A note on \textit{fsg} groups in \textit{p}-adically closed fields

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Abstract

Let \( G \) be a definable group in a \( p \)-adically closed field \( M \). We show that \( G \) has finitely satisfiable generics (\textit{fsg}) if and only if \( G \) is definably compact. The case \( M = \mathbb{Q}_p \) was previously proved by Onshuus and Pillay.

1 Introduction

Work in a monster model \( \mathbb{M} \) of some theory. Let \( G \) be a definable group. Say that a definable set \( X \subseteq G \) is \textit{left generic} or \textit{right generic} if \( G \) can be covered by finitely many left translates or right translates of \( X \), respectively. A definable group \( G \) is said to have \textit{finitely satisfiable generics (fsg)} if there is a small model \( M_0 \) and a global type \( p \in S_G(\mathbb{M}) \) such that every left translate \( g \cdot p \) is finitely satisfiable in \( M_0 \). This notion is due to Hrushovski, Peterzil, and Pillay \cite{HP}, who prove the following facts:

\textbf{Fact 1.1} (\cite{HP} Proposition 4.2). Suppose \( G \) has \textit{fsg}, witnessed by \( p \) and \( M_0 \).

1. A definable set \( X \subseteq G \) is left generic iff it is right generic.

2. Non-generic sets form an ideal: if \( X \cup Y \) is generic, then \( X \) is generic or \( Y \) is generic.

3. A definable set \( X \) is generic if and only if every left translate of \( X \) intersects \( G(M_0) \).

If \( G \) is a group definable in a nice o-minimal structure, then \( G \) has \textit{fsg} if and only if \( G \) is definably compact \cite{L} Remark 5.3]. Our main theorem is an analogue for \( p \)-adically closed fields:

\textbf{Theorem 1.2}. Let \( M \) be a \textit{p}-adically closed field and \( G \) be an \( M \)-definable group. Then \( G \) is definably compact if and only if \( G \) has \textit{fsg}.

The case \( M = \mathbb{Q}_p \) was proved by Onshuus and Pillay \cite{OP} Corollary 2.3].
1.1 Notation

We denote the theory of $p$-adically closed fields by $p\text{CF}$. In a $p$-adically closed field $M$, we will let $\Gamma$ or $\Gamma(M)$ denote the value group, and $\mathcal{O}$ or $\mathcal{O}(M)$ denote the valuation ring. The valuation will be denoted $v(x)$, and written additively, so $v(xy) = v(x) + v(y)$ and $v(x + y) \geq \min(v(x), v(y))$. If $D$ is a definable set, we write $S_D(M)$ for the set of complete types over $M$ concentrating on $D$.

1.2 Outline

In Section 2 we review the notion of definable compactness in definable groups in $p$-adically closed fields. In Section 3 we give the easy direction of Theorem 1.2: if $G$ has fsig then $G$ is definably compact. In Section 4 we show that definable compactness is a definable property—it varies definably in a definable family of definable groups. This ensures that every definably compact group is part of a 0-definable family of definably compact groups. In Section 5 we use the VC-theorem to show that fsig is witnessed in a very uniform manner for definably compact groups over $\mathbb{Q}_p$. In Section 6 we show that definable compactness implies fsig by using Section 4 to transfer the “uniform fsig” of Section 5 from $\mathbb{Q}_p$ to its elementary extensions.

2 Definable compactness in $p\text{CF}$

If $X$ is a definable set and $\tau$ is a topology on $X$, we say that $\tau$ is definable if there is a definable family $\{B_i\}_{i \in I}$ such that $\{B_i : i \in I\}$ is a basis for the topology. Two examples of definable topologies are the order topology on an o-minimal structure and the valuation topology on a $p$-adically closed field. A definable topological space is a definable set with a definable topology.

We will use the following abstract notion of definable compactness, which works well in $p$-adically closed fields and o-minimal structures:

**Definition 2.1** ([1, 5]). A definable topological space $X$ is **definably compact** if the following holds: if $\{F_i\}_{i \in I}$ is a definable family of non-empty closed subsets of $X$, and $\{F_i : i \in I\}$ is downwards-directed, then the intersection $\bigcap_{i \in I} F_i$ is non-empty.

A definable subset $Y \subseteq X$ is definably compact if it is compact with the induced subspace topology, which is definable.

Definable compactness has many properties analogous to compactness [5, Section 3.1]. We will need the following two trivial observations:

**Fact 2.2.**

1. If $f : X \to Y$ is a definable continuous function between definable topological spaces, and $D \subseteq X$ is definably compact, then the image $f(D) \subseteq Y$ is definably compact.

2. If $X$ is a definable topological space and $D_1, D_2 \subseteq X$ are definably compact, then $D_1 \cup D_2$ is definably compact.
\section{Definable manifolds}

Work in a \( p \)-adically closed field \( M \). A \textit{definable manifold} is a Hausdorff definable topological space \( X \) covered by finitely many definable opens \( U_1, \ldots, U_k \), such that each \( U_i \) is definably homeomorphic to an open subset of \( M^n \). Definable manifolds arise naturally in the study of definable groups by the following theorem of Pillay:

**Fact 2.3** (\cite{8}). If \( G \) is a definable group (in a \( p \)-adically closed field), then there is a unique definable topology \( \tau \) on \( G \) such that

- The group operations are continuous.

- \((G, \tau)\) is a definable manifold.

On definable manifolds, we can give a concrete characterization of definable compactness. First consider the case \( M^n \):

**Fact 2.4** (\cite{6}, Lemmas 2.4, 2.5). A definable set \( D \subseteq M^n \) is definably compact iff it is closed and bounded.

Here, a set \( D \) is “bounded” if there is \( N \in \Gamma \) such that \( v(x_i) > N \) for all \( x_i \in D \).

This can be generalized to other definable manifolds using the following notion:

**Definition 2.5.** Let \( X \) be a definable manifold. A \( \Gamma \)-exhaustion is a definable family \( \{W_\gamma\}_{\gamma \in \Gamma} \) such that

1. Each \( W_\gamma \) is an open, definably compact subset of \( X \).
2. \( \gamma \leq \gamma' \implies W_\gamma \subseteq W_{\gamma'} \).
3. \( X = \bigcup_{\gamma \in \Gamma} W_\gamma \).

For example, in \( M^n \), if \( B_\gamma(0) \) denotes the ball of radius \( \gamma \) around \( 0 \in M^n \), then \( \{B_{-\gamma}(0)\}_{\gamma \in \Gamma} \) is a \( \Gamma \)-exhaustion.

**Fact 2.6** (\cite{6}, Remark 2.8). Let \( X \) be a definable manifold.

1. There is at least one \( \Gamma \)-exhaustion on \( X \).
2. Suppose we write \( X \) as a finite union \( U_1 \cup \cdots \cup U_k \) of definable open sets. Suppose that for \( i < k \), the family \( \{W_{\gamma_i}^i\} \) is a \( \Gamma \)-exhaustion of \( U_i \). Let \( V_\gamma = \bigcup_{i=1}^k W_{\gamma_i}^i \). Then \( \{V_\gamma\} \) is a \( \Gamma \)-exhaustion of \( X \).

We can then characterize definable compactness as follows:

**Fact 2.7.** Let \( X \) be a definable manifold and \( \{W_\gamma\} \) be a \( \Gamma \)-exhaustion. Let \( D \subseteq X \) be a definable set. The following are equivalent:

1. \( D \) is definably compact.
2. $D$ is closed, and $D$ is bounded, in the sense that $D \subseteq W_\gamma$ for some $\gamma$.

3. For any definable continuous function $f : O \setminus \{0\} \to D$, there is a point $p \in D$ which is a cluster point of $f$ at 0, in the sense that for any neighborhood $U$ of $p$ and $V$ of 0, there is $x \in V \setminus \{0\}$ such that $f(x) \in U$.

4. If $r$ is a 1-dimensional definable type concentrating on $D$, then there is a point $p \in D$ such that $r$ specializes to $p$, in the sense that for any definable neighborhood $U$ of $p$, the type $r$ concentrates on $U$.

The equivalence of (1) and (2) follows from Proposition 2.9 and Fact 2.2(4–5) in [6]. The equivalence of (1) and (3) is [6, Proposition 2.15]. The equivalence of (1) and (4) is [6, Proposition 2.24].

Fact 2.8 ([6, Remark 2.12]). Let $X$ be a $\Q_p$-definable manifold and $Y \subseteq X$ be $\Q_p$-definable. Then $Y$ is definably compact if and only if $Y(\Q_p)$ is compact as a subset of the $p$-adic manifold $X(\Q_p)$.

Remark 2.9. Suppose $M \preceq N$ are two models of $p$CF, and $X = X(M)$ is a definable topological space in $M$. Let $X(N)$ denote the associated definable topological space in $N$. One can easily show from Definition 2.1 that $X(M)$ is definably compact if and only if $X(N)$ is definably compact. In other words, “definable compactness” is invariant under elementary extensions.

In particular, if $G = G(M)$ is a definable group in $M$, then $G(M)$ is definably compact in $M$ iff $G(N)$ is definably compact in $N$.

Consequently, we can move to a monster model without changing whether a definable group $G$ is definably compact.

3 $fsg$ implies definable compactness

Work in a monster model $\mathbb{M} \models p$CF.

Proposition 3.1. Let $G$ be a definable group. If $G$ has $fsg$, then $G$ is definably compact.

Proof. Take $p \in S_G(\mathbb{M})$ and a small model $M_0 \preceq \mathbb{M}$ witnessing $fsg$. Take a $\Gamma$-exhaustion $\{W_\gamma\}_{\gamma \in \Gamma}$. By definition, $G = \bigcup_{\gamma \in \Gamma} W_\gamma$. The set $G(M_0)$ is small, so by saturation there is $\gamma \in \Gamma = \Gamma(\mathbb{M})$ such that $G(M_0) \subseteq W_\gamma$. Let $D = W_\gamma$ and $D'$ be the complement $G \setminus D$. Then $D' \cap G(M_0) = \emptyset$. By Fact 1.1, the definable set $D'$ is not generic, and therefore $D$ is generic. Then finitely many left translates of $D$ cover $G$:

$$G = a_1 \cdot D \cup \cdots \cup a_k \cdot D.$$  

The maps $x \mapsto a_i \cdot x$ are continuous, so by Fact 2.2, $G$ is definably compact. \qed
4 Definable compactness is definable in families

Work in a monster model $\mathbb{M} \models p\text{CF}$.

**Proposition 4.1.** Let $\{G_i\}_{i \in I}$ be a definable family of definable groups. Then the set

$$\{i \in I : G_i \text{ is definably compact}\}$$

is definable.

If you stare carefully at Fact 2.4, Fact 2.6, and the equivalence between (1) and (2) in Fact 2.7, you can convince yourself that this is automatically true. But we include the details for completeness.

**Proof.** Let $n$ be the dimension of $G$. Take a small model $M_0$ defining the family $\{G_i\}$. Let $X = \{i \in I : G_i \text{ is definably compact}\}$. It suffices to show that both $X$ and $I \setminus X$ are $\lor$-definable, i.e., unions of $M_0$-definable sets. We consider $X$; the proof for $I \setminus X$ is similar.

Take some $i_0 \in X$, so that $G_{i_0}$ is definably compact. Fix the following data:

1. Finitely many open definable sets $U_1, \ldots, U_k$ covering $G_{i_0}$.
2. Open definable sets $V_j \subseteq M^n$ for $j \leq k$ and definable homeomorphisms $h_j : U_j \to V_j$.
3. For each $j$, a $\Gamma$-exhaustion $\{W^j_\gamma\}_{\gamma \in \Gamma}$ of $U_j$.

These exist by Facts 2.3 and 2.6[2]. Take a finite tuple $b_0 \in M^f$ over which the data in (1)–(3) are definable. We can define some $U^b_1$ for $b \in M^f$ such that $U^b_1$ depends $0$-definably on $b$, and $U_1 = U^b_1$. Define $U^b_j, V^b_j, h^b_j$ and $\{W^j_\gamma\}_{\gamma \in \Gamma}$ for $1 \leq j \leq k$ in a similar fashion.

There is an $L_{M_0}$ formula $\phi(x, y)$ such that $\phi(i, b)$ holds if and only if the following eight conditions hold:

- $i \in I$. (This can be expressed because $I$ is $M_0$-definable.)
- Each $U^b_j$ is a subset of $G_i$, and $G_i = \bigcup_{j=1}^k U^b_j$.
- Each $V^b_j$ is an open subset of $M^n$.
- Each $h^b_j$ is a bijection from $U^b_j$ to $V^b_j$.
- The collection of $h^b_j : U^b_j \to V^b_j$ for $j = 1, \ldots, k$ is an atlas making $G_i$ into a Hausdorff definable manifold.
- The group operations on $G_i$ are continuous with respect to the definable manifold structure.
- Each $\{W^j_\gamma\}_{\gamma \in \Gamma}$ is a $\Gamma$-exhaustion on $U^b_j$. (In order to express that $W^j_\gamma$ is definably compact, use the homeomorphism $h^b_j : U^b_j \to V^b_j$. Fact 2.4 shows how to express definable compactness for definable subsets of $V^b_j$.)
If we let $\tilde{W}^b_\gamma = \bigcup_{j=1}^k W^j b$, then there is some $\gamma$ such that $\tilde{W}^b_\gamma = G_i$.

**Claim 4.2.** $\phi(i_0, b_0)$ holds

*Proof.* All the bullet points are clear except the last one. Note that $\{\tilde{W}^b_\gamma\}$ is a $\Gamma$-exhaustion of $G_{i_0}$ by Fact 2.6(1). Then there is $\gamma$ such that $G_{i_0} \subseteq \tilde{W}^b_\gamma$, by the equivalence of parts (1) and (2) in Fact 2.7. □

**Claim 4.3.** If $\phi(i, b)$ holds, then $G_i$ is definably compact.

*Proof.* The definition of $\phi$ ensures that $i \in I$ and the sets $U^b_1, \ldots, U^b_k$ are an open cover of $G_i$. The family $\{\tilde{W}^b_\gamma\}_{\gamma \in \Gamma}$ appearing in the eighth bullet point is a $\Gamma$-exhaustion of $G_i$, by Fact 2.6(2). The eighth bullet point then ensures that $G_i$ is definably compact. □

Combining the two claims, we see that

$$i_0 \in \{i \in I \mid \exists b \in M^\ell : \phi(i, b)\} \subseteq X.$$ 

So there is an $M_0$-definable set containing $i_0$ and contained in $X$. As $i_0$ is an arbitrary element of $X$, it follows that $X$ is a union of $M_0$-definable sets.

A nearly identical argument shows that $I \setminus X$ is a union of $M_0$-definable sets. By saturation, $X$ is definable.

**Corollary 4.4.** Let $G$ be a definably compact definable group. Then there is a 0-definable family $\{G_i\}_{i \in I}$ such that $G = G_i$ for some $i$, and every $G_i$ is definably compact.

*Proof.* We can always find some 0-definable family of definable groups $\{G_i\}_{i \in J}$ containing $G$. Let $I$ be the set of $i \in J$ such that $G_i$ is definably compact. Then $I$ is $\text{Aut}(M/\emptyset)$-invariant, and definable by Proposition 4.1. Consequently, $I$ is 0-definable. Then the family $\{G_i\}_{i \in I}$ is 0-definable and contains $G$. □

## 5 Uniform witnesses to $fsg$

If $\phi(x, y)$ is an $L_{\text{rings}}$-formula, then the VC-dimension of $\phi(x, y)$ will denote the VC-dimension of $\phi(x, y)$ in $p\text{CF}$, i.e., the largest $n$ such that there is $M \models p\text{CF}$ and a set $\{a_1, \ldots, a_n\} \in M^{[x]}$ shattered by $\phi$, meaning that for any $S \subseteq \{a_1, \ldots, a_n\}$ there is $b \in M^{[y]}$ such that

$$S = \{a_1, \ldots, a_n\} \cap \phi(M, b).$$

The VC-dimension of $\phi$ is always finite, because $p\text{CF}$ is NIP.

Work in the standard model $\mathbb{Q}_p \models p\text{CF}$. For definable groups $G$, compactness is equivalent to definable compactness (Fact 2.8). Any compact definable group $G$ has a Haar measure $\mu = \mu_G$, which we normalize to make $\mu(G) = 1$. Any definable set is Borel, hence measurable.

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1For $I \setminus X$, change “some” to “no”.
2For $I \setminus X$, insert “not”.
3For $I \setminus X$, insert “not”.

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6
Proposition 5.1. Let $G$ be a compact definable group. Let $\mu$ be normalized Haar measure on $G$. Let $\phi(x, y)$ be a formula. Let $\varepsilon > 0$. There is $\{a_1, \ldots, a_N\} \in G$ such that for any $\phi$-set $D$ contained in $G$,

$$\mu(D) > \varepsilon \implies D \cap \{a_1, \ldots, a_N\} \neq \emptyset.$$ 

Moreover, $N$ can be chosen to depend only on the VC-dimension of $\phi$ and on $\varepsilon$, not on $G$.

This is essentially a direct consequence of the VC-theorem, but we include the details for lack of a reference that proves the exact statement we want. We will closely follow the argument from [9, Proposition 7.26].

Proof. We will need some notation from [9]. If $a_1, \ldots, a_n, b_1, \ldots, b_n \in G$, then

$$f_n(a_1, \ldots, a_n; b_1, \ldots, b_n) = \sup_{c \in \mathbb{Q}_p^{[\phi]}} |\text{Av}(a_1, \ldots, a_n; \phi(Q_p; c)) - \text{Av}(b_1, \ldots, b_n; \phi(Q_p; c))|.$$ 

The following is trivial:

Claim 5.2. Suppose $D$ is a $\phi$-set such that $D \cap \{a_1, \ldots, a_n\} = \emptyset \neq D \cap \{b_1, \ldots, b_n\}$. Then

$$f_n(a_1, \ldots, a_n; b_1, \ldots, b_n) \geq |\text{Av}(a_1, \ldots, a_n; D) - \text{Av}(b_1, \ldots, b_n; D)| = \text{Av}(b_1, \ldots, b_n; D).$$

Let $k$ be the VC-dimension of $\phi$. By the Sauer-Shelah Lemma [9, Lemma 6.4], the shatter function $\pi_\phi(n)$ is bounded by $\sum_{i=0}^{k} \binom{n}{i}$. (See [9, Section 6.1] for the definition of the shatter function.) Let $\delta = \varepsilon/2$. Let $N$ be large enough that $1/(N\varepsilon^2) < 1 - \delta$ and

$$4 \left( \sum_{i=0}^{k} \binom{N}{i} \right) \exp \left( - \frac{N\delta^2}{8} \right) < \delta.$$ 

Note that $\delta$ and $N$ can be chosen to depend only on $k$ and $\varepsilon$.

Applying [9, Lemma 7.24] to normalized Haar measure on $G^{2N}$, we see the following:

Claim 5.3. If $a_1, \ldots, a_N, b_1, \ldots, b_N$ are chosen randomly in $G$, then

$$\text{Prob}(f_N(\bar{a}, \bar{b}) > \delta) \leq 4\pi_\phi(N) \exp \left( - \frac{N\delta^2}{8} \right) < \delta.$$ 

By Fubini’s theorem, we can fix some $a_1, \ldots, a_N$ such that the following holds:

Claim 5.4. If $b_1, \ldots, b_N$ are chosen randomly in $G$, then

$$\text{Prob}(f_N(\bar{a}, \bar{b}) > \delta) < \delta.$$
We claim that $a_1, \ldots, a_N$ have the desired property. Otherwise, there is some $\phi$-set $D$ such that $\mu(D) > \varepsilon$ but $D \cap \{a_1, \ldots, a_N\} = 0$. Combining [5.2] and [5.3] we see

**Claim 5.5.** If $b_1, \ldots, b_N$ are chosen randomly in $G$, then

$$\text{Prob}(\text{Av}(b_1, \ldots, b_N; D) > \delta) < \delta.$$  

Since $\delta = \varepsilon/2$ and $\mu(D) > \varepsilon$, the following implication holds for any $b_i$:

$$\text{Av}(b_1, \ldots, b_N; D) \leq \delta \implies |\text{Av}(b_1, \ldots, b_N; D) - \mu(D)| \geq \varepsilon/2.$$  

The weak law of large numbers [9, Proposition B.4] shows that for random $\bar{b} \in G^N$,

$$\text{Prob}(\text{Av}(\bar{b}; D) \leq \delta) \leq \text{Prob}(|\text{Av}(\bar{b}; D) - \mu(D)| \geq \varepsilon/2) \leq \frac{1}{N \varepsilon^2} < 1 - \delta.$$  

The event $\text{Av}(\bar{b}; D) \leq \delta$ has probability less than $1 - \delta$, and the event $\text{Av}(\bar{b}; D) > \delta$ has probability less than $\delta$ (by Claim 5.5). This is absurd.  

**Remark 5.6.** We will let $N_{k,\varepsilon}$ denote the $N$ in Proposition 5.1 that works uniformly across all $\phi$-sets $D \subseteq G$ with $\mu_G(D) > \varepsilon$ and $\phi$ of VC-dimension $k$.

**Remark 5.7.** Proposition 5.1 can also be seen using facts about generically stable measures. Embed $\mathbb{Q}_p$ into a monster model $\mathbb{M}$. By [4, Theorem 6.3], the Haar measure $\mu$ on $G(\mathbb{Q}_p)$ is smooth, meaning that it has a unique extension to a global Keisler measure $\tilde{\mu}$ on $G(\mathbb{M})$. By [9, Lemma 7.17(i)], $\tilde{\mu}$ is generically stable. By [9, Theorem 7.29(ii)], for any formula $\phi$ and any $\varepsilon > 0$ there is $\{a_1, \ldots, a_N\} \in G(\mathbb{Q}_p)$ such that for any $\mathbb{Q}_p$-definable $\phi$-set $D \subseteq G$, we have

$$|\tilde{\mu}(D) - \frac{|D \cap \{a_1, \ldots, a_N\}|}{n}| < \varepsilon.$$  

This implies the conclusion of Proposition 5.1. Tracing through the proofs, one can see that $n$ depends only on $\varepsilon$ and the VC-dimension of $\phi$.

### 6 Definable compactness implies fsg

Let $\mathbb{Q}_p^\dagger$ be the expansion of $(\mathbb{Q}_p, +, \cdot)$ by the following data:

1. A new sort $\mathbb{R}$ with its full field structure.

2. For every 0-definable family $\mathcal{F} = \{G_i\}_{i \in I}$ of compact definable groups and every $\mathcal{L}_{\text{rings}}$-formula $\phi(x, y)$ a function $f_{\mathcal{F}, \phi} : I \times \mathbb{Q}_p^{[\mathbb{R}]} \to \mathbb{R}$ defined by

$$f_{\mathcal{F}, \phi}(i, b) = \mu_{G_i}(\phi(\mathbb{Q}_p; b) \cap G_i),$$

where $\mu_G$ denotes normalized Haar measure on $G_i$.  

8
Let $\mathbb{M}^\dagger = (\mathbb{M}, \mathbb{R}^*)$ be a monster model of $\mathbb{Q}_p^\dagger$, and let $\mathbb{M}$ be the reduct to $\mathcal{L}_{\text{rings}}$ (discarding the new sort $\mathbb{R}^*$). Then $\mathbb{M}$ is a monster model of $\rho\text{CF}$.

**Definition 6.2.** Suppose that $G$ is a definably compact group in $\mathbb{M}$ and $D \subseteq G$ is a definable subset. Let $\mathcal{F} = \{G_i\}_{i \in I}$ be a 0-definable family containing $G$. (This exists by Corollary 4.4.) Let $i \in I$ be such that $G = G_i$. Let $D = G \cap \phi(\mathbb{M}; b) = G \cap \psi(\mathbb{M}; c)$. Then

$$f_{\mathcal{F}, \phi}(i, b) = f_{\mathcal{F}, \psi}(j, c).$$  \hfill (1)

**Proof.** For fixed $\mathcal{F}, \mathcal{F}', \phi, \psi$, the second paragraph of the lemma can be expressed as a single first-order sentence in the language of $\mathbb{M}^\dagger$ and $\mathbb{Q}_p^\dagger$. The sentence holds in $\mathbb{Q}_p^\dagger$ by construction, so it holds in the elementary extension $\mathbb{M}^\dagger$. \hfill \Box

**Proposition 6.3.** Let $G$ be a definably compact group in $\mathbb{M}$, and let $D, D'$ be definable subsets of $G$.

1. $\mu_G(\emptyset) = 0$ and $\mu_G(G) = 1$.
2. $0 \leq \mu_G(D) \leq 1$.
3. $\mu_G(D \cup D') = \mu_G(D) + \mu_G(D') - \mu_G(D \cap D')$.
4. If $a \in G$, then $\mu_G(a \cdot D) = \mu_G(D)$.
5. For any $\mathcal{L}_{\text{rings}}$-formula $\phi(x; y)$ of VC-dimension $k$ and any $n$, if $N = N_{k, 1/n}$ is as in Remark 5.6, then there are $a_1, \ldots, a_N \in G$ such that if $D = \phi(\mathbb{M}; b) \subseteq G$ and $\mu_G(D) > 1/n$, then $D \cap \{a_1, \ldots, a_N\} \neq \emptyset$.

**Proof.** The proof is similar to Lemma 6.1 transfering things from $\mathbb{Q}_p^\dagger$ to $\mathbb{M}^\dagger$. Part (5) uses Proposition 5.1. \hfill \Box

**Proposition 6.4.** Let $G$ be a definably compact group in $\mathbb{M}$. Then $G$ has fsg.

**Proof.** For any definable set $D \subseteq G$, let $\mu^*(D)$ be the standard part of $\mu_G(D)$. Parts (1)–(4) of Proposition 6.3 imply that $\mu^*(D)$ is a left-invariant Keisler measure on $G$. By part (5), we can find a countable set $A \subseteq G$ such that if $\mu^*(D) > 0$, then $D \cap A \neq \emptyset$. Then $\mu^*$ and all of its left translates are finitely satisfiable in a small model. By [2, Remark 4.6], $G$ has fsg. \hfill \Box
The $M$ of this section is a monster model of the complete theory $pCF$, so Proposition 6.4 applies to definable groups in any model of $pCF$. Combined with Proposition 3.1, Theorem 1.2 follows.

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