Estimation and Inference for Multi-Kink Quantile Regression

Wei Zhonga, Chuang Wana, and Wenyang Zhangb

aMOE Key Lab of Econometrics, Wang Yanan Institute for Studies in Economics, Department of Statistics & Data Science, School of Economics and Fujian Key Lab of Statistics, Xiamen University, China; bGregory and Paula Chow Center for Economic Research, Xiamen University, China; cDepartment of Mathematics, The University of York, UK

ABSTRACT

This article proposes a new Multi-Kink Quantile Regression (MKQR) model which assumes different linear quantile regression forms in different regions of the domain of the threshold covariate but are still continuous at kink points. First, we investigate parameter estimation, kink points detection and statistical inference in MKQR models. We propose an iterative segmented quantile regression algorithm for estimating both the regression coefficients and the locations of kink points. The proposed algorithm is much more computationally efficient than the grid search algorithm and not sensitive to the selection of initial values. Second, asymptotic properties, such as selection consistency of the number of kink points and asymptotic normality of the estimators of both regression coefficients and kink effects, are established to justify the proposed method theoretically. Third, a score test based on partial subgradients is developed to verify whether the kink effects exist or not. Test-inversion confidence intervals for kink location parameters are also constructed. Monte Carlo simulations and two real data applications on the secondary industrial structure of China and the triceps skinfold thickness of Gambian females illustrate the excellent finite sample performances of the proposed MKQR model. A new R package Multikink is developed to easily implement the proposed methods.

1. Introduction

The kink regression model (Hansen 2017) or the bent line regression (Li et al. 2011) assumes that linear regression forms are separately modelled on two sides of an unknown threshold but still continuous at the threshold. It is a very useful tool to deal with the nonlinearity in data analysis. Let $Y_t$ be a response variable of interest, $X_t$ be a univariate threshold variable and $Z_t$ be a $p$ dimensional random vector of additional covariates, $t = 1, 2, \ldots, n$. Hansen (2017) considered the following kink regression model with an unknown threshold:

$$
Y_t = \alpha_0 + \alpha_1 (X_t - \delta) I(X_t \leq \delta) + \alpha_2 (X_t - \delta) I(X_t > \delta) + Z_t^\top + \epsilon_t,
$$

(1.1)

where $\epsilon_t$ is the random error with $E(\epsilon_t|X_t, Z_t) = 0$. The threshold variable $X_t$ has different slopes on different segments formed by $\delta$, but the regression function is continuous in $X_t$. This kind of nonlinear pattern is commonly referred to as kink effect (Hansen 2017) or bent line effect (Li et al. 2011). The parameter $\delta$ is called “kink point,” “change point,” or “threshold” exchangeably to represent the point where the regression function form changes. Compared with the linear models, the kink regression model relaxes the linearity assumption and is able to capture the necessary nonlinearity. Compared with the full nonparametric models, the kink regression model has the better interpretability by maintaining linear regression models in different regions of the domain of $X_t$. Thus, the kink regression model enjoys both the interpretability of linear models and the flexibility of nonparametric models.

In the literature, the kink regression models with a single unknown threshold point have been intensively studied. For example, Hansen (2017) combined the least-squares estimation and a grid search algorithm to estimate the regression coefficients and the kink point in Model (1.1). The $F$-type statistic test was also proposed for testing $H_0: \alpha_1 = \alpha_2$ in Model (1.1). In the application, Hansen (2017) demonstrated the famous inverted U-shaped relationship between the GDP growth rate and the ratio of debt to GDP (Reinhart and Rogoff 2010). Li et al. (2011) proposed a bent line quantile regression model and showed that the logarithm of maximal running speed of land mammals linearly increases with the logarithm of mass up to a certain point and then decreases as the mass rises. Zhang and Li (2017) studied estimation and hypothesis testing for a continuous threshold linear regression. Hidalgo, Lee, and Seo (2019) discussed whether there experiences a discontinuous jump or a continuous kink at the threshold point by using the quasi-likelihood-ratio test and constructed a robust confidence interval for the threshold.

However, the kink regression models with only one threshold point are not sufficient in some applications. In our second real data example, the logarithm of triceps skinfold thickness (TSF)
as an important measure of body density decreases with the age in the childhood up to about 10 years old, then experiences a growth spurt at adolescence up to about 18–20 years old and finally stays almost stable for adults (see Figure 4). The kink regression models with one threshold are clearly not appropriate for this case. It would be more sensible to consider the kink regression models with multiple threshold points. Muggeo and Adelfio (2010) studied a piecewise constant model in mean regression with multiple change points and used the penalized method to select change points. However, the mean regression is not robust to outliers and we find that the penalized method has relatively poor performance to select change points.

Moreover, to achieve the robustness to outliers and heavy-tailed errors which are often present in the data, we consider quantile regression to analyze data with heterogeneous conditional distributions. Quantile regression is able to provide a quantile regression to analyze data with heterogeneous conditional errors which are often present in the data, we consider that the MKQR model (1.2) is different from traditional threshold regression models (Hansen 2000; Caner 2002) that specify different regression functions in subsamples segmented by another threshold variable and jumps at the threshold points are allowed. However, the threshold variable is in the MKQR model is the predictor of interest in the regression and the regression curves are everywhere continuous on the domain of the covariates, especially when its upper or lower quantiles of the response are particularly of interest. Therefore, from the statistical inference perspective, we develop a score test based on partial subgradient of quantile objective function under the null hypothesis to verify whether the kink effects exist or not. A test-inversion confidence interval based on a smoothed rank score test for a kink location parameter is also proposed and can be extended to multiple kink parameters by sample splitting. Fifth, two real data applications on the secondary industrial structure of China and the triceps skinfold thickness for Gambian females are studied to identify the kink points, which would be of interest for economists and biologists, respectively. Last, a new R package MultiKink is developed to implement all the estimation and inference procedures, and is free to use.

The rest of the article is structured as follows. In Section 2, we describe the estimation procedures for the MKQR model and investigate the asymptotic properties. Section 3 presents a testing procedure for the existence of kink effects and construct the test-inversion confidence intervals for the kink locations. The finite sample performances of the proposed methods are evaluated via simulation experiments in Section 4. Section 5 presents two real data applications. Section 6 concludes the article. The technical proofs are presented in the appendix.

2. Estimation and Algorithm

2.1. Parameter Estimation When K is Given

Given K, the rth conditional quantile of Yt given Xt and Zt is denoted by

\[ Q_Y(t \mid X_t, Z_t) = \alpha_0 + \alpha_1 X_t + \sum_{k=1}^{K} \beta_k (X_t - \delta_k) I(X_t > \delta_k) + \gamma ^T Z_t + \epsilon_t, \quad (2.1) \]

where \( \beta_k \) represents the difference in slopes for \( X_t \) between two adjacent segments, \( \gamma \) is the coefficient vector of covariates \( Z_t \), which stays constant on the whole domain of \( Z_t \) and \( \epsilon_t \) are independent random errors such that \( P(\epsilon_t < 0 \mid X_t, Z_t) = \tau \). Note that \( \beta_k \neq 0 \) implies the existence of a kink effect at \( X_t = \delta_k \) and thus \( \{\delta_k, k = 1, \ldots, K\} \) represent the kink points or the locations where kink effects happen satisfying \( \delta_1 < \cdots < \delta_K \).

So there are \( K + 1 \) regimes in total divided by \( K \) kink points where the number of kink points \( K \) as well as their locations are both unknown. We emphasize that the regression for MKQR model is continuous everywhere. Note that all the unknown parameters depend on the quantile index \( \tau \), but we omit the subscript \( \tau \) for ease of notations throughout the article. We remark that the MKQR model (1.2) is different from traditional threshold regression models (Hansen 2000; Caner 2002) that specify different regression functions in subsamples segmented by another threshold variable and jumps at the threshold points are allowed. However, the threshold variable in the MKQR model is the predictor of interest in the regression and the regression curves are everywhere continuous on the domain of the covariates. We also remark that the MKQR models can be also regarded as a special semiparametric partially linear quantile regression model where the nonparametric function of \( X_t \) is specifically modelled by a continuous piecewise linear curve.

In this article, we focus on parameter estimation, kink points detection and statistical inference for the new MKQR model (1.2) where both the number of kink points and their locations are unknown using quantile regression. We contribute the literature in the following several aspects. First, the MKQR model (1.2) extends the existing kink regression with an unknown threshold to wider applications with unknown multiple kink points. We propose a Bootstrap Restarting Iterative Segmented Quantile (BRISQ) regression algorithm for estimating both regression coefficients and kink effects. It is much more computationally efficient than the grid search algorithm and not sensitive to the initial values due to the bootstrap restarting idea of Wood (2001). We theoretically demonstrate the asymptotical normality of the estimators for both regression coefficients and kink effects given the number of kink points. Second, we transform kink points detection into a model selection problem based on a strengthened quantile BIC and suggest a backward elimination algorithm to identify the number of kinks. The selection consistency of the estimated number of kink points is also established. Third, the MKQR model inherits the merits of quantile regression and thus is able to robustly model data with heterogeneous conditional distributions especially when upper or lower quantiles of the response are particularly of interest.
where \( \rho_t(u) = u[\tau - I(u < 0)] \). A common estimator for \( \theta \) is thereby

\[
\hat{\theta} = (\tilde{\eta}^T, \tilde{\gamma}^T)^T = \arg \min_{\eta \in \mathcal{B}, \delta \in \Gamma} S_\eta(\theta),
\]

(2.3)

where \( \mathcal{B} \subset \mathbb{R}^{2+K+p} \) and \( \Gamma \subset \Omega^K \) are compact sets, in which \( \Omega \) denotes the support of the threshold variable \( X_t \).

If all kink locations parameters \( \delta \) are known, a simple quantile regression can be directly used for (2.1). For the single kink regression with an unknown threshold, the greedy grid search algorithm (Li et al. 2011; Hansen 2017) can be used to exhaustively seek the kink point. However, without any prior information, we have to assume that both the number of kink points \( K \) and the kink locations vector \( \delta \) are unknown in Model (1.2). It makes the grid search approach inappropriate especially when \( K \) is large, because its computational cost grows at an exponential rate of \( K \). Since the minimization problem in (2.2) is non-convex with respect to \( \delta \), the convex optimization algorithms can not directly applied. Bai and Perron (2003) proposed a novel dynamic programming algorithm to detect change points for the linear model with multiple structural changes. Oka and Qu (2011) used the dynamic programming algorithm to estimate multiple structural changes in regression quantiles. However, the dynamic programming algorithms can not be directly applied to the kink models due to the continuity constraints. Next, we develop a new iterative segmented quantile regression algorithm to estimate regression coefficients and kink effects.

### 2.1. Bootstrap Restarting Iterative Segmented Quantile Algorithm

Given \( K \), each \((X - \delta_k)^T(I(X > \delta_k)) \) for \( k = 1, \ldots, K \) is not observable and still nondifferentiable at \( \delta_k \)'s. Given an initial location vector \( \delta^{(0)} = (\delta_1^{(0)}, \ldots, \delta_K^{(0)})^T \), we employ the first-order Taylor expansion to approximate \((X_t - \delta_k)^T(I(X_t > \delta_k)) \) around \( \delta_k^{(0)} \),

\[
(X_t - \delta_k)^T(I(X > \delta_k) \approx (X_t - \delta_k^{(0)})^T(I(X_t > \delta_k^{(0)})) - (\delta_k - \delta_k^{(0)})^T(I(X_t > \delta_k^{(0)})).
\]

Then, Model (2.1) can be approximated by

\[
Q_\mathcal{Y}(\tau; \theta | \mathbf{W}_t) \approx \alpha_0 + \alpha_1 X_t + \sum_{k=1}^K \beta_k \tilde{U}_{kt} + \sum_{k=1}^K \phi_k \tilde{V}_{kt} + \gamma^T Z_t,
\]

where \( \phi_k = \beta_k(\delta_k - \delta_k^{(0)}, \tilde{U}_{kt} = (X_t - \delta_k^{(0)})I(X_t > \delta_k^{(0)}) \) and \( \tilde{V}_{kt} = -I(X_t > \delta_k^{(0)}) \) are two new covariates with coefficients \( \beta_k \) and \( \phi_k \) respectively. Denote \( \beta = (\beta_1, \ldots, \beta_K)^T \) and \( \phi = (\phi_1, \ldots, \phi_K)^T \). This local linear approximation technique has been also used in the change-point detection for the piecewise constant model (Muggeo and Adelfio 2010) and nonconcave penalized regression (Zou and Li 2008).

By fitting the standard linear quantile model (2.4), a new estimator for \( \delta_k \) can be updated by \( \hat{\delta}^{(i+1)}_k = \hat{\delta}^{(i)}_k + \hat{\phi}_k / \hat{\beta}_k \) for \( k = 1, \ldots, K \). The estimator can be iteratively updated. However, if initial values are not appropriately chosen, some elements of \( \hat{\delta}^{(i)} \) at the \( i \)th iteration are possible to jump out of the support of the threshold covariate \( X_t \) or be very close to another kink point to make them hard to distinguish. We define the inadmissible set \( \mathcal{T} \) as

\[
\mathcal{T} = \left\{ \hat{\delta}_k : (\hat{\delta}_k \notin \Omega, \text{the support of } X_t \cup |(\hat{\delta}_k - \hat{\delta}_k')| < \Delta, \right\}
\]

where \( \Delta \) is small and \( k \neq k' \).

In practice, we need to discard all \( \hat{\delta}_k \) in the inadmissible set \( \mathcal{T} \). In our simulations, we choose \( \Delta \) to be one third of the range of the threshold variable \( X_t \). The underlying reason is that the local linear approximation technique is sensitive to the initial values \( \delta^{(0)} \), which makes the algorithm easy to get stuck in local optima. To deal with this drawback, we iteratively update the initial values using the bootstrap samples to make the new algorithm insensitive to the original initial values. It shares the similar spirit of the bootstrap restarting idea of Wood (2001). Thus, we call it Bootstrap Restarting Iterative Segmented Quantile (BRISQ) regression algorithm.

The main idea of the BRISQ algorithm is illustrated as follows. We first initialize parameters \( \hat{\delta}^{(0)} \) evenly dispersed on the domain of \( X_t \) given \( K \) and obtain the estimator \( \theta^{(0)} = (\hat{\eta}^{(0)}_T, \hat{\delta}^{(0)}_T)^T \) by iteratively fitting the working model (2.4). Then, we generate a bootstrap sample \( \mathcal{X}_t^* \) in the classic way that we randomly select original observations with replacement and estimate Model (2.4) again using \( \delta^{(0)} \) as the initial kink locations to obtain the bootstrap estimator \( \hat{\theta}^{* (1)} = (\hat{\eta}^{* (1)}_T, \hat{\delta}^{* (1)}_T)^T \). And then, we estimate Model (2.4) again based on the original sample using the bootstrap estimator \( \hat{\delta}^{* (1)}_k \), as initial values and obtain the new estimator \( \hat{\theta}^{(1)} = (\hat{\eta}^{(1)}_T, \hat{\delta}^{(1)}_T)^T \). Next, we compare \( S_n(\hat{\theta}^{(1)}) \) with \( S_n(\hat{\theta}^{(0)}) \). If \( S_n(\hat{\theta}^{(1)}) < S_n(\hat{\theta}^{(0)}) \), we update \( \hat{\theta}^{(1)} = \hat{\theta}^{(1)} \); otherwise, \( \hat{\theta}^{(1)} = \hat{\theta}^{(0)} \). Last, we repeat the previous procedure until convergence. The flowchart of this BRISQ algorithm is displayed in Figure 1. In practice, this algorithm can efficiently jump out of local minima and substantially improve the stability and accuracy of estimation, therefore less sensitive to the original initial values. The detailed procedures are summarized in Algorithm 1 and easily implemented using the newly developed R package MultiKink. The convergence of the BRISQ algorithm is studied in Section S.1 of the supplementary material.

### 2.1.2. Limiting Distribution

We then derive the asymptotic properties for \( \hat{\theta} \) given the true number of kink points. Denote \( \theta_0 = (\eta_0^T, \delta_0^T)^T \) as \( \arg \min \ S(\theta) \), where \( S(\theta) = E[I_{\{\tau \in \mathbb{R} \}}(I_{\{Y_t > Q_\mathcal{Y}(\tau; \theta | \mathbf{W}_t)\})] \). Define

\[
G_\theta = E \left\{ \left( \frac{\partial S_n(\theta)}{\partial \theta} \right) \left( \frac{\partial S_n(\theta)}{\partial \theta} \right)^T \right\} \bigg| _{\theta = \theta_0}
\]

\[= n^{-1} \sum_{i=1}^n \tau (1 - \tau) E[h(W_i; \theta_0) h^T(W_i; \theta_0)], \]

\[
D_\theta = E \left\{ \frac{\partial^2 S_n(\theta)}{\partial \theta \partial \theta^T} \right\} \bigg| _{\theta = \theta_0}
\]

\[= n^{-1} \sum_{i=1}^n E[\psi_{\tau}(Y_t - Q_\mathcal{Y}(\tau; \theta | \mathbf{W}_t)) h(W_i; \theta)] \bigg| _{\theta = \theta_0}, \]
where \( \psi_t(u) \equiv \tau - I(u \leq 0) \). To establish the asymptotic distribution of \( \hat{\theta} \), we need to introduce some notations. Denote \( h(W_t; \theta) = (1, X_t, X_t - \delta_1, \ldots, X_t - \delta_K, Z_t, -\beta_1 I(X_t > \delta_1), \ldots, -\beta_K I(X_t > \delta_K))^T \) and the \( r \)th conditional quantiles of \( e_t \) given \( W_t \) as \( F_{r}^{-1}(\tau|W_t) = \inf\{u : F(u|W_t) \geq \tau\} \). Then we make the following assumptions.

(A1) \( F_t \equiv F(\cdot|W_t) \) has a continuous density \( f_t(\cdot|W_t) \) that satisfies \( 0 < \inf f_t(\cdot) < \sup f_t(\cdot) < \infty \) at the point \( F^{-1}(\tau|W_t) \) for any sequence of values of \( W_t \).

(A2) The objective function \( S(\theta) \) has a unique global minimum at \( \theta_0 \).

(A3) The threshold variable \( X_t \) has a continuous density function with a compact support \([-M, M]\), where \( M \) is a positive constant.

(A4) \( \max_{t \leq n} \|Z_t\| = op(n^{1/2}) \) and \( E(\|Z\|^b) \) is bounded.

(A5) Given \( K \) and \( \beta \neq 0 \), there exist a nonnegative-definite matrix \( G \) and a full rank matrix \( D \), such that \( \lim_{n \to \infty} G_n = G \) and \( \lim_{n \to \infty} D_n = D \).

Assumption (A1) is generally assumed in quantile regression. Assumption (A2) ensures the identifiability of estimation. Assumptions (A3)–(A4) impose some conditions on the thresholds variable and other covariates, respectively, which can also be found in Li et al. (2011) and Zhang, Wang, and Zhu (2017). Assumptions (A1)-(A4) are used for the proof of consistency of \( \hat{\theta} \) and additional Assumption (A5) suffices for the asymptotical normality. The following theorem demonstrates the limiting distribution of the proposed estimator for \( \theta_0 \).

**Theorem 2.1.** Suppose the true number \( K_0 \) of kink points in Model (2.1) is given and Assumptions (A1)-(A5) hold, as \( n \to \infty \), we have

\[
\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \Sigma),
\]

where \( \Sigma = D^{-1}GD^{-1} \).

According to **Theorem 2.1**, the regression coefficients \( \eta \) and the threshold parameters \( \delta \) are jointly asymptotically normal with \( \sqrt{n} \) convergence rate. In conventional jump threshold model, the threshold parameter estimators converge to a nonstandard asymptotic distribution with \( n \) convergence rate, see Hidalgo, Lee, and Seo (2019) for more details.

Moreover, we estimate \( \Sigma \) by a plugging estimator \( \hat{\Sigma}_n = \hat{D}_n^{-1}\hat{G}_n\hat{D}_n^{-1} \), where \( \hat{G}_n = n^{-1} \sum_{t=1}^n (1 - \tau) h(W_t; \hat{\theta}) h^T(W_t; \hat{\theta}) \) and \( \hat{D}_n = n^{-1} \sum_{t=1}^n \hat{f}(\cdot) h(W_t; \hat{\theta}) h^T(W_t; \hat{\theta}) \). \( \hat{D}_n \) requires consistent estimate for conditional density function \( f_t(\cdot) \) of error term \( \epsilon_t \). We suggest using the method called Hendricks-Koenker Sandwich based on the difference quotients discussed by Hendricks and Koenker (1992). To select the bandwidth, two choices are often used. One is based on Edgeworth expansions of studentized quantiles described by Hall and Sheather (1988), the other is based on the minimum of the mean squared error of the density estimator suggested by Bofinger (1975). In our R package `MultiKink`, we provide both versions to estimate the covariance matrix.

![Figure 1. The flowchart of the BRISQ algorithm. Stage I contains Steps 1-3 and Stage II contains Step 4 in the Algorithm 1.](image-url)
2.2. Parameter Estimation When K is Unknown

The aforementioned BRISQ algorithm works well when the true number $K_0$ of kink points is given. However, the true number $K_0$ is usually unknown in practice. To estimate $K_0$, we first start with a large initial value $K_{\text{max}} (> K_0)$ and then iteratively fit the working model (2.4) using the BRISQ algorithm in which we discard all $\delta_k$’s and corresponding $\hat{U}_k$’s and $\hat{V}_k$’s if $\hat{\delta}_k \in T$ at each iteration. When it stops, we obtain an estimator for the number of kink points, denoted by $K_\ast (< K_{\text{max}})$.

However, we find that $K_\ast$ often overestimates the true value $K_0$ in practice. To improve the selection and estimation accuracy, one can evaluate each MKQR model with $k = 0, 1, \ldots, K_\ast$ kink points according to a prescribed information criterion and find the final model with the smallest information criterion. In our algorithm, we suggest a strengthened quantile Bayesian information criterion (sBIC) to refine the kink points detection,

$$\text{sBIC}(K) = \log (\hat{S}_n(\hat{\theta}_K)) + N_K \frac{\log n}{2n} C_n,$$  \hspace{1cm} (2.5)

where $\hat{\theta}_K$ denotes the estimator of parameters $\theta = (\gamma^T, \delta^T)^T$ with $K$ kink points, $N_K$ equals to $2 + p + 2K$ and $C_n$ is a positive constant that allows to approach infinity as $n$ increases. When $C_n = 1$, sBIC in Equation (2.5) becomes the standard quantile BIC studied by Lian (2012) for consistent model selection. When $C_n > 1$, it is similar to the modified BIC of Lee, Noh, and Park (2014). The BIC-type criteria have been widely used in model selection. For example, Wang, Li, and Tsai (2007) proved that the BIC tuning parameter selector is able to identify the true linear model consistently. Chen and Chen (2008) further proposed an extended BIC (EBIC) to take into account both the model complexity and the sample size for consistent model selection. Lee, Noh, and Park (2014) showed that a modified BIC is consistent in model selection for high-dimensional linear quantile regression. Fryzlewicz (2014) proposed a strengthened BIC for sequential change points detection.

We can estimate the number of the kink points by $\hat{K} = \arg\min_{K=0,1,\ldots,K_\ast} \text{sBIC}(K)$. When $K_\ast$ is large, it is possible to improve the computational efficiency by using the following backward elimination procedure (Chan, Yau, and Zhang 2015). Given $K_\ast$, we re-estimate the new MKQR model with $K_\ast - 1$ kink points using the BRISQ algorithm, and then compare the sBIC values of two models. This procedure is repeated until the sBIC values does not decrease. Then, the final estimators for $K_0$ and $\theta$ are obtained corresponding to the minimum sBIC. The detailed algorithm is summarized in Algorithm 2. Simulations will illustrate that this algorithm is able to identify the true kink points consistently.

We then show the selection consistency of $\hat{K}$ based on the sBIC selector. We make the two additional assumptions.

(A6) The matrix $E \left[ h(\hat{W}_i; \theta)^T (\hat{W}_i; \theta) \right]$ is finite and positive definite.

(A7) $C_n \log (n)/n \rightarrow 0$ as $n \rightarrow \infty$.

Assumption (A6) is similar to Assumption (A) in Lian (2012). Assumption (A7) requires that $C_n = o(n/ \log(n))$ which means $C_n$ cannot diverge too fast to infinity as $n$ increases to avoid underfitting the true model.

Algorithm 2: Backward Elimination Algorithm for Estimating $K$.

Step 1. Given $K_{\text{max}}$ initial kink points, repeat Step 2 of Algorithm 1 iteratively and remove any $\delta_k$ at each iteration if $\hat{\delta}_k \in T$ until convergence. $K_\ast$ denotes the resulting estimated number of kink points.

Step 2. Estimate the working model (2.4) with $K_\ast - 1$ initial kink points using Algorithm 1 to obtain $\theta_{K_\ast-1}$ and sBIC($K_\ast-1$).

Step 3. If $\text{sBIC}(K_\ast - 1) < \text{sBIC}(K_\ast)$, then update $K_\ast = K_\ast - 1$ and go to Step 2; If $\text{sBIC}(K_\ast - 1) \geq \text{sBIC}(K_\ast)$, then stop, set $\hat{K} = K_\ast$ and $\hat{\theta} = \theta_{K_\ast}$. If $K_\ast = 0$, then stop and there is no kink point.

Theorem 2.2. Under Assumptions (A1), and (A6) and (A7), let $\hat{R} = \arg\min_{K=0,\ldots,K_\ast} \text{sBIC}(K)$, we have $P(\hat{R} = K_0) \rightarrow 1$ as $n \rightarrow \infty$.

Theorem 2.2 shows that the quantile sBIC is able to consistently select the true number of kink points. This result plays a fundamental role since the limiting distribution of the parameter estimators $\hat{\theta}$ is established under the true number of kink points in Theorem 2.1.

3. Statistical Inference

3.1. Testing the Existence of Kink Effects

The kink effect estimation is meaningful if and only if the kink effect truly exists. In this section, we are interested in testing the existence of kink effects in the conditional quantiles. For $\tau \in (0,1)$, we consider the following null ($H_0$) and alternative ($H_1$) hypotheses for the MKQR model (2.1),

$$H_0 : \beta_k = 0, \text{ for all } k = 1, \ldots, K. \text{ v.s.} \quad H_1 : \beta_k \neq 0, \text{ for some } k = 1, \ldots, K. \hspace{1cm} (3.1)$$

Note that the parameters $\beta_k$’s depend on $\tau$. Under the null hypothesis, the MKQR model (2.1) degenerates to an ordinary quantile regression without any kink point. Under the alternative hypothesis, there exists at least one statistically significant kink point at the $\tau$ th quantile. Thus, we suggest the following score-based test statistic based on kink quantile regression with an unknown kink point:

$$T_n(\tau) = \sup_{\delta > 0} |R_n(\delta)|,$$  \hspace{1cm} (3.2)

where $R_n(\delta) = n^{-1/2} \sum_{t=1}^n \psi_\tau (Y_t - \alpha^T V_t) (X_t - \delta) I(X_t \leq \delta)$, $\psi_\tau (u) = \tau - I(u \leq 0)$, $\delta$ denotes the location of an unknown kink point, $V_t = (1, X_t, Z_t)^T$, $\alpha = (\alpha_0, \alpha_1, \gamma^T)^T$ and $\bar{\alpha} = \arg\min_{\alpha} \sum_{t=1}^n \psi_\tau (Y_t - \alpha^T V_t)$. Note that $R_n(\delta)$ is essentially the partial subgradient of the objective function with respect to $\beta_1$ evaluated at $\beta_1 = 0$ and $\alpha = \bar{\alpha}$ up to a constant in the model (2.1) with $K = 11$. $T_n(\tau)$ can be viewed as a weighted

\[^1\text{In fact, } R_\tau(\delta) \text{ is the partial subgradient of the quantile objective function with respect to } \beta_1 \text{ evaluated at } \beta_1 = 0 \text{ and } \alpha = \bar{\alpha} \text{ up to a constant for the model } Q_\tau(\tau; \theta | W_t) = \alpha_0 + \alpha_1 X_t + \beta_1 (X_t - \delta) I(X_t \leq \delta) + \gamma^T Z_t, \text{ which is essentially same as the model (2.1) with } K = 1 \text{ after simple reparameterization.}\]
CUSUM (cumulative-sum) type test statistic based on the signs of quantile residuals. Intuitively, under the null hypothesis, residuals $Y_t - \tilde{a}_t V_t$ are evenly located below or above zero which result in a relatively small value of $T_n(\tau)$. On the other hand, under the alternative hypothesis, the model is misspecified and the residuals would be consistently positive or negative which implies the large values of $T_n(\tau)$. The idea of subgradient-based tests has been studied in the literature. For example, Qu (2008) constructed the subgradient test statistic in quantile regression for testing the structural changes. Zhang, Wang, and Zhu (2014) proposed a score test based on the subgradient test for the jumping threshold effect in threshold models. Zhang and Li (2017) developed a related test for the continuous threshold effect in asymmetric least-square regression.

Theorem 3.1 in the following derives the asymptotic behavior of $T_n(\tau)$. To derive the large sample property for $T_n(\tau)$, we consider the local alternative model

$$Q_Y(\tau; W_t) = \alpha_0 + \alpha_1 X_t + n^{-1/2} \beta (X_t - \delta) I(X_t > \delta) + \gamma^T Z_t,$$

(3.3)

where $\delta$ denotes one kink point location, $\beta \neq 0$ is a nonzero constant and $n^{-1/2} \beta \neq 0$ is the corresponding kink effect to represent the local alternative signal which diverges to zero as the rate of $n^{1/2}$. Model (3.3) implies that there exists at least one kink effect. We also introduce some notations. Define $H_0 = n^{-1} \sum_{t=1}^n E \{ V_t V_t^T f_i(\epsilon_t) \}$, $H_{1n}(\delta) = n^{-1} \sum_{t=1}^n E \{ V_t (X_t - \delta) I(X_t \leq \delta) ] f_i(\epsilon_t) \}$ and $H_{2n}(\delta, \beta) = n^{-1} \sum_{t=1}^n E \{ V_t \beta (X_t - \delta) I(X_t > \delta) f_i(\epsilon_t) \}$. We further assume that

(A8) There exists a positive-definite matrix $H$ such that $\lim_{n \to \infty} H_n = H$ and another two matrices $H_1(\delta)$ and $H_2(\delta, \beta)$ such that $\lim_{n \to \infty} H_{1n}(\delta) = H_1(\delta)$ and $\lim_{n \to \infty} H_{2n}(\delta, \beta) = H_2(\delta, \beta)$ uniformly hold in $\delta, \beta$.

(A9) The density function of $\epsilon_i, f_i(\cdot)$ for $t = 1, \ldots, n$, has a bounded first-order derivative.

Theorem 3.1. Suppose Assumptions (A1), (A8) and (A9) hold. Under both the null hypothesis $H_0$ and the local alternative model (3.3), we have

$$T_n(\tau) \Rightarrow \sup_{\delta \in \Gamma} \{ R(\delta) + q(\delta, \beta) \},$$

(3.4)

where "$\Rightarrow$" denotes weak convergence, $R(\delta)$ is a Gaussian process with mean zeros and covariance function $W(\delta, \delta') = (1 - \tau)E[I(X_t \leq \delta) I(X_t \leq \delta') - H_1(\delta) H_1^T(\delta')(X_t - \delta)' I(X_t \leq \delta) - H_1^T(\delta') H_1(\delta)(X_t - \delta)]$, $\delta, \delta' \in \Gamma$ and $q(\delta, \beta) = -H_1(\delta) H_1^T H_2(\delta, \beta)$.

According to Theorem 3.1, under the null hypothesis, $q(\delta, \beta) = 0$ and $R_n(\delta)$ would converge to a Gaussian process $R(\delta)$ with mean zeros. When the kink effect exists under the alternatives, $q(\delta, \beta) \neq 0$ and $T_n(\tau)$ would be significantly larger than zero. Therefore, large values of $T_n(\tau)$ provide the evidence against the null hypothesis. The score-type statistic is only built on the null hypothesis without fitting models under the alternative hypothesis, so it can also be directly used to test the existence of multiple kink points. Since the asymptotical null distribution of $T_n(\tau)$ is nonstandard, we approximate the $P$-values using wild bootstrap (Feng, He, and Hu 2011). The detailed procedures are relegated to Section S.2 in the supplementary material.

3.2. Confidence Intervals for Kink Location Parameters

Next, we construct a test-inversion confidence interval for $\delta$ based on a smoothed rank score test. Consider the following hypotheses for a given $\delta$ in the domain of $X_t$ and $\tau \in (0, 1),$

$$H_0: \delta = \tilde{\delta} \text{ v.s. } H_1: \delta \neq \tilde{\delta}.$$  

(3.5)

Under $H_0$, we can obtain the estimator $\tilde{\delta}(\delta)$ of the regression coefficients $\eta$ given $\delta$ by fitting the standard linear quantile regression. Muggeo (2017) pointed that naive score statistic in threshold models may lower the test power due to the non-differentiable and non-smooth nature. To deal with the non-smoothness of the indicator function $I(X_t > \delta_k)$, we use the smoothed Gaussian distribution function $\Phi((X_t - \delta_k)/h_k)$ to approximate $I(X_t > \delta_k)$, where $h_k$ is the bandwidth. The smoothed objective function becomes

$$\tilde{Q}_Y (\tau; \eta, \delta; W_t) = \alpha_0 + \alpha_1 X_t + \sum_{k=1}^K \beta_k (X_t - \delta_k) \Phi((X_t - \delta_k)/h_k) + \phi(\cdot)$$

where $\Phi(\cdot)$ is the first derivative of $\Phi(\cdot)$. Motivated by the rank score tests in Gutenbrunner and Jurečková (1992), Gutenbrunner et al. (1997), and Zhang, Wang, and Zhu (2017), we define a smoothed rank score (SRS) test statistic as

$$\text{SRS}_n(\tau) = S^*_n V_n - I_{n},$$

(3.6)

where $S^*_n = -n^{-1/2} \sum_{t=1}^n P^*_t (\tau; \tilde{\eta}(\delta), \tilde{\delta}) \psi(\hat{\epsilon}_t)$, $\tilde{\epsilon}_t = Y_t - Q_Y (\tau; \tilde{\eta}(\delta), \tilde{\delta}; W_t)$ is the residual under $H_0$, and $V_n = n^{-1} \sum_{t=1}^n \tau (1 - \tau) P^*_t (\tau; \tilde{\eta}(\delta), \tilde{\delta})$.

Let $P^*_t (\tau; \tilde{\eta}(\delta), \tilde{\delta})$ is defined as follows. Let $M_1 (\delta) = (1, X_t - \delta, X_t^2 - \tilde{\delta}^2, X_t^3 - \tilde{\delta}^3, Z_t^2)$ and $M(\delta) = (M_1 (\delta), 1, M_0 (\delta))^T$. Let $P = (P_1 (\tau; \tilde{\eta}(\delta), \tilde{\delta}), \ldots, P_n (\tau; \tilde{\eta}(\delta), \tilde{\delta}))^T$, and define $P^* = (I_n - A) P$, where $I_n$ is an $n \times n$ identity matrix, $A = M(\delta)^T (\Psi M(\delta)-1) M(\delta)^T \Psi$, $\Psi = \text{diag}(\hat{\gamma}_1(\hat{\epsilon}_t), \ldots, \hat{\gamma}_n(\hat{\epsilon}_t))$. $P^*_t (\tau; \tilde{\eta}(\delta), \tilde{\delta})$ is defined as the rth row of the $n \times K$ matrix $P^*$ which is considered as the residuals by projecting the partial score vector $P$ on $\Psi M(\delta)$.

Intuitively, under $H_0$, the partial scores tend to be zero which implies that the test statistic is relatively small; otherwise, the large test statistic values provide the strong evidence against $H_0$. The following proposition demonstrates the null asymptotic distribution of $\text{SRS}_n(\tau)$ and its proof is shown in Section S.3 of the supplementary material.

Proposition 3.1. Suppose that Assumptions (A1), (A4) and (A5), and (A8) and (A9) hold. Under the null hypothesis $H_0$ in (3.5) for any $\tau \in (0, 1)$, as $n \to \infty$, we have $\text{SRS}_n(\tau) \Rightarrow \chi^2_K$.

For one kink point $\delta$, the confidence interval can be obtained by inverting the rank score test due to the fact that the test statistic $\text{SRS}_n(\tau)$ is convex in $\delta$. Specially, we first obtain the estimator $\tilde{\delta}$ for $\delta$ and then test $H_0: \delta = \tilde{\delta}$ for $\delta = \tilde{\delta} + \varrho$, where $\varrho$ is a small positive increment. If $H_0$ is not rejected, then increase
by $\tilde{\delta} = \tilde{\delta} + \tilde{\rho}$ and test $H_0 : \delta = \tilde{\delta}$ again. We repeat the previous testing procedure until $H_0$ is rejected and set the upper bound of the confident interval for $\delta$ as the minimum rejection point, denoted by $\tilde{\delta}_u$. In the similar way, we obtain the lower bound $\tilde{\delta}_l$. Thus, we can obtain a $(1 - \alpha)$th confidence interval for $\delta$, $[\tilde{\delta}_l, \tilde{\delta}_u]$. For multiple kink model (2.1), we separately construct the confidence interval for each kink location parameter by controlling other kink estimators. The details are summarized in Section S.3 of the supplementary material.

4. Monte Carlo Simulations

4.1. Parameters Estimation

We generate data from the following model:

$$Y_t = \alpha_0 + \alpha_1 X_t + \sum_{k=1}^{K} \beta_k (X_t - \delta_k) I(X_t > \delta_k) + \gamma Z_t + \sigma(X_t, Z_t) \epsilon_t, \quad t = 1, \ldots, n,$$

where $X_t \sim U(-5, 5)$, $Z_t \sim N(1, 1^2)$, $\epsilon_t \sim N(0, 1)$ or $t_3$ distribution and $\sigma(X_t, Z_t)$ controls the heteroscedasticity. Specially, $\sigma(X_t, Z_t)$ equals to 1 for a homoscedastic model and $1 + 0.2X_t$ for a heteroscedastic model. We set $\alpha_0 = 1$, $\alpha_1 = 1$, $\gamma = 1$ and consider three different cases for kink effects: (1) $K = 1$, $\beta_1 = -3$ and $\delta_1 = 0.5$; (2) $K = 2$, $(\beta_1, \beta_2) = (-3, 4)$ and $(\delta_1, \delta_2) = (-1, 2)$; (3) $K = 3$, $(\beta_1, \beta_2, \beta_3) = (-3, 4, -4)$ and $(\delta_1, \delta_2, \delta_3) = (-3, 0, 3)$.

We first check the selection consistency of Theorem 2.2. Since Condition (A7) requires that $C_n \log(n)/n \to 0$ as $n \to \infty$, we consider $C_n = 1, \log(\log(n))$ and $\log(n)$ in the definition of sBIC. We set the sample size $n = 500$. Table 1 reports the percentages of correctly selecting $\hat{K} = K$ based on 1000 replications under both homoscedastic and heteroscedastic models. All selection rates are very high and close to 100%. It shows that a diverging number for $\tilde{\delta}$ in accord with the results of Fryzlewicz (2014) for sequential change points detection. These results validate the selection consistency of Theorem 2.2.

| $\epsilon_t$ | $r$ | $K = 1$ | $K = 2$ | $K = 3$ |
|--------------|-----|---------|---------|---------|
| $t_3$        | 0.3 | 91.6%  | 99.8%  | 100.0%  |
|              | 0.5 | 94.2%  | 100.0% | 100.0%  |
|              | 0.7 | 91.1%  | 98.8%  | 98.8%   |

Next, we evaluate the finite sample performance of parameter estimators to check the validity of Theorem 2.1. For Case (1) with single kink effect, we compare the proposed estimation method with the bent line quantile estimators proposed by Li et al. (2011) and the kink regression least squares estimators proposed by Hansen (2017). Both existing methods assume there is only a single kink effect. We denote two methods as SKQR and SKLS, short for Single Kink Quantile Regression and Single Kink Least Square, respectively. We conduct the simulations 500 times and report the estimation biases (Bias), the empirical standard deviations (SD) and the mean square errors (MSE) for each parameter based on 500 estimates as well as their average estimated standard errors (SE) based on the asymptotical variance in Theorem 2.1. All simulation results are summarized in Table 2. All estimates have ignorable biases, and the standard deviations (SD) are close to the estimated standard errors (SE). In the homoscedastic model with normal random errors, the SKLS estimators have smaller mean square errors (MSE) than others. However, when the model is heteroscedastic or the errors follow $t_3$ distribution, our method works better than the other two in terms of MSE, which demonstrate the robustness and efficiency of the proposed estimators.

For the multi-kink models, both SKLS and SKQR methods are not able to detect the multiple kink points. Thus, we only present the simulation results of the proposed MKQR estimators for Case (2) with $K = 2$ in Table 3. All the biases are sufficiently close to zero and the SEs are compatible with the SDs for both homoscedastic and heteroscedastic errors. To save the space, we omit the similar simulation results for Case (3) with $K = 3$. These results demonstrate the validity of Theorem 2.1 for the multiple kink effects.

4.2. Power Analysis

We now assess the power performances for testing the existence of kink effects in Section 3.1. We generate the data from Case (1) except $\beta = n^{-1/2}c$, where $n = 1000$, $c = 0, 2, 4, 6, 8, 10$, and $c = 0$ corresponds to the null hypothesis. We compare our proposed test with two existing tests, the lack-of-fit (L.O.F) test proposed by He and Zhu (2003) and the F-type test proposed by Hansen (2017). The lack-of-fit test is a general test for checking model
specification, which was also used in Li et al. (2011). For our score-based test, we compute the $P$-values using wild bootstrap with 300 replicates. Figure 2 displays the power curves of three tests over different signal strength values of $c$. The L.O.F test cannot control the type I errors close to the nominal significance level 5% for homoscedastic hypothesis when $c$ (CIs) for the kink location parameter. However, the L.O.F test can not control the type I errors when there exists heteroscedasticity. As $c$ increases, that is, the kink effect gets enhanced, the empirical powers to identify the kink effect for all methods gradually increase to one for each scenario. Our proposed test have the higher empirical powers than the other two tests, especially when the errors follow $\alpha$-based CIs are constructed with $300$ replicates.

### 4.3. Confidence Intervals

Next, we evaluate the finite sample performances of the smoothed rank score (SRS) test-inversion confidence intervals (CIs) for the kink location parameter $\delta$. For a comparison purpose, we additionally consider the Wald-type CIs and the bootstrap CIs. The Wald-type $(1 - \alpha)$th CIs are constructed based on the asymptotic normality in Theorem 2.1, that is, $\hat{\delta}_k \pm z_{\alpha/2}SE(\hat{\delta}_k)$, for $k = 1, \ldots, K$, where $z_{\alpha/2}$ is the $\alpha/2$ upper tailed critical value of the standard normal distribution and $SE(\hat{\delta}_k)$ is the estimated standard error of $\delta_k$. The Bootstrap CIs are defined as $[\hat{\delta}_k - z_{\alpha/2}SE_1(\hat{\delta}_k), \hat{\delta}_k - z_{1-\alpha/2}]$, the $(\alpha/2)$th and $(1 - \alpha/2)$th quantiles of bootstrap estimators $[\hat{\delta}_k, b = 1, 2, \ldots, B]$ with $B$ paired bootstrap samples.

We generate data from the MKQR model of Case (2) with $K = 2$ and $n = 500$. To save the space, we only report the simulation results for the heteroscedastic model with the errors from $\tau_3$ distribution. Table 4 reports the coverage probabilities and the mean width of 95% confidence intervals as well as the average running time per replication at different quantile levels $\tau = 0.3, 0.5, 0.8$ based on 1000 simulations.

From Table 4, the coverage probabilities of Wald-type intervals are generally lower than the 95% nominal level. Hansen (2017) and Fong et al. (2017) also found that Wald-type CIs have poor finite sample performance, especially for threshold parameters due to the parameter-effects curvature. The bootstrap intervals have the highest coverage rates, but they have the largest interval lengths and need much more computing time. The bootstrap method is less computationally efficient. The proposed smoothed rank score (SRS) test-inversion CIs provide a balance between the estimation accuracy and the computation efficiency. They have higher coverage probabilities than the Wald-type intervals and also need much less computing time than the bootstrap intervals.

### 5. Empirical Analysis

#### 5.1. Secondary Industrial Structure of China

The past few decades have witnessed the miracle of China’s economic growth. Since China introduced the policy of reform and opening in 1978, GDP per capita has experienced a considerable growth and the industrial structure has also undergone tremendous changes. Classical development economic theory tells us that in the process of development, the proportion of first industry decreases while the tertiary industry instead increases gradually for one country. Meanwhile, the proportion of secondary industry experiences a process of increasing rapidly at first and then gradually stops growing or even decreases, which implies the presence of a kink pattern. The economic development model of China, as the biggest developing country

| $\tau$ | Homoscedasticity | Heteroscedasticity |
|-------|------------------|-------------------|
| $N$   |                  |                   |
| SKQR  | Bias             |                   |
| SD    | -0.020           | 0.054             |
| SE    | -0.014           | -0.002            |
| MSE   | -0.002           | -0.058            |
| SKLS  | Bias             |                   |
| SD    | 1.481            | 1.081             |
| SE    | 1.415            | 1.110             |
| MSE   | 0.218            | 0.117             |
| MKQR  | Bias             |                   |
| SD    | 1.124            | 1.056             |
| SE    | 1.132            | 1.056             |
| MSE   | 0.126            | 0.074             |
| $t_3$ | SKQR             |                   |
| SD    | 0.109            | 0.064             |
| SE    | 0.102            | 0.017             |
| MSE   | 0.023            | 0.001             |
| SKLS  | Bias             |                   |
| SD    | 1.539            | 1.173             |
| SE    | 1.606            | 1.304             |
| MSE   | 0.237            | 0.137             |
| MKQR  | Bias             |                   |
| SD    | 1.907            | 1.887             |
| SE    | 1.893            | 1.764             |
| MSE   | 0.362            | 0.355             |

| Estimation results (multiplied by a factor of 10) of the methods for Case (1). |
|-----------------|-------------------|-------------------|
| $\tau$          | Homoscedasticity  | Heteroscedasticity |
|                 | $\alpha_0$  $\alpha_1$  $\beta_1$  $\gamma$  $\delta$ | $\alpha_0$  $\alpha_1$  $\beta_1$  $\gamma$  $\delta$ |
| $N$ SKQR Bias   | -0.020           | 0.054             |
| SD              | 1.481            | 1.081             |
| SE              | 1.415            | 1.110             |
| MSE             | 0.218            | 0.117             |
| SKLS Bias       | 0.068            | 0.096             |
| SD              | 1.124            | 1.074             |
| SE              | 1.132            | 1.056             |
| MSE             | 0.126            | 0.116             |
| MKQR Bias       | 0.051            | 0.077             |
| SD              | 1.482            | 1.036             |
| SE              | 1.435            | 0.998             |
| MSE             | 0.219            | 0.107             |

| $t_3$ SKQR Bias | 0.109 | 0.064 |
|                 | 1.539 | 1.173 |
|                 | 1.606 | 1.304 |
|                 | 0.237 | 0.137 |
| SKLS Bias       | 0.025 | 0.025 |
|                 | 1.597 | 1.887 |
|                 | 1.907 | 1.764 |
|                 | 0.362 | 0.355 |
| MKQR Bias       | 0.063 | 0.063 |
|                 | 1.495 | 1.306 |
|                 | 1.560 | 1.080 |
|                 | 0.223 | 0.130 |

Biais: the empirical bias; SD: the empirical standard deviation; MSE: the mean square error; SE: the average estimated standard error. The minimum MSEs among three estimators are highlighted in bold.
Table 3. Estimation results (multiplied by a factor of 10) of the proposed MKQR method for Case (2) with \( K = 2 \).

| \( \tau \) | Bias  | \( a_0 \)  | \( a_1 \)  | \( \gamma \)  | \( \beta_1 \)  | \( \beta_2 \)  | \( \delta_1 \)  | \( \delta_2 \)  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0.2 | -0.001 | 0.000 | -0.001 | 0.000 | 0.000 | 0.000 | 0.000 |
|     | SD   | 0.006 | 0.020 | 0.014 | 0.037 | 0.044 | 0.024 | 0.019 |
|     | SE   | 0.006 | 0.020 | 0.014 | 0.038 | 0.045 | 0.024 | 0.019 |
|     | MEE  | 0.005 | 0.000 | 0.000 | 0.001 | 0.002 | 0.001 | 0.000 |
| 0.5 | -0.001 | 0.000 | -0.001 | 0.002 | 0.000 | 0.000 | 0.000 |
|     | SD   | 0.021 | 0.019 | 0.014 | 0.035 | 0.042 | 0.022 | 0.019 |
|     | SE   | 0.025 | 0.019 | 0.017 | 0.043 | 0.049 | 0.026 | 0.022 |
|     | MEE  | 0.044 | 0.000 | 0.000 | 0.001 | 0.002 | 0.000 | 0.000 |

| \( \alpha \)  | Wald  | Boot  | Score  |
|-----|-----|-----|-----|
| 0.3 | 0.252 | 0.970 | 0.315 |
| 0.5 | 0.297 | 0.975 | 0.333 |
| 0.8 | 0.320 | 0.982 | 0.367 |

Table 4. 95% confidence intervals for each kink point parameter in Case (2) in the heteroscedastic model with the \( t_3 \) errors.

| \( \tau \) | Type  | Coverage probability | Mean interval length |
|-----|-----|---------------------|---------------------|
|     | \( \alpha_1 \)  | \( \alpha_2 \)  | \( \alpha_1 \)  | \( \alpha_2 \)  |
| 0.3 | Wald  | 0.950 | 0.923 | 0.353 | 0.518 | 0.07 |
|     | Boot  | 0.960 | 0.970 | 0.391 | 0.689 | 377.81 |
|     | Score | 0.930 | 0.957 | 0.343 | 0.641 | 12.70 |
| 0.5 | Wald  | 0.923 | 0.933 | 0.307 | 0.436 | 4.45 |
|     | Boot  | 0.968 | 0.982 | 0.323 | 0.575 | 375.07 |
|     | Score | 0.927 | 0.953 | 0.303 | 0.530 | 11.38 |
| 0.8 | Wald  | 0.913 | 0.883 | 0.451 | 0.619 | 4.53 |
|     | Boot  | 0.970 | 0.974 | 0.501 | 0.801 | 378.38 |
|     | Score | 0.917 | 0.930 | 0.449 | 0.774 | 13.82 |

Wald: Wald-type CIs; Boot: bootstrap CIs; Score: SRS test-inversion CIs. Time is the average running time for one simulation.

Table 5 reports \( P \)-values for testing the existence of kink effects based on 1000 bootstrap replicates, the estimated number of kink points, the estimated parameters as well as their the standard errors, and the confidence intervals for kink locations.

2We also separately test the existence of kink effects between \( Y_t \) and \( Z_{t1} \) and \( Z_{t2} \) at different quantiles. The resulting \( p \)-values are all greater than 0.1 across all quantiles indicating no kink effect on FE and FAI.

in the world, has been aroused a great of research interest, see Song, Storesletten, and Zilibotti (2011), Brandt, Tombe, and Zhu (2013), Cao and Birchenall (2013), etc.

In this section, we aim to investigate whether there exist kink effects between secondary industrial structure and the economic growth from the quantile regression perspective using the prefecture-level cities data in China. After removing the missing values, we collect data for 280 Chinese prefecture-level cities of year 2016 from the Organisation for Economic Co-operation and Development (OECD) database available at https://insights.ceicdata.com/. We consider the MKQR model

\[
Q_{Y_t}(\tau | X_t, Z_t) = \alpha_0 + \alpha_1 X_t + \sum_{k=1}^{K} \beta_k (X_t - \delta_k) I(X_t > \delta_k) + \gamma^T Z_t, \quad t = 1, \ldots, 280, \tag{5.1}
\]

where \( Y_t \) represents the proportion of secondary industry of the \( t \)th city, \( X_t \) is the GDP per capita (10^4 Chinese Yuan) and \( Z_t \) includes the fiscal expenditure (FE) and fixed assets investment (FAI), which are generally deemed to be correlated with the industrial structure. To eliminate effect by the difference of economic scales, we divide the FE and FAI by the total GDP across all quantiles indicating nokin kink effecton FE and FAI.
Figure 2. Power comparison of the proposed test at $\tau = 0.5$ (black circle), the lack-of-fit test at $\tau = 0.5$ (brown triangle) and the F-type test (orange plus) at the significance level 5%.

Figure 3 displays the scatterplot between secondary industrial proportions of 280 cities in China and their GDP per capita with the fitted MKQR curves at different quantile levels. According to Table 5, $P$-values are 0 and $\hat{K} = 1$ for all different quantiles, which indicate that there exists a significant kink point. $\hat{\alpha}_1 > 0$ and $\hat{\beta}_1 < 0$ for all quantiles are statistically significant which means that the second industrial proportions $Y_t$ first quickly increase with GDP per capita and then stabilizes with a slow increasing rate of $\hat{\beta}_1 + \hat{\alpha}_1$ (e.g., $\hat{\beta}_1 + \hat{\alpha}_1 = 0.004$ for $\tau = 0.5$). This empirical finding demonstrates the classical economic theory about the process of development. It is also of interest to observe that the estimated kink points are around 35,000 to 45,000 Chinese Yuan (roughly 5000–6500 United States Dollar). Based on the Chenery industrialization stage theory (Chenery et al. 1986), GDP per capita in this interval indicates that an economic entity is going through an important turning period. During this period, if one economy can skip the threshold value and achieve economic restructuring, it will move into high-income group. Otherwise, the middle-income trap may loom. In addition, both regressors $Z_{t1}$ and $Z_{t2}$ are statistically significant based on the Wald-type test. It is confirmed that the proportions of secondary industry are indeed correlated to the government FE and the fixed asset investment.

5.2. Triceps Skinfold Thickness of Gambian Females

Triceps skinfold thickness (TSF) as an important measure for body density experiences the dynamic changes with the increase of the economic structure. He divided the structural transformation process of GDP per capita into three stages: Initial, Intermediate and Post-industrial stages, corresponding to the GDP per capita less than 1495 dollars, 1495–11214 dollars and greater than 11214 dollars.

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*Professor Hollis B. Chenery at Harvard University believed that modern economic growth can be understood as a comprehensive transformation.*
of age. People whose TSFs are above the 85th percentile are more likely to suffer from obesity, while those whose TSFs are lower the 20th percentile are usually skinny. Exploring the relationship between TSF and age at different quantiles has been of great interest in biological and human health studies. For instance, Durnin and Womersley (1974) divided 481 subjects aged from 16 to 72 into four subgroups based on ages and used linear regression to fit the logarithm of TSF and body densities for each subsample. The results showed that the regression coefficients of each group exhibited significant differences from the others. Cole and Green (1992) demonstrated that there existed cubic splines nonlinear pattern between the logarithm of TSF and age by using the smooth fitting curves. Perperoglou et al. (2019) fitted the Gambian females dataset using the spline regression to depict the nonlinearity between TSF and age. Although the spline regression captures the nonlinear trend, it does not provide any information concerning thresholds and is lack of interpretability in each segment. The nonparametric spline method is either not robust to outliers and heavy-tailed data.

We consider the dataset collected by Royston and Sauerbrei (2008) from an anthropometry survey at three Gambian villages in 1989, containing 892 women between the aged of 0 and 55. To investigate the relationship between their TSFs and ages and identify the potential kink points at different quantiles, the following MKQR model is considered

\[
Q_\tau(Y_t|X_t) = \alpha_0 + \alpha_1 X_t + \sum_{k=1}^{K} \beta_k (X_t - \delta_k) I(X_t > \delta_k),
\]

where \( Y_t = \log(\text{TSF}) \), \( X_t \) is the age, \( K \) is the unknown number of kink points. We set \( \tau = 0.1, 0.3, 0.5, 0.7, \) and 0.9 to study the different conditional quantiles of \( \log(\text{TSF}) \) on age.

Table 6 reports \( P \)-values for testing the existence of kink effects based on 1000 bootstrap replicates, the estimated number of kink points, the estimated parameters as well as their the standard errors, and the confidence intervals for kink locations.

![Figure 3. Scatterplot between secondary industrial structure of China and GDP per capita with the fitted MKQR curves at different quantile levels. ▲ denotes the estimated kink point.](image)

| \( \tau \) | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|
| \( \hat{\alpha}_0 \) | -0.012 (0.067) | 0.184 (0.057) | 0.090 (0.057) | 0.182 (0.057) | 0.182 (0.057) |
| \( \hat{\beta}_1 \) | 0.114 (0.022) | 0.059 (0.014) | 0.102 (0.019) | 0.084 (0.017) | 0.074 (0.015) |
| \( \hat{K} \) | 3.457 (0.307) | 4.900 (0.355) | 3.468 (0.227) | 3.571 (0.191) | 3.777 (0.301) |
| \( \gamma_1 \) | -0.985 (0.405) | -0.916 (0.229) | -0.711 (0.159) | -0.828 (0.303) | -0.687 (0.252) |
| \( \gamma_2 \) | 0.078 (0.024) | 0.086 (0.020) | 0.090 (0.015) | 0.098 (0.014) | 0.060 (0.012) |

The figures in parentheses denote the standard errors of estimators.
Table 6. Parameter estimation and test results of the MKQR model at different quantile levels for triceps skinfold thickness data for Gambian females.

| Parameter | τ = 0.1 | τ = 0.3 | τ = 0.5 | τ = 0.7 | τ = 0.9 |
|-----------|---------|---------|---------|---------|---------|
| P-values  | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| \( \hat{\alpha}_0 \) | 1.895(0.021) | 2.042(0.029) | 2.183(0.027) | 2.241(0.023) | 2.426(0.037) |
| \( \hat{\alpha}_1 \) | -0.041(0.002) | -0.046(0.005) | -0.047(0.009) | -0.090(0.012) | -0.086(0.013) |
| \( \hat{\beta}_1 \) | 0.096(0.013) | 0.106(0.010) | 0.129(0.010) | 0.136(0.012) | 0.144(0.015) |
| \( \hat{\beta}_2 \) | -0.056(0.0014) | -0.058(0.010) | -0.075(0.009) | -0.090(0.012) | -0.068(0.013) |
| Wald     | 9.781, 10.290 | 9.373, 10.861 | 9.430, 10.630 | 9.803, 11.467 | 7.679, 9.530 |
| Boot     | 8.206, 12.988 | 9.086, 12.235 | 9.425, 12.050 | 8.223, 12.202 | 7.373, 10.679 |
| Score    | 8.217, 13.930 | 8.979, 12.393 | 9.418, 12.478 | 7.663, 12.758 | 7.665, 10.967 |
| \( \hat{\delta}_1 \) | 10.035(0.130) | 10.117(0.379) | 10.030(0.306) | 10.635(0.425) | 8.604(0.472) |
| Wald     | 14.678, 26.150 | 16.700, 22.678 | 16.939, 21.047 | 17.307, 20.621 | 15.801, 21.639 |
| Boot     | 14.530, 47.470 | 17.280, 29.735 | 17.562, 23.226 | 18.067, 24.724 | 16.588, 24.821 |
| Score    | 17.487, 49.680 | 16.639, 42.566 | 17.945, 25.282 | 18.119, 25.728 | 15.742, 26.166 |

Figure 4. Scatterplot between the logarithm of TSF and ages for Gambian females with the fitted MKQR curves at different quantile levels. ▲ denotes the estimated kink point.

The resulting P-values are all close to zeros, implying that log(TSF) has significant kink effects on the age for all quantiles. We estimate the MKQR models at different quantiles by setting 10 initial kink points and identify \( \hat{K} = 2 \) kink points located round 10 years and 20 years. This result is in accord with the biological intuition. Two kink points split the domain of the age into the three growth periods of human beings: childhood, adolescence and adults. Figure 4 also displays the scatterplot between log(TSF) and their ages with the fitted MKQR curves at different quantile levels. One can observe that the logarithm of TSF decreases quickly with the age in the childhood up to about 8–11 years old, then experiences a growth spurt at adolescence up to about 18–21 years old and finally stays almost stable after then for adults. The variance of TSF increases with the age, which makes quantile regression necessary to handle with the heteroscedasticity. It is also interesting to notice that the kink points estimators are heterogeneous across different quantiles. For the higher quantiles such as \( \tau = 0.9 \), log(TSF) tend to experience the smaller kink points locations than other quantiles, for example \( \hat{\delta}_1 = 8.604 \) years for \( \tau = 0.9 \). It means that Gambian females with obesity reach the biological limits earlier, making their TSFs get changed sooner in the growth process.

As a comparison, we also analyze the dataset by using SKLS and SKQR methods. Both methods can only detect with a single kink point at around 6–8 years, which is much lower than our first threshold estimator \( \hat{\delta}_1 \). However, if the Wald-type test of Li et al. (2011) and the F-type test of Hansen (2017) are used to check the existence of kink effects using subsample in the second segment divided by the threshold estimator, we find that both tests reject the null hypothesis indicating that some potential kink effect is ignored. In contrast, our MKQR method is flexible and robust in practice to capture multi-kink effects.
6. Conclusion

In this article, we studied the flexible MKQR model without knowing the number of kink points. It is robust to outliers and heavy-tailed errors and more flexible for modelling data with heterogeneous conditional distributions. We proposed the BRIQ algorithm for estimating parameters. It is much more computationally efficient and not sensitive to the initial values. The selection consistency and the asymptotic normality were established and the statistical inference for kink effects were also developed. A R package MultiKink has been developed for all the estimation and inference procedures. In the future studies, several topics could be considered. It is interesting to test whether the regression curve is continuous or not at the change points and how to estimate the parameters if jumps happen at the change points. We may also consider some additional restrictions on the regression coefficients such as nondecreasing slopes in the MKQR model. We can also consider the composite quantile regression idea to enhance the estimation efficiency to identify the kink points under the commonality assumption of kink points across multiple quantile levels. Besides, extensions to other regressions such as generalized linear models, Cox proportional hazards models or censored models can be also considered for the future research.

Appendix

A.1. Proof of Theorem 2.1

To show the asymptotic normality of $\hat{\theta}$, we need derive its consistency at first.

Lemma A.1. Under Assumptions (A1)-(A4), $\hat{\theta}$ is a consistent estimator of $\theta_0$.

Proof of Lemma A.1. We first need to show that $\sup_{\theta \in \Theta} |S_n(\theta) - S(\theta)| \overset{p}{\to} 0$ as $n \to \infty$. Notice that $S(\theta)$ is continuous and has the following first derivative

$$\frac{\partial S(\theta)}{\partial \theta} = E \{\psi_n[I(Y_t - Q_{\tau}(\theta; W_t)]h(W_t; \theta)\}.$$  

By the Assumptions (A3) and (A4), we can get that $E \sup_{\theta \in \Theta} |h(W_t; \theta)| < \infty$. Together with $\sup_{\theta \in \Theta} |\psi_n[I(Y_t - Q_{\tau}(\theta; W_t)]h(W_t; \theta)|$ \leq \max(\tau, 1 - \tau)$, we can show that $E \sup_{\theta \in \Theta} |\psi_n[I(Y_t - Q_{\tau}(\theta; W_t)]h(W_t; \theta)|$ is finite. By using the mean-value theorem, for any $\theta^1, \theta^2 \in \Theta$, there exists a $\theta^*$ such that

$$S_n(\theta^1) - S_n(\theta^2) = \frac{1}{n} \sum_{t=1}^n \{\psi_n[I(Y_t - Q_{\tau}(\theta^*; W_t)]h(W_t; \theta^*)\}$$

By using Assumptions (A3) and (A4) again,

$$E \left| \frac{1}{n} \sum_{t=1}^n \{\psi_n[I(Y_t - Q_{\tau}(\theta^*; W_t)]h(W_t; \theta^*)\} \leq B_n < \infty,$$

where $B_n = E \sup_{\theta \in \Theta} \left| \frac{\max(\tau, 1 - \tau)}{\tau} \right| |h(W_t; \theta)|$. Hence $B_n = O_p(1)$ and $|S_n(\theta^1) - S_n(\theta^2)| \leq B_n||\theta^1 - \theta^2||$ for every $\theta^1$. By applying the Lemma 2.9 of Newey and McFadden (1994), we have $\sup_{\theta \in \Theta} |S_n(\theta) - S(\theta)| \overset{p}{\to} 0$ for $\theta \in \Theta$.

Since $S_n(\theta)$ is continuous w.r.t $\theta$, and $S(\theta)$ uniquely reaches its global minimum at $\theta_0$ (Assumption (A2)), together with $\sup_{\theta \in \Theta} |S_n(\theta) - S(\theta)| \overset{p}{\to} 0$, then we can immediately induce that $\hat{\theta} \overset{p}{\to} \theta_0$ as $n \to \infty$ by using the Theorem 2.1 of Newey and McFadden (1994).

The following lemma is sufficient for deriving the Bahadur representation of $\hat{\theta}$.

Lemma A.2. Suppose Assumptions (A1), and (A3) and (A4) hold, for any positive sequence $d_n$ converging to zero, we have

$$\sup_{|\theta - \theta_0| \leq d_n} \left| \sum_{t=1}^n \{\psi_n[I(Y_t - Q_{\tau}(\theta; W_t)]h(W_t; \theta)\} - \psi_n[I(Y_t - Q_{\tau}(\theta_0; W_t)]h(W_t; \theta_0)\} \right| = o_p(1),$$

Proof of Lemma A.2. Define $u_t(\theta, \theta_0) = \psi_n[I(Y_t - Q_{\tau}(\theta; W_t)]h(W_t; \theta) - \psi_n[I(Y_t - Q_{\tau}(\theta_0; W_t)]h(W_t; \theta_0).$

Note that $u_t(\theta, \theta_0)$ can be partitioned into several parts based on the range of $X_t$,

$$u_t(\theta, \theta_0) = \sum_{k=1}^{K_0+1} u_{t,k,1}(\theta, \theta_0) + \sum_{k=1}^{K_0} u_{t,k,2}(\theta, \theta_0) + \sum_{k=1}^{K_0} u_{t,k,3}(\theta, \theta_0),$$

where $\delta_{k,0}$ and $\delta_{K_0+1,0}$ denotes the minimum and maximum of $X_t$. To prove Lemma A.2, it is sufficient to show

$$\sup_{|\theta - \theta_0| \leq d_n} \left| \sum_{t=1}^n \left[ u_{t,k,j}(\theta, \theta_0) - E(u_{t,k,j}) \right] \right| = o_p(1)$$

for $k = 1, \ldots, K_0 + 1$ and $j = 1, \ldots, 3$. The proofs directly follow from the result of Lemma 4.6 in He and Shao (1996). We only take $u_{t,k,1}(\theta, \theta_0)$ for illustration and the remaining are the same. For this, we need to check the conditions (B1), (B3), and (B5) in He and Shao (1996).

The measurability is straightforward for (B1). For (B3), by using mean-value theorem, we have

$$\sum_{t=1}^n \mathbb{E}[u_{t,k,1}(\theta, \theta_0)]^2 \leq L_{\theta}d_n^{f_{\theta}^2} ||U_t||^3,$$

where $L$ is some positive constant, $U_t = (1, X_t, Z_t^T)^T$ and $f_{\theta}^2$ is some intermediate density satisfying $f_{\theta} \to f_0(\theta)$ almost surely when $n \to \infty$. It is obvious to obtain (B3). For (B5), let $A_n = L \sum_t f_{\theta}^2 ||U_t||^3$. Under Assumptions (A3) and (A4), we have $A_n = O_p(1/n)$, and $\max_{1 \leq t \leq n} |u_{t,k,1}(\theta, \theta_0)| = O_p(n^{1/2})$. Thus, (B5) is satisfied. By using Lemma 4.6 of He and Shao (1996), Lemma A.2 is therefore established.
Proof of Theorem 2.1. Based on Lemmas A.1 and A.2, we have
\[ n^{-1/2} \sum_{t=1}^{n} [\psi_{t}(Y_{t} - Q_{Y}(\tau; \hat{\theta})|W_{t})]h(W_{t}; \hat{\theta}) = op(1) \] (A.1) Applying the Taylor expansion, we obtain
\[ nD_{n}(\hat{\theta} - \theta) + Op((\hat{\theta} - \theta)^{2}) \] (A.2) where
\[ D_{n} = n^{-1} \sum_{t=1}^{n} \frac{\partial E[\psi_{t}(Y_{t} - Q_{Y}(\tau; \hat{\theta})|W_{t})]h(W_{t}; \hat{\theta})}{\partial \theta} \mid_{\theta = \theta_{0}} = n^{-1} \sum_{t=1}^{n} \left[ \frac{\partial E[F_{t}(Y_{t} - Q_{Y}(\tau; \hat{\theta})|W_{t})]h(W_{t}; \hat{\theta})}{\partial \theta} \right] \mid_{\theta = \theta_{0}} = n^{-1} \sum_{t=1}^{n} \left[ \frac{\partial E[F_{t}(Y_{t} - Q_{Y}(\tau; \hat{\theta})|W_{t})]h(W_{t}; \hat{\theta})}{\partial \theta} \right] \mid_{\theta = \theta_{0}} = n^{-1} \sum_{t=1}^{n} \left[ f_{t}(Y_{t} - Q_{Y}(\tau; \hat{\theta})|W_{t})]h(W_{t}; \hat{\theta})h^{T}(W_{t}; \theta_{0}) \right]. \]

Combined with the subgradient condition of quantile regression, we have
\[ n^{-1/2} \sum_{t=1}^{n} [\psi_{t}(Y_{t} - Q_{Y}(\tau; \hat{\theta})|W_{t})]h(W_{t}; \hat{\theta}) = op(1) \] (A.3) Together with Equations (A.1)–(A.3), we have
\[ n^{-1/2} \sum_{t=1}^{n} [\psi_{t}(Y_{t} - Q_{Y}(\tau; \theta_{0})|W_{t})]h(W_{t}; \theta_{0}) = op(1) \] (A.4)

Because the left-hand side term \( n^{-1/2} \sum_{t=1}^{n} [\psi_{t}(Y_{t} - Q_{Y}(\tau; \theta_{0})|W_{t})]h(W_{t}; \theta_{0}) \) is defined by Assumption (A5), we have \( ||\hat{\theta} - \theta_{0}|| = Op(n^{-1/2}). \) Therefore,
\[ n^{1/2}(\hat{\theta} - \theta_{0}) = -D_{n}(\hat{\theta} - \theta_{0}) + op((\hat{\theta} - \theta_{0})^{2}) + op(1). \] (A.5)

By Assumption (A5), it follows that \( n^{1/2}(\hat{\theta} - \theta_{0}) \) is asymptotically normal with mean zero and variance matrix \( D^{-1}G_{\theta}D^{-1} \). This completes the proof of Theorem 2.2. □

A.2. Proof of Theorem 2.2

The proof of Theorem 2.2 shares the similar strategy as Wang, Li, and Tsai (2007). Note that Theorem 2.2 is equivalent to
\[ P \left( \min_{K \neq K_{0}} sBIC(K) > sBIC(K_{0}) \right) \rightarrow 1. \] (A.5) To prove (A.5), we identify two different cases i.e Case 1 for \( K < K_{0} \) and Case 2 for \( K > K_{0} \). Denote \( \theta_{0}^{*} \) as the pseudo-true parameters for a given \( K_{0} \), that is \( \theta_{0}^{*} \equiv \arg \min_{K \neq K_{0}} E_{n}(\hat{\theta}_{K}). \)

Case 1: when \( K < K_{0} \), we need first prove \( s_{0}(\theta_{0}^{*}) = E_{\rho_{t}}[Y(\tau; \theta_{0}^{*})|W)] > s_{0}(\theta_{K_{0}}^{*}) = E_{\rho_{t}}[Y(\tau; \theta_{K_{0}}^{*})|W)] \). From Knight's identity, we can directly obtain that
\[ \rho_{t}[Y(\tau; \theta_{0}^{*})|W)] - \rho_{t}[Y(\tau; \theta_{K_{0}}^{*})|W)] \]
\[ = \{ Q_{Y}(\tau; \theta_{0}^{*})|W) - Q_{Y}(\tau; \theta_{K_{0}}^{*})|W) \} |I(\epsilon_{0} \leq 0) - \tau \}
\[ + \int_{0}^{\tau} Q_{Y}(\tau; \theta_{0}^{*})|W) - Q_{Y}(\tau; \theta_{K_{0}}^{*})|W) \[[I(\epsilon_{0} \leq s) - I(\epsilon_{0} \leq 0)]ds, \]
where \( \epsilon_{0} = Y(\tau; \theta_{K_{0}}^{*})|W) \) and therefore
\[ E[\rho_{t}[Y(\tau; \theta_{0}^{*})|W)] - \rho_{t}[Y(\tau; \theta_{K_{0}}^{*})|W)] \]
\[ = \int_{0}^{\tau} Q_{Y}(\tau; \theta_{0}^{*})|W) - Q_{Y}(\tau; \theta_{K_{0}}^{*})|W) \[[F_{0}(s) - F_{0}(0)]ds. \]

From Assumption (A1), density value \( f_{0}(\cdot;W) \) is always bounded away from zero. We can immediately obtain that \( E_{\int_{0}^{\tau} Q_{Y}(\tau; \theta_{0}^{*})|W) - Q_{Y}(\tau; \theta_{K_{0}}^{*})|W) \[[F_{0}(s) - F_{0}(0)]ds > 0 \) no matter \( Q_{Y}(\tau; \theta_{0}^{*})|W) - Q_{Y}(\tau; \theta_{K_{0}}^{*})|W) \) is positive or negative. Thus, \( S_{0}(\theta_{0}^{*}) > S_{0}(\theta_{K_{0}}^{*}) \) holds.

Denote \( \Theta_{K}^{*} = \arg \min_{K \neq K_{0}} S_{0}(\theta_{K}). \) By using similar argument in the proof of Lemma A.1, it is easy to show that \( \Theta_{K}^{*} \) is a consistent estimator for \( \theta_{0}^{*} \). Note that in the proof of Lemma A.1, we mainly use the Theorem 2.1 of Newey and McFadden (1994), which does not require the model to be correctly specified. Then we have
\[ sBIC(K) - sBIC(K_{0}) = \log \left( S_{0}(\Theta_{K}^{*}) \right) - \log \left( S_{0}(\Theta_{K_{0}}^{*}) \right) \]
\[ + (K - K_{0}) \log \frac{n}{C_{n}} \]

Since the first term can asymptotically dominate the second term and \( (K - K_{0}) \log \frac{n}{C_{n}} = o(1) \) by Assumption (A7),
\[ sBIC(K) - sBIC(K_{0}) \rightarrow \log \left( S_{0}(\Theta_{K}^{*}) \right) - \log \left( S_{0}(\Theta_{K_{0}}^{*}) \right) > 0, \]
as \( n \rightarrow \infty. \)

Therefore, \( sBIC(K) - sBIC(K_{0}) > 0 \) for \( K < K_{0} \) when \( n \) goes to the infinity.

Case 2: That is, when \( K > K_{0} \), by using the proposed BRiSQ algorithm, the estimator for \( \theta_{0}^{*} \) is denoted by \( \hat{\theta}_{K}^{*} = (\hat{\theta}_{K}^{*}, \hat{\delta}_{K}^{*}, \hat{\gamma}_{K}^{*})^{T}. \)
Recall that \( M_{t}(\delta_{K}) = (1, X_{t}(X_{t} - \delta_{1}) + \cdots + X_{t} - \delta_{K}) \right)^{T}. \) Then
\[ Q_{Y}(\tau; \theta_{K})|W) = M_{t}(\delta_{K})^{T} \hat{\theta}_{K}^{*} \] is a linear regression conditional on \( \delta_{K}. \) By using similar arguments in pages 121-122 of Koekner (2005), we can show that \( [\hat{\theta}_{K}^{*} - \theta_{K}^{*}] - \theta_{0}^{*} \)
\[ \sim Op(n^{1/2}) \] conditional on pseudo-true \( \delta_{K}^{*}. \) More specially, letting \( \hat{\theta}_{K}^{*} = \hat{\theta}_{K}^{*} + \hat{\delta}_{K}^{*} + \hat{\gamma}_{K}, \) where \( \hat{\delta}_{K}^{*} \) and \( \hat{\gamma}_{K}^{*} \) are the estimates of \( \delta_{K}^{*} \) and \( \gamma_{K}, \) we have
\[ \frac{\rho_{t}[Y_{t} - Q_{Y}(\tau; \hat{\theta}_{K}^{*}, \hat{\delta}_{K}^{*}, \hat{\gamma}_{K}|W_{t})| - \rho_{t}[Y_{t} - Q_{Y}(\tau; \theta_{0}^{*}, \delta_{K}^{*}|W_{t})]}{\sqrt{n}} \]
\[ = \frac{1}{\sqrt{n}} M_{t}(\hat{\delta}_{K}^{*})^{T} \hat{\gamma}_{K} \left[ I(\epsilon_{t} \leq 0) - \tau \right] \]
\[ + \int_{0}^{\tau} M_{t}(\hat{\delta}_{K}^{*})^{T} \hat{\gamma}_{K} \left[ I(\epsilon_{t} \leq s) - I(\epsilon_{t} \leq 0) \right] ds, \]
For the first term of right hand, we have \( \frac{1}{n} \sum_{i=1}^{n} \mathbf{M}_t(\mathbf{d}_K^* \theta^*_K)^T \mathbf{u}[l(\varepsilon_t \leq 0) - \tau] = O_p(1) \) by using central limit theorem. For the second term, it converges to a positive definite matrix. Thus we can obtain that \( \mathbf{d}_K^* \theta^*_K \) is a root-n consistent estimator and

\[
\left\| \sum_{t} \rho_t [Y_t - QY(\tau; \mathbf{d}_K^* \theta^*_K)] \right\| = O_p(1),
\]

In addition, since \( \mathbf{d}_K^* \theta^*_K \) is a consistent estimator for \( \theta^*_K \) and \( \sum_{t} \rho_t [Y_t - QY(\tau; \mathbf{d}_K^* \theta^*_K)] \) is continuous in \( \theta^*_K \), we have

\[
\left\| \sum_{t} \rho_t [Y_t - QY(\tau; \mathbf{d}_K^* \theta^*_K)] \right\| = O_p(1),
\]

Hence

\[
\left\| \sum_{t} \rho_t [Y_t - QY(\tau; \mathbf{d}_K^* \theta^*_K)] - \sum_{t} \rho_t [Y_t - QY(\tau; \mathbf{d}_K^* \theta^*_K)] \right\| = O_p(1),
\]

and

\[
sBIC(K) - sBIC(K_0) = \log \left( 1 + \frac{1}{n} \sum_{t=1}^{n} \rho_t [Y_t - QY(\tau; \mathbf{d}_K^* \theta^*_K)] \right)
\]

\[= \log \left( 1 + \frac{1}{n} \sum_{t=1}^{n} \rho_t [Y_t - QY(\tau; \theta^*_K)] \right)
\]

\[= \log \left( 1 + \frac{1}{n} \sum_{t=1}^{n} \rho_t [Y_t - QY(\tau; \theta^*_K)] \right)
\]

\[+ (K - K_0) \log \frac{n}{C_n}
\]

\[= O\left( \frac{1}{n} \right) + (K - K_0) \log \frac{n}{C_n}
\]

Since \( K - K_0 > 0 \), then \( sBIC(K) - sBIC(K_0) > 0 \). The proof of Theorem 2.2 is completed.

\section*{A.3. Proof of Theorem 3.1}

The following lemma is used for proving the Theorem 3.1.

\textbf{Lemma A.3.} Under the Assumptions (A1), (A3)-(A4) and (A8), as \( n \to \infty \), we have

\[
\sup_{\varepsilon_t} \left| \frac{1}{n} \sum_{t=1}^{n} \left[ \tilde{f}(\varepsilon_t) \varepsilon_t \varepsilon_t - H(\varepsilon_t) \right] \right| = O_p(1);
\]

\[
\sup_{\varepsilon_t} \left| \frac{1}{n} \sum_{t=1}^{n} \left[ \tilde{f}(\varepsilon_t) \varepsilon_t \varepsilon_t \varepsilon_t - H(\varepsilon_t) \right] \right| = O_p(1);
\]

\[
\sup_{\varepsilon_t} \left| \frac{1}{n} \sum_{t=1}^{n} \varepsilon_t \varepsilon_t - H \right| = O_p(1).
\]

\textbf{Proof of Lemma A.3.} We only give the proof for (I), since the proof for (II) and (III) are the same. For (I), it is sufficient to show that \( \sup_{\varepsilon_t} |H(\varepsilon_t) - H(\varepsilon_t) - 0| = O_p(1) \), where \( H(\varepsilon_t) = \frac{1}{n} \sum_{t=1}^{n} \tilde{f}(\varepsilon_t) \varepsilon_t \varepsilon_t - H(\varepsilon_t) \). We have

\[
H(\varepsilon_t) - H(\varepsilon_t) = \frac{1}{n} \sum_{t=1}^{n} \tilde{f}(\varepsilon_t) \varepsilon_t \varepsilon_t - H(\varepsilon_t) + \frac{1}{n} \sum_{t=1}^{n} \tilde{f}(\varepsilon_t) \varepsilon_t \varepsilon_t - H(\varepsilon_t) \leq \frac{1}{n} \sum_{t=1}^{n} \tilde{f}(\varepsilon_t) \varepsilon_t \varepsilon_t - H(\varepsilon_t)
\]

\[
\sup_{\varepsilon_t} |(a) + (b) + (c)| = O_p(1)
\]

which implies the desired result. \( \blacksquare \)

\textbf{Lemma A.4.} Under Assumptions (A1), (A3)-(A4) and (A8), the local alternative model \( (3.3) \), \( \hat{\alpha} \) has the following Bahadur representation:

\[
\hat{\alpha} = \alpha + O_p(1).
\]

\textbf{Proof of Lemma A.4.} By using Lemma 4.1 of He and Shao (1996), we have

\[
\left\| \sum_{t=1}^{n} \tilde{f}(\varepsilon_t) \varepsilon_t \varepsilon_t - H(\varepsilon_t) \right\| = O_p(1).
\]

where \( \varepsilon_t = Y_t - \alpha^T V_t - n^{-1/2} 2(\hat{\beta}_t - \beta) + o_p(1) \).
Based on the subgradient condition of quantile regression, we get
\[ n^{-1/2} \sum_{i=1}^{n} \psi_i(Y_i - \alpha^T V_t) V_t = o_p(1). \]

Hence,
\[ n^{-1/2} \sum_{i=1}^{n} [\psi_i(Y_i - \alpha^T V_t) - (\tau - F_t(Y_i - \alpha^T V_t))] V_t \]
\[ = n^{-1/2} \sum_{i=1}^{n} (F_t(Y_i - \alpha^T V_t) - \tau) V_t + o_p(1) \]
\[ = n^{-1/2} \sum_{i=1}^{n} f_i(Y_i - \alpha^T V_t - n^{-1/2} (X_t - \delta) I(X_t > \delta)) V_t V_t^T (\alpha - \alpha) - n^{-1} \]
\[ \sum_{i=1}^{n} f_i(Y_i - X_t^2 V_t - n^{-1/2} (X_t - \delta) I(X_t > \delta)) \]
\[ V_t V_t^T (\alpha - \alpha) - n^{-1} \sum_{i=1}^{n} f_i(Y_i - \alpha^T V_t - n^{-1/2} (X_t - \delta) I(X_t > \delta)) \]
\[ V_t V_t^T (\alpha - \alpha) - H_2(\delta, \beta) + o_p(1) + o_p \left( n^{-1/2} (\alpha - \alpha) \right) \].

Together with (A.7), the proof of Lemma A.4 is completed.

**Proof of Theorem 3.1.** Under the null hypothesis \( \beta = 0 \), \( q(\delta, \beta) = 0 \) and thus Theorem 3.1 holds under \( H_{0q} \). It remains to show that Theorem 3.1 holds under local alternative model (3.3). By Lemmas A.3 and A.4, and after some simple algebraic manipulation, it is easy to obtain that
\[ R_n(\delta) = n^{-1/2} \sum_{i=1}^{n} \psi_i(Y_i - \alpha^T V_t)(X_t - \delta) I(X_t \leq \delta) \]
\[ = n^{-1/2} \sum_{i=1}^{n} \psi_i(c_t + n^{-1/2} \beta_1(X_t - \delta) I(X_t > \delta)) \]
\[ - (\alpha - \alpha^T V_t)(X_t - \delta) I(X_t \leq \delta) \]
\[ = n^{-1/2} \sum_{i=1}^{n} \psi_i(c_t)(X_t - \delta) I(X_t \leq \delta) \]
\[ - H_1(\delta) H_2(\delta, \beta) + o_p(1) \]
\[ = R(\delta) + q(\delta, \beta) + o_p(1). \]

The weak convergence of \( R(\delta) \) can be obtained directly by following the proof of Stute (1997). This completes the proof of Theorem 3.1.

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**Supplementary Material**

The convergence of the BRISQ Algorithm, the Wild bootstrap algorithm for P-Values, the proof of Proposition 3.1 for the null asymptotic distribution of the smoothed rank score (SRS) test statistic and additional simulation results are included in a separate online supplemental file.

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