Reactive Planar Manipulation with Convex Hybrid MPC

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Abstract—This paper presents a reactive controller for planar manipulation tasks that leverages machine learning to achieve real-time performance. The approach is based on a Model Predictive Control (MPC) formulation, where the goal is to find an optimal sequence of robot motions to achieve a desired object motion. Due to the multiple contact modes associated with frictional interactions, the resulting optimization program suffers from combinatorial complexity when tasked with determining the optimal sequence of modes.

To overcome this difficulty, we formulate the search for the optimal mode sequences offline, separately from the search for optimal control inputs online. Using tools from machine learning, this leads to a convex hybrid MPC program that can be solved in real-time. We validate our algorithm with the problem of pushing a planar object of arbitrary shape with an arbitrary number of contact points.

I. INTRODUCTION

While humans naturally make use of sensing and feedback when manipulating objects, robot manipulators traditionally execute actions relying on open-loop control strategies. Given the uncertainty associated with frictional contact interactions [1], [2] and the inherent inaccuracies of contact models [3], [4], the use of feedback can play an important role to address model uncertainty. The long term goal of this work is to endow robots with real-time decision making capabilities to enable reactive manipulation.

This paper focuses on planar manipulation tasks where the physical interactions between manipulator, object, and environment can be modeled from first principles, using rigid body dynamics and Coulomb’s frictional law. A major challenge concerning feedback controller design for systems involving contact interactions is the presence of hybridness and underactuation [5]. Hybridness refers to the fact that frictional interactions between manipulator and object exhibit different contact modes (e.g. contact/separation, sticking/sliding, etc), while underactuation is a result of the limited set of forces and torques that can be transmitted by the robot to the object through frictional interactions. Figure 1 shows an animation of a hybrid manipulation task that exploits multiple contact modalities.

This paper’s main contribution is twofold. First, we present a controller design formulation that can be used to manipulate an object on a flat surface. The approach presented in this paper generalizes to multiple contact interactions between manipulator and object and for tracking trajectories in the plane. Second, we introduce a method to determine an effective contact mode sequence that leads to a convex hybrid Model Predictive Control (MPC) formulation. Due to the hybridness associated with frictional interactions, hybrid MPC formulations suffer from a combinatorial expansion due to unknown future contact interactions modes. To overcome this difficulty, we formulate the search for optimal modes separately from the search for optimal control inputs, by leveraging machine learning methods to determine effective mode sequences from prior experience. Once the mode sequences are selected, the control problem reduces to solving a convex quadratic program, which can be achieved at very high frequency.

We validate the proposed controller design formulation through a numerical simulation study, where we test the performance of the mode selection algorithm on various trajectory tracking applications. The simulation results show that the proposed algorithm achieves closed-loop tracking of a nominal trajectory, while eliminating the combinatorial complexity typically characteristic of hybrid and underactuated systems.

II. RELATED WORK

The mechanics of planar pushing manipulation tasks were first described by [6]. [7] introduced the concept of the limit surface, a useful geometric representation which maps the applied frictional forces on an object to its instantaneous velocity. Under the assumption of quasi-static interactions, the limit surface has been successfully used in simulation [6], planning [8], state estimation [9], and feedback control [5], [10], [11] applications. Due to the high computational costs associated with building a true limit surface, [12] proposed

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an ellipsoidal approximation which yields invertible models from force to motion. Recently, [13] exploited the convex properties of the limit surface to develop an efficient data-driven algorithm for its construction from contact interactions.

There is an ongoing effort to find planning frameworks that can effectively handle the underactuation and hybridness associated with contact models. [14] used sampled-based planning algorithms to plan robot motions for in-hand manipulation tasks. [15] developed a nonlinear trajectory optimization framework that includes frictional forces as decision variables within a nonlinear program and makes use of complementary constraints to encode the different contact interaction modes of the system. Another approach that shows promise is based on Differential Dynamic Programming (DDP), which iteratively approximates locally-quadratic models of the dynamics and cost functions to find a locally optimal path. DDP is best suited for systems with discontinuous dynamics. Most approaches in this line rely on approximating discontinuous dynamics with continuous relaxations. [16] used penalty methods to smooth contact interactions [11], [18], [19]. The control strategies presented for tact rich tasks has been limited to a small number of a priori knowledge of the contact mode sequencing. In [5], a feedback controller design is presented for the pusher-slider system using a Model Predictive Control framework, where a set of contact mode schedules are chosen such that they span a number of dynamic behaviors likely to occur. This method has been shown to work experimentally but requires heuristic methods in order to design candidate contact mode schedules. This paper aims to eliminate the need for contact mode enumeration based on human intuition by developing an algorithm that systematically selects contact modes sequences.

III. NOMENCLATURE

The notation used in the paper is described below:

- \( H \): Convex set representing the limit surface.
- \( p = [x \ y \ \theta]^{T} \): Object pose resolved in the body frame.
- \( w = [f_{x} \ f_{y} \ \tau]^{T} \): Applied wrench on the object resolved in the body frame.
- \( t = [\nu_{x} \ \nu_{y} \ \omega]^{T} \): Object twist resolved in the body frame.
- \( J_{c} \): Jacobian matrix associated with the contact point \( c \) resolved in the body frame.
- \( N = [n_{1}^{T} \ldots n_{c}^{T}]^{T} \): Matrix of object normal vectors at contact points resolved in body frame.
- \( D = [d_{1}^{T} \ldots d_{c}^{T}]^{T} \): Matrix of object tangent vectors at contact points resolved in body frame.
- \( f_{x} = [f_{x,1} \ldots f_{x,c}]^{T} \): Vector of applied normal forces at contact points resolved in body frame.
- \( \phi = [\phi_{1} \ldots \phi_{c}]^{T} \): Vector of relative angles of pusher relative to body frame.
- \( x = [x \ y \ \theta \ \phi]^{T} \): System state vector.
- \( u_{c} = [f_{n,c} \ f_{t,c} \ \phi_{c}]^{T} \): Vector of control inputs at contact point \( c \).
- \( u_{f} = [f_{n}^{T} \ f_{t}^{T}]^{T} \): Vector of commanded reaction forces resolved in body frame.
- \( u_{\phi} = \phi \): Vector of commanded angular velocities resolved in body frame.
- \( u = [u_{f}^{T} \ u_{\phi}^{T}]^{T} \): Control input.

IV. PLANAR PUSHING MODEL

This section introduces the motion model for planar manipulation tasks that is general to an arbitrary number of contact points and arbitrary object shapes. We adapt the modeling of planar pushing interactions from [20] that we briefly summarize below. All contacts in this work assume Coulomb friction interactions [21], uniform pressure distribution, and quasi-static interactions, where the inertial forces of the object are negligible.

A. Motion Model

The limit surface is a geometric representation that describes the relationship between the applied force on an object and its instantaneous velocity. In this paper, we use the ellipsoidal approximation to the limit surface [12], [20] due its simplicity and invertibility properties. The ellipsoidal limit surface can be expressed in convex quadratic form as

\[
H(w) = \frac{1}{2} w^{T} Aw.
\]

By the principle of maximal dissipation [7], the object twist is perpendicular to the limit surface for a given wrench

\[
t = \nabla H(w) = Aw,
\]

where the applied frictional wrench is

\[
w = \sum_{c=1}^{C} J_{c}^{T} (n_{c} f_{n,c} + d_{c} f_{t,c}).
\]

Consider the planar manipulation task with multiple contact points shown in Fig. 2. The unconstrained motion equations of the system can be expressed as

\[
x = f(x,u) = \begin{bmatrix} R t \\ u_{\phi} \end{bmatrix}, \quad R = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

assuming that all points maintain contact with the sliding object.

![Fig. 2: Free body diagram of a sliding object with \( c = 2 \) contact points.](image)
B. Frictional Constraints

The motion equations in (3) do not enforce that the reaction forces between the manipulator and sliding object are feasible. For example, following this equation, negative normal forces could be applied to the object although contact interactions cannot transmit such forces. To ensure that the motion equations are associated with physically reasonable behavior, we must impose constraints on the control input \( u \), ensuring that the motion model obeys contact interactions laws. Due to the hybrid nature of contact, the physical constraints that dictate the magnitude and direction of the frictional forces vary with the contact interaction mode. In accordance with Coulomb’s frictional law, the following constraints on the inputs must always be satisfied independently of the contact mode:

\[
\mathcal{C}_0 : \begin{cases} 
  f_n \geq 0, \\
  |f_t| \geq \mu_p f_n,
\end{cases}
\]

Fig. 3: Friction cone constraint. The applied force must remain within the blue shaded region.

implying the pusher can only exert a compressive force on the object and that the net frictional force applied on the object remains within the bounds of the friction cone in Fig. 4. In addition, we must enforce constraints that depend on the contact interaction mode.

**Sticking** When the pusher is sticking relative to the object, the tangential velocity is stationary, as in Fig. 4(a)

\[
\mathcal{C}_1 : \dot{\phi}_t = 0. 
\]

**Sliding Left** When the pusher is sliding left relative to the object, the tangential velocity is strictly positive and the frictional force must remain on the right hand side of the friction cone, as in Fig. 4(b)

\[
\mathcal{C}_2 : \begin{cases} 
  \dot{\phi}_t > 0, \\
  f_t, c = \mu_p f_n, 
\end{cases}
\]

**Sliding Right** When the pusher is sliding right relative to the object, the tangential velocity is strictly negative and the frictional force is constrained to remain on the left hand side of the friction cone, as in Fig. 4(c)

\[
\mathcal{C}_3 : \begin{cases} 
  \dot{\phi}_t < 0, \\
  f_t, c = -\mu_p f_n, 
\end{cases}
\]

V. HYBRID MODEL-PREDICTIVE CONTROL

This section presents a controller design framework for planar manipulation tasks. The proposed controller aims to stabilize the motion of a given object about a nominal trajectory. The approach presented follows a Model Predictive Control (MPC) formulation where the goal is to determine a sequence of control inputs over a receding horizon to minimize the error between the manipulated object and its desired motion. Due to the hybridness associated with Coulomb’s frictional law, the optimization program takes the form of a mixed-integer quadratic program (MIQP).

**Optimization Problem MPC (MIQP):** Given current error state \( \bar{x}_0 \) and nominal trajectory \( (x_n^*, u_n^*) \), solve

\[
\min_{\bar{x}, \bar{u}, z} \bar{x}_0^T Q_N \bar{x}_N + \sum_{i=0}^{N-1} \bar{x}_{i+1}^T Q \bar{x}_{i+1} + \bar{u}_i^T R_u + \bar{z}_i^T W_z
\]

subject to

\[
\bar{x}_{i+1} = \bar{x}_i + h \left[ A_i \bar{x}_i + B_i \bar{u}_i \right], \\
\bar{u}_{c,i} \in \mathcal{C}_0, \\
\bar{u}_{c,i} \in \mathcal{C}_1 \text{ if } c \text{ is sticking (i.e., } z_{1c,i} = 1), \\
\bar{u}_{c,i} \in \mathcal{C}_2 \text{ if } c \text{ is sliding left (i.e., } z_{2c,i} = 1), \\
\bar{u}_{c,i} \in \mathcal{C}_3 \text{ if } c \text{ is sliding right (i.e., } z_{3c,i} = 1), \\
z_{1c,i} + z_{2c,i} + z_{3c,i} = 1,
\]

with \( \bar{x}_i = x_i - x_n^*, \bar{u}_i = u_i - u_n^*, \) and \( z_i = [z_{1c,i}, z_{2c,i}, z_{3c,i}]^T \). The terms \( Q, Q_N, R_u, \) and \( W \) denote weight matrices associated with the error state, final error state, control input, and contact modes, respectively. We constrain the search to the linearized dynamics of the system and the contact constraints presented in sections IV where \( A_i = \frac{\partial}{\partial x} [x_n^*, u_n^*] \) and \( B_i = \frac{\partial}{\partial u} [x_n^*, u_n^*] \).

Fig. 5: Hybrid MPC framework. A sequence of control inputs is computed that will drive the predicted states to the reference trajectory while simultaneously finding the schedule of optimal hybrid mode transitions \( m = \{m_1, \ldots, m_M\} \). The control input \( \bar{u}_0 + u_0^* \) is applied to the system.
We introduce integer variables into the optimization program to denote the hybrid mode that is active at time step \( i \). The integer variables \( z_{1c,i}, z_{2c,i}, \) and \( z_{3c,i} \in \{0,1\} \), denote sticking, sliding left, and sliding right for contact point \( c \), respectively, where the integer variable takes the value of 1 if a contact mode is active and 0 otherwise. The resulting mixed-integer optimization program is simultaneously tasked with finding the optimal hybrid mode schedule during the prediction horizon and the optimal control sequence. We enforce that the sum of integers values must be unity at each time step to ensure that only one mode can be active at a time.

To speed up computation, it is often preferable to constraint adjacent time steps within a prediction horizon to have the same contact mode. This is shown in Fig. 5 where the agglomerated mode sequence \( m = \{m_1, \ldots , m_M\} \) is introduced, with \( m_m \in \{1,2,3\} \) denoting sticking, sliding left, and sliding right.

VI. OFFLINE MODE SCHEDULE LEARNING

We can visualize the feedback control architecture proposed in Section V in block diagram form in Fig. 6. Due to the non-convex nature of integer variables in Eq. (3), the solution time is typically slow and not appropriate for high bandwidth feedback control applications. In an effort to increase the control bandwidth, we aim to offload as much of the computational costs offline as possible. To accomplish this, we present a novel formulation that separates the search for the mode schedule selection from the optimal control sequence.

Consider the controller design architecture proposed in Figure 7. Suppose that given the state error \( \bar{x}_0 \), nominal trajectory \( (x^*, u^*) \), and mode schedule \( m \), solve

\[
\min_{\bar{x}_i, u_i} \quad \bar{x}_N^TQ_N\bar{x}_N + \sum_{i=0}^{N-1} \left( \bar{x}_{i+1}^TQ\bar{x}_{i+1} + u_{i+1}^TRu_{i+1} \right)
\]

subject to

\[
\bar{x}_{i+1} = \bar{x}_i + h[A,\bar{x}_i + B,u_{i}],
\]

\[
\bar{u}_{i+1} \in \mathcal{U}_0,
\]

\[
\bar{u}_{i+1} \in \mathcal{U}_1 \quad \text{if} \ c \text{ is sticking},
\]

\[
\bar{u}_{i+1} \in \mathcal{U}_2 \quad \text{if} \ c \text{ is sliding left},
\]

\[
\bar{u}_{i+1} \in \mathcal{U}_3 \quad \text{if} \ c \text{ is sliding right}.
\]

The main attraction of this approach is to convert a non-convex mixed-integer quadratic program into a convex quadratic program that can be solved in real-time.

Fig. 7: Block diagram of hybrid controller design with learned mode schedule classifier. The resulting MPC controller design is a convex quadratic program.

Machine learning algorithm is to train a candidate classifier model that minimizes the cross-entropy error function of the labelled training set. This new hybrid control architecture leads to a convex optimization program with prescribed mode sequence, referred to as MPC (learned modes).

Optimization Problem MPC (learned modes): Given current error state \( \bar{x}_0 \), nominal trajectory \( (x^*, u^*) \), and mode schedule \( m \), solve

\[
\min_{\bar{x}_i, u_i} \quad \bar{x}_N^TQ_N\bar{x}_N + \sum_{i=0}^{N-1} \left( \bar{x}_{i+1}^TQ\bar{x}_{i+1} + u_{i+1}^TRu_{i+1} \right)
\]

subject to

\[
\bar{x}_{i+1} = \bar{x}_i + h[A,\bar{x}_i + B,u_{i}],
\]

\[
\bar{u}_{i+1} \in \mathcal{U}_0,
\]

\[
\bar{u}_{i+1} \in \mathcal{U}_1 \quad \text{if} \ c \text{ is sticking},
\]

\[
\bar{u}_{i+1} \in \mathcal{U}_2 \quad \text{if} \ c \text{ is sliding left},
\]

\[
\bar{u}_{i+1} \in \mathcal{U}_3 \quad \text{if} \ c \text{ is sliding right}.
\]

The main attraction of this approach is to convert a non-convex mixed-integer quadratic program into a convex quadratic program that can be solved in real-time.

Fig. 8: Supervised learning framework for mode schedule selection. A dataset of \( E \) labelled datapoints is generated using the the MPC (MIQP) formulation. From the training examples, a classifier is trained to return the mode schedule based on the state error vector.
TABLE I: Simulation parameters.

| Property                              | Symbol | Value |
|---------------------------------------|--------|-------|
| Coefficient of friction (pusher-slider) | $\mu_p$ | 0.3   |
| Coefficient of friction (slider-table)  | $\mu_s$ | 0.35  |
| Mass of slider (simulation A), kg     | m      | 0.827 |
| Object radius (simulation A), m       | r      | 0.045 |
| Mass of slider (simulation B), kg     | m      | 0.827 |
| Object side length (simulation B), m  | a      | 0.09  |
| Line pusher width (simulation B), m   | d      | 0.03  |

TABLE II: Neural network parameters.

| Property                               | Value |
|----------------------------------------|-------|
| Number of hidden layers                | 4     |
| Neurons in hidden layer 1              | 32    |
| Neurons in hidden layer 2              | 200   |
| Neurons in hidden layer 3              | 200   |
| Neurons in hidden layer 4              | 256   |
| Activation functions                   | ReLu  |
| Loss function                          | Cross entropy |

VII. RESULTS

In this section, we implement a numerical study with the controller design presented in Section V along with the planar pushing model described in Section IV-A. In both of the simulations considered in this section, we parametrize the classifier model introduced in Fig. 7 with a deep neural network with the properties summarized in Table 1.

A. Case Study A: Single Point Pushing

First, we investigate the performance of the controller design in Fig. 7 on a planar manipulation system where the goal is to track a straight line trajectory at a constant velocity. We build the classifier model following the learning framework displayed in Fig. 8 where the training examples are generated using the MPC (MIQP) program presented in Eq. (8). We parametrize the classifier using a deep neural network, as detailed in Table 1. The physical properties of the circular object along with the frictional properties of the system are in Table 2. The controller design parameters used in the numerical simulations are $h = 0.3$ seconds, $N = 35$, $Q = 10 \text{ diag}\{1,30,1,0\}$, $Q_N = 2000 \text{ diag}\{1,30,1,0\}$, and $R = \text{ diag}\{1,1,0,1\}$. The prediction horizon is split into 8 parts during which the contact modes are held constant. The number of time steps associated with each contact mode section $m_n$ is $\{1,5,5,5,5,5,4\}$ with the associated contact mode weight matrix $W = 0.1 \text{ diag}\{0,3,1,1,1,0,0,0\}$.

Figure 9: Accuracy results of the neural network predictions on a test set of 70000 labelled data points. We evaluate the performance on each mode separately, as defined in Fig. 5. Figure 10 shows the performance on each mode separately, as defined in Fig. 5. Figure 11 compare the optimal contact mode associated with the first time step of the MPC (MIQP) with the prediction made by the classifier. The contact modes are generated as a function of initial state errors in $x$ and $y$ while holding $\theta = 0$ [deg] and $\phi = 0$ [deg]. The classifier model captures the important trends of the MPC (MIQP) optimal mode solutions.

![Optimal contact mode (MIQP)](a)

![Classifier prediction](b)

Fig. 10: Optimal contact mode for the first time step as a function of initial state errors in $x$ and $y$ while holding $\theta = 0$ [deg] and $\phi = 0$ [deg]. The classifier model captures the important trends of the MPC (MIQP) optimal mode solutions.

Figure 10 compare the optimal contact mode associated with the first time step of the MPC (MIQP) with the prediction made by the classifier. The contact modes are generated as a function of initial state errors in $x$ and $y$ while holding $\theta = 0$ [deg] and $\phi = 0$ [deg]. The regions shown in green, yellow, and blue denote the regions where the options actions are sliding left, sliding right, or sticking. From the figure, we notice that the optimal contact modes are separated into distinct region, thus justifying the search for contact mode as a classification problem and facilitating the learning process. We notice that the classifier model succeed in capturing the important trends of the MPC (MIQP) solutions.

Figure 11 shows the convergence of two simulations generated by applying the MPC (MIQP) and MPC (learned modes) controllers on a straight line trajectory tracking example with constant velocity of $v = 0.05$ [m/s] with the perturbed initial conditions $x = [0, 0.05, 0, 0]^T$. The novel convex MPC formulation with learned modes achieves good closed-loop performance, where the initial perturbations quickly driven to zero. The MPC (learned modes) approach

![Convergence](a)

(a) Optimal contact mode (MIQP). (b) Classifier prediction.
can be executed at a much higher bandwidth (250 Hz) than the non-convex MPC (MIQP) formulation (20 Hz).

**B. Case Study B: Pushing with Line Contact**

Figure 12 depicts the second numerical simulation, where the goal is to track a race-track like trajectory defined by lines and semicircles of radii 0.2 meters at a constant velocity of $v = 0.05$ [m/s]. We model the line pusher as 2 contact points that are constrained to move as a rigid-body, with the position of the center point of pusher denoted as $p_c$.

We define the state vector as $x = [x \ y \ \theta \ p_c]^T$ along with the perturbed initial conditions $x = [0, 0.05, 180, 0]^T$. Following a similar approach to that described in Section VII, we train a classifier model using 115000 labelled datapoints to predict the optimal mode schedule based on the error state of the system. The neural network properties used to parametrize the classifier model are related in Table 1 and 2, respectively. The controller design parameters are $h = 0.3$ seconds, $N = 35$, $Q = 10 \text{ diag}(1, 1, 1, 0)$, $Q_N = 2000 \text{ diag}(1, 1, 1, 0)$, and $R = 0.01 \text{ diag}(1, 1, 1, 1)$. We split the prediction horizon into 8 parts during which the contact modes are held constant. The number of time steps associated with each contact mode section $m_m$ is $\{1, 5, 5, 5, 5, 5, 5, 4\}$ with the associated weight matrix $W = 0.1 \text{ diag}(0, 3, 1, 1, 1, 0, 0, 0)$.

Figure 12 depicts the line pusher performing a fast initial sliding left motion to improve its control authority on the square object and drive it back towards the desired trajectory. The novel MPC (learned modes) controller performs comparably to the MPC (MIQP) formulation and succeeds in tracking the desired trajectory while having significantly faster control bandwidth (200 Hz vs. 15 Hz).

**VIII. CONCLUSION**

This paper presents a systematic methodology for feedback controller design of planar manipulation tasks. The control formulation is based on a model predictive control approach, where the hybridness and underactuation associated with contact are explicitly enforced as constraints within a mixed-integer optimization program.

In order to enable real-time implementation, we address the combinatorial complexity resulting from the hybrid expansion of contact modes by separating the search for optimal mode schedules (offline) from the search for optimal control inputs (online). This is made possible by formulating the contact mode selection as a supervised learning problem. This approach enables us to train a classifier model offline by harnessing the solutions returned by the mixed-integer optimization program and building a dataset of optimal mode schedules.

We validate the controller design methodology through numerical simulations, where it is shown that the proposed convex formulation controller achieves comparable performance as its non-convex alternative, while obtained a 10 fold improvement in the controller bandwidth. Furthermore, in contrast to mixed-integer MPC control formulations, the online component of the hybrid MPC formulation with learned modes has the potential to extend to more complex dynamical systems with additional contact interactions, as its associated convex program can be efficiently computed in real-time.

**REFERENCES**

[1] K.-T. Yu, M. Bauza, N. Fazeli, and A. Rodriguez, “More than a million ways to be pushed: a high-fidelity experimental dataset of planar pushing,” IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), Daejeon, Korea, October 9–14, 2016.
[2] M. Bauza and A. Rodriguez, “A probabilistic data-driven model for planar pushing,” IEEE International Conference on Robotics and Automation (ICRA) arXiv:1704.03033, 2017.
[3] N. Fazeli, E. Donlon, E. Drumwright, and A. Rodriguez, “Empirical evaluation of common contact models for planar impact,” IEEE International Conference on Robotics and Automation (ICRA), Singapore, May 29 – June 3, 2017.
[4] R. Kolbert, N. Chavan-Dafle, and A. Rodriguez, “Experimental validation of contact dynamics for in-hand manipulation,” International Symposium on Experimental Robotics (ISER), Spring Proceedings in Advanced Robots., vol. 1, 2016.
[5] F. R. Hogan and A. Rodriguez, “Feedback control of the pusher-slider system: a story of hybrid and underactuated contact dynamics.” In Proceedings of the 12th International Workshop on the Algorithmic Foundations of Robotics (WAFR), San Francisco, CA, USA, December 18–20, 2016.

[6] M. T. Mason, “Mechanics and planning of manipulator pushing operations,” The International Journal of Robotics Research, vol. 5, no. 3, pp. 53 – 71, 1996.

[7] S. Goyal, A. Ruina, and J. Papadopoulos, “Wear,” Planar sliding with dry friction Part 1. Limit surface and moment function, vol. 143, pp. 307 – 330, 1991.

[8] K. M. Lynch and M. T. Mason, “Stable pushing: mechanics, controllability, and planning,” The International Journal of Robotics Research, vol. 15, no. 6, pp. 533 – 556, 1996.

[9] K.-T. Yu, J. Leonard, and A. Rodriguez, “Shape and pose recovery from planar pushing.” IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), Hamburg, Germany, September 28 – October 02, 2016.

[10] K. Lynch, H. Maekawa, and K. Tanie, “Manipulation and active sensing by pushing using tactile feedback,” IEEE/RSJ International Conference on Intelligent Robots and Systems, Raleigh, USA, July 7 – 10, pp. 416–421, 1992.

[11] J. Zhou, J. Bagnell, and M. T. Mason, “A convex polynomial force-motion model for planar sliding: Identification and application.” IEEE International Conference on Robotics and Automation (ICRA), May 16–21, Stockholm, Sweden, May 16–21, 2016.

[12] N. Chavan-Dafle and A. Rodriguez, “Sampling-based planning for in-hand manipulations with external pushes.” International Symposium on Robotics Research (ISRR). Under Review, 2017.

[13] M. Posa, C. Cantu, and R. Tedrake, “A direct method for trajectory optimization of rigid bodies through contact.” The International Journal of Robotics Research, vol. 33, no. 1, pp. 69 – 81, 2014.

[14] Y. Tassa and E. Todorov, “Synthesis and stabilization of complex behaviors through online trajectory optimization.” IEEE/RSJ International Conference on Intelligent Robots and Systems, Vilamoura, Portugal, October 7 – 12, 2012.

[15] J. Pajarinen, V. Kyrki, M. Koval, S. Srinivasa, J. Peters, and G. Neumann, “Hybrid control trajectory optimization under uncertainty.” arXiv:1702.04396, 2017.

[16] J. Ryu, F. Ruggiero, and K. M. Lynch, “Control of nonprehensile rolling manipulation: Balancing a disk on a disk.” IEEE International Conference on Robotics and Automation (ICRA), St. Paul, MN, USA, May 14–18, pp. 3232–3237, 2012.

[17] M. Posa, S. Kuindersma, and R. Tedrake, “Optimization and stabilization of trajectories for constrained dynamical systems.” In Proceedings of the International Conference on Robotics and Automation (ICRA), Stockholm, Sweden, May 16–21, 2016.

[18] J. Zhou, J. Bagnell, and M. Mason, “A fast stochastic contact model for planar pushing and grasping: theory and experimental validation.” Robotics Science and Systems, Cambridge, MA, USA, July 12–16, 2017.

[19] M. T. Mason, Mechanics of robotic manipulation. Cambridge, Massachusetts: MIT Press, 2001.