Time-Optimal Control
of Linear Fractional Systems

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Abstract

Problem of time-optimal control of linear systems with fractional dynamics is treated in the paper from the convex-analytic standpoint. A linear system of fractional differential equations involving Riemann–Liouville derivatives is considered. A method to construct a control function that brings trajectory of the system to the terminal state in the shortest time is proposed in terms of attainability sets and their support functions.

1 Introduction

Optimal control of systems with fractional dynamics is a hard problem due to specific of fractional differentiation operators, e.g. lack of the semigroup property. The papers on this topic include [1], where necessary optimality conditions of Euler–Lagrange were derived, and [2], where the problem of time-optimal control is addressed.

Here the fractional time-optimal control problem [2] for a linear system with fractional dynamics is treated using technique of attainability sets and their support functions. This approach has its roots in some methods of the differential games theory [3][4].

2 Preliminary Results

Denote by $\mathbb{R}^n$ the $n$-dimensional Euclidean space and by $\mathbb{R}_+ = [0, \infty)$ the positive semi-axis. In what follows we will also denote by $x \cdot y$ the scalar (dot) product and by $\|x\|$ the Euclidean norm for any $x, y \in \mathbb{R}^n$.

Suppose $f: \mathbb{R}_+ \to \mathbb{R}^n$ is an absolutely continuous function. Let us recall that the Riemann–Liouville (left-sided) fractional integral and derivative of order $\alpha$, $0 < \alpha < 1$, are defined as follows:

$$J_{a+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > a,$$

$$D_{a+}^\alpha f(t) = \frac{d}{dt} J_{a+}^{1-\alpha} f(t), \quad t > a.$$
In what follows we will omit the lower limit of integration in the notation if it is equal to zero, i.e. $J_0^\alpha f(t) \triangleq J_{0+}^\alpha f(t)$, $D_0^\alpha f(t) \triangleq D_{0+}^\alpha f(t)$.

Along with the left-sided fractional integrals and derivatives, one can consider their right-sided counterparts:

$$
J_b^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha - 1} f(\tau) d\tau, \quad t < b,
$$

$$
D_b^{-\alpha} f(t) = -\frac{d}{dt} J_b^{1-\alpha} f(t), \quad t < b.
$$

In [5] the Mittag-Leffler generalized matrix function was introduced:

$$
E_\rho,\mu(B) = \sum_{k=0}^{\infty} \frac{B^k}{k!},
$$

where $\rho > 0$, $\mu \in \mathbb{C}$, and $B$ is an arbitrary square matrix of order $n$. It should be noted that $E_\rho,\mu(B)$ generalizes the matrix exponential as

$$
E_{1,1}(B) = e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!}.
$$

The matrix $\alpha$-Exponential function, introduced in [6], is closely related to the Mittag-Leffler generalized matrix function:

$$
e_{\alpha}^A t = t^{\alpha - 1} \sum_{k=0}^{\infty} \frac{A^k t^k}{\Gamma((k + 1)\alpha)} = t^{\alpha - 1} E_{\alpha,\alpha}(A t^\alpha).
$$

The both functions play important role in the theory of fractional differential equations (FDEs). In particular, consider a system of linear FDEs with constant coefficients

$$
D^\alpha z = Az + u, \quad 0 < \alpha < 1,
$$

where $z \in \mathbb{R}^n$, $A$ is a square matrix, and $u : \mathbb{R}_+ \to \mathbb{R}^n$ is a measurable and bounded function, under the initial condition

$$
J^{1-\alpha} z|_{t=0} = z^0.
$$

Then the solution to the Cauchy-type problem [3], [4] can be written down as follows [7]

$$
z(t) = t^{\alpha - 1} E_{\alpha,\alpha}(A t^\alpha) z^0 + \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha,\alpha}(A(t - \tau)^\alpha) u(\tau) d\tau
$$

or, in terms of matrix $\alpha$-exponential function, as [6]

$$
z(t) = e_{\alpha}^A t z^0 + \int_0^t e_{\alpha}^{A(t - \tau)} u(\tau) d\tau.
$$
Now we proceed with a homogeneous linear system involving right-sided fractional derivative in the sense of Riemann–Liouville

\[ D^\alpha_{b^{-}} z(t) = Az(t), \quad z \in \mathbb{R}^n, \quad t < b, \quad 0 < \alpha < 1, \]

under the boundary condition

\[ J^1_{b^{-}} z|_{t=b} = \hat{z}. \]

The following lemma holds true.

**Lemma 1.** Equation (7) under the condition (8) has a solution given by the following formula

\[ z(t) = \hat{z} e^{A(t-b) \alpha}. \]

**Proof.** Since \( e^{A(t-b) \alpha} \) is an entire function, the corresponding power series can be integrated and differentiated term-by-term. In view of the formulas

\[ D^\alpha_{b^{-}} (b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-t)^{\beta-\alpha-1}, \]

\[ J^\alpha_{b^{-}} (b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (b-t)^{\beta+\alpha-1}, \]

one can easily verify that (9) satisfies (7) by direct substitution. Moreover, (11) implies that \( J^1_{b^{-}} \hat{z} e^{A(t-b) \alpha} = \hat{z} E^{\alpha,1}_{\alpha} (A(t-b) \alpha) \) and condition (8) is also fulfilled.

The following properties of the matrix \( \alpha \)-exponential function are direct consequences of properties of the conjugate transpose:

\[ (e^{A t}_\alpha)^* = e^{A^* t}_\alpha \]

\[ \psi \cdot e^{A t}_\alpha u = e^{A^* t}_\alpha \psi \cdot u \]

Below we present some properties of set-valued maps used in the sequel.

Let us suppose that \( U \) is a nonempty compact (closed and bounded) set in \( \mathbb{R}^n \). Hereafter we will denote by \( U[0,t] \) the set of all measurable functions defined on \([0,t]\) and taking their values in \( U \).

\[ \sup_{u(\cdot)\in U[0,t]} \int_0^t f(\tau, u(\tau))d\tau = \int_0^t \max_{u\in U} f(\tau, u)d\tau \]

Denote by \( \text{co} \ M \) and \( \overline{\text{co}} \ M \) the convex hull and the closure of the convex hull of a set \( M \subset \mathbb{R}^n \), respectively.

For any continuous function \( F: [0, t] \times U \to \mathbb{R}^n \), the set-valued map \( F(\tau, U) \) possess the following property:

\[ \int_0^t F(\tau, U)d\tau = \left\{ \int_0^t F(\tau, U)d\tau : u(\cdot) \in U[0,t] \right\} = \overline{\text{co}} \int_0^t F(\tau, U)d\tau. \]
The integral \( \int_0^t F(\tau, U) d\tau \) is to be thought of in the sense of Aumann, i.e. as the set of integrals of all measurable selections of the set valued map \( F(\tau, U) \).

Here we recall definition of the support function and present a useful result of convex analysis.

Let \( M \in \mathbb{R}^n \) be a convex closed set, i.e. \( M = \overline{\text{co}} M \). Then the function

\[
\sigma_M(\psi) = \sup_{m \in M} \psi \cdot m, \quad \psi \in \mathbb{R}^n
\]

is called the support function of \( M \).

**Lemma 2** (**[4]**). Let \( X \) and \( M \) be convex closed sets. Moreover, assume that \( X \) is bounded. Then \( X \cap M = \emptyset \) if and only if there exist a vector \( \psi \in \mathbb{R}^n \) and a number \( \varepsilon > 0 \) such that

\[
\sigma_X(\psi) + \sigma_M(-\psi) \leq -\varepsilon.
\]

**Proof.** Since the sets \( X \) and \( M \) are disjoint, by assumptions of the lemma and in view of hyperplane separation theorem, there exist a vector \( \psi \) and a number \( \varepsilon > 0 \) such that

\[
\psi \cdot x \leq \psi \cdot m - \varepsilon \quad \forall x \in X, \ m \in M.
\]

Taking supremum in \( x \) on the left-hand side and infimum in \( m \) on the right-hand side, we obtain an equivalent inequality

\[
\sigma_X(\psi) \leq \inf_{m \in M} \psi \cdot m - \varepsilon = -\sup_{m \in M} (-\psi \cdot m) - \varepsilon = -\sigma_M(-\psi) - \varepsilon,
\]

which completes the proof. \( \square \)

**Corollary 1.** Let \( X = \overline{\text{co}} X \), \( M = \overline{\text{co}} M \) and \( X \) be bounded. Then \( X \cap M \neq \emptyset \) if and only if

\[
\lambda_{X,M} = \min_{\|\psi\|=1} \left[ \sigma_X(\psi) + \sigma_M(-\psi) \right] \geq 0.
\]

### 3 Optimal Control Problem

Consider a system of linear fractional differential equations (FDEs) with constant coefficients (**[3]**) under the initial condition (**[4]**).

Let us fix a point \( m \in \mathbb{R}^n \). Here we formulate the optimal control problem: find a control function \( u(\cdot) \), \( u : \mathbb{R}_+ \to U \), from a class of measurable functions taking their values in a nonempty compact set \( U \subset \mathbb{R}^n \), such that the corresponding trajectory of (**[3]**), (**[4]**) arrives at \( m \) in the shortest time \( T \).

If we fix some admissible control function \( u(\cdot) \in U[0, t] \), then the solution to the Cauchy-type problem (**[3]**), (**[4]**) is given by (**[6]**).

Consider the attainability set

\[
Z(t, z_0) = \left\{ e^{A_t} z_0 + \int_0^t e^{A(t-\tau)} u(\tau) d\tau : u(\cdot) \in U[0, t] \right\}
\]

\[
= e^{A_t} z_0 + \int_0^t e^{A(t-\tau)} U d\tau.
\]
According to properties of integrals of set-valued maps, in view of (15), the attainability set $Z(t, z^0)$ is closed and convex, while the boundedness of $U$ implies $Z(t, z^0)$ is also bounded.

Consider support function of the attainability set (18).

$$
\sigma_{Z(t, z^0)}(\psi) = \sup_{z \in Z(t, z^0)} (z \cdot \psi)
\begin{equation}
= \sup_{u(\cdot) \in U[0, t]} \left\{ \psi \cdot e^{At} z^0 + \int_0^t \psi \cdot e^{A(t-\tau)} u(\tau) d\tau \right\}
= \psi \cdot e^{At} z^0 + \int_0^t \sigma_U(e^{A^*(t-\tau)} \psi) d\tau.
\end{equation}
$$

(19)

Here we applied properties (12)–(14).

Let us introduce the function

$$
\lambda(t, z^0) = \min_{\|\psi\|=1} [\sigma_{Z(t, z^0)}(\psi) - m \cdot \psi]
$$

(20)

and denote

$$
T(z^0) = \min\{t \geq 0 : \lambda(t, z^0) \geq 0\}.
$$

(21)

Then the following theorem holds true.

**Theorem 1.** Trajectory of the system (3), (4) can be brought to the point $m$ at the minimal time $T = T(z^0)$, given by the formula (21), with the help of control function of the form

$$
\hat{u}(\tau) = \arg \max_{u \in U} u \cdot \psi(\tau),
$$

where $\psi(\tau)$ is a solution to the adjoint (co-state) system

$$
D_T^\alpha \psi = A^* \psi,
$$

(22)

$$
J_{T-}^{1-\alpha} \psi|_{t=T} = \hat{\psi}
$$

(23)

and $\hat{\psi} = \arg \min_{\|\psi\|=1} [\sigma_{Z(T, z^0)}(\psi) - \psi \cdot m]$.

**Proof.** Let $T = \min\{t \geq 0 : m \in Z(t, z^0)\}$. Here minimum is attained due to the closedness of $Z(t, z^0)$.

Moreover, $m$ is a boundary point of $Z(T, z^0)$, i.e. $m \in \partial Z(T, z^0)$. As a boundary point, $m$ is contained in at least a supporting hyperplane $H(\hat{\psi}) = \{x \in \mathbb{R}^n : \hat{\psi} \cdot x = \sigma_{Z(T, z^0)}(\hat{\psi})\}$. Hence, for some $\hat{\psi}$

$$
\hat{\psi} \cdot m = \sigma_{Z(T, z^0)}(\hat{\psi}).
$$

(24)

Thus, the control function $\hat{u}(\cdot)$ that ensures bringing trajectory of (3), (4) to the point $m$ is the function at which the maximum in (19) is attained. Therefore it must satisfy

$$
\hat{u}(\tau) = \arg \max_{u \in U} u \cdot e^{A^*(t-\tau)} \hat{\psi}, \ \tau \in [0, T].
$$

(5)
In view of Lemma 1, \( \psi(\tau) = e^{A^*(T-\tau)} \hat{\psi} \) is a solution to (22).

According to Corollary 1, \( m \in Z(t,z^0) \) if and only if \( \lambda(t,z^0) \geq 0 \), hence \( T = T(z^0) = \min\{t \geq 0 : \lambda(t,z^0) \geq 0\} \). Since \( \lambda(T,z^0) \geq 0 \), in virtue of (24), \( \hat{\psi} \) delivers minimum to the expression \( \sigma_{Z(T,z^0)}(\psi) - \psi \cdot m \).

Thus
\[
\hat{u}(\tau) = \arg \max_{u \in U} u \cdot \psi(\tau),
\]
where \( \psi(\tau) \) is a solution to (22) under the condition (23), which completes the proof.

4 Example

Let us illustrate the above results by a simple example.

Consider a system with fractional dynamics described by the equation
\[
\frac{D}{D^\alpha} z = u, \quad z \in \mathbb{R}^n, \quad \|u\| \leq 1, \quad 0 < \alpha < 1, \quad (25)
\]
under the initial condition (4). In this example, the matrix \( A \) and \( U \) is the unit ball centered at the origin.

Hence
\[
e^{\alpha t} e^{A^*t} = e^{A^*t} = t^{\alpha-1} \Gamma(\alpha) I
\]
and the support function of the attainability set has the form
\[
\sigma_{Z(t,z^0)}(\psi) = \psi \cdot e^{\alpha t} z^0 + \int_0^t \sigma_U(e^{A^*(t-\tau)} \psi) d\tau
\]
\[
= \psi \cdot \frac{t^{\alpha-1}}{\Gamma(\alpha)} z^0 + \int_0^t \left( \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \psi \right) d\tau
\]
\[
= \psi \cdot \frac{t^{\alpha-1}}{\Gamma(\alpha)} z^0 + \|\psi\|^2 \frac{t^\alpha}{\Gamma(\alpha+1)}.
\]

Suppose that \( m = 0 \), then
\[
\lambda(t,z^0) = \min_{\|\psi\|=1} [\sigma_{Z(t,z^0)}(\psi)] = \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \|z^0\|.
\]

Thus the minimum time \( T \), at which trajectory of the system (25) can reach the origin can be found as the smallest positive root of the equation
\[
\frac{t^\alpha}{\Gamma(\alpha+1)} = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \|z^0\|.
\]

At \( t = 0 \) the left-hand side of the latter equation is zero while its right-hand side is infinite, provided that \( \|z^0\| \neq 0 \). As \( t \to \infty \) the left-hand side increases without bound and the right-hand side approaches zero. Thus, the equation has a positive solution.
The system adjoint to (25), (4) has the form

\[ D_T^\alpha \psi = 0, \]

\[ J_{T-}^{1-\alpha} \psi \big|_{t=T} = -\frac{z^0}{\|z^0\|} \]

and its solution is

\[ \psi(t) = -\frac{(T-t)^{\alpha-1}}{\Gamma(\alpha)} \frac{z^0}{\|z^0\|}. \]

Finally the optimal control that ensures bringing trajectory of (25), (4) to the origin at the minimal time \( T \) is

\[ u(t) \equiv -\frac{z^0}{\|z^0\|}. \]

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