COMBINATORIC RESULTS FOR GRAPHICAL ENUMERATION AND THE HIGHER CATALAN NUMBERS

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ABSTRACT. We summarize some combinatoric problems solved by the higher Catalan numbers. These problems are generalizations of the combinatoric problems solved by the Catalan numbers. The generating function of the higher Catalan numbers appeared recently as an auxiliary function in enumerating maps and explicit computations of the asymptotic expansion of the partition function of random matrices in the unitary ensemble case. We give combinatoric proofs of the formulas for the number of genus 0 and genus 1 maps.

1. HIGHER CATALAN NUMBERS

The Catalan numbers solve a number of classical combinatoric problems such as the “Euler Polygon Division Problem”: how many ways are there to divide a marked polygon with \( j + 2 \) sides into triangles using edges and diagonals \([3, 7, 8, 12, 16]\) (see figure 1).

\[ \text{Figure 1. A polygon with } 4 + 2 = 6 \text{ sides divided into 4 triangles using edges and diagonals} \]

They count the number of right-left paths along a 1-Dimensional integer lattice which stay to the right of 0 and go from 0 to 0 in \( 2j \) steps; equivalently they count Dyck paths from \((0, 0)\) to \((2j, 0)\) \([1, 4, 15, 18]\).

They count the number of ways for \(2j\) customers to line up, with \(j\) customers having only a 2-dollar bill and \(j\) customers having only a 1-dollar bill, to purchase 1-dollar widgets, so that each customer receives exact change. They count the number of non-crossing handshakes possible across a round table between \(n\) people \([5]\).

In this paper we will explore a generalization of the Catalan numbers, the higher Catalan numbers. We will show that this generalization solves enumerative problems that are natural generalizations of the problems solved by the Catalan numbers. We will then highlight their appearance in recent results on map enumeration problems.

Let

\[
(1.1) \quad z(s) = 1 + \sum_{j=1}^{\infty} \zeta_j^{(\nu)} s^j
\]

be defined implicitly as the solution to the algebraic equation

\[
(1.2) \quad sz(s)^\nu - z(s) + 1 = 0,
\]

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which is regular at $s = 0$. This generating function was presented in [17, 20]. It is known that the $\zeta^\nu_j$ are (see [13, 20])

\begin{equation}
\zeta^\nu_j = \frac{1}{j \binom{\nu j}{j-1}}.
\end{equation}

The numbers $\zeta^\nu_j$ are called the higher Catalan numbers and solve a number of combinatoric problems.

The $\zeta^{(2)}_j$ are the Catalan numbers. The higher Catalan numbers count the number of ways to divide a marked $(\nu - 1)j + 2)$-sided polygon into $(\nu + 1)$-sided polygons using edges and diagonals as in figure 2.

**Figure 2.** A polygon with $2 \cdot 5 + 2 = 12$ sides divided into 5 squares, i.e. $\nu = 3$, using edges and diagonals.

They count the number of paths along the 1-Dimensional integer lattice which move to the right 1 unit per step, to the left $(\nu - 1)$ units per step, stay to the right of 0 and go from 0 to 0 in $\nu j$ steps [18]. They count the number of higher Dyck paths: paths which go from $(0, 0)$ to $(\nu j, 0)$ along the 2-D integer lattice with steps $(1, 1)$ and $(1, -(\nu - 1))$ which stay above the $x$-axis. They count the number of ways for $(\nu - 1)j$ customers with 1-dollar bills and $j$ customers with $\nu$-dollar bills to line up to buy 1-dollar widgets so that each customer receives correct change. They count the number of non-crossing handshakes possible across a round table between $n$ $\nu$-handed beings.

Remark 1.1. More generally we call generalized Dyck paths walks on the 2-D integer lattice restricted to the upper half plane with steps coming from a set of integer vectors. The generating function of the number of generalized Dyck paths satisfies a system of algebraic equations [14].

In addition to these combinatoric problems the coefficients of $z(s)$ satisfy the recursion relation

\begin{equation}
\zeta^\nu_j = \sum_{j_1 + j_2 + \cdots + j_\nu = j-1} \zeta^{\nu_{j_1}}_1 \zeta^{\nu_{j_2}}_2 \cdots \zeta^{\nu_{j_\nu}}_\nu.
\end{equation}

These numbers appeared in counting labeled maps using the partition function of random matrices [11]. We found, in solving the map enumeration problem for some simple cases, that the generating function could be written as a function of the generating function for the higher Catalan numbers.

A map $D$ on a compact, oriented connected surface $X$ is a pair $D = (K(D), [\iota])$ where

1. $K(D)$ is a connected 1-complex;
2. $[\iota]$ is an isotopical class of inclusions $\iota : K(D) \to X$;
3. the complement of $K(D)$ in $x$ is a disjoint union of open cells (faces);
4. the complement of $K_0(D)$ (vertices) in $K(D)$ is a disjoint union of open segments (edges).

Maps can be thought of as ribbon graphs (or fattened graphs) embedded on an oriented surface [2, 19, 21].

Define $e_g(t)$ to be the generating function of the number of connected maps of degree $2\nu$ embedded in a genus $g$ surface, in the following sense

\[ e_g(t) = \sum_{j=1}^{\infty} \kappa^g_{(2\nu)}(j) \frac{(-t)^j}{j!}, \]

where $\kappa^g_{(2\nu)}(j)$ is the number of planar maps with $j$ vertices of degree $2\nu$ embedded in a genus $g$ surface.
Define the class of functions: *iterated integrals of rational functions of* $z$, or *iir* of $z$, to be functions which are found by taking a finite number of anti-derivatives of a rational function of $z$. The particular subclass we are going to be concerned with are those *iir’s* which have singularities only at $z = 0, 1,$ or $\nu/(\nu - 1)$. One may check that this subclass of functions is closed under anti-differentiation.

To illustrate the results of [11] we present the explicit result we found for $g = 0$. Let

$$e_\nu = 2\nu \left(\frac{2\nu - 1}{\nu - 1}\right).$$

**Theorem 1.2** (Ercolani-McLaughlin-Pierce).

$$e_0(-t) = \mu_\nu (z(c_\nu t) - 1)(z(c_\nu t) - r_\nu) + \frac{1}{2} \log(z(c_\nu t)),$$

where

$$\mu_\nu = \frac{(\nu - 1)^2}{4\nu(\nu + 1)}$$

and

$$r_\nu = \frac{3(\nu + 1)}{\nu - 1}.$$

More generally we find that $e_g(-t)$ is an *iir* function of $z(c_\nu t)$ with singularities only possible at $z = 0, 1,$ or $\nu/(\nu - 1)$.

We will now outline the approach we used to prove this result. Define the partition function of random matrices

$$(1.5) \quad Z_N(t_1, t) = \int_{H_N} \exp \left( -N\text{Tr} \left( t_1 M + \frac{1}{2} M^2 + t M^{2\nu} \right) \right) dM,$$

where the integral is taken over the space of $N \times N$ Hermitian matrices, and $dM$ is the product of Lebesgue measures on the independent variables in $M$:

$$dM = \left[ \prod_{i<j} d\text{Re}(M_{ij}) d\text{Im}(M_{ij}) \right] \left[ \prod_i dM_{ii} \right].$$

Let

$$L = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The Toda lattice hierarchy is given by the differential equations

$$\frac{dL}{dt_j} = [B_j, L],$$

where $B_j = (L^j)_+ - (L^j)_-$, and where $(\cdot)_\pm$ indicates projection onto the upper (resp. lower) triangular parts. This hierarchy is an integrable hierarchy, possessing a complete family of independent commuting integrals. Solutions are generated as logarithmic derivatives of so called $\tau$-functions. The partition functions (1.5) are these $\tau$-functions.

Using this fact we showed that the leading order (as $N \to \infty$) of a second logarithmic derivative of the partition function was the generating function of the higher Catalan numbers.

**Theorem 1.3** (Ercolani-McLaughlin-Pierce).

$$z(-c_\nu t) + O\left(\frac{1}{N^2}\right) = \left[ \frac{1}{2N^2} \frac{\partial^2}{\partial t_1^2} \log \left[ \frac{1}{Z_N(0, 0)} Z_N(N^{1/2}t_1, N^{1-\nu}t) \right] \right]_{t_1 = 0}. $$

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The paper [11] presents two independent proofs of Theorem 1.2 in the case when $g = 0$: one is a direct calculation of the generalization of the method of steepest descent for computing the leading order of asymptotics of matrix integrals, the other uses the fact that the partition function (1.5) is a $\tau$-function of the Toda lattice hierarchy. Theorem 1.3 is found in either proof of Theorem 1.2 in the $g = 0$ case. The general result from this paper is that $e_g(t)$ is explicitly computable as a function of $z(s)$. Therefore the higher Catalan numbers play a central role in the combinatorics of maps.

1.1. Results. We will summarize our results for enumerations of genus 0 and genus 1 maps here. The generating function of the number of genus 0 maps is

$$e_0(-t) = \sum_{j=0}^{\infty} \kappa^{(2\nu)}_0(j) \frac{t^j}{j!},$$

while the generating function for genus 1 maps is

$$e_1(-t) = \sum_{j=0}^{\infty} \kappa^{(2\nu)}_1(j) \frac{t^j}{j!}.$$

We will give a combinatoric proof of the theorem from [11]:

**Theorem 1.4 (Ercolani-McLaughlin-Pierce).** The Taylor coefficients of $e_0$ are

$$\kappa^{(2\nu)}_0(j) = c_j (\nu j - 1)! \left( (\nu - 1)j + 2 \right)!,$$

and those of $e_1$ are

$$\kappa^{(2\nu)}_1(j) = \frac{(j - 1)!c_j}{12} \sum_{k=1}^{j} (\nu - 1)^{k-1} \binom{\nu j}{j-k}.$$

In section 2 we will review the combinatoric arguments showing that the higher Catalan numbers are generated by $z(s)$. Both the path counting problems analyzed in section 2 and the queuing problem mentioned are from Yaglom and Yaglom [20], and Sato and Cong [18]. The path counting problems have been cosmetically changed, but the method of solution is still the same. In section 3 we give the origin of theorems like 1.2 and our general result. In section 4 we give a combinatoric calculation starting from Theorem 1.2 which culminates in a formula for the number of planar maps with $j$ vertices of degree $2\nu$. In section 5 we carry out a similar calculation for counting genus 1 maps, giving a proof of Theorem 1.4.

2. Calculation of the Higher Catalan Numbers

The higher Catalan numbers satisfy a recursion relation which will tie together four types of combinatoric problems: the Taylor coefficient problem (1.1), the polygon division problem, and two problems we are about to introduce: a path counting problem and a queuing problem. The path counting problem is directly solvable by enumeration. This is then connected to the queuing problem and we show that both satisfy the same recursion (1.4) as $\zeta^{(\nu)}$ and the polygon division problem.

We give here a modification of the calculation outlined in Yaglom and Yaglom [20], and Sato and Cong [18]. In this section we will show that the higher Catalan numbers,

$$\zeta^{(\nu)}_j = \frac{1}{j} \binom{\nu j}{j-1},$$

count the number of right 1, left $(\nu - 1)$ walks on the 1-Dimensional lattice which go from 0 to 0 in $\nu j$ steps which stay to the right of 0.

Note that this 1-Dimensional counting problem is identical to counting the number of $(1, 1)$ and $(1, -(\nu - 1))$ paths on the 2-Dimensional Integer Lattice from $A = (0, 0)$ to $B = (\nu j, 0)$ which stay above $y = 0$, we call these higher Dyck paths.

As an example of how to count such paths we have the lemma
Lemma 2.1. The number of $(1,1)$ and $(1,-(\nu-1))$ paths from $P = (p_1, p_2)$ to $Q = (q_1, q_2)$ (assume $p_1 < q_1$) on the 2-Dimensional integer lattice is
\[ N_{PQ} = \binom{q_1 - p_1}{d}, \]
where $n, u,$ and $d$ solve the system of equations
\[ p_1 + u + d = q_1, \]
\[ p_2 + u - (\nu - 1)d = q_2. \]
If no integer solutions exist then there is no such path connecting these points.

To count the number of $(1,1)$, $(1,-(\nu-1))$ paths from $A$ to $B$ which stay above the $x$-axis we will count the number of such paths which pass below the $x$-axis. A path of this type going below the $x$-axis will go through one of the points
\[ D_k = (\nu k - 1, -1). \]
Let $A' = (0, -\nu)$.

Lemma 2.2. The number of $(1,1)$, $(1,-(\nu-1))$ paths from $A$ to $B$ which pass below the $x$-axis is
\[ (\nu-1)N_{A'B} = (\nu-1)\binom{\nu j}{j-1}. \]

Begin the proof of this lemma by computing
\[ N_{AD_k} = \binom{\nu k - 1}{k}, \]
and
\[ N_{A'D_k} = \binom{\nu k - 1}{k-1} = \frac{1}{\nu - 1} \binom{\nu k - 1}{k} = \frac{1}{\nu - 1} N_{AD_k}. \]
Now consider the number of paths from $A$ to $B$ which go through $D_1$
\[ N_{AD_1}N_{D_1B} = (\nu - 1)N_{A'D_1}N_{D_1B}, \]
which is $(\nu - 1)$ times the number of paths from $A'$ to $B$ through $D_1$. Then inductively assume that the number of paths from $A$ to $B$ through $D_1$ is $(\nu - 1)$ times the number of paths from $A'$ to $B$ through $D_l$ for $l < k$. Consider the number of paths from $A$ to $B$ which go through $D_k$ but not $D_l$ for $l < k$,
\[ N_{AD_k}N_{D_kB} - N_{AD_{k-1}}N_{D_{k-1}D_k}N_{D_kB} - \cdots = (\nu - 1)N_{A'D_k}N_{D_kB} - (\nu - 1)N_{A'D_{k-1}}N_{D_{k-1}D_k}N_{D_kB} - \cdots \]
The number of $(1,1)$, $(1,-(\nu-1))$ paths from $A$ to $B$ which stay above the $x$-axis is
\[ (2.1) \quad \binom{\nu j}{j} - (\nu - 1)\binom{\nu j}{j-1} = \frac{1}{j} \binom{\nu j}{j-1}, \]
which is the $j$’th $\nu$-higher Catalan number.

Consider the queuing problem: How many ways are there for $(\nu - 1)j$ customers with 1-dollar bills and $j$ customers with $\nu$-dollar bills to line up to buy 1-dollar widgets so that each customer receives correct change. In the path counting problem, in order for a step to the left to be placed on the path and stay to the right of 0 there must be $(\nu - 1)$ corresponding right steps that have come before on the path. In the queuing problem, in order for a customer with a $\nu$-dollar bill to receive correct change there must be $(\nu - 1)$ corresponding customers with 1-dollar bills in line ahead of the $\nu$-dollar bill.

To see that the numbers (2.1) agree with the coefficients (1.3) of $z(t)$ we argue that the number of right 1, left $(\nu - 1)$ paths which stay to the right of 0 satisfy the recursion relation (1.4). As argued above, to each left step in the path there must correspond $(\nu - 1)$ steps to the right. The path begins with a step to the right. This step corresponds, together with $(\nu - 2)$ other right steps, to a left step later on. This collection of $\nu$ steps divides the entire $\nu j$ steps into $\nu$ sub-paths of the type: steps right by 1, left by $(\nu - 1)$, from $k$ to $k$ which stay to the right of $k$. Therefore the recursion relation (1.4) is satisfied.

Now we argue that the number of ways to divide a marked $((\nu - 1)j + 2)$-sided polygon into $(\nu + 1)$-sided polygons using edges and diagonals satisfies the recursion relation (1.4). Take the sub-polygon which has the
marked outside edge as an edge. This sub-polygon divides the \((\nu - 1)j + 2\)-sided polygon into \(\nu\) polygons, therefore the recursion relation (1.3) is satisfied by this problem as well. Therefore these numbers are also the higher Catalan numbers.

3. The Enumeration of Planar Maps

Our interest in these combinatoric problems began with the function \(z(s)\) defined by (1.1). This function appeared as an auxiliary function in explicit calculations of the asymptotic expansion of the partition function of large random matrices. It is interesting that a generating function as classical as \(z(s)\) appeared as an auxiliary function in explicit calculations of the asymptotic expansion of the partition function of large random matrices. It is interesting that a generating function as classical as \(z(s)\) appeared as an auxiliary function in explicit calculations of the asymptotic expansion of the partition function of large random matrices. It is interesting that a generating function as classical as \(z(s)\) appeared as an auxiliary function in explicit calculations of the asymptotic expansion of the partition function of large random matrices.

In this section we highlight the information contained in the asymptotic expansion of the partition function (1.5) for large matrices. We have detailed data about the higher genus problems (11), however here we will concentrate on the genus zero (or planar) setting.

Recall that the partition function of random matrices we consider is (1.5):

\[
Z_N(t) = \int_{\mathcal{H}_N} \exp \left(-N \text{Tr} \left( \frac{1}{2} M^2 + t M^{2\nu} \right) \right) dM,
\]

where now we take \(t_1 = 0\). Ercolani and McLaughlin [10] showed that \(\log (Z_N(t)/Z_N(0))\) possesses an asymptotic expansion as \(N \to \infty\) inside a non-trivial \(t\) domain,

\[
\frac{1}{N^2} \log \left( \frac{Z_N(t)}{Z_N(0)} \right) = e_0(t) + \frac{1}{N^2} e_1(t) + \frac{1}{N^4} e_2(t) + \ldots
\]

where \(e_0(t)\) is analytic in a neighborhood of \(t = 0\) and is a counting function for genus \(g\) maps with \(2\nu\)-degree vertices. The idea of counting maps in this way originates with random matrix models of quantum field theories [2, 6, 19].

In this paper we will concentrate our attention on the genus 0 and genus 1 terms in (3.1). The genus 0 term is

\[
e_0(t) = \sum_{j=1}^{\infty} \kappa_0^{(2\nu)}(j) \frac{(-t)^j}{j!},
\]

where the \(\kappa_0^{(2\nu)}(j)\) are the number of genus 0 maps with \(j\) vertices of degree \(2\nu\). These numbers are calculated in [11] by explicit computation of the contour integral

\[
\kappa_0^{(2\nu)}(j) = \frac{j!}{2\pi i} \oint t^{-j-1} e_0(-t) dt,
\]

where the contour encircles \(t = 0\) and \(e_0\) is given in Theorem 1.2. To evaluate (3.2) one rewrites it as an integral with respect to \(z = z(e_0, t)\). The result is the first part of Theorem 1.4.

4. Computation of the Taylor Coefficients of \(e_0\)

We will now compute the Taylor coefficients of \(e_0\) from combinatoric arguments only. In section 2 we used combinatoric techniques to compute the Taylor coefficients of \(z(s)\). Theorem 1.2 gives \(e_0(-t)\) as a function of \(z(s)\). The calculation of the Taylor coefficients of \(e_0(-t)\) can be done from this theorem utilizing contour integration and a clever change of variables. Our goal in this note is to complete this calculation using only combinatoric arguments. To that end, we need to derive formulas for the coefficients of \(\log(z(s))\) and \((z(s) - 1)^2\).

4.1. Taylor Coefficients of powers of \((z(t) - 1)\) and \(\log(z(t))\). The coefficients of \((z(s) - 1)^i\) are given by

\[
\eta_j^{(\nu, i)} = \sum_{j_1 + j_2 + \ldots + j_i = j} \xi_1^{(\nu)} \xi_2^{(\nu)} \ldots \xi_i^{(\nu)}, \quad j \geq i.
\]

The \(\eta_j^{(\nu, i)}\) count the number of right 1, left \((\nu - 1)\) paths from 0 to 0 in \(\nu j\) steps which stay to the right of 0 and return to 0 at least \(i - 1\) times in between the ends of the path. This is equivalent to the queuing
problem: how many ways can \( j \) customers with \( \nu \)-dollar bills and \( (\nu - 1)j \) customers with 1-dollar bills form \( i \) lines to buy 1 dollar widgets so that each customer receives exact change.

This multi-line queuing problem is equivalent to: how many ways can \( j - i \) customers with \( \nu \)-dollar bills and \( (\nu - 1)j + 1 \) customers with 1-dollar bills form one line to buy 1 dollar widgets so that each customer receives exact change. This is done by shifting the lines together.

First, we notice that at the back of the first line is a \( \nu \)-dollar bill. We shift the problem in the following way: we change the \( \nu \)-dollar bill at the back of first line into a 1-dollar bill, then adjoin the second line to the back of the first. The first line now has \( j - 1 \) customers with \( \nu \)-dollar bills and \( (\nu - 1)j + 1 \) customers with 1-dollar bills (a \( \nu \)-dollar bill has been changed into a 1-dollar bill). We are now in the case of \( i - 1 \) lines and may shift again. After \( i - 1 \) repetitions of this process we have a single line arranged in such a way that the last customer has a \( \nu \)-dollar bill, we remove this customer from the process.

To show that the two counting problems are equivalent we show that the one-line queuing problem above can be transformed back into the multi-line problem. We begin with a line of \( j - i \) customers with \( \nu \)-dollar bills and \( (\nu - 1)j + 1 \) customers with 1-dollar bills in a single line to buy 1 dollar widgets so that each customer will receive exact change. First add a \( \nu \)-dollar bill to the back of the line. Starting from the back of the line and moving forward we will form sets of blocks of \( \nu \) customers. Each block has a customer with a \( \nu \)-dollar bill and \( \nu - 1 \) customers with 1-dollar bills. When we reach a customer with a 1-dollar bill for which there is no corresponding \( \nu \)-dollar bill behind, we have completed a line, and we remove it from the first line. We then turn the 1-dollar bill at the back of the first line into a \( \nu \)-dollar bill, and we find that \( \nu - 1 \) customers with 1-dollar bills form one line to buy 1 dollar widgets so that each customer receives exact change. This is done by shifting the lines together.

The path counting problem which corresponds to this shifted queuing problem is: how many ways can \( j \) customers with \( \nu \)-dollar bills and \( (\nu - 1)j + 1 \) customers with 1-dollar bills form one line to buy 1 dollar widgets so that each customer receives exact change. First add a \( \nu \)-dollar bill to the back of the line. Starting from the back of the line and moving forward we will form sets of blocks of \( \nu \) customers. Each block has a customer with a \( \nu \)-dollar bill and \( \nu - 1 \) customers with 1-dollar bills. When we reach a customer with a 1-dollar bill for which there is no corresponding \( \nu \)-dollar bill behind, we have completed a line, and we remove it from the first line. We then turn the 1-dollar bill at the back of the first line into a \( \nu \)-dollar bill, and we find that \( \nu - 1 \) customers with 1-dollar bills form one line to buy 1 dollar widgets so that each customer receives exact change. This is done by shifting the lines together.

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with

\[ A_j(\alpha, \nu) = \frac{\alpha}{(\alpha + \nu j)} \left( \begin{array}{c} \alpha + \nu j \\ j \end{array} \right). \]

In addition

\[ \frac{z^{\alpha+1}}{\nu - (\nu - 1)z} = \sum_{j=0}^{\infty} \left( \begin{array}{c} \alpha + \nu j \\ j \end{array} \right) s^j. \]

One can check that formula (4.3) is equivalent to (4.1).

5. The enumeration of genus one maps

In the paper [11] the authors show that there is a general construction of the functions \( e_g(t_{2\nu}) \) in the asymptotic expansion (3.1) as functions of \( z(c_{2\nu} t_{2\nu}) \). For example, we find that in the case of genus one maps

**Theorem 5.1 (Ercolani-McLaughlin-Pierce).** In the neighborhood of analyticity, the second term in the asymptotic expansion (3.1) is given by

\[ e_1(-t) = -\frac{1}{12} \log \left( \nu - (\nu - 1)z(c_{2\nu} t) \right), \]

where we choose the principal branch of the logarithm.

Our goal is to derive a formula for the Taylor coefficients of \( e_1(-t_{2\nu}) \) for general \( \nu \). This is a straightforward calculation: expand formula (5.1) as a powers series in \( (z(c_{2\nu} t) - 1) \);

\[ e_1(-t) = \sum_{k=1}^{\infty} \frac{1}{12k} \left( \nu - 1 \right)^k (z(c_{2\nu} t) - 1)^k. \]

Then insert the power series of \( (z - 1)^k \) into (5.2):

\[ e_1(-t) = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \frac{1}{12k} \left( \nu - 1 \right)^k \eta_j(\nu, k) t^j = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \frac{1}{12j} \left( \nu - 1 \right)^k \left( \begin{array}{c} \nu j \\ j - k \end{array} \right) t^j, \]

where the \( \eta_j(\nu, k) \) were computed in (4.1). Switching the order of summation in (5.3) we find that

\[ \kappa^j(2\nu)(j) = \frac{(j - 1)! c_{2\nu} \nu}{12} \sum_{k=1}^{j} \left( \nu - 1 \right)^k \left( \begin{array}{c} \nu j \\ j - k \end{array} \right). \]

This proves the second part of Theorem 1.4.

One is tempted to try to use the last part of Theorem 4.1 to find a more concise formula for \( \kappa^j(2\nu)(j) \). However using the relation

\[ z' = \frac{z^{\nu+1}}{\nu - (\nu - 1)z}, \]

derived from the implicit definition of \( z \) by formula (1.2), we see that the derivative of \( e_1(-t) \) with respect to \( t \) is

\[ e'_1(-t) = \frac{(\nu - 1)}{12} \frac{z(c_{2\nu} t)^{\nu+1}}{(\nu - (\nu - 1)z(c_{2\nu} t))^2}. \]

There does not seem to be a more concise formula for the coefficients of (5.5) than that found directly from \( j \) times \( \kappa^j(2\nu) \) in (5.4).
6. Conclusion

The higher Catalan numbers appeared naturally in the study of the enumeration of maps embedded on Riemann surfaces. We have gathered here a number of interesting facts about these numbers and the combinatoric problems related to them. It would be interesting to have a combinatoric argument which gives the form of Theorems 1.2 and 5.1.

Further work will study the fine structure of the asymptotic expansion of the partition function $Z_N$ when multiple time evolutions are involved. Additionally other partition functions over different families of Random Matrices encode similar combinatoric data. For example, in the case when

$$Z_N = \int_{S_N} \exp \left[ -N \text{Tr} \left( \frac{1}{4} M^2 + V(M) \right) \right] dM,$$

where the integral is taken over the space of $N \times N$ symmetric matrices, we find that the terms in the asymptotic expansion of $\log (Z_N)$ give generating functions for the number of unoriented maps partitioned by the Euler characteristic associated with the embedding of the map.

What connections these problems will have to the higher Catalan numbers or to other interesting (and classical) combinatoric problems remains to be seen.

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