DERIVATIONS AND AUTOMORPHISMS OF LOCALLY MATRIX ALGEBRAS

OKSANA BEZUSHCHAK

Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, 60 Volodymyrska Street, Kyiv 01033, Ukraine
bezusch@univ.kiev.ua

Abstract. We describe derivations and automorphisms of infinite tensor products of matrix algebras. Using this description we show that for a countable–dimensional locally matrix algebra $A$ over a field $F$ the dimension of the Lie algebra of outer derivations of $A$ and the order of the group of outer automorphisms of $A$ are both equal to $|F|^\aleph_0$, where $|F|$ is the cardinality of the field $F$.

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1. Introduction and Main Results

We study derivations and automorphisms of countable–dimensional locally matrix algebras.

Let $F$ be a ground field. Following [10], we call an associative $F$–algebra $A$ a locally matrix algebra if for each finite subset of $A$ there exists a subalgebra $B \subset A$ containing this subset such that $B \cong M_n(F)$ for some $n$. We call a locally matrix algebra $A$ unital if it contains a unit $1$.

J. G. Glimm [5] proved that every countable–dimensional unital locally matrix algebra is uniquely determined by its Steinitz number. In [2, 3], we showed that this is no longer true for unital locally matrix algebras of uncountable dimensions.

S. A. Ayupov and K. K. Kudaybergenov [1] constructed an outer derivation of the countable–dimensional unital locally matrix algebra of Steinitz number $2^\infty$ and used it as an example of an outer derivation in a von Neumann regular simple algebra. In [13], H. Strade studied derivations of locally finite–dimensional locally simple Lie algebras over a field of characteristic 0.

Recall that a linear map $d : A \rightarrow A$ is called a derivation if

$$d(xy) = d(x)y + xd(y)$$

for arbitrary elements \(x, y\) from \(A\).

For an element \(a \in A\) the adjoint operator

\[
\text{ad}_A(a) : A \to A, \quad x \mapsto [a, x],
\]

is an inner derivation of the algebra \(A\).

Let \(\text{Der}(A)\) be the Lie algebra of all derivations of the algebra \(A\) and let \(\text{Inder}(A)\) be the ideal of all inner derivations. The factor algebra

\[
\text{Outder}(A) = \text{Der}(A) / \text{Inder}(A)
\]

is called the algebra of outer derivations of \(A\).

Let \(\text{Aut}(A)\) and \(\text{Inn}(A)\) be the group of automorphisms and the group of inner automorphisms of the algebra \(A\), respectively. The factor group

\[
\text{Out}(A) = \text{Aut}(A) / \text{Inn}(A)
\]

is called the group of outer automorphisms of \(A\).

Along with automorphisms of the algebra \(A\) we consider the semigroup \(P(A)\) of injective endomorphisms (embeddings) of \(A\), \(\text{Aut}(A) \subseteq P(A)\).

In Sec. 2, we consider the Tykhonoff topology on the set \(\text{Map}(A, A)\) of all mappings \(A \to A\) and prove the following Theorem.

**Theorem 1.** Let \(A\) be a locally matrix algebra.

1. The ideal \(\text{Inder}(A)\) is dense in \(\text{Der}(A)\) in the Tykhonoff topology.
2. Let the algebra \(A\) contains 1. Then the completion of \(\text{Inn}(A)\) in \(\text{Map}(A, A)\) in the Tykhonoff topology is the semigroup \(P(A)\). In particular, \(\text{Inn}(A)\) is dense in \(\text{Aut}(A)\).

In Sec. 3, we describe derivations of an infinite tensor product of matrix algebras.

Let \(I\) be an infinite set and let \(\mathcal{P}\) be a system of nonempty finite subsets of \(I\). We say that the system \(\mathcal{P}\) is sparse if

1. for any \(S \in \mathcal{P}\) all nonempty subsets of \(S\) also lie in \(\mathcal{P}\),
2. an arbitrary element \(i \in I\) lies in no more than finitely many subsets from \(\mathcal{P}\).

Let

\[
\mathbf{A} = \bigotimes_{i \in I} A_i
\]

and all algebras \(A_i\) are finite-dimensional matrix algebras over \(\mathbb{F}\). For a subset \(S = \{i_1, \ldots, i_r\} \subset I\) the subalgebra

\[
A_S := A_{i_1} \otimes \cdots \otimes A_{i_r}
\]

is a tensor factor of the algebra \(\mathbf{A}\).
Let $P$ be a system of nonempty finite subsets of $I$. Let $f_S, S \in P$, be a system of linear operators $A \to A$. The sum

\[(2)\quad \sum_{S \in P} f_S\]

converges in the Tykhonoff topology if for an arbitrary element $a \in A$ the set

\[\{ S \in P \mid f_S(a) \neq 0 \}\]

is finite. In this case, the operator

\[a \mapsto \sum_{S \in P} f_S(a)\]

is a linear operator. Moreover, if every summand $f_S$ is a derivation of the algebra $A$ then the sum (2) is also a derivation of the algebra $A$.

Let $P$ be a sparse system. For each subset $S \in P$ choose an element $a_S \in A_S$. The sum

\[(3)\quad \sum_{S \in P} \text{ad}_A(a_S)\]

converges in the Tykhonoff topology to a derivation of $A$. Indeed, choose an arbitrary element $a \in A$. Let

\[a \in A_{i_1} \otimes \cdots \otimes A_{i_r}.\]

Because of the sparsity of the system $P$, for all but finitely many subsets $S \in P$ we have

\[\{i_1, \ldots, i_r\} \cap S = \emptyset, \quad \text{and therefore} \quad \text{ad}_A(a_S)(a) = 0.\]

Let $D_P$ be the vector space of all sums (3), $D_P \subseteq \text{Der}(A)$.

For each algebra $A_i, i \in I$, choose a subspace $A^0_i$ such that

\[(4)\quad A_i = \mathbb{F} \cdot 1_{A_i} + A^0_i\]

is a direct sum, $1_{A_i}$ is a unit element of $A_i$. Let $E_i$ be a basis of $A^0_i$. For a subset $S = \{i_1, \ldots, i_r\}$ of the set $I$ let

\[E_S := E_{i_1} \otimes \cdots \otimes E_{i_r} = \{ a_1 \otimes \cdots \otimes a_r \mid a_k \in E_{i_k}, 1 \leq k \leq r \}.\]

and

\[\text{ad}_A(E_S) = \{ \text{ad}_A(e) \mid e \in E_S \}.\]

A description of derivations of the algebra (1) is given by the following Theorem.
Theorem 2. (1) Suppose that the set $I$ is countable. Then

$$\text{Der}(A) = \bigcup_{\mathcal{P}} D_{\mathcal{P}},$$

where the union is taken over all sparse systems of subsets of $I$.

(2) Let $I$ be an infinite (not necessarily countable) set. Let $\mathcal{P}$ be a sparse system of subsets of $I$. Then the union of finite sets of operators

$$\bigcup_{S \in \mathcal{P}} \text{ad}_A(E_S)$$

is a topological basis of $D_{\mathcal{P}}$.

In Sec. 4, we prove the analog of the result of H. Strade [13] for locally finite-dimensional locally simple Lie algebras.

Theorem 3. Let $A$ be a countable-dimensional locally matrix algebra. Then the Lie algebra $\text{Outder}(A)$ is not locally finite-dimensional.

In Sec. 5, we describe automorphisms and unital injective endomorphisms of a countable-dimensional unital locally matrix algebra $A$. Remark, that by the result of A. G. Kurosh ([10], Theorem 10) the semigroup $P(A)$ of unital injective homomorphisms is strictly bigger than $\text{Aut}(A)$.

The starting point here is Koethe’s Theorem [9] stating that every countable-dimensional unital locally matrix algebra $A$ is isomorphic to a countable tensor product of matrix algebras. Therefore

$$A \cong \otimes_{i=1}^{\infty} A_i, \quad A_i \cong M_{n_i}(\mathbb{F}), \quad i \geq 1.$$  

Let $H_n$ be the subgroup of the group $\text{Inn}(A)$ generated by conjugations by invertible elements from

$$\otimes_{i \geq n} A_i.$$  

Clearly,

$$H_n \cong \text{Inn} \left( \otimes_{i \geq n} A_i \right)$$

and

$$\text{Inn}(A) = H_1 > H_2 > \cdots.$$  

For each $n \geq 1$ choose a system of representatives of left cosets $hH_{n+1}$, $h \in H_n$, and denote it as $\mathcal{X}_n$. We assume that each $\mathcal{X}_n$ contains the identical automorphism.

For an arbitrary sequence of automorphisms $\varphi_n \in \mathcal{X}_n$, $n \geq 1$, the infinite product $\varphi = \varphi_1\varphi_2 \cdots$ converges in the Tykhonoff topology. Clearly, $\varphi \in P(A)$. 

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**Theorem 4.** An arbitrary unital injective endomorphism \( \varphi \in P(A) \) can be uniquely represented as

\[
\varphi = \varphi_1 \varphi_2 \cdots,
\]

where \( \varphi_i \in \mathcal{X}_i \) for each \( i \geq 1 \).

We call a sequence of automorphisms \( \varphi_n \in H_n, \ n \geq 1 \), integrable if for an arbitrary element \( a \in A \) the subspace spanned by all elements

\[
\varphi_n \varphi_{n-1} \cdots \varphi_1(a), \quad n \geq 1,
\]

is finite-dimensional.

**Theorem 5.** An injective endomorphism

\[
\varphi = \varphi_1 \varphi_2 \cdots, \quad \text{where} \quad \varphi_n \in H_n, \quad n \geq 1,
\]

is an automorphism if and only if the sequence

(6)

\[
\{ \varphi_n^{-1} \}_{n \geq 1}
\]

is integrable.

**Example 1.** In each \( A_i, \ i \geq 1 \), choose an invertible element \( a_i \). Let \( \hat{a}_i \) be the conjugation automorphism by \( a_i \). Then the sequence

\[
\{ \hat{a}_i^{-1} \}_{i \geq 1}
\]

is integrable.

**Example 2.** Let \( e_{pq} \) denote a matrix unit. Let \( A_i \cong M_{n_i}(\mathbb{F}), \ i \geq 1 \), and assume that \( 1 \leq p, q \leq n_i \) so that \( e_{pq} \) can be thought of as a matrix unit of \( M_{n_i}(\mathbb{F}) \). Denote

\[
e_{pq}(i) = 1 \otimes \cdots \otimes 1 \otimes e_{pq} \otimes 1 \otimes \cdots \in A_i \subset A.
\]

Let

\[
a_i = e_{11}(i) e_{12}(i + 1).
\]

Clearly, \( a_i^2 = 0 \). Let \( \phi_i \) denote the conjugation by the element \( (1+a_i)^{-1} \). The sequence

(7)

\[
\{ \phi_i^{-1} \}_{i \geq 1}
\]

is not integrable. Hence, \( \phi = \phi_1 \phi_2 \cdots \) is an injective endomorphism that is not an automorphism.

**Remark 1.** This example provides another proof of Theorem 10 of A. G. Kurosh.
In Sec. 6, we determine dimensions of Lie algebras Der(A) and Outder(A) and orders of groups Aut(A) and Out(A), where A is a countable–dimensional locally matrix algebra.

We denote the cardinality of a set X as |X|. For two sets X and Y let Map(Y, X) denote the set of all mappings from Y to X. Given two cardinals α, β and sets X, Y such that |X| = α, |Y| = β we define αβ = |Map(Y, X)|. As always ℵ₀ stands for the countable cardinality.

**Theorem 6.** Let \( A = \bigotimes_{i \in I} A_i \), where \( I \) is an infinite set and each algebra \( A_i \) is a matrix algebra over a field \( F \) of the dimension > 1. Then

\[
\dim_F \text{Der}(A) = \dim_F \text{Outder}(A) = |F|^{|I|}.
\]

**Theorem 7.** Let \( A \) be a countable–dimensional locally matrix algebra over a field \( F \). Then

\[
\dim_F \text{Der}(A) = \dim_F \text{Outder}(A) = |F|^{|\aleph_0|}.
\]

**Remark 2.** For many uncountable cardinals \(|F|\) we have \(|F|^{|\aleph_0|} = |F|\). For example, this is the case when \(|F| = \lambda^\mu\) is a power of cardinals and \( \mu \geq \aleph_0 \). If \( F = F_0(z, z^{-1}) \) is the field of Laurent series over some field \( F_0 \), or its algebraic extension, then \(|F| = |F_0|^{|\aleph_0|}\), and therefore \(|F|^{|\aleph_0|} = |F|\).

**Remark 3.** A locally matrix algebra \( A \) over a field of zero characteristic gives rise to the locally finite–dimensional locally simple Lie algebra \( L = [A, A] \). Moreover, there are embeddings

\[
\text{Der}(A) \to \text{Der}(L), \quad \text{Inder}(A) \to \text{Inder}(L),
\]

\[
\text{Outder}(A) \to \text{Outder}(L),
\]

which imply

\[
\dim_F \text{Outder}(L) \geq \dim_F \text{Outder}(A).
\]

Therefore, Theorem 7 contradicts Theorem 3.2 from [13].

**Theorem 8.** Let \( A \) be a countable–dimensional locally matrix algebra over a field \( F \). Then

\[
|\text{Aut}(A)| = |\text{Out}(A)| = |F|^{|\aleph_0|}.
\]

2. **Tykhonoff topology**

Let \( X \) be an arbitrary set. The set Map(\( X, X \)) of mappings \( X \to X \) is equipped with the Tykhonoff topology (see [14]). For distinct elements \( a_1, \ldots, a_n \in X \) and arbitrary elements \( b_1, \ldots, b_n \in X, n \geq 1 \), consider the subset

\[
M(a_1, \ldots, a_n; b_1, \ldots, b_n) = \{ f : X \to X \mid f(a_i) = b_i, 1 \leq i \leq n \}.
\]
of \( \text{Map}(X,X) \). The Tykhonoff topology on \( \text{Map}(X,X) \) is generated by all open sets of this type \([11]\).

Thus, for a subset \( S \subseteq \text{Map}(X,X) \) a mapping \( f : X \to X \) lies in the completion \( \overline{S} \) of \( S \) if and only if for any \( n \geq 1 \) and for any elements \( a_1, \ldots, a_n \subseteq X \) there exists a mapping

\[
g : X \to X, \quad g \in S, \quad \text{such that} \quad f(a_i) = g(a_i), \quad 1 \leq i \leq n.
\]

**Proof of Theorem 1** (1) It is straightforward that the set of derivations \( \text{Der}(A) \) is closed in \( \text{Map}(X,X) \). It implies that

\[
\text{Inder}(A) \subseteq \text{Der}(A).
\]

To prove the assertion we need to show that for any derivation \( d : A \to A \) and arbitrary elements \( a_1, \ldots, a_n \subseteq A \) there exists an element \( b \subseteq A \) such that

\[
[b, a_i] = d(a_i), \quad 1 \leq i \leq n.
\]

Choose a subalgebra \( B_1 \subseteq A \) such that \( a_1, \ldots, a_n \subseteq B_1 \) and \( B_1 \cong M_k(\mathbb{F}) \). Then choose a subalgebra \( B_2 \subseteq A \) such that

\[
B_1 + d(B_1) \subseteq B_2 \quad \text{and} \quad B_2 \cong M_l(\mathbb{F}).
\]

The vector space \( B_2 \) is a \( B_1 \)-bimodule and \( d : B_1 \to B_2 \) is a bimodule derivation. Since any bimodule derivation of finite-dimensional matrix algebras over a field is inner (see \([11]\)) there exists an element \( b \subseteq B_2 \) such that \( d(a) = [b, a] \) for all elements \( a \subseteq B_1 \). This proves the part (1) of the Theorem.

(2) Let \( A \) be a unital locally matrix algebra. The set \( P(A) \) of unital injective endomorphisms is closed in the Tykhonoff topology. Hence

\[
\overline{\text{Im}(A)} \subseteq P(A).
\]

Now, let \( \varphi : A \to A \) be an injective endomorphism and \( \varphi(1) = 1 \). Let \( a_1, \ldots, a_n \subseteq A \). As above choose a subalgebra \( B_1 \subseteq A \) such that \( 1, a_1, \ldots, a_n \subseteq B_1 \) and \( B_1 \cong M_k(\mathbb{F}) \). Then choose another subalgebra \( B_2 \subseteq A \) such that

\[
B_1 + \varphi(B_1) \subseteq B_2 \quad \text{and} \quad B_2 \cong M_l(\mathbb{F}).
\]

By the Skolem–Noether Theorem (see \([6, 7]\)) there exists an invertible element

\[
a \subseteq B_2 \quad \text{such that} \quad \varphi(x) = a^{-1}xa \quad \text{for all elements} \quad x \in B_1.
\]

This finishes the proof of the Theorem. \( \square \)
3. Derivations of Tensor Products of Matrix Algebras

Recall that $I$ is an infinite set and let $A$ be a tensor product of the kind (I):

$$A = \otimes_{i \in I} A_i,$$

where all algebras $A_i$ are matrix algebras over $\mathbb{F}$, $\dim_{\mathbb{F}} A_i > 1$. Clearly, $A$ is a unital locally matrix algebra.

**Lemma 1.** For any $i \in I$ the subalgebra

$$C = \otimes_{j \neq i} A_j$$

is the centraliser of the subalgebra $A_i$ in $A$.

**Proof.** We have

$$A = A_i \otimes_{\mathbb{F}} C.$$

Clearly, $C$ lies in the centraliser of $A_i$ in $A$. Now, suppose that $x \in A$ and $[A_i, x] = \{0\}$. Let

$$x = \sum_{k=1}^{n} a_k \otimes c_k, \quad a_1, \ldots, a_n \in A_i, \quad c_1, \ldots, c_n \in C,$$

and the elements $c_1, \ldots, c_n$ are linearly independent. Then for an arbitrary element $a \in A_i$ we have

$$\sum_{k=1}^{n} [a, a_k] \otimes c_k = 0,$$

which implies

$$[a, a_1] = \cdots = [a, a_n] = 0.$$

Hence $a_1, \ldots, a_n \in \mathbb{F} \cdot 1_{A_i}$ and $x \in C$. □

**Lemma 2.** Let $i \in I$ and let $d \in \text{Der}(A)$. If $d(A_i) = \{0\}$ then the subalgebra

$$C = \otimes_{j \neq i} A_j$$

is $d$–invariant.

**Proof.** We have $[A_j, C] = \{0\}$. Hence

$$[A_i, d(C)] \subseteq d([A_i, C]) + [d(A_i), C] = \{0\}.$$

By Lemma 1 it implies $d(C) \subseteq C$. □

**Proof of Theorem 2** (1) Let $I = \{i_1, i_2, \ldots\}$. Let $d \in \text{Der}(A)$. Since the algebra of inner derivations $\text{Inder}(A)$ is dense in $\text{Der}(A)$, by Theorem 1(1), it follows that there exists an element $a_1 \in A$ such that

$$(d - \text{ad}_A(a_1))(A_{i_1}) = \{0\}.$$
There exists a finite subset $S_1 \subset I$ such that $a_1 \in A_{S_1}$. By Lemma 2, the derivation $d - \text{ad}_A(a_1)$ maps the subalgebra
\[
\otimes_{j \neq i_1} A_j
\]
into itself. Arguing as above, we find a finite subset $S_2 \subset I \setminus \{i_1\}$ and an element $a_2 \in A_{S_2}$ such that
\[
(d - \text{ad}_A(a_1) - \text{ad}_A(a_2))(A_{i_2}) = \{0\}
\]
and so on. We get a sequence $S_1, S_2, \ldots$ of nonempty finite subsets of $I$,
\[
S_n \subset I \setminus \{i_1, \ldots, i_{n-1}\}, \quad n \geq 2,
\]
and a sequence of elements $a_n \in A_{S_n}, n \geq 1$, such that
\[
d = \sum_{n=1}^{\infty} \text{ad}_A(a_n).
\]
Adding to the subsets $S_1, S_2, \ldots$ all their nonempty subsets, we get a sparse system $P$ and $d \in D_P$. This completes the proof of the part (1) of the Theorem.

(2) Let $I$ be an infinite, not necessarily countable set. Let $P$ be a sparse system of finite nonempty subsets of $I$. Choose a subset $S = \{i_1, \ldots, i_r\} \in P$ and element $a_k \in A_{i_k}, 1 \leq k \leq r$. Let
\[
a_k = \gamma_k \cdot 1_{A_{i_k}} + a^0_k, \quad \text{where} \quad \gamma_k \in \mathbb{F}, \quad a^0_k \in A^0_{i_k}.
\]
Expanding brackets in the tensor
\[
a_1 \otimes \cdots \otimes a_r = (\gamma_1 \cdot 1_{A_{i_1}} + a^0_1) \otimes \cdots \otimes (\gamma_r \cdot 1_{A_{i_r}} + a^0_r)
\]
we get
\[
a_1 \otimes \cdots \otimes a_r \in \mathbb{F} \cdot 1 + \sum_{k=1}^{r} 1 \otimes \cdots \otimes A^0_{i_k} \otimes \cdots \otimes 1 + \cdots + A^0_{i_1} \otimes \cdots \otimes A^0_{i_r}.
\]
Hence, the space $\text{ad}_A(A_s)$ is spanned by
\[
\bigcup_{\emptyset \neq S' \subseteq S} \text{ad}_A(E_{S'}).\]
This implies that an arbitrary element from $D_P$ can be represented as a converging sum
\[
\sum_k \alpha_k \text{ad}_A(e_k), \quad \text{where} \quad \alpha_k \in \mathbb{F} \quad \text{and} \quad \{e_k\}_k = \bigcup_{S \in P} E_S.
\]
Now we need to show that
\[
\sum_k \alpha_k \text{ad}_A(e_k) = 0
\]
implies $\alpha_k = 0$ for all $k$.

Let $S \in \mathcal{P}$, $i \in S$, $S^0 = S \setminus \{i\}$. An arbitrary element $e$ from $E_S$ can be represented (up to a permutation of tensors) as

$$e = e' \otimes e'', \quad \text{where} \quad e' \in E_i, \quad e'' \in E_{S^0}.$$ 

For an element $a \in A_i$ we have $[e, a] = [e', a] \otimes e''$.

Fix $i \in I$. Let $S_1, \ldots, S_t$ be all subsets from $\mathcal{P}$ that contain $i$. Let

$$S_j^0 = S_j \setminus \{i\}, \quad 1 \leq j \leq t.$$ 

If

$$e_k = e' \otimes e'', \quad \text{where} \quad e' \in E_i \quad \text{and} \quad e'' \in \bigcup_{j=1}^{t} E_{S_j^0},$$ 

or $e'' = 1$ if all $S_j = \{i\}$, then we denote $\alpha_{e', e''} := \alpha_k$. For an arbitrary element $a \in A_i$ we have

$$\left[ \sum_k \alpha_k e_k, a \right] = \sum \alpha_{e', e''} [e', a] \otimes e'' = 0,$$

where the summation runs over all

$$(e', e'') \in E_i \times \left( \bigcup_{j=1}^{t} E_{S_j^0} \bigcup \{1\} \right).$$

Hence

$$\left[ \sum_{e'} \alpha_{e', e''} e', a \right] = 0 \quad \text{for any} \quad e'' \in \bigcup_{j=1}^{t} E_{S_j^0} \bigcup \{1\}.$$ 

The element $\sum_{e'} \alpha_{e', e''} e'$ lies in $A_i^0$ and at the same time it lies in the center of the algebra $A_i$. Hence

$$\sum_{e'} \alpha_{e', e''} e' = 0$$

which implies $\alpha_{e', e''} = 0$. This completes the proof of Theorem 2. \hfill \Box

In what follows we will need the following Lemma.

**Lemma 3.** Let $A$ be a countable–dimensional locally matrix algebra. Let $0 \neq e \in A$ be an idempotent. Then every derivation of the subalgebra $eAe$ extends to a derivation of $A$.

**Proof.** Suppose at the first that the algebra $A$ is unital. By the Köthe’s Theorem \[9\], we can assume that algebra $A$ of the kind \[9\]:

$$A = \bigotimes_{i=1}^{\infty} A_i,$$
where each factor $A_i$ is a matrix algebra over $F$. There exists $n \geq 1$ such that $e \in A_1 \otimes \cdots \otimes A_n$. Replacing the first $n$ factors $A_1, \ldots, A_n$ by one factor $A_1 \otimes \cdots \otimes A_n$ we can assume that $e \in A_1$. Then
\[ eAe = eA_1e \otimes \left( \bigotimes_{j=2}^{\infty} A_j \right). \]

Let $d \in \text{Der}(eAe)$. By Theorem 2 (1), there exists a sparse system $\mathcal{P}$ of nonempty subsets of the set of positive integers such that
\[ d = \sum_{S \in \mathcal{P}} \text{ad}_{eAe}(a_S), \quad a_S \in (eAe)_S. \]

We have $(eAe)_S \subseteq A_S$. Since the system $\mathcal{P}$ is sparse it follows that the infinite sum
\[ \sum_{S \in \mathcal{P}} \text{ad}_A(a_S) \]
converges in the Tykhonoff topology on $\text{Map}(A, A)$ to a derivation that extends $d$.

Suppose now that $A$ is a countable–dimensional non unital locally matrix algebra. There exists a sequence of idempotents $e_1 = e, e_2, \ldots$ such that
\[ (12) \quad e_iAe_i \subset e_{i+1}Ae_{i+1}, \quad i \geq 1, \quad \text{and} \quad \bigcup_{i \geq 1} e_iAe_i = A. \]

Let $d \in \text{Der}(eAe)$. Remark, that for any idempotent $e' \in A$ the sub-algebra $e'Ae'$ is a unital locally matrix subalgebra. So, by the unital case of this Lemma (see above), there exist derivations $d_i \in \text{Der}(e_iAe_i)$ such that $d_{i+1}$ extends $d_i$, $i \geq 1$, and $d_1 = d$. The derivation
\[ \bigcup_{i=1}^{\infty} d_i \in \text{Der}(A) \]
extends the derivation $d$. \hfill \Box

4. The Lie Algebra of Outer Derivations is Not Locally Finite–Dimensional

Let $\mathbb{N}$ be the set of positive integers and let $M_{\infty}(\mathbb{F})$ be the algebra of all infinite $\mathbb{N} \times \mathbb{N}$ finitary matrices over $\mathbb{F}$, that is matrices that contain finitely many nonzero entries.

**Lemma 4.** Let $A$ be a countable–dimensional non unital locally matrix algebra such that for every idempotent $e \in A$ we have $\dim_F eAe < \infty$. Then $A \cong M_{\infty}(\mathbb{F})$. 
Proof. Since the algebra $A$ is countable–dimensional and non unital there exists a sequence of idempotents $e_1, e_2, \ldots$ such that (12) holds.

We claim that each subalgebra $e_i A e_i$ is isomorphic to a matrix algebra over $\mathbb{F}$. Indeed, since $\dim_\mathbb{F} e_i A e_i < \infty$ there exists a subalgebra $A' \subset A$ such that $e_i A e_i \subseteq A'$ and $A'$ is isomorphic to a matrix algebra. Let

$$
\psi : A' \to M_t(\mathbb{F})
$$

be an isomorphism. Let $n_i$ be the range of the matrix $\psi(e_i)$ in $M_t(\mathbb{F})$. Then

$$
e_i A e_i \cong \psi(e_i) M_t(\mathbb{F}) \psi(e_i) \cong M_{n_i}(\mathbb{F}).$$

Let

$$
id_{i,i+1} : e_i A e_i \to e_{i+1} A e_{i+1}
$$

be the embedding homomorphism, $i \geq 1$. It is easy to see that there exists a sequence of isomorphisms

$$
\varphi_i : M_{n_i}(\mathbb{F}) \to e_i A e_i, \quad n_i \geq 1, \quad i \geq 1,
$$

such that the embeddings

$$
\varphi_i^{-1} \circ \text{id}_{i,i+1} \circ \varphi_i \quad \text{of} \quad M_{n_i}(\mathbb{F}) \quad \text{into} \quad M_{n_{i+1}}(\mathbb{F})
$$

is diagonal, that is

$$
a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad a \in M_{n_i}(\mathbb{F}).
$$

The algebra $A$ is isomorphic to the direct limit of matrix algebras $M_{n_i}(\mathbb{F})$ with diagonal embeddings, that is, to $M_\infty(\mathbb{F})$. □

Proof of Theorem 3. Suppose at the first that the algebra $A$ is unital. Then by the Köthe’s Theorem [9], the algebra $A$ of the kind (5). We will assume that

$$
A = \otimes_{i=1}^\infty A_i, \quad A_i \cong M_{n_i}(\mathbb{F}), \quad i \geq 1, \quad n_i \geq 2.
$$

The algebras $A_i$ are embedded in $A$ via

$$
u_i : a \mapsto 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots, \quad a \in A_i.
$$

Let

$$
e_{pq}(i) := u_i(e_{pq}), \quad 1 \leq p, q \leq n_i,
$$

denote the image of the matrix unit $e_{pq} \in A_i \cong M_{n_i}(\mathbb{F})$. Since the images of $A_i$ and $A_j$, $i \neq j$, commute in $A$, we have

$$
[e_{pq}(i), e_{rs}(j)] = 0 \quad \text{for} \quad i \neq j.
$$
Consider the following derivations of the algebra $A$:

$$z = \sum_{i=1}^{\infty} \text{ad}(e_{12}(i)e_{11}(i + 1)) \in \text{ad}(A_{1,2}) + \text{ad}(A_{2,3}) + \cdots$$

and

$$y_k = \sum_{j=1}^{\infty} \text{ad}(e_{12}(j) \cdots e_{12}(j + k - 1)) \in \text{ad}(A_{1,k}) + \text{ad}(A_{2,k+1}) + \cdots,$$

where $k \geq 1$ and $[t, l]$ is the integer segment, $[t, l] = \{t, t + 1, \ldots, l\}$, $1 \leq t \leq l$.

We claim that $[z, y_k] = y_{k+1}$ for any $k \geq 1$. Indeed,

$$[z, y_k] = \sum_{i,j} \left[ \text{ad}(e_{12}(i)e_{11}(i + 1)), \text{ad}(e_{12}(j) \cdots e_{12}(j + k - 1)) \right] =$$

$$\sum_{i,j} \text{ad}\left( [e_{12}(i)e_{11}(i + 1), e_{12}(j) \cdots e_{12}(j + k - 1)] \right).$$

If $\{i, i + 1\} \cap \{j, \ldots, j + k - 1\} = \emptyset$ then each factor of $e_{12}(i)e_{11}(i + 1)$ commutes with each factor of $e_{12}(j) \cdots e_{12}(j + k - 1)$.

If $i \in \{j, \ldots, j + k - 1\}$ then

$$e_{12}(i)e_{11}(i + 1) \cdot e_{12}(j) \cdots e_{12}(j + k - 1) =$$

$$e_{12}(j) \cdots e_{12}(j + k - 1) \cdot e_{12}(i) e_{11}(i + 1) = 0$$

since $e_{12}(i)^2 = 0$.

It remains to consider only one case: $j = i + 1$. We have

$$e_{12}(i)e_{11}(i + 1) \cdot e_{12}(i + 1) \cdots e_{12}(i + k) = e_{12}(i)e_{12}(i + 1) \cdots e_{12}(i + k).$$

Multiplying these elements in the other order we get

$$e_{12}(i + 1) \cdots e_{12}(i + k) \cdot e_{12}(i)e_{11}(i + 1) = 0$$

since $e_{12}(i + 1)e_{11}(i + 1) = 0$. Finally,

$$[z, y_k] = \sum_{i=1}^{\infty} \text{ad}\left( [e_{12}(i)e_{11}(i + 1), e_{12}(i + 1) \cdots e_{12}(i + k)] \right) =$$

$$\sum_{i=1}^{\infty} \text{ad}(e_{12}(i)e_{12}(i + 1) \cdots e_{12}(i + k)) = y_{k+1}.$$

The Lie subalgebra of $\text{Der}(A)$ generated by elements $z$ and $y_1$ contains all elements $y_k$, $k \geq 1$.

Let us show that derivations $y_k$, $k \geq 1$, are linearly independent modulo $\text{Inder}(A)$. Recall that in each algebra $A_i$ we choose a subspace
Choose a subspace $A_i^0$ containing $e_{12}(i)$ and a basis $E_i$ in $A_i^0$ such that $e_{12}(i) \in E_i$. Then
\[
e_{12}(i) \cdots e_{12}(i + k - 1) \in E_{[i, i + k - 1]}.
\]
Suppose that $\alpha_1 y_1 + \cdots + \alpha_k y_k \in \text{Inder}(A)$ and $\alpha_1, \ldots, \alpha_k \in F$. Then there exists $p \geq 1$ such that
\[
\alpha_1 y_1 + \cdots + \alpha_k y_k \in \text{ad}_A(A_{[1, p]}).
\]
Without loss of generality, we will assume that $k \leq p$.

Consider a sparse system $P$ that consists of intervals $[i, i + p - 1]$, $i \geq 1$, and all their nonempty subsets. Let $E$ denote the topological basis of the vector space $D_P$ that corresponds to bases $E_i$ of subspaces $A_i^0$, see Theorem 2 (2). We have
\[
(13) \quad \alpha_1 y_1 + \cdots + \alpha_k y_k = \sum_{1 \leq j \leq k, 1 \leq i < \infty} \alpha_j \text{ad}_A(e_{12}(i) \cdots e_{12}(i + j - 1)).
\]
The operators $\text{ad}_A(e_{12}(i) \cdots e_{12}(i + j - 1))$ are distinct elements of the basis $E$. If at least one coefficient $\alpha_j$, $1 \leq j \leq k$, is not equal to 0, then the sum (13) contains infinitely many basis elements from $E$ with nonzero coefficients. Hence, it can not be equal to a finite linear combination of basic elements from $E$. Every element from $\text{ad}_A(A_{[1, p]})$ is a finite linear combination of basis elements. Therefore $\alpha_1 = 0, \ldots, \alpha_k = 0$. This proves the claim.

We showed that the Lie subalgebra of $\text{Der}(A)$ generated by derivations $z$ and $y_1$ is infinite–dimensional module $\text{Inder}(A)$. This completes the proof of the Theorem in the case when the algebra $A$ is unital.

Now, let $A$ be a countable–dimensional non unital locally matrix algebra. Suppose that there exists an idempotent $e \in A$ such that the unital algebra $eAe$ is infinite–dimensional. We have shown that there exist derivations $z$ and $y_1$ of the algebra $eAe$ such that the derivations
\[
y_k = [\underbrace{z, [z, \ldots, [z, y_1] \ldots]}_{k-1}], \quad k \geq 1,
\]
are linearly independent module $\text{Inder}(eAe)$.

By Lemma 3 there exist derivations $\tilde{z}, \tilde{y}_1 \in \text{Der}(A)$ that extend $z$ and $y_1$ respectively. Let us show that the derivations
\[
\tilde{y}_k = [\underbrace{\tilde{z}, [\tilde{z}, \ldots, [\tilde{z}, \tilde{y}_1] \ldots]}_{k-1}], \quad k \geq 1,
\]
are linearly independent module $\text{Inder}(A)$.

Suppose that
\[
d = \alpha_1 \tilde{y}_1 + \cdots + \alpha_n \tilde{y}_n \in \text{Inder}(A), \quad \alpha_1, \ldots, \alpha_n \in F.
\]
We will show that in this case $\alpha_1 y_1 + \cdots + \alpha_n y_n \in \text{Inder}(eAe)$. The derivation $\tilde{y}_i$ extends the derivation $y_i$. Hence, the subalgebra $eAe$ is invariant with respect to $d$. Since $d \in \text{Inder}(A)$ then there exists an element $u \in A$ such that $d(x) = [u, x]$ for an arbitrary element $x \in A$.

Consider the Peirce decomposition

$$u = eue + (1 - e)ue + eu(1 - e) + (1 - e)u(1 - e),$$

where $1$ is a formal unit. For an arbitrary element $x \in eAe$ we have

$$[u, x] = [eue, x] + (1 - e)ue x - x eu (1 - e).$$

The inclusion $[u, x] \in eAe$ implies $[u, x] = [eue, x]$.

We showed that the restriction of the derivation $d$ to $eAe$ is an inner derivation. Hence

$$\alpha_1 y_1 + \cdots + \alpha_n y_n \in \text{Inder}(eAe),$$

which implies $\alpha_1 = \cdots = \alpha_n = 0$.

By Lemma 4, if for an arbitrary idempotent $e \in A$ the subalgebra $eAe$ is finite–dimensional, then $A \cong M_\infty(\mathbb{F})$. Thus, it remains to verify that the Lie algebra of outer derivations $\text{Outder}(M_\infty(\mathbb{F}))$ is not locally finite–dimensional.

Infinite matrices

$$z = \sum_{i=1}^{\infty} e_{2i, 2i+2} \quad \text{and} \quad y_k = \sum_{i=1}^{\infty} e_{2i, 2i+2k-1}, \quad k \geq 1,$$

are not finitary, but

$$[z, M_\infty(\mathbb{F})] \subseteq M_\infty(\mathbb{F}) \quad \text{and} \quad [y_k, M_\infty(\mathbb{F})] \subseteq M_\infty(\mathbb{F}), \quad k \geq 1.$$

We have $[z, y_k] = y_{k+1}, \ k \geq 1$. The subalgebra generated by derivations $\text{ad}(z), \text{ad}(y_1) \in \text{Der}(M_\infty(\mathbb{F}))$ contains all derivations $\text{ad}(y_k), \ k \geq 1$. It is easy to see that the derivations $\text{ad}(y_k), \ k \geq 1$, are linearly independent modulo $\text{Inder}(M_\infty(\mathbb{F}))$. It completes the proof of Theorem 3.

5. AUTOMORPHISMS AND UNITAL INJECTIVE ENDOMORPHISMS

Proof of Theorem 4. Let $\varphi : A \to A$ be an injective endomorphism of the countable–dimensional unital locally matrix algebra (5), $\varphi(1) = 1$. There exists a finite subset $S_1 \subset \mathbb{N}$ such that $\varphi(A_1) \subseteq A_{S_1}$. Applying the Skolem–Noether Theorem (see [6, 7]), as we did in the proof of Theorem 1 we find an invertible element $a_1 \in A_{S_1}$ such that

$$\varphi(x) = a_1^{-1}xa_1 \quad \text{for all elements} \quad x \in A_1.$$

Let $a_1$ be the automorphism of conjugation by the element $a_1$. So, $a_1 \in H_1$. Let $\varphi_1 \in \mathcal{X}_1$ be a representative of the coset $a_1H_2$. The embedding $\psi_1 = \varphi_1^{-1} \varphi$ fixes all elements in the subalgebra $A_1$. 

For an arbitrary element \( a \in \otimes_{j \geq 2} A_j \) we have
\[
\{0\} = \psi_1([A_1, a]) = [\psi_1(A_1), \psi_1(a)] = [A_1, \psi_1(a)].
\]
Hence, the element \( \psi_1(a) \) lies in the centralizer of \( A_1 \). By Lemma 1
\[
\psi_1(a) \in \otimes_{j \geq 2} A_j.
\]
We showed that \( \psi_1 \) is an embedding of the algebra \( \otimes_{j \geq 2} A_j \) into itself.

Arguing as above, we find an automorphism \( \varphi_2 \in X_2 \) such that \( \varphi_2^{-1} \psi_1 \) fixes all elements in the subalgebra \( A_2 \), and so on. As a result, we represent \( \varphi \) as an infinite product
\[
\varphi = \varphi_1 \varphi_2 \cdots, \quad \varphi_i \in X_i, \quad i \geq 1.
\]

Now suppose that
\[
\varphi_1 \varphi_2 \cdots = \varphi'_1 \varphi'_2 \cdots, \quad \text{where} \quad \varphi'_i \in X_i, \quad i \geq 1.
\]
Applying both sides to elements from \( A_1 \), we see that
\[
\varphi_1\big|_{A_1} = \varphi'_1\big|_{A_1}.
\]
Let \( \varphi_1, \varphi'_1 \) be conjugations by invertible elements \( a, b \) respectively. Then the element \( a^{-1}b \) lies in the centralizer of \( A_1 \), hence in \( \otimes_{j > 1} A_j \). So, \( \varphi_1^{-1} \varphi'_1 \in H_2 \) and \( \varphi_1 = \varphi'_1 \). This implies
\[
\varphi_2 \varphi_3 \cdots = \varphi'_2 \varphi'_3 \cdots.
\]
Arguing as above, we see that \( \varphi_2 = \varphi'_2, \varphi_3 = \varphi'_3 \) and so on. \( \square \)

Proof of Theorem 5. Suppose that the sequence of automorphisms \( \{\varphi_i\} \) is integrable. Then for an arbitrary positive integer \( p \geq 1 \) the subspace spanned by
\[
\varphi_1^{-1} \cdots \varphi_i^{-1}(A_p), \quad i \geq 1,
\]
is finite–dimensional. Hence, there exists positive integer \( q \geq 1 \) such that
\[
\varphi_1^{-1} \cdots \varphi_i^{-1}(A_p) \subseteq A_{[1,q]} \quad \text{for any} \quad i \geq 1.
\]
This inclusion is equivalent to
\[
A_p \subseteq \varphi_1 \cdots \varphi_i(A_{[1,q]}), \quad i \geq 1.
\]
For \( i = q \) we have
\[
\varphi_1 \cdots \varphi_q(A_{[1,q]}) = \varphi(A_{[1,q]}),
\]
and therefore
\[
A_p \subseteq \varphi(A_{[1,q]}).
\]
We showed that the injective endomorphism \( \varphi \) is surjective, hence an automorphism.
Now suppose that the injective endomorphism $\varphi$ is surjective. Then for an arbitrary $p \geq 1$ there exists $q \geq 1$ such that

$$A_{[1,p]} \subseteq \varphi(A_{[1,q]}) = \varphi_1 \cdots \varphi_i(A_{[1,q]}) \text{ for } i \geq q.$$ 

Hence

$$\varphi_{i}^{-1} \cdots \varphi^{-1}_{1}(A_{[1,p]}) \subseteq A_{[1,q]} \text{ for } i \geq q.$$ 

It implies that the subspace spanned by

$$\varphi_{i}^{-1} \cdots \varphi^{-1}_{1}(A_{[1,p]}), \quad i \geq 1,$$

is finite–dimensional, hence the sequence (10) is integrable.

**Proof of Example 1.** For an arbitrary subalgebra $A_{i_1} \otimes \cdots \otimes A_{i_r}$ of the algebra $A$ and an arbitrary positive integer $j \geq 1$ we have

$$a_j \cdots a_1(A_{i_1} \otimes \cdots \otimes A_{i_r})a_1^{-1} \cdots a_j^{-1} = A_{i_1} \otimes \cdots \otimes A_{i_r}.$$ 

In particular, the subspace spanned by

$$\hat{a}_j^{-1} \cdots \hat{a}_1^{-1}(A_{i_1} \otimes \cdots \otimes A_{i_r}), \quad j \geq 1,$$

is finite–dimensional and the sequence $\hat{a}_i^{-1}, \quad i \geq 1$, is integrable.

**Proof of Example 2.** Recall that $a_i = e_{11}(i)e_{12}(i + 1)$ and the automorphism $\phi_i$ is a conjugation by $(1 + a_i)^{-1}, \quad i \geq 1$. Let $a_0 = e_{12}(1)$. Obviously, $(1 + a_i)^{-1} = 1 - a_i$ for $i \geq 0$. We claim that the sequence (7) is not integrable. We will use induction on $i$ to prove that

$$e_{12}(1) + e_{12}(1)e_{12}(2) + \cdots + e_{12}(1)e_{12}(2) \cdots e_{12}(i + 1).$$

For $i = 0$ the assertion is obvious. Consider the element

$$(1 + a_{i+1}) \left( \sum_{k=1}^{i+1} e_{12}(1) \cdots e_{12}(k) \right) (1 - a_{i+1}).$$

For an arbitrary $k, \quad 1 \leq k \leq i + 1$, we have

$a_{i+1}e_{12}(1) \cdots e_{12}(k)a_{i+1} = e_{11}(i + 1)e_{12}(i + 2)e_{12}(1) \cdots e_{12}(k)e_{11}(i + 1)e_{12}(i + 2) = 0,$

since

$$e_{12}(i + 2)^2 = 0.$$ 

Hence

$$(1 + a_{i+1}) \left( \sum_{k=1}^{i+1} e_{12}(1) \cdots e_{12}(k) \right) (1 - a_{i+1}) = \sum_{k=1}^{i+1} e_{12}(1) \cdots e_{12}(k) + \left[ a_{i+1}, \sum_{k=1}^{i+1} e_{12}(1) \cdots e_{12}(k) \right].$$
Since elements from different tensor factors commute, we get for $1 \leq k \leq i$
\[
\left[ e_{11}(i+1)e_{12}(i+2), e_{12}(1) \cdots e_{12}(k) \right] = 0.
\]
For $k = i + 1$
\[
\left[ e_{11}(i+1)e_{12}(i+2), e_{12}(1) \cdots e_{12}(i)e_{12}(i+1) \right] =
\]
\[
e_{12}(1) \cdots e_{12}(i) \left[ e_{11}(i+1), e_{12}(i+1) \right] e_{12}(i+2) =
\]
\[
e_{12}(1) \cdots e_{12}(i+2).
\]

So, (14) holds. Since the elements $e_{12}(1) \cdots e_{12}(i)$, $i \geq 1$, are linearly independent in the algebra $A$, we conclude that the subspace spanned by the elements
\[
\phi_i \cdots \phi_1(e_{12}(1)) =
\]
\[
(1 + a_i) \cdots (1 + a_i)e_{12}(1)(1 + a_i)^{-1} \cdots (1 + a_i)^{-1}, \quad i \geq 1,
\]
is infinite-dimensional. Hence, the sequence (7) is not integrable.

By Theorem 5 the injective endomorphism $\phi = \phi_1\phi_2 \cdots$ is not surjective. Hence, the subalgebra $B = \phi(A)$ is isomorphic to $A$ and $B \subseteq A$. This is another proof of Theorem 10 from [10].

In the next chapter we will use the following Lemma.

**Lemma 5.** Let $A$ be a countable–dimensional locally matrix algebra. Let $e \in A$ be an idempotent. Then every automorphism of $eAe$ extends to an automorphism of $A$.

**Proof.** At first, let us assume that the algebra $A$ is unital. Let $\varphi$ be an automorphism of the subalgebra $eAe$. If the automorphism $\varphi$ is inner then there exists an invertible element $x_e$ in the subalgebra $eAe$ such that
\[
\varphi(a) = x_e^{-1}ax_e \quad \text{for all elements} \quad a \in eAe.
\]

In this case, the element $x = x_e+(1-e)$ is invertible in $A$. The automorphism of conjugation $a \mapsto x^{-1}ax$, $a \in A$, extends the automorphism $\varphi$.

Let $\varphi$ not be an inner automorphism. Then choose subalgebras $A_1 \subseteq A_2 \subseteq A$ such that $1, e \in A_1$ and $A_1 \cong M_m(\mathbb{F})$ for some $m \geq 1$, and
\[
\varphi(eA_1e) \subseteq eA_2e \quad \text{and} \quad A_2 \cong M_n(\mathbb{F}) \quad \text{for some} \quad n \geq 1.
\]
Let $\varphi' := \varphi|_{eA_1e}$ be the restriction of $\varphi$ to the subalgebra $eA_1e$, so that
\[
\varphi' : eA_1e \to \varphi(eA_1e), \quad \text{and} \quad \varphi' : e \mapsto e.
\]

By the Skolem–Noether Theorem (see [6, 7]) there exists an invertible element $x_e \in eA_2e$ such that
\[
\varphi'(a) = x^{-1}_e ax_e \quad \text{for all elements} \quad a \in eA_1e.
\]
Now, let us consider the automorphism
\[ \psi' : eAe \to eAe, \quad a \mapsto x_e^{-1}ax_e. \]
As we have shown above, the inner automorphism \( \psi' \) of the subalgebra \( eAe \) extends to some automorphism \( \psi \) of the algebra \( A \). So, it is sufficient to show that the automorphism \( \psi' \circ \varphi \in \text{Aut}(eAe) \) extends to some automorphism \( \tilde{\varphi} \) of \( A \). Then the automorphism \( \varphi \) extends to the automorphism \( \psi \circ \tilde{\varphi} \) of \( A \).

Let \( \varphi_1 := \psi' \circ \varphi \). The composition \( \varphi_1 \) leaves every element from \( eA_1e \) fixed. Let \( C \) be the centralizer of the subalgebra \( A_1 \) in \( A \). Then
\[ A = A_1 \otimes_F C \quad \text{and} \quad eAe = eA_1e \otimes_F C. \]
Since the subalgebra \( e \otimes_F C \) is the centralizer of \( eA_1e \) in \( eAe \) it follows that \( e \otimes_F C \) is invariant with respect to \( \varphi_1 \). Hence, there exists an automorphism \( \theta \in \text{Aut}(C) \) such that
\[ \varphi_1(a \otimes c) = a \otimes \theta(c) \quad \text{for arbitrary elements} \quad a \in eAe, \quad c \in C. \]
Now, the automorphism
\[ \tilde{\varphi} : A \to A, \quad \tilde{\varphi}(a \otimes c) = a \otimes \theta(c), \quad a \in A_1, \quad c \in C, \]
extends \( \varphi_1 \).

We have proved the Lemma in the case when the algebra \( A \) is unital.

Now suppose that the algebra \( A \) is not unital. Then there exists a sequence of idempotents \( e_i \in A, i \geq 1 \), such that
\[ e_1 = e, \quad e_1Ae_1 \subset e_2Ae_2 \subset \cdots \quad \text{and} \quad \bigcup_{i \geq 1} e_iAe_i = A. \]
By what we proved above, there exists a sequence of automorphisms
\[ \varphi_i \in \text{Aut}(e_iAe_i), \quad \varphi_1 = \varphi \quad \text{and} \quad \varphi_{i+1} \big|_{e_iAe_i} = \varphi_i. \]
The union
\[ \tilde{\varphi} = \bigcup_{i \geq 1} \varphi_i \]
is an automorphism of \( A \) that extends \( \varphi \). \( \square \)

6. Dimensions of Lie Algebras of Derivations and Orders of Groups of Automorphisms

In the proofs of Theorems 6, 7 we will use the following nontrivial Theorem from Linear Algebra, that is due to P. Erdös and I. Kaplanskiy; see [8, 9].

Let \( V \) be a vector space over a field \( \mathbb{F} \) of infinite dimension \( d \). Let \( V^* \) be the dual space, that is the space of all functionals \( V \to \mathbb{F} \).

\[ ^1 \text{The author is grateful to V. V. Sergeichuk for this reference} \]
**Theorem 9** (P. Erdős, I. Kaplanskiy).

\[ \dim_F V^* = |F|^d. \]

*Proof of Theorem 6.* Consider the vector space \( \text{Lin}(A) \) of all linear transformations \( A \to A \). Obviously,

\[ \dim_F \text{Der}(A) \leq \dim_F \text{Lin}(A) \leq |\text{Lin}(A)|. \]

The dimension of the algebra \( A \) is equal to \(|I|\). The cardinality of the set \( \text{Lin}(A) \) does not exceed the cardinality of all \((I \times I)\)-matrices over the field \( F \), the latter being equal to

\[ |\text{Map}(I \times I, F)| = |F|^{|I \times I|} = |F|^{|I|}, \]

since \(|I|^2 = |I|\); see [12]. We proved that

\[ \dim_F \text{Der}(A) \leq |F|^{|I|}. \]

For an arbitrary index \( i \in I \) choose an element \( 0 \neq a_i \in A_0^I \). Let \( \mathcal{P} \) be the system of all one–element subsets of \( I \). Clearly, the system \( \mathcal{P} \) is sparse.

For an arbitrary mapping \( f : I \to F \) consider the derivation

\[ d_f = \sum_{i \in I} f(i) \text{ad}_A(a_i) \in D_{\mathcal{P}}. \]

By Theorem [2] (2), the mapping \( f \to d_f \) is an embedding of the vector space \( \text{Map}(I, F) \) into the vector space \( \text{Der}(A) \). By the Erdős–Kaplanskiy Theorem (see Theorem [9]) we have

\[ \dim_F \text{Map}(I, F) = |F|^{|I|}. \]

Hence

\[ |F|^{|I|} \leq \dim_F \text{Der}(A), \quad \text{and finally} \quad \dim_F \text{Der}(A) = |F|^{|I|}. \]

The dimension of the Lie algebra \( \text{Inder}(A) \) is equal to \(|I|, |I| < |F|^{|I|} \).

This implies that the equality \([8]\) holds.

*Proof of Theorem 7.* Let \( A \) be a countable–dimensional locally matrix algebra over a field \( F \). Assume at first, that the algebra \( A \) is unital. Then by the Köthe’s Theorem [9], the algebra \( A \) is isomorphic to a countable tensor product of finite–dimensional matrix algebras. Now, the Theorem immediately follows from Theorem [6].

Suppose now that the algebra \( A \) is not unital. As above,

\[ \dim_F \text{Der}(A) \leq \dim_F \text{Lin}(A) \leq |\text{Lin}(A)| = |F|^{|I|}. \]

Let \( e \) be an idempotent of the algebra \( A \) such that the subalgebra \( eAe \) is infinite–dimensional. By Lemma [3] every derivation of the subalgebra \( eAe \) extends to a derivation of the algebra \( A \). The algebra \( eAe \)
is countable–dimensional and unital. From what we proved above, it follows that
\[ |F|^\aleph_0 = \dim_F \text{Der}(eAe) \leq \dim_F \text{Der}(A). \]

We proved that in this case
\[ \dim_F \text{Der}(A) = |F|^\aleph_0. \]

Now, it remains only to consider the case when \( \dim_F eAe < \infty \) for all idempotents \( e \in A \). By Lemma 4, in this case \( A \cong M_\infty(F) \). For an arbitrary mapping \( f : \mathbb{N} \to F \) consider the infinite diagonal matrix
\[ d_f = \text{diag}(0, f(1), f(2), \ldots). \]

The matrix \( d_f \) is not necessarily finitary, but
\[ [d_f, M_\infty(F)] \subseteq M_\infty(F). \]

Hence,
\[ \text{ad}_{M_\infty(F)}(d_f) : x \mapsto [d_f, x], \]
is a derivation of the algebra \( M_\infty(F) \). The mapping
\[ f \to \text{ad}_{M_\infty(F)}(d_f) \]
is an embedding of vector spaces
\[ \text{Map}(\mathbb{N}, F) \to \text{Der}(M_\infty(F)). \]

By the Erdős–Kaplanskiy Theorem (see Theorem 9),
\[ \dim_F \text{Map}(\mathbb{N}, F) = |F|^\aleph_0. \]

Hence
\[ |F|^\aleph_0 \leq \dim_F \text{Der}(M_\infty(F)). \]

We proved that
\[ \dim_F \text{Der}(M_\infty(F)) = |F|^\aleph_0. \]

Since the Lie algebra \( \text{Inder}(A) \) is countable–dimensional and \( \aleph_0 < |F|^\aleph_0 \), it follows that
\[ \dim_F \text{Outder}(M_\infty(F)) = |F|^\aleph_0. \]

□

**Proof of Theorem 8** As above, we start with the case when the algebra \( A \) is unital. So,
\[ A \cong \bigotimes_{i=1}^\infty A_i, \quad A_i \cong M_{n_i}(F), \quad n_i \geq 2, \quad i \geq 1. \]

Let
\[ PGL(n_i, F) = GL(n_i, F) / F^*. \]
denote the projective linear group. Consider the set $$F$$ of mappings
\[ f : \mathbb{N} \to \bigcup_{i=1}^{\infty} \text{PGL}(n_i, \mathbb{F}) \]
such that $$f(i) \in \text{PGL}(n_i, \mathbb{F})$$ for all $$i \in \mathbb{N}$$. It is easy to see that $$|F| = \aleph_0$$. For an invertible element $$a \in A$$ let $$\hat{a}$$ denote the automorphism of conjugation by $$a$$. In Example 1, we showed that the sequence of inner automorphism $$f(i)^{-1}$$, $$i \geq 1$$, is integrable. Hence by Theorem 5, the infinite product
\[ \varphi_f = f(1)f(2) \cdots \]
is an automorphism of the algebra $$A$$.

Let us show that the mapping $$f \mapsto \varphi_f$$ is injective. Let $$f, g \in F$$ and $$\varphi_f = \varphi_g$$. Applying automorphisms $$\varphi_f, \varphi_g$$ to $$A_1$$, we see that
\[ f(1)|_{A_1} = g(1)|_{A_1}. \]
Hence $$f(1) = g(1)$$. Therefore $$f(2)f(3) \cdots = g(2)g(3) \cdots$$. Applying both sides to $$A_2$$, we get $$f(2) = g(2)$$ and so on. So, $$|\text{Aut}(A)| \geq |F|^{\aleph_0}$$.

On the other hand,
\[ |\text{Aut}(A)| \leq |\text{Lin}(A)| = |F|^{\aleph_0}. \]
We proved that for a unital algebra $$A$$
\[ |\text{Aut}(A)| = |F|^{\aleph_0}. \]

Now, let the algebra $$A$$ be not unital. Suppose that $$A$$ contains an idempotent $$e$$ such that $$\dim_e eAe = \aleph_0$$. The algebra $$eAe$$ is unital. Hence by what we proved above and by Lemma 5,
\[ |F|^{\aleph_0} = |\text{Aut}(eAe)| \leq |\text{Aut}(A)| \leq |\text{Lin}(A)| = |F|^{\aleph_0}, \]
which implies $$|\text{Aut}(A)| = |F|^{\aleph_0}$$.

It remains to consider the case, when $$\dim_e eAe < \infty$$ for all idempotents $$e \in A$$. By Lemma 4, $$A \cong M_\infty(\mathbb{F})$$. For an arbitrary mapping $$f : \mathbb{N} \to \mathbb{F}$$ consider the invertible infinite matrix
\[ a_f = \text{Id} + \sum_{i=1}^{\infty} f(i)e_{2i-1,2i}, \]
where $$\text{Id}$$ is the identity $$(\mathbb{N} \times \mathbb{N})$$-matrix and $$e_{i,j}$$ are matrix units. The matrices $$a_f$$ are not finitary but
\[ a_f^{-1}M_\infty(\mathbb{F})a_f = M_\infty(\mathbb{F}). \]
Let $$\hat{a}_f$$ denote the automorphism of conjugation by $$a_f$$. The mapping
\[ f \mapsto \hat{a}_f \in \text{Aut}(M_\infty(\mathbb{F})) \]
is injective since
\[ a_f^{-1}e_{1,2i-1}a_f = e_{1,2i-1} + f(i)e_{1,2i} \quad \text{for} \quad i \geq 1. \]
Hence \[ |\mathcal{F}|^{\aleph_0} = |\text{Map}(\mathbb{N}, \mathcal{F})| \leq |\text{Aut}(M_\infty(\mathcal{F}))| \leq |\text{Lin}_\mathcal{F}(A)| = |\mathcal{F}|^{\aleph_0}. \]

\[ \square \]

\section*{References}

[1] S. Ayupov and K. Kudaybergenov, Infinite dimensional central simple regular algebras with outer derivations, \textit{Lobachevskii Journal of Mathematics} \textbf{41} (no. 3) (2020).
[2] O. Bezushchak and B. Oliynyk, Unital locally matrix algebras and Steinitz numbers, \textit{J. Algebra Appl.} (2020) doi: 10.1142/S0219498820501807.
[3] O. Bezushchak and B. Oliynyk, Primary decompositions of unital locally matrix algebras, \textit{Bull. Math. Sci.} (2020) doi: 10.1142/S166436072050006X.
[4] O. Bezushchak and B. Oliynyk, Morita equivalent unital locally matrix algebras, \textit{Algebra Discrete Math.}, \textbf{29} (2020), no. 2, pp. 173–179.
[5] J. G. Glimm, On a certain class of operator algebras, \textit{Trans. Amer. Math. Soc.} \textbf{95} (no. 2) (1960) 318–340.
[6] Yu. A. Drozd, V. V. Kirichenko, Finite Dimensional Algebras, Springer-Verlag, Berlin–Heidelberg–New York (1994).
[7] I. N. Herstein, Noncommutative Rings, Cambridge University Press (1968).
[8] Jacobson N. Lectures in abstract algebra. Volume 2. Linear algebra, Springer-Verlag (1975).
[9] G. K"{o}the, Schiefk"{o}rper unendlichen Ranges uber dem Zentrum, \textit{Math. Ann.} \textbf{105} (1931) 15–39.
[10] A. Kurosh, Direct decompositions of simple rings, \textit{Rec. Math. [Mat. Sbornik] N.S.} \textbf{11} (53) (no. 3) (1942) 245–264.
[11] R. S. Pierce, Associative Algebras, Graduate Texts in Mathematics, \textbf{88}, Springer, New York (1982).
[12] Jean E. Rubin, Set Theory for the Mathematician, New York: Holden-Day (1967).
[13] Strade, H. Locally finite dimensional Lie algebras and their derivation algebras, \textit{Abh. Math. Sem. Univ. Hamburg} \textbf{69} (1999) 373–391.
[14] S. Willard, General Topology, Mineola, New York: Dover Publications (2004).