Natural frequencies of plates of medium thickness

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Abstract. A comparison is made of the frequencies obtained by the classical theory of thin plates and the refined theory of plates of medium thickness. Considered a rectangular plate with articulation along the contour. To determine the fundamental frequency, the Bubnov-Galerkin variational method is used. It is shown that with a relative plate thickness \(h/a = 0.2\), the calculation refinement is approximately 15-20\% for the values of the fundamental frequency of natural vibrations of a rectangular plate.

1. Introduction

A widespread element of building structures is rectangular plates with various fixings of the contour, the most important dynamic of which are the frequencies of natural vibrations.

The technical theory of plate bending, proposed by G. Kirchhoff, describes well the stress-strain state of fairly thin plates, however, with an increase in thickness, the results may not be entirely correct, and their work under load is much more accurately described by the theory of plates of medium thickness.

This area includes the work of a number of scientists (E. Reisner and others \([1 – 7]\)), including the theory developed by B.F. Vlasov \([8]\) and successfully implemented in a number of works \([9, 10]\).

2. State of the problem

In accordance with the B.F. Vlasov, the stress-strain state of a rectangular plate of average thickness under the action of a transverse load \(p(x,y)\) is reduced to the determination of two sought functions \(f(x,y)\) and \(\varphi(x,y)\) from the solution of a system of two differential equations

\[
\begin{align*}
D \nabla^2 \nabla^2 f &= p(x,y), \\
\nabla^2 \varphi &= \frac{2Gh}{D(1-\nu)} \varphi.
\end{align*}
\]

(1)

Here \(D = Eh^3/(12(1-\nu^2))^{-1}\) – is the cylindrical stiffness of the plate, \(E, G\) are the elastic and shear moduli of the plate material, \(h\) – its thickness, \(\nu\) – is the Poisson's ratio of the material, and \(\nabla^2\) – is the Laplace operator.

The deflection of the plate \(w(x,y)\) is related to the desired function \(f(x,y)\) as follows

\[
w = f - \frac{D}{Gh} \nabla^2 f
\]

(2)
When considering natural vibrations, in accordance with the d'Alembert principle, inertial forces are taken as the external load, equal to, where \( m \) is the distributed mass of the plate, and the points above the function \( w \) denote differentiation in time.

In this case, the first equation of system (1) takes the form
\[
DV^2 \nabla^2 f + m\ddot{w} = 0
\] (3)

Taking into account relation (2), equation (3) takes the form:
\[
DV^2 \nabla^2 f + m\left( \ddot{f} - \frac{D}{Gh} \nabla^2 f \right) = 0
\] (4)

In the considered setting, the function \( f \) depends both on coordinates and on time: \( f(x, y, t) \), therefore, in accordance with the Fourier method, we represent it as a product of two functions, one of which depends only on coordinates, and the other on time:
\[
f(x, y, t) = W(x, y) \cdot T(t)
\] (5)

Substituting expression (5) into equation (4), we obtain
\[
DV^2 \nabla^2 W \cdot T + m\left( W - \frac{D}{Gh} \nabla^2 W \right) \cdot \ddot{T} = 0
\] (6)

The separation of variables, traditional for the Fourier method, leads to the following relations
\[
\frac{D}{m} \frac{\nabla^2 \nabla^2 W}{W - \frac{D}{Gh} \nabla^2 W} = \frac{\ddot{T}}{T} = \bar{\omega}^2
\]
from which for the function \( T(t) \) we obtain the equation of harmonic oscillations:
\[
\ddot{T}(t) + \bar{\omega}^2 T(t) = 0
\]
and for the function \( W(x, y) \) – the equation
\[
\frac{D}{m} \frac{\nabla^2 \nabla^2 W - \bar{\omega}^2 \left( W - \frac{D}{Gh} \nabla^2 W \right)}{W - \frac{D}{Gh} \nabla^2 W} = 0
\] (7)

with which, depending on the type of boundary conditions, it is possible to determine the values of the frequencies of natural vibrations \( \bar{\omega}^2 \).

3. Analytical method of solving
Consider a rectangular plate hinged on all sides.

In this case, the deflection \( W(x, y) \) can be represented as a double trigonometric series satisfying all boundary conditions
\[
W(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} C_{ij} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}
\] (8)

Here \( a \) and \( b \) – are the dimensions of the plate in plan.

Substituting series (8) into equation (7), we obtain for each pair of numbers \((i, j)\) an equation of the form
\[
\frac{D\pi^2}{m} \left( \frac{i}{a} \right)^2 + \left( \frac{j}{b} \right)^2 - \bar{\omega}^2 \left( \frac{i}{a} \right)^2 + \left( \frac{j}{b} \right)^2 \right) = 0
\] (9)
from which it is possible to obtain the value of the corresponding frequency

$$\tilde{\omega}^2 = \frac{D\pi^4}{m} \left[ \left( \frac{i}{a} \right)^2 + \left( \frac{j}{b} \right)^2 \right]$$

(10)

Considering that

$$\frac{D}{Gh} = \frac{Eh^3}{12(1-v^2)} \frac{2(1+v)}{Eh} = \frac{h^2}{6(1-v)}$$

and taking out of the brackets one of the dimensions of the plate, for example, $a$, we get

$$\tilde{\omega}^2 = \frac{E}{ma} \frac{\pi^4}{12(1-v^2)} \left( \frac{h}{a} \right)^3 \left[ i^2 + (\alpha j)^2 \right]$$

$$\frac{\pi^2}{6(1-v)} \left( \frac{h}{a} \right)^2 \left[ i^2 + (\alpha j)^2 \right]$$

(11)

where $\alpha = a/b$.

For comparison, we present the formula by which the frequencies of rectangular plates are determined in accordance with the technical theory of G. Kirchhoff

$$\omega^2 = \frac{D\pi^4}{m} \left[ \left( \frac{i}{a} \right)^2 + \left( \frac{j}{b} \right)^2 \right] = \frac{E}{ma} \frac{\pi^4}{12(1-v^2)} \left( \frac{h}{a} \right)^3 \left[ i^2 + (\alpha j)^2 \right]$$

(12)

4. Examples of calculation

Let us determine for two types of plates the values of the basic dimensionless frequency ($i = j = 1$) in accordance with formulas (11) and (12)

$$\frac{ma}{E} \tilde{\omega}^2 = \frac{\pi^4}{12(1-v^2)} \left( \frac{h}{a} \right)^3 \left( 1 + \alpha^2 \right)^2$$

(13)

$$\frac{ma}{E} \omega^2 = \frac{\pi^4}{12(1-v^2)} \left( \frac{h}{a} \right)^3 \left( 1 + \alpha^2 \right)^2$$

(14)

In accordance with formulas (13) and (14), figure 1 shows the graphs of the dependence of the values of dimensionless frequencies $\frac{ma}{E} \tilde{\omega}^2$ and $\frac{ma}{E} \omega^2$ on the parameter $h/a$ for square plates ($\alpha = 1$).
Figure 1. Dependence of the value of natural vibration frequencies on \( h/a \) for square plates according to two theories:
1. on the technical theory of G. Kirchhoff,
2. according to B.F. Vlasov’s theory of plates of medium thickness.

Can be seen that the results obtained on the technical theory of thin plates with increasing relative thickness are overestimated.

To estimate the value of the resulting error, we compose the ratio of the squares of the frequencies obtained by the two theories under consideration

\[
\frac{\omega^2}{\bar{\omega}^2} = 1 + \frac{D\pi^2}{Gh} \left[ \left( \frac{i}{a} \right)^2 + \left( \frac{j}{b} \right)^2 \right] = 1 + \frac{D\pi^2}{Gh} \left[ \left( \frac{i}{a} \right)^2 + \left( \frac{j}{b} \right)^2 \right]
\]

which for the fundamental frequencies (for \( i = j = 1 \)) leads to the following result

\[
\frac{\omega^2}{\bar{\omega}^2} = 1 + \frac{\pi^2}{6(1-\nu)} \left( \frac{h}{a} \right)^2 \left( 1 + \alpha^2 \right)
\]

(15)

Figure 2. Ratio of the squares of natural frequencies obtained by two theories of bending of square plates at different values of the parameter \( h/a \).
Figure 2 shows a graph constructed in accordance with formula (15) and showing the ratio of the squares of the natural vibration frequencies obtained according to two bending theories for square plates (at $\alpha = 1$) at different values of the parameter $h/a$.

5. Conclusions
Comparison of the results obtained according to the classical theory of thin plates and the refined theory of plates of average thickness, showed that with a relative plate thickness $h/a = 0.2$, the refinement of the calculation is about 15-20% for the values of the fundamental frequency of natural vibrations.

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References
[1] Reissner E 1945 The effect of transverse shear deformation on the bending of elastic plates J Appl Mech 12 pp 68–77
[2] Green A E 1949 On Reissner’s theory of bending of elastic plates Quart Appl Math 7 (2) pp 223–8
[3] Reissner E 1975 On transverse bending of plates, including the effect of transverse shear deformation Int J Solids Struct 11 (5) pp 569–73
[4] Rychter Z 1986 An improved bound on the error in Reissner’s theory of plates Arch Mech (Warszawa) 38 (1 2) pp 209–213
[5] Huang C S, Chang M and Leissa A 2006 Vibrations of mindlin sectorial plates using the Ritz method considering stress singularities J of Vibration and Control 12 pp 635–57
[6] Gabbasov R F and Hoang T A 2014 Dynamic load calculation of a bending plate of average thickness using general equations of finite differences method. Vestnik MGSU 10 pp 16–23
[7] Blevins R D 2016 Natural frequency of plates and shells Formulas for Dynamics, Acoustics and Vibration pp 203–59
[8] Vlasov B F 1989 Bending equations for Medium thickness plates Proc. of the Univ. of Civil Eng. p 107-16
[9] Mardji M and Vlasov B F 1989 Calculation of rectangular plates on an elastic foundation according to the refined theory Proc. of the Univ. of Civil Eng. pp 116-20
[10] Vlasov B F and Papuch A V 1989 Proc. of the Univ. of Civil Eng. Pp 121-7