The Potts Model with a Reflecting Boundary

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Abstract

A Potts model with a reflecting boundary is introduced and it is shown that its partition function can be expressed as a Markov trace on the Temperley-Lieb Algebra of Coxeter type B.

1 Introduction

Statistical and quantum field theoretic models with boundary conditions have recently attracted increasing interest. We believe that the theory of knots and braids of Coxeter type B plays an equally important role in these situations as the ordinary knot theory does in models without reflecting boundaries.

In this paper we investigate a generalisation of the ordinary Potts model by including a reflecting boundary. Hence we have besides the usual lattice of sites a wall interacting with it. We visualise this as in Figure 1a. The dotted lines indicate interaction bonds with the wall while the solid lines are usual bonds between sites. Each site supports a ‘spin’ which may occupy one of $f$ states.

\begin{figure}[h]
\centering
\includegraphics{figure1.pdf}
\caption{(a) A lattice with boundary (b) A boundary graph}
\end{figure}
2 The Potts model on an arbitrary graph

It is useful to consider the lattice as a special kind of graph and consider Potts models defined on arbitrary graphs. We further need to allow some vertices to be located on the wall. Such configurations occur as intermediate states in the recursive procedure for calculating the partition function we are going to present.

A Potts model with boundary lives on a graph $G = (V, B)$ with a set of vertices $V$ and a set of edges (bonds) $B$. Some of the vertices are defined to lie on the boundary. They are bound with special bonds to the reflecting wall. Formally we write the set of bonds $B = B_1 \cup B_0$ as a disjoint union of inner bonds $B_1 \subset V \times V$ and boundary bonds $B_0 \subset V$. We also single out a subset $V_0 \subset V$ of vertices that lie on the wall. These vertices shall have no degree of freedom: The associated spin is always in the ground state.

The whole structure shall be called a graph with boundary.

In figure 1 dotted lines (resp solid lines) indicate boundary bonds (resp. internal bonds). Note that bonds ending at a vertex on the wall are internal bonds.

We allow each vertex in $V \setminus V_0$ to acquire one of $f$ states. Vertices in $V_0$ are fixed to occupy state 0. Hence the set of all states is $\mathcal{S} = \{S : V \to \{0, 1, \ldots, f-1\} | S(i) = 0, i \in V_0\}$. The partition function (with $k$ being the Boltzmann constant and $T$ the temperature) is

$$Z_G = \sum_{S \in \mathcal{S}} \exp \left( -\frac{E(S)}{kT} \right)$$

(1)

with the Hamiltonian ($\delta(x, y)$ is the Kronecker symbol with values 0, 1)

$$E(S) = \sum_{(i,j) \in B_1} \delta(S_i, S_j) + \kappa \sum_{i \in B_0} (1 - \delta(0, S_i))$$

(2)

The first term in $E(S)$ is the usual Hamiltonian of the Potts model. The second term introduces the boundary condition.

$$Z_G = \sum_{S \in \mathcal{S}} \exp \left( -\frac{1}{kT} \sum_{(i,j) \in B_1} \delta(S_i, S_j) - \frac{\kappa}{kT} \sum_{i \in B_0} 1 - \delta(0, S_i) \right)$$

$$= \sum_{S \in \mathcal{S}} \prod_{(i,j) \in B_1} \exp \left( -\frac{1}{kT} \delta(S_i, S_j) \right) \prod_{i \in B_0} \exp \left( \frac{\kappa}{kT} (\delta(0, S_i) - 1) \right)$$

$$= \sum_{S \in \mathcal{S}} \prod_{(i,j) \in B_1} \left( 1 + \delta(S_i, S_j)(e^{-\frac{kT}{\kappa}} - 1) \right) \prod_{i \in B_0} (e^{-\frac{kT}{\kappa}} + \delta(0, S_i)(1 - e^{-\frac{kT}{\kappa}}))$$

We introduce the following short cuts

$$A := 1 \quad B := e^{-\frac{kT}{\kappa}} - 1 \quad C := e^{-\frac{kT}{\kappa}} \quad D := 1 - C$$

(4)

and take a factor $C$ out of the sum

$$Z_G = \sum_{S \in \mathcal{S}} \prod_{(i,j) \in B_1} (1 + \delta(S_i, S_j)B) \prod_{i \in B_0} (C + \delta(0, S_i)D)$$

(5)
where the bond is contracted such that its site is located on the boundary. In
\( G \)essentially the same. We write the set of states of
\( G \) be the graph obtained from
\( G \) vertex then
\( \Delta = \sum \{ b \mid i, j \in B_1 \} (1 + \delta(S_i, S_j)B) \prod_{i \in B_0} (1 + \delta(0, S_i)D/C) \) (6)

From this expression it can easily be read off how \( Z_G \) changes when \( G \) is changed.

Let \( \Delta = \sum \{ b \mid i, j \in B_1 \} (1 + \delta(S_i, S_j)B) \prod_{i \in B_0} (1 + \delta(0, S_i)D/C) \)

\( Z_G = Z_{G_d} + BZ_{G_c} \)

(7)

It is irrelevant whether any of the two vertices of the bond lie on the wall.

A vertex \( i \in V \setminus V_0 \) without any bonds may freely occupy all states without contributing to the energy. So if \( G_1 \) is obtained from \( G \) by adding a single free vertex then \( Z_{G_1} = fZ_G \). A vertex \( i \in V_0 \) without any bonds can occupy only one state and doesn’t contribute to the energy at all. So it can safely be omitted.

Now fix a boundary bond \( b \in B_0 \) (connecting site \( i_b \) to the wall) and let \( G_D \)
be the graph obtained from \( G \) by deleting the bond and denote by \( G_C \) the graph where the bond is contracted such that its site is located on the boundary. In \( G_C \) \( i_b \) hence becomes a vertex in \( V_0 \).

\( Z_G = CZ_{G_D} + DZ_{G_C} \)

(8)

We shall give the short calculation establishing this relation. The proof of (7) is essentially the same. We write the set of states of \( G \) as a disjoint union \( S = S_b \cup S_0 \) with \( S_b := \{ S \mid S_{ib} = 0 \} \).

\[ Z_G = \sum_{S \in S} \prod_{(i,j) \in B_1} (1 + \delta(S_i, S_j)B) \prod_{i \in B_0 \setminus \{ i_b \}} (C + \delta(0, S_i)D)(C + D\delta(0, S_{ib})) \]

\[ = \sum_{S \in S_b} \prod_{(i,j) \in B_1} (1 + \delta(S_i, S_j)B) \prod_{i \in B_0 \setminus \{ i_b \}} (C + \delta(0, S_i)D)(C + D) \]

\[ + \sum_{S \in S_0} \prod_{(i,j) \in B_1} (1 + \delta(S_i, S_j)B) \prod_{i \in B_0 \setminus \{ i_b \}} (C + \delta(0, S_i)D)C \]

By definition \( \sum_{S \in S_b} \prod_{(i,j) \in B_1} (1 + \delta(S_i, S_j)B) \prod_{i \in B_0 \setminus \{ i_b \}} (C + \delta(0, S_i)D) \) is nothing but \( Z_{G_C} \). The second summand in the calculation of \( Z_G \) is almost \( CZ_{G_D} \). The difference is due to states that assign 0 to \( S_{ib} \). However the contribution of this states to \( Z_{G_D} \) is exactly the same as \( Z_{G_C} \) so that we can complete our calculation

\( Z_G = (C + D)Z_{G_C} + C(Z_{G_D} - Z_{G_C}) = DZ_{G_C} + CZ_{G_D} \).

The relations shown so far suffice to calculate the partition function for any boundary graph because we can break any bond and thus work towards trivial components.

Figure 2 displays these relations graphically.

3 Links and Temperley-Lieb Algebras of Coxeter Type B

To proceed we need some facts about type B knot theory. To every root system (of a simple Lie algebra) there exists the associated Weyl group (Coxeter group). For
root systems of type $A_n$ it is the permutation group. For type $B_n$ it is a semidirect product of a permutation group with $\mathbb{Z}^2_2$. It has generators $X_0, X_1, \ldots, X_{n-1}$ and relations $X_i^2 = 1, |i - j| > 1 \Rightarrow X_iX_j = X_jX_i, i + 1 = j > 0 \Rightarrow X_jX_i = X_iX_jX_i$ and $X_0X_1X_0 = X_1X_0X_1X_0$. Omitting the quadratic relations from the Coxeter presentations of these groups one obtains the braid group of the root system. T. tom Dieck initiated in [1] the systematic study of quotients of the group algebras of these braid groups.

**Definition 1** The braid group $\mathbb{Z}B_n$ of Coxeter type $B$ is generated by $X_0, X_1, \ldots, X_{n-1}$ and relations

- $X_0X_1X_0X_1 = X_1X_0X_1X_0 \quad (9)$
- $X_iX_jX_i = X_jX_iX_j \quad |i - j| = 1, i, j \geq 1 \quad (10)$
- $X_iX_j = X_jX_i \quad |i - j| > 1 \quad (11)$

We further need the group $\tilde{\mathbb{Z}}B_n$ which is the free group on generators $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}, \sigma'_0, \sigma'_1, \ldots, \sigma'_{n-1}$ with $\sigma_i \mapsto X_i, \sigma'_i \mapsto X_i^{-1}$ is a surjection.

Among the finite dimensional quotients of the group algebra of $\mathbb{Z}B_n$ there is the Temperley Lieb algebra of type $B$ that was studied in [1]. Here we use a slight generalisation that depends on more parameters.

**Definition 2** The Temperley-Lieb Algebra $TB_n$ of Coxeter type $B$ over a ring with parameters $c, c', d$ is generated by $e_0, e_1, \ldots, e_{n-1}$ and relations

- $e_1e_0e_1 = c' e_1 \quad (12)$
- $e_i e_{j} e_i = e_i \quad |i - j| = 1, i, j \geq 1 \quad (13)$
- $e_i e_j = e_j e_i \quad |i - j| > 1 \quad (14)$
\( e_0^2 = c e_0 \)  \( (15) \)
\( e_i^2 = d e_i \)  \( (16) \)

TB\(_n\) has a Markov trace recursively defined by:

\[
\text{tr}(1) = 1 \quad \text{tr}(e_0) = cd^{-1} \quad \text{tr}(ae_i b) := d^{-1}\text{tr}(ab) \quad \forall a, b \in \text{TB}\(_n\) \quad (17)
\]

There are two topological interpretations of these algebras. The first approach is to interpret the words in this algebras as symmetric tangles. A bra (or tangle) of type \( B \) in ZB\(_n\) has \( 2n \) strands starting and ending in two sets of points (each set numbered \( \{-n, \ldots, -1, 1, \ldots, n\} \)) and is symmetric under reflection at the middle plane.

Alternatively one may think of \( B \) type tangles as tangles which live in \( \mathbb{R}^3 \) where a fixed line (we call it the 0-strand) has been removed. We call this second picture the cylinder interpretation. Both interpretations are useful in our context and we illustrate them in Figure 3.

From such a purely topological point of view it is natural to require \( c = c' \) in the definition of TB\(_n\). Keeping \( c \) and \( c' \) independent means that we keep information about the angle between strands 0 and 1.

The map \( \phi : \widetilde{ZB}\(_n\) \rightarrow \text{TB}\(_n\) \) defined by

\[
\phi(\sigma_i) := \beta e_i + \alpha \quad \phi(\sigma'_i) := \alpha c_i + \beta \quad i \geq 1
\]
\[
\phi(\sigma_0) := \beta_0 e_0 + \alpha_0 \quad \phi(\sigma'_0) := \alpha_0 e_0 + \beta_0
\]

\( (18) \)
\( (19) \)

can be graphically interpreted as implementation of the skein relations shown in figure 3 (using the cylinder picture).
4 The Link of a Graph with Boundary

In analogy with Kauffman’s treatment of the ordinary Potts model [3] we associate a link diagram \( L(G) \) of type B with the boundary graph \( G \). To accomplish this we put a crossing on every bond (internal as well as boundary bonds) and connect their ends so that each of the cells we produce contains either a vertex or a point of the reflecting wall in which a boundary bond ends. Now we turn each crossing on an internal bond to an over crossing or under crossing according to the following rule: Colour each cell containing a vertex or a boundary bond ending point in black while the others remain white. If one traverses the crossing on one of its arcs in such a way that the black region is first at the right hand side and after the crossing on the left hand side then the arc lies on top of the other arc. The crossings on boundary bonds are turned into the new kind of braiding that is represented by the picture for \( X_0 \) in the cylinder picture. The 0-strand is considered to be the wall. If one traverses the crossing on a boundary bond on one of its arcs in such a way that the black region is first at the right hand side and after the crossing on the left hand side then the arc first over crosses the fixed 0-strand and then under crosses it to return. Vertices on the wall are represented by introducing the picture for the \( e_0 \) generator.

Figure 4 shows a simple example of this process.

It is important to recall that we have assigned link diagrams to boundary graphs. They may be deformed by isotopy, but no Reidemeister moves are allowed. This is the reason for the introduction of the group \( \tilde{Z}_B \).

5 The B Potts Polynomial

To each link diagram \( L \) of type B we associate a Polynomial \( W(L) \) which is an invariant of isotopy (without any Reidemeister moves). The partition function of a boundary Potts model on a boundary graph \( G \) is then given by

\[
Z_G = C^{B_0} d^V c^{-V_0} W(L(G))
\]

The B Potts bracket \( W(L) \) of a link diagram \( L \) is defined by the skein relations

\[
\begin{align*}
\times & = \beta \\
\cup & = \alpha + \beta \\
\downarrow \left( \begin{array}{c}
\circ \\
\end{array} \right) & = \beta_0 \\
\uparrow \left( \begin{array}{c}
= c' \\
\end{array} \right) & = \alpha_0 \\
\end{align*}
\]
The proof of (20) is done simply by checking that (20) changes in the right way (i.e. according to the rules found in the end of section 3) under changes of the underlying graph.

Adding a single free vertex in $V \setminus V_0$ enriches the link diagram by an extra unknotted circle. Thus $W$ picks up a factor $d$. An additional factor $d$ is explicitly multiplied on the right hand side of (20) so that $Z$ is multiplied by $d^2 = f$ as it should. If we add an additional vertex $i \in V_0$ on the wall the link diagram picks up a loop formed from $e_0$ i.e. a factor $c'$ according to the skein relations. But this factor is cancelled by the contribution $dc^{-1}$ arising from the counting of all vertices in (20) so that $Z$ remains unaltered.

Deleting an inner bond is the same as replacing a crossing by the trivial braid. Since this doesn’t change the number of vertices no factor is picked up and since the skein coefficients for this case agree ($A = \alpha = 1$) the first summand in (20) is reproduced correctly. Contracting a bond in the graph means introduction of parallel strands of the kind of the $e_i$ picture. Contracting a bond decreases the number of vertices in $V \setminus V_0$ by one. This is taken into account by setting $\beta := B/d$.

Now delete a boundary bond. In terms of the link diagram this means replacing a crossing of type $X_0$ by a trivial braid. The skein relations give for this a coefficient

\begin{align*}
\alpha &:= 1 \quad \alpha_0 := 1 \quad \text{(21)} \\
\beta &:= Bd^{-1} \quad \beta_0 := DC^{-1}c^{-1} \quad \text{(22)} \\
d &:= f^{1/2} \quad c := c'd \quad \text{(23)}
\end{align*}
\( \alpha_0 = 1 \). Inserting the bond back increases the number of boundary bonds by one so that (20) introduces an additional factor \( C \) as in (8). For the same reason contracting a boundary bond also yields a factor \( C \) but there is no such factor in the corresponding summand of (8) hence we have to introduce a factor \( C \) into the denominator of \( \beta_0 \). A factor \( D \) in the numerator is required by (8). There is however an additional effect: When deleting the bond we increase the number of sites on the wall by one. This means that the partition function for the original graph picks up a factor \( e^{-1} \) which we also have to supply in \( \beta_0 \).

Since these relations suffice to calculate \( Z \) we have shown that (20) indeed expresses the partition function considered in section 2.

These results can now be applied to the case of a rectangular lattice. From figure 6 it can be easily seen how the link diagram of such a lattice can be described in terms of the \( B \) braid generators.

\[
\tau_{n,m} := \tau'_m (\tau''_m \tau'_m)^n
\]  
\[\tau'_m := \sigma_0 \sigma_2 \cdots \sigma_{2m-2}\]  
\[\tau''_m := \sigma'_1 \sigma'_3 \cdots \sigma'_{2m-1}\]  
\[E_m := e_1 e_3 \cdots e_{2m-1}\]

\( E_m \tau_{n,m} E_m \) is obviously the B link diagram associated to a Potts model with boundary living on a \( m \times n \) lattice (with \( n \) boundary sites). Hence we have expressed its \( B \) Potts polynomial as

\[ W = d^m \text{tr}(E_m \phi(\tau_{n,m}) E_m) \]  
and hence its partition function as

\[ Z = C^n f^{nm/2} d^m \text{tr}(E_m \phi(\tau_{n,m}) E_m) \]

6 Comments

1. Knot theory of type \( B \) can also be used to find solutions of the reflection equation augmenting the parameter dependent Yang Baxter equations. This was done in [2] using a Birman Wenzl algebra of type \( B \). Together with the present work this supports our belief that a great part of the braid and knot program carried out for models on the plane can also be done for models on the half plane.

2. It is possible to use (29) to calculate the partition function for small lattices. This was done with a Mathematica program and provided a quick correctness check. The runtime behaviour of this approach in its direct incarnation is however worse than that of a direct summation according to (1).

References

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[3] L. H. Kauffman, Knots and Physics, World Scientific, 1991