Partial-state fidelity and quantum phase transitions induced by continuous level crossing

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(Dated: May 28, 2008)

The global-state fidelity cannot characterize those quantum phase transitions (QPTs) induced by continuous level crossing due to its collapse around each crossing point. In this paper, we take the isotropic Lipkin-Meshkov-Glick (LMG) model and the antiferromagnetic one-dimensional Heisenberg model as examples to show that the partial-state fidelity can signal such level-crossing QPTs. Extending to the thermodynamic limit we introduce the partial-state fidelity susceptibility and study its scaling behavior. The maximum of the partial-state fidelity susceptibility goes like N for the LMG model and N 3 for the Heisenberg model.

PACS numbers: 64.60.-i, 05.70.Fh, 75.10.-b

It has been an interesting issue for merging quantum phase transitions (QPTs) and fidelity. The former one is noticed by the observation of quantities undergoing structural changes around some critical points \[1\], while the latter one measures the amount of relevance for two quantum ground states of the system differed by some parameters \[2\]. A physical phenomenon is then associated with a pure quantum informational consideration \[3\]. The fidelity approach brings advantages to the characterization of QPTs, because by comparing the states, no a priori knowledge to the order parameter, symmetry, and type of QPTs of the system are required. The fidelity approach has been examined in various models and proved its ability in characterizing QPTs \[4, 5, 6, 7, 8, 9, 10, 11\].

This suggests experimental measurements of the quantum state itself which is a rather challenging task, leading order of the fidelity has been suggested as well \[12, 13\], for its critical exponents and divergence helps classification of the universality of the system \[14, 15, 16, 17, 18\].

However, there are still limitations to the fidelity approach. It is useful to study ground states of some continuous variables, but unable to describe discrete global ground states, i.e. states with fixed quantum numbers within a certain continuous range of parameters. Especially when the driving Hamiltonian commutes with the whole Hamiltonian, the leading order of the fidelity is not well-defined. To tackle this problem, the partial-state fidelity was introduced \[6, 19\]. It concerns the quantum relevance of part of a system with respect to a global change of the parameter. It has been investigated in characterizing the QPTs in the XY model, a three-body interacting model \[6\] and a conventional BCS superconductor with an inserted magnetic impurity system \[19\]. In addition, the operator fidelity susceptibility was also introduced and was shown that it can signal QPTs regardless of the degeneracy of the system \[20\]. However, attention to those phase transitions of continuous level crossing were not paid under their definitions.

In this paper, we put our attention on thermodynamic systems in which continuous level crossing occurs. One is the isotropic Lipkin-Meshkov-Glick (LMG) model introduced in nuclear physics \[21\], which is related to Bose-Einstein condensation and Josephson junctions. We make use of its exact spectrum to obtain the partial-state fidelity. The other one is the one-dimensional Heisenberg model, where we adopt the Bethe-Ansatz method to compute the ground state energy, as well as the required reduced density matrix. We show that the partial-state fidelity can be used to locate the critical point for these two models. We defined the corresponding fidelity susceptibility and perform scaling analysis.

Let a system be parameterized by \( h \), with its density operator \( \hat{\rho}(h) = | \Psi(h) \rangle \langle \Psi(h) | \) corresponds to the ground state \( | \Psi(h) \rangle \). When \( h \) is displaced by \( \delta h \) such that \( \tilde{h} = h + \delta h \), the density operator becomes \( \hat{\rho}(\tilde{h}) = | \Psi(\tilde{h}) \rangle \langle \Psi(\tilde{h}) | \), the fidelity is defined according to their respective density operator

\[
F(h, \delta h) = \text{Tr} \sqrt{\hat{\rho}(h) \hat{\rho}(\delta h) \hat{\rho}(h)^\dagger} = | \langle \Psi(h) | \Psi(\delta h) \rangle |. \tag{1}
\]

If the system is divided into two subsystems \( A \) and \( B \), the reduced density operator \( \hat{\rho}_A(h) = \text{Tr}_B \hat{\rho}(h) \) contributes to the partial-state fidelity

\[
F_A(h, \delta h) = \text{Tr} \sqrt{\hat{\rho}_A(h) \hat{\rho}_A(\delta h) \hat{\rho}_A(h)^\dagger}. \tag{2}
\]

The partial state of subsystem \( A \) can be a single-site state or a two-site state, or even a larger subsystem state. For convenience in this paper we consider tracing out all particles but one. So for a system with definite magnetization \( M \), one can make use of the on-site average magnetization basis \( \langle \sigma^z \rangle = 2M/N \), where \( N \) is the number of spins, to trace out the density operator. This left us the diagonal reduced density matrix \( \rho_A(h) \)

\[
\rho_A(h) = \frac{1}{2} \begin{pmatrix}
1 + \langle \sigma^z \rangle & 0 \\
0 & 1 - \langle \sigma^z \rangle
\end{pmatrix}. \tag{3}
\]

Consider a system of size \( N \) with a set of discrete ground state level-crossing points \( \{ h_j \} \), where \( j = 0, 1, 2... \) and \( h_j > h_{j-1} \). Let the partial state within a range \( h \in R_j \) = \( (h_j, h_{j-1}) \) with an average magnetization \( \langle \sigma_z \rangle \), the partial-state fidelity at \( h_j \) is defined by
$F_A(h_j) = \text{Tr} \left( \sqrt{ \frac{1}{4} \left( \begin{array}{cc} 1 + (\langle \sigma^x \rangle)^j & 0 \\ 0 & 1 - (\langle \sigma^x \rangle)^j \end{array} \right) \left( \begin{array}{cc} 1 + (\langle \sigma^x \rangle)^j & 0 \\ 0 & 1 - (\langle \sigma^x \rangle)^j \end{array} \right)} \right)
= \frac{1}{2} \sqrt{1 + (\langle \sigma_z \rangle)^j} (1 + (\langle \sigma_z \rangle)^j) \right) \), \quad (4)

It is the trace of the reduced density matrices at two sides. The non-unity of Eq. (4) signals the level crossing when $\langle \sigma^z \rangle$ changes.

The isotropic LMG model: The model reads

$H_{\text{LMG}} = -\frac{1}{N} \sum_{i<j} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) - h \sum_i \sigma_i^z$
$= -\frac{2}{N} (S_x^2 + S_y^2) - 2hS_z + \frac{1}{2}$
$= -\frac{2}{N} (S_x^2 - S_z^2 - N/2) - 2hS_z. \quad (5)$

For $\sigma_x (\kappa = x, y, z)$ are the Pauli matrices, and $S_x = \sum_i \sigma_i^x/2$. It describes a system of mutually interacting spins subjected to a transverse external field of strength $h$. The ground state lies in the maximum spin sector $S = N/2$. For it is diagonal in the basis $|N/2, M \rangle$, its eigenenergies are given by

$E(M, h) = \frac{2}{N} \left( M - \frac{hN^2}{2} \right) - \frac{N}{2} \left( 1 + h^2 \right). \quad (6)$

The ground state is determined by the minimum of the square,

$M_0 = \begin{cases} \left( \frac{N}{2} \right) & h \geq 1 \\ \left( \frac{hN}{2} \right) & 0 \leq h < 1 \end{cases}, \quad (7)$

where $I(x)$ gives the integer part of $x$. It can be shown that ground state level crossing occurs at some $h_j \equiv 1 - (2j + 1)/N$ and when $h \in (h_j, h_{j+1})$, the ground state is $M = N/2 - j$ \cite{18, 22}. In the thermodynamic limit $h = 1$ is the critical point. Since the model [Eq. (5)] is infinitely coordinated, we consider the partial state as a single particle state with density matrix still follows Eq. (3). The partial-state fidelity at $h_j$ has the exact form according to Eq. (4)

$F_A = \frac{1}{N} \left( \sqrt{(N-j)(N-j-1)} + \sqrt{j(j+1)} \right). \quad (8)$

Fig. 1 shows the plot based on the above formula. $F_A$ drops to a minimum at $h = h_0$, the level-crossing point closest to the critical point $h_c = 1$. Since there are no further level-crossing points for $h > 1$, the partial-state fidelity maintains the value one. As system size increases, the minimum of $F_A$ gets closer to one. It is because unlike ordinary treatment in the fidelity where $\delta h$ is fixed, we calculate the partial-state fidelity obtained by two nearest level-crossing points. When $\delta h$ becomes smaller, the similarity between states becomes higher and is reflected by the close-to-unity behavior, similar to that in global fidelity.

We emphasize the comparison between neighboring partial states, as a realization to continuous level crossing when $N \rightarrow \infty$. The partial-state fidelity helps extrapolating discrete level crossing to continuous level crossing.

The one dimensional Heisenberg model: The isotropic LMG model provides us an analytic form of the partial-state fidelity. Next we try to examine the one dimensional Heisenberg model, which is another system that exhibits ground state level crossing. The Hamiltonian reads

$H_{\text{Heisenberg}} = \sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z - 2hS_i^z, \quad (9)$

where $S_i^z$ is the spin-1/2 operator at site $i$. With a ring geometry $S_N^{z} = S_1^{z}$ imposed, one can solve the spectrum by the Bethe-Ansatz method \cite{23}

$E_0 = \frac{N}{4} - Nh + \sum_{j=1}^{N_1} \left( 2h - \frac{2}{\chi_j^2 + 1} \right), \quad (10)$

where $N_1$ is the number of down spins, and $\chi_j$ are spin rapidities. They satisfy the Bethe ansatz equations \cite{23}

$2N \tan^{-1} \chi_j = 2\pi I_j + 2 \sum_{i=1}^{N_1} \tan^{-1} \frac{x_j - x_i}{2}, \quad (11)$

where $I_j$ are quantum numbers and take values of $-(N_1 - 1)/2, \ldots, (N_1 - 1)/2$ for the ground state. By solving the Bethe-Ansatz equations numerically, the eigenenergies are obtained and the ground state is determined by the minimum eigenenergy. So are the level-crossing points. For the single-site subsystem $A$ can also be characterized by the on-site magnetization $\langle \sigma^z \rangle$, according to Eq. (4) again we compute the
partial-state fidelity at the level-crossing points by comparing the nearest partial states. The numerical result is shown in Fig. 2. We find similar features as in the LMG model such as a drop of $F_g$ at the critical point, the drop becomes sharper and the minimum gets closer to one as system size increases.

**Partial-state fidelity susceptibility:** Interested in the continuous level crossing that corresponds to the thermodynamic limit, we introduce the concept of partial-state fidelity susceptibility. It is because in such case an infinitesimal change of the parameter is sufficiently responsible for an obvious change of the fidelity. The partial-state fidelity susceptibility is defined in a similar manner as \cite{13}:

$$\chi_f^{(A)} = \lim_{\delta h \to 0} \frac{-2\ln F_A}{(\delta h)^2}. \quad (12)$$

The above formula combines the ability of partial-state fidelity in observing level-crossing transitions and the idea of global-state fidelity susceptibility that measures the leading response of fidelity to infinitesimal change of parameter.

In finite systems, we compute $\chi_f^{(A)}$ by taking the natural log of the partial-state fidelity at $h_j$, and divide it by the square of the modulus of the range $R_{j+1}^h$, i.e., $\delta h = h_j - h_{j+1}$. With this notion we arrive the analytic form of $\chi_f^{(A)}$ of the LMG model as $\delta h = 2/N$:

$$\chi_f^{(A)} = -\frac{N^2}{2} \ln \left[ \sqrt{\frac{1 - j}{N}} \left( 1 - \frac{j + 1}{N} \right) + \sqrt{\frac{j + 1}{N^2}} \right]. \quad (13)$$

The plot in Fig. 3 shows $\chi_f^{(A)}$ grows with system size, and arrives its maximum at $h_0$, the level-crossing point closest to the critical point. The response near the maximum becomes sharper for larger systems, indicating a divergence in the thermodynamic limit. It suggests $\chi_f^{(A)}$ as a smooth function of $h$ in the thermodynamic limit except at the critical point. It diverges at $h = h_c$ and drops to zero when $h > h_c$. The divergence of the maximum goes like $N$, since at $j = 0$, from Eq.

\begin{equation}
-\frac{N^2}{2} \ln \sqrt{1 - \frac{1}{N}} = \frac{N}{4}
\end{equation}

for large $N$.

We compute $\chi_f^{(A)}$ for the Heisenberg model and the result is shown in Fig. 4. The divergence is even sharper. Although the full analytic form of the $\chi_f^{(A)}$ is inaccessible as the spin rapidities $x_j$ form a set of transcendental equations, the critical exponent can be estimated by obtaining $h_0 - h_1$.

For $N_1 = 1$, from Eq. (11), we have

$$2N \tan^{-1} x_1 = 2N \tan^{-1} x_2 = \pi. \quad (15)$$

The ground state energy for $N_1 = 0$ is simply $\frac{N}{2} - Nh$ and that of $N_1 = 1$ is calculated by Eq. (10)

$$\frac{N}{4} - Nh + 2(h - 1), \quad (16)$$

in which $h_0 = 1$ is determined.

For $N_1 = 2$, Eq. (11) consists of two equations

$$2N \tan^{-1} x_1 = -\pi + 2 \tan^{-1} \left( \frac{x_1 - x_2}{2} \right),$$

$$2N \tan^{-1} x_2 = \pi + 2 \tan^{-1} \left( \frac{x_2 - x_1}{2} \right). \quad (17)$$

The value of $x_1$ and $x_2$ can be found, since by symmetry $x_1 = -x_2$, the above two equations become one

$$2N \tan^{-1} x_2 = \pi + 2 \tan^{-1} x_2, \quad x_2 = \tan \frac{\pi}{2(N - 1)}. \quad (18)$$

The ground state energy for $N_1 = 2$ is

$$\frac{N}{4} - Nh + 2 \left( 2h - \frac{2}{\frac{\pi}{2(N - 1)}^2 + 1} \right). \quad (19)$$
Then $h_1$ is determined when Eq. (16) equals to Eq. (19), that is

$$h_1 = -1 + \frac{2}{[\tan(\frac{\pi}{N})]^2 + 1}. \tag{20}$$

Expanding for large $N$ limit, for $\tan y \approx y$ for small $y$, we have $h_1 \approx 1 - \frac{\pi^2}{2(N-1)^2}$ and thus

$$\delta h = h_0 - h_1 \approx \frac{\pi^2}{2(N-1)^2}. \tag{21}$$

The partial-state fidelity susceptibility scales like

$$\chi_f^{(A)} = \frac{8(N-1)^4}{\pi^4} \ln \frac{N}{1-N} \propto N^3, \tag{22}$$

which is apparently different from that of the LMG model.

Let us make a remark. In many times, it is often to consider some averaged physical quantities to understand the intrinsic response to the driving agent. Fidelity susceptibility is one of them [14, 15]. But for the partial-state fidelity susceptibility, we have already focused on a part of the system. Such a local response to the global driving has already played a role as a certain averaged quantity. So we believe, supported by the two distinct models above, the divergence of the partial-state fidelity in continuous level crossing could be a general feature. Its divergence in the isotropic LMG model may be related to the $\gamma \to 1$ limit of the averaged fidelity susceptibility driven by external field in [18].

We introduced the partial-state fidelity susceptibility formalism derived from the partial-state fidelity and showed it is a suitable candidate to describe quantum phase transitions induced by continuous level crossing, which global-state fidelity cannot provide information to. Focusing on a subsystem, the partial-state fidelity susceptibility still diverges in the thermodynamic limit. The sudden drop-to-zero indicates the critical point of the system.

By examining two models, the isotropic LMG model and the one dimensional Heisenberg model, we started from the discrete level crossing and extrapolated to the thermodynamic limit which corresponds to continuous level crossing. We find the maximum of the partial-state fidelity susceptibility goes like $N$ for the LMG model and $N^3$ for the Heisenberg model, indicating they belong to different universality classes. The former one could be treated as a complement to the fidelity susceptibility analysis in the LMG model [18].

We demonstrated the calculation for a single-site partial state. However, the partial-state fidelity susceptibility shall not be limited to (sub)systems with definite magnetization, because it can still be well-defined for two-particle or many-particle partial states, yet not for all-particle (global) states. The partial-state fidelity is a new approach to tackle QPTs, studying its leading order which is independent of the small change of the driving parameter helps understanding the continuous level crossing QPTs as well as to determine the critical points. We hope this encourages discussions on the related topics.

Note added: After finishing this work, we received a preprint from XG Wang, in which the fidelity and its susceptibility of two-site partial state in the LMG model are studied [24].

We thank J. Vidal for comments on our work. This work is supported by the Direct grant of CUHK (A/C 2060344)

[1] S. Sachdev, Quantum Phase Transitions (Cambridge University Press, Cambridge, England, 1999).
[2] M. A. Nilesen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, England, 2000)
[3] H. T. Quan, Z. Song, X. F. Liu, P. Zanardi, and C. P. Sun, Phys. Rev. Lett. 96, 140604 (2006).
[4] P. Zanardi and N. Paunkovic, Phys. Rev. E 74, 031123 (2006).
[5] H. Q. Zhou and J. P. Barjaktarevic, arXiv: cond-mat/0701608; H. Q. Zhou, J. H. Zhao, and B. Li, arXiv:0704.2940.
[6] H. Q. Zhou, arXiv:0704.2945.
[7] P. Zanardi, M. Cozzini, and P. Giorda, J. Stat. Mech. 2, L02002 (2007).
[8] P. Buonsante and A. Vezzani, Phys. Rev. Lett. 98, 110601 (2007).
[9] M. Cozzini, P. Giorda, and P. Zanardi, Phys. Rev. B 75, 014439 (2007).
[10] M. Cozzini, R. Ionicioiu, and P. Zanardi, Phys. Rev. B 76, 104420 (2007).
[11] S. Chen, L. Wang, S. J. Gu, and Y. Wang, Phys. Rev. E 76, 061108 (2007); S. Chen, L. Wang, Y. Hao, and Y. Wang, Phys. Rev. A 77, 032111 (2008).
[12] P. Zanardi, P. Giorda, and M. Cozzini, Phys. Rev. Lett. 99, 100603 (2007).
[13] W. L. You, Y. W. Li, and S. J. Gu, Phys. Rev. E 76, 022101 (2007).
[14] L. Campos Venuti and P. Zanardi, Phys. Rev. Lett. 99, 095701 (2007).
[15] S. J. Gu, H. M. Kwok, W. Q. Ning, and H. Q. Lin, arXiv:0706.2945 [Phys. Rev. B, to appear].
[16] M. F. Yang, Phys. Rev. B 76, 180403 (R) (2007).
[17] Yu-Chin Tzeng and M. F. Yang, Phys. Rev. A 77, 012311 (2008).
[18] H. M. Kwok, W. Q. Ning, S. J. Gu, and H. Q. Lin, arXiv:0710.2581.
[19] N. Paunkovic et al., Phys. Rev. A 77, 052302 (2008).
[20] X. Wang, Z. Sun, and Z. D. Wang, arXiv:0803.2940.
[21] H. J. Lipkin, N. Meshkov, and A. J. Glick, Nucl. Phys. 62, 188 (1965).
[22] S. Dusuel and J. Vidal, Phys. Rev. B 71, 224420 (2005).
[23] H. A. Bethe, Z. Physik 71, 205 (1931).
[24] J. Ma, L. Xu, and X. G. Wang, arXiv:0805.4062.