DYNAMICS OF THE MULTICOLOR BOX-BALL SYSTEM WITH RANDOM INITIAL CONDITIONS VIA PITMAN’S TRANSFORMATION

KAZUKI KONDO

Contents

1. Introduction 1
2. Pitman transform 3
  2.1. One-sided Pitman transform 4
  2.2. Two-sided Pitman transform and its inverse 6
3. Path encodings of the multicolor BBS 11
  3.1. Vectors for path encodings 12
  3.2. Configuration of the one-sided multicolor BBS 13
  3.3. Carrier process for the one-sided multicolor BBS 15
  3.4. Action of the carrier for the one-sided multicolor BBS 16
  3.5. Two-sided multicolor BBS 19
  3.6. Inverse of the action 20
  3.7. Set of configurations 22
4. Random initial configurations 26
  4.1. Independent and identically distributed initial configuration 27
  4.2. Multicolor BBS on \( \mathbb{R} \) 28
  4.3. Brownian motion with drift 30
Acknowledgements 42
References 42

1. Introduction

The Box-Ball System (BBS) is a one-dimensional cellular automaton in \( \{0,1\}^\mathbb{Z} \) that was introduced by Takahashi and Satsuma in 1990 [7], and has been extensively studied from the viewpoint of integrable systems. In particular, it is connected with the KdV equation [5]

\[
\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad u = u(x,t), \quad x, t \in \mathbb{R},
\]

which is a non-linear partial differential equation giving a mathematical model for waves on shallow water surfaces. The BBS equation of motion is obtained from the KdV equation by applying an appropriate discretization and transform [15]. The KdV equation has soliton solutions whose shape and speed are conserved after collision with other solitons, and such a phenomenon is also observed in the BBS.

Now we present the original definition of the BBS from [7]. We denote a particle configuration by \( (\eta_n)_{n \in \mathbb{Z}} \in \{0,1\}^\mathbb{Z} \) for the two-sided case.
or \((\eta_n)_{n \in \mathbb{N}} \in \{0, 1\}^\mathbb{N}\) for the one-sided case. Specifically, we write \(\eta_n = 1\) if there is a particle at site \(n\), and \(\eta_n = 0\) otherwise. On the condition that there is a finite number of particles, that is, \(\sum_{n \in \mathbb{Z}} \eta_n < \infty\), the evolution of the BBS is described by an operator \(T : \{0, 1\}^\mathbb{Z} \rightarrow \{0, 1\}^\mathbb{Z}\) that is characterized by the following BBS equation of motion,

\[
(T\eta)_n = \min \left\{ 1 - \eta_n, \sum_{m=-\infty}^{n-1} (\eta_m - (T\eta)_m) \right\},
\]

where we suppose \((T\eta)_n = 0\) for \(n \leq \inf \{ l : \eta_l = 1 \}\), so the sums in the above definition are well-defined. In other words, the balls move sequentially from left to right, that is, from negative to positive, with each being transported to the leftmost unoccupied site to its right as follows.

\[
\eta = (\cdots 0 1 1 1 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 \cdots)
\]
\[
T\eta = (\cdots 0 0 0 0 1 1 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 \cdots)
\]
\[
T^2\eta = (\cdots 0 0 0 0 0 0 1 1 1 0 0 0 1 0 0 0 0 0 0 0 \cdots)
\]
\[
T^3\eta = (\cdots 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 0 0 0 0 0 \cdots)
\]
\[
T^4\eta = (\cdots 0 0 0 0 0 0 0 0 0 0 1 0 0 1 1 1 0 0 0 \cdots)
\]

This example exhibits a string of 3 consecutive balls, called a soliton, moving distance 3 in each time step when there is no interaction, and recovering its shape and speed after a collision with another soliton (of length 1).

In this paper we consider a generalization of the BBS that incorporates multiple colors of balls, that is, we assume that there are \(\kappa\)-color balls (particles) for some \(\kappa \in \mathbb{N}\). This model is called multicolor BBS and was introduced in [3], as a generalization of the original \(\kappa = 1\) BBS first introduced in [6]. In this model, particle configurations are given by \((\eta_n)_{n \in \mathbb{Z}} \in \{0, 1, \cdots, \kappa\}^\mathbb{Z}\), where we suppose that the numbers 1, \cdots, \(\kappa\) represent the colors of the balls and 0 represents the empty site. For each \(i = 1, \cdots, \kappa\), we define the operator \(T_i\) under which the balls of color \(i\) move from left to right, with each being transported to the leftmost unoccupied site to its right, with balls of other colors remaining static. The dynamics of the multicolor BBS are then defined by the operator \(T = T_\kappa \circ \cdots \circ T_1\).

For example, the evolution of the BBS with 3-color balls is as follows

\[
\eta = (\cdots 0 1 2 0 3 1 3 2 0 3 0 1 1 2 3 0 0 0 0 0 0 0 0 \cdots)
\]
\[
T_1\eta = (\cdots 0 0 2 1 3 0 3 2 1 3 0 0 0 2 3 1 1 0 0 0 0 0 0 0 \cdots)
\]
\[
T_2 \circ T_1\eta = (\cdots 0 0 0 1 3 2 3 0 1 3 2 0 0 0 3 1 1 2 0 0 0 0 0 0 \cdots)
\]
\[
T_3 \circ T_2 \circ T_1\eta = (\cdots 0 0 0 1 0 2 0 3 1 0 2 3 3 0 0 1 1 2 3 0 0 0 0 0 \cdots)
\]
\[
T^2\eta = (\cdots 0 0 0 0 1 0 2 0 3 1 0 0 0 2 3 3 0 0 1 1 2 3 0 0 0 \cdots)
\]

where \(T = T_3 \circ T_2 \circ T_1\). In the multicolor case, a string of consecutive balls of non-decreasing colors is called a soliton and shows the same behavior as in the 1-color case.

The multicolor BBS with finite number of balls has been well studied mostly in the context of integrable systems (see, e.g., the review article [13] or the textbook on the BBS [14]). Recently, [2] and [11], [8] considered the multicolor BBS with one-sided random initial configuration and derived scaling limits of probability measures on the space of -tuple of Young diagrams.
induced by the random configuration. Later, we introduce the two-sided version of the multicolor BBS, which is one of the main contributions of this paper.

The dynamics of the one-color BBS has been extended to two-sided infinite configurations and studied when the initial condition is random [4, 11]. In the paper [4] for the one-color BBS, the particle configuration is encoded by a certain path \( S = (S_n)_{n \in \mathbb{Z}} \) in \( \mathbb{Z} \) and the action \( T \) of the BBS is defined via an operation on the path space. Moreover, a formal inverse \( T^{-1} \) of \( T \) is defined, and the class of configurations \( S \) below such that \( TS \) and \( T^{-1}S \) are well-defined and reversible for all times, i.e.

\[
\{ S = (S_n)_{n \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z} : T^kS \text{ is well defined and } TT^{-1}(T^kS) = T^{-1}T(T^kS) = T^kS, \forall k \in \mathbb{Z} \},
\]

is precisely characterized. Within this framework, random initial conditions such that almost all paths are in the class is studied from the viewpoint of invariance under \( T \), the current of particles crossing the origin, and the speed of a single tagged particle.

Such an extended analysis was made possible thanks to connection that was identified between the BBS dynamics and Pitman’s transformation. Indeed, in [4], the action \( T \) on the path space is shown to correspond to the operation of reflection in the past maximum of the path, which is precisely the operation known Pitman transform. Pitman transform is introduced by [9] and appears in the well-known Pitman’s theorem, which states that if \( (B_t)_{t \geq 0} \) is a one-dimensional Brownian motion, then the stochastic process \( (2\sup_{0 \leq s \leq t} B_s - B_t)_{t \geq 0} \) is a three dimensional Bessel process, i.e. is distributed as the Euclidean norm of a three dimensional Brownian motion. This transform has been generalized to the multidimensional case by Biane [12], and in this paper, we show that the actions of the multicolor BBS can be described by the multidimensional Pitman transform.

We start by introducing the one-sided and two-sided Pitman transform for the multicolor BBS theory (Section 2.1, 2.2). Next, as in the case of the one-color BBS, we show that particle configurations of multicolor BBS can be encoded by a certain path in \( \mathbb{R}^\kappa \) (Section 3.1, 3.2) and the action \( T_\kappa \) corresponds to the composition of the extended Pitman transform and a certain operator (Section 3.4, 3.5). Moreover, we characterize the set of configurations for which the actions \( T_1, T_2, \ldots, T_\kappa \) are well-defined and reversible for all times (Section 3.7). Then, we give an example of a random initial condition that is invariant in distribution under the dynamics of the multicolor BBS (Section 4.1). Finally, we consider a generalization of the multicolor BBS, that is defined for continuous paths on \( \mathbb{R} \) (Section 4.2), and show that \( \kappa \)-dimensional Brownian motion with a certain drift is invariant under the action of the generalized multicolor BBS (Section 4.3).

Regarding notational conventions, we distinguish \( \mathbb{N} = \{1, 2, \ldots, \} \) and \( \mathbb{Z}_+ = \{0, 1, \ldots\} \).

2. Pitman transform

In this section, we prepare Pitman transform and the extended versions of it which will be used for the path encoding of the particle configuration in the subsequent sections. We start by defining one-sided Pitman transform
and studying its property (Section 2.1). Then, in Section 2.2 we define two-sided Pitman transform and examine its inverse on an appropriate set.

2.1. One-sided Pitman transform. We first see the definition of the multidimensional version of Pitman transform introduced by Biane [12].

**Definition 2.1.** Suppose that $\mathbb{R}^k$ is $k$-dimensional Euclidean space with dual space $V$ and let $\alpha \in \mathbb{R}^k, \alpha^* \in V$ be such that $\alpha^*(\alpha) = 2$. The Pitman transform $P_\alpha$ is defined on the set of continuous paths $\pi : [0, T] \rightarrow \mathbb{R}^k$, satisfying $\pi(0) = 0$, by the formula,

$$P_{\alpha, \alpha^*} \pi(t) = \pi(t) - \inf_{0 \leq s \leq t} \alpha^*(\pi(s))\alpha, \quad 0 \leq t \leq T.$$

For the multicolor BBS theory, we take the domain of $\pi$ as $\mathbb{Z}_+$ and $\alpha^*$ as the inner product with $\frac{\alpha}{|\alpha|^2}$ in the above definition, and define the one-sided Pitman transform.

**Definition 2.2.** Let $\alpha \in \mathbb{R}^k$ be such that $\alpha \neq 0$. The one-sided Pitman transform with respect to $\alpha$ is defined on the set of discrete paths $\pi : \mathbb{Z}_+ \rightarrow \mathbb{R}^k$, satisfying $\pi(0) = 0$, by the formula,

$$P_\alpha \pi(n) = \pi(n) - 2 \inf_{0 \leq m \leq n} \frac{\alpha \cdot \pi(m)}{|\alpha|^2} \alpha, \quad n \geq 0$$

where $\alpha \cdot \pi(m)$ is the inner product of $\alpha$ and $\pi(m)$, and $|\alpha|^2 = \alpha \cdot \alpha$.

**Example 2.3.** For any $\alpha \in \mathbb{R}$, $\alpha \neq 0$, the one-sided Pitman transform is given by

$$P_\alpha \pi(n) = \pi(n) - 2 \inf_{0 \leq m \leq n} \pi(m), \quad n \geq 0$$

for $\pi : \mathbb{Z}_+ \rightarrow \mathbb{R}$, satisfying $\pi(0) = 0$. Therefore the one-sided Pitman transform $P_\alpha$ on 1-dimensional Euclidean space does not depend on $\alpha$. We write it as $P_1$. (See Figure 1.)

**Definition 2.4.**

$$P_1 := P_\alpha \quad \text{for } \alpha \in \mathbb{R}, \ \alpha \neq 0.$$

That is,

$$P_1 \pi(n) = \pi(n) - 2 \inf_{0 \leq m \leq n} \pi(m), \quad n \geq 0$$

for $\pi : \mathbb{Z}_+ \rightarrow \mathbb{R}$, satisfying $\pi(0) = 0$. 
Next, we show the useful property of the one-sided Pitman transform for considering the actions of the BBS.

**Proposition 2.5.** Let $k \geq 2$ and $\pi_\alpha(n) := \frac{\alpha \cdot \pi(n)}{|\alpha|^2}$ for $\alpha \in \mathbb{R}^k$, $\pi : \mathbb{Z}_+ \rightarrow \mathbb{R}$ is decomposed into the sum of the vector projection of $\pi(n)$ along $\alpha$ and the vector orthogonal to $\alpha$:

$$\pi(n) = \pi_\alpha(n) \alpha + \{\pi(n) - \pi_\alpha(n) \alpha\}$$

for any $n \geq 0$. Then, it holds that

$$P_{\alpha} \pi(n) = \{P_1 \pi_\alpha(n)\} \alpha + \{\pi(n) - \pi_\alpha(n) \alpha\}.$$

(See figure 2.)

**Proof.**

$$P_{\alpha} \pi(n) = \pi(n) - 2 \inf_{0 \leq m \leq n} \frac{\alpha \cdot \pi(m)}{|\alpha|^2} \alpha$$

$$= \pi_\alpha(n) \alpha + \{\pi(n) - \pi_\alpha(n) \alpha\} - 2 \inf_{0 \leq m \leq n} \pi_\alpha(m) \alpha$$

$$= \left\{\pi_\alpha(n) - 2 \inf_{0 \leq m \leq n} \pi_\alpha(m) \right\} \alpha + \{\pi(n) - \pi_\alpha(n) \alpha\}$$

$$= \{P_1 \pi_\alpha(n)\} \alpha + \{\pi(n) - \pi_\alpha(n) \alpha\}.$$
2.2. Two-sided Pitman transform and its inverse. This section provides the two-sided Pitman transform and its inverse on an appropriate set.

**Definition 2.6.** Let $\alpha \in \mathbb{R}^k, \alpha \neq 0$. The two-sided Pitman transform with respect to $\alpha$ is defined on the set of discrete paths

$$\{\pi : \mathbb{Z} \to \mathbb{R}^k, \pi(0) = 0, \inf_{m \leq 0} \alpha \cdot \pi(m) > -\infty\}$$

by the formula,

$$P_\alpha \pi(n) = \pi(n) - 2 \inf_{m \leq n} \frac{\alpha \cdot \pi(m)}{||\alpha||^2} \alpha + 2 \inf_{m \leq 0} \frac{\alpha \cdot \pi(m)}{||\alpha||^2} \alpha, \quad n \in \mathbb{Z}.$$

Similarly to Example 2.3, it holds that

$$P_\alpha \pi(n) = \pi(n) - 2 \inf_{m \leq n} \pi(m) + 2 \inf_{m \leq 0} \pi(m), \quad n \in \mathbb{Z}$$

for any $\alpha \in \mathbb{R}, \alpha \neq 0$, and it does not depend on $\alpha$. Then we define

$$P_1 := P_\alpha \quad \text{for} \quad \alpha \in \mathbb{R}, \alpha \neq 0.$$

That is,

$$P_1 \pi(n) = \pi(n) - 2 \inf_{m \leq n} \pi(m) + 2 \inf_{m \leq 0} \pi(m), \quad n \in \mathbb{Z}$$

for $\pi : \mathbb{Z} \to \mathbb{R}$, satisfying $\pi(0) = 0, \inf_{m \leq 0} \pi(m) > -\infty$.

Next, we introduce a new transform which will be inverse of the two-sided Pitman transform on an appropriate set.

**Definition 2.7.** Let $\alpha \in \mathbb{R}^k, \alpha \neq 0$. Define the transform $P_\alpha^{-1}$ on the set of discrete paths

$$\{\pi : \mathbb{Z} \to \mathbb{R}^k, \pi(0) = 0, \inf_{m \geq 0} \alpha \cdot \pi(m) > -\infty\}$$
by the formula,
\[ P^{-1}_\alpha n = \pi(n) - 2 \inf_{m \geq n} \frac{\alpha \cdot \pi(m)}{|\alpha|^2} \alpha + 2 \inf_{m \geq 0} \frac{\alpha \cdot \pi(m)}{|\alpha|^2} \alpha, \quad n \in \mathbb{Z}. \]

In this case, it also holds that
\[ P^{-1}_\alpha n = \pi(n) - 2 \inf_{m \geq n} \pi(m) + 2 \inf_{m \geq 0} \pi(m), \quad n \in \mathbb{Z} \]
for any \( \alpha \in \mathbb{R}, \; \alpha \neq 0 \), and it does not depend on \( \alpha \). Then we define
\[ P^{-1}_1 := P^{-1}_\alpha \quad \text{for} \; \alpha \in \mathbb{R}, \; \alpha \neq 0. \]

That is,
\[ P^{-1}_1 \pi(n) = \pi(n) - 2 \inf_{m \geq n} \pi(m) + 2 \inf_{m \geq 0} \pi(m), \quad n \in \mathbb{Z} \]
for \( \pi: \mathbb{Z} \to \mathbb{R} \), satisfying \( \pi(0) = 0, \inf_{m \geq 0} \pi(m) > -\infty \).

**Remark 2.8.** With the same notation as Proposition 2.5, it holds that,
\[ P_\alpha \pi(n) = \{ P_1 \pi_\alpha(n) \} \alpha + \{ \pi(n) - \pi_\alpha(n) \alpha \} \]
\[ P^{-1}_\alpha \pi(n) = \{ P^{-1}_1 \pi_\alpha(n) \} \alpha + \{ \pi(n) - \pi_\alpha(n) \alpha \}. \]

Therefore, \( P_1 P^{-1}_1 = \text{id} \) on some set \( E_\alpha \) implies \( P_\alpha P^{-1}_\alpha = \text{id} \) on \{ \pi: \mathbb{Z} \to \mathbb{R}^k : \pi_\alpha \in E_\alpha \}, and \( P^{-1}_1 P_1 = \text{id} \) on some set \( F_\alpha \) implies \( P^{-1}_\alpha P_\alpha = \text{id} \) on \{ \pi: \mathbb{Z} \to \mathbb{R}^k : \pi_\alpha \in F_\alpha \}.

**Definition 2.9.** We define the domain of \( P_1 \) and \( P^{-1}_1 \), and their subsets,
\[ \mathcal{R}^{P_1} := \{ \pi: \mathbb{Z} \to \mathbb{R}, \; \pi(0) = 0, \inf_{m \leq 0} \pi(m) > -\infty \}, \]
(2.4)
\[ \mathcal{R}^{P^{-1}_1} := \{ \pi: \mathbb{Z} \to \mathbb{R}, \; \pi(0) = 0, \inf_{m \geq 0} \pi(m) > -\infty \}, \]
(2.5)
\[ \mathcal{R}^{P^{-1}_1 P_1} := \{ \pi \in \mathcal{R}^{P^{-1}_1} : |\pi(n+1) - \pi(n)| \in \{ 0, 1 \}, \forall n, \inf_{m \leq n} \pi(m) = \pi(n) \text{ i.o. as } n \to \infty \}, \]
(2.6)
\[ \mathcal{R}^{P_1 P^{-1}_1} := \{ \pi \in \mathcal{R}^{P^{-1}_1} : |\pi(n+1) - \pi(n)| \in \{ 0, 1 \}, \forall n, \inf_{m \geq n} \pi(m) = \pi(n) \text{ i.o. as } n \to -\infty \}. \]
(2.7)

We prepare following proposition to guarantee that \( P^{-1}_1 P_1 \) and \( P_1 P^{-1}_1 \) are well-defined on \( \mathcal{R}^{P_1} \) and \( \mathcal{R}^{P^{-1}_1} \) respectively.

**Proposition 2.10.** It holds that
\[ P_1 \left( \mathcal{R}^{P_1} \right) \subseteq \mathcal{R}^{P^{-1}_1}, \]
\[ P^{-1}_1 \left( \mathcal{R}^{P^{-1}_1} \right) \subseteq \mathcal{R}^{P_1}. \]
Proof. Suppose that \( n \geq 0 \) and \( \pi \in \mathcal{R}^{P_1} \). Since \( \inf_{m \leq n} \pi(m) \leq \inf_{m \leq 0} \pi(m) \), we have

\[
P_1 \pi(n) = \pi(n) - 2 \inf_{m \leq n} \pi(m) + 2 \inf_{m \leq 0} \pi(m) \geq \pi(n).
\]

On the other hand, since \( \inf_{m \leq n} \pi(m) \leq \pi(n) \), we have

\[
P_1 \pi(n) = \pi(n) - 2 \inf_{m \leq n} \pi(m) + 2 \inf_{m \leq 0} \pi(m) \geq -\pi(n) + 2 \inf_{m \leq 0} \pi(m).
\]

The above two inequalities show

\[
P_1 \pi(n) - \inf_{m \leq 0} \pi(m) \geq \pm \left\{ -\pi(n) + \inf_{m \leq 0} \pi(m) \right\},
\]

then

\[
P_1 \pi(n) \geq \inf_{m \leq 0} \pi(m).
\]

It shows the first claim and we can prove the second in the same way. \( \square \)

**Theorem 2.11.** It holds that

\[
P_1^{-1} P_1 = \text{id. on } \mathcal{R}^{P_1^{-1} P_1},
\]

\[
P_1 P_1^{-1} = \text{id. on } \mathcal{R}^{P_1 P_1^{-1}}.
\]

Proof. Let \( \pi \in \mathcal{R}^{P_1^{-1} P_1} \). Define the sequence

\[
\lambda_x = \inf_{m \in \mathbb{Z}} \{ m : \pi(m) = x \} \text{ for } x \in \mathbb{Z}
\]

with the convention that \( \inf \emptyset = \infty \). (See Figure 3,4.) Then, the sequence satisfies one of the following 4 conditions:

1. \( \cdots < \lambda_{x+1} < \lambda_x < \lambda_{x-1} < \cdots \)
2. \( -\infty = \lambda_s < \lambda_{s-1} < \cdots < \lambda_{x+1} < \lambda_x < \lambda_{x-1} < \cdots \)
3. \( \cdots < \lambda_{x+1} < \lambda_x < \lambda_{x-1} < \cdots < \lambda_{t+1} < \lambda_t < \lambda_{t-1} = \infty \)
4. \( -\infty = \lambda_s < \lambda_{s-1} < \cdots < \lambda_{x+1} < \lambda_x < \lambda_{x-1} < \cdots < \lambda_t < \lambda_{t-1} = \infty \)

where \( s = \lim_{n \to -\infty} \pi(n) \) when it is bounded and \( t = \lim_{n \to \infty} \pi(n) \) when it is bounded. The condition \( \inf_{m \leq n} \pi(m) = \pi(n) \) i.o. as \( n \to \infty \) implies \( s \leq t \), and if \( s = t \), it is the case that \( -\infty = \lambda_s = \lambda_t < \lambda_{t-1} = \infty \).

If (1) : \( n = \lambda_x \), for some \( x \), it holds that

\[
P_1 \pi(n) = P_1 \pi(\lambda_x) = -\pi(\lambda_x) + 2 \inf_{m \leq 0} \pi(m) = -\pi(n) + 2 \inf_{m \leq 0} \pi(m)
\]

and also it holds that

\[
P_1 \pi(\lambda_x) > P_1 \pi(\lambda_{x+1}) \text{ for any } -\infty < \lambda_{x+1} < \lambda_x < \infty.
\]

If (2) : \( -\infty < \lambda_{x+1} < n < \lambda_x < \infty \) for some \( x \), it holds that

\[
P_1 \pi(n) = \pi(n) - 2\pi(\lambda_x) + 2 \inf_{m \leq 0} \pi(m) \geq P_1 \pi(\lambda_x)
\]
It holds that

\[ P_1 \pi(n) = \pi(n) - 2s + 2 \inf_{m \leq n} \pi(m) \geq P_1 \pi(\lambda_{s-1}) - 1 \]
\[ = -(s - 1) + 2 \inf_{m \leq 0} \pi(m) - 1 \]

If (4) \( : n > \lambda_t \), it holds that

\[ P_1 \pi(n) = \pi(n) - 2t + 2 \inf_{m \leq 0} \pi(m). \]

and also it holds that

\[ P_1 \pi(n) = P_1 \pi(\lambda_t) \text{ i.o. as } n \to \infty. \]

From the above discussion, it holds that

\[ \inf_{m \geq n} P_1 \pi(m) = \begin{cases} -\pi(n) + 2 \inf_{m \leq 0} \pi(m), & \text{if (1)}, \\ -\pi(\lambda_x) + 2 \inf_{m \leq 0} \pi(m), & \text{if (2)}, \\ -s + 2 \inf_{m \leq 0} \pi(m), & \text{if (3)}, \\ -t + 2 \inf_{m \leq 0} \pi(m), & \text{if (4)}. \end{cases} \]

Therefore, if (1),

\[ P_1^{-1} P_1 \pi(n) = P_1 \pi(n) - 2 \inf_{m \geq n} P_1 \pi(m) + 2 \inf_{m \geq 0} P_1 \pi(m) \]
\[ = \left\{ -\pi(n) + 2 \inf_{m \leq 0} \pi(m) \right\} - 2 \left\{ -\pi(n) + 2 \inf_{m \leq 0} \pi(m) \right\} + 2 \inf_{m \geq 0} P_1 \pi(m) \]
\[ = \pi(n) - 2 \inf_{m \leq 0} \pi(m) + 2 \inf_{m \geq 0} P_1 \pi(m). \]

If (2),

\[ P_1^{-1} P_1 \pi(n) = P_1 \pi(n) - 2 \inf_{m \geq n} P_1 \pi(m) + 2 \inf_{m \geq 0} P_1 \pi(m) \]
\[ = \left\{ \pi(n) - 2\pi(\lambda_x) + 2 \inf_{m \leq 0} \pi(m) \right\} - 2 \left\{ -\pi(\lambda_x) + 2 \inf_{m \leq 0} \pi(m) \right\} + 2 \inf_{m \geq 0} P_1 \pi(m) \]
\[ = \pi(n) - 2 \inf_{m \leq 0} \pi(m) + 2 \inf_{m \geq 0} P_1 \pi(m). \]

If (3),

\[ P_1^{-1} P_1 \pi(n) = P_1 \pi(n) - 2 \inf_{m \geq n} \pi(m) + 2 \inf_{m \geq 0} P_1 \pi(m) \]
\[ = \left\{ \pi(n) - 2s + 2 \inf_{m \leq 0} \pi(m) \right\} - 2 \left\{ -s + 2 \inf_{m \leq 0} \pi(m) \right\} + 2 \inf_{m \geq 0} P_1 \pi(m) \]
\[ = \pi(n) - 2 \inf_{m \leq 0} \pi(m) + 2 \inf_{m \geq 0} P_1 \pi(m). \]

If (4),

\[ P_1^{-1} P_1 \pi(n) = P_1 \pi(n) - 2 \inf_{m \geq n} P_1 \pi(m) + 2 \inf_{m \geq 0} P_1 \pi(m) \]
\[ = \left\{ \pi(n) - 2t + 2 \inf_{m \leq 0} \pi(m) \right\} - 2 \left\{ -t + 2 \inf_{m \leq 0} \pi(m) \right\} + 2 \inf_{m \geq 0} P_1 \pi(m) \]
\[ = \pi(n) - 2 \inf_{m \leq 0} \pi(m) + 2 \inf_{m \geq 0} P_1 \pi(m). \]
Therefore it is enough to show that
\[
\inf_{m \leq n} \pi(m) = \inf_{m \geq 0} P_1 \pi(m),
\]
and it is obtained by following inequalities:
\[
\begin{align*}
\inf_{m \geq 0} P_1 \pi(m) &= \inf_{m \geq 0} \left\{ \pi(m) - 2 \inf_{l \leq m} \pi(l) + 2 \inf_{l \leq 0} \pi(l) \right\} \\
&\geq \inf_{m \geq 0} \left\{ \pi(m) - \left( \inf_{l \leq m} \pi(l) + \inf_{0 \leq l \leq m} \pi(l) \right) + 2 \inf_{l \leq 0} \pi(l) \right\} \\
&= \inf_{m \geq 0} \left\{ \pi(m) - \inf_{0 \leq l \leq m} \pi(l) \right\} + \inf_{l \leq 0} \pi(l) \\
&\geq \inf_{l \leq 0} \pi(l).
\end{align*}
\]
On the other hand, by the conditions on $\mathcal{R}^{P_1 P_1^{-1}}$, there exists $m_1 \geq 0$ such that $\pi(m_1) = \inf_{l \leq m_1} \pi(l) = \inf_{l \leq 0} \pi(l)$, then
\[
\begin{align*}
\inf_{m \geq 0} P_1 \pi(m) &= \inf_{m \geq 0} \left\{ \pi(m) - 2 \inf_{l \leq m} \pi(l) + 2 \inf_{l \leq 0} \pi(l) \right\} \\
&\leq \pi(m_1) - 2 \inf_{l \leq m_1} \pi(l) + 2 \inf_{l \leq 0} \pi(l) \\
&= \inf_{l \leq 0} \pi(l).
\end{align*}
\]
We can prove the second claim in the same way. ∎

Remark 2.12. The condition $|\pi(n+1) - \pi(n)| \in \{0,1\}$ in $\mathcal{R}^{P_1 P_1^{-1}}$ and $\mathcal{R}^{P_1 P_1^{-1}}$ can be replaced by $|\pi(n+1) - \pi(n)| \in \{0,c\}$ with any positive constant $c$ for Theorem 2.11 to hold.

Remark 2.13. The condition
\begin{equation}
\inf_{m \leq n} \pi(m) = \pi(n) \quad \text{i.o. as} \quad n \to \infty
\end{equation}

Figure 3. Example of the sequence $\{\lambda_x\}$ with $\pi(n), \inf_{m \leq n} \pi(m)$.
Figure 4. The sequence \( \{ \lambda_x \} \) in figure 3 with \( P_1 \pi(n) \).

Figure 5. Example of \( \pi \) not satisfying (2.8) and \( J_n := \inf_{m \leq n} \pi(m) \).

in \( \mathcal{R}^{P_1^{-1}P_1} \) is necessary for \( P_1^{-1}P_1 = \text{id} \). Indeed, one can check that if \( \pi \) does not satisfy (2.8), the increment of \( -\inf_{m \leq n} \pi(m) \) does not match that of \( \inf_{m \geq n} P_1 \pi(m) \). (See Figure 5, 6.)

**Corollary 2.14.** By Remark 2.8 it holds that

\[
P_\alpha^{-1}P_\alpha \pi = \pi, \quad \text{if} \quad \pi_\alpha \in \mathcal{R}^{P_1^{-1}P_1},
\]

\[
P_\alpha P_\alpha^{-1} \pi = \pi, \quad \text{if} \quad \pi_\alpha \in \mathcal{R}^{P_1P_1^{-1}},
\]

where \( \pi_\alpha(n) = \frac{\alpha \cdot \pi(n)}{|\alpha|^2} \).

3. Path encodings of the multicolor BBS

In the original paper [4], the particle configuration is corresponded to the nearest-neighbour walk path \( S \) on \( \mathbb{Z} \) in \( \mathbb{R} \), satisfying \( S_0 = 0 \) and \( S_n - S_{n-1} = 1 \) if \( \eta_n = 0 \) and \( S_n - S_{n-1} = -1 \) if \( \eta_n = 1 \). In this section, we extend this concept to the multicolor BBS with \( \kappa \)-color balls by considering the path \( S \) in \( \mathbb{R}^\kappa \) (Section 3.2). In particular, \( S \) satisfies \( S_0 = 0 \) and \( S_n - S_{n-1} = e_i \) if \( \eta_n = i \in \{0, 1, \ldots, \kappa\} \), where the vectors \( e_0, \ldots , e_\kappa \in \mathbb{R}^\kappa \) is obtained in
Section 3.1. Then we consider the dynamics of the one-sided multicolor BBS in terms of the carrier processes which pick up and drop a certain color ball moving on $\mathbb{Z}_+$ (Section 3.3), and Pitman transform on $S$ which describes the action $T_i$ (Section 3.4). In Section 3.5 we extend them to the case of two-sided multicolor BBS. Also we describe the inverse $T_i^{-1}$ and define the reversible set of $S$ for color $i$ such that $T_i^{-1}T_iS = T_iT_i^{-1}S = S$ (Section 3.6). Moreover, we investigate the set of configurations for which the actions $T_1, T_2, \cdots, T_\kappa$ are well-defined and reversible for all times. (Section 3.7).

From this section, we fix $\kappa \in \mathbb{N}$ the number of all colors and define the set of numbers representing colors $\mathcal{C} := \{1, \cdots, \kappa\}$.

3.1. Vectors for path encodings. In this subsection, we introduce a set of vectors which will be used for path encoding of the particle configuration.

**Definition 3.1.** Let vectors $e_0, e_1, \cdots, e_\kappa \in \mathbb{R}^\kappa$ represent the vertices of a regular $\kappa$-dimensional simplex center the origin, satisfying following conditions:

\begin{align}
(3.1) & \quad |e_i| = 1 \quad \forall i \in \mathcal{C} \cup \{0\}.
(3.2) & \quad e_i \cdot e_j = -\frac{1}{\kappa} \quad \forall i, j \in \mathcal{C} \cup \{0\}, i \neq j.
\end{align}
Proposition 3.2. The vectors $e_0, e_1, \ldots, e_\kappa$ have following properties, immediately obtained from (3.1) and (3.2), which will be useful in subsequent sections when it comes to defining the path encodings of the particle configuration and considering the actions of the multicolor BBS.

(i) $e_0 + e_1 + \cdots + e_\kappa = 0$

(ii) Let $a_i \in \mathbb{R}$ for $i \in \mathbb{C} \cup \{0\}$. It holds that

$$a_0 e_0 + a_1 e_1 + \cdots + a_\kappa e_\kappa = 0 \iff a_0 = a_1 = \cdots = a_\kappa.$$ 

(iii) Let $a_i, a'_i \in \mathbb{R}$ for $i \in \mathbb{C} \cup \{0\}$. Suppose that

$$a_0 e_0 + a_1 e_1 + \cdots + a_\kappa e_\kappa = a'_0 e_0 + a'_1 e_1 + \cdots + a'_\kappa e_\kappa.$$ 

Then there is a constant $c$ such that $a_i = a'_i + c$ for any $i$. In addition, suppose that

$$a_0 + a_1 + \cdots + a_\kappa = a'_0 + a'_1 + \cdots + a'_\kappa.$$ 

Then it is the case that $a_i = a'_i$ for any $i$.

(iv) Let $a_i \in \mathbb{R}$ for $i \in \mathbb{C} \cup \{0\}$, and $d_j \in \mathbb{R}$ for $j \in \mathbb{C}$. It holds that

$$a_0 e_0 + a_1 e_1 + \cdots + a_\kappa e_\kappa = d_i(e_i - e_0) + \sum_{j \in \mathbb{C}, j \neq i} d_j e_j$$ 

$$\iff d_j = a_j - \frac{a_0 + a_i}{2} \quad \forall j \in \mathbb{C}$$ 

for any $i \in \mathbb{C}$.

(v) Any set of $\kappa$ vectors in $\{e_0, e_1, \ldots, e_\kappa\}$ is the basis of $\mathbb{R}^\kappa$.

(vi) For any $v \in \mathbb{R}^\kappa$, there is an $\kappa + 1$-tuple $a_0, \ldots, a_\kappa$ of real numbers satisfying

$$v = a_0 e_0 + \cdots + a_\kappa e_\kappa, \quad a_0 + \cdots + a_\kappa = 0$$

3.2. Configuration of the one-sided multicolor BBS. In this section, we consider the one-sided multicolor BBS, and denote the particle configuration by $\eta = (\eta_n)_{n \in \mathbb{N}} \in \{0, 1, 2, \cdots, \kappa\}^\mathbb{N}$ As in the introduction, we write $\eta_n = i$ if there is a particle of color $i \in \mathbb{C}$ at site $n$, and $\eta_n = 0$ if there is no particle at site $n$. 

![Figure 7. $e_0, e_1, e_2 \in \mathbb{R}$, $e_0, e_1, e_2, e_3 \in \mathbb{R}^2$, $e_0, e_1, e_2, e_3 \in \mathbb{R}^3$](image)
We define a nearest-neighbour path in \( \mathbb{R}^\kappa \) as the path encoding of a particle configuration.

**Definition 3.3.** Given the particle configuration by \( \eta = (\eta_n)_{n \in \kappa} \in \{0, 1, 2, \cdots, \kappa\}^\mathbb{N} \), we define \( S = (S_n)_{n \in \mathbb{Z}_+} \) by setting

\[
S_0 = 0 \quad S_n - S_{n-1} = e_i \quad \text{if} \quad \eta_n = i.
\]

The \( S \) is called the path encoding of \( \eta \). We can describe it as

\[
S_n = a_0(n)e_0 + a_1(n)e_1 + \cdots + a_\kappa(n)e_\kappa
\]

for \( n \in \mathbb{Z}_+ \), where \( a_i(n) \in \mathbb{Z}_+ \), \( i \in \mathcal{C} \) is the number of the particles of color \( i \) at the sites located from 1 to \( n \), \( a_0(n) \in \mathbb{Z}_+ \) is the number of the empty sites located from 1 to \( n \), and \( a_i(0) = 0 \), \( i \in \mathcal{C} \cup \{0\} \). Also we define the path space in \( \mathbb{R}^\kappa \) as follows:

\[
S_\kappa := \{ S : \mathbb{Z}_+ \to \mathbb{R}^\kappa : S_0 = 0, \ S_{n+1} - S_n \in \{e_0, e_1, \cdots, e_\kappa\}, \ \forall n \in \mathbb{Z}_+ \}.
\]

**Example 3.4.** For \( \eta = (0, 1, 1, 2, \cdots) \), the path encoding \( S \) is given by

\[
S_0 = 0, \ S_1 = e_0, \ S_2 = e_0 + e_1, \ S_3 = e_0 + 2e_1, \ S_4 = e_0 + 2e_1 + e_2, \ \cdots
\]

**Remark 3.5.** By the definition, it clearly holds that the map from \( \eta = (\eta_n)_{n \in \mathbb{N}} \in \{0, 1, 2, \cdots, \kappa\}^\mathbb{N} \) to \( S \in S_\kappa \) is one to one. Also it holds that

\[
a_0(n) + a_1(n) + \cdots + a_\kappa(n) = n \quad \forall n.
\]

Therefore, from Proposition 3.3, the map from \( (a_0(n), a_1(n), \cdots, a_\kappa(n)) \in \mathbb{Z}_+^{\kappa+1} \) to \( S_n \in \mathbb{R}^\kappa \) is one to one for any \( n \in \mathbb{Z}_+ \).

For the subsequent sections, we introduce some operators of \( S \).

**Definition 3.6.** For \( i \in \mathcal{C} \), we define the function \( A_i : S_\kappa \to \mathbb{Z}^{\mathbb{Z}_+} \) given by

\[
A_iS_n = a_0(n) - a_i(n),
\]

for \( S_n = a_0(n)e_0 + a_1(n)e_1 + \cdots + a_\kappa(n)e_\kappa, \ n \in \mathbb{Z}_+ \).

**Remark 3.7.** For \( S_n = a_0(n)e_0 + a_1(n)e_1 + \cdots + a_\kappa(n)e_\kappa \), Proposition 3.2 shows

\[
S_n = \frac{1}{2} \{a_i(n) - a_0(n)\} (e_i - e_0) + \sum_{j \neq 0, i} d_j(n)e_j
\]

and (3.2) implies

\[
e_j \cdot (e_i - e_0) = 0 \quad \forall i, j \in \mathcal{C}, \ i \neq j.
\]

Therefore, the projection of \( S_n \) along \( (e_i - e_0) \) is equal to

\[
\frac{(e_i - e_0) \cdot S_n}{|e_i - e_0|^2}(e_i - e_0) = \frac{1}{2} \{a_i(n) - a_0(n)\} (e_i - e_0) = -\frac{1}{2} A_iS_n(e_i - e_0),
\]

then it holds that

\[
A_iS_n = -\frac{1}{2} \frac{(e_i - e_0) \cdot S_n}{|e_i - e_0|^2}.
\]
Remark 3.8. From Remark 3.7, we can write \( S_n \) as the sum of the vector projection on \((e_i - e_0)\) and the vector orthogonal to \((e_i - e_0)\) as following
\[
S_n = \frac{1}{2} A_i S_n (e_i - e_0) + \left( S_n + \frac{1}{2} A_i S_n (e_i - e_0) \right).
\]
Then, by Proposition 2.5, it holds that
\[
P_{e_i - e_0} S_n = P_{e_i - e_0} \left( -\frac{1}{2} A_i S_n (e_i - e_0) + \left( S_n + \frac{1}{2} A_i S_n (e_i - e_0) \right) \right)
\]
\[
= P_{e_i - e_0} \left( -\frac{1}{2} A_i S_n \right) (e_i - e_0) + \left( S_n + \frac{1}{2} A_i S_n (e_i - e_0) \right).
\]

Definition 3.9. We define the the permutation operator \( \tau_{(0,i)} : S_+ \to S_+ \) given by
\[
\tau_{(0,i)} S_n = a_i(n) e_0 + a_0(n) e_i + \sum_{j \neq 0, i} a_j(n) e_j
\]
for \( S_n = a_0(n) e_0 + a_1(n) e_1 + \cdots + a_n(n) e_n, n \in \mathbb{Z}_+ \).

Remark 3.10. Comparing
\[
S_n = \frac{1}{2} \{ a_i(n) - a_0(n) \} (e_i - e_0) + \sum_{j \neq 0, i} d_j(n) e_j
\]
\[
= -\frac{1}{2} A_i S_n (e_i - e_0) + \left( S_n + \frac{1}{2} A_i S_n (e_i - e_0) \right),
\]
and
\[
\tau_{(0,i)} S_n = \frac{1}{2} \{ a_0(n) - a_i(n) \} (e_i - e_0) + \sum_{j \neq 0, i} d_j(n) e_j
\]
\[
= \frac{1}{2} A_i S_n (e_i - e_0) + \left( S_n + \frac{1}{2} A_i S_n (e_i - e_0) \right),
\]
it is the case that \( \tau_{(0,i)} \) is the operator which multiply only the vector projection part of \( S \) along \( e_i - e_0 \) by \(-1\). Also it holds that
\[
\tau_{(0,i)} S_n = S_n + A_i S_n (e_i - e_0).
\]

3.3. Carrier process for the one-sided multicolor BBS. We introduce the concept of carrier with respect to particles of a certain color \( i \in \mathcal{C} \). It moves along \( \mathbb{Z}_+ \) from left to right picking up a particle of color \( i \) when it crosses one, and dropping off a particle of color \( i \) when it is holding at least one particle and sees an empty site. The dynamic \( T_i \) can be viewed in terms of this carrier. The carrier process is given as follows.

Definition 3.11. The carrier process \( W^{(i)} = \{ W_n^{(i)} \}_{n \in \mathbb{Z}_+} \) of the color \( i \) associated with \( \eta \in \{ 0, 1, 2, \cdots, \kappa \}^\mathbb{N} \) is defined by \( W_0^{(i)} = 0 \) and
\[
W_n^{(i)} = \begin{cases} 
W_{n-1}^{(i)} + 1, & \text{if } \eta_n = i, \\
W_{n-1}^{(i)}, & \text{if } \eta_n = j, j \neq 0, i \\
W_{n-1}^{(i)}, & \text{if } \eta_n = 0 \text{ and } W_{n-1}^{(i)} = 0, \\
W_{n-1}^{(i)} - 1, & \text{if } \eta_n = 0 \text{ and } W_{n-1}^{(i)} > 0.
\end{cases}
\]
W is obtained from S as following lemma.

**Lemma 3.12.** It holds that

\[ W_n^{(i)} = \sup_{0 \leq m \leq n} A_i S_m - A_i S_n, \quad \forall n \in \mathbb{Z}_+. \]

**Proof.** We prove it by induction. Clearly the result is true for \( n = 0 \). Suppose that \( W_{n-1}^{(i)} = \sup_{0 \leq m \leq n-1} A_i S_m - A_i S_{n-1} \) for some \( n \geq 1 \).

Now, if \( \eta_n = i \), then \( A_i S_n = A_i S_{n-1} \) and \( \sup_{0 \leq m \leq n} A_i S_m = \sup_{0 \leq m \leq n-1} A_i S_m \), and so

\[ \left\{ \sup_{0 \leq m \leq n} A_i S_m - A_i S_n \right\} - \left\{ \sup_{0 \leq m \leq n-1} A_i S_m - A_i S_{n-1} \right\} = 1. \]

If \( \eta_n = j \), \( j \neq i \), then \( A_i S_n = A_i S_{n-1} \) and \( \sup_{0 \leq m \leq n} A_i S_m = \sup_{0 \leq m \leq n} A_i S_m \), and so

\[ \left\{ \sup_{0 \leq m \leq n} A_i S_m - A_i S_n \right\} - \left\{ \sup_{0 \leq m \leq n-1} A_i S_m - A_i S_{n-1} \right\} = 0. \]

Moreover, if \( \eta_n = 0 \) and \( W_{n-1}^{(i)} = 0 \), then it is the case that \( \sup_{0 \leq m \leq n} A_i S_m = A_i S_n \), and so

\[ \left\{ \sup_{0 \leq m \leq n} A_i S_m - A_i S_n \right\} - \left\{ \sup_{0 \leq m \leq n-1} A_i S_m - A_i S_{n-1} \right\} = 0. \]

Similarly, if \( \eta_n = 0 \) and \( W_{n-1}^{(i)} > 0 \), then \( A_i S_n = A_i S_{n-1} + 1 \) and \( \sup_{0 \leq m \leq n} A_i S_m = \sup_{0 \leq m \leq n-1} A_i S_m \), and so

\[ \left\{ \sup_{0 \leq m \leq n} A_i S_m - A_i S_n \right\} - \left\{ \sup_{0 \leq m \leq n-1} A_i S_m - A_i S_{n-1} \right\} = -1. \]

Thus it holds that

\[ W_n^{(i)} - W_{n-1}^{(i)} = \left\{ \sup_{0 \leq m \leq n} A_i S_m - A_i S_n \right\} - \left\{ \sup_{0 \leq m \leq n-1} A_i S_m - A_i S_{n-1} \right\} \]

which by the inductive hypothesis implies

\[ W_n^{(i)} = \sup_{0 \leq m \leq n} A_i S_m - A_i S_n. \]

\[ \square \]

### 3.4. Action of the carrier for the one-sided multicolor BBS

In this section, we consider the action \( T_i \) on \( S \) given by (3.4). We fix the color \( i \in C \). From the viewpoint of the carrier process, we can write \( T_i \) as

\[ T_i(\eta)_n = 1_{\{W_n^{(i)} - W_{n-1}^{(i)} = 1\}}, \quad \forall n \in \mathbb{N}. \]

For \( j \neq i \), the numbers \( \{a_j(n)\}_{n \in \mathbb{Z}_+} \) do not change under the action \( T_i \), so the path encoding \( T_i S = (T_i S_n)_n \in \mathbb{Z}_+ \) of \( T_i \eta \) can be described as follows,

\[ T_i S_n = a_0'(n) + a_i'(n) + \sum_{j \neq 0, i} a_j(n) e_j \]

for some \( a_0'(n) \) and \( a_i'(n) \).

Then \( T_i \) satisfies the following formula.
Lemma 3.13. It holds that
\[ a_0'(n) - a_i'(n) = 2 \sup_{0 \leq m \leq n} \{a_0(m) - a_i(m)\} - \{a_0(n) - a_i(n)\}. \]
That is, from Definition 3.6
\[ A_i T_i S_n = 2 \sup_{0 \leq m \leq n} A_i S_m - A_i S_n \]
by using Pitman transform (2.2) in Definition 2.4.

Proof. It is easy to check that
\[ 21 \{A_i S_n - A_i S_{n-1} = 1\} = 1 + (A_i S_n - A_i S_{n-1}) - 1_{\{\eta_i \neq 0, i\}}. \]
This equation and Theorem 3.12 show that
\[ A_i T_i S_n - A_i T_i S_{n-1} \]
\[ = 1 - 21_{\{W_n^{(i)} = W_{n-1}^{(i)} - 1\}} - 1_{\{\eta_i \neq 0, i\}} \]
\[ = 1 - 21 \{A_i S_{n-1} = \sup_{0 \leq m \leq n-1} A_i S_m, A_i S_n - A_i S_{n-1} = 1\} - 1_{\{\eta_i \neq 0, i\}} \]
\[ = 1 - \left( 21 \{A_i S_{n-1} = 1\} - 21 \{A_i S_{n-1} = \sup_{0 \leq m \leq n-1} A_i S_m, A_i S_n - A_i S_{n-1} = 1\} \right) - 1_{\{\eta_i \neq 0, i\}} \]
\[ = - (A_i S_n - A_i S_{n-1}) + 21 \{A_i S_{n-1} = \sup_{0 \leq m \leq n-1} A_i S_m, A_i S_n - A_i S_{n-1} = 1\}. \]
Summing over the increments, we obtain
\[ A_i T_i S_n - A_i T_i S_0 \]
\[ = \sum_{m=1}^{n} (A_i T_i S_m - A_i T_i S_{m-1}) \]
\[ = A_i S_0 - A_i S_n + 2 \sum_{m=1}^{n} 1_{\{A_i S_{n-1} = \sup_{0 \leq m \leq n-1} A_i S_m, A_i S_n - A_i S_{n-1} = 1\}} \]
\[ = A_i S_0 - A_i S_n + 2 \left( \sup_{0 \leq m \leq n} A_i S_m - \sup_{0 \leq m \leq n} A_i S_m \right). \]
Since \(A_i S_0 = A_i T_i S_0 = \sup_{0 \leq m \leq 0} A_i S_m = 0\), the claim is proved. \(\square\)

Theorem 3.14. It holds that
\[ T_i S = \tau_{(0,i)} P_{e_i - e_0} S, \ \forall S \in S_+ \]
where \(P\) is one-sided Pitman transform defined in Definition 2.4.

Proof. By Remark 3.8 and Lemma 3.13 it holds that
\[ P_{e_i - e_0} S_n = P_1 \left( -\frac{1}{2} A_i S_n \right) (e_i - e_0) + \sum_{j \neq 0, i} d_j(n) e_j \]
\[ = \frac{1}{2} P_1 ( -A_i S_n ) (e_i - e_0) + \sum_{j \neq 0, i} d_j(n) e_j \]
Remark 3.15. The dynamic $T$ for the 1-color BBS in the paper equals and corresponds to Lemma 3.13. For the multicolor case, however, similarly the dynamic $T$ does not correspond to Remark 3.16. Therefore, we obtain the equation

$$
\tau = 2 \sup_{j \neq 0, i} d_j(n) e_j.
$$

Therefore, we obtain the equation $\tau_{(0,i)} T_i S_n = P_{e_i - e_0} S_n$ for any $n \in \mathbb{Z}_+$. □

Remark 3.15. The dynamic $T$ for the 1-color BBS in the paper is expressed as follows:

$$
TS_n = 2 \sup_{0 \leq m \leq n} S_m - S_n,
$$

where $S_n = a_0(n)e_0 + a_1(n)e_1 = a_0(n) - a_1(n)$. This is also called Pitman transform and corresponds to Lemma 3.13. For the multicolor case, however, the supremum expression is

$$
2 \sup_{0 \leq m \leq n} \frac{(e_0 - e_i) \cdot S_m}{|e_0 - e_i|^2} (e_0 - e_i) - S_n
$$

$$
= 2 \sup_{0 \leq m \leq n} \frac{(e_0 - e_i) \cdot S_m}{|e_0 - e_i|^2} (e_0 - e_i) - (a_0(n)e_0 + a_1(n)e_1 + \cdots + a_n(n)e_n)
$$

$$
= 2 \sup_{0 \leq m \leq n} \frac{a_0(m) - a_i(m)}{2} (e_0 - e_i) - \left( \frac{a_0(n) - a_i(n)}{2} (e_0 - e_i) + \sum_{j \neq 0, i} d_j(n) e_j \right)
$$

$$
= \frac{1}{2} \left( 2 \sup_{0 \leq m \leq n} \{a_0(m) - a_i(m)\} - \{a_0(n) - a_i(n)\} \right) (e_0 - e_i) - \sum_{j \neq 0, i} d_j(n) e_j,
$$

where $d_j(n) = a_j(n) - a_0(n) + a_i(n) = a_j(n) - a_0(n) + a_i(n)$, $j \neq 0, i$. Then this does not correspond to $T_i S_n$ because the sign of $d_j(n)$ is negative. This is the reason why we use infimum expression of Pitman transform.

Remark 3.16. From Theorem 3.14, it holds that

$$
T_2 T_1 = (\tau_{(0,2)} P_{e_2 - e_0}) (\tau_{(0,1)} P_{e_1 - e_0})
$$

$$
= \tau_{(0,1)} (\tau_{(1,2)} P_{e_2 - e_1}) P_{e_1 - e_0}
$$

$$
= \tau_{(0,1)} \tau_{(1,2)} P_{e_2 - e_1} P_{e_1 - e_0}.
$$

Similarly, the dynamic $T$ of the multicolor BBS is as follows:

$$
T = T_\kappa \cdots T_2 T_1 = \tau_{(0,1)} \tau_{(1,2)} \cdots \tau_{(\kappa-1,\kappa)} = P_{e_\kappa - e_{\kappa-1}} \cdots P_{e_2 - e_1} P_{e_1 - e_0}.
$$
3.5. **Two-sided multicolor BBS.** In this section, we extend the particle configuration to \( \eta = (\eta_n)_{n \in \mathbb{Z}} \in \{0, 1, 2, \ldots, \kappa\}^\mathbb{Z} \).

We can again obtain the path encoding \( S = (S_n)_{n \in \mathbb{Z}} \) of the \( \eta \) given by \((3.3)\) and \((3.4)\). In this case, for \( i \in \mathcal{C} \) and \( n \geq 1 \), \( a_i(n) \) means the number of the particles of color \( i \) at the sites located from 1 to \( n \), and, for \( i \in \mathcal{C} \) and \( n \leq -1 \), \(-a_i(n)\) means the same at the sites located from \( n + 1 \) to 0. The same is true for the number of the empty sites. Also we define that \( a_i(0) = 0 \) for \( i \in \mathcal{C} \cup \{0\} \). As in the case of one-sided multicolor BBS, it obviously holds
\[
a_0(n) + a_1(n) + \cdots + a_\kappa(n) = n \quad \forall n \in \mathbb{Z}.
\]

Also we define the path space in \( \mathbb{R}^\kappa \):
\[
S^0 := \{ S = (S_n)_{n \in \mathbb{Z}} : S_0 = 0, S_{n+1} - S_n \in \{e_0, e_1, \ldots, e_\kappa\}, \forall n \in \mathbb{Z} \}.
\]

Moreover, we define the function \( A_i : S^0 \rightarrow \mathbb{R}^\mathbb{Z} \) and the operator \( \tau_{(0,i)} : S^0 \rightarrow S^0 \) given by \((3.5)\) and \((3.7)\).

Whilst in the one-sided case, carrier process \( W \) and the actions \( T_i, i = 1, \ldots, \kappa \) are defined for any \( S \in S^+ \) (that is, for any configuration \( \eta \in \{0, 1, 2, \ldots, \kappa\}^\mathbb{N} \)), in the two-sided case, the following restriction on \( S \) is required to define the carrier and actions:
\[
\limsup_{n \to -\infty} A_i S_n < \infty.
\]

This condition can be transformed as follows:
\[
\limsup_{n \to -\infty} A_i S_n < \infty \iff \sup_{n \leq 0} A_i S_n < \infty
\]
\[
\iff \sup_{n \leq 0} \{a_0(n) - a_i(n)\} < \infty
\]
\[
\iff \inf_{n \leq 0} \{(-a_0(n)) - (-a_i(n))\} > -\infty
\]
\[
\iff -A_i S \in \mathbb{R}^{P_1}
\]
and this means that the number of particles of color \( i \) is not too much compared with the number of empty sites in the left side.

Indeed, in section 2.4 in the paper [4], two-sided multicolor BBS is understood with two-sided carrier process
\[
W_n^{(i)} = \sup_{m \leq n} A_i S_m - A_i S_n
\]
under the condition \((3.10)\).

Also the path encoding \( T_i S_n = a_0'(n)e_0 + a_i'(n)e_i + \sum_{j \neq 0, i} a_j(n)e_j \) of \( T_i \eta \), is obtained by the equation
\[
A_i T_i S_n = 2 \sup_{m \leq n} A_i S_m - A_i S_n + 2 \sup_{m \leq 0} A_i S_m
\]
under the condition \((3.10)\). Then, in the same way as proof of Theorem 3.14, it holds that
\[
T_i S = \tau_{(0,i)} P_{e_i - e_0} S
\]
where \( P \) is two-sided Pitman transform defined in Definition 2.6.

From the above discussion, the next set is obtained:
\[
S^{T_i} := \{ S \in S^0 : T_i S \text{ well-defined} \}
\[ S \in S^0 : \limsup_{n \to \infty} A_i S_n < \infty \]

### 3.6. Inverse of the action

In the previous section, we found that \( T_i = \tau_{(0,i)} P_{e_i - e_0} \) on \( S_T \). Then we can defined \( T_i^{-1} = P_{e_i - e_0}^{-1} \tau_{(0,i)} \) on an appropriate set, where \( P_{e_i - e_0}^{-1} \) is defined by Definition 2.7.

As in the proof of Theorem 3.10, \( T_i \) acts on \( S_n \) as follows:

\[
T_i S_n = \tau_{(0,i)} P_{e_i - e_0} S_n
= \tau_{(0,i)} \left( \frac{1}{2} P_1 (-A_i S)_n (e_i - e_0) + \sum_{j \neq 0,i} d_j(n) e_j \right).
\]

Therefore, Theorem 2.11 shows

\[
T_i^{-1} T_i S_n = T_i^{-1} \left( \frac{1}{2} P_1 (-A_i S)_n (e_i - e_0) + \sum_{j \neq 0,i} d_j(n) e_j \right)
= P_{e_i - e_0}^{-1} \tau_{(0,i)} \tau_{(0,i)} \left( \frac{1}{2} P_1 (-A_i S)_n (e_i - e_0) + \sum_{j \neq 0,i} d_j(n) e_j \right)
= P_{e_i - e_0}^{-1} \left( \frac{1}{2} P_1 (-A_i S)_n (e_i - e_0) + \sum_{j \neq 0,i} d_j(n) e_j \right)
= P_{e_i - e_0}^{-1} \left( \frac{1}{2} P_1 (-A_i S)_n (e_i - e_0) + \sum_{j \neq 0,i} d_j(n) e_j \right)
= \frac{1}{2} P_{e_i}^{-1} P_1 (-A_i S)_n (e_i - e_0) + \sum_{j \neq 0,i} d_j(n) e_j
= \frac{1}{2} (-A_i S)_n (e_i - e_0) + \sum_{j \neq 0,i} d_j(n) e_j
= S_n
\]

if and only if \(-A_i S \in R P_{e_i}^{-1} P_1\).

On the other hand, by Remark 3.10, \( T_i^{-1} \) acts on \( S_n \) as follows:

\[
T_i^{-1} S_n = P_{e_i - e_0}^{-1} \tau_{(0,i)} S_n
= P_{e_i - e_0}^{-1} \tau_{(0,i)} \left( -\frac{1}{2} (A_i S)_n (e_i - e_0) + \sum_{j \neq 0,i} d_j(n) e_j \right)
= P_{e_i - e_0}^{-1} \left( \frac{1}{2} (A_i S)_n (e_i - e_0) + \sum_{j \neq 0,i} d_j(n) e_j \right)
= P_{e_i - e_0}^{-1} \left( \frac{1}{2} (A_i S)_n (e_i - e_0) + \sum_{j \neq 0,i} d_j(n) e_j \right)
= \frac{1}{2} P_{e_i}^{-1} (A_i S)_n (e_i - e_0) + \sum_{j \neq 0,i} d_j(n) e_j
\]
Therefore, Theorem 3.6 and Remark 3.10 shows

\[ T_i T_i^{-1} S_n = \tau_{(0, i)} P_i - e_0 \left( \frac{1}{2} P_i^{-1} (A_i S)_n (e_i - e_0) + \sum_{j \neq 0, i} d_j(n) e_j \right) \]

\[ = \tau_{(0, i)} \left( P_i \left( \frac{1}{2} P_i^{-1} (A_i S) \right) \frac{1}{n} (e_i - e_0) + \sum_{j \neq 0, i} d_j(n) e_j \right) \]

\[ = \tau_{(0, i)} \left( \frac{1}{2} P_i P_i^{-1} (A_i S)_n (e_i - e_0) + \sum_{j \neq 0, i} d_j(n) e_j \right) \]

\[ = \tau_{(0, i)} \left( \frac{1}{2} A_i S_n (e_i - e_0) + \sum_{j \neq 0, i} d_j(n) e_j \right) \]

\[ = -\frac{1}{2} A_i S_n (e_i - e_0) + \sum_{j \neq 0, i} d_j(n) e_j \]

\[ = S_n \]

if and only if \( A_i S \in \mathcal{R}^{P_i P_i^{-1}} \).

Above discussion gives the following theorem characterizing the following sets:

\[ S^{T_i T_i^{-1}} := \{ S \in S^0 : T_i S, T_i^{-1} T S \text{ well-defined, } T_i^{-1} T_i S = S \} \]

\[ S^{T_i^{-T_i}} := \{ S \in S^0 : T_i^{-1} S, T_i T_i^{-1} S \text{ well-defined, } T_i T_i^{-1} S = S \} \].

**Theorem 3.17.** It holds that

\[ S^{T_i^{-T_i}} = \{ S \in S^0 : -A_i S \in \mathcal{R}^{P_i^{-1} P_i} \} \]

\[ = \{ S \in S^0 : \inf_{m \leq 0} (-A_i S_m) > -\infty, \inf_{m \leq n} (-A_i S_m) = -A_i S_n, \ i.o. \ as \ n \to \infty \} \]

\[ = \{ S \in S^0 : \sup_{m \leq 0} A_i S_m < \infty, \sup_{m \leq n} A_i S_m = A_i S_n, \ i.o. \ as \ n \to \infty \}, \]

and

\[ S^{T_i T_i^{-1}} = \{ S \in S^0 : A_i S \in \mathcal{R}^{P_i P_i^{-1}} \} \]

\[ = \{ S \in S^0 : \inf_{m \geq 0} A_i S_m > -\infty, \inf_{m \geq n} A_i S_m = A_i S_n, \ i.o. \ as \ n \to -\infty \}. \]

**Remark 3.18.** The above conditions can be transformed as follows:

\[ \sup_{m \leq n} A_i S_m = A_i S_n, \ i.o. \ as \ n \to \infty \iff \sup_{n \in \mathbb{Z}} A_i S_n = \limsup_{n \to \infty} A_i S_n, \]

\[ \inf_{m \geq n} A_i S_m = A_i S_n, \ i.o. \ as \ n \to -\infty \iff \inf_{n \in \mathbb{Z}} A_i S_n = \liminf_{n \to -\infty} A_i S_n. \]

Then it holds that

\[ S^{T_i^{-T_i}} = \{ S \in S^0 : M_0^{(i)} < \infty, \limsup_{n \to \infty} A_i S_n = M_0^{(i)} \}, \]
\[ S_{i}^{T; T^{-1}} = \{ S \in S^{0} : I_{0}^{(i)} > -\infty, \lim_{n \to -\infty} A_{i}S_{n} = I_{-\infty}^{(i)} \}. \]

where, we define

\[ M_{0}^{(i)} := \sup_{n \leq 0} A_{i}S_{n}, \quad M_{\infty}^{(i)} := \sup_{n \in \mathbb{Z}} A_{i}S_{n}, \]

\[ I_{0}^{(i)} := \inf_{n \geq 0} A_{i}S_{n}, \quad I_{-\infty}^{(i)} := \inf_{n \in \mathbb{Z}} A_{i}S_{n}. \]

Also we obtain the following set :

\[ S_{i}^{rev} := \{ S \in S^{0} : T_{i}S, T_{i}^{-1}S, T_{i}^{-1}TS, T_{i}T_{i}^{-1}S \text{ well-defined}, T_{i}^{-1}T_{i}S = T_{i}T_{i}^{-1}S = S \} = \{ S \in S^{0} : M_{0}^{(i)} < \infty, I_{0}^{(i)} > -\infty, \lim_{n \to -\infty} sup A_{i}S_{n} = M_{\infty}^{(i)}, \lim_{n \to -\infty} inf A_{i}S_{n} = I_{-\infty}^{(i)} \}. \]

3.7. Set of configurations. Even if \( S \in S_{i}^{rev} \) holds, it does not necessarily hold that \( T_{i}S \in S_{i}^{rev} \). In the paper [4] for the 1-color BBS, the set

\[ S_{i}^{inv} := \{ S \in S^{0} : T_{i}^{k}S \in S_{i}^{rev}, \forall k \in \mathbb{Z} \} \]

is characterized as following lemma.

Lemma 3.19. For any \( i \in \mathcal{C} \), it holds that

\[ S_{i}^{inv} = \bigcup_{*_{1}, *_{2} \in \{ sub-\text{critical}(i), \text{critical}(i) \}} \left( S_{*_{1}}^{-} \cap S_{*_{2}}^{+} \right), \]

where

\[ S_{\text{sub-critical}^{+}} := \left\{ S \in S^{0} : \lim_{n \to -\infty} A_{i}S_{n} = 1, \exists F_{i} \in \mathcal{F} \right\}, \]

\[ S_{\text{critical}}^{+} := \left\{ S \in S^{0} : \sup_{n \in \mathbb{Z}} W_{n}^{(i)} < \infty, \lim_{n \to -\infty} sup A_{i}S_{n} = \lim_{n \to -\infty} inf A_{i}S_{n} + \sup_{n \in \mathbb{Z}} W_{n}^{(i)} \in \mathbb{R} \right\}, \]

\[ \mathcal{F} := \{ F : \mathbb{Z} \to \mathbb{R} : \text{increasing function}, \lim_{n \to -\infty} F(n) = \infty, \lim_{n \to -\infty} F(n) = -\infty \}. \]

Moreover, it holds that

\[ \lim_{n \to -\infty} A_{i}S_{n} \frac{F_{i}(n)}{F_{i}(n)} = 1, \exists F_{i} \in \mathcal{F} \Rightarrow \lim_{n \to -\infty} A_{i}T_{i}S_{n} \frac{F_{i}(n)}{F_{i}(n)} = 1 \]

and

\[ \lim_{n \to -\infty} A_{i}S_{n} \frac{F_{i}(n)}{F_{i}(n)} = 1, \exists F_{i} \in \mathcal{F} \Leftrightarrow \lim_{n \to -\infty} \frac{A_{i}S_{n}}{\sup_{m \leq n} A_{i}S_{m}} = 1. \]

For the study of the multicolor BBS theory, it is natural to ask when \( T^{-1}TS = TT^{-1}S = S \) is true where \( T \) is any composition of \( T_{i}, i \in \mathcal{C} \) such as \( T = T_{e} \cdots T_{2}T_{1} \), \( T = T_{2}T_{1}T_{2} \) etc. In other words, what is the condition for \( S \) to be in the following set ?

\[ S_{i}^{inv} := \{ S \in S^{0} : TS \in \bigcap_{i \in \mathcal{C}} S_{i}^{rev} \text{ for any composition } T \text{ of } T_{i}, i \in \mathcal{C} \}. \]
One might expect that
\[ S^\text{inv} \supseteq \bigcap_{i \in \mathcal{C}} S^\text{inv}_i \]
but this is not true. (See Remark 3.22.) The main result of this section is the following theorem which gives a sufficient condition for \( S \) to be in the set \( S^\text{inv} \).

**Theorem 3.20.** Define the subset of \( \bigcap_{i \in \mathcal{C}} \left( S^\text{sub-critical(i)}_\text{good} \cap S^\text{sub-critical(i)}_\text{critical} \right) \) such that \( F_i \) and \( F_j \) have the same asymptotic behavior as \( n \to \pm\infty \) for any \( i, j \in \mathcal{C} \) as follows,
\[ S^\text{good}_\mathcal{C} := \left\{ S \in S^0 \mid \forall i \in \mathcal{C} \exists F_i \in \mathcal{F}, \lim_{n \to \pm\infty} \frac{A_i S_n}{F_i(n)} = 1 \text{ and } \limsup_{n \to \pm\infty} \frac{F_i(n)}{F_j(n)} < \infty \forall i, j \in \mathcal{C} \right\}. \]

It holds that
\[ S^\text{inv}_\mathcal{C} \supseteq S^\text{good}_\mathcal{C}. \]

To prove the above result, we prepare a simple lemma.

**Lemma 3.21.** For any \( i, j \in \mathcal{C}, i \neq j \), and \( S \in S^T \), it holds that
\begin{align*}
(3.14) \quad & A_j T_i S_n = A_j S_n + W^{(i)}_n - M^{(i)}_0 \\
(3.15) \quad & A_j T_i S_n = A_j S_n + \frac{1}{2} (A_i T_i S_n - A_i S_n)
\end{align*}

for any \( n \in \mathbb{Z} \).

**Proof.** Let \( S_n = a_0(n)e_0 + a_1(n)e_1 + \cdots + a_r(n)e_r \) and \( T_i S_n = a'_0(n)e_0 + a'_1(n)e_1 + \sum_{k \neq 0, i} a_k(n)e_k \). Then (3.11) shows
\[ a'_0(n) - a'_i(n) = 2 \sup_{m \leq n} \{ a_0(m) - a_i(m) \} - \{ a_0(n) - a_i(n) \} - 2M^{(i)}_0 \]
By adding \( a'_0(n) + a'_i(n) = a_0(n) + a_i(n) \) to the above equation, we have
\[ 2a'_0(n) = 2 \sup_{m \leq n} \{ a_0(m) - a_i(m) \} + 2a_i(n) - 2M^{(i)}_0. \]

Then it follows that
\[ a'_0(n) = a_0(n) + \sup_{m \leq n} \{ a_0(m) - a_i(m) \} - \{ a_0(n) - a_i(n) \} - M^{(i)}_0 \]
Since \( A_j T_i S_n = a'_0(n) - a_j(n) \) and \( \sup_{m \leq n} A^{(i)} S_m - A^{(i)} S_n = W^{(i)}_n \), the first claim is proved. Also \( a'_0(n) + a'_i(n) = a_0(n) + a_i(n) \) shows
\[ 2a'_0(n) - \{ a'_0(n) - a'_i(n) \} = 2a_0 - \{ a_0(n) - a_i(n) \} \]
then,
\[ a'_0(n) = a_0(n) + \frac{1}{2} (A_j T_i S_n - A_i S_n) \]
and this prove the second claim. \( \square \)
Proof of Theorem 3.20. Suppose that $S \in S^\text{good}_C$. It is enough to show that $T_i S \in S^\text{good}_C$ for any $i \in C$, so for that we show

$$\lim_{n \to \pm \infty} \frac{A_j T_i S_n}{F_j(n)} = 1$$

for any $i, j \in C$. From (3.15), we can write

$$\frac{A_j T_i S_n}{F_j(n)} = \frac{A_j S_n}{F_j(n)} + \frac{1}{2} \frac{F_j(n)}{F_i(n)} \left( \frac{A_i T_i S_n}{F_i(n)} - \frac{A_i S_n}{F_i(n)} \right).$$

By the assumption and (3.12), it holds that

$$\lim_{n \to \pm \infty} \frac{A_j S_n}{F_j(n)} = 1,$$

$$\lim_{n \to \pm \infty} \frac{A_i S_n}{F_i(n)} = 1,$$

$$\lim_{n \to \pm \infty} \frac{A_i T_i S_n}{F_i(n)} = 1.$$

Then the condition $\lim sup_{n \to \pm \infty} \frac{F_i(n)}{F_j(n)} < \infty$ shows the conclusion.

$\square$

Remark 3.22. Now we consider three examples of the configurations with $C = \{1, 2\}$. Each example shows one of the following three claims.

(a) $S^\text{inv}_C \nsubseteq \bigcap_{i \in C} \left( S^-_{\text{sub-critical}(i)} \cap S^+_{\text{sub-critical}(i)} \right)$,

(b) $S^\text{inv}_C \nsubseteq \bigcap_{i \in C} \left( S^-_{\text{critical}(i)} \cap S^+_{\text{critical}(i)} \right)$,

(c) $S^\text{inv}_C \supseteq S^\text{good}_C$.

(a) We give an example of $\eta$ whose path encoding $S$ satisfies

$S \in \bigcap_{i \in C} \left( S^-_{\text{sub-critical}(i)} \cap S^+_{\text{sub-critical}(i)} \right)$, $T_2 S \notin S^+_{\text{sub-critical}(1)}$, $T_2 S \notin S^+_{\text{critical}(1)}$

Let $\eta$ be as follows:

$$\eta = (\cdots 0 2(1) (0 1)(1) 0 (0 1)(2) 0 2(3) (0 1)(3) 0 (0 1)(4) 0 2(5) (0 1)(5) 0 (0 1)(6) \cdots$$

$$\cdots \cdots 0 2(2m-1) (0 1)(2m-1) 0 (0 1)(2m) \cdots),$$

where $i_{(k)} := i \ i \ \cdots \ i$ means $k$ consecutive $i$, and $(i \ j)_k := i \ j \ i \ j \ \cdots \ i \ j$ means that $i$ and $j$ alternately appear $k$ times. For simplicity, Figure 6 and Figure 7 show the graph of $A_2 S_n$ and $A_1 S_n$ skipping places where there is no increase or decrease where $S$ is path encoding of $\eta$. As seen in Figure 6, it holds that

$$1 \geq \lim sup_{n \to \infty} \frac{A_2 S_n}{\sup_{m \leq n} A_2 S_m}$$

$$\geq \lim inf_{n \to \infty} \frac{A_2 S_n}{\sup_{m \leq n} A_2 S_m}$$

$$= \lim_{k \to \infty} \frac{1 + (1 + 3) + (1 + 5) + \cdots + (1 + 2k - 1) - (2k - 1)}{1 + (1 + 3) + (1 + 5) + \cdots + (1 + 2k - 1)}$$

$$= 1.$$
As seen in Figure 7, it holds that $|\sup_{m \leq n} A_1 S_m - A_1 S_n| \leq 1$, for all $n$ and
\[ \lim_{n \to \infty} A_1 S_n = \infty, \] then
\[ \lim_{n \to \infty} \frac{A_1 S_n}{\sup_{m \leq n} A_1 S_m} = 1. \]
Also it clearly holds that
\[ \lim_{n \to \infty} A_1 S_n = \infty, \] then
\[ \lim_{n \to \infty} A_1 S_n \sup_{m \leq n} \leq A_1 S_m = 1. \]
Also it clearly holds that
\[ \lim_{n \to -\infty} A_1 S_n \sup_{m \leq n} \leq A_1 S_m = \lim_{n \to -\infty} A_2 S_n \sup_{m \leq n} = 1. \]
Therefore, by Lemma 3.19,
\[ S \in \bigcap_{i \in C} \left( S_{\text{sub-critical}(i)}^- \cap S_{\text{sub-critical}(i)}^+ \right). \]
However, the configuration of $T_2 \eta$ is as follows:
\[ T_2 \eta = (\cdots 0 \eta_0 = 0 0 0 0 1) (2 1) (1) 0 (0 1) (2) 0 0 0 0 0 1 0 0 (5 2 1) (5) 0 (0 1) (6) \cdots \]
\[ \cdots \cdots 0 0 (2 m-1) (2 1) (2 m-1) 0 (0 1) (2 m) \cdots. \]
Figure 8, the graph of $A_1 T_2 S_n$, shows that
\[ \liminf_{n \to \infty} A_1 T_2 S_n \sup_{m \leq n} \leq A_1 T_2 S_n = \frac{1}{2}. \]
Therefore, $T_2 S / \notin S_{\text{sub-critical}(1)}$.
Also $T_2 S / \notin S_{\text{critical}(1)}$ is obvious. Then, from Lemma 3.19, $T_2 S / \notin S_{\text{inv}}^1$. Such a phenomenon occurs because $W_n^{(2)}$ can be arbitrarily large and it causes a gap between the asymptotic behavior of $A_1 T_2 S_n$ and that of $A_1 S_n$ as $n \to \infty$ from the equation $A_1 T_2 S_n = A_1 S_n + W_n^{(2)} - M_0^{(2)}$ by (3.14).

Figure 8. The graph of $A_2 S_n$ skipping places where there is no increase or decrease.

Figure 9. The graph of $A_1 S_n$ skipping places where there is no increase or decrease.
Figure 10. The graph of $A_1T_2S_n$ skipping places where there is no increase or decrease.

(b) We give an example of $\xi$ whose path encoding $S^{(\xi)}$ satisfies

$$S^{(\xi)} \in \bigcap_{i \in \mathcal{C}} \left( S^-_{\text{critical}(i)} \cap S^+_{\text{critical}(i)} \right), \quad T_2S^{(\xi)} \notin S^+_{\text{sub-critical}(1)}, \quad T_2S^{(\xi)} \notin S^+_{\text{critical}(1)}.$$ 

Let $\xi$ be as follows:

$$\xi = (\cdots 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 \cdots).$$

Then,

$$T_2\xi = (\cdots 2 1 0 2 1 0 2 1 0 2 1 0 2 1 0 2 1 0 \cdots)$$

and they show above conditions.

(c) We give an example of $\zeta$ whose path encoding $S^{(\zeta)}$ satisfies

$$S^{(\zeta)} \in \bigcap_{i \in \mathcal{C}} \left( S^-_{\text{critical}(i)} \cap S^+_{\text{critical}(i)} \right), \quad S^{(\zeta)} \in S^{\text{inv}}_\mathcal{C}.$$ 

Let $\zeta$ be as follows:

$$\zeta = (\cdots 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 \cdots).$$

$T\zeta^{(\zeta)} \in \bigcap_{i \in \mathcal{C}} \left( S^-_{\text{critical}(i)} \cap S^+_{\text{critical}(i)} \right)$, where $T$ is any composition of $T_1$ and $T_2$, because the configuration of $T\zeta$ is always repeating $(012)$ or $(021)$. Therefore, it holds that $S^{(\zeta)} \in S^{\text{inv}}_\mathcal{C}$.

4. Random initial configurations

In this section, we consider the case when the initial configuration is random. Suppose that $\eta = (\eta_n)_{n \in \mathbb{Z}}$ is an ergodic sequence which is stationary with respect to the space shift. In particular, if we assume that the densities of the balls of color $i$

$$p_i = \mathbf{P}(\eta_0 = i) < p_0 = \mathbf{P}(\eta_0 = 0), \quad \forall i \in \mathcal{C}, \quad (4.1)$$

then ergodicity implies that $A_iS$ satisfies

$$\frac{A_iS}{n} = \frac{a_0(n) - a_i(n)}{n} \to p_0 - p_i > 0, \quad \mathbf{P}\text{-a.s.}$$
as $n \to \pm \infty$. Thus we obtain the following result, which yields that $(T^nS)_{k \in \mathbb{Z}}$ is well-defined and reversible by Theorem 4.2.

**Lemma 4.1.** If $\eta = (\eta_n)_{n \in \mathbb{Z}}$ is a stationary, ergodic sequence satisfying (4.1), then it holds that

$$A_i S \frac{(p_0 - p_i)n}{(p_0 - p_i)n} \to 1, \quad \mathbb{P}\text{-a.s.}$$

as $n \to \pm \infty$ for any $i \in \mathcal{C}$. In particular, $S \in S^{\text{good}}_C$, $\mathbb{P}$-a.s.

Next, it is natural for random initial configuration to ask whether the law of $\eta$ is preserved by $T_i$, that is, $T_i \eta \overset{d}{=} \eta$. We introduce the example of an invariant measure in Section 4.1. Moreover, we consider generalized multicolor BBS whose dynamic is defined for continuous path in $\mathbb{R}^\kappa$, and generalize each object appearing in the discrete case (Section 4.2). And in Section 4.3 we check that $\kappa$-dimensional Brownian motion with certain drift is invariant under the action of the multicolor BBS, and it is obtained by appropriate scaling limit of asymmetric random walk with distribution $\mathbb{P}(S_m - S_{m-1} = e_j) = \frac{1}{\kappa \kappa + 1} + \frac{e_j}{\kappa \kappa + 1}, j \in \mathcal{C} \cup \{0\}$ such that $c_0 > c_i, \forall i \in \mathcal{C}$, which represents a high density particle configuration.

### 4.1. Independent and identically distributed initial configuration.

Suppose that $\eta = (\eta_n)_{n \in \mathbb{Z}}$ is given by a sequence of i.i.d. random variables with following distribution

$$p_i = \mathbb{P}(\eta_0 = i) < p_0 = \mathbb{P}(\eta_0 = 0), \quad \forall i \in \mathcal{C},$$

then it satisfies (4.2) and the conditions in Lemma 4.1. Furthermore, $S$ is a random walk path in $\mathbb{R}^\kappa$ satisfying $S_0 = 0$ and

$$\mathbb{P}(S_m - S_{m-1} = e_j) = p_j, \quad \forall j \in \mathcal{C} \cup \{0\},$$

where the increments of $S$ are independent.

**Theorem 4.2.** If $\eta = (\eta_n)_{n \in \mathbb{Z}}$ is given by a sequence of i.i.d. random variables with (4.2), it holds that

$$T_i \eta \overset{d}{=} \eta$$

for any $i \in \mathcal{C}$.

**Proof.** We introduce some notations.

For each $n \in \mathbb{Z}$, define transform $f_n : \{0, 1, \ldots, \kappa\}^\mathbb{Z} \to \{1, \ldots, \kappa\}$ given by

$$f_n(\eta) = \begin{cases} \eta_m, & \text{if } \eta_m \not\in \{0, i\}, \\ i, & \text{if } \eta_m \in \{0, i\}. \end{cases}$$

Then it holds that $(f_n(T_i \eta))_{n \in \mathbb{Z}} = (f_n(\eta))_{n \in \mathbb{Z}}$, for each $\eta \in \{0, 1, \ldots, \kappa\}^\mathbb{Z}$.

For each $n \in \mathbb{Z}$ and $\eta \in \{0, 1, \ldots, \kappa\}^\mathbb{Z}$, define a subsequence $\{k_n(\eta)\}_{k \in \mathbb{Z}}$ of $\mathbb{Z}$ given by

$$k_n(\eta) = \begin{cases} \min \{m \in \mathbb{Z} : m > 0, \ \eta_m \not\in \{0, i\}\}, & \text{if } n = 0, \\ \min \{m \in \mathbb{Z} : m > k_{n-1}(\eta), \ \eta_m \in \{0, i\}\}, & \text{if } n \geq 1, \\ \max \{m \in \mathbb{Z} : m < k_{n+1}(\eta), \ \eta_m \in \{0, i\}\}, & \text{if } n \leq -1, \end{cases}$$

This $\{k_n(\eta)\}_{k \in \mathbb{Z}}$ is well-defined for $\eta$ almost everywhere.
For each $n \in \mathbb{Z}$, define transform $g_n : \{0, 1, \cdots, \kappa\}^\mathbb{Z} \to \{0, i\}$ given by
\[
g_n(\eta) = \eta_{k_n(\eta)}.
\]
Then it holds that $(g_n(T_\eta))_{n \in \mathbb{Z}} = T_i (g_n(\eta))_{n \in \mathbb{Z}}$, for each $\eta \in \{0, 1, \cdots, \kappa\}^\mathbb{Z}$. Moreover, [4] shows that
\[
T_i (g_n(\eta))_{n \in \mathbb{Z}} \overset{d}{=} (g_n(\eta))_{n \in \mathbb{Z}}.
\]

Denote the filtration,
\[
\mathcal{F} := \{f_n, n \in \mathbb{Z}\}, \quad \mathcal{G} := \{g_n, n \in \mathbb{Z}\}.
\]
It is obvious that $f_n(\eta)$ and $g_m(\eta)$ are independent for any $n, m \in \mathbb{Z}$, so is $\mathcal{F}$ and $\mathcal{G}$. Also $f_n(\eta)$ and $g_m(T_\eta)$ are independent.

Configuration $\eta$ is determined by $(f_n(\eta))_{n \in \mathbb{Z}}$ and $(g_n(\eta))_{n \in \mathbb{Z}}$, so there is a transform $\varphi$ such that
\[
\varphi((f_n(\eta))_{n \in \mathbb{Z}}, (g_n(\eta))_{n \in \mathbb{Z}}) = \eta, \quad \forall \eta \in \{0, 1, \cdots, \kappa\}^\mathbb{Z},
\]
which is measurable with respect to product measure $\mathcal{F} \times \mathcal{G}$.

Then it holds that
\[
T_i \eta = \varphi((f_n(T_i \eta))_{n \in \mathbb{Z}}, (g_n(T_i \eta))_{n \in \mathbb{Z}})
\]
\[
= \varphi((f_n(\eta))_{n \in \mathbb{Z}}, T_i ((g_n(\eta))_{n \in \mathbb{Z}}))
\]
\[
\overset{d}{=} \varphi((f_n(\eta))_{n \in \mathbb{Z}}, (g_n(\eta))_{n \in \mathbb{Z}})
\]
\[
= \eta.
\]

\begin{corollary}
As the same setting in Theorem 4.2, it holds that
\[
T \eta \overset{d}{=} \eta,
\]
where $T = T_\kappa \circ \cdots \circ T_1$.
\end{corollary}

4.2. Multicolor BBS on $\mathbb{R}$. In this section, we consider a generalization of the multicolor BBS, whose dynamic is defined for continuous path in $\mathbb{R}^\kappa$. At first, we define Pitman transform for continuous path.

\begin{definition}
Let $\alpha \in \mathbb{R}^k$, $\alpha \neq 0$. The two-sided Pitman transform $P_\alpha$ with respect to $\alpha$ is defined on the set
\[
\{ \pi : \mathbb{R} \to \mathbb{R}^k, \pi(0) = 0, \inf_{y \leq 0} \alpha \cdot \pi(y) > -\infty \}
\]
by the formula,
\[
P_\alpha \pi(x) = \pi(x) - 2 \inf_{y \leq x} \frac{\alpha \cdot \pi(y)}{\alpha} \alpha + 2 \inf_{y \geq 0} \frac{\alpha \cdot \pi(y)}{\alpha} \alpha, \quad x \in \mathbb{R}
\]
Similarly to discrete case, for $k = 1$, it holds that
\[
P_\alpha \pi(x) = \pi(x) - 2 \inf_{y \leq x} \pi(y) + 2 \inf_{y \geq 0} \pi(y), \quad x \in \mathbb{R}
\]
for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and it does not depend on $\alpha$. Then we define
\[
P_1 := P_\alpha \quad \text{for } \alpha \in \mathbb{R}, \alpha \neq 0.
\]
Also we define the transform $P_\alpha^{-1}$ on the set
\[
\{ \pi : \mathbb{R} \to \mathbb{R}^k, \pi(0) = 0, \inf_{y \geq 0} \alpha \cdot \pi(y) > -\infty \}
\]
by the formula,
\[ P_\alpha^{-1}(x) = \pi(x) - 2 \inf_{y \geq x} \alpha \cdot \pi(y) \alpha + 2 \inf_{y \geq 0} \alpha \cdot \pi(y) \alpha, \quad x \in \mathbb{R}, \]
and for \( k = 1 \),
\[ P_1^{-1}(x) = \pi(y) - 2 \inf_{y \geq x} \pi(y) + 2 \inf_{y \geq 0} \pi(y), \quad x \in \mathbb{R}. \]

Unlike the discrete case, we can not describe the particle configuration \( \eta \) directly, so we consider the dynamic for the path encoding \( S \) only. By analogy with the relevant discrete objects, define the path space
\[ S^0_c = \{ S : \mathbb{R} \to \mathbb{R}^\kappa : S_0 = 0, \ S \text{ is continuous} \}. \]

As the extension of (3.6) in Remark 3.7 and (3.8) in Remark 3.10, we define \( A_i \) and \( \tau \) as follows.

**Definition 4.5.** Define \( A_i \) and \( \tau_{(0,i)} \) as follows:
\[
A_iS_x = -2\frac{(e_i - e_0) \cdot S_x}{|e_i - e_0|^2},
\]
\[
\tau_{(0,i)}S_x = S_x + A_iS_x(e_i - e_0)
\]
for \( x \in \mathbb{R} \).

It is the case that the projection of \( S_x \) along \( e_i - e_0 \) is \(-\frac{1}{2}A_iS_x (e_i - e_0)\), and \( S_x \) is decomposed into the sum as follows:
\[ S_x = -\frac{1}{2}A_iS_x(e_i - e_0) + \left\{ S_x + \frac{1}{2}A_iS_x(e_i - e_0) \right\}, \]
and also it holds that
\[ \tau_{(0,i)}S_x = \frac{1}{2}A_iS_x(e_i - e_0) + \left\{ S_x + \frac{1}{2}A_iS_x(e_i - e_0) \right\}. \]

Then we can define the dynamics of the generalized multicolor BBS, given by
\[ T_i = \tau_{(0,i)}P_{e_i-e_0}, \quad \text{on } \{ S \in S^0_c : \limsup_{x \to -\infty} A_iS_x < \infty \}, \]
\[ T_i^{-1} = P_{e_i-e_0}^{-1}\tau_{(0,i)}, \quad \text{on } \{ S \in S^0_c : \liminf_{x \to -\infty} A_iS_x > -\infty \} \]
for each \( i \in \mathcal{C} \).

Moreover, the previous definitions of \( A_i \) and \( \tau_{(0,i)} \) yield the following alternative expression for \( T_i \).

**Theorem 4.6.** It holds that
\[ T_iS_x = S_x + \left( A_iS_x - \sup_{y \leq x} A_iS_y + \sup_{y \leq 0} A_iS_y \right)(e_i - e_0), \quad x \in \mathbb{R} \]
for any \( i \in \mathcal{C} \).
Proof. From (4.3) and (4.4), it holds that

\[ T_i S_x = \tau_{(0,i)} P_{e_i - e_0} S_x \]

\[ = \tau_{(0,i)} P_{e_i - e_0} \left( -\frac{1}{2} A_i S(e_i - e_0) + \left\{ S + \frac{1}{2} A_i S(e_i - e_0) \right\} \right) \]

\[ = \tau_{(0,i)} \left( P_{i} \left( -\frac{1}{2} A_i S \right) (e_i - e_0) + \left\{ S + \frac{1}{2} A_i S_x(e_i - e_0) \right\} \right) \]

\[ = \tau_{(0,i)} \left( \frac{1}{2} P_{i} (-A_i S) (e_i - e_0) + \left\{ S_x + \frac{1}{2} A_i S_x(e_i - e_0) \right\} \right) \]

\[ = -\frac{1}{2} P_{i} (-A_i S)_x (e_i - e_0) + \left\{ S_x + \frac{1}{2} A_i S_x(e_i - e_0) \right\} \]

\[ = -\frac{1}{2} \left\{ -A_i S_x - 2 \inf_{y \leq x} (-A_i S_y) + 2 \inf_{y \leq 0} (-A_i S_y) \right\} (e_i - e_0) + \left\{ S_x + \frac{1}{2} A_i S_x(e_i - e_0) \right\} \]

\[ = S_x + \left( A_i S_x - \sup_{y \leq x} A_i S_y + \sup_{y \leq 0} A_i S_y \right) (e_i - e_0). \]

As in the discrete case, it is natural to seek to characterize the set

\[ S_{\text{c},c}^{\text{inv}} := \{ S \in S_{\text{c}}^0 : TS \in \bigcap_{i \in \mathcal{C}} S_{\text{c},c}^{\text{rev}} \text{ for any composition } T \text{ of } T_i, \ i \in \mathcal{C} \}, \]

where

\[ S_{\text{c},c}^{\text{rev}} = \{ S \in S_{\text{c}}^0 : T_i S, T_i^{-1} S, T_i T_i^{-1} S \text{ well-defined, } T_i^{-1} T_i S = T_i T_i^{-1} S = S \}. \]

The following result is obtained by the similar argument in the discrete case.

**Theorem 4.7.** It holds that

\[ S_{\text{c},c}^{\text{inv}} \supseteq S_{\text{c},c}^{\text{good}}, \]

where

\[ S_{\text{c},c}^{\text{good}} := \left\{ S \in S_{\text{c}}^0 : \forall i \in \mathcal{C} \exists F_i \in \mathcal{F}_c, \lim_{x \to \pm \infty} \frac{A_i S_x}{F_i(x)} = 1 \text{ and } \limsup_{x \to \pm \infty} \frac{F_j(x)}{F_i(x)} < \infty \forall i, j \in \mathcal{C} \right\}, \]

and

\[ \mathcal{F}_c = \{ F : \mathbb{R} \to \mathbb{R} : \text{increasing function, } \lim_{x \to \infty} F(x) = \infty, \lim_{x \to -\infty} F(x) = -\infty \}. \]

### 4.3. Brownian motion with drift

Next, we consider a stochastic process whose path belongs to \( S_{\text{c},c}^{\text{good}} \) almost surely. As an example, let \( S = (S_x)_{x \in \mathbb{R}} \) be two-sided standard \( \kappa \)-dimensional standard Brownian motion with drift \( D \in \mathbb{R}^\kappa \). Namely, for \( x \geq 0 \), we define \( S_x = B_x^1 + xD, \ S_{-x} = - (B_x^2 + xD) \), where \( B^1, B^2 \) are independent standard Brownian motions in \( \mathbb{R}^\kappa \). Since

\[ A_i S_x = -2 \frac{(e_i - e_0) \cdot B_x^1}{|e_i - e_0|^2} - 2x \frac{(e_i - e_0) \cdot D}{|e_i - e_0|^2} \]

for \( x \geq 0 \), the condition \( \lim_{x \to \infty} \frac{A_i S_x}{F_i(x)} = 1 \), a.s. \( \exists F_i \in \mathcal{F}_c \) is satisfied if and only if \( (e_i - e_0) \cdot D < 0 \), and we can take \( F_i(x) = -2x \frac{(e_i - e_0) \cdot D}{|e_i - e_0|^2}, \ x \geq 0 \).
Similarly, \( \lim_{x \to -\infty} A_{iS} = 1 \), a.s. \( \exists F_i \in F_c \) if and only if \( (e_i - e_0) \cdot D < 0 \). Therefore, it holds that
\[ S \in S^\text{good}_{C,c}, \text{ a.s.} \iff (e_i - e_0) \cdot D < 0, \; \forall i \in C. \]

On the other hand, from Proposition 3.2, there is a \( \kappa + 1 \)-tuple \( c_0, \ldots, c_\kappa \) of real numbers for \( D \in \mathbb{R}^\kappa \) such that
\[ D = c_0e_0 + \cdots + c_\kappa e_\kappa, \; c_0 + \cdots + c_\kappa = 0, \]
and, by Proposition 3.2, it holds that
\[
(c_i - e_0) \cdot D = (c_i - e_0) \cdot \left( \frac{1}{2}(c_i - c_0)(e_i - e_0) + \sum_{j \neq i} \left( \frac{c_j - c_0}{2} \right) e_j \right) = (c_i - c_0) \frac{|e_i - e_0|^2}{2}.
\]
Thus we obtain the following set:
\[ D := \{ D \in \mathbb{R}^\kappa : (e_i - e_0) \cdot D < 0, \; \forall i \in C \} = \{ D \in \mathbb{R}^\kappa : D = c_0e_0 + \cdots + c_\kappa e_\kappa, \; c_0 > c_i, \; \forall i \in C, \; c_0 + \cdots + c_\kappa = 0 \}, \]
and it is the case that
\[ S \in S^\text{good}_{C,c}, \text{ a.s.} \iff D \in D. \]

The main theorem in this subsection is the following which implies any Brownian motion with drift belonging to \( S^\text{good}_{C,c} \) is invariant under the actions of the generalized multicolor BBS.

**Theorem 4.8.** If \( S \) is the two-sided \( \kappa \)-dimensional standard Brownian motion with drift \( D \in D \), then \( T_iS \overset{d}{=} S \) for each \( i \in C \).

**Corollary 4.9.** As the same setting in Theorem 4.8, it holds that
\[ TS \overset{d}{=} S, \]
where \( T = T_\kappa \circ \cdots \circ T_1 \).

Before prove this main theorem, we show that Brownian motion with drift is obtained by a simple random walk scaling limit. From now on, fix \( c_0, \cdots, c_\kappa \) satisfying \( c_0 > c_i, \; \forall i \in C, \; c_0 + \cdots + c_\kappa = 0 \) and define
\[ D = c_0e_0 + \cdots + c_\kappa e_\kappa, \]
and
\[ p_i^{(n)} = \frac{1}{\kappa + 1} + \frac{c_i}{\sqrt{n\kappa}}, \; i \in C \cup \{0\}. \]
for large enough \( n \) satisfying \( 0 < p_i^{(n)} < 1, \; \forall i \in C \). Then we introduce vector valued random variables \( \xi^{(n)} \) with distribution
\[
(4.6) \quad \mathbf{P}(\xi^{(n)} = e_i) = p_i^{(n)}, \; i \in C \cup \{0\}.
\]
Moreover, let \( \{ \xi_j^{(n)} \}_{j \in \mathbb{Z}} \) a sequence of independent identically distributed vector valued random variables and each \( \xi_j^{(n)} \) has the same distribution as \( \xi^{(n)} \). Also we define the sequence of partial sums

\[
S_{[x]}^{(n)} = \begin{cases} 
\xi_1^{(n)} + \cdots + \xi_{[x]}^{(n)}, & \text{if } [x] \geq 1, \\
0, & \text{if } [x] = 0, \\
-(\xi_{-1}^{(n)} + \cdots + \xi_{[x]}^{(n)}), & \text{if } [x] \leq -1,
\end{cases}
\]

and its linear interpolation

\[
Y_{[x]}^{(n)} = S_{[x]}^{(n)} + (x - [x]) \xi_{[x]+1}^{(n)}, \quad x \in \mathbb{R}.
\]

We introduce the notation \( \mu^{(n)} \) to represent the probability measure on \( S_0^k \) induced by the stochastic process \( (Y_{[x]}^{(n)})_{x \in \mathbb{R}} \). As shown in theorem 4.2, we have the invariance of \( \mu^{(n)} \) under \( T_i \) for any \( i \in C \). As explained above, let \( S = (S_x)_{x \in \mathbb{R}} \) be two-sided \( \kappa \)-dimensional Brownian motion with drift \( D \in \mathbb{R}^\kappa \) and denote \( \nu_D \) the probability measure on \( S_0^k \) induced by \( S = (S_x)_{x \in \mathbb{R}} \). Also we write \( \mu_{a,b} \) to be the scaled measure such that

\[
\mu_{a,b} (S \in A) = \mu (aS_b \cdot A),
\]

for a probability measure \( \mu \) on \( S_0^k \) and \( a, b > 0 \).

The following theorem is known as the Invariance Principle of Donsker.

**Theorem 4.10.** \( \nu_n := \frac{\mu^{(n)}}{\sqrt{n}} \) converges weakly to \( \nu_D \).

To prove this theorem, we prepare some lemmas.

**Lemma 4.11.** For any \( u \in \mathbb{R}^\kappa \) satisfying \( |u| = 1 \), it holds that

\[
(e_0 \cdot u)^2 + \cdots + (e_\kappa \cdot u)^2 = \frac{\kappa + 1}{\kappa}.
\]

**Proof.** By Proposition 3.2, there are \( a_0, \cdots, a_\kappa \in \mathbb{R} \) such that

\[
u := a_0 e_0 + \cdots + a_\kappa e_\kappa.
\]

The condition \( |u| = 1 \), (3.1) and (3.2) shows that

\[
\sum_{i=0}^\kappa a_i^2 - 2 \sum_{i \neq j} a_i a_j = 1.
\]

Then it holds that

\[
\sum_{i=0}^\kappa (e_i \cdot u)^2 = \sum_{i=0}^\kappa \left( a_i - \frac{1}{\kappa} \sum_{j \neq i} a_j \right)^2
\]

\[
= \sum_{i=0}^\kappa \left( a_i^2 - \frac{2a_i}{\kappa} \sum_{j \neq i} a_j + \frac{1}{\kappa^2} \left( \sum_{j \neq i} a_j \right)^2 \right)
\]

\[
= \frac{\kappa + 1}{\kappa} \sum_{i=0}^\kappa a_i^2 - \frac{2(\kappa + 1)}{\kappa^2} \sum_{i \neq j} a_i a_j
\]
\[ = \frac{\kappa + 1}{\kappa}. \]

**Lemma 4.12.** For any \( u, v \in \mathbb{R}^\kappa \) satisfying \( u \cdot v = 0 \), it holds that
\[
(e_0 \cdot u)(e_0 \cdot v) + \cdots + (e_\kappa \cdot u)(e_\kappa \cdot v) = 0.
\]

**Proof.** By Proposition 3.2, there are \( a_0, \ldots, a_\kappa, b_0, \ldots, b_\kappa \in \mathbb{R} \) such that
\[
u = a_0 e_0 + \cdots + a_\kappa e_\kappa,
\]
\[
v = b_0 e_0 + \cdots + b_\kappa e_\kappa.
\]
The condition \( u \cdot v = 0 \), (3.1) and (3.2) show that
\[
k \sum_{i=0}^{\kappa} a_i b_i - \frac{1}{\kappa} \sum_{i \neq j} a_i b_j = 0.
\]
Then it holds that
\[
k \sum_{i=0}^{\kappa} (e_i \cdot u)(e_i \cdot v) = \sum_{i=0}^{\kappa} \left( a_i - \frac{1}{\kappa} \sum_{j \neq i} a_j \right) \left( b_i - \frac{1}{\kappa} \sum_{j \neq i} b_j \right)
\]
\[
= \frac{\kappa + 1}{\kappa} \sum_{i=0}^{\kappa} a_i b_i - \frac{\kappa + 1}{\kappa^2} \sum_{i \neq j} a_i b_j = 0.
\]

**Lemma 4.13.** (a) For each \( i \in \mathcal{C} \cup \{0\} \), we denote the components of \( e_i \) as follows:
\[
e_0 = \begin{pmatrix} e_{0,1} \\ e_{0,2} \\ \vdots \\ e_{0,\kappa-1} \\ e_{0,\kappa} \end{pmatrix}, \quad e_1 = \begin{pmatrix} e_{1,1} \\ e_{1,2} \\ \vdots \\ e_{1,\kappa-1} \\ e_{1,\kappa} \end{pmatrix}, \quad \cdots, \quad e_\kappa = \begin{pmatrix} e_{\kappa,1} \\ e_{\kappa,2} \\ \vdots \\ e_{\kappa,\kappa-1} \\ e_{\kappa,\kappa} \end{pmatrix}.
\]
For any \( s, t \in \mathcal{C}, s \neq t \) it holds that
\[
e_{2,s} + \cdots + e_{2,s} = \frac{\kappa + 1}{\kappa},
\]
\[
e_{s,s} e_{0,t} + \cdots + e_{s,s} e_{\kappa,t} = 0.
\]
(b) For each \( n \), we denote the components of \( \xi^{(n)} \) by \( \xi^{(n)} = (\xi_1^{(n)}, \ldots, \xi_\kappa^{(n)}) \).
For any \( s, t \in \mathcal{C}, s \neq t \) it holds that
\[
\lim_{n \to \infty} E(\xi_{s,s}^{(n)}) = 0, \quad \lim_{n \to \infty} V(\xi_{s,s}^{(n)}) = \frac{1}{\kappa}, \quad \lim_{n \to \infty} E(\xi_{s,s}^{(n)} \xi_{s,s}^{(n)}) = 0.
\]
Proof. In Lemma 4.11 and 4.12, let \( u = (\delta_{s_1}, \ldots, \delta_{s_\kappa}) \) and \( v = (\delta_{t_1}, \ldots, \delta_{t_\kappa}) \) for \( s, t \in \mathcal{C}, s \neq t \), where \( \delta \) is the Kronecker delta. Then the two equations in (a) follow directly.

Assume that \( \Xi \) is vector valued random variable with distribution

\[
P(\xi = e_i) = \frac{1}{\kappa + 1}, \quad i \in \mathcal{C} \cup \{0\},
\]

and denote its components by \( \xi = (\xi_1, \ldots, \xi_\kappa) \). Then, by Proposition 3.2,

\[
E(\xi_s) = 0, \quad s \in \mathcal{C}
\]

where \( E \) is the expectation with respect to \( P \). Also above equations in (a) show that \( V(\xi_s) = 1/\kappa, s \in \mathcal{C} \) where \( V \) is the variance with respect to \( P \), and

\[
E(\xi_s \xi_t) = 0, \quad s, t \in \mathcal{C}, s \neq t.
\]

The distribution (4.6) and (4.8) imply that \( \xi(n) \) converges to \( \xi \) almost surely as \( n \to \infty \), and convergence theorem shows the claim (b). \( \square \)

Remark 4.14. Denote the components of \( D = c_0 e_0 + \cdots + c_\kappa e_\kappa \) by \( D = (D_1, \ldots, D_\kappa) \). Then it holds that

\[
E\left(\xi_j^{(n)}\right) = \frac{1}{\sqrt{n\kappa}} D_j, \quad j \in \mathcal{C}.
\]

It follows directly by (4.6).

To prove Theorem 4.10, it is enough to show following two claims.

1. The finite-dimensional distribution of \( \nu_n \) converges weakly to that of \( \nu_D \).
2. \( \{\nu_n\}_n \) is tight.

We prove (1) as Proposition 4.15 and show what is equivalent to (2) as Proposition 4.16. In the proof of Proposition 4.15 and 4.16, we write \( |\cdot| \) as the Euclidean norm.

Proposition 4.15. Define the stochastic process

\[
X^{(n)}_x = \frac{\sqrt{\kappa}}{\sqrt{n}} Y^{(n)}_{nx},
\]

where \( Y^{(n)} \) is given by (4.7). Then, for any \( 0 \leq x_1 < \cdots < x_d < \infty \),

\[
\left( X^{(n)}_{x_1}, \ldots, X^{(n)}_{x_d} \right) \xrightarrow{d} (B_{x_1} + x_1 D, \ldots, B_{x_d} + x_d D) \quad \text{as } n \to \infty
\]

where \( \{B_x\}_{x \geq 0} \) is a \( \kappa \)-dimensional Brownian motion. Also the same is true for \( x \leq 0 \).

Proof. We prove the case \( d = 2 \), that is

\[
\left( X^{(n)}_s, X^{(n)}_t \right) \xrightarrow{d} (B_s + s D, B_t + t D) \quad \text{for } 0 < s < t,
\]

and the other case proved similarly. Since

\[
\left| X^{(n)}_x - \frac{\sqrt{\kappa}}{\sqrt{n}} s^{(n)}_{nx} \right| \leq \frac{\sqrt{\kappa}}{\sqrt{n}} s^{(n)}_{nx} + 1 = \frac{\sqrt{\kappa}}{\sqrt{n}},
\]

...
we have by the Chebyshev inequality,

\[ P \left( \left| X^{(n)} - \frac{\sqrt{\kappa}}{\sqrt{n}} S^{(n)}_{[nx]} \right| > \varepsilon \right) \leq \frac{\kappa}{\varepsilon^2 n} \rightarrow 0 \]

as \( n \rightarrow \infty \). Then it is clear that

\[ \left( X^{(n)}(i), X^{(n)}(j) \right) \rightarrow \frac{\sqrt{\kappa}}{\sqrt{n}} (S_{[ns]}^{(n)}, S_{[nt]}^{(n)}) \rightarrow 0 \text{ in probability.} \]

Therefore, it is enough to show that

\[ \frac{\sqrt{\kappa}}{\sqrt{n}} (S_{[ns]}^{(n)}, S_{[nt]}^{(n)}) \xrightarrow{d} (B_s + sD, B_t + tD), \]

and it is equivalent to

\[ \frac{\sqrt{\kappa}}{\sqrt{n}} \left( \sum_{m=1}^{[ns]} \zeta_m^{(n)}, \sum_{m=[ns]+1}^{[nt]} \zeta_m^{(n)} \right) \xrightarrow{d} (B_s + sD, B_t - B_s + (t-s)D). \]

The independence of the random variables \( \{\zeta_m\}_{m=1}^{\infty} \) implies

\[
\mathbb{E} \left( \exp \left\{ \frac{\sqrt{\kappa}}{\sqrt{n}} \left( \sum_{m=1}^{[ns]} \zeta_m^{(n)} \cdot u + \sum_{m=[ns]+1}^{[nt]} \zeta_m^{(n)} \cdot v \right) \right\} \right) = \mathbb{E} \left( \exp \left\{ \frac{\sqrt{\kappa}}{\sqrt{n}} \sum_{m=1}^{[ns]} \zeta_m^{(n)} \cdot u \right\} \right) \mathbb{E} \left( \exp \left\{ \frac{\sqrt{\kappa}}{\sqrt{n}} \sum_{m=[ns]+1}^{[nt]} \zeta_m^{(n)} \cdot v \right\} \right),
\]

for any \( u = (u_1, \cdots, u_\kappa), v = (v_1, \cdots, v_\kappa) \in \mathbb{R}^\kappa \), and also it holds that

\[ \mathbb{E} \left( \exp \left\{ \frac{\sqrt{\kappa}}{\sqrt{n}} \sum_{m=1}^{[ns]} \zeta_m^{(n)} \cdot u \right\} \right) = \varphi_n \left( \frac{\sqrt{\kappa}}{\sqrt{n}} u \right)^{[ns]}, \]

where \( \varphi_n(\theta) \) is the characteristic function of \( \zeta^{(n)} \) given by

\[ \varphi_n(\theta) = \mathbb{E} \left( \exp \left\{ i\zeta^{(n)} \cdot \theta \right\} \right) \]

for \( \theta = (\theta_j)_{1 \leq j \leq \kappa} \in \mathbb{R}^\kappa \). The function \( \varphi_n \) satisfies

\[
\frac{\partial \varphi_n}{\partial \theta_j}(\theta) = \mathbb{E} \left( i\zeta_j^{(n)} \exp \left\{ i\zeta^{(n)} \cdot \theta \right\} \right) \quad j \in \mathcal{C},
\]

\[
\frac{\partial^2 \varphi_n}{\partial^2 \theta_j}(\theta) = \mathbb{E} \left( -\zeta_j^{(n)}^2 \exp \left\{ i\zeta^{(n)} \cdot \theta \right\} \right) \quad j \in \mathcal{C},
\]

\[
\frac{\partial \varphi_n}{\partial \theta_j \partial \theta_k}(\theta) = \mathbb{E} \left( -\zeta_j^{(n)} \zeta_k^{(n)} \exp \left\{ i\zeta^{(n)} \cdot \theta \right\} \right) \quad j, k \in \mathcal{C}, j \neq k.
\]

Remark 4.14 implies

\[ \frac{\partial \varphi_n}{\partial \theta_j}(0) = \frac{iD_j}{\sqrt{n\kappa}}, \]

and Lemma 4.13 shows, if \( n \rightarrow \infty \) and \( \theta \rightarrow 0 \), that is, \( \theta_j \rightarrow 0 \) for any \( j \in \mathcal{C} \),

\[ \frac{\partial \varphi_n}{\partial^2 \theta_j}(\theta) \rightarrow -\frac{1}{\kappa} \frac{\partial \varphi_n}{\partial \theta_j \partial \theta_k}(\theta) \rightarrow 0. \]
By Taylor’s theorem, there is a vector 

\[ u' = (u'_1, \cdots, u'_\kappa) \in \mathbb{R}^\kappa \]

for fixed 

\[ u \in \mathbb{R}^\kappa, \]

such that 

\[ 0 \leq u'_j \leq \frac{\sqrt{\kappa}}{\sqrt{n}}u_j \]

for any 

\[ j \in C \]

and 

\[ \varphi_n(\sqrt{\kappa}/\sqrt{n}u) = \varphi_n(0) + \sum_{j \in C} \frac{\sqrt{\kappa}}{\sqrt{n}}u_j \partial \varphi_n/\partial \theta_j(0) + \sum_{j \in C} \frac{1}{2} \left( \frac{\sqrt{\kappa}}{\sqrt{n}}u_j \right)^2 \partial^2 \varphi_n/\partial^2 \theta_j(u') + \sum_{j \neq k} \frac{1}{\kappa} \left( u_ju_k \right) \partial \varphi_n/\partial \theta_j \partial \theta_k(u') \]

Since 

\[ \log(1 + x) = x + o(x) \text{ as } x \to 0, \]

it holds that

\[ \log \left\{ \varphi_n(\sqrt{\kappa}/\sqrt{n}u) \right\} = \left\lfloor ns \right\rfloor \log \varphi_n(\sqrt{\kappa}/\sqrt{n}u) \]

\[ = \left\lfloor ns \right\rfloor \sum_{j \in C} Dju_j + \frac{\left\lfloor ns \right\rfloor}{2n} \sum_{j \in C} u_j^2 \frac{\kappa}{\sqrt{n}} \partial^2 \varphi_n/\partial^2 \theta_j(u') + \frac{\left\lfloor ns \right\rfloor \kappa}{n} \sum_{j \neq k} \left( u_ju_k \right) \partial \varphi_n/\partial \theta_j \partial \theta_k(u') \]

\[ \to isD \cdot u - \sum_{j \in C} su_j^2 \]

as 

\[ n \to \infty. \]

Thus,

\[ \lim_{n \to \infty} \mathbb{E} \left( \exp \left\{ \sqrt{\frac{\kappa}{n}} \sum_{m=1}^{\left\lfloor ns \right\rfloor} \zeta_m(n) \cdot u \right\} \right) = \exp \left( isD \cdot u - \sum_{j \in C} \frac{su_j^2}{2} \right). \]

Similarly,

\[ \lim_{n \to \infty} \mathbb{E} \left( \exp \left\{ \sqrt{\frac{\kappa}{n}} \sum_{m=\left\lfloor nt \right\rfloor+1}^{\left\lfloor nt \right\rfloor} \zeta_m(n) \cdot v \right\} \right) = \exp \left( i(t-s)D \cdot v - \sum_{j \in C} \frac{(t-s)u_j^2}{2} \right), \]

and the proof is complete. \( \square \)

The tightness of \( \{\nu_n\}_n \) is known to be equivalent to the following proposition [10, Theorem.2.4.10, 2.4.15].

**Proposition 4.16.** With the same setting in Proposition 4.15, it holds that

\[ \lim_{\lambda \uparrow \infty} \sup_{n \geq 1} \mathbb{P} \left( \left| X_0^{(n)} \right| > \lambda \right) = 0, \]

and, for any \( T > 0, \epsilon > 0, \)

\[ \lim_{\delta \downarrow 0} \max_{n \geq 1} \mathbb{P} \left( \sup_{|t-s| \leq \delta, 0 \leq s,t \leq T} \left| X_t^{(n)} - X_s^{(n)} \right| > \epsilon \right) = 0. \]
Proof. Since $X_0^{(n)} = 0$ for every $n$, (4.9) is obvious. We may replace $\sup_{n \geq 1}$ in (4.10) by $\limsup_{n \to \infty}$ because for a finite number of integers $n$ we can make the probability appearing in (4.10) as small as we choose by reducing $\delta$. Let $X_t^{(n)} = \left( X_t^{(n,1)}, \ldots, X_t^{(n,\kappa)} \right)$ for $t \geq 0$, it holds that

$$\max_{|t-s| \leq \delta, 0 \leq s, t \leq T} \left| X_t^{(n)} - X_s^{(n)} \right| = \max_{|t-s| \leq \delta, 0 \leq s, t \leq T} \left( \sum_{j=1}^{\kappa} \left| X_{t,j}^{(n)} - X_{s,j}^{(n)} \right| \right)^2 \leq \kappa \sum_{j=1}^{\kappa} \max_{|t-s| \leq \delta, 0 \leq s, t \leq T} \left| X_{t,j}^{(n)} - X_{s,j}^{(n)} \right|.$$ 

Thus,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P \left( \max_{|t-s| \leq \delta, 0 \leq s, t \leq T} \left| X_t^{(n)} - X_s^{(n)} \right| > \varepsilon \right) \leq \lim_{\delta \to 0} \limsup_{n \to \infty} P \left( \bigcup_{j \in \mathbb{C}} \max_{|t-s| \leq \delta, 0 \leq s, t \leq T} \left| X_{t,j}^{(n)} - X_{s,j}^{(n)} \right| > \varepsilon \right) \leq \kappa \lim_{\delta \to 0} \limsup_{n \to \infty} P \left( \max_{|t-s| \leq \delta, 0 \leq s, t \leq T} \left| X_{t,j}^{(n)} - X_{s,j}^{(n)} \right| > \varepsilon \right).$$

By the definition of $X^{(n)}$, $Y^{(n)}$ and $S^{(n)}$, it holds that

$$P \left( \max_{|t-s| \leq \delta, 0 \leq s, t \leq T} \left| X_t^{(n)} - X_s^{(n)} \right| > \varepsilon \right) = P \left( \max_{|t-s| \leq n\delta, 0 \leq s, t \leq nT} \left| Y_t^{(n)} - Y_s^{(n)} \right| > \frac{\varepsilon \sqrt{n}}{\kappa \sqrt{k}} \right),$$

and

$$\max_{|t-s| \leq n\delta, 0 \leq s, t \leq nT} \left| Y_t^{(n)} - Y_s^{(n)} \right| \leq \max_{|t-s| \leq [n\delta]+1, 0 \leq s, t \leq [nT]+1} \left| Y_{t,j}^{(n)} - Y_{s,j}^{(n)} \right| \leq \kappa \max_{1 \leq m \leq [n\delta]+1, 0 \leq k \leq [nT]+1} \left| S_{m+k,j}^{(n)} - S_{k,j}^{(n)} \right|,$$

where $Y_{x}^{(n)} = (Y_{x,1}^{(n)}, \ldots, Y_{x,k}^{(n)})$ and $S_{m}^{(n)} = (S_{m,1}^{(n)}, \ldots, S_{m,k}^{(n)})$. Therefore it is enough to show that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P \left( \max_{1 \leq m \leq [n\delta]+1, 0 \leq k \leq [nT]+1} \left| S_{m+k,j}^{(n)} - S_{k,j}^{(n)} \right| > \frac{\varepsilon \sqrt{n}}{\kappa \sqrt{k}} \right) = 0$$

for each $j \in \mathbb{C}$.

Recall the definition of $S^{(n)}$, the $j$-th component of it is as follows

$$S_{0,j}^{(n)} = 0, \quad S_{m,j}^{(n)} = \zeta_{1,j}^{(n)} + \cdots + \zeta_{m,j}^{(n)}, \quad m \geq 1,$$

where $\{ \zeta_{l,j}^{(n)} \}_{l \geq 1}$ are independent and

$$P \left( \zeta_{l,j}^{(n)} = e_{ij} \right) = \frac{1}{\kappa + 1} + \frac{c_i}{\sqrt{n \kappa}}, \quad i \in \mathbb{C},$$

$$E \left( \zeta_{l,j}^{(n)} \right) = \frac{D_{l,j}}{\sqrt{n \kappa}}.$$
from Remark 4.14. Now we define a new stochastic process,

\[ R_{m,j}^{(n)} = 0, \quad R_{m,j}^{(n)} = \left( \zeta_{1,j}^{(n)} - \frac{D_j}{\sqrt{nK}} \right) + \cdots + \left( \zeta_{m,j}^{(n)} - \frac{D_j}{\sqrt{nK}} \right), \quad m \geq 1. \]

Since

\[ |S_n^{(m+k,j)} - S_n^{(k,j)}| = \left| \left( R_{m+k,j}^{(n)} + (m+k) \frac{D_j}{\sqrt{nK}} \right) - \left( R_{k,j}^{(n)} + k \frac{D_j}{\sqrt{nK}} \right) \right| \leq \left| R_{m+k,j}^{(n)} - R_{k,j}^{(n)} \right| + \left| m \frac{D_j}{\sqrt{nK}} \right|, \]

it is enough to show

\[ \lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup \left( \max_{1 \leq m \leq [n\delta] + 1} \left| R_{m+k,j}^{(n)} - R_{k,j}^{(n)} \right| > \frac{\varepsilon \sqrt{n}}{2\kappa \sqrt{K}} \right) = 0, \]

and

\[ \lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup \left( \max_{1 \leq m \leq [n\delta] + 1} \left| \frac{m}{\sqrt{n}} D_j \right| > \frac{\varepsilon \sqrt{n}}{2\kappa \sqrt{K}} \right) = 0. \]

The first one is shown in [10, Lemma 2.4.19]. Also it holds that

\[ \lim_{n \to \infty} \sup \left( \max_{1 \leq m \leq [n\delta] + 1} \left| \frac{m}{\sqrt{n}} D_j \right| > \frac{\varepsilon \sqrt{n}}{2\kappa \sqrt{K}} \right) = 0, \]

and this shows the second one. \( \square \)

**Lemma 4.17.** Let \( a, b > 0, i \in \mathbb{C} \). If \( \mu \) is invariant under \( T_i \), then \( \mu_{a,b} \) is also invariant under \( T_i \).

**Proof.** Let \( S_x^{a,b} = a S_{bx} \) for \( a, b > 0 \) and \( x \in \mathbb{R} \). By using the expression (4.5), it holds that

\[ T_i S_x^{a,b} = a S_{bx} + \left( A_i a S_{bx} - \sup_{y \leq x} A_i a S_{by} + \sup_{y \leq 0} A_i a S_{by} \right) \left( e_i - e_0 \right) = (T_i S_x)^{a,b}, \]

and the claim follows. \( \square \)

**Lemma 4.18.** Suppose \( \{ \mu_n \} \) is a sequence of probability measures on \( S_x^0 \), each of which is invariant under \( T_i \), and \( \mu_n \) converges weakly to \( \mu \). Moreover, suppose that \( \mu_n \) satisfies for any \( z \in \mathbb{R} \),

\[ \lim_{x \to -\infty} \lim_{n \to \infty} \mu_n \left( \sup_{y \leq x} A_i S_y > A_i S_z \right) = 0. \]
and $\mu$ satisfies for any $z \in \mathbb{R}$,
\[
\lim_{x \to -\infty} \mu \left( \sup_{y \leq x} A_i S_y > A_i S_z \right) = 0.
\]

It then holds that $\mu$ is also invariant under $T_1$.

**Proof.** It is enough to show that for any $L > 0$ and continuous bounded function $f : C([-L, L], \mathbb{R}^\infty) \to \mathbb{R}$,
\[
\mu \left( f \left( S \big|_{[-L, L]} \right) \right) = \mu \left( f \left( T_i S \big|_{[-L, L]} \right) \right).
\]

Let
\[
M_x^{L'} := \begin{cases} 
A_i S_{-L'}, 
& \text{if } x < -L', \\
\sup_{-L' \leq y \leq x} A_i S_y, 
& \text{if } -L' \leq x \leq L', \\
\sup_{-L' \leq y \leq L} A_i S_y, 
& \text{otherwise.}
\end{cases}
\]

Also, define
\[
(T_i^{L'})_x := S_x + \left( A_i S_x - M_x^{L'} + M_0^{L'} \right) \left( e_i - e_0 \right), \quad x \in \mathbb{R}.
\]

Then, $T_i^{L'} : S^0_c \to S^0_c$ is continuous, and so
\[
\lim_{n \to \infty} \mu_n \left( f \left( (T_i^{L'}) \big|_{[-L, L]} \right) \right) = \mu \left( f \left( (T_i^{L'}) \big|_{[-L, L]} \right) \right),
\]
for any $L, L'$.

It is easy to verify that $(T_i^{L'}) \big|_{[-L, L]} = (T_i S) \big|_{[-L, L]}$ if $L < L'$ and $\sup_{y \leq -L'} A_i S_y \leq A_i S_{-L}$, by comparing (4.5) and (4.11). Therefore, for any $L' > L$,
\[
\left| \mu_n \left( f \left( (T_i^{L'}) \big|_{[-L, L]} \right) \right) - \mu_n \left( f \left( (T_i S) \big|_{[-L, L]} \right) \right) \right| \leq 2 \| f \|_\infty \mu_n \left( \sup_{y \leq -L'} A_i S_y > A_i S_{-L} \right).
\]

Hence, by assumption, we have that
\[
\lim_{L' \to \infty} \lim_{n \to \infty} \sup_{L' \leq L} \left| \mu_n \left( f \left( (T_i^{L'}) \big|_{[-L, L]} \right) \right) - \mu_n \left( f \left( (T_i S) \big|_{[-L, L]} \right) \right) \right| = 0,
\]
which implies, with (4.12),
\[
\lim_{n \to \infty} \sup \mu_n \left( f \left( (T_i S) \big|_{[-L, L]} \right) \right) = \lim_{L' \to \infty} \mu \left( f \left( (T_i^{L'}) \big|_{[-L, L]} \right) \right).
\]

Similarly it holds that
\[
\left| \mu \left( f \left( (T_i^{L'}) \big|_{[-L, L]} \right) \right) - \mu \left( f \left( (T_i S) \big|_{[-L, L]} \right) \right) \right| \leq 2 \| f \|_\infty \mu \left( \sup_{y \leq -L'} A_i S_y > A_i S_{-L} \right),
\]
and the assumption $\lim_{x \to -\infty} \mu \left( \sup_{y \leq x} A_i S_y > A_i S_{-L} \right) = 0$ for any $x$ implies that
\[
\lim_{L' \to \infty} \mu \left( f \left( (T_i^{L'}) \big|_{[-L, L]} \right) \right) = \mu \left( f \left( (T_i S) \big|_{[-L, L]} \right) \right).
\]

This is the right-hand side of (4.13). Also the left-hand side of (4.13) is equal to
\[
\lim_{n \to \infty} \sup \mu_n \left( f \left( S \big|_{[-L, L]} \right) \right) = \mu \left( f \left( S \big|_{[-L, L]} \right) \right),
\]
then the claim is proved. \qed
Finally, we check the assumptions of the previous result for \( \nu_n = \frac{\mu^{(n)}_{A \| x}}{\sqrt{n}} \) and \( \nu_D \).

**Lemma 4.19.** For any \( z \in \mathbb{R} \),

\[
\lim_{x \to -\infty} \limsup_{n \to \infty} \nu_n \left( \sup_{y \leq x} A_i S_y > A_i S_z \right) = 0
\]

and

\[
\lim_{x \to -\infty} \nu_D \left( \sup_{y \leq x} A_i S_y > A_i S_z \right) = 0.
\]

**Proof.** For \( S_x = B_x + (c_0 e_0 + \cdots + c_\kappa e_\kappa) x \) it holds that

\[
A_i S_x = -2 \frac{(e_i - e_0) \cdot S_x}{|e_i - e_0|^2}
\]

\[
= -2 \frac{(e_i - e_0) \cdot B_x}{|e_i - e_0|^2} + (c_0 - c_i) x
\]

\[
\to -\infty
\]

almost surely as \( x \to -\infty \). Therefore, the second claim of the lemma is obvious. To estimate the probability \( \nu_n \left( \sup_{y \leq x} A_i S_y > A_i S_z \right) \), first note that, for any \( x < z \),

\[
\nu_n \left( \sup_{y \leq x} A_i S_y > A_i S_z \right) = \mu^{(n)} \left( \sup_{y \leq nx} A_i S_y > A_i S_z \right)
\]

\[
\leq \mu^{(n)} \left( \sup_{y \leq \lfloor nx \rfloor + 1} A_i S_y > \min \{ A_i S_{\lfloor nz \rfloor}, A_i S_{\lfloor nz \rfloor + 1} \} \right)
\]

\[
= \mu^{(n)} \left( \sup_{y \leq \lfloor nz \rfloor + 1 - \lfloor nz \rfloor} A_i S_y > \min \{ A_i S_0, A_i S_1 \} \right)
\]

\[
\leq \mu^{(n)} \left( \sup_{y \leq \lfloor nz \rfloor + 1 - \lfloor nz \rfloor} A_i S_y \geq 0 \right),
\]

where \( \lfloor w \rfloor \) is the maximum integer not greater than \( w \). Thus we only need to show that

\[
\lim_{x \to -\infty} \limsup_{n \to \infty} \mu^{(n)} \left( \sup_{y \leq \lfloor nz \rfloor} A_i S_y \geq 0 \right) = 0.
\]

For any \( \ell \geq 1 \), we have

\[
\mu^{(n)} \left( \sup_{y \leq -\ell} A_i S_y \geq 0 \right) \leq \mu^{(n)} \left( A_i S_{-\ell} \geq 0 \right) + \sum_{k \leq -1} \mu^{(n)} \left( A_i S_{-\ell} = k \right) \left( \frac{p_1}{p_0} \right)^{-k}.
\]

Now, since \( A_i S_{-\ell} \stackrel{d}{=} -A_i S_\ell = -\sum_{k=1}^\ell (1_{\{y_k=0\}} - 1_{\{y_k=1\}}) \), we have

\[
\mu^{(n)} \left( A_i S_{-\ell} \geq 0 \right) = \mu^{(n)} \left( \sum_{k=1}^\ell (1_{\{y_k=0\}} - 1_{\{y_k=1\}}) \leq 0 \right)
\]
\[
\begin{align*}
&\leq \mu^{p(n)} \left( \frac{1}{\ell} \sum_{k=1}^{\ell} \left( 1_{\{y_k=0\}} - 1_{\{y_k=1\}} \right) - \frac{\ell (c_0 - c_i)}{\sqrt{n\kappa}} \right) \\
&\leq \mu^{p(n)} \left( \frac{1}{\ell} \sum_{k=1}^{\ell} \left( 1_{\{y_k=0\}} - 1_{\{y_k=1\}} - \frac{c_0 - c_i}{\sqrt{n\kappa}} \right) \right) \\
&\leq \frac{n\kappa}{\ell(c_0 - c_i)^2} E^{p(n)} \left( \left( 1_{\{y_k=0\}} - 1_{\{y_k=1\}} - \frac{c_0 - c_i}{\sqrt{n\kappa}} \right)^2 \right) \\
&= \frac{n\kappa}{\ell(c_0 - c_i)^2} V^{p(n)} \left( 1_{\{y_k=0\}} - 1_{\{y_k=1\}} \right) \\
&= \frac{n\kappa}{\ell(c_0 - c_i)^2} \left( \frac{2}{\kappa + 1} + \frac{c_0 + c_i}{\sqrt{n\kappa}} \right),
\end{align*}
\]
where \( E^{p(n)} \) is the expectation and \( V^{p(n)} \) is the variance with respect to \( \mu^{p(n)} \).

Moreover,
\[
\begin{align*}
&\sum_{k=1}^{\ell} \mu^{p(n)} \left( A_i S_{-\ell} = k \right) \left( \frac{p_i}{p_0} \right)^{-k} \\
&= \sum_{-\ell \leq k \leq -1} \mu^{p(n)} \left( A_i S_{-\ell} = k \right) \left( \frac{p_i}{p_0} \right)^{-k} \\
&= \sum_{-\ell \leq k \leq -1, 0 \leq j \leq \ell + \kappa} \binom{\ell}{j} \left( 1 - p_0 - p_i \right)^j \left( 1 - p_0 - p_i \right)^{\ell - j} \left( \frac{p_i}{p_0} \right)^{-k} \\
&= \sum_{-\ell \leq k \leq -1, 0 \leq j \leq \ell + \kappa} \binom{\ell}{j} \left( 1 - p_0 - p_i \right)^j \left( \frac{p_i}{p_0} \right)^{\ell - j} \\
&= \sum_{-\ell \leq k \leq -1} \mu^{p(n)} \left( A_i S_{-\ell} = -k \right) \\
&= \mu^{p(n)} \left( A_i S_{-\ell} \geq 1 \right) \\
&\leq \mu^{p(n)} \left( A_i S_{-\ell} \geq 0 \right),
\end{align*}
\]
where \( \binom{\ell}{j} \equiv 0 \) for \( q \notin \mathbb{N} \). Therefore, we have

\[
\lim_{x \to -\infty} \limsup_{n \to \infty} \mu^{p(n)} \left( \sup_{y \leq [nx]} A_i S_y \geq 0 \right) \leq \lim_{x \to -\infty} \limsup_{n \to \infty} \frac{2n\kappa}{n\kappa (c_0 - c_i)^2} \left( \frac{2}{\kappa + 1} + \frac{c_0 + c_i}{\sqrt{n\kappa}} \right) \\
= \lim_{x \to -\infty} \frac{2\kappa}{x (c_0 - c_i)^2} \frac{2}{\kappa + 1} = 0.
\]

\[ \square \]

**Proof of Theorem 4.8** Since \( \mu^{p(n)} \) is invariant under \( T \), so is \( \nu_n = \mu^{p(n)} \) by Lemma 4.11. Then Theorem 4.10, Lemma 4.18 and 4.19 show \( \nu_D \) is also invariant under \( T \). In other words, two-sided standard \( \kappa \)-dimensional Brownian motion with drift \( D \in \mathcal{D} \), given by

\[
D = c_0 e_0 + \cdots + c_\kappa e_\kappa, \quad c_0 > c_i, \quad \forall i \in \mathcal{C}, \quad c_0 + \cdots + c_\kappa = 0
\]
is invariant under $T_i$. 

Acknowledgements

The author appreciates M. Sasada for her guidance and constructive comments from beginning to end. He also thanks D. Croydon for useful advice.

References

[1] A. Kuniba and H. Lyu, *Large Deviations and One-Sided Scaling Limit of Randomized Multicolor Box-Ball System*, J Stat Phys 178, 38-74 (2020).

[2] Atsuo Kuniba, Hanbaek Lyu, and Masato Okado, *Randomized box-ball systems, limit shape of rigged configurations and thermodynamic Bethe ansatz*, Nuclear Physics B 937 (2018), 240-271.

[3] Daiju Takahashi, *On some soliton systems defined by using boxes and balls*, 1993 International Symposium on Nonlinear Theory and Its Applications, (Hawaii; 1993), 1993, pp. 555-558.

[4] D. Croydon, T. Kato, M. Sasada, and S. Tsujimoto, *Dynamics of the box-ball system with random initial conditions via Pitman’s transformation*, preprint appears at arXiv:1806.02147, 2018.

[5] D.J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag. (5) 39 (1895), no. 240, 422–443.

[6] D. Takahashi, *On Some soliton systems defined by using boxes and balls*, in 1993 International Symposium on Nonlinear Theory and Its Applications, (Hawaii; 1993), pages 555-558, 1993.

[7] D. Takahashi and J. Satsuma, *A soliton cellular automaton*. J. Phys. Soc. Japan, 59(10):3514-3519, 1990.

[8] Joel Lewis, Hanbaek Lyu, Pavlo Pylyavskyy, Arnab Sen, *Scaling limit of soliton lengths in a multicolor box-ball system*, preprint appears at arXiv:1911.04458 2019.

[9] J.W. Pitman, *One-dimensional Brownian motion and the three-dimensional Bessel process*, Advances in Appl. Probability 7 (1975), no. 3, 511–526.

[10] K. Ioannis and S. Steven, *Brownian Motion and Stochastic Calculus*, Graduate Texts in Mathematics.

[11] P. A. Ferrari, C. Nguyen, L. Rolla, and M. Wang, *Soliton decomposition of the box-ball system*, preprint appears at arXiv:1806.02798, 2018.

[12] P. Biane, P. Bougerol and N. O’Connell, *Littlemann paths and Brownian paths*, Duke Mathematical Journal, Volume 130, Number 1 (2005), 127-167.

[13] Rei Inoue, Atsuo Kuniba, and Taichiro Takagi, *Integrable structure of box-ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry*, Journal of Physics A: Mathematical and Theoretical 45 (2012), no. 7, 073001.

[14] T. Tokihiro, *Hakodamakei no suri*, Asakurasyoten, 2012.

[15] T. Tokihiro, D. Takahashi, J. Matsukidaira, and J. Satsuma, *From soliton equations to integrable cellular automata through a limiting procedure*, Phys. Rev. Lett. 76 (1996), no. 18, 3247–3250.