On Cusick-Cheon’s Conjecture About Balanced Boolean Functions in the Cosets of the Binary Reed-Muller Code

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Abstract—It is proved an amplification of Cusick-Cheon’s conjecture on balanced Boolean functions in the cosets of the binary Reed-Muller code $RM(k, m)$ of order $k$ and length $2^m$, in the cases where $k = 1$ or $k \geq (m - 1)/2$.

Index Terms—Boolean function, Reed-Muller code, coset of linear code, Walsh-Hadamard transform.

I. INTRODUCTION

For basic definitions and facts we refer to [8]. Let $RM(k, m)$ denote the $k$th-order Reed-Muller code of length $2^m$. This linear code consists of all binary vectors of length $2^m$ (truth tables) associated with Boolean functions in $m$ variables whose degree is less than or equal to $k$. A Boolean function is said to be balanced if its truth table contains equal number of zeroes and ones. We shall also call a truth table of a Boolean function balanced word. In [4], the authors of zeroes and ones. We shall also call a truth table of a balanced Boolean function balanced word. In [4], the authors have conjectured the following:

CONJECTURE 1.1: The code $RM(k, m)$, $k > 0$, considered as a coset in the quotient space $Q(k, m) \equiv RM(k + 1, m)/RM(k, m)$ has more balanced functions than any other coset in $Q(k, m)$.

This conjecture was verified in cases $k = 1, m - 1$ [5]. Based on it the authors of [4] derived very good upper and lower bounds on the number of balanced Boolean functions which are contained in $RM(k, m)$. For some particular values of $k$ and arbitrary $m$, explicit formulas for the number of balanced functions in $RM(k, m)$ are known [8], [10]. Apart from trivial cases ($k = 1, m - 1$ and $m$) there are such formulas for $k = 2$ and $k = m - 2$, as a part of the known weight-distribution of the corresponding Reed-Muller codes.

However, the problem of determining the weight-distribution of $RM(k, m)$ in general, seems to be difficult [8] and even partial results are welcomed [7]. For similar results in the context of cryptographic applications, see also [2], [3] and [11, Ch. 8].

This paper is organized as follows. In next section we summarize necessary background. In Section III we present proofs of the extension of Conjecture [1] for arbitrary coset of the $RM(k, m)$, if $k = 1$ or $k \geq (m - 1)/2$. Finally, in Section IV we give an example which illustrates these considerations.

II. BACKGROUND

Let us recall the so-called MacWilliams’s identity [8, p. 127].

Let $\mathbf{A}$ be a binary linear $(n, K)$ code and $(A_0, A_1, \ldots, A_n)$ denote the weight distribution of $\mathbf{A}$ i.e. the total number of vectors of weight $i$ in $\mathbf{A}$ is $A_i$ for each $i$. Then

$$
\sum_{i=0}^{n} A_i x^i = 2^{K-n} \sum_{i=0}^{n} B_i (1+X)^{n-i}(1-X)^i,
$$

where $B_i$ is the total number of vectors of weight $i$ in $A^\perp$, the orthogonal code of $\mathbf{A}$.

We make use also of the following result proven by Assmus and Mattson.

**Theorem 2.2:** ([11]) Let $\mathbf{A}$ be a binary linear $(n, K)$ code and $\mathbf{a}$ be an $n$-vector over $\mathbb{F}_2 = GF(2)$ not in $\mathbf{A}$. Let $(d_0, d_1, \ldots, d_n)$ denote the weight distribution of the coset $\mathbf{A} + \mathbf{a}$; thus the total number of vectors of weight $i$ in $\mathbf{A} + \mathbf{a}$ is $d_i$ for each $i$. Then

$$
\sum_{i=0}^{n} d_i x^i = 2^{K-n} \sum_{i=0}^{n} (2b_i - B_i)(1+X)^{n-i}(1-X)^i,
$$

where $b_i$ is defined as the number of vectors of weight $i$ in the orthogonal code $A^\perp$ that are also orthogonal to $\mathbf{a}$ and $B_i$ is the total number of vectors of weight $i$ in $A^\perp$.

The above results were stated for a linear code over an arbitrary finite field, but for our goals these particular versions are enough.

The following deep theorem is due to McEliece.

**Theorem 2.3:** ([9]) The weight of every codeword in $RM(k, m)$ is divisible by $2^{[(m-1)/k]}$.

Let us remind also the following definition.

**Definition 2.4:** [8, p. 151] For an arbitrary positive integer $n$ the Krawtchouk polynomial $P_k(x; n) = P_k(x)$ is defined as

$$
P_k(x; n) \equiv \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{n-x}{k-j},
$$

$k = 0, 1, 2, \ldots$, where as usual $x$ is a variable while the binomial coefficients are defined as in Ex. 18 [8, Ch. 1]. Note that $P_k(x)$ is a polynomial of degree $k$.

For the sake of completeness we recall the definitions of weight and Walsh-Hadamard transform of a Boolean function.

Below, \(\sum\) stands for the ordinary summation, while \(\text{"+"}\) is used for the modulo-2 summation.

The **weight** of a Boolean function $f$ is equal to the number of nonzero positions in the truth table of $f$ and is denoted by $wt(f)$. A Boolean function $f$ is uniquely determined by its
Walsh-Hadamard transform, which is a real-valued function over \( \mathbb{F}_2^n \) defined for all \( \omega \in \mathbb{F}_2^n \) as
\[
W_f(\omega) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) + x \cdot \omega} = 2^m - 2 \text{wt}(f + x \cdot \omega),
\]
Here the dot product or scalar product of the vectors \( x = (x_1, x_2, \ldots, x_m) \) and \( \omega = (\omega_1, \omega_2, \ldots, \omega_m) \) is defined as \( x \cdot \omega = x_1 \omega_1 + x_2 \omega_2 + \cdots + x_m \omega_m \).

It is easy to see that the Boolean function \( f \) is balanced if and only if \( W_f(0) = 0 \). We recall also, the so-called Parseval’s equation:
\[
\sum_{\omega \in \mathbb{F}_2^n} W_f(\omega)^2 = 2^{2m}
\]

III. THE PROOFS

First, we shall prove the following lemma.

**Lemma 3.1:** For an arbitrary even positive integer \( n \) and \( i = 0, 1, \ldots, n \), let us define the numbers \( K(i, n) \) as
\[
K(i, n) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} (n/2 - j).
\]

Then \( K(i, n) \) is equal to 0 for \( i \) odd, negative when \( i \equiv 2 \pmod{4} \) and positive when \( i \equiv 0 \pmod{4} \).

**Proof:** Note that \( K(i, n) \) is actually \( P_{n/2}(i) \), where \( P_{n/2}(x) \) is the Krawtchouk polynomial of degree \( n/2 \). Further we make use of the Ex. 46 [8, p. 153] which states that for arbitrary nonnegative integers \( i \) and \( j \) the following recurrent formula holds:
\[
(n - i)P_j(i + 1) = (n - 2j)P_j(i) - iP_j(i - 1),
\]
where \( P_j(x) \) is the Krawtchouk polynomial of degree \( j \). In our case \( j = n/2 \), thus we have:
\[
(n - i)P_{n/2}(i + 1) = -iP_{n/2}(i - 1)
\]
The initial values: \( P_{n/2}(0) = \binom{n}{n/2} \) and \( P_{n/2}(1) = 0 \) are easily computed (see e.g. equation 5.57 and Ex. 44 [8, pp. 151-153]). The proof follows by induction on \( i \) using recurrent relation 6.

Now, we shall prove an amplification of Cusick-Cheon’s conjecture in some special cases.

**Theorem 3.2:** Let \( B(k, m) \) be the number of balanced words in the binary Reed-Muller code \( RM(k, m) \), \( k \geq (m - 1)/2 \). Then any nontrivial coset of \( RM(k, m) \) contains less than \( B(k, m) \) balanced words.

**Proof:** Let \( a \) be a binary vector of length \( n = 2^m \) not in \( A = RM(k, m) \) and \( C = A + a \) be the considered coset. It is well-known that the dimension of \( A \) is \( K = \sum_{j=0}^{k} \binom{m}{j} \) and the orthogonal code \( A^\perp \) coincides to \( RM(m - k - 1, m) \). Let \( b_i \) be the number of vectors of weight \( i \) in \( A^\perp \) that are orthogonal to \( a \) and \( B_i \) is the total number of vectors of weight \( i \) in \( A^\perp \), \( 0 \leq i \leq n \). Applying Theorem 2.1 and Theorem 2.2 we get, respectively:
\[
B(k, m) = 2^{K-n} \sum_{i=0}^{n} B_i K(i, n)
\]
where \( d_n/2 \) is the number of balanced words in \( C \) and the numbers \( K(i, n) \) are defined in Lemma 3.1. So, we yield:
\[
B(k, m) - d_n/2 = 2^{K-n+1} \sum_{i=0}^{n} (B_i - b_i) K(i, n)
\]
Clearly, by definition of the numbers \( B_i \) and \( b_i \), we have: \( B_i \geq b_i \). Also, there exists at least one weight \( i \) for which last inequality holds strictly, since, otherwise the vector \( a \) must belong to \( A \). Furthermore, if \( k \geq (m - 1)/2 \) then \((m - 1)/(m - k - 1) \geq 2 \), and hence according to McEliee’s Theorem all weights of codewords in \( A^\perp \) are divisible by 4. Thus, by Lemma 3.1 the numbers \( K(i, n) \) are positive and consequently, the sum in equation 6 is positive as well, which completes the proof.

Finally, we shall prove the following extension of the Conjecture 1.1 in the case where \( k = 1 \):

**Proposition 3.3:** Any nontrivial coset of the first order binary Reed-Muller code \( RM(1, m) \) contains less than \( 2^{m+1} - 2 \) balanced words.

**Proof:** First, let us note that the number of balanced words in \( RM(1, 1) \) itself, is \( 2^m + 2 \), and the two unbalanced words are the all-zero and all-one vectors of length \( 2^m \).

Let \( f \) be an arbitrary non-affine function and \( f \) be its corresponding truth table. We consider the coset \( C = RM(1, m) + \omega \). By the Parseval’s equation there exists at least one \( \omega \), say \( \omega_0 \), such that \( W_f(\omega_0) \neq 0 \). Let \( g = f + x \cdot \omega_0 \). Clearly, \( W_f(\omega_0) = W_g(0) \) and therefore the function \( g \) is unbalanced, as well as \( g + 1 \), of course. Suppose, \( g \) and \( g + 1 \) are the only two unbalanced functions (words) in \( C \). Then obviously, \( W_f(\omega) = 0 \), for \( \omega \neq \omega_0 \) and by the Parseval’s equation \( W_f(\omega_0) = \pm 2^m \). Hence, according to 3, \( wt(g) \) is equal to either 0 or \( 2^m \), which means that either \( f = x \cdot \omega_0 \) or \( f = x \cdot \omega_0 + 1 \), a contradiction to a choice of \( f \). Consequently, \( C \) contains more than two unbalanced words which completes the proof.

IV. AN EXAMPLE

In this section we present an example which illustrates the above considerations. We shall use the same notations as in the proof of Theorem 3.3.

Consider the \((m - 2)\)th order Reed-Muller code \( RM(m - 2, m), m \geq 3 \), which is in fact the extended Hamming code of length \( n = 2^m \). The orthogonal code is the first-order Reed-Muller code \( RM(1, m) \) and consists of truth tables of the affine functions and the vectors 0, 1. So, the nonzero \( B_i \)’s are: \( B_{2m-1} = 2^{m+1} - 2 \) and \( B_i = 0 \) for \( i = 0, 2^m \). Applying equation 1 for the weight-distribution of the code \( H_{n,m} = RM(m - 2, m) \), we get the well-known (see e.g. [10]):
\[
\sum_{i=0}^{n} H_i X^i = 2^{-(m+1)}[(1 + X)^n + (2^{m+1} - 2)(1 - X^2)^{n/2} + (1 - X)^n]
\]

Thus, we have:
\[
B(m - 2, m) = H_{n/2} = \frac{1}{n} \left( \frac{n}{2} \right) + (n - 1) \left( \frac{n/2}{n/4} \right)
\]
Let \( a_1 \) be the following \( 2^m \)-vector of weight 2: \((0, 0, \ldots, 1, 1)\). This vector is associated with the Boolean function which is a product of the first \( m - 1 \) amongst the Boolean variables \( Y_1, Y_2, \ldots, Y_{m-1}, Y_m \) i.e. the function: \( Y_1 Y_2 \ldots Y_{m-1} \). It is easy to see that the truth table of an affine function is orthogonal to \( a_1 \) only if this function does not contain \( Y_m \) as an essential variable. So, \( b_{2^m-1} = 2^m - 2 \) and since the vectors \( 0, 1 \) are orthogonal to \( a_1 \) it follows \( b_0 = b_{2^m} = 1 \). Applying equation (2) for the weight-distribution of the coset \( C_1 = H_m + a_1 \), we get:

\[
\sum_{i=0}^{n} d_i X^i = 2^{-(m+1)}(1 + X)^n - 2(1 - X^2)^{n/2} + (1 - X)^n
\]

Therefore, we have:

\[
d_{n/2} = \frac{1}{m} \binom{n}{n/2} - \binom{n/4}{n/2}
\]

This result coincides with the outcome of computations given in [5]. In fact, it can be shown that all cosets of \( RM(m-2, m) \) in \( RM(m-1, m) \) are affine equivalent (see, e.g. [6]) and therefore they have the same weight-distribution (in particular, the same number of balanced functions).

Let, now \( a_2 \) be the following \( 2^m \)-vector of weight 1: \((0, 0, \ldots, 0, 1)\). This vector is associated with the Boolean function \( Y_1 Y_2 \ldots Y_{m-1} Y_m \). Of course, we can proceed as in the previous case, but the following simple arguments show that every word in the coset \( C_2 = H_m + a_2 \) is with odd weight. Indeed, if \( f \in H_m \) then \( wt(f) \) is an even number and \( wt(f + a_2) \) is equal to \( wt(f) \pm 1 \) accordingly to the value of the last coordinate of \( f \). Thus, there are no balanced functions in the coset \( C_2 \), and by similar arguments this is valid also for all cosets of \( RM(m-2, m) \) not in \( RM(m-1, m) \).

V. CONCLUSION

In this paper, we consider an extension of Cusick-Cheon’s conjecture on balanced Boolean functions on the cosets of the binary Reed-Muller code \( RM(k, m) \) and prove it in the special cases: \( k = 1 \) or \( k \geq (m - 1)/2 \). To our knowledge, the Conjecture 1.1 is still unproved (or disproved) in the remaining cases. Note also, that Theorem 3.2 is valid for any code of even length whose orthogonal is doubly-even code i.e. if the weights of all codewords in the orthogonal code are divisible by 4.

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