Smooth Surfaces via Nets of Geodesics

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June 4, 2024

Abstract

The goal of this study is to provide a method for computing the following: Given a network of curves in 3d (satisfying a condition at the intersection points), compute efficiently a smooth surface such that the curves are geodesics on it. This work can serve as a base for engineers who wish to implement computations of such surfaces in Computer Aided Design (CAD) software or other applications. The motivation for this study was the following hypothesis and observation together with the desire to improve CAD interfaces. The hypothesis and observation is that artists draw projections of geodesics to illustrate 3d objects: for example projections of nets of curves can be seen in drawings of Rembrandt. In addition, this observation is supported by research in cognitive sciences: in a seminal work by the late David Knill he suggested that the human visual system incorporates a geodesic constraint in the processing of reflected contours.

1 Introduction

The motivation for this work was to tackle the challenge of computing a smooth surface given a network of geodesic curves in 3d. In general, completion of surfaces from a network of curves is a natural task in CAD. The continuity-level or smoothness of the surface attained is of great importance from an aesthetic and performance-wise points of view in different industries, where those surfaces are actually manufactured. I speculate that nets of geodesics play a central role in encoding the geometry of common organic and synthetic surfaces along with an inherent inclination of such materials to remain as flat as possible. I believe that this work would have important applications in CAD interfaces, geometric modelling in architecture and industrial design and sketch-to-3d technology. The main hypothesis and observation which motivated this study was that illustrators draw projections of geodesics in order to illustrate an object: see for example Figure 1. This hypothesis is also supported by research in cognitive sciences: it has been suggested by Knill [15] that the human visual system uses projections from surface geodesics to infer the geometry of the underlying surface. In addition, this study include a theoretical analysis which is of its own interest: the analysis of minimal Gaussian curvature (squared) surfaces spanning a geodesic contour may have applications in engineering, physics of soft matter and more.

This work consists of two parts. The first part of this paper provides the mathematical analysis needed for the experimental part and serves as the theoretical foundation for future implementations. There we deal with finding surfaces in $\mathbb{R}^3$ which are as close as possible to being flat and span a given contour such that the contour is a geodesic on the sought surface. We look for a surface which minimizes the total Gaussian curvature squared. We show that by a change of coordinates the curvature of the optimal surface is controlled by a fourth order PDE. Then we show how to compute accurately the Dirichlet boundary condition and the Neumann boundary condition on the boundary of the domain on which we parametrize the unknown surface. We then state a system of PDEs for the function whose graph is the optimal surface. The analysis, where an arbitrary affine plane is considered for reference, is coordinate-free: the surface described by a solution to the PDE system will be independent of the choice of plane as long as the projection of the optimal surface on it is one-to-one. The geodesic constraint on the contour is expressed via the Neumann boundary condition which we show how to compute.

In the second part, based on the theoretical work shown before, I suggest an algorithm for computing smooth surfaces given nets of geodesics and I also show supporting results of numerical experiments. I also suggest a method based on the theory part to evaluate the error in the algorithm due to the heuristics involved. The novelty of the method is that it consists of the computation of each patch in
the net independently with the union of the patches being a smooth surface. The algorithm suggested may apply to the similar reconstruction problem raised by Sprynski et al. in [21]. We elaborate more about connections of this algorithm to existing works in the beginning of part two. Two illustrative videos of approximately minimal Gaussian curvature (squared) surfaces spanning non-trivial contours can be found on the corresponding arXiv page of this study under Ancillary files (direct links are at the end of the background section for the second part). Note that in the computation of these surfaces the Neumann boundary condition is taken to be zero – the plan is to use the accurate formula shown in future works.

2 Background: minimal Gaussian curvature surface

This part of the paper deals with the analysis relevant to finding surfaces in $\mathbb{R}^3$ which are as close as possible to being flat and span a given contour such that the contour is a geodesic on the sought surface. Explicitly:

Given a smooth closed simple curve in $\mathbb{R}^3$, we will state a system of PDEs for which its solution describes (as a graph) the surface of minimum total Gaussian curvature squared, that spans the given contour such the contour is a geodesic on that surface.

I wish to elaborate on the specific choice of energy used and the geodesic boundary requirement as this is a new concept. I wanted to be able to recover “nice” surfaces from networks of geodesic curves in 3d space. I was looking for an energy on a surface that will allow me to translate the geodesic requirement for the contour to the boundary conditions for an Euler-Lagrange equation. I could then consider each cell in the network of curves separately. I had in mind a piece of leather that is stretched on a shoe last or on a baseball – one starts with a flat piece of leather so it seemed reasonable to check the square of the Gaussian curvature (the seam on a baseball is not a geodesic but may have low geodesic curvature). The material, stretchable to some extent but intrinsically flat, would be inclined to stay flat. I was able to show that the geodesic boundary requirement translates to a formula for the Neumann boundary condition. This is explained in the following text.

In the following analysis we use a new concept where we consider two relatable coordinate charts of an ambient open surface containing the contour and the unknown optimal surface spanning it. Assuming there exists an affine plane such that the projection is one-to-one, we consider two coordinate charts defined on the surface and whose image are in a fixed arbitrary affine plane on which the projection is one-to-one. The first coordinate chart is the projection of the ambient surface on the affine plane. Next we assume that the ambient surface can covered by a single isothermal coordinate chart (I will explain exactly what it means shortly). The isothermal coordinates are adjusted so that they identify with the “projection coordinates” when restricted to the contour. We then consider two coordinate transformation maps which relate the projection coordinates to the isothermal coordinates. The affine plane can be arbitrary as long as the projection is one-to-one. The novelty is in having
the two charts, related by transformations: the projection chart which allows to ask for a function whose graph is the surface, and the isothermal chart where the Euler-Lagrange with its boundary conditions are computed and proved. The analysis is coordinate-free and does not depend on the affine plane considered for reference as long as the projection of the optimal surface on the affine plane is one-to-one.

In the numerical part of this study, we employ a heuristic approach: We do not use the full Euler-Lagrange equation, but we take its linear terms which consist of a bilaplacian operator applied to the conformal factor of the metric and that should equal zero. We show in the following how to accurately compute the boundary conditions for this biharmonic equation.

We remark very shortly about the mathematical context of this problem. A related question is the question of prescribing the Gaussian curvature of a surface. This question has been considered in the past. For an open set \( V \subset \mathbb{R}^2 \), and for a function \( u : V \subset \mathbb{R}^2 \to \mathbb{R} \) and a prescribed Gaussian curvature \( K : V \subset \mathbb{R}^2 \to \mathbb{R} \) the equation for \( u \) having curvature \( K \) is the following:

\[
\frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2} = K(x, y) \tag{1}
\]

Considering a general smooth \( K(x, y) \) defined in \( V \), this equation poses difficulties for even proving the existence of a solution. See Kazdan [14], Figalli [9] and references therein. We remark that this is a Monge-Ampère equation as it involves the determinant of the Hessian matrix.

This analysis was done with applications in computer graphics in mind. However, I hope that this analysis and the framework presented will be relevant to other fields in science and engineering and may also inspire related mathematical research.

## 3 Preliminary definitions and assumptions

We start with the following definitions.

**Definition 1.** A surface is a dimension 2 regular submanifold of \( \mathbb{R}^3 \). A surface inherits a Riemannian metric from the Euclidean metric on \( \mathbb{R}^3 \). We will regard a surface as a Riemannian manifold with this induced metric. (See [23], [24].)

**Definition 2.** We say that a surface \( S \) has the graphicality property with respect to an affine plane \( H \), if the projection of \( S \) on \( H \) is one-to-one.

**Definition 3.** A surface \( S \) which can be covered with one local coordinates patch is homeomorphic to \( \mathbb{R}^2 \). Denote this homeomorphism by \( \phi \). If \( \Gamma \) is a simple closed curve on \( S \) then we can use the Jordan curve theorem to define the interior of \( S \) with respect to \( \Gamma \) as the preimage under \( \phi \) of the interior component in \( \mathbb{R}^2 \) divided by \( \phi(\Gamma) \). We denote it by \( \tilde{S}_\Gamma \). We let \( S_\Gamma = \tilde{S}_\Gamma \cup \Gamma \).

Let \( \Gamma \) be a smooth, simple, closed curve in \( \mathbb{R}^3 \). Throughout the text \( \Gamma \) will denote such a curve. We sometimes regard \( \Gamma \) as a subset of \( \mathbb{R}^3 \) and sometimes as a parametrized curve. We use the notion of isothermal coordinates, which means a local chart on a surface for which the metric is of the form: \( e^{2f}(dx^2 + dy^2) \) for a smooth real function \( f \) defined on the image of the chart. We now make reasonable assumptions about \( \Gamma \). The first assumption is intended to deal with the fact that we will do our analysis on abstract 2-manifolds admitting an isothermal chart, and we do not currently know if they can always be smoothly embedded in \( \mathbb{R}^3 \). Here are the assumptions (which we call Property (1) and (2)):

1. There exists an open surface \( \tilde{S} \) containing \( \Gamma \), which can be covered with one isothermal coordinates chart with minimum total Gaussian curvature squared on the interior component of the surface with respect to \( \Gamma \). Denote this chart by \( (\tilde{S}, \phi) \). The minimum is taken over all the two-dimensional Riemannian manifolds which can be covered with one isothermal coordinates chart, such that the image of their charts contains, as a subset, the image under \( \phi \) of the interior component of \( \tilde{S} \) with respect to \( \Gamma \), that is \( \phi \left( \tilde{S}_\Gamma \right) \). We also require that the preimage of \( \phi(\Gamma) \) under their charts is a geodesic. In addition, for each manifold considered, the metric should agree with the metric induced on \( \Gamma \subset \tilde{S} \) by \( \mathbb{R}^3 \) at the preimage under \( \phi \) of a point in \( \phi(\Gamma) \) compared with the preimage under the chart of the manifold considered of the same point. The total
Gaussian curvature is computed on the "interior component" of the two-dimensional manifold under consideration (the preimage under its chart of the original "interior component").

2. There exists an affine plane \( \tilde{H} \) such that \( \tilde{S} \) has the graphicality property with respect to \( \tilde{H} \).

From now on we assume that \( \Gamma \) satisfies Properties (1) and (2).

**Definition 4.** \( \mathcal{E}_\Gamma(S) = \int_{S} K^2 \text{dvol}_g \) where \( g \) is the Riemann metric of \( S \) (inherited from \( \mathbb{R}^3 \)). \( K \) is the Gaussian curvature at each point of \( S \).

If Property (1) holds for \( \Gamma \) then \( \mathcal{E}_\Gamma(\tilde{S}) \leq \mathcal{E}_\Gamma(S) \) for any surface \( S \) which contains \( \Gamma \) as a geodesic and can be covered by one isothermal coordinates chart (see proof of Theorem 1).

### 4 Main analysis

Shortly, we will present the main theorem along with the derivation of the boundary conditions for the PDE controlling the curvature of the optimal surface, but we first give an exposition of the strategy that we use.

We assume that we are given \( \Gamma \) for which Properties (1) and (2) hold. Let \( \tilde{H} \) be an affine plane as in the description of Property (2). The strategy is to use the expression for the Gaussian curvature squared in isothermal coordinates. One big advantage in using isothermal coordinates is that there is a relatively simple expression for the Gaussian curvature, which makes the Euler-Lagrange computation feasible.

Assume for a minute that we know the optimal surface \( \tilde{S} \), and we look at the local coordinates which are given by the projection of \( \tilde{S} \) on \( \tilde{H} \) (identified with \( \mathbb{R}^2 \)). By a change of coordinates we can then assume that in local coordinates the Riemannian metric of the unknown surface is given by: \( g = e^{2f(x,y)}(dx^2 + dy^2) \), where \( f \) is a smooth real-valued function defined on the image of the chart. These coordinates are called isothermal coordinates. Moreover, we will argue that the isothermal coordinates can agree with the projection coordinates at each point of the contour \( \Gamma \), meaning that the chart functions, the projection chart and the isothermal chart, send each point in \( \Gamma \) to the same point in \( \tilde{H} \). In isothermal coordinates the Gaussian curvature is given by \( K(x, y) = -e^{-2f(x,y)} \Delta f(x, y) \), \( \Delta \) being the Laplacian operator (computed by The Theorema Egregium, see [24, p. 90], appears in Prof. Xianfeng David Gu’s lectures notes found online and also in a more general form in [11, Section 3.5: Yamabe equation]).

For the existence of isothermal coordinates locally for the case of a 2-manifold, see [20, p. 135–138], [22, p. 376–378] and the proof by Chern [4], however, we assume the existence of a single global chart.

We then compute the Euler-Lagrange equation for the functional which is the total Gaussian curvature squared, while expressing the curvature using the above formula which can be used when considering an isothermal chart and metric. In applications we will heuristically reduce the Euler-Lagrange equation to a biharmonic equation for the function \( f \) in the exponent of the metric. This function is called the conformal factor of the metric, its domain being the image of the chart covering the surface. The following PDE is the result of the Euler-Lagrange computation for \( \mathcal{E}_\Gamma(S) = \int_{S} K^2 \text{dvol}_g \) (see proof of Thm. 1 for more details):

\[
\begin{pmatrix}
f_{xx}^2 & f_{xx}f_{yy} & f_{xx}^2 & f_{xx}f_{xx} & f_{xx}^2 & f_{xx}f_{yy} & f_{xx}f_{xxx} & f_{xx}f_{xxy} & f_{xxx} & f_{xxx} & f_{xxx}
f_{xx}f_{yy} & f_{xx}f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{xyy} & f_{xx}f_{xyy} & f_{xyy} & f_{xyy}
f_{xx}f_{xx} & f_{xx}f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{xyy} & f_{xx}f_{xyy} & f_{xyy} & f_{xyy}
f_{xx}f_{yy} & f_{xx}f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{xyy} & f_{xx}f_{xyy} & f_{xyy} & f_{xyy}
f_{xxx} & f_{xxx} & f_{xxx} & f_{xxx} & f_{xxx} & f_{xxx} & f_{xxx} & f_{xxx} & f_{xxx} & f_{xxx} & f_{xxx}
f_{xxyy} & f_{xxyy} & f_{xxyy} & f_{xxyy} & f_{xxyy} & f_{xxyy} & f_{xxyy} & f_{xxyy} & f_{xxyy} & f_{xxyy} & f_{xxyy}
f_{xx}f_{xyy} & f_{xx}f_{xyy} & f_{xx}f_{xyy} & f_{xx}f_{xyy} & f_{xx}f_{xyy} & f_{xx}f_{xyy} & f_{xx}f_{xyy} & f_{xx}f_{xyy} & f_{xx}f_{xyy} & f_{xx}f_{xyy} & f_{xx}f_{xyy}
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f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy} & f_{yy}
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f_{xyy} & f_{xyy} & f_{xyy} & f_{xyy} & f_{xyy} & f_{xyy} & f_{xyy} & f_{xyy} & f_{xyy} & f_{xyy} & f_{xyy}
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f_{yy}f_{yy} & f_{yy}f_{yy} & f_{yy}f_{yy} & f_{yy}f_{yy} & f_{yy}f_{yy} & f_{yy}f_{yy} & f_{yy}f_{yy} & f_{yy}f_{yy} & f_{yy}f_{yy} & f_{yy}f_{yy} & f_{yy}f_{yy}
\end{pmatrix} \begin{pmatrix} f_{xx}^2 \\ f_{xx}f_{yy} \\ f_{xx}^2 \\ f_{xx}f_{xx} \\ f_{xx}^2 \\ f_{xx}f_{yy} \\ f_{xx}f_{xxx} \\ f_{xx}f_{xxy} \\ f_{xxx} \\ f_{xxx} \\ f_{xxx} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xxx} \\ f_{xx}f_{xxy} \\ f_{xxx} \\ f_{xxx} \\ f_{xxx} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xxx} \\ f_{xx}f_{xxy} \\ f_{xxx} \\ f_{xxx} \\ f_{xxx} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xxx} \\ f_{xx}f_{xxy} \\ f_{xxx} \\ f_{xxx} \\ f_{xxx} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xxx} \\ f_{xx}f_{xxy} \\ f_{xxx} \\ f_{xxx} \\ f_{xxx} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xxx} \\ f_{xx}f_{xxy} \\ f_{xxx} \\ f_{xxx} \\ f_{xxx} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xyy} \\ f_{xx}f_{xxx} \\ f_{xx}f_{xxy} \\ f_{xxx} \\ f_{xxx} \\ f_{xxx} \end{pmatrix} = 0.
\]
This equation looks intimidating but heuristically in applications, I replace it with the biharmonic equation:

$$f_{xxxx} + 2f_{xxyy} + f_{yyyy} = 0.$$  (3)

Which is obtained by discarding the terms which consist of two factors. This is a heuristic that needs to be checked in the future: One suggestion is to evaluate the discarded sum of terms using the solution to the biharmonic equation. Regardless, the strength of the analysis is in the accurate computation of the boundary conditions for the PDE. The biharmonic equation is an equation which is studied and appears in the literature in the context of clamped elastic thin plates and Stokes flow, see [16] for a review article which gives a historic overview of the topic. The similar context in which it appears in literature also suggests that this heuristic is valid.

The boundary conditions for the PDE constraining the function \( f \) are deduced by arguments which use the fact that the metric is isothermal and the isothermal chart and the projection chart agree on \( \Gamma \) (explained in the proof of Theorem 1). The Dirichlet boundary condition is computed and proved based on the relation at the contour between the isothermal metric and the metric induced by \( \mathbb{R}^3 \). The Neumann boundary condition is computed by a formula which uses the isothermal chart and the fact that \( \Gamma \) is a geodesic. In the next subsection we show how the formula is derived (I do not give a formal proof as some work is needed to make it fully rigorous).

Some preliminary notations: Recall that \( \Gamma \) is the original contour in \( \mathbb{R}^3 \), and let \( \bar{S} \) and \( \bar{H} \) be as in Property (1) and (2). We denote by \( \bar{\Gamma} \) the projection of \( \Gamma \) on the affine plane \( \bar{H} \). Let \( \pi_{\bar{\Gamma}} : \Gamma \to \bar{\Gamma} \) be the projection map. Let \( \gamma(t) \) be a parameterization (arc-length or not) of \( \bar{\Gamma} \) with length \( L \). In order to be able to take a derivative of \( \gamma \) at 0, we regard its domain as \( \mathbb{R}/L\mathbb{Z} \). We denote by \( \phi_{\bar{H}} : \bar{S} \to \mathbb{R}^2 \), the projection of \( \bar{S} \) on \( \bar{H} \) composed with an identification of \( \bar{H} \) with \( \mathbb{R}^2 \). Lastly, let \( \Omega = \phi_{\bar{H}}\left(\frac{\bar{\Gamma}}{\bar{H}}\right) \).

We prove the following theorem,

**Theorem 1.** Given a smooth simple closed curve \( \Gamma \) for which Properties (1) and (2) hold. Let \( \bar{S} \) and \( \bar{H} \) be as in the statement of Property (1) and (2), with respect to \( \Gamma \). Let \( g = e^{2f(x,y)}(dx^2 + dy^2) \) be a Riemannian metric on \( \bar{S} \) for a coordinate chart which maps \( \Gamma \) to \( \bar{\Gamma} \) by projection, then \( f \) satisfies the following PDE with the following boundary equation, where \( \bar{f}_{\Gamma} \) is a function which is approximated in the next subsection:

\[
\text{Equation (2) in } \Omega, \\
f(\gamma(t)) = \frac{1}{2} \cdot \log \left(\|(\pi_{\bar{\Gamma}}^{-1} \circ \gamma)'(t)\|\right), \quad t \in [0, L), \quad \text{on } \partial \Omega, \\
\frac{\partial f}{\partial \nu}(\gamma(t)) = \bar{f}_{\Gamma}(\gamma(t)) \quad \text{on } \partial \Omega. \tag{4}
\]

We state the PDE in \( \Omega \), and explain the Dirichlet and Neumann boundary conditions. For the Dirichlet boundary conditions we give a very concise proof. For the Neumann boundary condition, at the moment we state a heuristic formula with a solid theoretical base.

**Proof (PDE for \( f \) inside the domain).** The volume form for the surface \( \bar{S} \) in the isothermal coordinates is given by \( \sqrt{\det g} \, dx\, dy = e^{2f(x,y)} \, dx\, dy \), therefore the functional to be minimized is:

\[
\int_{\bar{S}} K^2 \text{dvol}_g = \int_{\Omega} e^{-2f(x,y)}(f_{xx} + f_{yy})^2 \, dx\, dy.
\]

Working out the Euler-Lagrange equation yields Equation (2).

**Proof (Dirichlet Boundary Condition).** For any two points \( x, y \in \partial \Omega \), we know the length on \( \bar{S} \) of the preimage of each of the two arcs in \( \partial \Omega \) connecting \( x \) to \( y \). These are the lengths of the segments of \( \Gamma \) connecting \( \pi_{\bar{\Gamma}}^{-1}(x) \) and \( \pi_{\bar{\Gamma}}^{-1}(y) \). The Dirichlet boundary condition is set to agree with this observation.
4.1 Formula for Neumann boundary condition

Assume $S$ is an (open) surface, and let $(S, \phi)$ be a global isothermal chart, assuming such exists. Let $\Gamma$ be a closed geodesic on $S$, we then want to compute the normal derivative of $\phi$, the conformal factor of the metric, on $\phi(\Gamma)$ at $x_0$. We do this by writing $\int_{\phi^{-1}(\text{chord})} ds = \int_{\phi^{-1}(\text{arc})} ds$ and taking the limit $l \to 0$, where $l$ is the length of an arc containing $x_0$ and the chord considered is between the two endpoints of the arc. The arc is an arc of the "osculating circle" of $x_0$ and the chord is a chord of that circle "shortcutting" $x_0$. The length of the preimage in $S$ of the chord is approximately:

$$\int_{\phi^{-1}(\text{chord})} ds \approx c \cdot e^{\left[f(x_0) - \frac{\partial f}{\partial \phi}(x_0) \cdot (r - r \cos \frac{l}{2r})\right]} = 2r \left(\sin \frac{l}{2r}\right) e^{\left[f(x_0) - \frac{\partial f}{\partial \phi}(x_0) \cdot (r - r \cos \frac{l}{2r})\right]},$$

(5)

and the length of the preimage in $S$ of the arc is approximately:

$$\int_{\phi^{-1}(\text{arc})} ds \approx l \cdot e^{f(x_0)}.$$  

(6)

Equating these equations and taking the limit as $l \to 0$, we obtain the following formula:

$$\frac{\partial f}{\partial \hat{n}}(x_0) \approx -\lim_{l \to 0} \frac{\ln \left(\frac{l}{2r \sin \frac{l}{2r}}\right)}{r - r \cos \frac{l}{2r}}.$$  

(7)

I will explain the reasoning for the approximations stated in (5) and (6) using the diagram in Figure 2. The image of a closed geodesic $\Gamma$, which lies on surface $S$, is a closed planar curve. We want to approximate the derivative with respect to the outward normal of the conformal factor $f$ at a point $x_0$, which is on the image of $\Gamma$ (normal pointing away from the image of the interior of $S$ w.r.t. $\Gamma$).

For this point $x_0$ we consider its "osculating circle", a circle which is tangent to the curve at $x_0$ and its radius is $r := 1/R$ where $R$ is the curvature of the curve at $x_0$. We look at chords on this circle which "shortcut" $x_0$. Looking at Figure 2 a chord marked by a dashed line is shown. The angle $\theta$ can be seen easily to be equal to $\frac{l}{2r}$ where $l$ is the length of the arc on the osculating circle corresponding to the chord considered (the circumference of the circle is $2\pi r$, take a portion of size $l$, and multiply by $2\pi$, full angle of a circle); also denote by $c$ the length of the chord. We assume by approximation that the conformal factor $f$ on the arc is equal to $\frac{\partial f}{\partial \phi}(x_0)$ at any point on the arc, and that the conformal factor on the chord is equal to $f(x_0) - \frac{\partial f}{\partial \phi}(x_0) \cdot (r - r \cos \frac{l}{2r})$ by linear approximation as the distance between the point $x_0$ on the arc and the chord is $r - r \cos \frac{l}{2r}$. This is a rough approximation which is used heuristically to approximate the Neumann boundary condition for the conformal factor evaluated at $x_0$ using a formula involving a limit stated in (7).

The logic behind the formula in (7) can be observed when considering a situation in which the normal derivative of the conformal factor is assumed positive. Noting that the conformal factor is smooth by definition and thus continuous. There would be a very small chord (arbitrarily small) such that the conformal factor at all points on the corresponding arc are positive. Heuristically the conformal factor on the chord would be smaller or equal than on the arc when considering a projection of the arc on the chord as a correspondence of points. Together with the fact that the length of the chord is shorter than the length of the arc, this would be a contradiction to the fact that $\Gamma$ is a geodesic on $S$ by comparing the lengths of the curves on $S$ which are the preimages of the chord and arc. We therefore conclude that the normal derivative of the conformal factor at $x_0$ is negative, we then decrease it until there is equality of the lengths of the preimages and we argue heuristically that this is the right value of the conformal factor.

4.2 PDE system for minimal Gaussian curvature surface

We now prove the following theorem which contains the statement for the PDE system for a minimal Gaussian curvature surface spanning a contour $\Gamma$:

**Theorem 2.** Given a smooth simple closed curve $\Gamma$ for which Properties (1) and (2) hold. Let $\bar{S}$ and $\bar{H}$ be as in the statement of the Property (1) and (2), with respect to $\Gamma$. Assume $\bar{H} \subset \mathbb{R}^2$, and let $\Omega$ be
Figure 2: Logic behind Neumann boundary condition formula.

the projection of $\hat{S}_\Gamma$ on $\mathbb{R}^2$. Let $h : \Omega \to \mathbb{R}$ be a function which is $C^\infty$ on $\Omega$ and continuous on $\overline{\Omega}$, such that the graph of $h$ is $\hat{S}_\Gamma$. (By Properties (1) and (2) such a function exists.) Let $f$ be the solution of Equation (4) in Theorem 1. Let $K(x, y) = -e^{-2f(x,y)}\Delta f(x, y)$ be a function on $\overline{\Omega}$. (Note that $x, y$ are the isothermal coordinates, where $u, v$ are the Cartesian coordinates). Let $E(u, v) = 1 + h_u^2(u, v)$, $F(u, v) = h_u(u, v)h_v(u, v)$ and $G(u, v) = 1 + h_v^2(u, v)$. Then there is a coordinate change $x(u, v), y(u, v)$, such that $h, x, y$ satisfy the following equations (for $(u, v) \in \Omega$ in the first three equations):

\[
\frac{h_{uu}h_{vv} - h_{uv}^2}{(1 + h_u^2 + h_v^2)^2} = K(x(u, v), y(u, v)),
\]

\[
\frac{\partial}{\partial v} Fx_u - Ex_v + \frac{\partial}{\partial u} Fx_v - Ex_u = 0,
\]

\[
\frac{\partial}{\partial v} Fy_u - Ey_v + \frac{\partial}{\partial u} Fy_v - Ey_u = 0,
\]

\[
x(u, v) = u \quad \text{on } \partial\Omega,
\]

\[
y(u, v) = v \quad \text{on } \partial\Omega.
\]

Proof. If the isothermal coordinates covering $\hat{S}$, which exist by Property (1), are given by $(\xi, \eta)$, then a coordinate change to new isothermal coordinates $(x, y)$ is given by the following equations: $x_{\xi \xi} + x_{\eta \eta} = 0, y_{\xi \xi} + y_{\eta \eta} = 0$ (see [20, p. 135–138]). By prescribing Dirichlet boundary values agreeing with the conditions in Theorem [4] for these Laplace equations, we can obtain isothermal coordinates $(x, y)$. This shows that isothermal coordinates, which agree with the conditions in Theorem [4], exist.

We assume that $(u, v)$ are the coordinates on $\hat{S}$ obtained by the projection of the surface on $\mathbb{R}^2$. If $h$ is the inverse function of this projection then $E, F, G$ are given by the identities stated. Then the maps $x(u, v), y(u, v)$ are the solutions of the two Laplace-Beltrami equations in the above system with the two Dirichlet boundary conditions. See [20, p. 135–138] for details.

If $f(x, y)$ satisfies Equation [4] then, by Property (1), $K(x, y) = -e^{-2f(x,y)}\Delta f(x, y)$ is the optimal Gauss curvature at $(x, y)$ of $\hat{S}$. The function $h$, whose graph is $\hat{S}_\Gamma$, is the solution of the Monge-Ampère equation stated.

5 Background: experimental part

In the first part we dealt with finding surfaces in $\mathbb{R}^3$, which are as close as possible to being flat and span a given contour such that the contour is a geodesic on the sought surface. We looked for a surface which minimizes the total Gaussian curvature squared. We then showed that by a change of coordinates the curvature of the optimal surface is controlled by a PDE which can be reduced, heuristically in applications, to the biharmonic equation with an easy-to-define Dirichlet boundary condition and Neumann boundary condition which can be approximated using an explicit formula. We then stated a system of PDEs for the function whose graph is the optimal surface. This result
allows us to consider each cell in the net of geodesics separately. Each cell in the net can be regarded as a contour. It is not smooth as we assumed in the first part of this study, as it has corners. However, it can be observed that the case of a polygonal contour is handled perfectly by the analysis for the boundary conditions in the first part. We remark that treating each cell separately allows a great simplification of the task of finding the complete smooth surface.

Works regarding the computation of surfaces with geodesic boundary curves appeared before in the context of CAD research. Most notably is [21], which had in mind a similar problem to the one that we are dealing with. The authors were looking to be able to process geodesic data acquired by a device that they are using. In that work they dealt with only two non-intersecting curves. Their work was later followed by [6, 7, 8]. All of these works consider Coons patches and Hermite interpolation in order to interpolate an unknown surface bounded by geodesics. In [6] an energy minimization approach is used but the choices of energy differ from the energy that I am using and the method is for constructing triangular patches which can be limiting in the applications considered here. My choice of energy intuitively encodes the tension of the surface spanning the bounding contour, where minimizing it corresponds to the natural inclination of many organic or synthetic materials which are inclined to stay as flat as possible, but allow stretching.

This part includes relevant numerical experiments which were performed throughout the study. I also describe an algorithm to construct an approximately optimal smooth surface when given a net of curves satisfying the following condition at the intersections. The condition is that the acceleration vectors of each two intersecting curves should be parallel at the intersection point. This is a necessary condition for the curves to be geodesics on a surface as the acceleration vector should be parallel to the normal of the surface everywhere for geodesics curves. (The smoothness of the result is based on the assumption that there is one smooth surface minimizing the considered energy and spanning the given net of curves such that the curves are geodesics on it.) For each cell, the algorithm finds an approximate solution for the PDE system presented in Eq. 5. This is done by considering a cell’s contour, heuristically fixing a plane such that the projection of the unknown minimal Gaussian curvature surface spanning the contour is with high probability one-to-one, solving a biharmonic equation for the conformal factor for the approximate optimal curvature by using a finite element method, and then solving the curvature Monge-Ampère equation for a function whose graph over the chosen affine plane has the prescribed curvature at each point and its trace equals the cell’s contour. Note that in the first part of this study we work in isothermal coordinates (coordinates for which the surface’s metric can be written as $ds^2 = e^{2f}(dx^2 + dy^2)$, where $f$ is a smooth real function on the parameterization domain) and we do not know how to compute the coordinate change maps from isothermal coordinates to “projection on a plane” coordinates. Inputting the approximate optimal curvature in Cartesian coordinates seems to work well and is based on the assumption that the isothermal chart is not too "wild" near the bounding contour. In addition, we are able to evaluate the error between the computed function and the solution of Eq. 5. This can be done by applying a discrete Laplace-Beltrami operator to evaluate the result. This will potentially allow a refinement of the net of geodesics in regions where the Laplace-Beltrami operator applied to each coordinate function of the projection (on the plane), with respect to the computed surface, is significantly larger in absolute value from zero. In addition, we can check if near the given curves, at the seam of two cells, the error is low. We plan to to implement this in a subsequent work which will put the emphasis on the implementation side of the method.

The most important contribution of this work is its prospects in applications in CAD interfaces, geometric modelling in architecture and industrial design, sketch-to-3d technology and more. In its very basic application, this method tackles the basic challenge of generically creating a smooth surface spanning given non-trivial closed contours in 3d, as shown in the two videos found on the arXiv page for this part of the study under Ancillary files. Here are the two clickable links to the videos:

https://arxiv.org/src/2109.01429v2/anc/example1.mp4
https://arxiv.org/src/2109.01429v2/anc/example2.mp4

6 The full PDE of a cell in the net

We consider a single cell in the net of geodesics and its contour $\Gamma$ in $\mathbb{R}^3$. We need to heuristically fix an affine plane for which the projection of the minimal Gaussian curvature surface spanning $\Gamma$ would be one-to-one. We then have a PDE system for the functions $h, x, y$, real functions defined on the
Figure 3: Computed minimal Gaussian curvature surfaces. Neumann boundary condition is naively set to zero allowing a simpler computing approach compromising accuracy.

region on that plane that is bounded by the projection of $\Gamma$. $\tilde{K}$ is the approximate optimal curvature that we compute after choosing an affine plane, its domain is the same as of $h, x, y$. We are specifically interested in the function $h$ (the height function with respect to the affine plane), whose graph over its domain is the minimal Gaussian curvature surface spanning $\Gamma$. In particular, its trace is $\Gamma$. The
PDE system was stated in the first part and is the following:

{\begin{align*}
\frac{h_{uu}h_{vv} - h^2_{uv}}{(1 + h^2_u + h^2_v)^2} &= \tilde{K}(x(u, v), y(u, v)), \\
\frac{\partial}{\partial v} \sqrt{E - G} + \frac{\partial}{\partial u} \sqrt{E - G} &= 0, \\
\frac{\partial}{\partial v} \sqrt{F - G} + \frac{\partial}{\partial u} \sqrt{F - G} &= 0, \\
x(u, v) &= u \quad \text{on } \partial \Omega, \\
y(u, v) &= v \quad \text{on } \partial \Omega.
\end{align*}}

(9)

Note that \( E, F, G \) are the elements of the first fundamental form of the surface we are looking for and therefore depend on \( h \). \( \Omega \) is the domain of \( h(x, y) \) and as mentioned, is a subset of some chosen affine plane. Note that the projection of \( \Gamma \) will be different for different choices of planes, and so will the boundary conditions considered in the next section, however the approximate computed optimal curvature will be similar due to the theory in first part of this study.

\( \tilde{K} \), the approximate optimal curvature, is a solution for the linear part of the Euler-Lagrange equation for the integral of the square of the Gauss curvature on the surface. It is constructed by solving a biharmonic equation \( \Delta^2 f = 0 \) with appropriate boundary conditions. We refer to \( f \) as the conformal factor of the corresponding Riemannian metric, \( e^{2f}(dx^2 + dy^2) \). We are prescribing the Gauss curvature given by \( \tilde{K}(x, y) = -e^{-2f(x, y)} \Delta f(x, y) \).

The analysis which yields the above derivation relies heavily on the usage of isothermal coordinates on surfaces. For related theory see [20] and [22].

7 Solving the biharmonic equation

We consider a single cell in the net of curves. One needs to guess an affine plane for which the minimal Gaussian curvature surface spanning the cell’s contour, can be projected one-to-one on. Heuristically, one can define the affine plane by any choice of three corner points of the cell. Let \( \partial \Omega \) be the projection of the cell’s contour on the chosen affine plane, and let \( \Gamma \) be the cell’s contour. Let \( \gamma : \partial \Omega \rightarrow \Gamma \) be a parameterization of the cell’s contour which is the inverse of the projection on the affine plane. As shown in the first part, the Dirichlet boundary condition of the biharmonic equation for the optimal curvature is given by \( \frac{1}{2} \log \| (\gamma \circ p(t))' \| \) at point \( p(t) \in \partial \Omega \) for a parameterization (arc-length or not) of \( \partial \Omega \). The Neumann boundary condition should be computed according the formula we have shown before.

For experimenting, we considered contours constructed by modulating a periodic function such as a sine or a cosine, on the unit circle in the \( x - y \) plane (we also take the \( x - y \) plane to be the reference affine plane). For these type of curves, one can compute the derivative and therefore have the Dirichlet boundary condition for the biharmonic equation for the conformal factor. In this part of the numerical experimentation I naively took the Neumann boundary condition to be zero. (In Figure 3 under each sub-figure we state the function modulated on the circle, which was used to form the contour.) We therefore had a well-defined biharmonic problem in the plane for each curve, that we had to solve. For each periodic function \( z(t) \), modulated on the unit circle, the Dirichlet boundary condition is given by \( \frac{1}{2} \log \sqrt{1 + z'^2(t)} \).

In order to solve such an equation we used FreeFEM++ [13]. One needs to put the PDE (biharmonic, in our case) in a variational formulation. I looked for a formulation which would also yield the Laplacian for the solution, as both the solution and its Laplacian are needed in order to compute the curvature. Such a formulation was given in the Ciarlet-Raviart approach to biharmonic problems [5], which is also due to Mercier [17]. In future implementations it would be better to use more modern approaches such as the \( C^0 \)-Interior Penalty method [2] or \( C^1 \)-Elements. The mixed methods formulation used here do not ensure convergence to the solution in non-convex domains. In this formulation it is also challenging to impose non-zero Neumann boundary condition.

Ciarlet-Raviart and Mercier’s formulation dealt with homogeneous boundary conditions, but by applying an elementary modification to their approach, one can consider the two unknowns \( u \) and \( \omega \).
(that will come out be equal to \(-\Delta u\)). We are looking for \(u \in H^1_g\) and for \(\omega \in H^1\) such that:

\[
\int_{\Omega} \nabla u \cdot \nabla \psi = \int_{\Omega} \omega \psi \quad \forall \psi \in H^1
\]

\[
\int_{\Omega} \nabla \omega \cdot \nabla \upsilon = \int_{\Omega} f \upsilon \quad \forall \upsilon \in H^1_0
\]

Generally \(f\) can be a non-zero function when looking for \(\Delta^2 u = f\). In our case, \(f = 0\). Note that now the particular solution (say, \(u_g\)) needed to construct \(H^1_g\) as \(H^1_g = u_g + H^1_0\), is easy to generate (just assigning the value of \(g\) at the boundary nodes and zero at internal nodes).

8 The curvature Monge-Ampère equation

In the previous section we showed how to obtain the approximate optimal curvature of the minimal Gauss curvature surface spanning a geodesic contour. We also remark that we input the curvature as if it were given in Cartesian coordinates, which may also lead to inaccuracies, however we can evaluate the result. We elaborate on this in the end of this section and in the next section.

We input the curvature and the contour to a finite-difference solver, which solves for a function which satisfies the Monge-Ampère curvature equation, and its trace is the inputted contour. The solver we used was generously made available to us by Brittany Froese Hemfeldt. The method used by the solver is described in [10] in general context and also utilized in Gaussian curvature context in [12]. It provided very good results with convergence of the finite-difference scheme to very low values dependent on the specified number of iterations. In addition, the method is meshfree - there are no structural constraints on the data points. A survey of methods for solving numerically Monge-Ampère equations can be found in the introduction section of [18].

Explicitly, the solver solves numerically the following equation:

\[
\frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2} = K(x, y).
\]

The solver’s input is the curvature on different nodes inside a domain, and the trace of \(h\) (the values of \(h\) on points on the boundary of the domain). Error estimates and empirical tests for the solver can be found in [10].

Note that in running the solver we inputted the curvature function found in Section 7 with no coordinate transformation. Thus we regarded the curvature as given in Cartesian coordinates. These are the same coordinates used for solving the biharmonic equation as described in the previous section. This is not compatible with the theory in the first part, which is also mentioned in Section 6, but is used to compute an approximation of the target surface. The underlying assumption is that the isothermal chart is close to the projection chart near the contour based on how we related the two chart in the first part of this study.

See Figure 4 for computations of minimal Gaussian surfaces for two nets of curves, each consisting of two cells. Note that the condition at the intersections does not hold, but it is a nice qualitative visual demonstration of the smoothness of the generated surface, even when the condition does not hold. The Neumann boundary condition is taken to be zero which is in accordance to the theory.

9 Applying the Laplace-Beltrami operator

Following the construction of a surface for a single cell, we may use the Laplace-Beltrami operator as described in [1] to obtain a value at each vertex of the mesh corresponding to the constructed surface. This part has not been experimented with yet, but the theory from the first part indicates that this should be a good evaluator of the approximation.

The method described in [1] is suitable for closed discrete surfaces or for surfaces with boundary, whose boundary is a geodesic. It is therefore suitable in our case. The recent work by Burman et al. [3], which is a finite element approximation of the Laplace-Beltrami operator on a surface with boundary, can be used alternatively.
The definition of the discrete Laplace-Beltrami that we used, is Definition 16 in [1]. (The weights defined are the same as the original weights defined in the classic work by Pinkall and Polthier [19], but assumes a preprocessing step of an intrinsic Delaunay triangulation of the mesh). The Laplace-Beltrami operator is given by $\Delta_g f : V \rightarrow \mathbb{R}^n$ at $x_i \in V$ (the set of vertices of the mesh):

$$\Delta_g f(x_i) = \sum_{x_j \in V : (x_i, x_j) \in E} \nu(x_i, x_j)(f(x_i) - f(x_j)), $$

where $E_D$ is the edge set of a Delaunay triangulation of $S$ and the weights are given by

$$\nu(x_i, x_j) = \begin{cases} \frac{1}{2} (\cot \alpha_{ij} + \cot \alpha_{ji}) & \text{for interior edges} \\ \frac{1}{2} \cot \alpha_{ij} & \text{for boundary edges} \end{cases}$$

The angles $\alpha_{ij}$ and $\alpha_{ji}$ are the $\alpha$-angles of an internal edge. Refer to Figure 1 in [1]. The metric $g$ is implicit, and only used for notation purposes.

When applying the discrete Laplace-Beltrami operator with respect to the constructed surface on coordinate projection functions $\pi_x, \pi_y$ with respect to any plane such that the projection is one-to-one (in particular, the reference affine plane chosen), we would hope to observe uniformly close to zero values on the vertices consisting of the discrete surface near the contour. If the surface constructed is an approximate minimal Gauss curvature surface, this would be consistent with the defining PDE system, Equations 9. It is unrealistic to expect that the operator will yield values close to zero everywhere on the surface as that would imply that a coordinates chart given by a projection map of a surface on the plane, is always an isothermal coordinates chart (see [22, 20] for a reference regarding the theory of isothermal coordinates).

10 Test case: the Neumann boundary condition formula

The test case considered in this section is the computation of the curvature for a minimal Guassian curvature (squared) surface spanning a circle in an affine plane in $\mathbb{R}^3$. In addition to minimizing the square of the Gauss curvature, the circle contour should be a geodesic on the target surface. Due to Gauss-Bonnet theorem the integral of the curvature of the surface spanning the circle contour (or any other smooth contour) such that it is a geodesic is equal to $2\pi$.

The PDE for the approximate conformal factor of the metric in this case is a two-dimensional homogeneous biharmonic equation inside a circle. The Dirichlet boundary condition can be seen to be zero (as the contour can be projected on the plane containing the circle), and the Neumann boundary condition is computed according to the formula in [4]. It is the same for all points on the circle, and it computes to be approximately -0.3636. After solving the PDE for $f$ we can compute the curvature according to the formula $K = -e^{2f}\Delta f$. Let $C = \{x^2 + y^2 \leq 1\}$ . Then we can compute $\int_S Kdvol_S = \int_C K \cdot e^{2f}dxdy = -\int_C \Delta f \cdot dxdy$ which should be equal approximately to $2\pi$. This should
be true even in the case that the minimizing 2-manifold of the considered energy is not embeddable in \( \mathbb{R}^3 \) as long as there exists a 2-manifold with a geodesic which is mapped under some chart to a circle such that the distance between any two points is equal to the length of the shorter arc connecting their images.

In Figure 5 you can observe the result of solving the homogeneous biharmonic equation with Dirichlet boundary condition zero everywhere, and Neumann boundary condition set to -0.3636 everywhere on the circle. This was done with a python script which uses the \( C^0 \)-Interior Penalty implementation by the NGSolve package. The script can be found here: [https://github.com/tomgilat/SolveBiharmonic](https://github.com/tomgilat/SolveBiharmonic). It also include the computation of the Laplacian of the solution.

The results are not perfect but I find them satisfactory at this point. I would like in the future to use \( C^1 \)-Elements which I hope would improve the results dramatically. The total curvature is computed to be 2.2891 which is far from \( 2\pi \), however the curvature near the boundary is 0.755 which is relatively close to 1 as in the case of a half-sphere spanning the contour. In a sense, this is an extreme example where no Dirichlet boundary conditions are involved and the minimizing surface is far from being flat. In view of that I find the result fairly good, however I would definitely try to use other numerical methods to check if I can get better results.

11 Method Summary

The intention of this section is to describe a complete potential method to create smooth surfaces from nets of geodesic curves. The theory and basic implementations of small scale examples presented in this paper, imply that such a system is feasible.

Given a net of curves in \( \mathbb{R}^3 \) (smooth or discretized), we treat each cell in the net independently. We perform the following process for each cell in the net. Looking at a specific cell, the method finds an affine plane such that the projection of the cell’s contour is one-to-one on that affine plane. Let \( \gamma(t) \) be an arc-length parameterization of the projection of the cell’s contour. Let \( z(\cdot) \) be the height function.
on the image of $\gamma$, such that $h(t) := z(\gamma(t))$ is the contour. One computes numerically $h'(\cdot)$, and uses it to define the Dirichlet boundary conditions for the curvature’s biharmonic equation as according to the first part of this study. Explicitly the Dirichlet boundary condition is $\frac{1}{2} \log \sqrt{1 + h'(t)^2}$ at $\gamma(t)$. These values are computed numerically on nodes along the boundary $\gamma(\cdot)$. Alternatively, if one has an analytic expression for the velocity of the cell’s contour segments, this can be used instead of the numerical computation.

The biharmonic equation described in the first part is solved by using the Ciarlet-Raviart/Mercier approach with a finite elements method. The Dirichlet boundary conditions are as described in the above paragraph. The Neumann boundary condition is computed according to our formula. The solution to the biharmonic equation is the conformal factor $f$. Use $C^0$-Interior Penalty or $C^1$-elements or a finite difference method to compute the solution, and then compute the Laplacian of the solution. We then have the approximate optimal curvature of the minimal Gaussian curvature surface spanning the cell’s contour - it is given by $K(x, y) = -e^{-2f(x,y)}\Delta f(x, y)$.

We then input the curvature we found and the cell’s contour to a solver that solves for a function that its graph is a surface with the specified curvature and specified trace. We regard the curvature as given in Cartesian coordinates.

We then apply a discrete Laplace-Beltrami operator with regards to the computed cell’s surface on the coordinates projection functions $\pi_x$ and $\pi_y$ (projecting to the chosen affine plane). If the application of the operator on the two functions, yields uniformly close to zero values, then we are done. If not, one suggestion is to do the following: we split the cell into two cells. This is done by finding a geodesic on the computed surface (running from two points chosen by a clever heuristic on the original cell’s contour, for example), which together with the original contour defines two new contours. We discard the original surface and compute new approximately optimal surfaces for the new contours.

12 Acknowledgments

I wish to thank Prof. Michel Bercovier from the Hebrew University for introducing me to FreeFEM++, and discussing different related topics with me.

I wish to thank Prof. Franco Brezzi for suggeting the Raviart-Ciarlet/Mercier formulation for the biharmonic equation, and its adaptation to non-homogeneous Dirichlet boundary condition.

I am deeply grateful and indebted to Prof. Brittany Froese Hemfeldt for her interest in the project, and most importantly her willingness to share her Matlab code with me.

I wish to thank Prof. Peter Monk for advising me to look at the C0IP method.

I wish to thank Prof. Michael Neilan for his amazing help with using the C0IP methods and with its theoretical aspects.

13 Data availability

No data was used in this study.

14 Disclosures

The author declares no conflicts of interest.

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