SIMPLICIAL ABEL-JACOBI MAPS AND RECIPROCITY LAWS

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ABSTRACT. We describe an explicit morphism of complexes that induces the cycle-class maps from (simplicially described) higher Chow groups to rational Deligne cohomology. The reciprocity laws satisfied by the currents we introduce for this purpose are shown to provide a clarifying perspective on functional equations satisfied by complex-valued di- and trilogarithms.

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1. INTRODUCTION

Abel-Jacobi maps for higher Chow groups

\begin{equation}
\text{AJ}^{p,n}_X : CH^p(X, n)_\mathbb{Q} \rightarrow H_{\mathcal{H}}^{2p-n}(X^n_\mathbb{C}, \mathbb{Q}(p))
\end{equation}

were introduced (for smooth quasi-projective $X$ over $k \subset \mathbb{C}$) in [Ke1, KLM] via an extension of Griffiths’s formula for $n = 0$ to a quasi-isomorphic subcomplex of the cubical Bloch complex $Z^p(X, \bullet)_\mathbb{Q}$. Together with their extension to motivic cohomology $H_{\mathcal{M}}^{2p-n}(X, \mathbb{Q}(n))$ in the singular case [KL], these AJ-maps have been used (for example) to interpret limits of normal functions of geometric origin [GGK], to study toric and Eisenstein symbols on families of Calabi-Yau varieties [DK], to compute a family of Feynman integrals [BKV], and to study torsion in $CH^p(X, n)$ [Pe] (though an integral moving lemma is still missing for this to work in general).

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Summary. The main purpose of this paper is to give an alternate formula (Theorem 3.2) for (1.1) on the simplicial Bloch complex $Z_p^\Delta(X, \bullet)_Q$, sending a precycle $\mathfrak{I}$ to a triple of currents $(T_3^\Delta, \Omega_3^\Delta, R_3^\Delta)$ (cf. (3.3)) on $X$. (We shall restrict for simplicity to the smooth projective case, so that the absolute Hodge cohomology in (1.1) is just Deligne cohomology; the generalization to quasi-projective is exactly as in §3 of [KL].) This had been a goal of the authors of [KLM], but seemed out of reach at the time. The basic currents on $\mathbb{P}^n$ we develop for this purpose in §2 (see (2.2), (2.6), and (2.7)) lead to two “simplicial” reciprocity laws (Theorems 4.5 and 4.8) for their integrals over subvarieties of projective space, which are applied to directly recover functional equations for complex-valued di- and tri-logarithms in §§5-6. The second of these laws takes a very intriguing form, and leads to a more straightforward proof of the Kummer-Spence relation (6.1) than for the real-valued trilogarithm in [Go3]. This paper is written in such a way that the reader interested only in these applications can skip §3 entirely.

The $AJ$ formulas of [KLM] were based on the cubical higher Chow complex for several reasons, including the greater ease of constructing good currents on $\square^n$ (not to mention explicit cycles in the cubical complex), the availability of bounding membranes (to provide a link to the extension class definition of $AJ$), and the greater naturality of cup-products in the cubical setting. On the other hand, the simplicial formulation of higher Chow groups allows for linear higher cycles, which provide direct links to the seminal work of Goncharov on polylogarithms (cf. [Go1]-[Go4]) and to the cohomology of the general linear group $[\mathbb{L}]$. These special features make a compelling argument for revisiting [KLM] from the simplicial point of view.

Moreover, we point out that, up to this point, there has not even been a correct real regulator formula on the simplicial level. While it was checked in §3.1 of [Ke1] that Goncharov’s currents in [Go1] yield the real regulator (i.e., composition of $AJ_p^n$ with $\pi_R : H^2p-n(\mathbb{H}_n) \rightarrow H^2p-n(\mathbb{P}^n)$) on the cubical complex, it turns out that the simplicial version constructed in §6.1 of [Go1] (or §2 of [Go2]) is neither well-defined nor a map of complexes. The main problem is that $Z_p^\Delta(X, n)_Q$ is a subgroup of $Z^p(X \times \mathbb{P}^n)_Q/Z^p(X \times \mathbb{H}_n)_Q$, where $\mathbb{H}_n \subset \mathbb{P}^n$ is the hyperplane defined by $X_0 + \cdots + X_n = 0$ and the currents of [op. cit.] do not vanish on $Z^p(X \times \mathbb{H}_n)_Q$. For more details, see Remark 3.4.

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1 We use the term “precycle” for an element of Bloch’s complex, and “[higher Chow] cycle” for an element of $CH^p(X, n)_Q$ (equivalence class of a closed precycle).

2 The Roman script $X$ and $Y$ are used throughout to denote varieties, while $X, Y$ denote projective coordinates; this convention is not followed for other letters.
Some motivation. On any $\mathbb{P}^m$, with coordinates $[Y_0 : \cdots : Y_m]$ (and $y_i := Y_i/Y_0$), one has a natural dlog-form
\[
\Omega_\Delta^m(Y_0: \cdots: Y_m) := \Omega_m \left( \frac{Y_1}{Y_0}, \ldots, \frac{Y_m}{Y_0} \right) := \frac{dy_1}{y_1} \land \cdots \land \frac{dy_m}{y_m}
\]
and real $m$-chain (see (2.4) for orientation)
\[
\mathcal{T}_m^\Delta(Y_0: \cdots : Y_m) := \{ y_1, \ldots, y_m \in \mathbb{R}_+ \};
\]
we may regard both as $m$-currents. The constructions of this paper center around the existence of sequences of $(m - 1)$-currents
\[
\mathcal{R}_m \in D^{m-1}(\mathbb{P}^m) \quad (m \geq 1)
\]
satisfying two properties. To motivate the first property ((1.3) below), consider the family
\[
X_s := \left\{ \frac{2^n}{\prod_{i=0}^{2n} X_i} - s \sum_{i=0}^{2n} X_i^{2n+1} = 0 \right\} \subset \mathbb{P}^{2n}
\]
of Calabi-Yau $(2n - 1)$-folds$^3$ and let $\mathbb{P}^n \cong P_t \subset \mathbb{P}^{2n}$ be a family of linear $n$-plane$^4$ with $\partial \Gamma_t = P_t - P_0$ ($\Gamma_t = (2n + 1)$-chain). Setting $\mathcal{Z}_{s,t} := P_t \cdot X_s$, we have $0 = [\mathcal{Z}_{s,t} - \mathcal{Z}_{s,0}] \in CH^n(X_s)$; in particular, writing $\mathcal{Z}_{s} := Res_{X_S} \left( \sum_{i=0}^{2n} (-1)^i dX_0 \land \cdots \land dX_i \land \cdots \land dX_{2n} \right)$, the Griffiths Abel-Jacobi integrals $\int_{\Gamma_t \cdot X_s} \mathcal{Z}_{s}$ vanish. Taking $s \to 0$ and writing $X_0 = \bigcup X_0^j$, where $X_0^j = \{ X_j = 0 \} \cong \mathbb{P}^{2n-1} \hookrightarrow \mathbb{P}^{2n}$, we obtain
\[
(1.2) \quad 0 = \int_{\Gamma_t \cdot X_0} \mathcal{Z}_{s} = \sum_{j=0}^{2n} (-1)^j \int_{\Gamma_t \cdot X_0^j} \Omega_\Delta^{2n-1} \left( X_0 : \cdots : \hat{X}_j : \cdots : X_{2n} \right)
\]
Now suppose that for each $m$
\[
(1.3) \quad d[\mathcal{R}_m] = \Omega_\Delta^m - (2\pi i)^m \delta_{T_\Delta^m} + 2\pi i \sum_{j=0}^{m} (-1)^j \rho_j \mathcal{R}_{m-1}
\]
holds. Then on $X_0$, $\omega_0 \equiv d[\sum (-1)^j \rho_j \mathcal{R}_{2n-1}]$ modulo $\mathbb{Z}(2n-1)$-valued currents, and by Stokes’s theorem (1.2) gives
\[
0 = \sum_{j=0}^{2n} (-1)^j \int_{3_{0,t}^j - 3_{0,0}^j} \mathcal{R}_{2n-1}
\]
(where $3_{0,t}^j := 3_{0,t} \cdot X_0^j$), which shows that
\[
(1.4) \quad \sum_{j=0}^{2n} (-1)^j \int_{3_{0,t}^j} \mathcal{R}_{2n-1} \quad \text{is constant.}
\]

$^3$We shall ignore the fact that this family is not semistable at $s = 0$.

$^4$Think of $t$ as varying in some neighborhood of 0 in a $\mathbb{C}^M$, with $\Gamma_t$ the union of $\{ P_t \}$ over a radial segment $0\hat{t}$. 

In fact, according to the first of the “reciprocity laws” in §4, this constant belongs to \( \mathbb{Z}(2n - 1) \), and the proof is simpler than the argument just given. (The second of the two laws, however, is more subtle.) If \( n = 1 \) and the \( P_t \) are lines, \([1.4]\) is just \( \log(x) - \log(y) + \log(y/x) \equiv 0 \).

To motivate the second property, suppose we would like to have lifts \( \tilde{\varepsilon}_n \in H^{2n-1}_{\text{meas}}(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{Z}(n)) \) of the Borel classes

\[
\varepsilon_n \in H^{2n-1}_{\text{cont}}(GL_n(\mathbb{C}), \mathbb{R}) := H^{2n-1}\{\text{Cont}(GL_n(\mathbb{C})^{\times (\bullet+1)})^{GL_n(\mathbb{C})}, \delta\}
\]

(e.g., \( \varepsilon_1(g_0, g_1) \propto \log |g_1/g_0| \) and

\[
\varepsilon_2(g_0, g_1, g_2, g_3) \propto D(CR([g_0v], [g_1v], [g_2v], [g_3v])),
\]

where \( D \) is the Bloch-Wigner function and \( v \in \mathbb{C}^2 \) is fixed). For instance, one might use such lifts to detect elements (particularly torsion ones) of \( H^{2n-1}(GL_n(\mathbb{P}), \mathbb{Z}) \) or to construct complex lifts of hyperbolic volume. Recall that Bloch’s higher Chow complexes were originally defined in their simplicial formulation: writing \( \Delta^n := \mathbb{P}^n \setminus H_n \), \( \partial \Delta^n := \bigcup_{j=0}^{n} \rho_j(\mathbb{P}^{n-1} \setminus H_{n-1}) \), \( (\partial)\Delta^n_X := X \times (\partial)\Delta^n \), the subgroups \( Z^p(\Delta^n_X, n) \leq Z^p(\Delta^n_X) = \frac{Z^p(\mathbb{C} \times H^n)}{(\mathbb{C} \times H^n)} \) (generated by subvarieties meeting faces of \( \partial \Delta^n_X \) properly) form a complex \( Z^p_X(X, \bullet)\mathbb{Q} \) under \( \partial = \sum (-1)^j \rho_j^* \) with homology \( CH^p(X, n)\mathbb{Q} \). The relevant case is where \( X = Spec(\mathbb{C}) \) is a point.

If the \( \{\mathfrak{R}_m\} \) satisfy the additional property

\[
(1.5) \quad \mathfrak{R}_m|_{H_n} = 0,
\]

then we can use them to induce Abel-Jacobi maps

\[
(1.6) \quad CH^n(Spec(\mathbb{C}), 2n - 1) \xrightarrow{AJ^\Delta} \mathbb{C}/\mathbb{Z}(n)
\]

by integrating \( (2\pi i)^{n-1} \mathfrak{R}_{2n-1} \) over a cycle \( \mathfrak{A} \). Composing this with the map

\[
H_{2n-1}(GL_n(\mathbb{C}), \mathbb{Z}) \longrightarrow CH^n(Spec(\mathbb{C}), 2n - 1)
\]

defined by fixing \( v \in \mathbb{C}^n \) and sending a tuple \( g := (g_0, \ldots, g_{2n-1}) \in GL_n(\mathbb{C})^{\times 2n} \) (in general position) to

\[
\mathfrak{A}_g := \left\{ \left( \begin{array}{cccc}
\uparrow & \cdots & \uparrow \\
g_0v & \cdots & g_{2n-1}v \\
\downarrow & \cdots & \downarrow \\
X_0 & \vdots & X_{2n-1}
\end{array} \right) | \begin{array}{c}
0 \\
0
\end{array} \right\} \subset \Delta^{2n-1},
\]

we apparently obtain a candidate for \( \tilde{\varepsilon}_n \). In fact, this is a bit glib as \( AJ^\Delta \) is not defined on all of \( Z^{2n-1}_{\Delta}(Spec(\mathbb{C}), \bullet) \), but only on a subcomplex \( Z^{2n-1}_{\Delta, \mathbb{R}}(Spec(\mathbb{C}), \bullet) \) of precycles well-behaved with respect to

\footnote{See [11] for details of this construction.}
the currents. So far we only know this subcomplex is rationally quasi-isomorphic (Proposition 3.1), and so (1.6) only maps to $\mathbb{C}/\mathbb{Q}(n)$. Nevertheless, the direct formula

$$\tilde{\epsilon}_n(g) := (2\pi i)^{1-n} \int_{\Delta_2} \mathfrak{R}_{2n-1}$$

appears to give a measurable cohomology class with $\mathbb{C}/\mathbb{Z}(n)$ coefficients, which should be investigated further.

Finally, to give the reader a flavor of what sort of concrete computation is possible with our $AJ$ formula in the simplest case, where $X$ is a point over a number field, consider the element $\mathfrak{3} \in Z^2_\Delta (\text{Spec}(\mathbb{Q}(\zeta)), 3)_Q$ ($\zeta = e^{2\pi i}$) defined by

$$Z \in \mathbb{Z}^2_\Delta(X, 2)$$

$$\mathfrak{3} := \mathfrak{3}_1 + \mathfrak{3}_2 :=$$

$$\begin{bmatrix}
-Z(Z - \zeta^2 W)^2 : W(Z - \zeta W)(Z - \zeta^2 W) : -W^3 : Z^3 \\
+ [3Z(Z + \zeta^2 W) : 3Z^2 : -W^2 : W^2]_{[Z:W] \in \mathbb{P}^1}
\end{bmatrix}$$

We have $\partial \mathfrak{3} = -[3\zeta : -1 : 1] + [3\zeta : -1 : 1] = 0$ and so this defines a higher Chow cycle $[\mathfrak{3}]$, whose image under the (simplicially defined) map

$$AJ^2: CH^2 (\text{Spec}(\mathbb{Q}(\zeta)), 3) \to \mathbb{C}/\mathbb{Q}(2)$$

is computed by integrating the 2-current

$$\mathfrak{R} := \frac{1}{2\pi i} R_3 \left( \frac{X_1+X_2+X_3}{-X_0}, \frac{X_2+X_3}{-X_1}, \frac{X_3}{-X_2} \right)$$

on $\mathbb{P}^3$ (cf. (2.2)) over $\mathfrak{3}$. Since $-\frac{X_3}{X_2} \equiv 1$ on $\mathfrak{3}_2$, we have

$$AJ([\mathfrak{3}]) = \int_{\mathfrak{3}_1} \mathfrak{R}$$

$$= \frac{1}{2\pi i} \int_{\mathbb{P}^3} R_3 \left( \frac{Z^3-W^3+W(Z-\zeta W)(Z-\zeta^2 W)}{Z(Z-\zeta^2 W)^2}, \frac{W^3-Z^3}{W(Z-\zeta W)(Z-\zeta^2 W)}, \frac{Z^3}{W^3} \right)$$

$$= \frac{1}{2\pi i} \int_{\mathbb{P}^3} R \left( \frac{t-\zeta}{t-\zeta^2}, 1-t, t^3 \right)$$

$$= 3 \int_{\zeta} \zeta^2 \log(1-t)d\log(t)$$

$$= 3 \left( Li_2(\zeta^2) - Li_2(\zeta) \right)$$

$$= -3\sqrt{3}L(\chi_{-3}, 2).$$

---

6See (3.1). Note that the intersections with coordinate hyperplanes which lie inside $H_3$ do not count, since $\rho^*_3 H_3 = H_2$ and $Z^2_{\Delta}(X, 2) \leq Z^2(X \times \mathbb{P}^2)/Z^2(X \times H_2)$. 

Remark. Throughout this paper, all cycle groups are taken with rational coefficients; henceforth, we drop the subscript \( \mathbb{Q} \) used above. This choice reflects the fact that we do not yet know how to prove Propositions 3.1 and 3.5 (or some substitute) integrally. (This will be necessary to enjoy the real benefits of the simplicial \( AJ \) map when \( X \) is the spectrum of a number field, since in this case the main point of lifting from \( \mathbb{R}(n - 1) \) to \( \mathbb{C}/\mathbb{Z}(n) \) is probably to extract torsion information.) Also note that in sections 3 and 6 we have relegated to appendices those technical details which we judged to interrupt the main line of argument (proofs of moving lemmas, etc.)

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2. Two classes of simplicial currents

The explicit formulas for Abel-Jacobi maps for higher Chow groups in \([KLM]\) were enabled by the construction of triples \((R_n, \Omega_n, T_n)\) of currents on each \((\mathbb{P}^1)^n\) with the telescoping property

\[
(2.1) \quad d[R_n] = \Omega_n - (2\pi i)^n \delta_{T_n} + 2\pi i \sum_{j=1}^{n} (-1)^j \left( \left( i_j^0 \right)_* R_{n-1} - \left( i_j^\infty \right)_* R_{n-1} \right),
\]

where \( i_j^\varepsilon : (\mathbb{P}^1)^{n-1} \hookrightarrow (\mathbb{P}^1)^n \) are the inclusions of the coordinate hyperplanes \( z_j = \varepsilon \). We briefly recall their definition: let \( T_f := f^{-1}(\mathbb{R}_{>0}) \) be oriented so that \( \partial T_f = (f) \), and \( \log(\cdot) \) denote the discontinuous function with \( \arg(\cdot) \in (-\pi, \pi] \). Writing \( \varepsilon := (-1)^{n-1} 2\pi i \), we set

\[
R_n := R_n(z_1, \ldots, z_n) := \sum_{j=1}^{n} \varepsilon^{j-1} \log(z_j) \frac{dz_{j+1}}{z_{j+1}} \wedge \cdots \wedge \frac{dz_n}{z_n} \cdot \delta_{T_{z_1 \cap \cdots \cap T_{z_{j-1} = 1}}} \in D^{n-1}((\mathbb{P}^1)^n),
\]

\[
(2.3) \quad \Omega_n := \Omega_n(z_1, \ldots, z_n) := \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n} \in D^{n,0}((\mathbb{P}^1)^n),
\]
and
\[ T_n := T_n(z_1, \ldots, z_n) := T_{z_1} \cap \cdots \cap T_{z_n} \in C^n_{\text{top}} \left( \mathbb{P}^1 \right) \].

Roughly speaking, (2.1) follows from \( d[\log z_j] = \frac{dz_j}{z_j} - 2\pi i\delta_{T_{z_j}} \) and
\[ d \left[ \frac{dz_j}{2\pi iz_j} \right] = \delta_{(z_j)} = d \left[ \delta_{T_{z_j}} \right] \).

The key point is (2.2), which was arrived at in [Ke1, Ke2] by formally applying P. Griffiths’s formula for \( AJ \) to relative cycles on the Cartesian product of a smooth projective \( d \)-fold \( X \) with
\[ (\Box^n, \partial \Box^n) := \left( (\mathbb{P}^1 \setminus \{1\})^n, \cup_{j,\epsilon} \epsilon_j \left( (\mathbb{P}^1 \setminus \{1\})^{n-1} \right) \right) \].

Writing \( p_j(z_1, \ldots, z_n) := (z_1, \ldots, \hat{z}_j, \ldots, z_n) \in \Box^{n-1} \), the cubical higher Chow precycles
\[ Z^p(X, n) \subset Z^p(X \times \Box^n) / \sum_j p_j^* Z^p(X \times \Box^{n-1}) \]
are the algebraic cycles meeting arbitrary intersections of the \( X \times \iota_j^n(\Box^{n-1}) \) properly, i.e. in the expected dimension (or less). The good precycles \( Z^p(X, n) \subset Z^p(X, n) \) are those which meet \( T_{z_1}, T_{z_1} \cap T_{z_2}, \ldots, T_{z_1} \cap \cdots \cap T_{z_n} \) and their arbitrary intersections with the \( X \times \iota_j^n(\Box^{n-1}) \) properly as well [KL]. Given \( 3 \in Z^p(X, n) \), the convergence of
\[ \int_X R_3 \wedge \omega := \int_3 R_n(z) \wedge \pi^*_X \omega \]
for arbitrary \( \omega \in A^{2d-2p+n+1}(X) \) defines a current \( R_3 \in D^{2p-n-1}(X) \); indeed, this holds on the level of summands of (2.2). Similarly, one defines \( \Omega_3 \in F^pD^{2p-n}(X) \) and
\[ T_3 := (\pi_X)^* \left( (X \times T_n) \cap 3 \right) \in C^{2p-n}_{\text{top}}(X; \mathbb{Q}) \],
and according to [KLM]
\[ d[R_3] = \Omega_3 - (2\pi i)^n \delta_{T_3} - 2\pi i R_{\partial \Omega_3}. \]

In particular, given \( f_1, \ldots, f_n \in \mathcal{O}_{\text{alg}}^*(U) \) (\( U \subset X \) Zariski open) for which the Zariski closure of
\[ \Gamma_f := \{(x, f_1(x), \ldots, f_n(x)) \mid x \in U\} \subset X \times \Box^n \]
is a good precycle, each term of \( R(f_1, \ldots, f_n) \) defines an \((n-1)\)-current on \( X \).

The relative dearth of coordinate hypersurfaces in projective space makes defining a telescoping sequence of currents a greater challenge than in the cubical case. Writing \( X_0 : \cdots : X_n \) for the projective
coordinates, the closure of \( \Gamma \left( \frac{x_1}{X_0}, \ldots, \frac{x_n}{X_0} \right) \) is not even a precycle. On
its own this is not necessarily a problem, but the non-integrability of \((\log z) \frac{dz}{z}\delta_{Tz}\) against 1 on \(D_\epsilon(0)\) means that along the hyperplane at
infinity, certain terms of \(R \left( \frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0} \right)\) (for \(n \geq 3\)) fail individually to yield currents on \(\mathbb{P}^n\). This must be corrected if any sort of computation
or manipulation is to take place. Moreover, it is not at all clear how to
generalize the construction of a bounding membrane in [Ke1, Ke2].

To get around the termwise-nonconvergence problem, there are two
natural choices on \(\mathbb{P}^n\):

\[
S_n^\Delta := S_n^\Delta(X_0 : \cdots : X_n) := R_n \left( -\frac{X_1}{X_0}, -\frac{X_2}{X_1}, \ldots, -\frac{X_n}{X_{n-1}} \right)
\]
and

\[
R_n^\Delta := R_n^\Delta(X_0 : \cdots : X_n) :=
\]

\[
R_n \left( -\frac{(X_1+\cdots+X_n)}{X_0}, -\frac{(X_2+\cdots+X_n)}{X_1}, \ldots, -\frac{X_n}{X_{n-1}} \right),
\]
with \(R_n(\cdots)\) as in (2.2), interpreted in the sense of \(\int_{\mathbb{P}^n} R_n(\cdots) \wedge \omega := \lim_{\epsilon \to 0} \int_{\mathbb{P}^n \setminus U_\epsilon} R_n(\cdots) \wedge \omega\) for \(U_\epsilon\) a tubular neighborhood of the singular
set of \(R_n(\cdots)\). The first version can be more convenient for reciprocity
laws, but the second is essential for defining Abel-Jacobi maps, as we
shall discover below.

**Lemma 2.1.** Each term of \(S_n^\Delta\) and \(R_n^\Delta\) belongs to \(\mathcal{D}^{n-1}(\mathbb{P}^n)\).

*Proof.* For \(S_n^\Delta\), we remark that no \(X_j\) appears more than twice in any
term, and that occurrences are always adjacent. This produces
singularities of the form \((\log z)\delta_{Tz}\) and \((\log z) \frac{dz}{z}\) (which are integrable
against smooth forms), and exterior products of such, while prohibiting
\(\frac{dz}{z}\delta_{Tz}\) and \((\log z) \frac{dz}{z}\delta_{Tz}\).

While \(R_n^\Delta\) appears to be more complicated, it turns out to be even
better behaved. Consider the cycle

\[
\Gamma_{f[n]} \in Z^n(\mathbb{P}^n \times \Box^n)
\]
where

\[
f[n] := f[X_0 : \cdots : X_n] := \left( -\frac{X_1 + \cdots + X_n}{X_0}, -\frac{X_2 + \cdots + X_n}{X_1}, \ldots, -\frac{X_n}{X_{n-1}} \right).
\]

Its intersections with \(\mathbb{P}^n \times \Box^j(\Box^{n-1})\) (\(j = 1, \ldots, n\)) and \(\mathbb{P}^n \times \Box^0(\Box^{n-1})\) are \(\Gamma_{f[X_0, \ldots, X_k \cdots : X_n]}\) for some \(k\). Those with \(\mathbb{P}^n \times \Box^j(\Box^{n-1})\) for \(j = 1, \ldots, n - 1\) are concentrated over the loci \(\mathbb{P}^{j-1} \cong \{X_j = \cdots = X_n = 0\} \hookrightarrow \mathbb{P}^n\), since away from this set, \(X_j + \cdots + X_n = 0 \implies\) one
of $-\frac{x_{j+1} + \cdots + x_n}{x_j}, \ldots, -\frac{x_n}{x_{n-1}}$ equals 1. In fact these intersections are proper and yield degenerate cycles, of the form $(I_j \times \delta^0_j)_* p^*_j, \ldots, n \Gamma_{f^{[j-1]}}$.

We conclude that $\Gamma_{f^{[n]}}$ is a precycle.

Moreover, writing $x_i := \frac{X_i}{X_0}$, it meets $(\mathbb{C}^*)^n \times T_{z_1}, (\mathbb{C}^*)^n \times (T_{z_1} \cap T_{z_2}), \ldots, (\mathbb{C}^*)^n \times T_n$ over the subsets of $(\mathbb{C}^*)^n$ defined by: $(x_1 + \cdots + x_n) > 0; x_1, (x_2 + \cdots + x_n) > 0; x_1, x_2, (x_3 + \cdots + x_n) > 0; \ldots; x_1, \ldots, x_n > 0$. The intersections with $\Gamma_{f^{[X_0; \ldots; X_k; \ldots; X_n]}}$ in $(\mathbb{C}^*)^{n-1} \subset \{X_k = 0\}$ behave similarly, and so we conclude that $\Gamma_{f^{[n]}} \in Z(\mathbb{P}^n, n)$. It follows immediately that (term for term) $R^\Delta_n = R_{\Gamma_{f^{[n]}}}^\Delta$ is a current.

To complete either (2.6) or (2.7) to a triple, the currents

\begin{equation}
\Omega^\Delta_n := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}, \quad T^\Delta_n := (\mathbb{R} > 0)^n.
\end{equation}

on $\mathbb{P}^n$ will be needed.

**Lemma 2.2.** We have $T^\Delta_n = T_n \left( -\frac{X_1}{X_0}, \ldots, -\frac{X_n}{X_{n-1}} \right) = T_n \left( f^{[n]} \right)$ and $\Omega^\Delta_n = \Omega_n \left( -\frac{X_1}{X_0}, -\frac{X_2}{X_1}, \ldots, -\frac{X_n}{X_{n-1}} \right) = \Omega_n (f^{[n]})$.

**Proof.** For the chains the first equality is clear, and $T^\Delta_n = T_n (f^{[n]})$ follows from the proof of 2.1. To illustrate the latter point for $n = 3$: in $T_3(f^{[3]}) = T_{- (x_1 + x_2 + x_3)} \cap T_{- (x_2 + x_3)} \cap T_{- \frac{x_2}{x_1}}$, we have $x_1 + x_2 + x_3 = a$, $x_2 + x_3 = bx_1$, and $x_3 = cx_2$ where $a, b, c > 0$. Hence $x_1 = \frac{a}{1+b} > 0$, $x_2 = \frac{b}{1+c}x_1 > 0$, and $x_3 = cx_2 > 0$. The reverse inclusion is clear.

The two equalities of $(n, 0)$-currents follows, via the dog log map from symbols to forms, from the following computation in Milnor $K$-theory of $\mathbb{C}(x_1, \ldots, x_n)$:

\[\left\{ -\frac{(X_1 + \cdots + X_n)}{X_0}, \frac{(X_2 + \cdots + X_n)}{X_1}, \ldots, -\frac{(X_{n-1} + X_n)}{X_{n-2}}, -\frac{X_n}{X_{n-1}} \right\} = \]
\[\left\{ -\frac{X_1}{X_0}, \frac{X_2}{X_1}, \left( 1 + \frac{X_3}{X_2} (1 + \cdots) \right), -\frac{X_2}{X_1}, \left( 1 + \frac{X_3}{X_2} (1 + \cdots) \right), \ldots, -\frac{X_{n-1}}{X_{n-2}}, \left( 1 + \frac{X_n}{X_{n-1}} \right), -\frac{X_n}{X_{n-1}} \right\} = \]
\[\left\{ -\frac{X_1}{X_0}, -\frac{X_2}{X_1}, \left( 1 + \frac{X_3}{X_2} (1 + \cdots) \right), -\frac{X_3}{X_2}, (1 + \cdots), -\frac{X_n}{X_{n-1}} \right\} = \cdots = \]
\[\left\{ -\frac{X_1}{X_0}, -\frac{X_2}{X_1}, -\frac{X_3}{X_2}, \ldots, -\frac{X_n}{X_{n-1}} \right\} = \cdots = \left\{ -\frac{X_1}{X_0}, -\frac{X_2}{X_1}, \ldots, -\frac{X_n}{X_0} \right\}, \]

where we have used in particular the relations $\{\ldots, a, \ldots, -a, \ldots\} = 1 = \{\ldots, a, \ldots, 1 - a, \ldots\}$. □
This brings us to the main point. Writing \( \rho_j [\xi_0 : \cdots : \xi_{n-1}] := [\xi_0 : \cdots : \xi_{j-1} : 0 : \xi_j : \cdots : \xi_{n-1}] \) for the inclusion of \( \{X_j = 0\} \) in \( \mathbb{P}^n \), we have

**Proposition 2.3.** Let \( \mathcal{R}_n \) stand for \( R^\Delta_n \) or \( S^\Delta_n \). Then \( d [\mathcal{R}_n] = \Omega^\Delta_n - (2\pi i)^n \delta_{T^\Delta_n} + 2\pi i \sum_{j=0}^n (-1)^j (\rho_j) \cdot \mathcal{R}_{n-1} \).

**Proof.** The computation of \( \Gamma_{f[n]} \cdot (\mathbb{P}^n \times t^\ell_k (\square^{n-1})) \) in the proof of Lemma 2.1 implies
\[
\partial_B \Gamma_{f[n]} = \sum (-1)^j (\rho_j) \cdot \Gamma_{f[n-1]},
\]
which together with (2.3) gives the result for \( R^\Delta_n \).

For \( S^\Delta_n \), the correct residues are suggested by the corresponding tame symbols in Milnor \( K \)-theory:
\[
\left\{ \frac{-X_1}{X_0}, \ldots, \frac{-X_{\ell}}{X_{\ell-1}}, \ldots, \frac{-X_n}{X_{n-1}} \right\} \equiv \left\{ \frac{-X_1}{X_0}, \ldots, \frac{-X_{\ell}}{X_{\ell-1}}, \ldots, \frac{-X_n}{X_{n-1}} \right\}.
\]

Since \( \Gamma \left( -\frac{x_1}{x_0}, \ldots, -\frac{x_n}{x_{n-1}} \right) \) isn’t a precycle, we must compute explicitly:

\[
S^\Delta_n (X_0 : \cdots : X_n) =
\]

\[
S^\Delta_{\ell-1} (X_0 : \cdots : X_{\ell-1}) \wedge \Omega^\Delta_{n-\ell+1} (X_{\ell-1} : X_\ell : \cdots : X_n)
\]

\[
+ (2\pi i)^{\ell-1} \delta_{T^\Delta_{\ell-1} (X_0 : \cdots : X_{\ell-1})} \cdot S^\Delta_2 (X_{\ell-1} : X_\ell : X_{\ell+1}) \wedge \Omega^\Delta_{n-\ell+1} (X_{\ell+1} : \cdots : X_n)
\]

\[
+ (2\pi i)^{\ell+1} \delta_{T^\Delta_{\ell+1} (X_0 : \cdots : X_{\ell+1})} \cdot S^\Delta_{n-\ell-1} (X_{\ell+1} : \cdots : X_n);
\]

\[
(1)^{\ell} \text{Res}(X_\ell) \text{ of which is}
\]

\[
S^\Delta_{\ell-1} (X_0 : \cdots : X_{\ell-1}) \wedge \Omega^\Delta_{n-\ell} (X_{\ell-1} : X_{\ell+1} : \cdots : X_n)
\]

\[
+ (2\pi i)^{\ell-1} \delta_{T^\Delta_{\ell-1} (X_0 : \cdots : X_{\ell-1})} \cdot \log \left(-\frac{X_{\ell+1}}{X_{\ell-1}}\right) \Omega^\Delta_{n-\ell-1} (X_{\ell+1} : \cdots : X_n)
\]

\[
+ (2\pi i)^{\ell} \delta_{T^\Delta_{\ell+1} (X_0 : \cdots : X_{\ell+1})} \cdot S^\Delta_{n-\ell-1} (X_{\ell+1} : \cdots : X_n) .
\]

\[
= S^\Delta_{n-1} (X_0 : \cdots : X_{\ell} : \cdots : X_n). \quad \text{Here the residue of } \Omega^\Delta_{n-\ell+1} \text{ follows from the } K \text{-theory computation, and the boundary of } T^\Delta_{\ell+1} (X_0 : \cdots : X_{\ell+1}) \text{ from the fact that it is just the closure of } \{x_1, \ldots, x_{\ell+1} \in \mathbb{R}_{>0}\}.
\]

Finally, using the fact that \( \log \left( \frac{f}{g} \right) = \log(f) - \log(g) \) where \( g \in \mathbb{R}_{>0} \), we have \( S^\Delta_2 (X_0 : X_1 : X_2) = \)

\[
\log (-x_1) \frac{dx_2}{x_2} + \left\{ -\log (-x_1) \frac{dx_1}{x_1} + 2\pi i \log(x_1) \delta_{T-x_1} \right\} - 2\pi i \log (-x_2) \delta_{T-x_2} .
\]

One checks that \( d \) of the bracketed current is zero, and so the only contribution to \( \text{Res}(X_1) \) (namely, \(-\log (-x_2)\)) comes from the last term. \( \square \)
For a dose of concreteness, here is a simple computation involving $S^\Delta_3$.

**Example 2.4.** Let $3 \subset \mathbb{P}^5$ be the $\mathbb{P}^2$ obtained by projectivizing the row space of
\[
\begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & -1 & 1 & 0 & 0 \\
a & -1 & 0 & 0 & 1 & 0
\end{pmatrix},
\]
with coordinates $[Y_0 : Y_1 : Y_2]$. Writing $z := \frac{Y_1}{Y_0}, w := \frac{Y_2}{Y_0}$, the pullback of $R_5\left(-\frac{X_1}{X_0}, -\frac{X_2}{X_1}, -\frac{X_3}{X_2}, -\frac{X_4}{X_3}, -\frac{X_5}{X_4}\right)$ to $3$ takes the form
\[
R_5\left(\frac{z-w}{1-aw}, \frac{z-1}{z-w}, \frac{w}{z}, \frac{1}{w}\right).
\]
$T_{\frac{z-w}{1-aw}} \cap T_{\frac{z-1}{z-w}}$ is a triangular membrane bounding on $z = w$, $z = 1$ and $w = \frac{1}{a}$, and for $a \not\in (1, \infty)$ we have $T_{\frac{z-w}{1-aw}} \cap T_{\frac{z-1}{z-w}} \cap T_{\frac{z}{z-1}} = \emptyset$. Hence the last two terms are zero on $3$, as are the first two by Hodge type, and
\[
\frac{1}{(2\pi i)^2} \int_3 S^\Delta_3 = \int_{u=1}^{a} \int_{v=1}^{w} \log\left(\frac{z}{z-1}\right) d\log\left(\frac{w}{z}\right) \wedge d\log\left(\frac{1}{w}\right).
\]
Substituting $u = \frac{1}{z}, v = \frac{1}{w}$, the above
\[
= -\int_{u=1}^{a} \int_{v=1}^{u} \log(1-u) d\log(u) \wedge d\log(v)
\]
\[
= \int_{v=1}^{u} \left( Li_2(v) - Li_2(1) \right) d\log(v)
\]
\[
= Li_3(a) - Li_3(1) - \log(a) Li_2(1)
\]
\[
= Li_3(a) - \zeta(3) - \frac{\pi^2}{6} \log(a).
\]

### 3. Abel-Jacobi maps for simplicial higher Chow groups

Let $X$ be a smooth projective variety. The complex
\[
C^\bullet_\ast(X; \mathbb{Q}(p)) := C^\bullet_{\text{top}} (X; (2\pi i)^p \mathbb{Q}) \oplus F^p D_\ast(X) \oplus D_{\ast-1}(X)
\]
of abelian groups with differential
\[
D (T, \Omega, R) := (-\partial T, -d[\Omega], d[R] - \Omega + \delta_T)
\]
computes the Deligne cohomology
\[
H^\ast_D (X; \mathbb{Q}(p)) := H^\ast \{ C^\bullet_\ast (X; \mathbb{Q}(p)) \}.
\]
These latter spaces are the targets for the $AJ$ (rational regulator) maps, whose explicit construction on the *simplicial* higher Chow complex is the subject of this section.
The idea is to replace $(\square^n, \partial \square^n)$ in the KLM-construction by

$$(\Delta^n, \partial \Delta^n) := \left( \mathbb{P}^n \setminus H_n, \bigcup_{j=0}^{n} \rho_j \left( \mathbb{P}^{n-1} \setminus H_{n-1} \right) \right)$$

where $H_n$ is the special hyperplane cut out by $X_0 + \cdots + X_n = 0$. We then define precycles (resp. good precycles)

$$Z^p_{\Delta, R}(X, n) \subset Z^p_{\Delta}(X, n) \subset Z^p(X \times \Delta^n)$$

to be those cycles meeting arbitrary intersections of the $X \times \rho_j(\Delta^{n-1})$ (resp. of these and the $T_{X_1 + \cdots + X_n} \cap T_{X_2 + \cdots + X_n}$, etc.) properly. The Bloch boundary map

$$(3.1) \quad \partial_b 3 := \sum_{j=0}^{n} (-1)^j \rho^*_j 3$$

makes these into quasi-isomorphic complexes:

**Proposition 3.1.** $H_n \left( Z^p_{\Delta, R}(X, \bullet) \right) \cong H_n \left( Z^p_{\Delta}(X, \bullet) \right) \cong CH^p(X, n)$

We shall define a morphism

$$(3.2) \quad Z^p_{\Delta, R}(X, -\bullet) \xrightarrow{AJ_{\Delta, X}} C^{2p+\bullet}(X; \mathbb{Q}(p)),$$

which then automatically induces (simplicial) $AJ$ maps

$$(3.3) \quad AJ_{\Delta, X} : CH^p(X, n) \longrightarrow H^{2p-n}_{D}(X, \mathbb{Q}(p)).$$

Namely, writing $\pi_\Delta, \pi_X$ for projections from the desingularization $\tilde{3}$ to $\Delta^n, X$, we set

$$R^\Delta_3 := (\pi_X)_* (\pi_\Delta)^* R_n^\Delta, \quad \Omega^\Delta_3 := (\pi_X)_* (\pi_\Delta)^* \Omega_n^\Delta,$$

$$T^\Delta_3 := (\pi_X)_* \left( (X \times T_n^\Delta) \cap \tilde{3} \right),$$

and

$$(3.4) \quad AJ_{\Delta, X}^p(\tilde{3}) := (2\pi i)^{p-n} \left( (2\pi i)^n T^\Delta_3, \Omega^\Delta_3, R^\Delta_3 \right).$$

**Theorem 3.2.** $AJ_{\Delta, X}^p$ is a well-defined morphism of complexes. The induced maps $AJ_{\Delta, X}^p$ recover Bloch’s cycle-class maps (in the sense of Definition 3.6 below).

---

The proof is deferred to the first Appendix to this section so as not to interrupt the main flow of ideas.
The proof is simple but somewhat formal, and so we shall preface it with a (probably more helpful) direct argument that (3.3) induces a map of complexes. First there is the question of whether it is well-defined, which splits into “algebraic” and “analytic” parts. The latter issue, of whether \( R^\Delta \) and \( \Omega^\Delta \) are actually in \( D^{2p-n-1}(X) \) resp. \( F^p D^{2p-n}(X) \) (since pullbacks \( \pi^*_\Delta \) need not preserve currents), is implicitly resolved in the proof below (by the relation to the cubical KLM currents). For reference, we have also included an explicit argument that \( R^\Delta \) is a current in the second appendix to this section.

Now \( Z^p(X \times \Delta^n) = Z^p(X \times \mathbb{P}^n) / Z^p(X \times H_n) \), and the “algebraic” well-definedness refers to the requirement that \( \tilde{AJ}^{p,n}_{\Delta,X} \) vanish on admissible precycles with support in \( X \times H_n \). In fact it suffices to check the following, writing \( \mathfrak{F} \in Z^p(X \times \mathbb{P}^{n+1}) \) for the closure of \( \mathfrak{F} \in Z^p(X \times \Delta^{n+1}) \). Given \( j \in \{0, \ldots, n+1\} \), \( \mathfrak{F} \in Z^p_{\Delta,R}(X, n+1) \), and \( \mathfrak{W} \) an irreducible component of \( \rho^*_\Delta \mathfrak{F} \) sitting inside \( X \times H_n \), we must have \( R^\Delta, \Omega^\Delta, T^\Delta \) all zero. But on \( H_n \) we have \( X_0 + \cdots + X_n \equiv 0 \), and on \( \mathfrak{W} \) we cannot have all \( X_i \equiv 0 \). If (say) \( X_0|_{\mathfrak{W}} \equiv \cdots \equiv X_{k-1}|_{\mathfrak{W}} \equiv 0 \) but \( X_k \) is not identically zero, then \(- x_{k+1} + \cdots + x_n |_{\mathfrak{W}} \equiv 1 \) and the currents are trivial as desired.

To verify that \( \tilde{AJ}^{p,n}_{\Delta,X} \) is a morphism of complexes, we use the formula in Proposition 2.3. This gives for each \( \mathfrak{F} \in Z^p_{\Delta,R}(X, n) \)

\[
d[R^\Delta] = \Omega^\Delta - (2\pi i)^n \delta_{T^\Delta} + 2\pi i \sum_{j=0}^{n} (-1)^j R^\Delta_{\rho_j^* \mathfrak{F}}
\]

(3.5)

while

\[
\begin{cases}
\partial T^\Delta_n = \sum_{j=0}^{n} (-1)^{j-1} \rho_j (T^\Delta_{n-1}) \\
 d[\Omega^\Delta_n] = 2\pi i \sum_{j=0}^{n} (-1)^{j-1} (\rho_j)_* \Omega^\Delta_{n-1}
\end{cases}
\]

\[
\Rightarrow
\begin{cases}
\partial T^\Delta_3 = -T^\Delta_{\partial_3} \\
 d[\Omega^\Delta_3] = -2\pi i \Omega^\Delta_{\partial_3}
\end{cases}
\]

and so

\[
D \left( (2\pi i)^n T^\Delta_3, \Omega^\Delta_3, R^\Delta_3 \right) = 2\pi i \left( (2\pi i)^{n-1} T^\Delta_{\partial_3}, \Omega^\Delta_{\partial_3}, R^\Delta_{\partial_3} \right)
\]

which yields \( D \circ \tilde{AJ} = \tilde{AJ} \circ \partial_3 \) as needed.

As mentioned in the Introduction, the simplicial \( AJ \) formula will be particularly natural for linear higher Chow cycles derived from elements of \( H_{2n-1}(GL_n(K), \mathbb{Z}) \) \( (K \) a number field). While we won’t pursue this application in the present paper, here is an example of what this will look like on an irreducible component of such a cycle.
Example 3.3. Let $\alpha \in \mathbb{C}\setminus \mathbb{R}_{\leq 0}$, and consider the linear precycle $3 := \{[\alpha Z - W : Z : -Z : W] \mid [Z : W] \in \mathbb{P}^1(\mathbb{C})\} \in Z_{\Delta, r}^2(Spec \mathbb{C}, 3),$

for $\alpha \in \mathbb{C}\setminus [1, \infty)$. It has boundary $\partial 3 = [1 : -1 : \alpha] - [\alpha : 1 : -1]$, and should be thought of as a simplicial analogue of the Totaro (pre)cycle.

Writing $z := \frac{z}{W}$ for the coordinate on $P^1$, $R_3^\Delta \in Ext^1_{MHS}(\mathbb{Q}(0), \mathbb{Q}(2)) \cong \mathbb{C}/\mathbb{Q}(2)$ is computed by

$$\int_{\partial 3} R_3^\Delta = \int_{P^1} R \left( -\frac{W}{\alpha Z - W}, -\frac{W - Z}{Z} \right)$$

$$= \int_{P^1} R \left( \frac{1}{1 - \alpha z}, 1 - \frac{1}{z}, \frac{1}{z} \right).$$

Since $T_{\frac{1}{1 - \alpha z}} = \left( \frac{1}{\alpha}, \infty \right) := \left\{ \frac{r}{\alpha} \mid r \in \mathbb{R}_{> 0} \right\}$ (oriented from $\frac{1}{\alpha}$ to $\infty$) and $T_{\frac{1}{1 - \alpha z}} \cap T_{\frac{1}{z}} = (\frac{1}{\alpha}, \infty) \cap (0, 1) = \emptyset$, this

$$= \int_{T_{\frac{1}{1 - \alpha z}}} \log \left( 1 - \frac{1}{z} \right) d\log \left( \frac{1}{z} \right)$$

$$= Li_2(\alpha).$$

Remark 3.4. The currents $S_n^\Delta$ are closer than the $R_n^\Delta$ to being invariant with respect to scaling the coordinates, which apparently makes them more suitable for studying reciprocity laws and functional equations of polylogarithms. However, they fail to yield well-defined $AJ$ maps, as they do not vanish on $H_n$.

The real $(n - 1)$-currents $r_n$ of $[Go1]$ more dramatically illustrate the problem, as they are actually invariant under scaling of coordinates, and are prevented by this property from vanishing on $H_n$, and hence from defining simplicial $AJ$ maps as claimed in [op. cit.]. That they do nevertheless produce $AJ$ maps on the cubical level, coinciding with the real or imaginary part of Bloch’s invariants, was checked in [Ke1, sec. 3.1.1].

It is instructive to demonstrate the issue for $r_3$. Let $a \in \mathbb{C}\setminus \mathbb{R}$. According to [Go1 Thm. 3.6],

$$\int_{P^1} r_3(z, 1 - z, z - a) = \frac{1}{2\pi i} D_2(a)$$
where $D_2$ is the Bloch-Wigner function. By [Go1, Prop. 3.2], this
\[
= \int_{\mathbb{P}^1} r_3 (z, 2 - 2z, z - a) = \int_{\mathbb{P}^1} r_3 \left( \frac{z}{a - 2}, \frac{2 - 2z}{a - 2}, \frac{z - a}{a - 2} \right)
= \int_{\mathbb{P}^1} r_3 (x_1, x_2, x_3),
\]
where $\mathcal{Z}_a = \{(a - 2 : z : 2 - 2z : z - a | z \in \mathbb{P}^1)\}$. But $(a - 2) + z + (2 - 2z) + (z - a) \equiv 0 \implies \mathcal{Z}_a \subset H_3 \implies \mathcal{Z}_a = 0 \in Z^2(\mathbb{C}, 3)$. So $3 \mapsto \int_3 r_3$
does not induce a well-defined map $\tilde{r} : Z^2(\mathbb{C}, 3) \to \mathbb{R}(1)$.

If one tries to make $\tilde{r}$ well-defined by insisting that it be “zero on zero”, another problem emerges: we do not obtain a map of complexes
\[
\cdots \to Z^2(\mathbb{C}, 4) \to Z^2(\mathbb{C}, 3) \to Z^2(\mathbb{C}, 2) \to \cdots \quad \downarrow \quad \tilde{r} \quad \downarrow \quad \cdots \to \quad \mathbb{R}(1) \quad \to \quad 0 \quad \to \quad \cdots.
\]
If we take $\mathcal{Z} := \{(V : (a - 2)(W + V) : Z - V : 2(W - Z) : Z - aW) | V : W : Z \in \mathbb{P}^1\}$
in $Z^2(\mathbb{C}, 4)$, then $\partial_0 \mathcal{Z} = \sum_{j=1}^4 (-1)^j \rho_j^0 \mathcal{Z}$ since $\rho_0^0 \mathcal{Z} = \mathcal{Z}_a = 0$. But the reciprocity properties of $r_3$ imply $\sum_{j=0}^4 (-1)^j \int_{\rho_j^0 \mathcal{Z}} r_3 = 0$, which gives
\[
\tilde{r}(\partial_0 \mathcal{Z}) = \sum_{j=1}^4 (-1)^j \int_{\rho_j^0 \mathcal{Z}} r_3 = - \int_{\rho_0^0 \mathcal{Z}} r_3 = - \frac{1}{2\pi i} D_2(a) \neq 0.
\]
So apparently, the only way to fix the problem is to replace $r_3 = r_3 \left( \frac{X_1}{X_0}, \frac{X_2}{X_0}, \frac{X_3}{X_0} \right)$ by something like $r_3 \left( - \frac{X_1 + X_2 + X_3}{X_0}, - \frac{X_2 + X_2}{X_1}, - \frac{X_3}{X_1} \right)$, which
affects its properties and calls into question (for example) the known proof that linear higher Chow groups of number fields surject onto the usual higher Chow cycles.

**Proof of Theorem 3.2** We shall need the subcomplexes of normalized precycles
\[
N^p_\Delta (X, n) := \bigcap_{j=0}^{n-1} \ker \rho_j^* \subset Z^p_\Delta (X, n)
\]
\[
N^p_{\Delta, \mathbb{R}} (X, n) := N^p_\Delta (X, n) \cap Z^p_{\Delta, \mathbb{R}} (X, n)
\]
and the following “moving lemma” (verified in the first appendix to this section):

---

8In fact, for our purposes it suffices to know that the integral is nonzero. This reduces to nonvanishing of $\int_{\mathbb{P}^1} \log |z - a| \log |z| \wedge \log |1 - z|$, which follows from that of $\int_{\mathbb{R}^2} \frac{\log |z - a| \log |z| \wedge \log |1 - z|}{|z|^{\gamma - \frac{d-1}{2}}} dA$ for $a \notin \mathbb{R}$.

9cf. Prop. 16 in [dJ]; we do expect that this can be fixed.
Proposition 3.5. \( H_n(N^p_\Delta, R) \cong H_n(N^p, X, \bullet) \cong H_n(Z^p_\Delta, X, \bullet) \).

With this, we may define the Bloch cycle-class map:

Definition 3.6. Let \( \xi \in CH^p(X, n) \) have normalized representative \( Z \in \ker(\partial) \subset N^p_\Delta(X, n) \); that is, all \( \rho^*_j Z = 0 \). Denoting \( X \times \Delta^n =: \Delta^n_X \), etc., the localization sequence for

\[(U^\Delta, \partial U^\Delta) := (\Delta^n_X \setminus |Z|, \partial \Delta^n_X \setminus \{|Z| \cap \partial \Delta^n_X\})\]

leads to an extension (with \( \mathbb{Q}(p) \)-coefficients)

\[
\begin{array}{c}
H^{2p-n-1}(X) \xrightarrow{\text{\(A(\lambda, \sigma)\)}} \mathbb{P}^\Delta \xrightarrow{\text{\(Q(-p) = \langle 3 \rangle\)}} \mathbb{Q}(\langle p \rangle)
\end{array}
\]

We define \( c_B(\xi) \in Ext^1_{\text{MHS}}(\mathbb{Q}(0), H^{2p-n-1}(X, \mathbb{Q}(p))) \cong H^{2p-n}_{\mathbb{Q}}(X, \mathbb{Q}(p)) \) to be the extension class of the top sequence.

The proof of the Theorem will now proceed in the three steps:

Step 1: The cube-to-simplex map. Recall that \( \Gamma_{f_{[n]}} \) (cf. (2.8)) is the restriction to \( \mathbb{P}^n \times \square^n \) of the correspondence in \( \mathbb{P}^n \times (\mathbb{P}^1)^n \) given by

\[
\begin{pmatrix}
\lambda_1 & \sigma_1 & \cdots & \sigma_1 & \sigma_1 \\
0 & \lambda_2 & \sigma_2 & \cdots & \sigma_2 & \sigma_2 \\
0 & 0 & \lambda_3 & \cdots & \sigma_3 & \sigma_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_n & \sigma_n
\end{pmatrix}
\]

\[
A(\lambda, \sigma)
\]

\[
\begin{bmatrix}
X_0 \\
X_1 \\
\vdots \\
X_n
\end{bmatrix}
\]

where \([\sigma_j : \lambda_j]\) are projective coordinates on \( \mathbb{P}^1_{z_j} \) \((z_j = \frac{\lambda_j}{\sigma_j})\). Observe that the \((n+1) \times (n+1)\) matrix \(B(\lambda, \sigma)\), obtained by adding a row of ones to \(A(\lambda, \sigma)\), has

\[
\det B(\lambda, \sigma) = \prod_{j=1}^n (\sigma_j - \lambda_j).
\]

Now (3.8) implies that

- \(A(\lambda, \sigma)\) has maximal rank, so that \(\Gamma_{f_{[n]}}\) induces a well-defined morphism from \(\square^n\) to \(\mathbb{P}^n\); and
- \(B(\lambda, \sigma)|X = 0\) has no nonzero solution, so that the image of this map avoids the hyperplane \(H_n\) (where \(X_0 + \cdots + X_n = 0\)).
We shall write

\[ F_n : \square^n \to \Delta^n \]

for this morphism and \( \Gamma_n \in Z^n(\Delta^n \times \square^n) \) for the associated correspondence. The explicit formula

\[
(3.9) \quad F_n ([\sigma_1 : \lambda_1], \ldots, [\sigma_n : \lambda_n]) = [\sigma_1(\sigma_2 - \lambda_2) \cdots (\sigma_n - \lambda_n) : \\
- \lambda_1 \sigma_2(\sigma_3 - \lambda_3) \cdots (\sigma_n - \lambda_n) : \lambda_1 \lambda_2 \sigma_3(\sigma_4 - \lambda_4) \cdots (\sigma_n - \lambda_n) \\
: \cdots : (-1)^{n-1} \lambda_1 \cdots \lambda_{n-1} \sigma_n : (-1)^n \lambda_1 \cdots \lambda_n]
\]

makes it clear that \( F_n(\partial \square^n) \subset \partial \Delta^n \). The induced map

\[
(F_n)_*: H_n(\square^n, \partial \square^n) \to H_n(\Delta^n, \partial \Delta^n)
\]

is an isomorphism since it sends representatives \( T_n \mapsto T_n^\Delta \), \( \Omega_n \mapsto \Omega_n^\Delta \). (This is essentially the same computation as in Lemma 2.2.) Hence for any (smooth projective) \( X \), denoting \( X \times \square^n =: \square^n_X \), etc.,

\[
F_{X,n}^*: H^m(\Delta^n_X, \partial \Delta^n_X) \to H^m(\square^n_X, \partial \square^n_X)
\]

is an isomorphism for any \( m \).

**Step 2 : Simplicial to cubical precycles.** The morphism \( F_n \) is the composition of an inclusion (of \( \square^n \) into a larger open subset of \( \mathcal{P}^1 \)) with a sequence of blow-ups at the smooth centers: \( X_1 = \cdots = X_n = 0 \); and (successive proper transforms of) \( X_2 = \cdots = X_n = 0, \ldots, X_{n-1} = X_n = 0 \). Its positive-dimensional fibers are contained in \( \cup_{j=1}^{n-1} I_j(\square^n - 1) \) and are degenerate in the sense that one or more \( z_i \)'s (in fact, \( z_j+1 \) thru \( z_n \)) are arbitrary. For cycles on \( X \times \Delta^n \) meeting the blow-up centers properly (which includes \( Z^n_\Delta(X, n) \)), the pullback under \( \text{id}_X \times F_n : X \times \square^n \to X \times \Delta^n \) is well-defined.\(^{10}\)

This yields a map

\[
\Gamma_{X,n}^*: Z^n_\Delta(X, n) \to Z^n(X \times \square^n)
\]

which we claim factors through \( Z^n_\square(X, n) \).

Let \( \tau \) be one of the real chains \( T_{x_1} \cap \cdots \cap T_{x_k} \cap \iota^\Delta_j(\square^n - |I|) \), say of (real) codimension \( c \). Inspection of (3.7) and (3.9) shows that \( \tau^\Delta := F_n(\tau) \) is one of the real chains \( T_{x_1 + \cdots + x_n} \cap \cdots \cap T_{x_r + \cdots + x_n} \cap \rho_I(\Delta^n - |I|) \), of codimension \( c^\Delta \geq c \). Since \( |\square| \cap (X \times \tau) \subset F^{-1}_n \left( |\square| \cap (X \times \tau^\Delta) \right) \), we have

\[
\dim_\mathbb{R} \left( |\square| \cap (X \times \tau) \right) \leq (c^\Delta - c) + \dim_\mathbb{R} \left( |\square| \cap (X \times \tau^\Delta) \right).
\]

\(^{10}\)That is, the “preimage” \( (\pi_1 \iota_j)_* (\pi_{12}^\Delta \cdot \pi_{23}^\Delta \Gamma_n) \) of an irreducible \( \square \) (where \( \pi_{ij} \) are the projections on \( X \times \Delta^n \times \square^n \)) already yields the proper transform, without having to throw out “exceptional” components contained in \( \cup_{j=1}^{n-1} I_j(\square^n - 1) \).
Moreover, as $Z \in Z_p \Delta X$, we have codim$^{[3]}_R \left( |Z| \cap (X \times \Delta) \right) \geq c \Delta$; it follows that codim$^{[3]}_R \left( |Z| \cap (X \times \tau) \right) \geq c$. Since $\tau$ was arbitrary, $Z \in Z_p \Delta X$.

Next we claim that

$$\Gamma^*_X : Z^p_{\Delta, R}(X, \bullet) \to Z^p_{\Delta}(X, \bullet)$$

is a map of complexes, i.e. that

$$\Gamma^*_X, n-1(\partial_B Z) = \sum_{j=0}^{n} (-1)^j \Gamma^*_n \rho^*_j Z$$

and

$$\partial_B(\Gamma^*_X Z) = \left[ -\sum_{j=1}^{n} (-1)^j (\iota_j^\infty)^* Z + (-1)^n (\iota^0_n)^* Z \right] + \sum_{j=1}^{n-1} (-1)^j (\iota_j^0)^* Z.$$  \hspace{1cm} (3.10)

agree. Inspection of (3.9) shows that $F_n$ restricts to $F_{n-1}$ on the facets $\iota_j^\infty(\square^{n-1}) (\to \rho_{j-1}(\Delta^{n-1}))$ (for $j = 1, \ldots, n$) and $\iota_n^0(\square^{n-1}) (\to \rho_n(\Delta^{n-1}))$, so that the right-hand side of (3.10) coincides with the square-bracketed term in (3.11). The restrictions of $F_n$ to the other facets map $\iota_j^0(\square^{n-1}) \to \rho_{j,\ldots,n}(\Delta^{j-1})$ (with degenerate fibers as mentioned above) for any $j = 1, \ldots, n-1$. Since $Z$ meets these $X \times \rho_{j,\ldots,n}(\Delta^{j-1})$ properly (in complex codim. $\geq n-j+1$), $Z$ meets $X \times \iota_j^0(\square^{n-1})$ in the $\psi_j$-image, which is degenerate. So the remaining terms on the right-hand side of (3.11) are zero in $Z^p_{\Delta}(X, n)$.

**Step 3: AJ and Bloch’s map.** Let

$$\widetilde{AJ}^p_X : Z^p_{\Delta}(X, \bullet) \to C^{2p+\bullet}(X; \mathbb{Q}(p))$$

be the map of complexes defined by sending a precycle $W \in Z^p_{\Delta}(X, n)$ to $(2\pi i)^{p-n} ((2\pi i)^n T_W, \Omega_W, R_W)$ \cite{KLM}. We claim that the composition

$$Z^p_{\Delta, \Delta}(X, \bullet) \xrightarrow{\Gamma_X} Z^p_{\Delta}(X, \bullet) \xrightarrow{\widetilde{AJ}^p_X} C^{2p+\bullet}(X; \mathbb{Q}(p))$$

is none other than the $\widetilde{AJ}^p_{\Delta, X}$ of (3.4), proving the first statement of Theorem 3.2. The point is that from Lemmas 2.1, 2.2 we have

\footnote{See the beginning of §2 for $T_W, \Omega_W, R_W$.}
\[ \Gamma_n(T_n, \Omega_n, R_n) = (T_n^\Delta, \Omega_n^\Delta, R_n^\Delta) \] so that

\[
\begin{align*}
(T_n^\Delta, \Omega_n^\Delta, R_n^\Delta) &= (3^\square)^* (T_n, \Omega_n, R_n) \\
&= 3^\square \Gamma^* (T_n, \Omega_n, R_n) \\
&= 3^\square (T_n^\Delta, \Omega_n^\Delta, R_n^\Delta) \\
&= (T_3^\Delta, \Omega_3^\Delta, R_3^\Delta)
\end{align*}
\]

(where the pullbacks of currents are well-defined by those lemmas and by \([KLM]\)).

Finally we let \(3\) be a normalized (simplicial) precycle as in Definition 3.6, with class \(\xi\). By the analysis in Step 2, we have that \(3^\square := \Gamma_{X,n}^* 3 \in Z_p^\square(X, n)\) belongs to \(\bigcap_i \ker(i^\square)\). Note that we may have \(3 \neq 0\) but \(3^\square = 0\). In this case, \(\Gamma_{X,n}\) yields a map from \((\square^n_X, \partial \square^n_X) \to (U^\Delta, \partial U^\Delta)\), which produces a splitting \(E^\square \to H^{2p-1}(\square^n_X, \partial \square^n_X) = H^{2p-n-1}(X)\). Hence \(c_B(\xi) = 0 = [AJ_X^p(0)] = [AJ_{\Delta X}^p(3)]\), finishing the proof in this case.

So assume that \(3^\square\) is nonzero. Writing

\[
(U^\square, \partial U^\square) := \left(\square^n_X \setminus \{3^\square\}, \partial \square^n_X \setminus \{3^\square \cap \partial \square^n_X\}\right),
\]

we get an extension

\[
\begin{array}{c}
H^{2p-1-n}(X) \ar[rr] \ar[dd] & & E^\square \ar[rr] \ar[dd] & & Q(-p) = 3^\square \\
\downarrow & & \downarrow & & \downarrow \\
H^{2p-1}(\square^n_X, \partial \square^n_X) & \ar[rr] & & H^{2p-1}(U^\square, \partial U^\square)
\end{array}
\]

analogous to (3.6). In fact, \(\Gamma_{X,n}\) restricts to a map from \(U^\square \to U^\Delta\) sending \(\partial U^\square \to \partial U^\Delta\), hence induces a map from the bottom row of (3.6) to the bottom row of (3.12). By the end of Step 1, this is an isomorphism on the left-hand terms. Since \(3^\square = \Gamma_{X,n} 3\), it also sends the \(Q(0)\) to the \(Q(0)\) and so gives an isomorphism of the top rows.

Hence \(c_B(\xi)\) is the extension class also of the top row of (3.12), which by \([KLM]\) Thm. 7.1 is computed by \(AJ_X^p(3^\square)\). Since \(AJ_{\Delta X}^p(3) = AJ_X^p(3^\square)\), we are done.

Remark 3.7. A (much longer) direct proof of Theorem 3.2 could also be given, basically by repeating the argument in §5.8 and §7 of \([KLM]\) in the simplicial setting.

Appendix I to §3: proof of moving lemmas 3.1-3.5. We preface the actual proof with some simplicial algebra. Recall the face maps \(\rho_i : \Delta^{n-1} \hookrightarrow \Delta^n\) and define degeneracy maps \(\sigma_i : \Delta^{n+1} \to \Delta^n\) by

\[
[X_0 : \cdots : X_{n+1}] \mapsto [X_0 : \cdots : X_{i-1} : X_i + X_{i+1} : X_{i+2} : \cdots : X_{n+1}].
\]
For all $i = 0, \ldots, n$, set
\[ \partial_i := (\text{id}_X \times \rho_i)^* : Z^p_\Delta(X, n) \to Z^p_\Delta(X, n - 1) \]
(so that $\partial = \sum_{i=0}^n (-1)^i \partial_i$) and
\[ s_i := (\text{id}_X \times \sigma_i)^* : Z^p_\Delta(X, n) \to Z^p_\Delta(X, n + 1). \]

One has the relations
\[
\begin{align*}
\partial_i \partial_j &= \partial_j - s_j \partial_i \quad \text{if} \quad i < j \\
\partial_i s_j &= \text{Id} \quad \text{if} \quad i = j \text{ or } i = j + 1 \\
\partial_i s_j &= s_j \partial_{i-1} \quad \text{if} \quad i > j + 1 \\
s_i s_j &= s_j s_i \quad \text{if} \quad i \leq j
\end{align*}
\] (3.13)

Also recall the normalized complex with terms
\[ N^p_\Delta(X, n) := \cap_{i=0}^{n-1} \ker(\partial_i) \subset Z^p_\Delta(X, n). \]

We introduce a filtration:
\[ Z^p_\Delta(X, \bullet) \supset F^0 Z^p_\Delta(X, \bullet) \supset F^1 Z^p_\Delta(X, \bullet) \supset \cdots \supset N^p_\Delta(X, \bullet) \]
as follows: for $\ell \geq 0$, put
\[ F^\ell Z^p_\Delta(X, n) = \left\{ \xi \in Z^p_\Delta(X, n) \mid \partial_i \xi = 0, \forall \ 0 \leq i < \min(n, \ell) \right\}. \]

Let
\[ \lambda_\ell : F^{\ell+1} Z^p_\Delta(X, \bullet) \subset F^\ell Z^p_\Delta(X, \bullet) \]
be the inclusion of chain complexes.

**Lemma 3.8.** $\lambda_\ell$ is a quasi-isomorphism.

**Proof.** Introduce
\[ \kappa_\ell : F^\ell Z^p_\Delta(X, \bullet) \to F^{\ell+1} Z^p_\Delta(X, \bullet), \]
by the formula
\[
\kappa_\ell(\xi) = \begin{cases} 
\xi & \text{if } \ell > n \\
\xi - s_\ell \partial_\ell(\xi) & \text{if } \ell \leq n
\end{cases}.
\] (3.15)

We claim that (3.14) is a morphism of complexes.

To see this, first observe that $\kappa_\ell$ is the identity for $\ell > n$, so it suffices to assume that $\ell \leq n$. Let $\xi \in F^\ell Z^p_\Delta(X, n)$. We must show that
\[ \kappa_\ell \partial_\ell \xi = \partial \kappa_\ell \xi, \]
i.e. that
\[ \sum_{j=\ell}^{n} (-1)^j \left( \partial_j \xi - s_\ell \partial_\ell \partial_j \xi \right) = \sum_{j=\ell+1}^{n} (-1)^j \left( \partial_j \xi - \partial_j s_\ell \partial_\ell \xi \right). \] (3.16)

\[ 12 \text{Obviously both sides are zero if } \ell > n. \]
For $j \geq \ell + 2$, we have

$$s_\ell \partial_\ell \partial_j = s_\ell \partial_{\ell - 1}\partial_\ell = \partial_\ell s_\ell \partial_\ell$$

from (3.13). Thus with regard to (3.16), we are reduced to showing that

$$(-1)^\ell \left[ \partial_\ell \xi - s_\ell \partial_\ell \partial_\ell \xi \right] + (-1)^{\ell + 1} \left[ \partial_{\ell + 1}\xi - s_\ell \partial_{\ell + 1}\xi \right] =$$

$$(-1)^{\ell + 1} \left[ \partial_{\ell + 1}\xi - \partial_{\ell + 1} s_\ell \partial_\ell \xi \right].$$

Using $\partial_{\ell + 1} s_\ell = \text{Id}$, this is reduced to the equation

$$\partial_\ell \partial_\ell - \partial_\ell \partial_{\ell + 1} = 0,$$

which follows from (3.13). The claim is established.

Next observe that $\kappa_\ell \circ \lambda_\ell$ is the identity on $F^\ell \Delta Z^p(X, \bullet)$. For $\xi \in F^\ell \Delta Z^p(X, n)$ we introduce the homotopy operator $T_\ell : F^\ell \Delta Z^p(X, n) \to F^\ell \Delta Z^p(X, n + 1)$ by the formula

$$T_\ell(\xi) = \begin{cases} 0 & \text{if } \ell > n \\ (-1)^\ell s_\ell(\xi) & \text{if } \ell \leq n \end{cases}.$$ 

We will check that

$$\partial T_\ell(\xi) + T_\ell \partial(\xi) = \xi - (\lambda_\ell \circ \kappa_\ell)(\xi),$$

which obviously implies that $\lambda_\ell \circ \kappa_\ell$ is homotopic to the identity on $F^\ell \Delta Z^p(X, n)$. Firstly, from (3.15), the right-hand side of (3.17) is given by

$$\xi - (\lambda_\ell \circ \kappa_\ell)(\xi) = s_\ell \partial_\ell \xi.$$

(Both sides of (3.18) are zero if $\ell > n$.) Next, the left-hand side of (3.17) is

$$(-1)^\ell [\partial s_\ell + s_\ell \partial](\xi).$$

But as $s_\ell : Z^p_\Delta(X, n) \to Z^p_\Delta(X, n + 1)$, for $\ell \leq n$, using (3.13) (and $\xi \in F^\ell$) gives

$$(-1)^\ell \partial s_\ell(\xi) = (-1)^\ell \sum_{j=0}^{n+1} (-1)^j \partial_j s_\ell(\xi) =$$

$$(-1)^\ell \sum_{j=0}^{\ell - 1} (-1)^j \partial_j s_\ell(\xi) + [\partial_\ell - \partial_{\ell + 1}] s_\ell(\xi) + (-1)^\ell \sum_{j=\ell + 2}^{n+1} (-1)^j \partial_j s_\ell(\xi) =$$

$$(-1)^\ell \sum_{j=\ell + 2}^{n+1} (-1)^j s_\ell \partial_{j - 1}(\xi) = (-1)^\ell \sum_{j=\ell + 1}^{n} (-1)^{j - 1} s_\ell \partial_j(\xi).$$
Next, and again using $\xi \in F^\ell$, 

$$(-1)^\ell s_\ell \partial(\xi) = s_\ell \partial(\xi) + (-1)^\ell \sum_{j=\ell+1}^{n} (-1)^j s_j \partial_j(\xi).$$

Then (3.19) becomes $s_\ell \partial(\xi)$, as required. 

Lemma 3.8 has the following corollary. Let $\kappa : Z^p_\Delta(X, \bullet) \to N^p_\Delta(X, \bullet)$ be the map of complexes defined by letting $\kappa^{(n)} : Z^p_\Delta(X, n) \to N^p_\Delta(X, n)$ be the composite $\kappa_{n-1} \circ \kappa_{n-2} \cdots \circ \kappa_0$. Then the inclusion $\lambda : N^p_\Delta(X, \bullet) \subset Z^p_\Delta(X, \bullet)$ induces an isomorphism on homology with inverse induced by $\kappa$; moreover, $\kappa \circ \lambda$ is the identity on $N^p_\Delta(X, \bullet)$. So we get $Z^p_\Delta(X, \bullet) \cong N^p_\Delta(X, \bullet) \oplus \ker \kappa$, where 

$$\ker \kappa = D^p_\Delta(X, \bullet) := \sum_{i=0}^{n-1} s_i (Z^p_\Delta(X, n - 1)), $$

and $D^p_\Delta(X, \bullet)$ is acyclic.

Turning to the proofs of our moving lemmas, we consider the commutative diagram 

\[
\begin{array}{ccc}
N^p_{\Delta,\mathbb{R}}(X, \bullet) & \xrightarrow{i_1} & Z^p_{\Delta,\mathbb{R}}(X, \bullet) \\
\downarrow i_2 & & \downarrow i_3 \\
N^p_\Delta(X, \bullet) & \xrightarrow{i_4} & Z^p_\Delta(X, \bullet),
\end{array}
\]

where $N^p_{\Delta,\mathbb{R}}(X, \bullet) := Z^p_{\Delta,\mathbb{R}}(X, \bullet) \cap N^p_\Delta(X, \bullet)$. We have seen that $i_4$ is a quasi-isomorphism.

We claim that $i_1$ is a quasi-isomorphism. To see this, let 

$$\tau = i_{n+1}^{k-j} := T_{x_{k+1}+\cdots+x_{n+1}} \cap \cdots \cap T_{x_{k+1}+\cdots+x_{n+1}} \cap \rho_j(\Delta^{n+1-|J|})$$

be one of the real chains in $\Delta^{n+1}$. Then one checks that $s_i(\tau) \subset \Delta^n$ is contained in a $\tau^{k-j}_{l'}$ of the same real dimension (as $s_i(\tau)$, not $\tau$). Reasoning as in Step 2 of the Proof of Theorem 3.2, we have that $s_\ell$ restricts to a map $Z^p_{\Delta,\mathbb{R}}(X, n) \to Z^p_{\Delta,\mathbb{R}}(X, n+1)$. By (3.15), it follows that $\kappa_\ell$ and $T_\ell$ also preserve “subscript $\mathbb{R}$”, so that the proof of Lemma 3.8 goes through with the real-intersection conditions, proving the claim.

It remains to show that $i_2$ is a quasi-isomorphism. The argument in [KL Appendix to 8.2] (cf. part (a)) proves exactly the same thing in the cubical context. Replacing cubes with simplices and $\mathcal{T}^n$ by the iterated double $T^\omega_\Delta := D\left(\Delta^\omega_X; \rho_0(\Delta^{n-1}_X), \ldots, \rho_n(\Delta^{n-1}_X)\right)$, the same proof (using

\footnote{This is a singular variety (resembling the union of facets of a polytope) with irreducible components all isomorphic to $\Delta^n_X$, and indexed by subsets of \{0, \ldots, n\}.}
ideas of Levine [Lv] goes through \textit{mutatis mutandis}. To give a flavor of
the proof, we summarize the steps for showing $i_2$ is “quasi-surjective”.
The idea is that any normalized cycle $\mathfrak{Z} \in \ker(\partial) \subset N^p_{\Delta}(X, n)$ can, up
to $\partial N^p_{\Delta}(X, n + 1)$, be described as the alternating pullback of a cycle on $T^1_X$. This cycle in turn may be obtained by intersecting with a cycle $\mathcal{W}$
on a homogeneous space for $GL_n(K)$, where $K$ is the field of definition of
$X$. Applying $g^*$ ($g \in GL_n(L)$, $L \supset K$) to $\mathcal{W}$ and pulling back to
$X$ yields a cycle $\mathfrak{Z}' \in N^p_{\Delta}(X_L, n)$ which still only differs from $\mathfrak{Z}$ by an
element of $\partial N^p_{\Delta}(X_L, n + 1)$. By a variant of Kleiman transversality (cf. [Lv]), one may choose $g$ so that $\mathfrak{Z}' \in N^p_{\Delta,R}(X_L, n)$; a norm argument
then produces $\mathfrak{Z}'' \in N^p_{\Delta,R}(X, n)$ in the same class as $\mathfrak{Z}$.

\textbf{Appendix II to \S3: verification that $R^\Delta_\mathfrak{Z} \in D^{2p-n-1}(X)$.} We consider
progressively more general cases, with $\mathfrak{Z} \subset X \times \mathbb{P}^n$ always irreducible and giving an element of $Z^p_{\Delta,R}(X, n)$:

\textbf{Case 1:} $p = n$, with $\pi_X(\mathfrak{Z}) = X$ and $\mathfrak{Z}$ generically of degree 1 over
$X$. Writing

$$f = (f_1, \ldots, f_n) := \mathfrak{Z}^* \left(-\frac{X_1 + \cdots + X_n}{X_0}, \ldots, -\frac{X_n}{X_{n-1}}\right),$$
we define subvarieties

$$H_f := |\mathfrak{Z}^* ((X_0 + \cdots + X_n))|,$$

$$Y_f := \bigcup_{j=1}^n |(1 - f_j)w|,$$ and

$$D_f := \bigcup_{j=1}^n |(f_j)|$$
of $X$. Let $\omega \in A^{2\dim(X) - n + 1}(X)$ be a $C^\infty$ test form; we must show that

$$\int_X R(f) \wedge \omega := \lim_{\epsilon \to 0} \int_{X \setminus \mathcal{N}(D_f)} R(f) \wedge \omega$$
is finite (where $\mathcal{N}(\cdot)$ denotes a small tubular neighborhood). Write
$\mathcal{E}_{f, \omega}$ for the union of irreducible components $W$ of $D_f$ along which
every term of $R(f) \wedge \omega$ has a factor of $dw$, $d\bar{w}$, $w$, or $\bar{w}$, where $w$ is
an algebraic (and locally holomorphic) function with $W$ in its zero-set.
More precisely, if $J_W := \{j \in \{1, \ldots, n\} ||(f_j)| \supset W\} = \{j_1, \ldots, j_k\}$,
then $R_f \wedge \omega$ breaks into terms

$$\log f_{j_1} d\log f_{j_{i+1}} \wedge \cdots \wedge d\log f_{j_k} \cdot \delta_{T_{f_1} \cap \cdots \cap T_{f_{j_{i-1}}}}$$
with $\alpha$ a monomial $C^\infty$ $(\dim(X) - k + 1)$-form in coordinates $\{z_1 = w, \ldots, z_n\}$, and we require that $\alpha$ contain a $w$, $\bar{w}$, $dw$, or $d\bar{w}$.

First assume that $D_f$ is a normal crossing divisor. In that event, it will suffice to bound (3.20) in a neighborhood of a general point of
each irreducible component of $D_f$, since the bounds near intersection
(higher codimension) points will break into products of codimension-1 bounds. The only possibilities for nonconvergence along \( W \) are terms of the form

\[
\log(w^b) \wedge C^\infty \quad \text{and} \quad \log(w^c) \wedge C^\infty.
\]

where without loss of generality one can take the integers \( a, b, c \) to be 1. Evidently the presence of a \( dw, \bar{d}w, w, \) or \( \bar{w} \) in each monomial term of the \( C^\infty \) expression makes (3.21) converge, so that we only need to worry about \( W \not\subseteq E_f, \omega \). But \( H_f \subset Y_f \subset E_f, \omega \), and \( Y_f \) also contains every \( W \) along which the numerator of \( f_1, f_2, \ldots, f_{n-1} \) vanishes, while outside \( H_f \) only one of \( X_0, \ldots, X_n \) can vanish in codimension one. Consequently, each component of \( D_f \not\subseteq E_f, \omega \) can only be contained in one \( |(f_j)| \) and (3.21) cannot occur.

If \( D_f \) does not have normal crossings, consider an embedded resolution

\[
\tilde{X} \xrightarrow{\beta} X
\]

\[
\tilde{D}_f \cup E_\beta \xrightarrow{\psi} D_f
\]

where \( \tilde{D}_f \) is the proper transform and \( E_\beta \) the exceptional divisor (with union a NCD). By a simple computation, \( E_{\beta^* f_\beta^* \omega} \supset \beta^{-1}(E_f, \omega) \) and we only need to consider components \( W = \{ w' = 0 \} \) of \( E_\beta \) in the preimage of \( X \setminus Y_f \). But then by the proper intersection conditions on \( 3 \), \( |J_{W'}| \) is bounded by \( c := \text{codim}_X(\beta(W')) \). In particular, if \( w_1 = \cdots = w_c = 0 \) locally cuts out \( \beta(W') \), we have in each term of \( R(f) \wedge \omega \) a \( dw_i, \bar{d}w_i, w_i, \) or \( \bar{w}_i \) factor \( (i \in \{ 1, \ldots, c \}) \), hence in each term of \( R(\beta^* f) \wedge \beta^* \omega \) a \( dw', \bar{d}w', w', \) or \( \bar{w}' \) factor. Conclude that \( W' \) hence \( E_\beta \) is contained in \( E_{\beta^* f_\beta^* \omega} \), proving convergence.

**Case 2:** Remove the degree-1 assumption (so \( 3 \) is simply finite over \( X \)). The above argument goes through for the branches of \( 3 \), when one considers that the expressions in (3.21) are not essentially different if we take \( a, b, c \in \mathbb{Q} \), and that codimension in \( 3 \) is codimension in \( X \).

**Case 3:** \( p > n \) and \( 3 \) generically finite over a subvariety \( V \) of \( X \). At first glance, one has to worry about the failure of proper intersection conditions for the base-change of \( 3 \) under a desingularization \( \tilde{V} \to V \). (Otherwise, we are reduced to Case 2.) But as in the end of Case 1, away from the sets \( 3 \cap (V \times \{ X_j + \cdots + X_n = 0 \}) (j = 0, \ldots, n-1) \), the number of singular \( \delta_T \) or \( d\log \) factors is bounded by the codimension
of the corresponding subvariety of \( \mathfrak{Z} \) (hence \( V \)), and then a similar argument holds.

**Case 4: general case.** Working locally, there is a finite projection of \( \mathfrak{Z} \) to \( V \times \mathbb{P}^k \) for some \( k < n \), and we are done by Case 3.

### 4. Milnor reciprocity laws

The telescoping property (Prop. 2.3) of the simplicial currents \( R^\Delta_n, S^\Delta_n \) makes them particularly suitable for the study of reciprocity laws arising from subvarieties of projective space. We shall begin, however, from a more general and “intrinsic” perspective, which is independent of the choice of simplicial vs. cubical.

Let \( X \) be a smooth complete curve over \( \mathbb{C} \), and \( f, g \in \mathbb{C}(X)^* \). Writing

\[
\begin{cases}
  \text{Tame}_p : K^M_2(\mathbb{C}(X)) \to K^M_1(\mathbb{C}) \cong \mathbb{C}^* \\
  \{f, g\} \mapsto \lim_{x \to p} (1)^{\nu_p(f) \nu_p(g) f(x) \nu_p(g) g(x) \nu_p(f)}
\end{cases}
\]

for \( p \in X(\mathbb{C}) \), **Weil reciprocity** states that the (finite) product

\[
\prod_{p \in X(\mathbb{C})} \text{Tame}_p \{f, g\} = 1.
\]

This result gives rise to several other reciprocity laws in higher dimension. For example, Parshin [resp. bilocal] reciprocity (cf. \[Ho\]) on an algebraic surface \( S \) is obtained by applying Weil reciprocity on a curve \( X \subset S \) to \( \text{Tame}_X \xi \) [resp. \( \{\text{Tame}_X \mu, \text{Tame}_X \eta\} \) for \( \xi \in K^M_3(\mathbb{C}(S)) \) [resp. \( \mu \otimes \eta \in K^M_2(\mathbb{C}(S)) \otimes \mathbb{Z}^2 \)]. Suslin reciprocity (cf. \[Ke3\]) generalizes Weil to higher \( K \)-theory, replacing (4.1) by \( \text{Tame}_p : K^M_3(\mathbb{C}(X)) \to K^M_2(\mathbb{C}) \).

The generalizations we pursue here take a different direction, and begin from the

**Proposition 4.1.** Let \( D = \{p_1, \ldots, p_t\} \subset X \) and \( X^* := X \setminus D \); then for each \( p \geq 2 \), the composition

\[
CH^p(X^*, 2p - 2) \xrightarrow{\oplus \alpha \text{Res}_{p_a}} \oplus \alpha CH^{p-1}(\mathbb{C}, 2p - 3) \xrightarrow{\text{AJ}} \oplus \alpha \mathbb{C}/\mathbb{Z}(p - 1)
\]

has image in the kernel of the augmentation map \( \oplus \alpha \mathbb{C}/\mathbb{Z}(p - 1) \xrightarrow{\sum} \mathbb{C}/\mathbb{Z}(p - 1) \).

This is easily proved from the localization sequence and its compatibility with the \( AJ \) map, or using Reciprocity Law A below. The case \( p = 2 \) is Weil reciprocity, while \( p = 3 \) [resp. 4, \ldots] is related to the dilogarithm [resp. trilogarithm, \ldots] at algebraic arguments and more generally special values of \( L \)-functions. So for polylogarithmic functional equations with variable arguments, this is not the way to go.
At the next stage of generalization, where $X/\mathbb{C}$ is any smooth projective variety, we encounter an unpleasant reality when $\dim X =: d > 1$. Consider a codimension-one subvariety $D \subset X$ with irreducible components $\{D_\alpha\}$ and smooth locus $\cup D_\alpha^*$, and write $X^* := X \setminus D$. Taking $p > d$, for any $\alpha$ the composition

$$CH^p(X^*, 2(p-d)) \xrightarrow{Res_\alpha} CH^{p-1}(D_\alpha^*, 2(p-d)-1) \xrightarrow{\text{Adj}} H^{2(d-1)}(D_\alpha^*, \mathbb{C}/\mathbb{Z}(p-1))$$

is zero unless $D_\alpha^* = D_\alpha$, so that integrating the image current does not give a well-defined number in $\mathbb{C}/\mathbb{Z}(p-1)$. So we are forced to work on the level of precycles, which yields

**Proposition 4.2.** Let $p > d \geq 1$, $n := 2(p-d)$, and $\bar{\mathfrak{Z}} \in Z^n_{\mathbb{Z}(\Delta)}(X, n)$ be a precycle with $\partial_B \bar{\mathfrak{Z}}$ supported on $D$. Then writing $\partial_B \bar{\mathfrak{Z}} =: \sum i_* D_\alpha \text{Res}_\alpha \mathfrak{Z}$ (with $\text{Res}_\alpha \mathfrak{Z} \in Z^{p-1}_{\mathbb{Z}(\Delta)}(D_\alpha, n-1)$), we have

$$\sum_\alpha \int_{D_\alpha} R^{(\Delta)}_{\text{Res}_\alpha \mathfrak{Z}} \equiv 0 \mod \mathbb{Z}(n-1).$$

**Proof.** Note that $R^{(\Delta)}_{\text{Res}_\alpha \mathfrak{Z}}$ is a current of top degree $2(p-1)-(n-1)-1 = 2(d-1)$ on $D_\alpha$. Since $p > d$, $F^p D^{2d}(X) = \{0\}$ and $\Omega^{(\Delta)} = 0$. So (3.5) becomes

$$d[R^{(\Delta)}_{\text{Res}_\alpha \mathfrak{Z}}] = -\left(2\pi i\right)^n \delta^{(\Delta)}_{\bar{\mathfrak{Z}}} + 2\pi i \sum_\alpha i_* D_\alpha R^{(\Delta)}_{\text{Res}_\alpha \mathfrak{Z}},$$

from which the result follows by Stokes’s theorem.

Restricting to the case $n = p = 2d$, suppose $F_0, \ldots, F_n \in \Gamma(X, \mathcal{O}_X(k))$ is an $n$-tuple of homogeneous functions such that

$$\Gamma_F := \{(x, [F_0(x) : \cdots : F_n(x)]) | x \in X(\mathbb{C})\} \in Z^n_{\mathbb{Z}(\Delta)}(X, n).$$

Writing $\sum m_{ij} D_{ij} := (F_i)$, one obtains

$$\partial_B \Gamma^\Delta_F = \sum_{i=0}^n (-1)^i \sum_j m_{ij} i^* D_{ij} \Gamma^\Delta_{[F_0; \ldots; F_i; \ldots; F_n]},$$

which together with Proposition 4.2 gives the

**Corollary 4.3.** We have

$$\sum_{i=0}^n (-1)^i \sum_j m_{ij} \int_{D_{ij}} R^\Delta(X_0 : \cdots : \tilde{X}_i : \cdots : X_n) \equiv 0 \mod (n-1).$$

\[14\] The parentheses $(\Delta)$ mean that we may work in either the simplicial or the cubical setting.
We leave to the reader the obvious analogue for the cubical Milnor regulator currents \( R(f_1, \ldots, f_i, \ldots, f_n) \). Note that the \( n = 2 \) case of this is Weil reciprocity for functions \( f_1, f_2 \in \mathbb{C}(X)^* \) with \( |f_1| \cap |f_2| = \emptyset \).

The Corollary has a natural "extrinsic" analogue for algebraic cycles in even-dimensional projective space. We lose no generality by stating this result, which is our first main point, for subvarieties.

**Definition 4.4.** We shall say that a subvariety of \( \mathbb{P}^M \) is in **general position** if it properly intersects all chains of the form \( \iota^*_{\mathbb{P}^M} - T_{x_k} \cap \cdots \cap T_{x_{k-1}} \) where \( \iota^* : \mathbb{P}^M - |J| \hookrightarrow \mathbb{P}^M \) sends \([Z_0 : \cdots : Z_{M-|J|}]\) to the projective \((M+1)\)-tuple obtained by inserting zeroes at the positions \( j_1, \ldots, j_{|J|} \).

**Theorem 4.5. (Reciprocity Law A)** Let \( \mathcal{R}_m \) stand for \( R^\Delta \) or \( S^\Delta \), and \( X \subset \mathbb{P}^{2d} \) be an irreducible subvariety of dimension \( d \), with \( Y_i := X \cdot (X_i) \) for \( i = 0, \ldots, 2d \). Assuming that \( X \) is in general position, we have

\[
\sum_{j=0}^{2d} (-1)^j \int_{X_j} \frac{1}{(2\pi i)^{d-1}} \mathcal{R}_{2d-1} (X_0 : \cdots : \widehat{X}_j : \cdots : X_{2d}) \equiv 0.
\]

**Proof.** The general position assumption allows us to pull back the result of Proposition 2.3. Noting that by Hodge type we have \( \iota^*_{\mathbb{P}^M} \Omega^\Delta_{2d} = 0 \), this gives

\[
d\iota^*_{\mathbb{P}^M} \mathcal{R}_{2d} + (2\pi i)^{2d} \delta_{X \cdot T^\Delta} = 2\pi i \sum_{j=0}^{2d} (-1)^j \iota^*_{\mathbb{P}^M} (\rho_j) \mathcal{R}_{2d-1}.
\]

Dividing by \((2\pi i)^d\) and integrating over \( X \) gives the result. \( \square \)

We have written it in this form because the first term of (say) \( S_{2d-1}(X_1 : \cdots : X_{2d}) \) whose pullback to \( Y_0^* \) does not vanish, is

\[
(2\pi i)^{d-1} \delta_{\mathbb{P}^d - x_k \cap \cdots \cap T_{x_{d-1}}} \log \left( \frac{-X_{d+1}}{X_d} \right) \log \left( \frac{X_{d+2}}{X_{d+1}} \right) \wedge \log \left( \frac{X_{2d}}{X_{2d-1}} \right).
\]

For \( X \approx \mathbb{P}^d \) a linear subvariety, one expects Theorem 4.5 to translate into functional equations for (a variant of) \( L_i_d \). It turns out that the \( S^\Delta_m \) version of the result, which allows for more singular integrals, is much more suited to making this connection.

There is also a natural "projective dual" to Theorem 4.5 which we shall only state for the \( S^\Delta_m \). (We do not know if an analogue of Lemma 4.7 holds for the \( R^\Delta_m \).) In first approximation, one would expect a
statement of the following form: given \( X \subset \mathbb{P}^{2d} \) general of dimension \( d - 1 \), the alternating sum

\[
\sum_{j=0}^{2d} (-1)^j \int_X \frac{1}{(2\pi i)^{d-1}} S_{2d-1}^\Delta \left( X_0 : \cdots : \hat{X}_j : \cdots : X_{2d} \right)
\]

is zero mod \( \mathbb{Z}(d) \). (Note that this morally involves projecting \( X \) to the coordinate hyperplanes in \( \mathbb{P}^{2d} \), rather than intersecting with them.)

This turns out to require correction terms, essentially because complex-valued regulator currents cannot be made exactly alternating multilinear in their arguments.

In order to make the corrections, we shall require two lemmas. Introduce the notation

\[
S_{alt}^k := \sum_{j=0}^{k+2} (-1)^j S_{k+1}^\Delta \left( X_0 : \cdots : \hat{X}_j : \cdots : X_{k+2} \right) \in D^k(\mathbb{P}^{k+2}),
\]

\[
I_*^{k+2} := \sum_{j=0}^{k+2} (-1)^j \rho_j^* : D^{*-2}(\mathbb{P}^{k+1}) \to D^*(\mathbb{P}^{k+2}),
\]

where we recall \( \rho_j^* : \mathbb{P}^{k+1} \hookrightarrow \mathbb{P}^{k+2} \) is the inclusion of the \( j \)th coordinate hyperplane. Note that \( I_*^{\ell+1} \circ I_*^\ell = 0 \). For \( k \) odd, let \( P_k \) denote a fixed \( \mathbb{P}^{k+5}_k \subset \mathbb{P}^{k+2} \). To motivate the first lemma, observe that on \( \mathbb{P}^2 \)

\[
S_{alt}^0 = \log \left( -\frac{X_2}{X_1} \right) - \log \left( \frac{X_2}{X_0} \right) + \log \left( \frac{X_1}{X_0} \right)
\]

\[
= : \pi i \delta_{\Gamma_{012}}
\]

takes values \( \pm \pi i \), making \( \Gamma_{012} \) an integral 4-chain (or \( \mathbb{Z} \)-valued 0-current). A computation shows that \( S_{alt}^1 = \)

\[
R_2 \left( -\frac{X_2}{X_1}, -\frac{X_3}{X_2} \right) - R_2 \left( -\frac{X_2}{X_0}, -\frac{X_3}{X_2} \right) + R_2 \left( -\frac{X_1}{X_0}, -\frac{X_3}{X_1} \right) - R_2 \left( -\frac{X_1}{X_0}, -\frac{X_2}{X_1} \right)
\]

\[
= d \left\{ \pi i \log \left( -\frac{X_3}{X_2} \right) \delta_{\Gamma_{012}} \right\} - \frac{1}{2} (2\pi i)^2 \delta_{\frac{x_0}{x_1}} \delta_{\Gamma_{123}},
\]

which forms the base case for

**Lemma 4.6.** There exists a sequence of currents \( \Xi^k \in D^k(\mathbb{P}^{k+3}) \) (\( k = 0, 1, 2, \ldots \)) and constants \( \alpha_1, \alpha_3, \alpha_5, \ldots \in \mathbb{C} \) such that for each \( k \geq 0 \)

\[
S_{alt}^{k+1} + 2\pi i I_*^{k+3} \Xi^{k-1} \equiv \begin{cases} 
-2\pi i k \Xi^k, & k \; \text{even} \\
-2\pi i k \Xi^k + \alpha_k \delta_{\Gamma_{k+3}^{k+1}} & k \; \text{odd}
\end{cases}
\]

modulo \( \mathcal{C}_{k+2} := \frac{1}{2} \mathbb{Z}(k+2) \)-valued chains.
Proof. By Proposition 2.3 and the fact that
\[ \sum_{j=0}^{n+3} (-1)^j \Omega_{n+2}^j \left( X_0 : \cdots : \widehat{X_j} : \cdots : X_{n+3} \right) = 0 \]
on \mathbb{P}^{n+3}, we have for each \( n \)
\[ dS_{alt}^{n+1} \equiv -2\pi i I_{n+3}^n S_{alt}^n \pmod{\mathfrak{C}_{n+2}}. \]
Inductively assuming (4.2) for \( k = n - 1 \), this gives
\[ dS_{alt}^{n+1} \equiv -2\pi i I_{n+3}^n \left\{ -2\pi i I_{n+2}^{n+2} \Xi^{n-2} + d\Xi^{n-1} \left[ + \alpha_{n-1} \delta_{I_{n+2}^p} \right] \right\} \]
\[ \equiv -2\pi id \left\{ I_{n+3}^{n+3} \Xi^{n-1} \right\} \]
\[ \implies S_{alt}^{n+1} + 2\pi i I_{n+3}^{n+3} \Xi^{n-1} \text{ is closed (mod } \mathfrak{C}_{n+2}). \]
If \( n \) is even, we are done since \( H^{n+1}(\mathbb{P}^{n+3}) = \{0\} \). Otherwise, noting that \( [I_{n+3}^{n+3} P_n] = [\mathbb{P}^{n+3}] \in H^{n+1}(\mathbb{P}^{n+3}) \), there exist \( \alpha \in \mathbb{C} \) and \( \Xi \in D^n(\mathbb{P}^{n+3}) \) such that
\[ S_{alt}^{n+1} \equiv -2\pi i I_{n+3}^{n+3} \Xi^{n-1} + d\Xi^n + \alpha \delta_{I_{n+3}^p} \pmod{\mathfrak{C}_{n+2}}. \]
\[ \square \]

In fact, a more detailed computation reveals that with the right choices of the \( \{ \Xi^k \} \), the \( \{ \alpha_k \} \) may be taken to be 0:

**Lemma 4.7.** One has for each \( k \geq 0 \)
\[ S_{alt}^{k+1} + 2\pi i I_{n+3}^{k+3} \Xi^{k-1} \equiv d\Xi^k \pmod{\mathfrak{C}_{k+2}}, \]
where
\[ \Xi^k = \pi i \sum_{\ell=0}^k (-2\pi i)^{\ell} \delta_T \log \left( \frac{-X_{\ell+3}}{X_{\ell+2}} \right) \, d\log \left( \frac{X_{\ell+4}}{X_{\ell+3}} \right) \wedge \cdots \wedge d\log \left( \frac{X_{k+3}}{X_{k+2}} \right) \]
and the codimension-\( \ell \) chain
\[ \Gamma^\ell = \sum_{j=0}^{\ell} \Gamma_{j,j+1,j+2} T_{-x_0} \cap \cdots \cap T_{-x_{j+1}} \cap T_{-x_{j+2}} \cap \cdots \cap T_{-x_{\ell+2}}. \]

**Proof.** (Sketch) The main step is to show directly that \( S_{alt}^{k+1} = \)
\[ \sum_{\ell=0}^{k+1} (-2\pi i)^{\ell} \left\{ \delta_T \log \left( \frac{-X_{\ell+2}}{X_{\ell+1}} \right) + (-1)^{\ell} \pi i \delta_T \cap \Gamma_{\ell+1,\ell+2} \right\} \, d\log \left( \frac{X_{\ell+3}}{X_{\ell+2}} \right) \wedge \cdots \wedge d\log \left( \frac{X_{k+3}}{X_{k+2}} \right), \]
\[ \text{15} \]The widehats mean that those two \( T \)'s are omitted from the intersection.
where \( T^\ell := T^\ell[0 : \cdots : \ell] = T - x_1 \cap \cdots \cap T - x_{\ell - 1} \) and

\[
T_{\text{alt}}^\ell := \sum_{j=0}^{\ell+1} (-1)^j T^\ell[0 : \cdots : j : \cdots : \ell].
\]

(Note that the term in braces is just \( \pi i \delta \Gamma_{012} \) for \( \ell = 0 \).) To verify (4.3), one then uses the formula

\[
\frac{1}{2} \partial \Gamma^0 = \frac{1}{2} \partial \Gamma_{012} = T - x_2 - x_0 - T - x_1 - x_0 - T - x_2 - x_1 = -T_{\text{alt}}^1,
\]

and \( \Gamma^{\ell-1} \cap T - x_{\ell+2} + T^\ell \cap \Gamma_{\ell,\ell+1,\ell+2} = \Gamma^\ell \). Details are left to the reader. \( \square \)

We can now state

**Theorem 4.8. (Reciprocity Law B)** Let \( X \subset \mathbb{P}^{2d} \) be an irreducible subvariety of dimension \( d - 1 \), with \( Y_i := X \cdot (X_i) \) for \( i = 0, \ldots, 2d \). Assuming that \( X \) and its projections to the coordinate hyperplanes are in general position, we have

\[
0 \equiv \sum_{j=0}^{2d} (-1)^j \int_X \frac{1}{(2\pi i)^{d-1}} S^\Delta_{2d-1} \left( X_0 : \cdots : X_j : \cdots : X_{2d} \right)
\]

\[
+ \sum_{j=0}^{2d} (-1)^j \int_{Y_j} \frac{1}{(2\pi i)^{d-2}} \Xi^{2d-4} \left( X_0 : \cdots : X_j : \cdots : X_{2d} \right)
\]

modulo \( \frac{1}{2} \mathbb{Z}(d) \).

**Proof.** Follows immediately from Lemma 4.1 with \( k = 2d - 3 \). \( \square \)

The correction terms \( \int_{Y_j} \Xi^{2d-3} \), as we shall see, may be thought of as "lower-weight" in the context of linear subvarieties and polylogarithms. In essence, one is trading off the formal simplicity of Reciprocity Law A for greater algebraic simplicity in the arguments of the expected \( \text{Li}_d \) terms \( \int_X S^\Delta_{2d-1} \).

5. **Functional equations for \( \text{Li}_2 \)**

To illustrate the different strengths of the two reciprocity laws of the last section, we shall apply both to obtain different forms of the 5-term relation for the dilogarithm

\[
\text{Li}_2(z) = -\int_0^1 \log(1 - z) \frac{dz}{z}.
\]
Reciprocity Law A involves intersecting an $X^d \subset \mathbb{P}^{2d}$ with the coordinate hyperplanes. Taking $d = 2$, let $X$ be the $\mathbb{P}^2 \subset \mathbb{P}^4$ obtained by projectivizing the row-space of

$$
\begin{pmatrix}
1 & 1 & -1 & 0 & 0 \\
\frac{1}{y} & 1 & 0 & -1 & 0 \\
\frac{1}{x} & 1 & 0 & 0 & -1
\end{pmatrix}.
$$

The intersections $Y_i$ ($i = 0, \ldots, 4$) are given by projectivizing the sub-row-spaces with $X_i = 0$ (and deleting the $i^{th}$ column):

$$
(5.1)
\begin{cases}
\begin{pmatrix}
\frac{x-1}{y} & 1 & -1 & 0 \\
\frac{x}{y-1} & 1 & 0 & -1 \\
\frac{y-x}{y} & 1 & 0 & -1
\end{pmatrix} & i = 0 \\
\begin{pmatrix}
\frac{1}{y} & 1 & -1 & 0 \\
\frac{1}{x} & 1 & 0 & -1 \\
\frac{1}{y} & 1 & 0 & -1
\end{pmatrix} & i = 1 \\
\begin{pmatrix}
1 & 1 & -1 & 0 \\
\frac{1}{y} & 1 & 0 & -1 \\
0 & 1 & 0 & -1
\end{pmatrix} & i = 2 \\
\begin{pmatrix}
\frac{1}{y} & 1 & -1 & 0 \\
\frac{1}{x} & 1 & 0 & -1 \\
0 & 1 & 0 & -1
\end{pmatrix} & i = 3 \\
\begin{pmatrix}
\frac{1}{y} & 1 & -1 & 0 \\
\frac{1}{x} & 1 & 0 & -1 \\
0 & 1 & 0 & -1
\end{pmatrix} & i = 4
\end{cases}
$$

Let $Y$ be the $\mathbb{P}^1 \subset \mathbb{P}^3$ given by

$$
\begin{pmatrix}
1 \\
-t
\end{pmatrix}
\begin{pmatrix}
a & c & -1 & 0 \\
b & d & 0 & -1
\end{pmatrix},
$$

where the notation means that $t$ parametrizes $Y$ by $t \mapsto [a - bt : c - dt : -1 : t]$. On $\mathbb{P}^3$, we have $\frac{1}{2\pi i} S^3_3 (X_0 : X_1 : X_2 : X_3) =$

$$
\frac{1}{2\pi i} \log \left(-\frac{X_1}{X_0}\right) \text{dlog} \left(\frac{X_2}{X_1}\right) \wedge \text{dlog} \left(\frac{X_3}{X_2}\right) + \log \left(-\frac{X_2}{X_1}\right) \text{dlog} \left(\frac{X_3}{X_2}\right) \delta_T \frac{x_i}{x_0}
$$

$$
+ 2\pi i \log \left(-\frac{X_3}{X_2}\right) \delta_T \frac{x_i}{x_0} \delta_T \frac{x_i}{x_1}.
$$

Only the middle term survives the pullback to $Y$, since $\text{dlog} \wedge \text{dlog} = 0$ and $T_{\frac{c - dt}{a - bt}} \cap T_{c - dt}$ is the closure of the intersection of two open arcs that do not meet. So we must compute

$$
\frac{1}{2\pi i} \int_Y S^3_3 = - \int_{T_{\frac{c - dt}{a - bt}}} \log(c - dt) \text{dlog}(t)
$$

$$
= - \int_{\frac{c}{t}}^\frac{c}{t} \left\{ \log c + \log \left(1 - \frac{d}{c} \frac{t}{c}\right) \right\} \text{dlog}(t)
$$
\[
\text{Li}_2(1) - \text{Li}_2 \left( \frac{ad}{bc} \right) + \log(c) \log \left( \frac{ad}{bc} \right).
\]

Taking the alternating sum over the 5 matrices (5), Theorem 4.5 gives the Abel-Spence relation\(^\text{16}\)

\[
0 = \text{Li}_2(x) - \text{Li}_2(y) + \text{Li}_2 \left( \frac{y}{x} \right) - \text{Li}_2 \left( \frac{y(1-x)}{x(1-y)} \right) + \text{Li}_2 \left( \frac{1-x}{1-y} \right)
- \text{Li}_2(1) + \log(x) \log \left( \frac{1-x}{1-y} \right).
\]

For a demonstration of Reciprocity Law B, we will need the integral of \(\frac{1}{2\pi i} S^\Delta_3\) over the most general form

\[
t \begin{pmatrix}
a_0 & a_1 & a_2 & a_3 \\
b_0 & b_1 & b_2 & b_3
\end{pmatrix}
\]

of \(Y \cong \mathbb{P}^1 \subset \mathbb{P}^3\). Using the substitution \(v = \frac{-a_3t+b_3}{a_2t+b_2}\) and denoting the minor \(a_ib_j - a_jb_i\) by \(|ij|\), this is

\[
\int_{T} \log \left( -\frac{a_2t+b_2}{a_1t+b_1} \right) d\log \left( -\frac{a_3t+b_3}{a_2t+b_2} \right) =
\]

\[
\int_{-\frac{|13|}{|12|}}^{\frac{|13|}{|12|}} \log \left( -\frac{|23|}{|12|v + |13|} \right) d\log(v) =
\]

\[
- \log \left( \frac{|23|}{|13|} \right) \log \left( \frac{|12||03|}{|13||02|} \right) - \text{Li}_2 \left( \frac{|12||03|}{|13||02|} \right) + \text{Li}_2(1)
\]

\[=: \mathcal{L}\{0123\}.
\]

Writing \(t_i := \frac{-b_i}{a_i}\), note that \(\frac{|12||03|}{|13||02|} = \frac{(t_0-t_3)(t_1-t_2)}{(t_0-t_2)(t_1-t_3)} =: CR(t_0, t_1, t_2, t_3).

Now consider a general \(X \cong \mathbb{P}^1\) in \(\mathbb{P}^4\) given by

\[
z \begin{pmatrix}
A_0 & A_1 & A_2 & A_3 & A_4 \\
B_0 & B_1 & B_2 & B_3 & B_4
\end{pmatrix},
\]

with projections to the coordinate \(\mathbb{P}^3\)’s (obtained simply by deleting a column) of the form (5.3). To apply Theorem 4.8, we will also have to evaluate the correction terms, or find some way to eliminate them.

\(^{16}\)combine (1.22) (with \(x \mapsto 1 - x\)) and (1.11) (with \(z = x\)) in \(\mathcal{L}\).
Again writing $|ij|$ for the minors, $\{y_j\} = Y_j = X \cdot (X_j)$, and recalling that on $\mathbb{P}^3 \cong^0 (X_0 : X_1 : X_2 : X_3) = \pi i \log \left( -\frac{X_3}{X_2} \right) \delta_{012}$, we find that

$$\sum_{j=0}^4 (-1)^j \Xi^0 (y_j) = \sum_{j=0}^4 (-1)^j \Xi^0 \left( |j0| : \cdots : |jj| : \cdots : |j4| \right)$$

$$=: \mathcal{H} \{01234\} \in \mathbb{C}$$

is anti-invariant under the permutation $\sigma := (04)(13)$ “flipping” (5.4).

On the other hand, noting that $(03)(12)$ fixes $z := \frac{12}{13\{02\}}$, and $\frac{23}{10} = 1 - z$, we have

$$\tilde{\mathcal{L}} \{0123\} := \frac{1}{2} \left( \mathcal{L} \{0123\} + \mathcal{L} \{3210\} \right)$$

$$= Li_2(1) - Li_2(z) - \frac{1}{2} \log(1 - z) \log(z)$$

$$=: L_2(z)$$

which is a version of the Rogers dilogarithm. Adding $\frac{1}{2}$ of

$$0 = \frac{1}{2\pi i} \int_X S^2_{alt} + \int_X I^4_{0}$$

$$= \sum_{j=0}^4 (-1)^j \mathcal{L} \{0 \cdots \hat{j} \cdots 4\} + \mathcal{H} \{01234\}$$

to $\sigma_*$ of itself therefore gives, with $z_j := z(y_j) = -\frac{B_j}{X_j}$,

(5.5) $$0 = \sum_{j=0}^4 (-1)^j L_2 \left( CR(z_0, \ldots, \hat{z}_j, \ldots, z_4) \right)$$

which is the other classic form of the 5-term relation.

Remark 5.1. The $\{f_j(X) := CR_X(z_0, \ldots, \hat{z}_j, \ldots, z_4)\}$ define 5 rational functions on $Gr(2, 5)$. Pulling them back to a suitable open $U \subset \mathbb{C}^2$ via

$$g : U \to Gr(2, 5)$$

$$(x, y) \mapsto \text{span} \left\{ \left( y^{-1}, 1, -1, 0, 1 \right) \right\}$$

produces the functions $\{F_j := f_j \circ g\} = \{x, y, \frac{y}{x} : \frac{(1-x)y}{x(1-y)} : \frac{1-x}{1-y} \}$.
whose level sets yield the Bol 5-web $B_5$\footnote{See [He] for basic material on webs.}. Clearly (5.5) pulls back to the variant
\begin{equation}
0 = \sum_{j=0}^{4} (-1)^j L_2 (F_j(x, y))
\end{equation}

of (5.2), which is the most interesting of the 6 independent abelian relations of $B_5$. Moreover, the terms of (5.6) are described by
\begin{equation}
2 \pi i L_2 (F_j(x, y)) = \int_{[y(x, y)]} S_3^\Delta \left( \cdots \tilde{X}_j \cdots \right) + \int_{\sigma[y(x, y)]} S_3^\Delta \left( \cdots \tilde{X}_j \cdots \right).
\end{equation}

6. A FUNCTIONAL EQUATION FOR $Li_3$

Turning to the trilogarithm $Li_3(z) = \int_0^z \frac{dz}{z}$, we will show that the Kummer-Spence relation\footnote{This form of the relation is obtained from [Le2 p. 177] by substituting $u = \frac{ab-b+1}{ab+1}$, $v = \frac{1}{ab+1}$; it is the complex-valued version of [Go3 (1.17)].}
\begin{equation}
-Li_3 \left( \frac{ab-b+1}{ab} \right) - Li_3 \left( \frac{ab-b+1}{a} \right) - Li_3 (a(ab - b + 1)) + 2 \left\{ Li_3(a) + Li_3(b) + Li_3(-ab) + Li_3(ab - b + 1) - Li_3(1) 
+ Li_3 \left( \frac{ab-b+1}{a} \right) + Li_3 \left( \frac{ab-b+1}{b} \right) \right\} = \log^2(a) \log(-ab) - \frac{\pi^2}{3} \log(a) - \frac{1}{3} \log^3(a)
\end{equation}

essentially follows from Reciprocity Law B. The “essentially” means that we will work modulo degenerate terms (i.e. products of log and $Li_2$ in rational-function arguments) and assume the relations
\begin{equation}
Li_3(y) = Li_3(\frac{1}{y}) + 2\zeta(2) \log(y) - \frac{1}{6} \log^3(y) - \frac{\pi^2}{2} \log^2(y)
\end{equation}
\begin{equation}
Li_3(x) + Li_3(1-x) + Li_3(\frac{x}{x-1}) = Li_3(1) + Li_2(1) \log(1-x) - \frac{1}{2} \log(-x) \log^2(1-x) + \frac{1}{6} \log^3(1-x)
\end{equation}
from [Le2 pp. 154-5]. We shall denote $Li_3(z) =: [z]$, so that (6.2) and (6.3) become $[y] \equiv [y^{-1}]$ and
\begin{equation}
[x] + [1-x] + [1 - \frac{1}{2} x] \equiv [1]
\end{equation}
modulo degenerates.

In contrast to the situation (of a $\mathbb{P}^1$ in $\mathbb{P}^4$) worked out in §5, the direct application of Reciprocity Law B to $X := \text{a completely general } \mathbb{P}^2 \text{ in } \mathbb{P}^6$
seems somewhat intractable. Working modulo degenerates allows us to eliminate the $\int_{\mathcal{P}} \Xi^2$ integrals, which (by Lemma 4.7) take the same form as the $S_3^\Delta$ integrals worked out in §5. At this point we can relax the notion of general position in Definition 4.4 to proper intersections for $k \geq 2$:

**Lemma 6.1.** Let $U \subset Gr(3,7)$ be the analytic open on which $X$ and its projections to the coordinate hyperplanes are general in this sense. (This is the complement of a real codimension-1 subset.) Then writing $S_{5,j}^\Delta := S_5^\Delta (X_0 : \cdots : \hat{X}_j : \cdots X_6)$, the integrals $\int_X S_{5,j}^\Delta$ are (complex) analytic as a function of $X \in U$.

With this relaxed notion, the projectivized row space $X_{a,b,c}(\cong \mathbb{P}^2)$ of

$$
\begin{pmatrix}
1 & 0 & c & -1 & 0 & 0 & 1 \\
x & a & 1 & 0 & -1 & 0 & 1 \\
y & 0 & b & 1 & -1 & 1 & 0
\end{pmatrix}
$$

is general in $\mathbb{P}^6$ for sufficiently general $(a, b, c) \in \mathbb{C}^3$. By Lemma 6.1, the seven integrals

$$
\mathcal{I}_j(a, b, c) := \frac{1}{(2\pi i)^2} \int_{X_{a,b,c}} S_{5,j}^\Delta, \quad j = 0, \ldots, 6
$$

are each analytic on the complement $U_j \subset \mathbb{C}^3$ of some real codimension-1 subset. (This is just the locus where the projection of $X_{a,b,c}$ to $\mathbb{P}_j^6$ is general.) Since we do not know if $\bigcap U_j \subseteq U := U \cap \mathbb{C}^3$ is connected, and we prefer to evaluate the $\mathcal{I}_j$ in different regions, we have to consider the “jumps” in the $\mathcal{I}_j$ as we cross over $\mathbb{C}^3 \setminus U_j$.

**Lemma 6.2.** The jumps in the $\{\mathcal{I}_j\}$ (across real codimension-1 components of $\mathbb{C}^3 \setminus U_j$) are degenerate.

Lemmas 6.1 and 6.2 are proved in the first appendix to this section. The upshot of this discussion is that we have

$$
\sum_{i=0}^{6} (-1)^j \mathcal{I}_j \equiv 0
$$

modulo degenerates, and that (in (6.7)) we may evaluate each $\mathcal{I}_j$ anywhere in $U_j$ and analytically continue the results to a common neighborhood in $U$. To apply Reciprocity Law B in this form, we shall begin by choosing real subloci $A_j \subset U_j \cap \mathbb{R}^3$ on which the integrand $\mathcal{I}_j$ has
only one nonvanishing term:

\[ A_0 := \{ a \in \mathbb{R}; b \in (\frac{1}{2}, 1); c \in (\frac{1}{1-b}, \infty) \} \]
\[ A_1 := \{ a \in (0, \frac{1}{2}); b \in \mathbb{R}; c \in (-\infty, 1 - \frac{1}{a}) \} \]
\[ A_2 := \{ a \in (0, 1); b \in (1, \frac{1}{1-a}); c \in \mathbb{R} \} \]
\[ A_3 = \cdots = A_6 := \{ a, b, c \in \mathbb{R}_{<0}; |abc| > 1 \} . \]

For example, \( \mathcal{I}_0 \) is the integral (on \( \mathbb{P}^2 \)) of

\[
\frac{1}{(2\pi i)^2} S_{\Delta}( x + by : c + y : -(1 + x + y) : y : x : 1 ) =
\begin{array}{cccc}
X_1 & X_2 & X_3 & X_4 \\
Y_1 & Y_2 & Y_3 & Y_4
\end{array}
\]

\[
\log \left( \frac{y}{1+x+y} \right) \, d\log \left( \frac{-x}{y} \right) \wedge d\log \left( \frac{-1}{x} \right) \cdot \delta_{T_{c+y}^{x+y+1}} \cap T_{c+y}^{1+x+y} \cap T_{c+y}^{1+x+y}
\]

and the boxed intersection is empty on \( A_0 \).

More uniformly, writing

\[
\tau_0 := T_{c+y}^{x+y+1} \cap T_{c+y}^{x+y+1},
\]
\[
\tau_1 := T_{c+y}^{x+y+1} \cap T_{c+y}^{x+y+1},
\]
\[
\tau_2 := T_{c+y}^{x+y+1} \cap T_{c+y}^{x+y+1},
\]
\[
\tau_3 = \cdots = \tau_6 := T_{c+y}^{x+y+1} \cap T_{c+y}^{x+y+1},
\]

we have that

\[
\tau_i \cap T_{c+y}^{x+y+1} = \emptyset \text{ on } A_i \ (i = 0, 1, 2)
\]
\[
\tau_3 \cap T_{c+y}^{x+y+1} = \emptyset \text{ on } A_3
\]
\[
\tau_i \cap T_{c+y}^{x+y+1} = \emptyset \text{ on } A_i \ (i = 4, 5, 6).
\]
Hence the $\mathcal{I}_j$ are integrals of $\log(\cdot)\, d\log(\cdot) \wedge d\log(\cdot)$-forms on the (positively-oriented) regions:

$$\mathcal{I}_0 = -\int_{y=-c}^{1/(b-1)} \int_{x=-by}^{x=-y=1} \frac{\log(y) - \log(x + y + 1)}{xy} \, dx \, dy,$$

$$-\mathcal{I}_1 = \int_{y=1-a/b}^{y=-x/b} \int_{x=-y-1}^{x=-y=1} \frac{\log(y) - \log(x + y + 1)}{xy} \, dx \, dy,$$

$$\mathcal{I}_2 = -\int_{x=-1/a}^{x=-1/a} \int_{y=x=-y-1}^{y=x=-y=1} \frac{\log(y) - \log(x + y + 1)}{xy} \, dx \, dy.$$
and

$$-J_3 + J_4 - J_5 + J_6 = \int_{\tau_3} \left\{ -\log \left( \frac{c+y}{c+y} \right) \ d\log \left( \frac{z+x}{z+y} \right) \land \ d\log \left( \frac{1}{x} \right) \\
+ \log \left( \frac{z+y+1}{c+y} \right) \ d\log \left( \frac{x}{x+y+1} \right) \land \ d\log \left( \frac{1}{x} \right) \\
- \log \left( \frac{z+y+1}{c+y} \right) \ d\log \left( \frac{y}{x+y+1} \right) \land \ d\log \left( \frac{1}{y} \right) \\
+ \log \left( \frac{z+y+1}{c+y} \right) \ d\log \left( \frac{y}{x+y+1} \right) \land \ d\log \left( \frac{1}{y} \right) \right\}$$

(6.11)

$$= \int_{y=\frac{1}{ab}}^{-c} \int_{x=-\frac{1}{a}}^{-by} \frac{\log(y) - \log(x + y + 1)}{xy} \ dx \ dy.$$

Evaluating these integrals as described in the second appendix below yields (mod degenerates)

$$J_0 \equiv [b] - [1 - b] + 2[c] + [1 - c] - \left[ \frac{bc-c+1}{b} \right] + [bc - c + 1] + \left[ \frac{bc-c+1}{bc} \right] - [1]$$

$$-J_1 \equiv [a] - [1 - c] - \left[ \frac{ac-a+1}{c} \right] + \left[ \frac{ac-a+1}{ac} \right] + [ac - a + 1]$$

$$J_2 \equiv [a] - \left[ \frac{1 - 1}{b} \right] - \left[ \frac{ab-b+1}{a} \right] + \left[ \frac{ab-b+1}{ab} \right] + [ab - b + 1]$$

$$-J_3 + J_4 - J_5 + J_6 \equiv 2[-ab] - 2[c] - \left[ \frac{ab-b+1}{ab} \right] + \left[ \frac{ab-b+1}{ab} \right] + 2 \left[ \frac{ab-b+1}{-b} \right]$$

$$+ \left[ \frac{bc-c+1}{b} \right] - [bc - c + 1] - \left[ \frac{bc-c+1}{bc} \right] - [a(ab - b + 1)]$$

$$+ [ab - b + 1] + \left[ \frac{ac-a+1}{c} \right] - \left[ \frac{ac-a+1}{ac} \right] - [ac - a + 1].$$

Adding these and making use of (6.4), all the terms involving $c$ cancel and we have

$$0 \equiv 2[a] + 2[b] + 2[-ab] - 2[1] + 2 \left[ \frac{ab-b+1}{ab} \right] + 2 \left[ \frac{ab-b+1}{-b} \right]$$

$$+ 2[ab - b + 1] - \left[ \frac{ab-b+1}{a} \right] - \left[ \frac{ab-b+1}{ab} \right] - [a(ab - b + 1)],$$

which recovers the $Li_3$ terms in (6.1).

Remark 6.3. (a) If we take $(a, b, c)$ equal in (6.8)-(6.11), they are just the integrals of $\log \left( \frac{y}{x+y+1} \right) \ dx \land \ dy \land \ dy$ over a sum of four canceling triangles. This gives a quicker proof of (6.1), but of course there is something to the fact that Reciprocity Law $B$ produces the right combination of triangles.

(b) For fixed $c$, (6.5) gives a map $G$ from $U \subset \mathbb{C}^2$ to $Gr(3, 7)$ analogous to $g$ in Remark 5.1. In contrast to the Bol 5-web situation, there is clearly no nice relationship between the leaves of the Kummer-Spence 9-web $\mathbb{P}^2$ and the $\int_{G(a,b)} S_{5,3}^\Delta$ integrals (of which there are only 7). One could still ask whether the functions $a, b, -ab, \frac{ab-b+1}{ab}, \frac{ab-b+1}{-b}, \frac{ab-b+1}{a}, \frac{ab-b+1}{ab^2}, a(ab - b + 1)$ are $G$-pullbacks of some natural functions on $Gr(3, 7)$, perhaps related to the higher cross-ratios (of 6 points on $\mathbb{P}^2$) of Goncharov [Go4].
(c) The cancellation of all terms involving $c$ was a surprise to the authors. We do expect that some variant of (6.5) should lead to a similar proof of Goncharov’s 22-term relation \cite{Go3}, but leave this as a problem for others.

**Appendix I to §6: Proof of Lemmas 6.1 and 6.2.** Write $U_j \subset Gr(3, 7)$ for the region on which the projection of $X$ to $\mathbb{P}^5_j$ is general in the weaker sense. Note that the $\tilde{J}_j(X) := \int_X S^\Delta_{5,j}$ are equisingular hence continuous for $X \in U_j$.\footnote{We don’t need to worry about the properness of intersection $X \cap T_{-\frac{f_j}{f_0}}$, because the terms $\int_{X \cap T_{-\frac{f_j}{f_0}}} \log \cdot \log \wedge d\log \wedge d\log$ vanish by Hodge type.} We will show that the restriction of $\tilde{J}_j$ to $P \cap U_j$ is holomorphic, for $P \subset Gr(3, 7)$ an arbitrary $\mathbb{P}^1$ (with coordinate $t$). Let $\mu \subset P \cap U_j$ be a small disk with boundary $\partial \mu =: \gamma$. By Morera’s theorem, it will suffice to check that $\int_{X \in \gamma} \tilde{J}_j(X)dt = 0$.

Pulling $S^\Delta_{5,j}$ back to the total space $\cup_{X \in U_j} X =: \tilde{X}_j \to U_j$, we compute using Proposition 2.3

\begin{equation}
\int_{\pi^{-1}(\gamma)} \frac{S^\Delta_{5,j}}{(2\pi i)^2} \wedge dt = \int_{\pi^{-1}(\mu)} d \left[ \frac{S^\Delta_{5,j}}{(2\pi i)^2} \right] \wedge dt = \sum_{j' \neq j} (\pm 1) \int_{\pi^{-1}(\mu)} (\rho_j')^* \frac{S^\Delta_{5,j'}}{(2\pi i)^2} \wedge dt.
\end{equation}

(Here we also use the fact that equisingularity $\implies T^\Delta_{5,j} \cap \pi^{-1}(\mu) = \emptyset$.) The terms of (6.12) take the form

$$\frac{1}{2\pi i} \int_{\mu \times \mathbb{P}^1} R_4 \left( -\frac{f_j}{f_0}, -\frac{f_j}{f_1}, -\frac{f_j}{f_2}, -\frac{f_j}{f_3} \right) \wedge dt,$$

where the $f_i$ are linear forms algebraic in $t \in \mu$ (and the $\mathbb{P}^1$ corresponds to $X \cap \mathbb{P}^5_j$). By Hodge type, the $\int_{\mu \times \mathbb{P}^1} \log \cdot d\log \wedge d\log \wedge dt$ and $\int_{\mu \times \mathbb{P}^1 \cap T_j} \log \cdot d\log \wedge d\log \wedge dt$ terms vanish; while the $\mu \times \mathbb{P}^1 \cap T \cap T \cap T$ vanish by equisingularity. In fact, the $\mu \times \mathbb{P}^1 \cap T \cap \frac{f_j}{f_0} \cap T \cap \frac{f_j}{f_1}$ vanish also, by equisingularity and linearity of the $f_i$ (so that $T \cap \frac{f_j}{f_0} \cap X \cap \mathbb{P}^5_j$ and $T \cap \frac{f_j}{f_1} \cap X \cap \mathbb{P}^5_j$ are open segments in $\mathbb{P}^1 \cong X \cap \mathbb{P}^5_j$ meeting only at an endpoint). Hence (6.12) is zero and Lemma 6.1 is proved.

The proof of Lemma 6.2 is similar. Pulling $S^\Delta_{5,j}$ back to the total space $\cup_{X_{a,b,c}} := \mathcal{X} \leftarrow \mathbb{C}^3 \subset Gr(3, 7)$, we have for $p, q \in U_j$

$$\int_{\pi^{-1}(p)} \frac{S^\Delta_{5,j}}{(2\pi i)^2} - \int_{\pi^{-1}(q)} \frac{S^\Delta_{5,j}}{(2\pi i)^2} = \int_{\pi^{-1}(q) \setminus \pi^{-1}(p)} d \left[ \frac{S^\Delta_{5,j}}{(2\pi i)^2} \right]$$
\[
\sum_{Q(3) \atop j' \neq j} (\pm 1) \int_{\pi^{-1}(\bar{q}p)} (p_{j'})^* \frac{S^\Delta_{j'j}}{2\pi i} \delta_T
\]
by Proposition 2.3. The only possible contributions to a jump arise when a \(\delta_T\)-term in \(S^\Delta_{j} \setminus U_j\) lies over a component of \(C^3 \setminus U_j\) crossed by \(\bar{q}p\), and then the contribution is a combination of \((2\pi i) \int_{\mathbb{P}^1} \log \cdot \log \cdot \delta_T\) and \((2\pi i)^2 \int_{\mathbb{P}^1} \log \cdot \delta_T \cdot \delta_T\) integrals, which are obviously degenerate.

Appendix II to §6: Evaluating (6.8) - (6.11). In the course of the computation, we must frequently evaluate integrals of the form

\[(6.13) \quad \int \frac{\log(a - x) \log(b - x)}{x} \, dx,\]

\[(6.14) \quad \int \text{Li}_2 \left( \frac{1}{a(1 + x)} \right) \frac{dx}{x}.\]

Begin by rewriting (6.13) as

\[(6.15) \quad \int \frac{\log^2(a - x) + \log^2(b - x)}{2x} \, dx - \int \frac{\log(\frac{a-x}{b-x})}{2x} \, dx.\]

The first integral may be done by parts twice (e.g. \(u = \log^2(a - x)\) and \(dv = \frac{dx}{x}\); then substitute \(t = a - x\) and take \(u = \log t, \quad dv = \frac{dt}{t} = (\log a) \frac{dt}{t} - d(\text{Li}_2(\frac{1}{a})))\), which yields

\[- \text{Li}_3 \left( 1 - \frac{x}{a} \right) + \text{Li}_2 \left( 1 - \frac{x}{a} \right) \log(a - x) + \frac{1}{2} \log(\frac{a-x}{a}) \log^2(a - x) - \text{Li}_3 \left( 1 - \frac{x}{b} \right) + \text{Li}_2 \left( 1 - \frac{x}{b} \right) \log(b - x) + \frac{1}{2} \log(\frac{b-x}{b}) \log^2(b - x)\]

For the second integral in (6.15), substituting \(y = \frac{a-x}{b-x}\) gives

\[-\frac{1}{2} (a-b) \int \frac{\log^2(y)}{(1-y)(a-yb)} \, dy,\]

whereupon repeated integration by parts (starting with \(u = \log^2 y, \quad dv = dy/((1-y)(a-yb)))\) yields

\[- \text{Li}_3(y) + \text{Li}_3(\frac{by}{a}) - \log(y) \text{Li}_2(\frac{by}{a}) - \frac{1}{2} \log^2(y) \log(1 - \frac{yb}{a}) + \text{Li}_2(y) \log(y) + \frac{1}{2} \log(1 - y) \log^2(y).\]

The \(\text{Li}_3\) terms from (6.13) are therefore

\[\left[ \frac{b(a-x)}{a(b-x)} \right] - \left[ \frac{a-x}{b-x} \right] - \left[ 1 - \frac{x}{a} \right] - \left[ 1 - \frac{x}{b} \right].\]
For (6.14), taking $u = \text{Li}_2\left(\frac{1}{a(1+x)}\right)$ and $dv = \frac{dx}{x}$ gives

$$\log(x)\text{Li}_2\left(\frac{1}{1+x}\right) - \int \frac{\log\left(\frac{1}{x+1}\right)\log(x)}{x+1}$$

whereupon substituting $t = x + 1$ puts the last integral in the form (6.13). This yields (6.14)$\equiv$

$$- \left[\frac{ax+a-1}{x}\right] + \left[\frac{ax+a-1}{ax}\right] + [1 - a(x + 1)] + [-x] + [x + 1].$$

So for example, $\mathcal{I}_0$ breaks into

$$- \int_{y=-c}^{1} \int_{x=-by}^{y-1} \frac{\log(y)}{xy} \, dx \, dy \equiv [c] - \left[\frac{1}{1-b}\right],$$

which is straightforward, and

$$\int_{y=-c}^{1} \int_{x=-by}^{y-1} \frac{\log(x+y+1)}{xy} \, dx \, dy =$$

$$\int_{y=-c}^{1} -\zeta(2) + \log(-1-y) \log(1+y) - \log(-by) \log(1+y) + \text{Li}_2\left(\frac{by}{1+y}\right)dy.$$
[Ho] I. Horozov, Reciprocity laws on algebraic surfaces via iterated integrals (with an appendix by I. Horozov and M. Kerr), J. of K-theory 14 (2014), 273-312.
[Ke1] M. Kerr, “Geometric construction of regulator currents with applications to algebraic cycles”, Princeton Univ. Ph.D. Thesis, 2003.
[Ke2] ———, A regulator formula for Milnor K-groups, K-theory 29 (2003), no. 3, 175-210.
[Ke3] ———, An elementary proof of Suslin reciprocity, Canad. Math. Bull. 48 (2005) v.2, 221-236.
[KL] M. Kerr and J. Lewis, The Abel-Jacobi map for higher Chow groups, II, Invent. Math. 170 (2007), 355-420.
[KLM] M. Kerr, J. Lewis and S. Müller-Stach, The Abel-Jacobi map for higher Chow groups, Compos. Math 142 (2006), no. 2, 374-396.
[Le] L. Lewin, “Polylogarithms and associated functions”, North-Holland, Amsterdam, 1981.
[Lv] M. Levine, Bloch’s higher Chow groups revisited, Asterisque 226 (1994), 235-320.
[Pe] O. Petras, Functional equations of the dilogarithm in motivic cohomology, J. of Number Theory 129 (2009), 2346-2368.
[Pi] L. Pirio, Abelian functional equations, planar web geometry and polylogarithms, Sel. math., New ser. 11 (2005), 453-489.

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