ABSTRACT. This paper generalises Mori’s famous theorem about “Projective manifolds with ample tangent bundles” [Mor79] to normal projective varieties in the following way:

A normal projective variety over \( \mathbb{C} \) with ample tangent sheaf is isomorphic to the complex projective space.

1. INTRODUCTION

In this paper we give a proof for the following theorem.

**Main Theorem.** A normal projective variety over \( \mathbb{C} \) with ample tangent sheaf is isomorphic to the projective space.

We work over the field of complex numbers \( \mathbb{C} \). Besides that restriction, the theorem is a generalisation to singular varieties of Mori’s famous result.

**Theorem ([Mor79]).** An \( n \)-dimensional projective manifold \( X \) over an algebraically closed field \( \mathbb{K} \) with ample tangent bundle is isomorphic to the projective space \( \mathbb{P}_n^\mathbb{K} \).

Mori’s work has been generalised over the years in various ways, for example by Andreattta and Wiśniewski [AW01]: For \( X \) being \( \mathbb{P}_n \) it suffices that \( T_X \) contains an ample subbundle. This has been altered by Aprodu, Kebekus and Peternell [AKP08, Section 4]. They add the assumption that \( X \) has Picard number 1, but an ample subsheaf (not necessarily locally free) of \( T_X \) then induces \( X \simeq \mathbb{P}_n \). Generalising those results, Liu [Liu16] recently showed that \( X \) is already the projective space if \( T_X \) contains an ample subsheaf (again not necessarily locally free). Kebekus [Keb02] even characterises \( \mathbb{P}_n \) only by using the anticanonical degree of all rational curves being greater than \( n \). All these efforts, besides Ballico’s article [Bal93], keep the preliminary that \( X \) is smooth. Ballico’s paper on the other hand treats mainly positive characteristic, as he requires the tangent sheaf to be locally free. Which, the Zariski-Lipman conjecture suggests, is most likely never the case over the complex numbers, if \( X \) is singular.
Outline of our proof. We consider a special desingularisation $\hat{X}$ of the given variety $X$ of dimension $\geq 2$ (normal curves are smooth) and prove that $\hat{X}$ is the projective space. As $\mathbb{P}_n$ is minimal, $X$ itself is already the projective space. To show that $\hat{X}$ is the projective space, we combine two strong results.

First, we relate $T_X$ to $T_{\hat{X}}$: For a suitable desingularisation $\pi: \hat{X} \to X$, there is a morphism $f: \pi^*T_X \to T_{\hat{X}}$ that is an isomorphism outside $\pi^{-1}(\text{Sing}(X))$ (Theorem 3.2).

Secondly, we use a corollary given by Cho, Miyaoka and Shepherd-Barron [CMSB02, Corollary 0.4 (11)] that Kebekus [Keb02] later proved directly (although he claims a weaker result): A uniruled manifold $\hat{X}$ is isomorphic to the projective space, if the anticanonical degree $-K_{\hat{X}} \cdot C$ is greater or equal $n+1$ for all rational curves $C$ through a general point $p$. The uniruledness of $\hat{X}$ follows from the negativity of $K_{\hat{X}}$ and the anticanonical degree is calculated using the splitting of $T_{\hat{X}}|_C$ on the normalisation of $C$ (Lemma 3.3). Hence $\hat{X} \simeq \mathbb{P}_n \simeq X$.

2. Preliminaries

Let us first recall the definition of the tangent sheaf for a proper variety, as it is a central term in this paper.

**Definition 2.1** (tangent sheaf). Let $X$ be a algebraic variety, then its tangent sheaf $T_X := \text{Hom}(\Omega^1_X, O_X)$ is the dual of the cotangent sheaf.

We want to work on a desingularisation $\hat{X}$ of the normal variety $X$, so we have to connect $T_X$ with $T_{\hat{X}}$:

**Theorem 2.2.** Let $X$ be a normal projective variety with tangent sheaf $T_X$. Then there is a desingularisation $\pi: \hat{X} \to X$ and an $O_X$-module isomorphism $T_X \to \pi_*T_{\hat{X}}$.

**Proof.** Graf and Kovács [GK14, Theorem 4.2] state that there is a resolution $\pi: \hat{X} \to X$ such that $\pi_*T_{\hat{X}}$ is reflexive. The sheaves $T_X$ and $\pi_*T_{\hat{X}}$ are reflexive, $X$ is normal and $\pi$ is an isomorphism outside the preimage of a set of codimension 2. Thus we obtain an isomorphism $T_X \to \pi_*T_{\hat{X}}$. □

**Remark.** For a more thorough understanding of the map $T_X \to \pi_*T_{\hat{X}}$ and the resolution $\pi$, see the paper of Greb, Kebekus and Kovács [GKK10, Section 4].
The most cited definition for ample sheaves is in Ancona’s paper [Anc82]. He defines ampleness and provides some equivalent characterisations, but gives very few properties. Kubota [Kub70] on the other hand works over graded $\mathcal{O}_X$-modules and gives some properties, but does not use the most modern language.

So we recall a definition and the most important properties we use throughout this work.

**Definition 2.3** (ample sheaf). Let $X$ be a proper algebraic variety and $\mathcal{E}$ a coherent sheaf on $X$. Then we say $\mathcal{E}$ is ample if for every coherent sheaf $\mathcal{F}$ on $X$ there exists an $n = n(\mathcal{F})$ such that $\mathcal{F} \otimes S^m\mathcal{E}$ is globally generated for $m \geq n$.

**Remark.** Other characterisations of ampleness can be found in [Anc82]. Note that an ample sheaf, unlike an ample vector bundle, on a proper variety $X$ does not yield that its support is projective, but only Moishezon [GPR94, Remark p. 244].

The following properties can be found in Debarre’s paper [Deb06, Section 2] or the proof in the vector bundle case (as in [Laz04]) carries over to coherent sheaves:

**Proposition 2.4.** Let $X$ and $Y$ be normal projective varieties, $f : Y \to X$ a finite morphism, $\mathcal{E}$, $\mathcal{E}_1$ and $\mathcal{E}_2$ sheaves of $\mathcal{O}_X$-modules and $\mathcal{E}$ ample, then

1. $f^*\mathcal{E}$ is ample (in particular restrictions of ample sheaves are ample)
2. every quotient of $\mathcal{E}$ is ample
3. $\mathcal{E}_1 \oplus \mathcal{E}_2$ is ample if and only if $\mathcal{E}_1$ and $\mathcal{E}_2$ are both ample

**Proposition 2.5 ([Laz04 6.4.17]).** Let $C$ be a smooth curve and $\mathcal{E}$ and $\mathcal{F}$ vector bundles on $C$. If $\mathcal{E}$ is ample and there is a homomorphism $\mathcal{E} \to \mathcal{F}$, surjective outside of finitely many points, then $\mathcal{F}$ is ample.

We need one further result which is, besides Theorem 2.2, the main ingredient for our result:

**Theorem 2.6 ([CMSB02 Corollary 0.4 (11)]).** A uniruled projective complex manifold $X$ of dimension $n$ with a dense open subspace $U$ such that for all $p \in U$ and all rational curves $C$ through $p$ the inequality $-K_X.C \geq n + 1$ holds, is isomorphic to $\mathbb{P}_n$. 

3. Projective varieties with ample tangent sheaves

Now we get to the main result of the paper:

**Theorem 3.1.** Let $X$ be a normal projective variety over $\mathbb{C}$ of dimension $n$ with ample tangent sheaf $T_X$, then

$$X \cong \mathbb{P}_n.$$ 

Before proving the main theorem we have to adapt the results given in Section 2.

**Theorem 3.2.** Let $X$ be a normal projective variety, then there is a desingularisation $\pi: \hat{X} \to X$ and an $\mathcal{O}_{\hat{X}}$-module homomorphism

$$f: \pi^* T_X \to T_{\hat{X}}$$

that is an isomorphism outside $\pi^{-1}(\text{Sing}(X))$.

**Proof.** Using Theorem [2.2], we obtain an isomorphism $T_X \to \pi_* T_{\hat{X}}$ for a suitable resolution $\pi: \hat{X} \to X$. The map $\pi$ is an isomorphism outside $\pi^{-1}(\text{Sing}(X))$ (one has to retrace the resolution guaranteed by [GK14 Theorem 4.2] to [Kol07 Theorem 3.45] for this property). Pulling back $T_X \to \pi_* T_{\hat{X}}$ and using the natural morphism $c: \pi^* \pi_* T_{\hat{X}} \to T_{\hat{X}}$, there is the diagram

$$\begin{array}{ccc}
\pi^* T_X & \xrightarrow{g} & \pi^* \pi_* T_{\hat{X}} \\
\downarrow{f} & & \downarrow{c} \\
T_{\hat{X}} & & T_{\hat{X}}
\end{array}$$

Considering the maps $g$ and $c$, it is easy to check that they, and therefore $f$, are isomorphisms outside $\pi^{-1}(\text{Sing}(X))$. \qed

**Remark.** The editor pointed out to the author that Kawamata [Kaw85 p. 14] made use of the map $f$ as well.

**Lemma 3.3.** Let $X$ be a normal projective variety of dimension $n$ with ample tangent sheaf $T_X$ and $C \subset X$ a closed curve that intersects $\text{Sing}(X)$ in at most finitely many points. Let $\pi: \hat{X} \to X$ be a desingularisation as in Theorem 3.2, $\hat{C}$ the strict transform of $C$ and $\eta: \hat{C} \to \check{C}$ the normalisation of $\hat{C}$. Accordingly, there is the following commutative diagram:

$$\begin{array}{ccc}
\hat{C} & \xrightarrow{\eta} & C \\
\downarrow{\nu} & & \downarrow{=} \\
\hat{X} & \xrightarrow{\pi} & X
\end{array}$$
Then $\nu^*T_X$ is an ample vector bundle and the anticanonical degree $-K_X.\hat{C}$ is positive. If $\hat{C}$ is a rational curve, $-K_X.\hat{C} \geq n + 1$.

**Proof.** The choice of $\pi$ yields the map $f : \pi^*T_X \to T_X$. Pulling back $f$ via $\nu$ and dividing out the kernel gives

$$\nu^*f : A \rightarrow \nu^*T_X$$

with $A := \nu^*\pi^*T_X/\ker(\nu^*f)$. The sheaf $A$ is ample, since $T_X$ is ample, $\pi \circ \nu$ is finite and quotients of ample sheaves are ample again. Moreover $A$ is locally free of rank $n$ because it is a torsion-free sheaf on a smooth curve, $\pi \circ \nu$ is an isomorphism outside of finitely many points and $\ker(\nu^*f)$ is supported on only finitely many points. Using Proposition 2.5, we deduce that $\nu^*T_X$ is an ample vector bundle. Because $-K_X.\hat{C} = \deg \nu^*T_X$, the anticanonical degree is certainly positive. Since $\nu^*T_X$ splits on $\mathbb{P}_1$ and a direct sum of ample vector bundles is ample only if all summands are ample, we obtain $\nu^*T_X \simeq \bigoplus_{i=1}^n O_{\mathbb{P}_1}(a_i)$ with $a_i \geq 1$ for all $i$. The dual of the homomorphism $\nu^*\Omega^1_X \to \Omega^1_{\hat{C}}$ is a non-trivial map $\mathcal{T}_{\mathbb{P}_1} \simeq O_{\mathbb{P}_1}(2) \to \nu^*T_X$. Thus $a_i \geq 2$ for at least one $i$ and we can conclude $-K_X.\hat{C} = \sum_{i=1}^n a_i \geq n + 1$. \hfill $\square$

Now we use Lemma 3.3 to show that the assumptions of Theorem 2.6 are fulfilled for $\hat{X}$ and hence $X$ is isomorphic to $\mathbb{P}_n$.

**Proof of Theorem 3.1** Normal curves are smooth, so we can assume that $n \geq 2$. Let $\pi : \hat{X} \to X$ be a desingularisation as in Lemma 3.3 and let $p \in \hat{X} \setminus \pi^{-1}(\text{Sing}(X))$ be any general point outside the exceptional locus. Since $\hat{X}$ is projective, there is an irreducible curve $\hat{C}$ through $p$. As $\hat{C}$ is the strict transform of a closed curve $C \subset X$, $K_{\hat{X}}.\hat{C} < 0$ according to Lemma 3.3. Therefore $\hat{X}$ is uniruled by [MM86, Theorem 1].

Any rational curve $\hat{C} \subset \hat{X}$ containing $p$ projects to a curve $C$ on $X$. The curve $C$ meets $\text{Sing}(X)$ in at most finitely many points, thus Lemma 3.3 applies and we have the assumptions of Theorem 2.6 fulfilled. So $\hat{X}$ is isomorphic to the projective space $\mathbb{P}_n$. Hence $X \simeq \mathbb{P}_n$ too. \hfill $\square$
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