ABOUT THE ALMOST EVERYWHERE CONVERGENCE
OF THE SPECTRAL EXPANSIONS OF FUNCTIONS
FROM $L^\alpha_1(S^N)$

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Abstract. In this paper we study the almost everywhere convergence of the expansions related to the self-adjoint extension of the Laplace-Beltrami operator on the unit sphere. The sufficient conditions for summability is obtained. The more general properties and representation by the eigenfunctions of the Laplace-Beltrami operator of the Liouville space $L^\alpha_1$ is used. For the orders of Riesz means, which greater than critical index $\frac{N-1}{2}$ we proved the positive results on summability of Fourier-Laplace series. Note that when order $\alpha$ of Riesz means is less than critical index then for establish of the almost everywhere convergence requests to use other methods form proving negative results. We have constructed different method of summability of Laplace series, which based on spectral expansions property of self-adjoint Laplace-Beltrami operator on the unit sphere.

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1. Introduction

Let $S^N$ is unit sphere in $R^N$

$$S^N = \{ x \in R^N : |x| = 1 \}$$

The sphere $S^N$ is naturally equipped with a positive measure $d\sigma(x)$ and with an elliptic second order differential operator $\Delta_s$, which named the Laplace-Beltrami operator on the sphere. This operator is symmetric and nonnegative, it extends to a nonnegative self-adjoint operator on the space $L^2(S^N)$ (by $L^p(S^N)$ we mean the $L^p$-space associated with the measure $d\sigma(x)$ on the sphere). Let $c$ be any positive number, and let $A = \Delta_s + 1$. We denote by $\text{Spec}(A) = \{ \lambda_k, k = 0, 1, 2, \ldots \}$ the spectrum of $A$. This spectrum is nondecreasing sequence of positive eigenvalues with finite multiplicities (and written as such) tending to infinity. We denote by $Y_j^k(x)$ an eigenfunction of the Laplace-Beltrami Operator.

$$\Delta_s Y_j^{(k)} = \lambda_k Y_j^{(k)}$$

where $\lambda_k = k(k + N - 1), k = 0, 1, 2, \ldots$. The system of eigendunctions of the Laplace-Beltrami operator is orthonormal basis in $L^1_2(S^N)$.

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One of the main problems of harmonic analysis is the reconstruction of functions from their expansion:

\[ f(x) \sim \sum_{k=0}^{\infty} Y_k(f, x) \]  

(1.1)

The main purpose of this article is the convergence problems of the partial sums of representation 1.1

\[ E_n f(x) = \sum_{k=0}^{n} Y_k f(x, y) \]  

(1.2)

in (1.1) and (1.2) denoted by \( Y_k(f, x) \)

\[ Y_k(f, x) = \int_{S^N} f(y) Z_k(x, y) d\sigma(y) \]

where \( Z_k(x, y) = \sum_{j=0}^{n} Y_j^{(k)}(x) Y_j^{(k)}(y) \) is Zonal harmonic of order \( k \). A spectral function \( \Theta(x, y, \lambda) \) of the Laplace-Beltrami operator on sphere is defined by

\[ \Theta(x, y, n) = \sum_{k=0}^{n} Z_k(x, y) \]  

(1.3)

Then the spectral expansions of the (1.2) can be rewritten as follow

\[ E_n f(x) = \int_{S^N} f(y) \Theta(x, y, n) d\sigma \]  

(1.4)

The Riesz means of the spectral expansions (1.2) is defined by the next expression

\[ E_n^\alpha f(x) = \int_{S^N} f(y) \Theta^\alpha(x, y, n) d\sigma \]  

(1.5)

where by \( \Theta^\alpha(x, y, \lambda) \) denoted the Riesz means of the spectral function (1.3):

\[ \Theta^\alpha(x, y, n) = \sum_{k=0}^{n} \left( 1 - \frac{\lambda_k}{\lambda_n} \right)^\alpha Z_k(x, y) \]  

(1.6)

Let us denote by \( T_n^\alpha f(x) \) the Cesaro means of the spectral expansions, which defined as follow

\[ T_n^\alpha f(x) = \int_{S^N} f(y) \Phi^\alpha(x, y, n) d\sigma \]  

(1.7)

with the kernel

\[ \Phi^\alpha(x, y, n) = \sum_{k=0}^{n} \frac{A_k^{\alpha}}{A_n^{\alpha}} Z_k(x, y) \]  

(1.8)

where

\[ A_n^\alpha = \frac{(\alpha + 1)(\alpha + 2)\cdots(\alpha + m)}{m!} \]
The maximal operator of the Integral operator (1.5) plays important role in establishing the almost everywhere convergent of spectral expansions
\[ E^\alpha \star f(x) = \limsup_{n \in \mathbb{N}} |E_n^\alpha f(x)| \]
Let \( L^\tau_p(S^N) \) is the class Liouville, which consists of all functions from \( L_p(S^N) \), for which
\[ \| f \|_{L^\tau_p(S^N)} = \| \sum_{k=0}^{\infty} \lambda_k Y_k(f, x) \|_{L_p} < \infty \] (1.9)
Main result of this paper is

\textbf{Theorem 1.1.} Let \( f \in L^\tau_1(S^N), \alpha + \tau > \frac{N-1}{2}, 0 \leq \alpha \leq \frac{N-1}{2}, \tau > 0 \), then
1) \( \| E^\alpha \star f(x) \|_{L_1(S^N)} \leq c_\alpha \| f \|_{L^\tau_1(S^N)} \)
2) The Riesz means \( E_n^\alpha f(x) \) almost everywhere on \( S^N \) converges to \( f(x) \).

Note that, if \( \tau > N \) then Riesz means of spectral expansions uniformly convergence to \( f \) (see [2]). In work [3] was investigated spectral expansions related to the Pseudo-differential operators and in the [4] considered spectral expansions of elliptic differential operators.

\section{2. Preliminaries}

In this section we recall some preliminaries concerning some properties of Liouville space and Riesz means.

\textbf{Lemma 2.1.} Let \( \tau \geq 0, \alpha \geq 0, p \geq 1 \), then for all \( f \in C^\infty(S^N) \) we have
\[ \| A^{\alpha/2} f \|_{L_p(S^N)} \leq c_\alpha \| f \|_{L_p^\alpha(S^N)} \]
The proof of this lemma it follows from embedding properties of Liouvil space. For more details see [2].

\textbf{Lemma 2.2.} Let \( \Theta^\alpha(x, y, n) \) is the Riesz means of the spectral function of the Laplace-Beltrami operator on sphere and
1) if spherical distance \( \gamma = \gamma(x, y) \) between \( x \) and \( y \) satisfied inequality \( |\frac{\pi}{2} - \gamma| < \frac{n}{n+1} \frac{\pi}{2} \) then we have
\[ \Theta^\alpha(x, y, n) = O(1) \frac{n^{(N-1)/2}}{(\sin \gamma)^{(N-1)/2}(\sin(\gamma/2))^{1+\alpha}} + \frac{n^{(N-3)/2}}{(\sin \gamma)^{(N+1)/2}(\sin(\gamma/2))^{1+\alpha}} + \frac{n^{-1}}{(\sin(\gamma/2))^{1+N}} \] (2.1)
2) if \( 0 \leq \gamma \leq \pi \), then we have
\[ \Theta^\alpha(x, y, n) = O(1)n^N; \]
3) if \( 0 < \gamma_0 \leq \gamma \leq \pi \), then we have
\[ \Theta^\alpha(x, y, n) = O(1)n^{N-\alpha}; \]
This lemma proved in our work [6].
Lemma 2.3. Let \( f \in C^\infty(S^N), \alpha + \tau > \frac{N-1}{2}, 0 \leq \alpha \leq \frac{N-1}{2}, \tau > 0 \) then for maximal operator of Riesz means we have

\[
\|E_\alpha^f\|_{L_1(S^N)} \leq c_\alpha \|f\|_{L_1^1(S^N)}
\]

Proof. For all \( \tau \geq 0 \) let us denote by \( \Theta_\alpha^{\tau}(x, y, n) \) the kernel of the integral operator \( A^{-\tau}E_\alpha^n \).

\[
\Theta_\alpha^{\tau}(x, y, n) = \sum_{k=1}^{n} \lambda_k^{-\tau} \left(1 - \frac{\lambda_k}{\lambda_n}\right)^{\alpha} Z_k(x, y)
\]

Let \( g = A^{\tau/2}f \). Using the equality

\[
E_\alpha^n f = A^{-\tau/2}E_\alpha^n A^{\tau/2} f
\]

we can rewrite the Riesz means as follow

\[
E_\alpha^n f(x) = \int_{S^N} \Theta_\alpha^{\tau/2}(x, y, n) g(y) d\sigma(y).
\]

The kernel \( \Theta_\alpha^{\tau/2}(x, y, n) \) we reduce to the next form

\[
\Theta_\alpha^{\tau/2}(x, y, n) = \sum_{k=1}^{n} \lambda_k^{-\tau/2} (\Theta_\alpha^{\tau/2}(x, y, k) - \Theta_\alpha^{\tau/2}(x, y, k - 1)) =
\]

\[
= \sum_{k=1}^{n} \left(\lambda_k^{-\tau/2} - \lambda_{k+1}^{-\tau/2}\right) \Theta_\alpha^{\tau/2}(x, y, k) + \lambda_n^{-\tau/2} \Theta_\alpha^{\tau/2}(x, y, n).
\]

Using this representation for the kernel \( \Theta_\alpha^{\tau/2}(x, y, n) \) we can estimate the Riesz means of the spectral expansions as follow:

\[
E_\alpha^n f(x) = \sum_{k=1}^{n} \left(\lambda_k^{-\tau/2} - \lambda_{k+1}^{-\tau/2}\right) \int_{S^N} \Theta_\alpha^{\tau/2}(x, y, k + 1) g(y) d\sigma(y) + \\
+ \lambda_n^{-\tau/2} \int_{S^N} \Theta_\alpha^{\tau/2}(x, y, n) g(y) d\sigma(y).
\]

Separating into four part the integral \( \int_{S^N} \Theta_\alpha^{\tau/2}(x, y, k + 1) g(y) d\sigma(y) \) and for estimate each part let us apply the lemma 2.2:

\[
\int_{S^N} \Theta_\alpha^{\tau/2}(x, y, k) g(y) d\sigma(y) = I_1 + I_2 + I_3 + I_4,
\]

where

\[
I_1 = \int_{\gamma < \frac{\pi}{4}} \Theta_\alpha^{\tau/2}(x, y, k) g(y) d\sigma(y)
\]

\[
I_2 = \int_{\frac{\pi}{4} < \gamma \leq \frac{\pi}{2}} \Theta_\alpha^{\tau/2}(x, y, k) g(y) d\sigma(y)
\]

\[
I_3 = \int_{\frac{\pi}{2} < \gamma \leq \pi} \Theta_\alpha^{\tau/2}(x, y, k) g(y) d\sigma(y)
\]

\[
I_4 = \int_{\pi < \gamma \leq \frac{3\pi}{2}} \Theta_\alpha^{\tau/2}(x, y, k) g(y) d\sigma(y)
\]
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\[ I_3 = \int_{\frac{\pi}{2} < \gamma \leq \pi - \frac{\pi}{k}} \Theta^\alpha(x, y, k)g(y)d\sigma(y) \]

\[ I_4 = \int_{\pi - \frac{\pi}{k} < \gamma \leq \pi} \Theta^\alpha(x, y, k)g(y)d\sigma(y). \]

For estimate \( I_1 \) and \( I_4 \) we use the asymptotic behavior 2.2 of the Riesz means of the spectral function: \( \Theta^\alpha(x, y, n) = O(1) \) and we get

\[ |I_1 + I_4| \leq C \left( k^N \int_{\gamma(x, y) \leq \frac{1}{k}} |g(y)|d\sigma(y) + k^N \int_{\gamma(x, y) = \pi - \gamma(x, y) \leq \frac{1}{k}} |g(y)|d\sigma(y) \right) \]

\[ \leq \left( g^*(x) + g^*(\overline{x}) \right) \]

here we denoted by \( g^*(x) \) the maximal function of the function \( d(x) \) which defined by the formula

\[ g^*(x) = \limsup_{r > 1} \frac{1}{\text{mes}B(x, r)} \int_{B(x, r)} |g(y)|d\sigma(y) \]

where \( B(x, r) \) is the ball in the unit sphere with the center at the point \( x \) and the radius \( r \):

\[ B(x, r) = \{ y \in S^N : \gamma(x, y) < r, r > 0 \}. \]

Due to estimation 1) of lemma 2.2 we have for \( I_2 \)

\[ |I_2| \leq Ck^{\frac{N-1}{2} - \alpha} \int_{\frac{1}{k} < \gamma \leq \frac{\pi}{2}} (\sin \gamma)^{-\frac{N+1}{2} - \alpha} |g(y)|d\sigma(y) + \]

\[ +Ck^{\frac{N-1}{2} - \alpha} \int_{\frac{1}{k} < \gamma \leq \frac{\pi}{2}} (\sin \gamma)^{-\frac{N+1}{2} - \alpha} |g(y)|d\sigma(y) + \]

\[ +Ck^{-1} \int_{\frac{1}{k} < \gamma \leq \frac{\pi}{2}} (\sin \gamma)^{-1-N} |g(y)|d\sigma(y) \leq Cg^*(x) \left( 1 + k^{\frac{N-1}{2} - \alpha} \right) \]

The \( I_3 \) after the denoting \( \gamma(\overline{x}, y) = \pi - \gamma(x, y) \) can be computed as \( I_2 \), and we have

\[ |I_3| \leq Cg^*(\overline{x}) \left( 1 + k^{\frac{N-1}{2} - \alpha} \right) \]

Such that for Riesz means of the spectral expansions we have

\[ E_n^\alpha f(x) \leq C \sum_{k=1}^{n} \left( \lambda_k^{-\tau/2} - \lambda_{k+1}^{-\tau/2} \right) \left( 1 + k^{\frac{N-1}{2} - \alpha} \right) (g^*(x) + g^*(\overline{x})) + \]

\[ +\lambda_n^{-\tau/2} \left( 1 + n^{\frac{N-1}{2} - \alpha} \right) (g^*(x) + g^*(\overline{x})). \]

We have to prove that the expression

\[ T_n = \sum_{k=1}^{n} \left( \lambda_k^{-\tau/2} - \lambda_{k+1}^{-\tau/2} \right) \left( 1 + k^{\frac{N-1}{2} - \alpha} \right) \]
is bounded with constant which does not depend of $n$. If note that eigenvalues 
$\lambda_k = k(k + N - 1), k = 1, 2, 3, ...$

$$T_n = \sum_{k=1}^{n} \left( k^{-\tau/2}(k + N - 1)^{-\tau/2} - (k + 1)^{-\tau/2}(k + N)^{-\tau/2} \right) k^{\frac{N-1}{2}-\alpha} =$$

$$= \sum_{k=1}^{n} \left( \left(1 + \frac{N-1}{k}\right)^{-\tau/2} - \left(1 + \frac{N+1}{k} + \frac{N}{k^2}\right)^{-\tau/2} \right) k^{\frac{N-1}{2}-\alpha-\tau}$$

and applying the Lagrange formula to function $J(x) = (1 + x)^{-\tau/2}$ in the segment $[\frac{N-1}{k}, \frac{N+1}{k} + \frac{N}{k^2}]$ then we obtain

$$J\left(\frac{N-1}{k}\right) - J\left(\frac{N+1}{k} + \frac{N}{k^2}\right) = -\left(\frac{2}{k} + \frac{N}{k^2}\right) J'(\xi)$$

where $\xi : \frac{N-1}{k} < \xi < \frac{N+1}{k} + \frac{N}{k^2}$. Then for Riesz means we have

$$|E_n^\alpha f(x)| \leq C = \sum_{k=1}^{\infty} k^{\frac{N-1}{2}-\alpha-\tau-1}$$ (2.2)

If $\alpha + \tau > \frac{N-1}{2}$ then numerical series 2.2 converges. Such that for Riesz means of order $\alpha$ we have proved estimation

$$|E_n^\alpha f(x)| \leq C \leq (g^*(x) + g^*(\overline{x}))$$ (2.3)

for all $f \in L_1^p(S^N)$ provided by the condition $\alpha + \tau > \frac{N-1}{2}$.

Let $p > 1$, then

$$\|E_n^\alpha f\|_{L_1} \leq C \|E_n^\alpha f\|_{L_p} \leq C \left(\|g^*(x)\|_{L_p} + \|g^*(\overline{x})\|_{L_p}\right) \leq C \|g\|_{L_p}$$

By virtue of the lemma 2.1 we have

$$\|g\|_{L_p} \leq C \|f\|_{L_p^\alpha}$$

It is well-known the embedding $L_{p+N(1-\frac{1}{p})+\epsilon} \subset L_1^\alpha$, if $\epsilon > 0$ and $p > 1$. Therefore finally we get the estimate

$$\|E_n^\alpha f\|_{L_1(S^N)} \leq C_0 \|f\|_{L_1(S^N)}$$ (2.4)

The assumption of lemma 2.3 is proved for all function $f \in C^\infty$. It is well-known, that Banach space $C^\infty$ is dense in $L_1^\tau$ for all $\tau > 0$. Therefore proof of the inequality 2.4 for all function from $L_1^\tau$ follows from density of $C^\infty$ in $L_1^\tau$. Lemma 2.3 is proved.

Let us prove second part of the theorem 1.1. Let $f \in L_1^\tau(S^N), \tau > 0, \alpha + \tau > \frac{N-1}{2}$. Fix any $\epsilon > 0$. From density of $C^\infty$, it follows that exists $h \in C^\infty$ such that $\|f - h\|_{L_1^\tau(S^N)} < \epsilon$. Riesz means $E_n^\alpha h(x)$ of spectral expansions of $h$ uniformly converges to $h(x)$. For any $\epsilon > 0$ there is $n_0(\epsilon)$ such that, if $n > n_0$ we have

$$|E_n^\alpha h(x) - h(x)| < \epsilon, x \in S^N.$$ 

Then for $n > n_0$ we have

$$|E_n^\alpha f(x) - f(x)| < E_n^\alpha (f - h)(x) + \epsilon + |h(x) - f(x)|.$$
Using first part of the theorem 1.1 it is not hard to see that
\[ \|E_n^\alpha f(x) - f(x)\|_{L_1} \leq \|h(x) - f(x)\|_{L_1(S^N)} + \epsilon. \]
The last inequality proves that
\[ \lim_{n \to \infty} E_n^\alpha f(x) = f(x) \]
holds almost everywhere in \( S^N \).
For Cesaro means the assertion of the theorem 1.1 proves analogously.

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