Generalization of order separability for free groups.

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Abstract

In this work the author studies the property close to property of order separability.

Key words: free groups, residual properties.

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1 Introduction.

Definition 1. We say that a group $G$ is order separable if, for each elements $g, h$ of $G$ such that $g$ and $h^{\pm 1}$ are not conjugate, there exists a homomorphism of $G$ onto a finite group such that the orders of the images of $g$ and $h$ are different.

In [1] it is proved that free groups are order separable. This result was generalized in [2] where it was shown that each free group $F$ is omnipotent, that is, for an arbitrary finite set of elements $g_1, ..., g_n$ of $F$ such that for each $i \neq j$ elements $g_i$ and $g_j$ have no nontrivial conjugate powers there exists a constant $k$ such that for each ordered sequence of positive integers $t_1, ..., t_n$ there exists a homomorphism $\varphi$ of $F$ onto a finite group such that the order of $\varphi(g_i)$ is $kl_i$.

It is known also that the property of order separability is inherited by free products [3]. The following theorem generalizes the property of order separability for free groups.

Theorem. Let $u_1, u_2$ be the elements of a free group $F$, which does not belong to conjugate cyclic subgroups. Then for each prime number $p$ and for each integer $m$ there exists a homomorphism $\varphi$ of $F$ onto a finite $p$-group such that $u_1$ and $u_2$ do not belong to the kernel of $\varphi$ and $|\varphi(u_1)| / |\varphi(u_2)| = p^m$.

2 Notations and definitions.

We shall use the correspondence between the actions of a free group $F(x, y)$ with basis $x, y$ and the graphs satisfying the following properties:

1) for each vertex $p$ of a graph there exist exactly one edge with label $x$ that goes away from this vertex and exactly one edge with label $y$ that goes into $p$;

2) we consider that for each labelled edge there exists the corresponding inverse edge and each two edges with labels are not mutually inverse;

Definition 2. We say that the graph is the action graph of a free group $F = F(x, y)$ if it satisfies the properties 1), 2). We shall consider additionally that all labelled edges in this graph are positively oriented and set the orientation of this graph.

If $\varphi$ is a homomorphism of a group $F$ then the Cayley graph of the group $\varphi(F)$ with the generating set $\{\varphi(x), \varphi(y)\}$ is the action graph for $F$ (we identify labels $\varphi(x), \varphi(y)$ with $x, y$ correspondingly in this graph).
Let $\Gamma$ be the action graph of the free group $F$. Fix a natural number $n$, vertex $p$ of the graph $\Gamma$ and element $z \in \{x^{\pm 1}, y^{\pm 1}\}$ of the group $F$. Consider $n$ copies of the graph $\Gamma : \Delta_1, \ldots, \Delta_n$. Let $p_i$ be the vertex of the graph $\Delta_i$ corresponding to the vertex $p$ of the graph $\Gamma$. If $z \in \{x, y\}$ then denote by $q_i$ the vertex such that there exists the edge $f_i$ with label $z$ which goes into $q_i$ from $p_i$. If $z \in \{x^{-1}, y^{-1}\}$ then denote by $q_i$ the vertex such that there exists the edge $f_i$ with label $z^{-1}$ which goes away from $q_i$ into $p_i$. We construct the graph $\Delta = \gamma_n(\Gamma; p; z)$ from graphs $\Delta_i, 1 \leq i \leq n$ deleting edges $f_i$ and connecting vertexes $p_i$ and $q_{i+1}$ by an edge whose label equals either $z$ when $z \in \{x, y\}$ or $z^{-1}$ when $z \in \{x^{-1}, y^{-1}\}$ (indices are modulo $n$). If $z \in \{x, y\}$ then this new edge goes away from the vertex $p_i$. If $z \in \{x^{-1}, y^{-1}\}$ then this new edge goes into the vertex $p_i$. The graph $\Delta$ is the action graph of the group $F$.

If $S$ is a path in a graph then $\alpha(S), \omega(S)$ are the beginning and the end of $S$ correspondingly. If $S = e_1 \ldots e_n$ is a path in the action graph of the group $F$, then $\text{Lab}(S) = \text{Lab}(e_1)' \ldots \text{Lab}(e_n)'$ is the label of the path $S$, where $\text{Lab}(e_i)' = \text{Lab}(e_i)$ is a label of edge $e_i$ in case $e_i$ is positively oriented and if the edge $e_i$ is negatively oriented then $\text{Lab}(e_i)' = \text{Lab}(e_i)^{-1}$, where edges $e_i, e_i'$ are mutually inverse.

Definition 3. Suppose we have two action graphs of the group $F$ — $\Gamma$ and $\gamma_n(\Gamma; q; z)$. If $p$ is a vertex of $\Gamma$ then $p'$ is a vertex in the graph $\gamma_n(\Gamma; q; z)$ which belongs to $i$-th copy of the graph $\Gamma$ and corresponds to the vertex $p$. If $S$ is a path in $\Gamma$ then $S^i$ is the path in $\gamma_n(\Gamma; q; z)$ which goes from the vertex $\alpha(S)^i$ and whose label is equal to the label of the path $S$.

Definition 4. Let $u$ be a cyclically reduced element of the free group $F$. If $S = e_1 \ldots e_n$ is a closed path without returnings whose label equals $u^k$ in the action graph of $F$ then we say that $S$ is the $u$-cycle whose length equals $k$ (we consider that there exists exactly one subpath of the path $S$ ending in $\omega(S)$ whose label equals $u$).

## 3 Proof of theorem.

Fix elements $u_1, u_2$ of $F$. We consider that these elements are cyclically reduced.

Suppose that $u_i = u_i^{p^m_i t_i}, i = 1, 2$, where $p \nmid t_i$. Denote by $v_i$ the elements $u_i^{t_i}, i = 1, 2$. Without loss of generality we may assume that $m_1 \geq m_2$. If there exists a homomorphism $\psi$ of $F$ onto a finite group such that $|\psi(v_1)| / |\psi(v_2)| = p^{n+m_1-m_2}$ then $|\psi(u_1)| / |\psi(u_2)| = p^n$. Hence we may assume that the subgroups $\langle u_1 \rangle$ and $\langle u_2 \rangle$ are $p'$-isolated.

There exists a homomorphism $\varphi$ of $F$ onto a finite $p'$-group such that the images of the elements $u_1, u_2$ are nonunit $p'$-elements [4]. Since all $p'$-isolated cyclic subgroups of free groups are separable in the class of finite $p'$-groups we may also assume that all $u_1$- and $u_2$-cycles in the Cayley graph $\Gamma$ of the group $\varphi(G)$ with the set of generators $x = \varphi(x'), y = \varphi(y')$ are simple. Without loss of generality we may consider that $|\varphi(u_1)| = p^k, |\varphi(u_2)| = p^{k+l}, k \geq 1$. Put $\Gamma_{-1} = \Gamma$. Let $u_1 = y_{-1}^{\varepsilon_1} \ldots y_{-1}^{\varepsilon_{-1}}$, $y_i \in \{x, y\}, \varepsilon_i \in \{-1, 1\}$ be the reduced form of the element $u_1$ in the basis $x, y$. Fix an arbitrary vertex $q$ in the graph $\Gamma_{-1}$.
For $i > -1$ we shall define by the induction the graph $\Gamma_i = \gamma_p(\Gamma_{i-1}; q_{i-1}; y_{i}^2)$ and the path $S_i$ in $\Gamma_i$ whose length equals $i + 1$, where $i'$ is the remainder from a division of $i$ on $k$. If $i > -1$ then $q_i$ is the vertex of the graph $\Gamma_i$ which is the end point of the path $S_i$. The vertex $q_{-1}$ is equal to the vertex $q$. If $i > -1$ we define a path $S_i$ in the graph $\Gamma_i$ in the following way. If $i = 0$ then $S_0$ is the first edge of the $u_1$-cycle which goes from the vertex $q_{-1}$ or its inverse. In case $i > 0$ we define the path $S_i$ as $S_{i-1} \cup f_i$ where $f_i$ is the edge of the graph $\Gamma_i$ one of whose endpoints coincides with $\omega(S_{i-1})$ and the label of $f_i$ or its inverse equals $y_i$. Also if $\varepsilon_i = 1$ then $f_i$ is positively oriented and has a label. If $\varepsilon_i = -1$ then $f_i$ is negatively oriented. It is easy to notice that the length of each $u_1$- or $u_2$-cycle in $\Gamma_i$ is the power of $p$. Since all $u_1$- and $u_2$-cycles are simple in the graph $\Gamma_{-1}$ this condition is also held for $\Gamma_i$ for each $i$. Besides for each $i$ the graph $\Gamma_i$ contains the maximal $u_1$-cycle which contains the path $S_i$.

Suppose that there exists $i$ such that the following conditions are true. In the graph $\Gamma_j$ each maximal $u_2$-cycle contains the path $S_j$ for all $j \leq i$ but not for $j = i + 1$ (if $i + 1 = 0$ we simply consider that in the graph $\Gamma_0$ not all maximal $u_2$-cycles contains $S_0$). Notice that if $j \leq i$ then in the graph $\Gamma_j$ the length of the maximal $u_1$-cycle coincide with $|\varphi(u_1)|p^{i+1}$, $i = 1, 2$. In the graph $\Gamma_{i+1}$ the length of the maximal $u_1$-cycle equals $|\varphi(u_1)|p^{i+1}$. Let's find the length of the maximal $u_2$-cycle in $\Gamma_{i+1}$. Fix a vertex $r$ of $\Gamma_i$. Then the length of the $u_2$-cycle which goes away from $r$ is equal to $|\varphi(u_2)|p^k$, $k \leq i$. Then for each $l = 1, ..., p$ in the graph $\Gamma_{i+1}$ the length of the $u_2$-cycle that goes away from the vertex $r_l$ is not more than $|\varphi(u_2)|p^{k+1}$. Suppose that the $u_2$-cycle $T$ of $\Gamma_i$ starting from $r$ is maximal, that is its length coincide with $|\varphi(u_2)|p^i$. The path $S_i$ is contained in $T$. The $u_2$-cycle $T'$ of $\Gamma_{i+1}$ contains the path $S_i$. From the condition on $i$ and the simplicity of $u_2$-cycles of $\Gamma_{i+1}$ we may deduce that $T'$ does not contain the edge $f_{i+1}$. This $u_2$-cycle does not also contain the edge connecting the first and the last copies of the graph $\Gamma_i$ because otherwise $T$ would have the self-intersection in the vertex $\omega(S_{i-1})$. Hence the length of $T'$ coincide with the length of $T$. It means that the length of the maximal $u_2$-cycle in $\Gamma_{i+1}$ coincide with $|\varphi(u_2)|p^i$. Since $u_1$ and $u_2$ does not belong to the conjugate cyclic subgroups then there exists $i$ that satisfies the conditions mentioned above.

The number of vertices in the graph $\Gamma_{i+1}$ equals $|\varphi(F)|p^{i+2}$. So there exists a homomorphism $\varphi_1$ of $F$ onto a finite $p$-group such that in the Cayley graph of the group $\varphi_1(F)$ with the set of generators $\varphi_1(x')$, $\varphi_1(y')$ all $u_1$- and $u_2$-cycles are simple. Besides $|\varphi_1(u_2)|/|\varphi_1(u_1)| = |\varphi(u_2)|/|\varphi(u_1)| \ast 1/p$.

If we construct graphs $\Gamma_i$ using the element $u_2$ instead of $u_1$ we obtain the homomorphism $\varphi_2$ that satisfies the same conditions concerning to the $u_1$- and $u_2$-cycles in the graph $\varphi_2(F)$ and the homomorphism $\varphi_1$ and $|\varphi_2(u_2)|/|\varphi_2(u_1)| = |\varphi(u_2)|/|\varphi(u_1)| \ast p$. In order to obtain the homomorphism of $F$ onto a finite $p$-group such that the ratio of orders of the images of $u_1$ and $u_2$ sufficient $p^n$ it is enough to apply these procedures several times. The theorem is proved.

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