Some Supports of Fourier Transforms of Singular Measures are not Rajchman

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Abstract. The notion of Riesz sets tells us that a support of Fourier transform of a measure with non-trivial singular part has to be large. The notion of Rajchman sets tells us that if the Fourier transform tends to zero at infinity outside a small set, then it tends to zero even on the small set. Here we present a new angle of an old question: Whether every Rajchman set should be Riesz.

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1. Introduction

The consideration of the properties of measures and their Fourier transforms is a classical area of Harmonic Analysis. In particular the following is well known.

Theorem 1.1 (Rajchman, 1929 [4]). If for a finite measure $\mu$ on the unit circle $T$ holds $\hat{\mu}(n) \to 0$ when $n \to -\infty$, then it holds also that $\hat{\mu}(n) \to 0$ when $n \to +\infty$.

This motivates the following.

Definition 1.2. We say that $\Lambda \subset \mathbb{Z}$ is a Rajchman set if as soon as $\hat{\mu}(n) \to 0$ when $|n| \to +\infty, n \in \mathbb{Z} \setminus \Lambda$, then $\hat{\mu}(n) \to 0$ when $|n| \to +\infty, n \in \Lambda$.

With this definition the Rajchman theorem says that the non-negative integers is a Rajchman set.

Now, given a (signed) Radon measure $\mu$ on the unit circle $T$, we can present it as $\mu = f \cdot m + \mu_s$, where $m$ is the Lebesgue measure and $\mu_s$ is the

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singular with respect to Lebesgue measure part of the measure $\mu$. We known
the following.

**Theorem 1.3 (F. and M. Riesz’s, 1916, [5]).** If a finite measure $\mu$ has the
property $\hat{\mu}(-n) = 0$ for $n = 1, \ldots$, then the measure is absolutely
continuous with respect to Lebesgue measure, i.e. $\mu = f \cdot m$, where $f \in L^1(\mathbb{T})$.

This result motivates the following definition.

**Definition 1.4.** We say that a subset $\Lambda \subset \mathbb{Z}$ is a Riesz set if it has the property,
that if $\text{supp}(\hat{\mu}) \subset \Lambda$ then $\mu$ has no singular part.

With this definition the F. and M. Riesz theorem says that the non-negative integers is a Riesz set.

**Theorem 1.5 (Host, Parreau, 1978 [1]).** A set $\Lambda \subset \mathbb{Z}$ is a Rajchman set iff
it doesn’t contain any shift of the Fourier support of a Riesz product, i.e. any
set $\Omega((n_j)) = \{\sum \epsilon_j n_j : \epsilon_j = -1, 0, 1; \sum |\epsilon_j| < \infty\}$, where $(n_j)$ is an infinite sequence.

Thus, any set which is not Rajchman, contains the support of the Fourier
transform of a singular measure, and thus is not Riesz (or, without negations,
that every Riesz set is a Rajchman set).

A natural question is following: Is every Rajchman set a Riesz set? (i.e.
Do the classes of Riesz and Rajchman sets coincide?) As far to the author’s
knowledge, this question was first raised by Pigno, 1978 [3].

As we are unable to answer the question, we want to diversify it:

**Definition 1.6.** We say that a closed set $E \subset \mathbb{T}$ is a parisian set if for every
non absolutely continuous measure $\mu \in M(E)$, the support of it’s Fourier
transform is not a Rajchman set.

The original question thus becomes: Is $\mathbb{T}$ a pariscian set?

While we are not able to answer the question above, we can show that
some pariscian sets do exist. As any subset of a pariscian set is pariscian, it
is clear that a positive answer on the original question would imply all the
results we prove here. Yet, there are good chances that the answer is negative
and a negative answer would give the study of the pariscian sets some interest.

It is natural to expect that the pariscian sets should be “small”. Thus
we try to construct a “big” pariscian set.

**Main Theorem A.** For any $\alpha < 1$ there exists a closed pariscian set $E$, such
that $\dim_H(E) \geq \alpha$, where $\dim_H(E)$ means the Hausdorff dimension of $E$.

**Main Theorem B.** For any $\alpha < 1$ there exists a Borel pariscian set $E$ such
that it is an additive subgroup of $\mathbb{T}$ and $\dim_H(E) \geq \alpha$.

**Notations.** In what follows we identify $\mathbb{T}$ with $(-1, 1]$, so that the Fourier
coefficients are $\hat{\mu}(n) = \frac{1}{2} \int e^{i\pi nx} d\mu(x)$.

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1It is actually proven in [1] not only for $\mathbb{T}$ but for any compact group.
2. Construction of a big parisian set

Let us first introduce a test to establish that a set is parisian.

**Lemma 2.1.** If there exist $\delta > 0$ and a sequence $(N_j)_{j=1}^\infty$ such that for every $j$ the set $E_j$ is a subset of $\frac{2}{N_j} \mathbb{Z} + [-1/2N_j^{1+\delta}, 1/2N_j^{1+\delta}]$, then the set $E$ is parisian.

**Proof.** Let us fix $\mu \in M_s(E)$. We want to show that $\text{supp}(\hat{\mu})$ contains a shift of a set $\Omega((n_j))$. Up to a shift of the Fourier transform we may assume without loss of generality that $\hat{\mu}(0) \neq 0$.

Here we construct the sequence $(n_j)$ as a subsequence of $(N_j)$ inductively. Assume that $(k - 1)$ first terms of the sequence $(n_j)$ are chosen. This means that for all combinations of $\epsilon_j$ the sum $\sum_{j=0}^{k-1} \epsilon_j n_j \in \text{supp}(\hat{\mu})$. Thus, we know that $\int e^{i\pi \sum_{j=1}^{k-1} \epsilon_j n_j x} \, d\mu(x) \neq 0$, for all combinations $(\epsilon_j = -1, 0, 1)_{j=1}^{k-1}$.

We can take $\gamma_{k-1}$ to be the minimum of the absolute value of the $3^{k-1}$ non-zero numbers, so that $|\int e^{i\pi \sum_{j=1}^{k-1} \epsilon_j n_j x} \, d\mu(x)| \geq \gamma_{k-1}$. We want to show that for some sufficiently large $n_k = N_{jk}$ for all combinations of $\epsilon_j$ holds

$$\int e^{i\pi \sum_{j=1}^k \epsilon_j n_j x} \, d\mu(x) \neq 0.$$ 

Indeed, as $E \subset 2\mathbb{Z}/N_m + [-1/N_m^{1+\delta}, 1/N_m^{1+\delta}]$, we know that $|e^{i\pi (\pm N_m x)} - 1| \leq \frac{\pi}{N_m}$, when $x \in E$. Now we see that

$$|\int_E e^{i\pi \sum_{j=1}^k \epsilon_j n_j x} \, d\mu(x) - \int_E e^{i\pi \sum_{j=1}^{k-1} \epsilon_j n_j x} \, d\mu(x)| \leq \int_E |d\mu||e^{i\pi \pm N_m x} - 1| \leq \|\mu\| \frac{1}{N_m^\delta}.$$ 

Thus, for sufficiently large $m$ we can be sure that the later is less than $\frac{1}{2} \gamma_{k-1}$.

Now, we see that by the triangle inequality $|\int_E e^{i\pi \sum_{j=1}^k \epsilon_j n_j x} \, d\mu(x)| \geq \frac{1}{2} \gamma_{k-1} > 0$ for all the combinations of $\epsilon_j = -1, 0, 1$, with $j = 1, \ldots, k$, and $n_k = N_m$. $\square$

A slight modification of the proof gives us the following.

**Lemma 2.2.** For an increasing sequence $(N_j) \subset \mathbb{N}$ and $\delta > 0$ the set $\tilde{E} = \{x \in \mathbb{T} : \sup_j (\text{dist}(x, 2\mathbb{Z}/N_j)/N_j^{1+\delta}) < \infty\}$ is a parisian set.

**Proof.** We start from observing that $\tilde{E} = \bigcup_{t \in \mathbb{N}} E_t$, where

$$E_t = \{x \in \mathbb{T} : \sup_j (\text{dist}(x, 2\mathbb{Z}/N_j)/N_j^{1+\delta}) \leq t\}$$

is an increasing sequence of closed sets.

Now, we start the proof exactly as the previous one, but after the choice of $\gamma_{k-1}$ and before the choice of $n_k$ we do one more step: We pick
$t_k$ large enough that $\mu_k = \mu|_{E_k}$ satisfies $\|\mu - \mu_k\| < \frac{1}{3}\gamma_k - 1$. Then we see that $|\int e^{i\pi \sum_{j=1}^{k-1} \varepsilon_j n_j x} \, d\mu_k(x)| \geq \frac{2}{5}\gamma_k - 1$. We proceed in the same way as before with $\mu_k$ in place of $\mu$, and find $n_k = N_{m_k}$ such that $|\int_E e^{i\pi \sum_{j=1}^{k} \varepsilon_j n_j x} \, d\mu_k(x)| \geq \frac{1}{3}\gamma_k - 1$. Then, $|\int_E e^{i\pi \sum_{j=1}^{k} \varepsilon_j n_j x} \, d\mu(x)| \geq \frac{1}{6}\gamma_k - 1 > 0$. □

**Remark 2.3.** The set $\tilde{E}$ is obviously an additive subgroup of $\mathbb{T}$ and thus either finite or dense in $\mathbb{T}$.

Let us now construct a set $E$ of large Hausdorff dimension which satisfies the hypothesis of the Lemma 2.1, and is thus parianian. As the constructed set is a subset of $\tilde{E}$ it will also give us the estimate on the Hausdorff dimension of $\tilde{E}$. Fix $\alpha \in (0, 1)$, and choose $\delta > 0$ so that $\delta = 1 - \alpha$. We will construct a rapidly increasing sequence $\{N_j\}$, and related sequence of closed sets $C_j \subset (-1, 1)$, such that the sets $C_j$ is the union of the closed intervals with centrum in $2\mathbb{Z}/N_j$, of length $1/N_j^{1+\delta}$ which are entirely contained in $\bigcap_{k=1}^{j-1} C_k$. We will let then the set $E = \bigcap_j C_j$, which is obviously closed. The set constructed in such a way is a Cantor-type set, and we show that provided the sequence $N_j$ grows quickly enough the dimension of such a set is at least $\alpha$.

**Lemma 2.4.** $\dim_H(E) \geq \alpha$.

**Proof.** In order to prove that the Hausdorff dimension of $E$ is at least $\alpha$ we will show that it is at least $s$ for any $0 < s < \alpha$, and to do so we construct a finite measure $\mu$ supported on $E$ such that $\mu(I) \leq c_s |I|^s$ for any interval $I$ (it is a standard fact of Geometric Measure Theory that a measure satisfying such an estimate should have support of Hausdorff dimension at least $s$, see for example [2]).

Let us take a subset $D_k$ of $\bigcap_{j=1}^{k} C_j$, which is a collection of intervals of length $1/N_k^{1+\delta}$. This collection is defined inductively: we know that every interval of length $1/N_k^{1+\delta}$ contains at least $N_k/2N_k^{1+\delta} - 1$ points of $2\mathbb{Z}/N_k$. Thus, every interval of $D_{k-1}$ contains (entirely) at least $M_k = N_k/2N_k^{1+\delta} - 3$ intervals with centrum in $2\mathbb{Z}/N_k$ and length $1/N_k^{1+\delta}$. (To make the estimates more simple we assume $(N_k)$ to grow so rapidly that $M_k \geq N_k/4N_k^{1+\delta}$.)

We pick from each interval of $D_{k-1}$ exactly $M_k$ such intervals. All together we will have picked $M_k \prod_{j=1}^{k-1} M_j$ intervals of length $1/N_k^s$. Then we take the probability measure $\mu_k$ equally distributed on the $\prod_{j=1}^{k} M_j$ intervals of $D_k$.

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This estimate is well known, but we give the proof for the sake of completeness.
We introduce $\mu$ as a weak limit point of $\mu_k$ (which has to be a probability measure supported by $E = \cap C_j$).

Let us estimate $\mu(I)$ where $1/N_{k-1} > |I| \geq 1/N_k$. The interval can intersect at most $N_k|I|/2 + 3$ intervals of $D_k$ (as $N_k|I| \geq 1$, we may use that it is at most $4N_k|I|$ intervals). As the measure of each interval of $D_k$ is $1/\prod_{j=1}^{k-1} M_j = (N_k/N_1)/(4^{k-1}(\prod_{j=1}^{k-1} N_j)^{\delta})$.

Thus, $\mu(I) \leq N_1 4^k (\prod_{j=1}^{k-1} N_j)^{\delta}|I| = N_1 4^k (\prod_{j=1}^{k-1} N_k)^{\delta}|I|^{1-s}|I|^s$.

Our task is fulfilled if we show that $c_{k,s} = N_1 4^k (\prod_{j=1}^{k-1} N_j)^{\delta}|I|^{1-s}$ is bounded above independently from $k$. We know that $|I| < 1/N_{k-1}$, and, as $\delta = 1 - \alpha$, we see that $c_{k,s} \leq N_1 4^k (\prod_{j=1}^{k-2} N_j)^{\delta}/N_{k-1}^{\alpha-s}$. It remains to take the sequence $(N_k)$ such that $(N_1 N_k^{k+2} (\prod_{j=1}^{k} N_j)^{\delta})^k < N_{k+1}$. For any fixed $s$ the sequence $c_{k,s}$ tends to zero, and so is bounded. (Notice that the bound $c_s = \sup_k \{c_{k,s}\}$ grows as $s \to \alpha$, but we only need it to be finite.)

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