Optimal re-centering bounds, with applications to Rosenthal-type concentration of measure inequalities

Iosif Pinelis

Department of Mathematical Sciences
Michigan Technological University
Houghton, Michigan 49931, USA
E-mail: ipinelis@mtu.edu

Abstract: For any nonnegative Borel-measurable function $f$ such that $f(x) = 0$ if and only if $x = 0$, the best constant $c_f$ in the inequality $E f(X - E X) \leq c_f E f(X)$ for all random variables $X$ with a finite mean is obtained. Properties of the constant $c_f$ in the case when $f = \cdot^p$ for $p > 0$ are studied. Applications to concentration of measure in the form of Rosenthal-type bounds on the moments of separately Lipschitz functions on product spaces are given.

AMS 2000 subject classifications: Primary 60E15; secondary 46B09.
Keywords and phrases: probability inequalities, Rosenthal inequality, sums of independent random variables, martingales, concentration of measure, separately Lipschitz functions, product spaces.

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1. Introduction

In many situations (as e.g. in [22]), one starts with zero-mean random variables (r.v.’s), which need to be truncated in some manner, and then the means no longer have to be zero. So, to utilize such tools as the Rosenthal inequality for sums of independent zero-mean r.v.’s, one has to re-center the truncated r.v.’s. Then one will usually need to bound moments of the re-centered truncated r.v.’s in terms of the corresponding moments of the original r.v.’s. To be more specific, let $Z$ be a given r.v., possibly (but not necessarily) of zero mean. Next, let $\tilde{Z}$ be a truncated version of $Z$ such that $|\tilde{Z}| \leq |Z|$; possibilities here include letting $\tilde{Z}$ equal $Z I\{Z \leq z\}$ or $Z I\{|Z| \leq z\}$ or $Z \land z$, for some $z > 0$; cf. [21, 16].
Assume that $E |\tilde{Z}| < \infty$. Then for any $p \geq 1$ one can use the inequalities $|x - y|^p \leq 2^{p-1}(|x|^p + |y|^p)$ and $(E |\tilde{Z}|)^p \leq E |\tilde{Z}|^p$, to write
\[ E |\tilde{Z} - E \tilde{Z}|^p \leq 2^p E |\tilde{Z}|^p \leq 2^p E |Z|^p, \] (1.1)
as is oftentimes done. However, the factor $2^p$ in (1.1) can be significantly improved, especially for $p \geq 2$. For instance, it is clear that for $p = 2$ this factor can be reduced from $2^2 = 4$ to 1. More generally, for every real $p > 1$ we shall provide the best constant factor $C_p$ in the inequality
\[ E |X - E X|^p \leq C_p E |X|^p \] (1.2)
for all r.v.’s $X$ with a finite mean $E X$. In particular, $C_p$ improves the factor $2^p$ more than 6 times for $p = 3$, and for large $p$ this improvement is asymptotically $\sqrt{8ep}$ times; see parts (vi) and (iv) of Theorem 2.3 and the left panel in Figure 2 in this paper. In fact, in Theorem 2.1 below we shall present an extended version of the exact inequality (1.2), for a quite general class of moment functions $f$ in place of the power functions $|·|^p$.

Another natural application of these results is to concentration of measure for separately Lipschitz functions on product spaces. In Section 3 of this paper, we shall give Rosenthal-type bounds on the moments of such functions. Similar extensions of the von Bahr–Esseen inequality were given in [17].

2. Summary and discussion

Let $f : \mathbb{R} \to \mathbb{R}$ be any nonnegative Borel-measurable function such that $f(x) = 0$ if and only if $x = 0$. Let $X$ stand for any random variable (r.v.) with a finite mean $E X$.

**Theorem 2.1.** One has
\[ E f(X - E X) \leq c_f E f(X), \] (2.1)
where
\[ c_f := \sup \left\{ \frac{af(b) + bf(-a)}{af(b - t) + bf(-a - t)} : a \in (0, \infty), b \in (0, \infty), t \in \mathbb{R} \right\} \] (2.2)
is the best possible constant factor in (2.1) (over all r.v.’s $X$ with a finite mean).

All necessary proofs will be given in Section 4.

Note that for all $a \in (0, \infty)$, $b \in (0, \infty)$, and $t \in \mathbb{R}$ both the numerator and the denominator of the ratio in (2.2) are strictly positive (since $f$ is nonnegative and vanishes only at 0). So, $c_f$ is correctly defined, with possible values in $(0, \infty]$.

It is possible to say much more about the optimal constant factor $c_f$ in the important case when $f$ is the power function $|·|^p$. To state the corresponding result, let us introduce more notation.
Take any \( a \in (0, \infty) \) and \( b \in (0, \infty) \), and let \( X_{a,b} \) be any zero-mean r.v. with values \( -a \) and \( b \), so that
\[
P(X_{a,b} = b) = \frac{a}{a+b} = 1 - P(X_{a,b} = -a).
\]
Note that
\[
X_{b,a} \overset{D}{=} -X_{a,b},
\]
where \( D \) denotes the equality in distribution.

Take any \( p \in (1, \infty) \) (2.3) and introduce
\[
R(p,b) := (b^{p-1} + (1-b)^{p-1})(b^{\frac{1}{p-1}} + (1-b)^{\frac{1}{p-1}})^{p-1}
\]
for any \( b \in [0,1] \). (2.4)

**Proposition 2.2.** If \( p \neq 2 \) then there exists \( b_p \in (0, \frac{1}{2}) \) such that

(i) \( \partial_b R(p,b) > 0 \) for \( b \in (0,b_p) \) and hence \( R(p,b) \) is (strictly) increasing in \( b \in [0,b_p] \);

(ii) \( \partial_b R(p,b) < 0 \) for \( b \in (b_p, \frac{1}{2}) \) and hence \( R(p,b) \) is decreasing in \( b \in [b_p, \frac{1}{2}] \).

So, \( b_p \) is the unique maximizer of \( R(p,b) \) over all \( b \in [0,\frac{1}{2}] \).

In Proposition 2.2 and in the sequel, \( \partial \cdot \) denotes the partial differentiation with respect to the argument in the subscript.

**Theorem 2.3.**

(i) Inequality (1.2) holds with the constant factor
\[
C_p := \epsilon_{|p|} = \sup_{b \in [0,1]} R(p,b) = \max_{b \in (0,1/2)} R(p,b) = R(p,b_p),
\]
where \( R(p,b) \) is as in (2.4) and \( b_p \) is as in Proposition 2.2. In particular, \( C_2 = R(2,b) = 1 \) for all \( b \in [0,1] \).

(ii) \( C_p \) is the best possible constant factor in (1.2). More specifically, the equality in (1.2) obtains if and only if one of the following three conditions holds:

(a) \( E|X|^p = \infty \);

(b) \( p = 2, E X^2 < \infty \), and \( E X = 0 \);

(c) \( p \neq 2 \) and \( X \overset{D}{=} \lambda(X_1-b_p,b_p - t_{b_p}) \) for some \( \lambda \in \mathbb{R} \), where
\[
tb := \frac{b - \frac{b^{1/(p-1)}}{b^{1/(p-1)} + (1-b)^{1/(p-1)}}}{b^{1/(p-1)} + (1-b)^{1/(p-1)}}
\]
for all \( b \in (0,1) \), and \( b_p \) is as in Proposition 2.2.
(iii) One has the symmetries

\[ C_p^{1/\sqrt{p-1}} = C_q^{1/\sqrt{q-1}} \quad \text{and} \quad b_p = b_q, \]

where \( q \) is dual to \( p \) in the sense of \( L^p \)-spaces:

\[ \frac{1}{p} + \frac{1}{q} = 1. \]

(iv) For \( p \to \infty \),

\[ C_p \sim \frac{2^p}{\sqrt{8\pi p}}; \quad (2.8) \]

as usual, \( A \sim B \) means that \( A/B \to 1 \).

(v) \( C_p \) is strictly log-convex and hence continuous in \( p \in (1, \infty) \); moreover, \( C_p \) decreases in \( p \in (1, 2] \) from 2 to 1 and increases in \( p \in [2, \infty) \) from 1 to \( \infty \).

(vi) The values of \( C_p, b_p, \) and \( t_{b_p} \) are algebraic whenever \( p \) is rational; in particular, \( C_3 = \frac{1}{12}(17 + 7\sqrt{7}) = 1.315... \), \( b_3 = \frac{1}{2} - \frac{1}{6} \sqrt{1 + 2\sqrt{7}} = 0.0819... \), and \( t_{b_3} = -\frac{1}{3} \sqrt{\frac{1}{2} (13\sqrt{7} - 34)} = -0.148... \).

By parts (vi) and (v) of Theorem 2.3, \( C_p \) can in principle be however closely bracketed for any real \( p \in (1, \infty) \). However, such a calculation may in many cases be inefficient. On the other hand, Proposition 2.2 allows one to bracket the maximizer \( b_p \) of \( R(b, p) \) however closely and thus, perhaps more efficiently, compute \( C_p \) with any degree of accuracy.

(A part of) the graph of \( C_p \) is shown in Figure 1, and those of \( 2^p/C_p \) and \( b_p \) are shown in Figure 2.

![Graph of C_p](image1.png)

Fig 1. \( C_p \) decreases in \( p \in (1, 2] \) from 2 to 1 and increases in \( p \in [2, \infty) \) from 1 to \( \infty \).
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2\(^p\)/C\(_p\)

\[b_p\]

Fig. 2. By (2.8), \(2^p/C_p \sim \sqrt{8e_p}\) as \(p \to \infty\). By (2.7), \(b_p = b_q\); note here also that \(p \in (1, 2] \Leftrightarrow q \in [2, \infty)\); by (4.16), \(b_p \sim (p - 1)/2\) as \(p \downarrow 1\).

Remark 2.4. What if, instead of the condition (2.3), one has \(p \in (0, 1]\)? It is easy to see that the inequality (1.2) holds for \(p = 1\) with \(C_1 = 2\) (cf. (1.1)), which is then the best possible factor, as seen by letting

\[X = X_{1-b,b} - b\] with \(b \downarrow 0\). (2.9)

However, the equality \(E|X - E X| = 2E|X|\) obtains only if \(X \overset{D}{=} 0\); one may also note here that, by part (v) of Theorem 2.1, \(C_{1+} = 2 = C_1\). As to \(p \in (0, 1)\), for each such value of \(p\) the best possible factor \(C_p\) in (1.2) is \(\infty\); indeed, consider \(X\) as in (2.9).

3. Application: Rosenthal-type concentration inequalities for separately Lipschitz functions on product spaces

It is well known that for every \(p \in [2, \infty)\) there exist finite positive constants \(c_1(p)\) and \(c_2(p)\), depending only on \(p\), such that for any independent real-valued zero-mean r.v.’s \(X_1, \ldots, X_n\)

\[E|Y|^p \leq c_1(p)A_p + c_2(p)B_p,\]

where \(Y := X_1 + \cdots + X_n\), \(A_p := E|X_1|^p + \cdots + E|X_n|^p\), and \(B := (E X_1^2 + \cdots + E X_n^2)^{1/2}\). An inequality of this form was first proved by Rosenthal [27], and has since been very useful in many applications. It was generalized to martingales [4, (21.5)], including martingales in Hilbert spaces [23] and, further, in 2-smooth Banach spaces [18]. The constant factors \(c_1(p)\) and \(c_2(p)\) were actually allowed in [23] and [18] to depend on certain freely chosen parameters, which provided for optimal in a certain sense sizes of \(c_1(p)\) and \(c_2(p)\), for any given positive value of the Lyapunov ratio \(A_p/B_p\). Best possible Rosenthal-type bounds for sums of independent real-valued zero-mean r.v.’s were given, under different conditions, by Utev [28] and Ibragimov and Sharakhmetov [6, 7]. Also for sums of independent real-valued zero-mean r.v.’s \(X_1, \ldots, X_n\), Latała [9] obtained an expression \(E\) in terms of \(p\) and the individual distributions of the \(X_i’s\) such that \(a_1E \leq \|Y\|_p \leq a_2E\) for some positive absolute constants \(a_1\) and \(a_2\).
Given a Rosenthal-type upper bound for real-valued martingales, one can use the Yurinskiĭ martingale decomposition \cite{Yurinskii} and (say) Theorem 2.3 to obtain a corresponding upper bound on the $p$th absolute central moment of the norm of the sum of independent random vectors in an arbitrary separable Banach space; even more generally, one can obtain such a measure-concentration inequality for separately Lipschitz functions on product spaces.

To state such a result, let $X_1, \ldots, X_n$ be independent r.v.’s with values in measurable spaces $\mathcal{X}_1, \ldots, \mathcal{X}_n$, respectively. Let $g: \prod \mathcal{X} \to \mathbb{R}$ be a measurable function on the product space $\prod \mathcal{X}_i = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$. Let us say (cf. \cite{Iosif19}) that $g$ is separately Lipschitz if it satisfies a Lipschitz-type condition in each of its arguments:

$$|g(x_1, \ldots, x_{i-1}, \tilde{x}_i, x_{i+1}, \ldots, x_n) - g(x_1, \ldots, x_n)| \leq \rho_i(\tilde{x}_i, x_i)$$

for some measurable functions $\rho_i: \mathcal{X}_i \times \mathcal{X}_i \to \mathbb{R}$ and all $i \in \overline{1, n}$, $(x_1, \ldots, x_n) \in \mathcal{X}$, and $\tilde{x}_i \in \mathcal{X}_i$. Take now any separately Lipschitz function $g$ and let

$$Y := g(X_1, \ldots, X_n).$$

Suppose that the r.v. $Y$ has a finite mean.

On the other hand, take any $p \in [2, \infty)$ and suppose that positive constants $c_1(p)$ and $c_2(p)$ are such that for all real-valued martingales $(\zeta_j)_{j=0}^n$ with $\zeta_0 = 0$ and differences $\xi_i := \zeta_i - \zeta_{i-1}$

$$E|\zeta_n|^p \leq c_1(p) \sum_{i=1}^n E|\xi_i|^p + c_2(p) \left( \sum_{i=1}^n E \rho_i(X_i, x_i)^2 \right)^{p/2},$$

where $E_j$ denotes the expectation given $\zeta_0, \ldots, \zeta_j$.

Then one has

**Corollary 3.1.** For each $i \in \overline{1, n}$, take any $x_i$ and $y_i$ in $\mathcal{X}_i$. Then

$$E|Y - EY|^p \leq C_p c_1(p) \sum_{i=1}^n E \rho_i(X_i, x_i)^p + c_2(p) \left( \sum_{i=1}^n E \rho_i(X_i, y_i)^2 \right)^{p/2},$$

where $C_p$ is as in (2.5).

An example of separately Lipschitz functions $g: \mathcal{X}^n \to \mathbb{R}$ is given by the formula

$$g(x_1, \ldots, x_n) = \|x_1 + \cdots + x_n\|$$

for all $x_1, \ldots, x_n$ in a separable Banach space $(\mathcal{X}, \|\cdot\|)$. In this case, one may take $\rho_i(\tilde{x}_i, x_i) \equiv \|	ilde{x}_i - x_i\|$. Thus, one immediately obtains

**Corollary 3.2.** Let $X_1, \ldots, X_n$ be independent random vectors in a Banach space $(\mathcal{X}, \|\cdot\|)$. Let here $Y := \|X_1 + \cdots + X_n\|$. For each $i \in \overline{1, n}$, take any $x_i$ and $y_i$ in $\mathcal{X}_i$. Then

$$E|Y - EY|^p \leq C_p c_1(p) \sum_{i=1}^n E \|X_i - x_i\|^p + c_2(p) \left( \sum_{i=1}^n E \|X_i - y_i\|^2 \right)^{p/2}.$$
Particular cases of separately Lipschitz functions more general than the norm of the sum as in (3.4) were discussed earlier in [25] and [24, pages 20–23].

For $p = 2$, it is obvious that the inequality (3.2) holds with $c_1(2) = 1$ and $c_2(2) = 0$, and then the inequalities (3.3) and (3.5) do so. Thus, for $p = 2$ (3.5) becomes

$$\text{Var} Y \leq \sum_{i=1}^{n} E \| X_i - x_i \|^2,$$

(3.6)

since $C_2 = 1$. The inequality (3.6) was presented in [24, page 29] and [26, Theorem 4], based on an improvement of the method of Yurinski˘ı [8]; cf. [14, 15, 1], [19, Section 4], and [18, Proposition 2.5]. The proof of Corollary 3.1 is based in part on the same kind of improvement.

The case $p = 3$ is also of particular importance in applications, especially to Berry–Esseen-type bounds; cf. e.g. [2, Lemma A1], [5, Lemma 6.3], and [22]. It follows from the main result of [23] that (3.2) holds for $p = 3$ with $c_1(3) = 1$ and $c_2(3) = 3$, whereas, by part (vi) of Theorem 2.3, $C_3 < 1.316$. Thus, one has an instance of (3.5) with rather small constant factors:

$$E |Y - E Y|^3 \leq 1.316 \sum_{i=1}^{n} E \| X_i - x_i \|^3 + 3 \left( \sum_{i=1}^{n} E \| X_i - y_i \|^2 \right)^{3/2}.$$

Similarly, the more general inequality (3.3) holds for $p = 3$ with 1.316 and 3 in place of $C_p c_1(p)$ and $c_2(p)$.

As can be seen from the proof given in Section 4, both Corollaries 3.1 and 3.2 will hold even if the separately-Lipschitz condition (3.1) is relaxed to

$$| E g(x_1, \ldots, x_{i-1}, \tilde{x}_i, X_{i+1}, \ldots, X_n) - E g(x_1, \ldots, x_i, X_{i+1}, \ldots, X_n) | \leq \rho_i(\tilde{x}_i, x_i).$$

(3.7)

Note also that in Corollaries 3.1 and 3.2 the r.v.’s $X_i$ do not have to be zero-mean, or even to have any definable mean; at that, the arbitrarily chosen $x_i$’s and $y_i$’s may act as the centers, in some sense, of the distributions of the corresponding $X_i$’s.

Other inequalities for the distributions of separately Lipschitz functions on product spaces were given in [1, 19, 17].

Clearly, the separate-Lipschitz (sep-Lip) condition (3.1) is easier to check than a joint-Lipschitz one. Also, sep-Lip (especially in the relaxed form (3.7)) is more generally applicable. On the other hand, when a joint-Lipschitz condition is satisfied, one can generally obtain better bounds. Literature on the concentration of measure phenomenon, almost all of it for joint-Lipschitz settings, is vast; let us mention here only [13, 11, 10, 3, 12].

4. Proofs

Proof of Theorem 2.1. It is well known that any zero-mean probability distribution on $\mathbb{R}$ is a mixture of zero-mean distributions on sets of at most two elements;
see e.g. [20, Proposition 3.18]. So, there exists a Borel probability measure \( \mu \) on the set
\[
S := \mathbb{R} \times (0, 1/2)
\]
such that
\[
\mathbb{E} g(X - \mathbb{E} X) = \int_S \mathbb{E} g(\lambda X_{1, b}) \mu(d\lambda \times db)
\tag{4.1}
\]
for all nonnegative Borel functions \( g \); the measure \( \mu \) depends on the distribution of the r.v. \( X - \mathbb{E} X \). Letting now
\[
S_0 := (\mathbb{R} \setminus \{0\}) \times (0, 1/2)
\tag{4.2}
\]
and using the condition \( f(0) = 0 \), one has
\[
\mathbb{E} f(X - \mathbb{E} X) = \int_S \mathbb{E} f(\lambda X_{1, b, b}) \mu(d\lambda \times db)
\leq \hat{c}_f \int_{S_0} \mathbb{E} f(\lambda X_{1, b, b} + \mathbb{E} X) \mu(d\lambda \times db)
\leq \hat{c}_f \int_{S_0} \mathbb{E} f(\lambda X_{1, b, b} + \mathbb{E} X) \mu(d\lambda \times db)
\leq \hat{c}_f \mathbb{E} f((X - \mathbb{E} X) + \mathbb{E} X) = \hat{c}_f \mathbb{E} f(X),
\tag{4.3}
\]
where
\[
\hat{c}_f := \sup \{ \hat{\rho}_f(\lambda, b, t) : (\lambda, b, t) \in S_0, t \in \mathbb{R} \} \quad \text{and} \quad \hat{\rho}_f(\lambda, b, t) := \frac{\mathbb{E} f(\lambda X_{1, b, b})}{\mathbb{E} f(\lambda (X_{1, b, b} - t))},
\tag{4.6}
\]
so that
\[
\hat{c}_f = c_f.
\tag{4.7}
\]
Now the inequality in (2.1) follows from the above multi-line display and (4.7), and (4.7) (together with (4.5) and (4.6)) also shows that \( c_f \) is the best possible constant factor in (2.1).

Proof of Proposition 2.2. It is straightforward to check the symmetry
\[
R(p, b)^{1/\sqrt{p-1}} = R(q, b)^{1/\sqrt{q-1}}
\tag{4.8}
\]
for all \( b \in [0, 1] \), where \( q \) is dual to \( p \).

So, it remains to consider \( p \in (1, 2) \). Also assume that \( b \in (0, 1/2) \) and introduce
\[
r := p - 1, \quad x := \frac{b}{1-b}, \quad \text{and} \quad z := -\frac{\ln x}{r},
\tag{4.9}
\]
so that
\[
r \in (0, 1), \quad x \in (0, 1), \quad \text{and} \quad z \in (0, \infty).
\]
Now introduce
\[ D_1(x) := D_1(r, x) := (1 - b) \frac{x^r + 1}{x^{r-1} - 1} \partial_b \ln R(p, b) = r - \frac{(x - x^{1/r})(1 + x^r)}{(x^r - x)(1 + x^{1/r})} \]  \hfill (4.10)
and
\[ D_2(x) := D_2(r, x) := rx^3(1 + x^{1/r})^2(x^{r-1} - 1)^2 D'_1(x), \]  \hfill (4.11)
so that \( D_1(x) \) and \( D_2(x) \) equal in sign to \( \partial_b \ln R(p, b) \) and \( D'_1(x) \), respectively. One can verify the identity
\[ D_2(x)e^{(1+r+r^2)z/2} = D_{21}(z) + (1-r)D_{22}(z), \]  \hfill (4.12)
where
\[
D_{21}(z) := r^2 sh((1-r)z) + sh(r(1-r)z) - r sh((1-r^2)z), \\
D_{22}(z) := h(z) - h(rz), \quad h(u) := sh ru - r sh u;
\]
we use \( sh \) and \( ch \) for sinh and cosh. Note that \( h'(u) = r(ch ru - ch u) < 0 \) for \( u > 0 \) and hence
\[ D_{22}(z) < 0. \]
Next,
\[
\frac{D'_{21}(z)}{(1-r)r} = \left( ch[(1-r)rz] - ch[(1-r^2)z] \right) + r \left( ch[(1-r)z] - ch[(1-r^2)z] \right) < 0,
\]
since \((1-r)r < 1 - r < 1 - r^2\). So, \( D_{21}(z) \) is decreasing \((z > 0)\) and, obviously, \( D_{21}(0+) = 0 \). Hence, \( D_{21}(z) < 0 \) as well. Thus, by (4.12), \( D_2(x) < 0 \), which shows that \( D'_1(x) < 0 \) and \( D_1(x) \) is decreasing \(- in x \in (0, 1) \). Moreover, \( D_1(0+) = r > 0 \) \( r - 1/r = D_1(1-) \). It follows, in view of (4.11), that \( D_1(x) \) changes in sign exactly once, from \(+ to -\), as \( x \) increases from 0 to 1. Equivalently, by (4.10), \( \partial_b \ln R(p, b) \) changes in sign exactly once, from \(+ to -\), as \( b \) increases from 0 to 1/2. This completes the proof of Proposition 2.2. \( \Box \)

**Proof of Theorem 2.3.**
(i) To begin the proof of part (i) of Theorem 2.3, note that the last two inequalities in (2.5) follow by the obvious symmetry
\[ R(p, b) = R(p, 1 - b) \quad \text{for all} \quad b \in [0, 1] \]  \hfill (4.13)
and Proposition 2.2.

Next, in view of the definition of \( C_p \) in (2.5), inequality (1.2) is a special case of (2.1). Moreover, by the definition of \( \hat{\rho} \) in (4.6) and the homogeneity of the power function \( \cdot |^p \),
\[ \hat{\rho}_{1,p}(\lambda, b, t) = \rho_p(b, t) := \frac{E |X_{1-b,b}|^p}{E |X_{1-b,b} - t|^p} \]  \hfill (4.14)
for all \((\lambda, b) \in S_0\) and \(t \in \mathbb{R}\), where \(S_0\) is as in (4.2). Next, the denominator \(\mathbb{E}|X_{1-b} - t|^p\) decreases in \(t \in (-\infty, b-1]\), increases in \(t \in [b, \infty)\), and attains its minimum over all \(t \in [b-1, b]\) (and thus over all \(t \in \mathbb{R}\)) only at \(t = t_b\), where \(t_b\) is as in (2.6). So,

\[
\max_{\lambda \in \mathbb{R} \setminus \{0\}, t \in \mathbb{R}} \hat{\rho}_t^p(\lambda, b, t) = \max_{t \in \mathbb{R}} \rho_\mu(b, t) = \rho_\mu(b, t_b) = R(p, b)
\]

(4.15) for all \(b \in (0, 1/2]\), in view of (2.4). Now (4.7), (4.5), and (4.13) yield

\[
c_{i,p} = \sup_{b \in [0, 1/2]} R(p, b) = \sup_{b \in [0, 1]} R(p, b).
\]

Thus, the proof of (2.5) and all of part (i) of Theorem 2.3 is complete.

(ii) That the equality in (1.2) obtains under either of the conditions (a) or (b) in part (ii) of Theorem 2.3 is trivial. If the condition (c) of part (ii) holds with \(\lambda = 0\), then \(X \overset{D}{=} 0\), and again the equality in (1.2) is trivial. If now (c) holds with some \(\lambda \in \mathbb{R} \setminus \{0\}\) – so that \(X \overset{D}{=} \lambda(X_{1-b} - t_b)\), then (2.5), (4.15), and (4.14) imply

\[
C_p = R(p, b_p) = \rho_\mu(b_p, t_{b_p}) = \frac{\mathbb{E}|X_{1-b_p} - t_{b_p}|^p}{\mathbb{E}|X_{1-b_p} - t_{b_p}|^p} = \frac{\mathbb{E}|X - \mathbb{E}X|^p}{\mathbb{E}|X|^p},
\]

whence the equality in (1.2) follows. Thus, for the equality in (1.2) to hold it is sufficient that one of the conditions (a), (b), or (c) be satisfied.

Let us now verify the necessity of one of these three conditions. W.l.o.g. condition (a) fails to hold, so that \(\mathbb{E}|X|^p < \infty\). If now \(p = 2\) then \(C_p = C_2 = 1\), and the necessity of the condition \(\mathbb{E}X = 0\) for the equality in (1.2) is obvious. It remains to consider the case when \(p \not= 2\) and \(\mathbb{E}|X|^p < \infty\). Suppose that one has the equality in (1.2) and let \(f = |\cdot|^p\). Then, by the definition of \(C_p\) in (2.5) and the equality (4.7), equalities take place in (4.3) and (4.4). In view of the condition \(\mathbb{E}|X|^p < \infty\), the integrals in (4.3) and (4.4) are both finite and equal to each other. So, the equality in (4.4) means that \(|\mathbb{E}X|^p \mu(\{0\} \times (0, 1/2]) = 0\). If now \(\mu(\{0\} \times (0, 1/2]) \not= 0\) then \(\mathbb{E}X = 0\), and the equality in (1.2) takes the form \(\mathbb{E}|X|^p = C_p \mathbb{E}|X|^p\); but, by part (v) of Theorem 2.3 (to be proved a bit later), the condition \(p \not= 2\) implies \(C_p > 1\), which yields \(\mathbb{E}|X|^p = 0\), and so, \(X \overset{D}{=} \lambda(X_{1-b_p} - t_{b_p})\) for \(\lambda = 0\). It remains to consider the case when \(p \not= 2\), \(\mathbb{E}|X|^p < \infty\), and \(\mu(\{0\} \times (0, 1/2]) = 0\). Then \(\mu(S_0) = \mu(S) = 1\), and the equality in (4.3) (again with \(f = |\cdot|^p\)), together with (2.5) and (4.7), will imply that \(\mathbb{E}|\lambda X_{1-b} - b|^p = C_p \mathbb{E}|\lambda X_{1-b} - b + \mathbb{E}X|^p\) for \(\mu\)-almost all \((\lambda, b) \in S_0\). In view of (4.14), (2.5), Proposition 2.2, and (4.15), this in turn yields

\[
\rho_\mu(b, -\mathbb{E}X/\lambda) = R(p, b_p) \geq R(p, b) = \rho_\mu(b, t_b)
\]

for \(\mu\)-almost all \((\lambda, b) \in S_0\). Now recall that for each \(b \in (0, 1/2]\) the maximum of \(\rho_\mu(b, t)\) in \(t \in \mathbb{R}\) is attained only at \(t = t_b\). It follows that for \(\mu\)-almost all \((\lambda, b) \in S_0\) one has
(i) \( R(p, b_p) = R(p, b) \) and hence, by Proposition 2.2, \( b = b_p \) and 
(ii) \(-\mathbb{E} X/\lambda = t_b = t_{b_p} \) or, equivalently, \( \lambda = -\mathbb{E} X/t_b = -\mathbb{E} X/t_{b_p} \) = \( \lambda_p \).

Therefore, \((\lambda, b) = (\lambda_p, b_p)\) for \( \mu \)-almost all \((\lambda, b) \in S_0\) and thus for \( \mu \)-almost all \((\lambda, b) \in S\). Now (4.1) shows that \( X + \lambda_p t_{b_p} = X - \mathbb{E} X \overset{D}{=} \lambda_p X_1 - b_p, b_p \) or, equivalently, \( X \overset{D}{=} \lambda_p (X_1 - b_p, \delta_p - t_{b_p}) \), which completes the proof of part (ii) of Theorem 2.3.

(iii) Part (iii) of Theorem 2.3 follows immediately by the symmetry (4.8) of \( R(p, b) \) in \( p \) and the definitions of \( C_p \) and \( b_p \) in (2.5) and Proposition 2.2, respectively.

(iv) As in (4.9), let \( r := p - 1 \), so that \( r \to \infty \). For a moment, take any \( k \in (0, \infty) \) and choose \( b = \frac{k}{r} \). Then, by (4.9), \( x \sim b = \frac{k}{r} \), and now (4.10) yields \( \partial_1 (x, r) \sim (1 - \frac{1}{2k}) r \), whence \( \partial_1 (r, x) \) is eventually (i.e., for all large enough \( r \)) positive or negative according as \( k \) is greater or less than \( \frac{1}{2} \). So, again by (4.9), for any real \( \tilde{k} \) and \( \tilde{k} \) such that \( 0 < \tilde{k} < \frac{1}{2} < \tilde{k} \), eventually \( \partial_b R(p, b) \mid_{b=\tilde{k}/\tilde{r}} < 0 < \partial_b R(p, b) \mid_{b=\tilde{k}/\tilde{r}} \). It follows by Proposition 2.2 that 

\[
 b_p \sim \frac{1}{2r},
\]

that is, \( b_p = k/r \) for some \( k \) varying with \( r \) so that \( k \to 1/2 \). Hence, 

\[
(1 - b_p)r + b_p^2 = (1 - k/r)r + (k/r)r \to e^{-1/2}.
\]

Next, \( \partial_1^{1/r} = (k/r)^{1/r} = \exp\left(\frac{1}{r} \ln \frac{k}{r}\right) = 1 + \frac{1}{r} \ln \frac{k}{r} + O\left(\left(\frac{1}{r} \ln \frac{k}{r}\right)^2\right) \) and 

\[
(1 - b_p)^{1/r} + b_p^{1/r} \sim \left[2 \exp\left\{\frac{1}{2r} \ln \frac{k}{r} + o\left(\frac{1}{r}\right)\right\}\right] \sim 2 \sqrt{\frac{k}{r}} \sim \frac{2p}{\sqrt{8p}}.
\]

Recalling now (2.5), (2.4), and (4.17), one obtains (2.8).

(v) Take any \( b \in (0, 1/2) \). Then 

\[
d_{2.1}(r) := \partial_r \partial_r \ln \left(b^r + (1 - b)^r\right) = \frac{(1 - b)^r b^r}{(b^r + (1 - b)^r)^2} \ln^2 \frac{1 - b}{b} > 0
\]

for all \( r > 0 \). Moreover, \( d_{2.2}(r) := \partial_r \partial_r \ln \left[(b^{1/r} + (1 - b)^{1/r})^r\right] = d_{2.1}(1/r)/r^3 > 0 \) for all \( r > 0 \). So, \( \partial_p \partial_r \ln R(p, b) = d_{2.1}(p - 1) + d_{2.2}(p - 1) > 0 \), which shows that \( R(p, b) \) is strictly log-convex in \( p \in (1, \infty) \). Also, \( \partial_p \ln R(p, b) \big|_{p=2} = 0 \), so that \( R(p, b) \) decreases in \( p \in (1, 2] \) and increases in \( p \in [2, \infty) \), with \( R(2, b) = 1 \). Therefore and in view of (2.5) – note in particular the attainment of the supremum there, \( C_p \) is strictly log-convex and hence continuous in \( p \in (1, \infty) \), and it also follows that \( C_p \) decreases in \( p \in (1, 2] \) and increases in \( p \in [2, \infty) \), with \( C_p = 1 \). Next, (2.8) shows that \( C_p \to \infty \) as \( p \to \infty \). Letting now \( p \downarrow 1 \) and using
(2.7), one has $q \to \infty$ and hence $C_p = C_q^{1/(q-1)} = (\frac{2q}{\sqrt{(8 + o(1))eq}})^{1/(q-1)} \to 2$. This completes the proof of part (v) of Theorem 2.3.

(vi) The proof of part (vi) of Theorem 2.3 is straightforward, in view of (2.5), Proposition 2.2, (2.4), and (2.6).

Proof of Corollary 3.1. The proof is based on ideas presented in [24, 26] concerning the use of the mentioned Yurinskiĭ martingale decomposition; similar ideas were also used e.g. in [1, 19, 17]. Consider the martingale defined by the formula

$$\zeta_j := \mathbb{E}_j(\tilde{Y} - \mathbb{E}\tilde{Y})$$

for $j \in 0, n$, where $\mathbb{E}_j$ stands for the conditional expectation given the $\sigma$-algebra generated by $(X_1, \ldots, X_j)$, with $\mathbb{E}_0 := \mathbb{E}$, and then consider the differences $\xi_i := \zeta_i - \zeta_{i-1}$. Next, for each $i \in 1, n$ introduce the r.v.

$$\eta_i := \mathbb{E}_i(Y - \tilde{Y}_i),$$

where $\tilde{Y}_i := g(X_1, \ldots, X_{i-1}, x_i, X_{i+1}, \ldots, X_n)$, so that $\xi_i = \eta_i - \mathbb{E}_{i-1}\eta_i$, since the r.v.'s $X_1, \ldots, X_n$ are independent. Also, in view of (3.1) or (3.7), for all $i \in 1, n$ and $z_i \in \mathbb{X}_i$ one has $|\eta_i| \leq \rho_i(X_i, z_i)$, whence, by (1.2),

$$E_{i-1} |\xi_i|^r = E_{i-1} |\eta_i - \mathbb{E}_{i-1}\eta_i|^r \leq C_r E_{i-1} |\eta_i|^r \leq C_r E_{i-1} \rho_i(X_i, z_i)^r = C_r \mathbb{E} \rho_i(X_i, z_i)^r$$

for all $r \in (1, \infty)$. Now (3.3) follows from (3.2), since $\zeta_n = Y - \mathbb{E}Y$ and $C_2 = 1$.}

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