Abstract

Lattice path matroids and bicircular matroids are two well-known classes of transversal matroids. In the seminal work of Bonin and de Mier about structural properties of lattice path matroids, the authors claimed that lattice path matroids significantly differ from bicircular matroids. Recently, it was proved that all cosimple lattice path matroids have positive double circuits, while it was shown that there is a large class of cosimple bicircular matroids with no positive double circuits. These observations support Bonin and de Miers’ claim. Finally, Sivaraman and Slilaty suggested studying the intersection of lattice path matroids and bicircular matroids as a possibly interesting research topic. In this work, we exhibit the excluded bicircular matroids for the class of lattice path matroids, and we propose a characterization of the graph family whose bicircular matroids are lattice path matroids. As an application of this characterization, we propose a geometric description of 2-connected lattice path bicircular matroids.

1 Introduction

Bicircular matroids are a minor closed class of matroids that arise from graphs. There is considerable amount of research towards studying certain subclasses of bicircular matroids. Matthews [9] characterized bicircular matroids which are also graphic matroids. In [12], ternary bicircular matroids are studied, and in [6], bicircular matroids representable over $GF(4)$ and $GF(5)$ are characterized. Neudauer [10] considered bicircular matroids that are fundamental transversal matroids. Finally, Sivaraman and Slilaty [13] characterize the class of 3-connected bicircular matroids whose duals are bicircular matroids as well. In the previous work, the authors suggested that studying the intersection of lattice path matroids and bicircular matroids might be an interesting research topic.

Bonin and de Mier [4] showed that lattice path matroids are a class of transversal matroids closed under minors and duality. Furthermore, they observed that these matroids have several strong structural properties. For instance, all connected lattice path matroids have a spanning circuit. In the same work, the authors claimed that bicircular matroids and lattice path matroids are significantly different classes of transversal matroids. In particular, they noticed that every uniform matroid is a lattice path matroid, while the only uniform bicircular matroids are $U_{3,5}$, $U_{3,6}$ and $U_{4,6}$, and the families $U_{1,n}$, $U_{2,n}$, $U_{n,n}$ and $U_{n,n+1}$ for $n \geq 0$. This remark yields a large class of lattice path matroids that are not bicircular matroids.

Recently, and in a different context, it was shown that every simple lattice path matroid has a positive coline [1]. Since the class of lattice path matroids is closed under duality, the previous statement implies that every cosimple lattice path matroid has a positive double circuit. On the contrary, we exhibited a large class of cosimple bicircular matroids with no positive double circuits [8]. This class is obtained by considering bicircular matroids of graphs with minimum degree at least 3 and with girth at least 5. So, the bicircular matroids of such graphs are not lattice path matroids, which yields a large class of bicircular matroids that are not lattice path matroids.
The observations of the two paragraphs above, motivate us to study the class of lattice path bicircular matroids. We show that indeed, lattice path bicircular matroids are a thin family of matroids. This is done by constructing five graph families \( F_i \) with \( i \in \{0, \ldots, 4\} \) such that the bicircular matroid of a graph \( G \in F_i \) is a lattice path matroid. This is one of our two main results.

**Theorem 1.** Let \( G \) be a graph such that \( B(G) \) is a connected matroid. The bicircular matroid \( B(G) \) is a lattice path matroid if and only if \( G \) belongs to \( F_0 \cup F_1 \cup F_2 \cup F_3 \cup F_4 \).

Moreover, we propose a characterization of lattice path bicircular matroids within the class of bicircular matroids. We do so by exhibiting the list of excluded bicircular minors to lattice path matroids.

**Theorem 2.** A bicircular matroid is a lattice path matroid if and only if it has no one of the following matroids as a minor:

\[
C^{2,4}, W^3, A^3, R^3, R^4, D^4, B^1, \text{ and } S^3.
\]

We denote by \( ex_B(L) \) the set that contains the eight matroids listed in Theorem 2. The proofs of Theorems 1 and 2 consists of three main blocks. In Section 3, we define the matroids listed in Theorem 2, and we show that these are excluded bicircular minors to lattice path matroids. In Section 4, we construct the graph families \( F_i \) for \( i \in \{0, \ldots, 4\} \), and show that each graph in these families has a lattice path bicircular matroid. Finally, in Section 5, we observe that if a connected bicircular matroid \( B(G) \) has no \( ex_B(L) \)-minor, then \( G \) belongs to some \( F_i \). Building on the work from Sections 3, 4 and 5, we prove Theorems 1 and 2 in Section 6 where we also discuss some of their implications. For instance, we propose a geometric description of 2-connected lattice path bicircular matroids.

We begin with Section 2, where we introduce bicircular matroids and lattice path matroids, together with certain properties of these that will be useful for our work.

## 2 Preliminaries

We assume familiarity with basic matroid and graph theory. A standard reference for matroid theory is [11] and for graph theory is [2]. In particular, given a graph \( G \) and a subset of edges \( I \), we denote by \( G[I] \) the subgraph of \( G \) induced by \( I \). That is, \( G[I] \) is the subgraph of \( G \) with edge set \( I \) and no isolated vertices.

Let \( G \) be a (not necessarily simple) graph with vertex set \( V \) and edge set \( E \). The **bicircular matroid of \( G \)** is the matroid \( B(G) \) with ground set \( E \) whose independent sets are the edge sets of \( G \) with edge set \( I \) and no isolated vertices.

Consider a pair \( P \) and \( Q \) of lattice paths from \( (0,0) \) to \( (m,r) \), where \( P \) never goes above \( Q \). Let \( P \) be the set of all lattice paths from \( (0,0) \) to \( (m,r) \) that do not go below \( P \) nor above \( Q \). For every \( i \in \{1, \ldots, r\} \) let \( N_i \) be the set \( \{ j : \text{ step } j \text{ is the } i-\text{th North step of some path in } P \} \). The matroid \( M[P,Q] \) is defined as the transversal matroid with ground set \( \{m+i : i \in \{1, \ldots, r\} \} \) as a presentation. We call \( (N_1, \ldots, N_r) \) the **standard presentation** of \( M[P,Q] \). A **lattice path matroid** is a matroid \( M \) isomorphic to \( M[P,Q] \) for some lattice paths \( P \) and \( Q \). Notice that in this case, the rank of \( M \) is \( r \) and its corank is \( m \).
Equivalently [1], a matroid $M$ with ground set $E$ is a lattice path matroid if and only if there is a linear ordering $e_1 \leq e_2 \leq \cdots \leq e_n$ of $E$, and a collection $\mathcal{N}$ of incomparable intervals of $(E, \leq)$ such that $M$ is isomorphic to the transversal matroid $(E, \mathcal{N})$. We call $e_1 \leq e_2 \leq \cdots \leq e_n$ the interval ordering of $\mathcal{N}$, and say that $(E, \mathcal{N})$ is an interval presentation of $M$. In particular, every standard presentation is an interval presentation.

Lattice path matroids have several nice structural properties investigated by Bonin and de Mier [4]. In particular, we are interested in the collection of fundamental flats. Let $X$ be a connected flat of a connected matroid $M$ for which $|X| > 1$ and $r(X) < r(M)$. We say that $X$ is a fundamental flat of $M$ if for some spanning circuit $C$ of $M$ the intersection $X \cap C$ is a basis of $X$. In [4], the authors assert that the fundamental flats of a lattice path matroid form two disjoint chains ordered by inclusion.

Proposition 3. [4] Assume $M[P, Q]$ is a connected lattice path matroid of rank $r$ and corank $m$. Let $X$ be a connected flat of $M[P, Q]$ with $|X| > 1$ and $r(X) < r$. Then $X$ is a fundamental flat of $M[P, Q]$ if and only if $X$ is an initial or a final segment of $[m + r]$.

Series extension

Sometimes, we will need to argue that if the bicircular matroid of some graph is a lattice path matroid, then the bicircular matroids of certain subdivisions of this graph are also lattice path matroids. Since edge subdivisions correspond to coparallel extensions, we begin by proving the following statement.

Lemma 4. Let $L$ be a lattice path matroid on $[n]$ with interval presentation $(N_1, \ldots, N_k)$. The matroid obtained by adding a coparallel element $a$ to 1 and a coparallel element $b$ to $n$ is a lattice path matroid with interval presentation $((a, 1), N_1, \ldots, N_k, \{n, b\})$ and interval ordering $a \leq 1 \leq \cdots \leq n \leq b$.

Proof. Since $(N_1, \ldots, N_k)$ is an ordered collection of incomparable intervals of $[n]$, then $((a, 1), N_1, \ldots, N_k, \{n, b\})$ is an ordered collection of incomparable intervals of $a \leq 1 \leq 2 \leq \cdots \leq k \leq b$. So, $((a, 1), N_1, \ldots, N_r, \{n, b\})$ is a lattice path matroid $L'$ with ground set $[n] \cup \{a, b\}$. To see that $a$ and 1 are coparallel elements in $L'$ suppose that there is a circuit $C$ that contains $a$ but does not contain 1. Then, $C - a$ is an independent set so, $C - a$ is a partial transversal of $(\{a, 1\}, N_1, \ldots, N_k, \{n, b\})$. But since $1 \notin C - a$, then $C - a$ is a partial transversal of $(N_1, \ldots, N_k, \{n, b\})$, and thus $C$ is a partial transversal of $(\{a, 1\}, N_1, \ldots, N_k, \{n, b\})$ which contradicts the choice of $C$. Analogously we show that $n$ and $b$ are coparallel elements in $L'$.

In particular, this implies that if $M$ is obtained from a uniform matroid as a series extension of at most two elements, then $M$ is a lattice path matroid. Also, since edge subdivisions in a graph $G$ correspond to series extensions in $B(G)$, the following lemma is a particular case of Lemma 4. Thus, we consider it to be proved.

Lemma 5. Let $G$ be a graph with a distinguished (colored) edge $e_r$. Suppose that $B(G)$ is a lattice path matroid, with an interval ordering such that $e_r$ is a minimal or a maximal element. If $G'$ is obtained from subdividing $e_r$ (into arbitrarily many edges), then $B(G')$ is a lattice path matroid.

2-sums

Consider a pair of matroids $M_1$ and $M_2$ such that $E(M_1) \cap E(M_2) = \{e\}$. The 2-sum of $M_1$ and $M_2$ over $e$, is the matroid $M_1 \oplus_e M_2$ with ground set $E(M_1) \cup E(M_2) - e$, whose circuits are defined as follows.

1. Every circuit of $M_1 - e$ and $M_2 - e$ is a circuit of $M_1 \oplus_e M_2$.

2. A set $C \subseteq E(M_1) \cup E(M_2) - e$ is a circuit if $C \cap E(M_i) \cup \{e\}$ is a circuit for both $i \in \{1, 2\}$.

In general, neither lattice path matroids nor bicircular matroids are closed under 2-sums. Nonetheless, there are some particular instances where the 2-sum of a pair of bicircular (resp. lattice path) matroids is a bicircular (resp. lattice path) matroid.
In particular, consider a pair of lattice path matroids $L_1$ and $L_2$ with interval presentations $(E_1, N)$ and $(E_2, N')$. Let $N = (N_1, \ldots, N_k)$, $N' = (N'_1, \ldots, N'_m)$, and $e_1 \leq \cdots \leq e_n$ and $e'_1 \leq \cdots \leq e'_m$ be the respective interval orderings of $N$ and of $N'$. Suppose that $E_1 \cap E_2 = \{ e \} = \{ e_n \} = \{ e'_1 \}$. In this case, the 2-sum $L_1 \oplus_e L_2$ is the lattice path matroid with interval presentation $(N_1, \ldots, N_{k-1}, (N_k \cup N'_1) - \{ e \}, N'_2, \ldots, N'_m)$, and interval ordering $e_1 \leq \cdots \leq e_{n-1} \leq e'_2 \leq \cdots e'_m$.

The previous assertion is a straightforward observation. It also follows from Lemma 3.2 in [3], and by noticing that the 2-sum $M_1 \oplus_e M_2$ of a pair of matroids can be obtained by deleting $e$ from the parallel connection of $M_1$ and $M_2$ over $e$.

Consider now a pair of graphs $G_1$ and $G_2$ such that $E(G_1) \cap E(G_2) = \{ e \}$, where $e$ is a loop incident with $v_1 \in V(G_1)$, and incident with $v_2 \in V(G_2)$. If $G$ is the graph obtained by gluing $G_1$ and $G_2$ over the vertices $v_1$ and $v_2$, and removing both copies of $e$, then the bicircular matroid $B(G)$ is the 2-sum $B(G_1) \oplus_e B(G_2)$. In this case, we say that $G$ is a loop sum of $G_1$ and $G_2$. Whenever any two loop sums of $G_1$ and $G_2$ are isomorphic we denote the unique (up to isomorphism) loop sum of $G_1$ and $G_2$ by $G_1 \oplus_l G_2$. The following claim is proved by the arguments in this brief subsection.

**Lemma 6.** Consider a pair of graphs $G_1$ and $G_2$ such that $E(G_1) \cap E(G_2) = \{ e \}$ where $e$ is a loop in $G_1$ and in $G_2$. Let $G$ be the loop sum of $G_1$ and $G_2$ over $e$ and suppose that $B(G_1)$ and $B(G_2)$ are lattice path matroids. If there are interval orderings of $E(G_1)$ and of $E(G_2)$ such that $e$ is a terminal element in both, then $B(G)$ is a lattice path matroid.

**Clones**

To conclude this section we introduce one more notion that will be useful in Section [4]. A pair of elements $e$ and $f$ in a matroid $M$ are called *clones* if the permutation of $e$ and $f$ is an automorphism of $M$. For instance, any pair of elements in a uniform matroid are a pair of clones. We say that a pair of edges $e$ and $f$ of a graph $G$ are clones if $e$ and $f$ are clones in $B(G)$. In particular, any pair of parallel edges are clone edges.

Consider a matroid $M$ and a set $S \subseteq E$ of clones in $M$. A *clone extension* of $M$ over $S$, is a matroid $M + f$ such that $f \in S \cup \{ f \}$ is a set of clones in $M + f$. In general, there might be more than one non-isomorphic clone extensions. For instance, if $S = \{ e \}$ the parallel and the coparallel extensions of $e$ are clone extensions of $M$ over $S$.

**Lemma 7.** Let $M$ be a matroid and $S$ a set of clones of $M$. If $S$ is a circuit of $M$, then there is a unique (up to isomorphism) clone extension of $M$ over $S$.

**Proof.** Consider a pair of clone extensions $M + e$ and $M + f$ of $M$ over $S$. We verify that $\varphi: M + e \to M + f$ defined as the identity on $E$ and $\varphi(e) = f$, is an isomorphism. We do so by observing that this function maps circuits to circuits, and the claim follows by symmetry. Clearly, every circuit of $M + e$ to which $e$ does not belong, is mapped to a circuit of $M + f$ to which $f$ does not belong. Similarly, if $C$ is a circuit of $M + e$ such that $e \in C$ but $s \not\in C$ for some $s \in S$, then $C - e + s$ is a circuit of $M + e$ (because $s$ and $e$ are clones). Since $C - e + s \subseteq E$, then $C - e + s$ is a circuit of $M + f$, and so, $(C - e + s) - s + f$ is a circuit of $M + f$, i.e., $\varphi[C] = C - e + f$ is a circuit of $M + f$. To conclude the proof, we argue that for every circuit $C$ of $M + e$ to which $e$ belongs, there is some $s \in S$ such that $s \not\in C$ (and thus, the claim will follow by the previously considered cases). Anticipating a contradiction, suppose that there is some circuit $C$ of $M + e$ such that $S \cup \{ e \} \subseteq C$. Since $M + e$ is an extension of $M$, then $S$ is a circuit of $M + e$. Hence, $C$ properly contains another circuit, which contradicts the fact that circuits are minimal dependent sets.

Clearly, any bicircular matroid $B(G)$ together with any set of three parallel edges of $G$, satisfy the hypothesis of Lemma [4]. We use this observation to prove the following result.

**Lemma 8.** Let $G$ be graph and let $f_1$, $f_2$ and $f_3$ be three different parallel edges. If $B(G)$ is a lattice path matroid, then $B(G + f)$ is a lattice path matroid whenever $f$ is added to the parallel class of $f_1$, $f_2$ and $f_3$. 


Moreover, if $B(G)$ has an interval ordering where $f_1$ is its minimum (resp. maximum), then $B(G + f)$ has an interval ordering where $f_1$ is its minimum (resp. maximum).

**Proof.** We begin with the following observation about transversal matroids. Consider a bipartite graph $H(E,Y)$, and let $T$ be the transversal matroid presented by $H$ (with ground set $E$). Let $e_1e_2e_3$ be a circuit of $T$. It is evident that $|N_H\{e_1,e_2,e_3\}| = 2$. Moreover, at least one of these elements has degree two in $H$; otherwise, by the pigeonhole principle, two of them would be leaves of $H$ with a common support, i.e., parallel elements in $T$. Also notice that any two elements $e, f \in E$ with the same neighborhood in $H$ are clones in $T$. In particular, any pair of parallel edges in a graph are clone elements in its bicircular matroid.

Let $G, f_1, f_2, f_3$ and $f$ be as in the hypothesis. In particular, $B(G + f)$ is a clone extension of $G$ over $\{f_1, f_2, f_3\}$. Let $(E,N)$ be an interval presentation of $B(G)$ with interval ordering $e_1 \leq \cdots \leq e_k \leq f_1 \leq \cdots \leq e_n$. By the choice of $f_1, f_2$ and $f_3$, and by the arguments in the previous paragraph, we assume that $f_1$ belongs to exactly two sets of $N$. Consider the presentation $(E \cup \{f\}, N')$, where $f$ belongs to the same two sets as $f_1$. Then, this is an interval presentation of a clone extension $B(G) + f$ over $\{f_1, f_2, f_3\}$ where any of the following is an interval ordering of $B(G) + f$

$$e_1 \leq \cdots \leq f_1 \leq f \leq e_k \leq \cdots \leq e_n, \text{ and } e_1 \leq \cdots \leq e_k-2 \leq f \leq f_1 \leq \cdots \leq e_n.$$

Since $B(G + f)$ and $B(G) + f$ are clone extensions of $B(G)$ over $\{f_1, f_2, f_3\}$, and $f_1f_2f_3$ is a circuit of $B(G)$, by Lemma 7 we conclude that $B(G + f) \cong B(G) + f$. Thus, $B(G + f)$ is a lattice path matroid. The moreover statement follows from the previously defined interval orderings. 

3 Excluded minors

In [3], Bonin exhibits an infinite family of excluded minors for lattice path matroids. In this section we introduce eight bicircular excluded minors for lattice path matroids. Some of these will have structural implications about those graphs with lattice path bicircular matroids. We begin by exhibiting the prism $C^{2,4}$ and the four graphs whose bicircular matroid is isomorphic to this prism.

![Figure 1: To the right, an affine representation of $C^{2,4}$. To the left, four graphs whose bicircular matroid is isomorphic to $C^{2,4}$.](image)

Recall that a block of a graph $G$ is a maximal 2-connected subgraph. An end block is a block of $G$ that is a leaf in the block tree of $G$; a middle block is a block of $G$ that is not an end block. In particular, if $B(G)$ is a connected matroid, then every end block of $G$ contains at least one cycle (i.e., $G$ has no loopless leaves). Furthermore, suppose that $G$ has three different end blocks. By contracting all of $G$ except these three end
blocks, we can notice that $G$ contains $P_i$ as a minor for some $i \in \{1, 2, 3, 4\}$ (these graphs are depicted in Figure 1). Therefore, if $G$ is a graph such that $B(G)$ is a connected matroid with no $C^{2,4}$-minor, then the block tree of $G$ is a path.

**Proposition 9.** Let $G$ be a graph such that $B(G)$ is a connected matroid. If $B(G)$ does not contain $C^{2,4}$ as a minor, then the block tree of $G$ is a path. Moreover, every middle block contains exactly two vertices.

**Proof.** In the paragraph preceding this proposition, we showed that the block tree of $G$ is a path. This also implies that every middle block contains exactly two cut vertices of $G$. To prove the moreover statement, and anticipating a contradiction, suppose that there is a middle block $MB$ of $G$ with at least three vertices. Without loss of generality, assume that $MB$ is the unique middle block of $G$; otherwise contract every middle block that is not $MB$. So, let $EB_1$, $MB$ and $EB_2$ be the blocks of $G$. Let $x \in MB \cap EB_1$ and $y \in MB \cap EB_2$ be the cut vertices of $G$ in $MB$. Since $B(G)$ is connected, $G$ has no loopless leaves. Thus, there are two possible scenarios for each end block $EB_i$: Either there is a cycle in $EB_i$, or $EB_i$ consist of an edge together with a loop incident with the end vertex. We consider the case when $EB_1$ and $EB_2$ contain a cycle. Since $MB$ is two connected and it is not an edge, there are two internally disjoint $xy$-paths $Q_1$ and $Q_2$. Furthermore, since $MB$ has at least three vertices, then we choose $Q_1$ such that it is not an edge. Finally, let $C_1$ and $C_2$ be a pair of cycles of the end blocks. Consider the following contractions of $G$: Contract $Q_1$ to a path of length 2; contract $Q_2$: contract $C_1$ to a pair of parallel edges; and contract $C_2$ to a pair of parallel edges. The previous minor of $G$ is isomorphic to the graph $P_2$ (Figure 1). Thus, $B(G)$ contains a $C^{2,4}$-minor. When one or both end blocks consist of an edge together with a loop, we find either of $P_3$ or $P_4$ as a minor of $G$, and the claim follows by similar arguments. □

Consider the four point line with ground set $\{1, 2, 3, 4\}$, and add three elements $1'$, $2'$, and $3'$, coparallel to 1, to 2 and to 3, respectively. Then, by contracting 4 we obtain $C^{2,4}$ as a contraction minor. Now, consider a uniform matroid $U_{r,n}$ where $2 \leq r \leq n - 1$. It is not hard to notice, that any four points of $U_{r,n}$ belong to a four point line minor of $U_{r,n}$. Therefore, if we add elements in series to at least three different points of $U_{r,n}$, we obtain a matroid $M$ with a $C^{2,4}$-minor. This arguments prove the following statement.

**Lemma 10.** Let $r$ and $n$ be a pair of non-negative integers and consider the uniform matroid $U_{r,n}$. If $2 \leq r \leq n - 1$, and $M$ is a series extension of at least three different elements of $U_{r,n}$, then $M$ contains a $C^{2,4}$-minor. □

The matroids listed in Theorem 2 are presented in Figures 1, 2, and 3. In each case, we depict a bicircular matroid and an affine representation of these matroids. With a brief look at Figure 2 in 3, the reader can realize that indeed, $C^{2,4}$, $W^3$, $A^3$, $R^3$, $R^4$, $D^4$ and $(S^1)^*$ are excluded minors for the class of lattice path matroids. It is not immediate to conclude from the description in 3 that $B^1$ is not a lattice path matroid. Nonetheless, this can easily be observed using Proposition 3.

**Proposition 11.** Each of the following matroids is a bicircular matroid that is not a lattice path matroid: $C^{2,4}$, $W^3$, $A^3$, $R^3$, $R^4$, $D^4$, $B^1$, and $S^3$.

**Proof.** From the corresponding illustration, it is clear that these matroids are bicircular matroids. As we already mentioned, with a brief look at 3, the reader can realize that indeed, $C^{2,4}$, $W^3$, $A^3$, $R^3$, $R^4$, $D^4$ and $(S^1)^*$ are excluded minors for the class of lattice path matroids. Since lattice path matroids are closed under duality 4, then $S^1$ is an excluded minor for the class of lattice path matroids. To see that $B^1$ is not a lattice path matroid it suffices to notice that their corresponding fundamental flats do not form two chains ordered by inclusion (Proposition 3). For instance, any four points of $B^1$ such that no three of them belong to a common line, are a spanning circuit. Thus, each three point line of $B^1$ is spanned by some spanning circuit, which implies that any three point line is a fundamental flat of $B^1$. So, there are three incomparable fundamental flats of $B^1$, which implies that the fundamental flats of $B^1$ do not form two disjoint chains ordered by inclusion. Thus, we conclude that $B^1$ is not a lattice path matroid. □
Figure 2: Six excluded minors to the class of lattice path matroids: $W^3$, $B^1$, $R^3$, $D^4$, $A^3$, and $R^4$. In each case there is an affine representation, together with all corresponding bicircular presentations.

Figure 3: To the left, three bicircular presentations of $S^1$. Since this is a rank 5 matroid, we choose to present an affine representation of its dual $(S^1)^*$. $S^1$ is an excluded bicircular minor to lattice path matroids.

4 Graph families

In this section, we define five graph families $F_i$ with $i \in \{0, \ldots, 4\}$ and we show that the graphs in these families have lattice path bicircular matroids. Throughout this section, we will heavily rely on Lemma 5 to argue that subdivisions of certain graphs have lattice path bicircular matroids. In particular, we will use that any series extension of two elements of a uniform matroid is a lattice path matroid. We will also rely on Lemma 8. This lemma asserts that if a graph $G$ has a lattice path bicircular matroid, then $G + f$ has a lattice path bicircular matroid whenever $f$ is added to a parallel class that contains three different edges.
**Family** $\mathcal{F}_0$

We begin with the simplest construction. Consider the graphs $K'_{2,3}$, $K''_{2,3}$ and $K^*_{2,3}$ obtained from $K_{2,3}$ by adding certain parallel edges or a loop as follows.

![Graphs K'_{2,3}, K''_{2,3}, and K^*_{2,3}](image)

For the sake of clarity in the upcoming arguments, we distinguish some edges (red and blue) of $K'_{2,3}$, of $K''_{2,3}$ and of $K^*_{2,3}$ as depicted above (we will use this technique through the rest of this work). A graph $G$ belongs to $\mathcal{F}_0$ if either of the following hold:

1. $G$ is a subdivision of $K_{2,3}$,
2. $G$ is a subdivision of blue and/or red edges of $K'_{2,3}$ of $K''_{2,3}$ or of $K^*_{2,3}$, or
3. $G$ is a subdivision of at most two edges of $K_4$.

In particular, notice that any subdivision of $K_{2,3}$ is a theta graph. Thus, $B(G)$ is the uniform matroid of rank $|E(G)| - 1$ and $|E(G)|$ elements. Clearly then, $B(G)$ is a lattice path matroid. Also, it is not hard to notice that the bicircular matroid of $K_{2,3}$ is $U_{3,6}$. Thus, by Lemma 5 the bicircular matroid of any subdivision $G'$ of any two edges of $K_4$ is a lattice path matroid.

**Proposition 12.** The bicircular matroid of each graph in $\mathcal{F}_0$ is a lattice path matroid.

**Proof.** Given the arguments above, we only need to show that when $G$ is a subdivision of blue and red edges of $K'_{2,3}$, of $K''_{2,3}$, or of $K^*_{2,3}$ then $B(G)$ is a lattice path matroid. The last follows by noticing that the bicircular matroid $B(K'_{2,3})$ is a series extension of $U_{3,6}$. Indeed, if we contract each pair of red and blue edges to a single red and blue edge, respectively, we obtain a graph $2K_3$ whose bicircular matroid is $U_{3,6}$. Thus, the bicircular matroid of any subdivision $G'$ of any two edges of $K_4$ is a lattice path matroid.

It is not hard to notice that there is a color preserving isomorphism from $B(K'_{2,3})$ to $B(K''_{2,3})$. So, to conclude this proof, it suffices to consider the case when $G$ is a subdivision of $K'_{2,3}$. Now, we propose the following lattice path presentation of the bicircular matroid of $K'_{2,3}$.

![Lattice Path Presentation](image)

It is not hard to notice that any color preserving labeling of the edges of $K'_{2,3}$ with $\{1, \ldots, 7\}$, yields an isomorphism from $B(K'_{2,3})$ to the lattice path matroid $L$. Moreover, this isomorphism defines an interval ordering of $E(K'_{2,3})$, where the minimum is a red edge, and the maximum is a blue edge. Thus, by Lemma 5 if $G$ is obtained from $K'_{2,3}$ by subdividing any blue and any red edge, then $B(G)$ is a lattice path matroid. Therefore, the statement of this proposition holds true.\[\square\]
**Family \( \mathcal{F}_1 \)**

Our most basic building block for the family \( \mathcal{F}_1 \) is the graph \( G_1 \), obtained from the 4 cycle by duplicating two non consecutive edges. We distinguish the parallel edges by thinking of one class as blue edges, and the other as red edges. Given three non-negative integers \( r, b \) and \( d \), we denote the graph \( G_1(r, d, b) \) obtained from \( G_1 \) by adding \( r \) red parallel edges (to the red edges), \( b \) blue parallel edges (to the blue edges), and \( d \) parallel diagonal edges. From left to right, the following is a depiction of \( G_1 = G_1(0, 0, 0) \), of \( G_1(1, 3, 1) \), and \( G(r, d, b) \), where the dashed edges represent \( r, d \) and \( b \) parallel copies.

The colors on the edges of \( G_1 \) are useful to define the family \( G_1(r, d, b) \). This colors are also convenient to define the family \( \mathcal{F}_1 \), but in this case, we extend this coloring when either of \( r, d \), or \( b \) equal 0. In Figure 4 we illustrate these edge colorings. With these figures in mind, we define the family \( \mathcal{F}_1 \). A graph \( G \) belongs to \( \mathcal{F}_1 \) if the following holds:

1. \( G \) is a subdivision of at most one red edge and at most one blue edge of \( G(r, d, b) \) for some non-negative integers \( r, d \) and \( b \) (see Figure 4).

![Graphs](image)

**Figure 4:** The generating set of graphs for \( \mathcal{F}_1 \). Each dashed edge represents \( r, d \) and \( b \) red, blue and diagonal parallel edges, respectively. Non-bended edges of \( G_1 \) are both blue and red.

Notice that \( G_1(r, d, b) \) is obtained by adding an edge to a parallel class to either of \( G_1(r - 1, d, b) \), \( G_1(r, d - 1, b) \) or \( G_1(r, d, b - 1) \). Thus, by Lemma 8 we conclude that if \( G_1(1, 3, 1) \) has a lattice path bicircular matroid, then \( G_1(r, d, b) \) has a lattice path bicircular matroid for every \( r, b \geq 1 \) and \( d \geq 3 \). Furthermore, it is obvious that if \( G_1(r, d, b) \) has a lattice path bicircular matroid, then \( G_1(r', d', b') \) has a lattice path bicircular matroid for any \( r' \leq r, d' \leq d \) and \( b' \leq b \). Both of these observations together, show that if \( G_1(1, 3, 1) \) has a lattice path bicircular matroid, then \( G_1(r, d, b) \) has a lattice path bicircular matroid for all non-negative integers \( r, d \) and \( b \).

**Lemma 13.** For all non-negative integers \( r, d \) and \( b \), the graph \( G_1(r, d, b) \) has a lattice path bicircular matroid, with an interval ordering where the minimum (resp. maximum) element is a red (resp. blue) edge.

**Proof.** Suppose that \( B(G_1(1, 3, 1)) \) is a lattice path matroid. By the arguments preceding this paragraph, we know that \( B(G_1(r, d, b)) \) is a lattice path matroid for all non-negative integers \( r, d \) and \( b \). Furthermore,
with the same arguments and the “moreover” statement of Lemma 8, we conclude that if \( B(G_1(1,3,1)) \) has an interval ordering where the minimum (resp. maximum) element is a red (resp. blue) edge, then so does \( B(G_1(r,d,b)) \). Thus, it suffices to show that this lemma holds for \( r = 1, d = 3 \) and \( b = 1 \). This follows from the lattice path presentation of \( B(G_1(1,3,1)) \) in Figure 5.

![Figure 5](image)

**Figure 5:** The graph \( G_1(1,3,1) \) together with a lattice path matroid \( L \), where the labels describe an isomorphism \( \varphi: B(G_1(1,3,1)) \rightarrow L \). Indeed, it is not hard to observe that \( \varphi \) defines a bijection between the families of circuits of rank at most 3, and that it defines a bijection on the collections of bases. Since \( B(G_1(1,3,1)) \) and \( L \) are rank 4 matroids, we conclude that \( \varphi \) defines a bijection on the family of all circuits.

Recall that a pair of elements in a matroid \( M \) are clones, if permuting these elements yields an automorphism of \( M \). It is not hard to see for each graph depicted in Figure 4, the blue and the red edges correspond to a pair of classes of clone elements in the corresponding bicircular matroid. Thus, for any of these graphs, any two subdivisions, \( G' \) and \( G'' \), of at most one red edge and at most one blue edge (into the same number of blue and red edges, respectively) it holds that \( B(G') \cong B(G'') \).

**Proposition 14.** The bicircular matroid of each graph in \( \mathcal{F}_1 \) is a lattice path matroid.

**Proof.** By Lemma 13, the bicircular matroid of each graph \( G_1(r,d,b) \) has an interval ordering where the minimum and maximum elements are a red and a blue edge, respectively. By the arguments in the paragraph above this lemma, any two subdivisions, \( G' \) and \( G'' \), of at most one red edge and at most one blue edge (into the same number of blue and red edges, respectively) it holds that \( B(G') \cong B(G'') \). Since we can choose an interval ordering of \( B(G_1(r,d,b)) \) where the minimum (resp. maximum) element is a red (resp. blue) edge, we conclude by Lemma 8 that if \( G' \) is a subdivision of at most one red edge and at most one blue edge of \( G_1(r,d,b) \), then \( B(G') \) is a lattice path matroid. Recall that each graph in \( \mathcal{F}_1 \) is obtained by subdividing at most one red edge and at most one blue edge of some \( G_1(r,d,b) \). The claim follows. \( \Box \)

**Family \( \mathcal{F}_2 \)**

This family is constructed in a similar fashion to the construction of \( \mathcal{F}_1 \). Denote by \( 2K_3 \) the graph obtained from \( K_3 \) by duplicating each edge. Again, we distinguish a pair of parallel classes, one by color red and the other one by blue. Similar to how we did in the previous subsection, for a pair of non-negative integers \( r \) and \( b \), we denote by \( 2K_3(r,b) \) the graph obtained from \( 2K_3 \) by adding \( r \) red parallel edges (to red edges) and \( b \) blue parallel edge (to blue edges). From left to right, the following is a depiction of \( 2K_3 = 2K_3(0,0) \), \( 2K_3(1,1) \), and of \( 2K_3(r,b) \), where the dashed edges represent \( r \) and \( b \) parallel copies.
As previously mentioned, we construct $F_2$ in a similar manner to how we constructed $F_1$. A graph $G$ belongs to $F_2$ if the following holds:

1. $G$ is a subdivision of at most one red edge and at most one blue edge of $2K_3(r, b)$ for some non-negative integers $r$ and $b$.

**Proposition 15.** The bicircular matroid of each graph in $F_2$ is a lattice path matroid.

**Proof.** We begin by proposing the following lattice path presentation of $B(2K_3(1,1))$.

Denote by $J$ the lattice path matroid on the right. One can notice that $B(2K_3(1,1))$ and $J$ are isomorphic matroids because the only non-spanning circuits in both matroids are 123 and 678. This lattice path presentation has an interval ordering where the minimum (resp. maximum) is a red (resp. blue) edge.

As mentioned before, any class of parallel edges is a set of cloned edges. Thus, any two subdivisions of $2K_3(r, b)$ of at most one red edge and at most one blue edge (into the same number of blue and red edges, respectively) have isomorphic bicircular matroids. By the arguments in the first paragraph, we can choose an interval ordering where the minimum (resp. maximum) is a red (resp. blue) edge.

As mentioned before, any class of parallel edges is a set of cloned edges. Thus, any two subdivisions of $2K_3(r, b)$ of at most one red edge and at most one blue edge (into the same number of blue and red edges, respectively) have isomorphic bicircular matroids. By the arguments in the first paragraph, we can choose an interval ordering where the minimum (resp. maximum) is a red (resp. blue) edge.

FAMILY $F_3$

Similar to previous cases, we begin by introducing the building blocks of $F_3$. Consider three non-negative integers $r$, $j$ and $l$. We denote by $K_3(r, j, l)$ the graph obtained from $K_3$ by adding $l$ loops incident with the same vertex $v$; by adding $j$ parallel edges to some edge of $K_3$ incident with $v$; and by adding $r$ parallel edges to the (unique) edge of $K_3$ not incident with $v$. In Figure 6 we depict $K_3(r, j, l)$ together with a couple of particular instances. Again, we use blue and red color to distinguish some edges, and this is useful for the definition of $F_3$.

In particular, notice that the bicircular matroids of $K_3(1,1,0)$ and $K_3(1,0,1)$ are isomorphic to the uniform matroid $U_{3,5}$. Also notice that for any pair of non-negative integers, $r$ and $j$, the graph $K_3(r, j, 0)$ is a subgraph of $2K_3(r, j)$. We construct $F_3$ in a similar fashion to how we constructed $F_1$. This is done by following the edge colorings depicted in Figure 6. A graph $G$ belongs to $F_3$ if either of the following statements hold:

1. $G$ is a subdivision of at most one red edge and at most one blue edge of $K_3(r, j, l)$ for some non-negative integers $r$, $j$ and $l$, with $r \geq 1$,
2. $G$ is a cycle with one chord $e$, and possibly some edges parallel to $e$, and at most one loop in each end point of $e$,
3. $G$ is a chordless cycle with arbitrarily many loops in at most one vertex $v$, and at most one loop in one neighbor of $v$, or
4. $G$ is a connected graph on two vertices.
Figure 6: The generating set of graphs for $F_3$. Each dashed edge represents $r$ and $j$ parallel edges, and the dashed loop represents $l$ loops incident in the same vertex. The non-bend edges of $K_3(1,1,0)$ and of $K_3(1,0,1)$ are both blue and red.

Lemma 16. For every graph $G$ in $F_3$ there is a graph $H \in F_1 \cup F_2$ such that $B(G)$ is a minor of $B(H)$.

Proof. We consider the four possible scenarios for a graph in $F_3$, and we begin with the last one. Suppose that $G$ is a connected graph on two vertices, $x$ and $y$. Let $l_x$ denote the number of loops on $x$, $l_y$ the number of loops on $y$, and $m$ the number of $xy$-edges. Consider the graph $H$ obtained from $G_1(l_x, m, l_y)$ by contracting one blue and one red edge. By removing one loop from each vertex of $H$ and two of the non-loop edges, we see that $G$ is a subgraph of $H$, and thus a minor of $G_1(l_x, m, l_y) \in F_1$. Below, we depict these graphs when $l_x = l_y = 1$ and $m = 2$.

Now suppose that $G$ satisfies the third statement of the definition of $F_3$. Let $m$ be the length of the cycle, $l_v$ denote the number of loops on $v$, and suppose that a neighbor $u$ of $v$ is incident with one loop; the case when no neighbor of $v$ has a loop, follows from the case we are now considering. In this case, consider the graph $H$ obtained by subdividing the top edge of $G(0,0,l_v−1)$ into $m−3$ new edges ($m−3 \geq 0$ since $m \geq 3$). Since the top edge of $G(0,0,l_v−1)$ is a red edge (see Figure 4), then $H \in F_1$. Finally, by contracting one of the red bend edges and one of the blue edges of $H$ we see that $G$ is a minor of $H$. We illustrate the case when $l_v = 3$ below, where the dotted edges represent a path of length $m−2$.

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Our next case is when $G$ is a cycle with two distinguished vertices $u$ and $v$, such that $u$ and $v$ have at most one loop each, and any chord of the cycle is a $uv$-edge. Analogous to the previous case, we only consider the case when $u$ and $v$ are incident with a loop each. Let $m$ be the number of $uv$-edges, $a_1$ the length of one of the $uv$-arcs of the cycle, and $a_2$ the length of the other. Let $H$ be the graph obtained form $G_1(0, m, 0)$ by subdividing the top and bottom edges into $a_1 - 1$ and $a_2 - 1$ edges, respectively. Since the top edge and bottom are red and blue edges of $G_1(0, m, 0)$ (see Figure 4), we conclude that $H \in \mathcal{F}_1$. In this case, $G$ is the minor of $H$ obtained by contracting of edge of each parallel class. We illustrate the case when $m = 2$ as follows, the dotted edges represent the corresponding subdivisions.

The final case is when $G$ is a subdivision of at most one red edge and at most one blue edge of $K_3(r, j, l)$. As we already observed, $K_3(r, j, 0)$ is a subgraph of $2K_3(r, l)$. Thus, $G$ is a subgraph of a subdivision $H$ of at most one red and at most one blue edge of $2K_3(r, j)$. The claim follows since $H \in \mathcal{F}_2$. Now suppose that $l \geq 1$, and let $H$ be the graph obtained from $G_1(r + 1, j + 1, l)$ by contracting one blue edge. In this case, $K_3(r, j, l)$ is a subgraph of $H$. We illustrate the case when $r = j = 1$ and $l = 2$ below.

By preserving the edge colors in the operations previously described, we can easily see that the red edges coincide with the colors described in Figure 6. So, if $G$ is a subdivision of at most one red edge of $K_3(r + 1, j + 1, l)$, then $G$ is a minor of a subdivision $H$ of at most one red edge of $G_1(r + 1, j + 1, l)$. Finally, notice that the only case when the set of blue edges of $K_3(r, j, l)$ is not empty, is when $l = 1$ (see Figure 6). Also notice, that when $l = 1$ then the blue edges and the unique loop in $K_3(r, j, l)$ are clone edges. Thus, subdividing a blue edge yields the same bicircular matroid as subdividing the loop. Following the operations previously defined, we can notice that blue edges of $G_1(r + 1, j + 1, l)$ are contracted to loop edges in $H$. Thus, for any subdivision $G$ of at most one red edge and at most one blue edge of $K_3(r, j, l)$, there is a subdivision $H$ of at most one red edge and at most one blue edge of some $G_1(r + 1, j + 1, l)$ such that $B(G)$ is a minor of $B(H)$. Since this covers the last case of the definition of $\mathcal{F}_3$, the claim is proved.

**Proposition 17.** The bicircular matroid of each graph in $\mathcal{F}_3$ is a lattice path matroid.

**Proof.** By Lemma 10, we know that the bicircular matroid of each graph in $\mathcal{F}_3$ is a minor of the bicircular matroid of some graph in $\mathcal{F}_1 \cup \mathcal{F}_2$. Since the class of lattice path matroid is a minor closed class [4], the claim follows by Propositions 14 and 15.

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**Family $\mathcal{F}_4$**

Our final family consists of certain graphs with cut vertices. The restrictions upon the middle blocks is that each middle block contains exactly two vertices (no restriction upon the edge set). To define the constraints over the end blocks of these graphs, we depict three possible end blocks. Again, we distinguish some red edges, and in this case we distinguish one vertex which corresponds to the cut vertex of the end block.

We first define an auxiliary family $\mathcal{F}_4'$. A graph $G$ belongs to $\mathcal{F}_4'$ if it is a loopless graph with the following properties:

1. the block graph of $G$ is a path,
2. every middle block of $G$ has exactly two vertices, and
3. each end block of $G$ is a subdivision of at most one red edge of either of the graphs above, where $x$ is a cut vertex.

An example of a graph in $\mathcal{F}_4'$ looks as follows, where one blue and one red edge might be subdivided (dashed edges represent multiple parallel copies, and dots represent arbitrarily many blocks of two vertices).

A graph belongs to $\mathcal{F}_4$ if it can be obtained from a graph in $\mathcal{F}_4'$ by adding loops to cut vertices and removing edges. In particular, any graph whose block tree is a path, and its blocks have exactly two vertices, belongs to $\mathcal{F}_4$. Also, any graph that satisfies items 1 and 2 above, and its end blocks are loopless cycles, also belongs to $\mathcal{F}_4$. Consider a graph $G$ obtained from some $H \in \mathcal{F}_4'$ by adding loops to one cut vertex. It is not hard to notice that $G$ can also be obtained from some $H' \in \mathcal{F}_4'$ by contracting one edge in some middle block. Thus, we inductively conclude that every graph in $\mathcal{F}_4$ is a minor of some graph in $\mathcal{F}_4'$.

**Proposition 18.** The bicircular matroid of every graph in $\mathcal{F}_4$ is a lattice path matroid.

**Proof.** Recall that lattice path matroids are closed under minors [4]. Thus, it suffices to show that all graphs in $\mathcal{F}_4'$ have lattice path bicircular matroids. To begin with, consider the following graphs.

On the one hand, the bicircular matroid of a subdivision $G$ of at most one red edge of $K_3(r, j, 1)$ is a lattice path matroid (Proposition 17). Clearly, all black parallel edges together with the loop, are a fundamental flat of $B(G)$. So, by Proposition 9 we know that there is an interval ordering where these edges (and the loop) are a final segment. Moreover, since this is a set of clone edges, we can choose an interval ordering of $E(G)$ where the loop is the maximum element. On the other hand, the bicircular matroid of $mK_2'$ is the uniform matroid $U_{2, m+2}$, so it is a lattice path matroid. Since any pair of elements in a uniform matroid are clones, then there is an interval ordering of the edge set of $mK_2'$ where the loops are the minimum and
maximum elements. Recall that the loop sum of a pair of graphs is obtained by gluing two graphs over a loop, and then deleting this loop. Now, notice that any graph in $F'_4$ is a loop sum of the form

$$EB_1 \oplus_l MB_1 \oplus_l \cdots \oplus_l MB_k \oplus_l EB_2,$$

where $EB_1$ and $EB_2$ are a subdivision of at most one red edge of either $K_3(r, j, 1)$, $K_3(1, 1, 1)$ or $K_3(1, 0, 1)$, and the graphs $MB_i$ are a graph on two vertices with exactly one loop in each vertex. Thus, by the arguments above, and by Lemma 6 the bicircular matroid of the loop sum above is a lattice path matroid. Therefore, the bicircular matroid of each graph in $F'_4$ is a lattice path matroid. So, the claim follows because every graph in $F'_4$ is a minor of some graph in $F_4$.

**Efficient recognition**

To conclude this section, we observe that it takes linear time to recognize the graph families $F_i$ with $i \in \{0, 1, 2, 3, 4\}$. Consider a connected graph $G$ and let $H$ be the graph obtained from $G$ after contracting all subdivided edges. By definition of each $F_i$, if $G$ belongs to some $F_i$ with $i \in \{0, 1, 2, 3, 4\}$, then $H$ must be either of the following graphs:

1. $K_{2,3}$, $K'_{2,3}$, $K''_{2,3}$, $K_{2,3}$ or $K_4$,
2. $G(r, d, b)$ for some non-negative integers $r$ and $b$,
3. $2K_3(r, b)$ for some non-negative integers $r$ and $b$,
4. $K_3(r, j, l)$ for some non-negative integers $r$, $j$ and $l$, or a graph on two vertices, or
5. a graph whose block tree is a path, every middle block has exactly two vertices, and each block has two vertices or it is some graph $K_3(r, j, l)$ for some non-negative integers $K_3(r, j, l)$.

Each of these classes can be recognized in linear time with respect to the size of the edge set of the input graph. Moreover, after contracting subdivided edges of graph $G$, and keeping track of which edges correspond to subdivision classes, we can determine if $G$ belongs to $\bigcup_{i=0}^4 F_i$ in linear time with respect to $|E(G)|$. By these arguments we observe the following.

**Observation 19.** Given an input graph $G$, there is a linear time algorithm (in $|E(G)|$) to decide if $G$ belongs to the union

$$F_0 \cup F_1 \cup F_2 \cup F_3 \cup F_4.$$

## 5 Excluded minors and graph families

We have arrived to the most technical section of this work. Here, we show if a connected bicircular matroid $B(G)$ is $ex_B(\mathcal{L})$-minor free, then $G$ belongs to a family $F_i$ for some $i \in \{0, 1, 2, 3, 4\}$. The set $ex_B(\mathcal{L})$ consists of the matroids introduced in Section 3 namely, the matroids

$\mathcal{W}_3$, $\mathcal{C}^{2,4}$, $R^3$, $R^4$, $D^1$, $A^3$, $B^1$, and $S^1$.

**Proposition 20.** Let $G$ be a graph. If $B(G)$ is an $ex_B(\mathcal{L})$-minor free connected matroid, then

$$G \in F_0 \cup F_1 \cup F_2 \cup F_3 \cup F_4.$$

**Proof.** This proof is divided in three main cases: Non-outerplanar graphs (Proposition 22), 2-connected graphs (Proposition 28), and outerplanar graphs with a cut vertex (Proposition 30). The claim follows from these three propositions. 

The proofs of Propositions 22, 28, and 30 are subdivided in several cases. Regardless of the particular subcase in turn, the proof idea is very simple: Given a graph $G$ we verify that if $G$ does not belong to some family $F_i$, then there is a graph $G'$ with a minor $H$ such that $B(G) \cong B(G')$ and $B(H) \in ex_B(\mathcal{L})$ (in most cases $G' = G$). In Figures 1, 2 and 3 we depict bicircular and affine presentations of matroids in $ex_B(\mathcal{L})$. 


Non-outerplanar graphs

A well-known characterization of outerplanar graphs states that a graph is an outerplanar graph if it does not contain a subdivision of $K_4$ nor a subdivision of $K_{2,3}$ [5]. We first consider the case when $G$ contains a subdivision of $K_4$, and later we consider the general case of non-outerplanar graphs.

Recall that the bicircular matroid of $K_4$ is $U_{4,6}$, and that the operation of subdividing an edge $e$ in a graph $G$, translates to series extension of $e$ in $B(G)$. Lemma [10] asserts that if $M$ is a series extension of at least three different elements of a uniform matroid (of rank at least two and positive corank), then $M$ contains a $C^{2,4}$-minor. So, we conclude that if $G$ contains a subdivision of at least three different edges of $K_4$, then $B(G)$ contains a $C^{2,4}$-minor.

Lemma 21. Let $G$ be a graph such that $B(G)$ is an $ex_B(\mathcal{L})$-minor free connected matroid. If $G$ contains a subdivision of $K_4$, then $G$ equals a subdivision of at most two edges of $K_4$. In particular, this implies that $G \in \mathcal{F}_0$.

Proof. By the arguments preceding this lemma, we know that any subdivision $G$ of at least three edges of $K_4$, satisfies that $B(G)$ has a $C^{2,4}$-minor. Thus, if $B(G)$ is an $ex_B(\mathcal{L})$-minor free matroid, and $G$ is a subdivision of $K_4$, then $G$ equals a subdivision of at most two edges of $K_4$. We proceed to show that if $G$ contains a subdivision of $K_4$, the $G$ is a subdivision of $K_4$. First, we observe that the bicircular matroids of the following graphs contain some minor in $ex_B(\mathcal{L})$.

Consider the minor of $K_4'$ obtained by contracting the edge $xy$. This minor is isomorphic to $A_3$ (Figure 2). Similarly, by contracting one of the $xz$-edges of $K_4'$ we also obtain $A_3$ as a minor of $K_4'$. Thus, $B(K_4')$ and $B(K_4'')$ contain $A^3$ as a minor. To conclude the proof, suppose that $G$ contains a subdivision $H$ of $K_4$, and $G \neq H$. So, there is an edge in $E(G) \setminus E(H)$, and by contracting $H$ to $K_4$, we find either $K_4'$ or $K_4''$ as a minor of $G$. This contradicts the fact that $B(G)$ is $ex_B(\mathcal{L})$-minor free, and so the claim follows.

To conclude this first scenario, we consider the general case of non-outerplanar graphs. We proceed by doing some case checking: We list all possible minimal supergraphs of $K_{2,3}$ that do not belong to $\mathcal{F}_0$, and we see that the bicircular matroids of these graphs have an $ex_B(\mathcal{L})$-minor. This (almost) exhaustive case checking is the same technique used in the rest of this section.

Proposition 22. Let $G$ be a non-outerplanar graph. If $B(G)$ is an $ex_B(\mathcal{L})$-minor free connected matroid, then $G$ belongs to $\mathcal{F}_0$.

Proof. Throughout this proof we assume that $G$ and $B(G)$ are as in the hypothesis. Also, if $G$ contains no subdivision of $K_{2,3}$ then it contains a subdivision of $K_4$, and the claim follows by Lemma 21. So, we also assume that $G$ contains a subdivision of $K_{2,3}$. Since any subdivision of $K_{2,3}$ belongs to $\mathcal{F}_0$, we suppose that $G$ contains a subdivision $H$ of $K_{2,3}$ plus some edge not in $E(H)$. In other words, $G$ contains a subdivision of some supergraph of $K_{2,3}$. It is not hard to notice that the minimal supergraphs of $K_{2,3}$ are $K_4$, $K_{2,3}', K_{2,3}'', K_{2,3}, T_1$ and $T_2$ depicted below.
In particular, if $G$ contains a subdivision of $K_4$, then $G \in F_0$ (Lemma 21). Also, observe that by contracting the edge $xy$ of $T_2$, we recover $P_2$ as a minor of $T_2$. Thus $B(T_2)$ contains a $C_2^2$-minor. Moreover, it is not hard to notice that $B(T_1) \cong B(T_2)$ so, $B(T_1)$ contains a $C_2^2$-minor as well. So, if $G$ is $exB(L)$-free then $G$ contains no subdivision of $T_1$ nor of $T_2$. We proceed to consider the case when $G$ contains a subdivision of $K'_{2,3}$. Again, we show that if $G$ is obtained from $K'_{2,3}$ by adding one edge, then $B(G)$ contains an $exB(L)$-minor. Recall that $T_1$ has a bicircular matroid with an $exB(L)$-minor. Thus, if $G'$ is obtained by adding a loop to $x$ (or $y$) then $B(G')$ has an $exB(L)$-minor. By considering the following cases we show that $G'$ plus any loop yields a bicircular matroid with an $exB(L)$-minor.

The graph to the left is $S'_1$, so $B(S'_1) = S^1 \in exB(L)$; and if we contract one of the $yw$-edges in the graph to the right, we recover $R''_1$. If $G$ is a spanning supergraph of $K'_{2,3}$ and does not contain any subgraph of the above, nor $K_4$, $T_1$ and $T_2$, then it contains one of the following subgraphs.

We see that $R''_1$ is a minor of the left most graph above, by contracting one of $wy$-edges. The middle left graph above is $S_1$ so, its bicircular matroid belongs to $exB(L)$. We can see that $A_3$ is a minor of the middle right graph by contracting one of the $yz$-edges and the edge $yw$. Finally, notice that the rightmost graph is $K'_{2,3}$. We already argued that if we add a loop to $K'_{2,3}$, then its bicircular matroid has an $exB(L)$-minor. Also, $T_2$ together with the previous four graphs show that if we add a non-loop edge to $K'_{2,3}$ we obtain either a copy of $K'_{2,3}$, or a graph whose bicircular matroid has an $exB(L)$-minor. This implies that if $G$ contains a subdivision $H$ of $K'_{2,3}$, but no subdivision of $K'_{2,3}$, then $H = G$ (otherwise, there is some edge in $G$ that is not in $H$, and with adequate contractions, we obtain one of the graphs above, which contradicts the fact that $B(G)$ is $exB(L)$-minor free).

Recall that we distinguished some blue and red edges in $K'_{2,3}$. In order to conclude the proof, we must show that if $H$ is a subdivision of some black edges of $K'_{2,3}$, then $B(H)$ contains an $exB(L)$-minor. In other words, the bicircular matroids of either of the following graph contain an $exB(L)$-minor.
Notice that if we contract one of the \( xz \)-edges in the graph to the left, we recover \( T_1 \) as a minor; on the graph to the right we find \( T_2 \) as a minor by contracting the \( yz \)-edge. We already argued that neither \( B(T_1) \) nor \( B(T_2) \) are \( \text{ex}_B(\mathcal{L}) \)-minor free. Thus, if \( B(G) \) is an \( \text{ex}_B(\mathcal{L}) \)-minor free connected matroid, and \( G \) is a graph that contains subdivision of \( K'_{2,3} \) but not of \( K'^*_{2,3} \), then \( G \in \mathcal{F}_0 \).

Suppose that \( G \) contains a subdivision of \( K^*_{2,3} \). Since \( K^*_{2,3} \) is a supergraph of \( K'_{2,3} \), then every subdivision of \( K^*_{2,3} \) contains a subdivision of \( K'_{2,3} \). So, by the arguments above, if \( G \) is a proper supergraph of \( K^*_{2,3} \), then \( B(G) \) has an \( \text{ex}_B(\mathcal{L}) \)-minor. Moreover, if \( H \) is a subdivision of \( K'_{2,3} \), then \( H \) is a subdivision of the blue and red edges of \( K^*_{2,3} \). Therefore, the statement of this proposition is settled when \( G \) contains a subdivision of \( K_4 \), of \( K'_{2,3} \) or a subdivision of \( K^*_{2,3} \). When \( G \) contains a subdivision of \( K''_{2,3} \) the proof follows analogous arguments to the ones above. Finally, if \( G \) contains no subdivision of \( K_4 \), of \( K'_{2,3} \), of \( K''_{2,3} \) nor of \( K^*_{2,3} \), then \( G \) contains a subdivision of \( T_1 \), of \( T_2 \) or \( G \) equals a subdivision of \( K_{2,3} \). Since \( B(G) \) is \( \text{ex}_B(\mathcal{L}) \)-minor free, we conclude that \( G \) is a subdivision of \( K_{2,3} \), so \( G \in \mathcal{F}_0 \). All possible cases for non-outerplanar graphs have now been considered.

**Subdivisions of \( G_1 \)**

We move on to the second case. In this case we consider 2-connected graphs. We begin by the subcase when \( G \) contains a subdivision of \( G_1 \). Even though this seems like an arbitrary assumption, the following lemma shows that this case settles a broad scenario.

**Lemma 23.** Let \( G \) be a 2-connected graph of minimum degree three. If \( G \) contains a pair of non-adjacent vertices and \( B(G) \) is \( \text{ex}_B(\mathcal{L}) \)-minor free, then \( G \) contains a subdivision of \( G_1 \).

**Proof.** We may assume that \( G \) has exactly four vertices. Indeed, suppose that \( G \) is a 2-connected graph, and let \( x \) and \( y \) be a pair of non-adjacent vertices. Since \( G \) is 2-connected, then there are a pair of disjoint \( xy \)-paths \( P \) and \( Q \) of length at least 2. Now, consider the subgraph of \( G \) induced by the vertices of \( P \) and \( Q \), and then contract \( P \) and \( Q \) to paths of length exactly 2. Thus, with out loss of generality, we assume that \( G \) is a graph on four vertices, and that \( x \) and \( y \) are a pair of non-adjacent vertices of \( G \). The following illustrates all edge minimal 2-connected graphs of minimum degree three, where \( xy \notin E \) (up to isomorphism).

![Graphs Illustrating Lemma 23](image)

To conclude the proof, we show that for each \( i \in \{1, \ldots, 6\} \), the bicircular matroid \( B(H_i) \) has an \( \text{ex}_B(\mathcal{L}) \)-minor. For instance, by removing any loop of \( H_1 \), we obtain a subdivision of \( W_3 \) (Figure 2), thus \( B(H_1) \) contains a \( W^3 \)-minor. Similarly, by contracting one of the \( yz \)-edges of \( H_2 \), we recover \( W_3 \) as a minor of \( H_2 \). Actually, by contracting one \( yz \)-edge in either of \( H_3, H_4 \) or \( H_6 \), we find \( A'_3 \) as a minor of these graphs. Finally, by contracting one \( yz \)-edge of \( H_5 \) we recover \( A_3 \). Therefore, if \( B(G) \) is \( \text{ex}_B(\mathcal{L}) \)-minor free, then \( G \) contains a subdivision of \( G_1 \).
At this point it is convenient to have a clear image of the structure of the graphs \( G_1(r, d, b) \) (see, for instance, Figure 4). With the same procedure as in previous occasions, we begin by studying which supergraphs of \( G_1 \) have a bicircular matroid with an \( ex_B(L) \)-minor. In this case, these graphs are the following ones.

By contracting one \( xy \)-edge in \( G_1' \) or in \( G_1'' \) we see that \( A_3 \) is a minor of these graphs, thus \( B(G_1') \) and \( B(G_1'') \) have an \( A^3 \)-minor. This implies that if \( G \) is a supergraph of \( G_1 \), and \( B(G) \) is a connected \( ex_B(L) \)-minor free matroid, then \( G \) equals \( G_1(r, d, b) \) for some non-negative integers \( r \), \( d \) and \( b \). Furthermore, if \( G \) contains a subdivision \( H \) of one of the previous graphs, then \( G = H \); otherwise, we could find a subdivision of either \( G_1' \) or of \( G_1'' \) in \( G \).

**Lemma 24.** Let \( G \) be a graph such that \( B(G) \) is an \( ex_B(L) \)-minor free connected matroid. If \( G \) contains a subdivision of \( G_1 \), then \( G \in \mathcal{F}_0 \cup \mathcal{F}_1 \).

**Proof.** Let \( G \) be as in the hypothesis. Above this statement, we argued that if \( G \) contains a subdivision of \( G_1 \), then \( G \) is a subdivision of a graph \( G_1(r, d, b) \) for some non-negative integers \( r \), \( d \) and \( b \).

To begin with, recall that \( G_1(0, 0, 0) = G_1 \) and \( B(G_1) \cong U_{4,6} \). So, if \( B(G) \) is an \( ex_B(L) \)-minor free matroid, then \( G \) is a subdivision of at most two edges of \( G_1(r, d, b) \); otherwise, \( B(G) \) contains a \( C^2,4 \)-minor (Lemma 10). We consider the minimal outerplanar subdivision of two edges of the graphs \( G_1(r, d, b) \) that are not subdivisions of one red and one blue edge. To this end, notice that subdividing any two parallel edges or any diagonal edge, creates a subdivision of \( K_{2,3} \), i.e., a non-outerplanar subdivision. Thus, any non-outerplanar subdivision of at most two edges of \( G_1 \) belongs to \( \mathcal{F}_1 \). (This settles the case when \( r = d = b = 0 \).)

Consider the case when \( r = b = 0 \) and \( d \geq 1 \). In this case, one (not black) color class corresponds to the edges on the top left side of the diagonal, and the other one to the bottom right edges. Subdividing a diagonal edge, creates a supergraph of \( K_{2,3} \), so this is taken care of by Proposition 22; this also happens if we subdivide a pair of parallel edges. Now we show that subdividing two non parallel edges on the same side of the diagonal yields a graph whose bicircular matroid has an \( ex_B(L) \)-minor. Up to symmetry, the following graph is the unique possible minimal case. To see that its bicircular matroid has an \( ex_B(L) \)-minor, notice that this graph contains \( P_2 \) as a minor: Remove the edge \( yz \), and contract the edges \( zw \) and \( zx \).

The second case we consider is when \( r \geq 1 \) and \( b = 0 \). Again, the subcases are when \( d = 0 \) and when \( d \geq 1 \). Suppose that \( d = 0 \) so, the red class consists of the \( r + 2 \) parallel edges, and the remaining edges are blue. By the same arguments as before, we do not consider subdivisions of a pair of parallel edges (this
creates a supergraph of \( K_{2,3} \). The minimal subdivisions of two edges of the same color class, but not of parallel edges of \( G \) are the following ones.

![Diagram](image)

By mapping the \( yz \)-edge (resp. the \( zw \)-path of length 2) on the left to the rightmost \( zw \)-edge (resp. the \( yz \)-path of length 2) on the right, we see that the bicircular matroids of these graphs are isomorphic. We see that \( R_4 \) is a minor of the graph on the left after contracting \( zw \) and either of the \( xy \)-edges. This implies that \( R_4 \) is a minor of \( B(G) \). Thus, if \( G \) is a subdivision of \( G_1(r,0,0) \), then \( G \) is a subdivision of at most one edge of each color class. To conclude this case, we consider the subcase when \( G \) is a subdivision of \( G_1(r,d,0) \) with \( d \geq 1 \). In this case, if \( G \) is a subdivision of two red or two blue edges, then \( G \) contains a subdivision of either two parallel edges, or of one of the graphs above. Thus, we only consider the case when \( G \) is a subdivision of some black non-diagonal edge. The following is the unique such subdivision.

![Diagram](image)

In this case, by contracting one of the \( xy \)-edges, and removing one \( zw \)-edge, we obtain \( R''_4 \) as a minor of the graph above. Therefore, if \( G \) is an outerplanar subdivision of either \( G_1(r,d,0) \) for some \( d \geq 1 \), then \( G \in F_1 \). By symmetry, and together with the previous subcase (when \( d = 0 \)), the claim holds when \( G \) is an outerplanar subdivision of \( G_1(r,d,0) \) or of \( G_1(0,d,b) \) for some non-negative integer \( d \).

The final case is when \( r \) and \( b \) are positive integers. In this case, the definitions of the red and blue edge do not depend on \( d \); each (not black) color class corresponds to a class of parallel edges. By the same argument as above, we do not consider the cases when \( G \) is a subdivision of a pair of parallel edges, nor when is a subdivision of some diagonal edges. Thus, it remains to show that if \( G \) is a subdivision of either the top or the bottom edge, then has an \( ex_B(L) \)-minor. Such a minimal subdivision, looks as follows, and in this case we see that it contains \( D_4 \) (Figure 2) as a minor by contracting the \( xw \)-edge.

![Diagram](image)

After this exhaustive case checking, we conclude that if \( G \) is an outerplanar graph that contains a subdivision of \( G_1 \), then \( G \) is a subdivision of at most one red edge and at most one blue edge of \( G_1(r,d,b) \) for some non-negative integers \( r \), \( d \) and \( b \). Therefore, \( G \in F_1 \) (recall that if \( G \) is not an outerplanar graph, then Proposition 22 asserts that \( G \in F_0 \)).

2-CONNECTED GRAPHS

The aim of this case is to settle Proposition 20 for 2-connected graphs. Notice that if \( G \) is a 2-connected graph, then \( G \) is a subdivision of some 2-connected graph \( H \) of minimum degree 3 (if \( G \) has minimum degree 3 then \( G = H \)). By Lemmas 23 and 24 if \( H \) has a pair of non-adjacent vertices, then \( H \) is a spanning supergraph of \( G_1 \) so, \( |V(H)| = 4 \). On the contrary, if every pair of vertices of \( H \) are adjacent, then \( H \) contains \( K_4 \) as a subgraph. Hence, by Lemma 21 we conclude that \( H = K_4 \). In both cases, we conclude that if \( H \) has at least four vertices, then it has exactly four vertices. Thus, we conclude the following statement.
Proposition 25. Let $G$ be a 2-connected graph. If $B(G)$ is an $ex_B(L)$-minor free matroid, then $G$ is a subdivision of a graph on at most 4 vertices. Moreover, if $G$ is a subdivision of a graph on 4 vertices of degree at least three, then $G \in F_0 \cup F_1$.

Proof. By the arguments in the paragraph above, we conclude that $G$ is a subdivision of a graph on at most 4 vertices. Moreover, with the same arguments we also notice that if $G$ is a subdivision of a graph on 4 vertices of minimum degree 3, then $G$ contains a subdivision of $K_4$ or of $G_1$. Thus, the claim follows by Lemmas 21 and 24.

Now we consider subdivisions of graphs on three vertices. Recall that the graph $2K_3$ is obtained by duplicating every edge of the triangle, and its bicircular matroid is the uniform matroid $U_{3,6}$. Notice that adding a loop to $2K_3$ creates a copy of $A_3$ as a subgraph. Also, adding one more parallel edge to each parallel class yields the graph $3K_3$. The bicircular matroids of $3K_3$ and of $A_3$ belong to $ex_B(L)$. Thus, if $G$ contains a subdivision of $2K_3$, and $B(G)$ is a connected $ex_B(L)$-minor free matroid, then $G$ is a subdivision of $2K_3(r,b)$ for some non-negative integers $b$ and $r$. We use these observations to prove the following lemma.

Lemma 26. Let $G$ be an outerplanar graph such that $B(G)$ is an $ex_B(L)$-minor free connected matroid. If $G$ contains a subdivision of $2K_3$, then $G \in F_2$.

Proof. Before this lemma, we argued that if $G$ contains a subdivision $H$ of $2K_3$, then $G = H$ and $H = 2K_3(r,b)$ for some non-negative integers $r$ and $b$. Recall that we distinguished some colored edges in $2K_3(r,b)$, we proceed to show that $G$ must be a subdivision of at most one red and at most one blue edge of $2K_3(r,b)$. As we have done before, we highlight that subdividing a pair of parallel edges of $2K_3$ creates a subdivision of $K_{2,3}$. Since we are considering outerplanar graphs, we do not consider such subdivisions.

First, consider the case when $r = b = 0$, i.e., when $G$ is a subdivision of $2K_3$, so $B(G)$ is a series extension of $U_{3,6}$. By Lemma 10 we know that if $G$ contains a subdivision of at least three different edges of $2K_3$, then $B(G)$ has a $C^{2,4}$-minor. Thus, $G$ is a subdivision of at most two non-parallel edges of $2K_3$, and so, $G \in F_2$.

Suppose now that $r \geq 1$ and $b = 0$. To show that $G$ is a subdivision of at most one red edge and at most one blue edge, we consider the minimal subdivision of $2K_3(1,0)$ that is not such a subdivision (nor a subdivision of two parallel edges).

By removing the $xz$-edge and contracting one $xy$-edge, we see that this graph contains $R''_0$ as a minor. Therefore, if $G$ is an outerplanar subdivision of $2K_3(1,0)$, then it is a subdivision of at most one red and at most one blue edge of $2K_3(r,0)$. By symmetric arguments, the claim also follows when $r = 0$ and $b \geq 1$. So, we conclude the proof by considering the case when $r$ and $b$ are positive integers. In this case, the red edges correspond to the $(r + 2)$-parallel edges, and the blue edges to the $(b + 2)$-parallel edges. Since we are considering outerplanar subdivisions, we show that the bicircular matroid of the following graph has an $ex_B(L)$-minor.

In this case, by removing the $xz$-edge, we see that $D_4$ is a subgraph of the graph above. So, its bicircular matroid has a $D^4$-minor. Hence, if $G$ is a subdivision of $2K_3(r,b)$ for some positive integers $r$ and $b$, then $G$ is a subdivision of at most one blue and at most one red edge. After considering all possible cases, we conclude that the statement of this proposition holds.
To conclude the case checking for outerplanar subdivisions of graphs on three vertices, we recall the definition of the graphs $K_{3}(r,j,l)$. These graphs are obtained from a triangle by adding $j$ parallel copies to some edge; adding $k$ parallel copies to another; and $l$ loops incident in some vertex that is not incident with both classes of parallel edges. We depict them as follows, together with the edges colorings used to define the family $\mathcal{F}_{3}$.

![Graphs K3(r,j,l), K3(1,1,0), K3(1,0,1)](image)

**Lemma 27.** Let $G$ be a 2-connected outerplanar graph such that $B(G)$ is an $ex_{B}(\mathcal{L})$-minor free matroid. If $G$ is a subdivision of a graph on three vertices of minimum degree three, then $G \in \mathcal{F}_{2} \cup \mathcal{F}_{3}$.

**Proof.** Let $H$ be the graph obtained from $G$ by contracting all subdivided edges. By the hypothesis of this lemma, we assume that $H$ has minimum degree three and $|V(H)| = 3$. If $H$ contains $2K_{3}$ as a subgraph, then $G$ contains a subdivision of $2K_{3}$ and the claim follows by Lemma 26. Now, we show that if $H$ does not contain a copy of $2K_{3}$, then $H = K_{3}(r,j,l)$ for some non-negative integers $r$, $j$ and $l$.

Notice that since $H$ is 2-connected (because is obtained from contracting subdivided edges of a 2-connected graph), then $H$ is a supergraph of $K_{3}$. So, if every vertex of $H$ has a loop, then $H$ contains $W_{3}$ as a subgraph, and so $B(G)$ has a $W_{3}$-minor. Thus, $H$ has at least one loopless vertex $v$. Since $d_{H}(v) \geq 3$, then $v$ is incident with a pair of parallel edges. Let $u$ be the other vertex incident in this same pair of parallel edges, and let $w$ be the remaining vertex. If both, $u$ and $w$ are looped vertices, then $H$ contains $A_{3}$ as a subgraph, which implies that $B(G)$ has an $A_{3}$-minor. Thus, $H$ has at most one looped vertex. Finally, if this vertex is also incident with two classes of parallel edges, $H$ contains a copy of $A_{3}$, which also implies that $B(G)$ has an $A_{3}$-minor. Since we are assuming that $B(G)$ is $ex_{B}(\mathcal{L})$-free, we conclude that $H$ has at most one looped vertex, and this vertex is not incident with two classes of parallel edges. Therefore, $H = K_{3}(r,j,l)$ for some non-negative integers $r$, $j$ and $l$, and so, $G$ is a subdivision of $K_{3}(r,j,l)$. Furthermore, since $H$ has minimum degree three, then $r \geq 1$ and $j + l \geq 1$.

Recall that the bicircular matroid of $K_{3}(1,1,0)$ and of $K_{3}(1,0,1)$ is the uniform matroid $U_{3,5}$. Hence, by Lemma 10 we know that any subdivision of at least three edges of $K_{3}(1,1,0)$ or of $K_{3}(1,0,1)$ has a bicircular matroid with a $C^{2,4}$-minor. By the edge symmetries of these graphs, any subdivision of at least three edges of $K_{3}(r,j,l)$ (where $r \geq 1$ and $j + l \geq 1$) has a bicircular matroid with a $C^{2,4}$-minor.

Let $G$ be a subdivision of at most two non-loop edges of $K_{3}(r,j,l)$. We begin by considering all cases when $l \leq 1$ (see the top two rows of Figure 6). In particular, if $r \geq 1$ and $j + l = 1$, then any subdivision of any pair of non-parallel edges of $K_{3}(r,j,l)$ belongs to $\mathcal{F}_{3}$ (this cases correspond to subdivisions of $K_{3}(1,1,0)$, of $K_{3}(1,0,1)$, of $K_{3}(r,0,1)$). Also, if $r = 1 = l$, then any subdivision of a pair of non-parallel edges of $K_{3}(1,j,1)$ belongs to $\mathcal{F}_{3}$, so the claim follows for subdivisions of $K_{3}(0,1,0)$, of $K_{3}(r,1,0)$, of $K_{3}(1,0,1)$, of $K_{3}(r,0,1)$ and of $K_{3}(1,j,1)$. The remaining subcases (of the case $l \leq 1$) are subdivisions of $K_{3}(r,j,l)$ where $r, j \geq 2$ (right most graphs in the top two columns of Figure 6). In these cases, we must show that $G$ is a subdivision of at most one edge in each parallel classes. Since we are working with outerplanar graphs, we assume that it is not a subdivision of edges in the same parallel class. Thus, it suffices to show that $G$ does not contain a subdivision of the bottom edge of $K_{3}(r,j,l)$. Again, we consider the minimal such subdivisions, and see that their bicircular matroids have an $ex_{B}(\mathcal{L})$-minor. We depict these cases below, and clearly, their bicircular matroid is $D^{4}$ (Figure 2) which belongs to $ex_{B}(\mathcal{L})$.

![Graphs K3(r,j,l) with subdivisions](image)

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1 Any subdivision of some loop of $K_{3}(r,j,l)$ yields a graph with a cut vertex.
In the paragraph above, we settled all cases when \( l \leq 1 \), now we assume that \( l \geq 2 \) (see the bottom row of Figure 6). We first observe that \( G \) is a subdivision of at most one non-loop edge of \( K_3(r, j, l) \). Since we are considering only outerplanar subdivisions, then we do not consider subdivisions of parallel edges. Thus, it suffices to show that the bicircular matroids of the following two graphs have an \( ex_B(\mathcal{L}) \)-minor.

Contracting one \( xy \)-edge in either of these graphs, shows that their bicircular matroids have an \( R^4 \)-minor. Since \( B(G) \) is an \( ex_B(\mathcal{L}) \)-minor free matroid, then \( G \) is a subdivision of at most one non-loop edge of \( K_3(r, j, l) \) (recall that we are assuming that \( r \geq 1 \) and \( l \geq 2 \)). In particular, since any subdivision of at most one edge of \( K_3(1, 0, l) \) belongs to \( F_3 \) (see left most graph in the bottom row of Figure 6), then the claim is settled for \( l \geq 2 \), \( r = 1 \) and \( j = 0 \).

To conclude the proof we consider the case when \( G \) is a subdivision of \( K_3(r, j, l) \), where \( r, l \geq 2 \). To show that \( G \in F_3 \), we argue that \( G \) is a subdivision of at most one red edge, i.e., an edge that is not incident with a loop. Again, we assume that \( G \) is not a subdivision of loop, nor of a pair of parallel edges. Hence, it suffices to show that the bicircular matroid of the following graph has an \( ex_B(\mathcal{L}) \)-minor.

Contract one of the \( xy \)-edges of the graph above. This minor is a triangle with two classes of parallel loops, i.e., this minor equals \( R_3 \). Thus, the bicircular matroid of the graph above has an \( R^4 \)-minor. Which implies that if \( G \) is an outerplanar subdivision of \( K_3(r, j, l) \) where \( r, l \geq 2 \), then \( G \) is a subdivision of at most one red edge. Therefore, \( G \in F_3 \). This completes all possible cases, so the claim follows.

Having proved all these technical lemmas, we are now ready to show that Proposition 20 holds for 2-connected graphs.

**Proposition 28.** Let \( G \) be a 2-connected graph. If \( B(G) \) is an \( ex_B(\mathcal{L}) \)-minor free connected matroid, then \( G \) belongs to \( F_0 \cup F_1 \cup F_2 \cup F_3 \).

**Proof.** When \( G \) is not an outerplanar graph, the claim follows by Proposition 22. Also, Proposition 25 asserts that if a 2-connected graph \( G \) satisfies that \( B(G) \) is an \( ex_B(\mathcal{L}) \)-minor free matroid, then \( G \) is a subdivision of a graph of at most four vertices. Moreover, the same proposition settles the case when \( G \) is a subdivision of a graph on four vertices of minimum degree three. Similarly, Lemma 27 shows that if \( G \) is a subdivision of a graph on 3 vertices of minimum degree three, then \( G \in F_2 \cup F_3 \). To conclude this case checking we consider the case when \( G \) is an outerplanar subdivision of a graph on at most two vertices. This case follows almost trivially.

A graph \( H \) on two vertices \( x \) and \( y \) consists of a class of parallel non-loop edges, some loops incident with \( x \) or \( y \). Subdividing any loop yields a graph with a cut vertex. Also, subdividing at least three different parallel edges creates a subdivision of \( K_{2,3} \). Thus, an outerplanar 2-connected subdivision of a graph on 2 vertices is a subdivision of at most two non parallel edges of \( H \). If each vertex of \( H \) has at most one loop, then any such subdivision belongs to \( F_3 \). The remaining cases are when exactly one vertex has at least two loops, and when both vertices have two loops. In the latter, we must verify that there are no subdivided edges, and in the former we must verify that there is at most one subdivided edge. This follows because the bicircular matroids of the following graphs are either \( R^3 \) or \( R^4 \) (see Figure 2), which both belong to \( ex_B(\mathcal{L}) \).
The final case is when $G$ is a 2-connected subdivision of a graph on one vertex. In this case, $G$ is a cycle with at most one looped vertex, and such a graph belongs to $F_3$. The claim now follows.

**Outerplanar graphs with cut vertices**

Propositions 22 and 28 settle Proposition 20 for non-outerplanar graphs and for 2-connected graphs. We conclude by considering the complement case. In particular, we consider graphs that do not contain subdivisions of $K_4$ nor of $K_{2,3}$. Also, in the paragraph before Lemma 26, we argued that if $G$ contains a subdivision $H$ of $2K_3$ and $B(G)$ is $ex_B(\mathcal{L})$-minor free, then $G = H$. Which implies that $G$ is a 2-connected graph. With analogous arguments, we conclude that if $G$ contains a subdivision of $G_1$ then $G$ is a 2-connected graph. These observations, together with Lemma 23 imply the following statement.

**Observation 29.** Let $G$ be an outerplanar graph such that $B(G)$ is an $ex_B(\mathcal{L})$-minor free connected matroid. If $G$ has a cut vertex and $EB$ is an end block of $G$, then one of the following holds:

1. $EB$ is a cycle with possible loops on the corresponding cut vertex,
2. $EB$ is a subdivision of a graph on 2 vertices of minimum degree 3, or
3. $EB$ is a subdivision of a graph on three vertices but contains no subdivision of $2K_3$.

Recall that, by Proposition 29, if the bicircular matroid of graph $G$ is $C^{2,4}$-minor free, then the block tree of $G$ is a path, and every middle block contains exactly two vertices. By definition of $F_4$, such a graph $G$ belongs to $F_4$ if the end blocks of $G$ are subgraphs of a subdivision of some red edge of the following graphs, where the cut vertex is corresponding cut vertex is $x$.

Proposition 30. Let $G$ be an outerplanar graph with some cut vertex. If $B(G)$ is an $ex_B(\mathcal{L})$-minor free connected matroid, then $G \in F_4$.

**Proof.** By the arguments in this subsection, it suffices to show the end blocks of $G$ are a subgraph of some subdivision of at most one red edge of either of $K_3(r, j, l)$, $K_3(1, 1, l)$, $K_3(1, 0, l)$ or $mK_2''$ depicted above. We divide this proof in the cases laid out by Observation 29. In particular, when both end blocks satisfy the first statement of Observation 29 then the claim follows immediately. Also, since $B(G)$ is a connected matroid, no end block of $G$ consists of exactly one edge. This implies that each end block can be contracted either to a pair of parallel edges, or to a leaf with a loop on the end vertex. We use this simple observation throughout this proof. Also, with out loss of generality we suppose that there are no loops incident with the cut vertex $x$ (any minor of $G$ that does not contain the loops on $x$, is also a minor of $G$).

Suppose that an end block $EB$ is a subdivision of a graph on two vertices of minimum degree three. Denote by $x$ the corresponding cut vertex and by $y$ the other vertex in $EB$ of degree at least 3. If $EB$ is an empty subdivision, i.e., $EB$ is a block on two vertices, then there is nothing to prove. Now we show that if $EB$ contains at least one subdivided edge, then $y$ has at most one loop and $EB$ has at most one subdivision class; in other words, $EB$ is a subdivision of at most one red edge of some $mK_2''$. First suppose that there are at least two loops incident with $y$. Since $x$ is a cut vertex and $B(G)$ is a connected matroid, then $G$ must contain one of the graphs as a minor.
Since the graph to the left is $R_4$ and the graph to the right is $R'_4$, we conclude that there is at most one loop incident with $y$. Finally, we show that there is at most one subdivision class. To this end, notice that by contracting the edge $xy$ in the following graphs, we obtain $P_2$ (on the left) and $P_3$ (on the right) as a minor of $G$.

In both cases, this contradicts the fact that $B(G)$ is $ex_B(L)$-minor free. So, if and end block $EB$ of $G$ satisfies the second statement of Observation 29 then $EB$ is a subgraph of a subdivision of at most one red edge of a graph $mK''_2$.

To conclude this proof we verify the case when $EB$ is a subdivision of a graph on three vertices of minimum degree 3. Denote by $x$, $y$ and $z$ the vertices of degree at least 3 of $EB$, with $x$ the cut vertex. Since $EB$ is a block, then $xy$, $yz$, $xz \in E(G)$. If $y$ or $z$ are incident with some loop, then by contracting $xy$ we find $R_3$ as a minor of $G$. Thus, $y$ and $z$ are loopless vertices. Moreover, since we are assuming that $EB$ contains no subdivision of $2K_3$, then without loss of generality, we suppose that $yz$ has no parallel edges.

These arguments show that $EB$ is a subdivision of $K_3(r,j,0)$ for some non-negative integer $j$ and $k$, with $j \geq 1$. Now we show that $EB$ is a subdivision of at most one red edge. Recall that subdividing a pair of parallel edges in $K_3(r,j,0)$ creates a subdivision of $K_{2,3}$. We do not consider such subdivisions because we are working with outerplanar graphs. To see that $EB$ has at most one subdivision class, we argue that the bicircular matroids of the following graphs have an $ex_B(L)$-minor.

We proceed from left to right. Contracting one $yz$-edge in the left most and second left most graphs, yields an $R''_4$-minor and an $R_4$-minor, respectively. The bicircular matroid of the middle right and of the right most graph are isomorphic. So, we conclude by noticing that we obtain $P_3$ as a minor of the right most graph by contracting the edges $yz$ and $zx$. This shows that the bicircular matroids of the graphs above have an $ex_B(L)$-minor.

To conclude the proof, we must consider the cases when $EB$ is a subdivision of $K_3(r,j,0)$ with $j \geq 1$ or with $r \geq 2$. In these cases, it suffices to observe that the bicircular matroid of the following graphs have an $ex_B(L)$-minor.

As we have done several times, we notice that each graph above contains a minor whose bicircular matroid belongs to $ex_B(L)$. Contracting one $yz$-edge of the left most graph above yields a graph isomorphic to $R_4$. Similarly, $R''_4$ is the minor obtained from the middle left graph by contracting one $yz$-edge. We see that $P_2$ is a minor of the middle right graph by removing the $xy$-edge and contracting the $xz$-edge. By applying analogous operations to the right most graph, we see that this graph contains $P_3$ as a minor.

After considering all these possible cases, we conclude that the end blocks of $G$ are subdivisions of at most one red edge of $K_3(r,j,l)$ or of $mK''_2$. Hence, we conclude that $G \in F_4$. 

□
6 Characterizations

Using the work of previous sections, we now prove Theorems 1 and 2, and we discuss some of their implications.

**Theorem 1.** Let \( G \) be a graph such that \( B(G) \) is a connected matroid. The bicircular matroid \( B(G) \) is a lattice path matroid if and only if \( G \) belongs to \( F_0 \cup F_1 \cup F_2 \cup F_3 \cup F_4 \).

**Proof.** By Propositions 12, 14, 15, 17, and 18 the bicircular matroid of any graph in \( F_0 \cup F_1 \cup F_2 \cup F_3 \cup F_4 \) is a lattice path matroid. On the other hand, if \( B(G) \) is a lattice path matroid, then it is \( ex_B(L) \)-minor free (Proposition 11). Thus, if \( B(G) \) is also connected, then by Proposition 20, we conclude that \( G \in F_0 \cup F_1 \cup F_2 \cup F_3 \cup F_4 \).

In this paragraph we claim and show that for any graph \( G \) in \( F_0 \), there is some graph in \( F_1 \cup F_2 \cup F_3 \cup F_4 \) whose bicircular matroid is isomorphic to \( B(G) \) — we do so to show that Theorem 1 implies Corollary 31.

For instance, any subdivision of at most two edges of \( K_4 \) has the same bicircular matroid as some subdivision of at most two edges of \( G_1 \). Similarly, any subdivision of blue and red edges of \( K_3^* \) has the same bicircular matroid as some subdivision of a pair of non parallel edges of \( 2K_3 \). Subdivisions of \( K_{2,3} \) are subgraphs of subdivisions of red and blue edges of \( K_{2,3}^* \), and the latter have the same bicircular matroids as subdivisions of red and blue edges of \( K_{2,3}^* \). To conclude this argumentation, we show that subdivisions of \( K_{2,3}^* \) have isomorphic bicircular matroids to bicircular matroids of some graph in \( F_4 \). To do so, it suffices to notice that the following graphs have isomorphic bicircular matroids (to the left, \( K_{2,3}^* \); to the right, a graph in \( F_4 \)).

**Corollary 31.** Let \( L \) be a connected lattice path matroid. Then, \( L \) is a bicircular matroid if and only if there is some graph \( G \in F_1 \cup F_2 \cup F_3 \cup F_4 \) such that \( L \cong B(G) \).

Recall that Proposition 22 asserts that \( F_0 \) contains all non-outerplanar graphs whose bicircular matroids are connected lattice path matroids.

**Corollary 32.** Let \( L \) be a lattice path matroid. If \( L \) is a bicircular matroid, then there is an outerplanar graph \( G \) such that \( L \cong B(G) \).

Observation 19 in Section 4 asserts that we can recognize the union \( \bigcup_{i=0}^4 F_i \) in linear time with respect to the edge set of the input graph. Thus, the following claim is an implication of Theorem 1.

**Corollary 33.** Given an input graph \( G \), there is a linear time algorithm (with respect to \(|E(G)|\)) that determines whether \( B(G) \) is a lattice path matroid.

To conclude this work, we exhibit the list of excluded bicircular minors to the class of lattice path matroids. We depict an affine presentations of these in Figure 7 and in Figure 8 we display a bicircular presentation of these matroids.

**Theorem 2.** A bicircular matroid is a lattice path matroid if and only if it has no one of the following matroids as a minor:

\[
C^{2,4}, \ W^3, \ A^3, \ R^3, \ R^4, \ D^4, \ B^1, \ \text{and} \ S^1.
\]

**Proof.** On the one hand, Proposition 11 shows that if a bicircular matroid has either of the above as a minor, then it is not a lattice path matroid. On the other hand, since lattice path matroids are closed under disjoint unions, then we can assume that the bicircular matroid \( B(G) \) is connected. Thus, by Proposition 20, if \( B(G) \) is \( ex_B(L) \)-minor free, then \( G \in F_1 \) for some \( i \in \{0, 1, 2, 3\} \). Now, the claim follows by Theorem 1. \( \square \)
It was recently proved that there are finitely many excluded minors to the class of bicircular matroids [7]. This, together with Theorem 2 implies that there are finitely many excluded minors to the class of lattice path bicircular matroids.

**Corollary 34.** There are only finitely many excluded minors to the class of lattice path bicircular matroids.

Allow us to discuss one more implication of the characterizations proposed in this section. By Corollary 31, every lattice path bicircular matroid admits a bicircular presentation by a graph in $\mathcal{F}_i$ for some $i \in \{1, 2, 3, 4\}$. By definition of $\mathcal{F}_4$, all graphs in this family have a cut vertex. This implies that their bicircular matroids are not 2-connected. Also, by Lemma 16, the bicircular matroids of graphs in $\mathcal{F}_3$ are minors of bicircular matroids of graphs in $\mathcal{F}_1 \cup \mathcal{F}_2$. Thus, every 2-connected lattice path bicircular matroid is a minor of the bicircular matroid of some graph in $\mathcal{F}_1 \cup \mathcal{F}_2$. 

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Recall that the bicircular matroid of some 2-connected graph \( G \) is cosimple whenever \( G \) has no subdivided edges. The subfamilies of graphs with no subdivided edges of \( F_1 \cup F_2 \) are the graph families \( G_1(r,b,d) \) and \( 2K_3(r,b) \). The bicircular matroids of these graphs have simple geometric descriptions. The bicircular matroid of \( 2K_3(r,b) \) is the rank 3 matroid with one \((r+2)\)-point line, one \((b+2)\)-point line, and two points in general position. The bicircular matroid of \( G_1(r,d,b) \) is the rank 4 matroid with two hyperplanes \( H_1 \) and \( H_2 \) intersecting in a \( d \)-point line \( \ell \), where \( H_1 \) and \( H_2 \) contain \( \ell \), one point that belongs to no 3-point line, and an \((r+2)\)-point line and a \((b+2)\)-point line, respectively. The following are affine representations of \( B(2K_3(2,1)) \) and of \( B(G_1(1,4,0)) \).

**Proposition 35.** For a 2-connected cosimple matroid \( M \) the following statements are equivalent:

1. \( M \) is a lattice path bicircular matroid,
2. \( M \) is a minor of \( B(G_1(r,b,d)) \) or of \( B(2K_3(r,b)) \), for some non-negative integers \( r \), \( b \) and \( d \).

**Proof.** This statement is proved in the two preceding paragraphs. \( \square \)

**Corollary 36.** The cosimplification of 2-connected lattice path bicircular matroid has rank at most 4.

7 Conclusions

Theorem 2 lists all excluded bicircular minors to the class of lattice path matroids. A natural extension of this work is to exhibit the list of excluded lattice path minors to the class of bicircular matroids. Moreover, Corollary 34 asserts that there are only a finite number of excluded minors for the class of lattice path bicircular matroids, which are these excluded minors?

**Problem.** Exhibit the excluded minors for lattice path bicircular matroids.

As Sivaraman and Sillyat [13] suggested, we also considered to study the intersection of bicircular matroids and multipath matroids; but it turned out to be (even more) technical than the present work. The interested reader could try to find nicer arguments to characterize multipath bicircular matroids. Also, the dual class of transversal matroids is the class of strict gammoids. Since lattice path matroids and bicircular matroids are duals of transversal matroids, studying the intersection of bicircular matroids and strict gammoids would be a nice extension of the present work and of the work of Sivaraman and Sillyat [13].

Finally, Corollary 32 asserts that every lattice path bicircular matroid is the bicircular matroid of some outerplanar graph; is there an interesting characterization of bicircular matroids of outerplanar graphs? We believe it could also be interesting to consider super classes of outerplanar graphs. For instance, studying the class \( \mathcal{C} \) of bicircular matroids of series-parallel graphs could yield a nice problem to investigate. In particular, all lattice path bicircular matroids belong to \( \mathcal{C} \). Also, if \( B(G) \) is representable over \( GF(4) \) then it is \( U_{4,6} \)-minor free, and thus \( G \) is \( K_4 \)-minor free, i.e., \( G \) is a series-parallel graph. We stress out that \( \mathcal{C} \) is not the class of \( U_{4,6} \)-minor free bicircular matroids since \( U_{4,6} \cong B(G_1) \) and \( G_1 \) is a series-parallel graph.

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