REVISITING MULTIFRACTAL ANALYSIS

FATHI BEN NASR* AND JACQUES PEYRIÈRE†‡♯

Abstract. New proofs of theorems on the multifractal formalism are given. They yield results even at points $q$ for which Olsen’s functions $b(q)$ and $B(q)$ differ. Indeed, we provide an example of measure for which functions $b$ and $B$ differ and for which the Hausdorff dimension of the sets $E_\alpha$ (the level sets of the local Hölder exponent) are given by the Legendre transform of $b$ and their packing dimension by the Legendre transform of $B$.

1. Introduction

The multifractal formalism aims at giving expressions of the dimension of the level sets of the local Hölder exponent of some set function $\mu$ in terms of the Legendre transform of some “free energy” function. If such a formula holds, one says that $\mu$ satisfies the multifractal formalism. At first, the formalism used “boxes”, or in other terms took place in a totally disconnected metric space. In this context, the closeness to large deviation theory is patent. To get rid of this boxes and have a formalism meaningful in Geometric Measure Theory, Olsen [5] introduced a formalism which nowadays is of common use. At this stage of the theory, whether it is dealt with or without boxes, the formalism was proven to hold when there exists an auxiliary measure, a so called Gibbs measure. Later on, it was shown that this formalism holds under the condition that the Olsen’s Hausdorff-like multifractal measure be positive (see [2] in the totally disconnected case, [3] in general); So, the situation when $b(q) = B(q)$ (in Olsen’s notation) is fairly well understood.

Here, we elaborate on the previous proofs. There is a vector version of Olsen’s constructs [6] and in particular of the functions $b$ and $B$; But, this time, they are functions of several variables. In this work, we show that the restriction to a suitable affine subspace of these functions allows to estimate the Hausdorff and Tricot dimensions of some level sets. In particular, this gives some results even in case when $b \neq B$. Despite the inherent complexity of notations, not only we provide a simple proof of already known results, but also we get new estimates.

2. Notations and definitions

We deal with a metric space $(X,d)$ having the Besicovitch property:

There exists an integer constant $C_B$ such that one can extract $C_B$ countable families $\{\{B_{j,k}\}_k\}_{1 \leq j \leq C_B}$ from any collection $B$ of balls so that

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* Département de Mathématiques, Faculté des Sciences de Monastir, Monastir 5000 Tunisie, fathi_bennasr@yahoo.fr.
† Department of Mathematics, Tsinghua University, Beijing 100084, P.R. China, peyriere@math.tsinghua.edu.cn.
‡ Université Paris-Sud, Mathématique bât. 425, CNRS UMR 8628, 91405 Orsay Cedex, France.
jacques.peyriere@math.u-psud.fr.
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(1) \( \bigcup B_{j,k} \) contains the centers of the elements of \( B \),
(2) for any \( j \) and \( k \neq k' \), \( B_{j,k} \cap B_{j,k'} = \emptyset \).

**Notations.** \( B(x, r) \) stands for the open ball \( B(x, r) = \{ y \in X : d(x, y) < r \} \).
The letter \( B \) with or without subscript will implicitly stand for such a ball. When
dealing with a collection of balls \( \{B_i\}_{i \in I} \) the following notation will implicitly be
assumed: \( B_i = B(x_i, r_i) \).

By a \( \delta \)-cover of \( E \subset X \), we mean a collection of balls of radii not exceeding \( \delta \)
whose union contains \( E \). A **centered cover** of \( E \) is a cover of \( E \) consisting in balls
whose centers belong to \( E \).

By a \( \delta \)-packing of \( E \subset X \), we mean a collection of disjoint balls of radii not exceeding \( \delta \)
centered in \( E \).

If \( E \) is a subset of \( X \), \( \dim_H E \) stands for its Hausdorff dimension and \( \dim_P E \) for
its packing dimension (introduced by Tricot \[7\]).

Let \( B \) stand for the set of balls of \( X \) and \( F \) for the set of maps from \( B \) to \([0, +\infty)\).

If \( \nu \in F \), one considers the outer measure \( \nu^\delta \) on \( X \) associated with \( \nu \) in the
following way:

\[
\nu^\delta(E) = \inf \left\{ \sum \nu(B_j) : E \subset \bigcup B_j \right\}.
\]

The set of \( \mu \in F \) such that \( \mu(B) = 0 \) implies \( \mu(B') = \) for all \( B' \subset B \) will be denoted
by \( F^\ast \). For such a \( \mu \), one defines its support \( S_\mu \) to be the complement of the set
\[
\bigcup \{ B \in B : \mu(B) = 0 \}.
\]

**Multifractal measures and separator functions.** For \( \mu = (\mu_1, \ldots, \mu_m) \in F^m \),
\( E \subset X \), \( q = (q_1, \ldots, q_m) \in \mathbb{R}^m \), \( t \in \mathbb{R} \), and \( \delta > 0 \), one sets

\[
F^\delta_{\mu,t}(E) = \sup \left\{ \sum r_j^t \prod_{k=1}^m \mu_k(B_j)^{q_k} : \{B_j\} \delta\text{-packing of } E \right\},
\]

where \( \ast \) means that one only sums the terms for which \( \prod_k \mu_k(B_j) \neq 0 \),

\[
F^\delta_{\mu,t}(E) = \lim_{\delta \downarrow 0} F^\delta_{\mu,t}(E),
\]

\[
F^q_{\mu,t}(E) = \inf \left\{ \sum \left( \sum r_j^{q_k} \prod_{k=1}^m \mu_k(B_j)^{q_k} \right) : \{B_j\} \text{ centered } \delta\text{-cover of } E \right\},
\]

and

\[
\mathcal{H}^\delta_{\mu,t}(E) = \inf \left\{ \sum r_j^t \prod_{k=1}^m \mu_k(B_j)^{q_k} : \{B_j\} \text{ centered } \delta\text{-cover of } E \right\},
\]

\[
\mathcal{H}^\delta_{\mu,t}(E) = \lim_{\delta \downarrow 0} \mathcal{H}^\delta_{\mu,t}(E),
\]

\[
\mathcal{H}^q_{\mu,t}(E) = \sup \left\{ \mathcal{H}^q_{\mu,t}(F) : F \subset E \right\}.
\]

When \( m = 1 \), these measures have been defined by Olsen \[5\]. When \( \mu \) is identically
1 these quantities do not depend on \( q \). They will simply be respectively
denoted by \( F^1_{\mu,t}(E) \), \( F(E) \), \( F^t(E) \), \( \mathcal{H}^q_{\mu,t}(E) \), \( \mathcal{H}^t_{\mu,t}(E) \), and \( \mathcal{H}^q_{\mu,t}(E) \). They are the
classical packing pre-measures and measures introduced by Tricot \[7\], and the Haus-
dorff centered pre-measures and measures.

Also, as usual, one considers the following functions

\[
\tau_{\mu,t}(q) = \inf \{ t \in \mathbb{R} : F^q_{\mu,t}(E) = 0 \} = \sup \{ t \in \mathbb{R} : F^q_{\mu,t}(E) = \infty \}
\]

\[
B_{\mu,t}(q) = \inf \{ t \in \mathbb{R} : F^q_{\mu,t}(E) = 0 \} = \sup \{ t \in \mathbb{R} : F^q_{\mu,t}(E) = \infty \},
\]

and
\[ b_{\mu,E}(q) = \inf \{ t \in \mathbb{R} : \mathcal{H}^{q,t}_{\mu}(E) = 0 \} = \sup \{ t \in \mathbb{R} : \mathcal{H}^{q,t}_{\mu}(E) = \infty \}. \]

It is well known [5, 6] that \( \tau \) and \( B \) are convex and that \( b \leq B \leq \tau \). Let \( J_{\tau}, J_{B}, \) and \( J_{b} \) stand for the interior of the sets where respectively \( \tau, B, \) and \( b \) are finite.

When \( \mu \) is identically 1 we will denote these quantities by \( \dim_{B} E, \dim_{P} E, \) and \( \dim_{H} E \). The first one is the Minkowski-Bouligand dimension (or box-dimension), the second is the Tricot (packing) dimension [7], and the last the Hausdorff dimension.

Here is an alternate definition of \( \tau_{\mu,E} \). Fix \( \lambda < 1 \) and define

\[ \tilde{\mathcal{P}}_{\mu}^{q,t}(E) = \sup \left\{ \sum_{i=0}^{\infty} r_{j}^{\gamma + \varepsilon} \prod_{k=1}^{m} \mu_{k}(B_{j})^{q_{k}} \mid \{B_{j}\} \text{ packing of } E \text{ with } \lambda \delta < r_{j} \leq \delta \right\}, \]

\[ \tilde{\mathcal{P}}_{\mu}^{q,t}(E) = \limsup_{\delta \searrow 0} \tilde{\mathcal{P}}_{\mu}^{q,t}(E), \]

and

\[ \tilde{\tau}_{\mu,E}(q) = \sup \{ t \in \mathbb{R} : \tilde{\mathcal{P}}_{\mu}^{q,t}(E) = \infty \}. \]

**Lemma 1.** One has \( \tilde{\tau}_{\mu,E} = \tau_{\mu,E} \).

**Proof.** Obviously \( \tilde{\mathcal{P}}_{\mu}^{q,t}(E) \leq \mathcal{P}_{\mu}^{q,t}(E) \), so \( \tilde{\tau}_{\mu,E} \leq \tau_{\mu,E} \). To prove the converse inequality, one only has to consider the case \( \tau_{\mu,E}(q) > -\infty \).

Choose \( \gamma < \tau_{\mu,E}(q) \) and \( \varepsilon > 0 \) such that \( \gamma + \varepsilon < \tau_{\mu,E}(q) \). There exists \( n_{0} \) such that, for all \( n > n_{0} \), there exists a \( \lambda^{n} \)-packing \( \{B_{j}\} \) of \( E \) such that

\[ \sum_{i=0}^{\infty} r_{j}^{\gamma + \varepsilon} \prod_{k=1}^{m} \mu_{k}(B_{j})^{q_{k}} > 1. \]

As

\[ \sum_{i=0}^{\infty} r_{j}^{\gamma + \varepsilon} \prod_{k=1}^{m} \mu_{k}(B_{j})^{q_{k}} = \sum_{i=0}^{\infty} \sum_{\lambda < r_{j} \lambda^{-(n+i)} \leq 1} r_{j}^{\gamma + \varepsilon} \prod_{k=1}^{m} \mu_{k}(B_{j})^{q_{k}}, \]

there exists \( i \geq 0 \) such that

\[ \sum_{\lambda < r_{j} \lambda^{-(n+i)} \leq 1} r_{j}^{\gamma + \varepsilon} \prod_{k=1}^{m} \mu_{k}(B_{j})^{q_{k}} > \lambda^{\varepsilon}(1 - \lambda^{\varepsilon}), \]

from which it follows

\[ \sum_{\lambda < r_{j} \lambda^{-(n+i)} \leq 1} r_{j}^{\gamma + \varepsilon} \prod_{k=1}^{m} \mu_{k}(B_{j})^{q_{k}} > \lambda^{-(n+i)\varepsilon} \lambda^{\varepsilon}(1 - \lambda^{\varepsilon}) = \lambda^{-n}(1 - \lambda^{\varepsilon}), \]

and \( \tilde{\mathcal{P}}_{\mu}^{q,t}(E) = \infty. \)

**Corollary 2.** For any \( \lambda < 1 \), one has

\[ \tau_{\mu,E}(q) = \lim_{\delta \searrow 0} \frac{-1}{\log \delta} \log \sup \left\{ \sum_{i=1}^{m} \prod_{k=1}^{m} \mu_{k}(B_{j})^{q_{k}} \mid \{B_{j}\} \text{ packing of } E \text{ with } \lambda \delta < r_{j} \leq \delta \right\}. \]
Level sets of local Hölder exponents.

Let \( \mu \) be an element of \( \mathcal{F}^* \). For \( \alpha, \beta \in \mathbb{R} \), one sets
\[
X_{\mu}(\alpha) = \left\{ x \in S_{\mu} ; \limsup_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r} \leq \alpha \right\},
\]
\[
X_{\mu}(\beta) = \left\{ x \in S_{\mu} ; \liminf_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq \beta \right\},
\]
and
\[
X_{\mu}(\alpha, \beta) = X_{\mu}(\alpha) \cap X_{\mu}(\beta),
\]
and
\[
X_{\mu}(\alpha) = X_{\mu}(\alpha) \cap X_{\mu}(\alpha).
\]

3. Results

First, one revisits Billingsley and Tricot lemmas [4, 7].

Lemma 3. Let \( E \) be a subset of \( \mathcal{F} \) and \( \nu \) an element of \( \mathcal{F} \).

a) If \( B_{\nu, E}(1) \leq 0 \), then
\[
\dim_H E \leq \sup_{x \in E} \liminf_{r \downarrow 0} \frac{\log \nu(B(x, r))}{\log r}, \tag{1}
\]
\[
\dim_P E \leq \sup_{x \in E} \limsup_{r \downarrow 0} \frac{\log \nu(B(x, r))}{\log r}. \tag{2}
\]

b) If \( \nu^\delta(E) > 0 \), then
\[
\dim_H E \geq \esssup_{x \in E, \nu^\delta} \liminf_{r \downarrow 0} \frac{\log \nu(B(x, r))}{\log r}, \tag{3}
\]
\[
\dim_P E \geq \esssup_{x \in E, \nu^\delta} \limsup_{r \downarrow 0} \frac{\log \nu(B(x, r))}{\log r}, \tag{4}
\]

where
\[
\esssup_{x \in E, \nu^\delta} \chi(x) = \inf \left\{ t \in \mathbb{R} ; \nu^\delta \left( E \cap \{ \chi > t \} \right) = 0 \right\}.
\]

Proof. Take \( \gamma > \sup_{x \in E} \liminf_{r \downarrow 0} \frac{\log \nu(B(x, r))}{\log r} \) and \( \eta > 0 \). Since \( B_{\nu, E}(1) \leq 0 \) there exists a partition \( E = \bigcup E_j \) such that \( \sum \nu^\delta_j(E_j) < 1 \). It results that \( \sum \nu^\delta_{\nu^\delta}(E_j) = 0 \).

Let \( F \) be a subset of \( E_k \) and \( \delta \) a positive number. For all \( x \in F \), there exists \( r \leq \delta \) such that \( \nu(B(x, r)) \geq r^\gamma \). By using the Besicovitch property there exists a centered \( \delta \)-cover \( \{ B_j \} \) of \( F \), which can be being decomposed in \( C_B \) packings, such that \( \nu(B_j) \geq r^\gamma_j \). We then have
\[
\sum r_j^{\gamma+\eta} \leq \sum r_j^\gamma \nu(B_j) \leq C_B \nu^\delta_{\nu^\delta}(E_k).
\]
Therefore we have, \( \mathcal{H}^{\gamma+\eta}(F) = 0 \), \( \mathcal{H}^{\gamma+\eta}(E_k) = 0 \), and finally \( \mathcal{H}^{\gamma+\eta}(E) = 0 \). Then (1) easily follows.

To prove (2), take \( \gamma > \sup_{x \in E} \limsup_{r \downarrow 0} \frac{\log \nu(B(x, r))}{\log r} \) and \( \eta > 0 \). As previously, there exists a partition \( E = \bigcup E_j \) such that \( \sum \nu^\delta_j(E_j) = 0 \).
For all \( x \in E \), there exists \( \delta > 0 \) such that, for all \( r \leq \delta \), one has \( \nu(B(x, r)) \geq r^\gamma \). Consider the set
\[
E(n) = \left\{ x \in E \; ; \; \forall r \leq 1/n, \; \nu(B(x, r)) \geq r^\gamma \right\}.
\]

Let \( \{B_j\} \) be a \( \delta \)-packing of \( E_k \cap E(n) \), with \( \delta \leq 1/n \). One has
\[
\sum r_j^{\gamma + \eta} \leq \sum r_j^\eta \nu(B_j) \leq \mathcal{P}_\nu^\eta(\nu, \delta)(E_k),
\]
from which \( \mathcal{P}_\nu^\eta(\nu, \delta)(E_k) \) follows.

So we have \( \mathcal{P}_\nu^\eta(\nu, \delta)(E(n)) = 0 \). Since \( E = \bigcup_{n \geq 1} E(n) \), one has \( \dim_P E \leq \gamma + \eta \).

Hence \( \mathfrak{H} \).

To prove \( \mathfrak{H} \), take \( \gamma < \text{ess sup}_{x \in E, \nu} \lim \inf_{r \to 0} \frac{\log \nu(B(x, r))}{\log r} \) and consider the set
\[
F = \left\{ x \in E \; ; \; \lim \inf_{r \to 0} \frac{\log \nu(B(x, r))}{\log r} > \gamma \right\}. \tag{4}
\]
Then, for all \( x \in F \), there exists \( \delta > 0 \) such that, for all \( r \leq \delta \), one has \( \nu(B(x, r)) \leq r^\gamma \). Consider the set
\[
F(n) = \left\{ x \in F \; ; \; \forall r \leq 1/n, \; \nu(B(x, r)) \leq r^\gamma \right\}.
\]

We have \( F = \bigcup_{n \geq 1} F(n) \). Since we assume \( \nu^\sharp(E) > 0 \), there exists \( n \) such that \( \nu^\sharp(F(n)) > 0 \). Then for any centered \( \delta \)-cover \( \{B_j\} \) of \( F(n) \), with \( \delta \leq 1/n \), one has
\[
0 < \nu^\sharp(F(n)) \leq \sum \nu^\sharp(B_j) \leq \sum \nu(B_j) \leq \sum r_j^\gamma.
\]
Therefore, \( \dim_H E \geq \dim_H F(n) \geq \gamma \) (one can compute the Hausdorff dimension by using centered covers).

To prove \( \mathfrak{H} \), take \( \gamma < \text{ess sup}_{x \in E, \nu} \lim \inf_{r \to 0} \frac{\log \nu(B(x, r))}{\log r} \) and consider the set
\[
F = \left\{ x \in E \; ; \; \lim \inf_{r \to 0} \frac{\log \nu(B(x, r))}{\log r} > \gamma \right\}. \tag{4}
\]
Let \( G \) be a subset of \( F \). Then, for all \( x \in G \), for all \( \delta > 0 \), there exists \( r \leq \delta \) such that \( \nu(B(x, r)) \leq r^\gamma \). Then for all \( \delta \), by using the Besicovitch property, there exist a collection \( \{\{B_{j,k}\} \} \) of \( \delta \)-packings of \( G \) which together cover \( G \) and such that \( \nu(B_{j,k}) \leq r_{j,k}^\gamma \). Then one has
\[
0 < \nu^\sharp(G) \leq \sum \nu^\sharp(B_{j,k}) \leq \sum \nu(B_{j,k}) \leq \sum r_{j,k}^\gamma.
\]
This implies that there exists \( k \) such that \( \sum r_{j,k}^\gamma \geq \frac{1}{C_B} \nu^\sharp(G) \). This implies
\[
\mathcal{P}_\nu^\gamma(G) \geq \frac{1}{C_B} \nu^\sharp(G). \tag{5}
\]
So if \( F = \bigcup_{j \in I} G_j \), one has
\[
\sum r_j^\gamma(G_j) \geq \frac{1}{C_B} \sum \nu^\sharp(G_j) \geq \frac{1}{C_B} \nu^\sharp(F) > 0,
\]
so \( \mathcal{P}_\nu^\gamma(F) > 0 \). Therefore, \( \dim_P F \geq \gamma \). Then \( \mathfrak{H} \) easily follows.

Lemma 4. Let \( \mu \) and \( \nu \) be elements of \( \mathcal{F}^* \) and \( \mathcal{F} \) respectively. Set \( \varphi(t) = B_{(\mu, \nu), S_\mu}(t, 1) \) and assume that \( \varphi(0) = 0 \) and \( \nu^\sharp(S_\mu) > 0 \). Then one has
\[
\nu^\sharp\left( X_{\mu}(-\varphi'_r(0), -\varphi'_l(0)) \right) = 0,
\]
where \( \varphi'_l \) and \( \varphi'_r \) are the left and right hand side derivatives of \( \varphi \).

The same result holds with \( \varphi(t) = \tau_{(\mu, \nu), S_\mu}(t, 1) \).
Proof. Take \( \gamma > -\varphi'(0) \), and consider the set
\[
E(\gamma) = \left\{ x \in \mathbb{S}_\mu : \limsup_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} > \gamma \right\}.
\]
If \( x \in E(\gamma) \), for all \( \delta > 0 \), there exists \( r \leq \delta \) such that \( \mu(B(x, r)) \leq r^\gamma \). Consider a partition of \( E(\gamma) \): \( E(\gamma) = \bigcup E_j \).
For \( \delta > 0 \), for all \( j \), one can find a \( \delta \)-cover \( \{B_{j,k}\} \) of \( E_j \) such that \( \mu(B_{j,k}) \leq r_{j,k}^\gamma \).
We have, for any \( t > 0 \),
\[
\nu^t(E_j) \leq \sum_k \nu^t(B_{j,k}) \leq \sum_k \nu(B_{j,k}) = \sum \mu(B_{j,k})^{-t} \mu(B_{j,k})^t \nu(B_{j,k}) \leq \sum \mu(B_{j,k})^{-t} r_{j,k}^\gamma \nu(B_{j,k}),
\]
which, together with the Besicovitch property, implies
\[
\nu^t(E(\gamma)) \leq C_B \mathcal{P}(-t,1,\gamma)(E_j)
\]
and
\[
\nu^t(E(\gamma)) \leq C_B \mathcal{P}(-t,1,\gamma)(\mathbb{S}_\mu).
\]
So, if \( \gamma t > \varphi(-t) \), we have \( \nu^t(E(\gamma)) = 0 \). But, since \( \gamma > -\varphi'(0) \), this happens for small enough positive \( t \).
We conclude that
\[
\nu^t \left( \left\{ x \in \mathbb{S}_\mu : \limsup_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} > -\varphi'(0) \right\} \right) = 0.
\]
In the same way, one proves that
\[
\nu^t \left( \left\{ x \in \mathbb{S}_\mu : \liminf_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} < -\varphi'(0) \right\} \right) = 0.
\]

Corollary 5. With the same notations and hypotheses as in the previous lemma, one has
\[
\dim_H X_\mu(-\varphi'(0), -\varphi'(0)) \geq \inf \left\{ \lim_{n \to \infty} \frac{\log \nu(B(x, r))}{\log r} : x \in X(-\varphi'(0), -\varphi'(0)) \right\}
\]
and
\[
\dim_P X_\mu(-\varphi'(0), -\varphi'(0)) \geq \inf \left\{ \lim_{n \to \infty} \frac{\log \nu(B(x, r))}{\log r} : x \in X(-\varphi'(0), -\varphi'(0)) \right\}.
\]

The previous lemmas contain the nowadays classical results on multifractal analysis \([6, 3, 0]\).
Indeed, let \( \mu \) be an element of \( \mathcal{P}^* \). Till the end of this section, we will write \( b, \tau, \) and \( B \) instead of \( \mu, \tau_\mu, S_\mu, \) and \( B_\mu, S_\mu \). For \( q \geq 0 \), take \( \nu(B) = \mu(B) q^\tau B(q) \).
Then the corresponding \( \varphi \) of Lemma 4 is \( B_{\mu,\nu}(l,1) = B(q + t) - B(q) \) and, for \( x \in \mathbb{X}_\mu(\alpha) \), one has
\[
\limsup_{r \searrow 0} \frac{\log \nu(B(x, r))}{\log r} = q \limsup_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} + B(q) \leq q \alpha + B(q).
\]
So, due to Lemma 3 (2) one gets
\[ \dim P X_\mu(\alpha) \leq \inf_{q \leq 0} q \alpha + B(q). \]

In the same way, we get
\[ \dim P X_\nu(\alpha) \leq \inf_{q \leq 0} q \alpha + B(q). \]

If moreover we assume that \( \mathcal{H}_\mu, B(q)(S_\mu) > 0 \), we have \( \nu^\sharp(S_\mu) > 0 \), and therefore, due to Lemma 4
\[ \nu^\sharp(\{ X_\mu(-B'_r(q), -B'_r(q)) \}) > 0. \]

Therefore, due to Lemma 6, we have
\[ \dim H \{ X_\mu(-B'_r(q), -B'_r(q)) \} \geq \begin{cases} -q B'_r(q) + B(q) & \text{if } q \geq 0, \\ -q B'_r(q) + B(q) & \text{if } q \leq 0. \end{cases} \]

Recall that the Legendre transform of a function \( \chi \) is defined to be \( \chi^*(\alpha) = \inf_{q \in \mathbb{R}} q \alpha + \chi(q) \).

All this gives a new proof of the following theorem (see \[ 2 \] in the totally disconnected case, \[ 3 \] in general).

**Theorem 6.** If \( B \) has a derivative at some point \( q \in J_B \) and if \( \mathcal{H}_\mu, B(q)(S_\mu) > 0 \), then
\[ \dim H X_\mu(-B'(q)) = B^*(B'(q)). \]

The same statement holds with \( \tau \) instead of \( B \).

In \[ 3 \] it is shown that if \( B'(q) \) exists and if \( \dim H X_\mu(-B'(q)) = B^*(-B'(q)) \), then \( b(q) = B(q) \).

We now deal with the case when \( b(q) \neq B(q) \). The following notations will prove convenient: for a real function \( \psi \), we set
\[ \psi^\sharp(q) = \limsup_{t \to 0} \frac{\psi(q - t) - \psi(q)}{-t} \quad \text{and} \quad \psi^\flat(q) = \limsup_{t \to 0} \frac{\psi(q + t) - \psi(q)}{t}. \]

**Lemma 7.** Let \( \mu \) and \( \nu \) be elements of \( \mathcal{F}^* \) and \( \mathcal{F} \) respectively. Set \( \varphi(t) = b_{(\mu, \nu), S_\mu}(t, 1) \) and assume that \( \varphi(0) = 0 \) and \( \nu^\sharp(S_\mu) > 0 \). Then one has
\[ \nu^\sharp \left( \left\{ x \in S_\mu : \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} > -\varphi^\sharp(0) \right\} \right) = 0 \]
and
\[ \nu^\sharp \left( \left\{ x \in S_\mu : \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} < -\varphi^\flat(0) \right\} \right) = 0. \]

**Proof.** Take \( \gamma > -\varphi^\flat(0) = \liminf_{r \to 0} \frac{\varphi(\frac{-t}{r})}{-t} \) and consider the set
\[ E = \left\{ x \in S_\mu : \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} > \gamma \right\}. \]

For all \( x \in E \), there exists \( \delta > 0 \) such that, for all \( r < \delta \), one has \( \mu(B(x, r)) \leq r^\gamma \).

Set \( E_\delta = \left\{ x \in S_\mu : \forall r \leq \delta, \mu(B(x, r)) \leq r^\gamma \right\} \). If \( \{ B_j \}_j \) is any centered \( \delta \)-cover of \( E_\delta \), one has, for any \( t > 0 \),
\[
\nu^\sharp(E_\delta) \leq \sum_j \nu^\sharp(B_j) \leq \sum_j \nu(B_j) \\
\leq \sum_j \mu(B_j)^{-\gamma} \mu(B_j)^{\gamma} \nu(B_j) \leq \sum_j \mu(B_j)^{-\gamma} r_j^\gamma \nu(B_j)
\]
Proposition 9. \( \text{and, for } q > 0 \text{ or } q < 0 \text{ or } q \geq b, \)

Due to the choice of \( \gamma \) there exists \( t > 0 \) such that \( H_{\gamma}^{-t}(S_{\mu}) = 0. \) This proves the first assertion. The second one is proved in the same way.

Proposition 8. Let \( \mu \) be an element in \( \mathcal{F} \). Suppose that for some \( q \in \mathbb{N}_0 \),

\[ H_{\mu}^{\mathbb{N}_0} \beta(q)(S_{\mu}) > 0, \]

and consider the set

\[ E = \left\{ x \in S_{\mu} : \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} \leq -b_\gamma(q) \text{ and } \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq -b_\gamma'(q) \right\}. \]

Then we have

\[ \dim_{\mu} E \geq b(q) - q b_\gamma(q), \quad \text{if } q \geq 0, \]

\[ b(q) - q b_\gamma'(q), \quad \text{if } q \leq 0. \]

In particular, if \( b(q) \) exists one has

\[ \dim_{\mu} \left\{ x \in S_{\mu} : \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} \leq -b(q) \leq \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} \right\} \geq b(q) - q b_\gamma(q). \]

Proof. This results from Lemma \([7]\) and Lemma \([5, 6]\).

4. An example

Now, we can deal with the example given in \([3]\) (Theorem 2.6). We take for \( \mathbb{X} \) the space \( \{0, 1\}^{2^n} \) endowed with the ultrametric which assigns diameter \( 2^{-n} \) to cylinders of order \( n \).

We are given two numbers such that \( 0 < p < \tilde{p} \leq 1/2 \) and a sequence of integers \( 1 = t_0 < t_1 < \cdots < t_n < \cdots \) such that \( \lim_{n \to \infty} t_n/t_{n+1} = 0. \)

We define a probability measure \( \mu \) on \( \{0, 1\}^{2^n} \): the measure assigned to the cylinder \( [\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n] \) is

\[ \mu([\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n]) = \prod_{j=1}^{n} w_j, \]

where

- if \( t_{2k-1} \leq j < t_{2k} \) for some \( k \), \( w_j = p \) if \( \varepsilon_j = 0, w_j = 1-p \) otherwise,
- if \( t_{2k} \leq j < t_{2k+1} \) for some \( k \), \( w_j = \tilde{p} \) if \( \varepsilon_j = 0, w_j = 1-\tilde{p} \) otherwise.

As a matter of fact, the measure considered in \([3]\) is obtained by taking the image of \( \mu \) under the natural binary coding of numbers in \( [0, 1] \) composed with the Gray code. The purpose of using the Gray code was to get a doubling measure on \([0, 1] \).

For \( q \in \mathbb{R} \), define

\[ \theta(q) = \log_2(p^q + (1-p)^q) \quad \text{and} \quad \tilde{\theta}(q) = \log_2(\tilde{p}^q + (1-\tilde{p})^q). \]

It results from \([3]\) that for \( 0 < q < 1 \) we have

\[ b(q) = \theta(q) < \tilde{\theta}(q) = B(q). \]

and, for \( q < 0 \) or \( q > 1 \),

\[ b(q) = \tilde{\theta}(q) < \theta(q) = B(q). \]

We have the following result.

Proposition 9. \( (1) \) For \( \alpha \in (-\log_2(1-\tilde{p}), -\log_2 \tilde{p}) \), we have

\[ \dim_H X_{\mu}(\alpha) = \inf_{q \in \mathbb{R}} b(q) + \alpha q. \]
(2) For $\alpha \in (-\log_2(1-\tilde{p}), -\log_2 \tilde{p}) \setminus \left([-B'(0), -B'_1(0)] \cup [-B'_1(1), B'_1(1)]\right)$, we have

$$\dim_p X_\mu(\alpha) = \inf_{q \in \mathbb{R}} B(q) + \alpha q.$$ 

Proof. We consider the measure $\nu$ constructed as $\mu$ with parameters $r$ and $\tilde{r}$ instead of $p$ and $\tilde{p}$. We impose the condition

$$r \log p + (1 - r) \log(1 - p) = \tilde{r} \log \tilde{p} + (1 - \tilde{r}) \log(1 - \tilde{p}).$$

(5)

As both $r$ and $\tilde{r}$ should belong to the interval $(0, 1)$, we must have

$$\log \frac{1 - p}{1 - \tilde{p}} < r \log \frac{1 - p}{p} < \log \frac{1 - p}{\tilde{p}}.$$ 

(6)

From Corollary 2 it is easy to compute $\varphi(x) = \tau_{(\nu, \nu), \mathcal{S}_\nu}$: we have

$$\varphi(x) = \log_2 \max \left\{ \left(p^x r + (1-p)^x(1-r)\right), \left(\tilde{p}^x \tilde{r} + (1-\tilde{p})^x(1-\tilde{r})\right) \right\}.$$ 

Then it results from Lemmas 4 and 3-b that

$$\varphi(0) = -r \log_2 p - (1-r) \log_2(1-p) = r \log_2 \frac{1-p}{p} - \log_2(1-p).$$

(7)

It results from (6) that $\alpha$ can take any value in the interval $(-\log_2(1-\tilde{p}), -\log_2 \tilde{p})$.

Besides, the strong law of large numbers shows that we have

$$\liminf_{n \to \infty} \frac{\log_2 \nu(B(x, 2^{-n}))}{-n} = \min\{h(r), h(\tilde{r})\}$$

and

$$\limsup_{n \to \infty} \frac{\log_2 \nu(B(x, 2^{-n}))}{-n} = \max\{h(r), h(\tilde{r})\}$$

for $\nu$-almost every $x$, where we set $h(r) = -r \log_2 r - (1-r) \log_2(1-r)$.

Then it results from Lemmas 4 and 3-b that

$$\dim_H X_\mu(\alpha) \geq \min\{h(r), h(\tilde{r})\}$$

(8)

and

$$\dim_p X_\mu(\alpha) \geq \max\{h(r), h(\tilde{r})\},$$

(9)

where $r$, $\tilde{r}$, and $\alpha$ are linked by Relations (5) and (7).

If $\alpha$ is defined by (7), we have

$$\alpha = -\bar{\theta}(q) \quad \text{if} \quad q = \frac{\log \frac{1-r}{p}}{\log \frac{1-p}{p}} \quad \text{and} \quad \alpha = -\bar{\theta}(\bar{q}) \quad \text{if} \quad \bar{q} = \frac{\log \frac{1-\tilde{r}}{p}}{\log \frac{1-\tilde{p}}{p}}.$$ 

(10)

Now, fix $q$ and $\bar{q}$ as above in (10). One can check that, for these values of $q$ and $\bar{q}$, one has

$$\theta(q) - \theta'(q) = h(r) \quad \text{and} \quad \tilde{\theta}(\bar{q}) - \tilde{\theta}'(\bar{q}) = h(\tilde{r}).$$

(11)

In order to have $\theta(q) = b(q)$, we must have $0 < q < 1$, which means

$$\log_2 \frac{1}{p^q(1-p)^{1-p}} < \alpha < \log_2 \frac{1}{\sqrt{p}(1-p)}.$$ 

(12)

In order to have $\tilde{\theta}(\bar{q}) = b(\bar{q})$, we must have $\bar{q} < 0$ or $\bar{\bar{q}} > 1$, which means

$$\alpha > \log_2 \frac{1}{\sqrt{\bar{p}(1-\bar{p})}}.$$ 

(13)
\[ \alpha < \log_2 \frac{1}{p^p(1 - p)^{1 - p}}. \] (14)

One can check that at least one of the conditions (12), (13) and (14) is fulfilled.

But for any \( q \) such that \( b'(q) \) exists, we have (see [5] or [1])
\[ \dim_H X_{\mu}(-b'(q)) \leq b(q) - q b'(q). \] (15)

The first assertion then results from (8), (11), and (15).

In order to have \( \theta(q) = B(q) \), we must have \( q < 0 \) or \( q > 1 \), which means
\[ \alpha > \log_2 \frac{1}{\sqrt{p(1 - p)}} = -B'_l(0) \] or
\[ \alpha > \log_2 \frac{1}{p^p(1 - p)^{1 - p}} = -B'_l(1). \]

In order to have \( \tilde{\theta}(\tilde{q}) = B(\tilde{q}) \), we must have \( 0 < \tilde{q} < 1 \), which means
\[ -B'_l(1) = \log_2 \frac{1}{p^p(1 - p)^{1 - p}} < \alpha < \log_2 \frac{1}{\sqrt{p(1 - p)}} = -B'_l(0). \]

Then Assertion (2) follows as previously.

5. The Vector Case

As in [6] instead of \( \mu(B) \) one may consider expressions of the form \( \exp -\langle q, \kappa(B) \rangle \), where \( \kappa \) takes its values in the dual \( E' \) of a separable Banach space \( E \) and \( q \in E \).

Let \( \nu \) be an element of \( E \). For \( E \subset X, \ q \in E, \ t \in \mathbb{R}, \) and \( \delta > 0 \), one sets
\[ \overline{\mathcal{P}}^{\delta,t}(E) = \sup \left\{ \sum r_j e^{-\langle q, \kappa(B_j) \rangle} \nu(B_j) ; \{ B_j \} : \delta \text{-packing of } E \right\}, \]
\[ \overline{\mathcal{P}}^{\delta}(E) = \lim_{\delta \searrow 0} \overline{\mathcal{P}}^{\delta,t}(E), \]
\[ \overline{\mathcal{P}}^{\delta,t}(E) = \inf \left\{ \sum \overline{\mathcal{P}}^{\delta,t}(E_j) ; E \subset \bigcup E_j \right\}, \]
and
\[ \overline{\mathcal{H}}^{\delta,t}(E) = \inf \left\{ \sum r_j e^{-\langle q, \kappa(B_j) \rangle} \nu(B_j) ; \{ B_j \} \text{ centered } \delta \text{-cover of } E \right\}, \]
\[ \overline{\mathcal{H}}^{\delta}(E) = \lim_{\delta \searrow 0} \overline{\mathcal{H}}^{\delta,t}(E), \]
\[ \overline{\mathcal{H}}^{\delta,t}(E) = \sup \left\{ \overline{\mathcal{H}}^{\delta,t}(F) ; F \subset E \right\}, \]

For a function \( \chi \) from \( E \) to \( \mathbb{R} \), and for \( v \in E \) of norm 1, one defines
\[ \partial_v \chi(0) = \lim_{t \searrow 0} \frac{\chi(tv) - \chi(0)}{t} \]
and
\[ \partial^*_v \chi(0) = \limsup_{t \searrow 0} -\frac{\chi(tv) - \chi(0)}{t}. \]

With these notations we have the following analogues of Lemmas [4] and [7]:

**Lemma 10.** Let \( \varphi(q) \) be one of the following functions:
\[ \inf \left\{ t ; \overline{\mathcal{P}}^{\delta,t}(X) = 0 \right\} \quad \text{or} \quad \inf \left\{ t ; \overline{\mathcal{H}}^{\delta,t}(X) = 0 \right\}. \]
Assume that $\varphi(0) = 0$ and that $\partial_v \varphi(0)$ at 0 is a lower semi-continuous function of $v$. Then one has

$$\nu^\sharp \left\{ x ; \liminf_{r \searrow 0} \frac{\langle v, \chi(B(x,r)) \rangle}{-\ln r} < -\partial_v \varphi(0) \text{ for some } v \in E \right\} = 0.$$  

**Lemma 11.** Set $\varphi(q) = \inf \{ t ; \mathcal{H}^{v,t}(X) = 0 \}$ and assume that $\varphi(0) = 0$ and that $\partial_v^* \chi(0)$ is a lower semi-continuous function of $v$. Then one has

$$\nu^\sharp \left\{ x ; \limsup_{r \searrow 0} \frac{\langle v, \chi(B(x,r)) \rangle}{-\ln r} < -\partial_v^* \varphi(0) \text{ for some } v \in E \right\} = 0.$$  

The proofs follow the same lines as above and as the proofs in [6]. As a corollary we get the following result (with the notations of [6]).

**Theorem 12.** Let $B(q) = \inf \{ t \in \mathbb{R} ; \mathcal{H}^{q,t}(X) = 0 \}$. Assume that, at some point $q$, the function $B$ is differentiable with derivative $B'(q)$ and that $\mathcal{H}^{q,B(q)}(X) > 0$. Then one has

$$\dim_H \left\{ x ; \forall v \in E, \lim_{r \searrow 0} \frac{\langle v, \chi(B(x,r)) \rangle}{\log r} = -B'(q)v \right\} = B(q) - B'(q)q.$$  

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