Critical behaviour in quantum gravitational collapse

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We study the gravitational collapse of an inhomogeneous scalar field with quantum gravity corrections associated with singularity avoidance. Numerical simulations indicate that there is critical behaviour at the onset of black hole formation as in the classical theory, but with the difference that black holes form with a mass gap.

Keywords: gravitational collapse, quantum gravity, critical phenomena, singularity avoidance.

I. INTRODUCTION

One of the outstanding problems in theoretical physics is the incomplete understanding at the quantum level of the formation, and subsequent evolution of black holes in a quantum theory of gravity. Although a subject of study for over three decades, it is fair to say that, in spite of partial results in string theory and loop quantum gravity, there is no widely accepted answer to many of the puzzles of black hole physics. This is largely because there has been no study of quantum dynamical collapse in these approaches. Rather, progress has focused mainly on explanations of the microscopic origin of the entropy of static black holes from state counting. A four-dimensional spacetime picture of black hole formation from matter collapse, and its subsequent evolution is not available in any approach to quantum gravity at the present time.

This paper describes an attempt to address this problem in the context of Hawking’s original derivation of black hole radiation: spherically symmetric gravity minimally coupled to a massless scalar field. This is a non-linear 2d field theory describing the coupled system of the metric and scalar field degrees of freedom.

Gravitational collapse in the classical theory in this model has been carefully studied numerically but its full quantization has never been addressed. The classical results are well known; the onset of black hole formation is characterized by a scaling law for black hole mass and a self-similar behavior of the field variables.

In semi-classical theory, Hawking’s calculation uses the eikonal approximation for the wave equation in a mildly dynamical background, where the dynamics centers on the surface of a star undergoing collapse. The essential content of it is the extraction of the phase of the ingoing mode from an outgoing solution of the scalar wave equation as a classically collapsing star crosses its Schwarzschild radius. According to this calculation, black holes create particles that originate near the event horizon. The approximation breaks down at the late stages of evaporation, where quantum gravity effects are expected to become important.

II. CLASSICAL COLLAPSE

The Einstein equations coupled to a massless minimally coupled scalar field take the form

$$R_{ab} = 8\pi\partial_a\phi\partial_b\phi$$

In spherical symmetry the resulting equations may be written with the metric ansatz

$$ds^2 = -f(r,t)^2dt^2 + g(r,t)^2dr^2 + r^2d\Omega^2,$$

with $$\phi = \phi(r,t)$$, or in double null coordinates as

$$ds^2 = -4\alpha(u,v)dudv + r^2(u,v)d\Omega^2.$$
The main result is that gravitational collapse at the threshold of black hole formation exhibits critical behaviour. For a parameter $a$ in the initial data for the scalar field, numerical simulations give a mass formula

$$M_{BH} \sim (a - a_*)^\gamma$$  \hspace{1cm} (4)$$

where $a_*$ is a critical value of $a$ above which black holes form. Both $a_*$ and the critical exponent $\gamma$ are numerically determined. Furthermore the field variables exhibit a discrete self-similarity of the form

$$\phi(r, t) = \phi(re^\Delta, te^\Delta)$$  \hspace{1cm} (5)$$

where the constant $\Delta$ is also numerically computed.

In the parametrization in eqn. (3) with $\alpha(u, v) := g(u, v) r'(u, v)$, where $r'$ denotes the derivative with respect to $v$, the field equations in four dimensions may be written in the compact form

$$\dot{r} = -\frac{\bar{g}}{2},$$  \hspace{1cm} (6)$$

$$\dot{h} = \frac{1}{2r^2} (h - \phi) (gr - 4\bar{g})$$  \hspace{1cm} (7)$$

where dot denotes partial derivative with respect to $u$, and we have defined

$$h = \phi + \frac{1}{4} r^2 \phi',$$  \hspace{1cm} (8)$$

$$g = \exp \left[ 8\pi \int_u^v \frac{1}{r} (h - \phi)^2 \, dv \right],$$  \hspace{1cm} (9)$$

$$\bar{g} = \frac{1}{2} \int_u^v g \, dv$$  \hspace{1cm} (10)$$

A numerical integration scheme for these equations proceeds by using a “space” $v$ discretization

$$h(u, v) \rightarrow h_i(u), \quad r(u, v) \rightarrow r_i(u)$$  \hspace{1cm} (11)$$

to obtain a set of coupled ODEs, where $i = 0, \cdots, N$ specifies the $v$ grid. Initial data for these two functions with, suitable boundary conditions, are prescribed on a constant $u = u_0$ initial slice, from which the functions $g$ and $\bar{g}$ are constructed using a Simpson’s rule integration. Evolution in $u$ “time” is performed using the forth order Runge-Kutta method.

For a physical picture, it is convenient to specify the initial scalar field configuration as $\phi(r(v, u_0), u_0)$, since the metric function $r$ is a dynamical variable in these coordinates. We use the data

$$r(u_0, v) = v$$  \hspace{1cm} (12)$$

$$\phi(u_0, r(u_0, v)) = ar^2 \exp \left[ -\left( \frac{r - r_0}{\sigma} \right)^2 \right].$$  \hspace{1cm} (13)$$

The initial data parameters are $a, \sigma, r_0$.

At each $u$ step of the evolution using Eqns. (6-7), the function

$$g^{ab} \partial_a \partial_b r = -2 r \frac{\dot{r}}{g}$$  \hspace{1cm} (14)$$

is computed. Its vanishing signals the formation of an apparent horizon. For each run of the code with fixed amplitude $a$, this function is scanned from larger to smaller radial values after each Runge-Kutta iteration. Evolution is terminated if a root is detected within a threshold value. The corresponding radial value is taken as the apparent horizon radius. This procedure is repeated for a range of $a$ values until its critical value $a_*$ is determined by a bisection method. For the subcritical case all radial grid points evolve to zero without horizon detection. This is the signal for complete reflection of the scalar field pulse.

Figure 1 displays typical results of a run of such a code for the Gaussian scalar field profile specified above. The results are in accord with other studies and serve as a useful check of our results. The slope of the line in this figure is $\gamma = 0.375$ giving a mass formula of the type (4) with the critical value $a^* = 0.0220025$ (for $\sigma = 1$ and $r_0 = 1$ in the initial data (13). The oscillation around the best-fit line is also a known feature of classical collapse. This data is actually produced from a code for the quantum gravity corrected equations (to be described below), but with the fundamental discreteness scale $L$ in these equations set to zero.
We turn now to describing an approach for obtaining quantum gravity modifications of the above equations that incorporate singularity avoidance.

III. QUANTUM GRAVITY CORRECTIONS

The origin of quantum corrections to gravitational collapse due to fundamental discreteness may be understood by considering quantum systems on a lattice. In spatial lattice based quantizations of field theory, local configuration variables are sampled discretely and momentum variables are realized indirectly via translation operators. The lattice may be uniform (equispaced) or irregular. One can construct a non-separable Hilbert space whose basis elements are labeled by the lattice chosen, in addition to other quantum numbers. Such a space may be viewed as the (infinite) sum of the Hilbert spaces, one for each lattice.

The kinematical Hilbert space of loop quantum gravity (LQG) is such a space. The classical variables used in LQG, the holonomy of a connection and a surface observable associated with its conjugate momentum, are non-local. Their quantization is realized on a space with basis states labelled by graphs embedded in a three-manifold. Operators may also be labeled by graphs, and hence connect states with different graph labels. An example in LQG is a proposal for a “graph changing” hamiltonian constraint\(^\textsuperscript{[14]}\), which is defined to carry an edge that gets attached to a graph state. Related quantizations are available for scalar field theory\(^\textsuperscript{[20]}\) and a similar construction exists for fermionic theories.

A. Quantum Mechanics on a Lattice

There are two types of quantum gravity effects that can arise from fundamental discreteness. These come from the way inverse configuration and momentum operators must be defined on a lattice. Both are readily demonstrated by considering the mechanics of a particle on a lattice.

A feature of spatial lattice theories is that there is a natural way to define inverse configuration operators via finite differencing. For a particle moving on the real line, a quantization on a uniform lattice sampled from the real line with points \(x_n = na, n \in \cdots -1, 0, 1 \cdots \) has an associated Hilbert space \(|n\rangle\) with inner product
\[
\langle m | n \rangle = \delta_{mn} \quad \text{(15)}
\]
on which we have the operators
\[
|\hat{x}|n\rangle = na |n\rangle \quad \text{(16)}
\]
\[
|\hat{U}_\lambda(p)|n\rangle \equiv \exp(i\lambda p) |n\rangle = |n - 1\rangle. \quad \text{(17)}
\]
We can define a densely defined operator corresponding to the variable \(|x|^{-1}\), or other inverse configuration operators, by realizing a lattice finite difference scheme for the derivative of \(f(x_n)\) such as
\[
\frac{df(x_n)}{dx} \to \frac{f(x_n + \lambda) - f(x_n - \lambda)}{2\lambda} \quad \text{(18)}
\]
on the Hilbert space\(^\textsuperscript{[20]}\). For example applying it to the function \(f(x) = \sqrt{|x|}\) gives the scheme
\[
\frac{1}{|x|} = \left(2 \frac{d\sqrt{|x|}}{dx}\right)^2 \quad \text{(19)}
\]
The right hand side of this expression is the eigenvalue of the operator
\[
\sqrt{|x|} : = \frac{1}{\lambda^2} \left(\hat{U}_\lambda \sqrt{|x|} \hat{U}_\lambda^\dagger - \hat{U}_\lambda^\dagger \sqrt{|x|} \hat{U}_\lambda\right)^2 \quad \text{(20)}
\]
in the basis \(|n\rangle\). The eigenvalue is bounded, and hence may be viewed as realizing an aspect of “singularity avoidance” in a theory with a fundamental discreteness scale.

The finite difference idea captures the essence of the definition on the inverse triad operator in LQG\(^\textsuperscript{[15]}\) and the inverse scale factor operator in loop quantum cosmology\(^\textsuperscript{[16]}\). In the general case, without symmetry reduction, such operators are not bounded unless these are restriction to suitably defined semiclassical subspaces\(^\textsuperscript{[19]}\).
Furthermore, it is clear that there are as many such operators as finite difference schemes in addition to the freedom in the choice of function \( f(x) \); for instance a more general case is \( f(x) = |x|^{1/k} \) for \( k > 1 \). These are all examples of quantization ambiguities.

So far we have seen an example, in a simplified context, of an operator and its eigenvalue that arises from the necessity of defining operators corresponding to inverse configuration variables. In quantum gravity there is a need to define operators corresponding to inverse metric functions such as the determinant of the 3-metric, which arises in the hamiltonian constraint. A second type of modification comes from the fact that momentum operators do not exist on a lattice. As already noted these must be defined indirectly using the translation operators \( \hat{U}_\lambda(p) \). One such definition is

\[
\hat{p}_\lambda := \frac{1}{i\lambda} \left( \hat{U}_\lambda - \hat{U}_\lambda^\dagger \right) \quad (21)
\]

Such expressions lead to “momentum corrections” due to fundamental discreteness, which in quantum gravity come from the variables canonically conjugate to the spatial metric – the ADM momentum. The momentum operators can also be used to define lattice creation and annihilation

\[
A_\lambda^{\pm} \equiv \hat{x} \pm ip_\lambda. \quad (22)
\]

In the following study of gravitational collapse of a scalar field, we consider only the former type of correction, with the understanding that a complete treatment would require both. The generalizations of the operators to the field theory case are similar.

### B. Gravity-Scalar Field Model

The application of this type of procedure to the gravity-scalar field collapse problem in spherical symmetry requires a lattice-based Hamiltonian quantization scheme. This has been developed recently\[2\]. It contains a prescription for defining inverse field operators to represent variables such as powers of \( 1/R(r,t) \), where \( R(r,t) \) is the metric variable that is the measure of the size of spheres. (\( r \) and \( t \) are radial and time coordinates.) Such factors arise in the Hamiltonian constraint and the field equations, as is evident from Eqn. \[1\], (where we have referred to this function as \( r(u,v) \)).

The quantization route we follow is unconventional in that field momenta are not represented as self-adjoint operators; rather only exponentials of momenta are realized on the Hilbert space. This is similar to what happens in a lattice quantization, except that, as we see below, every quantum state represents a lattice sampling of field excitations, with all lattices allowed. Any numerical computation is of course only possible if it is restricted to a fixed lattice subspace. In this sense non-separable Hilbert spaces are not computationally relevant.

The gravitational phase space variables for the model are the configuration variables \( (R, \phi) \) and their conjugate momenta \( (P_R, P_\phi) \). The basic Poisson bracket we consider is

\[
\{R_f, e^{i\lambda P_R(r)}\} \equiv \left\{ \int_0^\infty Rf \, dr, e^{i\lambda P_R(r)} \right\} = i2G\lambda f(r)e^{i\lambda P_R(r)}, \quad (23)
\]

where \( f \) is a suitable smearing function.

This Poisson bracket may be represented on a Hilbert space with basis

\[
|e^{i\sum_k a_k P_R(x_k)}, e^{iL^2\sum_i b_i P_\phi(y_i)}\rangle = |a_1\ldots a_{N_1}; b_1\ldots b_{N_2}\rangle, \quad (24)
\]

where the factors of \( L \) in the exponents reflect the length dimensions of the respective field variables, and \( a_k, b_i \) are real numbers which represent the excitations of the scalar quantum fields \( R \) and \( \phi \) at the radial locations \( \{x_k\} \) and \( \{y_i\} \). The inner product on this basis is

\[
\langle a_1\ldots a_{N_1}; b_1\ldots b_{N_2}|a'_1\ldots a'_{N_1}; b'_1\ldots b'_{N_2}\rangle = \delta_{a_1,a'_1}\ldots \delta_{b_{N_2},b'_{N_2}},
\]

if the states contain the same number of sampled points, and is zero otherwise.

The action of the basic operators are given by

\[
\hat{R}_f |a_1\ldots a_{N_1}; b_1\ldots b_{N_2}\rangle \equiv L^2\sum_k a_k f(x_k)|a_1\ldots a_{N_1}; b_1\ldots b_{N_2}\rangle, \quad (25)
\]

\[
e^{i\lambda P_R(x)}|a_1\ldots a_{N_1}; b_1\ldots b_{N_2}\rangle = |a_1\ldots a_j - \lambda_j,\ldots a_{N_1}; b_1\ldots b_{N_2}\rangle, \quad (26)
\]

where \( a_j \) is 0 if the point \( x_j \) is not part of the original basis state. In this case the action creates a new excitation at the point \( x_j \) with value \(-\lambda_j\). These definitions give the commutator

\[
[\hat{R}_f, e^{i\lambda P_R(x)}] = -\lambda f(x)L^2 e^{i\lambda P_R(x)}. \quad (27)
\]

Comparing this with \[23\], and using the Poisson bracket commutator correspondence \( \hbar \{, \} \leftrightarrow [\,,\,] \) gives \( L = \sqrt{2}l_P \), where \( l_P \) is the Planck length. There are similar operator definitions for the canonical pair \( (\phi, P_\phi) \).

This quantization is distinct from the LQG inspired “polymer” approach to field theory\[20\] where it is the configuration variables that are represented in exponential form, following the representation of holonomy operators in LQG. The present approach is more akin to conventional quantization in spatial lattice field theory. It may be viewed as the “dual” of the polymer approach.
C. Inverse R operators

The functionals

\[ R_f = \int_0^\infty df R \]

(28)

used as the configuration variables serve as our starting point for defining inverse operators. The procedure for doing this is similar to that of the particle example. Classical identities such as

\[ \frac{1}{|R_f|} \equiv \left( \frac{2}{i\lambda G f(r)} e^{-i\lambda P_\eta(r)} \left[ \sqrt{|R_f|}, e^{i\lambda P_\eta(r)} \right] \right)^2, \]

(29)

where the functions \( f \) do not have zeroes, may be used to define operators. For example a suitable choice is a sharply peaked Gaussian at the point \( r_k \), which also serves to localize the operator. (It is useful to choose the Gaussians such that \( f(r_k) = 1 \).

The representation for the quantum theory described above is such that the operator corresponding to \( R_f \) has a zero eigenvalue. Therefore we represent \( 1/R_f \) using the r.h.s. of (29). The corresponding operator

\[ \frac{1}{|R_f|} \equiv \left( \frac{2}{i\lambda P_\eta(x_j)} e^{-i\lambda P_\eta(x_j)} \left[ \sqrt{|R_f|}, e^{i\lambda P_\eta(x_j)} \right] \right)^2, \]

(30)

is densely defined and bounded. This may be illustrated with the basis state

\[ |a_0\rangle \equiv |e^{i\lambda P_\eta(r=0)}\rangle, \]

(31)

which represents an excitation \( a_0 \) of the quantum field \( \hat{R}_f \) at the coordinate origin:

\[ \hat{R}_f |a_0\rangle = (2i\lambda P_\eta f(0)) a_0 |a_0\rangle, \]

(32)

\[ \frac{1}{|R_f|} |a_0\rangle \equiv \frac{2}{(\lambda P_\eta f(0))^2} \left( |a_0|^{1/2} - |a_0 - 1|^{1/2} \right)^2 |a_0\rangle \]

(33)

which is clearly bounded. If there is no excitation of \( R_f \) at the coordinate origin, i.e. \( a_0 = 0 \), the upper bound on the eigenvalue of the inverse operator is \( 2/\lambda^2 f_\eta^2 \). (In the units we are using, \( P_\eta \) is dimensionless, so \( \lambda \) is a dimensionless number, which can be taken to be unity.)

A symmetrical version of this operator is defined as for the particle case in Eqn. (29). Its eigenvalue on the basis state above is of the same functional form as in Eqn. (19):

\[ \frac{1}{(\lambda P_\eta f(0))^2} \left( \sqrt{|a_0 + \lambda|} - \sqrt{|a_0 - \lambda|} \right)^2 \]

(33)

D. Semiclassical states

The basis states and their associated eigenvalues are not the ones we consider useful for quantum corrections to the collapse problem discussed in Sec. 2. Rather we would like to find suitable semiclassical states

\[ |R(r_k)\rangle_{sc}, \]

(34)

where \( R(r_k) \) is the sampling on a uniform radial lattice of any prescribed function \( R(r) \), with the properties that for each lattice point \( r_k \) we have

\[ \langle \hat{R}_f \rangle_{sc} = R(r_k) \]

(35)

and

\[ \langle \frac{1}{|R_f|} \rangle_{sc} = \frac{1}{(\lambda P_\eta f(r_k))^2} \times \left( \sqrt{|R(r_k) + \lambda|} - \sqrt{|R(r_k) - \lambda|} \right)^2 \]

(36)

The right hand side of the latter function has the property that for \( R(r_k) >> \lambda \), it behaves like \( 1/R(r_k) \) but has a different functional form for \( R(r_k) \sim \lambda \), just as in the particle mechanics case.

It is possible to explicitly construct such states. The idea is to associate semiclassical states for a single particle on a line, with the lattice points \( \{r_k\} \), such that a state at point \( r_i \) is peaked at coordinate value \( R(r_i) \). The state \( |R(r_k)\rangle \) is then defined to be the product of such single particle states. It represents a quantum state corresponding to the classical profile \( R(r) \). (It is convenient to take \( f(r_k) = 1 \) and set \( \lambda P_\eta = L \), which we do in the following.)

E. Modified Collapse Equations

A derivation of quantum gravity corrections to the field equations requires a number of inputs depending on the approach taken. One approach is to make use of semiclassical states peaked on classical configurations. Given a definition of constraint operators on a kinematical Hilbert space, one then computes expectation values of the constraints in such states. This results in quantum corrected “effective constraints,” which to leading order in \( \hbar \) are the classical constraints. The idea is to use these new constraints to derive evolution equations. For example, if \( H(\hat{x}, \hat{p}) \) is such an operator, its expectation value in a semiclassical state \( |\hat{x}, \hat{p}\rangle \) would give a function \( H(\hat{x}, \hat{p}) \). This function would then be used to derive quantum corrected equations. This is the approach we take. Similar methods have been used to derive effective equations in loop quantum cosmology and partly form the motivation for our work.

It is apparent from sample computations that this approach gives the classical equations in suitable limits. For example if there is a \( 1/x \) factor in a function \( H(x, p) \) (as is the case in the Hamiltonian constraint), the equation of motion also contains a power of \( 1/x \). The corresponding quantum corrected equation will have a power of the
spectrum function of the corresponding operator as in Eqn. (19). Given the form of this function, it is apparent that the large $x$ limit gives the classical equation as shown in Figure 2, and that the small $x$ region represents a “repulsion,” if the eigenvalue represents a potential energy.

A possible (and perhaps obvious) criticism of this approach to obtaining quantum corrected equations is that one is deviating from the ideal of solving the quantum constraints a’la Dirac and obtaining the physical Hilbert space before proceeding to study physical questions. This approach has however not yielded any results beyond simple mini-superspace reductions, in the metric or the connection variable constructions. In order to proceed it is important to see what quantum gravity effects can be extracted from a kinematical construction of constraint operators. The approach outlined above is one example of such a procedure, but it is an approximation which requires careful scrutiny especially with regard to the consistency of the quantum corrected constraints to the appropriate order.

The eigenvalue of the $1/R$ field operator on basis states contains the modulus of the eigenvalue of the radial field operator $\tilde{R}$. This causes a numerical problem in the quantum modified equations where it is necessary to compute derivatives with respect to a radial coordinate. To avoid this issue we replace the expectation value (36) by the smoothed version

$$\left\langle \frac{1}{R_f} \right\rangle \rightarrow \frac{1}{\tilde{R}} \left( 1 - e^{-\frac{(R/L)^2}{2}} \right)$$

for numerical calculations. This form has the same qualitative features as the actual eigenvalue, i.e. the large $R$ behaviour is $1/R$ with repulsion at small $R$, as in Fig. 2. It is possible that an exact computation of this expectation value in semiclassical states will provide a smooth function, so that such an approximation would become unnecessary. However we expect that the qualitative features of the result described below will remain unchanged.

Modified collapse equations may be obtained by first computing effective constraints and obtaining equations of motion from them, or by directly replacing the expectation value (37) in place of factors of $1/r$ into the equations from Sec. II. These two procedures of course give different effective equations. The procedure we used amounts to the latter since all the corrections from the former approach have not been incorporated into the code used. The modification is such that for large spheres (i.e. large $\tilde{R}$), the equations converge to the classical equations, with quantum gravity corrections confined to smaller $R$ values determined by the fundamental discreteness scale $L$. As already noted, we do not include the momentum corrections in the present work; this would amount to replacing momenta in the constraints such as $P_R$ by the compactified form $\sin(\lambda P_R) / \lambda$.

IV. RESULTS

The numerical procedure used for the quantum gravity corrected equations is identical to the one outlined in Sec. II. The simulations were performed for a range of $L$ values, with the integration lattice chosen such that time and space steps ranged from $10^{-3}$ to $10^{-5}$. The initial data used was the same as that for the classical case.

The results for horizon radius $R_{bh}$ as a function of the initial data parameter $a$ are given in Fig. 3, for fundamental discreteness scale $L$ values ranging from 0 to 0.2. There are a number of points to note here: (i) $L = 0$ gives no mass gap, and the expected classical result. (ii) Mass gaps at the onset of black hole formation are evident, with the critical value of the amplitude $a^*$ dependent on $L$; the gaps increase with $L$. (iv) For amplitudes $a >> a^*$ the points corresponding to different $L$ values begin to merge, i.e. for sufficiently massive initial data, the black hole masses become independent of $L$. This is expected intuitively because bigger masses should be less affected by smaller scale quantum effects than smaller masses. (v) The critical amplitude $a^*$ depends on $L$ in a rather unusual way: for non-zero $L$, $a^*$ decreases with increasing $L$. This means that black holes form more readily for larger $L$ but the effect is rather small as far as the parameter ranges are concerned.

Our results in Fig. 3 can be summarized in the black hole mass formula

$$M_{BH} = m_0(L,a) + k[a - a^*(L)]^{\gamma(L,a)},$$

in the supercritical region $a > a^*$, where $m_0$ is the mass gap and $k$ and $\gamma$ are numerically determined constants. It is apparent from the data that the mass gap $m_0$ increases with $L$, and that the exponent $\gamma$ is also a mild function of $L$. Dependence on the amplitude $a$ is also apparent in that for large values, the points appear to converge.

V. SUMMARY AND DISCUSSION

We have described a first exploration of possible quantum gravity corrections in the gravitational collapse of an inhomogeneous scalar field in spherical symmetry. Numerical simulations indicate that there is critical behavior at the onset of black hole formation, and that black holes form with a mass gap unlike in classical theory.

There are general grounds to expect the mass gap result given that there is a fundamental discreteness scale in quantum gravity. This is simply that any physical object must have mass and size in units of this scale. It is gratifying that this is borne out, especially in the details of its dependence on both this scale and on the amplitude of the initial data.

A mass gap in black hole formation has been noted before in a model where an exterior generalized Vaidya solution is patched to an interior Friedmann-Robertson-Walker mode. This is also related to singularity avoidance in that the interior incorporates the “inverse scale
factor” bound used in quantum cosmology\[16,17\]. We note that there is no critical behaviour in this model because there is only matter inflow, and a black hole always forms by construction regardless of the initial data. This of course is a feature of Oppenheimer-Snyder type models of gravitational collapse with pure inflow. Only models with both matter inflow and outflow (i.e. metric functions fully dependent on both $r, t$ or $u, v$) have the possibility of critical behaviour, with long time evolution leading either to black hole formation or full reflection.

There are a number of directions for further work based on the approach we have used. These include incorporating the momentum modification into the evolution equations, and a larger exploration of the parameter space. With this in place it would be especially interesting to explore the nature of the exactly critical solution $a = a^\ast$; in the classical theory this is a naked singularity that realizes a finely tuned violation of the cosmic censorship hypothesis. With quantum gravity corrections the critical solution might be a boson star\[20\].

A further challenge is to set up a numerical procedure that can be used to evolve data past horizon formation. This will likely require coordinates such as the flat slice ones\[22\] that have the potential to reveal how the scalar field and horizon evolve beyond its formation. If the horizon is found to shrink, it would be accompanied by a flux of scalar field away from the hole region, and should leave a clear numerical signature in the evolution of this field. An application of our approach to the possibly simpler 3-dimensional case with negative cosmological constant may be easier to study in this respect.

Another direction with regard to such questions is the path integral. There is a derivation of black hole radiance in this approach\[28\], and it would be interesting to see how quantum gravity corrections arise in the spin foam\[20\] approach. Given the understanding of 3-dimensional gravity\[29\] using spin foam methods, this may be a first example to consider.

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\[\text{FIG. 3: Black hole radius as a function of initial data amplitude } a \text{ for various values of the fundamental discreteness scale } L. \text{ Black hole mass gap increases with } L.\]

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