SIMULTANEOUS SIMILARITY CLASSES OF COMMUTING MATRICES OVER A FINITE FIELD

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Abstract. This paper concerns the enumeration of isomorphism classes of modules of a polynomial algebra in several variables over a finite field. This is the same as the classification of commuting tuples of matrices over a finite field up to simultaneous similarity. Let $c_{n,k}(q)$ denote the number of isomorphism classes of $n$-dimensional $F_q[x_1, \ldots, x_k]$-modules. The generating function $\sum_k c_{n,k}(q)t^k$ is a rational function. We compute this function for $n \leq 4$. We find that its coefficients are polynomial functions in $q$ with non-negative integer coefficients.

1. Introduction

1.1. Background. Let $F_q$ be a finite field of order $q$. Let $\mathcal{A}$ be a finite dimensional algebra (with unity) over $F_q$. Let $\mathcal{A}^*$ denote the group of units in $\mathcal{A}$. Let $k \geq 1$ be a positive integer, and $\mathcal{A}^{(k)}$ be the set of $k$-tuples of elements of $\mathcal{A}$, whose entries commute with each other, i.e.,

$$\mathcal{A}^{(k)} = \{ (a_1, \ldots, a_k) \in \mathcal{A}^k \mid a_ia_j = a_ja_i \text{ for } i \neq j \}.$$ 

Denote by $c_{\mathcal{A}^k}$, the number of orbits in $\mathcal{A}^{(k)}$ for the action of simultaneous conjugation of $\mathcal{A}^*$ on it, which is defined below:

Definition 1.1. For $(a_1, \ldots, a_k) \in \mathcal{A}^{(k)}$ and $g \in \mathcal{A}^*$:

$$g.(a_1, \ldots, a_k) = (ga_1g^{-1}, \ldots, ga_kg^{-1}).$$

This action is called simultaneous conjugation, and the orbits for this action are called simultaneous similarity classes.

For an element, $a \in \mathcal{A}$, let $Z_\mathcal{A}(a)$ denote the centralizer algebra of $a$, and $Z_{\mathcal{A}^*}(a)$ denote the group of units of $Z_\mathcal{A}(a)$. Let $h_\mathcal{A}(t)$ be the

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generating function of $c_{A,k}$ in $k$:

$$h_A(t) = 1 + \sum_{k=1}^{\infty} c_{A,k} t^k.$$ 

Then we have the theorem:

**Theorem 1.2.** For any finite dimensional algebra $A$ over $\mathbb{F}_q$, $h_A(t)$ is a rational function.

**Proof.** Given $a_1 \in A$, consider the $k$-tuple, $(a_1, a_2, \ldots, a_k) \in A^{(k)}$. That means, $(a_2, \ldots, a_k) \in Z_A(a_1)^{(k-1)}$. Thus, the map:

$$(a_1, a_2, \ldots, a_k) \mapsto (a_2, \ldots, a_k),$$

induces a bijection from the set of $A^*$-orbits in $A^{(k)}$, which contain an element whose 1st coordinate is $a_1$, onto the set of orbits in $Z_A(a_1)^{(k-1)}$ for the action of simultaneous conjugation by $Z_A^*(a_1)$ on it. Thus, we get:

$$c_{A,k} = \sum_{Z \subseteq A} s_Z c_{Z,k-1},$$

where $Z$ runs over subalgebras of $A$, $s_Z$ is the number of similarity classes in $A$ whose centralizer algebra is isomorphic to $Z$, and $c_{Z,k-1}$ is the number of orbits under the action of $Z^*$ (the group of units of $Z$) on $Z^{(k-1)}$ by simultaneous conjugation. Hence, we have

$$h_A(t) = 1 + \sum_{k=1}^{\infty} c_{A,k} t^k$$

$$= 1 + \sum_{k=1}^{\infty} \left( \sum_{Z \subseteq A} s_Z c_{Z,k-1} \right) t^k$$

$$= 1 + \sum_{Z \subseteq A} s_Z t \left( \sum_{k=1}^{\infty} c_{Z,k-1} t^{k-1} \right) \left( c_{Z,0} = 1 \text{ for } Z \subseteq A \right)$$

$$= 1 + \sum_{Z \subseteq A} s_Z t \left( 1 + \sum_{k=1}^{\infty} c_{Z,k} t^k \right)$$

$$= 1 + s_A t \left( 1 + \sum_{k=1}^{\infty} c_{A,k} t^k \right) + \sum_{Z \subseteq A} s_Z t \left( 1 + \sum_{k=1}^{\infty} c_{Z,k} t^k \right)$$

$$= 1 + s_A t h_A(t) + \sum_{Z \subseteq A} s_Z t h_Z(t)$$

( here, $h_Z(t) = 1 + \sum_{k=1}^{\infty} c_{Z,k} t^k$).
Therefore,

\[(1.1) \quad (1 - s_A t) h_A(t) = 1 + \sum_{Z \subseteq A} s_Z t h_Z(t).\]

The above identity establishes rationality when \(A\) is a commutative algebra. When \(A\) is commutative, \(A^{(k)} = A^k\). As \(A\) is commutative, \(Z_A(a) = A\) for all \(a \in A\). Each element of \(A\) is a similarity class in \(A\). Thus, \(s_A = |A|\), and \(s_Z = 0\) for \(Z \subsetneq A\). We have, \((1 - |A| t) h_A(t) = 1\), hence \(h_A(t) = \frac{1}{1 - |A|^t}\), which is a rational function.

If \(A\) is not commutative, then from identity (1.1), we are reduced to the case of algebras whose dimension is strictly less than that of \(A\). The rationality of \(h_A(t)\) follows by induction on the dimension of \(A\). When \(A\) is 1 dimensional, \(A = \mathbb{F}_q\), which is commutative. When \(\dim(A) > 1\), assuming induction for algebras with dimension \(<\dim(A)\), we get that, for \(Z \subsetneq A\), \(h_Z(t)\) is rational. As \(A\) is finite, it has only a finite number of subalgebras. Hence, from identity (1.1), \(h_A(t)\), being a sum of a finite number of rational functions, is rational. \(\square\)

1.2. Matrices Over Finite Fields. Let \(n, k\) be positive integers. Consider \(M_n(\mathbb{F}_q)\), the algebra of \(n \times n\) matrices over \(\mathbb{F}_q\), and \(M_n(\mathbb{F}_q)^{(k)}\), the set of \(k\)-tuples of commuting \(n \times n\) matrices over \(\mathbb{F}_q\). We have \(GL_n(\mathbb{F}_q)\) acting on \(M_n(\mathbb{F}_q)^{(k)}\) by simultaneous conjugation (as in Definition 1.1). Let \(c_{n,k}(q)\) denote the number of simultaneous similarity classes in \(M_n(\mathbb{F}_q)^{(k)}\).

Let \(h_n(t)\) denote the generating function of \(c_{n,k}(q)\) in \(k\).

\[h_n(t) = 1 + \sum_{k=1}^{\infty} c_{n,k}(q) t^k.\]

As \(M_n(\mathbb{F}_q)\) is a finite dimensional algebra over \(\mathbb{F}_q\), we get from Theorem 1.1:

**Theorem 1.3.** For any positive integer \(n\), \(h_n(t)\) is a rational function.

Next, we define the following:

**Definition 1.4.** We say that two simultaneous similarity classes of tuples of commuting matrices are of the same similarity class type (or just type), if their centralizers are isomorphic.
We will discuss these similarity class types in detail in Section 2.

While calculating $c_4(k, q)$ for $k \geq 1$ (Section 5), we come across some new types of simultaneous similarity classes of pairs of commuting matrices of $M_4(\mathbb{F}_q)$, i.e., the centralizers of pairs of these types are not isomorphic to the centralizers of any of the similarity classes in $M_4(\mathbb{F}_q)$. These new types are dealt with in the subsection 5.3 of Section 5.

In this paper, we compute $h_n(t)$ for $n = 2, 3, 4$, and the results are given in Table 1.

| $n$ | $h_n(t)$ |
|-----|----------|
| 1   | $\frac{1}{1-qt}$ |
| 2   | $\frac{1}{(1-qt)(1-q^2t)}$ |
| 3   | $\frac{1+q^2t^2}{(1-qt)(1-q^2t)(1-q^4t)}$ |
| 4   | $\frac{1+q^2t+2q^2t^2+q^3t^2+2q^4t^2+q^6t^3}{(1-qt)(1-q^2t)(1-q^4t)(1-q^6t)} - \frac{q^2+t^2+q^3t^2+2q^4t^3+2q^6t^3+q^{10}t^4}{(1-qt)(1-q^2t)(1-q^4t)(1-q^6t)}$ |

Table 1. Generating functions for $c_n,k$ for $n = 1, 2, 3, 4$

Our calculations are used to prove the following result:

**Theorem 1.5.** For each $n$ in $\{2, 3, 4\}$ and $k \geq 1$, there exists a polynomial, $Q_{n,k}(t) \in \mathbb{Z}[t]$, with non-negative integer coefficients such that $c_{n,k}(q) = Q_{n,k}(q)$, for every prime power $q$.

Let $R$ be a discrete valuation ring with maximal ideal $P$ and residue field $R/P \cong \mathbb{F}_q$. The results of Singla [9], Jambor and Plesken [6] show that $c_{n,2}(q)$ is the number of similarity classes of matrices in $M_n(R/P^2)$. Comparing the results in this paper with those of Avni, Onn, Prasad and Vaserstein [1], and Prasad, Singla and Spallone [8], we find that the number of similarity classes in $M_3(R/P^k)$ is equal to $c_{3,k}(q)$ for all $k$. The calculations of this paper and the results of the papers cited above lead us to conjecture the following:

- For all positive integers $n, k$, there exists a polynomial $Q_{n,k}(t)$ with non-negative integer coefficients such that $c_{n,k}(q) = Q_{n,k}(q)$.
- $c_{n,k}(q)$ is the number of conjugacy classes in $M_n(R/P^k)$.
**Notations.** We will be using these notations throughout this paper. \( \mathbb{F}_q \) denotes a finite field of order \( q \). Let \( n \) and \( k \) be positive integers. Then, for any matrix \( A \in M_n(\mathbb{F}_q) \), let \( Z(A) \) denote the centralizer algebra of \( A \), and \( Z(A)^* \) denote the centralizer group of \( A \). For any tuple \( (A_1, \ldots, A_k) \in M_n(\mathbb{F}_q)^{(k)} \), the common centralizer, \( \bigcap_{i=1}^k Z(A_i) \), of \( A_1, \ldots, A_k \), is denoted by \( Z(A_1, \ldots, A_k) \), and \( Z(A_1, \ldots, A_k)^* \) denotes the group of units of \( Z(A_1, \ldots, A_k) \).

2. **Similarity Class Types**

Given \( A \in M_n(\mathbb{F}_q) \) and \( x \in \mathbb{F}_q^n \), define for any polynomial \( f(t) \in \mathbb{F}_q[t] \), \( f(t).x = f(A)x \). This endows \( \mathbb{F}_q^n \) with an \( \mathbb{F}_q[t] \)-module structure, denoted by \( M^A \). It is easy to check that for matrices \( A \) and \( B \),

\[
M^A \cong M^B \iff A = gB g^{-1} \text{ for some } g \in GL_n(\mathbb{F}_q)
\]

We can easily see that, \( \text{End}_{\mathbb{F}_q[t]}(M^A) = Z(A) \), the centralizer of \( A \) in \( M_n(\mathbb{F}_q) \).

If \( A \) is a block diagonal matrix \( \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \), where \( B \) and \( C \) are square matrices whose characteristic polynomials are coprime, we write \( A \) as \( B \oplus C \), and

\[
M^{B \oplus C} = M^A \cong M^B \oplus M^C,
\]

and it can be easily shown that \( Z(A) \) is isomorphic to \( Z(B) \oplus Z(C) \).

Next, we have the Jordan decomposition of \( M^A \) for which we need:

**Definition 2.1.** Let \( p \) be an irreducible polynomial in \( \mathbb{F}_q[t] \), then the submodule

\[
M^{A_p} = \{ x \in M^A : p(t)^r.x = 0 \text{ for some } r \geq 1 \}
\]

is called the \textbf{p-primary} part of \( M^A \).

Let \( \text{Irr}(\mathbb{F}_q[t]) \) denote the set of irreducibles in \( \mathbb{F}_q[t] \). Then by the primary decomposition theorem, \( M^A \) has the decomposition,

\[
M^A = \bigoplus_{p \in \text{Irr}(\mathbb{F}_q[t])} M^{A_p},
\]

which is over a finite number of irreducibles since \( M^A \) is finitely generated. Then by Structure Theorem (see Dummit and Foote [2]) of finitely generated modules over a PID, for each \( p \), \( M^{A_p} \) has the decomposition,

\[
\mathbb{F}_q[t]/p^{\lambda_1} \oplus \mathbb{F}_q[t]/p^{\lambda_2} \oplus \cdots,
\]
where $\lambda_1 \geq \lambda_2 \geq \cdots$ are positive integers. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$. $\lambda$ is a partition. The primary decomposition together with the Structure Theorem decomposition of each primary part gives the decomposition:

$$\bigoplus_{p \in \text{Irr}(\mathbb{F}_q[t])} \left( \frac{\mathbb{F}_q[t]}{p^{\lambda_1}} \oplus \frac{\mathbb{F}_q[t]}{p^{\lambda_2}} \oplus \cdots \right).$$

This decomposition is the **Jordan decomposition**.

This gives a bijection between similarity classes in $M_n(\mathbb{F}_q)$ and the set of maps, $\nu$, from $\text{Irr}(\mathbb{F}_q[t])$ to the set of partitions, $\Lambda$.

Now, for any $\nu : \text{Irr}(\mathbb{F}_q[t]) \to \Lambda$, let $\text{Supp}(\nu)$ denote the set of irreducible polynomials $p(t)$ for which $\nu(p)$ is a non-empty partition. Clearly $\text{Supp}(\nu)$ is a finite set. For each partition, $\mu$, and each $d \geq 1$, let $r_{\nu}(\mu, d)$ be:

$$r_{\nu}(\mu, d) = |\{p(t) \in \text{Irr}(\mathbb{F}_q[t]) : \deg(p) = d \text{ and } \nu(p) = \mu\}|$$

This puts us in a position to define **Similarity Class Types**.

**Definition 2.2.** Let $A$ and $B$ be two similarity classes in $M_n(\mathbb{F}_q)$, and let $\nu^{(A)}$ and $\nu^{(B)}$ be the maps from $\text{Irr}(\mathbb{F}_q[t]) \to \Lambda$ corresponding to $A$ and $B$ respectively. We say that $A$ and $B$ are of the same **Similarity Class Type** if for each partition, $\lambda$, and each $d \geq 1$, $r_{\nu^{(A)}}(d, \lambda)$ is equal to $r_{\nu^{(B)}}(d, \lambda)$ (See Green [4]).

We shall denote a similarity class type by

$$\lambda^{(1)}_{d_1}, \ldots, \lambda^{(l)}_{d_l},$$

where $\lambda^{(1)}, \ldots, \lambda^{(l)}$ are partitions and $d_i \geq 1$ for $1 \leq i \leq l$, such that

$$\sum_{i=1}^{l} |\lambda^{(i)}|d_i = n.$$

For example, in $M_2(\mathbb{F}_q)$, there are four similarity class types, which are described in the table below:

| Type   | Description of the type |
|--------|-------------------------|
| $(1, 1)_1$ | $\lambda^{(1)} = (1, 1), \ d_1 = 1$ |
| $(2)_1$    | $\lambda^{(1)} = (2), \ d_1 = 1$ |
| $(1)_1(1)_1$ | $\lambda^{(1)} = (1), \ d_1 = 1$ $\lambda^{(2)} = (1), \ d_2 = 1$ |
| $(1)_2$    | $\lambda^{(1)} = (1), \ d_1 = 2$ |
So, for a similarity class, \( \nu : \text{Irr}(\mathbb{F}_q[t]) \to \Lambda \), such that \( \text{Supp}(\nu) = \{f_1, \ldots, f_l\} \), where \( \deg(f_i) = d_i \) and \( \nu(f_i) = \lambda^{(i)} \), the similarity class type is
\[
\lambda^{(1)}_{d_1}, \ldots, \lambda^{(l)}_{d_l}.
\]

**Definition 2.3.**

(1) We say that a matrix \( A \) is of the *Central type* if it is of the similarity class type
\[
(1, \ldots, 1)_{\underbrace{\text{n-ones}}}_{n-1}
\]
(2) And of the *Regular/Cyclic type* if it is of the class type
\[
\lambda^{(1)}_{d_1}, \ldots, \lambda^{(l)}_{d_l}
\]
where for each \( i = 1, \ldots, l \), the partition \( \lambda^{(i)} \) has only one part. For all such types, \( \mathbb{F}_q^n \) has a cyclic vector.

Before going to the next section, we shall define types for commuting tuples of matrices:

**Definition 2.4.** Let \( (A_1, \ldots, A_k) \) be a \( k \)-tuple and \( (B_1, \ldots, B_l) \), an \( l \)-tuple of commuting matrices. We say that they are of the same *similarity class type* if their respective common centralizers \( Z(A_1, \ldots, A_k) \) and \( Z(B_1, \ldots, B_l) \) are isomorphic in \( M_n(\mathbb{F}_q) \).

The above definition of types for tuples is a more precise version of Definition 1.4 and is consistent with the Definition 2.2 because, matrices \( A \) and \( B \) are of the same type if and only if their centralizers, \( Z(A) \) and \( Z(B) \), are isomorphic (see the definition of orbit-equivalent by Ravi S. Kulkarni in [7] or the definition of \( z \)-equivalent by Rony Gouraige [3]). If the centralizer, \( Z(A_1, \ldots, A_k) \), of a \( k \)-tuple, \( (A_1, \ldots, A_k) \), for \( k \geq 2 \), is isomorphic to that of a matrix, \( A \in M_n(\mathbb{F}_q) \) (of some type \( \tau \)), we say that the simultaneous similarity class of \( (A_1, \ldots, A_k) \) is of type \( \tau \). So if the centralizer, \( Z(B_1, \ldots, B_l) \), of \( (B_1, \ldots, B_l) \) is isomorphic to \( Z(A_1, \ldots, A_k) \), then it is isomorphic to the centralizer of \( A \); hence \( (B_1, \ldots, B_l) \) too is of type \( \tau \). If \( Z(A_1, \ldots, A_k) \) is not isomorphic to the centralizer of any matrix in \( M_n(\mathbb{F}_q) \), we have a new type of similarity class.

3. The \( 2 \times 2 \) Case

We shall examine the similarity classes of commuting \( k \)-tuples of \( 2 \times 2 \) matrices over \( \mathbb{F}_q \) in this section. Before going ahead, we shall define the *branch* of a similarity class type.
Definition 3.1. Given a matrix $A$ of a type $\tau$ in $M_n(\mathbb{F}_q)$, let $Z(A)$ be its centralizer. We saw in the proof of Theorem 1.2 that, counting the number of simultaneous similarity classes of pairs with the first coordinate $A$, is the same as counting the similarity classes in $Z(A)$ under the conjugation by its group of units, $Z(A)^*$. 

So, for each $B$ in $Z(A)$, its centralizer subalgebra in $Z(A)$ is the common centralizer, $Z(A,B)$, of $A$ and $B$. Let $\rho$ denote the class type of the similarity class of $(A,B)$ (in the sense of Definition 2.4). Then we say that the type $\rho$, is a branch of $\tau$.

Hence for a matrix, $A$, of type $\tau$, the number of branches of various types will be determined modulo the conjugation action of $Z(A)^*$ on $Z(A)$. We will use this same method in finding the branching rules in Sections 4 and 5.

In $M_2(\mathbb{F}_q)$, there are two kinds of similarity classes:

1. The Central type which is $(1,1)_1$.
2. The Regular/Cyclic types, where $\mathbb{F}_q^2$ has a cyclic vector.

Lemma 3.2. For a matrix, $A$, of the Central type, the branches are given in the table below:

| Type       | Number of Branches |
|------------|--------------------|
| Central    | $q$                |
| Regular    | $q^2$              |

Proof. When $A$ is of the Central type, $Z(A) = M_2(\mathbb{F}_q)$ and $Z(A)^* = GL_2(\mathbb{F}_q)$. So we only need to enumerate the similarity classes in $M_2(\mathbb{F}_q)$, which leads to the table shown in the statement of this lemma.

Lemma 3.3. A matrix of any of the Regular types has $q^2$ regular type of branches.

Proof. The centralizer algebra of a regular type of matrix say $A$ is

$$\{a_0I + a_1A : a_0, a_1 \in \mathbb{F}_q\}$$

which is a commutative algebra and thus each orbit under the conjugation action of $Z(A)^*$ on $Z(A)$ is a singleton. Hence, for any $B \in Z(A)$, $Z(A,B) = Z(A)$. There are $q^2$ such similarity classes; hence $q^2$ Regular branches.
So we see no new types of similarity classes here. Arranging the two types in the order: \{Central, Regular\}, we shall write down the branching matrix \(B_2 = [b_{ij}]\), indexed by the types. For each \(i\) and \(j\), \(b_{ij}\) is the number of type \(i\) branches of a tuple of type \(j\). So here, the branching matrix is:

\[
B_2 = \begin{pmatrix} q & 0 \\ q^2 & q^2 \end{pmatrix}
\]

We have

\[
c_{2,k}(q) = \left(\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right) B_2^k \left(\begin{array}{cc} 1 & 0 \end{array}\right)^T.
\]

From the entries of \(B_2\) and the above equation, it is clear that \(c_{2,k}(q)\) is a polynomial with non-negative integer coefficients. The generating function, \(h_2(t)\), of the \(c_{2,k}(q)\) is:

\[
h_2(t) = 1 + \sum_{k=1}^{\infty} c_{2,k}(q)t^k = \left(\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right) (I - tB_2)^{-1} \left(\begin{array}{cc} 1 & 0 \end{array}\right)^T,
\]

which is equal to

\[
\frac{1}{(1 - qt)(1 - q^2t)}.
\]

4. The 3 × 3 Case

In \(M_3(F_q)\) we have the following types of similarity classes:

1. The Central type, \((1,1,1)_1\).
2. The \((2,1)\) nilpotent type: \((2,1)_1\).
3. The \((2,1)\) semi-simple type: \((1,1)_1(1)_1\).
4. The Regular types, where \(F_3^q\) has a cyclic vector.

We now proceed to explain the branching rules.

**Lemma 4.1.** For a matrix \(A\) of the Central type, the branching rules are shown in the table below.

| Type          | Number of Branches |
|---------------|--------------------|
| Central       | \(q\)              |
| \((2,1)_1\)   | \(q\)              |
| \((1,1)_1(1)_1\) | \(q^2 - q\)       |
| Regular       | \(q^3\)            |

*Proof.* Since \(A\) is of Central type, \(Z(A)\) is \(M_3(F_q)\). Enumeration of the similarity class types in \(M_3(F_q)\) gives us the table above. \(\square\)
Lemma 4.2. For matrix, $A$, of the $(2,1)$-nilpotent type i.e., the type $(2,1)_1$, the branching rules are given in the table below.

| Type    | Number of Branches |
|---------|---------------------|
| $(2,1)_1$ | $q^2$              |
| Regular | $q^3 + q$           |

Proof. If $A$ is of type $(2,1)_1$, we shall consider its canonical form, 
\[
\begin{pmatrix}
  a & 1 & 0 \\
  0 & a & 0 \\
  0 & 0 & a
\end{pmatrix}, \ a \in \mathbb{F}_q.
\]
Thus,
\[
Z(A) = \left\{ \begin{pmatrix}
  a_0 & a_1 & b \\
  0 & a_0 & 0 \\
  0 & c & d
\end{pmatrix} \middle| a_0, a_1, b, c, d \in \mathbb{F}_q \right\}.
\]
Consider $B \in Z(A)$.
\[
B = \begin{pmatrix}
  a_0 & a_1 & b \\
  0 & a_0 & 0 \\
  0 & c & d
\end{pmatrix}.
\]
Let $X \in Z(A)^*$ be
\[
X = \begin{pmatrix}
  x_0 & x_1 & y \\
  0 & x_0 & 0 \\
  0 & z & w
\end{pmatrix},
\]
where $x_0, w \neq 0$. Let $B' = XBX^{-1}$. Then we have
\[
(4.1) \quad X \begin{pmatrix}
  a_0 & a_1 & b \\
  0 & a_0 & 0 \\
  0 & c & d
\end{pmatrix} = \begin{pmatrix}
  a'_0 & a'_1 & b' \\
  0 & a'_0 & 0 \\
  0 & c' & d'
\end{pmatrix} X
\]
From equation (4.1), we get $x_0a_0 = a'_0x_0$ and $wd = d'w$, so $a_0 = a'_0$ and $d = d'$, and we have the following equations:
\[
(4.2) \quad x_0a_1 + cy = a'_1x_0 + b'z \\
(4.3) \quad x_0b + yd = a_0y + b'w \\
(4.4) \quad a_0z + cw = c'x_0 + dz.
\]
So, we look at the two cases over here: \( a_0 = d \) and \( a_0 \neq d \).

**When \( a_0 = d \):** From equations (4.3) and (4.4), we get, \( x_0 b = b' w \)
and \( cw = c' x_0 \). So, we look at two sub cases here: \( b = c = 0 \) and \((b, c) \neq (0, 0)\).

When \( b = c = 0 \), equation (4.2) is reduced to \( x_0 a_1 = a_1' x_0 \), which gives us \( a_1' = a_1 \). So \( B \) is

\[
\begin{pmatrix}
a_0 & a_1 & 0 \\
0 & a_0 & 0 \\
0 & 0 & a_0
\end{pmatrix};
\]

therefore the centralizer, \( Z(A, B) \), of \( B \) in \( Z(A) \), is \( Z(A) \) itself. Therefore the pair \((A, B)\) is of similarity class type \((2, 1)_1\), and there are \( q^2 \) such similarity classes.

When \((b, c) \neq (0, 0)\): Suppose \( b \neq 0 \). Then, we can make \( b' = 1 \) in equation (4.3) by choosing a suitable \( x_0 \). So, letting \( b = b' = 1 \), we get \( x_0 = w \), and equation (4.4) gives us \( c = c' \). Equation (4.2) becomes \( x_0 a_1 + cy = a_1' x_0 + z \), so we choose \( z \) such that \( x_0 a_1 = 0 \); thus \( a_1 = 0 \). So

\[
B = \begin{pmatrix}
a_0 & 0 & 1 \\
0 & a_0 & 0 \\
0 & c & a_0
\end{pmatrix}
\]

whose centralizer in \( Z(A) \) is

\[
\left\{ \begin{pmatrix} x_0 & x_1 & y \\ 0 & x_0 & 0 \\ 0 & cy & x_0 \end{pmatrix} : x_0, x_1, y \in \mathbb{F}_q \right\},
\]

which is similar to the centralizer of a Regular nilpotent (type \((3)_1\)) type of matrix. This is because we can switch the 2nd and 3rd rows (resp. columns) to get a matrix that commutes with a Regular nilpotent matrix. Hence the branch \((A, B)\) is of Regular type and there are \( q^2 \) such branches.

If \( b = 0 \), then \( c \neq 0 \). In equation (4.4), choose \( w = x_0 / c \) to get \( c' = 1 \). Therefore, letting \( c = c' = 1 \), we get \( w = x_0 \) and equation (4.2) becomes \( x_0 a_1 + y = a_1' x_0 \). Now choose \( y = a_1' x_0 \) so that \( x_0 a_1 = 0 \) and
thus $a_1 = 0$. Thus

$$B = \begin{pmatrix} a_0 & 0 & 0 \\ 0 & a_0 & 0 \\ 0 & 1 & a_0 \end{pmatrix},$$

and its centralizer in $Z(A)$ is

$$\begin{cases} 
\begin{pmatrix} x_0 & x_1 & 0 \\ 0 & x_0 & 0 \\ 0 & z & x_0 \end{pmatrix} : x_0, x_1, z \in \mathbb{F}_q 
\end{cases},$$

which can again be seen as the centralizer of a Regular nilpotent\((3)_1\) type of matrix. We have $q$ more Regular branches.

When $a_0 \neq d$: In this case, in equation (4.3), we can find $y$ such that $b = 0$ and in equation (4.4), we can find a $z$ such that $c = 0$. Therefore, equation (4.2) is reduced to $x_0 a_1 = a_1' x_0$. Thus $a_1' = a_1$. So $B$ is

$$\begin{pmatrix} a_0 & a_1 & 0 \\ 0 & a_0 & 0 \\ 0 & 0 & d \end{pmatrix},$$

and its centralizer in $Z(A)$ is

$$\begin{cases} 
\begin{pmatrix} x_0 & x_1 & 0 \\ 0 & x_0 & 0 \\ 0 & 0 & w \end{pmatrix} : x_0, x_1, w \in \mathbb{F}_q 
\end{cases},$$

which is that of a matrix of a Regular type i.e., $(2)_1(1)_1$. We have $q^2(q-1)$ Regular branches. So we have a total of $(q^3-q^2)+q^2+q = q^3+q$ Regular branches.

Lemma 4.3. If $A$ is a matrix whose similarity class is of the type $(1, 1)_1(1)_1$ i.e., the $(2, 1)$-semisimple type, then it has

- $q^2$ branches of the $(2, 1)$ semisimple type, $(1, 1)_1(1)_1$.
- $q^3$ branches of the Regular types.

Proof. A matrix $A$ of similarity class which is of type $(1, 1)_1(1)_1$ is of the form $A' \oplus A''$ where $A'$ is a $2 \times 2$ matrix of the Central type and $A''$ is a $1 \times 1$ matrix. So the centralizer algebra of $A$ is of the form $Z(A') \oplus Z(A'')$ where $Z(A') = M_2(\mathbb{F}_q)$. Now $A'$ has $q$ branches of the Central type, and $q^2$ branches of the Regular type (see Lemma 3.2). And $A''$ has $q$ branches. The branches of $A$, being in $Z(A') \oplus Z(A'')$,
will be of the form $B' \oplus B''$, where $B'$ is a branch of $A'$, and $B''$ is a branch of $A''$. This leaves us with $q \times q = q^2$ branches of the type $(1,1)_1(1)$, and $q^2 \times q = q^3$ Regular branches. \[ \square \]

**Lemma 4.4.** If $A$ is a matrix of a Regular type, then it has $q^3$ branches of that same Regular type.

**Proof.** If $A$ is of a Regular type, its centralizer algebra $Z(A)$ is

\[ \{a_0I + a_1A + a_2A^2 : a_0, a_1, a_2 \in \mathbb{F}_q\}. \]

It is a commutative algebra of dimension 3. Thus for any $B \in Z(A)$, $Z(A, B) = Z(A)$. Therefore $(A, B)$ is of the Regular type and the number of such branches is $q^3$. \[ \square \]

We shall arrange the types in the order:

\{Central, $(2,1)_1, (1,1)_1(1), \text{Regular}\},

and write down the branching matrix, $B_3 = [b_{ij}]$ indexed by the types in that order. Here, an entry, $b_{ij}$, of $B_3$ is the number of type $i$ branches of a type $j$ similarity class. So,

\[
B_3 = \begin{pmatrix}
q & 0 & 0 & 0 \\
q & q & 0 & 0 \\
q^2 - q & 0 & q^2 & 0 \\
q^3 & q^3 + q & q^3 & q^3
\end{pmatrix}.
\]

To make things easier, we shall interpret the branching rules in terms of what we call *rational canonical form (rcf)* types, which we shall briefly discuss now.

The similarity class types in $M_n(\mathbb{F}_q)$ can be further classified into these rcf-types. The definition of rcf types is given below:

**Definition 4.5.** As $M^n$ is a finitely generated $\mathbb{F}_q[t]$-module, by the Structure Theorem (see Jacobson [5]), $M^n$ has the decomposition

\[
\mathbb{F}_q[t]_f(t) \oplus \cdots \oplus \mathbb{F}_q[t]_f(t),
\]

where $f_i(t) | f_{i-1}(t) | \cdots | f_1(t)$. Let $l_i$ be the degree of $f_i$. Then $\lambda = (l_1, \ldots, l_r)$ is a partition of $n$ and we say that $A$ is of *rational canonical form (rcf)*-type $\lambda$.

Let $A$ be a matrix with similarity class type, $\lambda^{(i)}_d_1, \ldots, \lambda^{(i)}_d_i$, where for each $i$, $\lambda^{(i)} = (\lambda^{(i)}_1, \lambda^{(i)}_2, \ldots)$. Then there are irreducible polynomials
$p_1(t), \ldots, p_l(t)$ with degrees $d_1, \ldots, d_l$ respectively such that

$$M^A = \bigoplus_{i=1}^{l} \left( \frac{\mathbb{F}_q[t]}{p_i(t)\lambda_i^{(i)}} \oplus \frac{\mathbb{F}_q[t]}{p_i(t)\lambda_i^{(2)}} \oplus \cdots \right)$$

Then, in the structure theorem decomposition of $M^A$ as given in equation (4.5), we have (see [5])

$$f_j(t) = p_1(t)^{\lambda_1^{(j)}} p_2(t)^{\lambda_2^{(j)}} \cdots p_l(t)^{\lambda_l^{(j)}}$$

Hence for each $j$, the degree $l_j$ of $f_j$ is

$$\sum_{i=1}^{l} \lambda_j^{(i)} d_i.$$ 

Hence, $(l_1, l_2, \ldots)$ is

$$\nu = \left( \sum_{i=1}^{l} \lambda_1^{(i)} d_i, \sum_{i=1}^{l} \lambda_2^{(i)} d_i, \ldots \right).$$

This partition $\nu$ is called the rcf-type of the similarity class type,

$$\lambda^{(1)} d_1, \ldots, \lambda^{(l)} d_l.$$ 

Thus, in the $3 \times 3$ case, the rcf types are $(1, 1, 1)$, $(2, 1)$ and $(3)$. We see that

(1) The Central type $(1, 1, 1)_1$ is the only class type with rcf type $(1, 1, 1)$.

(2) Similarity class types: $(2, 1)_1$ (i.e., the $(2, 1)$-nilpotent type) and $(1, 1)_1(1)_1$ (the $(2, 1)$-semisimple type) are of the rcf type $(2, 1)$.

(3) The Regular types are of rcf type $(3)$.

We know that there are $q^2$ classes with rcf-type $(2, 1)$ in $M_3(\mathbb{F}_q)$, of which $q^2 - q$ of them are of the semi-simple type $(1, 1)_1(1)_1$ and $q$ of them are of the nil-potent type $(2, 1)_1$. Hence a class of rcf type $(2, 1)$ is of type $(1, 1)_1(1)_1$ with probability $\frac{q-1}{q}$ and it is of type $(2, 1)_1$ with probability $\frac{1}{q}$.

So, the number of Regular branches that a matrix of rcf type $(2, 1)$ has on an average is

$$\frac{q-1}{q} \times q^3 + \frac{1}{q} \times (q^3 + q)$$
which is equal to $q^3 + 1$. The average number of rcf type $(2, 1)$ branches of the rcf type $(2, 1)$ is

$$\frac{q - 1}{q} \times q^2 + \frac{1}{q} \times q^2$$

which is equal to $q^2$. So, our branching matrix is reduced to

$$B_3 = \begin{pmatrix} q & 0 & 0 \\ q^2 & q^2 & 0 \\ q^3 & q^3 + 1 & q^3 \end{pmatrix}$$

In general, for a given rcf $\lambda$, let $p_\tau^\lambda$ be the probability of a class of rcf type $\lambda$, being of similarity class type $\tau$. Then, for rcf types, $\mu$ and $\lambda$, the average number of rcf-type $\mu$ branches of an rcf-type $\lambda$ similarity class is

$$b_{\mu\lambda} = \sum_{rcf(\tau) = \lambda} p_\tau^\lambda \left( \sum_{rcf(\gamma) = \mu} b_{\gamma\tau} \right).$$

Now that we have reduced $B_3$, we have the theorem:

**Theorem 4.6.** The number of similarity classes $c_{3,k}(q)$ of commuting $k$-tuples over $\mathbb{F}_q$ for $k \geq 2$ is given by

$$\left( \begin{array}{ccc} 1 & 1 & 1 \\ \end{array} \right) B_3^k \left( \begin{array}{cc} 1 & 0 \\ \end{array} \right)^T$$

Table 2 shows $c_{3,k}(q)$ calculated for $k = 1, 2, 3$.

| $k$ | $c_{3,k}(q)$ |
|-----|-------------|
| 1   | $q^3 + q^2 + q$ |
| 2   | $q^6 + q^5 + 2q^4 + q^3 + 2q^2$ |
| 3   | $q^9 + q^8 + 2q^7 + 2q^6 + 3q^5 + 2q^4 + 2q^3$ |

Table 2. $c_{3,k}(q)$ for $k = 1, 2, 3, 4$

As each entry of $B_3$ is a polynomial in $q$, with non-negative integer coefficients, we get that $c_{3,k}(q)$ is a polynomial in $q$ with non-negative integer coefficients. This proves Theorem 1.5 for this case. The generating function $h_3(q)$, of $c_{3,k}(q)$ is:

$$1 + \sum_{k=1}^{\infty} c_{3,k}(q) t^k = \left( \begin{array}{ccc} 1 & 1 & 1 \\ \end{array} \right)(I - B_3)^{-1} \left( \begin{array}{cc} 1 & 0 \\ \end{array} \right)^T,$$
which is:

\[
\frac{1 + q^2 t^2}{(1 - qt)(1 - q^3 t)(1 - q^4 t)}.
\]

5. The 4 × 4 Case

In the 4 × 4 case, we have 22 similarity class types, whose branching rules we need to check. Table 3 shows the rcf types and the similarity class types of each rcf-type listed below it.

| (1, 1, 1, 1)  | (2, 1, 1)  | (2, 2)   | (3, 1)   | (4)          |
|---------------|-----------|---------|---------|-------------|
| (1, 1, 1, 1)  | (2, 1, 1)  | (2, 2)  | (3, 1)  | Regular types, where \( \mathbb{F}_q^4 \) has a cyclic vector. |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |
|               | (1, 1, 1)  | (1, 1)  | (2, 1)  |             |

Table 3. rcf’s and similarity class types of 4 × 4 matrices

Before we move ahead, we shall give a broader definition of Regular type.

**Definition 5.1.** We say that a k-tuple of commuting matrices is of Regular type if its common centralizer algebra is a commutative algebra of dimension 4 or conjugate to the centralizer of a Regular type from \( M_4(\mathbb{F}_q) \) (centralizers of Regular types in \( M_n(\mathbb{F}_q) \) are 4-dimensional and commutative).

We shall first state the branching rules of the Regular and the Central types and discuss the branching rules of the other types in different subsections of this section.

**Lemma 5.2.** If \( A \) is a matrix of a Regular type, then it has \( q^4 \) branches of that same Regular type.

**Proof.** The centralizer \( Z(A) \) of \( A \), is the algebra of polynomials in \( A \) and it is a commutative algebra. Since the characteristic polynomial of \( A \) is of degree 4, the algebra \( Z(A) \) is 4-dimensional. So, for each \( B \in Z(A) \), \((A, B)\) is a branch of the Regular type. Therefore we have \( q^4 \) Regular branches. \( \square \)
**Lemma 5.3.** For $A$ of the Central type, its branches are given in the table below:

| Type                | No. of Branches | Type              | No. of Branches |
|---------------------|-----------------|-------------------|-----------------|
| Central             | $q$             | (3,1)$_1$         | $q$             |
| (2,1,1)$_1$         | $q$             | (2,1)$_1$(1)$_1$  | $q^2 - q$       |
| (1,1,1)$_1$(1)$_1$  | $q^2 - q$       | (1,1)$_1$(1)$_1$(1)$_1$ | $\frac{q(q-1)(q-2)}{2}$ |
| (2,2)$_1$           | $q$             | (1,1)$_1$(2)$_1$  | $q^2 - q$       |
| (1,1)$_1$, (1,1)$_1$| $\frac{q^2-q}{2}$ | (1,1)$_1$(1)$_2$  | $\frac{q^3-q^2}{2}$ |
| (1,1)$_2$           | $\frac{q^2-q}{2}$ | Regular           | $q^4$           |

**Proof.** As $A$ is of Central type, its centralizer algebra $Z(A)$ is the whole of $M_4(\mathbb{F}_q)$ and the centralizer group is the whole of $GL_4(\mathbb{F}_q)$. Enumerating the similarity classes of $M_4(\mathbb{F}_q)$ gives the above table.  

5.1. **Branching Rules of the non-primary, non-Regular types.**

Any non-primary similarity class type of $M_n(\mathbb{F}_q)$ is of the form

$$\lambda^{(1)}_{d_1} \cdots \lambda^{(l)}_{d_l}$$

where $l \geq 2$. Hence the centralizer algebra of matrices of such types consist of block matrices of the form

$$\begin{pmatrix}
X_1 & \cdots & O \\
\vdots \\
O & \cdots & X_l
\end{pmatrix}$$

where $X_i$ is in the centralizer of the primary type $\lambda^{(i)}_{d_i}$. Therefore, the branches of such types are of the form

$$(B_1 \oplus \cdots \oplus B_l)$$

where $B_i$ is a branch of $\lambda^{(i)}_{d_i}$, like we saw in Lemma 4.3. Thus, with the help of Lemmas 3.2, 3.3, 4.1 and 4.2 we have the following results:

**Lemma 5.4.** For $A$ of the type (1,1,1)$_1$(1)$_1$, its branching rules are given in the table below:

| Type              | Number of Branches |
|-------------------|--------------------|
| (1,1,1)$_1$(1)$_1$| $q^2$              |
| (2,1)$_1$(1)$_1$  | $q^2$              |
| (1,1)$_1$(1)$_1$(1)$_1$| $q^3 - q^2$      |
| Regular           | $q^4$              |
Lemma 5.5. If $A$ is of type $(2,1)_1(1)_1$, then it has $q^2$ branches of the type $(2,1)_1(1)_1$ and $q^4 + q^2$ branches of the Regular type.

Lemma 5.6. If $A$ is of similarity class type $(1,1)_1(1,1)_1$, then the branching rules are given in the table below

| Type                        | Number of Branches |
|-----------------------------|--------------------|
| $(1,1)_1(1,1)_1$            | $q^2$              |
| $(1,1)_1(2)_1$              | $2q^2$             |
| $(1,1)_1(1)_2$              | $q^2 - q^2$        |
| $(1,1)_1(1)_1(1)_1$         | $q^3 - q^2$        |
| Regular                     | $q^4$              |

Lemma 5.7. If $A$ is of type $(1,1)_1(2)_1$ then it has $q^3$ branches of the type $(1,1)_1(2)_1$ and $q^4$ Regular branches.

Lemma 5.8. If $A$ is of the type $(1,1)_1(1)_2$, then it has $q^3$ branches of the type $(1,1)_1(1)_2$ and $q^4$ Regular branches.

Lemma 5.9. If $A$ is of similarity class type $(1,1)_1(1)_1(1)_1$, then it has $q^3$ branches of the type $(1,1)_1(1)_1(1)_1$ and $q^4$ Regular branches.

5.2. Branching Rules of the Primary Types. We have three primary types of similarity classes in the $4 \times 4$ case: $(3,1)_1$, $(2,2)_1$ and $(2,1,1)_1$. The proofs of the lemmas here will be done by the method that was used in the proof prove Lemma 4.2.

Lemma 5.10. If $A$ is of the type, $(3,1)_1$, then it has $q^3$ branches of the type $(3,1)_1$ and $q^4 + q^2$ Regular branches.

Proof. A matrix, $A$ of type $(3,1)_1$, has the canonical form:

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

Any matrix $B \in Z(A)$ is of the form

$$
B = \begin{pmatrix}
a_0 & a_1 & a_2 & b \\
0 & a_0 & a_1 & 0 \\
0 & 0 & a_0 & 0 \\
0 & 0 & c & d
\end{pmatrix}
$$
Let $X$ be an invertible matrix in $Z(A)$.

\[
X = \begin{pmatrix}
x_0 & x_1 & x_2 & y \\
0 & x_0 & x_1 & 0 \\
0 & 0 & x_0 & 0 \\
0 & 0 & z & w
\end{pmatrix}, \text{ where } x_0, w \neq 0.
\]

Let $B' = \begin{pmatrix} a_0' & a_1' & a_2' & b' \\ 0 & a_0' & a_1' & 0 \\ 0 & 0 & a_0' & 0 \\ 0 & 0 & c' & d' \end{pmatrix}$ be the conjugate of $B$ by $X$, i.e., $XB = B'X$. Then we have the following:

\[
a_0' = a_0, \quad d' = d \quad \text{and} \quad a_1' = a_1
\]

With this, we have the following set of equations:

(5.1) \quad x_0a_2 + yc = a_2'x_0 + b'z
(5.2) \quad x_0b + yd = b'w + a_0y
(5.3) \quad za_0 + wc = c'x_0 + dz

We will count the number of branches by looking at the following cases:

$\mathbf{When} \ a_0 = d \ \text{and} \ a_0 \neq d.$

$\mathbf{When} \ b = c = 0$: In this case equation (5.1) boils down to $x_0a_2 = a_2'x_0$, therefore $a_2' = a_2$. Therefore, any matrix in $Z(A)$ commutes with $B$. Hence, $Z(A, B) = Z(A)$. Therefore $(A, B)$ is of the type $(3, 1)_1$. So, there are $q \times q \times q = q^3$ branches of this type.

$\mathbf{(b, c) \neq (0, 0)}$: First we assume that $b \neq 0$. Then equation (5.2) boils down to $x_0b = b'w$. As $b$ is non zero, choose $x_0 = w/b$ so that $b' = 1$. Letting $b = b' = 1$, we get $x_0 = w$. Therefore equation (5.3) boils down to $x_0c = c'x_0$, which implies: $c' = c$. Hence equation (5.1) boils down to $x_0a_2 + yc = a_2'x_0 + z$. So, choose a $z$ such that $a_2 = 0$. Then $B$ is
reduced to
\[
\begin{pmatrix}
  a_0 & a_1 & 0 & 1 \\
  0 & a_0 & a_1 & 0 \\
  0 & 0 & a_0 & 0 \\
  0 & 0 & c & a_0
\end{pmatrix}.
\]

Hence \( Z(A, B) \) is
\[
\left\{ \begin{pmatrix}
  x_0 & x_1 & x_2 & y \\
  0 & x_0 & x_1 & 0 \\
  0 & 0 & x_0 & 0 \\
  0 & 0 & cy & x_0
\end{pmatrix} : x_0, x_1, x_2, y \in \mathbb{F}_q \right\}.
\]

It is 4 dimensional and commutative (by a routine check). Hence we have a Regular branch and there are \( q \times q^2 = q^3 \) such Regular branches.

Next, we assume that \( b = 0 \) and \( c \neq 0 \). Then equation (5.3) boils down to \( wc = c'x_0 \). Like in the previous case, we can choose an appropriate \( w \) such that \( c' = 1 \). Letting \( c = c' = 1 \), we get \( x_0 = w \). Then equation (5.1) gets reduced to \( a'_2x_0 = a_2x_0 + y \). So, we can choose \( y \) such that \( a_2x_0 + y = 0 \). This gives us \( a'_2 = 0 \). Our \( B \) is reduced to,
\[
\begin{pmatrix}
  a_0 & a_1 & 0 & 0 \\
  0 & a_0 & a_1 & 0 \\
  0 & 0 & a_0 & 0 \\
  0 & 0 & 1 & a_0
\end{pmatrix}.
\]

Its centralizer in \( Z(A) \) is
\[
Z(A, B) = \left\{ \begin{pmatrix}
  x_0 & x_1 & x_2 & 0 \\
  0 & x_0 & x_1 & 0 \\
  0 & 0 & x_0 & 0 \\
  0 & 0 & z & x_0
\end{pmatrix} : x_0, x_1, x_2, z \in \mathbb{F}_q \right\}.
\]

\( Z(A, B) \) is 4 dimensional and commutative. The similarity class of \( (A, B) \) is of a Regular type and there are \( q^2 \) such branches.

**When** \( a_0 \neq d \): Using the fact that \( a_0 - d \neq 0 \), in equation (5.2), we can choose \( y \) such that \( b' \) becomes 0 and in equation (5.3), we can choose a suitable \( z \) such that \( c' \) becomes 0. Therefore, equation (5.1)
boils down to \( x_0 a_2 = a'_2 x_0 \). Thus giving us \( a_2 = a'_2 \). Hence, \( B \) is reduced to
\[
\begin{pmatrix}
a_0 & a_1 & a_2 & 0 \\
0 & a_0 & a_1 & 0 \\
0 & 0 & a_0 & 0 \\
0 & 0 & 0 & d
\end{pmatrix}
\]

Its centralizer in \( Z(A) \) is
\[
\left\{ \begin{pmatrix}
x_0 & x_1 & x_2 & 0 \\
0 & x_0 & x_1 & 0 \\
0 & 0 & x_0 & 0 \\
0 & 0 & 0 & w
\end{pmatrix} : x_0, x_1, x_2, w \in \mathbb{F}_q \right\},
\]

This centralizer is that of a matrix of similarity class type \((3)_{1}(1)_{1}\), which is a Regular type. This gives us \( q \times (q - 1) \times q^2 = q^4 - q^3 \) such Regular branches. So, adding up all the Regular branches, we get the total number of Regular branches \( A \) to be
\[
(q^4 - q^3) + q^3 + q^2 = q^4 + q^2.
\]

\[ \square \]

**Lemma 5.11.** For \( A \) of similarity class type, \((2,2)_{1}\), its branching rules are given in the table below.

| Type               | No. of Branches | Type               | No. of Branches |
|--------------------|-----------------|--------------------|-----------------|
| \((2,2)_{1}\)      | \(q^2\)         | New type NT2       | \(\frac{q^3 - q^2}{2}\) |
| Regular            | \(q^4\)         | New type NT3       | \(\frac{q^3 - q^2}{2}\) |
| New type NT1       | \(q^2\)         |                    |                 |

- The centralizer algebra of NT1 is:
\[
\left\{ \begin{pmatrix}
x_0 & x_1 & x_2 & x_3 \\
0 & x_0 & y_2 & y_3 \\
0 & 0 & x_0 & x_1 \\
0 & 0 & 0 & x_0
\end{pmatrix} : x_i, y_j \in \mathbb{F}_q \text{ for } i = 0, 1, 2, 3 \text{ and } j = 2, 3 \right\},
\]

and its group of units is therefore of size \(q^6 - q^5\).

- The centralizer algebra of NT2 is:
\[
\left\{ \begin{pmatrix}
p(C) & X \\
0 & p(C)
\end{pmatrix} : p(C) \in \mathbb{F}_q[C] \text{ and } X \in M_2(\mathbb{F}_q) \right\},
\]
and its group of units is therefore of size $q^6 - q^4$. Here $C$ is a $2 \times 2$ matrix of the type (1)₂.

• The centralizer algebra of $NT_3$ is

$$
\begin{pmatrix}
  x_0 & 0 & y_1 & y_2 \\
  0 & x_1 & y_3 & y_4 \\
  0 & 0 & x_0 & 0 \\
  0 & 0 & 0 & x_1
\end{pmatrix} : x_i, y_j \in \mathbb{F}_q \text{ for } i = 0, 1 \text{ and } j = 1, 2, 3, 4
$$

and its group of units is therefore of size $q^4(q - 1)^2$.

Proof. A matrix $A$ of similarity class type $(2, 2)_1$, is of the form

$$
A = \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0
\end{pmatrix}.
$$

Observe that, by conjugating $A$ by an elementary matrix such that its 2nd and 3rd rows (resp. columns) are switched, gives us

$$
\begin{pmatrix}
  O & I_2 \\
  O & O
\end{pmatrix},
$$

where $I_2$ is the $2 \times 2$ identity matrix. Thus $Z(A)$ is

$$
Z(A) = \left\{ \begin{pmatrix}
  C & D \\
  O & C
\end{pmatrix} : C, D \in M_2(\mathbb{F}_q) \right\}.
$$

Now, two matrices, $B = \begin{pmatrix}
  C & D \\
  O & C
\end{pmatrix}$ and $B' = \begin{pmatrix}
  C' & D' \\
  O & C'
\end{pmatrix} \in Z(A)$, are similar if there is an invertible matrix $\begin{pmatrix}
  X & Y \\
  O & X
\end{pmatrix}$ (where $X$ is invertible), such that

$$
\begin{pmatrix}
  C' & D' \\
  O & C'
\end{pmatrix} \begin{pmatrix}
  X & Y \\
  O & X
\end{pmatrix} = \begin{pmatrix}
  X & Y \\
  O & X
\end{pmatrix} \begin{pmatrix}
  C & D \\
  O & C
\end{pmatrix},
$$

which on expanding gives us

$$
\begin{pmatrix}
  C'X & C'Y + D'X \\
  O & C'X
\end{pmatrix} = \begin{pmatrix}
  XC & XD + YC \\
  O & XC
\end{pmatrix},
$$

where $\begin{pmatrix}
  X & Y \\
  O & X
\end{pmatrix}$ is invertible.
which means that $C'$ and $C$ have to be similar. Now, we shall see the similarity classes when

1. $C$ is of central type.
2. $C$ is of Regular type.

When $C$ is of central type, we have $D'X = XD$. So we only need to look at different types of $D$. Hence, to find out which matrix commutes with $(C D O C)$, we need to find the $X$ that commutes with $D$.

When $D$ is of the central type, then $X$ can be any $2 \times 2$ invertible matrix in $Z(A)$. Hence the centralizer algebra of $(C D O C)$ in $Z(A)$, is $Z(A)$ itself, which is isomorphic to the centralizer of the type $(2, 2)_1$. Therefore the branch $(A, B)$ is of type $(2, 2)_1$. The number of such branches is $q \times q = q^2$.

When $D$ is of the type $(2)_1$ i.e., $D = \begin{pmatrix} d & 1 \\ 0 & d \end{pmatrix}$, then $XD = DX$ if and only if $X = \begin{pmatrix} x_0 & x_1 \\ 0 & x_0 \end{pmatrix}$ Thus the centralizer algebra of $B$ in $Z(A)$ is

$$Z(A, B) = \left\{ \begin{pmatrix} x_0 & x_1 & y_1 & y_2 \\ 0 & x_0 & y_3 & y_4 \\ 0 & 0 & x_0 & x_1 \\ 0 & 0 & 0 & x_0 \end{pmatrix} : x_i, y_j \in \mathbb{F}_q \right\}.$$

Thus, the size of the centralizer group, $Z(A, B)^*$, is $(q-1) \times q^5 = q^6 - q^5$. But none of the known types of $4 \times 4$ matrices have centralizer groups of size $q^6 - q^5$. We thus have a new type of similarity class of pairs of commuting matrices. This is our new type NT1. There are $q^2$ such branches.

Next, if $D$ is of type, $(1)_2$, then the matrices, $X$, that commute with $D$ are polynomials in $D$, i.e., $xI_2 + yD$ where $x, y \in \mathbb{F}_q$. Therefore,

$$Z(A, B) = \left\{ \begin{pmatrix} xI_2 + yD \\ 0 \\ Y \\ xI_2 + yD \end{pmatrix} : x, y \in \mathbb{F}_q, Y \in M_2(\mathbb{F}_q) \right\}.$$

It can be shown that $xI + yD$ is invertible iff $(x, y) \neq (0, 0)$. Thus, the centralizer group, $Z(A, B)^*$, of $B$ in $Z(A)$ has $q^4 \times (q^2 - 1) = q^6 - q^4$.
matrices, which is not the size of the centralizer group of any known type in $M_4(\mathbb{F}_q)$. Thus we have, $\binom{q}{2} \times q = \frac{1}{2}(q^3 - q^2)$ branches of a new type which we shall refer to as NT2.

When $D$ is of type $(1)_{1}(1)_{1}$ i.e., $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, where $d_1 \neq d_2$: $X$ commutes with $D$ iff $X = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}$. So, the common centralizer, $Z(A, B)$, of $A$ and $B$ is

$$\left\{ \begin{pmatrix} x_0 & 0 & y_1 & y_2 \\ 0 & x_1 & y_3 & y_4 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_1 \end{pmatrix} : x_i, y_j \in \mathbb{F}_q \right\},$$

and the size of the centralizer group, $Z(A, B)^*$, is $(q - 1)^2 \times q^4$ which is the same as that of the centralizer group of $(3, 1)_{1}$. But $Z(A, B)$ is not isomorphic to the centralizer of $(3, 1)_{1}$, and there is no other similarity class type other than $(3, 1)_{1}$ in $M_4(\mathbb{F}_q)$, whose centralizer group is of size, $q^4(q - 1)^2$. Hence we have a new type which we call, NT3. There are $q \times \binom{q}{2} = \frac{1}{2}(q^3 - q^2)$ branches of this new type.

Now, when $C$ is any of the Regular types of matrices:

$C$ is of type $(2)_{1}$, $C$ is of the form $\begin{pmatrix} a_0 & 1 \\ 0 & a_0 \end{pmatrix}$ and so for $X$ to commute with $C$, we must have $X = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ where $x \neq 0$. So we have the following equation:

$$\begin{pmatrix} a_0 & 1 & c'_{_1} & c'_{_2} \\ 0 & a_0 & c'_{_3} & c'_{_4} \\ 0 & 0 & a_0 & 1 \\ 0 & 0 & 0 & a_0 \end{pmatrix} \begin{pmatrix} x & y & z_1 & z_2 \\ 0 & x & z_3 & z_4 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & x \end{pmatrix} = \begin{pmatrix} a_0 & 1 & c_1 & c_2 \\ 0 & a_0 & c_3 & c_4 \\ 0 & 0 & a_0 & 1 \end{pmatrix}.$$  

Then we get $c_3 = c'_{_3}$ and the following equations:

\begin{align*}
(5.4) & \quad c'_{_1}x + z_3 = xc_1 + yc_3 \\
(5.5) & \quad c'_{_2}x + c'_{_1}y + z_4 = xc_2 + yc_4 + z_1 \\
(5.6) & \quad c'_{_4}x + c'_{_3}y = xc_4 + z_3.
\end{align*}
Then, in equation (5.4) we can choose $z_3$ so that $c'_1 = 0$. Letting $c_1 = 0$, we have $z_3 = c_3y$. Then equation (5.6) becomes $c'_4x = xc_4$, and therefore $c'_4 = c_4$. In equation (5.5), we can choose $z_4$ such that $c'_2 = 0$. Thus, $B$ gets reduced to

$$B = \begin{pmatrix} a_0 & 1 & 0 & 0 \\ 0 & a_0 & c_3 & c_4 \\ 0 & 0 & a_0 & 1 \\ 0 & 0 & 0 & a_0 \end{pmatrix}. $$

Thus, $Z(A, B)$ is

$$\left\{ \begin{pmatrix} x & y & z_1 & z_2 \\ 0 & x & c_3y & c_4y + z_1 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & x \end{pmatrix} : x, y, z_1, z_2 \in \mathbb{F}_q \right\},$$

which is 4-dimensional and commutative (a routine check). Hence the pair $(A, B)$ is of Regular type, and there are $q^3$ such branches.

If $C$ is of type $(111)_1$, $C$ is of the form $\begin{pmatrix} a_0 & 0 \\ 0 & c \end{pmatrix}$, where $c \neq a_0$.

Any matrix that commutes with $C$ is of the form $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Now we have:

$$\begin{pmatrix} a_0 & 0 & d'_1 & d'_2 \\ 0 & c & d'_3 & d'_4 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \begin{pmatrix} x & 0 & z_1 & z_2 \\ 0 & y & z_3 & z_4 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 & z_1 & z_2 \\ 0 & y & z_3 & z_4 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix} \begin{pmatrix} a_0 & 0 & d_1 & d_2 \\ 0 & c & d_3 & d_4 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & c \end{pmatrix}. $$

This gives us $d'_1 = d_1$ and $d'_4 = d_4$, and the following equations:

(5.7) \quad c z_3 + d'_3 x = y d_3 + a_0 z_3,

(5.8) \quad a_0 z_2 + d'_2 y = x d_2 + z_2 c.

Using the fact that $c \neq a_0$, we can get rid of $d_2$ and $d_3$ and reduce $B$ to

$$B = \begin{pmatrix} a_0 & 0 & d_1 & 0 \\ 0 & c & 0 & d_4 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & c \end{pmatrix}. $$
Thus, $Z(A, B)$ is
\[
\begin{cases}
\begin{pmatrix}
x & 0 & z_1 & 0 \\
0 & y & 0 & z_4 \\
0 & 0 & x & 0 \\
0 & 0 & 0 & y
\end{pmatrix} : x, y, z_1, z_4 \in \mathbb{F}_q \\
\end{cases}.
\]

If we conjugate any matrix in the above algebra, by the elementary matrix such that the 2nd and 3rd rows (resp. columns) are switched, then we get
\[
\begin{cases}
\begin{pmatrix}
x & z_1 & 0 & 0 \\
0 & x & 0 & 0 \\
0 & z_4 & y & 0 \\
0 & 0 & 0 & y
\end{pmatrix} | x_0, y, z_1, z_4 \in \mathbb{F}_q \\
\end{cases},
\]
which is the centralizer algebra of the type $(2)_{1}(2)$. Hence, this branch is of the Regular type $(2)_{1}(2)$. The number of branches of this type is $q^2 \times \binom{q}{2} = \frac{1}{2}(q^4 - q^3)$.

When $C$ is of type, $(1)_2$, we may take $C$ to be the companion matrix of its characteristic polynomial, $f$ (a degree 2 irreducible polynomial over $\mathbb{F}_q$). Then from the equation below,
\[
\begin{pmatrix}
C_f & D' \\
0 & C_f
\end{pmatrix}
\begin{pmatrix}
X & Y \\
0 & X
\end{pmatrix}
= \begin{pmatrix}
X & Y \\
0 & X
\end{pmatrix}
\begin{pmatrix}
C_f & D \\
0 & C_f
\end{pmatrix},
\]
we have $C_f Y + D' X = X D + Y C_f$ (Here $X$ is a polynomial in $C_f$). We get 4 equations (by equating the 4 entries) and using the fact that the constant part of $f$ is non-zero (since it is irreducible), we can reduce
\[
\begin{pmatrix}
C_f & D \\
0 & C_f
\end{pmatrix}
\]
to
\[
B = \begin{pmatrix}
C_f & D' \\
0 & C_f
\end{pmatrix},
\]
where $D' = \begin{pmatrix}d_1 & 0 \\
d_2 & 0\end{pmatrix}$. So, $Z(A, B)$ is
\[
\begin{cases}
\begin{pmatrix}
x_0 I + x_1 C_f & x_1 D' + y_0 I + y_1 C_f \\
0 & x_0 I + x_1 C_f
\end{pmatrix} : x_0, x_1, y_0, y_1 \in \mathbb{F}_q \\
\end{cases}.
\]
Z(A, B) is 4-dimensional and commutative (again a routine check). Therefore we have a Regular type of branch, and there are \( q^2 \left( \frac{q}{2} \right) = \frac{q^4 - q^3}{2} \) such branches. Adding up the Regular branches gives us:

\[
\frac{q^4 - q^3}{2} + \frac{q^4 - q^3}{2} + q^3 = q^4 \text{ Regular branches.}
\]

\( \square \)

Lemma 5.12. For A of similarity class type \((2, 1, 1)_1\), the branching rules are given in Table 4.

| Type             | No. of Branches | Type             | No. of Branches |
|------------------|-----------------|------------------|-----------------|
| \((2, 1, 1)_1\)  | \(q^2\)        | NT1              | \(q\)          |
| \((3, 1)_1\)     | \(q^2 - q\)    | NT3              | \(q^2\)        |
| \((1, 1)(2)_1\)  | \(q^3 - q^2\)  | New type NT4    | \(q\)          |
| \((2, 1)(1)_1\)  | \(q^3 - q^2\)  | New type NT5    | \(q\)          |
| Regular          | \(q^4 + q^2\)   |                  |                 |

Table 4. Branching Rules of type \((2, 1, 1)_1\)

- The centralizer algebra of the new type, NT4, is of the form

\[
\begin{pmatrix}
 x_0 & x_1 & x_2 & x_3 \\
 0 & x_0 & 0 & 0 \\
 0 & z_1 & z_2 & z_3 \\
 0 & 0 & 0 & x_0
\end{pmatrix}
: x_i, z_j \in \mathbb{F}_q \text{ for } i = 0, 1, 2, 3 \text{ and } j = 1, 2, 3
\]

- The centralizer algebra of the new type, NT5, is of the form

\[
\begin{pmatrix}
 x_0 & 0 & x_1 & x_2 \\
 0 & x_0 & x_3 & x_4 \\
 0 & 0 & y_1 & y_2 \\
 0 & 0 & 0 & x_0
\end{pmatrix}
: x_i, y_j \in \mathbb{F}_q \text{ for } i = 0, 1, 2, 3, 4 \text{ and } j = 1, 2
\]
Proof. Matrix $A$ of the type $(2, 1, 1)_1$ has the canonical form

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

Any matrix, $B \in \mathbb{Z}(A)$, is of the form

\[
\begin{pmatrix}
a_0 & a_1 & a_2 & a_3 \\
0 & a_0 & 0 & 0 \\
0 & b_1 & b_2 & b_3 \\
0 & c_1 & c_2 & c_3
\end{pmatrix}.
\]

Conjugating $B$ by an elementary matrix (denote it by $E_{234}$), such that, the 2nd row (column) moves to the 3rd row (resp. column), the 3rd row (column) moves to the 4th row (resp. column) and the 4th row (column) moves to the 2nd row (resp. column), gives us:

\[
B = \begin{pmatrix}
a_0 & \begin{pmatrix} \vec{b}^T & a_1 \end{pmatrix} \\
0 & \begin{pmatrix} \vec{C} & \vec{d} \end{pmatrix} \\
0 & \begin{pmatrix} \vec{C}' & \vec{d}' \end{pmatrix}
\end{pmatrix},
\]

where $\vec{b}^T = [b_1 \ b_2]$, $\vec{d} = [d_1 \ d_2]$, and $C$ is a $2 \times 2$ matrix.

Let

\[
X = \begin{pmatrix}
x_0 & \begin{pmatrix} \vec{y}^T & x_1 \end{pmatrix} \\
0 & \begin{pmatrix} \vec{Z} & \vec{w} \end{pmatrix} \\
0 & \begin{pmatrix} \vec{0}^T & x_0 \end{pmatrix}
\end{pmatrix}
\]

be an invertible matrix in $\mathbb{Z}(A)$. Conjugate $B$ by $X$ to get $B'$, which is

\[
B' = \begin{pmatrix}
a_0 & \begin{pmatrix} \vec{b}'^T & a_1 \end{pmatrix} \\
0 & \begin{pmatrix} \vec{C}' & \vec{d}' \end{pmatrix} \\
0 & \begin{pmatrix} \vec{0}^T & a_0 \end{pmatrix}
\end{pmatrix}.
\]
Then $XB = B'X$ gives us the following:

(5.9) \[ C'Z = ZC, \]

(5.10) \[ a_0 \vec{y}^T + \vec{b}^T Z = x_0 \vec{b}^T + \vec{y}^T C, \]

(5.11) \[ C' \vec{w} + x_0 \vec{d} = Z \vec{d} + a_0 \vec{w}, \]

(5.12) \[ \vec{b}^T \cdot \vec{w} + a_1' x_0 = x_0 a_1 + \vec{y}^T \cdot \vec{d}. \]

To get the branches, we will analyze the different types of $C$. To begin with, there are two main cases of $C$.

1. $a_0$ is an eigenvalue of $C$. Here the types of $C$ are: $C$ is Central (i.e., $C = a_0 I$), $C$ is of the Regular types, $(2)_1$ and $(1)_1(1)_1$.

2. $a_0$ is not an eigenvalue of $C$. Here the types of $C$ are: $C$ is Central (i.e. $C = cI, c \neq 0$), $C$ is of the Regular types, $(2)_1$, $(1)_1(1)_1$ and $(1)_2$.

We can take $C' = C$ and therefore $Z$ is a matrix which commutes with $C$. So for each type of $C$, we will only need to see what $Z$ is and simplify $B$ only using equations (5.10) and (5.11).

**When $a_0$ is an eigenvalue of $C$:** We will first see the branching rules in the case where $a_0$ is an eigenvalue of $C$. So we have the following subcases:

- $\vec{b} = \vec{d} = \vec{0}
- (\vec{b}, \vec{d}) \neq (\vec{0}, \vec{0})

**Case:** $\vec{b} = \vec{d} = \vec{0}$. In this case, equation (5.12) is reduced to

\[ x_0 a_1' = x_0 a_1. \]

This implies $a_1' = a_1$. Now we see what happens for different types of $C$.

**When $C$ is central:** We have

\[ B = \begin{pmatrix} a_0 & 0 & a_1 \\ 0 & a_0 I & 0 \\ 0 & 0 & a_0 \end{pmatrix}. \]

Here, equations (5.10) and (5.11) are void. Thus any $X$ in $Z(A)$ commutes with $B$. $Z(A, B)$ is $Z(A)$ itself. Therefore $(A, B)$ is of type $(2, 1, 1)_1$ and the number of such branches is $q \times q = q^2$. 
When \( C \) is of type \((2)_{1}\): We have

\[
C = \begin{pmatrix} a_0 & 1 \\ 0 & a_0 \end{pmatrix}, \text{ and } B = \begin{pmatrix} a_0 & 0 & 0 & a_1 \\ 0 & a_0 & 1 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.
\]

So \( Z = \begin{pmatrix} z_1 & z_2 \\ 0 & z_1 \end{pmatrix} \), and equations (5.10) and (5.11) become

\[
\begin{pmatrix} a_0 y_1 & a_0 y_2 \\ a_0 w_1 & a_0 w_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ 0 & a_0 \end{pmatrix},
\]

\[
\begin{pmatrix} a_0 y_1 & a_0 y_2 \\ a_0 w_1 & a_0 w_2 \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 0 & a_0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.
\]

This gives us \( y_1 = w_2 = 0 \). Therefore, a matrix in \( Z(A, B) \) is of the form,

\[
\begin{pmatrix} x_0 & 0 & y_2 & x_1 \\ 0 & z_1 & z_2 & w_1 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix}.
\]

Conjugating this matrix in \( Z(A, B) \) by elementary matrices (by switching the 3rd and 4th rows (resp. columns)), gives us

\[
\begin{pmatrix} x_0 & 0 & x_1 & y_2 \\ 0 & z_1 & w_1 & z_2 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & z_1 \end{pmatrix},
\]

which is a matrix in the common centralizer of pair of commuting matrices of the new type, NT3. Thus the commuting pair \((A, B)\) is of similarity class type NT3, and there are \( q^2 \) branches of the new type NT3.
When $C$ is of type $(1)_{1}(1)_{1}$: $C = \begin{pmatrix} a_0 & 0 \\ 0 & c \end{pmatrix}$ where, $c \neq a_0$

and $B$ becomes

$$B = \begin{pmatrix} a_0 & 0 & 0 & a_1 \\ 0 & a_0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$ 

$Z$ commutes with $C$ iff $Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix}$.

From equations (5.10) and (5.11), we have the following:

$$\begin{pmatrix} a_0 y_1 & a_0 y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & c \end{pmatrix}$$

and

$$\begin{pmatrix} a_0 w_1 \\ a_0 w_2 \end{pmatrix} = \begin{pmatrix} a_0 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

which leaves us with $y_2 = w_2 = 0$ (since $a_0 \neq c$) and therefore, $Z(A, B)$ consists of $X$ of the form

$$X = \begin{pmatrix} x_0 & y_1 & 0 & x_1 \\ 0 & z_1 & 0 & w_1 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix},$$

Conjugating this by the matrix $E_{234}$, gives us

$$\begin{pmatrix} x_0 & x_1 & y_1 & 0 \\ 0 & x_0 & 0 & 0 \\ 0 & w_1 & z_1 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix},$$

which is in the centralizer algebra of a matrix of the type $(2, 1)_{1}(1)_{1}$. Hence the commuting pair $(A, B)$ is of the type $(2, 1)_{1}(1)_{1}$ and we have $q^3 - q^2$ branches of this type.
Case: $(\vec{b}, \vec{d}) \neq (\vec{0}, \vec{0})$: In this case, we can find a suitable $\vec{y}$ or $\vec{w}$ in equation (5.12) and get rid of the entry, $a_1$. So our $B$ is:

$$
\begin{pmatrix}
a_0 & \vec{b} & 0 \\
0 & C & \vec{d} \\
0 & 0 & a_0
\end{pmatrix}.
$$

When $C = a_0 I$: $Z$ is any $2 \times 2$ invertible matrix. We first assume $\vec{b} \neq \vec{0}$.

Equation (5.10) becomes

$$\vec{b}^T Z = x_0 \vec{b}^T.$$

We may replace $Z$ by $x_0^{-1} Z$ so that we have

$$\vec{b}^T Z = \vec{b}^T \text{ and } Z \vec{d} = \vec{d}' .$$

Since $\vec{b} \neq \vec{0}$ and $Z$ is invertible, we can find a suitable $Z$ such that $\vec{b}^T = (1 \ 0)$. Now, let $\vec{b}^T = \vec{b}'^T = (1 \ 0)$, then equation (5.10) gives us $Z = \begin{pmatrix} 1 & 0 \\ z_3 & z_4 \end{pmatrix}$. Hence, equation (5.11)

boils down to

$$
(5.13) \quad \begin{pmatrix} 1 & 0 \\ z_3 & z_4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} d'_1 \\ d'_2 \end{pmatrix},
$$

therefore

$$
\begin{pmatrix} d'_1 \\ d'_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ z_3 d_1 + z_4 d_2 \end{pmatrix}.
$$

If $\vec{d} \neq \vec{0}$, with $d_1 \neq 0$, then choose $z_3$ so that $z_4 d_2 + d_1 z_3 = 0$. Therefore, $d'_2 = 0$, and $B$ is reduced to

$$
\begin{pmatrix}
a_0 & 1 & 0 & 0 \\
0 & a_0 & 0 & d_1 \\
0 & 0 & a_0 & 0 \\
0 & 0 & 0 & a_0
\end{pmatrix},
$$

and any $X \in Z(A, B)$ is of the form

$$
X = \begin{pmatrix} x_0 & y_1 & y_2 & x_1 \\ 0 & x_0 & 0 & d_1 x_1 \\ 0 & 0 & z_4 & w_2 \\ 0 & 0 & 0 & x_0 \end{pmatrix}.
$$
Conjugate this by the elementary matrix (by switching the 3rd and 4th rows (resp. columns)). Then we get:

\[
\begin{pmatrix}
x_0 & y_1 & x_1 & y_2 \\
0 & x_0 & d_1 x_1 & 0 \\
0 & 0 & x_0 & 0 \\
0 & 0 & w_2 & z_4
\end{pmatrix},
\]

which is in the centralizer of a matrix of type, \((3, 1)_1\). Hence \((A, B)\) is of type \((3, 1)_1\). There are \(q(q - 1) = q^2 - q\) branches of this type.

Now when \(d \neq 0\) and \(d_1 = 0\), equation (5.13) becomes

\[
\begin{pmatrix}
d'_1 \\
d'_2
\end{pmatrix} = \begin{pmatrix}
0 \\
z_4 d_2
\end{pmatrix},
\]

which can be reduced to \(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\). Thus \(B\) is reduced to

\[
\begin{pmatrix}
a_0 & 1 & 0 & 0 \\
0 & a_0 & 0 & 0 \\
0 & 0 & a_0 & 1 \\
0 & 0 & 0 & a_0
\end{pmatrix},
\]

and a matrix in \(Z(A, B)\) is of the form

\[
X = \begin{pmatrix}
x_0 & y_1 & y_2 & x_1 \\
0 & x_0 & 0 & y_2 \\
0 & z_3 & x_0 & w_2 \\
0 & 0 & 0 & x_0
\end{pmatrix}.
\]

On conjugating \(X\) by the elementary matrices such that its 2nd and 3rd rows and columns are switched, we get

\[
\begin{pmatrix}
x_0 & y_2 & y_1 & x_1 \\
0 & x_0 & z_3 & w_2 \\
0 & 0 & x_0 & y_2 \\
0 & 0 & 0 & x_0
\end{pmatrix},
\]

which is in the centralizer of a pair of commuting matrices of the new type \(NT_1\). Hence \((A, B)\) is of type \(NT_1\), and we have
q branches of the new type NT1.

When \( \vec{d} = \vec{0} \), then

\[
B = \begin{pmatrix}
 a_0 & 1 & 0 & 0 \\
 0 & a_0 & 0 & 0 \\
 0 & 0 & a_0 & 0 \\
 0 & 0 & 0 & a_0 \\
\end{pmatrix}
\]

whose centralizer, \( Z(A, B) \) in \( Z(A) \) contains matrices of the form,

\[
X = \begin{pmatrix}
 x_0 & y_1 & y_2 & x_1 \\
 0 & x_0 & 0 & 0 \\
 0 & z_3 & z_4 & w_2 \\
 0 & 0 & 0 & x_0 \\
\end{pmatrix}
\]

So the common centralizer algebra of \( A \) and \( B \) is 7 dimensional. As there is no known type in \( M_4(\mathbb{F}_q) \) whose centralizer is 7 dimensional, we have a new type, which we call NT4. There are \( q \) branches of this type.

Next, when \( \vec{b} = \vec{0} \): Here \( \vec{d} \neq \vec{0} \), and from equation (5.11), we can find \( Z \) such that \( Z \vec{d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), and our \( B \) is reduced to

\[
\begin{pmatrix}
 a_0 & 0 & 0 & 0 \\
 0 & a_0 & 0 & 1 \\
 0 & 0 & a_0 & 0 \\
 0 & 0 & 0 & a_0 \\
\end{pmatrix}
\]

Thus \( Z(A, B) \) has matrices of the form

\[
\begin{pmatrix}
 x_0 & 0 & y_2 & x_1 \\
 0 & x_0 & z_2 & w_1 \\
 0 & 0 & z_4 & w_2 \\
 0 & 0 & 0 & x_0 \\
\end{pmatrix}
\]

Hence, the centralizer algebra of \( (A, B) \) is 7 dimensional, but it is not conjugate to the centralizer of NT4 and therefore, the branch is of a new type, which we shall call NT5. There are \( q \) such branches.
When $C$ is of type $(2)_1$ i.e., $C = \begin{pmatrix} a_0 & 1 \\ 0 & a_0 \end{pmatrix}$: We have $Z = \begin{pmatrix} z_1 & z_2 \\ 0 & z_1 \end{pmatrix}$ where $z_1 \neq 0$. From equation (5.10), we get:

$$b^T \begin{pmatrix} z_1 & z_2 \\ 0 & z_1 \end{pmatrix} + y^T (a_0I - C) = x_0 b^T$$

As $a_0I - C = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, the LHS in equation (5.14) above boils down to,

$$\begin{pmatrix} b'_1 z_1 & b'_1 z_2 + b'_2 z_1 - y_1 \end{pmatrix}.$$  

Choose $y_1$ so that $b^T = \begin{pmatrix} b'_1 z_1 & 0 \end{pmatrix}$. We now have two cases: $b'_1 \neq 0$ and $b'_1 = 0$.

When $b'_1 \neq 0$, we can choose $z_1$ so that $b^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Letting $b^T = b^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$, equation (5.10) becomes

$$\begin{pmatrix} z_1 & z_2 - y_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

which implies: $z_1 = 1$ and $y_1 = z_2$. So $y^T = \begin{pmatrix} z_2 & y_2 \end{pmatrix}$.

Then equation (5.11) is reduced to

$$\begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} d'_1 \\ d'_2 \end{pmatrix},$$

which implies that we can choose $w_2$ appropriately so that $\begin{pmatrix} d'_1 \\ d'_2 \end{pmatrix} = \begin{pmatrix} 0 \\ d_2 \end{pmatrix}$. Thus $B$ is reduced to

$$\begin{pmatrix} a_0 & 1 & 0 & 0 \\ 0 & a_0 & 1 & 0 \\ 0 & 0 & a_0 & d_2 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$
Thus, \( Z(A, B) \) is
\[
\begin{pmatrix}
  x_0 & y_1 & y_2 & x_1 \\
  0 & x_0 & y_1 & d_2y_2 \\
  0 & 0 & x_0 & d_2y_1 \\
  0 & 0 & 0 & x_0
\end{pmatrix} : x_0, x_1, y_1, y_2 \in \mathbb{F}_q,
\]
and it is conjugate to the centralizer of a Regular nilpotent \((4)_1\) type of matrix. This branch \((A, B)\) is of a Regular type, and there are \(q \times q = q^2\) such branches.

Now if \(b'_1 = 0\), then we have \(\vec{b}^T = \vec{0}^T\). Then equation (5.11) becomes
\[
\begin{pmatrix}
  z_1 & z_2 \\
  0 & z_2
\end{pmatrix}
\begin{pmatrix}
  d_1 \\
  d_2
\end{pmatrix} + \begin{pmatrix}
  0 & -1 \\
  0 & 0
\end{pmatrix}
\begin{pmatrix}
  w_1 \\
  w_2
\end{pmatrix} = \begin{pmatrix}
  d'_1 \\
  d'_2
\end{pmatrix}
\]
which gives us
\[
\vec{d} = \begin{pmatrix}
  z_1d_1 + z_2d_2 - w_2 \\
  z_1d_2
\end{pmatrix}
\]
choose \(w_2\) such that \(\vec{d}' = \begin{pmatrix}
  0 \\
  z_1d_2
\end{pmatrix}\).

If \(d_2 \neq 0\), we can scale it to 1 and thus we have
\[
B = \begin{pmatrix}
  a_0 & 0 & 0 & 0 \\
  0 & a_0 & 1 & 0 \\
  0 & 0 & a_0 & 1 \\
  0 & 0 & 0 & a_0
\end{pmatrix}
\]
so in this case \(Z(A, b)\) is:
\[
\begin{pmatrix}
  x_0 & 0 & 0 & x_1 \\
  0 & x_0 & z_2 & w_1 \\
  0 & 0 & x_0 & z_2 \\
  0 & 0 & 0 & x_0
\end{pmatrix} : x_0, x_1, w_1, z_2 \in \mathbb{F}_q.
\]
It is 4-dimensional and commutative. Therefore, this branch too is of a Regular type and the number of branches is \(q\). So we have a total of \(q^2 + q\) branches of this Regular type.

If \(d_2 = 0\), we are back to the case \(\vec{b} = \vec{d} = \vec{0}\).
When $C = \begin{pmatrix} a_0 & 0 \\ 0 & c \end{pmatrix}$ ($c \neq a_0$), $Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix}$. So equation (5.10) becomes

\[
\begin{pmatrix} b'_1 & b'_2 \end{pmatrix} \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & a_0 - c \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \end{pmatrix}.
\]

We get from this

\[
\begin{pmatrix} z_1b'_1 & z_2b'_2 + (a_0 - c)y_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \end{pmatrix}.
\]

As $a_0 - c \neq 0$, we can get rid of $b'_2$ so that $b'^T = \begin{pmatrix} z_1b'_1 & 0 \end{pmatrix}$.

If $b'_1 \neq 0$, then we can reduce $b'^T$ to $\begin{pmatrix} 1 & 0 \end{pmatrix}$. In equation (5.10), letting $\overrightarrow{d} = \overrightarrow{b'^T} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, we get $\begin{pmatrix} z_1 & (a_0 - c)y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

Thus $z_1 = 1$ and $y_2 = 0$. So $Z = \begin{pmatrix} 1 & 0 \\ 0 & z_4 \end{pmatrix}$.

Equation (5.11) becomes

\[
\begin{pmatrix} d'_1 \\ d'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z_4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a_0 - c \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.
\]

Using $a_0 \neq c$, we can reduce $\overrightarrow{d}$ to $\begin{pmatrix} d_1 \\ 0 \end{pmatrix}$. Thus

\[
B = \begin{pmatrix} a_0 & 1 & 0 & 0 \\ 0 & a_0 & 0 & d_1 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.
\]

Then $Z(A, B)$ is

\[
\left\{ \begin{pmatrix} x_0 & y_1 & 0 & x_1 \\ 0 & x_0 & 0 & d_1 y_1 \\ 0 & 0 & z_4 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix} \right\} x_0, x_1, y_1, z_4 \in \mathbb{F}_q
\]
Conjugating by the elementary matrices such that its 3rd and 4th rows and columns are switched, we get:

\[
\begin{pmatrix}
  x_0 & y_1 & x_1 & 0 \\
  0   & x_0 & d_1 y_1 & 0 \\
  0   & 0   & x_0   & 0 \\
  0   & 0   & 0     & z_4
\end{pmatrix}
\] \(x_0, x_1, y_1, z_4 \in \mathbb{F}_q\),

which is the centralizer of the Regular type \((3)_1(1)_1\). Therefore this branch is of Regular type. The number of such branches is \(q^2(q - 1) = q^3 - q^2\).

When \(b'_1 = 0\), then \(\vec{b}^T = \vec{0}^T\). Then equation (5.11) becomes

\[
\begin{pmatrix}
  d'_1 \\
  d'_2
\end{pmatrix}
= \begin{pmatrix}
  z_1 & 0 \\
  0   & z_4
\end{pmatrix}
\begin{pmatrix}
  d_1 \\
  d_2
\end{pmatrix}
+ \begin{pmatrix}
  0 & 0 \\
  0 & a_0 - c
\end{pmatrix}
\begin{pmatrix}
  w_1 \\
  w_2
\end{pmatrix}
= \begin{pmatrix}
  z_1 d_1 \\
  z_4 d_2 + (a_0 - c) w_2
\end{pmatrix}.
\]

As \(a_0 \neq c\), we can make \(z_4 d_2\) vanish by choosing \(w_2\) appropriately. So, \(\vec{b}^T = \begin{pmatrix}
  z_1 d_1 \\
  0
\end{pmatrix}\). If \(d_1 \neq 0\). Choose \(z_1\) so that

\[
\vec{d}' = \begin{pmatrix}
  1 \\
  0
\end{pmatrix}.
\]

Thus, \(B\) is reduced to

\[
\begin{pmatrix}
  a_0 & 0 & 0 & 0 \\
  0   & a_0 & 0 & 1 \\
  0   & 0   & c & 0 \\
  0   & 0   & 0 & a_0
\end{pmatrix}.
\]

So, here \(Z(A, B)\) is

\[
\begin{pmatrix}
  x_0 & 0 & 0 & x_1 \\
  0   & x_0 & 0 & w_1 \\
  0   & 0   & z_4 & 0 \\
  0   & 0   & 0   & x_0
\end{pmatrix}
\] \(x_0, x_1, w_1, z_4 \in \mathbb{F}_q\),

which is 4 dimensional and commutative. Thus the pair, \((A, B)\), is of Regular type and there are \(q(q - 1) = q^2 - q\) such branches.
So we have a total of
\[(q^3 - q^2) + (q^2 - q) + (q^2 + q) = q^3 + q^2\]
branches of the Regular type so far.

**When** \(a_0\) **is not an eigenvalue of** \(C\): Here, \(C - a_0 I\) is an invertible matrix. In equations (5.10) and (5.11), using the fact that \(C - a_0 I\) is invertible, we can reduce \(\vec{b}\) and \(\vec{d}\) to \(\vec{0}\). After this, equations (5.10) and (5.11) become.

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} (C - a_0 I) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
\[
(C - a_0 I) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Therefore \(\vec{y} = \vec{0}\) and equation (5.12) becomes \(a'_1 x_0 = x_0 a_1\), therefore \(a'_1 = a_1\). So \(B\) is of the form

\[
\begin{pmatrix} a_0 & \vec{0}^T \\ \vec{0} & C \end{pmatrix} \begin{pmatrix} a_1 \\ \vec{0} \end{pmatrix}.
\]

So the centralizers of such \(B\) in \(Z(A)\) are of the form

\[
Z(A, B) = \left\{ \begin{pmatrix} x_0 & \vec{0}^T \\ \vec{0} & Z \end{pmatrix} \begin{pmatrix} x_1 \\ \vec{0} \end{pmatrix} \mid x_0, x_1 \in \mathbb{F}_q, ZC = CZ \right\}.
\]

We can conjugate this by elementary matrices to get

\[
\left\{ \begin{pmatrix} x_0 & \vec{0}^T \\ \vec{0} & Z \end{pmatrix} \begin{pmatrix} x_1 \\ \vec{0} \end{pmatrix} \mid x_0, x_1 \in \mathbb{F}_q, ZC = CZ \right\}.
\]

When \(C\) is of the Central type i.e., \(C = cI\) where \(c \neq a_0\), we have \(Z\) to be any \(2 \times 2\) invertible matrix and thus \(Z(A, B)\) is the centralizer of a matrix of type \((1, 1)_{(2)}\). Therefore we have a branch of type \((1, 1)_{(2)}\) and, we have \(q^2(q - 1)\) such branches.

When \(C\) is of the Regular type whose eigenvalue is not \(a_0\), the centralizer, \(Z(A, B)\), of \(B\) in \(Z(A)\), consists of matrices of the form

\[
\begin{pmatrix} Y & 0 \\ 0 & p(C) \end{pmatrix}
\]

where \(p(C)\) is a polynomial in \(C\). This common centralizer of \(A\) and \(B\) is that of the type \((2)_{1}\) where \(\tau\) is one of \((2)_{1}\),
Lemma 5.13. If \( A \) is of type \((1, 1)_2\), then it has \( q^2 \) branches of the type \((1, 1)_2\) and \( q^4 \) Regular branches.

Proof. The proof is like that of the \((1, 1)_1\) case for \( 2 \times 2 \) matrices over \( \mathbb{F}_{q^2} \).

5.3. Branching Rules of the New types. While finding out the branching rules for the types, \((2, 1, 1)_1\) and \((2, 2)_1\), we got 5 new types of branches: NT1, NT2, NT3, NT4 and NT5. In this subsection, we will see the branching rules of those new types.

Lemma 5.14. For a pair \((A, B)\) of similarity class type NT1, the branching rules are given in the table below:

| Type        | Number of Branches |
|-------------|--------------------|
| NT1         | \( q^3 \)          |
| Regular     | \( q^4 - q^3 \)     |
| New Type NT6| \( q^4 - q^2 \)     |

The centralizer of the new type NT6 is

\[
\left\{ \begin{pmatrix} a_0I & C \\ 0 & a_0I \end{pmatrix} : a_0 \in \mathbb{F}_q, C \in M_2(\mathbb{F}_q) \right\}.
\]

Proof. In this case,

\[
Z(A, B) = \left\{ \begin{pmatrix} a_0 + a_1D & C \\ 0 & a_0I + a_1D \end{pmatrix} : C \in M_2(\mathbb{F}_q), a_0, a_1 \in \mathbb{F}_q \right\},
\]

where \( D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \).

To see the branching rules here, we will use a different approach from what we have been using so far. Let \( M = \begin{pmatrix} a_0 + a_1D & C \\ 0 & a_0I + a_1D \end{pmatrix} \) be...
an invertible matrix and \( X = \begin{pmatrix} x_0 + x_1 D & Y \\ 0 & x_0 I + x_1 D \end{pmatrix} \). We have

\[
MX = \begin{pmatrix} (a_0 I + a_1 D)(x_0 I + x_1 D) & (a_0 I + a_1 D)Y + C(x_0 + x_1 D) \\ 0 & (a_0 I + a_1 D)(x_0 I + x_1 D) \end{pmatrix},
\]

\[
XM = \begin{pmatrix} (x_0 I + x_1 D)(a_0 I + a_1 D) & (x_0 I + x_1 D)C + Y(a_0 + a_1 D) \\ 0 & (x_0 I + x_1 D)(a_0 I + a_1 D) \end{pmatrix}.
\]

So, \( XM = MX \) if and only if

\[
a_1 D Y + x_1 C D = x_1 D C + a_1 Y D,
\]

which implies

\[ (5.15) \quad [a_1 Y - x_1 C, D] = 0. \]

Thus we need to deal with 4 cases of what \( x_1 \) and \( Y \) are, in equation \((5.15)\).

**When** \( x_1 = 0 \) and \([Y, D] = 0\): There are \(qq^2 = q^3\) matrices \( X \) in this case and equation \((5.15)\) holds for any \( a_1 \) and any \( C \). Thus the centralizer group, \( Z(A, B, X)^* \), of \( X \) in \( Z(A, B)^* \) is \( Z(A, B)^* \) itself.

Thus, under conjugation by \( Z(A, B)^* \):

- Orbit size of \( X = 1 \).
- Number of orbits is \( \frac{q^3}{1} = q^3 \).

Thus \((A, B, X)\) is of type \( \text{NT1} \) and the number of branches is \( q^3 \).

**When** \( x_1 = 0 \) and \([Y, D] \neq 0\): The number of \( X \)'s is \( q(q^4 - q^2) \).

Thus, equation \((5.15)\) boils down to \( a_1 [Y, D] = 0 \). But \([Y, D] \neq 0\) implies \( a_1 = 0 \). Thus \( Z(A, B, X)^* \) is

\[
\left\{ \begin{pmatrix} a_0 I & C \\ 0 & a_0 I \end{pmatrix} : a_0 \neq 0 \right\}.
\]

The size of \( Z(A, B, X)^* \) is, \((q - 1)q^4 = q^5 - q^4\). But none of the types of similarity classes (the known types and the 5 new types), has a 5-dimensional centralizer algebra. So we now have another new type, \( \text{NT6} \). Therefore:

- Orbit size of \( X = \frac{q^6 - q^5}{q^5 - q^4} = q \).
- Number of orbits is \( \frac{q(q^4 - q^2)}{q} = q^4 - q^2 \).
Thus \((A, B, X)\) is of type NT6 and the number of branches is \(q^4 - q^2\).

When \(x_1 \neq 0\) and \([Y, D] = 0\): The number of \(X\)’s is \(q(q - 1)q^2 = q^4 - q^3\). Thus, equation (5.15) boils down to \(x_1[C, D] = 0\), which means that \([C, D] = 0\). Thus, \(C = b_0I + b_1D\). So

\[
Z(A, B, X)^* = \left\{ \begin{pmatrix} a_0I + a_1D & b_0I + b_1D \\ 0 & a_0I + a_1D \end{pmatrix} : a_0 \neq 0 \right\},
\]

and this centralizer group is a commutative group of size \(q^4 - q^3\). So \(Z(A, B, X)\) is 4-dimensional and commutative. Therefore:

- Orbit size of \(X = \frac{q^6 - q^5}{q^4 - q^3} = q^2\).
- Number of orbits is \(\frac{q^4 - q^3}{q^2} = q^2 - q\).

Thus \((A, B, X)\) is of a Regular type, and the number of branches is \(q^2 - q\).

When \(x_1 \neq 0\) and \([Y, D] \neq 0\): The number of \(X\)’s of this kind is \(q(q - 1)(q^4 - q^2)\). In this case, equation (5.15) remains as it is, i.e., \([a_1Y - x_1C, D] = 0\). This implies that \(x_1C - a_1Y \in \mathbb{F}_q[D]\). \(x_1 \neq 0\) implies \(C = x_1^{-1}a_1Y + b_0I + b_1D\). So,

\[
Z(A, B, X)^* = \left\{ \begin{pmatrix} a_0I + a_1D & x_1^{-1}a_1Y + b_0I + b_1D \\ 0 & a_0I + a_1D \end{pmatrix} : a_0 \neq 0 \right\}.
\]

It is of size \(q^4 - q^3\), and is commutative (a routine check). So \(Z(A, B, X)\) is 4-dimensional and commutative. Therefore:

- Orbit size of \(X = \frac{q^6 - q^5}{q^4 - q^3} = q^2\).
- Number of orbits is \(\frac{q(q - 1)(q^4 - q^2)}{q^2} = (q - 1)(q^3 - q)\).

Thus \((A, B, X)\) is of a Regular type and the number of branches is \(q^2 - q\).

Adding up the number of branches of all the Regular types, we get a total of \((q^2 - q) + (q - 1)(q^3 - q)\), which is equal to

\[(q - 1)(q + q^3 - q) = q^4 - q^3\] Regular branches.

Hence we have the table mentioned in the statement. \(\square\)
Lemma 5.15. For \((A, B)\) of similarity class type NT2, the branching rules are given in the table below

| Type     | Number of ranches |
|----------|-------------------|
| NT2      | \(q^3\)           |
| Regular  | \(q^4 - q^3\)     |
| NT6      | \(q^4 - q^3\)     |

Proof. \(Z(A, B)\) is equal to

\[
\left\{ \begin{pmatrix} a_0 I + a_1 C_f & D \\ 0 & a_0 I + a_1 C_f \end{pmatrix} : a_0, a_1 \in \mathbb{F}_q, D \in M_2(\mathbb{F}_q) \right\},
\]

where \(C_f\) is a \(2 \times 2\) matrix, whose characteristic polynomial is a degree 2 irreducible polynomial \(f\). A matrix in \(Z(A, B)\) is invertible iff \((a_0, a_1) \neq (0, 0)\) and hence the size of the \(Z(A, B)^*\) is \(q^6 - q^4\). To prove this lemma, we will follow the steps we used in the proof of Lemma 5.14. Let

\[
M = \begin{pmatrix} a_0 I + a_1 C_f & D \\ 0 & a_0 I + a_1 C_f \end{pmatrix}
\]

be invertible and let

\[
X = \begin{pmatrix} x_0 I + x_1 C_f & Y \\ 0 & x_0 I + x_1 C_f \end{pmatrix}
\]

Then \(M\) and \(X\) commute iff

\[(5.16) \quad [a_1 Y - x_1 D, C_f] = 0.\]

From equation (5.16), we have 4 cases for what \(x_1\) and \(Y\) should be: We shall analyze the cases:

When \(x_1 = 0\) and \([Y, C_f] = 0\): The number of \(X\)'s is \(qq^2 = q^3\).

Here, equation (5.16) holds for any \(a_1\) and any \(D\). Thus the centralizer group, \(Z(A, B, X)^*\), of \(X\) in \(Z(A, B)^*\) is the whole of \(Z(A, B)^*\). Thus there are \(q^3\) orbits under the conjugation by \(Z(A, B)^*\). Therefore the triple \((A, B, X)\) is of type NT2. Hence we have \(q^3\) branches of type NT2.

When \(x_1 = 0\) and \([Y, C_f] \neq 0\): The number of matrices \(X\) is \(q(q^4 - q^2)\). Equation (5.16) boils down to \(a_1[Y, C_f] = 0\) which implies \(a_1 = 0\). Thus

\[
Z(A, B, X)^* = \left\{ \begin{pmatrix} a_0 I & B \\ 0 & a_0 I \end{pmatrix} : a_0 \in \mathbb{F}_q, B \in M_2(\mathbb{F}_q) \right\}
\]

and its size is \((q - 1)q^4 = q^5 - q^4\). So \((A, B, X)\) is of class type NT6. From this we get:
• Orbit size of $X = \frac{q^6 - q^4}{q^5 - q^2} = q + 1$.

• Number of such orbits $= \frac{q(q^4 - q^2)}{q + 1} = q^4 - q^3$.

The number of branches of type NT6 is $q^4 - q^3$.

**When** $x_1 \neq 0$ and $[Y, C_f] = 0$: The number of matrices $X$ is 

$q(q - 1)q^2 = q^4 - q^3$.

From equation (5.16), $x_1[D, C_f] = 0$, which implies $[D, C_f] = 0$. Hence $D = d_0I + d_1C_f$ and therefore

$$Z(A, B, X)^* = \left\{ \begin{pmatrix} a_0I + a_1C_f & d_0I + d_1C_F \\ 0 & a_0I + a_1C_f \end{pmatrix} : (a_0, a_1) \neq (0, 0) \right\}.$$  

It is commutative and its size is $(q^2 - 1)q^2 = q^4 - q^2$. Hence, $Z(A, B, X)$ is of dimension 4. So, the triple $(A, B, X)$ is a branch of a Regular type. The size of the orbit of $X$ is $\frac{q^6 - q^4}{q^4 - q^2} = q^2$. There are

$$\frac{q^4 - q^3}{q^2} = q^2 - q$$

branches of this Regular type.

**When** $x_1 \neq 0$ and $[Y, C_f] \neq 0$: The number of matrices is 

$q(q - 1)(q^4 - q^2)$. Equation (5.16) gives us, $D \in x_1^{-1}a_1Y + \mathbb{F}_q[C_f]$. So, $Z(A, B, X)^*$ is:

$$\left\{ \begin{pmatrix} a_0I + a_1C_f & x_1^{-1}a_1Y + d_0I + d_1C_F \\ 0 & a_0I + a_1C_f \end{pmatrix} : (a_0, a_1) \neq (0, 0) \right\}.$$  

It is commutative and its size is $(q^2 - 1)q^2 = q^4 - q^2$. Thus, the algebra $Z(A, B, X)$ is of dimension 4. Thus, this branch too is Regular. The size of the orbit of $X$ in $Z(A, B)$ is $\frac{(q^6 - q^4)/(q^4 - q^2)}{q^2} = q^2$ and the number of orbits is therefore $q(q - 1)(q^4 - q^2)/q^2 = q(q - 1)(q^2 - 1)$. Therefore, the total number of Regular branches is

$$q(q - 1)(q^2 - 1) + (q^2 - q) = q^4 - q^3$$

Thus we have the table mentioned in the statement. \(\Box\)
Lemma 5.16. If \( A \) is of similarity class type NT3, then its branching rules are given in the table below:

| Type            | Number of Branches |
|-----------------|--------------------|
| NT3             | \( q^3 \)          |
| Regular         | \( q^4 - q^3 \)    |
| New Type NT6    | \( q^4 + q^3 \)    |

Proof. The centralizer algebra of a pair, \((A, B)\), of the new type NT3 is

\[
Z(A, B) = \left\{ \begin{pmatrix} D(c_0, c_1) & C \\ 0 & D(c_0, c_1) \end{pmatrix} \mid c_0, c_1 \in \mathbb{F}_q, C \in M_2(\mathbb{F}_q) \right\},
\]

where \( D(c_0, c_1) \) is a \( 2 \times 2 \) diagonal matrix with \( c_0 \) and \( c_1 \) as its diagonal entries. This \( D(c_0, c_1) \) can also be written as \( c_0 I + c_1 D(0, 1) \) (replace \( c_1 - c_0 \) by \( c_1 \)). Let \( X \) be:

\[
X = \begin{pmatrix} x_0 I + x_1 D(0, 1) & Y \\ 0 & x_0 I + x_1 D(0, 1) \end{pmatrix}
\]

and \( M \) be an invertible matrix in \( Z(A, B) \):

\[
M = \begin{pmatrix} c_0 I + c_1 D(0, 1) & C \\ 0 & c_0 I + c_1 D(0, 1) \end{pmatrix}.
\]

As \( M \) is invertible, \( c_0 \neq 0 \) and \( c_0 + c_1 \neq 0 \). So, \( XM = MX \) iff \([c_1 Y - x_1 D, D(0, 1)] = 0\). From this equation, we have four cases as to what \( x_1 \) and \( Y \) have to be, i.e.,

**When** \( x_1 = 0 \) and \([Y, D(0, 1)] = 0\): The number of such \( X \)'s is \( q^3 \).
Here \( c_1 \) can be anything and \( C \) can be any \( 2 \times 2 \) matrix. So the centralizer group, \( Z(A, B, X)^* \) of \( X \) in \( Z(A, B)^* \) is \( Z(A, B)^* \) itself. Therefore the orbit of \( X \) is of size 1 and there are \( q \times q^2 = q^3 \) such orbits. Hence \( q^3 \) branches of type NT3.

**When** \( x_1 = 0 \) and \([Y, D(0, 1)] \neq 0\): The number of such \( X \)'s is \( q(q^4 - q^2) \). \( c_1[Y, D(0, 1)] = 0 \) implies \( c_1 = 0 \). Thus,

\[
Z(A, B, X)^* = \left\{ \begin{pmatrix} c_0 I & C \\ 0 & c_0 I \end{pmatrix} : c_0 \neq 0 \right\}.
\]

Thus \((A, B, X)\) is of the type NT6. Its orbit size is \( \frac{q^4(q - 1)^2}{q^2(q - 1)} = q - 1 \) and there are \( q \times (q^4 - q^2) \) such matrices. Hence the number of orbits
is \( \frac{q^3(q^2 - 1)}{q - 1} = q^3(q + 1) = q^4 + q^3 \). We therefore have \( q^4 + q^3 \) branches of this new type.

**When** \( x_1 \neq 0 \) and \([Y, D(0, 1)] = 0\): There are \( q(q - 1)q^2 \) such matrices and we have \( x_1[D, D(0, 1)] = 0 \) which implies that \( C = d_0I + d_1D(0, 1) \). Hence,

\[
Z(A, B, X)^* = \left\{ \begin{pmatrix} c_0I + c_1D(0, 1) & d_0I + d_1D(0, 1) \\ 0 & c_0I + c_1D(0, 1) \end{pmatrix} : c_0 \neq 0, c_1 \neq -c_0 \right\}.
\]

Its size is \( q^2(q - 1)^2 \). It is of dimension 4 and it is commutative. Therefore, \((A, B, X)\) is a Regular branch of \((A, B)\). Each orbit is of size \( \frac{q^4(q - 1)^2}{q^2(q - 1)^2} = q^2 \) and therefore the number of branches is

\[
q(q - 1)q^2/q^2 = q^2 - q.
\]

**When** \( x_1 \neq 0 \) and \([Y, D(0, 1)] \neq 0\): There are \( q(q - 1)(q^4 - q^2) \) such \( X \) and \( D \in x_1^{-1}c_1Y + E_q[D(0, 1)] \). Thus \( C = x_1^{-1}c_1Y + d_0I + d_1D(0, 1) \) and so the \( Z(A, B, X)^* \) consists of matrices of the form

\[
\begin{pmatrix}
    c_0I + c_1D(0, 1) & x_1^{-1}c_1Y + d_0I + d_1D(0, 1) \\
    0 & c_0I + c_1D(0, 1)
\end{pmatrix}.
\]

Its size is \( q^2(q - 1)^2 \), it is of dimension 4 and it is commutative. Thus \((A, B, X)\) is a Regular branch. The size of its orbit is \( q^2 \) and there are a total of \( \frac{q(q - 1)(q^4 - q^2)}{q^2} = q(q - 1)(q^2 - 1) \). On adding up all the Regular branches, we have a total of

\[
q(q - 1)(q^2) + q(q - 1) = q^4 - q^3 \text{ Regular branches.}
\]

\[\square\]

**Lemma 5.17.** For the commuting pair \((A, B)\) of similarity class type NT4 or NT5, there are:

- \( q^3 \) branches of its own type.
- \( q^3 + q^2 \) branches of the new type NT6.
- \( q^4 \) branches of the Regular type.

**Proof.** The proof is the same for both NT4 and NT5. So it will suffice to prove for any one of them. We shall prove it for NT4.
$Z(A, B)$ consists of matrices of the form

$$M = \begin{pmatrix} a_0 & b_1 & b_2 & b_3 \\ 0 & a_0 & 0 & 0 \\ 0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & a_0 \end{pmatrix},$$

which on conjugation by elementary matrices (which switches the 2nd and 3rd rows and columns of $M$) becomes

$$M = \begin{pmatrix} a_0 & b_2 & b_1 & b_3 \\ 0 & c_2 & c_1 & c_3 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$

We shall rewrite $M$ as

$$M = \begin{pmatrix} a_0 & a_1 & b_1 & b_2 \\ 0 & b_0 & b_3 & b_4 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix},$$

and let $M'$ be a conjugate of $M$ in $Z(A, B)$:

$$M' = \begin{pmatrix} a'_0 & a'_1 & b'_1 & b'_2 \\ 0 & b'_0 & b'_3 & b'_4 \\ 0 & 0 & a'_0 & 0 \\ 0 & 0 & 0 & a'_0 \end{pmatrix}.$$

Then there is an invertible $X$ such that $XM = M'X$. Let

$$X = \begin{pmatrix} x_0 & x_1 & y_1 & y_2 \\ 0 & y_0 & y_3 & y_4 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix},$$
where $x_0, y_0 \neq 0$. Expanding $XM = M'X$ gives us $a'_0 = a_0$ and $b'_0 = b_0$ and the following equations:

\begin{align}
(5.17) \quad a_0 x_1 + a'_1 y_0 &= a_1 x_0 + x_1 b_0 \\
(5.18) \quad a'_1 y_3 + b'_1 x_0 &= x_0 b_1 + x_1 b_3 \\
(5.19) \quad a'_1 y_4 + b'_2 x_0 &= x_1 b_4 + b_2 x_0 \\
(5.20) \quad b_0 y_3 + b'_3 x_0 &= y_0 b_3 + y_3 a_0 \\
(5.21) \quad b_0 y_4 + b'_4 x_0 &= y_0 b_4 + y_4 a_0
\end{align}

We have two main cases: $a_0 \neq b_0$ and $a_0 = b_0$.

If $a_0 \neq b_0$. Then, in equation (5.17), using a suitable choice of $x_1$, we can make $a'_1 = 0$. With a suitable choice of $y_3$ in equation (5.20), we can make $b'_3 = 0$. Similarly, in equation (5.21), choose a suitable $y_4$ so that $b'_4 = 0$. Then from equations (5.18) and (5.19), we get $b'_1 = b_1$ and $b'_2 = b_2$. So

$$M = \begin{pmatrix} a_0 & 0 & b_1 & b_2 \\ 0 & b_0 & 0 & 0 \\ 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix}.$$ 

Its centralizer in $Z(A, B)$ is

$$Z(A, B, M) = \left\{ \begin{pmatrix} x_0 & 0 & y_1 & y_2 \\ 0 & y_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix} : x_0, y_0, y_1, y_2 \in \mathbb{F}_q \right\},$$

which is 4-dimensional and commutative. Therefore, this branch $(A, B, M)$ is of a Regular type and there are $q^3(q - 1) = q^4 - q^3$ such branches

If $a_0 = b_0$. Then equation (5.17) becomes $a'_1 y_0 = a_1 x_0$. Here again, there are two cases:

- $a'_1 \neq 0$, and $a_1 = 0$.
- $a'_1 = 0$. Then, from equations (5.20) and (5.21) we get $b'_3 = b_3$ and $b'_4 = b_4$. Equation (5.18) becomes $y_3 + b'_1 x_0 = x_0 b_1 + x_1 b_3$ and equation (5.19) becomes $y_4 + b'_2 x_0 = x_1 b_4 + b_2 x_0$. So we can choose $y_3$
and $y_4$ appropriately so that $b'_1 = b'_2 = 0$. So our $M$ reduces to
\[
\begin{pmatrix}
  a_0 & 1 & 0 & 0 \\
  0 & a_0 & b_3 & b_4 \\
  0 & 0 & a_0 & 0 \\
  0 & 0 & 0 & a_0
\end{pmatrix}.
\]

Hence,
\[
Z(A, B, M) = \left\{ \begin{pmatrix}
  x_0 & x_1 & y_1 & y_2 \\
  0 & x_0 & x_1b_3 & x_1b_4 \\
  0 & 0 & x_0 & 0 \\
  0 & 0 & 0 & x_0
\end{pmatrix} : x_0, x_1, y_1, y_2 \in \mathbb{F}_q \right\},
\]
which is 4 dimensional and commutative. Thus the branch, $(A, B, M)$, is of Regular type. The number of such branches is $q^3$. So we have a total of $q^4 - q^3 + q^3 = q^4$ Regular branches.

When $a_1 = 0$, equation (5.18) becomes $b'_1x_0 = x_0b_1 + x_1b_3$, equation (5.19) becomes $b'_2x_0 = x_0b_2 + x_1b_4$ and we have from equations (5.20) and (5.21), $b'_3x_0 = y_0b_3$ and $b'_4x_0 = y_0b_4$. So we can divide this into two cases.

$(b_3, b_4) = (0, 0)$ and $(b_3, b_4) \neq (0, 0)$

When $(b_3, b_4) = (0, 0)$ we have $b'_1 = b_1$ and $b'_2 = b_2$ and thus $M$ reduces to
\[
\begin{pmatrix}
  a_0 & 0 & b_1 & b_2 \\
  0 & a_0 & 0 & 0 \\
  0 & 0 & a_0 & 0 \\
  0 & 0 & 0 & a_0
\end{pmatrix}.
\]

Thus, $Z(A, B, M)$ is the whole of $Z(A, B)$. Thus $(A, B, M)$ is of the type NT4 and we have $q^3$ such branches.

When $(b_3, b_4) \neq (0, 0)$ and $b_3 \neq 0$. Then, in equation (5.20), using a suitable $y_0$, we can make $b'_3 = 1$. Letting $b'_3 = b_3 = 1$, we get $y_0 = x_0$ and therefore $b'_4 = b_4$. Equation (5.18) becomes $b'_1x_0 = x_0b_1 + x_1$, hence we can get $b'_1 = 0$. Letting $b_1 = b'_1 = 0$, we get $x_1 = 0$, and therefore
we get $b'_2 = b_2$ (from equation (5.19)). Thus $M$ is reduced to

$$
\begin{pmatrix}
a_0 & 0 & 0 & b_2 \\
0 & a_0 & 1 & b_4 \\
0 & 0 & a_0 & 0 \\
0 & 0 & 0 & a_0
\end{pmatrix},
$$

and $Z(A, B, M)$ is

$$
\left\{ \begin{pmatrix} x_0I & Y \\ O & x_0I \end{pmatrix} : x_0, \in \mathbb{F}_q, Y \in M_2(\mathbb{F}_q) \right\},
$$

which is that of the new type NT6. Therefore this branch, $(A, B, M)$ is of type NT6 and we have $q^3$ such branches.

If $b_3 = 0$ and $b_4 \neq 0$. Then we can make $b_4 = 1$ and by the arguments like in the above case, we can make $b_2 = 0$ and $b'_1 = b_1$. So

$$
M = \begin{pmatrix}
a_0 & 0 & b_1 & 0 \\
0 & a_0 & 0 & 1 \\
0 & 0 & a_0 & 0 \\
0 & 0 & 0 & a_0
\end{pmatrix}.
$$

So, $Z(A, B, M)$ is

$$
\left\{ \begin{pmatrix} x_0I & Y \\ O & x_0I \end{pmatrix} : x_0, \in \mathbb{F}_q, Y \in M_2(\mathbb{F}_q) \right\}.
$$

Thus this $(A, B, M)$ too is a branch of the new type NT6 and there are $q^2$ such branches. So in total we have $q^3 + q^2$ branches of the new type NT6.

**Lemma 5.18.** For a triple $(A, B, M)$ of similarity class type NT6, there are $q^5$ branches of the type NT6.

**Proof.** We know that

$$
Z(A, B, M) = \left\{ \begin{pmatrix} a_0I & C \\ 0 & a_0I \end{pmatrix} : a_0 \in \mathbb{F}_q \text{ and } C \in M_2(\mathbb{F}_q) \right\}.
$$

It is easy to see that this algebra is commutative. Hence, there is only one branch and it is of the type NT6 and there are $q^5$ of them. \qed

We therefore have no more new similarity class types.
5.4. Calculating $c_{4,k}(q)$. Now, that we have all the branching rules, we can form a matrix, $\mathcal{B}_4 = [b_{ij}]$, with rows and columns indexed by the types. For a given type $j$, $b_{ij}$ is the number of similarity class type $i$ branches of a tuple of similarity class type $j$. This $\mathcal{B}_4$ is our branching matrix. Table 3 lists the rcfs, and under each rcf, it has a list of the types with that rcf. Let each of the new types be treated as separate rcf's. By the averaging technique discussed in Section 4, we can reduce $\mathcal{B}_4$ to a $11 \times 11$ matrix indexed by the 5 rcfs and the 6 new types in the order:

$\{(1,1,1,1), (2,1,1), (2,2), (3,1), (4), \text{NT1}, \text{NT2}, \text{NT3}, \text{NT4}, \text{NT5}, \text{NT6}\}$.

rcf $(1,1,1,1)$: For rcf $(1,1,1,1)$, there is only one type, which is the Central type, $(1,1,1,1)_1$. It has $q$ branches of rcf $(1,1,1,1)_1$; $q^2$ branches each of rcf types, $(2,1,1)$ and $(2,2)$; $q^3$ branches with rcf $(3,1)$, and $q^4$ branches with rcf, $(4)$ (the Regular type of branches).

df (4): The Regular type of similarity class is of rcf-type, $(4)$. It has $q^4$ branches of rcf $(4)$.

rcf $(2,1,1)$: An element of rcf type $(2,1,1)$ is of class type $(1,1,1,1)(1)$ with probability $q^{-1}/q$ and of class type $(2,1,1)_1$ with probability $1/q$. So, on an average, a tuple of rcf type $(2,1,1)$ has:

- $q^2$ branches of rcf type $(2,1,1)$.
- $q^3 + q^2 - q - 1$ branches of rcf type $(3,1)$.
- $q^4 + q$ Regular (rcf type $(4)$) branches.
- 1 branch each of types NT1, NT4 and NT5.
- $q$ branches of type NT3.

rcf $(2,2)$: There are three similarity class types with rcf, $(2,2)$. They are $(1,1)_1, (1,1)_2, (2,2)_1$ and $(1,1)_2$. An element of rcf type, $(2,2)$, is of class type, $(1,1)_1(1,1)_1$, with probability, $\frac{(q-1)}{2q}$; of class type, $(2,2)_1$, with probability, $\frac{1}{q}$, and is of class type, $(1,1)_2$, with probability, $\frac{q-1}{2q}$. So on an average, a tuple of rcf-type $(2,2)$ has:

- $q^2$ branches of rcf type $(2,2)$.
- $q^3 - q^2$ branches of rcf $(3,1)$.
- $q^4$ Regular branches.
- $q$ branches of the new type NT1
- $(q^2 - q)/2$ branches each of the new types NT2 and NT3.
**rcf** (3, 1): The similarity class types with rcf (3, 1) are:

- (3, 1)₁
- (2, 1)₁₁(1)₁
- (1, 1)₁₁(2)₁
- (1, 1)₁₁(1)₂ and
- (1, 1)₁₁(1)₁₁(1)₁

Their probabilities are mentioned in the table below.

| Class Type | Probability |
|------------|-------------|
| (3, 1)₁     | \(q^{-1}\)   |
| (2, 1)₁₁(1)₁ | \(q^{-1}\)   |
| (1, 1)₁₁(2)₁ | \(q^{-1}\)   |
| (1, 1)₁₁(1)₂ | \(2q\)       |
| (1, 1)₁₁(1)₁₁(1)₁ | \((q-1)(q-2)\) |

All these types have branches of their own respective types and Regular branches. Hence we have on an average: \(q^3\) branches of rcf type (3, 1) and \(q^4 + q\) branches of rcf type (4).

So our branching matrix \(B₄\) is equal to

\[
\begin{pmatrix}
q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^2 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^2 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^3 & q^3+q & q^2 & q^4+q & q^4 & q^4 & q^4 & q^3 & q^3 & q^3 & q^2 & q^4 \\
0 & 1 & q & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{q^2}{q} & 0 & 0 & q^3 & 0 & 0 & 0 & 0 \\
0 & q & \frac{q^3}{q^2} & 0 & 0 & 0 & q^4 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & q^5 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & q^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q^4 & q^4 & q^4 & q^4 & q^4 & q^4 & q^3 & q^2 & q^5
\end{pmatrix}
\]

Let \(e_1\) denote the 11 × 1 column matrix with first entry being 1 and the rest, 0. Let \(1'\) denote the 1 × 11 row matrix, whose entries are all 1’s. Then we have

\[c_{4,k}(q) = 1'B_k e_1.\]
The table below lists $c_{4,k}(q)$ for $k = 1, 2, 3, 4$. The calculations were done using sage.

| $k$ | $c_{4,k}(q)$ |
|-----|-------------|
| 1   | $q^4 + q^3 + 2q^2 + q$ |
| 2   | $q^8 + q^7 + 3q^6 + 3q^5 + 5q^4 + 3q^3 + 3q^2$ |
| 3   | $q^{12} + q^{11} + 3q^{10} + 4q^9 + 8q^8 + 8q^7 + 11q^6 + 8q^5 + 5q^4 + 2q^3$ |
| 4   | $q^{16} + q^{15} + 3q^{14} + 5q^{13} + 9q^{12} + 12q^{11} + 16q^{10} + 17q^9 + 17q^8 + 13q^7 + 9q^6 + 4q^5 + 2q^4$ |

We can see that $c_{4,k}(q)$ is a polynomial in $q$ with non-negative integer coefficients for $k = 1, 2, 3, 4$. But, we can’t say the same about $c_{4,k}(q)$ for general $k$. So, we will have to use the generating function for $c_{4,k}(q)$, which is

$$h_4(q, t) = \sum_{k=0}^{\infty} c_{4,k}(q)t^k = 1'(I - tB_4)^{-1}e_1.$$ 

In the next subsection, we will carefully examine the expression, $h_4(q, t)$.

5.5. **Non-negativity of coefficients of $c_{4,k}(q)$**. Now it remains to check if the coefficients of $h_4(q, t)$ are non-negative. The rational generating function $h_4(q, t)$ is:

$$h_4(q, t) = \frac{r_+(q, t) - r_-(q, t)}{(1 - qt)(1 - q^2t)(1 - q^3t)(1 - q^4t)(1 - q^5t)},$$

where $r_+(q, t) = 1 + q^2t + 2q^2t^2 + q^3t^2 + 2q^4t^2 + q^6t^3$, and $r_-(q, t) = q^3t + q^7t^2 + q^7t^3 + 2q^7t^3 + 2q^9t^3 + q^{10}t^4$. We have

$$\frac{1}{(1 - qt)(1 - q^2t)(1 - q^3t)(1 - q^4t)(1 - q^5t)} = \left(\sum_{k=0}^{\infty} \left(\sum_{j=k}^{5k} p_{5,k}(j)q^j t^k\right)\right),$$

where $p_{5,k}(j)$ denotes the number of partitions of $j$ with $k$ parts, with the maximum part being $\leq 5$. With this,

$$h_4(q, t) = (r_+(q, t) - r_-(q, t)) \left[1 + \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{5k} p_{5,k}(j)q^j t^k\right)\right)\right].$$
Expanding this gives us

\[
(5.22) \quad \left( \sum_{k=0}^{\infty} \left( \sum_{j=k}^{5k} p_{5,k}(j)q^jt^k \right) \right) - \left( \sum_{k=0}^{\infty} \left( \sum_{j=k}^{5k} p_{5,k}(j)q^{j+5t}k^{+1} \right) \right)
\]

\[
+ \left( \sum_{k=0}^{\infty} \left( \sum_{j=k}^{5k} 2p_{5,k}(j)q^{j+2t}k^{2}+2k^{+2} \right) \right) - \left( \sum_{k=0}^{\infty} \left( \sum_{j=k}^{5k} p_{5,k}(j)q^{j+3t}k^{+3} \right) \right)
\]

\[
+ \left( \sum_{k=0}^{\infty} \left( \sum_{j=k}^{5k} p_{5,k}(j)q^{j+4t}k^{+4} \right) \right) - \left( \sum_{k=0}^{\infty} \left( \sum_{j=k}^{5k} 2p_{5,k}(j)q^{j+9t}k^{+3} \right) \right)
\]

\[
+ \left( \sum_{k=0}^{\infty} \left( \sum_{j=k}^{5k} p_{5,k}(j)q^{j+6t}k^{+3} \right) \right) - \left( \sum_{k=0}^{\infty} \left( \sum_{j=k}^{5k} p_{5,k}(j)q^{j+10t}k^{+4} \right) \right).
\]

The coefficient, \(d_{jk}\), of \(q^jt^k\) in equation (5.22) is

\[
(5.23) \quad d_{jk} = (p_{5,k}(j) - p_{5,k-1}(j - 5)) + (p_{5,k-1}(j - 2) - p_{5,k-2}(j - 7))
\]

\[
+ (2p_{5,k-2}(j - 2) - p_{5,k-3}(j - 3)) + (p_{5,k-2}(j - 3) - 2p_{5,k-3}(j - 7))
\]

\[
+ (2p_{5,k-2}(j - 4) - 2p_{5,k-3}(j - 9)) + (p_{5,k-3}(j - 6) - p_{5,k-4}(j - 10)).
\]

Here are some observations which will be enough to prove that equation (5.23) is non-negative.

**Lemma 5.19.** For any \(k \geq 1\), any \(j : k \leq j \leq 5k\), and any \(l\) such that, \(1 \leq l \leq 5\), \(p_{5,k}(j) \geq p_{5,k-1}(j - l)\).

**Proof.** We assume that \(j - l \leq 5(k - 1)\) so that \(p_{5,k-1}(j - l) \neq 0\). Given a partition of \(j - l\) with \(k - 1\) parts with maximal part \(\geq 5\), we can attach the part \(l\) to this partition to get a partition of \(j\) in \(k\) parts, with maximal part \(\leq 5\). Hence \(p_{5,k}(j) \geq p_{5,k-1}(j - l)\).

As a consequence of the above lemma, we have the following inequalities.

\[
(5.24) \quad p_{5,k}(j) \geq p_{5,k-1}(j - 5)
\]

\[
(5.25) \quad p_{5,k-1}(j - 2) \geq p_{5,k-2}(j - 7)
\]

\[
(5.26) \quad p_{5,k-2}(j - 2) \geq p_{5,k-3}(j - 3)
\]

\[
(5.27) \quad p_{5,k-2}(j - 3) \geq p_{5,k-3}(j - 7)
\]

\[
(5.28) \quad p_{5,k-2}(j - 4) \geq p_{5,k-3}(j - 9)
\]

\[
(5.29) \quad p_{5,k-3}(j - 6) \geq p_{5,k-4}(j - 10).
\]

**Lemma 5.20.** Let \(k \geq 4\). Then for \(j\) such that \(j - 7 \geq k - 3\) we have the following:

- If \(j - 7 = 5(k - 3)\), then

\[
(5.30) \quad (p_{5,k}(j) - p_{5,k-1}(j - 5)) + (p_{5,k-2}(j - 3) - 2p_{5,k-3}(j - 7)) \geq 0
\]
• If \(j - 7 < 5(k - 3)\) then

\[
(5.31) \quad p_{5,k-2}(j - 3) - 2p_{5,k-3}(j - 7) \geq 0
\]

Proof. When \(j - 7 = 5(k - 3)\), given the only partition of \(j - 7\) with \(k - 3\) parts, we can attach two 1’s to it, to get a partition of \(j - 5\) in \(k - 1\) parts. Hence \(p_{5,k-1}(j - 5) \geq p_{5,k-3}(j - 7)\).

Therefore \((p_{5,k}(j) - p_{5,k-1}(j - 5)) + (p_{5,k-2}(j - 3) - 2p_{5,k-3}(j - 7)) \geq p_{5,k}(j) - 2p_{5,k-1}(j - 5) + (p_{5,k-2}(j - 3) - p_{5,k-3}(j - 7))\).

Observe: \(j - 7 = 5k - 15 \Rightarrow j - 5 = 5k - 13 = 5(k - 1) - 8\). So any partition of \(j - 5\) with \(k - 1\) parts, with maximal part \(\leq 5\), will have atleast two parts, < 5. So, to each of these, we can either attach a 5, or add 1 each to the two parts which are less than 5 and attach 3 as the \(k\)th part. This gives 2 partitions of \(j\) having \(k\) parts. So, \(p_{5,k}(j) - 2p_{5,k-1}(j - 5) \geq 0\) and therefore

\[
(p_{5,k}(j) - p_{5,k-1}(j - 5)) + (p_{5,k-2}(j - 3) - 2p_{5,k-3}(j - 7)) \geq 0 \text{ Since } (p_{5,k-2}(j - 3) - p_{5,k-3}(j - 7)) \geq 0 \text{ (from ineq. (5.27)).}
\]

Hence inequality (5.30) holds.

When \(j - 7 < 5(k - 3)\), then, for any partition of \(j - 7\) with \(k - 3\) parts with each part being atmost 5, we have atleast one part which is strictly less than 5. Given any such partition, we can either, add 1 to the part that’s < 5 and attach a 3, or just attach a 4 to the existing partition, to get a partition of \(j - 3\) in \(k - 2\) parts. Hence we get two partitions of \(j - 3\) in \(k - 2\) parts. Therefore inequality (5.31) holds. □

Using Lemma 5.20 and inequalities (5.24) to (5.29), we can show that the coefficient of \(qt^k\) for each \(j,k \geq 0\), is non-negative. So for each \(k \geq 1\), the coefficients of \(c_{4,k}(q)\) are the coefficients of \(qt^k\) as \(j\) varies, which are non-negative. Therefore, the coefficients of \(c_{4,k}(q)\) are non-negative integers.

Thus, Theorem 1 is proved for \(n = 4\).

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