EXPANSION OF ITERATED STOCHASTIC INTEGRALS WITH RESPECT TO MARTINGALE POISSON MEASURES AND WITH RESPECT TO MARTINGALES BASED ON GENERALIZED MULTIPLE FOURIER SERIES

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ABSTRACT. We consider some versions and generalizations of the approach to the expansion of iterated Itô stochastic integrals of arbitrary multiplicity \( k \ (k \in \mathbb{N}) \) based on generalized multiple Fourier series. Expansions of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales were obtained. For the iterated stochastic integrals with respect to martingales, we have proved a theorem which gives a generalization of the expansion for iterated Itô stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series. Also we consider a modification of the mentioned expansion of iterated Itô stochastic integrals for the case of complete orthonormal systems of functions in the space \( L_2([t,T]) \). Mean-square convergence of the considered expansions is proved. An example of the expansion of iterated (double) stochastic integrals with respect to martingales using the system of Bessel functions is considered.

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References

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, let $\{\mathcal{F}_t, t \in [0,T]\}$ be a non-decreasing right-continuous family of $\sigma$-algebras of $\mathcal{F}$, and let $\mathbf{f}_t$ be a standard $m$-dimensional Wiener stochastic process which is $\mathcal{F}_t$-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{f}_t(i)$ ($i = 1, \ldots, m$) of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

$$x_t = x_0 + \int_0^t a(x_\tau, \tau)d\tau + \int_0^t B(x_\tau, \tau)d\mathbf{f}_\tau, \quad x_0 = x(0,\omega).$$

Here $x_t$ is some $n$-dimensional stochastic process satisfying the equation (1). The non-random functions $a : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let $x_0$ be an $n$-dimensional random variable which is $\mathcal{F}_0$-measurable and $\mathbb{E} \{ |x_0|^2 \} < \infty$ ($\mathbb{E}$ denotes a mathematical expectation). We assume that $x_0$ and $\mathbf{f}_t - \mathbf{f}_0$ are independent when $t > 0$.

It is well known [2]-[5] that Ito SDEs are adequate mathematical models of dynamic systems under the influence of random disturbances. One of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[17]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

$$(2) \quad J[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \ldots \int_t^{t_1} \psi_1(t_1)d\mathbf{w}^{(i_1)}_{t_{i_1}} \ldots d\mathbf{w}^{(i_k)}_{t_{i_k}},$$

$$(3) \quad J^*[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \ldots \int_t^{t_1} \psi_1(t_1)d\mathbf{w}^{(i_1)}_{t_{i_1}} \ldots d\mathbf{w}^{(i_k)}_{t_{i_k}},$$

where every $\psi_l(\tau)$ ($l = 1, \ldots, k$) is a non-random function on $[t,T]$, $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$. 

denote Ito and Stratonovich stochastic integrals, respectively (in \(4\), we use the definition of the Stratonovich stochastic integral from \([2]\)).

Note that \(\psi_l(\tau) \equiv 1\) (\(l = 1, \ldots, k\)) and \(i_1, \ldots, i_k = 0, 1, \ldots, m\) in \([2]-[7]\). At the same time \(\psi_l(\tau) \equiv (t-\tau)^{q_l}\) (\(l = 1, \ldots, k\), \(q_1, \ldots, q_k = 0, 1, 2, \ldots\)) and \(i_1, \ldots, i_k = 1, \ldots, m\) in \([8]-[17]\).

The problem of effective jointly numerical modeling (with respect to the mean-square convergence criterion) of iterated Ito and Stratonovich stochastic integrals \([2]\) and \([3]\) is difficult from theoretical and computing point of view \([2]-[14], [23]-[25], [31], [37], [38], [42], [47]-[49], [51], [54], [57], [58]\).

The only exception is connected with a narrow particular case, when \(i_1 = \ldots = i_k \neq 0\) and \(\psi_1(\tau), \ldots, \psi_k(\tau) \equiv \psi(\tau)\). This case can be investigated using the Ito formula \([2]-[4]\).

Note that even for the mentioned coincidence \((i_1 = \ldots = i_k \neq 0)\) but for different functions \(\psi_1(\tau), \ldots, \psi_k(\tau)\) the mentioned difficulties persist. As a result, relatively simple families of iterated Ito and Stratonovich stochastic integrals, which can be often met in the applications, can not be represented effectively in a finite form (with respect to the mean-square criterion of approximation) using the system of standard Gaussian random variables.

Usually, approaches to the expansion of iterated stochastic integrals \([2]\) and \([3]\) are based on the expansion of the Wiener process.

For example, in \([3]\) (also see \([2], [4]\)) Milstein G.N. proposed to expand \([2]\) or \([3]\) (the case \(k = 2\) and \(i_1 \neq i_2; i_1, i_2 = 1, \ldots, m\)) into the iterated series of products of standard Gaussian random variables by representing the Brownian bridge process as a trigonometric Fourier series with random coefficients (the version of the so-called Karhunen–Loeve expansion). To obtain the Milstein expansion of \([2]\) or \([3]\), the truncated Fourier expansions of components of the Wiener process \(f_i\) must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. The above procedure leads to iterative application of the operation of limit transition and does not lead to a general expansion of \([2]\) or \([3]\) which is valid for an arbitrary multiplicity \(k\). For this reason, only expansions of single, double, and triple stochastic integrals were presented in \([2]\) (\(k = 1, 2, 3\)) and in \([3], [4]\) (\(k = 1, 2\)) for the simplest case \(\psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \equiv 1; i_1, i_2, i_3 = 0, 1, \ldots, m\). Moreover, generally speaking, the convergence of approximations to the appropriate stochastic integrals \([3]\) is not proved rigorously for \(k = 3\) in \([2]\) (Sect. 5.8, pp. 202–204), \([5]\) (pp. 82–84), \([62]\) (pp. 438–439), \([63]\) (pp. 263–264) (see \([15]-[18]\) (Sect. 6.2), \([43], [45]-[48]\) for details).

Note that in \([60], [61]\) a method for the expansion of double \([60]\) and triple \([60]\) Ito stochastic integrals \([2]\) (\(k = 2, 3; \psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \equiv 1; i_1, i_2, i_3 = 0, 1, \ldots, m\)) based on the expansion of the Wiener process using Haar functions \([61]\) and trigonometric functions \([60], [61]\) has been considered. The restrictions of this method \([60], [61]\) are also connected with the iterated application of the operation of limit transition at least starting from the second or third multiplicity of iterated stochastic integrals.

A more effective and general approach to the expansion of iterated Ito stochastic integrals \([2]\) of arbitrary multiplicity \(k (k \in \mathbb{N})\) based on generalized multiple Fourier series (converging in the sense of norm in Hilbert space \(L_2([t, T]^k)\)) was proposed and developed by the author of this paper in \([10]\) (2006) (also see \([11], [36], [41], [50], [52], [55]\)). Hereinafter, this method is referred to as the method of generalized multiple Fourier series. As it turned out, the method of generalized multiple Fourier series can be adapted for the iterated Stratonovich stochastic integrals \([3]\) at least for the multiplicities 1 to 6 \([11], [15], [23], [25], [31], [37], [38], [42], [47]-[49], [51], [53], [57], [58]\). Expansions of these iterated Stratonovich stochastic integrals turned out simpler than the appropriate expansions for the iterated Ito stochastic integrals \([2]\).

The problem of iterative application of the operation of limit transition (see above) not appears in the method of generalized multiple Fourier series \([10], [36], [41], [50], [52], [55]\). The idea of this method is as follows: the iterated Ito stochastic integral \([2]\) of multiplicity \(k\) is represented as the multiple...
stochastic integral from the certain discontinuous non-random function of \( k \) variables defined on the hypercube \([t, T]^k\), where \([t, T]\) is the interval of integration of the iterated Ito stochastic integral (2). Then, the indicated non-random function is expanded in the hypercube into the generalized multiple Fourier series converging in the mean-square sense in the space \( L^2([t, T]^k) \). After a number of nontrivial transformations we come (see Theorems 1, 2 below) to the mean-square converging expansion of the iterated Ito stochastic integral (2) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of the generalized multiple Fourier series for the mentioned non-random function of \( k \) variables which can be calculated using the explicit formula regardless of the multiplicity \( k \) of the iterated Ito stochastic integral (2).

Recall that this method is referred to as the method of generalized multiple Fourier series.

Thus, we obtain the following new and useful possibilities of the method of generalized multiple Fourier series.

1. There is the explicit formula (see (3)) for calculation of expansion coefficients of the iterated Ito stochastic integral (2) with any fixed multiplicity \( k \).

2. We have new possibilities for exact calculation of the mean-square approximation error of the iterated Ito stochastic integral (2) of arbitrary multiplicity \( k \) [12]-[18], [26], [44].

3. Since the used multiple Fourier series is a generalized in the sense that it is constructed using various complete orthonormal systems of functions in the space \( L^2([t, T]) \), then we have new possibilities for approximation — we can use not only the trigonometric functions as in [2]-[4] but the Legendre polynomials.

4. As it turned out [10]-[36], [41]-[50], [52]-[55] it is more convenient to work with the Legendre polynomials for constructing the approximations of the iterated Ito stochastic integrals (2). Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions. Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in [15]-[18], [30], [41].

5. The approach based on the Karhunen–Loève expansion of the Brownian bridge process as well as the approach from [60], [61] lead to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorems 1, 2 (see below)) starting from the second multiplicity (in the general case) and third multiplicity (for the case \( \psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \equiv 1; i_1, i_2, i_3 = 1, \ldots, m \) of iterated Ito stochastic integrals. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as \( p_1, \ldots, p_k \)). For example, when \( p_1 = \ldots = p_k = p \rightarrow \infty \). For iterated series, the condition \( p_1 = \ldots = p_k = p \rightarrow \infty \) obviously does not guarantee the convergence of this series.

   However, in [2] (Sect. 5.8, pp. 202–204), [3] (pp. 82–84), [62] (pp. 438–439), [63] (pp. 263–264) the authors use (without rigorous proof) the condition \( p_1 = p_2 = p_3 = p \rightarrow \infty \) within the frames of the mentioned approach based on the Karhunen–Loève expansion of the Brownian bridge process (3) together with the Wong–Zakai approximation [64]-[66] (see [15]-[18] (Sect. 6.2), [43], [45]-[48] for details).

   The method of generalized multiple Fourier series allows some generalizations and modifications in several directions.

   Recently, the method of generalized multiple Fourier series (see Theorems 1, 2 below) was applied to the expansion and mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional \( Q \)-Wiener process [15]-[18] (Chapter 7), [32]-[35]. These results can be directly applied to the construction of high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations with non-linear multiplicative trace class noise [15]-[18] (Chapter 7), [32]-[35].

   In this article, we demonstrate that the method of generalized multiple Fourier series is essentially general and allows some transformations for other types of iterated stochastic integrals. We will
consider versions of the method of generalized multiple Fourier series for iterated stochastic integrals with respect to martingale Poisson measures and for iterated stochastic integrals with respect to martingales. The mentioned results are sufficiently natural according to general properties of martingales.

In Sect. 2, we formulate Theorem 1 on expansion of the iterated Itô stochastic integrals \((2)\) of arbitrary multiplicity \(k\) based on generalized multiple Fourier series (method of generalized multiple Fourier series) [10]-[36], [41]-[50], [52]-[55]. Sect. 3 is devoted to a generalization of Theorem 1 for the case of an arbitrary complete orthonormal system of functions in the space \(L_2([t,T])\) and \(\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])\). In Sect. 4, we define the stochastic integral with respect to the martingale Poisson measure and consider some properties of this integral. Sect. 5 is devoted to a version of Theorem 1 for the iterated stochastic integrals with respect to martingale Poisson measures. In Sect. 6, we consider a generalization of Theorem 1 for the case of iterated stochastic integrals with respect to martingales. Sect. 7 is devoted to versions of Theorem 1, 2 for the case of complete orthonormal with weight \(r(t_1) \ldots r(t_k) \geq 0\) systems of functions in the space \(L_2([t,T]^k)\).

In Sect. 8, we consider one modification of theorems from Sect. 6 and 7. Sect. 9 is devoted to an example of the application of results from Sect. 8.

We will say that the function \(f(x) : [t,T] \to \mathbb{R}^1\) satisfies the condition \((\ast)\), if it is continuous at the interval \([t,T]\) except may be for the finite number of points of the finite discontinuity as well as it is right-continuous at the interval \([t,T]\).

Let us suppose that \(\{\phi_j(x)\}_{j=0}^{\infty}\) is a complete orthonormal system of functions in the space \(L_2([t,T])\), each function \(\phi_j(x)\) of which for finite \(j\) satisfies the condition \((\ast)\). It is clear that complete orthonormal systems \(\{\phi_j(x)\}_{j=0}^{\infty}\) of continuous functions in the space \(L_2([t,T])\) satisfy the condition \((\ast)\).

Let us consider some examples of systems satisfying the condition \((\ast)\).

**Example 1.** The system of Legendre polynomials

\[
\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j \left( \left( \frac{x - \frac{T+t}{2}}{T-t} \right) \frac{2}{T-t} \right), \quad j = 0, 1, 2, \ldots, \quad x \in [t,T],
\]

where \(P_j(y), y \in [-1,1]\) is the Legendre polynomial

\[
P_j(y) = \frac{1}{2^j j!} \frac{d^j}{dy^j} (y^2 - 1)^j.
\]

**Example 2.** The system of trigonometric functions

\[
\phi_j(x) = \frac{1}{\sqrt{T-t}} \left\{ \begin{array}{ll}
1, & j = 0 \\
\sqrt{2} \sin \left( \frac{2\pi r(x-t)}{(T-t)} \right), & j = 2r - 1, \\
\sqrt{2} \cos \left( \frac{2\pi r(x-t)}{(T-t)} \right), & j = 2r
\end{array} \right.
\]

where \(x \in [t,T], \ r = 1, 2, \ldots\)

**Example 3.** The system of Haar functions

\[
\phi_0(x) = \frac{1}{\sqrt{T-t}}, \quad \phi_{nj}(x) = \frac{1}{\sqrt{T-t}} \varphi_{nj} \left( \frac{x-t}{T-t} \right), \quad x \in [t,T],
\]
where \( n = 0, 1, \ldots, j = 1, 2, \ldots, 2^n \), and the functions \( \varphi_{nj}(x) \) have the following form

\[
\varphi_{nj}(x) = \begin{cases} 
2^{n/2}, & x \in [(j - 1)/2^n, (j - 1)/2^n + 1/2^{n+1}) \\
-2^{n/2}, & x \in [(j - 1)/2^n + 1/2^{n+1}, j/2^n) \\
0, & \text{otherwise}
\end{cases}
\]

where \( n = 0, 1, \ldots, j = 1, 2, \ldots, 2^n \) (we choose the values of Haar functions in the points of discontinuity in order they will be right-continuous).

**Example 4.** The system of Rademacher–Walsh functions

\[
\phi_0(x) = \frac{1}{\sqrt{T - t}}, \\
\phi_{m_1 \ldots m_k}(x) = \frac{1}{\sqrt{T - t}} \varphi_{m_1}\left(\frac{x - t}{T - t}\right) \ldots \varphi_{m_k}\left(\frac{x - t}{T - t}\right), \quad x \in [t, T],
\]

where \( 0 < m_1 < \ldots < m_k, \ m_1, \ldots, m_k = 1, 2, \ldots, k = 1, 2, \ldots, \]

\[
\varphi_m(x) = (-1)^{\lfloor 2^m x \rfloor}, \quad x \in [0, 1], \ m = 1, 2, \ldots, \lfloor y \rfloor \text{ is an integer part of a real number } y.
\]

\[2. \text{ Method of Expansion of Iterated Ito Stochastic Integrals of Arbitrary Multiplicity Based on Generalized Multiple Fourier Series Converging in the Mean}\]

Suppose that every \( \psi_l(\tau) (l = 1, \ldots, k) \) is a non-random function from the space \( L_2([t, T]) \). Define the following function on the hypercube \([t, T]^k\)

\[
K(t_1, \ldots, t_k) = \begin{cases} 
\psi_1(t_1) \ldots \psi_k(t_k) & \text{for } t_1 < \ldots < t_k \\
0 & \text{otherwise}
\end{cases}, \quad t_1, \ldots, t_k \in [t, T], \quad k \geq 2
\]

and \( K(t_1) \equiv \psi_1(t_1) \) for \( t_1 \in [t, T] \).

Suppose that \( \{ \phi_j(x) \}_{j=0}^{\infty} \) is a complete orthonormal system of functions in the space \( L_2([t, T]) \). The function \( K(t_1, \ldots, t_k) \) belongs to the space \( L_2([t, T]^k) \). At this situation it is well known that the generalized multiple Fourier series of \( K(t_1, \ldots, t_k) \in L_2([t, T]^k) \) is converging to \( K(t_1, \ldots, t_k) \) in the hypercube \([t, T]^k\) in the mean-square sense, i.e.
\[
\lim_{p_1,\ldots,p_k \to \infty} \left\| K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right\|_{L_2([t,T]^k)} = 0,
\]

where

\begin{equation}
(6) \quad C_{j_k \ldots j_1} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k,
\end{equation}

\begin{equation}
(7) \quad t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta \tau_j \to 0 \quad \text{if} \quad N \to \infty, \quad \Delta \tau_j = \tau_{j+1} - \tau_j.
\end{equation}

Consider the partition \( \{\tau_j\}_{j=0}^{N} \) of \([t, T]\) such that

\begin{equation}
(8) \quad J[\psi^{(k)}]_{T,t} = \lim_{p_1,\ldots,p_k \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \phi_{j_l}(\tau_{l-1}) \Delta w_{\tau_{l-1}}^{(i_l)} \phi_{j_l}(\tau_{l-1}) \Delta w_{\tau_{l-1}}^{(i_l)} \right),
\end{equation}

where \( J[\psi^{(k)}]_{T,t} \) is defined by \((2)\),

\[ G_k = H_k \setminus L_k, \quad H_k = \{(l_1, \ldots, l_k): l_1, \ldots, l_k = 0, 1, \ldots, N-1\}, \]

\[ L_k = \{(l_1, \ldots, l_k): l_1, \ldots, l_k = 0, 1, \ldots, N-1; l_g \neq l_r (g \neq r); g, r = 1, \ldots, k\}, \]

\( \text{l.i.m. is a limit in the mean-square sense, } i_1, \ldots, i_k = 0, 1, \ldots, m, \)

\begin{equation}
(9) \quad \zeta_j^{(i)} = \int_t^T \phi_j(\tau) dW_t^{(i)}
\end{equation}

are independent standard Gaussian random variables for various \( i \) or \( j \) (if \( i \neq 0 \)), \( C_{j_k \ldots j_1} \) is the Fourier coefficient \((3)\), \( \Delta w_{\tau_{l-1}}^{(i_l)} = w_{\tau_{l-1}}^{(i_l)} - w_{\tau_{l-1}}^{(i_l)} \) \((i = 0, 1, \ldots, m\)), \( \{\tau_j\}_{j=0}^{N} \) is a partition of the interval \([t, T]\) which satisfies the condition \((7)\).

Let us consider transformed particular cases of Theorem 1 for \( k = 1, \ldots, 5 \) \([10, 36], [11, 50], [52, 55]\) (the cases \( k = 6 \) and \( 7 \) can be found in \([11, 17], [43]\)
(10) \[ J[\psi^{(1)}]_{T,t} = \text{l.i.m.} \sum_{j_1=0}^{p_1} C_{j_1} \psi^{(i_1)}_{j_1}, \]

(11) \[ J[\psi^{(2)}]_{T,t} = \text{l.i.m.} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2} \left( \psi^{(i_1)}_{j_1} \psi^{(i_2)}_{j_2} - 1_{\{i_1=i_2\neq 0\}} \psi^{(i_1)}_{j_1} \right), \]

(12) \[ J[\psi^{(3)}]_{T,t} = \text{l.i.m.} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3} \left( \psi^{(i_1)}_{j_1} \psi^{(i_2)}_{j_2} \psi^{(i_3)}_{j_3} - \right. \]

(13) \[ J[\psi^{(4)}]_{T,t} = \text{l.i.m.} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} \sum_{j_4=0}^{p_4} \sum_{j_5=0}^{p_5} C_{j_5} \left( \prod_{l=1}^{5} \psi^{(i_l)}_{j_l} - \right. \]
iterated Ito stochastic integrals (2). In order to do this, let us consider the unordered set

\[ \{ \{ i_1, i_2 \} \} \text{ (theses mean an ordered set).} \]

where \( A \) consists of the unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining pairs. \( A \) is the indicator of the set \( A \).

The convergence in the mean of degree \( 2n \) \((n \in \mathbb{N})\) is proved for approximations from Theorem 1 in \([11, 25, 43]\). In \([15, 17, 43, 45]\), the convergence with probability 1 (further w. p. 1) is proved for expansions of iterated Ito stochastic integrals of arbitrary multiplicity \( k \) \((k \in \mathbb{N})\) from Theorem 1 for the cases of Legendre polynomials and trigonometric functions.

As follows from Theorem 1, the expansion (13) is valid for approximations from Theorem 1 \([15, 17, 43, 45]\) satisfying the condition (\( \ast \)). For example, Theorem 1 is valid for the system of Haar functions as well as for the system of Rademacher–Walsh functions \([10, 25, 43]\).

3. Generalization of Theorem 1 to the Case of an Arbitrary Complete Orthonormal System of Functions in the Space \( L_2([t, T]) \) and \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T]) \)

Consider a generalization of formulas (10)–(14) for the case of arbitrary multiplicity \( k \) of the iterated Ito stochastic integrals \([2, 4]\). In order to do this, let us consider the unordered set \( \{ 1, 2, \ldots, k \} \) and separate it into two parts: the first part consists of \( r \) unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining \( k - 2r \) numbers. So, we have

\[
\sum_{\{ (\{ g_1, g_2 \}, \ldots, \{ g_{2r-1}, g_{2r} \}, \{ q_1, \ldots, q_{k-2r} \}) \} \in \{ 1, 2, \ldots, k \} \}} \left( (1) + 1_{\{ i_2 = i_3 \neq 0 \}} 1_{\{ j_2 = j_3 \}} 1_{\{ i_4 = i_5 \neq 0 \}} 1_{\{ j_4 = j_5 \}} \right) \right) + \\
1_{\{ i_2 = i_3 \neq 0 \}} 1_{\{ j_2 = j_3 \}} 1_{\{ i_4 = i_5 \neq 0 \}} 1_{\{ j_4 = j_5 \}} \right) \right),
\]

where \( 1_A \) is the indicator of the set \( A \).

The convergence in the mean of degree \( 2n \) \((n \in \mathbb{N})\) is proved for approximations from Theorem 1 in \([11, 25, 43]\). In \([15, 17, 43, 45]\), the convergence with probability 1 (further w. p. 1) is proved for expansions of iterated Ito stochastic integrals of arbitrary multiplicity \( k \) \((k \in \mathbb{N})\) from Theorem 1 for the cases of Legendre polynomials and trigonometric functions.

As follows from Theorem 1, the expansion (13) is valid for discontinuous complete orthonormal systems of functions in \( L_2([t, T]) \) satisfying the condition (\( \ast \)). For example, Theorem 1 is valid for the system of Haar functions as well as for the system of Rademacher–Walsh functions \([10, 25, 43]\).

Consider a generalization of formulas (10)–(14) for the case of arbitrary multiplicity \( k \) of the iterated Ito stochastic integrals \([2, 4]\). In order to do this, let us consider the unordered set \( \{ 1, 2, \ldots, k \} \) and separate it into two parts: the first part consists of \( r \) unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining \( k - 2r \) numbers. So, we have

\[
\sum_{\{ (\{ g_1, g_2 \}, \ldots, \{ g_{2r-1}, g_{2r} \}, \{ q_1, \ldots, q_{k-2r} \}) \} \in \{ 1, 2, \ldots, k \} \}} \left( (1) + 1_{\{ i_2 = i_3 \neq 0 \}} 1_{\{ j_2 = j_3 \}} 1_{\{ i_4 = i_5 \neq 0 \}} 1_{\{ j_4 = j_5 \}} \right) \right) + \\
1_{\{ i_2 = i_3 \neq 0 \}} 1_{\{ j_2 = j_3 \}} 1_{\{ i_4 = i_5 \neq 0 \}} 1_{\{ j_4 = j_5 \}} \right) \right),
\]

where \( \{ g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r} \} = \{ 1, 2, \ldots, k \} \), braces mean an unordered set, and parentheses mean an ordered set.

We will say that \( 157 \) is a partition and consider the sum with respect to all possible partitions

\[
\sum_{\{ (\{ g_1, g_2 \}, \ldots, \{ g_{2r-1}, g_{2r} \}, \{ q_1, \ldots, q_{k-2r} \}) \} \in \{ 1, 2, \ldots, k \} \}} a_{g_1 g_2 \ldots g_{2r-1} g_{2r} q_1 \ldots q_{k-2r}},
\]

where \( a_{g_1 g_2 \ldots g_{2r-1} g_{2r} q_1 \ldots q_{k-2r}} \in \mathbb{R} \).

Below there are several examples of sums in the form (16)

\[
\sum_{\{ (\{ g_1, g_2 \} \} (g_1, g_2) = (1, 2) \}} a_{g_1 g_2} = a_{12},
\]

\[
\sum_{\{ (\{ g_1, g_2 \}, \{ g_3, g_4 \}) (g_1, g_2, g_3, g_4) = (1, 2, 3, 4) \}} a_{g_1 g_2 g_3 g_4} = a_{12, 34} + a_{13, 24} + a_{23, 14},
\]

\[
\sum_{\{ (\{ g_1, g_2 \}, \{ q_1, q_2 \}) (g_1, g_2, q_1, q_2) = (1, 2, 3, 4) \}} a_{g_1 g_2 q_1 q_2} = a_{12, 34} + a_{13, 24} + a_{14, 23} + a_{23, 14} + a_{24, 13} + a_{34, 12},
\]
\[ a_{g_{12}, q_{12}, q_{31}} = a_{12, 345} + a_{13, 245} + a_{14, 235} + a_{15, 234} + a_{23, 145} + a_{24, 135} + a_{25, 134} + a_{34, 125} + a_{35, 124} + a_{45, 123}. \]

\[ a_{g_{12}, g_{34}, q_{11}} = a_{12, 345} + a_{13, 245} + a_{14, 235} + a_{15, 234} + a_{16, 234} + a_{23, 145} + a_{24, 135} + a_{25, 134} + a_{26, 134} + a_{27, 134} + a_{34, 125} + a_{35, 125} + a_{36, 125} + a_{37, 125} + a_{45, 123} + a_{46, 123} + a_{47, 123}. \]

Let us consider a generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space \( L_2([t, T]) \) and \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T]) \).

**Theorem 2** \cite{15} (Sect. 1.11), \cite{36}, \cite{13} (Sect. 15). Suppose that \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T]) \) and \( \{\phi_j(x)\}_{j=1}^\infty \) is an arbitrary complete orthonormal system of functions in the space \( L_2([t, T]) \). Then the following expansion

\[ J[\psi^{(k)}]_{T, t} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_1 \ldots j_k} \left( \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \right) + \sum_{r=1}^{[k/2]} (-1)^r \times \]

\[
\prod_{s=1}^{r} \left( i_{g_{2s-1}} \neq 0 \right) \left( j_{g_{2s-1}} = j_{g_{2s}} \right) \prod_{l=1}^{k-2r} \left( \sum_{q_{1l}} \zeta_{j_l}^{(i_l)} \right)
\]

converging in the mean-square sense is valid, where \([x]\) is an integer part of a real number \(x\), \( \prod_\emptyset \stackrel{\text{def}}{=} 1 \),

\[ \sum_\emptyset \stackrel{\text{def}}{=} 0; \] another notations are the same as in Theorem 1.

In particular from (17) for \( k = 5 \) we obtain

\[ J[\psi^{(5)}]_{T, t} = \lim_{p_1, \ldots, p_5 \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_5=0}^{p_5} C_{j_5 \ldots j_1} \left( \sum_{l=1}^{5} \zeta_{j_l}^{(i_l)} \right) - \sum_{\{(i_1, i_2), (i_1, i_2, q_3)\}} 1_{\left( i_1 = i_2 \neq 0 \right)} 1_{\left( j_{i_1} = j_{i_2} \neq j_{i_2} \right)} \prod_{l=1}^{3} \zeta_{j_l}^{(i_l)} + \]

\[ \sum_{\{(i_1, i_2), (i_3, i_4, q_1)\}} 1_{\left( i_1 = i_2 \neq 0 \right)} 1_{\left( j_{i_1} = j_{i_2} \neq j_{i_2} \right)} 1_{\left( i_3 = i_4 \neq 0 \right)} 1_{\left( j_{i_3} = j_{i_4} \right)} \zeta_{j_{i_1}}^{(i_l)} \]

The last equality obviously agrees with (14).

It should be noted that an analogue of Theorem 2 (the case \( i_1, \ldots, i_k = 1, \ldots, m \)) was considered in \cite{67} using the Hermite polynomials and Wick product. Note that we use another notations \cite{15}. 

EXPANSION OF ITERATED STOCHASTIC INTEGRALS

(Sect. 1.11), [36], [43] (Sect. 15) in comparison with [67]. Moreover, the proof from [67] is different from the proof given in [15] (Sect. 1.11), [36], [43] (Sect. 15). See Sect. 4 in [36] for details.

Below we demonstrate that an approach to the expansion of iterated Ito stochastic integrals considered in Theorems 1, 2 is essentially general and allows some transformations for other types of iterated stochastic integrals.

Note that Theorems 1, 2 allow to calculate exactly the mean-square approximation error of the iterated Ito stochastic integrals (2) of arbitrary multiplicity $k$ (see [13]–[18], [44]). In these papers we consider approximations of iterated Ito stochastic integrals as the expression on the right-hand side of (17) before passing to the limit with respect to $p_1,\ldots,p_k$.

4. Stochastic Integral with Respect to Martingale Poisson Measure

Let us consider the Poisson random measure in the space $[0,T] \times Y$ ($\mathbb{R}^n \overset{\text{def}}{=} Y$). We will denote the values of this measure at the set $\Delta \times A$ ($\Delta \subseteq [0,T], A \subseteq Y$) as $\nu(\Delta,A)$. Let us assume that

$$M \left\{ \nu(\Delta,A) \right\} = |\Delta|\Pi(A),$$

where $|\Delta|$ is the Lebesgue measure of $\Delta$, $\Pi(A)$ is a measure on $\sigma$-algebra $B$ of Borel sets of $Y$, and $B_0$ is a subalgebra of $B$ consisting of sets $A \subseteq B$ which satisfy the condition $\Pi(A) < \infty$.

Let us consider the martingale Poisson measure

$$\tilde{\nu}(\Delta,A) = \nu(\Delta,A) - |\Delta|\Pi(A).$$

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a fixed probability space, let $\{F_t, t \in [0,T]\}$ be a non-decreasing family of $\sigma$-algebras $F_t \subset \mathcal{F}$.

Assume that:

1. The random variables $\nu([0,t],A)$ are $F_t$-measurable for all $A \subseteq B_0$.
2. The random variables $\nu([t,t+h],A)$, $A \subseteq B_0$, $h > 0$ do not depend on $\sigma$-algebra $F_t$.

Let us define the class $H_2(\Pi,[0,T])$ of random functions $\varphi : [0,T] \times Y \times \Omega \to \mathbb{R}^1$, that are $F_t$-measurable for all $t \in [0,T], y \in Y$ and satisfy the following condition

$$\int_0^T \int_Y M \left\{ |\varphi(t,y)|^2 \right\} \Pi(dy)dt < \infty.$$

Let us consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[0,T]$ which satisfies the condition $\mathbb{U}$.

For $\varphi(t,y) \in H_2(\Pi,[0,T])$ let us define the stochastic integral with respect to the martingale Poisson measure as the following mean-square limit $\mathbb{I}$

\begin{equation}
\int_0^T \int_Y \varphi(t,y)\tilde{\nu}(dt,dy) \overset{\text{def}}{=} \lim_{N \to \infty} \int_0^T \int_Y \varphi^{(N)}(t,y)\tilde{\nu}(dt,dy),
\end{equation}

where $\varphi^{(N)}(t,y)$ is any sequence of step functions from the class $H_2(\Pi,[0,T])$ such that

$$\lim_{N \to \infty} \int_0^T \int_Y M \left\{ |\varphi(t,y) - \varphi^{(N)}(t,y)|^2 \right\} \Pi(dy)dt = 0.$$
It is well known [1] that the stochastic integral [13] exists, it does not depend on selection of the sequence \( \varphi^{(N)}(t, y) \) and it satisfies w. p. 1 the following properties

\[
M \left\{ \int_{0}^{T} \int_{Y} \varphi(t, y) \tilde{\nu}(dt, dy) \right\} = 0,
\]

\[
\int_{0}^{T} \int_{Y} (\alpha \varphi_{1}(t, y) + \beta \varphi_{2}(t, y)) \tilde{\nu}(dt, dy) = \alpha \int_{0}^{T} \int_{Y} \varphi_{1}(t, y) \tilde{\nu}(dt, dy) + \beta \int_{0}^{T} \int_{Y} \varphi_{2}(t, y) \tilde{\nu}(dt, dy),
\]

\[
M \left\{ \left| \int_{0}^{T} \int_{Y} \varphi(t, y) \tilde{\nu}(dt, dy) \right|^{2} \right\} = \int_{0}^{T} \int_{Y} M \left\{ |\varphi(t, y)|^{2} \right\} \Pi(dy) dt,
\]

where \( \alpha, \beta \) are some real constants and \( \varphi_{1}(t, y), \varphi_{2}(t, y), \varphi(t, y) \) from the class \( H_{2}(\Pi, [0, T]) \).

The stochastic integral

\[
\int_{0}^{T} \int_{Y} \varphi(t, y) \nu(dt, dy)
\]

with respect to the Poisson random measure will be defined as follows [1]

\[
\int_{0}^{T} \int_{Y} \varphi(t, y) \nu(dt, dy) = \int_{0}^{T} \int_{Y} \varphi(t, y) \tilde{\nu}(dt, dy) + \int_{0}^{T} \int_{Y} \varphi(t, y) \Pi(dy) dt,
\]

where we suppose that the right-hand side of the last equality exists.

According to the Ito formula for Ito processes with jump component, we obtain w. p. 1 [1]

\[
(z_{i})^{n} = \int_{0}^{t} \int_{Y} \left( (z_{\tau-} + \gamma(\tau, y))^{n} - (z_{\tau-})^{n} \right) \nu(d\tau, dy), \tag{19}
\]

where \( n \in \mathbb{N} \),

\[
z_{i} = \int_{0}^{t} \int_{Y} \gamma(\tau, y) \nu(d\tau, dy).
\]

We suppose that the function \( \gamma(\tau, y) \) satisfies the conditions of existence of the right-hand side of \( [19] [1] \).

Let us consider [1] the useful estimate for moments of the stochastic integral with respect to the Poisson random measure

\[
a_{n}(T) \leq \max_{j \in \{n, 1\}} \left\{ \left( \int_{0}^{T} \int_{Y} \left( \left( (b_{n}(\tau, y))^{1/n} + 1 \right)^{n} - 1 \right) \Pi(dy) d\tau \right)^{j} \right\}, \tag{20}
\]
where
\[ a_n(t) = \sup_{0 \leq \tau \leq t} M \left\{ |z_{\tau}|^n \right\}, \quad b_n(\tau, y) = M \left\{ |\gamma(\tau, y)|^n \right\}. \]

We suppose that the right-hand side of (20) exists. Since
\[ \tilde{\nu}(dt, dy) = \nu(dt, dy) - \Pi(dy) dt, \]
then according to the Minkowski inequality, we obtain
\[
\left( M \left\{ \left| \hat{z}_t \right|^n \right\} \right)^{1/2n} \leq \left( M \left\{ \left| z_t \right|^n \right\} \right)^{1/2n} + \left( M \left\{ \left| \hat{z}_t \right|^n \right\} \right)^{1/2n},
\]
where
\[ \hat{z}_t \overset{\text{def}}{=} \int_0^t \int_Y \gamma(\tau, y) \Pi(dy) d\tau \]
and
\[ \hat{z}_t = \int_0^t \int_Y \gamma(\tau, y) \tilde{\nu}(d\tau, dy). \]

The value \( M \left\{ \left| \hat{z}_t \right|^n \right\} \) can be estimated using the well known inequality [1]
\[
M \left\{ \left| \hat{z}_t \right|^n \right\} \leq t^{2n-1} \int_0^t M \left\{ \int_Y \varphi(\tau, y) \Pi(dy) \right\}^{2n} d\tau,
\]
where we suppose that
\[ \int_0^t M \left\{ \int_Y \gamma(\tau, y) \Pi(dy) \right\}^{2n} d\tau < \infty. \]

5. Expansion of Iterated Stochastic Integrals with Respect to Martingale Poisson Measures Based on Generalized Multiple Fourier Series

Let us consider the following iterated stochastic integrals
\[
P[\chi^{(k)}]_{T,t} = \int_t^T \int_X \chi_k(t_k, y_k) \ldots \int_t^{t_2} \int_X \chi_1(t_1, y_1) \tilde{\nu}(i_1)(dt_1, dy_1) \ldots \tilde{\nu}(i_k)(dt_k, dy_k),
\]
where \( i_1, \ldots, i_k = 0, 1, \ldots, m, \mathbb{R}^n \overset{\text{def}}{=} X, \)
\[ \chi_l(\tau, y) = \psi_l(\tau) \varphi_l(y) \quad (l = 1, \ldots, k), \]
every function \( \psi_l(\tau) : [t, T] \to \mathbb{R}^1 \) \((l = 1, \ldots, k)\) and every function \( \varphi_l(y) : X \to \mathbb{R}^1 \) \((l = 1, \ldots, k)\) is such that

\[
\chi_l(s, y) \in H_2(\Pi, [t, T]) \quad (l = 1, \ldots, k),
\]

where definition of the class \( H_2(\Pi, [t, T]) \) is given above,

\[
\nu^{(i)}(dt, dy) \quad (i = 1, \ldots, m)
\]

are independent Poisson random measures for various \( i \) which are defined on \([0, T] \times X\),

\[
\tilde{\nu}^{(i)}(dt, dy) = \nu^{(i)}(dt, dy) - \Pi(dy)dt \quad (i = 1, \ldots, m)
\]

are independent martingale Poisson measures for various \( i \),

\[
\tilde{\nu}^{(0)}(dt, dy) \overset{\text{def}}{=} \Pi(dy)dt.
\]

Let us formulate an analogue of Theorem 1 for the iterated stochastic integrals \((23)\).

**Theorem 3** \([12]-[18]\). Suppose that the following conditions are fulfilled:

1. Every \( \psi_l(\tau) \) \((l = 1, \ldots, k)\) is a continuous non-random function at the interval \([t, T]\).
2. \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of functions in the space \( L_2([t, T]) \), each function of which for finite \( j \) satisfies the condition \( * \) (see Sect. 1).
3. For \( l = 1, \ldots, k \) and \( q = 2k+1 \) the following condition is fulfilled

\[
\int_X |\varphi_l(y)|^q \Pi(dy) < \infty.
\]

Then, for the iterated stochastic integral with respect to martingale Poisson measures \( P[\chi^{(k)}]_{T,t} \) defined by \((23)\) the following expansion

\[
P[\chi^{(k)}]_{T,t} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{g=1}^{k} \pi^{(g,i_g)}_{j_g} \right) - \lim_{N \to \infty} \sum_{(l_1, \ldots, l_k) \in G_k} \prod_{g=1}^{k} \phi_{j_g}(\tau_{l_g}) \int_X \varphi_{g}(y)\tilde{\nu}^{(i_g)}([\tau_{l_g}, \tau_{l_g+1}], dy)
\]

\((24)\)

converging in the mean-square sense is valid, where \( \{\tau_j\}_{j=0}^{N} \) is a partition of the interval \([t, T]\) which satisfies the condition \([7]\).

\[
G_k = H_k \setminus L_k, \quad H_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1\}, \quad L_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1; l_g \neq l_r \ (g \neq r); \ g, r = 1, \ldots, k\},
\]

l.i.m. is a limit in the mean-square sense, \( i_1, \ldots, i_k = 0, 1, \ldots, m \), random variables

\[
\pi^{(g,i_g)}_{j_g} = \int_1^r \phi_j(\tau) \int_X \varphi_g(y)\tilde{\nu}^{(i_g)}(d\tau, dy)
\]
are independent for various \( i_y \neq 0 \) and uncorrelated for various \( j \),

\[
C_{j_k...j_1} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k
\]
is the Fourier coefficient,

\[
K(t_1, \ldots, t_k) = \begin{cases} 
\psi_1(t_1) \ldots \psi_k(t_k) & \text{for } t_1 < \ldots < t_k \\
0 & \text{otherwise}
\end{cases}, \quad t_1, \ldots, t_k \in [t, T], \quad k \geq 2
\]

and \( K(t_1) = \psi_1(t_1) \) for \( t_1 \in [t, T] \).

**Proof.** The scheme of the proof of Theorem 3 is the same as the scheme of the proof of Theorem 1 (see [10]-[25], [43] for details). Some differences will take place in the proof of the following lemmas (Lemmas 1, 2) and in the final part of the proof of Theorem 3.

**Lemma 1** [11]-[25]. Suppose that every \( \psi_l(\tau) \) (\( l = 1, \ldots, k \)) is a continuous function at the interval \( [t, T] \) and every function \( \varphi_l(y) \) (\( l = 1, \ldots, k \)) is such that

\[
\int_X |\varphi_l(y)|^2 \Pi(dy) < \infty.
\]

Then, the following equality

\[
P[\bar{\chi}^{(k)}]_{T,t} = \text{l.i.m.}_{N \to \infty} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_{2-1}} \prod_{l=1}^{k} \chi_l(\tau_{j_l}, y) \bar{\nu}^{(i)}([\tau_{j_l}, \tau_{j_l+1}], dy)
\]

is valid w. p. 1, where \( \{\tau_j\}_{j=0}^{N} \) is a partition of the interval \( [t, T] \) which satisfies the condition \( 7 \),

\[
\bar{\nu}^{(i)}([\tau, s], dy) = \begin{cases} 
\bar{\nu}^{(i)}([\tau, s], dy) & (i = 0, 1, \ldots, m), \\
\nu^{(i)}([\tau, s], dy) & \text{otherwise}
\end{cases}
\]

the integral \( P[\bar{\chi}^{(k)}]_{T,t} \) differs from the integral \( P[\chi^{(k)}]_{T,t} \) (see (23)) by the fact that in \( P[\bar{\chi}^{(k)}]_{T,t} \) we use \( \bar{\nu}^{(i)}(dt_l, dy) \) instead of \( \nu^{(i)}(dt_l, dy_l) \) (\( l = 1, \ldots, k \)).

**Proof.** Using the moment properties of stochastic integrals with respect to Poisson random measures (see above) and conditions of Lemma 1, it is easy to notice that the integral sum of the integral \( P[\bar{\chi}^{(k)}]_{T,t} \) under the conditions of Lemma 1 can be represented as a sum of the expression from the right-hand side of (25) before passing to the limit l.i.m. and the value which converges to zero in the mean-square sense if \( N \to \infty \).

Note that in the case when the functions \( \psi_l(\tau) \) (\( l = 1, \ldots, k \)) satisfy the condition \( * \) (see Sect. 1) we can suppose that among the points \( \tau_j, \ j = 0, 1, \ldots, N \) there are all points of jumps of the functions \( \psi_l(\tau) \) (\( l = 1, \ldots, k \)). Further, we can apply the argumentation as in Sect. 4 from [43] (also see [10]-[18]).
\[
\text{i.i.m. } N \to \infty \sum_{j_1, \ldots, j_k=0}^{N-1} \Phi(\tau_{j_1}, \ldots, \tau_{j_k}) \prod_{l=1}^{k} \int_{\mathbf{X}} \varphi_l(y) \tilde{\nu}^{(i_l)}([\tau_{j_l}, \tau_{j_l+1}], dy) \overset{\text{def}}{=} P[\Phi]^{(k)}_{T,t},
\]

where the sense of notations of the formula (25) is saved and \( \Phi(t_1, \ldots, t_k) : [t, T]^k \to \mathbb{R}^1 \) is a bounded non-random function.

Note that if the functions \( \varphi_l(y) (l = 1, \ldots, k) \) satisfy the conditions of Lemma 1 and the function \( \Phi(t_1, \ldots, t_k) \) is continuous in the domain of integration, then for the integral \( \hat{P}[\Phi]^{(k)}_{T,t} \) the equality similar to (25) is valid w. p. 1.

\textbf{Lemma 2} [11]-[25]. Assume that the following conditions are fulfilled:

\[ g_l(\tau, y) = h_l(\tau) \varphi_l(y) \quad (l = 1, \ldots, k), \]

where the functions \( h_l(\tau) : [t, T] \to \mathbb{R}^1 \) \( (l = 1, \ldots, k) \) satisfy the condition (\( \star \)) (see Sect. 1) and the functions \( \varphi_l(y) : \mathbf{X} \to \mathbb{R}^1 \) \( (l = 1, \ldots, k) \) satisfy the condition

\[ \int_{\mathbf{X}} |\varphi_l(y)|^p \Pi(dy) < \infty \quad \text{for} \quad p = 2^{k+1}. \]

Then

\[ \prod_{l=1}^{k} \int_{t}^{T} \int_{\mathbf{X}} g_l(s, y) \tilde{\nu}^{(i_l)}(ds, dy) = P[\Phi]^{(k)}_{T,t} \quad \text{w. p. 1}, \]

where \( i_l = 0, 1, \ldots, m \) \( (l = 1, \ldots, k) \) and

\[ \Phi(t_1, \ldots, t_k) = \prod_{l=1}^{k} h_l(t_l). \]

\textbf{Proof.} Let us introduce the following notations

\[ J[\tilde{g}_t]_N \overset{\text{def}}{=} \sum_{j=0}^{N-1} \int_{\mathbf{X}} g_t(\tau_j, y) \tilde{\nu}^{(i_j)}([\tau_j, \tau_{j+1}], dy), \]
\[ J[\tilde{g}_t]_{T,t} \overset{\text{def}}{=} \int_{t}^{T} \int_{\mathbf{X}} g_t(s, y) \tilde{\nu}^{(i_l)}(ds, dy), \]

where \( \{\tau_j\}_{j=0}^{N} \) is a partition of the interval \([t,T]\) satisfying the condition (7).

It is easy to see that

\[ \prod_{l=1}^{k} J[\tilde{g}_t]_N - \prod_{l=1}^{k} J[\tilde{g}_t]_{T,t} = \]
Using the Minkowski inequality and the inequality of Cauchy–Bunyakovs’ky together with estimates of moments of stochastic integrals with respect to Poisson random measures (see Sect. 4) and conditions of Lemma 2, we obtain

\[
\left( \frac{1}{k} \prod_{q=1}^{k-1} J[\bar{g}_{T,t}] - \frac{1}{k} \prod_{l=1}^{k-1} J[\bar{g}_{T,t}] \right)^{2} \leq C_{k} \sum_{l=1}^{k-1} \left( \left| J[\bar{g}_{N}] - J[\bar{g}_{T,t}] \right|^{4} \right)^{1/4},
\]

where \( C_{k} < \infty \).

We have

\[
J[\bar{g}_{N}] - J[\bar{g}_{T,t}] = \sum_{q=0}^{N-1} J[\Delta \bar{g}_{q}, \tau_{q+1}, \tau_{q}],
\]

where

\[
J[\Delta \bar{g}_{q}, \tau_{q+1}, \tau_{q}] = \int_{\tau_{q}}^{\tau_{q+1}} \int_{X} (g_{q}(\tau_{q}, y) - g_{q}(s, y)) \bar{\nu}^{(i)}(ds, dy).
\]

Let us introduce the notation

\[
h_{l}^{(N)}(s) = h_{l}(\tau_{q}), \quad s \in [\tau_{q}, \tau_{q+1}), \quad q = 0, 1, \ldots, N - 1.
\]

Then

\[
J[\bar{g}_{N}] - J[\bar{g}_{T,t}] = \sum_{q=0}^{N-1} J[\Delta \bar{g}_{q}, \tau_{q+1}, \tau_{q}]
\]

\[
= \int_{t}^{T} \left( h_{l}^{(N)}(s) - h_{l}(s) \right) \int_{X} \phi_{l}(y) \bar{\nu}^{(i)}(ds, dy).
\]

Applying the estimate (20) for \( n = 4 \) and the estimates (21), (22) for \( n = 2 \) to the value

\[
M \left\{ \int_{t}^{T} \left( h_{l}^{(N)}(s) - h_{l}(s) \right) \int_{X} \phi_{l}(y) \bar{\nu}^{(i)}(ds, dy) \right|^{4} \right\},
\]

taking into account (26) together with the conditions of Lemma 2 and the following estimate

\[
|h_{l}(\tau_{q}) - h_{l}(s)| < \varepsilon, \quad s \in [\tau_{q}, \tau_{q+1}], \quad q = 0, 1, \ldots, N - 1,
\]

where \( \varepsilon \) is an arbitrary small positive real number, we obtain that the right-hand side of (20) converges to zero when \( N \rightarrow \infty \). Considering this fact, we come to the statement of Lemma 2.

It should be noted that (27) is valid if the functions \( h_{l}(s) \) are continuous at the interval \([t, T]\), i.e., these functions are uniformly continuous at this interval. So, \( |h_{l}(\tau_{q}) - h_{l}(s)| < \varepsilon \) if \( s \in [\tau_{q}, \tau_{q+1}] \),
where \(|\tau_{q+1} - \tau_q| < \delta(\varepsilon), q = 0, 1, \ldots, N - 1 (\delta(\varepsilon) > 0 \text{ exists for any } \varepsilon > 0 \text{ and it does not depend on points of the interval } [t, T])\).

In the case when the functions \(h_l(s) (l = 1, \ldots, k)\) satisfy the condition (*) (see Sect. 1) we can suppose that among the points \(\tau_q, q = 0, 1, \ldots, N\) there are all points of jumps of the functions \(h_l(s) (l = 1, \ldots, k)\). Further, we can apply the argumentation as in Sect. 4 from [13] (also see [10]-[18]).

Obviously, if \(u_l = 0\) for some \(l = 1, \ldots, k\), then we also come to the statement of Lemma 2. Lemma 2 is proved.

Proving Theorem 3 according to the scheme used for the proof of Theorem 1 in [13] or Theorem 1.1 in [15]-[18] (also see [10] (Theorem 5.1, P. 236-237), [12] (Theorem 1, P. A.22-A.23), [13] (Theorem 5.1, P. A.250), [14] (Theorem 5.1, P. A.252-A.253)) and using Lemmas 1, 2 together with estimates for moments of stochastic integrals with respect to Poisson random measures (see Sect. 4), we obtain

\[
M \left\{ \left( R_{T,t}^{p_1, \ldots, p_k} \right)^2 \right\} \leq
\]

\[
C \prod_{l=1}^{k} \varphi_l^2(y) \Pi(dy) \sum_{(t_1, \ldots, t_k)} \left( K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_1 \ldots j_k} \prod_{l=1}^{k} \phi_l(t_l) \right)^2 \times
\]

\[
\int dt_1 \ldots dt_k =
\]

\[
= C \prod_{l=1}^{k} \varphi_l^2(y) \Pi(dy) \int_{[t,T]^k} \left( K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_1 \ldots j_k} \prod_{l=1}^{k} \phi_l(t_l) \right)^2 \times
\]

\[
\int dt_1 \ldots dt_k \leq
\]

\[
\leq C \int_{[t,T]^k} \left( K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_1 \ldots j_k} \prod_{l=1}^{k} \phi_l(t_l) \right)^2 dt_1 \ldots dt_k \to 0
\]

if \(p_1, \ldots, p_k \to \infty\), where constant \(C \) depends only on \(k\) (multiplicity of the iterated stochastic integral with respect to martingale Poisson measures). At that permutations \((t_1, \ldots, t_k)\) when summing

\[
\sum_{(t_1, \ldots, t_k)}
\]

in (28) are performed only in the values \(dt_1 \ldots dt_k\) and indexes near upper limits of integration are changed correspondently. Moreover, \(R_{T,t}^{p_1, \ldots, p_k}\) has the following form

\[
R_{T,t}^{p_1, \ldots, p_k} = \sum_{(t_1, \ldots, t_k)} \int_{t_1}^{t_2} \cdots \int_{t}^{T} K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_1 \ldots j_k} \prod_{l=1}^{k} \phi_l(t_l) \times
\]

\[
\int \varphi_1(y) \tilde{\phi}_1(t_1)(dt_1, dy) \ldots \int \varphi_k(y) \tilde{\phi}_k(t_k)(dt_k, dy),
\]

where permutations \((t_1, \ldots, t_k)\) when summing
in (29) are performed only in the values
\[ \varphi_1(y)\tilde{\nu}^{(i_1)}(dt_1, dy) \ldots \varphi_k(y)\tilde{\nu}^{(i_k)}(dt_k, dy). \]

At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if \( t_r \) swapped with \( t_q \) in the permutation \( (t_1, \ldots, t_k) \), then \( i_r \) swapped with \( i_q \) in the permutation \( (i_1, \ldots, i_k) \). Moreover, \( \varphi_r(y) \) swapped with \( \varphi_q(y) \) in the permutation \( (\varphi_1(y), \ldots, \varphi_k(y)) \). Theorem 3 is proved.

Let us consider an example of Theorem 3 usage. Suppose that \( i_1 \neq i_2, i_1, i_2 = 1, \ldots, m \). According to Theorem 3, we obtain
\[
\begin{aligned}
\int_t^T \int_\mathbb{X} \int_t^{t_2} \varphi_2(y_2) \int_t^{t_1} \varphi_1(y_1) \tilde{\nu}^{(i_1)}(dt_1, dy_1) \tilde{\nu}^{(i_2)}(dt_2, dy_2) = \\
= \frac{T - t}{2} \left( \pi_0^{(1,i_1), (2,i_2)} + \sum_{i=1}^{\infty} \frac{1}{\sqrt{4i^2 - 1}} \left( \pi_0^{(1,i_1), (2,i_2)} - \pi_0^{(1,i_1), (2,i_2)} \right) \right),
\end{aligned}
\]
where
\[
\pi_0^{(l,i)} = \int_t^T \int_\mathbb{X} \phi_j(\tau) \varphi_l(y) \tilde{\nu}^{(i_1)}(d\tau, dy) \quad (l = 1, 2)
\]
and \( \{\phi_j(\tau)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials in the space \( L_2([t, T]) \).

6. Expansion of Iterated Stochastic Integrals with Respect to Martingales

Let \((\Omega, \mathcal{F}, P)\) be a fixed probability space, let \(\{\mathcal{F}_t, t \in [0,T]\}\) be a non-decreasing family of \(\sigma\)-algebras \(\mathcal{F}_t \subset \mathcal{F}\), and let \(\mathcal{M}_2(\rho, [0,T])\) be a class of \(\mathcal{F}_t\)-measurable for each \(t \in [0,T]\) martingales \(M_t\) satisfying the conditions
\[
\begin{aligned}
\mathbb{M} \left\{ \left( M_s - M_t \right)^2 \right\} &= \int_t^s \rho(\tau)d\tau, \\
\mathbb{M} \left\{ |M_s - M_t|^p \right\} &\leq C_p|s - t|, \quad p = 3, 4, \ldots,
\end{aligned}
\]
where \(0 \leq t < s \leq T\), \(\rho(\tau)\) is a non-negative and continuously differentiable non-random function at the interval \([0, T]\), \(C_p < \infty\) is a constant.

Let us define the class \(H_2(\rho, [0,T])\) of stochastic processes \(\xi_t, t \in [0,T]\) which are \(\mathcal{F}_t\)-measurable for all \(t \in [0,T]\) and satisfy the condition
\[
\int_0^T M \left\{ |\xi_t|^2 \right\} \rho(t) dt < \infty.
\]

For any partition \( \{ \tau_j^{(N)} \}_{j=0}^N \) of the interval \([0,T]\) such that
\[
0 = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T, \quad \max_{0 \leq j \leq N-1} |\tau_{j+1}^{(N)} - \tau_j^{(N)}| \to 0 \quad \text{if} \quad N \to \infty
\]
we will define the sequence of step functions \( \xi^{(N)}(t,\omega) \) by the following relation
\[
\xi^{(N)}(t,\omega) = \xi_j(\omega) \ \text{w. p. 1 for} \quad t \in [\tau_j^{(N)}, \tau_{j+1}^{(N)}),
\]
where \( j = 0, 1, \ldots, N-1, \ N = 1, 2, \ldots. \)

Let us define the stochastic integral with respect to martingale from the process \( \xi(t,\omega) \in H_2(\rho, [0,T]) \) as the following mean-square limit [1]
\[
\lim_{N \to \infty} \sum_{j=0}^{N-1} \xi^{(N)}(\tau_j^{(N)},\omega) \left( M \left( \xi_{j+1}^{(N)},\omega \right) - M \left( \xi_j^{(N)},\omega \right) \right) = \int_0^T \xi_t dM_t,
\]
where \( \xi^{(N)}(t,\omega) \) is any step function from the class \( H_2(\rho, [0,T]) \) which converges to the function \( \xi(t,\omega) \) in the following sense
\[
\lim_{N \to \infty} \int_0^T M \left\{ \left| \xi^{(N)}(t,\omega) - \xi(t,\omega) \right|^2 \right\} \rho(t) dt = 0.
\]

It is well known [1] that the stochastic integral
\[
\int_0^T \xi_t dM_t
\]
exists and it does not depend on the selection of sequence \( \xi^{(N)}(t,\omega) \) and it satisfies w. p. 1 the following properties
\[
M \left\{ \int_0^T \xi_t dM_t \bigg| F_0 \right\} = 0,
\]
\[
M \left\{ \left| \int_0^T \xi_t dM_t \right|^2 \bigg| F_0 \right\} = M \left\{ \int_0^T \xi_t^2 \rho(t) dt \bigg| F_0 \right\},
\]
\[
\int_0^T (\alpha \xi_t + \beta \psi_t) dM_t = \alpha \int_0^T \xi_t dM_t + \beta \int_0^T \psi_t dM_t,
\]
where \( \xi_t, \phi_t \in H_2(\rho, [0, T]) \), \( \alpha, \beta \in \mathbb{R}^1 \).

Let \( Q_4(\rho, [0, T]) \) be the class of martingales \( M_t, t \in [0, T] \) which satisfy the following conditions:
1. \( M_t, t \in [0, T] \) belongs to the class \( M_2(\rho, [0, T]) \).
2. For some \( \alpha > 0 \) the following estimate is correct

\[
(33) \quad M \left\{ \int_t^\tau |g(s)|^\alpha ds \right\} \leq K_4 \int_t^\tau |g(s)|^\alpha ds,
\]

where \( 0 \leq t < \tau \leq T, \ g(s) \) is a bounded non-random function at the interval \([0, T]\), \( K_4 < \infty \) is a constant.

Let \( G_n(\rho, [0, T]) \) be the class of martingales \( M_t, t \in [0, T] \) which satisfy the following conditions:
1. \( M_t, t \in [0, T] \) belongs to the class \( M_2(\rho, [0, T]) \).
2. The following estimate is correct

\[
(34) \quad M \left\{ \int_t^\tau |g(s)| ds \right\} < \infty,
\]

where \( 0 \leq t < \tau \leq T, \ n \in \mathbb{N}, \ g(s) \) is the same function as in the definition of \( Q_4(\rho, [0, T]) \).

Let us remind that if \( (\xi_t)^n \in H_2(\rho, [0, T]) \) with \( \rho(t) \equiv 1 \), then the following estimate is correct \[1\]

\[
(35) \quad J[\psi^{(k)}]_{M,t}^M = \int_t^T \psi_k(t_k) \ldots \psi_1(t_1) dM_{t_1}^{(1,i_1)} \ldots dM_{t_k}^{(k,i_k)} \quad (i_1, \ldots, i_k = 0, 1, \ldots, m),
\]

where every \( \psi_l(\tau) \ (l = 1, \ldots, k) \) is a continuous non-random function at the interval \([t, T]\), \( M^{(r,i)} \)
\( (r = 1, \ldots, k) \) are independent martingales for various \( i = 1, \ldots, m \), \( M_{t}^{(r,0)} \) \( \equiv \tau \).

Let us formulate the following theorem.

**Theorem 4** [12] [18]. Suppose that the following conditions are fulfilled:
1. Every \( \psi_l(\tau) \ (l = 1, \ldots, k) \) is a continuous non-random function at the interval \([t, T]\).
2. \( \{\phi_j(x)\}^\infty_{j=0} \) is a complete orthonormal system of functions in the space \( L_2([t, T]) \), each function
   of which for finite \( j \) satisfies the condition (\*) (see Sect. 1).
3. \( M_{t}^{(l,i_l)} \in Q_4(\rho, [t, T]), G_n(\rho, [t, T]) \) with \( n = 2^{k+1}, i_l = 1, \ldots, m, \ l = 1, \ldots, k \).

Then, for the iterated stochastic integral \( J[\psi^{(k)}]_{M,t}^M \) with respect to martingales defined by [35] the following expansion

\[
J[\psi^{(k)}]_{M,t}^M = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_1 \ldots j_k} \left( \prod_{l=1}^k \xi^{(l,i_l)}_{j_l} \right) 
- \lim_{N \to \infty} \sum_{(l_1, \ldots, l_k) \in G_k} \phi_{j_1}(\tau_{l_1}) \Delta M_{\tau_{l_1}}^{(1,i_1)} \ldots \phi_{j_k}(\tau_{l_k}) \Delta M_{\tau_{l_k}}^{(k,i_k)}
\]
converging in the mean-square sense is valid, where \( i_1, \ldots, i_k = 0, 1, \ldots, m, \{\tau_j\}_{j=0}^N \) is a partition of the interval \([t, T]\) which satisfies the condition (7), \( \Delta M^{(r,i)}_j = M^{(r,i)}_{\tau_{j+1}} - M^{(r,i)}_j \) (\( i = 0, 1, \ldots, m, \ r = 1, \ldots, k \)),

\[
G_k = H_k \backslash L_k, \quad H_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1\},
\]

\[
L_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1; l_g \neq l_r (g \neq r); g, r = 1, \ldots, k\},
\]

l.i.m. is a limit in the mean-square sense,

\[
\xi^{(l,i)}_j = \int_t^T \phi_j(s) dM^{(l,i)}_s
\]

are independent for various \( i_1 = 1, \ldots, m, \ l = 1, \ldots, k \) and uncorrelated for various \( j \) (if \( \rho(\tau) \) is a constant, \( i_1 \neq 0 \) random variables,

\[
C_{j_1 \ldots j_k} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^k \phi_{j_l}(t_l) d t_1 \ldots d t_k
\]

is the Fourier coefficient,

\[
K(t_1, \ldots, t_k) = \begin{cases} 
\psi_1(t_1) \ldots \psi_k(t_k) & \text{for } t_1 < \ldots < t_k \\
0 & \text{otherwise}
\end{cases}, \quad t_1, \ldots, t_k \in [t, T], \quad k \geq 2
\]

and \( K(t_1) \equiv \psi_1(t_1) \) for \( t_1 \in [t, T] \).

**Remark 1.** Note that from Theorem 4 for the case \( \rho(\tau) \equiv 1 \) we obtain the variant of Theorem 1.

**Proof.** The scheme of the proof of Theorem 4 is the same with the scheme of the proof of Theorem 1 in [13] or Theorem 1.1 in [15-18] (also see [10-25, 43]). Some differences will take place in the proof of the following lemmas (Lemmas 3, 4) and in the final part of the proof of Theorem 4.

**Lemma 3.** Suppose that \( M^{(r,i)}_r \in M_2(\rho, [t, T]) \), \( M^{(r,0)}_r = \tau \) (\( i = 0, 1, \ldots, m, \ r = 1, \ldots, k \), and every \( \psi_l(\tau) \) (\( l = 1, \ldots, k \)) is a continuous non-random function at the interval \([t, T]\). Then

\[
J[\psi^{(k)}]_{M, T, \tau} = \lim_{N \to \infty} \sum_{\tau_j = 0}^{N-1} \sum_{j_1 = 0}^{j_2-1} \prod_{l=1}^k \psi_l(\tau_j) \Delta M^{(l,i)}_{\tau_{j+1}} \quad p. \ p. 1,
\]

where \( \{\tau_j\}_{j=0}^N \) is a partition of the interval \([0, T]\) satisfying the condition (7).

**Proof.** According to properties of the stochastic integral with respect to martingale, we have \[1\]

\[
M \left\{ \left( \int_t^\tau \xi_s dM^{(l,i)}_s \right)^2 \right\} = \int_t^\tau M \left\{ |\xi_s|^2 \right\} \rho(s) ds,
\]

\[
\xi^{(l,i)}_j = \int_t^T \phi_j(s) dM^{(l,i)}_s
\]
where \( \xi_s \in H_2(\rho, [0, T]) \), \( 0 \leq t < \tau \leq T \), \( i_l = 1, \ldots, m, \ l = 1, \ldots, k \). Then the integral sum of the integral \( J[\psi^{(l)}]_{T,t} \) under the conditions of Lemma 3 can be represented as a sum of the expression from the right-hand side of (36) before passing to the limit and the value which converges to zero in the mean-square sense if \( N \to \infty \). More detailed proof of the analogous lemma for the case \( \rho(\tau) \equiv 1 \) can be found in [10-25, 43].

In the case when the functions \( \psi_1(\tau) \) \( (l = 1, \ldots, k) \) satisfy the condition (\( \star \)) (see Sect. 1) we can suppose that among the points \( \tau_j, \ j = 0, 1, \ldots, N \) there are all points of jumps of the functions \( \psi_l(\tau) \) \( (l = 1, \ldots, k) \). Then can apply the argumentation as in Sect. 4 from [43] (also see [10-18]).

Let us define the following multiple stochastic integral

\[
(38) \quad M \left\{ \left( \int_t^\tau \xi_s ds \right)^2 \right\} \leq (\tau - t) \int_t^\tau M \left\{ |\xi_s|^2 \right\} ds,
\]

where \( \xi_s \in H_2(\rho, [0, T]) \), \( 0 \leq t < \tau \leq T \), \( i_l = 1, \ldots, m, \ l = 1, \ldots, k \). Then the integral sum of the integral \( J[\psi^{(l)}]_{T,t} \) under the conditions of Lemma 3 can be represented as a sum of the expression from the right-hand side of (36) before passing to the limit and the value which converges to zero in the mean-square sense if \( N \to \infty \). More detailed proof of the analogous lemma for the case \( \rho(\tau) \equiv 1 \) can be found in [10-25, 43].

In the case when the functions \( \psi_1(\tau) \) \( (l = 1, \ldots, k) \) satisfy the condition (\( \star \)) (see Sect. 1) we can suppose that among the points \( \tau_j, \ j = 0, 1, \ldots, N \) there are all points of jumps of the functions \( \psi_l(\tau) \) \( (l = 1, \ldots, k) \). Then can apply the argumentation as in Sect. 4 from [43] (also see [10-18]).

Let us define the following multiple stochastic integral

\[
(39) \quad \lim_{N \to \infty} \sum_{j_1, \ldots, j_k=0}^{N-1} \Phi(\tau_{j_1}, \ldots, \tau_{j_k}) \prod_{l=1}^k \Delta M^{(l,i_l)}_{\tau_{j_l}} \overset{\text{def}}{=} I[\Phi]_{T,t},
\]

where \( \{\tau_j\}_{j=0}^N \) is a partition of the interval \([0, T]\) satisfying the condition (7) and \( \Phi(t_1, \ldots, t_k) : [t, T]^k \to \mathbb{R}^1 \) is a bounded non-random function.

**Lemma 4.** Suppose that \( M^{(l,i_l)} \in Q_4(\rho, [t, T]), G_n(\rho, [t, T]) \) with \( n = 2^{k+1}, k \in \mathbb{N} \) \( (i_l = 0, 1, \ldots, m, \ l = 1, \ldots, k) \) and the functions \( g_1(s), \ldots, g_k(s) \) satisfy the condition (\( \star \)) (see Sect. 1). Then

\[
\prod_{l=1}^k \int_t^T g_l(s) dM^{(l,i_l)}_s = I[\Phi]_{T,t} \quad \text{w. p. 1},
\]

where

\[
\Phi(t_1, \ldots, t_k) = \prod_{l=1}^k g_l(t_l).
\]

**Proof.** Let us denote

\[
J[g_l]_N \overset{\text{def}}{=} \sum_{j=0}^{N-1} g_l(\tau_j) \Delta M^{(l,i_l)}_{\tau_j}, \quad J[g_l]_{T,t} \overset{\text{def}}{=} \int_t^T g_l(s) dM^{(l,i_l)}_s,
\]

where \( \{\tau_j\}_{j=0}^N \) is a partition of the interval \([t, T]\) satisfying the condition (7).

Note that

\[
\prod_{l=1}^k J[g_l]_N - \prod_{l=1}^k J[g_l]_{T,t} =
\]

\[
= \sum_{l=1}^k \left( \prod_{q=1}^{l-1} J[g_q]_{T,t} \right) \left( J[g_l]_N - J[g_l]_{T,t} \right) \left( \prod_{q=l+1}^k J[g_q]_N \right).
\]

**Proof.** Let us denote

\[
J[g_l]_N \overset{\text{def}}{=} \sum_{j=0}^{N-1} g_l(\tau_j) \Delta M^{(l,i_l)}_{\tau_j}, \quad J[g_l]_{T,t} \overset{\text{def}}{=} \int_t^T g_l(s) dM^{(l,i_l)}_s,
\]

where \( \{\tau_j\}_{j=0}^N \) is a partition of the interval \([t, T]\) satisfying the condition (7).

Note that

\[
\prod_{l=1}^k J[g_l]_N - \prod_{l=1}^k J[g_l]_{T,t} =
\]

\[
= \sum_{l=1}^k \left( \prod_{q=1}^{l-1} J[g_q]_{T,t} \right) \left( J[g_l]_N - J[g_l]_{T,t} \right) \left( \prod_{q=l+1}^k J[g_q]_N \right).
\]
Using the Minkowski inequality and the inequality of Cauchy-Bunyakovskiy as well as the conditions of Lemma 4, we obtain

\[
(40) \quad \left( \mathbb{M} \left\{ \left( \prod_{l=1}^{k} J[gi]_N - \prod_{l=1}^{k} J[gi]_{T,t} \right)^2 \right\} \right)^{1/2} \leq C_k \sum_{l=1}^{k} \left( \mathbb{M} \left\{ J[gi]_N - J[gi]_{T,t} \right\}^{4} \right)^{1/4},
\]

where \( C_k < \infty \) is a constant.

We have

\[
J[gi]_N - J[gi]_{T,t} = \sum_{q=0}^{N-1} J[\Delta gi]_{\tau_{q+1}, \tau_q},
\]

\[
J[\Delta gi]_{\tau_{q+1}, \tau_q} = \int_{\tau_q}^{\tau_{q+1}} (gi(t) - gi(s)) \, dM_s^{(l,ii)}.
\]

Let us introduce the notation

\[
g_l^{(N)}(s) = gi(\tau_q), \quad s \in [\tau_q, \tau_{q+1}), \quad q = 0, 1, \ldots, N - 1.
\]

Then

\[
J[\tilde{g}_l]_N - J[\tilde{g}_l]_{T,t} = \sum_{q=0}^{N-1} J[\Delta \tilde{g}_l]_{\tau_{q+1}, \tau_q} = \int_{t}^{T} \left( g_l^{(N)}(s) - g_l(s) \right) \, dM_s^{(l,ii)}.
\]

Applying the estimate (33), we obtain

\[
\mathbb{M} \left\{ \left( \int_{t}^{T} \left( g_l^{(N)}(s) - g_l(s) \right) \, dM_s^{(l,ii)} \right)^4 \right\} \leq K_4 \int_{t}^{T} \left| g_l^{(N)}(s) - g_l(s) \right|^\alpha \, ds = \sum_{q=0}^{N-1} \int_{\tau_q}^{\tau_{q+1}} \left| g_l(\tau_q) - g_l(s) \right|^\alpha \, ds < K_4 \varepsilon^\alpha \sum_{q=0}^{N-1} (\tau_{q+1} - \tau_q) = K_4 \varepsilon^\alpha (T - t).
\]

Note that deriving (41) we used the estimate

\[
(42) \quad |g_l(\tau_q) - g_l(s)| < \varepsilon, \quad s \in [\tau_q, \tau_{q+1}], \quad q = 0, 1, \ldots, N - 1,
\]

where \( \varepsilon \) is an arbitrary small positive real number.

Note that (42) is valid if the functions \( g_l(s) \) are continuous at the interval \([t, T]\), i.e. these functions are uniformly continuous at this interval. So, \( |g_l(\tau_q) - g_l(s)| < \varepsilon \) if \( s \in [\tau_q, \tau_{q+1}] \), where \( |\tau_{q+1} - \tau_q| < \delta(\varepsilon), q = 0, 1, \ldots, N - 1 (\delta(\varepsilon) > 0 \text{ exists for any } \varepsilon > 0 \text{ and it does not depend on points of the interval } [t, T]) \).
Thus, taking into account (41), we obtain that the right-hand side of (40) converges to zero when \( N \to \infty \). Considering this fact, we come to the statement of Lemma 4.

In the case when the functions \( g_l(s) \) (\( l = 1, \ldots, k \)) satisfy the condition (\( \ast \)) (see Sect. 1) we can suppose that among the points \( \tau_q, q = 0, 1, \ldots, N \) there are all points of jumps of the functions \( g_l(s) \) (\( l = 1, \ldots, k \)). Further, we can apply the argumentation as in Sect. 4 from [12] (also see [10] (Theorem 5.1, P. 236-237), [12] (Theorem 1, P. A.22-A.23), [13] (Theorem 5.1, P. A.250), [14] (Theorem 5.1, P. A.252-A.253)) and using Lemmas 3, 4 together with the estimates (37), (38) for moments of stochastic integrals with respect to martingales, we obtain

\[
M \left\{ \left( R_{T,t}^{p_1, \ldots, p_k} \right)^2 \right\} \leq \\
\leq C_k \sum_{(t_1, \ldots, t_k)} \int \cdots \int_t^{T} \left( K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 \times \\
\times \hat{\rho}_1(t_1)dt_1 \ldots \hat{\rho}_k(t_k)dt_k \leq \\
\leq C' k \sum_{(t_1, \ldots, t_k)} \int \cdots \int_t^{T} \left( K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k = \\
= C' k \int_{[t,T]} \left( K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k \to 0
\]

when \( p_1, \ldots, p_k \to \infty \), where constant \( C_k \) depends only on \( k \) (multiplicity of the iterated stochastic integral with respect to martingales) and \( \hat{\rho}_l(s) \equiv \rho(s) \) or \( \hat{\rho}_l(s) \equiv 1 \ (l = 1, \ldots, k) \). At that permutations \( (t_1, \ldots, t_k) \) when summing

\[
\sum_{(t_1, \ldots, t_k)}
\]

in (43) are performed only in the values \( dt_1 \ldots dt_k \) and indexes near upper limits of integration are changed correspondently. Moreover, \( R_{T,t}^{p_1, \ldots, p_k} \) has the following form

\[
R_{T,t}^{p_1, \ldots, p_k} = \sum_{(t_1, \ldots, t_k)} \int \cdots \int_t^{T} \left( K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right) \times \\
\times dM_{t_1}^{(1, t_1)} \ldots dM_{t_k}^{(k, t_k)}
\]

(44)

where permutations \( (t_1, \ldots, t_k) \) when summing
\[ \sum_{(t_1, \ldots, t_k)} \]

in (44) are performed only in the values

\[ dM_{t_1}^{(1, i_1)} \ldots dM_{t_k}^{(k, i_k)}. \]

At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if \( t_r \) swapped with \( t_q \) in the permutation \((t_1, \ldots, t_k)\), then \( i_r \) swapped with \( i_q \) in the permutation \((t_1, \ldots, i_k)\). Moreover, \( r \) swapped with \( q \) in the permutation \((1, \ldots, k)\).

Theorem 4 is proved.

7. **Expansion of Iterated Ito Stochastic Integrals Based on Generalized Multiple Fourier Series. The Case of Complete Orthonormal With Weight \( r(t_1) \ldots r(t_k) \geq 0 \) Systems of Functions in the Space \( L_2([t,T]^k) \)**

In this section, we consider modifications of Theorems 1, 2 for the case of complete orthonormal with weight \( r(t_1) \ldots r(t_k) \geq 0 \) systems of functions in the space \( L_2([t,T]^k) \), \( k \in \mathbb{N} \).

Let \( \{\Psi_j(x)\}_{j=0}^{\infty} \) be a complete orthonormal with weight \( r(x) \geq 0 \) system of functions in the space \( L_2([t,T]) \). It is well known that the Fourier series with respect to the system

\[ \{\Psi_j(x)\}_{j=0}^{\infty} \]

of the function \( f(x) \left( f(x) \sqrt{r(x)} \in L_2([t,T]) \right) \) converges to the function \( f(x) \) in the mean-square sense with weight \( r(x) \), i.e.

\[ \lim_{p \to \infty} \int_{t}^{T} \left( f(x) - \sum_{j=0}^{p} \tilde{C}_j \Psi_j(x) \right)^2 r(x) dx = 0, \]

where

\[ \tilde{C}_j = \int_{t}^{T} f(x) \Psi_j(x) r(x) dx \]

is the Fourier coefficient.

Obviously, the relation (45) can be obtained if we will expand the function \( f(x) \sqrt{r(x)} \in L_2([t,T]) \) into a usual Fourier series with respect to the complete orthonormal with weight 1 system of functions

\[ \{\Psi_j(x)\sqrt{r(x)}\}_{j=0}^{\infty} \]

in the space \( L_2([t,T]) \). Then

\[ \lim_{p \to \infty} \int_{t}^{T} \left( f(x) \sqrt{r(x)} - \sum_{j=0}^{p} \tilde{C}_j \Psi_j(x) \sqrt{r(x)} \right)^2 dx = \]

\[ = \lim_{p \to \infty} \int_{t}^{T} \left( f(x) - \sum_{j=0}^{p} \tilde{C}_j \Psi_j(x) \right)^2 r(x) dx = 0, \]
where $\tilde{C}_j$ has the form (46).

Let us consider an obvious generalization of this approach to the case of several variables. Let us expand the function $K(t_1, \ldots, t_k)$ such that

$$K(t_1, \ldots, t_k) \prod_{l=1}^{k} \sqrt{r(t_l)} \in L_2([t, T]^k)$$

using the complete orthonormal system of functions

$$\prod_{l=1}^{k} \Psi_{j_l}(t_l) \sqrt{r(t_l)}, \quad j_l = 0, 1, 2, \ldots, \quad l = 1, \ldots, k$$

in the space $L_2([t, T]^k)$ into the generalized multiple Fourier series.

It is well known that the mentioned generalized multiple Fourier series converges in the mean-square sense, i.e.

$$\lim_{p_1, \ldots, p_k \to \infty} \int_{[t, T]^k} \left( K(t_1, \ldots, t_k) \prod_{l=1}^{k} \sqrt{r(t_l)} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \prod_{l=1}^{k} \Psi_{j_l}(t_l) \sqrt{r(t_l)} \right)^2 \times \ dt_1 \ldots dt_k =$$

$$= \lim_{p_1, \ldots, p_k \to \infty} \int_{[t, T]^k} \left( K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \prod_{l=1}^{k} \Psi_{j_l}(t_l) \right)^2 \times \left( \prod_{j=1}^{k} r(t_j) \right) dt_1 \ldots dt_k = 0,$$

(48)

where

$$\tilde{C}_{j_k \ldots j_1} = \int_{[t, T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \left( \Psi_{j_l}(t_l) r(t_l) \right) dt_1 \ldots dt_k.$$

Let us consider the following iterated Ito stochastic integrals

$$(49) \quad \tilde{J}[\psi^{(k)}]_{T,t} = \int_{t}^{T} \psi_k(t_k) \sqrt{r(t_k)} \ldots \int_{t}^{T} \psi_1(t_1) \sqrt{r(t_1)} d\mathbf{w}^{(i_1)}_{t_k} \ldots d\mathbf{w}^{(i_k)}_{t_1},$$

where every $\psi_l(\tau)$ ($l = 1, \ldots, k$) is a non-random function on $[t, T]$, $\mathbf{w}^{(i)}_\tau = f^{(i)}_\tau$ for $i = 1, \ldots, m$, $\mathbf{w}^{(0)}_\tau = \tau$, and $i_1, \ldots, i_k = 0, 1, \ldots, m$.

So, we obtain the following modification of Theorem 1.

**Theorem 5** [14-17, 23]. Suppose that every $\psi_l(\tau)$ ($l = 1, \ldots, k$) is a continuous non-random function on $[t, T]$ and $\{\Psi_j(\sqrt{r(x)})\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$, each function $\Psi_j(\sqrt{r(x)})$ of which for finite $j$ satisfies the condition $(\ast)$ (see Sect. 1). Then
\[
J[p^{(k)}]_{T,t} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \right) - \lim_{N \to \infty} \sum_{(l_1, \ldots, l_k) \in G_k} \Psi_{j_1}(\tau_{l_1}) \sqrt{r(t_{l_1})} \Delta w_{\tau_{l_1}}^{(i_1)} \ldots \Psi_{j_k}(\tau_{l_k}) \sqrt{r(\tau_{l_k})} \Delta w_{\tau_{l_k}}^{(i_k)},
\]

(50)

where

\[G_k = H_k \setminus L_k, \quad H_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1\},\]

\[L_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1; l_g \neq l_r (g \neq r); g, r = 1, \ldots, k\},\]

\text{i.i.m. is a limit in the mean-square sense, } i_1, \ldots, i_k = 0, 1, \ldots, m,

\[\zeta_{j}^{(i)} = \int_{t}^{T} \Psi_{j}(s) \sqrt{r(s)} dw_{s}^{(i)}\]

are independent standard Gaussian random variables for various } i \text{ or } j \text{ (in the case when } i \neq 0),

\[\Delta w_{\tau_{l_i}}^{(i)} = w_{\tau_{l_i+1}}^{(i)} - w_{\tau_{l_i}}^{(i)} \quad (i = 0, 1, \ldots, m), \quad \{\tau_{j}\}_{j=0}^{N} \text{ is a partition of } [t, T] \text{ which satisfies the condition}\]

\[\text{Proof. According to Lemmas 1–3 in [43] or Lemmas 1.1–1.3 in [15]-[18] (also see [10]-[14]), we get the following representation w. p. 1}\]

\[J[p^{(k)}]_{T,t} = \]

\[= \sum_{(t_1, \ldots, t_k)} \int_{t}^{T} \ldots \int_{t}^{T} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \sqrt{r(t_l)} dw_{t_l}^{(i_1)} \ldots dw_{t_k}^{(i_k)} = \]

\[= \prod_{j_1=0}^{p_1} \ldots \prod_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \sum_{(t_1, \ldots, t_k)} \int_{t}^{T} \ldots \int_{t}^{T} \prod_{l=1}^{k} \left( \Psi_{j_l}(t_l) \sqrt{r(t_l)} \right) dw_{t_l}^{(i_1)} \ldots dw_{t_k}^{(i_k)} + \]
\[ + \tilde{R}_{T,t}^{p_1,\ldots,p_k} = \]
\[ = \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \times \]
\[ \times \lim_{N \to \infty} \sum_{l_1,\ldots,l_k=0}^{N-1} \Psi_{j_1}(\tau_{l_1}) \sqrt{r(\tau_{l_1})} \Delta w_{\tau_{l_1}}^{(i_1)} \cdots \Psi_{j_k}(\tau_{l_k}) \sqrt{r(\tau_{l_k})} \Delta w_{\tau_{l_k}}^{(i_k)} + \]
\[ = \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \times \]
\[ \times \left( \lim_{N \to \infty} \sum_{l_1,\ldots,l_k=0}^{N-1} \Psi_{j_1}(\tau_{l_1}) \sqrt{r(\tau_{l_1})} \Delta w_{\tau_{l_1}}^{(i_1)} \cdots \Psi_{j_k}(\tau_{l_k}) \sqrt{r(\tau_{l_k})} \Delta w_{\tau_{l_k}}^{(i_k)} \right) + \]
\[ - \lim_{N \to \infty} \sum_{(l_1,\ldots,l_k) \in G_k} \Psi_{j_1}(\tau_{l_1}) \sqrt{r(\tau_{l_1})} \Delta w_{\tau_{l_1}}^{(i_1)} \cdots \Psi_{j_k}(\tau_{l_k}) \sqrt{r(\tau_{l_k})} \Delta w_{\tau_{l_k}}^{(i_k)} + \]
\[ + \tilde{R}_{T,t}^{p_1,\ldots,p_k} , \]

where

\[ \tilde{R}_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int_{t_1}^{T} \cdots \int_{t_1}^{t_2} \left( K(t_1,\ldots,t_k) \prod_{l=1}^{k} \sqrt{r(t_l)} \right) - \]
\[ - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \prod_{l=1}^{k} \left( \Psi_{j_l}(t_l) \sqrt{r(t_l)} \right) dW_{t_1}^{(i_1)} \cdots dW_{t_k}^{(i_k)} , \]
Then the estimate on integral (49). Theorem 5 is proved.

At the same time the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if \( t_r \) swapped with \( t_q \) in the permutation \((t_1, \ldots, t_k)\), then \( t_r \) swapped \( i_q \) in the permutation \((i_1, \ldots, i_k)\).

Let us evaluate the remainder \( \tilde{R}_{t,T}^{p_1, \ldots, p_k} \) of the series.

According to Lemma 2 in [43] or Lemma 1.2 in [15] (also see [16]-[18]), we have

\[
M \left\{ \left( \tilde{R}_{t,T}^{p_1, \ldots, p_k} \right)^2 \right\} \leq C_k \sum_{(t_1, \ldots, t_k) \in \mathcal{T}} \int_t^T \cdots \int_t^{t_2} \left( K(t_1, \ldots, t_k) \prod_{l=1}^k \sqrt{r(t_l)} - \right.
\]

\[
- \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \prod_{l=1}^k \left( \Psi_{j_l}(t_l) \sqrt{r(t_l)} \right) \right)^2 dt_1 \ldots dt_k =
\]

\[
= C_k \int_{[t,T]^k} \left( K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \prod_{l=1}^k \Psi_{j_l}(t_l) \right)^2 \times
\]

\[
\times \left( \prod_{l=1}^k r(t_l) \right) dt_1 \ldots dt_k \to 0
\]

if \( p_1, \ldots, p_k \to \infty \), where constant \( C_k \) depends only on the multiplicity \( k \) of the iterated Ito stochastic integral [39]. Theorem 5 is proved.

Let us formulate the following theorem (the version of Theorem 3 in [43]).

**Theorem 6** [15]-[18]. Suppose that every \( \psi_l(\tau) (l = 1, \ldots, k) \) is a continuous non-random function on \([t, T] \) and \( \{ \Psi_j(\sqrt{r(x)}) \}_{0}^{\infty} \) \( r(x) \geq 0 \) is a complete orthonormal system of functions in the space \( L_2([t, T]) \), each function \( \Psi_j(\sqrt{r(x)}) \) of which for finite \( j \) satisfies the condition (*) (see Sect. 1). Then the estimate

\[
M \left\{ \left( \tilde{J}_{[\psi(l)]}^{[p(l)]}_{T,t} - \tilde{J}_{[\psi(l)]}^{[p(l)]}_{T,t} \right)^2 \right\} \leq
\]

\[
\leq k! \left( \int_{[t,T]^k} K^2(t_1, \ldots, t_k) \left( \prod_{l=1}^k r(t_l) \right) dt_1 \ldots dt_k - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1}^2 \right)
\]

is valid for the following cases:

1. \( i_1, \ldots, i_k = 1, \ldots, m \) and \( 0 < T - t < \infty \),
2. \( i_1, \ldots, i_k = 0, 1, \ldots, m, \) \( i_1^2 + \ldots + i_k^2 > 0 \), and \( 0 < T - t < 1 \),

where \( \tilde{J}_{[\psi(l)]}^{[p(l)]}_{T,t} \) is the stochastic integral [39], \( \tilde{J}_{[\psi(l)]}^{[p(l)]}_{T,t} \) is the expression on the right-hand side of (50) before passing to the limit \( \lim_{p_1, \ldots, p_k \to \infty} \); another notations are the same as in Theorem 5.

Consider the following generalizations of Theorems 5, 6.
**Theorem 7** [15] (Sect. 1.13), [43] (Sect. 17). Let \( \psi_1(x) \sqrt{r(x)}, \ldots, \psi_k(x) \sqrt{r(x)} \in L^2([t, T]) \), where \( r(x) \geq 0 \). Furthermore, let \( \{ \Psi_j(x) \sqrt{r(x)} \}_{j=0}^{\infty} \) is an arbitrary complete orthonormal system of functions in the space \( L^2([t, T]) \). Then, for the iterated Itô stochastic integral

\[
\mathcal{J}[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \sqrt{r(t_k)} \cdots \int_t^{t_2} \psi_1(t_1) \sqrt{r(t_1)} dw_{t_1}^{(i_1)} \cdots dw_{t_k}^{(i_k)}
\]

the following expansion

\[
\mathcal{J}[\psi^{(k)}]_{T,t} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \xi_l^{(i_l)} \right) + \left( \prod_{r=1}^{\lfloor k/2 \rfloor} (-1)^r \right) \times \sum_{\{((i_1, q_1), \ldots, (i_{k-1}, q_{k-1}), (i_k, q_k)) \neq (0, \ldots, 0)\}} \prod_{s=1}^{r} 1_{i_{q_{2s-1}} = i_{q_{2s}} \neq 0} 1_{j_{q_{2s-1}} = j_{q_{2s}}} \prod_{l=1}^{k-2r} \zeta_{l}^{(i_l)}
\]

that converges in the mean-square sense is valid, where \( i_1, \ldots, i_k = 0, 1, \ldots, m \), \( \zeta^{(i)} = \int_t^T \Psi_j(s) \sqrt{r(s)} dw_s^{(i)} \) are independent standard Gaussian random variables for various \( i \) or \( j \) (in the case when \( i \neq 0 \)),

\[
\hat{C}_{j_k \ldots j_1} = \int_{[t, T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \left( \Psi_j(t_l) r(t_l) \right) dt_1 \cdots dt_k
\]

is the Fourier coefficient, \( K(t_1, \ldots, t_k) \) is defined by [43]; another notations are the same as in Theorems 1, 2, 5.

**Theorem 8** [15] (Sect. 1.13), [43] (Sect. 17). Let \( \psi_1(x) \sqrt{r(x)}, \ldots, \psi_k(x) \sqrt{r(x)} \in L^2([t, T]) \), where \( r(x) \geq 0 \). Furthermore, let \( \{ \Psi_j(x) \sqrt{r(x)} \}_{j=0}^{\infty} \) is an arbitrary complete orthonormal system of functions in the space \( L^2([t, T]) \). Then the following estimate

\[
\mathbb{M} \left\{ \left( \mathcal{J}[\psi^{(k)}]_{T,t} - \mathcal{J}[\psi^{(k)}]_{T,t}^{p_1, \ldots, p_k} \right)^2 \right\} \leq k! \left( \int_{[t, T]^k} K^2(t_1, \ldots, t_k) \left( \prod_{l=1}^{k} r(t_l) \right) dt_1 \cdots dt_k - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \hat{C}_{j_k \ldots j_1}^2 \right)
\]

is valid for the following cases:

1. \( i_1, \ldots, i_k = 1, \ldots, m \) and \( 0 < T - t < \infty \),
2. \( i_1, \ldots, i_k = 0, 1, \ldots, m \), \( i_1^2 + \cdots + i_k^2 > 0 \), and \( 0 < T - t < 1 \).
where $\tilde{J}[\psi(k)]_{T,t}$ is the stochastic integral \cite{54}, \( \tilde{J}[\psi(k)]^{p_1,...,p_k}_{T,t} \) is the expression on the right-hand side of \cite{55} before passing to the limit \( \lim_{p_1,...,p_k \to \infty} \); another notations are the same as in Theorem 2, 7.

8. **One Modification of Theorems 4 and 5**

Let us compare \cite{52} and \cite{43}. If we suppose that $r(x) \geq 0$ and

\[
\frac{\rho(x)}{r(x)} \leq C < \infty,
\]

where $\rho(x)$ as in \cite{40}, then

\[
\int_{[t,T]^k} \left( K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1} \prod_{l=1}^{k} \Psi_{j_l}(t_l) \right)^2 \times
\]

\[
\times \rho(t_1)dt_1 \cdots \rho(t_k)dt_k =
\]

\[
= \int_{[t,T]^k} \left( K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1} \prod_{l=1}^{k} \Psi_{j_l}(t_l) \right)^2 \times
\]

\[
\times \frac{\rho(t_1)}{r(t_1)} \frac{\rho(t_k)}{r(t_k)} r(t_1)dt_1 \cdots \frac{\rho(t_k)}{r(t_k)} r(t_k)dt_k \leq
\]

\[
\leq C_k' \int_{[t,T]^k} \left( K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k\ldots j_1} \prod_{l=1}^{k} \Psi_{j_l}(t_l) \right)^2 \times
\]

\[
\times r(t_1)dt_1 \cdots r(t_k)dt_k,
\]

where $C_k'$ is a constant, \( \{\Psi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal with weight $r(x) \geq 0$ system of functions in the space $L_2([t,T])$, and the Fourier coefficient $\tilde{C}_{j_k\ldots j_1}$ has the form \cite{51}.

So, we obtain the following modification of Theorems 4 and 5.

**Theorem 9** \cite{15}, \cite{53}. Suppose that the following conditions are fulfilled:

1. Every $\psi_l(\tau)$ \( (l = 1, \ldots, k) \) is a continuous non-random function at the interval \([t,T]\).
2. $M_{r^{(l,i)}_{T,T}}(\rho, [t,T])$, $G_n(\rho, [t,T])$ with $n = 2^{k+1}$, $i_1 = 1, \ldots, m$, $l = 1, \ldots, k$ \( (k \in \mathbb{N}) \).
3. $\{\Psi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal with weight $r(\tau) \geq 0$ system of functions in the space $L_2([t,T])$, each function of which for finite $j$ satisfies the condition $(\ast)$ (see Sect. 1). Moreover,

\[
\frac{\rho(x)}{r(x)} \leq C < \infty.
\]

Then, for the iterated stochastic integral $\tilde{J}[\psi(k)]_{T,t}$ with respect to martingales defined by \cite{55} the following expansion
\[ J_{\Psi^{(k)}}^{\frac{M}{T,T}} = \lim_{p_1,\ldots,p_k \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \tilde{C}_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \xi_{l}^{(j_l i_l)} \right) \]

\[-1 \cdot \lim_{N \to \infty} \sum_{(l_1, \ldots, l_k) \in G_k} \Psi_{j_1}(\tau_{l_1}) \Delta M_{\tau_{l_1}}^{(1,i_1)} \cdots \Psi_{j_k}(\tau_{l_k}) \Delta M_{\tau_{l_k}}^{(k,i_k)} \]

converging in the mean-square sense is valid, where \( i_1, \ldots, i_k = 1, \ldots, m \), \( \{ \tau_j \}_{j=0}^{N} \) is a partition of the interval \([t,T]\) which satisfies the condition (31), \( \Delta M_{\tau_j}^{(r,i)} = M_{\tau_{j+1}}^{(r,i)} - M_{\tau_{j}}^{(r,i)} \) (\( i = 1, \ldots, m \), \( r = 1, \ldots, k \)),

\[ G_k = H_k \setminus L_k, \ H_k = \{ (l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1 \}, \]

\[ L_k = \{ (l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1; l_g \neq l_r \ (g \neq r) ; g, r = 1, \ldots, k \}, \]

\( \text{l.i.m. is a limit in the mean-square sense,} \)

\[ \xi_{j}^{(l,i)} = \int_{t}^{T} \Psi_{j}(s)dM_{s}^{(l,i)} \]

are independent for various \( i_l = 1, \ldots, m \) (\( l = 1, \ldots, k \)) and uncorrelated for various \( j \) (if \( i_l \neq 0 \), \( \rho(x) \equiv r(x) \)) random variables,

\[ \tilde{C}_{j_k \ldots j_1} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \left( \Psi_{j_l}(t_l)r(t_l) \right) dt_1 \cdots dt_k \]

is the Fourier coefficient,

\[ K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \cdots \psi_k(t_k) & \text{for } t_1 < \ldots < t_k \\ 0 & \text{otherwise} \end{cases} \]

and \( K(t_1) \equiv \psi_1(t_1) \) for \( t_1 \in [t,T] \).

**Remark 2.** Note that if \( \rho(\tau), r(\tau) \equiv 1 \) in Theorem 9, then we obtain the variant of Theorem 1.

9. **Example on Application of Theorem 9 for the System of Bessel Functions**

Let us consider the following boundary-value problem

\[ (p(x)\Phi'(x))' + q(x)\Phi(x) = -\lambda r(x)\Phi(x), \]

\[ \alpha \Phi(a) + \beta \Phi'(a) = 0, \quad \gamma \Phi(b) + \delta \Phi'(b) = 0, \]
where the functions \( p(x), q(x), r(x) \) satisfy the well known conditions and \( \alpha, \beta, \gamma, \delta, \lambda \) are real numbers.

It is well known (Steklov V.A.) that the eigenfunctions \( \Phi_0(x), \Phi_1(x), \ldots \) of this boundary-value problem form a complete orthonormal with weight \( r(x) \) system of functions in the space \( L_2([a,b]) \). It means that the Fourier series of the function \( \sqrt{r(x)}f(x) \in L_2([a,b]) \) with respect to the system of functions

\[
\sqrt{r(x)}\Phi_0(x), \quad \sqrt{r(x)}\Phi_1(x), \quad \ldots
\]

converges in the mean-square sense to the function \( \sqrt{r(x)}f(x) \) at the interval \([a,b]\). Moreover, the Fourier coefficients are defined by the formula

\[
C_j = \int_a^b r(x)f(x)\Phi_j(x)dx.
\]

(56)

It is known that when solving the problem on oscillations of a circular membrane (general case), a boundary-value problem arises for the following Euler–Bessel equation

(57)

\[
r^2 R''(r) + rR'(r) + \left(\lambda^2 r^2 - n^2\right)R(r) = 0 \quad (\lambda \in \mathbb{R}, \ n \in \mathbb{N}).
\]

The eigenfunctions of this problem, taking into account specific boundary conditions, are the following functions

(58)

\[
J_n\left(\mu_j \frac{r}{L}\right),
\]

where \( \tau \in [0, L] \) and \( \mu_j \) \((j = 0, 1, 2, \ldots)\) are positive roots of the Bessel function \( J_n(\mu) \) \((n = 0, 1, 2, \ldots)\) numbered in ascending order.

The problem on radial oscillations of a circular membrane leads to the boundary-value problem for the equation \( \text{(57)} \) for \( n = 0 \), the eigenfunctions of which are the functions \( \text{(58)} \) when \( n = 0 \).

Let us analyze the system of functions

(59)

\[
\Psi_j(\tau) = \frac{\sqrt{2}}{TJ_{n+1}(\mu_j)}J_n\left(\frac{\mu_j}{T} \tau\right), \quad j = 0, 1, 2, \ldots,
\]

where

\[
J_n(x) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{2}\right)^{n+2m} \frac{1}{\Gamma(m+1)\Gamma(m+n+1)}
\]

is the Bessel function of the first kind and

\[
\Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx
\]

is the gamma-function, \( \mu_j \) are positive roots of the function \( J_n(x) \) numbered in ascending order, and \( n \) is a natural number or zero.
Due to the well known properties of the Bessel functions, the system \( \{ \Psi_j(\tau) \}_{j=0}^{\infty} \) is a complete orthonormal system of continuous functions with weight \( \tau \) in the space \( L_2([0,T]) \).

Let us use the system of functions \( \{ \Psi_j \} \) in Theorem 9.

Consider the following iterated stochastic integral with respect to martingales

\[
\int_0^T \int_0^s dM^{(1)}_\tau dM^{(2)}_s,
\]

where

\[
M_s^{(i)} = \int_0^s \sqrt{\tau} d\bar{f}^{(i)}_\tau, \quad (i = 1, 2),
\]

\( f^{(i)}_\tau \) \((i = 1, 2)\) are independent standard Wiener processes, \( M_s^{(i)} \) \((i = 1, 2)\) are martingales (here \( \rho(\tau) \equiv \tau \)), \( 0 \leq s \leq T \). In addition, \( M_s^{(i)} \) has a Gaussian distribution.

It is obvious that the conditions of Theorem 9 are fulfilled for \( k = 2 \). Using Theorem 9, we obtain

\[
\int_0^T \int_0^s dM^{(1)}_\tau dM^{(2)}_s = \lim_{p_1, p_2 \to \infty} \frac{p_1}{p_1} \sum_{j_1=0}^{p_1} \frac{p_2}{p_2} \sum_{j_2=0}^{p_2} \tilde{C}_{j_2 j_1} \zeta^{(1)}_{j_1} \zeta^{(2)}_{j_2},
\]

where

\[
\zeta^{(i)}_{j} = \int_0^T \Psi_j(\tau) dM^{(i)}_\tau,
\]

are independent standard Gaussian random variables for various \( i \) or \( j \) \((i = 1, 2, \ j = 0, 1, 2, \ldots)\),

\[
\mathcal{M} \left\{ \zeta^{(1)}_{j_1} \zeta^{(2)}_{j_2} \right\} = 0,
\]

\[
\tilde{C}_{j_2 j_1} = \int_0^T \frac{p_1}{p_1} \Psi_{j_2}(s) \int_0^s \frac{p_2}{p_2} \Psi_{j_1}(\tau) d\tau d\tau.
\]

It is obvious that we can get this result using the another approach: we can use Theorems 1, 2 for the iterated Ito stochastic integral

\[
\int_0^T \sqrt{s} \int_0^s \sqrt{\tau} d\bar{f}^{(1)}_\tau d\bar{f}^{(2)}_s,
\]

and as a system of functions \( \{ \phi_j(s) \}_{j=0}^{\infty} \) in Theorems 1, 2 we can take

\[
\phi_j(s) = \frac{\sqrt{2s}}{T J_{n+1}(\mu_j)} J_n \left( \frac{\mu_j}{T} s \right), \quad j = 0, 1, 2, \ldots.
\]

As a result, we obtain
\[
\int_0^T \sqrt{s} \int_0^s \sqrt{t} \, df_T^{(1)} \, df_T^{(2)} = \lim_{p_1, p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2j_1} \zeta_j^{(1)} \zeta_j^{(2)},
\]

where

\[
\zeta_j^{(i)} = \int_0^T \phi_j(\tau) \, df_T^{(i)}
\]

are independent standard Gaussian random variables for various \( i \) or \( j \) (\( i = 1, 2, j = 0, 1, 2, \ldots \)),

\[
M \left\{ \zeta_j^{(1)} \zeta_j^{(2)} \right\} = 0, \quad C_{j_2j_1} = \int_0^T \sqrt{s} \phi_{j_2}(s) \int_0^s \sqrt{\tau} \phi_{j_1}(\tau) \, d\tau 
\]

is the Fourier coefficient. Obviously that \( C_{j_2j_1} = \tilde{C}_{j_2j_1} \).

Easy calculation demonstrates that

\[
\tilde{\phi}_j(s) = \frac{\sqrt{2(s-t)}}{(T-t)J_{n+1}(\mu_j)} J_n \left( \frac{\mu_j}{T-t} (s-t) \right), \quad j = 0, 1, 2, \ldots
\]

is a complete orthonormal system of functions in the space \( L_2([t,T]) \).

Then, using Theorems 1, 2, we obtain

\[
\int_t^T \sqrt{s-t} \int_t^s \sqrt{\tau-t} \, df_T^{(1)} \, df_T^{(2)} = \lim_{p_1, p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2j_1} \tilde{\zeta}_j^{(1)} \tilde{\zeta}_j^{(2)},
\]

where

\[
\tilde{\zeta}_j^{(i)} = \int_t^T \tilde{\phi}_j(\tau) \, df_T^{(i)}
\]

are independent standard Gaussian random variables for various \( i \) or \( j \) (\( i = 1, 2, j = 0, 1, 2, \ldots \)),

\[
M \left\{ \tilde{\zeta}_j^{(1)} \tilde{\zeta}_j^{(2)} \right\} = 0, \quad C_{j_2j_1} = \int_t^T \sqrt{s-t} \tilde{\phi}_{j_2}(s) \int_t^s \sqrt{\tau-t} \tilde{\phi}_{j_1}(\tau) \, d\tau 
\]

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