A Massive Quasi-normal Mode in the Holographic Lifshitz Theory

Chanyong Park

Institute for the Early Universe, Ewha womans University, Daehyun 11-1, Seoul 120-750, Korea

ABSTRACT

We investigate the holographic renormalization of the Einstein-Maxwell-dilaton theory which provides an asymptotic Lifshitz geometry dual to a Lifshitz field theory. In this case, the existence of a field combination with zero scaling dimension causes an ambiguity in fixing local counter terms. Nevertheless, we show that all possible local counter terms give rise to consistent thermodynamic quantities with the Lifshitz black brane results. In addition, we also study the retarded Green functions of the current and momentum operator of a non-relativistic Lifshitz field theory. In the non-zero momentum regime, the results show intriguingly that there exists a massive quasi-normal mode whose effective mass is linearly proportional to temperature and that even at zero temperature there exists a quasi-normal mode in the non-relativistic Lifshitz medium.

e-mail : cyong21@ewha.ac.kr
1 Introduction

After the Maldacena’s conjecture [1, 2, 3], there were a lot of works to understand strongly interacting
gauge theories via the dual classical gravity models. Recently, those works were further generalized to the
hyperscaling violation geometry in which the asymptotic geometry is deviated from the AdS space and
the boundary conformal symmetry is broken [4]-[15]. The study on the hyperscaling violation geometry is
an interesting topic to understand the gauge/gravity duality in depth and at the same time to apply the
holographic techniques to the real QCD [16, 17, 18, 19, 20, 21], nuclear matter [22, 23, 24, 25] or condensed
matter system [26, 27, 28, 29, 30, 31].

In the gauge/gravity duality point of view, one of the interesting field contents on the gravity side is a
nontrivial dilaton profile because it is dual to a running gauge coupling of the dual gauge theory. In addition,
it would be possible to understand the nontrivial RG flow of the strongly interacting systems through the
holographic renormalization in the corresponding classical gravity theory [32, 33, 34, 35]. There are several
gravity models including a nontrivial dilaton field. One is a relativistic non-conformal geometry which is the
vacuum solution of the Einstein-dilaton theory with a Liouville-type dilaton potential [4]-[15]. In this case, a
nontrivial dilaton profile breaks the scaling symmetry of the metric without breaking the boundary Lorentz
symmetry, so the dual theory represents a relativistic non-conformal field theory [11, 12, 13, 14, 15]. The
other interesting example is the Lifshitz geometry which was invented to understand the non-relativisitic
features by using the holographic techniques. There were several gravity models leading to the asymptotic
Lifshitz geometry, for examples, the gravity theory with various higher form fields [36] or with a massive
vector field [37, 38]. In this paper, we will concentrate on another model, the so-called Einstein-Maxwell-
dilaton theory, in which a bulk gauge field as well as a nontrivial dilaton profile are introduced [39, 40].

In an asymptotic AdS background without a dilaton field, a time component of a bulk gauge field is dual
to the number operator of matter in the dual field theory and its geometry can be described by the thermal
charged AdS for the confining phase at low temperature or the Reissner-Nordström black brane for the
deconfining phase at high temperature [22, 23, 24, 25]. In an asymptotic Lifshitz geometry, although there
exists a bulk gauge field, it does not provide a new hair to the black brane solution. Instead, it together
with the nontrivial dilaton field changes the asymptotic geometry to the Lifshitz one. Furthermore, the
anisotropic scaling symmetry of the Lifshitz geometry appears due to the breaking of the boost symmetry
between time and spatial coordinates caused by the background gauge field. As a result, a general solution
of the Einstein-Maxwell-dilaton theory is given by a Schwarzschild-type black brane with the Lifshitz-type
scaling symmetry. Its dual field theory can be reinterpreted as a gauge theory with Lifshitz matter whose
dispersion relation is governed by the Lifshitz-type field theory. In this case, since the boundary value of the
gauge field is not a free parameter, the density (or the chemical potential) of Lifshitz matter is fixed by the
intrinsic parameters of the Einstein-Maxwell-dilaton theory which may be reinterpreted as a microcanonical
ensemble.

In order to understand the thermodynamic properties and their RG flow, it is interesting to study the
holographic renormalization. The holographic renormalizations of other models describing the asymptotic
Lifshitz geometry have been already investigated [37, 38, 41, 42]. Here, we concentrate on the holographic
renormalization of the Einstein-Maxwell-dilaton theory. In this model, the local counter terms are not uniquely fixed due to the existence of a combination whose leading term at the asymptotic boundary has a zero scaling dimension. When we consider several lowest order counter terms, we find that all possible combinations in the local counter terms give rise to the same thermodynamics consistent with the Lifshitz black brane result. If we further regard the RG flow, the fact that the local counter term is not unique implies that the RG flow can not be also determined uniquely. Unfortunately, we do not have a clear idea yet how to resolve this problem. It may require more deep understanding about the microscopic aspects of the dual Lifshitz theory. We leave it as a future work.

In the zero momentum limit of the dual Lifshitz theory, the electric properties carried by Lifshitz matter and impurity were studied \[39, 40\]. Here, we further investigate the transport coefficients carried by Lifshitz matter in the non-zero momentum limit of the non-relativistic Lifshitz theory \((z = 2)\). Due to the existence of the background gauge field, a charge current is usually mixed with a momentum operator. We explicitly calculate the retarded Green functions of a charge current and momentum operator in the hydrodynamic limit. The results show that the shear viscosity is linearly proportional to temperature and saturates the lower bound of \(\eta/s\) \[39\]. Interestingly, the retarded Green functions of the non-relativistic Lifshitz theory show the existence of a massive quasi-normal mode which has a thermal effective mass linearly proportional to temperature and a constant momentum diffusion constant.

The rest of paper is organized as follows. In Sec. 2, we rederive the thermodynamics of the Lifshitz black brane by using the holographic renormalization. Although the local counter terms are not determined uniquely, we show that all possible counter terms lead to the consistent thermodynamics with the black brane result. On this background, we further investigate the transport coefficients carried by a Lifshitz matter in Sec. 3. Intriguingly, the current-current and momentum-momentum retarded Green functions show a massive quasi-normal mode whose mass is linearly proportional to temperature. In Sec. 4, we finish this work with some concluding remarks.

2 A holographic renormalization of a Lifshitz theory

Let us start with briefly summarizing the Lifshitz black brane geometry, its thermodynamics and our notations with a Lorentzian signature. The action for an Einstein-Maxwell-dilaton theory with a negative cosmological constant \(\Lambda\) is given by \[4, 39, 40\]

\[
S_{EMd} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\lambda \phi} F_{\mu \nu} F^{\mu \nu} \right),
\]

which is believed to be the dual gravity of a Lifshitz-type field theory. The Einstein and Maxwell equation are given by

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + g_{\mu\nu} \Lambda = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} g_{\mu\nu} (\partial \phi)^2 + \frac{1}{2} e^{\lambda \phi} F_{\mu \lambda} F^{\nu \lambda} - \frac{1}{8} g_{\mu\nu} e^{\lambda \phi} F^2,
\]

\[
\partial_\mu (\sqrt{-g} \partial^\mu \phi) = \frac{\lambda}{4} \sqrt{-g} e^{\lambda \phi} F^2.
\]
and the equation of motion for dilaton is

$$0 = \partial_{\mu}(\sqrt{-g}e^{\lambda \phi} F^{\mu \nu}). \quad (4)$$

These equations allow a Schwarzschild-type black brane solution

$$ds^2 = -r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2(dx^2 + dy^2),$$

$$\phi(r) = -\frac{4}{\lambda} \log r,$$

$$F_{rt} = \partial_r A_t = q r^{z+1}, \quad (5)$$

with

$$f(r) = 1 - \frac{r_h^{z+2}}{r^{z+2}},$$

$$\lambda = \frac{2}{\sqrt{z-1}},$$

$$q = \sqrt{2(z-1)(z+2)},$$

$$\Lambda = -\frac{(z+1)(z+2)}{2}, \quad (6)$$

where $r_h$ denotes an event horizon and $z$ is a dynamical exponent. We simply call it the Lifshitz black brane. Its Hawking temperature from the surface gravity is

$$T_H = \frac{z+2}{4\pi r_h}, \quad (7)$$

and the Bekenstein-Hawking entropy becomes

$$S_{BH} = \frac{V_2}{4G} r_h^{z+2}, \quad (8)$$

where $V_2$ means a spatial volume of the boundary space.

The Lifshitz black brane thermodynamics can be reinterpreted as that of the dual Lifshitz field theory following the gauge/gravity duality. From the thermodynamic law together with information of the Hawking temperature and Benkenstein-Hawking entropy, one can easily read other thermodynamic quantities. The internal energy $E$ and the free energy $F$ are

$$E = \frac{V_2}{8\pi G} r_h^{z+2},$$

$$F = -\frac{zV_2}{16\pi G} r_h^{z+2}. \quad (9)$$

In addition, the pressure is given by $P = -\partial F/\partial V_2$ which satisfies the Gibbs-Duhem relation, $E + PV_2 = T_H S_{BH}$.

The above thermodynamic results of the dual Lifshitz theory are just reinterpretation of the Lifshitz black brane geometry. Using the holographic renormalization method [14, 43, 44, 45], it was shown in a non-conformal geometry as well as an asymptotic AdS geometry that the boundary stress tensor gives rise
to the same thermodynamics as the black brane solution. Furthermore, as was shown in the different models for the Lifshitz geometry \cite{37, 41}, it is also true in the Einstein-Maxwell-dilaton gravity theory. These facts indicate that the assumption of the gauge/gravity duality is self-consistent even in the asymptotic Lifshitz geometry. In this paper, following the gauge/gravity duality assumption we will show that the holographic renormalization of the Einstein-Maxwell-dilaton gravity can reproduce the same thermodynamics of the Lifshitz black brane. To do so, we should first assume that similar to the AdS case the on-shell gravity action can be mapped to a boundary term proportional to the free energy of the dual field theory even in the non-AdS space. In this procedure, an appropriate holographic renormalization is required to remove UV divergences. After the holographic renormalization, we will show that the boundary stress tensor leads to the consistent thermodynamics in spite of an ambiguity in choosing the local counter terms.

From now on, we use the Euclidean signature \cite{14} because it is more convenient for later comparison. The Euclidean action can be easily obtained by applying the Wick rotation $t \rightarrow -i\tau$, in which the time component of the gauge field should be also rotated like $A_t \rightarrow iA_\tau$. Then, the Euclidean Einstein-Maxwell-dilaton action can be rewritten as

$$S_{EMd} = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^4 x \sqrt{g} \left( R - 2\Lambda - \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi - \frac{1}{4} e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu} \right),$$

where $g_{\mu\nu}$ is the Euclidean metric. Since the Euclidean action has the same as (10) up to an overall minus sign, all equations of motion are also the same as the previous ones in (2), (3) and (4). By explicit calculation, one can easily check that the Einstein and Maxwell equations are really satisfied by the Euclidean metric

$$ds^2 = r^{2z} f(r) \, d\tau^2 + \frac{dr^2}{r^2 f(r)} + r^2(dx^2 + dy^2),$$

and the Wick-rotated time component gauge field

$$A_\tau = -\frac{i}{z + 2} (r^{z+2} - r_h^{z+2}),$$

where the last constant term is introduced for a well-defined norm of $A_\tau$ at the black brane horizon.

For a well-defined metric variation of the action, the Gibbons-Hawking term must be added

$$S_{GH} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3 x \sqrt{\gamma} \, \Theta,$$

where $\Theta$ is an extrinsic curvature scalar and $\gamma_{ab}$ is an induced metric on the boundary. An extrinsic curvature tensor $\Theta_{\mu\nu}$ is defined by

$$\Theta_{\mu\nu} = -\frac{1}{2} (\nabla_\mu n_\nu + \nabla_\nu n_\mu),$$

where $\nabla_\mu$ and $n_\nu$ denote a covariant derivative and unit normal vector respectively. Since the Gibbons-Hawking term is a boundary term, it does not affect on the equations of motion. Furthermore, one can also add an additional boundary term called the Neumannizing term \cite{37, 46}

$$S_N = -\frac{c_N}{16\pi G} \int_{\partial\mathcal{M}} d^3 x \sqrt{\gamma} \, e^{\lambda\phi} \, n^\mu A_\nu F_{\mu\nu},$$
which determines the boundary condition of the bulk gauge field. For \( c_N = 0 \), the gauge field satisfies the Dirichlet boundary condition, whereas \( c_N = 1 \) when the Neumann boundary condition is imposed. Here, we choose a Dirichlet boundary condition, so our starting action for the Lifshitz theory is

\[
S = S_{EMd} + S_{GH}.
\] (16)

Although this action has a well-defined metric variation, it still suffers from the UV divergence at \( r_0 \to \infty \), where \( r_0 \) denotes the UV cutoff or the position of boundary. In addition, the boundary terms caused by the variations with respect to other matter fields can also yield the UV divergences. These divergences can be removed by the holographic renormalization with appropriate local counter terms. What are the correct local counter terms? Following the gauge/gravity duality, the renormalized on-shell gravity action is proportional to the free energy of the dual field theory and the boundary energy-momentum tensor derived from it can be identified with that of the dual field theory. Therefore, the renormalized action and its boundary energy-momentum tensor should be independent of the UV cutoff introduced by hand and finite even at \( r_0 \to \infty \).

The on-shell action without local counter terms at the UV boundary, after inserting solutions to (16), leads to

\[
S_{on} = \beta V_2 \left( -\frac{(1 + z)}{8\pi G} r_0^{z+2} + \frac{z}{16\pi G} r_h^{z+2} \right) + \mathcal{O}\left( \frac{1}{r_0^{z+2}} \right),
\] (17)

where \( \beta \) is the Euclidean time periodicity proportional to the inverse temperature and \( V_2 \) is a regularized spatial volume of the boundary space. The variation of the on-shell action with respect to the boundary metric gives rise to the boundary energy-momentum tensor

\[
T^a_b = -\frac{1}{8\pi G} \int d^2x \sqrt{\gamma} \gamma^{ac} (\Theta_{cb} - \gamma_{cb}\Theta),
\] (18)

where \( a \) and \( b \) means the boundary coordinates. Note that there is no contribution from the kinetic terms of the dilaton and vector field. The explicit energy and pressure of the dual Lifshitz theory read

\[
E = T^\tau_\tau = -\frac{V_2}{4\pi G} r_0^{z+2} + \frac{V_2}{4\pi G} r_h^{z+2} + \mathcal{O}\left( \frac{1}{r_0^{z+2}} \right),
\]

\[
P_i = \frac{T^i}{V_2} = \frac{(1 + z)}{8\pi G} r_0^{z+2} - \frac{z}{16\pi G} r_h^{z+2} + \mathcal{O}\left( \frac{1}{r_0^{z+2}} \right),
\] (19)

where an index \( i \) implies the spatial direction. It is worth to note that the free energy given by \( F = \frac{S_{on}}{\beta} \) is exactly the same as \(-P_i V_2\). The above results are unrenormalized ones so that they suffer from the UV divergences at \( r_0 \to \infty \), as mentioned before. Since the pressure is proportional to the free energy, at least two local counter terms are required to make the energy-momentum tensor finite at the UV cutoff.

Similar to the energy-momentum tensor, the variations with respect to the other matter fields also suffer from the UV divergences. Varying the unrenormalized on-shell action with respect to the matter fields gives rise to

\[
\frac{\partial S_{on}}{\partial \phi} = -\frac{\sqrt{z - 1}\beta V_2}{8\pi G} r_0^{z+2} + \frac{\sqrt{z - 1}\beta V_2}{8\pi G} r_h^{z+2} + \cdots,
\]

\[
\frac{\partial S_{on}}{\partial A_\tau} = -\frac{i\sqrt{z - 1}\sqrt{z + 2}\beta V_2}{8\sqrt{2\pi G}} + \cdots,
\] (20)
where the ellipses imply the higher order corrections which vanish for \( r_0 \rightarrow \infty \). For the well-defined variations and finiteness, these terms should be also removed by three more counter terms. As a result, the renormalization of the on-shell action requires five counter terms if all constraints for eliminating the UV divergences are independent. Actually, only three counter terms, as will be shown, are sufficient for the renormalization due to the redundancies of the constraints. Then, what kinds of the local counter term are possible?

In order to remove the UV divergences, the counter terms having the same leading divergence should be taken into account. There exist infinitely many possible counter terms generating the same divergence

\[
S_{\text{possible}} = \sum_{i=0}^{\infty} \frac{c_i}{8\pi G} \int d^3 x \sqrt{\gamma} \left( e^{\lambda \phi} A^2 \right)^i,
\]

where \( \sqrt{\gamma} \sim r_0^{z+2} \) and the leading term of \( e^{\lambda \phi} A^2 \) is given by a constant. Let us first assume that all constraints are independent, then the renormalized action contains at most five counter terms as previously mentioned. Since infinite many counter terms are possible, there exists an ambiguity in choosing five counter terms. Here, we simply choose the five lowest order counter terms

\[
S_{ct} = \sum_{i=0}^{4} \frac{c_i}{8\pi G} \int d^3 x \sqrt{\gamma} \left( e^{\lambda \phi} A^2 \right)^i.
\]

where \( c_i \) are coefficients to be determined. If adding more counter terms, the coefficients of them are not uniquely fixed because of the lack of constraints. Similarly, if the previous constraints are not independent, all coefficients in (22) can not be fixed exactly. This fact also generates the same ambiguity in choosing appropriate counter terms. As will be shown subsequently, only three constraints in the Lifshitz geometry are independent so that two coefficients can not be determined fully. In order to get rid of these two redundancies, one can simply set two of them in (22) to be zero. It should be noted that \( c_0 \) must have a non-vanishing value because the on-shell action of the AdS space \( (z = 1) \) is renormalized only by \( c_0 \). As a result, due to the non-vanishing \( c_0 \) there are six possibilities in choosing three counter terms.

Here, to show the redundancies explicitly let us start with five counter terms. The resulting renormalized action with an Euclidean signature is described by

\[
S_{\text{ren}} = S_{EMd} + S_{GH} + S_{ct},
\]

where the counter terms is given by (22). After substituting the solutions, (11) and (12), into the renormalized action and expanding it near the UV cutoff, the following five constraint equations are derived

\[
0 = -2z^4 - 16z^3 - 48z^2 - 64z - 32 + (z + 2)^4 c_0 + 2(z + 2)^3 (z - 1) c_1 - 12 \left( z^2 + z - 2 \right)^2 c_2
\]
\[
+ 40(z + 2)(z - 1)^3 c_3 - 112(z - 1)^4 c_4,
\]

\[
0 = -z^5 - 9z^4 - 32z^3 - 56z^2 - 48z - 16 + (z + 2)^4 c_0 - 2(z - 1)(z + 2)^3 c_1 + 4 \left( z^2 + z - 2 \right)^2 c_2
\]
\[
- 8(z - 1)^3 (z + 2) c_3 + 16(z - 1)^4 c_4,
\]

\[
0 = -z^4 - 8z^3 - 24z^2 - 32z - 16 - 4(z + 2)^3 c_1 + 16(z - 1)(z + 2)^2 c_2 - 48 \left( z^3 - 3z + 2 \right) c_3
\]

\[
0 = -z^5 - 9z^4 - 32z^3 - 56z^2 - 48z - 16 + (z + 2)^4 c_0 - 2(z - 1)(z + 2)^3 c_1 + 4 \left( z^2 + z - 2 \right)^2 c_2
\]
\[
- 8(z - 1)^3 (z + 2) c_3 + 16(z - 1)^4 c_4,
\]

\[
0 = -z^4 - 8z^3 - 24z^2 - 32z - 16 - 4(z + 2)^3 c_1 + 16(z - 1)(z + 2)^2 c_2 - 48 \left( z^3 - 3z + 2 \right) c_3
\]

6
\begin{align}
+128(z - 1)^3 c_4, \\
0 &= z^4 + 8z^3 + 24z^2 + 32z + 16 + 6(z + 2)^3 c_1 - 40(z - 1)(z + 2)^2 c_2 + 168(z^3 - 3z + 2) c_3 \\
&\quad - 576(z - 1)^3 c_4, \\
0 &= -z^4 - 8z^3 - 24z^2 - 32z - 16 - 4(z + 2)^3 c_1 + 16(z - 1)(z + 2)^2 c_2 - 48(z^3 - 3z + 2) c_3 \\
&\quad + 128(z - 1)^3 c_4. 
\end{align}

In the above, the first two equations describe vanishing of the divergence in energy and pressure. The third and fourth equations come from the well-defined variation with respect to the dilaton field. The remaining is the condition for the $A_{r}$ variation. Since the third constraint in (26) is the same as the fifth constraint in (28), all constraints are not independent. Furthermore, there exist another redundancy because the above constraints automatically satisfy the following relation

$$0 = (24) - (25) + (z - 1)(26).$$

Therefore, as mentioned before, only three of them are independent. This means that three counter terms are sufficient in the holographic renormalization of the Lifshitz theory.

Which coefficients can we set to be zero? This is an important question to fix the counter terms uniquely. However, at the present stage unfortunately we have no concrete idea for choosing three of them. In this paper, instead of resolving the uniqueness problem of the counter terms, we will investigate whether the physical properties of the dual theory crucially depend on the choice of the counter terms or not. In what follows, we summarize six different parameter solutions, which are only allowed cases when one starts with five lowest counter terms

1) For $c_3 = c_4 = 0$,

$$c_0 = \frac{13 + 3z}{8}, \quad c_1 = -\frac{3(2 + z)}{8}, \quad c_2 = -\frac{(z + 2)^2}{32(z - 1)}, \quad c_3 = 0, \quad \text{and} \quad c_4 = 0.$$  

2) For $c_2 = c_4 = 0$,

$$c_0 = \frac{1}{12}(5z + 19), \quad c_1 = -\frac{5}{16}(z + 2), \quad c_2 = 0, \quad c_3 = \frac{(z + 2)^3}{192(z - 1)^2}, \quad \text{and} \quad c_4 = 0.$$  

3) For $c_1 = c_4 = 0$,

$$c_0 = \frac{1}{8}(5z + 11), \quad c_1 = 0, \quad c_2 = \frac{5(z + 2)^2}{32(z - 1)}, \quad c_3 = \frac{(z + 2)^3}{32(z - 1)^2}, \quad \text{and} \quad c_4 = 0.$$  

4) For $c_2 = c_3 = 0$,

$$c_0 = \frac{1}{16}(7z + 25), \quad c_1 = -\frac{7}{24}(z + 2), \quad c_2 = 0, \quad c_3 = 0, \quad \text{and} \quad c_4 = -\frac{(z + 2)^4}{768(z - 1)^3}.$$  

5) For $c_1 = c_3 = 0$,

$$c_0 = \frac{1}{32}(21z + 43), \quad c_1 = 0, \quad c_2 = \frac{7(z + 2)^2}{64(z - 1)}, \quad c_3 = 0, \quad \text{and} \quad c_4 = -\frac{3(z + 2)^4}{512(z - 1)^3}.$$
6) For $c_1 = c_2 = 0$,

$$c_0 = \frac{1}{48}(35z + 61), \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = \frac{7(z + 2)^3}{96(z - 1)^2}, \quad \text{and} \quad c_4 = \frac{5(z + 2)^4}{256(z - 1)^3}. \quad (35)$$

Note that for $z = 1$, $A_r$ automatically vanishes so that only $c_0$ is required for the renormalization. In this case, $c_0$ reduces to 2 which is the case for the $AdS_4$ space [43]. In general cases, the internal energy and pressure depending on five coefficients read from the boundary energy-momentum tensor

\begin{align*}
E &= T^\tau_\tau \\
&= \frac{(z + 2) \left\{ 4(z + 2)^2 - (z + 2)^2c_0 - 6(z^2 + z - 2) c_1 + 60(z - 1)^2 c_2 \right\} - 280(z - 1)^3 c_3}{16\pi G(z + 2)^4} r_h^{z+2} \\
&\quad + \frac{1008(z - 1)^4 c_4}{16\pi G(z + 2)^4} r_h^{z+2}, \\
P_i &= -\frac{T^i_i}{V_2} \\
&= \frac{(z + 2) \left\{ -z(z + 2)^2 + (z + 2)^2c_0 - 6(z^2 + z - 2) c_1 + 20(z - 1)^2 c_2 \right\} - 56(z - 1)^3 c_3}{16\pi G(z + 2)^4} r_h^{z+2} \\
&\quad + \frac{144(z - 1)^4 c_4}{16\pi G(z + 2)^4} r_h^{z+2}. \quad (36)
\end{align*}

Using the above results given in (30) ~ (35), the resulting thermodynamic quantities, the free energy and energy-momentum tensor at $r_0 = \infty$, lead to the same result in all cases

\begin{align*}
F &= \frac{zV_2}{16\pi G} r_h^{z+2}, \\
E &= \frac{V_2}{8\pi G} r_h^{z+2}, \\
P_i &= \frac{z}{16\pi G} r_h^{z+2}, \quad (37)
\end{align*}

which are perfectly matched to the black brane thermodynamics in (19). These results show that although the local counter terms are not fixed uniquely, the holographic renormalization of the Lifshitz black brane leads to the correct boundary energy-momentum tensor of the dual Lifshitz theory. Furthermore, the holographic renormalization shows the self-consistency of the gauge/gravity duality even in the asymptotic Lifshitz geometry.

### 3 A massive quasi-normal mode in the non-relativistic theory

In the hydrodynamics of the relativistic quantum field theory, it was well-known that a momentum diffusion constant can be represented by the background thermodynamic quantities

$$D = \frac{\eta}{\epsilon + P}, \quad (38)$$

In the context of the non-relativistic theory, the diffusion constant $D$ can be expressed in terms of the shear viscosity $\eta$ and the energy density $\epsilon$.
where \( \eta, \epsilon \) and \( P \) are the shear viscosity, energy density and pressure. This result has been checked in the dual conformal field theory of the asymptotic AdS space by using the holographic hydrodynamics \[27, 28, 29, 30, 31\]. Furthermore, it was also shown that this relation is true even in the relativistic non-conformal theory dual to a non-AdS geometry \[14\]. However, we cannot expect that the above momentum diffusion constant is still valid in the asymptotic Lifshitz geometry because of the breaking of the Lorentz symmetry. Therefore, it is interesting to calculate the momentum diffusion constant of the dual Lifshitz theory. In this section, we will investigate the holographic hydrodynamics of the Lifshitz geometry, especially the non-relativistic case \((z = 2)\).

In general, if there exists a nonzero background gauge field, the shear mode of the metric fluctuation, \( g_{tx}^x \) and \( g_{ty}^y \), and the transverse mode of the gauge field fluctuation \( a_x \) are usually coupled to each other. In order to evaluate the momentum diffusion constant of the non-relativistic Lifshitz theory, one should turn on the gauge and metric fluctuations simultaneously. Now, let us expand all fluctuations as Fourier modes

\[
a_x(t,y,r) = \int \frac{d\omega}{(2\pi)^2} e^{-i\omega t + iky} a_x(\omega, k, r),
\]

\[
g_{tx}^x(t,y,r) = \int \frac{d\omega}{(2\pi)^2} e^{-i\omega t + iky} g_{tx}^x(\omega, k, r),
\]

\[
g_{ty}^y(t,y,r) = \int \frac{d\omega}{(2\pi)^2} e^{-i\omega t + iky} g_{ty}^y(\omega, k, r),
\]

(39)

For a general \( z \), shear modes are governed by

\[
0 = \frac{\omega}{r^{2z-2}} g_{tx}^x + k f g_{ty}^y + \frac{q \omega}{r^{z+3}} a_x,
\]

(40)

\[
0 = g_{tx}^{xx} + \frac{(5 - z)}{r} g_{tx}^x - \frac{k^2}{r^4 f} g_{tx}^x - \frac{k\omega}{r^4 f} g_{ty}^y + \frac{q}{r^{5-z}} a_x',
\]

(41)

\[
0 = \frac{g_{ty}^{xx}}{r f} + \frac{(z + 3) f}{r f} g_{ty}^y + \frac{\omega^2}{r^{2z+2}} g_{tx}^x + \frac{k\omega}{r^{2z+2}} g_{ty}^y + \frac{k}{r^2 f} g_{tx}^x,
\]

(42)

and the Maxwell equation for \( a_x \) is given by

\[
0 = a_{xx}'' + \frac{r f'}{r f} a_x' + \frac{r^2 \omega^2 - k^2 r^{2z} f^2}{r^{2z+4} f^2} a_x + \frac{q r^{3-z}}{f} g_{tx}^x.
\]

(43)

Note that for \( z = 1 \), since \( q = 0 \), the transverse mode is decoupled from shear modes. In equations of the shear modes, only two of them are independent because combining (40) and (41) leads to the rest one (42).

From now on we concentrate on the \( z = 2 \) case because its dual theory is described by a non-relativistic quantum field theory. Combining (40), (41) and (43) leads to

\[
0 = \Phi'' + \left( \frac{f'}{f} + \frac{7}{r} \right) \Phi' + \left( \frac{\omega^2}{r^6 f^2} + \frac{3 f'}{r f} - \frac{k^2}{r^4 f} - \frac{q^2}{r^2} + \frac{9}{r^2} \right) \Phi
\]

\[
+ \frac{q}{r^2} \left( 2a_x' + \frac{k^2}{r^3 f} a_x \right),
\]

(44)

where \( \Phi = g_{tx}^x \), and (43) reduces to

\[
0 = a_{xx}'' + \left( \frac{f'}{f} - \frac{1}{r} \right) a_x' + \left( \frac{\omega^2}{r^6 f^2} - \frac{k^2}{r^4 f} \right) a_x + \frac{q r}{f} \Phi.
\]

(45)
Near the event horizon, the leading solutions satisfying the incoming boundary condition read

\[ a_x(r_h) = \frac{f^{-1} \delta(k)}{4r_h}, \]
\[ \Phi(r_h) = \frac{G_0}{f^{-1} \delta(k)}, \] (46)

where \( F_0 \) and \( G_0 \) are two integration constants. Furthermore, in the hydrodynamic limit \( (\omega \sim k^2 \ll T_H) \), the perturbative solutions near the horizon can be expanded into

\[ a_x(r_h) = \frac{f^{-1} \delta(k)}{4r_h} \left( F_0 \delta(k) + \omega F_1(r) \delta(k) + k^2 F_2(r) + \cdots \right), \]
\[ \Phi(r_h) = \frac{G_0}{f^{-1} \delta(k)} \left( G_0 \delta(k) + \omega G_1(r) \delta(k) + k^2 G_2(r) + \cdots \right), \] (47)

where \( \delta(k) \) implies the zero momentum mode. The zero momentum modes do not coupled to \( g^y_x \). Solving (44) and (45) perturbatively, the solutions \( F_0 \) and \( G_0 \), which are regular at the horizon, are given by

\[ F_0 = \left( r^4 + r_h^4 \right) c_1, \]
\[ G_0 = -\frac{2\sqrt{2}}{r^3} \left( r^4 + r_h^4 \right) c_1, \] (48)

where \( c_1 \) is an undetermined integration constant. In order to satisfy (48) at the horizon, higher order solutions should be vanishing as well as regular. The solutions satisfying these constraints lead to

\[ F_1(r) = -\frac{i c_1}{4r_h^2} \left[ 2 \left( r^4 + r_h^4 \right) \left( 2 \log r + \log 2 \right) - \left( r^2 - r_h^2 \right)^2 - 2 \left( r^4 + r_h^4 \right) \log \left( r^2 + r_h^2 \right) \right], \]
\[ G_1(r) = \frac{ic_1}{\sqrt{2} r_h^3} \left[ 2 \left( r^4 + r_h^4 \right) \left( 2 \log r + \log 2 \right) - \left( r^2 - r_h^2 \right)^2 - 2 \left( r^4 + r_h^4 \right) \log \left( r^2 + r_h^2 \right) \right], \]
\[ F_2(r) = -\frac{c_1}{32r_h^2} \left[ \pi \left( r^4 + r_h^4 \right) + 2 \left( r^4 + 3r_h^4 \right) \left( r^4 - r_h^4 \right) + 2 \left( r^4 + r_h^4 \right) \right] \left\{ \text{Li}_2 \left( \frac{r^4}{r_h^4} \right) - 4\text{Li}_2 \left( \frac{r^2}{r_h^2} \right) \right\} \]
\[ -8 \log \left( \frac{r}{r_h} \right) \left\{ \left( r^4 + r_h^4 \right) \log \left( r^4 - r_h^4 \right) + 2r^2 r_h^2 \right\}, \]
\[ G_2(r) = \frac{c_1}{8\sqrt{2} r_h^3} \left[ \pi \left( r^4 + r_h^4 \right) + 2 \left( r^4 - r_h^2 \right)^2 + 2 \left( r^4 + r_h^4 \right) \right] \left\{ \text{Li}_2 \left( \frac{r^4}{r_h^4} \right) - 4\text{Li}_2 \left( \frac{r^2}{r_h^2} \right) \right\} \]
\[ -8 \log \left( \frac{r}{r_h} \right) \left\{ \left( r^4 + r_h^4 \right) \log \left( r^4 - r_h^4 \right) + 2r^2 r_h^2 \right\}, \] (49)

where \( \text{Li} \) denotes a polylogarithm function. The zero momentum modes \( F_0(r) \) and \( F_1(r) \) coincide with the results obtained in the zero momentum limit (40).

Near the horizon, the remaining \( g^x_y \) can be again written as sum of the zero and nonzero momentum mode

\[ g^x_y = \frac{f^{-1} \delta(k)}{4r_h} \left( H_0 \delta(k) + \omega H_1(r) \delta(k) + H_2(\omega, k, r) + \cdots \right). \] (50)

The zero momentum modes are governed by

\[ 0 = \psi'' + \left( \frac{f'}{f} + \frac{5}{r} \right) \psi' + \frac{\omega^2}{r^2 f^2} \psi. \] (51)
where $\psi(\omega, r)$ denotes the zero momentum modes, $\psi = f^{-i \frac{\omega}{4r_h^2}} (H_0 + \omega H_1(r) + O(\omega^2))$. This zero momentum mode is decoupled from others and determines the shear viscosity of the dual system. The regularity and vanishing condition fix $H_1 = 0$, so the zero momentum mode up to $\omega$ order becomes

$$\psi = d_1 f^{-i \frac{\omega}{4r_h^2}},$$

(52)

where $H_0 = d_1$ is an integration constant which will be determined by other boundary condition. The nonzero momentum mode $H_2$ is usually coupled to $g^x_t$ and $a_x$, which can be determined from (40) to be

$$H_2 = c_1 \omega k \int \frac{dr}{r_h} \int_r^{r_h} \frac{f^{-i \frac{\omega}{4r_h^2}}}{\sqrt{2} r (r^2 + r_h^2)} + \text{cc},$$

(53)

where cc is another integration constant. Due to the vanishing condition of $H_2$ at the horizon, cc should be zero. Although the analytic form of $H_2$ can not be fixed, one can still find a perturbative expansion form in the hydrodynamic limit which is sufficient to determine the hydrodynamic coefficient (see below).

Now, let us investigate the asymptotic behavior of solutions. Assuming the asymptotic forms of solutions as

$$a_x = A r^\alpha \quad \text{and} \quad \Phi = B r^\beta,$$

(54)

(44) and (45) are reduced to

$$0 = \alpha(\alpha - 2) A r^{\alpha - 2} + qB r^{\beta + 1},$$

$$0 = 2qA r^{\alpha - 5} + (\beta^2 + 6\beta + 9 - q^2) B r^{\beta - 2}.$$

(55)

In order to have nontrivial solutions with non-vanishing $A$ and $B$, $\beta$ should be $\beta = \alpha - 3$ which determines $\alpha$ as

$$\alpha = 1 \pm \sqrt{1 + q^2},$$

(56)

where $q^2 = 8$ for $z = 2$ and $B$ is related to $A$

$$B = -\frac{\alpha(\alpha - 2)}{q} A.$$

(57)

These results show that the asymptotic behaviors of $a_x$ and $g^x_t (= \int dr \Phi)$ are described by

$$a_x = a_0 r^4 (1 + \cdots) + \frac{a^*_0}{r^4} (1 + \cdots),$$

$$g^x_t = g_{t0} r^2 (1 + \cdots) + \frac{B g_{t0}^*}{r^4} (1 + \cdots),$$

(58)

where $a_0 = A$, $g_{t0} = B/2$ and ellipsis implies the higher order corrections. From (51), the asymptotic form of the zero momentum mode in $g^x_y$ can be easily determined to be

$$\psi = g_{y0} (1 + \cdots) + \frac{g_{y0}^*}{r^4} (1 + \cdots).$$

(59)
The asymptotic behavior of the nonzero momentum mode can be fixed by inserting the asymptotic solutions in (58) into the constraint equation (40), which leads

\[ H_2 = \tilde{C} - \frac{c_1 \omega k}{2\sqrt{2}r^2}, \]

where \( \tilde{C} \) is another integration constant. Since \( H_2 \) is the nonzero momentum mode, \( \tilde{C} \) should be zero. This result is consistent with (53) in the asymptotic region. In sum, the asymptotic behaviors of fluctuations are

\[ a_x \sim a_0 \ r^4, \]
\[ g_{t}^{x} \sim g_{t0} \ r^2, \]
\[ g_{y}^{x} \sim g_{y0}, \]

where coefficients, \( a_0, g_{t0} \) and \( g_{y0} \), correspond to sources of each fluctuations and should be determined by appropriate asymptotic boundary conditions. Comparing the near horizon solutions in (47) and (50) with the asymptotic behaviors of fluctuations in (61), these coefficients are related to undetermined parameters, \( c_1 \) and \( d_1 \),

\[ a_0 = \frac{c_1}{32r_h^2} \left( 32r_h^2 - 8i (2 \log 2 - 1) \omega + (\pi^2 - 2) k^2 \right), \]
\[ g_{t0} = -\frac{c_1}{16\sqrt{2}r_h^2} \left( 32r_h^2 - 8i (2 \log 2 - 1) \omega + (\pi^2 - 2) k^2 \right), \]
\[ g_{y0} = d_1. \]

These asymptotic behaviors of solutions are totally different from the AdS ones. On the AdS background, as mentioned previously, \( a_0 \) is independent with \( g_{t0} \) and \( g_{y0} \) because of the decoupling of the vector fluctuation from the others. The \( z = 2 \) Lifshitz geometry has a different situation in which the asymptotic value of \( g_{y}^{x} \) is independent from others while \( a_0 \) is proportional to \( g_{t0} \). This fact, as will be shown, leads to totally different retarded Green functions.

In order to evaluate the retarded Green function, we first evaluate the boundary term from the bulk action which reduces to

\[ S_B = \frac{1}{16\pi G} \int d^3 x \left( -r^5 g_{y0} g_{y}^{x'} + r^5 g_{t0} g_{t}^{x'} - r^3 a_0 a_x' \right). \]

From the boundary term together with (62), after discarding the contact terms, the resulting retarded Green functions of the non-relativistic Lifshitz theory read

\[ \langle J^x J^x \rangle = \frac{r_h^4 \left( 192i r_h^2 + (16r_h^2 + 192r_h^2 \log r_h - 27i \omega) k^2 \right)}{72\pi G \left( 32r_h^2 - 8i \omega (2 \log 2 - 1) + (\pi^2 - 2) k^2 \right)}, \]
\[ \langle T_t^x T_t^x \rangle = \frac{r_h^4 \left( 192i r_h^2 - (80r_h^2 - 192r_h^2 \log r_h + 9i \omega) k^2 \right)}{72\pi G \left( 32r_h^2 - 8i \omega (2 \log 2 - 1) + (\pi^2 - 2) k^2 \right)}, \]
\[ \langle T_y^x T_y^x \rangle = \frac{i \omega r_h^2}{16\pi G}, \]
\[ \langle J^x T_y^y \rangle = \frac{\sqrt{2}\omega k \ r_h^4}{\pi G \left( 32r_h^2 - 8i \omega (2 \log 2 - 1) + (\pi^2 - 2) k^2 \right)}, \]
\[
\langle T^y_x T^y_x \rangle = -\frac{\omega k r_h^4}{\pi G \left[ 32 r_h^2 - 8 i \omega (2 \log 2 - 1) + (\pi^2 - 2) k^2 \right]},
\]
\[
\langle J^x T^x_x \rangle = 0.
\] (64)

From the retarded Green function of the tensor mode \( T^y_x \), the shear viscosity is given by
\[
\eta \equiv \lim_{\omega, k \to 0} \frac{\langle T^y_x T^y_x \rangle}{i \omega} = \frac{r_h^2}{16 \pi G},
\] (65)

which shows the consistent result with the zero momentum calculation. Using the fact that the entropy density \( s \) is given by \( r_h^2 / 4G \) in (8), the shear viscosity of the dual non-relativistic Lifshitz theory saturates the lower bound of \( \eta/s \geq 1/4\pi \) [47]. As shown in [40], the DC conductivity leads to
\[
\sigma \equiv \lim_{\omega, k \to 0} \frac{\langle J^x J^x_x \rangle}{i \omega} = \frac{r_h^4}{12 \pi G}.
\] (66)

In terms of Hawking temperature, these results can be rewritten as
\[
\eta = \frac{1}{16 G} T_H,
\]
\[
\sigma = \frac{\pi}{12 G} T_H^2,
\] (67)

which shows that the shear viscosity and DC conductivity increases with Hawking temperature linearly and quadratically.

Interestingly, the above retarded Green functions indicate the existence of a massive quasi-normal mode whose dispersion relation is deviated from an usual Drude formula \( i \omega = D k^2 \). First, the current-current and momentum-momentum correlator have the same pole structure. Secondly, this quasi-normal mode has the following dispersion relation
\[
i \omega = \frac{4 r_h^2}{2 \log 2 - 1} + \frac{\pi^2 - 2}{8(2 \log 2 - 1)} k^2.
\] (68)

If one ignore the temperature-dependent part \( (T_H \sim r_h^2) \), it reduces to an usual Drude formula. In this case, the momentum diffusion constant is given by a constant which is the expected result due to the same scaling behavior of \( \omega \) and \( k^2 \) [40]. The ignored term is a new one which does not appear in the holographic dual of the relativistic (conformal) field theory. For understanding this result further, we rewrite the above dispersion relation as a Shr"{o}dinger-type equation
\[
i \left( i \frac{\partial}{\partial t} \right) \psi(t, y) = -\mathcal{D} \frac{\partial^2}{\partial y^2} \psi(t, y) + m \, \psi(t, y),
\] (69)

with
\[
\mathcal{D} = \frac{\pi^2 - 2}{8(2 \log 2 - 1)};
\]
\[
m = \frac{4 \pi T_H}{2 \log 2 - 1},
\] (70)

where an additional imaginary number on the left hand side denotes the instability of the wave function. Here we can identify \( m \) with an effective thermal mass, because it linearly depends on Hawking temperature,
and $D$ with a charge or momentum diffusion constant which is independent of Hawking temperature. Solving the Shr"odinger equation, the wave function describing a quasi-normal mode at a given temperature reads

$$\psi(t, y) \sim e^{-\left(m + Dk^2\right)t}e^{iky},$$

which allows to reinterpret a quasi-normal mode as a massive decaying mode. This unstable mode dissipates more rapidly with increasing temperature and momentum. Note that the hydrodynamic calculation is valid only in the range of $\omega \sim k^2 \ll T_H$. Another interesting thing in the non-relativistic Lifshitz medium is that there exists a quasi-normal mode even at zero temperature which follows the usual Drude formula.

4 Discussion

We have studied the holographic renormalization of the Einstein-Maxwell-dilaton theory, which is the dual of the Lifshitz field theory, and its hydrodynamics. For the finiteness of the boundary action and energy-momentum tensor, we have introduced appropriate local counter terms. There are infinitely many possibilities which make the on-shell gravity action finite. In this paper, we started with five lowest order counter terms. After imposing the non-vanishing value to $c_0$, which is the counter term for the AdS geometry, we investigated the holographic renormalization of the dual Lifshitz field theory. In this case, due to the redundancies of the constraints five coefficients are not fully determined. Although we can get rid of such redundancies by setting two of them to be zero by hand, there still remains an ambiguity in choosing them. In our set-up, six cases are possible and all cases, when the boundary is located at infinity, give rise to the same thermodynamics consistent with the Lifshitz black brane thermodynamics. If we further take into account the holographic renormalization flow of this system, since IR physics is usually depend on the local counter terms, the IR theory can not be uniquely determined without additional constraints. Those additional constraints would be related to the details of the dual Lifshitz theory or the renormalization scheme dependence of the field theory. So it would be interesting to study further the microscopic aspects of the dual Lifshitz field theory and its renormalization.

We also investigated the hydrodynamics of the non-relativistic Lifshitz theory ($z = 2$). In the nonzero momentum limit, the transverse mode of the gauge field couples to the shear mode of the metric fluctuation due to the nonzero background gauge field. This fact leads to nontrivial retarded Green functions between the current and momentum operator. In the non-relativistic medium, interestingly the current-current, current-momentum and momentum-momentum correlators show a massive quasi-normal mode unlike the relativistic cases, which is slightly deviated from the usual Drude formula. Its mass is linearly proportional to temperature and the momentum diffusion constant is independent of temperature. Moreover, this quasi-normal mode still remains even at zero temperature with the usual Drude formula, $i\omega = Dk^2$. It would be interesting to study whether the other Lifshitz models also generate a similar massive quasi-normal mode.

Acknowledgement
This work has been supported by the WCU grant no. R32-10130 and the Research fund no. 1-2008-2935-001-2 by Ewha Womans University. C. Park was also supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (NRF-2013R1A1A2A10057490).

References

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998) [hep-th/9802109].

[3] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) [hep-th/9802150].

[4] M. Taylor, arXiv:0812.0530 [hep-th].

[5] K. Goldstein, S. Kachru, S. Prakash and S. P. Trivedi, JHEP 1008, 078 (2010) [arXiv:0911.3586 [hep-th]].

[6] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis and R. Meyer, JHEP 1011, 151 (2010) [arXiv:1005.4690 [hep-th]].

[7] K. Goldstein, N. Iizuka, S. Kachru, S. Prakash, S. P. Trivedi and A. Westphal, JHEP 1010, 027 (2010) [arXiv:1007.2490 [hep-th]].

[8] X. Dong, S. Harrison, S. Kachru, G. Torroba and H. Wang, JHEP 1206, 041 (2012) [arXiv:1201.1905 [hep-th]].

[9] L. Huijse, S. Sachdev and B. Swingle, Phys. Rev. B 85, 035121 (2012) [arXiv:1112.0573 [cond-mat.str-el]].

[10] S. Sachdev and M. Mueller, J. Phys. Condens. Matter 21, 164216 (2009) [arXiv:0810.3005 [cond-mat.str-el]].

[11] B. -H. Lee, S. Nam, D. -W. Pang and C. Park, Phys. Rev. D 83, 066005 (2011) [arXiv:1006.0779 [hep-th]].

[12] B. -H. Lee, D. -W. Pang and C. Park, JHEP 1007, 057 (2010) [arXiv:1006.1719 [hep-th]].

[13] S. Kulkarni, B. -H. Lee, C. Park and R. Roychowdhury, JHEP 1209, 004 (2012) [arXiv:1205.3883 [hep-th]].

[14] C. Park, Adv. High Energy Phys. 2013, 389541 (2013) [arXiv:1209.0842 [hep-th]].

[15] S. Kulkarni, B. -H. Lee, J. -H. Oh, C. Park and R. Roychowdhury, JHEP 1303, 149 (2013) [arXiv:1211.5972 [hep-th]].
[16] J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, Phys. Rev. Lett. 95, 261602 (2005) [hep-ph/0501128].
[17] T. Sakai and S. Sugimoto, Prog. Theor. Phys. 113, 843 (2005) [hep-th/0412141].
[18] D. T. Son and A. O. Starinets, JHEP 0209, 042 (2002) [hep-th/0205051].
[19] G. Policastro, D. T. Son and A. O. Starinets, JHEP 0209, 043 (2002) [hep-th/0205052].
[20] G. Policastro, D. T. Son and A. O. Starinets, JHEP 0212, 054 (2002) [hep-th/0210220].
[21] P. Kovtun, D. T. Son and A. O. Starinets, JHEP 0310, 064 (2003) [hep-th/0309213].
[22] B.-H. Lee, C. Park and S.-J. Sin, JHEP 0907, 087 (2009) [arXiv:0905.2800 [hep-th]].
[23] C. Park, Phys. Rev. D 81, 045009 (2010) [arXiv:0907.0064 [hep-ph]].
[24] C. Park, Phys. Lett. B 708, 324 (2012) [arXiv:1112.0386 [hep-th]].
[25] B.-H. Lee, S. Mamedov, S. Nam and C. Park, JHEP 1308, 045 (2013) [arXiv:1305.7281 [hep-th]].
[26] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, Phys. Rev. Lett. 101, 031601 (2008) [arXiv:0803.3295 [hep-th]].
[27] S. A. Hartnoll, Class. Quant. Grav. 26, 224002 (2009) [arXiv:0903.3246 [hep-th]].
[28] C. P. Herzog, J. Phys. A 42, 343001 (2009) [arXiv:0904.1975 [hep-th]].
[29] J. McGreevy, Adv. High Energy Phys. 2010, 723105 (2010) [arXiv:0909.0518 [hep-th]].
[30] G. T. Horowitz, Lect. Notes Phys. 828, 313 (2011) [arXiv:1002.1722 [hep-th]].
[31] S. Sachdev, Lect. Notes Phys. 828, 273 (2011) [arXiv:1002.2947 [hep-th]].
[32] E. P. Verlinde and H. L. Verlinde, JHEP 0005, 034 (2000) [hep-th/9912018].
[33] M. Bianchi, D. Z. Freedman and K. Skenderis, Nucl. Phys. B 631, 159 (2002) [hep-th/0112119].
[34] K. Skenderis, Class. Quant. Grav. 19, 5849 (2002) [hep-th/0209067].
[35] I. Heemskerk and J. Polchinski, JHEP 1106, 031 (2011) [arXiv:1010.1264 [hep-th]].
[36] S. Kachru, X. Liu and M. Mulligan, Phys. Rev. D 78, 106005 (2008) [arXiv:0808.1725 [hep-th]].
[37] K. Balasubramanian and J. McGreevy, Phys. Rev. D 80, 104039 (2009) [arXiv:0909.0263 [hep-th]].
[38] Y. Korovin, K. Skenderis and M. Taylor, [arXiv:1306.3344 [hep-th]].
[39] D.-W. Pang, JHEP 1001, 120 (2010) [arXiv:0912.2403 [hep-th]].
[40] C. Park, [arXiv:1305.6690] [hep-th].

[41] S. F. Ross and O. Saremi, JHEP 0909, 009 (2009) [arXiv:0907.1846] [hep-th]].

[42] Y. Korovin, K. Skenderis and M. Taylor, JHEP 1308, 026 (2013) [arXiv:1304.7776] [hep-th]].

[43] V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208, 413 (1999) [hep-th/9902121].

[44] D. T. Son and A. O. Starinets, JHEP 0603, 052 (2006) [hep-th/0601157].

[45] A. Batrachenko, J. T. Liu, R. McNees, W. A. Sabra and W. Y. Wen, JHEP 0505, 034 (2005) [hep-th/0408205].

[46] A. Adams, K. Balasubramanian and J. McGreevy, JHEP 0811, 059 (2008) [arXiv:0807.1111] [hep-th]].

[47] N. Iqbal and H. Liu, Phys. Rev. D 79, 025023 (2009) [arXiv:0809.3808] [hep-th]].