Nonlinear Image Processing Based on Optimization of Generalized Information Methods

Anisa T. BAJKOVA
IAA RAS, Nab. Kutuzova, 10, 191187, St.-Petersburg, Russia

Abstract. A range of nonlinear image reconstruction procedures based on extremizing the generalized Shannon entropy, Kullback-Leibler cross-entropy and Renyi information measures and proposed by the author in early papers is presented. The “generalization” assumes search for the solution over the space of real bipolar or complex functions. Such an approach allows, first, to reconstruct signals of any type and physical nature and, secondly, to decrease nonlinear intensity image distortions caused by measurement errors. All the elaborated procedures are contained in VLBI “IMAGE” program package developed in IAA RAS.

1. Introduction

The main aim of this paper is consideration of a class of image deconvolution (reconstruction) algorithms based on optimization of different information measures. The most popular among them is the well known maximum entropy method in Shannon formulation. The generalized information methods based on extremizing entropic (Shannon, Kullback-Leibler) and Renyi measures were proposed by the author in several previous publications [1–3] for reconstruction of real bipolar and complex images. In this paper we give a brief review of the proposed techniques.

All the proposed methods are realized in ”IMAGE” program package elaborated in IAA RAS in framework of ”QUASAR” Russian Very Long Baseline Interferometry (VLBI) project.

2. Entropic Processing

2.1. Maximum Entropy Method

Maximum entropy method (MEM) is widely used for image reconstruction from incomplete and noisy Fourier data. In Fourier space the image reconstruction problem is equivalent to one of extrapolation and interpolation. The MEM produces maximally smooth, or unbiased, estimates [4]. If \( r_{ml} \) is a real non-negative signal (we consider...
two-dimensional sampling signals) to be estimated then the maximum entropy method assumes solving the following optimization problem with linear constraints dictated by data

$$\max \sum_m \sum_l (r_{ml} \ln(r_{ml})) - 1/2 \sum_n \sum_k (\eta_{nk}^2 + \eta_{nk}^2)/2, \quad r_{ml} \geq 0, \quad (1)$$

$$\sum_m \sum_l r_{ml}a_{ml}^n + \eta_{nk}^r = A_{nk}, \quad \sum_m \sum_l r_{ml}b_{ml}^i + \eta_{nk}^i = B_{nk}, \quad (2)$$

where $a_{ml}^n, b_{ml}^i$ are constants determined by linear image formation system (Fourier transform), $A_{nk}, B_{nk}$ are the real and imaginary parts of the Fourier data respectively, $\eta_{nk}^r, \eta_{nk}^i$ are noise components of data, $\sigma_{nk}^2$ are noise variances for each of the real and imaginary parts at sample $nk$. Note that the signal must be real and non-negative-definite for a real entropy functional to exist.

Using the Lagrange method it is easy to obtain a solution for $r_{ml}$ and noise components

$$r_{ml} = \exp(-\sum_n \sum_k \alpha_{nk}a_{ml}^n + \beta_{nk}b_{ml}^i - 1), \quad (3)$$

$$\eta_{nk}^r = \sigma_{nk}^2 \alpha_{nk}, \quad \eta_{nk}^i = \sigma_{nk}^2 \beta_{nk}, \quad (4)$$

where $\alpha_{nk}, \beta_{nk}$ are conventional Lagrange multipliers.

2.2.Kullback-Leibler Entropy Method

Kullback-Leibler information permits insertion of a prior bias function $a_{ml}$ into entropy functional (1) as follows

$$\sum_m \sum_l r_{ml} \ln(r_{ml}/a_{ml}), \quad r_{ml}, a_{ml} \geq 0. \quad (5)$$

The aim of such insertion is improving the entropic algorithm (1) using a priori information about an unknown signal [2]. Usually as a bias function $a_{ml}$ a first approximation to $r_{ml}$ obtained by inverse Fourier transform of input Fourier data is used. Missing data regions in Fourier space are filled out by zeros.

3.Renyi Information Processing

Another powerful method of reconstruction considered here is based on minimization of the following Renyi information measure [3]

$$\sum_m \sum_l r_{ml}^\alpha a_{ml}^{1-\alpha}, \quad r_{ml}, a_{ml} \geq 0, \quad \alpha \neq 0, \quad (6)$$

where $a_{ml}$ is a reference function.

4.Generalized Methods

Originally the generalized maximum entropy method (GMEM) was proposed by Bajkova [1] for reconstruction of complex coherent images formed by radio holography
principle. In [2] the GMEM idea was used by Frieden for generalization of Kullback-Leibler cross-entropy reconstruction of ISAR images. In [3] by Frieden the generalized form of entropy functionals was extended to minimum Renyi information method. Bajkova proposed the method of solution of corresponding generalized optimization problem [3].

4.1. Generalized Maximum Entropy Method

Here let us recall the basic aspects of the GMEM [1].

First, consider a real signal $r_{ml}$ which may take both positive and negative values. In this case the minimizing functional cannot be written as (1). Therefore it was proposed in [1] to represent entropy (1) as entropy of the absolute value of $r_{ml}$:

$$\sum_m \sum_l |r_{ml}| \ln(|r_{ml}|).$$

(7)

To represent the optimization problem in traditional form (1)-(2) let us represent the signal sought for $r_{ml}$ as a difference between two non-negative-definite functions:

$$r_{ml} = x_{ml} - y_{ml}, \quad x_{ml} \geq 0, \quad y_{ml} \geq 0.$$  

(8)

Thus the positive regions are represented by $x_{ml}$ and the negative by $y_{ml}$. Curves $x_{ml}$ and $y_{ml}$ do not have overlapping support regions. Nonoverlapping condition can be written as

if $r_{ml} > 0$ then $y_{ml} \to 0$ and $r_{ml} = x_{ml}$,

if $r_{ml} < 0$ then $x_{ml} \to 0$ and $r_{ml} = -y_{ml}$.

Then the problem of estimation of $r_{ml}$ can be replaced by one of estimation of $x_{ml}$ and $y_{ml}$ as

$$\max \sum_m \sum_l -(x_{ml} \ln(x_{ml}) + y_{ml} \ln(y_{ml})) - 1/2 \sum_n \sum_k (\eta_{nk}^2 + \eta_{nk}^2)/\sigma_{nk}^2.$$  

(9)

Jumping to a complex signal $u_{ml} = r_{ml} + jq_{ml}$, $j = \sqrt{-1}$, where both $r_{ml}$ and $q_{ml}$ are real bipolar signals, we can represent similarly to (8) the sought for sequences as the differences of non-negative-definite sequences as well:

$$r_{ml} = x_{ml} - y_{ml}, \quad q_{ml} = z_{ml} - v_{ml},$$  

(10)

where $x_{ml} \geq 0$, $y_{ml} \geq 0$, $z_{ml} \geq 0$, $v_{ml} \geq 0$.

Obviously, if the sequences $x_{ml}, y_{ml}$ and $z_{ml}, v_{ml}$ do not overlap, then $x_{ml}$ and $z_{ml}$ determine positive parts and $y_{ml}$ and $v_{ml}$ determine negative parts of the sequences $r_{ml}$ and $q_{ml}$ respectively.

Nonoverlapping condition mentioned above must be complemented by one for the signal $q_{ml}$:

if $q_{ml} > 0$ then $v_{ml} \to 0$ and $q_{ml} = z_{ml}$,

if $q_{ml} < 0$ then $z_{ml} \to 0$ and $q_{ml} = -v_{ml}$.
Then the minimized entropic functional for estimation of the complex signal can be written as the following functional

$$\sum_m \sum_l x_{ml} \ln(\alpha x_{ml}) + y_{ml} \ln(\alpha y_{ml}) + z_{ml} \ln(\alpha z_{ml}) + v_{ml} \ln(\alpha v_{ml})$$

(11)

with inserted a positive parameter $\alpha$ responsible for not overlapping positive and negative parts of the search for real bipolar sequences $r_{ml}$ and $q_{ml}$. Significance of the parameter $\alpha$ will be clear below from equations (15)-(17).

Linear constraints (2) derived from Fourier data must be rewritten in the way:

$$\sum_m \sum_l (x_{ml} - y_{ml})a_{nk}^{nl} - (z_{ml} - v_{ml})b_{ml}^{nk} + \eta^r_{nk} = A_{nk},$$

(12)

$$\sum_m \sum_l (x_{ml} - y_{ml})b_{ml}^{nk} + (z_{ml} - v_{ml})a_{nk}^{nl} + \eta^i_{nk} = B_{nk},$$

(13)

$$x_{ml} \geq 0, \quad y_{ml} \geq 0, \quad z_{ml} \geq 0, \quad v_{ml} \geq 0,$$

(14)

where $a_{nk}^{nl}, b_{ml}^{nk}$ are constant coefficients which are determined by Fourier transform, $A_{nk}, B_{nk}$ are measured real and imaginary parts of Fourier data respectively. The solutions of optimization problem (11)–(14) obtained using the Lagrange method for signals $(x_{ml}, y_{ml}, z_{ml}, v_{ml})$ look like

$$x_{ml} = \exp(-\sum_n \sum_k \alpha_{nk} a_{ml}^{nk} + \beta_{nk} b_{ml}^{nk} - 1 - \ln \alpha),$$

(15)

$$v_{ml} = \exp(-\sum_n \sum_k \alpha_{nk} b_{ml}^{nk} - \beta_{nk} a_{ml}^{nk} - 1 - \ln \alpha),$$

(16)

$$x_{ml} y_{ml} = z_{ml} v_{ml} = \exp(-2 - 2 \ln \alpha).$$

(17)

Making parameter $\alpha$ larger we can reach not overlapping effect. Usual value of $\alpha$ is 1000.

Unknown Lagrange multipliers can be found from the following dual optimization problem without supplementary conditions

$$\max \sum_m \sum_l - (x_{ml} + y_{ml} + z_{ml} + v_{ml}) - 1/2 \sum_n \sum_k (\eta^r_{nk} + \eta^i_{nk})^2/\sigma_{nk}^2$$

$$- \sum_n \sum_k (\alpha_{nk} A_{nk} + \beta_{nk} B_{nk}) - \sum_n \sum_k (\alpha_{nk} \eta^r_{nk} + \beta_{nk} \eta^i_{nk})$$

(18)

by substituting expressions (15)-(17) for $x_{ml}, y_{ml}, z_{ml}, v_{ml}$ and (4) for $\eta^r_{nk}, \eta^i_{nk}$.

Equations (15)-(17) show that for any real solution $\{\alpha_{nk}\}, \{\beta_{nk}\}$ to the problem, necessarily

$$x_{ml} \geq 0, \quad y_{ml} \geq 0, \quad z_{ml} \geq 0, \quad v_{ml} \geq 0.$$  

(19)

That is all the reconstructed signals are non-negative. This is important because non-negative-constrained solutions, being nonlinear in the data, can have higher bandwidth and hence higher resolution than the data. By (10) this is now true for real signals $r_{ml}$ and $q_{ml}$ and hence for complex signal $u_{ml}$ as well.

So, the GMEM approach allows to obtain a solution of the reconstruction problem in the space of complex functions. Because of non-negativity of the solutions (15)-(17)
the method is nonlinear and possesses a super resolution effect similarly to the classical maximum entropy method.

4.2. Generalized Kullback-Leibler Method

Generalized Kullback-Leibler method assumes prior knowledge of bias non-negative functions $a_{ml}, b_{ml}, c_{ml}, d_{ml}$ corresponding to the signals $x_{ml}, y_{ml}, z_{ml}, v_{ml}$ respectively. Then the entropic functional (11) is modified as

$$
\sum_m \sum_l x_{ml} \ln(\alpha x_{ml}/a_{ml}) + y_{ml} \ln(\alpha y_{ml}/b_{ml}) + z_{ml} \ln(\alpha z_{ml}/c_{ml}) + v_{ml} \ln(\alpha v_{ml}/d_{ml}).
$$

In this case, bias functions can be again the smoothed inverse Fourier transforms of Fourier data with zeros in the points where measurements are absent.

Using data constraints (12)-(13) and the Lagrange method we obtain the following solution to sought for non-negatively determined signals

$$
x_{ml} = (a_{ml}/e\alpha) \exp(-\sum_n \sum_k \alpha_{nk}a_{ml}^{nk} + \beta_{nk}b_{ml}^{nk} - 1),
$$

$$
v_{ml} = (d_{ml}/e\alpha) \exp(-\sum_n \sum_k \alpha_{nk}b_{ml}^{nk} - \beta_{nk}a_{ml}^{nk} - 1),
$$

$$
x_{ml}y_{ml} = a_{ml}b_{ml}e^{-2\alpha^{-2}},
$$

$$
z_{ml}v_{ml} = c_{ml}d_{ml}e^{-2\alpha^{-2}}.
$$

As seen, maximizing Kullback-Leibler entropy functional subject to the linear data constraints (12)-(13) produces solutions (21)-(24) that are linearly proportional to the input functions $a_{ml}, b_{ml}, c_{ml}, d_{ml}$ respectively.

4.3. Generalized Minimum Renyi Information Method

Similarly to maximum entropy method for complex signal $u_{ml} = r_{ml} + jq_{ml}$ we suggest [3] to minimize the sum of two Renyi measures of absolute values of $r_{ml}$ and $q_{ml}$. Then functional (6) is modified as

$$
\sum_m \sum_l -(|r_{ml}|^{\alpha}a_{ml}^{1-\alpha} + |q_{ml}|^{\alpha}b_{ml}^{1-\alpha}),
$$

where $a_{ml}$ and $b_{ml}$ are input modulus reference signals.

Representing the sought for sequences in the form of differences between positive and negative parts as in (10) we can rewrite previous functional (25) as follows

$$
\sum_m \sum_l -((x_{ml}a_{ml}^{1-\alpha} + y_{ml}b_{ml}^{1-\alpha} + z_{ml}c_{ml}^{1-\alpha} + v_{ml}d_{ml}^{1-\alpha}),
$$

where $a_{ml}, b_{ml}, c_{ml}, d_{ml}$ are corresponding reference signals.

Direct using the Lagrange method is very complicated. In order to simplify solution let us make the following substitution $\alpha = 1 + 1/2\mu$ and introduce the following new variables:

$$
t_{ml}(s_{ml}, h_{ml}, g_{ml}) = (x_{ml}(s_{ml}, h_{ml}, g_{ml})/a_{ml}(b_{ml}, c_{ml}, d_{ml}))^{1/2\mu}.
$$

$\mu$
Then the optimization problem (20), (12)-(14) can be rewritten with respect to new variables (27) as

\[
\begin{align*}
\max & \sum_m \sum_l - (a_{ml}t^{2\mu+1}_{ml} + b_{ml}s^{2\mu+1}_{ml} + c_{ml}h^{2\mu+1}_{ml} + d_{ml}g^{2\mu+1}_{ml}) \\
& - \frac{1}{2} \sum_n \sum_k (\eta_{nk}^2 + \eta_{nk}^2) / \sigma_{nk}^2, \\
\sum_m \sum_l (a_{ml}t^{2\mu}_{ml} - b_{ml}s^{2\mu}_{ml})a_{nk}^{nk} - (c_{ml}h^{2\mu}_{ml} - d_{ml}g^{2\mu}_{ml})b_{nk}^{nk} + \eta_{nk}^r = A_{nk}, \\
\sum_m \sum_l (a_{ml}t^{2\mu}_{ml} - b_{ml}s^{2\mu}_{ml})b_{nk}^{nk} + (c_{ml}h^{2\mu}_{ml} - d_{ml}g^{2\mu}_{ml})a_{nk}^{nk} + \eta_{nk}^i = B_{nk},
\end{align*}
\]

(28)

(29)

(30)

(31)

Using the Lagrange method we find the following solution for sought signals:

\[
\begin{align*}
t_{ml} &= -2\mu/(2\mu + 1) \sum_n \sum_k \alpha_{nk}a_{nk}^{nk} + \beta_{nk}b_{nk}^{nk}, \\
h_{ml} &= 2\mu/(2\mu + 1) \sum_n \sum_k \alpha_{nk}b_{nk}^{nk} - \beta_{nk}a_{nk}^{nk}, \\
t_{ml} &= -s_{ml}, \quad h_{ml} = -g_{ml}.
\end{align*}
\]

(32)

(33)

(34)

where \(\alpha_{nk}\) and \(\beta_{nk}\) are Lagrange multipliers.

Remembering condition (31) and taking (34) we can say that the negative and positive parts of sought for real solutions to \(r_{ml}\) and \(q_{ml}\) never overlap and no parameter (as \(\alpha\) in the GMEM) is required. That is an advantage of the generalized minimum Renyi information method.

Sought for a complex signal \(u_{ml}\) can be found by inverting equations (27) and using representation (10).

It is necessary to note that the quality of image reconstruction by Renyi method depends on choosing parameter \(\alpha\) in the functional. When \(\alpha = 2(\mu = 0.5)\) we have a conventional minimum intensity criterion. Experimentally established that when \(\alpha\) approaches 1 by increasing \(\mu\) we obtain a criterion with more strong nonlinear (extrapolation) features [5].

5. A problem of nonlinear distortions

Let us consider real non-negative images. Because of nonlinearity nature of considered above deconvolution procedures there is a problem connected with nonlinear image distortions due to noise in spectral data [6]. We investigated only the case of additive noise in data. As experiments show the generalized algorithms ensure much less artefacts than the classical ones. This phenomenon can be explained that noise can so degrade data that to the latter in general a real solution with both positive and negative values will correspond but not pure real non-negative one which is sought by conventional algorithms. Therefore we consider that seeking for a solution of image reconstruction problem in generalized form is more adequate.

6. The ”IMAGE” program package

"IMAGE” program package elaborated in IAA RAS [7] in framework of ”QUASAR”
VLBI project is intended for VLBI mapping natural (incoherent) and artificial (coherent, ISAR) radio sources and investigation of new image reconstruction procedures. At present the "IMAGE" program package contains except of the CLEAN and classical information nonlinear methods a number of generalized versions of information methods. For solution of related optimization problems both steepest descent and Newton-Raphson numerical methods are realized.

7. Conclusion

Maximum entropy, maximum Kullback-Leibler cross-entropy and minimum Renyi information methods having good extrapolation and interpolation features are generalized for reconstruction of signals of any type including complex ones. They may be used for reconstruction of real bipolar and complex signals or reconstruction of real non-negative images with minimal nonlinear distortions.

This work was supported by Russian Foundation for Basic Researches under grant N 96-02-19177.

References

[1] Bajkova A. The generalization of maximum entropy method for reconstruction of complex functions. *Astron. & Astroph. Tr.*, 1 (1992), 313–320.
[2] Frieden R., Bajkova A. Bayesian cross-entropy reconstruction of complex images. *Applied Optics*, 33 (Jan. 1994), 219–226.
[3] Frieden R., Bajkova A. Reconstruction of complex signals using minimum Renyi information. *Applied Optics*, 34 (July 1995), 4086–4093.
[4] Frieden R. Restoring with maximum likelihood and maximum entropy. *J. Opt. Soc. Am.*, 62 (1972), 511–518.
[5] Bajkova A. Renyi measure for image reconstruction in Astronomy. *Communications of IAA RAS*, No 59, 1994.
[6] Bajkova A. On the problem of nonlinear distortions of the maximum entropy method. *Izvestia Vuzov. Radiofizika*, 38 (1994), 1267–1277.
[7] Bajkova A. New version of “Image” program package in Linux for modeling and processing VLBI images. *Communications of IAA RAS*, No 120, 1998.