Spin-3 quasinormal modes of BTZ black hole

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Abstract

Using the operator approach, we obtain quasinormal modes (QNMs) of BTZ black hole in spin-3 topologically massive gravity by solving the first-order equation of motion with the transverse-traceless condition. We find that these are different from those obtained when solving the second-order differential equation for the third-rank tensor of spin-3 field subject to suitable boundary conditions and having the sign ambiguity of mass. However, it is shown clearly that two approaches to the left-moving QNMs are identical, while the right-moving QNMs of solving the second-order equation are given by descendants of the operator approach.

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1 Introduction

Recently, higher-spin theories on AdS$_3$ have been paid much attention because they admit a truncation to an arbitrary maximal spin $N$ \cite{1, 2}. Especially, the prototype of spin-3 model is a third-rank tensor of spin-3 field coupled to topologically massive gravity. The authors \cite{3} have discussed the traceless spin-3 fluctuations around AdS$_3$ spacetimes and, found that there exists a single massive propagating mode, besides left-moving and right-moving massless modes (gauge artifacts). Also, a trace part of spin-3 fluctuations on AdS$_3$ spacetimes has been studied in Ref. \cite{4}. However, such a massive trace mode has zero energy and becomes pure gauge at the chiral point. These are considered through extended analysis of spin-2 field in the cosmological topologically massive gravity \cite{5}.

Very recently, Datta and David \cite{6} have solved massive wave equations of arbitrary integer spin fields including spin-3 fields in the BTZ black hole background, and have obtained their quasinormal modes which are consistent with the location of the poles of the corresponding two-point function in the dual conformal field theory. This could be predicted by the AdS$_3$/CFT$_2$ correspondence. They have considered the second-order equation of $[□ - m^2 + 4/\ell^2] \Phi_{\rho\mu\nu} = 0$ for spin-3 fields with the ingoing modes at horizon and Dirichlet boundary condition at infinity. However, in this case, one confronts with sign ambiguity of mass $m$. Thus, in order to avoid this ambiguity, one could solve the first-order equation of $\epsilon^{\alpha\beta\rho}_a \nabla_\alpha \Phi_{\beta\mu\nu} + m \Phi_{\rho\mu\nu} = 0$ itself with the transverse and traceless (TT) gauge condition.

On the other hand, it was known that the operator approach (method) \cite{7} is very useful to derive the quasinormal modes of spin-2 fields in the non-rotating BTZ black hole background in the framework of cosmological topologically massive gravity. This method has been applied to new massive gravity to derive their quasinormal modes of the non-rotating BTZ black hole \cite{8}.

In this work, we obtain quasinormal modes of the non-rotating BTZ black hole in spin-3 topologically massive gravity by directly solving the first-order equation with the TT gauge condition in the operator approach. This method shows clearly how to derive quasinormal modes without sign ambiguity in mass.

2 Perturbation analysis for spin-3 fields

Since the spin-3 fluctuations on AdS$_3$ or BTZ background was formulated in \cite{3}, let us write down the perturbation equation for the spin-3 fields $\Phi_{\rho\mu\nu} = \epsilon_{\rho\mu\nu}^{ab} \epsilon^{\nu}_\alpha \epsilon^{\lambda}$ with $\epsilon_{\rho\mu\nu}$ spin-3
connection and $\hat{e}^{\mu}_{\nu}$ the background dreibein as
\begin{equation}
\Box \Phi^\alpha_{\beta} + \frac{1}{2\mu} \epsilon^{\mu\nu} \nabla_{\mu} \Box \Phi_{\nu}^{\alpha\beta} = 0. \tag{1}
\end{equation}
Similar to the perturbed equation of the (spin-2) graviton,
\begin{equation}
(\Box + \frac{2}{\ell^2}) h^\rho_{\sigma} + \frac{1}{\mu} \epsilon^{\rho\mu
u} \nabla_{\rho} (\Box + \frac{2}{\ell^2}) h_{\nu\sigma} = 0,
\end{equation}
the spin-3 fluctuation also satisfies a third-order differential equation.

In this work, we consider the non-rotating BTZ black hole with the mass $M = 1$ and the AdS$_3$ curvature radius $\ell = 1$ in global coordinates as
\begin{equation}
ds^2_{\text{BTZ}} = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\sinh^2 \rho d\tau^2 + \cosh^2 \rho d\phi^2 + d\rho^2,
\end{equation}
where the event horizon is located at $\rho = 0$, while the infinity is at $\rho = \infty$. Here we note that $\bar{g}_{\mu\nu} = \hat{e}^{\mu}_{\alpha} \hat{e}^{\nu}_{\beta} \eta_{ab}$. In terms of the light-cone coordinates $u/v = \tau \pm \phi$, the metric tensor $\bar{g}_{\mu\nu}$ takes the form of
\begin{equation}
\bar{g}_{\mu\nu} = \begin{pmatrix}
\frac{1}{4} & -\frac{1}{4} \cosh 2\rho & 0 \\
-\frac{1}{4} \cosh 2\rho & \frac{1}{4} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\end{equation}

Then the metric tensor (4) admits the Killing vector fields $L_k$ ($k = 0, -1, 1$) for the local $\text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$ algebra as
\begin{equation}
L_0 = -\partial_u, \quad L_{-1/1} = e^\mp u \left[ -\frac{\cosh 2\rho}{\sinh 2\rho} \partial_u - \frac{1}{\sinh 2\rho} \partial_v \mp \frac{1}{2} \partial_\rho \right],
\end{equation}
and $\bar{L}_0$ and $\bar{L}_{-1/1}$ are obtained by substituting $u \leftrightarrow v$. Locally, they form a basis of the $\text{SL}(2,\mathbb{R})$ Lie algebra as
\begin{equation}
[L_0, L_{\pm1}] = \mp L_{\pm1}, \quad [L_1, L_{-1}] = 2L_0.
\end{equation}

In the BTZ black hole background, the spin-3 field of $\Phi^{\rho\mu\nu}$ determined by a third-order differential equation (1) is totally symmetric and satisfies the TT gauge condition
\begin{equation}
\Phi^{\rho\mu\nu} = 0, \quad \bar{\nabla}^{\nu} \Phi_{\mu\rho\nu} = 0.
\end{equation}
Hence its number of propagating degrees of freedom is counted to be one as
\begin{equation}
10 - 3 - 6 = 1,
\end{equation}
which corresponds to a single massive propagating mode as that in AdS$_3$ background [3]. The third-order equation (1) can also be expressed as
\begin{equation}
(D^M D^L D^R \Phi)^{\rho\mu\nu} = 0
\end{equation}
in terms of mutually commuting operators of

\[(D^{L/R})_{\rho \sigma } = \delta_{\rho \sigma } + \frac{1}{2} \epsilon^{\rho \sigma \mu } \nabla_{\mu }, \quad (D^{M})_{\rho \sigma } = \delta_{\rho \sigma } + \frac{1}{2\mu } \epsilon^{\rho \sigma \mu } \nabla_{\mu }\]. \tag{10}

At the critical point of \( \mu = 1 \), the operators \( D^{M} \) and \( D^{L} \) degenerate. We note that Eq. (9) is reduced to Eq. (1) when using the BTZ background

\[\bar{R}_{\rho \sigma } = -(\bar{g}_{\rho \mu } \bar{g}_{\sigma \nu } - \bar{g}_{\rho \nu } \bar{g}_{\sigma \mu }), \quad \bar{R}_{\mu \nu } = -2\bar{g}_{\mu \nu }\], \tag{11}

together with the TT gauge condition and the relation of \([\nabla_{\mu }, \nabla_{\nu }]\Phi_{\rho \sigma \mu } = -4\Phi_{\rho \sigma }\]. Therefore, the third-order equation (11) can be decomposed into three first-order differential equations:

\[(D^{M}\Phi)_{\rho \mu \nu } = 0, \quad (D^{L}\Phi)_{\rho \mu \nu } = 0, \quad (D^{R}\Phi)_{\rho \mu \nu } = 0, \tag{12}\]

for a massive, a left-moving, and a right-moving degree of freedom, respectively.

Importantly, three first-order differential equations (12) can be simply rewritten in terms of a single massive first-order differential equation as

\[\epsilon_{\rho}^{\alpha \beta} \nabla_{\alpha} \Phi_{\beta \mu \nu } + m\Phi_{\rho \mu \nu } = 0 \] \tag{13}

with \( m = 2\mu, \ 2, \) and \(-2\). On the other hand, it could also be expressed in terms of a second-order differential equation [6] as

\[\square^{2} - m^{2} + 4\Phi_{\rho \mu \nu } = 0\]. \tag{14}

At this stage, we wish to point out the presence of sign ambiguity \( \pm m \) in the second-order equation (14). In order to avoid this ambiguity, one could directly solve the first-order equation (13) with the TT gauge condition.

Having the structure in mind, let us find quasinormal modes for the spin-3 field in the BTZ background by solving the equation of motion (13) with the TT gauge condition. In order to implement the operator method [7, 8], let us choose either the anti-chiral highest weight condition of \( L_{1}\Phi_{\rho \mu \nu } = 0 \) or the chiral highest weight condition of \( \bar{L}_{1}\Phi_{\rho \mu \nu } = 0 \), but not both simultaneously. Actually, we note that for a generic symmetric tensor \( \Phi_{\rho \mu \nu } \), the transversality condition of \( \nabla^{\mu} \Phi_{\mu \nu \rho } = 0 \) is not equivalent to choosing the chiral (anti-chiral) highest weight condition.

### 3 Left-moving quasinormal modes

The least damped \((n = 0)\) quasinormal mode can be found by considering the form

\[\Phi_{\rho \mu \nu}(u, v, \rho ) = e^{-i\omega \tau - i\phi} F_{\rho \mu \nu}(\rho ) = e^{-ihu - i\bar{h}v} F_{\rho \mu \nu}(\rho )\] \tag{15}
with $\omega = h + \bar{h}$ and $k = h - \bar{h}$. This is the primary field which satisfies
\[
L_0 \Phi_{\rho\mu
u}(u, v, \rho) = i\hbar \Phi_{\rho\mu
u}(u, v, \rho), \quad \bar{L}_0 \Phi_{\rho\mu
u}(u, v, \rho) = i\bar{\hbar} \Phi_{\rho\mu
u}(u, v, \rho).
\] (16)

Note here that the subscript $\rho$ in $\Phi_{\rho\mu
u}(u, v, \rho)$ is a dummy index, while $\rho$ in the argument is the radial coordinate in (3). It seems to be a formidable task to solve the first-order equation with the TT gauge condition without choosing a simplified form of $F_{\rho\mu\nu}$. Inspired by the lesson learned from the spin-2 analysis [7, 8], after tedious computations, we find the explicit solution
\[
F_{u\mu\nu}(\rho) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\] (17)
\[
F_{v\mu\nu}(\rho) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{2}{\sinh 2\rho} \\ 0 & \frac{2}{\sinh 2\rho} & \frac{1}{\sinh^2 2\rho} \end{pmatrix} F_{vvv}(\rho),
\] (18)
\[
F_{\rho\mu\nu}(\rho) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{\sinh 2\rho} & \frac{4}{\sinh^2 2\rho} \\ 0 & \frac{4}{\sinh^2 2\rho} & \frac{8}{\sinh^3 2\rho} \end{pmatrix} F_{vvv}(\rho),
\] (19)

which imply that $F_{vvv}(\rho)$ is a single massive propagating mode. Here Eq. (18) is similar to the spin-2 case [7], while Eqs. (17) and (19) represent new features of the spin-3 field.

Under the form of $\Phi_{\rho\mu\nu}(u, v, \rho)$ in Eq. (15) with $F_{\rho\mu\nu}(\rho)$ in Eqs. (17)-(19), the transversality condition of $\bar{\nabla}^\mu \Phi_{\mu\rho} = 0$ is equivalent to the anti-chiral highest weight condition of $L_1 \Phi_{\rho\mu\nu} = 0$, giving the constraint
\[
\sinh 2\rho \left[ \frac{d}{d\rho} F_{vvv}(\rho) \right] + 2i(\hbar + \hbar \cosh 2\rho) F_{vvv}(\rho) = 0.
\] (20)

We emphasize that Eq. (20) takes the form of the equation for the scalar field $F_{vvv}(\rho)$, not a third-rank tensor field. Then, its solution is given by
\[
F_{vvv}(\rho) = C(\sinh 2\rho)^{-i\hbar}(\tanh \rho)^{-i\hbar}
\] (21)
with a constant $C$. Finally, the equation of motion (13) determines $h$ as a function of $m$ of
\[
h = -ih_L(m), \quad h_L(m) = \frac{1}{2}(m - 2).
\] (22)

Thus, the solution is summarized as
\[
\Phi_{\rho\mu\nu}^L(u, v, \rho) = e^{ik(\tau - \phi) - 2h_L(m)\tau} F_{\rho\mu\nu}^L(\rho)
\] (23)
where $F_{\mu\nu}^L(\rho)$ is given by Eqs. (17)-(19) with

$$F_{\nu\nu}^L(\rho) = (\sinh \rho)^{-2h_L(m)}(\tanh \rho)^i k.$$  \hspace{1cm} (24)

Considering the form of quasinormal frequency

$$\omega = \omega_{\text{Re}} - i\omega_{\text{Im}},$$

we read off it from Eq. (23)

$$\omega_L = -k - 2ih_L(m).$$

Thus, the solution (23) corresponds to a left-moving massive quasinormal mode of the least damped ($n = 0$) case for $m = 2\mu$, leading to

$$h_L(\mu) = \mu - 1 > 0 \quad \mu > 1.$$ \hspace{1cm} (27)

As is expected by the anti-chiral gravity, we observe that there is no quasinormal modes ($h_L = 0$) at the anti-chiral point of $\mu = 1$. Also we observe that the asymptotic $\rho$-dependence of $F_{\nu\nu}^L$ takes the form of $F_{\nu\nu}^L \sim e^{2(1-\mu)\rho}$, which is compared to the spin-2 asymptotic dependence of $F_{\nu\nu} \sim e^{(1-\mu)\rho}$ for $m = \mu$.  \hspace{1cm} (9)

In order to derive the higher-order quasinormal modes, we act on the anti-chiral highest weight quasinormal modes with the operator of $\bar{L}_{-1}L_{-1}$. The effect of this will be to replace $\omega_{\text{Im}}$ by $\omega_{\text{Im}} + 2$ in Eq. (23). Hence one could expect to have

$$\Phi_{\mu\nu}^{(n)L}(u, v, \rho) = \left(\bar{L}_{-1}L_{-1}\right)^n \Phi_{\mu\nu}^L(u, v, \rho),$$

which are descendents of $\Phi_{\mu\nu}^L(u, v, \rho)$. Since $\bar{L}_{-1}L_{-1}$ commutes with the equation (13), $\Phi_{\mu\nu}^{(n)L}(u, v, \rho)$ is again the solution to the first-order equation with the same boundary condition of asymptotic fall-off as in $\Phi_{\mu\nu}^L(u, v, \rho)$. Hence, the complete tower of the left-moving spin-3 quasinormal modes could be generated from $\Phi_{\mu\nu}^L(u, v, \rho)$.

Consequently, the corresponding quasinormal frequencies are given by

$$\omega^n_L = -k - 2i\left(h_L(\mu) + n\right), \quad n \in \mathbb{Z}.$$ \hspace{1cm} (29)
4 Right-moving quasinormal modes

On the other hand, right-moving quasinormal modes of the least damped case can be obtained by substitution of $u \to v$, $h \to \bar{h}$, and $m \to -m$. Explicitly, they take the form of

$$F_{u\mu
u}(\rho) = \begin{pmatrix} 1 & 0 & \frac{2}{\sinh 2\rho} \\ 0 & 0 & 0 \\ \frac{2}{\sinh 2\rho} & 0 & \frac{4}{\sinh^2 2\rho} \end{pmatrix} F_{uuu}(\rho), \quad (30)$$

$$F_{v\mu
u}(\rho) = \begin{pmatrix} \frac{2}{\sinh 2\rho} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (31)$$

$$F_{\rho\mu
u}(\rho) = \begin{pmatrix} \frac{2}{\sinh 2\rho} & 0 & \frac{4}{\sinh^2 2\rho} \\ 0 & 0 & 0 \\ \frac{8}{\sinh^2 2\rho} & 0 & \frac{8}{\sinh^2 2\rho} \end{pmatrix} F_{uuu}(\rho), \quad (32)$$

which imply that $F_{uuu}(\rho)$ is a single massive propagating mode. Here Eq. (30) is similar to the spin-2 case [7], while Eqs. (31) and (32) represent new features of the spin-3 field.

The transversality condition of $\bar{\nabla}^\mu \Phi_{\mu\rho\nu} = 0$ in this case is now compatible with the chiral highest weight condition of $\bar{L}_1 \Phi_{\rho\mu\nu} = 0$, giving the differential equation of $F_{uuu}(\rho)$

$$\sinh 2\rho \left[ \frac{d}{d\rho} F_{uuu}(\rho) \right] + 2i(h + \bar{h} \cosh 2\rho) F_{uuu}(\rho) = 0.$$

The solution is given by

$$F_{uuu}(\rho) = D(\sinh 2\rho)^{-i\bar{h}}(\tanh \rho)^{-ih} \quad (34)$$

with a constant $D$. The equation of motion (13) determines $\bar{h}$ as a function of $m$

$$\bar{h} = -ih_R(m), \quad h_R(m) = \frac{1}{2}(m - 2).$$

Considering $m = -2\mu$, one finds

$$h_R(\mu) = -\mu - 1 \equiv \bar{\mu} - 1 > 0, \quad \mu < -1 \ (\bar{\mu} > 1). \quad (36)$$

Then, the $n = 0$ least damped right-moving solution is given by

$$\Phi_R^{\mu\nu}(u, v, \rho) = e^{-ik(\tau + \phi) - 2h_R(\mu)\tau} F_R^{\mu\nu}(\rho) \quad (37)$$

where $F_R^{\mu\nu}(\rho)$ is given by Eqs. (30)-(32) with

$$F_R^{\mu\nu}(\rho) = (\sinh \rho)^{-2h_R(\mu)}(\tanh \rho)^{-ik}. \quad (38)$$
Thus, its quasinormal mode can be read off as
\[ \omega_R = k - 2i h_R(\mu). \] (39)

As is expected by the chiral gravity, we observe that there is no quasinormal modes \((h_R = 0)\) at the chiral point of \(\mu = -1\).

Similarly, the higher-order quasinormal modes are obtained by acting the operator of \(\bar{L}_{-1}L_{-1}\) as
\[ \Phi_{\rho\mu\nu}^{(n)R}(u, v, \rho) = \left(\bar{L}_{-1}L_{-1}\right)^n \Phi_{\rho\mu\nu}^{R}(u, v, \rho), \] (40)
which are descendants of \(\Phi_{\rho\mu\nu}^{R}(u, v, \rho)\). Its quasinormal frequencies are given by
\[ \omega^n_R = k - 2i \left( h_R(\mu) + n \right). \] (41)

In Table 1, we have briefly summarized the results by comparing the spin-2 field in Ref. [7, 8] with the spin-3 topologically massive gravity. For the spin-2 field satisfying (2), the QNMs for the left-moving component exist for only \(\mu > 1\), while the QNMs for the right-moving one for only \(\mu < -1\) as in Ref. [7, 8]. Especially, the result of Ref. [7] is obtained by \(u \rightarrow v, h \rightarrow \bar{h}\), but not by \(m \rightarrow -m\). Instead, the authors gave the mass ranges for the QNMs as the left-moving (right-moving) component for \(\mu > 1\) \((\mu < -1)\), which are exactly the same with replacing \(m\) by \(-m\) in the equation of motion as shown in Table 1. The QNMs for the left-moving (right-moving) spin-3 field are, by the same token, valid for only \(\mu > 1\) \((\mu < -1)\). Here we also note that there are no QNMs at \(\mu = 1\) \((\mu = -1)\) for the left-moving (right-moving) spin-2 field, while at \(\mu = 1\) \((\mu = -1)\) for the left-moving (right-moving) spin-3 field, expected by anti-chiral (chiral) gravity, respectively.

5 Discussions

We have obtained quasinormal modes of BTZ black hole in spin-3 topologically massive gravity by directly solving the first-order equation with the transverse-traceless condition in the operator approach. We have found that there is no \(n = 0\) quasinormal modes \((h_{L/R} = 0)\) at the anti-chiral/chiral point of \(\mu = \pm 1\).

It seemed that these are different from those with \(T_{L/R} = \frac{1}{2\pi} \) [6]
\[ \omega_{sL}^{n} = k - 2\pi T_{L}i(2n + m + 1 - s), \quad \omega_{sR}^{n} = -k - 2\pi T_{R}i(2n + m + 1 + s), \] (42)
which are obtained when solving the second-order differential equations for the \(s\)-rank tensor of spin-\(s\) field imposed by the boundary conditions. We note that the signs \pm of real part...
Solutions of first-order differential equations

| spin-2 field | spin-3 field |
|--------------|--------------|
| $\epsilon^{\alpha\beta\mu}_{\lambda} \nabla_{\alpha} h_{\beta\nu} + m h_{\mu\nu} = 0$ | $\epsilon^{\alpha\beta\mu}_{\lambda} \nabla_{\alpha} \Phi_{\beta\mu\nu} + m \Phi_{\rho\mu\nu} = 0$ |
| $L_1 h_{\mu\nu} = 0$ | $L_1 \Phi_{\rho\mu\nu} = 0$ |
| $h_L(m) = \frac{m}{2} - \frac{1}{2} m_{=\mu} = \frac{m}{2} - \frac{1}{2}$ | $h_R(m) = \frac{m}{2} - \frac{1}{2} m_{=\mu} = -\frac{m}{2} - \frac{1}{2}$ |
| $\omega_L^n = -k - 2i (h_L(m) + n)$ | $\omega_R^n = k - 2i (h_R(m) + n)$ |

Table 1: Summary of the QNMs by comparing the spin-2 field in Ref. [8] with the spin-3 field in topologically massive gravity: The right-moving solution is obtained by solving the first-order equation of motion with the replacement of $u \to v$, $h \to \bar{h}$, and $m \to -m$. The equation of motion with $m$ allows only the left-moving (anti-chiral) solution, while the one with $-m$ gives only the right-moving (chiral) solution in Ref. [8].

$\omega_{sL/R}^n$ are different from $\mp$ of $\omega_{sL/R}^n$. We adhere to the convention of Ref. [7] for the left/right-moving modes. Comparing the left-moving QNMs in Eq. (29) with $\omega_{sL}^n$ with $m = 2\mu$ leads to the same expression for the imaginary sector. The same thing happens for the spin-2 when comparing $\omega_{sL}^n$ with $m = \mu$.

On the other hand, it seems that the right-moving QNMs in Eq. (41) are different from $\omega_{3R}^n$. However, $\omega_{3R}^n$ could be recovered from the descendants of $\Phi_{\rho\mu\nu}^{R}(u,v,\rho)$ [9]. That is, one has that imaginary $[\omega_{3R}^n]$=imaginary $[\omega_{sR}^{n+3}]$ for spin-3. Similarly, one has that imaginary $[\omega_{2R}^n]$=imaginary $[\omega_{sR}^{n+2}]$ for spin-2. We have constructed these descendants for the spin-2 case in Appendix A and the spin-3 case in Appendix B, whose asymptotic forms of relevant part are consistent with those obtained by solving the second-order differential equations.

For $\mu = \pm 1$, we expect to develop logarithmic modes of spin-3 field in the BTZ black hole background as did for spin-2 field [8] [10].

Consequently, the operator approach combined with the first-order equation is a useful method to derive QNMs of the BTZ black hole in the spin-s topologically massive gravity. We suggest that two approaches to the left-moving QNMs are identical, while the right-moving QNMs of solving second-order equation are given by descendants of the operator approach.
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Appendix: Descendants of spin-\(s\) field

A. Descendants of spin-2 field

It was known that by solving the first-order equation of \(\epsilon_{\mu}^{\alpha \beta} \nabla_{\alpha} h_{\beta \nu} + mh_{\mu \nu} = 0\) with the TT condition, one has the ingoing highest weight solution for the left-moving spin-2 field near the horizon \([8, 9]\)

\[
h_{\mu \nu}^{L} = e^{(1-\mu)t + ik(t-\phi)}(\sinh \rho)^{1-\mu}(\tanh \rho)^{ik} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{2}{\sinh 2\rho} \\ 0 & \frac{2}{\sinh 2\rho} & \frac{4}{\sinh^2 2\rho} \end{pmatrix}.
\]  
(43)

On the other hand, by the substitution of \(u \rightarrow v, h \rightarrow \bar{h},\) and \(m \rightarrow -m,\) we have the ingoing highest weight solution for the right-moving spin-2 field near the horizon

\[
h_{\mu \nu}^{R} = e^{(\mu + 1)t - ik(t+\phi)}(\sinh \rho)^{\mu+1}(\tanh \rho)^{-ik} \begin{pmatrix} 1 & 0 & \frac{2}{\sinh 2\rho} \\ 0 & 0 & 0 \\ \frac{2}{\sinh 2\rho} & \frac{4}{\sinh^2 2\rho} \end{pmatrix}.
\]  
(44)

By acting the operator of \(\bar{L}_{-1}L_{-1},\) the second descendent for the right-moving mode is given by

\[
h_{\mu \nu}^{(2)R} = \left(\bar{L}_{-1}L_{-1}\right)^2 h_{\mu \nu}^{R}
= \frac{1}{8} e^{-2kR(\mu)t - ik(t+\phi)}(\sinh \rho)^{\mu-3}(\tanh \rho)^{-ik} \begin{pmatrix} m_{uu} & 0 & \frac{m_{uo}}{\sinh 2\rho} \\ 0 & 0 & 0 \\ \frac{m_{uo}}{\sinh 2\rho} & 0 & \frac{m_{oo}}{\sinh^2 2\rho} \end{pmatrix},
\]  
(45)
where
\[
R = \frac{\mu}{2} + \frac{3}{2},
\] (46)
and
\[
m_{uu} = (\mu(1 + \mu) - k^2 - ik(1 + 2\mu)) \times \\
(16 - 13\mu + 3\mu^2 - 8k^2 + 4\mu(-3 + \mu) \cosh 2\rho + \mu(1 + \mu) \cosh 4\rho \\
- 8ik(-3 + \mu + \mu \cosh 2\rho)),
\] (47)
\[
m_{u\rho} = \frac{1}{8 \cosh^4 \rho} \{ -16 + 40\mu + 139\mu^2 - 150\mu^3 + 35\mu^4 - k^2(172 - 475\mu + 243\mu^2) \\
+ 48k^4 - ik(100 + 283\mu - 465\mu^2 + 150\mu^3 + 16k^2(10 - 11\mu)) \\
+ 4(2(1 - \mu)^2(-11 - 15\mu + 7\mu^2) - k^2(39 - 181\mu + 90\mu^2) + 16k^4 \\
+ ik(7 - 102\mu + 177\mu^2 - 58\mu^3 - 62k^2(1 - \mu))) \cosh 2\rho \\
+ 4(-2 + 6\mu + 21\mu^2 - 26\mu^3 + 7\mu^4 - k^2(25 - 73\mu + 35\mu^2) + 4k^4 \\
- ik(15 + 41\mu - 75\mu^2 + 26\mu^3 + 4k^2(6 - 5\mu))) \cosh 4\rho \\
- 4(1 + \mu)(2(1 - 2\mu^2 + \mu^3) + k^2(5 - 6\mu) - ik(1 - 9\mu + 6\mu^2 - 2k^2)) \cosh 6\rho \\
+ \mu(1 - \mu)(\mu(1 - \mu) + k^2 - ik(1 - 2\mu)) \cosh 8\rho \},
\] (48)
\[
m_{\rho\rho} = \frac{1}{4 \cosh^4 \rho} \{ 140 - 196\mu + 391\mu^2 - 210\mu^3 + 35\mu^4 - k^2(502 - 665\mu + 243\mu^2) \\
+ 48k^4 - ik(14 + 823\mu - 651\mu^2 + 150\mu^3 + 16k^2(14 - 11\mu)) \\
- 8(28 + 44\mu - 77\mu^2 + 42\mu^3 - 7\mu^4 + k^2(62 - 133\mu + 45\mu^2) - 8k^4 \\
- ik(86 - 153\mu + 129\mu^2 - 29\mu^3 - k^2(46 - 31\mu))) \cosh 2\rho \\
+ 4(28 - 44\mu + 75\mu^2 - 42\mu^3 + 7\mu^4 - k^2(82 - 121\mu + 35\mu^2) + 4k^4 \\
+ ik(2 - \mu)(1 - 71\mu + 26\mu^2 - 20k^2)) \cosh 4\rho \\
- 8(2 - \mu)(2 + 3\mu - 4\mu^2 + \mu^3 + k^2(5 - 3\mu) - ik(5 - 9\mu + 3\mu^2 - k^2)) \cosh 6\rho \\
+ (1 - \mu)(2 + \mu)((1 - \mu)(2 - \mu) - k^2 + ik(3 - 2\mu)) \cosh 8\rho \}. 
\] (49)

From \((\sinh \rho)^{\mu-3}\) in Eq. (45), its asymptotic form is given by
\[
h_{(2)R}^{(2)R} \sim e^{(\mu-3)\rho}, \quad \rho \to \infty,
\] (50)
which is consistent with that of \(n = 0\) right-moving quasinormal modes obtained by solving the second-order differential equation \([6]\). Explicitly, from Eq. (3.55) in Ref. \([6]\), we recover the same asymptotic form of \(R_{22}(\xi) \sim e^{(m-3)\xi}\) for \(\xi = \rho\) and \(m = \mu\).
B. Descendants of spin-3 field

As was done in the spin-2 field, from the right-moving highest weight solution for the spin-3 field in Eqs. (37) and (38), the third descendent quasinormal mode can be computed as

\[
\Phi^{(3)}_{R \rho \mu \nu} = \left( \bar{L}_{-1} L_{-1} \right)^3 \Phi^R_{\rho \mu \nu}
\]

\[
= \frac{1}{8} e^{-2h_R(\mu)t - ik(t+\phi)} (\sinh \rho)^{2(\mu-2)} (\tanh \rho)^{-ik} F^{(3)}_{\rho \mu \nu}(\rho),
\]

(51)

where

\[
h_R(\mu) = -\mu + 2,
\]

(52)

and

\[
F^{(3)}_{u \mu \nu}(\rho) = \begin{pmatrix}
m_{uuu} & 0 & m_{app} \\
0 & 0 & 0 \\
m_{app} \sinh 2\rho & 0 & m_{app} \sinh^2 2\rho
\end{pmatrix},
\]

(53)

\[
F^{(3)}_{v \mu \nu}(\rho) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
F^{(3)}_{p \mu \nu}(\rho) = \begin{pmatrix}
m_{app} \sinh 2\rho & 0 & m_{app} \sinh 2\rho \\
0 & 0 & 0 \\
m_{app} \sinh^2 2\rho & 0 & m_{app} \sinh^2 2\rho
\end{pmatrix}
\]

(53)

with

\[
m_{uuu} = \{ (1 + 2\mu)(4\mu(1 + \mu) - 3k^2) - ik(2(1 + 6\mu + 6\mu^2) - k^2) \} \times
\]

\[
2(-2 + \mu)(12 - 19\mu + 10\mu^2 - 12k^2) - 2ik(44 - 51\mu + 18\mu^2 - 4k^2)
\]

\[
+ 3\mu(29 - 33\mu + 10\mu^2 - 8k^2 - 16ik(-2 + \mu)) \cosh 2\rho
\]

\[
+ 6\mu(-2 + \mu - ik)(1 + 2\mu) \cosh 4\rho + \mu(1 + \mu)(1 + 2\mu) \cosh 6\rho\}
\]

(54)
\[ m_{\text{mup}} = \frac{1}{64 \cosh^6 \rho} \left\{ 84(1 - 2\mu)^2(27 + 90\mu - 25\mu^2 - 100\mu^3 + 44\mu^4) \\
+ 28k^2(115 - 912\mu - 1485\mu^2 + 4560\mu^3 - 2172\mu^4) \\
- 12(-6(9 - 42\mu - 523\mu^2 + 700\mu^3 + 636\mu^4 - 1120\mu^5 + 352\mu^6) \\
+ 4k^2(-278 + 819\mu + 1382\mu^2 - 4340\mu^3 + 2104\mu^4) \\
+ k^4(-214 + 2242\mu - 2244\mu^2) + 40k^6 \\
+ ik(-525 - 3181\mu + 6468\mu^2 + 7512\mu^3 - 17080\mu^4 + 6528\mu^5 \\
- k^2(477 + 1791\mu - 8822\mu^2 + 5792\mu^3) + 4k^4(-57 + 116\mu)) \cosh 2\rho \\
- 3(-6(1 - 2\mu)^2(108 + 336\mu - 169\mu^2 - 420\mu^3 + 220\mu^4) \\
+ 2k^2(-516 + 3296\mu + 5585\mu^2 - 18560\mu^3 + 9532\mu^4) \\
- 16k^4(29 - 282\mu + 276\mu^2) + 64k^6 \\
+ ik(576 - 6603\mu + 13888\mu^2 + 14768\mu^3 - 37760\mu^4 + 15600\mu^5 \\
- k^2(1168 + 3739\mu - 18272\mu^2 + 12308\mu^3) + 64k^4(-7 + 13\mu)) \cosh 4\rho \\
- 2(-36 + 216\mu + 2966\mu^2 - 5400\mu^3 - 2776\mu^4 + 8640\mu^5 - 3520\mu^6 \\
+ 4k^2(-236 + 843\mu + 1302\mu^2 - 4620\mu^3 + 2664\mu^4) \\
- 2k^4(158 - 951\mu + 942\mu^2) + 16k^6 \\
+ ik(-474 - 2723\mu + 7188\mu^2 + 6152\mu^3 - 20040\mu^4 + 9600\mu^5 \\
- k^2(498 + 2045\mu - 8442\mu^2 + 6112\mu^3) + 24k^4(-7 + 12\mu)) \cosh 6\rho \\
+ 6(-1 + 2\mu)(2(-9 + 77\mu^2 - 46\mu^3 - 116\mu^4 + 88\mu^5) \\
+ 4k^2(-46 + 36\mu + 358\mu^2 - 404\mu^3) + 4k^4(-4 + 11\mu) \\
- ik(-15 - 175\mu - 114\mu^2 - 468\mu^3 + 424\mu^4 \\
+ k^2(1 + 123\mu - 190\mu^2) + 4k^4)) \cosh 8\rho \\
+ 6\mu(-1 + 2\mu)(2\mu(5 - 10\mu - 8\mu^2 + 16\mu^3) + k^2(4 + 16\mu - 48\mu^2) + 2k^4 \\
+ ik(-5 + 18\mu + 28\mu^2 - 64\mu^3 + k^2(-3 + 16\mu)) \cosh 10\rho \\
+ (2\mu^2(1 - 4\mu^2)(1 - 4\mu^2 + 3k^2) + ik\mu(1 - 4\mu^2)(1 + 12\mu^2 - k^2)) \cosh 12\rho \right\} \\
\] (55)
\[
m_{\text{upp}} = \frac{1}{16 \cosh^6 \rho} \{2(12(189 + 284\mu - 518\mu^2 - 632\mu^3 + 2037\mu^4 - 1428\mu^5 + 308\mu^6)) - k^2(1010 - 4695\mu + 37881\mu^2 - 45384\mu^3 + 15204\mu^4) + k^4(841 - 2991\mu + 2118\mu^2 - 80k^6) - 2ik(2(2931 - 2812\mu - 5415\mu^2 + 24814\mu^3 - 22050\mu^4 + 5796\mu^5)) + k^2(210 - 12953\mu + 23325\mu^2 - 10678\mu^3) + 36k^4(-17 + 25\mu)) - 12(630 - 752\mu + 1790\mu^2 - 2108\mu^3 - 6848\mu^4 + 4848\mu^5 - 1056\mu^6) + k^2(-1167 - 1619\mu + 10438\mu^2 - 12608\mu^3 + 4208\mu^4) + k^4(-433 + 1643\mu - 1122\mu^2) + 20k^6 + ik(-779 - 1969\mu - 3124\mu^2 + 13816\mu^3 - 12360\mu^4 + 3264\mu^5) + k^2(397 - 3487\mu + 6434\mu^2 - 2896\mu^3) + 8k^4(-21 + 29\mu))) \cosh 2\rho - 3(-4(360 + 448\mu - 1075\mu^2 - 1198\mu^3 + 4033\mu^4 - 2940\mu^5 + 660\mu^6) + k^2(272 - 3999\mu + 23723\mu^2 - 28792\mu^3 + 9532\mu^4) - 16k^4(59 - 221\mu + 138\mu^2) + 32k^6 + ik(3536 - 4486\mu - 7450\mu^2 + 32004\mu^3 - 29100\mu^4 + 7800\mu^5) + k^2(496 - 7765\mu + 14255\mu^2 - 6154\mu^3) + 32k^4(-11 + 13\mu))) \cosh 4\rho - 2(810 - 1536\mu + 2434\mu^2 + 2628\mu^3 - 9728\mu^4 + 7440\mu^5 - 1760\mu^6) + k^2(-949 - 1677\mu + 13818\mu^2 - 16128\mu^3 + 5328\mu^4) + k^4(-599 + 1653\mu - 942\mu^2) + 8k^6 + ik(-1 + \mu)(717 + 2212\mu + 6256\mu^2 - 12600\mu^3 + 4800\mu^4) + k^2(-291 + 4318\mu - 3056\mu^2) + 144k^4) \cosh 6\rho + 6(-1 + \mu)(4(-15 - 19\mu + 35\mu^2 + 83\mu^3 - 128\mu^4 + 44\mu^5) - k^2(46 + 321\mu - 788\mu^2 + 404\mu^3) + k^4(-34 + 44\mu) - ik(2(-59 + 101\mu + 272\mu^2 - 518\mu^3 + 212\mu^4) + k^2(-66 + 267\mu - 190\mu^2) + 4k^4) \cosh 8\rho + 6(1 - 3\mu + 2\mu^2)(2(-3 + 7\mu + 4\mu^2 - 16\mu^3 + 8\mu^4) + k^2(-5 + 28\mu - 24\mu^2) + k^4 + ik(-5 - 14\mu + 52\mu^2 - 32\mu^3 + k^2(-5 + 8\mu))) \cosh 10\rho + (\mu(1 - 3\mu + 2\mu^2)(4\mu(1 - 3\mu + 2\mu^2) + 3k^2(1 - 2\mu) + ik(-2 + 12\mu - 12\mu^2 + k^2))) \cosh 12\rho},
\]

(56)
\[
m_{\rho \rho \rho} = \frac{1}{16 \cosh^4 \rho} \{ 4(3(3168 + 1488 \mu + 3737 \mu^2 - 13896 \mu^3 + 15848 \mu^4 - 7392 \mu^5 + 1232 \mu^6) \\
- k^2(9806 - 30354 \mu + 74763 \mu^2 - 58848 \mu^3 + 15204 \mu^4) \\
+ k^4(3362 - 7782 \mu + 4236 \mu^2 - 80 \mu^6) \\
- ik(32556 + 26819 \mu - 125136 \mu^2 + 194416 \mu^3 - 114240 \mu^4 + 23184 \mu^5) \\
+ k^2(7920 - 51493 \mu + 60576 \mu^2 - 21356 \mu^3) + 12k^4(-133 + 150 \mu) \\
+ 12(6(-1056 + 144 \mu + 1003 \mu^2 - 3972 \mu^3 + 4516 \mu^4 - 2112 \mu^5 + 352 \mu^6) \\
- 4k^2(-575 - 4652 \mu + 1043 + \mu^2 - 8268 \mu^3 + 2104 \mu^4) \\
+ 2k^4(878 - 2165 \mu + 1122 \mu^2) + 40 \mu^6 \\
- ik(-7452 + 5081 \mu - 36124 \mu^2 + 54952 \mu^3 - 32360 \mu^4 + 6528 \mu^5) \\
+ k^2(3696 - 14033 \mu + 16914 \mu^2 - 5792 \mu^3) + k^4(-444 + 464 \mu)) \cosh 2 \rho \\
+ 3(6(2139 + 498 \mu + 2407 \mu^2 - 9888 \mu^3 + 11224 \mu^4 - 5280 \mu^5 + 880 \mu^6) \\
- 2k^2(5561 - 22802 \mu + 49529 \mu^2 - 39024 \mu^3 + 9532 \mu^4) \\
+ 16k^4(247 - 602 \mu + 276 \mu^2) - 64 \mu^6 \\
- ik(20475 + 13397 \mu - 89648 \mu^2 + 133648 \mu^3 - 78640 \mu^4 + 15600 \mu^5) \\
+ k^2(6735 - 32439 \mu + 38748 \mu^2 - 12308 \mu^3) + 64k^4(-15 + 13 \mu)) \cosh 4 \rho \\
- 2(9486 - 5892 \mu - 10870 \mu^2 + 38952 \mu^3 - 44776 \mu^4 + 21120 \mu^5 - 3520 \mu^6 \\
+ 8k^2(8 - 3141 \mu + 7788 \mu^2 - 5754 \mu^3 + 1332 \mu^4) \\
- 2k^4(1283 - 2355 \mu + 942 \mu^2) + 16 \mu^6 \\
+ ik(-8913 + 15535 \mu - 56124 \mu^2 + 86072 \mu^3 - 49560 \mu^4 + 960 \mu^5) \\
+ k^2(4071 - 20363 \mu + 21054 \mu^2 - 6112 \mu^3) + 24k^4(-17 + 12 \mu)) \cosh 6 \rho \\
+ 6(-3 + 2 \mu)(2(-114 + 4 \mu - 165 \mu^2 + 538 \mu^3 - 396 \mu^4 + 88 \mu^5) \\
+ k^2(80 - 956 \mu + 1218 \mu^2 - 404 \mu^3) + k^4(-52 + 44 \mu) \\
- ik(-260 - 205 \mu + 1714 \mu^2 - 1604 \mu^3 + 424 \mu^4) \\
+ k^2(-184 + 411 \mu - 190 \mu^2) + 4k^4)) \cosh 8 \rho \\
+ 6(3 - 5 \mu + 2 \mu^2)(2(-21 + 11 \mu + 50 \mu^2 - 56 \mu^3 + 16 \mu^4) - 8k^2(5 - 12 \mu + 6 \mu^2) \\
+ 2k^4 + ik(7 - 130 \mu + 180 \mu^2 - 64 \mu^3 + k^2(-17 + 16 \mu)) \cosh 10 \rho \\
+ (3 - 11 \mu + 12 \mu^2 - 4 \mu^3)(6 - 22 \mu + 24 \mu^2 - 8 \mu^3 - 6k^2(1 - \mu) \\
+ ik(11 - 24 \mu + 12 \mu^2 - k^2)) \cosh 12 \rho \}. \quad (57)
\]

From \((\sinh \rho)^{2(\mu - 2)}\) in Eq. (51), its asymptotic form is given by
\[
\Phi_{\rho \rho \rho}^{(3)R} \sim e^{2(\mu - 2) \rho}, \quad (58)
\]
which coincides with that of \( n = 0 \) right-moving quasinormal modes obtained by solving the second-order differential equation \([6]\). Explicitly, from Eq. (B.32) in Ref. \([6]\), we recover the same asymptotic form of \( R_{222}(\xi) \sim e^{2(\frac{m}{\rho} - 2)\xi} \) for \( \xi = \rho \) and \( m = 2\mu \).

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