Linear quadratic problems for fully coupled forward-backward stochastic control systems

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Abstract

This paper is concerned with optimal control of stochastic fully coupled forward-backward linear quadratic (FBLQ) problems with indefinite control weight costs. In order to obtain the state feedback representation of the optimal control, we propose a new decoupling technique and obtain one kind of non-Riccati-type ordinary differential equations (ODEs). By applying the completion-of-squares method, we prove the existence of the solutions for the obtained ODEs under some assumptions and derive the state feedback form of the optimal control. For this FBLQ problem, the optimal control depends on the entire trajectory of the state process. Some special cases are given to illustrate our results.

Key words. fully coupled forward-backward stochastic differential equation, linear quadratic optimization control, stochastic maximum principle, completion-of-squares method

AMS subject classifications. 93E20, 60H10, 35K15

1 Introduction

The fully coupled forward-backward stochastic differential equations (FBSDEs) are an important class of stochastic differential equations and there are many literatures on the well-posedness of them. When the coefficients of a fully coupled FBSDE are deterministic and the diffusion coefficient of the forward equation is nondegenerate, Ma, Protter and Yong [13] proposed the four-step scheme approach. Under some monotonicity conditions, Hu and Peng [8] first obtained an existence and uniqueness result which was generalized by Peng and Wu [10]. Yong [24] developed this approach and called it the method of continuation. The

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The fixed point approach is due to Antonelli [1], Pardoux and Tang [17]. The readers may refer to Ma and Yong [15], Cvitanić and Zhang [5], Ma, Wu, Zhang and Zhang [14], Yong and Zhou [26] for the FBSDE theory.

As a well-defined dynamic system, it is appealing to investigate the optimal control of the fully coupled FBSDEs. In this paper, the optimal control of a linear fully coupled FBSDE with a quadratic criteria is investigated. We call this kind of problem the stochastic forward-backward linear-quadratic (FBLQ) problem.

It is well-known that the stochastic linear-quadratic (LQ) problems play an important role in optimal control theory. On one hand, many nonlinear control problems can be approximated by the LQ control problems; on the other hand, solutions to the LQ control problems show elegant properties because of their brief and beautiful structures. Stochastic LQ regulator problems have been first studied by Wonham [22] and by many researchers later [2, 10, 20, 21]. Most of them imposed the positiveness for the coefficient of the control in the cost functional. Chen, Li and Zhou found even when the coefficient is negative, the stochastic control problem is still well-posed (see [3, 4]). For stochastic LQ problems, one method is applying the stochastic maximum principle to obtain the optimal control and then solving the corresponding Hamiltonian system by a decoupling technique which leads to a Riccati equation. Finally the optimal control is expressed in the form of state feedback. Another method is the completion-of-squares method which yields the same Riccati equation and state feedback form of the optimal control. Dokuchaev and Zhou [6] first proposed the stochastic backward linear-quadratic (BLQ) problem in which the state equation is described by a backward stochastic differential equation (BSDE). Applying the completion-of-squares method and the decoupling method, Lim and Zhou [12] completely solved it and obtained the state feedback representation.

Up to our knowledge, there are only a few results for the stochastic FBLQ problem and except some special examples in the literatures, there are no systematical results related to the state feedback form of the optimal control. Our main contribution of this paper is to obtain the state feedback form of the optimal control for the FBLQ problem. After applying the stochastic maximum principle, we find that the decoupling technique for stochastic LQ and BLQ problems is no longer applicable to the FBLQ problem. In more details, for the stochastic FBLQ problem, the obtained Hamiltonian system (3.1) consists of two parts: \((\bar{X}(\cdot), m(\cdot))\) (the forward state process \(\bar{X}(\cdot)\) and its backward adjoint process \(m(\cdot)\)) and \((\bar{Y}(\cdot), h(\cdot))\) (the backward state process \(\bar{Y}(\cdot)\) and its forward adjoint process \(h(\cdot)\)). Both of them are fully coupled FBSDEs. Following the decoupling method for the stochastic LQ problem, we try to decouple the above Hamiltonian system by

\[
\begin{align*}
 h(t) &= P_1(t)\bar{X}(t) + P_2(t)\bar{Y}(t) + \varphi_1(t), \\
 m(t) &= P_3(t)\bar{X}(t) + P_4(t)\bar{Y}(t) + \varphi_2(t).
\end{align*}
\]

In other words, we want to use the state process \((\bar{X}(\cdot), \bar{Y}(\cdot))\) to represent the adjoint process \((m(\cdot), h(\cdot))\). But after calculation, we can’t get the Riccati-type equations for \(P_i(t), i = 1, 2, 3, 4\) through this decoupling approach. To overcome this difficulty, we propose the following new decoupling technique: we regard the forward stochastic differential equation (SDE) \((\bar{X}(\cdot), h(\cdot))\) as the state process, the BSDE \((\bar{Y}(\cdot), m(\cdot))\) as the
adjoint process and decouple the Hamiltonian system \((\mathbf{3.1})\) by

\[
\begin{align*}
m(t) &= P_1(t)\dot{X}(t) + P_2(t)\dot{h}(t) + \varphi_1(t), \\
\dot{Y}(t) &= P_2(t)\dot{X}(t) - P_3(t)h(t) + \varphi_2(t).
\end{align*}
\]

(1.1)

Using the above decoupling technique, we derive the equations for \(P_i(t), i = 1, 2, 3, \varphi_1(\cdot), \varphi_2(\cdot)\) and obtain the optimal control which can be explicitly expressed as a feedback form of the state process \((\tilde{X}(\cdot), \tilde{Y}(\cdot))\) (see Corollary \(\mathbf{3.3}\)).

Although we can decouple the Hamiltonian system \((\mathbf{3.1})\) by \((\mathbf{1.1})\), the obtained equations for \(P_i(t), i = 1, 2, 3\) are no longer Riccati-type ones. They are highly nonlinear ordinary differential equations (ODEs) and the solvability of them is challenging. In this paper, we propose a project to obtain the existence of the solutions \(P_i(t), i = 1, 2, 3\). We first construct a sequence of Riccati equations for \(i\bar{P}(t)\). Then, applying the completion-of-squares method, we establish the the relations between \(P_i(t), i = 1, 2, 3 \) and \(i\bar{P}(t)\) (see Theorem \(\mathbf{4.4}\)) which are different from the stochastic LQ and BLQ problems. With the help of these relations and the good properties of \(i\bar{P}(t)\), we obtain the existence of the solutions \(P_i(t), i = 1, 2, 3\). Especially, we relax the positiveness of the control weight in the cost functional as in Chen et al. \(\mathbf{[3, 4]}\). For this indefinite case, the control \(\bar{u}(\cdot)\) obtained by our decoupling technique is only a candidate of the optimal control. By applying the completion-of-squares method, it can be verified that \(\bar{u}(\cdot)\) is indeed the optimal control of the FBLQ problem. Furthermore, although the optimal control for the FBLQ problem may not be unique, we can still prove that the optimal state feedback optimal control law is unique (see Theorem \(\mathbf{5.2}\)). Finally, it is worth pointing out that we can’t solve the FBLQ problem by the decoupling method or the completion-of-squares method alone.

The rest of the paper is organized as follows. In Section 2, we give the preliminaries and the formulation of the FBLQ problem. A new decoupling technique is introduced in Section 3. Applying the completion-of-squares method, we prove the existence and uniqueness results for non-Riccati-type equations in Section 4. In Section 5, we obtain the feedback optimal control for the FBLQ problem. Several special cases are given to illustrate our results in Section 6.

2 Preliminaries and formulation of FBLQ problem

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space on which a standard \(d\)-dimensional Brownian motion \(B = (B_1(t), B_2(t), \ldots, B_d(t))^{\mathbb{T}}_{t \leq T}\) is defined. Assume that \(\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}\) is the \(P\)-augmentation of the natural filtration of \(B\), where \(\mathcal{F}_0\) contains all \(P\)-null sets of \(\mathcal{F}\). Denote by \(\mathbb{R}^n\) the \(n\)-dimensional real Euclidean space and \(\mathbb{R}^{n \times k}\) the set of \(n \times k\) real matrices. Let \(\langle \cdot, \cdot \rangle\) (resp. \(|\cdot|\)) denote the usual scalar product (resp. usual norm) of \(\mathbb{R}^n\) and \(\mathbb{R}^{n \times k}\). The scalar product (resp. norm) of \(M = (m_{ij}), N = (n_{ij}) \in \mathbb{R}^{n \times k}\) is denoted by \(\langle M, N \rangle = tr\{MN^\tau\}\) (resp. \(|M| = \sqrt{tr(MM^\tau)}\)), where the superscript \(\tau\) denotes the transpose of vectors or matrices.

For each given \(p \geq 1\), we introduce the following spaces.

- \(\mathbb{S}^n: \) the space of all \(n \times n\) symmetric matrices;
- \(\mathbb{S}^n_+: \) the subspace of all nonnegative definite matrices of \(\mathbb{S}^n;\)
Consider the case \( dF \), and minimizing the following cost functional

\[
\mathcal{S}_n^a := \mathbb{E}[|\eta|^p] < \infty;
\]

\( L^\infty(F_T; \mathbb{R}^n) \): the space of \( F_T \)-measurable \( \mathbb{R}^n \)-valued random vectors \( \eta \) such that

\[
||\eta||_{\infty} = \text{ess sup}_{\omega \in \Omega} |\eta(\omega)| < \infty;
\]

\( L^\infty(0; \mathbb{R}^{n \times k}) \): the space of essential bounded measurable \( \mathbb{R}^{n \times k} \)-valued functions;

\( C([0, T], \mathbb{R}^n) \): the space of continuos \( \mathbb{R}^n \)-valued functions;

\( L^p_F(0, T; \mathbb{R}^n) \): the space of \( F \)-adapted \( \mathbb{R}^n \)-valued stochastic processes on \([0, T]\) such that

\[
\mathbb{E} \left[ \int_0^T |f(r)|^p dr \right] < \infty;
\]

\( L^\infty_F(0, T; \mathbb{R}^n) \): the space of \( F \)-adapted \( \mathbb{R}^n \)-valued stochastic processes on \([0, T]\) such that

\[
||f(\cdot)||_{\infty} = \text{ess sup}_{(t, \omega) \in [0, T] \times \Omega} |f(t, \omega)| < \infty;
\]

\( L^p_F([0, T]; \mathbb{R}^n) \): the space of \( F \)-adapted \( \mathbb{R}^n \)-valued stochastic processes on \([0, T]\) such that

\[
||f(\cdot)||_{p, q} = \left\{ \mathbb{E} \left[ \left( \int_0^T |f(t)|^p dt \right)^{\frac{q}{p}} \right] \right\}^{\frac{1}{q}} < \infty;
\]

\( L^p_F(\Omega; C([0, T], \mathbb{R}^n)) \): the space of \( F \)-adapted \( \mathbb{R}^n \)-valued continuous stochastic processes on \([0, T]\) such that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |f(t)|^p \right] < \infty.
\]

Consider the following linear forward-backward stochastic control system

\[
\begin{cases}
    dX(t) = [A_1(t)X(t) + B_1(t)Y(t) + C_1(t)Z(t) + D_1(t)u(t)]dt \\
    \quad + [A_2(t)X(t) + B_2(t)Y(t) + C_2(t)Z(t) + D_2(t)u(t)]dB(t), \\
    dY(t) = -[A_3(t)X(t) + B_3(t)Y(t) + C_3(t)Z(t) + D_3(t)u(t)]dt + Z(t)dB(t), \\
    X(0) = x_0, \quad Y(T) = FX(T) + \xi,
\end{cases}
\]

and minimizing the following cost functional

\[
J(u(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle A_4(t)X(t), X(t) \rangle + \langle B_4(t)Y(t), Y(t) \rangle + \langle C_4(t)Z(t), Z(t) \rangle \\
\quad + \langle D_4(t)u(t), u(t) \rangle \right) dt + \langle GX(T), X(T) \rangle + \langle HY(0), Y(0) \rangle \right]
\]

where \( A_1(\cdot), B_1(\cdot), C_1(\cdot), D_1(\cdot) \) are deterministic matrix-valued functions of suitable sizes, \( \xi \in L^2(F_T; \mathbb{R}^m) \), \( F, G, H \) are \( \mathbb{R}^{m \times n} \), \( \mathbb{R}^{n \times n} \), \( \mathbb{R}^{m \times m} \)-valued matrices respectively. To simplify the presentation, we only consider the case \( d = 1 \). The results for \( d > 1 \) are similar. The solution to (2.1) is \((X(\cdot), Y(\cdot), Z(\cdot)) \in \ldots \)

\[ \ldots \]
The variable $\sigma^2$ stochastic maximum principle (see [7, 18, 23]), the optimal control $\bar{u}(\cdot)$ to the FBLQ problem (2.1)-(2.2). Let $u(\cdot)$ be an admissible control, and the corresponding state is $(X(\cdot), Y(\cdot), Z(\cdot))$.

Let $\bar{u}(\cdot)$ be an optimal control and $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$ be the corresponding optimal state. Then by stochastic maximum principle (see [7, 18, 23]), the optimal control $\bar{u}(\cdot)$ satisfies

$$
D_4(t)\bar{u}(t) + D_1(t)^\top m(t) + D_2(t)\bar{Y}(t) + D_3(t)h(t) = 0, \tag{2.3}
$$

where

$$
\begin{cases}
  dh(t) = [B_3(t)^\top h(t) + B_1(t)^\top m(t) + B_2(t)\bar{Y}(t) + B_4(t)\bar{Y}(t)] dt \\
  + [C_3(t)^\top h(t) + C_1(t)^\top m(t) + C_2(t)\bar{Y}(t) + C_4(t)\bar{Y}(t)] dB(t), \\
  dm(t) = -[A_3(t)^\top h(t) + A_1(t)^\top m(t) + A_2(t)^\top n(t) + A_4(t)\bar{X}(t)] dt \\
  + n(t)dB(t), \\
  h(0) = H\bar{Y}(0), m(T) = G\bar{X}(T) + Fh(T).
\end{cases}
$$

**Assumption 2.1** For any $u(\cdot) \in L^2(0,T;\mathbb{R}^k), (2.1)$ (resp. (2.4)) has a unique solution in $L^2(\Omega;C([0,T],\mathbb{R}^n)) \times L^2(\Omega;C([0,T],\mathbb{R}^m)) \times L^2(\Omega;C([0,T],\mathbb{R}^m)) \times L^2(\Omega;C([0,T],\mathbb{R}^n)) \times L^2(0,T;\mathbb{R}^n)$.

**Remark 2.2** It is well-known that there are many conditions which can guarantee the existence and uniqueness of (2.1) and (2.4) (see [15], [5], [19], [7]) such as monotonicity conditions or weakly coupled conditions and so on.

**Assumption 2.3** The data appearing in the FBLQ problem satisfy $A_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n×n}), B_1(\cdot), C_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n×m}), D_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{m×k}),$ for $i = 1, 2, A_3(\cdot) \in L^\infty(0,T;\mathbb{R}^{m×n}), B_3(\cdot), C_3(\cdot) \in L^\infty(0,T;\mathbb{R}^{m×m}), D_3(\cdot) \in L^\infty(0,T;\mathbb{R}^{m×k}), A_4(\cdot) \in L^\infty(0,T;\mathbb{S}^n), B_4(\cdot), C_4(\cdot) \in L^\infty(0,T;\mathbb{S}^m), D_4(\cdot) \in L^\infty(0,T;\mathbb{S}^k), F \in \mathbb{R}^{m×n}, G \in \mathbb{S}^n, H \in \mathbb{S}^m$.

Sometimes we need the data to satisfy the following assumptions:

**Assumption 2.4** $A_4(\cdot) \in L^\infty(0,T;\mathbb{S}^n), B_4(\cdot) \in L^\infty(0,T;\mathbb{S}_+^m), G \in \mathbb{S}^n_+, H \in \mathbb{S}^m_+.$

**Assumption 2.5** $C_4(\cdot) \in L^\infty(0,T;\mathbb{S}_+^m), D_4(\cdot) \in L^\infty(0,T;\mathbb{S}_+^k).$

Note that (2.3) becomes a sufficient condition for the optimal control under some positiveness assumptions on the coefficients.

**Theorem 2.6** (see [7, 18, 23]) Suppose that Assumptions 2.1, 2.3, 2.4 and 2.5 hold. If there exists an admissible control $\bar{u}(\cdot)$ satisfying (2.4), where $(h(\cdot), m(\cdot), n(\cdot))$ is defined in (2.4), then $\bar{u}(\cdot)$ is the unique optimal control for the FBLQ problem (2.1)-(2.4).

In the rest of this paper, sometimes we write $A$ for a (deterministic or stochastic) process, omitting the variable $t$, whenever no confusion arises. Under this convention, when $A \geq (>0)$ means $A(t) \geq (>0), \forall t \in [0,T]$.  

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3 A new decoupling technique for FBLQ problem

3.1 FBLQ problem with positive definite control weight cost

In this subsection, we only consider the FBLQ problem (2.1-2.2) with positive definite control weight cost. In other words, we assume that $D_4 > 0$. The Hamiltonian system for the FBLQ problem is

$$
\begin{aligned}
&d\tilde{X}(t) = [A_1(t)\tilde{X}(t) + B_1(t)\tilde{Y}(t) + C_1(t)\tilde{Z}(t) + D_1(t)\bar{u}(t)]dt \\
&\quad + [A_2(t)\tilde{X}(t) + B_2(t)\tilde{Y}(t) + C_2(t)\tilde{Z}(t) + D_2(t)\bar{u}(t)]dB(t), \\
&d\tilde{Y}(t) = -[A_3(t)\tilde{X}(t) + B_3(t)\tilde{Y}(t) + C_3(t)\tilde{Z}(t) + D_3(t)\bar{u}(t)]dt + \tilde{Z}(t)dB(t), \\
&dh(t) = [B_3(t)^T h(t) + B_1(t)^T m(t) + B_2(t)^T n(t) + B_4(t)^T \tilde{Y}(t)]dt \\
&\quad + [C_3(t)^T h(t) + C_1(t)^T m(t) + C_2(t)^T n(t) + C_4(t)^T \tilde{Z}(t)]dB(t), \\
&dm(t) = -[A_3(t)^T h(t) + A_1(t)^T m(t) + A_2(t)^T n(t) + A_4(t)^T \tilde{X}(t)]dt \\
&\quad + n(t)dB(t), \\
\tilde{X}(0) = x_0, \quad \tilde{Y}(T) = F\tilde{X}(T) + \xi, \quad h(0) = H\tilde{Y}(0), \quad m(T) = G\tilde{X}(T) + F^T h(T).
\end{aligned}
$$

(3.1)

Set

$$
\tilde{X}(\cdot) = (\tilde{X}(\cdot)^T, h(\cdot)^T)^T, \quad \tilde{Y}(\cdot) = (m(\cdot)^T, \tilde{Y}(\cdot)^T)^T, \quad \tilde{Z}(\cdot) = (n(\cdot)^T, Z(\cdot)^T)^T.
$$

Due to (2.3), we have $\bar{u}(t) = -D_1(t)^{-1}(D_1(t)^T m(t) + D_2(t)^T n(t) + D_3(t)^T h(t))$. Then the Hamiltonian system (3.1) can be rewritten as

$$
\begin{aligned}
&d\tilde{X}(t) = [\tilde{A}_1(t)\tilde{X}(t) + \tilde{B}_1(t)\tilde{Y}(t) + \tilde{C}_1(t)\tilde{Z}(t)]dt \\
&\quad + [\tilde{A}_2(t)\tilde{X}(t) + \tilde{B}_2(t)\tilde{Y}(t) + \tilde{C}_2(t)\tilde{Z}(t)]dB(t), \\
&d\tilde{Y}(t) = -[\tilde{A}_3(t)\tilde{X}(t) + \tilde{B}_3(t)\tilde{Y}(t) + \tilde{C}_3(t)\tilde{Z}(t)]dt + \tilde{Z}(t)dB(t), \\
\tilde{X}(0) = (x_0^T, (H\tilde{Y}(0))^T)^T, \quad \tilde{Y}(T) = F\tilde{X}(T) + \tilde{\xi},
\end{aligned}
$$

(3.2)

where

$$
\begin{aligned}
\tilde{A}_1(t) &= \begin{pmatrix} A_1(t) & -D_1(t)D_4(t)^{-1}D_3(t)^T \\ 0 & B_3(t)^T \end{pmatrix}, & \tilde{B}_1(t) &= \begin{pmatrix} -D_1(t)D_4(t)^{-1}D_1(t)^T \\ B_1(t)^T \end{pmatrix}, \\
\tilde{C}_1(t) &= \begin{pmatrix} -D_1(t)D_4(t)^{-1}D_2(t)^T & C_1(t) \\ B_2(t)^T & 0 \end{pmatrix}, & \tilde{A}_2(t) &= \begin{pmatrix} A_2(t) & -D_2(t)D_4(t)^{-1}D_3(t)^T \\ 0 & C_3(t)^T \end{pmatrix}, \\
\tilde{B}_2(t) &= \begin{pmatrix} -D_2(t)D_4(t)^{-1}D_1(t)^T & B_2(t) \\ C_1(t)^T & 0 \end{pmatrix}, & \tilde{C}_2(t) &= \begin{pmatrix} -D_2(t)D_4(t)^{-1}D_2(t)^T & C_2(t) \\ C_2(t)^T & C_4(t) \end{pmatrix}.
\end{aligned}
$$
where we conjecture that $\tilde{X}(\cdot)$ and $\tilde{Y}(\cdot)$ are related by

$$
\tilde{Y}(t) = Q(t)\tilde{X}(t) + \varphi(t)
$$

with $Q(\cdot) \in C([0,T], \mathbb{R}^{(n+m) \times (n+m)})$ and $\varphi(\cdot) \in L^2_p(\Omega; C([0,T], \mathbb{R}^{n+m}))$. Applying the same steps as in Section 4 of [23] or Appendix in [22], we obtain $Q(\cdot)$ satisfies the following matrix ODE

$$
\begin{aligned}
dQ(t) &= - \left[ Q(t)\tilde{A}_1(t) + Q(t)\tilde{B}_1(t)Q(t) + Q(t)\tilde{C}_1(t)K(t) + \tilde{A}_3(t) + \tilde{B}_3(t)Q(t) \\
& \quad + \tilde{C}_3(t)K(t) \right] dt, \\
Q(T) &= \tilde{F},
\end{aligned}
$$

and $(\varphi(\cdot), v(\cdot)) \in L^2_p(\Omega; C([0,T], \mathbb{R}^{n+m})) \times L^2_p(0,T; \mathbb{R}^{n+m})$ satisfies the following linear BSDE

$$
\begin{aligned}
d\varphi(t) &= - \left\{ \left[ Q(t)\tilde{B}_1(t) + \tilde{B}_3(t) - (I_{n+m} - Q(t)\tilde{C}_2(t))^{-1}Q(t)\tilde{B}_2(t) \right] \varphi(t) \\
& \quad + \left[ Q(t)\tilde{C}_1(t) + \tilde{C}_3(t) \right] (I_{n+m} - Q(t)\tilde{C}_2(t))^{-1}v(t) \right\} dt + v(t)dB(t), \\
\varphi(T) &= \tilde{\xi},
\end{aligned}
$$

where

$$
K(t) = (I_{n+m} - Q(t)\tilde{C}_2(t))^{-1} \left( Q(t)\tilde{A}_2(t) + Q(t)\tilde{B}_2(t)Q(t) \right).
$$

Set

$$
Q(t) = \begin{pmatrix}
Q_1(t) & Q_2(t) \\
Q_3(t) & -Q_4(t)
\end{pmatrix}, \quad K(t) = \begin{pmatrix}
K_1(t) & K_2(t) \\
K_3(t) & K_4(t)
\end{pmatrix}, \quad \varphi(\cdot) = \begin{pmatrix}
\varphi_1(\cdot) \\
\varphi_2(\cdot)
\end{pmatrix},
$$

$$
v(\cdot) = \begin{pmatrix}
v_1(\cdot) \\
v_2(\cdot)
\end{pmatrix}, \quad \begin{pmatrix}
J_1(t) & J_2(t) \\
J_3(t) & J_4(t)
\end{pmatrix} = (I_{n+m} - Q(t)\tilde{C}_2(t))^{-1}Q(t)\tilde{B}_2(t),
$$

$$
\begin{pmatrix}
I_1(t) & I_2(t) \\
I_3(t) & I_4(t)
\end{pmatrix} = (I_{n+m} - Q(t)\tilde{C}_2(t))^{-1}.
$$
where \( Q_1(\cdot), K_1(\cdot), J_1(\cdot), I_1(\cdot) \) are \( \mathbb{R}^{n\times n} \)-valued, \( Q_2(\cdot), K_2(\cdot), J_2(\cdot), I_2(\cdot) \) are \( \mathbb{R}^{n\times m} \)-valued, \( Q_3(\cdot), K_3(\cdot), J_3(\cdot), I_3(\cdot) \) are \( \mathbb{R}^{m\times n} \)-valued, \( Q_4(\cdot), K_4(\cdot), J_4(\cdot), I_4(\cdot) \) are \( \mathbb{R}^{m\times m} \)-valued, \( \varphi_1(\cdot) \in L^2_\mathbb{F}(\Omega; C([0, T], \mathbb{R}^n)), \varphi_2(\cdot) \in L^2_\mathbb{F}(\Omega; C([0, T], \mathbb{R}^m)), v_1(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n), v_2(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m) \).

**Theorem 3.1** Suppose that Assumptions \( A_1 \) \( A_2 \) and \( A_3 \) hold. Moreover, suppose that \( Q(\cdot) \) has a solution \( Q(\cdot) \in C([0, T]; \mathbb{R}^{(m+n)\times (m+n)}) \) such that \( I_m + HQ_4(0) \) is invertible and \((I_{n+m} - Q(t)\tilde{C}_2(t))^{-1} \in \mathbb{L}_\infty(0, T; \mathbb{R}^{(n+m)\times (n+m)})\). Then Problem \( B_1 \) - \( B_4 \) has a unique optimal control

\[
\bar{u}(t) = \begin{cases} 
-D_4(t)^{-1}(D_4(t)^T Q_1(t) + D_2(t)^T K_1(t)) X^*(t) \\
-D_4(t)^{-1}(D_4(t)^T Q_2(t) + D_2(t)^T K_2(t) + D_3(t)^T) h^*(t) \\
-D_4(t)^{-1}[D_4(t)^T \varphi_1(t) + D_2(t)^T (J_1(t)\varphi_1(t) + J_2(t)\varphi_2(t) + I_1(t)v_1(t) + I_2(t)v_2(t))]
\end{cases}
\]  

(3.5)

where \( \bar{X}^*(\cdot) := (X^*(\cdot)^T, h^*(\cdot)^T)^T \) is the solution to the following SDE

\[
\begin{align*}
\begin{cases} 
    d\bar{X}^*(t) = & \left\{ \left( \bar{A}_1(t) + \bar{B}_1(t)Q(t) + \bar{C}_1(t)K(t) \right) \bar{X}^*(t) \\
                        & + \bar{B}_1(t)\varphi(t) + \bar{C}_1(t)(I_{n+m} - Q(t)\tilde{C}_2(t))^{-1} \left[ Q(t)\tilde{B}_2(t)\varphi(t) + v(t) \right] \right\} dt \\
                        & + \left\{ \left( \bar{A}_2(t) + \bar{B}_2(t)Q(t) + \bar{C}_2(t)K(t) \right) \bar{X}^*(t) \\
                        & + \bar{B}_2(t)\varphi(t) + \bar{C}_2(t)(I_{n+m} - Q(t)\tilde{C}_2(t))^{-1} \left[ Q(t)\tilde{B}_2(t)\varphi(t) + v(t) \right] \right\} dB(t), \\
\end{cases}
\end{align*}
\]

(3.6)

Furthermore, the solution to \( S_1 \) with respect to \( \bar{u}(\cdot) \) defined in \( S_3 \) satisfies

\[
\begin{align*}
\bar{X}(t) & = \bar{X}^*(t), \ h(t) = h^*(t), \ \bar{Y}(t) = Q_3(t)X^*(t) - Q_4(t)h^*(t) + \varphi_2(t), \\
\bar{Z}(t) & = K_3(t)X^*(t) + K_4(t)h^*(t) + J_3(t)\varphi_1(t) + J_4(t)\varphi_2(t) \\
& + I_3(t)v_1(t) + I_4(t)v_2(t), \\
m(t) & = Q_1(t)X^*(t) + Q_2(t)h^*(t) + \varphi_1(t), \\
n(t) & = K_1(t)X^*(t) + K_2(t)h^*(t) + J_1(t)\varphi_1(t) + J_2(t)\varphi_2(t) \\
& + I_1(t)v_1(t) + I_2(t)v_2(t).
\end{align*}
\]  

(3.7)

**Proof.** By \((I_{n+m} - Q(t)\tilde{C}_2(t))^{-1} \in \mathbb{L}_\infty(0, T; \mathbb{R}^{(n+m)\times (n+m)}) \) and \( A_3 \) is a BSDE with Lipschitz coefficients. Then \( B_3 \) has a unique solution \((\varphi(\cdot), v(\cdot)) \in L^2_\mathbb{F}(\Omega; C([0, T], \mathbb{R}^n)) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^{n+m}) \). It yields that the stochastic differential equation \( B_3 \) admits a unique strong solution \((X^*(\cdot)^T, h^*(\cdot)^T)^T \in L^2_{\mathbb{F}}(\Omega; C([0, T], \mathbb{R}^{n+m})) \). Thus the control \( \bar{u}(\cdot) \) defined in \( B_3 \) is admissible. Putting this \( \bar{u}(\cdot) \) into \( B_1 \) and reversing the above decoupling technique, it can be verified that \( (\bar{X}(\cdot), h(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), m(\cdot), n(\cdot)) \) defined in \( B_4 \) solves \( B_1 \) and \( \bar{u}(\cdot) \) satisfies \( B_4 \). By Theorem \( 2.6 \) this \( \bar{u}(\cdot) \) is the unique optimal control. This completes the proof. \( \blacksquare \)
Remark 3.2 We give a sufficient condition which guarantee the existence of solution to (3.3) in Corollary

Corollary 3.3 (i) Under the same assumptions as in Theorem 3.1, if \(Q_4(\cdot)\) in (3.3) is invertible on \([0, T]\), then

\[ h(t) = Q_4(t)^{-1}Q_3(t)\dot{X}(t) - Q_4(t)^{-1}\dot{Y}(t) + Q_4(t)^{-1}\varphi_2(t) \]

and

\[ \ddot{u}(t) = -D_4(t)^{-1}\{D_1(t)^TQ_4(t) + D_2(t)^TK_1(t)
+ [D_1(t)^TQ_2(t) + D_2(t)^TK_2(t) + D_3(t)^TQ_3(t)]\dot{X}(t)
+ D_4(t)^{-1}[D_1(t)^TQ_2(t) + D_2(t)^TK_2(t) + D_3(t)^TQ_3(t)](\dot{Y}(t) - \varphi_2(t))
- D_4(t)^{-1}[D_1(t)^T\varphi_1(t) + D_2(t)^T(J_1(t)\varphi_1(t) + J_2(t)\varphi_2(t))
+ I_3(t)v_1(t) + I_2(t)v_2(t))\}. \]

(ii) If \(\xi = 0\), then the optimal control for the fully coupled forward-backward control system in Theorem 3.1
depends only on \((\dot{X}(\cdot), h(\cdot))\). Moreover, \(h(\cdot)\) has the following closed-form:

\[ h(t) = \left(\begin{bmatrix} I_m + HQ_4(0) \end{bmatrix}^{-1}H(Q_3(0)x_0 + \varphi_2(0))
+ \Phi(t) \int_0^t \Phi(s)^{-1}(b_1(s) - a_2(s)b_2(s))\dot{X}(s)ds
+ \Phi(t) \int_0^t \Phi(s)^{-1}b_2(s)\dot{X}(s)dB(s) \right), \]

where \(a_1(t) = B_3(t)^T - B_4(t)Q_4(t) + B_1(t)^TQ_3(t) + B_2(t)^TK_1(t), b_1(t) = B_4(t)Q_3(t) + B_1(t)^TQ_1(t) + B_2(t)^TK_1(t), a_2(t) = C_3(t)^T + C_4(t)K_4(t) + C_1(t)^TQ_2(t) + C_2(t)^TK_2(t), b_2(t) = C_4(t)K_3(t) + C_1(t)^TQ_1(t) + C_2(t)^TK_1(t)\), and \(\Phi(\cdot)\) is the solution of the following linear equation:

\[ d\Phi(t) = a_1(t)\Phi(t)dt + a_2(t)\Phi(t)dB(t), \quad \Phi(0) = I_m. \]

This corollary can be directly derived from Theorem 3.1. So we omit the proof.

Remark 3.4 By Corollary 3.3, the optimal control at time \(t\) depends on the entire past history of the state process \(X(\cdot)\). This is different from the classical stochastic LQ problems. Furthermore, if \(Q_4(\cdot)\) in (3.3) is invertible on \([0, T]\), then the optimal control at time \(t\) will depend only on the current state pair \((\dot{X}(t), \dot{Y}(t))\).

### 3.2 FBLQ problem with indefinite control weight cost

In this subsection, we relax the assumption \(D_4 > 0\) and deduce formally the following non-Riccati-type equations (3.19), (3.20) and (3.21) which play an important role in solving the FBLQ problem (see Section 5).

Set

\[ m(t) = P_1(t)\dot{X}(t) + P_2(t)h(t) + \varphi_1(t), \]

\[ \dot{Y}(t) = P_2(t)\dot{X}(t) - P_3(t)h(t) + \varphi_2(t), \]

(3.8)
Combining (3.9) and (3.10), we have following BSDE
\[
\begin{align*}
d\varphi_1(t) &= -\gamma_1(t)dt + v_1(t)dB(t), \\
d\varphi_2(t) &= -\gamma_2(t)dt + v_2(t)dB(t).
\end{align*}
\]
Applying Itô’s formula to \( \bar{Y}(\cdot) \), \( m(\cdot) \) in (3.8) and comparing with the diffusion terms of the equation (3.1), we have
\[
\begin{align*}
\bar{Z}(t) &= I_m(t) - \sum_{i=1}^{3} (P_i(t)C_i(t)^Tn(t)) \\
&\quad + (P_1(t)A_1(t) + P_2(t)B_2(t)P_2(t) + P_2(t)C_1(t)^TP_1(t))\bar{X}(t) \\
&\quad + (P_2(t)B_2(t)p_3(t) + P_3(t)C_3(t)^T + P_3(t)C_1(t)^TP_2(t)^T)h(t) + P_2(t)D_2(t)\bar{u}(t) \\
&\quad + P_2(t)B_2(t)\varphi_2(t) - P_3(t)C_1(t)^T\varphi_1(t) - P_3(t)C_2(t)^Tn(t) + v_2(t)
\end{align*}
\]
where
\[
L_1(t) = I_m(t) - P_2(t)C_2(t) + P_3(t)C_4(t).
\]
Combining (3.9) and (3.10), we have
\[
n(t) = L_2(t)^{-1}(L_3(t)\bar{X}(t) + L_4(t)h(t) + S_1(t)D_2(t)\bar{u}(t) + S_2(t)),
\]
where
\[
L_2(t) = I_n - P_2(t)^T C_2(t)^T + (P_1(t)C_2(t) + P_2(t)^T C_4(t)) L_1(t)^{-1} P_3(t) C_2(t)^T,
\]
\[
L_3(t) = P_1(t) A_2(t) + P_1(t) B_2(t) P_2(t) + P_2(t)^T C_1(t)^T P_1(t)
+ (P_2(t)^T C_4(t) + P_1(t) C_2(t)) L_1(t)^{-1}
\cdot (P_2(t) A_2(t) + P_2(t) B_2(t) P_2(t) - P_3(t) C_1(t)^T P_1(t)),
\]
\[
L_4(t) = P_2(t)^T C_3(t)^T + P_2(t)^T C_1(t)^T P_2(t)^T - P_1(t) B_2(t) P_3(t)
- (P_2(t)^T C_4(t) + P_1(t) C_2(t)) L_1(t)^{-1}
\cdot (P_2(t) B_2(t) P_3(t) + P_3(t) C_3(t)^T + P_3(t) C_1(t)^T P_2(t)^T),
\]
\[
S_1(t) = P_1(t) + (P_2(t)^T C_4(t) + P_1(t) C_2(t)) L_1(t)^{-1} P_2(t),
\]
\[
S_2(t) = P_1(t) B_2(t) \varphi_2(t) + P_2(t)^T C_1(t)^T \varphi_1(t) + v_1(t)
+ (P_1(t) C_2(t) + P_2(t)^T C_4(t)) L_1(t)^{-1}
\cdot [P_2(t) B_2(t) \varphi_2(t) - P_3(t) C_1(t)^T \varphi_1(t) + v_2(t)].
\]

Putting them into (2.3), we obtain
\[
\ddot{u}(t) = L_6(t) \ddot{X}(t) + L_7(t) \dot{h}(t) + S_3(t),
\] (3.14)

where
\[
L_5(t) = D_4(t) + D_2(t)^T L_2(t)^{-1} S_1(t) D_2(t),
\]
\[
L_6(t) = -L_5(t)^{-1} (D_1(t)^T P_1(t) + D_2(t)^T L_2(t)^{-1} L_3(t)),
\]
\[
L_7(t) = -L_5(t)^{-1} (D_1(t)^T P_2(t)^T + D_2(t)^T L_2(t)^{-1} L_4(t) + D_3(t)^T),
\]
\[
S_3(t) = -L_5(t)^{-1} [D_1(t)^T \varphi_1(t) + D_2(t)^T L_2(t)^{-1} S_2(t)].
\] (3.15)

**Remark 3.5** Instead of requiring $D_4 > 0$, here we assume that $L_5(t)$ is invertible.

From (3.14) - (3.15), we deduce that
\[
n(t) = L_8(t) \ddot{X}(t) + L_9(t) \dot{h}(t) + S_4(t),
\]
\[
\ddot{Z}(t) = L_{10}(t) \dddot{X}(t) + L_{11}(t) \dot{h}(t) + S_5(t),
\] (3.16)
and comparing with the drift term of the new form of (3.1), we have

\[ L_\delta(t) = L_2(t)^{-1} (L_3(t) + S_1(t)D_2(t)\lambda_0(t)), \]

\[ L_\alpha(t) = L_2(t)^{-1} (L_4(t) + S_1(t)D_2(t)L_7(t)), \]

\[ S_4(t) = L_2(t)^{-1} [S_1(t)D_2(t)S_3(t) + S_2(t)], \]

\[ L_{10}(t) = L_1(t)^{-1} [P_2(t)A_2(t) + P_2(t)B_2(t)P_2(t) - P_3(t)C_1(t)^TP_1(t) \]

\[ - P_3(t)C_2(t)^TL_\delta(t) + P_2(t)D_2(t)L_6(t)], \]

\[ L_{11}(t) = L_1(t)^{-1} [P_2(t)D_2(t)L_7(t) - P_3(t)C_2(t)^TL_9(t) \]

\[ - P_2(t)B_2(t)P_3(t) - P_3(t)C_3(t)^T - P_5(t)C_1(t)^TP_2(t)^T], \]

\[ S_5(t) = L_1(t)^{-1} [P_2(t)D_2(t)S_3(t) - P_3(t)C_2(t)^TS_4(t) + P_2(t)B_2(t)\varphi_2(t) \]

\[ - P_3(t)C_1(t)^T\varphi_1(t) + v_2(t)]. \]  

(3.17)

Now we determine the equations satisfied by \( P_i(t), i = 1, 2, 3 \). We first put (3.8), (3.14) and (3.10) into (3.1) and obtain a new form of the Hamiltonian system (3.1). Then applying Itô’s formula to \( m(t) \) in (3.8) and comparing with the drift term of the new form of (3.1), we have

\[
\dot{P}_1(t) \bar{X}(t) + P_1(t) \left\{ A_1(t) \bar{X}(t) + B_1(t) \left[ P_2(t) \bar{X}(t) - P_3(t)h(t) + \varphi_2(t) \right] \right. \\
+ C_1(t) \left[ L_{10}(t) \bar{X}(t) + L_{11}(t)h(t) + S_5(t) \right] + D_1(t) \left[ L_\alpha(t) \bar{X}(t) + L_7(t)h(t) + S_3(t) \right] \right\} \\
\left. + \dot{P}_2(t)^T h(t) + P_2(t)^T \left\{ B_3(t)^T h(t) + B_1(t)^T \left[ P_1(t) \bar{X}(t) + P_2(t)^T h(t) + \varphi_1(t) \right] \right. \right. \\
+ B_2(t)^T \left[ L_8(t) \bar{X}(t) + L_9(t)h(t) + S_4(t) \right] + B_4(t) \left[ P_2(t) \bar{X}(t) - P_3(t)h(t) + \varphi_2(t) \right] \right\} \\
- \gamma_1(t) \\
= - \left\{ A_3(t)^T h(t) + A_1(t)^T \left[ P_1(t) \bar{X}(t) + P_2(t)^T h(t) + \varphi_1(t) \right] \right. \\
+ A_2(t)^T \left[ L_8(t) \bar{X}(t) + L_9(t)h(t) + S_4(t) \right] + A_4(t) \bar{X}(t) \right\}.
\]

Hence, \( P_1(t), P_2(t)^T \) and \( \varphi_2(t) \) should be solutions of

\[
\begin{aligned}
dP_1(t) &= -\left\{ P_1(t)A_1(t) + P_1(t)B_1(t)P_2(t) + P_1(t)C_1(t)L_{10}(t) + P_1(t)D_1(t)L_\alpha(t) \right. \\
&\quad + P_2(t)^TB_1(t)^TP_1(t) + P_2(t)^TB_2(t)^TL_8(t) + P_2(t)^TB_4(t)P_2(t) + A_4(t) \\
&\quad + A_1(t)^TP_1(t) + A_2(t)^TL_8(t) \right\} dt, \\
P_1(T) &= G,
\end{aligned}
\]  

(3.18)

(3.19)
respectively. Applying Itô’s formula to $\bar{Y}(t)$ in (3.8) and comparing with the drift term of the new form of (5.1), we have
\[
\begin{align*}
\dot{P}_2(t)X(t) + P_2(t)\{A_1(t)\dot{X}(t) + B_1(t) [P_2(t)X(t) - P_3(t)h(t)] + \varphi_2(t)\} \\
+ C_1(t) \left[ L_{10}(t)X(t) + L_{11}(t)h(t) + S_5(t) \right] + D_1(t) \left[ L_6(t)X(t) + L_7(t)h(t) + S_3(t) \right] \\
- \dot{P}_3(t)h(t) - P_3(t)\{B_3(t)h(t) + B_4(t) [P_2(t)X(t) - P_3(t)h(t) + \varphi_2(t)] \} \\
+ B_2(t)^T \left[ L_8(t)X(t) + L_9(t)h(t) + S_4(t) \right] + B_4(t) \left[ P_2(t)X(t) - P_3(t)h(t) + \varphi_2(t) \right] \} \\
- \gamma_2(t) \\
= -\{A_3(t)\dot{X}(t) + B_3(t) [P_2(t)X(t) - P_3(t)h(t)] + \varphi_2(t)\} \\
+ C_3(t) \left[ L_{10}(t)X(t) + L_{11}(t)h(t) + S_5(t) \right] \\
+ D_3(t) \left[ L_6(t)X(t) + L_7(t)h(t) + S_3(t) \right].
\end{align*}
\] (3.22)

$P_2(\cdot)$, $P_3(\cdot)$ and $\varphi_2(\cdot)$ should be solutions of
\[
\begin{align*}
dP_2(t) \\
= -\{P_2(t)A_1(t) + P_2(t)B_1(t)P_2(t) + P_2(t)C_1(t)L_{10}(t) \\
+ P_2(t)D_1(t)L_6(t) - P_3(t)B_1(t)^TP_2(t) - P_3(t)B_2(t)^TL_8(t) \} \\
- P_3(t)B_4(t)P_2(t) + A_3(t) + B_3(t)P_2(t) + C_3(t)L_{10}(t) + D_3(t)L_6(t) \} dt, \\
P_2(T) = F,
\end{align*}
\] (3.23)
rigorously. Hence we will present the material in this subsection in an informal way although they can be verified (3.19), (3.23) and (3.24).

In this section, we study the existence and uniqueness results for solutions to non-Riccati-type equations

\[ \begin{aligned}
&d P_3(t) \\
= & -\{P_2(t) B_1(t) P_3(t) - P_2(t) C_1(t) L_{11}(t) - P_2(t) D_1(t) L_7(t) + P_3(t) B_3(t)^T \} \\
+ & P_3(t) B_1(t)^T P_2(t)^T + P_3(t) B_2(t)^T L_9(t) - P_3(t) B_4(t) P_3(t) \\
+ & B_3(t) P_3(t) - C_3(t) L_{11}(t) - D_3(t) L_7(t) \} dt,
\end{aligned} \tag{3.24} \]

\[ P_3(T) = 0, \]

\[ \begin{aligned}
&d \varphi_2(t) = -\{P_2(t) [B_1(t) \varphi_2(t) + C_1(t) S_n(t) + D_1(t) S_4(t)] \\
&- P_3(t) [B_1(t)^T \varphi_1(t) + B_2(t)^T S_4(t) + B_4(t) \varphi_2(t)] \\
&+ B_3(t) \varphi_2(t) + C_3(t) S_5(t) + D_3(t) S_3(t) \} dt + \varphi_2(t) dB(t),
\end{aligned} \tag{3.25} \]

\[ \varphi_2(T) = \xi \]

respectively. It can be verified that the equation (3.19), (3.21) are symmetric and (3.23) is indeed the transpose of (3.20).

**Remark 3.6** If \( D_4 > 0 \) and \( C_4 \geq 0 \), then the following relations hold:

\[ Q_1(t) = P_1(t), \quad Q_2(t) = P_2(t)^T, \]

\[ Q_3(t) = P_2(t), \quad Q_4(t) = P_3(t). \]

4 Non-Riccati-type equations

In this section, we study the existence and uniqueness results for solutions to non-Riccati-type equations (3.19), (3.23) and (3.24).

4.1 Auxiliary Riccati-type equations

Our aim of this subsection is to reveal the origin of the following auxiliary Riccati equation (4.5) and (4.8). Hence we will present the material in this subsection in an informal way although they can be verified rigorously.

We first introduce an auxiliary stochastic LQ problem which leads to a Riccati-type equation for \( \hat{P}(\cdot) \). Then the relations between \( P(\cdot) \) and \( \hat{P}(\cdot) \) are deduced and with the help of good properties of \( \hat{P}(\cdot) \), we will obtain the existence results for the solutions of (3.18) - (3.25).

Inspired by [11, 12, 15], for the FBLQ problem (2.1) - (2.2), we regard the BSDE as a controlled forward SDE and the term \( Z(\cdot) \) as a control. Thus, it becomes a forward LQ problem. Set \( \tilde{X}(t) = (X(t)^T, Y(t)^T)^T \) and \( \tilde{u}(t) = (u(t)^T, Z(t)^T)^T \). The state equation becomes

\[ d \tilde{X}(t) = \left[ \tilde{A}(t) \tilde{X}(t) + \tilde{B}(t) \tilde{u}(t) \right] dt + \left[ \tilde{C}(t) \tilde{X}(t) + \tilde{D}(t) \tilde{u}(t) \right] dB(t), \tag{4.1} \]
and the cost functional becomes

\[ J(u(\cdot), Z(\cdot)) = \frac{1}{2} E \left[ \int_0^T \left( \dot{X}(t)^T \hat{Q}(t) \dot{X}(t) + \tilde{u}(t)^T \tilde{R}(t) \tilde{u}(t) \right) dt + \langle GX(T), X(T) \rangle \right], \tag{4.2} \]

where

\[
\hat{A}(t) = \begin{pmatrix} A_1(t) & B_1(t) \\ -A_3(t) & -B_3(t) \end{pmatrix}, \quad \hat{B}(t) = \begin{pmatrix} D_1(t) & C_1(t) \\ -D_3(t) & -C_3(t) \end{pmatrix},
\]

\[
\hat{C}(t) = \begin{pmatrix} A_2(t) & B_2(t) \\ 0 & 0 \end{pmatrix}, \quad \hat{D}(t) = \begin{pmatrix} D_2(t) & C_2(t) \\ 0 & I_m \end{pmatrix},
\]

\[
\hat{Q}(t) = A_4(t) 0 \\
0  B_4(t)
\]

\[
\tilde{R}(t) = \begin{pmatrix} D_4(t) & 0 \\ 0 & C_4(t) \end{pmatrix}.
\]

Now we solve the above LQ problem by the completion-of-squares technique similar as in Theorem 3.1 in [4]. Suppose that \((\tilde{\varphi}(\cdot), \tilde{v}(\cdot))\) satisfies the following BSDE

\[ d\tilde{\varphi}(t) = -\tilde{\gamma}(t) dt + \tilde{v}(t) dB(t), \tag{4.3} \]

where \(\tilde{\gamma}(\cdot)\) will be determined later. For a function \(\hat{P}(\cdot)\) to be determined, applying Itô’s formula to

\[ \left( \dot{X}(t) - \tilde{\varphi}(t) \right)^T \hat{P}(t) \left( \dot{X}(t) - \tilde{\varphi}(t) \right) + \int_0^t \left( \dot{X}(s)^T \hat{Q}(s) \dot{X}(s) + \tilde{u}(s)^T \tilde{R}(s) \tilde{u}(s) \right) ds, \]

we have

\[
d\left[ \left( \dot{X}(t) - \tilde{\varphi}(t) \right)^T \hat{P}(t) \left( \dot{X}(t) - \tilde{\varphi}(t) \right) + \int_0^t \left( \dot{X}(s)^T \hat{Q}(s) \dot{X}(s) + \tilde{u}(s)^T \tilde{R}(s) \tilde{u}(s) \right) ds \right]
= \left\{ \left[ \tilde{u}(t) + M_4(t)^{-1} \left( M_2(t) \dot{X}(t) - M_3(t) \tilde{\varphi}(t) - M_4(t) \tilde{v}(t) \right) \right]^T M_1(t) \right.
- \left[ \tilde{u}(t) + M_4(t)^{-1} \left( M_2(t) \dot{X}(t) - M_3(t) \tilde{\varphi}(t) - M_4(t) \tilde{v}(t) \right) \right]
+ \dot{X}(t)^T \left[ \hat{P}(t) + \hat{D}(t) \hat{A}(t) + \hat{A}(t)^T \hat{P}(t) + \hat{C}(t)^T \hat{P}(t) \hat{C}(t) + \hat{Q}(t) \right]
- M_2(t)^T M_4(t)^{-1} M_2(t) \dot{X}(t)
+ \dot{X}(t)^T \left[ M_2(t)^T M_1(t)^{-1} \left( M_3(t) \tilde{\varphi}(t) - M_4(t) \tilde{v}(t) \right) - \hat{A}(t)^T \hat{P}(t) \tilde{\varphi}(t) - \hat{P}(t) \tilde{\varphi}(t)
- \hat{P}(t) \tilde{\gamma}(t) - \hat{C}(t)^T \hat{P}(t) \tilde{v}(t) \right]
+ \left[ M_2(t)^T M_1(t)^{-1} \left( M_3(t) \tilde{\varphi}(t) - M_4(t) \tilde{v}(t) \right) - \hat{A}(t)^T \hat{P}(t) \tilde{\varphi}(t) - \hat{P}(t) \tilde{\varphi}(t)
- \hat{P}(t) \tilde{\gamma}(t) - \hat{C}(t)^T \hat{P}(t) \tilde{v}(t) \right]^T \dot{X}(t)
+ M_5(t) \right\} dt + \{\ldots\} dB(t),
\]
where
\[
M_1(t) = \tilde{R}(t) + \bar{D}(t)\tilde{P}(t)\bar{D}(t), \quad M_2(t) = \bar{B}(t)\tilde{P}(t) + \bar{D}(t)\tilde{P}(t)\bar{C}(t),
\]
\[
M_3(t) = \bar{B}(t)\tilde{P}(t), \quad M_4(t) = \bar{D}(t)\tilde{P}(t),
\]
\[
M_5(t) = -\tilde{\gamma}(t)^T\tilde{P}(t)\tilde{\varphi}(t) - \tilde{\varphi}(t)^T\tilde{P}(t)\tilde{\gamma}(t) + \tilde{\varphi}(t)^T\tilde{P}(t)\tilde{v}(t) + \tilde{v}(t)^T\tilde{P}(t)\tilde{v}(t)
\]
\[-(M_3(t)\tilde{\varphi}(t) + M_4(t)\tilde{v}(t))^T M_1(t)^{-1} (M_3(t)\tilde{\varphi}(t) + M_4(t)\tilde{v}(t)).
\]
Thus, we can obtain the form of the Riccati equation and the optimal control \((\bar{u}(\cdot), \bar{Z}(\cdot))\) as following:
\[
\begin{align*}
\dot{\tilde{P}}(t) + \tilde{P}(t)\tilde{A}(t) + (\tilde{A}(t))^T\tilde{P}(t) + \tilde{C}(t)^T\tilde{P}(t)\tilde{C}(t) + & \tilde{Q}(t) \\
-M_2(t)^TM_1(t)^{-1}M_2(t) = 0, \\
\bar{R}(t) + \bar{D}(t)^T\tilde{P}(t)\bar{D}(t) > 0,
\end{align*}
\]
\[
(\bar{u}(t), \bar{Z}(t))^T = -M_1(t)^{-1} [M_2(t)(\bar{X}(t), \bar{Y}(t))^T - M_3(t)\tilde{\varphi}(t) - M_4(t)\tilde{v}(t)],
\]
and
\[
\dot{\tilde{\gamma}}(t) = \tilde{P}(t)^{-1} \left[ M_2(t)^TM_1(t)^{-1}(M_3(t)\tilde{\varphi}(t) - M_4(t)\tilde{v}(t)) - \tilde{A}(t)^T\tilde{P}(t)\tilde{\varphi}(t) \\
-\tilde{P}(t)\tilde{\varphi}(t) - \tilde{C}(t)^T\tilde{P}(t)\tilde{v}(t) \right].
\]
Set
\[
\tilde{P}(t) = \begin{pmatrix} \tilde{P}_1(t) & \tilde{P}_2(t)^T \\ \tilde{P}_2(t) & \tilde{P}_3(t) \end{pmatrix}, \quad \tilde{\varphi}(t) = \begin{pmatrix} \tilde{\varphi}_1(t) \\ \tilde{\varphi}_2(t) \end{pmatrix}, \quad \tilde{v}(t) = \begin{pmatrix} \tilde{v}_1(t) \\ \tilde{v}_2(t) \end{pmatrix}.
\]
By the relationship between the adjoint process and the state process for stochastic LQ problems, we have
\[
(m(t)^T, -h(t)^T)^T = \tilde{P}(t) \begin{pmatrix} \bar{X}(t) - \tilde{\varphi}(t) \end{pmatrix}.
\]
Comparing (4.7) with (3.3), we obtain the relations between \(\tilde{P}(\cdot), (\tilde{\varphi}(\cdot), \tilde{v}(\cdot))\) and \(P(\cdot), (\varphi(\cdot), v(\cdot))\) as following:
\[
P_1(t) = \tilde{P}_1(t) - \tilde{P}_2(t)^T\tilde{P}_3(t)^{-1}\tilde{P}_2(t),
\]
\[
P_2(t) = -\tilde{P}_3(t)^{-1}\tilde{P}_2(t),
\]
\[
P_3(t) = \tilde{P}_3(t)^{-1},
\]
\[
\varphi_1(t) = -P_1(t)\tilde{\varphi}_1(t), \quad v_1(t) = -P_1(t)\tilde{v}_1(t),
\]
\[
\varphi_2(t) = \tilde{\varphi}_2(t) - P_2(t)\tilde{\varphi}_1(t), \quad v_2(t) = \tilde{v}_2(t) - P_2(t)\tilde{v}_1(t),
\]
or the equivalent form
\[
\tilde{P}_1(t) = P_1(t) + P_2(t)^TP_3(t)^{-1}P_2(t),
\]
\[
\tilde{P}_2(t) = -P_3(t)^{-1}P_2(t),
\]
\[
\tilde{P}_3(t) = P_3(t)^{-1},
\]
\[16\]
\[
\hat{\varphi}_1(t) = -P_1(t)^{-1}\varphi_1(t), \quad \tilde{v}_1(t) = -P_1(t)^{-1}v_1(t), \\
\hat{\varphi}_2(t) = \varphi_2(t) - P_2(t)P_1(t)^{-1}\varphi_1(t), \quad \tilde{v}_2(t) = v_2(t) - P_2(t)P_1(t)^{-1}v_1(t).
\]

Note that \(P_3(T) = 0\) which makes \(P_3(T)^{-1}\) meaningless. So we need to modify the terminal conditions of \(\hat{P}(\cdot), \hat{\varphi}(\cdot)\) and \(P(\cdot), \varphi(\cdot)\). For \(i = 1, 2, \ldots\), consider the solutions

\[
i_P(t) = \begin{pmatrix} P_{1,i}(t) & P_{2,i}(t) \\ P_{2,i}(t) & P_{3,i}(t) \end{pmatrix}, \quad i\varphi(t) = \begin{pmatrix} \varphi_{1,i}(t) \\ \varphi_{2,i}(t) \end{pmatrix}, \quad iv(t) = \begin{pmatrix} v_{1,i}(t) \\ v_{2,i}(t) \end{pmatrix}
\]

to equations \(4.8, 4.9, 4.10, 4.11\) with the terminal conditions

\[
i_P(T) = \begin{pmatrix} G & F^T \\ F & \frac{1}{2}I_m \end{pmatrix}, \quad i\varphi(T) = \begin{pmatrix} 0 \\ \xi \end{pmatrix}. \tag{4.8}
\]

Correspondingly, we consider the Riccati equation \(4.8\) and \(4.9\) for

\[
i\hat{P}(t) = \begin{pmatrix} \hat{P}_{1,i}(t) & \hat{P}_{2,i}(t) \\ \hat{P}_{2,i}(t) & \hat{P}_{3,i}(t) \end{pmatrix}, \quad i\hat{\varphi}(t) = \begin{pmatrix} \hat{\varphi}_{1,i}(t) \\ \hat{\varphi}_{2,i}(t) \end{pmatrix}, \quad i\hat{v}(t) = \begin{pmatrix} \tilde{v}_{1,i}(t) \\ \tilde{v}_{2,i}(t) \end{pmatrix}
\]

with terminal conditions

\[
i\hat{P}(T) = \begin{pmatrix} G + iF^TF & -iF^T \\ -iF & \frac{1}{2}I_m \end{pmatrix}, \quad i\hat{\varphi}(T) = \begin{pmatrix} 0 \\ \xi \end{pmatrix}. \tag{4.9}
\]

**Remark 4.1** In fact, \(4.9\) and \(4.10\) with terminal conditions \(4.9\) correspond to the following stochastic control problem: the state equation is \(4.1\) and the cost functional is

\[
J_i(\mu(\cdot), Z(\cdot)) = \frac{1}{2}E \left[ \int_0^T \left( \Xi(t)^T\hat{Q}(t)\Xi(t) + \hat{u}(t)^T\hat{R}(t)\hat{u}(t) \right) \, dt + \langle GX(T), X(T) \rangle \right] \\
+ \frac{1}{2}E \left[ (Y(T) - FX(T) - \xi)^T(Y(T) - FX(T) - \xi) \right]. \tag{4.10}
\]

Theorem 4.1 justifies the above heuristic derivation.

**Assumption 4.2** There exist a natural number \(i_0\) such that for \(i \geq i_0, \ (4.3)\) has a positive definite solution \(i\hat{P}(\cdot)\) which satisfies the terminal condition \(4.9\).

**Remark 4.3** Under the assumption \(D_4 > 0\) and \(C_4 \geq 0\), it is easy to check that \(\hat{R} + \hat{D}^T\hat{D} > 0\). Then, by Theorem 4.1 in [4], Assumption 4.2 holds for \(i_0 = 1\).

Set

\[
L_{1,i}(t) = I_n - P_{2,i}(t)C_2(t) + P_{3,i}(t)C_4(t), \\
L_{2,i}(t) = I_n - P_{2,i}(t)^TC_2(t)^T + (P_{1,i}(t)C_2(t) + P_{2,i}(t)^TC_4(t)) L_{1,i}(t)^{-1}P_{3,i}(t)C_2(t)^T.
\]
**Theorem 4.4** Suppose that Assumptions 4.5, 4.4 and 4.2 hold. For \( i \geq i_0 \), define

\[
P_{1,i}(t) = \tilde{P}_{1,i}(t) - \tilde{P}_{2,i}(t)\tilde{P}_{3,i}(t)^{-1}\tilde{P}_{2,i}(t),
\]

\[
P_{2,i}(t) = -\tilde{P}_{3,i}(t)^{-1}\tilde{P}_{2,i}(t),
\]

\[
P_{3,i}(t) = \tilde{P}_{3,i}(t)^{-1},
\]

\[
\varphi_{1,i}(t) = -P_{1,i}(t)\varphi_{1,i}(t), \quad v_{1,i}(t) = -P_{1,i}(t)v_{1,i}(t),
\]

\[
\varphi_{2,i}(t) = \varphi_{2,i}(t) = P_{2,i}(t)\varphi_{1,i}(t), \quad v_{2,i}(t) = \tilde{v}_{2,i}(t) - P_{2,i}(t)v_{1,i}(t),
\]

where \( i\dot{\varphi}() \) and \( i\tilde{\varphi}() \) are solutions to (4.11) and (4.3). Suppose that \( L_{1,i}()^{-1} \) and \( L_{2,i}()^{-1} \) exist. Then the above defined \( (P_{1,i}(), P_{2,i}(), P_{3,i}()) \) solves 3.10, 3.23, 3.24 and \( (i\varphi(), i\tilde{\varphi}()) \) solves 3.21, 3.25 with 3.8 for each \( i \geq i_0 \).

We put the proof in Appendix 7.1.

**Lemma 4.5** Under the same assumptions as Theorem 4.4 for \( i \geq i_0 \), we have

\[
M_{1,i}(t)^{-1}M_{2,i}(t) = -\begin{pmatrix}
L_{6,i}(t) + L_{7,i}(t)P_{3,i}(t)^{-1}P_{2,i}(t) & -L_{7,i}(t)P_{3,i}(t)^{-1} \\
L_{10,i}(t) + L_{11,i}(t)P_{3,i}(t)^{-1}P_{2,i}(t) & -L_{11,i}(t)P_{3,i}(t)^{-1}
\end{pmatrix},
\]

\[
M_{1,i}(t)^{-1}[M_{3,i}(t)i\varphi(t) + M_{4,i}(t)i\tilde{\varphi}(t)] = \begin{pmatrix}
L_{7,i}(t)P_{3,i}(t)^{-1}\varphi_{2,i}(t) + S_{3,i}(t) \\
L_{11,i}(t)P_{3,i}(t)^{-1}\varphi_{2,i}(t) + S_{5,i}(t)
\end{pmatrix}.
\]

The proof is in Appendix 7.2. This lemma will be used in the proof of Theorem 4.2.

**Remark 4.6** If \( C_2 > 0 \) and \( D_4 > 0 \), then it can be verified that Assumption 4.2 holds. If \( C_2 = 0 \) and \( D_4 > 0 \), then \( L_{1,i}()^{-1} \) and \( L_{2,i}()^{-1} \) in Theorem 4.4 exist.

### 4.2 Existence and uniqueness results

In this subsection, we study the solvability of 3.10, 3.23, 3.24 by Theorem 4.4.

**Lemma 4.7** Suppose \( \tilde{P}_{1}() \) and \( \tilde{P}_{2}() \) are solutions to Riccati equation (4.3) with terminal conditions \( \tilde{P}_{1}(T) \geq \tilde{P}_{2}(T) \), then \( \tilde{P}_{1}(t) \geq \tilde{P}_{2}(t) \) for \( t \in [0,T] \).

**Proof.** By Theorem 6.1 in [26], the value function of the corresponding LQ problem is \( x^T\tilde{P}_{1}(t)x \) (resp. \( x^T\tilde{P}_{2}(t)x \)) for all \((t,x) \in [0,T] \times \mathbb{R}^n \). The proof can be obtained from \( \tilde{P}_{1}(T) \geq \tilde{P}_{2}(T) \). \( \blacksquare \)

**Theorem 4.8** Suppose that the same assumptions as Theorem 4.4 hold and \( (\tilde{R}()+\tilde{D}()\tilde{P}()\tilde{D}()^{-1}) \) is bounded for each \( i \geq i_0 \). Then \( P_{3,i}(t) \geq P_{3,i+1}(t) \geq 0 \), and \( P_{1,i+1}(t) \geq P_{1,i}(t) \geq 0 \) for \( i \geq i_0 \). Moreover, suppose that \( P_{1,i}() \) has upper bound and \( |P_{2,i}()| \), \( L_{1,i}()^{-1} \), \( L_{2,i}()^{-1} \) and \( L_{5,i}()^{-1} \) are uniformly bounded for each \( i \geq i_0 \). Then 3.12, 3.23, 3.24 have a unique solution \( (P_{1}(), P_{2}(), P_{3}()) \).
**Proof.** It can be verified that

\[
\begin{pmatrix}
G + (i + 1)F^T F & - (i + 1)F^T \\
-iF & (i + 1) \mathbb{I}_m
\end{pmatrix} 
\geq \begin{pmatrix}
G + iF^T F & -iF \\
-iF & i\mathbb{I}_m
\end{pmatrix}.
\]

By Lemma 4.7 we have \(i+1\hat{P}(t) \geq \hat{P}(t)\) which yields that \(\hat{P}_{3,i+1}(t) \geq \hat{P}_{3,i}(t)\) and \(i+1\hat{P}(t)^{-1} \leq \hat{P}(t)^{-1}\).

Moreover, note that

\[
\begin{pmatrix}
\hat{P}_{1,i}(t) & \hat{P}_{2,i}(t) \\
\hat{P}_{2,i}(t) & \hat{P}_{3,i}(t)
\end{pmatrix}^{-1} = \begin{pmatrix}
\hat{P}_{1,i}(t) - \hat{P}_{2,i}(t)^T \hat{P}_{3,i}(t)^{-1} & ... \\
... & ...
\end{pmatrix}.
\]

By the relationship (4.11), we obtain that \(P_{3,i+1}(t) \leq P_{3,i}(t)\) and \(P_{1,i+1}(t) \geq P_{1,i}(t)\).

Thus, \(\{P_{3,i}(t)\}_{i \geq i_0}\) is a bounded decreasing (resp. increasing) sequence in \(C([0,T]; S^m)\) (resp. \(C([0,T]; S^m)\) ) and therefore has a limit. The convergence of \(\{P_{2,i}(t)\}_{i \geq i_0}\) can be obtained by the following Proposition 4.11. Denote by \((P_1(\cdot), P_2(\cdot), P_3(\cdot))\) the limit of \(\{(P_{3,i}(\cdot), P_{2,i}(\cdot), P_{3,i}(\cdot))\}_{i \geq i_0}\). By the bounded convergence theorem, one can obtain that \((P_1(\cdot), P_2(\cdot), P_3(\cdot))\) is the solution to (3.11), (3.23), (3.24). This completes the proof. 

**Corollary 4.9** Suppose that Assumptions 2.3, 2.4 and 2.5 hold. Moreover, suppose that \(P_{1,i}(\cdot)\) has upper bound and \(|P_{2,i}(\cdot)|, L_{1,i}(\cdot)^{-1}\) and \(L_{2,i}(\cdot)^{-1}\) are uniformly bounded for each \(i \geq 1\). Then the equation (3.20) has a unique solution.

**Proof.** By Remark 4.6 Assumption 2.2 holds. Since \(D_1 > 0\), it is easy to verify that \(L_{5,i}(\cdot)^{-1} \leq D_i^{-1}\) for each \(i \geq 1\). By Remark 3.6 and Theorem 4.8 then the equation (3.3) has a unique solution. 

**Proposition 4.10** Suppose that all assumptions in Theorem 4.3 hold. Then for \(i \geq i_0\), we have

\[
|P_1(t) - P_{1,i}(t)| + |P_2(t) - P_{2,i}(t)| + |P_3(t) - P_{3,i}(t)| \leq C i^{-2},
\]

where \((P_1(\cdot), P_2(\cdot), P_3(\cdot))\) is the limit of \(\{P_{1,i}(\cdot), P_{2,i}(\cdot), P_{3,i}(\cdot)\}\) and \(C\) is a constant independent of \(i\).

**Proof.** Set \(\Delta_{1,i}(t) = P_{1,i+1}(t) - P_{1,i}(t), \Delta_{2,i}(t) = P_{2,i+1}(t) - P_{2,i}(t), \Delta_{3,i}(t) = P_{3,i+1}(t) - P_{3,i}(t)\). By (3.19), (3.23), (3.24) and the boundedness assumptions, we have

\[
|\Delta_{1,i}(t)| + |\Delta_{2,i}(t)| + |\Delta_{3,i}(t)| = \frac{m}{i+1} + C' \int_{i}^{T} (|\Delta_{1,i}(s)| + |\Delta_{2,i}(s)| + |\Delta_{3,i}(s)|) ds,
\]

where \(C'\) is a constant independent of \(i\). Then, by Gronwall’s inequality we have

\[
|\Delta_{1,i}(t)| + |\Delta_{2,i}(t)| + |\Delta_{3,i}(t)| \leq C i^{-2},
\]

where \(C = me^{C'T}\).
Thus, \((3.19), (3.23), (3.24)\) have a unique solution where \(C\) is a constant independent of \(i\). By Theorem 4.8, \(P\) is bounded. Then one can check that

\[
-\frac{1}{2} \left( (D_1(t) + D_2(t)C_3(t) + 1 + P_3(t)C_4(t))^{-1} \right) P_2(t)
+ (D_1(t) + D_2(t)A_2(t)) \Delta P_1(t)
\]

where

\[
\Delta = D_4(t) + D_2(t)^2 P_1(t) + C_4(t)D_2(t)^2 (1 + P_3(t)C_4(t))^{-1} P_2(t)^2.
\]

By Theorem 4.8, \(P_{3,i}(\cdot)\) is bounded. Then one can check that

\[
|P_{2,i}(t)| \leq |F| + C \int_0^T (|P_{2,i}(s)| + 1) \, ds,
\]

where \(C\) is a constant independent of \(i\). By Gronwall’s inequality, \(|P_{2,i}(\cdot)|\) is bounded. Because

\[
0 \leq P_{3,i}(t) \leq G + C \int_0^T (P_{3,i}(s) + 1) \, ds,
\]

where \(C\) is a constant independent of \(i\), we deduce that \(P_{3,i}(\cdot)\) has a upper bound by Gronwall’s inequality. Thus, \((3.19), (3.23), (3.24)\) have a unique solution \((P_1(\cdot), P_2(\cdot), P_3(\cdot))\) by Theorem 4.8.
5 Feedback optimal control for FBLQ problem

In this section, we prove the existence of optimal control without the positiveness of $C_\alpha(\cdot)$ and $D_\alpha(\cdot)$. We first give the following lemma.

**Lemma 5.1** Suppose all assumptions in Theorem 4.8 hold. Then $\left\{ \mathbb{E} \int_0^T |M_{5,i}(t)| dt \right\}_{t \geq i_0}$ is uniformly bounded, where $M_{5,i}(\cdot)$ is defined by replacing $\hat{P}(\cdot)$ with $\hat{P}(\cdot)$ in (4.4).

The proof is in Appendix 7.3.

**Theorem 5.2** Suppose Assumption 2 holds and all assumptions in Theorem 4.8 hold, and $P_1(t) = \lim_{i \to \infty} P_{3,i}(t) > 0$, for $t \in [0, T)$. Then there exists an optimal control $\bar{u}(\cdot)$ for the FBLQ problem (2.1) - (2.2). Furthermore, any optimal control $\bar{u}(\cdot)$ satisfies

$$\bar{u}(t) = \left[ L_6(t) + L_7(t)P_3(t)^{-1}P_2(t) \right] \bar{X}(t)$$

$$- L_7(t)P_3(t)^{-1}Y(t) + L_7(t)P_3(t)^{-1}\varphi_2(t) + S_3(t),$$

where $P_1(t) = \lim_{i \to \infty} P_{1,i}(t), P_2(t) = \lim_{i \to \infty} P_{2,i}(t)$, and $L_6(t), L_7(t), S_3(t)$ are defined in (5.2).

**Proof.** By Theorems 4.8, (3.1), (3.2), (3.3) solves the equations (5.1), (5.2), (5.3). Then there exists a unique solution $((\varphi_1(\cdot)^T, \varphi_2(\cdot)^T, v_1(\cdot)^T, v_2(\cdot)^T)^T)$ to the BSDE (5.2) - (5.3). Set

$$N_1(t) = A_1(t) + B_1(t)P_2(t) + C_1(t)L_{10}(t) + D_1(t)L_6(t), \quad N_2(t) = -B_1(t)P_3(t) + C_1(t)L_{11}(t) + D_1(t)L_7(t),$$

$$N_3(t) = B_1(t)\varphi_2(t) + C_1(t)S_5(t) + D_1(t)S_3(t), \quad N_4(t) = A_2(t) + B_2(t)P_2(t) + C_2(t)L_{10}(t) + D_2(t)L_6(t),$$

$$N_5(t) = -B_2(t)P_3(t) + C_2(t)L_{11}(t) + D_2(t)L_7(t), \quad N_6(t) = -B_2(t)\varphi_2(t) + C_2(t)S_5(t) + D_2(t)S_3(t),$$

$$N_7(t) = B_3(t)^TP_1(t) + B_2(t)^TL_8(t) + B_4(t)^TP_2(t), \quad N_8(t) = B_3(t)^T + B_1(t)^TP_2(t)^T + B_2(t)^TL_6(t) - B_4(t)^TP_3(t),$$

$$N_9(t) = B_1(t)^T\varphi_1(t) + B_2(t)^TS_4(t) + B_4(t)^T\varphi_2(t), \quad N_{10}(t) = C_1(t)^TP_1(t) + C_2(t)^TL_8(t) + C_4(t)^TL_{10}(t),$$

$$N_{11}(t) = C_3(t)^TP_2(t)^T + C_2(t)^TL_6(t) + C_4(t)^TL_{11}(t), \quad N_{12}(t) = C_1(t)^T\varphi_1(t) + C_2(t)^TS_4(t) + C_4(t)^TS_5(t).$$

Consider the following linear SDE for $(X^*(\cdot), h^*(\cdot))$:

$$dX^*(t) = \left[ N_1(t)X^*(t) + N_2(t)h^*(t) + N_3(t) \right] dt + \left[ N_4(t)X^*(t) + N_5(t)h^*(t) + N_6(t) \right] dB(t),$$

$$dh^*(t) = \left[ N_7(t)X^*(t) + N_8(t)h^*(t) + N_9(t) \right] dt + \left[ N_{10}(t)X^*(t) + N_{11}(t)h^*(t) + N_{12}(t) \right] dB(t),$$

$$X^*(0) = x_0, \quad h^*(0) = (I_m + HP_3(0))^{-1}H(P_2(0)x_0 + \varphi_2(0)).$$

Since (5.2) has bounded coefficients, it has a unique solution $(X^*(\cdot)^T, h^*(\cdot)^T) \in L^2_{\mathcal{G}}(\Omega; C([0, T], \mathbb{R}^{n+m}))$. Set

$$\bar{u}(t) = L_6(t)X^*(t) + L_7(t)h^*(t) + S_3(t)$$

(5.3)
which is an admissible control. It can be verified that

\[
X(t) = X^*(t), \quad h(t) = h^*(t),
\]

\[
\bar{Y}(t) = P_2(t)X(t) - P_3(t)h(t) + \varphi_2(t), \quad \bar{Z}(t) = L_{10}(t)\bar{X}(t) + L_{11}(t)\bar{h}(t) + S_6(t),
\]

(5.4)

solves the Hamiltonian system \[ \text{[III]} \]. Now we prove that \( \bar{u}(\cdot) \) is an optimal control in two steps.

**Step 1:** For \( t \in [0, T] \), set

\[
\tilde{P}(t) = \begin{pmatrix}
P_1(t) + P_2(t)^{\top} P_3(t)^{-1} P_2(t) & -P_2(t)^{\top} P_3(t)^{-1} \\
-P_3(t)^{-1} P_2(t) & P_3(t)^{-1}
\end{pmatrix}.
\]

For any given \( \varepsilon > 0 \), by Theorem 4.4, \( \tilde{P}(\cdot) \) solves the equation \[ \text{[IV]} \] on \([0, T - \varepsilon]\). By the completion-of-squares technique, we have

\[
J(\bar{u}(\cdot)) = \frac{1}{2} \left[ (x_0 - \bar{\varphi}_1(0))^T, (\bar{Y}(0) - \bar{\varphi}_2(0))^T \right] \tilde{P}(0) \left[ (x_0 - \bar{\varphi}_1(0))^T, (\bar{Y}(0) - \bar{\varphi}_2(0))^T \right] + \bar{Y}(0)^T H \bar{Y}(0)
\]

\[
+ \frac{1}{2} \mathbb{E} \left[ \tilde{X}(T)^T G \tilde{X}(T) - \left( \frac{X(T - \varepsilon) - \bar{\varphi}_1(T - \varepsilon)}{Y(T - \varepsilon) - \bar{\varphi}_2(T - \varepsilon)} \right)^T \tilde{P}(T - \varepsilon) \left( \frac{X(T - \varepsilon) - \bar{\varphi}_1(T - \varepsilon)}{Y(T - \varepsilon) - \bar{\varphi}_2(T - \varepsilon)} \right) \right] + \frac{1}{2} \mathbb{E} \int_{0}^{T - \varepsilon} M_5(t) dt
\]

\[
+ \frac{1}{2} \mathbb{E} \left[ \int_{0}^{T - \varepsilon} \left\{ \left( \frac{u(t)}{2\varepsilon(t)} \right) + M_1(t)^{-1} \left( \frac{\bar{X}(t)}{\bar{Y}(t)} \right) - M_3(t)\bar{\varphi}(t) - M_4(t)\bar{v}(t) \right\}^T M_1(t) \left( \frac{u(t)}{2\varepsilon(t)} \right) + M_1(t)^{-1} \left( \frac{\bar{X}(t)}{\bar{Y}(t)} \right) - M_3(t)\bar{\varphi}(t) - M_4(t)\bar{v}(t) \right\} dt
\]

\[
+ \frac{1}{2} \mathbb{E} \left[ \int_{0}^{T - \varepsilon} \left( \langle A_4(t)\bar{X}(t), \bar{X}(t) \rangle + \langle B_4(t)\bar{Y}(t), \bar{Y}(t) \rangle + \langle C_4(t)\bar{Z}(t), \bar{Z}(t) \rangle + \langle D_4(t)\bar{u}(t), \bar{u}(t) \rangle \right) dt \right]
\]

\[
= (I) + (II) + (III) + (IV) + (V).
\]

The part (I) is simplified as follows.

\[
\left( (x_0 - \bar{\varphi}_1(0))^T, (\bar{Y}(0) - \bar{\varphi}_2(0))^T \right) \tilde{P}(0) \left( (x_0 - \bar{\varphi}_1(0))^T, (\bar{Y}(0) - \bar{\varphi}_2(0))^T \right) + \bar{Y}(0)^T H \bar{Y}(0)
\]

\[
= R_1(\bar{Y}(0))^T \left( P_3(0)^{-1} + H \right) R_1(\bar{Y}(0)) + R_2,
\]

where

\[
R_1(y) = y - (I_m + P_3(0)H)^{-1} P_2(0)x_0 + \varphi_2(0),
\]

\[
R_2 = (x_0 + P_1(0)^{-1}\varphi_1(0))^T P_1(0) \left( x_0 + P_1(0)^{-1}\varphi_1(0) \right)
\]

\[
+ [P_2(0)x_0 + \varphi_2(0)]^T (I_m + HP_3(0))^{-1} H [P_2(0)x_0 + \varphi_2(0)].
\]

One can check that \( I_m - P_3(0) (I_m + HP_3(0))^{-1} H - (I_m + P_3(0)H)^{-1} = 0 \) which implies \( R_1(\bar{Y}(0)) = 0 \).
Then, we prove the part \((II)\) converges to 0 as \(\varepsilon \to 0\). Noting that \(\tilde{Y}(\cdot) = P_2(\cdot)\tilde{X}(\cdot) - \varphi_2(\cdot) = -P_3(\cdot)h(\cdot)\), we have
\[
\tilde{X}(T)^T G \tilde{X}(T) = \left(\frac{X(T-\varepsilon) - \tilde{X}(T-\varepsilon)}{Y(T-\varepsilon) - \tilde{Y}(T-\varepsilon)}\right)^T \tilde{P}(T-\varepsilon) \left(\frac{X(T-\varepsilon) - \tilde{X}(T-\varepsilon)}{Y(T-\varepsilon) - \tilde{Y}(T-\varepsilon)}\right)
\]
\[
= \langle \tilde{X}(T) \rangle^T G \tilde{X}(T) - \left(\tilde{X}(T) - \tilde{X}(T) + \tilde{P}(T-\varepsilon)^{-1} \varphi_1(T-\varepsilon)\right)^T P_1(T-\varepsilon) - \tilde{X}(T-\varepsilon) + \tilde{P}(T-\varepsilon)^{-1} \varphi_1(T-\varepsilon)\rangle^T P_1(T-\varepsilon)
\]
\[
\xrightarrow{\varepsilon \to 0} 0 \quad \text{as} \quad \varepsilon \to 0.
\]

The part \((V)\) converges to 0 as \(\varepsilon \to 0\) due to the integrability of \(\tilde{X}(\cdot), \tilde{Y}(\cdot)\) and \(\tilde{Z}(\cdot)\). By Lemma \([5.5, 5.1]\) and \([5.4]\), we deduce that the part \((IV)\) equals to 0.

Finally, by Lemma \([5.1]\) and letting \(\varepsilon \to 0\) on both sides of \([5.4]\), we have
\[
J(\tilde{u}(\cdot)) = \frac{1}{2} R_2 + \frac{1}{2} \mathbb{E} \int_0^T M_5(t) dt.
\]

**Step 2.** We first give a lower bound for the cost functional by the completion-of-squares technique. For an admissible control \(u(\cdot)\), let \((X(\cdot), Y(\cdot), Z(\cdot))\) be the corresponding state process. Set \(X(t) = (X(t)^T, Y(t)^T)^T\). Applying Itô’s formula to
\[
\left(\tilde{X}(t) - i\tilde{\varphi}(t)\right)^T i\tilde{P}(t) \left(\tilde{X}(t) - i\tilde{\varphi}(t)\right)
\]
and taking expectations, we have
\[
\mathbb{E} \left[ X(T)^T G X(T) + Y(0)^T H Y(0) \right] = R_{1,i}(0) + R_{2,i} + \mathbb{E} \left[ \int_0^T d \left( \tilde{X}(t) - i\tilde{\varphi}(t) \right)^T i\tilde{P}(t) \left( \tilde{X}(t) - i\tilde{\varphi}(t) \right) \right],
\]
where \(R_{1,i}(y)\) and \(R_{2,i}\) are defined by replacing \(P(0)\) with \(iP(0)\) in \([5.6]\). By the completion-of-squares technique,
\[
J(u(\cdot)) = \frac{1}{2} \left[ R_{1,i}(0)^T \left( P_{3,i}(0)^{-1} + H \right) R_{1,i}(0) + R_{2,i} \right] + \frac{1}{2} \mathbb{E} \int_0^T M_5(t) dt
\]
\[
+ \frac{1}{2} \mathbb{E} \int_0^T \left\{ \left[ \left( u(t)^T, Z(t)^T \right)^T + M_1(t)^{-1} \left( M_2(t) \tilde{X}(t) - M_3(t) \tilde{\varphi}(t) - M_4(t) \tilde{\psi}(t) \right) \right]^T \right. \\
\left. \left. M_1(t) \left[ \left( u(t)^T, Z(t)^T \right)^T + M_1(t)^{-1} \left( M_2(t) \tilde{X}(t) - M_3(t) \tilde{\varphi}(t) - M_4(t) \tilde{\psi}(t) \right) \right]^T \right] dt.
\]

Note that \(M_1(t) > 0\). Letting \(i \to \infty\) and appealing to Fatou’s lemma and Lemma \([5.1]\) we have
\[
J(u(\cdot)) \geq J(\tilde{u}(\cdot)) + \frac{1}{2} R_{1,i}(0)^T \left( P_{3,i}(0)^{-1} + H \right) R_{1,i}(0)
\]
\[
+ \frac{1}{2} \mathbb{E} \int_0^T \left\{ \left[ \left( u(t)^T, Z(t)^T \right)^T + M_1(t)^{-1} \left( M_2(t) \tilde{X}(t) - M_3(t) \tilde{\varphi}(t) - M_4(t) \tilde{\psi}(t) \right) \right]^T \right. \\
\left. \left. M_1(t) \left[ \left( u(t)^T, Z(t)^T \right)^T + M_1(t)^{-1} \left( M_2(t) \tilde{X}(t) - M_3(t) \tilde{\varphi}(t) - M_4(t) \tilde{\psi}(t) \right) \right]^T \right] dt
\]
\[
\geq J(\tilde{u}(\cdot)).
\]
In this section, we illustrate our results for the indefinite stochastic LQ, BLQ and deterministic FBLQ problems.

Some special cases

Example 5.3 Suppose that all variables are 1-dimensional. For the FBLQ problem (2.1)-(2.4), suppose that \( A_3(t) = B_1(t) = C_1(t) = B_2(t) = C_2(t) = F = \xi = 0 \) and \( D_1(t) + D_2(t)A_2(t) = 0 \). Then the solutions to (2.12), (2.23) and (3.24) are \( P_{1,i}(t) = G e^{tT} (2A_1(s) + A_2(s)^2) ds + \int_t^T A_4(s) e^{tT} (2A_1(r) + A_2(r)^2) dr ds, \) \( P_{2,i}(t) \equiv 0 \) and \( P_{3,i}(\cdot) \) satisfies

\[
\begin{cases}
  dP_{3,i}(t) = - \left\{ -B_4(t)P_{3,i}(t) + \left[ 2B_3(t) + C_3(t)^2(1 + P_{3,i}(t)C_4(t)^{-1}) \right] P_{3,i}(t) \\
  \quad + \frac{D_3(t)^2}{D_4(t) + D_2(t)P_{3,i}(t)} \right\} dt, \\
  P_{3,i}(T) = \hat{i}^{-1}.
\end{cases}
\]

Suppose that \( D_4 < 0, C_4 < 0, D_4(t) + D_2(t)^2P_{3,i}(t) \geq \delta > 0, D_2(t)^2 > 0 \) and \( 1 + \hat{P}_{3,i}(t)C_4(t) \geq \delta > 0 \) where

\[
\begin{cases}
  d\hat{P}_{3,i}(t) = - \left\{ \left[ 2B_3(t) + C_3(t)^2\delta^{-1} \right] \hat{P}_{3,i}(t) + \frac{D_3(t)^2}{D_4(t) + D_2(t)\hat{P}_{3,i}(t)} \right\} dt, \\
  \hat{P}_{3,i}(T) = \hat{i}^{-1}.
\end{cases}
\]

By Comparison theorem we have \( P_{3,i}(t) \leq \hat{P}_{3,i}(t) \) which leads to \( 1 + P_{3,i}(t)C_4(t) \geq \delta \). Then, by Theorem 4.5 (\( P_1(\cdot), P_2(\cdot), P_3(\cdot) \)) has a unique solution. Moreover,

\[
P_3(t) = \int_t^T \frac{\hat{D}_3(s)}{\hat{D}_4(s) + \hat{D}_2(s)^2\hat{P}_3(s)} e^{tT} (2\hat{A}_1(r) + \hat{C}_3(r)^2(1 + \hat{P}_3(r)\hat{C}_4(r)^{-1} - \hat{B}_4(r)\hat{P}_3(r))) dr ds.
\]

It is obvious that \( P_3(t) \geq 0 \) for \( t < T \). Thus, by Theorem 5.2 the optimal control is

\[
\hat{u}(t) = - \left( D_4(t) + D_2(t)^2P_1(t) \right)^{-1} \cdot \left[ (D_1(t) + A_2(t)D_2(t)) P_1(t) \hat{X}(t) - D_3(t)P_3(t)^{-1}\hat{Y}(t) \right].
\]

Remark 5.4 Although the forward-backward stochastic control system in the above example is completely decoupled, in order to obtain the optimal control \( \hat{u}(\cdot) \) in (5.7) we still need to solve a fully coupled FBSDE.

6 Some special cases

In this section, we illustrate our results for the indefinite stochastic LQ, BLQ and deterministic FBLQ problems.
6.1 Indefinite stochastic LQ problem

If \( A_i(\cdot) = D_i(\cdot) = B_i(\cdot) = C_i(\cdot) = F = H = \xi = 0, i = 2, 3, 4 \), then the FBLQ problem \((2.1)-(2.2)\) degenerates to the following indefinite stochastic LQ problem as in [3]: minimizing the following cost functional

\[
J(u(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle A_4(t)X(t), X(t) \rangle + \langle D_4(t)u(t), u(t) \rangle \right) dt + \langle GX(T), X(T) \rangle \right]
\]

subject to

\[
\begin{aligned}
&dX(t) = \left[ A_4(t)X(t) + D_4(t)u(t) \right] dt + \left[ A_2(t)X(t) + D_2(t)u(t) \right] dB(t), \\
&X(0) = x_0.
\end{aligned}
\]

By Theorem 5.2 the optimal control is

\[
\bar{u}(t) = -(D_4(t) + D_2(t)^T P_1(t) D_2(t))^{-1} (D_4(t)^T P_1(t) + D_2(t)^T P_1(t) A_2(t)) \bar{X}(t),
\]

where

\[
\begin{aligned}
\dot{P}_1(t) + A_1(t)^T P_1(t) + P_1(t) A_4(t) + A_2(t)^T P_1(t) A_2(t) \\
- (D_4(t)^T P_1(t) + D_2(t)^T P_1(t) A_2(t))^T (D_4(t) + D_2(t)^T P_1(t) D_2(t))^{-1} \\
(D_4(t)^T P_1(t) + D_2(t)^T P_1(t) A_2(t)) = 0,
\end{aligned}
\]

\[
P_1(T) = G,
\]

\[
D_4(t) + D_2(t)^T P_1(t) D_2(t) > 0.
\]

The state feedback representation of the optimal control and the Riccati equation for \( P_1(\cdot) \) are just the corresponding ones in Theorem 3.2 in Chen, Li and Zhou [3].

6.2 BLQ problem

If \( A_i(\cdot) = B_i(\cdot) = C_i(\cdot) = D_i(\cdot) = F = G = A_3(\cdot) = A_4(\cdot) = 0, i = 1, 2 \) and \( D_4(\cdot) > 0 \), then the problem \((2.1)-(2.2)\) degenerates to the following BLQ problem as in [12]: minimizing the following cost functional

\[
J(u(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle B_4(t)Y(t), Y(t) \rangle + \langle C_4(t)Z(t), Z(t) \rangle + \langle D_4(t)u(t), u(t) \rangle \right) dt \right]
\]

subject to

\[
\begin{aligned}
dY(t) = \left[ -B_4(t)Y(t) + C_4(t)Z(t) + D_3(t)u(t) \right] + Z(t) dB(t), \\
Y(T) = \xi.
\end{aligned}
\]

By Theorem 3.1 the optimal control is

\[
\bar{u}(t) = -D_4(t)^{-1} D_3(t)^T h(t),
\]

25
and the following relation holds:

$$\dot{Y}(t) = -Q_{4}(t)h(t) - \varphi_{2}(t),$$

where

$$\left\{
\begin{array}{l}
dQ_{4}(t) \\
= - \{ Q_{4}(t)B_{3}(t)\T + B_{3}(t)Q_{4}(t) - Q_{4}(t)B_{4}(t)Q_{4}(t) - D_{3}(t)D_{4}(t)^{-1}D_{3}(t)^{\T} \\
+ C_{3}(t)(I_{m} + Q_{4}(t)C_{4}(t))^{-1}Q_{4}(t)C_{3}(t)^{\T} \} \, dt, \\
Q_{4}(T) = 0,
\end{array}
\right.$$

$$\left\{
\begin{array}{l}
d\varphi_{2}(t) \\
= - \{ Q_{4}(t)B_{3}(t)\varphi_{2}(t) + B_{3}(t)\varphi_{2}(t) + C_{3}(t)(I_{m} + P_{3}(t)C_{4}(t))^{-1}v_{2}(t) \} \, dt \\
+ v_{2}(t)dB(t), \\
\varphi_{2}(T) = \xi.
\end{array}
\right.$$

The equation for $Q_{4}(\cdot)$ is just the Riccati equation (3.4) in Lim and Zhou [12]. And the optimal control is consistent with the one in Theorem 3.3 in [12].

**Remark 6.1** It is worth pointing out that our results in this paper can be also applied to the indefinite BLQ problem.

### 6.3 Deterministic FBLQ problem

If $C_{i}(\cdot) = A_{2}(\cdot) = B_{3}(\cdot) = C_{2}(\cdot) = D_{2}(\cdot) = C_{3}(\cdot) = C_{4}(\cdot) = \xi = 0$ and $D_{4}(\cdot) > 0$, then the problem (6.1) degenerates to a deterministic FBLQ problem. For this case, (3.19), (3.23), (3.24) become

$$\dot{P}_{1}(t) + P_{1}(t)A_{1}(t) + A_{1}(t)^{\T}P_{1}(t) + P_{1}(t)B_{1}(t)P_{2}(t) + P_{2}(t)^{\T}B_{1}(t)^{\T}P_{1}(t)
- P_{1}(t)D_{1}(t)D_{4}(t)^{-1}D_{1}(t)^{\T}P_{1}(t) + P_{2}(t)^{\T}B_{4}(t)P_{2}(t) + A_{4}(t) = 0,
$$

$$\dot{P}_{2}(t) + P_{2}(t)A_{1}(t) + B_{3}(t)P_{2}(t) - P_{3}(t)B_{4}(t)P_{2}(t)
- P_{2}(t)D_{1}(t)D_{4}(t)^{-1}D_{1}(t)^{\T}P_{1}(t) + P_{2}(t)B_{1}(t)P_{2}(t) - P_{3}(t)B_{1}(t)^{\T}P_{1}(t)
- D_{3}(t)D_{4}(t)^{-1}D_{1}(t)^{\T}P_{1}(t) + A_{4}(t) = 0,
$$

$$\dot{P}_{3}(t) + P_{3}(t)B_{3}(t)^{\T} + B_{3}(t)P_{3}(t) + P_{2}(t)B_{1}(t)P_{3}(t) + P_{3}(t)B_{1}(t)^{\T}P_{2}(t)^{\T}
- P_{3}(t)B_{4}(t)P_{3}(t) + (P_{2}(t)D_{1}(t) + D_{3}(t))D_{4}(t)^{-1}(P_{2}(t)D_{1}(t) + D_{3}(t))^{\T} = 0,$$

$$P_{1}(T) = G, \ P_{2}(T) = F, \ P_{3}(T) = 0.$$
Proposition 6.2 Suppose that Assumptions 2.1, 2.3, 2.4 and 2.5 hold. If (6.1) has a solution \((P_1(\cdot), P_2(\cdot), P_3(\cdot)) \in C([0, T]; S^n \times \mathbb{R}^{m \times n \times S^m})\) such that \(P_3(t) > 0\) for \(t < T\), then the above deterministic FBLQ problem has a unique optimal control

\[
\bar{u}(t) = -D_4(t)^{-1}\left\{D_1(t)P_1(t) + P_2(t)P_3(t)^{-1}P_2(t) + D_3(t)^TP_3(t)^{-1}P_2(t)\right\}X(t)
- \left(D_1(t)^TP_2(t)^T + D_3(t)^TP_3(t)^{-1}\bar{Y}(t)\right).
\]

For 1-dimensional case \((n = m = 1)\), if \(B_4(\cdot) = 0, B_1(\cdot) < 0\) and \(F\) is large enough such that \(P_2, i(\cdot)\) is non-negative and bounded, then (6.1) has a unique solution. The reason is that when \(B_4(\cdot) = 0, B_1(\cdot) < 0\) and \(P_2, i(\cdot) \geq 0\), we have

\[
P_{1,i}(t) \leq Ge^{\int_t^T 2A_1(s)ds} + \int_t^T A_4(s)e^{\int_t^s 2A_1(r)dr}ds.
\]

Thus, \(P_{1,i}(\cdot)\) is bounded and we obtain the desired result due to Theorem 4.8.

7 Appendix

This appendix is devoted to proofs of Theorem 4.4, Lemma 4.5 and Lemma 5.1. Before giving the proofs, let’s give some notations.

Set

\[
\left(\tilde{R}(t) + \tilde{D}(t)^T \tilde{P}(t) \tilde{D}(t)\right)^{-1} = \begin{pmatrix}
a_{11}(t) & a_{21}(t)^T \\
a_{21}(t) & a_{22}(t)
\end{pmatrix}, \tag{7.1}
\]

\[
\left(\tilde{B}(t)^T \tilde{P}(t) + \tilde{D}(t)^T \tilde{P}(t) \tilde{C}(t)\right)^T = \begin{pmatrix}
a(t) & b(t) \\
c(t) & d(t)
\end{pmatrix}. \tag{7.2}
\]
where

\[ a(t) = \tilde{P}_{1,i}(t)D_1(t) - \tilde{P}_{2,i}(t)D_2(t) + A_2(t)^\top \hat{P}_{1,i}(t)D_2(t), \]

\[ b(t) = \tilde{P}_{1,i}(t)C_1(t) - \tilde{P}_{2,i}(t)C_3(t) + A_2(t)^\top \hat{P}_{1,i}(t)C_2(t) + A_2(t)^\top \hat{P}_{2,i}(t)^\top, \]

\[ c(t) = \tilde{P}_{2,i}(t)D_1(t) - P_{3,i}(t)D_3(t) + B_2(t)^\top \hat{P}_{1,i}(t)D_2(t), \]

\[ d(t) = \tilde{P}_{2,i}(t)C_1(t) - P_{3,i}(t)C_3(t) + B_2(t)^\top \hat{P}_{1,i}(t)C_2(t) + B_2(t)^\top \hat{P}_{2,i}(t)^\top, \]

\[ a_{11}(t) = \left\{ D_4(t) + D_2(t)^\top \hat{P}_{1,i}(t)D_2(t) \right\}^{-1} \left( -D_2(t)^\top \left( \tilde{P}_{1,i}(t)C_2(t) + \tilde{P}_{2,i}(t)^\top \right) D(t)^{-1} \left( C_2(t)^\top \hat{P}_{1,i}(t) + \tilde{P}_{2,i}(t)^\top \right) D_2(t) \right\}^{-1}, \]

\[ a_{21}(t) = -D(t)^{-1} \left( C_2(t)^\top \hat{P}_{1,i}(t) + \tilde{P}_{2,i}(t)^\top \right) D_2(t)a_{11}(t), \]

\[ a_{22}(t) = D(t)^{-1} + D(t)^{-1} \left( C_2(t)^\top \hat{P}_{1,i}(t) + \tilde{P}_{2,i}(t)^\top \right) D_2(t)a_{11}(t)D_2(t)^\top \left( \tilde{P}_{1,i}(t)C_2(t) + \tilde{P}_{2,i}(t)^\top \right) D(t)^{-1}, \]

\[ D(t) = C_4(t) + \left( C_2(t)^\top \hat{P}_{1,i}(t) + \tilde{P}_{2,i}(t)^\top \right) C_2(t) + C_2(t)^\top \hat{P}_{2,i}(t)^\top + \tilde{P}_{3,i}(t). \]

(7.3)

And set

\[
\vec{B}(t)^\top \vec{P}(t) = \begin{pmatrix}
\vec{a}(t) & \vec{b}(t) \\
\vec{c}(t) & \vec{d}(t)
\end{pmatrix},
\]

(7.4)

where

\[
\vec{a}(t) = D_1(t)^\top \hat{P}_{1,i}(t) - D_3(t)^\top \hat{P}_{2,i}(t), \quad \vec{a}(t) = D_2(t)^\top \hat{P}_{1,i}(t),
\]

\[
\vec{b}(t) = D_1(t)^\top \hat{P}_{2,i}(t)^\top - D_3(t)^\top \hat{P}_{3,i}(t), \quad \vec{b}(t) = D_2(t)^\top \hat{P}_{2,i}(t)^\top,
\]

\[
\vec{c}(t) = C_1(t)^\top \hat{P}_{1,i}(t) - C_3(t)^\top \hat{P}_{2,i}(t), \quad \vec{c}(t) = C_2(t)^\top \hat{P}_{1,i}(t) + \tilde{P}_{2,i}(t),
\]

\[
\vec{d}(t) = C_1(t)^\top \hat{P}_{2,i}(t)^\top - C_3(t)^\top \hat{P}_{3,i}(t)^\top, \quad \vec{d}(t) = C_2(t)^\top \hat{P}_{2,i}(t)^\top.
\]

(7.5)

### 7.1 Proof of Theorem 4.4

Before we prove Theorem 4.4, we list the following relations which can be verified directly:

\[
P_{3,i}(t)C_2(t)^\top \left( C_2(t)^\top \hat{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right)^\top = P_{3,i}(t)D(t) - (I_m - P_{2,i}(t)C_2(t) + P_{3,i}(t)C_4(t)), \]

(7.6)

\[
\left( C_2(t)^\top \hat{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right)^\top D(t)^{-1} = L_{2,i}(t)^{-1} \left( C_2(t)^\top \hat{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right)^\top
\]

\[
\cdot (I_m - P_{2,i}(t)C_2(t) + P_{3,i}(t)C_4)^{-1} P_{3,i}(t),
\]

(7.7)

\[
P_{3,i}(t)d(t) = -(P_{2,i}(t)C_1(t) + C_3(t)) + P_{3,i}(t)B_2(t)^\top \left( C_2(t)^\top \hat{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right)^\top,
\]

(7.8)
\[ L_{2,i}(t)^{-1}S_{1,i}(t) = \hat{P}_{1,i}(t) - \left( C_2(t)^T \hat{P}_{1,i}(t) + \hat{P}_{2,i}(t) \right)^T D(t)^{-1} \left( C_2(t)^T \hat{P}_{1,i}(t) + \hat{P}_{2,i}(t) \right), \]  \tag{7.9}

\[ a_{11}(t) = \left( D_4(t) + D_2(t)^T L_{2,i}(t)^{-1}S_{1,i}(t)D_2(t) \right)^{-1}, \]  \tag{7.10}

\[ L_{2,i}(t)^{-1}L_{3,i}(t) = \left[ \hat{P}_{1,i}(t) - \left( C_2(t)^T \hat{P}_{1,i}(t) + \hat{P}_{2,i}(t) \right)^T D(t)^{-1} \left( C_2(t)^T \hat{P}_{1,i}(t) + \hat{P}_{2,i}(t) \right) \right] \]  
\[ \cdot \left( A_2(t) + B_2(t)P_{2,i}(t) - \left( C_2(t)^T \hat{P}_{1,i}(t) + \hat{P}_{2,i}(t) \right)^T D(t)^{-1}C_1(t)^T \hat{P}_{1,i}(t), \right. \]  \tag{7.11}

\[ L_{2,i}(t)^{-1}L_{4,i}(t) = \left( C_2(t)^T \hat{P}_{1,i}(t) + \hat{P}_{2,i}(t) \right)^T D(t)^{-1}d(t)^T \hat{P}_{3,i}(t)^{-1} - \hat{P}_{1,i}(t)B_2(t)\hat{P}_{3,i}(t)^{-1}, \]  \tag{7.12}

\[ b(t)^T + d(t)^TP_{2,i}(t) = \left( C_2(t)^T \hat{P}_{1,i}(t) + \hat{P}_{2,i}(t) \right) \left( A_2(t) + B_2(t)P_{2,i}(t) \right) + C_1(t)^T \hat{P}_{1,i}(t), \]  \tag{7.13}

\[ a(t)^T + c(t)^TP_{2,i}(t) = D_1(t)^TP_{1,i}(t) + D_2(t)^T \hat{P}_{1,i}(t) \left( A_2(t) + B_2(t)P_{2,i}(t) \right). \]  \tag{7.14}

**Proof of Theorem 4.4** The proof is divided into five steps. The first three steps we verify the relationship between \( iP(\cdot) \) and \( i\hat{P}(\cdot) \), that is,

\[ P_{1,i}(t) = \hat{P}_{1,i}(t) - \hat{P}_{2,i}(t)^T \hat{P}_{3,i}(t)^{-1} \hat{P}_{2,i}(t), \]

\[ P_{2,i}(t) = -\hat{P}_{3,i}(t)^{-1} \hat{P}_{2,i}(t), \]

\[ P_{3,i}(t) = \hat{P}_{3,i}(t)^{-1}. \]

or equivalently

\[ \hat{P}_{1,i}(t) = P_{1,i}(t) + P_{2,i}(t)^TP_{3,i}(t)^{-1}P_{2,i}(t), \]

\[ \hat{P}_{2,i}(t) = -P_{3,i}(t)^{-1}P_{2,i}(t), \]

\[ \hat{P}_{3,i}(t) = P_{3,i}(t)^{-1}. \]  \tag{7.15}

Recall that the equations satisfied by \( \hat{P}_i(\cdot) \) are

\[
\begin{align*}
\begin{cases}
\hat{P}_{1,i}(t) + \hat{P}_{1,i}(t)A_1 - \hat{P}_{2,i}(t)^TA_3(t) + A_1(t)^T \hat{P}_{1,i}(t) - A_3(t)^T \hat{P}_{2,i}(t)^T + A_2(t)^T \hat{P}_{1,i}(t)A_2(t) + A_4(t) \\
- [a(t)a_{11}(t)a(t)^T + b(t)a_{21}(t)a(t)^T + a(t)a_{21}(t)^Tb(t)^T + b(t)a_{22}(t)b(t)^T] = 0,
\end{cases}
\end{align*}
\]  \tag{7.16}

\[
\begin{align*}
\begin{cases}
\hat{P}_{2,i}(t) + \hat{P}_{2,i}(t)A_1 - \hat{P}_{3,i}(t)A_3(t) + B_1(t)^T \hat{P}_{1,i}(t) - B_3(t)^T \hat{P}_{2,i}(t) + B_2(t)^T \hat{P}_{1,i}(t)A_2(t) \\
- [c(t)a_{11}(t)a(t)^T + d(t)a_{21}(t)a(t)^T + c(t)a_{21}(t)^Tb(t)^T + d(t)a_{22}(t)b(t)^T] = 0,
\end{cases}
\end{align*}
\]  \tag{7.17}

\[
\begin{align*}
\begin{cases}
\hat{P}_{3,i}(t) + \hat{P}_{2,i}(t)B_1 - \hat{P}_{3,i}(t)B_3(t) + B_1(t)^T \hat{P}_{2,i}(t) - B_3(t)^T \hat{P}_{3,i}(t) + B_2(t)^T \hat{P}_{1,i}(t)B_2(t) + B_4(t) \\
- [c(t)a_{11}(t)c(t)^T + d(t)a_{21}(t)c(t)^T + c(t)a_{21}(t)^Td(t)^T + d(t)a_{22}(t)d(t)^T] = 0,
\end{cases}
\end{align*}
\]  \tag{7.18}

\[ \hat{P}_{3,i}(t) = 0. \]
\textbf{Step 1:} In this step we verify $\tilde{P}_{3,i}(\cdot) = \tilde{P}_{3,i}(\cdot)^{-1}$.

$\tilde{P}_{3,i}(\cdot)^{-1}$ satisfies the following equation:

\begin{equation}
\left( \tilde{P}_{3,i}(t)^{-1} \right)^\top - \tilde{P}_{3,i}(t)^{-1}\{ \hat{P}_{2,i}(t)B_1 - \hat{P}_{3,i}(t)B_3(t) + B_1(t)^\top \hat{P}_{2,i}(t)^\top \nonumber \\
- B_3(t)^\top \hat{P}_{3,i}(t) + B_2(t)^\top \hat{P}_{1,i}(t)B_2(t) + B_3(t) \nonumber \\
- \left[ c(t)a_{11}(t)c(t)^\top + d(t)a_{21}(t)c(t)^\top + c(t)a_{21}(t)d(t)^\top + d(t)a_{22}(t)d(t)^\top \right] \tilde{P}_{3,i}(t)^{-1} = 0.
\end{equation}

(7.19)

Note that $\tilde{P}(\cdot)$ and $P(\cdot)$ are governed by the same equations except the terminal conditions. Putting the relation (7.14) into (7.19) and comparing with (3.24), we need to verify

\begin{equation}
\begin{aligned}
& - P_{3,i}(t)B_2(t)^\top \left( P_{1,i}(t) + P_{2,i}(t)^\top P_{3,i}(t)^{-1} P_{2,i}(t) \right) B_3(t)P_{3,i}(t) \\
& + P_{3,i}(t) \left[ c(t)a_{11}(t)c(t)^\top + d(t)a_{21}(t)c(t)^\top + c(t)a_{21}(t)d(t)^\top + d(t)a_{22}(t)d(t)^\top \right] P_{3,i}(t) \\
& = - \left[ (P_{2,i}(t)C_1(t) + C_3(t)) L_{11,i}(t) - P_{3,i}(t)B_2(t)^\top L_{9,i}(t) + (P_{2,i}(t)D_1(t) + D_3(t)) L_{7,i}(t) \right].
\end{aligned}
\end{equation}

(7.20)

We compare the coefficients of $a_{11}(\cdot)$ and the remainder terms on both sides of the above equation. The coefficient of $a_{11}(\cdot)$ on the left hand side (LHS) is

$$P_{3,i}(t)c(t) - P_{3,i}(t)d(t)D(t)^{-1} \left( C_2(t)^\top \hat{P}_{1,i}(t) + \hat{P}_{2,i}(t) \right) D_2(t),$$

and the one on the right hand side (RHS) is

$$\left( P_{2,i}(t)C_1(t) + C_3(t) \right) D(t)^{-1} \left( C_2(t)^\top \hat{P}_{1,i}(t) + \hat{P}_{2,i}(t) \right) D_2(t) + P_{3,i}(t)D_1(t) + D_3(t) - P_{3,i}(t)B_2(t)^\top L_{2,i}(t)^{-1} S_{1,i}(t)D_2(t).$$

The remainder terms on the LHS is

$$- P_{3,i}(t)B_2(t)^\top \left( P_{1,i}(t) + P_{2,i}(t)^\top P_{3,i}(t)^{-1} P_{2,i}(t) \right) B_3(t)P_{3,i}(t) + P_{3,i}(t)d(t)D(t)^{-1}d(t)P_{3,i}(t),$$

and the ones on the RHS is

$$\left[ (P_{2,i}(t)C_1(t) + C_3(t)) (I_m - P_{2,i}(t)C_2(t) + P_{3,i}(t)C_4(t))^{-1} P_{3,i}(t)C_2(t)^\top + P_{3,i}(t)B_2(t)^\top \right] L_{2,i}(t)^{-1} L_{4,i}(t) + (P_{2,i}(t)C_1(t) + C_3(t)) (I_m - P_{2,i}(t)C_2(t) + P_{3,i}(t)C_4(t))^{-1}

\times \left( P_{2,i}(t)B_2(t)P_{3,i}(t) + P_{3,i}(t)C_3(t)^\top + P_{3,i}(t)C_1(t)^\top P_{2,i}(t)^\top \right).$$

By relations (7.8), (7.9), (7.12) and (7.15), we obtain that (7.20) holds.

\textbf{Step 2:} In this step we verify $P_{2,i}(t) = - \tilde{P}_{3,i}(t)^{-1} \tilde{P}_{2,i}(t)$. 

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By calculation, we need to prove

\[-\hat{P}_{3,i}(\cdot)^{-1}\hat{P}_{2,i}(\cdot)\]

satisfies the following equation:

\[
\left(-\hat{P}_{3,i}(t)^{-1}\hat{P}_{2,i}(t)\right)' - P_{3,i}(t)\{\hat{P}_{2,i}(t)A_1 - \hat{P}_{3,i}(t)A_3(t) + B_1(t)^T\hat{P}_{1,i}(t)
\]

\[-B_3(t)^T\hat{P}_{2,i}(t) + B_2(t)^T\hat{P}_{1,i}(t)A_2(t)
\]

\[-[c(t)a_{i1}(t)a(t)^T + d(t)a_{21}(t)a(t)^T + c(t)a_{21}(t)b(t)^T + d(t)a_{22}(t)b(t)^T)]
\]

\[-\{P_{2,i}(t)B_1(t)P_{3,i}(t) - P_{2,i}(t)C_1(t)L_{11,i}(t) - P_{2,i}(t)D_1(t)L_{7,i}(t)
\]

\[+P_{3,i}(t)B_3(t)^T + P_{3,i}(t)B_1(t)^TP_{2,i}(t)^T + P_{3,i}(t)B_2(t)^TL_{9,i}(t) - P_{3,i}(t)B_4(t)P_{3,i}(t)
\]

\[+B_3(t)P_{3,i}(t) - C_3(t)L_{11,i}(t) - D_3(t)L_{7,i}(t)\} \hat{P}_{2,i}(t) = 0.
\]

Putting the relation (7.15) into (7.21) and comparing with (7.22), we only need to verify

\[-P_{3,i}(t)\{B_1(t)^TP_{2,i}(t)^TP_{3,i}(t)^{-1}P_{2,i}(t) - B_3(t)^TP_{2,i}(t) + B_2(t)^TP_{1,i}(t)A_2(t)
\]

\[-[c(t)a_{i1}(t)a(t)^T + d(t)a_{21}(t)a(t)^T + c(t)a_{21}(t)b(t)^T + d(t)a_{22}(t)b(t)^T)]
\]

\[-\{P_{2,i}(t)C_1(t)L_{11,i}(t) - P_{2,i}(t)D_1(t)L_{7,i}(t) + P_{3,i}(t)B_3(t)^T + P_{3,i}(t)B_1(t)^TP_{2,i}(t)^T
\]

\[+P_{3,i}(t)B_2(t)^TL_{9,i}(t) - C_3(t)L_{11,i}(t) - D_3(t)L_{7,i}(t)\} \hat{P}_{2,i}(t)
\]

\[= [(P_{2,i}(t)C_1(t) + C_3(t)L_{10,i}(t) - P_{3,i}(t)B_2(t)^TL_{8,i}(t) + (P_{2,i}(t)D_1(t) + D_3(t)) L_{6,i}(t)].
\]

The coefficient of \(a_{i1}(\cdot)\) on the LHS is

\[P_{3,i}(t)[c(t) - d(t)D(t)^{-1}(C_2(t)^T\hat{P}_{1,i}(t) + \hat{P}_{2,i}(t))D_2(t)]a_{i1}(t)
\]

\[[a(t)^TD(t)^{-1}D(t)^{-1}b(t)]
\]

\[= [(P_{2,i}(t)C_1(t) + C_3(t)) D(t)^{-1}(C_2(t)^T\hat{P}_{1,i}(t) + \hat{P}_{2,i}(t))D_2(t) - P_{2,i}(t)D_1(t) - D_3(t)
\]

\[+P_{3,i}(t)B_2(t)^TL_{2,i}(t)^{-1}S_{1,i}(t)D_2(t)]a_{i1}(t)[c(t)^T - D_2(t)^T \left(\hat{P}_{1,i}(t)C_2(t) + \hat{P}_{2,i}(t)^T\right) D(t)^{-1}d(t)^T]P_{2,i}(t),
\]

and the one on the RHS is

\[[(P_{2,i}(t)C_1(t) + C_3(t)) D(t)^{-1}(C_2(t)^T\hat{P}_{1,i}(t) + \hat{P}_{2,i}(t))D_2(t) - P_{2,i}(t)D_1(t) - D_3(t)
\]

\[+P_{3,i}(t)B_2(t)^TL_{2,i}(t)^{-1}S_{1,i}(t)D_2(t)]a_{i1}(t)[c(t)^T + c(t)^TP_{2,i}(t)
\]

\[+d(t)^T P_{2,i}(t)]
\]

By calculation, we need to prove

\[P_{3,i}(t)[c(t) - d(t)D(t)^{-1}(C_2(t)^T\hat{P}_{1,i}(t) + \hat{P}_{2,i}(t))D_2(t)]
\]

\[= [(P_{2,i}(t)C_1(t) + C_3(t)) D(t)^{-1}(C_2(t)^T\hat{P}_{1,i}(t) + \hat{P}_{2,i}(t))D_2(t)
\]

\[+P_{2,i}(t)D_1(t) + D_3(t) - P_{3,i}(t)B_2(t)^TL_{2,i}(t)^{-1}S_{1,i}(t)D_2(t)],
\]

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which has already been verified in Step 1. The remainder terms on the LHS is

\[-P_{3,i}(t)B_2(t)^TP_{1,i}(t)A_2(t) + P_{3,i}(t)d(t)D(t)^{-1}b(t)^T\]

\[+(P_{2,i}(t)C_1(t) + C_3(t))D(t)^{-1}d(t)P_{3,i}(t)P_{2,i}(t) - P_{3,i}(t)B_2(t)^T L_{2,i}(t)^{-1} L_{4,i}(t) \tilde{P}_{2,i}(t)\]

and the ones on the RHS is

\[-(P_{2,i}(t)C_1(t) + C_3(t))D(t)^{-1} (b(t)^T + d(t)^TP_{2,i}(t)) - P_{3,i}(t)B_2(t)^T L_{2,i}(t)^{-1} L_{3,i}(t).\]

By relations (7.18), (7.19), (7.12), (7.13) and (7.15), we obtain that (7.22) holds.

**Step 3:** In this step we verify \(P_{1,i}(t) = \tilde{P}_{1,i}(t) - \tilde{P}_{2,i}(t)^TP_{3,i}(t)^{-1} \tilde{P}_{2,i}(t).\)

Since \(P_{2,i}(t) = -\tilde{P}_{3,i}(t)^{-1} \tilde{P}_{2,i}(t), P_{3,i}(t) = \tilde{P}_{3,i}(t)^{-1}, \) we have

\[\tilde{P}_{1,i}(t) - \tilde{P}_{2,i}(t)^TP_{3,i}(t)^{-1} \tilde{P}_{2,i}(t) = \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t)^TP_{2,i}(t).\]

Deriving on both sides of the above equation,

\[\left(\tilde{P}_{1,i}(t) - \tilde{P}_{2,i}(t)^TP_{3,i}(t)^{-1} \tilde{P}_{2,i}(t)\right)' + \tilde{P}_{1,i}(t)A_1 - \tilde{P}_{2,i}(t)^TP_{3,i}(t) + A_1(t)^TP_{1,i}(t) - A_3(t)\tilde{P}_{1,i}(t) + A_4(t)

\[-A_3(t)\tilde{P}_{1,i}(t)^TP_{2,i}(t) + A_2(t)^TP_{1,i}(t)A_2(t) + A_4(t)

\[-[a(t)a_{11}(t)a(t)^T + b(t)a_{21}(t)a(t)^T + a(t)a_{21}(t)b(t)^T + b(t)a_{22}(t)\tilde{b}(t)^T]

\[+(\tilde{P}_{2,i}(t)A_1 - \tilde{P}_{3,i}(t)A_3(t) + B_1(t)^TP_{1,i}(t) - B_3(t)^TP_{2,i}(t) + B_2(t)^TP_{1,i}(t)A_2(t)

\[+ \tilde{P}_{2,i}(t)^TP_{2,i}(t)^TP_{3,i}(t)^{-1} \tilde{P}_{2,i}(t) - \tilde{P}_{3,i}(t)B_2(t)^T L_{8,i}(t) - \tilde{P}_{3,i}(t)B_4(t)P_{2,i}(t)

\[+ A_3(t) + B_3(t)P_{2,i}(t) + C_3(t)L_{10,i}(t) + D_3(t)L_{6,i}(t) = 0.\]

Putting the relation (7.15) into (7.23) and comparing with (3.19), we need to verify

\[P_{2,i}(t)^TP_{3,i}(t)^{-1}P_{2,i}(t)A_1(t) + A_1(t)^TP_{2,i}(t)^TP_{3,i}(t)^{-1}P_{2,i}(t) - \tilde{P}_{2,i}(t)^TP_{3,i}(t) + A_3(t) - A_3(t)^TP_{2,i}(t)

\[+ A_2(t)^TP_{1,i}(t)A_2(t) - [a(t)a_{11}(t)a(t)^T + b(t)a_{21}(t)a(t)^T + a(t)a_{21}(t)b(t)^T + b(t)a_{22}(t)b(t)^T]

\[+(\tilde{P}_{2,i}(t)A_1 - \tilde{P}_{3,i}(t)A_3(t) + B_1(t)^TP_{2,i}(t)^TP_{3,i}(t)^{-1}P_{2,i}(t) - B_3(t)^TP_{2,i}(t) + B_2(t)^TP_{1,i}(t)A_2(t)

\[+ \tilde{P}_{2,i}(t)^TP_{2,i}(t)^TP_{3,i}(t)^{-1} \tilde{P}_{2,i}(t) - \tilde{P}_{3,i}(t)B_2(t)^T L_{8,i}(t) - \tilde{P}_{3,i}(t)B_4(t)P_{2,i}(t)

\[+ A_3(t) + B_3(t)P_{2,i}(t) + C_3(t)L_{10,i}(t) + D_3(t)L_{6,i}(t) = 0.\]
that is

\[
A_2(t)^T (P_{1,i}(t) + P_{2,i}(t)^TP_{3,i}(t)^{-1}P_{2,i}(t)) (A_2(t) + B_2(t)P_{2,i}(t)) \\
- \tilde{P}_{2,i}(t)^T (P_{2,i}(t)C_1(t) + C_3(t)) D(t)^{-1} (b(t)^T + d(t)^TP_{2,i}(t)) \\
- \tilde{P}_{2,i}(t)^T P_{3,i}(t)B_2(t)^TL_{2,i}(t)^{-1}L_{3,i}(t) \\
- [a(t)a_{11}(t)a(t)^T + b(t)a_{21}(t)a(t)^T + a(t)a_{22}(t)b(t)^T + b(t)a_{22}(t)b(t)^T] \\
- [c(t)a_{11}(t)a(t)^T + d(t)a_{21}(t)a(t)^T + c(t)a_{22}(t)b(t)^T + d(t)a_{22}(t)b(t)^T]^T P_{3,i}(t) \\
- \tilde{P}_{2,i}(t)^T \{(P_{2,i}(t)C_1(t) + C_3(t)) [a_{21}(t) (a(t)^T + c(t)^TP_{2,i}(t)) + D(t)^{-1} (C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t)) \\
\cdot D_2(t) - \{P_{1,i}(t)C_1(t) + D_3(t) - P_{3,i}(t)B_2(t)^TL_{2,i}(t)^{-1}S_1(t)D_2(t)\} \\
\cdot [a_{11}(t) (a(t)^T + c(t)^TP_{2,i}(t)) + a_{21}(t) (b(t)^T + d(t)^TP_{2,i}(t))\} \\
= \left\{ \begin{array}{c} P_{1,i}(t)C_1(t)D(t)^{-1} \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right) D_2(t) - P_{1,i}(t)D_1(t) \\
\quad - (A_2(t) + B_2(t)P_{2,i}(t))^T L_{2,i}(t)^{-1}S_1(t)D_2(t) \end{array} \right\} \\
a_{11}(t)[D_2(t)^T \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right)^T D(t)^{-1} (b(t)^T + d(t)^TP_{2,i}(t)) - a(t)^T - c(t)^TP_{2,i}(t)] \\
+ P_{1,i}(t)C_1(t)D(t)^{-1} (b(t)^T + d(t)^TP_{2,i}(t)) + (A_2(t) + B_2(t)P_{2,i}(t))^T L_{2,i}(t)^{-1}L_{3,i}(t).
\]

The coefficient of \(a_{11}(\cdot)\) on the LHS is

\[
- \left\{ b(t)D(t)^{-1} \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right) D_2(t) - a(t) + \tilde{P}_{2,i}(t)^T \\
\quad \cdot [P_{2,i}(t)C_1(t) + C_3(t)] D(t)^{-1} \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right) D_2(t) - P_{2,i}(t)D_1(t) \\
\quad - D_3(t) + P_{3,i}(t)B_2(t)^TL_{2,i}(t)^{-1}S_1(t)D_2(t) \} \\
a_{11}(t)[D_2(t)^T \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right)^T D(t)^{-1} (b(t)^T + d(t)^TP_{2,i}(t)) - a(t)^T - c(t)^TP_{2,i}(t)],
\]

and the one on the RHS is

\[
- \left\{ P_{1,i}(t)C_1(t)D(t)^{-1} \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right) D_2(t) - P_{1,i}(t)D_1(t) \\
\quad - (A_2(t) + B_2(t)P_{2,i}(t))^T L_{2,i}(t)^{-1}S_1(t)D_2(t) \} \\
a_{11}(t)[D_2(t)^T \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right)^T D(t)^{-1} (b(t)^T + d(t)^TP_{2,i}(t)) - a(t)^T - c(t)^TP_{2,i}(t)].
\]
In the completion-of-squares technique, we derive
\[
b(t) D(t)^{-1} \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right) D_2(t) - a(t) + \tilde{P}_{2,i}(t)[(P_{2,i}(t)C_1(t) + C_3(t)) D(t)^{-1} \\
\cdot \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right) D_2(t) - P_{2,i}(t) D_1(t) - D_3(t) + P_{3,i}(t) B_2(t)^T L_{2,i}(t)^{-1} S_{1,i}(t) D_2(t) ]
\]
\[= P_{1,i}(t) C_1(t) D(t)^{-1} \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right) D_2(t) - P_{1,i}(t) D_1(t) \]
\[- (A_2(t) + B_2(t) P_{2,i}(t))^T L_{2,i}(t)^{-1} S_{1,i}(t) D_2(t) \]
\[= [D_2(t)^T \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right) D(t)^{-1} (b(t)^T + d(t)^T P_{2,i}(t)) - a(t)^T - c(t)^T P_{2,i}(t) ]^T. \]

The remainder terms on the LHS is
\[
A_2(t)^T \left( P_{1,i}(t) + P_{2,i}(t)^T P_{3,i}(t)^{-1} P_{2,i}(t) \right) (A_2(t) + B_2(t) P_{2,i}(t)) + P_{2,i}(t)^T B_2(t)^T L_{2,i}(t)^{-1} L_{3,i}(t)
\]  
\[- \left[ b(t) + \tilde{P}_{2,i}(t) (P_{2,i}(t) C_1(t) + C_3(t)) \right] D(t)^{-1} (b(t)^T + d(t)^T P_{2,i}(t)), \]
and the ones on the RHS is
\[- P_{1,i}(t) C_1(t) D(t)^{-1} (b(t)^T + d(t)^T P_{2,i}(t)) + (A_2(t) + B_2(t) P_{2,i}(t))^T L_{2,i}(t)^{-1} L_{3,i}(t). \]

By the definition of \( b(t) \) and
\[
A_2(t)^T L_{2,i}(t)^{-1} L_{3,i}(t) = A_2(t)^T \left( P_{1,i}(t) + P_{2,i}(t)^T P_{3,i}(t)^{-1} P_{2,i}(t) \right) (A_2(t) + B_2(t) P_{2,i}(t)) \\
- A_2(t)^T \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right) D(t)^{-1} (b(t)^T + d(t)^T P_{2,i}(t)), \]
the remainder terms on both sides are consistent.

In the following two steps, we verify the relationship between \( i \varphi(\cdot) \) and \( i \tilde{\varphi}(\cdot) \), that is,
\[
\tilde{\varphi}_{1,i}(t) = - P_{1,i}(t)^{-1} \varphi_{1,i}(t), \quad \tilde{\varphi}_{2,i}(t) = \varphi_{2,i}(t) - P_{2,i}(t) P_{1,i}(t)^{-1} \varphi_{1,i}(t), \]
\[
\tilde{\nu}_{1,i}(t) = - P_{1,i}(t)^{-1} \nu_{1,i}(t), \quad \tilde{\nu}_{2,i}(t) = \nu_{2,i}(t) - P_{2,i}(t) P_{1,i}(t)^{-1} \nu_{1,i}(t). \]

Since \( d_i \tilde{\varphi}(t) = - i \tilde{\gamma}(t) dt + i \tilde{\nu}(t) dB(t) \), the above relations are equivalent to
\[
\tilde{\gamma}_{1,i}(t) = - \left\{ P_{1,i}(t)^{-1} \tilde{P}_{1,i}(t) P_{1,i}(t) \varphi_{1,i}(t) + P_{1,i}(t)^{-1} \gamma_{1,i}(t) \right\},
\]
and
\[
\tilde{\gamma}_{2,i}(t) = \gamma_{2,i}(t) + \tilde{P}_{2,i}(t) P_{1,i}(t)^{-1} \varphi_{1,i}(t) - \gamma_{2,i}(t) + \tilde{P}_{2,i}(t) P_{1,i}(t)^{-1} \gamma_{1,i}(t) \]  
\[= \gamma_{2,i}(t) + \tilde{P}_{2,i}(t) P_{1,i}(t)^{-1} \varphi_{1,i}(t) + P_{2,i}(t) \tilde{\gamma}_{1,i}(t). \]

In the completion-of-squares technique, \( i \tilde{\gamma}(t) \) satisfies
\[
\left( B(t)^T \tilde{P}(t) + \tilde{D}(t)^T \tilde{P}(t) \tilde{C}(t) \right)^T (\tilde{R}(t) + \tilde{D}(t)^T \tilde{P}(t) \tilde{D}(t))^{-1} \left( B(t)^T \tilde{P}(t) i \tilde{\varphi}(t) - \tilde{D}(t)^T \tilde{P}(t) i \tilde{\varphi}(t) \right) \\
- \tilde{A}(t)^T \tilde{P}(t) i \tilde{\varphi}(t) - \tilde{P}(t) i \tilde{\varphi}(t) - \tilde{P}(t) i \tilde{\varphi}(t) - \tilde{C}(t)^T \tilde{P}(t) i \tilde{\varphi}(t) = 0.
\]
Then
\[ \dot{\tilde{P}}(t) \tilde{\gamma}(t) = \left( -\left[ \dot{P}_{1,i}(t)P_{1,i}(t)^{-1}\varphi_{1,i}(t) + \gamma_{1,i}(t) \right] + P_{2,i}(t)^\top \left( \gamma_{2,i}(t) + \dot{P}_{2,i}(t)P_{1,i}(t)^{-1}\varphi_{1,i}(t) \right) \right)^\top \tilde{P}_3,i(t) \left( \gamma_{2,i}(t) + \dot{P}_{2,i}(t)P_{1,i}(t)^{-1}\varphi_{1,i}(t) \right), \]

and we need to verify the following two equalities:

\[ -[(a(t)a_{11}(t) + b(t)a_{21}(t))c(t)^\top + (a(t)a_{21}(t))^\top + b(t)a_{22}(t))d(t)^\top] \varphi_{2,i}(t) - a(t)S_{3,i}(t) - b(t)S_{5,i}(t) \]
\[ + A_1(t)^\top \left( \ddot{P}_{1,i}(t)\tilde{\varphi}_{1,i}(t) + \ddot{P}_{2,i}(t)^\top \tilde{\varphi}_{2,i}(t) \right) - A_3(t)^\top \left( \ddot{P}_{2,i}(t)\tilde{\varphi}_{1,i}(t) + \ddot{P}_{3,i}(t)^\top \tilde{\varphi}_{2,i}(t) \right) + \dot{\tilde{P}}_1,i(t)\tilde{\varphi}_{1,i}(t) \]
\[ + \dot{\tilde{P}}_2,i(t)\tilde{\varphi}_{2,i}(t) + A_2(t)^\top \left( \ddot{P}_1,i(t)\tilde{v}_{1,i}(t) + \ddot{P}_2,i(t)^\top \tilde{v}_{2,i}(t) \right) \]
\[ = \left( \ddot{P}_2,i(t)^\top \dot{P}_2,i(t)P_{1,i}(t)^{-1} - \dot{P}_1,i(t)P_{1,i}(t)^{-1} \right)\varphi_{1,i}(t) + \left( \ddot{P}_2,i(t)^\top \dot{P}_2,i(t) - P_{1,i}(t) \right) \]
\[ \cdot [B_1(t)\varphi_{1,i}(t) + C_1(t)S_{5,i}(t) + D_1(t)S_{5,i}(t)] \]
\[ + \dot{\tilde{P}}_3,i(t)^\top [B_3(t)\varphi_{3,i}(t) + C_3(t)S_{5,i}(t) + D_3(t)S_{5,i}(t)] - (A_1(t)^\top \tilde{\varphi}_{1,i}(t) + A_1(t)^\top S_{4,i}(t)) \]

and

\[ -[(c(t)a_{11}(t) + d(t)a_{21}(t))c(t)^\top + (c(t)a_{21}(t))^\top + d(t)a_{22}(t))d(t)^\top] \varphi_{2,i}(t) - c(t)S_{3,i}(t) - d(t)S_{5,i}(t) \]
\[ + B_1(t)^\top \left( \ddot{P}_{1,i}(t)\tilde{\varphi}_{1,i}(t) + \ddot{P}_{2,i}(t)^\top \tilde{\varphi}_{2,i}(t) \right) - B_3(t)^\top \left( \ddot{P}_{2,i}(t)\tilde{\varphi}_{1,i}(t) + \ddot{P}_{3,i}(t)^\top \tilde{\varphi}_{2,i}(t) \right) + \dot{\tilde{P}}_2,i(t)\tilde{\varphi}_{1,i}(t) \]
\[ + \dot{\tilde{P}}_3,i(t)\tilde{\varphi}_{2,i}(t) + B_2(t)^\top \left( \ddot{P}_1,i(t)\tilde{v}_{1,i}(t) + \ddot{P}_2,i(t)^\top \tilde{v}_{2,i}(t) \right) \]
\[ = \ddot{P}_3,i(t) \left[ P_2,i(t)B_1(t)\varphi_{3,i}(t) + C_1(t)S_{5,i}(t) + D_1(t)S_{5,i}(t) \right] \]
\[ - P_3,i(t) \left[ B_1(t)^\top \tilde{\varphi}_{1,i}(t) + B_2(t)^\top S_{4,i}(t) + B_4(t)\varphi_{2,i}(t) \right] \]
\[ + \ddot{P}_3,i(t)\ddot{P}_2,i(t)P_{1,i}(t)^{-1}\varphi_{1,i}(t). \]

**Step 4:** Verification of (7.25):

The equation (7.25) can be simplified to

\[ -[(c(t)a_{11}(t) + d(t)a_{21}(t))c(t)^\top + (c(t)a_{21}(t))^\top + d(t)a_{22}(t))d(t)^\top] \varphi_{2,i}(t) \]
\[ + B_1(t)^\top \left( \ddot{P}_{1,i}(t)\tilde{\varphi}_{1,i}(t) + \ddot{P}_{2,i}(t)^\top \tilde{\varphi}_{2,i}(t) \right) - B_3(t)^\top \left( \ddot{P}_{2,i}(t)\tilde{\varphi}_{1,i}(t) + \ddot{P}_{3,i}(t)^\top \tilde{\varphi}_{2,i}(t) \right) + \dot{\tilde{P}}_2,i(t)\tilde{\varphi}_{1,i}(t) \]
\[ + \dot{\tilde{P}}_3,i(t)\tilde{\varphi}_{2,i}(t) + B_2(t)^\top \left( \ddot{P}_1,i(t)\tilde{v}_{1,i}(t) + \ddot{P}_2,i(t)^\top \tilde{v}_{2,i}(t) \right) \]
\[ = \tilde{P}_3,i(t) \left[ P_2,i(t)B_1(t)\varphi_{3,i}(t) - P_3,i(t) \left[ B_1(t)^\top \tilde{\varphi}_{1,i}(t) + B_4(t)\varphi_{2,i}(t) \right] + B_3(t)\varphi_{2,i}(t) \right] \]
\[ + \ddot{P}_3,i(t)\ddot{P}_2,i(t)P_{1,i}(t)^{-1}\varphi_{1,i}(t) - B_2(t)^\top P_{2,i}(t)B_1(t)\varphi_{3,i}(t) + v_{2,i}(t) - B_3(t)^\top C_1(t)^\top \varphi_{1,i}(t) \]
\[ - B_2(t)^\top P_{2,i}(t)P_{3,i}(t)^{-1} [P_2,i(t)B_2(t)\varphi_{3,i}(t) + v_{2,i}(t) - P_{3,i}(t)C_1(t)^\top \varphi_{1,i}(t)]. \]

Then we compare the coefficients of \( \varphi_{1,i}(\cdot), \varphi_{2,i}(\cdot), v_{1,i}(\cdot) \) and \( v_{2,i}(\cdot) \) on both sides of the above equation.
The coefficient of $\varphi_{1,i}()$ on the LHS is

$$-B_1(t)^\top \left( \tilde{P}_{1,i}(t)P_{1,i}(t)^{-1} + \tilde{P}_{2,i}(t)^\top P_{2,i}(t)P_{1,i}(t)^{-1} \right) + B_3(t)^\top \left( \tilde{P}_{2,i}(t)P_{1,i}(t)^{-1} + \tilde{P}_{3,i}(t)P_{2,i}(t)P_{1,i}(t)^{-1} \right)$$

$$-\tilde{P}_{2,i}(t)P_{1,i}(t)^{-1} + \tilde{P}_{3,i}(t)P_{2,i}(t)P_{1,i}(t)^{-1}$$

and the one on the RHS is

$$-B_1(t)^\top + \tilde{P}_{3,i}(t)\tilde{P}_{2,i}(t)P_{1,i}(t)^{-1} - B_2(t)^\top P_{2,i}(t)^\top C_1(t)^\top + B_2(t)^\top P_{2,i}(t)^\top P_{3,i}(t)^{-1}P_{3,i}(t)C_1(t)^\top.$$ 

The coefficient of $v_{2,i}()$ on the LHS is

$$-[(c(t)a_{11}(t) + d(t)a_{21}(t))c(t)^\top + (c(t)a_{21}(t) + d(t)a_{22}(t))d(t)^\top]$$

$$+B_1(t)^\top \tilde{P}_{2,i}(t)^\top - B_3(t)^\top \tilde{P}_{1,i}(t) + \tilde{P}_{3,i}(t)$$

and the one on the RHS is

$$\tilde{P}_{3,i}(t)P_{2,i}(t)B_1(t) - B_3(t) + \tilde{P}_{3,i}(t)B_3(t) - B_2(t)^\top P_{1,i}(t)B_2(t) - B_2(t)^\top P_{2,i}(t)^\top P_{3,i}(t)^{-1}P_{3,i}(t)B_2(t).$$

The coefficient of $v_{1,i}()$ on the LHS is

$$-B_2(t)^\top \left( \tilde{P}_{1,i}(t)P_{1,i}(t)^{-1} + \tilde{P}_{2,i}(t)^\top P_{2,i}(t)P_{1,i}(t)^{-1} \right)$$

and the one on the RHS is $-B_2(t)^\top$. The coefficient of $v_{2,i}()$ on the LHS is $B_2(t)^\top \tilde{P}_{2,i}(t)^\top$ and the one on the RHS is $-B_2(t)^\top P_{2,i}(t)^\top P_{3,i}(t)^{-1}$. By the notations in (7.23) and (7.15), we obtain (7.25) holds.

**Step 5:** Verification of (7.24):

The equation (7.24) can be simplified to

$$-[(a(t)a_{11}(t) + b(t)a_{21}(t))c(t)^\top + (a(t)a_{21}(t) + b(t)a_{22}(t))d(t)^\top] \varphi_{2,i}(t)$$

$$+A_1(t)^\top \left( \tilde{P}_{1,i}(t)\tilde{P}_{1,i}(t)^{-1} + \tilde{P}_{2,i}(t)^\top \tilde{P}_{2,i}(t) \right) - A_3(t)^\top \left( \tilde{P}_{2,i}(t)\tilde{P}_{1,i}(t) + \tilde{P}_{3,i}(t)\tilde{P}_{2,i}(t) \right) + \tilde{P}_{1,i}(t)\varphi_{1,i}(t)$$

$$+\tilde{P}_{2,i}(t)^\top \varphi_{2,i}(t) + A_2(t)^\top \left( \tilde{P}_{1,i}(t)\tilde{P}_{1,i}(t)^{-1} + \tilde{P}_{2,i}(t)^\top \tilde{P}_{2,i}(t) \right)$$

$$= \left( \tilde{P}_{2,i}(t)^\top B_1(t)P_{1,i}(t)^{-1} - \tilde{P}_{1,i}(t)P_{1,i}(t)^{-1} \right) \varphi_{1,i}(t) + \left( \tilde{P}_{2,i}(t)^\top P_{2,i}(t) - P_{1,i}(t) \right) B_1(t)\varphi_{2,i}(t)$$

$$+\tilde{P}_{2,i}(t)^\top B_3(t)\varphi_{2,i}(t) - A_1(t)^\top \varphi_{1,i}(t) + A_2(t)^\top \left( -P_{2,i}(t)^\top P_{3,i}(t)^{-1} \right)$$

$$\cdot (P_{2,i}(t)B_2(t)\varphi_{3,i}(t) - P_3(t)C_1(t)^\top \varphi_{1,i}(t) + v_{3,i}(t)) - P_{1,i}(t)B_2(t)\varphi_{3,i}(t) - P_{2,i}(t)^\top C_1(t)^\top \varphi_{1,i}(t) - v_{1,i}(t).$$

By comparing the coefficients of $\varphi_{1,i}()$, $\varphi_{2,i}()$, $v_{1,i}()$ and $v_{2,i}()$ on both sides of the above equation, we deduce that (7.22) holds. ■
7.2 Proof of Lemma 4.5

From (4.3) and (4.6), the optimal control \( \bar{u}(\cdot), \bar{Z}(\cdot) \) has the following form

\[
\bar{u}(t) = \begin{bmatrix} a_{11}(t) \left( a(t)^T + c(t)^TP_{2,i}(t) \right) + a_{21}(t)^T \left( b(t)^T + d(t)^TP_{2,i}(t) \right) \end{bmatrix} X(t) \\
+ \begin{bmatrix} a_{11}(t)c(t)^T + a_{21}(t)d(t)^T \end{bmatrix} P_{3,i}(t) h(t) + a_{11}(t) \begin{bmatrix} \bar{b}(t)\varphi_{2,i}(t) - \bar{a}(t)\bar{\varphi}_{1,i}(t) \end{bmatrix} \\
- \bar{b}(t)\bar{\varphi}_{2,i}(t) + \bar{a}(t)B_{2}(t)\varphi_{2,i}(t) - \bar{a}(t)\bar{\nu}_{1,i}(t) - \bar{b}(t)\bar{\nu}_{2,i}(t) \\
+ a_{21}(t)^T \begin{bmatrix} \bar{a}(t)\varphi_{2,i}(t) - \bar{c}(t)\bar{\varphi}_{1,i}(t) - \bar{d}(t)\bar{\varphi}_{2,i}(t) + \bar{c}(t)B_{2}(t)\varphi_{2,i}(t) - \bar{c}(t)\bar{\nu}_{1,i}(t) - \bar{d}(t)\bar{\nu}_{2,i}(t) \end{bmatrix}.
\]

\[
\bar{Z}(t) = \begin{bmatrix} a_{21}(t)^T + c(t)^TP_{2,i}(t) \end{bmatrix} X(t) \\
+ \begin{bmatrix} a_{21}(t)c(t)^T + a_{22}(t)d(t)^T \end{bmatrix} P_{3,i}(t) h(t) + a_{21}(t) \begin{bmatrix} \bar{b}(t)\varphi_{2,i}(t) - \bar{a}(t)\bar{\varphi}_{1,i}(t) \end{bmatrix} \\
- \bar{b}(t)\bar{\varphi}_{2,i}(t) + \bar{a}(t)B_{2}(t)\varphi_{2,i}(t) - \bar{a}(t)\bar{\nu}_{1,i}(t) - \bar{b}(t)\bar{\nu}_{2,i}(t) \\
+ a_{22}(t) \begin{bmatrix} \bar{a}(t)\varphi_{2,i}(t) - \bar{c}(t)\bar{\varphi}_{1,i}(t) - \bar{d}(t)\bar{\varphi}_{2,i}(t) + \bar{c}(t)B_{2}(t)\varphi_{2,i}(t) - \bar{c}(t)\bar{\nu}_{1,i}(t) - \bar{d}(t)\bar{\nu}_{2,i}(t) \end{bmatrix}.
\]

The following relations can be verified directly:

\[
(a(t) + \bar{b}(t)P_{2,i}(t)) P_{1,i}(t)^{-1} = D_{2}(t)^T, \tag{7.26}
\]

\[
(c(t) + \bar{a}(t)P_{2,i}(t)) P_{1,i}(t)^{-1} = C_{2}(t)^T, \tag{7.27}
\]

\[
(a(t) + \bar{b}(t)P_{2,i}(t)) P_{1,i}(t)^{-1} = D_{1}(t)^T, \tag{7.28}
\]

\[
(c(t) + \bar{a}(t)P_{2,i}(t)) P_{1,i}(t)^{-1} = C_{1}(t)^T, \tag{7.29}
\]

\[
L_{1,i}(t)^{-1} \left( P_{2,i}(t) - P_{3,i}(t)C_{2}(t)^TP_{2,i}(t)^{-1}S_{1,i}(t) \right) = -D(t)^{-1} \left( C_{2}(t)^TP_{1,i}(t) + P_{2,i}(t) \right), \tag{7.30}
\]

\[
L_{2,i}(t)^{-1} \left[ P_{2,i}(t)^T - (P_{1,i}(t)C_{2}(t) + P_{2,i}(t)^TC_{4}(t))L_{1,i}(t)^{-1}P_{3,i}(t) \right] = - \left( C_{2}(t)^TP_{1,i}(t) + P_{2,i}(t) \right)^TD(t)^{-1}, \tag{7.31}
\]

\[
\left( P_{1,i}(t)C_{2}(t) + P_{2,i}(t)^T - P_{1,i}(t)C_{2}(t) - P_{2,i}(t)^TC_{4}(t) \right) L_{1,i}(t)^{-1} = -P_{2,i}(t)^TP_{3,i}(t)^{-1}, \tag{7.32}
\]

\[
[I_m - C_{2}(t)^TP_{2,i}(t)^T + C_{2}(t)^T \left( P_{1,i}(t)C_{2}(t) + P_{2,i}(t)^TC_{4}(t) \right) L_{1,i}(t)^{-1}P_{3,i}(t)] = D(t)L_{1,i}(t)^{-1}P_{3,i}(t). \tag{7.33}
\]
By notations in (7.3), (7.10), (7.13) and (7.14), it can be verified that

\[
L_{0,i}(t) = - [a_{11}(t) (a(t)^T + c(t)^T P_{2,i}(t)) + a_{21}(t) (b(t)^T + d(t)^T P_{2,i}(t))],
\]

\[
L_{3,i}(t) = (a_{11}(t) c(t)^T + a_{21}(t) d(t)^T) P_{3,i}(t),
\]

\[
L_{10,i}(t) = - [a_{21}(t) (a(t)^T + c(t)^T P_{2,i}(t)) + a_{22}(t) (b(t)^T + d(t)^T P_{2,i}(t))],
\]

\[
L_{11,i}(t) = (a_{21}(t) c(t)^T + a_{22}(t) d(t)^T) P_{3,i}(t).
\]

(7.34)

Before proving Lemma 4.5, we give the following lemma:

**Lemma 7.1** Under the same assumptions as Theorem 1.4 for \( i \geq i_0 \), we have

\[
S_{3,i}(t) = a_{11}(t) \left[ \tilde{b}(t) \varphi_{2,i}(t) - \tilde{a}(t) \tilde{\varphi}_{1,i}(t) - \tilde{b}(t) \tilde{\varphi}_{2,i}(t) + \tilde{a}(t) B_2(t) \varphi_{2,i}(t) - \tilde{a}(t) \tilde{v}_{1,i}(t) - \tilde{b}(t) \tilde{v}_{2,i}(t) \right] + a_{21}(t)^T \left[ \tilde{d}(t) \varphi_{2,i}(t) - \tilde{c}(t) \tilde{\varphi}_{1,i}(t) - \tilde{d}(t) \tilde{\varphi}_{2,i}(t) + \tilde{c}(t) B_2(t) \varphi_{2,i}(t) - \tilde{c}(t) \tilde{v}_{1,i}(t) - \tilde{d}(t) \tilde{v}_{2,i}(t) \right],
\]

\[
S_{5,i}(t) = a_{21}(t) \left[ \tilde{b}(t) \varphi_{2,i}(t) - \tilde{a}(t) \tilde{\varphi}_{1,i}(t) - \tilde{b}(t) \tilde{\varphi}_{2,i}(t) + \tilde{a}(t) B_2(t) \varphi_{2,i}(t) - \tilde{a}(t) \tilde{v}_{1,i}(t) - \tilde{b}(t) \tilde{v}_{2,i}(t) \right] + a_{22}(t) \left[ \tilde{d}(t) \varphi_{2,i}(t) - \tilde{c}(t) \tilde{\varphi}_{1,i}(t) - \tilde{d}(t) \tilde{\varphi}_{2,i}(t) + \tilde{c}(t) B_2(t) \varphi_{2,i}(t) - \tilde{c}(t) \tilde{v}_{1,i}(t) - \tilde{d}(t) \tilde{v}_{2,i}(t) \right].
\]

(7.35)

**Proof.** We first prove the equality for \( S_{3,i}(\cdot) \). Compare the coefficients of \( \varphi_{1,i}(\cdot), \varphi_{2,i}(\cdot), v_{1,i}(\cdot) \) and \( v_{2,i}(\cdot) \) for \( S_{3,i}(\cdot) \) in (7.34) with the ones in (7.15). By the notations in (7.3), (7.10) and (7.13), we obtain the equality for \( S_{3,i}(\cdot) \) holds.

Then, we prove the equality for \( S_{5,i}(\cdot) \). Putting \( S_{2,i}(\cdot), S_{3,i}(\cdot) \) and \( S_{4,i}(\cdot) \) into \( S_{5,i}(\cdot) \), the equality for \( S_{5,i}(\cdot) \) becomes

\[
S_{5,i}(t) = - \left[ \left[ P_{3,i}(t) C_1(t)^T + (P_{2,i}(t) - P_{3,i}(t) C_2(t)^T) L_{2,i}(t)^{-1} S_{1,i}(t) \right] D_2(t) a_{11}(t) D_3(t)^T \right] \varphi_{1,i}(t) - P_{2,i}(t) B_2(t) \varphi_{2,i}(t) - v_{2,i}(t) + \left[ P_{3,i}(t) C_2(t)^T L_{2,i}(t)^{-1} + (P_{2,i}(t) - P_{3,i}(t) C_2(t)^T) L_{2,i}(t)^{-1} S_{1,i}(t) \right] D_2(t) a_{11}(t) D_2(t)^T L_{2,i}(t)^{-1} S_{3,i}(t).
\]

(7.36)

Compare the coefficients of \( \varphi_{1,i}(\cdot), \varphi_{2,i}(\cdot), v_{1,i}(\cdot) \) and \( v_{2,i}(\cdot) \) for \( S_{5,i}(\cdot) \) in (7.35) with the ones in (7.30). By the notations in (7.3), (7.10) and (7.20)–(7.23), we obtain the equality for \( S_{5,i}(\cdot) \) holds.

**Proof of Lemma 4.5** By the notations in (7.1) and (7.2), we have

\[
\left( \hat{R}(t) + \hat{D}(t)^T \hat{P}(t) \hat{D}(t) \right)^{-1} \left( \hat{B}(t)^T \hat{P}(t) + \hat{D}(t)^T \hat{P}(t) \hat{C}(t) \right) = \begin{pmatrix}
  a_{11}(t) a(t)^T + a_{21}(t) b(t)^T & a_{11}(t) c(t)^T + a_{21}(t) d(t)^T \\
  a_{21}(t) a(t)^T + a_{22}(t) b(t)^T & a_{21}(t) c(t)^T + a_{22}(t) d(t)^T 
\end{pmatrix}.
\]

(7.34)

By the notations in (7.34), we obtain the first relation in Lemma 4.5 holds.
From the relationship of \(iP(\cdot), \bar{P}(\cdot), i\varphi(\cdot), \) and \(i\bar{\varphi}(\cdot)\), we have

\[ i\bar{\varphi}(t)^T i\bar{P}(t) i\bar{\varphi}(t) = \begin{bmatrix}
    -P_{1,i}(t)^{-1} & 0 \\
    -P_{2,i}(t)P_{1,1}(t)^{-1} & I_m
\end{bmatrix}
\begin{bmatrix}
    \varphi_{1,i}(t) \\
    \varphi_{2,i}(t)
\end{bmatrix}^T
\begin{bmatrix}
    \bar{P}_{1,i}(t) - \bar{P}_{2,i}(t) \\
    \bar{P}_{2,i}(t) - \bar{P}_{3,i}(t)
\end{bmatrix}
\begin{bmatrix}
    -P_{1,i}(t)^{-1} & 0 \\
    -P_{2,i}(t)P_{1,1}(t)^{-1} & I_m
\end{bmatrix}
\begin{bmatrix}
    \varphi_{1,i}(t) \\
    \varphi_{2,i}(t)
\end{bmatrix}
\]

\[ = \varphi_{1,i}(t)^TP_{1,1}(t)^{-1}\varphi_{1,i}(t) + \varphi_{2,i}(t)^TP_{3,1}(t)^{-1}\varphi_{2,i}(t). \]

Due to (7.34) and Lemma 7.1 we obtain

\[ \begin{aligned}
    &\left( \bar{R}(t) + \bar{D}(t)^T \bar{P}(t) \bar{D}(t) \right)^{-1} \begin{pmatrix}
        \bar{B}(t)^T \bar{P}(t) i\bar{\varphi}(t) + \bar{D}(t)^T \bar{P}(t) \bar{C}(t) i\bar{v}(t)
    \end{pmatrix}^T \\
    &= \begin{pmatrix}
        S_{3,i}(t) \\
        S_{5,i}(t)
    \end{pmatrix} + \begin{pmatrix}
        (a_{11}(t)c(t)^T + a_{21}(t)d(t)^T) \varphi_{2,i}(t) \\
        (a_{21}(t)c(t)^T + a_{22}(t)d(t)^T) \varphi_{2,i}(t)
    \end{pmatrix}
    \begin{pmatrix}
        S_{3,i}(t) + L_{7,i}(t)P_{3,1}(t)^{-1}\varphi_{2,i}(t) \\
        S_{5,i}(t) + L_{11,i}(t)P_{3,1}(t)^{-1}\varphi_{2,i}(t)
    \end{pmatrix},
\end{aligned} \]

This completes the proof. \(\blacksquare\)

### 7.3 Proof of Lemma 5.1

It can be verified that

\[ (P_{2,i}(t)C_2(t) - I_m) L_{1,i}(t)^{-1} P_{3,i}(t) = (P_{2,i}(t)C_2(t) - I_m - P_{3,i}(t)C_4(t)) L_{1,i}(t)^{-1} P_{3,i}(t) + P_{3,i}(t)C_4(t) L_{1,i}(t)^{-1} P_{3,i}(t), \]

\[ L_{2,i}(t)^{-1} [P_{2,i}(t)^T - (P_{2,i}(t)^T C_4(t) + P_{1,i}(t)C_2(t)) L_{1,i}(t)^{-1} P_{3,i}(t)] = -\left( C_2(t)^T \bar{P}_{1,i}(t) + \bar{P}_{2,i}(t) \right)^T D(t)^{-1}, \]

\[ S_{3,i}(t) = a_{11}(t)\lambda_{1,i}(t), \]
\[ L_{7,i}(t) = a_{11}(t)\lambda_{2,i}(t), \]
\[ S_{5,i}(t) = L_{2,i}(t)^{-1} S_{1,i}(t)D_2(t)a_{11}(t)\lambda_{1,i}(t), \]

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\[
S_{\alpha,i}(t) = -D(t)^{-1} \left( C_2(t)\bar{P}_{\alpha,i}(t) + \bar{P}_{\alpha,i}(t) \right) D_2(t)a_{11}(t)\lambda_1(t)
\]
\[
= -D(t)^{-1} \left( C_2(t)\bar{P}_{\alpha,i}(t) + \bar{P}_{\alpha,i}(t) \right) D_2(t)a_{11}(t)
\cdot \left[ \tilde{b}(t)\varphi_{\alpha,i}(t) - \bar{a}(t)\bar{v}_{\alpha,i}(t) - \bar{b}(t)\bar{v}_{\alpha,i}(t) + \bar{a}(t)B_2(t)\varphi_{\alpha,i}(t) - \bar{a}(t)\bar{v}_{\alpha,i}(t) - \bar{b}(t)\bar{v}_{\alpha,i}(t) \right]
+ D(t)^{-1} \left( C_2(t)\bar{P}_{\alpha,i}(t) + \bar{P}_{\alpha,i}(t) \right) D_2(t)a_{11}(t)D_2(t)^\top \left( \bar{P}_{\alpha,i}(t)C_2(t) + \bar{P}_{\alpha,i}(t)^\top \right) D(t)^{-1}
\cdot \left[ \bar{a}(t)\varphi_{\alpha,i}(t) - \bar{c}(t)\bar{v}_{\alpha,i}(t) - \bar{d}(t)\bar{v}_{\alpha,i}(t) + \bar{c}(t)B_2(t)\varphi_{\alpha,i}(t) - \bar{c}(t)\bar{v}_{\alpha,i}(t) - \bar{d}(t)\bar{v}_{\alpha,i}(t) \right]
+ \beta_i(t),
\]
\[
L_{11,i}(t) = -D(t)^{-1} \left( C_2(t)\bar{P}_{\alpha,i}(t) + \bar{P}_{\alpha,i}(t) \right) D_2(t)a_{11}(t)c(t)^\top P_{3,i}(t)
+ D(t)^{-1} \left( C_2(t)\bar{P}_{\alpha,i}(t) + \bar{P}_{\alpha,i}(t) \right) D_2(t)a_{11}(t)D_2(t)^\top \left( \bar{P}_{\alpha,i}(t)C_2(t) + \bar{P}_{\alpha,i}(t)^\top \right) D(t)^{-1}
\cdot d(t)^\top P_{3,i}(t) + \tilde{\beta}_i(t),
\]
\[
L_{2,i}(t)^{-1}S_{2,i}(t) = \beta_{5,i}(t)\varphi_{1,i}(t) + \beta_{6,i}(t)\varphi_{2,i}(t) + \beta_{7,i}(t)v_{1,i}(t) + \beta_{8,i}(t)v_{2,i}(t),
\]
where
\[
\lambda_{1,i}(t) = - \left[ D_1(t)^\top \varphi_{1,i}(t) + D_2(t)^\top L_{2,i}(t)^{-1}S_{2,i}(t) \right],
\]
\[
\lambda_{2,i}(t) = - \left[ D_1(t)^\top P_{3,i}(t)^\top + D_2(t)^\top L_{2,i}(t)^{-1}L_{4,i}(t) + D_3(t)^\top \right],
\]
\[
\beta_i(t) = D(t)^{-1} \left[ \bar{d}(t)\varphi_{\alpha,i}(t) - \bar{c}(t)\bar{v}_{\alpha,i}(t) - \bar{d}(t)\bar{v}_{\alpha,i}(t) + \bar{c}(t)B_2(t)\varphi_{\alpha,i}(t) - \bar{c}(t)\bar{v}_{\alpha,i}(t) - \bar{d}(t)\bar{v}_{\alpha,i}(t) \right]
= \beta_{1,i}(t)\varphi_{1,i}(t) + \beta_{2,i}(t)\varphi_{2,i}(t) + \beta_{3,i}(t)v_{1,i}(t) + \beta_{4,i}(t)v_{2,i}(t),
\]
\[
\beta_{1,i}(t) = D(t)^{-1} \left[ \bar{c}(t)P_{1,i}(t)^{-1} + \bar{a}(t)B_2(t)P_{1,i}(t) \right],
\]
\[
\beta_{2,i}(t) = D(t)^{-1} \left[ \bar{c}(t)B_2(t)P_{1,i}(t)^{-1} + \bar{d}(t)P_{2,i}(t) \right],
\]
\[
\beta_{3,i}(t) = D(t)^{-1} \left[ \bar{c}(t)P_{1,i}(t)^{-1} + \bar{d}(t)P_{2,i}(t) \right],
\]
\[
\beta_{4,i}(t) = -D(t)^{-1}d(t),
\]
\[
\tilde{\beta}_i(t) = D(t)^{-1}d(t)^\top P_{3,i}(t),
\]
\[
\beta_{5,i}(t) = L_{2,i}(t)^{-1} \left[ P_{2,i}(t)^\top C_1(t)^\top - (P_1(t)C_2(t) + P_2(t)^\top C_4(t))L_{1,i}(t)^{-1}P_{3,i}(t)C_1(t)^\top \right],
\]
\[
\beta_{6,i}(t) = L_{2,i}(t)^{-1} \left[ P_{1,i}(t)B_2(t) + (P_1(t)C_2(t) + P_2(t)^\top C_4(t))L_{1,i}(t)^{-1}P_{3,i}(t)B_2(t) \right],
\]
\[
\beta_{7,i}(t) = L_{2,i}(t)^{-1},
\]
\[
\beta_{8,i}(t) = L_{2,i}(t)^{-1}(P_1(t)C_2(t) + P_2(t)^\top C_4(t))L_{1,i}(t)^{-1}.
\]
Proof of Lemma 5.1: By Lemma 4.5 and the relations between \(iP(\cdot), \hat{iP}(\cdot), i\varphi(\cdot)\) and \(i\tilde{\varphi}(\cdot)\), we have

\[
M_{5,i}(t) = -\gamma(t)\xi \hat{\bar{P}}(t) i\varphi(t) - i\tilde{\varphi}(t)^T \hat{\bar{P}}(t) i\varphi(t) + i\tilde{\varphi}(t)^T \hat{\bar{P}}(t) i\tilde{\varphi}(t) \\
- \left( \hat{\bar{B}}(t)^T \hat{\bar{P}}(t) i\varphi(t) + \hat{\bar{D}}(t)^T \hat{\bar{P}}(t) i\tilde{\varphi}(t) \right)^T (\hat{\bar{R}}(t) + \hat{\bar{D}}(t)^T \hat{\bar{P}}(t) \hat{\bar{D}}(t))^{-1} \\
\cdot \left( \hat{\bar{B}}(t)^T \hat{\bar{P}}(t) i\varphi(t) + \hat{\bar{D}}(t)^T \hat{\bar{P}}(t) i\tilde{\varphi}(t) \right) \\
= -\gamma_{1,i}(t)^T P_{1,i}(t)^{-1} \varphi_{1,i}(t) - \varphi_{1,i}(t)^T P_{1,i}(t)^{-1} \hat{P}_{1,i}(t) P_{1,i}(t)^{-1} \varphi_{1,i}(t) - \varphi_{1,i}(t)^T P_{1,i}(t)^{-1} \gamma_{1,i}(t) \\
+ v_{1,i}(t)^T P_{1,i}(t)^{-1} v_{1,i}(t) - \gamma_{2,i}(t)^T P_{3,i}(t)^{-1} \varphi_{2,i}(t) - \varphi_{2,i}(t)^T P_{3,i}(t)^{-1} \hat{P}_{3,i}(t) P_{3,i}(t)^{-1} \varphi_{2,i}(t) \\
- \varphi_{2,i}(t)^T P_{3,i}(t)^{-1} \gamma_{2,i}(t) + v_{2,i}(t)^T P_{3,i}(t)^{-1} v_{2,i}(t) \\
+ \left[ \varphi_{1,i}(t)^T D_{1}(t) + \varphi_{2,i}(t)^T P_{3,i}(t)^{-1} (P_{2,i}(t) D_{1}(t) + D_{3}(t) + v_{1,i}(t)^T D_{2}(t) + v_{2,i}(t)^T P_{3,i}(t)^{-1} P_{2,i}(t) D_{2}(t)) \right] \\
\cdot \left( S_{3,i}(t) + L_{3,i}(t)^{-1} - C_{3}(t) \right) + \left[ \varphi_{1,i}(t)^T C_{3}(t) + \varphi_{2,i}(t)^T P_{3,i}(t)^{-1} (P_{2,i}(t) C_{1}(t) + C_{3}(t)) \right] \\
+ v_{1,i}(t)^T C_{2}(t) + v_{2,i}(t)^T P_{3,i}(t)^{-1} P_{2,i}(t) C_{2}(t) \right] \cdot \left( S_{5,i}(t) + L_{11,i}(t) \hat{P}_{3,i}(t)^{-1} \varphi_{2,i}(t) \right).
\]

It can be verified that the following two equalities hold

\[
\varphi_{2,i}(t)^T \left[ P_{3,i}(t)^{-1} (P_{2,i}(t) D_{1}(t) + D_{3}(t)) L_{7,i}(t) P_{3,i}(t)^{-1} + P_{3,i}(t)^{-1} (P_{2,i}(t) C_{1}(t) + C_{3}(t)) \right] \\
+ C_{3}(t)) L_{11,i}(t) P_{3,i}(t)^{-1} - P_{3,i}(t)^{-1} \hat{P}_{3,i}(t) P_{3,i}(t)^{-1} \varphi_{2,i}(t) \\
= \varphi_{2,i}(t)^T P_{3,i}(t)^{-1} \left[ P_{2,i}(t) B_{1}(t) P_{3,i}(t) + P_{3,i}(t) B_{3}(t)^T + P_{3,i}(t) B_{1}(t)^T P_{2,i}(t)^T \right] \\
+ P_{3,i}(t) B_{2}(t)^T L_{9,i}(t) - P_{3,i}(t) B_{4}(t) P_{3,i}(t) + B_{5}(t) P_{3,i}(t) \right] P_{3,i}(t)^{-1} \varphi_{2,i}(t),
\]
\[
\left\{ \left( \varphi_{1,i}(t)^T + \varphi_{2,i}(t)^T \tilde{P}_{1,i}(t)^{-1} P_{2,i}(t) \right) - \left( \varphi_{1,i}(t)^T + \varphi_{2,i}(t)^T \tilde{P}_{1,i}(t)^{-1} P_{2,i}(t) \right) - D(t) - 1 \left( C(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right) D(t) \right\} a_{11}(t) \lambda_{1,i}(t) \\
+ \left\{ \varphi_{1,i}(t)^T D(t) + v_{1,i}(t)^T D(t) + v_{2,i}(t)^T P_{3,i}(t)^{-1} P_{2,i}(t) D(t) \right\} \\
- \left[ \varphi_{1,i}(t)^T C_1(t) + v_{1,i}(t)^T C_2(t) + v_{2,i}(t)^T P_{3,i}(t)^{-1} P_{2,i}(t) C_2(t) - v_{2,i}(t)^T P_{3,i}(t)^{-1} \right] \\
\cdot D(t) - 1 \left( C_2(t)^T \tilde{P}_{1,i}(t) + \tilde{P}_{2,i}(t) \right) D(t) - (P_{2,i}(t) D(t) + D_3(t)) \\
+ P_{3,i}(t) B_2(t)^T L_{2,i}(t)^{-1} S_{1,i}(t) D_2(t) \right\} a_{11}(t) \lambda_{1,i}(t) \\
= \lambda_{1,i}(t) a_{11}(t) \lambda_{1,i}(t) - \varphi_{2,i}(t)^T B_2(t)^T L_{2,i}(t)^{-1} S_{1,i}(t) D_2(t) a_{11}(t) \lambda_{2,i}(t) P_{3,i}(t)^{-1} \varphi_{2,i}(t).
\]

By (7.37) and (7.38), we have

\[
M_{5,i}(t) = -\lambda_{1,i}(t) a_{11}(t) \lambda_{1,i}(t) - \gamma_{1,i}(t)^T P_{1,i}(t)^{-1} \varphi_{1,i}(t) - \varphi_{1,i}(t)^T P_{1,i}(t)^{-1} \gamma_{1,i}(t) + v_{1,i}(t)^T P_{1,i}(t)^{-1} v_{1,i}(t) \\
- \varphi_{1,i}(t)^T P_{1,i}(t)^{-1} \tilde{P}_{1,i}(t) P_{1,i}(t)^{-1} \varphi_{1,i}(t) - B_2(t)^T L_{2,i}(t)^{-1} S_{1,i}(t) B_2(t) + C_1(t) \beta_{1,i}(t) + C_2(t) \beta_{3,i}(t) \\
+ C_2(t)^T L_{2,i}(t)^{-1} (P_{1,i}(t) C_2(t) + P_{2,i}(t)^T C_4(t)) L_{1,i}(t)^{-1} + C_4(t) L_{1,i}(t)^{-1} \\
\cdot \left[ I_m - P_{3,i}(t) C_2(t)^T L_{2,i}(t)^{-1} (P_{1,i}(t) C_2(t) + P_{2,i}(t)^T C_4(t)) L_{1,i}(t)^{-1} \right] + B_1(t)^T + B_2(t)^T \beta_{5,i}(t) \\
+ C_1(t) \beta_{2,i}(t) + B_1(t) + \beta_{5,i}(t)^T B_2(t) + C_1(t) L_{1,i}(t)^{-1} \left\{ P_{2,i}(t) B_2(t) + P_{3,i}(t) C_2(t)^T L_{2,i}(t)^{-1} \right\} \\
- \left\{ P_{1,i}(t) C_2(t) + P_{2,i}(t)^T C_4(t) \right\} L_{1,i}(t)^{-1} P_{2,i}(t) \right\} B_2(t) \\
+ C_1(t) \beta_{3,i}(t) + C_2(t) \beta_{1,i}(t) + C_1(t) \beta_{3,i}(t) + \left[ I_m - C_4(t) L_{1,i}(t)^{-1} P_{3,i}(t) \right] (C_2(t)^T \beta_{5,i}(t) + C_1(t)^T) \\
+ C_2(t) \beta_{3,i}(t) + C_1(t) L_{1,i}(t)^{-1} P_{3,i}(t) - I_m \right\} C_2(t)^T L_{2,i}(t)^{-1} + B_2(t)^T \beta_{7,i}(t) + C_2(t) \beta_{2,i}(t) + \beta_{7,i}(t) B_2(t) \\
+ C_2(t) L_{1,i}(t)^{-1} \left\{ P_{2,i}(t) B_2(t) + P_{3,i}(t) C_2(t)^T L_{2,i}(t)^{-1} \right\} \left\{ P_{1,i}(t) C_2(t) + P_{2,i}(t)^T C_4(t) \right\} L_{1,i}(t)^{-1} P_{2,i}(t) \right\} B_2(t) \\
+ C_4(t) L_{1,i}(t)^{-1} \left\{ P_{2,i}(t) B_2(t) - P_{3,i}(t) C_2(t)^T \beta_{6,i}(t) \right\}.
\]
References

[1] F. Antonelli, Backward-forward stochastic differential equations, Ann. Appl. Probab. 3 (1993): pp. 777-793.

[2] A. Bensoussan,. Lectures on stochastic control. Nonlinear filtering and stochastic control. Springer, Berlin, Heidelberg, 1982. pp. 1-62.

[3] S. Chen, X. Li and X.Y. Zhou. Stochastic linear quadratic regulators with indefinite control weight costs. SIAM Journal on Control and Optimization 36(5) (1998): pp. 1685-1702.

[4] S. Chen, and X.Y. Zhou. Stochastic linear quadratic regulators with indefinite control weight costs. II. SIAM Journal on Control and Optimization, 39(4), (2000) pp.1065-1081.

[5] J. Cvitanić and J. Zhang, Contract theory in continuous-time models. Springer-Verlag, 2013.

[6] M. Dokuchaev and X.Y. Zhou, Stochastic controls with terminal contingent conditions. J. Math. Anal. Appl. 238(1) (1999): pp. 143-165.

[7] M. Hu, S. Ji and X. Xue, A global stochastic maximum principle for fully coupled forward-backward stochastic systems, SIAM J. Control Optim. 56(6) (2018): pp. 4309-4335.

[8] J. Huang and J. Shi, Maximum principle for optimal control of fully coupled forward-backward stochastic differential delayed equations. ESAIM: Control, optimisation and calculus of variations, 18(4) (2012): pp. 1073-1096.

[9] Y. Hu and S Peng, Solution of forward-backward stochastic differential equations. Probability Theory and Related Fields, 103(2) (1995): pp. 273-283.

[10] M. Kohlmann and S. Tang. Minimization of risk and linear quadratic optimal control theory. SIAM journal on control and optimization 42(3) (2003): pp. 1118-1142.

[11] M. Kohlmann and X. Y. Zhou, Relationship between backward stochastic differential equations and stochastic controls: A linear-quadratic approach, SIAM J. Control Optim., 38 (2000): pp. 1392–1407.

[12] A. Lim, and X.Y. Zhou. Linear-quadratic control of backward stochastic differential equations. SIAM journal on control and optimization 40(2) (2001): pp. 450-474.

[13] J. Ma, P. Protter and J. Yong, Solving forward–backward stochastic differential equations explicitly–a four step scheme, Probab. Theory Related Fields 98 (2) (1994): pp. 339–359.

[14] J. Ma, Z. Wu, D. Zhang and J.Zhang, On well-posedness of forward–backward SDEs–A unified approach. The Annals of Applied Probability, 25(4) (2015): pp. 2168-2214.

[15] J. Ma and J. Yong, Forward-backward stochastic differential equations and their applications. Springer Science & Business Media, 1999.

[16] Q. Meng, A maximum principle for optimal control problem of fully coupled forward-backward stochastic systems with partial information. Science in China Series A: Mathematics 52.7 (2009): pp. 1579-1588.
[17] E. Pardoux and S. Tang, Forward-backward stochastic differential equations and quasilinear parabolic PDEs. Probability Theory and Related Fields, 114(2) (1999):pp. 123-150.

[18] S. Peng, Backward stochastic differential equations and applications to the optimal control, Appl. Math. Optim., 27 (1993):pp. 125-144.

[19] S. Peng and Z. Wu. Fully coupled forward-backward stochastic differential equations and applications to optimal control. SIAM Journal on Control and Optimization 37(3) (1999):pp. 825-843.

[20] J. Sun, X. Li and J. Yong, Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems. SIAM Journal on Control and Optimization, 54(5) (2016):pp. 2274-2308.

[21] S. Tang, General linear quadratic optimal stochastic control problems with random coefficients: linear stochastic Hamilton systems and backward stochastic Riccati equations. SIAM journal on control and optimization. 42(1) (2003):pp. 53-75.

[22] W. M. Wonham, On a matrix Riccati equation of stochastic control. SIAM Journal on Control, 6(4) (1968):pp. 681-697.

[23] Z. Wu, Maximum principle for optimal control problem of fully coupled forward-backward stochastic systems. Syst. Sci. Math. Sci. 11 (1998):pp. 249-259.

[24] J. Yong, Finding adapted solution of forward–backward stochastic differential equations-method of continuation, Probab. Theory Related Fields 107 (4) (1997):pp. 537–572.

[25] J. Yong, Linear forward-backward stochastic differential equations with random coefficients. Probability theory and related fields 135(1) (2006):pp. 53-83.

[26] J. Yong and X. Y. Zhou, Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer-Verlag, New York, 1999.