Novikov super-algebras are related to quadratic conformal super-algebras which correspond to Hamiltonian pairs and play fundamental role in completely integrable systems. In this paper, we focus on quadratic Novikov super-algebras, which are Novikov super-algebras with associative non-degenerate super-symmetric bilinear forms. We show that quadratic Novikov super-algebras are associative and the associated Lie-super algebras of quadratic Novikov super-algebras are 2-step nilpotent. Moreover, we give some properties on quadratic Novikov super-algebras and classify the associated Lie-super algebras of quadratic Novikov super-algebras up to dimension 7.

Keywords: Novikov super-algebra; Novikov algebra; quadratic Novikov super-algebra; quadratic Novikov algebra; Lie-super algebra.

1. Introduction

Novikov super-algebras are super variant of Novikov algebras. They are closely connected to popular algebraic objects such as conformal super-algebras [5], vertex operator super-algebras [8] and super Gelfand–Dorfman bialgebras [7], which play important role in quantum field theory and the theory of completely integrable systems.

A Novikov super-algebra $A$ is a $\mathbb{Z}_2$-graded vector space $A = A_0 + A_1$ with a bilinear product $(x, y) \mapsto xy$ for any $x \in A_i$, $y \in A_j$, $i, j \in A$ satisfying

\begin{align}
(x, y, z) &= (-1)^{ij} (y, x, z), \\
(zx)y &= (-1)^{ij} (zy)x,
\end{align}

(1.1) (1.2)

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where \((x, y, z) = (xy)z - x(yz)\). The even part of a given Novikov super-algebra is what is said to be a Novikov algebra introduced in connection with the Poisson brackets of hydrodynamic type \([1]\) and Hamiltonian operators in the formal variational calculus \([2-4, 9, 10]\). The super-commutator
\[
[x, y] = xy - (-1)^{ij}yx, \quad \text{for any } x \in A_i, \ y \in A_j
\]
makes any Novikov super-algebra \(A\) a Lie-super algebra denoted \(g(A)\) in what follows. We call \(g(A)\) the associated Lie-super algebra of \(A\). As usual, a form \(f : A \times A \rightarrow \mathbb{F}\) is said to be super-symmetric if
\[
f(x, y) = (-1)^{ij}f(y, x), \quad \text{for any } x \in A_i, \ y \in A_j,
\]
and non-degenerate if
\[
f(x, y) = 0 \quad \text{for any } y \in A \Rightarrow x = 0 \quad \text{and} \quad f(y, x) = 0 \quad \text{for any } y \in A \Rightarrow x = 0,
\]
and even if
\[
f(x, y) = 0, \quad \text{for any } x \in A_0, \ y \in A_1.
\]
In this paper, we introduce the term quadratic Novikov super-algebra for denoting the pair \((A, f)\) where \(A\) is a Novikov super-algebra and the bilinear form \(f\) on \(A\) is non-degenerate, super-symmetric and associative, i.e.,
\[
f(xy, z) = f(x, yz), \quad \text{for any } x, y, z \in A.
\]
The motivation for studying quadratic Novikov super-algebras comes from the fact that Lie or associative algebras with forms have important applications in several areas of mathematics and physics, such as the structure theory of finite-dimensional semi-simple Lie algebras, the theory of complete integrable Hamiltonian systems and the classification of statistical models over two-dimensional graphs.

The main goal of this paper is to study quadratic Novikov super-algebras and their associated Lie-super algebras. The paper is organized as follows. In Sec. 2, we show that \(A\) is associative and the associated Lie-super algebra \(g(A)\) is 2-step nilpotent if \((A, f)\) is a quadratic Novikov super-algebra. Also we show that \((g(A), f)\) is a quadratic Lie-super algebra. In Sec. 3, assume that \(f\) is even. We obtain some properties on quadratic Novikov super-algebras and their associated Lie-super algebras. In Sec. 4, we give the classification of the associated Lie-super algebras of quadratic Novikov super-algebras with even forms up to dimension 7.

Throughout this paper we assume that the algebras are of finite dimension over \(\mathbb{C}\).

2. Quadratic Novikov Super-Algebras

Firstly, we give some definitions. Let \(A\) be a Novikov super-algebra. Define \(Z(A) = \{x \in A \mid xy = yx = 0, \text{ for any } y \in A\}\). As usual, the pair \((g, f)\) is called a quadratic Lie-super
Thus for any \( f \) on \( g \) is non-degenerate, super-symmetric and \( g \)-invariant, i.e.,
\[
f(x, [y, z]) = f([x, y], z), \quad \text{for any } x, y, z \in g.
\]
Let \((g, f)\) be a quadratic Lie-super algebra and \( H \) be an ideal of \( g \). As usual, \( H \) is said to be isotropic if \( f|_{H,H} = 0 \) and non-degenerate if \( f|_{H, g} \) is non-degenerate.

**Theorem 2.1.** Let \((A, f)\) be a quadratic Novikov super-algebra. Then \( A \) is associative.

**Proof.** For any \( x, y, z, d \in A \),
\[
f((x, y, z), d) = f((xy)z - x(yz), d)
= f((xy)z, d) - f(x(yz), d)
= f(xy, zd) - f(x, (yz)d)
= f(x, yzd) - (yz)d
= -f(x, (y, z, d)).
\]
Thus for any \( x \in A_i, y \in A_j, z \in A_k, d \in A_m \), we have that
\[
f((x, y, z), d) = -f(x, (y, z, d)) = (-1)^{i+j+k+m+jk}f((z, y, d), x)
= (-1)^{i+j+k+m+jk}f(z, (y, d), x),
= (-1)^{i+j+k+m+jk+i+j+m}f((y, d), x, z),
= (-1)^{i+j+k+m+jk+i+j+m+jm}f((d, y, x), z),
= (-1)^{i+j+k+m+jk+i+j+m+jm+k}f(d, (y, x), z),
= (-1)^{i+j+k+m+jk+i+j+m+jm+k}f((y, x, z), d),
= (-1)^{i+j+k+m+jk+i+j+m+jm+k}f((x, y, z), d),
= (-1)^{i+j+k+m+jk+i+j+m+k}f((x, y, z), d),
= -f((x, y, z), d).
\]
It follows that \( (x, y, z) = 0 \) by the non-degeneracy of \( f \).

**Theorem 2.2.** Let \((A, f)\) be a quadratic Novikov super-algebra. Then \([x, y] \in Z(A)\) for any \( x, y \in A \). In particular, \([x, [y, z]] = 0\) for any \( x, y, z \in A \).

**Proof.** For any \( x \in A_i, y \in A_j, z \in A_k \), by Theorem 2.1, we have that
\[
z[x, y] = z(xy) - (−1)^{ij}z(yx) = z(xy) - (−1)^{ij}(zy)x
= (zx)y - (−1)^{jk+jk}(zx)y = 0.
\]
Then for any \( x \in A_i, y \in A_j, z \in A_k, d \in A_m \), we have that
\[
f((x, y, z), d) = f((x, y), zd) = (−1)^{(i+m)(j+k)}f(z, d(x, y)) = 0.
\]
It follows that \([x, y]z = 0\) by the non-degeneracy of \( f \). Then \([x, y] \in Z(A)\) for any \( x, y \in A \).
Let (A, f) be a quadratic Novikov super-algebra and \( g(A) \) be the associated Lie-super algebra of A. Then \( (g(A), f) \) is a quadratic Lie-super algebra.

Proof. Since \( (A, f) \) is a quadratic Novikov super-algebra, we have that
\[
f(x, y, z) = f(xy - (-1)^{|y|}yx, z) = f(x, yz) - (-1)^{|y||z|} f(x, zy) = f(x, yz) - (-1)^{|y||z|} f(yx, x) = f(x, yz) - (-1)^{|y||z|} f(yx, x)
\]
for any \( x, y, z \in A \). Namely, \( (g(A), f) \) is a quadratic Lie-super algebra.

Let \( C(g) \) denote the center of a Lie-super algebra \( g \), i.e.,
\[
C(g) = \{ x \in g[[x, y] = 0, \text{ for any } y \in g \}.
\]

Proposition 2.4. Let \((g, f)\) be a quadratic Lie-super algebra. Then we have:

1. \( C(g) = [g, g]^{\perp} \).
2. Let \( H \) be an ideal of \( g \). Then \( H^{\perp} \) is an ideal of \( g \). Furthermore, assume that \( H \) is non-degenerate. Then \( H^{\perp} \) is also non-degenerate and \( g = H \oplus H^{\perp} \).

3. Properties on Quadratic Novikov Super-Algebras with Even Forms

Let \((A, f)\) be a quadratic Novikov super-algebra with \( f \) even and \( g(A) \) be the associated Lie-super algebra of \( A \). Let \( H \) be a subspace of \( A \), define
\[
H^{\perp} = \{ x \in A | f(x, y) = 0, \text{ for any } y \in H \}.
\]
In the following, we assume that \( f \) is even without special statements although most of the results exist when \( f \) is not even.

Lemma 3.1. Let \((A, f)\) be a quadratic Novikov super-algebra. Then \( Z(A) = (AA)^{\perp} \). Moreover, \( \dim Z(A) + \dim AA = \dim A \).

Proof. For any \( x \in Z(A) \) and \( y, z \in A \), we have that \( f(x, yz) = f(xy, z) = 0 \). Namely, \( Z(A) \subseteq (AA)^{\perp} \). For any \( x \in (AA)^{\perp} \) and \( y, z \in A \), \( f(x, yz) = 0 \). So \( f(xy, z) = 0 \). It follows that \( xy = 0 \) by the non-degeneracy of \( f \). By \( f(yx, z) = f(y, xz) = 0 \), \( xy = 0 \). Thus, \( x \in Z(A) \). Then \( Z(A) = (AA)^{\perp} \). Clearly \( \dim Z(A) + \dim AA = \dim A \) since \( f \) is non-degenerate.

Proposition 3.2. Let \((A, f)\) be a quadratic Novikov super-algebra. Then \( \dim g(A) \geq 2 \dim g(A) \).

Proof. By Proposition 2.4, we have \( \dim C(g(A)) + \dim [g(A), g(A)] = \dim g \). By Theorem 2.2, \( [g(A), g(A)] \subseteq C(g(A)) \). Then the theorem follows.

Proposition 3.3. Let \((A, f)\) be a quadratic Novikov super-algebra. Then \( [g(A), g(A)] \subseteq AA \subseteq C(g(A)) \) and \( [g(A), g(A)] \subseteq Z(A) \subseteq C(g(A)) \).
Proof. We only need to prove that $AA \subseteq C(g(A))$. For any $x, y, z, d \in A$.

$$f(x, [y, zd]) = f([x, y], zd) = f([x, y]z, d) = 0.$$ 

It follows that $[y, zd] = 0$ by the non-degeneracy of $f$. So $AA \subseteq C(g(A))$. \hfill \square

Proposition 3.4. Let $(A, f)$ be a quadratic Novikov super-algebra. If $C(g(A))$ is isotropic, then $[g(A), g(A)] = C(g(A))$ and $Z(A) = AA$. Furthermore $\dim g(A)$ is even.

Proof. If $C(g(A))$ is isotropic, then $C(g(A)) \subseteq C(g(A))^\perp = [g(A), g(A)]$. By Proposition 3.3, we have $[g(A), g(A)] = C(g(A))$ and $Z(A) = AA$. Then $g(A)$ is even by Proposition 2.4. \hfill \square

4. The Associated Lie-super Algebras of Quadratic Novikov Super-algebras

By Theorem 2.2, we know that the associated Lie-super algebra of a quadratic Novikov super-algebra is 2-step nilpotent. On the other hand:

Theorem 4.1. Let $(g, f)$ be a 2-step nilpotent quadratic Lie-super algebra. Then $g$ admits a quadratic Novikov super-algebra structure.

Proof. Define a bilinear product on $g$ by $xy = \frac{1}{2}[x, y]$. Under the product, $g$ is a Novikov super-algebra since $g$ is 2-step nilpotent as a Lie-super algebra. Moreover for any $x, y, z \in g$, $f(xy, z) = f(\frac{1}{2}[x, y], z) = f(x, \frac{1}{2}[y, z]) = f(x, yz)$. Namely, $g$ admits a quadratic Novikov super-algebra structure.

Thus to get the classification of associated Lie-super algebras of quadratic Novikov super-algebras, it is enough to get the classification of 2-step nilpotent quadratic Lie-super algebras. By Proposition 2.4, we have:

Statement 1. Let $(A, f)$ be a quadratic Novikov super-algebra. Then its associated Lie-super algebra $g(A)$ is a direct sum $g(A) \cong g_0 \oplus g_1$, where $g_0$ is an Abelian ideal with non-degenerate restriction $f$ on it and $g_1$ is an ideal with an isotropic center.

The statement 1 formulated above shows that the classification under consideration reduces to classification of quadratic Lie-super algebra with isotropic center. In the following, we will discuss the classification up to dimension $7$ with even forms. Firstly, we have a well-known fact:

Statement 2. Let $g = g_0 + g_1$ be a Lie-super algebra and $x = x_0 + x_1$ be an element of $C(g)$ with $x_0 \in g_0, x_1 \in g_1$. Then $x_0, x_1 \in C(g)$.

Theorem 4.2. Let $(A, f)$ be a quadratic Novikov super-algebra with $f$ even and $\dim A \leq 7$ such that the center of the associated Lie-super algebra $g$ is isotropic. Then $g$ is one of the following cases:

1. $\dim g = 4$ and there exists a basis $\{e_1, e_2, e_3, e_4\}$ of $g$ such that $[e_4, e_4] = e_4, [e_1, e_2] = e_3$, where $e_1, e_2 \in g_0$ and $e_3, e_4 \in g_1$;
2. $\dim g = 6$ and there exists a basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ of $g$ such that $[e_2, e_5] = e_3, [e_2, e_6] = e_4, [e_5, e_6] = e_1, [e_5, e_6] = ke_1$, where $e_1, e_2 \in g_0$ and $e_3, e_4, e_5, e_6 \in g_1$;
(3) $\dim \mathfrak{g} = 6$ and there exists a basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ of $\mathfrak{g} = \mathfrak{g}_0$ such that $[e_1, e_2] = e_4$, $[e_2, e_3] = e_5$, $[e_3, e_4] = e_6$.

**Proof.** Since $C(\mathfrak{g})$ is isotropic, we have that $\dim \mathfrak{g}$ is even by Proposition 3.4. That is, $\dim \mathfrak{g} = 2, 4$ or 6.

Case 1. It is easy to check that $\dim \mathfrak{g} \neq 2$.

Case 2. $\dim \mathfrak{g} = 4$. Since $C(\mathfrak{g})$ is isotropic, we have that $\dim C(\mathfrak{g}) = 2$.

If $C(\mathfrak{g}) \cap \mathfrak{g}_0 = 0$, then $C(\mathfrak{g}) \subseteq \mathfrak{g}_1$. Since $f$ is non-degenerate on $\mathfrak{g}_1$, we have that $\dim \mathfrak{g}_1 = 4$. Then $[\mathfrak{g}, \mathfrak{g}] = 0$ which contradicts to $\dim C(\mathfrak{g}) = \dim [\mathfrak{g}, \mathfrak{g}] = 2$.

Assume that $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 2$. Then $C(\mathfrak{g}) \subseteq \mathfrak{g}_0$. Similarly, we have that $\dim \mathfrak{g}_0 = 4$ since $f$ is non-degenerate on $\mathfrak{g}_0$. Then $\mathfrak{g}_0$ is a 2-step nilpotent Lie algebra, that is, $\mathfrak{g}$ is Abelian. It contradicts to $\dim C(\mathfrak{g}) = 2$.

So we must have that $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 1$. Thus there exists a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that $f(e_1, e_2) = f(e_2, e_1) = 1$ and $f(e_3, e_4) = -f(e_4, e_3) = 1$, where $e_1, e_2 \in C(\mathfrak{g})$, $e_3, e_4 \in \mathfrak{g}_0$ and $e_1, e_2, e_3, e_4 \in \mathfrak{g}_1$. We have that $[e_1, e_4] = a e_1$ since $[e_4, e_4] \in \mathfrak{g}_0$ and $\mathfrak{g}$ is 2-step nilpotent. Similarly, $[e_2, e_4] = c e_3$. Thus

$$a = f(e_2, [e_4, e_4]) = f([e_2, e_4], e_4) = f([e_2, e_4], e_4) = c.$$

Moreover $a = c \neq 0$ since $\dim [\mathfrak{g}, \mathfrak{g}] = 2$. We can assume that $a = c = 1$ by replacing $e_1$ by $ae_1$ and $e_3$ by $ae_3$.

Case 3. $\dim \mathfrak{g} = 6$. Since $C(\mathfrak{g})$ is isotropic, we have that $\dim C(\mathfrak{g}) = 3$.

If $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 0$, then $C(\mathfrak{g}) \subseteq \mathfrak{g}_1$. Thus $\dim \mathfrak{g}_1 = 6$ since $f$ is even and non-degenerate on $\mathfrak{g}_1$. It is a contradiction.

Support that $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 1$. Since $f$ is even and non-degenerate on $\mathfrak{g}$, we have that there exists a basis $\{e_1, e_2, \ldots, e_6\}$ such that

$$f(e_1, e_2) = f(e_2, e_1) = 1, \quad f(e_1, e_3) = -f(e_3, e_1) = f(e_4, e_3) = -f(e_1, e_4) = 1,$$

where $e_1, e_3, e_4 \in C(\mathfrak{g})$, $e_1, e_2 \in \mathfrak{g}_0$ and $e_3, e_4 \in \mathfrak{g}_1$. Let $V$ be a vector subspace of $\mathfrak{g}_1$ extended by $e_3$ and $e_4$. Assume that $[x, x] = 0$ for any $x \in V$. Then for any $y, z \in V$,

$$[y, z] = [z, y] = \frac{1}{2}([y, z] + [y, z] - [y, y] - [z, z]) = 0.$$

It contradicts $C(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$. Thus there exists a basis $\{e_7, e_8\}$ of $V$ such that

$$[e_7, e_8] = e_1, \quad [e_7, e_8] = [e_8, e_9] = 0, \quad [e_9, e_9] = [e_9, e_9] = k e_1.$$

At this time, $[e_2, e_9] = a_1 e_3 + a_2 e_4$, $[e_2, e_9] = b_1 e_3 + b_2 e_4$. Since $\dim [\mathfrak{g}, \mathfrak{g}] = 3$, we have that $e_2 = [e_2, e_2] = [e_2, e_2]$ are linear independent. That is, $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is a basis of $\mathfrak{g}$.

If $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 2$, then $\dim \mathfrak{g}_0 = 4$ since $f$ is non-degenerate and even. Thus $\mathfrak{g}_0$ is a 2-step nilpotent quadratic Lie algebra of dimension 4. Then $\mathfrak{g}_0$ is Abelian. It follows that $\dim [\mathfrak{g}, \mathfrak{g}] \leq 2$. It contradicts to $\dim [\mathfrak{g}, \mathfrak{g}] = 3$.

If $\dim C(\mathfrak{g}) \cap \mathfrak{g}_0 = 3$, then $\dim \mathfrak{g}_0 = 6$. Namely, $\mathfrak{g} = \mathfrak{g}_0$ is a 2-step nilpotent quadratic Lie algebra. Moreover $\mathfrak{g}$ is Abelian because $\dim [\mathfrak{g}, \mathfrak{g}] = 3$. Clearly, there exists a basis $\{e_1, \ldots, e_6\}$ such that $[e_1, e_2] = e_4$, $[e_2, e_3] = e_5$, $[e_3, e_4] = e_6$. \qed
By Theorem 4.2 and Statement 1, it is easy to classify the associated Lie-super algebras of quadratic Novikov super-algebras with even forms up to dimension 7.

5. Further Discussion
Propositions 3.2–3.4 mean that if $C(g)$ is isotropic, then it is a maximal isotropic subspace in $g$ and $C(g) = [g, g]$. Together with even form $f$, we can choose a basis $v_1, \ldots, v_n, w_1, \ldots, w_n$ in $g$ such that $w_i \in C(g)$ for any $i$, the subspace spanned by $v_1, \ldots, v_n$ is isotropic, and

$$f(v_i, w_j) = \delta_{ij} = (-1)^{p(v_i)}f(w_j, v_i),$$

where $p(v)$ denotes the parity of $v$. Then $[v_i, v_j] = \sum c^k_{ij}w_k$, where $c^k_{ij} = -(-1)^{p(v_i)p(v_j)}c^k_{ji}$ and $c^k_{ij} \neq 0$ only if $p(v_i) + p(v_j) = p(v_k)$. Since $f$ is $g$-invariant, we have the following equalities:

$$f([v_i, v_j], v_k) = (-1)^{p(v_i)}c^k_{ij} = f(v_i, [v_j, v_k]) = c^k_{ji} = -(-1)^{p(v_i)p(v_j)}c^k_{kj}$$

or $c^k_{ij} = -(-1)^{p(v_i)+1)p(v_j)}c^k_{kj}$.

Then the classification of quadratic Lie-super algebras with isotropic center reduces to a problem of Linear Super-algebra on a canonical form of a non-degenerate tensor $c^k_{ij}$ on a linear super-space $V$ with certain symmetry of each pair of indices. In particular, in the presence of a non-degenerate form on $V$, the symmetric in the purely even cases reduces to classification of 3-vectors, i.e., antisymmetric indices tensors $c^i_{jk}$. This problem is completely solved in [6]. But it remains unknown in the absence of a non-degenerate form on $V$.

Acknowledgments
We would like to express thanks to the referee and editor for their constructive suggestions. This work was completed when the first author visited Chern Institute of Mathematics, Nankai University. The first author thanks Prof. Ke Liang for inviting to visit Chern Institute of Mathematics and acknowledges Prof. Chengming Bai for helpful suggestions.

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