Quaternionic contact 4n + 3-manifolds and their 4n-quotients

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Received: 3 July 2020 / Accepted: 16 January 2021 / Published online: 3 March 2021
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Abstract
We study some types of qc-Einstein manifolds with zero qc-scalar curvature introduced by S. Ivanov and D. Vassilev. Secondly, we shall construct a family of quaternionic Hermitian metrics \((g_a, \{J_a\}_{a=1}^3)\) on the domain \(Y\) of the standard quaternion space \(\mathbb{H}^n\) one of which, say \((g_a, J_1)\) is a Bochner flat Kähler metric. To do so, we deform conformally the standard quaternionic contact structure on the domain \(X\) of the quaternionic Heisenberg Lie group \(\mathcal{M}\) to obtain quaternionic Hermitian metrics on the quotient \(Y\) of \(X\) by \(\mathbb{R}^3\).

Keywords Quaternionic contact structure · Quaternionic Hermitian structure · HyperKähler structure · Quaternionic Heisenberg Lie group · Spherical CR structure · Bochner geometry

Mathematics Subject Classification 53C55 · 57S25 · 51M10

1 Introduction
We study the quaternionic contact structure \([3]\) (qc-structure for short) on 4n + 3-manifolds \(X\) to construct quaternionic Hermitian 4n-manifolds as their quotients. In the previous paper \([2]\), we studied a qc-structure \((D, Q)\) whose \(\text{Im} \mathbb{H}\)-valued (globally defined) 1-form \(\omega\) representing \(D\) satisfies that each distribution defined by \(d\omega_a + 2\omega\beta \wedge \omega_\gamma = 0\) (\((\alpha, \beta, \gamma) \sim (1, 2, 3)\)) has the three-dimensional common kernel on \(X\). \((D, \omega, Q)\) is called a quaternionic CR-structure (cf. \([2, \text{Definition 2.1}]\)). It has shown in \([1, 2]\) that every positive definite quaternionic CR-structure \((X, (\omega, Q))\) induces a 3-Sasaki manifold. Then, \(X\) admits a (local) principal \(\text{Sp}(1)\)-bundle \(\text{Sp}(1) \to X \to X/\text{Sp}(1)\) over a quaternionic Kähler orbifold \(X/\text{Sp}(1)\). In particular, according to the results \([8, 9]\) of Biquard’s connection \([3]\), \(X\) is a qc-Einstein manifold with nonzero qc-scalar curvature. For the remaining case of vanishing qc-scalar curvature, there is no nondegenerate quaternionic CR-structure on \(X\) since the integrability of quaternionic CR-structure does not hold. Taking into account these results, we shall interpret a qc-Einstein manifold with vanishing qc-scalar curvature in terms of the
differentiable equations of the contact forms $\omega_a \ (a = 1, 2, 3)$. Given a quaternionic contact manifold $X$, let
\[ E = \{ \xi \in TX \mid d\omega_1(\xi, v) = d\omega_2(\xi, v) = d\omega_3(\xi, v) = 0, \forall v \in TX \} \]
be the distribution on $X$. If $E$ has the three-dimensional kernel, then we call $(\mathcal{D}, \omega, Q)$ a strict $qc$-structure on $X$.

When $X$ is a $qc$-Einstein manifold with vanishing $qc$-scalar curvature, it follows from Lemma 6.4 [8] (also (1) of Proposition 6.3) that the Reeb fields $\{\xi_a\}_{a=1,2,3}$ of $\omega$ are Killing and generate a (local) abelian Lie group (that is, $[\xi_a, \xi_b] = 0$), it is easy to see that $E = \{\xi_a\}_{a=1,2,3}$. Thus, a $qc$-Einstein manifold with vanishing $qc$-scalar curvature is a strict $qc$-manifold. Conversely, if $X$ is a strict $qc$-manifold, then we prove in Proposition 2.5 of Sect. 2 that $E$ generates a three-dimensional local abelian Lie group $\mathcal{K}$ and if $\mathcal{K}$ extends to a global $\mathbb{R}^3$-action on $X$, then there is a principal bundle $\mathcal{M} \rightarrow X$ over the hyper-Kähler manifold $X/\mathbb{R}^3$. (This holds always locally over an appropriate neighborhood of $X$ in case $\mathcal{K}$ is a local $qc$-action.) Since $X/\mathbb{R}^3$ is hyperKähler (locally in general), using the pullback by $\rho$, both $qc$-Ricci tensor and $qc$-scalar curvature of $X$ vanish by the definition (cf. [8]), so $X$ is a $qc$-Einstein manifold with vanishing $qc$-scalar curvature. Thus, a strict quaternionic contact manifold is the same as a $qc$-Einstein manifold with vanishing $qc$-scalar curvature. Indeed, we owe a lot to the referee who pointed out this equivalence in our earlier draft.

If a Lie group $G$ admits a left invariant strict $qc$-structure, then $G$ is called a strict $qc$-group. An example is the quaternionic Heisenberg nilpotent Lie group $\mathcal{M}$ with the standard $qc$-structure admitting a nontrivial central extension $\mathcal{M} \rightarrow \mathbb{R}^3 \rightarrow \mathcal{M}$ with $\mathcal{M} \times T^n$ where $k + \ell = n, T^n \leq \text{Sp}(n)$, (see Sect. 3.3, cf. [4]).

**Theorem A** If $G$ is a contractible unimodular strict $qc$-group, then $G$ is isomorphic to $\mathcal{M}(k, \ell)$.

A $4n + 3$-dimensional $qc$-manifold $X$ is uniformizable (or spherical) if $X$ is locally modeled over $(\text{PSp}(n + 1, 1), S^{4n+3})$. (This is the case $W_qc = 0$, see [10] also.) The pair $(\text{PSp}(n + 1, 1), S^{4n+3})$ is obtained from projective compactification of the complete simply connected quaternionic hyperbolic space $\mathbb{H}^{n+1}$ with $\text{Isom}(\mathbb{H}^{n+1}) = \text{PSp}(n + 1, 1)$.

Denote by $\text{Aut}_{qc}(X)$ the group of $qc$-transformations of $X$. If there exists a discrete subgroup $\Gamma \leq \text{Aut}_{qc}(X)$ acting properly with compact quotient $X/\Gamma$, then $X$ is said to be divisible (cf. Definition 4.3). The following result [13, Theorem 1.1] was proved for the compact case.

**Theorem B** Let $M$ be a $(4n + 3)$-dimensional compact uniformizable strict $qc$-manifold. Then, $M$ is $qc$-conformal to the quaternionic infranilmanifold $\mathcal{M}/\Gamma$ (some finite cover of which is a principal $T^3$-bundle over the quaternionic flat torus $T^6$).

The following uniqueness theorem characterizes especially the noncompact case (cf. Theorem 4.4).

**Theorem C** Let $(X, \mathcal{D}, \omega, \{J_a\}_{a=1}^3)$ be a noncompact simply connected uniformizable strict $qc$-manifold. Put $E = \{\xi_1, \xi_2, \xi_3\}$. Suppose $X$ is divisible by $\Gamma$. 

\[ \text{Annals of Global Analysis and Geometry (2021) 59:435–455} \]
(1) If $\text{Aut}_{qc}(X)$ leaves $E$ invariant, then the developing pair reduces to the equivariant immersion:

$$(\rho, \text{dev}) : (\text{Aut}_{qc}(X), X) \to (\text{Aut}_{qc}(M), M).$$

In addition,

(2) For any $\gamma \in \Gamma$ and $\alpha = 1, 2, 3$, suppose $\gamma_s \xi_\alpha = \sum_{\beta=1}^{3} a_{\alpha\beta} \xi_\beta$ for some function $a_{\alpha\beta} : X \to \text{SO}(3)$. Then,

(i) $\text{dev} : X \to M$ is a qc-diffeomorphism so that $\mathcal{R} = \mathbb{R}^3$.

(ii) There exists a strict qc-structure $(\mathcal{D}, \eta, \{J_a\}_{a=1}^{3})$ $qc$-conformal to $(\omega, \{J_a\}_{a=1}^{3})$. The quotient $(X/\mathbb{R}^3, \{\Theta_a, \tilde{J}_a\}_{a=1}^{3})$ is a hyperKähler manifold isometric to $\mathbb{H}^n$.

For the difference between Theorem B and Theorem C, we remark that in Theorem B there is a $T^3$-action on $X/\Gamma$ which lifts to $X$ an $\mathbb{R}^3$-action centralizing $\Gamma$, while in Theorem C $X$ is divisible by $\Gamma$, but the intersection $\mathbb{R}^3 \cap \Gamma$ is not necessarily uniform in $\mathbb{R}^3$, which does not imply to induce a $T^3$-action on $X/\Gamma$.

The second part of this paper treats the quaternionic Hermitian quotient in place of the hyperKähler quotient. We construct a noncompact qc-manifold to obtain a quaternionic Hermitian manifold $(Y, \tilde{\Omega}_a, \tilde{J}_a)_{a=1}^{3}$ such that one of $(\tilde{\Omega}_a, \tilde{J}_a)$’s is Kähler. (See Theorem 6.5, Theorem 7.2, Corollary 7.3.)

**Theorem D** There exists a uniformizable noncompact qc-manifold $X$ whose quotient by the $\mathbb{R}^3$-action gives a $4n$-dimensional quaternionic Hermitian manifold $(Y, \hat{\tilde{\Omega}}, \{\hat{\tilde{J}}_a\}_{a=1}^{3})$. Moreover,

1. $(Y, \hat{\tilde{\Omega}}, \{\hat{\tilde{J}}_1\})$ is a Bochner flat complex Kähler manifold.
2. $(Y, \hat{\tilde{\Omega}})$ is not Einstein. In particular, $Y$ is not isometric to any domain of the quaternionic euclidean space $\mathbb{H}^n$.
3. The quaternionic Hermitian isometry group $\text{Isom}_{g\mathbb{H}}(Y, \hat{\tilde{\Omega}}, \{\hat{\tilde{J}}_a\}_{a=1}^{3})$ is isomorphic to a $k$-torus $T^k$ for some $k$ where $n + 1 \leq k \leq 2n$.

The paper is organized as follows In Sect. 2, we give some basic facts on strict qc-structure. The fundamental property of strict qc-manifolds is proved in Proposition 2.5 which produces hyperKähler structures on their $\mathbb{R}^3$-quotients as mentioned. In Sect. 3, we review quaternionic Heisenberg nilpotent Lie group $\mathcal{M}$ where the group structure and qc-structure are explained explicitly. We give a nontrivial strict qc-group as a qc manifold in Theorem 3.3. From another viewpoint, we discuss strict qc manifolds in connection with spherical (uniformizable) qc geometry $(\text{PSp}(n + 1, 1), S^{4n+3})$ in Sect. 4. Theorem 4.4 gives a sufficient condition for a divisible group $\Gamma$ of the qc-automorphism group $\text{Aut}_{qc}(X)$ characterizing that the quotient $X/\mathbb{R}^3$ may be isometric to $\mathbb{H}^n$ as the standard hyperKähler manifold. In Sect. 5, we relax the condition strict on $E$ in order to get a quaternionic Hermitian structure $(\hat{\tilde{\Omega}}, \{\hat{\tilde{J}}_a\}_{a=1}^{3})$ on the quotient domain $Y = X/\mathbb{R}^3$ of $\mathbb{H}^n$. This can be achieved by the conformal change of the $\text{Im} \mathbb{H}$-valued one-form $\alpha_0$, which represents the standard qc-structure on $\mathcal{M}$. We can show that one of them, say $(\hat{\tilde{\Omega}}, \hat{\tilde{J}}_1)$ is
a Kähler metric on $Y$. Moreover, in Sect. 6 a prominent property of this construction is that $(Y, \hat{\Omega}_1, \hat{J}_1)$ admits a Bochner flat Kähler structure. In particular, $Y$ is not locally isometric to any domain of the flat space $\mathbb{H}^n$. In Sect 7, we discuss the quaternionic isometry group $\text{Isom}_{qH}(Y, \hat{g}, \{\hat{\omega}_1, \hat{J}_1\}_{a=1}^3)$. In course of discussion, we obtain a strictly pseudoconvex spherical pseudo-Hermitian structure $\{\hat{\omega}, \hat{J}\}$ on the $(4n + 1)$-quotient $X/\mathbb{R}_q^2$ such that the pseudo-Hermitian transformation group $\text{Psh}(X/\mathbb{R}_q^2)$ is isomorphic to $\mathbb{R} \times T^{2n}$. Theorem D is a consequence of the results of Sects. 6 and 7.1.

2 Strict quaternionic contact manifolds

The hypercomplex structure $\{J_a, J_\beta, J_\gamma\}$ on $D$ is defined by the following equation $((\alpha, \beta, \gamma) \sim (1, 2, 3))$:

$$d\omega_a(u, v) = d\omega_\beta(J_a u, v) \quad (u, v \in D).$$ (2.1)

There is the reciprocity on $D$:

$$d\omega_a(J_a u, v) = d\omega_\beta(J_a u, v) = d\omega_\gamma(J_a u, v) \quad ((\alpha, \beta, \gamma) \sim (1, 2, 3)).$$ (2.2)

It is easy to see from (2.2)

$$d\omega_a(J_a u, J_a v) = d\omega_a(u, v) \quad (\alpha = 1, 2, 3).$$ (2.3)

2.1 Strict qc-manifolds

Let $(X, D, \omega, \{J_a\}_{a=1}^3)$ be a strict $qc$-manifold with distribution $E$ (cf. (1.1)).

Lemma 2.1 $E$ generates a three-dimensional local abelian Lie group $\mathcal{R}$.

Proof Since $E$ is of dimension 3 and transverse to $D$, it follows $\omega(E) = \text{Im} \mathbb{H}$. There exist vector fields $\{\xi_a\}_{a=1, 2, 3} \subset E$ such that

$$\omega_a(\xi_a) = \delta_{a\beta}.$$ (2.4)

(Equivalently $\omega(\xi_a) = \omega_1(\xi_a)i + \omega_2(\xi_a)j + \omega_3(\xi_a)k = \delta_{1a}i + \delta_{2a}j + \delta_{3a}k$.) By (1.1), $2d\omega_a(\xi_\beta, \xi_\gamma) = -\omega_a([\xi_\beta, \xi_\gamma]) = 0$ and so $[\xi_\beta, \xi_\gamma] \in D$. For any $v \in D$, $0 = 2d\omega_a(\xi_\beta, v) = -\omega_a([\xi_\beta, v])$ ($\alpha = 1, 2, 3$) so $[\xi_\beta, v] \in D$ ($((\alpha, \beta, \gamma) \sim (1, 2, 3)$). Using the Jacobi identity,

$$2d\omega_a([\xi_\beta, \xi_\gamma], v) = -\omega_a([\xi_\beta, \xi_\gamma], v) = \omega_a([\xi_\beta, v], \xi_\gamma]) + \omega_a([v, \xi_\beta], \xi_\gamma)$$

$$= -2d\omega_a([\xi_\beta, v], \xi_\gamma) - 2d\omega_a([v, \xi_\beta], \xi_\gamma) = 0.$$

By the non-degeneracy of $d\omega_a$ on $D$, it follows $[\xi_\beta, \xi_\gamma] = 0$ for any $\beta, \gamma$. Thus, $E = \{\xi_a, \alpha = 1, 2, 3\}$ generates a local abelian Lie group.

Proposition 2.2 Denote by $\mathcal{L}_\xi$ the Lie derivative of a vector field $\xi$ on $X$. 
\begin{enumerate}
\item $\mathcal{L}_x \omega_\beta = 0$, $\mathcal{L}_x \delta \omega_\beta = 0$ ($\alpha, \beta = 1, 2, 3$). In particular, $\mathcal{L}_x D = D$.
\item $\mathcal{L}_x J_\beta = 0$ ($\alpha, \beta = 1, 2, 3$).
\end{enumerate}

\textbf{Proof} First note that $\mathcal{L}_x \omega_\alpha = (d_\nu + \iota_\nu \, d)\omega_\alpha = \iota_\nu \, d\omega_\alpha = 0$, $\mathcal{L}_x \delta \omega_\beta = d\mathcal{L}_x \omega_\beta = 0$ from (2.4), (1.1). For any $v \in D$, $0 = (\mathcal{L}_x \omega)(v) = -\omega(\mathcal{L}_x v)$ so $\mathcal{L}_x v \in D$. Thus, $(\mathcal{L}_x J_\beta)v = \mathcal{L}_x (J_\beta v) - J_\beta (\mathcal{L}_x v) \in D$. For $u, v \in D$, $(\mathcal{L}_x \delta \omega)(J_\beta J_u, v) = 0$, which equals

$$
\mathcal{L}_x (\delta \omega)(J_\beta J_u, v) = \mathcal{L}_x (\delta \omega)(J_u, v). 
$$

Similarly, it follows $\delta \omega_a((\mathcal{L}_x J_\beta) u, v) = 0$. By the non-degeneracy of $\delta \omega_a$ on $D$, it follows $\mathcal{L}_x J_u = 0$, $\mathcal{L}_x J_\beta = 0$.

Let $\mathcal{R}$ denote the local abelian group obtained from Lemma 2.1.

\textbf{Proposition 2.3} Suppose that $\mathcal{R}$ generates a global abelian group of a strict $qc$-manifold $X$. Then, $\mathcal{R}$ acts properly on $X$ as $qc$-transformations, that is a closed subgroup $\mathcal{R} \leq \text{Aut}_{qc}(X)$.

\textbf{Proof} By (1), (2) of Proposition 2.2, it follows $t^\ast \omega_a = \omega_a$, $t^\ast J_a = J_a$ for any $t \in \mathcal{R}$ ($\alpha = 1, 2, 3$). Define a Riemannian metric on $X$ by

$$
g(A, B) = \sum_{i=1}^{3} \omega_i(A) \cdot \omega_i(B) + \delta \omega_1(J_1 A, B) + \delta \omega_2(J_2 A, B). 
$$

(2.5)

(We may choose whichever $\delta \omega_a \circ J_a$ from the reciprocity $\delta \omega_1 \circ J_1 = \delta \omega_2 \circ J_2 = \delta \omega_3 \circ J_3$.) Then, note that $\mathcal{R} \leq \text{Isom}(X, g) \leq \text{Aut}_{qc}(X)$. If $\bar{\mathcal{R}}$ is the closure of $\mathcal{R}$ in $\text{Isom}(X, g)$, then it acts properly on $X$. Let $\tau$ be a vector field induced by a one-parameter subgroup of $\bar{\mathcal{R}}$. Then, there is a sequence of vector fields $\{\xi^{(n)}\} \subset E$ such that $\delta \omega_1(\tau, A) = \lim_{n \to \infty} \delta \omega_1(\xi^{(n)}, A) = 0$ ($\xi A \in TX$) by (1.1). And so $\tau \in E$. This implies $\bar{\mathcal{R}} = \mathcal{R}$.

For example, if $X$ is complete with respect to $g$ of (2.5), then $\mathcal{R}$ extends to a global action of $X$. If a strict $qc$-manifold $(X, \omega_a, D, \{J_a\}_{a=1}^{3})$ admits a global $\mathbb{R}^3$-action induced by $E$, then $\mathbb{R}^3$ acts properly by Proposition 2.3 and hence freely on $X$. There is a principal bundle over a 4n-dimensional manifold $Y = X/\mathbb{R}^3$; $\mathbb{R}^3 \to Y$ to $\mathcal{R}$. We will show that $\mathcal{R}$ admits a hyperKähler metric. Since each $t \in \mathbb{R}^3$ satisfies $J_a \cdot t = t \cdot J_a$ on $D$ by (2) of Proposition 2.2, $\mathbb{R}^3$ induces a well-defined almost complex structure $J_a$ on $Y$ such that $\pi_a \cdot J_a = J_a \cdot \pi_a : D \to TY$ at each point of $X$. $\{J_a\}_{a=1}^{3}$ constitutes a quaternionic structure on $Y$. Define a 2-form $\Omega_a (\alpha = 1, 2, 3)$ on $Y$ to be

$$
\pi^a \Omega_a = \delta \omega_a. 
$$

(2.6)

\textbf{Proposition 2.4} The 2-form $\Omega_a$ is a well-defined closed 2-form ($\alpha = 1, 2, 3$) satisfying the following equality:
\[ \Omega_1(\hat{J}_1 \hat{A}, \hat{B}) = \Omega_2(\hat{J}_2 \hat{A}, \hat{B}) = \Omega_3(\hat{J}_3 \hat{A}, \hat{B}) \quad (\hat{A}, \hat{B} \in TY). \]

Moreover,
\[ g(\hat{A}, \hat{B}) = \Omega_a(\hat{J}_a \hat{A}, \hat{B}) \quad (a = 1, 2, 3) \]
is a hyperKähler metric on \((Y, \{\hat{J}_a\}_{a=1,2,3})\).

**Proof** Let \( A = V + u, B = W + v \in TX \) \((^2 V, W \in \mathbb{E} = \mathbb{R}^3, \quad ^3 u, v \in \mathbb{D}). \) (Similarly, \( A' = V' + u', B' = W' + v' \)) Suppose \( \pi_a A_p = \pi_a A'_q, \pi_a B_p = \pi_a B'_q. \) Then, \( q = tp \in X \) for some \( t \in \mathbb{R}^3 \) in which \( A' = t_1 T + t_2 A, \quad B' = t_1 T' + t_3 B \) \((^3 T, T' \in \mathbb{R}^3 = \mathbb{E}_q). \) Since each element \( t \) leaves \( \omega_a \) invariant by (1) of Proposition 2.2, (1.1) shows \( d\omega_a(A', B') = d\omega_a(A, B), \) thus (2.6) is well-defined. In particular,
\[ d\omega_a = 0 \text{ on } Y. \]
Furthermore, as \( V, W \in \mathbb{E}, d\omega_a(A, B) = d\omega_a(u, v). \) Since \( \hat{A} = \pi_a A = \pi_a u, \quad \hat{B} = \pi_a B = \pi_a v, \) it follows \( d\omega_a(J_a u, v) = \pi_a^* \Omega_a(J_a u, v) = \Omega_a(\hat{J}_a \hat{A}, \hat{B}). \)
Since \( d\omega_1(J_1 u, v) = d\omega_2(J_2 u, v) = d\omega_3(J_3 u, v) \) is positive definite on \( \mathbb{D} \) [cf. (2.2)], we have a positive definite 2-form on \( Y: \)
\[ g(\hat{A}, \hat{B}) = \Omega_1(\hat{J}_1 \hat{A}, \hat{B}) = \Omega_2(\hat{J}_2 \hat{A}, \hat{B}) = \Omega_3(\hat{J}_3 \hat{A}, \hat{B}) \quad (\hat{A}, \hat{B} \in TY). \]
By (2.6), \( \Omega_a(\hat{J}_a \hat{A}, \hat{J}_a \hat{B}) = \Omega_a(\hat{A}, \hat{B}). \) It follows
\[ g(\hat{J}_a \hat{A}, \hat{J}_a \hat{B}) = g(\hat{A}, \hat{B}) \quad \text{on } Y (a = 1, 2, 3). \]
By the definition, \( g \) is a hyperKähler metric on \( Y. \)

In summary, we obtain the result implied in Introduction.

**Proposition 2.5** Let \((X, \mathcal{D}, (\omega, (J_a)_{a=1}^3))\) be a strict qc-manifold. Let \( \mathcal{R} \) be a local abelian group generated by the distribution \( \mathbb{E}. \) If \( \mathcal{R} \) extends to a global action of \( \mathbb{R}^3 \) on \( X, \) then the quotient manifold \( Y = X/\mathbb{R}^3 \) supports a hyperKähler structure \((g, (\Omega_a, \hat{J}_a)_{a=1}^3).\)

## 3 Quaternionic Heisenberg Lie group \( \mathcal{M} \)

### 3.1 Quick review of quaternionic parabolic geometry

We recall **parabolic quaternionic group** derived from the quaternionic hyperbolic group. The quaternionic hyperbolic space \( \mathbb{H}^{p+1} \) has a (projective) compactification whose boundary is diffeomorphic to \( S^{4n+3}. \) The isometric action of the quaternionic hyperbolic group \( \text{Isom}(\mathbb{H}^{p+1}) = \text{PSp}(n + 1, 1) \) extends to an analytic action on \( S^{4n+3}, \) which we may call a *quaternionic contact* action on \( S^{4n+3}. \) Let \( \infty \) be the point at infinity of \( S^{4n+3}. \) The standard sphere \( S^{4n+3} \) with \( \infty \) removed admits a qc-structure isomorphic to the quaternionic Heisenberg Lie group \( \mathcal{M} \) with \( \text{Aut}_{qc}(\mathcal{M}) = \mathcal{M} \rtimes (\text{Sp}(n) \cdot \text{Sp}(1) \times \mathbb{R}^+). \) Recall the definition of \( \mathcal{M} \) from [2]. Put \( t = (t_1, t_2, t_3), s = (s_1, s_2, s_3) \in \mathbb{R}^3 = \text{Im } \mathbb{H}, \) and \( z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{H}^n \) and so on. Then, \( \mathcal{M} \) is the product \( \mathbb{R}^3 \times \mathbb{H}^n \) with group law:
(t, z) \cdot (s, w) = (t + s - \text{Im}(z, w), z + w)

where \(\langle z, w \rangle = i\bar{z}w\) is the Hermitian inner product. \(\mathcal{M}\) is a nilpotent Lie group such that the center is the commutator subgroup \([\mathcal{M}, \mathcal{M}] = \mathbb{R}^3\) consisting of elements \((t, 0)\).

Each element \(h = (t, v, \sqrt{\bar{u}A\bar{a}}) \in \text{Aut}_{qc}(\mathcal{M})\) acts on \(\mathcal{M}\) as

\[
h(s, z) = (t + uas\bar{a} - \text{Im}(v, \sqrt{uA\bar{a}}), v + \sqrt{uA\bar{a}}) \quad (s, z) \in \mathcal{M}).
\]

(3.1)

In particular, \(\text{E}(\mathcal{M}) = \mathcal{M} \rtimes (\text{Sp}(n) \cdot \text{Sp}(1))\) is a normal subgroup of \(\text{Aut}_{qc}(\mathcal{M})\) acting properly and transitively on \(\mathcal{M}\) in the manner of (3.1).

3.2 The \(qc\)-structure of \(\mathcal{M}\)

The \(\text{Im}\mathbb{H}\)-valued 1-form on \(\mathcal{M}\) is defined by

\[
\omega_0 = dt_1 + dt_2j + dt_3k + \text{Im}(z, dz).
\]

(3.2)

Put \(\omega_0 = \omega_1 + \omega_2j + \omega_3k\). Let \(D_0 = \bigcap_{i=1}^3 \ker \omega_i = \ker \omega_0\) which denotes the codimension 3-subbundle on \(\mathcal{M}\) satisfying \(T\mathcal{M} = D_0 \oplus \left\langle \frac{d}{dt_1}, \frac{d}{dt_2}, \frac{d}{dt_3}\right\rangle\). Let \(x_\alpha (\alpha = 1, \ldots, 4n)\) be a real number so that \(\mathbb{R}^{4n}\) is identified with \(\mathbb{H}^n = (x_1 + ix_2 + jx_3 + kx_4; \ldots, x_{4n-3} + ix_{4n-2} + jx_{4n-1} + kx_{4n})\). A direct calculation shows

\[
\omega_1 = dt_1 + (x_1dx_2 - x_2dx_1) + (x_3dx_4 - x_4dx_3) + \cdots
\]
\[
\cdots + (x_{4n-3}dx_{4n-2} - x_{4n-2}dx_{4n-3}) + (x_{4n}dx_{4n-1} - x_{4n-1}dx_{4n}),
\]
\[
\omega_2 = dt_2 + (x_1dx_3 - x_3dx_1) + (x_2dx_4 - x_4dx_2) + \cdots
\]
\[
\cdots + (x_{4n-3}dx_{4n-1} - x_{4n-1}dx_{4n-3}) + (x_{4n}dx_{4n-2} - x_{4n-2}dx_{4n-2}),
\]
\[
\omega_3 = dt_3 + (x_1dx_4 - x_4dx_1) + (x_2dx_3 - x_3dx_2) + \cdots
\]
\[
\cdots + (x_{4n-3}dx_{4n-2} - x_{4n-2}dx_{4n-3}) + (x_{4n-1}dx_{4n-2} - x_{4n-2}dx_{4n-1}).
\]

(3.3)

The hypercomplex structure \(\{J_1, J_2, J_3\}\) on \(D_0\) is given as in (2.1). Alternatively if \(\pi : \mathcal{M} \rightarrow \mathbb{H}^n\) is the canonical projection (homomorphism), then \(\pi_* : D_0 \rightarrow T\mathbb{H}^n\) is an isomorphism at each point of \(\mathcal{M}\) for which each \(J_\alpha\) on \(D_0\) is defined by the commutative rule:

\[
\pi_* \circ J_\alpha = I_\alpha \circ \pi_*
\]

(3.4)

where \(\{I_\alpha\}_{\alpha=1}^3\) of the right hand side is the standard quaternionic structure \(\{i, j, k\}\) on \(\mathbb{H}^n\), respectively.

Proposition 3.1 \((\mathcal{M}, (D_0, \omega_0, \{J_\alpha\}_{\alpha=1}^3))\) is a strict \(qc\)-manifold for which

1. \(E_0 = \left\langle \frac{d}{dt_1}, \frac{d}{dt_2}, \frac{d}{dt_3} \right\rangle\) generates the center \(\mathbb{R}^3\) of \(\mathcal{M}\), transverse to \(D_0\).
2. There is a principal bundle: \(\mathbb{R}^3 \rightarrow \mathcal{M} \xrightarrow{\pi} \mathbb{H}^n\) whose \(qc\)-structure \((\omega_0, \{J_\alpha\}_{\alpha=1}^3)\) induces the standard hyperKähler structure on \(\mathbb{H}^n\).
Proof It follows \( d\omega_a \left( \frac{d}{dt_\beta} \right) = \delta_{a\beta}, \ d\omega_a \left( \frac{d}{dt_\beta}, X \right) = 0 \) (\( X \in T\mathcal{M} \)) by (3.3). And so 
\[
E_0 = \left\{ \frac{d}{dt_1}, \frac{d}{dt_2}, \frac{d}{dt_3} \right\}.
\]
The remaining follows from Proposition 2.5. Explicitly, if \( g_{\Omega} \) is the standard quaternionic euclidean metric on \( \mathbb{H}^n \), then (3.4), (3.2) show 
\[
d\omega_a(J_aX, Y) = g_{\Omega}(\pi_+X, \pi_+Y) (\forall X,Y \in D_0).
\]
\[\square\]

Remark 3.2 There is a canonical equivariant Riemannian submersion:
\[
\mathbb{R}^3 \to (E(M), \mathcal{M}, g_\omega) \xrightarrow{\pi} (E(\mathbb{H}^n), \mathbb{H}^n, g_{\Omega})
\]
where \( g_\omega = \sum_{a=1}^3 \omega_a \cdot \omega_a + d\omega_a \circ J_1 \) is a Riemannian metric (cf. (2.5)). Note that this metric is not a 3-Sasaki metric globally defined on \( \mathcal{M} \). Here, \( E(\mathbb{H}^n) = \mathbb{H}^n \rtimes \text{Sp}(n) \cdot \text{Sp}(1) \) is the quaternionic isometry group Isom(\( \mathbb{H}^n, g_{\Omega} \)).

From (3.1) (cf. [2]), we see that any element \( h = (t, u, \sqrt{uA \cdot a}) \in \text{Aut}_{qc}(\mathcal{M}) \) satisfies
\[
h^* \omega_0 = u \cdot a \omega_0 \bar{a}
\]
which thus preserves \( D_0 \). Thus, for \( h \in \mathcal{M} \rtimes \text{Sp}(n) \), it follows
\[
h^* \omega_0 = \omega_0 \text{ on } \mathcal{M}, \ h_3 J_a = J_a h_3 \text{ on } D_0.
\]
(3.7)
In particular, note that \( t_3 J_a = J_a t_3 \) (\( \forall t \in \mathbb{R}^3 = C(\mathcal{M}) \)).

3.3 Strict qc-group

Let \( (\mathbb{D}, \omega, \{J_a\}_{a=1}^3) \) be a qc-structure on \( X \). Put
\[
Psh_{qc}(X) = \{ h \in \text{Diff}(X) \mid h^* \omega = \omega, \ h_3 J_a = J_a h_3 \mid \mathbb{D}, \ a = 1, 2, 3 \}.
\]
(3.8)
It is a subgroup of \( \text{Aut}_{qc}(X) \). We apply the similar constriction for Sasaki groups (cf. [4]). Let \( \rho : \mathbb{H}^\ell \to \text{Sp}(k) \) be a non-trivial homomorphism (\( k + \ell = n \)). Define \( \mathbb{H}(k, \ell) \) to be the semidirect product \( \mathbb{H}^k \rtimes \rho \mathbb{H}^\ell \) which is canonically embedded to the group of hyperkähler isometries \( \mathbb{H}^n \rtimes \text{Sp}(n) \) of flat quaternionic space \( \mathbb{H}^n \). Since \( \rho(k, \ell) \) acts simply transitively on \( \mathbb{H}^n \), it is a flat hyperKähler group. (In fact, in view of [7, Theorem II], every flat hyperKähler Lie group contained in \( \mathbb{H}^n \rtimes \text{Sp}(n) \) may be conjugate to some \( \rho(k, \ell) \).) Let \( Psh_{qc}(\mathcal{M}) = \mathcal{M} \rtimes \text{Sp}(n) \) be the normal subgroup of \( E(\mathcal{M}) \). Take the pull-back \( \mathcal{M}(k, \ell) \) of \( \mathbb{H}(k, \ell) \) in the following central extension:
\[
1 \longrightarrow \mathbb{R}^3 \longrightarrow \mathcal{M} \rtimes \text{Sp}(n) \longrightarrow \mathbb{H}^n \rtimes \text{Sp}(n) \longrightarrow 1
\]
(3.9)
\[
1 \longrightarrow \mathbb{R}^3 \longrightarrow \mathcal{M}(k, \ell) \longrightarrow \mathbb{H}(k, \ell) \longrightarrow 1.
\]
Here, \( \mathcal{M}(n, 0) = \mathcal{M} \). Then, \( \mathcal{M}(k, \ell) \) is a simply connected solvable Lie group acting simply transitively by qc-transformations on the strict qc-manifold \( \mathcal{M} \). Thus, \( \mathcal{M}(k, \ell) \) admits a strict qc structure as a Lie group.

Theorem 3.3 Let \( G \) be a contractible unimodular strict qc Lie group. Then, \( G \) is isomorphic to \( \mathcal{M}(k, \ell) \).
Proof $G$ is viewed as a strict $qc$-manifold endowed with a left invariant strict $qc$-structure $(\omega, \{J_a\}_{a=1}^3)$. Then, $G \leq \text{Psh}_{qc}(G)$ by (3.8). If $\mathbb{R}^3$ is the abelian group generated by $E = (\xi_\alpha, \alpha = 1, 2, 3)$, then $\mathbb{R}^3 \leq \text{Psh}_{qc}(G)$ by Proposition 2.2. Let $\text{Isom}_{h_\alpha}(G/\mathbb{R}^3) = \{ h \in \text{Diff}(G/\mathbb{R}^3) | h^*\Omega_\alpha = \Omega_\alpha, h_\alpha J_a = J_a h_\alpha, \alpha = 1, 2, 3 \}$ be a subgroup of $\text{Isom}_{h_\alpha}(G/\mathbb{R}^3)$ of the hyperKähler manifold $G/\mathbb{R}^3$ as in Proposition 2.5. Denote by $\text{Isom}_{h_\alpha}(G/\mathbb{R}^3) \{ h \in \text{Diff}(G/\mathbb{R}^3) | h^*\Omega = \Omega, h J_a = J_a h \}$ the holomorphic isometry group of $G/\mathbb{R}^3$ as a Kähler manifold. Recall that $\text{Psh}(G/\mathbb{R}^3) = \{ h \in \text{Diff}(G/\mathbb{R}^3) | h^* \omega = \omega, h_\alpha J_a = J_a h_\alpha \}$ is the group of strictly pseudo-convex pseudo-Hermitian transformations of $(G/\mathbb{R}^2, (\alpha_1, J_1))$. There is the commutative diagram of central group extensions (cf. [4, Proposition 3.4]):

\[
\begin{array}{ccc}
1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \text{Psh}_{qc}(G) & \longrightarrow & \text{Isom}_{h_\alpha}(G/\mathbb{R}^3) \\
\text{\|} & \downarrow & \text{\|} & & \text{\|} & \downarrow \text{\|} & \\
1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \text{Psh}(G/\mathbb{R}^2) & \longrightarrow & \text{Isom}_{h_\alpha}(G/\mathbb{R}^4) & \longrightarrow & 1
\end{array}
\]  

(3.10)

It follows from [4, Theorem 2] that $\text{Isom}_{h_\alpha}(G/\mathbb{R}^3)^0 = (\mathbb{C}^k \rtimes U(k)) \times S_0$ for which $S_0$ is a semisimple Lie group of noncompact type. It acts transitively on the Kähler manifold $G/\mathbb{R}^3$ holomorphically isometric to the product $\mathbb{C}^k \times D$ where $D$ is the bounded symmetric domain.

Consider (1) $\mathbb{R}^3$ is normal in $G$. Then, $G/\mathbb{R}^3$ is a flat hyperKähler group by Hano’s theorem [7] and so $G/\mathbb{R}^3 = \mathbb{H}(k, \varepsilon)$ as above. Then, the pull back of this in (3.10) gives $G = \mathcal{M}(k, \varepsilon)$. Otherwise, (2) $\mathbb{R}^2$ is normal in $G$, where $k = 0, 1, 2$. Case (i) If $\mathbb{R}^2$ is normal in $G$, then the quotient group $\tilde{G} = G/\mathbb{R}^2$ is a Sasaki group for which $S^1 = \mathbb{R}/\mathbb{Z} \rightarrow \tilde{G} = \tilde{G}/\mathbb{Z} \rightarrow G/\mathbb{R}^3$ is a pseudo-Hermitian (Sasaki) bundle. As in the proof of [4, Theorem 2], $G/\mathbb{R}^3 = G/S^3$ is a bounded symmetric domain so that $G/\mathbb{R}^3 = \mathbb{H}^2_3$ with $\tilde{G} = \text{PSL}(2, \mathbb{R})$. It is impossible for $G/\mathbb{R}^3$ to admit a quaternionic structure. Case (ii) If $\mathbb{R}$ is normal in $G$, then put $\tilde{G} = G/\mathbb{R}$. The principal bundle $\mathbb{R}^2/\mathbb{Z}^2 \rightarrow \tilde{G} = \tilde{G}/\mathbb{Z}^2 \rightarrow G/\mathbb{R}^3$ becomes a principal bundle of homogeneous space $T^2 \rightarrow \tilde{G} \rightarrow G/\mathbb{R}^3$. As $G/T^2$ is a bounded symmetric domain, $\tilde{G}$ is locally isomorphic to $\text{PSL}(2, \mathbb{R}) \rtimes \text{PSL}(2, \mathbb{R})$, which is impossible since $\tilde{G}/T^2 = \text{PSL}(2, \mathbb{R})/S^1 \times \text{PSL}(2, \mathbb{R})/S^1 = \mathbb{H}^2_\mathbb{R} \times \mathbb{H}^2_\mathbb{R}$ has a positive scalar curvature which is not hyperKähler. Finally, Case (iii) $\tilde{G} = G/\mathbb{Z}^3$ and $\mathbb{R}^3 = \mathbb{Z}^3/\mathbb{Z}^3$ such that $G/\mathbb{R}^3 = \tilde{G}/T^3$ is a bounded symmetric domain. Thus, $\tilde{G}$ is locally isomorphic to $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ and $G/\mathbb{R}^3 = \mathbb{H}^2_\mathbb{R} \times \mathbb{H}^2_\mathbb{R} \times \mathbb{H}^2_\mathbb{R}$, which cannot be a quaternionic manifold.

\[\square\]

4 Spherical $qc$-manifolds

The following theorem is a supporting example to Proposition 2.5 which is implied by Schoen’s result [14]. (Compare [6, 12] for the proofs of the quaternionic case.)

Theorem 4.1 Let $(X, D, \{J_a\}_{a=1}^3)$ be a noncompact quaternionic contact manifold. If $\text{Aut}_{qc}(X)$ does not act properly on $X$, then $X$ admits the spherical $qc$-structure $qc$-conformal to the quaternionic Heisenberg Lie group $\mathcal{M}$.

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Let \((\tilde{\omega}, D_0, \{J_a\}_{a=1}^3)\) be the standard \(qCR\)-structure on \(S^{4n+3}\) such that \(\ker \tilde{\omega} = D_0\) (cf. [2]).

**Definition 4.2** A \(qc\)-manifold \((M, D, (\omega, \{J_a\}_{a=1}^3))\) is spherical (or uniformizable) with respect to \((\text{PSp}(n + 1, 1), S^{4n+3})\) if there exists a \(\rho\)-equivariant developing immersion from the universal covering manifold \(X\) of \(M\):

\[
(\rho, \text{dev}) : (\text{Aut}_{qc}(X), X) \rightarrow (\text{PSp}(n + 1, 1), S^{4n+3})
\]

such that

(i) \(\text{dev}^* \tilde{\omega} = \lambda \cdot \omega \cdot \tilde{\lambda} = u \cdot a \omega \tilde{a}\) for some map \(\lambda = \sqrt{u} \cdot a : X \rightarrow \mathbb{R}^+ \times \text{Sp}(1)\) \((u \in \mathbb{R}^+, a \in \text{Sp}(1))\).

(ii) If the conjugate by the map \(a : X \rightarrow \text{Sp}(1)\) represents the matrix \((a_{\alpha \beta}) : X \rightarrow \text{SO}(3)\), then \(\text{dev}_* \circ J_a = \sum \rho J_\beta \circ \text{dev}_* : D \rightarrow D_0\).

(iii) \(\rho : \text{Aut}_{qc}(X) \rightarrow \text{PSp}(n + 1, 1)\) is the holonomy homomorphism such that \(\text{dev} \circ h = \rho(h) \circ \text{dev} \quad (\forall h \in \text{Aut}_{qc}(X))\).

**Definition 4.3** A \(qc\)-manifold \(X\) is divisible if there exists a discrete subgroup \(\Gamma \leq \text{Aut}_{qc}(X)\) which acts properly discontinuously with compact quotient.

**Theorem 4.4** Let \((X, D, (\omega, \{J_a\}_{a=1}^5))\) be a simply connected noncompact uniformizable strict \(qc\)-manifold. Put \(E = \{\xi_\alpha, \alpha = 1, 2, 3\}\). Suppose \(X\) is divisible by \(\Gamma\).

1. If \(\text{Aut}_{qc}(X)\) leaves \(E\) invariant, then the developing pair reduces to the equivariant immersion:

\[
(\rho, \text{dev}) : (\text{Aut}_{qc}(X), X) \rightarrow (\text{Aut}_{qc}(M), M).
\]

2. For any \(\gamma \in \Gamma\) and \(\alpha = 1, 2, 3\), suppose \(\gamma_* \xi_\alpha = \sum_{\beta=1}^5 a_{\alpha \beta} \xi_\beta\) for some function \(a_{\alpha \beta} : X \rightarrow \text{SO}(3)\). Then,

(i) \(\text{dev} : X \rightarrow M\) is a \(qc\)-diffeomorphism so that \(R = \mathbb{R}^3\).

(ii) There exists a \(qc\)-structure \((\eta, \{J'_a\}_{a=1}^3)\) \(qc\)-conformal to \((\omega, \{J_a\}_{a=1}^3)\). The quotient \((X/\mathbb{R}^3, \{\Theta_\alpha J'_a\}_{a=1}^3)\) is a hyperKähler manifold isometric to \(\mathbb{H}^n\).

The method of proof is based on that of [13] by taking into account the results of [14] (cf. [6, 12]).

**Proof** Put \(\text{Aut}_{qc}(X) = \text{Aut}(X)\). Let \(G = \rho(\text{Aut}(X)) \in \text{PSp}(n + 1, 1)\). We first show that (i) \(G\) is not compact.

**Case 1.** Suppose \(G\) is compact. If \(G\) has no fixed point on \(S^{4n+3}\), then \(G\) has the unique fixed point at the origin \(0\) in \(\mathbb{H}^{4n+1}\) where \(S^{4n+3} = \partial \mathbb{H}^{4n+1}\). As in the proof of [13], \(\text{dev} : X \rightarrow S^{4n+3}\) is shown to be an isometry, which is excluded by the non-compactness of \(X\). So \(G\) has the fixed point set \(F\) in \(S^{4n+3}\). We may assume that \(\text{Aut}(X)\) acts properly on \(X\) by Theorem 4.1, so \(\text{dev}\) misses \(F\). It reduces to an immersion \(\text{dev} : X \rightarrow S^{4n+3} - F\). As \(\text{Aut}(S^{4n+3} - F)\) acts properly on \(S^{4n+3} - F\) by the result of [14], there is a Riemannian
metric on $S^{4n+3} - F$ invariant under $\text{Aut}(S^{4n+3} - F)$. Since $X$ is divisible, $X$ is complete with respect to the pullback metric, $\text{dev} : X \to S^{4n+3} - F$ is a covering map. On the other hand, if we note that the action of $G$ is linear on $S^{4n+3}$, $F$ must be a subbundle $S^k (0 \leq k < 4n + 3)$ such that the complement $S^{4n+3} - F$ is unknotted, that is, homeomorphic to $\mathbb{R}^{k+1} \times S^{4n+2-k}$. Moreover, it is shown in [13, Lemma 3.1] (also [5, p.77]) that $S^{4n+3} - F$ is either one of the following:

1. $S^{4n+3} - S^{m-1}\, \text{ where } m \leq n + 1$.
   $\text{Aut}(S^{4n+3} - S^{m-1}) = P(O(m, 1) \cdot \text{Sp}(1) \times \text{Sp}(n - m + 1)).$
   $S^{4n+3} - S^{m-1} = \mathbb{H}^m_\mathbb{R}$ (1 $\leq m \leq n + 1$).

2. $S^{4n+3} - S^{2m-1}\, \text{ where } m \leq n + 1$.
   $\text{Aut}(S^{4n+3} - S^{2m-1}) = P(U(m, 1) \cdot U(1) \times \text{Sp}(n - m + 1)).$
   $S^{4n+3} - S^{2m-1} = \mathbb{H}^{2m}_\mathbb{C}$ (1 $\leq m \leq n + 1$).

3. $S^{4n+3} - S^{4m-1}\, \text{ where } m \leq n$.
   $\text{Aut}(S^{4n+3} - S^{4m-1}) = \text{Sp}(m, 1) \cdot \text{Sp}(n - m + 1).$
   $S^{4n+3} - S^{4m-1} = \mathbb{H}^{4m}_\mathbb{R}$ (1 $\leq m \leq n$).

4. $S^{4n+3} - S^2\, \text{ where } S^2 = \mathbb{H}^1_{\text{Im}}$.
   $\text{Aut}(S^{4n+3} - S^2) = \text{SL}(2, \mathbb{C}) \cdot \text{Sp}(n)$.

This case reduces to (3).

In particular, $S^{4n+3} - F$ is simply connected in each case. Hence, $\text{dev} : X \to S^{4n+3} - F$ is diffeomorphic, so $\rho$ is an isomorphism. However, this case does not occur since $\text{Aut}(X)^0 \cong G = \rho(\text{Aut}(X)) = \text{Aut}(S^{4n+3} - F)^0$ which is noncompact (in fact $\text{Aut}(S^{4n+3} - F)^0$ contains a noncompact subgroup $O(m, 1)$, $U(m, 1)$ $(1 \leq m \leq n + 1)$, $\text{SL}(2, \mathbb{C})$ or $\text{Sp}(m, 1)$ $(1 \leq m \leq n)$, respectively.) As a consequence, the case $G$ is compact does not occur.

**Case 2.** Suppose $G$ is noncompact. Then, either $G$ has a common fixed point $\{\infty\}$ in $S^{4n+3}$ or $G$ leaves invariant a totally geodesic subspace of $\mathbb{H}^{4n+1}_\mathbb{R}$. (See [5, Theorem 4.4.1].) In the latter case, $G$ (possibly $\text{PSp}(n + 1, 1)$) is either one of the identity component of the above groups (1), (2), (3), (4). On the other hand, let $\mathcal{R}$ be the local abelian group generated by $E$ as before. The holonomy homomorphism $\rho$ maps $\mathcal{R}$ into $\text{PSp}(n + 1, 1)$. By Liouville’s theorem, $\rho(\mathcal{R})$ extends globally to a subgroup of $\text{PSp}(n + 1, 1)$ on $S^{4n+3}$. As $\text{Aut}_\text{dev}(X)$ leaves $E$ invariant, $\text{Aut}(X)^0$ normalizes $\mathcal{R}$. Thus, $G$ normalizes $\overline{\rho(\mathcal{R})}$ also. In particular, the radical of $G$ is nontrivial, so $G$ is not semisimple. On the other hand, $G$ is *semisimple* except for the case $m = 1$ of (1) such that $\text{Aut}(S^{4n+3} - S^0)^0 = \text{SO}(1, 1)^0 \times \text{Sp}(1) \cdot \text{Sp}(n) \cong \mathbb{R}^+ \times \text{Sp}(n) \cdot \text{Sp}(1)$. For this case, $G$ has exactly two fixed points $\{0, \infty\}$. Noting from Theorem 4.1, as in the argument of (i), $\text{dev} : X \to S^{4n+3} - \{0, \infty\}$ is $\mathbb{H}^1_{\mathbb{R}} \times \mathbb{R}^+$ is a diffeomorphism so that $\rho$ maps the radical of $\text{Aut}(X)^0$ isomorphically onto the radical $\mathbb{R}^+$. Since the radical contains $\rho(\mathcal{R})$ of dimension three, it is impossible. As a consequence, by the non-ellipticity of elements in $\text{PSp}(n + 1, 1)$, $G$ has a *unique* common fixed point $\{\infty\}$. Noting $\text{Aut}(X)$ acts properly on $X$, $\text{dev}$ misses $\{\infty\}$. This proves (1).

(2) Suppose some $\rho(\gamma) \in \rho(\Gamma)$ has a nontrivial summand in $\mathbb{R}^+ \times \text{Aut}(\mathcal{M})$. It follows from (3.6) that $\rho(\gamma)^* \omega_0 = \nu \cdot b \omega_0$ where $\lambda = \sqrt{\nu} \cdot b \in \text{Sp}(1) \times \mathbb{R}^+$. On the other hand, by our hypothesis (2), $\gamma \cdot \xi_a = \sum_{\beta=1}^3 a_{\alpha \beta} \xi_\beta$ for some function $a_{\alpha \beta} : X \to \text{SO}(3)$. Put $\text{dev} \gamma \cdot \xi_a = \tilde{\xi}_a$ on $\mathcal{M}$. As $\text{dev} \gamma = \rho(\gamma) \cdot \text{dev}$, letting $(i, j, k) \sim (i_\alpha, i_\beta, i_\gamma)$, it follows

\[
\rho(\gamma) \cdot (\xi_a i_\alpha) = a(\xi_a i_\alpha) \tilde{a}\]

where the conjugate of $a$ represents the matrix $(a_{\alpha \beta}) \in \text{SO}(3)$. Calculate

\[
\rho(\gamma)^* \omega_0(\xi_a i_\alpha) = a \omega_0(\xi_a i_\alpha) \tilde{a}\]

(4.1)
Taking the norm in $\mathbb{H}$, it follows
\[
|\omega_0(\xi_a i_a)| = |a\omega_0(\xi_a i_a)\tilde{a}| = |u \cdot b\omega_0(\xi_a i_a)\tilde{a}b| = u|\omega_0(\xi_a i_a)|.
\]
Hence, $u = 1$ on $X$. This implies $\rho(y) \in E(M)$ so that $\rho(T) \leq E(M)$. As usual, there is the $E(M)$-invariant Riemannian metric on $M$. Since $X$ is divisible, $X$ is complete with respect to the pullback metric. Thus, $(\rho, \text{dev}) : (\Gamma, X) \to (E(M), M)$ is an equivariant isometry. As $\rho : \text{Aut}_{qc}(X) \to \text{Aut}_{qc}(M)$ is an isomorphism, and $R$ is normalized by $\text{Aut}_{qc}(X)$, so does $\rho(R)$ in $\text{Aut}_{qc}(M)$. By the action of (3.1) and the group structure of $\text{Aut}_{qc}(M)$ we note $\rho(R) = \mathbb{R}^3$ which is the center of $M$. This proves (i). In particular, $\text{dev}_s E = E_0$.

(ii). Let $(\omega_0, \{J_a^\beta\}_{a=1}^3)$ be the standard (spherical) $qc$-structure on $M$ where $\omega_0 = \omega_j i + \omega_j j + \omega_j k$. By the definition, it satisfies $\text{dev}^* \omega_0 = u \cdot a\omega\tilde{a}$ for some $u \in \mathbb{R}^+$, $\alpha \in \text{Sp}(1)$. When $a$ represents $(a_\beta) \in \text{SO}(3)$ as before, it follows $\text{dev}_s J_a = \sum_\beta a_\beta J_\beta \text{dev}_s$ (cf. Definition 4.2). Put
\[
\eta = \text{dev}^* \omega_0 (\eta_a = \text{dev}^* \omega_a), \ J_a' = \sum_{\beta=1}^3 a_\beta J_\beta, (\alpha = 1, 2, 3).
\]

We check $(\mathcal{D}, \eta, \{J_a'\}_{a=1}^3)$ is a $qc$-structure $qc$-conformal to $(\mathcal{D}, \omega, \{J_a\}_{a=1}^3)$. For this, let $X, Y \in \mathcal{D}$ so that $\text{dev}_s X, \text{dev}_s Y \in \mathcal{D}_0$. Note that $J_a \text{dev}_s = \text{dev}_s \sum_\mu a_\mu J_\mu$ as above. Let $\tilde{J}_\gamma = (d\eta_\gamma)^{-1} \text{dev}_s \eta_\gamma = (\text{dev}_s^* \omega_0)$. By calculation,
\[
d\eta_\gamma(J_s u, v) = d\eta_\gamma(\text{dev}_s u, \text{dev}_s v) = \text{dev}_s d\eta_\gamma(\sum_\mu a_\mu J_\mu u, \text{dev}_s v) = \text{dev}_s (d\eta_\gamma(J_s u, v)).
\]

Noting $d\eta_\gamma(J_s u, v) = d\eta_a(u, v)$, the non-degeneracy of $d\eta_\gamma$ implies $\tilde{J}_\gamma = J_s$. The equations $\text{dev}^* \omega_0 = \eta, \text{dev}_s E = E_0$ imply $d\eta(E, A) = 0 (\forall A \in TX)$. Thus, $(\mathcal{D}, \eta, \{J_a'\}_{a=1}^3)$ is a strict $qc$-structure. Noting that $R$ acts properly and freely on $X$, we have a smooth manifold $Y = X/R$. As dev : $X \to M$ is $\rho$-equivariant, dev induces a diffeomorphism $\hat{\text{dev}} : Y \to H^n$ with the commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\text{dev}} & M \\
\pi \downarrow & & \downarrow \pi \\
Y & \xrightarrow{\text{dev}} & H^n.
\end{array}
\]

Applying Proposition 2.5 (cf. (2.6)), the 2-form $\Theta_a$ on $Y$ is defined by
\[
\pi^* \Theta_a = d\eta_a, (a = 1, 2, 3).
\]

A quaternionic structure $\{J_a'\}_{a=1}^3$ on $Y$ is also induced by
\[
J_a' \pi_s = \pi_s J_a'.
\]

Using the reciprocity $J_\gamma = (d\eta_\gamma)^{-1} \text{dev}_s \eta_\gamma$, it follows for any $\hat{A}, \hat{B} \in TY$
\[
\Theta_a(J_a' \hat{A}, J_a' \hat{B}) = \Theta_a(\hat{A}, \hat{B}) \quad (\alpha = 1, 2, 3).
\]

Thus, $\Theta_a$ is a Kähler form on $(Y, \{J_a'\}_{a=1}^3)$. The quaternionic Hermitian metric
\[ g(\hat{A}, \hat{B}) = \Theta_{\alpha}(\hat{J}^\alpha_1 \hat{A}, \hat{B}) = \Theta_{\beta}(\hat{J}^\beta_1 \hat{A}, \hat{B}) = \Theta_{\gamma}(\hat{J}^\gamma_1 \hat{A}, \hat{B}) \tag{4.5} \]

is a hyperKähler metric on \( (Y, \{ \hat{J}^\alpha_1 \}_{a=1}^3) \). Let \( (\mathbb{H}^n, g_{\mathbb{H}}, \{ I_a \}_{a=1}^3) \) be the standard euclidean metric as in Remark 3.2. Noting \( d\omega_0 \circ J^\alpha_1 = \pi^* g_{\mathbb{H}} \) and (4.2), a calculation shows

\[ \text{dev}^* g_0 = g, \text{dev}_* \circ J^\alpha_1 = I_a \circ \text{dev}_*. \tag{4.6} \]

This gives an isometry of \( (Y, g, \{ J^\alpha_1 \}_{a=1}^3) \) onto \( (\mathbb{H}^n, g_{\mathbb{H}}, \{ I_a \}_{a=1}^3) \).

**Remark 4.5** The new Kähler form \( \Theta_{\alpha} \) and \( \Theta = \Theta_{\alpha i} + \Theta_{\beta j} + \Theta_{\gamma k} \) are related to the original forms \( \Omega \) and \( \Omega_{\alpha} \) as follows. For some constant \( c > 0 \),

\[ \Theta = c \cdot \alpha \Omega \bar{\alpha}, \Theta_{\alpha} = c \cdot \sum_{\beta=1}^3 \alpha_{\alpha \beta} \Omega_{\beta}. \]

In fact, as we put \( \eta = \text{dev}^* \omega_0 = u \cdot a \omega \bar{a} \), it follows \( d\eta = u \cdot a \cdot d\omega \circ \bar{a} \) so that \( \Theta = \hat{u} \cdot a \Omega \bar{a} \) where \( \hat{u}, \alpha \) are induced functions on \( Y \). As usual, \( \Theta^2 = \hat{u}^2 \Omega^2 \) which shows that \( \hat{u} \) is a constant \( c > 0 \). We have \( \Theta_{\alpha} = c \sum_{\beta=1}^3 \alpha_{\alpha \beta} \Omega_{\beta} \). (\( \pi^* \alpha = a \), \( \pi^* \alpha_{\alpha \beta} = a_{\alpha \beta} \)).

## 5 Quotient quaternionic Hermitian manifolds

For the strict qc-structure \( (\mathcal{D}_{\xi}, \omega_0, \{ J_a \}_{a=1}^3) \) on the quaternionic Heisenberg Lie group \( \mathcal{M} \), we consider a qc-structure \( \eta = n_1 i + n_2 j + n_3 k \) which is qc-conformal to \( \omega \). Take a one-form, say \( \eta_1 \) to define a distribution:

\[ E_1 = \{ \xi \mid d\eta_1(\xi, A) = 0, \forall A \in TX \}. \tag{5.1} \]

\( E_1 \) does not induce a distribution such as \( E \). When \( E_1 \) generates a three-dimensional abelian Lie group \( \mathcal{R} \), we shall show that there is an invariant domain \( X \) such that the quotient \( X/\mathcal{R} \) admits a special kind of quaternionic Hermitian structure.

Choose numbers \( a_1, \ldots, a_n \) such that

\[ 0 < a_1 < a_2 < \cdots < a_n. \tag{5.2} \]

Let \( A_1 \) be the diagonal matrix

\[ \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, \ldots, e^{i\alpha_n}) \in T^n \leq \text{Sp}(n). \tag{5.3} \]

Define a homomorphism \( \rho_1 : \mathbb{R}^3 \to (\mathbb{R}^3, 0) \times \text{Sp}(n) \leq \mathcal{M} \times \text{Sp}(n) \) to be

\[ \rho_1(t_1) = \left( ((t_1, 0, 0), 0), A_{t_1} \right), \quad \rho_1(t_2) = \left( ((0, t_2, 0), 0), I \right), \quad \rho_1(t_3) = \left( ((0, 0, t_3), 0), I \right). \tag{5.4} \]

More precisely, this action is defined on \( \mathcal{M} = \mathbb{R}^3 \times \mathbb{H}^n \) as
\[ \rho_1(t_1)((s_1, s_2, s_3), z_1, \ldots, z_n) = \left((s_1 + t_1), s_2, s_3, e^{i\theta t_1}z_1, \ldots, e^{i\theta t_1}z_n\right), \]

\[ \rho_1(t_2)((s_1, s_2, s_3), z_1, \ldots, z_n) = \left((s_1, s_2 + t_2), s_3, z_1, \ldots, z_n\right), \]

\[ \rho_1(t_3)((s_1, s_2, s_3), z_1, \ldots, z_n) = \left((s_1, s_2, (s_3 + t_3)), z_1, \ldots, z_n\right). \]

In view of (5.4), the group \((\rho_1(t_1), \rho_1(t_2), \rho_1(t_3))\) forms a 3-dimensional abelian Lie group \(\rho_1(\mathbb{R}^3)\). If \(\xi^1\) is the vector field induced by \(\{\rho_1(t_1)\}_{t_1 \in \mathbb{R}}\), then it follows

\[ \xi^1 = \frac{\rho_1(t_1)}{dt} \Big|_{t_1=0} = \frac{d}{dt} + (a_1iz_1, \ldots, a_niz_n) = \frac{d}{dt} + \sum_{k=1}^n a_k(-x_{4k-2} + x_{4k-3}x_{4k-2} - x_{4k}x_{4k-1} + x_{4k-1}x_{4k}). \] (5.6)

Letting \(z_k = u_k + v_j = (x_{4k-3} + ix_{4k-2}) + (x_{4k-1} + ix_k)j \in \mathbb{H} \) \((k = 1, \ldots, n)\), a calculation using (3.3) shows that

\[ \omega_1((\xi^1)_p) = 1 + \sum_{k=1}^n a_k(x_{4k-3}^2 + x_{4k-2}^2 - x_{4k}^2 - x_{4k-1}^2) \]

\[ = 1 + \sum_{k=1}^n a_k(|u_k|^2 - |v_k|^2) \quad \text{at} \quad p = ((s_1, s_2, s_3), z_1, \ldots, z_n). \] (5.7)

The singular set \(S = \{p \in \mathcal{M} \mid \omega_1((\xi^1)_p) = 0\}\) is not entirely equal to \(\mathcal{M}\) from (5.7). Denote the domain of \(\mathcal{M}\) by

\[ X = \{p \in \mathcal{M} \mid \omega_1((\xi^1)_p) \neq 0\} \] (5.8)

(if necessary taking the component containing the origin \((0, 0) \in \mathcal{M}\)). Since \(\rho_1(\mathbb{R}^3)\) leaves \(S\) invariant, so does \(X\). Put \(Y = X / \rho_1(\mathbb{R}^3)\). Then, there is a commutative diagram of principal bundles.

\[ \rho_1(\mathbb{R}^3) \quad \longrightarrow \quad \mathcal{M} \quad \xrightarrow{\pi_1} \quad \mathbb{H}^n \]

\[ \| \quad \cup \quad \cup \quad \| \] (5.9)

\[ \rho_1(\mathbb{R}^3) \quad \longrightarrow \quad X \quad \xrightarrow{\pi_1} \quad Y. \]

The image \(\pi_1S = \{\pi_1(p) \in \mathbb{H}^n \mid \sum_{k=1}^n a_k(|u_k|^2 - |v_k|^2) = -1\}\) is a real hypersurface in \(\mathbb{R}^{4n} = \mathbb{H}^n\). Noting \(Y = \mathbb{H}^n - \pi_1S\), it follows each component of \(Y\) is simply connected in \(\mathbb{H}^n\).

### 5.1 Conformal change of \(\omega_0\)

Let \(\omega_0 = \omega_1i + \omega_2j + \omega_3k\) be the qc-form on \(\mathcal{M}\) (cf. Sect. 3.2). We introduce new 1-forms on \(X\):

\[ \eta_a = \frac{1}{\omega_1(\xi^1)}\omega_a (\alpha = 1, 2, 3). \] (5.10)

Put \(\eta = \eta_1i + \eta_2j + \eta_3k\) on \(X\). Since \(\eta\) is conformal to \(\omega_0\), it follows \(\ker \eta = \ker \omega_0 = D_0\) on \(X\). As \(\rho_1(\mathbb{R}^3)\) leaves \(\omega_a\) invariant, so does \(\eta_a\). The hypercomplex structure of
\( \eta, \{ \hat{J}_a = \frac{d\eta}{d_0}|D_0 \}^{-1} \circ (d\eta_a|D_0) \}_{a=1}^3 \) coincides with \( \{ J_a \}_{a=1}^3 \) of \( \omega_0 \) on \( D_0 \). Noting 
\[ \rho_1(t) J_a = J_a \rho_1(t), \quad (t \in \mathbb{R}^3) \] from (3.7), \( \{ J_a \} \) induces a quaternionic structure \( \{ \hat{J}_a \}_{a=1}^3 \) on \( Y \) such that 
\[ \pi_1^* \hat{J}_a(v) = \hat{J}_a \pi_1^*(v) \quad (\forall \, v \in D_0). \tag{5.11} \]

**Proposition 5.1** \( \{ \hat{J}_a \}_{a=1}^3 \) is a quaternionic Hermitian manifold.

**Proof** Define \( \hat{\Omega}_a (a = 1, 2, 3) \) to be 
\[ \hat{\Omega}_a(\pi_1^* u, \pi_1^* v) = d\eta_a(u,v)(\pi^* u, \pi^* v \in D_0). \tag{5.12} \]
Since \( \eta_a \) is \( \rho_1(\mathbb{R}^3) \)-invariant and the distribution by \( \rho_1(\mathbb{R}^3) \) is transverse to \( D_0 \), \( \hat{\Omega}_a \) is well-defined on \( Y \) (cf. Lemma 5.2). Put 
\[ \hat{g}(\hat{u}, \hat{v}) = \hat{\Omega}_a(\hat{J}_a \hat{u}, \hat{v}) \quad (\forall \, \hat{u}, \hat{v} \in TY). \tag{5.13} \]
As \( \hat{\Omega}_a \) is invariant under \( \hat{J}_a \), it follows \( \hat{\Omega}(\hat{J}_a \hat{u}, \hat{v}) = \hat{g}(\hat{u}, \hat{v}) \). Thus, \( \hat{g} \) is a quaternionic Hermitian metric on \( \{ \hat{J}_a \}_{a=1}^3 \).

As in (5.4), the distribution \[ \left\{ \xi_1, \frac{d}{d t_2}, \frac{d}{d t_3} \right\} \] generates \( \rho_1(\mathbb{R}^3) \leq \mathcal{M} \rtimes \text{Sp}(n) \). Note from (3.3) that 
\[ \eta_1(\xi_1) = 1, \quad \eta_1(\frac{d}{d t_2}) = 0, \quad \eta_1(\frac{d}{d t_3}) = 0. \tag{5.14} \]

**Lemma 5.2** \( E_1 = \langle \xi_1, \frac{d}{d t_2}, \frac{d}{d t_3} \rangle \).

**Proof** For any \( A \in TX \), we prove \( d\eta_1(\xi_1, A) = 0 \), \( d\eta_1(\frac{d}{d t_2}, A) = 0 \) \( (\beta = 2, 3) \). If \( A \in D_0 \), then \( [\xi_1, A] \in D_0 \). Since the distribution \( \langle \xi_1, \frac{d}{d t_2}, \frac{d}{d t_3} \rangle \) generates \( \mathbb{R}^3, [\xi_1, \frac{d}{d t_\beta}] = 0 (\beta = 2, 3) \).
Then, it is easy to see that \( d\eta_1(\xi_1, A) = d\eta_1(\xi_1, \frac{d}{d t_\beta}) = 0 \). As \( \frac{d}{d t_\beta} (\beta = 2, 3) \) are induced from the central subgroup \( (0, \mathbb{R}^2) \) of \( \mathbb{R}^3 \), (5.14) shows \( d\eta_1(\frac{d}{d t_\beta}, B) = 0 \), \( d\eta_1(\frac{d}{d t_\beta}, \frac{d}{d t_\gamma}) = 0 \)
(\( B \in D_0 \)). \( \square \)

**Lemma 5.3** Let \( \hat{\Omega}_1 \) be the 2-form on \( Y \) as in (5.12). Then, \( d\hat{\Omega}_1 = 0 \).

**Proof** It suffices to show 
\[ \pi_1^* \hat{\Omega}_1 = d\eta_1 \quad \text{on} \quad Y. \tag{5.15} \]
\( \pi_1^* \hat{\Omega}_1 = d\eta_1 \) on \( D_0 \) from (5.12). For any \( \xi \in E_1 \) and \( A \in TX \), \( d\eta_1(\xi, A) = 0 \) by Lemma 5.2.
As \( E_1 \oplus D_0 = TX \), it follows \( \pi_1^* \hat{\Omega}_1 = d\eta_1 \) on \( X \). \( \square \)
6 Pseudo-Hermitian structure

6.1 Heisenberg Lie group \( \mathcal{N} \)

Let \( \mathcal{N} \) be the \( 4n + 1 \)-dimensional Heisenberg Lie group which has a central group extension \( 1 \rightarrow \mathbb{R} \rightarrow \mathcal{N} \rightarrow \mathbb{C}^{2n} \rightarrow 1 \). A pseudo-Hermitian structure \( (\omega_{\mathcal{N}}, J_{\mathcal{N}}) \) consists of a contact form

\[
\omega_{\mathcal{N}} = dt + 3(\langle (z, w), (dz, dw) \rangle) = dt + 3(\bar{z}dz + \bar{w}dw) \quad ((z, w) \in \mathbb{C}^{2n})
\]

together with a complex structure \( J_{\mathcal{N}} \) on \( \ker \omega_{\mathcal{N}} \) which is isomorphic to the standard complex structure on \( \mathbb{C}^{2n} \) at each point of \( \mathcal{N} \) (cf. [11]). As \( \mathbb{R}^2 = \{(0, t_2, t_3)\} \) is a central subgroup of \( \mathbb{R}^3 = C(M) \), there is a quotient nilpotent Lie group \( \mathcal{M}/\mathbb{R}^2 \) with central group extension \( 1 \rightarrow \mathbb{R} \rightarrow \mathcal{M}/\mathbb{R}^2 \rightarrow \mathbb{H}^n \rightarrow 1 \). We shall find an explicit isomorphism to identify \( \mathcal{M}/\mathbb{R}^2 \) with \( \mathcal{N} \). For our use, let \( z + wj \in \mathbb{H}^n \) such that \( z, w \in \mathbb{C}^n \). Then, \( \mathcal{M}/\mathbb{R}^2 \) is the product \( \mathbb{R} \times \mathbb{H}^n \) with group law:

\[
(a, z + wj) \cdot (b, z' + w'j) = (a + b - 3(\bar{z}z' + \bar{w}w'), z + z' + (w + w')j).
\]

Define a diffeomorphism \( \varphi : \mathcal{M}/\mathbb{R}^2 = \mathbb{R} \times \mathbb{H}^n \rightarrow \mathcal{N} = \mathbb{R} \times \mathbb{C}^{2n} \) to be

\[
\varphi(a, (z + wj)) = (a, (z, \bar{w})).
\]

As we see that \( \varphi((a, z + wj) \cdot (b, z' + w'j)) = (a, (z, \bar{w})) \cdot (b, (z', \bar{w}')) \).

**Lemma 6.1** \( \varphi \) is a Lie group isomorphism of \( \mathcal{M}/\mathbb{R}^2 \) onto \( \mathcal{N} \).

Consider the projection

\[
\mathbb{R}^2 = \mathbb{R}j + \mathbb{R}k \longrightarrow \mathcal{M} \xrightarrow{\rho_1} \mathcal{M}/\mathbb{R}^2
\]

for which the subbundle \( \left( \frac{d}{dt_2}, \frac{d}{dt_3} \right) \) is tangent to the fiber \( \mathbb{R}^2 \). For \( \omega_0 = \omega_1i + \omega_2j + \omega_3k \), noting (3.3), \( \omega_1 \) induces a 1-form \( \hat{\omega}_1 \) on \( \mathcal{M}/\mathbb{R}^2 \) such that

\[
\rho_1^*\hat{\omega}_1 = \omega_1 \quad \text{on} \quad \mathcal{M}.
\]

As \( \hat{\omega}_1 = dt_1 + 3(\langle (z, \bar{w}), (dz, d\bar{w}) \rangle) \) from (3.2) \((z + wj \in \mathbb{H}^n)\), (6.1) shows

\[
\varphi^*\omega_{\mathcal{N}} = \hat{\omega}_1 \quad \text{on} \quad \mathcal{M}/\mathbb{R}^2.
\]

Let \( \hat{\varphi} : \mathbb{H}^n \rightarrow \mathbb{C}^{2n} \) be a diffeomorphism defined by

\[
\hat{\varphi}(z + wj) = (z, \bar{w}),
\]

with the following commutative diagram from (6.1):

\[
\begin{array}{ccc}
\mathcal{M}/\mathbb{R}^2 & \varphi \rightarrow & \mathcal{N} \\
\hat{\varphi} & \downarrow & \rho \\
\mathbb{H}^n & \rightarrow & \mathbb{C}^{2n}.
\end{array}
\]
Take the standard complex structure $J_1$ on $\mathbb{C}^n$ such that $J_1 u = u \bar{i}$. As $u \bar{w} = (z + wij)\bar{i} = (iz + iwj)$, it follows $\bar{i} \phi(u \bar{w}) = i \phi(u)$.

If we take the anti-complex structure $J'_C$ on $\mathbb{C}^{2n}$ such as $J'_C(v) = iv$, it follows

$$\phi_* \circ J_1 = J'_C \circ \phi_* : T\mathbb{C}^n \to T\mathbb{C}^{2n}. \quad (6.7)$$

As in (3.4) of Sect. 3.2, the almost complex structure $J_1$ on $D_0$ induces an almost complex structure $J_1$ on $p_{1*}D_0 = \ker \hat{\omega}_1$ such that

$$\hat{\pi}_* \circ J_1 = J_1 \circ \hat{\pi}_* : p_{1*}D_0 \to T\mathbb{C}^n. \quad (6.8)$$

Similarly, if $J'_N$ is the anti-complex structure on $\ker \omega_N$ of $N$, it follows

$$p_* \circ J'_N = J'_C \circ p_* : \ker \omega_N \to T\mathbb{C}^{2n}. \quad (6.9)$$

Then, (6.7), (6.8) and (6.9) imply

$$\phi_* \circ J_1 = J'_N \circ \phi_* \quad \text{on } \ker \hat{\omega}_1 (= p_{1*}D_0). \quad (6.10)$$

In particular, $J_1$ is a complex structure on $p_{1*}D_0 = \ker \hat{\omega}_1$. We have the principal bundle induced from (5.9):

$$\rho_1(\mathbb{R}) \longrightarrow X/\mathbb{R}^2 \overset{\hat{\pi}_1}{\longrightarrow} Y. \quad (6.11)$$

There is a CR-structure (ker $\hat{\omega}_1, J_1$) on $X/\mathbb{R}^2 \subset M/\mathbb{R}^2$ as above. Let $\hat{J}_1$ be the almost complex structure on $Y$ as in (5.11), that is $\hat{\pi}_1J_1 = \hat{J}_1 \hat{\pi}_1$ : $\ker \hat{\omega}_1 \to TY$.

**Lemma 6.2** $\hat{J}_1$ is integrable on $Y$.

**Proof** Let $\ker \hat{\omega}_1 \otimes \mathbb{C} = P^{1,0} \oplus P^{0,1}$ be the eigenspace decomposition for $J_1$. We have an isomorphism $\hat{\pi}_1 : \ker \hat{\omega}_1 \otimes \mathbb{C} \to TY \otimes \mathbb{C} = Q^{1,0} \oplus Q^{0,1}$ (the eigenspace decomposition for $\hat{J}_1$, respectively). Since $J_1$ is integrable, $[u, v] \in P^{1,0}$ for $u, v \in P^{1,0}$. Then, $\hat{\pi}_1([u, v]) = [\hat{\pi}_1(u), \hat{\pi}_1(v)]$. It follows $[\hat{\pi}_1(u), \hat{\pi}_1(v)] \in Q^{1,0}$. Thus, $\hat{J}_1$ is integrable. \qed

Combining this lemma with Lemma 5.3, we obtain

**Proposition 6.3** $(Y, \{\hat{\omega}_1, \hat{J}_1\})$ is a Kähler manifold.

### 6.2 Bochner flat structure on $(Y, \hat{J}_1)$

Put $e^{iat}z = (e^{iat}z_1, \ldots, e^{iat}z_n)$ for short, similarly for $e^{-iat}w$ for $a = (a_1, \ldots, a_n)$ satisfying (5.2). Let $\mathbb{R}$ act on $\mathcal{N}$ by

$$\rho(t)(s, (z, w)) = (t + s, (e^{iat}z, e^{-iat}w)) \quad (6.12)$$

such that $\rho(\mathbb{R}) \leq \mathbb{R} \times U(2n) \leq \mathcal{N} \leq U(2n) \leq \text{Aut}_{CR}(\mathcal{N})$. There induces another principal bundle $\rho(\mathbb{R}) \to \mathcal{N} \to \mathbb{C}^{2n}$. By (6.1) and (5.5),
\[
\varphi(\rho_1(t)(s,(z+wj))) = \varphi(s + t, (e^{i\alpha} z + e^{i\alpha}wj)) = (s + t, (e^{i\alpha} z, e^{-i\alpha}w)) = \rho(t)\varphi(s, (z+wj)),
\]

there is the bundle isomorphism (cf. (5.9)):

\[
\begin{array}{ccc}
\rho_1(\mathbb{R}) & \longrightarrow & \rho(\mathbb{R}) \\
\downarrow & & \downarrow \\
\mathcal{M}/\mathbb{R}^2 & \xrightarrow{\varphi} & \mathcal{N} \\
\hat{\pi}_1 & \downarrow & q \\
\mathbb{H}^n & \xrightarrow{\hat{\varphi}} & \mathbb{C}^{2n}.
\end{array}
\]

Let \( \xi \) be the vector field induced by \( \rho(\mathbb{R}) \) on \( \mathcal{N} \). Put \( p_1*\xi_1 = \hat{\xi}_1 \) from (5.6), (6.2), which is the Reeb field of \( (\mathcal{M}/\mathbb{R}^2, (\hat{\omega}_1, J_1)) \). We have \( \varphi_*\hat{\xi}_1 = \xi \). At \( p = ((s_1, s_2, s_3), z_1, \ldots, z_n) \in \mathcal{M} \) with \( z_k = u_k + v_kj \), (6.4), (6.3) and (5.7) imply

\[
\omega_\mathcal{N}(\xi) = \hat{\omega}_1(\hat{\xi}_1) = \omega_1(\xi_1) = 1 + \sum_{k=1}^{n} a_k(|u_k|^2 - |v_k|^2). \tag{6.14}
\]

Corresponding to \( X/\mathbb{R}^2 = \{ x \in p_1(X) \mid \hat{\omega}_1((\hat{\xi}_1)_x) \neq 0 \} \), the bundle isomorphism \( \varphi \) maps \( X/\mathbb{R}^2 \) onto the domain \( \mathcal{N}_1 = \{ z \in \mathcal{N} \mid \omega_\mathcal{N}(\xi_z) \neq 0 \} \) of \( \mathcal{N} \). As in (5.10), define the contact forms to be

\[
\hat{\eta}_1 = \frac{1}{\hat{\omega}_1(\hat{\xi}_1)} \hat{\omega}_1 \text{ on } X/\mathbb{R}^2, \quad \eta_{\mathcal{N}_1} = \frac{1}{\omega_\mathcal{N}(\xi)} \omega_\mathcal{N} \text{ on } \mathcal{N}_1. \tag{6.15}
\]

In particular, \( p_1*D_0 = \ker \hat{\eta}_1 \). Noting (6.14), (6.4) shows that

\[
\varphi^*\eta_{\mathcal{N}_1} = \hat{\eta}_1.
\]

Put \( J_{\mathcal{N}_1} = J_{\mathcal{N}/\mathcal{N}_1} \). Since \( (\eta_{\mathcal{N}_1}, J_{\mathcal{N}_1}) \) represents a spherical pseudo-Hermitian structure on \( \mathcal{N}_1 \), this equation together with (6.10) implies (cf. [11])

**Proposition 6.4** The pseudo-Hermitian structure \( (X/\mathbb{R}^2, \hat{\eta}_1, J_1) \) is anti-holomorphically isomorphic to the spherical CR-structure \( (\mathcal{N}_1, \eta_{\mathcal{N}_1}, J_{\mathcal{N}_1}) \).

The projection \( \hat{\pi}_1 : X/\mathbb{R}^2 \rightarrow Y \) of (6.11) induces the following from (5.15), (5.11):

\[
\begin{align*}
\hat{\pi}_1^*\hat{\Omega}_1 &= d\hat{\eta}_1 \text{ on } X/\mathbb{R}^2, \\
\hat{J}_a \circ \hat{\pi}_1^* &= \hat{\pi}_1 \circ J_a \text{ on } p_1*D_0 \ (a = 1, 2, 3). \tag{6.16}
\end{align*}
\]

**Theorem 6.5** \( (Y, \hat{\gamma}, (\hat{\Omega}_1, \hat{J}_1)) \) is an anti-holomorphic Bochner flat Kähler manifold for the quaternionic Hermitian manifold \( (Y, \hat{\gamma}, (\hat{\Omega}_a, \hat{J}_a)_{a=1}^3) \).

**Proof** Since \( d\hat{\eta}_1 = \hat{\pi}_1^*\hat{\Omega}_1 \) from \( X/\mathbb{R}^2 \) by (6.16), \( \rho_1(\mathbb{R}) \rightarrow X/\mathbb{R}^2 \xrightarrow{\hat{\pi}_1} Y \) is a pseudo-Hermitian bundle with the Reeb field \( \hat{\xi}_1 \), the Bochner curvature tensor of the Kähler manifold \( (Y, (\hat{\Omega}_1, \hat{J}_1)) \) coincides with the Chern–Moser curvature tensor on \( (X/\mathbb{R}^2, \hat{\eta}_1, J_1) \) by the

\[\text{ Springer}\]
pull-back of \( \hat{\pi} \) in (6.16) (cf. [16]). As \( X/\mathbb{R}^2 \) is spherical CR by Proposition 6.4, the Chern–Moser curvature tensor is zero and thus \( (Y, \{\hat{\Omega}_1, \hat{J}_1\}) \) is a Bochner flat manifold. \( \square \)

7 Pseudo-Hermitian group \( \text{Psh}(X/\mathbb{R}^2) \)

We determine the holomorphic isometry group \( \text{Isom}_h(Y) = \{ f \in \text{Diff}(Y) \mid f^*\hat{\Omega}_1 = \hat{\Omega}_1, f^*\hat{J}_1 = \hat{J}_1 \} \) of the Kähler manifold \( (Y, (\hat{\Omega}_1, \hat{J}_1)) \). In order to do so, consider the pseudo-Hermitian group of the pseudo-Hermitian manifold \( (X/\mathbb{R}^2, \hat{\eta}_1, J_1) \) (cf. [11]):

\[
\text{Psh}(X/\mathbb{R}^2) = \{ \tilde{f} \in \text{Diff}(X/\mathbb{R}^2) \mid \tilde{f}^*\hat{\eta}_1 = \hat{\eta}_1, \tilde{f}^*J_1 = J_1 \circ \tilde{f} \}.
\]

As \( H^1(Y; \mathbb{R}) = 0 \) (see the remark below (5.9)), it associates the exact sequence from [4, Proposition 3.4]:

\[
1 \longrightarrow \rho_1(\mathbb{R}) \overset{l}{\longrightarrow} \text{Psh}(X/\mathbb{R}^2) \overset{\varphi}{\longrightarrow} \text{Isom}_h(Y) \longrightarrow 1. \tag{7.1}
\]

Since \( \rho_1(\mathbb{R}) \) induces the Reeb field \( \hat{\xi}_1 \) of \( \hat{\eta}_1 \), \( \text{Psh}(X/\mathbb{R}^2) \) itself is the centralizer of \( \rho_1(\mathbb{R}) \) in \( \text{Psh}(X/\mathbb{R}^2) \) (cf. [4, 11]). If we recall the representation of \( \rho(\mathbb{R}) \) from (6.12):

\[
\rho(t) = (t, (e^{ia_1t}, \ldots, e^{ia_nt}, e^{-ia_1t}, \ldots, e^{-ia_nt})) \in \mathbb{R} \times U(2n) \leq \mathcal{N} \rtimes U(2n),
\]

under the equivariant diffeomorphism \( \varphi \) of (6.13), the condition (5.2) implies that

\[
\text{Psh}(X/\mathbb{R}^2) = \mathbb{R} \times T^n \times T^n \leq \mathbb{R} \times U(2n). \tag{7.2}
\]

We have from (7.1) that

**Proposition 7.1** \( \text{Isom}_h(Y) = T^{2n} \).

**Theorem 7.2** The quaternionic Hermitian manifold \( (Y, \hat{g}, \{\hat{J}_a\}_{a=1}^3) \) is not Einstein. Moreover, it is never holomorphically isometric to any domain of the quaternionic euclidean space \( \mathbb{H}^n \).

**Proof** When the Bochner flat manifold \( (Y, \hat{g}, (\hat{\Omega}_1, \hat{J}_1)) \) is Einstein, it is of constant holomorphic curvature by Tachibana’s result [15]. Thus, \( Y \) is locally holomorphically isometric to the flat space \( \mathbb{C}^{2n} \). We may assume that the origin \( 0 \) belongs to \( Y \) (cf. (5.7)). Then, the stabilizer at \( 0 \) is the maximal compact subgroup isomorphic to \( U(2n) \). Since \( T^{2n} \) is the full holomorphic isometry group of \( (Y, \hat{g}, \hat{J}_1) \) by Proposition 7.1, it is impossible. \( (Y, \hat{g}, \hat{J}_1) \) is not holomorphically isometric to any domain of \( (\mathbb{H}^n, g_{\mathbb{H}}) \) with the standard euclidean metric \( g_{\mathbb{H}} \). \( \square \)

7.1 Quaternionic Hermitian isometry group of \( (Y, \hat{g}) \)

The quaternionic Hermitian isometry group of the quaternionic Hermitian manifold \( (Y, \hat{g}, (\hat{\Omega}_a, \hat{J}_a)_{a=1}^3) \) may be denoted naturally by the following:
Isom_{qH}(Y) = \{ \hat{h} \in \text{Diff}(Y) \mid \hat{h}^*\hat{\Omega}_a = \sum_{\beta=1}^3 \hat{\Omega}_\beta\hat{a}_\beta, \hat{h}_*\hat{J}_a = \sum_{\beta=1}^3 \hat{a}_\beta\hat{J}_\beta\hat{h}_* \},

where \((\hat{a}_\beta)_{a,\beta=1,2,3} : Y \longrightarrow \text{SO}(3)\) are smooth maps.

For the abelian group \(\rho_1(\mathbb{R}^3)\) defined by (5.4), let \(N_{\text{Aut}(\mathbb{M})}(\rho_1(\mathbb{R}^3))\) be the normalizer of \(\rho_1(\mathbb{R}^3)\) in \(\text{Aut}(\mathbb{M})\). By the formula (5.3) of \(A_i\), the normalizer of \(\{A_i\}\) in \(\text{Sp}(n)\) is isomorphic to \(T^n\). Since the only subgroup \(S^1 = \langle e^{i\theta} \rangle\) of \(\text{Sp}(1)\) normalizes \(\rho_1(\mathbb{R}^3)\) in view of the actions (3.1) and (5.4), it follows that

\[
N_{\text{Aut}(\mathbb{M})}(\rho_1(\mathbb{R}^3)) = (\mathbb{R}^3, 0) \times T^n \cdot S^1 \leq \mathbb{M} \rtimes \text{Sp}(n) \cdot \text{Sp}(1) \tag{7.3}
\]

where \(T^n \cdot S^1 = T^n \times_{\{\pm 1\}} S^1\). Recall every element of \(\text{Aut}(X)\) extends to an element of \(\text{Aut}(\mathbb{M})\). Since each element of \(N_{\text{Aut}(\mathbb{M})}(\rho_1(\mathbb{R}^3))\) preserves both \(\omega_1\) and \(\xi_1\) from (5.8), it follows \(N_{\text{Aut}(\mathbb{M})}(\rho_1(\mathbb{R}^3)) = N_{\text{Aut}(\mathbb{X})}(\rho_1(\mathbb{R}^3)).\)

Let \(\eta = \eta_i + \eta_j + \eta_k\) be as before (cf. (5.10)). Note \(h^*\eta = a \cdot \eta \cdot \hat{a}\) for \(h = ((t, 0), A \cdot a) \in N_{\text{Aut}(\mathbb{X})}(\rho_1(\mathbb{R}^3))\). By (7.3), the projection \(\pi_1 : \mathbb{X} \longrightarrow Y\) of (5.9) induces an element \(\hat{h} : Y \longrightarrow Y\). Since \(\pi_1^*\hat{\Omega} = d\eta\) for \(\hat{\Omega} = \hat{\Omega}_i + \hat{\Omega}_j + \hat{\Omega}_k\), we have \(h^*\hat{\Omega} = a \cdot \hat{\Omega} \cdot \hat{a}\) for \(a \in S^1\). Thus, it assigns an element \(\hat{h} \in \text{Isom}_{qH}(Y, \hat{\xi}, \{\hat{\Omega}_a, \hat{J}_a\}_{a=1}^3) = \text{Isom}_{qH}(Y)\). Letting \(a = (a_{ab})_{a,\beta=2,3} \in \text{SO}(2)\), \text{Isom}_{qH}(Y)\) can be described as

\[
\{ \hat{h} \in \text{Diff}(Y) \mid \hat{h}^*\hat{\Omega}_1 = \hat{\Omega}_1, \hat{h}_*\hat{J}_a = \hat{J}_1 \circ \hat{h}_*\}, \quad \hat{h}^*\hat{\Omega}_a = \sum_{\beta=2,3} \hat{\Omega}_\beta a_{ab}, \quad \hat{h}_*\hat{J}_a = \sum_{\beta=2,3} a_{ab}\hat{J}_\beta \circ \hat{h}_* \}. \tag{7.4}
\]

Setting \(\bar{\phi}(h) = \hat{h}\), (7.3) gives an exact sequence:

\[
1 \longrightarrow \rho_1(\mathbb{R}^3) \longrightarrow N_{\text{Aut}(\mathbb{X})}(\rho_1(\mathbb{R}^3)) \overset{\bar{\phi}}{\longrightarrow} T^n \cdot S^1 \leq \text{Isom}_{qH}(Y). \tag{7.5}
\]

If \(i : \text{Isom}_{qH}(Y) \longrightarrow \text{Isom}_h(Y) = T^{2n}\) (cf. Proposition 7.1) is the natural inclusion (that is, forgetting the quaternionic structure but leaving the holomorphic structure as it is), then under the equivariant diffeomorphism \(\bar{\phi}\) of (6.5), it follows \(i(T^n \cdot S^1) = \{(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n) \times (e^{i\theta_1}, \ldots, e^{i\theta_n})\} \leq T^{2n}\) where \(z_i \in S^1 (i = 1, \ldots, n)\). We obtain

**Corollary 7.3** The quaternionic Hermitian isometry group \(\text{Isom}_{qH}(Y)\) is isomorphic to the torus \(T^k\) for some \(k (n + 1 \leq k \leq 2n)\).

**Remark 7.4** Since the forms \(\hat{\Omega}_2, \hat{\Omega}_3\) are not Kähler, the equation \(\pi_1^*\hat{\Omega}_a = d\eta_a\) does not hold on \(X\) (only on \(D_0\)), the method of [4, Propositions 3.4, 3.1, 3.2] cannot be applied to show the surjectivity of \(\bar{\phi}\) in (7.5).

**Acknowledgements** We thank the anonymous referee whose significant comments and suggestions greatly improved the exposition of the paper. This work was partially supported by the President Research Grants 2019 at Josai University.

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