Some Generalized BRS Transformations. I
The Pure Yang-Mills Case

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Abstract

Some generalized BRS transformations are developed for the pure Yang-Mills theory, and a form of quantum gravity. Unlike the usual BRS transformations: these are nonlocal; may be infinite formal power series in the gauge fields; and do not leave the action invariant, but only the product $e^{-S}$ with the Jacobian. Similar constructions should exist for many other field theory situations.
I. Introduction

Since the development of BRS transformations for the Yang-Mills theory, [1], they have played a major role in theoretical applications, such as to the study of renormalization and unitarity. BRS transformations have also been given for quantum gravity [2],[3],[4], and applied to study the renormalizability of higher derivative quantum gravity, [4]. Our interest was to develop a BRS transformation for a particular formulation of quantum gravity in a natural gauge to the theory, [5]. This led us to develop the generalized BRS transformations of this paper, and to apply them to the pure Yang-Mills theory. The Yang-Mills setting is a simpler arena to present the basic ideas, and hopefully generalized BRS transformations may have application to the Yang-Mills theory. There has been study of some aspects of the Yang-Mills theory by other generalizations of the BRS symmetry,[6].

For the pure Yang-Mills we write the action as follows:

\[
S = \int \text{Tr}[\alpha F_{\mu\nu}^2 + \frac{\beta}{2}(\partial_{\mu}A_{\mu})^2 + \gamma \bar{c}_i(\partial_{\mu}L_i)(\partial_{\mu}L_j + [A_{\mu},L_j])c_j].
\] (1)

The superscript L indicates differentiation is to the left, and \(c_i\) is the ghost field. Sum over repeated indices will always be understood, except where otherwise indicated.

The BRS transformations are then:

\[
A_{\mu}(x) \rightarrow A_{\mu}(x) + (\partial_{\mu}c_j(x))L_j\lambda + [A_{\mu}(x),L_j]c_j(x)\lambda.
\] (2)

\[
\bar{c}_j(x) \rightarrow \bar{c}_j(x) - \left(\frac{\beta}{\gamma}\right) \partial_{\mu}A_{\mu}^f(x)\lambda.
\] (3)

\[
c_j(x) \rightarrow c_j(x) + \frac{1}{2} \delta_{j\kappa\ell}c_\kappa(x)c_\ell(x)\lambda.
\] (4)

We work in Euclidean space, and the \(L_i\) are orthonormal in the trace inner product.
These transformations leave the action invariant and have (super-) Jacobian 1 (of course working to linear order in $\lambda$). The structure constants satisfy:

$$s_{ij\kappa} = \text{Tr}(L_i[L_j, L_\kappa]).$$

(5)

II Generalized BRS Transformations for the Pure Yang-Mills

In contrast to (2),(3),(4) the generalized BRS transformations for the pure Yang-Mills theory will involve a rather arbitrary formal gauge transformation and are given as:

$$A_\mu(x) \to A_\mu(x) + \frac{\partial}{\partial x^\mu}[c_j(x) + \int_y F_j(x, y)c_j(y)]L_j\lambda$$

$$+ [A_\mu(x), L_j][c_j(x) + \int_y F_j(x, y)c_j(y)]\lambda$$

(6)

$$\bar{c}_i(x) \to \bar{c}_i(x) - \left(\frac{\beta}{\gamma}\right)(\frac{\partial}{\partial x^\mu}A^\mu_i(x))\lambda - \left(\frac{\beta}{\gamma}\right)G_i(x)\lambda$$

(7)

$$c_j(x) \to c_j(x) + \frac{1}{2}s_{j\kappa\ell}c_\kappa(x)c_\ell(x)\lambda + \int_y \int_z Z_{j\kappa\ell}(x, y, z)c_\kappa(y)c_\ell(z)\lambda$$

(8)

Here $F_j(x, y)$ is an essentially arbitrary formal power series in the $A_\mu$ field with the lowest order term of degree 1.

$$F_j(x, y) = F^1_j(x, y) + F^2_j(x, y) + ...$$

(9)

$F^i_j(x, y)$ is of degree $i$. $F^1$, say, is of form:

$$F^1_j(x, y) = \int_z f^1_j(x, y, z)A^{\mu}_i(z)$$

(10)
where

\[ A_\mu(x) = \Sigma_i A^i_\mu(x)L_i \]  \hspace{1cm} (11)

The \( G_i \) and \( Z_{j\kappa\ell} \) are determined as formal power series in the \( A_\mu \), inductively by degree, as will be specified below. If \( F_j \equiv 0 \) one gets the usual BRS transformation. If to order one in \( \lambda \) we write:

\[ S \rightarrow S + \Delta S\lambda \]  \hspace{1cm} (12)

\[ J = 1 + \Delta J\lambda \]  \hspace{1cm} (13)

Where \( J \) is the Jacobian of the transformation (6) –(8), then we require:

\[ \Delta S - \Delta J = 0 \]  \hspace{1cm} (14)

which ensures invariance of \( \int e^{-S} \) (i.e. invariance of \( e^{-S} \) times integration measure density).

We write

\[ \Delta S = \Delta S_1 + \Delta S_2 \]  \hspace{1cm} (15)

\[ \Delta J = \Delta J_1 + \Delta J_2 \]  \hspace{1cm} (16)

where the subscripts 1 and 2 split the expressions into terms linear and quadratic in \( c_i(x) \). Eq. (14) becomes two equations:

\[ \Delta S_1 - \Delta J_1 = 0 \]  \hspace{1cm} (17)

\[ \Delta S_2 - \Delta J_2 = 0 \]  \hspace{1cm} (18)

It is easy to see:

\[ \Delta J_2 = 0 \]  \hspace{1cm} (19)

The equations (18)-(19) are just:

\[ \Delta S_2 = 0 \]  \hspace{1cm} (20)
which by a simple calculation holds for \( Z \) satisfying:

\[
\Delta_x Z_{i\kappa\ell}(x, y, z) + \frac{\partial}{\partial x^\mu} (A^\nu_\mu(x) Z_{s\kappa\ell}(x, y, z)) s_{irs} - \frac{\partial}{\partial x^\mu} \left[ \left( \frac{\partial}{\partial x^\mu} F_\kappa(x, y) \right) \delta(z - x) \right] s_{i\kappa\ell} - \frac{\partial}{\partial x^\mu} [A^s_\mu(x) F_\kappa(x, y) \delta(z - x)] s_{rs\kappa} s_{ir\ell} = 0 \tag{21}
\]

\( \delta(z - x) \) is a four dimensional delta function. Equation (21) may be solved inductively in degree for \( Z \) a formal power series in the fields \( A_\mu(x) \), similar to \( F_\kappa(x, y) \).

In equation (21) indices \( i, \kappa, \ell \) are never summed!

With \( F_\kappa(x, y) \) given, and \( Z_{i\kappa\ell}(x, y, z) \) now determined, \( G_i(x) \) is obtained from equation (17), completely specifying the generalized BRS transformation (6)–(8). Similarly to the derivation of \( Z_{i\kappa\ell}(x, y, z) \), equation (17) holds if \( G_i(x) \) satisfies:

\[
\Delta_y G_i(y) - \left( \frac{\partial}{\partial y^\mu} G_i(y) \right) A^\mu_j(y) s_{rji} + \int_x A^\nu_i(x, \mu, \nu) \frac{\partial}{\partial x^\mu} F_i(x, y) \nonumber
\]

\[
+ \int_x A^\nu_i(x, \mu, \nu) A^\mu_j(x) F_i(x, y) s_{rji} + \frac{1}{\beta} \int_x \frac{\delta}{\delta A^\mu_i(x)} \frac{\partial}{\partial x^\mu} F_i(x, y) \nonumber
\]

\[
+ \frac{1}{\beta} \int_x \frac{\delta}{\delta A^\mu_i(x)} \left\{ A^\nu_j(x) F_i(x, y) \right\} s_{rji} - \frac{1}{\beta} \int_x \left[ Z_{jji}(x, x, y) - Z_{jij}(x, y, x) \right] = 0. \tag{22}
\]

This equation may be solved inductively as a formal power series in \( A_\mu(x) \) for \( G_i \). The familiar notation for functional derivative has been used, and commas indicate partial derivatives. Index \( i \) is never summed over! The last term is eq.(22) is delicate to calculate... the nitty-gritty yet awaits a proper exegesis.
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