Motion of Quantized Vortices as Elementary Objects

Uwe R. Fischer

Lehrstuhl für Theoretische Festkörperphysik, Institut für Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14, D-72076 Tübingen, Germany

(June 14, 2021)

Abstract

The general local, nondissipative equations of motion for a quantized vortex moving in an uncharged laboratory superfluid are derived from a relativistic, co-ordinate invariant framework, having vortices as its elementary objects in the form of stable topological excitations. This derivation is carried out for a pure superfluid with isotropic gap at the absolute zero of temperature, on the level of a hydrodynamic, collective co-ordinate description. In the formalism, we use as fundamental ingredients that particle number as well as vorticity are conserved, and that the fluid is perfect. No assumptions are involved as regards the dynamical behaviour of the order parameter. The interaction of the vortex with the background fluid, representing the Magnus force, and with itself via phonons, giving rise to the hydrodynamic vortex mass, are separated. For a description of the motion of the vortex in a dense laboratory superfluid like helium II, two limits have to be considered: The nonrelativistic limit for the superfluid background is taken, and the motion of the vortex is restricted to velocities much less than the speed of sound. The canonical structure of vortex motion in terms of the collective co-ordinate is used for the quantization of this motion.
I. INTRODUCTION

Vortices are the fundamental line-like excitations of a fluid. Their motion governs a large number of properties of the fluid under given external conditions. Superfluids are peculiar in that the vortices can exist only in certain classes, distinguished by integers. This fact that vorticity can only arise quantized, results in far-reaching consequences for the character of vortex generation and motion in quantum fluids, as compared to simple perfect fluids. The aim of the present investigation lies in the derivation of the general local equations of motion for a quantized vortex in a laboratory superfluid at absolute zero, thereby starting from a minimum of very basic assumptions about the co-ordinate invariant nature of the hydrodynamic conservation laws in a relativistic perfect fluid and the Magnus force acting on the vortex. In particular, no assumptions will be made with respect to the underlying dynamics of the order parameter of the superfluid.

The most fundamental property of a vortex living in a superfluid is its topological stability. It is a topological line defect belonging to the first homotopy group \( \Pi_1(G/H) \), which contains the equivalence classes of loops around a singularity: The circle \( S^1 \) in real space is mapped onto the coset space \( G/H \), and the equivalence classes identified. Here, \( G \) is the symmetry group of the system under consideration, and \( H \) is the little group, a subgroup of \( G \), characterizing the partial symmetry of the system which remains after the symmetry of the larger group \( G \) has been broken. This coset space represents the manifold of the degenerate vacuum states after spontaneous symmetry breakdown, and can be identified with the order parameter. In this paper, we will deal exclusively with a very simple such coset space, namely the circle \( S^1 \) (the group \( G \) is U(1) and \( H \) is the identity), so that the mapping in question is between the circle in ordinary and that in order parameter space. The winding number \( N_v \in \mathbb{Z} = \Pi_1(S^1) \) classifying the paths around the line equivalent to each other, indicates the number of times one is ‘winding’ around the singular line in order parameter space, if one is going once around it in real space (see Fig. 1). The complex order parameter of a spontaneously broken U(1) symmetry, with phase \( \theta \), then gives rise to the quantization of the nonrelativistic superfluid circulation \( \Gamma_s \) in units of \( \kappa = \hbar/m \),

\[
\Gamma_s = \oint \vec{v}_s \cdot d\vec{s} = \frac{\hbar}{m} \oint \nabla \theta \cdot d\vec{s} = N_v \cdot \frac{\hbar}{m} \equiv N_v \cdot \kappa,
\]

provided the superfluid velocity is identified as \( \vec{v}_s = (\hbar/m)\nabla \theta \). For a vortex in the superfluid helium II, the superfluid particle’s mass \( m = 6.64 \cdot 10^{-27} \) kg, so that \( \kappa \simeq 10^{-7} \) m\(^2\)/s. The definition in (1) is the classical \textit{kinematical} definition of circulation [1,2]. We will see below that a relativistically invariant, \textit{dynamical} definition has to use the line integral of the momentum, rather than that of the velocity, to gain truly invariant meaning. However, in any case, be it relativistic or nonrelativistic, the stability (and very definability) of the quantized vortex as a fundamental object able to persist while moving in the superfluid stems essentially from its topological properties. The superfluid cannot be continuously deformed into a vortex-free state. On the other hand, this feature of topological stability has its limits if large quantum statistical fluctuations of the order parameter become important. In the dense, unpaired superfluid helium II, they become large on the same coherence length scales, on which the order parameter modulus varies, of the order of the inter-particle distance. Hence a vortex, having curvature scales of the singular line, which are of order the inter-particle distance, is not definable topologically and thus does not exist as a well-defined line...
defect in order parameter space. On these scales, it has to be defined using the full quantum many-body structure of the dense superfluid and the corresponding vortex wave functions.

This hydrodynamic level of accuracy can only be afforded by the fully relativistic description of the dense liquid we present in the section which follows, too, because the same problem of incorporating microscopic many-body structure excitations into the description remains, which exists already in the nonrelativistic superfluid. A hydrodynamic formalism is thus a necessity to describe vortex motion properly, as long as this problem of microscopic dynamics has not been solved. The covariant treatment, in turn, provides structural insights which are not given by more conventional treatments.
II. DUALITY OF VORTICES AND CHARGED STRINGS

We establish in this section the properties of a vortex as a stringlike fundamental object. For this purpose, the means of dual transformation will be instrumental (where dual is meant in the sense of being ‘equivalent as a physical system’). We will introduce the duality of interest here in its relativistic context (established in Refs. [3,4], elaborations in various directions can be found in Refs. [5]–[12]), subsequently reducing it to its nonrelativistic limit.

Conventions are that the velocity of light \( c \) is set equal to unity and the signature \( \eta_{\mu\nu} = \text{diag} (-1, 1, 1, 1) \) of the metric is being employed. Greek indices mean spacetime indices and take the values 0, 1, 2, 3. The sign convention we use for the Levi-Civita pseudotensor is \( \epsilon^{0123} = +1 = -\epsilon_{0123} \) in a Lorentz frame. The dual of a \( p \)-form \( \mathbf{f} \) in four-dimensional spacetime has the contravariant components \( *f^{\beta_1...\beta_p} = (1/p!)f_{\alpha_1...\alpha_p}\epsilon^{\alpha_1...\alpha_p\beta_1...\beta_p} \).

The hydrodynamic conservation law we will use as a first basic ingredient of our theory for an uncharged relativistic fluid is that of particle number. The most familiar mathematical form of nonrelativistic number conservation is provided by \( \partial_t \rho + \text{div}(\rho \mathbf{v}) = 0 \). This is expressed covariantly, using \( p \)-forms [17], as

\[
d \wedge \mathbf{j} = 0 \quad (d \equiv dx^\alpha \partial_\alpha). \quad (2)
\]

The number current one-form is defined \( \mathbf{j} \equiv \rho \mathbf{u} = \rho u_\alpha dx^\alpha \), where \( dx^\mu \) is a one-form basis dual to a co-ordinate basis and \( \rho \) the rest frame number density. We omit the subscript \( s \)
from the relativistic quantity $u$, whose normalisation will be required by its interpretation as a four-velocity to be $u^a u_a = -1$.

The momentum of a perfect fluid particle is defined as the one-form $p$, with components
\begin{equation}
 p_\alpha = \mu u_\alpha .
\end{equation}

The chemical potential $\mu = \partial \epsilon / \partial \rho$, where $\epsilon$ is the rest frame energy density, then plays the role of an effective rest mass in a Hamiltonian quadratic in the four-momentum. The momentum and number current density one-forms are thus related through
\begin{equation}
 \frac{\rho}{\mu} p_\alpha = \frac{K}{\hbar^2} p_\alpha = j_\alpha .
\end{equation}

The quantity $K = \hbar^2 (\rho / \mu)$ is the stiffness coefficient against variations of the order parameter phase (a quantity to be defined below) in the free energy, cf. \[18\] and section 3.1 in \[20\]. The speed of sound $c_s$ is related to the quantities introduced in the above, for a barotropic fluid \[13\], as $c_s^2 = d(\ln \mu) / d(\ln \rho) = (K/\hbar^2) d^2 \epsilon / dp^2$.

We define the vorticity as the dual of the exterior derivative of $p$:
\begin{equation}
 \omega = \ast d \wedge p \iff \ast \omega = -d \wedge p .
\end{equation}

The vorticity thus defined is required to be conserved, in the form
\begin{equation}
 d \wedge \ast \omega = 0 \iff d \wedge d \wedge p = 0 .
\end{equation}

Outside the cores of the vortices, where the quantity vorticity $\omega$ defined above equals zero, the momentum of the fluid always remains proportional to the (exterior) derivative of a scalar $\theta$, which we identify with the phase of the U(1) order parameter pertaining to the superfluid:
\begin{equation}
 p_\alpha = \hbar \partial_\alpha \theta .
\end{equation}

The circulation is in the relativistic case defined as the integral of the momentum. This choice of definition stems from the fact that a particle’s rest mass, occurring in the nonrelativistic definition of the circulation (1), is an undefined quantity in a fully relativistic, dense superfluid. The proper circulation thus reads
\begin{equation}
 \gamma_s = \oint p_\mu dx^\mu = N_v \hbar ,
\end{equation}

and is quantized into multiples of Planck’s quantum of action: The constant of proportionality $\hbar = \hbar / 2\pi$ in (7) stems from the covariant Bohr-Sommerfeld quantization of $\gamma_s$, \textit{i.e.} from the fact that we require $p$ to be a quantum variable. It may be worthwhile to point out in this context that particularly striking evidence for the fundamentality of the one-form quantity momentum, $p = p_\mu dx^\mu$, is given by consideration of superfluid rotation in the presence of gravity \[19\], \textit{e.g.}, in neutron stars. Superfluid irrotationality is equivalent, for axial symmetry, to $p_\phi = 0$, whereas the contravariant axial component $p^\phi$ and thus the contravariant axial velocity do not have to vanish.
The fact that quantization of circulation as a line integral of momentum is crucial for a proper understanding of the nature of the line defect vortex is not connected to the fact that the superfluid is relativistic or nonrelativistic. The point is rather that the vortex is in general nothing but a singular line of zeroes in the order parameter manifold, designated by a single number \( N_v \), which represents the order of the pole. This fundamental, intrinsic property of the vortex is not related to a quantity like mass, characterizing the matter with which it interacts, but is independent of matter properties.

The normalisation of \( u_\mu \), by using the metric and writing \( u_\mu u_\alpha = -1 \), is promoting \( u_\mu \) into a four-velocity. It leads, with (3) and (7), to the relation

\[
\mu = h \left( -\partial_\mu \theta \partial^\mu \theta \right)^{1/2}
\]

between the phase and the chemical potential. This is a relativistically covariant version of the Josephson equation \( \hbar \dot{\theta} = - \left( \mu + \frac{1}{2} m \bar{v}^2 \right) \), familiar from nonrelativistic condensed matter physics [23]. The Josephson equation expresses, generally, the conjugateness of the two canonical variables phase and number density in the hydrodynamic limit.

The dual of the current \( j_\mu \) is a 3-form \( \ast j = j_\mu \epsilon_{\mu\nu\alpha} \, dx^\nu \wedge dx^\alpha \wedge dx^\beta \), which is, according to the conservation law above, closed. We define the field strength \( H \) by

\[
\ast j = \rho \mu \ast p = h \rho \mu \ast d\theta = d \wedge b = H,
\]

The field \( H_{\mu\nu\alpha} \) is totally antisymmetric in its three indices and has, by definition, only four independent components. In a simply connected region, \( H \) is exact, i.e., it is the exterior derivative of a gauge 2-form \( b = b_{\mu\nu} \, dx^\mu \wedge dx^\nu \):

\[
H_{\alpha\beta\gamma} = \partial_\alpha b_{\beta\gamma} + \partial_\beta b_{\gamma\alpha} + \partial_\gamma b_{\alpha\beta},
\]

The field strength \( H \) is invariant under gauge transformations \( b \rightarrow b + d \wedge \Lambda \), where \( \Lambda = \Lambda_\alpha \, dx^\alpha \) is an arbitrary 1-form. Thus

\[
b_{\mu\nu} \rightarrow b_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \quad \Rightarrow \quad H_{\mu\nu\alpha} \rightarrow H_{\mu\alpha\nu}.
\]

We then have the following sequence of relations, which define the correspondences of \( \rho, \mu, \theta \) and \( b \):

\[
\ast j = \rho \mu \ast p = h \rho \mu \ast d\theta = d \wedge b = H,
\]

The dual transformation in the conventional sense [34] obtains if we neglect any possible variations of \( \rho/\mu = K/\hbar^2 \) in the spacetime domain of interest. Then, \( b \) has only one degree of freedom corresponding to the order parameter phase \( \theta \).

From the equation \( H = (\rho/\mu) \ast p \), it follows that

\[
\ast d \wedge \left( \frac{\mu}{\rho} \ast H \right) = \ast d \wedge \left( \frac{\hbar^2}{K} \ast H \right) = \ast d \wedge p = \omega,
\]

which is the general field equation for the motion of vortices with conserved vorticity \( \omega \). It is a field equation analogous to the inhomogeneous Maxwell equation. In case that the
stiffness ratio of density and chemical potential is a constant $K_0$ in space and time, and in Lorenz gauge $\mathbf{d} \wedge \mathbf{b} = 0$, the wave equation of the gauge field takes the familiar form

$$\Box \mathbf{b} = -(K_0/h^2) \mathbf{\omega},$$

(15)

where the d’Alembertian $\Box \equiv \partial_\mu \partial^\mu$. The homogeneous Maxwell equation for strings is, by definition, for any value of $\rho/\mu$,

$$\mathbf{d} \wedge \mathbf{H} = 0 \quad \text{(arbitrary } \rho/\mu = K/h^2).$$

(16)

The advantage of using $\mathbf{b}$ over $\theta$ consists in the fact that it can be chosen to be single-valued, which is to be contrasted with the necessary multivaluedness of $\theta$ in the presence of a vortex. The physical significance of $\mathbf{b}$ is that it represents a generalization of the stream function concept encountered in classical nonrelativistic hydrodynamics [2]. Namely, integrating

$$\int_{\partial \Omega} b_{\mu\nu} dx^\mu \wedge dx^\nu = \int_{\Omega} \rho u^\lambda \epsilon_{\lambda\mu\nu} dx^\mu \wedge dx^\nu \wedge dx^\alpha,$$

(17)

we see that the integral of $\mathbf{b}$ over any 2-surface $\partial \Omega$ enclosing a 3-surface $\Omega$ in spacetime is given by the superflow flux through the 3-surface (the symbol $| \cdots |$ indicates ordering of the indices contained between the vertical lines with increasing numerical value). The 2-form quantity $\mathbf{b}$ can thus be called a generalized stream tensor. The relation (17) expresses a generalization of the usual concept of stream function (for which the incompressibility condition $\rho = \rho_0$ holds), to flows for which the density can vary.

In particular, in the nonrelativistic case, which has $u^0 = 1$, setting $\rho = \rho_0$,

$$\frac{1}{2} \int_{\partial \Omega} b_{ij} dx^i \wedge dx^j = \rho_0 V(\Omega),$$

(18)

that is, the specified surface integral of the purely spatial part of $\mathbf{b}$ is the spatial volume enclosed by the surface $\partial \Omega$ times the density $\rho_0$, and thus the bulk number of particles contained in the volume $\Omega$.

The superfluid we will be dealing with later in section [11] is nonrelativistic. We thus have to rewrite (13) in a form having only Galilean invariance. This is accomplished by writing (13) (with $\rho = \rho_0$) for spatial and temporal indices separately,

$$\rho_0 u_0^\lambda \epsilon_{0ijk} = \partial_k b_{ij} + \partial_j b_{ki} + \partial_i b_{jk} = H_{ijk},$$

$$\rho_0 u_i^i \epsilon_{0ijk} = \partial_k b_{0j} - \partial_j b_{0k} + \frac{1}{c} \partial_t b_{jk} = H_{0jk},$$

(19)

and taking the Galilean limit $c \to \infty$ ($u^0 \to c$, $u^i \to v^i_s$; we have temporarily reinstated the velocity of light for this purpose). We then get the relations (in which $\epsilon_{0ijk} = -\epsilon_{ijk} = -\sqrt{g} n_{ijk}$):

$$-\sqrt{g} n_{ijk} \rho_0 = \partial_k b_{ij} + \partial_j b_{ki} + \partial_i b_{jk},$$

(20)

$$\sqrt{g} n_{ijk} v_s^i = \partial_k \psi_j - \partial_j \psi_k,$$

(21)

where the vectorial version of the usual stream function [3] is defined through

$$\psi_i \equiv b_0 i / \rho_0.$$

(22)

The determinant of the spatial co-ordinate system we are using is designated $g$, and $n_{ijk} = n_{[ijk]} = \pm 1$ is the unit antisymmetric symbol.
A. The Magnus force

The fundamental force acting on a superfluid vortex at the absolute zero of temperature is the Magnus force. This force is now shown to be equivalent to a stringy generalization of the Lorentz force. Given this identification, we will be able to write down electrodynamic correspondences in the next subsection.

The Magnus force acting on a vortex in a nonrelativistic superfluid has the standard form

$$\vec{F}_M = m \rho_0 \vec{\Gamma}_s \times \left( \vec{X} - \vec{v}_s \right),$$

where $\vec{v}_s$ is the superflow velocity far from the vortex center located at $X^i(t, \sigma)$ (i.e. the flow field at the vortex location without the contribution of the vortex itself), and $\sigma$ is the arc length parameter labeling points on the vortex string. The circulation vector $\vec{\Gamma}_s = N_V \kappa \vec{X}'$ of the vortex is, for positive $N_V$, pointing along the $z$-direction in a right-handed system.

In the second line, the force is written in a form manifestly independent of a ‘mass’ value. A relativistic generalization of the above expression then reads [21]:

$$\left( F_M \right)_\alpha = \gamma_s H_{\alpha \mu \nu} \vec{X}^\mu \vec{X}^\nu.$$  \hspace{1cm} (24)

We introduced the relativistic string co-ordinates $X^\mu = X^\mu(\tau, \sigma)$, and let $X'^\mu \equiv \partial X^\mu / \partial \sigma$ be the line tangent, as well as $\dot{X}^\mu \equiv \partial X^\mu / \partial \tau$ the vortex velocity. The essential features of the vortex as a two-dimensional object, living in spacetime, are represented in Figure 2.

FIG. 2. The vortex is represented by the string world sheet embedded in spacetime and hence described by the co-ordinates $X^\mu(\tau, \sigma)$. The tangent space basis vectors have components $\dot{X}^\mu$ in the timelike and $X'^\mu$ in the spacelike direction on the world sheet, where the metric is given by $\gamma_{ab} = g_{\mu \nu} \partial X^\mu / \partial \zeta^a \partial X^\nu / \partial \zeta^b$, with $a, b = 1, 2$, $\zeta^1 = \tau$, $\zeta^2 = \sigma$. 
It is to be stressed that both of the forces (23) and its relativistic counterpart (24) are forces per unit σ-length and are of topological origin. They do not depend on the local shape of a line segment, represented in the relativistic case by the world sheet metric γ_{ab}, but only on the local external field \( H_{αµν} \), generated by other line segments and an externally imposed flow field (cf. the equation for the corresponding action (25) and the discussion at the beginning of section III B).

The Magnus force can be derived from the variation of the action (cf., in particular, the Refs. [21,6,9,11])

\[
S_M = \gamma_s \int dτdσ b_{µν} X^µ X^ν \\
= \gamma_s \int d^4 x b_{µν} ω^{µν}, \tag{25}
\]

that is to say

\[
\frac{δS_M}{δX^α} = \gamma_s H_{αµν} X^µ X^ν = (F_M)_α. \tag{26}
\]

The singular vorticity tensor components are thence given by

\[
ω^{µν} = \gamma_s \int \int dτdσ \left( X^µ X^ν - \dot{X}^µ X^ν \right) δ(4)(x - X(t, σ)). \tag{27}
\]

From the vorticity tensor, we obtain the components of the usual vorticity vector by choosing \( X^0 = t \), a choice possible in any global Lorentz frame [11]:

\[
ω^{0i} = \gamma_s \int dσ \dot{X}^i δ(3)(x - X(t, σ)). \tag{28}
\]

For a rectilinear line in \( z \)-direction, \( ω^{02} = \gamma_s δ(x - X) δ(y - Y) \), which integrated over the \( x-y \) plane gives the circulation. The vorticity tensor components (27) are, then, to be understood as a generalization of a quantity surface density of circulation, in the sense that

\[
\gamma_s \iint d^2 S_{|µν|} |ω_{|αβ|}| \equiv \iint ω^{µν} d^2 S_{|µν|}. \]

Integration is over a (sufficiently small, i.e. local) 2-surface with element \( d^2 S_{|µν|} = -δ_{|αβ|} dx^α ∧ dx^β \), which is threaded by the vortex world sheet.

### B. Electromagnetism of vortex strings

It proves very suggestive to cast the laws we found into the language of Maxwell’s electrodynamics. From the familiar Lorentz force law

\[
(F_{Lorentz})_α = qF_{αµ} \dot{X}^µ, \tag{29}
\]

one can define that the ‘electric’ and ‘magnetic’ fields are represented by a projection of \( H_{µνα} \) on the local vortex axis, i.e. on the tangent space vector along the spacelike direction on the world sheet [14]:

\[
(F_{Lorentz})_α = qF_{αµ} \dot{X}^µ, \tag{29}
\]
\[ F_{\mu\nu} \equiv H_{\mu\nu\beta}X'^\beta = \epsilon_{\mu\nu\alpha\beta}J^\alpha X'^\beta, \]
\[ \vec{E} = \rho_0 \vec{u} \times \vec{X}', \quad \vec{B} = \epsilon_{0123} \rho_0 u_0 \vec{X}'. \] (30)

The second line is valid for a global Lorentz frame, in which, as already mentioned above, we can choose \( X'^0 = t \). In general, the density \( \rho \) in the field \( H_{\mu\nu\beta} \), and thus also \( F_{\mu\nu} \), is arbitrary; at the location of the line itself and for the Lorentz/Magnus force law, however, we set \( \rho = \rho_0 \). In our sign convention for \( \epsilon_{0123} = -1 \), the local ‘magnetic field’ is antiparallel to the local tangent.

The local ‘electromagnetic’ 3-potential will accordingly be defined via
\[ a_\mu = b_{\mu\sigma} \equiv b_{\mu\nu}X'^\nu. \] (31)

In an arrangement of cylindrical symmetry, fulfilled by a ring vortex, the 3-potential is the vector \( a^\mu = (a^0, a^r, a^z) = (-b_0, b_r, b_z) \). The component \( \psi_S \equiv b_0/\rho_0 \) corresponds for stationary flows and \( \rho = \rho_0 \) to Stokes’ stream function [2]. The function \( \psi_S \) fulfills
\[ 2\pi \psi_S = 2\pi \int r(u^r(r, z)dz - u^z(r, z)dr), \] according to (17). The static scalar potential of our ‘vortodynamics’ is thus in the cylindrically symmetric case given by \( a^0 = \rho_0 \int r(u^z dr - u^r dz) \). Integrating this relation to obtain the value of \( a^0(r_0, z_0) \) at some point \( r_0, z_0 \) in the superfluid, the line integrals run over the envelope of an arbitrary surface of revolution, which is generated by rotating the integration line, joining the point at \( r_0, z_0 \) and a point on the \( z \)-axis of symmetry [2].

In making the above definitions (30), we identified \( q \equiv \gamma_s = N_v \hbar \) with the ‘charge’ per unit length of vortex. This vortex ‘charge’ is thus of topological nature, quantized by the index of the homotopy group member (in units with \( \hbar = 1 \), the charge \( q = 2\pi N_v \)). The quantization of the charge vanishes in the purely classical limit of \( \hbar \to 0 \), whereas the ‘electromagnetic’ field strength \( F \) is understood to be derived from a (superfluid or not) conserved number current.

We can define the vortex current four-vector in a fashion analogous to the definition of the potential in (31), from the minimal coupling term represented in (25):
\[ \Omega^\mu \equiv \omega^{\mu\sigma} \equiv \omega^{\mu\nu}X'_\nu, \] (32)
that is, we project the vorticity current on the local vortex axis. This completes the picture of the electromagnetism analogy we wanted to develop. We have constructed, within the semiclassical realm, a fundamental topological object, the vortex, which moves according to familiar laws of electrodynamics, by projecting the string equations of motion on the local spacelike vortex tangent axis. The field equations have the form of Maxwell equations local in \( \sigma \),
\[ d \wedge F = 0, \quad ^* d \wedge (\hbar^2 F/K) = \Omega, \] (33)
where the field \( F \) is the usual 2-form and the current \( \Omega \) a 1-form, whose components are defined in (24) and (32), respectively. The vortices can be understood as fundamental, ‘charged’ objects, generating ‘electromagnetic’ fields, which in turn act on them by the local Lorentz force in (28).

\[ ^1 \text{We remark that we have chosen not to incorporate the factor } 4\pi, \text{ appearing in the cgs units} \]
The equations (33) are representations of the conservation laws of particle number, vorticity, and the equation (3) characterizing properties of the medium, which relates these conservation equations to each other. The homogeneous ‘Maxwell’ equation is a projection of number conservation, whereas the inhomogeneous ‘Maxwell’ equation is equivalent to a projection of (the conservation of) vorticity, combined with a property of the medium. *Vice versa*, we conclude that the Maxwell equations of ordinary electromagnetism can be cast into the form of conservation equations of relativistic perfect fluid hydrodynamics, provided we admit the identifications specified in (13).

C. Comparison of relativistic and nonrelativistic notions

The identifications we have made pertain to the vortex as a stringlike fundamental object and are suitable for a relativistic framework. In the remainder of this paper, however, we will only be concerned with nonrelativistic Eulerian flows. At this stage it is thus useful to compare and contrast the relativistic notions and those of classical nonrelativistic hydrodynamics [1,2].

The most important difference of the two frameworks occurs if we consider the two definitions (1) and (8) for the circulation. The *kinematical* quantity velocity suitable for its definition in nonrelativistic circumstances can not be used for its relativistic definition. The *dynamical* quantity average angular momentum per particle, though, which is quantized into units of $\hbar$ because of the existence of a macroscopic superfluid phase, is a well-defined quantity for any superfluid. Because of this fact, the definition in (8) is the truly invariant definition of circulation. The only possibility to use a ‘velocity line integral’ for the definition of circulation is to employ the perfect fluid relation \( p_\alpha = \mu u_\alpha \). Then, however, one has the complication of a possible position dependence of \( \mu \).

A prescription to translate from the fully relativistic superfluid to terms of standard nonrelativistic hydrodynamics is afforded by the limits \( \mu \to m, \ u_\alpha \to (1, v^s_i) \), and the replacement \( \gamma^s \to \Gamma^s \).

III. CANONICAL STRUCTURE OF VORTEX MOTION

A. Phonons and the Vortex Mass

In quantum electrodynamics, the fundamental entities interact between each other and with themselves by photons. The corresponding quanta in our view of a superfluid are Maxwell equations, into the definition of the vortex charge, *i.e.* wrote the ‘Maxwell’ equations in Heaviside-Lorentz units 22.

2The term ‘Eulerian’ is used for a fluid obeying in its (non-dissipative) dynamics a continuum version of Newton’s equation of motion, the Euler equation (there exist also generalizations of this equation for curved space-time backgrounds, cf. §22.3 in 17).
phonons, the excitations of the ‘vacuum ether’ surrounding the ‘charged’ strings. The role of these excitations is addressed in this section.

The vortices acquire a nonzero hydrodynamic mass from their self-interaction with these phonon excitations. This is most easily understood if one realizes that in a 2+1d superfluid there exists a direct correspondence between the genuinely relativistic electron-positron pair interaction via photons in quantum electrodynamics and a ‘relativistic’ point-vortex-point-antivortex pair interaction via phonons [24,25] (we will use throughout inverted commas to distinguish pseudorelativistic \(\equiv\) ‘relativistic’ behaviour from the actual Lorentz invariance of section II). In the light of the vortex – fundamental object correspondence expounded in the preceding section this is only natural: The only ‘vacuum’ excitations (‘vacuum’ in the sense of the spontaneously symmetry-broken superfluid vacuum), by means of which different vortex line segments can interact are, in the hydrodynamic limit, phonons. What remained to be done is to convert the quite direct 2+1d correspondence into a 3+1d correspondence by defining the respective quantities local on the string, \(i.e.\) as functions of \(\sigma\). Then, the phonons give ‘relativistic’ fields propagating on a nonrelativistic background of Eulerian superflow, and mediating interactions between (singular) vortex line segments.

To explain this further, we begin by considering that the unit circulation vortex carries, per unit length, the hydrodynamic self-energy

\[
E_{\text{self}} = \frac{\hbar^2 \rho_0}{4\pi m} \left[ \ln \left( \frac{8 R_c}{\xi_e C} \right) \right]
\]

with it. The energy \(E_{\text{self}}\) is the energy of a vortex sitting at some fixed place, that is, its rest energy in its rest frame.

The infrared cutoff in the logarithm is in the static limit equal to the mean distance of line elements, respectively, in the localized self induction approximation, proportional to the local curvature radius \(R_c\) of the line. The constant \(C\) parameterizes the core structure [26], and has order unity. In classical hydrodynamics, with a hollow core of radius \(\xi\), \(C = 2\), whereas in a model of constant core vorticity one has \(C = 7/4\). In the Gross-Pitaevskii framework ( [27], [28], for singular vorticity, it turns out that \(C = 1.615\) [26].

If we separate off in the order parameter phase two parts,

\[
\theta = \theta_{bg} + \theta_{ph},
\]

(35)

a part due to a background flow \(\theta_{bg}\), and a part which describes sound excitations \(\theta_{ph} \ll \theta_{bg}\) on this background, we can carry through the program of the last sections for \(\theta_{bg}\) and \(\theta_{ph}\) separately. For the phase \(\theta_{bg}\), we can not retain the Lorentz invariance of the equations we derived there (our fluid is very much Eulerian), and have to use the Galilei invariant set of equations (20)-(21). On the other hand, for the part \(\theta_{ph}\), a ‘relativistic’ equation from which to start the dual transformation derivation for \(\theta_{ph}\) is the wave equation

\[
\partial_{\mu} \partial^{\mu} \theta_{ph} \equiv \left[ -\frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} + \Delta \right] \theta_{ph} = 0.
\]

(36)

This is ‘relativistic’ in the sense that the speed of light \(c\) is replaced by the speed of sound \(c_s\) in a Lorentz invariant scalar wave equation for \(\theta_{ph}\). From this part of the phase we can derive ‘electromagnetic’ phonon wave field strengths obeying the ‘relativistic’ Maxwell equations.
The nonlinear equation of motion of the superfluid, the Euler equation, gives in its linearized version, together with the continuity equation, the above equation of motion (36) for $\theta_{ph} \ll \theta_{bg}$, if the background flow is at a velocity much less than that of sound. For general background flows with velocities $(\hbar/m) |\nabla \theta_{bg}| \lesssim c_s$, there results a scalar wave equation for sound in a curved Lorentzian signature background metric with nonzero curvature. See the nice discussion in [31] (also cf. section IV in [13]). We consider, however, ‘nonrelativistic’ background flows of small Mach number, so that these corrections are not of importance here (though they may be important in other contexts like for the Aharonov-Bohm interferences of phonons leading to the Iordanskii force [32]), and the ‘acoustic metric’ is taken to be that of a global Lorentz frame.

We now have defined the entities vortices as objects defined by the superfluid vacuum background and ‘relativistically’ interacting by the small density and phase perturbations in this vacuum medium. Hence, they obey the Einsteinian mass-energy relation

$$M_0 c_s^2 = E_{\text{self}}.$$  

The mass $M_0$ is the hydrodynamic vortex mass in the rest frame of a vortex in equilibrium. If we consider a neutral, unpaired superfluid, this mass is dominant compared to other possible sources as the so-called backflow mass or the core mass corresponding to the normal fluid in the core [33]. This is true because we are in the limit of $\xi \ll R_c$, in which the logarithmic divergence of the hydrodynamic mass dominates other possible contributions. In field theoretical terms, the energy contribution of the Goldstone boson field (i.e. the phonon field) is not screened by the gauge field like in superfluids with a local dynamical gauge field, which are charged in a conventional sense. This leads to the logarithmic divergence of the energy associated with the spontaneously broken global symmetry.

It is useful to compare the definition of the vortex mass above to that of the classical electron radius $r_c$ in electromagnetism. For a homogeneously charged electron of radius $r_c$, this definition reads $m_e c^2 = (3/5) e^2 / r_c$, where the factor $3/5$ stems from the electron (‘core’) structure, which was assumed to be homogeneous. In the case of the electron, the mass serves to define, via the expression for the field energy, the value of the classical electron radius. In the case of the vortex (a line ‘charge’), the field energy of flow around the vortex serves to define the renormalized, hydrodynamic vortex mass and a cutoff needs to be introduced because this field energy is logarithmically divergent.

The knowledge of the mass $M_0$ will be sufficient for our purposes, as the formalism laid down in these pages should apply for the actual, dense superfluid and consequently is restricted to velocities of the vortex and the background superfluid much less than the speed of sound, indeed even much less than the Landau critical velocity of roton creation. Also, we will consider scales much larger than $\xi$. Correspondingly, the (local) frequencies of vortex motion will be much less than the typical frequency $\omega_s \equiv c_s / \xi$. Scaling this frequency with parameters appropriate for helium II gives

$$\omega_s = \frac{c_s}{\xi} = 9.4 \cdot 10^{11} \text{Hz} \frac{c_s [240 \text{ m/s}]}{\xi [\sigma_{LJ}]} ,$$  

which turns out be very close to the roton minimum ($\omega_{\text{roton}} \simeq 1.13 \cdot 10^{12} \text{Hz at } p \simeq 1 \text{ bar}$). We have scaled the coherence length with the Lennard Jones parameter of the helium interaction, $\sigma_{LJ} = 2.556 \text{ Å}$, and assume $\xi [\sigma_{LJ}]$ is approximately unity. The condition of ‘nonrelativistic’
velocities on scales much larger than $\xi$ is then equivalent to $\omega \ll \omega_s$. For ‘relativistic’ vortex motion frequencies in the order of and above $\omega_s$, the vortex mass will be frequency and wave vector dependent \[29\]. These high frequencies, employed in a semiclassical treatment, may be of interest in dilute superfluids. In what follows, we however restrict ourselves to describe ‘nonrelativistic’ vortex motion, which is the case of interest in a dense superfluid.

In the context of the question of vortex mass, it is advisable to point out that it is unnecessary, and indeed misleading, to use any postulated dynamical behaviour for the order parameter in the Lagrangian form (for example, of the Gross-Pitaevskii or time dependent Ginzburg-Landau variety). If we discuss the proper way of deriving the mass (37) in a dense superfluid, we just need that the phase moves according to the Josephson relation $\hbar \dot{\theta} = - \left( \mu + \frac{1}{2} m \vec{v}_s^2 \right)$ \[29\]. The validity of the Josephson relation, in turn, is not dependent on a Lagrangian for the order parameter, and is derivable from (Hamiltonian) superfluid hydrodynamics alone \[30\].

**B. The Vortex Lagrangian**

We have argued that the vortex has mass. Being a line-like, stable topological object, we assume it further to develop restoring forces if it is deformed, starting from some equilibrium position. That is, the vortex is assumed to have elastic energy, arising from the (local) interaction of a particular line segment with adjacent line segments.

Let us first spend a few words on the relation of the self-action we will write down to ‘relativistic’ vortex motion properties. In the ‘nonrelativistic’ case under study, the differential string arc length will be written as $\sqrt{\gamma(t, \sigma)} d\sigma$ (we remind the reader that the inverted commas are used to distinguish pseudorelativistic from proper relativistic terms). The self-action of the singular Nambu string\(^3\) is proportional to the world sheet area \[36\]:

\[
S_{\text{Nambu}} = - \mu_T \int \int \sqrt{-\gamma(\zeta^0, \zeta^1)} \, d^2\zeta ,
\] (39)

which is given by the double integral of the square root of the negative world sheet metric determinant $\sqrt{-\gamma}$ over the timelike $\zeta^0 = \tau$ and spacelike $\zeta^1 = \sigma$ co-ordinates, parameterizing the world sheet (cf. Figure 2). The Nambu action is minus the constant string tension $\mu_T$ (i.e., a constant self energy per unit length), multiplied by this area \[21,36,37\]. The writing for the string arc length we will employ thus indicates that we are dealing with a ‘Galilean limit’ of the ‘world sheet area’ interval, in which the only metric element is the proper length interval of the line.

The Nambu string is structureless (i.e., the core extension is negligibly small on the scales of interest), and has constant string tension. The actual superfluid vortex has a (quantum

\(^3\)We refer to this elementary extended object for brevity simply as ‘Nambu’ string, though it has been introduced independently by Nambu and Gotô \[34,35\], and thus is (properly) also quite often referred to as ‘Nambu-Gotô’ string.
mechanical) core structure of extension $\xi$ which is comparatively large and, in addition, has a string tension (the vortex mass) that depends on vortex co-ordinates through the cutoff in the renormalization logarithm. Then, on the large ($\gg \xi$) scales we have to consider the vortex can not be considered as a ‘relativistic’ Nambu string living in the spontaneously broken symmetry vacuum of the dense superfluid. Given that the velocities the vortex reaches are on these scales always less than $c_s$, as a result, the velocity dependent part of the vortex self-action is dominated by that of the Magnus action. This corresponds to the dominance of the minimal coupling term in the momentum over the kinetic part, see equation (49) below.

1. Co-ordinate frames

The co-ordinates we will be using to describe the vortex self-action are defined by local right-handed basis vectors on the string. We can, for example, choose them to be the triad

$$\vec{e}_1(\sigma) = \vec{e}_2 \times \vec{\tau}, \quad \vec{e}_2(\sigma) = -\frac{1}{\gamma} \vec{X}''', \quad \vec{\tau} = \frac{1}{\sqrt{\gamma}} \vec{X}'', \quad (40)$$

that is, the binormal, negative normal and tangent unit vectors of the line. They are related by the Serret-Frenet formulas [38] to fundamental invariant properties of the line

$$\frac{1}{\sqrt{\gamma}} \vec{\tau}' = -\frac{1}{R_c} \vec{e}_2, \quad \frac{1}{\sqrt{\gamma}} \vec{e}_2' = \frac{1}{R_c} \vec{\tau} - T \vec{e}_1, \quad \frac{1}{\sqrt{\gamma}} \vec{e}_1' = T \vec{e}_2, \quad (41)$$

where $R_c^{-1}(\sigma) = |\vec{X}''/\gamma|$ is the curvature ($R_c$ the curvature radius), and $T(\sigma)$ the torsion of the line. For a three-dimensional superfluid a typical equilibrium line configuration is a circular vortex, for which $\vec{e}_1 = \vec{e}_Z, \vec{e}_2 = \vec{e}_R, \vec{\tau} = \vec{e}_\Phi$, in a conventional cylindrical co-ordinate system with the normalised triad above.

If it appears more convenient for representation purposes, we will use a more general system of non-normalised co-ordinate basis vectors, which are defined formally by the derivatives [17]

$$\vec{e}_a = \partial/\partial X^a, \quad \vec{e}_\sigma = \partial/\partial \sigma \quad (a = 1, 2), \quad (42)$$

in our non-curved Euclidean embedding space simply acting on the position vector $\vec{X}$, thereby creating the basis. We will freely use the more convenient system, indicating the type of system by indices $a = 1, 2$ for the first normalised triad and $a = 1, 2$ for the latter general co-ordinate basis.

---

4Whereby it is meant that the characteristic velocity at the core circumference, $v_L = \kappa/2\pi\xi$, is much less than the speed of sound, if we again take $\xi \sim \sigma_{LJ}$. 
FIG. 3. Co-ordinate ortho-basis on the vortex line. The displacement vectors \( \vec{Q}(\sigma_0) \) are lying in the plane spanned by \( \vec{e}_1(\sigma_0), \vec{e}_2(\sigma_0) \), perpendicular to the local tangent \( \vec{X}'(\sigma_0) \). Normalising these vectors, one obtains the basis (40).

2. The self-action

After this preparatory work of fixing conventions, we write down the vortex self-action as a sum of its static part and one quadratic in derivatives of perturbations from the equilibrium string configuration:

\[
S_{\text{self}}[\vec{Q}(t,\sigma)] = - \int dt d\sigma \sqrt{\gamma(t,\sigma)} \left\{ M_0 c_s^2 - \frac{1}{2} M_0 \dot{\vec{Q}}^2 + \frac{1}{2} \frac{\alpha}{\gamma} \dot{\vec{Q}}^2 \right\}. \tag{43}
\]

The displacements \( \vec{Q} \) perpendicular to the line have to be small. Only in this case of small perturbations from an equilibrium configuration the equations of motion of interacting vortex segments obey the Hamiltonian structure we wish to derive below [39]. In particular, self-crossings of a line bending back on itself have to be excluded. The first terms in the self action then represent ‘nonrelativistic’ terms of static and kinetic energy in the Lagrangian for a particle, weighted with the local differential string arc length, and integrated over the length of the string.

The cutoff related elastic coefficient \( \alpha \) (parameterized by the longitudinal as well as the transversal string core structure), depends on \( t,\sigma \) in general, just as the vortex effective hydrodynamic mass does. In accordance with our ‘nonrelativistic’ treatment, we do not consider short wavelength perturbations, where this dependence becomes significant, and
take $\alpha \equiv M_0 c_s^2 = E_{\text{self}}$. This specific choice in the elastic energy of the string is related to a cutoff choice in the localized self induction approximation of classical hydrodynamics \[40\]. It corresponds to the assumption that in the long wavelength limit we are using, only massless sound excitations propagating along the string can survive\[41\]. It also naturally accounts for the fact that $\alpha$ must remain finite in the incompressible limit $c_s \to \infty$ (whereas the hydrodynamic mass has to vanish).

For a simple straight line in $z$-direction, we can write for (43), Fourier analysing the co-ordinates $\vec{Q} = (1/(2\pi)^2) \int d\omega \int dk \exp[-i(\omega t - k z)] \vec{Q}(\omega, k)$, and neglecting the dependence of $M_0$ on the co-ordinates,

$$S_{\text{self}}/M_0 c_s^2 = - \int \int dt \, dz \frac{1}{(2\pi)^2} \int \int d\omega \, dk \, \frac{1}{2} \vec{Q}^2(\omega, k) \left\{ (\omega/c_s)^2 - k^2 \right\}. \quad (44)$$

If the wiggles of the line are occurring on scales $\ll R_c$, this should also hold for other shapes of line by summing up contributions of approximately straight segments.

3. The total vortex action and momentum

To the self-action, we have to add the interaction with the background. Using the conventions (22),(31),

$$S_M = \gamma_s \int \int dt \, d\sigma \left( \rho_0 \psi_i X^i + b_{ij} X^j \dot{X}^i \right) \quad (45)$$

$$\equiv q \int \int dt \, d\sigma \left( -a^0 + a_i \dot{X}^i \right).$$

The gauge potential $(\psi_i, b_{ij})$ is defined to belong entirely to the background:

$$b_{\mu\nu}(x) \equiv \frac{1}{(2\pi)^4} \int_{k_0} \hat{\vec{Q}}_{\mu\nu} \exp[i k_{\mu} x^\mu]. \quad (46)$$

The cutoff $(k_\mu)_0 = (\omega/c_s, k_i)_0$ indicates the separation line between what we consider as being a phonon, which is integrated out in the self-action, and what we lump together into a time dependent and inhomogeneous background. In what follows, we will neglect the remaining phonon fluctuations and approximate the background to be an incompressible superfluid. The separation line is fixed in a quite natural way by the infrared cutoff in the vortex energy \[34\], $|\vec{k}_0| = \exp C/8R_c$.

Summation of (45) and (43) yields the total vortex action

$$S_V = S_{\text{self}} + S_M. \quad (47)$$

\[5\]This is true given that we consider the kinetic and elastic parts isolated from the static energy. Neglecting the kinetic energy, we can obtain the Kelvin modes \[39\]. Apart from the fact that the vortex has a certain core structure (such that an ultraviolet cutoff is present), they are not related to compressibility, i.e. they exist even for infinite $c_s$. 
Taking the functional derivative of $S_V$ after the vortex velocity, we arrive at the following expression for the vortex momentum per $\sigma$-length interval of the vortex line ($\dot{\vec{X}} = \dot{\vec{Q}}$):

$$\vec{P} = \vec{P}^{\text{inc}} + \vec{P}^{\text{kin}} = q\vec{a} + M_0\sqrt{\gamma}\dot{\vec{X}}.$$  \hspace{1cm} (48)

It has, as expected, the same appearance as the canonical momentum for a nonrelativistically moving particle of charge $q$, which is subject to an external vector potential $\vec{a}$. We separated the momentum into a part due to the (incompressible) background $\vec{P}^{\text{inc}}$ and a kinetic (vortex matter) part $\vec{P}^{\text{kin}}$.

Corresponding to the dominance of the static energy in the self-action, the ratio of the contributions to the momentum is in order of magnitude

$$\frac{\left| \vec{P}^{\text{kin}} \right|}{\left| \vec{P}^{\text{inc}} \right|} \approx \frac{\Gamma_s |\dot{\vec{X}}|}{|\vec{X}| c_s},$$  \hspace{1cm} (49)

This depends on $|\dot{\vec{X}}|/c_s$ as well as $\kappa/(c_s|\vec{X}|)$ ($= O(\xi/|\vec{X}|)$ in helium II). Both quantities are necessarily $\ll 1$ if the vortex is to be described within the hydrodynamic formalism we presented. We used here that $|b_{ij}|$ is of order $|X^k|$ in a Cartesian frame (see below), and neglected the dependence of the self-energy logarithm on the vortex co-ordinates (which multiplies the right-hand side of the equation above).

C. Gauge dependence of the vortex momentum

The momentum (48) is gauge dependent through the choice for the vector potential $\vec{a}$. The relation (20) determines $\vec{a}$ in terms of the original field $b_{ij}$. This equation is, in turn, equivalent to the last expression in (30) in its nonrelativistic version,

$$\text{rot} \vec{a} = \vec{B} = -\rho_0 \vec{X}'.$$  \hspace{1cm} (50)

In a Cartesian frame, a possible solution for $b_{ij}$ is $b_{ij} = -(1/3)\rho_0\eta_{ijk}X^k$ (it is, for example, used to represent to vortex velocity dependent part of the Magnus action in [42]). This is an isotropic solution, which is convenient to describe the purely spatial part of the gauge 2-form in the bulk superfluid. In the presence of the vortex, however, we choose another solution more appropriate to a natural basis on the string. For a singly quantized vortex, the momentum components $P_a$ in the basis of (42), $\vec{e}_a$, $\vec{e}_\sigma = \vec{X}'$ ($a = 1, 2$), with determinant $g = \det[\vec{e}_a \cdot \vec{e}_j]$, obey [13]

$$\partial_2 P^{\text{inc}}_1 - \partial_1 P^{\text{inc}}_2 = h\rho_0\sqrt{g},$$  \hspace{1cm} (51)

as follows from (50). A factor $N_v$ in front of the right hand side enters for vortices of arbitrary winding number.

An isotropic solution of (51) in Cartesian co-ordinates reads $\vec{P}^{\text{inc}} = (1/2)h\rho_0\vec{X} \times \vec{X}'$. The simplest possible solutions are obtained if we gauge one of the $P^{\text{inc}}_i$'s to zero:
\[ P_{2}^{\text{inc}} = 0 \Rightarrow P_{1}^{\text{inc}} = \hbar \rho_{0} \int dX^{2} \sqrt{g}, \]
\[ P_{1}^{\text{inc}} = 0 \Rightarrow P_{2}^{\text{inc}} = -\hbar \rho_{0} \int dX^{1} \sqrt{g}. \] (52)

It is stressed that the canonical local momentum \( \vec{P}^{\text{inc}} \) is a gauge object, and not necessarily identical with a physical momentum of the vortex. A connection to a physical momentum can be established by using the first choice in the equation above for a circular vortex (\( \vec{e}_{1} = \vec{e}_{Z}, \vec{e}_{2} = \vec{e}_{R}, \vec{e}_{\sigma} = \vec{X}', \vec{e}_{\Phi} = \vec{e}_{Z} \) and \( \sqrt{g} = r \)), and integrating \( P_{1}^{\text{inc}} = \hbar \rho_{0} \int dX^{1} \sqrt{g} \) over \( \Phi = \sigma = [0 : 2\pi] \). We then obtain that \( \vec{P}^{\text{Kelvin}} = \hbar \rho_{0} \pi R^{2} \vec{e}_{Z} \) equals the Kelvin momentum. For a superfluid the Kelvin momentum is just the surface area of the vortex times the bulk superfluid density times Planck’s quantum of action. This result is also obtained by integrating \( \vec{P} = \int \int \int m \rho_{0} \vec{v} s dV = \hbar \int \int \rho_{0} \nabla \theta dV = \rho_{0} \hbar \int \theta dS \), and taking the 2\( \pi \)-discontinuity of \( \theta \) (cf. Figure 1) across the surface enclosed by the vortex line.

D. Quantization of Vortex Motion

The canonical quantization of the massive vortex string proceeds in the usual manner. We take as the canonical phase space the displacement vector \( \vec{Q} \) together with the momentum \( \vec{P} \) from (48). If we are to quantize the vortex motion, we have to impose the following canonical commutation relations, written for convenience in the basis (40),

\[ [Q^{a}(\sigma), P_{b}(\sigma')] = i \hbar \delta^{a}_{b} \delta(\sigma - \sigma'), \quad (a, b = 1, 2) \]
\[ [Q^{a}(\sigma), Q^{b}(\sigma')] = 0 \]
\[ [P_{a}(\sigma), P_{b}(\sigma')] = 0 \] (53)

It should be observed that in our ‘magnetic’ field, the kinetic momentum components, defined in (48), do not commute, as follows from the classical Poisson bracket, respectively from direct calculation, using the commutators above:

\[ [P_{a}^{\text{kin}}(\sigma), P_{b}^{\text{kin}}(\sigma')] = i \hbar q(-\rho_{0}) n_{ab} \delta(\sigma - \sigma'), \] (54)

which amounts to saying that the local velocity components of the string are not both determined to arbitrary accuracy. This statement is analogous to that for the velocity components of an electron in a magnetic field (cf. [14], §110), quite as it should, according to the analogy explained in section 113.
FIG. 4. The direction of the incompressible part of the vortex momentum in coordinate space depends on the choice of the gauge in (50) respectively (51). On every point $\sigma$ of the vortex line the momentum can point in a different direction of the local coordinate plane $\vec{e}_1, \vec{e}_2$. This direction is parameterized by the angle $\alpha(\sigma)$. The first choice in (52) corresponds to $\alpha = 0$, the second to $\alpha = \pi/2$.

If we neglect the part $\vec{P}_{\text{kin}}$ as compared to the dominant $\vec{P}_{\text{inc}}$ in (48) altogether, as indicated by (49), taking again a ring vortex as the simple archetypical example, the complete quantum dynamics of the (undeformed) vortex is effectively one-dimensional (has only one independent coordinate and momentum component), and given by the canonical commutator

$$[Z(\sigma), P_{\text{inc}}^{\text{Z}}(\sigma')] = \frac{1}{2} [Z, h \rho_0 R^2] = i\hbar \delta(\sigma - \sigma')$$

$$[Z, S] = i(2\pi \rho_0)^{-1},$$

(55)

where the second line is the version of the local commutator in the first line integrated along the string, so that the commutator involves the Kelvin momentum of the ring. Consequently, $S = \pi R^2$ represents a surface operator of the ring plane. In the limit of $\vec{P}_{\text{kin}} \to 0$, generally,
phase space and configuration space merge and become indistinguishable, the momentum components, then, becoming functions of the co-ordinates alone.

The momentum space counterpart of the Figure 3 is depicted in Figure 4. We see that on every point of the line, we are free to choose the direction of the momentum afresh, according to a solution of (51) with some local gauge.

\[ H_V = \oint d\sigma \sqrt{\gamma} \left[ M_0 c_s^2 + \frac{1}{2\gamma M_0} (\vec{P} - q\vec{a})^2 \right] + q \int d\sigma \left( \frac{1}{2} a_C^0 + a_u^0 \right). \]  

(56)

We separated Coulomb and background velocity parts of the scalar potential, \( i.e. \) wrote \( a^0 \equiv a_C^0 + a_u^0 \). This gives the correct factor of 1/2 in the Coulomb interaction energy with other vortices: The energy of these other vortices is contained in the background part of the energy, whereas \((1/2) qa_C^0\) is the energy solely pertaining to the vortex under consideration. The Hamiltonian \( H_V \) gives the total energy of this particular vortex and yields its equations of motion. This is true provided that the background is in the limit of infinite extension, such that its energy change as the vortex moves, expanding and contracting, is negligible.

The full Hamiltonian equations of motion

\[ \frac{d\vec{P}}{dt} = -\frac{\delta}{\delta \vec{Q}} H_V, \quad \frac{d\vec{Q}}{dt} = \frac{\delta}{\delta \vec{P}} H_V, \]  

(57)

in which \( \delta/\delta \vec{Q}, \delta/\delta \vec{P} \) formally indicate invariant functional derivatives, are taking the form (setting \( c_s \equiv 1 \)),

\[ \frac{d\vec{P}}{dt} = -\nabla_{\vec{Q}}(\sqrt{\gamma} M_0) - q \nabla_{\vec{Q}} \left( \frac{1}{2} a_C^0 + a_u^0 \right) - \frac{1}{2} (\vec{P} - q\vec{a})^2 \nabla_{\vec{Q}} \left( \frac{1}{\sqrt{\gamma} M_0} \right) \]

\[ + \frac{q}{\sqrt{\gamma} M_0} \left( \nabla_{\vec{Q}} \otimes \vec{a} \right) (\vec{P} - q\vec{a}) + \frac{\partial}{\partial \sigma} \left( \frac{M_0}{\sqrt{\gamma} \vec{Q}} \right) - \frac{1}{2} \dot{\vec{Q}}^2 \nabla_{\vec{Q}} \left( \frac{M_0}{\sqrt{\gamma}} \right) \]  

(58)

and, of course, \( \dot{\vec{Q}} = (\vec{P} - q\vec{a})/(\sqrt{\gamma} M_0) \). They are considerably complicated by the fact that we admit a dependence of the terms containing \( M_0, \gamma \) on the co-ordinates of the line element.

The two (vectorial) Hamiltonian equations of motion, being of first order in time, can be shown to be equivalent to the second order Lagrangian equations of motion as they follow from the corresponding action (47):
\[
\frac{\partial}{\partial t} \left( \frac{M_0}{\sqrt{\gamma}} \dot{Q} \right) - \frac{\partial}{\partial \sigma} \left( \frac{M_0}{\sqrt{\gamma}} \dot{Q}' \right) + \left( 1 - \frac{1}{2} \dot{Q}^2 \right) \nabla \dot{Q} (\sqrt{\gamma} M_0) + \frac{1}{2} \dot{Q}^2 \nabla \dot{Q} \left( \frac{M_0}{\sqrt{\gamma}} \right) \\
= q \left( \vec{E} + \dot{Q} \times \vec{B} \right), \tag{59}
\]

by making use of the identities
\[
\frac{d\vec{P}}{dt} = \frac{\partial}{\partial t} \left( \sqrt{\gamma} M_0 \dot{Q} \right) + q \dot{a} + q \dot{Q} (\nabla \otimes \vec{a}),
\]
\[
\dot{Q} \times \text{rot} \vec{a} = \dot{Q} (\nabla \otimes \vec{a}) - (\nabla \otimes \vec{a}) \dot{Q}, \tag{60}
\]
\[
\vec{E} = -\nabla \dot{Q} \left( \frac{1}{2} a_C^0 + a_u^0 \right) - \dot{a} \vec{B} = \text{rot} \vec{a}.
\]

In these equations, the expression \(\nabla \otimes \vec{a}\) means a second-rank tensor with components \(\partial_i a_j\) and a vector standing to the left or right of this tensor is contracted with the first or second index, respectively.

It is instructive to write down the equation of motion in the original hydrodynamic variables:
\[
\frac{\partial}{\partial t} \left( \sqrt{\gamma} \ln[\cdots] \dot{Q} \right) - \frac{\partial}{\partial \sigma} \left( \frac{1}{\sqrt{\gamma}} \ln[\cdots] \dot{Q}' \right) + \left( 1 - \frac{1}{2} \dot{Q}^2 \right) \nabla \dot{Q} (\sqrt{\gamma} \ln[\cdots]) \\
+ \frac{1}{2} \dot{Q}^2 \nabla \dot{Q} \left( \frac{1}{\sqrt{\gamma}} \ln[\cdots] \right) = (4\pi/\Gamma_s) \vec{X}' \times \left( \dot{\vec{Q}} - \vec{v}_s \right). \tag{61}
\]

The logarithm of the self-energy \([34]\) is abbreviated \(\ln[\cdots]\). The bulk superfluid density \(\rho_0\) disappears in this writing.

**IV. SUMMARY**

In this paper, we have developed a general hydrodynamic formalism to describe the zero temperature motion of vortices in compressible, conventional superfluids, and the fields emanating from and interacting with them during this motion. The assumptions pertaining to this formalism are as follows. There exists an entity vortex, whose center of topological stability, the singularity of the vortex center, can be described by the spacetime embedding of the vortex world sheet. The intrinsic property of the vortex is the circulation, defined as the line integral of the momentum \(p\), and quantized in units of Planck’s quantum of action \(h\) by the homotopy group index \(N_v\). In the underlying spacetime, there exists a conserved particle current density \(j(x)\), whose dual is the field \(H\) acting on the vortex. The fundamental degrees of freedom in this treatment are thus the position of the vortex \(X^\mu(\tau, \sigma)\) in spacetime, representing the order parameter defect location, and the current density \(j(x)\). In a perfect fluid, these two degrees of freedom are connected physically in the Magnus force action. This action is given by the linear coupling of the vorticity \(\omega = *d \wedge p\) and a gauge potential \(b\), whose exterior derivative gives the field strength \(H = d \wedge b\).

In the Galilei invariant, nonrelativistic fluid we are then dealing with, we have further separated the superfluid phase into the nonrelativistic part of the background fluid and a part related to small density and phase oscillations, namely sound waves. This latter part behaves
‘relativistic’ in that the equations governing its behaviour have pseudo-Lorentz invariance, that is, they are Lorentz invariant under the replacement $c \rightarrow cs$. Self-interaction of the vortex line with these ‘relativistic’ phonon excitations gives rise to the hydrodynamic vortex mass. In a dense superfluid, the ‘nonrelativistic’ motion of the vortex is of relevance, because large order parameter modulus (quantum) variations take over, long before $|\partial \vec{X}/\partial t| \lesssim cs$ is attained. The full canonical structure of vortex motion obtains by adding elastic energy in the self-action of the vortex string, related to the localized self induction approximation.

In the final description of the vortex as a massive, elastic string object, we have thus incorporated parts of the fundamental interaction of the bare vortex with the general flow field. The phonon fluctuations in this flow field give mass renormalization, whereas the interaction with adjacent line segments gives elasticity. The background flow potential which remains is well approximated to be coming from an incompressible fluid. This procedure yields equations of motion in which the role of the bare, small scale many-body quantum dynamics of the vortex is reduced to give an ultraviolet cutoff parameter in the mass and elasticity coefficients.

The local self-action of the laboratory vortex, considering only the influence of neighbouring segments on a particular line element, has to be modified in a nonlocal manner if more distant parts of the line are closely approaching each other, or when nontrivial topologies of the vortex line, knotted structures, are under consideration [43]. Given that possible vortex configurations are regular and topologically trivial in this sense, the equations of motion in subsection III E can, for example, be used to calculate the Euclidean action of vortex tunnelling motions under the influence of external flow fields [15,16].

Acknowledgements

I thank Nils Schopohl for structurally clarifying discussions. This research work was supported by the LGFG Baden-Württemberg.
REFERENCES

* Electronic address: uwe.fischer@uni-tuebingen.de

[1] Sir Horace Lamb, “Hydrodynamics”, Republication of the sixth edition 1932, Dover, 1945
[2] L. M. Milne-Thomson, “Theoretical Hydrodynamics”, Fifth Edition, Macmillan, 1968
[3] M. Kalb, P. Ramond, Classical direct interstring action, Phys. Rev. D 9 (1974), 2273
[4] F. Lund, T. Regge, Unified approach to strings and vortices with soliton solutions, Phys. Rev. D 14 (1976), 1524
[5] R. L. Davis, E. P. S. Shellard, Antisymmetric Tensors and Spontaneous Symmetry Breaking, Phys. Lett. B 214 (1988), 219
[6] R. L. Davis, E. P. S. Shellard, Global Strings and Superfluid Vortices, Phys. Rev. Lett. 63 (1989), 2021
[7] K. Lee, Dual formulation of cosmic strings and vortices, Phys. Rev. D 48 (1993), 2493
[8] A. Zee, Vortex strings and the antisymmetric gauge potential, Nucl. Phys. B421 (1994), 111
[9] B.-A. Gradwohl, G. Kälbermann, T. Piran, Global Strings and Superfluid Vortices: Analogies and Differences, Nucl. Phys. B338 (1990), 371
[10] U. Ben-Ya’acov, Strings, superfluid vortices and relativity, Phys. Rev. D 44 (1991), 2452
[11] U. Ben-Ya’acov, Relativistic superfluid vortices and Helmholtz’s theorem, J. Phys. A: Math. Gen. 27 (1994), 7165
[12] B. Carter, D. Langlois, Kalb-Ramond coupled vortex fibration model for relativistic superfluid dynamics, Nucl. Phys. B454 (1995), 402-424
[13] B. Carter, D. Langlois, Equation of state for cool relativistic two-constituent superfluid dynamics, Phys. Rev. D 51 (1995), 5855-64
[14] U. R. Fischer, Massive Charged Strings in the Description of Vortex Ring Quantum Nucleation, J. Low Temp. Phys. 110 (1998), 39-44
[15] U. R. Fischer, Geometric Laws of Vortex Quantum Tunneling, Phys. Rev. B 58 (1998), 105-108
[16] U. R. Fischer, On the theory of vortex quantum tunnelling in the dense Bose superfluid helium II, Physica B 255 (1998), 41-54
[17] C. W. Misner, K. S. Thorne, J. A. Wheeler, “Gravitation”, Freeman, 1973
[18] B. Carter, I. M. Khalatnikov, Equivalence of convective and potential variational derivations of covariant superfluid dynamics, Phys. Rev. D 45 (1992), 4536-44
[19] D. A. Kirzhnits, S. N. Yudin, Paradoxes of superfluid rotation, Phys.-Uspekhi 38 (1995), 1283-1288
[20] G. E. Volovik, “Exotic Properties of superfluid 3He”, World Scientific, 1992
[21] A. Vilenkin, E. P. S. Shellard, “Cosmic Strings and Other Topological Defects”, Cambridge University Press, 1994
[22] E. Schmutzer, “Grundlagen der Theoretischen Physik”, VEB Deutscher Verlag der Wissenschaften, 1989
[23] P. W. Anderson, Considerations on The Flow of Superfluid Helium, Rev. Mod. Phys. 38 (1966), 298; R. E. Packard, The role of the Josephson-Anderson equation in superfluid helium, Rev. Mod. Phys. 70 (1998), 641-651
[24] V. N. Popov, Quantum vortices and phase transitions in Bose systems, *JETP* 37 (1973), 341
[25] D. P. Arovas, J. A. Freire, Dynamical Vortices in Superfluid Films, *Phys. Rev B* 55 (1997), 1068
[26] P. H. Roberts, J. Grant, Motions in a Bose condensate I. The structure of the large circular vortex, *J. Phys. A: Gen. Phys.* 4 (1971), 55
[27] L. P. Pitaevskii, Vortex lines in an imperfect Bose gas, *JETP* 13 (1961), 451
[28] E. P. Gross, Structure of a Quantized Vortex in Boson Systems, *Nuovo Cimento* 20 (1961), 454
[29] J.-M. Duan, Mass of a vortex line in superfluid $^4$He: Effects of gauge-symmetry breaking, *Phys. Rev. B* 49 (1994), 12381
[30] I. M. Khalatnikov, “An Introduction to the Theory of Superfluidity”, Addison Wesley, 1965
[31] M. Visser, Acoustic black holes: horizons, ergospheres, and Hawking radiation, *Class. Quant. Grav.* 15 (1998), 1767-1791; M. Visser, Hawking radiation without black hole entropy, *Phys. Rev. Lett.* 80 (1998), 3436
[32] G. E. Volovik, Vortex versus spinning string: Iordanskii force and gravitational Aharonov-Bohm effect, *JETP Lett.* 67 (1998), 881-887
[33] G. E. Volovik, Vortex Mass in BCS systems: Kopnin and Baym-Chandler contributions, *JETP Lett.* 67 (1998), 528-532
[34] T. Gotô, Relativistic Quantum Mechanics of One-Dimensional Mechanical Continuum and Subsidiary Condition of Dual Resonance Model, *Prog. Theor. Phys.* 46 (1971), 1560-1569
[35] Y. Nambu, Strings, monopoles, and gauge fields, *Phys. Rev. D* 10 (1974), 4262-4268
[36] A. Vilenkin, Cosmic Strings and Domain Walls, *Phys. Rep.* 121 (1985), 263-315
[37] C. Teitelboim, Gauge Invariance for Extended Objects, *Phys. Lett. B* 167 (1986), 63
[38] R. Aris, “Vectors, Tensors and the Basic Equations of Fluid Mechanics”, Republication of Prentice-Hall Edition 1962, Dover, 1989
[39] A. L. Fetter, Quantum Theory of Superfluid Vortices. I. Liquid Helium II, *Phys. Rev.* 162 (1967), 143
[40] K. W. Schwarz, Three-dimensional vortex dynamics in superfluid $^4$He: Line-line and line-boundary interactions, *Phys. Rev. B* 31 (1985), 5782-5804
[41] D. Förster, Dynamics of Relativistic Vortex Lines and their Relation to Dual Theory, *Nucl. Phys.* B81 (1974), 84
[42] M. Rasetti, T. Regge, Vortices in HeII, Current Algebras and Quantum Knots, *Physica* 80A (1975), 217
[43] D. C. Samuels, C. F. Barenghi, R. L. Ricca, Quantized Vortex Knots, *J. Low Temp. Phys.* 110 (1998), 509-514
[44] L. D. Landau, E. M. Lifshitz, “Quantum Mechanics”, Pergamon Press, Second Edition 1965