ON ITERATION OF COX RINGS

JÜRGEN HAUSEN AND MILENA WROBEL

Abstract. We characterize all varieties with a torus action of complexity one that admit iteration of Cox rings.

1. Introduction

We consider normal algebraic varieties $X$ defined over the field $\mathbb{C}$ of complex numbers. If $X$ has finitely generated divisor class group $K$ and only constant invertible global regular functions, then one defines the $K$-graded Cox ring $R_1$ of $X$ as follows, see [2] for details:

$$R_1 = \bigoplus_K \Gamma(X, \mathcal{O}(D)).$$

If the Cox ring $R_1$ is a finitely generated $\mathbb{C}$-algebra, then one has the total coordinate space $X_1 := \text{Spec} R_1$. We say that $X$ admits iteration of Cox rings if there is a chain

$$X_p \xrightarrow{\mathcal{H}_{p-1}} X_{p-1} \xrightarrow{\mathcal{H}_{p-2}} \cdots \xrightarrow{\mathcal{H}_2} X_2 \xrightarrow{\mathcal{H}_1} X_1$$

dominated by a factorial variety $X_p$ where in each step, $X_{i+1}$ is the total coordinate space of $X_i$ and $H_i = \text{Spec} \mathbb{C}[K_i]$ the characteristic quasitorus of $X_i$, having the divisor class group $K_i$ of $X_i$ as its character group. Note that if the divisor class group $K$ of $X$ is torsion free, then $R_1$ is a unique factorization domain and iteration of Cox rings is trivially possible. As soon as $K$ has torsion, it may happen that during the iteration process a total coordinate space with non-finitely generated divisor class group pops up and thus there is no chain of total coordinate spaces as above, see [1, Rem. 5.12].

In [1] we studied normal, rational, $T$-varieties $X$ of complexity one, where the latter means that $X$ comes with an effective torus action $\mathbb{T} \times X \rightarrow X$ such that $\dim(\mathbb{T}) = \dim(X) - 1$ holds. We showed that for affine $X$ with $\Gamma(X, \mathcal{O})^T = \mathbb{C}$ and at most log terminal singularities, the iteration of Cox rings is possible. In the present article, we characterize all varieties $X$ with a torus action of complexity one that admit iteration of Cox rings.

First consider the case $\Gamma(X, \mathcal{O})^T = \mathbb{C}$. In order to have finitely generated divisor class group, $X$ must be rational and then the Cox ring of $X$ is of the form $R = \mathbb{C}[T_{ij}, S_k]/I$, with a polynomial ring $\mathbb{C}[T_{ij}, S_k]$ in variables $T_{ij}$ and $S_k$ modulo the ideal $I$ generated by the trinomial relations

$$T_0^l + T_1^{i_1} + T_2^{i_2}, \quad \theta_1 T_1^{d_1} + T_2^{d_2} + T_3^{d_3}, \quad \ldots, \quad \theta_r T_{r-2}^{d_{r-2}} + T_{r-1}^{d_{r-1}} + T_r^{d_r},$$

with $T_i^{l_i} = T_1^{l_i_1} \cdots T_{m_i}^{l_{m_i}}$. For each exponent vector $l_i$ set $l_i := \gcd(l_{i1}, \ldots, l_{im_i})$. We say that $R$ is hyperplatonnic if $l_0^{-1} + \ldots + l_r^{-1} > r - 1$ holds. After reordering $l_0, \ldots, l_r$ decreasingly, the latter condition precisely means that $l_i = 1$ holds for all $i \geq 3$ and $(l_0, l_1, l_2)$ is a platonic triple, i.e., a triple of the form

$$(5, 3, 2), \quad (4, 3, 2), \quad (3, 3, 2), \quad (x, 2, 2), \quad (x, y, 1), \quad x, y \in \mathbb{Z}_{\geq 1}.$$
Theorem 1.1. Let $X$ be a normal $T$-variety of complexity one with $\Gamma(X, O)^T = \mathbb{C}$. Then the following statements are equivalent.

(i) The variety $X$ admits iteration of Cox rings.
(ii) The variety $X$ is rational with hyperplatonic Cox ring.

We turn to the case $\Gamma(X, O)^T \neq \mathbb{C}$. Here, $O(X)^* = \mathbb{C}^*$ and finite generation of the divisor class group of $X$ force $\Gamma(X, O)^T = \mathbb{C}[T]$. In this situation, we obtain the following simple characterization.

Theorem 1.2. Let $X$ be a normal $T$-variety of complexity one with $\Gamma(X, O)^T \neq \mathbb{C}$. Then $X$ admits Cox ring iteration if and only if $X$ and its total coordinate space are rational. Moreover, if the latter holds, then the Cox ring iteration stops after at most one step.

As a consequence of the two theorems above, we obtain the following structural result, generalizing [1, Thm. 3], but using analogous ideas for the proof.

Corollary 1.3. Let $X$ be a normal, rational variety with a torus action of complexity one admitting iteration of Cox rings. Then $X$ is a quotient $X = X'/\!/G$ of a factorial affine variety $X' := \text{Spec}(R')$, where $R'$ is a factorial ring and $G$ is a solvable reductive group.

On our way of proving Theorem 1.1, we give in Proposition 2.6 an explicit description of the Cox ring of a variety Spec $R$ for a hyperplatonic ring $R$. This allows to describe the possible Cox ring iteration chains more in detail. After reordering the numbers $l_0, \ldots, l_r$ associated with $R$ decreasingly, we call $(l_0, l_1, l_2)$ the basic platonic triple of $R$.

Corollary 1.4. The possible sequences of basic platonic triples arising from Cox ring iterations of normal, rational varieties with a torus action of complexity one and hyperplatonic Cox ring are the following:

(i) $(1, 1, 1) \to (2, 2, 2) \to (3, 3, 2) \to (4, 3, 2)$,
(ii) $(1, 1, 1) \to (x, x, 1) \to (2x, 2, 2)$,
(iii) $(1, 1, 1) \to (x, x, 1) \to (x, 2, 2)$,
(iv) $(\frac{l_{01}^r l_{01}}{l_{01}^r l_{01}} l_{1}, 1) \to (l_0, l_1, 1)$, where $l_{01} := \text{gcd}(l_0, l_1) > 1$.

Contents

1. Introduction
2. Proof of Theorem 1.1
3. Proof of Theorem 1.2
References

2. Proof of Theorem 1.1

We will work in the notation of [3, 5], where the Cox ring of a rational $T$-variety of complexity one is encoded by a pair of defining matrices. Let us briefly recall the precise definitions we need from [5]; note that the setting will be slightly more flexible than the informal one given in the introduction.

Construction 2.1. Fix integers $r, n > 0$, $m \geq 0$ and a partition $n = n_0 + \ldots + n_r$. For every $i = 0, \ldots, r$, fix a tuple $l_i \in \mathbb{Z}_{>0}^n$ and define a monomial

$$T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \in \mathbb{C}[T_{ij}, S_k; 0 \leq i \leq r, 1 \leq j \leq n_i, 1 \leq k \leq m].$$
We will also write \( \mathbb{C}[T_{ij}, S_k] \) for the above polynomial ring. Let \( A := (a_0, \ldots, a_r) \) be a \( 2 \times (r + 1) \) matrix with pairwise linearly independent columns \( a_i \in \mathbb{C}^2 \). For every \( i = 0, \ldots, r - 2 \) we define

\[
g_i := \det \begin{bmatrix} T_{i+1} & T_{i+2} \\ a_i & a_{i+1} \\ a_{i+2} \end{bmatrix} \in \mathbb{C}[T_{ij}, S_k].
\]

We build up an \( r \times (n + m) \) matrix from the exponent vectors \( l_0, \ldots, l_r \) of these polynomials:

\[
P_0 := \begin{bmatrix} -l_0 & l_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -l_0 & 0 & l_r & 0 & \cdots & 0 \end{bmatrix}.
\]

Denote by \( P_0^* \) the transpose of \( P_0 \) and consider the projection

\[
Q_0, Z^{n+m} \to K_0 := Z^{n+m}/\text{im}(P_0^*).
\]

Denote by \( e_{ij}, e_k \in Z^{n+m} \) the canonical basis vectors corresponding to the variables \( T_{ij}, S_k \). Define a \( K_0 \)-grading on \( \mathbb{C}[T_{ij}, S_k] \) by setting

\[
\text{deg}(T_{ij}) := Q(e_{ij}) \in K_0, \quad \text{deg}(S_k) := Q(e_k) \in K_0.
\]

This is the finest possible grading of \( \mathbb{C}[T_{ij}, S_k] \) leaving the variables and the \( g_i \) homogeneous. In particular, we have a \( K_0 \)-graded factor algebra

\[
R(A, P_0) := \mathbb{C}[T_{ij}, S_k]/(g_0, \ldots, g_{r-2}).
\]

By the results of \[3, 4\] the rings \( R(A, P_0) \) are normal complete intersections, admit only constant homogeneous units and we have unique factorization in the multiplicative monoid of \( K_0 \)-homogeneous elements of \( R(A, P_0) \). Moreover, suitably downgrading the rings \( R(A, P_0) \) leads to the Cox rings of the normal rational \( T \)-varieties \( X \) of complexity one with \( \Gamma(X, O)^T = \mathbb{C} \), see \[4, 3, 5\].

In order to iterate a Cox ring \( R(A, P_0) \), it is necessary that \( \text{Spec } R(A, P_0) \) has finitely generated divisor class group. The latter turns out to be equivalent to rationality of \( \text{Spec } R(A, P_0) \). From \[1, \text{Cor. 5.8}\], we infer the following rationality criterion.

**Remark 2.2.** Let \( R(A, P_0) \) be as in Construction 2.4 and set \( l_i := \text{gcd}(l_{i1}, \ldots, l_{in_i}) \). Then \( \text{Spec } R(A, P_0) \) is rational if and only if one of the following conditions holds:

(i) We have \( \text{gcd}(l_i, l_j) = 1 \) for all \( 0 \leq i < j \leq r \), in other words, \( R(A, P_0) \) is factorial.

(ii) There are \( 0 \leq i < j \leq r \) with \( \text{gcd}(l_i, l_j) > 1 \) and \( \text{gcd}(l_u, l_v) = 1 \) whenever \( v \notin \{i,j\} \).

(iii) There are \( 0 \leq i < j < k \leq r \) with \( \text{gcd}(l_i, l_j) = \text{gcd}(l_i, l_k) = \text{gcd}(l_j, l_k) = 2 \) and \( \text{gcd}(l_u, l_v) = 1 \) whenever \( v \notin \{i,j,k\} \).

**Definition 2.3.** Let \( R(A, P_0) \) be as in Construction 2.4 such that \( \text{Spec } R(A, P_0) \) is rational. We say that \( P_0 \) is gcd-ordered if it satisfies the following two properties

(i) \( \text{gcd}(l_i, l_j) = 1 \) for all \( i = 0, \ldots, r \) and \( j = 3, \ldots, r \),

(ii) \( \text{gcd}(l_1, l_2) = \text{gcd}(l_0, l_1, l_2) \).

Observe that if \( \text{Spec } R(A, P_0) \) is rational, then one can always achieve that \( P_0 \) is gcd-ordered by suitably reordering \( l_0, \ldots, l_r \). This does not affect the \( K_0 \)-graded algebra \( R(A, P_0) \) up to isomorphism.

**Lemma 2.4.** Let \( R(A, P_0) \) be as in Construction 2.7 such that \( \text{Spec } R(A, P_0) \) is rational and \( P_0 \) is gcd-ordered. Then, with \( K_0 = Z^{n+m}/\text{im}(P_0^*) \), the kernel of
\[
\mathbb{Z}^{n+m} \to K_0/K_0^{\text{tors}}\] is generated by the rows of
\[
P_1 := \begin{bmatrix}
\frac{-1}{\gcd(t_0, t_1)} & \frac{-1}{\gcd(t_0, t_1)} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\frac{-1}{\gcd(t_0, t_2)} & 0 & \frac{-1}{\gcd(t_0, t_2)} & l_1 & 0 & 0 \\
-1 & 0 & 0 & l_3 & 0 & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & l_r & 0 & \cdots & 0
\end{bmatrix}.
\]

**Proof.** The arguments are similar as for [1 Cor. 6.3]. The row lattice of \(P_0\) is a sublattice of finite index of that of \(P_1\) and thus there is a commutative diagram

\[K_0 \xrightarrow{\cdot} K_0/K_0^{\text{tors}} \xrightarrow{\cdot} \mathbb{Z}^{n+m}/\text{im}(P_1^*)\]

We have to show, that \(\mathbb{Z}^{n+m}/\text{im}(P_1^*)\) is torsion free. Suitable elementary column operations on \(P_1\) reduce the problem to showing that for the \(r \times (r + 1)\) matrix
\[
\begin{bmatrix}
\frac{-1}{\gcd(t_0, t_1)} & \frac{-1}{\gcd(t_0, t_1)} & 0 & \cdots & 0 \\
\frac{-1}{\gcd(t_0, t_2)} & 0 & \frac{-1}{\gcd(t_0, t_2)} & l_1 & 0 \\
-1 & 0 & 0 & l_3 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & l_r
\end{bmatrix}
\]

the \(r\)-th determinantal divisor and therefore the product of the invariant factors equals one. Up to sign, the \(r \times r\) minors of the above matrix are
\[
\frac{1}{\gcd(t_0, l_0) \gcd(t_0, l_1) \cdots l_{i-1} l_{i+1} \cdots l_r}, \quad \text{where } i = 0, \ldots, r.
\]

Suppose that some prime \(p\) divides all these minors. Then \(p \nmid t_j\) holds for all \(j \geq 3\), because otherwise we find an \(i \neq j\) with \(p \mid t_i\), contradicting gcd-orderedness of \(P_0\). Thus, \(p\) divides each of the numbers
\[
\frac{t_0 l_2}{\gcd(t_0, l_1) \gcd(t_0, l_2)}, \quad \frac{t_1 l_2}{\gcd(t_0, l_1) \gcd(t_0, l_2)}, \quad \frac{l_0 l_1}{\gcd(t_0, l_1) \gcd(t_0, l_2)}.
\]

By the assumption of the lemma, \(i := \gcd(t_1, l_2)\) equals \(\gcd(t_0, l_1, l_2)\). Consequently, we obtain
\[
\gcd(l_0 l_2, t_0 l_1, t_1 l_2) = \gcd(l_0 l_1, l_1 l_2) = \gcd(t_0, l_1) \gcd(t_0, l_2).
\]

We conclude \(p = 1\); a contradiction. Being the greatest common divisor of the above minors, the \(r\)-th determinantal divisor equals one. \(\square\)

**Lemma 2.5.** Let \(R(A, P_0)\) be as in Construction 2.4 and \(X := \text{Spec} R(A, P_0)\) be rational. Assume that \(P_0\) is gcd-ordered. Then the number \(c(i)\) of irreducible components of \(V(X, T_{i,j})\) is given by

\[
\begin{array}{c|c|c|c|c}
i & 0 & 1 & 2 & \geq 3 \\
c(i) & \gcd(t_1, l_2) & \gcd(l_0, l_2) & \gcd(t_0, l_1) & \frac{1}{2} \gcd(t_1, l_2) \gcd(l_0, l_2) \gcd(l_0, l_1)
\end{array}
\]

**Proof.** The assertion is a direct consequence of [1 Lemma 6.4]. \(\square\)

We are ready for the main ingredience of the proof of Theorem 1.1, the explicit description of the iterated Cox ring.
Thus, for the first two copies of $P_0$ with the exponent vectors of the Cox ring $l$ times the exponent vector $W$, we may assume that $P_0$ is rational. If the total coordinate space of $X$ is computed in terms of the exponent vectors $l_0, \ldots, l_r$ of $R(A, P_0)$ according to the table below, where “$a \times l_i$” means that the vector $l_i$ shows up $a$ times:

| bpt of $R(A, P_0)$ | exponent vectors in $(A', P')$ |
|---------------------|---------------------------------|
| $(4,3,2)$           | $2 \times l_1, \frac{1}{2}l_0, \frac{1}{2}l_2$ and $2 \times l_i$ for $i \geq 3$ |
| $(3,3,2)$           | $3 \times l_2, \frac{1}{2}l_0, \frac{1}{2}l_1$ and $3 \times l_i$ for $i \geq 3$ |
| $(x,2,2)$ and $2 \nmid x$ | $2 \times \frac{1}{2}l_0, 2 \times \frac{1}{2}l_1, 2 \times \frac{1}{2}l_2$ and $4 \times l_i$ for $i \geq 3$ |
| $(x,2,2)$ and $2 \nmid x$ | $2 \times l_0, \frac{1}{2}l_1, \frac{1}{2}l_2$ and $2 \times l_i$ for $i \geq 3$ |
| $(x,y,1)$          | $\frac{1}{\gcd(x,y)}l_0, \frac{1}{\gcd(x,y)}l_1$ and $\gcd(x,y) \times l_i$ for $i \geq 2$ |

**Lemma 2.8.** Let $R(A, P_0)$, arising from Construction 2.7 be non-factorial and assume that $X := \text{Spec } R(A, P_0)$ is rational. If the total coordinate space of $X$ is rational as well, then $l_i > 1$ holds for at most three $0 \leq i \leq r$.

**Proof.** We may assume that $P_0$ is gcd-ordered. Then Proposition 2.6 provides us with the exponent vectors of the Cox ring $R(A', P_0)$ of $X$. As $R(A, P_0)$ is rational and non-factorial, Remark 2.2 leaves us with the following two cases.

**Case 1.** We have $\gcd(l_0, l_1) > 1$ and $\gcd(l_i, l_j) = 1$ whenever $j \geq 2$. This means in particular $l_0, l_1 > 1$. Assume that there are $2 \leq i < j \leq r$ with $l_i, l_j > 1$. According to Proposition 2.6 we find $c(i)$ times the exponent vector $l_i$ and $c(j)$ times the exponent vector $l_j$ in $P_0$. Lemma 2.5 tells us $c(j) = c(i) = \gcd(l_0, l_i)$ > 1. Thus, for the first two copies of $l_i$ and $l_j$, we obtain $\gcd(l_{i,1}, l_{j,1}) = l_i > 1$ and $\gcd(l_{j,1}, l_{j,2}) = l_j > 1$ respectively. Remark 2.2 shows that $\text{Spec } R(A', P_0')$ is not rational; a contradiction.
Case 2. We have \( \gcd(l_0, l_1) = \gcd(l_0, l_2) = \gcd(l_1, l_2) = 2 \). Assume that there is an index \( 3 \leq i \leq r \) with \( l_i > 1 \). Proposition 2.6 and Lemma 2.5 yield that the exponent vector \( l_i \) occurs \( c(k) = 4 \) times in the matrix \( P_0' \). As in the previous case we conclude via Remark 2.3 that the total coordinate space \( \text{Spec} \, R(A', P_0') \) is not rational; a contradiction.

Proof of Theorem 1.2. We prove “(ii)⇒(i)”. Then \( X \) is a rational and has a hyperplatonic Cox ring \( R(A, P_0) \) provided by Construction 2.1. If \( R(A, P_0) \) is factorial, then there is nothing to show. So, let \( R(A, P_0) \) be non-factorial. We may assume that \( P_0 \) is gcd-ordered. Then \((l_0, l_1, l_2)\) is the basic platonic triple of \( R(A, P_0) \). From Remark 2.7 we infer that \( X_1 := \text{Spec} \, R(A, P_0) \) has a Cox ring \( R(A', P_0') \). So, we can pass to \( X_2 := R(A', P_0') \) and so forth. The table of possible basic platonic triples given in Remark 2.7 shows that the iteration process terminates at a factorial ring.

We prove “(i)⇒(ii)”. Since \( X \) has a Cox ring, \( X \) must have finitely generated divisor class group. As for any \( T \)-variety of complexity one, the latter is equivalent to \( X \) being rational. The Cox ring of \( X \) is a ring \( R(A, P_0) \) as provided by Construction 2.1. If \( R(A, P_0) \) is factorial, then we are done. So, let \( R(A, P_0) \) be non-factorial. Then we may assume that \( P_0 \) is gcd-ordered and, moreover, \( l_{i_0} \neq 1 \). Since \( X_1 := \text{Spec} \, R(A, P_0) \) has a Cox ring \( R(A', P_0') \), it must be rational. By Lemma 2.8 we have \( l_j = 1 \) whenever \( j \geq 3 \) holds. Remark 2.2 leaves us with the following cases.

Case 1. We have \( l_{i_0} := \gcd(l_0, l_1) > 1 \) and \( \gcd(l_1, l_2) = 1 \) whenever \( j \geq 2 \) holds. Then we may assume \( l_0 \geq l_1 \).

1.1. Consider the case \( l_{i_0} > 3 \). By Lemma 2.6 the exponent vector \( l_2 \) occurs \( l_{i_0} \) times in the defining relations of the Cox ring \( R(A', P_0') \) of \( X_1 \). Since \( \text{Spec} \, R(A', P_0') \) is rational, Remark 2.2 yields \( l_2 \leq 2 \). Thus, \((l_0, l_1, l_2)\) is platonic.

1.2. Assume \( l_{i_0} = 3 \). Then \( l_2 \) occurs 3 times as exponent vector in the defining relations of \( R(A', P_0') \). Remark 2.2 shows \( l_2 \leq 2 \). Thus, \((l_0, l_1, l_2)\) is platonic.

1.3. Let \( l_{i_0} = 2 \). If \( l_0 = l_1 = 2 \) holds, then \((l_0, l_1, l_2)\) is a platonic triple for any \( l_2 \). So, assume \( l_0 > l_1 \) holds. As we are in Case 1, the number \( l_2 \) must be odd. If \( l_2 = 1 \) holds, then \((l_0, l_1, l_2)\) is a platonic triple. By Proposition 2.6 and Lemma 2.5 we find the exponent vectors \( 1/2 \, l_0 \) and \( 1/2 \, l_1 \) as well as twice \( l_2 \) in \( P_0' \). Since \( X_1 := \text{Spec} \, R(A, P_0') \) is rational and \( l_0 > l_1 \) holds, Lemma 2.8 shows \( l_1 = 2 \) and the triple of non-trivial gcd’s of exponent vectors of \( P_0' \) is \((l_0/2, l_2, l_2)\). After gcd-ordering \( P_0' \), we can apply Case 1.1 and with \( l_0/2 > 2 \) we obtain \( l_0 = 4 \) and \( l_2 = 3 \). In particular, \((l_0, l_1, l_2)\) is platonic.

Case 2: We have \( \gcd(l_0, l_1) = \gcd(l_0, l_2) = \gcd(l_1, l_2) = 2 \). Then we may assume \( l_0 \geq l_1 \geq l_2 \). Proposition 2.6 and Lemma 2.5 tell us that each of the exponent vectors \( 1/2 \, l_0, 1/2 \, l_1 \) and \( 1/2 \, l_2 \) occurs twice in \( P_0' \). Since \( \text{Spec} \, R(A', P_0') \) is rational, Lemma 2.8 yields \( l_1 = l_2 = 2 \). Thus, \((l_0, l_1, l_2)\) is platonic.

3. Proof of Theorem 1.2

As a first step we relate the total coordinate space of a rational variety with torus action of complexity one admitting non-constant invariant functions to the total coordinate space of one with only constant invariant functions; see Corollary 3.3. This allows us to characterize rationality of the total coordinate space using previous results; see Corollary 3.5. Then we determine in a similar manner as before, the iterated Cox ring; see Proposition 3.7. This finally allows us to prove Theorem 1.2.

We begin with recalling the necessary notions from [5].
Construction 3.1. Fix integers $r, n > 0$, $m \geq 0$ and a partition $n = n_1 + \ldots + n_r$. For each $1 \leq i \leq r$, fix a tuple $l_i \in \mathbb{Z}_{>0}^{n_i}$ and define a monomial
\[ T_i^l := T_{i1}^{l_1} \cdots T_{in_i}^{l_{n_i}} \in \mathbb{C}[T_{ij}, S_k; 1 \leq i \leq r, 1 \leq j \leq n_i, 1 \leq k \leq m]. \]
Let $A := (a_1, \ldots, a_r)$ be a list of pairwise different elements of $\mathbb{C}$. Define for every $i = 1, \ldots, r$ a polynomial
\[ g_i := T_i^l - T_i^{l+1} - (a_{i+1} - a_i) \in \mathbb{C}[T_{ij}, S_k]. \]
We build up an $r \times (n + m)$ matrix from the exponent vectors $l_1, \ldots, l_r$ of these polynomials:
\[ P_0 := \begin{bmatrix} l_1 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & l_r & 0 & \ldots & 0 \end{bmatrix}. \]
Similar to the case in Construction 2.4, the matrix $P_0$ defines a grading of the group $K_0 := \mathbb{Z}^{n+m}/\text{im}(P_0^r)$ on the ring $R(A, P_0) := \mathbb{C}[T_{ij}, S_k]/(g_1, \ldots, g_{r-1})$.

Following [3] we call a ring $R(A, P_0)$ arising from Construction 3.1 of Type 1 and a ring $R(A, P_0)$ as in Construction 2.4 of Type 2. According to [5], the suitable downgradings of the rings $R(A, P_0)$ of Type 1 yield precisely the Cox rings of the normal rational $T$-varieties $X$ of complexity one with $\Gamma(X, \mathcal{O})^T = \mathbb{C}[T]$.

Construction 3.2. Consider a ring $R(A, P_0)$ of Type 1. Set $l_1 := \text{gcd}(l_{i1}, \ldots, l_{in_i})$ and $\ell := \text{lcm}(l_1, \ldots, l_r)$. Then, writing $L_0$ for the column vector $- (\ell, \ldots, \ell) \in \mathbb{Z}^r$, we obtain a ring $R(\tilde{A}, \tilde{P}_0)$ of Type 2 with defining matrices
\[ \tilde{A} := \begin{bmatrix} -1 & a_1 & \cdots & a_r \\ 0 & 1 & \cdots & 1 \end{bmatrix}, \quad \tilde{P}_0 := [L_0, P_0]. \]

Proposition 3.3. Let $R(A, P_0)$ be a ring of Type 1 and $R(\tilde{A}, \tilde{P}_0)$ the associated ring of Type 2 obtained via Construction 3.2. Fix $\alpha_{ij} \in \mathbb{Z}$ with $l_i = \alpha_{1i} l_1 + \ldots + \alpha_{im} l_{in_i}$. Then one obtains an isomorphism of graded $\mathbb{C}$-algebras
\[ \psi: \mathbb{C}[\tilde{T}_{ij}, \tilde{S}_k][T_{01}, T_{01}]^{-1} \to \mathbb{C}[T_{ij}, S_k][T_{01}, T_{01}]^{-1}, \quad \tilde{T}_{ij} \mapsto T_{ij}, \quad \tilde{T}_{01} \mapsto T_{01}, \quad \tilde{T}_{ij} T_{01}^{\alpha_{ij}}, \quad \tilde{S}_k \mapsto S_k. \]

Proof. By construction, $R(\tilde{A}, \tilde{P}_0)$ is a factor algebra of $\mathbb{C}[\tilde{T}_{ij}, \tilde{S}_k]$ and $R(A, P_0)$ of $\mathbb{C}[T_{ij}, S_k]$. We have an isomorphism of $\mathbb{C}$-algebras
\[ \psi: \mathbb{C}[\tilde{T}_{ij}, \tilde{S}_k][T_{01}, T_{01}]^{-1} \to \mathbb{C}[T_{ij}, S_k][T_{01}, T_{01}]^{-1}, \quad \tilde{T}_{ij} \mapsto T_{ij}, \quad \tilde{T}_{01} \mapsto T_{01}, \quad \tilde{T}_{ij} T_{01}^{\alpha_{ij}}, \quad \tilde{S}_k \mapsto S_k. \]

Observe $\psi(T_i^l) = T_i^{l_i}$. We claim that $\psi$ is compatible with the gradings by $K_0$ on the l.h.s. and by $\mathbb{Z} \times K_0$ on the r.h.s., where the latter grading is given by
\[ \text{deg}(T_{01}) = (1, 0) \in \mathbb{Z} \times K_0, \quad \text{deg}(T_{ij}) = (0, e_{ij} + \text{im}(P_0^r)) \in \mathbb{Z} \times K_0. \]

Indeed, because of $\psi(\tilde{T}_{01}^{-\ell} T_i^l) = T_i^{l_i}$, the kernels of the respective downgrading maps
\[ \mathbb{Z}^{n+1+m} \to K_0, \quad \mathbb{Z}^{n+1+m} \to \mathbb{Z} \times K_0, \]
generated by the rows $\tilde{P}_0$ and $P_0$, correspond to each other under $\psi$. The defining ideal of $R(\tilde{A}, \tilde{P}_0)$ is generated by the polynomials $\tilde{g}_1, \ldots, \tilde{g}_{r-1}$, where
\[ \tilde{g}_i := \text{det} \begin{bmatrix} \tilde{T}_0^i & T_i^l & T_i^{l+1} \\ -1 & a_i & a_{i+1} \\ 0 & 1 & 1 \end{bmatrix}. \]
The above isomorphism sends $\tilde{g}_i$ to $T_i^l g_i$, where the $g_i$ are the generators of the defining ideal of $R(A, P_0)$, and thus induces the desired isomorphism. \qed
Lemma 3.6. Let \( X := \text{Spec} \, A(P_0) \) be the affine variety arising from a ring of Type 1 and \( \bar{X} := \text{Spec} \, A(\bar{P}_0) \) the one arising from the associated ring of Type 2. Then \( X \times \mathbb{C}^* \) is isomorphic to the principal open subset \( \bar{X}_{T_{01}} \subseteq \bar{X} \). In particular, \( X \) is rational if and only if \( \bar{X} \) is so.

Corollary 3.5. Let \( R(A,P_0) \) be a ring of Type 1. Then \( X = \text{Spec} \, R(A,P_0) \) is rational if and only if one of the following conditions holds:

(i) One has \( l_i = 1 \) for all \( 1 \leq i \leq r \), in other words, \( R(A,P_0) \) is factorial.

(ii) There is exactly one \( 1 \leq i \leq r \) with \( l_i > 1 \).

(iii) There are \( 1 \leq i < j \leq r \) with \( l_i = l_j = 2 \) and \( l_u = 1 \) whenever \( u \notin \{i,j\} \).

Proof. Combine Corollary 3.4 with the rationality criterion Remark 2.2. \( \square \)

Lemma 3.6. Let \( R(A,P_0) \) be of Type 1 with \( X := \text{Spec} \, R(A,P_0) \) rational and assume that \( (l_1, \ldots, l_r) \) is decreasingly ordered. Then the number \( c(i) \) of irreducible components of \( V(X, T_{ij}) \) is given as

\[
\begin{array}{ccc|c}
 & 1 & 2 & \geq 3 \\
 c(i) & l_1 & l_2 & l_1 l_2 \\
\end{array}
\]

Proof. Due to Corollary 3.4 we can realize \( X \times \mathbb{C}^* \) as a principal open subset of the associated variety \( \bar{X} \) of Type 2. Then the irreducible components of \( V(X, T_{ij}) \times \mathbb{C}^* \) are in one-to-one correspondence with the irreducible components \( X \cap V(X, T_{ij}) \). The assertions follows. \( \square \)

Proposition 3.7. Let \( R(A,P_0) \) be non-factorial of Type 1 with \( \text{Spec} \, R(A,P_0) \) rational and \( (l_1, \ldots, l_r) \) decreasingly ordered. Define numbers \( n' := c(1)n_1 + \ldots + c(r)n_r \) and

\[
n_{i,1}, \ldots, n_{i,c(i)} := n_i, \quad l_{i,1}, \ldots, l_{i,c(i)} := \frac{1}{l_i}.
\]

Then the vectors \( l_{i,a} \in \mathbb{Z}^{n_{i,a}} \) build up an \( r' \times (n' + m) \) matrix \( P_0' \). With a suitable matrix \( A' \) the affine variety \( \text{Spec} \, R(A', P_0') \) is the total coordinate space of the affine variety \( \text{Spec} \, R(A,P_0) \).

Proof. First observe that the kernel of \( \mathbb{Z}^{n+m} \to K_0/K_0^{\text{tors}} \) is generated by the rows of the following \( r \times (n + m) \) matrix:

\[
\begin{bmatrix}
\frac{1}{l_1} & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \frac{1}{l_r} & 0 & \ldots & 0
\end{bmatrix}
\]

Now one determines the Cox ring of \( X = \text{Spec} \, R(A,P_0) \) in the same manner as in the proof of [1, Prop. 6.6] by exchanging the matrix \( P_1 \) used there by the matrix above and applying Lemma 3.6. \( \square \)

Proof of Theorem 1.2. If \( R(A,P_0) \) is rational of Type 1, then Proposition 3.7 shows that the Cox ring of \( \text{Spec} \, R(A,P_0) \) is factorial. Thus, Cox ring iteration is possible for \( X \) if and only if the total coordinate space of \( X \) is rational. Moreover, if the latter holds then the Cox ring iteration ends with at most one step. \( \square \)

References

[1] M. Arzhantsev, L. Braun, J. Hausen, M. Wrobel Log terminal singularities, platonic tuples and iteration of Cox rings. Preprint. [arXiv:1703.03627]

[2] I. Arzhantsev, U. Derenthal, J. Hausen, A. Laface: Cox rings. Cambridge Studies in Advanced Mathematics, Vol. 144. Cambridge University Press, Cambridge, 2014.
[3] J. Hausen, E. Herppich: Factorially graded rings of complexity one. Torsors, étale homotopy and applications to rational points, 414–428, London Math. Soc. Lecture Note Ser., 405, Cambridge Univ. Press, Cambridge, 2013.

[4] J. Hausen, H. Süss: The Cox ring of an algebraic variety with torus action. Adv. Math. 225 (2010), no. 2, 977–1012.

[5] J. Hausen, M. Wrobel: Non-complete rational $T$-varieties of complexity one. Math. Nachr, to appear. DOI: 10.1002/mana.201600009

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail address: juergen.hausen@uni-tuebingen.de

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail address: milena.wrobel@math.uni-tuebingen.de