Abstract. The aim of this paper is to study the behavior of knot Floer homology under Murasugi sum. We establish a graded version of Ni’s isomorphism between the extremal knot Floer homology of Murasugi sum of two links and the tensor product of the extremal knot Floer homology groups of the two summands. We further prove that $\tau = g$ for each summand if and only if $\tau = g$ holds for the Murasugi sum (with $\tau$ and $g$ defined appropriately for multi-component links). Some applications are presented.

1. Introduction

The Murasugi sum is an operation that one can perform on isotopy classes of surfaces with non-empty boundary embedded in 3-manifolds. Applying it to Seifert surfaces yields an operation on isotopy classes of links. As the name suggests, the operation was introduced by Murasugi [Mur63, Mur58], whose motivation was a calculation of the genus of an alternating link by means of the degree of its Alexander polynomial. An important point to be made about the Murasugi sum is that it not a well-defined binary operation on the set of links. Indeed, many essential choices are made in its definition; not only the isotopy classes of the chosen Seifert surfaces, but also the polygons embedded therein along which the sum is performed. Despite these choices, one can easily show that the coefficient of the Alexander polynomial corresponding to the first betti number of the surfaces in question is multiplicative under Murasugi sum. Gabai later showed that Seifert genus behaves additively under Murasugi sum, extending the well-known special case of the additivity of genus under connected sum. Indeed, he showed that the Murasugi sum of two surfaces is minimal genus if and only the two summands are minimal genus [Gab83, Gab85]. Note, however, that pathology arises if one considers non-minimal genus surfaces; for instance, Thompson showed that one can sum two unknots along genus one surfaces to get a trefoil or sum two figure eights to get the unknot [Tho94], and Able and Hirasawa have recently shown that in fact any knot can be obtained as a Murasugi sum of any other two knots along (typically) non-minimal genus Seifert surfaces [AH].

In light of the connections between the knot Floer homology groups and both the Alexander polynomial (through their Euler characteristic [OS04b, Ras03]) and the genus (through their breadth [OS04a]), one might wonder about the behavior of the extremal knot Floer homology under Murasugi sum. Here, “extremal” refers to the knot Floer homology group in Alexander grading given by negative the genus of the surfaces used. (Modulo a well-understood grading shift, this is isomorphic to the knot Floer homology group in Alexander grading given by the genus, a group often referred to as the “top” group) For instance, the knot Floer homology groups of a connected sum are a bigraded tensor product of those of the summands [OS04b, Theorem 7.1]. Simple examples exploiting the non-uniqueness of Murasugi sums show that this cannot hold for general Murasugi sums and, in fact, there can be no closed formula for the knot Floer homology of the Murasugi sum of links in terms of the knot Floer homology of the summands. Despite this, it would be reasonable to conjecture that the extremal knot Floer homology of a Murasugi sum is a tensor product of the extremal terms of its summands. Ni proved that this is indeed the case for ungraded knot Floer homology groups with field coefficients [Ni06b, Theorems 1.1,4.5] (cf. [Juh08, Corollary 8.8]). Since an ungraded vector
space over a field is determined up to isomorphism by its dimension, this result is equivalent to saying that
the rank of the extremal knot Floer homology is multiplicative under Murasugi sum. It is natural to wonder
if Ni’s result extends in a (Maslov) graded fashion, and our first result confirms that this is indeed the case.
To state it, we define the index of a (possibly disconnected) surface \( R \) to be the quantity
\[
i(R) := \frac{\chi(R) - \partial(R)}{2}
\]

**Theorem 1.1.** For \( i \in \{1, 2\} \), let \( L_i \) be an \( l_i \)-component link and let \( R_i \) be a Seifert surface for \( L_i \). Let \( R_1 \ast R_2 \) be a Murasugi sum of \( R_1 \) and \( R_2 \), and let \( \partial(R_1 \ast R_2) = L = L_1 \ast L_2 \) be the corresponding \( l \)-component link. Then with \( \mathbb{F}_2 \)-coefficients, we have a graded isomorphism
\[
\widehat{HFK}(L, -i(R_1 \ast R_2)) [l - 1] \cong \widehat{HFK}(L_1, -i(R_1)) [l_1 - 1] \otimes \widehat{HFK}(L_2, -i(R_2)) [l_2 - 1].
\]

We should note that it is not clear how to extend Ni’s argument, nor the argument using the decomposition
theorem for sutured Floer homology presented by Juhasz, to yield the graded statement given above (despite some effort to do so). On a superficial level, though, our proof follows the same strategy as its antecedents; namely, we find particular Heegaard diagrams adapted to Seifert surfaces and their Murasugi sum, and then analyze the resulting Floer complexes in detail. The diagrams we end up using are more specialized, however, and yield more control over the combinatorics and homotopy theoretic aspects of the associated chain complexes. This increase in control further allows us to glean some information about the rest of the knot Floer homology filtration, in the form of the following result about the integer-valued concordance invariant \( \tau \) [OS03] (for the extension of \( \tau \) to links, see [Cav18, OSS15, HR20]).

**Theorem 1.2.** If the link \( L \) is the Murasugi sum of links \( L_1 \) and \( L_2 \) along minimal index Seifert surfaces, then \( \tau(L_i) = g(L_i) \) for all \( i \in \{1, 2\} \) if and only if \( \tau(L) = g(L) \). (Here \( \tau \) denotes \( \tau_{\text{top}} \) from [HR20] when discussing links with more than one component.)

A large class of links for which \( \tau(L) = g(L) \) is provided so-called *strongly quasipositive* links. These links possess a Seifert surface which is properly isotopic into the 4-ball onto a piece of an algebraic curve and which therefore minimizes the smooth 4-genus. Rudolph gave a partial extension of Gabai’s results to 4-genera, by showing that the Murasugi sum of links along Seifert surfaces is strongly quasipositive if and only if the two summands are. Our result strengthens the resulting implications for the 4-genus.

Our results lead to topological restrictions on which link types can be expressed as Murasugi sums of others along minimal index Seifert surfaces. Some of the complexity of this problem, and the restrictions offered by our theorems, can be algebraically distilled by defining a Grothendieck group of links. Recall that the Grothendieck group \( K(\text{M}) \), of a commutative monoid \( M \) is the quotient of the free abelian group on the set \( M \) by the relations \([x+y] = [x] + [y]\), where on the left “+” is taken with respect to the monoidal operation and on the right within the free abelian group. While the Murasugi sum \( \ast \) is not a monoidal operation on links (relying as it does on the choice of Seifert surface and embedded 2n-gon), we can nonetheless define a group
\[
K(\text{links}, \ast) = \mathbb{Z}(\text{Isotopy classes of links}) / [L_1 \ast L_2 = [L_1] + [L_2]]
\]
which we call the *Grothendieck group of links under Murasugi sum along minimal index surfaces*. It is simply the quotient of the free abelian group on the set of isotopy classes of links by the relations \([L_1 \ast L_2] = [L_1] + [L_2] \), where \( \ast \) denotes any Murasugi sum along any 2n-gon in any minimal index Seifert surface for the links in question. Thus \( K(\text{links}, \ast) \) consists of equivalence classes of links, where two links are equivalent if they become isotopic after iteratively Murasugi summing both of them with some collection \((R_1, \ldots, R_t)\) of minimal index Seifert surfaces (along any 2n-gons embedded therein, and in any order). Fibered links, endowed with their (unique) minimal index Seifert surface, form an important class of links which is closed under Murasugi sums by Gabai’s work [Gab83] (see also [Sta78] for the closure under plumbing). If one considers their associated Grothendieck subgroup \( K(\text{fibered links}, \ast) < K(\text{links}, \ast) \), a deep theorem arising
from the Giroux correspondence asserts that $K(\text{fibered links}, \ast) \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by the positive and negative Hopf links [GG06]. One might hope that all links could similarly be generated by a small family, given the complexity allowed by choices of Seifert surfaces and embedded 2n-gons.

Multiplicativity of the rank of the extremal knot Floer homology under Murasugi sum shows that $K(\text{links}, \ast) \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by the positive and negative Hopf links [GG06]. One might hope that all links could similarly be generated by a small family, given the complexity allowed by choices of Seifert surfaces and embedded 2n-gons.

Corollary 1.3. The Poincaré polynomial of the top group of knot Floer homology induces a homomorphism

$$P : K(\text{links}, \ast) \to \mathbb{Q}_{>0}(\mathbb{t})$$

where the codomain is the multiplicative group of rational functions in $\mathbb{t}$ with positive rational coefficients.

It would be interesting to identify the image of $P$, a problem in the realm of geography questions for knot Floer homology. In particular, we have the following natural question:

Question 1.4. Is every Laurent polynomial with $\mathbb{N}$ coefficients realized as the Poincaré polynomial of the top group of knot Floer homology for some link in the 3-sphere?

Obstructions for a bigraded collection of abelian groups to arise as knot Floer homology groups were obtained in [HW18, BVV18], but these place no restriction on the top group.

Despite a lack of understanding of the geography question for the top group of knot Floer homology, our results indicate that any collection of knots whose Poincaré polynomials are coprime are linearly independent in the Grothendieck group, even if their total rank is the same. In particular, the kernel of the homomorphism $\mathbb{Q}_{>0}(\mathbb{t}) \to \mathbb{Q}_{>0}(\mathbb{t})$ induced by setting $\mathbb{t}$ equal 1 intersects the image of $P$ non-trivially. Perhaps more concretely, we have

Corollary 1.5. Suppose the Poincaré polynomial of the top group of knot Floer homology of a link $L \subset S^3$ is irreducible, viewed as a Laurent polynomial over $\mathbb{Z}$. If $L$ is a Murasugi sum of links $L = L_1 \ast L_2$, then one of $L_i$ is fibered.

As another quick corollary, we can show that alternating links or, more generally, links with thin Floer homology, are far from generating all links under Murasugi sum.

Corollary 1.6. Suppose the top group of the knot Floer homology of $L \subset S^3$ is non-trivial in more than one Maslov grading. Then $L$ is not a Murasugi sum of alternating links nor is any link which contains $L$ as a Murasugi summand.

For instance, the top group of the Kenoshita-Terasaka knot and its mutant, the Conway knot, have Poincaré polynomials given by $1 + \mathbb{t}$, up to multiplication by $\mathbb{t}^k$ (with $k = 2$ for the KT knot and $k = 3$ for the Conway knot) [OS04d, Theorems 1.1 and 1.2]. Therefore neither can be realized as a Murasugi sum of alternating links, nor is there any way to iteratively Murasugi sum them with other links to eventually arrive at a Murasugi sum of alternating (or thin) links.

As a final corollary, our results can be used in conjunction with the literature to calculate the top group of an arbitrary cable knot:

Corollary 1.7. Let $K_{p,q}$ be the $(p, q)$ cable of a knot $K$ with Seifert genus $g$. Then for any $p > 0$, we have
The key observation, due to Neumann and Rudolph, is that $K_{\frac{p}{q}} \cong K_{\text{sign}(q)} \ast T_{p,q}$, where sign($q$) is $\pm 1$ depending on whether $q > 0$ or $q < 0$, the corollary can then be deduced if the top group is known for two particular examples of $K_{\frac{p}{q}}$: one with $p$ positive, and one with $q$ negative. But the results of [Hed05a, Hed09] (cf. [Hed05b]) indicate that the top group of $K_{p,-pn+1}$ is isomorphic to that of $K$ and the bottom group of $K_{p,-pn+1}$ is isomorphic to that of $K$, provided in both cases that $n \gg 0$. Together with the symmetry between the top and bottom groups of knot Floer homology, and the observations above, the corollary follows. This is essentially the argument for the special case of fibered cable knots from [Hed08].

We conclude this introduction by highlighting a few problems and questions raised by our work. Perhaps the most interesting and challenging is

**Problem 1.8.** Determine the isomorphism type of $K(\text{links}, \ast)$.

Solving this would, ideally, yield an explicit presentation for $K(\text{links}, \ast)$ by generators and relations. Note that Gabai’s work implies that the link invariant $b_{1\text{min}}$ obtained by minimizing the first Betti number over all Seifert surfaces for a given link, is additive under Murasugi sums. Hence, it descends to a homomorphism $B_{1\text{min}} : K(\text{links}, \ast) \to K(N^+) \cong \mathbb{Z}$. An affirmative answer to the following question would solve the problem:

**Question 1.9.** Is the homomorphism $P \oplus B_{1\text{min}} : K(\text{links}, \ast) \to \mathbb{Q}_{>0}^*(t) \oplus \mathbb{Z}$ an isomorphism?

Note that an affirmative answer would require an affirmative answer to the geography problem raised by Question 1.4. Moreover, combined with any of the known algorithms to compute knot Floer homology (e.g. [MOS09, Bel10, OS19]), one would also arrive at a solution to the isomorphism problem in $K(\text{links}, \ast)$ and, presumably, a presentation. While we are inclined to believe the answer is no, the restriction of $P \oplus B_{1\text{min}}$ to the subgroup generated by fibered links is an isomorphism onto its image. Indeed, the image of $P$ on the fibered subgroup is the multiplicative subgroup $\{t^n\}_{n \in \mathbb{Z}}$ and the power $n$ associated to a given fibered link is the Hopf invariant of the 2-plane field associated to its corresponding open book decomposition (up to normalization, the Hopf invariant is equal to Rudolph’s enhancement of the Milnor number [Rud87]). To conclude with a more tractable question, we leave the reader with:

**Question 1.10.** Does the Poincaré polynomial homomorphism $P : K(\text{links}, \ast) \to \mathbb{Q}_{>0}^*(t)$ contain an infinite rank subgroup in the kernel of the rank homomorphism obtained by setting $t = 1$ in the Poincaré polynomial?

**Outline:** The paper is organized as follow: In Section 2, we review the knot Floer homology for links, recall the definition for Murasugi sum, construct Heegaard diagrams associated to Seifert surfaces, and describe the Murasugi sum operation in terms of Heegaard diagrams. In Section 3, we study some local isotopies on Heegaard diagrams which will largely reduce the number of generators; moreover, we prove that there is a subcomplex that remains unchanged when applying these isotopies if some technical conditions are satisfied. In Section 4, we use the simplifications from the previous section to prove the main results.

**Acknowledgement.** This article started almost fifteen years ago, but was placed on hiatus multiple times due to a variety of reasons. We are grateful to Robert Lipshitz, Yi Ni, Peter Ozsváth, and Zoltán Szabó for many helpful conversations about this paper at various points in the past decade. During this period, Zhechi Cheng was supported from NSFC grant No. 12126101, NSF grants DMS-1609148, DMS-1564172 and Swedish Research Council under grant No. 2016-06596, Matthew Hedden was supported from NSF...
2. Heegaard diagrams adapted to Seifert surfaces

2.1. Heegaard diagrams. We begin with a quick review of Heegaard diagrams. Most of what follows extends in a straightforward manner to arbitrary closed connected oriented three-manifolds, but since we are primarily concerned with the operation of Murasugi sum in $S^3$ we will specialize our definitions and constructions to this situation. We begin by recalling the definition of a Heegaard diagram:

**Definition 2.1.** A Heegaard diagram for $S^3$ is a 4-tuple 
\[
\mathcal{H} = (\Sigma(g), \alpha^{(g+k-1)}, \beta^{(g+k-1)}, \omega^{(k)})
\]

where

- $\Sigma \subset S^3$ is an oriented surface of genus $g$ whose complement has two components, the closures of which are genus $g$ handlebodies $U_\alpha$ and $U_\beta$ with $\Sigma = \partial U_\alpha = -\partial U_\beta$;
- $\alpha^{(g+k-1)} = (\alpha_1, \ldots, \alpha_{g+k-1})$ (respectively, $\beta^{(g+k-1)} = (\beta_1, \ldots, \beta_{g+k-1})$) is a collection of disjoint simple closed curves on $\Sigma$, each bounding a disk in the handlebody $U_\alpha$ (respectively, $U_\beta$), such that $\Sigma \setminus \alpha$ (respectively, $\Sigma \setminus \beta$) has exactly $k$ components;
- the $\alpha$ circles are transverse to the $\beta$ circles;
- $w = (w_1, \ldots, w_k)$ is a collections of markings on $\Sigma$, such that each component of $\Sigma \setminus \alpha$ contains a $w$ marking, and each component of $\Sigma \setminus \beta$ contains a $w$ marking.

Unless otherwise mentioned, we will assume our Heegaard diagrams to satisfy a certain technical condition called (weak) admissibility [OS08, Definition 3.5] (cf. [OS04c, Definition 4.10]). A generator is a $(g+k-1)$-tuple $x = (x_1, \ldots, x_{g+k-1})$ of points in $\Sigma$, called the coordinates of $x$, such that each $\alpha$ and $\beta$ circle contain exactly one of the coordinates; we will denote the set of generator by $\mathcal{G}_\mathcal{H}$.

Let $L \subset S^3$ be an $l$-component link and $R$ a Seifert surface for $L$, which we assume to be oriented but not necessarily connected. We have the following notion of a diagram adapted to $R$ [OSz04, Ni06b, Juh08, HJS13],

**Definition 2.2.** A Heegaard diagram adapted to $R$ is a 6-tuple 
\[
\mathcal{H} = (\Sigma(g), \alpha^{(g+k-1)}, \beta^{(g+k-1)}, z^{(k)}, \omega^{(k)}, S)
\]
satisfying

- $(\Sigma, \alpha, \beta, w)$ and $(\Sigma, \alpha, \beta, z)$ are both Heegaard diagrams for $S^3$;
- $S \subset \Sigma$ is an oriented surface-with-boundary which is isotopic to $R$ in $S^3$;
- each generator has at most $(k - \chi(R))$ coordinates inside $R$;
- the $2k$ markings $z = (z_1, \ldots, z_k)$ and $w = (w_1, \ldots, w_k)$ all lie on $\partial S$;
- each component of $\partial S$ contains at least one marking, and on each component of $\partial S$, the $z$ markings and the $w$ markings alternate;
- the oriented arcs in $\partial S$ joining each $z$ marking to the next $w$ marking are disjoint from the $\alpha$ circles, and the arcs in $\partial S$ joining each $w$ marking to the next $z$ marking are disjoint from the $\beta$ circles.

Given a Seifert surface $R$ for an $l$-component link $L \subset S^3$, we can employ the following slightly enhanced version of the algorithm from [HJS13], or a further modification thereof, to construct a Heegaard diagram adapted to $R$.

**Algorithm 2.3.** Adapting a Heegaard diagram to a Seifert surface $R \subset S^3$. 
(H-1) Embed a graph \( G \) with \( n \) vertices and \((n - \chi(R))\) edges in the interior of the surface \( R \), such that \( R \) deform retracts to \( G \). Therefore, \( R \) is isotopic to \( \overline{\text{bd}_R(G)} \), the closure of a regular neighborhood of \( G \) in \( R \). This is essentially a band presentation of \( R \).

(E-2) Consider \( \overline{\text{bd}_{S^3}(G)} \), the closure of a regular neighborhood of \( G \) in \( S^3 \). Although \( \overline{\text{bd}_{S^3}(G)} \) is a union of handlebodies, its complement in \( S^3 \) is usually not. Rectify this by tunneling out some one-handles from the complement and adding them to \( \overline{\text{bd}_{S^3}(G)} \), so as to get a Heegaard decomposition of \( S^3 \).

(H-3) Let \( U_\alpha \) be the handlebody obtained from \( \overline{\text{bd}_{S^3}(G)} \) by adding these new handles, and let \( U_\beta \) be complementary handlebody. Let \( \Sigma \) be the dividing Heegaard surface, oriented as the boundary of \( U_\alpha \).

(H-4) Push off \( \overline{\text{bd}_R(G)} \) towards \( \Sigma \) to get a surface \( S \subset \Sigma \) in a way so as to ensure that the orientation on \( S \) induced by \( R \) agrees with the one induced by \( \Sigma \).

(H-5) Place \( 2k \) distinct markings \( z = (z_1, \ldots, z_k) \) and \( w = (w_1, \ldots, w_k) \) on \( \partial S \) such that each component of \( \partial S \) contains at least one \( z \) and \( w \) marking, and on each component of \( \partial S \), the \( z \) markings and the \( w \) markings alternate.

(H-6) If the surface \( \Sigma \) has genus \( g \), then draw \((g + k - 1)\) \( \alpha \) circles and \((g + k - 1)\) \( \beta \) circles on \( \Sigma \setminus (z \cup w) \) such that the following holds:

(a) The \( \alpha \) circles are disjoint from one another.
(b) The \( \beta \) circles are disjoint from one another.
(c) The \( \alpha \) circles intersect the \( \beta \) circles transversally.
(d) Each component of \( \Sigma \setminus \alpha \) contains one \( z \) marking and one \( w \) marking.
(e) Each component of \( \Sigma \setminus \beta \) contains one \( z \) marking and one \( w \) marking.
(f) Exactly \((k - \chi(R))\) \( \alpha \) circles intersect \( S \).

(H-7) From each \( w \) marking, as one travels along \( -\partial S \) to the next \( z \) marking, isotope all the \( \alpha \) circles that one encounters, by finger moves, across the \( z \) marking. Similarly, from each \( w \) marking, as one travel along \( \partial S \) to the next \( z \) marking, isotope all the \( \beta \) circles that one encounters, by finger moves, across the \( z \) marking.

(H-8) Finally, perform isostopies of the \( \alpha \) circles and the \( \beta \) circles in \( \Sigma \setminus (z \cup w) \) to make the diagram admissible.

Note that such a Heegaard diagram is indeed adapted to \( R \). In particular, (H-6f) ensures that each generator has at most \((k - \chi(R))\) coordinates inside \( R \).

We now spell out an explicit way of making all the choices alluded to in the previous list. The process is best understood in conjunction with an explicit example, as illustrated in Figure 2.1. At various points it will be useful to make minor alterations to these choices, but for the sake of brevity (and sanity), we will not explicitly describe all the choices made each time a Heegaard diagram is constructed.

**Algorithm 2.4.** Explicit diagram adapted to a planar projection of an embedded Seifert surface \( R \subset S^3 \).

(E-1) Given a Seifert surface \( R \) for an \( l \)-component link \( L \subset S^3 \), view it as a surface lying in \( \mathbb{R}^3 \). Consider a projection \( \pi : \mathbb{R}^3 \to \mathbb{R}^2 \), and assume that \( \pi|_R \) is generic and the image \( \pi(R) \) is connected.

(E-2) If \( R \) has \( n \) components, let \( G \) be a graph with \( n \) vertices and \((n - \chi(R))\) edges, embedded in the interior of \( R \), such that the following holds:

(a) \( R \) deform retracts to \( G \).
(b) The vertex \( v \) is a regular point of \( \pi|_R \).
(c) \( \pi|_G \) is an immersion with no triple points, and all the preimages of the double points lie in the interior of the edges.

(E-3) Let \( U_\alpha = \overline{\text{bd}_{S^3}(\pi(G))} \) be a genus \( g \) handlebody and let \( U_\beta = S^3 \setminus \overline{\text{bd}_{S^3}(\pi(G))} \) be the complementary handlebody. Let \( \Sigma = \partial U_\alpha \) be the Heegaard surface.

(E-4) Designate \( g \) of the \((g + 1)\) circles in \( \Sigma \cap \mathbb{R}^2 \) as \( \beta \) circles.
Figure 2.1. An algorithm for constructing a Heegaard diagram adapted to a Seifert surface. As usual, the red circles are $\alpha$ and the blue ones are $\beta$. The surface $S$ is orange. The magenta dots are $w$-markings and the green dots are $z$-markings. In the last diagram, the $\alpha$ circles are represented by train tracks, with the thin red lines denoting curves.
For each of the \((n - \chi(R))\) edges of \(G\), choose a point in the image of the interior of the edge that is not a double point, and draw an \(\alpha\) circle on \(\Sigma\) which is the boundary of a normal disk to \(G\) inside \(U_\alpha = \operatorname{bdry}_3(\pi(G))\). Draw an additional \(\alpha\) circle near each of the \((g + \chi(R) - 1)\) double points of \(\pi|_G\), such that the \(\alpha\) circle bounds a disk in \(U_\alpha\) near the double point, and if the disk were surgered out, then \(U_\alpha\) locally would have two components, corresponding to the two preimages of the double point, with the same crossing information.

For each vertex \(v_i\) of \(G\), let \(p_i \in \Sigma\) be the unique point such that \(\pi(p_i) = \pi(v_i)\) and \(|d\pi|_N(p_i)\) has the same sign as \(|d\pi|_N(v_i)\). Let \(D_i \subset \Sigma\) be a small disk containing \(p_i\); let \(D = \bigcup_i D_i\).

For each of the \((n - \chi(R))\) edges of \(G\), attach a band to \(D\) lying in \(\Sigma \setminus \{\alpha\text{ circles near the double points}\}\). Choose each band so that it deformation retracts onto an arc that projects to the corresponding edge under \(\pi\), and so that the surface framing of the band in \(\Sigma\) is same as the surface framing of the corresponding edge in \(R\). Let \(S \subset \Sigma\) be the surface obtained from \(D\) by adding the bands.

Put 2l markings \(z = (z_1, \ldots, z_l)\) and \(w = (w_1, \ldots, w_l)\) on \(\partial S \cap \partial D\), such that each component of \(\partial S\) contains exactly one \(z\) marking and exactly one \(w\) marking, and the l arcs \(b_1, \ldots, b_l \subset \partial S\), which join the \(w\) markings to the \(z\) markings, are supported inside \(\partial S \cap \partial D\).

For each disk \(D_i\), add an \(\alpha\) circle around all but one of the \(b_j\)'s supported in \(D_i\). This adds a total of \((l - y)\ \alpha\) circles.

For \(2 \leq i \leq l\), add a \(\beta\) circles around \(b_i\).

Perform finger moves on the \(\alpha\) circles, as described in (H-7), to obtain the final Heegaard diagram. One can check that the diagram thus obtained is admissible. Furthermore, since the surface \(S\) was disjoint from the \((g + \chi(R) - 1)\ \alpha\) circles near the double points, see (E-7), it remains disjoint from them even after the finger moves, and consequently, it only intersects \((g + l - 1) - (g + \chi(R) - 1) = (l - \chi(R))\ \alpha\) circles.

2.2. Knot Floer homology. We briefly recall the definition of the “tilde” version of Heegaard Floer homology, essentially following [OS08, Section 6.1] cf. [MOS09, Proposition 2.5]. Given a Heegaard diagram for \(S^3\), \(\mathcal{H} = (\Sigma(g), \alpha^{(g+k-1)}, \beta^{(g+k-1)}, w^{(k)})\), the chain complex \(\tilde{CF}_\mathcal{H}\) is the \(\mathbb{F}_2\)-module freely generated by the elements of \(\mathcal{G}_\mathcal{H}\).

Given generators \(x, y \in \mathcal{G}_\mathcal{H}\), a domain joining them is a 2-chain \(D\) generated by the elementary regions of \(\mathcal{H}\) such that \(\partial(\partial D \cap \alpha) = y - x\); here, an elementary region is the closure of a component of \(\Sigma \setminus (\alpha \cup \beta)\), and we are thinking of the generators as formal linear sums of their coordinates. The set of all the domains joining \(x\) to \(y\) is denoted by \(D(x, y)\). A domain \(D\) is said to be positive if all its coefficients are non-negative, and at least one of the coefficients is positive. Given a point \(p \in \Sigma \setminus (\alpha \cup \beta)\), let \(n_p(D)\) denote the coefficient of \(D\) at the elementary region containing the point \(p\); let \(n_w(D) = \sum_{i=1}^k n_{w_i}(D)\). Domains with \(n_w(D) = 0\) are called empty domains, and the set of all empty domains joining \(x\) to \(y\) is denoted by \(D_0(x, y)\). Elements of \(\mathcal{G}_\mathcal{H}\) carry a well-defined grading called the absolute Maslov grading \(M\), which serves as the homological grading of \(\tilde{CF}_\mathcal{H}\). The difference in Maslov gradings can be computed as

\[
M(x) - M(y) = \mu(D) - 2n_w(D),
\]

where \(D \in D(x, y)\) is any domain, and \(\mu(D)\) denotes its Maslov index.

After choosing a generic path of almost complex structures on \(\text{Sym}^{g+k-1}(\Sigma)\), sufficiently close to the constant path of one induced from a complex structure on \(\Sigma\), one can define the contribution function \(c\), from the set of all empty Maslov index one domains, to \(\mathbb{F}_2\), given by \(c(D) = [M(D)]/\mathbb{R}\), the number of points in a certain unparametrized moduli space. The function \(c\) has the property that it evaluates to 1 only if

(a) the domain is positive [OS04c, Lemma 3.2], and
(b) the closure of the union of the elementary regions where the domain is supported is connected [Ras03, Corollary 9.1].

Then the boundary map on the chain complex $\tilde{CF}_H$ is given by

$$\partial x = \sum_{y \in \mathcal{M}} \sum_{D \in D_0(x,y) : \mu(D) = 1} c(D)y.$$  

(The chain complex $\tilde{CF}_H$ usually depends on the choice of the path of almost complex structures on $\text{Sym}^{g+k-1}(\Sigma)$; nevertheless, we will suppress this from the notation.)

**Theorem 2.5.** [OS08, MOS09] The homology $HF_H$ of the chain complex $\tilde{CF}_H$ coming from a Heegaard diagram $H = (\Sigma(g), \alpha^{(g+k-1)}, \beta^{(g+k-1)}, \omega(k))$ for $S^3$ is isomorphic, as graded $\mathbb{F}_2$-modules, to $\otimes^{g+k-1}(\mathbb{F}_2 \otimes \mathbb{F}_2[-1])$, where $[i]$ denotes a grading shift by $i$.

The tilde version of knot Floer homology or link Floer homology [OS04b, Ras03, OS08] is a refinement of Heegaard Floer homology. Let $H = (\Sigma(g), \alpha^{(g+k-1)}, \beta^{(g+k-1)}, \omega(k), S)$ be a Heegaard diagram adapted to a Seifert surface $R$ of an $l$-component link $L \subset S^3$. Consider the Heegaard diagram $H_0 = (\Sigma(g), \alpha^{(g+k-1)}, \beta^{(g+k-1)}, \omega(k))$ obtained by forgetting $S$ and the $z$ markings. The set of generators $\mathcal{G}_H$ is same as $\mathcal{G}_{H_0}$, and they carry the same absolute Maslov grading. Given a 2-chain $D$ generated by the elementary regions of $H$, let $n_x(D) = \sum_{k=0}^h n_x(D)$. The elements of $\mathcal{G}_H$ carry another well-defined grading called the absolute Alexander grading $A$, such that for any domain $D \in D(x,y)$, $A(x) - A(y) = n_x(D) - n_y(D)$.

**Proposition 2.6.** If $x \in \mathcal{G}_H$ is a generator in $H = (\Sigma(g), \alpha^{(g+k-1)}, \beta^{(g+k-1)}, \omega(k), S)$, then its absolute Alexander grading is given by $A(x) = \text{(number of $x$-coordinates inside $S$)} - \frac{1}{2}(2k - l - \chi(R))$. In particular, the Alexander grading satisfies: $-\frac{1}{2}(2k - l - \chi(R)) \leq A(x) \leq \frac{1}{2}(l - \chi(R))$.

**Proof.** The Alexander grading of a generator $x \in \mathcal{G}_H$ is given by $\frac{1}{2}|c_1(s(x))|$, $[R, \partial R]$. If the generator has no coordinate inside $S$ (called outer in [Juh08]), we can evaluate it as $\frac{1}{2}c(S)$ where $c(S)$ is the quantity defined in [Juh08, Section 3]. Then using [Juh08, Lemma 3.9], we see that

$$c(S) = \chi(S) + I(S) - r(S) = \chi(S) - k - (k - l) = \chi(R) + l - 2k,$$

where $I(S)$ equals minus half the number of sutures (which are basepoints in our setting) and $r(S)$ is 0 if there is exactly one pair of sutures on each boundary and decreases by one for each additional pair of sutures. For generators which are not disjoint from $S$, we only need to notice that $c_1(s(x)) - c_1(s(y)) = 2\text{PD}(\alpha)$, where $\alpha = \partial D$ for any domain $D \in D(x,y)$. It is not hard to see that the algebraic intersection number of $\alpha$ with $\partial S$ equals the number of $x$-coordinates inside $S$ minus the number of $y$-coordinates inside $S$, and therefore,

$$A(x) = \frac{1}{2}(\chi(R) + l - 2k) + \text{(number of $x$-coordinates inside $S$)}.$$

This proves the lefthand side of the inequality. For the righthand side, we only need to use the following fact

$$\text{(number of $x$-coordinates inside $S$)} \leq \text{(number of $\alpha$ circles intersecting $S$)} = k - \chi(R).$$

In view of the above proposition, we make the following definitions.

**Definition 2.7.** Given a compact surface $R$ (possibly disconnected), define its index to be $i(R) = \frac{1}{2}(|\partial R| - \chi(R))$. Call a Seifert surface $R$ for a link $L$ minimal if it minimizes the index, and define the genus of the link, $g(L)$, to be this minimal index.
It is easy to see that the chain complex \( \widehat{CF}_\mathcal{H} \) is filtered by the Alexander grading. Let the filtration level \( \mathcal{F}_\mathcal{H}(m) \subseteq \widehat{CF}_\mathcal{H} \) denote the subcomplex generated by the generators with Alexander grading \( m \) or less. We call such an \((M,A)\)-bigraded complex, where the differential decreases \( M \) by one and does not increase \( A \), to be an \( M \)-graded-\( A \)-filtered complex.

**Theorem 2.8.** [OS04b, Ras03, OS08] To an \( l \)-component link \( L \subset S^3 \), one can associate (the filtered chain homotopy type of) an \( M \)-graded-\( A \)-filtered complex \( CFK(L) \) such that the chain complex \( \widehat{CF}_\mathcal{H} \), coming from any Heegaard diagram \( \mathcal{H} = (\Sigma(g), \alpha^{(g+k-1)}, \beta^{(g+k-1)}, z^{(k)}, w^{(k)}, S) \) adapted to any Seifert surface \( R \) for \( L \), is filtered chain homotopy equivalent to \( CFK(L) \otimes^{k-l} (F_2 \oplus F_2[-1,-1]) \), where \([i,j]\) denotes the \((M,A)\) bi-grading shift by \((i,j)\).

It is clear from Proposition 2.6 and Theorem 2.8 that the subcomplex of \( CFK(L) \) in Alexander grading less than \(-i(R)\) is filtered chain homotopy equivalent to zero. Let \( \widehat{CFK}(L, -i(R)) \) denote the subcomplex of \( CFK(L) \) in Alexander grading less than or equal to \(-i(R)\). If \( R \) is minimal, then its homology, \( \widehat{HFK}(L, -g(L)) \), is non-zero [OS04a, Ni06a] carrying a single grading coming from the Maslov grading, and is called the extremal knot Floer homology.

Instead of studying the full filtration on \( CFK(L) \), we will restrict our attention to the two-step filtration \( \widehat{CFK}(L, -i(R)) \subseteq CFK(L) \). Recall that a two-step filtered complex is simply a pair \((S,C)\) where \( C \) is a chain complex and \( S \subseteq C \) is a subcomplex. A filtered chain map \( f \) from \((S,C)\) to \((S',C')\) is a chain map \( f: C \to C' \) so that \( f(S) \subseteq S' \). A filtered chain map \( f \) from \((S,C)\) to \((S',C')\) is a quasi-isomorphism if both \( f: C \to C' \) and \( f|_S: S \to S' \) induce isomorphisms on homology. We will make use of the following corollary of Theorem 2.8.

**Corollary 2.9.** Let \( \mathcal{H} = (\Sigma(g), \alpha^{(g+k-1)}, \beta^{(g+k-1)}, z^{(k)}, w^{(k)}, S) \) be a Heegaard diagram adapted to a minimal Seifert surface \( R \) for \( L \). Then there is a quasi-isomorphism of pairs

\[
(\mathcal{F}_\mathcal{H}(-i(R) - k + l), \widehat{CF}_\mathcal{H}) \cong (\widehat{CFK}(L, -g(L)) \otimes^{k-l} (F_2[-1,-1]), CFK(L) \otimes^{k-l} (F_2 \oplus F_2[-1,-1])).
\]

In particular, the extremal knot Floer homology is isomorphic to the homology of \( \mathcal{F}_\mathcal{H}(-i(R) - k + l)[k-l] \). Moreover, the maps on homologies, \( \widehat{HFK}(L, -g(L)) \to H_\ast(CFK(L)) \) and \( H_\ast(\mathcal{F}_\mathcal{H}(-i(R) - k + l)) \to HF_\mathcal{H} \) have the same rank.

We conclude this section by describing how the extremal knot Floer homology is related to the \( \tau \)-invariant. Ozaváth-Szabó originally defined the \( \tau \)-invariant for knots in \( S^3 \); there are a number of generalizations of this invariant to links, and we will concentrate on \( \tau_{\text{bot}} \) and \( \tau_{\text{top}} \) which, by [HR20, Proposition 5.16] correspond to the smallest and largest of all the possible \( \tau \) invariants for links (For a knot \( K \), \( \tau(K) = \tau_{\text{bot}}(K) = \tau_{\text{top}}(K) \)). For now, we only need the following properties of these invariants.

**Proposition 2.10.** If \( m(L) \) denotes the mirror of \( L \), then \( \tau_{\text{bot}}(m(L)) = -\tau_{\text{top}}(L) \). The invariant \( \tau_{\text{bot}} \) satisfies \(-g(L) \leq \tau_{\text{bot}}(L) \leq g(L) \) with \( \tau_{\text{bot}}(L) = -g(L) \) if and only if the map \( \widehat{HFK}(L, -g(L)) \to H_\ast(CFK(L)) \) is non-zero.

**Proof.** The relationship between \( \tau_{\text{bot}} \) and \( \tau_{\text{top}} \) under mirroring follows from their definition, a duality property satisfied by generalized \( \tau \) invariants [HR20, Proposition 2.5]. That \( \tau_{\text{bot}} \) is bounded by the genus of \( L \) follows from the fact that it is correspondingly bounded by the “slice genus” [HR20, Proposition 5.14]. The final statement is a consequence of the definition of \( \tau_{\text{bot}} \), and the monotonicity of the \( \tau \) invariants for links established in [HR20, Proposition 5.16]. See [HR20, Theorem 2 and Section 5.3] for more details. \( \square \)
Proposition 2.11. \( H = (\Sigma(g), \alpha(g+k-1), \beta(g+k-1), \gamma(g+k-1), w^{(k)}; S) \) be a Heegaard diagram adapted to a minimal Seifert surface \( R \) for a knot \( L \subset S^3 \). Then \( \tau_{\text{tot}}(L) = -g(L) \) if and only if the map on homology \( H_*\mathcal{F}(\mathcal{H}(-g(L) - k + l)) \rightarrow \overline{\mathcal{H}} \) induced from the inclusion \( \mathcal{F}(\mathcal{H}(-g(L) - k + l)) \hookrightarrow \overline{\mathcal{H}} \) is non-trivial.

Proof. This follows immediately from Proposition 2.10 and Corollary 2.9. \hfill \Box

2.3. Triangle maps. We briefly introduce the definition of triangle maps in our restricted setting, once again following the original definitions from [OS04c]. Let \( H = (\Sigma(g), \alpha(g+k-1), \beta(g+k-1), \gamma(g+k-1), w^{(k)}) \) be a triple Heegaard diagram, that is: \( \mathcal{H}_{\alpha\beta} = (\Sigma, \alpha, \beta, w) \) and \( \mathcal{H}_{\gamma\beta} = (\Sigma, \gamma, \beta, w) \) are Heegaard diagrams for \( S^3 \); \( \alpha_i \) is disjoint from \( \gamma_j \) for \( i \neq j \); \( \alpha_i \) is transverse to \( \gamma_i \) and they intersect each other in exactly two points, none of which lies on the \( \beta \) curves; furthermore, if \( \alpha_j \) bounds a disk \( D_j \) in the \( \alpha \)-handlebody \( U_\alpha \), then \( \gamma_i \) is isotopic to \( \alpha_i \) in \( \overline{\text{bnd}} U_\alpha \cup (\bigcup_{j \neq i} D_j) \cup (\Sigma \setminus w) \), that is, \( \gamma_i \) can be isotoped to \( \alpha_i \) after sliding it over some other \( \alpha \) circles in the complement of the \( w \) markings. We will once again assume that the triple Heegaard diagram is admissible.

Orient \( \alpha_i \) arbitrarily, and then orient \( \gamma_i \) in the same direction, induced from the isotopy joining \( \gamma_i \) to \( \alpha_i \). Let \( \theta_i \) be the positive intersection point in \( \gamma_i \cap \alpha_i \), and let \( \theta = (\theta_1, \ldots, \theta_{g+k-1}) \). It is usually called the top generator. Note that \( (\Sigma, \gamma, \alpha, w) \) is a Heegaard diagram for \#\( S^1 \times S^2 \) on which \( (k-1) \) index 0/3 stabilizations have been performed, and \( \theta \) is its unique generator of highest Maslov grading.

Elementary regions of \( \mathcal{H} \) are closures of the components of \( \Sigma \setminus (\alpha \cup \beta \cup \gamma) \); a triangular domain joining a generator \( x \in \mathcal{G}_{\mathcal{H}_{\alpha\beta}} \) to a generator \( y \in \mathcal{G}_{\mathcal{H}_{\gamma\beta}} \) is a 2-chain \( D \) generated by the elementary regions such that \( \partial(\partial D \cap \alpha) = \theta \cap \alpha \) and \( \partial(\partial D \cap \beta) = \theta \cap \beta \); a triangular domain is said to be positive if all its coefficients are non-negative. Given a triangular domain \( D \), let \( n_w(D) = \sum_i n_{w_i}(D) \), where \( n_{w_i}(D) \) is the coefficient of the elementary region containing \( w_i \), in the 2-chain \( D \). Let \( T(x, y) \) be the set of all triangular domains joining \( x \in \mathcal{G}_{\mathcal{H}_{\alpha\beta}} \) to \( y \in \mathcal{G}_{\mathcal{H}_{\gamma\beta}} \); and let \( T_0(x, y) \) be the subset consisting of the empty triangular domains, that is, triangular domains with \( n_w = 0 \). The Maslov grading \( \mu(D) \) of any triangular domain \( D \in T(x, y) \) satisfies \( \mu(D) - 2n_w(D) = M(y) - M(x) \).

Choosing an appropriate family (parametrized by the 2-simplex) of almost complex structures on \( \text{Sym}^{g+k-1}(\Sigma) \), we can define a contribution function \( c \) from the set of all Maslov index zero triangular domains, to \( \mathbb{F}_2 \). Picking the family of almost complex structures to be integrable near a collection of hypersurfaces specified by basepoints in the elementary regions ensures that the contribution function has non-zero support only on the positive triangular domains. Then the following map is a graded quasi-isomorphism from \( \overline{\mathcal{H}}_{\mathcal{H}_{\alpha\beta}} \) to \( \overline{\mathcal{H}}_{\mathcal{H}_{\gamma\beta}} \).

\[
f(x) = \sum_{y \in \mathcal{G}_{\mathcal{H}_{\gamma\beta}}} \sum_{D \in T_0(x, y) \cap \mu(D) = 0} c(D)y.
\]

We will reprove a special case of this fact in Theorem 3.3.

2.4. Murasugi sum. We are now prepared to discuss the Murasugi sum operation. Let \( S^2 \subset S^3 \) be the standard 2-sphere. We will mentally ‘one-point-decompactify’ the picture, and draw it as \( \mathbb{R}^2 \subset \mathbb{R}^3 \). There are two components in \( S^3 \setminus S^2 \), the ‘inside’ \( B_1 \) and the ‘outside’ \( B_2 \), such that \( S^2 \) is oriented as \( \partial B_1 \). Let \( A_1A_2 \ldots A_{2n} \) be a 2n-gon lying on \( S^2 \). For \( i \in \{1, 2\} \), let \( R_i \) be a Seifert surface for an \( l_i \)-component link \( \overline{L}_i \subset \overline{B}_i \), such that: \( R_i \cap S^2 \) is the \( A_1A_2 \ldots A_{2n} \) with the same orientation; \( L_1 \cap S^2 \) is the union of the oriented segments \( A_1A_2, A_3A_4, \ldots, A_{2n-1}A_{2n} \); and \( L_2 \cap S^2 \) is the union of the oriented segments \( A_2A_3, A_4A_5, \ldots, A_{2n}A_1 \). Then the Murasugi sum \( L = L_1 \# L_2 \) is the link \((L_1 \cup L_2) \setminus S^2 \), and it bounds the Seifert surface \( R_1 \# R_2 = R_1 \cup R_2 \). The special cases when \( n = 1 \) is just the connected sum and \( n = 2 \) is a plumbing. The case \( n = 2 \) is illustrated in Figure 2.2.
Figure 2.2. The Murasugi sum operation. The link $L$ is obtained by plumbing the link $L_1$ below the plane with the link $L_2$ above the plane along the rectangle $A_1 A_2 A_3 A_4$.

We need the following quantities, $\delta_1, \delta_2, \delta$, which will simplify certain expressions later on. Define $\delta_1$ (respectively, $\delta_2, \delta$) to be $n$ minus the number of components of $L_1$ (respectively, $L_2, L$) that intersect the $2n$-gon. Note, $(l + \delta - n) = (l_1 + \delta_1 - n) + (l_2 + \delta_2 - n)$.

We will now describe how to draw Heegaard diagrams adapted to $R_1$ and $R_2$, and how they can be combined to form a Heegaard diagram for $R_1 \ast R_2$. The procedures closely follow the outline from Section 2.1 with a few differences. The most important feature of the following construction is that the roles of $\alpha$ and $\beta$ are reversed while constructing the Heegaard diagrams adapted to $R_1$ and $R_2$.

(M-1) We first embed a graph $G_i$ in $R_i$ so that $R_i$ deforms retract to $G_i$, and $G_i$ intersects the $2n$-gon in a single vertex with exactly $n$ edges going out to $n$ of its edges. This is easy to ensure, see Figure 2.3.

(M-2) Then consider the handlebody $B_{3-i} \cup \text{nbd}_{B_i}(G_i \cap B_i)$. Its complement need not be a handlebody, so we add a few tunnels, none intersecting $S^2$, to complete this to a Heegaard decomposition of $S^3$. Let $\Sigma_i$ be the resulting Heegaard surface, oriented so that its orientation agrees with the orientation of $S^2 = \partial B_1$ on $S^2 \cap \Sigma_i$. Let $U_{\alpha,i}$ and $U_{\beta,i}$ be the components of $S^3 \setminus \Sigma_i$, so that $\Sigma_i$ is oriented as the boundary of $U_{\alpha,i}$.

(M-3) Construct a surface $S_i \subset \Sigma_i$, such that $S_i$ is isotopic to $R_i$ and $S_i \cap S^2$ is the $2n$-gon $A_1 A_2 \ldots A_{2n} \cap \Sigma_i$.

(M-4) Put $z$ markings at $A_1, A_3, \ldots, A_{2n-1}$, and put $w$ markings at $A_2, A_4, \ldots, A_{2n}$. On every other component of $\partial S_i$, put a $z$ marking and a $w$ marking right next to one another, such that a small arc in $(-1)^i \partial S_i$ joins the $w$ marking to the $z$ marking. Therefore, the total number of $z$ (or $w$) markings is $l_i + \delta_i$.

(M-5) Then draw $\alpha$ circles and $\beta$ circles on $\Sigma_i \setminus (z \cup w)$, such that:

(a) The $\alpha$ circles and $\beta$ circles are transverse to each other and to $\partial S_i$.
(b) The $\alpha$ circles are pairwise disjoint and they span a half-dimensional subspace of $H_1(\Sigma_i)$; each component of $\Sigma_i \setminus \alpha$ contains a $z$ marking and a $w$ marking.
(c) The $\beta$ circles are pairwise disjoint and they span a half-dimensional subspace of $H_1(\Sigma_i)$; each component of $\Sigma_i \setminus \beta$ contains a $z$ marking and a $w$ marking.
Figure 2.3. **Embedding a graph $G$ in the Seifert surface.** In each case, the Seifert surface deform retract to a neighborhood of $G$ that contains $A_1 A_2 A_3 A_4$.

(d) Each component of $\Sigma_1 \setminus \alpha$ (respectively, $\Sigma_2 \setminus \beta$) has an oriented arc in $(-1)^i \partial S_i$ joining the $w$ marking to the $z$ marking.

(e) Exactly $(l_i + \delta_i - \chi(R_i)) \beta$ (respectively, $\alpha$) circles intersect $S_1$ (respectively, $S_2$).

(f) There are exactly $(n - 1) \alpha$ circles lying entirely inside the 2-sphere $S^2$, and they encircle the edges $A_3 A_4, A_4 A_5, \ldots, A_{2n-1} A_{2n}$. There are exactly $(n - 1) \beta$ circles lying entirely inside the 2-sphere $S^2$, and they encircle the intervals $A_4 A_5, A_6 A_7, \ldots, A_{2n} A_1$. Moreover, the consecutive $\alpha$ and $\beta$ circles intersect each other at exactly two points.

(g) Other than the above circles, there are no $\beta$ (respectively, $\alpha$) circle of $\Sigma_1$ (respectively, $\Sigma_2$) that intersects $S^2$. There could be some $\alpha$ (respectively, $\beta$) arcs of $\Sigma_1$ (respectively, $\Sigma_2$) that intersect $S^2$; in that case, their intersection with $S^2$ lies entirely inside $S_i \cap S^2$; and we can also ensure that there are at most $(n - 1)$ of such $\alpha$ (respectively, $\beta$) arcs.

(M-6) We then do Fiver moves on the $\beta$ (respectively, $\alpha$) circles on $\Sigma_1$ (respectively, $\Sigma_2$) to convert this to a Heegaard diagram $H_1$ (respectively, $H_2$) adapted to the Seifert surface $R_1$ (respectively, $R_2$). These final diagrams, in the case when $n = 2$, are shown in Figure 2.4. In the case when $n = 3$, the diagrams are also shown in the first two figures of the top row of Figure 4.1 (where $X$ denotes a handle going down into $B_1$ and $O$ denotes a handle coming up into $B_2$).

(M-7) The first two figures in the third row of Figure 4.1 represent slightly modified Heegaard diagrams $H_i''$ that are obtained from $H_i$ by deleting the $z$-markings, modifying the surface $S_i$, and performing small isotopies to reduce the number of intersections between $\alpha$ and $\beta$ circles. We will assume that we have already performed some isotopies on the $\alpha$ and $\beta$ circles on $H_i$ away from $S^2$ so as to ensure that these modified diagrams $H_i''$, and hence $H_i$ itself, is already admissible.

We can now "combine" the Heegaard diagram $H_1$ adapted to $R_1$ and $H_2$ adapted to $R_2$ to form a Heegaard diagram $H_1 \ast H_2$ adapted to $R_1 \ast R_2$. Recall that in $H_1$, the $\alpha$-handlebody $U_{\alpha,1}$ is obtained by tunneling out a few one-handles from $T_1$, and the $\beta$-handlebody $U_{\beta,1}$ is obtained by attaching those corresponding
Figure 2.4. The Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$. We continue to represent $w$ and $z$ markings by magenta and green dots, respectively. Once again, the thin lines are curves, and the thick lines are train tracks, representing some (possibly zero) curves running in parallel.

Figure 2.5. The Heegaard diagram $\mathcal{H}_1 \ast \mathcal{H}_2$. This diagram is obtained by combining the Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ from Figure 2.4.

one-handles to $B_2$. Similarly, in $\mathcal{H}_2$, the $\alpha$-handlebody $U_{\alpha,2}$ is obtained by attaching a few one-handles to $B_1$, and the $\beta$-handlebody $U_{\beta,2}$ is obtained by tunneling out those corresponding one-handles from $B_2$. In the ‘combined’ Heegaard diagram $\mathcal{H}_1 \ast \mathcal{H}_2$, the $\alpha$-handlebody $U_{\alpha,1} \ast U_{\alpha,2}$ is obtained from $B_1$ by tunneling out all the one-handles that were tunnelled out in $U_{\alpha,1}$ and by attaching all the one-handles that were attached in $U_{\alpha,2}$, and the $\beta$-handlebody $U_{\beta,1} \ast U_{\beta,2}$ is the closure of its complement. The Heegaard surface $\Sigma_1 \ast \Sigma_2$ is the oriented boundary of $U_{\alpha,1} \ast U_{\alpha,2}$. There is a surface $S_1 \ast S_2 \subset \Sigma_1 \ast \Sigma_2$, isotopic to $R_1 \ast R_2$, which
Figure 3.1. The disk $D$ in the Heegaard diagrams $\mathcal{H}_{\alpha\beta}$ and $\mathcal{H}_{\gamma\beta}$. The $\alpha$, $\beta$, and $\gamma$ arcs are represented by red, blue, and pink train tracks, respectively, with thin lines denoting curves. The number of arcs in each train track is also shown.

is obtained from $S_1 \subset \Sigma_1$ and $S_2 \subset \Sigma_2$. The $w$ and $z$ basepoints, and the $\alpha$ and $\beta$ circles on $\Sigma_1 \ast \Sigma_2$ are induced from the corresponding objects in $\Sigma_1$ and $\Sigma_2$. The Heegaard diagram $\mathcal{H}_1 \ast \mathcal{H}_2$, in the case when $n = 2$, looks like Figure 2.5, and in the case $n = 3$, looks like the third figure in the top row of Figure 4.1. Since the modified diagrams $\mathcal{H}''_1$ and $\mathcal{H}''_2$ are admissible, it follows that the corresponding modified diagram $(\mathcal{H}_1 \ast \mathcal{H}_2)''$ (third figure in the third row of Figure 4.1), and hence $\mathcal{H}_1 \ast \mathcal{H}_2$ itself, is also admissible.

3. Certain local isotopies

Let $\mathcal{H}_{\alpha\beta} = (\Sigma, \alpha^{(q+k-1)}, \beta^{(q+k-1)}, w^{(k)})$ be a Heegaard diagram for $S^3$, which possibly is non-admissible, and $S \subset \Sigma$ be an open subsurface. Let $A_{\mathcal{H}_{\alpha\beta}, S} \subseteq \mathcal{G}_{\mathcal{H}_{\alpha\beta}}$ be the set of all the generators, none of whose coordinates lie inside $S$, and let $B_{\mathcal{H}_{\alpha\beta}, S} = \mathcal{G}_{\mathcal{H}_{\alpha\beta}} \setminus A_{\mathcal{H}_{\alpha\beta}, S}$ denote the rest of the generators. Let us assume that $S$ contains a disk $D$ that looks like the first part of Figure 3.1 (with the train track convention): there are $b \beta$ arcs, all parallel to each other, with $b \geq 1$; there are $a_1 + 1 + a_2 \alpha$ arcs, all parallel to each other, with $a_1, a_2 \geq 0$, such that $a_2$ of them are disjoint from the $\beta$ arcs, and each of the $a_1 + 1$ others, intersect each of the $b \beta$ arcs in exactly two points; there is a $w$ marking, such that the oriented boundary of the component of $D \setminus (\alpha \cup \beta)$ containing the $w$ marking, is an $\alpha$ arc followed by a $\beta$ arc followed by an arc in $\partial D$. Note that these $b \beta$ arcs need not belong to $b$ different $\beta$ circles, and these $a_1 + 1 + a_2 \alpha$ arcs need not belong to $a_1 + 1 + a_2$ different $\alpha$ circles.

Let $\mathcal{H}_{\gamma\beta} = (\Sigma, \gamma, \beta, w)$ be the Heegaard diagram (also possibly non-admissible) obtained from $\mathcal{H}$ after the local isotopy as shown in Figure 3.1. The surface $\Sigma$, the $\beta$ circles, the $w$ markings, and the subsurface $S$ are unchanged. The $\alpha$ circles are replaced by the $\gamma$ circles, which for the most part, are small perturbations of the corresponding $\alpha$ circles, except for one of the arcs in the disk $D$. There are $a_1 + 1 + a_2 \gamma$ arcs in $D \subset S \subset \Sigma$ in $\mathcal{H}_{\gamma\beta}$, of which $1 + a_2$ of them are disjoint from the $\beta$ arcs, and each of the remaining $a_1$ of them intersect each of the $b \beta$ arcs in exactly two points. This is shown in the second part of Figure 3.1, with the $\gamma$ train tracks and arcs being denoted by thick and thin pink lines respectively. Once again, let $A_{\mathcal{H}_{\gamma\beta}, S} \subseteq \mathcal{G}_{\mathcal{H}_{\gamma\beta}}$ be the set of all the generators, none of whose coordinates lie inside $S$, and let $B_{\mathcal{H}_{\gamma\beta}, S} = \mathcal{G}_{\mathcal{H}_{\gamma\beta}} \setminus A_{\mathcal{H}_{\gamma\beta}, S}$ denote the rest of the generators. There is an obvious bijection $A_{\mathcal{H}_{\alpha\beta}, S} \cong A_{\mathcal{H}_{\gamma\beta}, S}$, and we will always
We will prove the contrapositive of the statement. The basic idea is that any positive domain
from $A$ we will prove this by showing that $s + m_i - m_0 \geq s + u_i$. implicitly identify them by this bijection; there is an obvious injection $B_{H_{x,y}} \hookrightarrow B_{H_{x,y}}$, and we will always implicitly treat $B_{H_{x,y}}$ as a subset of $B_{H_{x,y}}$ by this injection.

**Proposition 3.1.** If there are no empty positive domains from $A_{H_{x,y}}$ to $B_{H_{x,y}}$, then there are no empty positive domains from $A_{H_{x,y}}$ to $B_{H_{x,y}}$.

**Proof.** We will prove the contrapositive of the statement. The basic idea is that any positive domain from $A_{H_{x,y}}$ to $B_{H_{x,y}}$ induces a corresponding positive domain in the original diagram, simply by tracing multiplicities through the reversal of the isotopy. To make this precise, let us assume that $D_1 \in D_0(x,y)$ is a positive domain, from some generator $x \in A_{H_{x,y}}$ to some generator $y \in B_{H_{x,y}}$. Let $\bar{x} \in A_{H_{x,y}}$ and $\bar{y} \in B_{H_{x,y}}$ be the images of $x$ and $y$ under the bijection $A_{H_{x,y}} \to A_{H_{x,y}}$ and the injection $B_{H_{x,y}} \hookrightarrow B_{H_{x,y}}$, respectively. The empty domain $D_1$ gives rise to another empty domain $D_2 \in D_0(\bar{x}, \bar{y})$. The local coefficients of $D_1$ and $D_2$ are shown in Figure 3.2. We will simply show that $D_2$ is also a positive domain.

Towards this end, we will prove that none of the coefficients $m_1, \ldots, m_b$ are smaller than the coefficient $m_0$. We will prove this by showing that $u_i \leq m_i - m_0$, for all $0 \leq i \leq b$. Since each $u_i \geq 0$, this will complete the proof.

We will prove $u_i \leq m_i - m_0$ by an induction on $i$. Since, $u_0 = 0 = m_0 - m_0$, the base case is trivial. Assume by induction that the statement is true for $i$. Before we prove the statement for $i + 1$, let us make one small observation about the domain $D_1$.

Since $\partial(\partial D_1 \cap \beta) = x - y$, and since none of the coordinates of $x$ lie in the disk $D$, we therefore have that $\partial(\partial D_1 \cap \beta)$, viewed as a 0-chain, does not contain any point with positive sign in the local neighborhood $D$. Let $\tau$ be an oriented arc, which is a subspace of a $\beta$ circle, and is supported entirely inside $D$. Let the coefficient of $\partial D_1 \cap \beta$ near the beginning of $\tau$ be $c > 0$. The observation that $\partial(\partial D_1 \cap \beta)$ does not contain any positively signed point in $D$, implies that the coefficient of $\partial D_1 \cap \beta$ near the end of $\tau$ is greater than or equal to $c$. We summarize this observation by the statement that “$\partial D_1 \cap \beta$ does not stop inside $D$”.

Now, we are all set to prove the induction statement for $i + 1$. In the Heegaard diagram $H_{x,y}$, let $\beta_1$ be the $\beta$ circle that separates the elementary region with coefficient $m_{i+1}$ from the elementary region with coefficient $m_i$, and let $\tau$ be an oriented subarc of $\beta_1$, running from the elementary region with coefficient $m_i$ to the elementary region with coefficient $u_i$. Therefore, the signed coefficients of $\partial D_1 \cap \beta$ near the beginning
Figure 3.3. **The induction step.** Assuming \( u_i \leq m_i - m_0 \), we prove \( u_{i+1} \leq m_{i+1} - m_0 \) by showing \( u_{i+1} - u_i \leq m_{i+1} - m_i \). The coefficients of \( D_1 \) and the coefficients of \( \partial D_1 \cap \beta \) are shown.

and the end of \( \tau \) are \( m_i - m_{i+1} \) and \( u_i - u_{i+1} \) respectively, as shown in Figure 3.3. In light of the observation above that \( \partial D_1 \cap \beta \) does not stop inside \( D \), it follows that coefficient of \( \partial D_1 \cap \beta \) near the end of \( \tau \) is greater than or equal to that at the end:

\[
m_i - m_{i+1} \leq u_i - u_{i+1}.
\]

Subtracting \( m_0 \) from both sides of the inequality, and rearranging, we have:

\[
u_{i+1} + m_i - m_0 - u_i \leq m_{i+1} - m_0
\]

But the inductive hypothesis is that \( 0 \leq m_i - m_0 - u_i \), hence \( u_{i+1} \leq m_{i+1} - m_0 \), as desired.

Now, we prove a similar statement for triangular domains. Let \( \mathcal{H}_{\alpha \beta \gamma} = (\Sigma, \alpha, \beta, \gamma, w) \) be the triple Heegaard diagram, obtained by combining the above two diagrams; it is also possibly non-admissible. We
assume that \( \gamma_i \) is a small translate of \( \alpha_i \), intersecting it transversely in exactly two points, so that \( \gamma_i \) is disjoint from \( \alpha_j \) for \( i \neq j \). Let \( \alpha_1 \) be the \( \alpha \) circle that is changed to the \( \gamma \) circle \( \gamma_1 \) in Figure 3.4. Therefore, a neighborhood \( N_i \) of \( \alpha_i \) \( \cup \gamma_i \) for \( i \neq 1 \) looks like the first part of Figure 3.5, with none of the two intersection points in \( \alpha_i \cap \gamma_i \) lying in the neighborhood \( D \). A neighborhood \( N_1 \) of \( \alpha_1 \) \( \cup \gamma_1 \) looks like the second part of Figure 3.5, with both the intersection points in \( \alpha_1 \cap \gamma_1 \) lying in the neighborhood \( D \). The coordinates of the top generator \( \theta \) are shown.

**Proposition 3.2.** If there are no empty positive domains from \( \mathcal{A}_{H_{\alpha, \beta}} \) to \( \mathcal{B}_{H_{\alpha, \beta}} \), there are no empty positive (triangular) domains from \( \mathcal{A}_{H_{\alpha, \beta}} \) to \( \mathcal{B}_{H_{\alpha, \beta}} \).

**Proof.** Let \( D_1 \in \mathcal{T}_0(x, y) \) be a positive triangular domain, for some \( x \in \mathcal{A}_{H_{\alpha, \beta}} \) and \( y \in \mathcal{B}_{H_{\alpha, \beta}} \). Let \( \tilde{y} \in \mathcal{B}_{H_{\gamma, \beta}} \) be the image of \( y \) under the injection \( \mathcal{B}_{H_{\gamma, \beta}} \hookrightarrow \mathcal{B}_{H_{\alpha, \beta}} \). It is easy to see that there is a unique empty triangular domain \( D_2 \in \mathcal{T}_0(\tilde{y}, y) \), whose non-zero coefficients are supported inside the neighborhoods \( N_i \), such that \( \partial D_2 \cap \gamma = \partial D_1 \cap \gamma \). Then, the 2-chain \( D_3 = D_1 - D_2 \) is a domain in \( D_0(x, \tilde{y}) \). We will show that \( D_3 \) is also a positive domain, establishing the contrapositive of the given statement.

The coefficients of \( D_2 \) are zero outside \( \cup_i N_i \), and the neighborhoods \( N_i \) for \( i \neq 1 \) can be considered as special cases of the neighborhood \( N_1 \). Therefore, we only need to concentrate on the coefficients of \( D_3 \) in \( N_1 \). Figure 3.6 shows the coefficients of \( D_1 \), \( \partial D_1 \cap \gamma_1 \) and \( D_3 \) in this neighborhood. The coordinates of \( \theta \) and \( y \) on \( \gamma_1 \) are shown. The coefficient of \( \partial D_1 \cap \gamma_1 \) is either \( r \) or \( r + 1 \) (which need not be positive), as shown.

Since \( D_1 \) is a positive domain, and since the region with coefficient \( q_\ell \) in \( D_1 \) has \( \gamma_1 \) on its boundary with coefficient \( r + 1 \), we must have \( q_\ell \geq r + 1 \) for all \( 1 \leq \ell \leq d \). Similarly, we must have \( p_\ell \geq r \) for all \( 1 \leq \ell \leq c \) and \( m_0 \geq -r \). Therefore, in order to show that \( D_3 \) is also a positive domain, we only need to show that each of the coefficients \( m_1, \ldots, m_b \) are greater than or equal to \( -r \). However, exactly as in the proof of Proposition 3.1, using the fact that \( x \) has no coordinates inside the disk \( D \), we can show that none of the coefficients \( m_1, \ldots, m_b \) are smaller than \( m_0 \), and this completes the proof.

Let us henceforth assume that the Heegaard diagram \( \mathcal{H}_{\gamma, \beta} \) is admissible. Then \( \mathcal{H}_{\alpha, \beta} \) and \( \mathcal{H}_{\alpha, \gamma} \) are also admissible, and in that case, \( \mathcal{CF}_{\mathcal{H}_{\alpha, \beta}} \) and \( \mathcal{CF}_{\mathcal{H}_{\gamma, \beta}} \) are the chain complexes, freely generated over \( \mathbb{F}_2 \), by \( \mathcal{G}_{\mathcal{H}_{\alpha, \beta}} \).
Figure 3.6. Coefficients of $D_1$ (left) and $D_3$ (right) in the neighborhood $N_1$ of $\alpha_1 \cup \gamma_1$. The coordinates of $\theta$ and $y$ are shown by white dots. The coefficients of $\partial D_1 \cap \gamma_1$ are also shown on the left.

and $G_{\gamma_3}$, respectively. Let $\widehat{SF}_{H_{\alpha \beta},S}$ and $\widehat{SF}_{H_{\gamma \beta},S}$ be the $\mathbb{F}_2$-submodules, freely generated by $A_{H_{\alpha \beta},S}$ and $A_{H_{\gamma \beta},S}$, respectively.

**Theorem 3.3.** Assume that there are no empty positive domains from $A_{H_{\alpha \beta},S}$ to $B_{H_{\alpha \beta},S}$. Then $\widehat{SF}_{H_{\alpha \beta},S}$ is a subcomplex of $\widehat{CF}_{H_{\alpha \beta}}$, $\widehat{SF}_{H_{\gamma \beta},S}$ is a subcomplex of $\widehat{CF}_{H_{\gamma \beta}}$, and the chain map from $\widehat{CF}_{H_{\alpha \beta}}$ to $\widehat{CF}_{H_{\gamma \beta}}$ induces a chain map from $\widehat{SF}_{H_{\alpha \beta},S}$ to $\widehat{SF}_{H_{\gamma \beta},S}$. Furthermore, the chain maps $\widehat{CF}_{H_{\alpha \beta}} \to \widehat{CF}_{H_{\gamma \beta}}$ and $\widehat{SF}_{H_{\alpha \beta},S} \to \widehat{SF}_{H_{\gamma \beta},S}$ are quasi-isomorphisms.

**Proof.** Proposition 3.1 implies that there are no empty positive domains from $A_{H_{\alpha \beta},S}$ to $B_{H_{\alpha \beta},S}$, and Proposition 3.2 implies that there are no empty positive triangular domains from $A_{H_{\alpha \beta},S}$ to $B_{H_{\alpha \beta},S}$. Since the non-zero terms in the boundary maps on $\widehat{CF}_{H_{\alpha \beta}}$ and $\widehat{CF}_{H_{\gamma \beta}}$ come only from empty positive domains, and the non-zero terms in the chain map from $\widehat{CF}_{H_{\alpha \beta}}$ to $\widehat{CF}_{H_{\gamma \beta}}$ come only from empty positive triangular domains, $\widehat{SF}_{H_{\alpha \beta},S} \hookrightarrow \widehat{CF}_{H_{\alpha \beta}}$ is a subcomplex, $\widehat{SF}_{H_{\gamma \beta},S} \hookrightarrow \widehat{CF}_{H_{\gamma \beta}}$ is a subcomplex, and the chain map $\widehat{CF}_{H_{\alpha \beta}} \to \widehat{CF}_{H_{\gamma \beta}}$ induces a chain map $\widehat{SF}_{H_{\alpha \beta},S} \to \widehat{SF}_{H_{\gamma \beta},S}$, resulting in the following commuting square.

$$
\begin{array}{ccc}
\widehat{SF}_{H_{\alpha \beta},S} & \hookrightarrow & \widehat{CF}_{H_{\alpha \beta}} \\
\downarrow & & \downarrow \\
\widehat{SF}_{H_{\gamma \beta},S} & \hookrightarrow & \widehat{CF}_{H_{\gamma \beta}}
\end{array}
$$

We will now show that the vertical arrows induce isomorphisms on homology. A 2-cochain on a Heegaard diagram or a triple Heegaard diagram is a map which assigns real numbers to the elementary regions; a non-negative 2-cochain is a 2-cochain which only assigns non-negative numbers; and a positive 2-cochain is a 2-cochain which only assigns positive numbers. Since $H_{\gamma \beta}$ is admissible, by [OS04c, Lemma 4.12], there exists a positive 2-cochain $C_0$ on $H_{\gamma \beta}$, which evaluates to zero on all empty periodic domains in $H_{\gamma \beta}$.
A cochain in $\mathcal{H}_{\alpha\beta\gamma}$ induces cochains in $\mathcal{H}_{\alpha\beta}$ and $\mathcal{H}_{\gamma\beta}$, by forgetting the $\gamma$ circles and the $\alpha$ circles, respectively, as well as the coefficients of the cochain on the thin elementary regions that lie entirely inside the neighborhoods $N_i$. We will now construct a non-negative cochain $C$ on $\mathcal{H}_{\alpha\beta\gamma}$, such that: $C$ assigns zero precisely to the elementary regions that lie entirely in the neighborhoods $N_i$; and some of them might lie entirely in the neighborhoods $N_i$, but at least one of them does not. Choose non-negative real numbers $r_1, \ldots, r_n$, such that, $\sum_i r_i = r$ and $r_i = 0$ if and only if $R_i$ lies entirely in $\cup_i N_i$. Then assign the number $r_i$ to the elementary region $R_i$ in the 2-cochain $C$.

This non-negative 2-cochain $C$ gives rise to filtrations on the mapping cones $\tilde{SF}_{\mathcal{H}_{\alpha\beta}, S} \rightarrow \tilde{SF}_{\mathcal{H}_{\gamma\beta}, S}$ and $\tilde{CF}_{\mathcal{H}_{\alpha\beta}} \rightarrow \tilde{CF}_{\mathcal{H}_{\gamma\beta}}$, as follows: given any two generators $x, y \in G_{\mathcal{H}_{\alpha\beta}} \cup G_{\mathcal{H}_{\gamma\beta}}$, the relative filtration grading between them is $(C, D)$, for any $D$ in $D_0(x, y)$ or $D_0(x, y)$ as the case may be. On the associated graded level, we only count domains or triangular domains that lie entirely inside these neighborhoods $N_i$, and then it is a fairly straightforward to check that the associated graded maps on the associated graded objects are isomorphisms, and therefore, the original chain maps must have been quasi-isomorphisms as well. 

\section{4. Main Theorems}

\textbf{Proof of Theorem 1.1.} Following the notations from Section 2.4, let $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_1 \ast \mathcal{H}_2$, as shown in Figures 2.4 and 2.5, be the Heegaard diagrams adapted to Seifert surfaces $R_1$, $R_2$ and $R_1 \ast R_2$ for the links $L_1$, $L_2$ and $L = L_1 \ast L_2$, respectively. The corresponding Heegaard surfaces contain embedded subsurfaces $S_1$, $S_2$ and $S_1 \ast S_2$, which represent $R_1$, $R_2$, and $R_1 \ast R_2$, respectively. Furthermore, $\mathcal{H}_1$ has $(l_1 + \delta_1)$ $w$-markings, $\mathcal{H}_2$ has $(l_2 + \delta_2)$ $w$-markings.

Thanks to Corollary 2.9, we only need to produce an isomorphism of graded chain complexes

\begin{equation}
\mathcal{F}_{\mathcal{H}_1 \ast \mathcal{H}_2} \left( - \frac{1}{2}(l + 2\delta - \chi(R_1 \ast R_2)) \right) \llbracket l + \delta - 1 \rrbracket \cong \mathcal{F}_{\mathcal{H}_1} \left( - \frac{1}{2}(l_1 + 2\delta_1 - \chi(R_1)) \right) \llbracket l_1 + \delta_1 - 1 \rrbracket \otimes \mathcal{F}_{\mathcal{H}_2} \left( - \frac{1}{2}(l_2 + 2\delta_2 - \chi(R_2)) \right) \llbracket l_2 + \delta_2 - 1 \rrbracket,
\end{equation}

or in terms of the notation from Section 3, since $(l + \delta - n) = (l_1 + \delta_1 - n) + (l_2 + \delta_2 - n),

\begin{equation}
\tilde{SF}_{\mathcal{H}_1 \ast \mathcal{H}_2, S_1 \ast S_2} \llbracket n - 1 \rrbracket \cong \tilde{SF}_{\mathcal{H}_1, S_1} \llbracket n - 1 \rrbracket \otimes \tilde{SF}_{\mathcal{H}_2, S_2} \llbracket n - 1 \rrbracket.
\end{equation}

Let $(\mathcal{H}, S)$ denote any of $(\mathcal{H}_1, S_1)$, $(\mathcal{H}_2, S_2)$ or $(\mathcal{H}_1 \ast \mathcal{H}_2, S_1 \ast S_2)$, as shown in the first row of Figure 4.1. The sphere $S^2$ of the Murasugi sum is represented by the squares on the page, with $L_1$ lying below the page and $L_2$ lying above. Let $\Sigma$ be the Heegaard surface and let $\Sigma^1$ (respectively, $\Sigma^2$) be the portion of $\Sigma$ that lies inside (respectively, outside) the sphere $S^2$.

Let $\mathcal{H}'$ be the Heegaard diagrams for $S^3$ obtained from $\mathcal{H}$ by forgetting the $z$ markings, and let $S'$ be the open subsurface of the Heegaard surface of $\mathcal{H}$, obtained by slightly modifying $S$ in a neighborhood of the $2n$-gon $A_1A_2 \ldots A_{2n}$ so as to include all the intersections between $\alpha$ and $\beta$ circles near the erstwhile $z$-markings. These modified diagrams $(\mathcal{H}', S')$ are shown in the second row of Figure 4.1.

Since we have not changed the underlying Heegaard diagrams, clearly, $\tilde{CF}_{\mathcal{H}} = \tilde{CF}_{\mathcal{H}}$. Next we claim that any generator $x \in G_{\mathcal{H}}$ that does not have any coordinate in $S$ can not have any coordinate in $S'$ either. This is a simple counting argument. Indeed, let us say $\mathcal{H}_i$ has a total of $n_i$ $\alpha$-circles and $n_i$ $\beta$-circles; of them, exactly $(n - 1)$ $\alpha$-circles and $(n - 1)$ $\beta$-circles lie entirely inside the $S^2$. Then $\mathcal{H}_1 \ast \mathcal{H}_2$ has a total of
Figure 4.1. The Heegaard diagrams appearing in the proof of Theorems 1.1 and 1.2. The left, middle, right columns represent diagrams obtained from \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1*\mathcal{H}_2 \), respectively; the consecutive rows represent the diagrams \( \mathcal{H}, \mathcal{H}', \mathcal{H}'', \) and \( \mathcal{H}''' \); X and O denote handles going down (into the page) and up (towards the reader), respectively.
\( (n_1 + n_2 - n + 1) \alpha\)-circles and \((n_1 + n_2 - n + 1) \beta\)-circles, again with exactly \((n - 1) \alpha\)-circles and \((n - 1) \beta\)-circles lying entirely inside the \(S^2\). For \(\mathcal{H} \neq \mathcal{H}_2\), we see that \((n_1 - n + 1) \alpha\) circles of \(\mathcal{H}\) lie entirely within \(\Sigma^1 \cup S\), and so \(x\) must have at least \((n_1 - n + 1) \alpha\) coordinates in \(\Sigma^1\) (as it avoids the surface \(S\), by assumption). Similarly, for \(\mathcal{H} \neq \mathcal{H}_1\), we see that \((n_2 - n + 1) \beta\) circles of \(\mathcal{H}\) lie entirely within \(\Sigma^2 \cup S\), and so \(x\) must have at least \((n_2 - n + 1) \beta\) coordinates in \(\Sigma^2\). Therefore, in all cases, \(x\) has at most \((n - 1) \alpha\) and \((n - 1) \beta\) circles that lie entirely therein. Specifically, they must be the white or black dots as shown in the first row of Figure 4.1. Therefore, we get that \(\tilde{\mathcal{SF}}_{\mathcal{H},S} = \tilde{\mathcal{SF}}_{\mathcal{H},S'}\). Moreover, since there are no positive domains from generators of \(\tilde{\mathcal{SF}}_{\mathcal{H},S}\) to the other generators of \(\tilde{\mathcal{CF}}_{\mathcal{H}}\) due to Alexander grading, there are no positive domains from the generators of \(\tilde{\mathcal{SF}}_{\mathcal{H}',S'}\) to the generators of \(\tilde{\mathcal{CF}}_{\mathcal{H}'}\) as well, and we have the following identification of subcomplexes.

\[
\begin{align*}
\tilde{\mathcal{SF}}_{\mathcal{H},S} & \hookrightarrow \tilde{\mathcal{CF}}_{\mathcal{H}} \\
\text{q.i.} & \quad \text{q.i.} \\
\tilde{\mathcal{SF}}_{\mathcal{H}',S'} & \hookrightarrow \tilde{\mathcal{CF}}_{\mathcal{H}'}
\end{align*}
\]

As in Section 3, we perform local isotopies to separate the \(\alpha\) and \(\beta\) curves in the neighborhood of the erstwhile \(z\)-markings so that we obtain the Heegaard diagrams in the third row of Figure 4.1, which we denote by \((\mathcal{H}'', S')\). The aforementioned isotopies are supported inside \(S'\), and there are no domains from the generators of \(\tilde{\mathcal{SF}}_{\mathcal{H}',S'}\) to the other generators of \(\tilde{\mathcal{CF}}_{\mathcal{H}''}\). Recalling that the Heegaard diagrams were constructed to ensure that \(\mathcal{H}''\) is admissible (see (M-7)), we see that the hypotheses of Propositions 3.1 and 3.2 are satisfied. Applying the propositions, we obtain a quasi-isomorphism between the following two-step filtered complexes.

\[
\begin{align*}
\tilde{\mathcal{SF}}_{\mathcal{H}',S'} & \hookrightarrow \tilde{\mathcal{CF}}_{\mathcal{H}''} \\
\text{q.i.} & \quad \text{q.i.} \\
\tilde{\mathcal{SF}}_{\mathcal{H}'',S''} & \hookrightarrow \tilde{\mathcal{CF}}_{\mathcal{H}'''}
\end{align*}
\]

Next we claim that for any generator \(x\) of \(\tilde{\mathcal{CF}}_{\mathcal{H}''}\), all its coordinates in \(S^2\) must be the white or black dots from the third row of Figure 4.1. It is once again a counting argument, but with the roles of \(\alpha\) and \(\beta\) reversed. For \(\mathcal{H}'' \neq \mathcal{H}_2''\), we see that \((n_1 - n + 1) \beta\) circles of \(\mathcal{H}''\) have no intersections with \(\alpha\) circles in \(S^2\), so \(x\) must have at least \((n_1 - n + 1) \beta\) coordinates in \(\Sigma^1\). Similarly, for \(\mathcal{H}'' \neq \mathcal{H}_1''\), we see that \((n_2 - n + 1) \alpha\) circles of \(\mathcal{H}''\) have no intersections with \(\beta\) circles in \(S^2\), so \(x\) must have at least \((n_2 - n + 1) \alpha\) coordinates in \(\Sigma^2\). In all cases, \(x\) has at most \((n - 1) \alpha\) and \((n - 1) \beta\) circles that lie entirely in \(S^2\); moreover, they must be the white or black dots as shown in the third row of Figure 4.1.

After numbering the \(\alpha\) circles that lie entirely in \(S^2\) arbitrarily (but consistently across Heegaard diagrams) from 1 to \(n - 1\), each generator \(x\) can be represented as a pair \((\vec{a}, x^o)\), where \(\vec{a} = (a_1, \ldots, a_{n-1}) \in \{0,1\}^{n-1}\) with \(a_i = 0\) if and only if \(a_i\) contains the white dot, and \(x^o\) denotes the coordinates of \(x\) that lie outside \(S^2\). Consider the usual partial order on \(\{0,1\}^{n-1}\) with \(\vec{a} \leq \vec{b}\) if \(a_i \leq b_i\) for all \(i\), and the usual \(L^1\)-norm on \(\{0,1\}^{n-1}\) given by \(|\vec{a}| = \sum_i a_i\). Since empty positive domains are not allowed to pass through the \(w\) markings, we see that if \(D \in \mathcal{D}_0((\vec{a}, x^o), (\vec{b}, y^o))\) is an empty positive domain, then \(\vec{b} \leq \vec{a}\). Let \(\tilde{\mathcal{CF}}_{\mathcal{H}'''}^W\) denote the subcomplex of \(\tilde{\mathcal{CF}}_{\mathcal{H}'''}\) spanned by generators with \(\vec{a} = 0\), that is, generators with only white dots.
Then we have nested subcomplexes
\[ \widetilde{SF}_{\mathcal{H}, \mathcal{S}} \hookrightarrow \widetilde{CF}^W_{\mathcal{H}} \hookrightarrow \widetilde{CF}_{\mathcal{H}}. \]

For any generator \( x \) of \( \widetilde{CF}_{(\mathcal{H}_1 \ast \mathcal{H}_2), \mathcal{S}} \), let \( x^i \) be all its coordinates that lie in \( \Sigma^i \). Then the map
\[(\vec{0}, x^1, x^2) \mapsto (\vec{0}, x^1) \otimes (\vec{0}, x^2)\]
produces the following identification between the following chain groups:
\[
\begin{align*}
\widetilde{SF}_{(\mathcal{H}_1 \ast \mathcal{H}_2), \mathcal{S}} & \\ \cong & \\ \cong \\
\widetilde{SF}_{\mathcal{H}_1^i, \mathcal{S}_i} \otimes \widetilde{SF}_{\mathcal{H}_2^j, \mathcal{S}_j} & \quad \cong & \quad \cong \\
\widetilde{CF}^W_{\mathcal{H}_1^i} \otimes \widetilde{CF}^W_{\mathcal{H}_2^j}. 
\end{align*}
\]

Let us now prove that the vertical arrows are relative Maslov grading preserving chain maps—that is, the above identifications are identifications of chain complexes, up to a single absolute Maslov grading shift. Let \( (\vec{0}, x^1, x^2) \) and \( (\vec{0}, y^1, y^2) \) be two generators of \( \widetilde{CF}^W_{(\mathcal{H}_1 \ast \mathcal{H}_2), \mathcal{S}} \) and let \( D \in \mathcal{D}((\vec{0}, x^1, x^2), (\vec{0}, y^1, y^2)) \) be an empty positive domain in \( (\mathcal{H}_1 \ast \mathcal{H}_2, \mathcal{S}) \) connecting them. Such a domain has to be disjoint from \( S^2 \). Indeed, the fact the domain has no corner points in \( S^2 \) forces it to restrict to a periodic domain therein, which the admissibility condition then ensures is the trivial domain with zero multiplicities. It follows that \( D \) splits as a disjoint union \( D^1 \cup D^2 \) of empty positive domains, with \( D^i \in \mathcal{D}((\vec{0}, x^i), (\vec{0}, y^i)) \) in \( \mathcal{H}_i^i \). Recall from Section 2.2 that \( D \) can only contribute to the differential if one of \( D^1 \) and \( D^2 \) is the trivial domain. Furthermore, since such domains avoid \( S^2 \), we may choose complex structures (and their perturbations) for the three Heegaard diagrams so that they agree on \( \Sigma^1 \) and \( \Sigma^2 \) (and their corresponding symmetric products); therefore, if \( D^1 \) (respectively, \( D^2 \)) is the trivial domain, then the contribution of \( D \) will agree with the contribution of \( D^2 \) (respectively, \( D^1 \)). This is an instance of the localization principle [Ras03, Section 9.4], and it establishes that the vertical arrows are chain maps, and indeed chain isomorphisms. To see that the vertical arrows also respect the relative Maslov gradings, consider generators \((0, x^1), (0, y^1)\) of \( \widetilde{CF}^W_{\mathcal{H}_1^i} \), and choose empty domains \( D^i \) in \( \mathcal{H}_1^i \) (not necessarily positive) connecting them. In \( \mathcal{H}_1^i \) (respectively, \( \mathcal{H}_2^j \)) consider the \((n-1)\) \( \alpha \) (respectively, \( \beta \)) circles that lie entirely inside \( S^2 \); each of them bounds a disk also entirely inside \( S^2 \), and each such disk is comprised of two bigon-shaped elementary regions—one containing a basepoint, and one without. By adding some number of copies of these disks to the domain \( D^i \), we can get a domain \( E^i \) (not necessarily empty) connecting \((0, x^i)\) to \((0, y^i)\) in \( \mathcal{H}_i^i \), which has coefficient zero in the bigon region that does not contain the basepoint. Simply by adding the underlying 2-chains, these two domains \( E^1 \) and \( E^2 \) induce a domain \( E \) in \( (\mathcal{H}_1 \ast \mathcal{H}_2)^{\prime \prime} \) connecting \((0, x^1, y^1)\) to \((0, x^2, y^2)\). It follows from Lipshitz’ Maslov index formula [Lip06] that \( \mu(E) = \mu(E^1) + \mu(E^2) \), and it is immediate that \( n_w(E) = n_w(E^1) + n_w(E^2) \). Consequently, the relative Maslov grading is preserved. Therefore, in order to finish the proof, we only need to calculate the absolute Maslov grading shift under the given isomorphism of relatively \( \mathbb{Z} \)-graded chain complexes. We will calculate this shift using the triangle maps associated to handleslides of the circles in \( S^2 \) over curves in the remainder of the diagram.

Towards this end, modify the Heegaard diagrams \( \mathcal{H}^{i \prime} \) once more to get the Heegaard diagrams \( \mathcal{H}^{i \prime \prime} \) of the fourth row of Figure 4.1. Namely, we slide the \( \alpha \) circles inside \( S^2 \) off the attaching handles of \( \Sigma^2 \) (the ones marked O in Figure 4.1) and we slide the \( \beta \) circles inside \( S^2 \) off the attaching handles of \( \Sigma^1 \) (the ones marked X in Figure 4.1). There is an obvious identification between \( \alpha \) circles, \( \beta \) circles, and generators \((\vec{a}, x^i)\) of \( \mathcal{H}^{i \prime \prime} \) and the corresponding objects of \( \mathcal{H}^{i \prime \prime \prime} \); let \( \vec{\alpha}, \vec{\beta} \), and \((\vec{a}, x^i)\) denote the corresponding objects in \( \mathcal{H}^{i \prime \prime \prime} \). Then each \( \alpha \) circle only intersects the corresponding \( \vec{\alpha} \) circle, and does so at two points. So the Heegaard
diagram $(\Sigma, \alpha, \beta, \bar{\alpha})$ is of the type as described in Section 2.3. The top generator $\theta$ has coordinates just next the white dots of $\mathcal{H}''$ and $\mathcal{H}'''$, and indeed, there is a small Maslov index zero triangular domain connecting $(\bar{0}, x^o)$ and $(\bar{0}, \bar{x}^o)$. Therefore, the Maslov grading of $(\bar{0}, x^o)$ is same as the Maslov grading of $(\bar{0}, \bar{x}^o)$. A similar argument, but with the roles of $\alpha$ and $\beta$ reversed, proves that the Maslov grading is preserved under the handleslides of the $\beta$-circles as well.

Now $\widehat{\defining} \mathcal{H}'''$ decomposes into $2^{n-1}$ direct summands, one for each $\bar{a} \in \{0,1\}^{n-1}$. Moreover, the map $(0,x^o) \mapsto (\bar{a}, x^o)$ is an isomorphism between $\widehat{\defining} \mathcal{H}'''[\bar{a}]$ and the summand corresponding to $\bar{a}$. Let $\mathcal{H}''_d$ denote the Heegaard diagram destabilized $n - 1$ times, obtained from $\mathcal{H}'''$ by removing the $(n - 1)$ $\alpha$ and $\beta$ circles that lie inside $S^2$, and the $(n - 1)$ $w$-markings enclosed by them. Then by Theorem 2.5,

$$\widehat{\defining} \mathcal{H}'''[n-1] \cong \widehat{\defining} \mathcal{H}''_d$$

via the map $(\bar{0}, x^o) \mapsto x^o$.

Now the map $(x^1, x^2) \mapsto x^1 \otimes x^2$ produces an identification of chain groups following a similar but simpler argument of Equation 4.4:

$$\widehat{\defining} (\mathcal{H}_1 \ast \mathcal{H}_2)''_d \cong \widehat{\defining} \mathcal{H}_1'' \otimes \widehat{\defining} \mathcal{H}_2''.$$

Moreover, this map is a relative Maslov grading preserving chain map, using a similar (but easier) localization principle argument to that above. However, $(\mathcal{H}_1)''_d$, $(\mathcal{H}_2)''_d$, and $(\mathcal{H}_1 \ast \mathcal{H}_2)''_d$ are Heegaard diagrams for $S^3$ with $(l_1 + \delta_1 - n + 1)$, $(l_2 + \delta_2 - n + 1)$, and $(l + \delta - n + 1)$ basepoints, respectively; therefore, by Theorem 2.5 (and since $(l_1 + \delta_1 - n) + (l_2 + \delta_2 - n) = (l + \delta - n)$), either side of the equation has homology $\otimes_{l+\delta}(\mathbb{F}_2 \oplus \mathbb{F}_2[-1])$, so the chain isomorphism (4.6) preserves absolute Maslov grading as well.

Combining this with Equation (4.5) and the previous fact that corresponding generators in $\mathcal{H}''$ and $\mathcal{H}'''$ have equal Maslov gradings, we conclude that the isomorphism from Equation (4.4) shifts gradings by $n - 1$. Then with the aid of Equations (4.2) and (4.3), we arrive at the desired graded isomorphism Equation (4.1).

We turn now to Theorem 1.2, which states that $\tau^\top$ of a Murasugi sum is maximal if and only if $\tau^\top$ of each summand is maximal.

**Proof of Theorem 1.2.** By Proposition 2.10, we may take mirrors and prove the following statement for $\tau^\bot$: If $L$ is a Murasugi sum of links $L_1$ and $L_2$ along minimal index Seifert surfaces, then $\tau^\bot(L_i) = -g(L_i)$ for all $i \in \{1,2\}$ if and only if $\tau^\bot(L) = -g(L)$.

We will continue from the previous proof, and re-use the same notation. Thanks to Corollary 2.9 and Proposition 2.10, we only need to prove that, for both $i = 1,2$, the inclusion

$$\widehat{\defining} \mathcal{H}_i \hookrightarrow \widehat{\defining} \mathcal{H}_i$$

induces a non-zero map on homology if and only if the inclusion

$$\widehat{\defining} \mathcal{H}_1 \ast \mathcal{H}_2, S_1 \ast S_2 \hookrightarrow \widehat{\defining} \mathcal{H}_1 \ast \mathcal{H}_2$$

induces a non-zero map on homology.

Thanks to Equations (4.2) and (4.3), it is enough to prove the above for $(\mathcal{H}''_i, S'_i)$, that is,

$$\forall i(H_*(\widehat{\defining} \mathcal{H}''_i) \rightarrow H_*(\widehat{\defining} \mathcal{H}''_i) \text{ is non-zero}) \iff H_*(\widehat{\defining} \mathcal{H}''_1 \ast \mathcal{H}''_2, S'_1 \ast S'_2) \rightarrow H_*(\widehat{\defining} \mathcal{H}''_1 \ast \mathcal{H}''_2) \text{ is non-zero}.$$

Recall that we have nested subcomplexes

$$\widehat{\defining} \mathcal{H}'' \hookrightarrow \widehat{\defining} \mathcal{H}'' \hookrightarrow \widehat{\defining} \mathcal{H}''.$$
We claim the map $\widetilde{CF}^W_{H''} \to \widetilde{CF}_{H''}$ is injective on homology for $H'' = H'_1, H''_1$, or $H'_1 \ast H''_1$. For each of the three diagrams, the homology of $\widetilde{CF}^W_{H''}$ is isomorphic, up to a grading shift, to the homology of the $(n - 1)$-times destabilized diagrams $H'_d$ (recall Equation (4.5)); let $\omega$ denote its rank. By counting the number of basepoints, in each of the three cases, the homology of $\widetilde{CF}_{H''}$ has rank $2^{n-1}\omega$. As before, the generators of $\widetilde{CF}_{H''}$ can be represented as pairs $(\vec{a}, x^n)$ where $\vec{a} \in \{0, 1\}^{n-1}$, and the differential is filtered with respect to $\vec{a}$. The associated graded complex of this filtration has $2^{n-1}$ summands, each isomorphic to $\widetilde{CF}_{H''}$. By a spectral sequence argument, the homology of the quotient complex $\widetilde{CF}_{H''}/\widetilde{CF}^W_{H''}$ has rank at most $(2^{n-1} - 1)\omega$. Therefore, in the exact triangle

$$\xymatrix{ H_*(\widetilde{CF}_{H''}) \ar[r] & H_*(\widetilde{CF}_{H''}/\widetilde{CF}^W_{H''}) \ar[l] \ar[r] & H_*(\widetilde{CF}^W_{H''}) \ar[l] }$$

the three terms have ranks $\omega$, $2^{n-1}\omega$, and at most $(2^{n-1} - 1)\omega$, which implies that the map $H_*(\widetilde{CF}^W_{H''}) \to H_*(\widetilde{CF}_{H''})$ is injective.

Thanks to this, instead of Equation (4.7), it is enough to prove

$$\forall i(H_*(\tilde{SF}_{H''}, S^i) \to H_*(\widetilde{CF}_{H''}) \text{ is non-zero}) \Leftrightarrow H_*(\tilde{SF}_{H'' \ast S^1}, S^i) \to H_*(\widetilde{CF}_{H'' \ast S^1}) \text{ is non-zero},$$

which follows from Equation (4.4). \qed

**Remark 4.1.** We remark that one direction of the theorem (maximality of $\tau_{top}$ of a Murasugi sum implies maximality for its summands) could be deduced from the fact that maximality of $\tau_{top}$ is preserved under taking subsurfaces of a minimal index Seifert surfaces. The latter fact is a consequence of a bound satisfied by $\tau_{top}$ for cobordisms between links, analogous to [HR20, Theorem 1].

**References**

[AH] Jared Able and Mikami Hirasawa, Construction and decomposition of knots by Murasugi sums of Seifert surfaces, in preparation.

[Bel10] Anna Beliakova, A simplification of combinatorial link Floer homology, J. Knot Theory Ramifications **19** (2010), no. 2, 125–144. MR 2647050

[BVV18] John Baldwin and David Shea Vela-Vick, A note on the knot Floer homology of fibered knots, Algebr. Geom. Topol. **18** (2018), no. 6, 3669–3690. MR 3868231

[Cav18] Alberto Cavallo, The concordance invariant tau in link grid homology, Algebr. Geom. Topol. **18** (2018), no. 4, 1917–1951. MR 3797061

[Gab83] David Gabai, The Murasugi sum is a natural geometric operation, Low-dimensional topology (San Francisco, Calif., 1981), Contemp. Math., vol. 20, Amer. Math. Soc., Providence, RI, 1983, pp. 131–143. MR 718138

[Gab85] David Gabai, The Murasugi sum is a natural geometric operation. II, Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982), Contemp. Math., vol. 44, Amer. Math. Soc., Providence, RI, 1985, pp. 93–100. MR 813105

[GG06] Emmanuel Giroux and Noah Goodman, On the stable equivalence of open books in three-manifolds, Geom. Topol. **10** (2006), 97–114. MR 2207791

[Hed05a] Matthew Hedden, On knot Floer homology and cabling, Algebr. Geom. Topol. **5** (2005), 1197–1222. MR 2171808

[Hed05b] Matthew Hedden, On knot Floer homology and cabling, ProQuest LLC, Ann Arbor, MI, 2005, Thesis (Ph.D.)–Columbia University. MR 2707267

[Hed08] Matthew Hedden, Some remarks on cabling, contact structures, and complex curves, Proceedings of Gökova Geometry-Topology Conference 2007, Gökova Geometry/Topology Conference (GGT), Gökova, 2008, pp. 49–59. MR 2509749

[Hed09] Matthew Hedden, On knot Floer homology and cabling. II, Int. Math. Res. Not. IMRN (2009), no. 12, 2248–2274. MR 2511910

[HJS13] Matthew Hedden, András Juhász, and Sucharit Sarkar, On sutured Floer homology and the equivalence of Seifert surfaces, Algebr. Geom. Topol. **13** (2013), no. 1, 505–548. MR 3116378
[HR20] Matthew Hedden and Katherine Raoux, Knot Floer homology and relative adjunction inequalities, arXiv e-prints (2020), arXiv:2009.05462.

[HW18] Matthew Hedden and Liam Watson, On the geography and botany of knot Floer homology, Selecta Math. (N.S.) 24 (2018), no. 2, 997–1037. MR 3782416

[Juh08] András Juhász, Floer homology and surface decompositions, Geom. Topol. 12 (2008), no. 1, 299–350. MR 2390347

[Lip06] Robert Lipshitz, A cylindrical reformulation of Heegaard Floer homology, Geom. Topol. 10 (2006), 955–1097. MR 2240908

[MOS09] Ciprian Manolescu, P. Ozsváth, and S. Sarkar, A combinatorial description of knot Floer homology, Ann. of Math. (2) 169 (2009), no. 2, 633–660.

[Mur58] Kunio Murasugi, On the genus of the alternating knot. I, II, J. Math. Soc. Japan 10 (1958), 94–105, 235–248. MR 99664

[Mur63] , On a certain subgroup of the group of an alternating link, Amer. J. Math. 85 (1963), 544–550. MR 157375

[Ni06a] Yi Ni, A note on knot Floer homology of links, Geom. Topol. 10 (2006), 695–713. MR 2240902

[Ni06b] , Sutured Heegaard diagrams for knots, Algebr. Geom. Topol. 6 (2006), 513–537. MR 2220687

[NR87] Walter Neumann and Lee Rudolph, Unfoldings in knot theory, Math. Ann. 278 (1987), no. 1-4, 409–439. MR 909235

[OS03] Peter Ozsváth and Zoltán Szabó, Knot Floer homology and the four-ball genus, Geom. Topol. 7 (2003), 615–639. MR 2026543

[OS04a] , Holomorphic disks and genus bounds, Geom. Topol. 8 (2004), 311–334. MR 2023281

[OS04b] , Holomorphic disks and knot invariants, Adv. Math. 186 (2004), no. 1, 58–116. MR 2065507

[OS04c] , Holomorphic disks and topological invariants for closed three-manifolds, Ann. of Math. (2) 159 (2004), no. 3, 1027–1158. MR 2113019

[OS04d] Peter Ozsváth and Zoltán Szabó, Knot Floer homology, genus bounds, and mutation, Topology Appl. 141 (2004), no. 1-3, 59–85. MR 2058681

[OS08] Peter Ozsváth and Zoltán Szabó, Holomorphic disks, link invariants and the multi-variable Alexander polynomial, Algebr. Geom. Topol. 8 (2008), no. 2, 615–692. MR 2443092

[OS19] Peter Ozsvath and Zoltan Szabo, Algebras with matchings and knot Floer homology, arXiv e-prints (2019), arXiv:1912.01657.

[OS15] Peter S. Ozsváth, András I. Stipsicz, and Zoltán Szabó, Grid homology for knots and links, Mathematical Surveys and Monographs, vol. 208, American Mathematical Society, Providence, RI, 2015. MR 3381987

[OS09] Peter Ozsváth and Zoltán Szabó, Holomorphic disks and three-manifold invariants: properties and applications, Ann. of Math. (2) 159 (2004), no. 3, 1159–1245. MR 2113020 (2006b:57017)

[Ras03] Jacob Andrew Rasmussen, Floer homology and knot complements, ProQuest LLC, Ann Arbor, MI, 2003, Thesis (Ph.D.)–Harvard University. MR 2058507

[Rud87] Lee Rudolph, Isolated critical points of mappings from $\mathbb{R}^4$ to $\mathbb{R}^2$ and a natural splitting of the Milnor number of a classical fibered link. I. Basic theory; examples, Comment. Math. Helv. 62 (1987), no. 4, 630–645. MR 920062

[Sta78] John R. Stallings, Constructions of fibred knots and links, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978, pp. 55–60. MR 520522

[The94] Abigail Thompson, A note on Murasugi sums, Pacific J. Math. 163 (1994), no. 2, 393–395. MR 1262303