Euler equations in a 3+1 framework

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Abstract

In this paper we show how the non-relativistic transport equations for a simple fluid can be obtained using a 3+1 representation. A pseudo-galilean transformation is introduced in order to obtain the Euler conservation laws. The interpretation of such transformation as well as the implications and potential extensions of the formalism are briefly discussed.
I. INTRODUCTION

Kinetic theory provides a fundamental framework to obtain transport equation for fluids through the Enskog equation. Usually, in the foundations of irreversible thermodynamics [1, 2], these equations are obtained by considering a Maxwell-Boltzmann distribution as the local equilibrium function and using the fact that the relation between the hydrodynamic, molecular and chaotic velocities \( \vec{u}, \vec{v} \) and \( \vec{k} \) respectively-is given by \( \vec{v} = \vec{k} + \vec{u} \).

In this work we propose an alternative method to obtain the Euler equations for a simple non-relativistic gas. This scheme is based on the approach proposed by A. Sandoval and L. S. Garcia-Colin in Ref. [3] for the relativistic gas. The formalism we develop in this paper consists in working the non-relativistic case in a similar way as the relativistic fluid equations are obtained in special relativity [4]. We formulate the Enskog transport equation as a conservation law for four-fluxes, namely the particle flux and the energy-momentum tensor. In this scheme one should be able to obtain transport equations to any order in the gradients by calculating the fluxes in the comoving frame and using an appropriate transformation law.

In Section II we construct the 3+1 scheme and obtain the conservation laws for a fluid at rest. In Section III we propose a transformation law for the four-fluxes and obtain the transport equations using the Enskog equation for the transformed quantities. Finally, a discussion and final remarks are included in Section IV.

II. FLUXES IN THE COMOVING FRAME AND CONSERVATION LAWS

In kinetic theory, the conservation laws, given by the Enskog transport equation, are obtained from the Boltzmann equation which, for a simple fluid, reads

\[
\frac{df}{dt} = J(f f')
\]  

(1)
where $f$, the distribution function, gives the probability of finding a molecule in a given cell of phase space. The term on the right side is the collision kernel whose details are irrelevant for the purpose of this work. From Eq. (1) one can readily obtain the transport equation \[1, 2\]

$$\frac{\partial}{\partial t} [n \langle \psi \rangle] + \nabla \cdot [n \langle \vec{v} \psi \rangle] = 0 \quad (2)$$

where $\psi$ is a collision invariant and $n$ the number density, given by

$$n = \int f \, d^3v \quad (3)$$

Thus, for $\psi = 1$ the continuity equation is obtained and for $\psi = m \vec{v}$ and $\psi = mv^2/2$ the momentum and internal energy balances respectively.

As in the relativistic case, we propose now a 3+1 representation by considering a fourth (time) component of the position vector. That is, if

$$x^\mu = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}, \quad (4)$$

$$v^\mu = \begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix}, \quad (5)$$

, Eq. (2) can be written as

$$(n \langle v^\mu \psi \rangle)_{,\mu} = 0 \quad (6)$$

Here and throughout this work a semicolon represents a covariant derivative, latin indices run from 1 to 3, greek ones from 1 to 4 and the usual summation convention
is used. Next, we define the particle and energy-momentum four-fluxes as

\[ N^\alpha = n \langle v^\alpha \rangle \]  

(7)

and

\[ T^{\ell \nu} = n \langle v^\ell v^\nu \rangle \]  

(8)

respectively. Notice that \( T^{44} \) has been left unspecified. With these definitions, Eq. (6) is a statement of conservation of these quantities and can be written as

\[ N^\alpha_{\alpha} = 0 \]  

(9)

and

\[ T^\mu_{\nu} = 0 \]  

(10)

Equations (9) and (10) yield the particle number density and momentum conservation laws. However, the energy balance must be introduced apart, if one considers \( T^{4\nu} = n \langle v^4 v^\nu \rangle \) since in that case \( T^4_{\nu} = 0 \) only yields an identity. In order to obtain an energy balance equation from the conservation of an energy-momentum tensor, and following the ideas of the 3+1 formalism in special relativity, we propose instead

\[ T^{\mu \nu} = n \begin{pmatrix} \langle v^1 v^1 \rangle & \langle v^2 v^1 \rangle & \langle v^3 v^1 \rangle & \langle v^4 v^1 \rangle \\ \langle v^1 v^2 \rangle & \langle v^2 v^2 \rangle & \langle v^3 v^2 \rangle & \langle v^4 v^2 \rangle \\ \langle v^1 v^3 \rangle & \langle v^2 v^3 \rangle & \langle v^3 v^3 \rangle & \langle v^4 v^3 \rangle \\ \langle v^1 v^4 \rangle & \langle v^2 v^4 \rangle & \langle v^3 v^4 \rangle & E \end{pmatrix} \]  

(11)

where \( E \) is the total energy per particle, including a term corresponding to the rest energy. With these definitions one obtains, for the simple fluid at rest

\[ N^\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ n c \end{pmatrix} \]  

(12)
where $p = nkT$ is the hydrostatic pressure and $\varepsilon$ includes both internal energy and rest energy such that, for the ideal gas $\varepsilon = mc^2 + \frac{3}{2}kT$. From these four-fluxes one can directly obtain the conservation equations. This calculation is carried out in the laboratory frame in the next section where Euler’s equations are obtained.

### III. A PSEUDO-GALILEAN TRANSFORMATION

In this section we consider the case of a fluid with hydrodynamic velocity

$$u_\nu = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ c \end{pmatrix}$$

(14)

for which the conservation equations will be obtained by calculating Eqs. (9) and (10) for transformed fluxes. In order to obtain those quantities, we propose the use of a pseudo-galilean transformation which is the non-relativistic limit of the Lorentz transformation, namely

$$G^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & \frac{u_x}{c} \\ 0 & 1 & 0 & \frac{u_y}{c} \\ 0 & 0 & 1 & \frac{u_z}{c} \\ \frac{u_x}{c} & \frac{u_y}{c} & \frac{u_z}{c} & 1 + \frac{u^2}{2c^2} \end{pmatrix}$$

(15)
Notice that, for $\mathcal{G}^{44}$ we suggest keeping second order in $u/c$. This is necessary since the energy equation is quadratic in the velocities. Moreover, as will be shown in the next section, the zeroth order terms only account for the thermal part of the total energy while the second order terms include the mechanical effects. The implications of including this term will be discussed to some detail in the last section of this work.

As in the relativistic case [3, 4], the transformed quantities are

$$\tilde{N}^\mu = \mathcal{G}^\mu_\nu N^\nu$$

$$\tilde{T}^{\mu\nu} = \mathcal{G}^\mu_\alpha \mathcal{G}^\nu_\beta T^{\alpha\beta}$$

Using the transformation given by Eq. (15) one obtains

$$\tilde{N}^\ell = nu^\ell,$$ (16)

$$\tilde{N}^4 = nc \left(1 + \frac{u^2}{2c^2}\right),$$ (17)

$$\tilde{T}^{k\ell} = p\delta^{k\ell} + \frac{\varepsilon}{c^2} u^k u^\ell,$$ (18)

$$\tilde{T}^{4\ell} = \left[n\varepsilon \left(1 + \frac{u^2}{2c^2}\right) + p\right] \frac{u^\ell}{c},$$ (19)

$$\tilde{T}^{44} = n\varepsilon \left(1 + \frac{u^2}{2c^2}\right) + p \frac{u^2}{c^2}.$$ (20)

Calculating the four-divergence of $\tilde{N}^\alpha$ one obtains

$$\tilde{N}^\alpha_{,\alpha} = (nu^\ell)_{,\ell} + \frac{\partial n}{\partial t} + n \frac{u^\ell}{2c} \frac{\partial u}{\partial t} = 0,$$ (21)

where, neglecting the last term, using $u \ll c$, the usual continuity equation is obtained.
The momentum balance equation is obtained from $\bar{T}_{\nu}^{\mu} = 0$, namely

$$
\left( p\delta^{k\ell} + \frac{n\varepsilon}{c^2} u^k u^\ell \right) + \\
\frac{1}{c} \frac{\partial}{\partial t} \left\{ n\varepsilon \left( 1 + \frac{u^2}{2c^2} \right) + p \right\} \frac{u^k}{c} = 0.
$$

(22)

Since the fluid here considered is non-relativistic, the internal energy term is

$$
\varepsilon \sim m \left( 1 + \frac{3}{2} \frac{kT}{mc^2} \right) \sim m
$$

such that Eq. (22) is precisely the momentum balance

$$
\frac{\partial}{\partial t} \left( nu^k \right) + \left( p\delta^{k\ell} + nm u^k u^\ell \right) = 0.
$$

(23)

Finally, the energy balance is obtained from $T_{\nu}^{\mu} = 0$,

$$
\left\{ n\varepsilon \left( 1 + \frac{u^2}{2c^2} \right) + p \right\} \frac{u^\ell}{c} \left( \frac{3}{2} nkT + nm \frac{u^2}{2} \right) + \\
\frac{\partial}{\partial t} \left\{ n\varepsilon \left( 1 + \frac{u^2}{2c^2} \right) + p \frac{u^2}{2c^2} \right\} = 0.
$$

(24)

It is important to emphasize at this point that the terms that correspond to the mechanical energy in the total energy balance arise from the expression

$$
n\varepsilon \frac{u^2}{2c^2} \sim n \frac{mu^2}{2},
$$

(25)

which would not be present if the second order terms in $G^{44}$ are considered, as mentioned above. Thus, the energy balance is

$$
\frac{\partial}{\partial t} \left( \frac{3}{2} nkT + nm \frac{u^2}{2} \right) + \\
\left[ \left( \frac{3}{2} nkT + p + nm \frac{u^2}{2} \right) u^\ell \right] = 0
$$

(26)
or
\[
\frac{\partial T}{\partial t} + \bar{u} \cdot \nabla T + \frac{2}{3} T \nabla \cdot \bar{u} = 0,
\] (27)
which is the usual expression for the temperature evolution in Euler’s regime.

IV. DISCUSSION AND FINAL REMARKS

In the previous sections we have archived a 3+1 representation for Euler hydrodynamics. The link between fluxes in the comoving and laboratory frame is given by the pseudo-galilean transformation given in Eq. (11). This matrix is obtained as the non-relativistic limit of the Lorentz transformation. Notice that the \( \mathcal{G}_{\ell 4} \) components are odd in \( u/c \) and, as can be easily seen, the next correction is third order in this quantity. Thus, the proposed transformation is the non-relativistic limit of the Lorentz transformation to second order in \( u/c \). As pointed out before, by keeping these second order terms, the formalism is capable of producing the energy balance equation from the energy-momentum tensor conservation law. Notice that, under this transformation, the position vector in the laboratory frame is given by

\[
\bar{x}^\ell = x^\ell + u^\ell t
\]
(28)
\[
\bar{x}^4 = ct \left( 1 + \frac{u^2}{2c^2} \right) + \frac{u^\ell x_\ell}{c}
\]
(29)
from which we could characterize the transformation as purely galilean in space and semi-relativistic in time. This suggests that the usual galilean transformation is not the correct approximation for the non-relativistic limit but an “ultra non-relativistic limit”.

Besides providing an alternative method for obtaining the Euler equations for the non-relativistic fluid, the formalism proposed serves as a benchmark for the relativistic case. In special relativity, not only the transformation law for the velocities
is not a linear relation but also a relativistic local equilibrium distribution function must be considered. In that case the integrals involved in calculating the averages are not trivial to perform and the physical meaning can be lost in the lengthy and complicated algebra. However, the calculation of the fluxes in the comoving frame features no major complications. Thus, the formalism here developed seems a viable alternative for studying the dynamics of relativistic fluids in equilibrium.

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