Is there a $\rho$ in the $O(4) \lambda \phi^4$ theory?*

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Abstract

A Monte Carlo simulation of the $O(4) \lambda \phi^4$ theory in the broken phase is performed on a hypercubic lattice in search of an $I = 1$, $J = 1$ resonance. The region of the cutoff theory where the interaction is strong is investigated since it is there that a resonance would be expected to have a better chance to form. In that region the presence of an $I = 1$, $J = 1$ resonance with mass below the cutoff is excluded.

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The question of whether or not an \( I = 1, J = 1 \) resonance exists in the broken phase of the four component \( \lambda \phi^4 \) theory in four dimensions is in fact an old one. In the sixties, a quite phenomenological theory describing the low energy pion processes was developed by Gell–Mann and Levy \(^1\). The theory is described by a chiral Lagrangian, and involves the three pion fields \( \pi^1, \pi^2, \pi^3 \), the scalar field \( \sigma \), the two nucleons, their interactions, and the dimensionful pion decay constant \( f_\pi = 90 \text{ MeV} \). The \( \sigma \), although it has never really been seen, may exist in nature as a broad and quite heavy \((\approx 700 \text{ MeV})\) isospin \( I=0, \text{spin} \ J=0 \) resonance. In that setting, it was natural to ask whether the presence of the \( \rho \) resonance was a consequence of the pion interactions or of some higher energy physics. Since the theory of Gell–Mann and Levy becomes an \( O(4) \lambda \phi^4 \) theory if the nucleon fields are neglected, this question reduces to whether or not the \( O(4) \lambda \phi^4 \) theory can sustain an \( I = 1, J = 1 \) resonance in the broken phase. Earlier attempts \(^2\) to answer this question used analytical approximations such as Padé approximants and found that an \( I = 1, J = 1 \) resonance is present in the theory. It is not clear, however, whether or not the presence of this resonance is an artifact of the approximations used and therefore the question of the existence of an \( I = 1, J = 1 \) resonance in the theory has yet to receive a conclusive answer.

Today, in a very different setting, this same question is of interest again. The scalar sector of the Minimal Standard Model is also a four component \( \lambda \phi^4 \) theory in the broken symmetry phase. The equivalent of the \( \sigma \) resonance is the Higgs particle, the three pions are the Goldstone bosons, the pion decay constant is the weak scale \( f_G = 246 \text{ GeV} \), and the scattering of longitudinally polarized vector bosons behaves exactly the same way as a scaled up version of \( \pi - \pi \) scattering (equivalence theorem \(^3\)). It is possible that the Higgs, like the \( \sigma \), is quite heavy and broad and may avoid detection at SSC. On the other hand, if the four component \( \lambda \phi^4 \) theory does indeed contain an \( I = 1, J = 1 \) resonance, then since the \( \rho \) resonance, as it appears in nature, is quite strong, its equivalent in the scalar sector of the Minimal Standard Model may have a good chance to be detected at SSC. Therefore, it becomes very important to know if another “signature” of the scalar sector, besides the Higgs resonance, is awaiting discovery at SSC.

A direct Monte Carlo simulation was decided to be the best way to shed new light on this old question (a leading order large \( N \) calculation was not able to produce an \( I = 1, J = 1 \) resonance). The \( O(4) \lambda \phi^4 \) theory was simulated on the lattice in the \( \lambda \to \infty \) limit. In that limit, the theory has
the strongest interactions and a resonance probably has a better chance to form. The model has the lattice action

\[ S = \sum_{x \in \Lambda} \left\{ -k \sum_{\mu=1}^{4} \left( \bar{\Phi}_x \Phi_{x+\hat{\mu}} + \Phi_x \bar{\Phi}_{x-\hat{\mu}} \right) + \lambda (\bar{\Phi}_x \Phi_x - 1)^2 + \bar{\Phi}_x \Phi_x \right\} \tag{1} \]

and was simulated on a hypercubic lattice \( \Lambda \) of spatial extension \( L \) and time extension \( L_t \) using an incomplete heat bath algorithm [4] on the CM-2 machine at SCRI.

Since, on a finite lattice, the direction of the symmetry breaking changes from configuration to configuration, the same approach as in [5] was used to disentangle the massive scalar field \( \sigma \) from the Goldstone modes. For each configuration, the global O(4) coordinate system was rotated so that its first axis would be parallel to the direction of the symmetry breaking. Because there are many ways to perform this rotation, a “simple” one was consistently used throughout the simulation. The field, expressed in the new coordinate system, is \( \bar{\Phi}_x = (\sigma_x, \pi^1_x, \pi^2_x, \pi^3_x) \), with \( \sigma \) being the massive scalar field and \( \pi^1, \pi^2, \pi^3 \) the three massless pion fields.

To measure the mass \( m_\sigma \) of the \( \sigma \) field, the time slice connected correlation function \( C_{0,0}(t) = \langle O_\sigma(0), O_\sigma(t) \rangle_c \) of the zero 3–momentum operator

\[ O_\sigma(t) = \frac{1}{L_3^3} \sum_{x \in \Lambda_t} \sigma_x , \tag{2} \]

where \( \Lambda_t \) is the three dimensional time slice at Euclidean time \( t \), was fitted to

\[ C(t) = a \cosh \left[ M \left( t - \frac{L_t}{2} \right) \right] + b \tag{3} \]

by a three parameter correlated \( \chi^2 \)-fit with \( M = m_\sigma \). For each fit, the errors were obtained by varying \( \chi^2 \) by 1.

To measure the energy of the lowest laying state in the \( I = 1, J = 1 \) channel, an operator carrying these quantum numbers needs to be constructed. The simplest such operator is:

\[ O_{c,m}(t) = \frac{1}{L_3^3} \sum_{x \in \Lambda_t} \pi^a_x \pi^b_{x+\hat{m}} \epsilon_{abc} \tag{4} \]

where summation over repeated indices is assumed, \( a, b, c \) are the isospin indices, \( \epsilon_{abc} \) is the totally antisymmetric tensor, \( \hat{m} \) is the \( m \)’th Euclidean
unit vector of the time slice $\Lambda_t$, and $m \in [1, 2, 3]$ is the z-component spin index. Unfortunately, the time slice connected correlation function of this operator gives a very weak signal. To get a better signal an operator that extends over several lattice spacings needs to be constructed using a trial wave function for the two–pion state. The “bag” and “bound state” meson wave functions were considered [6]. The latter gave a better signal with an affordable cost in computer time and was therefore used for the simulation.

Using as a trial wave function for the two pions at relative position $\vec{R}$, the hydrogen wave function $\Psi_{n=2,l=1,m}(\vec{R})$, an $I = 1, J = 1$ operator with total 3-momentum zero was constructed:

\[
O_{c,m}(t) = \sum_{x \in \Lambda_t} \left\{ \sum_{\vec{R} \in B} |\vec{R}| \exp(-|\vec{R}|/2a_0) Y_{1,m}(\theta, \phi) \pi_x^a \pi_x^b \vec{R} \epsilon_{abc} \right\}
\]

where $B$ is a three-dimensional cubic box centered at the origin and contained in $\Lambda_t$, $a_0$ is the “Bohr radius” in lattice units, $\theta$ and $\phi$ are the azimuthal and polar angles of $\vec{R}$, and $Y_{1,m}$ is the $l = 1$ spherical harmonic (up to a multiplicative normalization constant). The parameter $a_0$ can be given any value. A very large $a_0$ will cause the exponential to decrease very slowly and then the sum over $\vec{R}$ will have to be carried out over a box $B$ as large as $\Lambda_t$. Since this can increase the computer time significantly, a smaller $a_0$ has to be used so that the size of the box that contains the important contribution from the exponential can be made smaller. However, $a_0$ cannot be made very small because the signal becomes weaker as $a_0$ decreases. An optimal choice of $a_0$ and $B$ was found to be $a_0 = 2$, and $B$ extending from $-3$ to $+3$ in each of the three directions. The operator $O_{c,m}(t)$ couples to the $I = 1, J = 1$ states. The energy of the lowest laying state in this channel can be found by looking at the time slice connected correlation function $C_{c,m}(t) = \langle O_{c,m}(0)O_{c,m}(t)^* \rangle_c$ of this operator.

The simulation was done on an $L = 8, L_t = 16$ and $L = 16, L_t = 16$ lattice and for three values of the hopping parameter $\kappa = 0.305, 0.310, 0.330$. These values were chosen so that a wide range of $m_\sigma$ will be covered (they also coincide with some of the values used in [3]). The expectation values of $C_{0,0}(t), C_{1,1}(t) = \frac{1}{9} \sum_{c,m} C_{c,m}(t)$, and also of the pion zero 3–momentum time slice correlations were measured. The pions will not concern us here except to mention that they are the massless Goldstone modes and found to behave as in [3].

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The number of sweeps, the number of measurements of $C_{0,0}(t)$ and $C_{1,1}(t)$, and the values of $m_\sigma$, for the various lattice sizes and $\kappa$’s, are given in table 1. The results for $L = 16$ are in good agreement with the results of [5].

$C_{1,1}(t)$ is plotted versus $t$ in figure 1 for the $L = 16$ lattice and the three values of $\kappa$. $C_{1,1}(t)$ was fitted with the expression in equation 3 with $b = 0$ for a few different ranges of $t$, and the resulting energies $E = M$, together with the $\chi^2$ per degree of freedom for each fit are given in table 2 for the $L = 16$ and $L = 8$ lattices. The dotted line in figure 1 is the fit for $\kappa = 0.310$, $L = 16$, and $2 \leq t \leq 8$ and is presented to give the reader a feeling of the quality of the fits.

### Table 1: Number of measurements and $m_\sigma$.

| $\kappa$ | $L$ | sweeps | measurements | $m_\sigma$ |
|----------|-----|---------|--------------|------------|
| 0.305    | 16  | $4 \times 10^6$ | $1.6 \times 10^4$ | 0.225(4)   |
| 0.310    | 16  | $1.6 \times 10^5$ | $1.6 \times 10^4$ | 0.424(6)   |
| 0.330    | 16  | $0.8 \times 10^5$ | $1.6 \times 10^4$ | 0.83(1)    |
| 0.305    | 8   | $17.5 \times 10^5$ | $7 \times 10^4$  | 0.343(2)   |
| 0.310    | 8   | $7 \times 10^5$  | $7 \times 10^4$  | 0.468(3)   |
| 0.330    | 8   | $1 \times 10^6$  | $2 \times 10^4$  | 0.91(2)    |

### Table 2: Energy $E$ in the $I = 1, J = 1$ channel.

| $\kappa$ | $L$ | $E$     | fit-range | $\chi^2$/d.o.f. |
|----------|-----|---------|-----------|-----------------|
| 0.305    | 16  | 0.94(1) | 2–8       | 8.8             |
| 0.305    | 16  | 0.76(3) | 3–8       | 1.8             |
| 0.310    | 16  | 0.94(1) | 2–8       | 0.7             |
| 0.310    | 16  | 0.91(3) | 3–8       | 0.7             |
| 0.330    | 16  | 0.94(1) | 2–8       | 4.2             |
| 0.330    | 16  | 0.87(2) | 3–8       | 2.4             |
| 0.305    | 8   | 1.549(5)| 1–5       | 1.7             |
| 0.310    | 8   | 1.531(6)| 1–5       | 1.7             |
| 0.330    | 8   | 1.501(9)| 1–4       | 0.4             |
The effective energy $E_{\text{eff}}(t)$, obtained by fitting $C_{1,1}(t)$ to the expression in equation 3, with $b = 0$ for the two time slices at $t - 1$ and $t$, is plotted versus $t$ and for the three values of $\kappa$ in figures 2 ($L = 16$) and 3 ($L = 8$). The values of $t$ omitted from those plots had an $E_{\text{eff}}(t)$ with very large error.

In a two-pion $I = 1$, $J = 1$ state with zero total 3–momentum, the lowest 3–momentum a pion can have has one component equal to $\frac{2\pi}{L}$ and two equal to 0. The next one has two components equal to $\frac{2\pi}{L}$ and one equal to 0. The energy spectrum in the $I = 1$, $J = 1$ channel is therefore expected to contain levels with energies close to the energies of these states, but slightly different because of the interaction. For the $L = 16$ lattice, these levels have energies $E_0 \simeq 0.78$ and $E_1 \simeq 1.09$ respectively, and are denoted by the dotted lines.
in figure 2. From this figure, it is clear that the observed energy levels are very close to the levels of two free pions. In fact, for smaller $t$, the levels are close to $E_1$, and for larger $t$ they are close to $E_0$. Because the free two–pion levels are not very well separated at $L = 16$, the observed levels are probably a mixture of the two lowest ones. In that sense, the energies in table 2 for the $L = 16$ lattice are probably a mixture as well. The fact that the observed levels correspond to a two–pion state and not to a resonance is also greatly supported by the fact that these levels change only slightly from $\kappa = 0.305$ to $\kappa = 0.330$. After all, in that range $m_\sigma$ varies from 0.225 to 0.91. Therefore, if a resonance is present it must have energy larger than $\approx 0.78$ and is either too heavy (for example, heavier than 1.09) to be observed with this method,
or is “hiding” between 0.78 and 1.09. That the latter is not the case can be seen by looking at the energy levels obtained for the \( L = 8 \) lattice. From figure 3 it is clear that there are no energy levels below \( \approx 1.4 \). In fact, since the two lowest free two–pion states are well separated in this lattice, the lowest energy level is clearly visible in figure 3. It is true that the limited statistics give \( E_{\text{eff}}(t) \) only up to \( t = 4 \), but because the lowest level is well separated from the next one, the correlated \( \chi^2 \) fit gives a good estimate of the energy of this level. The numbers given in table 2 are indeed very close to the lowest free two–pion energy \( E_0 \simeq 1.50 \) and do not seem to change much for the different values of \( \kappa \). From this analysis it is evident that if an \( I = 1, J = 1 \) resonance exists for \( \kappa \geq 0.305 \), it is unphysical since it must

\textbf{Figure 3:} \( E_{\text{eff}}(t) \) for the \( L = 8, L_t = 16 \) lattice.
have energy in lattice units greater than 1 (physical energy above the cutoff).

For the theory of Gell-Mann and Levy of low energy pion processes the \( \kappa \geq 0.305 \) region corresponds to where the \( \sigma \) particle mass is greater than approximately 180MeV (or equivalently the cutoff is less than approximately 1.3GeV). Since \( m_\sigma \) is expected to be much larger than 180MeV, it is clear from the results that the existence of the \( \rho \) resonance in nature cannot be accounted for by this low energy theory alone (in contrast to [2]).

For the scalar sector of the Minimal Standard Model the \( \kappa \geq 0.305 \) region corresponds to where the Higgs mass is greater than approximately 500GeV (or equivalently the cutoff is less than approximately 3.5TeV). The Higgs mass is of course not known but its upper bound is placed at around 650GeV (hypercubic lattice triviality bound). Thus the existence of an \( I = 1, J = 1 \) resonance can be excluded with confidence for values of the Higgs mass above \( \approx 500\text{GeV} \). For \( \kappa < 0.305 \) (Higgs mass < 500GeV) the strength of the interaction becomes weaker and hence it is safe to say that if a resonance could not form for \( \kappa \geq 0.305 \), where the interaction is stronger, it is unlikely that it will in this region either. For this reason it was not deemed necessary to investigate the \( \kappa < 0.305 \) region. A numerical simulation for \( \kappa < 0.305 \) not only is not necessary, but it would also be very costly since larger lattices will have to be used (the correlation length becomes larger than approximately \( \frac{1}{0.225} \approx 4.5 \)).

It must be emphasized that these conclusions are valid only within the realm of the scalar sector of the Minimal Standard Model. It is of course still possible that the physics that enters at higher energies may be able to produce such a resonance in very much the same way the physical \( \rho \) particle owes its existence to QCD. This resonance if it exists due to some higher energy theory it would have energy determined by that theory. In fact, it is possible that the energy of this resonance is determining the cutoff energy of the Minimal Standard Model.

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\footnote{The connection with the physical scale \( f_\pi \) (or \( f_G \)) is made through the renormalized coupling constant \( g_R = 3m_\sigma^2/f^2_\pi \). The value of \( g_R \) for this value of \( \kappa \) was taken from [3].}
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