Ricci Flow Equation on \((\alpha, \beta)\)-Metrics

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Abstract

In this paper, we study the class of Finsler metrics, namely \((\alpha, \beta)\)-metrics, which satisfies the un-normal or normal Ricci flow equation.

Keywords: Finsler metric, Einstein metric, Ricci flow equation.

1 Introduction

In 1982, R. S. Hamilton for a Riemannian metric \(g_{ij}\) introduce the following geometric evolution equation

\[
\frac{d}{dt}(g_{ij}) = -2Ric_{ij}, \quad g(t = 0) = g_0,
\]

where \(Ric_{ij}\) is the Ricci curvature tensor and is known as the un-normalised Ricci flow in Riemannian geometry [7]. Hamilton showed that there is a unique solution to this equation for an arbitrary smooth metric on a closed manifold over a sufficiently short time. He also showed that Ricci flow preserves positivity of the Ricci curvature tensor in three dimensions and the curvature operator in all dimensions [6]. The Ricci flow theory related geometric analysis and various applications became one of the most intensively developing branch of modern mathematics [5, 7, 8, 12, 17]. The most important achievement of this theory was the proof of W. Thurston’s geometrization conjecture by G. Perelman [14, 15, 16]. The main results on Ricci flow evolution were proved originally for (pseudo) Riemannian and Kähler geometries. Thus the Ricci flow theory became a very powerful method in understanding the geometry and topology of Riemannian and Kählerian manifolds.

On the other hand, Finsler geometry is a natural extension of Riemannian geometry without quadratic restriction [19]. But it is not simple to define Ricci flows of mutually compatible fundamental geometric structures on Finsler manifolds. The problem of constructing the Finsler-Ricci flow theory contains a number of new conceptual and fundamental issues on compatibility of geometrical and physical objects and their optimal configurations. The same equation can be used in the Finsler setting, since both the fundamental tensor \(g_{ij}\) and Ricci tensor \(Ric_{ij}\) have been generalized to that broader framework, albeit gaining a \(y\) dependence in the process [2, 18]. However, there are some reasons why we shall refrain from doing so: (i) not every symmetric covariant 2-tensor \(g_{ij}(x, y)\) arises

\[...\]
from a Finsler metric \( F(x, y); \) (ii) there is more than one geometrical context in which \( g_{ij} \) makes sense. Thus, Bao called this equation as an un-normalised Ricci flow for Finsler geometry. Using the elegance work of Akbar-Zadeh in [1], Bao proposed the following normalised Ricci flow equation for Finsler metrics

\[
\frac{d}{dt} \log F = -R + \frac{1}{Vol(SM)} \int_{SM} R \, dV, \quad F(t = 0) = F_0,
\]

where the underlying manifold \( M \) is compact [2].

In a series of papers, Vacaru studied Ricci flow evolutions of geometries and physical models of gravity with symmetric and nonsymmetric metrics and geometric mechanics, when the field equations are subjected to nonholonomic constraints and the evolution solutions, mutually transform as Riemann and Finsler geometries [20, 21, 22, 23, 24, 25, 26].

It is remarkable that, Chern had asked whether every smooth manifold admits a Ricci-constant Finsler metric? The weaker case of this question is whether every smooth manifold admits a Einstein Finsler metric? His question has already been settled in the affirmative for dimension 2 because, by a construction of Thurston, every Riemannian metric on a two-dimensional manifold admits a complete Riemannian metric of constant Gaussian curvature.

## 2 Preliminaries

Let \( M \) be an \( n \)-dimensional \( C^\infty \) manifold. Denote by \( T_xM \) the tangent space at \( x \in M \), and by \( TM = \bigcup_{x \in M} T_xM \) the tangent bundle of \( M \).

A Finsler metric on \( M \) is a function \( F : TM \to [0, \infty) \) which has the following properties:

(i) \( F \) is \( C^\infty \) on \( TM_0 := TM \setminus \{0\} \);

(ii) \( F \) is positively 1-homogeneous on the fibers of tangent bundle \( TM \),

(iii) for each \( y \in T_xM_0 \), the following form \( g_y \) on \( T_xM \) is positive definite,

\[
g_y(u, v) := \frac{1}{2} \left[ F^2(y + su + tv) \right]_{s, t = 0}, \quad u, v \in T_y M.
\]

For a Finsler metric \( F = F(x, y) \) on a manifold \( M \), the spray \( G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i} \) is a vector field on \( TM \), where \( G^i = G^i(x, y) \) are defined by

\[
G^i = g^{ij} \left\{ [F^2]_{x^i y^j} - [F^2]_{x^i} \right\}.
\]

Let \( x \in M \) and \( F_x := F|_{T_xM} \). To measure the non-Euclidean feature of \( F_x \), define \( C_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R} \) by

\[
C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \left[ g_{y + tw}(u, v) \right]_{t = 0}, \quad u, v, w \in T_xM,
\]

The family \( \mathbf{C} := \{ C_y \}_{y \in TM_0} \) is called the Cartan torsion. It is well known that \( \mathbf{C} = 0 \) if and only if \( F \) is Riemannian. For \( y \in T_xM_0 \), define mean Cartan torsion \( I_y \) by \( I_y(u) := I_y(u)^i, \) where \( I_i := g^{jk}C_{ijk}, \) \( C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} \) and \( u = u^i \frac{\partial}{\partial x^i} \). By Deicke’s Theorem, \( F \) is Riemannian if and only if \( I_y = 0 \).
Regarding the Cartan tensors of these metrics, M. Matsumoto introduced the notion of C-reducibility and proved that any Randers metric $F = \alpha + \beta$ and Kropina metric $F = \alpha^2 / \beta$ are C-reducible, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M$. Matsumoto-Hojó proved that the converse is true [10]. Furthermore, by considering Kropina and Randers metrics, Matsumoto introduced the notion of $(\alpha, \beta)$-metrics [9]. An $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha \varphi(s)$, where $s = \beta / \alpha$, $\varphi(\cdot)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha$ is a Riemannian metric and $\beta$ is a 1-form on $M$.

In [11], Matsumoto-Shibata introduced the notion of semi-C-reducibility by considering the form of Cartan torsion of a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ with dimension $n \geq 3$. A Finsler metric is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{p}{1 + n} \{h_{ij}I_k + h_{jk}I_i + h_{ki}J_j\} + \frac{q}{C^2}I_iI_jI_k,$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on $TM$, $h_{ij} := g_{ij} - F^{-2}y_iy_j$ is the angular metric and $C^2 = I^2I_i$. If $q = 0$, then $F$ is just C-reducible metric.

3 Ricci Flow Equation

In 1982, R. S. Hamilton introduce the following geometric evolution equation

$$\frac{d}{dt}(g_{ij}) = -2Ric_{ij}, \quad g(t = 0) = g_0$$

which is known as the un-normalised Ricci flow in Riemannian geometry [7]. The same equation can be used in the Finsler setting, since both the fundamental tensor $g_{ij}$ and Ricci tensor $Ric_{ij}$ have been generalized to that broader framework, albeit gaining a $y$ dependence in the process. However, there are some reasons why we shall refrain from doing so: (i) Not every symmetric covariant 2-tensor $g_{ij}(x, y)$ arises from a Finsler metric $F(x, y)$; (ii) There is more than one geometrical context in which $g_{ij}$ makes sense. Thus, Bao called this equation as an un-normalised Ricci flow for Finsler geometry.

Professor Chern had asked, on several occasions, whether every smooth manifold admits a Ricci-constant Finsler metric. It is hoped that the Ricci flow in Finsler geometry eventually proves to be viable for addressing Cherns question. How to formulate and generalize these constructions for non-Riemannian manifolds and physical theories is a challenging topic in mathematics and physics. Bao studied Ricci flow equation in Finsler spaces [2]. In the following a scalar Ricci flow equation is introduced according to the Bao’s paper.

A deformation of Finsler metrics means a 1-parameter family of metrics $g_{ij}(x, y, t)$, such that $t \in [-\epsilon, \epsilon]$ and $\epsilon > 0$ is sufficiently small. For such a metric $\omega = u_i dx^i$, the volume element as well as the connections attached to it depend on $t$. The same equation can be used in the Finsler setting. We can also use another Ricci flow equation instead of this tensor evolution equation [2]. By contracting $\frac{d}{dt}g_{ij} = -2Ric_{ij}$ with $y_i$ and $y_j$ gives, via Eulers theorem, we get

$$\frac{\partial F^2}{\partial t} = -2F^2 R,$$
where \( R = \frac{1}{F^2} \text{Ric} \). That is,

\[
\frac{d}{dt} \log F = -R, \quad F(t=0) = F_0.
\]

This scalar equation directly addresses the evolution of the Finsler metric \( F \), and makes geometrical sense on both the manifold of nonzero tangent vectors \( TM_0 \) and the manifold of rays. It is therefore suitable as an un-normalized Ricci flow for Finsler geometry.

4 Un-Normal Ricci Flow Equation on \((\alpha, \beta)\)-Metrics

Here, we study \((\alpha, \beta)\)-metrics satisfying un-normal Ricci flow equation and prove the following.

**Theorem 4.1.** Let \((M, F)\) be a Finsler manifold of dimension \( n \geq 3 \). Suppose that \( F = \Phi(\frac{\beta}{\alpha}) \alpha \) be an \((\alpha, \beta)\)-metric on \( M \). Then every deformation \( F_t \) of the metric \( F \) satisfying un-normal Ricci flow equation is an Einstein metric.

To prove the Theorem 4.1, we need the following.

**Lemma 4.2.** Let \( F_t \) be a deformation of an \((\alpha, \beta)\)-metric \( F \) on manifold \( M \) of dimension \( n \geq 3 \). Then the variation of Cartan tensor is given by following

\[
C_{ijk}^I I^j I^k = -\frac{2R(1 + nq)}{1 + n ||I||^2} - \frac{1}{2} F^2 R_{i,j,k} I^j I^k - 3 ||I||^2 I^m R_m
\]

where \( ||I||^2 = I_m I^m \).

**Proof.** First assume that \( F_t \) be a deformation of a Finsler metric on a two-dimensional manifold \( M \) satisfies Ricci flow equation, i.e.

\[
\frac{d}{dt} g_{ij} := g_{ij}^t = -2\text{Ric}_{ij}, \quad \frac{d}{dt} \log F := \frac{F^t}{F} = -R,
\]

where \( R = \frac{1}{F^2} \text{Ric} \). By definition of Ricci tensor, we have

\[
\text{Ric}_{ij} = \frac{1}{2} [RF^2]_{y^i y^j} = R g_{ij} + \frac{1}{2} F^2 R_{i,j} + R_{i,j} - R_{i,j}
\]

where \( R_{i,j} = \frac{\partial R}{\partial y^i} \) and \( R_{i,j} = \frac{\partial^2 R}{\partial y^i \partial y^j} \). Taking a vertical derivative of (1) and using \( y^i_{i,j} = g_{ij} \) and \( FF_k = y_k \) yields

\[
\text{Ric}_{ij,k} = 2R C_{ij,k} + \frac{1}{2} F^2 R_{i,j,k} + \{g_{jk} R_i + g_{ij} R_k + g_{k,j} R_i\} + \{R_{j,k} y_i + R_{i,j} y_k + R_{k,i} y_j\}.
\]

Contracting (5) with \( I^j I^k \) and using \( y_i I^i = y^i I_i = 0 \) implies that

\[
\text{Ric}_{ij,k} I^j I^k = 2R C_{ij,k} I^j I^k + \frac{1}{2} F^2 R_{i,j,k} I^j I^k + 3 ||I||^2 I^m R_m.
\]
The Cartan tensor of an \((\alpha, \beta)\)-metric on a \(n\)-dimensional manifold \(M\) is given by
\[
C_{ijk} = \frac{p}{1 + n} \left\{ h_{ij} I_k + h_{jk} I_i + h_{ki} I_j \right\} + \frac{q}{||I||^2} I_i I_j I_k,
\] (7)
where \(p = p(x, y)\) and \(q = q(x, y)\) are scalar function on \(TM\) with \(p + q = 1\). Multiplying (7) with \(I^i I^j I^k\) yields
\[
C_{ijk} I^i I^j I^k = \left( \frac{p}{1 + n} + q \right) ||I||^4 = \frac{1 + nq}{1 + n} ||I||^4.
\] (8)
Then by (6) and (8), we get
\[
Ric_{ijk} I^i I^j I^k = 2 \frac{R(1 + nq)}{1 + n} ||I||^4 + \frac{1}{2} F^2 R_{i,j,k} I^i I^j I^k + 3 ||I||^2 I^m R_{m,n}.
\] (9)
On the other hand, since \(F_t\) satisfies Ricci flow equation then
\[
C'_{ijk} = \frac{1}{2} \frac{\partial g'_{ij}}{\partial y^k} = \frac{1}{2} \frac{\partial (-2 Ric_{ij})}{\partial y^k} = -Ric_{i,j,k}.
\] (10)
By (9) and (10) we get (2).

**Lemma 4.3.** Let \(F_t\) be a deformation of an \((\alpha, \beta)\)-metric \(F\) on a \(n\)-dimensional manifold \(M\). Then \(C'_{ijk} I^i I^j I^k\) is a factor of \(||I||^2\).

**Proof.** Since \(g^{ij} g_{jk} = \delta_k^i\), then we have
\[
0 = (g^{ij} g_{jk})' = g'^{ij} g_{jk} + g^{ij} g'_{jk} = g^{ij} g_{jk} - 2 g^{ij} Ric_{jk},
\]
or equivalently \(g'^{ij} g_{jk} = 2 g^{ij} Ric_{jk}\) which contracting it with \(g^{ik}\) implies that
\[
g'^{il} = 2 Ric^{il}.
\] (11)
Then we have
\[
I'_i = (g^{ik} C_{ijk})' = (g^{ik})' C_{ijk} + g^{ik} C'_{ijk}
= 2 Ric^{ik} C_{ijk} - g^{ik} Ric_{ijk,i}
= 2 Ric^{ik} g_{jk,i} - (g^{ik} Ric_{jk})_i + g^{ik} Ric_{jk}
= - (g^{ik} Ric_{jk})_i = -\rho_i
\] (12)
where \(\rho := g^{ik} Ric_{jk}\) and \(\rho_i = \frac{\partial \rho}{\partial y^i}\). Thus
\[
I'^i = (g^{ij} I_j)' = (g^{ij})' I_j + g^{ij} I'_j
= 2 Ric^{ij} I_j - g^{ij} \rho_j
= 2 Ric^{ij} I_j - \rho^i.
\] (13)
The variation of \(y_i := FF_{y^i}\) with respect to \(t\) is given by
\[
y'_i = -2 Ric_{im} y^m.
\]
Therefore, we can compute the variation of angular metric $h_{ij}$ as follows

$$h'_{ij} = (g_{ij} - F^{-2} y_i y_j)' = -2\text{Ric}_{ij} - 2F^{-2} R_{y_i y_j} + 2F^{-2} (\text{Ric}_{im} + \text{Ric}_{jm}) y^m$$

where $\ell_i := F^{-1} y_i$. Thus

$$h'_{ij} = 2R_{ij} - 2R g_{ij} - 2\text{Ric}_{ij} + 2(\text{Ric}_{im} \ell_j + \text{Ric}_{jm} \ell_i) y^m. \quad (14)$$

Now, we consider the variation of Cartan tensor

$$C'_{ijkl} = \left\{ \frac{p}{1+n}[h_{ij} I_k + h_{jk} I_i + h_{ki} J_j] + \frac{q}{||I||^2} I_{ij} I_k \right\}'$$

$$= \left\{ \frac{p'}{1+n}[h_{ij} I_k + h_{jk} I_i + h_{ki} J_j] + \frac{q'}{||I||^2} I_{ij} I_k \right\}$$

$$+ \frac{p}{1+n}[h_{ij} I_k + h_{jk} I_i + h_{ki} J_j]' + q \left\{ \frac{1}{||I||^2} I_{ij} I_k \right\}' \quad (15)$$

We have

$$\left[ \frac{1}{||I||^2} I_{ij} I_k \right]' = \left( \frac{(n+q)(n+q')}{(n+1)||I||^2} + 3p_m I^m \right)||I||^2$$

Multiplying (16) with $I^i I^j I^k$ implies that

$$\left[ \frac{1}{||I||^2} I_{ij} I_k \right]' I^i I^j I^k = \left( \frac{(n+q)(n+q')}{(n+1)||I||^2} + 3p_m I^m \right)$$

$$= \left( \frac{2(nq+1)(p^m I^m - \text{Ric}_{pq} I_q)}{n+1} - 3p_m I^m \right)||I||^2 \quad (17)$$

On the other hand

$$[h_{ij} I_k + h_{jk} I_i + h_{ki} J_j]' = (-p h_{jk} + p_j h_{ik} + R_i g_{ij}) - 2R I_i g_{jk} + I_j g_{ik} + J_k g_{ij}$$

$$- 2I_i \text{Ric}_{jk} + I_j \text{Ric}_{ik} + I_k \text{Ric}_{ij}$$

$$+ 2R I_i h_{jk} + I_j h_{ik} + I_k h_{ij}$$

$$+ 2I_i \Lambda_{jk} + I_j \Lambda_{ik} + I_k \Lambda_{ij}, \quad (18)$$

where $\Lambda_{jk} := (\text{Ric}_{jk} - \text{Ric}_{ij} \ell_k)\ell_j$. Multiplying (18) with $I^i I^j I^k$ implies that

$$[h_{ij} I_k + h_{jk} I_i + h_{ki} J_j]' I^i I^j I^k = -3(p^m I^m + 2\text{Ric}_{pq} I^p I^q)||I||^2. \quad (19)$$

On the other hand, since $p' + q' = 0$ then we get

$$\left[ \frac{p'}{1+n} h_{ij} I_k + h_{jk} I_i + h_{ki} J_j \right]' I^i I^j I^k = \frac{nq'}{1+n} ||I||^4 \quad (20)$$

Putting (17), (19) and (20) in (15) implies that $C'_{ijkl} I^i I^j I^k$ is a factor of $||I||^2$. More precisely, we have the following

$$C'_{ijkl} I_i I_j I_k = \left[ \frac{nq'}{n+1} ||I||^2 - \frac{q}{||I||^2} \left( \frac{2(nq+1)(p^m I^m - \text{Ric}_{pq} I_q)}{(n+1)||I||^2} - 3p_m I^m \right) \right]$$

$$- \frac{3p}{n+1} (p^m I^m + 2\text{Ric}_{pq} I^p I^q)||I||^2. \quad (21)$$

This completes the proof.
Proof of Theorem 4.1. By Lemmas 4.2 and 4.3, it follows that $R_{i,j,k} I^k = A_{ij} I_k + B_i g_{jk}$.

It is remarkable that, since $R_{i,j,k}$ is symmetric with respect to indexes $i, j$ and $k$, then the order of indexes in this relation doesn't matter. Now, multiplying $R_{i,j,k}$ with $y^k$ or $y^j$ implies that $R_{i,j} = 0$. It means that $R = R(x)$ and then $F_t$ is an Einstein metric.

5 Normal Ricci Flow Equation on $(\alpha, \beta)$-Metrics

If $M$ is a compact manifold, then $S(M)$ is compact and we can normalize the Ricci flow equation by requiring that the flow keeps the volume of $SM$ constant. Recalling the Hilbert form $\omega := F_y dx^i$, that volume is

$$Vol_{SM} := \int_{SM} \frac{(-1)^{n(n-1)}}{(n-1)!} \omega \wedge (d\omega)^{n-1} := \int_{SM} dV_{SM}.$$  

During the evolution, $F$, $\omega$ and consequently the volume form $dV_{SM}$ and the volume $Vol_{SM}$, all depend on $t$. On the other hand, the domain of integration $SM$, being the quotient space of $TM_0$ under the equivalence relation $z \sim y$, $z = \lambda y$ for some $\lambda > 0$, is totally independent of any Finsler metric, and hence does not depend on $t$. We have

$$\frac{d}{dt}(dV_{SM}) = [g_{ij} \frac{d}{dt}g_{ij} - n \frac{d}{dt} \log F] dV_{SM}.$$  

A normalized Ricci flow for Finsler metrics is proposed by Bao as follows

$$\frac{d}{dt} \log F = -R + \frac{1}{Vol(SM)} \int_{SM} R dV, \quad F(t = 0) = F_0,$$  

where the underlying manifold $M$ is compact. Now, we let $Vol(SM) = 1$. Then all of Ricci-constant metrics are exactly the fixed points of the above flow. Let

$$Ric_{ij} = \frac{1}{2} (F^2 R).y^i.y^j$$  

and differentiating (22) with respect to $y^i$ and $y^j$ the following normal Ricci flow tensor evaluation equation is concluded

$$\frac{d}{dt} g_{ij} = -2Ric_{ij} + \frac{2}{Vol(SM)} \int_{SM} R dV g_{ij}, \quad g(t = 0) = g_0,$$  

Starting with any familiar metric on $M$ as the initial data $F_0$, we may deform it using the proposed normalized Ricci flow, in the hope of arriving at a Ricci constant metric.

Theorem 5.1. Let $(M, F)$ be a Finsler manifold of dimension $n \geq 3$. Suppose that $F = \Phi(\beta)\alpha$ be an $(\alpha, \beta)$-metric on $M$. Then every deformation $F_t$ of the metric $F$ satisfying normal Ricci flow equation is an Einstein metric.
Proof. Now, we consider Finsler surfaces that satisfies the normal Ricci flow equation. Then

$$\frac{dg_{ij}}{dt} = -2Ric_{ij} + 2 \int_{SM} R \, dV \, g_{ij}, \quad d \log F := \frac{F'}{F} = -R + \int_{SM} R \, dV. \quad (24)$$

By the same argument used in the un-normal Ricci flow case, we can calculate the variation of mean Cartan tensor as follows

$$I'_i = (g^{jk} C_{ijk})' = (g^{jk})' C_{ijk} + g^{jk} C'_{ijk}$$

$$= 2[Ric^{jk} - \int_{SM} R \, dV \, g^{jk}] C_{ijk} + g^{jk}[Ric_{jk,i} + 2 \int_{SM} R \, dV C_{ijk}]$$

$$= 2Ric^{jk} g_{jk,i} - (g^{jk} Ric_{jk})_i + g^{jk} Ric_{jk}$$

$$= -(g^{jk} Ric_{jk})_i$$

$$= -\rho_i \quad (25)$$

Then we have

$$I''_i = (g^{ij} I_j)' = (g^{ij})' I_j + g^{ij} I'_j$$

$$= 2[Ric^{ij} - \int_{SM} R \, dV \, g^{ij}] I_j - g^{ij} \rho_j \quad (26)$$

By the same way that we used in un-normal Ricci flow, it follows that

$$C'_{ijk} = \frac{-I'_m I_m + I'_m I'_m}{||I||^2} C_{ijk} - \frac{1}{||I||^2} (\rho_i I_j I_k + \rho_j I_i I_k + \rho_k I_i I_j) \quad (27)$$

Contracting it with $I_i I_j I_k$ yields

$$C'_{ijk} I_i I_j I_k = (\Omega ||I||^2 - 3\rho_m I_m) ||I||^2, \quad (28)$$

where

$$\Omega := \frac{-I'_m I_m + I'_m I'_m}{||I||^2} = \frac{2\rho_m I_m - 2Ric^{mj} I_j}{||I||^2} + 2 \int_{SM} R \, dV.$$

By Lemma 4.2, we deduce that $R_{i,j,k} I_i I_j I_k$ is a factor of $||I||^2$. By the same argument, it result that every deformation $F_t$ of the metric $F$ satisfying normal Ricci flow equation is an Einstein metric. \hfill \Box

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