ANALOGY BETWEEN THE CYCLOTOMIC TRACE MAP K → TC AND THE GROTHENDIECK TRACE FORMULA VIA NONCOMMUTATIVE GEOMETRY

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Dedicated to my Father

Abstract. In this article, we suggest a categorification procedure in order to capture an analogy between Crystalline Grothendieck-Lefschetz trace formula and the cyclotomic trace map $K \rightarrow TC$ from the algebraic $K$-theory to the topological cyclic homology $TC$. First, we categorify the category of schemes to the $(2, \infty)$-category of noncommutative schemes à la Kontsevich. This gives a categorification of the set of rational points of a scheme. Then, we categorify the Crystalline Grothendieck-Lefschetz trace formula and find an analogue to the Crystalline cohomology in the setting of noncommutative schemes over $\mathbb{F}_p$. Our analogy suggests the existence of a categorification of the $l$-adic cohomology trace formula in the noncommutative setting for $l \neq p$. Finally, we write down the corresponding dictionary.

1. Arithmetical side: Grothendieck trace formula

In this short expository article, we explore the notion of noncommutative algebraic space and use a categorification and stabilization procedures to make a surprising analogy between two formulas arising from, a priori, different areas of mathematics.

(1) Crystalline and $l$-adic Grothendieck-Lefschetz trace formula for smooth $\mathbb{F}_p$-schemes.

(2) The cyclotomic trace map $K \rightarrow TC$ from the algebraic $K$-theory to the topological cyclic homology for smooth $\mathbb{F}_p$-schemes.

(3) The bridge between the two precedent trace formulas is given via the categorification procedure from the category of schemes to the $(2, \infty)$-category of dg-categories (Morita equivalences), sending any scheme $X$ to the dg-category of perfect complexes $\text{Perf}(X)$.

Grothendieck-Lefschetz trace formula: $l$-adic cohomology. For more details we refer to [10]. We fix a finite field $\mathbb{F}_p$ with $p$ a prime and let $X$ be a nice scheme. An important challenge in arithmetical algebraic geometry is counting the number of rational points of $X$, namely the set $X(\mathbb{F}_p)$. Grothendieck has defined an appropriate cohomology theory ($l$-adic cohomology with compact support,
$l$ different from $p$) in order to mimic the famous Lefschetz fixed point

$$\sharp\{X(F_p^n)\} = \sum_{i=0}^{\infty} (-1)^i \text{trace}[F^n : H_{\text{ét}}^i(X, \mathbb{Q}_l)], l \neq p. \quad (1.1)$$

where $\overline{X} = X \otimes_{F_p} \mathbb{F}_p$ and $F^n : \overline{X} \to \overline{X}$ is the $n^{th}$-iterated Frobenius morphisms. In particular when $X$ is a smooth regular scheme of dimension $d$ the formula simplifies

$$\sharp\{X(F_p^n)\} = \sum_{i=0}^{2d} (-1)^i \text{trace}[F^n : H_{\text{ét}}^i(\overline{X}, \mathbb{Q}_l)], l \neq p. \quad (1.2)$$

The cohomology $H^*_\text{ét}(X, \mathbb{Q}_l)$ is defined by the formula

$$H^*_\text{ét}(X, \mathbb{Q}_l) := \lim_k H^*_\text{ét}(X, \mathbb{Z}/l^k\mathbb{Z}) \otimes \mathbb{Q}_l. \quad (1.3)$$

We denote the $l$-adic cohomology

$$H^*_\text{adic}(X) := \lim_k H^*_\text{ét}(X, \mathbb{Z}/l^k\mathbb{Z}) \quad (1.4)$$

and recall that

$$\sharp\{X(F_p^n)\} = \sharp\{x \in X(F_p)|F^n(x) = x\}. \quad (1.5)$$

It seems that the origin of the étale site and $l$-adic cohomology is closely related to realizing the analogue of Lefschetz formula for $F_p$-schemes. However, we notice that Grothendieck formula has a limitation namely the restriction $p \neq l$. The case $p = l$ seems to suggest the existence of another appropriated cohomology theory.

One of our main goals is to propose an analogue of $l$-adic cohomology (and Crystalline cohomology) in the context of algebraic noncommutative spaces which can be interpreted as noncommutative intersection theory.

**Grothendieck-Lefschetz trace formula: Crystalline cohomology.** For more details, we refer the reader to [8] and [3]. As we noticed, the Grothendieck-Lefschetz trace formula for $(enough good)$ $F_p$-scheme $X$ works when we use the $l$-adic cohomology for $l \neq p$. The natural question to ask is the following: What happens if $l = p$? do we still have a trace formula? The answer to this question is the Crystalline cohomology theory $H_{\text{cris}}^*(-)$.

$$\sharp\{X(F_p^n)\} = \sum_{i=0}^{\infty} (-1)^i \text{Trace}[F^n : H_{\text{cris}}^i(X) \otimes \mathbb{Z}_p, \mathbb{Q}_p] \quad (1.6)$$

where $\mathbb{Q}_p$ is fraction field of the total ring of Witt vectors $\lim_k W_k(F_p) \simeq W(F_p) \simeq \mathbb{Z}_p$ ($p$-adic integers) and

$$H_{\text{cris}}^i(X) = \lim_k H_{\text{cris}}^i(X/W_k(F_p)) \quad (1.7)$$

**Relation to de Rham-Witt complex.** The relation between de Rham-Witt cochain complex [2] and Crystalline cohomology is explained in [3, section 3 and 4], the hypercohomology of de Rham-Witt complex $W\Omega^*(-)$ verifies the following property for $(good enough)$ $F_p$-scheme $X$, namely the isomorphism

$$H_{\text{cris}}^i(X) \cong H^i(X, W\Omega^*(-)) \quad (1.8)$$
2. Homotopical side: Cyclotomic trace formula $K \to TC$

The other side of the picture comes from a homotopical approximation of the Algebraic $K$-theory. For more details and comprehension we refer to [4]. It is well known that the algebraic $K$-theory of schemes gives very important arithmetical informations. For any scheme $X$ we associate the the differential graded category of perfect complexes $\text{Perf}(X)$, we have a trace map from the spectrum of algebraic $K$-theory to the topological Hochschild homology

$$K(\text{Perf}(X)) := K(X) \to \text{THH}(X) := \text{THH}(\text{Perf}(X)).$$

(2.1)

The $\text{THH}$ has a natural action of the topological circle $S^1$ and the trace map factors trough the homotopy fixed points

$$K(X) \to \text{THH}(X)^{S^1} \to \text{THH}(X)^{hS^1} \to \text{THH}(X).$$

(2.2)

The problem is that the construction $\text{THH}(X)^{S^1}$ is not homotopy invariant and $\text{THH}(X)^{hS^1}$ is not a very good approximation in general. There is an other alternative which consists to construct an other spectra $\text{TH}(-)$ with the following properties [9]:

- $\text{TH}(-)$ is equivalent to $\text{THH}(-)$ as a naive $S^1$-spectra.
- It is a cyclotomic $S^1$-spectra.
- It comes equipped with restriction map $R$, Frobenius map $F$ and Verschiebung map $V$ (wich we don’t use explicitly).

Recall that rationally $\text{THH}(X)^{hS^1}$ coincides with Cyclic Homology. Now, we fix a prime $l$ and we take an infinite increasing chain of inclusions of finite cyclic groups

$$C_1 \subset C_2 \subset C_3 \subset \ldots S^1$$

and define a new homology theory [1]

$$\text{TR}(X, l) = \text{holim}_n \text{TH}(X)^{C_n}$$

(2.3)

using the restriction map $R : \text{TH}(X)^{C_{n+1}} \to \text{TH}(X)^{C_n}$. If the natural map $\text{TH}(X)^{C_i} \to \text{TH}(X)^{hC_i}$ is an equivalence after $l$-completion, then by Tsalidis theorem $\text{TH}(X)^{C_n} \to \text{TH}(X)^{hC_n}$ is also an equivalence after $l$-completion for any $n > 0$. This argument was intensively used in [11] in order to compute the 2- primary part of the $K$-theory of integers. In this case $\text{TR}(\mathbb{Z}, 2)$ is equivalent to $\text{THH}(\mathbb{Z})^{hS^1}$ after 2-completion. Now, we are ready to give the definition of the topological cyclic homology of $X$ at prime $l$.

**Definition 2.1.** The topological cyclic homology of $X$ at prime $l$ is given by [6]

$$\text{TC}(\text{Perf}(X), l) = \text{TC}(X, l) := \text{TR}(X, l)^{hF} = \text{TR}(\text{Perf}(X), l)^{hF},$$

(2.4)

where $F : \text{TR}(X, l) \to \text{TR}(X, l)$ is the Frobenius map and $\text{TR}(X, l)^{hF}$ is the spectra of the homotopy fixed points with respect to $F$.

**Remark 2.2.** The derived spectrum of maps from the sphere spectrum $S$ to $\text{TH}(X)$ in the homotopy category of $l$-cyclotomic spectra is naturally isomorphic to $\text{TC}(X, l)$ after $l$-completion [2, Theorem 1.4].

**Definition 2.3.** The trace map $K(X) \to \text{TH}(X)$ is $S^1$-equivariant (the $S^1$-action on $K(X)$ is trivial), we have a natural map

$$\text{trc} : K(X) \to \text{TC}(X, l) := \text{TR}(X, l)^{hF}$$

(2.5)

called the cyclotomic trace map.
When \( p = l \), and \( X \) is an affine smooth scheme over \( \mathbb{F}_p \), it turns out that \( trc \) is a very good approximation (after \( p \)-completion of the connective covers).

### 2.1. Categorification of Euler characteristic and Lefschetz formula.

Let \( X \) be any finite CW complex of dimension \( n \), the Euler characteristic is defined as

\[
\chi(X) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{Q}} H^i(X, \mathbb{Q}).
\]  

(2.6)

It follows that the Euler characteristic is defined only by using the numerical invariant i.e. the dimension of the vector spaces of the rational cohomology \( H^i(X, \mathbb{Q}) \).

A generalization of the Euler characteristic is the Lefschetz formula (in the case where \( X \) is a triangulated manifold of dimension \( n \) given by

\[
\Lambda(X, F) := \sum_{i=0}^{n} (-1)^i \text{trace}[F^* : H^i(X, \mathbb{Q})],
\]  

(2.7)

where \( F : X \to X \) is continuous endomorphism of \( X \). If \( \Lambda(X, F) \neq 0 \) then \( F \) has at least one fixed point. More precisely when the set of fixed point \( \text{Fix}(F) \) is finite then

\[
\Lambda(X, F) = \sum_{x \in \text{Fix}(F)} i(F, x)
\]

where \( i(F, x) \) is the index (integer) of \( F \) at \( x \).

We should remark that \( \Lambda(X, Id) = \chi(X) \). A reasonable categorification of Euler formula is the singular cochain complex \( C^\ast(X, \mathbb{Q}) \) or more artificially (at least a priori) we can say that the categorification of Euler characteristic is \( C^\ast(X, \mathbb{Q}) \cong C^\ast(X, \mathbb{Q})^{hId} \), this suggests that a natural categorification of the right side of the Lefschetz formula is given by

\[
C^\ast(X, \mathbb{Q})^{hF},
\]  

(2.8)

and the decategorification is given by

\[
\sum_{i=0}^{n} \text{trace}[F^* : H^i(X, \mathbb{Q})].
\]  

(2.9)

### 3. The Bridge: Noncommutative algebraic geometry

The previous categorification seems to suggests that the Grothendieck-Lefschetz formula should be categorified in the following way. For any scheme \( X \) over \( \mathbb{F}_p \), the set of rational points \( X(\mathbb{F}_p) \) is corepresentable in the category of \( \mathbb{F}_p \)-schemes, indeed

\[
\text{Map}_{\text{Sch}}(\text{spec}(\mathbb{F}_p), X) = X(\mathbb{F}_p).
\]  

(3.1)

Kontsevich has proposed a noncommutative point of view of schemes which was formalized later by Tabuada. Let \( \text{dgCat}_k \) denotes the model category of small dg-categories and dg-functors where the weak equivalences are Morita equivalences \[12\]. For any scheme \( X \) can associate the differential graded category \( \text{Perf}(X) \) which is the full dg-subcategory of compact object in enhanced derived category \( D^{dg}(X) \). The model category \( \text{dgCat}_k \) is actually a symmetric monoidal \((\infty, \infty)\)-category. Notice that the existence of the derived internal \( \text{RHom} \) in \( \text{dgCat}_k \) is due to Toën \[13\]. Any differential graded \( k \)-algebra \( A \) can be viewed as a differential graded category with one object in obvious way and its corresponding fibrant replacement
is the dg-category of perfect complexes $\text{Perf}(A)$. According to Tabuada, the model category $\text{dgCat}_k$ can be stabilized and localized such that we have a universal stabilization functor

$$U_{\text{add}} : \text{dgCat}_k \to \text{Mot}_{\text{add}}$$

where $\text{Mot}_{\text{add}}$ is stable model category (of additive noncommutative motives) and for any dg-categories $A$ and $B$, the derived spectral enrichment

$$\text{Map}_{\text{Mot}_{\text{add}}}(U_{\text{add}}(A), U_{\text{add}}(B))$$

is isomorphic to Waldhausen $K$-theory

$$K(\text{RHom}(A, B))$$

in the homotopy category of spectra. In particular when $A \simeq \text{Perf}(k)$ and $B = \text{Pref}(X)$ then

$$\text{Map}_{\text{Mot}_{\text{add}}}(U_{\text{add}}(\text{Perf}(k)), U_{\text{add}}(\text{Perf}(X))) \simeq K(\text{Perf}(X)) := K(X).$$ (3.2)

Therefore, the algebraic $K$-theory is corepresentable by $\text{Perf}(k)$ [12, Theorem 6.1]. It is natural to suggest that the categorification of $X(F_p)$ 3.1 is given by the ($E_\infty$-ring) spectra $K(\text{Pref}(X)) := K(X)$ 3.2.

Now, the idea is to find the the analogue of the $l$-adic cohomology of schemes in the context of the noncommutative geometry. First, we should notice that the $l$-adic Grothendieck trace formula is valid when $l \neq p$ while in the noncommutative side of the picture the cyclotomic trace map is a very good approximation when $l = p$. It becomes clear ones we understand the meaning of the homotopy groups of $TR(X, p)_p^\wedge$ for a smooth $F_p$-scheme $X$. It was proved [5, page 18] that there is an isomorphism of cochain complexes

$$\pi_* TR(-, p)_p^\wedge \cong W\Omega^*(-).$$ (3.3)

For the cochain complex structure on $\pi_* TR(X, p)_p^\wedge$ we refer to [9]. We conclude that the cohomology theory $TR(-, p)_p^\wedge$ is the presheaf on the category of smooth $F_p$-schemes of de Rham-Witt cochain complex. The decategorification of the algebraic $K$-theory $K(X)$ is $X(F_p)$ and the decategorification of $TR(X, p)_p^\wedge := TC(X, p)_p^\wedge$ gives the right side of Crystalline Grothendieck trace formula by applying the hypercohomology i.e.

$$\sum_{i=0} (-1)^i \text{Trace}[F : H^i_{\text{cris}}(X) \otimes_{Z_p} Q_p]$$ (3.4)

which is equivalent to

$$\sum_{i=0} (-1)^i \text{Trace}[F : H^i_{\text{cris}}(X) \otimes_{Z_p} Q_p],$$

since $\pi_* TR(-, p)_p^\wedge \cong W\Omega^*(-)$ 3.3 and $H^i_{\text{cris}}(X) \cong H^i(X, W\Omega^*(-))$ 1.8. Hence, by decategorification of the cyclotomic trace map $K(X) \to TR(X, p)_p^\wedge := TC(X, p)_p^\wedge$ we recover the Crystalline Grothendieck trace formula 1.6.

4. ANALOGY AND CONJECTURES

The passage from the commutative world (schemes) to the noncommutative one (dg-categories) is given by the functor

$$X \mapsto \text{Perf}(X)$$
followed by the universal additive stabilization (in the sense of Tabuada)

\[ \text{Perf}(X) \rightarrow U_{\text{add}}\text{Perf}(X). \]

| category of smooth schemes over $\mathbb{F}_p$ | $\infty$-stable category $\text{Mot}_{\text{add}}$ over $\mathbb{F}_p$ |
|---------------------------------------------|--------------------------------------------------|
| $X$                                         | $U_{\text{add}}\text{Perf}(X)$                 |
| Rational points                             | Motivic points                                  |
| $\text{Map}_{\text{Sch}}(\text{spec}(\mathbb{F}_p), X) = X(\mathbb{F}_p)$ | $\text{Map}_{\text{Mot}_{\text{add}}}(U_{\text{add}}(\text{Perf}(\mathbb{F}_p)), U_{\text{add}}(\text{Perf}(X))) \simeq K(X)$ |
| Crystalline cohomology $H^\text{cris}_i(X)$ | Cohomology theory $\text{TR}(X, p)_p^\wedge$   |
| $l$-adic cohomology $H^\text{add}_{i-adic}(X)$ | conjecture 4.1                                  |
| $\sum_{i=0}^\infty (-1)^i \text{trace}[F : H^i_{\text{cris}}(X) \otimes \mathbb{Q}_p]$ | $\text{TR}(X, p)^{\wedge}_p \rightarrow TC(X, p)^\wedge$ |
| Crystalline Grothendieck trace formula $\sharp\{X(\mathbb{F}_p)\} = \sum_{i=0}^\infty (-1)^i \text{trace}[F : H^i_{\text{cris}}(X) \otimes \mathbb{Q}_p]$ | Cyclotomic trace map $K(X)_p^\wedge \rightarrow TC(X, p)_p^\wedge$ |
| right side = left side                      | Cyclotomic trace map is an equivalence, conjecture 4.2 |

We still don’t have an analogue of the $l$-adic cohomology for noncommutative spaces, it was conjectured by Quillen and Lichtenbaum that the canonical map from $K$-theory to the étale $K$-theory is a very good approximation for (good enough) scheme $X$. More precisely, the natural map of spectra

\[ K(X) \rightarrow K^{\text{et}}(X) \]

induces an isomorphism on (stable) homotopy groups $\pi_i$ for sufficiently large $i$ (related to the dimension of $X$). From our previous discussion, we can say that this approximation does not fit in the spirit of our analogy. We propose the following alternative conjecture.

**Conjecture 4.1.** There exists a (co)homology theory $\text{TR}$ for dg-categories with the following properties:

1. the functor $\text{TR} : \text{dgCat}_{\mathbb{F}_p} \rightarrow \text{Sp}$ is Morita invariant i.e. if $C \rightarrow D$ is a Morita equivalence in $\text{dgCat}_{\mathbb{F}_p}$, then $\text{TR}(C) \rightarrow \text{TR}(D)$ is a stable equivalence of spectra.
2. The (co)homology theory $\text{TR}(-)$ is a localizing invariant in the sense of Tabuada [12, definition 5.1].
3. There is a natural transformation $K(-) \rightarrow \text{TR}(-)$.
4. For any $C \in \text{dgCat}_{\mathbb{F}_p}$, then $\pi_*\text{TR}(C)$ has a natural structure of cochain complex.
5. If $X$ is (good enough) $\mathbb{F}_p$-scheme and a prime $l$ different form $p$ then :
   - the hypercohomology $H^i(X, \pi_*\text{TR}(\text{Perf}(-))_p^\wedge)$ is isomorphic to the $l$-adic cohomology $H^i_{l-adic}(X)$.
   - There is a Frobenius natural transformation $F : \text{TR}(\text{Perf}(-)) \rightarrow \text{TR}(\text{Perf}(-))$
   - It recovers the $l$-adic trace formula a la Lefschetz-Grothendieck.
   - The natural transformation $K(\text{Perf}(-)) \rightarrow \text{TR}(\text{Perf}(-))$ factors as trace map $\text{trc} : K(\text{Perf}(-)) \rightarrow \text{TR}(\text{Perf}(-))^{\wedge}_p$, and this new trace map is a good (connective) approximation after $l$-completion for $l \neq p$. More precisely,
     $$\pi_i K(\text{Perf}(X)) \otimes \mathbb{Z}_l \cong \pi_i (\text{TR}(\text{Perf}(X))^{\wedge}_p) \otimes \mathbb{Z}_l$$
     for $i \geq 0$. 


Since the Crystalline trace formula \[1.6\] is true for any nice \(F_p\)-scheme \(X\) the analogy has a very natural prediction which we formulate in the following conjecture:

**Conjecture 4.2.** For any nice \(F_p\)-scheme \(X\), the cyclotomic trace map \(K(X) \to TC(X, p)\) is a weak equivalence (between connective covers) after \(p\)-completion.

### 5. Secondary K-theory and \(F_p\)-points

Based on our analogy \[4\] we want to illustrate the idea that the algebraic \(K\)-theory of a nice \(F_p\)-scheme \(X\) contains a substantial informations about the concrete set of rational points \(X(F_p)\). In order to establish the categorification procedure from rational points \(X(F_p)\) to \(K(X)\), we have used the fact that the point functor is corepresentable in the category of schemes by \(\text{spec}(F_p)\) and the algebraic \(K\)-theory is corepresentable by the noncommutative motive \(\mathcal{U}_{add}\text{Perf}(F_p)\).

**Definition 5.1.** The \(k\)-secondary algebraic \(K\)-theory for \(F_p\)-schemes is defined as

\[
K^{(k)}(X) = TR(X, p)^{hF^k}
\]

where \(F^k\) is the \(k\)-iterated Frobenius operator.

A natural question pop-up:

**Can we recover \(X(F_p^k)\) form \(K^{(k)}(X)\)?**

If \(k = 1\), a short answer is yes if the conjecture \[4.2\] is true by using \[3.4\] and the Crystalline Grothendieck-Lefschetz formula \[1.6\]. It seems that we can recover \(X(F_p^k)\) using the decategorification of the homotopy fixed points \(TR(X, p)^{hF^k}\) with respect to the \(k\)-iterated Frobenius operator \(F^k\) i.e.,

\[
\sum_{i=0} \left(-1\right)^i \left[F^k : H^i(X, \pi_*\text{TR}(-, p)^\wedge \otimes \mathbb{Z}_p \mathbb{Q}_p)\right].
\]

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