Duality symmetries and $G^{+++}$ theories

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Abstract
We show that the nonlinear realizations of all the very extended algebras $G^{+++}$, except the $B$ and $C$ series which we do not consider, contain fields corresponding to all possible duality symmetries of the on-shell degrees of freedom of these theories. This result also holds for $G^{++}$ and we argue that the nonlinear realization of this algebra accounts precisely for the form fields present in the corresponding supersymmetric theory. We also find a simple necessary condition for the roots to belong to a $G^{+++}$ algebra.

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1. Introduction
The IIA [1] and IIB [2–4] supergravity theories, by virtue of the large amount of supersymmetry that they possess, encode all the low energy effects of their corresponding string theories, while the 11-dimensional supergravity theory [5] is thought to be the low energy effective action for an as yet undefined theory called M theory. Given that there exists no complete formulation of string theory these supergravity theories have proved invaluable in our understanding of string theory and extensions of it to include branes. The result [6] that these supergravity theories can be formulated as nonlinear realizations led to the conjecture [7] that the nonlinear realization of the generalized Kac–Moody algebra $E_{11}$ was an extension of 11-dimensional supergravity. Indeed this algebra, when decomposed with respect to its $A_{10}$, or $SL(11)$, algebra associated with 11-dimensional gravity, has lowest level positive root generators which are in one-to-one correspondence with the graviton, the 3- and 6-form gauge fields, while the next generator corresponds to the dual graviton [7]. Nonlinear realizations of $E_{11}$ can also be used to describe the maximal ten-dimensional supergravity theories, but in this case the adjoint representation is decomposed into representations of $A_9$, or $SL(10)$, associated with ten-dimensional gravity. There are only two such $A_9$ sub-algebras and the two different choices were found to lead to nonlinear realizations which at low levels are the IIA and IIB supergravity theories in [7] and [8], respectively. It is striking to examine tables of the generators [9] listed in terms of increasing level and see how the generators associated with the field content of the IIA and IIB supergravity theories occupy precisely all the lower levels, before an infinite sea of generators whose physical significance was unknown at the time the tables of [9] were constructed.
The fact that the three maximal supergravity theories in ten and eleven dimensions could be formulated in terms of a single $E_{11}$ theory encouraged the belief [7] that this algebra might be a symmetry of the underlying M theory.

Amongst the set of $E_{11}$ generators appropriate to the IIA theory is one with nine anti-symmetrized indices which in the nonlinear realization leads to a field with the same index structure [9]. This 9-form generator occurs in the table of IIA generators at a place which is in amongst the generators associated with the fields of the IIA supergravity theory. A non-trivial value for this field is known to lead to the massive IIA supergravity theory [10] and as a result it was realized [9] that $E_{11}$ can incorporate the massive IIA theory.

However, there is a one-to-one correspondence between the fields of the nonlinear realizations of $E_{11}$ appropriate to the 11-dimensional, IIA and IIB theories and using this correspondence one finds that the 9-form of the IIA theory corresponds to a field with the index structure $A^{a,b,c_1,...,c_{10}}$ which occurs at a level which is beyond those of the supergravity fields of 11-dimensional supergravity theory [11]. As such $E_{11}$ provides an 11-dimensional origin of the massive IIA theory which involves one of the higher level fields and in this way the physical interpretation of at least one of the higher level fields in the 11-dimensional $E_{11}$ theory became apparent.

For quite some time the significance of the higher fields in the nonlinear realization, with the exception of the one field just mentioned above, was unclear. However, it was shown that the quadruplet and doublet space-filling forms of the IIB theory which were known [9] to be present in the $E_{11}$ formulation were found from a rather different perspective. Remarkably it was shown [12] that if one included the dual fields corresponding to all the physical fields of the IIB supergravity theory then the supersymmetry algebra could be closed in the presence of a set of 10-forms which were precisely the fields that were predicted by $E_{11}$. Furthermore, it was shown that there was a precise matching of coefficients for the gauge algebras of all these forms found on the one hand from closing the supersymmetry algebra and on the other hand from the $E_{11}$ algebra [13]. A similar story applies to the closure of the IIA supergravity algebra [14] and the space-filling forms predicted from $E_{11}$ [9].

Recently, considerable numbers of higher level $E_{11}$ fields were shown to have a physics meaning. First, it was shown [15] that an infinite class of the higher fields were just those required to realize all possible duality symmetries of the basic on-shell degrees of freedom of the theory. More recently all the form fields, that is those fields whose indices are totally anti-symmetrized, resulting from the dimensional reduction of the 11-dimensional nonlinear realization were calculated [16]. It emerged that the formulation of the maximal supergravities in the lower dimension $D$ which arose from $E_{11}$ was democratic meaning that the physical degrees of freedom of the theory were described by two fields whose field strengths were related by Hodge duality, except in the case of self-dual fields. Furthermore, the rank $(D-1)$-forms that arose could be used to classify all possible deformations of the maximal supergravity theories that arise from a Lagrangian formulation and we found that the results were in complete agreement with the known maximal gauged supergravity theories. The latter have been found over many years by considering the deformations of the massless maximal supergravity theory in the dimension of interest (see, for example, [17] and references therein). This result also provided an 11-dimensional origin for all the gauged, or massive, maximal supergravity theories. All the space-filling forms were also found in [16] which are important for the consistency of orientifold models [18]. The results of [16] were also found in [19] by analysing the $E_{11}$ theory directly in the dimension of interest.

Arguments similar to those advocated for 11-dimensional supergravity in [7] were applied to the effective action of the closed bosonic string $D$ dimensions [7], to gravity in $D$ dimensions [20], and the type I supergravity theory [21] and the underlying Kac–Moody algebras were
identified. It was realized that the algebras that arose in all these theories were of a special kind and were called very extended Kac–Moody algebras [22]. Indeed, for any finite-dimensional semi-simple Lie algebra \( \mathfrak{g} \) one can systematically extend its Dynkin diagram by adding three more nodes to obtain an indefinite Kac–Moody algebra denoted by \( \mathfrak{g}^{+++} \). In this notation \( E_{11}^8 \) is written as \( E_{11}^{+++} \). The Kac–Moody algebras that were conjectured to underlie the closed bosonic string, gravity and type I supergravity being \( D_{10}^{+++} \) [7], \( A_{17}^{+++} \) [20] and \( D_8^{+++} \) [21], respectively.

The ideas in [7, 20–22] were generalized in [23, 24] to consider the nonlinear realization of any \( \mathfrak{g}^{+++} \) algebra. For each very extended algebra \( \mathfrak{g}^{+++} \) one can find an \( A_{D-1} \) sub-algebra that is associated with a set of \( D-1 \) nodes which form a line in the \( \mathfrak{g}^{+++} \) Dynkin diagram. This line is called the gravity line and it must start with the very extended node. In the nonlinear realization the \( A_{D-1} \) sub-algebra is associated with the gravity sector of the theory and the resulting theory lives in \( D \) dimensions. In general there is more than one possible gravity line or \( A_{D-1} \) sub-algebra. By analysing \( \mathfrak{g}^{+++} \) with respect to the \( A_{D-1} \) sub-algebra it was shown [9] that the low level fields present in the nonlinear realization of \( \mathfrak{g}^{+++} \) contained gravity, form gauge fields and in some cases scalars which was the result anticipated by the above conjectures. This was also consistent with the oxidation points [24] of all the three-dimensional theories which possessed coset symmetries.

In this paper, we investigate if the \( \mathfrak{g}^{+++} \) algebras have some of the same features described above for the \( E_8^{+++} \) algebra. In particular, we will see that if the gravity sub-algebra of \( \mathfrak{g}^{+++} \) being considered is \( A_{D-1} \) then we can define dual generators to be those that have no blocks of anti-symmetrized \( D \) or \( D-1 \) indices. We will then find all dual generators and show that they are only those corresponding to the on-shell degrees of freedom of the theory together with those fields whose field strengths are related to these by Hodge duality, as well as generators that have the same index structure as these fields, but are decorated by any number of blocks of \( D-2 \) indices. We will show this result for all \( \mathfrak{g}^{+++} \) algebras with the exception of the \( B \) and \( C \) algebras which we do not consider. This result implies that the nonlinear realization encodes all possible duality symmetries of the on-shell degrees of freedom of these theories.

We also derive the form fields of the \( G_2^{+++} \) theory in five, four and three dimensions and discuss the possible resulting deformations of the theory. The results are completely compatible with the known literature on the subject. In two separate appendices, we also derive the form fields arising in any dimension from 8 to 4 for the \( E_6^{+++} \) theory and from 6 to 4 for the \( F_4^{+++} \) theory, while a third appendix is devoted to a discussion of the \( G_2^{+++} \) case.
of the properties of the Kac–Moody algebra are determined in terms of the $A_n$ sub-algebra [22, 25, 26] and recall how it works in detail for the case of $E_{11}$ [26]. We will then recover the dual generators of [15], but also give some new insights into how to determine the remaining generators of the algebra.

For an algebra where only one node, labelled by $c$, needs to be deleted to find an $A_n$ gravity sub-algebra, the simple roots of the Kac–Moody algebra can be written in terms of those of $A_n$, that is $\alpha_i, i = 1, \ldots, n$, and the simple root corresponding to the deleted node which can be written as [22]

$$\alpha_c = x - \nu$$  \hspace{1cm} (2.1)

where $\nu$ is given by

$$\nu = -\sum_i A_{ci} \lambda_i, \hspace{1cm} (2.2)$$

$x$ is a vector orthogonal to the $A_n$ weight lattice and $\lambda_i$ are the fundamental weights of the $A_n$ sub-algebra. Indices $a, b, \ldots$ run over the rank of the full Kac–Moody algebra, while $i, j, \ldots$ over the rank of the $A_n$ sub-algebra. We note that the simple roots do indeed replicate the Cartan matrix $A_{ab}$ of the Kac–Moody algebra, which if this is symmetric are given by $(\alpha_a, \alpha_b) = A_{ab}$. Here we used that $(\alpha_i, \lambda_j) = \delta_{ij}$ for the $A_n$ sub-algebra as indeed is the case for all simply laced finite-dimensional semi-simple Lie algebras. The quantity $x^2$ is determined by demanding that $\alpha_c^2$ have the correct value, which for the case of symmetric Cartan matrix is just $\alpha_c^2 = 2$.

Any root of the Kac–Moody algebra can be written, using equation (2.2), in the form

$$\alpha = \sum_i n_i \alpha_i + l \alpha_c = lx - \Lambda,$$

where $\Lambda = v - \sum_i n_i \alpha_i$.  \hspace{1cm} (2.3)

We recognize $l$ as the level. We see that $\Lambda$ belongs to the weight lattice of the $A_n$ sub-algebra. If a representation of $A_n$, with highest weight $\sum_i p_i \lambda_i$, where $p_j$ are the Dynkin indices, occurs then this highest weight must occur as one of the possible $\Lambda$`s as the roots of the Kac–Moody algebra vary. As such, a necessary condition for the adjoint representation of the generalized Kac–Moody algebra to contain the highest weight representation of $A_n$ with Dynkin indices $p_j$ [22, 25, 26] is that

$$\sum_j p_j \lambda_j = lv - \sum_i n_i \alpha_i, \hspace{1cm} (2.4)$$

Taking the scalar product with $\lambda_k$ leads to the condition

$$n_k = lv \cdot \lambda_k - \sum_j p_j (\lambda_j, \lambda_k). \hspace{1cm} (2.5)$$

We recall that $(\lambda_i, \lambda_j) = A^{-1}_{ij}$ for any simply laced algebra, where $A^{-1}_{ij}$ is the inverse Cartan matrix which is positive definite for a finite-dimensional semi-simple Lie algebra. The inverse Cartan matrix of the $A_{D-1}$ algebra is given by

$$A^{-1}_{ij} = \begin{cases} i(D - j), & i \leq j \\ D, & i = j \\ j(D - i), & j \leq i \end{cases} \hspace{1cm} (2.6)$$

In equation (2.5) the integers $l, n_k, p_j$ must be positive and so this places a necessary, but not sufficient, condition on the possible $A_n$ representations contained in the adjoint representation of the Kac–Moody algebra at level $l$. 

Taking the scalar product of the expression for \( \alpha \) of equation (2.3) we find that
\[
\alpha^2 = l^2 x^2 + \sum_{i,j} p_i(\lambda_i, \lambda_j) p_j = 2, 0, -2, \tag{2.7}
\]
We have used the fact that the lengths of the roots of a symmetric Kac–Moody algebra are constrained to take the values 2, 0, -2, ... [27].

Thus we find a second necessary, but not sufficient, constraint on the \( A_n \) representations contained in the adjoint representation of the Kac–Moody algebra. In fact the two constraints of equations (2.5) and (2.7) are not as strong as imposing the Serre relations on the multiple commutators, although almost all the solutions they possess are roots that actually appear in the Kac–Moody algebra.

Let us explain this procedure for the case of \( E_{11} \) [26] whose Dynkin diagram is given below

\[
\begin{array}{cccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
\]

Deleting the node 11 leaves an \( A_{10} \) sub-algebra. The simple roots of \( E_{11} \) are those of \( A_{10} \), i.e. \( \alpha_i, i = 1, \ldots, 10 \), as well as the simple root of the deleted node which is given by
\[
\alpha_{11} = x - \lambda_8, \quad x^2 = -\frac{2}{11}. \tag{2.8}
\]
The general root of \( E_{11} \) has the form
\[
\alpha = l\alpha_{11} + \sum_{i=1}^{10} n_i \alpha_i = lx - \Lambda \tag{2.9}
\]
where \( \Lambda = l\lambda_8 - \sum_{i=1}^{10} n_i \alpha_i \). This equation effectively rewrites the adjoint representation of \( E_{11} \) in terms of representations of \( A_{10} \). A necessary condition for a representation of \( A_{10} \) to occur is that the set of all \( \Lambda \)'s contain the highest weight of the representation in question, that is \( \sum_i p_i \lambda_i \), where \( p_i \) are the Dynkin indices. As such, we can set \( \Lambda = \sum_i p_i \lambda_i \) and taking the scalar product with \( \lambda_j \) we find that
\[
\sum_i p_i \lambda_i \cdot \lambda_j - l\lambda_8 \cdot \lambda_j = -n_j. \tag{2.10}
\]
While the square of the corresponding \( E_{11} \) root is given by equation (2.7), that is
\[
\alpha^2 = -\frac{2}{11} x^2 + \sum_{i,j} p_i(\lambda_i, \lambda_j) p_j = 2, 0, -2, \ldots. \tag{2.11}
\]
Any generator is found by taking the multiple commutator of the Chevalley generators and the level of a generator is the number of times the Chevalley generator corresponding to node 11 appears in this multiple commutator. The Chevalley generator associated with node 11 has three \( A_{10} \) indices, hence it adds three indices every time it appears in the multiple commutator. Therefore, any level \( l \) generator can be written with \( 3l \) indices. We note that the remaining Chevalley generators are contained in the \( K^{a,b} \) generators of \( A_{10} \) and so these do not change the number of indices. Given a generator with Dynkin index \( p_i \), the index contributes \( p_j \) blocks of \( 11 - j \) anti-symmetrized indices; as a result the total number of indices obeys the relation [15]
\[
3l = \sum_i (11 - i) p_i + 11 m. \tag{2.12}
\]
The last term corresponds to the possible presence of \( m \) blocks of fully anti-symmetrized rank 11 indices. Such blocks do not transform under the \( A_{10} \) sub-algebra.

If we substitute the level condition of equation (2.12) into the root length condition of equation (2.11) we find that

\[
\alpha^2 = \frac{1}{9} \sum_{j=1}^{10} (11 - j)(j - 2)p_j^2 + \frac{2}{9} \sum_{i<j} (11 - j)(i - 2)p_i p_j - \frac{4}{9} m \sum_i (11 - i)p_i - \frac{2.11}{9} m^2 \\
= 2, 0, -2, \ldots.
\]  (2.13)

Substituting the level condition of equation (2.12) into the root condition of equation (2.10) we find that the factors of \( \frac{1}{11} \) disappear and the values of \( n_i \) are given by

\[
n_j = \sum_{i,i<j} (j - i)p_i + jm, \quad j = 1, \ldots, 8, \\
n_9 = \frac{2}{3} \left( \sum_i (8 - i)p_i + 8m \right) + p_{10}, \\
n_{10} = \frac{1}{3} \left( \sum_i (8 - i)p_i + 8m \right).
\]  (2.14)

We see that \( n_i, i = 1, \ldots, 9 \), are positive as they must be. Furthermore, using the level condition of equation (2.12) one sees that \( \sum_j (8 - j)p_j + 8m \) is a multiple of 3 as \( 3(\sum_j p_j - s) = \sum_j (8 - j)p_j + 8m \) and so \( n_9 \) and \( n_{10} \) are integers which one can also show are positive.

Hence, we find the perhaps surprising result that the root condition of equation (2.10) is automatically satisfied if we use the level matching condition of equation (2.12). We recall that previously one found the possible roots of \( E_{11} \) by finding all the solution of equations (2.10) and (2.11). However, now we need only solve the level matching condition of equation (2.12) and the reformulation of the length squared condition of equation (2.13). Clearly, these are much simpler conditions that those of equations (2.10) and (2.11).

In [15] the concept of dual generators was introduced; these are generators which possess with no blocks of rank 10 or 11 totally anti-symmetrized indices. This means that \( m = 0 = p_1 \) and so now \( 3l = \sum_j (11 - j) \). We observe that in the root length condition of equation (2.13) all terms on the right-hand side become positive and that as \( p_2 \) is absent this Dynkin index has no condition placed on it. Taking \( p_2 = 0 \), the only allowed roots are given by

\[
\alpha_A = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \quad p_8 = 1 \\
\alpha_B = (0, 0, 0, 0, 0, 1, 2, 3, 2, 1, 2), \quad p_5 = 1 \\
\alpha_C = (0, 0, 0, 1, 2, 3, 4, 5, 3, 1, 3), \quad p_3 = p_{10} = 1.
\]  (2.15)

The other \( p_j \) are 0. These correspond in the nonlinear realization to the 3-form, 6-form and dual graviton generators, respectively, and as expected they occur with multiplicity one. The solution for \( p_2 = 1 \) is given by

\[
\gamma = (0, 0, 1, 2, 3, 4, 5, 6, 4, 2, 3), \quad p_2 = 1.
\]  (2.16)

This solution has multiplicity zero and so does not actually occur in the \( E_{11} \) algebra. It corresponds to a generator of the form \( R^{a_1 \ldots a_n} \).

The roots corresponding to all the dual generators can then be written in the form [15]

\[
\alpha_A(p_2) = \alpha_A + p_2 \gamma \quad \alpha_B(p_2) = \alpha_B + p_2 \gamma \quad \alpha_C(p_2) = \alpha_C + p_2 \gamma \quad \alpha_D(p_2) = p_2 \gamma.
\]  (2.17)
They are just found by taking multiple commutators of the multiplicity generator $R^{a_1 \ldots a_9}$ with each of the generators corresponding to the roots of equation (2.15).

The new fields found in the nonlinear realization corresponding to the roots of equation (2.17) differ from those of equation (2.15) by blocks of nine indices. However, as the little group of the massless states is $SO(9)$ these fields describe the same on-shell states. As a result, we find that the $E_{11}$ nonlinear realization includes all possible ways of describing the original degree of freedom of the theory and so we may conclude that $E_{11}$ encodes all possible duality transformations [15].

We now apply the above arguments to the other very extended algebras.

3. $A_{D-3}^{+++}$

It has been conjectured that a suitably extended version of pure gravity in $D$ dimensions can be expressed as a nonlinear realization of $A_{D-3}^{+++}$ [20]. Analysing this algebra with respect to its $A_{D-1}$ sub-algebra we find at level 0 the $A_{D-1}$ generators $K^{a_i}$, corresponding to the graviton, and at level 1 the generator $R^{a_1 \ldots a_2 , a_3}$ corresponding to the dual graviton field. The Dynkin diagram of $A_{D-3}^{+++}$ is given by

$$
\begin{array}{cccccccc}
D & - & - & 0 & - & - \\
| & | & | & | & | & | & |

0 & - & 0 & - & 0 & 0 & \cdots & 0 & - & 0 \\
1 & 2 & 3 & & & & & & & D & -1.
\end{array}
$$

For simplicity we will assume that $D > 4$. Deleting the node labelled $D$ we find the $A_{D-1}$ gravity sub-algebra. The simple roots of $A_{D-3}^{+++}$ can be written as the simple roots $\alpha_i, i = 1, \ldots, D - 1$ of $A_{D-1}$ as well as the simple root of the deleted node which is given by

$$
\alpha_D = x - \lambda_3 - \lambda_{D-1}.
$$

Given $\alpha^2 = 2$ we find $x^2 = \frac{4}{D} - 2$. As such, a general root of $A_{D-3}^{+++}$ can be written as

$$
\alpha = \sum_i n_i \alpha_i + l\alpha_D = lx - \Lambda,
$$

where $\Lambda = l(\lambda_3 + \lambda_{D-1}) - \sum_i n_i \alpha_i.

A highest weight representation of $A_{D-1}$ with Dynkin indices $p_i$ can occur if $\Lambda = \sum_i p_i \lambda_i$ occurs. Taking the scalar product of this equation with $\lambda_k$ gives an equation for the root components

$$
n_k = l((\lambda_3 , \lambda_k) + (\lambda_k , \lambda_{D-1})) - \sum_j p_j (\lambda_j , \lambda_k).
$$

Squaring the expression for $\alpha$ in (2.2) gives

$$
\alpha^2 = -\frac{2(D-2)}{D} l^2 + \sum_{i,j} p_i (\lambda_i , \lambda_j) p_j = 2, 0, -2, \ldots
$$

The Chevalley generator corresponding to the deleted node $D$ has a block of $D - 3$ anti-symmetrized $A_{D-1}$ indices, and a vector index, i.e. $R^{a_1 \ldots a_{D-1}, b}$, giving $D - 2$ indices overall. A level $l$ generator, the multiple commutator of which contains the generator corresponding to the deleted node $l$ times, will therefore have $(D - 2)l A_{D-1}$ indices in total. Hence, we get

$$
\sum_j p_j (D - j) + Dm = (D - 2)l
$$

(3.5)
where \( m \) is the number of blocks of \( D \) anti-symmetrized indices. Substituting this expression for \( l \) into (3.4) we find that

\[
\alpha^2 = \frac{1}{(D-2)} \left( \sum_j p_j^2(D-j)(j-2) + 2 \sum_{i<j} p_i p_j(D-j)(i-2) \right)
- \frac{4m}{(D-2)} \sum_i (D-i) p_i - \frac{2m^2D}{(D-2)} = 2, 0, -2, \ldots
\]  

(3.6)

We now show that any generator that satisfies (3.5) automatically satisfies the root condition (3.3), namely that it gives positive integer values for \( n_k \). Substituting (3.5) into (3.3) we find that the factors of \( \frac{1}{D} \) disappear and it gives

\[
n_j = \begin{cases} 
  p_{j-1} + jm, & j = 1, 2, 3 \\
  \sum_{i,i<j} (j-i) p_i + jm - (j-3)l, & j = 3, \ldots, D-1
\end{cases}
\]

(3.7)

where we have formally taken \( p_0 = 0 \).

The definition of a dual generator is that it has no blocks \( D \) or \( D-1 \) totally antisymmetric \( A_{D-1} \) indices. This may be written as \( p_1 = m = 0 \) and so the level matching condition becomes

\[
\sum_j p_j(D-j) = (D-2)l,
\]

(3.8)

while equation (3.6) becomes

\[
\alpha^2 = \frac{1}{(D-2)} \left( \sum_j p_j^2(D-j)(j-2) + 2 \sum_{i<j} p_i p_j(D-j)(i-2) \right) = 2, 0, -2, \ldots
\]

(3.9)

Two things are immediately obvious from this equation; \( p_2 \) is unconstrained, and both terms are positive definite, so \( \alpha^2 \) can only be 2 or 0. For \( \alpha^2 = 0 \), both terms must be 0, so only \( p_2 \) may be non-zero. Equation (3.5) then implies that \( l = p_2 \) and equation (3.7) gives the first \( D-1 \) coefficients \((n_1, \ldots, n_{D-1})\) of the root, while the last coefficient of the root is simply the level, \( n_D = l = p_2 \). Defining \( \gamma \) to correspond to the coefficients \( n_j = (0, 0, 1, \ldots, 1, \ldots, 1) \) gives the roots corresponding to all \( \alpha^2 = 0 \) dual generators in terms of the free parameter \( p_2 \) by

\[
\alpha_B(p_2) = p_2 \gamma.
\]

(3.10)

In fact, the generator with \( p_2 = 1 \) has multiplicity zero and so does not appear in the algebra.

The remaining solutions have \( \alpha^2 = 2 \). We note that \( p_3 = p_{D-1} = 1 \), with \( p_2 = 0 \), and all other \( p_i = 0 \) give a solution with \( \alpha^2 = 2 \). Using equation (3.7) we find that the corresponding root is given by

\[
\alpha_A = (0, 0, \ldots, 0, 1), \quad p_3 = 1 = p_{D-1}.
\]

(3.11)

This corresponds to the generator \( R^{a_1 \ldots a_{D-1} b} \) which has multiplicity one and so up to this level the theory contains gravity and the dual graviton [9, 20].

Since \( p_2 \) does not appear in equation (3.6) given any solution we may find a whole class of solutions that have the same Dynkin indices as before, but with the addition of a \( p_2 \) that can be any positive integer. Adding such a \( p_2 \) leads to a root also satisfies the level matching condition of equation (3.8) as \( p_2 \to p_2 + 1 \) just change \( l \to l + 1 \). We recall that equation (3.3)
is automatically satisfied if the level matching condition holds. Applying this to the solution of equation (3.11) we have new solutions which can be written in the form

\[ \alpha_A(p_2) = \alpha_A + p_2 \gamma. \]  
(3.12)

This line of argument will apply to all the \(G^{+++}\) algebras considered in this paper.

In fact, all possible dual generators are given in equations (3.10) and (3.12) and summarizing we find that all the dual generators present in the \(A_{D-3}^{+++}\) algebras may be written as

\[ \alpha_A(p_2) = \alpha_A + p_2 \gamma \quad \alpha_B(p_2) = p_2 \gamma. \]  
(3.13)

They encode all possible ways of describing the on-shell degrees of freedom of the theory which are just those of gravity. As such the dual fields encode all possible duality transformations.

We now explain why there are no other solutions. The contribution from a single \(p_i\) to \(\alpha^2\) is given by

\[ \alpha^2 = p_i^2 \left( (i - 2) - \frac{(i - 2)^2}{(D - 2)} \right). \]  
(3.14)

Let us suppose that a single \(p_i\) is non-zero and takes the value \(p_i = r\). The level matching condition of equation (3.8) becomes \((D - 2)l = (D - i)r\) which in turn implies that \((D - 2)(l - r) = -(i - 2)r\). Using this latter condition and that \(\alpha^2 = 2\) we find that \(2 = rl(i - 2)\) which has no solution that is compatible with the level matching condition.

Now let us suppose that two Dynkin indices are non-zero, i.e. \(p_i = 1 = p_j\) with \(i < j\). In this case, the contribution to \(\alpha^2\) is given by

\[ \alpha^2 = 3(i - 2) + (j - 2) - \frac{(i - j)^2}{(D - 2)}. \]  
(3.15)

However, since \(j - i < D - 3\) we find that this implies that \(\alpha^2 > 2i - 8\) and since \(\alpha^2 = 2\) we conclude that \(i = 3\) is the only allowed value. Thus we only find the one solution of equation (3.11).

4. \(E_7^{+++}\)

The nonlinearily realized \(E_7^{+++}\) theory contains at low levels either a ten-dimensional truncation of IIB supergravity or an eight-dimensional truncation of maximal \(D = 8\) supergravity, depending on the choice of gravity line [9]. Here we will only consider the ten-dimensional theory. The Dynkin diagram of \(E_7^{+++}\) is given by

\[
\begin{array}{cccccccccc}
0 & 10 \\
| \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9.
\end{array}
\]

The node labelled 3 is the affine node with nodes 1 and 2 being the over and very extended nodes, respectively.

In the case of \(E_7^{+++}\) node 10 is deleted to leave an \(A_9\) sub-algebra. The simple root corresponding to node 10 can be written as

\[ \alpha_{10} = x - \lambda_6 \]  
(4.1)
where $\lambda_6$ is a fundamental weight of the $A_9$ sub-algebra. Given that $\alpha_{10}^2 = 2$ we find $x^2 = -2/5$. A general root of $E_7^{++}$ can be written in terms of the simple roots

$$\alpha = \sum_{i=1}^{9} n_i \alpha_i + l \alpha_{10} = lx - \Lambda,$$

where $\Lambda = l \lambda_6 - \sum_i n_i \alpha_i$. (4.2)

A highest weight representation of $A_{D-1}$ with Dynkin indices $p_j$ can occur if we can find amongst the roots of $E_7^{++}$ one such that $\Lambda = \sum_j p_j \lambda_j$. Combining this with (4.2) and taking the scalar product with $\lambda_i$ implies that

$$n_k = l (\lambda_6, \lambda_k) - \sum_j p_j (\lambda_j, \lambda_k).$$ (4.3)

Squaring the root $\alpha$ of (4.2) and applying the knowledge of the known lengths of roots in simply laced Kac–Moody algebras one finds

$$\alpha^2 = -\frac{2}{5} l^2 + \sum_{i,j} p_i (\lambda_i, \lambda_j) p_j = 2, 0, -2, \ldots.$$ (4.4)

The Chevalley generator corresponding to the deleted node 10 has 4 indices and so a generator with level $l$ will have $4lA_{D-1}$ indices. Any block of ten fully anti-symmetrized indices will not transform under the $A_9$ sub-algebra. Using $m$ to denote the number of such blocks we may write

$$\sum_{j=2}^{9} p_j (10 - j) + 10m = 4l.$$ (4.5)

Substituting this into (4.4) gives

$$\alpha^2 = \frac{1}{8} \left[ \sum_{j=1}^{9} p_j^2 (10 - j)(j - 2) + 2 \sum_{i<j} p_i p_j (10 - j)(i - 2) \right]
- \frac{1}{2} m \sum_{j=1}^{9} (10 - j) p_j - \frac{5}{2} m^2 = 2, 0, -2, \ldots.$$ (4.6)

We now show that any generator that satisfies (4.5) will automatically satisfy the root condition (4.3). In particular, substituting (4.5) into (4.3) we find that the coefficients of the roots are given by

$$n_j = \sum_{i=j}^{9} p_i (j - i) + jm, \quad j = 1, \ldots, 6, \quad n_7 = \frac{3}{4} \left( \sum_{i=1}^{9} (6 - i) p_i + 6m \right) + 2p_9 + p_8,$$

$$n_8 = \frac{2}{4} \left( \sum_{i=1}^{9} (6 - i) p_i + 6m \right) + p_9, \quad n_9 = \frac{1}{4} \left( \sum_{i=1}^{9} (6 - i) p_i + 6m \right).$$ (4.7)

We note that $4l - 4 \sum_j p_j - 4m = \sum_{i=1}^{9} (6 - i) p_i + 6m$. Also, $n_j$ are all integers as required.

Dual generators are defined to be those with no blocks of nine or ten fully anti-symmetrized indices. This condition may be written as

$$p_1 = m = 0.$$ (4.8)

Substituting this into (4.6) gives

$$\alpha^2 = \frac{1}{8} \left[ \sum_{j}^{9} p_j^2 (10 - j)(j - 2) + 2 \sum_{i<j} p_i p_j (10 - j)(i - 2) \right] = 2, 0, -2, \ldots.$$ (4.9)
This equation is independent of $p_2$ and its terms are positive. Setting $p_2 = 0$ we find that the only solutions are given by

$$
\begin{align*}
\alpha_A &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 1), & p_6 &= 1 \\
\alpha_B &= (0, 0, 0, 1, 2, 3, 2, 1, 0, 2), & p_3 &= p_9 = 1
\end{align*}
$$

(4.10)

where all other $p_i = 0$. Solutions $A$ and $B$ correspond to the generators $R^{a_1...a_2}$ and $R^{a_1...a_1,b}$ respectively. These have multiplicity one. The resulting low level field content in the nonlinear realization is a self-dual 4-form $A_{a_1,...,a_7}$ and the dual graviton $A_{a_1,...,a_7,b}$ [9].

Setting $p_2 = 1$ we find the solution

$$
\gamma = (0, 0, 1, 2, 3, 4, 3, 2, 1, 2).
$$

(4.11)

The generator corresponding to this root has 0 multiplicity and so it does not appear in the algebra.

The roots corresponding to all possible dual generators in $E_7^{+++}$ can then be written in the form

$$
\begin{align*}
\alpha_A(p_2) &= \alpha_B + p_2 \gamma & \alpha_B(p_2) &= \alpha_C + p_2 \gamma & \alpha_C(p_2) &= p_2 \gamma
\end{align*}
$$

(4.12)

As in all the other cases, this corresponds to the presence of all possible duality transformations.

5. $D_{D-2}^{+++}$

The nonlinear realization of the very extended $D_{D-2}^{+++}$ algebras was conjectured as a symmetry of a suitably extended low energy effective action of the bosonic string in $D$ dimensions [20]. The Dynkin diagram of $D_{D-2}^{+++}$ is given by

$$
\begin{array}{cccccccc}
0 & D + 1 & & & & & & \\
\mid & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & D - 3 & D - 2 & D - 1.
\end{array}
$$

For simplicity we consider $D \geq 7$; the Dynkin diagram has a slightly different structure for smaller $D$. For the $D_{D-2}^{+++}$ series of algebras two nodes must be deleted to give an $A_{D-1}$ algebra. The simple roots of $D_{D-2}^{+++}$ are the simple roots of the $D_D$ algebra and the simple root of the deleted node $D + 1$ which may be written as

$$
\alpha_{D+1} = \gamma - l_4
$$

(5.1)

where $l_4$ denotes a fundamental weight of the $D_D$ algebra. Next we delete node $D$ to find an $A_{D-1}$ gravity sub-algebra. The simple roots of $D_{D-2}^{+++}$ are now given by the simple roots of the $A_{D-1}$ sub-algebra, the simple root of node $D$, which we may write as

$$
\alpha_D = x - \lambda_{D-2},
$$

(5.2)

and the simple root of equation (5.1). However, we may express the fundamental weight $l_4$ of $D_D$ in terms of the fundamental weights of $A_{D-1}$ by [22]

$$
l_4 = \lambda_4 + \frac{x}{x^2}(\lambda_{D-2}, \lambda_4).
$$

(5.3)

Noting that $\alpha_{D+1}^2 = \alpha_D^2 = 2$ we find that $x^2 = \frac{4}{D}, y^2 = -2$.

A general root of $D_{D-2}^{+++}$ can be written as

$$
\alpha = \sum_i n_i \alpha_i + l_x \alpha_D + l_y \alpha_{D+1} = l_y y + x (l_x - 2l_y) - \Lambda,
$$

where

$$
\Lambda = l_y \lambda_4 + l_x \lambda_{D-2} - \sum_i n_i \alpha_i
$$

(5.4)
where there are now two levels, $l_x$ and $l_y$, which refer to the deleted nodes $D$ and $D + 1$, respectively. A highest weight representation of $A_{D-1}$ with Dynkin indices $p_i$ can occur if the roots of the Kac–Moody algebra $D^{++}_{D-2}$ include the case when

$$\Lambda_1 = \sum p_i \lambda_i.$$ 

Taking the scalar product of both sides with $\lambda_k$ gives

$$n_k = l_x(\lambda_4, \lambda_k) + l_y(\lambda_{D-2}, \lambda_k) - \sum_j p_j(\lambda_j, \lambda_k). \quad (5.5)$$

The square of the corresponding root is given by

$$\alpha^2 = -2l_x^2 + \frac{4}{D}(l_x - 2l_y)^2 + \sum_{i,j} p_i(\lambda_i, \lambda_j)p_j = 2, 0, -2, \ldots \quad (5.6)$$

where $\alpha^2$ has been constrained to take the values given above as it belongs to a Kac–Moody algebra [27].

The Chevalley generator corresponding to node $D$ has $2A_{D-1}$ indices and the Chevalley generator corresponding to node $D + 1$ has $(D - 4)$ indices. By definition the multiple commutator of a generator with level $(l_x, l_y)$ contains the Chevalley generator corresponding to node $D l_x$ times, adding $2l_x$ indices, and the Chevalley generator corresponding to node $D + 1 l_y$ times, adding $(D - 4)l_y$ indices. In total this gives $2l_x + (D - 4)l_y$ indices. Using $m$ to denote the number of rank $D$ index blocks, we may write

$$\sum_j p_j(D - j) + Dm = (D - 4)l_y + 2l_x. \quad (5.7)$$

Substituting this into (5.6) gives

$$\alpha^2 = \frac{1}{D - 2} \left[ 4d^2 + \sum_i p_i^2(D - 2)(D - i) + 2 \sum_{i<j} p_i p_j(D - j)(i - 2) \right]$$

$$- \frac{4m}{(D - 2)} \sum_i p_i(D - i) - \frac{2m^2}{(D - 2)} = 2, 0, -2, \ldots \quad (5.8)$$

In deriving this equation we have used the identity

$$-Dl_x^2 + 2(l_x - 2l_y)^2 = \frac{1}{D - 2} \left[ 2D(l_x - l_y)^2 - ((D - 4)l_y + 2l_x)^2 \right] \quad (5.9)$$

where $d = l_x - l_y$.

In fact, any solution that satisfies equation (5.7) will automatically satisfy the root condition of equation (4.50) and one finds that

$$n_j = \begin{cases} 
\sum_{i,j} p_i(j - i) + mj, & j = 1, 2, 3, 4 \\
\sum_{i,j} p_i(j - i) + mj + l_y(j - 4), & j = 4, \ldots, D - 1 
\end{cases} \quad (5.10)$$

$$n_{D-1} = l_y + l_x - \sum_i p_i - m.$$ 

Dual generators are defined to be those with no blocks of $D$ or $D-1$ fully anti-symmetrized indices. This may be written as $p_1 = m = 0$ which when substituting this into equation (5.8) gives

$$\sum_j p_j(D - j) = (D - 4)l_y + 2l_x. \quad (5.11)$$
Equation (5.8) then becomes
\[
\alpha^2 = \frac{1}{D-2} \left[ 4d^2 + \sum_j p_j^2 (D-2)(j-2) + 2 \sum_{i<j} p_i p_j (D-2)(i-2) \right] = 2, 0, -2, \ldots 
\]
(5.12)

This equation is independent of \(p_2\) and all terms in the middle equation are positive definite. The general solution to this equation can be found following similar arguments to that deployed for the case of \(A^{++}_{D-3}\), in particular below equation (3.14).

Setting \(p_2 = 0\) we find the following roots:
\[
\begin{align*}
\alpha_A &= (0, \ldots, 0, 1, 0), & p_{D-2} &= 1, \\
\alpha_B &= (0, \ldots, 0, 1), & p_4 &= 1, \\
\alpha_C &= (0, 0, 0, 1, 0, 1, 1), & p_3 &= p_{D-1},
\end{align*}
\]
(5.13)

All other \(p_i\)'s are 0. The generators corresponding to the roots \(A, B\) and \(C\) are \(R^{a_1 a_2}\), \(R^{a_1 \ldots a_{D-4}}\) and \(R^{a_1 a_2 a_3 b}\), respectively. These all have multiplicity one.

Setting \(p_2 = 1\) we also find the solution
\[
\gamma = (0, 0, 1, 2, \ldots, 2, 1, 1, 1)
\]
(5.14)
corresponding to the generator \(R^{a_1 \ldots a_{D-2}}\) which has multiplicity one.

As a result, the nonlinear realization contains at low levels the fields of gravity and a dilaton, as the rank of \(D^{++}_{D-2}\) is one greater than that of \(D\) the dimension of spacetime, a 2-form \(A_{a_1 a_2}\), its dual \(A_{a_1 \ldots a_{D-4}}\), the dual graviton \(A_{a_1 \ldots a_{D-3} b}\) and the field \(A_{a_1 a_2 a_3 b}\), which is the dual of the dilaton. These are the on-shell states of the effective action of the bosonic string generalized to \(D\) dimensions [9, 20].

The roots corresponding to all dual generators may be written in the form
\[
\begin{align*}
\alpha_A(p_2) &= \alpha_A + p_2 \gamma, & \alpha_B(p_2) &= \alpha_B + p_2 \gamma, \\
\alpha_C(p_2) &= \alpha_C + p_2 \gamma, & \alpha_D(p_2) &= p_2 \gamma.
\end{align*}
\]
(5.15)

6. \(E_6^{++}\)

At low levels the nonlinear realization of the very extended \(E_6\) algebra has precisely the field content [9] of the oxidation endpoint of the non-supersymmetric \(E_6\) coset theory described in [28]. The Dynkin diagram of \(E_6^{++}\) is given by
\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
0 & 9 & & & & & & \\
| & | & & & & & & \\
0 & 8 & & & & & & \\
| & | & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
\end{array}
\]

As usual nodes 1, 2 and 3 are the very extended, over extended and affine nodes. In the case of \(E_6^{++}\) node 8 is deleted leaving the algebra \(A_7 \otimes A_1\). This decomposition is similar to the decomposition of \(E_11\) which is appropriate to IIB supergravity [8, 9]. The roots of \(E_6^{++}\) can be written as the roots of \(A_7, a_i, i = 1, \ldots, 7\), the root of the \(A_1\) algebra \(\beta\) and the deleted root \(\alpha_8\), which may be written as
\[
\alpha_8 = x - \lambda_5 - \mu
\]
(6.1)
where \( \lambda_5 \) is a fundamental weight of the \( A_7 \) algebra and \( \mu \) is the fundamental weight of the \( A_1 \) sub-algebra. Knowing \( \alpha_8^2 = 2 \), we find that \( x^2 = -\frac{3}{8} \). A general root of \( E_6^{+++} \) may be written as

\[
\alpha = \sum_{i=1}^{7} n_i \lambda_i + l \alpha_8 + r \beta = lx - \Lambda \quad \text{where} \quad \Lambda = l \lambda_5 + l \mu - \sum_{i=1}^{7} n_i \lambda_i - r \beta. \tag{6.2}
\]

A highest weight representation of \( A_7 \otimes A_1 \) with Dynkin indices \( p_i, q \) can occur if \( E_6^{+++} \) include the possibility \( \Lambda = \sum_i p_i \lambda_i + q \mu \). Dotting this with \( \lambda_k \) and \( \mu \) in turn gives the pair of equations

\[
n_k = l (\lambda_5, \lambda_k) - \sum_{j=1}^{7} p_j (\lambda_j, \lambda_k) \quad \text{and} \quad r = \frac{l - q}{2}.
\tag{6.3}
\]

Note that \( (\mu, \mu) = 1/2 \) and \( (\beta, \mu) = 1 \). Squaring the expression for \( \alpha \) in (6.2) gives

\[
\alpha^2 = -3/8 l^2 + \sum_{i,j} (\lambda_i, \lambda_j) p_i p_j + q^2/2. \tag{6.4}
\]

The Chevalley generator corresponding to node 8 has three \( A_7 \) indices. As a result, a level \( l \) generator, the multiple commutator for which by definition contains the Chevalley generator corresponding to node 8 \( l \) times, will have \( 3l \) \( A_7 \) indices in total. We note that the Chevalley generator corresponding to node 9 has no \( A_7 \) indices, so it does not contribute to the index count. As a result,

\[
\sum_j p_j (8 - j) + 8m = 3l \tag{6.5}
\]

where \( m \) denotes the number of blocks of eight fully anti-symmetrized indices.

Substituting for \( l \) from (6.5) into (6.4) gives

\[
\alpha^2 = \frac{1}{6} \left( \sum_j p_j^2 (8 - j) (j - 2) + 2 \sum_{i < j} p_i p_j (8 - j) (i - 2) - 16 m^2 - 4m \sum_j p_j (8 - j) \right) + \frac{1}{2} q^2
\]

\[
= 2, 0, -2, \ldots. \tag{6.6}
\]

We now show that any generator that satisfies (6.5) automatically satisfies the root condition (6.4). Substituting (6.5) into (6.4) we find

\[
n_j = \begin{cases} 
\sum_{i, i < j} p_i (j - i) + jm, & j \leq 5 \\
6 = \frac{2}{3} \left( \sum_i (5 - i) p_i + 5m \right) + p_7 \\
7 = \frac{1}{3} \left( \sum_i (5 - i) p_i + 5m \right) 
\end{cases} \tag{6.7}
\]

We note that \( 3l - 3 \sum_i p_i - 3m = \left( \sum_i (5 - i) p_i + 5m \right) \) and \( n_j \) are positive integers as required.

Dual generators are defined to be those with no blocks of seven or eight fully anti-symmetrized indices. This may be written as

\[
p_i = m = 0. \tag{6.8}
\]
and above.

In appendix A we derive all the forms arising from dimensional reduction to four dimensions.

\[ \alpha^2 = \frac{q^2}{2} + \frac{1}{6} \sum_j p_j^2 (8 - j)(j - 2) + \frac{1}{3} \sum_{i < j} p_i p_j (8 - j)(i - 2) = 2, 0, -2, \ldots \quad (6.9) \]

The middle terms in this equation are positive definite and \( p_2 \) is undetermined. Setting \( p_2 = 0 \) gives the solutions

\[
\begin{align*}
\alpha_A &= (0, 0, 0, 0, 0, 0, 1, 0), & q &= 1, & p_5 &= 1 \\
\alpha_B &= (0, 0, 0, 1, 2, 1, 0, 2), & q &= 0, & p_3 &= p_7 = 1 \\
\alpha_C &= (0, 0, 0, 0, 0, 0, 0, -1), & q &= 2.
\end{align*}
\]

All other \( p_i \)'s are zero. Generators with \( q = 0, q = 1 \) and \( q = 2 \) are singlets, doublets and triplets of \( SL(2, R) \), respectively. We note that the coefficients of the expression for a general root of equation (6.3) in terms of simple roots corresponding to roots 8 and 9 are \( l \) and \( r \), respectively. The generators corresponding to the roots \( A \) and \( B \) are \( R^{\alpha_{(a)} \beta_{(b)}} \) and \( R^{\alpha_{(a)} \beta} \), respectively, where \( \alpha, \beta = 1, 2 \) indices label the vector representation of \( SL(2, R) \). Root \( C \) corresponds to the generators \( R^{(a)(b)} \) which are none other than the generators of \( A_1 \) arising from node 9. Unlike the other representations of \( A_7 \otimes A_1 \) mentioned above which occur entirely within the negative level root space of \( E_6^{+++} \) with a copy in the positive root space this representation contains positive and negative levels. This is related to the fact that it belongs to the adjoint representation of \( A_7 \otimes A_1 \) with respect to which we are decomposing the \( E_6^{+++} \) algebra. Following the derivation of equation (6.2) and reading [26] one sees that in this case a negative value of \( r \) is allowed.

If \( p_2 = 1 \) we find the two solutions

\[
\begin{align*}
\alpha_D &= (0, 0, 1, 2, 3, 2, 1, 2), & q &= 2 \\
\gamma &= (0, 0, 1, 2, 3, 2, 1, 2), & q &= 0.
\end{align*}
\]

The corresponding generators are \( R^{a_{(a)} \beta_{(b)}} \) and \( R^{a_{(a)} \beta} \) and these have multiplicity one and zero, respectively.

The fields corresponding to the above generators are two scalars \( \phi \) and \( \chi \) that belong to the coset \( SL(2, R) \) with local sub-algebra \( SO(2) \) associated with root \( C \), their duals \( A_{a_{(a)} \beta_{(b)}} \) which are subject to one constraint and are associated with root \( D \), a doublet of 3-forms \( A_{a_{(a)} \beta_{(b)}} \), whose 4-form field strength is self-dual and are associated with root \( B \) and finally the dual graviton \( A_{a_{(a)} \beta} \) associated with root \( B \). This is the content of an eight-dimensional non-supersymmetric theory.

The roots of all dual generators may be written in the form

\[
\begin{align*}
\alpha_A(p_2) &= \alpha_A + p_2 \gamma \quad &\alpha_B(p_2) &= \alpha_B + p_2 \gamma \\
\alpha_C(p_2) &= \alpha_C + p_2 \gamma \quad &\alpha_D(p_2) &= \alpha_D + p_2 \gamma \\
\alpha_E(p_2) &= p_2 \gamma.
\end{align*}
\]

In appendix A we derive all the forms arising from dimensional reduction to four dimensions and above.
7. $G_2^{+++}$

The nonlinear realization of $G_2^{+++}$ contains at low levels the fields of $N = 1$ five-dimensional supergravity [9]. The Dynkin diagram of $G_2^{+++}$ is given by

\[
\begin{array}{c}
0 \\
\| \\
\| \\
0 - 0 - 0 - 0 \\
1 2 3 4
\end{array}
\]

The simple root vectors can be chosen to be

\[
(\alpha_5, \alpha_5) = 2/3, \quad (\alpha_i, \alpha_i) = 2, \quad i = 1, \ldots, 4,
\]

\[
(\alpha_i, \alpha_j) = -1, \quad i, j = 1, \ldots, 4, \quad |i - j| = 1
\]

(7.1)

All other scalar products are 0.

In this case, the node labelled 5 is deleted to leave an $A_4$ gravity sub-algebra. The deleted node can be written as

\[
\alpha_5 = x - \lambda_4.
\]

(7.2)

Given $\alpha_5^2 = 2/3$ we find $x^2 = -2/15$. A general root of $G_2^{+++}$ may be written as

\[
\alpha = \sum_{i=1}^{4} n_i \alpha_i + l \alpha_5 = lx - \Lambda, \quad \text{where} \quad \Lambda = l \lambda_4 - \sum_i n_i \alpha_i.
\]

(7.3)

A highest weight representation of $A_4$ with Dynkin indices $p_i$ can occur if we can choose $\Lambda = \sum_i p_i \lambda_i$. Dotting this with $\lambda_5$ we find

\[
n_k = l(\lambda_4, \lambda_5) - \sum_j p_j (\lambda_j, \lambda_5).
\]

(7.4)

Squaring the expression for $\alpha$ in (7.3) gives

\[
\alpha^2 = -\frac{2}{15} l^2 + \sum_{i<j} p_i p_j (\lambda_i, \lambda_j) = \frac{1}{3} (6, 2, 0, -4, -10, -16, -22, \ldots)
\]

(7.5)

where $\alpha^2$ is constrained to take the values above as it belongs to the Kac–Moody algebra $G_2^{+++}$ [27]. The Chevalley generator corresponding to node 5 has one $A_4$ index, so a level $l$ generator, the multiple commutator for which contains the generator corresponding to node 5 $l$ times, will have $lA_4$ indices in total. As a result,

\[
l = \sum_j (5 - j) p_j + 5m
\]

(7.6)

where $m$ denotes the number of blocks of five fully anti-symmetrized indices. Substituting this into (7.5) gives

\[
\alpha^2 = 1/3 \left( \sum_j p_j^2 (5 - j)(j - 2) + 2 \sum_{i<j} p_i p_j (5 - j)(i - 2) - 4m \sum_j (5 - j)p_j - 10m^2 \right).
\]

(7.7)
We now show that any generator that satisfies (7.6) possesses a root that automatically satisfies the root condition (7.4). Substituting (7.6) into (7.4) we find that

\[ n_j = \sum_{k<j} p_k (j - k) + m j. \]  

(7.8)

Note that it gives positive integer values for the root components \( n_k \).

Dual generators are defined to be those with no blocks of 4 or 5 totally antisymmetrized indices, which may be written as

\[ m = p_1 = 0. \]  

(7.9)

Substituting these into (7.7) gives

\[ \alpha^2 = \frac{1}{3} \left( \sum_j p_j^2 (5 - j)(j - 2) + 2 \sum_{i<j} p_i p_j (5 - j)(i - 2) \right) \]

\[ = \frac{1}{3} (6, 2, 0, -4, -10, -16, -22, \ldots). \]  

(7.10)

Note that \( p_2 \) is absent from this formula, and the terms in the middle equation are positive.

Taking \( p_2 = 0 \) in equation (7.10) we get the following solutions:

\[ \alpha_A = (0, 0, 0, 1), \quad p_4 = 1 \]

\[ \alpha_B = (0, 0, 1, 2), \quad p_3 = 1 \]

\[ \alpha_C = (0, 0, 1, 3), \quad p_3 = p_4 = 1. \]  

(7.11)

All other \( p_i \)'s are 0. The roots \( \alpha_A \) and \( \alpha_B \) correspond to the generators \( R^a \) and \( R^{a_1 a_2} \), respectively. These give rise to the 1-form \( A_a \) and 2-form \( A_{a_1 a_2} \) in the nonlinear realization. Since the theory is in five dimensions the latter is the dual field of the former. The root \( \alpha_C \) corresponds to the generator \( R^{a_1 a_2 : b} \). The corresponding field is the dual graviton. All these fields have multiplicity one. Thus, at lowest levels the nonlinear realization contains gravity and a vector field as its on-shell degree of freedom [9]. This is five-dimensional \( N = 1 \) (eight supercharges) supergravity. This theory was constructed in [29, 30] and is sometimes referred to as \( N = 2 \) as it gives an \( N = 2 \) theory when dimensionally reduced.

Setting \( p_2 = 1 \) gives the solution

\[ \gamma = (0, 0, 1, 2, 3). \]  

(7.12)

This root vector has multiplicity 0 so does not contribute a field to the nonlinear realization.

One finds that all dual generators have the roots

\[ \alpha_A (p_2) = p_2 \gamma + \alpha_A \quad \alpha_B (p_2) = p_2 \gamma + \alpha_B \]

\[ \alpha_C (p_2) = p_2 \gamma + \alpha_C \quad \alpha_D (p_2) = p_2 \gamma. \]  

(7.13)

We will now find the space-filling and next to space-filling forms of the \( G_2^{++} \) theory in five, four and three dimensions in the same way as was done for the \( E_8^{+++} \) theory in [16]. In five dimensions, that is the highest dimension the theory can exist, and the one considered above, we find that a 3-form generator appeared in equation (7.12), but this had multiplicity zero. Examining equation (7.7) generator appeared in equation (7.12), but this had multiplicity zero. Examining equation (7.7) generator appeared in equation (7.12), but this had multiplicity zero. However, there are no scalars in this theory and so there are no fields to which the field strength of the 3-form can be dual. As such it is not required. That there is no 4-form implies that there are no deformations and
so no gauged supergravities that arise from the field strength of the 4-form. There is a known
gauging of this theory [32]. However it gauges a $U(1)$ sub-algebra of a $USp(2)$ symmetry
that does not act on the bosonic fields, but only on the gravitino. Hence, one does not expect
the gauged theory to be associated with a deformation of the bosonic sector. To predict this
gauged theory from the nonlinear realization one must first introduce the gravitino as it is only
on this field that the $USp(2)$ symmetry acts.

Let us consider adding 3-form and 4-form gauge fields and extending the known
 supersymmetry. Given that there is no internal symmetry in the five-dimensional $G_2^{++}$
nonlinear realization, the only allowed supersymmetry variation of the 3-form is given
by $\delta A_{ai;0} = i \epsilon^{ij} \gamma_{ai} \psi_0 \epsilon_{ij}$. However, taking the commutator of two supersymmetry
transformations we find that the supersymmetry algebra does not close and so one should not
introduce a 3-form. The same conclusion holds for the 4-form. This is consistent with the fact
that $G_2^{++}$ assigns multiplicity zero to these fields.

We now consider the $G_2^{++}$ theory in four dimensions. This can be found by just
dimensionally reducing the theory in five dimensions or deleting node 4 of the $G_2^{++}$ Dynkin
diagram and analysing the content with respect to the remaining $A_1 \otimes A_1$ algebra. The latter $A_1$
factor is the internal symmetry group of the four-dimensional theory. The results are the same,
but we will consider the former approach. Using the tables of generators of [9, 31], we find
the fields: two scalars $(h^1, A_3)$ which belong to the coset $SL(2, R)$ with local sub-algebra
$SO(2)$, an $SL(2, R)$ quadruplet of vectors $(h^2_3, A_4, A_5, A_{5,5})$ which satisfy self-duality
conditions, a triplet of 2-forms $(A_{a1;2}, A_{a2;5}, A_{a1;2;5,5})$ which are subject to one constraint
and are dual to the two scalars. We also find a doublet of 3-forms $(A_{a1;2;5,5}, A_{a1;2;5,5})$ and
finally a triplet of 4-forms $(A_{a1;2;5,5}, A_{a1;2;5,5}, A_{a1;2;5,5})$.

This theory is $N = 2$ supergravity coupled to one $N = 2$ vector multiplet which was
constructed in [30] and more recently in [33]. However, the $G_2^{++}$ formulation is a democratic
formulation in that it includes the Hodge duals of all the field strengths. As it possesses a
doublet of 4-forms it predicts two possible deformations that is gauged supergravities. The
nonlinear realization and the supersymmetry closure of this theory are under investigation and
the preliminary results confirm the $G_2^{++}$ picture [34].

We now turn to the three-dimensional $G_2^{++}$ theory which results from deleting node 3
of the $G_2^{++}$ Dynkin diagram leaving a $G_2$ internal symmetry. We will compute the forms by
dimensionally reducing the five-dimensional theory. We find six scalars $(h^1, A_1, A_{ij})$ which
belong to the nonlinear realization $G_2$ with local sub-algebra $SU(3)$. There are 14 vectors
$(h^1_3, A_2, A_{4, j}, A_{4, ij, k}, A_{4, ij, kl}, A_{4, ij, kl, k})$. These all have multiplicity one and belong to
the adjoint representation of $G_2$ and are dual to the scalars taking into account that they satisfy some
constraints. The 2-forms belong to the 27-dimensional representation of $G_2$ and an additional
singlet of $G_2$ (in the original version of this paper, the singlet deformation was erroneously
missing, see the end of appendix B for details). In fact, to find these latter forms one has to
go slightly beyond the tables of [9, 31] and take into account the fact that some generators
have multiplicity two. Hence, this theory possesses a set of deformations parametrized by the 27-dimensional representation of $G_2$ and a singlet of $G_2$, and a corresponding set of gauged
supergravities.

In [35], it is pointed out that the $G_2^{++}$ theory does not satisfactorily encompass the known
gauged supergravity of [32]. We believe that the reason for this mismatch is due to the fact that
in [32] it is a $U(1)$ subgroup of the fermionic symmetry $USp(2)$ that is gauged, and given that
there is no $USp(2)$ inside $G_2^{++}$ in five dimensions, this result is totally expected. Introducing
the fermions in the nonlinear realization might lead to a solution of this problem. We will
discuss this in more detail in appendix C.
8. $F^{+++}_4$

The Dynkin diagram of $F^{+++}_4$ is given by

```
0  7
|   |
0  6
```

Given the Cartan matrix we may choose the root vectors to obey

\[(\alpha_i, \alpha_i) = 2, \quad i = 1, \ldots, 5 \]
\[(\alpha_6, \alpha_6) = (\alpha_7, \alpha_7) = 1 \]
\[(\alpha_i, \alpha_j) = -1, \quad i, j = 1, \ldots, 6, \quad |i - j| = 1 \]
\[(\alpha_6, \alpha_7) = -1/2. \quad (8.1)\]

All other scalar products are 0. In the case of $F^{+++}_4$ the node labelled 6 is deleted, leaving an $A_5$ gravity algebra and an $A_1$ algebra. The roots of $F^{+++}_4$ can be written as the roots of $A_5$, $\alpha_i$, $i = 1, \ldots, 5$, the root of $A_1$, $\beta$, and the deleted root $\alpha_6$. The deleted root may be written as

\[\alpha_6 = x - \lambda_5 - \mu \quad (8.2)\]

where $\lambda_5$ is a fundamental weight of the $A_5$ sub-algebra and $\mu$ is the fundamental weight of the $A_1$ sub-algebra. The root of the $A_1$ sub-algebra, $\beta$, is normalized to have length 1, hence $\mu^2 = 1/4$, $\mu\beta = 1/2$. We find $x^2 = -1/12$.

A general root of $F^{+++}_4$ can be written as

\[\alpha = l\alpha_6 + \sum_{i=1}^{5} n_i \alpha_i + r\beta = lx - \Lambda \quad \text{where} \quad \Lambda = l\lambda_5 + l\mu - \sum_i n_i \alpha_i - r\beta. \quad (8.3)\]

A highest weight representation of $A_5 \otimes A_1$ with Dynkin indices $p_i$, $q$, respectively, can occur if we can choose $\Lambda = \sum_i p_i \lambda_i + q\mu$. Dotting this with $\lambda_k$ and $\mu$ in turn gives the pair of equations

\[n_k = l(\lambda_5, \lambda_k) - \sum_i (\lambda_i, \lambda_k) \]
\[r = \frac{l - q}{2}. \quad (8.4)\]

Squaring the expression for $\alpha$ in (8.3) gives

\[\alpha^2 = -\frac{1}{12}l^2 + \sum_{i,j} p_i p_j (\lambda_i, \lambda_j) + q^2/4 = 2, 1, 0, -1, \ldots \quad (8.5)\]

where $\alpha^2$ is constrained to take the values above.

The Chevalley generator corresponding to node 6 has one $A_5$ index, so a level $l$ generator, the multiple commutator for which contains the generator corresponding to node 6 $l$ times, will have $lA_5$ indices in total. The Chevalley generator corresponding to node 7 has no $A_5$ indices, so it does not contribute to the index total. As a result,

\[l = \sum_j (6 - j)p_j + 6m \quad (8.6)\]
where \( m \) denotes the number of blocks of six fully anti-symmetrized indices. Substituting this into (8.5) gives

\[
\alpha^2 = \frac{1}{4} \sum_{j=1}^{5} p_j^2 (6 - j)(j - 2) + \frac{1}{2} \sum_{i<j} p_i p_j (6 - j)(i - 2) + \frac{q^2}{4} - m \sum_j (6 - j)p_j - 3m^2
\]

\[
= 2, 1, 0, -1, \ldots. \tag{8.7}
\]

We now show that any generator that satisfies (8.6) automatically satisfies the root condition (8.4). Substituting (8.6) into (8.4) we find

\[
n_j = \sum_{i<j} p_i (i - j) + mk, \quad j = 1, \ldots, 5. \tag{8.8}
\]

We note that \( n_7 = r \) and \( n_6 = l \).

Dual generators are defined to be those with no blocks of five or six fully anti-symmetrized \( A_5 \) indices. This may be written as

\[
m = p_1 = 0. \tag{8.9}
\]

Substituting these into (8.7) gives

\[
\alpha^2 = \frac{1}{4} \sum_{j=1}^{5} p_j^2 (6 - j)(j - 2) + \frac{1}{2} \sum_{i<j} p_i p_j (6 - j)(i - 2) + \frac{q^2}{4} = 2, 1, 0, -1, \ldots. \tag{8.10}
\]

We now find all the solutions to equation (8.10). The middle terms of this equation are positive definite with \( p_2 \) undetermined. Taking \( p_2 = 0 \) we get the following solutions for \( \alpha^2 = 1 \):

\[
\begin{align*}
\alpha_A &= (0, 0, 0, 0, 1, 0), \quad & p_5 &= 1, \quad & q &= 1 \\
\alpha_B &= (0, 0, 0, 1, 2, 3, 1), \quad & p_5 &= 1, \quad & q &= 1 \\
\alpha_C &= (0, 0, 0, 0, 1, 2, 1), \quad & p_4 &= 1, \quad & q &= 0 \tag{8.11} \\
\alpha_D &= (0, 0, 0, 0, 0, 0, -1), \quad & q &= 2.
\end{align*}
\]

All other \( p_i \)'s are 0, \( p_1 \)'s and \( q \) are the highest weight components of the \( A_5 \oplus A_1 \) representation. Generators with \( q = 0, 1, 2 \) are \( A_1 \) singlets, doublets and triplets, respectively.

The generators corresponding to roots \( A, B, C \) and \( D \) are \( R^{\alpha A}, R^{\alpha B}, R^{\alpha C} \) and \( R^{\alpha D} \), respectively, where \( \alpha, \beta = 1, 2 \). These have multiplicity one except for \( R^{\alpha A} \), which has multiplicity zero. The last generator is symmetric in \( \alpha \beta \) and is just the triplet of generators of \( A_1 \) itself. These are at level zero with respect to \( n_6 \) and contain generators at levels \( n_7 = 0 \pm 1 \). The appearance of a negative level is allowed for the same reason as it did in the case of \( E^{+++}_6 \). These generators give rise to the fields expected in the nonlinear realization except for the multiplicity zero generator which leads to no field and the generator \( R^{\alpha (a \beta)} \) which only gives rise to two scalars \( \phi \) and \( \chi \) due to the local symmetry which being the Cartan involution invariant sub-algebra includes the \( SO(2) \) part of \( A_1 \).

The solutions for \( \alpha^2 = 2 \), still with \( p_2 = 0 \), are given by

\[
\begin{align*}
\alpha_E &= (0, 0, 0, 0, 1, 2, 0), \quad & p_4 &= 1, \quad & q &= 2 \\
\alpha_F &= (0, 0, 0, 1, 2, 4, 2), \quad & p_3 &= p_5 = 1, \quad & q &= 0. \tag{8.12}
\end{align*}
\]

These both have multiplicity one and correspond to the generators \( R^{a_{12}2, (a \beta)} \) and \( R^{a_1 a_2 a_3, b} \). If we take \( p_2 = 1 \) we find the generators

\[
\begin{align*}
\alpha_G &= (0, 0, 1, 2, 3, 4, 1), \quad & p_2 &= 1, \quad & q &= 2 \\
\gamma &= (0, 0, 1, 2, 3, 4, 2), \quad & p_2 &= 1, \quad & q &= 0. \tag{8.13}
\end{align*}
\]
The second generator has multiplicity zero while the first has multiplicity one and corresponds to the generator $R^{\alpha_1...\alpha_4}_{a_1...a_4}$. Up to the levels considered the field content of the nonlinear realization is: two scalars $\phi, \chi$ that belong to the coset $SO(1, 2)$, with local sub-algebra $SO(2)$, together with their duals $A_{a_1...a_4}, \alpha$ (which are subject to one constraint), two vectors $A_{a, \alpha}$ in the spinor representation and their duals $A_{a_1...a_4}, \alpha$, two self-dual 2-forms and one anti-self dual 2-form and the dual graviton. This makes up $(1, 0)$ supergravity in six dimensions ($h_a A^{a_1...a_4}$) and coupled to two vector multiplets $(A_{a, \alpha})$ and two tensor multiplets $(A_{a_1...a_4}, \alpha)$ and as well as the dual graviton $R^{a_1...a_4}_{a_5...a_7}$ [9]. The ± on the 2-forms indicate their self-duality and we have not shown their $SO(1, 2)$ indices, but they combine together to form the vector representation of $SO(1, 2)$.

The roots of all dual generators may be written in the form

$$\alpha_I (p^2) = \alpha_I + p^2 \gamma, \quad I = A, B, C, D, E, F, G$$

$$\alpha_H (p^2) = p^2 \gamma.$$ 

Thus, as with all the other cases, we find that the nonlinear realization contains all possible dual descriptions of the on-shell degrees of freedom of the theory.

In appendix B, we derive all the forms resulting from the $F_{4+++}$ nonlinear realization in four dimensions and above.

9. Summary of results and discussion

In this paper, we have considered all the very extended algebras $G^{+++}$ with the exception of the $B$ and $C$ series. We have deleted one, and in some cases two nodes, from the Dynkin diagram to find a preferred $A_{D-1}$ algebra, with in two cases an additional $A_1$ algebra, and we have analysed the content of the algebra $G^{+++}$ in terms of this $A_{D-1}$ algebra. We have shown that all the generators of $G^{+++}$ can be written with a set of $A_{D-1}$ indices the total number of which obeys a condition that depends on the level, or levels, of the generator in question. We found that this level matching condition automatically solves, in all cases, the condition that a highest weight representation of $A_{D-1}$ occurred amongst the root space of $G^{+++}$. As a result, necessary conditions for the roots of the Kac–Moody algebra $G^{+++}$ become the condition on the length of the root squared and the level matching condition itself. Despite the fact that the level matching conditions varied from algebra to algebra the condition on the length of the root squared has a universal form given by

$$\alpha^2 = \frac{1}{(D-2)} \left( \sum_j p^2_j (D - j)(j - 2) + 2 \sum_{i<j} p_i p_j (D - j)(i - 2) \right) - \frac{4m}{(D-2)} \sum_i (D - i) p_i - \frac{2m^2 D}{(D-2)} \left( \frac{4d^2}{(D-2)} + \frac{4d^2}{(D-2)} \right) = 2, \ldots. $$

The constant $c$ relates to the presence of an additional $A_1$ algebra that survives the deletion and it is zero for all cases except for $E_6^{+++}$ and $F_4^{+++}$ where it is 2 and 1, respectively. The integer $d$ is zero for all cases except for $D^{+++}_{2d}$ where it is the difference in the two levels. For the case of a symmetric Kac–Moody algebra $\alpha^2$ can only take the values $2, 0, -2, \ldots$, for the non-symmetric case the possible values are given in this paper.

Consequently, the necessary condition for a representation of $A_{D-1}$ with Dynkin indices $p_j$ and $m$ blocks of $D$ totally anti-symmetrized indices to occur in the algebra $G^{+++}$ is the condition of equation (9.1) and the level matching condition. In all cases, we have also
found a formula for the $G^{+++}$ roots in terms of their Dynkin indices $p_j$ and $m$. Indeed, we can think of the set of integers $\hat{p}_j = (m, p_j)$ as belonging to a lattice and equation (2.13) as containing the scalar product on this lattice. The possible generators of the Kac–Moody algebra $G^{+++}$ correspond to points $\hat{p}_j$ of the lattice that have an allowed length squared and obey the rather trivial level matching condition. This, and the physical nature of the higher level fields, suggests that it may not be totally impossible to list what the generators of these very extended Kac–Moody algebras are.

We then defined the notion of dual generators which are those that have no blocks of $D$ and $D−1$ totally anti-symmetrized indices, that is $m = 0 = p_1$. Substituting these conditions into equation (9.1) we find that it simplifies, all the terms are positive and we were able to find all possible solutions. As is apparent from equation (9.1) we always have a class of solutions with $p_2$ taking any positive integer value all other $p_j = 0$. It is straightforward to verify that this always satisfies the level matching condition. The solutions with $p_2 = 0$ correspond in the nonlinear realization to a set of fields that give the simplest description of the on-shell degrees of freedom of the theory together with a set of dual fields whose fields strengths are related to those of the original fields. Thus, one finds in this sense a democratic formulation. The only other dual generators are all the just-mentioned solutions but with the addition of the Dynkin index $p_2$ which can take any positive integer value. In the nonlinear realization this corresponds to adding blocks of $D − 1$ totally antisymmetric indices. These encode all possible ways of writing the on-shell degrees of freedom of the theory and their presence means that the theory encodes all possible duality transformations of these on-shell degrees of freedom. With the assignment of indices to the generators given in this paper, the generators of the affine sub-algebra $G^+$ are just given by taking the indices to only take the values $0, \ldots, D − 3$. In this case, all generators with blocks of $D − 1$ and $D$ indices are absent and one is left with the dual generators with the restricted index range. Indeed, the existence of the dual generators can be seen as a consequence of the affine sub-algebra. One can think of the role of the dual generators as lifting the infinite number of duality relations found in two dimensions up to $D$ dimensions.

We have explained that the $G_2^{++}$ theory in five dimensions does not possess 3- and 4-form fields and why this is compatible with the known gauged supergravity theory, contrary to that claimed in [35]. We have also computed the form fields for the $G_2^{++}$ theory in three and four dimensions and predicted the corresponding deformations or gauged supergravities. A detailed analysis of all the forms arising in lower dimensions for the cases of $E_6^{++}$ and $F_4^{++}$ is performed in two separate appendices.

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Note Added. The referee has asked us to comment on the occurrence of an $SL(2)$ doublet and an $SL(2)$ quadruplet of 10-forms in the $E_{11}$ formulation of the IIB theory. Although these appeared for the first time in the tables of [9], it was in [36] that it was stressed that $E_{11}$ predicts that the RR 10-form of IIB belongs to a quadruplet. This prediction was considered unexpected and therefore was assumed to signal a failure of the $E_{11}$ nonlinear realization [36]. It was only later that a doublet and a quadruplet of 10-forms were shown to occur in the IIB supersymmetry algebra [12], thus giving a highly non-trivial check of the $E_{11}$ predictions.
Appendix A. \textit{E}_{6}^{+++} in lower dimensions

In this appendix, we determine all the forms that arise in the non-supersymmetric \textit{E}_{6}^{+++} nonlinear realization in dimensions from 8 to 4, with the exception of the 4-forms in four dimensions. The scalar manifold is \textit{Sl}(2)/\textit{SO}(2) in eight dimensions, \textit{Sl}(2)/\textit{SO}(2) × \mathbb{R} in seven, [\textit{Sl}(2) × \textit{Sl}(2)]/[\textit{SO}(2) × \textit{SO}(2)] × \mathbb{R} in six, [\textit{Sl}(3) × \textit{Sl}(3)]/[\textit{SO}(3) × \textit{SO}(3)] in five and \textit{Sl}(6)/\textit{SO}(6) in four. We proceed using the same strategy of [16], that is we list all the eight-dimensional fields that can give rise to forms after dimensional reduction. We use the same notation as in [16], labelling each field with numbers denoting the number of antisymmetric spacetime indices in eight dimensions. The index \( \alpha \) is in the fundamental of \textit{Sl}(2).

We give here the final result, where the first column denotes the highest dimension for which the corresponding fields give rise to forms after dimensional reduction:

| D | Fields |
|---|---|
| 8 | \( \mathbb{A}^3_8 \) \( \mathbb{A}^{\text{eff}}_{6,3} \) \( \mathbb{A}^\alpha_{7,1,1} \) \( \mathbb{A}^\alpha_{6,3} \) \( \mathbb{A}^{\text{eff}}_{7,1,1} \) |
| 7 | \( h^1_1 \) \( \mathbb{A}^\alpha_{3,1} \) \( \mathbb{A}^{\text{eff}}_{7,1,1} \) \( \mathbb{A}^\alpha_{6,1} \) \( \mathbb{A}^{\text{eff}}_{6,1} \) |
| 6 | \( \mathbb{A}^\alpha_{7,2} \) \( \mathbb{A}^\alpha_{3,2,1,1} \) |
| 5 | \( \mathbb{A}^\alpha_{6,3} \) \( \mathbb{A}^\alpha_{3,2,1,1} \) \( \mathbb{A}^{\text{eff}}_{8,3,1,1} \) \( \mathbb{A}^\alpha_{6,3,3,3,1} \) |
| 4 | \( \mathbb{A}^\alpha_{7,4,4,4} \) \( \mathbb{A}^{\text{eff}}_{7,4,4,4} \) |

Observe that some of the fields in the list have multiplicity higher than one. All these fields, as well as the corresponding multiplicities, were listed in [31].

Performing the dimensional reduction, one obtains the results that are summarized in the following table:

| D | G | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( A_5 \) | \( A_6 \) | \( A_7 \) | \( A_8 \) |
|---|---|---|---|---|---|---|---|---|---|
| 8 | \textit{Sl}(2) | 1 | 2 | 1 | 1 | 2 | 3 | |
| 7 | \textit{Sl}(2) ⊗ \mathbb{R}^+ | 2 × 1 | 2× 2 | 2× 1 | 1× 1 | 2× 2 | 1× 3 | |
| 6 | \textit{Sl}(2)^2 ⊗ \mathbb{R}^+ | 2× 1 | 2× 2 | 2× 1 | 1× 1 | 2× 2 | 1× 3 | |
| 5 | \textit{Sl}(3) ⊗ \textit{Sl}(3) | 3× 3 | 3× 3 | 8× 1 | 3× 6 | 15× 3 | |
| 4 | \textit{Sl}(6) | 20 | 35 | 70 | | | | |

Observe that some of the fields in the list have multiplicity higher than one. All these fields, as well as the corresponding multiplicities, were listed in [31].
The \((D - 1)\)-forms in the table correspond to the massive deformations that the nonlinear realization allows in dimension \(D\).

**Appendix B. \(F_{4}^{+++}\) in lower dimensions**

In this appendix, we perform for the supersymmetric \(F_{4}^{+++}\) case the same analysis that was carried out in the previous appendix for the \(E_{6}^{+++}\) case. We determine all the forms that arise in dimensions from 6 to 4, with the exception of the 4-forms in four dimensions.

In six dimensions, the theory describes the bosonic sector of the gravity multiplet together with two tensor multiplets and two vector multiplets. There are two scalars, belonging to the tensor multiplets, parametrizing the manifold \(SL(2)/SO(2)\). In five dimensions this corresponds to the gravity multiplet plus five vector multiplets, and the five scalars parametrize \(SL(3)/SO(3)\). Finally, in four dimensions the theory describes an \(N' = 2\) gravity multiplet together with six vector multiplets, and the 12 scalars parametrize the manifold \(Sp(6)/[SU(3) \otimes U(1)]\).

The list of all the fields that give rise to forms after dimensional reduction, together with the highest dimension for which this occurs, is given here:

| \(D\) | Fields |
|-------|--------|
| 6     | \(A^{\mu}_{1}\) | \(A^{\mu\nu}_{2}\) | \(A^{\mu}_{3}\) | \(A^{\mu\nu}_{4}\) | \(A^{\mu\nu\rho}_{5}\) | \(A^{\mu\nu\rho\sigma}_{6}\) |
| 5     | \(h_{1,1}^{1}\) | \(A_{3,1}^{\mu}\) | \(A_{5,1}^{\mu\nu}(\times 2)\) | \(A_{5,1}^{\mu\nu\rho}\) | \(A_{6,1}^{\mu\nu\rho\sigma}(\times 3)\) | \(A_{6,1}^{\mu\nu\rho\sigma}(\times 2)\) |
| 4     | \(A_{3,1,1}^{\mu\nu}(\times 2)\) | \(A_{6,1,1,1}^{\mu\nu}(\times 2)\) | \(A_{6,1,1,1}^{\mu\nu\rho}\) | \(A_{5,2,1}^{\mu\nu\rho\sigma}\) | \(A_{5,2,1}^{\mu\nu\rho\sigma}\) | \(A_{2,2,2}^{\mu\nu\rho\sigma}\) |

Observe that some of the fields in the list have multiplicity higher than one. Although most of the fields were listed in the tables of \([9, 31]\) up to multiplicity 6, the fields with more than six indices have not appeared in the literature.

By dimensional reduction, one can then obtain all the forms that arise. The results are summarized in the following table:

| \(D\) | \(G\) | \(A_{1}\) | \(A_{2}\) | \(A_{3}\) | \(A_{4}\) | \(A_{6}\) |
|-------|-------|-------|-------|-------|-------|-------|
| 6     | \(SL(2)\) | 2     | 3     | 2     | 3     | 4     | 3     |
| 5     | \(SL(3)\) | 5     | 6     | 8     | 3     | 15    | 24    |
| 4     | \(Sp(6)\) | 14    | 21    | 64    | ?     |

The table shows in particular that the six-dimensional theory possesses massive deformations in the \(2 \oplus 4\) of \(SL(2)\), the five-dimensional one in the \(3 \oplus 15\) of \(SL(3)\) and the four-dimensional one in the \(64\) of \(Sp(6)\).

In the original version of this paper, the forms listed in the last table of this appendix for \(F_{4}^{+++}\) and those listed in the last table of appendix A for \(E_{6}^{+++}\), as well as the forms
obtained in section 7 for $G_2^{+++}$, were computed using the tables of [9] and [31]. However, it turns out that for some of the higher rank forms these tables are not sufficiently exhaustive to contain all the required fields. We have used the program SimpLie [19], available on http://strings.fnms.rug.nl/SimpLie/, to compute the extra fields and have subsequently modified the tables and corrected section 7. We thank the referee for drawing this to our attention.

Appendix C. $G_2^{+++}$ and minimal gauged supergravity in five dimensions

The minimal supergravity multiplet in five dimensions contains the metric, a $U(1)$ vector $A_\mu$ and a $USp(2)$ doublet of gravitini $\psi_\mu^i$ satisfying symplectic Majorana conditions. The $USp(2)$ symmetry only acts on the gravitini, and in [32] it was shown that this theory admits a gauging of a $U(1)$ subgroup of the $USp(2)$ symmetry. This gauging is thus different from the ones that occur in maximal supergravity because it arises from a symmetry that only acts on the fermions.

The field equation for the vector at lowest order in the fermions, in both the massless and the gauged case, has the form

$$D_\mu F^{\mu\nu} = -\frac{1}{2\sqrt{6}} \epsilon_{\mu\nu\rho\sigma\tau} A_\mu F^{\rho\sigma} F^{\nu\tau}$$ \hspace{1cm} (C.1)$$

because of the presence of a Chern–Simons term $A \wedge F \wedge F$ in the Lagrangian, where $F_{\mu\nu}$ is the field strength of $A_\mu$. In the gauged theory the Lagrangian contains a minimal coupling $g$ of the gravitino to the vector as well as a mass term $g$ for the gravitino and a cosmological constant $-g^2$. The Lagrangian reads, up to quartic order fermi terms,

$$L = \sqrt{-g} \left[ -\frac{1}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \bar{\psi}_\mu^i \gamma^{\mu\nu\rho} D_\rho \psi^i_{\nu} + \frac{1}{2} g \bar{\psi}_\mu^i \gamma^{\mu\nu\rho} A_\nu \delta_{ij} \psi^j_{\rho} - i \frac{\sqrt{6}}{4} g \bar{\psi}_\mu^i \gamma^{\mu\nu\rho} \psi^j_{\nu} \delta_{ij} + 4g^2 \right] + \frac{3}{8\sqrt{6}} \epsilon_{\mu\nu\rho\sigma\tau} A_\mu F^{\nu\rho} F_{\sigma\tau\nu}.$$ \hspace{1cm} (C.2)

The fermionic terms that arise in the gauged theory, that is the order $g$ terms in equation (C.2), are proportional to $\delta_{ij}$, which breaks $USp(2)$ explicitly.

The authors of [35] extended the algebra of this model introducing a 2-form dual to the vector as well as 3-forms and 4-forms that are both non-propagating. In five dimensions a 3-form is dual to a scalars but in [35] the 3-forms are introduced regardless the fact that there are no scalars in the model, and therefore their field strength is required to vanish identically. The authors originally showed that the supersymmetry algebra allows the introduction of a 3-form and a 4-form whose field strength is dual to the coupling constant $g$ in the gauged theory. In the first version of this paper, we pointed out that this would have led to an explicit symmetry breaking of $USp(2)$ in the algebra in the ungauged theory, and an explicit breaking of the $R$ symmetry is inconsistent with supersymmetry. The authors then revised their paper showing that both the 3-form and the 4-form are actually contained in a $USp(2)$ triplet, and inserting a footnote to acknowledge our contribution.

The $G_2^{+++}$ nonlinear realization in five dimensions describes the bosonic sector of this model, and in particular it contains a vector and its dual 2-form [9]. No 3-forms and 4-forms are present, and the authors of [35] pointed out that the mass parameter of the gauged supergravity of [32] cannot be obtained as the dual of a 4-form arising in the $G_2^{+++}$ nonlinear realization, because no such forms are present in this theory in five dimensions.
We now summarize the results of [35]. In the ungauged theory, the supersymmetry algebra closes on the 2-form $B_{\mu\nu}$ as expected from $G_{+++}^2$, imposing that its field strength

$$G_{\mu\nu} = 3\partial_{[\mu}B_{\nu]} - \sqrt{6}A_{[\mu}F_{\nu]}$$  \hspace{1cm} (C.3)$$

is dual to the field strength of the vector $F_{\mu\nu}$. Taking the curl of the duality relation

$$G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\tau}F^{\rho\tau}$$  \hspace{1cm} (C.4)$$

one recovers the vector field equation (C.1). The $USp(2)$ triplet of 3-forms $C_{\mu\nu\rho}^{ij}$ not found in $G_{+++}^2$ transforms under supersymmetry as

$$\delta C_{\mu\nu\rho}^{ij} = i\epsilon^{lij}[\gamma_{[\mu\nu}\psi_{\rho]}]$$  \hspace{1cm} (C.5)$$

and the commutator of two supersymmetry transformations closes at lowest order in the fermions imposing that the field strength vanishes identically,

$$H_{\mu\nu\rho}^{ij} = \partial_{[\mu}C_{\nu\rho]}^{ij} = 0.$$  \hspace{1cm} (C.6)$$

The 4-forms $D_{\mu\nu\rho\sigma}^{ij}$ transform under supersymmetry as

$$\delta D_{\mu\nu\rho\sigma}^{ij} = \epsilon^{ijkl}[\gamma_{[\mu\nu\rho}\psi_{\sigma]}] - i\sqrt{6}A_{[\mu}\epsilon^{ijkl}\gamma_{\nu\rho\sigma}[\psi_{\delta]}$$  \hspace{1cm} (C.7)$$

and their field strength vanishes identically, i.e.,

$$L_{\mu\nu\rho\sigma\tau}^{ij} = \partial_{[\mu}D_{\nu\rho\sigma\tau]}^{ij} = 0.$$  \hspace{1cm} (C.8)$$

The supersymmetry transformation of the gravitino has the form [32]

$$\delta \psi_{\mu}^{i} = D_{\mu}\epsilon^{i} - \frac{1}{4\sqrt{6}}(\gamma_{\mu}^{\nu\rho} - 4\delta_{\mu}^{\nu}\gamma_{\rho})F_{\nu\rho}\epsilon^{i}$$  \hspace{1cm} (C.9)$$

and at lowest order in the fermions one has to only consider the variations of the gravitino in equations (C.5) and (C.7). This only produces gauge transformations, while the general coordinate transformations are generated from the fact that the field strengths of equations (C.6) and (C.8) vanish identically. Thus, the fact that the field strengths vanish identically implies that these fields are pure gauge, and thus supersymmetry closes on these fields in a rather trivial way, as the authors point out [35].

The procedures used in [35] and those used in the $E_{11}$ formulation of maximal supergravities are different for a number of reasons. First of all, in [35] the introduction of 3-forms and 4-forms is trivial in the sense that their field strengths of the 3-form and the 4-form do not contain any lower rank field. This is in contrast with the $E_{11}$ case, in which the field strengths of higher rank contain the fields of lower rank. Secondly, while in $E_{11}$ the $(D-2)$-forms are dual to scalars, and their field strengths are not identically zero, in the model of [35] there is no real democracy because there are no scalars and this makes the introduction of 3-forms somehow artificial. Finally, the 3-forms and 4-forms are triplets of the fermionic global symmetry $USp(2)$ and thus are not related to the bosonic symmetry that arises in the nonlinear realization (which is absent in this five-dimensional case). The closure of the supersymmetry algebra does rely on the properties of the $\gamma$ matrices. However, the 3-forms and 4-forms do not couple to the other fields and thus the closure is a rather trivial consequence of the $\gamma$ matrix identities. The democratic formulation of the supersymmetry algebra of maximal five-dimensional supergravity theory has recently been described in [37]. In that case the 3-forms are in the adjoint of $E_6$ and are dual to the scalars that realize nonlinearly $E_6$ with local sub-algebra $USp(8)$. One can see from that result that one needs the same $\gamma$ matrix identities to cancel the $F_{\mu\nu}$ terms in the supersymmetry commutator on the 3-forms, and thus putting to zero the scalars and the field strengths of the 3-forms is from this
point of view a singular limit, in which 3-forms arise because one is using only a part of the constraints that come from the algebra when the scalars are present.

We now study the gauged theory of [35]. The supersymmetry variation of the gravitino is modified by the addition of a term of the form [32]

$$\delta^\prime \psi_\mu^I = -g A_\mu \delta^{ij} \epsilon_j - i \frac{1}{\sqrt{6}} g \gamma_\mu \delta^{ij} \epsilon_j.$$  \hspace{1cm} (C.10)

This term does not affect the supersymmetry commutator on the 3-forms, while it does affect the commutator on the 4-forms, which now closes provided that the duality relation

$$L_{\mu\nu\rho\sigma\tau}^{ij} \sim g \epsilon^{\mu\nu\rho\sigma\tau} \delta^{ij}$$  \hspace{1cm} (C.11)

holds, where $L_{\mu\nu\rho\sigma\tau}^{ij}$ is the gauge-invariant field strength defined in (C.8). The supersymmetry algebra on the 2-form implies that the gauge-invariant field strength for the 2-form is now

$$G_{\mu\nu} = 3 \delta_{[\mu} B_{\nu]} - \sqrt{6} A_{[\mu} F_{\nu]} - 3 g \delta_{ij} C_{\mu\nu}^{ij}.$$  \hspace{1cm} (C.12)

Thus, the 2-form transforms with respect to the gauge parameter of the 3-form, which therefore can be used to gauge away the 2-form completely. Secondly, the field strengths of the 4-forms are dual to the coupling constant $g$. We observe that the field strengths of the 3-forms and the 4-forms are not modified with respect to the massless case, and in particular the 3-forms are pure gauge quantities also in the gauged case. Again, like in the massless case there is no real democracy, and in particular one can introduce the 4-forms dual to the mass parameters without needing to introduce the 2-form and the 3-forms. This is in contrast with what happens in the $E_{11}$ nonlinear realization. Finally, if one believes that $(D-1)$-forms are responsible for gauged supergravities, the existence of a triplet of 4-forms as proposed in [35] would predict three such theories. However, only one such theory exists. This is again in contrast with what happens in the $E_{11}$ case, in which the number of deformations and the number of $(D-1)$-forms coincide in any dimension.

We believe that there are exceptional cases for which the gauged theory is not accounted for by the bosonic degrees of freedom in $G^{+++}$ but by the fermionic partners as treated from the viewpoint of the nonlinear realization. This particular model is singular in the sense that it has no scalars and has a global symmetry that only acts on the fermions.

As a toy model showing that the analysis of higher rank forms becomes more singular in cases with fewer supersymmetry and no scalars, we consider the very well known theory of pure minimal supergravity in four dimensions. In this case the supergravity multiplet only contains the metric and a Majorana gravitino, and one can introduce a negative cosmological constant $-g^2$ and a mass term for the gravitino, that schematically correspond to the appearance in the action of the terms

$$\det e(g \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu + g^2)$$  \hspace{1cm} (C.13)

and supersymmetry is provided by the fact that the gravitino transforms as

$$\delta \psi_\mu = D_\mu \epsilon + g \gamma_\mu \epsilon.$$  \hspace{1cm} (C.14)

We first want to consider the introduction of 2-forms. These would be the equivalent of 3-forms in five dimensions. In the massless theory the supersymmetry algebra closes trivially on a 2-form $B_{\mu\nu}$ whose supersymmetry transformation is

$$\delta B_{\mu\nu} = \bar{\epsilon} \gamma_{[\mu} \psi_{\nu]}$$  \hspace{1cm} (C.15)

if one imposes

$$\partial_{[\mu} B_{\nu]} = 0.$$  \hspace{1cm} (C.16)
In the massive theory this field can no longer be introduced because the supersymmetry commutator produces a term of the form $\bar{\epsilon}_1 \gamma_{\mu \nu} \epsilon_2$ which cannot be interpreted as a gauge transformation of any kind.

We now move to the 3-forms. One can introduce in both the massless and the massive theory a 3-form $C_{\mu \nu \rho}$ whose supersymmetry variation is

$$\delta C_{\mu \nu \rho} = \bar{\epsilon} \gamma_{\mu \nu} \gamma_5 \psi_{\rho}$$

(C.17)

if one imposes

$$\partial_{[\mu} C_{\nu \rho]} = g \epsilon_{\mu \nu \rho}.$$  

(C.18)

This form is therefore dual to the mass parameter $g$. In addition to this, one can also introduce a 3-form $C'_{\mu \nu \rho}$ whose supersymmetry variation is

$$\delta C'_{\mu \nu \rho} = \bar{\epsilon} \gamma_{\mu \nu} \psi_{\rho}$$

(C.19)

if one imposes

$$\partial_{[\mu} C'_{\nu \rho]} = 0.$$  

(C.20)

This additional 3-form has therefore vanishing field strength in both the massless and the massive theory. Therefore, in this model one can introduce a 2-form only if $g = 0$, while for any value of the coupling constant one can introduce two 3-forms, one of them being dual to $g$. While it seems that for any massive supersymmetric theory one can introduce a $(D - 1)$-form whose field strength is dual to the mass deformation parameter, we think that this model reveals the singular nature of these manipulations in theories that are not enough constrained by supersymmetry.

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