A RIGOROUS TREATMENT OF THE PERTURBATION THEORY FOR MANY-ELECTRON SYSTEMS

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Abstract
Four point correlation functions for many electrons at finite temperature in periodic lattice of dimension $d \geq 1$ are analyzed by the perturbation theory with respect to the coupling constant. The correlation functions are characterized as a limit of finite dimensional Grassmann integrals. A lower bound on the radius of convergence and an upper bound on the perturbation series are obtained by evaluating the Taylor expansion of logarithm of the finite dimensional Grassmann Gaussian integrals. The perturbation series up to second order is numerically implemented along with the volume-independent upper bounds on the sum of the higher order terms in 2 dimensional case.

Keywords: Fermionic Fock space; the Hubbard model; Grassmann integral formulation; perturbation theory; numerical analysis.

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1 Introduction

The thermal average of an observable \( \mathcal{O} \) for many electrons in a solid is expressed as
\[
\text{Tr} \frac{e^{-\beta H} \mathcal{O}}{e^{-\beta H}}
\]
where \( H \) is a Hamiltonian representing the total energy of the system, \( \beta \) is the inverse temperature and the trace operation \( \text{Tr} \) is taken over the Fermionic Fock space, the Hilbert space of all the possible states of electrons. If the movements of electrons are confined in finite lattice sites under periodic boundary condition, the Fermionic Fock space becomes finite dimensional. The thermal average \( \text{Tr} \frac{e^{-\beta H} \mathcal{O}}{e^{-\beta H}} \) is defined as a quotient of finite sums over the orthonormal basis spanning the space. Though the expectation value \( \text{Tr} \frac{e^{-\beta H} \mathcal{O}}{e^{-\beta H}} \) has a clear mathematical meaning in this setting, to rigorously control its behavior for interacting electrons poses a challenge. The purpose of this paper is to analyze the thermal expectation value for 4 point functions modeling paired electrons’ condensation by means of the perturbation theory.

In the earlier article [12] Koma and Tasaki rigorously proved upper bounds on 2 point and 4 point correlation functions for the Hubbard model and concluded the decay properties in 1 and 2 dimensional cases. In an abstract general context, on the other hand, Feldman, Kn"orrer and Trubowitz gave a concise representation of the Schwinger functionals formulating the correlation functions via Grassmann integral and established upper bounds of the Schwinger functionals in [5]. Let us also remark the intensive renormalization group study by the same authors in [8], which analyzes the Grassmann integral formulation corresponding to the temperature zero limit of the correlation function for the momentum distribution function. The work [8] was presented as the 11th paper in the series of Feldman, Kn"orrer and Trubowitz’s 2-d Fermi liquid construction. A flow chart showing the hierarchical relation between these 11 papers is found in the digest [7].

In this paper we focus on the correlation function \( \text{Tr} \frac{e^{-\beta H} \mathcal{O}}{e^{-\beta H}} \) for 4 point functions \( \mathcal{O} = \psi_{x_1}^\dagger \psi_{x_2}^\dagger \psi_{y_2} \psi_{y_1} + \psi_{y_1}^\dagger \psi_{y_2}^\dagger \psi_{x_2} \psi_{x_1} \) and the Hubbard model \( H \) defined on a finite lattice. We expand the 4 point correlation function as a perturbation series with respect to the coupling constant and study the properties of the perturbation series. We especially aim at establishing upper bounds on the sum of higher order terms of the perturbation series so that one can numerically measure the error between the correlation function and the low order terms of the perturbation series. More precisely, our goal is set to
(1) find a constant $r > 0$ such that for any $U \in \mathbb{R}$ with $|U| \leq r$

$$\frac{\text{Tr}(e^{-\beta H} \mathcal{O})}{\text{Tr} e^{-\beta H}} = \sum_{n=0}^{\infty} a_n U^n,$$

where $U$ denotes the coupling constant and $a_n \in \mathbb{R}$ ($\forall n \in \mathbb{N} \cup \{0\}$), and to

(2) establish an inequality of the form that for any $U \in \mathbb{R}$ with $|U| \leq r$ and $m \in \mathbb{N} \cup \{0\}$

$$\left| \frac{\text{Tr}(e^{-\beta H} \mathcal{O})}{\text{Tr} e^{-\beta H}} - \sum_{n=0}^{m} a_n U^n \right| \leq R_{m+1}(|U|),$$

where $R_{m+1}(|U|) = O(|U|^{m+1})$ as $|U| \searrow 0$.

The inequality claimed in (2) is proved in Theorem 4.10 as our main result and a volume-independent $r$ required in (1)-(2) is obtained in Proposition 5.1 for 2 dimensional case.

Our strategy is based on the discretization of the integrals over the interval of temperature appearing in the temperature-ordered perturbation series. By replacing the integrals by finite Riemann sums we obtain a fully discrete analog of the perturbation series in which all the variables run in finite sets. The discretized perturbation series is formulated in a finite dimensional Grassmann Gaussian integral, which is rigorously defined as a linear functional on the finite dimensional linear space of Grassmann algebras. See [21] for another approach to the finite dimensional Grassmann integral formulation based on the Lie-Trotter type formula. We then rewrite the 4 point correlation function as the Taylor series expansion of logarithm of the Grassmann Gaussian integral. By evaluating the partial derivatives of logarithm of the Grassmann Gaussian integral, which were characterized as the tree expansion by Salmhofer and Wieczerkowski in [22], and passing the parameter defining the Riemann sum to infinity, we obtain an upper bound on each term of the perturbation series of the original correlation function. For completeness of the paper and convenience for readers, the derivation of the temperature-ordered perturbation series is presented in Appendices.

As a key lemma we make use of the volume- and temperature-independent upper bound on the determinant of the covariance matrix recently established by Pedra and Salmhofer in [18]. Pedra-Salmhofer’s determinant bound enables us to find a numerical upper bound on the Fermionic perturbation theory in a simple argument. As one aim, this paper intends to show a practical application of Pedra-Salmhofer’s determinant bound.

Let us note that the lower bound on the radius of convergence of the perturbation series proved in Theorem 4.10 and Proposition 5.1 below for 2 dimensional case is proportional to $\beta^{-3}$. By applying advanced multi-scale, renormalization techniques to the correlation functions of the 2 dimensional Hubbard model, Rivasseau [19] and Afchain, Magnen and Rivasseau [1] proved that a lower bound on the radius of convergence is proportional to $(\log \beta)^{-2}$, which is larger than our lower bound for large $\beta$, i.e, small temperature. In this article, however, we feature calculating
the quantities in a simple manner so that readers can verify the construction of the
theory by themselves, rather than improving the temperature-dependency of the
convergence of the perturbation theory via large machinery.

Our motivation to implement the perturbation theory for many electrons with
rigorous error estimate numerically was grown amid active research of numerical
analysis for high temperature superconductivity. The macroscopic behavior of elec-
tromagnetic fields around a type-II superconductor is governed by a system of non-
linear Maxwell equations called the macroscopic critical-state models. Prigozhin
initiated the variational formulation of the Bean critical-state model for type-II
superconductivity and reported numerical simulations by finite element method in
[17]. Following Prigozhin’s preceding work [17], finite element approximations of
various macroscopic models have been studied in rigorous levels up until today. See
[2], [11] for the latest developments on this subject. In a smaller length scale, the
density of superconducting charge carriers, the induced magnetic field and motions
of the quantized vortices in a type-II superconductor under an applied magnetic
field can be simulated by solving the mesoscopic Ginzburg-Landau models. Numer-
ical approximation schemes for the Ginzburg-Landau models such as finite element
method, finite difference method and finite volume method are summarized in the
review article [4], which also explains extensions of the Ginzburg-Landau models to
describe high temperature superconductivity characterized by d-wave pairing sym-
metry. We now turn our attention to microscopic models governing many electrons
in a solid and try to approximate the 4 point correlation functions, which are be-
lieved to exhibit the off-diagonal long-range order as explained by Yang in [24] if
superconductivity is happening in the system. However, the concept of error esti-
mate for the numerical computation of the correlation functions formulated in the
Fermionic Fock space is not yet seen in a mathematical literature as we can see for
the macroscopic critical-state models and the mesoscopic Ginzburg-Landau models
today. Hence, in this paper we attempt to propose an error analysis for the nu-
merical approximation of the correlation functions defined in microscopic quantum
theory and implement our numerical scheme in practice.

The contents of this paper are outlined as follows. In Section 2 the model Hamil-
tonian and the correlation function of our interest are defined. The perturbation
series of the correlation function is derived. In Section 3 the temperature-ordered
perturbation series of the partition function is discretized and the discretized par-
tition function is formulated in a finite dimensional Grassmann Gaussian integral.
In Section 4 each coefficient of the perturbation series of the correlation function is
evaluated and upper bounds on the sum over higher order terms are obtained as
our main result. In Section 5 the perturbation series up to 2nd order is numerically
implemented together with the error estimates between the 2nd order perturbation
and the correlation function in 2 dimensional case. In Appendix A the standard
properties of the Fermionic Fock space are reviewed. A self-contained proof for
the temperature-ordered perturbation series expansion is presented in Appendix B.
Finally, the temperature-discrete covariance matrix is diagonalized and its determi-
nant is calculated in Appendix C.


2 The perturbation theory

In this section we define the Hamiltonian operator, formulate 4 point correlation function governed by the Hamiltonian under finite temperature and expand the correlation function as a power series of the coupling constant. To analyze the properties of the power series of the 4 point correlation function derived in this section is set to be the main purpose of this paper.

2.1 The Hubbard model

First of all we define the Hubbard model $H$ as the field Hamiltonian operator on the Fermionic Fock space along with various notations and parameters treated in this paper.

The spacial lattice $\Gamma$ is defined by $\Gamma := \mathbb{Z}^d/(L \mathbb{Z})^d$, where $L (\in \mathbb{N})$ is the length of one edge of the rectangular lattice and $d (\in \mathbb{N})$ stands for the space dimension.

On any set $S$ we define Kronecker’s delta $\delta_{x,y}$ ($x, y \in S$) by $\delta_{x,y} := 1$ if $x$ is identical to $y$ in $S$, $\delta_{x,y} := 0$ otherwise. For example, $\delta_{(0,0),(L,L)} = 1$ for $(0,0),(L,L) \in \mathbb{Z}^2/(L \mathbb{Z})^2$.

For any proposition $A$ the function $1_A$ is defined by

$$1_A := \begin{cases} 1 & \text{if } A \text{ is true}, \\ 0 & \text{otherwise}. \end{cases}$$

Using the annihilation operator $\psi_{x\sigma}$ and the creation operator $\psi^*_{x\sigma}$, which is the adjoint operator of $\psi_{x\sigma}$, at site $x \in \Gamma$ and spin $\sigma \in \{\uparrow, \downarrow\}$, the free part $H_0$ and the interacting part $V$ of the Hubbard model $H$ are defined as follows.

$$H_0 := \sum_{x,y \in \Gamma} \sum_{\sigma,\tau \in \{\uparrow, \downarrow\}} F(x\sigma, y\tau)\psi^*_{x\sigma}\psi_{y\tau}, \quad V := U \sum_{x \in \Gamma} \psi^*_{x\uparrow}\psi_{x\downarrow}\psi_{x\downarrow}\psi_{x\uparrow}, \quad (2.1)$$

where

$$F(x\sigma, y\tau) := \delta_{\sigma,\tau} \left( -t \sum_{j=1}^{d} (\delta_{x,y-e_j} + \delta_{x,y+e_j}) \\
- t' \cdot 1_{d \geq 2} \sum_{j,k=1}^{d} (\delta_{x,y-e_j,-e_k} + \delta_{x,y,e_j+e_k} + \delta_{x,y,e_j-e_k} + \delta_{x,y+e_j,e_k} + \delta_{x,y+e_j,-e_k} - \mu \delta_{x,y} \right),$$

$$\quad (2.2)$$

the vectors $e_j \in \Gamma$ ($j \in \{1, \cdots, d\}$) are given by $e_j(l) = \delta_{j,l}$ for all $j,l \in \{1, \cdots, d\}$.

The parameters $t, t', \mu, U \in \mathbb{R}$ are called the nearest neighbor hopping amplitude, the next to nearest neighbor hopping amplitude, the chemical potential and the coupling constant, respectively. Note that the term representing the next to nearest neighbor hopping in $F(x\sigma, y\tau)$ is effective only for $d \geq 2$. 

\[5\]
The Hubbard model $H$ is defined by $H := H_0 + V$ and is a self-adjoint operator on the Fermionic Fock space $F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C}))$. We summarize the definitions and the basic properties of the Fermionic Fock space, the annihilation, creation operators in Appendix A. Here we note the fact that $\dim F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})) = 2^{2L^d} < +\infty$, which means that any linear operator on $F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C}))$ can be considered as a matrix.

Let us prepare some more notations used in this paper. For any linear operator $A : F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})) \to F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C}))$, $\text{Tr} A$ is defined by

$$\text{Tr} A := \sum_{l=1}^{2^{2L^d}} \langle \phi_l, A\phi_l \rangle_{F_f},$$

where $\langle \cdot, \cdot \rangle_{F_f}$ is the inner product of $F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C}))$ (see Appendix A) and $\{\phi_l\}_{l=1}^{2^{2L^d}}$ is any orthonormal system of $F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C}))$. The correlation function $\langle A \rangle$ under the finite temperature $T$ is defined by

$$\langle A \rangle := \frac{\text{Tr}(e^{-\beta H} A)}{\text{Tr} e^{-\beta H}},$$

where $\beta := 1/(k_B T) > 0$ with the Boltzmann constant $k_B > 0$.

The momentum lattice $\Gamma^*$ is defined by $\Gamma^* := (2\pi \mathbb{Z}/L)^d/(2\pi \mathbb{Z})^d$.

For any vectors $\alpha, \gamma$ of algebra of length $n$, let $\langle \alpha, \gamma \rangle$ denote $\sum_{l=1}^n \alpha(l)\gamma(l)$. Let $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ denote the inner product of $\mathbb{C}^n$ defined by $\langle u, v \rangle_{\mathbb{C}^n} := \sum_{l=1}^n u^*(l)v(l)$ for any $u, v \in \mathbb{C}^n$.

For any finite set $S$, $\#S$ stands for the number of elements contained in $S$.

Let $S_n$ denote the set of all the permutations on $n$ elements for $n \in \mathbb{N}$.

## 2.2 The correlation function

Our goal is to analyze the 4 point correlation function $\langle \psi_{x_1}^\uparrow, \psi_{x_2}^\uparrow, \psi_{y_2}^\downarrow, \psi_{y_1}^\downarrow \rangle$ by means of the perturbation method with respect to the coupling constant $U$. The correlation function of our interest can be derived from the logarithm of the partition function. Let us substitute real parameters $\{\lambda_{x,y,z,w}\}_{x,y,z,w \in \Gamma} (\subset \mathbb{R})$ into our Hamiltonian $H_\lambda$ and define the parameterized Hamiltonian $H_\lambda$ by

$$H_\lambda := H_0 + V_\lambda, \quad V_\lambda := \sum_{x,y,z,w \in \Gamma} U_{x,y,z,w} \psi_x^\uparrow \psi_y^\uparrow \psi_z^\downarrow \psi_w^\downarrow,$$

where we set

$$U_{x,y,z,w} := U \delta_{x,y} \delta_{z,w} \delta_{x,z} + \lambda_{x,y,z,w} + \lambda_{z,w,x,y},$$

for all $x, y, z, w \in \Gamma$. Note that $H_\lambda$ still keeps the self-adjoint property and that $H_\lambda |_{\lambda_{x,y,z,w} = 0, \forall \{x,y,z,w\} \in \Gamma} = H$.

To simplify notations, let $\mathcal{X}$ represent a vector in $\Gamma^4$ in our argument unless otherwise stated. From now we fix 4 sites $x_1, x_2, y_1, y_2 \in \Gamma$ to define the correlation
function \( \langle \psi_{x_1}^*, \psi_{x_2}^* \rangle \psi_{y_2} \psi_{y_1} + \psi_{y_2}^* \psi_{y_1} \rangle \psi_{x_1} \psi_{x_1} \) and write \( \tilde{x}_1 = (x_1, x_2, y_1, y_2) \) and \( \tilde{x}_2 = (y_1, y_2, x_1, x_2) \).

**Lemma 2.1.** The following equality holds.

\[
\langle \psi_{x_1}^*, \psi_{x_2}^* \rangle \psi_{y_2} \psi_{y_1} + \psi_{y_2}^* \psi_{y_1} \rangle \psi_{x_1} \psi_{x_1} = -\frac{1}{\beta} \frac{\partial}{\partial \lambda_{x_1}} \log \left( \frac{\text{Tr} e^{-\beta H_{\lambda}}} {\text{Tr} e^{-\beta H_{0}}} \right) \bigg|_{\lambda_{x_1} = 0},
\]

(2.5)

**Remark 2.2.** Since \( H_{\lambda} \) is self-adjoint, its spectrum \( \sigma(H_{\lambda}) \) is a subset of \( \mathbb{R} \). The spectral mapping theorem (see, e.g., Section VIII-7, Corollary 1) shows that \( \{e^{-\beta \lambda_{x}} \}_{x \in \sigma(H_{\lambda})} \) is the spectrum of \( e^{-\beta H_{\lambda}} \). Thus, \( \text{Tr} e^{-\beta H_{0}} > 0 \). For the same reason as above the inequality \( \text{Tr} e^{-\beta H_{\lambda}} > 0 \) holds. Therefore, \( \log(\text{Tr} e^{-\beta H_{\lambda}} / \text{Tr} e^{-\beta H_{0}}) \) is well-defined.

Let \( L(F_j(L^2(\Gamma \times \{1, 1\}; \mathbb{C}) \) denote the space of linear operators on \( F_j(L^2(\Gamma \times \{1, 1\}; \mathbb{C}) \). The proof of Lemma 2.1 is based on the following lemma.

**Lemma 2.3.** Let \( (a, b) \) be an interval of \( \mathbb{R} \). Assume that \( A : (a, b) \to L(F_j(L^2(\Gamma \times \{1, 1\}; \mathbb{C}) \) is an operator-valued \( C^1 \)-class function. The following equality holds. For all \( s \in (a, b) \)

\[
\frac{d}{ds} e^{A(s)} = \int_0^1 e^{(1-t)A(s)} \frac{d}{ds} A(s) e^{tA(s)} dt.
\]

**Proof.** Fix any \( s \in (a, b) \) and take small \( \varepsilon > 0 \) such that \( [s - \varepsilon, s + \varepsilon] \subset (a, b) \). For any \( s' \in (s - \varepsilon, s + \varepsilon) \)

\[
e^{A(s)} - e^{A(s')} = \left[-e^{(1-t)A(s)} e^{tA(s')} \right]_{t=0}^1 = \int_0^1 \frac{d}{dt} (e^{(1-t)A(s)} e^{tA(s')}) dt
\]

\[
= \int_0^1 e^{(1-t)A(s)} (A(s) - A(s')) e^{tA(s')} dt.
\]

Moreover, we see that

\[
\frac{d}{ds} e^{A(s)} = \lim_{s' \to s} \frac{e^{A(s)} - e^{A(s')}}{s - s'}
\]

\[
= \lim_{s' \to s} \int_0^1 e^{(1-t)A(s)} \frac{A(s) - A(s')}{s - s'} e^{tA(s')} dt = \int_0^1 e^{(1-t)A(s)} \frac{d}{ds} A(s) e^{tA(s)} dt,
\]

where we have used the inequality

\[
\left\| \frac{A(s) - A(s')}{s - s'} e^{tA(s')} \right\| \leq \sup_{\theta \in [s - \varepsilon, s + \varepsilon]} \left\| \frac{d}{ds} A(\theta) \right\| e^{\sup_{\theta \in [s - \varepsilon, s + \varepsilon]} \| A(\theta) \|}
\]

with the operator norm \( \| \cdot \| \) and Lebesgue’s dominated convergence theorem to exchange the order of the limit operation and the integral. \( \square \)
Proof of Lemma 2.2. Since the operator-valued function \( \lambda_{\hat{x}_1} \to H_{\lambda} \) is continuously differentiable on any interval containing 0 inside, we can apply Lemma 2.3 to have

\[
- \frac{1}{\beta} \left. \frac{\partial}{\partial \lambda_{\hat{x}_1}} \log \left( \frac{\text{Tr} e^{-\beta H_{\lambda}}}{\text{Tr} e^{-\beta H_0}} \right) \right|_{\lambda_{\hat{x}_1} = 0} = - \frac{1}{\beta} \frac{\text{Tr} \left( \int_0^1 e^{(1-t)(-\beta H)} \frac{\partial}{\partial \lambda_{\hat{x}_1}} (-\beta H_{\lambda}) \right) \bigg|_{\lambda_{\hat{x}_1} = 0} e^{t(-\beta H) dt} \right) \frac{\text{Tr} e^{-\beta H}}{\text{Tr} e^{-\beta H}} = \int_0^1 \frac{\text{Tr} \left( e^{(1-t)(-\beta H)} \psi_{x_1}^* \psi_{x_2}^* \psi_{y_{y_1}} \psi_{y_{y_1}} + \psi_{x_1}^* \psi_{x_2}^* \psi_{y_{y_1}} \psi_{y_{y_1}} \right) e^{t(-\beta H)}}{\text{Tr} e^{-\beta H}} dt
\]

where the constraint \( x_{2j-1} \) requires the variable \( x_{2j} \), to take the same value as \( x_{2j-1} \) for all \( j \in \{1, 2, \ldots, n\} \) and each component of the covariance matrix \( (C(x_j \sigma_j x_j, y_k \sigma_k x_k))_{1 \leq j, k \leq 2n} \) \( \forall x_j \in \{1, 2, \ldots, n\} \) is defined by

\[
C(x_\sigma x, y_\tau y) := \frac{\delta_{\sigma \tau}}{d} \sum_{k \in \Gamma^*} e^{i(k, y-x)} e^{-\frac{(y-x)^2}{2E_k}} \left( \frac{1_{y-x \leq 0}}{1 + e^{\beta E_k}} - \frac{1_{y-x > 0}}{1 + e^{-\beta E_k}} \right)
\]

with the dispersion relation

\[
E_k := -2t \sum_{j=1}^d \cos(k, e_j) - 4t' \sum_{d \geq 2} \sum_{j \leq k} \cos(k, e_j) \cos(k, e_k) - \mu.
\]
Lemma 2.1 indicates that we can construct the power series of $(\psi_{x,j}^* \psi_{y,j} + \psi_{x,j}^* \psi_{y,j})$ by substituting the series (2.6) into the Taylor series expansion of the function $\log(x)$ around $x = 1$. Since the radius of convergence of the Taylor series of $\log(z)$ around $1$ is $1$, we need to know when the inequality $|\text{Tr} e^{-\beta H_\lambda}/\text{Tr} e^{-\beta H_0} - 1| < 1$ holds beforehand. An answer will be given to this question in Proposition 2.7 below.

It will be more convenient for our analysis to generalize the problems so that the variables $U$, $\lambda_X \in \Gamma^4$, $\{U_X\}_{X \in \Gamma^4}$ are allowed to be complex. We will then recover the statements on our original problem by restricting the variables to be real. For \{U_X\}_{X \in \Gamma^4} \subset \mathbb{C}$ we define a function $P(\{U_X\}_{X \in \Gamma^4})$ by the power series of the right hand side of (2.6). Let us recall that the real function $\log(x)$ $(x > 0)$ is extended to be the complex analytic function $\log(z)$ in the domain \{z \in \mathbb{C} \mid |z - 1| < 1\} by the power series

$$\log(z) = \sum_{n=1}^\infty (-1)^{n-1} \frac{1}{n}(z - 1)^n.$$ 

In our argument to clarify when the inequality

$$|P(\{U_X\}_{X \in \Gamma^4}) - 1| < 1 \quad (2.9)$$

holds as well as in the proofs of other lemmas in this paper, the following lemma on the determinant bound on the covariance matrix plays essential roles.

Lemma 2.5. [18, Theorem 2.4] For any $n \in \mathbb{N}$, $(x_j, \sigma_j, x_j)$, $(y_j, \tau_j, y_j) \in \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta]$ $(\forall j \in \{1, \cdots, n\})$,

$$\sup_{u_j, v_j \in \mathbb{C}^n \text{ with } \|u_j\|_{\mathbb{C}^n}, \|v_j\|_{\mathbb{C}^n} \leq 1, \forall j \in \{1, \cdots, n\}} \left|\text{det}(\langle u_j, v_k \rangle_{\mathbb{C}^n} C(x_j \sigma_j x_j, y_k \tau_k y_k))_{1 \leq j, k \leq n}\right| \leq 4^n,$$

where $\|u\|_{\mathbb{C}^n} := \langle u, u \rangle_{\mathbb{C}^n}^{1/2}$ for all $u \in \mathbb{C}^n$.

Remark 2.6. The statement of [18, Theorem 2.4] is on the determinant bound of the covariance matrices independent of the spin coordinate. It is, however, straightforward to derive the bound claimed in Lemma 2.5 on our spin-dependent covariance matrix from [18, Theorem 2.4].

We can expand $-1/\beta \partial/\partial \lambda_X \log(P(\{U_X\}_{X \in \Gamma^4}))|_{\lambda_X = 0, \forall X \in \Gamma^4}$ as a power series of $U$ as follows.

Proposition 2.7. Assume that $U \in \mathbb{C}$ satisfies $|U| < \log 2/(16 \beta L^d)$. Then there exists $\varepsilon > 0$ such that if $\{\lambda_X\}_{X \in \Gamma^4}$ satisfies $|\lambda_X| \leq \varepsilon$ for all $X \in \Gamma^4$, the inequality (2.4) holds. Moreover, we have

$$-\frac{1}{\beta} \frac{\partial}{\partial \lambda_X} \log(P(\{U_X\}_{X \in \Gamma^4}))|_{\lambda_X = 0, \forall X \in \Gamma^4} = \sum_{n=0}^\infty a_n U^n, \quad (2.10)$$
where the coefficients \( \{a_n\}_{n=0}^{\infty} \) are given by

\[
a_n := \frac{1}{\beta} \frac{\partial}{\partial \lambda_{x_1}} \left( \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{j} \sum_{m_1 + \ldots + m_j = n+1} \sum_{m_k \geq 1, \forall k \in \{1, \ldots, j\}} \frac{j}{\prod_{k=1}^{j} G_{m_k}} \right) \bigg|_{\lambda_{x_1} = 0, \forall x \in \Gamma^4},
\]

(2.11)

with \( \{G_n\}_{n=1}^{\infty} \) defined by

\[
G_n := \frac{1}{n!} \prod_{j=1}^{n} \left( 1 - \sum_{x_j \in \{1, \ldots, \beta L^d\}} \sum_{y_j \in \{1, \ldots, \beta L^d\}} \int_{0}^{\beta} dx_{2j-1} \delta_{y_{2j-1}, 1} \delta_{x_{2j-1}, 1} \cdot \left( \delta_{x_{2j-1}, x_j} \delta_{y_{2j-1}, y_j} + \delta_{x_{2j-1}, 1} \delta_{y_{2j-1}, 1} + \delta_{y_{2j-1}, 1} \delta_{x_{2j-1}, 1} \right) \right) \cdot \det(C(x_j, \sigma_j, x_j, \sigma_k, x_j))_{1 \leq j, k \leq 2n} \bigg|_{x_j, y_j = \{1, \ldots, \beta L^d\}}.
\]

(2.12)

**Proof.** Let us fix \( U \in \mathbb{C} \) with \(|U| < 2/(16\beta L^d)\). Take any \( \varepsilon \in (0, \log 2/(16\beta L^d) - |U|/2) \) and assume that \( \{\lambda_x\}_{x \in \Gamma^4} \) satisfies \(|\lambda_x| \leq \varepsilon \) for all \( x \in \Gamma^4 \). Then, we see that for all \( x \in \Gamma^4 \)

\[
|U_x| < \frac{\log 2}{16\beta L^d}.
\]

(2.13)

By using the inequality (2.13) and Lemma 2.5, we observe that

\[
|P(\{U_x\}_{x \in \Gamma^4}) - 1| < \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{\beta L^d \cdot \log 2}{16\beta L^d} \cdot 16 \right)^n = e^{\log 2} - 1 = 1.
\]

(2.14)

The inequality (2.11) allows us to consider \( \log(P(\{U_x\}_{x \in \Gamma^4})) \) as an analytic function of the multi-variable \( \{U_x\}_{x \in \Gamma^4} \) in the domain (2.13). Moreover, we have

\[
\frac{-1}{\beta} \frac{\partial}{\partial \lambda_{x_1}} \log(P(\{U_x\}_{x \in \Gamma^4})) \bigg|_{\lambda_{x_1} = 0, \forall x \in \Gamma^4} = -1 \sum_{m=0}^{\infty} (-1)^m \left( P(\{U_x\}_{x \in \Gamma^4}) \bigg|_{\lambda_{x_1} = 0, \forall x \in \Gamma^4} - 1 \right)^m \frac{\partial}{\partial \lambda_{x_1}} P(\{U_x\}_{x \in \Gamma^4}) \bigg|_{\lambda_{x_1} = 0, \forall x \in \Gamma^4},
\]

(2.15)

where we used the equality that \( d \log(z)/dz = \sum_{m=0}^{\infty} (-1)^m (z - 1)^m \) (\( \forall z \in \mathbb{C} \) with \(|z - 1| < 1\)). Furthermore, we can write

\[
\frac{\partial}{\partial \lambda_{x_1}} P(\{U_x\}_{x \in \Gamma^4}) \bigg|_{\lambda_{x_1} = 0, \forall x \in \Gamma^4} = \sum_{n=1}^{\infty} \frac{\partial}{\partial \lambda_{x_1}} G_n \bigg|_{\lambda_{x_1} = 0, \forall x \in \Gamma^4} U^n - 1, 
\]

(2.16)
where $G_n$ ($n \in \mathbb{N}$) is defined in \ref{eq:2.12}. By substituting \ref{eq:2.10} into \ref{eq:2.16} we obtain

\[
\frac{1}{\beta} \frac{\partial}{\partial \lambda_{x_i}} \log(P(\{U_x\}_{x \in \Gamma^4})) \bigg|_{\lambda_{x_i} = 0 \quad \forall x_i \in \Gamma^4} \\
= -\frac{1}{\beta} \sum_{m=0}^{\infty} \left( -\sum_{n=1}^{\infty} G_n \bigg|_{\lambda_{x_i} = 0 \quad \forall x_i \in \Gamma^4} U^n \right) \sum_{n=1}^{\infty} \frac{\partial}{\partial \lambda_{x_i}} G_n \bigg|_{\lambda_{x_i} = 0 \quad \forall x_i \in \Gamma^4} U^{n-1}. 
\tag{2.17}
\]

Again by using Lemma 2.3 we can show that for $U \in \mathbb{C}$ with $|U| < \log 2/(16\beta L^d)\notag$

\[
\sum_{n=1}^{\infty} G_n \bigg|_{\lambda_{x_i} = 0 \quad \forall x_i \in \Gamma^4} |U|^n < 1, \quad \sum_{n=1}^{\infty} \frac{\partial}{\partial \lambda_{x_i}} G_n \bigg|_{\lambda_{x_i} = 0 \quad \forall x_i \in \Gamma^4} |U|^{n-1} < \infty. 
\tag{2.18}
\]

Since the radius of convergence of the power series $\sum_{m=0}^{\infty} z^m$ is 1, the inequalities \ref{eq:2.18} provide a sufficient condition to reorder the right hand side of \ref{eq:2.17} (see, e.g., [14, Theorem 3.1, Theorem 3.4] for products and compositions of convergent power series) to deduce

\[
\frac{1}{\beta} \frac{\partial}{\partial \lambda_{x_i}} \log(P(\{U_x\}_{x \in \Gamma^4})) \bigg|_{\lambda_{x_i} = 0 \quad \forall x_i \in \Gamma^4} \\
= -\frac{1}{\beta} \sum_{m=0}^{\infty} \left( \sum_{n=1}^{\infty} \frac{\partial}{\partial \lambda_{x_i}} G_n \bigg|_{\lambda_{x_i} = 0 \quad \forall x_i \in \Gamma^4} \right) \sum_{n=1}^{\infty} \sum_{j=1}^{n+l} \sum_{m_k=0}^{n+1-l} (-1)^k \left( G_{m_k} \bigg|_{\lambda_{x_i} = 0 \quad \forall x_i \in \Gamma^4} + \frac{\partial}{\partial \lambda_{x_i}} G_{n+1} \bigg|_{\lambda_{x_i} = 0 \quad \forall x_i \in \Gamma^4} \right) U^n. 
\tag{2.19}
\]

Arranging \ref{eq:2.19} yields \ref{eq:2.11}.

By restricting $U$ to be real in \ref{eq:2.10}, we obtain the power series expansion of the correlation function $\langle \psi_{x_1}^* \psi_{x_2}^* \psi_{y_1} \psi_{y_2} \rangle$. At this point, however, we only know that the series $\sum_{n=0}^{\infty} a_n U^n$ converges for $U \in \mathbb{C}$ with $|U| < \log 2/(16\beta L^d)$, which heavily depends on the volume factor $L^d$. With the aim of enlarging the radius of convergence and finding upper bounds of the power series $\sum_{n=0}^{\infty} a_n U^n$, we will construct our theory in the following sections.

\section{Grassmann Gaussian integral formulation}

In this section we discretize the integrals over $[0, \beta]$ contained in the perturbation series $P(\{U_x\}_{x \in \Gamma^4})$ so that the discretized perturbation series can be formulated in a Grassmann Gaussian integral involving only finite dimensional Grassmann algebras. Moreover, by showing that the discrete analog of $P$ uniformly converges to the original $P$, we characterize our partition function $\text{Tr} e^{-\beta H} / \text{Tr} e^{-\beta H_0}$ and the $4$
prop function $\langle \psi_{x_1}^* \psi_{x_1}^* \psi_{y_1} \psi_{y_1} | + \psi_{x_1}^* \psi_{y_1} \psi_{x_1} \psi_{y_1} \rangle$ as a limit of finite dimensional Grassmann integrals. The finite dimensional Grassmann Gaussian integral formulation will then enable us to apply the tree formula for the connected part of the exponential of Laplacian operator of the Grassmann left derivatives to express each term of the discretized perturbation series as a finite sum over trees in Section 3.1 Discretization of the integral over $[0, \beta]$

We define the fully discrete perturbation series by replacing the integral $\int_0^\beta dx$ in $P(\{U_X\}_{X \in \Gamma^4})$ by the Riemann sum. Let us introduce finite sets $[0, \beta)_h$ and $(-\beta, \beta)_h$ parameterized by $h \in \mathbb{N}/\beta$ as follows.

$$[0, \beta)_h := \left\{ 0, \frac{1}{h}, \frac{2}{h}, 0, \ldots, \beta - \frac{1}{h} \right\},$$

$$[-\beta, \beta)_h := \left\{ -\beta, -\beta + \frac{1}{h}, -\beta + \frac{2}{h}, \ldots, \beta - \frac{1}{h} \right\}.$$

Note that $\sharp[0, \beta)_h = \beta h$ and $\sharp[-\beta, \beta)_h = 2\beta h$. We define the function $P_h(\{U_X\}_{X \in \Gamma^4})$ of the multi-variable $\{U_X\}_{X \in \Gamma^4}(\subset \mathbb{C})$ by

$$P_h(\{U_X\}_{X \in \Gamma^4}) :=$$

$$1 + \sum_{n=1}^{L^4 \beta h} \frac{1}{n!} \prod_{j=1}^{n} \left( - \frac{1}{h} \sum_{x_{j-1}, x_j, y_{j-1}, y_j \in \Gamma, \sigma_{j-1}, \sigma_j \in \{1, \ldots, \} x_{j-1}, x_j, y_{j-1}, y_j \in [0, \beta)_h} \delta_{\sigma_{j-1}, \sigma_j} \delta_{x_{j-1}, x_j, y_{j-1}, y_j} \right) \det(C(x_j, \sigma_j, y_j, k, \sigma_k x_k))_{1 \leq j, k \leq 2n}.$$

(3.1)

Note that if $n > L^4 \beta h$, $\det(C(x_j, \sigma_j, y_j, k, \sigma_k x_k))_{1 \leq j, k \leq 2n} = 0$ for any $(x_j, \sigma_j, x_j), (y_j, \tau_j, y_j) \in \Gamma \times \{1, \ldots, \} \times [0, \beta)_h (j \in \{1, \ldots, 2n\})$, since $\sharp \Gamma \times \{1, \ldots, \} \times [0, \beta)_h = 2L^4 \beta h$.

Let us summarize the properties of the function $P_h$ in the same manner as in Proposition 2.4.

**Lemma 3.1.** Assume that $U \in \mathbb{C}$ satisfies $|U| < \log 2/(16 \beta L^4)$. The following statements hold.

(i) There exists $\epsilon > 0$ such that for any $\{\lambda_X\}_{X \in \Gamma^4}(\subset \mathbb{C})$ with $|\lambda_X| \leq \epsilon$ (\forall X \in \Gamma^4) and any $h \in \mathbb{N}/\beta$, the inequality $|P_h(\{U_X\}_{X \in \Gamma^4}) - 1| < 1$ holds.

(ii) For any $h \in \mathbb{N}/\beta$

$$-\frac{1}{\beta \partial \partial \lambda_X} \log(P_h(\{U_X\}_{X \in \Gamma^4}))(\lambda_X = 0)_{\forall X \in \Gamma^4} = \sum_{n=0}^{\infty} a_{h, n} U^n,$$
where the coefficients \( \{a_{n,h}\}_{n=0}^{\infty} \) are given by

\[
a_{n,h} := -\frac{1}{\beta} \frac{\partial}{\partial \lambda_x} \left( \sum_{j=1}^{n+1} (-1)^{j-1} \sum_{m_1 + \cdots + m_j = n+1 \atop m_k \geq 1, \forall k \in \{1, \ldots, j\}} \prod_{k=1}^{j} G_{h,m_k} \right)_{\lambda_x = 0, \psi \in \Gamma^4},
\]

with \( \{G_{h,n}\}_{n=1}^{\infty} \) defined by

\[
G_{h,n} := \frac{1}{n!} \prod_{j=1}^{n} \left( -\frac{1}{h} \sum_{x_{j-1}, x_j, y_{j-1}, y_j} \sum_{\sigma_{j-1}, \sigma_j \in \Gamma} \sum_{x_{j-1}, x_j, y_{j-1}, y_j} \delta_{\sigma_{j-1}} \delta_{\sigma_j} \delta_{x_{j-1}, x_j} \cdot \lambda x_{j-1}, x_j, y_{j-1}, y_j \right)
\]

\[
\cdot \det(C(x_j \sigma_j x_j, y_k \sigma_k x_k))_{1 \leq j, k \leq 2n}.
\]

(iii) For all \( n \in \mathbb{N} \cup \{0\} \), \( \lim_{h \to +\infty, h \in \mathbb{N}/\beta} a_{n,h} = a_n \), where \( \{a_n\}_{n=0}^{\infty} \) is defined in (2.11 - 2.12).

**Proof.** The proofs for the claims (i) and (ii) are parallel to that of Proposition 2.7 based on Lemma 2.5. By the definition (2.17) \( \det(C(x_j \sigma_j x_j, y_k \sigma_k x_k))_{1 \leq j, k \leq 2n} \) is piece-wise smooth with respect to the variables \( \{x_j\}_{j=1}^{2n} \), which implies that the Riemann sums over \( [0, \beta]_h \) in \( G_{h,n} \) all converge to the corresponding integrals in \( G_n \) as \( h \to +\infty \). Thus, the claim (iii) is true.

Lemma 3.3 (iii) tells us that establishing an \( h \)-dependent upper bound on \( |a_{n,h}| \) and showing that the upper bound converges as \( h \to +\infty \) lead to finding a bound on \( |a_n| \). This goal will be achieved in Section 4.

The main aim of this section is to formulate \( P_h \) as a finite dimensional Grassmann Gaussian integral, which will be used in the characterization of the coefficients \( \{a_{n,h}\}_{n=0}^{\infty} \) in Section 4. Though it is not directly required in our search for the upper bound on \( \sum_{n=0}^{\infty} a_n U^n \), to represent the original partition function \( P^n \) and the 4 point function \( \langle \psi_{x_1}^* \psi_{x_2} \psi_{x_3} \psi_{x_4} \rangle \) as a limit of the finite dimensional Grassmann integrals also interests us. The following uniform convergence property of \( P_h \) provides a framework to this purpose. The following proposition will be referred in the proof of our main theorem Theorem 4.10 as well.

**Proposition 3.2.** For any \( r > 0 \)

\[
\lim_{h \to +\infty} \sup_{U_X \in \mathbb{C} \text{ with } |U_X| \leq r} |P_h(\{U_X\} \in \Gamma^4) - P(\{U_X\} \in \Gamma^4)| = 0.
\]

**Remark 3.3.** For the same reason as for the convergence property Lemma 3.1 (iii), each term of the series \( P_h(\{U_X\} \in \Gamma^4) \) converges to the corresponding term of
Note that by using the property that \( E \) we present an elementary proof without employing the convergence theorem of the Lebesgue integration theory.

**Proof of Proposition 3.2** By using Lemma 2.5 and the inequality that \(|U_X| \leq r \, (\forall \lambda' \in \Gamma^4)\), we have

\[
|P(\{U_X \}_{x \in \Gamma^4}) - P_h(\{U_X \}_{x \in \Gamma^4})| \leq \sum_{n=\beta h+1}^{\infty} \frac{2}{n!} (r_β L^{4d})^n 4^{2n}
\]

\[
+ \sum_{n=2}^{\beta h} \frac{1}{n!} \prod_{j=1}^{n-1} \left( \sum_{x_{j-1}, s_{j-1}, y_j \in \Gamma, \sigma_{j-1}, \sigma_j \in \{\|, \|\}} \delta_{\sigma_{j-1}} \delta_{\sigma_j} \right)
\]

\[
\cdot \left( \prod_{j=1}^{\beta h} \left( \int_0^{\beta} ds_{j-1} \right) \right) \det(C(x_j \sigma_j s_j, y_k \sigma_k s_k))_{1 \leq j, k \leq 2n \mid \sigma_j = \sigma_j - 1, \forall \lambda' \in \{\|, \|\}}
\]

\[
- \sum_{n=2}^{\beta h} \frac{1}{n!} \prod_{j=1}^{n-1} \left( \sum_{x_{j-1} \in (0, \beta) h} \right) \det(C(x_j \sigma_j x_j, y_k \sigma_k x_k))_{1 \leq j, k \leq 2n \mid \sigma_j = \sigma_j - 1, \forall \lambda' \in \{\|, \|\}}
\]

\[
\leq \sum_{n=\beta h+1}^{\infty} \frac{2}{n!} (r_β L^{4d})^n 4^{2n}
\]

\[
+ \sum_{n=2}^{\beta h} \frac{1}{n!} (rL^{4d})^n \sup_{x_j, y_j \in \Gamma, \sigma_j \in \{\|, \|\}} \prod_{j=1}^{n-1} \left( \sum_{l_{j-1}=0}^{(l_{j-1})/h} \int_{l_{j-1}/h}^{l_{j-1}/h} ds_{j-1} \right)
\]

\[
\cdot \left( \det(C(x_j \sigma_j s_j, y_k \sigma_k s_k))_{1 \leq j, k \leq 2n \mid \sigma_j = \sigma_j - 1, \forall \lambda' \in \{\|, \|\}} - \det(C(x_j \sigma_j l_j/h, y_k \sigma_k l_k/h))_{1 \leq j, k \leq 2n \mid \sigma_j = \sigma_j - 1, \forall \lambda' \in \{\|, \|\}} \right)
\]

(3.5)

We especially need to show that the second term of the right hand side of the inequality (3.5) converges to 0 as \( h \to +\infty \).

Let us fix \( n \in \{2, 3, \ldots, \beta h\} \) and \( x_j, y_j \in \Gamma, \sigma_j \in \{\|, \|\} \, (\forall \lambda' \in \{1, \ldots, 2n\}) \).

There exists a function

\[
g : (-\beta, \beta)^{n(n-1)/2} \to \mathbb{R}, \ g \in C^\infty((-\beta, \beta) \setminus \{0\})^{n(n-1)/2}
\]

such that for all \( s_{j-1} \in [0, \beta) \, (\forall \lambda' \in \{1, \ldots, n\}) \)

\[
g(s_1 - s_3, s_1 - s_5, \ldots, s_1 - s_{2n-1}, s_3 - s_5, \ldots, s_3 - s_{2n-1}, \ldots, s_{2n-3} - s_{2n-1})
\]

\[
= \det(C(x_j \sigma_j s_j, y_k \sigma_k s_k))_{1 \leq j, k \leq 2n \mid \sigma_j = \sigma_j - 1, \forall \lambda' \in \{\|, \|\}}
\]

Note that by using the property that \( E_k = E_\lambda \) for all \( k \in \Gamma^* \), we can show \( C(x \sigma x, y \tau y) \in \mathbb{R} \) for all \( (x, \sigma, x), (y, \tau, y) \in \Gamma \times \{\|, \|\} \times [0, \beta) \). Thus, the function
$g$ is chosen to be real-valued. Then we see that

$$
\prod_{j=1}^{n} \left( \sum_{l_{2j-1}=0}^{\beta h-1} \int_{l_{2j-1}/h}^{(l_{2j-1}+1)/h} ds_{2j-1} \right)
\cdot \left| \det(C(x_j \sigma_j s_j, y_k \sigma_k s_k))_{1 \leq j, k \leq 2n} \right|
\cdot \left( \frac{\beta h}{\beta h-1} \prod_{l_{2j-1}=0}^{\beta h-1} \int_{l_{2j-1}/h}^{(l_{2j-1}+1)/h} ds_{2j-1} \right)
\cdot g(s_1 - s_3, \ldots, s_{2n-3} - s_{2n-1}) - \frac{g(l_1/h - l_3/h, \ldots, l_{2n-3}/h - l_{2n-1}/h)}{l_{2j-1} - l_{2k-1}/h}
$$

(3.6)

where the functions $\chi_1, \chi_{1,s}$ are defined by

$$
\chi_1 := \begin{cases} 
1 & \text{if there exist } j, k \in \{1, \ldots, n\} \text{ such that } j \neq k \text{ and } l_{2j-1} = l_{2k-1}, \\
0 & \text{otherwise},
\end{cases}
$$

and

$$
\chi_{1,s} := \begin{cases} 
1 & \text{if } s_{2j-1} - s_{2k-1} \neq l_{2j-1}/h - l_{2k-1}/h \\
0 & \text{otherwise}.
\end{cases}
$$

Let us fix $l = (l_1, l_3, \ldots, l_{2n-1})$ and $s = (s_1, s_3, \ldots, s_{2n-1})$ with

$$
l_{2j-1} \in \{0, 1, \ldots, \beta h - 1\}, s_{2j-1} \in (l_{2j-1}/h, (l_{2j-1} + 1)/h) \text{ for all } j \in \{1, \ldots, n\}
$$

satisfying $\chi_1 = 0$ and $\chi_{1,s} = 1$. In this case $l_{2j-1} \neq l_{2k-1}$ and $s_{2j-1} - s_{2k-1} \neq l_{2j-1}/h - l_{2k-1}/h$ for all $j, k \in \{1, \ldots, n\}$ with $j \neq k$. Note that if $l_{2j-1} < l_{2k-1}, l_{2j-1}/h - l_{2k-1}/h - s_{2k-1} \in (-\beta, 0)$. If $l_{2j-1} > l_{2k-1}, l_{2j-1}/h - l_{2k-1}/h - s_{2j-1} \in (0, \beta)$. Let us set the interval $I(b, c)$ for $b, c \in \mathbb{R}$ with $b \neq c$ by

$$
I(b, c) := [b, c] \text{ if } b < c, [c, b] \text{ if } b > c.
$$

Then we see that $I(s_{2j-1} - s_{2k-1}, l_{2j-1}/h - l_{2k-1}/h) \subset (-\beta, \beta) \setminus \{0\}$ for all $j, k \in \{1, \ldots, n\}$ with $j \neq k$. Since $g \in C_{\infty}(((-\beta, \beta) \setminus \{0\})^{n(n-1)/2}$, the mean value theorem ensures that for any $j, k \in \{1, \ldots, n\}$ with $j < k$ there exists $\theta_{2j-1,2k-1} \in I(s_{2j-1} - s_{2k-1}, l_{2j-1}/h - l_{2k-1}/h)$ such that

$$
g(s_1 - s_3, \ldots, s_{2n-3} - s_{2n-1}) - g(l_1/h - l_3/h, \ldots, l_{2n-3}/h - l_{2n-1}/h)
= \langle \nabla g(\theta_{1,3}, \ldots, \theta_{2n-3,2n-1}),
(s_1 - s_3 - (l_1/h - l_3/h), \ldots, s_{2n-3} - s_{2n-1} - (l_{2n-3}/h - l_{2n-1}/h)) \rangle,
$$

which leads to

$$
|g(s_1 - s_3, \ldots, s_{2n-3} - s_{2n-1}) - g(l_1/h - l_3/h, \ldots, l_{2n-3}/h - l_{2n-1}/h)|
\leq \frac{1}{h} \left( \frac{n(n-1)}{2} \right)^{1/2} \sup_{s \in ((-\beta, \beta) \setminus \{0\})^{n(n-1)/2}} |\nabla g(s)|.
$$

(3.7)
Moreover, by using Lemma 2.5 we see that for \( j < k \),

\[
\left| \frac{\partial}{\partial(s_{2j-1} - s_{2k-1})} g(s_1 - s_3, \ldots, s_{2n-3} - s_{2n-1}) \right| \leq \sum_{p_1=0}^{1} \sum_{p_2=0}^{1} \frac{\partial}{\partial(s_{2j-1} - s_{2k-1})} C(x_{2j-1+p_1} \sigma_{2j-1+p_1}s_{2j-1}, y_{2k-1+p_2} \sigma_{2k-1+p_2}s_{2k-1}) \\
+ \frac{\partial}{\partial(s_{2j-1} - s_{2k-1})} C(x_{2k-1+p_1} \sigma_{2k-1+p_1}s_{2k-1}, y_{2j-1+p_2} \sigma_{2j-1+p_2}s_{2j-1}) \right) 4^{2n-1} \leq 8 \cdot 4^{2n-1} \sup_{x \in \Gamma, x \in (-\beta, \beta) \setminus \{0\}} \left| \frac{\partial}{\partial x} C(x^\top x, 0^\top 0) \right|. \tag{3.8}
\]

By (3.7)–(3.8) we have

\[
\prod_{j=1}^{n} \left( \sum_{l_{2j-1}=0}^{\beta h - 2} \int_{l_{2j-1}/h}^{l_{2j-1}+1}/h ds_{2j-1} \right) (1 - \chi_1) \chi_{ls} \\
\cdot |g(s_1 - s_3, \ldots, s_{2n-3} - s_{2n-1}) - g(l_1/h - l_3/h, \ldots, l_{2n-3}/h - l_{2n-1}/h)| \leq \frac{n(n-1)3^n}{h} 4^{2n} \sup_{x \in \Gamma, x \in (-\beta, \beta) \setminus \{0\}} \left| \frac{\partial}{\partial x} C(x^\top x, 0^\top 0) \right|, \tag{3.9}
\]

On the other hand, note that

\[
|\{1/h \in [0, \beta]_h \mid \chi_1 = 1\}| = (\beta h)_n - |\{1/h \in [0, \beta]_h \mid \chi_1 = 0\}| = (\beta h)_n - \left( \frac{\beta h}{n} \right) n! \leq n^2 (\beta h)^{n-1}, \tag{3.10}
\]

where we used the inequality

\[
N^n - \left( \begin{array}{c} N \\ n \end{array} \right) n! \leq n^2 N^{n-1},
\]

which holds for all \( N \in \mathbb{N} \) and \( n \in \{0, 1, \ldots, N\} \). By using Lemma 2.5 and (3.10) we obtain

\[
\prod_{j=1}^{n} \left( \sum_{l_{2j-1}=0}^{\beta h - 2} \int_{l_{2j-1}/h}^{l_{2j-1}+1}/h ds_{2j-1} \right) \chi_1 \\
\cdot |g(s_1 - s_3, \ldots, s_{2n-3} - s_{2n-1}) - g(l_1/h - l_3/h, \ldots, l_{2n-3}/h - l_{2n-1}/h)| \leq 2h^{-n} n^2 (\beta h)^{n-1} 4^{2n} = \frac{2}{h^2 \beta^{n-1} 14^{2n}}. \tag{3.11}
\]

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Corollary 3.4. For all $U \in \mathbb{R}$

$$
(\psi_{x_1}^* \psi_{x_2}^* \psi_{y_1}^* \psi_{y_2}^* \psi_{x_2} \psi_{x_1}) = -\frac{1}{\beta} \lim_{h \to +\infty, h \to 0} \frac{\partial}{\partial \lambda_x} P_h(\{U_x\}_{x \in \Gamma^4})
$$

(3.12)

Proof. The relation (2.4) and Cauchy’s integral formula ensure that for any \( \{\hat{U}_x\}_{x \in \Gamma^4} \subset \mathbb{C} \) and \( r > 0 \)

$$
\frac{\partial}{\partial \lambda_x} (P_h - P)(\{U_x\}_{x \in \Gamma^4}) = \left( \frac{\partial}{\partial U_{\hat{x}_1}} + \frac{\partial}{\partial U_{\hat{x}_2}} \right) (P_h - P)(\{U_x\}_{x \in \Gamma^4})
$$

$$
= \frac{1}{2\pi i} \left( \oint_{|U_{\hat{x}_1} - U_{\hat{x}_1}| = r} dU_{\hat{x}_1} \frac{(P_h - P)(\{U_x\}_{x \in \Gamma^4})}{(U_{\hat{x}_1} - U_{\hat{x}_1})^2} \right. \left. + \oint_{|U_{\hat{x}_2} - U_{\hat{x}_2}| = r} dU_{\hat{x}_2} \frac{(P_h - P)(\{U_x\}_{x \in \Gamma^4})}{(U_{\hat{x}_2} - U_{\hat{x}_2})^2} \right) \bigg|_{U_x = \hat{U}_x}_{\forall x \in \Gamma^4}
$$

(3.13)

By applying Proposition 3.2 to (3.13) we can show that for any \( \hat{r} > 0 \) and any \( \{\hat{U}_x\}_{x \in \Gamma^4} \) with \( |\hat{U}_x| \leq \hat{r} \) (\( \forall x \in \Gamma^4 \))

$$
\left| \frac{\partial}{\partial \lambda_x} (P_h - P)(\{U_x\}_{x \in \Gamma^4}) \right| \leq \frac{2}{\hat{r}} \sup_{\{U_x\}_{x \in \Gamma^4} \subset \mathbb{C}, |U_x| \leq \hat{r}, \forall x \in \Gamma^4} \sup_{\hat{U}_x \leq \hat{r}} |P_h(\{U_x\}_{x \in \Gamma^4}) - P(\{U_x\}_{x \in \Gamma^4})| \to 0
$$

(3.14)

as \( h \to +\infty, h \to 0 \). Combining (3.14) with Lemma 2.1 yields (3.12).
3.2 The Grassmann Gaussian integral

To deal with the discretized partition function $P_h$ rather than $P$ is advantageous since the variables run in the finite set $\Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h$ in every term of the power series $P_h$. Accordingly, we can formulate $P_h$ as a Grassmann Gaussian integral on finite Grassmann algebras. Elementary calculus on finite Grassmann algebras has been summarized in the books [6], [20]. For a convenience of calculation, especially in order to refer to Proposition 6.7 shown in Appendix C, we assume that $h \in 2\mathbb{N}/\beta$ from now.

Let us number elements of the set $\Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h$ so that we can write $\Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h = \{(x_j, \sigma_j, x_j) | j \in \{1, \cdots, N\}\}$ with $N := 2L^d \beta h$. We then introduce a set of Grassmann algebras denoted by $\{\psi_{x_j, \sigma_j, x_j}, \overline{\psi}_{x_j, \sigma_j, x_j} | j \in \{1, \cdots, N\}\}$. Remind us that the Grassmann algebra $\{\psi_{x_j, \sigma_j, x_j}, \overline{\psi}_{x_j, \sigma_j, x_j} | j \in \{1, \cdots, N\}\}$ satisfies the anti-commutation relations

$$\psi_{x_j, \sigma_j, x_j} \psi_{x_k, \sigma_k, x_k} = -\psi_{x_k, \sigma_k, x_k} \psi_{x_j, \sigma_j, x_j}, \quad \overline{\psi}_{x_j, \sigma_j, x_j} \overline{\psi}_{x_k, \sigma_k, x_k} = -\overline{\psi}_{x_k, \sigma_k, x_k} \overline{\psi}_{x_j, \sigma_j, x_j},$$

for all $j, k \in \{1, \cdots, N\}$.

Let $\mathbb{C}[\psi_{x_j, \sigma_j, x_j}, \overline{\psi}_{x_j, \sigma_j, x_j}, j \in \{1, \cdots, N\}]$ denote the complex linear space spanned by all the monomials consisting of $\{\psi_{x_j, \sigma_j, x_j}, \overline{\psi}_{x_j, \sigma_j, x_j} | j \in \{1, \cdots, N\}\}$. As a linear functional on $\mathbb{C}[\psi_{x_j, \sigma_j, x_j}, \overline{\psi}_{x_j, \sigma_j, x_j}, j \in \{1, \cdots, N\}]$, the Grassmann integral

$$\int d\overline{\psi}_{x_N \sigma_N x_N} \cdots d\overline{\psi}_{x_1 \sigma_1 x_1} \psi_{x_N \sigma_N x_N} \cdots \psi_{x_1 \sigma_1 x_1}$$

is defined as follows.

$$\int \psi_{x_{1 \sigma_1 x_1}} \cdots \psi_{x_{N \sigma_N x_N}} \overline{\psi}_{x_1 \sigma_1 x_1} \cdots \overline{\psi}_{x_N \sigma_N x_N} d\overline{\psi}_{x_N \sigma_N x_N} \cdots d\psi_{x_1 \sigma_1 x_1} = 1,$$

$$\int \psi_{x_{j_1 \sigma_{j_1} x_{j_1}}} \cdots \psi_{x_{j_m \sigma_{j_m} x_{j_m}}} \overline{\psi}_{x_{k_1 \sigma_{k_1} x_{k_1}}} \cdots \overline{\psi}_{x_{k_m \sigma_{k_m} x_{k_m}}} d\overline{\psi}_{x_N \sigma_N x_N} \cdots d\psi_{x_1 \sigma_1 x_1} = 0$$

if $n \neq N$ or $m \neq N$, and linearly extended onto the whole space.

Let us simply write the vectors of the Grassmann algebras $(\psi_{x_1 \sigma_1 x_1}, \cdots, \psi_{x_N \sigma_N x_N})$, $(\overline{\psi}_{x_1 \sigma_1 x_1}, \cdots, \overline{\psi}_{x_N \sigma_N x_N})$ as

$$\psi_X = (\psi_{x_1 \sigma_1 x_1}, \cdots, \psi_{x_N \sigma_N x_N}), \quad \overline{\psi}_X = (\overline{\psi}_{x_1 \sigma_1 x_1}, \cdots, \overline{\psi}_{x_N \sigma_N x_N}).$$

In order to indicate the dependency on the parameter $h$, we write the covariance matrix as

$$C_h := (C(x_j \sigma_j x_j, x_k \sigma_k x_k))_{1 \leq j, k \leq N}$$

and define a $2N \times 2N$ skew symmetric matrix $C_h$ by

$$C_h := \begin{pmatrix} 0 & C_h \\ -C_h^t & 0 \end{pmatrix}.$$
The diagonalization of $C_h$ is presented in Appendix [C]. Here we note the fact that $\det C_h \neq 0$ proved in Proposition [C.7] to see that $C_h$ is invertible.

For any $f(\psi_X, \overline{\psi}_X) \in \mathbb{C}[\psi_{x_j}, \overline{\psi}_{x_j}, \psi_{x_k}, \overline{\psi}_{x_k} | j \in \{1, \cdots, N\}]$, $e^{f(\psi_X, \overline{\psi}_X)}$ is defined by

$$e^{f(\psi_X, \overline{\psi}_X)} := e^{f(0,0)} \left( \sum_{n=0}^{2N} \frac{1}{n!} (f(\psi_X, \overline{\psi}_X) - f(0,0))^n \right),$$

where $f(0,0)$ denotes the constant part of $f(\psi_X, \overline{\psi}_X)$.

Let us also write in short

$$d\psi_X = d\psi_{x_1} \cdots d\psi_{x_1} d\psi_{x_2} \cdots d\psi_{x_2}$$

Definition 3.5. As a linear functional on $C[\psi_{x_j}, \overline{\psi}_{x_j}, \psi_{x_k}, \overline{\psi}_{x_k} | j \in \{1, \cdots, N\}]$, the Grassmann Gaussian integral representation of $\mathbb{F}$ or any $F$ is as follows.

The denominator of (3.15) is non-zero. In fact, a direct calculation and the assumption $h \in 2\mathbb{N}/\beta$ show

$$\int e^{-\frac{1}{2} f(\psi_X, \overline{\psi}_X)} d\mu_{C_h}(\psi_X, \overline{\psi}_X) = (\det C_h)^{-1} (-1)^{N(N-1)/2} = (\det C_h)^{-1},$$

which takes a positive value independent of $h$ by Proposition [C.7].

The Grassmann Gaussian integral representation of $P_h$ is as follows.

**Proposition 3.7.** Assume that $\{U_X\}_{X \in \Gamma^4(\subset \mathbb{C})}$ satisfies the equality that for all $x, y, z, w \in \Gamma$

$$U_{x,y,z,w} = U_{z,w,x,y}. \tag{3.16}$$

The following equality holds.

$$P_h(\{U_X\}_{X \in \Gamma^4}) = \int e^{\sum_{x,y,z,w \in \Gamma} U_{x,y,z,w} V_{h,x,y,z,w}(\psi_X, \overline{\psi}_X)} d\mu_{C_h}(\psi_X, \overline{\psi}_X),$$

where

$$V_{h,x,y,z,w}(\psi_X, \overline{\psi}_X) := -\frac{1}{h} \sum_{x \in [0,\beta]} \overline{\psi}_{x|z} \psi_{y|z} \psi_{w|z} \psi_{x|z}.$$

**Proof.** By substituting the equalities $\int 1 d\mu_{C_h}(\psi_X, \overline{\psi}_X) = 1$ and

$$\int \overline{\psi}_{x_1} \psi_{x_2} \cdots \overline{\psi}_{x_n} \psi_{x_{n+1}} \cdots \overline{\psi}_{x_{2n}} \psi_{x_{2n+1}} d\mu_{C_h}(\psi_X, \overline{\psi}_X) = \det(C_h(x_{2p}, \sigma_{2p} x_{2q}, x_{2q} \sigma_{2p} x_{2q}))_{1 \leq p, q \leq n}$$

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for any \( j_1, \cdots, j_n, k_1, \cdots, k_n \in \{1, 2, \cdots, N\} \) (see \cite[Problem I.13]{Problem}) into (3.18), we have

\[
P_h((U_X) x \in \mathfrak{g}^*) = 1 + \sum_{n=1}^{L^d \beta_h} \sum_{\prod_j \frac{1}{n}} \sum_{x_{2j-1}, x_{2j} \in \Gamma} \sum_{y_{2j-1}, y_{2j} \in \Gamma} \sum_{x_{2j-1}, x_{2j} \in \{1, 1\}} \delta_{x_{2j-1},1} \delta_{x_{2j},1} \delta_{x_{2j-1},x_{2j}} U_{x_{2j-1}, x_{2j}, y_{2j-1}, y_{2j}}
\]

\[
\cdot \int \psi_{x_{2n}, \sigma_n, x_{2n},} \cdots \psi_{x_1, \sigma_1, x_1} \overline{\psi}_{y_1, \sigma_1, x_1} \cdots \overline{\psi}_{y_{2n}, \sigma_{2n}, x_{2n}} d\mu_{C_h}(\psi_X, \overline{\psi}_X)
\]

\[
= \int \left( 1 + \sum_{n=1}^{L^d \beta_h} \sum_{\prod_j \frac{1}{n}} \sum_{x_{2j-1}, x_{2j} \in \Gamma} \sum_{y_{2j-1}, y_{2j} \in \Gamma} \sum_{x_{2j-1}, x_{2j} \in \{1, 1\}} \delta_{x_{2j-1},1} \delta_{x_{2j},1} \delta_{x_{2j-1},x_{2j}} U_{x_{2j-1}, x_{2j}, y_{2j-1}, y_{2j}}
\]

\[
\cdot \psi_{x_{2j-1}, \sigma_{2j-1}, x_{2j-1}} \psi_{x_{2j-1}, \sigma_{2j-1}, x_{2j-1}} \overline{\psi}_{y_{2j-1}, \sigma_{2j-1}, y_{2j-1}} \overline{\psi}_{y_{2j-1}, \sigma_{2j-1}, y_{2j-1}} d\mu_{C_h}(\psi_X, \overline{\psi}_X)
\]

\[
= \int e^{-\frac{1}{\beta_h} \sum_{x, y, w \in \Gamma} \sum_{\prod_j \frac{1}{n}} U_{x, y, w} \overline{\psi}_{x, \sigma, y} \overline{\psi}_{w, \sigma, \psi} d\mu_{C_h}(\psi_X, \overline{\psi}_X).
\]

(3.17)

To obtain the last equality of (3.17) we used the equality (3.16). □

As a corollary, our original partition functions and the correlation function are represented as a limit of the finite dimensional Grassmann integrals.

**Corollary 3.8.** For any \( U \in \mathbb{R} \) and \( \{\lambda_X\} x \in \mathfrak{g}^* \subset \mathbb{R} \), the following equalities hold.

\[
\begin{align*}
\text{Tr} e^{-\beta H_h} / \text{Tr} e^{-\beta H_0} &= \lim_{h \to 0} \int e^{\frac{1}{h} \sum_{x, y, w \in \Gamma} U_{x, y, w} \overline{\psi}_{x, \sigma, y} \overline{\psi}_{w, \sigma, \psi} d\mu_{C_h}(\psi_X, \overline{\psi}_X),} \quad (3.18) \\
\text{Tr} e^{-\beta H} / \text{Tr} e^{-\beta H_0} &= \lim_{h \to 0} \int e^{\frac{1}{h} \sum_{x, y, w \in \Gamma} \sum_{\prod_j \frac{1}{n}} U_{x, y, w} \overline{\psi}_{x, \sigma, y} \overline{\psi}_{w, \sigma, \psi} d\mu_{C_h}(\psi_X, \overline{\psi}_X),} \quad (3.19) \\
\langle \psi_{x_1}^{\ast}, \psi_{x_2}^{\ast}, \psi_{y_1}^{\ast}, \psi_{y_2}^{\ast} \rangle &= \lim_{h \to 0} \int_{x \in \{0, 1\}^\beta} \psi_{x_1}^{\ast} \psi_{x_2}^{\ast} \overline{\psi}_{x_1} \overline{\psi}_{x_2} d\mu_{C_h}(\psi_X, \overline{\psi}_X) \quad (3.20)
\end{align*}
\]
Proof. Since the relation (2.4) implies the condition (3.16), we can apply Proposition 3.2 and Proposition 3.7 to deduce (3.18). The equality (3.19) is (3.18) for \( \lambda_X = 0 \) \((\forall X \in \Gamma^4)\). Note the fact that

\[
\frac{\partial}{\partial \lambda_1} e^{\sum_{X \in \Gamma^4} U_X V_{h,X} (\bar{\psi}_X, \bar{\psi}_X)} \bigg|_{\lambda_X = 0}
\]

\[
= -\frac{1}{h} \sum_{x \in \{0, \beta\}^4} \left\langle \bar{\psi}_{x_1} \bar{\psi}_{x_2} \bar{\psi}_{y_1} \bar{\psi}_{y_2} \bar{\psi}_{x_3} \bar{\psi}_{x_4} \bigg| \bar{\psi}_{x_1} \bar{\psi}_{x_2} \bar{\psi}_{y_1} \bar{\psi}_{y_2} \bar{\psi}_{x_3} \bar{\psi}_{x_4} \right\rangle
\]

\[
\quad \cdot e^{-\frac{1}{h} \sum_{x \in \Gamma} \sum_{x \in \{0, \beta\}^4} \bar{\psi}_{x_1} \bar{\psi}_{x_2} \bar{\psi}_{y_1} \bar{\psi}_{y_2} \bar{\psi}_{x_3} \bar{\psi}_{x_4}},
\]

where the differential operator \( \partial / \partial \lambda_1 \) is defined to act on every coefficient of Grassmann monomials in the expansion \( e^{\sum_{X \in \Gamma^4} U_X V_{h,X} (\bar{\psi}_X, \bar{\psi}_X)} \) (see [6, Problem I.3]). Moreover, by expanding \( e^{\sum_{X \in \Gamma^4} U_X V_{h,X} (\bar{\psi}_X, \bar{\psi}_X)} \) one can verify the equality

\[
\frac{\partial}{\partial \lambda_1} \int e^{\sum_{X \in \Gamma^4} U_X V_{h,X} (\bar{\psi}_X, \bar{\psi}_X)} \mu_{C_h}(\bar{\psi}_X, \bar{\psi}_X)
\]

\[
= \int \frac{\partial}{\partial \lambda_1} e^{\sum_{X \in \Gamma^4} U_X V_{h,X} (\bar{\psi}_X, \bar{\psi}_X)} \mu_{C_h}(\bar{\psi}_X, \bar{\psi}_X). \tag{3.22}
\]

The equality (3.20) follows from Corollary 3.4, Proposition 3.7 and (3.21)-(3.22). \( \square \)

4 Upper bound on the perturbation series

In this section we calculate upper bounds on our perturbation series \( \sum_{n=0}^{\infty} a_n U^n \) by evaluating the tree formula for the connected part of the exponential of Laplacian operator. In order to employ the Grassmann Gaussian integral formulation of \( P_h \) developed in Section 3.2, we assume that \( h \in 2\mathbb{N} / \beta \) throughout this section.

4.1 The connected part of the exponential of Laplacian operator

Our approach to find an upper bound on \( |a_n| \) of our perturbation series \( \sum_{n=0}^{\infty} a_n U^n \) is based on the characterization of the connected part of the exponential of the Laplacian operator of Grassmann left derivatives reported in [22]. Let us construct our argument step by step to reveal the structure of the problem.

The Grassmann integral \( \int e^{\sum_{X \in \Gamma^4} U_X V_{h,X} (\bar{\psi}_X, \bar{\psi}_X)} \mu_{C_h}(\bar{\psi}_X, \bar{\psi}_X) \) can be viewed as an analytic function of the multi-variable \( \{U_X\}_{X \in \Gamma^4} \). Since

\[
\int e^{\sum_{X \in \Gamma^4} U_X V_{h,X} (\bar{\psi}_X, \bar{\psi}_X)} \mu_{C_h}(\bar{\psi}_X, \bar{\psi}_X) \bigg|_{U_X = 0}
\]

\[
= 1,
\]

if \( |U_X| \) is sufficiently small for all \( X \in \Gamma^4 \), the inequality

\[
\left| \int e^{\sum_{X \in \Gamma^4} U_X V_{h,X} (\bar{\psi}_X, \bar{\psi}_X)} \mu_{C_h}(\bar{\psi}_X, \bar{\psi}_X) - 1 \right| < 1
\]
holds. Thus, we can define a function \( W_h(\{U_X\}_{X \in \Gamma^4}) \) by
\[
W_h(\{U_X\}_{X \in \Gamma^4}) := \log \left( \int_{\mathbb{C}^d} \sum_{\alpha \in \mathbb{N}_0^d} W_{\alpha} v_{\alpha, X} \psi_X \overline{\psi_X} d\mu_{C_h}(\psi_X, \overline{\psi_X}) \right),
\]
which is analytic in a neighborhood of 0 in \( \mathbb{C}^{4d} \).

**Lemma 4.1.** For all \( h \in 2\mathbb{N}/\beta \) and \( n \in \mathbb{N} \cup \{0\} \), the following equality holds.
\[
a_{h,n} = -\frac{1}{\beta n!} \sum_{z_1, \ldots, z_n \in \Gamma} \left( \frac{\partial^{n+1} W_h}{\partial U_{x_1, x_2, y_1, y_2} \partial U_{z_1, z_2, z_3, z_4} \cdots \partial U_{z_n, z_n, z_n, z_n}} \bigg|_{U_X = 0} \right)
+ \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{X_1, \ldots, X_n \in \Gamma^4} \frac{\partial^n W_h}{\partial U_{x_1} \cdots \partial U_{x_n}} \bigg|_{U_X = 0} U_{X_1} \cdots U_{X_n}. \tag{4.1}
\]

where \( a_{h,n} \) was defined in (4.2) and (4.3).

**Proof.** The Taylor expansion of \( W_h(\{U_X\}_{X \in \Gamma^4}) \) around 0 is given by
\[
W_h(\{U_X\}_{X \in \Gamma^4}) = W_h \bigg|_{U_X = 0} + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{X_1, \ldots, X_n \in \Gamma^4} \frac{\partial^n W_h}{\partial U_{x_1} \cdots \partial U_{x_n}} \bigg|_{U_X = 0} U_{X_1} \cdots U_{X_n}. \tag{4.2}
\]

Fix any \( U \in \mathbb{C} \) with \( |U| < \log 2/(16\beta L^{4d}) \). Let \( \varepsilon > 0 \) be the constant claimed in Lemma 3.1. By using a parameter \( \{X_X\}_{X \in \Gamma^4}(\subset \mathbb{C}) \) with \( |X_X| \leq \varepsilon \) \( \forall X \in \Gamma^4 \) we define the variable \( \{U_X\}_{X \in \Gamma^4} \) by the equality (2.3). Then the inequality \( |P_h(\{U_X\}_{X \in \Gamma^4}) - 1| < 1 \) holds by Lemma 3.1 and the condition (3.10) is satisfied. Thus, Proposition 5.1 ensures that \( W_h(\{U_X\}_{X \in \Gamma^4}) = \log(P_h(\{U_X\}_{X \in \Gamma^4})) \) for this \( \{U_X\}_{X \in \Gamma^4} \). Moreover, by the equalities (4.2) and (4.3) we have that
\[
-\frac{1}{\beta} \frac{\partial}{\partial x_1} \log(P_h) \bigg|_{X_X = 0} = -\frac{1}{\beta} \frac{\partial}{\partial x_1} W_h \bigg|_{X_X = 0} = -\frac{1}{\beta} \left( \frac{\partial}{\partial U_{x_1}} + \frac{\partial}{\partial U_{x_2}} \right) W_h \bigg|_{X_X = 0}
+ \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{z_1, \ldots, z_n \in \Gamma^4} \left( \frac{\partial^n W_h}{\partial U_{z_1} \partial U_{z_2} \cdots \partial U_{z_n} \partial U_{z_n} \partial U_{z_n} \partial U_{z_n}} \bigg|_{U_X = 0} \right). \tag{4.4}
\]

By uniqueness of Taylor series and Lemma 3.1(ii) we obtain (4.1). \( \square \)

A message from Lemma 4.1 is that upper bounds on \( |a_{h,n}| \) can be obtained by characterizing the partial derivatives of \( W_h \) at \( U_X = 0 \) \( \forall X \in \Gamma^4 \), which is the way we follow from now. Since \( |a_{h,0}| \) can be evaluated directly from (3.2) by using Lemma 2.5, let us study the equality (4.1) for \( n \geq 1 \). Fix any \( n \geq 1 \) and use the simplified notations defined as follows.
\[
Z_j := (z_j, z_j, z_j, z_j) \in \Gamma^4 \text{ for } z_j \in \Gamma \text{ (} \forall j \in \{1, \ldots, n\} \text{)}, \quad Z_0 := \bar{X}_1 \in \Gamma^4. \tag{4.3}
\]
Set $\mathbb{N}_{n+1} := \{0, 1, \cdots, n\}$. By noting that

$$
\prod_{j \in Q} \frac{\partial}{\partial U_{z_j}} \int e^{\frac{1}{2} \sum_{x \in \mathbb{N}} U_{x} V_{h,x}(\psi_{x}, \bar{\psi}_{x})} d\mu_{C_{h}}(\psi_{x}, \bar{\psi}_{x}) \bigg|_{U_{x}=0, \ \forall x \in \mathbb{N}} = \prod_{j \in \mathbb{N}_{n+1}} \frac{\partial}{\partial U_{z_j}} \int \prod_{j \in \mathbb{N}_{n+1}} (1 + U_{z_j} V_{h,z_j}(\psi_{x}, \bar{\psi}_{x})) d\mu_{C_{h}}(\psi_{x}, \bar{\psi}_{x}) \bigg|_{U_{z_j}=0, \ \forall j \in \mathbb{N}_{n+1}}
$$

for any $Q \subset \mathbb{N}_{n+1}$, we see that

$$
\prod_{j \in \mathbb{N}_{n+1}} \frac{\partial}{\partial U_{z_j}} W_{h} \bigg|_{U_{x}=0, \ \forall x \in \mathbb{N}} = \prod_{j \in \mathbb{N}_{n+1}} \frac{\partial}{\partial U_{z_j}} \log \left( \int \prod_{j \in \mathbb{N}_{n+1}} (1 + U_{z_j} V_{h,z_j}(\psi_{x}, \bar{\psi}_{x})) d\mu_{C_{h}}(\psi_{x}, \bar{\psi}_{x}) \right) \bigg|_{U_{z_j}=0, \ \forall j \in \mathbb{N}_{n+1}}
$$

$$
= \prod_{j \in \mathbb{N}_{n+1}} \frac{\partial}{\partial U_{z_j}} \log \left( 1 + \sum_{Q \subset \mathbb{N}_{n+1} \atop Q \neq \emptyset} \prod_{j \in Q} V_{h,z_j}(\psi_{x}, \bar{\psi}_{x}) d\mu_{C_{h}}(\psi_{x}, \bar{\psi}_{x}) \prod_{j \notin Q} U_{z_j} \right) \bigg|_{U_{z_j}=0, \ \forall j \in \mathbb{N}_{n+1}}.
$$

(4.4)

The Grassmann Gaussian integral contained in the right hand side of (4.4) can be rewritten as follows.

**Lemma 4.2.** Introduce Grassmann algebras $\{\psi^{q}_{x,j}, \bar{\psi}^{q}_{x,j} \mid j \in \{1, \cdots, N\}\}$ indexed by $q \in \mathbb{N}_{n+1}$ and write

$$
\psi_{x} = (\psi^{q}_{x,1x,1}, \cdots, \psi^{q}_{x,Nx,Nx}), \quad \bar{\psi}_{x} = (\bar{\psi}^{q}_{x,1x,1}, \cdots, \bar{\psi}^{q}_{x,Nx,Nx})
$$

for all $q \in \mathbb{N}_{n+1}$. Let $\partial / \partial \psi_{x}^{q}$, $\partial / \partial \bar{\psi}_{x}^{q}$ be the vectors of left derivatives associated with $\psi_{x}^{q}$, $\bar{\psi}_{x}^{q}$, respectively. Then, the following equality holds. For all $Q \subset \mathbb{N}_{n+1}$ with $Q \neq \emptyset$

$$
\int \prod_{q \in Q} V_{h,z_q}(\psi_{x}, \bar{\psi}_{x}) d\mu_{C_{h}}(\psi_{x}, \bar{\psi}_{x}) = e^{\Delta} \prod_{q \in Q} V_{h,z_q}(\psi_{x}^{q}, \bar{\psi}_{x}^{q}) \bigg|_{\psi_{x}=\bar{\psi}_{x}=0}, \quad (4.5)
$$

where the Laplacian operator $\Delta$ and its exponential $e^{\Delta}$ are defined by

$$
\Delta := - \sum_{p,q \in \mathbb{N}_{n+1}} \left\langle \left( \frac{\partial}{\partial \psi_{x}^{q}} \right)^{t}, C_{h} \left( \frac{\partial}{\partial \bar{\psi}_{x}^{q}} \right)^{t} \right\rangle, \quad e^{\Delta} := \sum_{l=0}^{2N(n+1)} \frac{1}{l!} \Delta^{l}.
$$

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Remark 4.3. When we introduce another set of Grassmann algebras, let us think that the complex linear space spanned by monomials of all the Grassmann algebras introduced up to this point is defined on the assumption of multiplication satisfying the anti-commutation relations between these algebras. The notion of the Grassmann integral \( \int d\psi_X d\bar{\psi}_X \) is naturally extended to be a linear map from the enlarged linear space of all the algebras to the subspace without \( \psi_X, \bar{\psi}_X \).

For a monomial \( \phi_j, \cdots \phi_{j_n} \) of Grassmann algebras \( \{ \phi_l \}_{l=1}^m \), the left derivative \( \partial/\partial \phi_l \phi_j, \cdots \phi_{j_n} \) is defined by

\[
\frac{\partial}{\partial \phi_l} \phi_j, \cdots \phi_{j_n} := \begin{cases} (-1)^{k-1} \phi_j, \cdots \phi_{j_k} \phi_{j_{k+1}} \cdots \phi_{j_n} & \text{if there uniquely exists } k \in \{1, \ldots, n\} \text{ s.t. } l = j_k, \\ 0 & \text{otherwise}. \end{cases}
\]

Then, the left derivative \( \partial/\partial \phi_l \) is extended to be a linear map on the linear space of monomials of the algebras \( \{ \phi_l \}_{l=1}^m \).

The concepts of Grassmann integrals and left derivatives are generally defined as operators on Grassmann algebra with coefficients in a superalgebra (see [0] Chapter I).

Proof of Lemma 4.4. We define another Grassmann algebra \( \{ \eta_X^q, \bar{\eta}_X^q \} \) indexed by the sets \( \Gamma = \{ \uparrow, \downarrow \} \times [0, \beta) \) and \( \mathbb{N}_{n+1} \) and the associated left derivative \( \partial/\partial \eta_X^q, \partial/\partial \bar{\eta}_X^q \) in the same way as \( \{ \psi_X^q, \bar{\psi}_X^q \} \) and \( \{ \partial/\partial \psi_X^q, \partial/\partial \bar{\psi}_X^q \} \). Then, we see that for any subset \( Q \subset \mathbb{N}_{n+1} \) with \( Q \neq \emptyset \)

\[
\int \prod_{q \in Q} V_h, z_q (\psi_X, \bar{\psi}_X) d\mu_{C_h}(\psi_X, \bar{\psi}_X)
\]

\[
= \int \prod_{q \in Q} V_h, z_q \left( \frac{\partial}{\partial \eta_X^q}, \frac{\partial}{\partial \bar{\eta}_X^q} \right) e^{((\eta_X^q, \bar{\eta}_X^q)^{\prime}, (\psi_X, \bar{\psi}_X)^{\prime})} d\mu_{C_h}(\psi_X, \bar{\psi}_X) \bigg|_{\eta_X^q=\bar{\eta}_X^q=0}^{\eta_X^q=\bar{\eta}_X^q=0}
\]

\[
= \prod_{q \in Q} V_h, z_q \left( \frac{\partial}{\partial \eta_X^q}, \frac{\partial}{\partial \bar{\eta}_X^q} \right) \int e^{((\eta_X^q, \bar{\eta}_X^q)^{\prime}, (\psi_X, \bar{\psi}_X)^{\prime})} d\mu_{C_h}(\psi_X, \bar{\psi}_X) \bigg|_{\eta_X^q=\bar{\eta}_X^q=0}^{\eta_X^q=\bar{\eta}_X^q=0}
\]

\[
= \prod_{q \in Q} V_h, z_q \left( \frac{\partial}{\partial \eta_X^q}, \frac{\partial}{\partial \bar{\eta}_X^q} \right) e^{-\sum_{p \in Q} ((\eta_X^p, \bar{\eta}_X^p)^{\prime})} \bigg|_{\eta_X^q=\bar{\eta}_X^q=0}^{\eta_X^q=\bar{\eta}_X^q=0}
\]

\[
= e^{\Delta} \prod_{q \in Q} V_h, z_q (\psi_X, \bar{\psi}_X) \bigg|_{\psi_X=\bar{\psi}_X=0}^{\psi_X=\bar{\psi}_X=0},
\]

where we have used the equality that

\[
\int e^{\sum_{q \in Q} ((\eta_X^q, \bar{\eta}_X^q)^{\prime}, (\psi_X, \bar{\psi}_X)^{\prime})} d\mu_{C_h}(\psi_X, \bar{\psi}_X) = e^{-\sum_{p \in Q} ((\eta_X^p, \bar{\eta}_X^p)^{\prime})}
\]

(see [6] Problem I.13). To verify the equalities (4.6) in more detail, see the books [0], [20] for the properties of left derivatives. \( \square \)
By combining (4.4) with (4.5) we obtain
\[
\Pi_{j \in \mathbb{N}^{n+1}} \frac{\partial}{\partial U_{Z_j}} W_h \bigg|_{U_X = 0 \forall X \in \Gamma^4} = \\
\Pi_{j \in \mathbb{N}^{n+1}} \frac{\partial}{\partial U_{Z_j}} \log \left( 1 + \sum_{Q \subseteq \mathbb{N}^{n+1}} e^{\Delta} \sum_{q \in Q} V_{h, z_q} (\psi_q^X, \overline{\psi}_q^X) \bigg|_{\psi_q^X = \overline{\psi}_q^X = 0} \prod_{p \in Q} U_{Z_p} \right) \bigg|_{U_{Z_p} = 0 \forall q \in \mathbb{N}^{n+1}}.
\]

(4.7)

In order to characterize the right hand side of (4.7), let us review the general theory developed in [22] by translating in our setting. Consider a map \( \alpha \) from the power set \( \mathcal{P}(\mathbb{N}^{n+1}) \) to \( \mathbb{C} \) defined by
\[
\alpha(\emptyset) := 1 \quad \text{and for } Q \in \mathcal{P}(\mathbb{N}^{n+1}) \{\emptyset\},
\]
\[
\alpha(Q) := e^{\Delta} \prod_{q \in Q} V_{h, z_q} (\psi_q^X, \overline{\psi}_q^X) \bigg|_{\psi_q^X = \overline{\psi}_q^X = 0}.
\]

By [22, Lemma 1] there uniquely exists a map \( \alpha_c : \mathcal{P}(\mathbb{N}^{n+1}) \rightarrow \mathbb{C} \) such that for all \( Q \in \mathcal{P}(\mathbb{N}^{n+1}) \{\emptyset\} \)
\[
\alpha(Q) = \sum_{Q_0 \subseteq Q} \alpha_c(Q_0) \alpha(Q \setminus Q_0),
\]

where \( \min Q \) stands for the smallest number contained in \( Q \). In [22, Lemma 2] it was proved that the right hand side of (4.7) is equal to \( \alpha_c(\mathbb{N}^{n+1}) \), which is called the connected part of the operator \( e^{\Delta} \). The formula for \( \alpha_c(\mathbb{N}^{n+1}) \) was given in [22, Theorem 3]. We summarize the result below.

**Lemma 4.4.** [22, Theorem 3] The following equality holds.
\[
\Pi_{j \in \mathbb{N}^{n+1}} \frac{\partial}{\partial U_{Z_j}} W_h \bigg|_{U_X = 0 \forall X \in \Gamma^4} = \\
\sum_{T \in \mathcal{T}(\mathbb{N}^{n+1})} \prod_{(q, q') \in T} (\Delta_{q, q'} + \Delta_{q', q}) \int_{[0,1]^n} \sum_{\pi \in S_{n+1}(T)} \phi(T, \pi, s) e^{\Delta(M(T, \pi, s))} \bigg|_{\psi_q^X = \overline{\psi}_q^X = 0} \\
\cdot \prod_{q \in \mathbb{N}^{n+1}} V_{h, z_q} (\psi_q^X, \overline{\psi}_q^X) \bigg|_{\psi_q^X = \overline{\psi}_q^X = 0},
\]

(4.8)

where \( \mathcal{T}(\mathbb{N}^{n+1}) \) is the set of all the trees (connected graphs without loop) on \( \mathbb{N}^{n+1} \),
\[
\Delta_{q, q'} := -\left( \left( \frac{\partial}{\partial \psi_q^X} \right)^t, C_h \left( \frac{\partial}{\partial \overline{\psi}_q^X} \right)^t \right),
\]

25
Then we observe that
\[ p, q \]
\[ M(T, \pi, s) \] is a subset of \( S_{n+1} \) depending on \( T \), \( \phi(T, \pi, s) \) is a
real-valued non-negative function of \( s \) depending on \( T \) and \( \pi \) with the property that
\[
\int_{[0,1]^n} ds \sum_{\pi \in S_{n+1}(T)} \phi(T, \pi, s) = 1,
\tag{4.9}
\]
\( M(T, \pi, s) \) is an \((n+1) \times (n+1)\) real symmetric non-negative matrix depending on
\( T \), \( \pi \), \( s \) satisfying \( M(T, \pi, s)_{q,q} = 1 \) for all \( q \in \mathbb{N}_{n+1} \) and the operator \( \Delta(M(T, \pi, s)) \)
is defined by
\[
\Delta(M(T, \pi, s)) := \sum_{p,q \in \mathbb{N}_{n+1}} M(T, \pi, s)_{p,q} \Delta_{p,q}.
\]

In order to bound \(|a_{h,n}|\), the tree formula (4.8) will be evaluated in the rest of
this section.

4.2 Evaluation of upper bounds

Here we evaluate the tree expansion given in Lemma 4.4. Let us first prepare
some necessary tools. The following lemma essentially uses the determinant bound
Lemma 2.5.

**Lemma 4.5.** For any \( l \in \mathbb{N} \) and any \( p_m, q_m \in \mathbb{N}_{n+1} \), \((x_{j_m}, \sigma_{j_m}, x_{j_m})\),
\((x_{k_m}, \sigma_{k_m}, x_{k_m}) \in \Gamma \times \{1, \ldots, l\} \times \mathbb{N} \) \( \forall m \in \{1, \ldots, l\} \)
\[
\left| e^{\Delta(M(T, \pi, s))} \sum_{\psi^p_1, \psi^q_1, \ldots, \psi^p_l, \psi^q_l} \prod_{j=1}^l \left( \frac{\psi^p_j}{x_{j_1}^{p_1} \ldots x_{j_j}^{p_j} x_{k_1}^{q_1} \ldots x_{k_k}^{q_k}} \right) \left| \psi^p_j \psi^q_j \right| = 0 \right|_{\psi^p_j, \psi^q_j \in \mathbb{N}_{n+1}} \leq 4^l.
\]

**Proof.** Since \( M(T, \pi, s) \) is a non-negative real symmetric matrix, there are constants
\( \gamma_q \geq 0 \) \((q \in \mathbb{N}_{n+1})\) and projection matrices \( P_q \) \((q \in \mathbb{N}_{n+1})\) satisfying that \( P_q P_q = 0 \)
for all \( p, q \in \mathbb{N}_{n+1} \) with \( p \neq q \) such that \( M(T, \pi, s) = \sum_{q=0}^n \gamma_q P_q \). Define an
\((n+1) \times (n+1)\) real matrix \( \tilde{M} \) by \( \tilde{M} := \sum_{q=0}^n \sqrt{\gamma_q} P_q \). Then we see that
\[
M(T, \pi, s) = \tilde{M} \tilde{M}.
\tag{4.10}
\]

By writing \( \tilde{M} = (v_0, \ldots, v_n) \) with vectors \( v_q \in \mathbb{R}^{n+1} \) \((q \in \mathbb{N}_{n+1})\), the equality
(4.10) implies that \( M(T, \pi, s)_{p,q} = \langle v_p, v_q \rangle \) for all \( p, q \in \mathbb{N}_{n+1} \). The property that
\( M(T, \pi, s)_{q,q} = 1 \) \((\forall q \in \mathbb{N}_{n+1})\) ensures that for all \( q \in \mathbb{N}_{n+1} \)
\[
|v_q| = 1.
\tag{4.11}
\]

Then we observe that
\[
e^{\Delta(M(T, \pi, s))} \sum_{\psi^p_1, \psi^q_1, \ldots, \psi^p_l, \psi^q_l} \prod_{j=1}^l \left( \frac{\psi^p_j}{x_{j_1}^{p_1} \ldots x_{j_j}^{p_j} x_{k_1}^{q_1} \ldots x_{k_k}^{q_k}} \right) \left| \psi^p_j \psi^q_j \right| = 0 \right|_{\psi^p_j, \psi^q_j \in \mathbb{N}_{n+1}} \leq 4^l
\]
\[
= (-1)^{(l-1)/2} \frac{1}{l!} \sum_{p,q \in \mathbb{N}_{n+1}} \langle v_p, v_q \rangle \Delta_{p,q} \prod_{m=1}^l \frac{\psi^{p_m}_{j_m} x_{j_m}^{m_1} \ldots x_{j_m}^{m_j} \psi^{q_m}_{k_m} x_{k_m}^{m_1} \ldots x_{k_m}^{m_k}}{\psi^{p_m}_{j_m} x_{j_m}^{m_1} \ldots x_{j_m}^{m_j} \psi^{q_m}_{k_m} x_{k_m}^{m_1} \ldots x_{k_m}^{m_k}} \left| \psi^{p_m}_{j_m} \psi^{q_m}_{k_m} = 0 \right|_{\psi^{p_m}_{j_m}, \psi^{q_m}_{k_m} \in \mathbb{N}_{n+1}}
\]
\[
= (-1)^{(l-1)/2} \det(\langle v_{p_s}, v_{q_t} \rangle C_h(x_{j_s} \sigma_{j_s} x_{j_1}, x_{k_t} \sigma_{k_t} x_{k_1}))_{1 \leq s, t \leq l}.
\tag{4.12}
\]
Proof. Set the vertex Lemma 4.7. For the number of the Grassmann algebras with index \( T \) let us always think that any tree in later generation. We see that more than 4 derivatives with respect to the Grassmann algebras indexed by \( d \). For any \( q \in \mathbb{N}_{n+1} \) let \( L_q(T) \) (\( \subseteq T \)) be the set of lines from the vertex \( q \) to the vertices of the later generation. We see that

\[
L_0(T) = d_0, \quad L_q(T) = d_q - 1, \quad \forall q \in \mathbb{N}_{n+1} \setminus \{0\}.
\]

We define the combinatorial factor \( N(T) \) we want to calculate as follows.

**Definition 4.6.** For any \( T \in \mathcal{T}(\mathbb{N}_{n+1}) \) the combinatorial factor \( N(T)(\in \mathbb{N}) \) is defined as the total number of monomials appearing in the expansion of

\[
\Pi_{\{q,q'\} \in T} (\Delta_{q,q'} + \Delta_{q',q}) \prod_{q \in \mathbb{N}_{n+1}} \bar{z}_q^1 x_q^1 z_q^2 x_q^2 \psi_{q}^3 z_q^3 x_q^3 \psi_{q}^4 z_q^4 x_q^4. \tag{4.13}
\]

Note that \( N(T) \) is independent of how to choose \( z_j^q \in \Gamma, \quad x_j^q \in [0,\beta)_h \quad (\forall j \in \{1, 2, 3, 4\}, \forall q \in \mathbb{N}_{n+1}) \).

The combinatorial factor \( N(T) \) is counted as follows.

**Lemma 4.7.** For \( T \in \mathcal{T}(\mathbb{N}_{n+1}) \) let \( d_q \) (\( q \in \mathbb{N}_{n+1} \)) denote the incidence number of the vertex \( q \) in \( T \). If there is \( q \in \mathbb{N}_{n+1} \) such that \( d_q > 4 \), \( N(T) = 0 \). Otherwise,

\[
N(T) = 4 \prod_{q \in \mathbb{N}_{n+1}} \left( d_q - 1 \right) (d_q - 1)!. \tag{4.14}
\]

**Proof.** Set \( W := \Pi_{q \in \mathbb{N}_{n+1}} \bar{z}_q^1 x_q^1 z_q^2 x_q^2 \psi_{q}^3 z_q^3 x_q^3 \psi_{q}^4 z_q^4 x_q^4 \). If there is \( p \in \mathbb{N}_{n+1} \) such that \( d_p > 4 \), every term in the expansion of the product \( \Pi_{\{q,q'\} \in T} (\Delta_{q,q'} + \Delta_{q',q}) \) contains more than 4 derivatives with respect to the Grassmann algebras indexed by \( p \). Since the number of the Grassmann algebras with index \( p \) in \( W \) is 4, \( \text{(4.13)} \) must vanish.

Let us consider the case that \( d_q \in \{1, 2, 3, 4\} \) for all \( q \in \mathbb{N}_{n+1} \). The operator \( \Delta_{q,q'} \) can be decomposed as \( \Delta_{q,q'} = \sum_{\sigma \in \{1,\ldots,4\}} \Delta_{q,q'}^\sigma \), where

\[
\Delta_{q,q'}^\sigma := - \sum_{x,y \in \Gamma} \sum_{x,y \in [0,\beta)_h} C_h(x, y) \frac{\partial}{\partial \psi_{x\sigma}^{\alpha}} \frac{\partial}{\partial \psi_{y\sigma}^{\beta}} \tag{4.14}
\]

for \( \sigma \in \{1,\ldots,4\} \). Note that for any \( \sigma \in \{1,\ldots,4\} \) and \( p, p', p'' \in \mathbb{N}_{n+1} \)

\[
\Delta_{q,q'}^\sigma \Delta_{p,p'}^\sigma W = \Delta_{q,q'}^\sigma \Delta_{p''}^\sigma \Delta_{p,p''}^\sigma W = 0. \tag{4.15}
\]

By changing the numbering of vertices if necessary we may assume the following condition on \( T \) without losing generality.
The distance between the vertex $p$ and the initial vertex 0 is less than or equal to that between the vertex $q$ and the vertex 0 for all $p, q \in \mathbb{N}_{n+1} \setminus \{0\}$ with $p \leq q$.

Note that
\[
\Pi_{q', q} \in T (\Delta_{q,q'} + \Delta_{q',q}) W = \Pi_{q' \in \mathbb{N}_{n+1} \setminus \{0\}} \Pi_{\ell_q(T) \neq \emptyset} (\Delta_{q,p} + \Delta_{q',p} + \Delta_{p,q} + \Delta_{p,q'}) W. \tag{4.16}
\]

Let us count $N(T)$ recursively with respect to $q \in \mathbb{N}_{n+1}$ as follows. The expansion of the product $\Pi_{(0,p) \in L_q(T)} (\Delta_{0,p} + \Delta_{0,p} + \Delta_{p,0} + \Delta_{p,0})$ is a sum of $4^{|L_q(T)|}$ terms, each of which is a product of $|L_q(T)|$ Laplacians. By the property (4.15) any term containing the products $\Delta_{q,q'} \Delta_{q',q'}$ or $\Delta_{q,q} \Delta_{q',q'}$ for some $\sigma \in \{\uparrow, \downarrow\}$, $\{0, q\}, \{0, q'\} \in L_q(T)$ does not contribute to the number of remaining monomials in (4.16), thus, can be eliminated. Therefore, we only need to count
\[
\left(\frac{4}{\#L_0(T)}\right)^{\#L_0(T)!}
\]
terms in the expansion of $\Pi_{(0,p) \in L_q(T)} (\Delta_{0,p} + \Delta_{0,p} + \Delta_{p,0} + \Delta_{p,0})$.

Take any $q \in \mathbb{N}_{n+1} \setminus \{0\}$ with $L_q(T) \neq \emptyset$. By the condition (♠) there uniquely exists $q' \in \mathbb{N}_{n+1}$ with $q' < q$ such that $\{q', q\} \in L_q(T)$. Thus, every term in the expansion of the product
\[
\prod_{j \in \mathbb{N}_{n+1} \setminus \{0\}, \ell_j(T) \neq \emptyset} \prod_{(j,p) \in L_j(T)} (\Delta_{j,p} + \Delta_{j,p} + \Delta_{p,j} + \Delta_{p,j})
\]
contains one of the Laplacians of the form $\Delta_{q,q'}, \Delta_{q',q} \ (\sigma \in \{\uparrow, \downarrow\})$. Therefore, by the property (4.15) only
\[
\left(\frac{3}{\#L_q(T)}\right)^{\#L_q(T)!}
\]
terms in the expansion of the product $\Pi_{(q,p) \in L_q(T)} (\Delta_{q,p} + \Delta_{q,p} + \Delta_{p,q} + \Delta_{p,q})$ need to be counted.

By repeating this argument recursively for all $q \in \mathbb{N}_{n+1} \setminus \{0\}$ we can calculate $N(T)$ as follows.

\[
N(T) = \left(\frac{4}{\#L_0(T)}\right)^{\#L_0(T)!} \Pi_{q \in \mathbb{N}_{n+1} \setminus \{0\}, \ell_q(T) \neq \emptyset} \left(\frac{3}{\#L_q(T)}\right)^{\#L_q(T)!}
\]

\[
= \left(\frac{4}{d_0}\right) d_0! \prod_{q \in \mathbb{N}_{n+1} \setminus \{0\}, d_q \geq 2} \left(\frac{3}{d_q - 1}\right) (d_q - 1)! = 4 \prod_{q \in \mathbb{N}_{n+1}} \left(\frac{3}{d_q - 1}\right) (d_q - 1)!,
\]

where we used the fact that
\[
\left(\frac{3}{0}\right) 0! = 1.
\]
Let $D_h$ denote the $L^1$-norm of the covariance matrix $C_h$, i.e.,

$$D_h := \frac{1}{h} \sum_{x \in \mathbb{R}^d} \sum_{x \in \{-\beta, \beta\}_h} |C_h(x, x, 0)|.$$ 

Now we can find an upper bound on $|a_{h,n}|$.

**Lemma 4.8.** For any $h \in 2\mathbb{N}/\beta$ and for all $n \in \mathbb{N} \cup \{0\}$ the following equality holds.

$$|a_{h,n}| \leq \frac{128}{3n+4} \left( \frac{3n+4}{n} \right)^n (4D_h)^n. \tag{4.17}$$

**Proof.** By using Lemma 2.5 and (3.2)-(3.3) we can directly evaluate $|a_{h,0}|$ to obtain $|a_{h,0}| \leq 32$, which is (4.17) for $n = 0$. Let us show (4.17) for $n \geq 1$. By (4.11) and (4.8), we need to evaluate

$$\frac{1}{\beta n!} \sum_{T \in T(\mathbb{N}_{n+1})} \sup_{\{q,q^\prime\} \in T} \left| e^{\Delta_q} \prod_{\{q,q^\prime\} \in T} (\Delta_q + \Delta_{q^\prime}^\prime) \cdot V_{h,z_0}(\psi_0, \psi_X) \prod_{q=1}^n V_{h,z_q}(\psi_X, \psi_X) \right|_{v_{\psi} = \sigma_{\psi} = 0}. \tag{4.18}$$

By using the property (4.9) we have

$$\frac{1}{\beta n!} \sum_{T \in T(\mathbb{N}_{n+1})} \sup_{\{q,q^\prime\} \in T} \left| e^{\Delta_q} \prod_{\{q,q^\prime\} \in T} (\Delta_q + \Delta_{q^\prime}^\prime) \cdot V_{h,z_0}(\psi_0, \psi_X) \prod_{q=1}^n V_{h,z_q}(\psi_X, \psi_X) \right|_{v_{\psi} = \sigma_{\psi} = 0}. \tag{4.19}$$

Take any $T \in T(\mathbb{N}_{n+1})$. If $T$ contains a vertex whose incidence number is larger than 4, by Lemma 4.7 the tree $T$ does not contribute to the sum in (4.19). Thus, we may assume that the incidence numbers $d_0, d_1, \cdots, d_n$ of $T$ are less than equal to 4. Moreover, as in the proof of Lemma 4.7 without losing generality we may assume the condition (4.10) on $T$.

Let $q_1, q_2, \cdots, q_l \in \mathbb{N}_{n+1} \setminus \{0\}$ be such that $q_1 < q_2 < \cdots < q_l$ and \{\{q_1, q_2, \cdots, q_l\} = \{q \in \mathbb{N}_{n+1} \setminus \{0\} \mid L_q(T) \neq 0\}. Every term of the expansion of the product $\prod_{\{q,q^\prime\} \in T} (\Delta_q + \Delta_{q^\prime}^\prime)$ has the form

$$\prod_{\{q,q^\prime\} \in T} \Delta_{\{q,q^\prime\}} \prod_{j=1}^{l-1} \Delta_{\{q_j,q_j^\prime\}}^{\tau_j} \prod_{j=1}^{l} \Delta_{\{q_j,q_j^\prime\}}^{\tau_j}, \tag{4.20}$$

where by using the notation (4.14), $\Delta_{\{q,p\}}^{\tau} \in \{\Delta_{\{q,p\}}^{\tau}, \Delta_{\{p,q\}}^{\tau} \mid \tau \in \{\uparrow, \downarrow\} \}$ for all $\{q,p\} \in \{\{q_1, q_2, \cdots, q_l\} = \{q \in \mathbb{N}_{n+1} \setminus \{0\} \mid L_q(T) \neq 0\}.$
$L_q(T)$ with $q \in \mathbb{N}_{n+1}$ satisfying $L_q(T) \neq \emptyset$. For any $\{0, p\} \in L_0(T)$ we set

$$C^\sigma_{\{0, p\}}(x_0, z_p x_p) :=$$

$$|C_h(z_p \uparrow x_p, x_1 \uparrow x_0)| \text{ if } \Delta^\sigma_{\{0, p\}} = \Delta_{p, 0}^\uparrow,$$

$$|C_h(z_p \downarrow x_0, x_2 \downarrow x_0)| \text{ if } \Delta^\sigma_{\{0, p\}} = \Delta_{p, 0}^\downarrow,$$

$$|C_h(y_1 \uparrow x_0, z_p \uparrow x_p)| \text{ if } \Delta^\sigma_{\{0, p\}} = \Delta_{p, p}^\uparrow,$$

$$|C_h(y_2 \downarrow x_0, z_p \downarrow x_p)| \text{ if } \Delta^\sigma_{\{0, p\}} = \Delta_{p, p}^\downarrow,$$

for all $x_0, x_p \in [0, \beta)_h$ and $z_p \in \Gamma$. For any $j \in \{1, 2, \ldots, l\}$ and any $\{q_j, p\} \in L_{q_j}(T)$ we define

$$C^\sigma_{\{q_j, p\}}(z_{q_j} x_{q_j}, z_p x_p) :=$$

$$|C_h(z_{q_j} \tau x_{q_j}, z_p \tau x_p)| \text{ if } \Delta^\sigma_{\{q_j, p\}} = \Delta^\tau_{q_j, p} \text{ for some } \tau \in \{\uparrow, \downarrow\},$$

$$|C_h(z_{q_j} \tau x_{q_j}, z_{q_j} x_{q_j})| \text{ if } \Delta^\sigma_{\{q_j, p\}} = \Delta^\tau_{p, q_j} \text{ for some } \tau \in \{\uparrow, \downarrow\}$$

for all $x_p, x_{q_j} \in [0, \beta)_h$ and $z_p, z_{q_j} \in \Gamma$. By noting that (1.20) is the product of $n$ Laplacians and using Lemma 4.3 and the condition (♠), we observe that

$$\left| e^{\Delta(M(T, \pi, \eta))} \prod_{\{0, p\} \in L_0(T)} \Delta^\sigma_{\{0, p\}} \prod_{j=1}^l \prod_{\{q_j, p\} \in L_{q_j}(T)} \Delta^\sigma_{\{q_j, p\}} \right|$$

$$\cdot V_h, z_0(\varphi^0_X, \varphi^0_X) \prod_{q \in \mathbb{N}_{n+1}} \sum_{x_q \in \Gamma} V_h, z_q(\varphi^q_X, \varphi^q_X) \left|_{\varphi^q_X = \varphi^0_X = 0} \right|$$

$$\leq \frac{4^{n+2}}{h} \sum_{x_0 \in [0, \beta)_h} \prod_{\{0, p\} \in L_0(T)} \left( \frac{1}{h} \sum_{x_q \in [0, \beta)_h} \sum_{z_q \in \Gamma} \prod_{\{q_j, p\} \in L_{q_j}(T)} C^\sigma_{\{q_j, p\}}(z_{q_j} x_{q_j}, z_p x_p) \right)$$

$$= \frac{4^{n+2}}{h} \sum_{x_0 \in [0, \beta)_h} \prod_{\{0, p\} \in L_0(T)} \left( \frac{1}{h} \sum_{x_p \in [0, \beta)_h} \sum_{z_p \in \Gamma} \prod_{\{q_j, p\} \in L_{q_j}(T)} C^\sigma_{\{0, p\}}(x_0, z_p x_p) \right)$$

$$\cdot \prod_{\{q_1, p_1\} \in L_{q_1}(T)} \left( \frac{1}{h} \sum_{x_{q_1} \in [0, \beta)_h} \sum_{z_{q_1} \in \Gamma} \prod_{\{q_1, p_1\} \in L_{q_1}(T)} C^\sigma_{\{q_1, p_1\}}(z_{q_1} x_{q_1}, z_{p_1} x_{p_1}) \right)$$

$$\cdot \prod_{\{q_2, p_2\} \in L_{q_2}(T)} \left( \frac{1}{h} \sum_{x_{q_2} \in [0, \beta)_h} \sum_{z_{q_2} \in \Gamma} \prod_{\{q_2, p_2\} \in L_{q_2}(T)} C^\sigma_{\{q_2, p_2\}}(z_{q_2} x_{q_2}, z_{p_2} x_{p_2}) \right)$$

$$\cdots \prod_{\{q_k, p_k\} \in L_{q_k}(T)} \left( \frac{1}{h} \sum_{x_{q_k} \in [0, \beta)_h} \sum_{z_{q_k} \in \Gamma} \prod_{\{q_k, p_k\} \in L_{q_k}(T)} C^\sigma_{\{q_k, p_k\}}(z_{q_k} x_{q_k}, z_{p_k} x_{p_k}) \right)$$
\[\leq 4^{n+2} \beta \prod_{\{0, p_0\} \in L_0(T)} \left( \sup_{x_0 \in (0, \beta), h} \frac{1}{h} \sum_{x_{p_0} \in (0, \beta), x_{p_0} \in \Gamma} C_{\{0, p_0\}}^{q}(x_0, z_{p_0}, x_{p_0}) \right)\]

\[\cdot \prod_{\{q_1, p_1\} \in L_{q_1}(T)} \left( \sup_{x_{q_1} \in (0, \beta), x_{q_1} \in \Gamma} \frac{1}{h} \sum_{x_{p_1} \in (0, \beta), x_{p_1} \in \Gamma} C_{\{q_1, p_1\}}^{q}(z_{q_1}, x_{q_1}, z_{p_1}, x_{p_1}) \right)\]

\[\cdot \prod_{\{q_2, p_2\} \in L_{q_2}(T)} \left( \sup_{x_{q_2} \in (0, \beta), x_{q_2} \in \Gamma} \frac{1}{h} \sum_{x_{p_2} \in (0, \beta), x_{p_2} \in \Gamma} C_{\{q_2, p_2\}}^{q}(z_{q_2}, x_{q_2}, z_{p_2}, x_{p_2}) \right)\]

\[\cdots \prod_{\{q_l, p_l\} \in L_{q_l}(T)} \left( \sup_{x_{q_l} \in (0, \beta), x_{q_l} \in \Gamma} \frac{1}{h} \sum_{x_{p_l} \in (0, \beta), x_{p_l} \in \Gamma} C_{\{q_l, p_l\}}^{q}(z_{q_l}, x_{q_l}, z_{p_l}, x_{p_l}) \right)\]

\[\leq 4^{n+2} \beta D_h^n. \quad (4.21)\]

To obtain the last inequality in (4.21) we have used the fact that for all \(\{0, p_0\} \in L_0(T)\) and all \(\{q_j, p_j\} \in L_{q_j}(T)\) \((\forall j \in \{1, 2, \cdots, l\})\)

\[\sup_{x_0 \in (0, \beta), h} \frac{1}{h} \sum_{x_{p_0} \in (0, \beta), x_{p_0} \in \Gamma} C_{\{0, p_0\}}^{q}(x_0, z_{p_0}, x_{p_0}) \leq D_h,\]

\[\sup_{x_{q_j} \in (0, \beta), x_{q_j} \in \Gamma} \frac{1}{h} \sum_{x_{p_j} \in (0, \beta), x_{p_j} \in \Gamma} C_{\{q_j, p_j\}}^{q}(z_{q_j}, x_{q_j}, z_{p_j}, x_{p_j}) \leq D_h.\]

By combining (4.21) with (4.19) we have

\[16 \leq (4D_h)^n \sum_{T \in \mathcal{T}(N_{n+1})} N(T), \quad (4.22)\]

where \(N(T)\) is defined in Definition 4.6.

By repeating the same argument as above we can show the inequality (4.22) for the case that \(\mathcal{Z}_0 = \mathcal{X}_2\). Thus, by recalling (4.1) we arrive at

\[|\alpha_{h,n}| \leq \frac{32}{n!} (4D_h)^n \sum_{T \in \mathcal{T}(N_{n+1})} N(T). \quad (4.23)\]

To complete the proof we calculate the sum \(\sum_{T \in \mathcal{T}(N_{n+1})} N(T)\) in (4.23). As characterized in Lemma 4.7, the number \(N(T)\) only depends on the incidence numbers of \(T\). By using Lemma 4.7 and Cayley’s theorem on the number of trees with
fixed incidence numbers (see, e.g., [23, Corollary 2.2.4]) we have
\[
\sum_{T \in \mathcal{T}(N_{n+1})} N(T) = \sum_{d_q \in \{1,2,3,4\}} 1^{\sum_{q \in N_{n+1}} d_q = 2n} \frac{(n-1)!}{\prod_{q \in N_{n+1}} (d_q - 1)!} \cdot \frac{4}{\prod_{q \in N_{n+1}} \left( \frac{3}{d_q - 1} \right)} (d_q - 1)!
\]

\[
= 4(n-1)! \sum_{d_q \in \{1,2,3,4\}} 1^{\sum_{q \in N_{n+1}} d_q = 2n} \prod_{q \in N_{n+1}} \left( \frac{3}{d_q - 1} \right).
\]

Moreover, by using Cauchy’s integral formula and the residue theorem we see that for a positive \(r > 0\)
\[
\sum_{T \in \mathcal{T}(N_{n+1})} N(T) = \frac{4(n-1)!}{(2n)!} \left( \frac{d}{dz} \right)^{2n} \left( \sum_{d=1}^{4} \left( \frac{3}{d-1} \right) z^d \right)^{n+1} \bigg|_{z=0}
\]
\[
= \frac{4(n-1)!}{2\pi i} \oint_{|z| = r} dz \frac{z^{n+1} (1+z)^{3(n+1)}}{z^{2n+1}} = 4(n-1)! \text{Res}_{z=0} \left( \frac{(1+z)^{3(n+1)}}{z^n} \right)
\]
\[
= 4(n-1)! \left( \frac{3n+3}{n-1} \right) = \frac{4n!}{3n+4} \left( \frac{3n+4}{n} \right).
\]

Combining (4.24) with (4.23) yields the result.

The inequality (4.17) motivates us to know the properties of the power series \(f(x)\) defined by
\[
f(x) := \sum_{n=0}^{\infty} \frac{4}{3n+4} \left( \frac{3n+4}{n} \right) x^n.
\]

As the last lemma before our main theorem, the properties of \(f(x)\) are summarized.

**Lemma 4.9.** The radius of convergence of the power series \(f(x)\) defined in (4.25) is \(4/27\) and \(f(4/27) = 81/16\). Moreover, for any \(x \in (0,4/27]\) the following equality holds.
\[
f(x) = 16 \left( \frac{\tan^{-1}\left( \frac{\sqrt{2/3}x - 1}{3} \right) + \pi}{9x^2} \right) \cos^4 \left( \frac{\tan^{-1}\left( \frac{\sqrt{2/3}x - 1}{3} \right) + \pi}{3} \right),
\]

where the function \(\tan^{-1}(\cdot)\) is defined as a bijective map from \(\mathbb{R}\) to \((-\pi/2, \pi/2)\) satisfying \(\tan^{-1}(\tan(\theta)) = \theta\) for all \(\theta \in (-\pi/2, \pi/2]\).

**Proof.** As a topic in generating functions the power series (4.25) is commonly studied (see, e.g., [3], pp. 200–201). However, we give a proof for the statements for completeness. Let us analyze the cubic equation
\[
X = 1 + xX^3
\]
for $x \in (0, 4/27)$. We see that for any $x \in (0, 4/27)$ and $z \in \mathbb{C}$ satisfying $|z-1| = 1/2$, the inequality $|xz^3| < |z-1|$ holds. Thus, the Lagrange inversion theorem (see, e.g., [13, Theorem 2.3.1]) implies that in the domain \( \{ z \in \mathbb{C} \mid |z-1| < 1/2 \} \) there is exactly one root $X = v(x)$ of (4.27) and

\[
v(x)^4 = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left( \frac{d}{dz} \right)^{n-1} (4z^3z^{3n}) \bigg|_{z=1} = f(x). \tag{4.28}
\]

On the other hand, by algebraically solving (4.27) and specifying a root contained in the domain \( \{ z \in \mathbb{C} \mid |z-1| < 1/2 \} \) we can determine the explicite form of $v(x)$ as follows.

\[
v(x) = \frac{2}{\sqrt{3x}} \cos \left( \tan^{-1} \left( \frac{\sqrt{\frac{27}{2\pi}} - 1 + \pi}{3} \right) \right), \tag{4.29}
\]

where $\tan^{-1}(\cdot)$ is defined as stated in Lemma 4.9.

The equalities (4.28)-(4.29) give (4.30) for $x \in (0, 4/27)$. The ratio test shows that the radius of convergence of $f$ is $4/27$. Moreover, by continuity we have $\lim_{x \to 4/27} f(x) = v(4/27)^4 = 81/16$, which completes the proof. \( \square \)

Define a constant $D > 0$ by

\[
D := \lim_{h \to +\infty, h \in \mathbb{N}/\beta} D_h = \int_{-\beta}^{\beta} dx \sum_{x \in \Gamma} |C(x, 0)|. \tag{4.30}
\]

Our main result is stated as follows.

**Theorem 4.10.** For any $x_1, x_2, y_1, y_2 \in \Gamma$ and $m \in \mathbb{N} \cup \{0\}$ and any $U \in \mathbb{R}$ with $|U| \leq 1/(27D)$, the following equality and inequalities hold.

\[
\langle \psi_{x_1}^* \psi_{x_2} \psi_{y_1} \psi_{y_2} \rangle = \sum_{n=0}^{\infty} a_n U^n, \tag{4.31}
\]

\[
|\langle \psi_{x_1}^* \psi_{x_2} \psi_{y_1} \psi_{y_2} \rangle| \leq R(|U|), \tag{4.32}
\]

\[
|\langle \psi_{x_1}^* \psi_{x_2} \psi_{y_1} \psi_{y_2} \rangle - \sum_{n=0}^{m} a_n U^n| \leq R(|U|) - \sum_{n=0}^{m} \frac{128}{3n+4} \left( \frac{3n+4}{n} \right) (4D|U|)^n, \tag{4.33}
\]

where \( \{a_n\}_{n=0}^{\infty} \) is given in (2.11)-(2.12) and

\[
R(|U|) := \begin{cases} 
32 & \text{if } U = 0, \\
\frac{32}{9D^2|U|^2} \cos^4 \left( \tan^{-1} \left( \frac{\sqrt{27D|U|} - 1 + \pi}{3} \right) \right) & \text{if } 0 < |U| \leq \frac{1}{27D}. 
\end{cases} \tag{4.34}
\]
with the function \( \tan^{-1}(\cdot) : \mathbb{R} \to (-\pi/2, \pi/2) \) satisfying \( \tan^{-1}(\tan \theta) = \theta \) for all \( \theta \in (-\pi/2, \pi/2) \).

**Proof.** Since by Lemma 3.1 (iii) and Lemma 4.8

\[
|a_n| = \lim_{h \to +\infty, |h| \neq 0} |a_{h,n}| \leq \frac{128}{3n+4} \left( \frac{3n+4}{n} \right) (4D)^n, \tag{4.35}
\]

Lemma 4.8 implies that for all \( U \in [-1/(27D), 1/(27D)] \)

\[
\sum_{n=0}^{\infty} |a_n||U|^n \leq R(|U|), \tag{4.36}
\]

where \( R(|U|) \) is defined in (4.34). The inequalities (4.32)-(4.33) follow from (4.31) and (4.35)-(4.36).

We show that the equality (4.31) holds for \( U \in \mathbb{R} \) with \( |U| \leq 1/(27|D|) \). Let us fix any \( \varepsilon \in (0, 1/(27D)) \). Since \( P|_{\lambda_X=0, X \in \Gamma^4} = \text{Tr} e^{-\beta H} / \text{Tr} e^{-\beta H_0} > 0 \) for all \( U \in \mathbb{R} \), Proposition 4.2 implies that there exists \( N_0 \in \mathbb{N} \) such that \( |P_h|_{\lambda_X=0, X \in \Gamma^4} > 0 \) for all \( h \in 2\mathbb{N}/\beta \) with \( h \geq N_0/\beta \) and all \( U \in \mathbb{R} \) with \( |U| \leq 1/(27D) - \varepsilon \). Moreover, \( P_h|_{\lambda_X=0, X \in \Gamma^4} \) is a polynomial of \( U \) we can take a simply connected domain \( O_h \subset \mathbb{C} \) containing the interval \([1/(27D) - \varepsilon, 1/(27D) - \varepsilon]\) inside such that \( |P_h|_{\lambda_X=0, X \in \Gamma^4} > 0 \) for all \( U \in O_h \). Thus, we see that \( \partial P_h / \partial \lambda_X / P_h|_{\lambda_X=0, X \in \Gamma^4} \) defines an analytic function of \( U \) in the domain \( O_h \). By Lemma 4.8 and Lemma 4.9 the series \( \sum_{n=0}^{\infty} a_{h,n} U^n \) converges for all \( U \in \mathbb{C} \) with \( |U| \leq 1/(27D_h) \). By choosing \( N_0 \) sufficiently large we may assume that \( 1/(27D) - \varepsilon \leq 1/(27D_h) \) for all \( h \in 2\mathbb{N}/\beta \) with \( h \geq N_0/\beta \). Therefore, Lemma 3.1 (iii) and the identity theorem for analytic functions ensure that

\[
- \frac{1}{\beta} \frac{\partial}{\partial \lambda_X} P_h(\{U_X\}_{X \in \Gamma^4})|_{\lambda_X=0} \sum_{n=0}^{\infty} a_{h,n} U^n \tag{4.37}
\]

for all \( U \in [1/(27D) + \varepsilon, 1/(27D) - \varepsilon] \).

Note that Lemma 4.8 implies

\[
|a_{h,n} U^n| \leq \frac{128}{3n+4} \left( \frac{3n+4}{n} \right) \left( \frac{4}{27} \right)^n \tag{4.38}
\]

for all \( U \in [1/(27D) + \varepsilon, 1/(27D) - \varepsilon] \) and the right hand side of (4.38) is summable over \( \mathbb{N} \cup \{0\} \). Thus, by Lemma 3.1 (iii), Corollary 5.4 and Lebesgue’s dominated convergence theorem for \( l^1(\mathbb{N} \cup \{0\}) \) we can pass \( h \to +\infty \) in (4.37) to deduce the equality (4.31) for all \( U \in [-1/(27D) + \varepsilon, 1/(27D) - \varepsilon] \). Then, by sending \( \varepsilon \to 0 \) and continuity we obtain (4.31) for all \( U \in [-1/(27D), 1/(27D)] \). \( \square \)

**Remark 4.11.** In Proposition 5.1 we give a volume-independent upper bound on the decay constant \( D \) in 2 dimensional case. One can straightforwardly extend the calculation of Proposition 5.1 to derive volume-independent upper bounds on \( D \) in any dimension. By replacing \( D \) by these upper bounds, Theorem 4.10 provides volume-independent upper bounds on the perturbation series \( \sum_{n=0}^{\infty} a_n U^n \).
5 Numerical results in 2D

In this section we compute the perturbation series of the correlation function
\[ \langle \psi_{x_1} \psi_{x_1} \psi_{y_2} \psi_{y_1} + \psi_{y_1} \psi_{y_1} \psi_{x_2} \psi_{x_1} \rangle \] up to 2nd order term in 2 dimensional case. We also implement the upper bound obtained in Theorem 4.10 and report the error between the correlation function and the 2nd order perturbation. Throughout this section it is assumed that \( d = 2 \).

5.1 The decay constant for \( d = 2 \)

In order to estimate the radius of convergence of the perturbation series \( \sum_{n=0}^{\infty} a_n U^n \) and compute the upper bound on the sum of the higher order terms numerically, first we need to evaluate the decay constant \( D \) defined in (4.30). The result is presented in the following proposition.

**Proposition 5.1.** The following inequality holds.

\[
D \leq \left( \frac{16}{\beta^2} + \frac{32\pi^2}{3\sqrt{3}\beta} + \frac{16\pi^4}{3\sqrt{3}} \right) \left( \frac{\beta^3}{2} + \frac{(2|t| + 4|t'|)e^{3\xi} \beta}{1 + e^{3\xi}} \right) + 3 \left( \frac{(2|t| + 4|t'|)e^{3\xi} \beta}{1 + e^{3\xi}} \right)^2 + \left( \frac{(2|t| + 4|t'|)e^{3\xi} \beta}{1 + e^{3\xi}} \right)^3, \tag{5.1}
\]

where \( \xi := 4|t| + 4|t'| + |\mu| \).

The derivation of the inequality (5.1) needs the following estimate.

**Lemma 5.2.** The following inequality holds.

\[
\sum_{x \in \Gamma} \frac{1}{1 + \sum_{l=1}^{L/2} |e^{2\pi x_l/L} - 1|^3 L^3/(8\pi^3 \beta^3)} \leq 4 + \frac{8\pi^2 \beta}{3\sqrt{3}} + \frac{4\pi^3 \beta^2}{3\sqrt{3}},
\]

where \( x = (x_1, x_2) \in \Gamma \).

**Proof.** For any \( y \in \mathbb{R} \) let \( |y| \) denote the largest integer which does not exceed \( y \). By using the inequality that \( |e^{i\theta} - 1| \geq 2|\theta|/\pi \) for any \( \theta \in [-\pi, \pi] \), we see that

\[
\sum_{x \in \Gamma} \frac{1}{1 + \sum_{l=1}^{L/2} |e^{2\pi x_l/L} - 1|^3 L^3/(8\pi^3 \beta^3)} \\
\leq 4 \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\infty} \frac{1}{1 + \sum_{l=1}^{L/2} |e^{2\pi x_l/L} - 1|^3 L^3/(8\pi^3 \beta^3)} \\
\leq 4 \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\infty} \frac{1}{1 + 8x_1^3/(\pi^3 \beta^3) + 8x_2^3/(\pi^3 \beta^3)} \\
= 4 + 8 \sum_{x_1=1}^{\infty} \frac{1}{1 + 8x_1^3/(\pi^3 \beta^3)} + 4 \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \frac{1}{1 + 8x_1^3/(\pi^3 \beta^3) + 8x_2^3/(\pi^3 \beta^3)}.
\]
Then, by using the equality
\[
\int_0^\infty \frac{dx}{1 + x^3} \geq 1 + \int_1^\infty \frac{dx}{x^2} = 2.
\]
(see [10]), we obtain the desired inequality.

**Proof of Proposition 5.1.** Let us define a linear operator \(d_{l,L} : C^\infty(\mathbb{R}^2) \rightarrow C^\infty(\mathbb{R}^2)\) \((l = 1, 2)\) by
\[
(d_{l,L}f)(k) := \frac{f(k + 2\pi \hat{e}_l/L) - f(k)}{2\pi L}
\]
for any \(f \in C^\infty(\mathbb{R}^2)\). Then, the mean value theorem shows that for all \(k = (k_1, k_2) \in \mathbb{R}^2\)
\[
|d_{l,L}f(k, k)| \leq \sup_{\hat{k}_1 \in [k_1, k_1 + 6\pi/L]} \left| \frac{\partial^3}{\partial k_1^3} f(\hat{k}_1, k_2) \right|,
\]
(5.2)
\[
|d_{l,L}f(k_1, k_2)| \leq \sup_{\hat{k}_2 \in [k_2, k_2 + 6\pi/L]} \left| \frac{\partial^3}{\partial k_2^3} f(k_1, \hat{k}_2) \right|.
\]
(5.3)

Define a function \(F(k, x) : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}\) by \(F(k, x) := e^{x E_k}/(1 + e^{3E_k})\). We see that for any \(x = (x_1, x_2) \in \Gamma, x \in [-\beta, \beta]\) and \(l \in \{1, 2\}\)
\[
\left( e^{i2\pi x_1/L} - 1 \right) \frac{C(x \uparrow x, 0 \uparrow 0)}{2\pi/L} = \frac{1}{L^d} \sum_{k \in \Gamma^*} e^{-i(k,x)} ((d_{l,L}^3 F(k, x) 1_{x \geq 0} - (d_{l,L})^3 F(k, x + \beta) 1_{x < 0}).
\]
(5.4)
Note that for $l = 1, 2$

$$\frac{\partial^3}{\partial k_l^3} F(k, x) = \frac{\partial^3 E_k}{\partial k_l^3} \frac{\partial}{\partial k_l} \left( \frac{e^{x E_k}}{1 + e^{\beta E_k}} \right) + 3 \frac{\partial E_k}{\partial k_l} \frac{\partial^2}{\partial k_l^2} \left( \frac{e^{x E_k}}{1 + e^{\beta E_k}} \right) + \left( \frac{\partial E_k}{\partial k_l} \right)^3 \frac{\partial^3}{\partial E_k^3} \left( \frac{e^{x E_k}}{1 + e^{\beta E_k}} \right) \tag{5.5}$$

and

$$|E_k| \leq 4|t| + 4|t'| + |\mu|, \quad \left| \frac{\partial E_k}{\partial k_l} \right|, \left| \frac{\partial^2 E_k}{\partial k_l^2} \right|, \left| \frac{\partial^3 E_k}{\partial k_l^3} \right| \leq 2|t| + 4|t'|. \tag{5.6}$$

Moreover, we can prove that for $m \in \{1, 2, 3\}$ and any $E \in \mathbb{R}$

$$\sup_{x \in [0, \beta]} \left( \frac{d}{dE} \right)^m \frac{e^{x E}}{1 + e^{\beta E}} \leq \left( \frac{\beta e^{\beta E}}{1 + e^{\beta E}} \right)^m. \tag{5.7}$$

By the equalities that $e^{x E}/(1 + e^{\beta E}) = e^{(\beta - x)(-E)}/(1 + e^{\beta(-E)})$ and $d/dE = -d/(d/-E)$ we may assume $E \geq 0$ without losing generality in the following argument. First let us study the case for $m = 1$.

$$\left| \frac{d}{dE} \frac{e^{x E}}{1 + e^{\beta E}} \right| = \left| \frac{e^{x E}}{1 + e^{\beta E}} \left( x - \frac{\beta e^{\beta E}}{1 + e^{\beta E}} \right) \right| \leq \frac{\beta e^{\beta E}}{1 + e^{\beta E}},$$

which is (5.7) for $m = 1$.

Next consider the case that $m = 2$. We see that

$$\left( \frac{d}{dE} \right)^2 \frac{e^{x E}}{1 + e^{\beta E}} = \frac{e^{x E}}{1 + e^{\beta E}} g_1(x),$$

with

$$g_1(x) := \left( x - \frac{\beta e^{\beta E}}{1 + e^{\beta E}} \right)^2 - \frac{\beta^2 e^{\beta E}}{(1 + e^{\beta E})^2}.$$

Note that $|g_1(0)|, |g_1(\beta e^{\beta E}/(1 + e^{\beta E}))|, |g_1(\beta)| \leq \beta^2 e^{\beta E}/(1 + e^{\beta E})$. Thus, we have that

$$\left| \frac{\partial^2}{\partial E^2} \frac{e^{x E}}{1 + e^{\beta E}} \right| \leq \frac{\beta e^{\beta E}}{1 + e^{\beta E}} \max\{|g_1(0)|, |g_1(\beta e^{\beta E}/(1 + e^{\beta E}))|, |g_1(\beta)|\} \leq \left( \frac{\beta^2 e^{\beta E}}{(1 + e^{\beta E})^2} \right)^2.$$

Finally, analyze the case for $m = 3$. A calculation shows that

$$\left( \frac{d}{dE} \right)^3 \frac{e^{x E}}{1 + e^{\beta E}} = \frac{e^{x E}}{1 + e^{\beta E}} g_2(x), \tag{5.8}$$

where

$$g_2(x) := x^3 - 3 \frac{\beta e^{\beta E}}{1 + e^{\beta E}} x^2 + \frac{3 \beta^2 (e^{\beta E} - 1) e^{\beta E}}{(1 + e^{\beta E})^2} x - \frac{\beta^3 e^{\beta E}}{(1 + e^{\beta E})^3} (e^{2\beta E} - 4 e^{\beta E} + 1).$$
The roots $x_1, x_2$ of the equation $g_2'(x) = 0$ are given by

$$x_1 = \frac{\beta e^{\beta E/2}}{1 + e^{\beta E}} (e^{\beta E/2} - 1), \quad x_2 = \frac{\beta e^{\beta E/2}}{1 + e^{\beta E}} (e^{\beta E/2} + 1).$$

Since $0 \leq x_1 < \beta \leq x_2$, we have $|g_2(x)| = \max\{|g_2(0)|, |g_2(x_1)|, |g_2(\beta)|\}$. Moreover, we see that

$$|g_2(0)| = \frac{\beta^3 e^{\beta E}}{(1 + e^{\beta E})^3} (e^{2\beta E} - 4e^{\beta E} + 1) \leq \frac{\beta^3 e^{\beta E}}{(1 + e^{\beta E})^3} (e^{2\beta E} + e^{\beta E}) = \frac{\beta^3 e^{2\beta E}}{(1 + e^{\beta E})^2},$$

$$|g_2(\beta)| = \frac{\beta^3}{(1 + e^{\beta E})^3} (e^{2\beta E} - 4e^{\beta E} + 1) \leq |g_2(0)|,$$

$$|g_2(x_1)| = \frac{\beta^3 e^{\beta E}}{(1 + e^{\beta E})^3} (e^{\beta E} + 2e^{\beta E/2} - 1) \leq \frac{\beta^3 e^{\beta E}}{(1 + e^{\beta E})^3} (e^{2\beta E} + e^{\beta E}) = \frac{\beta^3 e^{2\beta E}}{(1 + e^{\beta E})^2},$$

which imply that

$$\max_{x_0 \in [0, \beta]} |g_2(x_0)| \leq \frac{\beta^3 e^{2\beta E}}{(1 + e^{\beta E})^2}. \quad (5.9)$$

Combining (5.8) with (5.9) deduces (5.7) for $m = 3$.

Then by (5.4)-(5.7) we have for $l = 1, 2$

$$\left| \frac{(e^{2\pi x_1/L} - 1)^3}{2\pi/L} C(x|x, 0|0) \right| \leq \left( \frac{2|t| + 4|t'|}{1 + e^{2\xi}} \right)^3 \beta e^{\beta \xi} + \left( \frac{2|t| + 4|t'|}{1 + e^{2\xi}} \right)^2 \beta e^{\beta \xi}, \quad (5.10)$$

with $\xi := 4|t| + 4|t'| + |\mu|$. By using the inequality that $|C(x|x, 0|0)| \leq 1$, (5.10) and Lemma 5.2 we can derive the inequality (5.1).

5.2 The 2nd order perturbation

As a preparation for the numerical implementation of the 2nd order perturbation $a_0 + a_1 U + a_2 U^2$, we rewrite the formula (2.11) for $a_n$ in a more suitable form for practical computation.

By decomposing the determinant of the covariance matrix into the sums over permutations, $G_n$ defined in (2.12) can be written as follows.

$$G_n = \frac{1}{n!} \sum_{\pi, \tau \in S_n} \text{sgn}(\pi) \text{sgn}(\tau) g_n(\pi, \tau), \quad (5.11)$$
with

\[ g_n(\pi, \tau) := \prod_{j=1}^{\pi} \left( -\sum_{x_1, x_2, y_1, y_2 \in \Gamma} \int_0^\beta dx_j (\delta_{x_i, x_i' \delta_{y_i, y_i' \delta_{x_i', y_i'}} + \lambda_{x_i, x_i', y_i', y_i'} + \lambda_{y_i, y_i', x_i, x_i'}) \cdot C(x_1^j | x_j, y_1^{\pi(j)}) C(x_2^j | x_j, y_2^{\tau(j)}) \right) \cdot \right) \]

The formula (2.11) for \( a_0, a_1, a_2 \) is

\[ a_0 = -\frac{1}{\beta} \frac{\partial}{\partial \lambda_{\tilde{X}_1}} G_1 \bigg|_{\lambda_{\tilde{X}_1} = 0 \ \forall \ X \in \Gamma^4}, \quad a_1 = -\frac{1}{\beta} \frac{\partial}{\partial \lambda_{\tilde{X}_1}} \left( G_2 - \frac{1}{2} G_1^2 \right) \bigg|_{\lambda_{\tilde{X}_1} = 0 \ \forall \ X \in \Gamma^4}, \quad a_2 = -\frac{1}{\beta} \frac{\partial}{\partial \lambda_{\tilde{X}_1}} \left( G_3 - G_1 G_2 + \frac{1}{3} G_1^3 \right) \bigg|_{\lambda_{\tilde{X}_1} = 0 \ \forall \ X \in \Gamma^4}. \] (5.12)

By substituting (5.11) into (5.12) and canceling we can simplify the expressions for \( a_1, a_2 \) as follows.

\[ a_1 = -\frac{1}{\beta} \frac{\partial}{\partial \lambda_{\tilde{X}_1}} \frac{1}{2} \left( g_2 \left( \left( \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right) \right) - g_2 \left( \text{Id}_2, \left( \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right) \right) - g_2 \left( \left( \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right), \text{Id}_2 \right) \right) \bigg|_{\lambda_{\tilde{X}_1} = 0 \ \forall \ X \in \Gamma^4}, \]

\[ a_2 = -\frac{1}{\beta} \frac{\partial}{\partial \lambda_{\tilde{X}_1}} \frac{1}{3} \left( g_3 \left( \text{Id}_3, \left( \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \end{array} \right) \right) + g_3 \left( \left( \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \end{array} \right), \text{Id}_3 \right) + g_3 \left( \left( \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \end{array} \right), \left( \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \end{array} \right) \right) \bigg|_{\lambda_{\tilde{X}_1} = 0 \ \forall \ X \in \Gamma^4}, \]

where

\[ \text{Id}_2 = \left( \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right), \quad \left( \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right) \in S_2, \]

\[ \text{Id}_3 = \left( \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array} \right), \quad \left( \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array} \right), \quad \left( \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array} \right), \quad \left( \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array} \right), \quad \left( \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array} \right) \in S_3. \]

In practice we need to implement the permutation-dependent terms \( \partial/\partial \lambda_{\tilde{X}_1} g_n(\pi, \tau) \big|_{\lambda_{\tilde{X}_1} = 0 \ \forall \ X \in \Gamma^4} \). Let us describe its implementation below.
Note that the multiple integral $\Pi_{j=1}^{n} \int_0^\beta dx_j f(x_1, \ldots, x_n)$ of any integrable function $f$ can be decomposed into a sum of $n!$ integrals as follows.

\[
\prod_{j=1}^{n} \int_0^\beta dx_j f(x_1, \ldots, x_n) = \prod_{j=1}^{n} \int_0^\beta dx_j \sum_{\eta \in S_n} 1_{x_{n(1)} > x_{n(2)} > \cdots > x_{n(n)}} f(x_1, \ldots, x_n)
\]

\[
= \sum_{\eta \in S_n} \prod_{j=1}^{n} \int_0^\beta dx_{\eta(j)} f(x_1, \ldots, x_n),
\]

(5.13)

where $x_{\eta(0)} := \beta$.

By substituting the expression (2.7) of the covariance matrix and decomposing the integral $\prod_{j=1}^{n} \int_0^\beta dx_j$ as in (5.13), we can expand $\partial / \partial \lambda_{E} g_n(\pi, \tau)|_{\lambda_{E}=0, \forall \chi \in \Gamma^*}$ as sums over the momentum space $\Gamma^*$ as follows. For $n \in \mathbb{N}$

\[
\frac{\partial}{\partial \lambda_{E}} g_n(\pi, \tau) \bigg|_{\lambda_{E}=0} = \left( -1 \right)^n \sum_{k_1, \ldots, k_n, p_1, \ldots, p_n \in \Gamma^*} \prod_{j=1}^{n} \delta(k_j + p_j - k_{\pi^{-1}(j)} - p_{\pi^{-1}(j)})
\]

\[
\cdot \sum_{j=1}^{n} \prod_{\eta \neq j}^{\pi^{-1}(j)} \int_0^{\tau_{\eta(j)}} dx_{\eta(j)} e^{\tau_{\eta(j)}(E_{k_{\eta(j)}} + E_{p_{\eta(j)}} - E_{k_{\pi^{-1}(j)}} - E_{p_{\pi^{-1}(j)}})}
\]

\[
\cdot \prod_{j=1}^{n} \left( \frac{1}{1 + e^{\beta E_{k_j}}} - \frac{1}{1 + e^{-\beta E_{k_j}}} \right) \left( \frac{1}{1 + e^{\beta E_{p_j}}} - \frac{1}{1 + e^{-\beta E_{p_j}}} \right),
\]

where $x_{\eta(0)} = \beta$.

Let us sketch how to implement (5.14). We prepare a function of real variables $E_1, \ldots, E_n$ returning the exact value of $\Pi_{j=1}^{n} \int_0^{\tau_{j-1}} dx_j e^{\tau_j E_j}$ with $x_0 = \beta$ beforehand. Then we iterate the system with respect to the variables $k_1, \ldots, k_n$, $p_1, \ldots, p_n \in \Gamma^*$. For fixed $k_1, \ldots, k_n, p_1, \ldots, p_n \in \Gamma^*$ we iterate with respect to the permutation $\eta \in S_n$. For each $\eta \in S_n$ we substitute the variables $E_j = E_{k_{\eta(j)}} + E_{p_{\eta(j)}} - E_{k_{\pi^{-1}(j)}} - E_{p_{\pi^{-1}(j)}} (j = 1, \ldots, n)$ into the function $\Pi_{j=1}^{n} \int_0^{\tau_{j-1}} dx_j e^{\tau_j E_j}$ and its returning value is then multiplied by the constant

\[
1_{x_{n(1)} > \cdots > x_{n(n)}} \prod_{j=1}^{n} \left( \frac{1}{1 + e^{\beta E_{k_j}}} - \frac{1}{1 + e^{-\beta E_{k_j}}} \right) \left( \frac{1}{1 + e^{\beta E_{p_j}}} - \frac{1}{1 + e^{-\beta E_{p_j}}} \right).
\]

5.3 Numerical values

Here we display our numerical results. In our computation we fix the physical parameters $t, t', \mu, \beta$ to satisfy $t = t' = \mu = 0.01, \beta = 1$. In this configuration the
Table 1: Errors between the correlation function and the 2nd order perturbation

| $|U|$ | $1.0 \times 10^{-6}$ | $5.0 \times 10^{-6}$ | $1.0 \times 10^{-5}$ | $5.0 \times 10^{-5}$ |
|---|---|---|---|---|
| Error | $1.408 \times 10^{-7}$ | $1.773 \times 10^{-5}$ | $1.433 \times 10^{-4}$ | $1.942 \times 10^{-2}$ |

Table 2: 2nd order perturbation in the case that $x_{\text{upper bound on } D}$ to 18. Again we see that each of $a$ values of $x_L$ becomes sufficiently small, the 1st and 2nd order terms do not contribute to the function and the 2nd order perturbation is estimated as 1.

$a$ becomes sufficiently small, the 1st and 2nd order terms do not contribute to the function and the 2nd order perturbation is estimated as 1.

Let us fix $U = 1.0 \times 10^{-5}$. According to Table 1, the error between the correlation function and the 2nd order perturbation is estimated as $1.433 \times 10^{-4}$. Table 2 shows values of $a_0, a_1, a_2$ and $a_0 + a_1U + a_2U^2$ in the case that $x_1 = x_2 = y_1 = y_2 = (l, l)$ ($l \in \{0, 1, \cdots, 5\}$) for various lattice size $L$ from 10 up to 18. We observe that each of $a_0, a_1, a_2$ respectively takes the same value for any $L \in \{10, \cdots, 18\}$ and $l \in \{0, \cdots, 5\}$.

Table 3 shows the values of $a_0, a_1, a_2$ and $a_0 + a_1U + a_2U^2$ in the case that $x_1 = y_1 = (0, 0), x_2 = y_2 = (l, l)$ ($l \in \{1, \cdots, 5\}$) for various lattice size $L$ from 10 to 18. Again we see that each of $a_0, a_1, a_2$ respectively takes the same value for any $L \in \{10, \cdots, 18\}$ and $l \in \{1, \cdots, 5\}$. Since we have fixed a small $U$ so that Error becomes sufficiently small, the 1st and 2nd order terms do not contribute to the sum $a_0 + a_1U + a_2U^2$ much in these numerical simulations.

We also computed $a_0, a_1, a_2$ in the case that $x_1 = x_2 = (0, 0), y_1 = y_2 = (l, l)$ for $l \in \{1, \cdots, 5\}$ for $L = 10, 11, \cdots, 18$. The result shows that $|a_0|, |a_1|, |a_2| \leq 1.5 \times 10^{-5}$ for any $l \in \{1, \cdots, 5\}$ and $L \in \{10, 11, \cdots, 18\}$. In this case the values of $|a_0|, |a_1|, |a_2|$ are much smaller than those presented in Tables 2-3. This result
\[
\begin{array}{|c|c|}
\hline
L & 10, 11, \ldots, 18 \\
\hline
a_0 & 5.050 \times 10^{-1} \\
\hline
a_1 & -2.524 \times 10^{-1} \\
\hline
a_2 & 9.402 \times 10^{-2} \\
\hline
a_0 + a_1 U + a_2 U^2 & 5.050 \times 10^{-1} \\
\hline
\end{array}
\]

Table 3: 2nd order perturbation in the case that \( x_1 = y_1 = (0, 0), x_2 = y_2 = (l, l) \) \((l \in \{1, \ldots, 5\})\)

indicates that the 4 point correlation function takes small values if \( x_1 = x_2, y_1 = y_2 \) and \(|x_1 - y_1|\) is large and agrees with the decaying property of the 4 point correlation function for the 2 dimensional Hubbard model proved in [12].

Appendices

In this section we review the definitions of the Fermionic Fock space and the annihilation, creation operators, prove Proposition 2.4 and show that the covariance matrix \( C_h \) has a non-zero determinant independent of the parameter \( h \).

We write a matrix \( M \) indexed by finite sets \( S, S' \) with \( |S| = |S'| = n \) as \( M = (M(s, s'))_{s \in S, s' \in S'} \). In this notation let us think that each element of \( S \) and \( S' \) has already been given a number from 1 to \( n \) and \( M \) is defined by \( M = (M(s_j, s_k))_{1 \leq j, k \leq n} \) even if the numbering of \( S \) and \( S' \) is not specified in the context. The main results Proposition 2.4 and Proposition C.7 concluded after some argument involving such matrices in this Appendices are independent of how to number the index sets. For any finite set \( B \) let \( L^2(B; \mathbb{C}) \) denote the complex linear space consisting of complex-valued functions on \( B \), even when we do not introduce an inner product in \( L^2(B; \mathbb{C}) \).

A The Fermionic Fock space

The first part of Appendices reviews the definitions of the Fermionic Fock space on the lattice \( \Gamma \times \{\uparrow, \downarrow\} \) and the annihilation, creation operators \( \psi_{x\sigma}, \psi^*_{x\sigma} \).

For any \( n \in \mathbb{N} \) we consider the linear space \( L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C}) \) as a Hilbert space equipped with the inner product \( \langle \cdot, \cdot \rangle_{L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C})} \) defined by

\[
\langle \phi_1, \phi_2 \rangle_{L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C})} := \sum_{x_1, \ldots, x_n \in \Gamma, \sigma_1, \ldots, \sigma_n \in \{\uparrow, \downarrow\}} \phi_1(x_1 \sigma_1, \ldots, x_n \sigma_n)\phi_2(x_1 \sigma_1, \ldots, x_n \sigma_n).
\]

By convention we set \( L^2((\Gamma \times \{\uparrow, \downarrow\})^0; \mathbb{C}) := \mathbb{C} \).
For $n \in \mathbb{N}$ the anti-symmetrization operator $A_n : L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C}) \rightarrow L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C})$ is defined by

$$(A_n \phi)(x_1 \sigma_1, \ldots, x_n \sigma_n) := \frac{1}{n!} \sum_{\pi \in S_n} \text{sgn}(\pi) \phi(x_{\pi(1)} \sigma_{\pi(1)}, \ldots, x_{\pi(n)} \sigma_{\pi(n)}).$$

The operator $A_0$ is defined as the identity map on $\mathbb{C}$, i.e., $A_0 z := z$ for all $z \in \mathbb{C}$.

The subspace $A_n(L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C}))$ of $L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C})$ is called as the Fermionic $n$–particle space and is a Hilbert space equipped with the inner product of $L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C})$. Note that by anti-symmetry for any $n > 2L^d$, $A_n(L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C})) = \{0\}$.

The Fermionic Fock space $F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C}))$ is defined as the direct sum of $A_n(L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C}))$ ($n = 0, \ldots, 2L^d$) as follows.

$$F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})) := \bigoplus_{n=0}^{2L^d} A_n(L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C})).$$

The space $F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C}))$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{F_f}$ defined by

$$\langle \phi_1, \phi_2 \rangle_{F_f} := \sum_{n=0}^{2L^d} \langle \phi_1, \phi_2 \rangle_{L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C})},$$

for any vectors $\phi_1 = (\phi_{1,0}, \phi_{1,1}, \ldots, \phi_{1,2L^d})$, $\phi_2 = (\phi_{2,0}, \phi_{2,1}, \ldots, \phi_{2,2L^d}) \in F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C}))$.

Define a set of functions $\{\phi_{k\sigma}(x)\}_{(k, \sigma) \in \Gamma^* \times \{\uparrow, \downarrow\}} \subset L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})$ by $\phi_{k\sigma}(x) := \delta_{x, \tau} e^{-i(k \cdot x) / L^d / 2}$. We then define a function $\phi_{k_1\sigma_1} \cdots \phi_{k_n\sigma_n} \in L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C})$ by

$$\phi_{k_1\sigma_1} \cdots \phi_{k_n\sigma_n}(x_1 \tau_1, \cdots, x_n \tau_n) := \phi_{k_1\sigma_1}(x_1 \tau_1) \cdots \phi_{k_n\sigma_n}(x_n \tau_n).$$

An orthonormal basis of $F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C}))$ is given by $\bigcup_{n=0}^{2L^d} B_n$, where

$$B_0 := \{1\}(\subset \mathbb{C}),$$

$$B_n := \left\{ \sqrt{n!} A_n \left( \prod_{(k, \sigma) \in \Gamma^* \times \{\uparrow, \downarrow\}} \phi_{k\sigma}^{n_{k\sigma}} \right) \mid n_{k\sigma} \in \{0, 1\}, \sum_{k \in \Gamma^*} n_{k\sigma} = n \right\}$$

(A.1)

for $n \in \{1, 2, \ldots, 2L^d\}$. Thus, we see that $\dim F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})) = \sum_{n=0}^{2L^d} \sharp B_n = 2^{2L^d}$.

The annihilation operator $\psi_{x\sigma} : F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})) \rightarrow F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C}))$ ($x \in \Gamma, \sigma \in \{\uparrow, \downarrow\}$) is defined in the following steps. For any $n \in \mathbb{N} \cup \{0\}$ and any $\phi \in A_{n+1}(L^2((\Gamma \times \{\uparrow, \downarrow\})^{n+1}; \mathbb{C}))$, $\psi_{x\sigma} \phi \in A_n(L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C}))$ is defined by

$$(\psi_{x\sigma} \phi)(x_1 \sigma_1, \ldots, x_n \sigma_n) := \sqrt{n+1} \phi(x_1 \sigma_1, \ldots, x_n \sigma_n).$$

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For any \( z \in A_0(L^2((\Gamma \times \{\uparrow, \downarrow\})^0; \mathbb{C}) ) \), \( \psi_{x\sigma} z := 0 \). The domain of the operator \( \psi_{x\sigma} \) is then extended to the whole space \( F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})) \) by linearity.

The creation operator \( \psi_{x\sigma}^* : F_f(L^2((\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})) \to F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})) \) is the adjoint operator of \( \psi_{x\sigma} \) and characterized as follows. For any \( n \in \mathbb{N} \cup \{0\} \) and any \( \phi \in A_n(L^2((\Gamma \times \{\uparrow, \downarrow\})^n; \mathbb{C})) \), \( \psi_{x\sigma}^* \phi \in A_{n+1}(L^2((\Gamma \times \{\uparrow, \downarrow\})^{n+1}; \mathbb{C})) \) and

\[
\psi_{x\sigma}^* \phi(x_1 \sigma_1, \ldots, x_{n+1} \sigma_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{l=1}^{n+1} (-1)^{l-1} \delta_{x_l \sigma_l} \delta_{\sigma_l \sigma_l} \phi(x_1 \sigma_1, \ldots, \widehat{x_l \sigma_l}, \ldots, x_{n+1} \sigma_{n+1}),
\]

where the notation \( 'x_l \sigma_l' \) stands for the omission of the variable \( x_l \sigma_l \).

For any operators \( A, B \) let \( \{A, B\} = AB + BA \). The operators \( \psi_{x\sigma}, \psi_{x\sigma}^* \) \( (x \in \Gamma, \sigma \in \{\uparrow, \downarrow\}) \) satisfy the following canonical anti-commutation relations. For all \( x, y \in \Gamma, \sigma, \tau \in \{\uparrow, \downarrow\} \),

\[
\{\psi_{x\sigma}, \psi_{y\tau}^* \} = \{\psi_{x\sigma}^*, \psi_{y\tau} \} = 0, \quad \{\psi_{x\sigma}, \psi_{y\tau} \} = \delta_{x,y} \delta_{\sigma,\tau}.
\]

See, e.g., [3] for more detailed definitions of the Fermionic Fock space and the operators on it.

**B The temperature-ordered perturbation series**

In this section we present the derivation of the perturbation series (2.6). Propositions claimed here are standard tools in many-body theory (see, e.g., [10] Chapter 2, Chapter 3). This part of Appendices is devoted to show them in a mathematical context.

Let us fix notations used in the analysis below. Let \( H_0, V_\lambda \) be the operators defined in (2.1)-(2.2) and (2.3), respectively. In our argument in this section, however, we do not use the relation (2.4) or the condition (3.16) imposed on the parameter \( \{U_X\}_{X \in \Gamma^+} \). One can consider more general \( V_\lambda \) of the form (2.3) parameterized by any complex multi-variable \( \{U_X\}_{X \in \Gamma^+} \) in this section.

Define the operators \( V_\lambda(s), \psi_{x\sigma}(s), \psi_{x\sigma}^*(s) \) \( (s \in \mathbb{R}, x \in \Gamma, \sigma \in \{\uparrow, \downarrow\}) \) by \( V_\lambda(s) := e^{sH_0}V_\lambda e^{-sH_0} \), \( \psi_{x\sigma}(s) := e^{sH_0}\psi_{x\sigma} e^{-sH_0} \), \( \psi_{x\sigma}^*(s) := e^{sH_0}\psi_{x\sigma}^* e^{-sH_0} \).

For \( a \in \{0, 1\} \) the operator \( \psi_{x\sigma a}(s) \) is defined by

\[
\psi_{x\sigma a}(s) := \begin{cases} 
\psi_{x\sigma}^*(s) & \text{if } a = 1, \\
\psi_{x\sigma}(s) & \text{if } a = 0.
\end{cases}
\]

Next we define the ordering operators \( T_1, T_2 \).

**Definition B.1.** Consider linear operators \( M(s_1), \ldots, M(s_n) : F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})) \to F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})) \) parameterized by \( s_1, \ldots, s_n \in \mathbb{R} \). Assume that \( s_j \neq s_k \) for any \( j, k \in \{1, \ldots, n\} \) with \( j \neq k \). The operator \( T_1(M(s_1) \cdots M(s_n)) \) is defined by

\[
T_1(M(s_1) \cdots M(s_n)) := M(s_{\pi(1)}) \cdots M(s_{\pi(n)}),
\]
where \( \pi \in S_n \) is uniquely determined by the condition that
\[
s_{\pi(1)} > s_{\pi(2)} > \cdots > s_{\pi(n)}.
\]

Let us define a relation \('\sim\'\) in the set \( \{ \psi_{x_1}(s) \}_{x_1, \sigma, a, s} \in \Gamma \times \{1, \ldots, n\} \times \{0, 1\} \times \mathbb{R} \) as follows.
\[
\psi_{x_1}(s_1) \sim \psi_{y_1}(s_2) \text{ if } a_1 = a_2 \text{ and } s_1 = s_2.
\]
We see that \('\sim\') is an equivalence relation in \( \{ \psi_{x_1}(s) \}_{x_1, \sigma, a, s} \in \Gamma \times \{1, \ldots, n\} \times \{0, 1\} \times \mathbb{R} \). Let \( [\psi_{x_1}(s)] \) denote the equivalence class represented by an element \( \psi_{x_1}(s) \). We define relations \('\succ\'\) and \('\succeq\'\) in the quotient set \( \{ \psi_{x_1}(s) \}_{x_1, \sigma, a, s} \in \Gamma \times \{1, \ldots, n\} \times \{0, 1\} \times \mathbb{R}/\sim \) as follows.
\[
[\psi_{x_1}(s_1)] \succ [\psi_{y_1}(s_2)] \text{ if } s_1 > s_2, \text{ or } s_1 = s_2 \text{ and } a_1 > a_2,
\]
\[
[\psi_{x_1}(s_1)] \succeq [\psi_{y_1}(s_2)] \text{ if } [\psi_{x_1}(s_1)] \succ [\psi_{y_1}(s_2)] \text{ or } [\psi_{x_1}(s_1)] = [\psi_{y_1}(s_2)].
\]
The set \( \{ \psi_{x_1}(s) \}_{x_1, \sigma, a, s} \in \Gamma \times \{1, \ldots, n\} \times \{0, 1\} \times \mathbb{R}/\sim \) is totally ordered under the relation \('\succ\'\) and the relation \('\succeq\'\) is a strict order in this quotient set.

**Definition B.2.** For any \( \psi_{x_1, \sigma, a, s}(s_j) \) \( (j = 1, \ldots, n) \) the operator
\[
T_2(\psi_{x_1, \sigma, a, s}(s_1) \cdots \psi_{x_n, \sigma, a, s}(s_n))
\]
defined by
\[
T_2(\psi_{x_1, \sigma, a, s}(s_1) \cdots \psi_{x_n, \sigma, a, s}(s_n)) := \text{sgn}(\pi)\psi_{x_{\pi(1)} \sigma_{\pi(1)} a_{\pi(1)} n_{\pi(1)}}(\pi(1)) \cdots \psi_{x_{\pi(n)} \sigma_{\pi(n)} a_{\pi(n)} n_{\pi(n)}}(\pi(n)),
\]
where \( \pi \in S_n \) is uniquely determined by the conditions that
\[
[\psi_{x_{\pi(1)} \sigma_{\pi(1)} a_{\pi(1)} n_{\pi(1)}}(\pi(1))] \succeq [\psi_{x_{\pi(2)} \sigma_{\pi(2)} a_{\pi(2)} n_{\pi(2)}}(\pi(2))] \succeq \cdots \succeq [\psi_{x_{\pi(n)} \sigma_{\pi(n)} a_{\pi(n)} n_{\pi(n)}}(\pi(n))],
\]
and if there exist \( l_1, l_2 \in \{1, \ldots, n\} \) with \( l_1 < l_2 \) such that
\[
[\psi_{x_1, \sigma_1, a_1}(s_{l_1})] = [\psi_{x_2, \sigma_2, a_2}(s_{l_2})], \neq [\psi_{x_1, \sigma_1, a_1}(s_{j})], \forall j \in \{l_1 + 1, l_2 - 1\}
\]
and \( \pi(m) = l_1 \) with \( m \in \{1, \ldots, n\} \), then \( \pi(m + 1) = l_2 \).

Using the ordering operator \( T_1 \) we have the following expansion.

**Lemma B.3.** For any \( t_1, t_2 \in \mathbb{R} \) with \( t_1 < t_2 \),
\[
e^{-((t_2-t_1)(H_0+V_\lambda))} = e^{-(t_2-t_1)H_0} + e^{-t_2H_0} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{[t_1, t_2]^n} ds_1 \cdots ds_n T_1(V_\lambda(s_1) \cdots V_\lambda(s_n)) e^{t_1H_0}.
\]

**Remark B.4.** Though the operator \( T_1(V_\lambda(s_1) \cdots V_\lambda(s_n)) \) is defined only for \( (s_1, \ldots, s_n) \) with \( s_j \neq s_k \) \( (j \neq k) \), we can consider \( T_1(V_\lambda(s_1) \cdots V_\lambda(s_n)) \) as a Bochner integrable function over \( [t_1, t_2]^n \) since the Lebesgue measure of the set \( \{(s_1, \ldots, s_n) \in [t_1, t_2]^n \mid \exists j, k \in \{1, \ldots, n\} \text{ s.t. } j \neq k \text{ and } s_j = s_k \} \) is zero.
Proof of Lemma \([3.3]\). Since the operator-valued function \(\xi \mapsto e^{-(t_2-t_1)(H_0+\xi V_\lambda)}\) is analytic, we have

\[
e^{-(t_2-t_1)(H_0+\xi V_\lambda)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{d}{d\xi} \right)^n e^{-(t_2-t_1)(H_0+\xi V_\lambda)} \bigg|_{\xi=0}. \tag{B.1}
\]

It is sufficient to show that for all \(n \in \mathbb{N}\) and all \(\xi \in \mathbb{R}\)

\[
\left( \frac{d}{d\xi} \right)^n e^{-(t_2-t_1)(H_0+\xi V_\lambda)} = (-1)^n e^{-t_2(H_0+\xi V_\lambda)} \int_{[t_1, t_2]^n} ds_1 \cdots ds_n T_1(V_{\lambda,\xi}(s_1) \cdots V_{\lambda,\xi}(s_n)) e^{t_1(H_0+\xi V_\lambda)}, \tag{B.2}
\]

where \(V_{\lambda,\xi}(s) := e^{s(H_0+\xi V_\lambda)} V_\lambda e^{-s(H_0+\xi V_\lambda)}\). In fact, substituting \([3.2]\) for \(\xi = 0\) into \([3.1]\) gives the result. We show \([3.2]\) by induction on \(n\).

By Lemma \(2.3\) we have

\[
\frac{d}{d\xi} e^{-(t_2-t_1)(H_0+\xi V_\lambda)} = \int_0^{t_1} ds e^{-(1-s)(t_2-t_1)(H_0+\xi V_\lambda)}(t_1-t_2)V_\lambda e^{-(t_2-t_1)(H_0+\xi V_\lambda)} = -e^{-t_2(H_0+\xi V_\lambda)} \int_{t_1}^{t_2} ds V_{\lambda,\xi}(s)e^{t_1(H_0+\xi V_\lambda)},
\]

which is \([3.2]\) for \(n = 1\).

Let us assume that \([3.2]\) is true for \(n-1\) \((n \geq 2)\).

\[
\frac{d}{d\xi} e^{-(t_2-t_1)(H_0+\xi V_\lambda)} = (-1)^{n-1}(n-1)! e^{-t_2(H_0+\xi V_\lambda)} \int_{[t_1, t_2]^{n-1}} ds_1 ds_2 \cdots ds_{n-1} \cdot 1_{s_1 > s_2 > \cdots > s_{n-1}} V_{\lambda,\xi}(s_1) V_{\lambda,\xi}(s_2) \cdots V_{\lambda,\xi}(s_{n-1}) e^{t_1(H_0+\xi V_\lambda)} = (-1)^{n-1}(n-1)! \int_{t_1}^{t_2} ds_1 \int_{t_1}^{s_1} ds_2 \cdots \int_{t_1}^{s_{n-2}} ds_{n-1} e^{-(t_2-s_1)(H_0+\xi V_\lambda)} V_\lambda \cdot e^{-(s_1-s_2)(H_0+\xi V_\lambda)} V_\lambda \cdots e^{-(s_{n-2}-s_{n-1})(H_0+\xi V_\lambda)} V_\lambda e^{-(s_{n-1}-t_1)(H_0+\xi V_\lambda)}.
\]

By writing \(t_2 = s_0, t_1 = s_n\) and using Lemma \(2.3\) again we observe that

\[
\left( \frac{d}{d\xi} \right)^n e^{-(t_2-t_1)(H_0+\xi V_\lambda)} = (-1)^{n-1}(n-1)! \int_{s_n}^{s_0} ds_1 \cdots \int_{s_n}^{s_{n-2}} ds_{n-1} \sum_{j=0}^{n-1} e^{-(s_0-s_1)(H_0+\xi V_\lambda)} V_\lambda \cdots e^{-(s_0-s_{j+1})(H_0+\xi V_\lambda)} \frac{d}{d\xi} \left( e^{-(s_j-s_{j+1})(H_0+\xi V_\lambda)} \right) V_\lambda \cdots e^{-(s_{n-2}-s_{n-1})(H_0+\xi V_\lambda)} V_\lambda e^{-(s_{n-1}-s_n)(H_0+\xi V_\lambda)}
\]

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\[\begin{align*}
\mathcal{F}(\mathcal{A}) &= \mathcal{F}(\mathcal{B}) \\
&= (\mathcal{F}(\mathcal{C}) \circ \mathcal{D})(\mathcal{E}).
\end{align*}\]
Moreover, we see that for any linear operator $A : F_j(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C})) \to F_j(L^2(\Gamma \times \{\uparrow, \downarrow\}; \mathbb{C}))$, let $(A)_0$ denote $\text{Tr}(e^{-\beta H_0}A)/\text{Tr} e^{-\beta H_0}$. For a set of the operators $\{\psi_{x, \sigma, a_j(s_j)}\}_{j=1}^n$, let $\psi_{x_1, \sigma_1, a_1(s_1)} \cdots \psi_{x_j, \sigma_j, a_j(s_j)} \cdots \psi_{x_n, \sigma_n, a_n(s_n)}$ denote the product obtained by eliminating $\psi_{x_j, \sigma_j, a_j(s_j)}$ from the product $\psi_{x_1, \sigma_1, a_1(s_1)} \cdots \psi_{x_n, \sigma_n, a_n(s_n)}$.

**Lemma B.6.** If $n \in \mathbb{N}$ is odd, for any $(x_j, \sigma_j, a_j, s_j) \in \Gamma \times \{\uparrow, \downarrow\} \times \{0,1\} \times \mathbb{R}$ ($j = 1, \cdots, n$)

$$\langle \psi_{x_1, \sigma_1, a_1(s_1), \cdots, x_n, \sigma_n, a_n(s_n)} \rangle_0 = 0.$$ 

If $n \in \mathbb{N}$ is even, for any $(x_j, \sigma_j, a_j, s_j) \in \Gamma \times \{\uparrow, \downarrow\} \times \{0,1\} \times \mathbb{R}$ ($j = 1, \cdots, n$)

$$\langle \psi_{x_1, \sigma_1, a_1(s_1), \cdots, x_n, \sigma_n, a_n(s_n)} \rangle_0 = \sum_{j=2}^{n} (-1)^j \langle \psi_{x_1, \sigma_1, a_1(s_1), x_j, \sigma_j, a_j(s_j)} \rangle_0 \langle \psi_{x_2, \sigma_2, a_2(s_2), \cdots, x_j, \sigma_j, a_j(s_j)} \cdots \psi_{x_n, \sigma_n, a_n(s_n)} \rangle_0.$$  

(B.5)

Moreover,

$$\langle \psi_{x_1, \sigma_1, a_1(s_1), x_2, \sigma_2, a_2(s_2)} \rangle_0 = \sum_{y \in \Gamma} \sum_{\tau \in \{\uparrow, \downarrow\}} (I + e^{-\beta F(a_1)})^{-1}(x_1, \sigma_1, y, \tau) \langle \psi_{y, \tau, a_1(s_1), x_2, \sigma_2, a_2(s_2)} \rangle.$$  

(B.6)
Proof. By using the orthonormal basis $\bigcup_{m=0}^{2L^d} B_m$ defined in (A.1), we can write
\[ \text{Tr}(e^{-\beta H_0} \psi_{x_1,\sigma_1 a_1}(s_1) \cdots \psi_{x_n,\sigma_n a_n}(s_n)) = \sum_{m=0}^{2L^d} \sum_{\phi \in B_m} \langle \phi, e^{-\beta H_0} \psi_{x_1,\sigma_1 a_1}(s_1) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \phi \rangle_{F_j}. \] (B.7)

Since for all $s \in \mathbb{R}$ and $m \in \{0, 1, \ldots, 2L^d\}$
\[ e^{sH_0} \left( A_m(L^2((\Gamma \times \{1,1\})^m; \mathbb{C})) \right) \subset A_m(L^2((\Gamma \times \{1,1\})^m; \mathbb{C})), \]
we see that if $n$ is odd, for any $m \in \{0, 1, \ldots, 2L^d\}$ and $\phi \in B_m$
\[ e^{-\beta H_0} \psi_{x_1,\sigma_1 a_1}(s_1) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \phi = 0, \]
or
\[ e^{-\beta H_0} \psi_{x_1,\sigma_1 a_1}(s_1) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \phi \in A_l(L^2((\Gamma \times \{1,1\})^l; \mathbb{C})) \]
with $l \neq m$, which implies that
\[ \langle \phi, e^{-\beta H_0} \psi_{x_1,\sigma_1 a_1}(s_1) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \phi \rangle_{F_j} = 0. \] (B.8)

The first statement follows from (B.7) and (B.8).

Let us assume that $n$ is even. We see that
\[ \psi_{x_1,\sigma_1 a_1}(s_1) \psi_{x_2,\sigma_2 a_2}(s_2) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \]
\[ = \{ \psi_{x_1,\sigma_1 a_1}(s_1), \psi_{x_2,\sigma_2 a_2}(s_2) \} \psi_{x_3,\sigma_3 a_3}(s_3) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \]
\[ - \psi_{x_2,\sigma_2 a_2}(s_2) \psi_{x_1,\sigma_1 a_1}(s_1) \psi_{x_3,\sigma_3 a_3}(s_3) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \]
\[ = \{ \psi_{x_1,\sigma_1 a_1}(s_1), \psi_{x_2,\sigma_2 a_2}(s_2) \} \psi_{x_3,\sigma_3 a_3}(s_3) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \]
\[ - \psi_{x_2,\sigma_2 a_2}(s_2) \{ \psi_{x_1,\sigma_1 a_1}(s_1), \psi_{x_3,\sigma_3 a_3}(s_3) \} \psi_{x_4,\sigma_4 a_4}(s_4) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \]
\[ + \psi_{x_2,\sigma_2 a_2}(s_2) \psi_{x_3,\sigma_3 a_3}(s_3) \psi_{x_1,\sigma_1 a_1}(s_1) \psi_{x_4,\sigma_4 a_4}(s_4) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \]
\[ = \sum_{j=2}^{n} \{ \psi_{x_1,\sigma_1 a_1}(s_1), \psi_{x_j,\sigma_j a_j}(s_j) \} (-1)^j \psi_{x_2,\sigma_2 a_2}(s_2) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \]
\[ - (-1)^n \psi_{x_2,\sigma_2 a_2}(s_2) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \psi_{x_1,\sigma_1 a_1}(s_1), \]
which yields
\[ \langle \psi_{x_1,\sigma_1 a_1}(s_1) \psi_{x_2,\sigma_2 a_2}(s_2) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \rangle_0 \]
\[ = \sum_{j=2}^{n} \{ \psi_{x_1,\sigma_1 a_1}(s_1), \psi_{x_j,\sigma_j a_j}(s_j) \} (-1)^j \langle \psi_{x_2,\sigma_2 a_2}(s_2) \cdots \psi_{x_n,\sigma_n a_n}(s_n) \psi_{x_1,\sigma_1 a_1}(s_1) \rangle_0. \] (B.9)
On the other hand, by Lemma [B.5],
\[
\psi_{x,s} (s + \beta) = \sum_{y \in \Gamma} \sum_{\tau \in \{1, 1\}} e^{-\beta F(a)} (x, y^\tau) \psi_{y, \tau} (s).
\] (B.10)

By using [B.10] and the equality that Tr(AB) = Tr(BA) for any operators A, B, we observe that
\[
\langle \psi_{x, \sigma_1} (s_1) \cdots \psi_{x, \sigma_n} (s_n) \rangle_{0} = \langle \psi_{x, \sigma_1} (s_1 + \beta) \psi_{x, \sigma_2} (s_2) \cdots \psi_{x, \sigma_n} (s_n) \rangle_{0}
\]
\[
= \sum_{y \in \Gamma} \sum_{\tau \in \{1, 1\}} e^{-\beta F(a)} (x_1, \psi_{x_1, \sigma_1} (s_1)) \psi_{y, \tau} (s_1) \psi_{y, \tau} (s_2) \cdots \psi_{y, \tau} (s_n) \rangle_{0}.
\] (B.11)

By substituting [B.11] into [B.9] we obtain
\[
\sum_{y \in \Gamma} \sum_{\tau \in \{1, 1\}} (\delta_{x, y} \delta_{\gamma, \tau} + e^{-\beta F(a)} (x_1, \psi_{x_1, \sigma_1} (s_1)))
\]
\[
\cdot \langle \psi_{y, \gamma} (s_1) \psi_{y, \gamma} (s_2) \cdots \psi_{y, \gamma} (s_n) \rangle_{0}
\]
\[
= \sum_{j=2}^n \{ \psi_{x_1, \sigma_1} (s_1), \psi_{x, \sigma_j} (s_j) \} (-1)^j \langle \psi_{x, \sigma_1} (s_1) \psi_{x, \sigma_2} (s_2) \cdots \psi_{x, \sigma_j} (s_j) \psi_{x, \sigma_n} (s_n) \rangle_{0}.
\] (B.12)

Let us define a unitary matrix \( M = (M(k, \tau, x, \sigma))_{(k, \tau) \in \Gamma^* \times \{1, 1\}, (x, \sigma) \in \Gamma \times \{1, 1\}} \) by
\[
M(k, \tau, x, \sigma) := \frac{\delta_{x, \tau} \delta_{\gamma, \tau}}{L^{\frac{d}{2}}} e^{-i (k, x)}.
\]

Then we have for all \((k, \tau), (\hat{k}, \hat{\tau}) \in \Gamma^* \times \{1, 1\}\)
\[
\phi^{MF^*} (k, \tau, \hat{k}, \hat{\tau}) = \phi^{MF^*} (k, \tau, \hat{k}, \hat{\tau}) = \delta_{k, k} \delta_{\tau, \hat{\tau}} E_k.
\] (B.13)

where \( E_k \) is defined in (2.1). The equality (B.13) implies that
\[
\det (I + e^{-\beta F}) = \det (I + e^{-\beta MF^*}) \neq 0, \quad \det (I + e^{\beta F^*}) = \det (I + e^{\beta MF^*}) \neq 0.
\]

Thus, for \( a = 0, 1 \) the matrix \( I + e^{-\beta F(a)} \) is invertible. The equality (B.12) leads to
\[
\langle \psi_{x_1, \sigma_1} (s_1) \psi_{x_2, \sigma_2} (s_2) \cdots \psi_{x, \sigma_n} (s_n) \rangle_{0}
\]
\[
= \sum_{j=2}^n \sum_{y \in \Gamma} \sum_{\tau \in \{1, 1\}} \langle I + e^{-\beta F(a)} \rangle (-1)^j \langle \psi_{x_1, \sigma_1} (s_1), \psi_{x_2, \sigma_2} (s_2) \cdots \psi_{x, \sigma_j} (s_j) \psi_{y, \tau} (s_1) \rangle_{0}.
\] (B.14)

The equality (B.6) is (B.14) for \( n = 2 \). Then, by substituting (B.6) into (B.14) we obtain (B.5).
From now we show some lemmas involving the ordering operator $T_2$. To simplify notations, let $\psi_j$ denote $\psi_{x_j,\sigma_j,\alpha_j(s_j)}$ for fixed variables $(x_j, \sigma_j, \alpha_j, s_j) \in \Gamma \times \{\uparrow, \downarrow\} \times \{0,1\} \times \mathbb{R}$ ($j = 1, \ldots, n$).

**Lemma B.7.** For any $\pi \in S_n$,

$$T_2(\psi_1 \psi_2 \cdots \psi_n) = \text{sgn}(\pi)T_2(\psi_{\pi(1)} \psi_{\pi(2)} \cdots \psi_{\pi(n)}).$$  \hspace{1cm} (B.15)

**Proof.** It is sufficient to show (B.15) for any transposition $\pi$ as any permutation is a product of transpositions. Let us assume that $\pi = (j, k)$, $1 \leq j < k \leq n$. Let $\tau, \eta \in S_n$ be the unique permutations associated with the definitions of $T_2(\psi_1 \cdots \psi_n)$ and $T_2(\psi_{\pi(1)} \cdots \psi_{\pi(n)})$, respectively.

$$T_2(\psi_1 \cdots \psi_n) = \text{sgn}(\tau)\psi_{\tau(1)} \cdots \psi_{\tau(n)},$$  \hspace{1cm} (B.16)

$$T_2(\psi_{\pi(1)} \cdots \psi_{\pi(n)}) = \text{sgn}(\eta)\psi_{\pi(\eta(1))} \cdots \psi_{\pi(\eta(n))}. \hspace{1cm} (B.17)

First consider the case that $[\psi_j] \neq [\psi_k]$. Let $A, B \subset \{j+1, \ldots, n-1\}$ satisfy that $[\psi_j] = [\psi_\alpha]$ for any $\alpha \in A$, $[\psi_k] = [\psi_\gamma]$ for any $\gamma \in B$ and $[\psi_j]$, $[\psi_k] \neq [\psi_p]$ for any $p \in \{j+1, \ldots, n-1\} \backslash A \cup B$.

If $A, B \neq \emptyset$, we can write $A = \{\alpha_1, \ldots, \alpha_l\}$, $B = \{\gamma_1, \ldots, \gamma_m\}$ with $j+1 \leq \alpha_1 < \cdots < \alpha_l \leq k-1$, $j+1 \leq \gamma_1 < \cdots < \gamma_m \leq k-1$. By the definition of $T_2$ the product $\psi_{\pi(\eta(1))} \cdots \psi_{\pi(\eta(n))}$ is obtained by replacing $\psi_j \psi_{\alpha_1} \cdots \psi_{\alpha_l}$ and $\psi_k \psi_{\gamma_1} \cdots \psi_{\gamma_m}$ respectively in the product $\psi_{\tau(1)} \cdots \psi_{\tau(n)}$. Thus, if we define cycles $\zeta_1, \zeta_2 \in S_n$ by

$$\zeta_1 = \begin{pmatrix} j & j & \cdots & j \\ \alpha_1 & \alpha_2 & \cdots & \alpha_l \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_m \\ k & k & \cdots & k \end{pmatrix},$$

the permutation $\eta$ is written as

$$\eta = \pi^{-1}\zeta_1\zeta_2\tau. \hspace{1cm} (B.18)$$

On the other hand, Lemma B.5 (ii) ensures that

$$\psi_{\alpha_1} \cdots \psi_{\alpha_l} \psi_j = (-1)^l \psi_j \psi_{\alpha_1} \cdots \psi_{\alpha_l}, \quad \psi_k \psi_{\gamma_1} \cdots \psi_{\gamma_m} = (-1)^m \psi_{\gamma_1} \cdots \psi_{\gamma_m} \psi_k. \hspace{1cm} (B.19)$$

By (B.16)-(B.19) we see that

$$T_2(\psi_{\pi(1)} \cdots \psi_{\pi(n)}) = \text{sgn}(\pi^{-1}\zeta_1\zeta_2\tau)\psi_{\zeta_1(\zeta_2(\tau(1)))} \cdots \psi_{\zeta_1(\zeta_2(\tau(n)))}$$

$$= (-1)^{l+m} \text{sgn}(\tau)(-1)^{l+m} \psi_{\tau(1)} \cdots \psi_{\tau(n)} \hspace{1cm} (B.20)$$

$$= -T_2(\psi_1 \cdots \psi_n).$$

If $A = \emptyset$ or $B = \emptyset$, by setting $\zeta_1 = \text{Id}$ and $l = 0$ or $\zeta_2 = \text{Id}$ and $m = 0$, respectively, we see that the equalities (B.18) and (B.20) hold true.

Next consider the case that $[\psi_j] = [\psi_k]$. Let $\hat{A} \subset \{j+1, \ldots, n-1\}$ be such that $[\psi_j] = [\psi_\eta]$ for any $q \in \hat{A}$ and $[\psi_j] \neq [\psi_q]$ for any $q \in \{j+1, \ldots, n-1\} \backslash \hat{A}$.

If $\hat{A} \neq \emptyset$, we write $\hat{A}$ as $\hat{A} = \{q_1, \ldots, q_r\}$ with $j+1 \leq q_1 < \cdots < q_r \leq k-1$. By the definition of $T_2$ the product $\psi_{\pi(\eta(1))} \cdots \psi_{\pi(\eta(n))}$ is obtained by replacing
\[ \psi_j \psi_{q_1} \cdots \psi_{q_n} \psi_k \text{ by } \psi_k \psi_{q_1} \cdots \psi_{q_n} \psi_j \text{ in the product } \psi_{\tau(1)} \cdots \psi_{\tau(n)}. \] Thus, the permutation \( \eta \) satisfies the equality
\[ \eta = \tau. \] (B.21)

By Lemma B.5 (ii) the following equality holds.
\[ \psi_k \psi_{q_1} \cdots \psi_{q_n} \psi_j = -\psi_j \psi_{q_1} \cdots \psi_{q_n} \psi_k. \] (B.22)

By combining (B.16)-(B.17) with (B.21)-(B.22) we have
\[ T_2(\psi_{\tau(1)} \cdots \psi_{\tau(n)}) = \text{sgn}(\tau) \psi_{\tau(1)} \cdots \psi_{\tau(n)} \]
\[ = -\text{sgn}(\tau) \psi_{\tau(1)} \cdots \psi_{\tau(n)} \]
\[ = -T_2(\psi_1 \cdots \psi_n). \] (B.23)

By repeating the same argument as above without the term \( \psi_{q_1} \cdots \psi_{q_n} \) we can prove the equalities (B.23) for the case that \( \hat{A} = 0 \), which completes the proof. \( \square \)

**Lemma B.8.** Assume that \( n \in \mathbb{N} \) is even and \( |\psi_1| \geq |\psi_j| \) \( \forall j \in \{2, 3, \ldots, n\} \). The following equality holds.
\[ \langle T_2(\psi_1 \cdots \psi_n) \rangle_0 = \sum_{j=2}^{n} (-1)^j \langle T_2(\psi_1 \psi_j) \rangle_0 \langle T_2(\psi_2 \cdots \psi_j \cdots \psi_n) \rangle_0. \] (B.24)

**Proof.** For \( n = 2 \) the equality (B.24) is trivial. Assume that \( n \geq 4 \). Let \( \tau \in S_n \) be the unique permutation associated with the definition of \( T_2(\psi_1 \cdots \psi_n) \).
\[ T_2(\psi_1 \cdots \psi_n) = \text{sgn}(\tau) \psi_{\tau(1)} \cdots \psi_{\tau(n)}. \]

By assumption \( \tau(1) = 1 \). Moreover, Lemma B.6 ensures that
\[ \langle T_2(\psi_1 \cdots \psi_n) \rangle_0 = \text{sgn}(\tau) \sum_{j=2}^{n} (-1)^j \langle \psi_1 \psi_{\tau(j)} \rangle_0 \langle \psi_{\tau(2)} \cdots \psi_{\tau(j)} \cdots \psi_{\tau(n)} \rangle_0. \] (B.25)

Let us fix \( j \in \{2, 3, \ldots, n\} \). Let \( \pi \in S_n \) be such that
\[ (\pi(1), \pi(2), \pi(3), \ldots, \pi(n)) = (1, \tau(j), \tau(2), \ldots, \tau(j), \ldots, \tau(n)), \] (B.26)
where \( \tau(j) \) stands for the omission of the number \( \tau(j) \) from the row \( (\tau(2), \tau(3), \ldots, \tau(n)) \). Then, we have
\[ \text{sgn}(\pi) = (-1)^{j-2} \text{sgn}(\tau) = (-1)^j \text{sgn}(\tau). \] (B.27)

On the other hand, we can write \( \{1, \ldots, n\} \setminus \{1, \tau(j)\} = \{l_1, \ldots, l_{n-2}\} \) with \( 2 \leq l_1 < l_2 < \cdots < l_{n-2} \leq n \). There exists \( \eta \in S_{n-2} \) such that
\[ (l_{\eta(1)}, l_{\eta(2)}, \ldots, l_{\eta(n-2)}) = (\tau(2), \tau(3), \ldots, \tau(j), \ldots, \tau(n)). \] (B.28)
By \((B.30)\) and \((B.31)\) we obtain
\[
(\pi(1), \pi(2), \pi(3), \ldots, \pi(n)) = (1, \tau(j), l_{\eta(1)}, l_{\eta(2)}, \ldots, l_{\eta(n-2)}),
\]
which implies that
\[
\text{sgn}(\pi) = (-1)^{\tau(j)-2} \text{sgn}(\eta) = (-1)^{\tau(j)} \text{sgn}(\eta).
\]
\[(B.29)\]

By \((B.26)\) and \((B.28)\) we obtain
\[
\langle T_2(\psi_1\psi_{\tau(j)}), \psi_{\tau(j)} \rangle_0 = \langle T_2(\psi_1), \psi_{\tau(j)} \rangle_0 \langle T_2(\psi_2 \cdots \psi_{\tau(j)} \cdots \psi_n), \psi_{\tau(j)} \rangle_0.
\]
\[(B.31)\]

By substituting \((B.30)\) and \((B.31)\) into \((B.25)\) we see that
\[
\langle T_2(\psi_1 \cdots \psi_n), \psi_{\tau(j)} \rangle_0 = \sum_{j=2}^{n} (-1)^{\tau(j)} \langle T_2(\psi_1\psi_{\tau(j)}), \psi_{\tau(j)} \rangle_0 \langle T_2(\psi_2 \cdots \psi_{\tau(j)} \cdots \psi_n), \psi_{\tau(j)} \rangle_0
\]
\[
= \sum_{j=2}^{n} (-1)^{\tau(j)} \langle T_2(\psi_1\psi_{\tau(j)}), \psi_{\tau(j)} \rangle_0 \langle T_2(\psi_2 \cdots \psi_{\tau(j)} \cdots \psi_n), \psi_{\tau(j)} \rangle_0,
\]
which is \((B.24)\).

\[\square\]

Lemma B.9. For all \(x_j, y_j \in \Gamma, \sigma_j, \tau_j \in \{1, \downarrow\}, s_j, t_j \in \mathbb{R} (j = 1, 2, \ldots, n),\)
\[
\langle T_2(\psi^{*}_{x_1\sigma_1}(t_1) \cdots \psi^{*}_{x_n\sigma_n}(s_n)\psi_{y_n\tau_n}(t_n)), \psi_{\tau(j)} \rangle_0
\]
\[
= \det(\langle T_2(\psi^{*}_{x_j\sigma_j}(s_j)\psi_{y_j\tau_j}(t_k)), \psi_{\tau(j)} \rangle_0)_{1 \leq j,k \leq n}.
\]
\[(B.32)\]

Proof. We show \((B.32)\) by induction on \(n\). The equality \((B.32)\) is obviously true when \(n = 1\). Let us assume that \((B.32)\) is true for \(n - 1 (n \geq 2)\).

Lemma B.7 implies that for all \(\pi \in S_n,\)
\[
\langle T_2(\psi^{*}_{x_1\sigma_1}(s_1)\cdots \psi^{*}_{x_n\sigma_n}(s_n)\psi_{y_n\tau_n}(t_n)), \psi_{\tau(j)} \rangle_0
\]
\[
= \langle T_2(\psi^{*}_{x_1(1)\sigma_{1(1)}}(s_{\pi(1)})\cdots \psi^{*}_{x_n(\pi(n))\sigma_{\pi(n)}}(s_{\pi(n)})\psi_{y_{\pi(n)}\tau_{\pi(n)}}(t_{\pi(n)})), \psi_{\tau(j)} \rangle_0
\]
\[
= (-1)^n \langle T_2(\psi^{*}_{x_1(1)\tau_{\pi(1)}}(t_{\pi(1)})\cdots \psi^{*}_{x_n(\pi(n))\tau_{\pi(n)}}(s_{\pi(1)})\cdots \psi_{y_{\pi(n)}\tau_{\pi(n)}}(t_{\pi(n)})), \psi_{\tau(j)} \rangle_0
\]
\[(B.33)\]

and
\[
\det(\langle T_2(\psi^{*}_{x_j\sigma_j}(s_j)\psi_{y_j\tau_j}(t_k)), \psi_{\tau(j)} \rangle_0)_{1 \leq j,k \leq n}
\]
\[
= \det(\langle T_2(\psi^{*}_{x_j\sigma_j\tau_{\pi(k)}}(s_{\pi(k)})\psi_{y_{\pi(k)}\tau_{\pi(k)}}(t_{\pi(k)})), \psi_{\tau(j)} \rangle_0)_{1 \leq j,k \leq n}
\]
\[
= (-1)^n \det(\langle T_2(\psi^{*}_{x_{\pi(k)}\sigma_{\pi(k)}}(t_{\pi(k)})\psi_{y_{\pi(k)}\sigma_{\pi(k)}}(s_{\pi(k)})), \psi_{\tau(j)} \rangle_0)_{1 \leq j,k \leq n}.
\]
\[(B.34)\]
The equalities (B.33) and (B.34) enable us to assume that
\([\psi_{x_1,\sigma_1}(s_1)] \geq [\psi_{x_1,\sigma_1}(s_j)]\), \([\psi_{x_j,\tau_j}(t_j)]\) (\(\forall j \in \{1, \cdots, n\}\)) without losing generality in the following argument.

By using Lemma B.7, Lemma B.8, the hypothesis of induction, and the fact that \(\langle \psi_{x_0}(t)\psi_{x_0}(t') \rangle_0 = 0\), we have
\[
\langle T_2(\psi_{x_1,\sigma_1}(s_1)\psi_{x_1,\tau_1}(t_1) \cdots \psi_{x_n,\sigma_n}(s_n)\psi_{y_n,\tau_n}(t_n)) \rangle_0
= \sum_{j=1}^{n} \langle T_2(\psi_{x_1,\sigma_1}(s_1)\psi_{y,\tau_j}(t_j)) \rangle_0
\cdot \langle T_2(\psi_{y_1,\tau_1}(t_1)\psi_{x_2,\sigma_2}(s_2)\psi_{y_2,\tau_2}(t_2) \cdots \psi_{x_j,\sigma_j}(s_j)\psi_{y_j,\tau_j}(t_j) \cdots \psi_{x_n,\sigma_n}(s_n)\psi_{y_n,\tau_n}(t_n)) \rangle_0
= \sum_{j=1}^{n} (-1)^{j-1} \langle T_2(\psi_{x_1,\sigma_1}(s_1)\psi_{y,\tau_j}(t_j)) \rangle_0 \text{det}(\langle T_2(\psi_{x_1,\sigma_1}(s_1)\psi_{y_k,\tau_k}(t_k)) \rangle_0)_{1 \leq k \leq n}^{1 \leq j, k \leq n}
= \text{det}(\langle T_2(\psi_{x_1,\sigma_1}(s_j)\psi_{y_k,\tau_k}(t_k)) \rangle_0)_{1 \leq j, k \leq n},
\]
which concludes the proof. □

**Lemma B.10.** For all \(x, y \in \Gamma, \sigma, \tau \in \{\uparrow, \downarrow\}\), \(x, y \in \mathbb{R}\)
\[
\langle T_2(\psi_{x\sigma}(x)\psi_{y\tau}(y)) \rangle_0 = C(x\sigma x, y\tau y),
\]
(B.35)
where \(C(x\sigma x, y\tau y)\) is defined in (2.30).

**Proof.** By the definition of \(T_2\), we have
\[
\langle T_2(\psi_{x\sigma}(x)\psi_{y\tau}(y)) \rangle_0 = \langle \psi_{x\sigma}(x)\psi_{y\tau}(y) \rangle_0 1_{x-y \geq 0} - \langle \psi_{y\tau}(y)\psi_{x\sigma}(x) \rangle_0 1_{x-y < 0}.
\]
(B.36)

By Lemma B.5 (ii), (B.6) and (B.13), we see that
\[
\langle \psi_{x\sigma}(x)\psi_{y\tau}(y) \rangle_0
= \sum \sum (I + e^{\beta F^e})^{-1}(x\sigma, x'\sigma')\langle \psi_{x'\sigma'}(x)\psi_{y\tau}(y) \rangle
= \sum \sum (I + e^{\beta F^e})^{-1}(x\sigma, x'\sigma')e^{(x-y)F}(y\tau, x'\sigma')
= \left((I + e^{\beta F^e})^{-1}e^{(x-y)F}\right)(x\sigma, y\tau)
= \left(M'\left(I + e^{\beta F^e M'^e}\right)^{-1}e^{(x-y)M'}\right)(x\sigma, y\tau)
= \frac{\delta_{\sigma,\tau}}{L^d} \sum_{k,k} \delta_{k,k}e^{-i(x,k)e^{(y,k)}\frac{e^{(x-y)E_k}}{1 + e^{\beta E_k}}} = \frac{\delta_{\sigma,\tau}}{L^d} \sum_{k} e^{i(y-x)\frac{e^{(y-x)E_k}}{1 + e^{\beta E_k}}},
\]
(B.37)

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By (B.39), Lemma B.5 (ii), the definition of $T$

By combining (B.37) and (B.38) with (B.36), we obtain (B.35).

Proof of Proposition 2.4.

We have prepared all the lemmas necessary to prove Proposition 2.4.

**Proof of Proposition 2.4** By applying Lemma B.3 for $t_1 = 0$, $t_2 = \beta$ we have

$$
e^{-\beta H_\lambda} = e^{-\beta H_0} + e^{-\beta H_0} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{[0, \beta]} ds_1 \cdots ds_n T_1(V_\lambda(s_1) \cdots V_\lambda(s_n))$$

$$= e^{-\beta H_0} + e^{-\beta H_0} \sum_{n=1}^{\infty} (-1)^n \int_{[0, \beta]} ds_1 \cdots ds_n 1_{s_1 > \cdots > s_n} V_\lambda(s_1) \cdots V_\lambda(s_n).$$

(B.39)

By (B.39), Lemma B.3 (ii), the definition of $T_2$ and Lemma B.7 we see that

$$\frac{\text{Tr } e^{-\beta H_\lambda}}{\text{Tr } e^{-\beta H_0}}$$

$$= 1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \left( - \sum_{x_1, y_1, \ldots, y_n} \int_{0}^{\beta} ds_j U_{x_j, y_j, z_j, w_j} \right) 1_{s_1 > \cdots > s_n}$$

$$\cdot \langle \psi_{s_1}^* I_1 (s_1) \psi_{y_1}^* (s_1) \psi_{w_1}^* (s_1) \psi_{s_1} (s_1) \cdots \psi_{s_n}^* (s_n) \psi_{y_n} (s_n) \psi_{w_n} (s_n) \psi_{s_n} (s_n) \rangle_0$$

$$= 1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \left( - \sum_{x_1, x_2, y_1, y_2} \int_{0}^{\beta} dx_{2j-1} U_{x_{2j-1}, y_{2j-1}, y_j} \right) 1_{x_1 > x_2 > \cdots > x_{n-1}} (-1)^{n} \langle \psi_{x_1}^* I_1 (x_1) \psi_{y_2}^* (x_1) \psi_{y_1} (x_1) \psi_{y_2} (x_1) \cdots \psi_{x_2}^* (x_2n-1) \psi_{y_{2n-1}} (x_2n-1) \psi_{y_{2n-1}} (x_2n-1) \rangle_0$$
Lemma C.1. For any $M$ and $n$ in Proposition C.7, verifies the well-posedness of the Grassmann Gaussian integral that the determinant of the covariance matrix is non-zero, which is to be proved. In this part of Appendices we diagonalize the covariance matrix.

Then by using Lemma B.9 and Lemma B.10 we obtain the series (2.6).

\[ 1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \left( - \sum_{x_{2j-1}, x_{2j}, y_{2j-1}, y_{2j} \in \Gamma} \int_{0}^{\beta} dx_{2j-1} U_{x_{2j-1}, x_{2j}, y_{2j-1}, y_{2j}} \right) \cdot 1_{x_1 > x_3 > \ldots > x_{2n-1}} (T_2(\psi_{x_1}^n(x_1) \psi_{y_1}^n(x_1) \psi_{x_2}^n(x_1) \psi_{y_2}^n(x_1) \cdots \psi_{x_{2n-1}}^n(x_{2n-1}) \psi_{y_{2n-1}}^n(x_{2n-1}) \psi_{x_{2n}}^n(x_{2n}) \psi_{y_{2n}}^n(x_{2n}))) \]

\[ = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \left( - \sum_{x_{2j-1}, x_{2j}, y_{2j-1}, y_{2j} \in \Gamma} \sum_{\sigma_{2j-1}, \sigma_{2j} \in \{1, 2\}} \int_{0}^{\beta} dx_{2j-1} \delta_{\sigma_{2j-1}, 1} U_{x_{2j-1}, x_{2j}, y_{2j-1}, y_{2j}} \right) \cdot 1_{x_1 > x_3 > \ldots > x_{2n-1}} (T_2(\psi_{x_1}^n(x_1) \psi_{y_1}^n(x_1) \psi_{x_2}^n(x_2) \psi_{y_2}^n(x_2) \cdots \psi_{x_{2n-1}}^n(x_{2n-1}) \psi_{y_{2n-1}}^n(x_{2n-1}) \psi_{x_{2n}}^n(x_{2n}) \psi_{y_{2n}}^n(x_{2n}))) \]

Then by using Lemma [3.9] and Lemma [3.10] we obtain the series (2.6) \[ \Box \]

C Diagonalization of the covariance matrix

In this part of Appendices we diagonalize the covariance matrix $(C_h(x, x), y, y)_{(x, x), (y, y) \in \Gamma \times \{1, 2\} \times [0, h)}$ and calculate its determinant. The fact that the determinant of the covariance matrix is non-zero, which is to be proved in Proposition C.7, verifies the well-posedness of the Grassmann Gaussian integral defined in Definition 3.5.

For convenience of calculation we assume that $h \in 2\mathbb{N}/\beta$. Define the sets $W_h$ and $M_h$ by

\[ W_h := \left\{ \omega \in \frac{\pi \mathbb{Z}}{\beta} \mid -\pi h \leq \omega < \pi h \right\}, \quad M_h := \left\{ \omega \in \frac{\pi \mathbb{Z} + 1}{\beta} \mid -\pi h < \omega < \pi h \right\}. \]

Note that $\sharp W_h = 2\beta h$ and $\sharp M_h = \beta h$. The assumption that $h \in 2\mathbb{N}/\beta$ ensures the equality

\[ M_h = W_h \backslash \frac{2\pi \mathbb{Z}}{\beta}. \quad (C.1) \]

The set $M_h$ is seen as a set of the Matsubara frequencies with cut-off.

For $f \in L^2([-\beta, \beta)_h; \mathbb{C})$ we define $\hat{f} \in L^2(W_h; \mathbb{C})$ by

\[ \hat{f}(\omega) := \frac{1}{h} \sum_{t \in [-\beta, \beta)_h} e^{-i\omega t} f(t). \]

Lemma C.1. For any $f \in L^2([-\beta, \beta)_h; \mathbb{C})$

\[ f(t) = \frac{1}{2\beta} \sum_{\omega \in W_h} e^{i\omega t} \hat{f}(\omega), \quad \forall t \in [-\beta, \beta)_h. \]
Proof. If \( t = -\beta + s/h \) with \( s \in \{0, \cdots, 2\beta h - 1\} \),

\[
\frac{1}{2\beta} \sum_{\omega \in W_h} e^{i\omega t} \hat{f}(\omega) = \frac{1}{2\beta h} \sum_{\omega \in W_h} \sum_{u \in [-\beta,\beta)_h} e^{i\omega t} e^{-i\omega u} f(u)
\]

\[
= \frac{1}{2\beta h} 2^{2\beta h - 1} \sum_{m=0}^{2\beta h - 1} e^{i(-\pi h + \pi m/\beta)(s/h - l/h)} f \left( -\beta + \frac{l}{h} \right)
\]

\[
= \frac{1}{2\beta h} \sum_{l=0}^{2\beta h - 1} e^{-i\pi(s-l)} \sum_{m=0}^{2\beta h - 1} e^{i\pi m(s-l)/(\beta h)} f \left( -\beta + \frac{l}{h} \right)
\]

\[
= \sum_{l=0}^{2\beta h - 1} e^{-i\pi(s-l)} \delta_{s,l} f \left( -\beta + \frac{l}{h} \right) = f \left( -\beta + \frac{s}{h} \right) = f(t).
\]

\[\Box\]

Lemma C.2. If \( f \in L^2([-\beta,\beta)_h; \mathbb{C}) \) satisfies \( f(t) = -f(t + \beta) \) for all \( t \in [-\beta,\beta)_h \) with \( t < 0 \),

\[ f(t) = \frac{1}{2\beta} \sum_{\omega \in M_h} e^{i\omega t} \hat{f}(\omega), \ \forall t \in [-\beta,\beta)_h. \]  

(C.2)

Proof. Take any \( \omega \in W_h \cap 2\pi \mathbb{Z}/\beta \). By assumption we see that

\[
\hat{f}(\omega) = \frac{1}{h} \sum_{t \in [-\beta,\beta)_h} e^{-i\omega t} f(t) + \frac{1}{h} \sum_{t \in [0,\beta)_h} e^{-i\omega t} f(t)
\]

\[
= -\frac{1}{h} \sum_{t \in [-\beta,\beta)_h \setminus [0,\beta)_h} e^{-i\omega t} f(t + \beta) + \frac{1}{h} \sum_{t \in [0,\beta)_h} e^{-i\omega t} f(t) \quad \text{(C.3)}
\]

\[
= -\frac{1}{h} \sum_{t \in [0,\beta)_h} e^{-i\omega(t+\beta)} f(t) + \frac{1}{h} \sum_{t \in [0,\beta)_h} e^{-i\omega t} f(t) = 0.
\]

Then, by (C.1), (C.3) and Lemma C.1 we obtain (C.2). \[\Box\]

Let us define \( g_k \in L^2([-\beta,\beta)_h; \mathbb{C}) \) (\( k \in \Gamma^* \)) by

\[ g_k(t) := e^{tE_k} \left\{ \begin{array}{ll}
\frac{1}{1 + e^{\beta E_k}} & \text{if } t \geq 0 \\
\frac{1}{1 + e^{-\beta E_k}} & \text{if } t < 0
\end{array} \right\}.
\]

Note that the function \( g_k \) satisfies the anti-periodic property \( g_k(t) = -g_k(t + \beta) \) for all \( t \in [-\beta,\beta)_h \) with \( t < 0 \).

Lemma C.3. For all \( t \in [-\beta,\beta)_h \)

\[ g_k(t) = \frac{1}{\beta} \sum_{\omega \in M_h} e^{i\omega t} \frac{e^{i\omega t}}{h(1 - e^{-\omega/h + E_k/h})}. \]  

(C.4)
Proof. By Lemma C.2

\[ g_k(t) = \frac{1}{2\beta} \sum_{\omega \in M_h} e^{i\omega t} \hat{g}_k(\omega). \]

Moreover, we observe that for \( \omega \in M_h \)

\[ \hat{g}_k(\omega) = -\frac{1}{\hbar} \sum_{t \in [-\beta, \beta)_h \setminus [0, \beta)_h} e^{-i\omega t} \frac{e^{tE_k}}{1 + e^{\beta E_k}} + \frac{1}{\hbar} \sum_{t \in (0, \beta)_h} e^{-i\omega t} \frac{e^{tE_k}}{1 + e^{\beta E_k}} \]

\[ = -\frac{1}{\hbar} \sum_{t \in [0, \beta)_h} e^{-i\omega(t-\beta)} \frac{e^{tE_k}}{1 + e^{\beta E_k}} + \frac{1}{\hbar} \sum_{t \in (0, \beta)_h} e^{-i\omega t} \frac{e^{tE_k}}{1 + e^{\beta E_k}} \]

\[ = \frac{2}{\hbar} \sum_{t \in [0, \beta)_h} e^{-i\omega t} \frac{e^{tE_k}}{1 + e^{\beta E_k}} = \frac{2}{\hbar(1 + e^{\beta E_k})} \sum_{t \in [0, \beta)_h} e^{t(-i\omega + E_k)} \]

\[ = \frac{2}{\hbar(1 - e^{-i\omega/h + E_k/h})}. \]

The equality (C.4) follows from (C.5) and (C.6). \( \square \)

By substituting the characterization of \( g_k \) given in Lemma C.3 into

\[ C_h(x\sigma x, y\tau y) = \frac{\delta_{\sigma,\tau}}{L^d} \sum_{k \in \Gamma^*} e^{i(k\cdot y - x)} g_k(x - y) \]

we obtain

Lemma C.4. For any \((x, \sigma, x), (y, \tau, y) \in \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h\),

\[ C_h(x\sigma x, y\tau y) = \frac{\delta_{\sigma,\tau}}{\beta L^d} \sum_{k \in \Gamma^*} \sum_{\omega \in M_h} e^{i(k\cdot y - x)} e^{-i\omega(y - x)} \frac{1}{h(1 - e^{-i\omega/h + E_k/h})}. \]

In order to diagonalize \( C_h \), we define a matrix

\[ Y = (Y(k\tau \omega, x\sigma x))_{(k, \tau, \omega) \in \Gamma^* \times \{\uparrow, \downarrow\} \times M_h, (x, \sigma, x) \in \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h} \]

by

\[ Y(k\tau \omega, x\sigma x) := \frac{\delta_{\sigma,\tau}}{\sqrt{\beta h L^d}} e^{i(k\cdot x)} e^{-i\omega x}. \]

Lemma C.5. The matrix \( Y \) is unitary.

Proof. Assume that \( \omega = -\pi h + \pi / \beta + 2\pi m / \beta, \tilde{\omega} = -\pi h + \pi / \beta + 2\pi \tilde{m} / \beta \) with \( m, \tilde{m} \in \{0, 1, \cdots, \beta h - 1\} \). Then we observe that

\[ YY^*(k\tau \omega, \hat{k}\tilde{\omega}) = \frac{\delta_{\tau,\tilde{\tau}}}{\beta h L^d} \sum_{x \in \Gamma} \sum_{x \in [0, \beta)_h} e^{i(x\cdot k - \tilde{k})} e^{-ix(\omega - \tilde{\omega})} \]

\[ = \frac{\delta_{\tau,\tilde{\tau}}}{\beta h} \sum_{k \in \Gamma} e^{-i2\pi l(m - \tilde{m})/(\beta h)} \]

\[ = \delta_{\tau,\tilde{\tau}} \delta_{k,\tilde{k}} \delta_{m,\tilde{m}} = \delta_{\tau,\tilde{\tau}} \delta_{k,\tilde{k}} \delta_{\omega,\tilde{\omega}}. \]

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Let $x = s/h$, $\hat{x} = \hat{s}/h$ with $s, \hat{s} \in \{0, 1, \ldots, \beta h - 1\}$.

\[
Y^* Y(x_\sigma x, \hat{x}_{\hat{\sigma}} \hat{x}) = \frac{\delta_{\sigma, \hat{\sigma}}}{\beta h} \sum_{k \in \Gamma^*} \sum_{\omega \in M_h} e^{-i(k, x - \hat{x})} e^{i\omega(x - \hat{x})}
\]
\[
= \frac{\delta_{\sigma, \hat{\sigma}} \delta_{x, \hat{x}}}{\beta h} \sum_{m=0}^{\beta h - 1} e^{i(-\pi h + \pi/\beta + 2\pi m/\beta)(s/h - \hat{s}/h)}
\]
\[
= \frac{\delta_{\sigma, \hat{\sigma}} \delta_{x, \hat{x}}}{\beta h} \sum_{m=0}^{\beta h - 1} e^{i2\pi m(s - \hat{s})/(\beta h)}
\]
\[
= \delta_{\sigma, \hat{\sigma}} \delta_{x, \hat{x}} \delta_{s, \hat{s}} e^{i(-\pi h + \pi/\beta)(s/h - \hat{s}/h)} \delta_{s, \hat{s}} e^{i(-\pi h + \pi/\beta)(s/h - \hat{s}/h)} \delta_{s, \hat{s}}
\]

By using the matrix $Y$ and (C.7) we can diagonalize $C_h$ as follows.

**Lemma C.6.** For all $(k, \tau, \omega), (\hat{k}, \hat{\tau}, \hat{\omega}) \in \Gamma^* \times \{\uparrow, \downarrow\} \times M_h$,

\[
(Y C_h Y^*)(k_\tau \omega, \hat{k}_{\hat{\tau}} \hat{\omega}) = \delta_{\tau, \hat{\tau}} \delta_{k, \hat{k}} \delta_{\omega, \hat{\omega}} \frac{1}{1 - e^{-i\omega/h + E_k/h}}.
\]

Finally we calculate the determinant of the covariance matrix $C_h$.

**Proposition C.7.** For any $h \in 2\mathbb{N}/\beta$

\[
\det C_h = \frac{1}{\prod_{k \in \Gamma^*} (1 + e^{\beta E_k})^2}.
\]

**Proof.** Since \(e^{-i\omega/h + E_k/h} \mid \omega \in M_h\) is the set of all the $\beta h$th roots of $-e^{\beta E_k}$,

\[
z^{\beta h} + e^{\beta E_k} = \prod_{\omega \in M_h} (z - e^{-i\omega/h + E_k/h})
\]

for all $z \in \mathbb{C}$. Especially,

\[
\prod_{\omega \in M_h} (1 - e^{-i\omega/h + E_k/h}) = 1 + e^{\beta E_k}.
\]  

(C.8)

By Lemma C.5, Lemma C.6 and (C.8), we see that

\[
\det C_h = \det(Y C_h Y^*) = \prod_{k \in \Gamma^*} \prod_{\sigma \in \{\uparrow, \downarrow\}} \prod_{\omega \in M_h} \frac{1}{1 - e^{-i\omega/h + E_k/h}} = \frac{1}{\prod_{k \in \Gamma^*} (1 + e^{\beta E_k})^2}.
\]

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