Since the current algebra relation (6) is a frame independent relation, it can then be used to extract the condensate from the LF calculation. However, since \( \langle 0 | \bar{\psi} \psi | 0 \rangle \) has a discontinuous LF limit, it would be misleading to draw conclusions about the vacuum structure from its value obtained directly on the LF.

Another lesson that one should learn from this exercise is that one should

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7. It should be emphasized that the LF calculation preceded the ET calculation.

8. Another way of thinking about this procedure is to imagine a gradual boost to infinite momentum.
always be careful when defining operators and observables on the LF. Unless
proven otherwise, one should always be prepared that nontrivial renormal-
izations occur and that the canonical expressions are no longer valid.

Finally, a warning should be issued at this point: in the ’t Hooft model, one
obtains the correct spectra and (after some detours) even the correct chiral
condensate “for free”, i.e. by using the naive (canonical) LF Hamiltonian. It
turns out that this is an exception rather than the rule, i.e. in most theories
the canonical LF Hamiltonian yields incorrect results. This point will be
explored in the following sections.

3.2 Self-Interacting Scalar Fields

In the ’t Hooft model discussed above, the naive (canonical) LF calculation
automatically gave the correct meson spectrum. It turns out that this is
an exception rather than the rule! In most theories one obtains different
spectra when one diagonalizes the canonical ET Hamiltonian and canonical
LF Hamiltonian. In fact, one does not have to look very hard to find such
examples — disagreement between LF and ET calculations arise already with
self-interacting scalar fields, described by the generic Lagrangian

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \]  

where

\[ V(\phi) = \sum_{n>1} \frac{\lambda_n}{n!} \phi^n. \]  

The main difference between the LF formulation and the ET formulation is
that generalized tadpoles (a typical example is shown in Fig.4), i.e. Feynman
diagrams where one piece of the diagram is connected to the rest of the
diagram only at one point, cannot be generated by a LF Hamiltonian: in
time ordered perturbation theory, at least one of the vertices in a generalized
tadpole diagram has all lines coming out of or disappearing into the vacu-
um (Fig.5) — which is forbidden on the LF (without zero-modes).

So the bad news is that all generalized tadpole diagrams are zero on the
LF and they are nonzero in ET quantization, i.e. there is a difference between
the perturbation series generated by the two formulations [18], [19].

However, there are two good news. The first good news is that in self-
interacting scalar theories, it is only in generalized tadpole diagrams where
such a difference occurs. The second good news is that all tadpole sub-
diagrams are just constants, i.e. they do not depend on the momenta of
any external legs. The reason is that there is no momentum flowing through
them. Since generalized tadpoles are only constants, their absence can be
compensated for by local counter-terms in the interaction.

\[ \text{To my knowledge, there is no strict proof of this result, but it is based on hand-}
\]  

\[ \text{waving arguments as well as on a thorough three loop analysis.} \]
Fig. 4. Generalized tadpole (Feynman-) diagram in $\phi^4$ theory.

Fig. 5. Same as Fig. 4 but as LF-time ordered diagrams. At least one of the vertices has all lines popping out of or disappearing into the vacuum.

In other words, the difference between LF quantization and ET quantization arises only if one compares calculations done with the same bare parameters! Suppose one would start with parameters that have been chosen so that the bare parameters on the LF include already the counter-terms that are necessary to compensate for the absence of tadpoles. Then ET and LF formulation should give the same results for all n-point Green’s functions, i.e. physical observables should be the same. But how can one find the appropriate counter-terms without having to refer to an ET calculation? There is nothing easier than that: simply by using only physical input parameters to fix the bare parameters! For example, if one matches the physical masses of a few particles between an ET calculation and a LF calculation then the bare parameters that one needs in order to get these masses will be different in ET and LF quantization. The difference will be just such that it compensates for
the absence of tadpoles on the LF and hence all further observables will be the same. In other words there is no problem at all with the LF formulation.

Beyond this happy ending, there is one more very interesting aspect to this story, which has to do with vacuum condensates. For example, every generalized tadpole diagram in $\phi^4$ theory is numerically equal to a diagram that contributes to $\langle 0 | \phi^2 | 0 \rangle$. For example, the tadpole in Fig. 4 is proportional to a term that contributes to $\langle 0 | \phi^2 | 0 \rangle$ to second order in $\lambda$. In fact, after working out the details one finds that the additional LF counter-term, necessary to obtain equivalence between ET and LF quantization is a mass counter-term \[ \Delta m^2 = \lambda \langle 0 | \sqrt{\phi}^2 | 0 \rangle, \] (12)

where $\lambda$ is the four point coupling and the vacuum expectation value (VEV) on the r.h.s. is to be evaluated in normal coordinates.

This result can be readily generalized to an arbitrary polynomial interaction. One finds the following dictionary: perturbation theory based on a canonical equal time Hamiltonian with

\[ L_{\text{int}}^{ET} = \sum_n \lambda_n \langle 0 | \phi^n | 0 \rangle, \] (13)

and perturbation theory based on a canonical light-front Hamiltonian with

\[ L_{\text{int}}^{LF} = \sum_n \tilde{\lambda}_n \langle 0 | \phi^n | 0 \rangle, \] (14)

are equivalent if

\[ \tilde{\lambda}_n = \sum_{k \leq n} \lambda_{n-k} \langle 0 | \phi^k | 0 \rangle. \] (15)

In Ref.[19] this fundamental result was derived perturbatively and the prescription for constructing the effective Hamiltonian was tested non-perturbatively by calculating physical masses of "mesons" and solitons in the sine-Gordon model.

At this point it is very tempting to conjecture that this dictionary (15) also holds for non-perturbative condensates (such as condensates which arise after spontaneous symmetry breaking). While a general proof is still missing, it has indeed been possible to demonstrate for a few specific models that the conjecture is correct [20].

It should also be emphasized that these equivalence considerations hold irrespective of the number of space-time dimensions, i.e. they apply to $1+1$ as well as $2+1$ and $3+1$ dimensional theories. One must only be careful to use commensurate cutoffs when comparing ET and LF quantized theories. An example would be a transverse lattice cutoff, which can be employed both in ET quantization as well as in LF quantization.
Fig. 6. Generalized tadpole diagrams for scalar field theories with higher polynomial interactions. Both are set to zero in LF quantization without zero-modes. Both are proportional to $\langle 0|\phi^4|0 \rangle$. The diagram in a.) gives rise to a mass renormalization counter-term and b.) renormalizes the four-point interaction.

What makes all these results particularly interesting is that they show how non-perturbative effects can "sneak" into the LF formalism and how one can resolve the apparent conflict between trivial LF vacua and nontrivial vacuum effects.

### 3.3 Fermions with Yukawa Interactions

Eventually, we are interested to understand chiral symmetry breaking in QCD, i.e. we need to understand fermions. As a first step in this direction let us consider a Yukawa model in 1+1 dimensions

$$L = \bar{\psi} (i \partial - m_F - g \phi \gamma_5) \psi - \phi (\Box + m_B^2) \phi. \quad (16)$$

The main difference between scalar and Dirac fields in the LF formulation is that not all components of the Dirac field are dynamical: multiplying the Dirac equation

$$i \partial \psi^\pm = (m_F + g \phi \gamma_5) \gamma^\pm \psi^\mp, \quad (18)$$

by $\frac{1}{2} \gamma^\pm$ yields a constraint equation (i.e. an "equation of motion" without a time derivative)

$$i \partial_\psi^\pm = (m_F + g \phi \gamma_5) \gamma^\mp \psi^\pm, \quad (18)$$

where

$$\psi^\pm \equiv \frac{1}{2} \gamma^\mp \gamma^\pm \psi. \quad (19)$$

For the quantization procedure, it is convenient to eliminate $\psi^\mp$ from the classical Lagrangian before imposing quantization conditions, yielding

$$L = \sqrt{2} \psi^\dagger_\psi^\dagger \partial^\dagger \psi^\dagger - \phi (\Box + m_B^2) \phi - \psi^\dagger \frac{m_F}{\sqrt{2}i \partial_\psi^\mp} \psi^\mp \quad (20)$$

$$- \psi^\dagger_\psi^\dagger \left( g \phi \frac{m_F \gamma_5}{\sqrt{2}i \partial_\psi^\mp} + \frac{m_F \gamma_5}{\sqrt{2}i \partial_\psi^\pm} g \phi \right) \psi^\mp - \psi^\dagger_\psi^\dagger g \phi \frac{1}{\sqrt{2}i \partial_\psi^\mp} g \phi \psi^\mp.$$
The rest of the quantization procedure very much resembles the procedure for self-interacting scalar fields. In particular, we must be careful about generalized tadpoles, which might cause additional counter-terms in the LF Hamiltonian. In the Yukawa model one usually (i.e. in a covariant formulation) does not think about tadpoles. However, after eliminating $\psi(-)$, we are left with a four-point interaction in the Lagrangian, which does give rise to time-ordered diagrams that resemble tadpole diagrams (Fig.7). In fact, the four-point interaction gives rise to diagrams where a fermion emits a boson, which may or may not self-interact, and then re-absorb the boson at the same LF-time.\footnote{There are also tadpoles, where the fermions get contracted. But those only give rise to an additional boson mass counter-term, but not to a non-covariant counter-term that we investigate here.}

As we discussed in detail in the previous section, such interactions cannot be generated by a LF Hamiltonian, i.e. the LF formalism defines such tadpoles to be zero.

For self-interacting scalar fields, the difference between ET and LF perturbation theory which thus results can be compensated by a redefinition of parameters that appear already in the Lagrangian. In the Yukawa model, the situation is a little more complicated. The missing tadpoles have the same operator/Lorentz structure as the so called kinetic mass term

$$P_{\text{kin}} = \psi_+^\dagger \frac{m^2}{\sqrt{2i\partial_-}} \psi_+.$$  \hfill (21)

One obtains this result by contracting the two scalar fields in the four-point interaction. More details can be found in Ref. [21]. The important point here is that there is no similar counter-term for the term linear in the fermion
mass \(m_F\). Thus the difference between ET and LF quantization cannot be compensated by tuning the bare masses differently. The correct procedure requires to renormalize the kinetic mass term (the term \(\propto m_F^2\)) and the vertex mass term (the term \(\propto m\)) independent from each other. More explicitly this means that one should make an ansatz for the renormalized LF Hamiltonian density of the form

\[
P^{-} = \frac{m_B^2}{2} \phi^2 + \psi_{(+)}^\dagger \frac{c_2}{\sqrt{2i\partial_-}} \psi_{(+)} + c_3 \psi_{(+)}^\dagger \left(\phi \frac{\gamma_5}{\sqrt{2i\partial_-}} + \frac{\gamma_5}{\sqrt{2i\partial_-}} g \phi \right) \psi_{(+)} + c_4 \psi_{(+)}^\dagger \frac{1}{\sqrt{2i\partial_-}} \phi \psi_{(+)},
\]

where the \(c_i\) do not necessarily satisfy the canonical relation \(c_2^3 = c_2 c_4\). However, this does not mean that the \(c_i\) are completely independent from each other. In fact, only for specific combinations of \(c_i\) will Eq.(22) describe the Yukawa model. It is only that we do not know the relation between the \(c_i\).

Thus the bad news is that the number of parameters in the LF Hamiltonian has increased by one (compared to the Lagrangian). The good news is that a wrong combination of \(c_i\) will in general give rise to a parity violating theory. This is good news because one can thus use parity invariance for physical observables as an additional renormalization condition to determine the additional “free” parameter.

In fact, in Ref. [22], it was shown that utilizing parity constraints as renormalization conditions is practical. The observable considered in that work was the vector transition form factor (in a scalar Yukawa theory in 1+1 dimensions) between physical meson states of opposite C-parity (and thus supposedly opposite parity)

\[
\langle p', n | j^\mu | p, m \rangle = \varepsilon^\mu_\nu q_\nu F_{mn}(q^2),
\]

where \(q = p' - p\). When writing the r.h.s. in terms of one invariant form factor, use was made of both vector current conservation and parity invariance. A term proportional to \(p^\mu + p'^\mu\) would also satisfy current conservation, but has the wrong parity. A term proportional to \(\varepsilon^{\mu\nu} p_\nu + p'_\nu\) has the right parity, but is not conserved and a term proportional to \(q^\mu\) is both not conserved and violates parity. Other vectors do not exist for this example. The Lorentz structure in Eq. (23) has nontrivial implications even if we consider only the

\[\text{As an example, consider the free theory, where the correct relation (}c_3^3 = c_2 c_4\) follows from a covariant Lagrangian. Any deviation from this relation can be described on the level of the Lagrangian (for free massive fields, equivalence between LF and covariant formulation is not an issue) by addition of a term of the form \(\delta L = \bar{\psi} \frac{\gamma_5}{\partial^-} \psi\), which is obviously parity violating, since parity transformations result in \(A^\pm \to A^\mp\) for Lorentz vectors \(A^\mu\); i.e. \(\delta L \to \bar{\psi} \frac{\gamma_5}{\partial^-} \psi \neq \delta L\).]
“good” component of the vector current \(^{12}\), yielding

\[
\frac{1}{q^2} \langle p', n | j^+ | p, m \rangle = F_{mn}(q^2). \tag{24}
\]

That this equation implies nontrivial constraints can be seen as follows: as a function of the longitudinal momentum transfer fraction \(x \equiv q^+/p^+\), the invariant momentum transfer reads (\(M^2_m\) and \(M^2_n\) are the invariant masses of the in and outgoing meson)

\[
q^2 = x \left( M^2_m - \frac{M^2_n}{1-x} \right) \tag{25}
\]

Typically, there are two values of \(x\) that lead to the same value of \(q^2\). It is highly nontrivial to obtain the same form factor for both values of \(x\). In Ref. [22], the coupling as well as the physical masses of both the fermion and the lightest boson where kept fixed, while the “vertex mass” was tuned (note that this required re-adjusting the bare kinetic masses). Figure 8 shows a typical example. In that example, the calculation of the form factor was repeated for three values of the DLCQ parameter \(K\) (24, 32 and 40) in order to make sure that numerical approximations did not introduce parity violating artifacts. For the “magic value” of the vertex mass one finds that the parity condition (24) is satisfied over the whole range of \(q^2\) considered. This provides a strong self-consistency check, since there is only one free parameter, but the parity condition is not just one condition but a condition for every single value of \(q^2\) (i.e. an infinite number of conditions). In other words, keeping the vertex mass independent from the kinetic mass is not only necessary, but also seems sufficient in order to properly renormalize Yukawa\(_{1+1}\).

\(^{12}\) In the context of LF calculations, currents that are bilinear in the dynamical component \(\psi_{(+)\rangle}\) are usually easiest to renormalize and calculate. Other combinations, such as \(\psi_{(+)\rangle}^\dagger \psi_{(--)}\) involve interactions when expressed in terms of the dynamical components and are thus terrible to handle — hence they are often referred to as “bad” components.
Fig. 8. Inelastic transition form factor (24) between the two lightest meson states of the Yukawa model, calculated for various vertex masses $m_v$ and for various DLCQ parameters $K$. The physical masses for the fermion and the scalar meson have been renormalized to the values $(m_F^{phys})^2 = (m_{\phi}^{phys})^2 = 4$. All masses and momenta are in units of $\sqrt{\lambda} = \sqrt{\epsilon_4/2\pi}$. In this example, only for $m_v^2 \approx 5$ one obtains a form factor that is a unique function of $Q^2$, i.e. only for $m_v^2 \approx 5$, the result is consistent with Eq. (24). Therefore, only for this particular value of the vertex mass, is the matrix element of the current operator consistent with both parity and current conservation.
3.4 A Model with Spontaneous Breakdown of Chiral Symmetry

The discussion in the previous section showed that the renormalization of fermions on the LF requires the introduction of non-covariant counter-terms. The question is: are a finite number of such counter-terms to the LF Hamiltonian sufficient to describe a physical situation with spontaneous breakdown of chiral symmetry? In order to investigate this question, I was looking for a nontrivial model where the mechanism for chiral symmetry breaking (χSB) is understood\(^{13}\) and which is also suitable for a LF formulation. In addition, the model should allow to work with the same cutoffs and approximations in the conventional formulation and the LF formulation, since this facilitates a comparison the two frameworks at each possible step in the calculations.

One model that has all the above features consists of fermions coupled to the transverse component of a non-Abelian vector field and which is described by the Lagrangian

\[\mathcal{L} = \bar{\psi}_k \left( i\sigma^\mu - m \right) \psi_k - \eta \sqrt{\frac{N}{N_C}} \bar{\psi}_k \gamma^\perp A_{kl}^\perp \psi_l - \frac{1}{2} A_{kl}^\perp \left( \Box + \lambda^2 \right) A_{kl}^\perp. \tag{26}\]

\(k, l\) can be interpreted as “color” indices and the \(N \to \infty\) limit is taken. Without going into details, the motivation for choosing such a bizarre model was the following: the coupling of the fermion to a vector field is chirally invariant; the longitudinal components of the vector field was omitted completely since vector couplings with \(A^\perp = 0\) are notoriously difficult on the LF; a non-Abelian not-self-interacting was chosen since this results in a solvable model in the \(N_C \to \infty\) limit.

Despite certain similarities, the above model is not a gauge theory and there is nothing wrong with imposing momentum cutoffs. A cutoff that can be used both in “normal coordinates” as well as on the LF is a transverse momentum cutoff (sharp momentum cutoff or transverse lattice or similar).

In normal coordinates, one can easily solve above model by solving the Dyson-Schwinger (DS) equations in the rainbow approximation (exact in the \(N_C \to \infty\) limit). What we will demonstrate in the following subsections is that a standard LF calculation\(^{14}\) with appropriate counter-terms yields the same spectrum as a conventional calculation. This result holds for all values of the quark mass\(^{15}\) which makes it particularly interesting since the model that we are considering exhibits spontaneous breakdown of chiral symmetry in the limit \(m \to 0\).

Since the details of this proof are rather lengthy and formal, readers not interested in details should immediately proceed to the Summary.

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\(^{13}\) By the way, this excludes \(QCD_{3+1}\) since \(\chiSB\) is not yet understood there — even in “normal” coordinates.

\(^{14}\) Such as DLCQ without zero-modes, but in the limit of large harmonic resolution.

\(^{15}\) Except strictly zero, but this point can easily be approached via a limiting procedure.
Schwinger-Dyson Solution: Because the above toy model lacks full covariance (there is no symmetry relating longitudinal and transverse coordinates) the full fermion propagator is of the form

\[ S_F(p^\mu) = \gamma_L S_L(p_L^2, p_L^3) + \hat{p}_\perp S_L(p_L^2, p_\perp^2) + S_0(p_L^2, p_\perp^2), \]

where \( k_L \equiv k_0 \gamma^0 + k_3 \gamma^3 \) and \( k_\perp \equiv k_1 \gamma^1 + k_2 \gamma^2 \). On very general grounds, it should always be possible to write down a spectral representation for \( S_F^{16} \)

\[ S_i(p_L^2, p_\perp^2) = \int_0^\infty dM^2 \frac{\rho(M^2, p_\perp^2)}{p_L^2 - M^2 + i\epsilon}, \]

where \( i = L, \perp, 0 \). Note that this spectral representation differs from what one usually writes down as a spectral representation in that we are not assuming full covariance here. Note that in a covariant theory, one usually writes down spectral representations in a different form, namely \( S = \int_0^\infty dM^2 \rho(M^2)/(p_\perp^2 - M^2) \), i.e. with \( p_\perp^2 \) in the denominator. This is a special case of Eq. (28) with \( \rho(M^2, p_\perp^2) = \int_0^\infty dM^2 \rho(M^2) \delta(M^2 - M^2 - p_\perp^2) \).

Using ansatz from above (28) for the spectral densities, one finds for the self-energy

\[ \Sigma(p^\mu) \equiv i g^2 \int \frac{d^4 k}{(2\pi)^4} \gamma_\perp S_F(p^\mu - k^\mu) \gamma_\perp \frac{1}{k^2 - \lambda^2 + i\epsilon} = \gamma_L \Sigma_L(p_L^2, p_L^3) + \Sigma_0(p_L^2, p_\perp^2), \]

where

\[ \Sigma_L(p_L^2, p_\perp^2) = g^2 \int_0^\infty dM^2 \int_0^1 dx \frac{d^2 k_\perp}{8\pi^3} \frac{(1 - x) \rho_L(M^2, (p - k)^2)}{D} \]

\[ \Sigma_0(p_L^2, p_\perp^2) = -g^2 \int_0^\infty dM^2 \int_0^1 dx \frac{d^2 k_\perp}{8\pi^3} \frac{\rho_0(M^2, (p - k)^2)}{D}. \]

and

\[ D = x(1 - x)p_L^2 - xM^2 - (1 - x)(k_L^2 + \lambda^2) \]

Note that \( \Sigma_\perp \) vanishes, since \( \sum_{i=1,2} \gamma_i \gamma_j \gamma_i = 0 \) for \( j = 1, 2 \). Self-consistency then requires that

\[ S_F = \frac{1}{\gamma_L \left[ 1 - \Sigma_L(p_L^2, p_\perp^2) \right] + \gamma_\perp - [m + \Sigma_0(p_L^2, p_\perp^2)]} \]

\[ ^{16} \text{What we need is that the Green’s functions are analytic except for poles and that the location of the poles are consistent with longitudinal boost invariance (which is manifest in our model). The fact that the model is not invariant under transformations which mix } p_L \text{ and } p_\perp \text{ does not prevent us from writing down a spectral representation for the dependence on } p_L. \]
In the above equations we have been sloppy about cutoffs in order to keep the equations simple, but this can be easily remedied by multiplying each integral by a cutoff on the fermion momentum, such as $\Theta(\Lambda^2 - (p - k)^2_\perp)$.

In principle, the set of equations [Eqs. (29),(30),(32)] can now be used to determine the spectrum of the model. But we are not going to do this here since we are more interested in the LF solution to the model. However, we would still like to point out that, for large enough $g$, one obtains a self-consistent numerical solution to the Euclidean version of the model which has a non-vanishing scalar piece — even for vanishing current quark mass $m$, i.e. chiral symmetry is spontaneously broken and a dynamical mass is generated for the fermion in this model.

**LF Solution:** A typical framework that people use when solving LF quantized field theories is discrete light-cone quantization (DLCQ) [23]. Since it is hard to take full advantage of the large $N_C$ limit in DLCQ, we prefer to use a Green’s function framework based on a 4 component formulation of the model. The Green’s function approach has the advantage that in the above toy model more things can be done analytically. In addition, it allows us to work in the continuum limit, where a comparison with the Schwinger-Dyson calculation can be done without further extrapolations. I should emphasize that we did verify that the Green’s function approach (with momentum integrals discretized) yielded the same physical masses as a DLCQ calculation [26].

In a LF formulation of the model, the fermion propagator (to distinguish the notation from the one above, we denote the fermion propagator by $G$ in this subsection) should be of the form

$$G(p^\mu) = \gamma^+ p^- G_+(2p^+ p^-, p^2_\perp) + \gamma^- p^+ G_-(2p^+ p^-, p^2_\perp) + k^\perp_\perp G_0(2p^+ p^-, p^2_\perp).$$  

(33)

Again we can write down spectral representations

$$G_i(2p^+ p^-, p^2_\perp) = \int_0^\infty dM^2 \frac{\rho_{\perp}^{LF}(M^2, p^2_\perp)}{2p^+ p^- - M^2 + i\varepsilon},$$  

(34)

where $i = +, -, \perp, 0$. This requires some explanation: On the LF, one might be tempted to allow for two terms in the spectral decomposition that are proportional to $\gamma^+$, namely

$$tr(\gamma^- G) \propto \int_0^\infty dM^2 \frac{p^- \rho_{\perp}(M^2, p^2_\perp)}{2p^+ p^- - M^2 + i\varepsilon}.$$  

(35)

However, upon writing

$$\frac{1}{p^+} = \frac{1}{p^+ M^2} (M^2 - 2p^+ p^-) + \frac{2p^-}{M^2}$$  

(36)

Note that in a LF formulation, $G_+$ and $G_-$ are not necessarily the same.
one can cast Eq. (35) into the form

\[ \text{tr}(\gamma^- G) \propto \int_0^\infty dM^2 p^- \frac{\rho_0(M^2, p^2_\perp)}{2p^+ p^- - M^2 + i\varepsilon} - \frac{1}{p^-} \int_0^\infty dM^2 \rho_0(M^2, p^2_\perp) \]

which is of the form in Eq.(34) plus an energy independent term. The presence of such an additional energy independent term would spoil the high energy behavior of the model [24]: In a LF Hamiltonian, not all coupling constants are arbitrary. In many examples, 3-point couplings and the 4-point couplings must be related to one another so that the high energy behavior of scattering via the 4-point interaction and via the iterated 3-point interaction cancel [24]. If one does not guarantee such a cancelation then the high-energy behavior of the LF formulation differs from the high-energy behavior in covariant field theory and in addition one often also gets a spectrum that is unbounded from below. In Eq. (37), the energy independent constant appears if the coupling constants of the "instantaneous fermion exchange" interaction in the LF Hamiltonian and the boson-fermion vertex are not properly balanced.

In the following we will assume that one has started with an ansatz for the LF Hamiltonian with the proper high-energy behavior, i.e. we will assume that there is no such energy independent piece in Eq. (37).

The LF analog of the self-energy equation is obtained by starting from an expression similar to Eq.(30) and integrating over \( k^- \). One obtains

\[ \Sigma^\text{LF} = \gamma^+ \Sigma_+^\text{LF} + \gamma^- \Sigma_-^\text{LF} + \Sigma_0^\text{LF}, \]

where

\[
\Sigma_+^\text{LF}(p) = g \int_0^\infty dM^2 \int_0^{p^+} dp^+ \int_0^{k^+} dk^+ \frac{d^2 k_\perp}{16\pi^2} \frac{p^- - \frac{\lambda^2 + k^2_\perp}{2k^+}}{k^+(p^+ - k^+)} \rho_+^\text{LF}(M^2, (p - k)^2_\perp) + \text{CT} \\
\Sigma_-^\text{LF}(p) = g \int_0^\infty dM^2 \int_0^{p^+} dp^+ \int_0^{k^+} dk^+ \frac{d^2 k_\perp}{16\pi^2} \frac{(p^+ - k^+)^2 \rho_-^\text{LF}(M^2, (p - k)^2_\perp)}{k^+(p^+ - k^+)} D^\text{LF} \\
\Sigma_0^\text{LF}(p) = -g^2 \int_0^\infty dM^2 \int_0^{p^+} dp^+ \int_0^{k^+} dk^+ \frac{d^2 k_\perp}{16\pi^2} \frac{\rho_0^\text{LF}(M^2, (p - k)^2_\perp)}{k^+(p^+ - k^+)} D^\text{LF}. \]

where

\[ D^\text{LF} = p^- - \frac{M^2}{2(p^+ - k^+)} - \frac{\lambda^2 + k^2_\perp}{2k^+} \]

and CT is an energy \( (p^-) \) independent counter-term. The determination of this counter-term, such that one obtains a complete equivalence with the Schwinger Dyson approach, is in fact the main achievement of this paper. First we want to make sure that the counter-term renders the self-energy finite. This can be achieved by performing a “zero-energy subtraction” with
a free propagator, analogous to adding self-induced inertias to a LF Hamiltonian, yielding

\[ CT = g^2 \int_0^{p^+} \frac{dk^+}{k^+} \int \frac{d^2 k_\perp}{16\pi^3} \frac{\lambda^2 + k^2}{2k^+} k^+ (p^+ - k^+) D^L_0 \frac{\Delta m_{ZM}^2}{2p^+} . \] (41)

where

\[ D^L_0 = - \frac{M_0^2 + (p - k)^2}{2(p^+ - k^+)} \frac{\lambda^2 + k^2}{2k^+} \] (42)

and where we denoted the finite piece by \( \Delta m_{ZM}^2 \) (for zero-mode), since we suspect that it arises from the dynamics of the zero-modes. \( M_0^2 \) is an arbitrary scale parameter. We will construct the finite piece \( \frac{\Delta m_{ZM}^2}{2} \) so that there is no dependence on \( M_0^2 \) left in CT in the end.

At this point, only the infinite part of CT is unique [24], since it is needed to cancel the infinity in the \( k^+ \) integral in Eq. (39), while the finite (w.r.t. the \( k^+ \) integral) piece (i.e. \( \Delta m_{ZM}^2 \)) seems arbitrary. \(^{18}\) Below we will show that it is not arbitrary and only a specific choice for \( \Delta m_{ZM}^2 \) leads to agreement between the SD and the LF approach.

Note that the equation for the self-energy can also be written in the form

\[ \Sigma^{LF}_+(p) = g^2 \int_0^{p^+} \frac{dk^+}{k^+} \int \frac{d^2 k_\perp}{8\pi^3} p_F^+ G_0 \left( 2p_F^+ p_F^-, p_{\perp F}^2 \right) + CT \]

\[ \Sigma^{LF}_-(p) = g^2 \int_0^{p^+} \frac{dk^+}{k^+} \int \frac{d^2 k_\perp}{8\pi^3} p_F^- G_0 \left( 2p_F^- p_F^+, p_{\perp F}^2 \right) \]

\[ \Sigma^L_0(p) = - g^2 \int_0^{p^+} \frac{dk^+}{k^+} \int \frac{d^2 k_\perp}{8\pi^3} G_0 \left( 2p_F^+ p_F^- p_{\perp F}^2 \right) , \] (43)

where

\[ p_F^+ \equiv p^+ - k^+ \]
\[ p_F^- \equiv p^- - \frac{\lambda^2 + k^2}{2k^+} \]
\[ p_{\perp F} \equiv p_{\perp} - k_{\perp} \] (44)

One can prove this by simply comparing expressions! Bypassing the use of the spectral function greatly simplifies the numerical determination of the Green’s function in a self-consistent procedure.

\(^{18}\) Note that what we called the “finite piece” w.r.t. the \( k^+ \) integral is still divergent when one integrates over \( d^2 k_\perp \) without a cutoff!
Comparing the LF and SD solutions: Motivated by considerations in Ref. [25], we make the following ansatz for \( Z_M \):

\[
\Delta m_{ZM}^2 = g^2 \int_0^\infty dM^2 \int \frac{d^2k_\perp}{8\pi^3} \rho_{L}^{LF} \left( M^2, \mathbf{p}_{\perp} \right) \ln \frac{M^2}{M_0^2 + \mathbf{p}_{\perp}^2}. \tag{45}
\]

The motivation for this particular ansatz becomes obvious once we rewrite the expression for \( \Sigma_+^{LF} \). For this purpose, we first note that

\[
\frac{p^- - \frac{x^2 + k^2}{2k}}{k^+(p^+ - k^+)D^{LF}_0} + \frac{\frac{x^2 + k^2}{2k}}{k^+(p^+ - k^+)D^{LF}_0} = \frac{p^- - \frac{x^2 - k^2}{2k}}{k^+(p^+ - k^+)D^{LF}_0} - \frac{1}{p^+} \frac{\partial}{\partial k^+} \ln \left[ \frac{D^{LF}}{D^{LF}_0} \right]. \tag{46}
\]

Together with the normalization condition

\[
\int_0^\infty dM^2 \rho^{LF}_+ (M^2, \mathbf{k}^2) = 1,
\]

this implies

\[
\Sigma_+^{LF}(p) = g^2 \int_0^\infty dM^2 \int \frac{d^2k_\perp}{8\pi^3} \rho_{L}^{LF} \left( M^2, \mathbf{p}_{\perp} \right) \ln \frac{M^2}{M_0^2 + \mathbf{p}_{\perp}^2} + \frac{\Delta m_{ZM}^2}{2p^+}, \tag{47}
\]

where we used our particular ansatz for \( \Delta m_{ZM}^2 \) [Eq. (45)]. Thus, with our particular choice for the finite piece of the kinetic energy counter term, the expressions for \( \Sigma_+^{LF} \) and \( \Sigma^L \) are almost the same — the only difference being the replacement of \( \rho_+^{LF} \) with \( \rho_+^L \) and an overall factor of \( \frac{p^-}{p^+} \). Furthermore, and this is the most important result of this paper, a direct comparison (take \( x = k^+/p^+ \)) shows that the same spectral densities that provide a self-consistent solution to the SD equations (30) also yield a self-consistent solution to the LF equations, provided one chooses

\[
\rho_+^{LF} (M^2, \mathbf{k}^2) = \rho_-^{LF} (M^2, \mathbf{k}^2), \quad \rho_0^{LF} (M^2, \mathbf{k}^2) = \rho_0 (M^2, \mathbf{k}^2). \tag{48}
\]

In particular, the physical masses of all states (in the sector with fermion number one) must be the same in the SD and the LF framework.

In the formal considerations above, we found it convenient to express \( \Delta m_{ZM}^2 \) in terms of the spectral density. However, this is not really necessary since one can express it directly in terms of the Green’s function

\[
\Delta m_{ZM}^2 = g^2 p^+ \int_0^\infty dp^- \frac{d^2p_\perp}{4\pi^3} \left[ G_+ (2p^+ p^- \cdot \mathbf{p}_{\perp}^2) - \frac{1}{2p^+ p^-} \frac{1}{\mathbf{p}_{\perp}^2 - M_0^2} \right]. \tag{49}
\]
Analogously, one can also perform a "zero-energy subtraction" in Eq. (43) with the full Green’s function, i.e. by choosing

\[ CT = -g^2 \int_0^{\infty} \frac{p^+dk^+}{k^+} \int \frac{d^2k_{\perp}}{8\pi^3} \bar\psi_F G_+ (2p^+_F \bar\psi_F, p^2_{\perp}) , \]

(50)

with \( \bar\psi_F = -\left(\lambda^2 + k_{\perp}^2\right)/2k^+ \). This expression turns out to be very useful when constructing the self-consistent Green’s function solution. We used both ansätze [Eqs. (49) and (50)] to determine the physical masses of the dressed fermion. In both cases, numerical agreement with the solution to the Euclidean SD equations was obtained.

Note that, in a canonical LF calculation (e.g. using DLCQ) one should avoid expressions involving \( G_+ \), since it is the propagator for the unphysical ("bad") component of the fermion field that gets eliminated by solving the constraint equation. However, since the model that we considered has an underlying Lagrangian which is parity invariant, one can use \( G_+ = G_- \) for the self-consistent solution and still use Eq. (49) or Eq. (50) but with \( G_+ \) replaced by \( G_- \). In doing so, we obviously used a method to determine \( \Delta m_{ZM}^2 \) that is not always applicable. However, the important point here is not getting the actual numerical value for \( \Delta m_{ZM}^2 \) in the above toy model, but to demonstrate explicitly that such a value exists which leads to non-perturbative equivalence between covariant and LF calculations.

**Conclusion:** There are several things one can learn from this simple toy model.

- Most importantly, even though the model exhibits spontaneous breakdown of chiral symmetry, a LF calculation without zero-modes still gives the same physics (i.e. spectral density) as a covariant calculation. Thus there is no conflict between trivial vacua on the LF and spontaneous breakdown of chiral symmetry.
- The equivalence between LF and covariant approach does not come for free. It is necessary to introduce an additional renormalization parameter — which could however be fixed by imposing conditions derived from parity invariance.
- As a byproduct, one can also infer from the above results that the vertex mass (which multiplies the only chirally odd term in the LF Hamiltonian) is to be identified with the current quark mass in the covariant approach. This result has been known from perturbative considerations, but the above example demonstrates its validity in a non-perturbative context that even includes spontaneous \( \chi_{SB} \) as \( m \to 0 \). This result may seem surprising at first since the chirally odd vertex mass term is also the only term which lifts the degeneracy of the \( \pi \) and the \( \rho \) (\( J_z = 0 \)). However, hadron wave functions typically have a rather singular end point behavior in the chiral limit so that matrix elements of the vertex mass term don’t necessarily have to vanish.
3.5 Demise of the Zero-Modes

Recently, there has been a considerable effort to include explicit zero-mode degrees of freedom into LF calculations in order to account for non-trivial vacuum effects (see for example Refs. [27], [28] and references therein). There are several comments that I would like to make about these zero-modes. First, I agree that vacuum effects must have to do with the \( k^+ \to 0 \) region.

However, the examples discussed here and in the references show that it is not always necessary to include explicit zero-mode degrees of freedom in order to obtain the right results: with a little extra effort to properly renormalize, we got away without any explicit zero-mode degrees of freedom at all.

Would the renormalization be simpler (i.e. less “independent” coefficients) if zero-mode degrees of freedom were included? In the examples discussed above the answer is no! Including explicit zero-modes in the above examples still requires an infinite kinetic mass counter-term that is not accompanied by an infinite vertex mass counter-term. This leaves the finite part of the kinetic mass counter-term ambiguous, i.e. there is the same number of renormalization constants. The root of this perhaps surprising result can be seen best by approaching LF coordinates using \( \varepsilon \) coordinates [14]. In this approach, LF quantization is defined via a limiting procedure — very much like a careful infinite momentum boost. If one performs this limiting transition on a finite interval of length \( L \) then the following pattern is observed: if one takes the limit \( L \to \infty \) first and the LF limit next then one gets complete equivalence with the covariant formulation — without any non-covariant counter-terms. However, when one takes the limits in opposite order (first go to the LF and then \( L \to \infty \)) one still needs to allow for independent renormalization of vertex mass and kinetic mass — even if the zero-mode is included. From a practical point-of-view nothing is thus gained by including the zero-mode. Including the zero-modes does not simplify the renormalization procedure.

Going back to the first approach (with the ordering: \( L \to \infty \) first, LF next) shows that any nontrivial zero-mode effects do NOT arise from one single zero-mode, but from an infinite number of modes in an infinitesimal vicinity of \( k^+ = 0 \). It is thus foolish to believe that inclusion of LF zero modes (finite intervals on the LF) will automatically and properly take care of the \( k^+ \to 0 \) physics. Constrained zero modes are not sufficient to account for all (or even most) aspects of vacuum structure [28].

Since I am criticizing the contemporary zero-mode approaches, I should also point out the limitations of the conclusions that one can draw from this work: all the zero modes that played a role in the examples above were so called constrained zero-modes [29]. The proper definition of this terminology can be found in Ref. [29], but what it means roughly speaking is zero modes that have to satisfy some complicated nonlinear constraint equation, but not a true dynamical equation of motion. I don’t understand yet what role dynamical zero-modes play on the LF. However, one result that emerges
from this work is that constrained zero-modes can probably be omitted completely. Proper renormalization takes care of them automatically.

Examples for fields that have constrained zero-modes are scalar fields, fermions and the transverse component of the gauge fields. One example for fields with dynamical zero-modes is the longitudinal component $A^+$ of a gauge field.

4 Summary

Light-Front Hamiltonians without zero-modes have a trivial vacuum. Nevertheless, and this is the most important message of these lectures, nontrivial vacuum structure and trivial LF vacua are not contradictions in terms.

Studies in $QCD_{1+1}(N_c \to \infty)$ showed that spectra and coupling constants obtained from a LF calculation based on a trivial LF vacuum still allow to calculate the correct value for the chiral condensate. Studies in scalar field theories $1+1$ and $3+1$ dimensions showed how condensates can appear as explicit parameters in the effective LF Hamiltonian. These examples were very helpful to understand why the apparent conflict between trivial LF vacua and spontaneous symmetry breaking is no conflict after all.

The main difference between chiral symmetry breaking and spontaneous symmetry breaking in scalar theories is that the order parameter in scalar fields has the same quantum numbers as the dynamical degrees of freedom, which facilitates incorporating dynamical symmetry breaking in the effective Hamiltonian. The calculations in a 3+1 dimensional toy model with $\chi$SB, which I presented in these lectures, were thus very important to confirm that proper renormalization of LF Hamiltonians leads to agreement with conventional calculations — even when chiral symmetry is spontaneously broken.

It should be emphasized that the successful LF quantization of the above 1+1 and 3+1 dimensional models has been accomplished without any explicit zero-mode degrees of freedom. Since the zero-modes degrees of freedom that would appear in a DLCQ analysis of these models would be so called constrained zero-modes, it seems at this point that at least the constrained zero-modes are unnecessary.

In all results that I have presented here, it was very important that the renormalization was properly done. In the models that I mentioned during this lecture, it seems that we understand now what this means in practice. However, even though there are now several 3+1 dimensional models where this aspect is understood, the big remaining challenge is still to construct an ansatz for the renormalized Hamiltonian for $\text{LF-QCD}_{3+1}$.

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The reason I included the word “probably” here is because I was able to show that they are unnecessary only by means of examples but not by means of a rigorous proof.
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