KLEINIAN GROUPS VIA STRICT HYPERBOLIZATION

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Abstract. In this paper, we construct Kleinian groups $\Gamma < \text{Isom}(\mathbb{H}^n)$ from the direct product of $n$ copies of the rank 2 free group $F_2$ via strict hyperbolization. We give a description of the limit set and its topological dimension. Such construction can be generalized to other right-angled Artin groups.

1. Introduction

A Kleinian group is a discrete isometry group of the $n$-dimensional hyperbolic space $\mathbb{H}^n$. There are a lot of ways to construct Kleinian groups. The most common ones are to use the Poincaré fundamental polyhedron theorem (see e.g. [11, 12]), the Klein-Maskit Combination Theorem (see e.g. [9, 11]), and to construct arithmetic groups and their subgroups (see e.g. [10, 13]). One can also deform a given Kleinian group or to find limits of sequences of Kleinian groups (see, e.g. [7] for a survey). In this paper, we construct Kleinian groups from the direct product of $n$ copies of the rank 2 free group $F_2 \times \cdots \times F_2 = F_n^2$ via the strict hyperbolization.

The strict hyperbolization introduced by Charney and Davis is a procedure which associates to a simplicial complex $K$ a piecewise hyperbolic space $G_X(K)$ of curvature $\leq -1$ [4]. One can use one of Gromov’s techniques to construct a polyhedron $\mathcal{H}(K)$ associated to $K$ which is a cubical cell complex where each cube is isometric to a regular Euclidean cube [6]. The key ingredient in the strict hyperbolization procedure is to replace the Euclidean cube in $\mathcal{H}(K)$ by an appropriate face of some compact, connected, orientable, hyperbolic manifold $X$ with corner. The manifold $X$ is obtained by cutting an arithmetic hyperbolic manifold $M$ along a system of codimension one submanifolds [4]. Our construction of the Kleinian groups in this paper relies on the arithmetic hyperbolic manifold $M$. In particular, we need to take some finite cover of the manifold $M$ if necessary to ensure the normal injectivity radii of some closed geodesics are large enough. For simplicity, we still denote the finite cover by $M$.

The direct product of $n$ copies of the rank 2 free group $F_2$ is the fundamental group of the direct product of $n$ copies of a wedge of two circles, which we denote by $W^n$. The complex $W^n$ is actually the Salvetti complex defined for the right-angled Artin group $F_n^2$. We refer the reader to the note [3] for an introduction of right-angled Artin groups and the Salvetti complex. The $n$-dimensional complex $W^n$ corresponds to an $n$-dimensional complex $Z$ embedded in the $2n$-dimensional arithmetic manifold $M^{2n}$ used in the strict hyperbolization. The inclusion map $f^* : Z \to M^{2n}$ induces a map $f'^* : \pi_1(Z) \to \pi_1(M^{2n})$. We prove:

**Theorem 1.1.** The map $f'^* : \pi_1(Z) \to \pi_1(M^{2n})$ is injective. Hence $\Gamma_n = \pi_1(Z)$ is a torsion free Kleinian group, i.e. a discrete torsion free isometry subgroup of Isom$(\mathbb{H}^{2n})$.

In general, one can use the method in the paper to construct Kleinian groups from other right-angled Artin groups such as the free abelian group $Z^m$ or the right-angled Artin groups represented by the $m$-gons.
In contrast to Kleinian groups in dimension 3, there is no comprehensive structure theory for higher dimensional Kleinian groups, i.e. \( n \geq 4 \). One way to study higher dimensional Kleinian groups is to see the geometric and topological properties of the limit set which is the accumulation set of an orbit in the visual boundary. For example, groups with zero dimensional limit sets are relatively well understood. We refer the readers to [7] for more details about the study of higher dimensional Kleinian groups. The limit set of the Kleinian groups \( \Gamma_n \) we construct in the paper is the closure of countably infinite many \((n-1)\)-dimensional spheres \( S^{n-1} \), i.e.

\[
\Lambda(\Gamma_n) = S \cup E
\]

where \( S \) is the union of the \((n-1)\)-dimensional spheres \( S^{n-1} \), and \( E \) is the rest of points in the limit set with the cardinality of the continuum. The points in the limit set are endpoints of piecewise geodesic rays which are uniform quasi-geodesics. For the detailed description, see Section 3 and Section 4.

The Kleinian group \( \Gamma_n \) is constructed via the right-angled Artin group \( F_2^n \). The boundary of \( F_2^n \) here is defined to be the visual boundary of the universal cover of \( W^n \), which is the join of \( n \) copies of the Cantor set. In fact, the boundary of \( F_2^n \) is well-defined independent of choice of the CAT(0) space on which the group acts geometrically [13]. This does not hold for general right-angled Artin groups [5]. On the contrary, for any nonelementary Kleinian group \( \Gamma \), its limit set \( \Lambda(\Gamma) \) cannot be a join of Cantor sets. However, we prove that the limit set \( \Lambda(\Gamma_n) \) contains the join of \( n \) copies of \( K_3 \) where \( K_3 \) is a set of three points, hence it cannot be embedded in \( \mathbb{R}^{2n-2} \), see [2, Lemma 9].

**Theorem 1.2.** The limit set \( \Lambda(\Gamma_n) \) cannot be embedded in \( \mathbb{R}^{2n-2} \).

On the other hand, It is interesting to ask what properties of the set \( \partial_\infty F_2^n \) is preserved in \( \Lambda(\Gamma_n) \). For example one can ask:

**Question 1.3.** Whether the support of the simplicial homology of \( \partial_\infty F_2^n \) is the same as the support of simplicial homology (or Čech cohomology) of \( \Lambda(\Gamma_n) \).

The homology group \( H_i(\partial_\infty F_2^n) \) is nonzero if and only if \( i = 0 \) or \( n - 1 \). We prove that:

**Theorem 1.4.** The topological dimension of \( \Lambda(\Gamma) \) equals \( n - 1 \).

**Corollary 1.5.** For any \( n \geq 2 \), \( H_{n-1}(\Lambda(\Gamma_n)) \) is nontrivial.

For \( 0 < i < n \), we cannot determine whether \( H_i(\Lambda(\Gamma_n)) \) vanishes or not.

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2. **Strict hyperbolization**

In this section, we review the strict hyperbolization introduced by Charney and Davis [4], which is used to construct the higher dimensional Kleinian groups \( \Gamma_n \subset \text{Isom}(\mathbb{H}^{2n}) \) corresponding to the right-angled Artin group \( F_2^n \).

Let \( B_n \) denote the symmetric group of the \( n \)-dimensional cube, and let \( r_i \) denote the linear reflection across the hyperplane \( x_i = 0 \) in \( \mathbb{R}^n \). The group \( B_n \) has standard action on \( \mathbb{R}^n \) generated by permutations of coordinates and the reflections \( r_i \).
Theorem 2.1. [4, Theorem 6.1] For each \( n \geq 0 \), there is a closed connected hyperbolic \( n \)-dimensional manifold \( M^n \), a system \( \mathcal{Y} = \{Y_1, \cdots, Y_n\} \) of closed connected submanifolds of codimensional one in \( M^n \), and an isometric action of \( B_n \) on \( M^n \), stabilizing \( \mathcal{Y} \), such that the following properties hold:

1. \( Y_i \) is a component of the fixed point set of \( r_i \) on \( M^n \).
2. Each \( Y_i \) is totally geodesic in \( M^n \).
3. The \( Y_i \)'s intersect orthogonally.
4. \( Y_1 \cap \cdots \cap Y_n \) is a single point \( y \).
5. \( B_n \) fixes \( y \) and the representation of \( B_n \) on \( T_y M^n \) is equivalent to the standard representation.
6. \( M^n \), as well as each \( Y_i \) is orientable.

Remark 2.2. The group \( B_n \) normalizes the group \( \pi_1(M^n) \) by the construction of \( M^n \), see [4, Section 6]. Hence, it acts on \( M^n \) as isometries. The key point to ensure \( Y_1 \cap \cdots \cap Y_n \) is a single point is to prove that \( \pi_1(M^n) \) is a torsion free congruence subgroup of some cocompact lattice \( O(\phi) < O(n,1) \). For details, see [4, Lemma 6.6].

Proposition 2.3. Given a constant \( R > 0 \), there exists a closed connected hyperbolic \( n \)-dimensional manifold \( M' \) and a closed geodesic \( \gamma \subset M' \) such that the normal injective radius of \( \gamma \) in \( M' \) is at least \( R \), and \( M' \) satisfies all the conditions in Theorem 2.1.

Proof. We start from the discrete cocompact lattice \( \Gamma = SO(n,1) \cap O(\phi) \) which is used to construct the arithmetic manifold \( M^n \) in Theorem 2.1. Choose a loxodromic isometry \( g \in \Gamma \), and let \( A \subset \mathbb{H}^n \) be an axis of \( g \) such that \( \gamma_1 = A/\langle g \rangle \) is a closed geodesic in \( M^n = \mathbb{H}^n / \Gamma \). Let \( C \) denote the \( R \)-neighborhood of \( \gamma_1 \) in \( M^n \). There are only finitely many isometries \( h_i \in \Gamma \) such that \( h_i(C) \cap C \neq \emptyset \) where \( i \in \{1, \cdots, m\} \) and \( h_i \notin \langle g \rangle \). One can choose a congruence subgroup \( \Gamma' < \Gamma \) which does not contain any of the finitely many commutators \([g,h_i]\). Then none of \( h_i \) is a power of \( g \) in the quotient \( \Gamma/\Gamma' \). The subgroup \( \Gamma' \) does not contain any of the finitely many isometries \( h_i \), but contains some power of \( g \) (say \( g^k \)). Since the subgroup \( \Gamma' \) is a congruence subgroup, the quotient manifold \( M' = \mathbb{H}^n / \Gamma' \) satisfies all the conditions in Theorem 2.1. The closed geodesic \( \gamma = A/\langle g^k \rangle \) has normal injective radius at least \( R \) in \( M' \).

By the same argument, we have:

Corollary 2.4. Given a constant \( R > 0 \) and \( m > 0 \), there exists a closed connected hyperbolic \( n \)-dimensional manifold \( M' \) and closed geodesics \( \gamma_1, \cdots, \gamma_m \subset M' \) such that their normal injective radii in \( M' \) are all at least \( R \), and \( M' \) satisfies all the conditions in Theorem 2.1.

3. The construction of Kleinian groups

Let \( W \) be the wedge of two circles whose fundamental group is the rank 2 free group \( F_2 \) generated by \( v_0, v_1 \). Let \( a_0, a_1 \) denote the two circles in \( W \). There is a map \( f : W^n = W \times W \times \cdots \times W \to T^{2n} \) where \( T^{2n} \) is the \( 2n \)-dimensional torus. Note that \( W^n \) is a \( n \)-dimensional CAT(0) complex and the map \( f \) induces \( f_* : \pi_1(W^n) = F_2^n \to \pi_1(T^{2n}) \). Note that \( f_* \) is not injective since \( F_2^n \) is not commutative.

We claim that \( W^n \) corresponds to an \( n \)-dimensional CAT(0) complex \( Z \) embedded in the arithmetic manifold \( M \) in Theorem 2.1.
Recall that for the arithmetic manifold \( M \), there is a smooth map \( \phi : M \to T^{2n} \) such that \( \mathcal{Y} \) is the transverse inverse image of the standard system of subtori in \( T^{2n} \). Let \( \Sigma_{i_1 \cdots i_n} = \phi^{-1}(f(a_{i_1} \times a_{i_2} \times \cdots \times a_{i_n})) \) where \( i_j \in \{0,1\} \). Note that \( f(a_{i_1} \times \cdots \times a_{i_n}) = T^n \subset T^{2n} \), and the submanifold \( \Sigma_{i_1 \cdots i_n} \) is an \( n \)-dimensional totally geodesic submanifold in \( M \). There are \( 2^n \) such \( n \)-dimensional submanifolds, and the intersection of any two such submanifolds is a totally geodesic submanifold \( \phi^{-1}(T^m) \) for some \( 0 \leq m \leq n - 1 \). The intersection of all these \( 2^n \) submanifolds is a single point, denoted by \( O \in M \). We let \( Z \) denote the union of these \( 2^n \) totally geodesic submanifolds \( \Sigma_{i_1 \cdots i_n} \).

By the discussion above, if the subscripts of two submanifolds \( \Sigma_{i_1 \cdots i_n} \) and \( \Sigma_{i'_1 \cdots i'_n} \) are different except at the \( k \)-th entry, i.e. \( i_j \neq i'_j \) for \( j \neq k \), and \( i_k = i'_k \). Then the intersection of \( \Sigma_{i_1 \cdots i_n} \) and \( \Sigma_{i'_1 \cdots i'_n} \) is a closed geodesic, denote by \( \gamma_k \) or \( \gamma'_k \) depending on the \( k \)-entry is 0 or 1. Then we have \( 2^n \) such closed geodesics and they intersect orthogonally with each other.

By Corollary 2.4 we assume that the normal injectivity radii of the \( 2^n \) closed geodesics \( \gamma_i, \gamma'_i \) are all at least \( 3L \) where \( L = 2\cosh^{-1}(2\sqrt{2}) + 1 \) in \( M \) up to some finite index. The constant \( L \) is the same constant as the one in \( \mathbb{S} \) Proposition 7.2) by letting \( \theta = \pi/2 \).

In analogue to the map \( f \), there exists a map \( f' : Z \to M \) and an induced map \( f'_1 : \pi_1(Z) \to \pi_1(M) = \Gamma \). In contrast to the map \( f_* \), we prove that \( f'_1 \) is injective.

**Proof of Theorem 1.1:** Pick an element \( \omega \in \pi_1(Z,O) \) corresponding to the geodesic loop \( w \). Then we can write \( \omega = \omega_1 \omega_2 \cdots \omega_k \) where \( \omega_i \in \pi_1(\Sigma_i, O) \) and \( \Sigma_i \) is one of the \( 2^n \) submanifolds \( \Sigma_{i_1 \cdots i_n} \). Write the loop \( w = w_1 \ast w_2 \ast \cdots \ast w_k \) where each \( w_i \) is a geodesic loop in \( \Sigma_i \) based on \( O \). Now we prove that \( f'_1(\omega) \) is not the identity for any nontrivial element \( \omega \).

Consider the universal cover \( \mathbb{H}^{2n} \) of \( M \), and a lift \( \tilde{w} \) of the geodesic loop \( w \) in \( \mathbb{H}^{2n} \). The bi-infinite path \( \tilde{w} = \tilde{w}_1 \ast \tilde{w}_2 \ast \cdots \ast \tilde{w}_k \ast \cdots \) is a piecewise geodesic path such that each segment \( \tilde{w}_i \) is a geodesic segment in a lift \( \tilde{\Sigma}_i \) of \( \Sigma_i \). Note that two consecutive segments \( \tilde{w}_i \) and \( \tilde{w}_{i+1} \) meet at one lift \( \tilde{O}_i \) of \( O \), and the intersection of the corresponding lifts of \( \Sigma_i \) and \( \Sigma_{i+1} \) is either a single point \( \tilde{O}_i \) or contains a geodesic which is a lift of one of the closed geodesics \( \gamma_k, \gamma'_k \). Moreover, these two lifts intersect orthogonally \( \mathbb{H} \) Corollary 6.2].

If the two lifts \( \tilde{\Sigma}_i \) and \( \tilde{\Sigma}_{i+1} \) intersect at a single point \( \tilde{O}_i \), then the consecutive geodesic segments \( \tilde{w}_i \) and \( \tilde{w}_{i+1} \) meet at \( \tilde{O}_i \) with angle \( \pi/2 \). If instead the intersection of these lifts contains a geodesic \( \tilde{\gamma}_i \), the angle between \( \tilde{w}_i \) and \( \tilde{w}_{i+1} \) can be arbitrarily small as in Figure 1. In this case, we need to replace \( \tilde{w}_{i+1} \) by a new path which is homotopic to \( \tilde{w}_{i+1} \). There are two cases depending on the intersection of the lifts \( \tilde{\Sigma}_{i+1} \) and \( \tilde{\Sigma}_{i+2} \) (a lift where \( \tilde{w}_{i+2} \) lies in):

- **Case (1):** Suppose that \( \tilde{\Sigma}_{i+1} \) intersects \( \tilde{\Sigma}_{i+2} \) at a single point \( \tilde{O}_{i+1} \). Then we replace \( \tilde{w}_{i+1} \) by the path \( \tilde{O}_i a_{i+1} \ast a_{i+1} \tilde{O}_{i+1} \) where \( \tilde{O}_{i+1} a_{i+1} \) is perpendicular to \( \tilde{\gamma}_i \) at \( a_{i+1} \).
- **Case (2):** The intersection of \( \tilde{\Sigma}_{i+1} \) and \( \tilde{\Sigma}_{i+2} \) contains a geodesic \( \tilde{\gamma}_{i+1} \). Note that both \( \tilde{O}_{i+1} \) and \( \tilde{O}_i \) are lifts of \( O \), so there exists an element \( g \in \Gamma \) such that \( \tilde{O}_{i+1} = g(\tilde{O}_i) \). Consider the geodesic \( g(\tilde{\gamma}_i) \). Then \( \tilde{\gamma}_{i+1} \) either is identified with \( g(\tilde{\gamma}_i) \) or intersects \( g(\tilde{\gamma}_{i+1}) \) orthogonally at \( \tilde{O}_{i+1} \). Let \( a_i b_i \) denote the shortest geodesic which is othogonal to both \( \tilde{\gamma}_i \) and \( g(\tilde{\gamma}_i) \). Then we replace \( \tilde{w}_{i+1} \) by \( \tilde{O}_i a_i \ast a_i b_i \ast b_i \tilde{O}_{i+1} \), which is homotopic to \( \tilde{w}_{i+1} \). Note that the distance between \( \tilde{\gamma}_i \) and \( g(\tilde{\gamma}_i) \) is at least \( 3L \) by construction. Hence the length of \( a_i b_i \) is at least \( 3L \). By repeating this process for each segment \( \tilde{w}_i \), we replace the bi-infinite path \( \tilde{w} \) by a new piecewise geodesic path \( \tilde{w}' \) which is homotopic to \( \tilde{w} \). We claim that \( \tilde{w}' \) is a quasi-geodesic. By the Morse lemma, the isometry \( \omega \) represented by \( \tilde{w}' \) and \( \tilde{w} \) is nontrivial.

Observe that the piecewise geodesic path \( \tilde{w}' \) contains long geodesics which arise from the large normal injectivity radii of the geodesics \( \gamma_k, \gamma'_k \). The remaining geodesic segments might be very short. Note that every two consecutive geodesic segments meet at the angle \( \pi/2 \) by the construction. These short geodesic segments locally look like the ones (e.g.
Suppose that the length of the long geodesic segments $3L \leq d(a_i, b_i) < 6L$. Actually if $d(a_i, b_i) \geq 6L$, take the point $a_{i1} \in a_ib_i$ such that $d(a_i, a_{i1}) = 3L$. If $d(a_{i1}, b_i) \geq 6L$, we continue the process until we get $a_{ij} \in a_ib_i$ such that $3L \leq d(a_{ij}, b_i) < 6L$. Thus we get a new partition of the piecewise geodesic path $\tilde{w}'$ such that the long geodesic segment has length in $[3L, 6L)$, and consecutive arcs meet either at the angle $\pi$ or the angle $\pi/2$. For the short arcs, if the length is $\geq L$, the path $\tilde{w}'$ is a uniform quasi-geodesic by [8, Proposition 7.2]. Hence we assume that the lengths of some short segments are $< L$.

In order to prove that $\tilde{w}'$ is $(A, B)$-quasigeodesic, with $A \geq 1$ and $B \geq 0$, we need to verify the inequality that

$$\frac{1}{A} \cdot \text{length}(\tilde{w}'|_{[t_a,t_b]}) - B \leq d(a,b) \leq A \cdot \text{length}(\tilde{w}'|_{[t_a,t_b]}) + B$$

for all pair of points $a, b \in \tilde{w}'$ where $\tilde{w}'(t_a) = a$ and $\tilde{w}'(t_b) = b$. The upper bound (for arbitrary $A \geq 1$ and $B \geq 0$) follows from the triangle inequality and we only need to establish the lower bound.

Consider the subpath $a_ib_i * b_i\tilde{O}_{i+1} * \tilde{O}_{i+1}a_{i+1} * a_{i+1}b_{i+1}$ with long segments $a_ib_i$ and $a_{i+1}b_{i+1}$. By the triangle inequality,

$$d(a_i, a_{i+1}) \geq 3L - L - L = L.$$ 

Observe that $a_{i+1}b_{i+1}$ and $a_i{a_{i+1}}$ meet at $a_{i+1}$ with angle $\pi/2$. The bisectors of the arc $a_{i+1}b_{i+1}$ and $a_i{a_{i+1}}$ are at least distance 2 apart by the similar argument of [S Proposition 7.2].

Suppose that the points $a = \tilde{w}'(t_a), b = \tilde{w}'(t_b)$ in $\tilde{w}'$ are terminal points of geodesic segments $\tilde{w}'_i, \tilde{w}'_j, i < j$. Note that $\tilde{w}'|_{[t_a,t_b]}$ contains at least $(j - i - 4)/3$ long geodesic segments like $a_ib_i$ in Figure 1. If two consecutive segments are both long segments, by the proof of [S Proposition 7.2], their bisectors are at least distance 2 apart. If there are 2 short geodesic segments (e.g. $b_i\tilde{O}_{i+1}, \tilde{O}_{i+1}a_{i+1}$) lying between the long geodesic segments (e.g. $a_ib_i$ and $a_{i+1}b_{i+1}$), then we consider the bisectors of $a_i{a_{i+1}}$ and $a_{i+1}b_{i+1}$ which are...
also at least distance 2 apart. Every pair of these bisectors divides \( ab \) into a small segment with length at least 2. By adding these lengths together, we obtain the inequality
\[
d(a, b) \geq \frac{2}{6}(j - i - 4),
\]
while
\[
\text{length}(\tilde{w}'|_{[t_a, t_b]}) \leq 6(j - i + 1)L.
\]
Putting these inequalities together, we obtain
\[
d(a, b) \geq \frac{1}{18L}\text{length}(\tilde{w}'|_{[t_a, t_b]}) - 4.
\]
Lastly, for general points \( a, b \in \tilde{w}', \tilde{w}_j \), they are within distance \( < 6L \) from the terminal endpoints of \( a', b' \) of these segments. Hence,
\[
d(a, b) \geq d(a', b') - 12L \geq \frac{1}{18L}\text{length}(\tilde{w}'|_{[t_a, t_b]}) - 4 - 12L \geq \frac{1}{18L}\text{length}(\tilde{w}'|_{[t_a, t_b]}) - (4 + 12L).
\]
\[
\square
\]

4. The limit set of the Kleinian group

By Theorem \ref{thm:1.1} \( \Gamma_n = \pi_1(Z) \) is a Kleinian group. In this section, we study the properties of the limit set \( \Lambda(\Gamma_n) \). Recall that \( Z \) is a CAT(0)-complex, and we let \( \hat{Z} \) denote its universal cover which is the union of the lifts of the \( n \)-dimensional submanifolds \( \Sigma_{i_1...i_n} \) in \( \mathbb{H}^{2n} \).

Given a point \( \hat{O} \in \hat{Z} \), two geodesic rays \( \rho_1 \) and \( \rho_2 \) in \( \hat{Z} \) are called asymptotic if they are at finite Hausdorff distance. The ideal boundary of the metric space \( \hat{Z} \) is the collection of equivalence classes of geodesic rays, and we denote it by \( \partial Z_{\text{CAT}(0)} \).

The visual topology \( \tau^t_{\hat{O}} \) on \( \partial Z_{\text{CAT}(0)} \) is generated by the basis of neighborhoods
\[
\{N'((\rho, \epsilon, R) \mid \rho \in \partial \hat{O} Z, \epsilon > 0, R > 0)\},
\]
where
\[
N'(\rho, \epsilon, R) = \{\rho' : d(\rho(R), \rho'(R)) < \epsilon\}, \text{ with } R \gg 1, \epsilon \ll 1
\]
and
\[
\partial \hat{O} Z := \{\rho : \rho \text{ is a geodesic ray in } \hat{Z} \text{ with } \rho(0) = \hat{O}\}.
\]

Fix a lift \( \hat{O} \) of \( O \) in \( \hat{Z} \). Consider the lifts of the \( n \)-dimensional submanifolds \( \Sigma_{i_1...i_n} \) passing through \( \hat{O} \). Each lift is one copy of the \( n \)-dimensional plane \( \mathbb{H}^n \) whose visual boundary is \( S^{n-1} \). We let \( S_{i_1...i_n} \) denote the visual boundary of the lift of \( \Sigma_{i_1...i_n} \) passing through \( \hat{O} \). Let \( S \) be the union of the \( 2^n \) spheres \( S_{i_1...i_n} \). Then we have
\[
\partial Z_{\text{CAT}(0)} = \overline{\Gamma(S)} = \Gamma(S) \cup E
\]
where \( E \) denote the remaining points not in \( \Gamma(S) \).

Each point in the visual boundary corresponds to a geodesic ray \( \rho \) emanating from \( \hat{O} \). By the construction of the universal cover, the geodesic rays travel along the lifts of the submanifolds \( \Sigma_{i_1...i_n} \). The points in \( \Gamma(S) \) correspond to the geodesic rays that stay in one lift after some time \( t \). Otherwise, if the geodesic rays keep travelling along different lifts as \( t \to \infty \), the endpoints lie in \( E \).

There is a natural surjective map \( i : \partial Z_{\text{CAT}(0)} \to \partial Z_{\mathbb{H}^{2n}} \) where \( \partial Z_{\mathbb{H}^{2n}} \) denotes the visual boundary of \( \hat{Z} \) embedded in \( \mathbb{H}^{2n} \), and actually this is the same as the limit set \( \Lambda(\Gamma_n) \) by Theorem \ref{thm:1.1}. We first prove that \( \partial Z_{\text{CAT}(0)} \) cannot be embedded in \( \partial \mathbb{H}^m \) for any \( m < 2n \), see Theorem \ref{thm:1.2}. We also compare \( \partial Z_{\text{CAT}(0)} \) and \( \partial Z_{\mathbb{H}^{2n}} \), proving that \( i \) is a homeomorphism, see Theorem \ref{thm:4.1}.
Proof of Theorem 1.2: Recall that a finite graph is planar if and only if it does not contain a subgraph that is a subdivision of the complete graph \( K_5 \) or the complete bipartite graph \( K_{3,3} \), which is known as Kuratowski’s theorem. In general, the complex \( *^nK_3 \) which is the join of \( n \) copies of three points \( K_3 \) cannot be embedded in \( \mathbb{R}^{2n-2} \), see [2], Lemma 9.

It suffices to prove that the limit set \( \Lambda(\Gamma_n) \) contains the complex \( *^nK_3 \).

We first consider the case that \( n = 2 \). Recall that every lift of the surface \( \Sigma_{i_1i_2} \) is a copy of the 2-dimensional plane \( \mathbb{H}^2 \) with ideal boundary \( S^1 \) where \( i_j \in \{0, 1\} \). Let \( \tilde{O} \) denote one lift of \( O \). The configuration of the ideal boundary of the lifts of the four surfaces \( \Sigma_{i_1i_2} \) passing through \( \tilde{O} \) is shown as in Figure 2, and the limit set \( \Lambda(\Gamma_2) \) contains this configuration. It is not hard to see that in Figure 2, the vertices \( A, B, C, D, E, F \) consist of a complete bipartite graph \( K_{3,3} \), hence, it is not planar. Therefore, \( \Lambda(\Gamma_2) \) cannot be embedded in \( \mathbb{R}^2 \).

We next use the induction on \( n \) to show that the lifts of the \( 2^n \)-dimensional submanifolds \( \Sigma_{i_1\ldots i_n} \) passing through \( \tilde{O} \) contains the subcomplex \( *^{n-1}K_3 \). Assume the claim holds for \( n-1 \). Recall the lifts of an \( n \)-dimensional submanifold \( \Sigma_{i_1\ldots i_n} \) are copies of the \( n \)-dimensional planes \( \mathbb{H}^n \) with ideal boundary \( S^{n-1} \) where \( i_j \in \{0, 1\} \). Consider the ideal boundary \( S_{i_1\ldots i_{n-1}0} \) of the lifts of the submanifolds \( \Sigma_{i_1\ldots i_{n-1}0} \) passing through \( \tilde{O} \). By the construction of the complex \( Z \) in Section 3, the intersection \( \bigcap S_{i_1\ldots i_{n-1}0} \) consists of two points \( A, A' \) which are the endpoints of the lift of closed geodesic \( \gamma_n \) passing through \( \tilde{O} \). By the assumption of the induction, \( \bigcup S_{i_1\ldots i_{n-1}0} \) contains \( *^{n-1}K_3 \), which indicates that it also contains the set \( *^{n-1}K_3 \setminus \{A, A'\} \). By the same reason, \( \bigcap S_{i_1\ldots i_{n-1}1} \) consists of two points \( B, B' \) which are the endpoints of the lift of the closed geodesic \( \gamma_0 \) passing through \( \tilde{O} \), and \( \bigcup S_{i_1\ldots i_{n-1}1} \) contains the complex \( *^{n-1}K_3 \). Hence, \( \bigcup S_{i_1\ldots i_n} \) contains \( *^{n-1}K_3 \setminus \{A, A', B, B'\} \), therefore it contains the complex \( *^nK_3 \).

\[ \square \]

\[ \text{Figure 2. local link} \]

**Theorem 4.1.** The map \( i : \partial Z_{\text{CAT}(0)} \to \partial Z_{\mathbb{H}^{2n}} \) is homeomorphic.

**Proof.** We first prove that the map \( i \) is injective. Consider two different geodesic rays \( \rho_1 \) and \( \rho_2 \) with different endpoints \( \xi_1, \xi_2 \in \partial Z_{\text{CAT}(0)} \). We first suppose that both \( \rho_1(t), \rho_2(t) \) keep staying in some lifts of the submanifolds \( \Sigma_{i_1\ldots i_n} \), i.e. \( \rho_1|_{[t, \infty)}, \rho_2|_{[t, \infty]} \) stay in some copies of
hyperbolic planes $\mathbb{H}^n$, respectively. Suppose that $\xi_1 = \xi_2 \in \partial \mathbb{H}^{2n}$. Then the intersection of these two lifts is non-empty, and we let $A$ denote one intersection point. Then the geodesic ray $A\xi$ lies in both of the lifts. By the $\delta$-hyperbolicity, there exists a constant number $K > 0$ such that the geodesic rays $p_1|_{[t, \infty)}$, $p_2|_{[t, \infty)}$ are within the $K$-neighborhoods of $A\xi$. Hence, the Hausdorff distance of $p_1|_{[t, \infty)}$ and $p_2|_{[t, \infty)}$ in the CAT(0) complex $Z$ is bounded by $2K$ which contradicts to our assumption that $\xi_1 \neq \xi_2 \in \partial Z_{\text{CAT}(0)}$.

Now we consider the case that $p_1$ keeps travelling along different lifts as $t \to \infty$. Suppose that there exists a lift $P_0$ such that the intersections $p_1 \cap P_0$ and $p_2 \cap P_0$ are nonempty and $p_1, p_2$ won’t stay in the same lift after $P_0$. Let $t_0$ denote the time when the geodesic ray $p_1$ starts to enter another lift $P_1$ different from $P_0$. Note that $p_2$ may enter another lift $P_2 \neq P_1$ which we call type 2 or stay in the same lift $P_0$ for the rest of the time which we call type 1. We claim that in both cases, we form a new bi-infinite piecewise geodesic path which is a quasi-geodesic with endpoints $\xi_1, \xi_2 \in \partial \mathbb{H}^{2n}$. By the Morse lemma, $\xi_1 \neq \xi_2$ in $\partial \mathbb{H}^{2n}$.

The intersection of $P_1$ and $P_0$ is either a point or contains a geodesic $\tilde{\gamma}_i$ as in the proof of Theorem 1.1. Assume that $p_2$ is type 1. Then we make a new bi-infinite piecewise geodesic path

$$p_3 = p_1|_{[t_0, \infty)} \ast ab \ast b p_2(\infty)$$

where $a = p_1(t_0)$ and $b$ is the unique intersection point of $P_0$ and $P_1$ or $bp_2(\infty)$ meets $\tilde{\gamma}_i$ orthogonally at $b$. By replacing the segments in $p_3$ as what we do in Theorem 1.1, we get the new path $p_3'$ which is a uniform quasi-geodesic. By $\delta$-hyperbolicity of the lifts, $p_3$ is within bounded neighborhood of $p_3'$. Hence, $p_3(\infty) = p_3'(\infty)$ and $p_3(-\infty) = p_3'(\infty)$. By the Morse lemma, $p_3'(\infty)$ is different from $p_3'(\infty)$ which means that $p_1(\infty)$ is different from $p_2(\infty)$.

If $p_2$ is type 2, assume that $p_2(t_0')$ is the starting point of the geodesic segment in $P_2$. We form an bi-infinite piecewise geodesic path

$$p_3 = p_1|_{[t_0, \infty)} \ast ab \ast b p_2[|t_0', \infty)$$

where $a = p_1(t_0)$ and $b = p_2(t_0')$. By the similar argument above, we have a new bi-infinite piecewise geodesic path $p_3'$ which is a quasi-geodesic and $p_3'(\infty) = p_3'(\infty), p_3'(\infty) = p_3(\infty)$.

Hence, the two endpoints of $p_1$ and $p_2$ are different.

We last check the case that both the geodesic rays $p_1, p_2$ are type 2, and they travel along the same lifts for all $t \in [0, \infty)$. By the similar argument to the previous case, there are piecewise geodesic paths $p_1', p_2'$ which are both quasi-geodesics in $\mathbb{H}^{2n}$ such that $p_i$ is within a uniform bounded neighborhood of $p_i'$ for $i = 1, 2$. Hence, $p_1'(\infty) = p_1(\infty)$ and $p_2'(\infty) = p_2(\infty)$. If $\xi_1 = \xi_2$ in $\partial X_{\mathbb{H}^{2n}}$, then $\partial \xi_1 = \partial \xi_2$, and the Hausdorff distance between $p_1'$ and $p_2'$ is also uniformly bounded by the Morse lemma. Therefore, there exist sufficiently large time $t_1, t_2, t_1', t_2'$ such that $p_1|_{[t_1, t_2]}$ and $p_2[|t_1', t_2']$ both lie in the same lift and the Hausdorff distance between these segments in the CAT(0)-complex is uniformly bounded which contradicts to the assumption that $p_1$ and $p_2$ are geodesic rays which are not equivalent in the CAT(0)-space.

We have proved that the map $i$ is one-to-one. It suffices to prove that the inverse map $i^{-1}: \partial Z_{\mathbb{H}^{2n}} \to \partial Z_{\text{CAT}(0)}$ is continuous in order to see that $i$ is a homeomorphism since both $\partial Z_{\mathbb{H}^{2n}}$ and $Z_{\text{CAT}(0)}$ are compact sets. We briefly recall the visual topology $\tau_{\tilde{O}}$ on $\partial Z_{\text{CAT}(0)}$, which is generated by the basis of neighborhood

$$\{ N'(\rho, \epsilon, R) \mid \rho \in \partial \tilde{O}, \epsilon > 0, R > 0 \}.$$
are within uniform neighborhoods of geodesic rays $\rho(0)\rho(\infty)$. This means that $i$ maps $N'(\rho, \epsilon, R)$ to an open set $N(\rho(0)\rho(\infty), \epsilon', R)$ in $\partial Z_{\mathbb{H}^{2n}}$, which indicates that $i^-$ is continuous. □

**Corollary 4.2.** The topological dimension of $\Lambda(\Gamma_n)$ equals $n - 1$.

*Proof.* Note that the topological dimension of $\partial Z_{\text{CAT}(0)}$ equals $n - 1$ [1, Theorem 1.7]. Then the corollary follows straightforward from Theorem 4.1. □

**Corollary 4.3.** For any $n \geq 2$, $H_{n-1}(\Lambda(\Gamma_n))$ is nontrivial.

*Proof.* Recall that any $(n - 1)$-dimensional sphere $S_{i_1\cdots i_n}$ generates a cycle, and it is nontrivial in homology by Corollary 4.2. □

**References**

[1] M. Bestvina. Local homology properties of boundaries of groups. *Michigan Math. J.*, 43(1):123–139, 1996.

[2] M. Bestvina, M. Kapovich, and B. Kleiner. Van Kampen’s embedding obstruction for discrete groups. *Invent. Math.*, 150(2):219–235, 2002.

[3] R. Charney. An introduction to right-angled Artin groups. *Geom. Dedicata*, 125:141–158, 2007.

[4] R. M. Charney and M. W. Davis. Strict hyperbolization. *Topology*, 34(2):329–350, 1995.

[5] C. B. Croke and B. Kleiner. Spaces with nonpositive curvature and their ideal boundaries. *Topology*, 39(3):549–556, 2000.

[6] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[7] M. Kapovich. Kleinian groups in higher dimensions. In *Geometry and dynamics of groups and spaces*, volume 265 of *Progr. Math.*, pages 487–564. Birkhäuser, Basel, 2008.

[8] M. Kapovich and B. Liu. Geometric finiteness in negatively pinched Hadamard manifolds. *Ann. Acad. Sci. Fenn. Math.*, 44(2):841–875, 2019.

[9] S. L. Krushkal, B. N. Apanasov, and N. A. Gusevskii. *Kleinian groups and uniformization in examples and problems*, volume 62 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1986. Translated from the Russian by H. H. McFaden, Translation edited and with a preface by Bernard Maskit.

[10] C. Maclachlan and A. W. Reid. *The arithmetic of hyperbolic 3-manifolds*, volume 219 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.

[11] B. Maskit. *Kleinian groups*, volume 287 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1988.

[12] J. G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, Cham, third edition, [2019] ©2019.

[13] K. E. Ruane. Boundaries of $\text{CAT}(0)$ groups of the form $\Gamma = G \times H$. *Topology Appl.*, 92(2):131–151, 1999.

[14] E. B. Vinberg and O. V. Shvartsman. Discrete groups of motions of spaces of constant curvature. In *Geometry, II*, volume 29 of *Encyclopaedia Math. Sci.*, pages 139–248. Springer, Berlin, 1993.

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