Transmutation of Pure 2-D Supergravity Into Topological 2-D Gravity and Other Conformal Theories

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Abstract We consider the BRST and superconformal properties of the ghost action of 2-D supergravity. Using the background spin structure on the worldsheet, we show that this action can be transformed by canonical field transformations to reach other conformal models such as the 2-D topological gravity or the chiral models for which the gauge variation of the action reproduces the left or right conformal anomaly. Our method consists in using the gravitino and its ghost as fundamental blocks to build fields with different conformal weights and statistics. This indicates in particular that the twisting of a conformal model into another one can be classically interpreted as a change of "field representation" of the superconformal symmetry.
1 Introduction

Many conformal field theories are based on energy momentum tensors which, at the classical level, are not more than quadratic in the dynamical fields and their first and second order derivatives. Their action is essentially a free one and can be often written as a superposition of $b - c$ systems. In various cases a relationship occurs between the quantum energy momentum tensor operators of different theories. They can be ”twisted” one into each other by the addition of the derivative of an abelian current [1], and moreover an $N = 2$ supersymmetry is present [2]. Here, we investigate whether these phenomena rely on classical properties, independently on the quantization of the fields. We consider conformal theories with equal number of commuting and anticommuting fields, which is a necessary condition to get some kind of supersymmetry. Observing that the only meaningful differences which can exist at the classical level between different models are the conformal weights and statistics of the fields, we find it reasonable to believe that a root theory should exist, where the fields belong to what one could call the ”fundamental representation”, the other ”representations” being obtainable by suitable canonical field transformations. Since the conformal symmetry is supposed to be maintained in any ”representation”, the Lagrangians, energy momentum tensors and other conserved currents such as the ghost number and BRST Noether currents should be also related by these changes of field variables. The Ward identity expressing the conformal symmetry should have the same expression in all representations, the only freedom in a given theory being the value of the coefficient of the possible anomalous term. It is of course understandable that the anomaly coefficient escapes the classical property that systems which are related by canonical field transformations are equivalent, since its value must be computed at the quantum level and involves a regularisation. One also expects that for two different ”representations” the expressions of physical observables, computed at the level of field polynomials from the BRST cohomology of the theory or from BRST exact terms, be related by the canonical changes of variables, but the interpretations and values of their expectation values, as the anomaly coefficient, differ when one goes from one theory to the other.
Our point of view forces us to work with a non trivial reference metric on the worldsheet, and moreover to introduce a gravitino, in order to have a spin structure on the worldsheet. Indeed, we find it natural to use the gravitino ghost field, which is a commuting object with conformal weight one half, as a building block to generate by field multiplications higher order conformal field ”representations”, and the background gravitino field, which is an anticommuting object, to possibly change the field statistics. The conformal gauge results are easily obtained by setting the Weyl independent parts of the background metric and gravitino fields equal to zero. The use of metric and gravitino background fields has also the advantage of simplifying the derivation of the various properties of the energy momentum and supersymmetry tensors, since they they are the sources of these objects [3] [4].

The paper is organized as follows. We study the ghost action of 2-D supergravity in the gauges where the metric and gravitino are conformally set equal to non vanishing background metric and gravitino fields, a particular case of which is the conformal gauge. In addition to the ordinary supergravity BRST symmetry, present by construction, we find a background local supersymmetry, acting on the supergravity ghost, antighost and background metric and gravitino fields, with a generator which anticommutes with the basic BRST and ghost number symmetries. In both holomorphic and antiholomorphic sectors, the formulae fall in a very simple $N=1$ superfield formalism, although the background is non trivial. The $N=2$ conformal supersymmetry [2] of the ghost action appears as an accident of the case when the background gravitino is set equal to zero and the Beltrami parameter is choosen constant. Indeed, with these choices of the background gauge, the ghost current can be splitted into two independently conserved abelian currents and this provides the additional U(1) symmetry which is necessary to extend the fundamental background $N=1$ supersymmetry into an $N=2$ conformal supersymmetry. Then comes our main observation. The presence of the background gravitino and of its ghost permits one to do canonical field redefinitions, which change the conformal weights as well as the statistics. The redefined fields can still be seen as ”realizing” the original symmetries and can be used to define other BRST and conformally invariant actions with different values of the anomaly coefficient, although the starting actions are related by
canonical transformations. At the classical level the new energy momentum tensors differ from the one of the ghost action of 2-D supergravity by terms which are derivatives of $U(1)$ currents. As an interesting application, we show the relationship between the ghost action of 2-D supergravity and the action of topological 2-D gravity in the gauge of Labastida, Pernici and Witten [5]. Further field redefinitions permit one to introduce the chiral actions whose gauge variations reproduce the left or right conformal anomaly with an arbitrary coefficient, that is left-right asymmetric anomaly compensating actions, which can be combined to other systems [6] [7] [8]. All our formula are given for the holomorphic sector and those of the antiholomorphic sector are trivially obtainable by conjugation. One has the amusing possibility of doing different transmutations in the left and right sectors, starting from supergravity.

2 Conformally invariant ghost system of 2-D supergravity

The basic fields of 2-D supergravity are a zweibein and its supersymmetric partner, the gravitino. It is possible to extract from these fields objects which only depend on the superconformal classes, called moduli and supermoduli, that is the Beltrami and superBeltrami parameters. In the holomorphic sector one denotes them as $\mu^z_\tau$ and $\alpha^{\theta}_\tau$. On a given Riemann surface, up to a conformal factor, the line element is simply $ds^2 = |dz + \mu^z d\tau|^2$. General super-reparametrizations transform $\mu^z_\tau$ and its supersymmetric partner $\alpha^{\theta}_\tau$ with the rules of a closed algebra, in relation with the superVirasoro algebra. This algebra can thus be expressed as a nilpotent BRST algebra by changing the local diffeomorphism parameter into the anticommuting ghost $c^z$ and the local supersymmetry parameter into the commuting ghost $\gamma^\theta$. The action of its graded differential BRST operator $s$ is defined as [3] [9]

$$s\mu^z_\tau = \partial_\tau c^z + c^z \partial_\tau \mu^z_\tau - \mu^z_\tau \partial_\tau c^z - \frac{1}{2} \alpha^{\theta}_\tau \gamma^\theta$$

$$sc^z = c^z \partial_\tau c^z - \frac{1}{4} \gamma^\theta \gamma^\theta$$
\[
\begin{align*}
&so^\theta = -\partial_z \gamma^\theta + c^z \partial_z \alpha^\theta + \frac{1}{2} (\alpha^\theta \partial_z c^z + \mu^\theta \partial_z \gamma^\theta - \frac{1}{2} \gamma^\theta \partial_z \mu^\theta \\
&s\gamma^\theta = c^z \partial_z \gamma^\theta - \frac{1}{2} \gamma^\theta \partial_z c^z 
\end{align*}
\]  

(2.1)

There is a global $N = 1$ supersymmetry inherent to the supergravity. Let $\sigma$ its generator, defined as

\[
\begin{align*}
\sigma \mu^\theta &= \alpha^\theta \\
\sigma c^z &= \gamma^\theta 
\end{align*}
\]  

(2.2)

The way the symmetries (anti)commute with the space derivatives is $\sigma d + d\sigma = ds + sd = 0$, where $d = dz \partial_z + d\bar{\zeta} \partial_{\bar{\zeta}}$. The consistency of the underlying symmetries implies

\[
s^2 = 0 \quad \sigma^2 = \partial_z \quad \sigma s + s\sigma = 0 \quad (2.3)
\]

These formula can be unified in a superfield formalism. Let us interpret $\theta$ as the single Grassman variable of the $N = 1$ supersymmetry in the holomorphic sector and define the superfields

\[
\begin{align*}
M^z &= dz + \mu^\theta \bar{\zeta} + \theta \alpha^\theta \bar{\zeta} \\
C^z &= c^z + \theta \gamma^\theta 
\end{align*}
\]  

(2.4)

One finds easily that $\sigma$ is the following graded differential operator

\[
\sigma = \partial_\theta + \theta \partial_z 
\]  

(2.5)

and that the BRST equations can be written in superfield notations as

\[
\begin{align*}
&\text{s}M^z = -dC^z + C^z \partial_z M^z + M^z \partial_z C^z - \frac{1}{2} \sigma C^z \sigma M^z \\
&\text{s}C^z = C^z \partial_z C^z - \frac{1}{4} (\sigma C^z)^2 
\end{align*}
\]  

(2.6)

One has a further simplifications in the notation where the ghost number and the form degree are unified in a bigrading (to read the formula, just expand them in form and ghost
number). Indeed the definition of the operator $s$ can be written as the more geometrical equation

$$(d + s)(M^z + C^z) = (M^z + C^z)\partial_z(M^z + C^z) - \frac{1}{4}\sigma(M^z + C^z)\sigma(M^z + C^z) \quad (2.7)$$

Moreover one can also verify

$$(d + s)\sigma(M^z + C^z) = (M^z + C^z)\partial_z\sigma(M^z + C^z) - \frac{1}{2}\sigma(M^z + C^z)\partial_z(M^z + C^z) \quad (2.8)$$

which shows that $\sigma(M^z + C^z)$ has holomorphic weight $\frac{1}{2}$, as it should, since it is the supersymmetric partner of the object $M^z + C^z$ which has weight 1.

3 The ghost action of 2-D supergravity in non trivial background gauges

Consider now the problem of gauge fixing a superconformal invariant action in the gauge in which the supermetric of the worldsheet is conformally set equal to a given background supermetric, with superBeltrami parametrization $\mu^z_{\bar{z}0}$ and $\alpha^\theta_{\bar{z}0}$. In the holomorphic sector, this generates the following Faddeev-Popov action

$$I = \int d^2z \left\{ k_{zz}(\mu_z^z - \mu_{\bar{z}0}^z) + k_{z\theta}(\alpha_{z}^\theta - \alpha_{\bar{z}0}^\theta) \right\} \quad (3.1)$$

$b_{zz}$ and $b_{z\theta}$ are antighosts with the following BRST transformations

$$sb_{zz} = k_{zz} \quad sk_{zz} = 0$$

$$sb_{z\theta} = k_{z\theta} \quad sk_{z\theta} = 0 \quad (3.2)$$

Of course the BRST operator $s$ does not act at this level on the background fields, $s\mu^z_{\bar{z}0} = s\alpha^\theta_{\bar{z}0} = 0$. The action is automatically BRST invariant, since $s^2 = 0$ on all fields. Expanding the action from the definition of $s$ gives

$$I = \int d^2z \left\{ k_{zz}(\mu_z^z - \mu_{\bar{z}0}^z) + k_{z\theta}(\alpha_{z}^\theta - \alpha_{\bar{z}0}^\theta) 
- b_{zz}(\partial_z\bar{c}^z + c^z\partial_z\mu_{\bar{z}0}^z - \mu_{\bar{z}0}^z\partial_z c^z - \frac{1}{2}\alpha_{z}^{\theta}\gamma^{\theta}) 
+ \beta_{z\theta}(-\partial_z\gamma^{\theta} + c^z\partial_z\alpha_{\bar{z}0}^\theta + \frac{1}{2}\alpha_{z}^{\theta}\partial_z c^z + \mu_{\bar{z}0}^z\partial_z\gamma^{\theta} - \frac{1}{2}\gamma^{\theta}\partial_z\mu_{\bar{z}0}^z) \right\} \quad (3.3)$$
Eliminating the Lagrange multipliers $k_{zz}$ and $k_{z\theta}$, one gets $\mu_{\bar{z}z} = \mu_{\bar{z}z_0}$ and $\alpha_{\bar{z}}^\theta = \alpha_{\bar{z}}^{\theta_0}$ and thus

\[ I = \int d^2z \{ -b_{zz}(\partial_z \bar{c}^\gamma + c^z \partial_{\bar{z}} \mu_{\bar{z}z} - \mu_{\bar{z}z_0} \partial_z c^z - \frac{1}{2} \alpha_{\bar{z}z_0}^{\theta} \gamma^\theta) \\
+ \beta_{z\theta}(-\partial_z \gamma^\theta + c^z \partial_{\bar{z}} \alpha_{\bar{z}z_0}^\theta + \frac{1}{2} \alpha_{\bar{z}z_0}^{\theta} \partial_z c^z + \mu_{\bar{z}z_0} \partial_z \gamma^\theta - \frac{1}{2} \gamma^\theta \partial_{\bar{z}} \mu_{\bar{z}z_0}) \} \quad (3.4) \]

In this equation, as a result of the gauge-fixing $\mu_{\bar{z}}$ and $\alpha_{\bar{z}}^\theta$ have been set equal to the classical backgrounds $\mu_{\bar{z}z_0}$ and $\alpha_{\bar{z}}^{\theta_0}$. Only the ghosts and antighosts are dynamical and the BRST symmetry operator $s$ has been changed into $s_0$, which acts now in a non trivial way on the background metric and gravitino, with

\[ I = \int d^2z \left\{ -b_{zz} s_0 \mu_{\bar{z}z_0} + \beta_{z\theta} s_0 \alpha_{\bar{z}}^{\theta_0} \right\} \quad (3.5) \]

where $s_0$ is defined as

\[ s_0 \mu_{\bar{z}z_0} = \partial_z c^z + c^z \partial_{\bar{z}} \mu_{\bar{z}z_0} - \mu_{\bar{z}z_0} \partial_z c^z - \frac{1}{2} \alpha_{\bar{z}z_0}^{\theta} \gamma^\theta \]

\[ s_0 c^z = c^z \partial_{\bar{z}} c^z - \frac{1}{4} \gamma^\theta \gamma^\theta \]

\[ s_0 \alpha_{\bar{z}z_0}^\theta = -\partial_z \gamma^\theta + c^z \partial_{\bar{z}} \alpha_{\bar{z}z_0}^\theta + \frac{1}{2} \alpha_{\bar{z}z_0}^{\theta} \partial_z c^z + \mu_{\bar{z}z_0} \partial_z \gamma^\theta - \frac{1}{2} \gamma^\theta \partial_{\bar{z}} \mu_{\bar{z}z_0} \]

\[ s_0 \gamma^\theta = c^z \partial_{\bar{z}} \gamma^\theta - \frac{1}{2} \gamma^\theta \partial_{\bar{z}} c^z \quad (3.6) \]

and

\[ s_0 b_{zz} = s_0 \beta_{z\theta} = 0 \quad (3.7) \]

The new BRST invariance of the action, $s_0 I = 0$, is obvious from $s_0^2 = 0$.

We define the $\sigma$ supersymmetry transformations of the antighosts as

\[ \sigma \beta_{z\theta} = b_{zz} \quad \sigma b_{zz} = \partial_z \beta_{z\theta} \quad (3.8) \]

which implies that

\[ B_{z\theta} = \beta_{z\theta} + \theta b_{zz} \quad (3.9) \]

is a superfield with $s_0 B_{z\theta} = 0$ and

\[ \sigma I = 0 \quad (3.10) \]
The difference between the operators $s$ and $s_0$ originates in the elimination of the Lagrange multiplier fields. From now on, for the sake of notational simplicity, we will skip the index ”$0$”, keeping in mind that the $\mu^z$ and $\alpha^z$ are background fields defined on the worldsheet.

The following expressions of the action $I$ and the various anticommutation relations of $s_0$ and $\sigma$, similar to 2.3, make obvious its invariances under $s$ and $\sigma$ transformations

$$I = \int d^2z \left( b_{zz} \mu^z + \beta_{z\theta} \alpha^z \right) = \int d^2z \left( s \sigma \beta_{z\theta} \mu^z \right) = \int d^2zd\theta s(B_{z\theta}M^z) \quad (3.11)$$

4 Conformal properties of 2-D supergravity BRST symmetry

We now look for the transformations properties under superconformal transformations of the field system discussed in the previous section.

We define ”holomorphic superconformal transformations” by the action of the following graded generator

$$\delta \mu^z = \partial_z l^z + l^z \partial_z \mu^z - \mu^z \partial_z l^z - \frac{1}{2} \alpha_\theta^\lambda \lambda^\theta$$

$$\delta c^z = l^z \partial_z c^z + c^z \partial_z l^z - \frac{1}{2} \lambda^\theta \gamma^\theta$$

$$\delta \alpha_\theta^z = -\partial_z \lambda^\theta + l^z \partial_z \alpha_\theta^z + \frac{1}{2} \alpha_\theta^\lambda \partial_z l^z + \mu^z \partial_z l^z - \frac{1}{2} \lambda^\theta \partial_z \mu^z$$

$$\delta \gamma^\theta = l^z \partial_z \gamma^\theta - \frac{1}{2} \gamma^\theta \partial_z l^z + c \partial_z \lambda^\theta - \frac{1}{2} \lambda^\theta \partial_z c^z \quad (4.1)$$

To obtain the convenient grading, namely that $\delta$ is odd, we have ”ghostified” the parameters of the transformations, which means that the local diffeomorphism and supersymmetry parameter $l^z$ and $\lambda^\theta$ are respectively anticommuting and commuting.

When $l^z = 0$ and $\lambda^\theta = constant$, the local symmetry defined from $\delta$ reduces to the global symmetry with generator $\sigma$ that we discussed in the previous section. The $\delta$ transformations form a closed algebra. Moreover, by introducing the super-parameter

$$\Lambda^z = l^z + \theta \lambda^\theta \quad (4.2)$$
and using the ghost form degree bigrading, we get for the $\delta$ transformations,

$$
\delta(M^z + C^z) = -d\Lambda^z + \Lambda^z \partial_z (M^z + C^z) + (M^z + C^z) \partial_z \Lambda^z - \frac{1}{2} \sigma \Lambda^z \sigma (M^z + C^z)
$$

(4.3)

The way $\delta$ acts on the BRST transformed fields is instructive. Using

$$
s(M^z + C^z) = -d(M^z + C^z) + (M^z + C^z) \partial_z (M^z + C^z) - \frac{1}{4} \sigma (M^z + C^z) \sigma (M^z + C^z)
$$

(4.4)

one gets after a simple computation

$$
\delta(s(M^z + C^z)) = \Lambda^z \partial_z s(M^z + C^z) + s(M^z + C^z) \partial_z \Lambda^z - \frac{1}{2} \sigma \Lambda^z \sigma s(M^z + C^z)
$$

(4.5)

This equation shows that the background superconformal grade $d$ operator $\delta$ is compatible with the BRST operator $s$, that is, $s$ and $\delta$ anticommute,

$$
\delta s + s \delta = 0
$$

(4.6)

In order that the action $I$ be invariant under the transformations $\delta$, we define the $\delta$ transformation of the antighosts $B_{z\theta} = \beta_{z\theta} + \theta b_{zz}$ as

$$
\delta B_{z\theta} = \Lambda^z \partial_z B_{z\theta} + \frac{3}{2} B_{z\theta} \partial_z \Lambda^z - \frac{1}{2} \sigma B_{z\theta} \sigma \Lambda^z
$$

(4.7)

that is in components

$$
\delta b_{zz} = l^z \partial_z b_{zz} - 2b_{zz} \partial_z l^z + \frac{3}{2} b_{zz} \partial_z \lambda^\theta + \frac{1}{2} \beta_{z\theta} \partial_z \lambda^\theta
$$

$$
\delta \beta_{z\theta} = l^z \partial_z \beta_{z\theta} + \frac{3}{2} \beta_{z\theta} \partial_z l^z - \frac{1}{2} b_{zz} \lambda^\theta
$$

(4.8)

Indeed, this definition implies

$$
\delta(s(B_{z\theta} M^z)) = \partial_z (\ldots) + \sigma (\ldots)
$$

(4.9)

To simplify the formulae, one could interpret $\Lambda^z$ as a superghost. In this way the differential operator $\delta$ becomes a background BRST operator, with its own ghost number, and one gets the unified equation $(d + s + \delta)(M^z + C^z + \Lambda^z) = (M^z + C^z + \Lambda^z) \partial_z (M^z + C^z + \Lambda^z) - \frac{1}{4} \sigma (M^z + C^z + \Lambda^z)^2$, which makes particularly easy the demonstration of most formulae.
which proves the $\delta$ invariance of the supergravity ghost action

$$\delta I = \delta \int d^2zd\theta s(B_\theta M^z) = 0 \quad (4.10)$$

Another way to see the possibility of the $\delta$ invariance of the action is to examine $I = \int d^2z s(b_{zz}(\mu_z-\mu_{z0})+\beta_z(\alpha^\theta_z-\alpha^\theta_{z0}))$ before the elimination of the Lagrange multipliers which are the BRST transformed of the antighosts. Since $\mu_z$ and $\mu_{z0}$ on the one hand, $\alpha^\theta_z$ and $\alpha^\theta_{z0}$ on the other hand, transform identically under $\delta$, the form of $\delta(b_{zz}(\mu_z-\mu_{z0})+\beta_z(\alpha^\theta_z-\alpha^\theta_{z0})) = \partial_z(\ldots)$, which implies the $\delta$ invariance of the action. The transformation laws under $\delta$ of the Lagrange multipliers must correspond to those of the antighosts. Thus, all relations of the type 2.3 can be extended in the antighost sector.

The Ward identity of the $\delta$ invariance permits a direct derivation of the expression of supersymmetry and energy momentum tensors $G_{z\theta}$ and $T_{zz}$, defined as

$$T_{zz} = \frac{\delta I}{\delta \mu_z} \quad G_{z\theta} = \frac{\delta I}{\delta \alpha^\theta_z} \quad (4.11)$$

The $\delta$ invariance of the action means

$$\int d^2z(T_{zz}\delta \mu_z + G_{z\theta}\delta \alpha^\theta_z + \frac{\delta I}{\delta b_{zz}}\delta b_{zz} + \frac{\delta I}{\delta c^z}\delta c^z + \frac{\delta I}{\delta \beta_z}\delta \beta_z + \frac{\delta I}{\delta \gamma^\theta}\delta \gamma^\theta) = 0 \quad (4.12)$$

This equation can be separated in two identities, corresponding to independent values of the parameters $l^z$ and $\lambda^\theta$. Up to the equations of motion of propagating fields, that is of the ghosts, 4.12 gives

$$\int d^2z(T_{zz}\delta \mu_z + G_{z\theta}\delta \alpha^\theta_z) = 0 \quad (4.13)$$

Inserting the expression of $\delta \mu_z$ and $\delta \alpha^\theta_z$ and using the fact that one has identities true for all possible values of the parameters $l^z$ and $\lambda^\theta$, we obtain

$$(\partial_z - \mu_z^z\partial_z - 2\partial_z\mu_z^z)T_{zz} - (\frac{1}{2}\alpha^\theta_z\partial_z + \frac{3}{2}\partial_z\alpha^\theta_z)G_{z\theta} = 0$$

$$(\partial_z - \mu_z^z\partial_z - \frac{3}{2}\partial_z\mu_z^z)G_{z\theta} - \frac{1}{2}\alpha^\theta_zT_{zz} = 0 \quad (4.14)$$

These Ward identities, valid in the presence of general values of the background metric $\mu_z^z$ and gravitino $\alpha^\theta_z$ of the worldsheet, are the covariant generalization of the analyticity conditions of $T_{zz}$ and $G_{z\theta}$ in the conformal gauge $\mu_z^z = \alpha^\theta_z = 0$. 

10
In superfield notations, the "super energy-momentum tensor" is

\[ \tilde{T}_{z\theta} = G_{z\theta} + \theta T_{zz} \]  \hspace{1cm} (4.15)

One easily verifies

\[ \tilde{T}_{z\theta} = \frac{\delta L}{\delta \mu^z_\theta} \int d^2zd\theta(B_{z\theta} sM^z) = C^z \partial_z B_{z\theta} + \frac{3}{2} B_{z\theta} \partial_z C^z - \frac{1}{2} \sigma C^z \sigma B_{z\theta} \]  \hspace{1cm} (4.16)

and another way to write the Ward identity (4.14) is

\[ (\partial_z - M^z \partial_z - \frac{3}{2} \partial_z M^z - \frac{1}{2} \sigma M^z \sigma) \tilde{T}_{z\theta} = 0 \]  \hspace{1cm} (4.17)

Since the action is linear in \( \mu^z_\theta \) and \( \alpha^{\theta z} \), \( T_{zz} \) and \( G_{z\theta} \) are independent on these fields.

Let us check that our definitions truly give the known definitions of \( T_{zz} \) and \( G_{z\theta} \) in the conformal gauge for which \( \mu^z_\theta = \alpha^{\theta z} = 0 \). One has indeed

\[ T_{zz} = \frac{\delta L}{\delta \mu^z_\theta} = \int d^2z(b_{zz} \frac{\delta}{\delta \mu^z_\theta} s\mu^z_\theta + \beta_{z\theta} \frac{\delta}{\delta \mu^z_\theta} s\alpha^{\theta z}) = c^z \partial_z b_{zz} - 2b_{zz} \partial_z c^z + \frac{1}{2} \gamma^\theta \partial_z \beta_{z\theta} + \frac{3}{2} \beta_{z\theta} \partial_z \gamma^\theta \]  \hspace{1cm} (4.18)

and

\[ G_{z\theta} = \frac{\delta L}{\delta \alpha^{\theta z}} = G_{z\theta}^+ + G_{z\theta}^- \]  \hspace{1cm} (4.19)

where

\[ G_{z\theta}^+ = \int d^2zb_{zz} \frac{\delta}{\delta \alpha^{\theta z}} s\mu^z_\theta = \frac{1}{2} b_{zz} \gamma^\theta \]

\[ G_{z\theta}^- = \int d^2z\beta_{z\theta} \frac{\delta}{\delta \alpha^{\theta z}} s\alpha^{\theta z} = \frac{1}{2} \beta_{z\theta} \partial_z c^z + \partial_z(\beta_{z\theta} c^z) \]  \hspace{1cm} (4.20)

If one switches to a classical Hamiltonian formalism, the form of the action indicates that the antighost field is the conjugate momenta of the ghost field operator. Moreover, the action of a conformal super-reparametrization \( \delta \) with super-parameter \( \Lambda^z \) on any given dynamical field \( X \) can be written as \( \delta X = \frac{1}{2\pi i} \{ \oint dzd\theta \Lambda^z \tilde{T}_{z\theta}, X \}_+ \), where the anti-bracket \( \{ , \}_+ \) occurs if \( X \) has an odd grading. The group structure of the transformations \( \delta \) implies the anticommutation relation

\[ \{ \oint dzd\theta \Lambda^z \tilde{T}_{z\theta}, \oint dzd\theta \Lambda^{z'} \tilde{T}_{z\theta} \}_+ = \oint dzd\theta(\Lambda^z \partial_z \Lambda^{z'} + \frac{3}{2} \Lambda^z \partial_z \Lambda^{z'} - \frac{1}{2} \sigma \Lambda^z \sigma \Lambda^{z'}) \tilde{T}_{z\theta} \]  \hspace{1cm} (4.21)
where $\Lambda^z = l^z + \theta \Lambda^z$ and $\Lambda^{z'} = l^{z'} + \theta \Lambda^{z'}$ stand for the super-parameters of two $\delta$ transformations.

One may decompose the last equation by projection over the various possibilities of the component content of the super-parameters $\Lambda^z$. This gives the graded commutation relations of the holomorphic N=1 superconformal algebra

$$\{T_{zz}, T_{zz}\} \sim T_{zz} \quad \{G_{z\theta}, T_{zz}\} \sim G_{z\theta} \quad \{G_{z\theta}, G_{z\theta}\}_+ \sim T_{zz} \quad (4.22)$$

where we do not make explicit the structure coefficients for the sake of notational simplicity. Since $\{G_{z\theta^+}, G_{z\theta^+}\}_+ = \{G_{z\theta^-}, G_{z\theta^-}\}_+ = 0$, one has

$$\{G_{z\theta^-}, G_{z\theta^+}\}_+ \sim T_{zz} \quad (4.23)$$

For any given value of the background fields $\mu^z$ and $\alpha^\theta$, one has an obviously classically conserved abelian current, the ghost current

$$J_{\text{ghost}}^z = b_{zz} c^z + \beta_{z\theta} \gamma^\theta \quad (4.24)$$

However, for $\alpha^\theta = 0$ one can observe that the two currents $b_{zz} c^z$ and $\beta_{z\theta} \gamma^\theta$ are separately conserved due to the vanishing of mixing terms in the invariant action, $b_{zz} \alpha_{z^\theta} \gamma^\theta$ and $\beta_{z\theta}(c^\theta \partial_z \alpha_{z^\theta} + \frac{1}{2} \alpha_{z^\theta} \partial_z c^\theta)$. Moreover, the nilpotent transformation $\sigma^+$ associated to $G_{z\theta^+}$, namely

$$\sigma^+ c^z = \gamma^\theta \quad \sigma^+ \gamma^\theta = 0$$

$$\sigma^+ \beta_{z\theta} = b_{zz} \quad \sigma^+ b_{zz} = 0 \quad (4.25)$$

that is $\sigma^+ = \partial_\theta$ in superfield notation, is a symmetry of the action if and only if

$$\alpha^\theta = 0 \quad \partial_z \mu_{z^\theta} = 0 \quad (4.26)$$

But in this case the action is also invariant under the action of $G_{z\theta^-}$ since it is generally invariant under the fundamental symmetry associated to $G_{z\theta} = G_{z\theta^+} + G_{z\theta^-}$. Thus, when the background gauge is restricted as in (4.26) and in particular in the conformal gauge case $\mu_{z^\theta} = 0$, one has two supersymmetries of the action associated to the generators
$G_{z\theta}^+$ and $G_{z\theta}^-$. This gives the known $N = 2$ superconformal supersymmetry, for which $J_z^+ = b_{zz}c^z$ or $J_z^- = b_{zz}\gamma^\theta$ plays the role of the abelian part, while $G_{z\theta}^+$ and $G_{z\theta}^-$ are the two fermionic generators [2]. From our point of view this symmetry appears as rather accidental, the basic symmetry being the one associated to $G_{z\theta}$, $T_{zz}$ and $J_{z\theta}^{ghost}$.

After quantization of the action, which means either doing path integral over the dynamical fields or changing the fields into operators and Poisson bracket into commutators, the anomaly can be understood as a consistent term generated by loop corrections which can be substituted to zero at the right hand side of the superconformal Ward identity 4.14 or 4.17. After a quick look to the structure equations one finds that the consistency of the symmetry equations implies that the Ward identity 4.14 or 4.17 can be made anomalous only under the following form

$$(\partial_z - M^z \partial_z - \frac{3}{2} \theta \partial_z M^z - \frac{1}{2} \sigma M^z) \tilde{T}_{z\theta} = c \partial_z (\partial_\theta + \theta \partial_\theta) \partial_z M^z$$

$$= c (\partial_z^2 \alpha_z^\theta + \theta \partial_z^3 \mu_z)$$

(4.27)

The value of the coefficient $c$ of the anomaly depends on the explicit form of $T_{zz}$. It must be computed at the quantum level, using one of the many methods available. One gets $c = -26 + 11 = -15$ for the case of the 2-D supergravity ghost action.

5 Transmutations to other conformal theories

The ghost action which stems from the conformal gauge fixing to a non trivial background structure of 2-D supergravity possesses the super-reparametrization invariance in addition to its BRST invariance. It is thus conceivable to think of other conformal theories that one would obtain by introducing new fields which are composites of the original ghosts of the supergravity, obtained by products or more subtle combinations of these fields, with the possibility of a dependence on the backgrounds fields $\mu_z^z$ and $\alpha_z^\theta$, so that one has a priori many options to choose from for the holomorphic weights as well as the statistics. The new fields would be well defined under the local background super-reparametrization invariance since $\delta$ acts as differential operator. The theories stemming
from BRST invariant actions built from these fields would also have conformal properties, since \( s \) and \( \delta \) anticommute. The values of their anomaly coefficients would differ, since the weights of the fields would be different, but their Lagrangians would be related by canonical changes of field variables, as well as their classical energy momentum tensors, BRST Noether currents and all other conserved currents.

To understand this construction, we will explain how the action of the topological 2-D gravity in the type of gauge used by Labastida Pernicci and Witten [3] is simply related, through a canonical change of variable, to the one of 2-D supergravity discussed in the previous section. Let us redefine

\[
\begin{align*}
\Psi^z &= -\frac{1}{2} \gamma^\theta \alpha^\theta \\
\Phi^z &= \frac{1}{4} \gamma^\theta \gamma^\theta \\
\Phi_{zz} &= -2 \beta z \gamma^\theta \gamma^{\theta^{-1}}
\end{align*}
\]  

(5.1)

while we keep unchanged \( b_{zz} \) and \( c^z \). It is easy to check that this change of variable is canonical, which means that the Poisson bracket of the redefined fields \( \Phi_{zz} \) and \( \Phi^z \) is equal to that of the conjugate fields \( \beta_{z \theta} \) and \( \gamma^\theta \).

To determine the BRST algebra of the redefined fields, we insert in the basic supergravity BRST equations 2.7 and 2.8 the change of variable 5.1, observing that it means

\[
\begin{align*}
(d + s)(dz + \mu^z d\bar{z} + c^z) &= (dz + \mu^z d\bar{z} + c^z) \partial_z (dz + \mu^z d\bar{z} + c^z) + \Psi^z + \Phi^z \\
(d + s)(\Psi^z + \Phi^z) &= (dz + \mu^z d\bar{z} + c^z) \partial_z (\Psi^z + \Phi^z) + (\Psi^z + \Phi^z) \partial_z (dz + \mu^z d\bar{z} + c^z)
\end{align*}
\]  

(5.2)

Expanded in ghost number, these equations mean

\[
\begin{align*}
\mu^z &= \Psi^z + \partial_z c^z + c^z \partial_z \mu^z - \mu^z \partial_z c^z \\
c^z &= \Phi^z + c^z \partial_z c^z \\
\Psi^z &= -\partial_z \Phi^z + \Phi^z \partial_z \mu^z - \mu^z \partial_z \Phi^z + c^z \partial_z \Psi^z + \Psi^z \partial_z c^z \\
\Phi^z &= c^z \partial_z \Phi^z - \Phi^z \partial_z c^z
\end{align*}
\]  

(5.3)
One recognizes the BRST operator of topological 2-D gravity, as expressed in [6]. Thus the BRST symmetry of 2-D supergravity has been transformed into that of topological 2-D gravity by the change of variables [5]. It is quite interesting that the combination of the anticommuting physical gravitino $\alpha_{\theta z}$ with its commuting ghost $\gamma_{\theta}$ produces the anticommuting ghost $\Psi_{z}$, that is the topological ghost partner of the Beltrami variable $\mu_{z}$, while the square of $\gamma_{\theta}$ produces the topological ghost of ghost $\Phi_{z}$.

Consider now the BRST invariant action

$$I_{\text{top}} = \int d^{2}z \{ b_{zz}(\mu_{z}^{2} - \mu_{z0}^{2}) + \Psi_{z}(\Psi_{z} + \frac{1}{2} \alpha_{\tau z0} \gamma_{\theta}) \}$$

(5.4)

The action of $s$ on the antighosts is

$$sb_{zz} = \lambda_{zz} \quad s\lambda_{zz} = 0$$

$$s\Phi_{zz} = \eta_{zz} \quad s\eta_{zz} = 0$$

(5.5)

Expanding the action $I_{\text{top}}$ we see that $\lambda_{zz}$ plays the role of a commuting Lagrange multiplier for the condition $\mu_{z}^{2} = \mu_{z0}^{2}$ while $b_{zz}$ is an anticommuting Lagrange multiplier for the condition $\Psi_{z} = -\partial_{z}c^{z} - c^{z}\partial_{z}\mu_{z0}^{2} + \mu_{z0}^{2}\partial_{z}c^{z} + \frac{1}{2} \alpha_{\tau z0} \gamma_{\theta}$. Thus, only the second term of $I_{\text{top}}$ gives rise to a dynamical part. If furthermore we choose the background gravitino $\alpha_{\tau z0} = 0$, what remains of the action after the elimination of the Lagrange multiplier fields is

$$I_{\text{top}} = \int d^{2}z \{ -\frac{1}{2} \alpha_{\tau z0} \gamma_{\theta} \}$$

(5.6)

It is easy to verify that $I_{\text{top}}$ is invariant under the background symmetry $\delta$. The expression of $\delta$ can be obtained from that we derived in 2-D supergravity, eqs. [11] 4.8 through our changes of variables. The transformations laws of the redefined fields under $\delta$ permits of course to verify that their conformal weights are truly what is indicated by our indices.

As far as its interpretation is concerned, the action $I_{\text{top}}$ is of the Labastida-Pernici-Witten type [3], expressed in the Beltrami parametrization. It can be considered as a
holomorphic gauge fixed version of the topological invariant \( \int d^2x \sqrt{g} R \), by mean of the
gauge condition \( \mu \tilde{z} = \mu_{\tilde{z}0} \). Before quantization, the energy momentum tensor, can be
equivalently computed by differentiation of the topological gravity action \[5.6\] with respect
to \( \mu_{\tilde{z}0} \), or by doing the changes of field variables \[5.1\] in the energy momentum tensor of
2-D supergravity \[4.18\]. If the fields are identified, the two energy momentum tensors
differ by a term of the type \( \partial_{\tilde{z}} (ab_{zz} \gamma^\theta + a' \beta_{zz} c^z) \). The BRST symmetry Noether currents
of both theories are also related by the canonical changes of variables. One still has the
Ward identities \[4.14\] with the same possible anomalous term as in \[4.27\]. The value of the
anomaly coefficient \( c \), not predicted from classical considerations, is now zero, due to an
exact compensation between the contributions of commuting and anticommuting ghosts.
The discussion about the \( N = 2 \) conformal supersymmetry of the redefined action can
be repeated exactly as in the last section, when the background is such that \( \mu_{\tilde{z}0} = 0 \).

The mechanism presented here, which allows the change of the holomorphic weight
and statistics of the propagating fields, by mean of canonical field redefinition involving
the gravitino and its ghost, that is the spin structure of the worldsheet, indicates a possibly
deep relationship between two actions with different physical interpretations. The canonical
changes on field variables, \((\eta_{zz}, c^z) = (b_{zz}, c^z)\) and \((\Phi_{zz}, \Phi^z) = (-2 \beta_{zz} \gamma^\theta^{-1}, -\frac{1}{4} \gamma^\theta \gamma^\theta)\)
connects the symmetries, the field equations, the energy momentum tensor and other
conserved currents of both actions. On the other hand, the anomaly coefficient, whose
computation relies on quantum effects changes. It is intriguing to find out whether the
mechanism which governs the change of the values of the anomaly coefficient is linked to
an ordering problem or to the singularity occurring at \( \gamma^\theta = 0 \) in the field redefinitions.

Having seen the existence of the transmutation mechanism on a given example, we
may go further and consider the possibility of introducing other \( b - c \) systems, the root
being the ghost system of 2-D supergravity that one transforms by canonical field redefi-
nitions. We introduce at the classical level a pair of new fields by the following canonical
change of variables

\[
\exp -\frac{L}{a} = \Phi^z + c^z \partial_z c^z
\]
\[ M = -\frac{\Phi_{zz}}{a} \exp -\frac{L}{a} \]  

(5.7)

\( a \) is an arbitrarily fixed real number. In terms of the redefined fields, the action is

\[ I_{\text{top}} = \int d^2 z \{ \varpi_{zz}(\partial_z \mu_{\varpi_0} \partial_z + \partial_z \mu_{\varpi_0}) c^z - M(\partial_z L - \mu_{\varpi_0} \partial_z L - a \partial_z \mu_{\varpi_0}) \} \]  

(5.8)

The BRST symmetry of the action has been changed into

\[
egin{align*}
\delta L &= l^z \partial_z L - a \partial_z l^z \\
\delta M &= \partial_z (l^z M)
\end{align*}
\]  

(5.10)

From a mathematical point of view, the meaning of the fields \( L \) and \( M \) is not clear when \( a \) is not integer or half integer. However, we have the background conformal symmetry

\[
egin{align*}
\delta L &= l^z \partial_z L - a \partial_z l^z \\
\delta M &= \partial_z (l^z M)
\end{align*}
\]  

(5.10)

which leaves invariant the action. We may think to define the nature of the fields from these equations. Moreover, the physical meaning of the action 5.8 is quite clear. The ghost part \( \pi_{zz}(\partial_z \mu_{\varpi_0} \partial_z + \partial_z \mu_{\varpi_0}) c^z \) means that the Beltrami parameter of the worldsheet has been set equal to the background value \( \mu_{\varpi_0} \) in a BRST invariant way. Interpreting the field \( M \) as a Lagrange multiplier, the part \( M((\partial_z \mu_{\varpi_0} \partial_z) L - a \partial_z \mu_{\varpi_0}) \) means that one has a field \( L \) which satisfies the constraint (5.8)

\[
(\partial_z - \mu_{\varpi_0} \partial_z) L = a \partial_z \mu_{\varpi_0}
\]  

(5.11)

We have the following property

\[
s(\partial_z L(\partial_z - \mu_{\varpi_0} \partial_z) L - 2 \partial_z \mu_{\varpi_0} \partial_z L) = a \partial_z c^z \partial_z^2 \mu_{\varpi_0}
\]  

(5.12)
which holds true as a consequence of (5.3), independently, of the constraint \((\partial_\xi - \mu_{\xi_0} \partial_\xi) L = a \partial_\xi \mu_{\xi_0} \). If, moreover, this constraint is satisfied, we have
\[
s(\partial_\xi \mu_{\xi_0} \partial_\xi L) = -a \partial_\xi c^2 \partial_\xi^2 \mu_{\xi_0} \tag{5.13}\]
Thus, the constrained field \(L\) can be used to compensate the conformal anomaly, since the gauge variation of the action \(\partial_\xi \mu_{\xi_0} \partial_\xi L\) is proportional to the conformal anomaly \(\partial_\xi c^2 \partial_\xi^2 \mu_{\xi_0}\). Moreover, at the quantum level, we have an admissible invariant counterterm, of the cosmological term type, which can be added to the action,
\[
\mathcal{I}_{ct} = cte \int d^2 z \ exp \left( - \frac{L}{a} \right) \tag{5.14}
\]
since the \(s\) and \(\delta\) variation of the integrand is a pure derivative. All these properties indicate that we may call the field \(L\) ”half a Liouville field” as in [9].

To summarize, through our successive changes of field variables, we have reached the action
\[
\mathcal{I}_{WZ} = \int d^2 z \{ \eta_{zz} (\partial_\xi - \mu_{\xi_0} \partial_\xi + \partial_\xi \mu_{\xi_0}) c^2 ) \\
- M( (\partial_\xi - \mu_{\xi_0} \partial_\xi) L - a \partial_\xi \mu_{\xi_0} ) + b \partial_\xi \mu_{\xi_0} \partial_\xi L \} \tag{5.15}
\]
where \(b\) is an arbitrarily given real number, independent of \(a\). The energy momentum tensor is
\[
T_{zz} = \frac{\delta \mathcal{I}_{WZ}}{\delta \mu_{\xi_0}} = -\eta_{zz} \partial_\xi c^2 - \partial_\xi (\eta_{zz} \partial_\xi c^2 ) + M \partial_\xi L - a \partial_\xi M + b \partial_\xi^2 L \tag{5.16}
\]
An easy computation shows that the contribution of all the fields to the anomaly coefficient is
\[
c = -26 + 12ab + 1 \tag{5.17}
\]
If one is interested to get a model with a vanishing anomaly coefficient, \(c = 0\), there are many choices for the values of the pairs \(a, b\). All possible values of \(a\) are admissible, provided
\[
b = \frac{25}{12a} \tag{5.18}
\]
As far as observables are concerned, one relies on the criteria of BRST invariance for their selection \[\text{[10]}\]. But since the BRST symmetry is the same in all these models, up to field redefinitions, we expect a universality in the definition of the products of fields of which one should take the expectation values. As an example, in the topological phase, we have the BRST-exact cocycle \(\Phi^z + c^z \partial_z c^z\) with ghost number two as an observable, which corresponds in the Liouville phase to the BRST invariant "cosmological term" \(\exp - \frac{L}{\alpha}\), through the redefinition \[\text{5.7}\]. Notice that in the topological phase a supersymmetry breaking mechanism should occur in order that \(\langle \Phi^z + c^z \partial_z c^z \rangle\) be not zero, due to its ghost number. The values of the expectation values of these physical operators, as well as the methods of computation, should differ when one goes from one model to the other. It is quite interesting that the reduced cohomology introduced at the algebraic level in \[\text{[11]}\], can be directly deduced from the BRST charge corresponding to the action \[\text{5.15}\].

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