Continuum mechanics and thermodynamics in the Hamilton and the Godunov-type formulations

Received: 15 September 2017 / Accepted: 9 January 2018 / Published online: 18 January 2018
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Abstract Continuum mechanics with dislocations, with the Cattaneo-type heat conduction, with mass transfer, and with electromagnetic fields is put into the Hamiltonian form and into the form of the Godunov-type system of the first-order, symmetric hyperbolic partial differential equations (SHTC equations). The compatibility with thermodynamics of the time reversible part of the governing equations is mathematically expressed in the former formulation as degeneracy of the Hamiltonian structure and in the latter formulation as the existence of a companion conservation law. In both formulations the time irreversible part represents gradient dynamics. The Godunov-type formulation brings the mathematical rigor (the local well posedness of the Cauchy initial value problem) and the possibility to discretize while keeping the physical content of the governing equations (the Godunov finite volume discretization).

Keywords Godunov · GENERIC · Hyperbolic · Hamiltonian · Continuum thermodynamics · Non-equilibrium thermodynamics

1 Introduction

Results of experimental observations are seen in mathematical models as properties of solutions of their governing equations. Universality of some results translates in the models into the universality of the mathematical
structure of the equations. For instance one of the universal experimental observations is the approach of externally unforced macroscopic systems to the thermodynamic equilibrium states at which the behavior is found to be well described by the classical equilibrium thermodynamics. The structure responsible for this observation has been identified to be the following. The irreversible part of the time evolution is of gradient type generated by a potential called entropy. This potential is then preserved in the reversible part of the time evolution that is Hamiltonian, since it is the part of the time evolution directly inherited from the underlying microscopic mechanics. The time-evolution equations involving both the Hamiltonian and the gradient part have appeared first in [19], in [41] (that was presented at the AMS-IMS-SIAM Joint Summer Research Conference in the Mathematical Sciences on Fluids and Plasmas “Geometry and Dynamics”, held at the University of Colorado, Boulder, CO, USA, 17–23 July 1983) and in [5,42,47,56,65,69,70]. In [47,70], the abstract equation (14) has been called GENERIC.

In the smaller pool of governing equations that arise in continuum mechanics, we may expect to identify even stronger common structure. In particular, what is missing in GENERIC is the guarantee that the initial value problem is well posed at least locally in time. This is indeed one of the basic validations of a model. The time evolution is seen to exist and thus the time evolution seen in the model should also exist. Another aspect that is not included in GENERIC is the adaptability of the governing equation to searching their solutions numerically (although numerical schemes for partial differential equations exploiting the Hamiltonian structure are getting more and more attention [57]).

A structure addressing both of these aspects have been identified by Godunov [34,35] for a system of the first-order partial differential equations having the form of local conservation laws (i.e., equations having the form of the time derivative of a field equals divergence of a flux). The compatibility with thermodynamics of these equations (representing the time reversible part of the time evolution) is expressed by the existence of a companion local conservation law for a new field that is an appropriately regular and convex function of the fields playing the role of state variables. It is very interesting that this requirement of the agreement with experimental observations implies then directly the mathematical well posedness [the companion local conservation law enables to symmetrize the system of the first-order hyperbolic equations which then implies their well posedness (local in time)]. Godunov has also introduced a discretization (the modification of the governing equations needed when computers are brought to assistance in the process of finding details of their solutions) preserving the physical content of the original partial differential equations [33].

In this paper we introduce a large family of partial differential equations that:
1. Govern the time evolution of fluids, solids without and with dislocations, the Cattaneo-type heat conduction and the electromagnetic fields, all coupled together.
2. Possess the GENERIC structure.
3. Possess the SHTC structure that extends to Godunov structure to a larger class of the first-order partial differential equations.

The GENERIC route begins with the well-known Hamiltonian structure of the Euler hydrodynamic equations [1,15]. By adopting the field of labels into the set of state variables, we pass to the Lagrangian formulation and to the solid mechanics without dislocations [43] and (by using the extension developed in this paper) with dislocations. This Hamiltonian formulation of the reversible part of the continuum mechanics is then coupled to the Hamiltonian formulation of the Cattaneo-like heat conduction and to the Hamiltonian formulation of the time evolution of electromagnetic fields.

The Godunov route begins with the Lagrangian formulation in which the time reversible governing equations have the form of the local conservation laws and the original Godunov analysis thus applies to them. By passing to the Eulerian formulation (the Lagrange → Euler passage is worked out in detail in this paper) the governing equations do not remain to be, at least in general, local conservation laws. Their weakly non-local extension is, however, shown to possess a new structure (called SHTC structure) that also allows symmetrization, and thus the mathematical well posedness, as well as the applicability of the Godunov discretization, is guaranteed for them. The weakly non-local extension (having the physical meaning, for example, of including dislocations into the consideration) consists of:
1. Constructing new local conservation laws for new fields that are appropriate spatial derivatives of the original fields included in the set of the state variables (e.g., the vorticity) by appropriately combining the original system of equations with equations obtained by applying appropriate spatial derivatives on the original system of equations.
2. Adopting the new fields (that are appropriate spatial derivatives of the fields included in the original set of the state variables) as new independent fields.
Both the GENERIC and the Godunov routes lead to the same equations. Having proven that the equations possess both the GENERIC and the SHTC structures, all the results and techniques that are available for these structures (in particular the physical, the mathematical, and the numerical well posedness) are now available for the governing equations investigated in this paper.

Because the paper discusses two quite different approaches which are based on different theoretical apparatuses, we have tried to provide all essential ingredients of the approaches important for their understanding. This, however, results in an excessive length of the paper which does not fit to the standards of an ordinary journal article. Therefore, all technical details are kept in the e-print version of the paper [76], which can be considered as the supplementary materials.

2 Preliminaries

2.1 SHTC framework, symmetric hyperbolic equations, well posedness, causality

The origin of the SHTC formulation of continuum mechanics should be attributed to the work of Godunov [34], where he considered “an interesting class” of nonlinear conservation laws. The key motivation (from the words of S.K. Godunov) for this seminal paper was the will to understand what physical principles may guarantee the well posedness of the initial value problem (IVP), at least local in time, for a nonlinear system of time-dependent PDEs. Indeed, when one deals with nonlinear dynamical phenomena, and in particular with nonlinear time-dependent partial differential equations (PDEs), perhaps, the first examination a new model has to pass is to verify if the IVP is well posed, at least locally, i.e., whether the solution to the system of PDEs with given initial data exists, is unique and stable (depends continuously on the initial data). We emphasize that the well posedness of the IVP should not be considered as a purely mathematical requirement but as a fundamental physical observation about the time evolution of physical systems, i.e., exactly as we consider causality, conservation and thermodynamic principles, Galilean or Lorentz invariance to be essential features of macroscopic time evolution. In other words, a model describing the time evolution of a physical system and having an ill-posed IVP should be regarded as wrong. Moreover, the well posedness of the IVP is a fundamental property of time-dependent PDEs in order for them to be numerically solvable.

Not all nonlinear systems of PDEs, of course, have well-posed Cauchy problem. This even cannot be guaranteed for a first-order quasilinear system [4,18,67], or for models which were believed to be consistently derived from microscopic theories as, for example, the Burnett equations derived from the gas kinetic theory [6,91,92]. So how can one guarantee that the IVP for a new nonlinear continuum mechanics model is well posed?

One possibility is to develop the model within a very important subclass of first-order systems

\[ \mathbf{A}(\mathbf{p}) \frac{\partial \mathbf{p}}{\partial t} + \mathbf{B}_k(\mathbf{p}) \frac{\partial \mathbf{p}}{\partial x_k} = 0, \]

where \( \mathbf{A}^T = \mathbf{A} > 0 \) and \( \mathbf{B}_k^T = \mathbf{B}_k \), for which local well posedness is known to hold true [4,55,67,88]. This subclass is called symmetric hyperbolic systems of PDEs, which is a generalization of Friedrichs-symmetrizable linear systems [26]. However, one should naturally question how restrictive it is for a model to be symmetric hyperbolic and, at the same time, represent diverse physical phenomena admitting continuum mechanics description? As it was shown by Godunov [34–36] and later by others [7,27,89] [see also an interesting historical remark in [90], pp. 39–40] there is, in fact, an intimate connection between the symmetric hyperbolicity and thermodynamics for a specific subclass of system of conservation laws.\(^1\)

In particular, if a first-order system of conservation laws\(^2\) for conservative state variables \( \mathbf{q} = (q_1, q_2, \ldots, q_n) \)

\[ \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}^k(\mathbf{q})}{\partial x_k} = 0 \]

\(^1\) It is also important to recall that apart of the local well posedness of the IVP the hyperbolicity also naturally accounts for another fundamental observation about time evolution of physical systems, namely the finite velocity for any perturbation propagation, i.e., causality.

\(^2\) For simplicity, we do not consider production algebraic terms on the right-hand side of system (2). However, it is important to remark that such source terms being low-order terms do not affect Godunov’s observation and can be easily added. Their addition should, however, be consistent with some fundamental physical principles like the energy conservation principle, see discussion in Sect. 5. What is really important for original Godunov’s observation is that the equations should have the so-called divergence form.
admits an extra conservation law
\[ \frac{\partial E(q)}{\partial t} + \frac{\partial G^k(q)}{\partial x_k} = 0 \] (3)
for a strictly convex potential \( E(q) \), which plays the role of the total energy\(^3\) for the system, then such a system can be parametrized with new state variables \( p = (p_1, p_2, \ldots, p_n) \) and a potential \( L(p) \) as follows (which shall be referred to as the Godunov form of conservation laws)
\[ \frac{\partial L_p}{\partial t} + \frac{\partial L^k_p}{\partial x_k} = 0, \] (4)

where, as throughout this paper, \( L_p \) and \( L^k_p \) are the vectors with components \( L_{pi} = \partial L / \partial p_i, L^k_{pi} = \partial L^k / \partial p_i \), see below for the concrete form of these vectors for the conservative system. It is obvious that (4) can be rewritten in a symmetric quasilinear form
\[ L_{pi,pj} \frac{\partial p_j}{\partial t} + L^k_{pi,pj} \frac{\partial p_j}{\partial x_k} = 0. \] (5)

Moreover, the conservative state variables \( q \) and the potential \( E(q) \) are related to the new variables \( p \) and potential \( L(p) \) as
\[ q = L_p, \quad E(q) = p \cdot L_p - L(p) = p \cdot q - L(p) \] (6)
and vice versa,
\[ p = E_q, \quad L(p) = q \cdot E_q - E(q) = q \cdot p - E(q). \] (7)

The relations \( p = E_q \) and \( q = L_p \) are one-to-one relations because we assume that \( E(q) \) is convex and hence, the Hessian \( \frac{\partial^2}{\partial q \partial q} E = E_{qq} = (L_{pp})^{-1} > 0 \). Thus, the convexity of \( E(q) \), in fact, means that (5) is symmetric hyperbolic. This result is sometimes referred to as the Godunov–Friedrichs–Lax (GFL) theorem [28]; however, it rather should be called Godunov–Boillat theorem because the Godunov [34–36] and Boillat [7] symmetrization does not imply a change of the original equations while the Friedrichs-Lax symmetrization implies a multiplication of the original PDE system by a matrix which in particular means that weak solutions to the original system and the symmetrized system are different in general.

The key point of the Godunov observation is that the system (2) and conservation law (3) together form an overdetermined system of PDEs, i.e., the number of PDEs is \( n + 1 \) while the number of state variables is \( n \) (\( E \) is not an unknown but the potential \( E(q) \)), and thus, in order such a system has a solution, one of the conservation laws should be a consequence of the others. Therefore, the simultaneous validity of (2) and (3) implies that
\[ E_q \cdot \left( \frac{\partial q}{\partial t} + \frac{\partial F^k(q)}{\partial x_k} \right) \equiv \frac{\partial E(q)}{\partial t} + \frac{\partial G^k(q)}{\partial x_k}, \] (8)

where the dot denotes the scalar product. After introducing the potentials \( L^k(p) = E_q \cdot F^k(q) - G^k(q) = p \cdot F^k(q) - G^k(q) \) and noting that \( L_{pi}^k = F^k \), the energy conservation (3) can be rewritten as
\[ \frac{\partial (p_i L_{pi} - L)}{\partial t} + \frac{\partial (p_i L_{pi}^k - L^k)}{\partial x_k} = 0. \] (9)

In other words, \( L \) is the Legendre-conjugate energy and \( L^k \) is Legendre conjugation of energy flux. Therefore, returning back to our last question, such a subclass of symmetric hyperbolic PDEs can be associated with the thermodynamically compatible systems of first-order nonlinear conservation laws.

However, it turns out that there are some important continuum mechanics models, like the ideal magnetohydrodynamics (MHD) equations, for example, which seems to be represented by systems of conservation laws (2) with an additional conservation law (3) but still can not be symmetrizable in such a way. A more attentive analysis by Godunov of this inconsistency of the MHD equations in [36] shows that, in fact, the MHD equations written in a classical conservative form are not compatible with the energy conservation, i.e., the energy conservation law is not the consequence of the mass, momentum, entropy and magnetic field conservation laws, like in (8). More precisely, the energy flux of the MHD equation is inconsistent with the

\(^3\) See also an important remark about a possible interpretation of the accompanying law at the end of this Sect. 2.
other fluxes unless the non-conservative product $B_i v_i \frac{\partial B_k}{\partial x_k} \equiv 0$ is added to the left-hand side of (8), where $v_i$ and $B_i$ are the velocity and magnetic fields accordingly. We discuss this in more details in Sect. 3.2.3. Godunov concludes that the ideal MHD equations do not admit the original Godunov structure (4) but generalize it in the following way

$$\frac{\partial L_p}{\partial t} + \frac{\partial(v_k L)}{\partial x_k} + C_k \frac{\partial p}{\partial x_k} = 0,$$

(10)

where $C^T_k = C_k$ are symmetric matrices and $v_k$ being the velocity field.

Godunov and Romenski subsequently analyzed (in the nineties) various equations (ideal MHD equations, nonlinear elasticity, electrodynamics of slowly moving medium, superfluid flow, superconductivity) studied in the books of Landau and Lifshitz [59,60], see papers [20,29,31,38–40,87] (see also a discussion of the similar question in [8] by Boillat). They found that for a proper understanding of the structure of the continuum mechanics models it is important to distinguish the Lagrangian and Eulerian frames of reference, and that the original Godunov structure (4) is, in fact, rather inherent to the conservation laws written in the Lagrangian frame, while their Eulerian counterparts are inherently non-conservative time evolutions and have more complicated structure (10).

The departure point of the SHTC framework is the invariance under arbitrary rotations of the conservation laws in the Lagrangian frame which were proposed in [29] and can be derived from Hamilton’s principle of stationary action, see Sect. 3.1.5. Here, we also demonstrate for the first time that the Lagrangian SHTC equations can be generated by canonical Poisson brackets, see Sect. 4.1. Then, Eulerian non-conservative SHTC equations (10) are the direct consequence of the Lagrange-to-Euler transformation applied to the Lagrangian conservation laws.

It is the primary goal of this paper to demonstrate that the Eulerian non-conservative SHTC equations (10) are fully compatible with the GENERIC framework presented in the subsequent section. Another goal of the paper is to provide the reader with some non-trivial details of the Lagrange-to-Euler transformation between the conservative Lagrangian SHTC equations and their non-conservative Eulerian counterparts which have never been published before. Moreover, we review all properties of the SHTC formulation that were previously published rather separately in a series of paper by Godunov and Romenski [20,29,31,38,38–40,87].

2.2 GENERIC

General Equation for Non-Equilibrium Reversible-Irreversible Coupling (GENERIC) was first introduced in [47,70]. The fundamental idea behind GENERIC is that evolution equations for chosen state variables are generated from four building blocks: a Poisson bracket $\{\cdot, \cdot\}$, Hamiltonian $H = \int E d\mathbf{r}$, dissipation potential $\mathcal{S}$ and entropy functional $\mathcal{S} = \int S d\mathbf{r}$, where the potentials $E$ and $S$ are the total energy and entropy densities. Once these building blocks are at hand, the evolution equations follow by direct calculation.

The building blocks are required to fulfill certain degeneracies and conditions:

1. Entropy should be a Casimir of the Poisson bracket, which means that $\{\mathcal{S}, \mathcal{S}\} = 0$ for all functionals $\mathcal{S}$ of the state variables.
2. Energy is not changed by the dissipation potential, i.e., $\langle \mathcal{S}_E, \mathcal{S}_S \rangle = 0$, and
3. The dissipation potential is convex, i.e., $\langle \mathcal{S}_E, \mathcal{S}_S \rangle = 0$ for all entropies.

Apart from these conditions, the Poisson bracket has the following properties:

1. Antisymmetry, $\{\mathcal{A}, \mathcal{B}\} = -\{\mathcal{B}, \mathcal{A}\}$,
2. Leibniz rule, $\{\mathcal{A}, \mathcal{B} \cdot \mathcal{C}\} = \{\mathcal{A}, \mathcal{C}\} \mathcal{B} + \{\mathcal{B}, \mathcal{C}\} \mathcal{A}$,
3. Jacobi identity, $\{\mathcal{A}, \mathcal{B} \cdot \mathcal{C}\} + \{\mathcal{B}, \mathcal{C} \cdot \mathcal{A}\} + \{\mathcal{C}, \mathcal{A} \cdot \mathcal{B}\} = 0$.

Moreover, the Poisson bracket is constructed from a Poisson bivector $L^{ij}$ (antisymmetric twice contravariant tensor field) as

$$\{\mathcal{A}, \mathcal{B}\} = \mathcal{A}_{ij} L^{ij} \mathcal{B}_{kj}.$$

(11)

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4. These lengthy technical details are available as the supplementary materials [76].
5. $\langle \cdot, \cdot \rangle$ denoting integration ($L^2$ scalar product).
6. Note that subscripts denote functional derivatives $\mathcal{S}_x = \frac{\partial \mathcal{S}}{\partial x}$. In the special case when the functional is a spatial integral over a density which is a real-valued function of the state variables, the functional derivative becomes the usual derivative of the density with respect to the state variable. For example, $\mathcal{S} = \int E(x) d\mathbf{r}$ and $\mathcal{S}_x = \frac{\partial \mathcal{S}}{\partial x}$ if $E$ is a real-valued function of real variables.
Note that summation over an index can be interpreted as integral over fields. Finally, Hamiltonian evolution of functionals is given by

$$\frac{\partial A}{\partial t} = \{A, \mathcal{E}\}. \quad (12)$$

**GENERIC** then gives the following evolution equations for functionals of state variables $x$:

$$\frac{\partial A}{\partial t} = \{A, \mathcal{E}\} + \left\langle A_x, \frac{\delta \Xi}{\delta x} \right\rangle. \quad (13)$$

From antisymmetry of the Poisson bracket and from the degeneracy of the dissipation potential, it follows that energy is conserved, $\partial \mathcal{E}/\partial t = 0$. From the degeneracy of the Poisson bracket and from convexity of the dissipation potential, it follows that entropy grows, $\partial \mathcal{S}/\partial t \geq 0$. Moreover, when energy, entropy and the dissipation potential are taken as even functions with respect to time-reversal transformation, the Hamiltonian part generates reversible evolution while the dissipative part generates irreversible evolution [72]. Finally, Onsager–Casimir reciprocal relations are satisfied automatically [69,72]. These results manifest compatibility of GENERIC with thermodynamics.

Instead of working with evolution of functionals, which can be interpreted as a sort of weak formulation, one can recover the evolution equations of the state variables easily. Taking $A = x^i$, we obtain

$$\frac{\partial x^i}{\partial t} = L_{ij} E^j + \Xi_{,x^i}. \quad (14)$$

This last equation shows how to write down the evolution equations (usually partial differential equations) implied by GENERIC.

Since Poisson brackets are a cornerstone of GENERIC, a question often arises how the Poisson brackets can be derived from first principles. For example, the canonical Poisson bracket that generates Hamilton canonical equations in classical mechanics is generated by the canonical two-form present on the cotangent bundle of classical mechanics (positions and momenta of all particles). The canonical Poisson bracket can be thus interpreted as a geometric result of construction of the cotangent bundle. Another Poisson brackets can be derived from some more detailed Poisson brackets by projection [71], or directly from the underlying geometrical structure of the state variables (e.g., as Lie–Poisson brackets) [62].

It is, in fact, often more difficult to derive the dissipation potential. In the simplest cases dissipation potentials can be quadratic functions of derivatives of entropy. That is the setting of classical irreversible thermodynamics [48,69]. The matrix of second derivatives of the dissipation potential is then referred to as the dissipative matrix $M^{ij}$, and the evolution equations then become

$$\frac{\partial x^i}{\partial t} = L^{ij} E^j + M^{ij} \mathcal{S}_{,x^j} \quad \text{for} \quad \Xi = \frac{1}{2} \mathcal{S}_{,x^i} M^{ij} (x) \mathcal{S}_{,x^j}. \quad (15)$$

Apart from quadratic dissipation potentials, exponential dissipation potentials (like hyperbolic cosine) are used in case of chemical reactions and Boltzmann equation [44,47], and such dissipation potentials have also statistical reasoning [64].

Let us also discuss the roles of energy and entropy within the GENERIC and SHTC frameworks. In both frameworks, the vector field generating the time evolution (right-hand side of the evolution equations (15))) is a gradient of both the entropy and the energy transformed into a vector by two geometrical structures. The first one, the Hamiltonian structure, “ignores” the entropy and the second one, the gradient structure, “ignores” the energy. By “ignoring a geometrical structure” we mean that the structure is degenerate. In other words, in the mechanical (i.e., the Hamiltonian) part of the time evolution the entropy is conserved for any choice of energy while in the thermodynamic (i.e., the gradient) part it plays the role of the generating potential that grows during the time evolution. The energy is on the other hand preserved in both parts of the time evolution but for two very different reasons. In the mechanical part, it is preserved due to its role as the generating potential and due to the essential property of the Hamiltonian structure (the antisymmetry of the Poisson bracket), which leads to energy conservation as a companion conservation law. In the thermodynamic part of the time evolution, the energy is preserved due to the degeneracy of the gradient structure. The energy is a mechanical concept while the entropy is a non-mechanical concept that emerges in mesoscopic theories in order to account for the ignorance of microscopic details. In SHTC, entropy makes it possible to conserve energy even when dissipative terms are added—it preserves the companion law of energy conservation, and it also arises in the development
of shocks. A clarification of the relation between the shock formation and the emergence of dissipation is one of the questions that we intend to investigate in the future.

In this paper, we limit ourselves to the particular case in which the local entropy field is in a one-to-one relation to the energy field (or vice versa). We are making this simplifying assumption because only with the interchangeability of the local energy and the local entropy we are able to find in the illustrative examples that we discuss in this paper the Hamiltonian structures with the required degeneracies. From the physical point of view, the simplifying assumption is a weaker version of the local equilibrium assumption. We recall that the local equilibrium assumption consists of requiring: (i) that the state variables are the same as in the classical equilibrium thermodynamics, (ii) that they are also related to each other in the same way as in the equilibrium thermodynamics, (iii) but, contrary to the state variables in the classical equilibrium thermodynamics, they are fields, i.e., they depend on the position vector. We, on the other hand, allow other fields to be included into the set of state variables, but we still require that the fields of the energy and the entropy are among the fields forming the set of state variables and we also require that they share with the equilibrium thermodynamic energy and entropy the property that the derivative of the energy with respect to the entropy is always positive. In the classical equilibrium thermodynamics, the derivative is the absolute temperature that indeed is always positive.

In summary, the GENERIC framework is an alternative approach to non-equilibrium thermodynamics, which does not rely on classical balance laws and their extensions unlike classical non-equilibrium thermodynamics [48], rational thermodynamics [67] or extended irreversible thermodynamics [54]. The reversible (Hamiltonian) part of the evolution can be constructed by various methods, for example, by means of differential geometry (Lie groups, Lie algebras, semidirect products) or projections. The irreversible part is constructed as gradient dynamics satisfying the first and second laws of thermodynamics. Applicability of GENERIC is very wide, covering for example particle mechanics, kinetic theory, motions of fluids and solids, complex fluids, electromagnetism and mixtures.

3 SHTC formulation of continuum mechanics

In this section we present the SHTC formulation of continuum mechanics and discuss its properties in both Lagrangian and Eulerian frames. The Lagrangian SHTC equations are fully compatible with original Godunov’s structure (4). The Lagrangian SHTC equations can be also seen as Hamiltonian dynamics generated by a canonical Poisson bracket of the cotangent bundle of two vector fields, see Sect. 4.1. We then proceed with presenting the Eulerian SHTC equations, which are the equations of major interest in this paper. The derivation of Poisson brackets for the Eulerian SHTC equations and thus demonstrating their compatibility with GENERIC is presented in Sect. 4.

3.1 Lagrangian frame

In the series of papers [20,29,31,38,38–40,87], Godunov and Romenski investigated the question of whether time evolution equations of various continuum mechanics models share some common elements of a general structure. More specifically, in [29], they examined whether it is possible to conclude something meaningful about the structure of a PDE system representing a continuum mechanics model without specifying the physical settings but being based merely on universally accepted fundamental physical principles such as invariance principles, conservation principles, causality principle, the laws of thermodynamics and well posedness of the IVP. In [29], this question was considered in the Lagrangian coordinates and the following requirements to PDEs were imposed:

- PDEs are invariant under any rotations of the space.
- PDEs are compatible with an extra conservation law.
- PDEs are generated by only one thermodynamic potential.
- PDEs are nonlinear but have well posed (locally in time) IVP (e.g., they are first-order symmetric hyperbolic).
- PDEs are conservative and the fluxes are generated by invariant differential operators only, such as div, grad and curl.

Then, based on the analysis of various continuum mechanics models and the group representation theory, a quite general system (system (25) in [29]) of Lagrangian conservation laws satisfying all the above requirements
was proposed. Moreover, it has appeared that even a smaller system of PDEs is sufficient for many practical applications, and it was proposed to consider a sub-system which we shall refer to as the master system and is discussed in the following section.

3.1.1 Energy formulation

As will be shown throughout this paper, one of the apparent common things between the SHTC and GENERIC frameworks is the use of the state variables and their conjugates (by Legendre transformations) with respect to a potential, see also [75]. In the SHTC framework, the primary state variables are the density-like fields and the primary potential is the total energy. The conjugate state variables and potential have the meaning of the fluxes and a generalized pressure accordingly. In this section we present the Energy formulation of the SHTC Lagrangian equations, i.e., they are written in terms of the primary density fields and the total energy potential. The conjugate formulation is discussed in Sect. 3.1.2.

The master system is formulated in terms of the following state variables

\[ q = (m, F, h, e, w, \sigma) \]

and the thermodynamic potential

\[ U = U(q), \]

where \( m = [m_i] \) is a vector field with the meaning of the generalized momentum, e.g., see [17], \( F = [F_{ij}] \) is the deformation gradient, \( e = [e_j] \) and \( h = [h_i] \) are vector fields which can be interpreted as electromagnetic fields [17], \( \sigma \) is a scalar field which is transported according to the vector field \( w = [w_j] \). The later two are used for modeling of the mass and heat transfer where \( \sigma \) has the meaning of mass concentration or entropy, accordingly.

In fact, we do not need to give a precise meaning to the fields (16) and the further analysis is independent of specific physical interpretations. The meanings of the fields (16) depend on the specification of the generating potential \( U \) in the SHTC framework. For instance, in the example Sect. 6, we show that the same equations can describe the heat and mass transfer and the represented physics is fully determined by the way we define the generating potential.

Throughout the paper, we use the summation over the repeated indexes and we denote the partial derivative of the potentials with respect to the state variables as \( U_{qi} = \partial U / \partial q_i, U_q = [U_{qi}], U_{mi} = \partial U / \partial m_i \), etc.

A master system that fulfills all of the requirements listed in Sect. 2.1 was proposed in [29]. Although a more general master system can be constructed, see [29], the minimal system has been selected after analysis of a large number of continuum mechanics models. Such a minimal master system reads

\[ \begin{aligned}
\partial_t m_i - \partial_j U_{F_{ij}} &= 0, \\
\partial_t F_{ij} - \partial_j U_{m_i} &= 0, \\
\partial_t h_i + \varepsilon_{ijk} \partial_j U_{e_k} &= 0, \\
\partial_t e_i - \varepsilon_{ijk} \partial_j U_{h_k} &= 0, \\
\partial_t w_j + \partial_j U_{\sigma} &= 0, \\
\partial_t \sigma + \partial_j U_{w_j} &= 0,
\end{aligned} \]

where (as well as in Sect. 3.1) \( \partial_t = d/dt \) denotes the standard material derivative, \( \partial_j = \partial / \partial y_j \) are the spatial Lagrangian derivatives.

In addition, an extra conservation (companion conservation law) law is the consequence of the above PDEs

\[ \partial_t U - \partial_j (U_{e_i} U_{F_{ij}} + \varepsilon_{ijk} U_{e_i} U_{h_k} - U_{\sigma} U_{w_j}) = 0. \]

It can be obtained as the sum of the governing PDEs multiplied by the corresponding fluxes:

\[ (19) \equiv U_{m_i} \cdot (18a)_1 + U_{F_{ij}} \cdot (18a)_2 + U_{e_i} \cdot (18b)_1 + U_{h_i} \cdot (18b)_2 + U_{\sigma} \cdot (18c)_1 + U_{w_j} \cdot (18c)_2. \]

The potential \( U \) plays usually the role of the total energy density (per unit mass), e.g., \( U = \rho_0 \mathcal{W} \) with \( \mathcal{W} \) being the total specific energy per unit mass and \( \rho_0 \) a reference mass density.

The first pair of equations usually represents the equations of motion while the second pair, (18b), may represent a nonlinear generalization of the Maxwell equations. However, as we already mentioned, they may have a dramatically different interpretations. For example, in [75], the equations with the structure of (18b) were used to describe the dynamics of flow defects, while in Sect. 3.2.6, the Eulerian counterpart of (18b) is used to describe a weakly non-local dynamics.
3.1.2 Conjugate formulation and symmetric form

System (18) is a system of conservation laws and it is compatible with the accompanying conservation law (19), and hence, according to the Godunov–Boillat theorem, discussed in Sect. 2, it can be written in the Godunov form (4) and hence is symmetric hyperbolic. In order to see this explicitly, it is necessary to introduce fluxes of (18) as new state variables
\[ p = [p_i], \]
and a potential \( M(p) \) (it has the meaning close to the pressure).

The new variables and the potential \( (p, M) \) are thermodynamically conjugate to the pair \( (q, U) \) in the following sense
\[ p = U_q, \quad M(p) = q_i U_{q_i} - U = q_i p_i - U, \]
(i.e., \( M(p) \) is the Legendre transformation of \( U(q) \). Of course, we can go also in the opposite way,
\[ q = M_p, \quad U(q) = p_i M_{p_i} - M = q_i p_i - M. \]
In order to have a one-to-one relation between (21) and (22), it is necessary that \( U(q) \) be a convex function (\( M(p) \) is then also convex because of the properties of the Legendre transformation), i.e., \( U_{qq} = (M_{pp})^{-1} > 0 \).

3.1.3 Complimentary structure and odd–even parity

One may clearly note a complimentary structure of Eq. (18). A field under the time derivative, say \( m_i \) in (18a)1, appears in the flux of the complimentary equation (18a)2 in the partial derivative \( U_{m_i} \) of the generating potential \( U \). Thus Eq. (18) is split into complimentary pairs for \( (m_i, F_{ij}), (h_i, e_i) \) and \( (w_i, \sigma) \). Therefore, when one constructs a specific model in the SHTC framework, it is impossible to take only one PDE in a pair because its flux would be undetermined. However, an arbitrary number of ordinary differential equations
\[ \dot{q}_i = f_i(q), \]
can be added to system (18). Here, \( q_i \) might be scalars or entries of a vector or tensor field, \( f_i(q) \) are sufficiently smooth functions of the entire vector of state variables. For example, Eq. (18a) and ordinary differential equations of type (23) constitute the nonlinear elastoplasticity model studied in [30].

The complementarity structure of the SHTC equations can be also seen from the point of view of the odd–even parity with respect to time-reversal transformation (TRT) [69, 72], which has been playing a central role in thermodynamics for very long time. The odd–even parity of the state variables is also evident from the geometric form of the Lagrangian SHTC equations presented in Sect. 4.1.

3.1.4 Involution constraints

Solutions to the master system satisfy some stationary laws that are conditioned by the structure of the fluxes:
\[ \partial_k F_{ij} - \partial_j F_{ik} = const, \quad \partial_i h_i = const, \quad \partial_i e_i = const, \quad \partial_k w_j - \partial_j w_k = const, \]
In fact, these stationary laws impose constraints on the initial data and hold for every \( t > 0 \) if they were satisfied at \( t = 0 \). Indeed, applying the divergence operator, for instance, to Eq. (18b)2 we obtain \( \partial_t (\partial_i e_i) = 0 \). Thus, the stationary laws (24) should be considered as involution constraints.

3.1.5 Variational nature of the SHTC equations

The master system was proposed in [29] based merely on the requirements listed at the beginning of Sect. 3.1. In supplementary materials [76], we show that the master system can be also obtained from Hamilton’s principle of stationary action if the Lagrangian is taken as a partial Legendre transformation of the total energy potential and stationary laws (24) have zeros on the right-hand sides. Partly, these results were published in Chapter 23 of the Russian edition [38] of the book [39] as well as in the recent paper [17].

\[ ^7 \text{See also e-print version [76] for an extended discussion.} \]
3.2 Eulerian frame

We now present the Eulerian SHTC equations, which later will be shown to be compatible with GENERIC in Sect. 4. We first present the equations which are directly obtained from the Lagrangian master system (18) by means of the Lagrange-to-Euler change of the spatial variables. In particular, the time and spatial derivatives transform as follows,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k}, \quad \frac{\partial}{\partial y_j} = F_{kj} \frac{\partial}{\partial x_k}. \quad (25)$$

The resulting Eulerian master system has a more complicated structure of PDEs than its Lagrangian counterpart (18). The main difference is that, in contrast to the fully conservative Lagrangian master system, the Eulerian equations are inherently non-conservative and do not admit the original Godunov structure (4).

Instead, the Eulerian equations have a generalized structure (from now on, $\partial_k$ stands for the partial derivatives with respect to the Eulerian coordinates $x_k$)

$$\partial_t L_p + \partial_k (v_k L)_p + C_k \partial_k p = 0, \quad (26)$$

where one can recognize the Godunov structure (4) in the first two terms, and where $C_k^T = C_k$ are some symmetric matrices. The reason for the insufficiency of the original conservative Godunov structure (4) is the fact that the involution constraints in the Eulerian frame are not just auxiliary equations as in the Lagrangian frame, but they are an intrinsic part of the structure of the Eulerian SHTC equations and contribute into matrices $C_k$.

As in the previous section, we start the discussion with the Eulerian SHTC equations in terms of the total energy potential and density-type state variables. We then discuss the role of the involution constraints, and only for pedagogical reasons we present a fully conservative reformulation of the SHTC equations in Sect. 3.2.2. This reformulation violates Galilean invariance of the Eulerian equations, and thoughtless use of the conservative formulation should be avoided in practice. We conclude the section by presenting the conjugate formulation demonstrating how the involution constraints essentially affect the symmetrization procedure of the SHTC equations in the Eulerian frame.

3.2.1 Energy formulation

Transformation from the Lagrangian SHTC master system (18) to the Eulerian equations has been done in [29] while omitting non-trivial details of the calculation, which can be now found in Appendix of [76]. The resulting non-conservative Eulerian system of equations is

$$\partial_t m_i + \partial_k (m_i v_k + \delta_{ik} (\rho E_\rho + \sigma E_\sigma + m_j E_m + e_l E_e + h_l E_h) - E) - e_l E_e_l - h_k E_h_k + v_j E_w_j + A_{kj} E_\alpha_j = 0, \quad (27a)$$

$$\partial_t A_{ik} + \partial_k (A_{ij} v_l) + v_j (\partial_j A_{ik} - \partial_k A_{ij}) = 0, \quad (27b)$$

$$\partial_t h_l + \partial_k (h_j v_k - v_l h_k + \epsilon_{jkl} E_e_l) + v_l \delta_l h_k = 0, \quad (27c)$$

$$\partial_t e_l + \partial_k (e_l v_k - v_l e_k - \epsilon_{jkl} E_h_l) + v_l \delta_l e_k = 0, \quad (27d)$$

$$\partial_t w_k + \partial_k (v_j w_l + E_w) + v_j (\partial_j w_k - \partial_k w_j) = 0, \quad (27e)$$

$$\partial_t \sigma + \partial_k (\sigma v_k + E_w) = 0, \quad (27f)$$

$$\partial_t \rho + \partial_k (\rho v_k) = 0. \quad (27g)$$

We note that it is always implied in the SHTC framework (as well as in the GENERIC) that the velocity field $v_l$ and the total momentum field are conjugate fields, i.e., $v_l = E_{mi}$. Thus, in all the equations of system (27), $v_l$ can be substituted by $E_{mi}$, but we prefer to keep $v_l$ explicitly written for historical reasons. On the other hand, in Sect. 4 we shall always use $E_{mi}$ instead of $v_l$.

Although we use the same notations for the state variables in systems (18) and (27), they represent different fields related to each other by the formulas

$$m' = w m, \quad F = A^{-1}, \quad \rho_0 = w \rho, \quad w = \det(F), \quad (28a)$$

$$e' = w A e, \quad h' = w A h, \quad (28b)$$

$$\sigma' = w \sigma, \quad w' = F^\top w. \quad (28c)$$
while the Lagrangian total energy density $U$ is related to Eulerian total energy density $E$ as $U = \rho E$. In these relations, $m'$, $F$, $h'$, $e'$, $w'$ and $\sigma'$ are the Lagrangian fields, i.e., exactly those fields in Eq. (18), while $m$, $A$, $h$, $e$, $w$ and $\sigma$ are the fields in the Eulerian equations (27). Reference (Lagrangian) mass density is denoted by $\rho_0$.

Exactly as in the Lagrangian framework [see summation rule (20)], the conservation of the total energy density $E$,

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_k} \left( v_k E + v_l \left[ (\rho E_{,l} + \sigma E_{,l} + m_l E_{m,l} + e_l E_{e,l} + h_l E_{h,l} - E) \right] \right)$$

$$\delta_{ik} - e_k E_{e,i} - h_k E_{h,i} + w_i E_{w,k} + A_{jk} E_{A,j} \right)$$

$$+ \epsilon_{ijk} E_{e,i} E_{h,j} = 0,$$

is just a consequence$^8$ of the governing PDEs (27). Namely, it is the sum of PDEs (27) multiplied by the corresponding factors, the conjugate state variables:

$$(29) \equiv E_m \cdot (27a) + E_{A_{ik}} \cdot (27b) + E_{h_i} \cdot (27c) + E_{e_i} \cdot (27d) + E_{w_k} \cdot (27e) + E_{\sigma} \cdot (27f) + E_{\rho} \cdot (27g). \quad (30)$$

As discussed in Sect. 5, this companion conservation law also plays a key role in the introduction of the dissipative terms in the SHTC framework.

There is no need to use the mass conservation equation in the Lagrangian frame because the mass density is $\rho = \rho_0/\det F$. In the Eulerian settings, the mass conservation (27g) is, in fact, the consequence of the time evolution (27b) for the distortion matrix $A$, see [75]. But for many reasons, it is convenient to treat $\rho$ as an independent state variable governing by its own time evolution. In this sense, the complimentary pair of Lagrangian equations (18a) corresponds to three Eqs. (27a), (27b) and (27g) of the Eulerian SHTC master system. Also, remark that an arbitrary number of equations with the structure of (27g) can be added to system (27). In practical cases, the entropy conservation law for the ideal fluids or time evolution for the volume fraction in multi-phase models have the same structure as (27g), e.g., see Sect. 6.5.2. Thus, (27g) symbolizes various equations with such a structure.

Eventually, we mention a few examples in which Eq. (27) were used, see also more details in Sect. 6. Equations (27a), (27b) were used in [2,3,13,22,36,39,85] for modeling of elastic and elastoplastic deformations in metals, and the same equations constitute the unified formulation of continuum mechanics proposed in [16,17,77,78] describing fluid flows. Equations (27a)–(27c) were used in [17,86] to describe the electromodynamics of moving medium, Eqs. (27a), (27e) and (27f) can be used to model multi-phase flows, heat conduction, superfluid helium flows, see [81–83,86].

### 3.2.2 Involution constraints

As we already mentioned, the involution constraints play an exceptional role in the Eulerian SHTC equations. This role will show up in all the subsequent sections. Let us start the discussion by demonstrating that the following stationary constraints$^6$

$$\nabla \times A = B, \quad \nabla \cdot h = Q, \quad \nabla \cdot e = R, \quad \nabla \times w = \Omega$$

are compatible with system (27), where the quantities $B$, $Q$, $R$ and $\Omega$ satisfy the following conservation laws

$$\partial_t B_{ij} + \delta_k (B_{ij} v_k - v_j B_{ik}) + v_j \partial_k B_{ik} = 0, \quad (32a)$$

$$\partial_t R + \delta_k (R v_k) = 0, \quad (32b)$$

$$\partial_t Q + \delta_k (Q v_k) = 0, \quad (32c)$$

$$\partial_t \Omega_j + \delta_k (\Omega_j v_k - v_j \Omega_k) + v_j \partial_k \Omega_k = 0. \quad (32d)$$

Indeed, the following involution constraints

$$\partial_t (\nabla \times A - B) \equiv 0, \quad \partial_t (\nabla \cdot e - R) \equiv 0, \quad \partial_t (\nabla \cdot h - Q) \equiv 0, \quad \partial_t (\nabla \times w - \Omega) \equiv 0,$$

$^8$ For details, see Appendix B in the e-print version [76] of the paper.

$^9$ The curl operator is applied to matrix $A$ in the row-wise manner.
are satisfied on the solutions to (27) if they hold true at the initial moment of time, since \( \nabla \times A, \nabla \cdot e, \nabla \cdot h \) and \( \nabla \times w \) satisfy the following evolution equations\(^{10}\)

\[
\frac{\partial}{\partial t}(\nabla \times A)_{ij} + \frac{\partial}{\partial x_k}(\nabla \times A)_{kJ} v_k - v_j (\nabla \times A)_{ik} + v_j \frac{\partial}{\partial x_k}(\nabla \times A)_{ik} = 0, \quad (34a)
\]

\[
\frac{\partial}{\partial t}(\nabla \cdot h) + \frac{\partial}{\partial x_k}(v_k(\nabla \cdot h)) = 0, \quad (34b)
\]

\[
\frac{\partial}{\partial t}(\nabla \cdot e) + \frac{\partial}{\partial x_k}(v_k(\nabla \cdot e)) = 0, \quad (34c)
\]

\[
\frac{\partial}{\partial t}(\nabla \times w)_j + \frac{\partial}{\partial x_k}((\nabla \times w)_j v_k - v_j (\nabla \times w)_k) + v_j \frac{\partial}{\partial x_k}(\nabla \times w)_k = 0. \quad (34d)
\]

By comparing (32) and (34), one may conclude that (33) is satisfied if it was true at \( t = 0 \). We emphasize that even though the non-conservative terms in (34a) and (34d) are zeros, they cannot be omitted because this would violate the Galilean invariance property of these equations and also change the characteristic structure.

In some cases, at \( t = 0 \), one may assume that \( B = 0, h = 0, e = 0 \) and \( \Omega \neq 0 \) and hence, (32) has trivial solution, and (33) becomes

\[
\nabla \times A \equiv 0, \quad \nabla \cdot h \equiv 0, \quad \nabla \cdot e \equiv 0, \quad \nabla \times w \equiv 0 \quad (35)
\]

for any \( t > 0 \). However, in some applications we have to set \( B \neq 0, h \neq 0, e \neq 0 \) or \( \Omega \neq 0 \) in the initial conditions and hence, system (32) has non-trivial time-dependent solution. Thus, it is necessary to note that quantities \( B, h, e \) and \( \Omega \) may have a certain physical meaning. For example, \( \nabla \times A \) has the meaning of the density of microscopic defects (dislocation density tensor) \([38,39,75]\) in the elastoplasticity theory. Thus, if the material has suffered from plastic deformations in the past \( (t < 0) \) then at \( t = 0 \) we have \( \nabla \times A \neq 0 \). For example, the impact of the dislocation density on the dispersive properties of the elastic waves was studied in \([84]\) where it was assumed that the material has nonzero initial concentration of defects.

As we shall see later, the involution constraints (33) and (35) have a strong impact on the symmetrization of system (27). Thus, if constraints (35) hold then the symmetrization is much simpler than in the case of (33). If only (33) holds, then system (27) can be symmetrized in an extended sense. Namely, we shall demonstrate that system (27) extended by adding to it the constraints (32) can be symmetrized. This is, in fact, a very interesting feature of the SHTC formulation because one may note that the extended system (27), (32) is \textit{weakly non-local} because the quantities \( \nabla \times A, \nabla \cdot e \), etc., represent a coarser scale than the state variables \( A, e \), etc. This will be even more pronounced when we shall discuss the irreversible dynamics within the SHTC framework in Sect. 5 because in the presence of dissipative processes the quantities \( \nabla \times A, \nabla \cdot e, \nabla \times w \) are not zeros even if they were zero at \( t = 0 \).

### 3.2.3 Weakly non-local conservative but energy incompatible non-symmetrizable formulation

In contrast to the Lagrangian master system (18), the Eulerian system (27) is not a system of \textit{conservation laws} due to the presence of the non-conservative differential terms,

\[
v_j (\partial_j A_{ik} - \partial_k A_{ij}), \quad v_i \partial_i h_k, \quad v_i \partial_k e, \quad v_j (\partial_j w_k - \partial_k w_j), \quad (36)
\]

in Eqs. (27b), (27c), (27d) and (27e) accordingly. Therefore, there is a temptation to rewrite Eulerian SHTC equations in a fully conservative form using the results of Sect. 3.2.2 and to symmetrize it in a way discussed in Sect. 2.1. Unfortunately, such an idea is not viable, and the pedagogical goal of this section is to demonstrate this.

For the sake of simplicity, we consider the ideal MHD equations in this section. However, all the analysis can be applied to the entire system (27) without any changes. Thus, let us consider the MHD equations which using the results of Sect. 3.2.2 can be written as

\[
\partial_t m + \partial_k (m_i v_k + \delta_l k (\rho E_p + \sigma E_o + m_l E_{ml} + B_l E_{Bl} - E) - B_k E_{Bl}) = 0, \quad (37a)
\]

\[
\partial_t B_l + \partial_k (B_l v_k - v_l B_k) = -v_l Q, \quad (37b)
\]

\[
\partial_t Q + \partial_k (Q v_k) = 0, \quad (37c)
\]

\[
\partial_t \rho + \partial_k (\rho v_k) = 0, \quad (37d)
\]

\[
\partial_t \sigma + \partial_k (\sigma v_k) = 0, \quad (37e)
\]

\(^{10}\) For the derivation of (34a) and (34d), see Appendix C in [76], while Eqs. (34c) and (34b) can be easily obtained by applying \( \partial / \partial x_1 \) to the SHTC equations (27d) and (27c) accordingly.
\[ \partial_t E + \partial_k \left( v_k E + v_i \left[ \left( \rho E_\rho + \sigma E_\sigma + m_i E_{m_i} + B_i E_{B_i} - E \right) \delta_{ik} - B_k E_{w_i} \right] \right) = 0, \]  

(38)

which look like the system of conservation laws (i.e., all the differential terms are written in a conservative manner) with the algebraic source term \(-v_j Q\) in the PDE for \(B_j\) which can be ignored in the symmetrization process as a low-order term. Here, instead of the field \(h_i\) used in system (27), we use a conventional notation \(B_i\) for the magnetic field, \(\rho\) is the mass density, \(\sigma\) is treated as the entropy density. As discussed in Sect. 3.2.2, on the solution of the system, the involution constraint

\[ \partial_t (\nabla \cdot \mathbf{B} - Q) = 0 \]  

(39)

is fulfilled. This involution constraint can be reduced to \(\partial_t Q = 0\) because \(\nabla \cdot \mathbf{B} = 0\) which has the meaning of the absence of magnetic monopoles. Therefore, one may even suggest to replace (37b) with

\[ \partial_t B_i + \partial_k \left( B_i v_k - v_i B_k \right) = 0, \]  

(40)

as it is done in many works, e.g., see discussion in [79]. However, neither (37) nor (40) is compatible with the energy conservation law (38). Indeed, Eqs. (37) and (38) form an overdetermined system of PDEs. Hence, the energy conservation should be a combination of Eq. (37) or (40). However, as it was shown by Godunov in [36], the energy flux cannot be obtained as a combination of the conservative fluxes (37) or (40) because the energy flux is exactly (see details in Appendix B in [76])

\[ \text{flux}(38) \equiv E_{m_i} \cdot \text{flux}(37a) + E_{B_i} \cdot \text{flux}(37b) + E_{Q} \cdot \text{flux}(37c) + E_{\rho} \cdot \text{flux}(37d) + E_{\sigma} \cdot \text{flux}(37e) + E_{B_i} v_i \partial_i B_k, \]  

(41)
i.e., it is the combination not only of the fluxes of (37), but also the term \(E_{B_i} v_i \partial_i B_k = 0\) should be added as well.

Therefore, the energy conservation is not a combination of either (37) or (40) and hence the symmetrization is impossible by means of the Godunov–Boillat theorem discussed in Sect. 2.1. Other ways of symmetrization of the MHD equations are unknown. Eventually, none of the formulations (37) and (40) is Galilean invariant, e.g., see [79], and should be avoided in practical use.

The MHD formulation which is compatible with the energy conservation (38) is the following non-conservative system

\begin{align*}
\partial_t m_i + \partial_k \left( m_i v_k + \delta_{ik} \left( \rho E_\rho + \sigma E_\sigma + m_i E_{m_i} + B_i E_{B_i} - E \right) - B_k E_{B_i} \right) &= 0, \\
\partial_t B_i + \partial_k \left( B_i v_k - v_i B_k \right) + v_i \partial_k B_k &= 0, \\
\partial_t \rho + \partial_k (\rho v_k) &= 0, \\
\partial_t \sigma + \partial_k (\sigma v_k) &= 0.
\end{align*}

(42a) (42b) (42c) (42d)

This formulation is non-conservative due to the presence of the non-conservative differential term \(v_i \partial_i B_k\) in the PDE for \(B_i\). However, exactly this formulation is compatible with the energy law (38), it is symmetrizable as was shown in [36], it has the structure of Eq. (27c) of the SHTC master system (27), it is Galilean invariant, and exactly Eq. (42b) can be obtained from the Lagrangian equation for the magnetic field

\[ \partial_t B_i = 0, \]  

(43)

see details in Appendix A.3 of [76].

In summary, it is important not only to pay attention to casting the equations into the form of conservation laws, but also to the companion conservation law, which has to be compatible with the rest of the evolution equations. Otherwise, the Godunov–Boillat theorem cannot be applied.

### 3.2.4 Conjugate formulation

In Sect. 3.2.1, the governing equations were formulated in terms of the state variables \(q\) and the total energy density \(E(q)\). In this section, we also provide another, thermodynamically conjugate, formulation in terms of the conjugate state variables \(p\) (flux fields) and a potential \(L(p)\) which is the Legendre transform of \(E(q)\) and has the physical meaning of a generalized pressure. This formulation allows to emphasize the exceptional role of the generating potentials \(L\) and \(E\) and shall be used to prove the symmetric hyperbolicity of the governing PDEs in Sects. 3.2.6 and 3.2.5 within the SHTC formalism.

As in the Lagrangian framework, we introduce the conjugate variables

\[ p = (r, v_i, \alpha_{ik}, b_i, d_i, \eta_k, \theta) \]  

(44)
as the partial derivatives of the total energy density \( E(q) \) with respect to the conservative state variables
\[
v_i = E_{m_i}, \quad \alpha_{ik} = E_{A_{ik}}, \quad b_i = E_{b_i}, \quad d_i = E_{d_i}, \quad \eta_k = E_{\eta_k}, \quad \theta = E_{\theta}, \quad r = E_{r},
\]
and the new potential
\[
L(p) = q \cdot p - E = r \rho + v_i m_i + \alpha_{ij} A_{ij} + d_i e_i + b_i h_i + \eta_i w_i + \theta \sigma - E
\]
as the Legendre transformation of \( E \). In terms of the new variables \( p \) and potential \( L(p) \), system (27) can be rewritten as
\[
\begin{align}
&\partial_t L_{v_i} + \partial_k \left[ (v_k L)_{v_i} + \alpha_{mk} L_{\alpha_{mk}} - \delta_{ik} \alpha_{mn} L_{\alpha_{mn}} - d_i L_{dk} - b_i L_{bk} + \eta_k L_{\eta_k} - \delta_{ik} \eta \eta_n L_{\eta_n} \right] = 0, \\
&\partial_t L_{a_{ik}} + \partial_k \left[ (v_k L)_{a_{ik}} + v_j \left( \partial_j L_{a_{ik}} - \partial_k L_{a_{ij}} \right) \right] = 0, \\
&\partial_t L_{b_i} + \partial_k \left[ (v_k L)_{b_i} - v_i L_{bk} + \varepsilon_{ik} d_i \right] + v_i \partial_k L_{bk} = 0, \\
&\partial_t L_{d_i} + \partial_k \left[ (v_k L)_{d_i} - v_i L_{dk} - \varepsilon_{ik} b_i \right] + v_i \partial_k L_{dk} = 0, \\
&\partial_t L_{\eta_k} + \partial_k \left[ (v_k L)_{\eta_k} + \theta \right] + v_j \left( \partial_j L_{\eta_k} - \partial_k L_{\eta_j} \right) = 0, \\
&\partial_t L_\theta + \partial_k \left[ (v_k L)_{\theta} + \eta_k \right] = 0, \\
&\partial_t L_r + \partial_k (v_k L)_r = 0,
\end{align}
\]
while the energy conservation (29) now reads (the proof that it is the consequence of Eq. (46) is given in Appendix B [76])
\[
\begin{align}
&\partial_t \left( r L_r + v_i L_{v_i} + \alpha_{ij} L_{a_{ij}} + d_i L_{d_i} + b_i L_{b_i} + \theta L_\theta + \eta_k L_{\eta_k} - L \right) \\
&\quad + \partial_k \left( v_k \left( r L_r + v_i L_{v_i} + \alpha_{ij} L_{a_{ij}} + d_i L_{d_i} + b_i L_{b_i} + \theta L_\theta + \eta_k L_{\eta_k} - L \right) \right) \\
&\quad + v_i \left[ (L - \alpha_{mk} L_{\alpha_{mk}} - d_i L_{dk} - b_i L_{bk} + \eta_k L_{\eta_k} - L) \right] \\
&\quad = 0.
\end{align}
\]
It is still not clear, however, whether (46) can be written as a symmetric hyperbolic system because of the non-trivial structure of the fluxes and non-conservative differential terms. Apparently, the Godunov–Boillat theorem discussed in Sect. 2.1 cannot be applied because (46) is not a system of conservation laws. Moreover, it cannot be treated as a conservative system in the extended sense as discussed in Sect. 3.2.3 because energy conservation would be incompatible with such conservative equations\(^{11}\) and the theorem would still be inapplicable.

The source of the problem is in the following. Unlike the Lagrangian frame, it has been shown in [32, 36, 86, 87] that in the Eulerian frame the involution constraints (33) play an important role in the symmetrization of system (46). If the initial conditions are chosen in such a way that (35) holds true, then the symmetrization is much easier and is discussed in Sect. 3.2.5 while the general case for non-curl-free and divergence-free involution constraints (33) is discussed in Sect. 3.2.6.

3.2.5 Symmetric form of conjugate formulation for curl-free and divergence-free involution constraints

As in the Lagrangian frame, the parametrization of the governing equations in terms of the conjugate variables \( p \) and generating potential \( L(p) \) allows to rewrite the system in a symmetric quasilinear form. However, in the Eulerian frame it is not that straightforward. In order to show that (46) is equivalent to a quasilinear symmetric system, it is necessary to add the following linear combination of the involution constraints (35)
\[
\alpha_{jk} \left( \partial_i L_{a_{ijk}} - \partial_k L_{a_{ij}} \right) + d_i \partial_k L_{d_i} + b_i \partial_k L_{b_i} + \eta_k \left( \partial_i L_{\eta_i} - \partial_k L_{\eta_k} \right) = 0
\]
to Eq. (46a) and formally brake its conservative form. Then after some simple rearrangements of the terms in the rest of the equations, (46) can be rewritten as a symmetric quasilinear system, (see details in the e-print version [76] of the paper)
\[
A \frac{\partial p}{\partial t} + B_k \frac{\partial p}{\partial x_k} + C_k \frac{\partial p}{\partial x_k} = 0
\]
where \( A(p) = L_{pp} \) and \( B_k(p) = (v_k L)_{pp} \) are second-order derivatives of the potentials \( L \) and \( v_k L \), and hence are symmetric matrices, while \( C_k(p) \) is symmetric but consist of non-conservative terms, see details in [76]. Moreover, if \( L \) is a convex potential, then system (46) is symmetric hyperbolic because \( A = L_{pp} > 0 \). Recall that due the properties of the Legendre transformation, the convexity of \( L(p) \) in \( p \) is equivalent to convexity of \( E(q) \) in \( q \).

\(^{11}\) See details in Appendix B in [76], where we prove that if the non-conservative terms are ignored then the energy flux is not a consequence of the other fluxes.
3.2.6 General symmetric form of conjugate weakly non-local formulation

In the previous section, we have demonstrated that conjugate form (46) of master equations (27) can be written as a symmetric quasilinear system. The key step was a modification of the momentum equation (46a) by formally breaking its conservative form via adding to it the linear combination of involution constraints (48). But in the case of non-curl-free and non-divergence-free involution constraints (33), such a step is not valid.

However, the symmetrization is still possible, though in an extended sense where the PDEs (32) for the new state variables $B$, $Q$, $R$ and $\Omega$ are added to the master system. Because these quantities coincide with $\nabla \times A$, $\nabla \cdot h$, $\nabla \cdot e$ and $\nabla \times w$ accordingly at each time instant, the extended system should be treated as a weakly non-local extension system of the original Eulerian SHTC equations (27). Here, we follow the idea presented in [20], and for the sake of simplicity we demonstrate the symmetrization algorithm for the complimentary couple $(w_i, \sigma)$, or in fact, for their conjugate couple $(\eta_j, \theta)$. As will be shown in Sect. 6, these equations constitute an important class of equations based on which heat and mass transfer hyperbolic models are developed. All the steps of the algorithm then can be applied for symmetrizing the PDEs for the couples $(m_i, A_{ij})$ and $(e_i, h_i)$ and eventually for the entire system (27).

Consider the following system for the $(w_i, \sigma)$-pair coupled with the evolution PDE for the vector field $\Omega_i = \varepsilon_{ijk} \partial_j w_k$,

$$
\partial_t m_i + \partial_k \left( m_i v_k + \delta_{i k} \left( \rho E_{\mu} + \sigma E_{\sigma} + m_l E_{m_l} + \Omega_l E_{\Omega_l} - E \right) - \Omega_k E_{\Omega_l} + w_l E_{w_l} \right) = 0, \quad (50a)
$$

$$
\partial_t w_k + \partial_k (w_{l a} v_{l a} + E_{\sigma}) + v_j (\partial_j w_k - \delta_k w_j) = 0, \quad (50b)
$$

$$
\partial_t \sigma + \partial_k (\sigma v_k + E_{w_k}) = 0, \quad (50c)
$$

$$
\partial_t \Omega_i + \partial_k (\Omega_l v_k - \Omega_k v_l) + v_i \partial_k \Omega_k = 0, \quad (50d)
$$

$$
\partial_t \rho + \partial_k (\rho v_k) = 0. \quad (50e)
$$

Here, it is implied that we have extended the set of state variables by adding to it the vector field $\Omega$. However, the addition of a new state variable and its evolution PDE cannot be done arbitrarily within the SHTC framework because the equations might become incompatible with the companion energy conservation law. In order to make them compatible, we have to add the extra stress $-\Omega_k E_{\Omega_l}$ in the momentum flux; otherwise, the summation rule (30) would not give the energy conservation.

The energy density and the conjugate potential

$$
E = E(m_i, w_i, \sigma, \Omega_i, \rho), \quad L = L(v_i, \eta_i, \theta, \omega_i, \tau)
$$

now also depend on $\Omega_i$ and its conjugate $\omega_i = E_{\Omega_i}$ accordingly. The entire system (50) including new equation for $\Omega_i$ forms a sub-system of the master system (27). For example, one may clearly notice that the new PDE for $\Omega_i$ has the same but slightly reduced structure as the PDE (27c) for the field $h_i$ which, in turn, also gives the corresponding extra stress $-h_k E_{h_i}$. The following involution constraints are implied as well

$$
\partial_t \left( \varepsilon_{ijk} \partial_j w_k - \Omega_i \right) = 0, \quad \partial_t (\partial_k \Omega_k) = 0. \quad (52)
$$

Therefore, the extended system (53) is still formulated within the SHTC framework.

The conjugate formulation of (50) is

$$
\partial_t L_{v_i} + \partial_k \left( (v_k L)_{v_i} + \eta_k L_{\eta_k} - \delta_{i k} \eta_k L_{\eta_k} - \omega_k L_{\omega_k} \right) = 0, \quad (53a)
$$

$$
\partial_t L_{\eta_k} + \partial_k (L_{\eta_k} v_{l a} + \theta) + v_j (\partial_j L_{\eta_k} - \partial_k L_{\eta_j}) = 0, \quad (53b)
$$

$$
\partial_t L_{\theta} + \partial_k ((v_k L)\theta + \eta_k) = 0, \quad (53c)
$$

$$
\partial_t L_{\omega_k} + \partial_j (L_{\omega_k} v_j - L_{\omega_j} v_i) + v_i \partial_k L_{\omega_k} = 0, \quad (53d)
$$

$$
\partial_t L_{\tau} + \partial_k (v_k L)_{\tau} = 0, \quad (53e)
$$

while the involution constraints (52) are

$$
\partial_t \left( \nabla \times L_{\eta} - L_{\omega} \right) \equiv 0, \quad \partial_t \left( \nabla \cdot L_{\omega} \right) \equiv 0. \quad (54)
$$

The energy conservation law for such an extended formulation now reads as

$$
\partial_t \left( r L_{\tau} + v_i L_{v_i} + \omega_l L_{\omega_l} + \theta L_{\theta} + \eta_k L_{\eta_k} - L \right) + \partial_k \left( v_k \left( r L_{\tau} + v_i L_{v_i} + \omega_l L_{\omega_l} + \theta L_{\theta} + \eta_k L_{\eta_k} - L \right) + v_i \left[ (L - \eta_k L_{\eta_k}) \delta_{i k} - \omega_k L_{\omega_k} \right] + \theta \eta_k \right) = 0, \quad (55)
$$
which, we emphasize, has the fluxes fully compatible with the spatial differential terms of the governing equations (53), i.e., it can be obtained by the summation rule (30).

Now, we are in position to demonstrate that the conjugate formulation (53) is indeed symmetrizable. In the first step of the symmetrization algorithm, we open the brackets in the PDE for \( L_{\eta_k} \), which becomes

\[
\partial_t L_{\eta_k} + L_{\eta_k} \partial_k v_\alpha + v_j \partial_j L_{\eta_k} + \partial_k \theta = 0.
\]

(56)

In the second step, we modify this PDE by adding to it \( L_{\eta_k} \partial_j v_j - L_{\eta_k} \partial_j v_j \equiv 0 \). We have

\[
\partial_t L_{\eta_k} + \partial_k (v_k L)_{\eta_k} - L_{\eta_k} \partial_k v_k + L_{\eta_k} \partial_t v_\alpha + \partial_t \theta = 0.
\]

(57)

In the last step, we only formally “destroy” the conservative form of the momentum equation by adding to it the linear combination of the constraints (54)

\[
0 \equiv \eta_k (\partial_t L_{\eta_k} - \partial_k L_{\eta_k}) - \epsilon_{ijk} \eta_j L_{\omega_k} + \omega_i \partial_k L_{\omega_k}.
\]

(58)

Eventually, we have

\[
\begin{align*}
\partial_t L_{v_i} + \partial_k (v_k L)_{v_i} + L_{\eta_k} \partial_k \eta_k - L_{\eta_k} \partial_k v_\alpha - L_{\omega_k} \partial_k \omega_i &= \epsilon_{ijk} \eta_j L_{\omega_k}, \\
\partial_t L_{\eta_i} + \partial_k (v_k L)_{\eta_i} - L_{\eta_k} \partial_k v_k + L_{\eta_k} \partial_t v_\alpha + \partial_t \theta &= 0, \\
\partial_t L_{\omega_i} + \partial_k (v_k L)_{\omega_i} - L_{\omega_k} \partial_k v_k &= 0,
\end{align*}
\]

(59a, 59b, 59c)

\[
\begin{align*}
\partial_t L_{\theta} + \partial_k (v_k L)_{\theta} + \partial_k \eta_k &= 0, \\
\partial_t L_{r} + \partial_k (v_k L)_{r} &= 0,
\end{align*}
\]

(59d, 59e)

which can be written as a symmetric quasilinear system (49) with an algebraic source term \( \mathbf{S} = (\epsilon_{ijk} \eta_j L_{\omega_k}, 0, 0, 0, 0) \),

\[
A \frac{\partial \mathbf{p}}{\partial t} + B_k \frac{\partial \mathbf{p}}{\partial x_k} + C_k \frac{\partial \mathbf{p}}{\partial x_k} = \mathbf{S},
\]

(60)

where symmetric matrices \( C_k \) consist of the non-conservative differential terms in (59). The key idea of such a symmetrization is that the source term \( \epsilon_{ijk} \eta_j L_{\omega_k} \) in the momentum equation (59a) is an algebraic term due to the fact that we treat \( \nabla \times \mathbf{w} \) as a new independent state variable \( \mathbf{W} = L_{\omega} \). This source term is therefore treated as a low-order term and does not affect the hyperbolic character of the PDEs (i.e., it does not contribute into the structure of the coefficient matrices \( A, B_k \) and \( C_k \)).

Let us demonstrate the symmetric structure of the matrix \( C_k \). For example, for \( k = 1 \) the submatrix of \( C_1 \) corresponding to the variables \( v_i, \eta_1, \theta \) and \( \omega_1 \) is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & -L_{\eta_2} & -L_{\eta_3} & -L_{\omega_1} & 0 & 0 & 0 \\
0 & 0 & 0 & L_{\eta_2} & 0 & 0 & 0 & -L_{\omega_1} & 0 & 0 \\
0 & 0 & 0 & L_{\eta_3} & 0 & 0 & 0 & 0 & -L_{\omega_1} & 0 \\
0 & -L_{\eta_2} & L_{\eta_3} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-L_{\eta_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-L_{\eta_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-L_{\omega_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -L_{\omega_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -L_{\omega_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

(61)

The equations for the pairs \( (m, A) \) and \( (h, e) \) are symmetrizable exactly in the same way. For example, in order to symmetrize the equations for the \( (m, A) \)-pair, one has to add the time evolution (32a) for \( \mathbf{B} = \nabla \times \mathbf{A} \) to the system, extend the set of state variables by considering the total energy as \( E = E(m, A, B, \rho) \) and repeat the above steps.

In summary, extending the idea from [20], we have demonstrated that in case the constraints \( \nabla \times A = 0, \nabla \cdot h = 0, \nabla \cdot e = 0 \) and \( \nabla \times w = 0 \) are not met, the Eulerian SHTC equations (27) can be symmetrized only after a weakly non-local extension when the quantities \( \nabla \times A, \nabla \cdot h, \nabla \cdot e \) and \( \nabla \times w \) start to play the role of the state variables governed by their own time evolutions (32).
4 GENERIC form of SHTC equations

After having recalled the SHTC equations, let us now turn to the question whether the reversible and irreversible parts of the SHTC equations, Eq. (27), can be seen as Hamiltonian and gradient evolutions, respectively. The answer to that question is affirmative, and the SHTC equations are thus fully compatible with the GENERIC framework. Moreover, focusing on the Poisson bracket provides an alternative way of derivation and generalization of the SHTC equations.

Firstly, it is demonstrated in Sect. 4.1 that the Lagrangian SHTC equations can be generated by canonical Poisson brackets while the rest of this section is dedicated to the central and more challenging question of whether the Eulerian SHTC system (27) can be seen as a Hamiltonian evolution.

4.1 Lagrangian SHTC system as Hamiltonian evolution

The Lagrangian SHTC evolution equations (18) can be seen as canonical Hamiltonian evolution of a triplet of cotangent bundles (see details in the e-print version of the paper [76]). The three cotangent bundles are equipped with three canonical Poisson brackets, and simultaneous evolution within the cotangent bundles is given by the sum of the three canonical Poisson bracket. Then the triplet of operations gradient, divergence and curl, which form a sort of complete set of operations in three-dimensional space, see p. 181 of book [24] and [29], induce three projections, one for each of the cotangent bundles. After such a transformation, the sum of the three Poisson brackets generates the Lagrangian SHTC equations (18).

4.2 Hydrodynamics of ideal fluids

Let us now turn to the Eulerian evolution equations. Hamiltonian form of hydrodynamics of ideal fluids has been known for a long time. The Poisson bracket for Euler equations of ideal fluids, i.e., for fields \(( m, \rho, s )\), is (e.g., see [1,15,42,62,66,71])

\[
\{ A, B \}^{(Euler)} = \int d\boldsymbol{r} m_j (B_m \partial_j A_m - A_m \partial_j B_m) 
+ \int d\boldsymbol{r} \rho (B_m \partial_i A_\rho - A_\rho \partial_i B_m) 
+ \int d\boldsymbol{r} s (B_m \partial_i A_s - A_s \partial_i B_m),
\]

where \( A(m, \rho, s) \) and \( B(m, \rho, s) \) are two arbitrary functionals and \( A_\rho, B_\rho, A_m, B_m, \) etc. are the functional derivatives, i.e., \( A_\rho = \frac{\delta A}{\delta \rho} \cdot A_m = \frac{\delta A}{\delta m} \), etc.

The Poisson bracket expresses kinematics of the state variables, which are the fields \( x = (m, \rho, s) \). Taking an arbitrary functional \( A(m, \rho, s) \), Hamiltonian evolution of the functional is given by the Poisson bracket

\[
\partial_t A = \{ A, \mathcal{H} \}^{(Euler)},
\]

together with the Hamiltonian (or total energy)

\[
\mathcal{H}(m, \rho, s) = \int E(m, \rho, s)d\boldsymbol{r},
\]

where the integrand \( E(m, \rho, s) \) is the total energy potential density. Poisson bracket and the Hamiltonian thus generate reversible evolution of functionals of state variables, and the evolution equations for the fields \( (m, \rho, s) \) are

\[
\partial_t m_j = \{ m_j, \mathcal{H} \}^{(Euler)}, \quad \partial_t \rho = \{ \rho, \mathcal{H} \}^{(Euler)}, \quad \partial_t s = \{ s, \mathcal{H} \}^{(Euler)}.
\]

Alternatively, evolution of a functional \( A(x) = \int A(x)d\boldsymbol{r} \) can be given by

\[
\partial_t A = \int \frac{\delta A}{\delta x^i} \frac{\partial x^i}{\partial t} d\boldsymbol{r},
\]
and hence, by collecting terms multiplying derivatives of the functional $\mathcal{A}$ in the Poisson bracket, i.e., by rewriting the bracket as

$$\{\mathcal{A}, \mathcal{H}\}^{(Euler)} = \int \left[ \mathcal{A}_{m_i} \cdot (\cdots) + \mathcal{A}_\rho \cdot (\cdots) + \mathcal{A}_s \cdot (\cdots) \right] d\mathbf{r}, \quad (69)$$

one can directly read the evolution equations of the state variables denoted as “...” in (69).

Using either (67) or (69), one can obtain the Euler equations of compressible hydrodynamics

$$\partial_t m_i = - \partial_j (m_i E_{m_j}) - \rho \partial_i E_{\rho} - m_j \partial_i E_{m_j} - s \partial_i E_s = - \partial_j (m_i E_{m_j}) - \partial_i p, \quad (70a)$$

$$\partial_t \rho = - \partial_i (\rho E_{m_i}), \quad (70b)$$

$$\partial_t s = - \partial_i \left( s E_{m_i} \right), \quad (70c)$$

where the pressure is identified as $p = \rho E_\rho + s E_s + m_j E_{m_j} - E$. This construction of pressure is general within the context of Hamiltonian dynamics and SHTC formalism. Compressible Euler equations can be thus seen as Hamiltonian evolution.

4.3 Distortion matrix

The goal of this section is to formulate the Poisson brackets generating the time evolutions (27a), (27b) and (27g) for the momentum, distortion and the mass density fields of the SHTC formulation, and hence to demonstrate that these SHTC equations are compatible with GENERIC.\footnote{Note that the solid dynamics has already been formulated within GENERIC. For instance in [51] an evolution equation for the deformation tensor is generated by a Poisson bracket which is constructed by using the requirements of antisymmetry and momentum conservation.} The distortion field is a very important concept, which is used to describe elastic and inelastic deformation of continuum (viscous flows, elastoplastic deformations) in the SHTC theory. Before discussing the dynamics of distortion matrix, let us first consider dynamics of labels advected by the fluid particles.

4.3.1 Dynamics of labels

The aim here is to find evolution equations for solids capturing balance of mass, momentum and entropy (or energy) as well as dynamics of individual points of the continuum (particle labels). Therefore, the state variables will be density $\rho$, momentum density $m$, entropy density $s$ and mass density of the labels $a$. The label vector field $a$ expresses where the material at a given Eulerian position belongs.

Poisson bracket governing evolution of variables $\rho, m, s$ and $b = \rho a$ is the standard hydrodynamic Poisson bracket (62) which is now coupled with the passively advected vector field $b$ expressing volume density of labels. Subsequent transformation to $a = b/\rho$, i.e., to the mass density of labels, gives the overall bracket\footnote{See the e-print version [76] for more details.}

$$\{\mathcal{A}, \mathcal{B}\}^{(Lin)} = \{\mathcal{A}, \mathcal{B}\}^{(Euler)} + \int \partial_j a_i \left( \mathcal{A}_{m_j} \mathcal{B}_{ai} - \mathcal{B}_{m_j} \mathcal{A}_{ai} \right) d\mathbf{r}. \quad (71)$$

This Poisson bracket was first introduced in [43], where it was referred to as the Lin Poisson bracket. The evolution equations implied by this Poisson bracket are fully compatible with the standard evolution equations of continuum mechanics in the Eulerian frame if $a_i$ are interpreted as the Lagrangian coordinates.

4.3.2 Dynamics of general distortion matrix

Let us now reformulate the dynamics solely in terms of the gradient $\nabla a$, which is referred to as the distortion matrix (following [16,37–39,77]),

$$A_{ij} = \partial_j a_i, \quad (72)$$

and expresses local variations of the field of labels. This definition of the distortion matrix implies the integrability conditions

$$\partial_k A_{ij} = \partial_j A_{ik} \forall i, j, k, \quad \text{that is} \quad \epsilon_{ijk} \partial_j A_{ik} = 0 \forall I \iff \nabla \times A = 0. \quad (73)$$
We first propose a Poisson bracket for the time evolution of the compatible distortion, i.e., for which the integrability conditions are fulfilled and which is thus the gradient of the field of labels. Our main goal, however, is to provide a Poisson bracket for generally incompatible distortion fields, for which \( \nabla \times \mathbf{A} \neq 0 \).

The incompatible distortion field is the most important case because such distortions are used to describe irreversible deformations of viscous fluids and elastoplastic solids [16,39,77,78], see also examples in Sect. 6, and non-Newtonian fluids, see Sect. 4.3.3.

Transforming Poisson bracket (71) into variables \( (\mathbf{m}, \mathbf{A}, \rho, s) \) by letting the two functionals depend only on the new variables leads to the Poisson bracket of hydrodynamics with the compatible distortion matrix,

\[
\{ \mathcal{A}, \mathcal{B} \}^{(Euler+A)} = \{ \mathcal{A}, \mathcal{B} \}^{(Euler)} + \int A_{ij} \left( \mathcal{B}_{m_j} \partial_t \mathcal{A}_{A_{ik}} - \mathcal{A}_{m_j} \partial_t \mathcal{B}_{A_{ik}} \right) \, \mathrm{d} \mathbf{r},
\]

(74)

This bracket is a Poisson bracket, in particular it fulfills Jacobi identity (checked with program [58]), if and only if the integrability conditions (72) hold. However, it can be expected that irreversible evolution violates these conditions and hence the distortion matrix loses the connection (73) with the field of labels. Indeed, having \( \nabla \times \mathbf{A} \neq 0 \) also means that integral of \( \mathbf{A} \) over a closed loop does not necessarily give zero, which means that there might be a discontinuity in the field of labels – the material undergoes irreversible deformation and identity of the particles (the labels) are altered or lost. We need to extend the Poisson bracket so that it fulfills the Jacobi identity also for the incompatible distortion fields.

The extension should be done in such a way that for curl-free distortion matrices bracket (74) is recovered, which means that the extra terms (to be added to the bracket) should thus be multiplied by \( \nabla \times \mathbf{A} \). Moreover, in order to generate reversible evolution, the extra terms should provide additional coupling between \( \mathbf{A} \) (even with respect to time reversal) and momentum \( \mathbf{m} \) (odd) because \( \mathbf{m} \) is the only variable with opposite parity than \( \mathbf{A} \), see Sect. 3.1.3 and [73]. The new distortion matrix bracket then reads

\[
\{ \mathcal{A}, \mathcal{B} \}^{(DM)} = \{ \mathcal{A}, \mathcal{B} \}^{(Euler+A)} + \int \left( \partial_k A_{ij} - \partial_j A_{ik} \right) \left( \mathcal{A}_{A_{ik}} \mathcal{B}_{m_j} - \mathcal{A}_{m_j} \mathcal{B}_{A_{ik}} \right) \, \mathrm{d} \mathbf{r},
\]

(75)

which fulfills the Jacobi identity unconditionally (checked with program [58]). Note that the extra terms cannot be multiplied by any number or function without violating the Jacobi identity, which makes the choice of the extension unique. This Poisson bracket leads to evolution equations

\[
\begin{align*}
\partial_t m_i &= -\partial_j (m_j E_{m_j}) - \rho \partial_t E_\rho - m_j \partial_t E_{m_j} - s \partial_t s - \partial_t E_{A_{ij}} - \partial_t A_{ji} E_{A_{jk}} - (\partial_t A_{ji} - \partial_t A_{jk}) E_{A_{jk}} \quad (76a) \\
\partial_t E_\rho &= -\partial_j (m_j E_{m_j}) + \partial_j \sigma_{ij} \quad (76b) \\
\partial_t \sigma_{ij} &= -\partial_k (A_{ij} E_{m_j}) + (\partial_k A_{ij} - \partial_j A_{ik}) E_{m_j} \quad (76c) \\
\partial_t s &= -\partial_j (s E_{m_j}) \quad (76d)
\end{align*}
\]

where the generalized pressure \( p \) and extra stress tensor \( \sigma_{ij} \) were identified as (taken \( \mathcal{H} = \int E(\mathbf{m}, \mathbf{A}, \rho, s) \, \mathrm{d} \mathbf{r} \))

\[
p = \rho E_\rho + s E_s + m_i E_{m_i} + A_{ij} E_{A_{ij}} - E, \quad \sigma_{ij} = -A_{kl} E_{A_{ij}} + A_{kl} E_{A_{ij}} \delta_{ij} \quad (77)
\]

Equations (76) are the same as the Eulerian SHTC equations (27a), (27b).

In summary, the hydrodynamic Poisson bracket was first enriched with the field of labels (71), kinematics of which is pure advection. That is the setting of classical continuum mechanics in the Eulerian frame. Due to the required invariance with respect to spatial shifts, only spatial gradients of the field of labels (the distortion matrix) were kept among the state variables, and a new Poisson bracket was obtained (74). Such a bracket expresses the kinematics of the inverse deformation gradient, which is standard in continuum mechanics. As such, bracket (74) fulfills the Jacobi identity only for curl-free distortion matrices, and to satisfy the identity also for distortion matrices with nonzero curl, it was necessary to add (in a unique way) extra terms to the bracket (74). Poisson bracket (75) expressing kinematics of non-curl-free distortion matrices has then been found, which generates the same evolution equations (27a), (27b) and (27g) of the SHTC framework. To the best of our knowledge, these results are new.
4.3.3 Dynamics of the left Cauchy-Green (or Finger) tensor

One of the main motivations for developing the GENERIC framework was modeling of rheological properties of complex fluids [47, 70], where it is popular to include a symmetric positive definite tensor into the set of state variables, e.g., the conformation tensor. Here we choose the left Cauchy-Green tensor (or Finger tensor)

\[ B_{ij} = A^{-1}_{ik} A^{-1}_{kj}, \] (78)

which has been shown useful when describing complex fluids [50, 61, 80].

In particular, it can be demonstrated (see the e-print version [76] of the paper for the details) that a Poisson bracket for the electromagnetic field itself is

\[ \{ E_e, E_e \} = \mathcal{A} E_e \partial_j \mathcal{A}_D_j - \mathcal{A}_m \partial_j \mathcal{A} D_j \] dr

where the bracket for the electromagnetic field itself is

\[ \{ \mathcal{A}, \mathcal{B} \}^{(EM)} = \int (\mathcal{A}_{D_i} \mathcal{B}_{j} \partial_j \mathcal{A}_{B_k} - \mathcal{B}_{m_i} \mathcal{A}_{B_k} \partial_j \mathcal{B}_{B_k}) \] dr.

14 See details in the e-print version [76] of the paper.
Note that the momentum density $m$ is the total momentum of matter and electromagnetic field as in [21] and [17].

By taking a Hamiltonian $\mathcal{H} = \int E(m, B, D, \rho, s)dr$ and rewriting bracket (80) in a form similar to (69), one directly obtains the following evolution equations
\begin{align}
\partial_t m_i &= - \partial_j (m_i E_{mj}) - \rho \partial_i E_\rho - m_j \partial_i E_{mj} - s \partial_i E_s - D_j \partial_i E_{Dj} - B_j \partial_i E_{Bj} + \partial_j \left( D_j E_{Dj} + B_j E_{Bj} \right), \\
\partial_t B_j &= - \partial_j (B_j E_{mj} - E_{mj} B_j + \epsilon_{ijk} E_{Dk}) - E_{mj} \partial_j B_j, \\
\partial_t D_j &= - \partial_j (D_j E_{mj} - E_{mj} D_j - \epsilon_{ijk} E_{Bk}) - E_{mj} \partial_j D_j, \\
\partial_t \rho &= - \partial_i \left( \rho E_{mj} \right), \\
\partial_t s &= - \partial_i \left( s E_{mj} \right),
\end{align}
where $E(m, B, D, \rho, s)$ being the total energy density. These evolution equations are also equipped with two constraints imposed on the Poisson bracket $\text{div} \ (D) = z \epsilon \rho / \epsilon_0$, $\text{div} \ (B) = 0$, where $z$ is number of elementary charges per particle, $\epsilon$ is the elementary charge and $\epsilon_0$ permittivity of vacuum.

By subtracting and adding $\partial_i E$ to the momentum equation, it can be transformed exactly into the SHTC momentum conservation law (27a), while the equations for $B$ and $D$ already have exactly the structure of Eqs. (27c) and (27d) accordingly. Thus, the entire GENERIC formulation and the SHTC formulation for electrodynamics of moving medium are fully compatible.

4.5 Ballistic transport

The goal of this section is to present a Poisson bracket which generates equations (27a), (27f) and (27e), and thus to prove their compatibility with GENERIC. In applications, these equations can be used for modeling of heat and mass transfer if equipped with appropriate dissipative algebraic source terms (see Sects. 5 and 6 for details). However, in this section we shall use the terminology for heat conduction and we shall call the scalar field $s$ the entropy density while the vector field $w$ as the conjugate entropy flux. Absence of the dissipative terms, i.e., absence of the interaction between the mass or heat carriers and the medium, corresponds to the most non-equilibrium state, ballistic transport, which explains the title of this Section.

Consider first a rigid heat-conducting body. We wish to prescribe a field of entropy density $s$ expressing local thermodynamic state of the body. Let us now address dynamics of the field. Taking a Lagrangian $\Lambda(s, \dot{s})$, the Legendre transformation introduces the corresponding Hamiltonian,
\begin{equation}
\frac{\delta}{\delta \dot{s}} \left( -\Lambda(s, \dot{s}) + \int \psi(r) \dot{s}(r) dr \right) = 0 \quad \Leftrightarrow \quad A_\dot{s} = \psi, \tag{83}
\end{equation}
and the conjugate scalar field $\psi(r, t)$. The couple $(\psi, s)$ is equipped with the canonical Poisson bracket
\begin{equation}
\{A, B \}^{(\psi, s)} = \int (\partial_k A_k B - \partial_k B_k A_k) d^3 r. \tag{84}
\end{equation}
However, the aim is to work with a vector (covariant) field that plays the role of conjugate entropy flux. Therefore, taking functionals dependent only on $w = -\nabla \psi$, the canonical bracket becomes
\begin{equation}
\{A, B \}^{(w, s)} = \int (\partial_k A_k B_{wk} - \partial_k B_k A_{wk}) d^3 r, \tag{85}
\end{equation}
and the implied evolution equations generated by bracket (85) are
\begin{align}
\partial_t s &= - \partial_k E_{wk}, \tag{86a} \\
\partial_t w_k &= - \partial_k E_s, \tag{86b}
\end{align}
equipped with the condition that $w$ is a potential vector field, or that
\begin{equation}
\partial_t w_j = \partial_j w. \tag{86c}
\end{equation}
From (86a) it is clear why \(w\) is referred to as the conjugate entropy flux. Equation (86) express evolution of entropy within rigid heat conductors.

How can one couple the dynamics of heat with hydrodynamics, i.e., with transport of matter? The hydrodynamic bracket for \((m, \rho)\) is a Lie–Poisson bracket on the Lie algebra dual represented by the hydrodynamic fields. Action of hydrodynamics on elements of the cotangent bundle \((\psi, s)\) is minus Lie derivative with respect to the fluid velocity. Semidirect product of the hydrodynamic fields and the cotangent bundle is then equipped with Poisson bracket

\[
\{\mathcal{A}, \mathcal{B}\}^{(m, \rho) \times (\psi, s)} = \{\mathcal{A}, \mathcal{B}\}^{(Euler)} + \int (\mathcal{A}_\psi \mathcal{B}_s - \mathcal{A}_s \mathcal{B}_\psi) \, d\mathbf{r} + \int (\mathcal{B}_\psi \partial_j \mathcal{A}_\psi \partial_j \mathcal{B}_\psi - \mathcal{B}_s \partial_j \mathcal{A}_s \partial_j \mathcal{B}_s) \, d\mathbf{r},
\]

see, e.g., [76] or [21]. Taking functionals dependent only on gradients of \(\psi\), denoted by \(w_i = -\partial_i \psi\), this last bracket becomes

\[
\{\mathcal{A}, \mathcal{B}\}^{(m, \rho) \times (w, s)} = \{\mathcal{A}, \mathcal{B}\}^{(Euler)} + \int (\partial_j \mathcal{A}_w \partial_j \mathcal{B}_w - \partial_j \mathcal{A}_s \partial_j \mathcal{B}_s) \, d\mathbf{r} + \int w_j \left( \partial_i \mathcal{A}_{w_i} \partial_j \mathcal{B}_{w_j} - \partial_i \mathcal{B}_{w_i} \partial_j \mathcal{A}_{w_j} \right) \, d\mathbf{r}.
\]  

(87)

This Poisson bracket expresses kinematics of variables \((m, \rho, w, s)\) provided \(w\) is a potential vector field, i.e., \(\nabla \times w = 0\). Indeed, Jacobi identity is fulfilled only when this involution constraint is met (as in the case of the transition from kinematics of labels to kinematics of distortion matrix).

Validity of Jacobi identity for bracket (87) even for fields \(w\) with nonzero curl can be achieved by extending the Poisson bracket as in the case of distortion matrix in Sect. 4.3.2. A new term has to be added to the bracket proportional to \(\partial_i w_j - \partial_j w_i\) (the term cannot be multiplied by any constant). Moreover, the term has to provide coupling between \(w\), which is an odd variable, and another odd field, for example, with the field \(m\). The only way to extend the bracket (87) is then as follows (the Jacobi identity for this bracket has been checked by program [58])

\[
\{\mathcal{A}, \mathcal{B}\} = \{\mathcal{A}, \mathcal{B}\}^{(m, \rho) \times (w, s)} + \int \left( \partial_i w_j - \partial_j w_i \right) \left( \mathcal{A}_{w_i} \mathcal{B}_{w_j} - \mathcal{B}_{w_i} \mathcal{A}_{w_j} \right) \, d\mathbf{r},
\]  

(88)

which\(^{15}\) after rearranging the terms as in (69), leads to the time evolutions equations

\[
\begin{align*}
\partial_t m_i &= -\partial_k (m_i E_{mk}) - \rho \partial_i E_\rho - s \partial_i E_s - m_k \partial_i E_{mk} - w_j \partial_i E_{w_k} - (\partial_i w_j - \partial_j w_i) E_{w_k}, \\
\partial_t w_k &= -\partial_k (w_j E_{mj} + E_s) - (\partial_j w_k - \partial_k w_j) E_{mj}, \\
\partial_t s &= -\partial_k (s E_{mk} + E_s), \\
\partial_t \rho &= -\partial_k (\rho E_{mk}).
\end{align*}
\]

(89)

which coincide with the Eulerian SHTC equations (27a), (27e), (27f) and (27g).

In summary, Poisson bracket (87) has been derived as a result of coupling between the Lie algebra dual of classical hydrodynamics and the cotangent bundle of dynamics of entropy. The Jacobi identity is valid only for potential vector fields \(w\). Validity of Jacobi identity can be extended to fields \(w\) with nonzero curl by extending the bracket by a term coupling fields \(w\) and \(m\), and such an extension can be carried out in only one way. The final extended Poisson bracket is bracket (88), which generates the SHTC evolution equations (89) for ballistic transport of the scalar field \(s\).

An alternative Poisson bracket for hyperbolic heat conduction can be proposed based on [46], see [76]. That alternative bracket is very similar to the SHTC bracket (88), but it contains extra terms expressing inertia of the field \(w\).

4.6 Complete Poisson bracket

The complete Poisson bracket generating SHTC Eulerian equations (27) is then the combination of brackets (62), (75), (80) and (88),

\[
\{\mathcal{A}, \mathcal{B}\}^{(SHTC)} = -2\{\mathcal{A}, \mathcal{B}\}^{(Euler)} + \{\mathcal{A}, \mathcal{B}\}^{(DM)} + \{\mathcal{A}, \mathcal{B}\}^{(EMHD)} + \{\mathcal{A}, \mathcal{B}\}^{(Heat)}.
\]

(90)

Bracket \(\{\mathcal{A}, \mathcal{B}\}^{(Euler)}\) is contained in the last three terms, and therefore we have to subtract it twice so that it is contained only once in the final Poisson bracket. Validity of the Jacobi identity for this bracket was checked by the program developed in [58].

\(^{15}\) This bracket was first proposed in [68].
5 Dissipation

Dissipative evolution is usually meant as the evolution that raises entropy. In the sense of the hyperbolic conservation laws (SHTC), entropy density is among state variables and entropy is thus conserved unless dissipative terms are added to equations (27). In the sense of GENERIC, entropy is required to be a Casimir of the Poisson bracket, i.e., \( \{ \mathcal{S}, \mathcal{E} \} = 0 \) for all possible energies \( \mathcal{E} \), and it is thus also conserved unless some dissipative terms are added. It is the goal of this section to compare the way dissipation is added in the SHTC and GENERIC formalisms.

Before introducing concrete forms of dissipative terms, let us make a comment on reversibility and irreversibility in the sense of time reversal. An evolution equation is reversible if it is intact by the time-reversal transformation (TRT), which inverts velocities of all particles and magnetic field as well as the sign of the time increment in time derivatives. Evolution equations in non-equilibrium thermodynamics usually contain both reversible and irreversible parts. Assuming that the left-hand sides of the evolution equations are constituted by only partial time derivatives of the state variables, the reversible parts of the right-hand sides transform under TRT exactly as the left-hand sides of the corresponding equations (partial time derivatives of the state variable) while the irreversible parts of the right-hand sides gain opposite signs than the corresponding left-hand sides. Reversible evolution is invariant with respect to TRT while the irreversible changes its sign.

In order to be able to apply TRT to an equation, however, one has to first introduce parities with respect to TRT. A quantity is called even with respect to TRT (parity equal to 1) if TRT does not alter the quantity, and it is called odd (parity equal to \(-1\)) if TRT changes sign of the quantity. How can we determine whether a quantity is even or odd? The odd quantities are those that change its sign when velocities of particles and magnetic field are inverted. For example, density is even, momentum is odd, entropy and energy are even. Electric displacement field is even while magnetic field is odd. As usually in non-equilibrium thermodynamics, we require that the reversible parts of evolution equation do not change entropy (i.e., are non-dissipative) while the irreversible terms raise entropy (are dissipative), and the adjectives irreversibility and dissipativity will be then considered equivalent. This requirement is satisfied both in the SHTC equations and in the GENERIC framework, see, e.g., [73].

In general, dissipative (or irreversible) evolution within the SHTC framework is constructed by means of a quadratic dissipation potential, i.e., the irreversible terms are linear in derivatives of energy (conjugate state variables within SHTC). Within GENERIC we also prefer using a dissipation potential although some authors prefer using a possibly non-symmetric dissipative bracket [52,69], which makes the distinctions between reversible and irreversible dynamics based on TRT and entropy growth non-equivalent. The geometric character and statistical interpretation of dissipation potentials [64], discussed also in [53], convince us to prefer dissipation potentials. The dissipation potential is, however, used in a slightly different way in SHTC than within GENERIC, which differ by the way conjugate variables are constructed from the state variables. In the SHTC formalism conjugate variables are constructed as derivatives of energy while in GENERIC they are derivatives of entropy. Let us now discuss all the mentioned possibilities.

5.1 Irreversible dynamics

5.1.1 Energy representation

In the SHTC framework, irreversible dynamics is modeled via only algebraic (independent of spatial gradients) dissipative terms which do not harm the hyperbolic type of the PDEs. In the light of our recent results [16,17,77] on modeling of dissipative processes such as viscous momentum, heat and charge transfer, the modeling of dissipative phenomena with algebraic terms of relaxation type does not seem to be something restrictive.

In what follows we provide a recipe on how to introduce dissipative terms into system (27). It is strongly motivated by the key fact around the SHTC equations, the summation rule (30). In this consideration we do not consider the momentum and mass conservation as they are pure reversible time evolutions. Thus, we add, so far, arbitrary functions of the state variables into the right-hand side of each of the remaining equations:

\[
\begin{align*}
\partial_t A_{ik} + \partial_k (A_{il} v_l) + v_j (\partial_j A_{ik} - \partial_k A_{ij}) &= S_{ik}^A, \\
\partial_t h_i + \partial (h_i v_k - v_i h_k + \epsilon_{ikl} E_{el}) + v_i \partial_k h_k &= S_{i}^h, \\
\partial_t e_i + \partial_k (e_i v_k - v_i e_k - \epsilon_{ikl} E_{hl}) + v_i \partial_k e_k &= S_{i}^e,
\end{align*}
\]
\[ \begin{align*}
\partial_t w_k + \partial_k (v_j w_l + E_s) + v_j (\partial_j w_k - \partial_k w_j) &= S_k^w, \\
\partial_t s + \partial_k (sv_k + E_w) &= S^s.
\end{align*} \]  
(91d, 91e)

Because the functions \( S_{ij}^A, S_i^h, S_i^e, S_i^w \) and \( S^s \) are quite arbitrary, we cannot guaranty that
\[ A_i^s S_{ij}^A + h_i^s S_i^h + e_i^s S_i^e + w_i^s S_i^w + s^s S^s \equiv 0, \]
(92)

which is required by the summation rule (30) and energy conservation (29), and where conjugate variables are denoted as follows,
\[ A_i^s = E_A_{ij}, \quad e_i^s = E_{e_i}, \quad h_i^s = E_{h_i}, \quad s^s = E_s, \quad w_i^s = E_{w_i}. \]
(93)

We shall use this notation hereafter.

The central question is now how to introduce the dissipative sources in (91) so that first law (29) and the second law of thermodynamics (entropy grows) are satisfied? It seems that the only degree of freedom we have is to assign to one of the variables the role of the entropy. Bearing in mind the Hamiltonian nature of the reversible part of the SHTC equations, only a Casimir of the corresponding Poisson bracket (90) can be chosen as the entropy. The field \( s \) indeed fulfills this criterion. We emphasize that the introduction of the entropy in the SHTC framework is not a spontaneous step, but it is fully motivated by the necessity to respect the first law of thermodynamics.

We now show how the entropy field can be used to fix the energy conservation, i.e., to guarantee the zero on the right-hand side of (92). It is evident that if we define the entropy source term as
\[ S^s = -\frac{1}{s^s} \left( A_i^s S_{ij}^A + h_i^s S_i^h + e_i^s S_i^e + w_i^s S_i^w \right) \]
(94)

then (92) holds automatically (recall that \( s^s = E_s \) is usually interpreted as the temperature). However, there is also the second law of thermodynamics which postulates the positivity of the entropy production, \( S^s \geq 0 \). How can we guarantee the second law is fulfilled? Perhaps, the simplest (but not exhaustive) solution is as follows. Let us introduce the vectors of state variables and conjugate variables
\[ Q = (q, s), \quad Q^* = (q^*, s^*), \]
(95)

where \( q = (A_{ij}, h_i, e_i, w_i) \) and \( q^* = (A_i^s, h_i^s, e_i^s, w_i^s) \), i.e., they do not include \( s \) and \( s^s \) accordingly. Then, we define the dissipative sources as the partial derivatives
\[ S_{ij}^A = -\frac{\partial \Psi}{\partial A_{ij}}, \quad S_i^h = -\frac{\partial \Psi}{\partial h_i^s}, \quad S_i^e = -\frac{\partial \Psi}{\partial e_i^s}, \quad S_i^w = -\frac{\partial \Psi}{\partial w_i^s}, \]
(96)
of a potential \( \Psi(q^*) \) which does not explicitly depend on \( s^s \) and is taken as
\[ \Psi(q^*) = \frac{1}{2} q^* \Lambda q^*, \]
(97)

where \( \Lambda = \Lambda(Q) \) being a symmetric positive semidefinite matrix, \( \Lambda^T = \Lambda \geq 0 \), entries of which may depend on the primary state variables \( Q \) (as well as on other state variables whose time evolutions are reversible and are omitted in this section, e.g., \( \rho \)) and will be associated with the inverse of the characteristic dissipation times in Sect. 6. Therefore, with the choice (96), (97), the second law, i.e., the inequality
\[ S^s \geq 0, \]
(98)
is automatically fulfilled.

We note that the main nonlinearity of the dissipative processes is not defined by the nonlinearity of the dissipation potential (97) or nonlinearity of the conjugate variables \( q^* \) but it is, in fact, hidden in the entries of \( \Lambda(Q) \). For example, when dealing with irreversible deformations in solids, the entries of \( \Lambda \) are associated with the inverse strain relaxation times \([3, 13, 22, 30, 37–39, 85] \) which may vary over several orders of magnitude.

Therefore, we have shown that the SHTC irreversible time evolution can be expressed in terms of the quadratic dissipation potential \( \Psi(q^*) \) (97) as
\[ (\partial_t q_i)_{irr} = -\Psi_{q_i^*}, \]
(99a)

except for the right-hand side of the evolution equation for entropy density, which is constructed as
\[ (\partial_t s)_{irr} = \frac{1}{s} q_i^* \Psi_{q_i^*} \geq 0. \]
(99b)

This particular choice of the dissipative source terms respects the both laws of thermodynamics.
5.1.2 Entropic representation and gradient dynamics

In the preceding section we have shown that the irreversible part of the SHTC evolution of all the state variables can be generated by a potential \( \Psi(q^*) \) which does not depend explicitly on \( s^* \), and \( q^* \) were defined as the conjugate variables with respect to the energy, \( q^* = E_q \). Within GENERIC, on the other hand, conjugate variables in the irreversible part are usually identified as derivatives of entropy with respect to the state variables and irreversible time evolution can be formulated as nonlinear gradient dynamics with a prefactor playing the role of temperature

\[
(\partial_t q)_{irr} = \frac{1}{s_E} \frac{\delta \mathcal{E}}{\delta q} \quad \text{and} \quad (\partial_t E)_{irr} = \frac{1}{s_E} \frac{\delta \mathcal{E}}{\delta s_E},
\]

where \( \delta/\delta \) stands for a functional derivative and \( E \) is the total energy density. The prefactor was introduced for example in [73], and here it is used merely to make the evolution completely compatible with SHTC. Without the prefactor the equations would be nearly the same (up to the temperature prefactor). Let us now demonstrate the compatibility of the irreversible SHTC and GENERIC evolutions.

But first we shall recall transformations between the energetic representation, where state variables are \((q, s)\), and entropic representation, where state variables are \((q, E)\). In the energetic representation conjugate variables are identified with derivatives of energy with respect to the state variables, \((q^*, s^*) = (E_q, E_s)\), while in the entropic representation the conjugate variables are derivatives of entropy, \((p^*, E^*) = (s_q, s_E)\).

The two representations thus have different conjugate variables.

The relations between conjugate variables in the two representations are:

\[
\left( \frac{\partial E}{\partial q_i} \right)_s = -\left( \frac{\partial E}{\partial q_i} \right)_q \cdot \left( \frac{\partial s}{\partial q_i} \right)_E \quad \text{and} \quad \frac{\partial E}{\partial s} = \frac{1}{\delta s/\delta q},
\]

or

\[
q^*_i = -s^*_i p^*_i \quad \text{and} \quad s^* = \frac{1}{E^*}.
\]

From (101) it follows that

\[
\left( \frac{\partial q^*_i}{\partial p^*_j} \right)_{E^*} = -s^*_i \delta_{ij}, \quad \left( \frac{\partial q^*_i}{\partial E^*} \right)_{p^*} = \frac{p^*_i}{(E^*)^2} \quad \text{(103a)}
\]

\[
\left( \frac{\partial s^*}{\partial p^*_i} \right)_{E^*} = 0 \quad \text{and} \quad \left( \frac{\partial s^*}{\partial E^*} \right)_{p^*} = -\frac{1}{(E^*)^2}. \quad \text{(103b)}
\]

Derivatives of the dissipation potential then transform as

\[
\frac{\partial \mathcal{E}}{\partial p^*_i} = \frac{\partial \mathcal{E}}{\partial q^*_j} \frac{\partial q^*_j}{\partial p^*_i} + \frac{\partial \mathcal{E}}{\partial s^*} \frac{\partial s^*}{\partial p^*_i} = -s^* \frac{\partial \mathcal{E}}{\partial q^*_i}. \quad \text{(104a)}
\]

\[
\frac{\partial \mathcal{E}}{\partial E^*} = \frac{\partial \mathcal{E}}{\partial q^*_j} \frac{\partial q^*_j}{\partial E^*} + \frac{\partial \mathcal{E}}{\partial s^*} \frac{\partial s^*}{\partial E^*} = -s^* \left( q^*_j \frac{\partial \mathcal{E}}{\partial q^*_j} + s^* \frac{\partial \mathcal{E}}{\partial s^*} \right). \quad \text{(104b)}
\]

Irreversible GENERIC evolution equations (100) can be then reformulated as

\[
\left( \frac{\partial q^*_i}{\partial t} \right)_{irr} = -(s^*)^2 \mathcal{E}_{q^*_i}. \quad \text{(105a)}
\]

\[
\left( \frac{\partial E}{\partial t} \right)_{irr} = -(s^*)^2 \left( q^*_j \frac{\partial \mathcal{E}}{\partial q^*_j} + s^* \frac{\partial \mathcal{E}}{\partial s^*} \right). \quad \text{(105b)}
\]

Since the SHTC dissipation potential \( \Psi \) is always assumed algebraic, i.e., does not contain spatial gradients, the sought corresponding GENERIC dissipation potential \( \mathcal{E} \) should be also algebraic. Energy conservation then requires that the right-hand side of the evolution equation for total energy density disappears, i.e.,

\[
\mathcal{E}_{E^*} = 0 = q^*_j \frac{\partial \mathcal{E}}{\partial q^*_j} + s^* \frac{\partial \mathcal{E}}{\partial s^*}. \quad \text{(105c)}
\]
which means that the dissipation potential must be a zero-homogeneous function in the energetic representation. These evolution equations imply the irreversible evolution of entropy density,

\[
\left( \frac{\partial s}{\partial t} \right)_{irr} = \frac{p_j^s}{E^s} \Xi_{p_j^s} = q_j^s s^s \Xi_{q_j^s}.
\] (105d)

We have arrived at the algebraic energy-conserving irreversible GENERIC evolution equations in the energetic representation.

Let us now go back to the irreversible evolution formulated within the SHTC framework, Eq. (99), which are generated by the dissipation potential \( \Psi \), which is independent of \( s^s \), and by the requirement of energy conservation. In order to find the gradient dynamics in the entropic representation that is compatible with the SHTC irreversible evolution, we need to find the dissipation potential \( \Xi \) such that Eq. (105) is the same as Eq. (99). This is achieved by choosing

\[
\Xi(q^s, s^s) = \Psi(q^s)|_{q^s:=q^s/s^s}.
\] (106)

Such a dissipation potential clearly conserves energy because it can be written as a function of \( p^s \) only (not \( E^s \)). Derivatives of this dissipation potential are

\[
\Xi_{q_j^s} = \frac{1}{s^s} \Psi_{q_j^s}|_{q^s:=q^s/s^s} \quad \text{and} \quad \Xi_{s^s} = -\frac{q_i^s}{(s^s)^2} \Psi_{q_i^s}|_{q^s:=q^s/s^s},
\] (107)

and evolution equations (105) become

\[
\left( \frac{\partial q_j^s}{\partial t} \right)_{irr} = -s^s \Psi_{q_j^s}|_{q^s:=q^s/s^s}, \quad \left( \frac{\partial s}{\partial t} \right)_{irr} = q_i^s \Psi_{q_i^s}|_{q^s:=q^s/s^s},
\] (108)

which are exactly the same as Eq. (99) for quadratic potentials \( \Psi \). For non-quadratic potentials it gives extra temperature prefactors. Non-quadratic dissipation potentials have been successfully used to describe chemical or electrochemical reactions and the Boltzmann collision operator [44,47,74].

In summary, energy-conserving algebraic irreversible GENERIC gradient dynamics with the temperature prefactor is completely compatible with the irreversible SHTC equations provided the SHTC dissipation potential is quadratic. If the potential is not quadratic, extra dependence on temperature pops out, which does not play any crucial role because of the possible dependence of the phenomenological equations on temperature. One can even start with purely gradient dynamics without any prefactor, as is usual within GENERIC, and the resulting evolution equations in the energetic representation would be compatible with the SHTC evolution up to a prefactor given by a power of temperature.

5.2 Involution constraints in the case of dissipation

It is important to remark that there is an intimate connection between the dissipation and involution constraints discussed in Sect. 3.2.2. The presence of the dissipative algebraic source terms \( S_{ik}^A, S_{ik}^B, S_{ik}^c \) and \( S_{ik}^w \) in (91) also affects the time evolutions of \( B = \nabla \times A, Q = \nabla \cdot h, R = \nabla \cdot e \) and \( \Omega = \nabla \times w \) in the following way

\[
\begin{align*}
\partial_t B_{ij} + \partial_k (B_{ij} v_k - v_j B_{ik} + \varepsilon_{jkl} S_{ik}^A) + v_j \partial_k B_{ij} &= 0, \quad (109a) \\
\partial_t Q + \partial_k (Q v_k + S_{ik}^B) &= 0, \quad (109b) \\
\partial_t R + \partial_k (R v_k + S_{ik}^c) &= 0, \quad (109c) \\
\partial_t \Omega_j + \partial_k (\Omega_j v_k - v_j \Omega_k + \varepsilon_{jkl} S_{ik}^w) + v_j \partial_k \Omega_k &= 0, \quad (109d)
\end{align*}
\]

i.e., the dissipative sources emerge as constitutive fluxes in the time evolutions for the fields \( B, Q, R \) and \( \Omega \). Therefore, in contrast to the reversible dynamics, these fields deviate from zero values even if initially \((t = 0)\) \( B = 0, Q = 0, R = 0 \) and \( \Omega = 0 \) and hence, may serve as indicators of irreversible dynamics. See also Sect. 6 with some examples for more discussions.
5.3 Dispersive dynamics

It is not obligatory in the SHTC theory that addition of the algebraic source terms to the master equations results in that the dynamics becomes irreversible, i.e., rise the entropy. See also [69] for similar situations within GENERIC. In fact, the source terms can be added in such a way that both, the entropy and the total energy, are conserved. This can be achieved by introducing new state variables representing the dynamics at a microscale and designing the source terms which models the energy conversion between the macro- and micro- scales without any lost. For example, the impact of the dislocation density on the dispersive properties of the elastic waves was studied in [84] where it was assumed that the material has nonzero initial concentration of defects and the energy converts reversibly between the kinetic energy carried by the elastic waves and the energy stored in the microscopic defects.

We recall that in the classical continuum mechanics, the dispersive dynamics, as well as the dissipative dynamics, is modeled by means of high-order PDEs. However, as demonstrated in [16, 17, 77], the classical dissipative models can be successfully modeled with first-order PDEs, the dispersive properties of the continuous media can be also modeled with merely first-order PDEs, e.g., see [23, 63, 84].

6 Examples

6.1 Dynamic MaxEnt reduction

In non-equilibrium thermodynamics one often deals with the problem of how to reduce some detailed evolution to a less detailed evolution. In particular, one often has a fast state variable which is to be eliminated from the evolution equations (enslaved by the other variables). A standard procedure for elimination of the fast variable is the Chapman–Enskog reduction, see, e.g., [48]. However, the reduction often relies on demanding calculations and rigorous origin of the method can also be questioned.

An alternative method to the Chapman–Enskog reduction was proposed in [73]. The method relies on identification of state and conjugate variables in the evolution equations. Note that the Chapman–Enskog reduction does not use this extra information. It is becoming fashionable to formulate non-equilibrium thermodynamics within contact geometry, where the conjugate variables have their independent meaning and own evolution equations, see, e.g., [45]. In this sense, it is reasonable to treat conjugate and state variables differently. For example, in the reduction proposed in [73], let us refer to it as to the dynamic MaxEnt reduction, the fast variable is set to the corresponding value given by maximization of entropy while the conjugate fast variable is set to a value solving the stationary fast evolution equation. This way the conjugate fast variable becomes a function of the other state variables, and due to the coupling between the fast and slow variables, evolution equations for the slow variables obtain irreversible terms caused by relaxation of the fast variable. Let us now demonstrate this reduction on particular examples.

6.2 Navier–Stokes dynamics

Compressible Navier–Stokes (NS) equations consist of a reversible and irreversible part. The reversible part, the compressible Euler equations, is of course implied by both the SHTC and GENERIC frameworks. Taking, for example, Poisson bracket (62), the implied evolution equations are Eq. (70), which represent the Euler equations.

The irreversible part of NS equations is more difficult to obtain. The NS equations cannot be incorporated into the SHTC framework because of the parabolic character of the irreversible terms (Laplacian of velocity). Within GENERIC, the NS equations can be recovered by taking a quadratic dissipation potential, or a dissipative bracket, which is composed of second derivatives of the potential, as in [69],

$$\Sigma(x^*) = \frac{1}{2} S_{ij} M^{ij} S_{x^j},$$

(110)

where state variables are \(x = (m, \rho, E)\) and \(x^*\) are the corresponding conjugate (with respect to entropy) variables.

Our goal, however, is to see the Navier–Stokes equation as an approximation of the SHTC equations. This, in particular, was demonstrated in [16, 77], but this time we would like also to demonstrate the dynamic...
MaxEnt reduction. For this purpose, let us consider dynamics of the Finger (or left Cauchy-Green) tensor from Sect. 4.3.3, which is a consequence of kinematics of the generalized distortion matrix. Let the energy potential depend on the fields through

$$ E = \varepsilon(\rho, s) + \frac{1}{2} \alpha^2 (\text{dev} \mathbf{B})^2 + \frac{1}{2} \beta^2 (\text{tr} \mathbf{B} - 3)^2 + \frac{m^2}{2 \rho}, \quad (111) $$

derivative of which with respect to the Finger tensor is

$$ E_\mathbf{B} = \alpha^2 \text{dev} \mathbf{B} + \beta^2 (\text{tr} \mathbf{B} - 3) \mathbf{I}. \quad (112) $$

Note that \( \text{dev} \mathbf{B} \) is the deviatoric (traceless) part of tensor \( \mathbf{B} \) and \( \alpha \) and \( \beta \) are assumed to be material-dependent constants (typically sound speeds, e.g., see [16]). The derivative \( E_\mathbf{B} \) is zero for \( \mathbf{B} = \mathbf{I} \). Similarly, one can express entropy as function of \( (\mathbf{m}, \mathbf{B}, \rho, E) \) and derivative of entropy with respect to \( \mathbf{B} \) will be zero only for Finger tensor equal to the unit matrix. The MaxEnt value of \( \mathbf{B} \) is thus the unit matrix \( \mathbf{I} \).

Putting the MaxEnt value of \( \mathbf{B} \) into the evolution equations (79), and adding the shear and volume dissipative terms\(^\text{16}\) \(-\tau_S^{-1} \text{dev} \mathbf{B}^* \) and \(-\tau_V^{-1} 1/3(\text{tr} \mathbf{B}^*) \mathbf{I} \) to the right-hand side of the equation for \( \mathbf{B} \), leads to

$$ \partial_t m_i = - \partial_j (m_j m_i^*) - \rho \partial_i \rho^* - m_j \partial_i m_j^* - s \partial_i s^* + 2 \partial_k B^*_{ik} $$
$$ 0 = \partial_j m_j^* + \partial_j m_i^* - \frac{1}{\tau_S} \text{dev} B^*_{ij} - \frac{1}{3 \tau_V} \text{tr} (\mathbf{B}^*) \delta_{ij}. \quad (113a, 113b) $$

The last equation means that

$$ \text{dev} B^*_{ij} = \tau_S \left( \partial_j m_j^* + \partial_j m_i^* - \frac{2}{3} \partial_k m_k^* \delta_{ij} \right) \quad \text{and} \quad \frac{1}{3} \text{tr} (\mathbf{B}^*) = \frac{2}{3} \tau_V \partial_k m_k^*, \quad (114) $$

which, after substitution into the equation for \( \mathbf{m} \) and evaluating the conjugate variables as the corresponding derivatives of energy, leads to the Navier–Stokes equations

$$ \frac{\partial m_j}{\partial t} = - \partial_j (m_i E_{m_i}) - \rho \partial_i E_{\rho} - m_j \partial_i E_{m_j} - s \partial_i E_s \quad \text{and} \quad \partial_k \left( 2 \tau_S \left( \partial_i E_{m_k} + \partial_k E_{m_i} - \frac{2}{3} \partial_l E_{m_l} \delta_{ik} \right) \right) + \partial_i \left( \frac{4}{3} \tau_V \partial_k E_{m_k} \right), \quad (115) $$

where \( \tau_S \) and \( \tau_V \) are related to shear and volume viscosities.

Note that if the volumetric relaxation time, \( \tau_V \), goes to infinity, it follows from Eq. (113b) that the spherical part of the \( \mathbf{B} \) tensor remains undetermined and the condition of incompressibility, \( \nabla \cdot \mathbf{E}_m = 0 \), is enforced after the dynamic MaxEnt reduction. This way the incompressible Navier–Stokes equation can be recovered, and the undetermined spherical part of \( \mathbf{B} \) plays the role of pressure in incompressible fluids.

In summary, compressible (and also incompressible) Navier–Stokes equation is an approximation of the evolution of the Finger tensor, which is implied by evolution of the generalized distortion matrix, compatible with the SHTC equations.

### 6.3 Elastic and elastoplastic solids

Classical evolution of elastic solids can be formulated by means of the distortion matrix easily [37–39]. The distortion matrix is then interpreted as inverse of the deformation gradient, and evolution equations (76) or (27a), (27b), (27g) become equivalent with the balance of mass, balance of momentum, and evolution equation for the inverse deformation gradient in the Eulerian frame. Recall that the fact that the distortion matrix is the inverse deformation gradient is expressed by the integrability condition \( \nabla \times \mathbf{A} = 0 \).

Equation (76) or (27b) is, however, much more general than the mentioned Eulerian equations of classical continuum theory because matrix \( \mathbf{A} \) is allowed to have nonzero curl, in which case the interpretation as a \( \text{global} \) deformation field becomes invalid. The case of distortion matrix with nonzero curl corresponds to

\(^{16}\) The dissipative terms can be seen as derivatives of a quadratic dissipation potential with respect to \( \mathbf{B}^* \), see Sect. 5. It should be also borne in mind that \( \mathbf{B}^* \) is interpreted as an element of the dual space to the space where \( \mathbf{B} \) lives, but in the precise variant of the evolution, Eq. (79), it can be identified with derivative of energy with respect to \( \mathbf{B} \), i.e., \( \mathbf{B}^* = E_\mathbf{B} \).
irreversible deformations, where indeed one cannot reconstruct the original Lagrangian frame precisely, e.g., see [37–39, 85]. Nonzero curl of $A$ is caused by dissipative term $\frac{1}{\tau}E_A$ [see (109a)] in the evolution equation for the distortion matrix because the time evolution for the Burgers tensor $B = \nabla \times A$ (dislocation density tensor) is, e.g., see Appendix C in [76] or [37–39, 75],

$$\partial_t B_{ij} + \partial_k \left( B_{ij} v_k - v_j B_{ik} + \epsilon_{jkm} \frac{1}{\tau} E_{A_{im}} \right) + v_j \partial_k B_{ik} = 0,$$

from which it follows that even if $B_{ij} = 0$ initially it will, in general, distinct from zero at later times. One can clearly see that this equation has the SHTC structure of Eq. (27c). However, to have the full SHTC and GENERIC structure, this equation should be accompanied by a time evolution for a complimentary tensor field $D$, exactly as $h$ is the complimentary field to $e$, see Sect. 3.1.3. In the theory of dislocations, the complimentary field $D$ has the meaning of the flux dislocation density, e.g., see a discussion in [75].

It is interesting to note that if $B \neq 0$, i.e., in the case of irreversible deformations, the SHTC Eulerian equations (27a), (27b) can be symmetrized only in the extended weakly non-local sense, see Sect. 3.2.6, i.e., when the Burgers tensor $B$ and the flux dislocation density tensor $D$ are considered as new independent state variables and the total energy potential $E = E(m, A, B, D, \rho, s)$ depends on the extended set of state variables. The later leads to the appearing of extra stress in the momentum flux (27a) due to dislocation motion:

$$\partial_t m_i + \partial_k \left( m_i E_{mk} + \delta_{ik} \left( \rho E_\rho + \sigma E_\sigma + m_i E_{mj} + B_{jl} E_{B jl} + D_{jl} E_{D jl} - E \right) - B_{jk} E_{B ji} - D_{jk} E_{D ji} + A_{jk} E_{A ji} \right) = 0.$$

Thus, evolution of the distortion matrix $A$ gives, in fact, exact evolution of the Burgers tensor, defining dynamics of dislocations [37–39]. The evolution equation of the distortion matrix coupled with mass, momentum and energy conservation, thus represents an Eulerian framework covering elastic and elastoplastic-plastic solids as well as fluids, see also [3, 13, 16, 22, 77, 78, 85]. In practice, the transition between the elastic and plastic response of solids is governed by the dependence of the strain relaxation time on the state parameters, $\tau = \tau(A, B, D, \rho, s)$, see details in [3, 13, 22, 77, 85].

6.4 Non-Newtonian fluid dynamics

Dynamics of viscoelastic fluids (or solids) can be also formulated within the SHTC and GENERIC formalisms by the proper choice of the dissipative source term in the time evolution for the distortion matrix:

$$\partial_t A_{ik} + \partial_k (A_{im} v_m) + v_j (\partial_j A_{ik} - \partial_k A_{ij}) = -\frac{1}{\tau'} E_{A_{ik}}$$

or by using the Finger tensor as in Sect. 4.3.3. Here, the relaxation function $\tau' = \tau'(\tau)$ depends on the real relaxation time $\tau$ and typically is defined as $\tau' \sim \tau b^2$, while its further specification depends on the specific choice of the total energy, see [16, 77]. Here, $b$ is the shear sound speed (the speed of propagation of small transversal perturbations). For the further discussion, let us recall the physical interpretation of the strain dissipation time $\tau$. In the unified flow theory [16, 77, 78], $\tau$ is understood as a continuum interpretation of the seminal idea of the so-called particle settled life time of Frenkel [25], who applied it to describe the ability of liquids to flow, see also recent promising experimental and theoretical advances [9–12, 14] confirming and further developing Frenkel’s idea. Thus, in our continuum approach, the time $\tau$ is the time taken by a given continuum particle (finite volume) to “escape” from the cage composed of its neighbor particles, i.e., the time taken to rearrange with one of its neighbors. The more viscous a fluid is, the larger the time $\tau$, i.e., the longer the continuum particles stay in contact with each other.

Such a concept of the strain dissipation time, in contrast to the phenomenological viscosity concept, allows to cover the entire spectrum of material responses, from ideal fluids ($\tau = 0$) to elastic solids ($\tau = \infty$) through the realm of dissipative flows, either Newtonian or non-Newtonian ($0 < \tau < \infty$). Note that the viscosity concept does not allow to model deformable solids. Moreover, in the proposed framework, the division on the Newtonian and non-Newtonian fluids becomes artificial because it does not rely on any constitutive laws designed specifically for either Newtonian or non-Newtonian flows. For example, to apply our model to modeling Newtonian flows one may even not know that Newton’s viscous law exists. All flows are treated as being non-equilibrium and no assumptions on the closeness to the global equilibrium was assumed in both
SHTC and GENERIC approaches. Thus, according to our approach, simple fluids as water, for example, exhibit elastic response at time scales $T < \tau$, viscoelastic at time scales $T \sim \tau$ and Newtonian response at time scales $T \gg \tau$, where $T$ is the characteristic flow time scale. We emphasize that the viscoelastic and even solid-like properties of simple liquids at small time scales is an experimental fact, see e.g., [9–12,14] and was predicted by Frenkel [25].

In practice, as in the modeling of elastoplastic solids, the user have to provide a suitable function $\tau = \tau(A, \rho, s)$ to cover diverse non-Newtonian responses. We emphasize that, in contrast to the viscosity approach, there is no need to use the dependence of $\tau$ on the strain rate because our relaxation model is rate dependent by the construction, e.g., see Fig. 1 in [77]. Moreover, in contrast to the nonlinear viscosity, the switching from $\tau = \text{const}$ to nonlinear $\tau(A, \rho, s)$ does not increase the complexity of the model. Thus, in classical phenomenological approach the substitution of the constant viscosity onto a nonlinear viscosity changes the dissipative source terms and coupled with momentum and mass conservation, i.e., the mathematical structure of the equations.

We shall demonstrate that Eqs. (27e), (27f) equipped with proper universality of the energy potential in (123) that in the leading terms, the Fourier law of heat conduction is recovered as

$$q_{i} = E_{s}w_{i} = \theta E_{s}w_{i} \quad \theta := E_{s}$$

where the scalar $\sigma$ from (119) is now treated as the entropy density $s$, and $w_{i}$ is the so-called thermal impulse (the momentum density of the heat carriers) with units $[w_{i}] = \text{K s/m}$. By taking the energy potential in the most simple form

$$E = E_{\text{hyd}r-o}(\rho, s) + \frac{\alpha^2}{2} w^2 + \frac{1}{2\rho} m^2$$

and expanding the solution in a series in $\tau$, e.g., $w = w_{0} + \tau w_{1} + \tau^2 w + \ldots$, one can show (exactly as in [16]) that in the leading terms, the Fourier law of heat conduction is recovered as

$$q = -\rho \theta \tau \alpha^2 \nabla \theta.$$ 

From this relation, it is easy to recognize the heat conductivity $\kappa$ as $\kappa = \rho \theta \tau \alpha^2$. 

6.5 Heat and mass transfer

In this section we demonstrate the viewpoint discussed in the Introduction section on the universality of the mathematical structure of the equations. We shall demonstrate that Eqs. (27e), (27f) equipped with proper dissipative source terms and coupled with momentum and mass conservation, i.e.,

$$\partial_t m_{i} + \partial_{k}(m_{i}v_{k} + [\rho E_{\rho} + \sigma E_{\sigma} + m_{i}E_{m_{i}} - E] \delta_{ik} + w_{i}E_{w_{k}}) = 0, \quad (119a)$$

$$\partial_t w_{k} + \partial_{k}(w_{i}v_{i} + E_{\sigma}) + v_{j}(\partial_{j}w_{k} - \partial_{k}w_{j}) = -\frac{1}{\tau}E_{w_{k}}, \quad (119b)$$

$$\partial_t \sigma + \partial_{k}(\sigma v_{k} + E_{w_{k}}) = \frac{1}{E_{\sigma} \tau}E_{w_{i}}E_{w_{i}}, \quad (119c)$$

$$\partial_t \rho + \partial_{k}(\rho v_{k}) = 0, \quad (119d)$$

does not increase the complexity of the model. Thus, in classical phenomenological approach the substitution of the constant viscosity onto a nonlinear viscosity changes the differential part of the equations and hence transforms a linear parabolic PDE system into a nonlinear parabolic system, which is a way more difficult problem from both, theoretical as well as numerical viewpoint. In our case, $\tau(A, \rho, s)$ does not affect the differential part (reversible part of the time evolution) of the model at all.

6.5.1 Heat conduction

Energy conservation law (29) for system (119) reduces to

$$\partial_t E + \partial_{k}(v_{k}E + v_{i}[(\rho E_{\rho} + sE_{s} + m_{i}E_{m_{i}} - E)\delta_{ik} + w_{i}E_{w_{k}}] + E_{s}E_{w_{k}}) = 0, \quad (120)$$

from which one can recognize the heat flux

$$q_{i} = E_{s}E_{w_{i}} = \theta E_{w_{i}}, \quad \theta := E_{s}$$

where the scalar $\sigma$ from (119) is now treated as the entropy density $s$, and $w_{i}$ is the so-called thermal impulse (the momentum density of the heat carriers) with units $[w_{i}] = \text{K s/m}$. By taking the energy potential in the most simple form

$$E = E_{\text{hyd}r-o}(\rho, s) + \frac{\alpha^2}{2} w^2 + \frac{1}{2\rho} m^2$$

and expanding the solution in a series in $\tau$, e.g., $w = w_{0} + \tau w_{1} + \tau^2 w + \ldots$, one can show (exactly as in [16]) that in the leading terms, the Fourier law of heat conduction is recovered as

$$q = -\rho \theta \tau \alpha^2 \nabla \theta.$$ 

From this relation, it is easy to recognize the heat conductivity $\kappa$ as $\kappa = \rho \theta \tau \alpha^2$. 

Let us now demonstrate that similar results can be obtained with the help of dynamic MaxEnt approach (see Sect. 6.1) which we have already applied to the Navier–Stokes equations in Sect. 6.2. Again, for this purpose, it is more convenient to use equivalent form (89) of (119). Thus, the goal is to reduce the Cattaneo-type hydrodynamics, represented by the equations

\[
\partial_t m_i = -\partial_j (m_i E_{mj}) - \rho \partial_i E_\rho - s \partial_i E_s - m_k \partial E_{mk} - w_i \partial_k w^*_k - w^*_k (\partial_k w_i - \partial_i w_k),
\]
\[
\partial_t w_k = -\partial_k E_s - w_j \partial_k E_{mj} - E_{mj} \partial_j w_k - \frac{1}{\tau} w^*_k,
\]
\[
\partial_t s = -\partial_k (s E_{mk} + w^*_k) + \frac{1}{\tau} \frac{1}{s^*} w^*_k w^*_k
\]

into hydrodynamics with Fourier heat conduction. Here, we use the notation \( w^*_k = E_{wk} \) for the \( w \) conjugate field.

As always, the energy is assumed to be a convex function of \( w \) and thus the entropy is a concave function of \( w \). Derivative of entropy with respect to \( w \) is zero for \( w = 0 \), which is the value to which field \( w \) is set in the MaxEnt reduction. This leads to equations

\[
\partial_t m_i = -\partial_j (m_i E_{mj}) - \rho \partial_i E_\rho - m_j \partial_i E_{mj} - s \partial_i E_s \quad \text{(125a)}
\]
\[
0 = -\partial_k E_s - \frac{1}{\tau} w^*_k \quad \text{(125b)}
\]
\[
\partial_t s = -\partial_k (s E_{mk} + w^*_k) + \frac{1}{\tau} \frac{1}{E_s} E_{wi} E_{wi} \quad \text{(125c)}
\]

The second equation gives

\[
w^*_k = -\tau \partial_k \theta, \quad \theta = E_s. \quad \text{(125d)}
\]

Plugging the relaxed value of \( w^* \) into the equation for entropy yields

\[
\partial_t m_i = -\partial_j (m_i E_{mj}) - \rho \partial_i E_\rho - m_j \partial_i E_{mj} - s \partial_i E_s \quad \text{(126a)}
\]
\[
\partial_t s = -\partial_k (s E_{mk} - \tau \partial_k \theta) + \frac{\tau}{\theta} \partial_k \theta \partial_i \theta \quad \text{(126b)}
\]

which represent Euler equations with Fourier heat conduction.

Eventually we note that, by combining the Cattaneo-type hydrodynamics with the dynamics of the Finger tensor, Sect. 4.3.3, one can obtain the full Navier–Stokes–Fourier (NSF) system of equations by the MaxEnt reduction from the SHTC framework although the NSF system is not part of the framework itself.

### 6.5.2 Mass transfer

In this section we continue exploiting equations (119) and, this time, show that the classical Fickian diffusion usually described by parabolic PDE can be successfully modeled by Eq. (119). However, for this purpose we shall prescribe different meanings to the variables \( (w, \sigma) \).

Let us consider a mixture of two inviscid fluids. We characterize the mixture morphology by three independent scalars, the mass density of the mixture \( \rho \), and the mass and volume fractions of one of the phases (denoted by the index “1”) \( c \) and \( \alpha \) correspondingly. These three scalars relate to each other through the phase mass densities \( \rho_1 \) and \( \rho_2 \) as \( \rho = \alpha \rho_1 + (1 - \alpha) \rho_2 \) and \( c = \frac{\alpha c_1}{\rho} \). Obviously, the mass and volume fraction of the second phase are \( 1 - c \) and \( 1 - \alpha \) accordingly.

The kinematics of the mixture is characterized by two velocity fields \( v = cv^1 + (1 - c)v^2 \) and \( w = v^1 - v^2 \), which are the mixture velocity and the relative velocity of the phases.

Thus, if one identifies the scalar \( \sigma \) in (119) with \( \rho c \) and introduces the specific total energy \( E = E/\rho \) then system (119) can be written as

\[
\partial_t (\rho v_i) + \partial_k (\rho v_i v_k + \rho^2 E_\rho \delta_{ik} + \rho w_i E_{wk}) = 0, \quad \text{(127a)}
\]
\[
\partial_t w_k + \partial_k (w_j v_j + E_i) + v_j (\partial_j w_k - \partial_k w_j) = -\frac{1}{\tau E_w} E_{wk}, \quad \text{(127b)}
\]
\[
\partial_t (\rho c) + \partial_k (\rho c v_k + \rho E_{wk}) = 0, \quad \text{(127c)}
\]
where also we add the time evolution of $\rho \alpha$, while the entropy time evolution now has the structure of not as in the heat conduction case in the previous section, i.e., as in the time evolution for the scalar $\sigma$ in (119), but as the mass conservation. Recall that PDEs with such a simple structure of fluxes as in (127d), (127e) and (127f) can be added in an arbitrary amount, see Sect. 3.1.3.

To close the system (127), one has to provide the specific total energy potential $E$. For example, it can be taken as

$$E = e^1 (\rho_1, s) + (1 - c) e^2 (\rho_2, s) + \frac{1}{2} (1 - c) \mathbf{w}^2 + E^{\text{mix}} (\rho, s, \alpha, c) + \frac{1}{2} \rho \mathbf{v}^2,$$

where $e^1$ and $e^2$ are the internal energies of the phases, $E^{\text{mix}}$ defines the chemical interaction of the phase molecules and can be left unspecified for our purposes. Then, the mixture chemical potential is defined as the derivative of the energy with respect to the mass fraction

$$E_{\alpha} = e^1 + \rho e^1_\alpha - e^2 - \rho_2 e^2_\alpha + \frac{1}{2} (1 - 2c) \mathbf{w}^2 + E^{\text{mix}}_{\alpha} = \mu_1 - \mu_2 + \frac{1}{2} (1 - 2c) \mathbf{w}^2 + E^{\text{mix}}_{\alpha}.$$

where $\mu_i = e^i + \frac{p_i}{\rho_i}$, $i = 1, 2$ are the chemical potentials of the phases.

Taking the derivatives of the total energy with respect to $\alpha$, we obtain the pressure relaxation term in the volume fraction time evolution $E_{\alpha} = (p_2 - p_1) / \rho$, where $p_1 = \rho_1 \frac{\partial e^1}{\partial \rho_1}$ are the phase pressures.

Now, we are in position to show that the Fick law of diffusion is an inherent property of the model (127). As in the case of the Fourier law of heat conduction (e.g., see details in [16]), it can be shown that the Fick law is recovered in our hyperbolic model in the leading terms for small relaxation time $\tau^{(w)}$ when the mass fraction time evolution reduces to the nonlinear diffusion equation

$$\rho \frac{dc}{dt} = \nabla \cdot \left[ \rho \tau^{(w)} \nabla \left( \mu_1 - \mu_2 + E^{\text{mix}}_{\alpha} \right) \right]$$

from which one can recognize the Fick law [see (129)]

$$\rho \frac{dc}{dt} = \nabla \cdot \mathbf{J}, \quad \mathbf{J} = \eta \nabla E_{\alpha}, \quad \eta = \rho \tau^{(w)}$$

Eventually, we recall that the viscous momentum transfer is modeled by the distortion field in the SHTC formulation [16, 17, 77]. Thus, if one wants to generalize the above equations to model the viscous properties of mixtures then it is sufficient to add only one distortion field representing the distortion of the mixture elements. Then, the strain relaxation time, see [16], should be composed of the phase relaxation times. We hope to discuss this in a subsequent paper dedicated to modeling of multi-phase flows.

Recall also that there is a Poisson bracket leading to the reversible part of Eq. (127), see Sect. 4.5. The bracket, however, is not compatible with the two-fluid bracket presented, for example, in [71], since the velocity $\mathbf{w}$ has no self-inertia in Eq. (127).

6.6 Electrodynamics of slowly moving medium

The electrodynamics of slowly moving medium ($|v| \ll c$, where $c$ is the speed of light) can be modeled by Eqs. (27a), (27d), (27c) and (27g), see [17, 21]. By means of adding the dissipative source term $-\frac{1}{\tau} E_{\alpha}$ to the right-hand side of Eq. (27d) which represents the dynamics of the electric field, one can describe dielectrics ($\eta \to \infty$), ideal conductors ($\eta \to 0$), and resistive conductors ($0 < \eta < \infty$) as particular cases, where $\eta$ is the resistivity, see details in [17, 21].
7 Concluding remarks

Continuum mechanics of fluids and solids with dislocations, with Cattaneo-type heat conduction, with mass transfer, and with electromagnetic fields is formulated in the Hamiltonian and the Godunov-type form. In order to emphasize that the Godunov-type structure that we consider is an extension of the original Godunov structure of local conservation laws, we call it the SHTC structure. The Hamiltonian structure guarantees the physical well posedness and the Godunov-like structure the mathematical and numerical well posedness.

From the microscopic point of view, the physical systems under consideration are mechanical systems, and their microscopic governing equations thus possess the Hamiltonian structure. This structure then passes also to the time-reversible part of the dynamics in the continuum formulation. However, due to the ignorance of microscopic details in the continuum formulation, a new dynamics, that is time irreversible and of gradient type, emerges. The governing equations of continuum mechanics are thus an appropriate combination of the Hamiltonian and the gradient dynamics (called GENERIC).

The governing equations in the SHTC form are first-order (symmetric) hyperbolic partial differential equations admitting a companion local conservation law. We have seen that the Hamiltonian continuum equations can be cast into the SHTC form. The companion local conservation law in the SHTC formulation appears in the Hamiltonian formulation as a consequence of antisymmetry of the Poisson bracket. The time-irreversible part of the continuum dynamics is in both the GENERIC and the SHTC formulations of gradient type. In the SHTC formulation, the time-irreversible part is moreover required to be free of spatial derivatives. The advantage of the SHTC formulation is the mathematical well posedness in the sense that the partial differential equations can be symmetrized (due to the existence of the companion local conservation law) and as such the Cauchy initial value problem for them is well posed at least locally in time. The numerical well posedness means that the Godunov finite volume discretization method that keeps the physical content of the governing equations also in their discrete form can be applied.

Acknowledgements I.P. acknowledges a financial support from ANR-11-LABX-0040-CIMI within the program ANR-11-IDEX-0002-02. M.P. and M.G. were supported by Czech Science Foundation, Project No. 17-15498Y, by Natural Sciences and Engineering Research Council of Canada (NSERC), grant numbers RGPIN-2014-06504-CRSNG and RGPAS462034-2014-CRSNG. This work has been supported by Charles University Research Program No. UNCE/SCI/023. E.R. acknowledges a partial support by the Program N15 of the Presidium of RAS (Project 121) and the Russian Foundation for Basic Research (Grant No. 16-29-15131).

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