Lévy’s martingale characterization and reflection principle of $G$-Brownian motion

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Abstract. In this paper, we obtain Lévy’s martingale characterization of $G$-Brownian motion without the nondegenerate condition. Base on this characterization, we prove the reflection principle of $G$-Brownian motion. Furthermore, we use Krylov’s estimate to get the reflection principle of $\tilde{G}$-Brownian motion.

Key words. $G$-expectation, $G$-Brownian motion, martingale characterization, reflection principle

AMS subject classifications. 60H10, 60E05

1 Introduction

Motivated by the model uncertainty in financial market, Peng in [9, 10] first introduced the notions of $G$-expectation, conditional $G$-expectation and $G$-Brownian motion via the following $G$-heat equation:

$$\partial_t u - G(D_x^2 u) = 0, \ u(0, x) = \varphi(x),$$

where $G : \mathbb{S}(d) \to \mathbb{R}$ is a monotonic and sublinear function. In [11], Peng further extended the definition of $G$-expectation and $G$-Brownian motion to $\tilde{G}$-expectation and $\tilde{G}$-Brownian motion via the following PDE:

$$\partial_t u - \tilde{G}(D_x^2 u) = 0, \ u(0, x) = \varphi(x),$$

where $\tilde{G} : \mathbb{S}(d) \to \mathbb{R}$ is monotonic and dominated by $G$. Based on $G$-Brownian motion, Peng established the related Itô’s stochastic calculus theory, for example, Itô’s integral, Itô’s formula, $G$-martingale and $G$-stochastic differential equation ($G$-SDE).

It is well-known that the reflection principle for classical Brownian motion is an important result. In this paper, we study the reflection principle for $G$-Brownian motion or, more generally, $\tilde{G}$-Brownian motion, i.e., calculate the distribution of the following process:

$$\sup_{s \leq t} (B_s - B_t), \ t \geq 0,$$
where $B$ is a $G$-Brownian motion or $\tilde{G}$-Brownian motion. In order to obtain the reflection principle, we first investigate the Lévy’s martingale characterization of $G$-Brownian motion.

In [16, 17], Xu et al. obtained the Lévy’s martingale characterization theorem for 1-dimensional symmetric $G$-martingale under the nondegenerate condition. Song [14] also proved the martingale characterization theorem for 1-dimensional symmetric $G$-martingale under the nondegenerate condition by a different method. In this paper, we consider general symmetric martingales, which may not be $G$-martingale, without the nondegenerate condition. We introduce a new kind of discrete product space method for martingale (see Lemmas 4.2 and 4.3) to deal with the degenerate case, and obtain the corresponding Lévy’s martingale characterization of $G$-Brownian motion. In the symmetric $G$-martingale case, we give a simpler and more direct method. Based on the martingale characterization of $G$-Brownian motion, we obtain the reflection principle for $G$-Brownian motion.

However, the martingale characterization method does not hold for $\tilde{G}$-Brownian motion, because we do not have the Lévy martingale characterization of $\tilde{G}$-Brownian motion. In this case, we first use the estimate in [4] to give a discrete approximation of process $(\text{sgn}(B_t))_{t\leq T}$ (see Lemma 5.9) under the nondegenerate condition. Then we obtain the related reflection principle under the nondegenerate condition. Finally, we apply the Krylov’s estimate to get the related reflection principle without the nondegenerate condition.

The paper is organized as follows. In section 2, we recall some basic notions and results of sublinear expectation and $G$-Brownian motion. In section 3, we give the definition of consistent sublinear expectation space, and extend Peng’s definition of stochastic calculus for $G$-Brownian motion to a kind of martingales on the consistent sublinear expectation space. The Lévy’s martingale characterization of $G$-Brownian motion is stated and proved in section 4. In section 5, we obtain the reflection principle for $G$-Brownian motion and $\tilde{G}$-Brownian motion.

## 2 Preliminaries

In this section, we recall some basic notions and results of sublinear expectation and $G$-Brownian motion. The readers may refer to [1–3, 9–11] for more details.

Let $\Omega$ be a given set and $\mathcal{H}$ be a linear space of real-valued functions on $\Omega$ such that $c \in \mathcal{H}$ for all constants $c$ and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. We further suppose that if $X_1, \ldots, X_n \in \mathcal{H}$, then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{b, \text{Lip}}(\mathbb{R}^n)$, where $C_{b, \text{Lip}}(\mathbb{R}^n)$ denotes the set of all bounded and Lipschitz functions on $\mathbb{R}^n$. $\mathcal{H}$ is considered as the space of random variables. $X = (X_1, X_2, \ldots, X_n)^T$, $X_i \in \mathcal{H}$, is called an $n$-dimensional random vector, denoted by $X \in \mathcal{H}^n$, where $^T$ denotes the transpose.

**Definition 2.1** A sublinear expectation is a functional $\hat{E} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for each $X, Y \in \mathcal{H}$, we have

1. **Monotonicity:** $X \geq Y$ implies $\hat{E}[X] \geq \hat{E}[Y]$;
2. **Constant preserving:** $\hat{E}[c] = c$ for $c \in \mathbb{R}$;
3. **Sub-additivity:** $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$;
4. **Positive homogeneity:** $\hat{E}[\lambda X] = \lambda \hat{E}[X]$ for $\lambda > 0$. 


The triple \((\Omega, \mathcal{H}, \mathbb{E})\) is called a sublinear expectation space. If (1) and (2) are satisfied, \(\mathbb{E}\) is called a nonlinear expectation and the triple \((\Omega, \mathcal{H}, \mathbb{E})\) is called a nonlinear expectation space.

**Remark 2.2** If the inequality in (3) becomes equality, \(\mathbb{E}\) is called a linear expectation and \((\Omega, \mathcal{H}, \mathbb{E})\) is called a linear expectation space.

For each given \(p \geq 1\), in order to define the \(p\)-norm on \(\mathcal{H}\), suppose \(|X|^p \in \mathcal{H}\) for each \(X \in \mathcal{H}\). We denote by \(L^p(\Omega)\) the completion of \(\mathcal{H}\) under the norm \(|X|_{L^p} := (\mathbb{E}[|X|^p])^{1/p}\). It is easy to verify that \(L^p(\Omega) \subset L^{p'}(\Omega)\) for each \(1 \leq p' \leq p\). Note that \(|\mathbb{E}[X] - \mathbb{E}[Y]| \leq \mathbb{E}[|X - Y|]\), then \(\mathbb{E}\) can be continuously extended to the mapping from \(L^1(\Omega)\) to \(\mathbb{R}\). One can check that \((\Omega, L^1(\Omega), \mathbb{E})\) forms a sublinear expectation space, which is called a complete sublinear expectation space.

We say that a sequence \(\{X_n\}_{n=1}^\infty\) converges to \(X\) in \(L^p\), denoted by \(X = \lim_{n \to \infty} X_n\), if

\[
\lim_{n \to \infty} \mathbb{E}[|X_n - X|^p] = 0.
\]

Now we give the notions of identical distribution and independence.

**Definition 2.3** Let \(X_1\) and \(X_2\) be two \(n\)-dimensional random vectors on complete sublinear expectation spaces \((\Omega_1, L^1(\Omega_1), \mathbb{E}_1)\) and \((\Omega_2, L^1(\Omega_2), \mathbb{E}_2)\), respectively. They are called identically distributed, denoted by \(X_1 \overset{d}{=} X_2\), if for each \(\varphi \in C_{b,Lip}(\mathbb{R}^n)\),

\[
\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)].
\]

**Definition 2.4** Let \((\Omega, L^1(\Omega), \mathbb{E})\) be a complete sublinear expectation space. An \(m\)-dimensional random vector \(Y\) is said to be independent from an \(n\)-dimensional random vector \(X\) if for each \(\varphi \in C_{b,Lip}(\mathbb{R}^{n+m})\),

\[
\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)]|_{x=X}].
\]

In the following, we review \(G\)-normal distribution and \(G\)-Brownian motion.

Let \(G : \mathbb{S}(d) \to \mathbb{R}\) be a given monotonic and sublinear function, i.e., for \(A, A' \in \mathbb{S}(d)\),

\[
\begin{align*}
G(A) &\geq G(A') \text{ if } A \geq A', \\
G(A + A') &\leq G(A) + G(A'), \\
G(\lambda A) &= \lambda G(A) \text{ for } \lambda \geq 0,
\end{align*}
\]

where \(\mathbb{S}(d)\) denotes the collection of all \(d \times d\) symmetric matrices. By the Hahn-Banach theorem, one can check that there exists a bounded, convex and closed subset \(\Gamma \subset \mathbb{S}_+(d)\) such that

\[
G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma A] \text{ for } A \in \mathbb{S}(d),
\]

(2.1)

where \(\mathbb{S}_+(d)\) denotes the collection of nonnegative elements in \(\mathbb{S}(d)\). From (2.1), it is easy to deduce that there exists a constant \(C > 0\) such that for each \(A, A' \in \mathbb{S}(d)\),

\[
|G(A) - G(A')| \leq C|A - A'|.
\]
Definition 2.5 Let \((\Omega, \mathbb{L}^1(\Omega), \mathbb{E})\) be a complete sublinear expectation space. A d-dimensional random vector \(X\) is called G-normally distributed, denoted by \(X \overset{d}{=} N(0, \Gamma)\), if for each \(\varphi \in C_b\text{Lip}(\mathbb{R}^d)\),
\[
\mathbb{E}[\varphi(X)] = u^\varphi(1,0),
\]
where \(u^\varphi\) is the viscosity solution of the following G-heat equation
\[
\begin{cases}
\partial_t u - G(D^2 u) = 0, \\
u(0,x) = \varphi(x).
\end{cases}
\] (2.2)

Remark 2.6 If \(G(A) = \frac{1}{2}\text{tr}[A]\) for \(A \in \mathbb{S}(d)\), then \(\Gamma = \{I_d\}\) and G-normal distribution is the classical standard normal distribution.

Definition 2.7 Let \((\Omega, \mathbb{L}^1(\Omega), \mathbb{E})\) be a complete sublinear expectation space. A d-dimensional process \((B_t)_{t \geq 0}\) with \(B_t \in (\mathbb{L}^1(\Omega))^d\) for each \(t \geq 0\) is called a G-Brownian motion if the following properties are satisfied:

(i) \(B_0 = 0;\)

(ii) For each \(t, s \geq 0, B_{t+s} - B_t \overset{d}{=} N(0, s\Gamma)\) and is independent from \((B_{t_1}, \ldots, B_{t_n})\), for each \(n \in \mathbb{N}\) and \(0 \leq t_1 \leq \cdots \leq t_n \leq t.\)

Remark 2.8 It is easy to verify that for each \(\varphi \in C_b\text{Lip}(\mathbb{R}^d), \mathbb{E}[\varphi(B_{t+s} - B_t)] = u^\varphi(s, 0)\), where \(u^\varphi\) is the solution of the G-heat equation (2.2). If \(G(A) = \frac{1}{2}\text{tr}[A]\) for \(A \in \mathbb{S}(d)\), then G-Brownian motion is classical standard Brownian motion.

For each \(t \geq 0\), set
\[
L_{ip}(\Omega_t) = \{\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}) : n \in \mathbb{N}, 0 \leq t_1 < \cdots < t_n \leq t, \varphi \in C_b\text{Lip}(\mathbb{R}^{n \times d})\}
\]
and
\[
L_{ip}(\Omega) := \bigcup_{m=1}^{\infty} L_{ip}(\Omega_m).
\]

For each \(p \geq 1\), we denote by \(L^p_G(\Omega_t)\) (resp. \(L^\infty_G(\Omega)\)) the completion of \(L_{ip}(\Omega_t)\) (resp. \(L_{ip}(\Omega)\)) under the norm \(\|X\|_{L^p_G} := (\mathbb{E}[\|X\|^p])^{\frac{1}{p}}\) for each \(t \geq 0\). For each \(\xi = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}) \in L_{ip}(\Omega),\) define the conditional expectation of \(\xi\) at \(t_i, i \leq n,\) by
\[
\mathbb{E}_{t_i}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] = \psi(B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}),
\] (2.3)
where \(\psi(x_1, \ldots, x_i) = \mathbb{E}[\varphi(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_n} - B_{t_{n-1}})].\) It is easy to check that \(\mathbb{E}_{t_i}\) still satisfies (1)-(4) in Definition 2.1 and can be continuously extended to the mapping from \(L^1_G(\Omega_t)\) to \(L^1_G(\Omega_{t_i}).\)

\((\Omega, L^1_G(\Omega_t), L^\infty_G(\Omega_t))_{t \geq 0}, (\mathbb{E}_{t_i})_{t \geq 0}\) is called the G-expectation space.

3 Consistent sublinear expectation space

In this section, we first give the definition of consistent sublinear expectation space, and then we extend Peng’s definition of stochastic calculus for G-Brownian motion to a kind of martingales on the consistent
sublinear expectation space. We only consider the continuous time case, and the definitions and results still hold for discrete time case.

Let $\Omega$ be a given set and $(\mathcal{H}_t)_{t \geq 0}$, $\mathcal{H}$ be a family of linear space of real-valued functions on $\Omega$ such that

(i) $\mathcal{H}_0 = \mathbb{R}$ and $\mathcal{H}_s \subset \mathcal{H}_t$ for each $0 \leq s \leq t$;

(ii) If $X \in \mathcal{H}_t$ (resp. $\mathcal{H}$), then $|X| \in \mathcal{H}_t$ (resp. $\mathcal{H}$);

(iii) If $X_1, \ldots, X_n \in \mathcal{H}_t$ (resp. $\mathcal{H}$), then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}_t$ (resp. $\mathcal{H}$) for each $\varphi \in C_{b,Lip}(\mathbb{R}^n)$.

**Definition 3.1** A consistent sublinear expectation on $(\mathcal{H}_t)_{t \geq 0}$ is a family of mappings $\hat{E}_t : \mathcal{H} \rightarrow \mathcal{H}_t$ which satisfies the following properties: for each $X, Y \in \mathcal{H}$,

1. **Monotonicity:** $X \geq Y$ implies $\hat{E}_t[X] \geq \hat{E}_t[Y]$;
2. **Constant preserving:** $\hat{E}_t[\eta] = \eta$ for $\eta \in \mathcal{H}_t$;
3. **Sub-additivity:** $\hat{E}_t[X + Y] \leq \hat{E}_t[X] + \hat{E}_t[Y]$;
4. **Positive homogeneity:** $\hat{E}_t[\eta X] = \eta \hat{E}_t[X]$ for each positive and bounded $\eta \in \mathcal{H}_t$;
5. **Consistency:** $\hat{E}_s[\hat{E}_t[X]] = \hat{E}_s[X]$ for $s \leq t$.

The triple $(\Omega, \mathcal{H}, (\hat{E}_t)_{t \geq 0})$ is called a consistent sublinear expectation space and $\hat{E}_t$ is called a sublinear conditional expectation.

**Remark 3.2** We always denote $\hat{E} := \hat{E}_0$ in the above definition. Obviously, $(\Omega, \mathcal{H}, \hat{E})$ forms a sublinear expectation space.

**Remark 3.3** One can check that the G-expectation space $(\Omega, L^1_G(\Omega), (L^1_G(\Omega)_t)_{t \geq 0}, (\hat{E}_t)_{t \geq 0})$ satisfies (1)-(5) in Definition 3.1.

For each given $p \geq 1$, in order to define the $p$-norm on $\mathcal{H}_t$ (resp. $\mathcal{H}$), suppose $|X|^p \in \mathcal{H}_t$ (resp. $\mathcal{H}$) for each $X \in \mathcal{H}_t$ (resp. $\mathcal{H}$), and we denote by $L^p(\Omega)$ (resp. $L^p(\mathcal{H}_t)$) the completion of $\mathcal{H}_t$ (resp. $\mathcal{H}$) under the norm $\|X\|_p := (\hat{E}[|X|^p])^{1/p}$. Note that $\|\hat{E}_t[X] - \hat{E}_t[Y]\| \leq \|X - Y\|$, then $\hat{E}_t$ can be continuously extended to the mapping from $L^1(\Omega)$ to $L^1(\Omega)$. It is easy to check that $(\Omega, L^1(\Omega), (L^1(\Omega)_t)_{t \geq 0}, (\hat{E}_t)_{t \geq 0})$ forms a consistent sublinear expectation space, which is called a complete consistent sublinear expectation space.

The following properties are very useful in sublinear analysis.

**Proposition 3.4** Let $(\Omega, L^1(\Omega), (L^1(\Omega)_t)_{t \geq 0}, (\hat{E}_t)_{t \geq 0})$ be a complete consistent sublinear expectation space. Assume $X \in (L^1(\Omega)_t)^n$ and $Y \in (L^1(\Omega))^m$. Then

$$\hat{E}_t[\varphi(X, Y)] = \hat{E}_t[\varphi(x, Y)]|_{x=X}$$

for each $\varphi \in C_{b,Lip}(\mathbb{R}^{n+m})$. \hspace{1cm} (3.1)

Specially, for each bounded $\eta \in L^1(\Omega)$,

$$\hat{E}_t[\eta Z] = \eta^+ \hat{E}_t[Z] + \eta^- \hat{E}_t[-Z]$$

for each $Z \in L^1(\Omega)$. \hspace{1cm} (3.2)
Proof. For each fixed $N > 0$, denote $B_N(0) := \{ x \in \mathbb{R}^n : |x| \leq N \}$. For each integer $n \geq 1$, by partition of unity theorem, there exist $h_i^n \in C_c^\infty(\mathbb{R}^n), i = 1, \ldots, k_n$, such that the diameter of support $\lambda(\text{supp}(h_i^n)) \leq 1/n$, $0 \leq h_i^n \leq 1$ and $\int_{B_N(0)} x \leq \sum_{i=1}^{k_n} h_i^n(x) \leq 1$. Choose $x_i^n \in \mathbb{R}^n$ such that $h_i^n(x_i^n) > 0$ and $L > 0$ such that $|\varphi| \leq L$. Note that

$$\sum_{i=1}^{k_n} h_i^n(X)E_{\varphi}[\varphi(x_i^n, Y)] - \hat{E}_t[\varphi(X, Y)]$$

$$\leq \sum_{i=1}^{k_n} h_i^n(X) \left| \hat{E}_t[\varphi(x_i^n, Y)] - \hat{E}_t[\varphi(X, Y)] \right| + \left( 1 - \sum_{i=1}^{k_n} h_i^n(X) \right) \left| \hat{E}_t[\varphi(X, Y)] \right|$$

$$\leq \sum_{i=1}^{k_n} h_i^n(X) \left| \hat{E}_t[\varphi(x_i^n, Y)] - \hat{E}_t[\varphi(X, Y)] \right| + \frac{L|X|}{N}$$

$$\leq C_\varphi \sum_{i=1}^{k_n} h_i^n(X)|X - x_i^n| + \frac{L|X|}{N}$$

and

$$\left| \sum_{i=1}^{k_n} h_i^n(X)\hat{E}_t[\varphi(x_i^n, Y)] - \hat{E}_t[\varphi(x_i^n, Y)]_{x=X} \right|$$

$$\leq \sum_{i=1}^{k_n} h_i^n(X) \left| \hat{E}_t[\varphi(x_i^n, Y)] - \hat{E}_t[\varphi(x, Y)]_{x=X} \right| + \frac{L|X|}{N}$$

$$\leq C_\varphi \sum_{i=1}^{k_n} h_i^n(X)|X - x_i^n| + \frac{L|X|}{N}$$

$$\leq \frac{C_\varphi}{n} + \frac{L|X|}{N},$$

where $C_\varphi$ is the Lipschitz constant of $\varphi$, then we obtain

$$\left| \hat{E}_t[\varphi(X, Y)] - \hat{E}_t[\varphi(x_i^n, Y)]_{x=X} \right| \leq \frac{2C_\varphi}{n} + \frac{2L|X|}{N}.$$ 

Letting $n \to \infty$ first and then $N \to \infty$, we get $\hat{E}_t[\varphi(X, Y)] = \hat{E}_t[\varphi(x, Y)]_{x=X}$. By positive homogeneity of $\hat{E}_t$, we can deduce $\hat{E}_t[\eta Z] = \eta^+ \hat{E}_t[Z] + \eta^- \hat{E}_t[-Z]$. □

Proposition 3.5 Let $(\Omega, L^1(\Omega), (L^1(\Omega))_{t \geq 0}, (\hat{E}_t)_{t \geq 0})$ be a complete consistent sublinear expectation space. Assume $Y \in L^1(\Omega)$ such that $\hat{E}_t[Y] = -\hat{E}_t[-Y]$ for some $t \geq 0$. Then, for each $X \in L^1(\Omega)$ and bounded $\eta \in L^1(\Omega)$, we have

$$\hat{E}_t[X + \eta Y] = \hat{E}_t[X] + \eta \hat{E}_t[Y].$$

Proof. By Proposition 3.4 and $\hat{E}_t[Y] = -\hat{E}_t[-Y]$, we get

$$\hat{E}_t[\eta Y] = \eta^+ \hat{E}_t[Y] + \eta^- \hat{E}_t[-Y] = \eta^+ \hat{E}_t[Y] - \eta^- \hat{E}_t[-Y] = \eta \hat{E}_t[Y].$$

Similarly, we have $\hat{E}_t[-\eta Y] = -\eta \hat{E}_t[Y]$. Then by the sub-additivity of $\hat{E}_t$, we obtain

$$\hat{E}_t[X + \eta Y] \leq \hat{E}_t[X] + \hat{E}_t[\eta Y] = \hat{E}_t[X] + \eta \hat{E}_t[Y].$$
and
\[ \mathbb{E}_t[X] + \eta \mathbb{E}_t[Y] = \mathbb{E}_t[X] - \mathbb{E}_t[-\eta Y] \leq \mathbb{E}_t[X + \eta Y]. \]
Thus
\[ \mathbb{E}_t[X + \eta Y] = \mathbb{E}_t[X] + \eta \mathbb{E}_t[Y]. \]

□

Now we give the definitions of adapted processes and martingales on the consistent sublinear expectation space.

**Definition 3.6** Let \((\Omega, \mathcal{L}^1(\Omega), (L^1(\Omega_t))_{t \geq 0}, (\mathbb{E}_t)_{t \geq 0})\) be a complete consistent sublinear expectation space. A \(d\)-dimensional adapted process is a family of random vectors \((X_t)_{t \geq 0}\) such that \(X_t \in (L^1(\Omega_t))^d\) for each \(t \geq 0\).

Two \(d\)-dimensional adapted processes \(X^i = (X^i_t)_{t \geq 0}, i = 1, 2,\) on complete consistent sublinear expectation spaces \((\Omega, L^1(\Omega), (L^1(\Omega_t))_{t \geq 0}, (\mathbb{E}_t)_{t \geq 0})\) are called identically distributed, denoted by \(X^1 \equiv X^2\), if for each \(n \in \mathbb{N}\), \(0 \leq t_1 < \cdots < t_n < \infty\), \(\varphi \in C_{b, \text{Lip}}(\mathbb{R}^{n \times d})\),

\[ \mathbb{E}^1[\varphi(X^1_{t_1}, X^1_{t_2} - X^1_{t_1}, \ldots, X^1_{t_n} - X^1_{t_1 - 1})] = \mathbb{E}^2[\varphi(X^2_{t_1}, X^2_{t_2} - X^2_{t_1}, \ldots, X^2_{t_n} - X^2_{t_n - 1})]. \]

**Definition 3.7** Let \((\Omega, \mathcal{L}^1(\Omega), (L^1(\Omega_t))_{t \geq 0}, (\mathbb{E}_t)_{t \geq 0})\) be a complete consistent sublinear expectation space. A \(d\)-dimensional adapted process \(M_t = (M^1_t, \ldots, M^d_t)^T, t \geq 0,\) is called a martingale if \(\mathbb{E}_s[M^i_t] = M^i_s\) for each \(s \leq t\) and \(i \leq d\). Furthermore, a \(d\)-dimensional martingale \(M\) is called symmetric if \(\mathbb{E}[M^i_t] = \mathbb{E}[-M^i_t]\) for each \(t \geq 0\) and \(i \leq d\).

**Remark 3.8** In the above definition, \(\mathbb{E}[M^i_t] = -\mathbb{E}[-M^i_t]\) implies \(\mathbb{E}_s[M^i_t] = -\mathbb{E}_s[-M^i_t]\) for each \(s \in [0, t]\), which is deduced by \(\mathbb{E}_s[M^i_t] + \mathbb{E}_s[-M^i_t] \geq 0\) and

\[ \mathbb{E}[\mathbb{E}_s[M^i_t] + \mathbb{E}_s[-M^i_t]] \leq \mathbb{E}[\mathbb{E}_s[M^i_t]] + \mathbb{E}[\mathbb{E}_s[-M^i_t]] = \mathbb{E}[M^i_t] + \mathbb{E}[-M^i_t] = 0. \]

In the following, we construct the stochastic calculus with respect to a kind of martingales. For simplicity, we first discuss the 1-dimensional case.

Let \(M\) be a symmetric martingale such that \(M_0 = 0, M_t \in L^2(\Omega_t)\) for each \(t \geq 0\) and

\[ \mathbb{E}_t[|M_{t+s} - M_t|^2] \leq C s, \quad \text{for each } t, s \geq 0, \quad (3.2) \]

where \(C > 0\) is a constant.

Let \(T > 0\) be given. Set

\[ M^{2,0}(0, T) := \{ \eta_t = \sum_{i=0}^{n-1} \xi_i I_{[t_i, t_{i+1})}(t) : 0 = t_0 < \cdots < t_n = T, \xi_i \in L^2(\Omega_t) \}. \]

We denote by \(M^{2,0}(0, T)\) the completion of \(M^{2,0}(0, T)\) under the norm \(|\eta|_{M^{2,0}} = (\mathbb{E}[\int_0^T |\eta|^2 dt])^{1/2}\).

For each \(\eta \in M^{2,0}(0, T)\) with \(\eta = \sum_{i=0}^{n-1} \xi_i I_{[t_i, t_{i+1})}(t)\), define the stochastic integral with respect to \(M\) as,

\[ \mathcal{I}(\eta) = \int_0^T \eta_t dM_t := \sum_{i=0}^{n-1} \xi_i (M_{t_{i+1}} - M_{t_i}). \]

The proof of the following Lemma is the same to the proof of Lemma 3.5 in [11]. We omit it here.
Lemma 3.9 The mapping \( I : \mathbb{M}^{2,0}(0,T) \to \mathbb{L}^2(\Omega_T) \) is a continuous linear mapping and can be continuously extended to \( I : \mathbb{M}^{2}(0,T) \to \mathbb{L}^2(\Omega_T) \). For each \( \eta \in \mathbb{M}^{2}(0,T) \), \( \int_0^t \eta_t dM_t \) is a symmetric martingale and

\[
\hat{\mathbb{E}}[\int_0^T \eta_t dM_t | \mathcal{F}_t] \leq C \hat{\mathbb{E}}[\int_0^T |\eta_t|^2 dt].
\]

(3.3)

For multi-dimensional case, let \( M = (M^1, \ldots, M^d)^T \) be a \( d \)-dimensional symmetric martingale with \( M^i \) satisfying \( \mathcal{L} \) for \( i = 1, \ldots, d \). Then for each \((n \times d)\)-dimensional process \( \eta = (\eta^i) \) with \( \eta^i \in \mathbb{M}^{2}(0,T) \), the definition of the stochastic integral \( \int_0^T \eta_t dM_t \) can be defined through 1-dimensional integral just like the classical case.

Now we consider the quadratic variation process of the 1-dimensional symmetric martingale \( M \) satisfying \( \mathcal{L} \). For any \( t \geq 0 \), let \( \pi^a_t = \{ t^n_0, \ldots, t^n_n \} \) be a partition of \([0,t]\) and denote

\[
\mu(\pi^a_t) := \max \{|t^n_{i+1} - t^n_i| : i = 0, 1, \ldots, n - 1\}.
\]

Note that

\[
\sum_{i=0}^{n-1} (M^n_{t^n_{i+1}} - M^n_{t^n_i})^2 = \sum_{i=0}^{n-1} (M^n_{t^n_{i+1}} - M^n_{t^n_i})^2 - 2 \sum_{i=0}^{n-1} M^n_{t^n_i} (M^n_{t^n_{i+1}} - M^n_{t^n_i})
\]

\[= M_t^2 - 2 \sum_{i=0}^{n-1} M^n_{t^n_i} (M^n_{t^n_{i+1}} - M^n_{t^n_i}),\]

and

\[
\hat{\mathbb{E}}\left[\int_0^t \sum_{i=0}^{n-1} |M^n_{t^n_i} I_{(t^n_i, t^n_{i+1})}(s) - M_s|^2 ds\right] = \hat{\mathbb{E}}\left[\sum_{i=0}^{n-1} \int_{t^n_i}^{t^n_{i+1}} |M^n_{t^n_i} - M_s|^2 ds\right]
\]

\[\leq \sum_{i=0}^{n-1} \int_{t^n_i}^{t^n_{i+1}} \hat{\mathbb{E}}[|M^n_{t^n_i} - M_s|^2] ds
\]

\[\leq \frac{C}{2} \mu(\pi^a_t),\]

then \( \sum_{i=0}^{n-1} (M^n_{t^n_{i+1}} - M^n_{t^n_i})^2 \) converges to \( M_t^2 - 2 \int_0^t M_s dM_s \) in \( \mathbb{L}^2 \) as \( \mu(\pi^a_t) \to 0 \). We call the limit the quadratic variation of \( M \) and denote it by \( \langle M \rangle_t \).

It is clear that \( \langle M \rangle \) is an increasing process. By Lemma 3.9 and \( \mathcal{L} \), we can obtain for each \( t, s \geq 0 \),

\[
\hat{\mathbb{E}}_t[\langle M \rangle_t] = \hat{\mathbb{E}}_t[\langle M \rangle_{t+s} - \langle M \rangle_t] = \hat{\mathbb{E}}_t[|M_{t+s} - M_t|^2] \leq C s.
\]

For symmetric martingales \( M \) and \( M' \) satisfying \( \mathcal{L} \), we know that \( M + M' \) and \( M - M' \) are also symmetric martingales satisfying \( \mathcal{L} \). As

\[
\sum_{i=0}^{n-1} (M^n_{t^n_{i+1}} - M^n_{t^n_i})(M'_{t^n_{i+1}} - M'_{t^n_i})
\]

\[= \frac{1}{4} \sum_{i=0}^{n-1} \{(M^n_{t^n_{i+1}} + M'_{t^n_{i+1}}) - (M^n_{t^n_i} + M'_{t^n_i})^2 - [(M^n_{t^n_{i+1}} - M'_{t^n_{i+1}}) - (M^n_{t^n_i} - M'_{t^n_i})]^2\},
\]

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we can define the mutual variation process by
\[ \langle M, M' \rangle_t := \mathbb{L}^2 - \lim_{\mu(\pi^n) \to 0} \sum_{i=0}^{n-1} (M_{i+1}^\nu - M_i^\nu)(M'_{i+1}^\nu - M_i^\nu) \]
\[ = \frac{1}{4}[(M + M')_t - (M - M')_t]. \]

For a \( d \)-dimensional symmetric martingale \( M = (M^1, \ldots, M^d)^T \) with \( M^i \) satisfying (4.2) for \( i = 1, \ldots, d \), we define the quadratic variation by
\[ (M)_t := ((M^i, M^j)_t)_{1 \leq i,j \leq d}. \]

4 Lévy’s martingale characterization of \( G \)-Brownian motion

The following theorem is the Lévy’s martingale characterization of \( G \)-Brownian motion on the complete consistent sublinear expectation space.

**Theorem 4.1** Let \( (\Omega, \mathbb{L}^1(\Omega), (\mathbb{L}^1(\Omega_t))_{t \geq 0}, (\mathbb{E}_t)_{t \geq 0}) \) be a complete consistent sublinear expectation space and \( G : \mathcal{S}(d) \to \mathbb{R} \) be a given monotonic and sublinear function. Assume \( (M_t)_{t \geq 0} \) is a \( d \)-dimensional symmetric martingale satisfying \( M_0 = 0, M_t \in (\mathbb{L}^2(\Omega))^d \) for each \( t \geq 0 \) and \( \sup \{ \mathbb{E}[|M_t + \delta - M_t|^2] : t \leq T \} = o(\delta) \) as \( \delta \downarrow 0 \) for each fixed \( T > 0 \). Then the following conditions are equivalent:

1. \( (M_t)_{t \geq 0} \) is a \( G \)-Brownian motion;
2. \( \mathbb{E}_t[|M_{t+s} - M_t|^2] \leq Cs \) for each \( t, s \geq 0 \) and the process \( \frac{1}{2} \text{tr} A(M)_t - G(A)t, t \geq 0, \) is a martingale for each \( A \in \mathcal{S}(d) \), where \( C > 0 \) is a constant;
3. The process \( \frac{1}{2} \text{tr} A(M_t, M_t) - G(A)t, t \geq 0, \) is a martingale for each \( A \in \mathcal{S}(d) \).

In order to prove the above theorem, we need the following lemmas. For simplicity, we only prove the theorem for the case \( d = 1 \), and the proof for \( d > 1 \) is the same. Under the case \( d = 1 \),
\[ G(a) = \frac{1}{2}(\sigma^2 a^+ - \sigma^2 a^-) \quad \text{for} \quad a \in \mathbb{R}, \]
where \( 0 \leq \sigma^2 \leq \sigma^2 < \infty \). We suppose \( \sigma^2 > 0 \), since \( M \equiv 0 \) if \( \sigma^2 = 0 \), which is trivial.

The main idea to prove the above main theorem is to use PDE method, which needs that the solution \( u \) for \( G \)-heat equation (2.2) belongs to \( C^{1,2} \). But, when \( \sigma^2 = 0 \), \( u \) may not be in \( C^{1,2} \). So we introduce the following auxiliary space.

Let \( (\tilde{\Omega}, \mathbb{L}^1(\tilde{\Omega}), \mathbb{E}) \) be a complete linear expectation space and \( (W_t)_{t \geq 0} \) be a standard \( 1 \)-dimensional classical Brownian motion on it. Define
\[ \tilde{\Omega} := \{ \tilde{\omega} = (\omega, \tilde{\omega}) : \omega \in \Omega, \tilde{\omega} \in \Omega \}. \]

Let \( T > 0 \) be fixed. For any given partition \( \pi = \{ t_0, \ldots, t_n \} \) of \( [0, T] \), i.e., \( 0 = t_0 < \cdots < t_n = T \), set \( \tilde{\mathcal{H}}^T_{t_i} := \mathbb{R} \) and for \( i = 1, \ldots, n, \)
\[ \tilde{\mathcal{H}}^T_{t_i} := \{ \xi(\tilde{\omega}) = \phi(\omega, W_{t_1}(\tilde{\omega}), \cdots, W_{t_i}(\tilde{\omega}) - W_{t_{i-1}}(\tilde{\omega})) : \tilde{\omega} = (\omega, \tilde{\omega}) \in \tilde{\Omega}, \phi \in L^2_{t_i} \}, \]
where \( L^2_{t_i} \) is the space of functions \( \phi : \Omega \times \mathbb{R} \to \mathbb{R} \) satisfying the following properties:
(1) $\phi$ is bounded and $\phi(., x) \in L^1(\Omega_{t_i})$ for each $x \in \mathbb{R}^1$;

(2) There exists a constant $K > 0$ such that $|\phi(\omega, x) - \phi(\omega, x')| \leq K|x - x'|$ for each $\omega \in \Omega$, $x, x' \in \mathbb{R}^1$.

It is easy to verify that $\hat{H}_n^\pi \subset \hat{H}_n$ for $0 \leq i < j \leq n$. Now we want to define a sublinear conditional expectation $\tilde{E}_n^\pi : \hat{H}_n^\pi \to \hat{H}_n^\pi$ such that $(\Omega, (\hat{H}_n^\pi)_{n=0}^\infty, (\tilde{E}_n^\pi)_{n=0}^\infty)$ forms a discrete consistent sublinear expectation space. For this purpose, we first define the operator $\tilde{\pi} : H_{t+1}^N \to H_{t}^N$, $i = 0, \ldots, n - 1$. For each $\xi(\omega) = (\phi(\omega, W_{t+1}(\omega) - W_t(\omega)))) \in \hat{H}_{t+1}$, define

$$\tilde{E}_{t+1}^\pi[\xi](\omega) := \varphi(\omega, W_t(\omega), \ldots, W_{t+1}(\omega) - W_t(\omega)),$$

where

$$\varphi(\cdot, x_1, \cdots, x_i) := \tilde{E}_t[\psi(\cdot, x_1, \cdots, x_i)],$$

and

$$\psi(\omega, x_1, \cdots, x_i) = \tilde{E}[\phi(\omega, x_1, \cdots, x_i, W_{t+1} - W_t)] \text{ for } \omega \in \Omega.$$  

In the following, we will show that $\tilde{E}_{t+1}^\pi[\xi] \in \hat{H}_{t+1}$. Now we define $\tilde{E}_n^\pi : \hat{H}_n^\pi \to \hat{H}_n$ backwardly from $i = n$ to $i = 0$ as follows: for each $X \in \hat{H}_n^\pi$, define

$$\tilde{E}_n^\pi[X] := \tilde{E}_{n+1}^\pi[\tilde{E}_{n+1}^\pi[X]], \quad i = 0, \ldots, n - 1,$$

where

$$\tilde{E}_n^\pi[X] = X.$$

**Lemma 4.2** Let $(\hat{H}_n^\pi)_{n=0}^\infty$ be defined in (4.2) and $(\tilde{E}_n^\pi)_{n=0}^\infty$ be defined in (4.7). Then $(\tilde{\pi}, (\hat{H}_n^\pi)_{n=0}^\infty, (\tilde{E}_n^\pi)_{n=0}^\infty)$ forms a discrete consistent sublinear expectation space. Furthermore, if $\xi \in \hat{H}_n^\pi$ is independent of $\omega$ (resp. $\omega$), then $\tilde{E}_n^\pi[\xi](\omega, \bar{\omega}) = \tilde{E}_n[\xi](\omega)$ (resp. $\tilde{E}_n^\pi[\xi](\omega, \bar{\omega}) = \tilde{E}_n[\xi](\bar{\omega})$).

**Proof.** We only prove that $\tilde{E}_{t+1}^\pi$ in (4.3) is well-defined, the other properties can be easily verified. For this, we only need to show that $\psi(\cdot, x) \in L^1(\Omega_{t+1})$ for each $x = (x_1, \cdots, x_i) \in \mathbb{R}^1$. For each given integer $N > 0$, set $x_j^N = -N + \frac{2j}{N}$, $j = 0, \ldots, N^2$. Then, for each $\omega \in \Omega$,

$$\left| \tilde{E}[\phi(\omega, x, W_{t+1} - W_t)] - \tilde{E} \left[ \sum_{j=0}^{N^2-1} \phi(\omega, x_j^N, x_j^N) I_{[x_j^N, x_{j+1}^N]}(W_{t+1} - W_t) \right] \right| \leq \frac{2K}{N} + L\tilde{E}[I_{[N, \infty]}(|W_{t+1} - W_t|)] $$

$$\leq \frac{2K}{N} + \frac{L}{N} \tilde{E}[|W_{t+1} - W_t|],$$

where $L > 0$ such that $|\phi| \leq L$. From this, we can deduce that

$$\left| \psi(\cdot, x) - \sum_{j=0}^{N^2-1} a_j^N \phi(\cdot, x_j^N) \right| \leq \frac{2K}{N} + L\tilde{E}[|W_{t+1} - W_t|],$$

where $a_j^N = \tilde{E}[I_{[x_j^N, x_{j+1}^N]}(W_{t+1} - W_t)]$. Noting that $\phi(., x_1, \cdots, x_i, x_j^N) \in L^1(\Omega_{t+1})$, we obtain $\psi(., x_1, \cdots, x_i) \in L^1(\Omega_{t+1})$ by (4.7). $\square$

The completion of $(\tilde{\Omega}, (\hat{H}_n^\pi)_{n=0}^\infty, (\tilde{E}_n^\pi)_{n=0}^\infty)$ is denoted by $(\tilde{\Omega}, (L^1(\hat{H}_n^\pi))_{n=0}^\infty, (\tilde{E}_n^\pi)_{n=0}^\infty)$. 

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Lemma 4.3  For each $\eps \in (0,1)$, set
\[ M^\eps_t(\omega, \bar{\omega}) := M_t(\omega) + \eps W_t(\bar{\omega}) \text{ for } t \geq 0, (\omega, \bar{\omega}) \in \bar{\Omega}. \]
If $\frac{1}{2}a(M_t)^2 - G(a) t$, $t \geq 0$, is a martingale for each $a \in \mathbb{R}$, then, under $(\bar{\Omega}, (\bar{\Omega}_t^\eps)_{t=0}^n, (\bar{E}_t^\eps)_{t=0}^n)$, we have
\begin{enumerate}
    \item $(M^\eps_t)_{t=0}^n$ is a discrete symmetric martingale;
    \item $\left(\frac{1}{2}a(M^\eps_t)^2 - G(\eps(a) t_i)\right)_{i=0}^n$ is a discrete martingale, where $G_\eps(a) = G(a) + \frac{1}{2}\eps^2 a$ for $a \in \mathbb{R}$.
\end{enumerate}

Proof. (1) By the definition of $\bar{E}_t^\eps$, we have, for $i = 0, \ldots, n - 1$,
\[ \bar{E}_t^\eps[M^\eps_{t+i}](\omega, \bar{\omega}) = \bar{E}_t^\eps[M^\eps_{t+i}](\omega) + \eps \bar{E}[W_{t+i} - W_t] + \eps W_t(\bar{\omega}) = M^\eps_t(\omega, \bar{\omega}) \]
and
\[ \bar{E}_t^\eps[-M^\eps_{t+i}](\omega, \bar{\omega}) = \bar{E}_t^\eps[-M^\eps_{t+i}](\omega) + \eps \bar{E}[-(W_{t+i} - W_t)] - \eps W_t(\bar{\omega}) = -M^\eps_t(\omega, \bar{\omega}). \]
Thus $(M^\eps_t)_{t=0}^n$ is a discrete symmetric martingale.

(2) Since $\frac{1}{2}a(M_t)^2 - G(a) t$, $t \geq 0$, is a martingale, we have
\[ \bar{E}_t^\eps[a(M_{t+i})^2 - a(M_t)^2] = 2G(a)(t_i + 1 - t_i). \]
By the fact that $(M_t)_{t \geq 0}$ is a symmetric martingale, we also get
\[ \bar{E}_t^\eps[a(M_{t+i})^2 - a(M_t)^2] = \bar{E}_t^\eps[a(M_{t+i} - M_t)^2]. \]
Thus
\[ \bar{E}_t^\eps[a(M_{t+i} - M_t)^2] = 2G(a)(t_i + 1 - t_i). \]
It follows from (1) and the definition of $\bar{E}_t^\eps$ that
\[ \bar{E}_t^\eps[M^\eps_{t+i}(M^\eps_{t+i} - M^\eps_t)] = \bar{E}_t^\eps[-M^\eps_t(M^\eps_{t+i} - M^\eps_t)] = 0 \]
and
\[ \bar{E}_t^\eps[a(M_{t+i} - M_t)^2](\omega, \bar{\omega}) = \bar{E}_t^\eps[a(M_{t+i} - M_t)^2](\omega) + \eps^2 \bar{E}[(W_{t+i} - W_t)^2] = 2G(a)(t_i + 1 - t_i) + \eps^2(t_i + 1 - t_i). \]
Then
\[ \bar{E}_t^\eps[a(M_{t+i} - M_t)^2] = a(M^\eps_{t+i})^2 + \bar{E}_t^\eps[a(M^\eps_{t+i} - M^\eps_t)^2] = a(M^\eps_{t+i})^2 + 2G_\eps(a)(t_i + 1 - t_i), \]
which implies that $\left(\frac{1}{2}a(M^\eps_t)^2 - G(\eps(a) t_i)\right)_{i=0}^n$ is a discrete martingale. \qed

Remark 4.4 When $\sigma^2 = 0$, $G_\eps(a) = \frac{1}{2}(\sigma^2 + \eps^2)a^+ - \eps^2 a^-$ is non-degenerate. So we can consider $M^\eps$ in Theorem 4.2.

Consider the following two PDEs:
\begin{align}
\partial_t u + G(\partial_x^2 u) &= 0, \quad u(T, x) = \varphi(x), \quad (4.8) \\
\partial_t u^\eps + G_\eps(\partial_x^2 u^\eps) &= 0, \quad u^\eps(T, x) = \varphi(x). \quad (4.9)
\end{align}
Lemma 4.5 For each \( \varepsilon \in (0, 1) \) and \( \varphi \in C_{b, \text{Lip}}(\mathbb{R}) \), let \( u \) and \( u^{\varepsilon} \) be the solution of PDEs (4.8) and (4.9) respectively. Then, for \( t, t' \in [0, T] \), \( x \in \mathbb{R} \),

\[
|u^{\varepsilon}(t, x) - u^{\varepsilon}(t', x)| \leq C_{\varphi} \sqrt{2(\sigma^2 + 1)/\pi} \sqrt{|t - t'|}
\]

and

\[
|u^{\varepsilon}(t, x) - u(t, x)| \leq C_{\varphi} \sqrt{2(T - t)/\pi \varepsilon},
\]

where \( C_{\varphi} \) is the Lipschitz constant of \( \varphi \).

**Proof.** Let \((B^1_t, B^2_t)_{t \geq 0}\) be a 2-dimensional \( G \)-Brownian motion on a complete sublinear expectation space \((\Omega, L^1(\Omega), \hat{E})\) with

\[
\hat{G} \left( \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right) = G(a) + \frac{1}{2} c \text{ for } a, b, c \in \mathbb{R}.
\]

It is easy to verify that \((B^1_t)_{t \geq 0}\) is a 1-dimensional \( G \)-Brownian motion and \((B^1_t + \varepsilon B^2_t)_{t \geq 0}\) is a 1-dimensional \( G_{\varepsilon} \)-Brownian motion. By Remark 2.8, we deduce that, for \((t, x) \in [0, T] \times \mathbb{R} \),

\[
u(t, x) = \mathbb{E}[\varphi(x + B^1_t - B^1_0)] \text{ and } u^{\varepsilon}(t, x) = \mathbb{E}[\varphi(x + B^1_t + \varepsilon B^2_t - B^1_0 - \varepsilon B^2_0)].
\]

Thus, by Proposition 1.5 in Chapter III of [11], we obtain

\[
|u^{\varepsilon}(t, x) - u^{\varepsilon}(t', x)| \leq C_{\varphi} \mathbb{E}[|B^1_t + \varepsilon B^2_t - B^1_{t'} - \varepsilon B^2_{t'}|] = C_{\varphi} \sqrt{2(\sigma^2 + \varepsilon^2)/\pi} \sqrt{|t - t'|}
\]

and

\[
|u^{\varepsilon}(t, x) - u(t, x)| \leq C_{\varphi} \mathbb{E}[|B^1_t - B^2_t|] = C_{\varphi} \sqrt{2(T - t)/\pi \varepsilon}.
\]

\[\square\]

**Proof of Theorem 4.1.** (1)\(\Rightarrow\)(2) If \((M_t)_{t \geq 0}\) is a \( G \)-Brownian motion, then, by Proposition 1.4 in Chapter IV of [11], we get \( \mathbb{E}[|M_{t+s} - M_t|^2] = \sigma^2 s, \langle M \rangle_t = \sigma^2 t \) and \( \mathbb{E}[M_t^2] = \langle M \rangle_t \), \( t \geq 0 \), are two martingales, which implies that \( \frac{1}{2} a(M_t) - G(a)t, t \geq 0 \), is a martingale for each \( a \in \mathbb{R} \).

(2)\(\Rightarrow\)(3) Note that \( M^2_t = 2 \int_0^t M_s dM_s + \langle M \rangle_t \) and \( \int_0^t M_s dM_s \) is a symmetric martingale, then \( \frac{1}{2} a(M_t)^2 - G(a)t, t \geq 0 \), is a martingale for each \( a \in \mathbb{R} \).

(3)\(\Rightarrow\)(1) By Proposition 4.4, we only need to prove that

\[
\mathbb{E}[[\varphi(M_T - M_t)] = u(t, 0),
\]

where \( 0 < t < T, \varphi \in C_{b, \text{Lip}}(\mathbb{R}) \) and \( u \) is the solution of PDE (4.8). The proof is divided into three steps.

**Step 1: Taylor’s expansion.** For each fixed \( \varepsilon \in (0, 1) \) and \( h \in (0, t) \), let \( v^{\varepsilon} \) be the solution of the following PDE:

\[
\partial_t v^{\varepsilon} + G_{\varepsilon}(\partial^2_{xx} v^{\varepsilon}) = 0, \quad v^{\varepsilon}(T + h, x) = \varphi(x).
\]

It is clear that \( v^{\varepsilon}(s, x) = u^{\varepsilon}(s - h, x) \) for \( (s, x) \in [h, T + h] \times \mathbb{R} \), where \( u^{\varepsilon} \) is the solution of PDE (4.9).

Since PDE (4.10) is uniformly parabolic and \( G_{\varepsilon}(\cdot) \) is a convex function, by the interior regularity of \( v^{\varepsilon} \) (see [12, 13]),

\[
|v^{\varepsilon}|_{C^{1,\alpha/2,2+\alpha/2}([0, T] \times \mathbb{R})} < \infty \text{ for some } \alpha \in (0, 1).
\]

\[\text{\begin{equation} (4.11) \end{equation}}\]
For each \( n \geq 1 \), set \( t^n_i = t + n^{-1}i(T - t) \), \( i = 0, \ldots, n \). Let \( (M^\varepsilon_t)_{t \geq 0} \) be defined as in Lemma 4.3 Denote \( M^\varepsilon_{t_2} - M^\varepsilon_{t_1} \) for \( 0 \leq t_1 \leq t_2 \), then

\[
v^\varepsilon(T, M^\varepsilon_T) - v^\varepsilon(t, 0) = \sum_{i=0}^{n-1} [v^\varepsilon(t^n_{i+1}, M^\varepsilon_{t^n_{i+1}}) - v^\varepsilon(t^n_{i}, M^\varepsilon_{t^n_{i+1}})] \\
= \sum_{i=0}^{n-1} [J^n_i + I^n_i],
\]

where

\[
J^n_i = \partial_t v^\varepsilon(t^n_i, M^\varepsilon_{t^n_i}) (t^n_{i+1} - t^n_i) + \partial_x v^\varepsilon(t^n_i, M^\varepsilon_{t^n_i}) M^\varepsilon_{t^n_{i+1}} + \frac{1}{2} \partial^2_{xx} v^\varepsilon(t^n_i, M^\varepsilon_{t^n_i}) \left(M^\varepsilon_{t^n_{i+1}}\right)^2
\]

and

\[
I^n_i = (t^n_{i+1} - t^n_i) \int_0^1 [\partial_t v^\varepsilon(t^n_i + \alpha(t^n_{i+1} - t^n_i), M^\varepsilon_{t^n_{i+1}}) - \partial_x v^\varepsilon(t^n_i, M^\varepsilon_{t^n_i})] d\alpha
\]

\[
+ (M^\varepsilon_{t^n_{i+1}})^2 \int_0^1 \int_0^1 \alpha \partial^2_{xx} v^\varepsilon(t^n_i, M^\varepsilon_{t^n_i}) + \alpha \beta M^\varepsilon_{t^n_{i+1}} - \partial^2_{xx} v^\varepsilon(t^n_i, M^\varepsilon_{t^n_i})] d\beta d\alpha.
\]

Step 2: Estimation of residual terms. Set \( \pi_n = \{0, t^n_0, t^n_1, \ldots, t^n_n\} \), by (4.12) and Lemma 4.2 we get

\[
\mathbb{E}^\varepsilon_n \left[ |v^\varepsilon(T, M^\varepsilon_T)| - v^\varepsilon(t, 0) - \mathbb{E}^\varepsilon_n \left[ \sum_{i=0}^{n-1} J^n_i \right] \right] \leq \mathbb{E}^\varepsilon_n \left[ \sum_{i=0}^{n-1} I^n_i \right].
\]

It follows from Proposition 3.4 and Lemma 4.3 that

\[
\mathbb{E}^\varepsilon_n \left[ J^n_i \right] = \mathbb{E}^\varepsilon_n \left[ \partial_t v^\varepsilon(t^n_i, x)(t^n_{i+1} - t^n_i) + \partial_x v^\varepsilon(t^n_i, x) M^\varepsilon_{t^n_{i+1}} + \frac{1}{2} \partial^2_{xx} v^\varepsilon(t^n_i, x) (M^\varepsilon_{t^n_{i+1}})^2 \right] = 0.
\]

Since \( v^\varepsilon \) satisfies PDE (4.10), we obtain \( \mathbb{E}^\varepsilon_n \left[ J^n_i \right] = 0 \). Thus

\[
\mathbb{E}^\varepsilon_n \left[ \sum_{i=0}^{n-1} J^n_i \right] = \mathbb{E}^\varepsilon_n \left[ \sum_{i=0}^{n-2} J^n_i + \mathbb{E}^\varepsilon_n \left[ J^n_{n-1} \right] \right] = \mathbb{E}^\varepsilon_n \left[ \sum_{i=0}^{n-2} J^n_i \right] = \cdots = 0.
\]

Combining (4.13) with (4.14), we conclude that

\[
\mathbb{E}^\varepsilon_n \left[ |v^\varepsilon(T, M^\varepsilon_T)| - v^\varepsilon(t, 0) \right] \leq \sum_{i=0}^{n-1} \mathbb{E}^\varepsilon_n [I^n_i].
\]

As \( v^\varepsilon \) satisfying (4.11), we can easily get

\[
|I^n_i| \leq C_1 \left\{ |t^n_{i+1} - t^n_i|^{1+\alpha/2} + |M^\varepsilon_{t^n_{i+1}}|^{2+\alpha} \right\},
\]

where \( C_1 > 0 \) is a constant depending on \( \varepsilon, h, T, G \) and \( \varphi \). By Lemma 4.2 we have

\[
\mathbb{E}^\varepsilon_n \left[ M^\varepsilon_{t^n_{i+1}} \right] \leq 2^{1+\alpha} \left\{ \mathbb{E}^\varepsilon_n \left[ |M_{t^n_{i+1}} - M_T|^{2+\alpha} \right] + \mathbb{E}^\varepsilon_n \left[ |W_{t^n_{i+1}} - W_T|^{2+\alpha} \right] \right\} \leq 2^{1+\alpha} \left\{ \mathbb{E} \left[ |M_{t^n_{i+1}} - M_T|^{2+\alpha} \right] + \mathbb{E} \left[ |W_{t^n_{i+1}} - W_T|^{2+\alpha} \right] \right\}.
\]
Thus we get
\[ \bar{E}^n \| I^n \| \leq C_2 \left\{ |t^n_{i+1} - t^n_i|^{1+\alpha/2} + \bar{E} \left| M^n_{t^n_{i+1}} - M^n_t \right|^{2+\alpha} \right\}, \]  
(4.16)
where \( C_2 > 0 \) is a constant depending on \( \varepsilon, h, T, G \) and \( \varphi \). By Hölder’s inequality, we obtain
\[ \bar{E} \left| M^n_{t^n_{i+1}} - M^n_t \right|^2 \leq \left( \bar{E} \left| M^n_{t^n_{i+1}} - M^n_t \right|^2 \right)^{1-\alpha} \left( \bar{E} \left| M^n_{t^n_{i+1}} - M^n_t \right|^{\alpha} \right). \]  
(4.17)
Note that \( \sup \{ \bar{E} \| M_{t+\delta} - M_t \| : t \leq T \} = o(\delta) \) as \( \delta \downarrow 0 \). By Lemma 4.6, we obtain
\[ \bar{E} \left| M^n_{t^n_{i+1}} - M^n_t \right|^2 = 2G(1)(t^n_{i+1} - t^n_i) \]  
and
\[ t^n_{i+1} - t^n_i = n^{-1}(T - t), \]
then, by (4.16) and (4.17), we get
\[ \sum_{i=0}^{n-1} \bar{E}^n \| I^n_i \| = o(1). \]  
(4.18)

**Step 3: Approximation.** From (4.15) and (4.18), we obtain
\[ \lim_{n \to \infty} \bar{E}^n \left[ \left| \bar{E}^n_T [v^\varepsilon(T, M^t_T)] - v^\varepsilon(t, 0) \right| \right] = 0. \]  
(4.19)
It follows from Lemma 4.4 that
\[ |v^\varepsilon(T, x) - v^\varepsilon(T+h, x)| = |u^\varepsilon(T-h, x) - u^\varepsilon(T, x)| \leq C_\varphi \sqrt{2(\sigma^2 + 1)/\pi} \sqrt{h} \]
and
\[ |v^\varepsilon(t, 0) - u(t, 0)| = |u^\varepsilon(t-h, 0) - u(t, 0)| \]
\[ \leq |u^\varepsilon(t-h, 0) - u^\varepsilon(t, 0)| + |u^\varepsilon(t, 0) - u(t, 0)| \]
\[ \leq C_\varphi \sqrt{2(\sigma^2 + 1)/\pi h} + C_\varphi \sqrt{2(T-t)/\pi \varepsilon}. \]

Since \( v^\varepsilon(T+h, x) = \varphi(x) \), by Lemma 4.2, we deduce
\[ \bar{E} \left[ \left| \bar{E}^n_I [\varphi(M_T - M_t)] - u(t, 0) \right| \right] \]
\[ = \bar{E}^n \left[ \left| \bar{E}^n_I [\varphi(M_T - M_t)] - u(t, 0) \right| \right] \]
\[ \leq \bar{E}^n \left[ |\varphi(M_T - M_t)| - \varphi(M^x_T) \right] + |v^\varepsilon(T+h, M^x_T) - v^\varepsilon(T, M^x_T)| \]
\[ + \bar{E}^n \left[ \left| \bar{E}^n_I [v^\varepsilon(T, M^x_T)] - v^\varepsilon(t, 0) \right| \right] + |v^\varepsilon(0) - u(t, 0)| \]
\[ \leq \bar{E}^n \left[ \left| \bar{E}^n_I [v^\varepsilon(T, M^x_T)] - v^\varepsilon(t, 0) \right| \right] + C_\varphi \sqrt{2(\sigma^2 + 1)/\pi \sqrt{h}} + C_\varphi \sqrt{2(T-t)/\pi \varepsilon} \]
\[ \leq \bar{E}^n \left[ \left| \bar{E}^n_I [v^\varepsilon(T, M^x_T)] - v^\varepsilon(t, 0) \right| \right] + 2C_\varphi \sqrt{2(\sigma^2 + 1)/\pi \sqrt{h}} + 2C_\varphi \sqrt{2(T-t)/\pi \varepsilon}. \]
Taking \( n \to \infty \), by (4.19), we get
\[ \bar{E} \left[ \left| \bar{E}^n_I [\varphi(M_T - M_t)] - u(t, 0) \right| \right] \leq 2C_\varphi \sqrt{2(\sigma^2 + 1)/\pi \sqrt{h}} + 2C_\varphi \sqrt{2(T-t)/\pi \varepsilon}. \]

Letting \( h \to 0 \) and \( \varepsilon \to 0 \), we obtain \( \bar{E}^n [\varphi(M_T - M_t)] = u(t, 0) \). \( \square \)

We now consider the Lévy’s martingale characterization of G-Brownian motion on the G-expectation space. In this case, we do not need the assumptions \( M_t \in (L^3(\Omega))^d \) and \( \sup \{ \bar{E} \| M_{t+\delta} - M_t \| : t \leq T \} = o(\delta) \) as \( \delta \downarrow 0 \) as in Theorem 4.1.
**Theorem 4.6** Let \( \bar{G} : \mathcal{S}(d') \to \mathbb{R} \) and \( G : \mathcal{S}(d) \to \mathbb{R} \) be two given monotonic and sublinear functions, and let \( (\Omega, L_G^1(\Omega), (L_G^1(\Omega_t))_{t \geq 0}, (\mathbb{E}_t)_{t \geq 0}) \) be a \( G \)-expectation space. Assume \( (M_t)_{t \geq 0} \) is a \( d \)-dimensional symmetric martingale satisfying \( M_0 = 0, M_t \in (L^2_G(\Omega_t))^d \) for each \( t \geq 0 \). Then the following conditions are equivalent:

1. \( (M_t)_{t \geq 0} \) is a \( G \)-Brownian motion;
2. The process \( \frac{1}{2}(AM_t, M_t) - G(A)t, t \geq 0, \) is a martingale for each \( A \in \mathcal{S}(d) \);
3. \( \mathbb{E}_t[|M_{t+s} - M_t|^2] \leq Cs \) for each \( t, s \geq 0 \) and the process \( \frac{1}{2}tr[A(M)_t] - G(A)t, t \geq 0, \) is a martingale for each \( A \in \mathcal{S}(d) \), where \( C > 0 \) is a constant.

**Proof.** We only prove the case \( d = 1 \). The case \( d > 1 \) is similar. The proof for (1) \( \Rightarrow \) (2) is the same as Theorem 4.3.

(2) \( \Rightarrow \) (3) Taking \( A = 1 \), we get \( M_t^2 - \sigma^2 t \) is a martingale, where \( G(a) = \frac{1}{2}(\sigma^2 a^+ - \sigma^2 a^-) \) for \( a \in \mathbb{R} \). From this, we have

\[
\mathbb{E}_t[|M_{t+s} - M_t|^2] = \mathbb{E}_t[M_{t+s}^2 - M_t^2 - 2M_t(M_{t+s} - M_t)] = \sigma^2(t - s).
\]

Noting that \( (M)_t = M_t^2 - 2\int_0^t M_s dM_s \), we obtain the desired result.

(3) \( \Rightarrow \) (1) By Proposition 3.4, we only need to prove that

\[
\mathbb{E}_t[\varphi(M_T - M_t)] = u(t, 0),
\]

where \( 0 < t < T, \varphi \in C_{b, Lip}(\mathbb{R}) \) and \( u \) is the solution of PDE (4.3).

Let \( (B_t)_{t \geq 0} \) be the \( \bar{G} \)-Brownian motion. Following Section 2 in Chapter III in [11], we can construct an auxiliary \( \bar{G} \)-expectation space \( (\bar{\Omega}, L_{\bar{G}}^1(\bar{\Omega}), (L_{\bar{G}}^1(\bar{\Omega}_t))_{t \geq 0}, (\bar{\mathbb{E}}_t)_{t \geq 0}) \) such that

1. \( \bar{\Omega} = \Omega \times C_0([0, \infty)), \) where \( C_0([0, \infty)) \) is the space of real-valued continuous paths \( (\omega_t)_{t \geq 0} \) with \( \omega_0 = 0; \)
2. \( (B_t, \bar{B}_t)_{t \geq 0} \) is a \( \bar{G} \)-Brownian motion, where \( \bar{B} \) is the canonical process on \( C_0([0, \infty)) \), and

\[
\bar{G}(A) = G(A') + \frac{1}{2}c \quad \text{for} \quad A = \begin{pmatrix} A' & b \\ b & c \end{pmatrix} \in \mathcal{S}(d' + 1).
\]

For each fixed \( \varepsilon \in (0, 1) \), define \( M^\varepsilon_t = M_t + \varepsilon \bar{B}_t \). One can easily check that \( (M^\varepsilon_t)_{t \geq 0} \) is a symmetric martingale. By Corollary 5.7 in Chapter III in [11], we can deduce that \( \frac{1}{2}a(M^\varepsilon)_t - G(x)t, t \geq 0, \) is a martingale. For each fixed \( h \in (0, t) \), let \( v^\varepsilon \) be the solution to the following PDE:

\[
\partial_t v^\varepsilon + G(x)(\partial_{xx} v^\varepsilon) = 0, \quad v^\varepsilon(T + h, x) = \varphi(x).
\]

By the interior regularity of \( v^\varepsilon \) (see [13]),

\[
||v^\varepsilon||_{C^{1+\alpha, 2+\alpha}([0, T] \times \mathbb{R})} < \infty \quad \text{for some} \quad \alpha \in (0, 1).
\]
By martingale representation theorem for symmetric martingale (see Theorem 4.8 in [13]), applying Itô’s formula to \( v^\varepsilon(s,M^\varepsilon,t) \) on \([t,T]\), where \( M^\varepsilon_t = M^\varepsilon - M^\varepsilon_t \), we get

\[
v^\varepsilon(T, M^\varepsilon_T) = v^\varepsilon(t,0) + \int_t^T \partial_x v^\varepsilon(s, M^\varepsilon_s) dM^\varepsilon_s + \frac{1}{2} \int_t^T \partial_{xx}^2 v^\varepsilon(s, M^\varepsilon_s) d(M^\varepsilon)^2_s + \int_t^T \partial_t v^\varepsilon(s, M^\varepsilon_s) ds
\]

\[
= v^\varepsilon(t,0) + \int_t^T \partial_x v^\varepsilon(s, M^\varepsilon_s) dM^\varepsilon_s + \frac{1}{2} \int_t^T \partial_{xx}^2 v^\varepsilon(s, M^\varepsilon_s) d(M^\varepsilon)^2_s - \int_t^T G_\varepsilon(\partial_{xx}^2 v^\varepsilon(s, M^\varepsilon_s)) ds.
\]

Taking conditional expectation \( \mathbb{E}_t \) on both sides, we have

\[
\mathbb{E}_t[v^\varepsilon(T, M^\varepsilon_T)] = v^\varepsilon(t,0) + \mathbb{E}_t \left[ \frac{1}{2} \int_t^T \partial_{xx}^2 v^\varepsilon(s, M^\varepsilon_s) d(M^\varepsilon)^2_s - \int_t^T G_\varepsilon(\partial_{xx}^2 v^\varepsilon(s, M^\varepsilon_s)) ds \right].
\]

Noting that \( \frac{1}{2}a(M^\varepsilon_t) - G_\varepsilon(a)t, t \geq 0, \) is a martingale, by Proposition 1.4 in Chapter IV in [11], we know that

\[
\mathbb{E}_t \left[ \frac{1}{2} \int_t^T \partial_{xx}^2 v^\varepsilon(s, M^\varepsilon_s) d(M^\varepsilon)^2_s - \int_t^T G_\varepsilon(\partial_{xx}^2 v^\varepsilon(s, M^\varepsilon_s)) ds \right] = 0.
\]

Thus

\[
\mathbb{E}_t[v^\varepsilon(T, M^\varepsilon_T)] = v^\varepsilon(t,0).
\]

Similar to the proof of Theorem 4.1, we get

\[
\mathbb{E} \left[ \left| \mathbb{E}_t[v^\varepsilon(M_T - M_t)] - u(t,0) \right| \right] \leq 2C_\varphi \sqrt{2(\sigma^2 + 1)/\pi} \sqrt{t} + 2C_\varphi \sqrt{2(1-t)/\pi} \varepsilon,
\]

where \( C_\varphi \) is the Lipschitz constant of \( \varphi \). Letting \( h \to 0 \) and \( \varepsilon \to 0 \), we obtain the desired result. \( \square \)

**Remark 4.7** It is important to note that we can easily construct a continuous symmetric martingale \((M^\varepsilon_t)_{t \geq 0}\) and use Itô’s formula on the \( G \)-expectation space. However, we can only construct a discrete symmetric martingale \((M^\varepsilon_i^n)_{i=0}^n\) and use Taylor’s expansion on the complete consistent sublinear expectation space.

## 5 Reflection principle of G-Brownian motion

In this section, let \( \Omega = C_0([0, \infty)) \) be the space of real-valued continuous paths \((\omega_t)_{t \geq 0}\) with \( \omega_0 = 0 \). The canonical process \((B_t)_{t \geq 0}\) is defined by

\[
B_t(\omega) := \omega_t \text{ for } \omega \in \Omega.
\]

For each given \( 0 \leq a^2 \leq \sigma^2 \) with \( \sigma^2 > 0 \), define

\[
G(a) := \frac{1}{2}(\sigma^2 a^+ - \sigma^2 a^-) \text{ for } a \in \mathbb{R}.
\]

Peng in [11] constructed a sublinear expectation \( \mathbb{G}[\cdot] \) called \( G \)-expectation on \( L_{ip}(\Omega) \), under which \((B_t)_{t \geq 0}\) is a 1-dimensional \( G \)-Brownian motion. Furthermore, for any given \( \tilde{G} : \mathbb{R} \to \mathbb{R} \) such that

\[
\begin{align*}
\tilde{G}(0) &= 0; \\
\tilde{G}(a) &\leq \tilde{G}(b) \text{ if } a \leq b; \\
\tilde{G}(a) - \tilde{G}(b) &\leq G(a - b) \text{ for } a, b \in \mathbb{R}.
\end{align*}
\]

\begin{align*}
\begin{cases}
\tilde{G}(a) &\leq \tilde{G}(b) \text{ if } a \leq b; \\
\tilde{G}(a) - \tilde{G}(b) &\leq G(a - b) \text{ for } a, b \in \mathbb{R}.
\end{cases}
\end{align*}
By using the following PDE:

$$\partial_t u - \bar{G}(\partial_x^2 u) = 0, \quad u(0, x) = \varphi(x),$$  \tag{5.3}$$

Peng constructed a nonlinear expectation $\hat{\mathbb{E}}^G[\cdot]$ called $\bar{G}$-expectation on $L_{ip}(\Omega)$ satisfying the following relation:

$$\hat{\mathbb{E}}^G[\varphi(B_t)] = u^\varphi(t, 0) \text{ for } t \geq 0,$$

$$\hat{\mathbb{E}}^G[X] - \hat{\mathbb{E}}^G[Y] \leq \hat{\mathbb{E}}^G[|X - Y|] \text{ for } X, Y \in L_{ip}(\Omega),$$  \tag{5.4}$$

where $u^\varphi$ is the solution of PDE (5.3). Similar to the definition of $\hat{\mathbb{E}}^G[\cdot]$ in (2.3), Peng also define the nonlinear conditional expectation $\hat{\mathbb{E}}^G_t[\cdot]$, which still satisfies the relation (5.4). Under the nonlinear expectation space $(\Omega, L_{ip}(\Omega), (L_{ip}(\Omega_t))_{t \geq 0}, (\hat{\mathbb{E}}^G_t)_{t \geq 0})$, $(B_t)_{t \geq 0}$ is a process with stationary and independent increments, which is called a 1-dimensional $\bar{G}$-Brownian motion. It is important to note that $G$ satisfies (5.2), which implies that the $\bar{G}$-Brownian motion is a generalization of the $G$-Brownian motion. By (5.4), we can easily obtain

$$|\hat{\mathbb{E}}^G[X] - \hat{\mathbb{E}}^G[Y]| \leq \hat{\mathbb{E}}^G[|X - Y|] \text{ for } X, Y \in L_{ip}(\Omega).$$  \tag{5.5}$$

Thus $\hat{\mathbb{E}}^G[\cdot]$ can be continuously extended on $L^1_G(\Omega)$.

Consider the following space of simple processes: for $p \geq 1,

$$M^p_G(0, T) := \left\{ \eta = \sum_{i=0}^{N-1} \xi_i I_{[t_i, t_{i+1})}(t) : 0 = t_0 < \cdots < t_N = T, \xi_i \in L^p_G(\Omega_t) \right\}.$$ 

Denote by $M^p_G(0, T)$ (resp. $\hat{M}^p_G(0, T)$) the completion of $M^p_G(0, T)$ under the norm $||\eta||_{M^p_G} := \left( \hat{\mathbb{E}}^G \left[ \int_0^T |\eta|^p dt \right] \right)^{1/p}$ (resp. $||\eta||_{\hat{M}^p_G} := \left( \hat{\mathbb{E}}^G \left[ \int_0^T |\eta|^p d\langle B \rangle_t \right] \right)^{1/p}$), where $(\langle B \rangle_t)_{t \geq 0}$ is the quadratic variation process of $G$-Brownian motion $(B_t)_{t \geq 0}$. Peng in [11] showed that

$$x^2 s \leq \langle B \rangle_{t+s} - \langle B \rangle_t \leq \sigma^2 s \text{ for } t, s \geq 0.$$ 

Thus $M^2_G(0, T) \subset \hat{M}^2_G(0, T)$ and $M^p_G(0, T) = \hat{M}^p_G(0, T)$ under $\sigma^2 > 0$. For each $\eta = \sum_{i=0}^{N-1} \xi_i I_{[t_i, t_{i+1})}(t) \in M^2_G(0, T)$, define the Itô integral

$$\int_0^T \eta_t dB_t := \sum_{i=0}^{N-1} \xi_i (B_{t_{i+1}} - B_{t_i}).$$

Peng in [11] obtained the following Itô equality

$$\hat{\mathbb{E}}^G \left[ \left( \int_0^T \eta_t dB_t \right)^2 \right] = \hat{\mathbb{E}}^G \left[ \int_0^T |\eta_t|^2 d\langle B \rangle_t \right].$$

Thus the Itô integral can be continuously extended on $\hat{M}^2_G(0, T)$.

The purpose of this section is to obtain the distribution of $\sup_{s \leq t} B_s - B_t$, for this we need to use the Itô-Tanaka formula. In [2, 18], the authors obtained the Itô-Tanaka formula and related properties for $G$-Brownian motion. Here we use the representation theorem for $G$-expectation to study the Itô-Tanaka formula.

The following theorem is the representation theorem for $G$-expectation.
Theorem 5.1 (see [1, 3]) There exists a weakly compact set of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that
\[ \hat{E}^G[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \text{ for all } \xi \in L^1_G(\Omega). \]

$\mathcal{P}$ is called a set that represents $\hat{E}^G$.

Let $\mathcal{P}$ be a weakly compact set that represents $\hat{E}^G$. For this $\mathcal{P}$, we define capacity $c(A) := \sup_{P \in \mathcal{P}} P(A)$, $A \in \mathcal{B}(\Omega)$.

A set $A \subset \mathcal{B}(\Omega)$ is called polar if $c(A) = 0$. A property holds "quasi-surely" (q.s.) if it holds outside a polar set. In the following, we do not distinguish two random variables $X$ and $Y$ if $X = Y$ q.s.

The following theorem is the well-known Krylov’s estimate (see [5, 8, 12]).

Theorem 5.2 (Krylov’s estimate) Let $(B_t)_{t \geq 0}$ be a 1-dimensional $G$-Brownian motion. Then, for $T > 0$, $p \geq 1$ and for each Borel function $g$,
\[ \hat{E}^G \left[ \int_0^T |g(B_t)| dB_t \right] \leq C \left( \int_{\mathbb{R}} |g(x)|^p dx \right)^{1/p}, \]
where $C = (\hat{E}^G[|B_T|])^{(p-1)/p}(\hat{E}^G[|B_T|])^{1/p}$.

Proof. For reader’s convenience, we give a probabilistic proof. For any $P \in \mathcal{P}$, it is easy to check that $(B_t)_{t \geq 0}$ is a martingale under $P$. By the occupation times formula, we obtain
\[ \int_0^T |g(B_t)| dB_t = \int_{\mathbb{R}} |g(a)|^p L_P^T(a) da, \text{ P-a.s.,} \tag{5.6} \]
where $L_T^P(a)$ is the local time in $a$ of $B$ under $P$. On the other hand, by the Itô-Tanaka formula, we have
\[ |B_T - a| = |a| + \int_0^T \text{sgn}(B_t - a) dB_t + L_T^P(a), \text{ P-a.s.} \tag{5.7} \]
Taking expectation on both sides, we get
\[ 0 \leq E_P[L_T^P(a)] = E_P[|B_T - a| - |a| \leq E_P[|B_T|] \leq \hat{E}^G[|B_T|]]. \tag{5.8} \]
Combining (5.6) and (5.8), by Hölder’s inequality, we obtain
\[ E_P \left[ \int_0^T |g(B_t)| dB_t \right] \leq C_1 \left( E_P \left[ \int_0^T |g(B_t)|^p dB_t \right] \right)^{1/p} \leq C \left( \int_{\mathbb{R}} |g(x)|^p dx \right)^{1/p}, \]
where $C_1 = (\hat{E}^G[|B_T|])^{(p-1)/p}$ and $C = C_1(\hat{E}^G[|B_T|])^{1/p}$. Since $C$ is independent of $P$, we get the desired result by taking supremum over $P \in \mathcal{P}$ in the above inequality. □

Similar to Theorems 4.15 and 4.16 in [4], we have the following proposition which contains the case $\mathfrak{s}^2 = 0$. The proof is omitted.
Proposition 5.3 Let \((B_t)_{t \geq 0}\) be a 1-dimensional \(G\)-Brownian motion. For each \(T > 0\), we have the following results:

1. If \(\varphi\) is in \(L^p(\mathbb{R})\) with \(p \geq 1\), then \((\varphi(B_t))_{t \leq T} \in M^1_G(0, T)\). Moreover, for each \(\varphi' = \varphi\), a.e., we have \((\varphi'(B_t))_{t \leq T} = (\varphi(B_t))_{t \leq T}\) in \(M^1_G(0, T)\).

2. Let \((\varphi^k)_{k \geq 1}\) be a sequence of Borel measurable functions such that \(|\varphi^k(x)| \leq \bar{c} (1 + |x|^l)\), \(k \geq 1\), for some positive constants \(\bar{c}\) and \(l\). If \(\varphi^k \to \varphi\), a.e., then for each \(p \geq 1\),
   \[
   \lim_{k \to \infty} \hat{E}^G \left[ \int_0^T |\varphi^k(B_t) - \varphi(B_t)|^p \, dB_t \right] = 0.
   \]

3. If \(\varphi\) is a Borel measurable function of polynomial growth, then \((\varphi(B_t))_{t \leq T} \in M^2_G(0, T)\).

Now we can give the Itô-Tanaka formula on the \(G\)-expectation space. For each \(P \in \mathcal{P}\), we have the following Itô-Tanaka formula under \(P\)

\[
|B_t - a| = |a| + \int_0^t \text{sgn}(B_s - a) \, dB_s + L_t^P(a), \quad P\text{-a.s.}
\]  (5.9)

By Proposition 5.3, we have \((\text{sgn}(B_s - a))_{s \leq t} \in M^2_G(0, t)\), which implies that \(\int_0^t \text{sgn}(B_s - a) \, dB_s \in L^2_G(\Omega_t)\) for \(t \geq 0\). Set

\[
L_t(a) = |B_t - a| - |a| - \int_0^t \text{sgn}(B_s - a) \, dB_s \in L^2_G(\Omega_t).
\]

Then, by (5.9), we obtain the following Itô-Tanaka formula on the \(G\)-expectation space

\[
|B_t - a| = |a| + \int_0^t \text{sgn}(B_s - a) \, dB_s + L_t(a), \quad \text{q.s.},
\]  (5.10)

and \(L_t(a)\) is called the local time in \(a\) of \(B\) under \(\hat{E}^G[\cdot]\).

Lemma 5.4 Let \((B_t)_{t \geq 0}\) be a 1-dimensional \(G\)-Brownian motion. Then \(\int_0^t \text{sgn}(B_s) \, dB_s\), for \(t \geq 0\), is still a \(G\)-Brownian motion.

Proof. By Proposition 5.3, we have \((\text{sgn}(B_s))_{s \leq t} \in M^2_G(0, t)\) for each \(t \geq 0\). Then we obtain that \(\int_0^t \text{sgn}(B_s) \, dB_s \in L^2_G(\Omega_t)\), \(t \geq 0\), is a symmetric martingale, and

\[
\langle \int_0^t \text{sgn}(B_s) \, dB_s \rangle_t = \int_0^t |\text{sgn}(B_s)|^2 \, dB_s = \langle B \rangle_t.
\]

By Theorem 4.6, we get the desired result. \(\square\)

The following theorem is the reflection principle for \(G\)-Brownian motion \(B\).

Theorem 5.5 Let \((B_t)_{t \geq 0}\) be a 1-dimensional \(G\)-Brownian motion and \((L_t(0))_{t \geq 0}\) be the local time of \(B\) under \(\hat{E}^G[\cdot]\). Then

\[
(S_t - B_t)_{t \geq 0} \overset{d}{=} \langle |B_t|, L_t(0) \rangle_{t \geq 0}
\]

under \(\hat{E}^G[\cdot]\), where \(S_t = \sup_{s \leq t} B_s\) for \(t \geq 0\).
Remark 5.6 Specially, \( S_t - B_t \overset{d}{=} |B_t| \), i.e., \( \sup_{s \leq t} (B_s - B_t) \overset{d}{=} |B_t| \).

In order to prove this theorem, we need the following well-known Skorokhod lemma. Let \( \mathcal{D}_0([0, \infty)) \) be the space of real-valued right continuous with left limit (RCLL) paths \((\omega_t)_{t \geq 0}\) with \(\omega_0 = 0\).

Lemma 5.7 (Skorokhod) Let \( x \in \mathcal{D}_0([0, \infty)) \) be given. Then there exists a unique pair \((y, z) \in \mathcal{D}_0([0, \infty); \mathbb{R}^2)\) such that

(a) \( z(t) = x(t) + y(t) \geq 0 \) for \( t \geq 0 \);
(b) \( y \) is increasing with \( y(0) = 0 \);
(c) \( \int_0^\infty z(t)dy(t) = 0 \).

Moreover, \((y, z)\) can be expressed as

\[
y(t) = \sup_{0 \leq s \leq t} (-x(s)), \quad z(t) = x(t) + \sup_{0 \leq s \leq t} (-x(s)), \quad t \geq 0.
\] (5.11)

Proof of Theorem 5.5. For each \( x \in \mathcal{D}_0([0, \infty)) \), let \((y, z) \in \mathcal{D}_0([0, \infty); \mathbb{R}^2)\) be defined as in (5.11). Define the mapping \( F : \mathcal{D}_0([0, \infty)) \rightarrow \mathcal{D}_0([0, \infty); \mathbb{R}^2) \) as

\[
F(x) = (y, z).
\]

By the Itô-Tanaka formula on the \( G \)-expectation space, we have

\[
|B_t| = \int_0^t \text{sgn}(B_s)dB_s + L_t(0), \quad \text{q.s.}
\]

Applying Itô’s formula to \(|B_t|^2\) and \(|B_t|^2 = 2 \int_0^t B_sdB_s + \langle B \rangle_t\), we can get

\[
\int_0^\infty |B_s|dL_s(0) = 0, \quad \text{q.s.}
\]

Thus, by Lemma 5.7, we obtain

\[
F\left[ \left\{ \int_0^t \text{sgn}(B_s)dB_s \right\}_{t \geq 0} \right] = (L_t(0), |B_t|)_{t \geq 0}, \quad \text{q.s.}
\] (5.12)

On the other hand,

\[
S_t - B_t = -B_t + S_t.
\]

Noting that \( S_t = \sup_{0 \leq s \leq t} (-(-B_s)) \), by Lemma 5.4, we have

\[
F\left[ (-B_t)_{t \geq 0} \right] = (S_t, S_t - B_t)_{t \geq 0}.
\] (5.13)

By Lemma 5.4, we know

\[
\left( \int_0^t \text{sgn}(B_s)dB_s \right)_{t \geq 0} \overset{d}{=} (B_t)_{t \geq 0} \overset{d}{=} (-B_t)_{t \geq 0}.
\] (5.14)

By (5.11), it is easy to check that \( F \) is Lipschitz continuous under the uniform topology. Thus, by (5.12), (5.13) and (5.14), we can deduce the desired result. \( \Box \)

Let \( \tilde{G} : \mathbb{R} \rightarrow \mathbb{R} \) satisfy (5.2). In the following, we extend the reflection principle to \( \tilde{G} \)-Brownian motion \( B \). We need the following lemma.
Lemma 5.8 Let \((B_t)_{t \geq 0}\) be a 1-dimensional \(\bar{G}\)-Brownian motion. Then \(\int_0^T \text{sgn}(B_s)dB_s, \ t \geq 0,\) is still a \(\bar{G}\)-Brownian motion.

Under nonlinear expectation space, we do not obtain the Lévy’s martingale characterization theorem. The key problem lies in the fact that the solution of the following PDE
\[
\partial_t u - \bar{G} (\partial^2 u) = \frac{1}{2} \sigma^2 \partial^2 u = 0, \ u(0, x) = \varphi(x)
\]
may not be regular. In the following, we give a new proof. First we need the following lemma.

Lemma 5.9 Suppose \(\sigma^2 > 0\) in \(G\). Let \(\pi_n = \{t_0^n, t_1^n, \ldots, t_n^n\}\) be a partition of \([0,T]\) and \(\mu(\pi_n)\) be the diameter of \(\pi_n\). Then
\[
\mathbb{E}^G \left[ \int_0^T \sum_{i=0}^{n-1} \text{sgn}(B_{t_i^n}) I_{[t_i^n, t_{i+1}^n)}(t) - \text{sgn}(B_t) \right] \to 0 \text{ as } \mu(\pi_n) \to 0.
\]

Proof. For each fixed \(\varepsilon > 0\), define
\[
\phi_\varepsilon(x) = I_{[\varepsilon, \infty)}(x) + \frac{x}{\varepsilon} I_{(-\varepsilon, -\varepsilon)}(x) - I_{(-\infty, -\varepsilon)}(x) \text{ for } x \in \mathbb{R}.
\]
It is easy to verify that \(|\phi_\varepsilon(x) - \phi_\varepsilon(x')| \leq \frac{1}{\varepsilon}|x - x'|\) and \(|\text{sgn}(x) - \phi_\varepsilon(x)| \leq I_{(-\varepsilon, \varepsilon)}(x)\). By Example 3.9 in \cite{4}, we have
\[
\mathbb{E}^G[I_{(-\varepsilon, \varepsilon)}(B_t)] \leq \exp(\frac{1}{2\sigma^2}) \frac{1}{t^{\alpha}},
\]
where \(\alpha = \frac{\sigma^2}{\varepsilon^2} \in (0, \frac{1}{2})\). Then
\[
\begin{align*}
\mathbb{E}^G \left[ \int_0^T \sum_{i=0}^{n-1} \text{sgn}(B_{t_i^n}) I_{[t_i^n, t_{i+1}^n)}(t) - \text{sgn}(B_t) \right] & \leq 3 \left( \mathbb{E}^G \left[ \int_0^T |I_{11}^{\varepsilon, n}(t)|^2 dt \right] + \mathbb{E}^G \left[ \int_0^T |I_{22}^{\varepsilon, n}(t)|^2 dt \right] + \mathbb{E}^G \left[ \int_0^T |I_{33}^{\varepsilon, n}(t)|^2 dt \right] \right) \\
& \leq 3 \left\{ \sum_{i=0}^{n-1} \mathbb{E}^G \left[ I_{(-\varepsilon, \varepsilon)}(B_{t_i^n}) \right] (t_i^n - t_{i+1}^n) + \frac{1}{\varepsilon} \sum_{i=0}^{n-1} \mathbb{E}^G \left[ |B_{t_i^n} - B_{t_{i+1}^n}|^2 \right] dt + \int_0^T \mathbb{E}^G \left[ I_{(-\varepsilon, \varepsilon)}(B_t) \right] dt \right\} \\
& \leq 3 \left\{ \exp(\frac{1}{2\sigma^2}) \varepsilon^{2\alpha} \sum_{i=0}^{n-1} \frac{1}{(t_i^n - t_{i+1}^n)} + \frac{\sigma^2}{\varepsilon^2} \mu(\pi_n) T + \exp(\frac{1}{2\sigma^2}) \varepsilon^{2\alpha} \int_0^T \frac{1}{L^n} dt \right\},
\end{align*}
\]
where
\[
I_{11}^{\varepsilon, n}(t) = \sum_{i=0}^{n-1} \text{sgn}(B_{t_i^n}) I_{[t_i^n, t_{i+1}^n)}(t) - \sum_{i=0}^{n-1} \phi_\varepsilon(B_{t_i^n}) I_{[t_i^n, t_{i+1}^n)}(t),
\]
\[
I_{22}^{\varepsilon, n}(t) = \sum_{i=0}^{n-1} \phi_\varepsilon(B_{t_i^n}) I_{[t_i^n, t_{i+1}^n)}(t) - \phi_\varepsilon(B_t),
\]
\[
I_{33}^{\varepsilon, n}(t) = \phi_\varepsilon(B_t) - \text{sgn}(B_t).
\]
Letting $\mu(\pi_n) \to 0$ in the above inequality, we get
\[
\limsup_{\mu(\pi_n) \to 0} \hat{E}^G \left[ \left( \int_0^T \sum_{i=0}^{n-1} \sgn(B_{t_i}) I_{[t_i, t_{i+1})}(t) - \sgn(B_t) \right)^2 dt \right] \\
\leq 3 \exp \left( \frac{1}{2\sigma^2} \right) e^{2\alpha} \left\{ \lim_{\mu(\pi_n) \to 0} \sum_{i=0}^{n-1} \frac{1}{(t_{i+1}^n - t_i^n)} + \int_0^T \frac{1}{t} dt \right\} \\
= 6 \exp \left( \frac{1}{2\sigma^2} \right) e^{2\alpha} \int_0^T \frac{1}{t} dt.
\]
Taking $\varepsilon \to 0$, we obtain the desired result. $\square$

**Proof of Lemma 5.8** By the proof of Lemma 5.3 we know that $\int_t^T \sgn(B_s) dB_s \in L^2_G(\Omega_T)$ for $0 \leq t < T < \infty$. By the property of $\hat{E}^G_t[\cdot]$, we only need to prove that
\[
\hat{E}^G_t[\varphi(\int_t^T \sgn(B_s) dB_s)] = u^\varphi(T - t, 0),
\]
where $0 < t < T$, $\varphi \in C_b, L_{lip}(\mathbb{R})$ and $u^\varphi$ is the solution of PDE (5.3). The proof is divided into two steps.

**Step 1:** We first prove (5.16) under the case $\sigma^2 > 0$. By Lemma 5.9 we have
\[
\hat{E}^G_t \left[ \left\{ \int_t^T \varphi(\sum_{i=0}^{n-1} \sgn(B_{t_i}) (B_{t_{i+1}^n} - B_{t_i^n})) - \hat{E}^G_t[\varphi(\int_t^T \sgn(B_s) dB_s)] \right\}^2 \right] \\
\leq C^2_{\varphi} \hat{E}^G_t \left[ \sum_{i=0}^{n-1} \sgn(B_{t_i}) (B_{t_{i+1}^n} - B_{t_i^n}) - \int_t^T \sgn(B_s) dB_s \right]^2 \\
\leq C^2_{\varphi} \sigma^2 \hat{E}^G_t \left[ \int_t^T \sum_{i=0}^{n-1} \sgn(B_{t_i}) I_{[t_i, t_{i+1})}(s) - \sgn(B_s) \right]^2 ds \\
\to 0, \text{ as } n \to \infty,
\]
where $C_{\varphi}$ is the Lipschitz constant of $\varphi$. On the other hand, noting that $-(B_{t_{i+1}^n} - B_{t_i^n}) \overset{d}{=} (B_{t_{i+1}^n} - B_{t_{i+1}^n})$, by the property of $\hat{E}^G_t[\cdot]$, we get
\[
\hat{E}^G_t[\varphi(\sgn(B_{t_i})(B_{t_{i+1}^n} - B_{t_i^n}))] = \hat{E}^G_t[\varphi(B_{t_{i+1}^n} - B_{t_i^n})] \text{ for any } \varphi \in C_b, L_{lip}(\mathbb{R}).
\]
From this, we can easily obtain
\[
\hat{E}^G_t[\varphi(\sum_{i=0}^{n-1} \sgn(B_{t_i^n})(B_{t_{i+1}^n} - B_{t_i^n}))] = \hat{E}^G_t[\varphi(B_T - B_t)] = u^\varphi(T - t, 0).
\]
Combining (5.17) and (5.18), we deduce (5.16).

**Step 2:** Set $\hat{\Omega} = C_0([0, \infty); \mathbb{R}^2)$, the corresponding canonical process is denoted by $(B_t, \hat{B}_t)_{t \geq 0}$. Define
\[
G' \left( \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right) = G(a) + \frac{1}{2} c, \quad \hat{G}' \left( \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right) = \hat{G}(a) + \frac{1}{2} c \text{ for } a, b, c \in \mathbb{R}.
\]
Following Peng [11], we can construct $G^\epsilon$-expectation $(\hat{E}_t^G)^{\epsilon,0}$ and $\hat{G}^\epsilon$-expectation $(\hat{E}_t^{\hat{G}})^{\epsilon,0}$ via the following PDE:

\[
\begin{align*}
\partial_t u - G(\partial_{xx} u) - \frac{1}{2} \partial_{yy} u &= 0, \quad u(0, x, y) = \psi(x, y), \\
\partial_t u - \hat{G}(\partial_{xx} u) - \frac{1}{2} \partial_{yy} u &= 0, \quad u(0, x, y) = \psi(x, y).
\end{align*}
\]

It is easy to verify that $\hat{E}_t^G[\xi] = \hat{E}_t^G[\xi], \hat{E}_t^{\hat{G}}[\xi] = \hat{E}_t^{\hat{G}}[\xi]$ for each $\xi \in L^1_t(\Omega)$ and $(B_t+\delta B_t)_{t \geq 0}$ is a $G_\delta$-Brownian motion under $\hat{E}_t^G[\cdot]$ for each $\delta > 0$, where $G_\delta(a) = G(a) + \frac{1}{\delta^2} a$ for $a \in \mathbb{R}$. By Step 1, we have

\[
\begin{align*}
\hat{E}_t^G[\varphi\left(\int_t^T \text{sgn}(B_s + \delta B_s) d(B_s + \delta B_s)\right)] &= \hat{E}_t^G[\varphi(B_T + \delta B_T - B_t - \delta B_t)] \\
&= \hat{E}_t^{\hat{G}}[\varphi(B_T + \delta B_T - B_t - \delta B_t)].
\end{align*}
\]

(5.19)

For each fixed $\epsilon > 0$, define $\phi_\epsilon$ as in (5.15). By Theorem 5.2, we have

\[
\begin{align*}
\hat{E}_t^G &\left[\hat{E}_t^G[\varphi\left(\int_t^T \text{sgn}(B_s + \delta B_s) d(B_s + \delta B_s)\right)] - \hat{E}_t^G[\varphi\left(\int_t^T \text{sgn}(B_s) dB_s\right)]\right]^2 \\
&\leq C_\varphi^2 \hat{E}_t^G \left[\int_t^T \text{sgn}(B_s + \delta B_s) d(B_s + \delta B_s) - \int_t^T \text{sgn}(B_s) dB_s\right]^2 \\
&\leq 4C_\varphi^2 \hat{E}_t^G \left[\int_t^T I_1^\epsilon(s) d(B_s + \delta B_s)\right]^2 + \int_t^T I_2^\epsilon(s) dB_s\right]^2 + \delta \int_t^T \phi_\epsilon(B_s + \delta B_s) dB_s\right|^2 + \int_t^T I_3^\epsilon(s) dB_s\right]^2 \\
&\leq 4C_\varphi^2 \left\{\hat{E}_t^G \left[\int_t^T I_1^\epsilon(s) d(B_s + \delta B_s)\right]^2 + \frac{\delta^2 \sigma^2 T^2}{2\epsilon^2} + \delta^2 T + \hat{E}_t^G \left[\int_t^T I_1^\epsilon(s) d(B_s)\right]\right\} \\
&\leq 4C_\varphi^2 \left\{2\hat{E}_t^G[\|B_T + \hat{B}_T\|^2] + \frac{\sigma^2 T^2 + 2\epsilon^2 T}{2\epsilon^2}\delta^2 + 2\hat{E}_t^G[\|\hat{B}_T\|\epsilon]\right\},
\end{align*}
\]

where

\[
\begin{align*}
I_1^\epsilon(s) &= \text{sgn}(B_s + \delta B_s) - \phi_\epsilon(B_s + \delta B_s), \\
I_2^\epsilon(s) &= \phi_\epsilon(B_s + \delta B_s) - \phi_\epsilon(B_s), \\
I_3^\epsilon(s) &= \phi_\epsilon(B_s) - \text{sgn}(B_s).
\end{align*}
\]

Taking $\delta \downarrow 0$ in the above inequality, we obtain

\[
\begin{align*}
\limsup_{\delta \downarrow 0} \hat{E}_t^G &\left[\hat{E}_t^G[\varphi\left(\int_t^T \text{sgn}(B_s + \delta B_s) d(B_s + \delta B_s)\right)] - \hat{E}_t^G[\varphi\left(\int_t^T \text{sgn}(B_s) dB_s\right)]\right]^2 \\
&\leq 4C_\varphi^2 \left\{2\hat{E}_t^G[\|B_T + \hat{B}_T\|^2] + 2\hat{E}_t^G[\|\hat{B}_T\|\epsilon]\right\},
\end{align*}
\]

which implies

\[
\begin{align*}
\lim_{\delta \downarrow 0} \hat{E}_t^G &\left[\hat{E}_t^G[\varphi\left(\int_t^T \text{sgn}(B_s + \delta B_s) d(B_s + \delta B_s)\right)] - \hat{E}_t^G[\varphi\left(\int_t^T \text{sgn}(B_s) dB_s\right)]\right]^2 = 0
\end{align*}
\]

(5.20)
by letting $\varepsilon \downarrow 0$. It is easy to check that
\[
\lim_{\delta \downarrow 0} \left| \mathbb{E}^\mathcal{G}[\varphi(B_T + \delta \hat{B}_T - B_t - \delta \hat{B}_t)] - \mathbb{E}^\mathcal{G}[\varphi(B_T - B_t)] \right| = 0.
\] (5.21)

By (5.19), (5.20) and (5.21), we obtain (5.16). □

Similar to the proof of Theorem 5.5, we can immediately obtain the following reflection principle for \( \hat{\mathcal{G}} \)-Brownian motion \( B \). The proof is omitted.

**Theorem 5.10** Let \((B_t)_{t \geq 0}\) be a 1-dimensional \( \hat{\mathcal{G}} \)-Brownian motion and \((L_t(0))_{t \geq 0}\) be the local time of \( B \) under \( \mathbb{E}^\mathcal{\hat{G}}[\cdot] \). Then
\[
(S_t - B_t, S_t)_{t \geq 0} \overset{d}{=} (|B_t|, L_t(0))_{t \geq 0},
\]
under \( \mathbb{E}^\mathcal{\hat{G}}[\cdot] \), where \( S_t = \sup_{s \leq t} B_s \) for \( t \geq 0 \).

**Remark 5.11** In particular, \((\sup_{s \leq t} (B_s - B_t))_{t \geq 0} \overset{d}{=} (|B_t|)_{t \geq 0}\) under \( \mathbb{E}^\mathcal{\hat{G}}[\cdot] \).

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