SYSTEMATIC ERRORS OF BOUND-STATE PARAMETERS EXTRACTED BY MEANS OF SVZ SUM RULES

Wolfgang Lucha  
Institute for High Energy Physics, Austrian Academy of Sciences,  
Nikolsdorfgasse 18, A-1050, Vienna, Austria  

Dmitri Melikhov  
Institute for High Energy Physics, Austrian Academy of Sciences,  
Nikolsdorfgasse 18, A-1050, Vienna, Austria  
and  
Nuclear Physics Institute, Moscow State University,  
119991, Moscow, Russia  

Silvano Simula  
INFN, Sezione di Roma III,  
Via della Vasca Navale 84, I-00146, Roma, Italy

Abstract

This talk presents the results of our study of systematic errors of the ground-state parameters obtained by Shifman–Vainshtein–Zakharov (SVZ) sum rules. We use the harmonic-oscillator potential model as an example: in this case we know the exact solution for the polarization operator, which allows us to obtain both the OPE to any order and the parameters (masses and decay constants) of the bound states. We extract the parameters of the ground state by making use of the standard procedures of the method of QCD sum rules, and compare the obtained results with their known exact values. We show that if the continuum contribution to the polarization operator is not known and is modelled by some effective continuum threshold, the standard procedures adopted in sum rules do not allow one to gain control over the systematic errors of the extracted ground-state parameters.
A QCD sum-rule calculation of hadron parameters\(^1\) involves two steps: one first constructs the operator product expansion (OPE) series for a relevant correlator and then extracts the parameters of the ground state by a numerical procedure. Each of these steps leads to certain uncertainties in the final result.

The first step lies entirely within QCD and, in the case of SVZ sum rules, allows for a rigorous treatment of the uncertainties: the correlator is not known precisely because of uncertainties in quark masses, condensates, \(\alpha_s\), etc., but all corresponding errors in the correlator may be controlled. [Complications arising in light-cone sum rules are discussed in our second talk\(^2\).]

The second step lies beyond QCD: even if several terms of the OPE for the correlator were known precisely, the hadronic parameters might be extracted by a sum rule only within some error, which may be treated as a systematic error of the method.

Here we present the results of our recent study of systematic uncertainties of the sum-rule procedures\(^3\)\(^4\). To this end, a quantum-mechanical harmonic-oscillator (HO) potential model is a perfect tool: in this case both the spectrum of bound states (i.e., masses and wave functions) and the exact correlator (and hence its OPE to any order) are known precisely. Therefore, one may apply the sum-rule machinery for extracting parameters of the ground state and test the accuracy of the extracted values by comparing with the known exact results. In this way the accuracy of the method can be probed. For a detailed discussion of various aspects of sum rules in quantum mechanics, we refer to Refs. [5–9].

To illustrate the essential features of the QCD calculation, we consider a non-relativistic model with a confining potential,

\[
V(r) = \frac{m\omega^2 r^2}{2}, \quad r = |r|,
\]

and analyze the Borel transform \(\Pi(\mu)\) of the polarization operator \(\Pi(E)\), which gives the evolution operator in the imaginary time \(1/\mu\):

\[
\Pi(\mu) = \left(\frac{2\pi}{m}\right)^{3/2} \left\langle 0 \left| \exp\left(-\frac{H}{\mu}\right) \right| r_f = 0 \right\rangle.
\]

For the HO potential\(^1\), the exact analytic expression for \(\Pi(\mu)\) is well known:

\[
\Pi(\mu) = \left(\frac{\omega}{\sinh(\omega/\mu)}\right)^{3/2}.
\]
Expanding the above expression in inverse powers of $\mu$, we get the OPE series

$$\Pi_{\text{OPE}}(\mu) \equiv \Pi_0(\mu) + \Pi_1(\mu) + \Pi_2(\mu) + \cdots$$

$$= \mu^{3/2} \left( 1 - \frac{\omega^2}{4\mu^2} + \frac{19}{480} \frac{\omega^4}{\mu^4} + \cdots \right);$$

higher power corrections may be derived from the exact result (4).

The “phenomenological” representation for $\Pi(\mu)$ is obtained by using the basis of hadronic eigenstates of the model, namely,

$$\Pi(\mu) = \sum_{n=0}^{\infty} R_n \exp \left( \frac{-E_n}{\mu} \right),$$

where $E_n$ is the energy of the $n$th bound state and $R_n$ [the square of the leptonic decay constant of the $n$th bound state] is given by

$$R_n = \left( \frac{2\pi}{m} \right)^{3/2} |\Psi_n(r = 0)|^2.$$  

For the lowest states, one finds from (3)

$$\begin{align*}
E_0 &= \frac{3}{2} \omega, \quad R_0 = 2\sqrt{2}\omega^{3/2}, \\
E_1 &= \frac{7}{2} \omega, \quad R_1 = 3\sqrt{2}\omega^{3/2}, \quad \ldots.
\end{align*}$$

The sum rule is just the equality of the correlator calculated in the “quark” basis and in the “hadron” basis:

$$R_0 \exp \left( \frac{-E_0}{\mu} \right) + \int_{z_{\text{cont}}}^{\infty} dz \rho_{\text{phen}}(z) \exp \left( \frac{-z}{\mu} \right)$$

$$= \int_{0}^{\infty} dz \rho_0(z) \exp \left( \frac{-z}{\mu} \right) \mu^{3/2} \left( -\frac{\omega^2}{4\mu^2} + \frac{19}{480} \frac{\omega^4}{\mu^4} + \cdots \right).$$

Following Ref. [1], we use explicit expressions for the power corrections, but for the zeroth-order free-particle term we use its expression in terms of the spectral integral.

Let us introduce an “effective” continuum threshold $z_{\text{eff}}(\mu)$, different from the physical $\mu$-independent continuum threshold $z_{\text{cont}}$, by the relation

$$\Pi_{\text{cont}}(\mu) = \int_{z_{\text{cont}}}^{\infty} dz \rho_{\text{phen}}(z) \exp \left( \frac{-z}{\mu} \right) = \int_{z_{\text{eff}}(\mu)}^{\infty} dz \rho_0(z) \exp \left( \frac{-z}{\mu} \right).$$
The spectral densities $\rho_{\text{phen}}(z)$ and $\rho_0(z)$ are different functions. Thus the two
sides of (9) can be equal to each other only if the effective continuum threshold,
$z_{\text{eff}}(\mu)$, depends on $\mu$ in an appropriate way. In our model, we can calculate
$\Pi_{\text{cont}}$ precisely and, therefore, we can obtain the function $z_{\text{eff}}(\mu)$ by solving (9).
In the general case of an actual QCD sum-rule analysis, the effective continuum
threshold is not known and constitutes one of the essential fitting parameters.

Making use of (9), we now rewrite the sum rule (8) in the form
\begin{equation}
R_0 \exp \left( - \frac{E_0}{\mu} \right) = \Pi(\mu, z_{\text{eff}}(\mu)),
\end{equation}
where the cut correlator $\Pi(\mu, z_{\text{eff}}(\mu))$ reads
\begin{equation}
\Pi(\mu, z_{\text{eff}}(\mu)) \equiv \frac{2}{\sqrt{\pi}} \int_0^{z_{\text{eff}}(\mu)} dz \sqrt{z} \exp \left( - \frac{z}{\mu} \right) + \mu^{3/2} \left( - \frac{\omega^2}{4\mu^2} + \frac{19}{480} \omega^4 + \cdots \right).
\end{equation}
As is obvious from (10), the cut correlator satisfies the equation
\begin{equation}
- \frac{d \log \Pi(\mu, z_{\text{eff}}(\mu))}{d(1/\mu)} = E_0.
\end{equation}
The cut correlator $\Pi(\mu, z_{\text{eff}}(\mu))$ is the quantity that actually governs the ex-
traction of the ground-state parameters.

The “fiducial” range of $\mu$ is defined as the range where, on the one
hand, the OPE reproduces the exact expression with better than some chosen
accuracy (for instance, within, say, 0.5%) and, on the other hand, the ground
state is expected to give a sizable contribution to the correlator. If we include
only the first three power corrections (that is, $\Pi_1$, $\Pi_2$, and $\Pi_3$) we must require
$\omega/\mu < 1.2$. Since we know the ground-state parameters, we fix $\omega/\mu > 0.7$,
where the ground state contributes more than 60% of the full correlator. So
our fiducial range is $0.7 < \omega/\mu < 1.2$.

We shall be interested in situations where the hadronic continuum is not
known — which is typical for heavy-hadron physics and in the discussion of the
properties of exotic hadrons. Can we extract the ground-state parameters?

We denote the values of the ground-state parameters extracted from the
sum rule (10) by $E$ and $R$. The notations $E_0$ and $R_0$ are reserved for the exact
values. In many interesting cases the ground-state energy may be determined,
Figure 1: The effective continuum threshold $z_{\text{eff}}(\mu)$ obtained by solving (8) for $E = E_0$ and $R = 0.7 R_0$ [long-dashed (blue) line], $R = R_0$ [solid (red) line] and $R = 1.15 R_0$ [dash-dotted (green) line].

e.g., from experiment. However, setting $E = E_0$ does not help: still, for any $R$ within a broad range, one finds a function $z_{\text{eff}}(\mu, R)$ [Fig. 1] which solves the sum rule (10) exactly. Therefore, we conclude that in a limited range of $\mu$ the OPE alone cannot say much about the ground-state parameters. What really matters is the continuum contribution, or, equivalently, $z_{\text{eff}}(\mu)$. Only by making some assumptions about $z_{\text{eff}}(\mu)$ one is able to extract $R$.

Typically, one assumes $z_{\text{eff}}(\mu)$ to be constant and imposes some criteria to fix its value. Rigorously speaking, a constant effective continuum threshold $z_{\text{eff}}(\mu) = z_c = \text{const}$ is incompatible with the sum rule (10). Nevertheless, such an Ansatz may work well, especially in our HO model: As seen from Fig. 1, the exact $z_{\text{eff}}(\mu)$ is almost flat in the fiducial interval. Therefore, the HO model represents a very favorable situation for applying the QCD sum-rule machinery.

Now, how to determine $z_c$? A widely used procedure is to calculate

$$E(\mu, z_c) \equiv -\frac{d \log \Pi(\mu, z_c)}{d(1/\mu)}, \quad (13)$$

which now depends on $\mu$ due to approximating $z_{\text{eff}}(\mu)$ by a constant. Then, one determines $\mu_0$ and $z_c$ as the solution to the system of equations

$$E(\mu_0, z_c) = E_0, \quad \frac{\partial}{\partial \mu} E(\mu, z_c) \bigg|_{\mu=\mu_0} = 0, \quad (14)$$

yielding $z_c = 2.454 \omega, \mu_0/\omega = 1$ [Fig. 2]. Finally, one takes the value $R(\mu_0, z_c)$ as the sum-rule estimate for the quantity $R$. The error of $R$ is usually obtained
Figure 2: Constant effective continuum threshold $z_c$: $E(\mu)$ for three different values of $z_c$ (a) and the corresponding $R(\mu)$ (b).

by looking at the range covered by $R(\mu, z_c)$ when one allows for a variation of $\mu$ within the fiducial range. Following this procedure, one obtains in our case a good central-value estimate: $R/R_0 = 0.96$. Since $R(\mu, z_c)$ is extremely stable in the fiducial range, one expects its true value to be rather close to the extracted value and, accordingly, assigns a very small error to the sum-rule estimate.

Note, however, a dangerous point: (i) a perfect description of $\Pi(\mu)$ with an accuracy better than 1%, (ii) a deviation of $E(\mu, z_c)$ from $E_0$ at the level of only 1%, and (iii) an extreme stability of $R(\mu)$ in the entire fiducial range conspire to lead to a 4% error in the extracted value of $R$! Clearly, this error could not be guessed on the basis of the other numbers obtained, and it would be wrong to try to estimate the error from, e.g., the range covered by $R$ when varying the Borel parameter $\mu$ within the fiducial interval.
Let us summarize the lessons we have learnt from the above investigation:

1. The knowledge of the correlator to any accuracy within a limited range of the Borel parameter $\mu$ is not sufficient for an extraction of the ground-state parameters since rather different models for the correlator, generically of the form of a ground state plus an effective continuum, lead to the same correlator.

2. Modelling the hadron continuum by a constant effective continuum threshold $z_c$ allows one to determine the value of $z_c$ by, e.g., requiring the average energy $E(\mu)$ to be close to $E_0$ in the region of stability of the sum rule. In the model under discussion this leads to a good estimate, $R/R_0 = 0.96$, with almost $\mu$-independent $R$. The unpleasant feature of this extraction procedure is that the deviation of $R$ from $R_0$ is much larger than the variations of $E(\mu)$ and $R(\mu)$ over the fiducial interval of $\mu$. In particular, it would be wrong to assign the systematic error on the basis of the range covered by $R(\mu)$ when $\mu$ is varied within the fiducial interval. This means that the standard procedures adopted in QCD sum rules do not allow one to control the systematic errors. Consequently, no rigorous systematic errors for hadronic parameters extracted by sum rules can be provided. Let us also stress that the independence of the extracted values of the hadron parameters from the Borel mass $\mu$ does not guarantee the extraction of their true values.

Finally, in the model under consideration sum rules provide a rather good estimate for $R_0$, even though its error cannot be determined on the basis of the standard procedures adopted in sum-rule analyses. This may be a consequence of the following features of the model: (i) a large gap between ground state and the first excitation contributing to the sum rule; (ii) an almost constant exact effective continuum threshold. Whether or not the same good accuracy may be achieved in QCD, where the features mentioned above are absent, is not obvious at all and requires more detailed investigations.

We would like to point out that with respect to the problem of assigning systematic errors to the extracted hadron parameters, the method of QCD sum rules faces very similar problems as the application of approaches based on the constituent quark picture: for instance, the relativistic dispersion approach\cite{11} yields very successful predictions for the form factors of exclusive $D$ decays and provides many predictions for the form factors of weak decays of $B$ mesons\cite{12}. However, assigning rigorous errors to these predictions could not be done so far.
Acknowledgements. We thank R. Bertlmann, B. Grinstein, and B. Stech for interesting discussions and inspiring comments, and the Austrian Science Fund (FWF) for financial support under project P17692.

References

1. M. Shifman, A. Vainshtein, and V. Zakharov, Nucl. Phys. B 147, 385 (1979).
2. W. Lucha, D. Melikhov, and S. Simula, "Systematic errors of transition form factors extracted by means of light-cone sum rules", arXiv:0712.0178.
3. W. Lucha, D. Melikhov, and S. Simula, Phys. Rev. D 76, 036002 (2007); Phys. Lett. B 657, 148 (2007).
4. W. Lucha, D. Melikhov, and S. Simula, Phys. Rev. D 75, 096002 (2007); W. Lucha and D. Melikhov, Phys. Rev. D 73, 054009 (2006); Phys. Atom. Nucl. 70, 891 (2007).
5. V. Novikov, M. Shifman, A. Vainshtein, and V. Zakharov, Nucl. Phys. B 237, 525 (1984).
6. A. I. Vainshtein, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, Sov. J. Nucl. Phys. 32, 840 (1980).
7. V. A. Novikov et al., Phys. Rep. 41, 1 (1978); M. B. Voloshin, Nucl. Phys. B 154, 365 (1979); J. S. Bell and R. Bertlmann, Nucl. Phys. B 177, 218 (1981); Nucl. Phys. B 187, 285 (1981); V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B 191, 301 (1981).
8. A. Le Yaouanc et al., Phys. Rev. D 62, 074007 (2000); Phys. Lett. B 488, 153 (2000); Phys. Lett. B 517, 135 (2001).
9. D. Melikhov and S. Simula, Phys. Rev. D 62, 074012 (2000).
10. M. Jamin and B. Lange, Phys. Rev. D 65, 056005 (2002).
11. D. Melikhov, Phys. Rev. D 53, 2460 (1996); Phys. Rev. D 56, 7089 (1997); Eur. Phys. J. direct C4, 2 (2002) [hep-ph/0110087]; D. Melikhov and S. Simula, Eur. Phys. J. C 37, 437 (2004).
12. D. Melikhov and B. Stech, Phys. Rev. D 62, 014006 (2000).