COMPACT EMBEDDED MINIMAL SURFACES OF POSITIVE GENUS WITHOUT AREA BOUNDS

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ABSTRACT. Let $M^3$ be a three-manifold (possibly with boundary). We will show that, for any positive integer $\gamma$, there exists an open nonempty set of metrics on $M$ (in the $C^2$-topology on the space of metrics on $M$) for each of which there are compact embedded stable minimal surfaces of genus $\gamma$ with arbitrarily large area. This extends a result of Colding and Minicozzi, who proved the case $\gamma = 1$.

INTRODUCTION

Throughout this paper, we use the $C^2$-topology on the space of metrics on a manifold. Our main result is the following theorem.

**Theorem 1:** Let $M^3$ be a three-manifold (possibly with boundary), and let $\gamma$ be a positive integer. There exists an open nonempty set of metrics on $M$ for each of which there are compact embedded minimal surfaces of genus $\gamma$ with arbitrarily large area. In fact, these can be chosen to be stable, i.e., with Morse index zero.

Although the theorem ensures that there are “many” metrics for which we can embed compact genus $\gamma$ minimal surfaces of arbitrarily large area, the result is false for a large class of metrics. Namely, a result of Choi and Wang (see [CW]) asserts that for any metric in which $M$ has Ricci curvature bounded below by a positive constant, there is an upper bound on the area of compact embedded minimal surfaces of genus $\gamma$, depending on $\gamma$ and the lower bound for $\text{Ric}_M$.

Colding and Minicozzi (see [CM1]) have already proved Theorem 1 for $\gamma = 1$. In Section 2 we will prove the theorem for $\gamma = 2$, with an argument borrowing heavily from the genus one case. The theorem will then be extended easily to genus greater than two in Section 3.

It remains an open question as to whether or not the theorem remains valid for genus zero, i.e., embedded minimal 2-spheres.

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1. The genus 2 case

Let $\Sigma_2$ denote the standard genus two surface. This has fundamental group

$$\pi_1(\Sigma_2) = \langle x_1, y_1, x_2, y_2 \mid x_1y_1x_1^{-1}y_1^{-1}x_2y_2x_2^{-1}y_2^{-1} \rangle$$

where $x_1$ and $x_2$ are freely homotopic to meridians of the two handles, where the meridians have the same orientation, and $y_1$ and $y_2$ are freely homotopic to lines of latitude of the two handles, where the lines of latitude have the same orientation. Let $\Omega_2$ be a solid genus two surface with a solid genus two surface and two solid tori removed, where the solid tori lie in the same handle of the ambient genus two surface. This can be pictured as in Figure 1, where the top and bottom of the picture are identified.

![Figure 1. $\Omega_2$. The top and bottom of the picture are identified.](image)

The fundamental group of $\Omega_2$ is

$$\pi_1(\Omega_2) = \langle a, b, c_1, d_1, c_2, d_2 \mid ad_1a^{-1}d_1^{-1}, bd_1b^{-1}d_1^{-1}, c_1d_1c_1^{-1}d_1^{-1}c_2d_2c_2^{-1}d_2^{-1} \rangle$$

where the generators are as follows:

(i): $a$ and $b$ are freely homotopic to meridians of the two removed solid tori (clockwise rotation around the two removed solid tori in Figure 1).

(ii): $c_1$ is freely homotopic to a meridian of the handle of the removed solid genus two in the same handle of the ambient solid genus two as the removed solid tori (clockwise rotation around the left handle of the removed solid genus two in Figure 1).

(iii): $d_1$ is freely homotopic to a line of latitude of the left handle of the ambient solid genus two in Figure 1.

(iv): $c_2$ is freely homotopic to a meridian of the removed solid genus two in the other handle from the meridian which is freely
homotopic to \( c_1 \) (clockwise rotation around the right handle of the removed solid genus two in Figure 1).

**(v):** \( d_2 \) is freely homotopic to a line of latitude of the right handle of the ambient solid genus two in Figure 1, with the same orientation as the line of latitude which is freely homotopic to \( d_1 \).

Before we give the proof of Theorem 1 for the genus two case, we need the following proposition, whose proof is inspired by a calculation in [Es].

**Proposition 1:** Let \( \Omega^n \) be a compact Riemannian manifold with boundary and dimension \( n \geq 3 \). Then, the set of metrics on \( \Omega \) in which \( \Omega \) is strictly mean convex is open and nonempty.

**Proof:** The set of such metrics is clearly open, by the definition of strictly mean convex. To show it is nonempty, let \( g \) be any metric on \( \Omega \), and let \( \tilde{g} = e^{2f}g \) be a metric conformally related to \( g \). Let \( \{e_1, ..., e_n\} \) be a framing for \( \Omega \) so that \( g_{ij} = \delta_{ij} \) and \( e_n \) is the unit normal to \( \partial \Omega \) in \( g \) (and therefore, \( e^{-f}e_n \) is the unit normal to \( \partial \Omega \) in \( \tilde{g} \)). Fix a point \( p \in \partial \Omega \), and choose coordinates \( \{x_1, ..., x_n\} \) at \( p \) so that, at \( p \), \( e_i = \frac{\partial}{\partial x_i} \) for all \( i \). Then, the second fundamental form of \( \partial \Omega \) in \( g \) at \( p \) is given by, for \( i, j = 1, ..., n - 1 \),

\[
h_{ij} = g(\nabla_{e_i} e_j, e_n) = \sum_{k=1}^{n} g(\Gamma^k_{ij} e_k, e_n) = \Gamma^n_{ij}.
\]

and the mean curvature of \( \partial \Omega \) in \( g \) at \( p \) is given by

\[
h = \frac{1}{n-1} \sum_{i,j=1}^{n-1} g^{ij} h_{ij} = \frac{1}{n-1} \sum_{i,j=1}^{n-1} \delta_{ij} \Gamma^n_{ij} = \frac{1}{n-1} \sum_{i=1}^{n-1} \Gamma^n_{ii}.
\]

The second fundamental form of \( \partial \Omega \) in \( \tilde{g} \) at \( p \) is given by, for \( i, j = 1, ..., n - 1 \),

\[
\tilde{h}_{ij} = \tilde{g}(\tilde{\nabla}_{e_i} e_j, e^{-f} e_n) = e^{2f} g(\tilde{\nabla}_{e_i} e_j, e^{-f} e_n) = \sum_{k=1}^{n} e^{f} g(\tilde{\Gamma}^k_{ij} e_k, e_n) = e^{f} \tilde{\Gamma}^n_{ij}.
\]

Now,

\[
\tilde{\Gamma}^n_{ij} = \frac{1}{2} \sum_{l=1}^{n} (\tilde{g}_{jl,i} + \tilde{g}_{il,j} - \tilde{g}_{ij,l}) \tilde{g}^{ln}
\]

where, for example,

\[
\tilde{g}_{jl,i} = \frac{\partial}{\partial x_i} \tilde{g}_{jl}
\]
So, at \( p \),

\[
\tilde{\Gamma}_{ij} = \frac{1}{2}e^{-2f}\sum_{l=1}^{n}(\tilde{g}_{jl,i} + \tilde{g}_{il,j} - \tilde{g}_{ij,l}) g_{ln}
\]

\[
= \frac{1}{2}e^{-2f}\sum_{l=1}^{n}(\tilde{g}_{jl,i} + \tilde{g}_{il,j} - \tilde{g}_{ij,l}) \delta_{ln}
\]

\[
= \frac{1}{2}e^{-2f}(\tilde{g}_{jn,i} + \tilde{g}_{in,j} - \tilde{g}_{ij,n})
\]

\[
= \frac{1}{2}e^{-2f}\left(2 \frac{\partial f}{\partial x_i} e^{2f} g_{jn} + e^{2f} g_{jn,i} + 2 \frac{\partial f}{\partial x_j} e^{2f} g_{in} + e^{2f} g_{in,j}
- 2 \frac{\partial f}{\partial x_n} e^{2f} g_{ij} - e^{2f} g_{ij,n}\right)
\]

\[
= \frac{1}{2}(g_{jn,i} + g_{in,j} - g_{ij,n}) + \frac{\partial f}{\partial x_i} \delta_{jn} + \frac{\partial f}{\partial x_j} \delta_{in} - \frac{\partial f}{\partial n} \delta_{ij}
\]

\[
= \Gamma_{ij} - \frac{\partial f}{\partial n} \delta_{ij}
\]

since \( i, j < n \), where \( \frac{\partial f}{\partial n} \) is the normal derivative of \( f \) with respect to the unit normal \( e_n \). Therefore, we have

\[
\tilde{h}_{ij} = e^f \tilde{\Gamma}_{ij} = e^f \Gamma_{ij} - e^f \frac{\partial f}{\partial n} \delta_{ij} = e^f \Gamma_{ij} - \frac{\partial}{\partial n}(e^f) \delta_{ij}.
\]

The mean curvature of \( \partial \Omega \) in \( \tilde{g} \) at \( p \) is then given by

\[
\tilde{h} = \frac{1}{n-1} \sum_{i,j=1}^{n-1} \tilde{g}^{ij} \tilde{h}_{ij}
\]

\[
= \frac{1}{n-1} \sum_{i,j=1}^{n-1} e^{-2f} \delta_{ij} \left(e^f \Gamma_{ij} - \frac{\partial}{\partial n}(e^f) \delta_{ij}\right)
\]

\[
= \frac{1}{n-1} \sum_{i=1}^{n-1} \left(e^{-f} \Gamma_{ii} - e^{-2f} \frac{\partial}{\partial n}(e^f)\right)
\]

\[
= e^{-f} \left(\frac{1}{n-1} \sum_{i=1}^{n-1} \Gamma_{ii}\right) - e^{-2f} e^f \frac{\partial f}{\partial n}
\]

\[
= e^{-f} \left(h - \frac{\partial f}{\partial n}\right)
\]

We have shown that this relation holds at an arbitrarily chosen point of \( \partial \Omega \), and so it holds everywhere on \( \partial \Omega \) since all quantities involved are tensorial. Let \( m \) be the minimum of \( h \) on \( \partial \Omega \), which exists since
∂Ω is compact. Choose \( f \) so that \( \frac{\partial f}{\partial n} < m \) everywhere on \( \partial \Omega \) and \( f \equiv 0 \) outside a small tubular neighborhood around \( \partial \Omega \). Then, \( \tilde{g} = g \) except for a small tubular neighborhood around \( \partial \Omega \), and \( \tilde{h} > 0 \) everywhere on \( \partial \Omega \), so \( \tilde{g} \) is a metric in which \( \Omega \) is strictly mean convex. This completes the proof of Proposition 1.

To prove Theorem 1, we will also need the following lemma.

Lemma 1: Let \( N^3 \) be a compact Riemannian manifold, and let \( \{M_n\} \subset N \) be a sequence of stable, compact, connected, embedded minimal surfaces without boundary such that the following conditions hold:

- (i): there exists a constant \( C_1 > 0 \) such that \( \text{Area}(M_n) \leq C_1 \) for all \( n \).
- (ii): there exists a constant \( C_2 > 0 \) such that

\[
\sup_{M_n} |A_n|^2 \leq C_2
\]

for all \( n \), where \( A_n \) is the second fundamental form of \( M_n \).

Then, a subsequence of \( \{M_n\} \) converges to a compact, connected, embedded minimal surface without boundary \( M \subset N \) of finite multiplicity.

Proof: Take a finite covering \( \{B_r(y_j)\} \) of \( N \) so that \( \{B_{r/2}(y_j)\} \) is still a covering of \( N \). Then, by [CM2], a subsequence of \( \{M_n\} \) converges in each \( B_{r/2}(y_j) \) to a lamination \( M \) with minimal leaves. By taking a diagonal subsequence, we have a subsequence of \( \{M_n\} \), which we still call \( \{M_n\} \), converging to \( M \) everywhere. \( M \) is clearly minimal, and it is embedded by the maximum principle.

We claim that the number of leaves of \( M_n \) in each \( B_r(y_j) \) which intersect \( B_{r/2}(y_j) \) has an upper bound which is uniform in \( n \) and \( j \). Let \( \Gamma_{n,j} \) be any such leaf. Then, there exists \( x_j \in \Gamma_{n,j} \cap \partial B_{r/2}(y_j) \). So, \( B_{r/2}(x_j) \subset B_r(y_j) \), and by monotonicity of area, there exists a constant \( C > 0 \) so that

\[
\text{Area}(\Gamma_{n,j} \cap B_{r/2}(x_j)) \geq C \left( \frac{r}{2} \right)^2.
\]

So, each \( \Gamma_{n,j} \) is of at least some fixed positive area, and so the area bound (i) gives an upper bound for the number of such leaves which is uniform in \( n \) and \( j \). We can take a subsequence so that the number of leaves of \( M_n \) is the same in each \( B_{r/2}(y_j) \) for all \( n, j \). Then, the limit \( M \) must have finite multiplicity, although the multiplicity may be different in each connected component of \( M \). We have shown that each connected component of \( M \) is a closed surface. The diameter of
M is bounded, since M is covered by finitely many balls $B_{r/2}(y_j) \cap M$. So, M is compact. M is without boundary since each $M_n$ is without boundary.

It remains to show that M is connected, which would imply that $M_n \to M$ with fixed finite multiplicity. Suppose M is not connected, and let A and B be distinct connected components of M. Then, $\epsilon = \text{dist}(A, B) > 0$. Let $R = \{ x \in N | \frac{\epsilon}{3} < \text{dist}(x, A) < \frac{2\epsilon}{3} \}$. So, R is disjoint from both A and B. Since $M_n \to M$, for large enough n we have $M_n \cap A \neq \emptyset$ and $M_n \cap B \neq \emptyset$, but $M_n \cap R = \emptyset$, contradicting the connectedness of $M_n$. So, M is connected.

Therefore, $M_n$ converges to a compact, connected, embedded minimal surface without boundary $M \subset N$ of finite multiplicity. This completes the proof of Lemma 1.

Proof of Theorem 1 for $\gamma = 2$:

Given any three-manifold M, we can embed $\Omega_2$ in M. Choose a metric $g$ on M so that $\Omega_2$ is strictly mean convex (by Proposition 1, the set of such g’s is open and nonempty). Let $f_n : \Sigma_2 \to \Omega_2$ be a map such that the induced map $f_n^\# : \pi_1(\Sigma_2) \to \pi_1(\Omega_2)$ is the following:

$$f_n^\#(x_1) = (ba)^nc_1(ba)^{-n}a^{-1}ba$$
$$f_n^\#(y_1) = d_1$$
$$f_n^\#(x_2) = c_2$$
$$f_n^\#(y_2) = d_2$$

It is easy to see that there exist such maps $f_n$ which are embeddings.

One can check that $f_n^\#(x_1)$ minimizes the word metric for its conjugacy class: by conjugating $f_n^\#(x_1)$ by any element of $\pi_1(\Omega_2)$ and using the relations of $\pi_1(\Omega_2)$, one can not decrease the length of $f_n^\#(x_1)$ in the word metric (for the definition of word metric, see [CM1]). So, for each n, we have an embedded incompressible genus two surface $\Sigma_{2,n} = f_n(\Sigma_2)$. By [ScY], there are immersed least-area (minimal) genus two surfaces $\Gamma_{2,n} \subset \Omega_2$ with $\Gamma_{2,n} \cap \partial \Omega_2 = \emptyset$ so that $\Gamma_{2,n}$ and $\Sigma_{2,n}$ induce the same mapping from $\pi_1(\Sigma_2)$ to $\pi_1(\Omega_2)$ for each n. Since the $\Sigma_{2,n}$ are embedded, [FHS] implies that the $\Gamma_{2,n}$ are embedded.

We claim that the areas of the $\Gamma_{2,n}$’s are unbounded. Assume not. Then, there exists a constant $C_1 > 0$ such that $\text{Area}(\Gamma_{2,n}) \leq C_1$ for all n. The $\Gamma_{2,n}$ are stable since they are area-minimizing. So, by [Sc], we get a uniform curvature estimate: there exists a constant $C_2 > 0$ such that, for small enough r and all $\sigma \in (0, r]$,
for all \( n \) and all balls \( B_{r-\sigma} \subset \Omega_2 \), where \( A_n \) is the second fundamental form of \( \Gamma_{2,n} \). Since the \( \Gamma_{2,n} \) are all without boundary, we get a uniform curvature estimate on all of \( \Gamma_{2,n} \), instead of just on balls. Therefore, by Lemma 1, a subsequence of \( \{ \Gamma_{2,n} \} \) converges to a compact, connected, embedded minimal surface without boundary \( \Gamma_2 \subset \Omega_2 \) of finite multiplicity.

For large \( n \), the \( \Gamma_{2,n} \) are coverings of \( \Gamma_2 \) by the maximum principle, and the degree of the covering is proportional to \( n \). Let \( n \to \infty \). Then, \( \Gamma_2 \) has infinite multiplicity, a contradiction. Therefore, the areas of the \( \Gamma_{2,n} \)'s are unbounded. This completes the proof of Theorem 1 for the case \( \gamma = 2 \).

2. The general case: \( \gamma \geq 2 \)

We now move to the general case. The arguments for fixed genus \( \gamma \geq 2 \) are essentially the same as in the genus 2 case.

Let \( \Sigma_\gamma \) denote the standard genus \( \gamma \) surface, \( \gamma \geq 2 \). This has fundamental group

\[
\pi_1(\Sigma_\gamma) = \langle x_1, y_1, \ldots, x_\gamma, y_\gamma | [x_1y_1] \cdots [x_\gamma y_\gamma] \rangle
\]

where the \( x_i \) are freely homotopic to meridians of the handles, all with the same orientation, the \( y_i \) are freely homotopic to lines of latitude of the handles, all with the same orientation, and \([x_i y_i] = x_i y_i x_i^{-1} y_i^{-1}\) for \( i = 1, \ldots, \gamma \). Let \( \Omega_\gamma \) be a solid genus \( \gamma \) surface with a solid genus \( \gamma \) surface and two solid tori removed, where the solid tori both lie in one of the end handles of the ambient genus \( \gamma \) surface (the case \( \gamma = 3 \) is shown in Figure 2, where the top and bottom of the picture are identified).

![Figure 2. \( \Omega_3 \). The top and bottom of the picture are identified.](image-url)
The fundamental group of $\Omega_{\gamma}$ is

$$\pi_1(\Omega_{\gamma}) = \langle a, b, c_1, d_1, \ldots, c_{\gamma}, d_{\gamma} | [ad_1], [bd_1], [c_1d_1] \cdots [c_\gamma d_\gamma] >$$

where the generators are defined as in the case $\gamma = 2$ (so, $a$, $b$, and all $c_i$ are freely homotopic to meridians with the same orientation, and all $d_i$ are freely homotopic to lines of latitude with the same orientation).

**Proof of Theorem 1:**

Given any three-manifold $M$, we can embed $\Omega_{\gamma}$ in $M$. Choose a metric $g$ on $M$ so that $\Omega_{\gamma}$ is strictly mean convex (by Proposition 1, the set of such $g$’s is open and nonempty). Let $f_n : \Sigma_{\gamma} \rightarrow \Omega_{\gamma}$ be a map such that the induced map $f_n^# : \pi_1(\Sigma_{\gamma}) \rightarrow \pi_1(\Omega_{\gamma})$ is the following:

\[
\begin{align*}
  f_n^#(x_1) &= (ba)^nc_1(ba)^{-n}a^{-1}ba \\
  f_n^#(x_i) &= c_i \text{ for } i = 2, \ldots, \gamma \\
  f_n^#(y_i) &= d_i \text{ for } i = 1, \ldots, \gamma
\end{align*}
\]

It is easy to see that there exist such maps $f_n$ which are embeddings. The proof then proceeds exactly as in the genus 2 case. The results of [ScY], [FHS], and [Sc] again apply.

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