New results on Hunt’s hypothesis (H) for Lévy processes

Ze-Chun Hu
Department of Mathematics, Nanjing University, Nanjing, 210093, China
E-mail: huzc@nju.edu.cn

Wei Sun
Department of Mathematics and Statistics, Concordia University,
Montreal, H3G 1M8, Canada
E-mail: wei.sun@concordia.ca

Jing Zhang
Department of Mathematics and Statistics, Concordia University,
Montreal, H3G 1M8, Canada
E-mail: waangel520@gmail.com

Abstract  In this paper, we present new results on Hunt’s hypothesis (H) for Lévy processes. We start with a comparison result on Lévy processes which implies that big jumps have no effect on the validity of (H). Based on this result and the Kanda-Forst-Rao theorem, we give examples of subordinators satisfying (H). Afterwards we give a new necessary and sufficient condition for (H) and obtain an extended Kanda-Forst-Rao theorem. By virtue of this theorem, we give a new class of Lévy processes satisfying (H). Finally, we construct a type of subordinators that does not satisfy Rao’s condition.

Keywords  Hunt’s hypothesis (H), Getoor’s conjecture, Lévy process, subordinator.

Mathematics Subject Classification (2010)  Primary: 60J45; Secondary: 60G51

Contents

1  Introduction 2

2  A comparison result on Lévy processes 4

3  Examples of subordinators satisfying (H) 7
3.1 Special subordinators ........................................... 7
3.2 Locally quasi-stable subordinators ......................... 8
3.3 Further examples .............................................. 9

4 A new necessary and sufficient condition for (H) and an extended Kanda-Forst-Rao theorem 11

5 A type of subordinators that does not satisfy Rao’s condition 15
5.1 Construction of the example ................................. 16
5.2 Discussions ................................................... 20

1 Introduction

Let $X$ be a time-homogeneous Markov process. Hunt’s hypothesis (H) says that “every semipolar set of $X$ is polar”. This hypothesis plays a crucial role in the potential theory of (dual) Markov processes. To illustrate its importance, let us recall some potential-theoretic principles.

Suppose that $E$ is a locally compact space with a countable base. Let $(X, P^x)$ and $(\hat{X}, \hat{P}^x)$ be a pair of dual standard Markov processes on $E$ as described in Blumenthal and Getoor [2, VI]. Denote by $B^n$ the family of all nearly Borel measurable subsets of $E$. For $D \subset E$, we define the first hitting time of $D$ by

$$\sigma_D := \inf\{t > 0 : X_t \in D\}.$$ 

A set $D \subset E$ is called polar (respectively, essentially polar) if there exists a set $C \in B^n$ such that $D \subset C$ and $P^x(\sigma_C < \infty) = 0$ for every $x \in E$ (respectively, almost every $x \in E$ with respect to the reference measure). $D$ is called a thin set if there exists a set $C \in B^n$ such that $D \subset C$ and $P^x(\sigma_C = 0) = 0$ for every $x \in E$. $D$ is called semipolar if $D \subset \bigcup_{n=1}^{\infty} D_n$ for some thin sets $\{D_n\}_{n=1}^{\infty}$.

Denote by $E^x$ the expectation with respect to $P^x$. Let $\alpha > 0$. A finite $\alpha$-excessive function $f$ on $E$ is called a regular potential provided that $E^x\{e^{-\alpha T_n}f(X_{T_n})\} \to E^x\{e^{-\alpha T}f(X_T)\}$ for $x \in E$ whenever $\{T_n\}$ is an increasing sequence of stopping times with limit $T$. Denote by $(U^\alpha)_{\alpha > 0}$ the resolvent operators for $X$.

- **Bounded positivity principle** $(P^x_\nu)$: If $\nu$ is a finite signed measure such that $U^\alpha\nu$ is bounded, then $\nu U^\alpha\nu \geq 0$, where $\nu U^\alpha\nu := \int_E U^\alpha\nu(x)\nu(dx)$.

- **Bounded energy principle** $(E^x_\nu)$: If $\nu$ is a finite measure with compact support such that $U^\alpha\nu$ is bounded, then $\nu$ does not charge semipolar sets.

- **Bounded maximum principle** $(M^x_\nu)$: If $\nu$ is a finite measure with compact support $K$ such that $U^\alpha\nu$ is bounded, then $\sup\{U^\alpha\nu(x) : x \in E\} = \sup\{U^\alpha\nu(x) : x \in K\}$.  


• **Bounded regularity principle** \((R_\alpha)\): If \(\nu\) is a finite measure with compact support such that \(U^n\nu\) is bounded, then \(U^n\nu\) is regular.

• **Polarity principle** (Hunt’s hypothesis \((H)\)): Every semipolar set is polar.

**Proposition 1.1** Assume that all \(1\)-excessive (equivalently, all \(\alpha\)-excessive, \(\alpha > 0\)) functions are lower semicontinuous. Then

\[
(P_\alpha^*) \iff (E_\alpha^*) \iff (M_\alpha^*) \iff (R_\alpha^*) \iff (H).
\]

**Proof.** \((R_\alpha^*) \iff (H)\) is proved in Blumenthal and Getoor \([2]\) and \((M_\alpha^*) \iff (H)\) is proved in Blumenthal and Getoor \([3]\). \((P_\alpha^*) \Rightarrow (M_\alpha^*)\) is proved in Rao \([21]\) and \((M_\alpha^*) \Rightarrow (P_\alpha^*)\) is proved in Fitzsimmons \([5]\). By \([3\), Proposition (2.1)\], \((E_\alpha^*) \Rightarrow (M_\alpha^*)\). By \([3\, Proposition (5.1)\] and the equivalence of \((M_\alpha^*)\) and \((H)\), \((M_\alpha^*) \Rightarrow (E_\alpha^*)\). \(\Box\)

Hunt’s hypothesis \((H)\) is also equivalent to some other important properties of Markov processes. For example, Blumenthal and Getoor \([3\, Proposition (4.1)\] and Glover \([10,\, Theorem (2.2)\] showed that \((H)\) holds if and only if the fine and cofine topologies differ by polar sets; Fitzsimmons and Kanda \([7]\) showed that \((H)\) is equivalent to the dichotomy of capacity.

In spite of its importance, \((H)\) has been verified only in some special situations. Some forty years ago, Getoor conjectured that essentially all Lévy processes satisfy \((H)\).

From now on we let \((\Omega, \mathcal{F}, P)\) be a probability space and \(X = (X_t)_{t \geq 0}\) be an \(\mathbb{R}^n\)-valued Lévy process on \((\Omega, \mathcal{F}, P)\) with Lévy-Khintchine exponent \(\psi\), i.e.,

\[
E[\exp\{i\langle z, X_t \rangle\}] = \exp\{-t\psi(z)\}, \quad z \in \mathbb{R}^n, \quad t \geq 0,
\]

where \(E\) denotes the expectation with respect to \(P\) and \(\langle \cdot, \cdot \rangle\) denotes the Euclidean inner product of \(\mathbb{R}^n\). The classical Lévy-Khintchine formula tells us that

\[
\psi(z) = i\langle a, z \rangle + \frac{1}{2} \langle z, Q z \rangle + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{i\langle z, x \rangle} + i\langle z, x \rangle 1_{\{|x|<1\}} \right) \mu(dx),
\]

where \(a \in \mathbb{R}^n, Q\) is a symmetric nonnegative definite \(n \times n\) matrix, and \(\mu\) is a measure (called the Lévy measure) on \(\mathbb{R}^n\setminus\{0\}\) satisfying \(\int_{\mathbb{R}^n\setminus\{0\}} (1 \wedge |x|^2) \mu(dx) < \infty\). Hereafter, we use \(\text{Re}(\psi)\) and \(\text{Im}(\psi)\) to denote the real and imaginary parts of \(\psi\), respectively, and use \((a, Q, \mu)\) to denote \(\psi\).

Let us recall some important results obtained so far for Getoor’s conjecture. When \(n = 1\), Kesten \([18]\) (cf. also Bretagnolle \([4]\) showed that if \(X\) is not a compound Poisson process, then every \(\{x\}\) is non-polar if and only if

\[
\int_0^\infty \text{Re}([1 + \psi(z)]^{-1}) dz < \infty.
\]

Port and Stone \([20]\) proved that for the asymmetric Cauchy process on the line every \(x\) is regular for \(\{x\}\), and thus \((H)\) holds in this case. Further, Blumenthal and Getoor \([3]\) showed that all stable processes with index \(\alpha \in (0, 2)\) on the line satisfy \((H)\).
Kanda [16] and Forst [8] proved that (H) holds if \( X \) has bounded continuous transition densities (with respect to the Lebesgue measure \( dx \)) and the Lévy-Khintchine exponent \( \psi \) satisfies \( |\text{Im}(\psi)| \leq M(1 + \text{Re}(\psi)) \) for some positive constant \( M \). Rao [21] gave a short proof of the Kanda-Forst theorem under the weaker condition that \( X \) has resolvent densities. In particular, for \( n \geq 1 \), all stable processes with index \( \alpha \neq 1 \) satisfy (H). Kanda [17] proved that (H) holds for stable processes on \( \mathbb{R}^n \) with index \( \alpha = 1 \) if we assume that the linear term vanishes. Silverstein [23] extended the Kanda-Forst condition to the non-symmetric Dirichlet forms setting, Fitzsimmons [6] extended it to the semi-Dirichlet forms setting and Han et al. [12] extended it to the positivity-preserving forms setting. Glover and Rao [11] proved that \( \alpha \)-subordinates of general Hunt processes satisfy (H) (cf. Theorem 3.1 below). Rao [22] proved that if all 1-excessive functions of \( X \) are lower semicontinuous and \( |\text{Im}(\psi)| \leq (1 + \text{Re}(\psi))f(1 + \text{Re}(\psi)) \), where \( f \) is an increasing function on \([1, \infty)\) such that \( \int_N^\infty (\lambda f(\lambda))^{-1}d\lambda = \infty \) for every \( N \geq 1 \), then \( X \) satisfies (H).

Let \( X \) be a Lévy process on \( \mathbb{R}^n \) with Lévy-Khintchine exponent \((a, Q, \mu)\). In [15], we showed that if \( Q \) is non-degenerate then \( X \) satisfies (H); if \( Q \) is degenerate then, under the assumption that \( \mu(\mathbb{R}^n \setminus \sqrt{Q}\mathbb{R}^n) < \infty \), \( X \) satisfies (H) if and only if the equation

\[
\sqrt{Q}y = -a - \int_{\{x \in \mathbb{R}^n : \sqrt{Q}\mathbb{R}^n : |x| < 1\}} x\mu(dx)
\]

has at least one solution \( y \in \mathbb{R}^n \). We also showed that if \( X \) is a subordinator and satisfies (H) then its drift coefficient must be 0.

In this paper, we will continue to explore (H) for Lévy processes. The rest of the paper is organized as follows. In Section 2, we present a comparison result on Lévy processes which shows that big jumps have no effect on the validity of (H) in some sense. Based on this result and the Kanda-Forst-Rao theorem, in Section 3, we give examples of subordinators satisfying (H). In Section 4, we give a new necessary and sufficient condition for (H) and obtain an extended Kanda-Forst-Rao theorem. By virtue of this theorem, we give a new class of Lévy processes satisfying (H). In section 5, we construct a type of subordinators that does not satisfy Rao’s condition. To the best of our knowledge, no existing criteria can be applied to this example. It suggests that maybe new ideas and methods are needed in order to completely solve Getoor’s conjecture even for the case of subordinators.

## 2 A comparison result on Lévy processes

In this section, we prove a comparison result on Lévy processes which implies that big jumps have no effect on the validity of (H) in some sense.

Let \( X \) be a Lévy process on \( \mathbb{R}^n \) with Lévy-Khintchine exponent \((a, Q, \mu)\). Suppose that \( \mu_1 \) is a finite measure on \( \mathbb{R}^n \setminus \{0\} \) such that \( \mu_1 \leq \mu \). Denote \( \mu_2 := \mu - \mu_1 \) and let \( X' \) be a Lévy process on \( \mathbb{R}^n \) with Lévy-Khintchine exponent \((a', Q, \mu_2)\), where

\[
a' := a + \int_{\{|x| < 1\}} x\mu_1(dx).
\]
**Theorem 2.1** Let $X$ and $X'$ be Lévy processes defined as above. Then

(i) they have same semipolar sets.

(ii) they have same essentially polar sets.

(iii) if both $X$ and $X'$ have resolvent densities, then $X$ satisfies (H) if and only if $X'$ satisfies (H).

**Proof.** Denote by $\psi$ and $\psi'$ the Lévy-Khintchine exponents of $X$ and $X'$, respectively. Then,

$$
\psi'(z) = i\langle a', z \rangle + \frac{1}{2} \langle z, Qz \rangle + \int_{\mathbb{R}^n} (1 - e^{i\langle z, x \rangle} + i\langle z, x \rangle 1_{|x|<1}) \mu_2(dx),
$$

$$
\psi(z) = i\langle a, z \rangle + \frac{1}{2} \langle z, Qz \rangle + \int_{\mathbb{R}^n} (1 - e^{i\langle z, x \rangle} + i\langle z, x \rangle 1_{|x|<1}) \mu(dx)
$$

$$
= \psi'(z) + \int_{\mathbb{R}^n} (1 - e^{i\langle z, x \rangle}) \mu_1(dx). \tag{2.1}
$$

(i) Suppose that $Y$ is a compound Poisson process with Lévy measure $\mu_1$ and is independent of $X'$. By (2.1), $X$ has the same law as that of $X' + Y$. Let $T_1$ be the first jumping time of $Y$. Then $T_1$ possesses an exponential distribution and thus $P(T_1 > 0) = 1$. Hence, for any set $A$ and any point $x \in \mathbb{R}^n$, $x$ is a regular point of $A$ relative to $X$ if and only if it is a regular point of $A$ relative to $X'$. Therefore $X$ and $X'$ have same semipolar sets.

(ii) Set $C := \mu_1(\mathbb{R}^n \setminus \{0\})$. By (2.1), we get

$$
\text{Re} \psi'(z) \leq \text{Re} \psi(z) \leq \text{Re} \psi'(z) + C \tag{2.2}
$$

and

$$
|\text{Im} \psi(z)| \leq |\text{Im} \psi'(z)| + C, \quad |\text{Im} \psi'(z)| \leq |\text{Im} \psi(z)| + C. \tag{2.3}
$$

For $\lambda > 0$, we have

$$
\text{Re} \left( \frac{1}{\lambda + \psi'(z)} \right) = \frac{\lambda + \text{Re} \psi'(z)}{(\lambda + \text{Re} \psi'(z))^2 + (\text{Im} \psi'(z))^2}, \tag{2.4}
$$

$$
\text{Re} \left( \frac{1}{\lambda + \psi(z)} \right) = \frac{\lambda + \text{Re} \psi(z)}{(\lambda + \text{Re} \psi(z))^2 + (\text{Im} \psi(z))^2}. \tag{2.5}
$$

By (2.2) and (2.3), we find that if $\lambda \geq \sqrt{2C}$ then

$$
\frac{\lambda + \text{Re} \psi'(z)}{(\lambda + \text{Re} \psi'(z))^2 + (\text{Im} \psi'(z))^2} \geq \frac{\lambda + \text{Re} \psi'(z)}{(\lambda + \text{Re} \psi'(z))^2 + (|\text{Im} \psi'(z)| + C)^2}
$$

$$
\geq \frac{\lambda + \text{Re} \psi'(z)}{2[(\lambda + \text{Re} \psi'(z))^2 + 2C^2 + (\text{Im} \psi'(z))^2]}
$$

$$
\geq \frac{1}{4} \frac{\lambda + \text{Re} \psi'(z)}{(\lambda + \text{Re} \psi'(z))^2 + (\text{Im} \psi'(z))^2}. \tag{2.6}
$$
Corollary 2.2

Let

\[ \frac{\lambda + \text{Re}\psi'(z)}{(\lambda + \text{Re}\psi'(z))^2 + (\text{Im}\psi'(z))^2} \geq \frac{\lambda + \text{Re}\psi(z) - C}{(\lambda + \text{Re}\psi(z))^2 + (\text{Im}\psi(z))^2 + 2C^2 + 2(\text{Im}\psi(z))^2} \geq \frac{1}{4} \frac{\lambda + \text{Re}\psi(z)}{(\lambda + \text{Re}\psi(z))^2 + (\text{Im}\psi(z))^2}. \]  

(2.7)

By (2.4)-(2.7), we obtain that if \( \lambda \geq 2C \) then

\[ \frac{1}{4} \text{Re}\left(\frac{1}{\lambda + \psi'(z)}\right) \leq \text{Re}\left(\frac{1}{\lambda + \psi(z)}\right) \leq 4 \text{Re}\left(\frac{1}{\lambda + \psi'(z)}\right). \]  

(2.8)

By (2.8) and Hawkes [14, Theorem 3.3], we obtain that a set is essentially polar for \( X \) if and only if it is essentially polar for \( X' \).

(iii) This is a direct consequence of (i), (ii) and [14, Theorem 2.1].

For \( \delta > 0 \), we define

\[ B_\delta := \{ x \in \mathbb{R}^n : 0 < |x| < \delta \}. \]

Corollary 2.2 Let \( X_\delta \) be a Lévy process on \( \mathbb{R}^n \) with Lévy-Khintchine exponent \((a_\delta, Q, \mu|_{B_\delta})\), where

\[ a_\delta := \begin{cases} 
    a + \int_{\{\delta \leq |x| < 1\}} x \mu(dx), & \text{if } 0 < \delta < 1, \\
    a, & \text{if } \delta \geq 1.
\end{cases} \]

Then, all the assertions of Theorem 2.1 hold with \( X' \) replaced by \( X_\delta \).

Remark 2.3 If \( \int_{|x| \leq 1} |x| \mu(dx) < \infty \), then \( \psi \) can be expressed by

\[ \psi(z) = i\langle d, z \rangle + \frac{1}{2} \langle z, Qz \rangle + \int_{\mathbb{R}^n} (1 - e^{i\langle z, x \rangle}) \mu(dx), \]

where \(-d\) is called the drift of \( X \). In this case, we call \((d, Q, \mu)\) the Lévy-Khintchine exponent of \( X \). For \( \delta > 0 \), we define \( B_\delta \) and \( X_\delta \) as above. Let \( X'_\delta \) be a Lévy process on \( \mathbb{R}^n \) with Lévy-Khintchine exponent \((d, Q, \mu|_{B_\delta})\). We claim that \( X_\delta \) and \( X'_\delta \) have the same law and then all the assertions of Theorem 2.1 hold with \( X' \) replaced by \( X'_\delta \). In fact, we have

\[ d = a + \int_{\{|x| < 1\}} x \mu(dx). \]

(2.9)

If \( 0 < \delta < 1 \), then

\[ a_\delta + \int_{\{|x| < 1\}} x \mu|_{B_\delta}(dx) = \left(a + \int_{\{\delta \leq |x| < 1\}} \mu(dx)\right) + \int_{\{|x| \leq \delta\}} x \mu(dx) = d; \]

(2.10)

if \( \delta \geq 1 \), then

\[ a_\delta + \int_{\{|x| < 1\}} x \mu|_{B_\delta}(dx) = a + \int_{\{|x| < 1\}} x \mu(dx) = d. \]

(2.11)

By (2.9)-(2.11), we know that \( X_\delta \) and \( X'_\delta \) have the same Lévy-Khintchine exponent \((d, Q, \mu|_{B_\delta})\) and thus have the same law.
3 Examples of subordinators satisfying (H)

In this section, we will present new examples of subordinators satisfying (H) by virtue of the comparison result given in Section 2 and the Kanda-Forst-Rao theorem. To the best of our knowledge, which subordinators satisfy (H) is unknown in general. To appreciate the importance of the validity of (H) for subordinators, let us recall the following remarkable result of Glover and Rao.

Theorem 3.1 (Glover and Rao [11]) Let \((X_t)_{t \geq 0}\) be a standard process on a locally compact space with a countable base and \((T_t)_{t \geq 0}\) be an independent subordinator satisfying Hunt’s hypothesis (H). Then \((X_{T_t})_{t \geq 0}\) satisfies (H).

Let \(X\) be a subordinator. Then, its Lévy-Khintchine exponent \(\psi\) can be expressed by

\[
\psi(z) = -idz + \int_{(0, \infty)} (1 - e^{izx}) \mu(dx), \quad z \in \mathbb{R},
\]

where \(d \geq 0\) (called the drift coefficient) and \(\mu\) satisfies \(\int_{(0, \infty)} (1 \wedge x) \mu(dx) < \infty\). In [15], we have proved the following result.

Proposition 3.2 If \(X\) is a subordinator and satisfies (H), then \(d = 0\).

By Proposition 3.2, when we consider (H) for subordinators, we may concentrate on the case that \(d = 0\). Hereafter we use \(c_1, c_2, \ldots\) to denote constants whose values can change from one appearance to another.

3.1 Special subordinators

Let \(X\) be a subordinator. Recall that the potential measure \(U\) of \(X\) is defined by

\[
U(A) = E \left[ \int_0^\infty 1_{\{X_t \in A\}} dt \right], \quad A \subset [0, \infty).
\]

For \(\alpha > 0\), the \(\alpha\)-potential measure \(U^\alpha\) of \(X\) is defined by

\[
U^\alpha(A) = E \left[ \int_0^\infty e^{-\alpha t} 1_{\{X_t \in A\}} dt \right], \quad A \subset [0, \infty).
\]

\(X\) is called a special subordinator if \(U\big|_{(0, \infty)}\) has a decreasing density with respect to the Lebesgue measure.

Theorem 3.3 Let \(X\) be a special subordinator. Then \(X\) satisfies (H) if and only if \(d = 0\).
**Proof.** By Proposition 3.2 we need only prove the sufficiency. Suppose that \( d = 0 \). If \( \mu \) is a finite measure, then \( X \) is a compound Poisson process and thus satisfies (H).

Now we consider the case that \( \mu \) is an infinite measure. By Bretagnolle [4, Theorem 8], \( X \) does not hit points, i.e., any single point set \( \{ x \} \) is a polar set of \( X \), which together with the assumption that \( U|_{[0,\infty)} \) has a decreasing density with respect to the Lebesgue measure, implies that \( U|_{[0,\infty)} \) has a density with respect to the Lebesgue measure. Since for any \( \alpha > 0 \), \( U^\alpha(\cdot) \leq U(\cdot) \), we obtain that for any \( \alpha \geq 0 \), \( U^\alpha \) is absolutely continuous with respect to the Lebesgue measure. Then by Hawkes [14, theorem 2.1], we know that for any \( \alpha \geq 0 \), all \( \alpha \)-excessive functions are lower semicontinuous. Therefore, by the fact that \( X \) does not hit points and Blumenthal and Getoor [3, Proposition (5.1), Theorem (5.3)], following the same argument for stable subordinators [3, page 140], we obtain that \( X \) satisfies (H). \( \square \)

### 3.2 Locally quasi-stable subordinators

Let \( S \) be a stable subordinator of index \( \alpha \), \( 0 < \alpha < 1 \). Then, its Lévy-Khintchine exponent \( \psi_S \) has the form

\[
\psi_S(z) = c|z|^\alpha(1 - i \operatorname{sgn}(z) \tan(\pi \alpha/2)), \quad z \in (-\infty, \infty),
\]

where \( c > 0 \). Its Lévy measure \( \mu_S \) is absolutely continuous with respect to the Lebesgue measure \( dx \) and can be expressed by

\[
\mu_S(dx) = \begin{cases} 
  c^+x^{-\alpha - 1}dx, & \text{if } x > 0, \\
  0, & \text{if } x \leq 0,
\end{cases}
\]

(3.1)

where \( c^+ > 0 \).

**Definition 3.4** Let \( X \) be a subordinator with drift 0 and Lévy measure \( \mu \). We call \( X \) a locally quasi-stable subordinator if there exist a stable subordinator \( S \) with Lévy measure \( \mu_S \), positive constants \( c_1, c_2, \delta \), and finite measures \( \mu_1 \) and \( \mu_2 \) on \( (0, \delta) \) such that

\[
c_1\mu_S - \mu_1 \leq \mu \leq c_2\mu_S + \mu_2 \quad \text{on } (0, \delta).
\]

**Proposition 3.5** Any locally quasi-stable subordinator satisfies (H).

**Proof.** Let \( X, S, \mu_1, \mu_2 \) and \( \delta \) be as in Definition 3.4. By Theorem 2.1 and Remark 2.3 we assume without loss of generality that \( \mu|_{(0,\infty)} = 0 \) and \( \mu_1 = 0 \). Denote by \( \psi \) and \( \psi_S \) the Lévy-Khintchine exponents of \( X \) and \( S \), respectively. Let \( \mu_S \) be as in (3.1). Then

\[
\operatorname{Re}\psi(z) = \int_0^\infty (1 - \cos(zx))\mu(dx) \\
\geq c_1 \int_0^\delta (1 - \cos(zx))\mu_S(dx) \\
= c_1 \left( \int_0^\infty (1 - \cos(zx))\mu_S(dx) - \int_\delta^\infty (1 - \cos(zx))\mu_S(dx) \right) \\
= c_1 \operatorname{Re}\psi_S(z) - K_1 \\
= c'|z|^{\alpha - K_1},
\]

(3.2)
where \( c_1, c', K_1 \) are positive constants.

\[
|\text{Im} \psi(z)| \leq \int_0^\infty |\sin(zx)| \mu(dx) \\
\leq c_2 \int_0^\delta |\sin(zx)| \mu_S(dx) \\
= c_2 \int_0^\infty |\sin(zx)| \mu_S(dx) - c_2 \int_\delta^\infty |\sin(zx)| \mu_S(dx) \\
\leq c_2 c \left\{ \int_0^{1/|z|} |\sin(zx)| x^{-\alpha} dx + \int_{1/|z|}^\infty |\sin(zx)| x^{-\alpha} dx \right\} + K_2 \\
\leq c_2 c \left\{ |z| \int_0^{1/|z|} x^{-\alpha} dx + \int_{1/|z|}^\infty x^{-\alpha} dx \right\} + K_2 \\
= c'' |z|^\alpha + K_2,
\]

where \( c_2, c'', K_2 \) are positive constants. By (3.2) and (3.3) we know that the Kanda-Forst condition holds for \( \psi \). By (3.2) and Hartman and Wintner \[13\], we know that \( X \) has bounded continuous transition densities. Therefore, \( X \) satisfies (H) by the Kanda-Forst theorem.

\[ \Box \]

**Corollary 3.6** Let \( \varphi \) be a Lévy-Khintchine exponent and \( \mu \) be a Lévy measure of some special subordinator with drift 0 or some locally quasi-stable subordinator. Then, the Lévy process with Lévy-Khintchine exponent

\[
\Phi(z) := \int_{(0, \infty)} \left( 1 - e^{-\varphi(z)x} \right) \mu(dx)
\]

satisfies (H).

**Proof.** Let \( X \) be a Lévy process with Lévy-Khintchine exponent \( \varphi \) and \( (T_t)_{t \geq 0} \) be a subordinator with drift 0 and Lévy measure \( \mu \), which is independent of \( X \). Then \( Y_t := X_{T_t} \) has the Lévy exponent \( \Phi \) defined by (3.4). Therefore, by Theorem 3.1 Theorem 3.3 and Proposition 3.5, we obtain that \( Y \) satisfies (H).

\[ \Box \]

### 3.3 Further examples

In this subsection, we give further examples of subordinators satisfying (H) by virtue of the comparison result given in Section 2 and the following theorem of Rao.

**Theorem 3.7** (Rao \[22\]) Let \( X \) be a Lévy process such that all 1-excessive functions are lower semicontinuous. Suppose there is an increasing function \( f \) on \([1, \infty)\) such that \( \int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty \) for any \( N \geq 1 \) and \( |1 + \psi| \leq (1 + \text{Re}(\psi)) f(1 + \text{Re}(\psi)) \). Then (H) holds.
Let $0 < \alpha < 1$ and $0 < \delta < 1$. We define
\[ \mu_T(dx) := \frac{1}{-\log(x)x^{1+\alpha}}dx, \quad 0 < x < \delta \]
and
\[ \mu_V(dx) = \frac{-\log(x)}{x^{1+\alpha}}dx, \quad 0 < x < \delta. \]
Let $X$ be a subordinator with drift 0 and Lévy measure $\mu$.

(i) If $c_1\mu_T - \mu_1 \leq \mu \leq c_2\mu_S + \mu_2$ on $(0, \delta)$ for some positive constants $c_1, c_2$ and finite measures $\mu_1, \mu_2$ on $(0, \delta)$, then $X$ satisfies (H).

In fact, by Theorem 2.1 and Remark 2.3, we may assume without loss of generality that $\mu|_{[\delta, \infty)} = 0$ and $\mu_1 = 0$. For any $z \in \mathbb{R}$ with $|z| > 1$, we have
\[
\text{Re}\psi(z) = \int_0^\infty (1 - \cos(zx))\mu(dx) \\
\geq c_1 \int_0^\delta (1 - \cos(zx))\mu_T(dx) \\
= c_1 \int_0^\infty (1 - \cos(zx))\mu_T(dx) - c_1 \int_\delta^\infty (1 - \cos(zx))\mu_T(dx) \\
\geq c_1 \int_0^{1/|z|} (1 - \cos(zx)) \left\{ \frac{x^2}{-\log(x)x^{1+\alpha}} \right\} dx - K_3 \\
\geq c_1' z^2 \int_{1/2|z|}^{1/|z|} \frac{x^2}{-\log(x)x^{1+\alpha}} dx - K_3 \\
\geq c_1'' \frac{|z|^{1/\alpha}}{\log(2|z|)} \int_0^{1/|z|} \frac{x^2}{x^{1+\alpha}} dx - K_3 = c_1'' \frac{|z|^{1/\alpha}}{\log(2|z|)} - K_3,
\]
where $c_1', c_2', K_3$ are positive constants. By (3.3) and (3.5), we obtain that $|\text{Im}\psi(z)| \leq c^*(1 + \text{Re}\psi(z))\log(1 + \text{Re}\psi(z))$ for some positive constant $c^*$. By Hartman and Wintner [13] and (3.5), we know that $X$ has bounded continuous transition densities. Therefore, $X$ satisfies (H) by Theorem 3.7.

(ii) If $c_1\mu_S - \mu_1 \leq \mu \leq c_2\mu_V + \mu_2$ on $(0, \delta)$ for some positive constants $c_1, c_2$ and finite measures $\mu_1, \mu_2$ on $(0, \delta)$, then $X$ satisfies (H).

In fact, by Theorem 2.1 and Remark 2.3, we may assume without loss of generality that $\mu|_{[\delta, \infty)} = 0$ and $\mu_1 = 0$. For any $z \in \mathbb{R}$ with $|z| > 1/\delta$, we have
\[
|\text{Im}\psi(z)| \leq \int_0^\infty |\sin(zx)|\mu(dx) \\
\leq c_2 \int_0^\delta |\sin(zx)|\mu_V(dx) + K_4
\]
where \( c_2, c''_2, K_4 \) are positive constants. By (3.2) and (3.6), we obtain that
\[
\operatorname{Im} \psi(z) \leq c^{**} \operatorname{Re} \psi(z) \log(\operatorname{Re} \psi(z))
\]
for some positive constant \( c^{**} \). By (3.2) and Hartman and Wintner [13], we know that \( X \) has bounded continuous transition densities. Therefore, \( X \) satisfies (H) by Theorem 3.7.

4 A new necessary and sufficient condition for (H) and an extended Kanda-Forst-Rao theorem

Let \( X \) be a Lévy process on \( \mathbb{R}^n \). From now on we assume that all 1-excessive functions are lower semicontinuous, equivalently, \( X \) has resolvent densities. Define
\[
A := 1 + \operatorname{Re}(\psi), \quad B := |1 + \psi|.
\]

**Theorem 4.1** (Rao [22]) Let \( \nu \) be a finite measure of finite 1-energy, i.e.,
\[
\int_{\mathbb{R}^n} B^{-2}(z)A(z)|\hat{\nu}(z)|^2dz < \infty.
\]
Then
\[
\lim_{\lambda \to \infty} \int_{\mathbb{R}^n} |\hat{\nu}(z)|^2(\lambda + \operatorname{Re}\psi(z))|\lambda + \psi(z)|^{-2}dz
\]
exists. The limit is zero if and only if \( U^1 \nu \) is regular.

Based on Theorems 4.1 and 3.7, we can prove the following result.

**Lemma 4.2** Let \( \nu \) be a finite measure of finite 1-energy and \( f \) be an increasing function on \([1, \infty)\) such that \( \int_{N}^{\infty}(\lambda f(\lambda))^{-1}d\lambda = \infty \) for some \( N \geq 1 \). Then \( U^1 \nu \) is regular if and only if
\[
\lim_{\lambda \to \infty} \sum_{k=1}^{\infty} \int_{\{B(z)>A(z)f(A(z)), k \leq \frac{|\operatorname{Im}\psi(z)|}{A(z)} < k+1, A(z) \leq \lambda < (k+1)|\operatorname{Im}\psi(z)|\}} \frac{\lambda}{\lambda^2 + (|\operatorname{Im}\psi(z)|)^2} |\hat{\nu}(z)|^2dz = 0.
\]
Proof. Since \( f \) is an increasing function on \([1, \infty)\), \( \int_1^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty \) for some \( N \geq 1 \) if and only if \( \int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty \) for any \( N \geq 1 \). From the proof of Theorem 3.7 (see Rao 22), we know that the limit

\[
\lim_{\lambda \to \infty} \int_{A(z) \leq \lambda} \frac{\lambda}{\lambda^2 + B^2(z)} |\dot{\nu}(z)|^2 \, dz
\]

exists and equals the limit in (4.1). We now show that the limit in (4.2) equals 0 if and only if

\[
\lim_{\lambda \to \infty} \int_{\{A(z) \leq \lambda, B(z) > A(z)f(A(z))\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\dot{\nu}(z)|^2 \, dz = 0. \tag{4.3}
\]

To this end, we need only show that (4.3) implies that

\[
\lim_{\lambda \to \infty} \int_{A(z) \leq \lambda} \frac{\lambda}{\lambda^2 + B^2(z)} |\dot{\nu}(z)|^2 \, dz = 0. \tag{4.4}
\]

Suppose that (4.3) holds. Then, the limit

\[
\lim_{\lambda \to \infty} \int_{\{A(z) \leq \lambda, B(z) \geq A(z)f(A(z))\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\dot{\nu}(z)|^2 \, dz
\]

exists since the limit in (4.2) always exists. Note that

\[
\int_1^\infty \lambda^{-1} f(\lambda)^{-1} \, d\lambda \int_{\{A(z) \leq \lambda, B(z) \geq A(z)f(A(z))\}} \lambda(\lambda^2 + B^2(z))^{-1} |\dot{\nu}(z)|^2 \, dz
\]

\[
= \int_{\{B(z) \leq A(z)f(A(z))\}} |\dot{\nu}(z)|^2 \, dz \int_1^\infty [f(\lambda)(\lambda^2 + B^2(z))]^{-1} \, d\lambda
\]

\[
\leq \frac{\pi}{2} \int_{\{B(z) \leq A(z)f(A(z))\}} [B(z)f(A(z))]^{-1} |\dot{\nu}(z)|^2 \, dz
\]

\[
\leq \frac{\pi}{2} \int_{\mathbb{R}^d} B^{-2}(z) A(z)|\dot{\nu}(z)|^2 \, dz
\]

\[
< \infty.
\]

Since \( \int_1^\infty \lambda^{-1} f(\lambda)^{-1} \, d\lambda = \infty \),

\[
\lim_{\lambda \to \infty} \int_{\{A(z) \leq \lambda, B(z) \leq A(z)f(A(z))\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\dot{\nu}(z)|^2 \, dz = 0.
\]

Therefore, (4.4) holds by (4.3).

For each \( k \in \mathbb{N} \), we have

\[
1 \{k \leq \frac{\text{Im} \psi(z)}{A(z)} < k+1, \lambda \geq (k+1)|\text{Im} \psi(z)|\} \frac{\lambda}{\lambda^2 + (\text{Im} \psi(z))^2} |\dot{\nu}(z)|^2
\]

\[
\leq 1 \{k \leq \frac{\text{Im} \psi(z)}{A(z)} < k+1, \lambda \geq (k+1)|\text{Im} \psi(z)|\} \frac{1}{\lambda} |\dot{\nu}(z)|^2
\]

\[
\leq \frac{1}{k+1} \{k \leq \frac{\text{Im} \psi(z)}{A(z)} < k+1\} |\dot{\nu}(z)|^2.
\]
We assume without loss of generality that $f(1) = \sqrt{2}$. Note that $B(z) > A(z)f(A(z))$ implies that $B(z) \leq \sqrt{2} \Im \psi(z)$. Then, we obtain by $\int_{\mathbb{R}^n} B^{-2}(z)A(z)|\hat{\nu}(z)|^2dz < \infty$ that

$$\sum_{k=1}^{\infty} \frac{1}{2(k+1)} \int_{\{B(z)>A(z)f(A(z)), k \leq \frac{\Im \psi(z)}{A(z)} < k+1\}} \frac{|\hat{\nu}(z)|^2}{|\Im \psi(z)|}dz < \infty. \quad (4.6)$$

By (4.5), (4.6) and the dominated convergence theorem, we get

$$\lim_{\lambda \to \infty} \sum_{k=1}^{\infty} \int_{\{B(z)>A(z)f(A(z)), k \leq \frac{\Im \psi(z)}{A(z)} < k+1, A(z) < \lambda < (k+1)|\Im \psi(z)|\}} \frac{\lambda}{\lambda^2 + (\Im \psi(z))^2} |\hat{\nu}(z)|^2dz = 0. \quad (4.7)$$

Therefore, the proof is complete by noting (4.3). \qed

Note that if $\nu$ is a finite measure such that $U^1 \nu$ is bounded then $\nu$ has finite 1-energy (cf. Rao [22, page 622]). By Lemma 4.2 and Proposition 1.1, we obtain the following necessary and sufficient condition for (H).

**Theorem 4.3** Let $f$ be an increasing function on $[1, \infty)$ such that $\int_{N}^{\infty} (\lambda f(\lambda))^{-1}d\lambda = \infty$ for some $N \geq 1$. Then (H) holds if and only if

$$\lim_{\lambda \to \infty} \sum_{k=1}^{\infty} \int_{\{B(z)>A(z)f(A(z)), k \leq \frac{\Im \psi(z)}{A(z)} < k+1, A(z) < \lambda < (k+1)|\Im \psi(z)|\}} \frac{\lambda}{\lambda^2 + (\Im \psi(z))^2} |\hat{\nu}(z)|^2dz = 0 \quad (4.7)$$

for any finite measure $\nu$ with compact support such that $U^1 \nu$ is bounded.

**Remark 4.4** Theorem 4.3 indicates that the validity of (H) is closely related to the behavior of $\psi(z)$ where $\Im(\psi(z))$ is not well controlled by $\Re(\psi(z))$, which is possible and can be seen from the uniform motion on $\mathbb{R}$ and the example given in Section 5.

By virtue of Theorem 4.3, we obtain the following result extending the Kanda-Forst-Rao theorem on (H).

**Theorem 4.5** (H) holds if the following extended Kanda-Forst-Rao condition ((EKFR) for short) holds:

(EKFR) There are two measurable functions $\psi_1$ and $\psi_2$ on $\mathbb{R}^n$ such that $\Im(\psi) = \psi_1 + \psi_2$, and

$$|\psi_1| \leq Af(A), \quad \int_{\mathbb{R}^n} \frac{|\psi_2(z)|}{(1 + \Re \psi(z))^2 + (\Im \psi(z))^2}dz < \infty, \quad (4.8)$$

where $f$ is an increasing function on $[1, \infty)$ such that $\int_{N}^{\infty} (\lambda f(\lambda))^{-1}d\lambda = \infty$ for some $N \geq 1$.

**Remark 4.6** If $\psi_2 = 0$, then the (EKFR) condition is just Rao’s condition. In particular, if $f = 1$, then it is just the Kanda-Forst condition. An integrability condition similar to (4.8) has been used in Glover [2], Theorem 3.1.
Proof of Theorem 4.5. By Theorem 4.3, we need only show that the limit in (4.7) equals 0. We assume without loss of generality that \( f(1) = 1/3 \). Note that \( B(z) > 3\sqrt{2}A(z)f(A(z)) \) implies that \( |\text{Im}\psi(z)| > A(z) \) and \( |\text{Im}\psi(z)| > B(z)/\sqrt{2} \), and \( |\psi_2(z)| > 2A(z)f(A(z)) \) implies that \( |\psi_2(z)| > |\text{Im}\psi(z)|/2 \). Then, by (4.8), the fact that \( A(z) \leq c(1 + |z|^2) \) for some positive constant \( c \) and the dominated convergence theorem, we obtain that

\[
\sum_{k=1}^{\infty} \int \left\{ B(z) > 3\sqrt{2}A(z)f(A(z)), k \leq \frac{|\text{Im}\psi(z)|}{A(z)} < k+1, A(z) \leq \lambda < (k+1)|\text{Im}\psi(z)| \right\} \frac{\lambda}{\lambda^2 + (\text{Im}\psi(z))^2} |\hat{\nu}(z)|^2 dz
\]
\[
\leq \sum_{k=1}^{\infty} \int \left\{ |\text{Im}\psi(z)| > 3A(z)f(A(z)), k \leq \frac{|\text{Im}\psi(z)|}{A(z)} < k+1, A(z) \leq \lambda < (k+1)|\text{Im}\psi(z)| \right\} \frac{1}{2|\text{Im}\psi(z)|} |\hat{\nu}(z)|^2 dz
\]
\[
\leq \sum_{k=1}^{\infty} \int \left\{ |\psi_2(z)| > 2A(z)f(A(z)), k \leq \frac{|\text{Im}\psi(z)|}{A(z)} < k+1, A(z) \leq \lambda < (k+1)|\text{Im}\psi(z)| \right\} \frac{|\psi_2(z)|}{|\text{Im}\psi(z)|^2} |\hat{\nu}(z)|^2 dz
\]
\[
\leq \sum_{k=1}^{\infty} \int \left\{ k \leq \frac{|\text{Im}\psi(z)|}{A(z)} < k+1, A(z) \leq \lambda < (k+1)^2A(z) \right\} \frac{2|\psi_2(z)|}{B^2(z)} |\hat{\nu}(z)|^2 dz
\]
\[
\leq \sum_{k=1}^{\infty} \int \left\{ k \leq \frac{|\text{Im}\psi(z)|}{A(z)} < k+1, \lambda < c(k+1)^2(1+|z|^2) \right\} \frac{2|\psi_2(z)|}{B^2(z)} |\hat{\nu}(z)|^2 dz
\]
\[
\rightarrow 0 \text{ as } \lambda \rightarrow \infty.
\]

The proof is complete. \( \square \)

In the following, we give an application of Theorem 4.5.

**Theorem 4.7** Let \( \gamma > 0 \) and \( X \) be a Lévy process on \( \mathbb{R} \) satisfying

\[
\liminf_{|z| \to \infty} \frac{\text{Re}\psi(z)}{|z| \log^\gamma(|z|)} > 0.
\]

(4.9)

Then \( X \) satisfies (H).

**Proof.** By (4.9), we get

\[
\lim_{|z| \to \infty} \frac{\text{Re}\psi(z)}{\log(1 + |z|)} = \infty.
\]

Hence \( X \) has bounded continuous transition densities by Hartman and Wintner [13]. Let \( f(\lambda) = \log(\lambda) \) for \( \lambda \in [1, \infty) \) and set \( \psi_1(z) := 1_{\{|\text{Im}\psi(z)| \leq A(z)f(A(z))\}}|\text{Im}\psi(z)|, \psi_2(z) := 1_{\{|\text{Im}\psi(z)| > A(z)f(A(z))\}}|\text{Im}\psi(z)| \) for \( z \in \mathbb{R} \). Condition (4.9) implies that there exists a constant \( c > 0 \) such that

\[
|\psi_2(z)| \geq c1_{\{|\psi(z)| > A(z)f(A(z))\}}|z| \log^{1+\gamma}(|z|)
\]

when \( |z| \) is sufficiently large. Therefore, (4.8) holds and the proof is complete by Theorem 4.5. \( \square \)
**Example 4.8** By Theorem 4.7, Theorem 2.1 and Corollary 2.2, we obtain a new class of 1-dimensional Lévy processes satisfying (H). Let $X$ be a Lévy process on $\mathbb{R}$ with Lévy-Khintchine exponent $(a, Q, \mu)$. Suppose that there exist constants $\gamma > 0$, $0 < \delta < 1$, $c > 0$, and a finite measure $\mu'$ on $\{x \in \mathbb{R}^n : 0 < |x| < \delta\}$ such that
\[
d\mu \geq c \left( \frac{- \log(|x|)^\gamma}{x^2} \right) - d\mu' \quad \text{on} \quad \{x \in \mathbb{R} : 0 < |x| < \delta\}.
\]
Similar to (3.5), we can show that (4.9) holds. Then, $X$ satisfies (H). Note that in this example it does not matter if $a$ or $Q$ equals 0.

Let $Y$ be another 1-dimensional Lévy process which is independent of $X$. Theorem 4.7 implies that the perturbed process $Y + X$ also satisfies (H).

**Remark 4.9** Blumenthal and Getoor introduced in [1] the following index $\beta''$ defined by
\[
\beta'' = \sup \left\{ \tau \geq 0 : \frac{\Re \psi(z)}{|z|^\tau} \to \infty \text{ as } |z| \to \infty \right\}.
\]
Let $X$ be a Lévy process on $\mathbb{R}$. Then, Theorem 4.7 implies that (H) holds when $\beta'' > 1$. This result is also a direct consequence of the following proposition.

**Proposition 4.10** Let $X$ be a Lévy process on $\mathbb{R}$. Suppose that
\[
\liminf_{|z| \to \infty} \frac{|\psi(z)|}{|z| \log^{1+\gamma} |z|} > 0
\]
for some constant $\gamma > 0$. Then (H) holds.

**Proof.** Let $f \equiv 1$ and set $\psi_1(z) := 1_{\{|\Im \psi(z)| \leq A(z)f(A(z))\}} \Im \psi(z)$, $\psi_2(z) := 1_{\{|\Im \psi(z)| > A(z)f(A(z))\}} \Im \psi(z)$ for $z \in \mathbb{R}$. Condition (4.11) implies that
\[
\limsup_{|z| \to \infty} \left\{ \frac{|\psi_2(z)|}{(1 + \Re \psi(z))^2 + (\Im \psi(z))^2} \cdot |z| \log^{1+\gamma} |z| \right\} < \infty.
\]
Therefore, (4.8) holds and the proof is complete by Theorem 4.5.

We remark that Proposition 4.10 can also be proved by Theorem 4.1. In fact, the limit in (4.1) equals the limit in (4.2) and hence equals 0 by (4.11) and the dominated convergence theorem.

5 **A type of subordinators that does not satisfy Rao’s condition**

As pointed out in Rao [22], from the proof of Theorem 3.7 it seems that the condition $B \leq Af(A)$ is not far from being necessary. In this section, however, we will construct a type of subordinators that does not satisfy Rao’s condition.
5.1 Construction of the example

We fix an $\alpha$ such that $\frac{1}{2} < \alpha < 1$. In the sequel, we define a function $\rho$ on $\mathbb{R}$ which will be used as the density function of a Lévy measure $\mu$.

First, we set $n_1 = 2$. Define a function $\rho_1$ on $\mathbb{R}$ as follows.

$$\rho_1(x) = \frac{1}{x^1+\alpha}, \quad \text{if } \frac{1}{2n_1^2} < x < \frac{1}{n_1^2}; \quad 0, \quad \text{otherwise.}$$

We define $\mu_1(dx) = \rho_1(x)dx$ and denote by $\psi_1$ the Lévy-Khintchine exponent of $\mu_1$. Then, for $z \in \left[\frac{n_1}{2}, 2n_1\right]$, we have

$$\text{Re } \psi_1(z) = \int_0^1 (1 - \cos(zx)) \mu_1(dx) \leq \int_{1/2n_1^2}^{1/n_1^2} \frac{z^2x^2}{x^1+\alpha} dx \leq \frac{2n_1^{2\alpha-2}}{2 - \alpha} \leq 2 \quad (5.1)$$

and

$$\text{Im } \psi_1(z) = \int_0^1 \sin(zx) \mu_1(dx) = \int_{1/2n_1^2}^{1/n_1^2} \sin(zx) \mu_1(dx) \geq \int_{1/2n_1^2}^{1/n_1^2} \frac{zx}{2x^{1+\alpha}} dx \geq \frac{1}{8}n_1^{2\alpha-1}. \quad (5.2)$$

We increase $n_1$ so that $\frac{1}{8}n_1^{2\alpha-1} > \frac{6}{1-\alpha}$.

For any $z \in \mathbb{R}$, we have

$$\text{Re } \psi_1(z) = \int_0^1 (1 - \cos(zx)) \mu_1(dx) \leq \int_{1/2n_1^2}^{1/n_1^2} \frac{1}{x^{1+\alpha}} dx \leq \frac{2\alpha n_1^{2\alpha}}{\alpha} \leq 4n_1^{2\alpha} \quad (5.3)$$

and

$$|\text{Im } \psi_1(z)| \leq \int_0^1 |\sin(zx)| \mu_1(dx) \leq \int_{1/2n_1^2}^{1/n_1^2} \frac{1}{x^{1+\alpha}} dx \leq \frac{2\alpha n_1^{2\alpha}}{\alpha} \leq 4n_1^{2\alpha}. \quad (5.4)$$

We choose an $n_2 \in \mathbb{N}$ such that $n_2^2 > 2n_1^2$. We define a function $\rho_2$ on $\mathbb{R}$ as follows.

$$\rho_2(x) = \frac{1}{x^{1+\alpha}}, \quad \text{if } \frac{1}{2n_2^2} < x < \frac{1}{n_2^2}; \quad 0, \quad \text{otherwise.}$$
Note that there is no overlap between $\rho_1$ and $\rho_2$. We define $\mu_2(dx) = \rho_2(x)dx$ and denote by $\psi_2$ the Lévy-Khintchine exponent of $\mu_2$. Then, similar to the above, we can show that for $z \in \left[\frac{n_1}{2}, 2n_2\right]$

\[
\text{Re}\psi_2(z) \leq 2 \quad \text{and} \quad \text{Im}\psi_2(z) \geq \frac{1}{8}n_2^{2\alpha-1} \left(\frac{6}{1 - \alpha}\right). \tag{5.5}
\]

Note that for $z \in \left[\frac{n_1}{2}, 2n_1\right]$ we have

\[
\text{Re}\psi_2(z) = \int_0^1 (1 - \cos(zx))\mu_2(dx)
\leq \frac{1}{2} \int_{1/2n_2}^{1/n_2} z^2 x^2 \frac{1}{x^{1+\alpha}}dx
\leq \frac{2n_1^2n_2^{2\alpha-4}}{2 - \alpha} \tag{5.6}
\]
and

\[
|\text{Im}\psi_2(z)| \leq \int_0^1 |\sin(zx)|\mu_2(dx)
\leq \int_{1/2n_2}^{1/n_2} |\sin(zx)| \frac{1}{x^{1+\alpha}}dx
\leq \int_{1/2n_2}^{1/n_2} 2n_1 x \frac{1}{x^{1+\alpha}}dx
\leq \frac{2n_1n_2^{2\alpha-2}}{1 - \alpha}. \tag{5.7}
\]

We increase $n_2$ (with $n_1$ fixed) so that $n_2 \geq n_1^{5/(2 - 2\alpha)}$. By (5.6) and (5.7), we get

\[
\text{Re}\psi_2(z) \leq \frac{2}{(1 - \alpha)n_1^2}, \quad |\text{Im}\psi_2(z)| \leq \frac{2}{(1 - \alpha)n_1^2}, \quad z \in \left[\frac{n_1}{2}, 2n_1\right]. \tag{5.8}
\]

Then, by (5.1), (5.2) and (5.8), we obtain that for $z \in \left[\frac{n_1}{2}, 2n_1\right]$,

\[
\text{Re}\psi_1(z) + \text{Re}\psi_2(z) \leq 2 + \frac{2}{(1 - \alpha)n_1^2} \tag{5.9}
\]
and

\[
\text{Im}\psi_1(z) + \text{Im}\psi_2(z) \geq \frac{1}{8}n_1^{2\alpha-1} - \frac{2}{(1 - \alpha)n_1^2}. \tag{5.10}
\]

We further increase $n_2$ so that $n_2 \geq (96)^{1/(2\alpha - 1)}n_1^{(4 + 2\alpha)/(2\alpha - 1)}$ which ensures that for any $z \in \mathbb{R}$ (cf. (5.3), (5.4) and (5.5))

\[
\text{Re}\psi_1(z) \leq \frac{1}{3n_1^2} \text{Im}\psi_2\left(\frac{n_2}{2}\right), \quad |\text{Im}\psi_1(z)| \leq \frac{1}{3n_1^2} \text{Im}\psi_2\left(\frac{n_2}{2}\right). \tag{5.11}
\]
By (5.5) and (5.11), we obtain that for $z \in \left[ \frac{n_2}{2}, 2n_2 \right]$, 

$$\text{Re}\psi_1(z) + \text{Re}\psi_2(z) \leq \frac{1}{3n_1^4} \text{Im}\psi_2 \left( \frac{n_2}{2} \right) + 2$$  \hspace{1cm} (5.12) 

and 

$$\text{Im}\psi_1(z) + \text{Im}\psi_2(z) \geq \left( 1 - \frac{1}{3n_1^4} \right) \text{Im}\psi_2 \left( \frac{n_2}{2} \right).$$  \hspace{1cm} (5.13) 

Define 

$$\vartheta := \max \left\{ \frac{5}{2 - 2\alpha}, \frac{4 + 2\alpha}{2\alpha - 1} \right\}. \hspace{1cm} (5.14)$$

We can set $n_2$ to be $cn_1^\vartheta$, for some positive constant $c$ depending only on $\alpha$, such that (5.9), (5.10), (5.12) and (5.13) hold.

For any $z \in \mathbb{R}$, we have 

$$\text{Re}\psi_2(z) = \int_0^1 (1 - \cos(zx))\mu_2(dx) \leq \int_{1/2n_2^2}^1 \frac{1}{x^{1+\alpha}} dx \leq \frac{2^{\alpha}n_2^{2\alpha}}{\alpha} \leq 4n_2^{2\alpha}$$  \hspace{1cm} (5.15) 

and 

$$|\text{Im}\psi_2(z)| \leq \int_0^1 |\sin(zx)|\mu_2(dx) \leq \int_{1/2n_2^2}^1 \frac{1}{x^{1+\alpha}} dx \leq \frac{2^{\alpha}n_2^{2\alpha}}{\alpha} \leq 4n_2^{2\alpha}. \hspace{1cm} (5.16)$$

We choose an $n_3 \in \mathbb{N}$ such that $n_3^3 > 2n_2^2$. We define a function $\rho_3$ on $\mathbb{R}$ as follows. 

$$\rho_3(x) = \begin{cases} \frac{1}{x^{1+\alpha}}, & \text{if } \frac{1}{2n_3^2} < x < \frac{1}{n_3^2}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that there is no overlap among $\rho_1$, $\rho_2$ and $\rho_3$. We define $\mu_3(dx) = \rho_3(x)dx$ and denote by $\psi_3$ the Lévy-Khintchine exponent of $\mu_3$. Then, similar to the above, we can show that for $z \in \left[ \frac{n_2}{2}, 2n_3 \right]$, 

$$\text{Re}\psi_3(z) \leq 2 \quad \text{and} \quad \text{Im}\psi_3(z) \geq \frac{1}{8}n_3^{2\alpha-1}$$ \hspace{1cm} (5.17) 

and for any $z \in \mathbb{R}$, 

$$\text{Re}\psi_3(z) \leq 4n_3^{2\alpha}, \quad |\text{Im}\psi_3(z)| \leq 4n_3^{2\alpha}.$$ 

Similar to (5.6) and (5.7), we obtain that for $z \in \left[ \frac{n_1}{2}, 2n_1 \right]$, 

$$\text{Re}\psi_3(z) \leq \frac{2n_1^{2\alpha-4}}{2 - \alpha}, \quad |\text{Im}\psi_3(z)| \leq \frac{2n_1n_3^{2\alpha-2}}{1 - \alpha}$$ \hspace{1cm} (5.18) 

and for $z \in \left[ \frac{n_1}{2}, 2n_2 \right]$, 

$$\text{Re}\psi_3(z) \leq \frac{2n_2^{2\alpha-4}}{2 - \alpha}, \quad |\text{Im}\psi_3(z)| \leq \frac{2n_2n_3^{2\alpha-2}}{1 - \alpha}.$$ \hspace{1cm} (5.19) 

We increase $n_3$ (with $n_1, n_2$ fixed) so that $n_3 \geq n_2^{5/(2 - 2\alpha)}$. By (5.18) and (5.19), we get 

$$\text{Re}\psi_3(z) \leq \frac{2}{(1 - \alpha)n_2^2}, \quad |\text{Im}\psi_3(z)| \leq \frac{2}{(1 - \alpha)n_2^4}, \quad z \in \left[ \frac{n_1}{2}, 2n_1 \right] \cup \left[ \frac{n_2}{2}, 2n_2 \right].$$ \hspace{1cm} (5.20)
Hence, by (5.9), (5.10) and (5.20), we obtain that for \( z \in \left[ \frac{n_2}{2}, 2n_1 \right] \),
\[
\text{Re}\psi_1(z) + \text{Re}\psi_2(z) + \text{Re}\psi_3(z) \leq 2 + \frac{2}{(1 - \alpha)n_1^4} + \frac{2}{(1 - \alpha)n_2^4} \tag{5.21}
\]
and
\[
\text{Im}\psi_1(z) + \text{Im}\psi_2(z) + \text{Im}\psi_3(z) \geq \frac{1}{8} n_2^{2\alpha - 1} - \frac{2}{(1 - \alpha)n_1^4} - \frac{2}{(1 - \alpha)n_2^4}. \tag{5.22}
\]
By (5.12), (5.13), (5.20) and (5.5), we obtain that for \( z \in \left[ \frac{n_2}{2}, 2n_1 \right] \),
\[
\text{Re}\psi_1(z) + \text{Re}\psi_2(z) + \text{Re}\psi_3(z) \leq \frac{2}{3n_1^4} \text{Im}\psi_2 \left( \frac{n_2}{2} \right) + 2 + \frac{2}{(1 - \alpha)n_2^4} \tag{5.23}
\]
and
\[
\text{Im}\psi_1(z) + \text{Im}\psi_2(z) + \text{Im}\psi_3(z) \geq \left( 1 - \frac{1}{3n_1^4} - \frac{1}{3n_2^4} \right) \text{Im}\psi_2 \left( \frac{n_2}{2} \right). \tag{5.24}
\]

We further increase \( n_3 \) so that \( n_3 \geq (192)^{1/(2\alpha - 1)} n_2^{(4+2\alpha)/(2\alpha - 1)} \) which ensures that for any \( z \in \mathbb{R} \) (cf. (5.3), (5.4), (5.15), (5.16) and (5.17)),
\[
\text{Re}\psi_1(z), \text{Re}\psi_2(z), |\text{Im}\psi_1(z)|, |\text{Im}\psi_2(z)| \leq \frac{1}{6n_2^4} \text{Im}\psi_2 \left( \frac{n_3}{2} \right). \tag{5.25}
\]

Therefore, we obtain by (5.17) and (5.25) that for \( z \in \left[ \frac{n_3}{2}, 2n_3 \right] \),
\[
\text{Re}\psi_1(z) + \text{Re}\psi_2(z) + \text{Re}\psi_3(z) \leq \frac{1}{3n_2^4} \text{Im}\psi_3 \left( \frac{n_3}{2} \right) + 2 \tag{5.26}
\]
and
\[
\text{Im}\psi_1(z) + \text{Im}\psi_2(z) + \text{Im}\psi_3(z) \geq \left( 1 - \frac{1}{3n_1^4} - \frac{1}{3n_2^4} \right) \text{Im}\psi_3 \left( \frac{n_3}{2} \right). \tag{5.27}
\]

We set \( n_3 \) to be \( 2^{1/(2\alpha - 1)} cn_2^\vartheta \), where \( \vartheta \) and \( c \) are as the same as above.

Continue in this way, we define \( \rho_4, \rho_5, \ldots \) All of these functions have no overlap and we have estimates similar to (5.21)-(5.24), (5.26) and (5.27). Now we define

\[
\rho = \sum_{i=1}^\infty \rho_i.
\]

One finds that \( \mu(dx) = \rho(x)dx \) is the Lévy measure of a subordinator \( X \) with the Lévy-Khintchine exponent

\[
\psi = \sum_{i=1}^\infty \psi_i.
\]

Moreover, we have that for \( k \geq 2 \),
\[
n_k = (k - 1)^{1/(2\alpha - 1)} cn_k^\vartheta, \tag{5.28}
\]
and for $z \in \left[\frac{n_k}{2}, 2n_k\right]$,
\[
\text{Im}\psi_k(z) \geq \frac{1}{8}n_k^{2\alpha - 1},
\]
(5.29)
\[
\text{Re}\psi(z) \leq \frac{1}{3n_k^4} \text{Im}\psi_k \left(\frac{n_k}{2}\right) + 2 + \frac{2}{1 - \alpha} \sum_{k=1}^{\infty} \frac{1}{n_k^3},
\]
(5.30)
and
\[
\text{Im}\psi(z) \geq \left(1 - \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{n_k^4}\right) \text{Im}\psi_k \left(\frac{n_k}{2}\right).
\]
(5.31)

5.2 Discussions

In this subsection, we make discussion about the subordinators constructed in Subsection 5.1. Below we use $c_1, c_2, \ldots$ to denote positive constants depending only on $\alpha$.

1. By the estimates (5.30) and (5.31), we can show that Rao’s condition does not hold for the subordinators. In fact, by (5.28), there exists a constant $c_1 > 1$ such that
\[
n_k > c_1, \quad k \in \mathbb{N}.
\]
By (5.29), (5.30) and (5.31), we find that there exist constants $c_2, c_3, c_4 > 0$ such that for any $k \geq 2$,
\[
\frac{\text{Im}\psi(z)}{1 + \text{Re}\psi(z)} \geq c_2 n_k^4 \leq c_3 n_k^{3/\alpha} \geq c_3 \left(\frac{z}{2}\right)^{3/\alpha}, \quad \forall z \in \left[\frac{n_k}{2}, 2n_k\right].
\]
(5.33)
\[
\text{Re}\psi(z) \leq c_4 n_k^{\alpha - 3}, \quad \forall z \in \left[\frac{n_k}{2}, 2n_k\right].
\]
(5.34)
The estimates (5.33) and (5.34) imply that there does not exist an increasing function $f$ on $[1, \infty)$ satisfying $\int_N^\infty (\Lambda f(\lambda))^{-1} d\lambda = \infty$ for some $N \geq 1$ and $|1 + \psi| \leq (1 + \text{Re}(\psi))f(1 + \text{Re}(\psi))$. That is, Rao’s condition does not hold for the subordinators constructed in Subsection 5.1.

By Theorem 2.1, we can modify the Lévy measure $\mu$ defined in Subsection 5.1 by a finite measure and hence obtain a subordinator which does not satisfy Rao’s condition and whose Lévy measure $\mu$ has a smooth density $\rho$ with respect to the Lebesgue measure on $(0, \infty)$.

2. Besides the index $\beta''$ (see (4.10)), Blumenthal and Getoor introduced also in [11] the indexes $\beta$ and $\sigma$ defined by
\[
\beta = \inf \left\{ \tau > 0 : \int_{\{|x| < 1\}} |x|^\tau \mu(dx) < \infty \right\}
\]
and
\[
\sigma = \sup \left\{ \tau \leq 1 : \int_1^\infty \frac{x^{\tau-1}}{\int_0^\infty (1 - e^{-xy}) \mu(dy)} dx < \infty \right\}.
\]
From the construction of the subordinators given in Subsection 5.1, we obtain by [11] Theorem 6.1 that
\[
\sigma = \beta = \alpha.
\]
20
By (5.28) and (5.30) (cf. (3.3)), we get
\[ \beta'' \leq \alpha - \frac{4}{q}. \]

3. Take \( \alpha = 3/4 \). For the subordinators constructed in Subsection 5.1, we claim that there exists a finite signed measure \( d\nu = g_1 dx - g_2 dx \) with \( g_1, g_2 \in L^1_+ (\mathbb{R}; dx) \) such that
\[
\int_{\mathbb{R}} B^{-2}(z) A(z) |\hat{\nu}(z)|^2 \, dz < \infty
\]
but
\[
\lim_{\lambda \to \infty} \int_{\mathbb{R}} |\hat{\nu}(z)|^2 (\lambda + \text{Re} \psi(z)) |\lambda + \psi(z)|^{-2} \, dz = \infty.
\]

Let \( \omega \) be a sufficiently large number. We define
\[
\zeta_\omega(x) := \left\{ 1 - \frac{1 - 1/\omega^{0.1}}{\omega} \cdot |x| \right\}, \quad \text{if } |x| \leq \omega; \quad \frac{1}{|x|^{0.1}}, \quad \text{otherwise},
\]
and
\[
\eta_\omega(x) := \left\{ 1 - \frac{1 - 1/\omega^{0.1}}{\omega} \cdot |x| \right\} \lor 0, \quad x \in \mathbb{R}.
\]
By Polya’s theorem (cf. Lukacs [19, Theorem 4.3.1]), both \( \zeta_\omega \) and \( \eta_\omega \) are characteristic functions of absolutely continuous symmetric distributions. Define \( \varsigma_\omega := \eta_\omega - \zeta_\omega \). Then, \( \varsigma_\omega(x) = 0 \) if \( |x| \leq \omega; \)
\( \varsigma_\omega(x) = 1/|x|^{0.1} \) if \( |x| \geq (1.1)\omega \); and \( 0 \leq \varsigma_\omega(x) \leq 1/|x|^{0.1} \) otherwise.

Let \( k_0 \in \mathbb{N} \) be a sufficiently large number. For \( k \geq k_0 \), we define \( \xi_k := \varsigma_{k/2} - \varsigma_{2k/1} \). We find that \( \xi_k \) is a characteristic function of the difference of two functions \( g_1^k, g_2^k \in L^1_+ (\mathbb{R}; dx) \) with \( \|g_1^k\|_{L^1}, \|g_2^k\|_{L^1} \leq 2 \). Define \( g_1 := \sum_{k=1}^{\infty} g_1^k/2^k \), \( g_2 := \sum_{k=1}^{\infty} g_2^k/2^k \) and \( d\nu := g_1 dx - g_2 dx \). By applying (5.14), (5.28), (5.32) and the first inequality of (5.33) to \( B(z)/A(z) \) and applying (5.29), (5.31) to \( B(z) \), we find that there exists a constant \( c_5 > 0 \) such that
\[
\int_{\mathbb{R}} B^{-2}(z) A(z) |\hat{\nu}(z)|^2 \, dz = \int_{\mathbb{R}} \frac{1}{B(z)/A(z) \cdot B(z)} |\hat{\nu}(z)|^2 \, dz
\]
\[
\leq c_5 \sum_{k=1}^{\infty} \frac{4^{-1}}{n_k} \cdot \frac{1}{n_k (n_k^{2.9} \cdot 2^{2k})^{\frac{1}{n_k} \cdot 2^{2k} \cdot 1/2} \int_{n_k/2}^{2n_k} 1/2 \, dz}
\]
\[
= c_5 \sum_{k=1}^{\infty} \frac{1}{n_k^{9/11}} \cdot \frac{1}{2^{2k}} \int_{n_k/2}^{2n_k} 1/2 \, dz
\]
\[
< \infty.
\]

However, there exists a constant \( c_6 > 0 \) such that (cf. (3.3) and (5.32))
\[
\int_{\mathbb{R}} |\hat{\nu}(z)|^2 \left( \frac{n_k^4}{(n_k^2)^2} + (\text{Im} \psi(z))^2 \right) \, dz \geq c_6 \frac{1}{n_k^{4/2} \cdot 2^{2k}} \int_{(0.55)n_k}^{2n_k} 1/2 \, dz
\]
\[
\to \infty \quad \text{as } k \to \infty,
\]
which implies (5.36).

By (5.35) and (5.36) we can also conclude that Rao’s condition does not hold for the subordinators constructed in Subsection 5.1. In fact, from the proof of Theorem 3.7 (see Rao [22]), we can see that under Rao’s condition,

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}} |\hat{\nu}(z)|^2 (\lambda + \text{Re} \psi(z)) |\lambda + \psi(z)|^{-2} dz = 0$$

holds for any finite signed measure of finite 1-energy.

It is interesting to compare (5.35) and (5.36) with the following result, which is a consequence of Theorem 4.1.

**Theorem 5.1** Let $X$ be a Lévy process on $\mathbb{R}^n$ such that all 1-excessive functions are lower semi-continuous. Then (H) holds if and only if

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}^n} |\hat{\nu}(z)|^2 (\lambda + \text{Re} \psi(z)) |\lambda + \psi(z)|^{-2} dz = 0$$

(5.37)

for any finite measure $\nu$ of finite 1-energy.

**Proof.** By Theorem 4.1, Rao [22, Remark, page 622] and Blumenthal and Getoor [2, VI. (4.8)], we need only prove the necessity. Suppose that (H) holds for $X$. Let $\nu$ be a finite measure of finite 1-energy and $\kappa$ be the standard Gaussian measure on $\mathbb{R}^n$. Then, $\nu + \kappa$ has finite 1-energy, which implies that

$$\int_{\mathbb{R}^n} U^1(\nu + \kappa)d(\nu + \kappa) < \infty.$$ 

(5.38)

By (5.38), $\kappa(\{x : U^1(\nu + \kappa)(x) = \infty\}) = 0$. Hence $U^1(\nu + \kappa)$ is locally integrable (with respect to the Lebesgue measure $dx$) by [2, VI. (2.3)]. By (H) and [2, VI. (4.9)], we find that $U^1(\nu + \kappa)$ is regular. Therefore, (5.37) holds by Theorem 4.1 and the proof is complete.

So far we have not been able to prove or disprove that (H) holds for the subordinators constructed in Subsection 5.1. This example suggests that maybe completely new ideas and methods are needed for resolving Getoor’s conjecture.

**Acknowledgments**

We thank Professor Fengyu Wang for helpful comments that improved a previous version of the paper. We are grateful to the support of NNSFC, Jiangsu Province Basic Research Program (Natural Science Foundation) (Grant No. BK2012720), and NSERC.

**References**

[1] Blumenthal R.M., Getoor R.K.: Sample functions of stochastic processes with stationary independent increments. J. Math. Mech., 10, 493-516 (1961).
[2] Blumenthal R.M., Getoor R.K.: Markov Processes and Potential Theory. Academic Press, New York and London (1968).

[3] Blumenthal R.M., Getoor R.K.: Dual processes and potential theory. Proc. 12th Biennial Seminar of the Canadian Math. Congress, 137-156 (1970).

[4] Bretagnolle J.: Résults de Kesten sur les processus à accroissements indépendants. Séminare de Probabilités V, Lect. Notes in Math., Vol. 191, Springer-Verlag, Berlin, 21-36 (1971).

[5] Fitzsimmons P.J.: On the equivalence of three potential principles for right Markov processes. Probab. Th. Rel. Fields 84, 251-265 (1990).

[6] Fitzsimmons P.J.: On the quasi-regularity of semi-Dirichlet forms. Potential Anal. 15, 151-185 (2001).

[7] Fitzsimmons P.J., Kanda M.: On Choquet’s dichotomy of capacity for Markov processes. Ann. Probab. 20, 342-349 (1992).

[8] Forst G.: The definition of energy in non-symmetric translation invariant Dirichlet spaces. Math. Ann. 216, 165-172 (1975).

[9] Glover J.: Energy and the maximum principle for nonsymmetric Hunt processes. Probability Theory and Its Applications, XXVI, 4, 757-768 (1981).

[10] Glover J.: Topics in energy and potential theory. Seminar on Stochastic Processes, 1982, Birkhäuser, 195-202 (1983).

[11] Glover J., Rao M.: Hunt’s hypothesis (H) and Getoor’s conjecture. Ann. Probab. 14, 1085-1087 (1986).

[12] Han X.-F., Ma Z.-M. and Sun W.: $\hat{h}\hat{h}$-transforms of positivity preserving semigroups and associated Markov processes. Acta Math. Sinica, English Series 27, 369-376 (2011).

[13] Hartman P., Wintner A.: On the infinitesimal generators of integral convolutions. Amer. J. Math. 64, 273-298 (1942).

[14] Hawkes J.: Potential theory of Lévy processes. Proc. London Math. Soc. 3, 335-352 (1979).

[15] Hu Z.-C. and Sun W.: Hunt’s hypothesis (H) and Getoor’s conjecture for Lévy processes. Stoch. Proc. Appl. 122, 2319-2328 (2012).

[16] Kanda M.: Two theorems on capacity for Markov processes with stationary independent increments. Z. Wahrsch. verw. Gebiete 35, 159-165 (1976).

[17] Kanda M.: Characterisation of semipolar sets for processes with stationary independent increments. Z. Wahrsch. verw. Gebiete 42, 141-154 (1978).

[18] Kesten H.: Hitting probabilities of single points for processes with stationary independent increments. Memoirs of the American Mathematical Society, No. 93, American Mathematical Society, Providence, R.I. (1969).
[19] Lukacs E.: Characteristic Functions. 2ed., Griffin, London (1970).

[20] Port S.C., Stone C.J.: The asymmetric Cauchy process on the line. Ann. Math. Statist. 40, 137-143 (1969).

[21] Rao M.: On a result of M. Kanda. Z. Wahrsch. verw. Gebiete 41, 35-37 (1977).

[22] Rao M.: Hunt’s hypothesis for Lévy processes. Proc. Amer. Math. Soc. 104, 621-624 (1988).

[23] Silverstein M.L.: The sector condition implies that semipolar sets are quasi-polar. Z. Wahrsch. verw. Gebiete 41, 13-33 (1977).