DECOMPOSITION OF HYPERCUBES INTO SUNLET GRAPHS

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Abstract
For any positive integer $k \geq 3$, the sunlet graph of order 2$k$, denoted by $L_{2k}$, is the graph obtained by adding a pendant edge to each vertex of a cycle of length $k$. In this paper, we prove that the necessary and sufficient condition for the existence of an $L_{16}$-decomposition of the $n$-dimensional hypercube $Q_n$ is $n = 4$ or $n \geq 6$. Also, we prove that for any integer $m \geq 2$, $Q_{mn}$ has an $L_{2k}$-decomposition if $Q_n$ has a $C_k$-decomposition.

Keywords: decomposition, hypercube, sunlet graph

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1 Introduction

All graphs under consideration are simple and finite. For any positive integer $n$, the hypercube of dimension $n$, denoted by $Q_n$, is a graph with vertex set $\{x_1x_2\cdots x_n : x_i = 0 \text{ or } 1 \text{ for } i = 1, 2, \cdots, n\}$ and any two vertices are adjacent in $Q_n$ if and only if they differ at exactly one position. The Cartesian product of graphs $G$ and $H$, denoted by $G \square H$, is a graph with vertex set $V(G) \times V(H)$, and two vertices $(x, y)$ and $(u, v)$ are adjacent in $G \square H$ if and only if either $x = u$ and $y$ is adjacent to $v$ in $H$, or $x$ is adjacent to $u$ in $G$ and $y = v$. It is well-known that $Q_n$ is the Cartesian product of $n$ copies of the complete graph $K_2$. Note that $Q_n$ is an $n$-regular and $n$-connected graph with $2^n$ vertices and $n2^{n-1}$ edges.

Let $k \geq 3$ be an integer. A cycle of length $k$ is denoted by $C_k$. The sunlet graph of order 2$k$, denoted by $L_{2k}$, is obtained by adding a pendant edge to each vertex of the cycle $C_k$. Note that $L_{2k}$ has $2k$ vertices and $2k$ edges. The sunlet graph of order sixteen $L_{16}$ is shown in Figure 1.

Figure 1. The sunlet graph $L_{16}$
A decomposition of a graph $G$ is a collection of edge-disjoint subgraphs of $G$ such that the edge set of the subgraphs partitions the edge set of $G$. For a given graph $H$, an $H$-decomposition of $G$ is a decomposition into subgraphs each isomorphic to $H$.

The problem of decomposing the given graph into the sunlet graphs is studied for various classes of regular graphs in the literature \cite{Fu1999, Fu2000, Anitha2001, Akwu2003, Sowndhariya2005, Sonawane2006}. Fu et al. \cite{Fu1999} proved that if $k = 6, 10, 14$ or $2^m$ ($m \geq 2$), then there exists an $L_{2k}$-decomposition of $K_n$ if and only if $n \geq 2k$ and $n(n-1) \equiv 0$ (mod $4k$). The existence of an $L_{10}$-decomposition of the complete graph $K_n$ for $n \equiv 0, 1, 5, 16$ (mod 20) is guaranteed by Fu, Huang and Lin \cite{Fu2000}. Anitha and Lekshmi \cite{Anitha2001} established that the complete graph $K_{2n}$, the complete bipartite graph $K_{2n, 2n}$ and the Harary graph $H_{4,2n}$ have $L_{2n}$-decompositions for all $n \geq 3$. Akwu and Ajayi \cite{Akwu2003} proved that for even $m \geq 2$, odd $n \geq 3$ and odd prime $p$, the lexicographic product of $K_n$ and the graph $K_m$ consisting of only $m$ isolated vertices has an $L_{2p}$-decomposition if and only if $\frac{1}{2}n(n-1)m^2 \equiv 0$ (mod $2p$). Sowndhariya and Muthusamy \cite{Sowndhariya2005} gave necessary and sufficient conditions for the existence of an $L_8$-decomposition of tensor product and wreath product of complete graphs. Sowndhariya and Muthusamy \cite{Sowndhariya2006} studied an $L_8$-decomposition of the graph $K_n \square K_m$ and proved that such a decomposition exists if and only if $n$ and $m$ satisfy one of the specific eight conditions. Sonawane and Borse \cite{Sonawane2006} proved that the $n$-dimensional hypercube $Q_n$ has an $L_8$-decomposition if and only if $n$ is 4 or $n \geq 6$.

In this paper, we consider the problem of decomposing the hypercube $Q_n$ into the sunlet graphs. In Section 2, we prove that the necessary and sufficient condition for the existence of an $L_{16}$-decomposition of $Q_n$ is $n = 4$ or $n \geq 6$. In Section 3, we prove that if $Q_n$ has a $C_k$-decomposition, then $Q_{mn}$ has an $L_{2k}$-decomposition for $m \geq 2$.

## 2 An $L_{16}$-decomposition of hypercubes

In this section, we prove that the necessary and sufficient condition for the existence of an $L_{16}$-decomposition of $Q_n$ is $n = 4$ or $n \geq 6$.

We need a corollary of the following result due to El-Zanati and Eyinden \cite{El-Zanati2013}. They considered the cycle decomposition of the Cartesian product of cycles each of length power of 2 and obtained the result, which is stated below.

**Theorem 2.1.** Let $n, k_1, k_2, \ldots, k_n \geq 2$ be integers and let $G$ be the Cartesian product of the cycles $C_{k_1}, C_{k_2}, \cdots C_{k_n}$. Then there exists a $C_s$-decomposition of $G$ if and only if $s = 2^t$ with $2 \leq t \leq k_1 + k_2 + + k_n$. 

The following result is a corollary of the above theorem as $Q_n$ is the Cartesian product of $\frac{n}{2}$ cycles of length 4 for any even integer $n \geq 2$.

**Corollary 2.2.** For any even integer $n \geq 2$, there exists a $C_s$-decomposition of $Q_n$ if and only if $s = 2^t$ with $2 \leq t \leq 2^n$.

In the next lemma, we prove that the necessary condition for the existence of an $L_{16}$-decomposition of $Q_n$ is $n = 4$ or $n \geq 6$.

**Lemma 2.3.** There does not exist an $L_{16}$-decomposition of $Q_n$ if $n \in \{1, 2, 3, 5\}$.

**Proof.** Contrary assume that $Q_n$ has an $L_{16}$-decomposition for some $n \in \{1, 2, 3, 5\}$. Then the number of edges of $L_{16}$ must divide the number of edges of $Q_n$. Hence 16 divides $n2^{n-1}$. This shows that $n \geq 4$ and so, $n = 5$. Since $Q_5$ has 80 edges, there are five copies of the graph $L_{16}$ in the $L_{16}$-decomposition of $Q_5$. Every vertex of $Q_5$ has degree 5 whereas $L_{16}$ has eight vertices of degree 3 and eight of degree 1. Therefore, a degree 3 vertex of any copy of $L_{16}$ in the decomposition cannot be a degree 3 vertex of another copy of $L_{16}$. This implies that $Q_5$ has at least 40 vertices, a contradiction. \(\blacksquare\)

In the next lemma, we give decomposition of $C_k \square C_k$ into spanning sunlet subgraphs for any even integer $k \geq 4$.

**Lemma 2.4.** For any even integer $k \geq 4$, the graph $C_k \square C_k$ has an $L_{k^2}$-decomposition.

**Proof.** Let $V(C_k) = Z_k$ such that a vertex $i$ is adjacent to a vertex $i + 1 \pmod{k}$. Then $V(C_k \square C_k) = \{(i, j) : i, j = 1, 2, \cdots, k\}$. We construct two vertex-disjoint cycles $Z_1$ and $Z_2$ of length $\frac{k^2}{2}$ in $C_k \square C_k$ as $Z_1 = \langle (1, 1), (1, 2), \cdots, (1, \frac{k}{2}), (2, \frac{k}{2} + 1), \cdots, (2, k - 1), (3, k - 1), (3, 1), \cdots, (3, \frac{k}{2} - 2), \cdots, (k, 1) \rangle$ and $Z_2 = \langle (1, \frac{k}{2} + 1), (1, \frac{k}{2} + 2), \cdots, (1, k), (2, k), (2, 1), \cdots, (2, \frac{k}{2} - 1), (3, \frac{k}{2} - 1), (3, \frac{k}{2}), \cdots, (3, k - 1), \cdots, (k, \frac{k}{2} + 1) \rangle$. Now we adjoin a pendant edge to each vertex of $Z_1$ and $Z_2$ in the lexicographic order as per the availability of the vertex, so that we get two edge-disjoint spanning subgraphs of $C_k \square C_k$ which are isomorphic to $L_{k^2}$. This completes the proof. \(\blacksquare\)

For an illustration, an $L_{64}$-decomposition of $C_8 \square C_8$ is shown in Figure 2. For convenience, edges of the cycles $C_{32}$ are shown by lines and edges with the pendant vertices by dotted lines in both the copies of $L_{64}$.
The following result is a corollary of the above lemma.

**Corollary 2.5.** For any integer \( n \geq 1 \), there exists an \( L_{2^n} \)-decomposition of \( Q_{4n} \). In other words, \( Q_{4n} \) has a decomposition into the spanning sunlet graphs for any integer \( n \geq 1 \).

**Proof.** We can write \( Q_{4n} = Q_{2n} \sqcap Q_{2n} \). By Corollary 2.2, \( Q_{2n} \) has a decomposition into Hamiltonian cycles. Let \( Z_1, Z_2, \ldots, Z_n \) be Hamiltonian cycles in \( Q_{2n} \) such that the collection \( \{ Z_1, Z_2, \ldots, Z_n \} \) decomposes \( Q_{2n} \). Then \( Z_1 \sqcap Z_1, Z_2 \sqcap Z_2, \ldots, Z_n \sqcap Z_n \) are edge-disjoint spanning subgraphs of \( Q_{4n} \) and their collection decomposes \( Q_{4n} \). By Lemma 2.4, each \( Z_i \sqcap Z_i \) has an \( L_{16} \)-decomposition. Hence \( Q_{4n} \) has an \( L_{2^{2n}} \)-decomposition. \( \square \)

Now we prove the necessary condition for the existence of an \( L_{16} \)-decomposition of \( Q_n \) is also sufficient.

We need the following four lemmas to prove the sufficient condition.

**Lemma 2.6.** There exists an \( L_{16} \)-decomposition of \( Q_6 \).

**Proof.** Write \( Q_6 = Q_4 \sqcap C_4 \) as \( C_4 = C_2 \). Thus \( Q_6 \) is obtained by replacing each vertex of \( C_4 \) by a copy of \( Q_4 \) and replacing each edge of \( C_4 \) by a matching between two copies of \( Q_4 \) corresponding to the end vertices of that edge. Let \( C_4 = \langle 0, 1, 2, 3, 0 \rangle \) and \( Q_4^1, Q_4^2, Q_4^3, Q_4^0 \) be copies of \( Q_4 \) in \( Q_6 \) corresponding to vertices 0, 1, 2, 3 of \( C_4 \), respectively. For \( i \in \{ 0, 2 \} \), \( Q_4^i \) has an \( L_{16} \)-decomposition by Lemma 2.4 as each \( Q_4^i \) can be written as the Cartesian product of cycles of length 4. For \( i \in \{ 1, 3 \} \), from each vertex of \( Q_4^i \), exactly two cycles of length eight
passes as $Q_1^3$ has a $C_8$-decomposition by Corollary 2.2. Adjoin each vertex of one of two cycles to the corresponding vertex in $Q_4^2$, and adjoin each vertex of the other cycle to the corresponding vertex in $Q_4^2$. So, from each copy of the cycle of length eight, we get a copy of $L_{16}$. This completes the proof. □

**Lemma 2.7.** There exists an $L_{16}$-decomposition of $Q_7$.

*Proof.* Write $Q_7$ as $Q_7 = Q_4 \Box Q_3$. Let $D$ be a directed graph obtained from $Q_3$ by giving directions to the edges, as shown in Figure 3.

![Figure 3.](image)

In $D$, there are two vertices with in-degree 3 and out-degree 0, and the in-degrees and out-degrees of remaining all vertices are 1 and 2, respectively. The graph $Q_7$ is obtained by replacing each vertex of $Q_3$ with a copy of $Q_4$ and replacing each edge of $Q_3$ by a matching between two copies of $Q_4$ corresponding to the end vertices of that edge. Consider an $L_{16}$-decomposition of copies of $Q_4$ corresponding to each vertex of $D$ with out-degree 0, and a $C_8$-decomposition of copies of $Q_4$ corresponding to each vertex of $D$ with out-degree 2. In a $C_8$-decomposition of copies of $Q_4$, exactly two cycles pass from each vertex. Adjoin a pedant edge to each vertex of copies of $Q_4$ of a vertex corresponding the out-degree 2, to one of the vertices of its nearest copy of $Q_4$ according to the direction of the corresponding edge in $D$. Then we get $L_{16}$ from each $C_8$ from a $C_8$-decomposition of each copy of $Q_4$ of a vertex corresponding to the out-degree 2. Hence we get an $L_{16}$-decomposition of $Q_7$. □

**Lemma 2.8.** There exists an $L_{16}$-decomposition of $Q_9$.

*Proof.* Write $Q_9$ as $Q_9 = Q_6 \Box Q_3$. Let $D$ be a directed graph obtained from $Q_3$ by giving directions to the edges, as shown in Figure 4.
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In $D$, there are four vertices with out-degree 0, and the out-degree of the remaining four vertices is 3. The graph $Q_9$ is obtained by replacing each vertex of $Q_3$ with a copy of $Q_6$ and replacing each edge of $Q_3$ by a matching between two copies of $Q_6$ corresponding to the end vertices of that edge. Consider an $L_{16}$-decomposition of copies of $Q_6$ of vertices corresponding to the out-degree 0 and a $C_8$-decomposition of copies of $Q_6$ of vertices corresponding to the out-degree 3. In a $C_8$-decomposition of copies of $Q_6$, exactly three cycles pass from each vertex. Adjoin a pendant edge to each vertex of copies of $Q_6$ corresponding to each vertex with out-degree 3, to one of the vertices of its nearest copy of $Q_6$ according to the direction of the corresponding edge in $D$. Then we get a copy of $L_{16}$ from each copy of $C_8$ from a $C_8$-decomposition of each copy of $Q_6$ corresponding to each vertex with out-degree 3. Hence we get an $L_{16}$-decomposition of $Q_9$.

The following lemma follows from the definition of the Cartesian product of graphs.

**Lemma 2.9.** If the graphs $G_1$ and $G_2$ each has an $H$-decomposition, then the graph $G_1 \square G_2$ has an $H$-decomposition.

In the following lemma, we prove that the sufficient condition for the existence of an $L_{16}$-decomposition of $Q_n$ is $n = 4$ or $n \geq 6$.

**Lemma 2.10.** There exists an $L_{16}$-decomposition of $Q_n$ if $n = 4$ or $n \geq 6$.

**Proof.** We prove the result by induction on $n$. For $n = 4$, the result holds as $Q_4$ has an $L_{16}$-decomposition by Lemma 2.4. For $n = 8$, we write $Q_8 = Q_4 \square Q_4$ and the result holds by Lemma 2.9. For $n \in \{6, 7, 9\}$, the result follows by Lemmas 2.6, 2.7 and 2.8. Suppose that $n \geq 10$. Assume that the result holds for the $k$-dimensional hypercube for any integer $k$ with $6 \leq k \leq n - 1$. Write $Q_n = Q_{n-4} \square Q_4$. By induction hypothesis, $Q_{n-4}$ has an $L_{16}$-decomposition as $n - 4 \geq 6$. Hence $Q_n$ has an $L_{16}$-decomposition by Lemma 2.9. This completes the proof.

The following result follows from Lemmas 2.3 and 2.10.
Theorem 2.11. The necessary and sufficient condition for the existence of an \( L_{16} \)-decomposition of \( Q_n \) is \( n = 4 \) or \( n \geq 6 \).

3 An \( L_{2k} \)-decomposition of hypercubes

In this section, we prove that \( Q_{mn} \) has an \( L_{2k} \)-decomposition if \( Q_n \) has a \( C_k \)-decomposition for \( m \geq 2 \). In next two lemmas, we prove the result for \( m = 2 \) and \( m = 3 \). Note that a \( C_k \)-decomposition of \( Q_n \) is possible only for an even integer \( n \geq 2 \). For \( n = 2 \), \( Q_n = C_4 \).

Lemma 3.1. If \( Q_n \) has a \( C_k \)-decomposition, then \( Q_{2n} \) has an \( L_{2k} \)-decomposition.

Proof. Suppose \( Q_n \) has a \( C_k \)-decomposition. Note that in the \( C_k \)-decomposition of \( Q_n \), from each vertex of \( Q_n \) exactly \( \frac{n}{2} \) cycles passes. We can write \( Q_{2n} = Q_n \square Q_n \). Let \( W_0, W_1, \ldots, W_{2^n-1} \) be copies of \( Q_n \) in \( Q_{2n} \) replaced by vertices of \( Q_n \). Then each \( W_i \) has a \( C_k \)-decomposition. Also, there are \( n \) copies of \( W_j \)'s that are adjacent to \( W_i \) for each \( i \).

Since \( Q_n \) is a regular and connected graph with even degree \( n \), there is a directed Eulerian circuit in \( Q_n \) in which each of in-degree and out-degree of each vertex is \( \frac{n}{2} \). In a \( C_k \)-decomposition of each \( W_i \), adjoin each vertex of each cycle to exactly one vertex of the nearest copy \( W_j \) of \( W_i \) in \( Q_{2n} \), if there is a directed edge in the directed Eulerian circuit from the vertex \( i \) to the vertex \( j \). From a \( C_k \)-decomposition of each \( W_i \)'s, we get edge-disjoint copies of \( L_{2k} \). This completes the proof.

We need concepts of even and odd parity vertex in the proof of the following lemma. A vertex \( v = x_1x_2\cdots x_n \) of \( Q_n \) is said to be a vertex with even (odd) parity if there are even (odd) number of \( x_i \)'s are 1 in \( v \). Let \( X \) and \( Y \) be subsets of vertex set of \( Q_n \) containing vertices with even parity and odd parity, respectively and \( X \cup Y = V(Q_n) \). Then \( (X,Y) \) is a bipartition of the bipartite graph \( Q_n \).

Lemma 3.2. If \( Q_n \) has a \( C_k \)-decomposition, then \( Q_{3n} \) has an \( L_{2k} \)-decomposition.

Proof. We can write, \( Q_{3n} = Q_{2n} \square Q_n \). Let \( W_0, W_1, \ldots, W_{2^n-1} \) be copies of \( Q_{2n} \) in \( Q_{3n} \) replaced by vertices of \( Q_n \). Let \( D \) be a digraph obtained from \( Q_n \) such that out-degree of each vertex with even parity is \( n \) and odd parity is 0. By Lemma 3.1, each \( W_j \) corresponding to vertex of \( Q_n \) with odd parity, has an \( L_{2k} \)-decomposition. Consider a \( C_k \)-decomposition of \( W_j \) corresponding to vertex of \( Q_n \) with even parity. Note that in the \( C_k \)-decomposition of \( W_j \), from each vertex exactly \( n \) edge-disjoint cycles passes. By adjoining exactly one vertex to each cycle in \( W_j \).
corresponding to vertex of $Q_n$ with even parity, we get copies of $L_{2k}$ corresponding to each $C_k$ in the $C_k$-decomposition of $W_j$. This completes the proof. 

Now, we have the following result.

**Theorem 3.3.** If $Q_n$ has a $C_k$-decomposition, then $Q_{mn}$ has an $L_{2k}$-decomposition for $m \geq 2$.

**Proof.** If $m$ is multiple of 2, the result holds by Lemmas 2.9 and 3.1 as $Q_{mn}$ is the Cartesian product of $m/2$ copies of $Q_{2n}$. Similarly, the result holds by Lemmas 2.9 and 3.2 if $m$ is multiple of 3 as $Q_{mn}$ is the Cartesian product of $m/3$ copies of $Q_{2n}$. For $m = 5$ and 7, we can write $Q_{mn}$ as $Q_{5n} = Q_{2n} \square Q_{3n}$ and $Q_{7n} = Q_{4n} \square Q_{3n}$, respectively. Thus the result holds by Lemmas 2.9, 3.1 and 3.2 for $m = 5, 7$. It follows that the result holds for $m$ with $2 \leq m \leq 10$. Suppose that $m \geq 11$, and $m$ is not multiple of 2 and 3. Then either $m = 6q + 5$ for some $q \geq 1$ or $m = 6q + 1$ for some $q \geq 2$. Suppose $m = 6q + 5$ for $q \geq 1$. Then we can write $Q_{mn}$ as $Q_{mn} = Q_{6qn} \square Q_{5n}$. Suppose $m = 6q + 1$ for $q \geq 2$. Then we can write $Q_{mn}$ as $Q_{mn} = Q_{6(q-1)n} \square Q_{7n}$. Note that for any $r \geq 1$, $Q_{6rn}$ has an $L_{2k}$-decomposition by both Lemmas 3.1 and 3.2. Thus by Lemma 2.9, $Q_{mn}$ has an $L_{2k}$-decomposition.

As a consequence of Theorem 3.3, we have the following result.

**Corollary 3.4.** Let $m \geq 2$ be an integer and $n \geq 4$ be an even integer.

1. $Q_{mn}$ has an $L_{2t+1}$-decomposition for $2 \leq t \leq n-1$.
2. $Q_{mn}$ has an $L_{2n}$-decomposition.
3. $Q_{mn}$ has an $L_{4n}$-decomposition.
4. $Q_{mn}$ has an $L_{8n}$-decomposition.
5. $Q_{mn}$ has an $L_{n2^{k+1}}$-decomposition for $2n \leq n2^k \leq \frac{2n}{n}$.

**Proof.** We have following $C_k$-decompositions of $Q_n$ for an even integer $n \geq 4$.

1. Zanati and Eynden [12] proved that $Q_n$ has a $C_{2^t}$-decomposition for $2 \leq t \leq n-1$.
2. Ramras [7] proved that $Q_n$ has a $C_n$-decomposition.
3. Mollard and Ramras [6] proved that $Q_n$ has a $C_{2n}$-decomposition.
4. Tapadia, Borse and Waphare [11] obtained that $Q_n$ has a $C_{4n}$-decomposition.

5. Axenovich, Offner and Tompkins [3] established that $Q_n$ has a $C_{n2^k}$-decomposition for $2n \leq n2^k \leq \frac{2n}{n}$.

By applying Theorem 3.3 to each of above $C_k$-decompositions of $Q_n$, we get the desired $L_{2k}$-decomposition of $Q_{mn}$.

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