STRONG $I$ AND $I^*$-STATISTICALLY PRE-CAUCHY DOUBLE SEQUENCES IN PROBABILISTIC METRIC SPACES

PRASANTA MALIK*, ARGHA GHOSH* AND MANOJIT MAITY**

* Department of Mathematics, The University of Burdwan, Golapbag, Burdwan-713104, West Bengal, India. Email: pmjupm@yahoo.co.in., buagbh@yahoo.co.in
** Boral High School, Kolkata-700154, India. Email: mepsilon@gmail.com

Abstract. In this paper we consider the notion of strong $I$-statistically pre-Cauchy double sequences in probabilistic metric spaces in line of Das et. al. [6] and introduce the new concept of strong $I^*$-statistically pre-Cauchy double sequences in real line as well as in probabilistic metric spaces. We mainly study inter relationship among strong $I$-statistical convergence, strong $I$-statistical pre-Cauchy condition and strong $I^*$-statistical pre-Cauchy condition for double sequences in probabilistic metric spaces and examine some basic properties of these notions.

Key words and phrases: Probabilistic metric space, strong $I$-statistical convergence, strong $I$-statistical pre-Cauchy condition, strong $I^*$-statistical pre-Cauchy condition.

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1. Introduction

The notion of probabilistic metric (PM) was introduced by K. Menger [10] under the name of “statistical metric spaces” by considering the distance between two points $x$ and $y$ as a distribution function $F_{xy}$ instead of a non-negative real number and the value of the function $F_{xy}$ at any $t > 0$ i.e. $F_{xy}(t)$ is interpreted as the probability that the distance between the points $x$ and $y$ is $\leq t$. After Menger, the theory of probabilistic metric was developed by Schweizer and Sklar ([16], [17], [18], [19]), Tardiff [23], Thorp [24] and many others. A thorough development of probabilistic metric spaces (can be seen from the famous book Schweizer and Sklar [20]. Many different topologies and it is the main tool of our paper.
The idea of usual notion of convergence of real sequences was extended to statistical convergence by Fast [8] and Schoenberg [15] independently. For the last few years a lot of work has been done on this convergence (see [2], [8], [9], [14], [22] etc.). In [2] the notion of statistically pre-Cauchy sequences of real numbers was introduced and it was shown that statistically convergent sequences are always statistically pre-Cauchy and the converse statement holds under certain conditions. The notion of statistical convergence was further extended to I-convergence [13] using the ideals of \( \mathbb{N} \) and also to I-statistical convergence in [14]. The notion of strong I-statistical convergence for double sequences of real numbers had been introduced by Belen et. al. in [1]. Recently in [6] Das and Savas introduced the notion of I-statistically pre-Cauchy sequence of real numbers as a generalization of I-statistical convergence. They proved that every I-statistically convergent sequence is I-statistically pre-Cauchy and the converse is true under certain sufficient conditions.

Following the line of Das and Savas [6], in this paper we introduce the notion of strong I-statistically pre-Cauchy double sequences in probabilistic metric spaces. We also introduce the new notion of strong \( I^* \)-statistically pre-Cauchy double sequences in probabilistic metric spaces. We show that every strong I-statistically convergent double sequence is strong I-statistically pre-Cauchy and every strong \( I^* \)-statistically pre-Cauchy double sequence is strong I-statistically pre-Cauchy and the converse of each of the results holds under certain conditions.

1. Preliminaries

We now recall some definition and notation

**Definition 1.1.** If \( K \) is a subset of the set of positive integer \( \mathbb{N} \), then \( K_n \) denotes the set \( \{ k \in K : k \leq n \} \) and \( |K_n| \) denotes the number of elements in \( K_n \). The "natural density" of \( K \) [4] is given by \( \delta(K) = \lim_{n \to \infty} \frac{|K_n|}{n} \).

**Definition 1.2.** [25] Let \( X \neq \emptyset \). A class \( I \subseteq 2^X \) of subsets of \( X \) is said to be an ideal in \( X \) provided that \( I \) satisfies these conditions:

(i) \( \emptyset \in I \)
(ii) \( A, B \in I \Rightarrow A \cup B \in I \),
(iii) \( A \in I, B \subseteq A \Rightarrow B \in I \)

An ideal is called non-trivial if \( X \notin I \).

**Definition 1.3.** [25] Let \( X \neq \emptyset \). A non-empty class \( F \subseteq 2^X \) of subsets of \( X \) is said to be a filter in \( X \) provided that:

(i) \( \emptyset \notin F \)
(ii) \( A, B \in F \Rightarrow A \cap B \in F \),
(iii) \( A \in F, B \supseteq A \Rightarrow B \in F \).
The following result expresses a relation between the notions of ideal and filter:

**Result 1.** Let $I$ be a non-trivial ideal in $X$, $X \neq \phi$. Then the class $\mathcal{F}(I)=\{M \subseteq X : \exists A \in I : M = X \setminus A\}$ is a filter on $X$ (we will call $\mathcal{F}(I)$ the filter associated with $I$).

The proof of result 1 is easy and so it can be left.

**Definition 1.4.** [5] A non-trivial ideal $I$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $I$ for each $i \in \mathbb{N}$.

Clearly every admissible ideal of $\mathbb{N} \times \mathbb{N}$ is strongly admissible.

We can also use the concept of porosity of subsets of a metric space.

The concept of statistical convergence and the study of similar types of convergence ([1], [2], [3], [4]) lead us to introduced the notion of $I$-convergence of sequences. This notion gives a unifying look at many types of convergence related to statistical convergence.

**Definition 1.5.** [25]: Let $I$ be a non-trivial ideal of $\mathbb{N}$. A sequence $x = \{x_n\}_{n=1}^{\infty}$ of real numbers is said to be $I$-convergent to $\zeta \in \mathbb{R}$ if for every $\epsilon > 0$ the set $A(\epsilon) = \{n : |x_n - \zeta| \geq \epsilon\} \in I$.

If $x = \{x_n\}_{n=1}^{\infty}$ is $I$-convergent to $\zeta$ we write $\lim_{n \to \infty} x_n = \zeta$ (or $I - \lim x = \zeta$) and the number $\zeta$ is called the $I$-limit of $x$.

We now recall some basic concepts related to statistical convergence of double sequences (see [7] for more details). Let $K \subset \mathbb{N} \times \mathbb{N}$. Let $K(n, m)$ be the number of $(j, k) \in K$ such that $j \leq n, k \leq m$. The number $d_2(K) = \lim\sup_{m \to \infty, n \to \infty} \frac{K(n, m)}{nm}$ is called the upper double natural density of $K$. If the sequence $\{\frac{K(n, m)}{nm}\}_{n, m \in \mathbb{N}}$ has a limit in Pringsheim’s sense, then we say that $K$ has the double natural density and it is denoted by $d_2(K) = \lim_{m \to \infty, n \to \infty} \frac{K(n, m)}{nm}$.

**Definition 1.6.** [7] A double sequence $x = \{x_{jk}\}_{j, k \in \mathbb{N}}$ of real numbers is said to be statistically convergent to $\xi \in \mathbb{R}$ if for every $\epsilon > 0$, we have $d_2(A(\epsilon)) = 0$ where $A(\epsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : |x_{jk} - \xi| \geq \epsilon\}$. In this case we write $st - \lim x_{jk} = \xi$.

A statistically convergent double sequence of elements in a metric space $(X, \rho)$ is defined essentially in the same way using $\rho(x_{jk}, \xi) \geq \epsilon$ instead of $|x_{jk} - \xi| \geq \epsilon$. 
Definition 1.7. [7] Let \((X, \rho)\) be a metric space. A double sequence \(x = \{x_{jk}\}_{j,k \in \mathbb{N}}\) in \(X\) is said to be statistically Cauchy if for every \(\epsilon \geq 0\), there exist natural numbers \(N = N(\epsilon)\) and \(M = M(\epsilon)\) such that for all \(j, p \geq N\) and \(k, q \geq M\),
\[
d_2(\{(j, k) \in \mathbb{N} \times \mathbb{N} : \rho(x_{jk}, x_{pq}) \geq \epsilon\}) = 0
\]

Theorem 1.1. [7] A double sequence \(x = \{x_{jk}\}_{j,k \in \mathbb{N}}\) in a metric space \((X, \rho)\) is statistically convergent to \(\xi \in X\) if and only if there exists a subset \(K = \{(j, k) \in \mathbb{N} \times \mathbb{N}\}\) of \(\mathbb{N} \times \mathbb{N}\) such that \(d_2(K) = 1\) and \(\lim_{j \to \infty, k \to \infty} x_{jk} = \xi\).

Theorem 1.2. [7] For a sequence \(x = \{x_{jk}\}_{j,k \in \mathbb{N}}\) in a metric space \((X, \rho)\), the following statements are equivalent:
1. \(x\) is statistically convergent to \(l \in X\).
2. \(x\) is statistically Cauchy.
3. There exists a subset \(M \subset \mathbb{N} \times \mathbb{N}\) such that \(d_2(M) = 1\) and \(\{x_{jk}\}_{(j,k) \in M}\) converges to \(l\).

2. Basic concepts of Probabilistic Metric Spaces.

First we recall some basic concepts related to the probabilistic metric spaces (or PM spaces) which can be studied in details from the fundamental book [21] by Schweizer and Sklar.

Definition 2.1. A non-decreasing function \(F : \mathbb{R} \to [0, 1]\) defined on \(\mathbb{R}\) with \(F(-\infty) = 0\) and \(F(\infty) = 1\), where \(\mathbb{R} = [-\infty, \infty]\), is called a distribution function.

The set of all left continuous distribution function over \((-\infty, \infty)\) is denoted by \(\Delta\).

We consider the relation \(\leq\) on \(\Delta\) defined by \(G \leq F\) if and only if \(G(x) \leq F(x)\) for all \(x \in \mathbb{R}\). Clearly it can be easily verified that the relation ‘\(\leq\)’ is a partial order on \(\Delta\).

Definition 2.2. For any \(p \in [-\infty, \infty]\) the unit step at \(p\) is denoted by \(\epsilon_p\) and is defined to be a function in \(\Delta\) given by
\[
\epsilon_p(x) = \begin{cases} 
0, & -\infty \leq x \leq p \\
1, & p < x \leq \infty.
\end{cases}
\]

Definition 2.3. A sequence \(\{F_n\}_{n \in \mathbb{N}}\) of distribution functions converges weakly to a distribution function \(F\) and we write \(F_n \stackrel{w}{\rightharpoonup} F\) if and only if the sequence \(\{F_n(x)\}_{n \in \mathbb{N}}\) converges to \(F(x)\) at each continuity point \(x\) of \(F\).

Definition 2.4. The distance between \(F\) and \(G\) in \(\Delta\) is denoted by \(d_L(F, G)\) and is defined as the infimum of all numbers \(h \in (0, 1]\) such that the inequalities
\[ F(x - h) - h \leq G(x) \leq F(x + h) + h \]
and \[ G(x - h) - h \leq F(x) \leq G(x + h) + h \]
hold for every \( x \in \left(-\frac{1}{h}, \frac{1}{h}\right) \).

It is known that \( d_L \) is a metric on \( \Delta \) and for any sequence \( \{F_n\}_{n \in \mathbb{N}} \) in \( \Delta \) and \( F \in \Delta \), we have
\[ F_n \xrightarrow{w} F \text{ if and only if } d_L(F_n, F) \to 0. \]

Here we will be interested in the subset of \( \Delta \) consisting of those elements \( G \) that satisfy \( G(0) = 0 \).

**Definition 2.5.** A non-decreasing function \( G \) defined on \( \mathbb{R}^+ = [0, \infty] \) that satisfies \( G(0) = 0 \) and \( G(\infty) = 1 \) and is left continuous on \((0, \infty)\) is called a distance distribution function.

The set of all distance distribution functions is denoted by \( \Delta^+ \).

**Theorem 2.1.** Let \( F \in \Delta^+ \) be given. Then for any \( t > 0, F(t) > 1 - t \) if and only if \( d_L(F, \epsilon_0) < t \).

**Definition 2.6.** A triangle function is a binary operation \( \tau \) on \( \Delta^+ \), \( \tau : \Delta^+ \times \Delta^+ \to \Delta^+ \) which is commutative, nondecreasing, associative in each place, and \( \epsilon_0 \) is the identity.

**Definition 2.7.** A probabilistic metric space, briefly PM space, is a triplet \((S, F, \tau)\) where \( S \) is a nonempty set whose elements are the points of the space; \( F \) is a function from \( S \times S \) into \( \Delta^+ \), \( \tau \) is a triangle function, and the following conditions are satisfied for all \( x, y, z \in S \):

1. \( \mathfrak{F}(x, x) = \epsilon_0 \)
2. \( \mathfrak{F}(x, y) \neq \epsilon_0 \) if \( x \neq y \)
3. \( \mathfrak{F}(x, y) = \mathfrak{F}(y, x) \)
4. \( \mathfrak{F}(x, z) \geq \tau(\mathfrak{F}(x, y), \mathfrak{F}(y, z)) \).

From now on we will denote \( \mathfrak{F}(x, y) \) by \( F_{xy} \) and its value at \( a \) by \( F_{xy}(a) \).

**Definition 2.8.** Let \((S, \mathfrak{F}, \tau)\) be a PM space. For \( x \in S \) and \( t > 0 \), the strong \( t \)-neighbourhood of \( x \) is defined as the set
\[ N_x(t) = \{ y \in S : F_{xy}(t) > 1 - t \}. \]

The collection \( \mathfrak{N}_x = \{ N_x(t) : t > 0 \} \) is called the strong neighbourhood system at \( x \), and the union \( \mathfrak{N} = \bigcup_{x \in S} \mathfrak{N}_x \) is called the strong neighbourhood system for \( S \).

By Theorem 2.1, we can write
\[ N_x(t) = \{ y \in S : d_L(F_{xy}, \epsilon_0) < t \}. \]
If $\tau$ is continuous, then the strong neighbourhood system $\mathcal{N}$ determines a Hausdorff topology for $S$. This topology is called the strong topology for $S$.

**Definition 2.9.** Let $(S, \mathcal{F}, \tau)$ be a PM space. Then for any $t > 0$, the subset $\mathcal{U}(t)$ of $S \times S$ given by

$$\mathcal{U}(t) = \{(x, y) : F_{xy}(t) > 1 - t\}$$

is called the strong $t$-vicinity.

**Theorem 2.2.** Let $(S, \mathcal{F}, \tau)$ be a PM space and $\tau$ be continuous. Then for any $t > 0$, there is an $\eta > 0$ such that

$$\mathcal{U}(\eta) \circ \mathcal{U}(\eta) \subset \mathcal{U}(t)$$

where $\mathcal{U}(\eta) \circ \mathcal{U}(\eta) = \{(x, z) : \text{for some } y, (x, y) \text{ and } (y, z) \in \mathcal{U}(t)\}$.

**Note 2.1.** From the hypothesis of Theorem 2.2 we can say that for any $t > 0$, there is an $\eta > 0$ such that $F_{ab}(t) > 1 - t$ whenever $F_{ac}(\eta) > 1 - \eta$ and $F_{cb}(\eta) > 1 - \eta$. Equivalently it can be written as: for any $t > 0$, there is an $\eta > 0$ such that $d_L(F_{ab}, \epsilon_0) < t$ whenever $d_L(F_{ac}, \epsilon_0) < \eta$ and $d_L(F_{cb}, \epsilon_0) < \eta$.

In a PM space $(S, \mathcal{F}, \tau)$, if $\tau$ is continuous then the strong neighbourhood system $\mathcal{N}$ determines a Kuratowski closure operation which is called the strong closure, and for any subset $A$ of $S$ the strong closure of $A$ is denoted by $\kappa(A)$ and for any nonempty subset $A$ of $S$

$$\kappa(A) = \{x \in S : \text{for any } t > 0, \text{ there is a } y \in A \text{ such that } F_{xy}(t) > 1 - t\}.$$ 

**Remark 2.1.** Throughout the rest of the article, in a PM space $(S, \mathcal{F}, \tau)$, we always assume that $\tau$ is continuous and $S$ is endowed with the strong topology.

**Definition 2.10.** Let $(S, \mathcal{F}, \tau)$ be a PM space. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in $S$ is said to be strong convergent to a point $x \in S$ if for any $t > 0$, there exists a natural number $N$ such that $x_n \in N_x(t)$ where $n \geq N$ and we write $x_n \to x$ or $\lim_{n \to \infty} x_n = x$.

Similarly a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $S$ is called a strong Cauchy sequence if for any $t > 0$, there exists a natural number $N$ such that $(x_m, x_n) \in \mathcal{U}(t)$ whenever $m, n \geq N$.

By the convergence of a double sequence we mean convergence in Pringsheim’s sense [16]. A real double sequence $x = \{x_{jk}\}_{j, k \in \mathbb{N}}$ is said to converge to a real number $a$ if for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that $|x_{jk} - a| < \epsilon$ whenever $j, k \geq N$.

A real double sequence $x = \{x_{jk}\}_{j, k \in \mathbb{N}}$ is said to be a Cauchy Sequence if for every $\epsilon > 0$, there exists $N, M \in \mathbb{N}$ such that for all $j, p \geq N ; k, q \geq M$, $|x_{jk} - x_{pq}| < \epsilon$.

**Definition 2.11.** Let $(S, \mathcal{F}, \tau)$ be a PM space. A double sequence $x = \{x_{jk}\}_{j, k \in \mathbb{N}}$ in $S$ is said to be strong convergent to a point $\xi \in S$ if for any $t > 0$, there exists a natural number $K$ such that $x_{jk} \in N_\xi(t)$ whenever $j, k \geq K$. 


In this case we write \( x_{jk} \to \xi \) or \( \lim_{j \to \infty, k \to \infty} x_{jk} = \xi \).

Similarly a double sequence \( x = \{ x_{jk} \}_{j,k \in \mathbb{N}} \) in \( S \) is called a strong Cauchy double sequence if for any \( t > 0 \), there exist natural numbers \( N, M \) such that for all \( j, p \geq N \); \( k, q \geq M \), \( (x_{jk}, x_{pq}) \in \mathcal{U}(t) \).

3. Main Results

In this section, we are concerned with ideal statistical pre-Cauchy and ideal statistical convergence for double sequences in a PM space.

**Definition 3.1.** [11] A double sequence \( (x_{jk})_{j,k \in \mathbb{N}} \) of real numbers is \( I \)-statistically convergent to a real numbers \( L \), and we write \( x_{jk} \overset{I^*}{\to} L \), provided that for \( \epsilon > 0 \) and \( \delta > 0 \)

\[
\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \{ (j,k) : |x_{jk} - L| \geq \epsilon, j \leq m, k \leq n \} \right| \geq \delta \} \in I
\]

**Definition 3.2.** [11] A double sequence \( (x_{jk})_{j,k \in \mathbb{N}} \) of real numbers is said to be \( I \)-statistically pre-Cauchy if for any \( \epsilon > 0 \) and \( \delta > 0 \)

\[
\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \{ (j,k) : |x_{jk} - x_{pq}| \geq \epsilon, j,p \leq m; k,q \leq n \} \right| \geq \delta \} \in I.
\]

Now we introduce some definitions in a Probabilistic Metric space.

**Definition 3.3.** Let \( (S, \mathcal{F}, \tau) \) be a PM space. A double sequence \( (x_{jk})_{j,k \in \mathbb{N}} \) in \( S \) is strong \( I \)-statistically convergent to \( p \) in \( S \), and we write \( x_{jk} \overset{str-I^*}{\to} p \), provided that for \( t > 0 \) and \( \delta > 0 \)

\[
\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \{ (j,k) : (x_{jk} - x_{pq}) \notin \mathcal{N}_p(t), j \leq m, k \leq n \} \right| \geq \delta \} \in I
\]

**Definition 3.4.** Let \( (S, \mathcal{F}, \tau) \) be a PM space. A double sequence \( (x_{jk})_{j,k \in \mathbb{N}} \) in \( S \) is said to be strong \( I \)-statistically pre-Cauchy if for any \( t > 0 \) and \( \delta > 0 \)

\[
\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \{ (j,k) : (x_{jk} - x_{pq}) \notin \mathcal{U}(t); j,p \leq m; k,q \leq n \} \right| \geq \delta \} \in I.
\]

**Note 3.1.** In a PM space a double sequence which is strong statistically convergent is clearly strong \( I \)-statistically convergent but converse is not true for this we consider the following example. Consider the equilateral PM space \( (S, \mathcal{F}, \tau) \) where \( \mathcal{F} \) is defined by

\[
\mathcal{F}_{pq} = \begin{cases} 
F & \text{if } p \neq q \\
\epsilon_0 & \text{if } p = q
\end{cases}
\]

and \( M \) is the maximal triangular function. Here \( F \in \Delta^+ \) is fixed and distinct from \( \epsilon_0 \) and \( \epsilon_\infty \). Let \( I \) is an ideal of \( \mathbb{N} \times \mathbb{N} \). We let \( A \in I \) such that \( A = \{(t_m, t_n) : (m, n) \in \mathbb{N} \times \mathbb{N} \} \). Let \( B \) be an subset of \( \mathbb{N} \times \mathbb{N} \) such that \( d_2(B) = 0 \). Now fora fixed \( p, q \in S \) we define

\[
x_{t_mt_n} = \begin{cases} 
p & \text{if } (m, n) \in B \\
q & \text{if } (m, n) \notin B
\end{cases}
\]
and \( x_{mn} = p \) if \((m,n) \notin A\).

Then \( \{x_{mn}\}_{m,n \in \mathbb{N}} \) is \( I \)-statistically convergent to \( p \) but is not statistically convergent.

**Theorem 3.1.** An strong \( I \)-statistically convergent double sequence is strong \( I \)-statistically pre-Cauchy double sequence in a PM space.

**Proof.** Let \( \{x_{jk}\}_{j,k \in \mathbb{N}} \) be strong \( I \)-statistically convergent to \( a \) in \( S \). Let \( t > 0 \) and \( \delta > 0 \). Choose \( \delta_1 > 0 \) such that \( 1 - (1 - \delta_1)^2 < \delta \).

Now for that \( t > 0 \), there exists an \( \eta > 0 \) such that for all \( a, b, c \in S \) we have

\[
d_L(F_{ac}, \epsilon_0) < t \quad \text{whenever} \quad d_L(F_{ab}, \epsilon_0) < \eta \quad \text{and} \quad d_L(F_{bc}, \epsilon_0) < \eta \quad \text{...................}(1).
\]

Now for \( \delta_1 > 0 \) and \( \eta > 0 \) we have

Let \( C = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ (j,k) : x_{jk} \notin N_a(\eta), j \leq m, k \leq n \right\} \right| \geq \delta_1 \} \in I \).

Let \((m,n) \in C^c\) then

\[
\frac{1}{mn} \left| \left\{ (j,k) : x_{jk} \notin N_a(\eta), j \leq m, k \leq n \right\} \right| < \delta_1
\]

\[
\Rightarrow \frac{1}{mn} \left| \left\{ (j,k) : x_{jk} \in N_a(\eta), j \leq m, k \leq n \right\} \right| > 1 - \delta_1.
\]

Let \( B_{mn} = \{(j,k) : x_{jk} \in N_a(\eta), j \leq m, k \leq n\} \).

Then for \((j,k), (p,q) \in B_{mn}, \; d_L(F_{x_{jk}x_{pq}}, \epsilon_0) < \eta \) and \( d_L(F_{x_{pq}a}, \epsilon_0) < \eta \) which implies \( d_L(F_{x_{jk}x_{pq}}, \epsilon_0) < t \) from (1).

This implies

\[
|B_{mn}|^2/m^2n^2 \leq \frac{1}{mn^2} \left| \left\{ (j,k) : (x_{jk}, x_{pq}) \in U(t); j, p \leq m; k, q \leq n \right\} \right|
\]

Thus for all \((m,n) \in C^c\) we have

\[
(1 - \delta_1)^2 < |B_{mn}|^2/m^2n^2 \leq \frac{1}{mn^2} \left| \left\{ (j,k) : (x_{jk}, x_{pq}) \in U(t); j, p \leq m; k, q \leq n \right\} \right|
\]

\[
\Rightarrow \frac{1}{mn} \left| \left\{ (j,k) : (x_{jk}, x_{pq}) \notin U(t); j, p \leq m; k, q \leq n \right\} \right| \leq 1 - (1 - \delta_1)^2 < \delta. \]

We see that for all \((m,n) \in C^c\),

\[
\frac{1}{mn} \left| \left\{ (j,k) : (x_{jk}, x_{pq}) \notin U(t); j, p \leq m; k, q \leq n \right\} \right| < \delta
\]

and so

\[
\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ (j,k) : (x_{jk}, x_{pq}) \notin U(t); j, p \leq m; k, q \leq n \right\} \right| \geq \delta \} \subset C.
\]

Since \( C \in I \),

\[
\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn^2} \left| \left\{ (j,k) : (x_{jk}, x_{pq}) \notin U(t); j, p \leq m; k, q \leq n \right\} \right| \geq \delta \} \in I.
\]

Hence \( x \) is strong \( I \)-statistically pre-Cauchy. \( \square \)

The next result gives a necessary and sufficient condition for a double sequence to be \( I \)-statistically pre-Cauchy.
**Theorem 3.2.** Let \( x = \{x_{jk}\} \) be double sequence in a PM space \((S, \mathcal{S}, r)\). A double sequence \( x = \{x_{jk}\} \) is strong \( I \)-statistically pre-Cauchy if and only if 
\[
I - \lim_{m,n \rightarrow \infty} \frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) = 0.
\]

**Proof.** First we assume that \( I - \lim_{m,n \rightarrow \infty} \frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) = 0 \). Note that for \( t > 0 \) and \((m, n) \in \mathbb{N} \times \mathbb{N} \) we have \( \frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq t \left( \frac{1}{m+n} \right) \left| \{(j, k) : d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq t; j, p \leq m; k, q \leq n \} \right| \).

Therefore for any \( \delta > 0 \),
\[
A = \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq \delta \right\}
\]

\[
\subseteq \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq \delta t \right\}.
\]

Since \( I - \lim_{m,n \rightarrow \infty} \frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) = 0 \) thus right hand side belongs to \( I \) which implies that

\[
\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m+n} \left| \{(j, k) : d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq t; j, p \leq m; k, q \leq n \} \right| \geq \delta \} \in I.
\]

This shows that \( x \) is strong \( I \)-statistically pre-Cauchy.

Conversely assume that \( x \) is strong \( I \)-statistically pre-Cauchy double sequence in a PM space and let \( \delta > 0 \) has been given. Choose \( t > 0 \) and \( \delta_1 > 0 \) such that \( \frac{t}{2} + \delta_1 < \delta \). Then for each \((m, n) \in \mathbb{N} \times \mathbb{N}, \)

\[
\frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0)
\]

\[
= \frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) + \frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \leq \frac{t}{2} + \frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} \left| \{(j, k) : d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq \frac{t}{2}; j, p \leq m; k, q \leq n \} \right| \cdot \left[ \text{Since } d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \leq 1 \right].
\]

Now since \( x \) is strong \( I \)-statistically pre-Cauchy for that \( \delta_1 > 0 \)

\[
\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m+n} \left| \{(j, k) : d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq \frac{t}{2}; j, p \leq m; k, q \leq n \} \right| \geq \delta_1 \} \in I.
\]

Then for \((m, n) \in A^c, \)

\[
\frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) < \frac{t}{2} + \delta_1 < \delta. \]

We see that for all \((m, n) \in A^c \)

\[
\frac{1}{m+n} \sum_{j,p \leq m} \sum_{k,q \leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \leq \delta. \]
\[
\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m+n} \sum_{j,p \leq m, k,q \leq n} d_L(F_{x_{jk}}, \epsilon_0) \geq \delta_1 \right\} \subset A.
\]

Since \( A \in I \) so
\[
\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m+n} \sum_{j,p \leq m, k,q \leq n} d_L(F_{x_{jk}}, \epsilon_0) \geq \delta_1 \right\} \in I.
\]

Therefore \( I - \lim_{n \to \infty} \frac{1}{m+n} \sum_{j,p \leq m, k,q \leq n} d_L(F_{x_{jk}}, \epsilon_0) = 0 \). This proves the necessity of the theorem.

Now we give a sufficient condition under which a strong \( I \)-statistically pre-Cauchy double sequence can be a strong \( I \)-statistically convergent.

**Definition 3.5.** [11] Let \( I \) be an admissible ideal of \( \mathbb{N} \times \mathbb{N} \) and \( x = \{x_{jk}\}_{j,k \in \mathbb{N}} \) be a real double sequence. Let \( A_x = \{\alpha \in \mathbb{R} : \{(j,k) : x_{jk} < \alpha\} \notin I\} \). Then \( I \)-limit inferior of \( x \) is given by
\[
I - \lim \inf x = \left\{ \begin{array}{ll}
\inf A_x & \text{if } A_x \neq \phi \\
\infty & \text{if } A_x = \phi
\end{array} \right.
\]

It is known (Theorem 3, [28]) that \( I \)-lim inf \( x = \alpha \) (finite) if and only if for arbitrary \( \epsilon > 0 \),
\[
\{ (j,k) : x_{jk} < \alpha + \epsilon \} \notin I \text{ and } \{ (j,k) : x_{jk} < \alpha - \epsilon \} \in I
\]

**Definition 3.6.** Let \( x = \{x_{ij}\}_{i,j \in \mathbb{N}} \) be a double sequence of a PM space \( (S, \mathcal{F}, \tau) \). Let \( K \) be a subset of \( \mathbb{N} \times \mathbb{N} \) such that for each \( (i,j) \in \mathbb{N} \times \mathbb{N} \), there exists a \( (m,n) \in K \) such that \( (m,n) > (i,j) \) with respect to dictionary order. Then we define \( \{x\}_K = \{x_{ij}\}_{i,j \in K} \) as a subsequence of \( x \).

**Theorem 3.3.** Let \( (S, \mathcal{F}, \tau) \) be a PM space and \( x = \{x_{jk}\}_{j,k \in \mathbb{N}} \) be a strong \( I \)-statistically pre-Cauchy double sequence in \( S \). If \( x = \{x_{jk}\}_{j,k \in \mathbb{N}} \) has a subsequence \( \{x_{t_j,t_k}\}_{j,k \in \mathbb{N}} \) which is strong converges to \( L \) and
\[
0 < I - \lim_{n \to \infty} \inf_{m \to \infty} \left| \{(t_j,t_k) : t_j \leq m, t_k \leq n ; j,k \in \mathbb{N}\} \right| < \infty,
\]
then \( x \) is strong \( I \)-statistically convergent to \( L \).

**Proof.** Let \( I - \lim_{n \to \infty} \inf_{m \to \infty} \left| \{(t_j,t_k) : t_j \leq m, t_k \leq n ; j,k \in \mathbb{N}\} \right| = r. \) Then \( 0 < r < \infty \). Let \( t > 0 \) and \( \delta > 0 \) be given. We choose \( \delta_1 > 0 \) such that \( \frac{2\delta_1}{\epsilon_0} < \delta \). Now for that \( t > 0 \) there exists \( \eta > 0 \) such that for all \( a,b,c \in S \) we have \( d_L(F_{ab}, \epsilon_0) < t \) whenever \( d_L(F_{bc}, \epsilon_0) < \eta \) and \( d_L(F_{ac}, \epsilon_0) < \eta \)..........(1). Now select \( n_0 \in \mathbb{N} \) such that \( t_j > n_0 \), \( t_k > n_0 \) for some \( j,k \in \mathbb{N} \) then \( d_L(F_{x_{t_j,t_k}}, \epsilon_0) < \eta \). Let
Let \( A = \{(t_j, t_k) : t_j > n_0, t_k > n_0; j, k \in \mathbb{N} \} \) and \( B(t) = \{(j, k) : d_L(F_{x,jk}, \epsilon_0) \geq t\} \).

But we have from (1) that
\[
\frac{1}{m \cdot n} \sum_{j, p \leq m, k \leq n} |\{(j, k) : d_L(F_{x,jk}, \epsilon_0) \geq \eta; j, p \leq m, k \leq n\}| \geq \frac{1}{m \cdot n} \sum_{j, p \leq m, k \leq n} \chi_{A \times B(t)}((j, k) \times (p, q)) = \frac{1}{mn} |\{(t_j, t_k) \in A : t_j \leq m, t_k \leq n\}| \times \frac{1}{mn} \sum_{t \geq 1} \left|\{(p, q) : d_L(F_{x,pq}, \epsilon_0) \geq t; p \leq m, q \leq n\}\right|.
\]

Since \( x \) is strong \( I \)-statistically pre-Cauchy so for \( \delta_1 > 0 \) and \( \eta > 0 \)
\[
C = \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m \cdot n} \left|\{(j, k) : d_L(F_{x,jk}, \epsilon_0) \geq \eta; j, p \leq m, k, q \leq n\}\right| \geq \delta_1 \right\} \in I.
\]

Therefore for every \( (m, n) \in C^c \) we have \( \frac{1}{m \cdot n} \left|\{(j, k) : d_L(F_{x,jk}, \epsilon_0) \geq \eta\}\right| < \delta_1 \). Now since \( I - \lim_{n \to \infty} \inf \frac{1}{m \cdot n} \left|\{(j, k) : t_j \leq m, t_k \leq n; j, k \in \mathbb{N}\}\right| = r \), so the set \( D = \{(m, n) : \frac{1}{m \cdot n} \left|\{(j, k) : t_j \leq m, t_k \leq n; j, k \in \mathbb{N}\}\right| < \frac{r}{2} \} \in I \). So every \( (m, n) \in D^c \) we have \( \frac{1}{m \cdot n} \left|\{(j, k) : t_j \leq m, t_k \leq n; j, k \in \mathbb{N}\}\right| \geq \frac{r}{2} \).

Now from (ii), (iii) we get for every \( (m, n) \in C^c \cup D^c = (C \cup D)^c \),
\[
\frac{1}{mn} \left|\{(p, q) : d_L(F_{x,pq}, \epsilon_0) \geq t\}\right| < \frac{2r}{mn} < \delta.
\]

This implies \( \{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left|\{(j, k) : d_L(F_{x,jk}, \epsilon_0) \geq t; j \leq m, k \leq n\}\right| \geq \delta \} \subseteq (C \cup D) \). Since \( C, D \in I \) thus \( C \cup D \in I \) and so
\[
\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left|\{(j, k) : d_L(F_{x,jk}, \epsilon_0) \geq t; j \leq m, k \leq n\}\right| \geq \delta \} \in I.
\]

This shows that \( x \) is strong \( I \)-statistically convergent to \( L \). \( \Box \)

To give an example of a sequence which is strong \( I \)-statistically pre-Cauchy but not strong \( I \)-statistically convergent we first observe that every strong \( I \)-statistically convergent sequence must have a strong convergent subsequence which is convergent in the usual sense. Since it is not straight forward so we give proof below.

Let \( x = \{x_{jk}\} \) be a strong \( I \)-statistically convergent double sequence in a PM space convergent to \( a \). For \( \delta = t = 1 \) we have
\[
C = \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left|\{(j, k) : d_L(F_{x,jk,a}, \epsilon_0) \geq 1; j \leq m, k \leq n\}\right| \geq 1 \right\} \in I.
\]

Since \( I \) is an non-trivial ideal of \( \mathbb{N} \times \mathbb{N} \) so \( C \neq \mathbb{N} \times \mathbb{N} \) thus there exists \( (m_1, n_1) \in C^c \) so that
\[
\frac{1}{m_1 n_1} \left|\{(j, k) : d_L(F_{x,jk,a}, \epsilon_0) \geq 1; j \leq m_1, k \leq n_1\}\right| < 1 \\
\Rightarrow \frac{1}{m_1 n_1} \left|\{(j, k) : d_L(F_{x,jk,a}, \epsilon_0) < 1; j \leq m_1, k \leq n_1\}\right| > 0.
\]

So there exists \( j_1 \leq m_1 \) and \( k_1 \leq n_1 \) such that \( d_L(F_{x,j_1k_1,a}, \epsilon_0) < 1 \). Again taking \( \delta = t = \frac{1}{2} \) we have
\[
D = \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left|\{(j, k) : d_L(F_{x,jk,a}, \epsilon_0) \geq \frac{1}{2}; j \leq m, k \leq n\}\right| \geq \frac{1}{2} \right\} \in I.
\]
Let $I$ be a strongly admissible ideal then we have $D \cup (\mathbb{N} \times \{1, 2, \ldots, 4n_1\}) \cup (\{1, 2, \ldots, 4n_1\} \times \mathbb{N}) \in I$. Since $I$ is non-trivial choose $(m_2, n_2)$ such that $(m_2, n_2) \notin D$ and $m_2 > 4n_1, n_2 > 4n_1$. Then

\[
\frac{1}{m_2n_2} \sum_{j < k} \left\{ \left( j, k, \delta, \epsilon \right) \right\} = \frac{1}{m_2n_2} \left\{ \left( j, k, \delta, \epsilon \right) \right\} \leq \frac{1}{2} \cdot \frac{1}{2}.
\]

Now that if $d_L(F_{x_{jk}a}, \epsilon_0) \geq \frac{1}{2}$ for all $m_1 < j < m_2$ and for all $n_1 < k < n_2$. Then

\[
\frac{1}{m_2n_2} \sum_{j < k} \left\{ \left( j, k, \delta, \epsilon \right) \right\} \leq \frac{m_1n_1}{m_2n_2} < \frac{1}{2}.
\]

Consequently there exist $m_1 < j < m_2$ and $n_1 < k < n_2$ such that $d_L(F_{x_{jk}a}, \epsilon_0) < \frac{1}{2}$. Now we write $j = j_2$ and $k = k_2$ then clearly $j_1 < j_2$ and $k_1 < k_2$. Proceeding in this way we get a set $K = \{ (j_1, k_1), (j_2, k_2), \ldots \}$ with $j_1 < j_2, k_1 < k_2$ and $d_L(F_{x_{jk}a}, \epsilon_0) < \frac{1}{2}$. This shows that $x$ has a subsequence $\{x\}_K$ which is strongly convergent to $a$.

We now construct the following example

**Example 1:** Let $(S, d)$ be the Euclidean line and $H(x) = 1 - e^{-x}$ where $H \in \Delta^+$. Consider the simple space $(S, d, H)$ which is generated by $(S, d)$ and $H$. Then this space becomes a PM space $(S, G)$ under the continuous triangle function $\tau_M$, which is in fact a Menger space, where $G$ is defined on $S \times S$ by

\[
G(p, q)(t) = F_{pq}(t) = H(t) = 1 - e^{-1/t} \quad \text{for all } p, q \in S \text{ and } t \in \mathbb{R}^+.
\]

Here we make the convention that $H(t/0) = H(\infty) = 1$ for $t > 0$, and $H(0) = H(\frac{1}{2}) = 0$. Now let $x = \{x_{jk}\}$ be a double sequence in the PM space $S$ defined by

\[
x_{jk} = \sum_{u=1}^{m} \sum_{v=1}^{n} \frac{1}{u} + \sum_{v=1}^{n} \frac{1}{v}.
\]

Where $(m - 1)! < j < m!$ and $(n - 1)! < k < n!$. Clearly $x$ has no strong convergent subsequence by construction of the sequence. But $x$ is strongly statistically pre-Cauchy since for given $t_1 > 0$ we have the following we define

\[
G_m(t) = 1 - e^{-t/m} \quad \text{then clearly } G_m(t) \text{ is a d.d.f and for } t > 0, G_m(t) \text{ weakly convergent to } \epsilon_0 \text{ as } m \to \infty. \quad \text{So for that } t_1 > 0 \text{ there exists an positive integer } M \text{ such that for all } m \geq M \text{ we have } d_L(G_m(t), \epsilon_0) < t_1. \quad \text{Choose } m > M \text{ and } m < n \text{ then for } m! < m_1 \leq (m + 1)! \text{ and } n! < n_1 \leq (n + 1)!, \text{ also we have } (m - 1)! < j, p \leq m_1, (n - 1)! < k, q \leq n_1 \text{ then } \frac{|x_{jk} - x_{pq}|}{m} < \frac{2}{m}. \quad \text{It follows that for } t_1 > 0 \text{ and } m! < n_1 \leq (m + 1)! \text{, } n! < n_1 \leq (n + 1)!. \quad \text{Then}
\]

\[
\frac{1}{m_1n_1} \sum_{j < k} \left\{ \left( j, k, \delta, \epsilon \right) \right\} \geq \frac{1}{m_1n_1} \left| |m_1 - (m - 1)|^2 |n_1 - (n - 1)|^2 \right| \geq \frac{1}{m_1^2} \frac{|1 - \frac{1}{m}|^2}{1 - \frac{1}{m}^2} = 1,
\]

it follows that $x$ is strong statistically pre-Cauchy hence strong $I$-statistically pre-Cauchy.
**Definition 3.7.** Let \((S, \mathcal{F}, \tau)\) be a PM space and \(I\) be a strongly admissible ideal of \(\mathbb{N} \times \mathbb{N}\). A double sequence \(x = \{x_{jk}\}_{j,k \in \mathbb{N}}\) in \(S\) is said to be strong \(I^*\)-statistically pre-Cauchy if there exists a set \(M \in \mathcal{F}(I)\) such that \(\{x\}_M\) is strong statistically pre-Cauchy double sequence i.e.;

\[
\lim_{m, n \to \infty} \prod_{(m, n) \in M} \left\{ (j, k) : d_L(F_{x_{jk}, \epsilon_0}) \geq t; j, p \leq m; k, q \leq n \right\} = 0.
\]

**Theorem 3.4.** Let \((S, \mathcal{F}, \tau)\) be a PM space and \(x = \{x_{jk}\}_{j,k \in \mathbb{N}}\) which is strong \(I^*\)-statistically pre-Cauchy double sequence in \(S\) then \(x\) is strong \(I\)-statistically pre-Cauchy.

**Proof.** Let \(t > 0\) and \(\delta > 0\) be given. Since \(x = \{x_{jk}\}_{j,k}\) is strong \(I^*\)-statistically pre-Cauchy so there exists a set \(M \in \mathcal{F}(I)\) such that

\[
\lim_{m, n \to \infty} \prod_{(m, n) \in M} \left\{ (j, k) : d_L(F_{x_{jk}, \epsilon_0}) \geq t; j, p \leq m; k, q \leq n \right\} = 0.
\]

Then there exists \(n_0 \in \mathbb{N}\) such that for \((m, n) \in M\) with \(m \geq n_0, n \geq n_0\) we have

\[
\prod_{(m, n) \in M} \left\{ (j, k) : d_L(F_{x_{jk}, \epsilon_0}) \geq t; j, p \leq m; k, q \leq n \right\} < \delta.
\]

Let \(K = \{1, 2, \ldots, n_0 - 1\}.\) Then obviously

\[A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \prod_{(m, n) \in M} \left\{ (j, k) : d_L(F_{x_{jk}, \epsilon_0}) \geq t; j, p \leq m; k, q \leq n \right\} \geq \delta\} \subseteq (\mathbb{N} \times \mathbb{N} \setminus M) \cup (K \times \mathbb{N}) \cup (\mathbb{N} \times K)\]\n
... (i). Since \(I\) is strongly admissible ideal and \(A \in I\) so the set on the right side of (i) belongs to \(I\). Hence \(x = \{x_{jk}\}_{j,k \in \mathbb{N}}\) is strong \(I\)-statistically pre-Cauchy double sequence in a PM space \(S\).

**Definition 3.8.** [24] We say an admissible ideal \(I \subset 2^{\mathbb{N} \times \mathbb{N}}\) satisfies the property \((AP_2)\) if for every countable family of mutually disjoint sets \(\{A_1, A_2, \ldots\}\) belonging to \(I\), there exists a countable family of sets \(\{B_1, B_2, \ldots\}\) such that \(A_j \Delta B_j\) is included in the finite union of rows and columns in \(\mathbb{N} \times \mathbb{N}\) for each \(j \in \mathbb{N}\) and \(B = \bigcup_{j=1}^{\infty} B_j \in I\). Here \(\Delta\) denotes the symmetric difference between two sets.

**Lemma 3.5.** [24] Let \(\{P_i\}\) be a countable collection of subsets of \(\mathbb{N} \times \mathbb{N}\) such that \(P_i \in \mathcal{F}(I)\) for each \(i\), where \(\mathcal{F}(I)\) is a filter associated with a strong admissible ideal \(I\) with the property \((AP_2)\). Then there exists a set \(P \in \mathbb{N} \times \mathbb{N}\) such that \(P \in \mathcal{F}(I)\) and the set \(P \setminus P_i\) is finite for all \(i\).

**Theorem 3.6.** Let \((S, \mathcal{F}, \tau)\) be a PM space. If \(I\) is a strong admissible ideal of \(\mathbb{N} \times \mathbb{N}\) with the property \((AP_2)\) then the notions of strong \(I\)-statistically pre-Cauchy and \(I^*\)-statistically pre-Cauchy coincide.

**Proof.** From the theorem[3.4] it is sufficient to prove that a strong \(I\)-statistically pre-Cauchy double sequence \(x = \{x_{jk}\}_{j,k \in \mathbb{N}}\) in a PM space \(S\) is strong \(I^*\)-statistically pre-Cauchy. Let \(x = \{x_{jk}\}_{j,k \in \mathbb{N}}\) in \(S\) be a strong \(I\)-statistically pre-Cauchy double sequence. Let \(t > 0\) be given. For each \(i \in \mathbb{N}\), let \(P_i = \{j \in \mathbb{N} : \exists k \in \mathbb{N} : d_L(F_{x_{jk}, \epsilon_0}) \geq t; j, p \leq m; k, q \leq n\} \in \mathcal{F}(I)\). Then, \(\prod_{(m, n) \in M} \left\{ (j, k) : d_L(F_{x_{jk}, \epsilon_0}) \geq t; j, p \leq m; k, q \leq n \right\} = 0\) for some \(M \in \mathcal{F}(I)\).
\[(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m^2n^2} \left| \{(j, k) : (x_{jk}, x_{pq}) \notin \mathbb{U}(t); j, p \leq m; k, q \leq n\} \right| < \frac{1}{r} \right). \] Then \( P_i \in \mathbb{F}(I) \) for each \( i \in \mathbb{N} \). Since \( I \) has the property \((AP_2)\), then by lemma[3.5] there exists a set \( P \subseteq \mathbb{N} \times \mathbb{N} \) such that \( P \in \mathbb{F}(I) \) and \( P \setminus P_i \) is finite for all \( i \in \mathbb{N} \).

Now we show that

\[
\lim \frac{1}{m^2n^2} \left| \{(j, k) : d_L(F_{x_{jk}, x_{pq}}, \epsilon_0) \geq t; j, p \leq m; k, q \leq n\} \right| = 0.
\]

Let \( \epsilon > 0 \) be given then there exists a \( j \in \mathbb{N} \) such that \( j > \frac{1}{\epsilon} \). Let \((m, n) \in P \) and since \( P \setminus P_j \) is a finite set, so there exists \( k = k(j) \in \mathbb{N} \) such that \((m, n) \in P_j \) for all \( m, n \geq k(j) \). Therefore for all \((m, n) \in P \) with \( m, n \geq k(j) \) we have

\[
\lim \frac{1}{m^2n^2} \left| \{(j, k) : d_L(F_{x_{jk}, x_{pq}}, \epsilon_0) \geq t; j, p \leq m; k, q \leq n\} \right| < \frac{1}{j} < \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, so

\[
\lim \frac{1}{m^2n^2} \left| \{(j, k) : d_L(F_{x_{jk}, x_{pq}}, \epsilon_0) \geq t; j, p \leq m; k, q \leq n\} \right| = 0.
\]

Therefore \( x \) is strong \( I^* \)-statistically pre-Cauchy. \( \square \)

Next we present an interesting property of \( I \)-statistically pre-Cauchy double sequences of real numbers in line of Theorem 2.4 [8].

**Theorem 3.7.** Let \( x = \{x_{jk}\}_{j,k \in \mathbb{N}} \) be a double sequence of real numbers and \((\alpha, \beta)\) is an open interval such that \( x_{jk} \notin (\alpha, \beta) \), for all \((j, k) \in \mathbb{N} \times \mathbb{N} \). We write \( A = \{(j, k) : x_{jk} \leq \alpha\} \) and \( B = \{(j, k) : x_{jk} \geq \beta\} \) and further assume that the following property is satisfied

\[
\limsup_{m, n \to \infty} D_{mn}(A) - \liminf_{m, n \to \infty} D_{mn}(A) < r.
\]

for some \( 0 \leq r \leq 1 \). If \( x \) is \( I \)-statistically pre-Cauchy then either \( I - \lim_{m, n \to \infty} D_{mn}(A) = 0 \) or \( I - \lim_{m, n \to \infty} D_{mn}(B) = 0 \), where \( D_{mn}(A) = \frac{1}{mn} \left| \{(j, k) \in A : j \leq m, k \leq n\} \right| \).

**Proof.** Here \( B = \mathbb{N} \times \mathbb{N} \setminus A \) and so \( D_{mn}(B) = 1 - D_{mn}(A) \), for all \((m, n) \in \mathbb{N} \times \mathbb{N} \). To complete the proof it is sufficient to show that either \( I - \lim_{m, n \to \infty} D_{mn}(A) = 0 \) or \( 1 \). Note that

\[
\chi_{A \times B}(j, k, (p, q)) \leq \left| \{(j, k) : |x_{jk} - x_{p,q}| \geq \beta - \alpha\} \right| \quad \text{.........(1)}
\]

Since \( x \) is \( I \)-statistically pre-Cauchy, so

\[
I - \lim_{m, n \to \infty} \frac{1}{mn} \left| \{(j, k) : |x_{jk} - x_{p,q}| \geq \beta - \alpha; j, p \leq m; k, q \leq n\} \right| = 0.
\]

But from (1) we get,

\[
0 = L.H.S = I - \lim_{m, n \to \infty} D_{mn}(A)[D_{mn}(B)] = I - \lim_{m, n \to \infty} D_{mn}(A)[1 - D_{mn}(A)].
\]

Now from the definition of \( I \)-convergence it follows that
\{ (m, n) \in \mathbb{N} \times \mathbb{N} : D_{mn}(A)[1 - D_{mn}(A)] \geq \frac{1}{25} \} \in I.

Say \( M = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : D_{mn}(A)[1 - D_{mn}(A)] < \frac{1}{25} \} \in \mathcal{F}(I) \). Clearly for all \( (m, n) \in M \) either \( D_{mn}(A) < \frac{1}{5} \) or \( D_{mn}(A) > \frac{4}{5} \). If \( D_{mn}(A) < \frac{1}{5} \) for all \( (m, n) \in M \subset M \) for some \( M \in \mathcal{F}(I) \) then \( I = -\lim_{m,n \to \infty} D_{mn}(A) = 0 \). This is because for any \( \epsilon > 0, 0 < \epsilon < \frac{1}{5} \), from the definition of \( I \)-convergence we get \( \{ (m, n) \in \mathbb{N} \times \mathbb{N} : D_{mn}(A)[1 - D_{mn}(A)] \geq \frac{1}{25} \} = M_2(say) \in \mathcal{F}(I) \). Taking \( M_0 = M_1 \cap M_2 \), we see that \( M_0 \in \mathcal{F}(I) \) and \( D_{mn}(A) < \epsilon \), for all \( (m, n) \in M_0 \).

Therefore
\[
\{ (m, n) : D_{mn}(A) \geq \epsilon \} \subset (\mathbb{N} \times \mathbb{N} \setminus M_0).
\]

Since \( (\mathbb{N} \times \mathbb{N} \setminus M_0) \in I \) so \( \{ (m, n) : D_{mn}(A) \geq \epsilon \} \in I \) and hence \( I = -\lim_{m,n \to \infty} D_{mn}(A) = 0 \). Similarly if \( D_{mn}(A) > \frac{4}{5} \) for all \( (m, n) \in M_3 \subset M \) for some \( M_3 \in \mathcal{F}(I) \) then we can show that \( I = -\lim_{m,n \to \infty} D_{mn}(A) = 1 \).

If neither of above cases happen then considering dictionary order on \( \mathbb{N} \times \mathbb{N} \), we can find an increasing sequence
\[
\{ (m_1, n_1) < (m_2, n_2) < \ldots \}
\]
from \( M \) such that
\[
D_{m_i, n_i} < \frac{1}{5} \quad \text{when} \quad i \quad \text{is odd integer}.
\]
\[
D_{m_i, n_i} > \frac{4}{5} \quad \text{when} \quad i \quad \text{is even integer}.
\]

Then clearly
\[
\limsup_{m,n \to \infty} D_{mn}(A) - \liminf_{m,n \to \infty} D_{mn}(A) \geq \frac{3}{5}.
\]

We again start the above process and see that
\[
\{ (m, n) \in \mathbb{N} \times \mathbb{N} : D_{mn}(A)[1 - D_{mn}(A)] \geq \frac{1}{25} \} = M_4(say) \in \mathcal{F}(I).
\]

Which consequently implies as above that either \( I = -\lim_{m,n \to \infty} D_{mn}(A) = 1 \). or \( I = -\lim_{m,n \to \infty} D_{mn}(A) = 0 \). or if neither holds then
\[
\limsup_{m,n \to \infty} D_{mn}(A) - \liminf_{m,n \to \infty} D_{mn}(A) \geq \frac{4}{5}.
\]

Proceeding in this way we observe that the process stops only when we get either \( I = -\lim_{m,n \to \infty} D_{mn}(A) = 0 \). or \( I = -\lim_{m,n \to \infty} D_{mn}(A) = 1 \). and if it does not stop at a finite step then we will have
\[
\limsup_{m,n \to \infty} D_{mn}(A) - \liminf_{m,n \to \infty} D_{mn}(A) \geq \lim_{k \to \infty} \frac{k-2}{k}.
\]
Which contradicts to our assumption that $0 \leq r \leq 1$. This completes the proof of the theorem. \hfill \Box

Remark 1: For $A \subset \mathbb{N} \times \mathbb{N}$ if $I - \lim_{m,n \to \infty} \frac{1}{mn} | \{(j,k) \in A : j \leq m, k \leq n\}|$ exists we can say that the $I$-asymptotic density of $A$ exists and denote it by $d_I(A)$. Therefore the above result can be re-phrased as: Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence of real numbers and $(\alpha, \beta)$ is an open interval such that $x_{jk} \notin (\alpha, \beta)$, for all $(j,k) \in \mathbb{N} \times \mathbb{N}$. We write $A = \{(j,k) : x_{jk} \leq \alpha\}$ and further assume that the following property is satisfied
\[\limsup_{m,n \to \infty} D_{mn}(A) - \liminf_{m,n \to \infty} D_{mn}(A) < r.\]
for some $0 \leq r \leq 1$. If $x$ is $I$-statistically pre-Cauchy then either $d_I(A) = 0$ or $d_I(A) = 1$. It should be mentioned in this context that for $I = I_{\text{fin}}$, the above result holds without any additional assumption since it is easy to see thus we omitted. For our final result we assume that $I$ is such an ideal and $x$ is such that the above result holds without any additional assumption i.e;
(\*) If $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ is $I$-statistically pre-Cauchy double sequence of real numbers and $x_{jk} \notin (\alpha, \beta)$ for all $(j,k) \in \mathbb{N} \times \mathbb{N}$ then either $d_I(\{(j,k) : x_{jk} \leq \alpha\}) = 0$ or $d_I(\{(j,k) : x_{jk} \geq \beta\}) = 0$.

Before we prove our final result, we introduce the following definition.

Definition 3.9. A real number $\xi$ is said to be an $I$-statistical cluster point of a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ of real numbers if for any $\epsilon > 0$
\[d_I(\{(j,k) : |x_{jk} - \xi| < \epsilon\}) \neq 0.\]

Theorem 3.8. Let $x = \{x_{jk}\}$ be an $I$-statistically pre-Cauchy double sequence of real numbers. If the set of limit points of $x$ is no-where dense and $x$ has an $I$-statistical cluster point. Then $x$ is $I$-statistically convergent under the hypothesis (\*). 

Proof. Suppose $x$ has a $I$-statistical cluster point $\xi \in \mathbb{R}$. So for any $\epsilon > 0$ we have $d_I(\{(j,k) : |x_{jk} - \xi| < \epsilon\}) \neq 0$. Assume that $x$ is $I$-statistically pre-Cauchy satisfying the hypothesis (\*) but $x$ is not $I$-statistically convergent, so there is an $\epsilon_0 > 0$ such that $d_I(\{(j,k) : |x_{jk} - \xi| \geq \epsilon_0\}) \neq 0$. Without loss of generality we assume that $d_I(\{(j,k) : x_{jk} \leq \xi - \epsilon_0\}) \neq 0$. We claim that every point of $(\xi - \epsilon_0, \xi]$ is a limit point of $x$. If not, then we can find an interval $(\alpha, \beta) \subset (\xi - \epsilon_0, \xi]$ such that $x_{jk} \notin (\alpha, \beta)$ for all $(j,k) \in \mathbb{N} \times \mathbb{N}$. From above it immediately follows that both $d_I(\{(j,k) : x_{jk} \leq \alpha\}) = 0$ and $d_I(\{(j,k) : x_{jk} \geq \beta\}) = 0$. But this contradicts the hypothesis (\*). Hence every point of $(\xi - \epsilon_0, \xi]$ is a limit point of $x$ which contradicts that the set of limit points of $x$ is a no-where dense set. Hence $x$ is $I$-statistically convergent. \hfill \Box

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