Dipolar Dark Matter as an Effective Field Theory

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Dipolar Dark Matter (DDM) is an alternative model motivated by the challenges faced by the standard cold dark matter model to describe the right phenomenology at galactic scales. A promising realisation of DDM was recently proposed in the context of massive bigravity theory. The model contains dark matter particles, as well as a vector field coupled to the effective composite metric of bigravity. This model is completely safe in the gravitational sector thanks to the underlying properties of massive bigravity. In this work we investigate the exact decoupling limit of the theory, including the contribution of the matter sector, and prove that it is free of ghosts in this limit. We conclude that the theory is acceptable as an Effective Field Theory below the strong coupling scale.

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I. INTRODUCTION

We are witnesses of centenaries. The year 2015 marked the 100th anniversary of Albert Einstein’s elaborate theory of General Relativity (GR), while 2016 celebrated the centenary of the first paper on gravitational waves by the announcement of their experimental detection [1]. GR meets the requirements of the underlying physics in a broad range of scales, from black hole to solar system size. It stood up to intense scrutiny and prevailed against all alternative competitors. It constitutes the bedrock upon which our fundamental understanding of gravity relies. However, some important questions remain.

The lack of renormalizability motivates the modifications of gravity in the ultraviolet (UV), that incorporate the quantum nature of gravity. The singularities present in the classical theory could be regularized by the new physics [2]. The UV modifications might also dictate a different scenario for the early Universe as an alternative to inflation [3]. The inflaton field in the standard picture might be just a remnant of the modification of gravity in the UV.

From a more observational point of view, GR faces additional challenges on cosmological scales. In order to account for the observed amount of ingredients of the Universe, it is necessary to introduce dark matter and dark energy despite of their unclear origin. Notwithstanding of remarkable efforts, the dark matter has so far not been directly detected. Concerning the dark energy, the standard model in form of a cosmological constant \(\Lambda\) accounts for most of the observations even though it faces the unnaturality problem [4]. Combined with the non-baryonic cold dark matter (CDM) component, the model explains remarkably well the observed fluctuations of the cosmic microwave background and the formation of large scale structures.

Albeit the many successes of the \(\Lambda\)-CDM model at large scales, it has difficulties to explain the observations of dark matter at galactic scales. For instance, it is not able to account for the tight correlations between dark and luminous matter in galaxy halos [5, 6]. In this remark, the first unsatisfactory discrepancy comes from the observed Tully-Fisher relation between the baryonic mass of spiral galaxies and their asymptotic rotation velocity. Another discrepancy, perhaps more fundamental, comes from the correlation between the presence of dark matter and the acceleration scale [7, 8]. The prevailing view regarding these problems is that they should be resolved once we understand the baryonic processes that affect galaxy formation and evolution [9]. However, this explanation is challenged by the fact that galactic data are in excellent agreement with the MOND (MOdified Newtonian Dynamics) empirical formula [10, 12]. From a phenomenological point of view, this formula accommodates remarkably well all observations at galactic scales. Unfortunately, extrapolation of the MOND formula to the larger scale of galaxy clusters confronts an incorrect dark matter distribution [13, 17].

The ideal scenario would be to have a hybrid model in which the properties of the \(\Lambda\)-CDM model are naturally incorporated on large scales, whereas the MOND formula would take place on galactic scales. There have been many attempts to embed the physics beyond the MOND formula into an approved relativistic theory, either via invoking new propagating fields without dark matter [18, 24], or by considering MOND as an emergent phenomenology [25, 34].

Here we consider a model of the latter class, called Dipolar Dark Matter (DDM) [27, 28, 31]. The most compelling version of DDM has been recently developed, based on the formalism of massive bigravity theory [35, 36]. To describe the potential interactions between the two metrics of bigravity the model uses the...
effective composite metric introduced in Refs. [37–39]. Two species of dark matter particles are separately coupled to the two metrics, and an internal vector field that links the two dark matter species is coupled to the effective composite metric. The MOND formula is recovered from a mechanism of gravitational polarization in the non relativistic approximation. The model has the potential to reproduce the physics of the Λ-CDM model at large cosmological scales.

In the present paper we address the problem of whether there are ghost instabilities in this model. The model itself [35–39] will be reviewed in Sec. [I] The model is safe in the gravitational sector because it uses the ghost-free framework of massive bigravity. The interactions of the matter fields with the effective metric reintroduce a ghost in the matter sector beyond the strong coupling scale, as found in [37–38]. In our model, apart from this effective coupling the different species of matter fields interact with each other via an internal vector field. This additional coupling might spoil the property of ghost freedom within the strong coupling scale. We therefore investigate, in Sec. [II], the exact decoupling limit (DL) of our model, crucially including the contributions coming from the matter sector and notably from the internal vector field. The model dictates what are the relevant scalings of the matter fields in terms of the Planck mass in the DL. Using that, we shall prove that the theory is free of ghosts in the DL and conclude that it is acceptable as an Effective Field Theory below the strong coupling scale. We end the paper with a few concluding remarks in Sec. [IV].

II. DIPOLAR DARK MATTER

The model that we would like to study in this work is the dark matter model proposed in Ref. [35] where the Dipolar Dark Matter (DDM) at small galactic scales is connected to bimetric gravity based on the ghost-free biformalism of massive gravity [30, 31]. The action of a successful realisation was investigated in [36] and we would like to push forward the analysis performed there. The Lagrangian is the sum of a gravitational term, based on massive bigravity theory, plus a matter part: \( \mathcal{L} = \mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{mat}} \). The gravitational part reads

\[
\mathcal{L}_{\text{grav}} = \frac{M_{\text{eff}}^2}{2} \sqrt{-g} R_g + \frac{M_{\text{eff}}^2}{2} \sqrt{-f} R_f + 2 M^2_{\text{eff}} \sqrt{-g_{\text{eff}}} ,
\]

(2.1)

where \( R_g \) and \( R_f \) denote the Ricci scalars of the two metrics \( g_{\mu\nu} \) and \( f_{\mu\nu} \), with the corresponding Planck scales \( M_g \) and \( M_f \) and the interactions carrying another Planck scale \( M_{\text{eff}} \), together with the graviton’s mass \( m \). In this formulation, the ghost-free potential interactions between the two metrics are defined as the square root of the determinant of the effective composite metric [37–39]

\[
g_{\mu\nu}^{\text{eff}} = \alpha^2 g_{\mu\nu} + 2\alpha \beta g_{\text{eff}}^{\mu\nu} + \beta^2 f_{\mu\nu} ,
\]

(2.2)

with the arbitrary dimensionless parameters \( \alpha \) and \( \beta \) (typically of the order of one). Here \( g_{\mu\nu}^{\text{eff}} \) denotes the effective metric in the previous DDM model [31], given by \( g_{\mu\nu}^{\text{eff}} = g_{\mu\nu} X^g_\rho \) where \( X = \sqrt{g^{-1}f} \), or equivalently \( g_{\mu\nu}^{\text{eff}} = f_{\mu\nu} Y^g_\rho \) where \( Y = \sqrt{f^{-1}g} \). It is trivial to see that the square root of the determinant of this effective metric \( g_{\mu\nu}^{\text{eff}} \) corresponds to the allowed ghost-free potential interactions [37].

The matter part of the model will consist of ordinary baryonic matter and a dark sector including dark matter particles. The crucial feature of the model is the presence of a vector field \( A_\mu \) in the dark sector, that is sourced by the mass currents of dark matter particles and represents a “graviphoton” [42]. This vector field stabilizes the DDM medium and ensures a mechanism of “gravitational polarisation”. The matter action reads

\[
\mathcal{L}_{\text{mat}} = -\sqrt{-g} (\rho_{\text{mat}} + \rho_g) - \sqrt{-g} \rho_f
\]

\[
+ \sqrt{-g_{\text{eff}}} \left[ A_\mu (j_\mu^g - j_\mu^f) + \lambda M_{\text{eff}}^2 W(\mathcal{X}) \right] . \quad (2.3)
\]

Note the presence of a non-canonical kinetic term for the vector field in form of a function \( W(\mathcal{X}) \) of

\[
\mathcal{X} = -\frac{\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}}{4\lambda} , \quad (2.4)
\]

with the field strength defined by \( \mathcal{F}_{\mu\nu} \equiv g_{\text{eff}}^{\mu\rho} \mathcal{F}^{\rho\sigma} \mathcal{F}_{\sigma\nu} \) where \( \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). The form of the function \( W(\mathcal{X}) \) has been determined by demanding that the model reproduces the MOND phenomenology at galactic scales [31, 36, 43]. This corresponds to the limit \( \lambda \to 0 \) and we have

\[
W(\mathcal{X}) = \mathcal{X} - \frac{2}{3} (\alpha + \beta)^2 \mathcal{X}^{3/2} + O(\mathcal{X}^2) , \quad (2.5)
\]

so that the leading term in the action (2.3) is

\[
\lambda M_{\text{eff}}^2 W(\mathcal{X}) = -\frac{M_{\text{eff}}^2}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + O(\mathcal{F}^3) . \quad (2.6)
\]

Hence, we observe that the coupling scale of the vector field is dictated by \( M_{\text{eff}} \), while the parameter \( \lambda \) enters into higher-order corrections. In order to recover the correct MOND regime for very weak accelerations of baryons in the ordinary \( g \) sector, i.e. below the MOND acceleration scale \( a_0 \), these constants have been determined as [30]

\[
M_{\text{eff}} = \sqrt{2} r_g M_{\text{Pl}} \quad \text{and} \quad \lambda = \frac{a_0^2}{2} . \quad (2.7)
\]

Here \( M_{\text{Pl}} \) represents the standard Planck constant of GR and the constant \( r_g \) is defined below. It is worth mentioning that the standard Newtonian limit in the ordinary \( g \)

\[1\] Recall also that the MOND acceleration \( a_0 \) is of the order of the cosmological parameters, and thus is extremely small in Planck units, \( a_0 \sim 10^{-63} M_{\text{Pl}} \).
sector is obtained by imposing the relation
\[ M_\gamma^2 + \frac{\alpha^2}{\beta^2} M_f^2 = M_{\text{Pl}}^2. \] (2.8)
Thus, in this model the three mass scales \( M_g, M_f \) and \( M_{\text{eff}} \) are of the order of the Planck mass.

We represent the scalar energy densities of the ordinary pressureless baryons, and the two species of pressureless dark matter particles by \( \rho_\text{bar}, \rho_g \) and \( \rho_f \) respectively. Such densities are conserved in the usual way with respect to their respective metrics, hence \( \nabla_\mu (\rho_\text{bar} u_\mu) = 0 \), \( \nabla_\mu (\rho_g u_\mu) = 0 \) and \( \nabla_\mu (\rho_f u_\mu) = 0 \), with the four velocities being normalized as \( g_{\mu\nu} u_\mu u_\nu = -1 \), \( g_{\mu\nu} u_\mu g_\nu = -1 \) and \( f_{\mu\nu} u_\nu f_\mu = -1 \). The respective stress-energy tensors are defined as \( T^\mu_\nu = \rho_\text{bar} u_\nu u_\mu \), \( T^\mu_\nu = \rho_g u_\nu u_\mu \) and \( T^\mu_\nu = \rho_f u_\nu u_\mu \). The pressureless baryonic fluid obeys the geodesic law of motion \( a_\mu = \nabla_\mu u_\mu = 0 \), hence \( \nabla_\nu T^\nu_\mu = 0 \). On the other hand, because of their coupling to the vector field, the dark matter fluids pursue a non-geodesic motion:
\[ \nabla_\nu T^\nu_\mu = J^\nu_\mu \mathcal{F}_{\mu\nu}, \] (2.9a)
\[ \nabla_\nu T^\nu_\mu = -J^\nu_\mu \mathcal{F}_{\mu\nu}, \] (2.9b)
where the dark matter currents \( j^\mu_\nu \) and \( j^\mu_\nu \) are related to those appearing in Eq. (2.3) by
\[ J^\mu_\nu = \frac{\sqrt{-g}}{\sqrt{-g}} j^\mu_\nu \quad \text{and} \quad J^\mu_\nu = \frac{\sqrt{-g}}{\sqrt{-g}} J^\mu_\nu. \] (2.10)
It remains to specify the link between these currents and the scalar densities \( \rho_g \) and \( \rho_f \) of the particles. This is provided by \( J^\mu_\nu = r_g \rho_g u_\nu u_\mu \) and \( J^\mu_\nu = r_f \rho_f u_\nu u_\mu \), where \( r_g \) and \( r_f \) are two constants of the order of one, which can be interpreted as the ratios between the “charge” of the particles (with respect to the vector interaction) and their inertial mass. For correctly recovering MOND we must have \( \alpha \rho_g = \beta r_f \).

Whereas, the stress-energy tensor of the vector field \( A_\mu \) is obtained by varying (2.3) with respect to \( g^\text{eff}_{\mu\nu} \) (holding the \( g \) and \( f \) metrics fixed) and corresponds to
\[ T^\mu_\nu = M^2_{\text{eff}} \left[ W_X F^\mu_\nu F^\nu_\rho + \lambda W g^\text{eff}_{\mu\nu} \right], \quad (2.11) \]
where \( W_X \equiv dW/d\mathcal{X} \). The evolution of the vector field is dictated by the Maxwell law
\[ \nabla^\nu \left[ W_X F^{\mu\nu} \right] = \frac{1}{M^2_{\text{eff}}} \left( j^\mu_\nu - j^\mu_\nu \right), \] (2.12)
where the covariant derivative associated with \( g^\text{eff} \) is denoted by \( \nabla^\nu \). Together with the conservation of the currents, \( \nabla_\mu j^\mu_\nu = 0 \) and \( \nabla_\mu j^\mu_\nu = 0 \), the equations of motion for the vector field can also be expressed as
\[ \nabla^\nu \left[ T^\mu_\nu \right] = -(j^\mu_\nu - j^\mu_\nu) \mathcal{F}_{\mu\nu}, \] (2.13)
and we can combine these equations of motion all together into a “global” conservation law
\[ \sqrt{-g} g^\text{eff}_{\mu\nu} T^\mu_\nu + \sqrt{-g} \nabla^\nu (g^\text{eff}_{\mu\nu} F^\mu_\nu) + \sqrt{-f} \nabla^\nu (f^\mu_\nu F^\mu_\nu) = 0. \] (2.14)

III. DECOUPLING LIMIT

Being based on massive bigravity theory, the gravitational sector of the model, Eq. (2.1), is ghost-free up to any order in perturbation theory [40, 41]. In addition, the baryonic and dark matter particles can be coupled separately to either the \( g \) metric or \( f \) metric without changing this property [37]. The case of the pure matter coupling between the vector field \( A_\mu \) and the effective composite metric \( g^\text{eff}_{\mu\nu} \) in Eqs. (2.3), (2.4), is not trivial. In that case, it was shown in Ref. [37] that the coupling is ghost-free in the mini-superspace and in the decoupling limit. Furthermore it is known that such coupling to the composite metric is unique in the sense that it is the only non-minimal matter coupling that maintains ghost-freedom in the decoupling limit [14, 16].

However, in our model the vector field is also coupled to the \( g \) and \( f \) particles, through the standard interaction term \( \propto A_\mu (j^\mu_\nu - j^\mu_\nu) \). This term plays a crucial role for the dark matter model to work. This coupling introduces a supplementary, indirect interaction between the two metrics of bigravity, via the \( g \) and \( f \) particles coupled together by the term \( A_\mu (\partial^\mu \phi - \partial^\mu \phi) \). See Fig. 1 for a schematic illustration of the interactions in the model. As a result it was found in Ref. [37] that a ghost is reintroduced in the dark matter sector in the full theory. The aim of this paper is to investigate the occurrence and mass of this ghost, and whether or not the decoupling limit (DL) is maintained ghost-free. If the latter is true, then the model can be used in a consistent way as an Effective Field Theory valid below the strong coupling scale.

We now detail the analysis of the DL interactions in the graviton and matter sectors. We follow the preliminary work [35] and investigate the scale of the reintroduced Boulware-Deser (BD) ghost [37]. We first decouple the interactions below the strong coupling scale from those entering above it, and concentrate on the pure interactions of the helicity-0 mode of the massive graviton. Using the St"uckelberg trick, we restore the broken gauge invariance in the \( f \) metric by replacing it by
\[ \tilde{f}_{\mu\nu} = f_{\mu\nu} - \partial_\mu \phi \partial_\nu \phi \]
where \(a, b = 0, 1, 2, 3\) and the four Stückelberg fields \(\phi^a\) are decomposed into the helicity-0 mode \(\pi\) and the helicity-1 mode \(A^a\),
\[
\phi^a = x^a - \frac{M^a_\pi}{\Lambda_3} - f^{ab}_\pi \partial_b \pi . \tag{3.2}
\]
Here \(\Lambda_3 = (m^2 M_{\text{eff}})^{1/3}\) denotes the strong coupling scale. Note, that we define it with respect to \(M_{\text{eff}}\) here, since the potential interactions scale with \(m^2 M_{\text{eff}}^2\) in our case.

It is well known that the would-be BD ghost in the DL, would come in the form of higher derivative interactions of the helicity-0 mode \(\pi\) and neglect the interactions of the helicity-1 mode \(A^a\). For simplicity we do not write the tilde symbol over the Stückelbergized version of the metric \((3.1)\). Thus, considering also the helicity-2 mode in the \(g\) metric, we have\(^2\)
\[
g_{\mu\nu} = \left( \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_g} \right)^2 , \tag{3.3a}
\]
\[
f_{\mu\nu} = \left( \eta_{\mu\nu} - \frac{\Pi_{\mu\nu}}{\Lambda_3^2} \right)^2 , \tag{3.3b}
\]
where we introduced the notation \(\Pi_{\mu\nu} \equiv \partial_\mu \partial_\nu \pi\) for convenience, and raised and lowered indices with the Minkowski metric \(\eta_{\mu\nu}\). The effective metric reads then
\[
g_{\mu\nu}^{\text{eff}} = \left( (\alpha + \beta) \eta_{\mu\nu} + K_{\mu\nu} \right)^2 , \tag{3.4}
\]
in which we have introduced as a short-cut notation the linear combination
\[
K_{\mu\nu} = \frac{\alpha}{M_g} h_{\mu\nu} - \frac{\beta}{\Lambda_3^2} \Pi_{\mu\nu} . \tag{3.5}
\]
We will as next investigate the different contributions in the gravitational and matter sectors.

### A. Gravitational sector

There is no contribution of the Einstein-Hilbert term to the helicity-0 mode, since this is invariant under diffeomorphisms. On the other hand, there will be different contributions coming from the ghost-free potential interactions. The allowed potential interactions between the metrics \(g\) and \(f\) have been chosen in our model to be given by the square root of the determinant of the composite metric \((3.4)\), which becomes in this case
\[
\sqrt{-g_{\text{eff}}} = \sum_{n=0}^{4} (\alpha + \beta)^4 n e_{(n)}(K) , \tag{3.6}
\]
where \(e_{(n)}(K)\) denote the usual symmetric polynomials associated with the matrix \(K_{\mu\nu}^g = n^{\mu\nu} K_{\mu\nu}\), and given by products of antisymmetric Levi-Cevita tensors,
\[
e_{(0)}(K) = -\frac{1}{24} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} , \tag{3.7a}
\]
\[
e_{(1)}(K) = -\frac{1}{6} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} K^\lambda_\rho , \tag{3.7b}
\]
\[
e_{(2)}(K) = -\frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} K_\rho^3 K_\sigma^\lambda , \tag{3.7c}
\]
\[
e_{(3)}(K) = -\frac{1}{6} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} K_\rho^3 K_\sigma^3 K^\lambda_\rho , \tag{3.7d}
\]
\[
e_{(4)}(K) = -\frac{1}{24} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} K_\rho^3 K_\sigma^3 K^\lambda_\rho K^\lambda_\sigma . \tag{3.7e}
\]
In particular, we see that \(e_{(4)}(K) = \det(K)\).

First of all, the pure helicity-0 mode in the ghost-free potential interactions \((3.6)\) will come in the form of total derivatives \([40, 49]\). Indeed, as is clear from their definitions \((3.7)\) in terms of antisymmetric Levi-Cevita tensors, the symmetric polynomials \(e_{(n)}(\Pi) \equiv \mathcal{L}_{(n)}^{\text{eff}}(\Pi)\) fully encode the total derivatives at that order, and thus will not contribute to the equation of motion of the helicity-0 mode. In fact, in Ref. \([49]\), this very same property of total derivatives of the leading contributions at each order was used to build the ghost-free interactions away from \(h = 0\). Secondly, there will be the pure interactions of the helicity-2 mode, obtained by setting \(\Pi = 0\), and these will come with the corresponding inverse powers of \(M_g\). Finally, there will be the mixed interactions between the helicity-2 and helicity-0 modes.

We are after the leading interactions in the DL, which correspond to sending all the Planck scales to infinity,
\[
M_{\text{Pl}} \to \infty , \quad M_g \to \infty , \quad M_{\text{eff}} \to \infty , \quad M_f \to \infty , \tag{3.8}
\]

### B. Matter sector

Together with the graviton’s mass \(m \to 0\), while keeping
\[
\{ \Lambda_3^2 = m^2 M_{\text{eff}} , \quad M_g / M_{\text{Pl}} , \quad M_{\text{eff}} / M_{\text{Pl}} , \quad M_f / M_{\text{Pl}} \} = \text{const} . \tag{3.9}
\]
Taking into account the factor \(m^2 M_{\text{eff}}^2\) in front of the potential interactions, one immediately observes that the pure non-linear interactions of the helicity-2 modes do not contribute to the DL. As we already mentioned, the pure helicity-0 mode interactions do not contribute either. So it remains the mixed terms, for which the only surviving terms will be linear in the helicity-2 mode, and we finally obtain
\[
m^2 M_{\text{eff}}^2 \sqrt{-g_{\text{eff}}} = \sum_{n=1}^{3} \frac{a_n}{3^{n}(n-1)!} \h^{\mu\nu} P_{\mu\nu}^{(n)}(\Pi) + O \left( \frac{1}{M_g} \right) , \tag{3.10}
\]
where \( a_n = (\frac{M_m}{M_p})^{n+1} \alpha(-\beta)^n (\alpha + \beta)^{3-n} \) and we posed
\[
P_{\mu\nu}^{(n-1)}(\Pi) = \frac{\partial e_{(n)}(\Pi)}{\partial \Pi_{\mu\nu}}.
\] (3.11)

In arriving at Eq. (3.10) we have removed the trivial constant term in (3.6), and ignored the “ tadpole” which is simply proportional to the trace \( h = h^\mu_\mu \) and can be eliminated by choosing an appropriate de Sitter background (see, e.g., a discussion in [59]).

We can then write the total contribution of the gravitational sector in the DL, including that coming from the Einstein-Hilbert term of the g metric, which enters only at the leading quadratic order in \( h_{\mu\nu} \),
\[
\mathcal{L}_{\text{DL}}^{\text{grav}} = -h_{\mu\nu} \mathcal{E}_{\mu\nu} h_{\rho\sigma} + \sum_{n=1}^{3} \frac{a_n}{\Lambda_3^{n-1}} h_{\mu\nu} P_{\mu\nu}^{(n)}(\Pi),
\] (3.12)

where \( \mathcal{E}_{\mu\nu} \) is the usual Lichnerowicz operator on a flat background as defined by
\[
-2\mathcal{E}_{\mu\nu} h_{\rho\sigma} = \Box(h_{\mu\nu} - \eta_{\mu\nu} h) + \partial_\mu \partial_\nu h - 2\partial_\rho h_{\mu\nu} + \eta_{\mu\nu} \partial_\rho H^\rho,
\] (3.13)

with \( h = [h] = h_{\mu}^\mu \) and \( H^\rho = \partial_\rho \partial_\mu H^\mu \). The symmetric tensors \( P_{\mu\nu}^{(n)} \) are conserved, i.e., \( \partial_\rho P_{\mu\nu}^{(n)} = 0 \). For an easier comparison with the literature we give them as the product of two Levi-Civita tensors appropriately contracted with the second derivative of the helicity-0 field,
\[
P_{\mu\nu}^{(1)}(\Pi) = -\frac{1}{2} \varepsilon^\lambda_\mu \varepsilon_\nu\lambda\pi\sigma \Pi^\pi_\sigma,
\] (3.14a)
\[
P_{\mu\nu}^{(2)}(\Pi) = -\frac{1}{2} \varepsilon^\lambda_\mu \varepsilon_\nu\lambda\pi\pi_\pi^\pi_\tau \Pi^\pi_\tau,
\] (3.14b)
\[
P_{\mu\nu}^{(3)}(\Pi) = -\frac{1}{6} \varepsilon^\lambda_\mu \varepsilon_\nu\lambda\pi\pi^\pi_\pi_\pi^\pi_\tau \Pi^\pi_\pi^\pi_\tau.
\] (3.14c)

The first two interactions between the helicity-0 and helicity-2 fields in the Lagrangian (3.12) can be removed by the change of variable, defining
\[
\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{a_1}{2} \pi \eta_{\mu\nu} + \frac{a_2}{2A_3} \partial_\mu \pi \partial_\nu \Pi.
\] (3.15)

In this way the Lagrangian of the gravitational sector in the decoupling limit becomes [59]
\[
\mathcal{L}_{\text{grav}}^{\text{DL}} = -\hat{h}_{\mu\nu} \mathcal{E}_{\mu\nu} \hat{h}_{\rho\sigma} + \sum_{n=1}^{3} \frac{b_n}{\Lambda_3^{n-1}} (\partial_\pi)^2 e_{(n)}(\Pi)
+ \frac{a_3}{\Lambda_3} \hat{h}_{\mu\nu} P_{\mu\nu}^{(3)}(\Pi).\n\] (3.16)

We see in the first line the appearance of the ordinary Galileon terms up to quintic order [we denote \( (\partial_\pi)^2 \equiv \partial_\mu \pi \partial_\nu \pi \)]. The coefficients \( b_n \) are given by certain combinations of the \( a_n \)'s.

The last term of Eq. (3.16) is the remaining mixing between the helicity-0 and helicity-2 modes and is not removable by any local field redefinition like in (3.15).

The contribution of the gravitational sector to the equation of motion of the helicity-2 field gives
\[
\frac{\delta \mathcal{L}_{\text{grav}}^{\text{DL}}}{\delta h_{\mu\nu}} = -2 \mathcal{E}_{\mu\rho} \hat{h}_{\rho\sigma} + \frac{a_3}{\Lambda_3} P_{\mu\nu}^{(3)}(\Pi),
\] (3.17)

while its contribution to the equation of motion of the helicity-0 field reads
\[
\frac{\delta \mathcal{L}_{\text{grav}}^{\text{DL}}}{\delta \pi} = -2 \sum_{n=1}^{3} \frac{n b_{n-1}}{\Lambda_3^{n-1}} e_{(n)}(\Pi) + \frac{a_3}{\Lambda_3} Q_{\mu\nu}^{(2)}(\Pi) \partial_\rho \partial_\sigma h_{\rho\sigma},
\] (3.18)

where we posed
\[
Q_{\mu\nu}^{(2)}(\Pi) \equiv \frac{\partial P_{\mu\nu}^{(3)}}{\partial \Pi_{\rho\sigma}} = -\frac{1}{2} \varepsilon^\rho_\mu \varepsilon_\nu\sigma \pi\tau \Pi^\pi_\pi^\tau \Pi.\n\] (3.19)

The second-order nature of the equations of motion in the gravity sector is apparent. This is the standard property of the ghost-free massive gravity interactions [49] [50].

## B. Matter sector

As next, we shall control the contributions in the matter sector due to both the helicity-0 and helicity-2 fields. To this aim it is important to properly identify the matter degrees of freedom that are metric independent.

These are provided by the coordinate densities defined as \( \rho^g = \sqrt{-g} \rho_{\bar{g}} \) and \( \rho^f = \sqrt{-f} \rho_{\bar{f}} \), and by the ordinary (coordinate) velocities \( v^g_\mu = u^g_\mu / u^g_0 \) and \( v^f_\mu = u^f_\mu / u^f_0 \). The associated currents \( J^g_\mu = \rho^g v^g_\mu \) and \( J^f_\mu = \rho^f v^f_\mu \) are conserved in the ordinary sense, \( \partial_\mu J^g_\mu = 0 \) and \( \partial_\mu J^f_\mu = 0 \), and are related to the classical currents by
\[
J^g_\mu = \sqrt{-g} J^g_\mu \quad \text{and} \quad J^f_\mu = \sqrt{-f} J^f_\mu.
\] (3.20)

When varying the action we must carefully impose that the independent matter degrees of freedom are the metric independent currents \( J^g_\mu \) and \( J^f_\mu \). After variation we may restore the manifest covariance by going back to the classical currents using (3.20).

Next we must specify how the matter variables will behave in the DL when we take the scaling limits (3.8)–(3.9). In the DL we want to keep intact the coupling between the helicity-2 mode \( h_{\mu\nu} \) and the particles living in the g sector, therefore we impose
\[
T^\mu_\nu = M_g T^\mu_\nu \quad \text{and} \quad T^g_\mu = M_g T^g_\mu,
\] (3.21)

with \( T^\mu_\nu \) and \( T^g_\mu \) remaining constant in the DL. As for the \( f \) particles, in a similar way we demand that \( T^f_\mu = M_f T^f_\mu \) with \( T^f_\mu \) being constant.

The next important point concerns the internal vector field \( A_\mu \). As we have seen this vector field is a graviphoton [32], i.e. its scale is given by the Planck mass, witness

\[\text{Namely, } b_0 = -\frac{3}{4} a_1^2, b_1 = -\frac{3}{4} a_1 a_2, b_2 = -\frac{1}{2} a_2^2 - \frac{1}{4} a_1 a_3 \text{ and } b_3 = -\frac{5}{4} a_2 a_3.\]
the factor $M_{\text{eff}}^2$ in front of the kinetic term of the vector field (2.6), see also the factor $M_{\text{eff}}^2$ in front of the stress-energy tensor of the vector field, Eq. (2.11). For the model to work $M_{\text{eff}}$ must be of the order of the Planck mass, as determined in (2.7). This means that we have to canonically normalize the vector field $A_\mu$, according to

$$A_\mu = \frac{\hat{A}_\mu}{M}, \quad (3.22)$$

and keep $\hat{A}_\mu$ constant in the DL. Thus $T^{\mu\nu}_{\text{gbar}} = \hat{T}^{\mu\nu}_{\text{gbar}}$ should be considered constant in that limit.

A general variation of the matter action with respect to the two metrics reads

$$\delta L_{\text{mat}} = \frac{\sqrt{-g}}{2} (T^{\mu\nu}_{\text{bar}} + T^{\mu\nu}_{g}) \delta g_{\mu\nu} + \frac{\sqrt{-g}}{2} f^{\mu\nu} \delta f_{\mu\nu}$$

$$+ \frac{\sqrt{-g}}{2} T^{\mu\nu}_{\text{gbar}} \delta g^{\mu\nu}. \quad (3.23)$$

We insert Eqs. (3.3)–(3.4) and change the helicity-2 variable according to (3.15) to obtain the contribution of the matter action to the field equation for the helicity-2 field (in guise $h^{\mu\nu}$) as

$$\frac{\delta L_{\text{mat}}}{h^{\mu\nu}} = \frac{1}{M_g} \sqrt{-g} (T^{\rho\mu}_{\text{bar}} + T^{\rho\mu}_{g}) (\delta^{\rho}_{\mu} + \frac{\eta^{\rho}_{\mu}}{M_g})$$

$$+ \frac{\alpha}{Mg} \sqrt{-g_{\text{eff}}} T^{\rho\mu}_{\text{gbar}} \left( (\alpha + \beta) \delta^{\rho}_{\mu} + K^{\rho}_{\mu} \right). \quad (3.24)$$

Taking the DL with the postulated scalings (3.21)–(3.22) we find that the helicity-2 mode of the massive graviton is just coupled in this limit to the baryons and $g$ particles, and the internal vector field $A_\mu$. However, because of the scaling (3.22), which we recall is appropriate for the graviphoton whose coupling scale is given by the Planck mass, the vector field strength actually scales like $F^{\mu\nu} = \hat{F}^{\mu\nu}/M_{\text{eff}}$ in the DL limit. This fact kills all the interactions between the helicity-0 mode and the vector field in the DL, since they come with an inverse power of $M_{\text{eff}}$. Thus the direct contribution (3.28) is identically zero in the DL, and only the contribution (3.26b) is surviving, while (3.26a) is also zero. After further simplification with the matter equations of motion, we obtain

$$\frac{\delta L_{\text{DL}}}{\delta h^{\mu\nu}} = T^{\mu\nu}_{\text{bar}} + \hat{T}^{\mu\nu}_{g}, \quad (3.25)$$

where the (rescaled) stress-energy tensors $\hat{T}^{\mu\nu}_{\text{bar}}$ and $\hat{T}^{\mu\nu}_{g}$ in the DL are computed with the Minkowski background.

We next consider the contributions of the matter sector to the equation of motion of the helicity-0 field. We find three contributions, two coming from the field redefinition (3.15),

$$\frac{\delta L_{\text{mat}}}{\delta \pi} \biggr|_{(1a)} = \frac{a_1}{2M_g} \sqrt{-g} \left( T^{\mu\nu}_{\text{bar}} + T^{\mu\nu}_{g} \right) \left( \eta^{\mu\nu} + \frac{h^{\mu\nu}}{M_g} \right)$$

$$+ \frac{a_2}{M_3 A_3} \frac{\partial^2}{\partial \nu} \left[ \sqrt{-g} \left( T^{\mu\nu}_{\text{bar}} + T^{\mu\nu}_{g} \right) \left( \delta^{\rho}_{\mu} + \frac{\eta^{\rho}_{\mu}}{M_g} \right) \right], \quad (3.26a)$$

$$\frac{\delta L_{\text{mat}}}{\delta \pi} \biggr|_{(1b)} = \frac{a_1 a_2}{2M_g} \sqrt{-g_{\text{eff}}} T^{\mu\nu}_{\text{gbar}} \left( (\alpha + \beta) \eta^{\mu\nu} + K^{\mu\nu} \right)$$

$$+ \frac{a_2}{M_3 A_3} \frac{\partial^2}{\partial \nu} \left[ \sqrt{-g_{\text{eff}}} T^{\mu\nu}_{\text{gbar}} \left( (\alpha + \beta) \delta^{\rho}_{\mu} + K^{\rho}_{\mu} \right) \right], \quad (3.26b)$$

and the third one being “direct”, and already investigated in [36] with result

$$\frac{\delta L_{\text{mat}}}{\delta \pi} \biggr|_{(2)} = -\frac{1}{A_3} \partial_{\nu} \partial_{\mu} \sqrt{-f} T^{\rho\nu}_{f} \left( \delta^{\rho}_{\mu} - \frac{\Pi^{\rho}_{\mu}}{A_3} \right). \quad (3.27)$$

The latter contribution might look worrisome in the DL, but it becomes finite after using the equation of motion for the $f$ particles, Eq. (2.9b), and that for the vector field, Eq. (2.13). The calculation proceeds similarly to the one using Eqs. (3.29)–(3.32) in Ref. [37]. Finally the result can be brought into the form [36]

$$\frac{\delta L_{\text{mat}}}{\delta \pi} \biggr|_{(2)} = \frac{1}{M_3^2} \partial_{\nu} \left[ f^{\rho\nu} F_{\rho\sigma} \left( \eta^{\mu\nu} - \frac{\Pi^{\mu\nu}}{A_3} \right) \right]^{-1}$$

$$+ \beta \left( J^{g\nu} - J^{\nu}_{\text{fbar}} \right) F_{\rho\nu} \left( (\alpha + \beta) \eta^{\mu\nu} + K^{\mu\nu} \right)^{-1}], \quad (3.28)$$

where we describe the matter degrees of freedom by means of the coordinate currents (3.20).

The results (3.27) and (3.28) are general at this stage, and involve couplings between both the helicity-0 and helicity-2 modes with the matter fields — and $f$ particles, and the internal vector field $A_\mu$. However, because of the scaling (3.22), which we recall is appropriate for the graviphoton whose coupling scale is given by the Planck mass, the vector field strength actually scales like $F^{\mu\nu} = \hat{F}^{\mu\nu}/M_{\text{eff}}$ in the DL limit. This fact kills all the interactions between the helicity-0 mode and the vector field in the DL, since they come with an inverse power of $M_{\text{eff}}$. Thus the direct contribution (3.28) is identically zero in the DL, and only the contribution (3.26b) is surviving, while (3.26a) is also zero. After further simplification with the matter equations of motion, we obtain

$$\frac{\delta L_{\text{DL}}}{\delta \pi} \biggr|_{(2)} = \frac{a_1}{2} (\hat{T}^{\mu\nu}_{\text{bar}} + \hat{T}^{\mu\nu}_{g}) \partial_{\mu} \partial_{\nu} \pi. \quad (3.29)$$

Recapitulating, we find that the DL of the model consists of the following equation for the helicity-2 mode, i.e.

$$\delta L_{\text{DL}}/\delta h^{\mu\nu} = 0 \quad \text{or equivalently}$$

$$-2\epsilon^{\mu\nu\rho\sigma} \hat{h}_{\rho\sigma} + \frac{a_3}{A_3} \left[ \partial_{\mu} \left( T^{\mu\nu}_{\text{bar}} + \hat{T}^{\mu\nu}_{g} \right) \partial_{\nu} \pi \right]. \quad (3.30)$$

which is of second-order nature. Thus, the contributions of the gravitational and matter sector to the equations of motion of the helicity-2 mode in the DL are ghost-free. Note, that the Bianchi identity of this equation (taking the divergence of it) is identically satisfied, since the particles actually follow geodesics in the DL. Indeed, using (3.21)–(3.22) together with the equations of motion [e.g. (2.9)], we have $\partial_{\mu} \hat{T}^{\mu\nu}_{\text{bar}} = \partial_{\nu} \hat{T}^{\mu\nu}_{g} = 0$ (the particles move on Minkowski straight lines).

\footnote{Note that if we do not impose the scaling $F^{\mu\nu} = \hat{F}^{\mu\nu}/M_{\text{eff}}$ the equation (3.24) for the helicity-2 field diverges in the DL. Similarly for Eq. (3.26a).}
In addition we have the total equation of motion of the helicity-0 mode, namely $\delta \mathcal{L}_{\text{DL}} / \delta \pi = 0$ which reads

$$
-2 \sum_{n=1}^{4} \frac{n h_n^{-1}}{\Lambda_3(n-1)} e_n(\Pi) + \frac{a_3}{\Lambda_3} O^{(2)}(\Pi) \partial_\mu \partial_\nu \hat{h}^{\mu\nu} \quad (3.31)
$$

$$
= -\frac{a_1}{2} (\bar{T}_\text{bar} + \bar{T}_g) - \frac{a_2}{\Lambda_3^2} (\bar{T}_\text{bar}^{\mu\nu} + \bar{T}_g^{\mu\nu}) \partial_\mu \partial_\nu \pi.
$$

Since this equation is perfectly of second-order in the derivatives of the $\pi$ field, we conclude our study by stating that the model is safe (ghost-free) up to the strong coupling scale. Below that scale the theory is perfectly acceptable as an Effective Field Theory, and its consequences can be worked out using perturbation theory as usual. For instance, solving at linear order the helicity-0 equation (3.31) we obtain the usual well-posed (hyperbolic-like) equation

$$
\Box \pi = \frac{a_1}{4a_0} (\bar{T}_\text{bar} + \bar{T}_g) + \mathcal{O}(\pi^2),
$$

which can then be perturbatively iterated to higher order. With this we have proved, that the coupling of the dark matter particles with the internal vector field does not introduce any ghostly contribution in the DL.

### IV. CONCLUSIONS

This work was dedicated to the detailed study of the decoupling limit interactions of the dark matter model proposed in [35, 36]. This model is constructed via a specific coupling of two copies of dark matter particles to two metrics in the framework of massive bigravity. Furthermore, an internal vector field links the two dark matter species. This enables us to implement a mechanism of gravitational polarization, which induces the MOND phenomenology on galactic scales (with the specific choice of parameters studied in [36]). Note that, since our model successfully reproduces all aspects of that phenomenology, it will be in agreement with the recent observations of the MOND mass-discrepancy-acceleration relation in [51].

Some theoretical and phenomenological consequences of this model were studied in detail in Ref. [36], but it was also pointed out that the decoupling limit of the theory may be problematic, with higher derivative terms occurring in the equation of motion of the helicity-0 mode of the massive graviton.

In the present work, we studied the complete DL interactions crucially including the contributions of the matter sector, and we showed that by necessary rescaling of the vector field (as appropriate for a vector field with Planckian coupling constant) the theory is free from ghosts in the DL, and hence can be used as a valid Effective Field Theory up to the strong coupling scale.

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