MASS CONCENTRATION PHENOMENON TO THE 2D CAUCHY PROBLEM OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we consider the global strong solutions to the Cauchy problem of the compressible Navier-Stokes equations in two spatial dimensions with vacuum as far field density. It is proved that the strong solutions exist globally if the density is bounded above. Furthermore, we show that if the solutions of the two-dimensional (2D) viscous compressible flows blow up, then the mass of the compressible fluid will concentrate on some points in finite time.

1. Introduction. This paper concerns the Cauchy problem of the 2D compressible Navier-Stokes equations:

\[ \begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u + \nabla P &= 0,
\end{align*} \tag{1} \]

where \( x = (x_1, x_2) \in \mathbb{R}^2 \) is the spatial coordinate and \( t \geq 0 \) is the time. The unknown functions \( \rho, u = (u^1, u^2) \) and \( P = P(\rho) \) denote the density, velocity field and pressure, respectively. The aim of the present paper is to show the existence of the global strong solutions and the formation of singularities in finite time. The equation of state is given by

\[ P = A\rho^\gamma, \tag{2} \]

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with $A$ being a positive constant and $\gamma > 1$. The real parameters $\mu$ and $\lambda$ are the shear viscosity and the bulk viscosity coefficients, respectively, which satisfy the following physical restrictions:

$$\mu > 0, \quad \mu + \lambda \geq 0.$$  \hspace{1cm} (3)

System (1) will be investigated with initial conditions

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in \mathbb{R}^2,$$  \hspace{1cm} (4)

and the far-field conditions

$$(\rho, u)(x, t) \rightarrow (0, 0) \text{ as } |x| \rightarrow \infty.$$  \hspace{1cm} (5)

There are huge literatures on the global regularity criterion for the multi-dimensional compressible Navier-Stokes equations. In the absence of vacuum for the initial density, the local well-posedness theory of classical solutions has been well developed, see [11, 23, 24] and the references therein. When vacuum is allowed for the initial density, it can be proved that the strong solutions exist locally in time if one adds suitable compatibility conditions [3, 4, 5]. For large initial data which may contain vacuum, the major breakthrough is due to Lions [22] (see also Feireisl [10] and Jiang-Zhang [17, 18]), which proved that weak solutions exist globally in time when the exponent $\gamma$ is suitably large. However, the uniqueness of such weak solutions remains completely open even in the 2D case. To our knowledge, Vaigant-Kazhikhov [29] proved the first result of the global well-posedness to the periodic boundary problem with large initial data, under the assumption that the initial density is uniformly away from vacuum and viscous coefficients $\mu$ is constant, $\lambda = \rho^\beta$ with $\beta > 3$. More recently, Huang-Li [13] and Jiu-Wang-Xin [19] have extended the global well-posedness of strong solutions for periodic case to the Cauchy problem of 2D Navier-Stokes equations with vacuum when the viscosity coefficients depend on the density $\rho$ and $\beta$ is suitably large. But, if both the shear and bulk viscosities are positive constants, the global well-posedness of classical solutions to the Cauchy problem of 2D baratropic compressible Navier-Stokes system with vacuum and large initial data remains open.

On the other hand, due to the results of Xin [32], who showed that there is no global smooth solutions to the Cauchy problem (1) at least in 1D, as long as the initial density has compact support. That is to say, the Navier-Stokes equations may produce singularities in finite time. Therefore, it is important to study the mechanism of blowup and structure of possible singularities to the compressible Navier-Stokes equation. Starting with the pioneering works by Beal-Kato-Madja [1] and Serrin [26], many articles have dedicated to provide sufficient conditions for the global regularity of incompressible and compressible viscous fluids (see [6, 7, 8, 9, 12, 14, 15, 16, 27, 28, 30, 31] and the references therein). There are a series of blowup criteria for the 2D compressible Navier-Stokes equations, especially, Sun-Zhang [28] gave a blowup criterion for strong solutions in a bounded domain of $\mathbb{R}^2$. More precisely, they proved that if $T^*$ is the life span of the strong solution to system (1), then

$$\lim_{T \to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} = \infty.$$  \hspace{1cm}

The goal of this paper is to extend the blowup criterion for the 2D compressible Navier-Stokes equations in bounded domain to unbounded domain (Cauchy problem). The main difficulty lies in the fact that the Brezis-Waigner's inequality [2] fails for the 2D Cauchy problem, and it seems difficult to estimate $\|u\|_{L^q(\mathbb{R}^2)}$ for any
One way to overcome this difficulty is to estimate the momentum $\rho u$ instead of the velocity $u$, since $\rho$ decays for large $x$. However, the momentum $\rho u$ decays faster than $u$ itself. Furthermore, we use the variant of Gagliardo-Nirenberg inequality and a finer estimate for the convective term $\rho u \cdot \nabla u$. Moreover, the high order estimates on $\rho$ and $u$ will not be improved as that in the bounded domain case. Hence, we introduce the Hardy-type inequality to control the $L^q$-norm of $\dot{\rho}$. Furthermore, the initial density vacuum is allowed in this paper.

Before stating the main result, we first introduce the following simplified notations

$$
\int f dx = \int_{\mathbb{R}^2} f dx,
$$

and

$$
L^r = L^r(\mathbb{R}^2), \quad W^{k,r} = W^{k,r}(\mathbb{R}^2), \quad H^k = W^{k,2}(\mathbb{R}^2),
$$

for $1 \leq r \leq \infty$, $k \geq 0$.

Without loss of generality, we assume that the initial density $\rho_0$ satisfies

$$
\int \rho_0 dx = 1. \tag{6}
$$

The local strong solutions for the 2D Navier-Stokes system with vacuum was obtained in [21]. Here we write down one of those results in $\mathbb{R}^2$.

**Proposition 1.** For given positive constants $0 < \eta_0 \leq 1$, $q > 2$ and $a \in (1,2)$, we define

$$
\bar{x} \triangleq (e + |x|^2)^{\frac{a}{2}} \log^{1+\eta_0} (e + |x|^2), \tag{7}
$$

and the initial data satisfy (6) and

$$
\rho_0 \geq 0, \quad \bar{x}^a \rho_0 \in L^1 \cap H^1 \cap W^{1,q}, \quad \nabla u_0 \in L^2, \quad \sqrt{\rho} u_0 \in L^2. \tag{8}
$$

Then there exists a positive constant $T_0$ such that the Cauchy problem (1)-(5) has a unique strong solution $(\rho, u)$ in $\mathbb{R}^2 \times (0,T_0]$ satisfying

$$
\begin{cases}
\rho \in C([0,T_0]; L^1 \cap H^1 \cap W^{1,q}), \quad \bar{x}^a \rho \in L^\infty([0,T_0]; L^1 \cap H^1 \cap W^{1,q}), \\
\sqrt{\rho} u, \nabla u, \bar{x}^{-1} u, \sqrt{\nabla \rho} u \in L^\infty(0,T_0; L^2) \\
\nabla u \in L^\infty(0,T_0; H^1) \cap L^{\frac{4}{3}+1} (0,T_0; W^{1,q}), \quad \sqrt{\nabla u} \in L^2(0,T_0; W^{1,q}), \\
\sqrt{\rho} u_t, \sqrt{\nabla u}_t, \sqrt{\bar{x}^{-1} u}_t \in L^2(\mathbb{R}^2 \times (0,T_0)),
\end{cases} \tag{9}
$$

and

$$
\inf_{0 \leq t \leq T_0} \int_{B_N} \rho(x,t) dx \geq \frac{1}{4}, \tag{10}
$$

for some constant $N > 0$ and $B_N \triangleq \{x \in \mathbb{R}^2 \mid |x| < N\}$.

The main result of this paper can be stated as follows.

**Theorem 1.1.** Suppose that the initial data $(\rho_0, u_0)$ satisfy (6) and (8), and $(\rho, u)$ be a strong solution to the Cauchy problem (1)-(5) satisfying (9)-(10) in $\mathbb{R}^2 \times (0,T^*)$. If $T^* < \infty$ is the maximal existence time of the strong solution, then there exists some positive constant $s_0$ such that

$$
\lim_{T \to T^*} \| \rho \|_{L^\infty(0,T; L^s)} = \infty, \tag{11}
$$

for all $s \geq s_0$. 

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Remark 1. Indeed, in view of continuity equation
\[ \rho_t + u \cdot \nabla \rho + \rho \text{div} u = 0, \]
which implies that as long as the velocity field is regular, we can define the characteristic line as
\[ \frac{dy}{ds}(s; x, t) = u(y(s; x, t), s), \quad y(t; x, t) = x, \]
and the density can be denoted as
\[ \rho(x, t) = \rho_0(y(0; x, t)) \exp \left( -\int_0^t \text{div} u(y(s; x, t), s) ds \right). \]
This implies that if the singularity of the solution to the compressible Navier-Stokes equations (1) formulates in finite time \( T^* \), there may hold for the density:

1. The density may concentrate, namely,
\[ \lim_{T \to T^*} \| \rho \|_{L^\infty(0, T; L^\infty)} = \infty; \quad (12) \]
2. Vacuum states may appear in the non-vacuum region: there exist some \( x_1 \in \mathbb{R}^2 \) and \( x_1(t) \) satisfying
\[ \rho_0(x_1) > 0 \quad \text{and} \quad y(0; x_1(t), t) = x_1, \]
such that
\[ \lim_{t \to T^*} \rho(x_1(t), t) = 0. \quad (13) \]
3. Vacuum states may vanish: there exist some \( x_0 \in \mathbb{R}^2 \) and \( x_0(t) \) satisfying
\[ \rho_0(x_0) = 0 \quad \text{and} \quad y(0; x_0(t), t) = x_0, \]
such that
\[ \lim_{t \to T^*} \rho(x_0(t), t) \geq c_0 > 0; \quad (14) \]

Then one may ask: which one or some of (12)-(14) will happen when the singularity formulates? In view of the mass conservation equation (1) \( 1 \) and (6), Theorem 1.1 gives an answer to this question and shows that the mass of the fluid will concentrate on some points before other cases (2) and (3) happen.

Remark 2. The approach can also be adapted to deal with the bounded domain in \( \mathbb{R}^2 \). In particular, it would be interesting to study whether the upper bound of density is a necessary condition.

Remark 3. The mass conservation equation (1) \( 1 \) and (6) imply the boundedness of \( \| \rho \|_{L^\infty(0, T; L^1)} \) directly. It follows from the standard interpolation inequality that the bound of \( \| \rho \|_{L^\infty(0, T; L^{q_1})} \leq C \| \rho \|_{L^\infty(0, T; L^{q_2})} \leq C \) for any \( q_2 \in (1, q_1) \). Hence, it suffices to prove the result of Theorem 1.1 holds for \( s = s_0 \).

The remain of this paper is organized as follows. Some important inequalities and auxiliary lemmas will be given in Section 2. We prove the main result Theorem 1.1 in Section 3.
2. Preliminaries. In this section, some elementary lemmas will be used later. One of which is the variant of Gagliardo-Nirenberg inequality, the proof is referred to [25].

Lemma 2.1. Assuming \( f \in W^{1, m}(\mathbb{R}^2) \cap L^r(\mathbb{R}^2) \), it holds that
\[
\|f\|_{L^r} \leq C\|\nabla f\|_{L^m}^{\theta} \|f\|_{L^r}^{1-\theta},
\]
where \( \theta = \left( \frac{1}{r} - \frac{1}{m} \right) / \left( \frac{1}{r} - \frac{1}{m} + \frac{1}{2} \right) \), and if \( m < 2 \), then \( q \in \left[ r, \frac{2m}{2-m} \right] \), if \( m = 2 \), then \( q \in [2, \infty) \), if \( m > 2 \), then \( q \in [2, \infty] \) and constant \( C \) depends on \( q, m \) and \( r \).

Next, the material derivative \( \dot{f} \), the effective viscous flux \( G \), and the vorticity \( w \) are defined as follows.
\[
\dot{f} = f_t + u \cdot \nabla f, \quad G = (2\mu + \lambda) \text{div} u - P(\rho), \quad w = \text{rot} u = \partial_{x_1} u_2 - \partial_{x_2} u_1.
\]
Therefore, there are two key elliptic system of \( G \) and \( w \).
\[
\Delta G = \text{div} (\rho \dot{u}), \quad \mu \Delta w = \text{rot} (\rho \dot{u}),
\]
where we have used (1)_4 and (16).

From the standard \( L^p \)-estimate of elliptic system (17), we have the following estimate.

Lemma 2.2. Let \( (G, w) \) be a strong solutions of (17). Then there exists a generic positive constant \( C \) depending only on \( \mu, \lambda \) and \( p \), such that
\[
\|\nabla G\|_{L^p} + \|\nabla w\|_{L^p} \leq C\|\rho \dot{u}\|_{L^p},
\]
where \( 1 < p < \infty \).

Remark 4. In particular, taking \( p_0 \in (1, 2) \), using Hölder’s inequality, we have
\[
\|\nabla G\|_{L^{p_0}} + \|\nabla w\|_{L^{p_0}} \leq C\|\sqrt{\rho} \dot{u}\|_{L^2} \|\sqrt{\rho}\|_{L^{\frac{2p_0}{2-p_0}}}.\]

In addition, in order to estimate \( \|\nabla u\|_{L^p} \), we introduce the following inequality, which is crucial to the estimate in 2D Cauchy problem (see [19] for the detailed proof).

Lemma 2.3. Suppose that \( u \in C_0^\infty(\mathbb{R}^2) \) for any \( p \in (1, \infty) \). There exists a constant \( C \) depending only on \( p \), such that
\[
\|\nabla u\|_{L^p} \leq C (\|\text{div} u\|_{L^p} + \|u\|_{L^p}).
\]

The following Hardy-type inequality plays a crucial role in the estimate, the proof of which can be found in [21].

Lemma 2.4. Let \( \bar{x} \) and \( \eta_0 \) be as in (7) and \( B_{N_1} = \{ x \mid |x| < N_1 \} \) with \( N_1 \geq 1 \). Assuming that \( \rho \in L^1 \cap L^\gamma \) is a non-negative function such that
\[
\int_{B_{N_1}} \rho dx \geq M_1, \quad \int_{\mathbb{R}^2} \rho^{\gamma} dx \leq M_2,
\]
holds for positive constants \( M_1 \) and \( M_2 \). Then there exists a positive constant \( C \) depending on \( M_1, M_2, \gamma, N_1 \) and \( \eta_0 \) such that
\[
\|v\bar{x}^{-\gamma-1}\|_{L^2} \leq C\|\sqrt{\rho} v\|_{L^2} + C\|\nabla v\|_{L^2},
\]
for any \( v \in \tilde{D}^{1,2} = \{ v \in H^{1,0}_{\text{loc}}(\mathbb{R}^2) \mid \nabla v \in L^2 \} \). Furthermore, for \( \epsilon > 0 \) and \( \eta > 0 \), there exists a positive constant \( C \) depending on \( \epsilon \), \( \eta \), \( M_1 \), \( M_2 \), \( \gamma \), \( N_1 \) and \( \eta_0 \) such that every function \( v \in \tilde{D}^{1,2} \) satisfies
\[
\| v^{\frac{2}{3}} \|_{L^2}^2 \leq C \| \sqrt{\rho} v \|_{L^2}^2 + C \| \nabla v \|_{L^2}^2,
\]
with \( \tilde{\eta} = \min\{1, \eta\} \).

Finally, in order to estimate the term \( \| \nabla u \|_{L^\infty} \), we introduce the following Beal-Kato-Majda-type inequality, which was first proved in [1] when \( \text{div} u = 0 \). The proof for a general situation can be found in [20].

**Lemma 2.5.** Suppose that \( \nabla u \in L^2(\mathbb{R}^2) \cap W^{1,\eta}(\mathbb{R}^2) \) for any \( \eta \in (2, \infty) \), there exists a constant \( C \) depending only on \( \eta \), such that
\[
\| \nabla u \|_{L^\infty} \leq C(\| \text{div} u \|_{L^\infty} + \| u \|_{L^\infty}) \log(\epsilon + \| \nabla^2 u \|_{L^\infty}) + C(1 + \| u \|_{L^\infty}).
\]

**3. Proof of the main results.** Let \( (\rho, u) \) be a strong solution of (1)-(3) on \( \mathbb{R}^2 \times [0, T^*) \). We will prove our main result by contradiction arguments.

Otherwise, for some sufficiently large \( 1 < s_0 < \infty \), one has
\[
\lim_{T \rightarrow T^*} \| u \|_{L^\infty(0, T; L^2)} \leq M < \infty.
\]
In addition, in view of (6), the mass conservation equation (1) yields
\[
\int \rho dx = \int \rho_0 dx = 1.
\]

First, from the standard energy estimate for \( (\rho, u) \), it is easy to obtain the boundedness of \( \| \nabla u \|_{L^2(0, T; L^2)} \).

**Lemma 3.1.** There exists a positive constant \( N_1 \) such that
\[
\sup_{0 \leq t \leq T} \left( \int (\rho |u|^2 + \rho^2) \right) dx + \int_0^T \int |\nabla u|^2 dx dt \leq C,
\]
and
\[
\inf_{0 \leq t \leq T} \int_{B_{N_1}} \rho dx \geq \frac{1}{4},
\]
for any \( T \in [0, T^*) \). Here and after, \( C \) and \( C_i \) \((i = 1, 2, \ldots)\) denote the generic positive constants depending on \( M \), \( \mu \), \( \lambda \), \( T^* \), \( N_1 \) and initial data.

**Proof.** First, the proof of (26) is standard. Multiplying the momentum equation by \( u \) and using the first equation in (1), we can obtain the estimate (26).

Next, for \( N_1 > 1 \), a cutoff function \( \eta_{N_1}(x) \in C_0^{\infty}(\mathbb{R}^2) \) is defined by
\[
0 \leq \eta_{N_1}(x) \leq 1, \quad \eta_{N_1}(x) = \begin{cases} 1, & \text{if } |x| \leq N_1, \\ 0, & \text{if } |x| \geq 2N_1, \quad |\nabla \eta_{N_1}| \leq \frac{2}{N_1}. \end{cases}
\]
Multiplying (1) by \( \eta_{N_1} \) and integrating by parts give
\[
\frac{d}{dt} \int \rho \eta_{N_1} dx = \int \rho u \cdot \nabla \eta_{N_1} dx
\geq -2N_1^{-1} \left( \int \rho dx \right)^{\frac{1}{2}} \left( \int \rho |u|^2 dx \right)^{\frac{1}{2}} \geq -C_1 N_1^{-1},
\]
due to Hölder’s inequality and (26).
Combining (31) with (32), choosing $q$ then integrating the resulting equation lead to

$$\inf_{0 \leq t \leq T} \int \rho \eta_{N_1} \, dx \geq \int \rho_0 \eta_{N_1} \, dx - C_1 N_1^{-1} T, \tag{29}$$

which together with (25) imply,

$$\inf_{0 \leq t \leq T} \int_{B_{N_1}} \rho \, dx \geq \frac{1}{4},$$

for $N_1$ suitably large. The proof of Lemma 3.1 is completed. $\square$

Next, there is an important estimate about $\rho u$, the similar arguments of the following estimates come from [28].

**Lemma 3.2.** Under condition (24), there exist some positive constants $a \in (1, 2)$, $q_0 \in \left(2, \min \left(\frac{4p + 2\lambda}{\mu + \lambda}, 4\right)\right)$ and $s_0 \geq s_1 \equiv \frac{4 + 2q_0 - 2q_0}{4q_0}$, such that

$$\sup_{0 \leq t \leq T} \int (\rho |u|^{q_0} + \rho^{\bar{a}}) \, dx \leq C, \tag{30}$$

for any $T \in [0, T^*)$.

**Proof.** Multiplying the momentum equations (1.1)$_2$ by $q|u|^{q-2}u$ ($2 < q < 4$) and then integrating the resulting equation lead to

$$\frac{d}{dt} \int \rho |u|^q \, dx + q \int |u|^{q-2} \left[\mu \nabla u^2 + (\mu + \lambda)(\text{div} u)^2 + \mu(q - 2)|\nabla |u|^2|\right] \, dx$$

$$= - (\mu + \lambda)q(q - 2) \int |u|^{q-3} u \cdot \nabla |u|^2 \, dx + q \int \text{div}(|u|^{q-2} u) \rho \, dx$$

$$\leq (\mu + \lambda)q(q - 2) \int |u|^{q-2} |\nabla |u|| |\text{div} u| \, dx + C \int \rho^\gamma |u|^{q-2} |\nabla |u| \, dx$$

$$\leq (\mu + \lambda)q(q - 2) \int |u|^{q-2} |\nabla |u|| |\text{div} u| \, dx + C \int \rho^{\frac{4 + 2q - 2q_0}{4q_0}} \, dx$$

$$+ C \left(\int \rho^{\frac{4 + 2q - 2q_0}{4q_0}} \, dx\right)^{\frac{4q}{4q_0}} \left(\int \rho |u|^q \, dx\right)^{\frac{q}{q_0}} \left(\int |\nabla u|^2 \, dx\right)^{\frac{1}{2}}$$

$$\leq (\mu + \lambda)q(q - 2) \int |u|^{q-2} |\nabla |u|| |\text{div} u| \, dx + C \int \rho^{\frac{4 + 2q - 2q_0}{4q_0}} \, dx$$

$$+ C \int \rho |u|^q \, dx + C \|\nabla u\|^2_{L^2},$$

due to Hölder’s inequality and the fact $|\nabla u| \geq |\nabla |u||$.

In addition, it is easy to see that

$$|u|^{q-2} \left[\mu \nabla u^2 + (\mu + \lambda)(\text{div} u)^2 + \mu(q - 2)|\nabla |u|^2|\right] - (\mu + \lambda)(q - 2)|\nabla |u|| \text{div} u|$$

$$\geq |u|^{q-2} \left[\frac{\mu + \lambda}{2} (4 - q)(\text{div} u)^2 + \left(\mu - \frac{\mu + \lambda}{2} (q - 2)\right)|\nabla u|^2\right]. \tag{32}$$

Combining (31) with (32), choosing $q_0 \in \left(2, \min \left(\frac{4p + 2\lambda}{\mu + \lambda}, 4\right)\right)$, then

$$\sup_{0 \leq t \leq T} \int \rho |u|^{q_0} \, dx \leq C. \tag{33}$$
Multiplying (1) by \( \bar{x}^a \) and integrating the resulting equation over \( \mathbb{R}^2 \), integrating by parts and (22), we obtain
\[
\frac{d}{dt} \int \rho \bar{x}^a \, dx \leq C \int \rho |u| \bar{x}^{a-1} \log^{1+q_0} (e + |x|^2) \, dx
\]
\[
\leq C \int \bar{x}^{a-\frac{1}{2}} \rho \frac{2a-1}{2a} \bar{x}^{-\frac{1}{2}} |u| \bar{x}^{-\frac{1}{2}} \log^{1+q_0} (e + |x|^2) \, dx
\]
\[
\leq \| \bar{x}^{-\frac{1}{2}} \|_{L^\infty} \left( \int \rho \bar{x}^a \, dx \right)^{\frac{2a-1}{2a}} \left( \int |u|^{2a} \bar{x}^{-\frac{2a}{2}} \, dx \right)^{\frac{1}{2a}}
\]
\[
\leq C \left( \int \rho \bar{x}^a \, dx \right)^{\frac{2a-1}{2a}} \left( \| \sqrt{p} u \|_{L^2} + \| \nabla u \|_{L^2} \right)
\]
\[
\leq C \left( 1 + \| \nabla u \|_{L^2}^2 \right) \left( \int \rho \bar{x}^a \, dx + 1 \right),
\]
which together with Gronwall’s inequality and (26) yield
\[
\sup_{0 \leq t \leq T} \int \rho \bar{x}^a \, dx \leq C.
\] (34)
Furthermore, inequalities (34) and (35) yield (30).

Proof. Multiplying the momentum equation (1) by \( u_i \) and integrating over \( \mathbb{R}^2 \) give
\[
\frac{1}{2} \frac{d}{dt} \int (\mu |\nabla u|^2 + (\mu + \lambda)(\text{div} u)^2) \, dx + \int \rho |\dot{u}|^2 \, dx
\]
\[
= \int P \text{div} u_i \, dx + \int \rho \dot{u}(u \cdot \nabla) u \, dx = \sum_{i=1}^2 I_i.
\] (37)
Moreover, the pressure \( P \) satisfies the following equation
\[
P_t + \text{div} (Pu) + (\gamma - 1) P \text{div} u = 0,
\] (38)
which can be derived by the continuity equation and the equation of state directly.

For the first term on the right-hand side of (37), assuming \( p_0 \in \left( \frac{q_0}{q_0 - 1}, 2 \right) \), that is \( \frac{p_0}{p_0 - 2} \leq q_0 \leq 4 \). Let \( s_2 \equiv \max \left\{ 4 \gamma, \frac{q_0 p_0 - p_0}{q_0 - p_0}, \frac{p_0}{\frac{p_0}{q_0} - 2} \right\} \), we obtain
\[
I_1 = \frac{d}{dt} \int P \text{div} u \, dx - \int P_t \text{div} u \, dx
\]
\[
= \frac{d}{dt} \int P \text{div} u \, dx + \int \text{div} (Pu) \text{div} u \, dx + (\gamma - 1) \int P (\text{div} u)^2 \, dx
\]
\[
= \frac{d}{dt} \int P \text{div} u \, dx - \int Pu \cdot \nabla \text{div} u + (\gamma - 1) \int P (\text{div} u)^2 \, dx
\]
where we have also used (16), (18), (19), (26), (38), Sobolev inequality, interpolation inequality to (15), (16), (18), (19), (20), (30) and Hölder’s inequality.

On the other hand, using Young’s inequality, we have

\begin{align*}
&\leq \frac{d}{dt} \int P \text{div} u dx + C \|\rho^{1/q_0} u\|_{L^{q_0}} \|\nabla G\|_{L^{q_0}} \|\rho^{\gamma-1/q_0}\|_{L^{q_0}} \frac{\gamma q_0-1}{\gamma q_0-\gamma_0} \\
&\quad + C \|P\|_{L^{p_0}} \|\nabla u\|_{L^2} + C \|P\|_{L^{p_0}} \|\nabla u\|_{L^2} \|G\|_{L^{2p_0}} \\
&\leq \frac{d}{dt} \int P \text{div} u dx + C \|\nabla G\|_{L^{q_0}} \|\rho\|_{L^{q_0}} \left(\frac{\gamma q_0-1}{\gamma q_0-\gamma_0}\right)^{2/q_0} \\
&\quad + C \|\rho\|_{L^{q_0}} \|\nabla u\|_{L^2} + C \|\rho\|_{L^{q_0}} \|\nabla u\|_{L^2} \|G\|_{L^{2p_0}} \\
&\leq \frac{d}{dt} \int \rho \|u\|^2 |\nabla u|^2 dx + C \|\nabla u\|_{L^2} + C \|\nabla u\|_{L^2} + C \\
&\leq \frac{d}{dt} \int \rho \|u\|^2 |\nabla u|^2 dx + \varepsilon C \|\nabla G\|_{L^{q_0}}^2 + C \|\nabla u\|_{L^2}^2 + C
\end{align*}

Substituting (39) and (40) into (37), choosing \(\varepsilon\) sufficiently small, we get

\begin{align*}
&\left(\frac{\mu}{2} |\nabla u|^2 + \frac{\mu + \lambda}{2} (\text{div} u)^2 - P \text{div} u\right) dx + \frac{1}{2} \int \rho |\dot{u}|^2 dx \\
&\leq C \|\nabla u\|_{L^2}^2 + C \int \rho |u|^2 |\nabla u|^2 dx + C
\end{align*}

Next, we will deal with the second term on the right-hand side of (41), choosing

\(p_1 \in \left(\frac{q_0}{q_0-1}, 2\right), s_3 = \frac{p_1(q_0-2)}{2(q_0 p_1 - q_0 - p_1)}\) such that

\begin{align*}
&\int \rho |u|^2 |\nabla u|^2 dx \\
&\leq C \left(\int \rho |u|^{q_0} dx\right)^{2/q_0} \left(\int |\nabla u|^{2p_1/(2-p_1)} dx\right)^{2/p_1-1} \left(\int \rho^{s_3} dx\right)^{(q_0-2)/q_0 s_3} \\
&\leq C \left(\int |\nabla u|^{2p_1/(2-p_1)} dx\right)^{2/p_1-1} \\
&\leq C \varepsilon \left(\|G\|_{L^{2p_1/(2-p_1)}}^2 + \|u\|_{L^{2p_1/(2-p_1)}}^2\right) + C \\
&\leq \varepsilon \left(\|\nabla G\|_{L^{2p_1}}^2 + \|\nabla u\|_{L^{2p_1}}^2\right) + C \\
&\leq \varepsilon C \|\rho \dot{u}\|_{L^{2p_1}}^2 + C \\
&\leq \varepsilon C \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C,
\end{align*}

due to (15), (16), (18), (19), (20), (30) and Hölder’s inequality.
Hence, choosing $\varepsilon$ sufficiently small, combining (41) with (42) yields
\[
\frac{d}{dt} \int \frac{\mu}{2} |\nabla u|^2 + \frac{\mu + \lambda}{2} (\text{div} u)^2 - P\text{div} u \, dx + \frac{1}{4} \int \rho |\dot{u}|^2 \, dx
\leq C\|\nabla u\|_{L^2}^2 + C, \tag{43}
\]
and noting that
\[
\left| \int P\text{div} u \, dx \right| \leq \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C.
\]
Thus, by using Gronwall’s inequality, (43) yields
\[
\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \int \rho |\dot{u}|^2 \, dx \, dt \leq C,
\]
which together with (21) and (26) gives
\[
\sup_{0 \leq t \leq T} \|x^{-1} u\|_{L^2} \leq C \sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} u\|_{L^2} + \|\nabla u\|_{L^2} \right) \leq C. \tag{44}
\]
Next, we can improve the regularity estimates on $\rho$ and $u$.

**Lemma 3.4.** Under condition (24), and constant $s_0 \geq s_2$ with which defined in Lemma 3.3, there holds
\[
\sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + t \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \right) + \int_0^T \int (t|\dot{x}^{-1} \dot{u}|^2 + |\nabla \dot{u}|^2 + t|\nabla \dot{u}|^2) \, dx \, dt \leq C,
\]
for any $T \in [0, T^*)$.

**Proof.** Applying the operator $\dot{u}^j [\partial_j + \text{div}(u \cdot)]$ to (1) \(_2 \) \(_{j = 1, 2} \) and using the first equation of (1), integrating by parts over $\mathbb{R}^2$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 \, dx = - \int \dot{u}^j \left[ \partial_j P_i + \text{div}(u \partial_j P) \right] \, dx + \mu \int \dot{u} \left[ \Delta u_i + \text{div}(u \Delta u^j) \right] \, dx
\]
\[
+ (\mu + \lambda) \int \dot{u} \left[ \partial_j \text{div} u_t + \text{div}(u \partial_j \text{div} u) \right] \, dx
\]
\[
= \sum_{i=1}^3 N_i. \tag{45}
\]
First, integrating by parts, we have
\[
N_1 = - \int \dot{u}^j \left[ \partial_j P_i + \text{div}(u \partial_j P) \right] \, dx
\]
\[
= \int \left( \partial_j \dot{u} \right) P' (\rho) \rho_i + \partial_k \dot{u}^j u^k \partial_j P \, dx
\]
\[
= \int \left( -\rho P'(\rho) \partial_j \dot{u} \text{div} u - \partial_j \dot{u}^j u^k \partial_k P + \partial_k \dot{u}^j u^k \partial_j P \right) \, dx
\]
\[
= \int \left( -\rho P'(\rho) \partial_j \dot{u} \text{div} u + P \partial_k (\partial_j \dot{u}^j u^k) - P \partial_j (\partial_k \dot{u}^j u^k) \right) \, dx
\]
\[
= \int \left( -\rho P'(\rho) \partial_j \dot{u} \text{div} u + P \partial_j \dot{u}^j u^k - P \partial_k \dot{u}^j \partial_j u^k \right) \, dx
\]
\[
\leq C\|\nabla u\|_{L^4} \|\nabla \dot{u}\|_{L^2} \|P\|_{L^4}
\]
\[
\leq C\|\nabla u\|_{L^4} \|\nabla \dot{u}\|_{L^2} \|\rho\|_{L^4}
\]
\[
\leq \varepsilon \|\nabla \dot{u}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^2,
\]
due to (24) and Hölder’s inequality. For $N_2$ and $N_3$, we use the similar arguments as that in [15] and obtain

$$N_2 \leq -\frac{7\mu}{8}\|\nabla \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4,$$

(47) and

$$N_3 \leq -\frac{7(\mu + \lambda)}{8}\|\text{div} \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 \leq C\|\nabla u\|_{L^4}^4.$$

(48)

Using (15), (18), (19), (24) and (36), we obtain

$$\|\nabla u\|_{L^4}^4 \leq C\left(\|\text{div} u\|_{L^4}^4 + \|w\|_{L^4}^4\right)$$

$$\leq C\left(\|G\|_{L^4}^4 + \|w\|_{L^4}^4\right)$$

$$\leq C\left(1 + \|\rho \dot{u}\|_{L^2}^p\right)$$

$$\leq C\left(1 + \|\rho \dot{u}\|_{L^2}^p\right)$$

(49)

for some $p_0 \in (\frac{4}{3}, 2)$. Substituting (46)-(49) into (45), we have

$$\frac{d}{dt} \int \rho |\dot{u}|^2 dx + \mu \int |\nabla \dot{u}|^2 dx \leq C\|\sqrt{\rho \dot{u}}\|_{L^2}^4 + C,$$

(50)

together with (36) yields

$$\sup_{0 \leq t \leq T} \|\sqrt{\rho \dot{u}}\|_{L^2}^2 + \int_0^T \int |\nabla \dot{u}|^2 dx dt \leq C.$$

(51)

Using Gronwall’s inequality, and then multiplying (50) by $t$ and integrating over $(0, T)$ lead to

$$\sup_{0 \leq t \leq T} t\|\sqrt{\rho \dot{u}}\|_{L^2}^2 + \int_0^T \int t|\nabla \dot{u}|^2 dx dt \leq C.$$  

Incorporating the above estimates with (21), we get

$$\int_0^T t \|\bar{x}^{-1} \dot{u}\|_{L^2}^2 dt \leq C \int_0^T \left( t\|\sqrt{\rho \dot{u}}\|_{L^2}^2 + t\|\nabla \dot{u}\|_{L^2}^2 \right) dt \leq C.$$  

Next, we will utilize Lemma 3.1–Lemma 3.4 to prove the boundedness of $\|\rho\|_{L^\infty(0, T; L^\infty)}$.

**Lemma 3.5.** Under the conditions of Theorem 1.1, let $s_0 \geq s_4$ ($s_4$ can be chosen in the proof), then

$$\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^\infty} \leq C,$$

(52)
holds for any $T \in [0, T^*)$. 

Proof. For any $p > 1$, multiplying $(1)_1$ by $p \rho^{p-1}$ and integrating by parts over $\mathbb{R}^2$, we have
\[
\frac{d}{dt} \int \rho^p dx = - \int (u \cdot \nabla \rho^p + p \rho^p \text{div} u) dx \\
= (1 - p) \int \rho^p \text{div} u dx \\
= \frac{1 - p}{2\mu + \lambda} \int \rho^p G dx + \frac{1 - p}{2\mu + \lambda} \int \rho^p P dx \\
\leq \frac{p - 1}{2\mu + \lambda} \|G\|_{L^\infty} \int \rho^p dx.
\]

Moreover, for some $p_0 > 2$, $\varepsilon > \frac{a - 1}{2 - a}$, taking $\eta = \frac{1}{p_0}$ and $s_4 = \frac{(2 + \varepsilon)(ap_0 - 1)}{a(1 + \varepsilon) - 2 - \varepsilon}$, we have
\[
\|G\|_{L^\infty} \leq C\|G\|_{L^2} + C\|\nabla G\|_{L^{p_0}} \\
\leq C\|\nabla G\|_{L^{p_0}} + C \\
\leq C\|\rho \tilde{u}\|_{L^{p_0}} + C \\
\leq C\|\rho\|_{L^{\frac{p_0}{1 + a}}} \|\rho^{1/a}\|_{L^\infty} \|\tilde{u}\|_{L^\frac{2 + \varepsilon}{2}} + C \\
\leq C (\|\sqrt{\rho} \tilde{u}\|_{L^2} + \|\nabla \tilde{u}\|_{L^2} + 1) \\
\leq C (\|\nabla \tilde{u}\|_{L^2} + 1),
\]
combining $(15)$, $(16)$, $(22)$, $(24)$, $(30)$ with $(24)$ and using Hölder’s inequality yield
\[
\frac{d}{dt} \|\rho\|_{L^p} \leq \frac{C(p - 1)}{p} (\|\nabla \tilde{u}\|_{L^2} + 1) \|\rho\|_{L^p} \\
\leq C (\|\nabla \tilde{u}\|_{L^2}^2 + 1) \|\rho\|_{L^p},
\]
with constant $C$ independent of $p$.

By $(54)$, $(44)$ and Gronwall’s inequality, we get
\[
\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^p} \leq C.
\]

Let $p \to \infty$, the proof of Lemma 3.5 has finished. \hfill \square

Remark 5. In view of Lemma 3.2-3.5, we can choose a positive constant $s_0 = \max\{s_1, s_2, s_3, s_4\}$, which depends on $\lambda$, $\mu$ and $\gamma$.

Finally, the following Lemma gives the bounds for the first-order derivative of density $\rho$ and the second spatial derivative of velocity $u$.

Lemma 3.6. Under condition $(24)$, it holds that for some $q \in (2, \infty)$ and $0 \leq t < T^*$,
\[
\sup_{0 \leq t \leq T} (\|\rho\|_{H^1 \cap V^1} + \|\nabla u\|_{H^1}) + \int_0^T \left( \|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + t \|\nabla^2 u\|_{L^q}^2 \right) dt \leq C. \tag{55}
\]

Proof. First, $|\nabla \rho|^q$ satisfies
\[
(|\nabla \rho|^q)_t + \text{div} (|\nabla \rho|^q u) + (q - 1)|\nabla \rho|^q \text{div} u \\
+ q|\nabla \rho|^{q-2}(\nabla \rho)^{tr} \nabla u (\nabla \rho) + q\rho|\nabla \rho|^{q-2} \nabla \rho \cdot \nabla \text{div} u = 0,
\]
for any $q > 2$, which implies
\[
\frac{d}{dt} \|\nabla \rho\|_{L^q} \leq C (1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^q} + C \|\nabla^2 u\|_{L^q}. \tag{56}
\]
Next, we estimate terms $\|\nabla u\|_{L^\infty}$ and $\|\nabla^2 u\|_{L^q}$, respectively. In fact, the standard $L^p$-estimate ($p > 1$) of elliptic system (1) yields

$$\|\nabla^2 u\|_{L^p} \leq C (\|\rho \dot{u}\|_{L^p} + \|\nabla \rho\|_{L^p}),$$

which together with Lemma 2.5 gives

$$\|\nabla u\|_{L^\infty} \leq C (\|\text{div} u\|_{L^\infty} + \|w\|_{L^\infty}) \log (e + \|\nabla^2 u\|_{L^q}) + C.$$  

In addition, we have the following estimate

$$\|\text{div} u\|_{L^\infty} + \|w\|_{L^\infty} \leq C (\|G\|_{L^\infty} + \|w\|_{L^\infty} + \|P\|_{L^\infty}) \leq C(q) (1 + \|\nabla \dot{u}\|_{L^2}^{2(q-1)} + \|\nabla w\|_{L^q}^{2(q-1)})$$

due to (15), (18), (36) and interpolation inequality.

Similar to Lemma 3.5, we have

$$\|\rho \dot{u}\|_{L^q} \leq C (1 + \|\nabla \dot{u}\|_{L^2})$$

for any $q > 2$. Substituting (57)-(60) into (56), we get

$$\frac{d}{dt} \|\nabla \rho\|_{L^q} \leq C (1 + \|\nabla \dot{u}\|_{L^2}^2) (e + \|\nabla \rho\|_{L^q}) \log (e + \|\nabla \rho\|_{L^q}),$$

or

$$\frac{d}{dt} \log (e + \|\nabla \rho\|_{L^q}) \leq C (1 + \|\nabla \dot{u}\|_{L^2}^2) \log (e + \|\nabla \rho\|_{L^q}),$$

using Gronwall’s inequality and (44) yield

$$\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^q} \leq C.$$  

Furthermore, we have

$$\int_0^T \left( \|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + t \|\nabla^2 u\|_{L^q}^2 \right) dt \leq C,$$

by (57) and (60).

Finally, using (15), $\|\nabla \rho\|_{L^2}$ satisfies

$$\frac{d}{dt} \|\nabla \rho\|_{L^2} \leq C (1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^2} + \|\nabla^2 u\|_{L^2}$$

which together with (44), (57), (60), (62) and using Gronwall’s inequality imply

$$\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^2} \leq C.$$  

Moreover, the standard $L^2$-estimate of elliptic system (1)$_2$ gives

$$\|\nabla^2 u\|_{L^2} \leq C (\|\sqrt{\rho} \dot{u}\|_{L^2} + \|\nabla \rho\|_{L^2}) \leq C.$$  

Hence, the desired estimate (55) follows from (62)-(64). 

**Lemma 3.7.** With the assumption (24), it holds that for some $q \in (2, \infty)$,

$$\sup_{0 \leq t \leq T} \|\bar{\rho} \|_{H^1 \cap W^{1,q}} \leq C.$$  

(65)
Proof. It follows from (1.1) that \( \bar{x}^a \rho \) satisfies
\[
(\bar{x}^a \rho)_t + u \cdot \nabla (\bar{x}^a \rho) - a \bar{x}^a \rho u \cdot \nabla \log \bar{x} + \bar{x}^a \rho \text{div} u = 0.
\]
Hence, for \( p \in [2, q] \), \( |\nabla (\bar{x}^a \rho)|^p \) satisfies
\[
\begin{align*}
&\left( |\nabla (\bar{x}^a \rho)|^p \right)_t + \text{div}(|\nabla (\bar{x}^a \rho)|^p u) + (p-1)|\nabla (\bar{x}^a \rho)|^p \text{div} u \\
&+ p|\nabla (\bar{x}^a \rho)|^{p-2} (\nabla (\bar{x}^a \rho))^{tr} \nabla u (\nabla (\bar{x}^a \rho)) + p \bar{x}^a \rho \nabla (\bar{x}^a \rho) |\nabla (\bar{x}^a \rho)|^{p-2} \nabla (\bar{x}^a \rho) \cdot \nabla \text{div} u \\
&- ap|\nabla (\bar{x}^a \rho)|^p u \cdot \nabla \log \bar{x} - a p \bar{x}^a \rho \nabla (\bar{x}^a \rho) |\nabla (\bar{x}^a \rho)|^{p-2} (\nabla (\bar{x}^a \rho))^{tr} \nabla u \nabla \log \bar{x} + u \nabla^2 \log \bar{x}
\end{align*}
\]
which implies
\[
\frac{d}{dt} \| \nabla (\bar{x}^a \rho) \|_{L^p} \leq C \left( 1 + \| \nabla u \|_{L^\infty} + \| u \cdot \nabla \log \bar{x} \|_{L^\infty} \right) \| \nabla (\bar{x}^a \rho) \|_{L^p} \\
+ C \| \nabla \text{div} u \|_{W^{1,4}} \| \nabla (\bar{x}^a \rho) \|_{L^p} \\
+ C \| \nabla u \|_{L^p} \left( \| \nabla \nabla \log \bar{x} \|_{L^p} + \| u \nabla^2 \log \bar{x} \|_{L^p} + \| \nabla u \|_{L^p} \right) \\
\leq C \left( 1 + \| \nabla u \|_{W^{1,4}} \right) \| \nabla (\bar{x}^a \rho) \|_{L^p} \\
+ C \| \nabla u \|_{L^p} \left( \| \nabla \nabla \log \bar{x} \|_{L^p} + \| u \nabla^2 \log \bar{x} \|_{L^p} + \| \nabla u \|_{L^p} \right),
\]
where we have used (15), (22), (36), (55) and the following facts
\[
\| u \cdot \nabla \log \bar{x} \|_{L^\infty} \leq C \| u \bar{x}^{-\frac{1}{2}} \|_{L^\infty} \\
\leq C \left( \| u \bar{x}^{-\frac{1}{2}} \|_{L^3} + \| \nabla \left( u \bar{x}^{-\frac{1}{2}} \right) \|_{L^3} \right) \\
\leq C \left( \| u \bar{x}^{-\frac{1}{2}} \|_{L^3} + \| \nabla u \|_{L^3} \right) \\
\leq C \left( 1 + \| \nabla u \|_{H^1} \right) \\
\leq C,
\]
incorporating (55) with (30) yield
\[
\| \rho \bar{x}^a \|_{L^\infty} \leq C \left( \| \rho \bar{x}^a \|_{L^2} + \| \nabla (\rho \bar{x}^a) \|_{L^4} \right) \\
\leq C \left( \| \rho \bar{x}^a \|_{L^2}^{\frac{1}{2}} \| \rho \bar{x}^a \|_{L^\infty}^{\frac{1}{2}} + \| \nabla (\rho \bar{x}^a) \|_{L^4} \right) \\
\leq C \left( \| \rho \bar{x}^a \|_{L^1} + \| \rho \bar{x}^a \|_{L^\infty} + \| \nabla (\rho \bar{x}^a) \|_{L^4} \right) \\
\leq C \left( 1 + \varepsilon \right) \| \rho \bar{x}^a \|_{L^\infty} + \| \nabla (\rho \bar{x}^a) \|_{L^4},
\]
we get
\[
\| \rho \bar{x}^a \|_{L^\infty} \leq C \left( 1 + \| \nabla (\rho \bar{x}^a) \|_{L^4} \right),
\]
by letting \( \varepsilon \) suitably small.
Choosing \( p = q \) in (66), it follows from Gronwall’s inequality and (55) that
\[
\sup_{0 \leq t \leq T} \| \nabla (\rho \bar{x}^a) \|_{L^4} \leq C. \tag{67}
\]
Furthermore, taking \( p = 2 \) in (66), in view of (55) and (67), we get
\[
\sup_{0 \leq t \leq T} \| \nabla (\rho \bar{x}^a) \|_{L^2} \leq C.
\]
Remark 6. For the convenience of the readers, we state a detailed proof to the regularity of terms $\sqrt{\rho u_t}$ and $\nabla u_t$ from the estimates on $\dot{u}$ as follows.

Firstly, for any $p \geq 2$, it follows from (22), (30) and (52), we obtain the following fact.
\[
\|\sqrt{\rho u}\|_{L^p} \leq \|\rho\|_{L^1}^{\frac{1}{2} - \frac{1}{2p}} \|\rho u_t\|_{L^1}^{\frac{1}{2}} \|\dot{u}\|_{L^p} \leq C \left(\|\sqrt{\rho u}\|_{L^2} + \|\nabla u\|_{L^2}\right). \tag{68}
\]

Differentiating (1)$_2$ with respect to $t$ yields
\[
\rho u_{tt} + \rho u \cdot \nabla u_t - \mu \Delta u_t - (\mu + \lambda)\nabla \text{div} u_t
= -\rho \dot{u} - \rho u_t \cdot \nabla u - \nabla P_t. \tag{69}
\]

Furthermore, multiplying (69) by $u_t$ and integrating the resulting equation over $\mathbb{R}^2$. Combining with (1)$_1$, Hölder’s inequality, (15), (36), (44), (52), (55), (65) and (68) give
\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 \, dx + \int \left(\mu |\nabla u_t|^2 + (\mu + \lambda)(\text{div} u)^2\right) \, dx
= -\int \rho u \cdot \nabla u \cdot u_t \, dx - \int \rho u_t \cdot \nabla u \cdot u_t \, dx + \int P_t \text{div} u_t \, dx
\leq C \left(\|\sqrt{\rho u}\|_{L^2} \|\nabla u_t\|_{L^2} + \|\dot{u}\|_{L^2} + \|\text{div} u_t\|_{L^2}\right)
\leq C \left(\|\sqrt{\rho u}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\text{div} u_t\|_{L^2}\right)
\leq C \left(\|\sqrt{\rho u}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\text{div} u_t\|_{L^2}\right)
\leq C \left(\|\sqrt{\rho u}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\text{div} u_t\|_{L^2}\right)
\leq C \left(\|\sqrt{\rho u}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\text{div} u_t\|_{L^2}\right)
\leq C \left(\|\sqrt{\rho u}\|_{L^2} + \|\nabla u_t\|_{L^2}\right) + C,
\]
then
\[
\frac{d}{dt} \int \rho |u_t|^2 \, dx + \int \left(\mu |\nabla u_t|^2 + (\mu + \lambda)(\text{div} u)^2\right) \, dx 
\leq C \left(\|\sqrt{\rho u}\|_{L^2} + \|\nabla u\|_{L^2}\right) + C,
\]
which together with Gronwall’s inequality and (44) yields
\[
\sup_{0 \leq t \leq T} \|\sqrt{\rho u_t}\|_{L^2}^2 + \int_0^T \|\nabla u_t\|_{L^2}^2 \, dt \leq C.
\]

Proof of Theorem 1.1. With the estimates in Lemma 3.1-Lemma 3.7 and local existence in Proposition 1, we can extend the local strong solutions of $(\rho, u)$ beyond $T^*$ in the same way as that in [15], which contradicts the maximality of $T^*$, we thus complete the proof.

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