The diffusive competition problem with a free boundary in strong heterogeneous environment and weak heterogeneous environment

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Abstract. In this paper, we consider the diffusive competition problem consisting of an invasive species with density $u$ and a native species with density $v$. We assume that $v$ undergoes diffusion and growth in $[0, \infty)$, and $u$ exists initially in $[0, h_0)$, but invades into the environment with spreading front $x = h(t)$. To understand the effect of the dispersal rate $d_1$, the initial occupying habitat $h_0$, the initial density $u_0(x)$ of invasive species ($u$), and the parameter $\mu$ (the ratio of the invasion speed of the free boundary and the invasive species gradient at the expanding front) on the dynamics of this free boundary problem, we divide the heterogeneous environment into two cases: strong heterogeneous environment and weak heterogeneous environment. A spreading-vanishing dichotomy is obtained and some sufficient conditions for the invasive species spreading and vanishing is provided both in the strong heterogeneous environment and weak heterogeneous environment. Moreover, when spreading of $u$ happens, some rough estimates of the spreading speed are also given.

Keywords: Diffusive competition problem; Free boundary; Spreading-vanishing dichotomy; Invasive population; heterogeneous environment; Sharp criteria.

AMS subject classifications (2000): 35K57, 35K61, 35R35, 92D25.

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1 Introduction

In this paper, we study the behavior of the solution \((u(t, x), v(t, x), h(t))\) to the following reaction-diffusion problem with a free boundary in the heterogeneous environment

\[
\begin{cases}
  u_t - d_1 u_{xx} = u(m_1(x) - u - c(x)v), & t > 0, \quad 0 < x < h(t), \\
  v_t - d_2 v_{xx} = v(m_2(x) - b(x)u - v), & t > 0, \quad 0 < x < \infty, \\
  u_x(t, 0) = v_x(t, 0) = 0, u(t, x) = 0, & t > 0, \quad h(t) \leq x < \infty, \\
  h'(t) = -\mu u_x(t, h(t)), & t > 0, \\
  h(0) = h_0, u(0, x) = u_0(x), & 0 \leq x \leq h_0, \\
  v(0, x) = v_0(x), & 0 \leq x < \infty.
\end{cases}
\]

(1.1)

where \(m_i(x) (i = 1, 2)\) accounting for the resources of the environment will be discussed later; the positive constants \(d_1\) and \(d_2\) are dispersal rates of \(u\) and \(v\), respectively; \(b(x), c(x) \in C^1([0, \infty)) \cap L^\infty([0, \infty))\) are strictly positive functions; and the initial function \(u_0\) and \(v_0\) satisfy

\[
\begin{cases}
  u_0 \in C^2([0, h_0]), \quad u_0(0) = u_0(h_0) = 0, \quad u_0 > 0 \quad \text{in} \ (0, h_0), \\
  v_0 \in C^2([0, \infty)) \cap L^\infty([0, \infty)), \quad v_0(0) = 0, \quad v_0 \geq 0 \quad \text{in} \ [0, \infty), \quad \text{and} \quad v_0 \neq 0.
\end{cases}
\]

(1.2)

Ecologically, this problem describes the dynamical process of a new competitor invading into the habitat of a native species. The first species \((u)\), which exists initially on a region \([0, h_0]\), stands for the species in the very early stage of its introduction, and disperses through random diffusion over an expanding front \(h(t)\), evolves according to the free boundary condition

\[
h'(t) = -\mu u_x(t, h(t)),
\]

(1.3)

where \(\mu\) is a given positive constant. The second species \((v)\) is native, which undergoes diffusion and growth in the entire one-dimensional available habitat. The equation (1.3) is a special case of the well-known Stefan condition, which has been used in the modeling of a number of applied problems [1, 2, 28].

In the absence of a native species, namely \(v \equiv 0\), the problem (1.1) reduces to the following diffusive logistic problem with a free boundary in the heterogeneous environment

\[
\begin{cases}
  u_t - d_1 u_{xx} = u(m_1(x) - u), & t > 0, \quad 0 < x < h(t), \\
  u_x(t, 0) = 0, u(t, x) = 0, & t > 0, \quad h(t) \leq x < \infty, \\
  h'(t) = -\mu u_x(t, h(t)), & t > 0, \\
  h(0) = h_0, u(0, x) = u_0(x), & 0 \leq x \leq h_0,
\end{cases}
\]

(1.4)

which has been studied in [4], where the authors divided the heterogeneous environment into strong heterogeneous environment and weak heterogeneous environment. By choosing \(d_1\) and \(\mu\) as variable parameters, they derived sufficient conditions for species spreading (resp. vanishing) in strong heterogeneous environment, while in weak heterogeneous environment, they obtained sharp criteria for the spreading and vanishing. Moreover, when spreading happens, they gave an estimate for the asymptotic spreading speed of the free boundary. There are many related research about diffusive logistic problem with a free boundary in the homogeneous or heterogeneous environment.
In particular, Du and Lin [9] are the first ones to study the spreading-vanishing dichotomy of species in the homogeneous environment of dimension one, which has been extended in [10] to the situation of higher dimensional space in a radially symmetric case. Other theoretical advances can also be seen in [13, 15, 26, 5, 12, 14, 22, 31, 34] and the references therein.

Recently, Du and Lin [11] considered the following two-species model in higher dimensional case with radically symmetry in the heterogeneous environment

\[
\begin{aligned}
&u_t - d_1 \Delta u = u(a_1 - b_1 u - c_1 v), \quad t > 0, \quad 0 < r < h(t), \\
v_t - d_2 \Delta v = v(a_2 - b_2 u - c_2 v), \quad t > 0, \quad 0 < r < \infty, \\
u_r(t, 0) = v_r(t, 0) = 0, \quad u(t, r) = 0, \quad t > 0, \quad h(t) \leq r < \infty, \\
h'(t) = -\mu u_r(t, h(t)), \quad t > 0, \\
v(0, r) = v_0(r), \quad 0 \leq r < \infty,
\end{aligned}
\]

\[(1.5)\]

where \(u\) and \(v\) represent the invasive and native species, respectively, and \(a_1, b_1, c_1 \quad (i = 1, 2)\) are positive constants. They showed that a spreading-vanishing dichotomy holds when \(u\) is a superior competitor. And when \(u\) is a inferior competitor, the dynamical behavior of (1.5) is similar to that of (1.5) in a fixed domain. Moreover, when spreading of the invasive species \((u)\) happens, some rough estimates of the spreading speed were also given. We remark that similar Lotka-Votterra competitive type problems with a free boundary were introduced in [19, 20, 26]. Other studies of Lotka-Votterra prey-predator problems with a free boundary can be found in [2, 3, 30, 32, 33].

Problem (1.1) is a variation of the following diffusive Lotka-Votterra competition problem, which is often considered over a a bounded spatial domain with suitable boundary conditions [17, 13, 27]

\[
\begin{aligned}
&u_t - d_1 \Delta u = u(m_1(x) - u - cv), \quad \text{in } \Omega \times (0, \infty), \\
v_t - d_2 \Delta v = v(m_2(x) - bu - v), \quad \text{in } \Omega \times (0, \infty), \\
\nabla u \cdot n = \nabla v \cdot n = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
u(0, x) = u_0(x), \quad v(0, x) = v_0(x) \quad \text{in } \Omega,
\end{aligned}
\]

\[(1.6)\]

where the spatial heterogeneity is reflected in \(m_i(x) > 0 \quad (i = 1, 2)\) on \(\Omega \) and \(\neq\) Constant; the intraspecific competition coefficients are normalized to be 1; \(\Omega\) is a bounded smooth domain of \(\mathbb{R}^N\) with \(N \geq 1\); and \(n\) is the outward unit normal vector on \(\partial \Omega\). The authors showed that if \(d_1 < d_2\), the steady-state for the \(u\)-only equation is the global attractor under suitable assumptions about \(m_i(x)\), i.e., all solutions to (1.6) with \(u(0, x) \neq 0\) converge over time to the semi-trivial steady state \((u^*(x), 0)\) of (1.6). Moreover, when \(m_i\) change sign in \(\Omega\) and \(b = c = 1\) in (1.6), He and Ni [16] showed that for all \(d_1\) small and \(d_2\) large, the semi-trivial steady state \((u^*(x), 0)\) of (1.6) is globally asymptotically stable under suitable assumptions about \(m_i(x)\).

Motivated by the above works, we will divide the environment into two different circumstances: strong heterogeneous environment and weak heterogeneous environment, where if \(m_i(x)\) satisfies the following assumptions

\[(H_1) \quad m_i(x) \in C^1((0, \infty)) \cap L^\infty((0, \infty)), \quad i = 1, 2, \]

and \(m_1(x)\) changes sign in \((0, h_0)\), \(m_2(x)\) changes sign in \((0, \infty), \]

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then it is called strong heterogeneous environment for population, and if $m_i(x)$ satisfies

$$(H_2) \quad m_i(x) \in C^1([0, \infty)), \quad 0 < m_i \leq m_i(x) \leq \bar{m}_i < \infty \text{ for } x \in [0, \infty), \quad i = 1, 2,$$

with $m_i$ and $\bar{m}_i$ being positive constants, then it is called weak heterogeneous environment for population.

The aim of this paper is to study the dynamics of problem (1.1) in the strong (resp. weak) heterogeneous environment. To best of our knowledge, the present paper seems to be the first attempt to consider the strong heterogeneous environment in the moving domain problem about the diffusive Lotka-Volterra competitive problem. It should be pointed out here that the arguments developed in the previous work [11, 19] do not work in the situation of strong heterogeneous environment, since $m_i(x)$ are admitted to change sign in $[0, \infty)$. We derive some sufficient conditions to ensure that spreading and vanishing occur, which yield the spreading-vanishing dichotomy, and sharp criteria governing spreading and vanishing both in the strong heterogeneous environment and weak heterogeneous environment. Furthermore, since dispersal is an important aspect of the evolution of many species, which can affect the persistence of species and mediate interactions between species. We employ the dispersal rate $d_1$, in addition to $h_0$, $\mu$ and $w_0(x)$, as the varying parameters to study problem (1.1) when $m_i(x)$ ($i = 1, 2$) change sign (resp. $0 < m_i(x) < \infty$, $x \in [0, \infty)$). More specifically, under suitable assumptions about $m_i(x)$, we give that slow diffusion, large diffusion and big initial density, large occupying habitat, and big initial density of invasive species ($u$) are benefit for invasive species ($u$) to survive in the new environment. Finally, we also extend the asymptotic spreading speed of a free boundary in weak heterogeneous environment, when spreading of invasive species ($u$) happens, to strong heterogeneous environment.

We remark that similar free boundary conditions to (1.3) have been used in ecological models over bounded spatial domains in several earlier papers, for example, [3, 23, 24, 25].

The rest of our paper is arranged as follows. In Section 2, we exhibit some fundamental results, including the global existence and uniqueness of the solution of problem (1.1) and the comparison principle in the moving domain; An eigenvalue problem under some suitable assumptions is given in Section 3; In Section 4, we investigate the dynamics of problem (1.1) in strong (resp. weak) heterogeneous environment. Section 5 is devoted to studying the asymptotic spreading speed of the free boundary when spreading of invasive species ($u$) occurs, and finally we give a short discussion in the last section.

## 2 Preliminaries

In this section, we give some fundamental results on solutions of problem (1.1) under the following assumption

$$(H_0) \quad m_i(x) \in C^1([0, \infty)) \cap L^\infty([0, \infty)), \quad i = 1, 2,$$

and $m_1(x)$ is positive somewhere in $(0, h_0)$, $m_2(x)$ is positive somewhere in $(0, \infty)$.

Clearly, the assumption $(H_0)$ is much weaker than $(H_1)$ and $(H_2)$, so the results obtained under $(H_0)$ are also true if $(H_1)$ or $(H_2)$ is satisfied. Throughout this section, $(H_0)$ is assumed to hold even if it is not explicitly mentioned.
Lemma 2.1. For any given \((u_0, v_0)\) satisfying (1.2) and any \(\alpha \in (0, 1)\), there is a \(T > 0\) such that problem (1.1) admits a unique bounded solution
\[
(u, v, h) \in C^{(1+\alpha)/2, 1+\alpha}(D_T) \times C^{(1+\alpha)/2, 1+\alpha}(D_T^\infty) \times C^{1+\alpha/2}([0, T]).
\]
Moreover,
\[
\|u\|_{C^{(1+\alpha)/2, 1+\alpha}(D_T)} + \|v\|_{C^{(1+\alpha)/2, 1+\alpha}(D_T^\infty)} + \|h\|_{C^{1+\alpha/2}([0, T])} \leq C,
\]
where \(D_T = \{(t, x) \in R^2 : t \in [0, T], x \in [0, h(t)]\}\), \(D_T^\infty = \{(t, x) \in R^2 : t \in [0, T], x \in [0, \infty)\}\), \(C\) and \(T\) only depend on \(h_0, \alpha, \|u_0\|_{C^2([0, h_0])}, \|v_0\|_{C^2([0, \infty])}\).

Proof. The essential idea of this proof is to construct a contraction mapping, and the desired would then follow from the contraction mapping theorem andshauder fixed point theorem. For brevity, we omit the burdensome process and list [11, 20] for consultation.

Remark 2.1. It follows from the uniqueness of the solution to (1.1) and a standard compactness argument that the unique solution \((u, v, h)\) depends continuously on the parameters appearing in (1.1). This fact will be used in the later sections.

To prove the global existence of the solution to (1.1) obtained in Lemma 2.1, we need the following estimates.

Lemma 2.2. Let \((u, v, h)\) be a unique and uniformly bounded solution of problem (1.1) defined on an open interval \((0, T)\) for some \(T \in (0, \infty)\), then there exist constants \(C_1\) and \(C_2\) independent of \(T\) such that
\[
0 < u(t, x) \leq C_1, \quad \text{for } t \in (0, T], \quad 0 < x < h(t),
\]
\[
0 < v(t, x) \leq C_2, \quad \text{for } t \in (0, T], \quad 0 < x < \infty.
\]
Moreover, there exists a constant \(C_3\) such that
\[
0 < h'(t) \leq C_3 \quad \text{for } t \in (0, T].
\]
Furthermore, (1.1) does not have any unbounded solution.

Proof. By Lemma 2.1, (1.1) has a unique bounded solution defined on \([0, T]\). Applying the strong maximum principle to the equation of \(u\), we immediately obtain \(0 < u(t, x), u_x(t, h(t)) < 0\) for \(0 < t < T\) and \(0 \leq x < h(t)\). Combining the inequality with the stefan condition, i.e., \(h'(t) = -\mu u_x(t, h(t))\), one easily sees that \(h'(t) > 0\) for \(t \in (0, T]\). Using the maximum principle again, we obtain
\[
u(t, x) \leq C_2 \triangleq \max\{\|m_2(x)\|_{L^\infty([0, \infty))}, \|v_0(x)\|_{L^\infty([0, h_0])}\}, \quad \text{for } t \in (0, T] \quad \text{and} \quad x \in [0, h(t)].
\]
Similarly we have \(0 < v(t, x)\) for \(t > 0, 0 \leq x < \infty\), and
\[
u(t, x) \leq C_2 \triangleq \max\{\|m_2(x)\|_{L^\infty([0, \infty))}, \|v_0(x)\|_{L^\infty([0, \infty))}\}, \quad \text{for } t \in (0, T] \quad \text{and} \quad x \in [0, \infty).
\]

It remains to show that \(h'(t) \leq C_3\) for \(t \in (0, T]\). We define
\[
\Omega_M \triangleq \{(t, x) : 0 < t < T, h(t) - M^{-1} < x < h(t)\},
\]
and construct an auxiliary function
\[ \bar{u}(t, x) \triangleq C_1 [2M(h(t) - x) - M^2(h(t) - x)^2]. \]
We will choose \( M > \frac{1}{h_0} \) so that \( \bar{u}(t, x) \geq u(t, x) \) holds over \( \Omega_M \).

Direct calculations yield that, for \((t, x) \in \Omega_M\),
\[
\begin{align*}
\bar{u}_t - d_1 \bar{u}_{xx} &= 2MC_1h'(t)[1 - M(h(t) - x)] + 2d_1C_1M^2 \geq 2d_1C_1M^2 \geq u(m_1 - u - c(x)v), \\
\bar{u}(t, h(t)) - M^{-1} &= C_1 \geq u(t, h(t) - M^{-1}), \\
\bar{u}(t, h(t)) &= 0 = u(t, h(t)),
\end{align*}
\]
provided \( M^2 \geq \frac{\|m_1(x)\|_{L^\infty(0, \infty)}}{2d_1} \). On the other hand, we calculate
\[ \bar{u}_x(0, x) = -2C_1M[1 - M(h_0 - x)] \leq -C_1M, \text{ for } x \in [h_0 - (2M)^{-1}, h_0]. \]
Therefore, by choosing
\[ M \triangleq \max \left\{ \frac{1}{h_0}, \sqrt{\frac{\|m_1(x)\|_{L^\infty(0, \infty)}}{2d_1} \frac{4\|u_0\|_{C^1([0, h_0])}}{3C_1}} \right\}, \]
we will have \( \bar{u}_x(0, x) \leq u_0(x) \) for \( x \in [h_0 - (2M)^{-1}, h_0] \). Since \( \bar{u}(0, h_0) = u_0(h_0) = 0 \), the above inequality implies
\[ \bar{u}(0, x) \geq u_0(x), \text{ for } x \in [h_0 - (2M)^{-1}, h_0]. \]
Moreover, for \( x \in [h_0 - M^{-1}, h_0 - (2M)^{-1}] \), we have
\[ \bar{u}(0, x) \geq \frac{3}{4} C_1, u_0(x) \leq \|u_0\|_{C^1([0, h_0])}M^{-1} \leq \frac{3}{4} C_1. \]
Therefore, \( u_0(x) \leq \bar{u}(0, x) \) for \( x \in [h_0 - (2M)^{-1}, h_0] \).

Applying the maximum principle to \( \bar{u} - u \) over \( \Omega_M \) gives that \( u(t, x) \leq \bar{u}(t, x) \) for \((t, x) \in \Omega_M\), which indicates that
\[ -2MC_1 = \bar{u}_x(t, h(t)) \leq u_x(t, h(t)), \quad h'(t) = -\mu u_x(t, h(t)) \leq C_3 \triangleq 2MC_1\mu \quad \text{for } t \in (0, T). \]

We next show that any solution of (1.1) is bounded, namely, there exists \( C_4 > 0 \) such that \( u, v \leq C_4 \) in the range they are defined, whenever \((u, v, h)\) is a solution to (1.1) defined in some maximal interval \( t \in (0, T) \). Indeed, Let \( U(x) \) be the unique boundary blow-up solution of
\[ -d_1U_{xx} = U(m_1(x) - U), \quad 0 < x < 1, U(1) = \infty, \]
(see Theorem 2.3 of [7]). Then it is easily checked by using the comparison principle that \( u \leq \|u_0\|_{C^1([0, h_0])} + U(0) \) in the range that \( u \) is defined. Similarly we can show \( v \leq \|v_0\|_{C^1([0, h_0])} + V(0) \), where \( V(x) \) is the unique boundary blow-up solution of
\[ -d_2V_{xx} = V(m_2(x) - V), \quad 0 < x < 1, V(1) = \infty. \]

Based on the above estimates, we are now in a position to prove that the solution of (1.1) is actually a global solution.
Theorem 2.1. The solution of problem (1.1) exists and is unique for all \( t \geq 0 \).

Proof. Let \([0, T_{\text{max}}]\) be the maximal time interval in which the solution exists. In view of Lemma 2.1, it suffices to show that \( T_{\text{max}} = \infty \). Suppose for contradiction that \( T_{\text{max}} < \infty \). Fix \( \delta \in (0, T_{\text{max}}) \), and \( \hat{T} > T_{\text{max}} \). By standard parabolic regularity, we can find \( C_5 > 0 \) depending only on \( \delta, \hat{T}, C_1, C_2 \) and \( C_3 \) such that

\[
\|u(t, \cdot)\|_{C^2([0, h(t))] + \|v(t, \cdot)\|_{C^2([0, \infty))}} \leq C_5, \quad \text{for } t \in [\delta, T_{\text{max}}).
\]

It has then follows from the proof of Lemma 2.1 that there exists a \( \tau > 0 \) depending only on \( C_1, C_2, C_3 \) and \( C_5 \) such that the solution of (1.1) with initial time \( T_{\text{max}} - \frac{\tau}{2} \) can be extended uniquely to the time \( T_{\text{max}} - \frac{\tau}{2} + \tau \). But this contradicts to the assumption about \( T_{\text{max}} \). \( \square \)

In what follows, we discuss the comparison principle for (1.1). For a given pair of functions \( u := (u, v) \) and \( \bar{u} := (\bar{u}, \bar{v}) \), we denote

\[
[u, \bar{u}] = \{ u := (u, v) \in [C([0, T] \times [0, \infty))]^2 : (u, v) \leq (u, \bar{v}) \},
\]

where by \((u_1, v_1) \leq (u_2, v_2)\), we mean \( u_1 \leq u_2 \) and \( v_1 \leq v_2 \).

Lemma 2.3. (The Comparison Principle) Suppose that \( T \in (0, \infty), \bar{h}, \bar{v} \in C([0, T]) \), \( u \in C(D_T^r) \cap C^{1,2}(D_T^s) \) with \( D_T^r \triangleq \{(t, x) \in \mathbb{R}^2 : t \in (0, T), x \in (0, \bar{h}(t))\} \), \( u \in C(D_T^r) \cap C^{1,2}(D_T^s) \) with \( D_T^r \triangleq \{(t, x) \in \mathbb{R}^2 : t \in (0, T), x \in (0, \bar{h}(t))\} \), \( u, \bar{v} \in (L^\infty \cap C)((0, T] \times [0, \infty)) \cap C^{1,2}((0, T] \times [0, \infty)) \) and

\[
\begin{align*}
\bar{u}_t - d_1 \bar{u}_{xx} & \geq \bar{u}(m_1(x) - \bar{u} - c(x)u), \quad 0 < t < T, \quad 0 < x < \bar{h}(t), \\
u_t - d_1 u_{xx} & \geq u(m_1(x) - u - c(x)v), \quad 0 < t < T, \quad 0 < x < h(t), \\
\bar{v}_t - d_2 \bar{v}_{xx} & \geq \bar{v}(m_2(x) - b(x)u - \bar{v}), \quad 0 < t < T, \quad 0 < x < \infty, \\
v_t - d_2 v_{xx} & \leq v(m_2(x) - b(x)v - u), \quad 0 < t < T, \quad 0 < x < \infty, \\
u_x(t, 0) = v_x(t, 0) = 0, & \quad \bar{u}(t, x) = 0, \quad 0 < t < T, \quad \bar{h}(t) \leq x < \infty, \quad h'(t) \leq -\mu \bar{u}_x(t, h(t)), \quad \bar{h}'(t) \geq -\mu \bar{u}_x(t, \bar{h}(t)), \quad 0 < t < T, \\
\bar{h}(0) \leq h_0 \leq \bar{h}(0), & \quad \bar{u}(0, x) \leq u_0(x) \leq \bar{u}(0, x), \quad 0 \leq x \leq h_0, \\
u(0, x) \leq v(0, x) \leq \bar{v}(0, x), & \quad 0 \leq x \leq \infty.
\end{align*}
\]

Let \((u, v, h)\) be the unique solution of (1.1), then

\[
\begin{align*}
h(t) \leq \bar{h}(t) & \quad \text{in } (0, T], \quad u(t, x) \leq \bar{u}(t, x), \quad v(t, x) \leq v(t, x) \quad \text{for } (t, x) \in (0, T] \times [0, \infty), \\
h(t) \geq \underline{h}(t) & \quad \text{in } (0, T], \quad u(t, x) \leq \underline{u}(t, x), \quad v(t, x) \leq \bar{v}(t) \quad \text{for } (t, x) \in (0, T] \times [0, \infty).
\end{align*}
\]

Proof. Denote \( f(u, v) = u(m_1(x) - u - c(x)v), g(u, v) = v(m_2(x) - b(x)u - v). \) We can easily see that \( f \) is nonincreasing in \( v \) for fixed \( u \), \( g \) is nonincreasing in \( u \) for fixed \( v \). That is, a function pair \((f(u, v), g(u, v))\) is quasimonotone nonincreasing. Moreover, \((f(u, v), g(u, v))\) is Lipschitz continuous in \([u, \bar{u}]\), with \( f(0, v) = g(u, 0) = 0 \). According to Lemma 2.6 in [11], we have the corresponding comparison principle to problem (1.1). \( \square \)
3 Some eigenvalue problems

In this section, we mainly study an eigenvalue problem and analyze the property of its principle eigenvalue. These results play an important role in later sections.

Consider the following eigenvalue problem

\[
\begin{cases}
d\varphi_{xx} + k(x)\varphi + \lambda \varphi = 0, & 0 < x < h, \\
\varphi_x(0) = \varphi(h) = 0,
\end{cases}
\]

where \(d, h\) are positive constants, and \(k(x)\) satisfies

\[k(x) \in C^1([0, \infty)) \cap L^\infty([0, \infty))\] is positive somewhere in \((0, h)\). \hspace{1cm} \text{(3.2)}

Let \(\lambda_1(d, h, k(x))\) denote the principle eigenvalue of the problem (3.1). It is well known that \(\lambda(d, h, k(x))\) exists uniquely and corresponding eigenfunction, denote by \(\varphi_1\), can be chosen positive in \([0, h]\) and normalized by \(\|\varphi_1\|_{L^2} = 1\). By variational method, \(\lambda_1(d, h, k(x))\) can be characterized by the following variational form

\[
\lambda_1(d, h, k(x)) = \inf_{\varphi \in \mathcal{F}} \left\{ \int_0^h (d\varphi_x^2 - k(x)\varphi^2)dx \right\},
\]

where \(\mathcal{F} \triangleq \{ \varphi \in W^{1,2}([0, h]), \varphi_x(0) = \varphi(h) = 0, \int_0^h \varphi^2dx = 1 \}\). For fixed \(h\) and varying \(d\), we write \(\lambda_1(d, h, k(x)) = \lambda_1(d, k(x))\) for brevity. Similarly, we write \(\lambda_1(d, h, k(x)) = \lambda_1(h, k(x))\) for fixed \(d\) and varying \(h\). First, we present the property of \(\lambda_1(d, k(x))\).

**Theorem 3.1.** (see [4]) Assume that (3.2) holds and fix \(h > 0\), the following conclusions regarding \(\lambda_1(d, k(x))\) hold

(i) \(\lambda_1(d, k(x))\) is a strictly monotone increasing function of \(d\);

(ii) \(\lambda_1(d, k(x)) \to \hat{\lambda} \triangleq -\max_{x \in [0, h]} k(x) < 0\) as \(d \to 0\);

(iii) \(\lambda_1(d, k(x)) \to +\infty\) as \(d \to +\infty\).

The above theorem implies the following result.

**Corollary 3.1.** Assume that (3.2) holds and fix \(h > 0\), then there exists \(d^*(h, k(x)) \to +\infty\) such that

\(\lambda_1(d, k(x)) < 0\) if \(0 < d < d^*(h, k(x))\), \(\lambda_1(d, k(x)) = 0\) if \(d = d^*(h, k(x))\), and \(\lambda_1(d, k(x)) > 0\) if \(d > d^*(h, k(x))\).

The following results are the counterpart of Theorem 3.1, and we recommend [4] and [31] for a detailed proof.

**Theorem 3.2.** Assume that (3.2) holds and fix \(d > 0\), the following conclusions regarding \(\lambda_1(h, k(x))\) hold

(i) \(\lambda_1(h, k(x))\) is a strictly monotone decreasing function of \(h\);

(ii) \(\lambda_1(h, k(x)) \to +\infty\) as \(h \to 0\).

Now, we need extra condition in addition to (3.2):

\[
0 < k_* \triangleq \liminf_{x \to \infty} k(x) \leq k^* \triangleq \limsup_{x \to \infty} k(x) < \infty,
\]

where \(k_*\) and \(k^*\) are positive constants.

The following corollary is a direct consequence of Theorem 3.2, and Remark 3.1 in [31].
Corollary 3.2. Assume that (3.2) and (3.4) hold and fix $d > 0$, then there exists $h^*(d, k(x)) > 0$ such that $\lambda_1(h, k(x)) > 0$ if $0 < h < h^*(d, k(x))$, $\lambda_1(h, k(x)) = 0$ if $h = h^*(d, k(x))$, and $\lambda_1(h, k(x)) < 0$ if $h > h^*(d, k(x))$.

If we replace the constant $h$ in (3.3) by $h(t)$, i.e., let
\[
\lambda_1(d, h(t), k(x)) = \inf_{\varphi \in \mathcal{F}(t)} \left\{ \int_0^{h(t)} (d\varphi_x^2 - k(x)\varphi^2)dx \right\},
\]
where $\mathcal{F}(t) \triangleq \{ \varphi \in W^{1,2}((0, h(t))), \varphi_x(0) = \varphi(h(t)) = 0, \int_0^{h(t)} \varphi^2 dx = 1 \}$. Let $\lambda_1(h(t), k(x))$ denote $\lambda_1(d, h(t), k(x))$ for fixed $d$. From Lemma 2.2, Theorem 3.2, and Corollary 3.2, we get the following result.

Corollary 3.3. Assume that (3.2) holds and fix $d > 0$, $\lambda_1(h(t), k(x))$ is a strictly monotone decreasing function of $t$. Moreover, if (3.4) also holds and $h_{\infty} \triangleq \lim_{t \to \infty} h(t) = +\infty$, then $\lambda_1(h(t), k(x)) < 0$ for a sufficiently large $t$.

4 Strong and weak heterogeneous environment

In this section, we mainly study problem (1.1) in the strong (resp. weak) heterogeneous environment (i.e., under the assumptions $(H_1)$ and $(H_2)$, respectively). Actually, our results hold under general assumptions $(H_0)$ and $(H_0')$.

\[(H_0') \quad 0 < m_{i,s} \triangleq \lim_{x \to \infty} m_i(x) \leq m_i^* \triangleq \lim_{x \to \infty} \sup m_i(x) < \infty, \quad i = 1, 2,
\]
where $m_{i,s}$ and $m_i^*$ are positive constants. Clearly, this condition allow $m_i(x)$ to change sign in a finite interval. So the results obtained under $(H_0)$ and $(H_0')$ are also true if $(H_1)$ and $(H_0')$ or $(H_2)$ are satisfied.

Note that $c(x) \in C^1([0, \infty)) \cap L^\infty([0, \infty))$ is a strictly positive functions. Now, we denote $c_*$ and $c^*$ as follows
\[
c_* \triangleq \liminf_{x \to \infty} c(x), \quad c^* \triangleq \limsup_{x \to \infty} c(x),
\]
and we assume
\[
m_1^* - c_* m_{2,s} \geq m_{1,s} - c^* m_2^* > 0. \quad (4.1)
\]

Throughout this paper, $(H_0)$, $(H_0')$ and (4.1) are assumed to hold even if they are not explicitly mentioned. We will give the dynamics of problem (1.1) both in the strong heterogeneous environment and weak heterogeneous environment.

4.1 Spreading-vanishing dichotomy

In this subsection, we prove the spreading-vanishing dichotomy. In view of Lemma 2.2, we see that the free boundary $h(t)$ is a strictly increasing function with respect to time $t$. Thus, either $h_{\infty} < \infty$ or $h_{\infty} = \infty$ holds. We first prove that if the habitat of the invasive species is limited in the long run, then the invasive species $(u)$ vanishes.
Lemma 4.1. If $h_\infty < \infty$, then $\limsup_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0$ and $\lim_{t \to \infty} v(t, x) = V(x)$ uniformly in any bounded subset of $[0, \infty)$, where $V(x)$ is the unique positive solution of
\[
\begin{align*}
-d_2 z_{xx} &= V(m_2(x) - V), \quad 0 < x < \infty, \\
V_x(0) &= 0.
\end{align*}
\] (4.2)

Proof. Since $m_2$ satisfies the assumption $(H'_0)$, Theorem 2.3 in [8] is available, and then the existence and uniqueness of $V(x)$ can be established. Similar to the proof of Lemma 3.3 in [20], one can show that if $h_\infty < \infty$, then $h'(t) \to 0$ as $t \to \infty$.

We now argue indirectly. We assume that $\limsup_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = \delta > 0$, then there exists a sequence $(t_n, x_n)$ with $0 < t_n < \infty, 0 \leq x_n < h(t_n)$ such that $u(t_n, x_n) \geq \frac{\delta}{2}$ for all $n \in \mathbb{N}$. Since $0 \leq x_n < h_\infty$, there exists a subsequence of $x_n$, denoted by itself, and $x_0 \in [0, h_\infty)$, such that $x_n \to x_0$ as $n \to \infty$. We claim that $x_0 < h_\infty$. If this is not true, then $x_n - h(t_n) \to 0$ as $n \to \infty$. According to Lemma 2.1 and the above assumption, we have
\[
\left| \frac{\delta}{2(x_n - h(t_n))} \right| \leq \left| \frac{u(t_n, x_n)}{x_n - h(t_n)} \right| = \left| \frac{u(t_n, x_n) - u(t_n, h(t_n))}{x_n - h(t_n)} \right| = |u_x(t_n, \bar{x}_n)| \leq C,
\]
where $\bar{x}_n \in (x_n, h(t_n))$. It is a contradiction since $x_n - h(t_n) \to 0$. Without loss of generality, we assume $x_n \to x_0 \in [0, h_\infty - \sigma]$ as $n \to \infty$ for some $\sigma > 0$.

Define
\[
u_n(t, x) = u(t + t_n, x) \quad \text{and} \quad v_n(t, x) = v(t + t_n, x) \quad \text{for} \quad (t, x) \in D_n,
\]
with $D_n := \{(t, x) \in \mathbb{R}^2 : t \in (-t_n, \infty), \ x \in [0, h(t_n + t_n)]\}$.

It follows from Lemma 2.1 that $\{(u_n, v_n)\}$ has a subsequence $\{(u_{n_i}, v_{n_i})\}$ such that
\[
\|(u_{n_i}, v_{n_i}) - (\bar{u}, \bar{v})\|_{C^1; z(D_{n_i}) \times C^1; z(D_{n_i})} \to 0 \quad \text{as} \ i \to \infty,
\]
and $(\bar{u}, \bar{v})$ satisfies
\[
\begin{align*}
\bar{u}_t - d_1 \bar{u}_{xx} &= \bar{u}(m_1(x) - \bar{u} - c(x)\bar{v}), \quad t \in (-\infty, \infty), \quad 0 < x < h_\infty, \\
\bar{v}_t - d_2 \bar{v}_{xx} &= \bar{v}(m_2(x) - b(x)\bar{u} - \bar{v}), \quad t \in (-\infty, \infty), \quad 0 < x < \infty, \\
\bar{u}(t, h_\infty) &= 0, \quad t \in (-\infty, \infty).
\end{align*}
\]
Since $\bar{u}(0, x_0) \geq \frac{\delta}{2}$, the maximum principle infers that $\bar{u} > 0$ in $(-\infty, \infty) \times [0, h_\infty)$. Thus we can apply the Hopf boundary lemma to conclude that $\sigma_0 := \bar{u}_x(0, h_\infty) < 0$. It follows that $u_x(t_n, h(t_n)) = \partial u_n(0, h(t_n)) \leq \frac{\sigma_0}{2} < 0$ for all large $i$, and hence $h'(t_n) \geq -\mu \sigma_0/2 > 0$ for all large $i$. This is a contradiction.

Next we prove $\lim_{t \to \infty} v(t, x) = V(x)$ uniformly in any bounded subset of $[0, \infty)$. In what follows, we use a squeezing argument developed in [7] to prove our result. The proof can be done by modifying the arguments of [3, 7]. We provide the details of proof here for the reader’s convenience.

Since $\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0$ for $t \geq 0$ and $u(t, x) = 0$ for $x \geq h(t)$, then for any small $\varepsilon > 0$ there exists $T > 0$ such that $0 < b(x)u(t, x) \leq \|b\|_{L^\infty} u(t, x) \leq \varepsilon$ for $t \geq T$ and $x \in [0, \infty)$. For any $L > 0$, we consider the following problem
\[
\begin{align*}
-d_2 z_{xx} &= z(m_2(x) - \varepsilon - z), \quad 0 < x < L, \\
z_x(0) &= z(L) = 0.
\end{align*}
\] (4.3)
Since \( m_2(x) \) satisfies the condition \((H'_0)\), we have \( \Sigma_{d_2} = \{ h > 0 : \lambda_1(d_2, h, m_2(x)) = 0 \} \neq \emptyset \) by Corollary 3.2. Thus, we may assume \( L_0 \in \Sigma_{d_2} \), and then \( \lambda_1(d_2, L, m_2(x)) < 0 \) for any \( L > L_0 \). Since \( \lambda_1(d, h, k(x)) \) is a strictly decreasing continuous function in \( k(x) \), then \( \lambda_1(d_2, L, m_2(x) - \varepsilon) < 0 \) for small \( \varepsilon \). Hence we can construct a lower solution for (4.3) as \( z = \varepsilon \varphi_1 \), where \( \varphi_1 \) is the eigenfunction of \( \lambda_1(d_2, L, m_2(x) - \varepsilon) \) and \( \varepsilon \) is a small constant. On the other hand, any constant \( K \geq \| m_2(x) \|_{L^\infty} \) is an upper solution for (4.3). By the standard upper and lower solution method, (4.3) has at least one positive solution. It follows from the comparison principle (Lemma 2.1 in [7]) that (4.3) has at most one positive solution. Therefore, for any \( L > L_0 \), (4.3) has a unique positive solution, denoted by \( z_L^* \).

We next consider the following boundary blow-up problem

\[
\begin{cases}
-d_2 w_{xx} = w(m_2(x) - w), & 0 < x < L, \\
w_x(0) = 0, & w(L) = \infty.
\end{cases}
\]

Similarly as above, we want to show that (4.4) has a unique positive solution \( w_L \) for any \( L \gg 1 \). In fact, by using the same arguments in Lemma 2.3 in [7], one can prove that

\[
\begin{cases}
-d_2 \bar{w}_{xx} = \bar{w}(\bar{m_2}(x) - \bar{w}), & -L < x < L, \\
\bar{w}(\pm L) = \infty,
\end{cases}
\]

admits a unique positive solution \( \bar{w} \), where

\[
\bar{m_2}(x) = \begin{cases}
m_2(x), & x \in (0, L), \\
m_2(-x), & x \in (-L, 0).
\end{cases}
\]

Since (4.5) is invariant under the transformation \( x \mapsto -x \), we see that \( \bar{w}|_{x \geq 0} \) is a positive solution of (4.4). Moreover, it follows from the uniqueness of \( \bar{w} \) that (4.4) has at most one positive solution.

So \( \bar{w}_L \) is unique and \( w_L = \bar{w}|_{x \geq 0} \).

Using the comparison principle (Lemma 2.1 in [7]) again, we see that as \( \varepsilon \to 0 \) and \( L \to \infty \), \( z_L^* \) increases to the unique positive solution \( V^e(x) \) of (4.2) with \( m_2(x) \) replaced by \( m_2(x) - \varepsilon \) and \( w_L \) decreases to \( V(x) \) of (4.2).

Now we choose a decreasing sequence \( \{ \varepsilon_n \} \) and an increasing sequence \( \{ L_n \} \) such that \( \varepsilon_n > 0, L_n > h_0 \) for all \( n \) and \( \varepsilon_n \to 0, L_n \to \infty \) as \( n \to \infty \). Clearly, both \( z_{L_n}^e \) and \( w_{L_n} \) converge to \( V(x) \) as \( n \to \infty \), and for each \( n \), there exists \( T_n > 0 \) such that \( h(t) \geq L_n \) for \( t \geq T_n \). From the choice of \( d_2 \) and \( L_n \) we know that the following problem

\[
\begin{cases}
Z_t - d_2 Z_{xx} = Z(m_2(x) - \varepsilon_n - Z), & t \geq T_n, \ 0 < x < L_n, \\
Z_x(t, 0) = Z(t, L_n) = 0, & t \geq T_n, \\
Z(T_n, x) = v(T_n, x), & 0 < x < L_n,
\end{cases}
\]

admits a unique positive solution \( Z_n(t, x) \) satisfying

\[
Z_n(t, x) \to z_{L_n}^e(x) \quad \text{uniformly for } x \in [0, L_n] \quad \text{as } t \to \infty.
\]

It follows from the comparison principle that

\[
Z_n(t, x) \leq v(t, x) \quad \text{for } t \geq T_n \quad \text{and } x \in [0, L_n].
\]
Hence
\[ \liminf_{t \to \infty} v(t, x) \geq z_{L_n}^n \text{ uniformly for } x \in [0, L_n]. \]
By Letting \( n \to \infty \) in the above inequality, we attain
\[ \liminf_{t \to \infty} v(t, x) \geq V(x) \text{ locally uniformly for } x \in [0, \infty). \] (4.6)
Similarly one can prove
\[ \limsup_{t \to \infty} v(t, x) \leq w_{L_n} \text{ uniformly for } x \in [0, L_n], \]
which implies (by sending \( n \to \infty \))
\[ \limsup_{t \to \infty} v(t, x) \leq V(x) \text{ locally uniformly for } x \in [0, \infty). \] (4.7)
The desired result would then follow directly (4.6) and (4.7). \( \square \)

**Lemma 4.2.** If \( h_\infty = \infty \), then \( U(x) \leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \hat{U}(x) \) uniformly in any compact subset of \([0, \infty)\), where \( U(x) \) is the unique positive solution of
\[
\begin{cases}
- d_1 U_{xx} = U(m_1(x) - c(x)V(x) - U), & 0 < x < \infty, \\
U_x(0) = 0, 
\end{cases}
\] (4.8)
and \( \hat{U}(x) \) is the unique positive solution of
\[
\begin{cases}
- d_1 \hat{U}_{xx} = \hat{U}(m_1(x) - \hat{U}), & 0 < x < \infty, \\
\hat{U}_x(0) = 0. 
\end{cases}
\] (4.9)
**Proof.** By Theorem 1 in [6], we have
\[ m_{2, *} \leq \liminf_{x \to \infty} V(x) \leq \limsup_{x \to \infty} V(x) \leq m_2^*, \]
where \( V(x) \) satisfies (4.2). Thus, there exists a positive constant \( K \) such that \( 0 < K \leq V(x) \leq \frac{1}{K} \).
Moreover, since (4.1) holds, then we know that
\[ 0 < m_{1, * - c^*m_2^*} \leq \liminf_{x \to \infty} (m_1(x) - c(x)V(x)) \leq \limsup_{x \to \infty} (m_1(x) - c(x)V(x)) \leq m_1^* - c_*m_{2, *} < \infty. \] (4.10)
Define
\[ \bar{v}(t, x) = (1 + He^{-Kt})V(x), \]
where \( H = \frac{1}{K} ||v_0||_\infty \) and \( V \) satisfies (4.2). Direct calculations yield
\[
\bar{v}_t - d_2 \bar{v}_{xx} - \bar{v}(m_2(x) - \bar{v}(x)) = He^{-Kt}V(x)[-K + (1 + He^{-Kt})V(x)] \\
\geq He^{-Kt}V(x)[-K + K] = 0,
\]
and \( \bar{v}(0, x) = (1 + H)V(x) = (1 + \frac{1}{K} ||v_0||_\infty)V(x) > ||v_0||_\infty > v_0(x) \) for any \( x \geq 0 \). Since \( \lim_{t \to \infty} \bar{v}(t, x) = V(x) \) uniformly in \([0, \infty)\), for any given \( 0 < \varepsilon < 1 \), there exists \( T_\varepsilon > 0 \) such that \( \bar{v}(t, x) \leq V(x) + \varepsilon \) for any \( t \geq T_\varepsilon, x \in [0, \infty) \). By the comparison principle, we have \( v(t, x) \leq V(x) + \varepsilon \) for any \( t \geq T_\varepsilon, x \in [0, \infty) \).

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Since $h_\infty = \infty$, then for any given $0 < \varepsilon < 1$ and $L \gg 1$, there exists a large $T \geq T_\varepsilon$, such that

$$h(t) > L, \quad v(t, x) < V(x) + \varepsilon, \quad \text{for any } x \in [0, L], \ t \in [T, \infty).$$

Let $u^*_L(t, x)$ be the unique solution of

$$\begin{cases}
    u_t - d_1 u_{xx} = u(m_1(x) - c(x)(V(x) + \varepsilon) - u), & t \geq T, \ 0 < x < L, \\
    u_x(t, 0) = 0 = u(t, L), & t \geq T, \\
    u(T, x) = u(T, x), & 0 < x < L.
\end{cases}$$

Then $u(t, x) \geq u^*_L(t, x)$ for $x \in [0, L], t \in [T, \infty)$. Since $L \gg 1$, we can deduce that $u^*_L(t, x) \to U^*_L(x)$, where $U^*_L(x)$ is the unique positive solution of

$$\begin{cases}
    -d_1 u_{xx} = u(m_1(x) - c(x)(V(x) + \varepsilon) - u), & 0 < x < L, \\
    u_x(0) = 0 = u(L).
\end{cases}$$

Hence, $\liminf_{t \to \infty} u(t, x) \geq U^*_L(x)$ uniformly in $[0, L]$. By Lemma 4.2 in [31], we know that $\lim_{L \to \infty} U^*_L(x) = U^*(x)$ uniformly in any compact subset of $[0, \infty)$, where $U^*(x)$ is the unique positive solution of

$$\begin{cases}
    -d_1 u_{xx} = u(m_1(x) - c(x)(V(x) + \varepsilon) - u), & 0 < x < \infty, \\
    u_x(0) = 0.
\end{cases}$$

Letting $\varepsilon \to 0^+$, it follows that $\liminf_{t \to \infty} u(t, x) \geq U(x)$ uniformly in any compact subset of $[0, \infty)$, where $U(x)$ satisfies (4.8).

On the other hand, from Lemma 2.2, we have

$$\begin{align*}
0 < u(t, x) &\leq C_1, \quad \text{for } t \in [0, \infty), \ x \in [0, h(t)), \\
0 < v(t, x) &\leq C_2, \quad \text{for } t \in [0, \infty), \ x \in [0, \infty).
\end{align*}$$

Therefore $u$ satisfies

$$\begin{cases}
    u_t - d_1 u_{xx} \leq u(m_1(x) - u), & t > 0, \ 0 < x < h(t), \\
    u_x(t, 0) = 0, u(t, x) = 0, & t > 0, \ h(t) \leq x < \infty, \\
    h'(t) = -\mu u_x(t, h(t)), & t > 0, \\
    h(0) = h_0, u(0, x) = u_0(x), & 0 \leq x \leq h_0.
\end{cases}$$

Now we consider the following problem

$$\begin{cases}
    \bar{u}_t - d_1 \bar{u}_{xx} = \bar{u}(m_1(x) - \bar{u}), & t > 0, \ 0 < x < \bar{h}(t), \\
    \bar{u}_x(t, 0) = 0, \bar{u}(t, x) = 0, & t > 0, \ \bar{h}(t) \leq x < \infty, \\
    \bar{h}'(t) = -\mu \bar{u}_x(t, \bar{h}(t)), & t > 0, \\
    \bar{h}(0) = h_0, \bar{u}(0, x) = u_0(x), & 0 \leq x \leq h_0.
\end{cases}$$

(4.11)

It follows from the comparison principle that

$$0 \leq u(t, x) \leq \bar{u}(t, x) \quad \text{and} \quad h(t) \leq \bar{h}(t) \quad \text{for } t \geq 0, \ 0 \leq x < h(t).$$
Since $h_\infty = \infty$, then we have $h_\infty = \infty$. By Theorem 4.2 in [31], we have $\lim_{t \to \infty} \bar{u}(t, x) = \bar{U}(x)$ in $C_{\text{loc}}([0, \infty))$, where $\bar{U}(x)$ is defined in (4.9). Thus, $\liminf_{t \to \infty} v(t, x) \leq \bar{U}(x)$ uniformly in any compact subset of $[0, \infty)$, which completes the proof of Lemma 4.2. \hfill \Box

The following result gives a sufficient condition for spreading and an estimate of $h_\infty$ when $h_\infty < \infty$.

**Lemma 4.3.** If $h_\infty < \infty$, then $h_\infty \leq h^*(d_1, m_1(x) - c(x)V(x))$, where $V(x)$ is the unique positive solution of (4.2).

**Proof.** By Corollary 3.2, we know that under the assumption (4.1), there exists $h^*(d_1, m_1(x) - c(x)V(x)) > 0$ such that $\lambda_1(d_1, h^*(d_1, m_1(x) - c(x)V(x)))$, $m_1(x) - c(x)V(x)) = 0$.

We assume $h_\infty > h^*(d_1, m_1(x) - c(x)V(x))$ to get a contradiction. Note that $h^*(d_1, k(x))$ is a strictly decreasing continuous function in $k(x)$, and due to Lemma 4.1, it is easily to see that for any given $0 < \varepsilon \ll 1$ there exists $T_\varepsilon > 1$ such that

$$h(T_\varepsilon) > \max\{h_0, h^*(d_1, m_1(x) - c(x)(V(x) + \varepsilon))\},$$

$$v(t, x) \leq V(x) + \varepsilon, \quad \forall t \geq T_\varepsilon, \quad x \in [0, h_\infty].$$

Set $L = h(T_\varepsilon)$, then $L > h^*(d_1, m_1(x) - c(x)(V(x) + \varepsilon))$. Let $u(t, x)$ be the unique positive solution of the following initial boundary value problem with fixed boundary

\[
\begin{cases}
  u_t - d_1u_{xx} = u(m_1(x) - c(x)V(x) + \varepsilon) - u, & t > T_\varepsilon, \quad 0 < x < L, \\
  u_x(t, 0) = u(t, L) = 0, & t > T_\varepsilon, \\
  u(T_\varepsilon, x) = u(T_\varepsilon, x), & 0 \leq x \leq L.
\end{cases}
\]

By the comparison principle

$$u(t, x) \geq \underline{u}(t, x), \quad \forall t > T_\varepsilon, \quad 0 \leq x \leq L.$$

Since $\lambda_1(d_1, L, m_1(x) - c(x)(V(x) + \varepsilon)) < \lambda_1(d_1, h^*(d_1, m_1(x) - c(x)(V(x) + \varepsilon)), m_1(x) - c(x)(V(x) + \varepsilon)) = 0$, we know that $\underline{u}(t, x) \to \underline{u}^*(x)$ as $t \to \infty$ uniformly for $x \in [0, L]$ (see Proposition 3.3 in [21]), where $\underline{u}^*(x)$ is the unique positive solution of

\[
\begin{cases}
  -d_1u_{xx} = u(m_1(x) - c(x)(V(x) + \varepsilon) - u), & 0 < x < L, \\
  u_x(0) = 0, u(L) = 0.
\end{cases}
\]

Hence, $\liminf_{t \to \infty} u(t, x) \geq \lim_{t \to \infty} \underline{u}(t, x) = \underline{u}^*(x) > 0$ in $[0, L]$. This contradicts to Lemma 4.1. \hfill \Box

According Lemma 4.3, we directly have

**Corollary 4.1.** If $h_0 > h^*(d_1, m_1(x) - c(x)V(x))$, then $h_\infty = \infty$.

Combining Lemma 4.1 - 4.3, we have the following dichotomy theorem.

**Theorem 4.1.** Let $(u(t, x), v(t, x), h(t))$ be any solution of (1.1). Then, the following alternative holds:

Either (i) spreading: $h_\infty = \infty$ and $U(x) \leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \bar{U}(x)$ uniformly in any compact subset of $[0, \infty)$;

or (ii) vanishing: $h_\infty \leq h^*(d_1, m_1(x) - c(x)V(x))$ and $\limsup_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t))]} = 0.$
4.2 Sharp criteria for spreading and vanishing

In this subsection, we will establish sharp criteria by select $d_1$, $h_0$, $u_0(x)$ and $\mu$ as varying parameters to distinguish the spreading-vanishing dichotomy for the invasive species ($u$). The following theorem shows that the invader cannot establish itself and the native species always survives the invasion if the dispersal rate $d_1$ of invasive species is large and the initial density $u_0(x)$ is small.

**Theorem 4.2.** If $d_1 > d^*(h_0, m_1(x))$ and $\|u_0(x)\|_{C(\{0, h_0\})}$ is small, then
\begin{enumerate}[(i)]  \item $h_\infty < \infty$, \lim_{t \to \infty} \|u(t, x)\|_{C(\{0, h_0(t)\})} = 0$;  \item $\lim_{t \to \infty} v(t, x) = V(x)$ uniformly in any bounded subset of $[0, \infty)$, where $V(x)$ satisfies (4.2). \end{enumerate}

**Proof.** In (4.11), we have known that $u(t, x) \leq \bar{u}(t, x)$ and $h(t) \leq \bar{h}(t)$ for $t \geq 0$ and $0 \leq x < h(t)$. According to Theorem 4.2 in [4], we have that $\lim_{t \to \infty} \|\bar{u}(t, x)\|_{C(\{0, \bar{h}(t)\})} = 0$, $\bar{h}(t) < \infty$ for $t \geq 0$ under the assumptions that $d^*(h_0, m_1(x)) < d_1$ and $\|u_0(x)\|_{C^2(\{0, h_0\})}$ is small. It implies $\lim_{t \to \infty} \|u(t, x)\|_{C(\{0, h(t)\})} = 0$ and $h(t) < \infty$ for $t \geq 0$.

On the other hand, we can use the same way as the proof of Lemma 4.1 to deduce that $\lim_{t \to \infty} v(t, x) = V(x)$ uniformly in any bounded subset of $[0, \infty)$ under the above assumptions, where $V(x)$ is the unique positive solution of (4.2). \hfill \Box

Actually, due to Theorem 4.3 in [4], we can prove a more general result by using the same arguments as Theorem 4.2.

**Theorem 4.3.** If $d_1 > d^*(h_0, m_1(x))$, then there exists $\mu_0 > 0$ depending on $u_0$ such that when $0 < \mu < \mu_0$
\begin{enumerate}[(i)]  \item $h_\infty < \infty$, \lim_{t \to \infty} \|u(t, x)\|_{C(\{0, h(t)\})} = 0$;  \item $\lim_{t \to \infty} v(t, x) = V(x)$ uniformly in any bounded subset of $[0, \infty)$, where $V(x)$ is the unique positive solution of (4.2). \end{enumerate}

**Remark 4.1.** In Theorem 4.2 and 4.3, we can replace $d_1 > d^*(h_0, m_1(x))$ by $h_0 < h^*(d_1, m_1(x))$. In fact, by the strictly monotony of $\lambda_1(d, h, m_1(x))$ in $d$ and $h$ (see Theorem 3.1 (i) and Theorem 3.2 (i)), we know $d_1 > d^*(h_0, m_1(x))$, $\lambda_1(d_1, h_0, m_1(x)) > 0$ and $h_0 < h^*(d_1, m_1(x))$ are equivalent.

Next, we show that the invasive species can spread successfully if the dispersal rate of invasive species is small.

**Theorem 4.4.** If $0 < d_1 \leq d^*(h_0, m_1(x) - c(x)V(x))$, then $h_\infty = \infty$, which implies spreading of the invasive species happens, where $V(x)$ is the unique positive solution of (4.2).

**Proof.** First, we prove the case $0 < d_1 < d^*(h_0, m_1(x) - c(x)V(x))$.

Note that in Lemma 4.2 we have defined $\bar{v}(t, x) \triangleq (1 + He^{-Kt})V(x) = (1 + \frac{1}{K^\mu}||v_0||_{\infty} e^{-Kt})V(x)$. We use $\lambda_1$ and $\varphi_1$ to denote the principle eigenvalue and the corresponding eigenfunction of problem (3.1) with $d = d_1$, $h = h(T_\epsilon)$, $k(x) = m_1(x) - c(x)V(x)$, respectively. Since $\lambda_1(d^*(h_0, m_1(x) - c(x)V(x)), h_0, m_1(x) - c(x)V(x)) = 0$ and $h(t)$ is strictly increasing with respect to $t$, it follows from Corollary 3.1 and 3.3 that $\lambda_1 < 0$.

Now we set
\[ u(t, x) = \begin{cases} c\varphi_1(x), & \text{for } t \geq T_\epsilon, \ x \in [0, h(T_\epsilon)], \\ 0, & \text{for } t \geq T_\epsilon, \ x > h(T_\epsilon). \end{cases} \]
Choose $\varepsilon > 0$ and $\epsilon > 0$ so small that

$$
\varepsilon \varphi_1 \leq -\lambda_1 - \varepsilon \|c\|_{\infty} V(x) \quad \text{and} \quad \varepsilon \varphi_1(x) \leq u(T_\varepsilon, x) \quad \text{for} \quad x \in [0, h(T_\varepsilon)].
$$

Then by direct calculation yields

$$
\begin{align*}
\frac{d}{dt} h(T_\varepsilon) &= 0, \quad t > T_\varepsilon, \quad x = h(T_\varepsilon), \\
\frac{d}{dt} h(T_\varepsilon) &= \frac{d}{dt} (h, m_1(x) - c(x)V(x)), \quad t > T_\varepsilon, \quad x > h(T_\varepsilon), \\
\frac{d}{dt} h(T_\varepsilon) &= 0, \quad t > T_\varepsilon, \quad x = h(T_\varepsilon).
\end{align*}
$$

By the comparison principle, we have

$$
u(t, x) \geq \nu(t, x) \quad \text{in} \quad [T_\varepsilon, \infty) \times [0, h(T_\varepsilon)].$$

It follows that

$$\liminf_{t \to \infty} \|v(t, \cdot)\|_{C([0, h(T_\varepsilon))]} \geq \varepsilon \varphi_1(0) > 0.$$ 

According to Lemma 4.1, we see that $h_\varepsilon = \infty$. Hence, by Lemma 4.2, spreading happens.

While for $d_1 = d^*(h_0, m_1(x) - c(x)V(x))$, we have $\lambda_1 = \lambda_1(d_1, h_0, m_1(x) - c(x)V(x)) = 0$. Using the monotonically of $h(t)$ again, we can select $t^* > 0$ such that $h(t^*) > h_0$. It follows from Corollary 3.3 that $\lambda_1(d_1, h(t^*), m_1(x) - c(x)V(x)) < \lambda_1(d_1, h_0, m_1(x) - c(x)V(x)) = 0$. Therefore, after replacing $h_0$ with $h(t^*)$, the same method employed above can obtain the desired result again.

In order to determine completely the effects of parameters $d_1, h_0, \mu$ and initial density $u_0(x)$ of the invasive species $u$, we first give a sufficient condition for the spreading of $u$ provided the principle eigenvalue $\lambda_1(d_1, m_1(x) - c(x)V(x)) > 0$, where $V(x)$ is the unique positive solution of (4.2).

**Theorem 4.5.** If $\lambda_1(d_1, h_0, m_1(x) - c(x)V(x)) > 0$, then $h_\varepsilon = \infty$ if $\|u_0(x)\|_{C([0, h_0])}$ is sufficiently large or if $\mu \geq \mu^0$, where $\mu^0$ depending on $u_0, v_0$ and $h_0$.

**Proof.** Note that in (4.10) we have

$$0 < m_{1, \ast} - c^* m_2^2 \leq \liminf_{x \to \infty} (m_1(x) - c(x)V(x)) \leq \limsup_{x \to \infty} (m_1(x) - c(x)V(x)) \leq m_1^1 - c \ast m_2, \ast < \infty.$$ 

Thus, by Corollary 3.2, we have $\Sigma_{d_2} = \{h > 0 : \lambda_1(d_2, h, m_1(x) - c(x)V(x)) = 0\} \neq \emptyset$ and

$$\lim_{L \to \infty} \lambda_1(d_1, \sqrt{L}, m_1(x) - c(x)V(x)) < 0.$$ 

Therefore, there exists $L^* > 0$ such that $\lambda_1(d_1, \sqrt{L^*}, m_1(x) - c(x)V(x)) < 0$.

Next, we construct a suitable lower solution to problem (1.1). Note that in Lemma 4.2 we have defined $\tilde{v}(t, x) \triangleq (1 + He^{-K_1 t})V(x) = (1 + \frac{1}{\sqrt{h}}\|v_0\|_{\infty} e^{-K_1 t})V(x)$, then $\tilde{v}(t, x) \leq \frac{1}{\sqrt{h}}(1 + \frac{1}{\sqrt{h}}\|v_0\|_{\infty}).$

Let $\lambda$ be the eigenvalue of

$$
\begin{align*}
-\frac{d}{dx} \varphi_{xx} - \frac{1}{2} \varphi_x &= \lambda \varphi, \quad 0 < x < 1, \\
\varphi'(0) &= \varphi(1) = 0.
\end{align*}
$$
the corresponding eigenfunction $\varphi > 0$ and $\varphi_x \leq 0$ in $[0, 1)$ and $\|\varphi\|_{L^\infty([0,1))} = 1$.

Defining

$$
\begin{cases}
    h(t) = \sqrt{t + \delta}, & t \geq 0, \\
    u(t, x) = \frac{M}{(t + \delta)^{k+1}} \varphi\left(\frac{x}{\sqrt{t + \delta}}\right), & t \geq 0, \quad 0 \leq x \leq \sqrt{t + \delta},
\end{cases}
$$

where $\delta, k, M$ are positive constants to be chosen later, we are now in a position to show that $(u, \bar{v}, h)$ is a lower solution of problem (1.1).

It follows from Lemma 2.2 that $0 \leq u \leq C_1$ for $t \geq 0, x \in [0, h(t)]$. Then there exists a positive constant $Q$ such that $m_1(x) - u - c(x)\bar{v} \geq \inf_{0 \leq x < \infty} m(x) - C_1 - \frac{1}{2} \left(1 + \frac{1}{t^2}\|v_0\|_{\infty}\right)\|c\|_{\infty} \geq -Q$. Direct calculations yield

$$
u_x - d_1 \nu_{xx} - \nu(m_1(x) - u - c(x)\bar{v}) = - \frac{M}{(t + \delta)^{k+1}} \left[k\varphi + \frac{\varphi_x^2}{2\sqrt{t + \delta}} + d_1\varphi_{xx} + (t + \delta)\varphi(m_1(x) - u - c(x)\bar{v})\right]$$

$$\leq - \frac{M}{(t + \delta)^{k+1}} \left[k\varphi + \frac{\varphi_x^2}{2\sqrt{t + \delta}} + d_1\varphi_{xx} - Q(t + \delta)\varphi\right].$$

Choosing $0 < \delta \leq 1, \lambda + Q(L + 1) < k$, we obtain

$$u_x - d_1 u_{xx} - u(m_1(x) - u - c(x)\bar{v}) \leq - \frac{M}{(t + \delta)^{k+1}} \left(\frac{1}{2}\varphi_x + d_1\varphi_{xx} + \lambda\varphi\right) = 0,$$

for $0 < x < h(t)$ and $0 < t \leq L^*$. (i) We may choose $0 < \delta \leq h_0^2$ and select $\mu > 0$ as a large value such that $\mu \geq \mu^0 \triangleq -\frac{(L^*+1)^2}{2M\varphi_x(1)}$, then we have

$$h'(t) + \mu u_x(t, h(t)) = \frac{1}{2\sqrt{t + \delta}} + \frac{\mu M \varphi_x(1)}{(t + \delta)^{k+1/2}} \leq 0 \quad \text{for } 0 < t \leq L^*. \quad (4.12)$$

Moreover, we select $M > 0$ being sufficiently small such that

$$u(0, x) = \frac{M}{\delta^k} \varphi\left(\frac{x}{\sqrt{\delta}}\right) < u_0(x) \quad \text{in } [0, \sqrt{\delta}]. \quad (4.13)$$

(ii) We may select $M$ and $\|u_0\|_{C([0, h_0])}$ being sufficiently large such that (4.12) and (4.13) hold.

Either by (i) or (ii), we have

$$
\begin{cases}
    u_x - d_1 u_{xx} \leq u(m_1(x) - u - c(x)\bar{v}), & 0 < t \leq L^*, \quad 0 < x < h(t), \\
    u_x(t, 0) = 0, u(t, h(t)) = 0, & 0 < t \leq L^*, \\
    h'(t) + \mu u_x(t, h(t)) \leq 0, & 0 < t \leq L^*, \\
    u(0, x) \leq u_0(x), & 0 \leq x \leq \sqrt{\delta}.
\end{cases}
$$

By the comparison principle to conclude that $h(t) \leq h(t)$ in $[0, L^*]$. Specially, we derive $h(L^*) \geq h(L^*) = \sqrt{L^* + \delta} \geq \sqrt{L^*}$. Since $\lambda_1(d_1, \sqrt{L^*}, m_1(x) - c(x)V(x)) < 0$, according to the strictly monotone decreasing of $\lambda_1(d_1, h, m_1(x) - c(x)V(x))$ in $h$, we have $h(L^*) \geq \sqrt{L^*} > h^*(d_1, m_1(x) - c(x)V(x))$, which implies $h_\infty > h^*(d_1, m_1(x) - c(x)V(x))$. From Lemma 4.3, we obtain $h_\infty = \infty$. □

According to the strict monotone increasing of $\lambda^*(d_1, h(t), m_1(x) - c(x)V(x))$ in $d_1$, and Theorem 4.5, we have
**Corollary 4.2.** If \( d_1 > d^*(h_0, m_1(x)) - c(x)V(x) \), then \( h_{\infty} = \infty \) if \( \|u_0\|_{C([0,T])} \) is sufficiently large or if \( \mu > \mu^0 \), where \( \mu^0 \) depending on \( u_0, v_0 \) and \( h_0 \).

Similarly, due to the strict monotone decreasing of \( \lambda^*(d_1, h(t), m_1(x) - c(x)V(x)) \) in \( h(t) \), and Theorem 4.5, we obtain

**Corollary 4.3.** If \( 0 < h_0 < h^*(d_1, m_1(x) - c(x)V(x)) \), then \( h_{\infty} = \infty \) if \( \|u_0\|_{C([0,T])} \) is sufficiently large or if \( \mu > \mu^0 \), where \( \mu^0 \) depending on \( u_0, v_0 \) and \( h_0 \).

If \( h_0 \) is fixed, the sharp criteria for spreading-vanishing of an invasive species \( (u) \) depends on the diffusion rate \( d_1 \), the initial occupying habitat \( h_0 \) and the initial number \( u_0(x) \) of the invasive species. It follows from Theorem 4.2, 4.4 and Corollary 4.2.

**Theorem 4.6.** There exist \( d^*(h_0, m_1(x)) \) and \( d^*(h_0, m_1(x) - c(x)V(x)) \) defined in \((0, \infty)\) such that

(i) For small initial value \( u_0(x) \), vanishing occurs if \( d_1 > d^*(h_0, m_1(x)) \);

(ii) Spreading happens if one of the following results holds:

(a) if \( 0 < d_1 \leq d^*(h_0, m_1(x) - c(x)V(x)) \);

(b) if \( d_1 > d^*(h_0, m_1(x) - c(x)V(x)) \) and \( \|u_0\|_{C([0,T])} \) is sufficiently large.

Similarly, if \( d_1 \) is fixed, the sharp criteria for spreading-vanishing of an invasive species \( (u) \) depends on \( h_0 \) of the initial occupying habitat and the initial number \( u_0(x) \) of the species \( (u) \). It follows from Corollary 4.1, 4.3, and Remark 4.1 that

**Theorem 4.7.** There exist \( h^*(d_1, m_1(x)) \) and \( h^*(d_1, m_1(x) - c(x)V(x)) \) defined in \((0, \infty)\) such that

(i) For small initial value \( u_0(x) \), vanishing occurs if \( h_0 > h^*(d_1, m_1(x)) \);

(ii) spreading happens if one of the following holds:

(a) if \( h_0 \geq h^*(d_1, m_1(x) - c(x)V(x)) \);

(b) if \( 0 < h_0 < h^*(d_1, m_1(x) - c(x)V(x)) \) and \( \|u_0\|_{C([0,T])} \) is sufficiently large.

Next, if \( d_1 \) and \( h_0 \) are fixed, the initial number \( u_0(x) \) governs the spreading and vanishing of the invasive species. Then we can derive the sharp criteria for spreading-vanishing of an invasive species \( (u) \) from Theorem 4.6 and 4.7, by the same arguments as Theorem 5.7 in [5].

**Theorem 4.8.** For any \( d_1 > 0, h_0 > 0 \) and given \( v_0 \), which satisfies (1.2), if \( u_0(x) = \epsilon \theta(x) \) for some \( \epsilon > 0 \) and \( \theta(x) \) such that \( u_0 \) satisfies (1.2), then \( \epsilon^* \) exists depending on \( \theta, v_0 \) and \( d_1 \) such that spreading occurs if \( \epsilon > \epsilon^* \), and vanishing happens if \( 0 < \epsilon \leq \epsilon^* \). Moreover, \( \epsilon^* = 0 \) if \( 0 < d_1 \leq d^*(h_0, m_1(x) - c(x)V(x)) \) (resp. \( h_0 \geq h^*(d_1, m_1(x) - c(x)V(x)) \)), \( \epsilon^* \geq 0 \) if \( d_1 > d^*(h_0, m_1(x) - c(x)V(x)) \) (resp. \( 0 < h_0 < h^*(d_1, m_1(x) - c(x)V(x)) \)), and \( \epsilon^* > 0 \) if \( d_1 > d^*(h_0, m_1(x)) \) (resp. \( h_0 < h^*(d_1, m_1(x)) \)).

Now we can derive the sharp criteria for spreading-vanishing of an invasive species \( (u) \) from Theorem 4.3, Corollary 4.2, 4.3, and Remark 4.1 by choosing the expansion capability \( \mu \) as parameter.

**Theorem 4.9.** For any \( d_1 > 0, h_0 > 0 \) and given \( v_0 \), which satisfies (1.2), \( \mu^* \) exists depending on \( u_0, v_0, h_0 \) and \( d_1 \) such that spreading occurs if \( \mu > \mu^* \), and vanishing occurs if \( 0 < \mu < \mu^* \). Moreover, \( \mu^* = 0 \) if \( 0 < d_1 \leq d^*(h_0, m_1(x) - c(x)V(x)) \) (resp. \( h_0 \geq h^*(d_1, m_1(x) - c(x)V(x)) \)), \( \mu^* \geq 0 \) if \( d_1 > d^*(h_0, m_1(x) - c(x)V(x)) \) (resp. \( 0 < h_0 < h^*(d_1, m_1(x) - c(x)V(x)) \)), and \( \mu^* > 0 \) if
$d_1 > d^*(h_0, m_1(x))$ (reap. $h_0 < h^*(d_1, m_1(x))$).

5 Estimates of the Spreading Speed

In this section, we give some rough estimates on the spreading speed of $h(t)$ for the case that spreading of $u$ happens. We first recall Proposition 3.1 of [10], whose proof is given in [34].

**Proposition 5.1.** For any given constants $a > 0$, $d > 0$ and $K \in [0, 2\sqrt{ad}]$, the problem

$$-dU'' + KU' = aU - U^2 \quad \text{in}(0, \infty), \quad U(0) = 0$$

admits a unique positive solution $U = U_K(x)$ satisfying $U'_K(x) > 0$ for $x \geq 0$. Moreover, for each $\mu > 0$, there exists a unique $K_0 = K_0(\mu, a) \in (0, 2\sqrt{ad})$ such that $\mu U'_K(0) = K_0$.

Making use of the function $K_0(\mu, a)$, we have the following estimate for the spreading speed of $h(t)$.

**Theorem 5.1.** Assume (4.1) holds. If $h_\infty = +\infty$, then

$$K_0(\mu, m_{1,*} - c^* m_2^*) \leq \liminf_{t \to +\infty} \frac{h(t)}{t} \leq \limsup_{t \to +\infty} \frac{h(t)}{t} \leq K_0(\mu, m_1^*).$$

**Proof.** Consider the following auxiliary problem

$$
\begin{align*}
\bar{u}_t - d_1 \bar{u}_{xx} &= \bar{u}(m_1(x) - \bar{u}), \quad t > 0, \quad 0 < x < \bar{h}(t), \\
\bar{u}_x(t, 0) &= 0, \quad \bar{u}(t, x) = 0, \quad t > 0, \quad \bar{h}(t) \leq x < \infty, \\
\hat{h}'(t) &= -\mu \bar{u}_x(t, \hat{h}(t)), \quad t > 0, \\
\bar{u}(0, x) &= u_0(x), \quad 0 \leq x \leq h_0.
\end{align*}
$$

By the comparison principle, it follows that $\tilde{h}(t) \geq h(t) \to +\infty$ as $t \to \infty$. By Theorem 6.1 in [4],

$$\limsup_{t \to +\infty} \frac{\tilde{h}(t)}{t} \leq K_0(\mu, m_1^*).$$

Thus, we have

$$\limsup_{t \to +\infty} \frac{h(t)}{t} \leq \limsup_{t \to +\infty} \frac{\tilde{h}(t)}{t} \leq K_0(\mu, m_1^*).$$

Next, we prove that $K_0(\mu, m_{1,*} - c^* m_2^*) \leq \liminf_{t \to +\infty} \frac{h(t)}{t}$.

As in the proof of Lemma 4.2, we know for any small $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that $v(t, x) \leq V(x) + \varepsilon$ for $t \geq T_\varepsilon$ and $x \in [0, \infty)$. Since $h_\infty = +\infty$, there exists $T'_\varepsilon > 0$ such that $h(T'_\varepsilon) > h^*(d_1, m_1(x) - c(x)V(x) + \varepsilon))$. Taking $\bar{T}_\varepsilon \triangleq \max\{T_\varepsilon, T'_\varepsilon\}$, then $(u, h)$ satisfies

$$
\begin{align*}
u_t - d_1 v_{xx} &\geq v(m_1(x) - c(x)V(x) + \varepsilon) - u, \quad t > \bar{T}_\varepsilon, \quad 0 < x < \bar{h}(t), \\
u_x(t, 0) &= 0, \quad v(t, x) = 0, \quad t > \bar{T}_\varepsilon, \quad h(t) \leq x < \infty, \\
v'(t) &= -\mu v_x(t, h(t)), \quad t > \bar{T}_\varepsilon, \\
u(T_\varepsilon, x) &= u(T_\varepsilon, x) > 0, \quad 0 < x \leq h(\bar{T}_\varepsilon).
\end{align*}
$$

This implies that $(u, h)$ is an upper solution to the problem

$$
\begin{align*}
u_t - d_1 \nu_{xx} &= \nu(m_1(x) - c(x)V(x) + \varepsilon) - \nu, \quad t > \bar{T}_\varepsilon, \quad 0 < x < \bar{h}(t), \\
u_x(t, 0) &= 0, \quad \nu(t, x) = 0, \quad t > \bar{T}_\varepsilon, \quad \bar{h}(t) \leq x < \infty, \\
u'(t) &= -\mu \nu_x(t, h(t)), \quad t > \bar{T}_\varepsilon, \\
u(T_\varepsilon, x) &= u(T_\varepsilon, x) > 0, \quad \bar{h}(\bar{T}_\varepsilon) = h(\bar{T}_\varepsilon), \quad 0 < x \leq h(\bar{T}_\varepsilon).
\end{align*}
$$
By Corollary 3.1, $h_\infty = \infty$ since $h(T_\varepsilon) = h(T_\varepsilon) > h^*(d_1, m_1(x) - c(x)(V(x) + \varepsilon))$. Moreover, by Theorem 1 in [6], we have $\limsup_{x \to \infty} c(x)V(x) \leq m_2^* c^*$. It follows from [4] that $\liminf_{t \to +\infty} \frac{h(t)}{t} \geq K_0(\mu, m_1, s - (m_2^* + \varepsilon)c^*)$, which implies that $K_0(\mu, m_1, s - m_2^* c^*) \leq \liminf_{t \to +\infty} \frac{h(t)}{t}$ for any $\varepsilon > 0$. Let $\varepsilon \to 0$ and using the continuity of $K_0$ with respect to its components, we immediately obtain the desired result. \hfill \square

6 Discussions

In this paper, we use a free boundary problem to describe a Lotka-Volterra type competition model with the spreading of an invasive species $(u)$ and native species $(v)$ in a one-dimensional habitat. We take into account the environmental heterogeneity and divide it into two cases: strong heterogeneous environment and weak heterogeneous environment. In both cases, sufficient conditions for two species $u$ and $v$ spreading or vanishing are derived. Furthermore, we obtain sharp criteria for spreading and vanishing. When spreading occurs, we take an estimate for the asymptotic spreading speed of the free boundary.

Compared with the previous work about free boundary problem, we are the first one to consider two species problem with a free boundary in strong heterogeneous environment and weak heterogeneous environment. Moreover, our conclusions provide a different way to understand the dynamics of problem (1.1) by choosing the diffusion rate $d_1$, the initial occupying habitat $h_0$, the initial value $u_0(x)$ of the species $(u)$ and the expansion capability $\mu$, which plays significant roles in population dynamics as variable parameters. In particular, we first obtain a spreading-vanishing dichotomy and sharp criteria for spreading and vanishing in the strong (resp. weak) heterogeneous environment of problem (1.1) by developing some new arguments. Also, we make contribution to extend the asymptotic spreading result from the original weak heterogeneous environment to some special cases of strong heterogeneous environment. These results are quite different from that of the corresponding problem in a fixed domain.

For the higher dimensional and radially symmetric case of (1.1), the methods of this paper are still valid and the corresponding results can be retained based on the results in [5]. The double free boundaries case for problem (1.1) will be considered in the forthcoming paper.

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