Breaking the Communication-Privacy-Accuracy Trilemma

Wei-Ning Chen  
Department of Electrical Engineering  
Stanford University  
wnchen@stanford.edu

Peter Kairouz  
Google  
kairouz@google.com

Ayfer Özgür  
Department of Electrical Engineering  
Stanford University  
aozgur@stanford.edu

Abstract

Two major challenges in distributed learning and estimation are 1) preserving the privacy of the local samples; and 2) communicating them efficiently to a central server, while achieving high accuracy for the end-to-end task. While there has been significant interest in addressing each of these challenges separately in the recent literature, treatments that simultaneously address both challenges are still largely missing. In this paper, we develop novel encoding and decoding mechanisms that simultaneously achieve optimal privacy and communication efficiency in various canonical settings. In particular, we consider the problems of mean estimation and frequency estimation under $\varepsilon$-local differential privacy and $b$-bit communication constraints. For mean estimation, we propose a scheme based on Kashin’s representation and random sampling, with order-optimal estimation error under both constraints. For frequency estimation, we present a mechanism that leverages the recursive structure of Walsh-Hadamard matrices and achieves order-optimal estimation error for all privacy levels and communication budgets. As a by-product, we also construct a distribution estimation mechanism that is rate-optimal for all privacy regimes and communication constraints, extending recent work that is limited to $b = 1$ and $\varepsilon = O(1)$. Our results demonstrate that intelligent encoding under joint privacy and communication constraints can yield a performance that matches the optimal accuracy achievable under either constraint alone.

1 Introduction

The rapid growth of large-scale datasets has been stimulating interest in and demands for distributed learning and estimation, where datasets are often too large and too sensitive to be stored on a centralized machine. When data is distributed across multiple devices, communication cost often becomes a bottleneck of modern machine learning tasks [32]. This is even more so in federated learning type settings, where communication occurs over bandwidth-limited wireless links [25]. Moreover, as more personal data is entrusted to data aggregators, in many applications it carries sensitive individual information, and hence finding ways to protect individual privacy is of crucial importance. In particular, local differential privacy (LDP) [15, 16, 28, 43] is a widely adopted privacy paradigm, which guarantees that the outcome from a privatization mechanism will not release too much individual information statistically. In this paper, we study the relationship between utility (often in forms of accuracy for certain statistical tasks), privacy, and communication jointly.

Preprint. Under review.
At first glance, privacy and communication may seem to be in conflict with each other: achieving privacy requires the addition of noise, therefore increasing the entropy of the data and making it less compressible. For instance, consider the mean estimation problem, which appears as a fundamental subroutine in many distributed optimization tasks, e.g. distributed stochastic gradient descent (SGD). Here, the goal is to estimate the empirical mean of a collection of $d$-dimensional vectors. If we first privatize each vector via $\text{PrivUnit}$ in [10] (which is optimal under LDP constraints) and then quantize via the RandomSampling quantizer in [18] (which is optimal under communication constrains), a tedious but straightforward calculation shows that the resulting $\ell_2$ estimation error grows with $d^2$. However, this is far from matching the error rate under each constraint separately, which has a linear dependence on $d$. A similar phenomenon happens in the distribution estimation problem, where each client’s data is drawn independently from a discrete distribution $p$ with support size $d$. One can satisfy both constraints by first perturbing the data via the Subset Selection (SS) mechanism [45] (which is optimal under LDP constraints) and then quantizing the noised data to $b$ bits. Again, it can be shown that under such strategy, the $\ell_2$ estimation error of $p$ has a quadratic dependence on $d$. This leaves a huge gap to the lower bounds under each constraint separately, which have a linear dependence on $d$. See Section [B] in the appendix for a detailed discussion.

While there has been significant recent progress on understanding how to achieve optimal accuracy under separate privacy [9, 45] and communication [36, 46] constraints, as illustrated above a simple concatenated application of these optimal schemes can yield a highly suboptimal performance. Recent works that attempt to break this communication-privacy-accuracy trilemma have been either limited to specific regimes or, as we show, are far from optimal. For example, [1] provides a 1-bit $\varepsilon$-LDP scheme for distribution estimation which is order-optimal only in the low communication regime ($b = O(1)$) and high privacy regime ($\varepsilon = O(1)$), while [18] tries to address both constraints in the mean estimation setting, but the error rate achieved under their mechanism is quadratic in $d$ and therefore does not improve on the above baseline.

This paper closes the above gaps for any given privacy level $\varepsilon$ and communication budget $b$. Indeed, our results show that the fundamental trade-offs are determined by the more stringent of the two constraints, and with careful encoding we can satisfy the less stringent constraint for free, thus breaking the privacy-communication-accuracy trilemma. For the same privacy level $\varepsilon$, this allows us to achieve the accuracy of existing mechanisms in the literature with drastically smaller communication budget, or equivalently, for the same communication budget to achieve higher privacy. It also explains, for example, why 1-bit communication budget is sufficient under the high privacy regime [18]. We will demonstrate this phenomenon in various canonical tasks and answer the following question: “given arbitrary privacy budget $\varepsilon$ and communication budget $b$, what are the fundamental limits for estimation accuracy?” We next formally define the settings and the problem formulations we consider in this paper.

### 1.1 Problem Formulation

The general distributed statistical tasks we consider in this paper can be formulated as follows: each one of the $n$ clients has local data $X_i \in \mathcal{X}$ and sends a message $Y_i \in \mathcal{Y}$ to the server, who upon receiving $Y^n$ aims to estimate some pre-specified quantity of $X^n$. Note that $X^n$ are not necessarily drawn from some distribution. At client $i$, the message $Y_i$ is generated via some mechanism (a randomized mapping that possibly uses shared randomness across participating clients) denoted by a conditional probability $Q_i(y|X_i)$ satisfying the following constraints.

**Local differential privacy** Let $(\mathcal{Y}, \mathcal{B})$ be a measurable space, and $Q(\cdot|x)$ be probability measures for all $x \in \mathcal{X}$, with $\{Q(\cdot|x) | x \in \mathcal{X}\}$ dominated by some $\sigma$-finite measure $\mu$ so that the density $Q(y|x)$ exists. A mechanism $Q$ is $\varepsilon$-LDP if

$$\forall x, x' \in \mathcal{X}, y \in \mathcal{Y}, \frac{Q(y|x)}{Q(y|x')} \leq e^{\varepsilon}.$$  

**$b$-bit communication constraint** $\mathcal{Y}$ satisfies $b$-bit communication constraint if each of its elements can be described by $b$ bits, i.e. $|\mathcal{Y}| \leq 2^b$.

The goal is to jointly design a mechanism (at clients’ sides) and an estimator (at the server side) so that the accuracy of estimating some target function $\sum_{i=1}^n f(X_i)$ is maximized. In this paper, we are mainly interested in the distribution-free framework, that is, we do not assume any underlying
distribution on $X_i$, but we also demonstrate that our results can be extended to probabilistic settings. To this end, we will focus on the following three canonical tasks.

**Mean estimation** For real-valued data, we consider the $d$-dimensional unit euclidean ball $\mathcal{X} = B_d(0,1)$ and are interested in estimating the mean $\bar{X} = \frac{1}{n} \sum_i X_i$. The goal is to minimize the worst-case $\ell_2$ estimation error defined as

$$r_{\text{ME}}(\ell_2, \varepsilon, b) \triangleq \min_{(\hat{X}, Q^n)} \max_{X^n \in \mathcal{X}^n} \mathbb{E} \left[ \left\| \bar{X} - \hat{X} \right\|_2^2 \right],$$

where $Q^n$ satisfies $\varepsilon$-LDP and $b$-bit communication constraints. When the context is clear, we may omit $\varepsilon$ and $b$ in $r_{\text{ME}}(\ell, \varepsilon, b)$.

**Frequency estimation** When $\mathcal{X}$ consists of categorical data, i.e. $\mathcal{X} = [d] = \{1, ..., d\}$, we are interested in estimating $D_{X^n}(x) \triangleq \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i = x\}}$ for $x \in [d]$. With a slight abuse of notation, $D_{X^n}$ is viewed as a vector $(D_{X^n}(1), ..., D_{X^n}(d))$ lying in the $d$-dimensional probability simplex. The worst-case estimation error is defined by

$$r_{\text{FE}}(\ell, \varepsilon, b) \triangleq \min_{(\hat{D}, Q^n)} \max_{D^n \in \Delta_d^d} \mathbb{E} \left[ \ell(D, \hat{D}, Q^n) \right],$$

where $\ell = \| \cdot \|_1$, or $\| \cdot \|_2^2$ and again $Q^n$ satisfies $\varepsilon$-LDP and $b$-bit communication constraints.

**Distribution estimation** A closely related setting is that of discrete distribution estimation, where we assume that the $X_i$’s are drawn independently from a discrete distribution $p$ on the alphabet $\mathcal{X} = [d]$, and the goal is to estimate $p$. In this case, the worst-case error is given by

$$r_{\text{DE}}(\ell, \varepsilon, b) \triangleq \inf_{(\hat{p}, Q^n)} \sup_{p \in \mathcal{P}_d} \mathbb{E} \left[ \ell(\hat{p}, p) \right],$$

where $\mathcal{P}_d$ is the $d$-dimensional probability simplex.

We note that these canonical tasks serve as fundamental subroutines in many distributed optimization and learning problems. For instance, the convergence rate of distributed SGD is determined by the $\ell_2$ error of estimating the mean of the local gradient vectors (see [3] for more on this connection). Lloyd’s algorithm [29] for k-means clustering or the power-iteration method for PCA can also be reduced to the mean estimation task.

**Remark 1.1** In the absence of shared randomness\(^1\), one can show that $\log d$ bits are necessary to achieve an error rate that decays with $n$, see Section A in the appendix for more details. Therefore, we assume the availability of public randomness for both frequency and mean estimation. For distribution estimation, on the other hand, we show that private randomness suffices.

### 1.2 Relation to Prior Work

Previous works in the mean estimation problem [4, 7, 18, 36, 38, 44] mainly focus on reducing communication cost, for instance, by random rotation [36] and sparsification [4, 11, 42, 44]. Among them, [18] considers LDP simultaneously. It proposes vector quantization and takes privacy into account, developing a scheme for $\varepsilon = \Theta(1)$ and $b = \Theta(\log d)$ with estimation error $O(d^2/n)$. In contrast, the scheme we develop in Theorem 2.1 achieves an estimation error $O(d/n)$ when $\varepsilon = \Theta(1)$ and $b = \Theta(\log d)$. Moreover, our scheme is applicable for any $\varepsilon$ and $b$ and achieves the optimal estimation error, which we show by proving a matching information theoretic lower bound. A key step in our scheme is to pre-process the local data via Kashin’s representation [30]. While various compression schemes, based on quantization, sparsification and dithering have been proposed in the recent literature and Kashin’s representation for communication efficiency has been also explored in a few works [13, 17, 34, 35], it is particularly powerful in the case of joint communication and privacy constraints as it helps spread the information in a vector evenly in every dimension. This

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\(^1\)We will use the terms “shared randomness” and “public-coin” interchangeably. In a public-coin scheme, the server and clients can access a shared random variable $U$ that is independent of the data; otherwise we call it a private-coin scheme.
Privacy \( \forall \varepsilon \)
Comm. \( \forall b \)
\( \ell_2 \) error \( \frac{d}{n \min(\varepsilon^2, \varepsilon, b)} \)

Privacy \( \varepsilon \in (0, 1) \)
Comm. \( \varepsilon \in (1, \log d) \)
\( \varepsilon \) \( b \)
\( \ell_2 \) error \( \Theta \left( \frac{d}{n \min(\varepsilon^2, \varepsilon, b)} \right) \)

Table 1: Comparison between our mean estimation scheme and vqSGD [18], where Cp and Sp refer to the Cross-polytope and Simplex methods in [18]. Our scheme applies to general communication and privacy regimes, and achieves optimal estimation error for all scenarios.

Frequency estimation under local differential privacy has been studied in [40], where they propose schemes for estimating the frequency of an individual symbol and minimizing the variance of the estimator. Some of their schemes, while matching the information-theoretic lower bound on \( \ell_2 \) estimation error under privacy constraints, require large communication. For instance, the scheme Optimal Unary Encoding (OUE) achieves optimal \( \ell_2 \) estimation error, but the communication required is \( O(d) \) bits, which, as we show in this work, can be reduced to \( O(\min(\lceil \varepsilon \rceil, \log d)) \) bits. We do this by developing a new scheme for frequency estimation under joint privacy and communication constraints. We establish the optimality of our proposed schemes by deriving matching information theoretic lower bounds on \( r_{FE}(\ell_2, \varepsilon, b) \).

Frequency estimation is also closely related to heavy hitter estimation [1, 8, 9, 12, 23, 33, 47], where the goal is to discover symbols that appear frequently in a given data set and estimate their frequencies. This can be done if the error of estimating the frequency of each individual symbol can be controlled uniformly (i.e. by a common bound), and thus is equivalent to minimizing the \( \ell_\infty \) error of estimated frequencies, i.e. \( r_{FE}(\ell_\infty, \varepsilon, b) \). It is shown in [9] that in the high privacy regime \( \varepsilon = O(1) \), \( r_{FE}(\ell_\infty, \varepsilon, b) = \Theta(\sqrt{\log d/\varepsilon^2}) \), and this rate can be achieved via a 1-bit public-coin scheme that has a runtime almost linear in \( n \). An extension, which we describe

The recent works of [31, 41] also consider estimating empirical mean under \( \varepsilon \)-LDP. They show that if the data is from \( d \)-dimensional unit \( \ell_\infty \) ball, i.e. \( X_i \in [-1, 1]^d \), then directly quantizing, sampling and perturbing each entry can achieve optimal \( \ell_\infty \) estimation error that matches the LDP lower bound in [14]. Nevertheless, their approach does not yield good \( \ell_2 \) error in general. Indeed, as in the case of separation schemes discussed in Section B, the \( \ell_2 \) error of their scheme can grow with \( d^2 \). We emphasize that in many applications the \( \ell_2 \) estimation error (i.e. MSE) is a more appropriate measure than \( \ell_\infty \). For instance, [3] shows a direct connection between the MSE in mean estimation and the convergence rate of distributed SGD.

| Loss | Estimation error | Communication |
|------|-----------------|--------------|
| OUE [40] | \( \ell_2 \) \( \Theta \left( \frac{d}{n \min(\varepsilon^2, \varepsilon, b)} \right) \) | \( d \) bits |
| Thm 3.1 | \( \ell_2 \) \( \Theta \left( \frac{d}{n \min(\varepsilon^2, \varepsilon, b)} \right) \) | \( \min(\lceil \varepsilon \rceil, \log d) \) bits |
| Thm 3.1 (Heavy hitter) | \( \ell_\infty \) \( \Theta \left( \sqrt{\frac{\log d}{n \min(\varepsilon^2, \varepsilon^2)}} \right) \) | \( \lceil \varepsilon \rceil \) bits |

Table 3: Comparison of different frequency estimation schemes.
in Section E.4 of the appendix, generalizes the achievability in [9] to arbitrary $\varepsilon$ and $b$, achieving $r_{\text{ME}}(\ell_2, \varepsilon, b) = O(\sqrt{\log d/n \min(\varepsilon^2, \varepsilon, b)})$. We compare our scheme and existing results in Table 3.

If we further assume $X^n$ are drawn from some discrete distribution $p$, then the problem falls into distribution estimation under local differential privacy [12, 14, 24, 39, 45, 47] and limited communication [6, 11, 19, 21, 22, 46]. Tight lower bounds are given separately: for instance [2, 45] shows $r_{\text{DE}}(\ell_1, \varepsilon, \log d) = \Omega(\sqrt{d^2/n \min((\varepsilon - 1)^2, \varepsilon)})$ and [21] shows $r_{\text{ME}}(\ell_1, \infty, b) = \Omega(\sqrt{d^2/n^2b})$.

We show that these lower bounds can be achieved simultaneously (Theorem 3.2). Our result recovers the result of [1] when $b = 1$ and $\varepsilon = O(1)$ as a special case. See Table 2 for a comparison.

1.3 Our Contributions and Techniques

To summarize, our main technical contributions include:

- For mean estimation, we characterize the optimal $\ell_2$ error $r_{\text{ME}}(\ell_2) = \Theta(d/n \min(\varepsilon^2, \varepsilon, b))$, by designing a public-coin scheme, Subsampled and Quantized Kashin’s Response (SQKR), and proving its optimality by deriving matching information theoretic bounds (Theorem 2.1). Our encoding scheme is based on Kashin’s representation [30] and random sampling, which allow the server to construct unbiased estimator of each $X_i$ privately and with little communication. This significantly improves on [18], which focuses on the special case $\varepsilon = O(1)$, $b = \log d$ and achieves quadratic dependence on $d$ in that case.

- For frequency estimation, we characterize the optimal $\ell_1$ and $\ell_2$ errors under both constraints (in Theorem 3.1) and propose an order-optimal public-coin scheme called Recursive Hadamard Response (RHR). Our result shows that the accuracy is dominated only by the worst-case constraint, and this implies that one can achieve the less stringent constraint for free. The proposed scheme RHR is based on Hadamard transform, but unlike previous works using Hadamard transform, e.g. [8], we crucially leverage the recursive structure of the Hadamard matrix, which allows us to make the estimation error decay exponentially as $\varepsilon$ and $b$ grow. RHR is computationally efficient, and the decoding complexity is $O(n + d \log d)$. We establish its optimality by showing matching lower bounds on the performance.

- We show that RHR easily leads to an optimal scheme for distribution estimation [1][2][45], in which case it does not require shared randomness and achieves order-optimal $\ell_1$ and $\ell_2$ error for all privacy regimes and communication budgets. We also provide empirical evidence that our scheme requires significantly less communication while achieving the same accuracy and privacy levels as the state-of-the-art approaches. See Section 4 for more results.

2 Mean Estimation

In the mean estimation problem, each client has a $d$-dimensional vector $X_i$ from the Euclidean unit ball, and the goal is to estimate the empirical mean $\bar{X} = \frac{1}{n} \sum_i X_i$ under $\varepsilon$-LDP and $b$ bits communication constraints. This problem has applications in private and communication efficient distributed SGD. The following theorem characterizes the optimal $\ell_2$ estimation error for this setting.

**Theorem 2.1** For mean estimation under $\varepsilon$-LDP and $b$-bit communication constraints, we can achieve

$$r_{\text{ME}}(\ell_2, \varepsilon, b) \leq d/n \min(\varepsilon^2, \varepsilon, b).$$

Moreover, if $\min(\varepsilon^2, \varepsilon, b) = o(d)$ and $n \cdot \min(\varepsilon^2, \varepsilon, b) > d$, the above error is optimal.

Note that by taking $\varepsilon \rightarrow \infty$ for a fixed $b$, or by taking $b \rightarrow \infty$ for a fixed $\varepsilon$ in part (i), Theorem 2.1 provides the optimal error when we have the corresponding constraint alone. Furthermore, for finite $\varepsilon$ and $b$ we see that the optimal error is dictated by the error due to one of these constraints, the one that leads to larger error, and hence the less stringent constraint is satisfied for free. This also implies that to achieve the optimal accuracy under $\varepsilon$-LDP constraints, we do not need more than $\lceil \varepsilon \rceil$ bits. We note that the two conditions for optimality in the theorem are standard and are needed to restrict the problem to the interesting parameter regime.

The lower bounds are obtained by connecting the problem to a specific parametric estimation problem with a distribution supported on the unit ball. The lower bounds $\Omega(\frac{d}{n\varepsilon^2})$ and $\Omega(\frac{d}{nb})$ appear
At a high-level, SQKR resembles vqSGD \cite{18} as both schemes seek a suitably designed representation to match this lower bound, we propose a public-coin scheme, Subsampled and Quantized Kashin’s Response (SQKR), based on Kashin’s representation \cite{30} and random sampling.

2.1 Subsampled and Quantized Kashin’s Response

For each observation \( X_i \), we aim to construct an unbiased estimator \( \hat{X}_i \), which is \( \varepsilon \)-LDP, can be described in \( b \) bits, and has small variance. Towards this goal, our general strategy is to quantize, subsample, and privatize the data \( X_i \). However before this, it is crucial to pre-process each \( X_i \) by a carefully designed mechanism to increase the robustness of the signal to noise introduced by sampling and privatization.

Pre-processing via Kashin’s representation We first introduce the idea of a tight frame in Kashin’s representation. A tight frame is a set of vectors \( \{ u_j \}_{j=1}^N \in \mathbb{R}^d \) that satisfy Parseval’s identity, i.e. \( \|x\|^2 = \sum_{j=1}^N \langle u_j, x \rangle^2 \) for all \( x \in \mathbb{R}^d \). A frame can be viewed as a generalization of the notion of an orthogonal basis in \( \mathbb{R}^d \) for \( N > d \). To increase robustness, we wish the information to be spread evenly across different coefficients. Thus, we say that the expansion \( x = \sum_{j=1}^N a_j u_j \) is a Kashin’s representation of \( x \) at level \( K \) if \( \max_j |a_j| \leq \frac{K}{\sqrt{N}} \|x\|_2 \) \cite{27}. \cite{30} shows that if \( N > (1 + \mu) d \) for some \( \mu > 0 \), then there exists a tight frame \( \{ u_j \}_{j=1}^N \) such that for any \( x \in \mathbb{R}^d \), one can find a Kashin’s representation at level \( K \). This implies that we can represent each \( X_i \) with coefficients \( \{ a_j \}_{j=1}^N \in [-c/\sqrt{d}, c/\sqrt{d}]^{c'd} \) for some constants \( c \) and \( c' \).

Quantization Each client \( i \) computes the Kashin’s representation \( \{ a_j \}_{j=1}^N \in [-c/\sqrt{d}, c/\sqrt{d}]^{c'd} \) of \( X_i \), and then quantizes each \( a_j \) into a 1-bit message \( q_j \in \{-c/\sqrt{d}, c/\sqrt{d}\} \) with \( \mathbb{E}[q_j] = a_j \). This yields an unbiased estimator of \( \{ a_j \}_{j=1}^N \), which can be described in \( \Theta(d) \) bits in total. Moreover, due to the small range of each \( a_j \), the variance of \( q_j \) is bounded by \( O(1/d) \).

Sampling and privatization To further reduce \( \{ q_j \} \) to \( k = \min(\lceil \varepsilon \rceil, b) \) bits, client \( i \) draws \( k \) independent samples from \( \{ q_j \}_{j=1}^N \) with the help of shared randomness, and privatizes its \( k \) bits message via \( 2^k \)-RR mechanism \cite{26,43}, yielding the final privatized report of \( k \) bits, which it sends to the server.

Upon receiving the report from client \( i \), the server can construct unbiased estimators \( \hat{a}_j \) for each \( \{ a_j \}_{j=1}^N \), and hence reconstruct \( \hat{X}_i = \sum_{j=1}^N \hat{a}_j u_j \), which yields an unbiased estimator of \( X_i \). We show that the variance of \( \hat{X}_i \) can be controlled by \( O \left( \frac{d}{\min(\varepsilon^2, \varepsilon, b)} \right) \). Therefore \( \frac{1}{N} \sum_i \hat{X}_i \) achieves the order-optimal \( \ell_2 \) estimation error, establishing the upper bound in Theorem 2.1. We provide a detailed description of the scheme and its performance analysis in Section C.

At a high-level, SQKR resembles vqSGD \cite{18} as both schemes seek a suitably designed representation for \( X_i \) before quantizing it. vqSGD represents \( X_i \) by a basis \( B = \{ b_1, \ldots, b_K \} \subset \mathbb{R}^d \) where \( B \) is chosen in such a way that its convex hull contains the unit \( \ell_2 \) ball. Therefore we can write \( X_i = \sum_{j=1}^N a_j b_j \) with \( \sum_j a_j = 1 \). Equivalently, the pre-processing step of vqSGD corresponds to a linear transformation that embeds the \( d \)-dim \( \ell_2 \) unit ball into a \( N \)-dim \( \ell_1 \) ball. In contrast, Kashin’s representation above embeds the \( d \)-dim \( \ell_2 \) unit ball into an \( N \)-dim \( \ell_\infty \) ball. Therefore, while both schemes have a pre-processing step of a similar flavor, what is achieved by these steps is quite different. The representation of vqSGD is most efficient when it concentrates the information in a few coefficients, while Kashin’s representation spreads the information evenly across different coefficients. The first representation serves us well when we only seek to quantize the signal. However, the quantized signal becomes very sensitive to privatization noise. Therefore vqSGD ends up with \( O(d^2) \) error in the case of both privacy and communication constraints, while we can achieve \( O(d) \) error.
3 Frequency Estimation

Recall that in the frequency estimation problem, given $X_1, \ldots, X_n \in [d]$, we want to estimate the empirical frequency $D_{X^n}(x)$ under $\varepsilon$-LDP and $b$ bits communication budgets on each $X_i$. The following theorem characterizes the optimal estimation error achievable in this setting.

**Theorem 3.1** For frequency estimation under $\varepsilon$-LDP and $b$ bits communication constraint, we can achieve

(i) $r_{\text{FE}}(\ell_2) \leq \frac{d}{n \min \{e^\varepsilon, (e^\varepsilon - 1)^2, 2^b, d\}}$, and $r_{\text{FE}}(\ell_1) \leq \frac{d}{\sqrt{n \min \{e^\varepsilon, (e^\varepsilon - 1)^2, 2^b, d\}}}$;

(ii) $r_{\text{FE}}(\ell_\infty) \leq \sqrt{\frac{\log d}{n \min \{e^\varepsilon, \varepsilon, b\}}}$.

Moreover, if $\min \{e^\varepsilon, (e^\varepsilon - 1)^2, 2^b\} = o(d)$ and $n \min \{e^\varepsilon, (e^\varepsilon - 1)^2, 2^b\} \geq d^2$, the errors in (i) are order-optimal.

Note that, similar to Theorem 2.1, Theorem 3.1 shows that for finite $\varepsilon$ and $b$, the error is determined by the error due to one of these constraints, and hence the other less stringent constraint is satisfied for free. It also implies that to achieve the optimal accuracy under $\varepsilon$-LDP constraints, we do not need more than $\min \{\lfloor \log_2 e \cdot \varepsilon \rceil, \log d\}$ bits. In the rest of the section, we overview the scheme we develop to achieve the optimal error in (i).

We next overview the scheme that achieves the error in (i) of Theorem 3.1. We call this scheme Recursive Hadamard Response (RHR) as it builds on the recursive structure of the Hadamard matrix. The formal description of the scheme and complete proof of Theorem 3.1 can be found in Section E.

3.1 Recursive Hadamard Response

For notational convenience, we will view $D_{X^n}$ as a $d$-dimensional vector $(D_{X^n}(1), \ldots, D_{X^n}(d))$ and assume $X_i$ is one-hot encoded, i.e. $X_i = e_j$ for some $j \in [d]$, so $D_{X^n} = \frac{1}{n} \sum X_i$. We further assume, without of loss of generality, that $d = 2^m$ for some $m \in \mathbb{N}$. Recall that a Hadamard matrix $H_d \in \{-1, +1\}^{d \times d}$ can be constructed in a recursive fashion as

$$H_m = \begin{bmatrix} H_{m/2} & H_{m/2} \\ H_{m/2} & -H_{m/2} \end{bmatrix},$$

where $H_1 = [1]$. It can be easily shown that $H_d^{-1} = H_d/d$.

Instead of directly estimating $D_{X^n}$, our strategy is to first estimate $H_d \cdot D_{X^n}$ and then perform the inverse transform $H_d^{-1}$ to get an estimate for $D_{X^n}$. So each client will transmit information about $Y_i \triangleq H_d \cdot X_i \in \{-1, 1\}^d$ rather than its original data $X_i$.

**The 1-bit case** In this case, each client transmits a uniformly at random chosen entry of $Y_i$ via any 1-bit LDP channel (for instance, using the 2-randomized response (RR) scheme [24, 26, 43]). Once receiving all the bits of the clients, the server can construct an unbiased estimator of $Y_i$ (since the randomness is public the server knows which entry is chosen for communication by each client). It turns out that this simple 1-bit scheme achieves optimal $\ell_1$ (and $\ell_2$) error $\Theta(\sqrt{d^2/n \varepsilon^2})$ in the high privacy regime $\varepsilon < 1$. This idea is not new and has been used in heavy hitter estimation [8] and distribution estimation [1]. However, a key question remains: how do we minimize the error given an arbitrary communication budget $b$ and privacy level $\varepsilon$?

**Moving beyond the 1-bit case** A natural way to extend the 1-bit scheme above to the case when each client can transmit $b$-bits is to have each client communicate $b$ randomly chosen entries of its transformed data $Y_i$ instead of a single entry. This will boost the sample size by a factor of $b$, equivalently decrease the $\ell_2$ error by a factor of $b$ ($\sqrt{b}$ for $\ell_1$). Instead, we argue next that we can exploit the recursive structure of the Hadamard matrix to boost the sample size by a factor of $2^b$, equivalently decrease the error by an exponential factor.
Consider $b \leq \lfloor \log d \rfloor$ and let $B = d/2^b-1$. Note that $H_d = H_{2b-1} \otimes H_B$, where $\otimes$ denotes the Kronecker product. To visualize, for $b = 3$, $H_d$ has the following structure:

$$Y_i = H_d X_i = \begin{bmatrix} H_B & H_B & H_B & H_B \\ H_B & -H_B & H_B & -H_B \\ H_B & H_B & -H_B & -H_B \\ H_B & -H_B & -H_B & H_B \end{bmatrix} \begin{bmatrix} X_i^{(1)} \\ X_i^{(2)} \\ X_i^{(3)} \\ X_i^{(4)} \end{bmatrix},$$

where for $l = 1, \ldots, 2^{b-1}$, $X_i^{(l)}$ denotes the $l$'th block of $X_i$ of length $B = d/2^b-1$. Therefore, in order to communicate $Y_i$, we can equivalently communicate $H_B X_i^{(l)}$ for $l = 1, \ldots, 2^{b-1}$. Since $H_{2b-1}$ is known, this is sufficient to reconstruct $Y_i$. We next observe that while communicating $Y_i$, requires $d = B \times 2^{b-1}$ bits, communicating $\{H_B X_i^{(l)}, l = 1, \ldots, 2^{b-1}\}$ requires $B + (b - 1)$ bits. This is because $X_i$ is one-hot encoded and all but one of the $2^{b-1}$ vectors $\{H_B X_i^{(l)}, l = 1, \ldots, 2^{b-1}\}$ are equal to zero. It suffices to communicate the index $l$ of the non-zero vector, by using $(b - 1)$ bits, and its $B$ entries by using additional $B$ bits. This is the key observation that RHR builds on.

When each client has only $b$ bits, they cannot communicate sufficient information for fully reconstructing $Y_i$, i.e. all $\{H_B X_i^{(l)}, l = 1, \ldots, 2^{b-1}\}$. Instead, each client chooses a random index $r_i \in [B]$ and communicates the $r_i$'th row of $\{H_B X_i^{(l)}, l = 1, \ldots, 2^{b-1}\}$, equivalently $\{(H_B)_{r_i} X_i^{(l)}, l = 1, \ldots, 2^{b-1}\}$ where $(H_B)_{r_i}$ denotes the $r_i$'th row of $H_B$. Note that as before, only one of the $2^{b-1}$ numbers $\{(H_B)_{r_i} X_i^{(l)}, l = 1, \ldots, 2^{b-1}\}$ is non-zero and therefore these numbers can be communicated by using $b$ bits, $b - 1$ bits to represent the index of the non-zero number and a single bit to communicate its value. When there is a privacy constraint, client $i$ perturbs its $b$ bits by a $2^b$-RR mechanism with privacy level $\varepsilon$, and this yields the privatized report of $b$ bits.

Upon receiving the reports from clients, the server constructs an unbiased estimator for $Y_i$. To do this, it first constructs an unbiased estimator for $\{H_B X_i^{(l)}, l = 1, \ldots, 2^{b-1}\}$ and then employs the structure $H_d = H_{2b-1} \otimes H_B$. Note that since the randomness is shared the server knows the index $r$ chosen by each client, and since the clients choose their indices independently and uniformly at random, roughly speaking, they communicate information about different rows of $\{H_B X_i^{(l)}, l = 1, \ldots, 2^{b-1}\}$. Finally, an unbiased estimator $\hat{Y}_i$ for $Y_i$ yields an unbiased estimator for $X_i$ through the transformation $\hat{X}_i = \frac{1}{d} H_d \cdot \hat{Y}_i$, and due to the orthogonality of $H_d$, it can be shown that the variance of $\hat{X}_i$ is the same as the variance of $\hat{Y}_i$ divided by $d$.

A subtle issue is that if $e^\varepsilon < 2^b$, the noise due to $2^b$-RR mechanism may be too large, so instead of using all $b$ bits, we perform the above encoding and decoding procedure with $b' \triangleq \min \left( \lfloor \log_2 e \cdot \varepsilon \rfloor \right)$. We defer the details and the formal proof to Section E.1.

Note that this careful construction based on the recursive structure of the Hadamard matrix is only required in the case when there are joint privacy and communication constraints. When only one constraint is present, the optimal error can be achieved in a much simpler fashion. When there is only a $b$ bit constraint, [21] shows that the optimal error can be achieved by simply having each client communicate a subset of the entries of its data vector $X_i$ (without requiring Hadamard transform). When there is only a privacy constraint $\varepsilon$, the optimal error can be achieved by a number of schemes, such as subset selection ($2^b$-SS) [45] and Hadamard response (HR) [2].

The encoding mechanism above involves two operations: 1) sampling a random index $r_i$ from $[B]$ at each client with the help of a public coin, and 2) computing $(H_d)_{r_i} \cdot X_i$. Since $X_i$ is one-hot, the encoding complexity is $O(\log d)$. On the other hand, in order to efficiently decode, the server first computes the joint histogram of client $i$'s report and $r_i$ in $O(n)$ time, which in turn allows us to calculate $\frac{1}{d} \sum_i Y_i$, and then apply the Fast Walsh-Hadamard transform (FWHT) to obtain the estimator of empirical frequency in $O(d \log d)$ time. Hence the overall decoding complexity is $O(n + d \log d)$. See Algorithm 3 and Algorithm 4 in Section E for details.

### 3.2 Application to distribution estimation

For frequency estimation, RHR requires shared randomness so that the server can construct an unbiased estimator. However, for distribution estimation where $X_1, \ldots, X_n \sim p$, we can replace the
random sampling with deterministic one and circumvent the use of shared randomness. This gives us the following theorem:

**Theorem 3.2** For distribution estimation under $\varepsilon$-LDP and $b$ bits communication constraint, we can achieve

$$r_{DE}(\ell_2) \asymp \frac{d}{n \min \{ e^\varepsilon, (e^\varepsilon - 1)^2, 2^b, d \}}, \text{ and } r_{DE}(\ell_1) \asymp \frac{d}{\sqrt{n \min \{ e^\varepsilon, (e^\varepsilon - 1)^2, 2^b, d \}}}.$$  

Moreover, if $n \cdot \min \{ e^\varepsilon, (e^\varepsilon - 1)^2, 2^b, d \} \geq d^2$, the above errors are optimal.

The lower bounds follow directly from the results of [45] (under LDP constraint) and [6, 21] (under communication constraint). We leave the formal proof of the achievability to Section F.

## 4 Experiments

In this section, we implement our mean estimation and frequency estimation schemes and present our experimental results. More detailed results can be found in Section C.

### 4.1 Mean estimation

We implement our mean estimation scheme Subsampled and Quantized Kashin’s Response (SQKR) as in Section 2 and compare it with a baseline, a concatenation of privUnit [10] (which is order-optimal under $\varepsilon$-LDP) and the quantizer based on Kashin’s representation [30] (which is optimal up to a logarithmic factor, under $b$-bit communication constraint). Note that privUnit mechanism (Algorithm 1 in [10]) samples a vector from the unit sphere with proper probability density (which depends on $X_i$), and scales it by a factor of $O(\sqrt{d})$ in order to make it unbiased. It can be shown that such direct concatenation will result in $\tilde{O}(d^2)$ error rate (see Section C in appendix for more details).

**Generating the data** In order to capture the distribution-free setting, we generate data independently but non-identically; in particular, we set $Z_1, ..., Z_{n/2} \sim_{i.i.d.} N(1, 1)^{\otimes d}$ and $Z_{n/2+1}, ..., Z_n \sim_{i.i.d.} N(10, 1)^{\otimes d}$ (this also makes the data non-central, i.e. $\mathbb{E} \sum Z_i \neq 0$). Since each sample has bounded $\ell_2$ norm, we normalize each $Z_i$ by setting $X_i = Z_i / \|Z_i\|_2$.

The code can be found in [https://github.com/WeiNingChen/Kashin-mean-estimation](https://github.com/WeiNingChen/Kashin-mean-estimation) (for the SQKR scheme) and [https://github.com/WeiNingChen/RHR](https://github.com/WeiNingChen/RHR) (for the RHR scheme).

![Figure 1: $\ell_2$ error with $n = 10^5$ and different dimensions $d$. In order to better emphasize the dependence to $d$, on the right-hand side we only plot the $\ell_2$ error of SQKR.](image.png)
Generating the tight frame  We construct the tight frame by using the random partial Fourier matrices in [30]. Specifically, we set \( N = 2^{\lceil \log_2 d \rceil + 1} = \Theta(d) \), and choose the basis \( U = \left\{ \frac{1}{\sqrt{N}}, -\frac{1}{\sqrt{N}} \right\}^{N \times d} \) by selecting the first \( d \) rows of \( H_N \cdot D \), where \( H_N \) is a \( N \times N \) Hadamard matrix and \( D \) is a random diagonal matrix with each diagonal entry generated from uniform \( \{+1, -1\} \). It can be shown that the tight frame based on \( U \) has Kashin’s level \( K = \tilde{O}(1) \).

In Figure 1, we fix the sample size to \( n = 10^5 \) and \( \varepsilon, b \), and increase the dimension \( d \). From the result, we see that SQKR has linear dependence on \( d \), whereas the baseline (labeled as “Separation” since it is based on the idea of separately coding for privacy and communication efficiency) has super-linear dependence. Therefore the performance differs drastically when \( d \) increases.

4.2 Frequency estimation

For frequency estimation problem, we experimentally compare our scheme, Recursive Hadamard Response (RHR), with SS [45], HR [2] and 1-bit HR [1]. We set \( d = \{1000, 10000\}, \varepsilon = \{2, 5\} \), and evaluate the \( \ell_1 \) estimation errors on the truncated and normalized geometric distribution with \( \lambda = 0.8 \). For each point (i.e., for each parameter \( n, \varepsilon, d \)), we repeat the simulation 30 times and average the \( \ell_2 \) errors. Figure 2 shows that our schemes can achieve the same performance as HR but is significantly more communication efficient. For instance, in Figure 2 with \( d = 10000, \varepsilon = 5 \), RHR uses only half of the communication budget for HR and achieves better performance. In all settings, SS has the best statistical performance, but this comes with drastically higher communication and computation cost.

![Figure 2: \( \ell_1 \) error with \( d = 5000 \) and \( d = 10000 \), under (truncated) Geo(0.8) and different \( \varepsilon \).](image)

5 Conclusion

We have investigated frequency and mean estimation under \( \varepsilon \)-LDP and \( b \)-bit communication constraints. A significant advantage of the approaches we presented is that they achieve the privacy and communication constraints simultaneously at the cost of the harsher one. Many interesting questions remain to be addressed, including investigating if we can reduce the amount of shared randomness, deriving decoding schemes with optimal runtimes, and applying our results to distributed SGD.

6 Acknowledgments

The authors would like to thank Jakub Konečný for bringing Kashin’s representation to their attention. This was helpful in achieving order-optimality for mean estimation.

\footnote{For HR, we use the codes from [2] (https://github.com/zitengsun/hadamard_response)}
References

[1] J. Acharya and Z. Sun. Communication complexity in locally private distribution estimation and heavy hitters. In International Conference on Machine Learning, pages 51–60, 2019.

[2] J. Acharya, Z. Sun, and H. Zhang. Hadamard response: Estimating distributions privately, efficiently, and with little communication. In The 22nd International Conference on Artificial Intelligence and Statistics, pages 1120–1129, 2019.

[3] N. Agarwal, A. T. Suresh, F. X. X. Yu, S. Kumar, and B. McMahan. cpgsd: Communication-efficient and differentially-private distributed sgd. In Advances in Neural Information Processing Systems, pages 7564–7575, 2018.

[4] D. Alistarh, D. Grubic, J. Li, R. Tomioka, and M. Vojnovic. Qsgd: Communication-efficient sgd via gradient quantization and encoding. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, Advances in Neural Information Processing Systems 30, pages 1709–1720, Curran Associates, Inc., 2017.

[5] L. P. Barnes, W.-N. Chen, and A. Ozgur. Fisher information under local differential privacy. arXiv preprint arXiv:2005.10783, 2020.

[6] L. P. Barnes, Y. Han, and A. Ozgur. Lower bounds for learning distributions under communication constraints via fisher information, 2019.

[7] L. P. Barnes, H. A. Inan, B. Isik, and A. Ozgur. rtop-k: A statistical estimation approach to distributed sgd, 2020.

[8] R. Bassily, K. Nissim, U. Stemmer, and A. Thakurta. Practical locally private heavy hitters. In Proceedings of the 31st International Conference on Neural Information Processing Systems, NIPS’17, page 2285–2293, Red Hook, NY, USA, 2017. Curran Associates Inc.

[9] R. Bassily and A. Smith. Local, private, efficient protocols for succinct histograms. In Proceedings of the Forty-Seventh Annual ACM Symposium on Theory of Computing, STOC ’15, page 127–135, New York, NY, USA, 2015. Association for Computing Machinery.

[10] A. Bhownick, J. Duchi, J. Freudiger, G. Kapoor, and R. Rogers. Protection against reconstruction and its applications in private federated learning. arXiv preprint arXiv:1812.00984, 2018.

[11] M. Braverman, A. Garg, T. Ma, H. L. Nguyen, and D. P. Woodruff. Communication lower bounds for statistical estimation problems via a distributed data processing inequality. In Proceedings of the forty-eighth annual ACM symposium on Theory of Computing, pages 1011–1020, 2016.

[12] M. Bun, J. Nelson, and U. Stemmer. Heavy hitters and the structure of local privacy. In Proceedings of the 37th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, SIGMOD/PODS ’18, page 435–447, New York, NY, USA, 2018. Association for Computing Machinery.

[13] S. Caldas, J. Konečny, H. B. McMahan, and A. Talwalkar. Expanding the reach of federated learning by reducing client resource requirements. arXiv preprint arXiv:1812.07210, 2018.

[14] J. C. Duchi, M. I. Jordan, and M. J. Wainwright. Local privacy and statistical minimax rates. In 2013 IEEE 54th Annual Symposium on Foundations of Computer Science, pages 429–438. IEEE, 2013.

[15] C. Dwork, F. McSherry, K. Nissim, and A. Smith. Calibrating noise to sensitivity in private data analysis. In Theory of cryptography conference, pages 265–284. Springer, 2006.

[16] A. Evfimievski, J. Gehrke, and R. Srikant. Limiting privacy breaches in privacy preserving data mining. In Proceedings of the twenty-second ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems, pages 211–222, 2003.

[17] J.-J. Fuchs. Spread representations. In 2011 Conference Record of the Forty Fifth Asilomar Conference on Signals, Systems and Computers (ASILOMAR), pages 814–817. IEEE, 2011.

[18] V. Gandikota, D. Kane, R. K. Maity, and A. Mazumdar. vqsgd: Vector quantized stochastic gradient descent, 2019.

[19] A. Garg, T. Ma, and H. Nguyen. On communication cost of distributed statistical estimation and dimensionality. In Advances in Neural Information Processing Systems, pages 2726–2734, 2014.
[20] Y. Han, J. Jiao, and T. Weissman. Minimax estimation of discrete distributions. In 2015 IEEE International Symposium on Information Theory (ISIT), pages 2291–2295. IEEE, 2015.

[21] Y. Han, P. Mukherjee, A. Özgür, and T. Weissman. Distributed statistical estimation of high-dimensional and nonparametric distributions. In 2018 IEEE International Symposium on Information Theory (ISIT), pages 506–510. IEEE, 2018.

[22] Y. Han, A. Özgür, and T. Weissman. Geometric lower bounds for distributed parameter estimation under communication constraints. arXiv preprint arXiv:1802.08417, 2018.

[23] J. Hsu, S. Khanna, and A. Roth. Distributed private heavy hitters. In Proceedings of the 39th International Colloquium Conference on Automata, Languages, and Programming - Volume Part I, ICALP'12, page 461–472, Berlin, Heidelberg, 2012. Springer-Verlag.

[24] P. Kairouz, K. Bonawitz, and D. Ramage. Discrete distribution estimation under local privacy. In Proceedings of The 33rd International Conference on Machine Learning, volume 48, pages 2436–2444, New York, New York, USA, 20–22 Jun 2016.

[25] P. Kairouz, H. B. McMahan, B. Avent, A. Bellet, M. Bennis, A. N. Bhagoji, K. Bonawitz, Z. Charles, G. Cormode, R. Cummings, et al. Advances and open problems in federated learning. arXiv preprint arXiv:1912.04977, 2019.

[26] P. Kairouz, S. Oh, and P. Viswanath. Extremal mechanisms for local differential privacy. The Journal of Machine Learning Research, 17(1):492–542, 2016.

[27] B. Kashin. Section of some finite-dimensional sets and classes of smooth functions (in russian) izv. Acad. Nauk. SSSR, 41:334–351, 1977.

[28] S. P. Kasiviswanathan, H. K. Lee, K. Nissim, S. Raskhodnikova, and A. Smith. What can we learn privately? SIAM Journal on Computing, 40(3):793–826, 2011.

[29] S. Lloyd. Least squares quantization in pcm. IEEE transactions on information theory, 28(2):129–137, 1982.

[30] Y. Lyubarskii and R. Vershynin. Uncertainty principles and vector quantization. IEEE Transactions on Information Theory, 56(7):3491–3501, 2010.

[31] T. T. Nguyên, X. Xiao, Y. Yang, S. C. Hui, H. Shin, and J. Shin. Collecting and analyzing data from smart device users with local differential privacy, 2016.

[32] F. Niu, B. Recht, C. Re, and S. J. Wright. Hogwild! a lock-free approach to parallelizing stochastic gradient descent. In Proceedings of the 24th International Conference on Neural Information Processing Systems, NIPS’11, page 693–701, Red Hook, NY, USA, 2011. Curran Associates Inc.

[33] Z. Qin, Y. Yang, T. Yu, I. Khalil, X. Xiao, and K. Ren. Heavy hitter estimation over set-valued data with local differential privacy. In Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security, CCS ’16, page 192–203, New York, NY, USA, 2016. Association for Computing Machinery.

[34] M. Safaryan, E. Shulgin, and P. Richtárik. Uncertainty principle for communication compression in distributed and federated learning and the search for an optimal compressor. arXiv preprint arXiv:2002.08958, 2020.

[35] C. Studer, W. Yin, and R. G. Baraniuk. Signal representations with minimum $\ell_\infty$-norm. In 2012 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pages 1270–1277. IEEE, 2012.

[36] A. T. Suresh, F. X. Yu, S. Kumar, and H. B. McMahan. Distributed mean estimation with limited communication. In Proceedings of the 34th International Conference on Machine Learning - Volume 70, ICML’17, page 3329–3337. JMLR.org, 2017.

[37] M. J. Wainwright. High-dimensional statistics: A non-asymptotic viewpoint, volume 48. Cambridge University Press, 2019.

[38] H. Wang, S. Sievert, S. Liu, Z. Charles, D. Papailiopoulos, and S. Wright. Atomo: Communication-efficient learning via atomic sparsification. In Advances in Neural Information Processing Systems, pages 9850–9861, 2018.

[39] S. Wang, L. Huang, P. Wang, Y. Nie, H. Xu, W. Yang, X.-Y. Li, and C. Qiao. Mutual information optimally local private discrete distribution estimation, 2016.
A Impossibility Results without Public Coin

Without access to the public randomness, [1] shows that at least $\Theta(d)$ bits of communication is required for heavy hitter estimation in order to obtain a consistent estimator. We state their result here:

**Lemma A.1 ([1] Theorem 4)** Let $b \leq \log d - 2$. For all private-coin schemes $(Q^n, \hat{D})$ with only private randomness and $b$ bits communication budgets, there exists a data sets $X_1, \ldots, X_n$ with $n > \frac{12(2^b + 1)^2}{2^b + 2 + 4}$, such that
\[
E \left[ \left\| \hat{D} (Q^n) - D_{X^n} \right\|_\infty \right] \geq \frac{1}{2^{b+2} + 4}.
\]

Based on this, we claim that without public coin, each client needs to transmit at least $\Theta(\log d)$ bits in order to construct consistent schemes for frequency estimation or mean estimation.

**Frequency estimation** We lower bound $\ell_1$ and $\ell_2$ error by $\ell_\infty$ and apply Lemma A.1:
\[
E \left[ \left\| \hat{D} (Q^n) - D_{X^n} \right\|_1 \right] \geq E \left[ \left\| \hat{D} (Q^n) - D_{X^n} \right\|_\infty \right] \geq \frac{1}{2^{b+2} + 4},
\]
and
\[
E \left[ \left\| \hat{D} (Q^n) - D_{X^n} \right\|_2 \right] \geq E \left[ \left\| \hat{D} (Q^n) - D_{X^n} \right\|_\infty \right]^2 \geq \left( \frac{1}{2^{b+2} + 4} \right)^2.
\]

This implies that it is impossible to construct consistent schemes with less than $\log d - 2$ bits per client in the absence of a public randomness. On the other hand, given $\log d$ bits, one can readily achieve the optimal estimation accuracy without any public randomness, for instance, by using Hadamard response [2] (see also the discussion in [1]). Therefore, the problem of frequency estimation is somewhat trivialized in the absence of public randomness.

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4An estimator is consistent if it has vanishing estimation error as $n$ tends to infinity.
We first group $T_i \in [d]$ be one-hot encoded, so $X_i \in B_d(0, 1)$. Then (2) implies the $\ell_2$ error of mean estimation is at least $1 / \left(2^{b+2} + 4\right)^2$. Thus with less than $\log d - 2$ bits, it is also impossible to construct a consistent scheme for mean estimation.

**B Separate Quantization and Privatization Is Strictly Sub-optimal**

**Distribution estimation** First let us recap the subset selection (SS) scheme proposed by [45]. Assume $X_1, \ldots, X_n \overset{i.i.d.}{=} p = (p_1, \ldots, p_d)$. Client $i$ maps the local data $X_i$ into $y \in \mathcal{Y}_{d,w} \triangleq \{ y \in \{0,1\}^d : \sum_j y_j = w \}$ with the transitional probability

$$Q_{SS}(y|X = j) = \frac{e^\varepsilon y_j + (1 - y_j)}{e^\varepsilon (w^{-1}) + (d - w)}.$$

The estimator for $p_j$ is defined by

$$\hat{p}_j \triangleq \left( (d - 1)e^\varepsilon + \frac{(d - 1)(d - w)}{w} \right) \frac{T_j}{n} - \frac{(w - 1)e^\varepsilon + d - w}{(d - w)e^\varepsilon - 1},$$

where $T_j \triangleq \sum_{i=1}^n Y_i(j)$. Note that by picking $w = \lceil \frac{d}{e^\varepsilon + 1} \rceil$, SS is order-optimal for all privacy regimes.

To demonstrate that separating privatization and quantization is strictly sub-optimal, we analyze the estimation error of directly concatenating the $2^b$-SS mechanism with the grouping-based quantization in [21]. Note that both schemes are known to be optimal under the corresponding constraints, privacy and communication respectively. However, their direct combination yields an $\ell_2$ error of order $O\left( d^2 \right)$, which is far from the optimal accuracy established in Theorem 3.1.

We first group $[d]$ into $s = d / 2^b$ equal-sized groups $G_1, \ldots, G_s$, and each client is only responsible to send information about one particular group. That is, let $Y_i$ be the outcome of the $2^b$-SS mechanism, i.e. $Y_i \sim Q_{SS} (\cdot | X_i)$, and client $i$ only transmits $\{ Y_i(j) | j \in G_s' \}$, for some $s' \in [s]$. Since the server estimates each component of $p$ separately as in [3], this grouping strategy reduces the effective sample size from $n$ to $n' = n 2^b / d$. Plugging $n'$ into the $\ell_2$ error (see Proposition III.1 in [45]), we conclude that the error grows as

$$O\left( \frac{d^2}{n 2^b \min \left( \varepsilon', (e^\varepsilon - 1)^2 \right)} \right).$$

Note that since each $Y_i$ contains exactly $w$ ones, the required communication budget to describe $\{ Y_i(j), j \in G_s \}$ may be larger than $b$ bits. But this is fine since it implies that even given more than $b$ bits, the estimation error still grows with $d^2$. In Theorem 3.2 on the other hand, we show that the optimal $\ell_2$ error is linear in $d$, so this demonstrates that separate quantization and privatization is sub-optimal.

**Mean estimation** For the mean estimation problem, a straightforward combination is using the PrivUnit mechanism (see Algorithm 1 in [10]) to perturb the local data $X_i \in B_d(0, 1)$, and then using RandomSampling quantization in (Theorem 6 in [18]) to compress the perturbed data. Both schemes are known to be optimal under the corresponding constraints, privacy and communication respectively. (Note that in Section 4 we replaced the RandomSampling quantization with a Kashin’s quantizer, since implementing the theoretically optimal RandomSampling quantization is computationally infeasible.)

By Proposition 4 in [10], the output of PrivUnit, denoted as $Z_i = \text{PrivUnit}(X_i, \varepsilon)$, has $\ell_2$ norm of order $\Theta\left( \frac{d}{\min(\varepsilon, e^\varepsilon)} \right)$. However, if we further apply RandomSampling to $b$ bits, by Theorem 6 in [18], the $\ell_2$ estimation error grows as

$$\Theta\left( \frac{\|Z_i\|_2}{n \cdot b} \right) = \Theta\left( \frac{d^2}{n b \min(\varepsilon, e^\varepsilon, b)} \right),$$

showing a quadratic dependence in $d$. By Theorem 2.1 nevertheless, we can construct a better scheme with $O(\frac{d}{n} \min(\varepsilon, e^\varepsilon, b))$ dependence under both constraints.
C More Experimental Results

C.1 Mean estimation

We generate the data as well as the tight frame as described in Section 4.

**Compare to privUnit [10]** We first compare our scheme SQKR with privUnit [10], which is order-optimal under $\varepsilon$-LDP. Since the outcome of privUnit is a $d$-dimensional vector lying in a radius $O(\sqrt{d})$ sphere, in general we need $32d$ bits to represent it (where we assume each float requires 32 bits). Figure 3 shows that SQKR achieves similar performance with significantly communication budgets. For instance, when $\varepsilon = 5$ and $d = 50$, the communication cost of privUnit is $2K$ bits, while SQKR uses only 5 bits but attains similar performance.

![Figure 3: $\ell_2$ error of privUnit and SQKR with different dimensions $d = 50, 200$.](image)

Next, we compare SQKR with a combination of privUnit and an optimal quantizer.

**Baseline: a direct concatenation of privUnit, Kashin’s quantizer and sampling** For each $X_i$ in unit $\ell_2$ ball, privUnit maps it to a vector $\tilde{X}_i$ with length $\|\tilde{X}_i\|_2 = \Theta(\sqrt{d/\min(\varepsilon, \varepsilon^2)})$. If we quantize $\tilde{X}_i$ according to its Kashin’s representation and then subsample $b$ bits from it as in Section 2, then the $\ell_2$ error (i.e. variance) will be

$$\tilde{O}\left(\frac{d}{b} \|\tilde{X}_i\|^2\right) = \tilde{O}\left(\frac{d^2}{b \min(\varepsilon, \varepsilon^2)}\right).$$

Therefore, averaging over $n$ clients, the $\ell_2$ error of estimating the empirical mean is

$$\tilde{O}\left(\frac{d^2}{n \cdot b \min(\varepsilon, \varepsilon^2)}\right).$$

However, in Theorem 2.1, we see that with a more sophisticated design, we can achieve smaller $\ell_2$ error

$$O\left(\frac{d}{n \cdot \min(\varepsilon, \varepsilon^2, b)}\right).$$

**Setup** In the experiment, we mainly focus on the high-privacy low-communication setting where $\varepsilon = b = 1$, and the low-privacy high-communication setting where $\varepsilon = b = 5$. We consider different dimensions $d$ and plot the (log-scale) $\ell_2$ estimation error (i.e. mean square error) with sample size $n$. For each point, i.e. each combination of parameters $\varepsilon, b, d, n$, we repeat the simulation for 8 iterations and compute the average. In Figure 4 we see that SQKR drastically outperforms the baseline (labeled as "Separation" since it is based on the idea of separately coding for privacy and communication efficiency). The gain increases in higher dimensions or with more stringent privacy/communication constraints.
Figure 4: Log-scale $\ell_2$ error with different dimensions $d = 20, 50, 80$ and different privacy and communication budgets.

In order to study the dependence on $d$, we fix the sample size to $n = 10^5$ and $\varepsilon, b$, and increase the dimension $d$. In Figure 5, we see that SQKR has linear dependence on $d$, and Separation has super-linear dependence. Therefore the performance differs drastically when $d$ increases.
Figure 5: $\ell_2$ error with $n = 10^5$ and different dimensions $d$. In order to better emphasize the dependence to $d$, on the right-hand side we only plot the $\ell_2$ error of SQKR.

C.2 Frequency estimation

For frequency estimation, we compare our scheme, Recursive Hadamard Response (RHR), with SS [45], HR [2] and 1-bit HR [1]. We set $d = \{1000, 5000, 10000\}$, $\epsilon \in \{0.5, 2, 5\}$ and $n = \{50000, 100000, ..., 500000\}$, and evaluate the $\ell_1$ estimation errors on uniform distribution and truncated and normalized geometric distribution with $\lambda = 0.8$. For each point (i.e. for each parameter $n, \epsilon, d$), we repeat the simulation 30 times and average the $\ell_2$ errors. Figure 6 and Figure 7 show that RHR can achieve the same performance as HR but is significantly more communication efficient. For instance, in Figure 7 with $d = 10000$, $\epsilon = 5$, RHR uses only half of the communication budget for HR and achieves better performance. In all settings, $k$-SS has the best statistical performance, but this comes with drastically higher communication and computation cost.
Figure 6: $\ell_1$ error with $d = 1000$. Left are Geo(0.8) and right are Uniform.
In Figure 8, we record the decoding time for each scheme. The decoding complexity of RHR is similar to HR and 1-bit HR, which are all much more computationally efficient than SS.

Figure 7: $\ell_1$ error with $d = 5000$ and $d = 10000$, under (truncated) $\text{Geo}(0.8)$ and different $\varepsilon$.

Figure 8: Left: time complexity with $d = 10000$, $\varepsilon = 7$ right: time complexity with $d = 5000$, $\varepsilon = 2$. 
D Proof of Theorem 2.1

D.1 achievability

In this section, we prove that Subsampled and Quantized Kashin’s Response (SQKR) achieves optimal ℓ2 estimation error. For each observation \( X_i \), we will construct an unbiased estimator \( \hat{X}_i \) (i.e. \( \mathbb{E} [\hat{X}_i | X_i] = X_i \)), where \( \hat{X}_i \) is \( \varepsilon \)-LDP, can be described by \( k \) bits, and has small variance. The encoding scheme consists of three main steps: (1) obtaining a Kashin’s representation for a tight frame \([30]\), (2) subsampling and (3) privatization.

Kashin’s representation We begin with introducing tight frames and Kashin’s representation \([30]\).

**Definition D.1 (Tight frame)** A tight frame is a set of vectors \( \{u_j\}_{j=1}^N \in \mathbb{R}^d \) that obeys Parseval’s identity
\[
\|x\|_2^2 = \sum_{j=1}^N (u_j, x)^2, \text{ for all } x \in \mathbb{R}^d.
\]

A frame can be viewed as a generalization of an orthogonal basis in \( \mathbb{R}^d \), which can improve the encoding stability by adding redundancy to the representation system when \( N > d \). To increase robustness, we wish the information to spread evenly in each coefficient, which motivates the following definition of a Kashin’s representation:

**Definition D.2 (Kashin’s representation)** For a set of vectors \( \{u_j\}_{j=1}^N \), we say the expansion
\[
x = \sum_{j=1}^N a_j u_j, \text{ with } \max_j |a_j| \leq \frac{K}{\sqrt{N}} \|x\|_2
\]
is a Kashin’s representation of vector \( x \) at level \( K \).

Therefore, if we can obtain unbiased estimators \( \{\hat{a}_j\}_{j=1}^N \) of the Kashin’s representation of \( X \) with respect to a tight frame \( \{u_j\}_{j=1}^N \), then the MSE can be controlled by
\[
\mathbb{E} \left[ (\hat{X} - X)^2 \right] = \mathbb{E} \left[ \left\| \sum_{j=1}^N (\hat{a}_j - a_j) u_j \right\|_2^2 \right] \leq \mathbb{E} \left[ \sum_{j=1}^N (\hat{a}_j - a_j)^2 \right] = \sum_{j=1}^N \text{Var}(\hat{a}_j), \tag{4}
\]
where (a) is due to the Cauchy–Schwarz inequality and the definition of a tight frame. Recall that \( X \) is deterministic, so here the expectation is taken with respect to the randomness on \( \hat{a}_j \). Notice that the cardinality \( N \) of the frame determines the compression (i.e. quantization) rate, and Kashin’s level \( K \) affects the variance. Hence we are interested in constructing tight frames with small \( N \) and \( K \).

By Theorem 3.5 and Theorem 4.1 in [30], we have the following lemma:

**Lemma D.1 (Uncertainty principle and Kashin’s Representation)** For any \( \mu > 0 \) and \( N > (1 + \mu) d \), there exists a tight frame \( \{u_j\}_{j=1}^N \) with Kashin’s level \( K = O \left( \frac{1}{\mu^2 \log \frac{1}{\mu^2}} \right) \). Moreover, for each \( X \), finding Kashin’s coefficient requires \( O(d N \log N) \) computation.

For our purpose, we choose \( \mu \) to be a constant, i.e. \( \mu = \Theta(1) \), so \( N = \Theta(d) \), \( K = \Theta(1) \), and we can obtain a representation of \( X = \sum_{j=1}^N a_j u_j \), with \( |a_j| \leq \frac{K}{\sqrt{N}} = \frac{c}{\sqrt{d}} \) for some constant \( c \). Therefore, we quantize each \( a_j \) as follows:
\[
q_j \overset{\Delta}{=} \begin{cases} 
\frac{-c}{\sqrt{d}}, & \text{with probability } \frac{c/\sqrt{d} - a_j}{2c/\sqrt{d}} \\
\frac{c}{\sqrt{d}}, & \text{with probability } \frac{a_j + c/\sqrt{d}}{2c/\sqrt{d}}.
\end{cases} \tag{5}
\]

\( q \overset{\Delta}{=} (q_1, ..., q_N) \) yields an unbiased estimator \( a \overset{\Delta}{=} (a_1, ..., a_N) \) and can be described by \( N = \Theta(d) \) bits.
As in the converse part of Theorem 3.1, the lower bound can be obtained by constructing a prior distribution \( P \) on \( X_1, ..., X_n \) and analyzing the statistical mean estimation problem. Therefore, we will impose a prior distribution \( P \) on \( X_1, ..., X_n \) and lower bound the \( \ell_2 \) error of estimating the mean \( \theta(P) \), where \( P \) is a distribution supported on the \( d \)-dimension unit ball.

**Sampling**  
To further reduce the communication cost, we sample \( k \) bits uniformly at random from \( q \) using public randomness. Let \( s_1, ..., s_k \) be the indices of the sampled elements, and define the sampled message as

\[
Q(q, (s_1, ..., s_k)) = (q_{s_1}, ..., q_{s_k}) \in \left\{ -c/\sqrt{d}, c/\sqrt{d} \right\}^k.
\]

Then \( Q \) can be described in \( k \) bits, and each of \( q_{s_m} \) yields an independent and unbiased estimator of \( a^k \):

\[
E[N \cdot q_{s_m} \cdot 1_{(j=s_m)}] = E[E[N \cdot q_{s_m} \cdot 1_{(j=s_m)}|q_1, ..., q_N]] = E[q_{s_m}] = a_j, \quad \forall j \in [N].
\]

**Privatization**  
Each client then perturbs \( Q \) via \( 2^k \)-RR mechanism (as a \( k \)-bit string):

\[
\tilde{Q} = \begin{cases} Q, & \text{with probability } \frac{e^{\varepsilon}}{e^{\varepsilon} + 2^k - 1} \\ Q' \in \left\{ -c/\sqrt{d}, c/\sqrt{d} \right\} / \{Q\}, & \text{with probability } \frac{1}{e^{\varepsilon} + 2^k - 1}. \end{cases}
\]

Since

\[
\sum_{Q' \in \left\{ -c/\sqrt{d}, c/\sqrt{d} \right\} / \{Q\}} Q' = -Q,
\]

it is not hard to see \( \left( e^{\varepsilon} + 2^k - 1 \right) \tilde{Q} \) yields an unbiased estimator of \( Q \). Indeed, if we write \( \tilde{Q} = (\tilde{q}_1, ..., \tilde{q}_k) \), then

\[
E \left[ \left( \frac{e^{\varepsilon} + 2^k - 1}{e^{\varepsilon} - 1} \right) \tilde{q}_m | q_1, ..., q_N, s_1, ..., s_k \right] = q_{s_m},
\]

or equivalently

\[
E \left[ \left( \frac{e^{\varepsilon} + 2^k - 1}{e^{\varepsilon} - 1} \right) \tilde{Q} | Q \right] = Q.
\]

**Estimation and the \( \ell_2 \) error**  
Given \( \tilde{Q} = (\tilde{q}_1, ..., \tilde{q}_k) \), define

\[
\hat{a}_j = \frac{N_k}{k} \left( \frac{e^{\varepsilon} + 2^k - 1}{e^{\varepsilon} - 1} \right) \sum_{m=1}^{k} \tilde{q}_m \cdot 1_{(j=s_m)}.
\]

According to (7) and (8), \( E[\hat{a}_j] = a_j \), and hence \( \tilde{X} \left( \tilde{Q}, (s_1, ..., s_k) \right) \triangleq \sum_{j=1}^{N} \hat{a}_j u_j \) gives us an unbiased estimator of \( X \).

**Claim D.1**  
The MSE of \( \tilde{X} \) can be bounded by

\[
E \left[ \left\| \tilde{X} - X \right\|_2^2 \right] \leq C \left( \frac{e^{\varepsilon} + 2^k - 1}{e^{\varepsilon} - 1} \right)^2 \frac{d}{k}.
\]

Finally, each client encodes its data \( X_i \) independently, and the server computes \( \frac{1}{n} \sum_i \tilde{X}_i \). Since \( \hat{X}_i \) is unbiased and by Claim D.1, we get

\[
E \left[ \left\| \frac{1}{n} \sum_{j=1}^{n} \tilde{X}_j - \tilde{X} \right\|_2^2 \right] = \frac{1}{n^2} \sum_{j=1}^{n} E \left[ \left\| \tilde{X}_i - X_i \right\|_2^2 \right] \leq C \left( \frac{e^{\varepsilon} + 2^k - 1}{e^{\varepsilon} - 1} \right)^2 \frac{d}{n^2 k}.
\]

Finally, picking \( k = \min (\lfloor \log_2 e \rfloor \varepsilon, b) \) gives us the desired upper bound.

**D.2 Lower Bound of Theorem 2.1**  
As in the converse part of Theorem 3.1, the lower bound can be obtained by constructing a prior distribution on \( X_i \) and analyzing the statistical mean estimation problem. Therefore, we will impose a prior distribution \( P \) on \( X_1, ..., X_n \) and lower bound the \( \ell_2 \) error of estimating the mean \( \theta(P) \), where \( P \) is a distribution supported on the \( d \)-dimension unit ball.
For any $\hat{X}$, observe that
\[
\mathbb{E}_{X,X^* \sim P} \left[ \|\hat{X} - \bar{X}\|^2 \right] \geq \mathbb{E} \left[ \left( \|\hat{X} - (\theta \circ P)\| - \|\bar{X} - (\theta \circ P)\| \right)^2 \right] \\
\geq \mathbb{E} \left[ \|\hat{X} - (\theta \circ P)\|^2 \right] - 2\mathbb{E} \left[ \|\hat{X} - (\theta \circ P)\| \|\bar{X} - (\theta \circ P)\| \right] \\
\geq \mathbb{E} \left[ \|\hat{X} - (\theta \circ P)\|^2 \right] - 2\sqrt{\mathbb{E} \left[ \|\hat{X} - (\theta \circ P)\|^2 \right] \mathbb{E} \left[ \|\bar{X} - (\theta \circ P)\|^2 \right]},
\]
where (a) and (b) follow from the triangular inequality and the Cauchy-Schwartz inequality respectively. Since $X_i$ and $\theta(P)$ are supported on the unit ball, $\mathbb{E} \left[ \|\hat{X} - (\theta \circ P)\|^2 \right] \asymp 1/n$, so it remains to find a distribution $P^*$ such that
\[
\min_X \mathbb{E} \left[ \|\hat{X} - (\theta \circ P^*)\|^2 \right] \geq \frac{d}{n \min (\varepsilon^2, \varepsilon, b)}.
\]

Consider the product Bernoulli model $Y \sim \prod_{j=1}^d \text{Ber}(\theta_j)$. If we set $\Theta = [1/2 - \varepsilon, 1/2 + \varepsilon]^d$ for some $\frac{1}{2} > \varepsilon > 0$, then it can be shown that both variance and sub-Gaussian norm of the score function of this model is $\Theta(1)$ [6 Corollary 4]. Therefore, applying [6 Corollary 8] and [5 Proposition 2, Proposition 4] yields
\[
\min_{\hat{\theta}} \mathbb{E} \left[ \|\hat{\theta} - \theta\|^2 \right] \geq \frac{d^2}{n \min (\varepsilon^2, \varepsilon, b)}.
\]
Finally, if we set $X_i = Y_i / \sqrt{d}$, then each $X_i$ is supported on the unit ball and $\mathbb{E} \left[ X_i \right] = \theta / \sqrt{d}$. Therefore
\[
\min_X \mathbb{E} \left[ \frac{\|\hat{X} - \theta\|^2}{\sqrt{d}} \right] \geq \frac{d}{n \min (\varepsilon^2, \varepsilon, b)}.
\]
Plugging into (9), as long as $\min(\varepsilon^2, \varepsilon, k) = o(d)$, the first term dominates and we get the desired lower bound. \hfill \square

**Remark D.1** [14 Proposition 4] and [36 Theorem 5] give lower bounds $\Omega(n^{-d/2})$ and $\Omega(n^{-d/2})$ for distributional mean estimation with $P$ supported on the unit ball. The lower bound $\Omega(n^{-d/2})$ is new.

### E Proof of Theorem 3.1

#### E.1 Achieving optimal $\ell_1$ and $\ell_2$ error (part (i) of Theorem 3.1)

In this section, we show that Recursive Hadamard Response (RHR) achieves optimal $\ell_1$ and $\ell_2$ estimation error.

**Decomposition of Hadamard matrix** Let us set $B = d/2^{k-1}$. Since $H_d = H_{2^{k-1}} \otimes H_B$, for any $j \in [B]$ and $m \in [2^{k-1}]$, if $j' = (m - 1)B + j$ (and thus $j \equiv j' \pmod{B}$), we must have $(H_d)_{j'} = (H_{2^{k-1}})_m \otimes (H_B)_j$, where $\otimes$ is the Kronecker product. This allows us to decompose the $j'$-th component of $H_d \cdot X_i$ into
\[
(H_d)_{j'} \cdot X_i = ((H_{2^{k-1}})_m \otimes (H_B)_j) \cdot X_i = \sum_{l=1}^{2^{k-1}} (H_{2^{k-1}})_{m,l} (H_B)_j \cdot X_i^{(l)}:
\]
where $X_i^{(l)}$ is the $l$-th block of $X_i$, i.e. $X_i^{(l)} \triangleq X_i[(l - 1)B + 1 : lB]$. Therefore, as long as we know $(H_B)_j \cdot X_i^{(l)}$ for $l = 1, \ldots, 2^{k-1}$, we can reconstruct $(H_d)_{j'} \cdot X_i$, for all $j' \equiv j \pmod{B}$.
Encoding mechanism  Let \( r_i \sim \text{Uniform}(B) \) be generated from the shared randomness, and consider the following quantizer

\[
Q(X_i, r_i) = \left( (H_B)_{r_i} \cdot X_i^{(l)} \right)_{l=1,...,2^{k-1}} \in \{-1, 0, 1\}^{2^{k-1}}.
\]

Since \( X_i \) is one-hot encoded, there is exactly one non-zero \( X_i^{(l)} \), so \( Q(X_i, r_i) \) can be described by a \( k \)-bit string (with \( k - 1 \) bits indicating the location of the non-zero entry and 1 bit indicating its sign).

Given \( Q(X_i, r_i) \), by (10) we can recover \( 2^{k-1} \) coordinates of \( Y_i = H_d \cdot X_i \):

\[
Y_i(r') = (H_d)_{r_i} \cdot X_i = \sum_{l=1}^{2^{k-1}} (H_{2^{k-1}})_{m,l} (H_B)_{r_i} \cdot X_i^{(l)} = (H_{2^{k-1}})_{m} \cdot Q(X_i, r_i),
\]

for any \( r' = (m - 1)B + r_i \). Therefore, if we define

\[
\hat{Y}_i(Q(X_i, r_i), r_i) = \begin{cases} \frac{1}{2}Y_i(r'), & \text{if } r' \equiv r_i \\ 0, & \text{else,} \end{cases}
\]

then \( \mathbb{E}[\hat{Y}_i] = \frac{1}{2}H_d \cdot X_i \), where the expectation is taken with respect to \( r_i \).

To protect privacy, client \( i \) then perturbs \( Q(X_i, r_i) \) via \( 2^k \)-RR scheme, since \( Q \) takes values on an alphabet of size \( 2^k \), denoted by \( \mathcal{Q} = \{ \pm e_1, \ldots, \pm e_{2^{k-1}} \} \),

\[
\tilde{Q}_i = \begin{cases} Q(X_i, r_i), & \text{w.p. } \frac{e^\varepsilon}{e^{2^k - 1}}, \\ Q' \in \mathcal{Q} \setminus \{Q(X_i, r_i)\}, & \text{w.p. } 1 - \frac{e^\varepsilon}{e^{2^k - 1}}, \end{cases}
\]

where \( e_l \) denotes the \( l \)-th coordinate vector in \( \mathbb{R}^{2^k-1} \).

Client \( i \) then sends the \( k \)-bit report \( \tilde{Q}_i \) to the server, and with \( \tilde{Q}_i \), the server can compute an estimate of \( Q_i \) since

\[
\mathbb{E}\left[ \hat{Q}_i \mid Q(X_i, r_i) \right] = e^{\frac{e^\varepsilon - 1}{e^{2^k-1}}} Q(X_i, r_i).
\]

Constructing estimator for \( \hat{D} \)  For a given \( \tilde{Q}_i \), we estimate \( Y_i \) by \( \hat{Y}_i \left( \frac{e^{e^\varepsilon - 1}}{e^\varepsilon + 2^k - 1} \tilde{Q}_i, r_i \right) \), where \( \hat{Y}_i \) is given by (11) and (12), with \( Q(X_i, r_i) \) in (11) replaced by \( \tilde{Q}_i \).

Claim E.1 \( \hat{Y}_i \) is an unbiased estimator of \( Y_i \).

The final estimator of \( D_{X^n} = \frac{1}{n} \sum X_i \) is given by

\[
\hat{D} \left( \left( \tilde{Q}_i, r_i \right)_{i=1,...,n} \right) = \frac{1}{n} \sum_{i=1}^{n} H_d \cdot \hat{Y}_i \left( \frac{e^\varepsilon + 2^k - 1}{e^\varepsilon - 1} \tilde{Q}_i, r_i \right).
\]

Note that by Claim E.1, \( \hat{D} \) is an unbiased estimator for \( D_{X^n} \). Finally picking \( k = \min\left( b, \lfloor \varepsilon \log_2 e \rfloor, \lfloor \log d \rfloor \right) \) yields the following bounds.

Claim E.2 The estimator \( \hat{D} \) in (13) achieves the optimal \( \ell_1 \) and \( \ell_2 \) errors:

\[
\mathbb{E} \left[ \left\| \hat{D} - D_{X^n} \right\|_2^2 \right] \lesssim \frac{d}{n \left( \min \left\{ e^\varepsilon, (e^\varepsilon - 1)^2, 2^k, d \right\} \right)} \quad \text{and}
\]

\[
\mathbb{E} \left[ \left\| \hat{D} - D_{X^n} \right\|_1 \right] \lesssim \frac{d}{\sqrt{n \left( \min \left\{ e^\varepsilon, (e^\varepsilon - 1)^2, 2^k, d \right\} \right)}}.
\]

This establishes the achievability part of Theorem 3.1.
### E.2 Algorithms

We summarize our proposed scheme RHR scheme below:

**Algorithm 1:** Encoding mechanism $\tilde{Q}_i$ (at each client)

**Input:** client index $i$, observation $X_i$, privacy level $\varepsilon$, alphabet size $d$

**Result:** Encoded message $(\tilde{\text{sign}}, \tilde{\text{loc}})$

Set $D = 2'^{(\log d)}$, $k = \min(b, \lceil \varepsilon \log_2 e \rceil)$, $B = D/2^{k-1}$;

Draw $r_i$ from uniform $(B)$ using public-coin;

begin

\[
\text{loc} \leftarrow \left\lceil \frac{X_i}{B} \right\rceil; \\
\text{sign} \leftarrow (H_d)_{r_i, X_i}; \\
(\text{sign}, \text{loc}) \leftarrow 2^k \cdot \text{RR}_\varepsilon((\text{sign}, \text{loc})) \quad /\!\!\!\!\!\!/ (\text{sign}, \text{loc}) \text{ as a } k\text{-bit string} \; /\!
\]

end

Notice that computing any entry of $H_d$ takes $O(\log d)$ Boolean operations, and uniformly sampling a $k$-bit string takes $O(k)$ time. Therefore the computation cost at each client is $O(\log d)$ time. Also note that the encoded message is a $k$-bit binary string, and therefore the communication cost at each client is $k = \min(b, \lceil \varepsilon \log_2 e \rceil) \leq b$.

Once receiving the $k$-bit messages from all clients, the server does the following operation:

**Algorithm 2:** Estimator of $D_{X^n}$ (at the server)

**Input:** $(\tilde{\text{sign}}[1:n], \tilde{\text{loc}}[1:n])$, privacy level $\varepsilon$, alphabet size $d$

**Result:** $\hat{D}$

Set $D = 2'^{(\log d)}$, $k = \min(b, \lceil \varepsilon \log_2 e \rceil)$, $B = D/2^{k-1}$;

Partition messages into groups $G_1, \ldots, G_B$, with message $i$ in $G_r$;

forall $j = 1, \ldots, B$ do

\[
G^+_j \leftarrow \{\text{loc}(i) \mid i \in G_j, \text{sign}(i) = +1\}; \\
G^-_j \leftarrow \{\text{loc}(i) \mid i \in G_j, \text{sign}(i) = -1\}; \\
\text{Emp}_j \leftarrow (\text{empirical distribution}(G^+_j) - \text{empirical distribution}(G^-_j)) \cdot \frac{\varepsilon^2 2^k - 1}{\varepsilon^2 - 1}; \\
\text{forall } l = 0, \ldots, 2^{k-1} - 1 \text{ do} \\
\quad \hat{E}[l \cdot B + j] \leftarrow \text{FWHT(Emp}_j)[l]\quad /* \text{fast Walsh-Hadamard transform} */
\]

end

$\hat{D} \leftarrow \frac{1}{d} \cdot \text{FWHT} \left( \hat{E} \right)$;

Partitioning $n$ samples into $B$ groups and computing the empirical distribution of each group takes $O(n)$ time, and the fast Walsh-Hadamard transform can be implemented in $O(d \log d)$ time. Hence the decoding complexity is $O(n + d \log d)$.

### E.3 Lower Bound on $\ell_1$ and $\ell_2$ errors in Theorem 3.1

We can bound the error by considering the worst case Bayesian setting, i.e. by imposing a prior distribution $p$ on $X_1, \ldots, X_n$ and applying the converse part of Theorem 3.2 in Section 3.2.
Let \( X_1, \ldots, X_n \overset{i.i.d.}{\sim} p \). Then for any \( \hat{D}(X^n) \), we must have
\[
\max_{X^n \sim p} \mathbb{E} \left[ \left\| \hat{D} - D_{X^n} \right\|_2^2 \right] \geq \max_p \mathbb{E} \left[ \left( \left\| \hat{D} - p \right\|_2 - \left\| D_{X^n} - p \right\|_2 \right)^2 \right] \\
\geq \max_p \left( \mathbb{E} \left[ \left\| \hat{D} - p \right\|_2^2 \right] - 2\mathbb{E} \left[ \left\| \hat{D} - p \right\|_2 \left\| D_{X^n} - p \right\|_2 \right] \right) \\
\geq \max_p \left( \mathbb{E} \left[ \left\| \hat{D} - p \right\|_2^2 \right] - 2\sqrt{\mathbb{E} \left[ \left\| \hat{D} - p \right\|_2^2 \right] \mathbb{E} \left[ \left\| D_{X^n} - p \right\|_2^2 \right]} \right)
\] (14)

where (a) and (b) follow from the triangular inequality and the Cauchy-Schwarz inequality respectively. By Theorem 3.2, there exists a worst case \( p^* \) such that
\[
c^d \frac{1}{n} \min \left\{ e^\varepsilon, (e^\varepsilon - 1)^2, 2^b \right\} \leq \mathbb{E} \left[ \left\| \hat{D} - p^* \right\|_2^2 \right] \leq C^d \frac{1}{n} \min \left\{ e^\varepsilon, (e^\varepsilon - 1)^2, 2^b \right\}, \] (15)

for some constants \( c \) and \( C \). On the other hand, the \( \ell_2 \) convergence of \( D(X^n) \) to \( p \) is \( O\left(1/n\right) \) for any \( p \), which gives us
\[
\mathbb{E} \left[ \left\| D_{X^n} - p^* \right\|_2^2 \right] \leq c' \frac{1}{n}. \] (16)

Plugging (15) and (16) back into (14) yields
\[
\max_{X^n \sim p} \mathbb{E} \left[ \left\| \hat{D} - D_{X^n} \right\|_2^2 \right] \geq C_1 \frac{d}{n} \left( \frac{1}{\min \left\{ e^\varepsilon, (e^\varepsilon - 1)^2, 2^b \right\}} \right) - C_2 \frac{1}{n} \sqrt{\frac{d}{\min \left\{ e^\varepsilon, (e^\varepsilon - 1)^2, 2^b \right\}}}.
\]

Thus as long as \( \min \left\{ e^\varepsilon, (e^\varepsilon - 1)^2, 2^b \right\} = o(d) \), the first term dominates and the desired \( \ell_2 \) lower bound follows.

For the case of \( \ell_1 \), we similarly have
\[
\max_{X^n \sim p} \mathbb{E} \left[ \left\| \hat{D} - D_{X^n} \right\|_1 \right] \geq \max_p \mathbb{E} \left[ \left\| \hat{D} - p \right\|_1 \right] - \mathbb{E} \left[ \left\| D_{X^n} - p \right\|_1 \right] \] (17)

It is well-known that \( \mathbb{E} \left[ \left\| D(X^n) - p \right\|_1 \right] \leq \sqrt{d/n} \) (for instance, see [20]), and by the converse part of Theorem 3.2
\[
\max_p \mathbb{E} \left[ \left\| \hat{D} - p \right\|_1 \right] \geq \frac{d^2}{n \min \left\{ e^\varepsilon, (e^\varepsilon - 1)^2, 2^b \right\}}.
\]

Plugging this into (17) yields the \( \ell_1 \) lower bound. \( \square \)

E.4 Achieving optimal \( \ell_\infty \) error (part (ii) of Theorem 3.1)

To obtain an upper bound on \( \ell_\infty \) error, we extend the TreeHist protocol in [8], a 1-bit LDP heavy hitter estimation mechanism, to communicate \( b \) bits and satisfy a desired privacy level \( \varepsilon \). A simpler version of TreeHist protocol, which is not optimized for computational complexity, is as follows: we first perform Hadamard transform on \( X_i \), and sample one random coordinate with public randomness \( r_i \). The 1-bit message is then passed through a binary \( \varepsilon \)-LDP mechanism. We can show that from the perturbed outcomes, the server can construct an unbiased estimator of \( X_i \) with bounded sub-Gaussian norm, and the \( \ell_\infty \) error will be \( O(\sqrt{\log d/n\varepsilon^2}) \).

To extend this scheme to an arbitrary privacy regime and an arbitrary communication budget of \( b \) bits, we independently and uniformly sample the Hadamard transform of \( X_i \) for \( k = \min \left( b, \left\lceil \varepsilon \right\rceil \right) \) times. Each 1-bit sample is then perturbed via a \( \varepsilon' \)-LDP mechanism with \( \varepsilon' \triangleq \varepsilon/k \).
Note that under the distribution-free setting, the randomness comes only from the sampling and the privatization steps, so we could view each re-sampled and perturbed message as generated from a fresh new copy of $X_i$ since $X_i$ is not random. Equivalently, this boils down to a frequency estimation problem with $n' = nk$ clients and under $\varepsilon' = \varepsilon/k$ and gives us the $\ell_\infty$ error

$$O \left( \sqrt{\frac{\log d}{n'} \left( \frac{\varepsilon}{k} \right)^2} \right) = O \left( \sqrt{\frac{\log d}{n \min (\varepsilon^2, \varepsilon, k)}} \right).$$

Below we describe the details.

**Encoding mechanism** Set $k = \min (b, \lfloor \varepsilon \rfloor)$. For each $X_i$, we randomly sample $(H_d)_{X_i}$ (i.e. the $X_i$-th column of $H_d$) $k$ times, identically and independently by using the shared randomness. Let $r_i^{(1)}, \ldots, r_i^{(k)}$ be the sampled coordinates, which are known to both the server and node $i$, and $(H_d)_{X_i, r_i^{(t)}}$ be the sampling outcomes. Then due to the orthogonality of $H_d$, for all $j \in [d], \ell \in [k],

$$E \left[ (H_d)_{j, r_i^{(t)}} \cdot (H_d)_{X_i, r_i^{(t)}} \right] = \begin{cases} 1, & \text{if } j = X_i \\ 0, & \text{if } j \neq X_i \end{cases}$$

where the expectation is taken over $r_i^{(t)}$.

We then pass $\{(H_d)_{X_i, r_i^{(t)}} \mid \ell = 1, \ldots, k\}$ through $k$ binary $\varepsilon'$-LDP channels sequentially, with $\varepsilon' \triangleq \varepsilon/k$. By the composition theorem of differential privacy, the privatized outcomes, denoted as $\{(\tilde{H}_d)_{X_i, r_i^{(t)}}\}$, satisfy $\varepsilon$-LDP.

**Estimation of $D_{X^n}$** Observe that

$$E \left[ \left( \frac{e^{\varepsilon'} + 1}{e^{\varepsilon'} - 1} \right) (\tilde{H}_d)_{X_i, r_i^{(t)}} (H_d)_{X_i, r_i^{(t)}} \right] = (H_d)_{X_i, r_i^{(t)}},$$

where the expectation is with respect to the privatization. Therefore

$$\hat{X}_i^{(t)}(j) \triangleq \left( \frac{e^{\varepsilon'} + 1}{e^{\varepsilon'} - 1} \right) (\tilde{H}_d)_{j, X_i} (H_d)_{X_i, r_i^{(t)}}$$

defines an unbiased estimator of $X_i(j)$. Moreover,

$$\left| \hat{X}_i^{(t)}(j) - X_i(j) \right| \leq \left( \frac{e^{\varepsilon'} + 1}{e^{\varepsilon'} - 1} + 1 \right) \text{ a.s.,}$$

so $\hat{X}_i^{(t)}(j)$ has sub-Gaussian norm bounded by

$$\sigma \leq 2 \frac{e^{\varepsilon'} + 1}{e^{\varepsilon'} - 1}. \tag{19}$$

Finally, we estimate $D_{X^n}(j)$ by

$$\hat{D}(j) = \frac{1}{nk} \sum_{i=1}^{n} \sum_{\ell=1}^{k} \hat{X}_i^{(t)}(j).$$

Observe that

$$\hat{D}(j) - D_{X^n}(j) = \frac{1}{nk} \sum_{i=1}^{n} \sum_{\ell=1}^{k} \left( \hat{X}_i^{(t)}(j) - X_i(j) \right) \tag{20}$$

has sub-Gaussian norm bounded by $\sigma/\sqrt{nk}$, where $\sigma$ is given by \[19\].

To bound the $\ell_\infty$ norm, we apply the maximum bound (see, for instance, \[37\] Chapter 2) for sub-Gaussian random variables (note that for $j, j', \hat{D}(j)$ and $\hat{D}(j')$ are not independent):

$$E \left[ \max_{j \in [d]} \left| \hat{D}(j) - D_{X^n}(j) \right| \right] \leq 2 \sqrt{\sigma^2 \log d} = 4 \sqrt{\frac{\log d}{nk}} \leq \sqrt{\frac{\log d}{n \min (\varepsilon^2, \varepsilon, k)}}. \tag{21}$$
where (a) holds since if \( \varepsilon = o(1) \), then \( k = 1 \) and hence

\[
\left( \frac{e^{\varepsilon} + 1}{e^{\varepsilon} - 1} \right)^2 \approx \frac{1}{\varepsilon^2};
\]

otherwise \( \varepsilon = \Omega(1) \) and \( \varepsilon' = \Omega(1) \), so

\[
\left( \frac{e^{\varepsilon'} + 1}{e^{\varepsilon'} - 1} \right)^2 \approx 1.
\]

Both cases are upper bounded by \((21)\), so the result follows. \( \square \)

**Remark E.1** Notice that in the high privacy regime \( \varepsilon = o(1) \), the upper bound matches the lower bound in \([9]\). For general privacy regimes with limited communication, however, we do not know whether the upper bound is tight or not. This remains as an open question.

### F Proof of Theorem 3.2

The construction of the distribution estimation scheme mainly follows Section E.1 except we replace the random sampling step by a deterministic grouping idea. We will use the same notation as in Section E.1.

**Encoding mechanism** We group \( n \) samples into \( B \) equal-sized groups, each with \( n' = n/B \) samples. For sample \( X_i \in G_j \), we quantize it to a \( 2^{k-1} \)-dimensional \( \{1, 0, -1\} \) vector:

\[
Q_j(X_i) = \begin{bmatrix}
(H_B)_j \cdot X_i^{(1)} \\
(H_B)_j \cdot X_i^{(2)} \\
\vdots \\
(H_B)_j \cdot X_i^{(2^{k-1})}
\end{bmatrix} \in \{-1, 0, 1\}^{2k-1}.
\]

Since \( X_i \) is one-hot encoded, there is only one \( l \in \{1, \ldots, 2^{k-1}\} \) such that \( (H_B)_j \cdot X_i^{(l)} \neq 0 \), so \( Q_j(X_i) \) can be described by \( k \) bits (1 bit for the sign and \( (k-1) \) bits for the location of the non-zero element). Also notice that

\[
E [Q_j(X_i)] = \begin{bmatrix}
(H_B)_j \cdot p^{(1)} \\
(H_B)_j \cdot p^{(2)} \\
\vdots \\
(H_B)_j \cdot p^{(2^{k-1})}
\end{bmatrix},
\]

where \( p^{(l)} \triangleq p[(l-1)B + 1 : lB] \). By \((10)\), the estimator \( \hat{q}_{j'} = \langle (H_{2^{k-1}})_m, Q_j(X_i) \rangle \) is unbiased for \( q_{j'} \) (where \( j' = (m-1)B + j \)).

We further perturb \( Q_j \) via \( 2^k \)-RR scheme, since \( Q \) takes values on an alphabet of size \( 2^k \), denoted by \( Q = \{ \pm e_1, \ldots, \pm e_{2^k-1} \} \).

\[
\tilde{Q}_j = \begin{cases}
Q_j, \text{ w.p. } \frac{e^\varepsilon - 1}{e^\varepsilon + 2^k - 1} \\
Q' \in Q \setminus \{Q_j\}, \text{ w.p. } \frac{1}{e^\varepsilon + 2^k - 1}
\end{cases},
\]

where \( e_l \) denotes the \( l \)-th coordinate vector in \( \mathbb{R}^{2^{k-1}} \). This gives us

\[
E [\tilde{Q}_j] = \frac{e^\varepsilon - 1}{e^\varepsilon + 2^k - 1} E [Q_j].
\]

Therefore \( \frac{e^\varepsilon + 2^k - 1}{e^\varepsilon - 1} \tilde{Q}_j \) yields an unbiased estimator of

\[
\begin{bmatrix}
(H_B)_j \cdot p^{(1)} \\
(H_B)_j \cdot p^{(2)} \\
\vdots \\
(H_B)_j \cdot p^{(2^{k-1})}
\end{bmatrix}.
\]
Constructing the estimator for $p$  For each $j' \equiv j \pmod{B}$, we estimate $(H_{2k-1})_m, Q_j(X_i), i \in \mathcal{G}_j$ (recall that $j' = j + (m-1)B$). Define the estimator
\[
\hat{q}_{j'}(\{X_i, i \in \mathcal{G}_j\}) = \frac{1}{|\mathcal{G}_j|} \sum_{i \in \mathcal{G}_j} (H_{2k-1})_m \cdot \left(\frac{e^\varepsilon + 2^k - 1}{e^\varepsilon - 1}\right) \hat{Q}_j(X_i)
\]
\[
= \frac{B}{n} \left(\frac{e^\varepsilon + 2^k - 1}{e^\varepsilon - 1}\right) \sum_{i \in \mathcal{G}_j} (H_{2k-1})_m \hat{Q}_j(X_i).
\]

The MSE of $\hat{q}_{j'}$ can be obtained by
\[
\mathbb{E} \left[ (\hat{q}_{j'} - q_{j'})^2 \right] \overset{(a)}{=} \text{Var} (\hat{q}_{j'}) \overset{(b)}{=} \frac{d}{n2^{k-1}} \left(\frac{e^\varepsilon + 2^k - 1}{e^\varepsilon - 1}\right)^2 \text{Var} \left( (H_{2k-1})_m \cdot \hat{Q}_j(X_i) \right) \overset{(c)}{\leq} \frac{d}{n2^{k-1}} \left(\frac{e^\varepsilon + 2^k - 1}{e^\varepsilon - 1}\right)^2,
\]
(22)
where (a) is due to the unbiasedness of $\hat{q}_{j'}$, (b) is due to the independence across $X_i$, and (c) is because $((H_{2k-1})_m, Q_j)$ only takes value in $\{-1, 1\}$.

Finally, let $\hat{p}$ be the inverse Hadamard transform of $\hat{q}$, the MSE is
\[
\mathbb{E} \|\hat{p} - p\|_2^2 = \mathbb{E} \left[ (\hat{p} - p, \hat{p} - p) \right] = \mathbb{E} \left[ (q - q)^T (H^{-1}_a)^T H^{-1}_a (q - q) \right] = \frac{1}{d} \mathbb{E} \|\hat{q} - q\|_2^2 \leq \frac{d}{n2^{k}} \left(\frac{e^\varepsilon + 2^k - 1}{e^\varepsilon - 1}\right)^2 = O \left(\frac{d}{n2^{k}} \left(\frac{e^\varepsilon + 2^k}{e^\varepsilon - 1}\right)^2\right),
\]
where the last inequality holds due to (22).

Picking $k = \min(b, \lceil \varepsilon \log_2 e \rceil, \lfloor \log d \rfloor)$ yields
\[
\mathbb{E} \|\hat{p} - p\|_2^2 = O \left(\frac{d}{n \min(2^b, e^\varepsilon, d)} \left(\frac{e^\varepsilon}{e^\varepsilon - 1}\right)^2\right).
\]

Observe that if $e^\varepsilon = O(2^b)$, then $e^\varepsilon \leq 2^b$, so $\mathbb{E} \|\hat{p} - p\|_2^2 = O \left(\frac{d e^\varepsilon}{n(e^\varepsilon - 1)^2}\right)$. On the other hand, if $e^\varepsilon = \Omega(2^b)$, then $\frac{e^\varepsilon}{e^\varepsilon - 1} = \Theta(1)$, and $\mathbb{E} \|\hat{p} - p\|_2^2 = O \left(\frac{d}{n \min(2^b, d)}\right)$.

Therefore we conclude that
\[
\mathbb{E} \|\hat{p} - p\|_2^2 \leq \max \left(\frac{d}{n \min(2^b, d)}, \frac{d e^\varepsilon}{n (e^\varepsilon - 1)^2}\right) = \frac{d}{n} \left(\frac{1}{\min\{e^\varepsilon, (e^\varepsilon - 1)^2, 2^b, d\}}\right).
\]

Finally, by Jensen’s inequality and Cauchy-Schwarz inequality, we also have
\[
\mathbb{E} \|\hat{p} - p\|_1 \leq \left(\mathbb{E} \|\hat{p} - p\|_1^2\right)^{\frac{1}{2}} \leq d \cdot \mathbb{E} \|\hat{p} - p\|_2^2 \frac{1}{2} \leq \frac{d}{\sqrt{n}} \left(\min\{e^\varepsilon, (e^\varepsilon - 1)^2, 2^b, d\}\right),
\]
establishing the achievability part of Theorem 3.2. □
F.1 Algorithms and analysis

Each client runs the following algorithm:

Algorithm 3: Encoding mechanism (at each client)

Input: client index $i$, observation $X_i$, privacy level $\varepsilon$, alphabet size $d$
Result: Encoded message $(\tilde{\text{sign}}, \tilde{\text{loc}})$

Set $D = 2^{\lceil \log d \rceil}$. Set $k = \min \left( b, \lceil \varepsilon \log_2 e \rceil \right)$, $B = D/2^{k-1}$;

begin
  $j \leftarrow i \mod B$ /* assign user $i$ to group $j$ */;
  $\text{loc} \leftarrow \lceil X_i B \rceil$;
  $\text{sign} \leftarrow (H_d)_{j,i}$;
  $(\tilde{\text{sign}}, \tilde{\text{loc}}) \leftarrow kRR_\varepsilon ((\text{sign}, \text{loc}))$;
end

As in Algorithm 1, the computation cost at each client is $O(\log d)$. Also note that the encoded message is a $k$-bit binary string, and therefore the communication cost at each client is $k = \min \left( b, \varepsilon \log_2 e \right) \leq b$.

Upon receiving the privatized $k$-bit messages from the clients, the server runs the following algorithm:

Algorithm 4: Estimation of $p$ (at the server)

Input: $(\tilde{\text{sign}}[1:n], \tilde{\text{loc}}[1:n])$, privacy level $\varepsilon$, alphabet size $d$
Result: $\hat{p}$

Set $D = 2^{\lceil \log d \rceil}$, $k = \min \left( b, \lceil \varepsilon \log_2 e \rceil \right)$, $B = D/2^{k-1}$;

Partition messages into groups $G_1, \ldots, G_B$, with message $i$ in $G_j$ if $i \equiv j \pmod{B}$;

forall $j = 1, \ldots, B$ do
  $G^+_j \leftarrow \{ \tilde{\text{loc}}(i) \mid i \in G_j, \tilde{\text{sign}}(i) = +1 \}$;
  $G^-_j \leftarrow \{ \tilde{\text{loc}}(i) \mid i \in G_j, \tilde{\text{sign}}(i) = -1 \}$;
  $D_j \leftarrow (\text{empirical distribution}(G^+_j) - \text{empirical distribution}(G^-_j)) \cdot \frac{\varepsilon + 2^{k-1}}{e \varepsilon - 1}$;
  forall $l = 0, \ldots, 2^{k-1} - 1$ do
    $\hat{q}[l \cdot B + j] \leftarrow \text{FWHT}(D_j)[l]$;
end

$\hat{p} \leftarrow \frac{1}{n} \cdot \text{FWHT}(\hat{q})$;

Partitioning $n$ samples into $B$ groups and computing the empirical distribution of each group takes $O(n)$ time, and the fast Walsh-Hadamard transform can be performed in $O(d \log d)$ time. Hence the decoding complexity is $O(n + d \log d)$. 

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G Proof of Claims

G.1 Proof of Claim D.1

Proof. According to (4), it suffices to control \( \text{Var}(\hat{a}_j) \). To bound the variance, consider

\[
\text{Var}(\hat{a}_j) = \frac{N^2}{k^2} \cdot \left( \frac{e^\varepsilon + 2k - 1}{e^\varepsilon - 1} \right)^2 \text{Var} \left( \sum_{m=1}^{k} \tilde{q}_m \cdot \mathbb{1}_{(j = s_m)} \right)
\]

\[
\leq \frac{N^2}{k^2} \cdot \left( \frac{e^\varepsilon + 2k - 1}{e^\varepsilon - 1} \right)^2 \mathbb{E} \left[ \left( \sum_{m=1}^{k} \tilde{q}_m \cdot \mathbb{1}_{(j = s_m)} \right)^2 \right]
\]

\[\leq \frac{N^2}{k^2} \cdot \left( \frac{e^\varepsilon + 2k - 1}{e^\varepsilon - 1} \right)^2 \left( \mathbb{E} \left[ \sum_{m=1}^{k} \mathbb{1}_{(j = s_m)} \right]^2 \right)
\]

\[\leq C \frac{N}{k^2} \left( \frac{e^\varepsilon + 2k - 1}{e^\varepsilon - 1} \right)^2 \left( \frac{k^2}{N^2} + \frac{k}{N} \right)
\]

\[= C \left( \frac{e^\varepsilon + 2k - 1}{e^\varepsilon - 1} \right)^2 \left( \frac{1}{N} + \frac{1}{k} \right),
\]

where (a) is due to \( |\tilde{q}_m| = \frac{1}{\sqrt{d}} \), and (b) is due to the second moment bound on Binomial\((k, 1/N)\) and the fact \( N = \Theta(d) \). Therefore by (4),

\[
\mathbb{E} \left[ \left\| \hat{X} - X \right\|_2^2 \right] \leq C_0 \sum_{i=1}^{N} \text{Var}(\hat{a}_i) \leq C_1 \left( \frac{e^\varepsilon + 2k - 1}{e^\varepsilon - 1} \right)^2 \frac{d}{k},
\]

establishing the claim. ■

G.2 Proof of Claim E.1

Proof. \( \hat{Y}_i \) yields an unbiased estimator since

\[
\mathbb{E} \left[ \hat{Y}_i \left( \frac{e^\varepsilon + 2k - 1}{e^\varepsilon - 1} \hat{Q}_i, r_i \right) \right] = \mathbb{E} \left[ \mathbb{E} \left[ \hat{Y}_i \left( \frac{e^\varepsilon + 2k - 1}{e^\varepsilon - 1} \hat{Q}_i, r_i \right) \mid r_i \right] \right]
\]

\[= \hat{Y}_i \left( \mathbb{E} \left[ \frac{e^\varepsilon + 2k - 1}{e^\varepsilon - 1} \hat{Q}_i \mid r_i \right] , r_i \right)
\]

\[= \mathbb{E} \left[ \hat{Y}_i \left( Q(X_i, r_i), r_i \right) \right]
\]

\[= \frac{1}{d} H_d X_i,
\]

where (a) holds since conditioning on \( r_i \), \( \hat{Y}_i(Q, r_i) \) is a linear function of \( Q \). ■

G.3 Proof of Claim E.2

Proof. The \( \ell_2 \) error is

\[
\mathbb{E} \left[ \left\| \hat{D} - D \right\|_2^2 \right] = \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| H_d \hat{Y}_i - H_d \mathbb{E} \left[ \hat{Y}_i \right] \right\|_2^2 \right]
\]

\[= \frac{d}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| \hat{Y}_i - \mathbb{E} \left[ \hat{Y}_i \right] \right\|_2^2 \right].
\]

It remains to bound \( \mathbb{E} \left[ \left\| \hat{Y}_i - \mathbb{E} \left[ \hat{Y}_i \right] \right\|_2^2 \right] \). Observe that

\[
\left| \mathbb{E} [\hat{Y}_i] \right| = \left| \frac{H_d \cdot X_i}{d} \right| = [1/d, \ldots, 1/d]^T,
\]

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and from expression (12), given \( r_i \), there are only \( 2^{k-1} \) non-zero coordinates, each with value bounded by \( (\varepsilon + 2^{k-1})/2^{k-1} \). Therefore we have

\[
E \left[ \| \hat{Y}_i - E [\hat{Y}_i] \|_2^2 \right] = E \left[ E \left[ \| \hat{Y}_i - E [\hat{Y}_i] \|_2^2 \mid r_i \right] \right] \\
\leq 2 \left( d \left( \frac{1}{d} \right)^2 + 2^{k-1} \left( \frac{\varepsilon + 2^k - 1}{2^{k-1} (\varepsilon - 1)} \right)^2 \right).
\]

Plugging this in to (24), we arrive at

\[
E \left[ \| \hat{D} - D_{X^n} \|_2^2 \right] \leq \frac{2 \left( d \left( \frac{1}{d} \right)^2 + 2^{k-1} \left( \frac{\varepsilon + 2^k - 1}{2^{k-1} (\varepsilon - 1)} \right)^2 \right)}{n^{2k-1}}.
\]

Picking \( k = \min (b, \lceil \varepsilon \log_2 \varepsilon \rceil, \lfloor \log d \rfloor) \) yields

\[
E \left[ \| \hat{D} - D_{X^n} \|_2^2 \right] = O \left( \frac{d}{n \min (2^b, \varepsilon, d)} \left( \frac{\varepsilon}{\varepsilon - 1} \right)^2 \right).
\]

Observe that

(i) if \( \varepsilon = O(2^b) \), then \( \varepsilon \leq 2^b \), so \( E \left[ \| \hat{D} - D_{X^n} \|_2^2 \right] = O \left( \frac{d \varepsilon}{n (\varepsilon - 1)^2} \right) \).

(ii) If \( \varepsilon = \Omega(2^b) \), then \( \frac{\varepsilon}{\varepsilon - 1} = \theta(1) \), and \( E \left[ \| \hat{D} - D_{X^n} \|_2^2 \right] = O \left( \frac{d}{n \min (2^b, d)} \right) \).

Therefore we conclude that

\[
E \left[ \| \hat{D} - D_{X^n} \|_2^2 \right] \leq \max \left( \frac{d}{n \min (2^b, d)}, \frac{d \varepsilon}{n (\varepsilon - 1)^2} \right) = \frac{d}{n} \left( \frac{1}{\min \{ \varepsilon, (\varepsilon - 1)^2, 2^b, d \}} \right).
\]

By Jensen’s inequality and Cauchy-Schwarz inequality, we also have

\[
E \left[ \| \hat{D} - D_{X^n} \|_1 \right] \leq \left( E \left[ \| \hat{D} - D_{X^n} \|_2^2 \right] \right)^{\frac{1}{2}} \leq \left( d \cdot E \| \hat{D} - D_{X^n} \|_2^2 \right)^{\frac{1}{2}} \\
\leq \frac{d}{\sqrt{n \min \{ \varepsilon, (\varepsilon - 1)^2, 2^b, d \}}}.
\]