Inverse electromagnetic scattering problems by a doubly periodic structure

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Abstract

Consider the problem of scattering of electromagnetic waves by a doubly periodic structure. The medium above the structure is assumed to be inhomogeneous characterized completely by an index of refraction. Below the structure is a perfect conductor or an imperfect conductor partially coated with a dielectric. Having established the well-posedness of the direct problem by the variational approach, we prove the uniqueness of the inverse problem, that is, the unique determination of the doubly periodic grating with its physical property and the index of refraction from a knowledge of the scattered near field by a countably infinite number of incident quasi-periodic electromagnetic waves. A key ingredient in our proofs is a novel mixed reciprocity relation derived in this paper.

Keywords: Uniqueness, Maxwell’s equations, inhomogeneous medium, doubly periodic structure, mixed boundary conditions, mixed reciprocity relation, inverse problem.

1 Introduction

Scattering theory in periodic structures has many applications in micro-optics, radar imaging and non-destructive testing. We refer to [20] for historical remarks and details of these applications. This paper is concerned with direct and inverse problems of electromagnetic scattering by a doubly periodic structure. The medium above the structure is assumed to be inhomogeneous. Below the structure is a perfect conductor which may be partially coated with a dielectric.

Let the doubly periodic structure be described by the doubly periodic surface

$$\Gamma_1 := \{ x \in \mathbb{R}^3 \mid x_3 = f(x_1, x_2) \},$$

where $f \in C^2(\mathbb{R}^2)$ is a $2\pi$-periodic function of $x_1$ and $x_2$:

$$f(x_1 + 2n_1\pi, x_2 + 2n_2\pi) = f(x_1, x_2) \quad \forall n = (n_1, n_2) \in \mathbb{Z}^2.$$

Assume that the medium above the structure $\Gamma_1$ is filled with an inhomogeneous, isotropic, conducting or dielectric medium of electric permittivity $\epsilon > 0$, magnetic permeability $\mu > 0$ and electric conductivity $\sigma \geq 0$. Suppose the medium is non-magnetic, that is, the magnetic
permeability $\mu$ is a fixed constant in the region above $\Gamma_1$ and the field is source free. Then the electromagnetic wave propagation is governed by the time-harmonic Maxwell equations (with the time variation of the form $e^{-i\omega t}$, $\omega > 0$)

$$\text{curl } E - i\omega \mu H = 0, \quad \text{curl } H + i\omega(\epsilon + i\sigma/\omega)E = 0,$$

where $E$ and $H$ are the electric and magnetic fields, respectively. Suppose the inhomogeneous medium is $2\pi$-periodic with respect to the $x_1$ and $x_2$ directions, that is, for all $n = (n_1, n_2) \in \mathbb{Z}^2$,

$$\epsilon(x_1 + 2\pi n_1, x_2 + 2\pi n_2, x_3) = \epsilon(x_1, x_2, x_3), \quad \sigma(x_1 + 2\pi n_1, x_2 + 2\pi n_2, x_3) = \sigma(x_1, x_2, x_3).$$

Suppose above the structure $\Gamma_1$ is another doubly periodic surface defined by

$$\Gamma_0 := \{ x \in \mathbb{R}^3 \mid x_3 = g(x_1, x_2) \},$$

where $g \in C^2(\mathbb{R}^2)$ is a $2\pi$-periodic function of $x_1$ and $x_2$:

$$g(x_1 + 2n_1 \pi, x_2 + 2n_2 \pi) = g(x_1, x_2) \quad \forall n = (n_1, n_2) \in \mathbb{Z}^2,$$

which separates the region above $\Gamma_1$ into two parts:

$$\Omega_0 := \{ x \in \mathbb{R}^3 \mid x_3 > g(x_1, x_2) \},$$

$$\Omega_1 := \{ x \in \mathbb{R}^3 \mid f(x_1, x_2) < x_3 < g(x_1, x_2) \}.$$

Assume further that $\epsilon(x) = \epsilon_0$, $\sigma = 0$ for $x \in \Omega_0$ (which means that the medium above the layer is lossless) and that the doubly periodic surface $\Gamma_1$ is a perfectly conductor coated partially with a dielectric.

Consider the scattering of the electromagnetic plane wave

$$E^i(x) = p e^{ik_0 x \cdot d}, \quad H^i(x) = r e^{ik_0 x \cdot d}$$

incident on the doubly periodic structure $\Gamma_0$ from the top region $\Omega_0$, where $k_0 = \sqrt{\epsilon_0 \mu} \omega$ is the wave number, $d = (\alpha_1, \alpha_2, -\beta) = (\cos \theta_1 \cos \theta_2, \cos \theta_1 \sin \theta_2, -\sin \theta_1)$ is the incident wave vector whose direction is specified by $\theta_1$ and $\theta_2$ with $0 < \theta_1 \leq \pi$, $0 < \theta_2 \leq 2\pi$ and the vectors $p$ and $r$ are polarization directions satisfying that $p = \sqrt{\mu/\epsilon_0}(r \times d)$ and $r \perp d$. The problem of scattering of time-harmonic electromagnetic waves in this model leads to the following problem (the magnetic field $H$ is eliminated):

$$\text{curl curl } E - k_0^2 E = 0 \quad \text{in } \Omega_0, \quad (1.1)$$

$$\text{curl curl } E - k_0^2 q(x) E = 0 \quad \text{in } \Omega_1, \quad (1.2)$$

$$\nu \times E|_+ = \nu \times E|_-, \quad \nu \times \text{curl } E|_+ = \lambda_0 \nu \times \text{curl } E|_- \quad \text{on } \Gamma_0, \quad (1.3)$$

$$\nu \times E = 0 \quad \text{on } \Gamma_1, \quad (1.4)$$

$$\nu \times \text{curl } E - i\rho (\nu \times E) \times \nu = 0 \quad \text{on } \Gamma_1, \quad (1.5)$$

where $q(x) = (\epsilon(x) + i\sigma(x)/\omega)/\epsilon_0$ is the refractive index, $\nu$ is the unit normal at the boundary, $E = E^i + E^s$ is the total field in $\Omega_0$ with $E^s$ being the scattered electric field, $\Gamma_1 = \Gamma_{1,D} \cup \Gamma_{1,I}$, $\lambda_0$ and $\rho$ are two positive constants.

We require the scattered field $E$ to be $\alpha$-quasi-periodic with respect to $x_1$ and $x_2$ in the sense that $E(x_1, x_2, x_3) e^{-i\alpha \cdot x}$ is $2\pi$ periodic with respect to $x_1$ and $x_2$, where $\alpha = (\alpha_1, \alpha_2, 0) \in \mathbb{R}^3$. It
is further required that the scattered field $E$ satisfies the following Rayleigh expansion radiation condition as $x_3 \to +\infty$:

$$E^s(x) = \sum_{n \in \mathbb{Z}^2} E_n e^{i(\alpha_n \cdot x + \beta_n x_3)}, \quad x_3 > g_+ := \max_{x_1, x_2} g(x_1, x_2),$$  \hspace{1cm} (1.6)

where $\alpha_n = (\alpha_1 + n_1, \alpha_2 + n_2, 0) \in \mathbb{R}^3$, $E_n = (E_n^{(1)}, E_n^{(2)}, E_n^{(3)}) \in \mathbb{C}^3$ are the Rayleigh coefficients and

$$\beta_n = \begin{cases} 
(k_0^2 - |\alpha_n|^2)^{1/2} & \text{if } |\alpha_n|^2 \leq k_0^2, \\
(i(|\alpha_n|^2 - k_0^2)^{1/2} & \text{if } |\alpha_n|^2 > k_0^2
\end{cases}$$

with $i^2 = -1$. From the fact that $\text{div} E^s(x) = 0$ it is clear that

$$\alpha_n \cdot E_n + \beta_n E_n^{(3)} = 0.$$  

Throughout this paper we assume that $\beta_n \neq 0$ for all $n \in \mathbb{Z}^2$.

The direct problem is to compute the scattered field $E^s$ in $\Omega_0$ and $E$ in $\Omega_1$ given the incident wave $E^i$, the diffraction grating profiles $\Gamma_0$ and $\Gamma_1$ with the corresponding boundary conditions and the refractive index $q(x)$. Our inverse problem is to determine the grating profile $\Gamma_1$ together with the impedance coefficient $\rho$ in the case when the interface grating profile $\Gamma_0$ is known and the refractive index $q$ in the case when the grating surfaces $\Gamma_0$ and $\Gamma_1$ are known and flat, utilizing the knowledge of the incident wave $E^i$ and the total tangential electric field $\nu \times E$ on a plane $\Gamma_h = \{x \in \mathbb{R}^3 \mid x_3 = h\}$ above the inhomogeneous layer.

Problems of scattering of electromagnetic waves by a doubly periodic structure have been studied by many authors using both integral and variational methods. The reader is referred to, e.g. [1, 3, 4, 5, 11, 12, 18, 21] for results on existence, uniqueness, and numerical approximations of solutions to the direct problems. Compared with the direct problem, not much attention has been paid to inverse problems from doubly periodic structures although they are not only mathematically interesting but have many important applications. For the case when $\Gamma_1, I = \emptyset$ and the medium above the periodic structure $\Gamma_1 = \Gamma_{1,D}$ is homogeneous, the inverse scattering problem has been considered in [2, 7, 6]. If the medium is lossy above the perfectly reflecting periodic structure, Ammari [2] proved a global uniqueness result for the inverse problem with one incident plane wave. If the medium is lossless above the perfectly reflecting periodic structure, a local uniqueness result was obtained in [7] for the inverse problem with one incident plane wave by establishing a lower bound of the first eigenvalue of the curl-curl operator with the boundary condition (1.4) in a bounded, smooth convex domain in $\mathbb{R}^3$. The stability of the inverse problem was also studied in [7]. Recently in [9], for the class of perfectly reflecting doubly periodic polyhedral structures global uniqueness results have been established in [6] for the inverse problem in the case of lossless medium above the structure, using only a minimal number (though unknown) of incident plane waves. Further, for a general Lipschitz, bi-periodic, partly coated structure $\Gamma_1$ a global uniqueness result was proved in [13] for the inverse problem in the case of a lossless, homogeneous medium above the structure, using infinitely many incident dipole sources.

On the other hand, for the case when $\Gamma_{1,I} = \emptyset$ (i.e. $\Gamma_1 = \Gamma_{1,D}$), $\lambda_0 = 1$ and the grating surfaces $\Gamma_0$ and $\Gamma_1$ are known and flat, a global uniqueness result was obtained in [14] for reconstructing the refractive index $q$, using all electric dipole incident waves (see [15] for the corresponding result in the 2D case).
In this paper, we prove global uniqueness results for the inverse problem of recovering a general smooth bi-periodic profile with a mixed boundary condition and a known bi-periodic interface from a knowledge of near field measurements above the known interface with a countably infinite number of quasi-periodic incident waves $E^1(x;m) = (1/k^2_0)\text{curl} \text{curl} [p \exp(\imath \alpha_m \cdot x - \imath \beta_m x_3)]$, $m = (m_1, m_2) \in \mathbb{Z}^2$. Further, we also establish a global uniqueness result for the inverse problem of determining the refractive index $q$ which depends on only one direction ($x_1$ or $x_2$) for the case when $\Gamma_{1,1} = \emptyset$ (i.e. $\Gamma_1 = \Gamma_{1,D}$) and the grating surfaces $\Gamma_0$ and $\Gamma_1$ are known and flat, using a countably infinite number of quasi-periodic incident waves $E^1(x;m)$. This is an improvement to the result of [14]. A key ingredient in our proofs is a novel mixed reciprocity relation derived in this paper for bi-periodic structures.

The rest of the paper is organized as follows. In Section 2 we introduce some suitable quasi-periodic function spaces and the Dirichlet-to-Neumann map on an artificial boundary above the domain. In Section 3, we establish the well-posedness of the scattering problem (1.1)-(1.6), employing a variational approach. Section 4 is devoted to the inverse problems. In Subsection 4.2 novel mixed reciprocity relations are established for doubly periodic structures, which play a key role in the proofs of the uniqueness results for our inverse problems. Subsection 4.3 is concerned with the unique reconstruction of the refractive index $q$ for the periodic cell of the grating profiles. To this end and for the subsequent analysis, we use $\Omega_h = \{x \in \mathbb{R}^3 | 0 < x_1, x_2 < 2\pi, f(x_1, x_2) < x_3 < h\}$ for $h > \text{max}\{f(x') | x' \in \mathbb{R}^2\}$.

We now introduce some vector quasi-periodic Sobolev spaces. Let

$$H(\text{curl}, \Omega_h) = \{E(x) = \sum_{n \in \mathbb{Z}^2} E_n(x_3) \exp(\imath \alpha_n \cdot x) | E \in (L^2(\Omega_h))^3, \text{curl } E \in (L^2(\Omega_h))^3\}$$

with the norm

$$||E||^2_{H(\text{curl}, \Omega_h)} = ||E||^2_{L^2(\Omega_h)} + ||\text{curl } E||^2_{L^2(\Omega_h)}$$

Note that the $\alpha$-quasi-periodic space $H(\text{curl}, \Omega_h)$ is a subset of the classical vector space $\mathbb{H}(\text{curl}, \Omega_h)$ defined by

$$\mathbb{H}(\text{curl}, \Omega_h) = \{E \in (L^2(\Omega_h))^3 | \text{curl } E \in (L^2(\Omega_h))^3\}$$
Recalling the trace theorem on $H^1$ where
\[ e^{2\pi i\alpha_1} E(0, x_2, x_3) \times e_1 = E(2\pi, x_2, x_3) \times e_1, \]
\[ e^{2\pi i\alpha_2} E(x_1, 0, x_3) \times e_2 = E(x_1, 2\pi, x_3) \times e_2, \]
where $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$.

To deal with the mixed boundary conditions (1.4) and (1.5), we introduce the subspace of $H^1$:
\[ X := \{ E \in H^1 | \nu \times E|_{\Gamma_{1, D}} = 0, \nu \times E|_{\Gamma_{1, I}} \in L^2(\Gamma_{1, I}) \} \]
with the norm
\[ ||E||^2_X = ||E||^2_{H^1} + ||\nu \times E||^2_{L^2(\Gamma_{1, I})} \]
where $L^2(\Gamma_{1, I}) = \{ E \in (L^2(\Gamma_{1, I}))^3 \mid \nu \cdot E = 0 \text{ on } \Gamma_{1, I} \}$.

For $x' = (x_1, x_2, h) \in \Gamma_h$, $s \in \mathbb{R}$ define
\[ H^s_t(\Gamma_h) = \{ E(x') = \sum_{n \in \mathbb{Z}^2} E_n \exp(i\alpha_n \cdot x') \mid E_n \in \mathbb{C}^3, e_3 \cdot E = 0, \}
\[ ||E||^2_{H^s_t(\Gamma_h)} = \sum_{n \in \mathbb{Z}^2} (1 + |\alpha_n|^2 s) |E_n|^2 < +\infty \}
\[ H^s_t(\text{div}, \Gamma_h) = \{ E(x') = \sum_{n \in \mathbb{Z}^2} E_n \exp(i\alpha_n \cdot x') \mid E_n \in \mathbb{C}^3, e_3 \cdot E = 0, \}
\[ ||E||^2_{H^s_t(\text{div}, \Gamma_h)} = \sum_{n \in \mathbb{Z}^2} (1 + |\alpha_n|^2 s (|E_n|^2 + |E_n \cdot \alpha_n|^2) < +\infty \}
\[ H^s_t(\text{curl}, \Gamma_h) = \{ E(x') = \sum_{n \in \mathbb{Z}^2} E_n \exp(i\alpha_n \cdot x') \mid E_n \in \mathbb{C}^3, e_3 \cdot E = 0, \}
\[ ||E||^2_{H^s_t(\text{curl}, \Gamma_h)} = \sum_{n \in \mathbb{Z}^2} (1 + |\alpha_n|^2 s (|E_n|^2 + |E_n \times \alpha_n|^2) < +\infty \}
and write $L^2_t(\Gamma_h) = H^0_t(\Gamma_h)$. We have the duality result:
\[ (H^s_t(\text{div}, \Gamma_h))^\prime = H^{-s-1}_t(\text{curl}, \Gamma_h). \]

Recalling the trace theorem on $H^1$, we have
\[ H^{-1/2}_t(\text{div}, \Gamma_h) = \{ e_3 \times E|_{\Gamma_h} \mid E \in H^1 \} \]
and that the trace mapping from $H_t$ to $H^{-1/2}_t(\text{div}, \Gamma_h)$ is continuous and surjective (see $\S$ and the references there). We also need the trace space $Y(\Gamma_0)$ and its duality space $Y'(\Gamma_0)$:
\[ Y(\Gamma_0) = \{ f \in H^{-1/2}_t(\Gamma_0) \mid \nabla_{\Gamma_0} \cdot f \in H^{-1/2}_t(\Gamma_0) \}
\[ Y'(\Gamma_0) = \{ f \in H^{-1/2}_t(\Gamma_0) \mid \nabla_{\Gamma_0} \times f \in H^{-1/2}_t(\Gamma_0) \}, \]
where \( \nabla_{\Gamma_0} \) denotes the surface gradient on \( \Gamma_0 \). Note that the trace space \( Y(\Gamma_0) \) can also be defined as follows (see [10] and [17, p. 58-59]):

\[
Y(\Gamma_0) = \{ f \in (H^{-1/2}(\Gamma_0))^3 \mid \text{there exists } E \in H(\text{curl}, \Omega_h \backslash \overline{\Omega}_1) \text{ with } \nu \times E = f \text{ on } \Gamma_0, \nu \times E = 0 \text{ on } \Gamma_h \}
\]

for \( h > g_+ \), with norm

\[
\| f \|_{Y(\Gamma_0)} = \inf \{ \| E \|_{H(\text{curl}, \Omega_h \backslash \overline{\Omega}_1)} \mid E \in H(\text{curl}, \Omega_h \backslash \overline{\Omega}_1), \nu \times E \big|_{\Gamma_0} = f, \nu \times E \big|_{\Gamma_h} = 0 \}.
\]

We assume throughout this paper that \( q \) satisfies the following conditions:

- (A1) \( q \in C^1(\overline{\Omega}_1) \) and \( q(x) = 1 \) for \( x \in \Omega_0 \);
- (A2) \( \text{Im} [q(x)] \geq 0 \) for all \( x \in \overline{\Omega}_1 \) if \( \Gamma_{1,I} \neq \emptyset \) and \( \text{Im} [q(x_0)] > 0 \) for some \( x_0 \in \overline{\Omega}_1 \) if \( \Gamma_{1,I} = \emptyset \);
- (A3) \( \text{Re} [q(x)] \geq \gamma \) for all \( x \in \overline{\Omega}_1 \) for some positive constant \( \gamma \).

### 3 The direct scattering problem

In this section we will establish the solvability of the scattering problem (1.1)-(1.6), employing the variational method. To this end, we propose a variational formulation of the scattering problem in a truncated domain by introducing a transparent boundary condition on \( \Gamma_h \) for \( h > g_+ \).

Let \( x' = (x_1, x_2, h) \in \Gamma_h \) for \( h > g_+ \). For \( \tilde{E} \in H^{t-1/2}(\text{div}, \Gamma_h) \) with

\[
\tilde{E}(x') = \sum_{n \in \mathbb{Z}^2} \tilde{E}_n \exp(i \alpha_n \cdot x')
\]

define the Dirichlet-to-Neumann map \( R : H^{t-1/2}(\text{div}, \Gamma_h) \to H^{t-1/2}(\text{curl}, \Gamma_h) \) by

\[
(R \tilde{E})(x') = (e_3 \times \text{curl } E) \times e_3 \quad \text{on } \Gamma_h,
\]

where \( E \) satisfying the Rayleigh expansion condition (1.6) is the unique quasi-periodic solution to the problem

\[
\text{curl curl } E - k^2 E = 0 \quad \text{for } x_3 > h, \quad \nu \times E = \tilde{E}(x') \quad \text{on } \Gamma_h.
\]

The map \( R \) is well-defined and can be used to replace the radiation condition (1.6) on \( \Gamma_h \). The scattering problem (1.1)-(1.6) can then be transformed into the following boundary value problem in the truncated domain \( \Omega_h \):

\[
\begin{align*}
\text{curl curl } E - k^2_0 E & = 0 \quad \text{in } \Omega_0, \quad (3.1) \\
\text{curl curl } E - k^2_0 q E & = 0 \quad \text{in } \Omega_1, \quad (3.2) \\
\nu \times E |_+ - \nu \times E |_- & = f_1, \quad \nu \times \text{curl } E |_+ - \lambda_0 \nu \times \text{curl } E |_- = f_2 \quad \text{on } \Gamma_0, \quad (3.3) \\
\nu \times E & = f_3 \quad \text{on } \Gamma_{1,D}, \quad \nu \times \text{curl } E - i \rho (\nu \times E) \times \nu = f_4 \quad \text{on } \Gamma_{1,I}, \quad (3.4) \\
(\text{curl } E)_T - \mathcal{R}(e_3 \times E) & = 0 \quad \text{on } \Gamma_h, \quad (3.5)
\end{align*}
\]

where \( f_1 = -\nu \times E |_{\Gamma_0} \in Y(\Gamma_0), \ f_2 = -\nu \times \text{curl } E |_{\Gamma_0} \in Y(\Gamma_0)', \ f_3 = f_4 = 0 \) and, for any vector \( F, \ (F)_T = (\nu \times F) \times \nu \) denotes its tangential component on a surface.
Remark 3.1. In the case when \( k_0^2 q(x) \equiv k_1^2 \) is a constant and the incident field is given by the electric dipole source \( E^i(x) = G_1(x, y_0) r \) for \( y_0 \in \Omega_1 \) and \( r \in \mathbb{R}^3 \) (e.g. in the problem (4.2)-(4.8) of Lemma 3.1), we have \( f_1 = \nu \times E^i|\Gamma_0 \in Y(\Gamma_0), f_2 = \lambda_0 \nu \times \text{curl} E^i|\Gamma_0 \in Y(\Gamma_0)', f_3 = -\nu \times E^i|\Gamma_{1,D} \in Y(\Gamma_{1,D}), f_4 = -\nu \times \text{curl} E^i|\Gamma_{1,I} + i \rho (\nu \times E^i) \times \nu|\Gamma_{1,I} \in L^2_2(\Gamma_{1,I}) \) in the problem (3.1)-(3.5), where \( Y(\Gamma_{1,D}) \) is defined in the same way as \( Y(\Gamma_0) \) with \( \Gamma_0 \) replaced by \( \Gamma_{1,D} \) (see [13]).

Define

\[
Y := \{ E \in H(\text{curl}, \Omega_1) \cap H(\text{curl}, \Omega_h \setminus \Omega_1) | \nu \times E|\Gamma_{1,D} = f_3, \\
\text{\quad} \nu \times E|\Gamma_{1,I} \in L^2_2(\Gamma_{1,I}), \nu \times E|_+ - \nu \times E|_- = f_1 \text{ on } \Gamma_0 \}
\]

with the norm

\[
\| E \|_Y^2 = \| E \|_{H(\text{curl}, \Omega_1)}^2 + \| E \|_{H(\text{curl}, \Omega_h \setminus \Omega_1)}^2 + \| \nu \times E \|_{L^2_2(\Gamma_{1,I})}^2.
\]

Then the variational formulation for the problem (3.1)-(3.5) is given as follows: find \( E \in Y \) such that

\[
A(E, F) = B(F) \quad \forall \ F \in X,
\]

where

\[
A(E, F) := \lambda_0 \int_{\Omega_1} (\text{curl} \ E \cdot \text{curl} \bar{F} - k_0^2 q E \cdot \bar{F}) dx \\
+ \int_{\Omega_h \setminus \Omega_1} (\text{curl} \ E \cdot \text{curl} \bar{F} - k_0^2 E \cdot \bar{F}) dx \\
- i\lambda_0 \rho \int_{\Gamma_{1,I}} E_T \cdot \bar{F}_T ds - \int_{\Gamma_h} R(\nu \times E) \cdot (\nu \times \bar{F}) ds,
\]

\[
B(F) := \int_{\Gamma_0} f_2 \cdot \bar{F}_T ds + \lambda_0 \int_{\Gamma_{1,I}} f_4 \cdot \bar{F}_T ds.
\]

If \( f_1 \in Y(\Gamma_0) \), then there exits \( \tilde{E}_0 \in H(\text{curl}, \Omega_h \setminus \Omega_1) \) such that \( \nu \times \tilde{E}_0|\Gamma_0 = f_1, \nu \times \tilde{E}_0|\Gamma_h = 0 \).

Similarly, for \( f_3 \in Y(\Gamma_{1,D}) \) there exists a function \( E_1 \in H(\text{curl}, \Omega_h) \) such that \( \nu \times E_1|\Gamma_{1,D} = f_3, \nu \times E_1|\Gamma_h = 0 \) and \( \nu \times E_1|\Gamma_{1,I} \in L^2_2(\Gamma_{1,I}) \). Let \( \tilde{E} = E - E_0 - E_1 \), where \( E_0 \) is a function in \( \Omega_h \) satisfying that \( E_0|_{\Omega_h \setminus \Omega_1} = \tilde{E}_0 \) and \( E_0|_{\Omega_1} = 0 \). Then \( \tilde{E} \in X \) and the variational problem (3.6) is equivalent to the problem: find \( \tilde{E} \in X \) such that

\[
A(\tilde{E}, F) = \tilde{B}(F) \quad \forall \ F \in X,
\]

where \( \tilde{B}(F) = B(F) - A(E_0, F) - A(E_1, F) \).

Theorem 3.2. Assume that the conditions \( \{A1\}-\{A3\} \) are satisfied. Then the problem (3.1)-(3.5) has a unique solution \( E \in Y \) for any \( f_1 \in Y(\Gamma_0), f_2 \in Y(\Gamma_0)', f_3 \in Y(\Gamma_{1,D}) \) and \( f_4 \in L^2_2(\Gamma_{1,I}) \). Furthermore, we have

\[
\| E \|_Y \leq C(||f_1||_{Y(\Gamma_0)} + ||f_2||_{Y(\Gamma_0)'} + ||f_3||_{Y(\Gamma_{1,D})} + ||f_4||_{L^2_2(\Gamma_{1,I})}),
\]

where \( C \) is a positive constant depending only on \( \Omega_h \).
Proof. It is enough to prove that the problem (3.7) has a unique solution $\tilde{E} \in X$ with the required estimate.

We first prove the uniqueness of solutions. To this end, let $f_j = 0$, $j = 1, 2, 3, 4$ and let $F = \tilde{E}$ in (3.7). Then $A(\tilde{E}, \tilde{E}) = 0$, that is,

$$
\lambda_0 \int_{\Omega_1} (|\text{curl} \tilde{E}|^2 - k_0^2 q|\tilde{E}|^2) dx + \int_{\Omega_h \setminus \Gamma_1} (|\text{curl} \tilde{E}|^2 - k_0^2 |\tilde{E}|^2) dx
$$

$$
- i\lambda_0 \rho \int_{\Gamma_{1,I}} |\tilde{E}_T|^2 ds - \int_{\Gamma_h} \mathcal{R}(\nu \times \tilde{E}) \cdot (\nu \times \tilde{E}) ds = 0.
$$

Taking the imaginary part of the above equation and noting that the imaginary part of the last integral in the above equation is non-negative (see [13, Equation (16)]), we deduce that

$$
k_0^2 \int_{\Omega_1} \text{Im}(q)|\tilde{E}|^2 dx + \rho \int_{\Gamma_{1,I}} |\tilde{E}_T|^2 ds \leq 0. \tag{3.8}
$$

If $\Gamma_{1,I} = \emptyset$, then by (3.8) and the condition (A2) we have $\tilde{E} \equiv 0$ in a small ball $B(x_0; \delta) \subset \Omega_1$. By [9, Theorem 6] we have $\tilde{E} \in (H^1(\Omega_1))^3$. Thus, by the unique continuation principle (see [19, Theorem 2.3]) we have $\tilde{E} \equiv 0$ in $\Omega_1$. This, together with the transmission condition (3.3) and Holmgren’s uniqueness theorem, implies that $\tilde{E} \equiv 0$ in $\Omega_h \setminus \Gamma_1$. If $\Gamma_{1,I} \neq \emptyset$, then (3.8) and the boundary condition (3.4) yield that $\nu \times \tilde{E}|_{\Gamma_{1,I}} = 0$ and $\nu \times \text{curl} \tilde{E}|_{\Gamma_{1,I}} = 0$. By unique continuation principle again we have $\tilde{E} \equiv 0$ in $\Omega_1$. Again, from the transmission condition (3.3) and Holmgren’s uniqueness theorem it follows that $\tilde{E} \equiv 0$ in $\Omega_h \setminus \Gamma_1$. The uniqueness of solutions is thus proved for both cases.

Now, arguing similarly as in the proof of Theorem 4.1 in [14] or Theorem 3.1 in [13] (see [14, 13] for details) we can prove that the problem (3.7) has a solution $\tilde{E} \in X$ satisfying the estimate

$$
\|\tilde{E}\| \leq C(\|f_2\|_Y(\Gamma_0) + \|f_4\|_{L^2(\Gamma_{1,I})} + \|\tilde{E}_0\|_{H(\text{curl} \Omega_h \setminus \Gamma_1)} + \|E_1\|_X)
$$

with $C$ depending only on $\Omega_h$. Since $\tilde{E} = E - E_0 - E_1$, and by taking the infimum over all $\tilde{E}_0 \in H(\text{curl} \Omega_h \setminus \Gamma_1)$ such that $\nu \times \tilde{E}_0|_{\Gamma_0} = f_1$ and $\nu \times \tilde{E}_0|_{\Gamma_h} = 0$ and over all $E_1 \in H(\text{curl} \Omega_h)$ such that $\nu \times E_1|_{\Gamma_{1,D}} = f_3$, $\nu \times E_1|_{\Gamma_h} = 0$ and $\nu \times E_1|_{\Gamma_{1,I}} \in L^2(\Gamma_{1,I})$ the desired estimate follows (on taking into account the definition of the norm on $Y(\Gamma_0)$ and $Y(\Gamma_{1,D})$).

\[\square\]

4 The inverse problems

In this section we consider the inverse problems of determining the doubly periodic grating profile $f$ with its physical property and the refractive index $q$ from a knowledge of the incident and scattered fields. To this end, we need the free-space quasi-periodic Green’s function

$$
G_0(x, y) = \frac{1}{8\pi^2} \sum_{n \in \mathbb{Z}^2} \frac{1}{i\beta_n} \exp(i\alpha_n \cdot (x - y) + i\beta_n|x_3 - y_3|)
$$

provided $\beta_n \neq 0$ for all $n \in \mathbb{Z}^2$ (see [18]). In the neighborhood of $x = y$, $G_0$ can be represented in the form $G_0(x, y) = \Phi(x, y) + a(x - y)$, where $\Phi(x, y) = \exp(ik_0|x - y|)/(4\pi|x - y|)$ is the
fundamental solution to the three-dimensional Helmholtz equation \((\Delta + k_0^2)u = 0\) and \(a(x - y)\) is a \(C^\infty\) function (see [16] for the 2D case). We now introduce the quasi-periodic Green’s tensor \(G_0 \in \mathbb{C}^{3 \times 3}\) for the time-harmonic Maxwell equations:

\[
G_0(x, y) = G_0(x, y)I + \frac{1}{k_0^2} \nabla_x \text{div}_x (G_0(x, y)I), \quad x \neq y,
\]  (4.1)

where \(I\) is a 3 \times 3 identity matrix. Consider the following incident dipole source located at \(z \in \mathbb{R}^3\) with polarization \(p\) (\(|p| = 1\)):

\[
E^i(x) = G_0(x, z)p, \quad x \neq z.
\]

Clearly, we have

\[
\text{curl curl } E^i(x) - k_0^2 E^i(x) = 0, \quad x \neq z.
\]

### 4.1 Mixed reciprocity relations

We establish two mixed reciprocity relations for the doubly periodic structure, which play a key role in the proofs of uniqueness results for the inverse problems.

**Lemma 4.1.** Assume that \(k_0^2g(x) \equiv k_1^2\) is a constant. For \(m = (m_1, m_2) \in \mathbb{Z}^2\) let \(E^i(x; m) = (1/k_0^2)\text{curl curl } [p \exp(\ii \alpha_m \cdot x - \ii \beta_m x_3)]\) and let \(E(x; m)\) (which is the sum \(E^i(x; m) + E^s(x; m)\) in \(\Omega_0\)) be the solution to the the scattering problem [11] - [16] with \(E^i(x) = E^i(x; m)\). On the other hand, define \(\widehat{\alpha} := -\alpha\) and for \(y_0 \in \Omega_1\) and \(r \in \mathbb{R}^3\) let \(E^i(x; y_0) = \widehat{G}_1(x, y_0)r\) and let \(\widehat{E}(x; y_0)\) solve the scattering problem:

\[
\text{curl curl } \widehat{E} - k_0^2 \widehat{E} = 0 \quad \text{in } \Omega_0, \quad (4.2)
\]

\[
\text{curl curl } \widehat{E} - k_0^2 \widehat{E} = 0 \quad \text{in } \Omega_1 \setminus \{y_0\}, \quad (4.3)
\]

\[
\nu \times \widehat{E}|_+ = \nu \times \widehat{E}|_- \quad \nu \times \text{curl } \widehat{E}|_+ = \lambda_0 \nu \times \text{curl } \widehat{E}|_- \quad \text{on } \Gamma_0, \quad (4.4)
\]

\[
\nu \times \widehat{E} = 0 \quad \text{on } \Gamma_{1,D}, \quad \nu \times \text{curl } \widehat{E} - \ii \rho (\nu \times \widehat{E}) \times \nu = 0 \quad \text{on } \Gamma_{1,I}, \quad (4.5)
\]

\[
\widehat{E}(x; y_0) = E^i(x; y_0) + \widehat{E}^s(x; y_0) \quad \text{in } \Omega_1 \setminus \{y_0\}, \quad (4.6)
\]

\[
\widehat{E}(x; y_0) = \sum_{n \in \mathbb{Z}^2} \widehat{E}_n(y_0) \exp(\ii \alpha_n \cdot x + \beta_n x_3) \quad \text{in } x_3 > g_+, \quad (4.7)
\]

\[
\widehat{\alpha}_n \cdot \widehat{E}_n + \widehat{\beta}_n \cdot \widehat{E}_n^{(3)} = 0. \quad (4.8)
\]

Here, \(\widehat{\alpha}_n\) and \(\widehat{\beta}_n\) are defined by

\[
\widehat{\alpha}_n = (-\alpha_1 + n_1, -\alpha_2 + n_2, 0) \quad \text{and} \quad \widehat{\beta}_n = \begin{cases} \sqrt{k_0^2 - |\alpha_n|^2} & \text{for } |\alpha_n|^2 \leq k_0^2, \\ i\sqrt{|\alpha_n|^2 - k_0^2} & \text{for } |\alpha_n|^2 > k_0^2 \end{cases}
\]

and \(\widehat{G}_1(x, y_0)\) is defined by [4.11] with \(\alpha_n\) and \(k_1^2\) replaced by \(\widehat{\alpha}_n\) and \(k_0^2\), respectively. Then we have

\[
r \cdot E(y_0; m) = \frac{8\pi^2 \ii}{\lambda_0} \widehat{\beta}_m \widehat{E}_m(y_0) \cdot p. \quad (4.9)
\]
Proof. Note first that $E(x; m)$ and $\hat{E}(x; y_0)$ are well-defined by the well-posedness of the direct scattering problem. Applying Green’s theorem in $\Omega \setminus B(y_0, \delta)$ and using the fact that contributions of the vertical line integrals cancel out due to the periodicity, we have

$$0 = \int_{\Omega \setminus B(y_0, \delta)} \left\{ \text{curl} \ E(x; m) \cdot [\hat{G}_1(x, y_0)r] - E(x; m) \cdot \text{curl} [\hat{G}_1(x, y_0)r] \right\} dx$$

$$= \left[ \int_{\Gamma_0} - \int_{\Gamma_1} + \int_{\partial B(y_0)} \right] \left\{ \nu \times \text{curl} E(x) \cdot [\hat{G}_1(x, y_0)r] - \nu \times \text{curl} [\hat{G}_1(x, y_0)r] \cdot E(x) \right\} ds$$

$$:= I_1 + I_2 + I_3, \quad (4.10)$$

where $B(y_0, \delta)$ is a small ball centered at $y_0$ with the radius $\delta$ such that $B(y_0, \delta) \subset \Omega$.

We now analyze the asymptotic behavior of $I_3$ as $\delta \to 0$. From the definition that $\hat{G}_1(x, y_0) = \hat{G}_1(x, y_0) \mathbb{I} + k_1^{-2} \nabla_x \text{div}_x (\hat{G}_1(x, y_0) \mathbb{I})$ it follows that

$$I_3 = \int_{\partial B(y_0, \delta)} \left\{ \nu \times \text{curl} E(x; m) \cdot k_1^{-2} \nabla_x \text{div}_x (\hat{G}_1(x, y_0)r) - \nu \times \text{curl} [\hat{G}_1(x, y_0)r] \cdot E(x; m) \right\} ds$$

$$+ \int_{\partial B(y_0, \delta)} \nu \times \text{curl} E(x; m) \cdot r \hat{G}_1(x, y_0) ds$$

$$:= I_4 + I_5. \quad (4.11)$$

The regularity of $E(x; m)$ and the singularity of $\hat{G}_1(x, y_0)$ at $x = y_0$ imply that $I_5 \to 0$ as $\delta \to 0$. On the other hand, by the divergence theorem on $\partial B(y_0, \delta)$ it can be seen that

$$I_4 = \int_{\partial B(y_0)} \left[ \nu \times \text{curl} E(x; m) \cdot \frac{1}{k_1^2} \text{Grad} \text{div} (\hat{G}_1(x, y_0)r) - \nu \times \text{curl} [\hat{G}_1(x, y_0)r] \cdot E(x; m) \right] ds$$

$$= \int_{\partial B(y_0)} \left[ \text{Div}(-\nu \times \text{curl} E(x; m)) \frac{1}{k_1^2} \text{div} (\hat{G}_1(x, y_0)r) - \nu \times \text{curl} [\hat{G}_1(x, y_0)r] \cdot E(x; m) \right] ds$$

$$= \int_{\partial B(y_0)} \left[ (\nu \cdot \text{curl} E(x; m)) \frac{1}{k_1^2} \text{div} (\hat{G}_1(x, y_0)r) - \nu \times \text{curl} [\hat{G}_1(x, y_0)r] \cdot E(x; m) \right] ds$$

$$= \int_{\partial B(y_0)} \left[ (\nu \cdot E(x; m)) \text{div} (\hat{G}_1(x, y_0)r) + \nu \times E(x; m) \cdot \text{curl} [\hat{G}_1(x, y_0)r] \right] ds$$

$$= \int_{\partial B(y_0)} \left[ (\nu \cdot E(x; m)) \nabla \hat{G}_1(x, y_0) - \nabla \hat{G}_1(x, y_0) \times (\nu \times E(x; m)) \right] ds \cdot r$$

$$\to -r \cdot E(y_0; m)$$

as $\delta \to 0$. This combined with (4.10) and (4.11) implies that

$$r \cdot E(y_0; m) = \left( \int_{\Gamma_0} - \int_{\Gamma_1} \right) \left[ \nu \times \text{curl} E(x) \cdot [\hat{G}_1(x, y_0)r] - \nu \times \text{curl} [\hat{G}_1(x, y_0)r] \cdot E(x) \right] ds. \quad (4.12)$$

Similarly, we have on noting the regularity of $\hat{E}^s(x; y_0)$ that

$$\left( \int_{\Gamma_0} - \int_{\Gamma_1} \right) \left[ \nu \times \text{curl} E(x; m) \cdot \hat{E}^s(x; y_0) - \nu \times \text{curl} \hat{E}^s(x; y_0) \cdot E(x; m) \right] ds = 0 \quad (4.13)$$
Combine (4.12) with (4.13) to conclude that

\[
r \cdot E(y_0; m) = \left( \int_{\Gamma_0} - \int_{\Gamma_1} \right) \left[ \nu \times \text{curl} E(x; m) \cdot \widehat{E}(x; y_0) - \nu \times \text{curl} \widehat{E}(x; y_0) \cdot E(x; m) \right] ds.
\]

Making use of the boundary conditions on \( \Gamma_j \) \((j = 0, 1)\) and Green’s theorem in \( \Omega_h \setminus \Omega_1 \) we obtain that

\[
r \cdot E(y_0; m) = \int_{\Gamma_0} \left[ \nu \times \text{curl} E(x; m)|_- \cdot \widehat{E}(x; y_0)|_- - \nu \times \text{curl} \widehat{E}(x; y_0)|_- \cdot E(x; m)|_- \right] ds
\]

\[
= \frac{1}{\lambda_0} \int_{\Gamma_0} \left[ \nu \times \text{curl} E(x; m)|_+ \cdot \widehat{E}(x; y_0)|_+ - \nu \times \text{curl} \widehat{E}(x; y_0)|_+ \cdot E(x; m)|_+ \right] ds
\]

\[
= \frac{1}{\lambda_0} \int_{\Gamma_h} \left[ \nu \times \text{curl} E(x; m) \cdot \widehat{E}(x; y_0) - \nu \times \text{curl} \widehat{E}(x; y_0) \cdot E(x; m) \right] ds
\]

Now, by the Rayleigh expansion radiation condition and the divergence-free property for \( E^s(x; m) \) and \( \widehat{E}(x; y_0) \) we have, on noting that \( \beta_n(\alpha) = \tilde{\beta}_{-n}(\tilde{\alpha}) \), that

\[
\int_{\Gamma_h} \left[ \nu \times \text{curl} E^s(x; m) \cdot \widehat{E}(x; y_0) - \nu \times \text{curl} \widehat{E}(x; y_0) \cdot E^s(x; m) \right] ds = 0.
\]

This implies that

\[
r \cdot E(y_0; m) = \frac{1}{\lambda_0} \int_{\Gamma_h} \left[ \nu \times \text{curl} E^i(x; m) \cdot \widehat{E}(x; y_0) - \nu \times \text{curl} \widehat{E}(x; y_0) \cdot E^i(x; m) \right] ds.
\]

Insert \( E^i(x; m) = k_0^{-2} \text{curl} \left[ \rho \exp(i\alpha_m \cdot x - i\beta_m x_3) \right] \) and \( \widehat{E}(x; y_0) = \sum_{n \in \mathbb{Z}^2} \widehat{E}_n \exp\{i\alpha_n \cdot x + i\beta_n x_3\} \) into the above equation to get

\[
r \cdot E(y_0; m) = \frac{i}{\lambda_0} \sum_{n \in \mathbb{Z}^2} \left\{ [\widehat{E}_n(y_0) \times e_3] \times (\alpha_m; -\beta_m) - e_3 \times [(\tilde{\alpha}_n; \tilde{\beta}_n) \times \widehat{E}_n(y_0)] \right\} e^{i(\tilde{\beta}_n - \beta_m)h}
\]

\[
\cdot \rho_m \int_0^{2\pi} \int_0^{2\pi} e^{i(\tilde{\alpha}_n + \alpha_m) \cdot x} dx_1 dx_2,
\]

where \((\tilde{\alpha}; b)\) is defined as \((\tilde{\alpha}; b) := \tilde{\alpha} + (0, 0, b)\) and \( p_m = p - [(\alpha_m; -\beta_m) \cdot p/k_0^2](\alpha_m; -\beta_m) \).

Finally, use the fact that \( \tilde{\alpha}_n + \alpha_m = (n + m, 0) \), \( \tilde{\beta}_l(\tilde{\alpha}) = \beta_l(\alpha) \) for all \( l \in \mathbb{Z}^2 \) and
\[
\int_0^{2\pi} \int_0^{2\pi} e^{i(n+m,0) \cdot x} \, dx_1 \, dx_2 = 0 \text{ for } n + m \neq (0,0) \text{ to conclude that }
\]
\[
r \cdot E(y_0; m) = 4\pi^2 \frac{i}{\lambda_0} \left\{ \left[ \hat{E}_m(y_0) \times e_3 \right] \times (\alpha_m; -\beta_m) - e_3 \times \left[ (\hat{\alpha}_m; \hat{\beta}_m) \times \hat{E}_m(y_0) \right] \right\} \cdot p_m
\]
\[
= 4\pi^2 \frac{i}{\lambda_0} \left\{ e_3 \times \left[ (\hat{\alpha}_m; \hat{\beta}_m) \times \hat{E}_m(y_0) \right] \right\} \cdot p_m
\]
\[
= 4\pi^2 \frac{i}{\lambda_0} \left\{ (\alpha_m; -\beta_m) \times \hat{E}_m(y_0) \right\} \cdot p_m
\]
\[
= 8\pi^2 \frac{i}{\lambda_0} (\alpha_m; -\beta_m) \cdot \hat{E}_m(y_0) \cdot p,
\]
where we have used the fact that
\[
(\alpha_m; -\beta_m) \cdot \hat{E}_m = (\alpha_m; -\beta_m) \cdot p_m = 0,
\]
\[
(\alpha_m; -\beta_m) \cdot \hat{E}_m = (-\hat{\alpha}_m; -\hat{\beta}_m) \cdot \hat{E}_m = 0.
\]
This completes the proof. \( \square \)

If \( y_0 \in \Omega_0 \), define the total field \( \hat{E}(x; y_0) = \hat{E}^s(x; y_0) + \hat{G}_0(x, y_0) \) in \( \Omega_0 \), where \( \hat{G}_0(x, y_0) \) is an \( \hat{\alpha} \)-quasi-periodic Green tensor defined in (1.1) with \( \alpha \) replaced with \( \hat{\alpha} \). Then arguing similarly as in the proof of Lemma 4.1 we can prove the following result.

**Lemma 4.2.** For \( m = (m_1, m_2) \in \mathbb{Z}^2 \) let \( E^i(x; m) = (1/k_0^2) \text{curl curl} \left\{ p \exp(i\alpha_m \cdot x - i\beta_m x_3) \right\} \) and let \( E(x; m) \) (which is the sum \( E^i(x; m) + E^s(x; m) \) in \( \Omega_0 \)) be the solution to the the scattering problem (1.1) - (1.6) with \( E^i(x) = E^i(x; m) \). For \( y_0 \in \Omega_0 \), \( \hat{\alpha} = -\alpha \) and \( r \in \mathbb{R}^3 \) let \( E^i(x; y_0) = \hat{G}_0(x, y_0) \) and let \( \hat{E}(x; y_0) \) (which equals to the sum \( E^i(x; y_0) + \hat{E}^s(x; y_0) \) in \( \Omega_0 \{y_0\} \)) satisfy the Maxwell equations \( \text{curl curl} \hat{E} - k_0^2 \hat{E} = 0 \) in \( \Omega_0 \{y_0\} \) and \( \text{curl curl} \hat{E} - k_0^2 q \hat{E} = 0 \) in \( \Omega_1 \) together with the transmission condition
\[
\nu \times \hat{E}_+ = \nu \times \hat{E}_-, \quad \nu \times \text{curl} \hat{E}_+ = \lambda_0 \nu \times \text{curl} \hat{E}_- \text{ on } \Gamma_0,
\]
the boundary condition
\[
\nu \times \hat{E} = 0 \text{ on } \Gamma_{1,D}, \quad \nu \times \text{curl} \hat{E} - i\rho(\nu \times \hat{E}) \times \nu = 0 \text{ on } \Gamma_{1,t}
\]
and the Rayleigh expansion radiation condition
\[
\hat{E}^s(x; y_0) = \sum_{n \in \mathbb{Z}^2} \hat{E}_n(y_0) \exp(i\hat{\alpha}_n \cdot x + i\hat{\beta}_n x_3) \text{ for } x_3 > g_+,
\]
where
\[
\hat{\alpha}_n \cdot \hat{E}_n + \hat{\beta}_n \cdot \hat{E}_n^{(3)} = 0.
\]
Then we have
\[
r \cdot E(y_0; m) = 8\pi^2 i\hat{\beta}_m \hat{E}_m(y_0) \cdot p.
\]
4.2 Unique determination of the impenetrable profile $f$

We now consider the unique determination of the impenetrable grating profile $f$, assuming that the interface profile $g$ is known and $k_0^2q(x) \equiv k_1^2$ is a constant. A key ingredient in our proof is the mixed reciprocity relation for the doubly periodic structure (see Lemma 3.1).

**Theorem 4.3.** Assume that $\beta_n \neq 0$ for all $n \in \mathbb{Z}^2$, the interface profile $g$ is known and $k_0^2q(x) \equiv k_1^2$ is a constant. Let $f_1, f_2 \in C^2(\mathbb{R}^2)$ be $2\pi$-periodic, let $\rho_1, \rho_2$ be two constants and let $h > \max_{x \in \mathbb{R}^2}\{f_1(x), f_2(x)\}$. If $\nu \times E_{1,m}^i\mid_{\Omega_h} = \nu \times E_{2,m}^s\mid_{\Omega_h}$ for all incident waves $E_{m}^i(x) = (1/k_0^2)\text{curl curl} [e_1 \exp(0x_\cdot x - i\beta_3 x_3)]$ with $m \in \mathbb{Z}^2$ and $l = 1, 2, 3$, then

$$f_1 = f_2, \quad \Gamma_{f_1,D} = \Gamma_{f_2,D}, \quad \Gamma_{f_1,l} = \Gamma_{f_2,l}, \quad \rho_1 = \rho_2,$$

where $e_1$ is the unit vector in the direction $x_1$, $l = 1, 2, 3$. Here, $E_{j,m} = E_{m}^i + E_{j,m}^s$ in $\Omega_0$ and $E_{j,m} \in \Omega_{f_j}$ are the unique quasi-periodic solution of the scattering problem (4.1) - (4.3) with $E^s = E_{m}^s$, $\rho = \rho_j$ and $f = f_j$, where $\Omega_{f_j} = \{x \in \mathbb{R}^3 \mid f_j(x_1, x_2) < x_3 < g(x_1, x_2)\}$, $j = 1, 2, 3$.

**Proof.** We assume without loss of generality that $f_1 \neq f_2$ and there exists a $z^* = (z^*_1, z^*_2, z^*_3) \in \Gamma_{f_1}$ with $f_1(z^*_1, z^*_2) > f_2(z^*_1, z^*_2)$, where $\Gamma_{f_2} = \{x \in \mathbb{R}^3 \mid x_3 = f_j(x_1, x_2)\}$. We choose $\varepsilon > 0$ such that $z_\varepsilon := z^* + \varepsilon e_3 \in \Omega_{f_1} \cap \Omega_{f_2}$.

Let $\hat{E}_{e,j}$ be the unique quasi-periodic solution to the scattered problem (4.12) - (4.3) with $y_0 = z_\varepsilon$, $\rho = \rho_j$, $f = f_j$. By Lemma 3.1 we have

\begin{align}
 r \cdot E_{1,m}(z_\varepsilon) &= \frac{8\pi^2 i}{\lambda_0} \hat{\beta}_{-m} \hat{E}_{1,-m}(z_\varepsilon) \cdot e_l, \quad (4.15) \\
 r \cdot E_{2,m}(z_\varepsilon) &= \frac{8\pi^2 i}{\lambda_0} \hat{\beta}_{-m} \hat{E}_{2,-m}(z_\varepsilon) \cdot e_l, \quad (4.16)
\end{align}

where $\hat{E}_{j,n}(z_\varepsilon)$ are the Rayleigh coefficients for $\hat{E}_{e,j}$.

On the other hand, from the Rayleigh expansion radiation condition and the assumption that $\nu \times E_{1,m}^i\mid_{\Omega_h} = \nu \times E_{2,m}^s\mid_{\Omega_h}$ we conclude by the unique continuation principle that $E_{2,m}^s = E_{2,m}^i$ in $\Omega_0$. This, together with the transmission condition on $\Gamma_0$ and Holmgren’s uniqueness theorem, implies that $E_{1,m} = E_{2,m}$ in $\Omega_{f_1} \cap \Omega_{f_2}$, so $E_{1,m}(z_\varepsilon) = E_{2,m}(z_\varepsilon)$. It then follows from (4.15) and (4.16) that

$$\frac{8\pi^2 i}{\lambda_0} \hat{\beta}_{-m} \hat{E}_{1,-m}(z_\varepsilon) \cdot e_l = \frac{8\pi^2 i}{\lambda_0} \hat{\beta}_{-m} \hat{E}_{2,-m}(z_\varepsilon) \cdot e_l \quad \text{or} \quad \hat{E}_{1,-m}(z_\varepsilon) = \hat{E}_{2,-m}(z_\varepsilon).$$

Thus, by the Rayleigh expansion radiation condition we have $\hat{E}_{e,1}(x) = \hat{E}_{e,2}(x)$ for $x_3 > g_+$. By the unique continuation principle, the transmission condition on $\Gamma_0$ and Holmgren’s uniqueness theorem again we obtain that

$$\hat{E}_{e,1}(x) = \hat{E}_{e,2}(x) \quad \text{in} \quad \overline{\Omega}_0 \quad \text{and} \quad \hat{E}_{e,1}(x) = \hat{E}_{e,2}(x) \quad \text{in} \quad \overline{\Omega}_{f_1} \cap \overline{\Omega}_{f_2}.$$ 

Without loss of generality we may assume that $z^*$ lies on the coated part of $\Gamma_{f_1}$. Since $z^*$ has a positive distance from $\Gamma_{f_2}$, then the well-posedness of the direct problem implies that there exists $C > 0$ (independent of $\varepsilon$) such that

$$|\langle \nu \times \text{curl} \hat{E}_{e,1} - i\rho_1 \nu \times \hat{E}_{e,1} \times \nu \rangle(z^*)| = |\langle \nu \times \text{curl} \hat{E}_{e,2} - i\rho_1 \nu \times \hat{E}_{e,2} \times \nu \rangle(z^*)| \leq C.$$
However, from the boundary condition on $\Gamma_{f_1}$ it is seen that
\[
|(\nu \times \text{curl} \, \tilde{E}_{e,1}^s - i \rho_1 \nu \times \tilde{E}_{e,1}^s \times \nu)(z^*)|
= |(\nu \times \text{curl} \, [\hat{G}_1\cdot, z_t]r - i \rho_1 \nu \times [\hat{G}_1\cdot, z_t]r \times \nu)(z^*)| \to +\infty
\]
as $\epsilon \to 0$. This is a contradiction, which implies that $f_1 = f_2$, that is, $\Omega_{f_1} = \Omega_{f_2}$ and $\Gamma_{f_1} = \Gamma_{f_2}$. Hence, we have $E_{1,m} = E_{2,m}$ in $\Omega_{f_1}$. We claim that $\Gamma_{f_1,D} \cap \Gamma_{f_2,I}$ must be empty (so $\Gamma_{f_1,D} = \Gamma_{f_2,D}$ and $\Gamma_{f_1,I} = \Gamma_{f_2,I}$) since, otherwise, a similar argument as below deduces that the total field $E_{1,m}$ vanishes in $\Omega_{f_1}$, which is impossible.

Now let $f = f_1 = f_2$. Then by the boundary condition we deduce that
\[
i(\rho_1 - \rho_2)(\nu \times E_{1,m}) \times \nu = 0 \quad \text{on} \quad \Gamma_{1,I}.
\]

If $\rho_1 \neq \rho_2$, then the above equation implies that $\nu \times E_{1,m} = 0$ on $\Gamma_{1,I}$, so by the boundary condition again $\nu \times \text{curl} \, E_{1,m} = 0$ on $\Gamma_{1,I}$. Thus, by Holmgren’s uniqueness theorem, $E_{1,m} = 0$ in $\Omega_1$. By the transmission condition on $\Gamma_0$ and Holmgren’s uniqueness theorem again it follows that $E_{1,m} = E_{1,m}^i + E_{1,m}^s = 0$ in $\Omega_0$, which is a contradiction. The proof is thus completed. \(\square\)

4.3 Unique determination of the refractive index

We now consider the inverse problem of recovering the refractive index $q$. We only consider the case that $\Gamma_{1,I} = \emptyset$, that is, the grating surface $\Gamma_1$ is a perfect conductor. However, we expect the result to hold in a more general case by constructing special solutions of the Maxwell equations. Throughout this section we assume that the transmission constant $\lambda_0$ is known and the shape of the grating surfaces $\Gamma_0$ and $\Gamma_1$ is also known and flat, that is, for two known constants $b > c$, $g(x') \equiv b$ and $f(x') \equiv c$ for all $x' \in \mathbb{R}^2$.

We have the following global uniqueness result for the inverse problem.

**Theorem 4.4.** Assume that $q = q_j$ satisfies the conditions $(A1) - (A3)$ and that $q_j$ depends only on $x_1$ or $x_2$, $j = 1, 2$. Let $h > b$. If
\[
\nu \times E_{1,m}^i|_{\Gamma_h} = \nu \times E_{2,m}^i|_{\Gamma_h}
\]
for all incident waves $E_{1,m}^i(x) = (1/k_m^n)\text{curl} \, \text{curl} \, [\hat{E}_l \exp(i \alpha_m \cdot x - i \beta_m x_3)]$ with $m \in \mathbb{Z}^2$ and $l = 1, 2, 3$, then we have $q_1 = q_2$. Here, $E_{j,m} = E_{m}^i + E_{j,m}^s$ in $\Omega_0$ and $E_{j,m}$ in $\Omega_1$ are the unique quasi-periodic solution of the scattering problem (1.1) - (1.6) with $E^i = E^i_m$ and $q = q_j$, $j = 1, 2$.

**Remark 4.5.** Theorem 4.4 improves the result in [14] Theorem 5.4, where only the special case $\lambda_0 = 1$ is considered and incident waves of the form (1.17) below are used for all $r \in L^2_0(\Gamma_h)$.

To prove Theorem 4.4 we need the following denseness result which is related to the incident waves of the form
\[
E^i(x;r) = \int_{\Gamma_h} \hat{G}_0(x,y)r(y)ds(y), \quad x_3 < h, \quad (4.17)
\]
where $r \in L^2_0(\Gamma_h)$. This result was proved in [14] Lemma 5.2 for the case $\lambda_0 = 1$, and the general case can be proved similarly (see the proof of Lemma 5.2 in [14]).
Lemma 4.6. The operator $F$ has a dense range in $H^{-1/2}_t(\text{div}, \Gamma_0)$. Here, $F : L^2_t(\Gamma_h) \to H^{-1/2}_t(\text{div}, \Gamma_0)$ is defined by $(Fr)(x) = e_3 \times \hat{E}(x;r)|_\Gamma$ on $\Gamma_0$, where $\hat{E}(x;r)$ is the solution of the scattering problem (1.11) – (1.10) with the incident wave $E^i(x) = E^i(x;r)$ given by (1.17).

Proof of Theorem 4.4. For any $r \in L^2_t(\Gamma_h)$ and $y \in \Gamma_h$ we have by Lemma 4.2 that
\[
 r(y) \cdot E^s_j(y;m) = 8\pi^2 i \beta_{-m} \hat{E}_{j,-m}(y) \cdot e_l, \quad j = 1, 2, \quad l = 1, 2, 3, \tag{4.18}
\]
where $\hat{E}_{j,-m}(y)$ are the Rayleigh coefficients of the scattered field $\hat{E}^s_j(\cdot;y)$ corresponding to $q = q_j$ and the incident wave $E^i(x) = \hat{G}_0(x,y)r(y)$. It follows from (4.18) that
\[
 \int_{\Gamma_h} r(y) \cdot E^s_j(y;m)ds(y) = 8\pi^2 i \beta_{-m} \int_{\Gamma_h} \hat{E}_{j,-m}(y)ds(y) \cdot e_l. \tag{4.19}
\]
Denote by $\hat{E}^s_j(x;r)$ and $\hat{E}_j(x;r)$ the scattered and total electric fields, respectively, corresponding to $q = q_j$ and the incident wave $E^i(x) = E^i(x;r)$, $j = 1, 2$. Then from the definition (4.17) of $E^i(x;r)$ it is seen that
\[
 \int_{\Gamma_h} \hat{E}_{j,-m}(y)ds(y) \text{ are the Rayleigh coefficients of } \hat{E}^s_j(x;r). \tag{4.20}
\]
On the other hand, from the Rayleigh expansion radiation condition and the assumption that $\nu \times E^i_1(x;m) = \nu \times E^s_2(x;m)$ on $\Gamma_h$ we conclude on using the unique continuation principle that $E^i_1(x;m) = E^s_2(x;m)$ in $\overline{\Omega}_0$. This, together with (4.19) and (4.20), implies that
\[
 \int_{\Gamma_h} \hat{E}_{1,-m}(y)ds(y) = \int_{\Gamma_h} \hat{E}_{2,-m}(y)ds(y).
\]
From this, the Rayleigh expansion radiation condition and the unique continuation principle it follows that
\[
 \hat{E}^s_1(x;r) = \hat{E}^s_2(x;r) \quad \text{or} \quad \hat{E}_1(x;r) = \hat{E}_2(x;r) \quad \text{in } \Omega_h \setminus \Omega_1.
\]
With the help of the transmission conditions on $\Gamma_0$, we get
\[
 \nu \times \hat{E}_1(x;r)|_\Gamma = \nu \times \hat{E}_2(x;r)|_\Gamma \quad \text{on } \Gamma_0, \quad \nu \times \text{curl } \hat{E}_1(x;r)|_\Gamma = \nu \times \text{curl } \hat{E}_2(x;r)|_\Gamma \quad \text{on } \Gamma_0.
\]
Now define $E(x) := \hat{E}_1(x;r) - \hat{E}_2(x;r)$ in $\overline{\Omega}_1$. Then $E$ satisfies the equation
\[
 \text{curl curl } E - k_0^2 q_2 E = k_0^2 (q_1 - q_2) \hat{E}_1(x;r) \quad \text{in } \Omega_1
\]
and the boundary conditions
\[
 \nu \times E = 0, \quad \nu \times \text{curl } E = 0 \quad \text{on } \Gamma_0, \quad \nu \times E = 0 \quad \text{on } \Gamma_1.
\]
Thus it follows from Green’s vector formula that
\[
\frac{k_0^2}{2} \int_{\Omega_1} (q_1 - q_2) \vec{E}_1(x; r) \cdot \vec{E}_2(x) dx = \int_{\Omega_1} (\text{curl curl} \ E - k_0^2 q_2 E) \cdot \vec{E}_2(x) dx = \int_{\Omega_1} (\text{curl curl} \ \vec{E}_2(x) - k_0^2 q_2 \vec{E}_2(x)) \cdot E(x) dx = 0
\]
for any \( r \in L^2_t(\Gamma_h) \), where \( E_2 \in H(\text{curl}, \Omega_1) \) satisfies the Maxwell equation (1.2) with \( q = q_2 \) and the boundary condition \( \nu \times E_2|_{\Gamma_1} = 0 \).

Now by Lemma 4.6 and (4.21) we obtain that
\[
\int_{\Omega_1} (q_1 - q_2) E_1(x) \cdot \vec{E}_2(x) dx = 0,
\]
where \( E_1 \) satisfies of the Maxwell equation (1.2) with \( q = q_1 \) and the boundary condition \( \nu \times E_1|_{\Gamma_1} = 0 \).

Finally, using the orthogonal relation (4.22) and arguing in exactly the same way as in the proof of Theorem 5.4 in [14], we can easily prove that \( q_1 = q_2 \). The proof is thus completed. \( \square \)

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