Optimal indecomposable witnesses without extremality or the spanning property

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Abstract

One of the interesting problems of optimal indecomposable entanglement witnesses is whether there exists an optimal indecomposable entanglement witness which neither has the spanning property nor is associated with an extremal positive linear map. Here, we answer this question in the negative by examining the extremality of the positive linear maps constructed by Qi and Hou (2011 J. Phys. A: Math. Theor. 44 215305).

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1. Introduction

A most general approach for distinguishing entanglement from separable states may be a criterion based on the notion of entanglement witnesses [1, 2]. A Hermitian operator $W$ acting on a complex Hilbert space $\mathcal{H} \otimes \mathcal{K}$ is called an entanglement witness (EW) if $W$ is not positive and $\text{Tr}(W \rho) \geq 0$ holds for all separable states $\rho$. Thus, if $W$ is an EW, then there exists an entangled state $\rho$ such that $\text{Tr}(W \rho) < 0$ (in this case, we say that $\rho$ is detected by $W$). It is well known [1] that a state is entangled if and only if it is detected by some entanglement witness.

For finite dimensional Hilbert spaces, this criterion is closely connected to the duality theory [3] between the cone of separable matrices and the cone of block positive matrices, which is in turn isomorphic to the cone of positive linear maps. Note that the Jamiołkowski–Choi isomorphism [4, 5] gives rise to an entanglement witness

$$W = \frac{1}{m} C_\Phi = \frac{1}{m} \sum_{i,j=1}^{m} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|),$$

for every positive linear map $\Phi : M_n \to M_n$ which is not completely positive, where $M_n$ denotes the $C^*$-algebra of all $n \times n$ matrices over the complex field $\mathbb{C}$ and the block matrix $C_\Phi$ is the Choi matrix of $\Phi$. We denote $W_\Phi = 1/m \, C_\Phi$ for the entanglement witness associated with the positive map $\Phi$. 
Important classes of positive linear maps from $M_m$ to $M_n$ come from elementary operators together with the transpose map:

$$\phi_V : X \rightarrow V^\dagger X V, \quad \phi^V : X \rightarrow V^\dagger X^t V,$$

where $V$ is an $m \times n$ matrix. The convex sums of the first (respectively second) types are said to be completely positive (respectively completely copositive) linear maps, and the convex sums of completely positive linear maps and completely copositive linear maps are said to be decomposable positive linear maps. If a positive linear map is not decomposable, we call it indecomposable. It is well known that decomposable positive linear maps give decomposable entanglement witnesses which take the general form $W = P + Q^\dagger$, where $P, Q \succeq 0$ and $Q^\dagger$ denotes the partial transpose of $Q$. If a given witness can not be written in this form, we call it indecomposable. Of course, indecomposable EWs are associated with indecomposable positive linear maps [6–8].

To characterize the set of EWs, the notion of optimality is important. An entanglement witness which detects a maximal set (in the sense of inclusion) of entangled states is said to be optimal (see [6] for a more precise formulation). Since every witness can be optimized [6], optimal EWs are sufficient to detect all the entangled states. So, it is significant to characterize the set of optimal EWs. Although there has been a considerable effort in this direction [7, 9–25], complete characterization and classification of optimal EWs are far from satisfactory.

In [6], it was shown that: (1) $W$ is an optimal EW if and only if $W - Q$ is no longer an EW for an arbitrary positive semidefinite (psd) matrix $Q$; (2) $W$ is an optimal EW if $W$ has a spanning property, that is $P_W = \{ |\xi, \eta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n : \langle \xi, \eta|W|\xi, \eta\rangle = 0 \}$ spans the whole space $\mathbb{C}^m \otimes \mathbb{C}^n$. From the criterion (1), we see that an EW associated to an extremal positive linear map is optimal. By an extremal positive linear map, we mean a positive linear map which generates an extremal ray of the convex cone consisting of all positive linear maps. That is, a positive linear map $\phi$ is said to be extremal if $\phi = \phi_1 + \phi_2$ with positive linear maps $\phi_i$, should imply $\phi_i = \lambda_i \phi$ with nonnegative real numbers $\lambda_i$. In the case of indecomposable EW, the Choi map [26, 27] and its variations [28–31, 20] are extremal and give rise to optimal EWs. Although the extremality of a positive linear map gives us a sufficient condition for the optimality of the associated EW, it is very difficult to check whether a positive linear map is extremal. On the other hand, the criterion (2) is a very practical way of checking the optimality of witnesses. In fact, almost all known optimal EWs are investigated with this criterion. (See [14, 18] and references therein.) However, the spanning property is also not a necessary condition for the optimality of an EW. In fact, the extremal Choi map [26, 27] introduces an optimal EW that has no spanning property. See [16] for examples of optimal decomposable EWs without the spanning property.

Recently, in order to examine the optimality of EWs without the spanning property, two kinds of method have been provided with examples of optimal indecomposable EWs which have no spanning property. Qi and Hou’s approach is based on the reinterpretation of optimal EWs in terms of a positive map [19, 32]. The first author and Kye [23] checked optimality by examining the facial structure of the convex body containing the positive linear map associated with the target EW. It remains to be shown whether the examples in [19, 23] are associated with extremal positive linear maps. To the best of the authors’ knowledge, only known examples of optimal indecomposable EWs without the spanning property are associated with positive linear maps which are variations of the Choi map. Then these positive linear maps turn out to be extremal, apart from the examples in [19, 23]. Therefore, it is natural to ask whether every optimal indecomposable EW without the spanning property is associated with an extremal positive linear map. The primary aim of this paper is to clarify this point.
For this purpose, we study the extremality of the indecomposable positive linear map \( \Phi^{(n,k)} \) constructed by Qi and Hou [32]. Then, we answer this question negatively by showing that \( \Phi^{(n,k)} \) is not extremal whenever \( n \) and \( k \) have common divisors greater than 1, that is, \( \gcd(n, k) > 1 \). Note that the optimality of associated entanglement witness \( W_{\Phi^{(n,k)}} \) with no spanning property is already known [19]. It was also observed [25] that \( W_{\Phi^{(n,k)}} \) is a PPTES entanglement witness [25] (that is, nd-OEW in the sense of [6]) since \( W_{\Phi^{(n,k)}} \) has the spanning property. See [22] for PPTES entanglement witnesses. Consequently, \( W_{\Phi^{(n,k)}} \) (with \( \gcd(n, k) > 1 \)) becomes the first example of optimal indecomposable EW, which neither has the spanning property nor is associated with an extremal positive linear map. For the case of \( \gcd(n, k) = 1 \), we try to show that \( \Phi^{(n,k)} \) is extremal. First, we show that \( \Phi^{(n,k)} \) is extremal if and only if \( \Phi^{(n,1)} \) is extremal when \( \gcd(n, k) = 1 \). Then we show that \( \Phi^{(4,1)} \) and so \( \Phi^{(4,3)} \) are indeed extremal. For general \( n \), we think that the extremality of \( \Phi^{(n,1)} \) can be dealt with similarly. Our approach to tackle extremality is based on Choi and Lam’s method [26, 27, 29, 33] using the correspondence between psd biquadratic forms and positive linear maps. Through the decomposition of a biquadratic form corresponding to \( \Phi^{(n,n/2)} \), we also reprove that \( \Phi^{(n,n/2)} \) is decomposable when \( n \) is an even integer greater than 2.

In the next section, we recall the positive linear maps \( \Phi^{(n,k)} \) and explain how to check the extremality of those maps according to Choi and Lam’s method [26, 27, 29, 33]. After we explore some extremal psd forms in section 3, we analyze the extremality of \( \Phi^{(n,k)} \) in the last section.

Throughout this note, \( \sigma_k : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \) denotes the permutation defined by \( \sigma_k(i) = i + k \mod n \).

2. Preliminaries

First, we recall [32] the positive linear map \( \Phi^{(n,k)} : M_n \to M_n \) for each \( k = 1, 2, \ldots, n-1 \) defined by

\[
\Phi^{(n,k)}([a_{ij}]) = \text{diag}(b_1, b_2, \ldots, b_n) - [a_{ij}]
\]

for \([a_{ij}] \in M_n\), where \( b_i = (n - 1)a_{ii} + a_{\sigma(k)(i), \sigma(k)(i)} \) for each \( i = 1, 2, \ldots, n \) (\( n \geq 3 \)).

Qi and Hou [32] showed that \( \Phi^{(n,k)} \) are indecomposable positive linear maps whenever either \( n \) is odd or \( k \neq n/2 \). They also showed [19] that the associated EWs \( W_{\Phi^{(n,k)}} \) are optimal EWs which have no spanning property whenever \( k \neq n/2 \), and \( W_{\Phi^{(n,n/2)}} \) is decomposable and not optimal when \( n \) is an even integer greater than 2. Recently, it was shown [25] that \( W_{\Phi^{(n,k)}} \) is indeed an optimal PPTES witness whenever \( k \neq n/2 \) (that is, nd-OEW in the sense of [6]). Therefore, \( W_{\Phi^{(n,k)}} \) detects a maximal set of entangled states with positive partial transposes in the sense of [22], \( \Phi^{(3,1)} \) and \( \Phi^{(3,2)} \), especially, are extremal Choi maps [26]. So these maps can be considered as extensions of extremal Choi map in the \( n \)-dimensional cases. Thus, we may expect these maps to be extremal. But, in general, these maps are not extremal. Although we can show that \( \Phi^{(4,1)} \) and \( \Phi^{(4,3)} \) are extremal, \( \Phi^{(4,2)} \) is not extremal since \( W_{\Phi^{(4,2)}} \) is not optimal. We will also show that \( \Phi^{(n,k)} \) is not extremal if \( \gcd(n, k) \neq 1 \). Note that \( W_{\Phi^{(n,k)}} \) is still optimal in the case of \( \gcd(n, k) \neq 1 \) as long as \( k \neq n/2 \). This is the aim of this work.

We note that \( \Phi^{(n,k)} \) maps \( M_n(\mathbb{R}) \) into itself. Therefore, we can use Choi and Lam’s method [27] (see also [29]) to check the extremality of \( \Phi^{(n,k)} \). For each \( n \geq 4 \) and \( k = 1, 2, \ldots, n-1 \), we define psd biquadratic forms \( B_{\Phi^{(n,k)}} \) by

\[
B_{\Phi^{(n,k)}}(x_1, x_2, \ldots, x_n) = y^t[\Phi^{(n,k)}(xx^t)]y = \sum_{j=1}^{n} x_j^2 + \sum_{i=1}^{n} x_{\sigma(k)(i)}^2 + 2 \sum_{1 \leq i < j \leq n} x_i x_j y_i y_j,
\]

(2)
where \( x = (x_1, x_2, \ldots, x_n) \), \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \). (In the same way, we can define the psd biquadratic form \( B_\Phi \) corresponding to a positive linear map \( \Phi \).) Let \( \mathcal{P}_{n,m} \) be the set of all psd real forms in \( n \) variables of degree \( m \). Then each \( B_{\Phi_{(n,k)}} \) belongs to \( \mathcal{P}_{2n,4} \) since \( \Phi_{(n,k)} \) is a positive linear map. A form \( F \in \mathcal{P}_{n,m} \) is said to be extremal if \( F = F_1 + F_2 \), \( F_i \in \mathcal{P}_{n,m} \), should imply \( F_i = \lambda_i F \) with nonnegative real numbers \( \lambda_i \). If we write \( \mathcal{E}(\mathcal{P}_{n,m}) \) for the set of all extremal psd forms in \( \mathcal{P}_{n,m} \), an elementary result in the theory of convex bodies shows that \( \mathcal{P}_{n,m} \) is the convex hull of \( \mathcal{E}(\mathcal{P}_{n,m}) \). From now on, we write \( F_1 \leq F_2 \) to indicate the fact that the form \( F_2 - F_1 \) is psd for \( F_1, F_2 \in \mathcal{P}_{n,m} \). In this ordering relation, if a form \( F \in \mathcal{P}_{n,m} \) is written by \( F = F_1 + F_2 \) for some \( F_i \in \mathcal{P}_{n,m} \), we see that \( 0 \leq F_i \leq F \) for \( i = 1, 2 \). Thus, to show that a form \( F \in \mathcal{P}_{n,m} \) is extremal, it suffices to show that \( 0 \leq F_i \leq F \) implies that \( F_i = \lambda_i F \) with nonnegative real numbers \( \lambda_i \).

It is well known [27, 29] that if a positive linear map \( \Phi : M_n \to M_n \) maps \( M_n(\mathbb{R}) \) into itself, then the corresponding biquadratic form \( B_\Phi \in \mathcal{E}(\mathcal{P}_{2n,4}) \) implies that \( \Phi \) is extremal in the convex cone consisting of all positive linear maps. Therefore, to show the extremality of \( \Phi_{(n,k)} \), it suffices to prove \( B_{\Phi_{(n,k)}} \in \mathcal{E}(\mathcal{P}_{2n,4}) \).

We also note that a psd biquadratic form \( B \) gives rise to a positive linear map \( \Phi \) such that \( B = B_\Phi \). Let \( B(X : Y) \) be a biquadratic form where \( X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \). Since this biquadratic form can be considered as a quadratic form with respect to each variable \( Y \) (as well as \( X \)), we can write it in the form \( \langle Y | S_X | Y \rangle \) where \( S_X \in M_n \) is a symmetric matrix. Thus we get a map sending each one-dimensional projection \( XX^t \in M_n \) to \( S_X \). Using linearity and hermiticity, we can extend it to a map which preserve hermiticity. It was shown by Choi that, given any psd form, this corresponding linear map is a positive linear map [26, 33].

For example, for a given psd biquadratic form \( B(X : Y) = (x_1y_3 - x_3y_1)^2 \) with \( X, Y \in \mathbb{R}^3 \), we get a symmetric matrix \( S_X \) of the form

\[
S_X = \begin{pmatrix}
x_1^2 & 0 & -x_1x_3 \\
0 & 0 & 0 \\
-x_3x_1 & 0 & x_3^2
\end{pmatrix}.
\]

Consequently, we obtain a positive (in fact, completely copositive) linear map \( \Phi : M_3 \to M_3 \) defined by

\[
\Phi([a_{ij}]) = a_{33}[1]\{1\} - a_{13}[1]\{3\} - a_{31}[3]\{1\} + a_{11}[3]\{3\} = V[a_{ij}]V^t
\]

for \([a_{ij}] \in M_3 \) and \( V = [3]\{1\} - [1]\{3\} \). We will use this correspondence between psd biquadratic forms and positive linear maps to show that if a biquadratic form \( B_{\Phi_{(n,k)}} \) is decomposed into the sum of psd biquadratic forms then the corresponding map \( \Phi_{(n,k)} \) is not extremal.

3. Extremal positive semidefinite forms

In this section, we explore some psd forms needed to show the extremality of \( B_{\Phi_{(4,1)}} \). For the positive linear map \( \Phi_{(4,1)} \), we define a quaternary octic psd form \( O_{(4,1)}(x, y, z, w) \) by

\[
O_{(4,1)}(x, y, z, w) := B_{\Phi_{(4,1)}} \begin{pmatrix}
yz & zw & zwx & wxy & xyz \\
x & y & z & w
\end{pmatrix}
\]

\[
= x^4z^2w^2 + y^4x^2w^2 + z^4x^2y^2 + w^4y^2z^2 - 4x^2y^2z^2w^2.
\]

We show that \( O_{(4,1)}(x, y, z, w) \) is extremal in \( \mathcal{P}_{4,8} \). Assume that there is a psd form \( F \in \mathcal{P}_{4,8} \) such that \( 0 \leq F \leq O_{(4,1)} \).

We note that the only possible monomial of \( F \) divisible by \( x^4 \) is \( x^4z^2w^2 \). To see this, we divide both sides of \( 0 \leq F \leq O_{(4,1)} \) by \( x^2 \) and take a limit of \( x \to \infty \) for each fixed
Lemma 2. We have $H = aO_{(4, 1)}$ in the identity (3).

Proof. We note that $\sum G(x, \epsilon_1 y, \epsilon_2 z, \epsilon_3 w) = 0$, where the sum is taken over all values $\epsilon_i \in \{-1, 1\}$. So we have

$$H(x, y, z, w) = \frac{1}{8} \sum_{\epsilon_i \in \{1, -1\}} F(x, \epsilon_1 y, \epsilon_2 z, \epsilon_3 w).$$

y, z and w. Then we have $\lim_{t \to \infty} \left[ F(x, y, z, w)/t^5 \right] = 0$ and so there is no monomial of $F$ divisible by $t^5$. Therefore, the sum of all monomials of $F$ divisible by $t^5$ is $x^5 \tilde{F}(y, z, w)$ where $\tilde{F}(y, z, w) = \lim_{t \to \infty} \left[ F(x, y, z, w)/t^5 \right]$. Since $0 \leq \tilde{F} \leq \lim_{t \to \infty} \left[ O_{(4, 1)}/t^5 \right] = z^2 w^2$, we know that $\tilde{F}(y, z, w)$ has no monomial divisible by $y$. By dividing both sides of $0 \leq \tilde{F} \leq z^2 w^2$ by $z^4$ (respectively $w^4$) and then taking the limit as $z \to \infty$ (respectively $w \to \infty$), we also see that $\tilde{F}$ has no monomial divisible by $z^3$ (respectively $w^3$). Since $\tilde{F}$ is quartic, we conclude that $\tilde{F}(y, z, w) = z^2 w^2$.

By applying the same idea for $y, z$ and $w$ we can write

$$F(x, y, z, w) = \tilde{H}(x, y, z, w) = \tilde{G}(x, y, z, w),$$

where $\tilde{H}(x, y, z, w) = ax^2 y^2 z^2 + bx^2 y^2 + cz^2 y^2 + dw^4 y^2 z^2 + ew^2 y^2 z^2$ and $\tilde{G}(x, y, z, w) = F(x, y, z, w) - \tilde{H}(x, y, z, w)$. From the identity (3), we see that every monomial in $\tilde{G}$ contains at least one variable on which the degree of the monomial is odd. We write

$$\tilde{G}(x, y, z, w) = \gamma_{x, 3}(y, z, w)x^3 + \gamma_{x, 2}(y, z, w)x^2 + \gamma_{x, 1}(y, z, w)x + \gamma_{x, 0}(y, z, w)$$

and examine $\gamma_{x, 3}$.

Lemma 1. $\gamma_{x, 3}(y, z, w)$ does not have monomials $y^3 w^2, y^3 z w, y^3 z^2, y^2 z^2 w, y^2 z^3, y z w^3, z^2 w^2$ and $z^2 w^3$. Thus $\gamma_{x, 3}(y, z, w)$ has only the monomials $y^2 w^2, y z^3 w$, and $y z w^3$.

Proof. From the inequality

$$F(y^2, y, z, w) \leq O_{(4, 1)}(y^2, y, z, w) = (1 + z^4)w^2 y^8 + (z^2 - 4w^2)z^2 y^6 + y^2 z^2 w^4,$$

we know that $\gamma_{x, 3}(y, z, w)$ (briefly, $\gamma_{x, 3}$) does not have the monomials $y^3 w^2, y^3 z w, y^3 z^2, y^2 w^3, y^2 z^3$ by considering the highest degree. To see that $\gamma_{x, 3}$ does not have the monomial $y^2 z^2 w$, we divide both sides of the above inequality by $y^8$ and take limit as $y \to \infty$, and then divide both sides by $w^4$ and take limit as $w \to 0$.

Inequality $F(x, y, z, w^2) \leq O_{(4, 1)}(x, y, z, w^2)$ implies that $\gamma_{x, 3}$ does not have monomials $z^2 w^2$ and $yz w^3$. From the inequality $\lim_{y \to 0} y^2 F(x, y, 1/y, w) \leq \lim_{y \to 0} y^2 O_{(4, 1)}(x, y, 1/y, w)$, we also see that $\gamma_{x, 3}$ does not have the monomial $z^3 w^2$.

Consequently, we have $\gamma_{x, 3}(y, z, w) = yz w(q_{11} z^2 + q_{12} z w + q_{13} y w)$ for some $q_{ij} \in \mathbb{R}$.

Like the previous lemma 1, we can check which monomials do not appear in $\gamma_{x, 3}(x, y, z, w, y, z)$, $\gamma_{x, 3}(x, y, w)$, and $\gamma_{w, 3}(x, y, z, w)$. That is, we can easily see that

$$\gamma_{x, 3}(x, y, z, w) = \gamma_{x, 3}(x, y, w) = \gamma_{x, 3}(y, z, w) = \gamma_{x, 3}(y, w) = \gamma_{x, 3}(w, x, y) = \gamma_{x, 3}(w, y, x) = \gamma_{y, 2}(x, z, w),$$

and

$$\gamma_{y, 3}(x, y, z, w) = \gamma_{y, 3}(x, y, w) = \gamma_{y, 3}(y, z, w) = \gamma_{y, 3}(y, w) = \gamma_{y, 3}(w, x, y) = \gamma_{y, 3}(w, y, x) = \gamma_{y, 2}(x, z, w),$$

for some $q_{ij} \in \mathbb{R}$. From the above identities on $\gamma_{x, 3}$ and (3), we have that

$$G(x, y, z, w) = x y z w + x y z s(x^2 + y^2) + y w(s x^2 + s y^2) + (s x^2 + s y^2) x^2 + s x y^2 + s y z x^2 + s z y x^2 + s_{10} x y^2,$$

where $s_i \in \mathbb{R}$.

Lemma 2. We have $H = aO_{(4, 1)}$ in the identity (3).
Therefore, we see that $0 \leq H \leq O_{(4,1)}$. Now, $O_{(4,1)}(x, x, x, x) = 0$ implies that

$$H(x, x, x, x) = x^4(a + b + c + e) = 0 \implies e = a + b + c + d.$$ 

Then, we get $H(x, x, z, z) = x^2z^2(x - z)(x + z)(bh^2 - dz^2) \geq 0$, and so $b = d$. In a similar way, we can show that $a = b = c = d$. Consequently, we have $H = aO_{(4,1)}$ with $0 \leq a \leq 1$.

From the lemma 2, the identity (3) is reduced to

$$F(x, y, z, w) = aO_{(4,1)}(x, y, z, w) + xyw[x(z(s_1)^2 + s_2w^2) + yw(s_3x^2 + s_4z^2)
+ (s_5x^2z^2 + s_6y^2w^2) + s_7xyz^2 + s_8yzw^2 + s_9zw^x^2 + s_{10}xw^y^2].$$

(5)

To arrive at the goal $O_{(4,1)} \in \mathcal{E}(\mathcal{P}_{4,8})$, we need two more lemmas.

**Lemma 3.** For $0 \leq a \leq 1$, if $F_1 = aO_{(4,1)} + x^2yz^2w(ay^2 + \beta y^2w^2) \in \mathcal{P}_{4,8}$, then $\alpha = \beta = 0$.

**Proof.** Because $F_1(x, x, x, x) = (\alpha + \beta)x^8$ and $F_1(x, x, x, -x) = -(\alpha + \beta)x^8$, we see that $\alpha + \beta = 0$. Then, $F_1(z, z, z, w) = a^2(z^2 - w^2)[a(z^2 - w^2) + \alpha zw] \geq 0$ implies $\alpha = 0$.

By the same argument, we have the following.

**Lemma 4.** For $0 \leq a \leq 1$, if $F_1 = aO_{(4,1)} + xyw(\alpha x^2z^2 + \beta y^2w^2) \in \mathcal{P}_{4,8}$, then $\alpha = \beta = 0$.

Now, we can show that $O_{(4,1)} \in \mathcal{E}(\mathcal{P}_{4,8})$.

**Theorem 5.** The quaternary octic $O_{(4,1)}(x, y, z, w)$ is extremal, i.e. $O_{(4,1)} \in \mathcal{E}(\mathcal{P}_{4,8})$.

**Proof.** Suppose $0 \leq F \leq O_{(4,1)}$ and define a form

$$F_{yw}(x, y, z, w) = \frac{1}{4} \sum_{\epsilon \in \{1, -1\}} F(\epsilon x, y, \epsilon z, w).$$

Then, from the identity (5), we see that

$$F_{yw} = aO_{(4,1)} + x^2yz^2w(s_1y^2 + s_2w^2)$$

and $0 \leq F_{yw} \leq O_{(4,1)}$.

Then by lemma 3, $s_1 = s_2 = 0$. In a similar way, we can show $s_i = 0$ ($1 \leq i \leq 10$). So we have $F = aO_{(4,1)}$. This completes the proof.

Now, we define the quaternary octic form $O_{(4,1)}$ and the senary quartic form $Q_{(4,1)}$ by

$$O_{(4,1)}(x, y, z, w) = w^8 + x^2y^2z^2 + y^2x^2z^2 + z^2x^2y^2 - 4x^2y^2z^2w^2,$$

$$Q_{(4,1)}(p, q, s, t, u, v) = B_{\Phi_{(4,1)}}(p, q, s, t, u, v)$$

$$= u^4 + 2(p^2q^2 + s^2t^2 + w^2v^2) + q^2s^2 + r^2t^2 + p^2u^2 - 2pqst - 4pquv - 4stuw.$$ 

Since $\Phi_{(4,1)}$ is a positive linear map, we see that $Q_{(4,1)}$ is a psd form, that is, $Q_{(4,1)} \in \mathcal{P}_{6,4}$.

From the arithmetic-geometric inequality, we can show that $O_{(4,1)} \in \mathcal{P}_{4,8}$. We also note that

$$O_{(4,1)}(x^2, yw, zw^2, zyw) = x^2z^2O_{(4,1)}(x, y, z, w),$$

(6)

$$Q_{(4,1)}(zw^3, xyw^2, xzw^2, xzw, w^3) = w^3O_{(4,1)}(x, y, z, w).$$

(7)

Now, we show that $O_{(4,1)} \in \mathcal{P}_{4,8}$ is extremal psd. Suppose $O_{(4,1)} \geq F \in \mathcal{P}_{4,8}$, Then we have

$$x^2z^2O_{(4,1)}(x, y, z, w) = O_{(4,1)}(x^2, yw, zw^2, zyw) \geq F(x^2, xyw, zw^2, zyw) \geq 0.$$
Since the left-hand side is extremal, we must have
\[ F(xx^2, xyw, zw^2, xzw) = \alpha \mathcal{O}(x, y, z, w) \]
for some \( \alpha \in \mathbb{R} \). We replace \( x, y, z \) and \( w \) by
\[
\left( \frac{w^2}{z^2}, \frac{z^2}{y^2}, \frac{y^2}{x^2}, \frac{x^2}{w^2} \right),
\]
respectively. Then equation (8) becomes \( F(x, y, z, w) = \alpha \mathcal{O}(x, y, z, w) \). This completes the following proposition.

**Proposition 6.** The quaternary octic \( \mathcal{O}(x, y, z, w) \) is extremal, i.e. \( \mathcal{O}(x, y, z, w) \in \mathcal{E}(\mathcal{P}_{4,8}) \).

We proceed to examine the extremality of \( \mathcal{Q}(x, y, z, w) \in \mathcal{P}_{4,4} \). From the equation (7) and the extremality of \( \mathcal{O}(x, y, z, w) \), it follows that whenever \( \mathcal{Q}(x, y, z, w) \in \mathcal{P}_{4,4} \), we have
\[
F(zw^3, yzw^2, xzw, w^4) = \alpha \mathcal{Q}(x, y, z, w)
\]
for some \( \alpha \in \mathbb{R} \). Replacing \( x, y, z \) and \( w \) by
\[
x = tv^{-3/4}, \quad y = qw^{1/4}, \quad z = pu^{-3/4}, \quad w = v^{1/4}
\]
respectively, equation (9) becomes
\[
F(p, q, pq/t, t, pq/v, v) = \alpha \mathcal{Q}(x, y, z, w).
\]

We consider \( G := F - \alpha \mathcal{Q}(x, y, z, w) \). Since \( G(p, q, pq/t, t, pq/v, v) = 0 \), we see that \( G \) is of the form
\[
G(p, q, s, t, u, v) = (pq - st)G_1(p, q, s, t, u, v) + (pq - uv)G_2(p, q, s, t, u, v)
\]
for some senary quadratic \( G_1 \) and \( G_2 \). Using the equality \( \mathcal{Q}(x, y, z, w) = \mathcal{Q}(x, y, z, w) \) and looking at the leading coefficient of each variable, we can get
\[
G = a_1(stv^2 - 2uv^3 + pqv^2) + a_2(-uv^2q - uv^2s + pqsv + stqv)
\]
\[
+ a_3(-2u^2v^2 + stu^2 + pquv) + a_4(pqt - u^2v^2) + a_5(-2qsuv + pq^2s + s^2tq)
\]
\[
+ a_6(p^2qu - pu^2v - vu^2 + st^2 + at^2)(p^2q^2 + s^2t^2 - stu - pquv).
\]

Note that \( \mathcal{Q}(x, y, z, w) = 0 \) on the set \( S = \{(p, q, s, t, u, v)|pq = st, pq = uv, st = uv\} \). Thus \( F = 0 \) on the set \( S \) and so \( F \) has local minima on the set \( S \). From \( \partial F/\partial p = \partial(G + \alpha \mathcal{Q}(x, y, z, w))/\partial p = 0 \) on the set \( S \), we get \( a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0 \) and \( a_7 = -a_4 - a_7 \). Compute \( F = \alpha G_1(x, y, z, w) \) when \( u = 1/v, t = v^3, s = 1/v^3 \) and \( p = q = 0 \), then we get \( a_7 = 0 \).

Therefore, we have that
\[
0 \leq F = a_3(st - uv)(pq - uv) + a_4 \mathcal{Q}(x, y, z, w) \leq \mathcal{Q}(x, y, z, w) \quad \text{with} \quad 0 \leq a \leq 1.
\]

Now, we will show that \( a_7 = 0 \) from the condition \( F \geq 0 \). When \( t = v^2 \) and \( u = 1 \), the discriminant \( D(F, p) \) of \( F \) on the variable \( p \) should be less than or equal to 0. That is, \( -D(F, p) \geq 0 \). From the identity
\[
-D(F, p) = (4\alpha^2q^3 + 8\alpha^2q^4 + 8\alpha^2v^4 - \alpha^2q^2v^4 + 4\alpha^2q^2v^4 + 12\alpha^2q^2v^4)sl
\]
\[
-4\alpha^2\alpha q^3 + 16\alpha^2v^3 - 2\alpha^2q^2v^3 + 4\alpha^2q^2v^3 + 48\alpha^2q^2v^3
\]
\[
+ 4\alpha^2\alpha q^3 + 8\alpha^2v^3 - \alpha^2q^2v^3 + 8\alpha^2v^3 + 16\alpha^2q^2v^3,
\]
we compute the condition on which the discriminant \( -D(F, p) \) should be less than or equal to 0. Since the coefficient of the highest degree of \( q, q^2 \), is \( 32\alpha^2v^2(a_7^2 - 16\alpha^2q^2) \) in \( D(-D(F, p), s) \), it follows that \( 32\alpha^2v^2(a_7^2 - 16\alpha^2q^2) \leq 0 \) for all \( v \neq 0 \). Consequently, we have \( a_7 = 0 \). This completes the proof of the following proposition.

**Proposition 7.** The senary quartic \( \mathcal{Q}(x, y, z, w) \) is extremal, i.e. \( \mathcal{Q}(x, y, z, w) \in \mathcal{E}(\mathcal{P}_{4,4}) \).
4. Extremality for Qi and Hou’s map

In this section, we show that $\Phi_{(n,k)}$ is not extremal whenever $n$ and $k$ are not relatively prime, that is, $\text{gcd}(n, k) > 1$. This answers the question on the existence of optimal EWs without extremality or the spanning property. For the case of $\text{gcd}(n, k) = 1$, we think that $\Phi_{(n,k)}$ may be extremal. Here, we give the details of the proof for the extremality of $\Phi_{(4,1)}$ and $\Phi_{(4,3)}$.

We begin with showing that $B_{\Phi_{(n,k)}}$ is not extremal whenever $\text{gcd}(n, k) > 1$. Let $S_n$ be the symmetric group consisting of all bijection (permutation) from the set $\{1, 2, \ldots, n\}$ onto itself. For any integer $q$, define $\sigma_q \in S_n$ by

$$\sigma_q(j) = j + q \quad (\mod n).$$

First, we consider the case when $k$ divides $n$ and $(n/k) > 1$. Note that $\sigma_k$ is a product of disjoint $k$ cycles. We recall the biquadratic form $B_{\Phi_{(n,k)}}$

$$B_{\Phi_{(n,k)}} = (n-2)\sum_{i=1}^{n} x_i^2 y_i^2 - 2 \sum_{1 \leq i < j \leq n} x_i y_i x_j y_j + \sum_{i=1}^{n} \sum_{q \in \mathbb{Z}} \sigma_q(i) \sigma_q(j) x_i^2 y_j^2,$$  

(11)

and define biquadratic forms

$$F_{n,d} = \left(\frac{n}{k}-2\right) \sum_{n/d \text{mod } k}^{n/k} x_i^2 y_i^2 - 2 \sum_{1 \leq i < j \leq n/k} x_i y_i x_j y_j + \sum_{i=1}^{n/k} \sum_{q \in \mathbb{Z}} \sigma_q(i) \sigma_q(j) x_i^2 y_j^2,$$  

(12)

for each $d = 1, \ldots, k$. Then we can easily check that

$$B_{\Phi_{(n,k)}} = \sum_{d=1}^{k} F_{n,d} + \sum_{d \text{mod } k}^{n/k} (x_i y_i - x_i y_j)^2.$$  

(13)

Now, we see that all the biquadratic forms of $F_{n,d}$ in (13) are equivalent to the biquadratic form $B_{\Phi_{(n,k)}}$, That is, by renaming $x_{d+i+m}$ by $x_{d+i}$ and $y_{d+i+m}$ by $y_{d+i}$ in the biquadratic form $F_{n,d}$, we get the $B_{\Phi_{(n,k)}}$

$$B_{\Phi_{(n,k)}} = \left(\frac{n}{k}-2\right) \sum_{i=1}^{n/k} x_i^2 y_i^2 - 2 \sum_{1 \leq i < j \leq n/k} x_i y_i x_j y_j + \sum_{i=1}^{n/k} \sum_{q \in \mathbb{Z}} \sigma_q(i) \sigma_q(j) x_i^2 y_j^2,$$  

(14)

where $\sigma_1$ is a permutation in $S_{n/k}$ defined by $\sigma_1(j) = j + 1 \; (\mod n/k)$. Thus $F_{n,d}$ is equivalent to $B_{\Phi_{(n,k)}}$. Furthermore, we can conclude that each $F_{n,d}$ in (13) is a psd quadratic form since $B_{\Phi_{(n,k)}}$ is psd. Consequently, we have the following result.

**Proposition 8.** If $n$ is divisible by $k$, then the psd biquadratic form $B_{\Phi_{(n,k)}}$ is decomposed as a sum of psd biquadratic forms as in the identity (13). Furthermore, each $F_{n,d}$ in (13) can be considered as $B_{\Phi_{(n,k)}}$ by renaming.
Proposition 9. Let \( \gcd(n, q) = k \geq 1 \). Then, \( B_{\Phi(n,q)} \) is extremal if and only if \( B_{\Phi(n,k)} \) is extremal.

Finally, we can show that the main result.

Theorem 10. If \( \gcd(n, k) \neq 1 \), then \( \Phi^{(n,k)} \) is not an extremal positive linear map.

Proof. From the proposition 9, it suffices to consider the case when \( n \) is divisible by \( k \). In this case, we know that the biquadratic form \( B_{\Phi(n,k)} \) is the sum of psd biquadratic forms from the proposition 8. Therefore the corresponding map \( \Phi^{(n,k)} \) is the sum of positive linear maps as explained in the last paragraph of section 2. That is, \( \Phi^{(n,k)} \) is not extremal. \( \square \)

As a byproduct, we have the following corollaries.

Corollary 11. There exists an optimal EW which neither has the spanning property nor is associated with an extremal positive linear map. In fact, \( W_{\Phi(n,k)} \) is such an optimal EW whenever \( \gcd(n, k) \neq 1 \).

Corollary 12. If \( \gcd(n, k) = 1 \), then \( \Phi^{(n,k)} \) is extremal if and only if \( \Phi^{(n,1)} \) is extremal.

Proof. From the proof of theorem 10, we see the non-extremality of \( B_{\Phi(n,k)} \) implies the non-extremality of \( \Phi^{(n,k)} \). By combining the results of propositions 8, 9 and theorem 10, the proof is completed. \( \square \)

Corollary 13. A positive linear map \( \Phi^{(n,n/2)} \) is decomposable when \( n \) is an even natural number greater than 2.

Proof. Since \( n \) is divisible by \( n/2 \), we have the decomposition of \( B_{\Phi(n,n/2)} \) as in (13). We also see that each \( F_{\alpha_2,d} \) in (12) is of the form

\[
F_{\alpha_2,d} = \frac{-2x_dy_ay(bd+n/2)y_1y_2}{x_d^2} + \frac{x_d^2y_1^2}{y_3}\frac{y_2}{y_4} = (x_d)_{d+n/2} - (x_d+n/2)Y_{d/2}
\]

Since the positive linear map corresponding to the psd biquadratic form \((x_d)_{d+n/2} - (x_d+n/2)Y_{d/2}\)^2 is completely copositive and the map corresponding to \((x_1,y_1-x_1y_1)^2\) is completely positive, we can conclude that the positive linear map \( \Phi^{(n,n/2)} \) is decomposable. \( \square \)

We now turn to the extremality of \( \Phi^{(4,k)} \). In this case, we can show that \( \Phi^{(4,1)} \) and \( \Phi^{(4,3)} \) are extremal from the extremality of the senary quartic form \( Q_{(4,1)} \) (recall the proposition 7).

Theorem 14. \( \Phi^{(4,k)} \) is an extremal positive linear map if and only if \( k = 1 \) or 3.

Proof. From the theorem 10, we know that \( \Phi^{(4,2)} \) is not extremal. We also know that \( \Phi^{(4,1)} \) is extremal if and only if \( \Phi^{(4,3)} \) is extremal by corollary 12. Therefore, it suffices to show that \( B_{\Phi^{(4,1)}} \) is an extremal psd biquadratic form as stated in section 2. Suppose \( F \) is a biquadratic form such that \( B_{\Phi} \geq F \geq 0 \). Then

\[
Q_{(4,1)}(p, q, s, t, u, v) = B_{\Phi^{(4,1)}} \left( \begin{array}{cc} p & s \\ q & t \end{array} \right) \left( \begin{array}{cc} u & v \\ u & v \end{array} \right) \geq F \left( \begin{array}{cc} p & s \\ q & t \end{array} \right) \left( \begin{array}{cc} u & v \\ u & v \end{array} \right) = 0.
\]

From the extremeness of \( Q_{(4,1)} \), we have

\[
F \left( \begin{array}{cc} p & s \\ q & t \end{array} \right) \left( \begin{array}{cc} u & v \\ u & v \end{array} \right) = \lambda_1 Q_{(4,1)}(p, q, s, t, u, v).
\]
In this paper, we have studied the extremality of the positive linear map \( \Phi_{(n,k)} \), constructed by Qi and Hou [32], and those associated entanglement witnesses \( W_{\Phi_{(n,k)}} \) are known as optimal indecomposable entanglement witnesses without the spanning property. One of the interesting problems on optimal indecomposable entanglement witnesses is whether an optimal indecomposable witness \( W \) exists such that the associated positive linear map is not extremal and corresponding \( \mathcal{P}_W \) do not span the Hilbert space fully. Here, we answer this question negatively by showing that \( \Phi_{(n,k)} \) is not extremal whenever \( \gcd(n, k) \neq 1 \). As a byproduct of

5. Conclusion

In this paper, we have studied the extremality of the positive linear map \( \Phi_{(n,k)} \) constructed by Qi and Hou [32], those associated entanglement witnesses \( W_{\Phi_{(n,k)}} \) are known as optimal indecomposable entanglement witnesses without the spanning property. One of the interesting problems on optimal indecomposable entanglement witnesses is whether an optimal indecomposable witness \( W \) exists such that the associated positive linear map is not extremal and corresponding \( \mathcal{P}_W \) do not span the Hilbert space fully. Here, we answer this question negatively by showing that \( \Phi_{(n,k)} \) is not extremal whenever \( \gcd(n, k) \neq 1 \). As a byproduct of

Since \( B_{\Phi_{(n,k)}} \) is invariant under the cyclic permutation (1234) applied to the subscripts of \((x_1, x_2, x_3, x_4)\) and \((y_1, y_2, y_3, y_4)\) simultaneously, we see that

\[
Q_{(4,1)}(p, q, s, t, u, v) = B_{\Phi_{(n,k)}} \left( \begin{array}{c} p \ s \ u \ v \\ q \ t \ v \ u \ q \end{array} \right) = B_{\Phi_{(n,k)}} \left( \begin{array}{c} p \ s \ u \ v \\ u \ q \ t \ v \end{array} \right).
\]

(16)

So we can similarly show that

\[
F \left( \begin{array}{c} s \ u \ v \ p \\ t \ v \ u \ q \end{array} \right) = \lambda_2 Q_{(4,1)}(p, q, s, t, u, v),
\]

\[
F \left( \begin{array}{c} u \ v \ p \ s \\ v \ u \ q \ t \end{array} \right) = \lambda_3 Q_{(4,1)}(p, q, s, t, u, v),
\]

\[
F \left( \begin{array}{c} v \ p \ s \ u \\ u \ q \ t \ v \end{array} \right) = \lambda_4 Q_{(4,1)}(p, q, s, t, u, v).
\]

(17)

By comparing the coefficients, we get \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \). In fact, we see that

\[
Q_{(4,1)}(s, t, s, t, s, t) = Q_{(4,1)}(t, s, t, t, s, t) = 2(s^2 - t^2)^2,
\]

and we get the following identities

\[
F \left( \begin{array}{c} s \ t \ s \ t \\ s \ t \ s \ t \end{array} \right) = \lambda_1 Q_{(4,1)}(s, t, s, t, s, t) = \lambda_2 Q_{(4,1)}(t, s, t, t, s, t)
\]

\[
= \lambda_3 Q_{(4,1)}(s, t, s, t, s, t) = \lambda_4 Q_{(4,1)}(t, s, t, t, s, t)
\]

from (15) and (17). This gives rise to \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \).

Now, for any fixed nonzero real numbers \( y_1, y_2, y_3 \) and \( y_4 \), we define a quadratic form \( f(x_1, x_2, x_3, x_4) \) by

\[
f(x_1, x_2, x_3, x_4) := (F - \lambda_1 B_{\Phi_{(n,k)}}) \left( \begin{array}{c} x_1 \ x_2 \ x_3 \ x_4 \\ y_1 \ y_2 \ y_3 \ y_4 \end{array} \right).
\]

From the identities (16) and (17), we see that

\[
f(x_1, x_2, y_4, y_3) \equiv 0, \ f(x_1, y_3, y_2, x_4) \equiv 0, \ f(y_2, y_1, x_3, x_4) \equiv 0, \ f(y_4, x_2, x_3, y_1) \equiv 0.
\]

Note that \( f(x_1, x_2, y_4, y_3) \equiv 0 \) implies that \( f \) is divisible by \( y_3 x_3 - y_4 x_4 \). Similarly, we see that \( f \) is divisible by \( y_2 x_2 - y_3 x_3 \) and \( y_1 x_1 - y_2 x_2 \). Since the degree of \( f \) is 2, this leads to \( f \equiv 0 \). In other words, we have

\[
(F - \lambda_1 B_{\Phi_{(n,k)}}) \left( \begin{array}{c} x_1 \ x_2 \ x_3 \ x_4 \\ y_1 \ y_2 \ y_3 \ y_4 \end{array} \right) \equiv 0
\]

whenever \( y_1, y_2, y_3 \) and \( y_4 \) are nonzero real numbers. By continuity, we conclude that \( F = \lambda_1 B_{\Phi_{(n,k)}} \).

Therefore, \( B_{\Phi_{(n,k)}} \) is extremal among biquadratic forms. □

5. Conclusion

In this paper, we have studied the extremality of the positive linear map \( \Phi_{(n,k)} \) constructed by Qi and Hou [32], those associated entanglement witnesses \( W_{\Phi_{(n,k)}} \) are known as optimal indecomposable entanglement witnesses without the spanning property. One of the interesting problems on optimal indecomposable entanglement witnesses is whether an optimal indecomposable witness \( W \) exists such that the associated positive linear map is not extremal and corresponding \( \mathcal{P}_W \) do not span the Hilbert space fully. Here, we answer this question negatively by showing that \( \Phi_{(n,k)} \) is not extremal whenever \( \gcd(n, k) \neq 1 \). As a byproduct of
our proof using the correspondence between psd biquadratic forms and positive linear maps, we have reproved that $\Phi_{(n,n/2)}^{(n,n/2)}$ is decomposable when $n$ is even.

For the case of $\gcd(n, k) = 1$, we showed that $\Phi_{(n,n/2)}^{(n,k)}$ is extremal if and only if $\Phi_{(n,1)}^{(n,1)}$ is extremal. In particular, we proved that $\Phi_{(4,1)}^{(4,1)}$ and $\Phi_{(4,3)}^{(4,3)}$ are extremal. Our proof for the extremality seems to be applicable for general $(n, k)$ with $\gcd(n, k) = 1$. But, it is too laborious since we should check the extremality of each $B_{(n,k)}$. So a new approach which can be applicable for all cases at the same time is needed.

Recently, Chruściński and Wudarski [34] gave another variant $\Phi[a, b, c, d]$ of the extremal Choi map between $M_4$. Entanglement witnesses arising from these maps are known to be indecomposable optimal entanglement witnesses which have both the spanning property and the co-spanning property. We say that an entanglement witness $W$ has the co-spanning property if $W^T$ has the spanning property [21, 22]. Among them, $\Phi[1, 1, 1, 0]$ and $\Phi[1, 0, 1, 1]$ are expected to be extremal. But, in this case, our method is not directly applicable since the corresponding quaternary octic psd form $O_4$ is not extremal. Therefore, it would be interesting to investigate the extremality of these maps. We also note that the extremality of maps $\Phi[1, 0, p_0 - 1; \theta]$, $\Phi[1, p_0 - 1, 0; \theta]$ [23] remains open to question.

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