On a conjecture of Braverman and Kazhdan

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Abstract

In this paper a proof of Conjecture 9.12 of Braverman–Kazhdan in their article $\gamma$-functions of representations and lifting on the acyclicity of their $\ell$-adic $\gamma$-sheaves over certain affine spaces is given for $GL(n)$.

Introduction

Classical Fourier transforms on vector spaces over local fields and adelic rings have found remarkable connections with the standard $L$-functions $L(s, \pi, \text{std})$ of $GL(n)$ since Tate [T50] for $n = 1$ and Godement–Jacquet [GJ72] for general $n$.

More generally, for each reductive group $G$ which is quasi-split over a nonarchimedean local field $F_v$ and each representation $\rho$ of its dual group $L^G$ satisfying some mild technical conditions, there exists a $\rho$-analogue of the Fourier transform which is essentially the operator of convolution by an invariant distribution $\Phi_{\psi, \rho}$ on $G(F_v)$ where $\psi$ is a fixed additive character of $F_v$, whose operator-valued Mellin transform $M(\Phi_{\psi, \rho} * (\bullet))$ is the scalar operator of multiplication by the $\gamma$-function $\gamma_{\psi, \rho}(\pi_v)$ investigated by Braverman–Kazhdan in [BK00]. Similar ideas have been developed further by L. Lafforgue, see for example [L13].

Incarnations $\Phi_{\psi, \rho}$ of $\Phi_{\psi, \rho}$ as $\ell$-adic perverse sheaves over finite fields have been constructed and studied by Braverman–Kazhdan in the last section of [BK00] and subsequently in [BK02]. The purpose of this paper is to establish Conjecture 9.12 in [BK00] for $GL(n)$. The argument generalizes that of Braverman–Kazhdan in [BK02] for $GL(2)$. Following the classical paradigm the generalization from $GL(2)$ to $GL(n)$ involves mirabolic groups as an essential ingredient.

Conventions   In this paper $k$ is an algebraic closure of a finite field $k_0$ with $q$ elements of characteristic $p$. Let $\ell$ be a prime number which is distinct from $p$, let $\overline{Q}_\ell$ be an algebraic closure of the field of $\ell$-adic numbers.
If \( X \) is a \( k \)-scheme, let \( D^b_c(X) \) denote the derived category of complexes of \( \overline{\mathbb{Q}}_\ell \)-étale sheaves on \( X \) with bounded constructible cohomology, let \([d] \) denote the \( d \)th translation functor on \( D^b_c(X) \). If \( f \) is a \( k \)-linear morphism of \( k \)-schemes, the six functors \( f^*, f_*, f^!, f_!, \otimes_X \) and \( \mathcal{H}om_X \) are understood in the derived sense. If \( j \) is the morphism of inclusion of an open \( k \)-subscheme, let \( j_\ast \) denote the intermediate extension functor of Goresky–MacPherson for \( \overline{\mathbb{Q}}_\ell \)-perverse sheaves (see [BBD82]).

We will denote by \( \mathbb{G}_a \) the additive group defined over \( k \). It has the Artin–Schreier covering, which is a torsor under the finite group \( k_0 \), given by the Lang isogeny \( L_{\mathbb{G}_a} : \mathbb{G}_a \to \mathbb{G}_a \) with \( L_{\mathbb{G}_a}(t) = t^q - t \). We fix a nontrivial character \( \psi : k_0 \to \overline{\mathbb{Q}}_\ell^\times \) and denote by \( L_\psi \) the rank one \( \overline{\mathbb{Q}}_\ell \)-local system attached to \( \psi \) obtained by pushing out the Artin–Schreier covering.

Similarly, we denote by \( L_c \) the the rank one \( \overline{\mathbb{Q}}_\ell \)-local system on the multiplicative group \( \mathbb{G}_m \) obtained by pushing out the Kummer covering \( L_{\mathbb{G}_m} : \mathbb{G}_m \to \mathbb{G}_m \) with \( L_{\mathbb{G}_m}(t) = t^{q-1} \).

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1 Katz’s hypergeometric sheaves

Let $T$ be a torus defined over $k$ and $\lambda : \mathbb{G}_m \to T$ a nontrivial cocharacter. We note that $\lambda$ is then necessarily a finite morphism so that we have

$$\Psi(\lambda) = \lambda_*(j^*L_\psi[1]) = \lambda_*(j^*L_\psi[1]).$$

where $j : \mathbb{G}_m \to \mathbb{G}_a$ is the inclusion morphism from the multiplicative group into the additive group. We will call $\Psi(\lambda)$ the hypergeometric sheaf attached to $\lambda : \mathbb{G}_m \to T$. They are perverse sheaves on $T$.

Following Katz [K90, Chapter 8], we construct general hypergeometric sheaves on $T$ by convolving the $\Psi(\lambda)$. We recall that convolution products on $T$ are constructed as direct images with respect to the multiplication morphism $\mu : T \times T \to T$ with $\mu(t_1, t_2) = t_1 t_2$. There are two convolution products attached to the direct image functors with or without the compact support condition: for every $F, G \in D^b_c(T)$ we define

$$F \star G = \mu_!(F \boxtimes G)$$
$$F \bullet G = \mu_*(F \boxtimes G)$$

related by the morphism of functors

$$F \star G \to F \bullet G$$

that consists in forgetting the compact support condition. For every collection of possibly repeated nontrivial cocharacters $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r)$, we consider the convolution products

$$\Psi_{\underline{\lambda}} = \Psi(\lambda_1) \cdots \Psi(\lambda_r)$$
$$\Psi^*_{\underline{\lambda}} = \Psi(\lambda_1) \cdots \Psi(\lambda_r)$$

and the forget support morphism

$$\Psi_{\underline{\lambda}} \to \Psi^*_{\underline{\lambda}}. \quad (1)$$

If we denote by $p_{\underline{\lambda}} : \mathbb{G}_m^r \to T$ the homomorphism given by

$$p_{\underline{\lambda}}(t_1, \ldots, t_r) = \prod_{i=1}^r \lambda_i(t_i)$$

and by $\text{tr} : \mathbb{G}_m^r \to \mathbb{G}_a$ the addition morphism $\text{tr}(t_1, \ldots, t_r) = \sum_{i=1}^r t_i$, then

$$\Psi_{\underline{\lambda}} = p_{\underline{\lambda}} \cdot \text{tr}^* L_\psi[r]$$
$$\Psi^*_{\underline{\lambda}} = p_{\underline{\lambda}}^* \cdot \text{tr}^* L_\psi[r].$$
For $T = \mathbb{G}_m$ and $\lambda_i : \mathbb{G}_m \to \mathbb{G}_m$ being the identity for all $i$, $\Psi_\Delta$ is the $r$-fold Kloosterman sheaf considered by Deligne in [D77s]. In general, this is what Braverman and Kazhdan have called $\gamma$-sheaves on tori in [BK02].

We will restrict ourselves in a setting where the morphism (1) is an isomorphism. Let $\sigma : T \to \mathbb{G}_m$ be a character. A cocharacter $\lambda : \mathbb{G}_m \to T$ is said to be $\sigma$-positive if the composition $\sigma \circ \lambda : \mathbb{G}_m \to \mathbb{G}_m$ is of the form $t \mapsto t^n$ where $n$ is a positive integer.

**Proposition 1.1.** Assume that $\lambda_1, \ldots, \lambda_n$ are $\sigma$-positive. Then the forget-support morphism $\Psi_\Delta \to \Psi_\Delta^*$ is an isomorphism. Moreover, $\Psi_\Delta$ is a perverse local system over the image of $p_\Delta$, which is a subtorus of $T$.

**Proof.** See Appendix B.

Let $\Sigma_\Delta$ denote the subgroup of the symmetric group $\mathfrak{S}_r$ consisting of permutations $\tau \in \mathfrak{S}_r$ such that for all $i \in \{1, \ldots, r\}$, we have $\lambda_{r(i)} = \lambda_i$. This subgroup is of the form $\Sigma_\Delta = \mathfrak{S}_{r_1} \times \cdots \times \mathfrak{S}_{r_m}$ where $(r_1, \ldots, r_m)$ is the partition of $r$ corresponding to positive number of occurrences in $\{\lambda_1, \ldots, \lambda_r\}$.

**Proposition 1.2.** The group $\Sigma_\Delta$ acts on $\Psi_\Delta$ via the sign character.

**Proof.** This is [D77s, Proposition 7.20].

**Proposition 1.3.** Let $\mathcal{L}$ be a Kummer local system on $T$. Then if $\lambda_1, \ldots, \lambda_n$ are $\sigma$-positive, we have

$$H^i_c(\Psi_\Delta \star \mathcal{L}) = 0$$

for $i \neq 0$ and $\dim H^0_c(\Psi_\Delta \star \mathcal{L}) = 1$. Moreover, there is a canonical isomorphism

$$\Psi_\Delta \star \mathcal{L} = H^0_c(\Psi_\Delta \star \mathcal{L}) \otimes \mathcal{L}.$$

**Proof.** This is [BK02, Theorem 4.8].

## 2 Braverman–Kazhdan’s $\gamma$-sheaves

Let $G$ be a reductive group over $k$. Let $T$ be a maximal torus of $G$, $B$ a Borel subgroup containing $T$ and $U$ the unipotent radical of $B$. Let $W = \text{Nor}_G(T)/T$ denote the Weyl group of $G$, $\text{Nor}_G(T)$ being the normalizer of $T$ in $G$. The group of cocharacters $\Lambda = \text{Hom}(\mathbb{G}_m, T)$ is a free abelian group of finite type equipped with an action of $W$. The complex dual group $\hat{G}$ is equipped with a maximal torus $\hat{T}$ and a Borel subgroup $\hat{B}$ containing $\hat{T}$. We have $\Lambda = \text{Hom}(\hat{T}, \mathbb{C}^\times)$. 
We will recall the construction, due to Braverman and Kazhdan, of the $\gamma$-sheaf attached to a representation of the dual group $\hat{G}$. Let $\rho : \hat{G} \to \text{GL}(V_\rho)$ be an $r$-dimensional representation of $\hat{G}$. The restriction of $\rho$ to $\hat{T}$ is diagonalizable i.e. there exists a finite set of weights

$$\{\lambda_1, \ldots, \lambda_m\} \subset \Lambda = \text{Hom}(\hat{T}, \mathbb{C}^\times)$$

such that there is a decomposition into direct sum of eigenspaces

$$V_\rho = \bigoplus_{i=1}^m V_{\lambda_i},$$

with $\hat{T}$ acting on $V_{\lambda_i}$ by the character $\lambda_i$. The integers $r_i = \dim(V_{\lambda_i})$ define a partition $r = r_1 + \cdots + r_m$. We will denote

$$\underline{\lambda} = (\underbrace{\lambda_1, \ldots, \lambda_1}_{r_1}, \ldots, \underbrace{\lambda_m, \ldots, \lambda_m}_{r_m}) \in \Lambda^r$$

where $\lambda_1, \ldots, \lambda_m$ appear in $\underline{\lambda}$ with multiplicity $r_1, \ldots, r_m$ respectively.

By choosing a basis $A_i = \{v_{i,j}, 1 \leq j \leq r_i\}$ of each $V_{\lambda_i}$, we obtain a basis $A = A_1 \sqcup \ldots \sqcup A_m$ of $V_\rho$. The Weyl group of $\text{GL}(V_\rho)$ can be identified with the symmetric group $\text{Perm}(A) = \mathfrak{S}_r$ of permutations of the finite set $A$. Let

$$\Sigma_{\underline{\lambda}} = \mathfrak{S}_{r_1} \times \cdots \times \mathfrak{S}_{r_m} \subset \mathfrak{S}_r$$

denote the subgroup consisting of $\tau \in \text{Perm}(A)$ such that $\tau(A_i) = A_i$.

Let $\Sigma'_{\underline{\lambda}}$ denote the subgroup of $\text{Perm}(A)$ consisting of permutations $\tau$ such that there exists a permutation $\xi \in \mathfrak{S}_r$ such that $\tau(A_i) = A_{\xi(i)}$ for all $i \in \{1, \ldots, m\}$. The application $\tau \mapsto \xi$ defines a homomorphism $\Sigma'_{\underline{\lambda}} \to \mathfrak{S}_m$ whose kernel is $\Sigma_{\underline{\lambda}}$. Its image consists of permutations $\xi \in \mathfrak{S}_m$ preserving the function $i \mapsto r_i$.

The Weyl group $W$ operates on $\Lambda$ and its action preserves the subset $\{\lambda_1, \ldots, \lambda_m\}$ of $\Lambda$. It induces a homomorphism $W \to \mathfrak{S}_m$. Its image is contained in the subgroup of $\mathfrak{S}_m$ of permutations preserving the function $i \mapsto r_i$ so that there is a canonical homomorphism

$$\rho_W : W \to \Sigma'_{\underline{\lambda}}/\Sigma_{\underline{\lambda}}.$$
We derive an extension $W'$ of $W$ by $\Sigma_\lambda$ fitting into the diagram

$$
\begin{array}{cccc}
1 & \rightarrow & \Sigma_\lambda & \rightarrow & W' & \rightarrow & W & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \Sigma_\lambda & \rightarrow & \Sigma'_\lambda & \rightarrow & \Sigma'_\lambda/\Sigma_\lambda & \rightarrow & 1
\end{array}
$$

where an element $w' \in W'$ consists of a pair $(w, \xi)$ with $w \in W$ and $\xi \in \Sigma'_\lambda$ such that $\rho_W(w) = \xi \mod \Sigma_\lambda$. One can check that the homomorphism $p_\lambda : \mathbb{G}_m^r \to T$, and its dual $\rho_{|T'} : \check{\mathbb{T}} \to (\mathbb{C}^\times)^r$, are $W'$-equivariant.

As in Section 1, the finite sequence of $\sigma$-positive weights $\Delta \in \Lambda^r$ gives rise to a hypergeometric sheaf $\Psi_\lambda$ on $T$, equipped with an action of $\Sigma_\lambda$. This hypergeometric sheaf is well behaved under certain positivity condition that can be phrased in the present circumstance as follows. Let $\sigma : G \to \mathbb{G}_m$ be a character of $G$, we also denote $\sigma : \mathbb{C}^\times \to \hat{G}$ the dual cocharacter. A representation $\rho : \check{\mathbb{G}} \to \text{GL}(V_\rho)$ is said to be $\sigma$-positive if for every weight $\lambda_i$ occurring in $V_\rho$, $\lambda_i \circ \sigma : \mathbb{C}^\times \to \mathbb{C}^\times$ is of the form $t \mapsto t^n$ where $n$ is a positive integer. We will assume that $\rho$ is $\sigma$-positive. We will also assume that the homomorphism $p_\lambda : \mathbb{G}_m^r \to T$ is surjective. Under these assumptions, we know that $\Psi_\lambda$ is a local system on $T$ with the degree shift $[\dim(T)]$.

As the homomorphism $p_\lambda : \mathbb{G}_m^r \to T$ is $W'$-equivariant and the morphism $\text{tr} : \mathbb{G}_m^r \to \mathbb{G}_a$ is invariant under the action of $W'$, we have an action of $W'$ on the hypergeometric sheaf $\Psi_\Delta = p_\Delta^* \text{tr}^* \mathcal{L}_\psi[r]$ compatible with the action of $W'$ on $T$ via $W' \to W$: for every $w' = (w, \xi)$ with $w \in W$ and $\xi \in \Sigma'_\lambda$ having the same image in $\Sigma'_\lambda/\Sigma_\lambda$, by [BK02, Proposition 6.2] we have an isomorphism,

$$
\iota_{w'}^r : w^* \Psi_\Delta \to \Psi_\Delta.
$$

We also know that the restriction of this action to $\Sigma_\lambda$ is the sign character i.e. for $w' = (1, \xi)$ with $\xi \in \Sigma_\lambda$, we have $\iota_{w'}^r = \text{sign}(\xi)$, $\xi$ being considered as a permutation of the finite set $A$. For every $w' = (w, \xi)$ we set

$$
\iota_{w'} = \text{sign}_r(\xi) \text{sign}_W(w) \iota_{w'}^r : w^* \Psi_\Delta \to \Psi_\Delta
$$

where $\text{sign}_r : \mathbb{G}_r \to \{\pm 1\}$ and $\text{sign}_W : W \to \{\pm 1\}$ are the sign characters of $\mathbb{G}_r$ and $W$ respectively. We can then check that $\iota_{w'}$ depends only on $w$ so that we get an action of $W$ on $\Psi_\lambda$.

Now we recall the Grothendieck–Springer simultaneous resolution of the
fibers of the Steinberg morphism (see [S65, Section 6]) $c : G \to S = T/W$:

$$
\begin{array}{ccc}
\tilde{G} & \xrightarrow{\tilde{c}} & T \\
\downarrow{\tilde{q}} & & \downarrow{q} \\
G & \xrightarrow{c} & S
\end{array}
$$

where $\tilde{G}$ is the variety of pairs $(g, h) \in G \times G/B$ such that $h^{-1}gh \in B$, the morphism $\tilde{c}$ given by $(g, h) \mapsto h^{-1}gh \mod U$ is smooth, and the morphism $\tilde{q}$ given by $(g, h) \mapsto g$ is proper and small in the sense of Goresky–MacPherson. If $T^{rss}$ is the largest open subset of $T$ where $W$ acts freely and $S^{rss} = T^{rss}/W$, then the diagram is Cartesian over $S^{rss}$. In particular $\tilde{G}^{rss} \to G^{rss}$ is a $W$-torsor, where $j^{rss} : G^{rss} \to G$ denotes the base change of the inclusion morphism $S^{rss} \subset S$ to $G$.

Recall that the induction functor $\text{Ind}^G_T : \text{D}^b_c(T) \to \text{D}^b_c(G)$ is defined by

$$
\text{Ind}^G_T(F) = \tilde{q}^*\tilde{c}^*F[d]
$$

where $d = \text{dim}(G) - \text{dim}(T)$. Because $q$ is a small map and $\Psi_\lambda$ is a perverse local system on $T$,

$$
\text{Ind}^G_T(\Psi_\lambda) = \tilde{q}^*\tilde{c}^*\Psi_\lambda[d]
$$

is a perverse sheaf isomorphic to the intermediate extension of its restriction to $G^{rss}$:

$$
\text{Ind}^G_T(\Psi_\lambda) = j^{rss!*}\text{Ind}^G_T(\Psi_\lambda).
$$

For $\tilde{G}^{rss} \to G^{rss}$ is a $W$-torsor and $\Psi_\lambda$ is $W$-equivariant, $W$ operates on $j^{rss!*}\text{Ind}^G_T(\Psi_\lambda)$. By functoriality of the intermediate extension functor, this action of $W$ can be extended to the perverse sheaf $\text{Ind}^G_T(\Psi_\lambda)$.

**Definition 2.1.** The $\gamma$-sheaf attached to $\rho$ is the $W$-invariant direct factor of the perverse sheaf $\text{Ind}^G_T(\Psi_\lambda)$

$$
\Phi_\rho = \text{Ind}^G_T(\Psi_\lambda)^W.
$$

As a direct factor of $\text{Ind}^G_T(\Psi_\lambda)$, $\Phi_\rho$ is also isomorphic to the intermediate extension of its restriction to $G^{rss}$. There is thus a slightly different way to construct it: we start by descending the restriction of $\Psi_\lambda$ to $T^{rss}$, which is a $W$-equivariant perverse local system on $T^{rss}$, to a perverse local system $\Phi_{\lambda,S^{rss}}$ on $S^{rss}$. Then we have

$$
\Phi_\rho = j^{rss!*}c^{rss!*}\Phi_{\lambda,S^{rss}}[d].
$$
The induction functor admits a left adjoint \( \text{Res}_G^T : D^b_c(G) \to D^b_c(T) \), the restriction functor, defined by
\[
\text{Res}_G^T(F) = \pi_! i^*(F) \quad (3)
\]
where \( \pi : B \to B/U = T \) and \( i : B \to G \) denote the quotient and inclusion morphisms. More generally \( \text{Res}_M^G : D^b_c(G) \to D^b_c(M) \) could be defined if we replace \( B \) by a standard parabolic subgroup \( P \) of \( G \) and \( T \) by the Levi component \( M \) of \( P \) in (3). The adjunction between restriction and induction and Frobenius reciprocity imply the following

**Proposition 2.2.** Let \( \Phi_{\rho,M} \) denote the perverse sheaf \( \Phi_{\rho'} \) on \( M \) where \( \rho' \) denotes the restriction of \( \rho \) from \( \hat{G} \) to \( \hat{M} \), then
\[
\Phi_{\rho,M} \simeq \text{Res}_M^G(\Phi_{\rho}).
\]

**Proof.** This is the first statement of [BK02, Theorem 6.6]. \( \square \)

In [BK00, Conjecture 9.2] Braverman and Kazhdan have conjectured the following vanishing property of \( \Phi_\rho \).

**Conjecture 2.3.** Let \( \pi_U : G \to G/U \) denote the quotient map. For every \( \sigma \)-positive representation \( \rho \) of \( \hat{G} \), \( \pi_U_* \Phi_\rho \) is supported on the closed subset \( T = B/U \) of \( G/U \).

The conjecture can be reformulated as follows. For every geometric point \( g \in G - B \), we conjecture that
\[
H^*_c(gU, i^* \Phi_\rho) = 0
\]
where \( i : gU \to G \) denotes the inclusion map.

Braverman and Kazhdan have verified their conjecture for groups of semisimple rank one, and for \( G = \text{GL}(n) \), \( \sigma = \det \) and \( \rho \) the standard representation of \( \text{GL}_n(\mathbb{C}) \).

**Theorem 2.4.** The above conjecture holds for \( G = \text{GL}(n) \), \( \sigma = \det \) and an arbitrary \( \sigma \)-positive representation of \( \rho \) of \( \text{GL}_n(\mathbb{C}) \).

Our argument applies in the case when \( p_{\Delta} : G^\vee_n \to T \) is surjective, however for \( \sigma \)-positive \( \rho \) the only other possibility is when \( \rho \) factors through \( \det : \text{GL}_n(\mathbb{C}) \to \mathbb{C}^\times \), which implies that \( \Phi_\rho \) is supported on the center of \( \text{GL}(n) \), so the theorem holds trivially in this case as well.
3 Conjugation action of the mirabolic group

Our proof of the Braverman–Kazhdan conjecture for $G = \text{GL}(n)$ is based on the geometry of the conjugation action of the mirabolic. This geometry has been described by Bernstein in [B84, 4.1-4.2]. What we aim for here is to put Bernstein’s description into a form suitable for our purpose. For consistency of notations with [B84] we will follow Bernstein and consider left group actions and left quotients for the rest of this paper. In particular our version of Theorem 2.4 applies to $U\backslash G$ instead of $G/U$.

Let $V$ be the standard $n$-dimension $k$-vector space with the standard basis $e_1, \ldots, e_n$. We consider the filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = V$ where $F_i$ is the subvector space generated by $e_1, \ldots, e_i$. We will denote $E_i = V/F_i$.

We denote by $Q$ the mirabolic subgroup of $G = \text{GL}(V)$ consisting of elements $g \in G$ fixing the line generated by $e_1$, and $Q_1$ the subgroup consisting of elements $g \in G$ fixing the vector $e_1$. We consider the $Q$-equivariant stratification of $G$

$$G = \bigsqcup_{m=1}^{n} X_m$$

where $X_m$ is the locally closed subset of $V$ consisting of $g \in G$ such that the subspace $F_x$ of $V$ generated by the vectors $v, xv, x^2v, \ldots$ is of dimension $m$. We will prove that $[Q_1 \backslash X_m]$ “looks like” a similar quotient $[Q_1 \backslash G]$ in lower rank. This statement can be made precise as follows.

**Theorem 3.1.** There exists a smooth surjective morphism

$$\phi_m : [\text{GL}(n-m) \backslash (A^m \times \text{GL}(n-m) \times A^{n-m})] \to [Q_1 \backslash X_m]$$

where $g \in \text{GL}_{n-m}$ acts on $(a, x', v) \in A^m \times \text{GL}(n-m) \times A^{n-m}$ by the formula

$$g(a, x', v) = (a, gx'g^{-1}, vg^{-1}).$$

Moreover, if $\phi_m(a, x', v) = x$ then

$$c(x) = a_t c(x')$$

where $c(x)$ and $c(x')$ are the characteristic polynomials of $x$ and $x'$ respectively, written with the formal variable $t$, and where $a_t$ is the polynomial

$$a_t = t^m + a_1 t^{m-1} + \cdots + a_m$$

for every $a = (a_1, \ldots, a_m) \in A^m$. 
Theorem 3.1 implies, through an induction on the rank $n$, that the mirabolic group acts on the space of matrices $x \in G$ of a given characteristic polynomial with finitely many orbits.

**Proof.** For each $m$, we have a fibration

$$
\pi_m : X_m \to \text{Gr}_m
$$

where $\text{Gr}_m$ is the Grassmannian of $m$-dimensional subspaces $F_x$ of $V$ containing $v$. It maps $x \in X_m$ to the subspace $F_x$ of $V$ generated by the vectors $v, xv, x^2v, \ldots$ which is of dimension $m$.

For each $m$ we also have the subspace $F_m$ and the quotient space $E_m$ of $V$. When there is no possible confusion on $m$, we will drop the index $m$ and write simply $F$ for $F_m$, $E$ for $E_m$. We consider the parabolic subgroup $P$ of $G$ consisting of elements $g \in G$ such that $gF_m = F_m$. An element $g \in P$ can be described as a block matrix

$$
g = \begin{bmatrix} g_F & v \\ 0 & g_E \end{bmatrix}
$$

where $g_F \in \text{GL}(F)$ and $g_E \in \text{GL}(E)$. The group $Q_1$ acts transitively on $\text{Gr}_m$ and its stabilizer at a point $F \in \text{Gr}_m$ is $P \cap Q_1$. The intersection $P \cap Q_1$ can be described by the condition $g_F \in Q_{F,1}$ where $Q_{F,1}$ is the subgroup of $\text{GL}(F)$ defined by $g_F e_1 = e_1$.

The fiber $\pi_m^{-1}(F)$ is the open subset of $P$ consisting of block matrices

$$
x = \begin{bmatrix} x_F & y \\ 0 & x_E \end{bmatrix}
$$

such that $v$ is a cyclic vector of $F$ with respect to the action of $x_F$. The group $P \cap Q_1$ acts on $\pi_m^{-1}(F)$ by conjugation and we have

$$
X_m = \pi_m^{-1}(F) \times^{P \cap Q_1} Q_1.
$$

For $x_F$ as above, $v, x_F v, \ldots, x_F^{m-1} v$ form a basis of $F$. It follows that there exists a unique $g_F \in Q_{F,1}$ such that $g_F^{-1} x_F g_F$ has the form of a companion matrix

$$
\begin{bmatrix}
0 & \cdots & 0 & -a_m \\
\vdots & & \ddots & \vdots \\
I_{m-1} & & & -a_1
\end{bmatrix}
$$

(5)
where $I_{m-1}$ is the identity matrix of size $m-1$, and $a_1, \ldots, a_m$ are the coefficients of the characteristic polynomial of $x_F$. It follows that

$$X_m = Y_m \times^{H_m} Q_1$$

where $Y_m$ is the space of matrices of the form

$$x = \begin{bmatrix} x_F & y \\ 0 & x_E \end{bmatrix}$$  \hspace{1cm} (6)$$

where $x_F$ is a companion matrix as in (5). $H_m$ is the subgroup of $P$ of matrices of the form

$$h = \begin{bmatrix} I_F & v \\ 0 & g_E \end{bmatrix}$$

acting on $Y_m$ by conjugation. The group $H_m$ has the structure of a semidirect product

$$H_m = U_P \rtimes GL(E)$$

where $U_P$, the unipotent radical of $P$, consists of matrices of the form

$$u = \begin{bmatrix} I_F & v \\ 0 & I_E \end{bmatrix}.$$  

It will be convenient to regard $u - I_V$ as a linear application $v \in \text{Hom}(E,F)$. The action of $U_P$ on $X_F$ can be written down as follows

$$\begin{bmatrix} I_F & -v \\ 0 & I_E \end{bmatrix} \begin{bmatrix} x_F & y \\ 0 & x_E \end{bmatrix} \begin{bmatrix} I_F & v \\ 0 & I_E \end{bmatrix} = \begin{bmatrix} x_F & y + x_Fv - vx_E \\ 0 & x_E \end{bmatrix}.$$  

In other words, the action of $v \in \text{Hom}(E,F)$ on the variable $y$ consists in a translation by $x_Fv - vx_E$.

Now, the rank of the linear transformation of $\text{Hom}(E,F)$ given by

$$v \mapsto x_Fv - vx_E$$

depends on the number of common eigenvalues of $x_F$ and $x_E$, in particular it is an isomorphism if $x_F$ and $x_E$ have no common eigenvalues. Thus if $x_F$ and $x_E$ have no common eigenvalues, $U_P$ acts simply transitively on the fiber of $Y_m$ over $(x_F, x_E)$ by conjugation. However, for our purpose, we will need a statement uniform with respect to $(x_F, x_E)$, no matter whether they have common eigenvalues or not.
Lemma 3.2. Let $0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{m-1} \subset F$ be a filtration of $F$ with $\dim(F_j) = j$. Let $x_F \in \text{GL}(F)$ be a linear transformation such that $x_F(F_j) \subset F_{j+1}$ and the induced application $F_j/F_{j-1} \to F_{j+1}/F_j$ is an isomorphism for every $j$ in the range $1 \leq j \leq m - 1$. Let $x_E \in \text{GL}(E)$ be an arbitrary linear transformation.

We consider two subgroups of $U_P = \text{Hom}(E,F)$:

- $U_1 = \text{Hom}(E,F_1)$.
- $U_{m-1} = \text{Hom}(E,F_{m-1})$.

Then the action of $U_1 \times U_{m-1}$ on the space of matrices of the form

$$x = \begin{bmatrix} x_F & y \\ 0 & x_E \end{bmatrix}$$

given by

$$(u_1, u_{m-1})x = u_1u_{m-1}x u_{m-1}^{-1}$$

is simply transitive.

Proof. We write

$$u_1 = \begin{bmatrix} I_F & v_1 \\ 0 & I_E \end{bmatrix} \quad \text{and} \quad u_{m-1} = \begin{bmatrix} I_F & v_{m-1} \\ 0 & I_E \end{bmatrix}$$

with $u_1 \in \text{Hom}(E,F_1)$ and $u_{m-1} \in \text{Hom}(E,F_{m-1})$. Then we have

$$u_1u_{m-1}x u_{m-1}^{-1} = \begin{bmatrix} x_F & y + v_1 + v_{m-1}x_E - x_Fv_{m-1} \\ 0 & x_E \end{bmatrix}$$

The lemma is now equivalent to saying that the linear application

$$\text{Hom}(E,F_1) \times \text{Hom}(E,F_{m-1}) \to \text{Hom}(E,F)$$

given by $(v_1, v_{m-1}) \mapsto v_1 + v_{m-1}x_E - x_Fv_{m-1}$ is an isomorphism. This is equivalent to proving that the map

$$A = \text{Hom}(E,F_{m-1}) \to \text{Hom}(E,F/E_1) = B$$

given by

$$\phi(v_{m-1}) = v_{m-1}x_E - x_Fv_{m-1} \mod \text{Hom}(E,F_1)$$

is an isomorphism.
We consider the filtration \(0 \subset A_1 \subset \cdots \subset A_{m-1} = A\) with \(A_j = \text{Hom}(E, F_j)\) and the filtration \(0 \subset B_1 \subset \cdots \subset B_{m-1} = B\) with \(B_j = \text{Hom}(E, F_{j+1}/F_1)\). We observe that \(\phi(A_j) \subset B_j\) for all \(j\) and the induced map on the associated graded \(A_j/A_{j-1} \to B_j/B_{j-1}\) is an isomorphism. Indeed the linear application \(\phi_E : v_{m-1} \mapsto v_{m-1} x_E\) satisfies \(\phi_E(A_j) \subset B_{j-1}\) and hence induces the zero map on the associated graded \(A_j/A_{j-1} \to B_j/B_{j-1}\). On the other hand, \(\phi_F : v_{m-1} \mapsto x_F v_{m-1}\) satisfies \(\phi_F(A_j) \subset B_j\) and induces an isomorphism on the associated graded \(A_j/A_{j-1} \to B_j/B_{j-1}\) by assumption on \(x_F\). It follows that \(\phi\) is an isomorphism.

We infer from the lemma the existence of a canonical isomorphism
\[
Y_m = A^m \times \text{GL}(E) \times \text{Hom}(E, F_1) \times \text{Hom}(E, F_{m-1})
\]
mapping \(x \mapsto (x_F, x_E, v_1, v_{m-1})\) with \(x_F\) a companion matrix as in [5], \(x_E \in \text{GL}(E)\), \(v_1 \in \text{Hom}(E, F_1)\) and \(v_{m-1} \in \text{Hom}(E, F_{m-1})\) such that
\[
x = \begin{bmatrix}
I_F & v_1 \\
0 & I_E
\end{bmatrix}
\begin{bmatrix}
I_F & v_{m-1} \\
0 & I_E
\end{bmatrix}
\begin{bmatrix}
x_E & 0 \\
0 & x_E
\end{bmatrix}
\begin{bmatrix}
I_F & -v_{m-1} \\
0 & I_E
\end{bmatrix}.
\]

In these new coordinates, the action of the subgroup \(U_{m-1} \rtimes \text{GL}(E)\) of \(H_m = U_P \rtimes \text{GL}(E)\) can be described as follows: the action \((v'_{m-1}, g_E) \in \text{Hom}(E, F_{m-1}) \rtimes \text{GL}(E)\) on \(x = (x_F, x_E, v_1, v_{m-1})\) is given by:
\[
(v'_{m-1}, g_E) x = (x_F, g_E x_E g_E^{-1}, v_1 g_E^{-1}, v_{m-1} + v'_{m-1}).
\]

One should note that \(U_{m-1} \rtimes \text{GL}(E)\) is not a normal subgroup of \(U_P \rtimes \text{GL}(E)\) and the action of the full \(U_P\) is unfortunately very complicated in these coordinates. Nevertheless, we have a smooth surjective morphism
\[
[(U_{m-1} \rtimes \text{GL}(E)\backslash Y_m) \to [H_m \backslash Y_m].
\]

This completes the proof of Theorem 3.1.

4 Action of \(U_Q\) by left translation

The unipotent radical \(U_Q\) of the mirabolic group \(Q\) consists of matrices of the form
\[
u = \begin{bmatrix}
1 & v \\
0 & I_{n-1}
\end{bmatrix}
\]
where \(v \in \text{Hom}(E_1, F_1)\). The action of \(U_Q\) on \(G\) by left translation \(g \mapsto u g\) respects the stratification \(G = \bigsqcup_m X_m\). In this section, we will pay
particular attention to the evaluation of the characteristic polynomial on
left cosets of \( U_Q \) i.e. the function \( u \mapsto c(ux) \) for \( x \) in each stratum \( X_m \).

The characteristic polynomial \( c(x) \) of \( x \in G \), with formal variable \( t \), is
of the form \( c(ux) = t^n + a_1 t^{n-1} + \cdots + a_n \) where \( a_1, \ldots, a_n \) are \( G \)-invariant
functions of \( x \). It can be regarded as a morphism \( c : G \to \mathbb{A}^n \) with \( c(ux) = (a_1, \ldots, a_n) \).

**Proposition 4.1.** For every \( x \in G \), the morphism \( l_x : U_Q = \text{Hom}(E_1, F_1) \to \mathbb{A}^n \) given by \( l_x(u) = c(ux) - c(x) \) is linear. If \( x \in X_m \), the linear application \( l_x \) is of rank \( m - 1 \).

For every \( x \in X_m \) is \( Q \)-conjugate to a matrix of the form \((6)\) with \( x_F \)
being a companion matrix as in \((5)\), we can assume that the matrix \( x \) is of
this special form. In particular we have \( x \in P \) where \( P \) the parabolic group
which preserves the subspace \( F_m \). Let \( U_P \) and \( L_P \) denote respectively its
unipotent radical and the standard Levi component. Every element \( u \in U_Q \)
can be written uniquely in the form \( u = u_L u_U \) where \( u_L \in U_Q \cap L_P \) and
\( u_U \in U_Q \cap U_P \) where \( u_L \) is a matrix of the form

\[
\begin{bmatrix}
1 & v & 0 \\
0 & I_{m-1} & 0 \\
0 & 0 & I_{n-m}
\end{bmatrix}
\]

where \( v = (v_1, \ldots, v_{m-1}) \in \mathbb{A}^{m-1} \) is a row vector. The proposition can now
be derived from a matrix calculation.

**Lemma 4.2.** For \( x \in X_m \) of the form \((6)\) with \( x_F \) being a companion matrix
as in \((5)\) and \( u \in U_Q \) with \( u = u_L u_U \) as above, we have

\[
c(ux) = c(u_L x_F) c(x_F).
\]

Moreover, if we write \( u_L \) in coordinates \((v_1, \ldots, v_{m-1}) \in V = \mathbb{A}^{m-1} \) as
above, and write \( c(u_L x_F) - c_L(x_F) \) in coordinates \((a_1, \ldots, a_m) \in A = \mathbb{A}^m \)
that are coefficients of the characteristic polynomials, then the application
\( u_L \mapsto c(u_L x_F) - c_L(x_F) \) induces an linear isomorphism between \( V \) and the
subspace of \( A \) defined by the equation \( a_m = 0 \).

**Proof.** By direct calculation, we find the following formula for the characteristic polynomial of

\[
u_L x_F = \begin{bmatrix}
1 & -v_1 & \cdots & -v_{m-1} \\
0 & I_{m-1} & & \\
\vdots & & & \\
0 & & & I_{m-1}
\end{bmatrix}
\begin{bmatrix}
0 & \cdots & 0 & -a_m \\
& & & -a_{m-1} \\
& & & \vdots \\
& & & -a_1
\end{bmatrix}.
\]
We have
\[ c(u L x F) = t^m + b_1 t^{m-1} + \cdots + b_{m-1} t + b_m \]
where \( a_m = b_m \) and for \( 1 \leq r \leq m - 1 \)
\[ b_r = a_r + \sum_{i=1}^{r-1} a_i v_{r-i} + v_r. \]
The lemma follows.

5 Proof of Theorem 2.4

We will deduce Theorem 2.4 from the analogous statement for the mirabolic subgroup \( Q \) that \((\dagger)\) if \( g \in G - Q \), then \( H_c^*(U_Q g, \Phi_\rho|_{U_Q g}) = 0 \).

To this end take \( g \in G - B \), there are two cases: \( g \in Q \) or \( g \notin Q \).

If \( g \notin Q \), by \((\dagger)\) we know that \( H_c^*(U_Q g, \Phi_\rho|_{U_Q g}) = 0 \). In fact for all \( u \in U_B \), \( ug \notin Q \) so that more generally we have \( H_c^*(U_Q ug, \Phi_\rho|_{U_Q ug}) = 0 \) for all \( u \in U_B \). Now one can establish the vanishing of \( H_c^*(U_B g, i^* \Phi_\rho) \) by using the Leray spectral sequence associated with the morphism \( U_B g \to U_Q \setminus (U_B g) \).

Now we consider the case \( g \in Q \). Let \( L_Q \) denote the standard Levi factor of \( Q \), \( g_L \) the image of \( g \) in \( L_Q \). We have \( g_L \notin B \cap L_Q \). Using the definition of the restriction functor, we have
\[ H_c^*(U_B g, i^* \Phi_\rho) = H_c^*((U_B \cap L) g_L, \text{Res}_L^G(\Phi_\rho)|_{(U_B \cap L) g_L}) \]
where \( \text{Res}_L^G(\Phi_\rho) = \Phi_{L \rho|_L} \) by Proposition 2.2. At this point we can conclude by an induction argument.

It remains to establish \((\dagger)\). For convenience of induction we will prove the following equivalent proposition:

**Proposition 5.1.** Let \( G \) be a direct product of general linear groups and \( Q \) a mirabolic subgroup of \( G \) of the form
\[ Q = \prod_{i \neq j} \text{GL}(n_i) \times Q_j \subset \prod_i \text{GL}(n_i) = G, \]
let \( \rho \) be a \( \sigma \)-positive representation of \( \tilde{G} \) where \( \sigma \) denotes the product of the characters \( \det_i : \text{GL}_{n_i}(\mathbb{C}) \to \mathbb{C}^\times \). If \( x \) is a geometric point of \( G - Q \), then
\[ H_c^*(U_Q x, i^* \Phi_\rho) = 0 \]
where \( i : U_Q x \to G \) denotes the inclusion map.
Proof. Argue by induction on the semisimple rank of $G$. In the base case $G$ is a torus, hence the proposition holds vacuously.

Otherwise $n_j \geq 2$, consider the stratification induced by (4) on $\text{GL}(n_j)$:

$$G = \bigsqcup_m X_m = \bigsqcup_{m=1}^{n_j} \left( \prod_{i \neq j} \text{GL}(n_i) \times X_{j,m} \right).$$

For $x \notin Q$ with $Q = X_1$, we have $x \in X_m$ for $2 \leq m \leq n_j$.

We first consider the case $x \in X_{n_j}$. For $X_{j,n_j}$ is contained in the open subset $\text{GL}(n_j)^{\text{reg}}$ of $\text{GL}(n_j)$ (see [S65]), we have a Cartesian diagram

$$
\begin{array}{ccc}
X_{n_j} & \xrightarrow{\tilde{c}} & \prod_{i \neq j} \text{GL}(n_i) \times T_j \\
\downarrow q & & \downarrow q \\
\hat{X}_{n_j} & \xrightarrow{c} & \prod_{i \neq j} \text{GL}(n_i) \times T_j/W_j
\end{array}
$$

where $c$ is smooth and $q$ is finite. It follows that the restriction of the $\gamma$-sheaf $\Phi_{\rho}$ to $X_{n_j}$ can be identified with a pullback by the characteristic polynomial map

$$\Phi_{\rho}|_{X_{n_j}} = c^* q_* \Phi_{\rho'} W_j$$

where $\rho'$ denotes the restriction of $\rho$ to the subgroup

$$\prod_{i \neq j} \text{GL}(n_i)(\mathbb{C}) \times \tilde{T}_j \subset \tilde{G}.$$

We recall that the coordinate ring $T_j/W_j = k^{n_j-1} \times \mathbb{G}_m$ is the ring of $\text{GL}(n_j)$-invariant functions on $\text{GL}(n_j)$, the projection $\sigma_{W_j} : T_j/W_j \rightarrow \mathbb{G}_m$ corresponds to the determinant function. By Lemma (4.2), the restriction of $c$ to $U_Q x$ induces an isomorphism between $U_Q x$ and the fiber of the determinant map on the $j$th component

$$\sigma_{W_j} : \prod_{i \neq j} \text{GL}(n_i) \times T_j/W_j \rightarrow \prod_{i \neq j} \text{GL}(n_i) \times \mathbb{G}_m$$

over the image of $x$. Thus, to prove the proposition, it is enough to prove that

$$\sigma|\Phi_{\rho'} W_j = \sigma_{W_j} q_* \Phi_{\rho'} W_j = 0$$

where

$$\sigma : \prod_{i \neq j} \text{GL}(n_i) \times T_j \rightarrow \prod_{i \neq j} \text{GL}(n_i) \times \mathbb{G}_m$$

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is the determinant map on the $j$th component.

Now recall the definition of the hypergeometric sheaf $\Psi_\Lambda = p_\Lambda^! \text{tr}^* L_\psi$ with homorphism $p_\Lambda : \mathbb{P}^r \to T$ given by

$$p_\Lambda(t_1, \ldots, t_r) = \prod_{i=1}^r \lambda_i(t_i).$$

We have

$$\sigma_1 \Phi_{\rho'} = \sigma_1 \text{Ind}^G_T \Psi_\Lambda \prod_{i \neq j} W_i$$

where $G' = \prod_{i \neq j} \text{GL}(n_i) \times T_j$.

With the definition of the action of $W$ on $\Psi_\Lambda$ given by (2), we see that the induced action of $W$ on $\sigma_1 \Phi_{\rho'}$ is through the character $\text{sign}_{n_j} : W_j \to \{\pm 1\}$. It follows that $\sigma_1 \Phi_{\rho'} W_j = 0$ because for $n_j \geq 2$ the sign character $\text{sign}_{n_j}$ is nontrivial. This concludes the case $x \in X_{n_j}$.

We now consider the general case $x \in X_m$ with $2 \leq m \leq n_j$. By $Q$-conjugation we can assume that the $j$th component $x_j$ of $x$ is of the form (6) with $x_{j,F}$ being a companion matrix as in (5). Let $P$ denote the standard parabolic of block matrices as in (6), $L$ the standard Levi factor of $P$ and $U_P$ its unipotent radical. By applying the result obtained above in the generic case to $\text{GL}(F)$, we get

$$H^*_c(U_L x_L, \Phi_{L,\rho}|_{U_L x_L}) = 0$$

where $x_L$ is the image of $x$ in $L$, $U_L$ consists of unipotent matrices of the form (7), and $\Phi_{L,\rho}|_L$ is the $\gamma$-sheaf on $L$ associated to the restriction to $\tilde{L}$ of the representation $\rho$ of $\tilde{G}$.

For $\Phi_{L,\rho}|_L = \text{Res}^L_G(\Phi_{\rho})$ by Proposition 2.2, (8) implies that

$$H^*_c(U_P U_L x, \Phi_{\rho}|_{U_P U_L x}) = 0$$

where $U_P$ and $U_L$ commute. With the help of Lemma 3.2, we see that the morphism

$$U_Q \times U_{m-1} \to U_P U_L x$$

given by

$$(u_{m-1}, u_Q) \mapsto u_{m-1} u_Q x u_{m-1}^{-1}$$

is an isomorphism. Now using the fact that $\Phi_{\rho}$ is equivariant under the adjoint action, (9) implies that

$$H^*_c(U_Q x, \Phi_{\rho}|_{U_Q x}) = 0.$$

This concludes the proof of Proposition 5.1 and therefore Theorem 2.4. □
A  The case of parabolic subgroups

In this appendix we give a proof for the extension of Theorem 2.4 to arbitrary parabolic subgroups $P$ of $G = \text{GL}(n)$. By a similar argument involving the Leray spectral sequence as in the beginning of the proof of Theorem 2.4, we are reduced to the case when $P$ is a maximal parabolic subgroup.

**Proposition A.1.** Let $P$ denote the maximal standard parabolic subgroup of $G = \text{GL}(n)$ consisting of block matrices of size $(n_1, n_2)$ where $n_1 + n_2 = n$ and $U_P$ its unipotent radical, let $\rho$ be a $\sigma$-positive representation of $\hat{G}$ where $\sigma = \text{det}$. If $g$ is a geometric point of $G - P$, then

$$H^*(U_P g, i^* \Phi_\rho) = 0$$

where $i : U_P g \to G$ denotes the inclusion map.

**Proof.** Argue by induction on $n_1$. In the base case when $n_1 = 1$ we are reduced to Proposition 5.1.

Otherwise $n_1 \geq 2$, let $P'$ denote the maximal standard parabolic subgroup of $\text{GL}(n)$ consisting of block matrices of size $(n_1 - 1, n_2 + 1)$ and $U_{P'}$ its unipotent radical. Let

$$g' = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a block matrix of size $(n_1, n_2)$ such that the $n_2 \times n_1$-matrix $c$ is of the form

$$b = \begin{bmatrix} * & \cdots & * & 1 \\ * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & 0 \end{bmatrix},$$

then by an analogous computation as in the proof of Lemma 3.2 and the end of the proof of Proposition 5.1, we see that acyclicity of $\Phi_\rho$ over $U_{P'} g'$:

$$H^*_c(U_{P'} g', \Phi_\rho|_{U_{P'} g'}) = 0$$

implies acyclicity over the subcoset $(U_P \cap U_{P'}) g'$:

$$H^*_c((U_P \cap U_{P'}) g', \Phi_\rho|_{(U_P \cap U_{P'}) g'}) = 0,$$

which then implies the proposition by the Levy spectral sequence.

Now it remains to prove (10). To this end it suffices to observe that $g$ is conjugate, under the action of the standard Levi factor $L$ of $P$, to a matrix of the form $g'$, which is in addition not contained in $P'$. This follows from the fact that the $L$-orbits on $c$ are classified by the rank of $c$. Hence we are done by induction. 

\[ \square \]
B Positive hypergeometric sheaves

In this appendix we give a proof for Proposition 1.1 concerning hypergeometric sheaves $\Psi_\lambda$ when $\lambda$ is $\sigma$-positive. The first half of Proposition 1.1 is due to Braverman–Kazhdan in [BK02, Theorem 4.2]:

**Proposition B.1.** If $\lambda$ is $\sigma$-positive, then the forget support morphism $\Psi_\lambda \to \Psi^*_\lambda$ is an isomorphism.

This is proved by restricting to the smooth neighborhood $p_\lambda : G_m^r \to T$ and then applying the classical Fourier–Deligne transform on $G_m^r$. The same idea is also crucial in the proof of the second half of Proposition 1.1:

**Proposition B.2.** If $\lambda$ is $\sigma$-positive, then $\Psi_\lambda$ is isomorphic to a shift of a local system on the image of $p_\lambda : G_m^r \to T$.

Without loss of generality we can assume that $p_\lambda$ is surjective. Then we can also factorize $p_\lambda$ into the product of a homomorphism with connected fibers and an isogeny. Since isogenies preserve local systems under push-forward by proper-smooth base change, without loss of generality we can assume that $p_\lambda$ has connected fibers.

We will deduce smoothness of $\Psi_\lambda$ from universal local acyclicity by Théorème 5.3.1 in [D77a] together with the fact that $\Psi_\lambda$ is a perverse sheaf by Proposition B.1, for this we need a compactification of $p_\lambda$.

Let $\Gamma$ be the normalization of the closure of the graph of $p_\lambda$ in $(G_m^r)^r \times T \subset (\mathbb{P}^1)^r \times T$, let $j_\lambda : (G_m)^r \to \Gamma$ be the open inclusion and $p_\lambda : \Gamma \to T$ the projection, then $p_\lambda$ compactifies $p_\lambda$.

We are therefore reduced to the following lemma:

**Lemma B.3.** If $\lambda$ is $\sigma$-positive and $p_\lambda$ is surjective with connected fibers, then $p_\lambda$ is universally locally acyclic with respect to $j_\lambda ! \tau^* L_\psi$.

**Proof.** The following argument is essentially due to Katz–Laumon in [KL85].

Let $\pi : (\mathbb{P}^1)^r \to (\mathbb{P}^1)^r$ be the completed Artin–Schreier covering defined by $[X : Y] \mapsto [X^q - XY^{q-1} : Y^q]$ on the homogeneous coordinates on each factor $\mathbb{P}^1$ and $\hat{\pi}$ the base change of $\pi$ along the projection $\Gamma \to (\mathbb{P}^1)^r$.

It follows from the decomposition theorem and total ramification of $\pi$ at infinity that the $l$-extension of $\tau^* L_\psi$ to $(\mathbb{P}^1)^r$ is a direct summand of $\pi_* Q_\ell$ over $(\mathbb{P}^1)^r - (0, \ldots, 0)$.

Now by assumption $\lambda$ is $\sigma$-positive, hence the image of $\Gamma$ in $(\mathbb{P}^1)^r$ is contained in $(\mathbb{P}^1)^r - (0, \ldots, 0)$ which implies that $j_\lambda ! \tau^* L_\psi$ is a direct summand of $\hat{\pi}_* Q_\ell$. Therefore it suffices to show instead that $\hat{p_\lambda}$ is universally locally acyclic with respect to $\hat{\pi}_* Q_\ell$. 


Let $K$ be the kernel of $p_\Delta$ which we may assume to be connected, let $\overline{K}$ be the normalization of the closure of $K$ in $\mathbb{P}^1)^r$ and $\overline{\pi} : (\mathbb{G}_m)^r \times \overline{K} \to \overline{\Gamma}$ the morphism which extends the multiplication map from $(\mathbb{G}_m)^r \times K$ to $(\mathbb{G}_m)^r$. Then the diagram

\[
\begin{array}{ccc}
(\mathbb{G}_m)^r \times \overline{K} & \xrightarrow{\overline{\pi}} & \overline{\Gamma} \\
pr \downarrow & & \downarrow pr_\Delta \\
(\mathbb{G}_m)^r & \xrightarrow{pr_\Delta} & T
\end{array}
\]

is Cartesian by the theory of toric varieties (see [F93] for example). Let $\hat{\pi}$ be the base change of $\tilde{\pi}$ along $\overline{\pi}$. We may assume that $p_\Delta$ is surjective hence smooth. Since universal local acyclicity is a local property with respect to the smooth topology, it suffices to show that $pr$ is universally locally acyclic with respect to $\hat{\pi}^* \overline{Q}_\ell$. Then by properness of $\hat{\pi}$ we are further reduced to showing that the composite $pr \circ \hat{\pi}$ is universally locally acyclic with respect to the constant sheaf $\overline{Q}_\ell$ on the source.

Let $U$ be an open subscheme of $(\mathbb{P}^1)^r$ of the form $U = U_1 \times \cdots \times U_r$ where each Cartesian factor $U_i$ is an open subscheme of $\mathbb{P}^1$ equal to either $\mathbb{G}_a$ or $\mathbb{P}^1 - 0$. Let $\tilde{U}$ be the inverse image of $U$ under $\pi : (\mathbb{P}^1)^r \to (\mathbb{P}^1)^r$ and $\overline{K}_U$ the intersection of $\overline{K}$ with $U$. We will check universal local acyclicity of $pr \circ \tilde{\pi}$ by restricting to the open subscheme $\tilde{U} \times_U ((\mathbb{G}_m)^r \times \overline{K}_U)$ of the source of $\tilde{\pi}$ and then calculating “by hand” with explicit coordinates.

To this end let $t$ be a coordinate on $\mathbb{G}_m$, $X$ a coordinate on $\mathbb{G}_a$ and $Y^{-1}$ a coordinate on $\mathbb{P}^1 - 0$, hence $(\ldots, X_i, \ldots, Y_j^{-1}, \ldots)$ are coordinates on $U$. By slight abuse of notation let $p_\Delta(\ldots, X_i, \ldots, Y_j^{-1}, \ldots) = 1$ denote a system of polynomial equations such that

\[
\overline{K}_U \simeq \operatorname{Spec} \left( \operatorname{Int} \left( \frac{k[X_i, \ldots, Y_j, \ldots]}{(p_\Delta(\ldots, X_i, \ldots, Y_j^{-1}, \ldots) = 1)} \right) \right)
\]

where $\operatorname{Int}$ denotes the integral closure, then modulo base change with respect to a normalization, we have that $\tilde{U} \times_U ((\mathbb{G}_m)^r \times \overline{K}_U)$ is isomorphic to

| Spec \left( \frac{k[X_i, \ldots, Y_j, \ldots]}{(\overline{X}_q - X_i = t_i X_i, (1 - Y_j q^{-1})^{-1} Y_j q = t_j^{-1} Y_j, p_\Delta(\ldots, X_i, \ldots, Y_j^{-1}, \ldots) = 1)} \right), |
which by substituting $s_j$ for $(1 - \tilde{Y}_j q^{-1})^{-1} t_j$ becomes isomorphic to

$$\text{Spec} \left( \frac{k[\ldots,t_i,t_i^{-1},\ldots,s_j,s_j^{-1},\ldots,\tilde{X}_i,\ldots,\tilde{Y}_j,(1 - \tilde{Y}_j q^{-1})^{-1},\ldots]}{(p_\Delta(\ldots,t_i^{-1}(\tilde{X}_i q - \tilde{X}_i),\ldots,(s_j \tilde{Y}_j q)^{-1},\ldots) = 1)} \right).$$

The morphism $\tilde{X} \mapsto \tilde{X}^q - \tilde{X}$ is étale and the morphism $\tilde{Y} \mapsto \tilde{Y}^q$ is finite surjective radicial, both are universally locally acyclic with respect to $\overline{\mathbb{Q}}_\ell$ on the source. By Corollaire 2.16 in [D77a] the projection $pr$ from

$$\text{Spec} \left( \frac{k[\ldots,t_n,t_n^{-1},\ldots,X_i,\ldots,Y_j,\ldots]}{(p_\Delta(\ldots,t_i^{-1}X_i,\ldots,(t_j Y_j)^{-1},\ldots) = 1)} \right) \simeq (\mathbb{G}_m)^r \times K_U$$

to $(\mathbb{G}_m)^r$ is universally locally acyclic with respect to $\overline{\mathbb{Q}}_\ell$ on the source. Hence so is their composite, the lemma follows.

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