DEGENERATION OF KÄHLER-RICCI SOLITONS

GANG TIAN AND ZHENLEI ZHANG

Abstract. Let \((Y, d)\) be a Gromov-Hausdorff limit of \(n\)-dimensional closed shrinking Kähler-Ricci solitons with uniformly bounded volumes and Futaki invariants. We prove that off a closed subset of codimension at least 4, \(Y\) is a smooth manifold satisfying a shrinking Kähler-Ricci soliton equation. A similar convergence result for Kähler-Ricci flow of positive first Chern class is also obtained.

1. Introduction

The degeneration of manifolds with bounded Ricci curvature has been extensively studied, cf. [9] [10] [11] [12] [6] [13], or see [7] for a survey of related results.

Let \((Y, d)\) be a Gromov-Hausdorff limit of a sequence of manifolds \((M_i, g_i)\) whose Ricci curvature is bounded uniformly from below. In the noncollapsing case, in [9] the authors proved that the singular set of \(Y\) is of codimension at least 2; if the Ricci curvature admits a two-sided uniform bound, the singular set is closed and, combining with the result of Anderson [2], the regular set is a \(C^{1,\alpha}\) Riemannian manifold. The cases when \(M_i\) has special holonomy or \(L^p\) bounded curvature were considered in [12], [6] and [13]. In [9]-[11], same structure theorems were proved even for the collapsing case.

When \((M_i, g_i)\) is Einstein, the metric is smooth and Einstein on the regular part of \(Y\) and the convergence takes place smoothly there. For shrinking Ricci solitons, under certain curvature conditions, the degeneration property has been studied in [31] [28] [27] [32]. These results gave generalizations of orbifold compactness theorem of Einstein manifolds [1] [3] [25]. Recently, in [33], the second-named author observed that the curvature condition is somehow unnecessary: if \(M_i\) are shrinking Ricci solitons without any curvature assumption in a prior, then \(Y\) has closed singular set whose codimension is at least 2 and the regular part is a smooth manifold satisfying the shrinking Ricci soliton equation.

The first author is supported in part by National Science Foundation grants DMS-0847524 and DMS-0804095.

The second author is supported by National Science Foundation grant of China 09221010056.
In this short note, we will consider the shrinking Kähler-Ricci solitons. Here a shrinking Kähler-Ricci soliton means a Kähler manifold \((M, g)\) which satisfies

\[
\begin{align*}
R_{ij} + \nabla_i \nabla_j u &= g_{ij}, \\
\nabla_i \nabla_j u &= \nabla_i \nabla_j u = 0,
\end{align*}
\]

for some smooth function \(u\). Associated to the shrinking Kähler-Ricci soliton, the Futaki invariant, evaluated at \(\nabla u\), is given by \(\int_M |\nabla u|^2 dv\).

We are going to prove the following theorem.

**Theorem 1.1.** Let \((M_i, g_i)\) be a sequence of \(n\)-dimensional compact shrinking Kähler-Ricci solitons such that \(c_1(M_i)^n \leq C\) and the Futaki invariant \(\leq C\) for a uniform constant \(C\). Then, by taking a subsequence if necessary, \((M_i, g_i) \xrightarrow{dGH} (Y, d)\), where \((Y, d)\) is a path metric space with a closed singular set \(S\) of codimension at least 4. On the regular set \(Y \setminus S\), \(d\) is induced by a smooth Kähler metric which satisfies a Kähler-Ricci soliton equation. Furthermore, the convergence takes place smoothly on \(Y \setminus S\).

We give some remarks about our theorem.

**Remark 1.2.** The hypothesis in the theorem implies a uniform upper bound on the diameter of \(M_i\). In general, the diameter is difficult to control; while on the other hand, the Futaki invariant is much easier to handle, since it is defined on the finite dimensional vector space generated by holomorphic vector fields. We also remark that the upper bound on Futaki invariant can be replaced by an upper bound of \(F_{-\nabla u_i}(-\nabla u_i) = \int_{M_i} |\nabla u_i|^2 e^{u_i} dv\), the modified Futaki invariant defined in [26].

**Remark 1.3.** Due to a theorem from algebraic geometry, given \(n\), there exist only finitely many families of \(M\) with \(c_1(M) > 0\). This should imply that there is an upper bound on both \(c_1(M)^n\) and the Futaki invariant. Hence, the conditions in the theorem hold automatically.

**Remark 1.4.** If the curvature of \(M_i\) admits a uniform \(L^2\) bound, then, following the proof of Theorem 1.23 in [12], one can also show that the \((2n-4)\)-dimensional Hausdorff measure \(\mathcal{H}^{2n-4}(S) < \infty\).

In §2, we recall and prove some preliminaries about shrinking Ricci solitons, then in §3, we provide a proof of the theorem. In the last section §4, we prove a theorem on the degeneration of a Kähler-Ricci flow \((M, g(t))\) on a compact manifold with positive first Chern class. In the course of the proof in §4, we show the \(L^\infty\) estimate for the minimizer of \(\mu(g(t), \frac{1}{2})\).

2. Preliminaries about shrinking Ricci solitons

We recall and prove some basic estimates about compact shrinking Kähler-Ricci solitons in this section. Let \((M, g)\) be a compact shrinking Kähler-Ricci
soliton with potential function $u$. We suppose throughout this note that $u$ is normalized such that

$$\int_M (2\pi)^n e^{-u} dv = 1.$$  

2.1. Some basic results. It’s well-known that the following identities

$$R + \Delta u = n,$$

$$R + |\nabla u|^2 = u - \bar{u} + n$$  

hold, where $R$ denotes the scalar curvature of $g$ and $\bar{u} = \int_M u(2\pi)^{-n} e^{-u} dv$. Obviously by definition $\inf u \leq \bar{u}$; while on the other hand, by a result of Ivey \[19\], $R > 0$, thus equation (4) implies that $\inf u > \bar{u} - n$. Thus,

$$\bar{u} - n < \inf u \leq \bar{u},$$

$$\int_M (-\Delta u + |\nabla u|^2)e^{-u} dv = \int_M (u - \bar{u})e^{-u} dv = 0.$$  

2.2. Perelman’s entropy functional. Recall that Perelman’s entropy functional for a closed Kähler manifold $(M,g)$ is defined by \[20\]

$$W(g, f, \tau) = \int_M (2\tau(R + |\nabla f|^2) + f - 2n)(4\pi\tau)^{-n} e^{-f} dv,$$

where $f \in C^\infty$ and $\tau > 0$ is any constant. Then define the $\mu$ functional via

$$\mu(g, \tau) = \inf \{ W(g, f, \tau) | \int_M (4\pi\tau)^{-n} e^{-f} dv = 1 \}.$$  

We remark that, according to Perelman’s monotonicity theorem about $W$ along the Ricci flow, if $(M,g)$ is a shrinking Kähler-Ricci soliton with potential function $u$, then

$$\mu(g, \frac{1}{2}) = \int_M (R + |\nabla u|^2 + u - 2n)(2\pi)^{-n} e^{-u} dv.$$  

Applying formulas (3) and (6), we can rewrite (9) as

$$\mu(g, \frac{1}{2}) = \bar{u} - n.$$  

The entropy relates the local collapsing information of a shrinking Ricci soliton. More precisely, by an argument as in \[23\], we have the following lemma.

**Lemma 2.1.** There exist a function $D = D(A, V, n)$ for any $A, V > 0$ and integer $n$ satisfying the following. Let $(M, g)$ be an $n$-dimensional shrinking Kähler-Ricci soliton such that $\mu(g, \frac{1}{2}) \geq -A$ and $\text{Vol}(M) \leq V$, then its diameter $\text{diam}(M) \leq D$. 
2.3. **Bound \( \mu \) in terms of Futaki invariant.** When \((M, g)\) is a shrinking Kähler-Ricci soliton, the Futaki invariant, evaluated at the gradient vector field of \(u\), is given by

\[
F = F(\nabla u) = \int_M |\nabla u|^2 dv.
\]

We will prove the following lemma.

**Lemma 2.2.** There exists a positive constant \( c = c(n) \) such that for any compact shrinking Kähler-Ricci soliton \((M, g)\), we have

\[
\mu(g, \frac{1}{2}) \geq -c(1 + F).
\]

**Proof.** Noting that the volume of a compact shrinking Kähler-Ricci soliton always admits a lower bound by a constant depending only on \( n \), so in view of relation (10), it suffices to show that

\[
\bar{u} \geq \min \{-4n, 2 \ln \text{Vol}(M) - 4n \ln(2\pi), -\frac{8F}{\text{Vol}(M)}\}.
\]

Assume that \( \bar{u} \leq -4n \). First of all, we claim that

\[
\text{Vol}\{x \in M | u(x) \leq \frac{\bar{u}}{2}\} \leq (2\pi)^n e^{\frac{\bar{u}}{2}}.
\]

Actually, this follows from the normalizing condition (2):

\[
1 = \int_M (2\pi)^{-n} e^{-u} dv = \int_{\{u \leq \frac{\bar{u}}{2}\}} (2\pi)^{-n} e^{-u} dv \geq (2\pi)^{-n} e^{-\frac{\bar{u}}{2}} \text{Vol}\{u \leq \frac{\bar{u}}{2}\}.
\]

Suppose that \( \bar{u} \leq 2 \ln \text{Vol}(M) - 4n \ln(2\pi) \), then \( \bar{u} \leq 2 \ln \frac{\text{Vol}(M)}{2} - 2n \ln(2\pi) \). We have \( \text{Vol}(M) \geq 2(2\pi)^n e^{\frac{\bar{u}}{2}} \geq 2 \text{Vol}\{u \leq \frac{\bar{u}}{2}\} \), which implies that

\[
\text{Vol}\{u \geq \frac{\bar{u}}{2}\} = \text{Vol} (M \{u < \frac{\bar{u}}{2}\}) \geq \frac{1}{2} \text{Vol}(M).
\]
Integrating equation (4) we get

\[ \int_M |\nabla u|^2 dv = \int_M (u - \bar{u} + n - R) dv \]
\[ = \int_M (u - \bar{u} + \Delta u) dv \]
\[ = \int_M (u - \bar{u}) dv \]
\[ = \int_{\{u \geq \frac{\bar{u}}{2}\}} (u - \bar{u}) dv + \int_{\{u \leq \frac{\bar{u}}{2}\}} (u - \bar{u}) dv \]
\[ \geq -\frac{\bar{u}}{2} \text{Vol}\{u \geq \frac{\bar{u}}{2}\} - n \text{Vol}\{u \leq \frac{\bar{u}}{2}\} \]
\[ \geq \left( -\frac{\bar{u}}{4} - \frac{n}{2} \right) \text{Vol}(M) \]
\[ \geq -\frac{\bar{u}}{8} \text{Vol}(M), \]

where we used (5) in the second inequality and the assumption \(\bar{u} \leq -4n\) in the last inequality. This finishes the proof of the lemma. \(\square\)

2.4. **Bound the Ricci curvature up to a conformal change.** Let \((M, g)\) be a Kähler-Ricci soliton whose diameter is less than \(D\). By an easy computation, we have the following bound,

\[ \|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R\|_{C^0} \leq C_1 \]

for some \(C_1 = C_1(n, D)\), which does not depend on the specified Kähler-Ricci soliton; see [33] §3.1 or [23] for a proof. As a consequence, the Ricci curvature of \(\tilde{g} = e^{-\frac{n}{n-1}}g\), which is a conformal Hermitian metric of \(g\), admits a two-sided bound [33] §3.2]

\[ |\text{Ric}(\tilde{g})|_{\tilde{g}} \leq C_2 \]

by a constant \(C_2 = C_2(n, D)\). The other observation is that \(\tilde{g}\) and \(g\) itself are uniformly equivalent to each other.

3. **Proof of Theorem 1.1**

Now let \((M_i, g_i)\) be a sequence of shrinking Kähler-Ricci solitons which satisfies a uniform upper bound on both \(c_1(M_i)^n\) and the Futaki invariant, then by Lemma [2.1] and [2.2] there exists \(D\) independent of \(i\) such that \(\text{diam}(M_i) \leq D\). By [33], there exists a compact path metric space \((Y, d_\infty)\) such that

\[ (M_i, g_i) \xrightarrow{d_H} (Y, d_\infty) \]
along a subsequence. Denote by $\mathcal{R}$ the smooth part of $Y$ and $g_\infty$ the smooth metric on $\mathcal{R}$ which induces $d_\infty$. From [33], the convergence takes place smoothly on $\mathcal{R}$. Thus, $\mathcal{R}$ admits a natural complex structure for which the metric $g_\infty$ is Kählerian and satisfies a shrinking Kähler-Ricci soliton equation on $\mathcal{R}$.

Following Cheeger and Colding [9], denote by $S_k$ the set of points $y \in Y$ any of whose tangent cone splits off at most a $k$-dimensional Euclidean space. By [9], together with Claim 3.9 in [33], we have $\dim(S_k) \leq k$, $S_k \subset S_{k+1}$ for all $k$. Furthermore, according to [33, Theorem 1.1], the singular set equals $S_{2n-2}$. So, to prove the Theorem 1.1, it suffices to show that both $S_{2n-2}\setminus S_{2n-3}$ and $S_{2n-3}\setminus S_{2n-4}$ are empty sets.

3.1. The tangent cone of $Y$. From now on, we fix a singular point $y \in S = Y\setminus \mathcal{R}$ and a tangent cone at $y$, which is given by

$$
(Y, d, o) = \lim_{j \to \infty} (Y, \rho_j^{-1}d_\infty, y)
$$

where $\rho_j \to 0$ is a sequence of positive numbers and the convergence is taken in the pointed Gromov-Hausdorff topology. Then $Y_y = \mathbb{R}^k \times C(X)$ for a path metric space $X$ with $\text{diam}(X) \leq 2\pi$, where $\mathbb{R}^k$ denotes the $k$-Euclidean space. We assume that, in the splitting, $k$ is maximal. Denote by $\mathcal{R}_y$ the regular part of $Y_y$.

Let $y_i \in M_i$ such that $y_i \to y$. Combining with (15), along a subsequence $i_j \to \infty$, we have

$$
(M_{i_j}, \rho_j^{-2}g_{i_j}, y_{i_j}) \to (Y, d_y, o),
$$

as $j \to \infty$. By the argument in [33], using that $\rho_j^{-2}g_{i_j}$ is a shrinking Ricci soliton, one can check that the singular set $Y_y\setminus \mathcal{R}_y$ is closed and has Hausdorff dimension $\leq 2n - 2$. Denote by $g_y$ the metric on $\mathcal{R}_y$ which induces $d_y$.

Claim 3.1. Passing a subsequence if necessary, the convergence in (17) is smooth on $\mathcal{R}_y$.

Proof. Basically, the proof is the same as the proof of Theorem 1.1 in [33]. We sketch it here; the main step is to show the $C^\alpha$ convergence on $\mathcal{R}_y$.

Let $u_i$ be the associated potential function of $g_i$. Then, passing a subsequence if necessary, $u_i$ converges to a Lipschitz function $u_\infty$ on $Y$. Denote by $\tilde{g}_i = e^{\frac{1}{\rho_i^2}(u_i(y_i) - u_i)}g_i$ the conformal Hermitian metric on $M_i$. The important feature is that $\tilde{g}_i$ has a two-sided bound on Ricci curvature; see (14). Thus,

$$
|\text{Ric}(\rho_j^{-2}\tilde{g}_{i_j})| \to 0 \text{ as } j \to \infty.
$$
Now applying the uniform bound of \( u_i \), cf. (13), one verifies that (passing a subsequence if necessary)
\[
(M_{ij}, \rho_j^{-2}g_{ij}, y_{ij}) \longrightarrow (Y_y, d_y, o)
\]
as \( j \to \infty \); see the proof of Claim 3.9 in [33] for details. By [2], the later convergence is in the \( C^{1,\alpha} \) topology on \( \mathcal{R}_y \). More precisely, for any compact subset \( K \subset \mathcal{R}_y \), there exist a sequence of diffeomorphic embeddings \( \phi_j : K \to M_{ij} \) such that
\[
\|\phi_j^*(\rho_j^{-2}g_{ij}) - g_y\|_{C^{1,\alpha}(K)} \to 0
\]
as \( j \to \infty \). With respect to the metric \( \rho_j^{-2}g_{ij} \) on \( \phi_j(K) \), by the \( C^1 \) estimate of the potential function (13), we have
\[
\|u_{ij}(y_{ij}) - u_{ij}\|_{C^0(\phi_j(K))} + \|\nabla u_{ij}\|_{C^0(M_{ij})} \to 0, \quad \text{as } j \to \infty.
\]
Passing to \( g_{ij} \), the pull-backed metric
\[
\phi_j^*(\rho_j^{-2}g_{ij}) = \phi_j^*(e^{-\frac{1}{n-1}(u_{ij}(y_{ij}) - u_{ij})}(\rho_j^{-2}g_{ij})) = e^{-\frac{1}{n-1}(u_{ij}(y_{ij}) - u_{ij} \circ \phi_j)} \phi_j^*(\rho_j^{-2}g_{ij})
\]
has a uniform \( C^\alpha \) bound on \( K \). Hence, combining this with (20) and (21) we get the \( C^\alpha \) convergence of \( \phi_j^*(\rho_j^{-2}g_{ij}) \) to \( g_y \) on \( K \).

Being known the \( C^\alpha \) convergence, the \( C^\infty \) convergence of (17) (along a subsequence) follows from a regularity argument based on Perelman’s pseudolocality theorem [20] (see also Theorem 4.2 in [14]) and Shi’s gradient estimate [24, 13] for Ricci flow, since the metrics in consideration are shrinking Ricci solitons and can be seen as time slices of the corresponding Ricci flows. See the proof of Claim 3.10 in [33], for example. We omit the details here.

Let \( J_{ij} \) be the canonical complex structure on \( M_{ij} \). By the smooth convergence, passing a subsequence if necessary, there exist a family of compact subsets \( K_j \) satisfying \( K_j \subset K_{j+1}, \cup K_j = \mathcal{R} \), and embeddings
\[
\phi_j : K_j \longrightarrow M_{ij}
\]
such that \( \phi_j^*(\rho_j^{-2}g_{ij}) \xrightarrow{C^\infty} g_y \) and \( \phi_j^*J_{ij} \xrightarrow{C^\infty} J_\infty \) for a complex structure \( J_\infty \) on \( \mathcal{R}_y \). Obviously \( J_\infty \) is parallel with respect to \( g_y \). We claim that \( J_\infty \) splits as a product complex structure on the regular part \( \mathcal{R}_y \), which implies that, in the splitting of \( Y_y \), \( k \) is even.

We need the following assertion.

**Claim 3.2.** Let \( \nabla \) and \( \tilde{\nabla} \) be the Levi-Civita connections of \( \rho_j^{-2}g_{ij} \) and \( \rho_j^{-2}g_{ij} = e^{-\frac{1}{n-1}(u_{ij}(y_{ij}) - u_{ij})}(\rho_j^{-2}g_{ij}) \) respectively. Then we have
\[
\|\tilde{\nabla} - \nabla\|_{C^0(M_{ij})} \to 0, \quad \text{as } j \to \infty,
\]
where the $C^0$ norm is taken with respect to the metric $\rho_j^{-2} \tilde{g}_{ij}$.

**Proof.** Let $\tilde{\Gamma}$ and $\Gamma$ be the Christoffel symbols of the Levi-Civita connections of $\rho_j^{-2} \tilde{g}_{ij}$ and $\rho_j^{-2} g_{ij}$. Then, in local coordinate, the connections have the difference

$$\nabla - \nabla = (\tilde{\Gamma}^s_{pq} - \Gamma^s_{pq}) dx^p \otimes dx^q \otimes \frac{\partial}{\partial x^s},$$

where $\nabla - \nabla \to 0$, while $J$ is an almost isometric endomorphism of $TM$ with metric $\rho_j^{-2} \tilde{g}_{ij}$ whenever $j$ is large enough. Then, in view of the convergence (19), by the explanation in [13, Page 399], if $v$ is the gradient of a harmonic function which induces an almost splitting, so does $Jv$. For more details, see the proof of Theorem 9.1 in [12]; note that the vector field $v$ satisfies assumption of Lemma 9.14 in [12] if and only if $Jv$ does, up to a modification of the constants $c$ and $\delta$ there. As a direct consequence, passing to the limit space $Y_y$, we have that

(24) $S_{2l+1} \setminus S_{2l} = \emptyset$, for all integer $l$.

In particular,

(25) $S_{2n-3} \setminus S_{2n-4} = \emptyset$.

Furthermore, the Euclidean factor $\mathbb{R}^k = \mathbb{C}^k$ with respect to $J_\infty$ and on the regular part of $Y_y$, we have the splitting structure

$$\mathcal{R}_y = \mathbb{C}^k \times \mathcal{R}(C(X)),$$

where $\mathcal{R}(C(X))$ denotes the regular part of $C(X)$.

In the next subsection, we will make use of the convergences (17) and (19) to remove the singularities of type $S_{2n-2} \setminus S_{2n-3}$ when $M_i$ are Kählerian.
3.2. The slice argument and regularity. As in [12, 6], we will show the following regularity property, which is sufficient to prove the main theorem in view of the convergence (17).

Let \( y \in S_{2n-2} \setminus S_{2n-3} \) be a typical singular point. Then the tangent cone \( Y = \mathbb{R}^{2n-2} \times C(S_t) \) where \( S_t \) is a round circle with circumference \( t < 2\pi \), \( t \in \mathbb{R}^{2n} \). A typical point of \( Y_y \) can be expressed as \((z, r, x)\) where \( r \geq 0 \) denotes the radial coordinate on \( C(S_t) \).

Proposition 3.3. For all \( \eta > 0 \), there exists \( j_0 \) such that

\[
  d_{GH}(B_{\rho_j^{-2}g_{ij}}(y, 1), B(0, 1)) \leq \eta, \quad \forall j \geq j_0.
\]

Here, \( B(0, 1) \) denotes the unit ball in the 2n-Euclidean space.

Let \( \Psi = \Psi(j, l, \delta, D, n) \) be a positive constant that may vary in the following argument such that for any fixed \( n, \delta \) and \( D \),

\[
  \Psi(j, l, \delta, D, n) \to 0, \quad \text{as } j \to \infty \text{ and } l \to \infty.
\]

Proof of Proposition 3.3. For simplicity, we fix a specified manifold \( (M, g) = (M_i, \rho_j^{-2}g_{ij}) \) and let

\[
  \tilde{g} = e^\frac{1}{\rho_j(t - u_{ij})} \rho_j^{-2}g_{ij}, \quad \bar{g} = e^\frac{1}{\rho_j(t - u_{ij})} \rho_j^{-2}g_{ij}.
\]

Let \( u = u_{ij} \) be the potential function and \( \rho = \rho_j \) be the rescaling factor. The metric \( \bar{g} \) is introduced for the convenience of computing the Chern class.

Since (19) holds, for any \( l \), we have

\[
  d_{GH}(B\tilde{g}(y, l), B(0, 1)) < l^{-1}
\]

whenever \( j \) is large enough. Let \( F : B\tilde{g}(y, l) \to Y \) be a Gromov-Hausdorff approximation realizing the convergence (19). We remark that by the proof of Claim 3.1, \( F \) also realizes the convergence (17). So by the smooth convergence on \( \mathbb{R}^{2n-2} \times (C(S_t) \setminus \{x^*\}) \), we may assume that \( F \) is holomorphic and diffeomorphic on the range outside of \( \mathbb{R}^{2n-2} \times B(x^*, \frac{1}{3}) \). Recall that by [12], see also [6], there exist smooth functions

\[
  \Lambda = (\Phi, r) : B\tilde{g}(y, 3) \to \mathbb{R}^{2n-2} \times \mathbb{R}_+
\]

such that

\[
  \|\Phi - z \circ F\|_{C^0} + \|r - r \circ F\|_{C^0} \leq \Psi.
\]

Furthermore, if we set

\[
  \Sigma_{z, r} = \Phi^{-1}(z) \cap \Lambda^{-1}(B(0, 1) \times [0, r]),
\]
then, there exist subsets $B_l \subset B(0,1)$ and $D_l \subset B(0,1) \times [0,1]$ consisting of regular values of $\Phi$ and $\Lambda$ such that
\begin{align}
(31) \quad \text{Vol}(B_l) \geq (1 - \Psi) \text{Vol}(B(0,1)), \\
(32) \quad \text{Vol}(D_l) \geq (1 - \Psi) \text{Vol}(B(0,1) \times [0,1]),
\end{align}
and that for all $z \in B_l$ and $(z,r) \in D_l$,
\begin{align}
(33) \quad |\text{Vol}_{\tilde{g}}(\Sigma_{z,r}) - \frac{r^2}{2} t| \leq \Psi. \\
(34) \quad |\text{Vol}_{\tilde{g}}(\Lambda^{-1}(z,r)) - rt| \leq \Psi.
\end{align}
Note that if $(z,r) \in D_l$, then $\Lambda^{-1}(z,r) = \partial \Sigma_{z,r}$.

Then one can use the differential character of the first Chern class on suitable slices of $\Lambda^{-1}(z,r)$ to give a proof of the proposition; see \cite{13}, \cite{7} and \cite{6} for details. The idea to use the differential character first appeared in \cite{12}.

In the following, we give an alternative and direct proof of the proposition based on the transgression theory. The argument is also inspired by the work of Cheeger, Colding and Tian for Kähler manifolds with bounded Ricci curvatures, cf. \cite{12}, \cite{13}, \cite{7} and \cite{6}.

Choose $(z,r) \in D_l$ such that $\frac{1}{2} \leq r \leq 1$ and let $L$ be the determinant bundle of $TM$ restricted to $\Sigma_{z,r}$. Denote also by $\tilde{g}$ the induced Hermitian metric on $L$. Denote by $\tilde{\Theta}$ the curvature form associated to the Chern connection of $\tilde{g}$. Let $S\Sigma_{z,r}$ be the unit circle bundle and $\pi : S\Sigma_{z,r} \to \Sigma_{z,r}$ be the projection. Then there exist connection form $\omega$ and curvature form $\Omega$ on $S\Sigma_{z,r}$ such that
\begin{equation}
(35) \quad \Omega = \pi^* \tilde{\Theta} = d\omega.
\end{equation}

Note that the curvature form $\tilde{\Theta}$ gives the first Chern class of $L$:
\begin{align}
\sqrt{-1} \frac{\tilde{\Theta}}{2\pi} &= P_{c_1}(L)|_{\Sigma_{z,r}} \\
&= -\sqrt{-1} \frac{\partial_p \partial_q \log \det(\tilde{g}) dz^p \wedge dz^q|_{\Sigma_{z,r}}}{2\pi} \\
&= \sqrt{-1} \frac{R_{pq} + \partial_p \partial_q u dz^p \wedge dz^q|_{\Sigma_{z,r}}}{2\pi} \\
&= \sqrt{-1} \frac{\rho^2 g_{pq} dz^p \wedge dz^q|_{\Sigma_{z,r}}}{2\pi},
\end{align}
whose norm tends to 0 uniformly as $j \to \infty$. Now choose a transversal section $s : \Sigma_{z,r} \to L$ with finite zero set $\{p_k\}_{k=1}^N$ such that $s = x \circ F$ on $\partial \Sigma_{z,r} = \Lambda^{-1}(z,r)$ with respect to local trivialization by $F$ on a neighborhood of $\Lambda^{-1}(z,r)$. Then we have $(\frac{\cdot}{|s|})^* \Omega = \tilde{\Theta}$ outside of the zero set, so by (33)
and using the $C^1$ equivalence of $\tilde{g}$ and $\tilde{\tilde{g}}$,

\[
\Psi = \frac{\sqrt{-1}}{2\pi} \int_{\Sigma_{z,r}} \tilde{\Theta}
\]

\[
= \lim_{\epsilon \to 0} \frac{\sqrt{-1}}{2\pi} \int_{\Sigma_{z,r} \setminus \cup_k D(p_k,\epsilon)} \tilde{\Theta}
\]

\[
= \lim_{\epsilon \to 0} \frac{\sqrt{-1}}{2\pi} \int_{\Sigma_{z,r} \setminus \cup_k D(p_k,\epsilon)} \left( \frac{s}{|s|} \right)^* \Omega
\]

\[
= \lim_{\epsilon \to 0} \frac{\sqrt{-1}}{2\pi} \int_{\Sigma_{z,r} \setminus \cup_k D(p_k,\epsilon)} d\left( \frac{s}{|s|} \right)^* \omega
\]

\[
= \sum_k \lim_{\epsilon \to 0} \frac{\sqrt{-1}}{2\pi} \int_{\partial D(p_k,\epsilon)} \left( \frac{s}{|s|} \right)^* \omega + \frac{\sqrt{-1}}{2\pi} \int_{\Lambda^{-1}(z,r)} \left( \frac{x}{|x|} \circ F \right)^* \omega
\]

Here, $D(p_k,\epsilon)$ denotes an $\epsilon$ ball around $p_k$ in $\Phi^{-1}(z)$. In the last formula, each limit term $\lim_{\epsilon \to 0} \frac{\sqrt{-1}}{2\pi} \int_{\partial D(p_k,\epsilon)} \omega$ is an integer since locally $\omega$ can be written as $d \log x + \tilde{\theta}$ where $\tilde{\theta}$ denotes the horizontal term of $\omega$; while $\frac{\sqrt{-1}}{2\pi} \int_{\Lambda^{-1}(z,r)} \omega$ approximates $\frac{1}{2\pi}$, the corresponding integration on the circle $\{(r,x)|x \in S_t\}$ in the cone $C(S_t)$, as $j \to \infty$ since the metric $\tilde{\tilde{g}}$ approaches $g_{\#}$ uniformly in the $C^1$ topology around $\Lambda^{-1}(z,r)$ when $(z,r) \in D_t$ with $\frac{1}{2} \leq r \leq 1$. Notice that the connection form $\omega$ is determined by first derivative of the metric. This is sufficient to show that $\frac{t}{2\pi}$ is an integer; then applying the relation (34) we infer that $t = 2\pi$.

This completes the proof of the proposition. \qed

Now, the proof of Theorem 1.1 follows easily.

**Proof of Theorem 1.1.** By (25) and Proposition 3.3, $S = S_{2n-4}$. As the convergence on the regular part is smooth, it satisfies a shrinking Kähler-Ricci soliton equation there. The proof is completed. \qed

### 4. On the Kähler-Ricci flow with positive $c_1$

In this section, we use the above method to study the degeneration of a Kähler-Ricci flow on a Kähler manifold $M$ with positive $c_1(M)$. Let $g(t)$, $t \in [0,\infty)$, be a solution to the Kähler-Ricci flow on $M$

\[
\frac{\partial}{\partial t} g_{ij} = -R_{ij} + g_{ij}
\]
whose Kähler form lies in the Kähler class $\pi c_1(M)$. It is well-known that the Kähler-Ricci flow preserves the volume, say $V = \text{Vol}_{g(t)}(M)$. It is also known that $g(t)$ has the same Kähler class. Thus, Hodge theory gives a family of potentials $\phi(t)$ such that $g(t) = g(0) + \partial \bar{\partial} \phi(t)$. The Kähler-Ricci flow is then equivalent to the complex Monge-Ampère equation for $\phi(t)$

$$\frac{\partial \phi(t)}{\partial t} = \log \frac{\det(g(0) + \partial \bar{\partial} \phi(t))}{\det(g(0))} + \phi(t).$$

Let $u(t)$ be associated Ricci potential in the sense that

$$R_{ij}(t) + \nabla_i \nabla_j u(t) = g_{ij}(t).$$

Suppose $u(t)$ is normalized such that

$$\int e^{-u(t)} dv_{g(t)} = (2\pi)^n.$$

Perelman proved the following estimates (see [23] for the proof)

$$\text{diam}(M, g(t)) + \|u(t)\|_{C^0} + \|\nabla u(t)\|_{C^0} + \|R(g(t))\|_{C^0} \leq C_1$$

for some $C_1$ independent of $t$. It is also shown in [23] that $\mu(g(t), \frac{1}{2})$, the function defined in §2.2, has a uniform bound

$$|\mu(g(t), \frac{1}{2})| \leq C_1.$$

In [22], Sesum studied the convergence of Kähler-Ricci flow with bounded Ricci curvature. One essential point in her argument is that the metric derivative in time is uniformly bounded (in terms of the Ricci curvature) and thus the metrics under the Kähler-Ricci flow are locally equivalent to each other. Here, what we are concerning is a Kähler-Ricci flow with uniformly bounded $(2, 0)$-part of Hess$(u(t))$

$$|\nabla_{\frac{\partial}{\partial z_i}} \nabla_{\frac{\partial}{\partial z_j}} u(t)| \leq C_2.$$

As we will see, the convergence result is same as the case of bounded Ricci curvature. The argument partially follows Sesum’s line.

We first observe that (42) implies an immediate bound of $\widetilde{Ric} + \text{Hess}(u)$:

$$|\text{Ric}(t) + \text{Hess}(u)(t)| \leq C_3$$

for some $C_3 = C_3(C_2, n)$. For any $g = g(t)$, let $\tilde{g} = e^{-\frac{u}{n-1}} g$ be a conformal Hermitian metric and denote by $\tilde{\text{Ric}}$ its Ricci curvature. Then, cf. [4],

$$\tilde{\text{Ric}} = \text{Ric} + \text{Hess}(u) + \frac{1}{2n-2} du \otimes du + \frac{1}{2n-2} (\Delta u - |\nabla u|^2) g.$$

Together with $R + \Delta u = n$, $\tilde{\text{Ric}}$ can be bounded as follows

$$|\text{Ric}|_{\tilde{g}} \leq C_4 e^{\frac{u}{n-1}} (1 + |\nabla u|^2 + |R|)$$
Theorem 4.2. Assume as above. There exist universal constants for some (49) Vol
a space-time point such that each (47) \( \partial \) independent of \( k \)
under the Kähler-Ricci flow. By [33], for any sequence of times (46)
for some \( C_5 \) independent of \( t \). Notice that the volume \( \text{Vol}(M, g) \)
is constant under the Kähler-Ricci flow. By [33], for any sequence of times \( t_k \to \infty \),
there exists a compact path metric space \((Y, d)\) such that \((M, g(t_k)) \xrightarrow{d_{GH}} (Y, d)\). The space \( Y \) has a singular set \( S \) of codimension at least 2. Furthermore, on the regular part \( R = Y \setminus S \), \( d \) is induced by a \( C^\alpha \) metric \( g_\infty \) and the convergence takes place \( C^\alpha \) there.

As we are on a Kähler-Ricci flow, the metric \( g_\infty \) should have more regular property. Indeed we can prove

**Theorem 4.1.** Let \( g(t), t \in [0, \infty) \), be a solution to the Kähler-Ricci flow
on a Kähler manifold \( M \) with positive \( c_1(M) \). Let \( u(t) \) be a family of Ricci potentials of \( g(t) \). If (42) holds for some \( C_2 \) independent of \( t \), then along a subsequence \((M, g(t)) \xrightarrow{d_{GH}} (Y, d)\), where \((Y, d)\) is a path metric space with a closed singular set \( S \) of codimension at least 4. On the regular set \( Y \setminus S \), \( d \) is induced by a smooth Kähler metric which satisfies a Kähler-Ricci soliton equation. Furthermore, the convergence takes place smoothly on \( Y \setminus S \).

Inspired by the work of [22], to prove the higher regularity of \( g_\infty \) on \( R \), we need to consider the Ricci flow \( g_k(t) = g(t_k + t) \). For simplicity, we consider the sequence of flows \( h_k(t) = \phi_{k,t}^* g_k(t), t \in [-t_k, \infty) \), instead of \( g_k(t) \) where \( \phi_{k,t} \) is a family of diffeomorphisms of \( M \) generated by the (real) gradient vector field \(-\frac{1}{2} \nabla u(t_k + t)\), with \( \phi_{k,0} = id_M \). By an easy computation, for each \( k \), \( h_k(t) \) satisfies the evolution

(47) \[ \frac{\partial}{\partial t} h_k(t) = \phi_{k,t}^* (-\text{Ric}(g_k(t)) - \text{Hess}(u_k(t))) + h_k(t), \quad t \in [-t_k, \infty), \]

where \( u_k(t) = u(t_k + t) \). By (43) and boundedness of \( |\nabla u(t)| \), one gets the uniform bound of derivative of \( h_k(t) \):

(48) \[ |\frac{\partial}{\partial t} h_k(t)| \leq C_6 \]

for some \( C_6 \) independent of \( k \) and \( t \).

The essential step is to prove the following pseudolocality theorem.

**Theorem 4.2.** Assume as above. There exist universal constants \( \delta_0, \epsilon_0 > 0 \)
independent of \( k \) with the following property. Let \((\bar{x}_0, \bar{t}_0) \in M \times [-t_k, \infty)\) be a space-time point such that

(49) \[ \text{Vol}_{h_k(t)}(B_{h_k(t)}(x, r)) \geq (1 - \delta_0) \text{Vol}(B(r)) \]

for all metric ball \( B_{h_k(t)}(x, r) \subset B_{h_k(t)}(\bar{x}_0, r_0) \) with \( t \in [\bar{t}_0, \bar{t}_0 + (\epsilon_0 r_0)^2] \), where \( B(r) \) denotes the metric ball of radius \( r \) in \( 2n \)-Euclidean space and \( \text{Vol}(B(r)) \)
denotes its volume, then the Riemannian curvature tensor satisfies
\begin{equation}
|\text{Rm}(h_k(t))(x)| \leq (t - \bar{t}_0)^{-1}
\end{equation}
whenever
\begin{equation}
\text{dist}_{h_k(t)}(\bar{x}_0, x) < \epsilon_0 r_0 \text{ and } \bar{t}_0 < t \leq \bar{t}_0 + (\epsilon_0 r_0)^2.
\end{equation}
In particular,
\begin{equation}
|\text{Rm}(h_k(t))(x_0)| \leq (t - \bar{t}_0)^{-1}, \quad \text{for } \bar{t}_0 < t \leq \bar{t}_0 + (\epsilon_0 r_0)^2.
\end{equation}

**Proof.** The argument is same as that of Theorem 4.2 in \[14\]. Rescale the flow $h_k(t)$ such that $r_0 = 1$. We may assume that under the rescaling the metric derivative $\frac{\partial}{\partial t} h_k(t)$ still satisfies the bound \(18\). This is fulfilled if the initial $r_0$ is less than 1.

Denote by $\overline{M}_k$ the set of space-time points $(x, t)$ satisfying (51) but the curvature $|\text{Rm}(h_k(t))(x)| \geq (t - \bar{t}_0)^{-1}$. Suppose $\overline{M}_k \neq \emptyset$, then as did in Claim 1 and Claim 2 of \[20\], one can choose another space-time point $(\bar{x}_k, \bar{t}_k) \in \overline{M}$ with $\bar{t}_k < \epsilon_0^2$ and $\text{dist}_{h_k(\bar{t}_k)}(\bar{x}_0, \bar{x}_k) \leq \frac{1}{10}$ such that $|\text{Rm}(h_k(t))(x)| \leq 4Q_k$ whenever
\begin{equation}
\bar{t}_k - \frac{1}{2n}Q_k^{-1} \leq t \leq \bar{t}_k, \quad \text{dist}_{h_k(\bar{t}_k)}(\bar{x}_k, x) \leq \frac{1}{1000n}(Q_k\epsilon_0^2)^{-1/2},
\end{equation}
where $Q_k = |\text{Rm}(h_k(\bar{t}_k))(\bar{x}_k)|$. From the metric derivative bound \(18\), the space-time point $(x, t)$ with (53) satisfies
\begin{equation}
\text{dist}_{h_k(t)}(\bar{x}_0, x) \leq \left(\frac{1}{10} + \frac{1}{1000n}(Q_k\epsilon_0^2)^{-1/2}\right)Ce_0^2 \leq \frac{1}{5}Ce_0^2 \leq \frac{1}{2}
\end{equation}
whenever $\epsilon_0$ is chosen small enough.

Suppose the theorem does not hold, then there exist sequences of positive numbers $r_k > 0$, $\epsilon_k \to 0$, $\delta_k \to 0$ and space-time points $(\bar{x}_k, \bar{t}_k)$ with $\bar{t}_k > -t_k$ such that (49) is fulfilled but (50) is not for all points in (51). As above we rescale the flow such that the radius $r_k = 1$. Reconstruct the base space-time point $(\bar{x}_k, \bar{t}_k)$ as in above process and consider the sequence of rescaled pointed flow
\begin{equation}
(\tilde{B}_{h_k(\bar{t}_k)}(\bar{x}_k, \frac{1}{1000n}(Q_k\epsilon_k^2)^{-1/2}), Q_k h_k(Q_k^{-1}t + \bar{t}_k, \bar{x}_k).
\end{equation}
One should keep in mind that this flow is really a Ricci flow, up to diffeomorphic actions. Actually, the family of Riemannian manifolds
\begin{equation}
(\phi_{-1,k}^{-1}B_{h_k(\bar{t}_k)}(\bar{x}_k, \frac{1}{1000n}(Q_k\epsilon_k^2)^{-1/2}), Q_k(\phi_{-1,k}^{-1}t + \bar{t}_k) \ast h_k(Q_k^{-1}t + \bar{t}_k), \phi_{-1,k}(\bar{x}_k))
\end{equation}
is really part of the rescaled Kähler-Ricci flow $(M, Q_kg_k(Q_k^{-1}t + \bar{t}_k))$ on the time interval $t \in [-\frac{1}{2n}, 0]$. Note that by construction the curvature of this sequence is uniformly bounded (less than 4) and the radius of the ball
equals $\frac{\varepsilon^4}{1000n}$ which tends to $\infty$ as $k \to \infty$. Thus, by Hamilton’s compactness theorem for Ricci flow [17], this sequence will converge along a subsequence to another pointed Kähler-Ricci flow $(M_\infty, \bar{g}_\infty(t), x_\infty)$ on the time interval $(-\frac{1}{2n}, 0]$. The curvature $|Rm(\bar{g}_\infty(0))|(x_\infty) = 1$; furthermore, by (10) the volume growth has Euclidean lower bound

$$\text{Vol}_{\bar{g}_\infty(0)}(B(x_\infty, r)) \geq \text{Vol}(B(r)), \forall r > 0.$$ 

Next we will show that $\bar{g}_\infty(0)$ is Ricci flat. Since $Q_k g_k(Q_k^{-1}t + \bar{t}_k) \to \bar{g}_\infty(t)$ smoothly and the scalar curvature of $Q_k g_k(Q_k^{-1}t + \bar{t}_k)$ tends to zero, the scalar curvature of $\bar{g}_\infty(t)$ vanishes identically. Thus,

$$\frac{\partial}{\partial t} \bar{g}_\infty(t) = -\text{Ric}(\bar{g}_\infty(t)).$$

Under this evolution, the scalar curvature satisfies

$$\frac{\partial}{\partial t} R(\bar{g}_\infty(t)) = \triangle R(\bar{g}_\infty(t)) + |\text{Ric}(\bar{g}_\infty(t))|^2.$$ 

It follows immediately that $\text{Ric}(\bar{g}_\infty(t)) \equiv 0$ for $t \in (-\frac{1}{2n}, 0]$. Together with Bishop-Gromov volume comparison theorem the volume growth condition implies that $\bar{g}_\infty(0)$ is actually flat. This contradicts with $|Rm(\bar{g}_\infty(0))|(x_\infty) = 1$. The contradiction proves the theorem.

As direct consequences, $g_\infty$ is Kähler and, by the argument in §3, the singular set $S$ has codimension at least 4.

It is time to claim the smooth convergence on $\mathcal{R}$. By $C^\alpha$ convergence, there exist embeddings $\psi_k : \mathcal{R} \to M$ such that $\psi_k^* g_k \stackrel{C^\alpha}{\longrightarrow} g_\infty$ uniformly on any compact subsets of $\mathcal{R}$.

**Claim 4.3.** Adjusting the embeddings $\psi_k$ if necessary we have $\psi_k^* g_k \stackrel{C^\infty}{\longrightarrow} g_\infty$ on $\mathcal{R}$.

**Proof.** By the uniqueness of Gromov-Hausdorff limit, it suffices to show that for any $K_\rho = \{x \in \mathcal{R}| \text{dist}(x, S) \geq \rho\}$ with $\rho > 0$, the metric $g_k$ is $C^\infty$ uniformly bounded on any $\psi_k(K_\rho)$ whenever $k$ is large enough.

Let $\varepsilon_0, \delta_0$ be the constants given in Theorem [12]. Fix one $\rho > 0$. Then by the $C^\alpha$ convergence of $g_k$ on $\mathcal{R}$, there exists $r_\rho \leq \frac{1}{2}\rho$ such that

$$\text{Vol}_{g_k}(B_{g_k}(x, r)) \geq (1 - \frac{1}{2}\delta_0) \text{Vol}(B(r)), \forall B_{g_k}(x, r) \subset K_\rho - r_\rho,$$

whenever $k$ is large enough and that

$$e^{-2nC_4(\varepsilon_0\rho)^2}(1 - \frac{1}{2}\delta_0) \geq 1 - \delta_0, \quad \frac{1}{2}e^{C_4(\varepsilon_0\rho)^2} \leq 1.$$
Fix any \( x_0 \in \psi_k(K_\rho) \). By the metric derivative estimate \( 48 \), we have

\[
\frac{d}{dt} \log \text{dist}_{h_k(t)}(p, q) \leq C_4 \tag{56}
\]

for all \( p, q \in M \) and

\[
\left| \frac{d}{dt} \log \text{Vol}_{h_k(t)}(U) \right| \leq \sup |\text{tr}_{h(t)} \frac{\partial h(t)}{\partial t}| \leq nC_4 \tag{57}
\]

for any domain \( U \subset M \). From these estimates one derives in particular that

\[
\text{Vol}_{h_k(t)}(B_{h_k(t)}(x, r)) \geq \text{Vol}_{h_k(t)}(B_{g_k}(x, e^{-C_4(t_k-t)}r)) \geq e^{-nC_4(e_0r_\rho)^2} \text{Vol}_{g_k}(B_{y_k}(x, e^{-C_4(t_k-t)}r)) \geq e^{-nC_4(e_0r_\rho)^2(1 - \frac{1}{2}\delta_0)} \text{Vol}(B(x, e^{-C_4(t_k-t)}r)) \geq (1 - \delta_0) \text{Vol}(B(x, r))
\]

for all \( - (e_0r_\rho)^2 \leq t \leq 0 \), \( B_{h_k(t)}(x, r) \subset B_{h_k(t)}(x_0, \frac{1}{2}r_\rho) \).

By Theorem \( 4.2 \) the curvature of \( h_k(t) \) satisfies the bounded

\[
|\text{Rm}(h_k(t))(x_0)| \leq \frac{1}{4} (e_0r_\rho)^2, \quad \forall x_0 \in \psi_k(K_\rho), t \in [-\frac{1}{2} (e_0r_\rho)^2, 0] \tag{58}
\]

whenever \( k \) is large enough. Now recall that \( h_k(t) = \phi_k^*g_k(t) \) for a family of diffeomorphisms \( \phi_k \) whose variation \( \frac{1}{2} \nabla u(t_k + t) \) is uniformly bounded. Thus, the curvature of \( g_k(t) \) satisfies

\[
|\text{Rm}(g_k(t))(x_0)| \leq \frac{1}{4} (e_0r_\rho)^2, \quad \forall x_0 \in \psi_k(K_\frac{1}{2}), t \in [-\delta (e_0r_\rho)^2, 0] \tag{59}
\]

for some uniform \( \delta > 0 \) and any \( k \) large enough. The derivative estimate for the curvature tensor follows directly from Shi’s gradient estimate \( 24 \), see also \( 18 \). The smooth convergence of \( g_k \) follows directly from Hamilton’s compactness theorem for Ricci flow \( 17 \).

The proof of the claim is completed. \( \square \)

Finally we show that \( g_\infty \) satisfies the Kähler-Ricci soliton equation. We shall apply Perelman’s monotonicity of \( \mu(g(t), \frac{1}{2}) \) under the Kähler-Ricci flow \( g(t) \). The \( L^\infty \) estimate to the minimizer of \( \mu(g(t), \frac{1}{2}) \) will play an essential role.

Let \( f_k \) be a normalized minimizer of \( \mu(g_k, \frac{1}{2}) \) such that

\[
\int e^{-f_k} dv_{g_k} = (2\pi)^n \tag{60}
\]
One verifies easily the following variation identity for $f_k$, cf. \[22\],
\begin{equation}
2\Delta f_k - |\nabla f_k|^2 + f_k = \mu(g_k, \frac{1}{2}) - R + 2n.
\end{equation}

Denote $v_k = e^{-f_k/2}$, then (61) is equivalent to
\begin{equation}
\triangle v_k = \frac{v_k}{4} \left( R(g_k) - 2n - \mu(g_k, \frac{1}{2}) - \log v_k \right).
\end{equation}

**Lemma 4.4.** There is a constant $C_7$ independent of $k$ such that
\begin{equation}
\inf_M f_k \geq -C_7.
\end{equation}

**Proof.** It suffices to prove a uniform upper bound of $v_k$. It is pointed out by Rothaus \[21\] that, $\sup_M v_k$ admits a universal upper bound by an easy iteration argument, using that $R(g_k)$ and $\mu(g_k, \frac{1}{2})$ are both uniformly bounded. We just mention that, according to the independent work of Ye \[29\] and Zhang \[30\], the Sobolev constant under the Kähler-Ricci flow has a uniform bound. In particular,
\begin{equation}
\left( \int_M \phi^\frac{2n}{n-1} dv_{g_k} \right)^\frac{n-1}{n} \leq C_8 \int_M (|\nabla \phi|^2 + \phi^2)dv_{g_k}, \quad \forall \phi \in C^\infty(M),
\end{equation}
for a uniform constant $C_8$. Thus, the iteration process works uniformly for all $k$. We will not give the details here. \qed

The uniform upper bound of $f_k$ is more interesting. The following proof relies on the Sobolev inequality and the Poincaré inequality under the Kähler-Ricci flow. The first step is to show an $L^2$ estimate of $f_k$.

**Lemma 4.5.** There exists $C_9$ independent of $k$ such that
\begin{equation}
\int_M f_k^2 dv_{g_k} \leq C_9.
\end{equation}

**Proof.** Integrating the identity (61) over $M$ and by the bound of $\mu(g_k, \frac{1}{2})$ and $R$ from (40) we get the estimate
\begin{equation}
\int_M |\nabla f_k|^2 dv_{g_k} = \int_M (f_k + R - \mu(g_k, \frac{1}{2}) - 2n) dv_{g_k}
\leq \int_M f_k dv_{g_k} + 2C_1 \text{Vol}_g(V).
\end{equation}

Another observation is that from the normalization $\int_M e^{-f_k} dv_{g_k} = (2\pi)^n$,
\begin{equation}
\text{Vol}_g(\{ f_k \leq \log V \}) \geq e^{-C_7}
\end{equation}
where $C_7$ is the constant in (63).
Denote the unit measure \( d\mu_k = (2\pi)^{-n} e^{-u_k} dv_{g_k} \) where \( u_k = u(t_k) \) is the normalized Ricci potential of \( g_k \). Recall the Poincaré inequality on a Kähler manifold which satisfies (38), cf. [16],

\[
\int_M \phi^2 d\mu_k \leq \int_M |\nabla \phi|^2 d\mu_k + \left( \int_M \phi d\mu_k \right)^2, \quad \forall \phi \in C^\infty(M).
\]

Set \( A = \max(\log V, C_7) \). Substituting \( f_k \) into the Poincaré inequality gives the estimate

\[
\int_M |\nabla f_k|^2 d\mu_k \geq \int_M f_k^2 d\mu_k - \left( \int_M |f_k| d\mu_k \right)^2 \\
\geq \int_M f_k^2 d\mu_k - \left( \int_{\{f_k > A\}} |f_k| d\mu_k + AV \right)^2 \\
\geq \int_M f_k^2 d\mu_k - \left( \int_{\{f_k > A\}} |f_k| d\mu_k \right)^2 - 2AV \int_M |f_k| d\mu_k - A^2 V^2.
\]

By Schwarz inequality

\[
\left( \int_{\{f_k > A\}} |f_k| d\mu_k \right)^2 \leq \int_{\{f_k > A\}} f_k^2 d\mu_k \cdot \int_{\{f_k > A\}} d\mu_k
\]

where \( \int_{\{f_k > A\}} d\mu_k \) can be estimated as follows

\[
\int_{\{f_k > A\}} d\mu_k = 1 - \int_{\{f_k \leq A\}} (2\pi)^{-n} e^{-u_k} dv_{g_k} \\
\leq 1 - (2\pi)^{-n} e^{-C_1 \text{Vol}_{g_k}\{f_k \leq A\}} \\
\leq 1 - (2\pi)^{-n} e^{-C_1 - C_7}.
\]

Plugging the estimate of \( \int_M |\nabla f_k|^2 d\mu_k \) into (69) gives

\[
\int_M |\nabla f_k|^2 d\mu_k \geq (2\pi)^{-n} e^{-C_1 - C_7} \int_M f_k^2 d\mu_k - 2AV \int_M |f_k| d\mu_k - A^2 V^2.
\]

Let \( C_{10} = (2\pi)^n e^{C_1 + C_7} \). Then, combining with (68) and (72) yields

\[
\int_M f_k^2 dv_{g_k} \leq C_{10} \int_M |\nabla f_k|^2 dv_{g_k} + 2AV C_{10} \int_M |f_k| dv_{g_k} + A^2 V^2 C_{10} \\
\leq C_{10} (2AV + 1) \int_M |f_k| dv_{g_k} + C_{10} (A^2 V^2 + 2C_1 V) \\
\leq C_{10} (2AV + 1) V^{1/2} \left( \int_M f_k^2 dv_{g_k} \right)^{1/2} + C_{10} (A^2 V^2 + 2C_1 V).
\]

One can derive easily a uniform upper bound of \( \int_M f_k^2 dv_{g_k} \). \( \square \)

The \( L^\infty \) bound of \( f_k \) follows from a standard Moser’s iteration argument.
Lemma 4.6. There exists $C_{11}$ independent of $k$ such that
\[
\text{sup } f_k \leq C_{11}.
\]

Proof. Let $\tilde{f}_k = f_k + C_7 + 1$ for $C_7$ in (63). It suffices to show a uniform upper bound of $\tilde{f}_k$. Obviously from (65),
\[
2\Delta \tilde{f}_k - |\nabla \tilde{f}_k|^2 + \tilde{f}_k = \mu_k - R + 2n + C_7 + 1,
\]
where $\mu_k = \mu(g_k, \frac{1}{2})$. Notice that $\tilde{f}_k \geq 1$. Multiplying $\tilde{f}_k^{p-1}$ for any $p > 1$ onto above identity and integrating by parts, we get
\[
\int_M (\mu_k - R + 2n + C_7 + 1) \tilde{f}_k^{p-1} dv_{g_k} = \int_M (2\Delta \tilde{f}_k - |\nabla \tilde{f}_k|^2 + \tilde{f}_k) \tilde{f}_k^{p-1} dv_{g_k}
\]
\[
\leq \int_M (2\tilde{f}_k^{p-1} \Delta \tilde{f}_k + \tilde{f}_k^p) dv_{g_k}
\]
\[
= -8(p - 1) \int_M |\nabla \tilde{f}_k^{p/2}|^2 dv_{g_k} + \int \tilde{f}_k^p dv_{g_k}.
\]
Rearranging the terms and using the bound of $\mu_k$ and $R$ by (40),
\[
\int_M |\nabla \tilde{f}_k^{p/2}|^2 dv_{g_k} \leq pC_{12} \int_M \tilde{f}_k^p dv_{g_k}, \quad \forall p \geq 2,
\]
where $C_{12}$ is a constant independent of $k$ and $p \geq 2$.

Define a sequence of positive numbers $p_i = 2 \cdot (\frac{n}{n-1})^i$, $i = 0, 1, 2, \cdots$, and apply the inequality (77) to each $p_i$. Applying the Sobolev inequality (64) to $\tilde{f}_k$ we obtain
\[
\left( \int_M \tilde{f}_k^{p_i} dv_{g_k} \right)^{\frac{1}{p_i}} \leq C_8 \left( \int_M |\nabla \tilde{f}_k^{p_i/2}|^2 + \tilde{f}_k^{p_i} \right) dv_{g_k}
\]
\[
\leq p_i C_{13} \int_M \tilde{f}_k^p dv_{g_k}
\]
where $C_{13}$ is a uniform constant independent of $k$ and $i$. It implies,
\[
\|\tilde{f}_k\|_{L^{p_{i+1}}} \leq p_i^{\frac{1}{p_i}} (2C_{13})^{\frac{1}{p_i}} \|\tilde{f}_k\|_{L^{p_i}}, \quad \forall i \geq 0.
\]
An iteration argument yields the estimate
\[
\|\tilde{f}_k\|_{L^\infty} \leq C_{14} \int_M \tilde{f}_k^2 dv_{g_k},
\]
for a uniform constant $C_{14}$. Combining with the $L^2$ estimate of $f_k$ in above lemma gives the desired upper bound of $\tilde{f}_k$ in this case. \qed

Let $g_k(t) = g(t_k + t)$ be the sequence of Kähler-Ricci flow as before and $f_k(t)$ be the associated solution to the backward heat equation
\[
\frac{\partial}{\partial t} f_k = -\triangle f_k + |\nabla f_k|^2 - R + n
\]
with initial value at 0 the minimizer $f_k$. Then the maximal principle derives the two-sided bound
\begin{equation}
\sup |f_k(t)| \leq C_{16}, \quad \forall -1 \leq t \leq 0
\end{equation}
for some uniform constant $C_{16}$.

Now we are ready to show that $g_\infty$ is a Kähler-Ricci soliton.

**Claim 4.7.** $g_\infty$ satisfies the Kähler-Ricci soliton equation on $\mathcal{R}$.

**Proof.** This follows essentially from the monotonicity of Perelman’s entropy functional $\mu(g(t), \frac{1}{2})$ under the original Kähler-Ricci flow $g(t)$ on $M$.

As before, let $f_k$ be one associated minimizer of $\mu(g_k, \frac{1}{2})$ and $v_k = e^{-f_k/2}$. Let $K_\rho = \{ x \in \mathcal{R} | d(x, S) \geq \rho \}$ for any $\rho > 0$. Then, by (59) and Shi’s local gradient estimate for curvature $[24, 18]$, the geometry of $\psi(K_\rho)$ is $C^\infty$ uniformly bounded whenever $k$ is large enough. The elliptic regularity theory to (52) gives the $C^\infty$ bound of $v_k$ as well as $f_k$ on $\psi(K_\rho)$. Hence, passing a subsequence and letting $\rho \to 0$, $f_k$ will converge smoothly to a function $f_\infty$ on $\mathcal{R}$.

Another implication of (59) is that by Hamilton’s compactness theorem for Ricci flow, for any fixed $\rho$ there exists $\tau_\rho > 0$ such that, $(\psi_k(K_\rho), g_k(t))$ converges along a subsequence to a Kähler-Ricci flow $(M, g_{\rho, \infty}(t))$ on some time interval $[-\tau_\rho, 0]$. Since $(\psi_k(K_\rho), g(t_k)) \xrightarrow{C^\infty} (K_\rho, g_\infty)$ by Claim 4.3, we may assume that, up to an isometry, $M_\rho = K_\rho$ and $g_{\rho, \infty}(0) = g_\infty$ on $K_\rho$.

The remaining parts of the proof are standard, cf. [22, 23]: Let $f_k(t)$ be the solution to (81) on the time interval $[-\tau_\rho, 0]$. By (82), $f_k$ is $C^0$ uniformly bounded on $M \times [-\tau_\rho, 0]$. The regularity theory for (local) heat equation (81) gives the bound
\begin{equation}
\|\nabla^l f_k\|_{\psi_k(K_\rho)} \leq C_{l, \rho}, \quad \forall t \in [-\tau_\rho, 0]
\end{equation}
whenever $k$ is large enough, where $C_{l, \rho}$ is a uniform constant depending only on $l$ and $\rho$. In particular, $f_k(t)$ converges along a subsequence to a family of smooth functions $f_{\rho, \infty}(t)$ on $K_i \times [-\tau_\rho, 0]$ with $f_\infty(0) = f_\infty|_{K_\rho}$.

By Perelman’s monotonicity formula [20], also see [23],
\[
\frac{d}{dt} \mathcal{W}(g(t), f_k(t), \frac{1}{2}) = \int_M \left( |\nabla f_k + \nabla_\rho \nabla f_k - g_{ij} |^2 + 2 |\nabla_\rho \nabla f_k |^2 \right) (2\pi)^{-n} e^{-f_k} dv_{g(t)}.
\]
In particular, $\mu(g(t), \frac{1}{2})$ is increasing and
\[
\int_{\tau_k - \rho}^{\tau_k} \int_M \left( |\nabla f_k + \nabla_\rho \nabla f_k - g_{ij} |^2 + 2 |\nabla_\rho \nabla f_k |^2 \right) (2\pi)^{-n} e^{-f_k} dv dt
\leq \mu(g(t_k), \frac{1}{2}) - \mu(g(t_k - \tau_\rho), \frac{1}{2}).
\]
By the monotonicity and boundedness of \( \mu(g(t), \frac{1}{2}) \),
\[
\mu(g_k(0), \frac{1}{2}) - \mu(g_k(-\tau_\rho), \frac{1}{2}) = \mu(g(t_k), \frac{1}{2}) - \mu(g(t_k - \tau_\rho), \frac{1}{2}) \to 0, \quad \text{as } k \to \infty.
\]
Passing to the limit space \( (K_\rho, g_{\rho, \infty}) \), this implies that for any \( \rho > 0 \),
\[
\int_{-\tau_\rho}^{0} \int_{K_\rho} (|\text{Ric} + \nabla \bar{\nabla} f_{\infty} - g_{\rho, \infty}|^2 + 2|\nabla \nabla f_{\infty}|^2)(2\pi)^{-n} dv dt
\leq e^{C_{16}} \lim_{k \to \infty} \int_{-\tau_\rho}^{0} \int_{\psi_k(K_\rho)} (|\text{Ric} + \nabla \bar{\nabla} f_k - g_k|^2 + 2|\nabla \nabla f_k|^2)(2\pi)^{-n} e^{-f_{k, \infty}} dv dt
\leq e^{C_{16}} \lim_{k \to \infty} \left( \mu(g_k(0), \frac{1}{2}) - \mu(g_k(-\tau_\rho), \frac{1}{2}) \right) = 0.
\]
where \( C_{16} \) is the constant in (82). That is, the couple \((g_{\rho, \infty}, f_{\rho, \infty})\) satisfies
\[
\begin{align*}
\text{Ric} + \nabla \bar{\nabla} f_{\rho, \infty} &= g_{\rho, \infty}, \\
\nabla \nabla f_{\rho, \infty} &= 0,
\end{align*}
\]
on the space time \( K_\rho \times [-\tau_\rho, 0] \). In particular, at time 0,
\[
\begin{align*}
\text{Ric} + \nabla \bar{\nabla} f_{\infty} &= g_{\infty}, \\
\nabla \nabla f_{\infty} &= 0,
\end{align*}
\]
which means that \( g_{\infty} \) satisfies the shrinking Kähler-Ricci soliton equation on each \( K_\rho \) with potential \( f_{\infty} \). The proof then follows from the arbitrariness of \( \rho \). \( \square \)

Summing up Claims 4.3 and 4.7 proves our theorem 4.1.

We finally show that the limits of \( f_k \) and \( u_k \) coincide. By the \( C^1 \) bound of \( u_k = u(t_k) \), cf. (40), the Ricci potentials \( u_k \) will converge along a subsequence to a Lipschitz function \( u_\infty \) on \( Y \). Applying the elliptic regular theory to \( \triangle u(t_k) + R(g_k) = n \) shows that \( u_\infty \) is actually smooth on \( \mathcal{R} \). It is clear that \( u_\infty \) is the Ricci potential of \( g_\infty \):
\[
R_{ij}(g_\infty) + \nabla_i \nabla_j u_\infty = g_\infty,_{ij}.
\]
We have the following proposition.

**Proposition 4.8** (Compare Proposition 14 in [23]). \( f_\infty = u_\infty \) on \( \mathcal{R} \). In particular, \( f_\infty \) can be extended to be a globally Lipschitz function on \( Y \).

**Proof.** We first show that \( f_\infty \) is globally Lipschitz on \( \mathcal{R} \) and so admits a natural extension over \( Y \). From the Kähler-Ricci soliton equation (85),
using second Bianchi identity, one verifies easily the following identity, cf. 
\[18\] for example,
\[ R(g_\infty) + |\nabla f_\infty|^2 = f_\infty + \text{const.}, \quad \text{on } \mathcal{R}. \tag{87} \]

We mention that one should apply the connectedness of \( \mathcal{R} \) to derive the identity. By Perleman’s estimate to \( R(g_k) \) and estimate \(82\), we have the uniform bound \( |R(g_\infty)| \leq C_1 \) and \( |f_\infty| \leq C_16 \) on \( \mathcal{R} \) because of the smooth convergence there. It concludes the uniform bound of \( |\nabla f_\infty| \) on \( \mathcal{R} \).

We next show that \( \nabla (u_\infty - f_\infty) = 0 \) on \( \mathcal{R} \). Combining with the normalization
\[ \int_{\mathcal{R}} e^{-f_\infty} dv_{g_\infty} = \int_{\mathcal{R}} e^{-u_\infty} dv_{g_\infty} = (2\pi)^n \tag{88} \]
which follows directly from the limiting process, we conclude that \( u_\infty = f_\infty \).

To prove \(88\), it suffices to show \( \int_{\mathcal{R}} |\nabla (u - f)|^2 dv_{g_\infty} = 0 \). To this aim, choose a sequence of compact submanifolds \( V_i \subset \mathcal{R} \) such that \( \bigcup V_i = \mathcal{R} \) and \( \operatorname{Vol}(\partial V_i) \to 0 \) as \( i \to \infty \). This can be done since the boundary \( \partial \mathcal{R} = \mathcal{S} \) has codimension at least 4. Then take integration by parts,
\[ \int_{V_i} |\nabla (u_\infty - f_\infty)|^2 dv_{g_\infty} = - \int_{\partial V_i} (u_\infty - f_\infty) \langle \nabla (u_\infty - f_\infty), \mu \rangle \\
- \int_{V_i} (u_\infty - f_\infty) \Delta (u_\infty - f_\infty) dv_{g_\infty} \leq \sup |(u_\infty - f_\infty) \nabla (u_\infty - f_\infty)| \operatorname{Vol}(\partial V_i) \tag{89} \]
which tends to 0 as \( i \to \infty \). Here we used that \( \Delta u_\infty = \Delta f_\infty \) by equations \(84\) and \(85\). The proof is now completed. \qed

We end the paper with several remarks.

**Remark 4.9.** By the arguments in \[22\], one can show that the metric \( d \) (respectively \( g_\infty \)) can be extended to be a family of metrics \( d_t, t \in (-\infty, \infty), \) with \( d_0 = d \) (respectively \( g_\infty(t) \)) such that \( (M, g(t_k + t)) \xrightarrow{d_{C^\infty}} (Y, d_t) \) (respectively \( \psi_k^* g(t_k + t) \xrightarrow{C^\infty} g_\infty(t) \)) at each time \( t \).

**Remark 4.10.** It should be true that the limit \( g_\infty(t) \) is independent of the choice of the sequence \( t_k \).

**Remark 4.11.** Let \( (M, g(t)) \) be a Kähler-Ricci flow without assumption \(42\) in a priori. Then \( (M, g(t)) \) converges subsequentially to a compact metric space \( (Y, d) \). Suppose \( N \subset Y \) is a smooth subset on which \( d \) is induced by a smooth metric \( g_\infty \). If \( g(t) \xrightarrow{C^\infty} g_\infty \), then \( g_\infty \) satisfies a shrinking Kähler-Ricci soliton equation on \( N \).
REFERENCES

[1] M. T. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds*, Jour. Amer. Math. Soc., 2 (1989), 455-490.

[2] M. T. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math., 102 (1990), 429-445.

[3] S. Bando, A. Kasue and H. Nakajima, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth*, Invent. Math., 97 (1989), 313-349.

[4] A. L. Besse, *Einstein Manifolds*, Berlin Heidelberg, Springer-Verlag, (2007).

[5] H.D. Cao and N. Sesum, *A compactness result for Kähler Ricci solitons*, Advan. Math., 211 (2007), 794-818.

[6] J. Cheeger, *Integral bounds on curvature, elliptic estimates and rectifiability of singular sets*, Geom. Funct. Anal., 13 (2003), 20-72.

[7] J. Cheeger, *Degeneration of Einstein metrics and metrics with special holonomy*, Surv. Differ. Geom., VIII, 29-73. Int. Press, Somerville, MA, 2003.

[8] J. Cheeger, *Degeneration of Riemannian Metrics under Ricci Curvature Bounds*, Lezione Fermiane, Academia Nazionale dei Lincei, Pisa: Scuola Normale Superiore, 2001.

[9] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. I.*, J. Diff. Geom., 45 (1997), 406-480.

[10] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. II.*, J. Diff. Geom., 54 (2000), 13-35.

[11] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. III.*, J. Diff. Geom., 54 (2000), 37-74.

[12] J. Cheeger, T. H. Colding and G. Tian, *On the singularities of spaces with bounded Ricci curvature*, Geom. Funct. Anal., 12 (2002), 873-914.

[13] J. Cheeger and G. Tian, *Anti-self-duality of curvature and degeneration of metrics with special holonomy*, Comm. Math. Phys., 255 (2005), 391-417.

[14] F.Q. Fang, Y.G. Zhang and Z.L. Zhang, *Maximal solutions of normalized Ricci flow on 4-manifolds*, Comm. Math. Phys., 283 (2008), 1-24.

[15] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Berlin Heidelberg, Springer-Verlag, (2001).

[16] A. Futaki, *Kähler-Einstein Metrics and Integral Invariants*, Lect. Notes Math., 1314, Springer-Verlag, Berlin, 1988.

[17] R. S. Hamilton, *A compactness property for solutions of the Ricci flow*, Amer. Jour. Math., 117 (1995), 545-572.

[18] R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surv. Diff. Geom., 2 (1995), 7-136.

[19] T. Ivey, *Ricci solitons on compact three-manifolds*, Diff. Geom. Appl., 3 (1993), 301-307.

[20] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*. arXiv:math.DG/0211159

[21] O. S. Rothaus, *Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators*, Jour. Funct. Anal., 42 (1981), 110-120.

[22] N. Sesum, *Convergence of a Kähler-Ricci flow*, Math. Res. Letters, 12 (2005), -632.

[23] N. Sesum and G. Tian, *Bounding scalar curvature and623 diameter along the Kähler Ricci flow (after Perelman) and some applications*, Jour. Inst. Math. Juss. 7 (2008), 575-587.
[24] W.X. Shi, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J. Diff. Geom., 30 (1989), 303-394.

[25] G. Tian, *On Calabi’s conjecture for complex surfaces with positive first Chern class*, Invent. Math., 101 (1990), 101-172.

[26] G. Tian and X.H. Zhu, *A new holomorphic invariant and uniqueness of Kähler-Ricci solitons*, Comment. Math. Helv., 77 (2002), 297-325.

[27] B. Wang and X.X. Chen, *Space of Ricci flows (I)*, arXiv:0902.1545 [math.DG]

[28] B. Weber, *Convergence of compact Ricci solitons*, arXiv:0804.1158 [math.DG]

[29] R.G. Ye, *The logarithmic Sobolev inequality along the Ricci flow*, arXiv:0707.2424

[30] Q.S. Zhang, *A uniform Sobolev inequality under Ricci flow*, Int. Math. Res. Notices, 2007 (2007), 1-17.

[31] X. Zhang *Compactness theorems for gradient Ricci solitons*, Jour. Geom. Phys., 56 (2006), 2481-2499.

[32] Z.L. Zhang, *Compactness, finiteness and rigidity of closed shrinking Ricci solitons*, preprint.

[33] Z.L. Zhang, *Degeneration of shrinking Ricci solitons*, Int. Math. Res. Notices, doi:10.1093/imrn/rnq020.

School of Mathematics, Peking University, Beijing, China

Department of Mathematics, Princeton University, Princeton NJ 08544

E-mail address: tian@math.princeton.edu

School of Mathematics, Capital Normal University, Beijing, 100048, China

Department of Mathematics, Princeton University, Princeton NJ 08544

E-mail address: zhleigo@yahoo.com.cn