ON LOCAL LAWS FOR NON-HERMITIAN RANDOM MATRICES
AND THEIR PRODUCTS

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Abstract. The aim of this paper is to prove a local version of the circular law for non-Hermitian random matrices and its generalization to the product of non-Hermitian random matrices under weak moment conditions. More precisely we assume that the entries $X_{jk}^{(q)}$ of non-Hermitian random matrices $X^{(q)}$, $1 \leq j, k \leq n, q = 1, \ldots, m, m \geq 1$ are i.i.d. r.v. with $E X_{jk} = 0$, $E X_{jk}^2 = 1$ and $E |X_{jk}|^{4+\delta} < \infty$ for some $\delta > 0$. It is shown that the local law holds on the optimal scale $n^{-1+2a}, a > 0$, up to some logarithmic factor. We further develop a Stein type method to estimate the perturbation of the equations for the Stieltjes transform of the limiting distribution. We also generalize the recent results [8], [47] and [37]. An extension to the case of non-i.i.d. entries is discussed.

1. Introduction and main result

One of the main questions of the Random matrix theory (RMT) is to investigate the limiting behaviour of spectra of random matrices from different ensembles. In the current paper we shall study the case of products of non-Hermitian random matrices. More precisely, we consider a set of random non-Hermitian matrices

$$X^{(q)} = [X^{(q)}_{jk}]_{j,k=1}^n, q = 1, \ldots, m, \quad m \in \mathbb{N}.$$ 

Assume that $X^{(q)}_{jk}, 1 \leq j, k \leq n, q = 1, \ldots, m$, are independent random variables (r.v.) with zero mean. Note that the distribution of $X^{(q)}_{jk}$ may depend on $n$. Denote by $(\lambda_1(X), \ldots, \lambda_n(X))$ – the eigenvalues of the matrix

$$X \overset{\text{def}}{=} \frac{1}{n} \prod_{q=1}^m X^{(q)}.$$

For any set $B \in \mathcal{B}(\mathbb{C})$ we introduce the counting function of the eigenvalues in $B$:

$$N_B \overset{\text{def}}{=} N_B(X) \overset{\text{def}}{=} \#\{1 \leq k \leq n : \lambda_k(X) \in B\}.$$ 

It is also convenient to denote by $\mu_n(\cdot)$ – the empirical spectral distribution of $X$:

$$\mu_n(B) \overset{\text{def}}{=} \frac{1}{n} N_B, \quad B \in \mathcal{B}(\mathbb{C}).$$

We first assume that $m = 1$. Denote

$$p^{(1)}(z) \overset{\text{def}}{=} \frac{1}{\pi} \mathbb{1}[|z| \leq 1], \quad z \in \mathbb{C}, \quad (1.1)$$

and let $A(\cdot)$ be the Lebesgue measure on $\mathbb{C}$. By $\overset{\text{w}}{\rightarrow}$ we denote weak convergence of probability measures. We first assume that $m = 1$. Then the following result is the well-known **circular law**.

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Theorem 1.1 (Macroscopic circular law). Let \( X_{jk}, 1 \leq j, k \leq n \) be i.i.d. complex r.v. with \( \mathbb{E} X_{jk} = 0, \mathbb{E} |X_{jk}|^2 = 1 \). Then \( \mu_n \xrightarrow{w} \mu^{(1)} \) a.s. as \( n \) tends to infinity, where
\[
d\mu^{(1)}(z) = p^{(1)}(z)dA(z).
\]

The circular law was first proven by Ginibre [21] in 1965 in the case when \( X_{jk} \) are standard complex Gaussian r.v. His proof was based on the joint density of \( (\lambda_1(X), \ldots, \lambda_n(X)) \). If \( X_{jk} \) are complex (real) Gaussian r.v. we say that \( X \) belongs to complex (resp. real) Ginibre ensemble of random matrices. Here we also refer to the book of M. Mehta [35]. Later on the circular law was extended to more general classes of random entries by V. Girko [22]. Therefore the circular law is often referred to as the Girko–Ginibre circular law. It has been further extended in a number of papers, for instance in [5], [31], [32], [42], [45], [46]. In particular, F. Götze and A. Tikhomirov, see [32], established the circular law under the assumption that \( \max_{j,k} \mathbb{E} |X_{jk}|^2 \log^{1+\eta}(1 + |X_{jk}|) < \infty \) for any \( \eta > 0 \). They also generalized it to the case of sparse random matrices. That is, let us define \( X_{jk} \) i.i.d. Bernoulli r.v. with parameter \( p \). It follows from [32], that one may take \( p \geq c \log n/n \) for some \( c > 0 \). A result with optimal moment conditions, see Theorem 1.1, was established by T. Tao and V. Vu in [46]. The progress made in [32], [42], [45], [46] was based on bounds for the least singular value of shifted matrices \( X - zI, z \in \mathbb{C} \), due to M. Rudelson and R. Vershynin, see e.g. [44]. For a detailed account we refer the interested reader to the overview [7].

In applications the case of the non-homogeneous circular law is of considerable interest, which means dropping the assumption of identical distribution of entries, while still assuming that \( \mathbb{E} X_{jk}^{(1)} = \sigma_{jk}^2 \). In particular, the papers [32], [46] already deal with non i.i.d. entries but under the additional assumption that all \( \sigma_{jk}^2 = 1 \). An extended model one would require some appropriate conditions on the matrix \( \Sigma \). For example, one may assume that \( \Sigma \) is doubly-stochastic. See e.g. [1], [12], [4].

The circular law may be further generalized to the case of dependent r.v. The typical example here is the case of matrices from Girko’s elliptic ensemble. Here the pairs \( (X_{jk}, X_{kj}), 1 \leq j < k \leq n \), are i.i.d. random vectors, and \( \mathbb{E} X_{jk}X_{kj} = \rho \), for some \( \rho : |\rho| \leq 1 \). The global limiting distribution for spectra of elliptic random matrices is given by a uniform law in the ellipsoid with the semi-axes equal to 1 + \( \rho \) and 1 – \( \rho \) resp. We refer the interested reader to the papers to [23], [36], [38], [27]. In the case \( \rho = 1 \) we get Wigner’s semicircle law, [49]. In the case \( |\rho| < 0 \) and under additional assumption that \( X \) belongs to Ginibre’s ensemble we again arrive at the circular law. For other models of non-Hermitian random matrices with dependent entries, see for instance [1], [6], [2].

The main emphasis of the current paper is concerned with the generalization of the circular law to the case of arbitrary \( m \geq 1 \). We denote
\[
p^{(m)}(z) \overset{df}{=} \frac{1}{\pi m} |z|^{\frac{2m}{m+1} - 2} \mathbb{I} [|z| \leq 1]. \tag{1.2}
\]

It is straightforward to check that \( p^{(m)}(z) \) is the density of \( m \)-th power of a uniform distribution on the unit circle. We state the following theorem in the macroscopic scale.

Theorem 1.2 (Products of random matrices, macroscopic regime). Let \( m \in \mathbb{N} \) and \( X_{jk}^{(q)}, 1 \leq j, k \leq n, q = 1, \ldots, m \), be i.i.d. complex r.v. with \( \mathbb{E} X_{jk}^{(q)} = 0, \mathbb{E} |X_{jk}^{(q)}|^2 = 1 \). Then \( \mu_n \xrightarrow{w} \mu^{(m)} \) in probability as \( n \) tends to infinity, where
\[
d\mu^{(m)}(A) = p^{(m)}(z)dA(z).
\]

We refer here to the results of F. Götze and A. Tikhomirov [29] and S. O’Rourke and A. Soshnikov [41]. For product of Girko’s elliptic random matrices, see [39] and [26].

The circular law and its generalisation to the product of random matrices are valid, in particular, for all circles \( B(z_0, r) \) with centre at \( z_0 \) and finite radius \( r > 0 \) independent of \( n \). Such sets typically contain a macroscopically large number of eigenvalues, which means a number of order \( n \). In particular, the statement of Theorem 1.1–1.2 may be formulated as follows:
\[
\frac{1}{nr^2} \mathbb{E} N_{B(z_0, r)} = \frac{1}{r^2} \int_{B(z_0, r)} p^{(m)}(z)dA(z) + \frac{R_n}{r^2}, \tag{1.3}
\]
where
\[ \lim_{n \to \infty} R_n = 0. \] (1.4)
(similar statement may be formulated for \( N(B_{(2r_n)^2}) \)). Unfortunately for smaller radius, when \( r \) tends to zero as \( n \) goes to infinity, the number of eigenvalues cease to be macroscopically large. In this case it is essential to describe the second term in (1.3) more precisely, rather then (1.4). We say that the local law holds if the second term in (1.3) tends to zero as \( r = r(n) \) tends to zero. The series of the results in that direction was recently proved by P. Bourgade, H.-T. Yau and J. Yin \cite{BourgadeYauYin2018}, \cite{TaoYauVu2018} and L. Erdős and V. Vu \cite{ErdosVu2018} in the case of \( m = 1 \). They derived the local version of Theorem 1.1 up to the optimal scale \( n^{-1+2\alpha}, \alpha > 0 \). In \cite{TaoYauVu2018}, \cite{ErdosVu2018} the local circular law was proved under the assumption of sub-exponential tails for the distribution of entries (or assuming finite moments of all orders). In \cite{ErdosVu2018} it was proved under similar assumptions by means of the so-called fourth moment theorem, which requires that the first four moments of \( X_{jk} \) match the corresponding moments of the standard Gaussian distribution. We also refer to the recent results \cite{BourgadeYauYin2018} and \cite{TaoVu2018}. The general case of \( m \geq 1 \) was proved by Y. Nemish \cite{Nemish2018} who obtained a local version of Theorem 1.2 under sub-exponential assumptions. For a more detailed discussion of these result, see the next section after Theorem 2.3.

The aim of the current paper is to relax the above assumptions and prove local versions of Theorem 1.1 and Theorem 1.2 under weak moment condition. More precisely we assume that \( 4 + \delta \) moments are finite for some \( \delta > 0 \). See the following section 2 for precise statements. This work continues the previous results of authors \cite{ErdosSchleinYau2010}, \cite{TaoVu2010}, where the local semicircle law for Hermitian random matrices was proved under similar moment conditions.

We continue to use Stein type methods for the estimation of perturbations of the equation for Stieltjes transforms of the limiting distribution, since it turn out to be very flexible and useful. In this context we provide a general result, i.e. Lemma 7.2, which may be of independent interest. In particular, as a consequence of this lemma one may derive among others a Rosenthal type inequality for moments of linear forms (e.g. \cite{Rosenthal1970}[Theorem 3] and \cite{Stam1975}[Inequality (A)]), inequality for moments of quadratic forms (e.g. \cite{ErdosSchleinYau2010}[Proposition 2.4] or \cite{TaoVu2010}[Lemma A.1]) with precise values of all constants involved. We also jointly apply the additive descent method introduced by L. Erdős, B. Schlein, H.-T. Yau and et al., see \cite{TaoVu2010}, \cite{ErdosSchleinYau2010}, \cite{ErdosSchleinYau2011}, \cite{TaoYauVu2018}, \cite{TaoVu2018}, \cite{TaoYauVu2018}, \cite{TaoVu2018} among others, together with multiplicative descent methods introduced in \cite{ErdosSchleinYau2010} and further developed in \cite{ErdosSchleinYau2011}, \cite{TaoVu2010}. See Lemma 5.2 for details.

We finish this section discussing some related results. In particular, we have already mentioned the local semicircle law. Significant progress in studying the local semicircular law was made in a series of papers by L. Erdős, B. Schlein, H.-T. Yau and et al., \cite{TaoVu2010}, \cite{ErdosSchleinYau2010}, \cite{ErdosSchleinYau2011}, \cite{TaoYauVu2018}, \cite{TaoVu2018}, \cite{TaoYauVu2018}, \cite{TaoVu2018}. We also refer to the more recent results \cite{ErdosSchleinYau2010}, \cite{ErdosSchleinYau2011}, \cite{TaoVu2010}. An extension to the elliptic random matrix ensembles, which generalizes both ensembles considered above would be of interest. This applies as well to local versions of the elliptic law and its extension to products of such matrices. See \cite{ErdosSchleinYau2011} and \cite{TaoVu2010} for the limiting behaviour in the macroscopic regime. In particular, it would be interesting to study the so-called weak non-Hermiticity limit, i.e. the case \( \rho \) tends to one, see \cite{TaoVu2010} and recent result \cite{ErdosSchleinYau2010}.

1.1. Notations. Throughout the paper we will use the following notations. We assume that all random variables are defined on a common probability space \((\Omega, F, \mathbb{P})\) writing \( \mathbb{E} \) for the mathematical expectation with respect to \( \mathbb{P} \). We denote by \( \mathbb{R} \) and \( \mathbb{C} \) the set of all real and complex numbers. We also introduce \( \mathbb{C}^+ \defeq \{ z \in \mathbb{C} : \Im z \geq 0 \} \).

1. We denote by \( \mathbb{1}[A] \) the indicator function of the set \( A \).
2. By \( C \) and \( c \) we denote some positive constants. If we write that \( C \) depends on \( \delta \) we mean that \( C = C(\delta, \mu_{4+\delta}) \).
3. For an arbitrary square matrix \( A \) taking values in \( \mathbb{C}^{n \times n} \) (or \( \mathbb{R}^{n \times n} \)) we define the operator norm by \( \| A \| \defeq \sup_{x \in \mathbb{R}^n, \| x \| = 1} \| Ax \|_2 \), where \( \| x \|_2 \defeq (\sum_{j=1}^n |x_j|^2)^{1/2} \). We use the Hilbert-Schmidt (Frobenius) norm given \( \| A \|_F \defeq \| A_1 \|_2^{1/2} A A^* = (\sum_{j,k=1}^n |A_{jk}|^2)^{1/2} \).
4. For a vector \( x = (x_1, \ldots, x_n)^T \) we denote \( |x| \defeq \max_{1 \leq k \leq n} |x_k| \).
5. For an arbitrary function from \( L_1(\mathbb{C}) \)-space we denote \( \| f \|_{L_1} \defeq \int_{\mathbb{C}} |f(z)| \, dz \).
For an arbitrary function $f$ we denote $\|f\| \overset{\text{df}}{=} \sup_{z \in \mathbb{C}} |f(z)|$.

(7) Define the Laplace operator in two dimensions as $\Delta \overset{\text{df}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

(8) We write $f \sim g$ if there exist positive constants $c_1, c_2$, such that $c_1|g| \leq |f| \leq c_2|g|$. We shall often write $f \lesssim g$ which mean that there exists positive constant $c$ such that $|f| \leq c|g|$.

2. Main result

Without loss of generality we will assume in what follows that $X^{(q)}$ are real non-symmetric matrices. Our results proven below apply to the case of complex matrices as well. Here we may additionally assume for simplicity that $\text{Re} X^{(q)}$ and $\text{Im} X^{(q)}$ are independent r.v. for all $1 \leq j, k \leq n, q = 1, \ldots, m$. Otherwise one needs to extend the moment inequalities for linear and quadratic forms in complex r.v. (see [24][Theorem A.1-A.2]) to the case of dependent real and imaginary parts, the details of which we omit.

We will often refer to the following conditions.

**Definition 2.1** (Conditions (C0)). We say that conditions (C0) hold if:

- $X^{(q)}_{jk}, 1 \leq j, k \leq n, q = 1, \ldots, m$, are independent real random variables;
- $\mathbb{E} X^{(q)}_{jk} = a^{(q)}_{jk}, \mathbb{E} |X^{(q)}_{jk}| = [\sigma^{(q)}_{jk}]$;
- $\max_{j,k,q,n} \mathbb{E} |X^{(q)}_{jk} + \delta|^{1+\delta} \overset{\text{df}}{=} m \delta < \infty$ for some $\delta > 0$;
- $|a^{(q)}_{jk}| \leq \tau^{-\epsilon}$, $1 - [\sigma^{(q)}_{jk}]^2 \leq \tau^{-\epsilon}$ for some $\epsilon > 0$.

**Definition 2.2** (Conditions (C1)). We say that conditions (C1) hold if:

- (C0) hold;
- There exists $\phi = \phi(\delta) > 0$ such that $|X^{(q)}_{jk}| \leq Dn^{1/2-\phi}$ for all $1 \leq j, k \leq n$ and some $D > 0$.

Here one may take $0 < \phi \leq \delta/(2(4 + \delta))$.

Let $f(z)$ be a smooth non-negative function with compact support, such that $\|f\| \leq C, \|f'\| \leq nC$ for some constant $C$ independent of $n$. Following [8], we define for any $a \in (0, 1/2)$ and $z_0 \in \mathbb{C}$ the function $f_{z_0}(z) \overset{\text{df}}{=} n^{2a} f((z - z_0)n^a)$ ($f_{z_0}$ is a smoothed delta-function at the point $z_0$). The main result of the current paper is the following theorem which provides a local version of Theorem 1.1 and Theorem 1.2 under weak moment conditions (C1).

**Theorem 2.3** (Local regime). Assume that the conditions (C1) hold. Let $z_0 : |z_0| - 1 \geq \tau > 0$. Then for any $Q > 0$ there exists $c > 0$ such that with probability at least $1 - n^{-Q}$

$$\left| \frac{1}{n} \sum_{j=1}^{n} f_{z_0}(\lambda_j) - \int f_{z_0}(z) d\mu^{(n)}(z) \right| \leq \frac{q(n)}{n^{1-2\alpha}} \|\Delta f\|_{L^1}, \tag{2.1}$$

where $q(n) \leq c \log^5 n$.

An immediate corollary of the main theorem is the following statement.

**Corollary 2.4.** Assume that the conditions (C0) hold. Then the inequality (2.1) holds with probability at least $1 - n^{-c(\delta)}$, where $c(\delta)$ is some positive constant.

**Proof of Corollary 2.4.** Let $\tilde{X}^{(q)}$ be $X^{(q)}$ with $X^{(q)}_{jk}$ replaced by $X^{(q)}_{jk} 1 \{|X^{(q)}_{jk}| \leq Dn^{1/2-\phi'}\}$, where $0 < \phi' < \phi$. Applying Markov’s inequality we obtain

$$\mathbb{P}(X^{(q)} \neq \tilde{X}^{(q)}) \leq \sum_{j,k=1}^{n} \mathbb{P}(|X^{(q)}_{jk}| \geq Dn^{1/2-\phi'}) \leq n^{-c(\delta)}.$$ 

This inequality implies the statement of Corollary 2.4. \qed
Remark. It still remains one challenging open problem, namely extending the bounds to weaken the moment condition to $\delta = 0$. Furthermore, it is not unlikely that the power of the logarithmic factor in the upper bound for $q(n)$ may be reduced.

It seems that the bound $n^{-c(\delta)}$ of Corollary 2.4 cannot be improved in general. The main difficulty here is to estimate the least singular value, see (4.4). The required bound should be faster than any polynomial. The proof of such bound is based on the result [44] and requires to control the largest singular value, see (4.3). Unfortunately, this requires to assume high finite moments of matrix entries. Another way is to assume that the matrix entries have absolutely continuous and bounded densities, see [4].

It is possible to consider the case when $z$ is near the edge of the unit circle and extend the results [9], [51], but this topic leaves the scope of the current paper.

We finish this section comparing our result with [8] in the case $m = 1$ and [37] for $m > 1$. In these papers the authors assume instead of condition (3) in (C0) that the uniform sub-exponential decay condition is satisfied:

$$\exists \theta > 0 : \max_{1 \leq q \leq m} \max_{1 \leq j, k \leq n} \mathbb{P}(|X_{jk}^{(q)}| \geq t) \leq \theta^{-1} e^{-t^\theta}.$$  

They also extended the latter to the case of finite moments of all orders. Another difference is in the upper bound for $q(n)$ in (2.1). It was proved that $q(n) \leq n^\varepsilon$ for any small $\varepsilon > 0$. In the case $m = 1$ in the paper [4] conditions (1) and (2) were replaced by assumption that $X_{jk}^{(1)}$ may be non-i.i.d. and $c_1 \leq \mathbb{E}|X_{jk}^{(1)}|^2 \leq c_2$ for some $c_1, c_2 > 0$, but one needs to assume that $\mathbb{E}|X_{jk}^{(q)}| \leq \mu_l < \infty$ for all $l \in \mathbb{N}$ and $X_{jk}^{(q)}$ have bounded density.

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4. Proof of Theorem 2.3

4.1. Linearization. We linearise the problem considering the following block matrix (see e.g. [10]):

$$W \overset{def}{=} \frac{1}{\sqrt{n}} \begin{bmatrix} O & X^{(1)} & O & \ldots & O \\ O & O & X^{(2)} & \ldots & O \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ X^{(m)} & O & O & \ldots & O \end{bmatrix}.$$  

It is straightforward to check that the eigenvalues of $W_m$ are $\lambda_1(X), \ldots, \lambda_n(X)$ with multiplicity $m$. Hence, the following identity holds:

$$\frac{1}{n} \sum_{j=1}^{n} f_{z_0}(\lambda_j(X)) - \int f_{z_0}(z) \mu^{(m)}(dz) = \frac{1}{nm} \sum_{j=1}^{nm} f_{z_0}(\lambda_j^{(m)}(W)) - \int f_{z_0}(z^m) d\mu^{(1)}(z) = \frac{1}{nm} \sum_{j=1}^{nm} \tilde{f}(\lambda_j(W)) - \int \tilde{f}(z) d\mu^{(1)}(z),$$  

(4.1)

where $\tilde{f}(z) \overset{def}{=} f_{z_0}(z^m)$.

Let us consider a r.v. $\zeta$ uniformly distributed in the unit circle and independent of all other r.v. Then for any $r > 0$ the eigenvalues of $W - r \zeta I$ are

$$\lambda_j(W) - r \zeta.$$  

We denote the counting measure of $\lambda_j(W) - r\zeta$, $j = 1, \ldots, nm$, by $\mu_n^{(r)}$. It follows that $\mu_n^{(0)} = \mu_n$. Since $\|f\| \leq n^C$ we get the following bound

$$
\frac{1}{n} \sum_{j=1}^{n} f_{\lambda_j}(X) - \int f_{\lambda_j}(z) \mu^{(m)}(dz) = \frac{1}{nm} \sum_{j=1}^{nm} \tilde{f}(\lambda_j(W) - r\zeta) - \int \tilde{f}(z) d\mu^{(1)}(z) + R_n(r),
$$

where $|R_n(r)| \leq rn^C$. Choosing $r$ small enough the term $R_n(r)$ will be negligible. In what follows we assume that $r \overset{\text{def}}{=} n^{-c\log n}$.

Together with the eigenvalues of $W - r\zeta I$ we will be also interested as well in the singular values of shifted matrices $W(z, r) \overset{\text{def}}{=} W - r\zeta I - zI$, $z \in \mathbb{C}$. Let $s_j(z, r) \overset{\text{def}}{=} s_j(W(z, r)), j = 1, \ldots, nm$, be the singular values of $W(z, r)$ arranged in the non-increasing order, i.e.

$$
s_1(z, r) \geq s_2(z, r) \geq \ldots \geq s_{nm}(z, r).
$$

We shall consider as well the following matrix

$$
V(z, r) \overset{\text{def}}{=} \begin{bmatrix} 0 & W(z, r) \\ W^*(z, r) & 0 \end{bmatrix}.
$$

It is easy to check that $\pm s_j(z, r), j = 1, \ldots, nm$, are the eigenvalues of $V(z, r)$. Introduce the empirical spectral distribution (ESD) of $V(z, r)$:

$$
F_n(z, x, r) \overset{\text{def}}{=} \frac{1}{2nm} \sum_{j=1}^{nm} \mathbb{I}[s_j(z, r) \leq x] + \frac{1}{2nm} \sum_{j=1}^{nm} \mathbb{I}[-s_j(z, r) \leq x].
$$

4.2. The logarithmic potential approach. A common tool to deal with non-Hermitian random matrices is the logarithmic potential, which is defined as follows. Let $\nu$ be an arbitrary (probability) measure on $\mathbb{C}$. Then the logarithmic potential of $\nu$ is given by

$$
U_\nu(z) \overset{\text{def}}{=} - \int_{\mathbb{C}} \log |z - w| d\nu(w).
$$

For any $f \in C^2_0(\mathbb{C})$ we have

$$
\int f(z) dU_\nu(z) = \frac{1}{2\pi} \int \Delta f(z) U_\nu(z) dA(z).
$$

Applying (4.2), we obtain

$$
\frac{1}{n} \sum_{j=1}^{n} f_{\lambda_j}(X) - \int f_{\lambda_j}(z) d\mu^{(m)}(z) = \frac{1}{2\pi} \int \Delta \tilde{f}(z) [U_n^{(r)}(z) - U_n^{(1)}(z)] dA(z) + R_n(r),
$$

where $U_n^{(r)}, U_n^{(1)}$ are the logarithmic potentials of $\mu_n^{(r)}, \mu_n^{(1)}$ respectively.

We observe that $\tilde{f}(z) = 0$ for all $z \in \mathcal{M} \overset{\text{def}}{=} \{z : |z^n - z_0| < Cn^{-a}\}$. For any $z \in \mathcal{M}$ we introduce the following event

$$
\Omega_n \overset{\text{def}}{=} \Omega_n(z) \overset{\text{def}}{=} \{\omega \in \Omega : s_{nm}(z, r) \geq n^{-c\log n}, \|W\| \leq K\}
$$

for some large $K$. It follows from [44] (see also [29][Lemma 5.1], [40][Theorem 31]) and [48] (see also [25][Lemma A.1]) that

$$
\mathbb{P}(\Omega_n) \leq n^{-c\log n}.
$$
We rewrite $U_n^{(r)}(z)$ as follows

$$U_n^{(r)} = -\frac{1}{nm} \sum_{j=1}^{nm} \log |\lambda_j(W) - r\zeta - z| \mathbb{I}[\Omega_n] - \frac{1}{nm} \sum_{j=1}^{nm} \log |\lambda_j(W) - r\zeta - z| \mathbb{I}[\Omega_n^c].$$

$$\equiv \mathcal{U}_n^{(r)}(z) + \mathcal{U}_n^{(r)}(z).$$

Let us investigate the difference $\mathcal{U}_n^{(r)}(z) - U_{\mu(1)}(z)$. Following Girko [22] we use his hermitization trick and rewrite $U_n^{(r)}$ as the logarithmic moment of $F_n(z, x, r)$:

$$U_n^{(r)}(z) = -\frac{1}{nm} \log |\det W(z, r)| = -\frac{1}{2nm} \log |\det \mathbf{V}(z, r)| = -\int_{-\infty}^{\infty} \log |x| dF_n(z, x, r).$$

Moreover, it was proved in [32] that there exists a d.f. $G(z, x)$ such that

$$U_{\mu(1)} = -\int_{-\infty}^{\infty} \log |x| dG(z, x).$$

These equations imply

$$|\mathcal{U}_n^{(r)}(z) - U_{\mu(1)}(z)| \leq I_1 + I_2 + I_3,$$

where

$$I_1 \equiv \left| \int_{|x| \leq N^{-c \log n}} \log |x| dG(z, x) \right|, \quad I_2 \equiv \left| \int_{N^{-c \log n} \leq |x| \leq K} \log |x| d(F_n(z, x, r) - G(z, x, r)) \right|,$n

$$I_3 \equiv \left| \int_{|x| \geq K} \log |x| dG(z, x) \right|.$$

We recall some properties of the limiting distribution and introduce additional notations. Let us denote $\alpha \equiv \sqrt{1 + 8|z|^2}$. Define

$$w_{1,2}^2 \equiv \frac{(\alpha \pm 3)^3}{8(\alpha \pm 1)^3},$$

and $\lambda_+ \equiv |w_1|, \lambda_- \equiv |w_2|$. Moreover, let

$$\mathcal{J}(z) \equiv \begin{cases} x \in \mathbb{R} : x \in [-\lambda_+, -\lambda_-] \cup [\lambda_-, \lambda_+], & \text{if } |z| > 1, \\ x \in \mathbb{R} : x \in [-\lambda_+, \lambda_+], & \text{if } |z| < 1. \end{cases}$$

(4.6)

It is known (e.g. [32]) that $\mathcal{J}(z)$ is the support of $G(z, x)$. Moreover, $G(z, x)$ has an absolutely continuous symmetric density $\gamma(z, x)$, which is bounded and at the endpoints $\pm \lambda_\pm$ of the support $\mathcal{J}(z)$ it behaves as follows $g(z, x) \sim \sqrt{\gamma(x)}$, where

$$\gamma(u) \equiv \begin{cases} \min(||u| - \lambda_+, ||u| - \lambda_-|), & \text{if } |z| > 1, \\ ||u| - \lambda_+|, & \text{if } |z| < 1. \end{cases}$$

(4.6)

Returning to $I_1$ and $I_3$ we may conclude that

$$I_1 \leq n^{-1} \quad \text{and} \quad I_3 = 0.$$ 

(4.7)

Let us consider the second term $I_2$. Applying integration by parts we obtain

$$I_2 \leq \Delta_n^*(z, r) \log^2 n,$$

(4.8)

where $\Delta_n^*(z, r) \equiv \sup_{x \in \mathbb{R}} |F_n(z, x, r) - G(z, x)|$. It is easy to check that

$$\Delta_n^*(z, r) \leq \Delta_n^*(z, 0) + Cr.$$ 

We proceed by application of the smoothing inequality of Corollary B.3. Let us denote $\mathcal{J}_\varepsilon \equiv \{ x \in \mathbb{R} : \gamma(x) \geq \varepsilon \}$ and introduce the following region in $\mathbb{C}_+$:

$$\mathcal{D}(z) \equiv \{ w = u + iv \in \mathbb{C}_+ : u \in \mathcal{J}_{\varepsilon/2}(z), v_0/\sqrt{\gamma(u)} \leq v \leq V_0 \},$$

(4.9)
where
\[ v_0 \overset{\text{def}}{=} A_0 n^{-1} \log^2 n \] (4.10)
and \( V \geq 1, A_0 > 0 \) are some constants defined later in section 6. Denote the Stieltjes transform of \( F_n(z, x) \overset{\text{def}}{=} F_n(z, x, 0) \) by \( m_n(z, w) \). It is known that \( m_n(z, w) \) converges a.s. to the Stieltjes transform \( s(z, w) \), which is a solution of the following cubic equation
\[ \Delta(z, w) = -\frac{w + s(z, w)}{(w+s(z, w))^2 - |z|^2}, \] (4.11)
see, for instance, [32]. Moreover, \( s(z, w) \) is the Stieltjes transform of the d.f. \( G(z, x) \). For detailed properties of \( s(z, w) \) we refer to [8][Lemma 4.1, 4.2]. Let us denote
\[ \Lambda_n(z, u + iv) \overset{\text{def}}{=} m_n(z, u + iv) - s(z, u + iv), \]
We may conclude from Theorem 5.1 below that there exists \( C > 0 \) such that
\[ \mathbb{P} \left( \bigcap_{z \in M} \bigcap_{w \in D} \left\{ |\Lambda_n(z, w)| \leq \frac{C \log^2 n}{nw} \right\} \right) \geq 1 - n^{-Q}. \] (4.12)
Applying the smoothing inequality, Corollary B.3, to \( \Delta_n \), we get the following bound
\[ \Delta_n^*(z, 0) \leq C_1 \int_{-\infty}^{\infty} |\Lambda_n(z, u + iv)| \, du + C_2 \sup_{x \in \mathbb{R}/2} \left| \int_{\mathbb{R}/2} \Lambda_n(z, x + iv) \, dv \right| + C_3 v + C_4 \varepsilon^2, \] (4.13)
\( v' = v_0/\sqrt{\varepsilon(x)} \). The proof of this inequality repeats the proof of its analogue in the case of the semi-circular law (see [30][Corollary 2.3]). For the readers convenience we include the arguments in the appendix. Let us take in this inequality \( \varepsilon \overset{\text{def}}{=} (2v_0a)^{2/3} \). Then \( C_3v_0 + C_4\varepsilon^{3/2} \leq Cn^{-1} \log^2 n. \) It follows from (4.5), (4.7), (4.8) and (4.13) that
\[ \int |\Delta \tilde{f}(z)| \overline{U_n}(z) - U(z) | \, dA(z) \leq C_1 \log^2 n \int |\Delta \tilde{f}(z)| \int_{-\infty}^{\infty} |\Lambda_n(z, u + iv)| \, du \, dA(z) \]
\[ + C_2 \log^2 n \int |\Delta \tilde{f}(z)| \sup_{x \in \mathbb{R}/2} \left[ \int_{\mathbb{R}/2} \Lambda_n(z, x + iv) \, dv \right] \, dA(z) \]
\[ + C_3 \log^3 n \log n. \] (4.14)
Inequality (4.12) implies that with probability at least \( 1 - n^{-Q} \)
\[ \sup_{z \in M} \sup_{x \in \mathbb{R}/2} \left| \int_{\mathbb{R}/2} \Lambda_n(z, x + iv) \, dv \right| \lesssim n^{-1} \log^3 n. \]
Hence,
\[ \int |\Delta \tilde{f}(z)| \sup_{x \in \mathbb{R}/2} \left[ \int_{\mathbb{R}/2} \Lambda_n(z, x + iv) \, dv \right] \, dA(z) \lesssim \|\Delta \tilde{f}\|_{L_1} n^{-1} \log^3 n \] (4.15)
with probability at least \( 1 - n^{-Q} \). We conclude from Lemma 5.4 that
\[ \mathbb{E}^n |\Lambda_n(z, u + iv)|^p \leq C \frac{Cp |s(z, u + iv)|^{p + 1}}{n}, \]
which holds for all \( w = u + iv, u \in \mathbb{R} \). Hence,
\[ \mathbb{E}^n \left[ \int_{-\infty}^{\infty} |\Lambda_n(z, u + iv)| \, du \right]^p \leq \int_{-\infty}^{\infty} \mathbb{E}^n |\Lambda_n(z, u + iv)|^p \, du \]
\[ \leq \frac{Cp}{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du dG(z, x) \leq n^{-1} \log^2 n. \]
It is straightforward to check that
\[
\mathbb{E}\left[ \int |\Delta \tilde{f}(z)| \int_{-\infty}^{\infty} |A_n(z, u + iV)| \, du \, dA(z) \right]^{p} \leq \| \Delta \tilde{f} \|_{L^p}^{p} \sup_{z \in \mathcal{M}} \mathbb{E} \left[ \int_{-\infty}^{\infty} |A_n(z, u + iV)| \, du \right]^{p} 
\leq \| \Delta \tilde{f} \|_{L^p}^{p} n^{-p} \log^{2p} n.
\]
Markov’s inequality implies that with probability at least \(1 - n^{-Q}\)
\[
\int |\Delta \tilde{f}(z)| \int_{-\infty}^{\infty} |A_n(z, u + iV)| \, du \, dA(z) \lesssim \| \Delta \tilde{f} \|_{L^1} n^{-1} \log^{2} n.
\]
Combining now (4.14), (4.15) and (4.16) we conclude that with probability at least \(1 - n^{-Q}\)
\[
\int |\Delta \tilde{f}(z)||\tilde{U}^{(r)}_n(z) - U^{(r)}_{\mu}(z)| \, dA(z) \lesssim \| \Delta \tilde{f} \|_{L^1} n^{-1} \log^{5} n.
\]
It remains to estimate
\[
\int |\Delta \tilde{f}(z)||\tilde{U}^{(r)}_n(z)| \, dA(z).
\]
Let us consider \(\tilde{U}^{(r)}_n(z)\). We get
\[
\mathbb{E} |\tilde{U}^{(r)}_n(z)|^{p} \leq \frac{1}{nm} \sum_{j=1}^{nm} \mathbb{E} \left[ \log^{2p} |\lambda_j - r\zeta - z| \right] P^{\hat{f}}(\Omega^j_n).
\]
We fix \(j = 1, \ldots, nm\) and write
\[
\frac{1}{2\pi} \int_{|\zeta| \leq 1} \log^{2p} |\lambda_j - r\zeta - z| \, d\zeta \leq J_1 + J_2 + J_3,
\]
where
\[
J_1 = \frac{1}{2\pi} \int_{|\zeta| \leq 1, |\lambda_j - r\zeta - z| \leq \varepsilon} \log^{2p} |\lambda_j - r\zeta - z| \, d\zeta,
\]
\[
J_2 = \frac{1}{2\pi} \int_{|\zeta| \leq 1, \varepsilon < |\lambda_j - r\zeta - z| \leq 1/\varepsilon} \log^{2p} |\lambda_j - r\zeta - z| \, d\zeta,
\]
\[
J_3 = \frac{1}{2\pi} \int_{|\zeta| \leq 1, |\lambda_j - r\zeta - z| \geq 1/\varepsilon} \log^{2p} |\lambda_j - r\zeta - z| \, d\zeta.
\]
It is easy to see that
\[
J_2 \leq \log^{p}(1/\varepsilon).
\]
To estimate \(J_1\) we first note that for any \(b > 0\), the function \(-u^b \log u\) is not decreasing on the interval \(0 < u < e^{-1/b}\). Hence, for any \(0 < u \leq \varepsilon < e^{-1/b}\) we obtain
\[
-u^b \log u \geq \varepsilon u^b \log(1/\varepsilon).
\]
We take \(b\) such that \(bp = 1\). Then
\[
J_1 \leq \frac{1}{2\pi r^2} \varepsilon^{-bp} \log^{p}(1/\varepsilon) \int_{|\zeta| \leq \varepsilon} |\zeta|^{-bp} \, d\zeta \leq \log^{p}(1/\varepsilon) \varepsilon^2 r^{-2}.
\]
Choosing \(\varepsilon = r\) we arrive at the inequality
\[
J_1 \leq \log^{p}(1/\varepsilon).
\]
It remains to estimate \(J_3\). It is straightforward to check that \(\log^{2p} u \leq \varepsilon^2 u^2 \log^{2p} \varepsilon\) for \(x \geq 1/\varepsilon\) and \(p\) of order \(\log n\) (we recall that \(\varepsilon = n^{-1/\log n}\)). Hence,
\[
\frac{1}{nm} \sum_{j=1}^{nm} \mathbb{E} J_3 \leq mn r^2 (2 + |z|^2) \log^{2p} \varepsilon.
\]
These bounds together imply that for $p$ of order $\log n$

$$\sup_{z \in M} \mathbb{E} |\tilde{U}_n^{(r)}(z)|^p \leq n^{-c \log n}.$$  

Repeating the same arguments as in the proof of (4.16) we conclude the estimate

$$\int |\Delta \tilde{f}(z)||\tilde{U}_n^{(r)}(z)|\,dA(z) \leq \frac{\|\Delta \tilde{f}\|_{L_1}}{n}, \tag{4.18}$$

which holds with probability at least $1 - n^{-Q}$. Combining (4.17) and (4.18) we come to the following bound

$$\left| \frac{1}{2\pi} \int \Delta \tilde{f}(z)[U_n^{(r)}(z) - U_{\mu(1)}(z)]\,dA(z) \right| \leq \frac{g(n)\|\Delta \tilde{f}\|_{L_1}}{n^{1-2\alpha}},$$

which holds with probability at least $1 - n^{-Q}$. The last inequality implies the claim of Theorem 2.3.

5. Local law for shifted matrices

The following theorem provides the estimate for $\Lambda_n(z, w)$ up to the optimal scale $v_0$ (see definition (4.10)). The proof of this result will be given later on in section.

**Theorem 5.1** (Local law for eigenvalues of $V(z)$). Assume that (C0) hold. Let $Q > 0$ be an arbitrary number. There exists $C > 0$ such that

$$\mathbb{P} \left( \bigcap_{z \in M} \bigcap_{w \in D} \left\{ |\Lambda_n(z, w)| \leq \frac{C\log^2 n}{nv} \right\} \right) \geq 1 - n^{-Q}.$$

By standard truncation arguments (see [28][Lemmas D.1-D.3]) in what follows we may assume that conditions (C1) hold and $a_{jk}^{(q)} = 0$, for all $j, k = 1, \ldots, n, q = 1, \ldots, m$. For simplicity we will also assume that $X_{jk}^{(q)} = 1$, $j, k = 1, \ldots, n, q = 1, \ldots, m$ are i.i.d. r.v. In this case one may also show (see [28][Lemmas D.1-D.3]) that it is possible to assume that $|\sigma_{jk}^{(q)}|^2 = 1$ for all $j, k = 1, \ldots, n, q = 1, \ldots, m$. The proof in the non i.i.d. case in the same. One needs to add additional $\varepsilon_j$ term in (5.8) which will be small due to the assumption that $|1 - |\sigma_{jk}^{(q)}|^2| \leq n^{-1} - \varepsilon$.

5.1. Bound for the distance between Stieltjes transforms. We start from the general lemma, which is motivated by the additive descent approach introduced and further developed by L. Erdős, B. Schlein, H.-T. Yau and et al., see [17], [16], [18], [13], [14], [15], [34] among others. Recall that

$$\Lambda_n(z, w) \overset{\text{def}}{=} m_n(z, w) - s(z, w). \tag{5.1}$$

For $w = u + iv \in \mathbb{C}^+$ we define

$$R(z, w) \overset{\text{def}}{=} (V(z) - w\mathbf{1})^{-1}.$$  

It is easy to see that $m_n(z, w) = \frac{1}{2\pi} \Tr R(z, w)$. Denote $j_\alpha \overset{\text{def}}{=} (\alpha - 1)n + j$. Introduce the following partial traces of resolvent $m_n^{(\alpha)}(z, w) \overset{\text{def}}{=} \frac{1}{n} \sum_{j=1}^n R_{j, j_\alpha}$ and

$$\Lambda \overset{\text{def}}{=} (\Lambda_1^{(1)}, \ldots, \Lambda_m^{(2m)})^T, \quad \Lambda_n^{(\alpha)} = m_n^{(\alpha)}(z, w) - s(z, w).$$

It is easy to check that

$$\Lambda_n = \frac{1}{m} \sum_{\alpha=1}^m \Lambda_n^{(\alpha)} = \frac{1}{m} \sum_{\alpha=m}^{m} \Lambda_n^{(m+n)}.$$  

Moreover,

$$|\Lambda_n(z, w)| \leq |\Lambda(z, w)|.$$
Let \( w = u + iv \in \mathbb{D} \)

\[
I(v) \overset{\text{def}}{=} I_N(z, u + iv) \overset{\text{def}}{=} \prod_{k=0}^{K_\nu} \mathbb{I} \left[ |A_n(z, u + iv^k)| \leq \gamma \text{Im}(s(z, u + iv^k)) \right],
\]

(5.2)

where \( K_\nu = \min\{ l : vs^l \geq V \} \) and \( s \geq 1 \). The exact value of \( s \) will be defined later in section 6. Let \( C \) be a positive constant. We take \( \gamma, A_1 \) sufficiently small and \( A_0 \) sufficiently large such that

\[
\frac{Cp}{nv} \leq \gamma \text{Im}(s(z, w))
\]

(5.3)

for any \( w = u + iv \in \mathbb{D} \) and \( 1 \leq p \leq A_1 \log n \). The exact values of \( \gamma, A_0, A_1 \) will be defined later in section 6.

The next lemma is crucial for the proof of Theorem 5.1.

**Lemma 5.2.** Let \( w \in \mathbb{D} \) and \( \gamma \) be some fixed number. Assume that for all \( v \geq v_0/\sqrt{\gamma(u)} \) and \( 1 \leq p \leq A_1 \log n \)

\[
E[|A_n(z, u + iv)|^p I(v)] \leq \frac{C p^p}{(nv)^p},
\]

(5.4)

and

\[
P \left( |A_n(z, u + iv)| \geq \gamma \text{Im}(s(z, u + iv)) \right) \leq \frac{C}{nv^\alpha}.
\]

(5.5)

Then for any \( v_0/\sqrt{\gamma(u)} \leq v \leq V \)

\[
P \left( |A_n(z, u + iv)| \geq \gamma \text{Im}(s(z, u + iv)) \right) \leq \frac{C}{nv^\alpha}.
\]

(5.6)

**Proof.** Let \( \kappa = \kappa_n \) be such that

\[
|A_n(z, u + iv) - A_n(z, u + iv + \kappa)| \leq \frac{\gamma}{2} \text{Im}(s(z, u + iv)).
\]

(5.7)

It easy to check that one may take, for example, \( \kappa_n = n^{-3} \). Denote \( v' = v_0/\sqrt{\gamma(u)} \). We split \([v', V]\) into \( N = (V - v')/\kappa \) intervals and denote \( v_k = v' + k\kappa \). Assume that we have already proved (5.6) for all \( v_k \leq v \leq V \) and prove it for any \( v \) up to \( v_k-1 \). For example, for \( v = v_N = V \) it follows from (5.5).

We fix \( v : v_{k-1} \leq v \leq v_k \). Taking \( p = A_1 \log n \) and \( K : K^{-p} \leq Cn^{-Q} \) we get

\[
P \left( |A_n(z, u + iv_k)| \geq \frac{KCp^2}{nv_k} \right) \leq E \mathbb{I} \left[ |A_n(z, u + iv_k)| \geq \frac{KCp^2}{nv_k} \right] I(v_k)
\]

\[
+ \sum_{l=0}^{K_\nu} \mathbb{P} \left( |A_n(z, u + iv_k s^l)| \geq \gamma \text{Im}(s(z, u + iv_k s^l)) \right)
\]

\[
\leq \left( \frac{C K^2}{nv_k} \right)^{-p} E[|A_n(z, u + iv_k)|^p I(v_k)] + \frac{C}{nv^\alpha} \leq \frac{C}{nv^\alpha}.
\]

Here we also used (5.4). Since \( v_k \geq v \geq v' \) we get that \( \frac{KCp^2}{nv_k} \leq \frac{1}{2} \gamma \text{Im}(s(z, u + iv)) \). Hence, using (5.7) we obtain

\[
P \left( |A_n(z, u + iv)| \geq \gamma \text{Im}(s(z, u + iv)) \right) \leq \frac{C}{nv^\alpha}.
\]

\[\square\]

It follows from Lemma that we need to check conditions (5.4)–(5.5).
5.2. Stieltjes transform and self-consistent equations. In this section we investigate $m_n(z, w)$ and show that it satisfies a cubic equation (see (5.9) below), which is a perturbation of the corresponding equation (4.11) for $s(z, w)$.

Let $R_{j_0, j_0}$ (resp. $R_{j_0, j_m}$) be the resolvent matrix of $V(z)$ with $j_0$-th row and column deleted (resp. $j_0$- and $j_{m+\alpha}$-th row and column deleted). Applying Schur’s inverse formula, we may write, for all $j = 1, \ldots, n$ and $\alpha = 1, \ldots, m$, that

$$R_{j_0, j_0} = \frac{1}{1 - w - m_n^{(\alpha+1)+m}} \left(1 - \varepsilon_{j_0}, R_{j_0, j_0}\right),$$

(5.8)

where $m_n^{(\alpha)}(z, w) \overset{\text{def}}{=} \frac{1}{n} \sum_{j=1}^{n} R_{j_0, j_0}$ and

$$\varepsilon_{j_0} \overset{\text{def}}{=} \varepsilon_{j_0,1} + \ldots + \varepsilon_{j_0,4},$$

(5.11)

Here $\varepsilon_{j_0} \overset{\text{def}}{=} \varepsilon_{j_0,1} + \ldots + \varepsilon_{j_0,4}$, where

$$\varepsilon_{j_0,1} \overset{\text{def}}{=} \frac{1}{n} \sum_{k \in \mathbb{T}} R_{j_0, k_{[\alpha+1]+m}} - \frac{1}{n} \sum_{k \in \mathbb{T}} R_{j_0, k_{[\alpha]+m+k_{[\alpha+1]+m}^+}},$$

$$\varepsilon_{j_0,2} \overset{\text{def}}{=} -\frac{1}{n} \left(\sum_{j \neq k \in \mathbb{T}} X_j^{(\alpha)} X_k^{(\alpha)} R_{j_0, k_{[\alpha]+m+k_{[\alpha+1]+m}^+}}\right),$$

$$\varepsilon_{j_0,3} \overset{\text{def}}{=} -\frac{1}{n} \left(\sum_{l \in \mathbb{T}} \left(|X_l^{(\alpha)}|^2 - 1\right) R_{j_0, l_{[\alpha]+m+l_{[\alpha+1]+m}^+}}\right),$$

$$\varepsilon_{j_0,4} \overset{\text{def}}{=} \sqrt{n} \sum_{l \in \mathbb{T}} X_l^{(\alpha)} R_{j_0, l_{[\alpha]+m+l_{[\alpha+1]+m}^+}}.$$
Here, $a \equiv s^{-2}(z, w)$, $b \equiv \left(\frac{|z|^2}{|w + s(z, w)|}\right)x$. Substituting $s(z, w)$ from the both sides of equations (5.9)-(5.10) we come to the following linear system:

$$\mathbf{A}\mathbf{A}_n = \mathbf{r}_n + s^{-1}\mathbf{T}_n,$$

where

$$\|\mathbf{r}_n\| \leq |\mathbf{A}_n|^2 \left(\frac{|z|^2}{|w + s|^2} \sum_{\alpha=1}^{m} \frac{1}{|w + m_n^{(\alpha)}|^2}\right)^{\frac{1}{2}} \left(1 + \frac{|\mathbf{A}_n|}{|s|}\right) + \frac{|z|^2}{|w + s|^2} \left(1 + \frac{|z|^2}{|w + s|^2}\right).$$

Permuting the rows and columns of $\mathbf{A}$ we may come to the matrix from [37][Equation 5.18].

### 5.3. Validity of condition (5.4).

Define

$$\mathcal{A}(z, v, q) \equiv \max_{(j, k) \in J_n} \max_{\alpha_1, \ldots, \alpha_m} \mathbb{E}\left| \sum_{k=0}^{K} \mathbb{E}^{(K)} R_{j, k}^{(j, k)}(z, w) I(v) \right|,$$

$$\mathcal{E}(q) \equiv \mathbb{E} s(z, w) + \mathcal{A}(z, v, q) \left(1 + \sum_{k=0}^{K} \frac{1}{s^{kp}} \mathbb{E}^{(K)} R_{j, k}^{(j, k)}(z, w) I(v) \right).$$

See (6.7) for the definitions of $J, K, J_n, T_j$. The next lemma shows that the condition (5.4) holds.

**Lemma 5.3.** Assume that for all $w \in D$ and $1 \leq p \leq A_1 \log n$

$$\mathbb{E}|s(z, iv)|^p I(v) \lesssim \frac{C^p \rho^{2p} \mathcal{E}(s, w)}{(nv)^{p}}$$

and

$$\mathcal{A}(s, w) \leq C^p \mathcal{E}(s, w)$$

for some $k > 0$. Then

$$\mathbb{E}|\mathbf{A}_n(z, iv)|^p I(v) \lesssim \left(\frac{C^p \rho^p}{nv}\right)^p.$$

We apply Stein’s method and “leave one out” idea to estimate $\mathbf{T}_n$. Here we follow the ideas introduced in [30] and further developed in [24]. It is clear that the bound for $\mathbf{T}_n$ requires estimation of the moments of $\mathbf{R}_{j,j}$. We do it section 6, where we also introduce the general principle to estimate the moments of so-called $k$-descent function (see definition 6.1). Moreover, in this section we show that (5.17) holds.

**Proof of Lemma 5.3.** We may rewrite (5.12) as follows

$$\mathbf{A} = \mathbf{A}^{-1}\mathbf{r}_n + s^{-1}\mathbf{A}^{-1}\mathbf{T}_n.$$

It follows that

$$\|\mathbf{A}_n\| \leq \|\mathbf{A}^{-1}\|\|\mathbf{r}_n\| + |s|^{-1}\|\mathbf{A}^{-1}\|\|\mathbf{T}_n\|$$

We may write

$$|w + m_n^{(\alpha)}(z, w)|I(v) \geq (|w + s(z, w)| - |\mathbf{A}_n^{(\alpha)}|)I(v) \geq (1 - \gamma)|w + s(z, w)|I(v)$$

Moreover, using the definition of $I(v)$ we obtain

$$|\mathbf{A}_n|^2 I(v) \leq \gamma|\mathbf{A}_n| I(v).$$
Taking into account the last two inequalities and definition (5.13) of \( r \) we obtain
\[
\|r\|_1 \leq \gamma |\Im s(z, w)| A_n I(v).
\]
It follows from \[37\] Proposition 5.5 then \( \|A^{-1}\| \leq \Im^{-1} s(z, w) \). Taking expectation of the both sides of (5.18) and applying (5.16), (5.17) we get the claim of this lemma.

5.4. **Validity of condition (5.5).** This conditions in a consequence of the the next lemma.

**Lemma 5.4.** For any \( w = u + iV, u \in \mathbb{R}, V \geq 1 \) and all \( p \geq 1 \) the following bound holds
\[
\mathbb{E}_P |A_n(z, u + iV)|^p \leq C_p |s(z, u + iV)| \frac{1}{n^p}.
\]

**Proof.** The proof is similar to the proof of the analogous inequality in \([25]\)** Inequality 2.8] in the semi-circle law case.

5.5. **Proof of Theorem 5.1.**

**Proof of Theorem 5.1.** The proof is the direct corollary of Lemmas 5.2 and 5.3. Indeed, taking \( p = A_1 \log n \) we may write
\[
\mathbb{P} \left( |A_n(z, u + iv)| \geq \frac{K C p^2}{n v} \right) \leq \mathbb{E} 1 \left[ |A_n(z, u + iv)| \geq \frac{K C p^2}{n v} \right] I(v) + \sum_{k=0}^{V} \mathbb{P} \left( |A_n(z, u + iv s^k)| \geq \gamma |\Im s(z, u + iv s^k)| \right) \leq \left( \frac{C K p^2}{n v} \right)^{-p} \mathbb{E}[|A_n|^p I(v)] + C \frac{C}{n^q} \leq C \frac{C}{n^q}.
\]
This inequality implies the claim of the theorem.

6. **Bound for functions with \( k \)-descent property**

As it was already mentioned that the estimation of \( T_n \) requires to bound the high moments of \( R_{j,j}(z, u + iv) \) and \( \Im R_{j,j}(z, u + iv) \) for \( j = 1, \ldots, nm \) up to the optimal value \( v_0 \) of \( v \). Here we are going to apply multiplicative descent method introduced in \([11]\) and further developed in the series of papers \([24], [28]\) by the authors. This method requires the small number of steps, usually of the logarithmic order. One may compare with additive descent method, Lemma 5.2, where one needs to make polynomial number of steps.

6.1. **Class of descent function.** We start from rather general definition and proposition which are essential for multiplicative descent. Let us introduce the following class of functions.

**Definition 6.1.** Let \( k \geq 1 \). We say that a function \( f(w), w = u + iv \in \mathbb{C}^+ \), satisfies the \( k \)-descent property if for any \( v > 0 \)
\[
\left| \frac{\partial}{\partial u} \log f(u + iv) \right| \leq \frac{k}{v}
\]

Let us denote by \( D(k) \overset{\text{def}}{=} \{ f : \mathbb{C}^+ \to \mathbb{C} : f \text{ satisfies} k \text{-descent property} \} \). The following statement collects the main properties of \( k \)-descent functions.

**Proposition 6.2.** The following statements hold:

1. If \( f \in D(k) \) then \( f^{-1} \in D(k) \);
2. If \( f \in D(k), g \in D(l) \) then \( fg \in D(k + l) \).
3. For any \( f \in D(k) \) and for any \( s \geq 1 \)
\[
|f(u + iv/s)| \leq s^k |f(u + iv)| \quad \text{and} \quad |f(u + iv)| \leq s^k |f(u + iv/s)|
\]
Proof. The proof of (1) and (2) are trivial. To prove (3) it is enough to mention that
\[ \left| \log f(u + iv/s) - \log f(u + iv) \right| \leq k \log s. \]
\[ \square \]

It is easy to check that \( |R_{jj}(z, w)|, \Im R_{jj}(z, w) \) are examples of functions with 1-descent property w.r.t. \( w \).

6.2. Bound for moments of some functions of the resolvent matrix. Recall the definition of \( I_\gamma(w) \) for \( w = u + iv \in \mathcal{D} \)
\[
I(w) \overset{\text{def}}{=} I_\gamma(z, u + iv) \overset{\text{def}}{=} \prod_{k=0}^{K_v} \mathbb{I} \left[ |A_n(z, u + ivs^k)| \leq \gamma \Im s(z, u + ivs^k) \right],
\]
where \( K_v \overset{\text{def}}{=} \min\{l : vs^l \geq V\} \). Here \( V \geq 1 \) and \( s \geq 1 \) are some constants defined later. It is easy to see that
\[
I(u + iv) \leq I(u + is_0v). \tag{6.1}
\]
In what follows for simplicity we shall often omit \( w = u + iv \) from all notations and write only imaginary part \( v \).

Lemma 6.3. Let \( V_0 \) be some fixed number. There exist a positive constant \( C_0 \) depending on \( V_0 \), \( z \) and positive constants \( A_0, A_1, \gamma \) depending on \( C_0 \) such that
\[
\max_{j=1,\ldots,nm} \mathbb{E} |R_{jj}(z, u + iv)|^p I(u + iv) \leq C_0^p, \tag{6.2}
\]
\[
\max_{j=1,\ldots,nm} \mathbb{E} \Im^p R_{jj}(z, u + iv)I(u + iv) \leq C_0^p \Im^p s(z, u + iv). \tag{6.3}
\]
for all \( u + iv \in \mathcal{D} \) and \( 1 \leq p \leq A_1 \log n \).

Lemma 6.4. Let \( V_0 \) be some fixed number. There exist a positive constant \( H_0 \) depending on \( V_0 \) and positive constant \( \gamma \) depending on \( H_0 \) such that the following inequalities hold:
\[
\max \left( \frac{1}{|w + m^{(\alpha)}(z, w)|}, \frac{1}{|w + m^{(\alpha+1)}(z, w)|} \right) I(v) \leq H_0 \tag{6.4}
\]
for all \( w = u + iv \in \mathcal{D} \) and \( \alpha = 1, \ldots, m \).

Remark. The statement of Lemma remains valid if one replaces \( m^{(\alpha)}(z, w) \) by \( m^{(\alpha, l, K)}(z, w) \).

Proof of Lemma. Assume that \( I(v) = 1 \). In the opposite case the claim is trivial. Then
\[
\frac{1}{|w + m^{(\alpha)}(z, w)|} \leq \frac{1}{|w + s(z, w)|} + \frac{|A_n^{(\alpha)}|}{|w + s(z, w)||w + m^{(\alpha)}(z, w)|} \leq c + \frac{\gamma c}{|w + m^{(\alpha)}(z, w)|},
\]
here \( c \) depends on \( z \) and \( V \). The last inequality implies
\[
\frac{1}{|w + m^{(\alpha)}(z, w)|} \leq \frac{c}{1 - \gamma c}.
\]
If we take, say, \( H_0 \geq 2c \), then we may find sufficiently small \( \gamma \), such that the r.h.s. of the previous inequality is bounded by \( H_0 \). Similarly we may prove.
\[ \square \]

Proof of Lemma 6.3. The proof of is more involved then the proof of the previous lemma. The general idea how to prove these results follows the idea of [11] about multiplicative descent approach developed in [24]. We briefly discuss these ideas on the bound (6.2) for \( \mathbb{E} |R_{jj}(z, v)|^p I(v) \) (the same will be true for (6.3)).

We prove below in Lemma 6.5 that the bound
\[
\max_{j \in T} \mathbb{E} |R_{jj}(v)|^p I(v) \leq C_0^p \tag{6.5}
\]
holds for all \( w \in \mathcal{D} \) and \( 1 \leq p \leq A_1(nv)^{(1-2\alpha)/2} \). We are interested in \( p \) of the order \( \log n \). Denote \( v_1 \overset{\text{def}}{=} n^{-1}\log^{2/(1-2\alpha)}n \) and take \( p = A_1 \log n \) (It is sufficient to consider only such values of \( p \). For all \( 1 \leq q \leq p \) we may apply the Lyapunov inequality). It is easy to see that (6.5) holds for all \( v : v_1 \leq v \leq V \) with \( p = A_1 \log n \). Let us fix \( v : v_0 \leq v \leq v_1 \) and let \( l_0 \overset{\text{def}}{=} \min\{l \geq 1 : v_0^l \geq v_1 \} \). We take \( s \overset{\text{def}}{=} l_0^{-1} \). It is clear that \( s \leq \log^{1/2-\alpha}n \). Applying Proposition 6.2, (6.5) and (6.1) we may show that for all \( v \geq v_0 \)

\[
\max_{j \in \mathbb{T}} \mathbb{E} |R_{jj}(v)|^p I(v) \leq C_0^n \log \left( \frac{4\pi}{\Re\gamma} \right) p n.
\]

It remains to remove the log factor on the right hand side of the previous inequality. To this aim we shall adopt the moment matching technique which has been successfully used recently by Lee and in Yin in [34](see Lemma 5.2 and Lemma 5.3). We denote by \( Y_{jk} \), \( 1 \leq j \leq k \leq n \) a triangular set of random variables such that \( |Y_{jk}| \leq D \), for some \( D \) chosen later, and \( E X_{jk} = E Y_{jk} \) for \( s = 1, \ldots, 4 \).

It follows from [34][Lemma 5.2] that such a set of random variables exists. Let us denote \( W^Y := \frac{1}{\sqrt{n}} Y, R^Y := (W^Y - zI)^{-1} \) and \( m_n^Y(z) := \frac{1}{n} \text{Tr} R^Y(z) \). Then, repeating the proof of [28][Lemma 3.5] we show that for all \( v \geq v_0 \) and \( 5 \leq p \leq A_1 \log n \) there exist positive constants \( C_1, C_2 \) such that

\[
\mathbb{E} |R_{jj}(v)|^p I(v) \leq C_0^p + C_2 \mathbb{E} |R_{jj}^y(v)|^p I(v).
\]

It is easy to see that \( Y_{jk} \) are sub-Gaussian random variables. Repeating the proof of Lemma 6.5 below for sub-Gaussian random variables one may show that

\[
\mathbb{E} |R_{jj}^y(v)|^p I(v) \leq C_0^p
\]

for all \( w \in \mathbb{D} \) and \( 1 \leq p \leq A_1 n v \). Here one needs to replace Lemmas A.1–A.4 by the Hanson-Wright inequality (see, for example, [28][Lemma A.4–A.7]). For details see the proof of the corresponding result in [28][Lemma 4.1]).

**Lemma 6.5.** Let \( V_0 \) be some fixed number. There exist a positive constant \( C_0 \) depending on \( V_0, z \) and positive constants \( A_0, A_1 \) depending on \( C_0 \) such that

\[
\max_{j, k = 1, \ldots, n} \mathbb{E} |R_{jk}(z, u + iv)|^p I(u + iv) \leq C_0^p
\]

for all \( A_0 n^{-1} \leq v \leq V_0, u \in \mathbb{J}_e \) and \( 1 \leq p \leq A_1(nv)^{1-2\alpha} \).

**Remark.** In Lemma 6.5 we bound the off-diagonal entries as well. We use the bound for off diagonal entries to show that (6.6) holds. See [28][Lemma 3.5] for details.

Let us define \( J_{\alpha} : \alpha = 1, \ldots, m \) as an arbitrary subsets of \( T \). Here \( J_{\alpha} \) will correspond to the indices of rows deleted from \( X(\alpha) \). Similarly we define \( K_{\alpha} : \alpha = 1, \ldots, m \) as the indices of columns deleted from \( X(\alpha) \). Moreover, let \( |J_{\alpha} \cap K_{\alpha}| = 0 \) or \( 1 \). For a particular choice of these sets we define

\[
\mathbb{J} \overset{\text{def}}{=} \{ J_{\alpha} \subset T, \alpha = 1, \ldots, m \}, \quad \mathbb{K} \overset{\text{def}}{=} \{ K_{\alpha} \subset T, \alpha = 1, \ldots, m \}.
\]

Handling now all possible \( J_{\alpha}, K_{\alpha} : \alpha = 1, \ldots, m \) we define \( J_L \) as follows

\[
J_L \overset{\text{def}}{=} \{ \mathbb{J}, \mathbb{K} : |\mathbb{J}| \leq L, |\mathbb{J} \setminus \mathbb{K}| \leq 1 \}, \quad L \geq 0.
\]

We also define \( T_{J_{\alpha}} \overset{\text{def}}{=} (\alpha - 1) m + T \setminus J_{\alpha} \). Similarly we may define \( T_{K_{\alpha}} \). Let \( X(\alpha, J_{\alpha}, K_{\alpha}) \) be a sub-matrix of \( X(\alpha) \) with entries \( X_{jk}(\alpha), j \in T_{J_{\alpha}}, k \in T_{K_{\alpha}} \). Then we may define \( W^{J,K} \) as \( W \) with all \( X^0 \) replaced by \( X^{\alpha, J_{\alpha}, K_{\alpha}} \). Similarly we define \( V^{J,K}, R^{J,K} \) and all other quantities. Denote

\[
I_{J,K}^{(\gamma \otimes)}(v) \overset{\text{def}}{=} \prod_{k=1}^{K_v} \left[ |A_{J,K}^{(\gamma \otimes)}(u + i s_0^k v) | \leq \gamma \text{Im} s(z, u + is_0^k v) \right].
\]

It is easy to see that

\[
I_{J,K}^{(\gamma \otimes)}(v) \leq I_{J,K}^{(\gamma \otimes)}(s_0 v)
\]

(6.8)
for any $\gamma_1: \gamma_1 \geq \gamma$.

**Lemma 6.6.** Assume that the conditions (C1) hold. Let $C_0$ and $s_0$ be arbitrary numbers such that $C_0 \geq \max(1/\gamma, H_0) s_0 \geq 2^{1/\gamma}$. There exist sufficiently large $A_0$ and small $A_1$ depending on $C_0, s_0, \gamma$ only such that the following statement holds. Fix some $\tilde{v}: v_0 s_0 / \sqrt{\gamma(u)} \leq \tilde{v} \leq V$. Suppose that for some integer $K > 0$, all $u, v, q$ such that $\tilde{v}/s_0 \leq v \leq V, u \in \mathbb{J}_x, 1 \leq q \leq A_1 nunv$.

\[
\min_{u, v, q} \max_{(u,v) \in \mathbb{J}_x} \max_{\mathbb{J}_x} \mathbb{E} \left| R^{(j,K)}_{\tilde{u}, \tilde{v}} (v') \right|^2 \frac{1}{\gamma(v')} (v) \leq C_0^2 \tag{6.9}
\]

Then for all $u, v, q$ such that $\tilde{v}/s_0 \leq v \leq V, u \in \mathbb{J}_x, 1 \leq q \leq A_1 nunv$.

**Proof.** Let $\alpha = 1, \ldots, m$. Without loss of generality we assume that $\mathbb{J} = \mathbb{K}$. We fix some $j \in \mathbb{T} \setminus J_\alpha$ and denote $\mathbb{J} = \{ J_\beta \neq \alpha, J_\beta \cup \{j\} \}$.

Similarly we define $\tilde{\mathbb{K}}$. We first consider the diagonal entries. Applying Schur’s inverse formula, we may write

\[
R^{(j,K)}_{j_\alpha, j_\alpha} = -\frac{1}{w - m_{\alpha}^{-1} + m_{\alpha} j_{\alpha K}} (1 - \tilde{\mathbb{E}}_{j_\alpha} R^{(j,K)}_{j_\alpha, j_\alpha}),
\]

where

\[
\tilde{\mathbb{E}}_{j_\alpha} = \frac{1}{w + m_{\alpha}^{-1}} \left( \mathbb{E} \left[ R^{(j,K)}_{j_\alpha, j_\alpha} \right] + \mathbb{E} \left[ R^{(j,K)}_{j_\alpha, j_\alpha} \right] \right) - \frac{1}{w + m_{\alpha}^{-1}} \left( \mathbb{E} \left[ R^{(j,K)}_{j_\alpha, j_\alpha} \right] \right).
\]

Here $\tilde{\mathbb{E}}_{j_\alpha} = \tilde{\mathbb{E}}_{j_\alpha}^{(1)} + \ldots + \tilde{\mathbb{E}}_{j_\alpha}^{(m)}$, where

\[
\tilde{\mathbb{E}}_{j_\alpha}^{(1)} = \frac{1}{w + \mathbb{T} \setminus K_{\alpha + 1}} \left( \mathbb{E} \left[ R^{(j,K)}_{j_\alpha, j_\alpha} \right] + \mathbb{E} \left[ R^{(j,K)}_{j_\alpha, j_\alpha} \right] \right) - \frac{1}{w + \mathbb{T} \setminus K_{\alpha + 1}} \left( \mathbb{E} \left[ R^{(j,K)}_{j_\alpha, j_\alpha} \right] \right).
\]

and $\tilde{\mathbb{E}}_{j_\alpha}^{(m)} = \tilde{\mathbb{E}}_{j_\alpha}^{(1)} + \ldots + \tilde{\mathbb{E}}_{j_\alpha}^{(m)}$, where

\[
\tilde{\mathbb{E}}_{j_\alpha}^{(1)} = \frac{1}{w + \mathbb{T} \setminus K_{\alpha + 1}} \left( \mathbb{E} \left[ R^{(j,K)}_{j_\alpha, j_\alpha} \right] + \mathbb{E} \left[ R^{(j,K)}_{j_\alpha, j_\alpha} \right] \right) - \frac{1}{w + \mathbb{T} \setminus K_{\alpha + 1}} \left( \mathbb{E} \left[ R^{(j,K)}_{j_\alpha, j_\alpha} \right] \right).
\]

We conclude from (5.8) and Lemma 6.4 that there exist a positive constant $H_0$ depending on $u_0, V_0, z$ and positive constant $A$ depending on $H_0$ such that the following inequality holds:

\[
\left| R^{(j,K)}_{j_\alpha, j_\alpha} (v) \right| I^{-1}_v (v) \leq H_0 \left( 1 + \tilde{\mathbb{E}}_{j_\alpha} R^{(j,K)}_{j_\alpha, j_\alpha} I^{-1}_v (v) \right).
\]
Hence,  
\[
\mathbb{E} \left[ R_{j_0, j_0}^{(J,K)}(v) \right] \leq 2^q H_0^q \left( 1 + \mathbb{E}^\frac{1}{2} \left[ R_{j_0, j_0}^{(J,K)}(v) \right] \right) \mathbb{E}^\frac{1}{2} \left[ R_{j_0, j_0}^{(J,K)}(v) \right].
\]

It follows from Proposition 6.2, (6.8) and (6.9) that  
\[
\mathbb{E}^\frac{1}{2} \left[ R_{j_0, j_0}^{(J,K)}(v) \right] \leq s_0^q C_0^q.
\]

The Cauchy-Schwartz inequality and Lemma imply  
\[
\mathbb{E} \left[ |c_{j,j}^{(J,K)}|^2 \right] \leq 2^{2q} \mathbb{E} \left[ |c_{j,j}^{(J,K)}|^2 \right] + 2^{2q} H_0^q |z|^4 \mathbb{E}^\frac{1}{2} \left[ |c_{j,j}^{(J,K)}|^4 \right] \mathbb{E}^\frac{1}{2} \left[ R_{j_0, j_0}^{(J,K)} \right].
\]

Similarly to (6.10),  
\[
\mathbb{E}^\frac{1}{2} \left[ |c_{j,j}^{(J,K)}|^2 \right] \leq \mathbb{E} \left[ |c_{j,j}^{(J,K)}|^2 \right] \leq (C_0 s_0)^{2q}. \tag{6.11}
\]

Applying (6.8) we obtain  
\[
\mathbb{E} \left[ |c_{j,j}^{(J,K)}|^2 \right] \leq \mathbb{E} \left[ |c_{j,j}^{(J,K)}|^2 \right] \leq (C_0 s_0)^{2q}. \tag{6.11}
\]

It is easy to see from Lemmas A.1–A.4 in the appendix that the moment bounds for \(\tilde{c}_{j,j}^{(J,K)}\) and \(\tilde{c}_{j,j}^{(J,K)}\) depend on the moments of off-diagonal entries of resolvent which are non \(k\)-descent function. Here we may use  
\[
R(w_1) - R(w_2) = (w_1 - w_2)R(w_1)R(w_2), \quad w_1, w_2 \in \mathbb{C}^+, \tag{6.12}
\]

which gives us that  
\[
|R_{j_0, j_0}^{(J,K)}(z, v)| \leq |R_{j_0, j_0}^{(J,K)}(z, sv)| + s|R_{j_0, j_0}^{(J,K)}(z, sv)| \leq |R_{j_0, j_0}^{(J,K)}(z, v)|^\frac{1}{2}.
\]

Now the desired bound follows from Proposition 6.2 and assumption (6.9). Lemmas A.1–A.4 in the appendix imply  
\[
\mathbb{E} \left[ |c_{j,j}^{(J,K)}|^2 \right] \leq \left( \frac{CC_0 s_0^2 q}{(nv)^{1-\frac{1}{2}}} \right)^{2q}.
\]

Here, \(C\) depends on \(z\) as well. Similarly, applying Lemmas A.5–A.7 from the appendix we may estimate  
\[
\mathbb{E}^\frac{1}{2} \left[ |c_{j,j}^{(J,K)}|^4 \right] \leq \left( \frac{CC_0 s_0^3 q}{(nv)^{1-\frac{1}{2}}} \right)^{2q}.
\]

The last two inequalities yield the following bound:  
\[
\mathbb{E} \left[ |c_{j,j}^{(J,K)}|^2 \right] \leq \left( \frac{CC_0 s_0^3 q}{(nv)^{1-\frac{1}{2}}} \right)^{2q}.
\]

Hence, choosing sufficiently large \(A_0\) and small \(A_1\) we may show that  
\[
\mathbb{E} \left[ |R_{j_0, j_0}^{(J,K)}(v)|^q \right] \leq C_0^q.
\]

To deal with off-diagonal entries we use the following representation  
\[
R_{j_0, k_\beta}^{(J,K)} = -\frac{1}{\sqrt{n}} \sum_{l \in T \setminus K_{[n+1]}} X_{jl} (\alpha) R_{l_0, l_0}^{(J,K)} R_{j_0, j_0}^{(J,K)} - \frac{z}{\sqrt{n}} \sum_{l \in T \setminus K_{[n-1]}} X_{jl} (\alpha-1) R_{l_0, l_0}^{(J,K)} R_{j_0, j_0}^{(J,K)}.
\]

Applying now Rosenthal’s inequality (e.g. [43][Theorem 3] and [33][Inequality (A)]) and assumption (6.9) we may show that one may choose sufficiently large \(A_0\) and small \(A_1\) such that  
\[
\mathbb{E} \left[ |R_{j_0, k_\beta}^{(J,K)}(v)|^q \right] \leq C_0^q.
\]
Proof of Lemma 6.3. Let us choose some sufficiently large constant $C_0 > \max(1/V, H_0)$ and fix $s_0 \equiv 2C_0$. Here $H_0$ is defined in Lemma 6.4. We also choose $A_0$ and $A_1$ as in Lemma 6.6. We fix $u \in \mathbb{J}_e$. Let $L \equiv \log_{s_0} \left( V \sqrt{\gamma(u)/v_0} \right) + 1$. Since $\|R^{(j)}(V)\| \leq V^{-1}$ we may write

$$\max_{(J,K) \in J_L} \max_{\alpha, \beta = 1, \ldots, m} \max_{I_{\alpha, \beta} \in I_{\alpha, \beta} \in I_{\alpha, \beta}} \mathbb{E} \left[ R^{(j)}_{I_{\alpha, \beta}}(V)^q I_{(L+1)}^{(j)} \right] (V) \leq C_0^q$$

for all $1 \leq p \leq A_1(nV)^{1-2\alpha}$. Fix arbitrary $v : V/s_0 \leq v \leq V$ and $p : 1 \leq p \leq A_1(nv)^{1-2\alpha}$. Lemma 6.6 yields that

$$\max_{l, k = 1, \ldots, m} \max_{(J,K) \in J_{L-1}} \max_{\alpha, \beta = 1, \ldots, m} \max_{I_{\alpha, \beta} \in I_{\alpha, \beta} \in I_{\alpha, \beta}} \mathbb{E} \left[ R^{(j)}_{I_{\alpha, \beta}}(v)^q I_{(L)}^{(j)} \right] (v) \leq C_0^q$$

for $1 \leq l \leq A_1(nV/s_0)^{1-2\alpha}$, $v \geq V/s_0$. We may repeat this procedure $L$ times and finally obtain

$$\max_{l, k = 1, \ldots, m} \mathbb{E} \left[ R_{I_{\alpha, \beta}}(v)^p I_{(L)}(v) \right] \leq C_0^p$$

for $1 \leq l \leq A_1(nV/s_0)^{1-2\alpha} \leq A_1(nv_0)$ and $v \geq v_0/V\sqrt{\gamma(u)}$. □

7. Estimation of $T_n$

In this section we prove the following theorem.

**Theorem 7.1.** For any $w \in D$ and all $1 \leq p \leq A_1 \log n$

$$\max_{1 \leq \alpha \leq m} \mathbb{E} \left| T_n^{(\alpha)} \right|^p \leq \frac{C_p p^2 p \mathbb{E}(KP)}{(nv)^p},$$

where $\mathbb{E}(q)$ is defined in (5.15) and $\kappa$ is a positive constant depending on $\delta$ only.

We shall proceed as in [24] applying Stein’s method.

7.1. Framework for moment bounds of some statistics of r.v. We start from the following lemma, which provide a framework to estimate the moments of some statistics of independent random variables.

Let $X_1, \ldots, X_n$ be independent r.v. and denote

$$\mathfrak{M} \equiv \sigma \{ X_1, \ldots, X_n \}, \quad \mathfrak{M}^{(j)} \equiv \sigma \{ X_1, \ldots, X_{j-1}, X_j, X_{j+1}, \ldots, X_n \}.$$  

For simplicity we introduce $E_j(\cdot) \equiv \mathbb{E}(\cdot | \mathfrak{M}^{(j)})$. Assume that $\xi_j, f_j, j = 1, \ldots, n$, are $\mathfrak{M}$-measurable r.v. and

$$E_j(\xi_j) = 0. \quad (7.1)$$

We consider the following statistic:

$$T_n^* \equiv \sum_{j=1}^n \xi_j f_j + \mathcal{R},$$

where $\mathcal{R}$ is some $\mathfrak{M}$ measurable function. Moreover, let $\tilde{f}_j$ an arbitrary $\mathfrak{M}^{(j)}$-measurable r.v. and

$$\tilde{T}_n^{(j)} \equiv E_j(T_n^*).$$

**Lemma 7.2.** For all $p \geq 2$ there exist some absolute constant $C$ such that

$$\mathbb{E} \left| T_n^* \right|^p \leq C p \left( \mathcal{A}^p + p^2 B^p + p^p C + p^p D + \mathbb{E} |\mathcal{R}|^p \right),$$

where $\mathcal{A}, B, C, D$ are some positive constants. □
where

\[ A \overset{\text{def}}{=} \mathbb{E}^\sharp \left( \sum_{j=1}^n \mathbb{E}_j [\xi_j(f_j - \hat{f}_j)] \right)^p, \]

\[ B \overset{\text{def}}{=} \mathbb{E}^\sharp \left( \sum_{j=1}^n \mathbb{E}_j (|\xi_j(T_n^* - \tilde{T}_n^{(j)})|)|\hat{f}_j| \right)^{\frac{p}{2}}, \]

\[ C \overset{\text{def}}{=} \sum_{j=1}^n \mathbb{E} |\xi_j||T_n^* - \tilde{T}_n^{(j)}|^{p-1}|\hat{f}_j|, \]

\[ D \overset{\text{def}}{=} \sum_{j=1}^n \mathbb{E} |\xi_j||f - \hat{f}_j||T_n^* - \tilde{T}_n^{(j)}|^{p-1}. \]

\textbf{Remark.} We conclude the statement of the last lemma by several remarks.

1. It follows from the definition of \( A, B, C, D \) that instead of estimation of high moments of \( \xi_j \) one needs to estimate conditional expectation \( \mathbb{E}_j |\xi_j|^\alpha \) for some small \( \alpha \). Typically, \( \alpha \leq 4; \)
2. Moreover, to get the desired bounds one needs to choose an appropriate approximation \( \hat{f}_j \) of \( f_j \) and estimate \( T_n^* - \tilde{T}_n^{(j)}; \)
3. This lemma may be generalized as follows. We may assume that

\[ T_n^* \overset{\text{def}}{=} \sum_{\nu=1}^m \sum_{j=1}^n \xi_{j\nu} f_{j\nu} + \mathcal{R}, \]  \hspace{1cm} (7.2)

where \( \xi_{j\nu}, f_{j\nu}, j = 1, \ldots, n, \nu = 1, \ldots, m, \) are \( \mathfrak{M} \)-measurable r.v. such that

\[ \mathbb{E}_j (\xi_{j\nu}) = 0, \quad \nu = 1, \ldots, m. \]

Repeating the previous calculations we obtain

\[ \mathbb{E} |T_n^*|^p \leq C^p \left( \sum_{\nu=1}^m (A_\nu^p + p^{\frac{p}{2}} B_\nu^p + p^p C_\nu) + \mathbb{E} |\mathcal{R}|^p \right), \]

where \( A_\nu, B_\nu, C_\nu \) are defined similarly to the corresponding quantities in Lemma 7.2.

\textbf{Proof of Lemma 7.2.} Let us introduce the following function:

\[ \varphi(\zeta) \overset{\text{def}}{=} \zeta |\zeta|^{p-2}. \]  \hspace{1cm} (7.3)

In these notations \( \mathbb{E} |T_n^*|^p \) may be rewritten as follows

\[ \mathbb{E} |T_n^*|^p = \mathbb{E} T_n^* \varphi(T_n^*) = \sum_{j=1}^n \xi_j f_j \varphi(T_n^*) + \mathcal{R} \varphi(T_n^*) = \sum_{l=1}^4 A_l, \]

where

\[ A_1 \overset{\text{def}}{=} \sum_{j=1}^n \mathbb{E} \xi_j \hat{f}_j \varphi(\tilde{T}_n^{(j)}), \quad A_2 \overset{\text{def}}{=} \sum_{j=1}^n \mathbb{E} \xi_j \hat{f}_j (\varphi(T_n^*) - \varphi(\tilde{T}_n^{(j)})), \]

\[ A_3 \overset{\text{def}}{=} \sum_{j=1}^n \mathbb{E} \xi_j (f_j - \hat{f}_j) \varphi(T_n^*), \quad A_4 \overset{\text{def}}{=} \mathcal{R} \varphi(T_n^*). \]

It follows from (7.1) that \( A_1 = 0 \). Applying the following useful inequality

\[ (x + y)^q \leq ex^q + (q + 1)^q y^q, x, y > 0, \quad q \geq 1, \]  \hspace{1cm} (7.4)
we estimate $A_4$ by the sums of the following terms

$$
A_{31} \overset{\text{def}}{=} e \sum_{j=1}^{n} \mathbb{E} |\xi_j| |f_j - \hat{f}_j| |\tilde{T}^{(j)}_n|^{p-1},
$$

$$
A_{32} \overset{\text{def}}{=} p^{p-1} \sum_{j=1}^{n} \mathbb{E} |\xi_j| |f_j - T^{*}_n - \tilde{T}^{(j)}_n|^{p-1}.
$$

The term $A_{32}$ we remain unchanged. It will appear in the final bound. H"older’, Jensen’ and Young’s inequalities imply

$$
A_{31} \leq \mathbb{E} \sum_{j=1}^{n} \mathbb{E} \left|\xi_j \right| \left|f_j - \hat{f}_j\right| \left|\tilde{T}^{(j)}_n\right|^{p-1},
$$

$$
A_{32} \leq \mathbb{P} \sum_{j=1}^{n} \mathbb{E} \left|\xi_j \right| \left|f_j - T^{*}_n - \tilde{T}^{(j)}_n\right| |\tilde{T}^{(j)}_n|^{p-1}.
$$

It follows from the Taylor formula that

$$
A_{2} = \sum_{j=1}^{n} \mathbb{E} \xi_j \hat{f}_j (T^{*}_n - \tilde{T}^{(j)}_n) \varphi' (\tilde{T}^{(j)}_n) + \theta (T^{*}_n - \tilde{T}^{(j)}_n)),
$$

where $\theta$ is a uniformly distributed on $[0, 1]$ r.v., independent of $X_j, j = 1, \ldots, n$. Taking absolute values and using (7.4) we get

$$
|A_2| \leq A_{21} + A_{22},
$$

where

$$
A_{21} \overset{\text{def}}{=} e p \sum_{j=1}^{n} \mathbb{E} \mathbb{E}_{j} (|\xi_j| |T^{*}_n - \tilde{T}^{(j)}_n|) |\hat{f}_j| |\tilde{T}^{(j)}_n|^{p-2},
$$

$$
A_{22} \overset{\text{def}}{=} p^{p-1} \sum_{j=1}^{n} \mathbb{E} \mathbb{E}_{j} (|\xi_j| |T^{*}_n - \tilde{T}^{(j)}_n|^{p-1}) |\hat{f}_j|.
$$

Applying Hölder’s and Jensen’s inequalities we obtain

$$
A_{21} \leq C p \mathbb{E} \mathbb{E}^{\frac{1}{p}} \mathbb{E}^{\frac{1}{p-2}} \left( \sum_{j=1}^{n} \mathbb{E} (|\xi_j| |T^{*}_n - \tilde{T}^{(j)}_n|) |\hat{f}_j| |\tilde{T}^{(j)}_n|^{p-2} \right)^{\frac{1}{p}} |T^{*}_n|^{p}.
$$

Now Young’s inequality implies

$$
A_{21} \leq C p \mathbb{E} \mathbb{E}^{\frac{1}{p}} \mathbb{E}^{\frac{1}{p-2}} \left( \sum_{j=1}^{n} \mathbb{E} (|\xi_j| |T^{*}_n - \tilde{T}^{(j)}_n|) |\hat{f}_j| \right)^{\frac{1}{p-2}} + \mathbb{E} |T^{*}_n|^{p}. \tag{7.6}
$$

Finally, for the term $A_4$ we may write

$$
A_4 \leq C p \mathbb{E} |R|^p + \mathbb{E} |T^{*}_n|^p. \tag{7.7}
$$

Inequalities (7.5), (7.6) and (7.7) yield the claim of the lemma. □
7.2. Proof of Theorem 7.1.

Proof of Theorem 7.1. We consider the case $\alpha = 1$ only. For simplicity we shall write $T_n = T_n^{(1)}$. First we mention that $T_n$ is of the kind (7.2). Indeed, here

$$\xi_j = \tilde{\xi}_j + \tilde{\xi}_2 + \ldots + \tilde{\xi}_u,$$

$$f_j = R_{jj}, \quad f_2 = -\frac{|z|^2 R_{jj} R_{j(\alpha)j}^2}{w + \gamma m_n(z, w)},$$

$$R = \frac{1}{n} \sum_{j=1}^n \tilde{\xi}_j f_j + \frac{1}{n} \sum_{j=1}^n \tilde{\xi}_j f_2.$$ We introduce the following smoothed version of $I(v)$. We denote

$$h_{\alpha, \beta}(x) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } 0 < x < \text{Im} s(z, w), \\ 1 - \frac{x - \alpha \text{Im} s(z, w)}{(\beta - \alpha) \text{Im} s(z, w)}, & \text{if } \alpha \text{ Im} s(z, w) < x < \beta \text{ Im} s(z, w), \\ 0, & \text{otherwise}, \end{cases}$$

and write $H(v) \overset{\text{def}}{=} \prod_{k=1}^{K_v} \prod_{\alpha=1}^{m_n} h_{\gamma_k, \tilde{\gamma}_k}(|A\alpha(v)|)$. It is easy to see that

$$I(v) \leq H(v) \leq I_{3/2, \gamma}(v) \leq I_{1/2}^{(1)}(v). \quad (7.8)$$

To simplify all notations below we shall often omit the bottom index from $I_\gamma(v)$ and all its counterparts. We will also write $I(v) \leq I_1(v)$ having in mind that $I_\gamma(v) \leq I_\gamma^{(1)}(v)$ for some fixed $\gamma' > \gamma$. Applying the notation of $H(v)$ we write

$$E|T_n|^p I(v) \leq E|T_n|^p H^p(v).$$

For simplicity we set $T_{n, h} \overset{\text{def}}{=} T_n H(v)$. Recall the definition (7.3) of $\varphi(\zeta)$. We rewrite the r.h.s. of the previous inequality as follows

$$E|T_{n, h}|^p = \frac{1}{n} \sum_{j=1}^n \sum_{\nu=1}^2 E\xi_{j\nu} f_{j\nu} H(v)\varphi(T_{n, h}) + E RH(v)\varphi(T_{n, h}).$$

We denote

$$A \overset{\text{def}}{=} \frac{1}{n} \sum_{j=1}^n \sum_{\nu=1}^2 E\xi_{j\nu} f_{j\nu} H(v)\varphi(T_{n, h}).$$

In these notations $E|T_{n, h}|^p$ may be rewritten as follows

$$E|T_{n, h}|^p = A + E RH(v)\varphi(T_{n, h}).$$

7.3. Estimate of $E RH(v)\varphi(T_{n, h})$. Simple calculations imply

$$R = \frac{1}{n^2} \sum_{j=1}^n \sum_{l=1}^n |R_{jl}|^2 - \frac{|z|^2}{n^2} \sum_{j=1}^n \sum_{l=1}^n |R_{j(\alpha)j}^2| R_{j(\alpha)j}^2,$$

$$- \frac{|z|^2}{n^2} \sum_{j=1}^n \sum_{l=1}^n |R_{j(\alpha)j}^2| R_{j(\alpha)j}^2.$$

Applying (7.8), Lemma 6.4 and Lemma A.12 in the appendix we conclude that

$$|R|H(v) \leq \frac{\text{Im} s(z, w)}{n^2} + \sum_{j=1}^n \text{Im} R_{jj}\varphi(T_{n, h}) |I(v)| + \text{Im} R_{jj}\varphi(T_{n, h}) |I(v)|.$$
Using now Hölder’s inequality and Young’s inequality we come to the following inequality
\[
\mathbb{E} \mathcal{R}H(v)\varphi(T_{n,h}) \leq C^p \mathbb{E} |\mathcal{R}|^p H(v)^p + \rho \mathbb{E} |T_{n,h}|^p \\
\leq C^p (1 + |z|)^{2p} \mathcal{A}^p(p) + \rho \mathbb{E} |T_{n,h}|^p.
\]

7.4. Estimate of \( A \). The estimation of \( A \) is more involved. Let us introduce conditional expectations \( \mathbb{E}_{j}(\cdot) \overset{\text{def}}{=} \mathbb{E}(\cdot|\mathfrak{M}^{(j)}) \) (resp. \( \mathbb{E}_{j,l}(\cdot) \overset{\text{def}}{=} \mathbb{E}(\cdot|\mathfrak{M}^{(j,l)}) \)) with respect to \( \sigma \)-algebras \( \mathfrak{M}^{(j)} \) (resp. \( \mathfrak{M}^{(j,l)} \)). Here \( \mathfrak{M}^{(j)} \) (resp. \( \mathfrak{M}^{(j,l)} \)) is formed from all \( X_{lh}^{(\alpha)}, l, k = 1, \ldots, n, \alpha = 1, \ldots, m, \) except \( X_{jk}^{(1)}, k = 1, \ldots, n \) (resp. except \( X_{jk}^{(1)}, X_{lj}^{(1)}, k, l = 1, \ldots, n \)). Moreover, we introduce the following notations
\[
\begin{aligned}
\tilde{T}_{n,h}^{(j)} &\overset{\text{def}}{=} \mathbb{E}(T_{n,h}^{(j)}) , & \tilde{T}_{n,h}^{(j,l)} &\overset{\text{def}}{=} \mathbb{E}(T_{n,h}^{(j,l)}), \\
\tilde{A}^{(j)}_{n} &\overset{\text{def}}{=} \mathbb{E}(A_{n}^{(j)}), & \tilde{A}^{(j,l)}_{n} &\overset{\text{def}}{=} \mathbb{E}(A_{n}^{(j,l)}).
\end{aligned}
\]

We rewrite \( A \) as follows \( A = A_{1} + \ldots + A_{5} \), where
\[
\begin{aligned}
A_{1} &\overset{\text{def}}{=} \frac{1}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} \xi_{j\nu} j_{j\nu} H^{(j)}(v) \varphi(\tilde{T}_{n,h}^{(j)}), \\
A_{2} &\overset{\text{def}}{=} \frac{1}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} \xi_{j\nu} j_{j\nu} [H(v) - \tilde{H}^{(j)}(v)] \varphi(\tilde{T}_{n,h}^{(j)}), \\
A_{3} &\overset{\text{def}}{=} \frac{1}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} \xi_{j\nu} j_{j\nu} H(v) \varphi(T_{n,h}) - \varphi(\tilde{T}_{n,h}^{(j)}), \\
A_{4} &\overset{\text{def}}{=} \frac{1}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} \xi_{j\nu} j_{j\nu} H(v) \varphi(T_{n,h}).
\end{aligned}
\]

Moreover, it is easy to check that \( A_{1} = 0 \).

7.4.1. Bound for \( A_{2} \). Taking conditional expectation and applying Hölder’s inequality it is straightforward to check that
\[
A_{2} \leq \mathbb{E} \left\{ \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}_{j}\left( |\xi_{j\nu} j_{j\nu}| H(v) - \tilde{H}^{(j)}(v) \right) \right\}^p.
\]

Moreover, Young’s inequality implies that
\[
A_{2} \leq \rho \mathbb{E} |T_{n,h}|^p + C^p \mathbb{E} \left( \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}_{j}\left( |\xi_{j\nu} j_{j\nu}| H(v) - \tilde{H}^{(j)}(v) \right) \right)^p. \tag{7.9}
\]

Let us denote for simplicity
\[
B_{2} \overset{\text{def}}{=} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ \mathbb{E}_{j}\left( |\xi_{j\nu} j_{j\nu}| H(v) - \tilde{H}^{(j)}(v) \right) \right]^p.
\]

To estimate the r.h.s. of (7.9) it is enough to bound \( B_{2} \). We may use Lemma 7.3 to estimate the difference \( H(v) - \tilde{H}^{(j)}(v) \). We get
\[
B_{2} \leq \frac{C^p \rho^p}{n} \sum_{j=1}^{n} \sum_{\alpha=1}^{m} \sum_{k=1}^{K_{v}} \frac{1}{\text{Im} s(z, v_{k})} \mathbb{E} \left[ \mathbb{E}_{j}\left( |\xi_{j\nu} j_{j\nu}| H^{(\alpha)}(v_{k}) - \tilde{\Lambda}_{n}^{(\alpha,j)}(v_{k}) I(v) \right)^{2p} \right], \tag{7.10}
\]

where \( v_{k} \overset{\text{def}}{=} v s^{k}, k \geq 0 \). We also used the fact that \( K_{v}^{p} \leq p^{p} \) and \( \tilde{f}_{j\nu} \) is \( \mathfrak{M}^{(j,l)} \)-measurable. We fix \( j, \alpha \) and \( k \) and study
\[
\mathbb{E} \left[ \mathbb{E}_{j}\left( |\xi_{j\nu} j_{j\nu}| H^{(\alpha)}(v_{k}) - \tilde{\Lambda}_{n}^{(\alpha,j)}(v_{k}) I(v) \right)^{2p}. \right.
\]
Applying Lemma 7.4 we get
\[ E^z \left[ \mathbb{E}_{j,v} \left( |\xi_{j,v}| A_n^{(\nu)}(v_k) - \tilde{\Lambda}_n^{(\alpha,j)}(v_k) I(v) \right)^{2p} \right] \leq \frac{C P \mathbb{E} \exp(\xi)}{(nv)^p}. \]

Since \( \text{Im} s(z, v) \geq (nv)^{-1} \) for \( w \in D \), the last inequality and (7.10) imply
\[ A_2 \leq p E |T_{n,h}|^p + \frac{C p \mathbb{E} \exp(\xi)}{(nw)^p}. \]

7.4.2. Bound for \( A_3 \). Applying Taylor’s formula
\[ |\varphi(T_{n,h}) - \varphi(T_{n,h}^j)| \leq p |T_{n,h}^j + \theta(T_{n,h} - T_{n,h}^j)|^p - 2 |T_{n,h} - T_{n,h}^j| \]

It is easy to check that
\[ T_{n,h}^j = T_{n,h}^{(j)} - \mathbb{E}_{j,v}[(T_{n,h} - T_{n,h}^j) H] \]

and
\[ T_{n,h} - T_{n,h}^j = T_{n,h}^{(j)} + T_{n,h} (H - \tilde{H}^{(j)}) \]

Hence,
\[ |\varphi(T_{n,h}) - \varphi(T_{n,h}^j)| \leq p |T_{n,h}^{(j)}|^p - 2 |T_{n,h} - T_{n,h}^j| \]

We obtain that \( A_3 \leq A_{31} + \ldots + A_{34} \), where
\[ A_{31} \overset{\text{def}}{=} \frac{p}{n} \sum_{j=1}^{n^2} \sum_{\nu=1}^{2} E \left| \xi_{j,v} f_{j,v} \right| |T_{n,h}^{(j)}|^p - 2 |T_{n,h} - T_{n,h}^j| H(v), \]
\[ A_{32} \overset{\text{def}}{=} \frac{p^{p-1}}{n} \sum_{j=1}^{n^2} \sum_{\nu=1}^{2} E \left| \xi_{j,v} f_{j,v} \right| |T_{n} - \tilde{T}_{n,h}^{(j)}|^p - 2 |T_{n,h} - T_{n,h}^j| H(v), \]
\[ A_{33} \overset{\text{def}}{=} \frac{p^{p-1}}{n} \sum_{j=1}^{n^2} \sum_{\nu=1}^{2} E \left| \xi_{j,v} f_{j,v} \right| |T_{n} - \tilde{T}_{n,h}^{(j)}|^p - 2 |T_{n,h} - T_{n,h}^j| H(v), \]
\[ A_{34} \overset{\text{def}}{=} \frac{p^{p-1}}{n} \sum_{j=1}^{n^2} \sum_{\nu=1}^{2} E \left| \xi_{j,v} f_{j,v} \right| \mathbb{E}_{j,v} \left[ \left| T_{n} - \tilde{T}_{n,h}^{(j)} \right|^p |H| |T_{n,h} - \tilde{T}_{n,h}^{(j)}| H(v) \right]. \]

It follows from these representations that \( A_{31} \leq A_{311} + \ldots + A_{313} \), where
\[ A_{311} \overset{\text{def}}{=} \frac{p}{n} \sum_{j=1}^{n^2} \sum_{\nu=1}^{2} E \left| \xi_{j,v} f_{j,v} \right| |T_{n} - \tilde{T}_{n,h}^{(j)}|^p |I(v)|, \]
\[ A_{312} \overset{\text{def}}{=} \frac{p}{n} \sum_{j=1}^{n^2} \sum_{\nu=1}^{2} E \left| \xi_{j,v} f_{j,v} \right| |T_{n} - \tilde{T}_{n,h}^{(j)}| |H|, \]
\[ A_{313} \overset{\text{def}}{=} \frac{p}{n} \sum_{j=1}^{n^2} \sum_{\nu=1}^{2} E \left| \xi_{j,v} f_{j,v} \right| |T_{n} - \tilde{T}_{n,h}^{(j)}|^p |I(v)|. \]
Let us consider the term $A_{311}$. We may apply Lemma 7.5 and bound this term by the sum of three terms:

$$A_{311} \lesssim \frac{p \text{Im}_s}{n} \sum_{\alpha=1}^{2n} \sum_{j=1}^{n} |E| \left| \xi_{j\nu} \tilde{f}_{j\nu} \right|^2 \left| T_{n,h} \right|^{p-2} |\Lambda_n - \tilde{\Lambda}_n^{(\alpha,j)}(I(v))|^2$$

$$+ \frac{p}{n(n+1)^2} \sum_{j=1}^{n} \sum_{\nu=1}^{2} |E| \left| \xi_{j\nu} \tilde{f}_{j\nu} \right|^2 \left| T_{n,h} \right|^{p-2} \left( \text{Im}_j R_{jj}^2 + \text{Im}_j R_{jj}^2 \right) |I(v)|$$

$$+ \frac{p}{n(n+1)^2} \sum_{j=1}^{n} \sum_{\nu=1}^{2} |E| \left| \xi_{j\nu} \tilde{f}_{j\nu} \right|^2 \left| T_{n,h} \right|^{p-2} \text{E}_j \left( \left| \Lambda_n^{(\alpha)}(v_k) - \tilde{\Lambda}_n^{(\alpha,j)}(v_k) \right| \right).$$

The last two terms, $A_{311}^{(2)}, A_{311}^{(3)}$, may be easily bounded as follows

$$A_{311}^{(2)} \leq \frac{C p^2 s}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} |E| \left| \xi_{j\nu} \tilde{f}_{j\nu} \right|^2 \left| T_{n,h} \right|^{p-2} \left( \text{Im}_j R_{jj}^2 + \text{Im}_j R_{jj}^2 \right) |I(v)|.$$ 

Indeed, one may apply Hölder’s inequality and Young’s inequality. For the estimation of $A_{311}^{(1)}$ we first use Young’s inequality and get

$$A_{311}^{(1)} \leq \rho E \left| T_{n,h} \right|^p + \frac{C p^2 s}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} |E| \left| \xi_{j\nu} \tilde{f}_{j\nu} \right|^2 \left| T_{n,h} \right|^{p-2} \left( \text{Im}_j R_{jj}^2 + \text{Im}_j R_{jj}^2 \right) |I(v)|.$$ 

Applying Lemma 7.4 we get

$$A_{311}^{(1)} \leq \rho E \left| T_{n,h} \right|^p + \frac{C p^2 s}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} |E| \left| \xi_{j\nu} \tilde{f}_{j\nu} \right|^2 \left| T_{n,h} \right|^{p-2} \left( \text{Im}_j R_{jj}^2 + \text{Im}_j R_{jj}^2 \right) |I(v)|.$$ 

It is easy to see that similarly one may estimate the term $A_{311}$. To finish estimation of $A_{311}$ it remains to estimate $A_{312}$. Applying Lemma 7.3 we obtain

$$A_{312} \lesssim \frac{p}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \sum_{\alpha=1}^{n} \sum_{k=0}^{K} \frac{1}{\text{Im}_s(z_{jk})} \text{E} \left| \xi_{j\nu} \tilde{f}_{j\nu} \right|^2 \left| T_{n,h} \right|^{p-2} \left( \text{Im}_j R_{jj}^2 + \text{Im}_j R_{jj}^2 \right) |I(v)|.$$ 

Applying Let us denote

$$I_{j,k,\alpha,v} \begin{aligned} & A_{312} \lesssim \rho E \left| T_{n,h} \right|^p + \frac{C p^2 s}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} |E| \left| \xi_{j\nu} \tilde{f}_{j\nu} \right|^2 \left| T_{n,h} \right|^{p-2} \left( \text{Im}_j R_{jj}^2 + \text{Im}_j R_{jj}^2 \right) |I(v)|. \end{aligned}$$

Using the Cauchy-Schwarz inequality we obtain

$$I_{j,k,\alpha,v} \lesssim \rho E \left| T_{n,h} \right|^p + \frac{C p^2 s}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} |E| \left| \xi_{j\nu} \tilde{f}_{j\nu} \right|^2 \left| T_{n,h} \right|^{p-2} \left( \text{Im}_j R_{jj}^2 + \text{Im}_j R_{jj}^2 \right) |I(v)|.$$ 

It is easy to show that

$$|\Lambda_n^{(\alpha)}(v_k) - \Lambda_n^{(\alpha,j)}(v_k)| \lesssim \frac{1}{n} \left( \delta_{\alpha,1} |R_{jj}(v_k)| + \delta_{\alpha,m+1} |R_{jj}^{(m+1)}(v_k)| \right).$$

We may use Lemmas A.2–A.4 and Young’s inequality to get

$$A_{312} \lesssim \rho E \left| T_{n,h} \right|^p + \frac{C p^2 s}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} |E| \left| \xi_{j\nu} \tilde{f}_{j\nu} \right|^2 \left| T_{n,h} \right|^{p-2} \left( \text{Im}_j R_{jj}^2 + \text{Im}_j R_{jj}^2 \right) |I(v)|.$$ 

(7.13)
Let us consider \( A_{321} \). Applying we may may estimate it by the sum of the following terms

\[
A_{321} \overset{\text{def}}{=} \frac{p^{p-1}}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} [\xi_{j\nu} \tilde{f}_{j\nu} | T_{n}|^{p-2} | H - \tilde{H}(j,j) |^{p-2} | T_{n} - \tilde{T}_{n}(j,j) | \tilde{H}(j,j) H(v),
\]

\[
A_{322} \overset{\text{def}}{=} \frac{p^{p-1}}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} [\xi_{j\nu} \tilde{f}_{j\nu} | T_{n}|^{p-1} | H - \tilde{H}(j,j) |^{p-1} H(v),
\]

\[
A_{321} \overset{\text{def}}{=} \frac{p^{p-1}}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} [\xi_{j\nu} \tilde{f}_{j\nu} | T_{n}|^{p-2} | H - \tilde{H}(j,j) |^{p-2} \mathbb{E}_{j,j} [(T_{n} - \tilde{T}_{n}(j,j)) H] H(v).
\]

All three terms may be bounded similarly. We turn our attention to the first term only. To deal with it we use the following inequality

\[
| T_{n}|^{p-2} I(v) \leq C \rho \text{Im}^{p-2} s(z, w),
\]

which may be deduced from equation (5.12). Hence,

\[
A_{321} \leq \frac{C \rho p^{p-1} \text{Im}^{p-2} s(v)}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \sum_{\alpha=1}^{K} \sum_{k=0}^{n} \mathbb{E} [\xi_{j\nu} \tilde{f}_{j\nu} | (\Lambda_{n}(\alpha)) (v_{k}) - \tilde{\Lambda}_{n}(\alpha,j,j) (v_{k}) |^{p-2} | T_{n} - \tilde{T}_{n}(j,j) | I(v).
\]

It follows from (7.13) and Lemma 7.4 that

\[
A_{321} \leq \frac{C \rho p^{p} E(p)(\rho)}{(n\rho)^{p}}.
\]

Let us consider \( A_{331} \). Applying (7.11)–(7.12) we get, where

\[
A_{331} = \frac{p^{p-1}}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} [\xi_{j\nu} \tilde{f}_{j\nu} | T_{n} - \tilde{T}_{n}(j,j) |^{p-1} H(v),
\]

\[
A_{331} = \frac{p^{p-1}}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} [\xi_{j\nu} \tilde{f}_{j\nu} | T_{n} - \tilde{T}_{n}(j,j) |^{p-2} | H - \tilde{H}(j,j) | | T_{n}, h |
\]

\[
A_{331} = \frac{p^{p-1}}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} [\xi_{j\nu} \tilde{f}_{j\nu} | T_{n} - \tilde{T}_{n}(j,j) |^{p-2} \mathbb{E}_{j,j} [(T_{n} - \tilde{T}_{n}(j,j)) H] H(v).
\]

We estimate \( A_{331} \) only. All other terms may be bounded similarly. We get

\[
A_{331} \leq \frac{C \rho p^{p-1} \text{Im}^{p-2} s(v)}{n(n\rho)^{p}} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} [\xi_{j\nu} \tilde{f}_{j\nu} | T_{n} - \tilde{T}_{n}(j,j) | H(v),
\]

Applying Lemmas 7.4 and 7.5 we get

\[
A_{331} \leq \frac{C \rho p^{p} E(p)(\rho)}{(n\rho)^{p}}.
\]

The term \( A_{34} \) may be estimated similarly to \( A_{33} \). We omit the details. Collecting all bounds we obtain the following estimate for \( A_{3} \):

\[
A_{3} \leq \rho \mathbb{E} | T_{n}, h |^{p} + \frac{C \rho p^{p} E(p)(\rho)}{(n\rho)^{p}}.
\]
7.4.3. Bound for $A_4$. Recall that

$$A_4 = \frac{1}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} \xi_{j\nu} [f_{j\nu} - \tilde{f}_{j\nu}] H(v) \varphi(T_{n,h}).$$

We may estimate it as follows

$$A_{41} \overset{\text{def}}{=} \frac{c}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} |\xi_{j\nu}| |f_{j\nu} - \tilde{f}_{j\nu}| T_{n,h}^{(i,j)} |p-1| H(v),$$

$$A_{42} \overset{\text{def}}{=} \frac{p-1}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{2} \mathbb{E} |\xi_{j\nu}| |f_{j\nu} - \tilde{f}_{j\nu}| T_{n,h} - T_{n,h}^{(i,j)} |p-1| H(v).$$

We may choose

$$\tilde{f}_{j1} \overset{\text{def}}{=} -\frac{1}{w - m_{n}(m+2,j,j)}(z, w) + \frac{|z|^{2}}{w + m_{n}^{(m,j,j)}(z,w)},$$

$$\tilde{f}_{j2} \overset{\text{def}}{=} -\frac{|z|^{2} \tilde{f}_{j1}}{|w + m_{n}^{(m,j,j)}(z,w)|^{2}}$$

To estimate the difference $f_{j\nu} - \tilde{f}_{j\nu}$ we may apply representation (5.8) and inequality (7.13). Repeating all arguments from the previous section we may conclude the bound

$$A_4 \leq \rho \mathbb{E} |T_{n,h}|^{p} + \frac{C p^{2} p^{2} \mathbb{E} (kp)}{(w v k)^{p}}.$$

Collecting now all bounds above we get the claim of the theorem. \hfill \square

7.5. Auxiliary lemmas. We finish this section by several important lemmas.

**Lemma 7.3.** Let $v_k = v s^{k}$, $k \geq 0$. The following inequality holds

$$|H(A_n) - H(\tilde{A}_n^{(j_\alpha,j_\beta)})| \leq \frac{1}{\gamma} \sum_{\beta=1}^{2m} \sum_{k=0}^{K_{\gamma}} \frac{1}{\mathbb{E} s(z, v_k)} |\Lambda_{n}^{(\beta)}(v_k) - \tilde{\Lambda}_{n}^{(j_\alpha,j_\beta)}(v_k)| I(v).$$

The same is true if one replaces $\tilde{\Lambda}_{n}^{(j_\alpha,j_\beta)}$ by $\Lambda_{n}^{(j_\alpha,j_\beta)}$.

**Proof.** The proof follows from the simple inequality $|\prod_{j=1}^{n} a_j - \prod_{j=1}^{n} b_j| \leq \sum_{j=1}^{n} |a_j - b_j|$ and direct calculations. \hfill \square

**Denote** $I(v, v') \overset{\text{def}}{=} I(v) I(v')$.

**Lemma 7.4.** Let $w = u + iv \in D, w' = u + iv' \in D$. Moreover, we assume that $v' \geq v$. Let $g_{j\beta}(w, w'), g_{2j\beta}(w, w')$ be some positive r.v.'s such that $\mathbb{E} [g_{j\beta}] < \infty, k = 1, 2$, for $1 \leq q \leq C \log n$. Then for any $\alpha = 1, \ldots, m$ and $j = 1, \ldots, n$

$$\max_{1 \leq \beta \leq 2m} \mathbb{E}[\xi_{j\alpha}(v), \xi_{j\beta}(v')] |\Lambda_{n}^{(\beta)}(v') - \tilde{\Lambda}_{n}^{(j_\alpha,j_\beta)}(v')| g_{1,j\alpha}(v, v') I(v, v') \lesssim \frac{A_{j\alpha}^{1/2}(v) B_{j\alpha}^{1/2}(v') g_{2j\beta}(v, v')}{(nv^{1/2}(nv')^{3/2}}.$$

where

$$A_{j\alpha}(v) \overset{\text{def}}{=} \max \left\{ \text{Im} s(v), \mathbb{E}_{j\alpha,j\beta}^{(j_\alpha)} [\text{Im}^{4} R_{j\alpha,j\beta}]_{j_\alpha+1, j_\alpha+1} I(v) \right\},$$

$$B_{j\alpha}(v') \overset{\text{def}}{=} \max \left\{ \text{Im} s(v'), \mathbb{E}_{j\alpha,j\beta}^{(j_\alpha)} [\text{Im}^{4} R_{j\alpha,j\beta}]_{j_\alpha+1, j_\alpha+1} I(v'), \mathbb{E}_{j\alpha,j\beta}^{(j_\alpha)} [\text{Im}^{4} R_{j\alpha,j\beta}]_{j_\alpha+1, j_\alpha+1} I(v') \right\},$$

$$\mathbb{E}_{j\alpha,j\beta}^{(j_\alpha)} [\text{Im}^{4} R_{j_\alpha+1, j_\alpha+1}] I(v), \mathbb{E}_{j\alpha,j\beta}^{(j_\alpha)} [\text{Im}^{4} R_{j_\alpha+1, j_\alpha+1}] I(v').$$
Proof. We start from the representation for $\Lambda_n^{(\beta)} - \tilde{\Lambda}_n^{(\beta,\nu_0,\nu_\alpha)}$. We rewrite it as follows

$$\Lambda_n^{(\beta)} - \tilde{\Lambda}_n^{(\beta,\nu_0,\nu_\alpha)} = \Lambda_n^{(\beta)} - \tilde{\Lambda}_n^{(\beta,\nu_0)} + \tilde{\Lambda}_n^{(\beta,\nu_0)} - \tilde{\Lambda}_n^{(\beta,\nu_0,\nu_\alpha)}.$$  

Let us introduce the following notations:

$$I_1 \overset{\text{def}}{=} E_{j_0,\nu_0} \left[ |g_{j_0}(v)| \Lambda_n^{(\beta)}(v') - |g_{j_0}(v)| \tilde{\Lambda}_n^{(\beta,\nu_0)}(v',v') \right],$$

$$I_2 \overset{\text{def}}{=} E_{j_0,\nu_0} \left[ |g_{j_0}(v)| \tilde{\Lambda}_n^{(\beta,\nu_0)}(v') - |g_{j_0}(v)| \tilde{\Lambda}_n^{(\beta,\nu_0,\nu_\alpha)}(v',v') \right].$$

We start from $I_1$. We first mention that $\Lambda_n^{(\beta)} - \tilde{\Lambda}_n^{(\beta,\nu_0)} = m_n^{(\beta)} - m_n^{(\beta,\nu_0)} - E_{j_0}(m_n^{(\beta)} - m_n^{(\beta,\nu_0)}).$ Let us consider $m_n^{(\alpha)} - m_n^{(\alpha,\nu_0)}$ and rewrite it as follows

$$m_n^{(\beta)} - m_n^{(\beta,\nu_0)} = \frac{1}{n} \left( \delta_{\alpha,\beta} R_{j_0,\nu_0} + \sum_{l \in \mathbb{T} \setminus \nu_0} (R_{l,\beta} - R_{l,\nu_0}) \right).$$

Writing down the decomposition for the diagonal entries of resolvent we get

$$m_n^{(\beta)} - m_n^{(\beta,\nu_0)} = \frac{1}{n} \left( \delta_{\alpha,\beta} R_{j_0,\nu_0} + \sum_{l \in \mathbb{T} \setminus \nu_0} R_{j_0,\beta}^{-1} [R_{j_0,\beta}]^2 \right)$$

$$= \frac{1}{n} \left( \delta_{\alpha,\beta} + \sum_{l \in \mathbb{T} \setminus \nu_0} \left[ \frac{1}{n} \sum_{k=1}^n X_{jk}^{(\alpha)} R_{k_{\nu_0},l_{\beta}}^{(j_0)} - \frac{1}{n} R_{j_0,l_{\beta}} \right] \right)^2 R_{j_0,\beta}$$

$$= \frac{1}{n} (\delta_{\alpha,\beta} + \eta_{j_0}) R_{j_0,\nu_0},$$

where $\eta_{j_0} \overset{\text{def}}{=} \eta_{j_0,0} + \ldots + \eta_{j_0,3}$ and

$$\eta_{j_0,0} \overset{\text{def}}{=} \sum_{l \in \mathbb{T} \setminus \nu_0} [R_{j_0,l_{\beta}}^2]^{1/2} - \frac{1}{n} \sum_{k=1}^n R_{j_0,l_{\beta}}^{(j_0)}$$

$$\eta_{j_0,1} \overset{\text{def}}{=} \sum_{k=1}^n X_{jk}^{(\alpha)} R_{k_{\nu_0},l_{\beta}}^{(j_0)}$$

$$\eta_{j_0,2} \overset{\text{def}}{=} \frac{1}{n} \sum_{k=1}^n [X_{jk}^{(1)}]^2 - 1] R_{k_{\nu_0},l_{\beta}}^{(j_0)}$$

$$\eta_{j_0,3} \overset{\text{def}}{=} - 2 \sum_{k=1}^n X_{jk}^{(\alpha)} R_{k_{\nu_0},l_{\beta}}^{(j_0)}.$$  

Here $R_{j_0,l_{\beta}}^{(j_0)} \overset{\text{def}}{=} \sum_{l \in \mathbb{T} \setminus \nu_0} R_{j_0,l_{\beta}}^{(j_0)} R_{k_{\nu_0},l_{\beta}}^{(j_0)}$. Using this representation we write

$$\Lambda_n^{(\beta)} - \tilde{\Lambda}_n^{(\beta,\nu_0)} = \frac{1}{n} (\delta_{\alpha,\beta} + \eta_{j_0,0}) [R_{j_0,\nu_0} - E_{j_0}(R_{j_0,\nu_0})] + \frac{1}{n} \widetilde{\eta}_{j_0} R_{j_0,\nu_0} - \frac{1}{n} E_{j_0}(\widetilde{\eta}_{j_0} R_{j_0,\nu_0}),$$

where $\widetilde{\eta}_{j_0} \overset{\text{def}}{=} \eta_{j_0,1} + \ldots + \eta_{j_0,3}$ and $g_{1,j_0}$. Moreover, it is straightforward to check that

$$\frac{1}{n} (\delta_{\alpha,\beta} + |\eta_{j_0,0}(v')|) I(v') \leq \frac{C}{n^{1/2}} (\text{Im } s(z, v') + \text{Im } R_{j_0,\nu_0}^{(j_0)}(z_{\nu_0}, v')) I(v').$$

Introduce the following approximation for $R_{j_0,\nu_0}$:

$$Q_{j_0} \overset{\text{def}}{=} \frac{1}{-w' - m_{\nu_0}} \left( \frac{1}{w' + m_{\nu_0}} \right)^2.$$
Then $R_{j_n,j_n} - Q_{j_n} = \xi_{j_n} f_{j_n} + \xi_{j_n} f_{j_n}^{-1}$. We conclude from the previous facts that

$$I_1 \leq \frac{1}{n} \mathbb{E}_{j_n,j_n} \left[ (\delta_{j_n} + |\eta_{j_n}(v')|) |\xi_{j_n} v(v')| g_{j_n} (v,v') I(v,v') \right]$$

$$+ \frac{1}{n} \mathbb{E}_{j_n,j_n} \left[ (\delta_{j_n} + |\eta_{j_n}(v')|) |\xi_{j_n} v(v')| g_{j_n} (v,v') I(v,v') \right]$$

$$+ \frac{1}{n} \mathbb{E}_{j_n,j_n} \left[ (\delta_{j_n} + |\eta_{j_n}(v')|) |\xi_{j_n} v(v')| [\mathbb{E}_{j_n} (|\xi_{j_n} v(v')| f_{j_n} (v'))] g_{j_n} (v,v') I(v,v') \right]$$

$$+ \frac{1}{n} \mathbb{E}_{j_n,j_n} \left[ (\delta_{j_n} + |\eta_{j_n}(v')|) |\xi_{j_n} v(v')| [\mathbb{E}_{j_n} (|\xi_{j_n} f_{j_n} (v')| f_{j_n} (v'))] g_{j_n} (v,v') I(v,v') \right]$$

Here $g_{j_n,j_n}$ is some positive function for bounded moments up to the order $C \log n$.

All terms in the upper bound for $I_1$ may be estimated directly. We show how to deal with one term only, all other terms may be estimated similarly. For example, we estimate the second last term.

Hölder's inequality implies

$$\frac{1}{n} \mathbb{E}_{j_n,j_n} \left[ |\xi_{j_n} v(v')| \tilde{\eta}_{j_n} (v') |g_{j_n} (v,v') I(v,v')| \right] \leq \frac{1}{n} \mathbb{E}_{j_n,j_n} \left[ (|\xi_{j_n} v(v')|^{2} I(v,v')) \mathbb{E}_{j_n,j_n} \left[ (|\tilde{\eta}_{j_n} (v')|)^{2} I(v,v') \right] \right]$$

Here, $\beta \equiv (4 + \delta)/4$. It remains to apply Lemmas A.9–A.11 to get the desired bound as stated in lemma.

Let us consider $I_2$. It is easy to check that

$$\tilde{A}_{n}(\beta,j_n,j_n) - \tilde{A}_n(\beta,j_n,j_n) = \mathbb{E}_{j_n} \left( m_n^{(\beta)} - m_n^{(\beta,j_{m,n})} \right) - \mathbb{E}_{j_n,j_n} \left( m_n^{(\beta)} - m_n^{(\beta,j_{m,n})} \right)$$

Similarly to (7.14) we may show that

$$m_n^{(\beta)} - m_n^{(\beta,j_{m,n})} = \frac{1}{n} \left( \sum_{\beta,j_{m,n}} R_{j_{m,n} + 1}^{-1} R_{j_{m,n}} = \sum_{\beta,j_{m,n}} R_{j_{m,n} + 1}^{-1} R_{j_{m,n}} \right)$$

$$= \frac{1}{n} \left( \sum_{\beta,j_{m,n}} R_{j_{m,n} + 1}^{-1} R_{j_{m,n}} \right)$$

$$= \frac{1}{n} \left( \sum_{\beta,j_{m,n}} R_{j_{m,n} + 1}^{-1} R_{j_{m,n}} \right)$$

$$+ \frac{1}{n} \sum_{\beta,j_{m,n}} \Theta_R R_{j_{m,n} + 1}^{-1} R_{j_{m,n}} = \tilde{\eta}_{j_n} + \frac{1}{n} \sum_{\beta,j_{m,n}} \Theta_R R_{j_{m,n} + 1}^{-1} R_{j_{m,n}}$$

Here, $\tilde{\eta}_{j_{m,n}} \equiv \tilde{\eta}_{j_{m,n},0} + \ldots + \tilde{\eta}_{j_{m,n}}^{4}$.
and \( R_{k,l}^{(j_0)} \) \( \stackrel{\text{def}}{=} \sum_{j \in \mathcal{T} \setminus J_0} R_{k_{(j_0+1)},l}^{(j_0)} R_{k_{(j_0)},l}^{(j_0+1)} \). Moreover,

\[
\Theta_I \stackrel{\text{def}}{=} \left[ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{jk}^{(\alpha-1)} [ R_{k_{(j_0+1)},l}^{(j_0+1)} - R_{k_{(j_0)},l}^{(j_0+1)} ] \right]^2 + 2 \left[ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{jk}^{(\alpha-1)} [ R_{k_{(j_0+1)},l}^{(j_0+1)} - R_{k_{(j_0)},l}^{(j_0+1)} ] \right] \left[ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{jk}^{(\alpha-1)} [ R_{k_{(j_0+1)},l}^{(j_0+1)} - \tau R_{k_{(j_0)},l}^{(j_0+1)} ] \right].
\]

It is easy to see that \( \tilde{h}_{j_{m+1},0} \) is \( \mathcal{M}(\gamma_{j_0}) \)-measurable. Hence,

\[
\tilde{\Lambda}_n^{(\beta_0)} - \tilde{\Lambda}_n^{(\beta_0+j_0)} = \frac{1}{n} (\delta_{\beta_0,j_0} + \tilde{h}_{j_{m+1},0}(v')) E_{j_0} \left[ \gamma_{j_{m+1},1}(v') f_{j_{m+1},1}(v') \right] g_{j_0}^{\beta_0}(v, v') I(v, v')
\]

Here \( \tilde{\gamma}_{j_{m+1}} \) \( \stackrel{\text{def}}{=} \tilde{h}_{j_{m+1},1} + \ldots + \tilde{h}_{j_{m+1},4} \). All this facts imply the following bound for \( I_2 \):

\[
I_2 \leq \frac{1}{n} E_{j_0,j_0} \left[ \xi_{j_0}(v') \right] \left[ \delta_{\beta_0,j_0} + \tilde{h}_{j_{m+1},0}(v') \right] E_{j_0} \left[ \gamma_{j_{m+1},1}(v') f_{j_{m+1},1}(v') \right] g_{j_0}^{\beta_0}(v, v') I(v, v')
\]

One may proceed similarly to the estimate of \( I_1 \). We only consider the second last term. It is straightforward to check that

\[
| \Theta_I | \leq \left[ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{jk}^{(\alpha-1)} R_{k_{(j_0+1)},j_0}^{(j_0+1)} - R_{k_{(j_0+1)},j_0}^{(j_0+1)} \right] \left[ R_{k_{(j_0+1)},j_0}^{(j_0+1)} || R_{k_{(j_0+1)},j_0}^{(j_0+1)} || - 2 \right] + 2 \left[ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{jk}^{(\alpha-1)} R_{k_{(j_0+1)},j_0}^{(j_0+1)} \right] \left[ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{jk}^{(\alpha-1)} R_{k_{(j_0+1)},j_0}^{(j_0+1)} - \tau R_{k_{(j_0+1)},j_0}^{(j_0+1)} \right] \left[ R_{k_{(j_0+1)},j_0}^{(j_0+1)} || R_{k_{(j_0+1)},j_0}^{(j_0+1)} || - 1 \right].
\]

Let us introduce the following notations:

\[
\zeta_{j_0,1} \stackrel{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{jk}^{(\alpha-1)} R_{k_{(j_0+1)},j_0}^{(j_0+1)} \quad \zeta_{j_0,2} \stackrel{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{jk}^{(\alpha-1)} R_{k_{(j_0+1)},j_0}^{(j_0+1)}.
\]
Lemma A.1. Estimations of $\zeta$ and $\theta$. Moreover, $\zeta_j$ be estimated as the sum of three terms. We will estimate the second term only. It is easy to see that both statements are consequence of the equation for $\zeta_j$.

Now we may estimate the second last term in the bound for $I_2$. Using the previous inequality it may be estimated as the sum of three terms. We will estimate the second term only. It is easy to see that $\zeta_{j,2}$ is $M^{(j_0)}$-measurable. Hence,

$$2 \mathbb{E}_{j_0, j_0} \left[ |\zeta_{j_0,2}(v)| \left( \frac{1}{n} \sum_{l \in \mathbb{T} \setminus j_0} |\zeta_{l,j_0}|^2 \right)^{1/2} \right]$$

$$\times \mathbb{E}_{j_0} \left[ |\zeta_{j_0,1}(v)| \left( \frac{1}{n} \sum_{l \in \mathbb{T} \setminus j_0} |\zeta_{l,j_0}|^2 \right)^{1/2} \right]$$

$$\times \mathbb{E}_{j_0, j_0} \left[ |\zeta_{j_0,2}(v)|^2 \right] \mathbb{E}_{j_0} \left[ |\zeta_{j_0,1}(v)|^2 \right] \mathbb{E}_{j_0, j_0} \left[ |\zeta_{j_0,3}(v)|^4 \right] \mathbb{E}_{j_0, j_0} \left[ |\zeta_{j_0,4}(v)|^4 \right] \mathbb{E}_{j_0, j_0} \left[ |\zeta_{j_0,5}(v)|^4 \right]$$

Applying now Rosenthal\'s inequality to $\zeta_{j_0,k}, k = 1, 2$, we conclude the bound as required by the statement of the lemma.

Lemma 7.5. For any $w \in \mathcal{D}$ the following inequality holds:

$$\max_{1 \leq \alpha \leq m} |T_n^{(\alpha)} - T_n^{(\alpha, j_0, j_0)}|I(v) \leq \frac{\text{Im} s}{\nu^2}$$

Moreover,

$$\max_{1 \leq \alpha \leq m} |T_n^{(\alpha)} - T_n^{(\alpha, j_0, j_0)}|I(v) \leq \text{Im} s, w \max_{1 \leq \beta \leq 2m} |\Lambda_n^{(\beta)} - \tilde{\Lambda}_n^{(\beta, j_0, j_0)}|I(v)$$

$$+ \frac{1}{(\nu^2)} \max_{1 \leq \alpha \leq m} \left[ \frac{\text{Im}^2 R_{j_0, j_0}}{|R_{j_0, j_0}|^2} + \frac{\text{Im}^2 R_{j_0, j_0}}{|R_{j_0, j_0}|^2} \right] I(v)$$

$$+ \frac{1}{(\nu^2)} \mathbb{E}_{j_0, j_0} \left[ \max_{1 \leq \alpha \leq m} \left( \frac{\text{Im}^2 R_{j_0, j_0}}{|R_{j_0, j_0}|^2} + \frac{\text{Im}^2 R_{j_0, j_0}}{|R_{j_0, j_0}|^2} \right) I(v) \right].$$

Proof. Both statements are consequence of the equation for $\Lambda_n$.

Appendix A. Inequalities for linear and quadratic forms

In this section we present some inequalities for linear and quadratic forms.

A.1. Estimations of $\zeta_j$, for $j \in \mathbb{T}$. In this section we estimate the moments of $\zeta_j^{(J,K)}$ for $J, K \in \mathbb{T}$ and $\nu = 1, \ldots, 5$ and $n_{J,j}^{(J,K)}, \nu = 1, \ldots, 5$. In what follows for any $J \subset \mathbb{T}$ and $j \in \mathbb{T}_J$ we denote $\tilde{J} \equiv J \cup \{j\}$. We also introduce $\sigma$-algebra $\mathcal{M}(\tilde{J}, K) \equiv \sigma\{X_{kl}, k \in \tilde{J}, l \in \mathbb{K}\}$, $J, K \subset \mathbb{T}$, and denote

$$E^*(\cdot) \equiv E(\cdot|\mathcal{M}(\tilde{J}, K)).$$

Lemma A.1. For any $j \in \mathbb{T} \setminus J_0$

$$|\tilde{\zeta}_j| \leq (\nu^2)^{-1}.$$
Proof. Since
\[ R_{[j,K]}^{(j,K)} = R_{[j,K]_1}^{(j,K)} + R_{[j,K]_2}^{(j,K)} \]
we get that
\[ |z|^{(j,K)}_{1} \leq \frac{1}{n^2} \| R_{j,K}^{(j,K)} \|^{n \nu} \leq (n^2)^{-1}. \]
Thus Lemma A.1 is proved. □

Lemma A.2. There exist a positive constant \( C \) such that for any \( j \in 2 \nabla J_\alpha \) and all \( p \geq 2 \)
\[ E^* |z|^{(j,K)}_{p,j,1} \leq C \| z \|^{\frac{p}{n-1}} \| R_{j,K}^{(j,K)} \|^{n \nu} + \mu_p \left( \frac{C}{n} \right)^{\frac{p}{2}} \| R_{j,K}^{(j,K)} \|^{n \nu} \]
Proof. Repeating the arguments of the proof of Lemma A.3 one gets the statement of this lemma.

Lemma A.8. For any $j \in \mathbb{T}$

$$|\eta_{j,0}| \lesssim \frac{1}{v} (\text{Im} m^{(j_n)}_{n}) + |z|^2 \text{Im} \Re^{(j_n)}_{j_{m+a}, j_{m+a}}.$$

Proof. The proof follows from Lemma A.12. □

Lemma A.9. Under conditions (C0) for any $j \in \mathbb{T}$ and all $p : 2 \leq p \leq 4$

$$\mathbb{E}^* |\eta_{j,1}|^p \lesssim \frac{1}{(nv)^{\frac{p}{2}}} \text{Im}^{\frac{p}{2}} m^{(j_n)}_{n}.$$

Proof. Applying moment inequality for quadratic forms, e.g. \[\text{[Proposition 2.4]}\] or \[\text{[Lemma A.1]},\] and Lemma A.12 we get the proof. □

Lemma A.10. Under conditions (C0) for any $j \in \mathbb{T}$ and all $p : 2 \leq p \leq \frac{1}{\alpha}$

$$\mathbb{E}^* |\eta_{j,2}|^p \lesssim \frac{1}{(nv)^{\frac{p}{2}}} \text{Im}^{\frac{p}{2}} m^{(j_n)}_{n}.$$

Proof. The proof follows from the Rosenthal type inequality for linear forms, e.g. \[\text{[Theorem 3]}\] and \[\text{[Inequality (A)]}\] and Lemma A.12. □

Lemma A.11. Under conditions (C0) for any $j \in \mathbb{T}$ and all $p : 2 \leq p \leq 4$

$$\mathbb{E}^* |\eta_{j,3}|^p \lesssim \frac{|z|^p \text{Im}^{\frac{p}{2}} \Re^{(j_n)}_{j_{m+a}, j_{m+a}}}{n^{1/p}} \text{Im}^{\frac{p}{2}} m^{(j_n)}_{n}.$$

Proof. Similar to the proof of previous lemma. □

A.2. Inequalities for resolvent matrices.

Lemma A.12. Let $0 \leq K < n$. For all $(J, K) \in J_K$

$$\frac{1}{mn} \text{Tr} |\Re^{(J,K)}|^2 \leq \frac{1}{v} \text{Im} m^{(J,K)}_{n} + \frac{|J| - |K|}{2mnv}. \quad (A.2)$$

For all $j = 1, \ldots, mn$ such that $j \notin J$

$$\sum_k^{*} |\Re^{(J,K)}_{jk}|^2 \leq \frac{1}{v} \text{Im} \Re^{(J,K)}_{jj}, \quad (A.3)$$

where $\sum_k^{*}$ is the sum over all $k = 1, \ldots, 2mn$ such that $k \notin J \cup K$.

Proof. The proof of (A.2) follows from the following inequality

$$\frac{1}{mn} \text{Tr} |\Re^{(J,K)}|^2 - \frac{1}{mnv} \text{Im} \text{Tr} \Re^{(J,K)} \leq \frac{1}{v} \text{Im} m^{(J,K)}_{n} + \frac{|J| - |K|}{2mnv}.$$

The bound (A.3) follows from the eigenvalue decomposition of $V(z)$. □

Appendix B. Bounds for the Kolmogorov distance between distribution functions via Stieltjes transforms

We reformulate the following smoothing inequality proved in \[\text{[Corollary 2.3]},\] which allows to relate distribution functions to their Stieltjes transforms. Let $G(x)$ be an arbitrary distribution function which support is an interval or union of non-intersecting intervals, say $J = \text{supp} G(x) = \bigcup_{\alpha=1}^{m} J_{\alpha}$ and $J_{\alpha} = [a_{\alpha}, b_{\alpha}]$. Additionally we assume that $G(x)$ has an absolutely continuous density which is bounded and for any end-point $c$ of the support $J$ it behaves as $g(x) \sim (x - c)^{-\frac{1}{2}}$. For any $x \in J$ we define $\gamma(x) \overset{\text{def}}{=} \min \{ |x - a_{\alpha}|, |b_{\alpha} - x| \}$. Given $\frac{1}{2} \min_{\alpha} \{b_{\alpha} - a_{\alpha}\} > \varepsilon > 0$ introduce the interval
\(J_{\varepsilon}(z) = \{x \in [a_\varepsilon, b_\varepsilon] : \gamma(x) \geq \varepsilon\}\) and \(J'_{\varepsilon} = \bigcup_{n=1}^{m} J'_{\varepsilon}(\varepsilon/2)\). For a distribution function \(F\) denote by \(\xi\) its Stieltjes transform,
\[
S_F(z) \overset{\text{def}}{=} \int_{-\infty}^{\infty} \frac{1}{x-z}dF(x).
\]
We also denote
\[
a \overset{\text{def}}{=} \sqrt{2} + 1. \quad \text{(B.1)}
\]

**Proposition B.1.** Let \(v_0 > 0\) and \(\frac{1}{2} > \varepsilon > 0\) be positive numbers such that
\[
v = 2v_0 \leq \varepsilon^{3/2}. \quad \text{(B.2)}
\]
Denote \(v' = v/\sqrt{\varepsilon}\). If \(G\) denotes the distribution function satisfying conditions above, and \(F\) is any distribution function, there exist some absolute constants \(C_1\) and \(C_2\) such that
\[
\Delta(F,G) \overset{\text{def}}{=} \sup_x |F(x) - G(x)| \leq 2 \sup_{x \in J'_{\varepsilon}} \left| \int_{-\infty}^{x} (S_F(u + iv') - S_G(u + iv'))du \right| + C_1v + C_2\varepsilon^{3/2}.
\]

**Remark.** For any \(x \in J_{\varepsilon}\) we have \(\gamma = \gamma(x) \geq \varepsilon\) and according to condition (B.2), \(\frac{\varepsilon}{\sqrt{\varepsilon}} \leq \frac{2}{x}\).

**Lemma B.2.** Let \(0 < v \leq \frac{\varepsilon^{3/2}}{2\varepsilon}\) and \(V > v\). Denote \(v' \overset{\text{def}}{=} v/\sqrt{\varepsilon}\). The following inequality holds
\[
\sup_{x \in J'_{\varepsilon}} \left| \int_{-\infty}^{x} \text{Im}(S_F(u + iv') - S_G(u + iv'))du \right| \leq \int_{-\infty}^{\infty} |S_F(u + iV) - S_G(u + iV)|du + \sup_{x \in J'_{\varepsilon}} \int_{v'}^{V} (S_F(x + iu) - S_G(x + iu))du.
\]

**Proof.** Let \(x \in J'_{\varepsilon}\) be fixed. Let \(\gamma = \gamma(x)\). Put \(z = u + iv'\). Since \(v' = \frac{v}{\sqrt{\varepsilon}} \leq \frac{2}{x}\), see (B.2), we may assume without loss of generality that \(v' \leq 4\) for \(x \in J'_{\varepsilon}\). Since the functions of \(S_F(z)\) and \(S_G(z)\) are analytic in the upper half-plane, it is enough to use Cauchy's theorem. We can write for \(x \in J'_{\varepsilon}\)
\[
\int_{-\infty}^{x} \text{Im}(S_F(z) - S_G(z))du = \text{Im} \left( \lim_{L \to \infty} \int_{-L}^{x} (S_F(u + iv') - S_G(u + iv'))du \right).
\]
By Cauchy's integral formula, we have
\[
\int_{-L}^{x} (S_F(z) - S_G(z))du = \int_{-L}^{x} (S_F(u + iV) - S_G(u + iV))du + \int_{v'}^{V} (S_F(-L + iu) - S_G(-L + iu))du - \int_{v'}^{v'} (S_F(x + iu) - S_G(x + iu))du.
\]
Denote by \(\xi\) (resp. \(\eta\)) a random variable with distribution function \(F(x)\) (resp. \(G(x)\)). Then we have
\[
|S_F(-L + iu)| = |E(\xi + L - iu)^{-1}| \leq (v')^{-1} P(|\xi| > L/2) + 2/L,
\]
for any \(v' \leq u \leq V\). Similarly,
\[
|S_G(-L + iu)| \leq v'^{-1} P(|\eta| > L/2) + 2/L.
\]
These inequalities imply that
\[
\int_{v'}^{V} (S_F(-L + iu) - S_G(-L + iu))du \to 0 \quad \text{as} \quad L \to \infty,
\]
which completes the proof. \(\square\)

Combining the results of Proposition B.1 and Lemma B.2, we get

\(\Gamma\)
**Corollary B.3.** Under the conditions of Proposition B.1 the following inequality holds

\[
\Delta(F, G) \leq 2 \int_{-\infty}^{\infty} |S_F(u + iV) - S_G(u + iV)| \, du + C_1 v + C_2 \varepsilon^2
\]

\[
+ 2 \sup_{x \in \mathbb{R}^+, \varepsilon > 0} \int_{\varepsilon}^{V} |S_F(x + iu) - S_G(x + iu)| \, du,
\]

where \( v' = \frac{\varepsilon}{\sqrt{\gamma}} \) and \( C_1, C_2 > 0 \) denote absolute constants.

**B.1. Proof of Bounds for the Kolmogorov Distance.**

**Proof. Proposition B.1.** The proof of Proposition B.1 is a straightforward adaptation of the proof from [30][Lemma 2.1]. We include it here for the sake of completeness. First we note that

\[
\sup_{x} |F(x) - G(x)| = \sup_{x \in \mathbb{J}} |F(x) - G(x)| = \max \left\{ \sup_{x \in \mathbb{J}} |F(x) - G(x)|, \sup_{x \in [a, a + \varepsilon]} |F(x) - G(x)|, \sup_{x \in [b_\varepsilon, b_\alpha]} |F(x) - G(x)|, \alpha = 1, \ldots, m \right\}.
\]

Without loss of generality we shall assume that \( a_1 \leq b_1 < a_2 \leq b_2 < \cdots \leq a_m \leq b_m \). Consider \( x \in [a_1, a_1 + \varepsilon] \) we have

\[
-G(a_\varepsilon) \leq F(x) - G(x) \leq F(a_1 + \varepsilon) - G(a_1 + \varepsilon) + G(a_1 + \varepsilon)
\]

This inequality yields

\[
\sup_{x \in [a_1, a_1 + \varepsilon]} |F(x) - G(x)| \leq \sup_{x \in \mathbb{J}} |F(x) - G(x)| + G(a_1 + \varepsilon).
\]

Let \( x \in [a_\alpha, a_\alpha + \varepsilon] \) for \( \alpha = 2, \ldots, m \). Note that \( G(b_\alpha - 1) = G(a_\alpha) \). We may write

\[
F(b_\alpha - 1) - G(b_\alpha - 1) - (G(a_\alpha + \varepsilon) - G(a_\alpha)) \leq F(x) - G(x)
\]

\[
\leq (F(a_\alpha + \varepsilon) - G(a_\alpha + \varepsilon)) + (G(a_\alpha + \varepsilon) - G(a_\alpha)).
\]

From here it follows that

\[
\sup_{x \in [a_\alpha, a_\alpha + \varepsilon]} |F(x) - G(x)| \leq \sup_{x \in \mathbb{J}} |F(x) - G(x)| + \sup_{x \in \mathbb{J}} |F(x) - G(x)|
\]

\[
+ (G(a_\alpha + \varepsilon) - G(a_\alpha)).
\]

By induction we get for any \( \alpha = 1, \ldots, m \)

\[
\sup_{x \in \mathbb{J}} |F(x) - G(x)| \leq m \max_{1 \leq i \leq m} \sup_{x \in \mathbb{J}} |F(x) - G(x)| + m \max_{1 \leq i \leq m} (G(a_i + \varepsilon) - G(a_i)).
\]

Similarly we get

\[
\sup_{x \in [b_\varepsilon, b_\alpha]} |F(x) - G(x)| \leq m \max_{1 \leq i \leq m} \sup_{x \in \mathbb{J}} |F(x) - G(x)| + m \max_{1 \leq i \leq m} (G(b_i + \varepsilon) - G(b_i)).
\]

Note that \( G(a_\alpha + \varepsilon) - G(a_\alpha) \leq Ce^{3/2} \) with some absolute constant \( C > 0 \). Combining all these relations we get

\[
\sup_{x} |F(x) - G(x)| \leq \Delta \epsilon(F, G) + C e^{3/2}, \tag{B.3}
\]
where $\Delta_x(F, G) = \sup_{x \in J_x} |F(x) - G(x)|$. We denote $v' \overset{\text{def}}{=} v/\sqrt{\pi}$. For any $x \in J'_x$

\[
\left| \frac{1}{\pi} \text{Im} \left( \int_{-\infty}^{\infty} (S_F(u + iv') - S_G(u + iv'))du \right) \right| \geq \frac{1}{\pi} \text{Im} \left( \int_{-\infty}^{\infty} (S_F(u + iv') - S_G(u + iv'))du \right)
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} v'd(F(y) - G(y))dy \right) du = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} 2v'(y-u)(F(y) - G(y))dy \right) du
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{F(x - v'y) - G(x - v'y)}{y^2 + 1} dy, \quad \text{by change of variables.} \tag{B.4}
\]

Furthermore, using the definition (B.1) of $a$ and $\Delta(F, G)$ we note that

\[
\frac{1}{\pi} \int_{|y| > a} \frac{|F(x - v'y) - G(x - v'y)|}{y^2 + 1} dy \leq (1 - \beta)\Delta(F, G). \tag{B.5}
\]

Since $F$ is non decreasing, we have

\[
\frac{1}{\pi} \int_{|y| \leq a} \frac{F(x - v'y) - G(x - v'y)}{y^2 + 1} dy \geq \frac{1}{\pi} \int_{|y| \leq a} \frac{F(x - v'a) - G(x - v'y)}{y^2 + 1} dy
\]

\[
\geq (F(x - v'a) - G(x - v'a))\beta - \frac{1}{\pi} \int_{|y| \leq a} |G(x - v'y) - G(x - v'a)|dy.
\]

These inequalities together imply (using a change of variables in the last step)

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{F(x - v'y) - G(x - v'y)}{y^2 + 1} dy \geq \beta(F(x - v'a) - G(x - v'a))
\]

\[
- \frac{1}{\pi} \int_{|y| \leq a} |G(x - v'y) - G(x - v'a)|dy - (1 - \beta)\Delta(F, G) \geq \beta(F(x - v'a) - G(x - v'a))
\]

\[
- \frac{1}{\pi} \int_{|y| \leq v'a} |G(x - y) - G(x - v'a)|dy - (1 - \beta)\Delta(F, G). \tag{B.6}
\]

Note that according to Remark B, $x \pm v'a \in J'_x$ for any $x \in J_x$. Assume first that $x_n \in J'_x$ is a sequence such that $F(x_n) - G(x_n) \to \Delta_x(F, G)$. Then $x_n' \overset{\text{def}}{=} x_n + v'a \in J'_x$. Using (B.4) and (B.6), we get

\[
\sup_{x \in J'_x} \left| \text{Im} \int_{-\infty}^{\infty} (S_F(u + iv') - S_G(u + iv'))du \right|
\]

\[
\geq \text{Im} \int_{-\infty}^{x'_n} (S_F(u + iv') - S_G(u + iv'))du \geq \beta(F(x'_n - v'a) - G(x'_n - v'a))
\]

\[
- \frac{1}{\pi v} \sup_{x \in J'_x} \sqrt{\pi} \int_{|y| \leq 2v'a} |G(x + y) - G(x)|dy - (1 - \beta)\Delta(F, G)
\]

\[
= \beta(F(x_n) - G(x_n)) - \frac{1}{\pi v} \sup_{x \in J'_x} \sqrt{\pi} \int_{|y| < 2v'a} |G(x + y) - G(x)|dy - (1 - \beta)\Delta(F, G). \tag{B.7}
\]

Assume for definiteness that $y > 0$. Recall that $\varepsilon \leq 2\gamma$, for any $x \in J'_x$. By Remark B with $\varepsilon/2$ instead of $\varepsilon$, we have $0 < y \leq 2v'a \leq \sqrt{2\varepsilon}$, for any $x \in J'_x$. By conditions of Proposition, we have,

\[
|G(x + y) - G(x)| \leq y \sup_{u \in [x, x+y]} G'(u) \leq yC\sqrt{\gamma + \varepsilon}
\]

\[
\leq C\sqrt{\gamma + 2v'a} \leq Cy\sqrt{\gamma + \varepsilon} \leq Cy\sqrt{\gamma}.
\]

This yields after integrating in $y$

\[
\frac{1}{\pi v} \sup_{x \in J'_x} \sqrt{\pi} \int_{0 \leq y \leq 2v'a} |G(x + y) - G(x)|dy \leq \frac{C}{v} \sup_{x \in J'_x} \gamma v'^2 \leq Cv. \tag{B.8}
\]
Similarly we get that
\[
\frac{1}{\pi x} \sup_{y \in J_{\varepsilon}} \sqrt{\gamma} \int_{0 \geq y' \geq -2 \gamma v} |G(x + y) - G(x)| dy \leq \frac{C}{v^2} \sup_{y \in J_{\varepsilon}} \gamma v^2 \leq Cv. \tag{B.9}
\]
By inequality (B.3)
\[
\Delta_{\varepsilon}(F, G) \geq \Delta(F, G) - C\varepsilon^\frac{3}{2}. \tag{B.10}
\]
The inequalities (B.7), (B.10) and (B.8), (B.9) together yield as \(n\) tends to infinity
\[
\sup_{x \in J_{\varepsilon}} \left| \int_{-\infty}^{\infty} (S_F(u + iv') - S_G(u + iv')) du \right| \geq (2\beta - 1) \Delta(F, G) - Cv - C\varepsilon^\frac{3}{2}, \tag{B.11}
\]
for some constant \(C > 0\). Similar arguments may be used to prove this inequality in case there is a sequence \(x_n \in J_{\varepsilon}\) such \(F(x_n) - G(x_n) \to -\Delta_{\varepsilon}(F, G)\). In view of (B.11) and \(2\beta - 1 = 1/2\) this completes the proof. \(\square\)

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