THE DYNKIN INDEX AND \( \mathfrak{sl}_2 \)-SUBALGEBRAS OF SIMPLE LIE ALGEBRAS

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INTRODUCTION

The ground field \( \mathbb{k} \) is algebraically closed and of characteristic zero. Let \( G \) be a connected semisimple algebraic group with Lie algebra \( \mathfrak{g} \). In 1952, Dynkin classified all semisimple subalgebras of semisimple Lie algebras \([2]\). As a tool to distinguish different (non-conjugate) embeddings of the same algebra, Dynkin introduced the index of a homomorphism of simple Lie algebras. It will be convenient for us to split this into the notions of (1) the index of a simple subalgebra of a simple Lie algebra and (2) the index of a representation of a simple Lie algebra. After Mal’cev and Kostant, it is known that the conjugacy classes of the \( \mathfrak{sl}_2 \)-subalgebras of \( \mathfrak{g} \) are in a one-to-one correspondence with the nonzero nilpotent \( G \)-orbits in \( \mathfrak{g} \). Therefore, one can define the index of a nilpotent element (orbit) as the Dynkin index of any associated \( \mathfrak{sl}_2 \)-subalgebra. As nilpotent orbits are related to the variety of intriguing problems in representation theory, the indices of \( \mathfrak{sl}_2 \)-subalgebras of \( \mathfrak{g} \) are most interesting for us. A simple Lie algebra has three distinguished nilpotent orbits: the principal (regular), subregular, and the minimal ones. It was noticed by Dynkin that in the last case the corresponding \( \mathfrak{sl}_2 \)-index equals 1 (cf. \([2, \text{Theorem 2.4}]\)). In \([9]\), we gave a general formula for the index of a principal \( \mathfrak{sl}_2 \)-subalgebra of \( \mathfrak{g} \).

This note can be regarded as a continuation of \([9]\). Here we provide simple formulae for the index of all nilpotent orbits (\( \mathfrak{sl}_2 \)-subalgebras) in the classical Lie algebras (Theorem 2.1) and a new formula for the index of the principal \( \mathfrak{sl}_2 \) (Theorem 3.2). Then we compute the difference, \( D \), of the indices of principal and subregular \( \mathfrak{sl}_2 \)-subalgebras. Our formula for \( D \) involves some data related to the McKay correspondence for \( \mathfrak{g} \), see Theorem 3.4 and Eq. (3.3). The index of a simple subalgebra \( s \) of \( \mathfrak{g} \), \( \text{ind}(s \hookrightarrow \mathfrak{g}) \), can be computed via any non-trivial representation of \( \mathfrak{g} \), and taking different representations of \( \mathfrak{g} \), one gets different formal expression for \( \text{ind}(s \hookrightarrow \mathfrak{g}) \). For \( s \simeq \mathfrak{sl}_2 \) and classical \( \mathfrak{g} \), we obtain essentially different formulae using the simplest and adjoint representations of \( \mathfrak{g} \), and the Jordan normal form of nonzero nilpotent elements of \( s \). This yields three series of interesting combinatorial identities parameterised by partitions, see Section 2.1. We also prove that the index of a nilpotent orbit strictly decreases under the passage to the boundary of orbits (Proposition 2.2).
1. The Dynkin Indices of Representations and Subalgebras

Let \( g \) be a simple finite-dimensional Lie algebra of rank \( n \). Let \( t \) be a Cartan subalgebra, and \( \Delta \) the set of roots of \( t \) in \( g \). Choose a set of positive roots \( \Delta^+ \) in \( \Delta \). Let \( \Pi \) be the set of simple roots and \( \theta \) the highest root in \( \Delta^+ \). As usual, \( \rho = \frac{1}{2} \sum_{\gamma > 0} \gamma \). The \( \mathbb{Q} \)-span of all roots is a \( \mathbb{Q} \)-subspace of \( t^* \), denoted \( E \). Following Dynkin, we normalise a non-degenerate invariant symmetric bilinear form \(( \cdot, \cdot )_g\) on \( g \) as follows. The restriction of \(( \cdot, \cdot )_g\) to \( t \) is non-degenerate, hence it induces the isomorphism of \( t \) and \( t^* \) and a non-degenerate bilinear form on \( E \). We then require that \((\theta, \theta)_g = 2\), i.e., \((\beta, \beta)_g = 2\) for any long root \( \beta \) in \( \Delta \).

**Definition 1** (Dynkin [2, §2]). Let \( \phi : s \to g \) be a homomorphism of simple Lie algebras. For \( x, y \in s \), the bilinear form \((x, y) \mapsto (\phi(x), \phi(y))_g\) is proportional to \((\cdot, \cdot)_s\) and the index of \( \phi \) is defined by the equality \((\phi(x), \phi(y))_g = \text{ind}(s \xrightarrow{\phi} g) \cdot (x, y)_s\), \( x, y \in s \).

- In particular, if \( s \) is a simple subalgebra of \( g \), then the **Dynkin index of \( s \) in \( g \)** is \[
\text{ind}(s \hookrightarrow g) := \frac{(x, x)_g}{(x, x)_s}, \quad x \in s.
\]

- If \( \nu : g \to \mathfrak{sl}(M) \) is a representation of \( g \), then the **Dynkin index of the representation \( \nu \)**, denoted \( \text{ind}_D(g, M) \) or \( \text{ind}_D(g, \nu) \), is defined by

\[
(1.1) \quad \text{ind}_D(g, M) := \text{ind}(g \xrightarrow{\nu} \mathfrak{sl}(M)).
\]

It is not hard to verify that, for the simple Lie algebra \( \mathfrak{sl}(M) \), the normalised bilinear form is given by \((x, x)_{\mathfrak{sl}(M)} = \text{tr}(x^2), \ x \in \mathfrak{sl}(M)\). Therefore, a more explicit expression for the Dynkin index of a representation \( \nu : g \to \mathfrak{sl}(M) \) is

\[
\text{ind}_D(g, M) = \frac{\text{tr}(\nu(x)^2)}{(x, x)_g}.
\]

The following properties easily follow from the definition:

**Multiplicativity**: If \( h \subset s \subset g \) are simple Lie algebras, then

\[
\text{ind}(h \hookrightarrow s) \cdot \text{ind}(s \hookrightarrow g) = \text{ind}(h \hookrightarrow g).
\]

**Additivity**: \( \text{ind}_D(g, M_1 \oplus M_2) = \text{ind}_D(g, M_1) + \text{ind}_D(g, M_2) \).

It is therefore sufficient to determine \( \text{ind}_D(g, \cdot) \) for the irreducible representations.

**Theorem 1.1** (Dynkin, [2, Theorem 2.5]). Let \( V_\lambda \) be a simple finite-dimensional \( g \)-module with highest weight \( \lambda \). Then

\[
\text{ind}_D(g, V_\lambda) = \frac{\dim V_\lambda}{\dim g} (\lambda, \lambda + 2\rho)_g.
\]
Although it is not obvious from the definition, the Dynkin index of a homomorphism is an integer [2, Theorem 2.2]. Dynkin’s original proof relied on classification results. In 1954, he gave a better proof that is based on a topological interpretation of the index. A short algebraic proof is given in [8, Ch. I, §3.10].

Conversely, the index of a simple subalgebra can be expressed via indices of representations. By the multiplicativity of index and Eq. (1.1), for a simple subalgebra \( s \subset g \) and a non-trivial representation \( \nu : g \to \mathfrak{sl}(M) \), we have

\[
\text{ind}(s \hookrightarrow g) = \frac{\text{ind}(s \hookrightarrow \mathfrak{sl}(M))}{\text{ind}(g \hookrightarrow \mathfrak{sl}(M))} = \frac{\text{ind}_D(s, M)}{\text{ind}_D(g, M)}.
\]

A nice feature of this formula is that one can use various \( M \) to compute the index of a given subalgebra.

Example 1.2.
(1) Let \( \mathcal{R}_d \) be the simple \( \mathfrak{sl}_2 \)-module of dimension \( d + 1 \). Then \( \text{ind}_D(\mathfrak{sl}_2, \mathcal{R}_d) = \left( \frac{d+2}{3} \right) \).

(2) Recall that \( \theta \) is the highest root in \( \Delta^+ \). By Theorem 1.1,

\[
\text{ind}_D(g, ad_g) = (\theta, \theta + 2\rho)_g = (\theta, \theta)_g(1 + (\rho, \theta^\vee)_g) = 2(1 + (\rho, \theta^\vee)_g).
\]

Note that \( (\rho, \theta^\vee)_g \) does not depend on the normalisation of the bilinear form on \( \mathcal{E} \). The integer \( 1 + (\rho, \theta^\vee)_g \) is customarily called the dual Coxeter number of \( g \), and we denote it by \( h^*(g) \). Thus, \( \text{ind}_D(g, ad_g) = 2h^*(g) \). In the simply-laced case, \( h^*(g) = h(g) \)—the usual Coxeter number. For the other simple Lie algebras, we have \( h^*(B_n) = 2n - 1, h^*(C_n) = n + 1, h^*(F_4) = 9, h^*(G_2) = 4 \). Applying this to Eq. (1.2) with \( M = g \) and \( \nu = ad_g \), we obtain

\[
\text{ind}(s \hookrightarrow g) = \frac{1}{2h^*(g)} \cdot \text{ind}_D(s, g).
\]

More generally, we have

Lemma 1.3. If \( s \subset g \) are simple Lie algebras and \( \nu : g \to \mathfrak{sl}(M) \) is a representation, then

\[
\text{ind}_D(s, M) = \frac{1}{2h^*(g)} \cdot \text{ind}_D(s, g) \cdot \text{ind}_D(g, M).
\]

Proof. By the multiplicativity and Eq. (1.3), we have

\[
\text{ind}_D(s, M) = \text{ind}(s \hookrightarrow g) \cdot \text{ind}(g \hookrightarrow \mathfrak{sl}(M)) = \frac{1}{2h^*(g)} \cdot \text{ind}_D(s, g) \cdot \text{ind}_D(g, M).
\]

Remark 1.4. The “strange formula” of Freudenthal-de Vries relates the scalar square of \( \rho \) with \( \dim g \). If \( \langle , \rangle \) is the canonical bilinear form on \( \mathcal{E} \) with respect to \( \Delta \), then \( \langle \rho, \rho \rangle = \dim g/24 \) [3, 47.11]. The canonical bilinear form is characterised by the property that \( \langle \gamma, \gamma \rangle = 1/h^*(g) \) for a long root \( \gamma \in \Delta \). It follows that if \( \langle , \rangle \) is any nonzero \( W \)-invariant bilinear form on \( \mathcal{E} \) and \( \langle \gamma, \gamma \rangle = c \), then \( \langle \rho, \rho \rangle = \frac{\dim g}{24} h^*(g)c \).
2. THE INDEX OF $\mathfrak{sl}_2$-SUBALGEBRAS AND COMBINATORIAL IDENTITIES

If $e \in \mathfrak{g}$ is nonzero and nilpotent, then there exists a subalgebra $a \subset \mathfrak{g}$ such that $a \simeq \mathfrak{sl}_2$ and $e \in a$ (Morozov, Jacobson)[1, 3.3]. All $\mathfrak{sl}_2$-subalgebras associated with a given $e$ are $G_e$-conjugate and we write $A_1(e)$ for such a subalgebra. In this section, we give explicit formulae for the indices $\text{ind}(A_1(e) \hookrightarrow \mathfrak{g})$ and some applications.

Let $\mathfrak{g}(V)$ be a classical simple Lie algebra (i.e., one of $\mathfrak{sl}(V)$, $\mathfrak{sp}(V)$, $\mathfrak{so}(V)$). The nilpotent elements (orbits) in $\mathfrak{g}(V)$ are parameterised by partitions of $\dim V$, and we give the formulae in terms of partitions. For $e \in \mathfrak{g}(V)$, let $\lambda(e) = (\lambda_1, \lambda_2, \ldots)$ be the corresponding partition. For $\mathfrak{sp}(V)$ or $\mathfrak{so}(V)$, $\lambda(e)$ satisfies certain parity conditions [4],[1, 5.1], which are immaterial at the moment. And, of course, $\dim V$ is even in the symplectic case.

**Theorem 2.1.** For a nonzero nilpotent $e \in \mathfrak{g}(V)$, with partition $\lambda(e)$, we have

(i) $\text{ind}(A_1(e) \hookrightarrow \mathfrak{sl}(V)) = \text{ind}(A_1(e) \hookrightarrow \mathfrak{sp}(V)) = \sum_i (\lambda_i + 1)$;
(ii) $\text{ind}(A_1(e) \hookrightarrow \mathfrak{so}(V)) = \frac{1}{2} \sum_i (\lambda_i + 1)$.

**Proof.** In all cases, we have $\mathfrak{V}|_{A_1(e)} = \bigoplus_i \mathcal{R}_{\lambda_i - 1}$.

(i) By formulae of Section 1, we have

$$\text{ind}(A_1(e) \hookrightarrow \mathfrak{sl}(V)) = \text{ind}_D(A_1(e), \mathfrak{V}) = \sum_i \text{ind}_D(A_1(e), \mathcal{R}_{\lambda_i - 1}) = \sum_i \binom{\lambda_i + 1}{3}.$$  

By the multiplicativity of the index,

$$\text{ind}(A_1(e) \hookrightarrow \mathfrak{sl}(V)) = \text{ind}(A_1(e) \hookrightarrow \mathfrak{sp}(V)) \cdot \text{ind}(\mathfrak{sp}(V) \hookrightarrow \mathfrak{sl}(V)).$$

Using Theorem 1.1, one easily computes that $\text{ind}(\mathfrak{sp}(V) \hookrightarrow \mathfrak{sl}(V)) = \text{ind}_D(\mathfrak{sp}(V), \mathfrak{V}) = 1$.

(ii) Likewise, we use the fact that $\text{ind}(\mathfrak{so}(V) \hookrightarrow \mathfrak{sl}(V)) = \text{ind}_D(\mathfrak{so}(V), \mathfrak{V}) = 2$. \hfill $\square$

For the exceptional Lie algebras, Dynkin already computed the index for all $\mathfrak{sl}_2$-subalgebras [2, Tables 16–20]. His calculations can be verified as follows. *First*, for any nilpotent element $e \in \mathfrak{g}$, the Jordan normal formal of $e$ in the simplest representation of $\mathfrak{g}$ is determined in [7]. *Second*, using Theorem 1.1, one obtains that the indices of the embeddings associated with the simplest representations of exceptional Lie algebras are:

$$\text{ind}(E_6 \hookrightarrow \mathfrak{sl}_{27}) = 6; \quad \text{ind}(E_7 \hookrightarrow \mathfrak{sp}_{56}) = 12; \quad \text{ind}(E_8 \hookrightarrow \mathfrak{so}_{248}) = 30;$$

$$\text{ind}(F_4 \hookrightarrow \mathfrak{so}_{26}) = 3; \quad \text{ind}(G_2 \hookrightarrow \mathfrak{so}_7) = 1.$$

Combining these data with formulae of Theorem 2.1, one readily computes the indices of all $\mathfrak{sl}_2$-subalgebras.

**Proposition 2.2.** If $e, e' \in \mathfrak{g}$ are nilpotent and $Ge' \subset \overline{Ge} \setminus Ge$, then

$$\text{ind}(A_1(e') \hookrightarrow \mathfrak{g}) < \text{ind}(A_1(e) \hookrightarrow \mathfrak{g}).$$
Proof. First, we prove this for $\mathfrak{g} = \mathfrak{sl}(\mathbb{V})$, and then derive the general assertion.

1) $\mathfrak{g} = \mathfrak{sl}(\mathbb{V})$. It suffices to consider the case in which $Ge'$ is dense in an irreducible component of $\overline{Ge} \setminus Ge$.

Here $\lambda(e')$ is obtained from $\lambda(e)$ via one of the following procedures. If $\lambda_i \geq \lambda_{i+1} + 2$, then $(\ldots, \lambda_i, \lambda_{i+1}, \ldots)$ can be replaced with $(\ldots, \lambda_i - 1, \lambda_{i+1} + 1, \ldots)$. Or, a fragment $(\ldots, a + 1, a, \ldots, a, a - 1, \ldots)$ in $\lambda(e)$ can be replaced with $(\ldots, a_{k+1}, \ldots) [4, \text{Prop. 3.9}].$

In both cases, one sees that the RHS in Theorem 2.1(i) strictly decreases.

2) For an arbitrary simple $\mathfrak{g}$, we consider a non-trivial representation $\nu : \mathfrak{g} \to \mathfrak{sl}(\mathbb{V})$. If $Ge' \subset \overline{Ge} \setminus Ge$, then $SL(\mathbb{V})e' \subset \overline{SL(\mathbb{V})e}$. By a result of Richardson [10], each irreducible component of $SL(\mathbb{V})e \cap \mathfrak{g}$ is a (nilpotent) $G$-orbit. This also implies that $SL(\mathbb{V})e' \neq SL(\mathbb{V})e$.

Hence

$$\text{ind}(\mathfrak{A}_1(e') \hookrightarrow \mathfrak{g}) = \frac{\text{ind}(\mathfrak{A}_1(e') \hookrightarrow \mathfrak{sl}(\mathbb{V}))}{\text{ind}(\mathfrak{g} \hookrightarrow \mathfrak{sl}(\mathbb{V}))} < \frac{\text{ind}(\mathfrak{A}_1(e) \hookrightarrow \mathfrak{sl}(\mathbb{V}))}{\text{ind}(\mathfrak{g} \hookrightarrow \mathfrak{sl}(\mathbb{V}))} = \text{ind}(\mathfrak{A}_1(e) \hookrightarrow \mathfrak{g}). \quad \Box$$

The index of a subalgebra can be used for obtaining non-trivial combinatorial identities. Taking different $\mathfrak{g}$-modules $M$ in Eq. (1.2) yields different expressions for $\text{ind}(\mathfrak{s} \hookrightarrow \mathfrak{g})$. If $\mathfrak{g} = \mathfrak{g}(\mathbb{V})$, then $\text{ind}(\mathfrak{s} \hookrightarrow \mathfrak{g})$ can be related to $\text{ind}_D(\mathfrak{s}, \mathbb{V})$ and there are two natural choices of test representations: the simplest representation, $M = \mathbb{V}$, and the adjoint representation, $M = \mathfrak{g}(\mathbb{V})$. Alternatively, one can apply Lemma 1.3 to $\mathfrak{g} = \mathfrak{g}(\mathbb{V})$ and $M = \mathbb{V}$. Anyway, the output is as follows:

- If $\mathfrak{g} = \mathfrak{sl}(\mathbb{V})$, then $\nu = \text{id}$, $\text{ind}_D(\mathfrak{s}, \mathbb{V}) = 1$, $h^*(\mathfrak{sl}(\mathbb{V})) = \dim \mathbb{V}$, and

$$\text{ind}_D(\mathfrak{s}, \mathbb{V}) = \frac{\text{ind}_D(\mathfrak{s}, \mathfrak{sl}(\mathbb{V}))}{2 \dim \mathbb{V}} . \quad (2.1)$$

- If $\mathfrak{g} = \mathfrak{sp}(\mathbb{V})$ and $\nu : \mathfrak{sp}(\mathbb{V}) \to \mathfrak{sl}(\mathbb{V})$, then $\text{ind}_D(\mathfrak{s}, \mathbb{V}) = 1$, $h^*(\mathfrak{sp}(\mathbb{V})) = \frac{1}{2} \dim \mathbb{V} + 1$, and

$$\text{ind}_D(\mathfrak{s}, \mathbb{V}) = \frac{\text{ind}_D(\mathfrak{s}, \mathfrak{sp}(\mathbb{V}))}{\dim \mathbb{V} + 2} . \quad (2.2)$$

- If $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ and $\nu : \mathfrak{so}(\mathbb{V}) \to \mathfrak{sl}(\mathbb{V})$, then $\text{ind}_D(\mathfrak{s}, \mathbb{V}) = 2$, $h^*(\mathfrak{so}(\mathbb{V})) = \dim \mathbb{V} - 2$, and

$$\text{ind}_D(\mathfrak{s}, \mathbb{V}) = \frac{\text{ind}_D(\mathfrak{s}, \mathfrak{so}(\mathbb{V}))}{\dim \mathbb{V} - 2} . \quad (2.3)$$

2.1. Combinatorial identities related to $\mathfrak{g}(\mathbb{V})$ and $\mathfrak{s} \simeq \mathfrak{sl}_2$.

If $\mathfrak{s} \simeq \mathfrak{sl}_2$ and a nonzero nilpotent element of $\mathfrak{s}$ has the Jordan normal form with partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, then $\sum_i \lambda_i = \dim \mathbb{V}$ and $\mathbb{V}|_{\mathfrak{s}} = \bigoplus_i R_{\lambda_i - 1}$. In particular, $\text{ind}_D(\mathfrak{s}, \mathbb{V}) = \sum_i (\lambda_i + 1)$, regardless of the type of $\mathfrak{g}(\mathbb{V})$. For each $\mathfrak{g}(\mathbb{V})$, we use below the simple relation between the $\mathfrak{g}(\mathbb{V})$-modules $\mathbb{V}$ and $\mathfrak{g}(\mathbb{V})$. 

1) $\mathfrak{g} = \mathfrak{sl}(\mathbb{V})$. Using the Clebsch-Gordan formula, we obtain
\[
\mathfrak{gl}(\mathbb{V})|_s = \mathbb{V} \otimes \mathbb{V}^*|_s = \bigoplus_{i,j} (\mathcal{R}_{\lambda_i-1} \otimes \mathcal{R}_{\lambda_j-1}) = \bigoplus_{i,j} \bigoplus_{k=0}^{\min\{\lambda_i-1, \lambda_j-1\}} \mathcal{R}_{\lambda_i+\lambda_j-2-2k}.
\]
Since $\mathfrak{gl}(\mathbb{V})$ and $\mathfrak{sl}(\mathbb{V})$ differ by a trivial $\mathfrak{g}$-module, we have $\text{ind}_D(s, \mathfrak{gl}(\mathbb{V})) = \text{ind}_D(s, \mathfrak{sl}(\mathbb{V}))$. Then using Eq. (2.1), we obtain, for an arbitrary partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, the identity
\[
\sum_i \binom{\lambda_i + 1}{3} = \frac{1}{2} \sum_i \lambda_i \sum_{i,j} \sum_{k=0}^{\min\{\lambda_i-1, \lambda_j-1\}} \binom{\lambda_i + \lambda_j - 2k}{3}.
\]
In particular, for a principal nilpotent element $e \in \mathfrak{sl}(\mathbb{V})$, we have $\lambda(e) = (\dim \mathbb{V}) = (N)$, and the identity reads
\[
\binom{N + 1}{3} = \frac{1}{2N} \sum_{k=0}^{N-1} \binom{2N - 2k}{3}.
\]

2) $\mathfrak{g} = \mathfrak{sp}(\mathbb{V})$. Here
\[
\mathfrak{sp}(\mathbb{V})|_s = S^2(\mathbb{V}|_s) = \bigoplus_{i,j} (\mathcal{R}_{\lambda_i-1} \otimes \mathcal{R}_{\lambda_j-1}) \oplus \bigoplus_i S^2(\mathcal{R}_{\lambda_i-1})
\]
and $S^2(\mathcal{R}_m) = \mathcal{R}_{2m} \oplus \mathcal{R}_{2m-4} \oplus \ldots$ by a variation of the Clebsch-Gordan formula. Using Eq. (2.2), we then obtain the “symplectic identity”
\[
\sum_i \binom{\lambda_i + 1}{3} = \frac{1}{(\sum_i \lambda_i) + 2} \left(\sum_{i,j} \sum_{k=0}^{\lambda_i-1} \binom{\lambda_i + \lambda_j - 2k}{3} + \sum_i \sum_{k=0}^{[\lambda_i/2]} \binom{2\lambda_i - 4k}{3}\right),
\]
where we use the fact that $\min\{\lambda_i - 1, \lambda_j - 1\} = \lambda_j - 1$ if $i < j$. For instance, $\lambda(e) = (\dim \mathbb{V}) = (2n)$ for a principal nilpotent element $e \in \mathfrak{sp}(\mathbb{V})$, and the identity reads
\[
\binom{2n + 1}{3} = \frac{1}{2n + 2} \sum_{k=0}^{n-1} \binom{4n - 4k}{3}.
\]

3) $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$. Here $\mathfrak{so}(\mathbb{V}) \simeq \wedge^2(\mathbb{V})$ and $\wedge^2(\mathcal{R}_m) = \mathcal{R}_{2m-2} \oplus \mathcal{R}_{2m-6} \oplus \ldots$. Then using Eq. (2.3) we obtain the “orthogonal identity”
\[
\sum_i \binom{\lambda_i + 1}{3} = \frac{1}{(\sum_i \lambda_i) - 2} \left(\sum_{i<j} \sum_{k=0}^{\lambda_j-1} \binom{\lambda_i + \lambda_j - 2k}{3} + \sum_i \sum_{k=1}^{[\lambda_i/2]} \binom{2\lambda_i + 2 - 4k}{3}\right).
\]
In particular, if $\dim \mathbb{V} = 2n$, then $\lambda(e) = (2n - 1, 1)$ for a principal nilpotent element $e \in \mathfrak{so}(\mathbb{V})$, and the identity is
\[
\binom{2n}{3} = \frac{1}{2n - 2} \left(\binom{2n}{3} + \sum_{k=1}^{n-1} \binom{4n - 4k}{3}\right).
3. On the Index of Principal and Subregular $\mathfrak{sl}_2$-Subalgebras

If $e \in \mathfrak{g}$ is a principal (= regular) nilpotent element, then the corresponding $\mathfrak{sl}_2$-subalgebras are also called principal. We refer to [2, n. 29] and [5, Sect. 5] for properties of principal $\mathfrak{sl}_2$-subalgebras. The set of non-regular nilpotent elements contains a dense $G$-orbit [1, 4.2]. The elements of this orbit and corresponding $\mathfrak{sl}_2$-subalgebras are said to be subregular. Write $(\mathfrak{sl}_2)^{pr}$ (resp. $(\mathfrak{sl}_2)^{sub}$) for a principal (resp. subregular) $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$. In [9], we obtained a uniform expression for $\text{ind}( (\mathfrak{sl}_2)^{pr} \hookrightarrow \mathfrak{g} )$. To recall it, we need some notation.

Let $\theta_s$ denote the short dominant root in $\Delta^+$. (In the simply-laced case, we assume that $\theta = \theta_s$.) Set $r = ||\theta||^2/||\theta_s||^2 \in \{1, 2, 3\}$. Along with $\mathfrak{g}$, we also consider the Langlands dual algebra $\mathfrak{g}^\vee$, which is determined by the dual root system $\Delta^\vee$. Since the Weyl groups of $\mathfrak{g}$ and $\mathfrak{g}^\vee$ are isomorphic, we have $h(\mathfrak{g}) = h(\mathfrak{g}^\vee)$. However, the dual Coxeter numbers can be different (cf. $B_n$ and $C_n$). The half-sum of the positive roots for $\mathfrak{g}^\vee$ is $\rho^\vee := \frac{1}{2} \sum_{\gamma > 0} \gamma^\vee = \sum_{\gamma > 0} \gamma / (\gamma, \gamma)_\theta$.

It is well-known (and easily verified) that $(\rho^\vee, \gamma)_\theta = \text{ht}(\gamma)$ for any $\gamma \in \Delta^+$. (This equality does not depend on the normalisation of a bilinear form on $\mathcal{E}$.) It follows that $h^*(g^\vee) = 1 + (\rho^\vee, \theta_s) = 1 + \text{ht}(\theta_s)$. Our first uniform expression is

**Theorem 3.1** ([9, Theorem 3.2]). $\text{ind}( (\mathfrak{sl}_2)^{pr} \hookrightarrow \mathfrak{g} ) = \frac{\dim \mathfrak{g}}{6} h^*(\mathfrak{g}^\vee)r$.

Below, we give yet another expression for this index. Let $\Delta^+_l$ (resp. $\Delta^+_s$) be the set of long (resp. short) positive roots. In the simply-laced case, all roots are assumed to be short and $r = 1$.

**Theorem 3.2.** $\text{ind}( (\mathfrak{sl}_2)^{pr} \hookrightarrow \mathfrak{g} ) = 2(\rho^\vee, \rho^\vee)_\theta = \sum_{\gamma \in \Delta^+_l} \text{ht}(\gamma) + r \sum_{\gamma \in \Delta^+_s} \text{ht}(\gamma)$.

**Proof.** In view of our choice of the form $(\ , \ )_\theta$, we have

$$2 \rho^\vee = \sum_{\gamma \in \Delta^+} 2 \gamma / (\gamma, \gamma)_\theta = \sum_{\gamma \in \Delta^+_l} \gamma + r \sum_{\mu \in \Delta^+_s} \mu.$$ 

Consequently,

$$2(\rho^\vee, \rho^\vee)_\theta = (\rho^\vee, \sum_{\gamma \in \Delta^+_l} \gamma + r \sum_{\mu \in \Delta^+_s} \mu)_\theta = \sum_{\gamma \in \Delta^+_l} \text{ht}(\gamma) + r \sum_{\gamma \in \Delta^+_s} \text{ht}(\gamma),$$

which yields the second equality.

Now, we obtain another expression for $(\rho^\vee, \rho^\vee)_\theta$ applying the “strange formula” of Freudenthal-de Vries to $\Delta^\vee$ and $\mathfrak{g}^\vee$, cf. Remark 1.4. If $\mu \in \Delta_s$, then $\mu^\vee$ is a long root in
\[\Delta^\vee \text{ and } (\mu^\vee, \mu^\vee)_0 = 2r. \text{ Therefore, } 2(\rho^\vee, \rho^\vee)_0 = \frac{2\dim(\mathfrak{g}^\vee)}{24} 2r h^*(\mathfrak{g}^\vee) = \frac{\dim \mathfrak{g}}{6} r h^*(\mathfrak{g}^\vee), \text{ which is exactly the index of } (\mathfrak{sl}_2)^{pr}. \]

Remark 3.3. It was noticed in [9] that the index of \((\mathfrak{sl}_2)^{pr}\) is preserved under the unfolding procedure \(\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}\) applied to the multiply laced Dynkin diagram, the four pairs \((\mathfrak{g}, \tilde{\mathfrak{g}})\) being \((\mathbf{C}_n, \mathbf{A}_{2m-1}), (\mathbf{B}_n, \mathbf{D}_{n+1}), (\mathbf{F}_4, \mathbf{E}_6), (\mathbf{G}_2, \mathbf{D}_4)\). Using Theorem 3.2, we may look at this coincidence from another angle. Let \(\tilde{\Delta}\) be the root system of \(\tilde{\mathfrak{g}}\) with respect to a Cartan subalgebra \(\tilde{\mathfrak{t}}\). The embedding \(\mathfrak{t} \rightarrow \tilde{\mathfrak{t}}\) induces a surjective map \(\pi : \tilde{\Delta}^+ \rightarrow \Delta^+\) such that \(\pi^{-1}(\Delta^+_i) \rightarrow \Delta^+_i\) is one-to-one and \(#\pi^{-1}(\gamma) = r\) for \(\gamma \in \Delta^+_i\). Furthermore, \(\pi\) is height-preserving. Thus, we get the natural equality \(\sum_{\gamma \in \Delta^+_i} \text{ht}(\gamma) + r \sum_{\gamma \in \Delta^+_j} \text{ht}(\mu) = \sum_{\tilde{\gamma} \in \tilde{\Delta}^+_i} \text{ht}(\tilde{\gamma})\), which again “explains” the coincidence of two indices.

Our next goal is to provide a simple uniform expression for the difference of the indices of subalgebras \(\mathfrak{sl}_2^{pr}\) and \((\mathfrak{sl}_2)^{\text{sub}}\). To this end, we need the relationship between the structure of \(\mathfrak{g}\) as the module over \((\mathfrak{sl}_2)^{pr}\) or \((\mathfrak{sl}_2)^{\text{sub}}\), see e.g. [11, Ch. 7]. Let \(m_1, \ldots, m_n\) be the exponents of \(\mathfrak{g}\). As was shown by Kostant [5],

\[(3.1) \quad \mathfrak{g}|_{(\mathfrak{sl}_2)^{pr}} = \bigoplus_{i=1}^n \mathcal{R}_{2m_i}.\]

To deal with the subregular \(\mathfrak{sl}_2\)-subalgebras, we may assume that \(n = \text{rk}(\mathfrak{g}) \geq 2\) and also \(1 = m_1 < m_2 \leq \ldots \leq m_{n-1} < m_n = h(\mathfrak{g}) - 1\). Then

\[(3.2) \quad \mathfrak{g}|_{(\mathfrak{sl}_2)^{\text{sub}}} = \left( \bigoplus_{i=1}^{n-1} \mathcal{R}_{2m_i} \right) \oplus \mathcal{R}_{a-2} \oplus \mathcal{R}_{b-2} \oplus \mathcal{R}_{h(\mathfrak{g})-2},\]

where \(a + b = h(\mathfrak{g}) + 2\). Assume that \(a \leq b\) and note that \((a, b, h(\mathfrak{g}))\) are just \((w_r, w_{r+1}, w_{r+2})\) in [11, p. 112]. Below, we write \(h\) and \(h^*\) for \(h(\mathfrak{g})\) and \(h^*(\mathfrak{g})\), respectively.

Theorem 3.4. \(D := \text{ind}((\mathfrak{sl}_2)^{pr} \hookrightarrow \mathfrak{g}) - \text{ind}((\mathfrak{sl}_2)^{\text{sub}} \hookrightarrow \mathfrak{g}) = \frac{h}{h^*} \left( \binom{h}{2} + \frac{(a - 2)(b - 2)}{4} \right).\)

Proof. If \(\mathfrak{g}|_{\mathfrak{sl}_2} = \bigoplus_j \mathcal{R}_{n_j}\), then Eq. (1.3) shows that \(\text{ind}(\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}) = \frac{1}{2 h^*} \sum_j \binom{n_j + 2}{3}\). Therefore, by Eq. (3.1) and (3.2), the difference \(D\) equals

\[\frac{1}{2 h^*} \left( \binom{2h}{3} - \binom{h}{3} - \binom{a}{3} - \binom{b}{3} \right).\]

Then routine transformations, where we repeatedly use the relation \((a - 1) + (b - 1) = h\), simplify this expression to the desired form. For instance, we first transform \(\binom{a}{3} + \binom{b}{3}\) into \(\frac{h}{6}(h^2 - 3(a - 1)(b - 1) - 1)\), etc. \(\square\)

In the following table, we gather the relevant data for all simple Lie algebras.
THE DYNKIN INDEX AND \( sl_2 \)-SUBALGEBRAS OF SIMPLE LIE ALGEBRAS

| \( g \) | \( A_n \) | \( B_n \) | \( C_n, n \geq 3 \) | \( D_n, n \geq 4 \) | \( E_6 \) | \( E_7 \) | \( E_8 \) | \( F_4 \) | \( G_2 \) |
|---|---|---|---|---|---|---|---|---|---|
| ind\(( sl_2^{pr} \to g )\) | \( \binom{n+2}{3} \) | \( \frac{1}{2} \binom{2n+2}{3} \) | \( \frac{1}{2} \binom{2n+1}{3} \) | \( \frac{1}{2} \binom{2n}{3} \) | 156 | 399 | 1240 | 156 | 28 |
| \( D \) | \( \binom{n+1}{2} \) | \( 2n^2 \) | \( 4n(n-1) \) | \( 2n(n-2) \) | 72 | 168 | 480 | 96 | 24 |
| \( a \) | 2 | 2 | 4 | 4 | 6 | 8 | 12 | 6 | 4 |
| \( b \) | \( n+1 \) | \( 2n \) | \( 2n-2 \) | \( 2n-4 \) | 8 | 12 | 20 | 8 | 4 |
| \( D/b \cdot \text{rk}(g) \) | \( 1/2 \) | 1 | 2 | 1 | \( 3/2 \) | 2 | 3 | 3 | 3 |

Remark 3.5. The numbers \((a, b)\) frequently occur in the study of the McKay correspondence and finite subgroups of \( SL_2 \), see e.g. [6]. Recall that Slodowy associates a finite subgroup of \( SL_2 \) to any \( g \) (not only of type A-D-E) [11, 6.2]. Let \( \tilde{\Gamma} \subset SL_2 \) be the finite subgroup corresponding to \( g \). Then (i) \( ab/2 = \# \tilde{\Gamma} \), (ii) \( \{a, b, h\} \) are the degrees of basic invariants for the associated 2-dimensional representation of \( \tilde{\Gamma} \), and (iii) the Poincaré series of this ring of invariants is \( \frac{1 + T^h}{(1 - T^a)(1 - T^b)} \). Using the first relation, one can also write

\[
D = \frac{h}{h^*} \cdot \frac{h(h - 2) + \# \tilde{\Gamma}}{2}.
\]

Remark 3.6. Let us point out some curious observations related to \( D \).

- It is always true that \( D \leq 2h \cdot \text{rk}(g) \), and the equality holds if and only if \( g \) is of type \( G_2, F_4, E_8 \). Furthermore, if \( h \) is even (which only excludes the case of \( A_{2n} \)), then \( D/\text{rk}(g) \) is an integer.

- It is always true that \( D \leq 3b \cdot \text{rk}(g) \), and the equality holds if and only if \( g \) is of type \( G_2, F_4, E_8 \). Moreover, for each classical series, the ratio \( D/b \cdot \text{rk}(g) \) is constant.

It might be interesting to find an explanation for these properties and understand the meaning of the constant \( D/b \cdot \text{rk}(g) \).

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