Number Theory

Eisenstein cohomology and ratios of critical values of Rankin–Selberg
$L$-functions

Cohomologie d’Eisenstein et rapports de valeurs critiques des fonctions $L$
de Rankin–Selberg

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This is an announcement of results on rank-one Eisenstein cohomology of $GL_N$, with
$N \geq 3$ an odd integer, and algebraicity theorems for ratios of successive critical values of certain
Rankin–Selberg $L$-functions for $GL_n \times GL_{n'}$ when $n$ is even and $n'$ is odd.

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Résumé

Cette Note annonce des résultats sur la cohomologie d’Eisenstein de rang 1 de $GL_N$, avec
$N \geq 3$ un entier impair, et donne des théorèmes d’algébricité pour les rapports de valeurs
critiques successives de certaines fonctions $L$ de Rankin–Selberg pour $GL_n \times GL_{n'}$ lorsque $n$
est pair et $n'$ est impair.

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Version française abrégée

Soit $\sigma_f \in \text{Coh}(GL_n, \lambda)$, ce qui signifie que $\sigma_f$ est un $GL_n(A_f)$-facteur de la cohomologie intérieure $H^*_c(S^\text{GL}_n, E_\lambda)$ d’un espace $S^\text{GL}_n := GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{A}) / \mathbb{K}^0 F_f$ à coefficients dans le faisceau $E_\lambda$ provenant d’une représentation irréductible algébrique de plus haut poids $\lambda$, cf. Section 1. Quand $n$ est pair et $\lambda$ est régulier, un tel $\sigma_f$ apparaît deux fois dans $H^*_c(S^\text{GL}_n, E_\lambda)$ pour

$$\bullet = \frac{n^2}{4}.$$ En comparant ces deux copies de $\sigma_f$, on en déduit une période $\Omega^4(\sigma_f, t) \in \mathbb{C}^\times$, où $t$ est un plongement du corps de rationalité de $\sigma_f$ dans la clôture algébrique de $\mathbb{Q}$ dans $\mathbb{C}$, cf. définition 2.1.

Soit maintenant $\sigma_f' \in \text{Coh}(GL_{n'}, \lambda')$ pour un entier impair $n'$. Posons $N = n + n'$. Soit $m \in \frac{1}{2} + \mathbb{Z}$ tel que $m$ et $m + 1$ soient critiques pour la fonction $L$ de Rankin–Selberg $L(\sigma_f \times \sigma_f^{\text{tr}}', t, s)$. En supposant la validité d’un certain lemme combinatoire (voir Conjecture 5.1) notre résultat principal sur les valeurs critiques affirme que

$$\frac{1}{2} \Lambda(\sigma_f \times \sigma_f^{\text{tr}}', t, m) \Lambda(\sigma_f \times \sigma_f^{\text{tr}}', t, m + 1)$$

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est algébrique et Galois-équivariant, cf. Théorème 4.1. Ici \( \Lambda(\sigma_f \times \sigma'_f, \iota, s) \) est la fonction \( L \) complétée.

Le Théorème 4.1 se démontre en étudiant l’image (appelée « cohomologie d’Eisenstein ») de la cohomologie globale \( H^\ast(S^\ast G, E_\lambda) \) dans la cohomologie \( H^\ast(\delta S^\ast G, E_\mu) \) de la frontière de Borel–Serre \( \delta S^\ast G \) de \( S^\ast G \). Nous étudions en particulier ceci pour la cohomologie en degré \( * = (N^2 - 1)/4 \) et pour un plus haut poids \( \mu \) qui dépend des poids \( \lambda \) et \( \lambda' \) via le lemme combinatoire. Le Théorème 5.2 donne une caractérisation de cette image.

1. The general situation

Let \( G/Q \) be a connected split reductive algebraic group over \( Q \) whose derived group \( G^{(1)}/Q \) is simply connected. Let \( Z/Q \) be the center of \( G \) and let \( S \) be the maximal \( Q \)-split torus in \( Z \). Let \( C_\infty \) be a maximal compact subgroup of \( G(R) \) and let \( K_\infty = C_\infty S(R)^0 \). The connected component of the identity of \( K_\infty \) is denoted \( K'_\infty \) and \( K'/K'_\infty = \pi_0(K_\infty) \to \pi_0(G(R)) \). Let \( K_f = \bigcap \pi_p \subset G(\mathbb{A}_f) \) be an open compact subgroup; here \( \mathbb{A} \) is the adèle ring of \( Q \) and \( \mathbb{A}_f \) is the ring of finite adèles. The locally symmetric space of \( G \) with level structure \( K_f \) is defined as

\[
\mathcal{S}^G_{K_f} := G(Q) \backslash \mathcal{A}(\mathbb{A})/K'_\infty K_f.
\]

(For the following see Harder [6, Chapter 3, Sections 2, 2.1, 2.2] for details.) For a dominant integral weight \( \lambda \), let \( E_\lambda \) be an absolutely irreducible finite-dimensional representation of \( G/Q \) with highest weight \( \lambda \), and let \( \mathcal{E}_\lambda \) denote the associated sheaf on \( \mathcal{S}^G_{K_f} \). We have an action of the Hecke-algebra \( \mathcal{H} = \mathcal{H}^G_{K_f} = \bigotimes_p \mathcal{H}_p \) on the cohomology groups \( H^\ast(\mathcal{S}^G_{K_f}, \mathcal{E}_\lambda) \).

We always fix a level, but sometimes drop it in the notation. For any finite extension \( F/Q \), let \( E_{\lambda, F} := E_\lambda \otimes_Q F \), then \( E_{\lambda, F} \) is the corresponding sheaf on \( \mathcal{S}^G_{K_f} \).

Let \( \mathcal{S}^G_{K_f} \) be the Borel–Serre compactification of \( \mathcal{S}^G_{K_f} \), i.e., \( \mathcal{S}^G_{K_f} = \mathcal{S}^G_{K_f} \cup \partial_p \mathcal{S}^G_{K_f} \), with \( P \) running through the conjugacy classes of proper parabolic subgroups defined over \( Q \). The sheaf \( E_{\lambda, F} \) naturally extends, using the definition of the Borel–Serre compactification, to a sheaf on \( \mathcal{S}^G_{K_f} \), which we also denote by \( E_{\lambda, F} \). Restriction from \( \mathcal{S}^G_{K_f} \) to \( \mathcal{S}^G_{K_f} \) in cohomology induces an isomorphism \( H^\ast(\mathcal{S}^G_{K_f}, \mathcal{E}_\lambda) \to H^\ast(\mathcal{S}^G_{K_f}, \mathcal{E}_\lambda) \).

Our basic object of interest is the following long exact sequence of \( \pi_0(K_\infty) \times \mathcal{H} \)-modules

\[
\cdots \longrightarrow H^\ast_c(\mathcal{S}^G_{K_f}, \mathcal{E}_\lambda) \overset{\iota^\ast}{\longrightarrow} H^\ast(\mathcal{S}^G_{K_f}, \mathcal{E}_\lambda) \overset{\iota^\ast}{\longrightarrow} H^\ast(\partial \mathcal{S}^G_{K_f}, \mathcal{E}_\lambda) \longrightarrow H^{\ast+1}(\mathcal{S}^G_{K_f}, \mathcal{E}_\lambda) \longrightarrow \cdots.
\]

The image of cohomology with compact supports inside the full cohomology is called inner or interior cohomology and is denoted \( H^\ast_\iota := \text{Image}(\iota^\ast) = \text{Im}(H^\ast \to H^\ast) \). The theory of Eisenstein cohomology is designed to describe the image of the restriction map \( \iota^\ast \). Our goal is to study the arithmetic information contained in the above exact sequence.

The inner cohomology is a semi-simple module for the Hecke-algebra. (See Harder [6, Chap. 3, 3.3.5].) After a suitable finite extension \( F/Q \), where \( Q \subset F \subset \bar{Q} \subset C \), we have an isotypical decomposition

\[
H^1(\mathcal{S}^G_{K_f}, \mathcal{E}_{\lambda, F}) = \bigoplus_{\pi_f \in \text{Coh}(G, K_f, \lambda)} H^1(\mathcal{S}^G_{K_f}, \mathcal{E}_{\lambda, F})(\pi_f)
\]

where \( \pi_f \) is an isotypism type of an absolutely irreducible \( \mathcal{H} \)-module, i.e., an \( F \)-vector space \( H_{\pi_f} \) with an absolutely irreducible action of \( \mathcal{H} \). The local factors \( \mathcal{H}_p \) are commutative outside a finite set \( V = V_{K_f} \) of primes and the factors \( \mathcal{H}_p \) and \( \mathcal{H}_q \), for \( p \neq q \), commute with each other. Hence for \( p \notin V \) the commutative algebra \( \mathcal{H}_p \) acts on \( H_{\pi_p} \) by a homomorphism \( \pi_p : \mathcal{H}_p \to \mathbb{F} \). Let \( H_{\pi_p} \) be the one dimensional vector space \( F \) with basis \( 1 \in F \) with the action \( \pi_p \) on it. Then \( H_{\pi_p} = \bigotimes_{p \in V} H_{\pi_p} \bigotimes_{p \notin V} H_{\pi_p} = \bigotimes_{p \in V} H_{\pi_p} \). The set of isomorphism classes which occur in the above decomposition is called the ‘spectrum’ \( \text{Coh}(G, K_f, \lambda) \). If we restrict the elements of the Galois group \( \text{Gal}(\bar{Q}/Q) \) to \( F \) we get the conjugate embeddings of \( F \) into \( \bar{Q} \); we introduce \( \mathcal{I}(F) = \{ i : F \to \bar{Q} \} = \{ i : F \to \bar{Q} \} \). For \( i \in \mathcal{I}(F) \) define \( i \circ \pi_f \) as \( H_{\pi_f} \otimes_{F, i} \mathbb{C} \). We define the rationality field of \( \pi_f \) as \( \mathbb{Q}(\pi_f) = \{ x \in F \mid i(x) = i' \} \) if \( i \circ \pi_f = i' \circ \pi_f \).

2. The case of \( \text{GL}_n \) and the definition of relative periods when \( n \) is even

Let \( T/Q \) be a maximal \( Q \)-split torus in \( G \), let \( T^{(1)} = T \cap T^{(1)} \). Let \( X^\ast(T) \) be its group of characters then restriction of characters gives an inclusion \( X^\ast(T) \subset X^\ast(T^{(1)}) \oplus X^\ast(Z) \) and after tensoring by \( Q \) this becomes an isomorphism. Any \( \lambda \in X^\ast(T) \) can be written as \( \lambda = \delta + \delta' \in X^\ast(T^{(1)}) \oplus X^\ast(Z) = X^\ast_G(T^{(1)}) \).

Consider the case \( G = \text{GL}_n/Q \). Take a regular essentially self-dual dominant integral highest weight \( \lambda \). Let \( \rho \in X^\ast_G(T^{(1)}) \) be half the sum of positive roots, and write \( \lambda + \rho = a_1 \gamma_1 + \cdots + a_{n-1} \gamma_{n-1} + d \cdot \det \), which is an equation in \( X^\ast_G(T) \); the \( \gamma_i \in X^\ast_G(T) \) restrict to the fundamental weights in \( X^\ast(T^{(1)}) \) and are trivial on the center \( Z \). Regular, dominant and integral mean that \( a_i \geq 2 \) are integers, and essentially self-dual means \( a_i = a_{n-i} \). Further, for such a weight \( \lambda \) we have \( 2d \in \mathbb{Z} \) and it satisfies the parity condition:

\[
2d \equiv w + n - 1 \pmod{2}
\]

where \( w = w(\lambda) := \sum a_i \) is the ‘motivic weight’; see below.
Given such a $\lambda$, there is a unique essentially unitary Harish-Chandra module $H_{\pi,\lambda}$ such that the relative Lie algebra cohomology group $H^* (\mathfrak{g}, K, H_{\pi,\lambda} \otimes \mathfrak{e}) \neq 0$. Let $L^2 (G(\mathbb{Q}) \backslash G(\mathbb{A})/K_f, \omega_{E,\lambda}^{-1})$ denote the discrete spectrum for $G(\mathbb{A})$ in the space of $L^2$-automorphic forms with level structure $K_f$ on which $Z(\mathbb{R})^\circ$ acts via the inverse of the central character of $E, \lambda$. For $\pi_f \in \text{Coh}(G(\mathbb{F}, \lambda)$ and $t \in \mathcal{F}_f$ we consider

$$ W\left(\pi^\circ_{\infty} \otimes t \circ \pi_f\right) = \text{Hom}_{(G, K, \mathfrak{g})} \left( (H_{\pi,\lambda} \otimes (H_{\pi,\lambda} \otimes F, i) \otimes C), L^2 (G(\mathbb{Q}) \backslash G(\mathbb{A})/K_f, \omega_{E,\lambda}^{-1}) \right) $$

which is one-dimensional due to multiplicity-one for the discrete spectrum of $G_\mathbb{A}$; the image is in fact the cuspidal spectrum by regularity of $\lambda$. (See, for example, Schwermer [11, Corollary 2.3].) We choose a generator $\Phi$ for $W(\pi^\circ_{\infty} \otimes t \circ \pi_f)$.

The summand $H^* (S_{K_f}^C, \epsilon, \lambda, \pi_f) (\pi_f)$ can be decomposed for the action of $\pi_0 (G(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ as

$$ H^* (S_{K_f}^C, \epsilon, \lambda, \pi_f) (\pi_f) = \bigoplus_{\epsilon : \pi_0 (G(\mathbb{R})) \rightarrow \mathbb{Z}/2\mathbb{Z}} H^* (S_{K_f}^C, \epsilon, \lambda, \pi_f) (\pi_f) (\epsilon). $$

The action of $\pi_0 (G(\mathbb{R})) = \pi_0 (K_{\mathbb{A}}) = K_{\mathbb{A}} / K_\mathbb{O}$ is via its action on $H^* (\mathfrak{g}, K, H_{\pi,\lambda} \otimes \mathfrak{e})$. (See, for example, Borel and Wallach [1, 1.5].) Therefore, we get

$$ \bigoplus_{\epsilon} W\left(\pi^\circ_{\infty} \otimes t \circ \pi_f\right) \otimes H^* (\mathfrak{g}, K, H_{\pi,\lambda} \otimes \mathfrak{e}) (\epsilon) \otimes H_{\pi,\lambda} \otimes F, i) \otimes C \rightarrow \bigoplus_{\epsilon} H^* (S^C_{K_f}, \epsilon, \lambda, \pi_f) (\pi_f) \otimes F, i) \otimes C (\epsilon). $$

Let $b_n = n^2 / 4$ if $n$ is even, and $(n^2 - 1) / 4$ if $n$ is odd. Since $\pi$ is cuspidal, it is well known (see, for example, Clozel [2]) that $\pi^\circ_{\infty}$ is irreducibly induced from essentially discrete series representations and that

$$ H^* (S^C_{K_f}, \epsilon, \lambda, \pi_f) (\pi_f) = \begin{cases} H^* (\mathfrak{g}, K, H_{\pi,\lambda} \otimes \mathfrak{e})_+ + H^* (\mathfrak{g}, K, H_{\pi,\lambda} \otimes \mathfrak{e})_+ & \text{if $n$ is even;} \\ H^* (\mathfrak{g}, K, H_{\pi,\lambda} \otimes \mathfrak{e})_- & \text{if $n$ is odd,} \end{cases} $$

where each piece on the right-hand side is one-dimensional, and $\epsilon$ is a canonical sign (see [10, Section 3.3]).

Now let $n$ be even. We will define certain periods that we call relative periods. We define consistent choices of generators

$$ \omega_+ \in \text{Hom}_{K_{\mathbb{A}}} \left( (A^b_n (\mathfrak{g}, t), H_{\pi,\lambda} \otimes \mathfrak{e})_+ \right), \quad \omega_- \in \text{Hom}_{K_{\mathbb{A}}} \left( (A^b_n (\mathfrak{g}, t), H_{\pi,\lambda} \otimes \mathfrak{e})_- \right). $$

from which we get isomorphisms

$$ (\Phi \otimes \omega_+ : t \circ \pi_f \rightarrow H^* (S^C_{K_f}, \epsilon, \lambda, \pi_f) (\pi_f) \otimes F, i) \otimes C. $$

Composing the inverse of one with the other gives a canonical transcendental isomorphism

$$ T^{\text{trans}} (\pi_f, t) = (\Phi \otimes \omega_-) \circ (\Phi \otimes \omega_+)^{-1} : H^* (S^C_{K_f}, \epsilon, \lambda, \pi_f) (\pi_f) \otimes F, i) \otimes C \rightarrow H^* (S^C_{K_f}, \epsilon, \lambda, \pi_f) (\pi_f) \otimes F, i) \otimes C. $$

This isomorphism does not depend on the choice of $\Phi$ or the pair $(\omega_+, \omega_-)$ because these are unique up to scalars which cancel out. On the other hand, we have an arithmetic isomorphism of $H^* (S^C_{K_f})$-modules

$$ T^{\text{arith}} (\pi_f) : H^* (S^C_{K_f}, \epsilon, \lambda, \pi_f) (\pi_f) \rightarrow H^* (S^C_{K_f}, \epsilon, \lambda, \pi_f) (\pi_f) $$

(3) which is unique up to an element in $\mathbb{Q}(\pi_f)$. Comparing (2) with (3) we get the following definition:

**Definition 2.1.** There is an array of complex numbers $\Omega (\pi_f) = (\ldots, \Omega (\pi_f, t), \ldots)_{t \in \mathcal{F}_f}$ defined by

$$ \Omega (\pi_f, t) T^{\text{trans}} (\pi_f, t) = T^{\text{arith}} (\pi_f) \otimes F, i) \otimes C. $$

Changing $T^{\text{arith}} (\pi_f)$ by an element $a \in \mathbb{Q}(\pi_f)$ changes the array into $(\ldots, \Omega (\pi_f, t) (a), \ldots)_{t \in \mathcal{F}_f} \otimes C$.

If we pass from $\lambda$ to $\lambda - l \cdot \det$ for an integer $l$, then we have a canonical isomorphism

$$ H^* (S^C_{K_f}, \epsilon, \lambda, \pi_f) (\pi_f) \rightarrow H^* (S^C_{K_f}, \epsilon, \lambda - l \cdot \det, \pi_f) (\pi_f) \otimes l \otimes l $$

under which the $\pm$ components are switched by $(-1)^l$. We get the following period relation:

$$ \Omega (\pi_f, t) = \Omega (\pi_f \otimes l, t) (-1)^l. $$

(4)
Remark 1. Since cuspidal automorphic representations of $\mathrm{GL}_n$ are globally generic we can also define periods by comparing rational structures on Whittaker models and cohomological realizations. The periods were denoted $p^\pm(\pi_f)$ in Raghuram and Shahidi [10] and they appear in algebraicity results for the central critical value of Rankin–Selberg $L$-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$; see Raghuram [9, Theorem 1.1]. The periods $p^\pm(\pi_f)$ depend on a choice of a nontrivial character of $\mathbb{Q}\setminus \mathbb{A}$ which is implicit in any discussion concerning Whittaker models. However, one may check that if we change this character then the period changes only by an element of $\mathbb{Q}(\pi_f)$. Further, it is an easy exercise to see that $\Omega(\pi_f) = p^+(\pi_f)/p^-(\pi_f)$ up to elements in $\mathbb{Q}(\pi_f)$. On the other hand, the definition of the relative periods $\Omega(\pi_f)$ does not require Whittaker models suggesting that it is far more intrinsic to the representation viewed as a Hecke-summand of global cohomology.

3. The case $G = \mathrm{GL}_n \times \mathrm{GL}_{n'}$ with $n$ even and $n'$ odd

Let $\sigma_f \in \text{Coh}(\mathrm{GL}_n, \lambda)$ and $\sigma_f' \in \text{Coh}(\mathrm{GL}_{n'}, \lambda')$. The level structures will be suppressed from our notation from now on. As before, the weights are written as $\lambda + \rho = a_1 \gamma_1 + \cdots + a_{n-1} \gamma_{n-1} + d \cdot \rho'$ and similarly $\lambda' + \rho' = a'_1 \gamma'_1 + \cdots + a'_{n'-1} \gamma'_{n'-1} + d' \cdot \rho'$, where $a_i = a_{n-i}$, $a'_i = a'_{n'-i}$, and again we assume regularity for both the weights. Let $G = \mathrm{GL}_n \times \mathrm{GL}_{n'}$, $\mu = \lambda + \lambda'$ and $\pi_f = \sigma_f \times \sigma_f'$. By the Künneth formula we get

$$H^*(S^G, E_{\mu, f})(\pi_f) = H^*(S^{\mathrm{GL}_n}, E_{\lambda, f})(\sigma_f) \otimes H^*(S^{\mathrm{GL}_{n'}}, E_{\lambda', f})(\sigma_f').$$

Using Grothendieck’s conjectural theory of motives, one supposes that there are motives $M_{\text{eff}}$ (resp., $M_{\text{eff}}'$) that are conjecturally attached to $\sigma_f$ (resp., $\sigma_f'$). (See, for example, [7].) We call a pair of integers $(p, q)$ a Hodge-pair for a motive $M$ if the Hodge number $h^{p,q}(M) \neq 0$. The Hodge-pairs of the motives $M_{\text{eff}}$ (resp., $M_{\text{eff}}'$) are expected to be $\{(0, 0), (w - a_1, a_1), \ldots, (0, 0)\}$ (resp., $\{(0, 0), (w - a'_1, a'_1), \ldots, (0, 0)\}$) where $w = \sum_{i=1}^{n-1} a_i$ (resp., $w' = \sum_{i=1}^{n'-1} a'_i$) are the motivic weights. The motives $M_{\text{eff}}$ (resp., $M_{\text{eff}}'$) are suitable Tate-twists of the motives expected to be attached to $\sigma_f$ (resp., $\sigma_f'$) as in Clozel [2, Conjecture 4.5]. The assertion about Hodge pairs may be verified by working with the representations at infinity and their associated local $L$-factors which determine the $\Gamma$-factors at infinity. The set of Hodge-pairs for $M_{\text{eff}} \otimes M_{\text{eff}}'$ are all the pairs of the form $(w - a_1- \cdots - a_i + w' - a'_i - \cdots - a'_j, a_1 + \cdots + a_i + a'_i + \cdots + a'_j)$. The motivic $L$-function $L(M_{\text{eff}} \otimes M_{\text{eff}}', s)$ is defined as in Deligne [3, (1.2.2)]. Intimately related to it is a ‘cohomological’ $L$-function $L^{\text{coh}}(\sigma_f \times \sigma_f', s)$ which is defined as an Euler product, where each Euler factor is expressed in terms of eigenvalues of certain normalized Hecke-operators acting on integral cohomology groups. Assume that the middle Hodge number of $M_{\text{eff}} \otimes M_{\text{eff}}'$ is zero, i.e., $h^{(w+w')/2, (w+w')/2} = 0$. Put $p(\mu) := \min \{p \mid w + w' \geq p > (w + w')/2, h^{p, w+w'-p} \neq 0\}$. Let $\sigma'\nu$ denote the contragredient of $\sigma'$. The critical points of $L^{\text{coh}}(\sigma_f \times \sigma_f', \iota, s)$ are the integers

$$\{p(\mu), p(\mu) - 1, \ldots, w + w' + 1 - p(\mu)\}.$$  

Note that this decreasing list of integers is centered around $(w + w' + 1)/2$ which is the center of symmetry of the cohomological $L$-function. The total number of critical integers is $2p(\iota) - (w + w')$. The cohomological $L$-function is up to a shift in the $s$-variable the usual automorphic Rankin–Selberg $L$-function $L(\sigma_f \times \sigma_f', \iota, s) := L((\iota \circ \sigma_f) \times (\iota \circ \sigma_f'), s)$ for which the functional equation is between $s$ and $1 - s$. More precisely, we have

$$L^{\text{coh}}(\sigma_f \times \sigma_f', \iota, s) = \frac{L(\sigma_f \times \sigma_f', s, \iota - \frac{w + w'}{2} + a(\mu))}{a(\mu)}$$

where $a(\mu) = d - d'$. The parity condition (1) when applied to both the weights $\lambda$ and $\lambda'$ implies that the shift $\frac{w + w'}{2} + a(\mu)$ in the $s$-variable is always a half-integer. Observe that the cohomological $L$-function is invariant under changing $\sigma$ to $\sigma \otimes | \iota |$ or $\sigma'$ to $\sigma' \otimes | \iota |$.

A celebrated conjecture of Deligne predicts the existence of two periods $\Omega(\mathbb{L}(M_{\text{eff}} \otimes M_{\text{eff}}', 1, s))$ obtained from the Betti and de Rham realizations of this motive that capture, up to prescribable powers of $(2\pi i)$, the possibly transcendentals of the critical values of $L(M_{\text{eff}} \otimes M_{\text{eff}}', 1, s)$. See [3, Conjecture 2.7, (3.1.2) and (5.1.8)] for a precise statement. Our main result on $L$-values is to be viewed from this perspective.

4. The main result on ratios of critical $L$-values

Theorem 4.1. Let $\sigma_f \in \text{Coh}(\mathrm{GL}_n, \lambda)$ and $\sigma_f' \in \text{Coh}(\mathrm{GL}_{n'}, \lambda')$. Assume that $n$ is even and $n'$ is odd. Let $m = 1/2 + m_0 \in 1/2 + \mathbb{Z}$ be a half-integer such that both $m$ and $m + 1$ are critical for $L(\sigma_f \times \sigma_f', \iota, s)$. Assuming the validity of a Combinatorial Lemma (see below) we have

$$\frac{1}{\Omega(\sigma_f, \iota) \Omega(\sigma_f', \iota) \Lambda(\sigma_f \times \sigma_f', \iota, m)} \in \mathcal{I}(F),$$

for any $\iota \in \mathcal{I}(F)$. Moreover, for all $\tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
parabolic subgroup of $\tilde{G}$ since $w \in E$.

Eisenstein cohomology gives a description of the image of the restriction map $\text{Res}$. We deduce that the standard intertwining operator $\text{Int}(\tau, \eta)$ for $GL_n$ runs through all the possible values $(n = 2)$ and it is actually an isotypical subspace. Hence, there is a Hecke-invariant projector $R_{\pi_f}$ to this subspace. The theory of Eisenstein cohomology gives a description of the image of the restriction map $r^*: H^b(\tilde{G}, E) \to H^b(\tilde{S}, E_{\tilde{\mu}})$.

Our main result on Eisenstein cohomology is the following:

**Theorem 5.2.** The image of $R_{\pi_f} \circ r^*$ is given by

$$R_{\pi_f} \circ r^*(H^b(\tilde{G}, E_{\tilde{\mu}})) \otimes \mathbb{C} = \left\{ \psi + \frac{C(\mu)}{\Omega(\sigma_f, \tau(w_0))} \cdot \frac{\Lambda^\text{coh}(\sigma_f \times j^{\omega_0}, \tau(w_0))}{\Lambda^\text{coh}(\sigma_f \times j^{\omega_0}, \tau(w_0) + 1)} \cdot T^{\text{arith}}(\pi_f, \tau(w_0))(\psi) \right\},$$

where $\psi$ is any class in $H^b(\tilde{G}, E_{\tilde{\mu}})(\pi_f)$ with $\pi_f = \sigma_f \otimes \sigma_{\tilde{\mu}}'$; the operator $T^{\text{arith}}(\sigma_f, \tau) = 1_{\sigma_f'}$ after using the Künneth-formula; $C(\mu)$ is a non-zero rational number; and the point of evaluation is $w_0 = \frac{w + w'}{2} - (m + N)$. (Note that $\Lambda^\text{coh}(\sigma_f \times j^{\omega_0}, \tau(w_0) + 1) = \Lambda^\text{coh}(\sigma_f \times j^{\omega_0}, \tau(w_0) + 1 - N/2))$.

Theorem 5.2 implies the rationality result stated in Theorem 4.1 for $m = -N/2$ because the ratio of $L$-values together with the period is the ‘slope’ of a rationally defined map. For an integer $l$, let us change $\sigma$ to $\sigma \otimes |y^l|$, then $\lambda$ changes to $-l \cdot \det$ and $a(\mu)$ changes to $a(\mu) - l$, however the possibilities for $l$ are restricted by the inequalities in the Combinatorial Lemma since $w, w'$ and $p(\mu)$ do not change. It may be verified using (5) that as $a(\mu)$ runs through all the possible values it can be taken as prescribed by the Combinatorial Lemma, the pair of numbers $v_0$ and $v_0 + 1$ run through all the successive critical arguments: Theorem 4.1 follows while using the period relations (4) for $\sigma_f$. The Combinatorial Lemma says that the theory of Eisenstein cohomology allows one to prove a rationality result for a ratio of successive $L$-values exactly when both the $L$-values are critical. (See also [5].)

The condition on $\mu$ imposed by the Combinatorial Lemma has certain strong implications on the situation that underlies Eisenstein cohomology. First, using Speh’s results (see, for example, [8, Theorem 10b]) on reducibility for induced representations for $GL_n(\mathbb{R})$, one sees that the representation $\text{Ind}_{P_{\text{P}}}^{GL_n(\mathbb{R})}((\pi_\lambda \otimes \pi_{\lambda}) \otimes (\pi_{\lambda}'))$ of $GL_n(\mathbb{R})$ obtained by un-normalized parabolic induction is irreducible. Next, using Shahidi’s results [12] on local factors and the fact that $v_0$ and $v_0 + 1$ are critical, we deduce that the standard intertwining operator $A_{\lambda}$ from the above induced representation to the representation similarly induced from $Q_\infty$ is both holomorphic and nonzero at $s = v_0$. The choice of bases $\omega_\pm$ fixes a basis for the one-dimensional...
space $H^N(gl_N, K_{\infty}) \otimes \text{Ind}_{\mathbb{A}_{\infty}}^{GL_N(R)}(\sigma^\lor_{\infty} \otimes \sigma_{\infty}^\lor \otimes E_{\hat{\mu}})$. The map induced by $A_{\infty}$ at the level of $(gl_N, K_{\infty})$-cohomology is then a nonzero scalar. This scalar is a power of $(2\pi i)$ times a rational number $C(\mu)$. The power of $(2\pi i)$ gives the ratio of $L$-factors at infinity hence giving us a statement for completed $L$-functions, and the quantity $C(\mu)$ is expected to be a simple number as was verified for $GL_3$ by Harder [4].

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