ON FINDING THE SURFACE ADMITTANCE OF AN OBSTACLE VIA THE TIME DOMAIN ENCLOSURE METHOD

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Abstract. An inverse obstacle scattering problem for the electromagnetic wave governed by the Maxwell system over a finite time interval is considered. It is assumed that the wave satisfies the Leontovich boundary condition on the surface of an unknown obstacle. The condition is described by using an unknown positive function on the surface of the obstacle which is called the surface admittance. The wave is generated at the initial time by a volumetric current source supported on a very small ball placed outside the obstacle and only the electric component of the wave is observed on the same ball over a finite time interval. It is shown that from the observed data one can extract information about the value of the surface admittance and the curvatures at the points on the surface nearest to the center of the ball. This shows that a single shot contains a meaningful information about the quantitative state of the surface of the obstacle.

1. Introduction and statement of the results. In this paper, we pursue further the possibility of the time domain enclosure method [10] for the Maxwell system developed in [11, 12]. We consider an inverse obstacle scattering problem for the wave governed by the Maxwell system in the time domain, in particular, over a finite time interval unlike the time harmonic reduced case, see [5, 15, 18].

Let us formulate the problem more precisely. We denote by $D$ the unknown obstacle. We assume that $D$ is a non empty bounded open set of $\mathbb{R}^3$ with $C^2$-boundary such that $\mathbb{R}^3 \setminus D$ is connected.

We assume that the electric field $E = E(x,t)$ and magnetic field $H = H(x,t)$ are generated only by the current density $J = J(x,t)$ at the initial time located not far away from the unknown obstacle. There should be several choices of current density $J$ as a model of antenna [2, 4]. In this paper, as considered in [11, 12] we assume that $J$ takes the form

\begin{equation}
J(x,t) = f(t)\chi_B(x)\mathbf{a},
\end{equation}

where $\mathbf{a}$ is an arbitrary unit vector; $B$ is a (very small) open ball satisfying $\overline{B} \cap \overline{D} = \emptyset$ and $\chi_B$ denotes the characteristic function of $B$; $f \in C^1[0, T]$ with $f(0) = 0$.

Let $0 < T < \infty$. In this paper, we assume that

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Assumption 1. (i) the pair \((E(t), H(t)) \equiv (E(\cdot, t), H(\cdot, t))\) belongs to \(C^1([0, T]; L^2(\mathbb{R}^3 \setminus \mathcal{D})^3 \times L^2(\mathbb{R}^3 \setminus \mathcal{D})^3)\) as a function of \(t\);
(ii) for each \(t \in [0, T]\), the pair \((\nabla \times E(t), \nabla \times H(t))\) belongs to \(L^2(\mathbb{R}^3 \setminus \mathcal{D})^3 \times L^2(\mathbb{R}^3 \setminus \mathcal{D})^3\);
(iii) it holds that
\[
\begin{aligned}
\frac{d}{dt} E - \epsilon^{-1} \nabla \times H &= \epsilon^{-1} J, \\
\frac{d}{dt} H + \mu^{-1} \nabla \times E &= 0, \\
E(0) &= 0, \\
H(0) &= 0;
\end{aligned}
\]
(iv) for each \(t \in [0, T]\), \((E(t), H(t))\) satisfies, in the sense of trace \([14]\)
\[
\nu \times H(t) - \lambda \nu \times (E(t) \times \nu) = 0 \text{ on } \partial \mathcal{D},
\]
where \(\lambda \in C^1(\partial \mathcal{D})\) and satisfies \(\inf_{x \in \partial \mathcal{D}} \lambda(x) > 0\).

Note that \(\nu\) denotes the unit outward normal to \(\partial \mathcal{D}\). The obstacle is embedded in a medium like air (free space) which has constant electric permittivity \(\epsilon(>0)\) and magnetic permeability \(\mu(>0)\). The boundary condition \((3)\) is called the Leontovich boundary condition \([1, 5, 15, 18]\) and see also \([16]\) for the case when \(\lambda\) is constant. The quantity \(1/\lambda\) is called the surface impedance, see \([1]\) and thus \(\lambda\) is called the admittance. The existence of the admittance \(\lambda\) causes the loss of the energy of the solution on the surface of the obstacle after stopping of the source supply.

In \([14]\) it is stated that the existence of \((E(t), H(t))\) satisfying (i)-(iv) can be derived from the theory of \(C_0\) contraction semigroups \([20]\). However, therein the detailed proof is not given as pointed out in \([12]\). To make the logical relation clear, here we assume that our pair \((E(t), H(t))\) satisfies (i)-(iv). This is our starting assumption. It should be pointed out that Assumption (\(\lambda\)) in \([12]\) implies the existence of such \((E(t), H(t))\) which ensures that conditions (i)-(iv) has a sense.

We consider the following problem.

**Problem.** Fix a large (to be determined later) \(T < \infty\). Observe \(E(t)\) on \(B\) over the time interval \([0, T]\). Extract information about the geometry of \(\mathcal{D}\) and the values of \(\lambda\) on \(\partial \mathcal{D}\) from the observed data.

First of all let us recall the previous reslult on this problem. Denote the solution of the system \((2)\) in the case when \(\mathcal{D} = \emptyset\) by \((E_0(t), H_0(t))\) with \(J\) given by \((1)\). Note that in this case, the solvability has been ensured by applying theory of \(C_0\) contraction semigroups \([20]\).

Define the indicator function
\[
I_J(\tau, T) = \int_B f(x, \tau) \cdot (W_e - V_e) dx, \quad \tau > 0
\]
where
\[
W_e(x, \tau) = \int_0^T e^{-\tau t} E(x, t) dt, \quad V_e(x, \tau) = \int_0^T e^{-\tau t} E_0(x, t) dt
\]
and

\[ f(x, \tau) = -\frac{\tau}{\epsilon} \int_0^T e^{-\tau t} J(x, t) \, dt. \]

And also, to describe another assumption, we introduce here

\[ W_m(x, \tau) = \int_0^T e^{-\tau t} H(x, t) \, dt. \]

Using the same argument as that of [12] under Assumption 1, we know the following facts.

• The pair \((W_e, W_m)\) belongs to \(L^2(\mathbb{R}^3 \setminus \overline{D})^3 \times L^2(\mathbb{R}^3 \setminus \overline{D})^3\) with \((\nabla \times W_e, \nabla \times W_m) \in L^2(\mathbb{R}^3 \setminus \overline{D})^3 \times L^2(\mathbb{R}^3 \setminus \overline{D})^3\).

• We have

\[
\begin{align*}
\nabla \times W_e + \tau \mu W_m &= -e^{-\tau T} \mu H(x, T) \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \\
\nabla \times W_m - \tau \epsilon W_e - \epsilon \frac{\tau}{\epsilon} f(x, \tau) &= e^{-\tau T} \epsilon E(x, T) \quad \text{in } \mathbb{R}^3 \setminus \overline{D}.
\end{align*}
\]

• The boundary condition (3) remains valid in the sense of the trace [14] as mentioned above if \((E(t), H(t))\) is replaced with \((W_e, W_m)\).

• It holds that

\[
\begin{align*}
\nabla \cdot W_m &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \\
\nabla \cdot W_e &= 0 \quad \text{in } (\mathbb{R}^3 \setminus \overline{D}) \setminus \overline{B}.
\end{align*}
\]

Note that, at this stage, each term on (3) does not have a point-wise meaning. What we know is: the left-hand side on (3) just belongs to the dual space of \(H^{1/2}(\partial D)^3\). In this paper, we introduce another assumption which states a regularity up to boundary.

**Assumption 2.** The functions \(W_e\) and \(W_m\) above belong to \(H^1\) on the intersection of an open neighbourhood of \(\partial D\) in \(\mathbb{R}^3\) with \(\mathbb{R}^3 \setminus \overline{D}\).

This assumption makes us possible to treat vector-valued functions appeared in a dual paring pointwise. Note that Assumption 2 is a special version of Assumption \((R)\) introduced in [12] by virtue of (6). However, for our purpose, it suffices to assume Assumption 2 instead of Assumption \((R)\). We believe that Assumption 2 should be removed.

Now, by Assumption 2, we have that both \(W_e\) and \(W_m\) belong to \(H^1\) in the intersection of an open neighbourhood of \(\partial D\) with \(\mathbb{R}^3 \setminus \overline{D}\). Then, we see that the boundary condition (3) is satisfied in the sense of the usual trace in \(H^{1/2}(\partial D)^3\):

\[ \nu \times W_m - \lambda \nu \times (W_e \times \nu) = 0 \quad \text{on } \partial D. \]

Note that this is equivalent to

\[ \nu \times (W_m \times \nu) + \lambda \nu \times W_e = 0 \quad \text{on } \partial D. \]

Moreover, note also that: since \(W_m \in L^2(\mathbb{R}^3 \setminus \overline{D})^3\) satisfies \(\nabla \times W_m \in L^2(\mathbb{R}^3 \setminus \overline{D})^3\), from the first equation on (6) and by applying Corollary 1.1 on page 212 and Remark 2 on page 213 in [6] one can conclude that \(W_m \in H^1(\mathbb{R}^3 \setminus \overline{D})^3\).

Set

\[ \lambda_0 = \sqrt{\frac{\mu}{\epsilon}}. \]
In [11], we introduced two conditions (A.I) and (A.II) on \( \lambda \) listed below:

(A.I) \( \exists C > 0 \lambda(x) \geq \lambda_0 + C \) for all \( x \in \partial D \);

(A.II) \( \exists C > 0 \exists C' > 0 \ C' \leq \lambda(x) \leq \lambda_0 - C \) for all \( x \in \partial D \).

Roughly speaking, we can say that: the condition (A.I)/(A.II) means that the admittance \( \lambda \) is greater/less than the special value \( \lambda_0 \) which is the admittance of free space [1].

Define dist \((D, B) = \inf_{x \in D, y \in B} |x - y|\).

Under assumptions 1-2 we have already known that the following statement is true.

**Theorem 1.1.** ([12]). Let \( a_j, j = 1, 2 \) be two linearly independent unit vectors. Let \( J_j(x, t) = f(t) \chi_B(x) a_j \) and \( f \) satisfy

\[
(8) \quad \exists \gamma \in \mathbb{R} \liminf_{\tau \to \infty} \tau \gamma \left| \int_0^T e^{-\tau t} f(t) \, dt \right| > 0.
\]

Then, we have:

\[
\lim_{\tau \to \infty} e^{\tau T} \sum_{j=1}^2 I_{J_j}(\tau, T) = \begin{cases} 0, & \text{if } T \leq 2\sqrt{\mu \epsilon} \text{dist} (D, B), \\ \infty, & \text{if } T > 2\sqrt{\mu \epsilon} \text{dist} (D, B) \text{ and (A.I) is satisfied}, \\ -\infty, & \text{if } T > 2\sqrt{\mu \epsilon} \text{dist} (D, B) \text{ and (A.II) is satisfied}. \end{cases}
\]

Moreover, if \( \lambda \) satisfies (A.I) or (A.II), then for all \( T > 2\sqrt{\mu \epsilon} \text{dist} (D, B) \)

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log \left| \sum_{j=1}^2 I_{J_j}(\tau, T) \right| = -2\sqrt{\mu \epsilon} \text{dist} (D, B).
\]

**Remark 1.** As described in Theorem 1.2 in [12], all the statements of Theorem 1.1 are valid if \( V_e \) in \( I_J(\tau, T) \) is replaced with the unique weak solution \( V \in L^2(\mathbb{R}^3) \) with \( \nabla \times V \in L^2(\mathbb{R}^3) \) of

\[
(9) \quad \frac{1}{\mu \epsilon} \nabla \times \nabla \times V + \tau^2 V + f(x, \tau) = 0 \quad \text{in } \mathbb{R}^3.
\]

In what follows, we denote by \( V^0_e \) the weak solution. Roughly speaking, the reason why such a replacement is possible is the following. Introduce another indicator function by the formula

\[
(10) \quad \tilde{I}_f(\tau, T) = \int_B f(x, \tau) \cdot (W_e - V_0^e) \, dx.
\]

Using the simple facts

\[
\|V_e - V_0^e\|_{L^2(\mathbb{R}^3 \setminus \overline{D})} = O(\tau^{-1} e^{-\tau T})
\]

and

\[
\|f\|_{L^2(B)} = O(\tau^{-1/2}),
\]

one has

\[
(11) \quad I_J(\tau, T) = \tilde{I}_f(\tau, T) + O(\tau^{-3/2} e^{-\tau T}).
\]

Thus, one can transplant all the results in Theorem 1.1 into the case when the indicator function is given by (10). This version’s advantage is: no need of time domain computation of \( E_0 \) in \( V_e \).
Remark 2. From Theorem 1.1 one can obtain another formula which has a similarity to the original version of the enclosure method developed in [8]. See (15) in [12].

The main purpose of this paper is to go further beyond Theorem 1.1 under Assumptions 1 and 2. Especially, we consider how to extract quantitative information about the state of the surface of an unknown obstacle using the time domain enclosure method. For the purpose, we clarify the leading profile of the indicator functions (4) or (10) as \( \tau \to \infty \).

In what follows, we denote by \( B_r(x) \) the open ball centered at \( x \) with radius \( r \). Set \( d_{\partial D}(p) = \inf_{y \in \partial D} |y - p| \) and \( \Lambda_{\partial D}(p) = \{ y \in \partial D \mid |y - p| = d_{\partial D}(p) \} \). To describe the formula, we recall some notion in differential geometry. Let \( q \in \partial D \). Let \( S_q(\partial D) \) and \( S_q(\partial B_{d_{\partial D}(p)}(p)) \) denote the shape operators (or Weingarten maps) at \( q \) of \( \partial D \) and \( \partial B_{d_{\partial D}(p)}(p) \) with respect to \( \nu_q \) and \( -\nu_q \), respectively (see [19] for the notion of the shape operator). Because \( q \) attains the minimum of the function: \( \partial D \ni y \mapsto |y - p| \), we have always \( S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D) \geq 0 \) as the quadratic form on the common tangent space at \( q \).

Now we are ready to state the main result in this paper.

Theorem 1.2. Assume that \( \partial D \) is \( C^4 \) and \( \lambda \in C^1(\partial D) \). Let \( f \) satisfy (8) and \( T > 2\sqrt{\pi} \, \text{dist}(D,B) \). Assume that the set \( \Lambda_{\partial D}(p) \) consists of finite points and

\[
\det(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D)) > 0 \quad \forall q \in \Lambda_{\partial D}(p).
\]

And assume also that there exists a point \( q \in \Lambda_{\partial D}(p) \) such that \( \lambda(q) \neq \lambda_0 \) and \( \nu_q \times a \neq 0 \).

Then, we have

\[
\lim_{\tau \to \infty} \tau^2 e^{2\tau \sqrt{\pi} \, \text{dist}(D,B)} \frac{\tilde{I}_f(\tau, T)}{\tilde{f}(\tau)^2} = \frac{\pi}{2} \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 \lambda_0^2 \sum_{q \in \Lambda_{\partial D}(p)} k_q(p) \frac{\lambda(q) - \lambda_0}{\lambda(q) + \lambda_0} |\nu_q \times a|^2,
\]

where

\[
\tilde{f}(\tau) = \int_0^T e^{-\tau t} f(t) \, dt
\]

and

\[
k_q(p) = \frac{1}{\sqrt{\det(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D))}}.
\]

Note that in Theorem 1.2 neither jump condition (A.I) nor (A.II) is assumed.

Once we have the formula (13), as done in [13] for the scalar wave equation case, we immediately obtain the following corollary. To indicate the dependence of the indicator function on the surface admittance we write

\[
\tilde{I}_f(\tau, T; \lambda) = \tilde{I}_f(\tau, T; \lambda).
\]

Corollary 1. Assume that \( \partial D \) is \( C^4 \). Let \( \lambda_1 \) and \( \lambda_2 \) belong to \( C^1(\partial D) \). Let \( f \) satisfy (8) and \( T > 2\sqrt{\pi} \, \text{dist}(D,B) \). Assume that: the set \( \Lambda_{\partial D}(p) \) consists of finite points and satisfies (12); for each \( j = 1, 2 \) there exists a point \( q \in \Lambda_{\partial D}(p) \).
such that \( \lambda_j(q) \neq \lambda_0 \) and \( \nu_q \times a \neq 0 \). Then, we have

\[
\lim_{\tau \to \infty} \frac{\tilde{I}_J(\tau, T; \lambda_2)}{\tilde{I}_J(\tau, T; \lambda_1)} = \sum_{q \in \Lambda_{\partial D}(p)} k_q(p) \frac{\lambda_2(q) - \lambda_0}{\lambda_2(q) + \lambda_0} |\nu_q \times a|^2
\]

provided the denominator on the right-hand side does not vanish. Moreover if \( \lambda_1 \) and \( \lambda_2 \) satisfies \( \min_{q \in \Lambda_{\partial D}(p)} (\lambda_1(q), \lambda_2(q)) > \lambda_0 \) or \( \max_{q \in \Lambda_{\partial D}(p)} (\lambda_1(q), \lambda_2(q)) < \lambda_0 \), then we have

\[
\min_{q \in \Lambda_{\partial D}(p)} \frac{\lambda_2(q) - \lambda_0}{\lambda_1(q) + \lambda_0} \leq \lim_{\tau \to \infty} \frac{\tilde{I}_J(\tau, T; \lambda_2)}{\tilde{I}_J(\tau, T; \lambda_1)} \leq \max_{q \in \Lambda_{\partial D}(p)} \frac{\lambda_2(q) - \lambda_0}{\lambda_1(q) + \lambda_0}.
\]

In particular, if \( \Lambda_{\partial D}(p) \) consists of a single point \( q \in \partial D \) such that \( \nu_q \times a \neq 0 \) and \( (\lambda_1(q) - \lambda_0)(\lambda_2(q) - \lambda_0) \neq 0 \), then we have

\[
\lim_{\tau \to \infty} \frac{\tilde{I}_J(\tau, T; \lambda_2)}{\tilde{I}_J(\tau, T; \lambda_1)} = \frac{\lambda_2(q) - \lambda_0}{\lambda_1(q) + \lambda_0}.
\]

Estimates (14) and formula (15) are remarkable since they do not require information about the curvatures of the surface of the obstacle in advance. Note that if we know a point \( q \in \Lambda_{\partial D}(p) \), then, all the intermediate points \( p' \) on the segment connecting \( q \) and \( p \), satisfy \( \Lambda_{\partial D}(p') = \{q\} \) and \( \det (S_q(\partial B_{\partial D}(p')) - S_q(\partial D)) > 0 \). Thus, one gets immediately the following corollary in which the set \( \Lambda_{\partial D}(p) \) can be an infinite one, even, continuum.

**Corollary 2.** Assume that \( \partial D \) is \( C^4 \). Let \( \lambda_1 \) and \( \lambda_2 \) belong to \( C^1(\partial D) \). Let \( p \) be an arbitrary point in \( \mathbb{R}^3 \setminus \overline{D} \) and \( q \in \Lambda_{\partial D}(p) \). Let \( p' \) be an arbitrary point on the open segment \( \{sq + (1-s)p | 0 < s < 1\} \) and \( B' \) an open ball centered at \( p' \) satisfying \( \overline{B'} \cap \overline{D} = 0 \). Let \( f \) satisfy (8) and \( T > 2 \sqrt{\mu \epsilon} \text{dist}(D, B') \). Let \( J' \) be the \( J \) given by (1) in which \( B \) is replaced with \( B' \). And assume also that \( (\lambda_1(q) - \lambda_0)(\lambda_2(q) - \lambda_0) \neq 0 \) and \( \nu_q \times a \neq 0 \).

Then, we have

\[
\lim_{\tau \to \infty} \frac{\tilde{I}_J(\tau, T; \lambda_2)}{\tilde{I}_J(\tau, T; \lambda_1)} = \frac{\lambda_2(q) - \lambda_0}{\lambda_1(q) + \lambda_0}.
\]

Thus formula (16) can be used for monitoring of the quantitative state of the surface, that is, the change of \( \lambda_1 \) to \( \lambda_2 \) of the surface admittance at a given monitoring point \( q \) on the surface.

All the results mentioned above can be transplanted as follows.

**Corollary 3.** Theorem 1.2 and Corollaries 1-2 remain valid if \( \tilde{I}_+ \) is replaced with \( I_+ \).
This can be seen as follows. From (8) we have
\[
\frac{\tau^2 e^{2\tau \sqrt{\mu} \text{dist}(D,B)}}{f(\tau)^2} \tau^{-3/2} e^{-\tau T} = \frac{\tau^{2\gamma+1/2} e^{-\tau(T-2\sqrt{\mu} \text{dist}(D,B))}}{\tau^2 f(\tau)^2} = O(\tau^{2\gamma+1/2} e^{-\tau(T-2\sqrt{\mu} \text{dist}(D,B))}).
\]
Thus if \( T \) satisfies \( T > 2\sqrt{\mu} \text{dist}(D,B) \), then (11) and (17) ensure both quantities
\[
\frac{\tau^2 e^{2\tau \sqrt{\mu} \text{dist}(D,B)}}{f(\tau)^2} \tilde{I}_s
\]
and
\[
\frac{\tau^2 e^{2\tau \sqrt{\mu} \text{dist}(D,B)}}{f(\tau)^2} I_s
\]
have the same leading profile as \( \tau \to \infty \).

Finally, we show that Theorem 1.2 suggests us a procedure for finding curvatures and \( \lambda \) at an arbitrary point \( q \) on \( \Lambda_{\partial D}(p) \). It is a translation of the procedure described in [13] in which the scalar wave equation is considered.

**Step 1.** Choose three points \( p_j, j = 1, 2, 3 \) on the segment connecting \( p \) and \( q \). Denote by \( B_j \) three open balls with very small radiiues centered at \( p_j \) such that \( \overline{B}_1 \cup \overline{B}_2 \cup \overline{B}_3 \subset \mathbb{R}^3 \setminus \overline{D} \). Note that we have \( \Lambda_{\partial D}(p_j) = \{q\} \) and \( \det(S_q(\partial B_{\partial \Lambda_{\partial D}(p_j)}(p_j)) - S_q(\partial D)) > 0 \).

**Step 2.** Fix \( T > 2 \max_j \sqrt{\mu} \text{dist}(D,B_j) \) and generate \( E \) and \( H \) on \( B_j \) by the source \( J_j = f(t) \chi_{B_j} a \) for a fixed unit vector \( a \) with \( a \times \nu_q \neq 0 \) and observe \( E \) on \( B_j \) over the time interval \([0, T]\).

**Step 3.** Compute \( \tilde{I}_J(\tau, T) \) from the observation data in Step 2.

**Step 4.** Apply Theorem 1.2 to the case \( B = B_j \). Then, we have
\[
\lim_{\tau \to \infty} \frac{\tau^2 e^{2\tau \sqrt{\mu} \text{dist}(D,B_j)}}{f(\tau)^2} \tilde{I}_J(\tau, T) = \frac{\pi}{2} \left( \frac{\eta}{d_{\partial D}(p_j)} \right)^2 \frac{\lambda_0}{c^2} \nu_q \times a \left| a \right|^2 \mathcal{F}_j,
\]
where
\[
\mathcal{F}_j = \frac{\lambda(q) - \lambda_0}{\sqrt{\det(S_q(\partial B_{\partial \Lambda_{\partial D}(p_j)}(p_j)) - S_q(\partial D))}}, \quad j = 1, 2, 3.
\]

**Step 5.** Use the expression
\[
\left\{ \begin{array}{l}
\det(S_q(\partial B_{\partial \Lambda_{\partial D}(p_j)}(p_j)) - S_q(\partial D)) = s_j^2 - 2H_{\partial D}(q)s_j + K_{\partial D}(q), \\
\frac{s_j}{1/d_{\partial D}(p_j)},
\end{array} \right.
\]
where \( H_{\partial D}(q) \) and \( K_{\partial D}(q) \) denote the mean and Gauss curvatures at \( q \) of \( \partial D \) with respect to \( \nu_q \). From \( \mathcal{F}_j \) we have
\[
\left( \begin{array}{c}
-(s_1 \mathcal{F}_1^2 - s_2 \mathcal{F}_2^2) \\
-(s_2 \mathcal{F}_2^2 - s_3 \mathcal{F}_3^2)
\end{array} \right) \left( \begin{array}{c}
\mathcal{F}_1^2 - \mathcal{F}_2^2 \\
\mathcal{F}_2^2 - \mathcal{F}_3^2
\end{array} \right) \left( \begin{array}{c}
2H_{\partial D}(q) \\
K_{\partial D}(q)
\end{array} \right) = \left( \begin{array}{c}
\mathcal{F}_1^2 s_1^2 - \mathcal{F}_2^2 s_2^2 \\
\mathcal{F}_2^2 s_2^2 - \mathcal{F}_3^2 s_3^2
\end{array} \right).
\]
Solving this linear system numerically, we may obtain $H_{\partial D}(q)$ and $K_{\partial D}(q)$.

**Step 6.** From $F_j$ one has

$$
\left( \frac{\lambda(q) - \lambda_0}{\lambda(q) + \lambda_0} \right)^2 = \frac{1}{3} \sum_{j=1}^{3} F_j^2(s_j^2 - 2H_{\partial D}(q)s_j + K_{\partial D}(q)).
$$

**Step 7.** From the signature of one of $F_j$ one can know whether $\lambda(q) > \lambda_0$ or $\gamma(q) < \lambda_0$.

**Step 8.** From Steps 6 and 7 we may obtain $\lambda(q)$.

This paper is organized as follows. In Section 2, we give a proof of Theorem 1.2. The proof is based on a rough asymptotic formula of the indicator function as $\tau \to \infty$ as stated in Lemma 2.1 which has been established in [12]. The formula consists of two terms and remainder. The treatment of the remainder is not a problem. And the first term is explicitly given by (23) as a Laplace type surface integral of $V_0^e$ and its rotation over $\partial D$. Thus the key point is the profile of the second term which is the energy integral of the so-called reflected solutions given by (24). Its asymptotic profile is stated as Theorem 2.2 which tells us that the leading profile is also given as a Laplace type surface integral of $V_0^e$ and its rotation. Then, using the leading profile of those two terms which is described in Lemma 2.3 as an application of the Laplace method, we obtain the reading profile of the indicator function as stated in Theorem 1.2.

The proof of Theorem 2.2 is given in Section 3. First we construct a candidate of the leading term of the reflected solutions. For the purpose we employ a combination of the reflection principle which has been established in [12] and a cut-off argument in a neighbourhood of $\partial D$ with a cut-off parameter $\delta$. Then the first and second terms of integral (24) is extracted as (49) in Lemma 3.1. To show that the first term is the reading profile we have to prove that the second term is small compared with first term. We see that it is true if $\delta$ is properly chosen according to the size of $\tau$. The essence is described as Lemma 3.2. The proof of Lemma 3.2 which is given in Section 4, is a modification of the Lax-Phillips reflection argument [17] originally developed for the study of the leading singularity of the scattering kernel for the scalar wave equation in the context of the Lax-Phillips scattering theory, however, our version of the argument is rather straightforward.

2. **Proof of Theorem 1.2.** Define

$$
V_0^m = -\frac{1}{\tau \mu} \nabla \times V_e^0.
$$

From this and (9) we have

$$
\begin{cases}
\nabla \times V_e^0 + \tau \mu V_0^m = 0 & \text{in } \mathbb{R}^3, \\
\nabla \times V_0^m - \tau \epsilon V_0^e - \frac{\epsilon}{\tau} f(x, \tau) = 0 & \text{in } \mathbb{R}^3.
\end{cases}
$$

It is a due course to deduce that $V_0^m \in H^1(\mathbb{R}^3)^3$ and $V_0^e$ belongs to $H^1$ in a neighbourhood of $\overline{D}$.

Define

$$
\begin{cases}
R_e = W_e - V_0^e, \\
R_m = W_m - V_0^m.
\end{cases}
$$
Note that \((R_e, R_m)\) belongs to \(L^2(\mathbb{R}^3 \setminus \overline{D})^3 \times L^2(\mathbb{R}^3 \setminus \overline{D})^3\) with \((\nabla \times R_e, \nabla \times R_m)\) belongs to \(L^2(\mathbb{R}^3 \setminus \overline{D})^3 \times L^2(\mathbb{R}^3 \setminus \overline{D})^3\); \(R_m \in H^1(\mathbb{R}^3 \setminus \overline{D})^3\) and \(R_e\) belongs to \(H^1\) in a neighbourhood of \(\partial D\).

From (19) and (5) we see that \(R_e\) and \(R_m\) satisfy

\[
\begin{align*}
\nabla \times R_e + \tau \mu R_m &= -e^{-\tau T} \mu H(x, T) \quad \text{in} \mathbb{R}^3 \setminus \overline{D}, \\
\nabla \times R_m - \tau \epsilon R_e &= e^{-\tau T} \epsilon E(x, T) \quad \text{in} \mathbb{R}^3 \setminus \overline{D}.
\end{align*}
\]

It follows from (7) that

\[
\nabla \times (R_m \times \nu) + \lambda \nu \times R_e = -\nu \times (V_0^m \times \nu) - \lambda \nu \times V_0^m.
\]

From [12], we have a rough asymptotic formula of the indicator function.

Lemma 2.1. ([12]). We have, as \(\tau \rightarrow \infty\)

\[
\tilde{I}_f(\tau, T) = J(\tau) + E(\tau) + O(e^{-\tau T} \tau^{-1}),
\]

where

\[
J(\tau) = \frac{1}{\mu \epsilon} \int_{\partial D} (\nu \times V_0^m) \cdot \nabla \times V_0^e \, dS - \frac{\tau}{\epsilon} \int_{\partial D} \frac{1}{\lambda} |V_0^m \times \nu|^2 \, dS
\]

and

\[
E(\tau) = \frac{\tau}{\epsilon} \left\{ \int_{\mathbb{R}^3 \setminus \overline{D}} (\tau \mu |R_m|^2 + \tau \epsilon |R_e|^2) \, dx + \int_{\partial D} \frac{1}{\lambda^2} |R_m \times \nu|^2 \, dS \right\}.
\]

Thus, the essential part of the proof of Theorem 1.2 should be the study of the asymptotic behaviour of \(J(\tau)\) and \(E(\tau)\) as \(\tau \rightarrow \infty\). The asymptotic behaviour of \(J(\tau)\) can be reduced to that of a Laplace-type integral [3]. See [12]. For that of \(E(\tau)\), we have the following result, which enables us to make a reduction of the study to a Laplace-type integral.

Theorem 2.2. Assume that \(\partial D\) is \(C^4\) and \(\lambda \in C^2(\partial D)\). Assume that \(\lambda\) has a positive lower bound, the set \(\Lambda_{\partial D}(p)\) consists of finite points, and (12) is satisfied; there exists a point \(q \in \Lambda_{\partial D}(p)\) such that \(\lambda(q) \neq \lambda_0\) and that

\[
\nu \times (a \times \nu) \neq 0.
\]

Let \(f\) satisfy (8) and

\[
T > \sqrt{\mu \epsilon \text{ dist}(D, B)}.
\]

Then, we have \(J^*(\tau) > 0\) for all \(\tau >> 1\) and

\[
\lim_{\tau \rightarrow \infty} \frac{E(\tau)}{J^*(\tau)} = 1,
\]

where

\[
J^*(\tau) = \frac{\tau}{\epsilon} \int_{\partial D} \frac{\lambda - \lambda_0}{\lambda + \lambda_0} (\nu \times V_0^m) \cdot V_0^e \, dS
\]

and

\[
V_0^e = \nu \times (V_0^e \times \nu) - \frac{1}{\lambda} \nu \times V_0^m.
\]
The proof of Theorem 2.2 is given in Section 3. Assumption (25) means that vector $a$ is not parallel to $\nu_q$ at $q$. Thus (25) is equivalent to the condition $|\nu_q \times a| \neq 0$. Note also that $|\nu_q \times (a \times \nu_q)|^2 = |\nu_q \times a|^2 = 1 - (\nu_q \cdot a)^2$.

Note that the factor 2 in the restriction $T > 2 \sqrt{\mu \epsilon \text{dist} (D, B)}$ in Theorem 1.2 is dropped in (26). The quantity $\sqrt{\mu \epsilon \text{dist} (D, B)}$ corresponds to the first arrival time of the wave generated at $t = 0$ on $B$ and reached at $\partial D$ firstly. The asymptotic formula (27) clarifies the effect on the leading profile of the energy of the reflected solutions $R_e$ and $R_m$ in terms of the deviation of the surface admittance from that of free-space admittance and the energy density of the incident wave.

To complete the proof of Theorem 1.2 we need the following asymptotic formulae of $J(\tau)$ and $J^*(\tau)$ as $\tau \to \infty$.

**Lemma 2.3.** We have

$$
\lim_{\tau \to \infty} \tau^2 e^{2\tau \sqrt{\mu \epsilon \text{dist} (D, B)}} \frac{J(\tau)}{f(\tau)^2} = \pi \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 \sum_{q \in \Lambda_{\partial D}(p)} k_q(p) \left( \frac{1}{\lambda_0} - \frac{1}{\lambda(q)} \right) |\nu_q \times (a \times \nu_q)|^2
$$

and

$$
\lim_{\tau \to \infty} \tau^2 e^{2\tau \sqrt{\mu \epsilon \text{dist} (D, B)}} \frac{J^*(\tau)}{f(\tau)^2} = \pi \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 \sum_{q \in \Lambda_{\partial D}(p)} k_q(p) \left( \frac{1}{\lambda(q)} - \frac{1}{\lambda_0} \right) \left( \frac{1}{\lambda(q)} - \frac{1}{\lambda_0} \right) |\nu_q \times (a \times \nu_q)|^2.
$$

**Proof.** Using (18), (29) and a simple computation in vector analysis, one can rewrite the right-hand side on (23) as

$$
J(\tau) = \tau \epsilon \int_{\partial D} \nu \times V_0^m \cdot V_{enm}^0 dS.
$$

Set

$$
v(x, \tau) = \frac{e^{-\tau \sqrt{\mu \epsilon}|x-p|}}{|x-p|}.
$$

By (18) in [11] we have already shown that $V_0^e$ has the form

$$
V_0^e = K(\tau) \tilde{f}(\tau) e M(a) \text{ in } \mathbb{R}^3 \setminus \mathcal{B},
$$

where

$$
K(\tau) = \frac{\mu \tau \varphi(\tau \sqrt{\mu \epsilon})}{(\tau \sqrt{\mu \epsilon})^3},
$$

$$
\varphi(\xi) = \xi \cosh \xi - \sinh \xi,
$$

$$
M = M(x; \tau) = A I_3 - B \omega_x \otimes \omega_x,
$$

$$
A = A(x, \tau) = 1 + \frac{1}{\tau \sqrt{\mu \epsilon}} \left( \frac{1}{|x-p|} + \frac{1}{\tau \sqrt{\mu \epsilon}|x-p|^2} \right),
$$

$$
B = B(x, \tau) = 1 + \frac{3}{\tau \sqrt{\mu \epsilon}} \left( \frac{1}{|x-p|} + \frac{1}{\tau \sqrt{\mu \epsilon}|x-p|^2} \right)
$$
and
\[ \omega_x = \frac{x - p}{|x - p|}. \]
This yields
\[ \nabla \times V_c^0 = -\tau \sqrt{\mu \epsilon} K(\tau) \tilde{f}(\tau) v \left( 1 + \frac{1}{\tau \sqrt{\mu \epsilon} |x - p|} \right) \omega_x \times a \]
and thus (18) gives
\[ V_m^0 = \lambda_0 K(\tau) \tilde{f}(\tau) v \left( 1 + \frac{1}{\tau \sqrt{\mu \epsilon} |x - p|} \right) \omega_x \times a. \]
A combination of (33) and (34) gives
\[ V_{em}^0 = \nu \times (V_c^0 \times \nu) - \frac{1}{\lambda} \nu \times V_m^0 \]
(35)
\[ = \lambda_0 K(\tau) \tilde{f}(\tau) v \nu \times \left\{ \frac{1}{\lambda_0} Ma \times \nu - \frac{1}{\lambda} \left( 1 + \frac{1}{\tau \sqrt{\mu \epsilon} |x - p|} \right) \omega_x \times a \right\} \]
\[ = \lambda_0 K(\tau) \tilde{f}(\tau) v \nu \times (D(x)a + O(\tau^{-1})), \]
where \( O(\tau^{-1}) \) means uniformly with respect to \( x \in \partial D \) and
\[ D(x)a = \frac{1}{\lambda_0} a \times \nu - \frac{1}{\lambda_0} (a \cdot \omega_x) \omega_x \times \nu - \frac{1}{\lambda} \omega_x \times a. \]
Thus we obtain
\[ \nu \times V_{m}^0 \cdot V_{em}^0 \]
(36)
\[ = \lambda_0^3 K^2(\tau) \tilde{f}(\tau)^2 v^2 \left\{ \nu \times (\omega_x \times a) \cdot \nu \times (D(x)a + O(\tau^{-1})) \right\} \]
and (32) gives
\[ \frac{J(\tau)}{\tilde{f}(\tau)^2} = \frac{\lambda_0^2}{\epsilon} \tau K^2(\tau) \int_{\partial D} \left\{ \nu \times (\omega_x \times a) \cdot \nu \times (D(x)a + O(\tau^{-1})) \right\} v^2 dS. \]
Note that if \( x \in \Lambda_{\partial D}(p) \), then \( \nu \) at \( x \) coincides with \( -\omega_x \). Thus we have
\[ D(x)a = \left( \frac{1}{\lambda_0} - \frac{1}{\lambda} \right) a \times \nu \]
and hence
\[ \nu \times (\omega_x \times a) \cdot \nu \times (D(x)a) = \left( \frac{1}{\lambda_0} - \frac{1}{\lambda} \right) |\nu \times (a \times \nu)|^2. \]

It is well known that the Laplace method under the assumption that \( \Lambda_{\partial D}(p) \) is finite and satisfies (12), yields
\[ \lim_{\tau \to \infty} \tau e^{2\pi d_{\partial D}(p)} \int_{\partial D} A(x) e^{-2\tau |x - p|} |x - p|^2 dS = \frac{\pi}{d_{\partial D}(p)^2} \sum_{q \in \Lambda_{\partial D}(p)} k_q(p) A(q), \]
where \( A \in C^1(\partial D) \). See [3], for example. The point is that the Hessian matrix of the function \( \partial D \ni x \mapsto |x - p| \) at \( q \in \Lambda_{\partial D}(p) \) is given by the operator \( S_q(\partial B_{d_{\partial D}(p)}(p)) \).
See, for example, [9] for this point. Replacing \( \tau \) above with \( \tau \sqrt{\mu \epsilon} \), we obtain
\[
\lim_{\tau \to \infty} \tau e^{2\pi \sqrt{\mu \epsilon} d_{\partial D}(p)} \int_{\partial D} A(x)v^2 dS = \frac{\pi}{\sqrt{\mu \epsilon} d_{\partial D}(p)^2} \sum_{q \in \Lambda_{\partial D}(p)} k_q(p)A(q).
\]
Note also that
\[
K(\tau) \sim \tau^{-1} \frac{\eta e^{\pi \sqrt{\mu \epsilon}}}{2\epsilon}
\]
and thus
\[
\frac{\lambda_0^2}{\epsilon} \tau K^2(\tau) = \frac{\tau}{\mu} K^2(\tau)
\]
\[
\sim \frac{\tau}{\mu} \left( \frac{\tau^{-1} \eta e^{\pi \sqrt{\mu \epsilon}}}{2\epsilon} \right)^2
\]
\[
= \frac{1}{\mu} \left( \frac{\eta}{2\epsilon} \right)^2 \tau^{-1} e^{2\pi \sqrt{\mu \epsilon}}.
\]
Applying (39) to (37) and noting (38) and (40), we obtain
\[
J(\tau) \sim \frac{\lambda_0}{\epsilon} \tau K^2(\tau) \sim \frac{\tau}{\mu} \left( \frac{\tau^{-1} \eta e^{\pi \sqrt{\mu \epsilon}}}{2\epsilon} \right)^2
\]
\[
\sim \frac{1}{\mu} \left( \frac{\eta}{2\epsilon} \right)^2 \tau^{-1} e^{2\pi \sqrt{\mu \epsilon}}.
\]
This is nothing but (30). Similarly, from (28), (36), (38), (39) and (40) we obtain (31).

Note that the formula (31) ensures \( J^*(\tau) > 0 \) for all \( \tau >> 1 \), which is stated in Theorem 2.2 since we have
\[
\frac{\lambda(q) - \lambda_0}{\lambda(q) + \lambda_0} \left( \frac{1}{\lambda_0} - \frac{1}{\lambda(q)} \right) = \frac{(\lambda(q) - \lambda_0)^2}{(\lambda(q) + \lambda_0)\lambda_0\lambda(q)}.
\]

Now we are ready to finish the proof of Theorem 1.2. Write (22) as
\[
\tilde{I}_f(\tau, T) = J(\tau) + J^*(\tau) \cdot \frac{E(\tau)}{J^*(\tau)} + O(e^{-\tau T} \tau^{-1}).
\]
Note that, under the assumption (8) we have
\[
\frac{\tau^2 e^{2\pi \sqrt{\mu \epsilon} \text{dist}(D, B)}}{\tilde{f}(\tau)^2} \tau^{-1} e^{-\tau T} = \frac{\tau^{1+2\gamma} e^{-\tau(T-2\sqrt{\mu \epsilon} \text{dist}(D, B))}}{\tau^{2\gamma} \tilde{f}(\tau)^2} = O(\tau^{1+2\gamma} e^{-\tau(T-2\sqrt{\mu \epsilon} \text{dist}(D, B))}).
\]
We have also
\[
\left( \frac{1}{\lambda_0} - \frac{1}{\lambda(q)} \right) + \frac{\lambda(q) - \lambda_0}{\lambda(q) + \lambda_0} \left( \frac{1}{\lambda_0} - \frac{1}{\lambda(q)} \right) = \frac{2}{\lambda_0} \cdot \frac{\lambda(q) - \lambda_0}{\lambda(q) + \lambda_0}.
\]
Applying this together with (27), (30) and (31) to the right-hand side on (41), we obtain (13). This completes the proof of Theorem 1.2.

3. Proof of Theorem 2.2. We denote by \( x' \) the reflection across \( \partial D \) of the point \( x \in \mathbb{R}^3 \setminus D \) with \( d_{AD}(x) < 2\delta_0 \) for a sufficiently small \( \delta_0 > 0 \). It is given by \( x' = 2q(x) - x \), where \( q(x) \) denotes the unique point on \( \partial D \) such that \( d_{AD}(x) = |x - q(x)| \). Note that \( q(x) \) is \( C^2 \) for \( x \in \mathbb{R}^3 \setminus D \) with \( d_{AD}(x) < 2\delta_0 \) if \( \partial D \) is \( C^3 \) (see [7]). Define \( \tilde{\lambda}(x) = \lambda(q(x)) \) for \( x \in \mathbb{R}^3 \setminus D \) with \( d_{AD}(x) < 2\delta_0 \). The function \( \tilde{\lambda} \) is \( C^2 \) therein and coincides with \( \lambda(x) \) for \( x \in \partial D \).

Choose a cutoff function \( \phi_\delta \in C^2(\mathbb{R}^3) \) with \( 0 < \delta < \delta_0 \) which satisfies \( 0 \leq \phi_\delta(x) \leq 1 \): \( \phi_\delta(x) = 1 \) if \( d_{AD}(x) < \delta \); \( \phi_\delta(x) = 0 \) if \( d_{AD}(x) > 2\delta \); \( |\nabla \phi_\delta(x)| \leq C\delta^{-1} \); \( |\nabla^2 \phi_\delta(x)| \leq C\delta^{-2} \).

Using the reflection across the boundary \( \partial D \), in [11] we have already constructed from \( V_e^0 \) in \( D \) the vector field \( (V_e^0)^* \) for \( x \in \mathbb{R}^3 \setminus \overline{D} \) with \( d_{AD}(x) < 2\delta_0 \) and another one

\[
(V_m^0)^* = -\frac{1}{\tau\mu} \nabla \times \{ (V_e^0)^* \}
\]

which satisfy

\[
(V_e^0)^* = -V_e^0 \quad \text{on } \partial D
\]

and

\[
(42) \quad \nu \times (V_m^0)^* = \nu \times V_m^0 \quad \text{on } \partial D.
\]

Define

\[
R_e^0 = \frac{\tilde{\lambda} - \lambda_0}{\lambda + \lambda_0} \phi_\delta(V_e^0)^*
\]

and

\[
R_m^0 = \frac{\tilde{\lambda} - \lambda_0}{\lambda + \lambda_0} \phi_\delta(V_m^0)^*.
\]

The pair \((R_e^0, R_m^0)\) belongs to \( H^1(\mathbb{R}^3 \setminus \overline{D})^3 \times H^1(\mathbb{R}^3 \setminus \overline{D})^3 \) and depends on \( \delta \).

Define

\[
\begin{cases}
R_e^1 = R_e - R_e^0, \\
R_m^1 = R_m - R_m^0.
\end{cases}
\]

Since \( R_e \) and \( R_m \) satisfy (21), we obtain

\[
\nu \times R_m^1 - \lambda \nu \times (R_e^1 \times \nu) = \nu \times (\nu \times (R_m^0 \times \nu)) - \lambda \nu \times (R_e^1 \times \nu)
\]

\[
= -\lambda \nu \times (\nu \times R_e^1) - \nu \times \left\{ \frac{2\lambda}{\lambda + \lambda_0} \left\{ \nu \times (V_m^0 \times \nu) + \lambda_0 \nu \times V_e^0 \right\} \right\}
\]

\[
= -\lambda \nu \times (R_e^1 \times \nu)
\]

\[
= -\frac{2\lambda}{\lambda + \lambda_0} \left\{ \nu \times V_m^0 - \lambda_0 \nu \times (V_e^0 \times \nu) \right\}.
\]

Define

\[
V_1 = \frac{2\lambda}{\lambda + \lambda_0} \left\{ \nu \times V_m^0 - \lambda_0 \nu \times (V_e^0 \times \nu) \right\} \quad \text{on } \partial D.
\]
It follows from (20) and (45) that $R_e^1$ and $R_m^1$ satisfy

\begin{align}
\nabla \times R_e^1 + \tau \mu R_m^1 &= - (\nabla \times R_e^0 + \tau \mu R_m^0) - e^{-\tau T} \mu H(x, T) \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \\
\nabla \times R_m^1 - \tau \epsilon R_e^1 &= - (\nabla \times R_m^0 - \tau \epsilon R_e^0) + e^{-\tau T} \epsilon E(x, T) \quad \text{in } \mathbb{R}^3 \setminus \overline{D}
\end{align}

and

\begin{align}
\nu \times R_m^1 - \lambda \nu \times (R_e^1 \times \nu) &= - V_1 \quad \text{on } \partial D.
\end{align}

Now we are ready to state an asymptotic formula of $E(\tau) - J^*(\tau)$ as $\tau \to \infty$ which extracts the main term involving $\nu \times R_m^1$ on $\partial D$.

**Lemma 3.1.** We have, as $\tau \to \infty$

\begin{align}
E(\tau) &= J^*(\tau) + \frac{T}{\epsilon} \int_{\partial D} \nu \times (R_m \times \nu) \cdot (\nu \times R_m) \, dS \\
&\quad + O(e^{-\tau T} (\tau^{-2} e^{-\tau \sqrt{\epsilon}} \operatorname{dist}(D, B) |\tilde{f}(\tau)| + \tau^{-1} e^{-\tau T})).
\end{align}

**Proof.** Recall (40) in [12]:

\begin{align}
\int_{\mathbb{R}^3 \setminus \overline{D}} (\tau \mu |R_m|^2 + \tau \epsilon |R_e|^2) \, dx
\end{align}

\begin{align}
= \int_{\partial D} \nu \times (R_m \times \nu) \cdot (\nu \times R_m) \, dS \\
- \epsilon^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{D}} (\mu H(x, T) \cdot R_m + \epsilon E(x, T) \cdot R_e) \, dx.
\end{align}

It follows from this and (24) that

\begin{align}
E(\tau) &= \frac{T}{\epsilon} \int_{\partial D} \left\{ \nu \times (R_m \times \nu) \cdot (\nu \times R_m) + \frac{1}{\lambda} |R_m \times \nu|^2 \right\} \, dS \\
&\quad - \epsilon^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{D}} (\mu H(x, T) \cdot R_m + \epsilon E(x, T) \cdot R_e) \, dx.
\end{align}

From (21) we have

\begin{align}
\nu \times R_e &= - \nu \times V_e^0 - \frac{1}{\lambda} \left\{ \nu \times (V_m^0 \times \nu) + \nu \times (R_m \times \nu) \right\}.
\end{align}

This gives

\begin{align}
- \nu \times (R_m \times \nu) \cdot (\nu \times R_e)
\end{align}

\begin{align}
= \nu \times (R_m \times \nu) \cdot \left\{ \nu \times V_e^0 + \frac{1}{\lambda} \left\{ \nu \times (V_m^0 \times \nu) + \nu \times (R_m \times \nu) \right\} \right\}
\end{align}

\begin{align}
= \frac{1}{\lambda} |R_m \times \nu|^2 + V_e^0 \cdot (R_m \times \nu).
\end{align}

Substituting $R_m = R_m^0 + R_m^1$ into the second term on this right-hand side and using (42) and (44), we obtain
\[ \nu \times (R_m \times \nu) \cdot (\nu \times R_e) + \frac{1}{\lambda} |R_m \times \nu|^2 \]
\[ = -V_{em}^0 \cdot (R_m \times \nu) \]
\[ = \frac{\lambda - \lambda_0}{\lambda + \lambda_0} V_{em}^0 \cdot (\nu \times V_m^0) + V_{em}^0 \cdot (\nu \times R_{m}^1). \]

Thus (50) becomes
\[ E(\tau) = \frac{\tau}{\epsilon} \int_{\partial D} \frac{\lambda - \lambda_0}{\lambda + \lambda_0} V_{em}^0 \cdot (\nu \times V_m^0) dS + \frac{\tau}{\epsilon} \int_{\partial D} V_{em}^0 \cdot (\nu \times R_{m}^1) dS \]
\[ - e^{-\tau T} \int_{\mathbb{R}^3 \setminus \mathcal{D}} (\mu H(x, T) \cdot R_m + \epsilon E(x, T) \cdot R_e) dx. \]

By Lemma 3.2 in [12] we have
\[ \|R_e\|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})} = \|R_m\|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})} = O(\tau^{-1/2} \|V_{em}^0\|_{L^2(\partial D)} + \tau^{-1} e^{-\tau T}). \]

Here we make use of the following asymptotic formula which can be shown similarly as formulae in Lemma 2.3 by using (35):
\[ \lim_{\tau \to \infty} \tau^3 e^{2\tau \sqrt{\text{dist}(D, B)}} \int_{\partial D} |V_{em}^0|^2 dS \]
\[ = \frac{\pi}{4} \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 \frac{\lambda^3}{\epsilon^3} \sum_{q \in \Lambda_{\partial D}(p)} k_q(p) \left( \frac{1}{\lambda_0} - \frac{1}{\lambda(q)} \right)^2 |\nu_q \times (a \times \nu_q)|^2. \]

Applying this to the right-hand side on (52), we obtain
\[ \|R_e\|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})} = \|R_m\|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})} = O(e^{-\tau T} \sqrt{\text{dist}(D, B)} |f(\tau)| + \tau^{-1} e^{-\tau T}). \]

Now a combination of (51) and (54) yields (49). 

Thus, the problem is: clarify the asymptotic behaviour of \( \nu \times R_{m}^1 \) on \( \partial D \) as \( \tau \to \infty \). The point is the choice of \( \delta \).

**Lemma 3.2.** Choose \( \delta = \tau^{-1/2} \). We have
\[ \lim_{\tau \to \infty} \tau^3 e^{2\tau \sqrt{\text{dist}(D, B)}} \|\nu \times R_{m}^1\|^2_{L^2(\partial D)} = 0. \]

The proof of Lemma 3.2 is given in Section 4.

Now choose \( \delta \) in the pair \((R_0^0, R_{m}^0)\) as that of Lemma 3.2.

Write
\[ \left| \frac{\tau}{\epsilon} \int_{\partial D} V_{em}^0 \cdot (\nu \times R_{m}^1) dS \right| \]
\[ \leq \frac{\tau^{3/2} e^{\tau \sqrt{\text{dist}(D, B)}} \|V_{em}^0\|_{\partial D} \cdot \tau^{3/2} e^{\tau \sqrt{\text{dist}(D, B)}} \|\nu \times R_{m}^1\|_{L^2(\partial D)}}{\epsilon^2 e^{2\tau \sqrt{\text{dist}(D, B)}} |J^*(\tau)|}. \]
Applying (31), (53) and (55) to this right-hand side, we obtain

\[
\lim_{\tau \to \infty} \frac{\tau}{\epsilon} \int_{\partial D} \frac{V^0_{\epsilon m} \cdot (\nu \times R^1_m)}{J^*(\tau)} \, dS = 0.
\]

Write

\[
O(e^{-\tau T}(\tau^{-2}e^{-\tau \sqrt{\mu} \text{dist} (D,B)}|\tilde{f}(\tau)| + \tau^{-1}e^{-\tau T}))
\]

\[
= \frac{O(e^{-\tau T}(e^{\tau \sqrt{\mu} \text{dist} (D,B)}|\tilde{f}(\tau)|^{-1} + \tau e^{-\tau T}e^{2\tau \sqrt{\mu} \text{dist} (D,B)}|\tilde{f}(\tau)|^{-2}))}{\tau^2 e^{2\tau \sqrt{\mu} \text{dist} (D,B)}J^*(\tau)|\tilde{f}(\tau)|^{-2}}.
\]

Note that, if \( f \) and \( T \) satisfy (8) and (26), respectively, then we have, as \( \tau \to \infty \)

\[
e^{-\tau T}(e^{\tau \sqrt{\mu} \text{dist} (D,B)}|\tilde{f}(\tau)|^{-1} + \tau e^{-\tau T}e^{2\tau \sqrt{\mu} \text{dist} (D,B)}|\tilde{f}(\tau)|^{-2})
\]

\[
= \tau^{-1}e^{-\tau T}e^{2\tau \sqrt{\mu} \text{dist} (D,B)}\tau^{-\gamma}|\tilde{f}(\tau)|^{-1} + \tau^{-1}e^{-\tau T}e^{2\tau \sqrt{\mu} \text{dist} (D,B)}\tau^{-2\gamma}|\tilde{f}(\tau)|^{-2}
\]

\[
\to 0.
\]

Now, applying this to (57) with the help of (31), we see that the left-hand side on (57) converges to 0 as \( \tau \to \infty \). Applying this and (56) to the right-hand side on (49), we obtain (27).

4. **Proof of Lemma 3.2.** In this section, we denote by \( C \) several positive constants independen of \( \delta \) and \( \tau \).

First we give an upper estimate on \( \| \nu \times R^1_m \|_{L^2(\partial D)} \) by using \( V_1, R^0_e \) and \( R^0_m \) which are explicitly given in terms of \( V^0_{\epsilon m}, V^0_\epsilon \). See (43), (44) and (46) in Section 3.

**Lemma 4.1.** We have

\[
\| \nu \times R^1_m \|_{L^2(\partial D)}^2 
\]

\[
\leq C(\| V_1 \|_{L^2(\partial D)}^2 + \tau^{-1}\| F_1 \|_{L^2(U_3)}^2 + \tau^{-1}\| F_2 \|_{L^2(U_3)}^2) + O(\tau^{-1}e^{-2\tau T}),
\]

where \( U_3 = \{ x \in \mathbb{R}^3 \setminus \mathcal{T} \mid d_{\partial D}(x) < 2\delta \} \) and

\[
\begin{cases}
    F_1 = \nabla \times R^0_e + \tau \mu R^0_m,
    \\
    F_2 = \nabla \times R^0_m - \tau \epsilon R^0_e.
\end{cases}
\]

**Proof.** Taking the inner product of the both sides of the first equation on (47) with \( R^1_m \), we obtain

\[
(\nabla \times R^1_e) \cdot R^m_m + \tau \mu |R^1_m|^2 = - F_1 \cdot R^1_m - e^{-\tau T} \mu H(x,T) \cdot R^1_m.
\]

Taking the inner product of the both sides of the second equation on (47) with \( R^1_e \), we obtain

\[
(\nabla \times R^1_m) \cdot R^1_e - \tau \epsilon |R^1_e|^2 = - F_2 \cdot R^1_e + e^{-\tau T} \epsilon E(x,T) \cdot R^1_e.
\]

By virtue of the fact that \( R^1_m \in H^1(\mathbb{R}^3 \setminus \mathcal{T})^3 \) and \( R^1_e \in H(\text{curl}, \mathbb{R}^3 \setminus \mathcal{T}) \), we have

\[
\int_{\mathbb{R}^3 \setminus \mathcal{T}} \nabla \cdot (R^1_e \times R^1_m) \, dx = \langle R^1_e \times \nu, \nu \times (R^1_m \times \nu) \rangle_{-1/2,1/2}.
\]
where this right-hand side denotes the value of the bounded linear functional $R_c^1 \times \nu$ on $H^{1/2}(\partial D)^3$ of $\nu \times (R_m^1 \times \nu) \in H^{1/2}(\partial D)^3$. However, $R_c^1$ belongs to $H^1$ in a neighbourhood of $\partial D$ this coincides with the integral

$$-\int_{\partial D} \nu \times (R_m^1 \times \nu) \cdot (\nu \times R_c^1) \, dS.$$ 

Note also that

$$\nabla \cdot (R_c^1 \times R_m^1) = (\nabla \times R_c^1) \cdot R_m^1 - (\nabla \times R_m^1) \cdot R_c^1.$$ 

From these, (59), (60) and (61) we obtain

$$\int_{\mathbb{R}^3 \setminus \overline{D}} (\tau \mu |R_m^1|^2 + \tau \epsilon |R_c^1|^2) \, dx$$

$$= \int_{\partial D} \nu \times (R_m^1 \times \nu) \cdot (\nu \times R_c^1) \, dS$$

(62)

$$- \int_{\mathbb{R}^3 \setminus \overline{D}} F_1 \cdot R_m^1 \, dx + \int_{\mathbb{R}^3 \setminus \overline{D}} F_2 \cdot R_c^1 \, dx$$

$$- e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{D}} (\mu H(x, T) \cdot R_m^1 + \epsilon E(x, T) \cdot R_c^1) \, dx.$$ 

Since we have

$$\nu \times (R_m^1 \times \nu) \cdot \nu \times R_c^1 = -\{ \nu \times (R_c^1 \times \nu) \} \cdot (\nu \times R_m^1),$$

from (48) one gets

$$\nu \times (R_m^1 \times \nu) \cdot \nu \times R_c^1 = -\lambda |\nu \times (R_c^1 \times \nu)|^2 + \nu \times (R_c^1 \times \nu) \cdot V_1.$$ 

Thus (62) becomes

$$\int_{\mathbb{R}^3 \setminus \overline{D}} (\tau \mu |R_m^1|^2 + \tau \epsilon |R_c^1|^2) \, dx + \int_{\partial D} \lambda |\nu \times (R_c^1 \times \nu)|^2 \, dS$$

(63)

$$+ e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{D}} (\mu H(x, T) \cdot R_m^1 + \epsilon E(x, T) \cdot R_c^1) \, dx$$

$$= \int_{\partial D} V_1 \cdot \{ \nu \times (R_c^1 \times \nu) \} \, dS + F(\tau),$$

where

$$F(\tau) = - \int_{\mathbb{R}^3 \setminus \overline{D}} F_1 \cdot R_m^1 \, dx + \int_{\mathbb{R}^3 \setminus \overline{D}} F_2 \cdot R_c^1 \, dx.$$ 

Rewrite (63) further as

$$\int_{\mathbb{R}^3 \setminus \overline{D}} \left( \tau \mu |R_m^1|^2 + \frac{F_1 + e^{-\tau T} \mu H(x, T)}{2 \mu \tau} \right)^2 \, dx$$

$$+ \int_{\partial D} \left| \nu \times (R_c^1 \times \nu) - \frac{V_1}{2 \lambda} \right|^2 \, dS$$

$$= \int_{\mathbb{R}^3 \setminus \overline{D}} \left( \frac{|F_1 + e^{-\tau T} \mu H(x, T)|^2}{4 \mu \tau} + \frac{|-F_2 + e^{-\tau T} \epsilon E(x, T)|^2}{4 \epsilon \tau} \right) \, dx + \int_{\partial D} \frac{|V_1|^2}{4 \lambda} \, dS.$$
This immediately yields
\[
\|\nu \times (R_{\nu}^2 \times \nu)\|_{L^2(\partial D)}^2 \\
\leq C(\|V_1\|_{L^2(\partial D)}^2 + \tau^{-1} \|F_1\|_{L^2(U_\delta)}^2 + \tau^{-1} \|F_2\|_{L^2(U_\delta)}^2) + O(\tau^{-1}e^{-2\tau T}).
\]
Then, the boundary condition (48) yields (58).

In order to make use of the right-hand side on (58), we prepare the following two lemmas.

**Lemma 4.2.** We have, as \(\tau \to \infty\)
\[
\lim_{\tau \to \infty} \tau^3 e^{2\tau \sqrt{\mu \tau}} \frac{\|V_1\|_{L^2(\partial D)}^2}{f(\tau)^2} = 0.
\]

*Proof.* From (29) and (46) we have the expression
\[
V_1 = -\frac{2\lambda_0 \lambda}{\lambda + \lambda_0} V_{0\text{em},\lambda=\lambda_0}.
\]
Thus (35) yields
\[
\tau^3 e^{2\tau \sqrt{\mu \tau}} \frac{\|V_1\|_{L^2(\partial D)}^2}{f(\tau)^2} \\
\leq C K(\tau)^2 e^{2\tau \sqrt{\mu \tau}} \int_{\partial D} v^2 \nu \times (D(x)|_{\lambda=\lambda_0} \lambda + O(\tau^{-1}))^2 dS
\]
\[
= C K(\tau)^2 e^{-2\tau \eta \sqrt{\mu \tau}} \tau e^{2\tau \sqrt{\mu \tau} d_{\partial D}(p)} \int_{\partial D} v^2 \nu \times (D(x)|_{\lambda=\lambda_0} \lambda + O(\tau^{-1}))^2 dS.
\]
Note that the term \(O(\tau^{-1})\) is uniform with respect to \(x \in \partial D\). Since \(D(x)|_{\lambda=\lambda_0} \lambda = 0\) for all \(x \in \Lambda_{\partial D}(p)\), it follows from (39) that
\[
\lim_{\tau \to \infty} \tau e^{-2\tau \eta \sqrt{\mu \tau} d_{\partial D}(p)} \int_{\partial D} v^2 \nu \times (D(x)|_{\lambda=\lambda_0} \lambda + O(\tau^{-1}))^2 dS = 0.
\]
Then, from (40) we obtain the desired conclusion. \(\square\)

**Lemma 4.3.** We have
\[
\|F_1\|_{L^2(U_\delta)} \leq C \delta^{-1} \tau^{-1} J_\infty(\tau)^{1/2}
\]
and
\[
\|F_2\|_{L^2(U_\delta)} \leq C (\tau^{-1} + \delta + \delta^{-1} \tau^{-1}) J_\infty(\tau)^{1/2},
\]
where
\[
J_\infty(\tau) = \frac{\tau}{\epsilon} \int_D (\tau \mu |V_m^0|^2 + \tau \epsilon |V_e^0|^2) dx.
\]

*Proof.* This is an application of a reflection argument developed in [17]. First of all, we compute both \(\nabla \times (V_m^0)^*\) and \(\nabla \times (V_e^0)^*\). From the definition we have
\[
\nabla \times (V_m^0)^* = -\tau \mu (V_m^0)^*
\]
and hence
\[
\nabla \times (V_m^0)^* = -\frac{1}{\tau \mu} \nabla \times \nabla \times (V_e^0)^*.
\]
Define

\[ \tilde{\phi}_\delta(x) = \frac{\lambda(x) - \lambda_0}{\lambda(x) + \lambda_0} \phi_\delta(x). \]

We have

\[ \nabla \times \mathbf{R}_e^0 = \tilde{\phi}_\delta \nabla \times (\mathbf{V}_e^0)^* + \nabla \tilde{\phi}_\delta \times (\mathbf{V}_e^0)^* \]

and from (68) one gets the expression

\[ F_1 = \nabla \tilde{\phi}_\delta \times (\mathbf{V}_e^0)^*. \]

This gives

\[ \|F_1\|_{L^2(U_\delta)} \leq C \delta^{-1} \|(\mathbf{V}_e^0)^*\|_{L^2(U_\delta)}. \]

Using the change of variables \( y = x^\tau \), one has

\[ \|(\mathbf{V}_e^0)^*\|_{L^2(U_\delta)} \leq C \|\mathbf{V}_e^0\|_{L^2(D)}. \]

We have

\[ \|(\mathbf{V}_e^0)^*\|_{L^2(D)} \leq \tau^{-1} J_\infty(\tau)^{1/2}. \]

Thus, from (70), (71) and (72), we obtain (65).

From (9) we know that \( \mathbf{V}_e^0 \) satisfies

\[ \frac{1}{\mu \epsilon} \nabla \times \nabla \times \mathbf{V}_e^0 + \tau^2 \mathbf{V}_e^0 = 0 \quad \text{in } D. \]

Applying Proposition 3 in [11] to this case, we have

\[ \frac{1}{\mu \epsilon} \nabla \times \nabla \times (\mathbf{V}_e^0)^* + \tau^2 (\mathbf{V}_e^0)^* \]

\[ = \text{terms from } \mathbf{V}_e^0(x^\tau) \text{ and } (\mathbf{V}_e^0)^*(x^\tau) + 2d_{\partial D}(x) \times \text{terms from } (\nabla^2 \mathbf{V}_e^0)(x^\tau) \]

\[ \equiv Q(x). \]

Note that all the coefficients of \( \mathbf{V}_e^0(x^\tau) \), \( (\mathbf{V}_e^0)^*(x^\tau) \) and \( (\nabla^2 \mathbf{V}_e^0)(x^\tau) \) in \( Q(x) \) are independent of \( \tau \) and continuous in a tubular neighbourhood of \( \partial D \); in particular, the coefficients of \( (\nabla^2 \mathbf{V}_e^0)(x^\tau) \) in \( Q(x) \) are \( C^1 \) therein.

From (69) we have

\[ \nabla \times \mathbf{R}_m^0 = \tilde{\phi}_\delta \nabla \times (\mathbf{V}_m^0)^* + \nabla \tilde{\phi}_\delta \times (\mathbf{V}_m^0)^* \]

\[ = -\frac{\epsilon}{\tau} \tilde{\phi}_\delta \frac{1}{\mu \epsilon} \nabla \times \nabla \times (\mathbf{V}_e^0)^* + \nabla \tilde{\phi}_\delta \times (\mathbf{V}_m^0)^*. \]

Thus (73) gives

\[ F_2 = -\frac{\epsilon}{\tau} \tilde{\phi}_\delta Q(x) + \nabla \tilde{\phi}_\delta \times (\mathbf{V}_m^0)^*. \]

This yields

\[ \|F_2\|_{L^2(U_\delta)} \leq C(\tau^{-1} \|Q\|_{L^2(U_\delta)} + \delta^{-1} \|(\mathbf{V}_m^0)^*\|_{L^2(U_\delta)}). \]

From the form of \( Q \) and the change of variables, we have

\[ \|Q\|_{L^2(D)} \leq C(\|\mathbf{V}_e^0\|_{L^2(D)} + \|(\mathbf{V}_e^0)^*\|_{L^2(D)} + \delta \|\nabla^2 (\mathbf{V}_e^0)\|_{L^2(D)}). \]

From the definition of \( (\mathbf{V}_m^0)^* \) and a change of variables we have

\[ \|(\mathbf{V}_m^0)^*\|_{L^2(U_\delta)} \leq C \tau^{-1} \|(\mathbf{V}_e^0)^*\|_{L^2(D)}. \]
Here we claim
\begin{equation}
\| (V^0_e)' \|_{L^2(D)} \leq CJ_\infty(\tau)^{1/2}
\end{equation}
and
\begin{equation}
\| \nabla^2 (V^0_e) \|_{L^2(D)} \leq C\tau J_\infty(\tau)^{1/2}.
\end{equation}

The estimate (77) has been established as (27) of Lemma 2.2 in [11] since from (18)
we have another expression
\begin{equation*}
J_\infty(\tau) = \frac{1}{\mu \epsilon} \int_D |\nabla \times V^0_e|^2 \, dx + \tau^2 \int_D |V^0_e|^2 \, dx.
\end{equation*}

The estimate (78) is proved using the explicit form (33). More precisely, we have
\begin{equation*}
(V^0_e)'(x) = K(\tau) \hat{f}(\tau) \{v(x)(M(x;p)a)' + (M(x;p)a) \otimes \nabla v(x)\}
\end{equation*}
and hence, for \( j = 1, 2, 3 \)
\begin{equation*}
\begin{split}
\frac{\partial}{\partial x_j} (V^0_e)'(x) &= \frac{(x_j - p_j)}{|x - p|^3} V^0_e(x) \otimes \omega_x - \hat{\tau} \left( 1 + \frac{1}{\hat{\tau}|x - p|} \right) \{ \frac{\partial}{\partial x_j} V^0_e(x) \} \otimes \omega_x \\
&\quad - \hat{\tau} \left( 1 + \frac{1}{\hat{\tau}|x - p|} \right) \frac{\partial}{\partial x_j} \omega_x \\
&\quad + K(\tau) \hat{f}(\tau) \frac{\partial}{\partial x_j} v(x)(M(x;p)a)' + K(\tau) \hat{f}(\tau) v(x) \frac{\partial}{\partial x_j} (M(x;p)a)'.
\end{split}
\end{equation*}

Then, it is easy to see that, there exists a positive constant \( C \) independent of \( \tau \) such that, for all \( x \in D \) and \( \tau > 0 \), we have
\begin{equation*}
|\nabla^2 (V^0_e)(x)| \leq C(\tau + 1)(|V^0_e(x)| + |(V^0_e)'(x)| + K(\tau) |\hat{f}(\tau)| v(x)|).
Thus choosing \( \theta \rightarrow \infty \) and \( F \) we know from (24) in [11] that, for all \( x \in \mathbb{R}^3 \setminus B \)

\[
|V_\epsilon^0(x)|^2 \geq \tau^{-2} \frac{K(\tau)^2 \tilde{f}(\tau)^2 v(x)^2}{\mu \epsilon |x-p|^2}.
\]

This yields

\[
\|\nabla^2(V_\epsilon^0)\|_{L^2(D)}^2 \leq C'\tau^2(\|V_\epsilon^0\|_{L^2(D)} + \|(V_\epsilon^0)'\|_{L^2(D)}).
\]

Then, applying (72) and (77) to this right-hand side we obtain (78).

Now choosing (79) and hence

\[
\frac{L}{\epsilon}(U_\delta) \leq C(1 + \delta \tau)J_{\infty}(\tau)^{1/2}.
\]

Then, this together with (74), (76) and (77) yields (66).

Lemma 4.3 tells us that, to obtain the upper bound to the terms involving \( F_1 \) and \( F_2 \) on the right-hand side of (58) it suffices to give an estimate on \( J_{\infty}(\tau) \) as \( \tau \rightarrow \infty \). A combination of (19) in \( D \) and (67), we have

\[
J_{\infty}(\tau) = -\frac{\tau}{\epsilon} \int_{\partial D} \nu \cdot (V_\epsilon^0 \times V_\epsilon^0_m) \, dS
\]

\[
= \frac{\tau}{\epsilon} \int_{\partial D} \nu \times (V_\epsilon^0 \times \nu) \cdot \nu \times V_\epsilon^0_m \, dS.
\]

Since this last expression means that \( J_{\infty}(\tau) = J(\tau)|_{\lambda=\infty} \), similarly to (30), we have

\[
\lim_{\tau \rightarrow \infty} \tau^{-2} e^{2\tau \sqrt{\epsilon \tau}} \text{dist}(D,B) J_{\infty}(\tau) \frac{f(\tau)}{\tilde{f}(\tau)^2}
\]

\[
= \pi \left( \frac{\eta}{a_{BD}(p)} \right)^2 \sum_{q \in \lambda_{BD}(p)} k_q(p) |\nu_q \times (a \times \nu_q)|^2.
\]

This gives, as \( \tau \rightarrow \infty \)

\[
e^{2\tau \sqrt{\epsilon \tau}} \text{dist}(D,B) J_{\infty}(\tau) \frac{f(\tau)}{\tilde{f}(\tau)^2} = O(\tau^{-2}).
\]

From this together with (65) and (66), we have

\[
\tau^{-1} \|F_1\|_{L^2(U_\delta)}^2 + \tau^{-1} \|F_2\|_{L^2(U_\delta)}^2
\]

\[
\leq C(\delta^{-2} \tau^{-5} + \tau^{-5} + \delta^2 \tau^{-3}) e^{-2\tau \sqrt{\epsilon \tau}} \text{dist}(D,B) \tilde{f}(\tau)^2
\]

and hence

\[
\tau^3(\tau^{-1} \|F_1\|_{L^2(U_\delta)}^2 + \tau^{-1} \|F_2\|_{L^2(U_\delta)}^2)
\]

\[
\leq C(\delta^{-2} \tau^{-2} + \tau^{-2} + \delta^2) e^{-2\tau \sqrt{\epsilon \tau}} \text{dist}(D,B) \tilde{f}(\tau)^2.
\]

Now choosing \( \delta = \tau^{-\theta} \) with \( \theta > 0 \), we have

\[
\delta^{-2} \tau^{-2} + \tau^{-2} + \delta^2 = \tau^{-2(1-\theta)} + \tau^{-2} + \tau^{-2\theta}.
\]

Thus choosing \( \theta = 1/2 \), we have \( 1 - \theta = \theta \) and (79) becomes

\[
\tau^3(\tau^{-1} \|F_1\|_{L^2(U_\delta)} + \tau^{-1} \|F_2\|_{L^2(U_\delta)}) \leq C \tau^{-1} e^{-2\tau \sqrt{\epsilon \tau}} \text{dist}(D,B) \tilde{f}(\tau)^2.
\]

Now applying this and (64) to the right-hand side on (58) together with (8) and (26), we obtain (55). This completes the proof of Lemma 3.2.
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