Big Bang Nucleosynthesis: an accurate
determination of light element yields

S. Esposito, G. Mangano, G. Miele, and O. Pisanti,

Dipartimento di Fisica, Università di Napoli "Federico II", and INFN, Sezione di Napoli,
Mostra D’Oltremare Pad. 20, I-80125 Napoli, Italy

Abstract

We report the results of a new accurate evaluation of light nuclei yields in primordial nucleosynthesis. All radiative effects, finite nucleon mass, thermal and plasma corrections are included in the proton to neutron conversion rates. The relic densities of $^4$He, $^3$He, $^7$Li have been numerically obtained via a new updated version of the standard BBN code. In particular the theoretical uncertainty on $^4$He is reduced to the order of 0.1 %.

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1 Introduction

Big Bang Nucleosynthesis (BBN) still represents one of the key subjects of modern cosmology even if its clear understanding traces back to over 25 years ago [1]. The reason for this relies on the fact that BBN is one of the most powerful tools to study fundamental interactions, since light nuclei abundances are crucially depending on many elementary particle properties. As a well-known example, the $^{4}\text{He}$ abundance is strongly affected by the number of effective neutrino degrees of freedom, but others fascinating phenomena such as neutrino degeneracy or oscillation phenomena can be studied too, using the universe few seconds after the bang as a laboratory.

In the recent years, the experimental accuracy of the light primordial nuclei abundances, mainly the one of $^{4}\text{He}$, underwent a sort of revolution. The qualitative results of not too many years ago, suggesting that the $^{4}\text{He}$ mass fraction $Y_4$ was of the order of 0.25, recently turned in measurements with accuracies of the order of one percent. A similar good improvement has been obtained in both Deuterium ($D$) and $^7\text{Li}$ abundances $Y_2 \equiv D/H$ and $Y_7 \equiv ^7\text{Li}/H$. In particular for $D$, measurements in distant Quasars Absorption line Systems (QAS) now represent a reliable estimate of the primordial value for $Y_2$, which is only lowered by subsequent stellar processing. Paradoxically, the refinement of these experimental techniques, due to the uncertainties in the models describing stellar activity, is at the basis of large discrepancies between different set of results. Such discrepancies are possibly of systematic origin, or may reveal new aspects of cosmological evolution of the universe. The observations of $Y_4$ from regression to zero metallicity in blue compact galaxies in two independent surveys still produce two incompatible results, a low value [2],

$$Y_4^{(l)} = 0.234 \pm 0.002 \pm 0.005 \ , \quad (1.1)$$

and a significantly higher one [3],

$$Y_4^{(h)} = 0.243 \pm 0.003 \ . \quad (1.2)$$

A similar situation occurs in $D$ measurements, where observations in different QAS, both at red shift larger than 3, give two results at bias for one order of magnitude [4, 5]

$$Y_2^{(l)} = (3.4 \pm 0.3) \times 10^{-5} \ , \quad (1.3)$$
\[ Y_2^{(h)} = (1.9 \pm 0.4) \times 10^{-4} \quad (1.4) \]

For the \(^7\text{Li}\) abundance, the almost constant \textit{Spite plateau} observed in the halo of POP II stars \cite{6, 7}

\[ Y_7^{(l)} = (1.6 \pm 0.36) \times 10^{-10} \quad (1.5) \]

is generally considered a reliable estimate of primordial abundance. Nevertheless, the observation of stars similar to the ones contributing to the Spite plateau, but with no traces of \(^7\text{Li}\) \cite{6, 8}, seems to imply the presence of a depletion mechanism. A recent analysis based on a sample of 41 stars does not find any evidence of depletion mechanism or post-BBN creation and yields the primordial abundance \cite{9}

\[ Y_7^{(h)} = (1.73 \pm 0.21) \times 10^{-10} \quad (1.6) \]

A brief summary of the complete experimental situation on primordial abundances can be found in Ref. \cite{10}.

Probably, future measurements or a better understanding of the present data will clarify about the systematics. Nevertheless what is emerging from the above results is that the \(^4\text{He}\) data are reaching a precision of the order of few percents. This fact requires a similar effort in the theoretical analysis, in order to reduce the uncertainty on the predictions at least at the same level of magnitude. In a previous paper \cite{11} we performed a thoroughly analysis of all corrections to the proton/neutron conversion rates,

\[
\begin{align*}
(a) \quad & \nu_e + n \rightarrow e^- + p \quad , & \quad (d) \quad & \bar{\nu}_e + p \rightarrow e^+ + n \\
(b) \quad & e^- + p \rightarrow \nu_e + n \quad , & \quad (e) \quad & n \rightarrow e^- + \bar{\nu}_e + p \\
(c) \quad & e^+ + n \rightarrow \bar{\nu}_e + p \quad , & \quad (f) \quad & e^- + \bar{\nu}_e + p \rightarrow n 
\end{align*}
\quad (1.7)
\]

which fix at the freeze out temperature \(\sim 1 \text{MeV}\) the neutron to proton density ratio. The Born rates, obtained in the tree level \(V - A\) limit and with infinite nucleon mass, have been corrected to take into account basically three classes of relevant effects:

i) electromagnetic radiative corrections, which largely contribute to the rates of the fundamental processes, in particular in the low temperature regime, \(T \leq 0.1 \text{MeV}\);
ii) finite nucleon mass corrections, which are of the order of $T/M_N$ or $m_e/M_N$, with $m_e, M_N$ the electron and nucleon mass, respectively;

iii) plasma effects, proportional to the surrounding plasma temperature, which both affect the microscopic process rates $(a)-(f)$, as well as the neutrino to photon temperature ratio through $e^\pm, \gamma$ equations of state.

The other main source of theoretical uncertainty comes from the partial knowledge of nuclear rates relevant for nuclei formation. Their numerical expressions, obtained by a convolution of the experimental data with a Boltzmann distribution, are affected by uncertainties of the order of 10% (see references quoted in [12]). More crucially, in many cases, these fits are known to well describe the data in a temperature interval which is only partially overlapping the one relevant for BBN, $0.01\,MeV \leq T \leq 10\,MeV$. However, both a Montecarlo analysis to sample the error distribution of the reaction cross sections [13], and a more recent method based on linear error propagation [14], show that, in particular for $^4He$ mass fraction, the effect is at most as large as the one due to the uncertainty on neutron lifetime $\tau_n$, and smaller than 1%. Therefore it is theoretically justified to look, as in [11], for all sources of theoretical uncertainty up to this level of precision. The situation gets worse with D and $^7Li$, where the uncertainties due nuclear reactions can be as large as $(10 \div 30)\%$ [4].

This paper represents the natural companion to [11]. We have built a new updated version of the standard BBN code, which is available since many years [1, 12], where all corrections i)-iii) have been included. In particular we have also included the modified $e^\pm, \gamma$ equations of state due to electromagnetic mass renormalization. In Section 2 we review the corrections to $n \leftrightarrow p$ Born rates, while in Section 4 we discuss the numerical method we have used to integrate the set of equations relevant for BBN, which are described in Section 3. The numerical results for light nuclei abundances, as functions of the final baryon to photon density ratio, $\eta$, the number of effective neutrino degrees of freedom, $N_\nu$, and the neutron lifetime, $\tau_n$, are reported in Section 5, where they are discussed and compared with the experimental data. We have also performed a fit of these abundances with a precision of the order of 0.1% in the interesting range for the parameters $\eta, N_\nu$. 

2 Corrections to $n \leftrightarrow p$ Born rates

As is well known, the key parameter in determining the primordial $^4He$ mass fraction, $Y_4$, is the value of the neutron to proton density ratio at the freeze-out temperature $T \sim 1$ MeV, since almost all residual neutrons are captured in $^4He$ nuclei due to its large binding energy per nucleon. In order to make an accurate theoretical prediction for $Y_4$ it is necessary, though not sufficient, to have a reliable evaluation of the rates for the processes (1.7). An effort in this direction has been pursued in the last ten years by many authors. Recently, the entire set of corrections to the Born rates $\omega_B$ at the level of 1% accuracy have been recalculated in [11] and [15], with quite compatible results. In this Section we shortly summarize the main corrections $\Delta \omega/\omega_B$ coming from considering radiative, finite nucleon mass, and thermal effects. This short review is here included for the sake of completeness and to fix the notation. A detailed discussion of the subject can be found in our paper [11].

2.1 The Born rates

Let us consider as an example, the thermal averaged rate per nucleon for the neutron decay process $(e)$. In the simple $V - A$ tree level, and in the limit of infinite nucleon mass, which we will refer to as Born approximation, one has

$$\omega_B(n \rightarrow e^- + \nu + p) = \frac{G_F^2 (C_V^2 + 3C_A^2)}{2\pi^3} \int_0^\infty d|p'| |p'|^2 q_0^2 \Theta(q_0) [1 - F_{\nu}(q_0)] [1 - F_e(p'_0)] ,$$

where $G_F$ is the Fermi coupling constant, $C_V$ and $C_A$ the nucleon vector and axial coupling. In our notation $p'$ and $p'_0$ are the electron momentum and energy, and $q_0$ the neutrino energy. The integration limits are imposed by the $\Theta$-function, $q_0 \geq 0$. For reaction $(e)$ we have $q_0 = M_n - M_p - p'_0 \equiv \Delta - p'_0$. The Fermi statistical distributions for $e^\pm$ and neutrinos in the comoving frame, neglecting chemical potentials, are

$$F_e(p'_0) = \left[ e^{\beta p'_0} + 1 \right]^{-1} , \quad F_{\nu}(q_0) = \left[ e^{\beta \nu(q_0)} + 1 \right]^{-1} ,$$

and $\tau_n$. Finally in Section 6 we give our conclusions.
with $\beta = 1/T$ and $\beta_\nu = 1/T_\nu$. All other rates for processes $(a) - (d), (f)$ can be simply obtained from (2.1) properly changing the statistical factors and the expression for $q_0$.\footnote{The ratio $T_\nu/T$ is fixed by entropy conservation and using the neutrino decoupling temperature [1] (see Section 2.4.1).}

The accuracy of Born approximation can be tested by comparing, for example, the prediction for neutron lifetime with the experimental value $\tau_{n}^{\text{ex}} = (886.7 \pm 1.9)\ s$ [16]. Using $C_V = 0.9751 \pm 0.0006$ and $C_A/C_V = 1.2601 \pm 0.0025$ [19], Eq. (2.1) in the vanishing density limit gives $\tau_n \simeq 961\ s$. Therefore, to recover the experimental value, a correction of about 8% is expected to come from radiative and/or finite nucleon mass effects. In the same way these corrections are also expected to contribute to all six processes $(a)-(f)$ relevant for BBN. In addition to these, microscopic $n \leftrightarrow p$ reactions taking place in the early universe, also feel the presence of the surrounding plasma of $\gamma$ and $e^\pm$ pairs in thermodynamical equilibrium. Emission and absorption of real $\gamma$ or $e^\pm$ from the thermal bath can be taken into account using the finite temperature field theory in the real time formalism. This has been considered by several authors [17], and recently in [11].

2.2 Electromagnetic radiative corrections

It is customary to separate the electromagnetic radiative corrections to the Born amplitudes for processes (1.7) in outer and inner terms. The first ones involve the nucleon as a whole and consist in a multiplicative factor to the modulus squared of transition amplitude of the form

$$1 + \frac{\alpha}{2\pi} g(p'_0, q_0) , \quad (2.3)$$

The function $g(p'_0, q_0)$ [18] depends on electron and neutrino energies and describes the deformation in the electron spectrum. Its effect on a freely decaying neutron is such to reduce the Born prediction for the lifetime of about 1.6%.

Inner corrections are sensible to nucleon structure details, and thus much more difficult to handle. They have been estimated in [19], studying corrections to the quark weak currents. Translating the quark–based description in the hadronic language, the inner corrections result in the additional multiplicative factor

$$1 + \frac{\alpha}{2\pi} \left( 4 \ln \frac{M_Z}{M_p} + \ln \frac{M_p}{M_A} + 2C + A_g \right) , \quad (2.4)$$
where the first term is the short-distance contribution and $A_g = -0.34$ is a perturbative QCD correction. The other two terms are related to the axial–induced contributions, with $M_A = (400 \div 1600) MeV$ a low energy cut-off in the short-distance part of the $\gamma W$ box diagram, and $C$ related to the remaining long distance term.

The global effect of these two kind of corrections, improved by resumming all leading logarithmic corrections $\alpha^n \ln^n(M_Z)$ [20], is via the multiplicative factor

$$G(p'_0, q_0) = \left[ 1 + \frac{\alpha}{2\pi} \left( \ln \frac{M_p}{M_A} + 2C \right) + \frac{\alpha(M_p)}{2\pi} \left[ g(p'_0, q_0) + A_g \right] S(M_p, M_Z) \right],$$

(2.5)

where $\alpha(\mu)$ is the QED running coupling constant defined in the $\overline{MS}$ scheme and $S(M_p, M_Z)$ a short distance rescaling factor, defined in [11].

Another effect to be considered, which can be in fact as large as few percents of the Born rates, is the so-called Coulomb correction, due to the rescattering of the electron in the field of the proton and leading to the Fermi function for Coulomb scattering

$$\mathcal{F}(p'_0) \simeq \left( 1 + \alpha \pi \frac{p'_0}{|p'|} \right).$$

(2.6)

However, this effect is only present when both electron and proton are in either the initial or final states, namely it only corrects the amplitudes of processes $(a), (b), (e)$ and $(f)$.

One may wonder if including the effects given by (2.5) and (2.6) the theoretical prediction for neutron decay is now compatible with the experimental results. Evaluating numerically the integral over the phase space one finds $\tau_{n}^{\text{th}} = 893.9 s$, still at variance with the experiment. Even adding all known sub–leading effects the agreement does not really improve [21]. As in Ref. [11] we take the point of view of rescaling all the rates (1.7), after including finite nucleon mass corrections (see next section), by the constant factor $1 + \delta_\tau = \tau_{n}^{\text{th}} / \tau_{n}^{\text{ex}} = 1.008$, which should be regarded as an energy independent correction to the weak process rates. This renormalization of the coupling guarantees the correct prediction for $\tau_{n}$.

In Fig. 1 we report the Born rates $\omega_B$ for $n \leftrightarrow p$ processes, while in Fig. 2 we plot the corresponding radiative corrections, $\Delta \omega / \omega_B$. Their effect is particularly large, up to $\sim 8\%$, at low temperature.
2.3 Finite nucleon mass corrections

There are three additional contributions to the $n \leftrightarrow p$ rates which appear when one relaxes the approximation of infinitely massive nucleons. The leading effects are proportional to $m_e/M_N$ or $T/M_N$, which, in the temperature range relevant for BBN, can be as large as the radiative corrections considered in the previous Section. This has been first pointed out in [22] and then also numerically evaluated in [11]. At order $1/M_N$ there are new couplings appearing in the expression of the weak hadronic current, the larger one coming from the weak magnetic moments of nucleons

$$J_{\mu}^{wm} = i\frac{G_F}{\sqrt{2}} \frac{f_2}{M_N} p_{\mu} (p - q')^{\nu} u_n(q') ,$$

(2.7)

where, from CVC, $f_2 = V_{ud}(\mu_p - \mu_n)/2 = 1.81 V_{ud}$. Both scalar and pseudoscalar contributions can be shown to be much smaller and negligible for the accuracy we are interested in. At the same order in inverse nucleon mass power it is also necessary to include the deformation of the allowed phase space for the relevant scattering and decay processes, due to nucleon recoiling. The sum of these two corrections for $n \leftrightarrow p$ rates with respect to the Born values, $\Delta \omega_M/\omega_B$, is plotted in Fig. 3.

The third effect is due to the initial nucleon thermal distribution. In the infinite nucleon mass limit, the average of weak rates over nucleon distribution is in fact trivial, since the nucleon is at rest in any frame. For finite $M_N$, by considering only $1/M_N$ terms, the effect of the thermal average over the thermal spreading of the nucleon velocity produces a purely kinetic correction $\Delta \omega_K$, whose expression can be reduced to a one dimensional integral over electron momentum which can be numerically evaluated. The explicit expression, which we do not report for brevity can be found in Section 4.2 and Appendix C of [11]. The ratios $\Delta \omega_K/\omega_B$ for $n \leftrightarrow p$ are reported in Fig. 4. Their size is rapidly growing with temperature, since they are proportional to the ratio $T/M_N$.

2.4 Thermal-Radiative corrections

The $n \leftrightarrow p$ rates, calculated as the processes would occur in vacuum, get slight corrections from the presence of the surrounding plasma of $e^\pm$ pairs and $\gamma$. These are the so-called thermal-radiative effects.
To compute these corrections one may use the standard real time formalism for finite temperature field theory. According to this scheme, field propagators get additional contributions proportional to the number density of that particular species in the surrounding medium. For $\gamma$ and $e^\pm$ we have

$$
\left(i \Delta \mu^\nu(k) = - \left[ \frac{i}{k^2 + 2\pi \delta(k^2)} \right] g^\mu\nu = - \left[ \frac{i}{k^2 + 2\pi \delta(k^2)} B(k_0) \right] g^\mu\nu \right),
$$

(2.8)

$$
\left(\Delta \mu (p) = \frac{i}{p' - m_e} \left[ - 2\pi \delta(p'^2 - m_e^2) \right] F_e(p_0) \left( p' + m_e \right) \right).
$$

(2.9)

The entire set of thermal corrections $\Delta \omega_{TR}$, at first order in its typical scale factor, i.e. $\alpha T/m_e$, have been computed by several authors [17] with quite different results. We have recently performed this lengthy calculation in [11], to which we refer for all details, and we have found a good agreement with the original result of first reference in [17], namely that they contribute to correct the Born rates only for less than 1%.

### 2.4.1 Radiative corrections on neutrino temperature

By assuming a sharp neutrino decoupling at $T_D = 2.3$ MeV [23], the ratio $T_\nu/T$ can be evaluated using entropy conservation [11]. This leads to the expression

$$
\frac{T_\nu}{T} = \left\{ \begin{array}{ll}
\left( \frac{\mathcal{I}(x_\gamma) + 2\mathcal{I}(x_\gamma)}{1} \right)^{1/3} & T \leq T_D \\
1 & T > T_D
\end{array} \right.,
$$

(2.10)

with

$$
\mathcal{I}(x) = \int_0^\infty \left( y^2 + 2yx \right)^{1/2} \left( 4y^2 + 8yx + 3x^2 \right) \left[ \exp(x + y) \pm 1 \right]^{-1} dy.
$$

(2.11)

According to our notation $x_\alpha \equiv m_\alpha^R/T$ and $x_{\alpha}^D \equiv m_\alpha^R/T_D$ with $\alpha = \gamma, e$ (+ or − in the above integrand is for fermions or bosons, respectively). Note that $m_\gamma^R$ and $m_e^R$ are the effective masses that photons and $e^\pm$ acquire in the heat bath due to their interactions with the background plasma (see Appendix A for details).

In [11] the neutrino temperature $T_\nu$ as a function of photon temperature $T$ was evaluated by using in (2.10) the approximated expressions $m_\gamma^R \simeq 0$ and $m_e^R \simeq m_e + \alpha T^2/m_e$. This simplified expression for $T_\nu$ has been used in all previous sections in order to obtain the Born rates and their corrections as a function of $T$ only. The difference between the neutrino temperature evaluated with the correct renormalized masses (A.1), (A.2), and
the one obtained with the simplified expressions results to be smaller than 0.01%. The corresponding effect on the rates [1.7] due to this small change in neutrino/photon temperature ratio, which can be seen as a further sub-leading thermal-radiative correction to Born rates, can be neglected at the level of precision we are interested in.

All thermal-radiative corrections to Born rates $\Delta\omega_{TR}/\omega_B$ are reported in Fig. 5. As evident from this plot, around the freeze-out temperature $T \sim 1 \text{ MeV}$, $\Delta\omega_{TR}$ only contribute as $\sim 0.4\%$ to the total rates. Thus they are clearly subdominant. Note that changing $T_D$ in the range $(2 \div 3)\text{MeV}$ only affects $T_\nu/T$ for less than 0.2%.

### 2.5 The total rates for $n \leftrightarrow p$ reactions

In Fig. 6 we report the total corrections $\Delta\omega$ to Born rates $\omega_B$. In order to use the corrected $n \leftrightarrow p$ rates $\omega = \omega_B + \Delta\omega$ in the BBN code, it is useful to fit their expressions as a function of the adimensional inverse photon temperature $z \equiv m_e/T,$

\[
\begin{align*}
\omega(n \rightarrow p) &= \frac{1}{\tau_n^{ex}} \exp(-q_{np} z) \sum_{l=0}^{13} a_l z^{-l}, \quad 0.01 \text{ MeV} \leq T \leq 10 \text{ MeV} \quad (2.12) \\
\omega(p \rightarrow n) &= \left\{ \frac{1}{\tau_n^{ex}} \exp(-q_{pn} z) \sum_{l=1}^{13} b_l z^{-l} \right. \quad 0.1 \text{ MeV} \leq T \leq 10 \text{ MeV} \\
& \left. \quad 0.01 \text{ MeV} \leq T < 0.1 \text{ MeV} \quad (2.13) \right.
\end{align*}
\]

with

\[
\begin{align*}
a_0 &= 1 & a_3 &= -0.401109 \cdot 10^2 & a_1 &= 0.160615 & a_2 &= 0.456817 \cdot 10^1 \\
a_4 &= 0.137254 \cdot 10^3 & a_5 &= -0.583644 \cdot 10^2 \\
a_6 &= 0.655938 \cdot 10^2 & a_7 &= -0.162185 \cdot 10^2 & a_8 &= 0.371109 \cdot 10^1 \\
a_9 &= -0.378497 & a_{10} &= 0.223840 \cdot 10^{-1} & a_{11} &= 0.723091 \cdot 10^{-5} \\
a_{12} &= -0.462476 \cdot 10^{-4} & a_{13} &= 0.186287 \cdot 10^{-5} & q_{np} &= 0.340994
\end{align*}
\]

(2.14)

\[
\begin{align*}
b_1 &= 0.199695 \cdot 10^2 & b_2 &= -0.671993 \cdot 10^2 & b_3 &= 0.109230 \cdot 10^3 \\
b_4 &= -0.295891 \cdot 10^1 & b_5 &= 0.407831 \cdot 10^2 & b_6 &= -0.225830 \cdot 10^1 \\
b_7 &= 0.146751 & b_8 &= -0.185408 \cdot 10^{-2} & b_9 &= -0.205210 \cdot 10^{-3} \\
b_{10} &= 0.158424 \cdot 10^{-5} & b_{11} &= 0.369573 \cdot 10^{-6} & b_{12} &= -0.130731 \cdot 10^{-9} \\
b_{13} &= -0.329060 \cdot 10^{-9} & q_{pn} &= 2.89858
\end{align*}
\]

(2.15)

The fit has been obtained requiring that the fitting functions differ by less than 0.1% from the numerical values, while it is also a good approximation to consider a vanishing rate $\omega(p \rightarrow n)$ for $T \leq 0.1 \text{ MeV}$, see Eq. (2.13), since it is a rapidly decreasing function with $T \rightarrow 0.$
3 The set of equations for BBN

Let us consider $N_{\text{nuc}}$ species of nuclides, whose number densities, $X_i = n_i/n_B$, are normalized with respect to the total baryon density $n_B$. The different nuclides are ordered in the following way: $n, H, D, ^3\text{He}, ^4\text{He}, ^6\text{Li}, ^7\text{Li}, ^7\text{Be}, ...$ (for the complete list see Ref. [12]). Denoting with $R(t)$ the universe scale factor, the BBN set of equations as functions of $R, n_B, T, X_i$, and of the electron chemical potential $\phi_e \equiv \mu_e/T$ reads

\[
\frac{\dot{R}}{R} = \sqrt{\frac{8\pi}{3M_P^2}} \left[ \rho_\gamma + \rho_e + \rho_\nu + \rho_B \right]^{1/2}, \tag{3.1}
\]

\[
\frac{\dot{n}_B}{n_B} = -\frac{3}{2} \frac{\dot{R}}{R} = -\sqrt{\frac{24\pi}{M_P^2}} \left[ \rho_\gamma + \rho_e + \rho_\nu + \rho_B \right]^{1/2}, \tag{3.2}
\]

\[
L\left(\frac{m_e}{T}, \phi_e\right) = \frac{\pi^2 n_B}{2T^3} \sum_j Z_j X_j, \tag{3.3}
\]

\[
\dot{T} = -\left\{3 \frac{\dot{R}}{R} \left[ \rho_\gamma + p_\gamma + \rho_e + p_e + \Theta(T-T_D) (\rho_\nu + p_\nu) + p_B \right] + \frac{\partial \rho_e}{\partial \phi_e} \left( \sum_j \frac{\partial \phi_e}{\partial X_j} \dot{X}_j - 3 \frac{\dot{R}}{R} n_B \frac{\partial \phi_e}{\partial n_B} \right) + n_B \sum_j \left( \Delta M_j + \frac{3}{2} \frac{\partial \phi_e}{\partial n_B} \right) \right\}^{-1}, \tag{3.4}
\]

\[
\dot{X}_i = \sum_{j,k,l} N_i \left( \Gamma_{kl \rightarrow ij} X_i^{N_j} X_k^{N_l} - \Gamma_{ij \rightarrow kl} \frac{X_i^{N_i} X_j^{N_j}}{N_i! N_j!} \right) \equiv \Gamma_i (X_j). \tag{3.5}
\]

In the previous relations $\rho$ and $p$ denote the energy density and the pressure of an arbitrary particle specie. The function $L(z, y)$ in (B.3) is defined as

\[
L(z, y) = \frac{1}{2} \int_{z}^{\infty} dx \, x \sqrt{x^2 - z^2} \left( \frac{e^y}{e^x + e^y} - \frac{e^{-y}}{e^x + e^{-y}} \right), \tag{3.6}
\]

$i, j, k, l = 1, \ldots, N_{\text{nuc}}$, and the $i$-th nuclide, with charge and atomic number $(Z_i, A_i)$, has mass $M_i$ and mass excess ($M_u$ is the atomic mass unit)

\[
\Delta M_i = M_i - A_iM_u. \tag{3.7}
\]

Moreover, in (B.3) we are considering in the sum a reaction between $N_i$ nuclides of type $i$ and $N_j$ of type $j$ which results in $N_l$ nuclides of type $l$ and $N_k$ of type $k$, with its reverse

\footnote{We are using natural units $\hbar = c = k_B = 1.$}
reaction. The energy density and the pressure of baryons take the form

\[ \rho_B = n_B \left[ M_u + \sum_j \left( \Delta M_j + \frac{3}{2} T \right) X_j \right] , \quad (3.8) \]

\[ p_B = n_B T \sum_j X_j . \quad (3.9) \]

Eq. (3.1) is easily recognized as the Friedmann equation where we have neglected for simplicity the cosmological constant. Eq. (3.2) rules the scaling on \( n_B \), whereas (3.3) states the neutrality of primordial plasma. From entropy conservation one gets (3.4), and (3.5) are the Boltzmann equations for the \( N_{\text{nuc}} \) nuclide number densities. Note that the presence of the \( \Theta \)-function in (3.4) is connected with neutrino decoupling at \( T = T_D \).

In the set of equations (3.1)-(3.5) one can safely substitute Eq. (3.3) with an analogous relation, obtained expanding the l.h.s. of (3.3) with respect to \( \phi_e \),

\[ L(z, y) \simeq y \int_z^\infty dx \ x \sqrt{x^2 - z^2} \frac{e^x}{(e^x + 1)^2} \equiv y f^{-1}(z) . \quad (3.10) \]

In this case, Eq. (3.3) provides an explicit expression for \( \phi_e = \phi_e(T, n_B, X_j) \),

\[ \phi_e \simeq \frac{\pi^2}{2} f \left( \frac{m_e}{T} \right) \frac{n_B}{T^3} \sum_j Z_j X_j . \quad (3.11) \]

The consistency of this approach has been tested by means of an iterative check.

The set of equations (3.1)-(3.5) can be transformed in a set of \( N_{\text{nuc}} + 1 \) differential equations with the dimensionless variable \( z = m_e/T \) as the evolution parameter. For numerical reasons, it is also better to turn the variable \( n_B \) into the dimensionless quantity \( \hat{h} \equiv n_B/T^3 \), which varies more slowly with \( z \) than \( n_B \). In terms of these new variables the BBN set of equations becomes

\[ \frac{d\hat{h}}{dz} = \left[ 1 - \bar{H}(z, \hat{h}, X_j) \ G(z, \hat{h}, X_j) \right] \frac{3\hat{h}}{z} , \quad (3.12) \]

\[ \frac{dX_i}{dz} = G(z, \hat{h}, X_j) \ \frac{\bar{\Gamma}_i}{z} , \quad (3.13) \]

where the function \( G \) is

\[ G(z, \hat{h}, X_j) = \begin{cases} \frac{\sum_{\alpha}(4\hat{\rho}_\alpha - z\frac{\partial \hat{\rho}_\alpha}{\partial z}) + 4\Theta(z_D - z)\hat{\rho}_\nu \frac{3}{4} \frac{\hat{h}}{z} \sum_j X_j}{3 \left[ \sum_{\alpha}(\hat{\rho}_\alpha + \hat{p}_\alpha) + \frac{3}{4} \Theta(z_D - z)\hat{\rho}_\nu + \hat{h} \sum_j X_j \right] \bar{H} + \hat{h} \sum_j \left( z\Delta M_j + \frac{3}{2} \right) \bar{\Gamma}_j} \end{cases} \quad (3.14) \]

\(^3\)We have assumed that all neutrinos decouple at the same temperature \( T_D \). Actually muon and tau neutrinos decouple at a slightly larger temperature of 3.5 MeV, but nevertheless our approximation is largely consistent with the required precision on \(^4\)He yields.
In the previous equations $z_D = m_e/T_D$, $\alpha = e, \gamma$, and we have considered the dimensionless Hubble parameter $\hat{H} = H/m_e$,

$$\hat{H}(z, \hat{h}, X_j) = \sqrt{\frac{8\pi m_e}{3 M_P^2}} \left[ \hat{\rho}_\gamma + \hat{\rho}_e + \hat{\rho}_\nu + \hat{h} \left( z\hat{M}_u + \sum_j \left( z\Delta\hat{M}_j + \frac{3}{2} \right) X_j \right) \right]^{1/2},$$

and the quantities $\hat{M}_u = M_u/m_e$, $\Delta\hat{M}_j = \Delta M_j/m_e$, $\hat{\Gamma}_j = \Gamma_j/m_e$. Energy densities and pressures have been adimensionalized dividing by $T^4$. In Eq.s (3.12) and (3.13) we have neglected the terms containing the derivatives of chemical potential. In Appendix A we report the expressions for $\hat{p}_\alpha$ and $\hat{\rho}_\alpha$ evaluated taking into account the $\gamma$ and $e^\pm$ electromagnetic mass renormalization. As already mentioned in the previous section this effect, changing the $\gamma$ and $e^\pm$ equations of state, slightly modifies the $T_\nu/T$ ratio too. In order to speed up the numerical computation a fit of $\hat{p}_\alpha$ and $\hat{\rho}_\alpha$ as functions of $z$ has been performed and is also reported in Appendix A.

The initial value for (3.12) is provided in terms of the final baryon to photon density ratio $\eta$ according to the equation

$$\hat{h}_{in} = \frac{2\zeta(3)}{\pi^2} \eta_{in} = \frac{11}{4} \frac{2\zeta(3)}{\pi^2} \eta.$$

The condition of Nuclear Statistical Equilibrium (NSE), which is satisfied with high accuracy at the initial temperature $T_{in} = 10$ MeV, is then fixing the initial nuclide relative abundances. From NSE one gets, for an arbitrary $i$-th nuclide with $g_i$ internal degrees of freedom,

$$X_i(T_{in}) = \frac{g_i}{2} \left( \frac{\zeta(3)}{\pi} \right)^{A_i-1} A_i^{\frac{3}{2}} \left( \frac{T_{in}}{M_N} \right)^{\frac{4}{3}(A_i-1)} \eta^{A_i-1} X_p^{Z_i} X_n^{A_i-Z_i} \exp \left\{ \frac{B_i}{T_{in}} \right\},$$

where $B_i$ denotes the binding energy.

### 4 Numerical Method

The most critical part of the BBN code concerns the method of numerical resolution of the set of differential equations (3.12), (3.13). In fact, since at high temperatures nuclear reactions proceed in both forward and reverse directions with almost equal rapidity, the r.h.s. of (3.13) results to be a small difference of large numbers. When this occurs the
Table 1: The reduced network of nuclear reactions.

|   | Reaction                                                                 |
|---|--------------------------------------------------------------------------|
| 1 | \( n \leftrightarrow p \)                                               |
| 2 | \( T \leftrightarrow ^3He \)                                            |
| 3 | \( p + n \leftrightarrow D + \gamma \)                                  |
| 4 | \( n + D \leftrightarrow T + \gamma \)                                  |
| 5 | \( n + ^3He \leftrightarrow ^4He + \gamma \)                            |
| 6 | \( n + ^6Li \leftrightarrow ^7Li + \gamma \)                            |
| 7 | \( n + ^3He \leftrightarrow T + p \)                                     |
| 8 | \( n + ^7Be \leftrightarrow ^7Li + p \)                                  |
| 9 | \( n + ^6Li \leftrightarrow ^4He + T \)                                  |
|10 | \( n + ^7Be \leftrightarrow ^4He + ^4He \)                              |
|11 | \( p + D \leftrightarrow ^3He + \gamma \)                               |
|12 | \( p + T \leftrightarrow ^4He + \gamma \)                               |
|13 | \( p + ^6Li \leftrightarrow ^7Be + \gamma \)                            |
|14 | \( p + ^6Li \leftrightarrow ^4He + ^3He \)                              |
|15 | \( p + ^7Li \leftrightarrow ^4He + ^4He \)                              |
|16 | \( D + ^4He \leftrightarrow ^6Li + \gamma \)                            |
|17 | \( T + ^4He \leftrightarrow ^7Li + \gamma \)                            |
|18 | \( ^3He + ^4He \leftrightarrow ^7Be + \gamma \)                          |
|19 | \( D + D \leftrightarrow ^3He + n \)                                     |
|20 | \( D + D \leftrightarrow T + p \)                                        |
|21 | \( D + T \leftrightarrow ^4He + n \)                                     |
|22 | \( D + ^3He \leftrightarrow ^4He + p \)                                  |
|23 | \( ^3He + ^3He \leftrightarrow ^4He + p + p \)                           |
|24 | \( D + ^7Li \leftrightarrow ^4He + ^4He + n \)                           |
|25 | \( D + ^7Be \leftrightarrow ^4He + ^4He + p \)                           |

numerical problem is said **stiff**. As a consequence, the step size is limited more severely by the requirement of stability than by the accuracy of the numerical technique. In other words, to preserve integration stability it is required to use a shorter step size than what would be dictated by accuracy only. In order to manage the problem, we use a NAG routine implementing a method belonging to the class of Backward Differentiation Formulas (BDFs) [24]. This is quite a new approach for BBN codes. In fact the standard code [1, 12] uses instead the implicit differentiating method (backward Euler scheme) [24] for writing the r.h.s. of (5.13) and a Runge-Kutta solver.

Few comments on the different numerical methods are in order. Let us consider the
differential equation
\[
\frac{dy(t)}{dt} = f(t, y(t)) \quad (4.1)
\]
In the Runge-Kutta methods the solution at \( t_{i+1} \) is completely determined by its value at \( t_i \) (one-step methods), namely the solver has no memory. A different approach is provided by a wide class of numerical methods referred to as multistep methods like BDFs. Here, the values of the solution at \( t_k \) \((k = i, i-1, \ldots, i-p)\), \( y(t_k) \equiv y_k \), previously computed, and the unknown value \( y(t_{i+1}) \equiv y_{i+1} \), are interpolated by a polynomial, \( P(t; y_{i+1}, y_i, \ldots) \), in order to approximate the solution and its derivative. Substituting in the differential equation,
\[
\frac{dP}{dt}(t_{i+1}; y_{i+1}, y_i, \ldots) \simeq f(t_{i+1}, y_{i+1}) \quad (4.2)
\]
one obtains a family of BDFs,
\[
(t_{i+1} - t_i) f(t_{i+1}, y_{i+1}) \simeq P(t_{i+1}; y_{i+1}, y_i, \ldots) - P(t_i; y_{i+1}, y_i, \ldots) = a_0 y_{i+1} + a_1 y_i + \ldots. \quad (4.3)
\]
Two methods can be used for solving the previous equation in the implicit case, \( a_0 \neq 0 \): functional iteration and Newton’s method. In the former case some initial guess is taken for \( y_{i+1} \) and refined by iteration. In the latter case, one linearizes Eq. (4.3) by expanding \( f \) around \( y_i \). The new point, \( y_{i+1} \), is then found by inverting a matrix, in a way similar to the backward Euler scheme. The NAG routine implements both methods and incorporates an error control test, which drives the step-size adjustment.

The nuclear reaction network used in the code includes all the 88 reactions between the 26 nuclides present in the standard code [1, 12]. We used the same nuclear rate data of the standard code, which are collected and updated in [25]. In order to reduce the computation time one can also choose a reduced network, made of the 25 reactions between 9 nuclides listed in Table 1. Using the complete network we have verified that the reduced one affects the abundances for no more than 0.01%, while it greatly reduces the evaluation time.
5 Results on Light Element Abundances

The reliable numerical code just discussed can now be used to study the effect of the different corrections to $n \leftrightarrow p$ Born rates on light elements abundances. By definition

$$Y_2 = \frac{X_3}{X_2}, \quad Y_3 = \frac{X_5}{X_2}, \quad Y_4 = \frac{M_6}{\sum_j M_j} \frac{X_6}{X_j}, \quad Y_7 = \frac{X_8}{X_2}. \quad (5.1)$$

In the first two rows of Table 2 are shown the predictions for $Y_2$, $Y_3$, $Y_4$ and $Y_7$ at $\eta = 5 \cdot 10^{-10}$, corresponding to the complete $n \leftrightarrow p$ rates, $\omega_{Tot}$, and to the Born approximation, $\omega_B$. As it is clear from Table 2, the main effect of the corrections, which results into the enhancement of the $n \rightarrow p$ conversion rate, is to allow a smaller number of neutrons to survive till the onset of nucleosynthesis. This ends up in a smaller fraction of elements which fix neutrons with respect to hydrogen.

The effects on light element yields due to the various corrections with respect to the Born predictions are also reported in Table 2. For all nuclides the pure radiative correction $\Delta \omega_R$ provides the dominant contribution, while the finite nucleon mass effects, the kinetic and the thermal-radiative ones almost cancel each other. Finally the last row reports the further contribution due to the additional term required to recover the experimental neutron lifetime [11].

If we make use of the results of [14] to quantify the uncertainties coming from nuclear reaction processes, we observe that only for $Y_4$ the radiative correction affects the Born result by an amount larger than the theoretical uncertainties, including nuclear reactions. For $^4He$ mass fraction in fact, the theoretical uncertainty due to nuclear reaction rates is estimated to be of the order of 0.1% and thus comparable with the uncertainty due to the experimental error on neutron lifetime. For $D$, $^3He$ and $^7Li$ the uncertainty due to the poor knowledge of nuclear reaction rates is estimated to be of the order of $(10 \div 30)$% [14], thus completely covering any radiative/thermal correction on $n \leftrightarrow p$ rates.

In Fig. 7 the predictions on $Y_4$ are shown versus $\eta$ for $N_\nu = 2, 3, 4$ and for a 1 $\sigma$ variation of $\tau_n^{ex}$. The two experimental estimates for the primordial $^4He$ mass fraction, $Y_4^{(l)}$ and $Y_4^{(h)}$, as horizontal bands, are also reported. Fig.s 8 and 9 show the predictions

\footnote{Note that, according to our notation, with $\omega_B$ we denote the pure Born predictions for $n \leftrightarrow p$ rates without any constant rescaling of coupling to account for the experimental value of neutron lifetime (see Section 2.2).}
Table 2: The predictions on light element abundances obtained with the numerical code for $\eta = 5 \cdot 10^{-10}$ and $N_\nu = 3$. In the lower rows the effect of the various corrections is reported.

|          | $Y_2$         | $Y_3$         | $Y_4$         | $Y_7$         |
|----------|---------------|---------------|---------------|---------------|
| $\omega_{\text{Tot}}$ | $0.3638 \cdot 10^{-4}$ | $0.1175 \cdot 10^{-4}$ | $0.2446$ | $0.2814 \cdot 10^{-9}$ |
| $\omega_B$   | $0.3727 \cdot 10^{-4}$ | $0.1184 \cdot 10^{-4}$ | $0.2550$ | $0.2873 \cdot 10^{-9}$ |
| $\Delta \omega_R$ | $-2.3\%$ | $-2.8\%$ | $-3.8\%$ | $-1.9\%$ |
| $\Delta \omega_M$ | $.2\%$ | $.1\%$ | $.3\%$ | $.2\%$ |
| $\Delta \omega_K$ | $.2\%$ | $.1\%$ | $.3\%$ | $.2\%$ |
| $\Delta \omega_{TR}$ | $-0.6\%$ | $-0.1\%$ | $-0.7\%$ | $-0.4\%$ |
| $\delta \tau \omega_T$ | $-0.3\%$ | $-0.1\%$ | $-0.6\%$ | $-0.3\%$ |

Table 2: The predictions on light element abundances obtained with the numerical code for $\eta = 5 \cdot 10^{-10}$ and $N_\nu = 3$. In the lower rows the effect of the various corrections is reported.

for $D$ and $^7Li$ abundances. Note that, due to the negligible variation of $Y_2$ and $Y_7$ on small $\tau_n$ changes, no splitting of predictions for $1\sigma$ variation of $\tau_n^{\text{ex}}$ is present.

A fit, up to $1\%$ accuracy, of the relevant observables $Y_2$, $Y_3$, $Y_4$ and $Y_7$ as a function of $x = \log_{10} (10^{10} \eta)$, $N_\nu$ and $\tau_n$ has been performed. The following expressions have been obtained

\[
10^3Y_2 = \left[ \sum_{i=0}^{4} a_i x^i + a_5 (N_\nu - 3) \right] \exp \left\{ a_6 x + a_7 x^2 \right\}, \quad (5.2)
\]
\[
10^5Y_3 = \left[ \sum_{i=0}^{4} a_i x^i + a_5 (N_\nu - 3) \right] \exp \left\{ a_6 x \right\}, \quad (5.3)
\]
\[
10^5Y_4 = \sum_{i=0}^{5} a_i x^i + a_6 (\tau - \tau_{\text{ex}}) + a_7 (N_\nu - 3) + a_8 x (\tau - \tau_{\text{ex}}) + a_9 x (N_\nu - 3) \quad (5.4)
\]
\[
10^9Y_7 = \left[ \sum_{i=0}^{3} a_i x^i + a_4 (N_\nu - 3) + a_5 x (N_\nu - 3) \right] \exp \left\{ \sum_{i=1}^{4} a_{5+i} x^i \right\}, \quad (5.5)
\]

where $\tau_{\text{ex}} = 886.7 s$ and the values of the fit coefficients are reported in Table 3.

Neutrino decoupling has been shown by many authors \cite{26} to be a process which still...
Table 3: Values of the fit coefficients (5.2)-(5.5) for light element abundances.

| Coeff. | $10^5 Y_2$ | $10^5 Y_3$ | $10 Y_4$ | $10^9 Y_7$ |
|--------|------------|------------|----------|------------|
| $a_0$  | 0.4854     | 3.325      | 2.209    | 0.5419     |
| $a_1$  | 0.2919     | 0.1496     | 0.5548   | −0.5981    |
| $a_2$  | −0.3516    | 1.597      | −0.6491  | −1.914     |
| $a_3$  | 0.5048     | −1.923     | 0.7661   | 4.521      |
| $a_4$  | −0.4269    | 1.312      | −0.5366  | 0.1587     |
| $a_5$  | $0.7772 \cdot 10^{-1}$ | 0.1782      | 0.1614   | −0.3256    |
| $a_6$  | −4.397     | −1.705     | $0.2059 \cdot 10^{-2}$ | −4.102     |
| $a_7$  | 0.5925     | /          | 0.1300   | 5.072      |
| $a_8$  | /          | /          | −0.4156 $\cdot 10^{-4}$ | −1.209     |
| $a_9$  | /          | /          | 0.7433 $\cdot 10^{-2}$ | −0.6269    |

takes place when $e^{\pm}$ pairs annihilate. This implies that neutrinos are in fact slightly reheated during this annihilation process and their final distribution in momentum space shows an interesting non equilibrium shape. In Ref. [27] it is estimated that the effect on $Y_4$ due to the inclusion of this slight neutrino heating is very small, $\delta Y_4 \sim 1.5 \cdot 10^{-4}$, in the whole range $10^{-10} \leq \eta \leq 10^{-9}$. We have included this constant correction to $Y_4$ prediction.

From the fit reported in Eq. (5.4) it is easy to quantify the theoretical error on $Y_4$. Since this is basically due to the uncertainty on $\tau_n$ we have

$$\frac{\Delta Y_4}{Y_4} = \frac{(a_6 + a_8 x) \Delta \tau_n}{10Y_4} \leq 0.1\% \quad (5.6)$$

In Fig. 10, our prediction for $Y_4$ of Eq. (5.4) with $N_{\nu} = 3$ and $\tau_n = 885.3$ s, is compared with an analogous fit, $Y'_4$, performed in [10]. The agreement between the two expressions
obtained by independent codes is up to 1% in the relevant range for \( \eta \).

In Fig. 11 we present the results of a likelihood analysis of the theoretical predictions obtained with the numerical code for the four combinations of experimental results: a) high \( D \), low \(^4\)He; b) high \( D \), high \(^4\)He; c) low \( D \), low \(^4\)He; d) low \( D \), high \(^4\)He, and using the low value for \(^7\)Li abundance. In particular, we plot the product of gaussian distribution for \( D \), \(^4\)He and \(^7\)Li centered around the measured values and with their corresponding experimental errors,

\[
L(N_\nu, x) = \exp \left\{ \frac{-(Y_2(N_\nu, x) - Y_{2,ex})^2}{2\sigma_2^2} \right\} \exp \left\{ \frac{-(Y_4(N_\nu, x) - Y_{4,ex})^2}{2\sigma_4^2} \right\} \times \exp \left\{ \frac{-(Y_7(N_\nu, x) - Y_{7,ex})^2}{2\sigma_7^2} \right\} .
\]

Notice that all functions have been normalized to unity in the maximum. As is clear from the plots, the analysis prefers the high value of \( D \) (plots a) and b)). In both cases the distributions are centered in the range \( x \in 0.2 \div 0.4 \), but at \( N_\nu \sim 3 \) for low \(^4\)He and \( N_\nu \sim 3.5 \) for high \(^4\)He. For low \( D \) the compatibility with experimental data is worse. Note that c) and d) distributions have been multiplied by a factor of 25 and 100 respectively and centered in the range \( x \in 0.6 \div 0.8 \), and at \( N_\nu \sim 2 \) for low \(^4\)He and \( N_\nu \sim 3 \) for high \(^4\)He. The better agreement at 1\( \sigma \) of the data set a) and b) with the theoretical predictions is basically due to the effect of \(^7\)Li data which corresponds to values for \( \eta \) compatible with low \( D \) data of c) and d) at 2\( \sigma \) only. It should be mentioned however that these results only take into account experimental errors, so that the confidence level regions in the \( N_\nu - x \) plane would be broader by convoluting the considered distributions with the ones containing the theoretical error.

6 Conclusions

In this paper a detailed study of the effects on light element yields of the radiative, finite nucleon mass, thermal and plasma corrections to Born rates (1.7) has been carried out. The aim of such an analysis was to reduce the error on, basically, \( Y_4 \) to less than 1%, which is motivated by the most recent experimental determinations for \(^4\)He abundance. This accurate analysis has been performed using an update version of the BBN standard code [1, 12]. A different numerical approach, based on BDF techniques has been implemented.
to solve the stiff Boltzmann equations for nuclei densities. The numerical results for $^4He$ mass fraction almost confirm the computation reported in Ref. [10], while the theoretical error, also including the propagation of uncertainties on nuclear processes, as estimated in [14], is of the order of 0.1%. Our analysis shows that the preferred experimental values are high value for $D$ and low one for $^4He$, in which case the distribution is centered at $x \sim 0.3$ and $N_\nu \sim 3$. 
A Radiative corrections to $e^\pm$, $\gamma$ equations of state

In an accurate description of the primordial plasma it is important to consider the electromagnetic correction to the $e^\pm$ and $\gamma$ equations of state induced by the $e^\pm$ and $\gamma$ mass renormalization.

As well known, the photon renormalized mass, up to first order correction in the electromagnetic coupling constant $\alpha$, reads [28]

$$m_R^\gamma(z) \simeq m_e \frac{2}{z} \sqrt{\frac{\alpha}{\pi}} \left[ \int_z^\infty dx \frac{\sqrt{x^2 - z^2}}{1 + e^x} \right]^{1/2},$$

(A.1)

and for $e^\pm$ [17]

$$m_R^{e}(z, y) \simeq m_e \left\{ 1 + \frac{\alpha}{\pi z^2} \left[ \frac{\pi^2}{3} + \int_z^\infty \left( \frac{2\sqrt{x^2 - z^2}}{2\sqrt{y^2 - z^2}} + \frac{z^2}{2\sqrt{y^2 - z^2}} \log \Lambda \right) \frac{dx}{1 + e^x} \right] \right\},$$

(A.2)

where $z \equiv m_e/T$, $y \equiv E_e/T$ and

$$\Lambda(x, y, z) = \frac{x^2 y^2 - \left( z^2 + \sqrt{x^2 - z^2} \sqrt{y^2 - z^2} \right)^2}{x^2 y^2 - \left( z^2 - \sqrt{x^2 - z^2} \sqrt{y^2 - z^2} \right)^2}.$$  

(A.3)

Note that in Eq.s (A.1) and (A.2) one can neglect the contribution of electron chemical potential, $\phi_e \equiv \mu_e/T$, due to its small value.

By using (A.1) and (A.2) in the expressions of $\rho_\gamma$, $p_\gamma$, $\rho_e$, $p_e$ one gets the latter quantities as functions of $z$ only. Since the $e^\pm$ and $\gamma$ energy densities and pressures have to be used in a BBN code, in order to speed up the computation one can fit these quantities as function of $z$ and use these fits in the evolution equations. The fitted expressions for the dimensionless electron energy density and pressure, $\hat{\rho}_e = \rho_e/T^4$ and $\hat{p}_e = p/T^4$, in the range $z \in [0.05, 8.52]$ ($\hat{\rho}_e = \hat{p}_e = 0$ for $z > 8.52$), result to be

$$\hat{\rho}_e(z) = 1.145 + 0.33981 \cdot 10^{-1} z - 0.14543 z^2 + 0.25507 \cdot 10^{-1} z^3 - 0.54168 \cdot 10^{-3} z^4$$
$$- 0.11263 \cdot 10^{-3} z^5 - 0.29742 \cdot 10^{-5} z^6 + 0.38331 \cdot 10^{-6} z^7 + 0.45263 \cdot 10^{-7} z^8$$
$$+ 0.19241 \cdot 10^{-8} z^9 - 0.96597 \cdot 10^{-10} z^{10} - 0.19505 \cdot 10^{-10} z^{11} - 0.14079 \cdot 10^{-12} z^{12},$$

(A.4)

$$\hat{p}_e(z) = 0.3786 + 0.19126 \cdot 10^{-1} z - 0.63895 \cdot 10^{-1} z^2 + 0.32085 \cdot 10^{-1} z^3 - 0.48501 \cdot 10^{-2} z^4$$
$$- 0.16611 \cdot 10^{-3} z^5 + 0.82922 \cdot 10^{-4} z^6 + 0.79884 \cdot 10^{-5} z^7 - 0.60619 \cdot 10^{-6} z^8.$$
\[ -0.19568 \times 10^{-6} z^9 - 0.10921 \times 10^{-7} z^{10} + 0.38564 \times 10^{-8} z^{11} \] \( e^{-0.13145 z^2} \).

(A.5)

Moreover, in the considered temperature range, one can show that \( \hat{\rho}_\gamma = \rho_\gamma / T^4 \) only varies between 0.6580 and 0.6573, and \( \hat{p}_\gamma = p_\gamma / T^4 \) between 0.2193 and 0.2187. Thus, for simplicity, one can assume \( \hat{\rho}_\gamma \) constant and equal to the average value 0.6577, and to 0.2190 for \( \hat{p}_\gamma \).

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Figure 1: The total Born rates, $\omega_B$, for $n \rightarrow p$ (solid line) and $p \rightarrow n$ transitions (dashed line). This notation is adopted hereafter.
Figure 2: The radiative corrections to Born rates, $\Delta \omega_R/\omega_B$, for $n \leftrightarrow p$ transitions.
Figure 3: The finite nucleon mass corrections to Born rates, $\Delta \omega_M/\omega_B$, for $n \leftrightarrow p$ transitions.
Figure 4: The kinetic corrections to Born rates, $\Delta \omega_K/\omega_B$, for $n \leftrightarrow p$ transitions.
Figure 5: The thermal-radiative corrections to Born rates, $\Delta \omega_{TR}/\omega_B$, for $n \leftrightarrow p$ transitions.
Figure 6: The total corrections to Born rates for $n \leftrightarrow p$ transitions.
Figure 7: The $^4$He mass fraction, $Y_4$, versus $\eta$. The three solid lines are, from larger to lower values of $Y_4$, the predictions corresponding to $N_\nu = 3$ and $\tau_\text{ex}^{\nu} = 888.6 \, s$, $886.7 \, s$, $884.8 \, s$, respectively. Analogously, the dashed lines correspond to $N_\nu = 4$ and the dotted ones to $N_\nu = 2$. The dotted and dashed horizontal band are the experimental values of Ref.s [2] and [3], respectively, with 1$\sigma$ interval.
Figure 8: The quantity $Y_2$ versus $\eta$ is reported. The notation used is the same of Fig. 8. Due to the negligible dependence of $Y_2$ on small variations of $\tau_n^{\text{ex}}$ no splitting of lines is present. The horizontal bands dashed and dotted are the experimental values of Ref.s [4, 5].
Figure 9: The quantity $Y_7$ versus $\eta$. The notation used is the same of Fig. 8. There is no splitting of lines related to $\Delta \tau_n$, due to the negligible dependence of $Y_7$ on small variations of $\tau_n^{ex}$. The horizontal bands dashed and dotted are the experimental values of Ref.s [6, 7] and [9], respectively.
Figure 10: The ratio \((Y_4 - Y'_4)/Y_4\) versus \(\log_{10}(10^{10}\eta)\) for \(N_\nu = 3\) and \(\tau_n = 885.3\) s [10] (see Section 5).
Figure 11: The likelihood distributions for the light element yields $Y_2$, $Y_4$, $Y_7$ are shown as functions of $N_\nu$ and $\log_{10}(10^{10})\eta$, normalized to unity in correspondence of the experimental values. From left to right and from top to bottom the following cases are considered: a) high $D$, low $^4He$; b) high $D$, high $^4He$; c) low $D$, low $^4He$; d) low $D$, high $^4He$. The plots for cases c) and d) are rescaled by a factor 25 and 100 times, respectively, compared to the one of a) and b).