ABSTRACT

A set of new adaptive control algorithms is presented. The algorithms are applicable to linearly parameterized systems and to nonlinearly parameterized systems satisfying a certain monotonicity requirement. A variational perspective based on the Bregman Lagrangian (Wibisono, Wilson, & Jordan, 2016) is adopted to define a general framework that systematically generates higher-order in-time speed gradient algorithms. These general algorithms reduce to a recently developed method as a special case, and naturally extend to composite methods that combine perspectives from both adaptive control and system identification. A provocative connection between adaptive control algorithms for nonlinearly parameterized systems and algorithms for isotonic regression and provable neural network learning is utilized to extend these higher-order algorithms to the realm of nonlinearly parameterized systems. Modifications to all presented algorithms inspired by recent progress in distributed stochastic gradient descent algorithms are developed. Time-varying learning rate matrices based on exponential forgetting/bounded gain forgetting least squares can be stably employed in the higher-order context, and conditions for their applicability in the nonlinearly parameterized setting are provided. A consistent theme of our contribution is the exploitation of strong connections between classical nonlinear adaptive control techniques and recent progress in optimization and machine learning.

1 Introduction

Adaptive control is an online learning problem concerned with controlling an unknown dynamical system. This task is accomplished by constructing an approximation \( \hat{f}(x, \hat{a}, t) \) to the true dynamics \( f(x, a, t) \) through the adjustment of a set of parameters \( \hat{a}(t) \). The fundamental goal of adaptive control is stable concurrent learning and control of dynamical systems.

Adaptive control theory is a mature field, and many results exist tailored to specific system structures (Ioannou & Sun, 2012; Narendra & Annaswamy, 2005; J.-J. Slotine & Li, 1991). An adaptive control algorithm typically consists of a parameter estimator coupled in feedback to the controlled system. While often strongly inspired by gradient-based optimization algorithms, an added complexity is that the estimator must not only be convergent, but must be stable when connected with the system in feedback.

Despite the difficulty of the problem, significant success has been achieved even for nonlinear systems in the linearly parameterized setting, where the dynamics approximation is of the form \( \hat{f} = Y(x, t)\hat{a} \) for some known regressor \( Y(x, t) \). Prominent examples include the adaptive robot trajectory controller of J.-J. E. Slotine and Li (1987) and the neural network-based controller of Sanner and Slotine (1992), which employs a mathematical expansion in physical basis functions to uniformly approximate the unknown dynamics.

Unlike its linear counterpart, solutions to the adaptive control problem in the general nonlinearly parameterized setting \( \hat{f} = f(x, \hat{a}, t) \) have remained elusive. Intuitively, this is unsurprising because gradient-based algorithms generally have guarantees only for convex loss functions: in the linearly parameterized setting, this requirement will be satisfied, but when the parameters appear nonlinearly, the problem is immediately in the difficult realm of non-convex optimization. Nevertheless, progress has been made in specific cases, such as with a convex or concave parameterization (Ai-Poh Loh, Annaswamy, & Skantze, 1999; Annaswamy, Skantze, & Loh, 1998; Kojić & Annaswamy, 2002), with a monotonicity condition (I. Tyukin, 2011; Tyukin, Prokhorov, & van Leeuwen, 2007), through the Immersion
and Invariance (I&I) approach [Astolfi & Ortega, 2003; Liu, Ortega, Su, & Chu, 2010], and through the speed gradient method (A. L. Fradkov, 1999; Fradkov, 1979; Ortega, 1996).

Our work continues in a recent tradition that utilizes a continuous-time view to analyze optimization algorithms; see, for example, Betancourt, Jordan, and Wilson (2018); Bofill and Slotine (2020); Diakonikolas and Jordan (2019); Maddison, Paulin, Teh, O’Donoghue, and Doucet (2018); Muehlebach and Jordan (2019); Wibisono et al. (2016). While the continuous-time view of optimization has seen a resurgence after it was used by Su, Boyd, and Candès (2016) to provide an intuitive justification for Nesterov’s accelerated gradient method (Nesterov, 1983), continuous-time differential equations were used as early as 1964 by Polyak to derive the classical momentum or “heavy ball” optimization method (B. Polyak, 1964). Given the gradient-based nature of many adaptive control algorithms, the continuous-time view of optimization provides a natural bridge from modern optimization to modern adaptive control.

Continuous-time often affords simpler proofs, and it enables the application of physical intuition when reasoning about optimization algorithms, but finding the limiting differential equations may still be a daunting task. In a significant advance, Wibisono et al. (2016) showed that many accelerated methods in optimization can all be derived via a variational point of view from a single mathematical object known as the Bregman Lagrangian.

In this paper, we contribute to the linearly and nonlinearly parameterized (under the monotonicity assumptions of (Tyukin et al., 2007)) problems. We utilize the Bregman Lagrangian in tandem with the speed gradient formalism (A. L. Fradkov, 1999; Fradkov, 1979) to define a general methodology to generate higher-order in-time (Morse, 1992) speed gradient algorithms. This contribution generalizes and extends a recently developed algorithm (Gaudio, Gibson, Annaswamy, & Bolender, 2019), which can be seen as a special case of one of our higher-order speed gradient laws. Based on the first-order speed gradient methodology, these higher-order laws lead naturally to the development of composite higher-order adaptation algorithms for linearly parameterized systems (J.-J. Slotine & Li, 1991). By use of a proportional-integral (PI) form, these composite laws are driven directly by the function approximation error ̂f = f – f itself, and do not require any explicit filtering of the system dynamics. Much like the well-known reduced-order Luenberger observer, the PI form enables obtaining ̂f in the adaptation law despite the fact that this signal is not explicitly measured (Luenberger, 1979).

By analogy between the nonlinearly parameterized law presented by Tyukin et al. (2007) and recent results in isotonic regression (Goel & Klivans, 2017; Goel, Klivans, & Meka, 2018; Kakade, Kalai, Kanade, & Shamir, 2011), we extend these higher-order algorithms to the nonlinearly parameterized setting. In a similar vein, we draw an orthogonal connecting thread to machine learning, and demonstrate a stable modification to our algorithms inspired by the Elastic Averaging Stochastic Gradient Descent (EASGD) algorithm (Zhang, Choromanska, & LeCun, 2014). We then show how to combine all of our algorithms with time-dependent learning rates through the bounded gain forgetting formalism (J.-J. Slotine & Li, 1991).

The paper is organized as follows. In Sec. 2, we present some required mathematical background. This includes a basic review of direct adaptive control (Sec. 2.1), the speed gradient formalism (Sec. 2.2), the Bregman Lagrangian and a higher-order adaptive control algorithm (Sec. 2.3), and the reduced-order Luenberger observer (Sec. 2.4). Sec. 3 presents our main contributions, with our higher-order speed gradient laws in Sec. 3.1 our first-order non-filtered composite laws in Sec. 3.2 our higher-order composite laws and higher-order laws for nonlinearly parameterized systems in Sec. 3.3 the elastic modification in Sec. 3.4 and our extension to time-dependent learning rates in Sec. 3.5. We conclude with some closing remarks and future directions in Sec. 4.

2 Preliminaries

2.1 Direct adaptive control

We begin with an introduction to the formalism of direct adaptive control, and describe the systems to which our results apply. For simplicity, we restrict ourselves to the class of nth-order nonlinear systems

\[ x^{(n)} + f(x, a, t) = u \]  

(1)

where \( x^{(i)} \in \mathbb{R} \) denotes the \( i \)-th derivative of \( x \), \( x = (x, x^{(1)}, \ldots, x^{(n-1)}) \in \mathbb{R}^n \) is the overall system state, \( a \in \mathbb{R}^p \) is a vector of unknown parameters, \( f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}_+ \rightarrow \mathbb{R} \) is of known functional form but is unknown through its dependence on \( a \), and \( u \in \mathbb{R} \) is the control input. We seek to design a feedback control law \( u = u(x, a) \) that depends on a set of estimated parameters \( \hat{a} \in \mathbb{R}^n \) and ensures that \( x(t) \rightarrow x_d(t) \) where \( x_d(t) \in \mathbb{R}^n \) is a known desired trajectory. We require that, along the way, the system remains stable and all system signals remain bounded. The estimated parameters \( \hat{a} \) are updated according to a learning rule or adaptation law

\[ \dot{\hat{a}} = g(a, \hat{a}, x) \]  

(2)
where \( g : \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^n \rightarrow \mathbb{R}^p \) must be implementable solely in terms of known system signals despite its potential dependence on \( a \). Hence, adaptive control is fundamentally an online learning problem where the data-generating process is a nonlinear dynamical system coupled in feedback to the learning process. For \( n \)th order systems as considered in (1), a common approach is to define the sliding variable (J.-J. Slotine & Li, 1991)

\[
s = \left( \frac{d}{dt} + \lambda \right)^{n-1} \tilde{x} = \tilde{x}^{(n-1)} - \tilde{x}_r^{(n-1)}
\]

where we have defined \( \tilde{x}(t) = x(t) - x_d(t) \) and \( \tilde{x}_r^{(n-1)} \) as the remainder. According to the definition (3), \( s \) obeys the differential equation

\[
\dot{s} = u - f(x, a, t) - \tilde{x}_r^{(n-1)}.
\]

Hence, from (4), we may choose \( u = f(x, \hat{a}, t) + \tilde{x}_r^{(n-1)} - \eta s \) to obtain the stable first-order linear filter

\[
\dot{\hat{s}} = -\eta s + f(x, \hat{a}, t) - f(x, a, t).
\]

For future convenience, we define \( \hat{f}(x, \hat{a}, a, t) = f(x, \hat{a}, t) - f(x, a, t) \) and we will omit its arguments when clear from the context. From the definition of \( s \) in (5), \( s = 0 \) defines the dynamics

\[
\left( \frac{d}{dt} + \lambda \right)^{n-1} \tilde{x} = 0.
\]

Equation (6) is a stable \((n-1)\)th-order filter which ensures that \( \tilde{x} \rightarrow 0 \) exponentially. For systems of the form (1), it is thus sufficient to consider the two first-order dynamics (2) and (5), and the adaptive control problem has been reduced to finding a learning algorithm that ensures \( s \rightarrow 0 \).

**Remark 2.1.** Systems in the matched uncertainty form

\[
\dot{x} = Ax + b(u - f(x, a, t))
\]

where the constant pair \((A, b)\) is controllable and the constant parameter vector \( a \) in the nonlinear function \( f(x, a, t) \) is unknown, can always be put in the form (1) by using a state transformation to the second controllability canonical form — see Luenberger (1979), Chapter 8.8. After such a transformation, the new state variables \( z \) satisfy \( \ddot{z}_i = \dot{z}_{i+1} \) for \( i < n \) and \( \ddot{z}_n = -\sum_{i=1}^{n-1} a_i z_i + u - f(x, a, t) \). Defining \( s \) as in (3) and correctly computing \( u \) leads to (5). Hence, all results in this paper extend immediately to such systems.

**Remark 2.2.** The fundamental utility of defining the variable \( s \) is its conversion of the adaptive control problem for the \( n \)th-order system (1) to an adaptive control problem for the first-order system (5). Our results may be simply extended to other error models (Ai-Poh Loh et al., 1999; Narendra & Annaswamy, 2005) of the form (5), or error models with similar input-output guarantees, as summarized by Lemma A.1.

**Remark 2.3.** We will use \( f \) to denote the equivalent first-order system to (1). \( \hat{x} = f(x, a, t) + u \), where \( f = (x_1, x_2, \ldots, f(x, a, t)) \) and \( u = (0, 0, \ldots, u) \).

A classic setting for adaptive control is when the unknown nonlinear dynamics depends linearly on the set of unknown parameters, that is

\[
f(x, a, t) = Y(x, t)a
\]

with \( Y(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{1 \times p} \) a known function. In this setting, a well-known algorithm is the adaptive controller of J.-J. Slotine and Coetsee (1986), given by

\[
\dot{\hat{a}} = -sPY^T
\]

and its extension to multi-input adaptive robot control (J.-J. E. Slotine & Li, 1987), where \( P = P^T > 0 \in \mathbb{R}^{p \times p} \) is a constant positive definite matrix of learning rates. Consideration of the Lyapunov-like function \( V = \frac{1}{2}s^2 + \frac{1}{2}a^TP^{-1}a \) shows stability of the feedback interconnection of (5) and (7) via an application of Barbalat’s Lemma (J.-J. Slotine & Li, 1991). We will refer to (7) as the Slotine and Li controller.

In this work, we make a mild additional assumption that simplifies some of the proofs. It requires the following definition.

**Definition 2.1.** A function \( f(x, \hat{a}, t) \) is said to be locally bounded in \( \hat{x} \) and \( \hat{a} \) uniformly in \( t \) if it is bounded whenever \( \|x\| \) and \( \|\hat{a}\| \) are finite.

Following Definition 2.1, we make the following assumption.

**Assumption 2.1.** The dynamics \( f(x, \hat{a}, t) \) is locally bounded in \( \hat{x} \) and \( \hat{a} \) uniformly in \( t \).
2.2 Speed gradient algorithms

We now provide a brief introduction to a class of adaptive control methods known as speed gradient algorithms \( A. L. Fradkov \) [1999, Fradkov, 1979, Ortega, 1996]. Speed gradient algorithms are applicable to nonlinearly parameterized systems that satisfy a convexity requirement described in Assumption 2.4. In their most basic form, these algorithms are specified by a “local” goal functional \( Q(x, t) \) in the following proposition (A. L. Fradkov, 1999).

**Assumption 2.2.** \( Q(x, t) \) is non-negative and radially unbounded: \( Q \geq 0 \) for all \( x, t \) and \( Q(x, t) \to \infty \) when \( \|x\| \to \infty \). \( Q(x, t) \) is also uniformly continuous in \( t \) whenever \( x \) is bounded.

**Assumption 2.3.** There exists an ideal set of controller parameters \( \hat{a} \) such that the origin of the system \( (\ref{eq:1}) \) is globally asymptotically stable when the control is evaluated at \( \hat{a} \). Furthermore, \( Q \) is a Lyapunov function for the system when the control is evaluated at \( a \). That is, there exists a strictly increasing function \( \rho \) such that \( \rho(0) = 0 \) with \( Q(x, a, t) \leq -\rho(Q) \).

**Assumption 2.4.** The time-derivative of \( Q \) is convex in the control parameters \( \hat{a} \). The first-order condition for convexity

\[
\dot{Q}(x, a_1, t) \geq \dot{Q}(x, a_2, t) + (a_1 - a_2)^T \nabla_{a_2} \dot{Q}(x, a_2, t) \tag{8}
\]

must be satisfied for all \( a_1, a_2 \).

When Assumptions 2.2-2.4 are satisfied, the speed gradient adaptive control law is defined as

\[
\dot{\hat{a}} = -P\nabla_{\hat{a}} \dot{Q}(x, \hat{a}, t). \tag{9}
\]

where \( P = P^T > 0 \) is a positive definite matrix of learning rates. Its properties are summarized in the following proposition (A. L. Fradkov, 1999).

**Proposition 2.1.** Consider the speed gradient algorithm (9) with local goal functional \( Q(x, t) \) satisfying Assumptions 2.2-2.4 Then, all solutions \( (x(t), \hat{a}(t)) \) given by (10) and (9) remain bounded, and

\[
\lim_{t \to \infty} Q(x(t), t) = 0
\]

for all \( x(0) \in \mathbb{R}^n \).

The proof follows by consideration of the Lyapunov function \( V = Q + \frac{1}{2} \hat{a}^T P^{-1} \hat{a} \). Intuitively, while the goal functional \( Q \) may only depend on the control parameters \( \hat{a} \) indirectly through \( x \), its time derivative will depend explicitly on \( \hat{a} \) through \( x \). The adaptation law (9) ensures that \( \hat{a} \) moves in a direction to decrease \( \dot{Q} \). Under the conditions specified by Assumptions 2.2-2.4 this causes \( \hat{Q} \) to be negative for long enough to accomplish the desired goal (A. L. Fradkov, 1999).

**Remark 2.4.** If \( Q \) is chosen so that \( \dot{Q} \) depends on \( \hat{a} \) only through \( f(x, \hat{a}, t) \) and \( f \) is linearly parameterized, then Assumption 2.4 will immediately be satisfied by convexity of affine functions. Indeed, consider defining the goal functional \( Q = \frac{1}{2} s^2 \) for system (1). It is clear that this proposed goal functional satisfies Assumptions 2.2 and 2.3.

Then \( \dot{Q} = -\eta s^2 + s \tilde{f} \), and (9) exactly recovers the Slotine and Li controller (7), originally derived based on Lyapunov considerations.

In this sense, speed gradient algorithms represent a flexible class of methods that contain as particular cases some pre-existing approaches.

Rather than a local functional, one may instead specify an integral goal functional of the form \( Q(x, \hat{a}, t) = \int_0^t R(x(t'), \hat{a}(t'), t') dt' \). In this case, (9) takes the form

\[
\dot{\hat{a}} = -P\nabla_{\hat{a}} R(x, \hat{a}, t). \tag{10}
\]

Equation (10) is a gradient flow algorithm on the loss function \( R(x, \hat{a}, t) \). Replacing Assumptions 2.2 and 2.3 by the slightly modified setting:

**Assumption 2.5.** \( R \) is a non-negative function: \( R(x(t), \hat{a}(t), t) \geq 0 \) for all \( t \).

**Assumption 2.6.** There exists an ideal set of controller parameters \( a \) and a scalar function \( \mu \) such that \( \int_0^\infty \mu(t') dt' < \infty \), \lim_{t \to \infty} \mu(t) = 0 \), and \( R(x(t), a, t) \leq \mu(t) \) for all \( t \).

The properties of algorithm (10) are summarized in the following proposition (A. L. Fradkov, 1999).

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1 Strictly speaking, \( \frac{1}{2} s^2 \) does not have to be uniformly continuous in \( t \) for bounded \( x \). This is a technical condition useful for the proof, but in many cases stability may still be shown via other means.

2 Note that \( s \tilde{f} \) is still convex in \( \hat{a} \) despite the fact that \( s \) may change sign because \( \tilde{f} \) is linear in \( \hat{a} \) by assumption.
Proposition 2.2. Consider the speed gradient algorithm (10) where the goal functional $Q$ satisfies Assumptions 2.4-2.6. Then $Q(t) \leq \alpha$ where
\[
\alpha = \frac{1}{2} \hat{a}(0)^T P^{-1} \hat{a}(0) + \int_0^\infty \mu(t') dt'
\]
and $R \to 0$ for any bounded solution $x(t)$.

Integral functionals allow the specification of a control goal that depends on all past data and that may have an explicit dependence on $\hat{a}$. $R$ is chosen so that it does not depend on the structure of the dynamics. Local functionals, on the other hand, result in adaptation laws that do have an explicit dependence on the dynamics through the appearance of the term $\left(\frac{\partial Q}{\partial x}\right)^T x$ in $Q$.

Integral functionals can be particularly useful if $R \to 0$ implies the desired control goal, and in this work, we will generally focus on the setting where $R = \frac{1}{2} f^2$ is the function approximation error. Goal functionals can also be written as a sum of local and integral functionals with similar guarantees, and these approaches will lead to composite algorithms in the subsequent sections; the interested reader is referred to A. L. Fradkov (1999), Chapter 3 for more details.

2.3 The Bregman Lagrangian and accelerated optimization algorithms

Beginning with the seminal paper of Su et al. (2016), there has been a recent revival of interest in the analysis of optimization algorithms as discretizations of continuous-time ordinary differential equations. From this angle, the analysis of an optimization algorithm can be broken into two steps: first, an understanding of the quantitative convergence rates of the continuous-time differential equation, and second, a search over numerical discretization techniques that preserve these convergence rates. In Wibisono et al. (2016), the Bregman Lagrangian was shown to generate a suite of accelerated optimization algorithms by appealing to the Euler Lagrange equations through the principle of least action.

Consider the speed gradient algorithm (10) where the goal functional $\alpha$ satisfies Assumptions 2.4-2.6. Then $Q(t) \leq \alpha$ where
\[
\alpha = \frac{1}{2} \hat{a}(0)^T P^{-1} \hat{a}(0) + \int_0^\infty \mu(t') dt'
\]
and $R \to 0$ for any bounded solution $x(t)$.

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The authors in Gaudio et al. (2019) recently utilized the Bregman Lagrangian to derive a momentum-like adaptive control algorithm. To do so, they defined $\alpha = \log(\beta N)$, $\beta = \log \left( \frac{\beta N}{\alpha} \right)$, and $\gamma = \int e^{\alpha} d\Gamma$. Here, $\gamma \geq 0$ and $\beta \geq 0$ are non-negative scalar hyperparameters and $N(t)$ is a context-dependent normalizing signal. With these definitions, and in the setting presented in Sec. 2.1, (11) reduces to
\[
L \left( \hat{a}, \dot{a}, t \right) = e^{\int_0^t \beta N(t) dt} \frac{1}{\beta N} \left( \frac{1}{2} \hat{a}^T \hat{a} - \gamma \beta N \frac{d}{dt} \left[ \frac{1}{2} e^{\gamma} \right] \right)
\]
\[\text{Note that these conditions validate the second ideal scaling condition but not the first. As mentioned above, the second ideal scaling condition is required only by the choice of Lyapunov function in the original work, which was used to derive convergence rates for optimization algorithms (Wibisono et al., 2016). In this sense, they are not required for adaptive control.}\]
\[\text{The authors in Gaudio et al. (2019) consider a linear dynamical system, where the form of (12) differs slightly but identical considerations apply.} \]
Comparing (11) and (12), it is clear that the loss function \( f(x) \) in (11) has been replaced by \( \frac{1}{2} s^2 \) in (12). Following Remark 2.4, this is precisely the \( Q \) speed gradient functional that gives rise to the Slotine and Li controller. For (12), the Euler-Lagrange equations lead to the adaptation law

\[
\dot{a} + a\left(\beta N - \frac{\dot{N}}{N}\right) = -\gamma \beta N s Y^T.
\]  

(13) may be understood as a higher-order version of the Slotine and Li adaptive controller. It may also be re-written as two first-order systems

\[
\dot{v} = -\gamma s Y^T, \quad (14)
\]

\[
\dot{a} = \beta N (\dot{v} - \dot{a}). \quad (15)
\]

(14) & (15) are useful for proving stability. The properties of (13) are summarized in the following proposition.

**Proposition 2.3.** Consider the higher-order adaptation algorithm (13) or its equivalent representation (14) & (15) with \( N = 1 + \mu \| Y \|^2 \) and \( \mu > \frac{1}{2} \). Then, all trajectories \((x, v, a)\) remain bounded, \( s \in L_{\infty} \cap L_2 \), \((\dot{a} - \dot{v}) \in L_2\), \( s \to 0 \) and \( x \to x_d \).

The proof follows by consideration of the Lyapunov function \( V = \frac{1}{2} \left( s^2 + \frac{1}{\gamma} \| \dot{v} \|^2 + \frac{1}{\gamma} \| \dot{a} \|^2 \right) \) (Gaudio et al., 2019).

**Remark 2.5.** While this transformation to a system of two first-order systems may seem somewhat ad-hoc, it is readily apparent by use of the Bregman Hamiltonian

\[
H(a, p) = \frac{1}{2} \beta Ne^{-\tau} \| p \|^2 + \gamma e^\tau \left[ \frac{d}{dt} \frac{1}{2} s^2 \right], \quad (16)
\]

which, via Hamilton’s equations, leads to

\[
\dot{p} = -\frac{\partial H}{\partial a} = -\gamma e^\tau s Y^T,
\]

\[
\dot{a} = \frac{\partial H}{\partial p} = \beta Ne^{-\tau} p.
\]

Defining \( \dot{v} = e^{-\tau} p + \dot{a} \) leads immediately to (14) & (15). As is typical in classical mechanics, the Bregman Hamiltonian may be obtained from a Legendre transform of the Bregman Lagrangian. The Hamiltonian equations may be useful for discrete-time algorithm development through application of symplectic discretization techniques (Betancourt et al., 2018; França, Sulam, Robinson, & Vidal, 2019; Shi, Du, Su, & Jordan, 2019).

### 2.4 Reduced-order observers and proportional-integral adaptation laws

The reduced-order Luenberger observer (Luenberger, 1979) is a key tool in linear systems theory. Consider an \( n \)-dimensional completely observable dynamical system

\[
\begin{align*}
\dot{x}_1 &= A_{11} x_1 + A_{12} x_2 + B_1 u \\
\dot{x}_2 &= A_{21} x_1 + A_{22} x_2 + B_2 u \\
y &= x_2
\end{align*}
\]

where \( x = (x_1^T, x_2^T)^T \in \mathbb{R}^n \) is the state, \( x_1 \in \mathbb{R}^{n-m} \), \( x_2 \in \mathbb{R}^m \), \( y \in \mathbb{R}^m \) is the system output, and the \( A_{ij} \) and \( B_i \) matrices are known. Define an auxiliary variable

\[
z = x_1 - L x_2, \quad (17)
\]

where \( L \) in (17) is an arbitrary design matrix. Then \( z \) obeys the dynamics

\[
\dot{z} = (A_{22} - LA_{12}) z + (A_{22} - LA_{12}) Ly + (A_{21} - LA_{11}) y + (B_2 - LB_1) u \quad (18)
\]

where it is clear that all quantities in (13) are known except for \( z \) itself. Constructing an observer of identical form

\[
\dot{\hat{z}} = (A_{22} - LA_{12}) \hat{z} + (A_{22} - LA_{12}) Ly + (A_{21} - LA_{11}) y + (B_2 - LB_1) u
\]
ensures that the error \( \dot{z} = \ddot{z} - \dot{z} \) obeys the dynamics
\[
\dot{z} = (A_{22} - LA_{12}) \ddot{z}.
\]

Because the original system is completely observable, \( A_{22} - LA_{12} \) can be selected by choice of \( L \) to have arbitrary eigenvalues by the pole placement theorem. The original state variables can then be reconstructed as \( \dot{x}_1 = \ddot{z} + Ly \).

While \( x_2 \) is unknown through its dependence on \( x_1 \), \( x_2 \) itself is known. Intuitively, unknown quantities contained in \( x_2 \) can thus be obtained in the observer dynamics through a proportional term containing \( x_2 \). Similar concepts can be extended to nonlinear observers; see Lohmiller and Slotine (1993), Sec. 4.1. This idea of gaining a “free” derivative has also been used in adaptive control, with particular success when applied to nonlinear parameterizations. Proportional-integral adaptive laws of this type have been known as algorithms in finite form (A. L. Fradkov, 1999; I. Y. Tyukin, 2003) and appear in the well-known I&I framework (Astolfi & Ortega, 2003; Liu et al., 2010). This approach will be the basis for our algorithms for nonlinearly parameterized systems.

3 Main contributions

In this section, we present the main contributions of this work. We begin by noting that the Bregman Lagrangian generates higher-order speed gradient algorithms, of which the adaptation law (13) is a special case. We prove some general conditions under which these higher-order algorithms will achieve tracking. By analogy with integral speed gradient functionals, we derive a proportional-integral scheme to implement a first-order composite adaptation law (J.-J. Slotine & Li, 1991) driven directly by the function approximation error rather than its filtered version. We subsequently fuse the generating functional for the composite law with the Bregman Lagrangian to construct a higher-order composite algorithm.

Combining a connection between the techniques of isotonic regression (Goel & Klivans. 2017; Goel et al., 2018; Kakade et al., 2011) and algorithms for monotone nonlinear parameterizations (I. Tyukin, 2011; Tyukin et al., 2007), we demonstrate how to modify our higher-order speed gradient framework to derive higher-order algorithms for nonlinearly parameterized systems. We follow this development by discussing a new form of high-order algorithm inspired by the Elastic Averaging Stochastic Gradient Descent (EASGD) algorithm (Boffi & Slotine, 2020; Zhang et al., 2014) and extensions to distributed adaptation (Wensing & Slotine, 2017). We conclude by demonstrating how to use time-varying learning rates based on the bounded gain forgetting technique with our presented algorithms (J.-J. Slotine & Li, 1991).

3.1 Accelerated speed gradient algorithms

As noted in Sec. 2.3, the Bregman Lagrangian (12) that generates the higher-order algorithm (13) contains the local speed gradient functional \( \dot{Q}(x) = \frac{1}{2} x^T a \) that gives rise to the Slotine and Li controller. Based on this observation, we define local and integral higher-order speed gradient algorithms via the Bregman Lagrangian. We begin with the local functional
\[
\mathcal{L} \left( \dot{a}, \ddot{a}, t \right) = e^{\int_0^t \beta N(t) dt} \frac{1}{\beta N(t)} \left( \frac{1}{2} \dot{a}^T \dot{a} - \gamma \beta N(t) \frac{d}{dt} \dot{Q}(x, t) \right),
\]
which generates the higher-order law
\[
\dddot{a} + \dot{a} \left( \beta N - \frac{\dot{N}}{N} \right) = -\gamma \beta N \frac{\partial \dot{Q}(x, \dot{a}, t)}{\partial \dot{a}}.
\]  
(19)

Algorithm (19) can be re-written as two first-order systems
\[
\dot{v} = -\gamma \frac{\partial \dot{Q}(x, \dot{a}, t)}{\partial \dot{a}} \quad \text{(20)}
\]
\[
\dot{\dot{a}} = \beta N \left( \dot{v} - \dot{a} \right). \quad \text{(21)}
\]

To achieve the control goal, we require the following technical assumption in addition to Assumptions 2.2 & 2.4. This assumption replaces Assumption 2.3 for first-order speed gradient algorithms.

**Assumption 3.1.** There exists a time-varying normalizing signal \( N(t) \) and non-negative scalar values \( \beta \geq 0, \mu \geq 0 \) such that the time-derivative of the goal functional evaluated at the true parameters, \( \dot{Q}(x, \dot{a}, t) \), satisfies the following inequality:
\[
\dot{Q}(x, \dot{a}, t) - \frac{\beta \mu}{\gamma} N(t) \left\| \dot{a} - \dot{v} \right\|^2 + 2 (\dot{a} - \dot{v})^T \frac{\partial \dot{Q}(x, \dot{a}, t)}{\partial \dot{a}} \leq -\rho(Q)
\]  
(22)
where \( \rho(\cdot) \) is positive definite, uniformly continuous in \( Q \), and satisfies \( \rho(0) = 0 \).

With Assumption 3.1 in hand, we can state the following proposition.

**Proposition 3.1.** Consider the algorithm [19] or its equivalent form [20] & [27], and assume \( Q \) satisfies Assumptions 2.2, 2.3 and 3.7. Then, all solutions \( (x(t), v(t), \hat{a}(t)) \) remain bounded, \( (\hat{a} - \hat{v}) \in L_2 \), and \( \lim_{t \to \infty} Q = 0 \).

**Proof.** Consider the Lyapunov-like function

\[
V = Q(x, t) + \frac{1}{2\gamma} \hat{v}^T \hat{v} + \frac{1}{2\gamma} \hat{a}^T (\hat{a} - \hat{v}) .
\]

Equation (23) implies that, with \( N(t) = 1 + \mu N(t) \),

\[
\dot{V} = \dot{Q}(x, \hat{a}, t) - \hat{a}^T \frac{\partial \dot{Q}}{\partial \hat{a}} - \frac{\beta}{\gamma} \| \hat{a} - \hat{v} \|^2 - \frac{\beta \mu}{\gamma} N(t) \| \hat{a} - \hat{v} \|^2 + 2 (\hat{a} - \hat{v})^T \frac{\partial \dot{Q}}{\partial \hat{a}} \\
\leq \dot{Q}(x, a, t) - \frac{\beta}{\gamma} \| \hat{a} - \hat{v} \|^2 - \frac{\beta \mu}{\gamma} N(t) \| \hat{a} - \hat{v} \|^2 + 2 (\hat{a} - \hat{v})^T \frac{\partial \dot{Q}}{\partial \hat{a}} \\
\leq -\rho(Q) - \frac{\beta}{\gamma} \| \hat{a} - \hat{v} \|^2.
\]

By radial unboundedness of \( Q(x, t) \) in \( x \), (23) & (24) show that \( x \) remains bounded. Similarly, radial unboundedness of \( V \) in \( \hat{v} \) and \( \hat{a} - \hat{v} \) show that \( \hat{v} \) and \( \hat{a} \) remain bounded. Integrating (24) shows that \( \frac{2}{\gamma} \int_0^\infty \| \hat{a} - \hat{v} \|^2 dt \leq V(0) - V(\infty) < \infty \), so that \( (\hat{a} - \hat{v}) \in L_2 \). An identical argument shows that \( \int_0^\infty \rho(Q) dt < \infty \). Now, because \( x \) and \( \hat{a} \) are bounded, and because \( \bar{f}(x, \hat{a}, t) \) is locally bounded in \( x \) and \( \hat{a} \) uniformly in \( t \) by assumption, writing \( x(t) - x(s) = \int_s^t \bar{f}(x, \hat{a}, t) dt \) shows that \( x(t) \) is uniformly continuous in \( t \). Because \( Q(x, t) \) is uniformly continuous when \( x \) is bounded, and because \( \rho \) is uniformly continuous in \( Q \), \( \rho \) is uniformly continuous in \( t \) and \( \lim_{t \to \infty} \rho(Q(x(t), t)) = 0 \) by Barbalat’s lemma. This shows that \( \lim_{t \to \infty} Q(x(t), t) = 0 \).

**Remark 3.1.** The uniform continuity assumptions on \( \rho(\cdot) \) and \( Q \) used in the general setting handled by the proof of Prop. 3.1 are not strictly necessary. Without them, we can conclude existence of the integral \( \int_0^\infty \rho(Q) dt \) but not that \( Q \to 0 \). In many cases, signal chasing arguments based on the finiteness of this integral are sufficient, as will be shown in the coming sections.

By taking \( Q = \frac{1}{2} s^2 \) in Prop. 3.1 we immediately recover Prop. 2.4. In this sense, Prop. 3.1 elucidates the underlying structure exploited by the Bregman Lagrangian to generate higher-order algorithms.

We now consider the integral functional

\[
\mathcal{L} \left( \hat{a}, \dot{a}, t \right) = e^{\int_0^t \dot{N}(t) dt} \frac{1}{\beta \dot{N}(t)} \left( \frac{1}{2} \hat{a}^T - \gamma \beta \dot{N}(t) \frac{d}{dt} \int_0^t R(x(t'), \hat{a}(t'), t') dt' \right) ,
\]

which generates the higher-order law

\[
\ddot{a} + \dot{a} \left( \beta \dot{N} - \frac{\dot{N}'}{\dot{N}} \right) = -\gamma \beta \dot{N} \frac{\partial R(x, \hat{a}, t)}{\partial \hat{a}} .
\]

We again re-write (25) as two first-order systems

\[
\dot{v} = -\gamma \frac{\partial R(x, \hat{a}, t)}{\partial \hat{a}} ,
\]

\[
\dot{\hat{a}} = \beta \dot{N} (\hat{a} - \hat{v}) .
\]

We now require a modified version of Assumption 3.1

---

5Following Remark 3.1 we conclude that \( s \in L_2 \cap L_\infty \), and \( \dot{s} \in L_\infty \) by local boundedness of \( \bar{f} \) in \( x \) and \( \hat{a} \) uniformly in \( t \), showing \( s \to 0 \) by Barbalat’s lemma.
Assumption 3.2. $R$ is a non-negative function: $R(x, \hat{a}, t) \geq 0$ for all $x, \hat{a},$ and $t$. Furthermore, there exists a time-dependent normalizing signal $N(t)$ and non-negative scalar values $\beta \geq 0, \mu \geq 0$ such that

$$R(x, a, t) - R(x, \hat{a}, t) - \frac{\beta \mu}{\gamma} N(t) \| \hat{a} - \bar{v} \|^2 + 2 (\hat{a} - \bar{v})^T \nabla a R(x, \hat{a}, t) \leq -k R(x, \hat{a}, t)$$

for some constant $k > 0$.

With Assumption 3.2, we can state the following proposition.

Proposition 3.2. Consider algorithm (25) or its equivalent form (26) & (27) along with Assumptions 2.4 & 3.2. Then, for any bounded trajectory $x(t)$ of (7), $\dot{v}$ and $\hat{a}$ remain bounded, $(\hat{a} - \bar{v}) \in L_2$, and $\int_0^\infty R(x(t'), \hat{a}(t'), t') dt' < \infty$.

Proof. Consider the Lyapunov-like function

$$V = \frac{1}{2\gamma} \dot{\hat{v}}^T \dot{\hat{v}} + \frac{1}{2\gamma} (\hat{a} - \bar{v})^T (\hat{a} - \bar{v}).$$

Equation (28) implies that, with $N(t) = 1 + \mu N(t)$,

$$\dot{V} = -\hat{a}^T \frac{\partial R(x, \hat{a}, t)}{\partial a} - \frac{\beta}{\gamma} \| \hat{a} - \bar{v} \|^2 - \frac{\beta \mu}{\gamma} N(t) \| \hat{a} - \bar{v} \|^2 + 2 (\hat{a} - \bar{v})^T \frac{\partial R(x, \hat{a}, t)}{\partial a}$$

$$\leq R(x, a, t) - R(x, \hat{a}, t) - \frac{\beta}{\gamma} \| \hat{a} - \bar{v} \|^2 - \frac{\beta \mu}{\gamma} N(t) \| \hat{a} - \bar{v} \|^2 + 2 (\hat{a} - \bar{v})^T \frac{\partial R(x, \hat{a}, t)}{\partial a}$$

$$\leq -k R(x, \hat{a}, t) - \frac{\beta}{\gamma} \| \hat{a} - \bar{v} \|^2$$

(28) & (29) show that, for any bounded solution $x$, $\dot{v}$ and $\hat{a}$ remain bounded. Furthermore, integrating (29) shows that $\int_0^\infty R(x(t'), \hat{a}(t'), t') dt' < \infty$ and $(\hat{a} - \bar{v}) \in L_2$.

As mentioned in Sec. 2.2, we will be particularly interested in Prop. 3.2 when $R = \frac{1}{2} f^2$, which will generate composite adaptation algorithms.

Remark 3.2. Classically, Lyapunov functions used in adaptive control consist of a sum of tracking and parameter estimation error terms, with $\hat{a}$ chosen to cancel a term of unknown sign. Several Lyapunov functions in this work consist only of parameter estimation error terms, such as (28). From a mathematical point of view, all that matters is that $\dot{V}$ is negative semi-definite and contains signals related to the tracking error. Integrating $\dot{V}$ allows the application of tools from functional analysis to ensure that the control goal is accomplished.

3.2 Composite adaptation laws

Here we consider the linearly parameterized setting $f(x, a, t) = Y(x, t) a$, and derive new first- and second-order composite adaptation laws. Composite adaptation laws are driven by two sources of error: the tracking error itself, as summarized by $s$ in the Slotine and Li controller, and a prediction error generally obtained from an algebraic relation which is itself constructed by filtering the dynamics (J.-J. Slotine & Li [1991]). A starting point for our first proposed algorithm is to consider a hybrid local and integral speed gradient functional

$$Q(x, t) = \frac{\gamma}{2} s^2 + \frac{\kappa}{2} \int_0^t \tilde{f}^2(x(t'), \hat{a}(t'), t') dt'$$

(30)

where $\kappa > 0, \gamma > 0$ are positive design parameters weighting the contributions of each term. As discussed in Sec. 2.2, the first term leads to the Slotine and Li controller. The second can be clearly seen to satisfy Assumptions 2.5 and 2.6 with $\mu(t) = 0$. It also satisfies Assumption 2.4, as $\tilde{f}^2$ is a quadratic function of $\hat{a}$ for linear $\tilde{f}$. Following the speed gradient formalism, the resulting adaptation law is given by

$$\dot{\hat{a}} = -P (\gamma s + \kappa Y \hat{a}) Y^T$$

(31)

which is a composite adaptation law simultaneously driven by $s$ and the instantaneous function approximation error $Y \hat{a} = \tilde{f}$. Equation (31) depends on the function approximation error $\tilde{f}$, which is not measured and hence cannot be
used directly in an adaptation law. Nevertheless, it can be obtained through a PI law as discussed in Sec. 2.4. To do so, we define

$$
\xi(x, x_d, t) = -\kappa P s(x, x_d)Y^T(x, t)
$$

(32)

and

$$
\rho(x, x_d, t) = \kappa P \int_{x_n(t_0)}^{x_n(t)} s(x, x_d) \frac{\partial Y(x, t)\partial x_n}{\partial x_n} dx_n
$$

(33)

$$
\dot{a} = \dot{\pi} + \xi(x, x_d, t) + \rho(x, x_d, t)
$$

(34)

$$
\dot{\pi} = - (\kappa \eta + \gamma) s P Y^T + \kappa s \sum_{i=1}^{n-1} P \frac{\partial Y}{\partial x_i} \dot{x}_i - \sum_{i=1}^{n-1} \left( \frac{\partial \rho}{\partial x_i} \right)^T \dot{x}_i
$$

(35)

Computing \(\dot{a}\) demonstrates that (31) is obtained despite its dependence on \(Y\dot{a}\) through only the known signals contained in (32)-(35). A few remarks concerning the algorithm (31)-(35) are in order.

**Remark 3.3.** The \(Y\dot{a}\) term may also be obtained by following the I&I formalism (Astolfi & Ortega, 2003; Liu et al., 2010). To our knowledge, this discussion is the first that demonstrates the possibility of using a PI law in combination with a standard Lyapunov-stability motivated adaptation law to obtain a composite law.

**Remark 3.4.** More error signals may be used for additional terms in the adaptation law. For example, a prediction error obtained by filtering the dynamics may also be employed, leading to a three-term composite algorithm.

**Remark 3.5.** Much like the standard composite law obtained by filtering the dynamics, rearranging (31) shows that \(\dot{a} + P (Y\dot{a}) Y^T = -P \dot{s} Y^T\), so that the additional term can be seen to add a damping term that smooths adaptation (J.-J. Slotine & Li, 1991).

**Remark 3.6.** As mentioned in Sec. 2.1 for clarity of presentation we have restricted our discussion to the \(n^{th}\)-order system (1). In general, the PI form (34) leads to unwanted unknown terms contained in \(\left( \frac{\partial \xi(x, x_d)}{\partial x} \right)^T \dot{x}\) in addition to the desired unknown term. In this case, the desired unknown term is \(-\kappa P (Y\dot{a}) Y^T\) while the undesired unknown term is \(-\kappa P s \frac{\partial Y}{\partial x_n} \dot{x}_n\). Indeed, the purpose of introducing the additional proportional term \(\rho(x, x_d)\) in (32) is to cancel the undesired unknown term. In general, cancellation of the undesired terms can be obtained by choosing \(\rho\) to solve a PDE, and solutions to this PDE will only exist if the undesired term is the gradient of an auxiliary function. \(\rho\) is then set to this auxiliary function. In some cases, the PDE can be avoided, such as through dynamic scaling techniques (Karagiannis, Sassano, & Astolfi, 2009) or the similar embedding technique of Tukin (I. Tukin, 2011).

The properties of the adaptive law (31) may be summarized with the following proposition.

**Proposition 3.3.** Consider the adaptation algorithm (31) with a linearly parameterized unknown, \(f(x, a, t) = Y(x, t)a\). Then all trajectories \((x, \dot{a})\) remain bounded, \(s \in L_2 \cap L_\infty\), \(\dot{f} \in L_2\), \(s \to 0\), and \(x \to x_d\).

**Proof.** Consider the Lyapunov-like function

$$
V = \frac{1}{2} s^2 + \frac{1}{2\gamma} \dot{a}^T \dot{a}
$$

which has time derivative

$$
\dot{V} = -\eta s^2 - \frac{\kappa}{\gamma} \dot{f}^2
$$

This immediately shows \(s \in L_\infty\) and \(\dot{a} \in L_\infty\). Because \(s \in L_\infty\), \(x \in L_\infty\) by definition of the sliding variable (J.-J. Slotine & Li, 1991). Integrating \(\dot{V}\) shows that \(s \in L_2\) and \(\dot{f} \in L_2\). The result follows by application of Lemma [A.1] or directly by Barbalat’s Lemma.

Following the accelerated speed gradient approach of Sec. 5.1 we now obtain a higher-order composite algorithm, and give a PI implementation. We again consider a hybrid local and integral speed gradient functional, so that (11) takes the form

$$
\mathcal{L} (\dot{a}, \dot{a}, t) = e^{\int_{0}^{t} \beta N(t) dt} \frac{1}{\beta N(t)} \left[ \frac{1}{2} \dot{a}^T \dot{a} - \beta N(t) \frac{d}{dt} \left[ \frac{\gamma}{2} s^2 + \frac{\kappa}{2} \int_{0}^{t} \dot{f}^2(x(t'), \dot{a}(t'), t') dt' \right] \right]
$$

(36)
where $\gamma > 0$ and $\kappa > 0$ are positive constants weighting the two error terms. The Euler-Lagrange equations then lead to the higher-order composite system

$$\ddot{a} + \left(\beta N - \frac{N}{N}\right) \dot{a} = -\beta N (\gamma s + \kappa Y \tilde{a}) Y^T.$$ (37)

As in Sec. 2.3 (37) may be implemented as two first-order systems

$$\dot{\tilde{v}} = -(\gamma s + \kappa Y \tilde{a}) Y^T$$ (38)

$$\dot{\tilde{a}} = \beta N (\tilde{v} - \tilde{a})$$ (39)

where now (38) is obtained through the PI form $\tilde{v} = \tilde{v} + \xi(x, x_d, t) + \rho(x, x_d, t)$ with $\xi$, $\rho$, and $\tilde{v}$ given by (32), (33), and (35) respectively with $P = I$. The properties of the higher-order composite adaptation law (37) are stated in the following proposition.

**Proposition 3.4.** Consider the higher-order composite adaptation algorithm (37) or its equivalent (38) & (39) for a linearly parameterized unknown, $f(x, a, t) = Y(x, t)a$. Set $N = 1 + \mu\|Y\|^2$ and $\mu > \frac{\gamma}{\beta} \left(\frac{1}{\sqrt{\gamma}} + \frac{\beta}{\gamma}\right)$. Then all trajectories $(x, \tilde{v}, \tilde{a})$ remain bounded, $\|\tilde{v} - \tilde{a}\| \in L_2$, $s \in L_\infty \cap L_2$, $\tilde{f} \in L_\infty \cap L_2$, $s \to 0$, and $x \to x_d$.

For clarity of exposition, the proof is deferred to the appendix.

**Remark 3.7.** By following the proof, the normalizing signal $N$ may be chosen alternatively to be matrix-valued as $N = I + \mu Y^T Y$.

**Remark 3.8.** The new $Y \tilde{a}$ term may be used in isolation, by consideration of the Lyapunov function $V = \frac{1}{2} \tilde{v}^T P^{-1} \tilde{v} + \frac{1}{2} (\tilde{a} - \tilde{v})^T P^{-1} (\tilde{a} - \tilde{v})$.

### 3.3 Higher-order algorithms for nonlinearly parameterized adaptive control

We now utilize the development in Sec. 3.2 to present a new higher-order algorithm applicable when the unknown parameters appear nonlinearly in the dynamics. Significant progress has been made in this direction with the assumption of monotonicity, and several notions of monotonicity have appeared in the literature [Astolfi & Ortega, 2003; Liu et al., 2010; Ortega, Gromov, Nuño, Pyrkin, & Romero, 2019; I. Tyukin, 2011; Tyukin et al., 2007]. We consider one notion of monotonicity presented by Tyukin, which is captured in the following assumption.

**Assumption 3.3.** There exists a known time- and state-dependent function $\alpha : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ such that

$$\tilde{a}^T \alpha(x, t) (f(x, \tilde{a}, t) - f(x, a, t)) \geq 0,$$ (40)

$$|\alpha(x, t)^T \tilde{a}| \geq \frac{1}{D_1} |f(x, \tilde{a}, t) - f(x, a, t)|.$$ (41)

where $D_1 > 0$ is a positive scalar.

This assumption is satisfied, for example, by all functions $f$ of the form

$$f(x, a, t) = \lambda(x, t) f_m(x, \phi(x, t)^T a, t)$$ (42)

where $\lambda : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\phi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^p$, $f_m : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, and where $f_m$ is monotonic and Lipschitz in $\phi(x)^T \tilde{a}$. In this setting, $\alpha(x, t)$ may be taken as $\alpha(x, t) = (-1)^p D_1 \lambda(x, t) \phi(x, t)$ where $p = 0$ if $f_m$ is non-decreasing in $\phi^T \tilde{a}$ and $p = 1$ if $f_m$ is non-increasing in $\phi^T \tilde{a}$ [I. Tyukin, 2011; Tyukin et al., 2007].

Under Assumption 3.3 Tyukin showed that the adaptation law

$$\dot{\tilde{a}} = -\tilde{f}(x, \tilde{a}, t) P \alpha(x, t)$$ (43)
with $P = P^T > 0$ ensures that $\tilde{f} \in L_2 \cap L_\infty$ and that all trajectories $(x, \tilde{a})$ remain bounded. Equation (43) may be implemented in a manner similar to [32]-[35]

$$\xi(x, x_d, t) = -Ps(x, x_d)\alpha(x, t), (44)$$

$$\rho(x, x_d, t) = P \int_{x_n(t_i)}^{x_n(t_i)} s(x, x_d) \frac{\partial \alpha(x, t)}{\partial x_n} dx_n, (45)$$

$$\hat{a} = \hat{\pi} + \xi(x, x_d, t) + \rho(x, x_d, t), (46)$$

$$\pi = -\eta_s \alpha + Ps \sum_{i=1}^{n} \frac{\partial \xi}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial \rho}{\partial x_i} + (\frac{\partial \rho}{\partial x_d})^T \dot{x}_d - \frac{\partial \xi}{\partial t} - \frac{\partial \rho}{\partial t}. (47)$$

Algorithm (43) is similar to a gradient flow algorithm. If $f(x, a, t)$ has the form (42) and is non-decreasing, gradient flow on the loss function $L(x, \tilde{a}, t) = \frac{1}{2} \tilde{f}^2(x, \tilde{a}, t)$ with a gain matrix $D_1 P$ leads to

$$\hat{a} = -\dot{f}(x, \tilde{a}, t) f'_m(x, \phi^T \tilde{a}, t) P \alpha$$

where ’ denotes differentiation with respect to the second argument. $f'_m(x, \phi^T \tilde{a}, t)$ is of unknown sign, but by monotonicity of $\tilde{f}$ in the sense of (44), it is sufficient to set it to one to ensure that the search proceeds in the correct direction despite non-convexity of the square loss in this setting. Similarly, if $f$ is non-increasing, we find

$$\hat{a} = f(x, \tilde{a}, t) f'_m(x, \phi^T \tilde{a}, t) P \alpha$$

and it is sufficient to set $f'_m$ to negative one to ensure the search proceeds in the correct direction.

In machine learning, specifically the area of isotonic regression, remarkably similar methods have been used to estimate generalized linear models (GLMs). GLM regression is an extension of linear regression where the data is assumed to be generated by a function of the form $f(x) = u(w^T x)$ for known function $u$ and unknown parameters $w$. The first computationally and statistically efficient algorithm for this problem – the GLM-Tron of [Kakade et al. 2011] – assumes that $u$ is Lipschitz and monotonic, much like Assumption [3,3]

The GLM-Tron algorithm was recently extended to the setting of kernel methods and was subsequently used to provably learn two hidden layer neural networks by [Goel and Klivans 2017]. In the kernel GLM setting handled by the Alphatron, the function to be approximated is assumed to be of the form $f(x) = u(\sum_{i=1}^{m} w_i K(x, x_i))$ where $K$ is the kernel function for a Reproducing Kernel Hilbert Space (RKHS) $H$. $K$ is thus given by the RKHS inner product of a feature map $\psi$, $K(x, y) = \langle \psi(x), \psi(y) \rangle_H$. As in the setting handled by the GLM-Tron, $u$ is assumed to be monotonic and Lipschitz. The Alphatron initializes all weights to zero, and given a batch of labeled training data $(x_i, f(x_i))_{i=1}^{m}$, updates them with a learning rate $\lambda > 0$ according to the iteration

$$\hat{w}^{t+1} = \hat{w}^t - \frac{\lambda}{m} \left( \tilde{f}(\hat{w}^t, x_i) - f(x_i) \right). (48)$$

We demonstrate a surprising equivalence between Tyukin’s adaptation law (43) and the Alphatron weight update (48) in the following proposition.

**Proposition 3.5.** The adaptation law (43) is an online variant of the Alphatron algorithm.

**Proof.** Defining the vector $\hat{v}^t = \sum_{i=1}^{m} \hat{w}^t_i \psi(x_i)$, (48) implies the iteration on $\hat{v}$,

$$\hat{v}^{t+1} = \hat{v}^t - \frac{\lambda}{m} \sum_{i=1}^{m} \left( \tilde{f}(\hat{w}^t, x_i) - f(x_i) \right) \psi(x_i). (49)$$

(49) shows that at time $t$,

$$\hat{v}^t = -\frac{\lambda}{m} \sum_{i=1}^{m} \left( \sum_{j=1}^{t} \hat{f}^j_i \right) \psi(x_i) (50)$$

where $\hat{f}^j_i$ in (50) is the function approximation error on the $i$th input example at iteration $j$, $\hat{f}^j_i = \tilde{f}(\hat{w}^j, x_i) - f(x_i)$.

Now, assuming that for the adaptive control problem $f(x, \alpha, t) = u(\alpha^T(x, t) a)$, setting $P = \lambda I$, $\hat{a}(0) = 0$, and integrating both sides of (43), we see that at time $t$,

$$\hat{a}(t) = -\lambda \int_0^t f(x(t'), a, t') \alpha(x(t'), t') dt'. (51)$$
Comparison of (50) to (51) shows that in (51) the data appears continuously, the batch size $m$ is one, and the data is not revisited. The current function approximation $\hat{f}$ at time $t$ for the parameters in (51) can then be written

$$\hat{f}(t) = u(\alpha^T(x, t) \hat{a}) = u \left( \int_0^t -\lambda \tilde{f}(x(t'), a, t') \alpha^T(x(t), t) \alpha(x(t'), t') dt' \right)$$

$$= u \left( \int_0^t c(t') K(t, t') dt' \right)$$

(52)

where we have defined $c(t') = -\lambda \tilde{f}(x(t'), a, t')$ and $K(t, t') = \alpha^T(x(t), t) \alpha(x(t'), t')$. Similarly, in the case of the Alphatron, the current approximation at iteration $t$ is given by

$$\tilde{f}(\hat{w}^t, x) = u \left( \langle \hat{v}^t, \psi(x) \rangle_H \right) = u \left( \sum_{i=1}^m \left( \sum_{j=1}^{t-1} -\frac{\lambda}{m} \tilde{f}_j^{t} \right) \langle \psi(x), \psi(x_i) \rangle \right)$$

$$= u \left( \sum_{i=1}^m \hat{w}_i^t \mathcal{K}(x, x_i) \right)$$

(53)

where we have noted that with $\hat{w}_i^0 = 0$ for all $i$, $\hat{w}_i^t = \sum_{j=1}^{t-1} -\frac{\lambda}{m} \tilde{f}_j^{t}$. The correspondence between (52) and (53) is clear.

Proposition 3.5 demonstrates an equivalence between techniques in non-linearly parameterized adaptive control and non-convex learning. Given the well-recognized importance of momentum in non-convex learning problems (Attouch, Goudou, & Redont, 2000; Jin, Netrapalli, & Jordan, 2017; O’Neill & Wright, 2017), this correspondence immediately suggests the higher-order version of (43)

$$\ddot{\hat{a}} + \left( \beta \mathcal{N} - \frac{\dot{\hat{v}}}{\mathcal{N}} \right) \dot{\hat{a}} = -\gamma \beta \mathcal{N} \tilde{f} \alpha,$$

(54)

which, as before, admits an equivalent representation in terms of two first-order systems,

$$\dot{\hat{v}} = -\gamma \tilde{f} \alpha,$$

(55)

$$\dot{\hat{a}} = \beta \mathcal{N} (\dot{\hat{v}} - \dot{\hat{a}}).$$

(56)

Equation (54) may be implemented through (55) & (56) via the PI form (44)-(47) applied to the $\dot{\hat{v}}$ variable. Equation (54) may be obtained via the Bregman Lagrangian (36) for higher-order speed gradient laws by choosing only the integral term. It is then necessary to modify the resulting Euler-Lagrange equations by setting $f'_m$ to $\pm 1$ based on monotonicity of $\tilde{f}$. (55) and (56) can also be obtained from the corresponding Bregman Hamiltonian, following Remark 2.5 and again setting $f'_m$ according to the monotonicity properties of $\tilde{f}$. The following proposition summarizes the properties of (55) & (56).

**Proposition 3.6.** Consider the algorithm (54) or its equivalent form (55) & (56) under Assumption 3.3 with $\mathcal{N} = 1 + \mu \|\alpha(x, t)\|^2$ and $\mu > \frac{\gamma \beta}{\alpha}$. Then, all trajectories $(x, \hat{a}, \hat{v})$ remain bounded, $\tilde{f} \in L_2$, $(\hat{a} - \hat{v}) \in \mathcal{L}_2$, $s \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $s \to 0$ and $x \to x_d$.

For clarity of exposition, the proof is deferred to the appendix.

**Remark 3.9.** As noted in Remark 3.7 by following the proof of Prop. 3.6 one may also take $\mathcal{N}$ to be matrix-valued as $\mathcal{N} = 1 + \mu \alpha(x, t) \alpha(x, t)^T$.

Proposition 3.6 shows that the Bregman Lagrangian-based framework for generating higher-order algorithms is quite general, and may also be applied to the non-linearly parameterized setting, despite non-convexity of the corresponding optimization problems.

Predominantly inspired by deep learning, there has recently been strong interest in non-convex models that are nevertheless amenable to gradient-based or gradient-inspired optimization. The development in this section suggests that progress in learning algorithm development for these models should be followed by the adaptive control community and used to develop new adaptive control algorithms. In the reverse direction, our higher-order algorithm (54) can immediately be converted into an online or batch learning algorithm with momentum for kernel-based GLMs by assuming that $\tilde{f}$ is measured, rather than differentially related to $s$ as in (5).
3.4 The elastic modification

We now consider a modification to the previously discussed adaptive control laws inspired by the Elastic Averaging SGD (EASGD) algorithm (Boffi & Slotine, 2020; Zhang et al., 2014). EASGD is an algorithm intended for distributed training of deep neural networks across \( p \) graphics processing units (GPUs). Each GPU is used to train a local copy of the deep network model, and each local copy maintains its own set of parameters \( \hat{a}^{(i)} \). These parameters are updated according to the iteration

\[
\hat{a}^{(i)}_{t+1} = \hat{a}^{(i)}_t - \lambda g^{(i)}_t + k \lambda \left( \overline{a}_t - \hat{a}^{(i)}_t \right),
\]

\[
\overline{a}_{t+1} = \overline{a}_t + \lambda k \left( \frac{1}{p} \sum_{i=1}^{p} \hat{a}^{(i)}_t - \overline{a}_t \right),
\]

where \( \lambda \) is the learning rate, \( g^{(i)}_t \) is the stochastic gradient approximation computed by the \( i \)th agent at timestep \( t \), \( k \) is the coupling strength, and \( \overline{a} \) is the quorum variable (Russo & Slotine, 2011; Tabareau, Slotine, & Pham, 2010). Equation (58) takes the form of a low-pass filter of the instantaneous average of the set of local parameters.

It was observed by Boffi and Slotine (2020) that in the non-distributed \( (p = 1) \) case, (57) & (58) do not reduce to standard stochastic gradient descent, and that application of EASGD in this setting has different generalization properties than standard SGD when used to train deep neural networks. In this sense, the \( p = 1 \) reduction of EASGD is a new higher-order optimization algorithm. In a similar vein, by construction of suitable Lyapunov functions, we now show that adding a quorum-like variable to the adaptive laws considered in previous sections maintains their stability. This immediately gives rise to a new class of higher-order adaptive control algorithms. Interestingly, these algorithms do not seem to admit an equivalent representation in terms of a single second-, third-, or fourth-order system for \( \hat{a} \), but must be written as a system of first-order equations.

**Remark 3.10.** The algorithms considered in this subsection also apply to the case of cloud-based adaptation for networked robotic systems (Wensing & Slotine, 2017), where the quorum variable is allowed to have its own dynamics as in (58).

3.4.1 Elastic modification for the Slotine & Li controller

We first apply the elastic modification to the Slotine & Li adaptive controller (7) for linearly parameterized unknown dynamics \( \tilde{f} = Y\tilde{a} \). These results extend trivially to the non-filtered composite algorithm of Sec. 3.2. To this end, we define the adaptation law

\[
\dot{\hat{a}} = -sPY^T + k (\overline{a} - \hat{a}) ,
\]

\[
\dot{\overline{a}} = k (\hat{a} - \overline{a}) ,
\]

whose basic stability properties are summarized in the following proposition.

**Proposition 3.7.** Consider the adaptation law (59) & (60). Then all trajectories \( (x, \hat{a}, \overline{a}) \) remain bounded, \( s \in L_2 \cap L_\infty \), \( \dot{s} \in L_2 \), \( s \to 0 \) and \( x \to x_d \).

**Proof.** The Lyapunov function

\[
V = \frac{1}{2} \left( \tilde{a}^T P^{-1} \tilde{a} + \tilde{\alpha}^T P^{-1} \tilde{\alpha} + s^2 \right)
\]

has time derivative

\[
\dot{V} = -\eta s^2 - k \left( \overline{a} - \hat{a} \right)^T P^{-1} \left( \overline{a} - \hat{a} \right).
\]

This shows that \( s, \hat{a}, \) and \( \overline{a} \) remain bounded. The remaining conclusions of the proposition are immediately drawn by integrating \( \dot{V} \). \( \square \)

3.4.2 Elastic modification for nonlinearly parameterized systems

We now apply the elastic modification to the algorithm (43) for nonlinearly parameterized unknown dynamics satisfying Assumption 3.3. As in (59) & (60), we define

\[
\dot{\hat{a}} = -f P \alpha + k (\overline{a} - \hat{a}) ,
\]

\[
\dot{\overline{a}} = k (\hat{a} - \overline{a}) .
\]
Proposition 3.8. Consider the adaptation law (61) & (62) implemented with appropriate modifications to (44)-(47). Then all trajectories \((x, \dot{x}, \ddot{x})\) remain bounded, \(\dot{f} \in L_2 \cap L_\infty, (\ddot{a} - \dddot{a}) \in L_2, s \in L_\infty \cap L_2, s \to 0\) and \(x \to x_d\).

Proof. The Lyapunov-like function

\[ V = \frac{1}{2} \left( \ddot{a}^T P^{-1} \dot{a} + \dddot{a}^T P^{-1} \dddot{a} \right) \]

has time derivative

\[ \dot{V} \leq -\frac{1}{D_1} \ddot{f}^2 - k (\ddot{a} - \dddot{a})^T P^{-1} (\ddot{a} - \dddot{a}). \]

This shows that \(\dot{a}\) and \(\dddot{a}\) remain bounded. Integration of \(\dot{V}\) shows \(\dot{f} \in L_2\) and \((\ddot{a} - \dddot{a}) \in L_2\). Application of Lemma A.1 completes the proof.

3.4.3 Elastic modification for higher-order algorithms

We now turn to consider the higher-order algorithms presented in Secs. 3.2 and 3.3. In the higher-order setting, there are three clear possibilities for the elastic modification: coupling to a quorum variable for the \(\dot{a}\) variable, or coupling to quorum variables in both \(\dot{a}\) and \(\ddot{a}\). We prove stability for all three possibilities only in the nonlinearly parameterized setting described by Assumption 3.3. The results extend naturally to the higher-order composite algorithm for linearly parameterized systems presented in Sec. 3.2. All proofs are deferred to the appendix. We begin with the first possibility,

\[ \dot{\ddot{a}} = \beta N (\ddot{v} - \dddot{a}) + k \beta N (\dddot{a} - \dddot{a}), \]

\[ \dddot{a} = k \beta N (\dddot{a} - \dddot{a}). \]

The basic stability properties of the algorithm (63)-(65) are summarized in the following proposition.

Proposition 3.9. Consider the higher-order algorithm with elastic modification in the \(\dot{a}\) variable (63)-(65) under Assumption 3.3. Set \(\frac{1}{k} \leq k < 1, N = 1 + \mu \|\alpha(x, t)\|^2\), and \(\mu > \frac{2D_1}{\rho(1-k)}\). Then all trajectories \((x, \dot{x}, \ddot{a}, \dddot{a})\) remain bounded, \(\dot{f} \in L_2 \cap L_\infty, s \in L_2 \cap L_\infty, (\ddot{a} - \dddot{a}) \in L_2, s \to 0\) and \(x \to x_d\).

We now consider the second possibility of adding a quorum variable in the \(\ddot{a}\) variable,

\[ \dot{\dddot{a}} = \beta N (\dddot{v} - \dddot{a}), \]

\[ \dddot{a} = \beta N (\dddot{a} - \dddot{a}). \]

The basic stability properties of (66)-(68) are summarized in the following proposition.

Proposition 3.10. Consider the higher-order algorithm with elastic modification in the \(\ddot{a}\) variable (66)-(68) under Assumption 3.3. Set \(\rho < 2\beta, N = 1 + \mu \|\alpha(x, t)\|^2\), and \(\mu > \frac{2D_1}{\rho(1-k)}\). Then all trajectories \((x, \dot{x}, \ddot{a}, \dddot{a})\) remain bounded, \(\dot{f} \in L_2 \cap L_\infty, s \in L_2 \cap L_\infty, (\ddot{a} - \dddot{a}) \in L_2, s \to 0\) and \(x \to x_d\).

Finally, we consider adding coupling to quorum variables in both \(\dot{a}\) and \(\ddot{a}\),

\[ \dot{\dddot{a}} = \beta N (\dddot{v} - \dddot{a}), \]

\[ \dddot{a} = \beta N (\dddot{a} - \dddot{a}). \]

The basic stability properties of (69)-(72) are summarized in the following proposition.

Proposition 3.11. Consider the higher-order algorithm with the elastic modification in both the \(\ddot{a}\) and \(\dddot{a}\) variables (69)-(72) under Assumption 3.3. Set \(\rho < \beta(1-k), \frac{1}{k} \leq k < 1, N = 1 + \mu \|\alpha(x, t)\|^2\), and \(\mu > \frac{2D_1}{\rho(1-k)}\). Then, all trajectories \((x, \dot{x}, \ddot{a}, \dddot{a})\) remain bounded, \(\dot{f} \in L_2 \cap L_\infty, s \in L_2 \cap L_\infty, (\ddot{a} - \dddot{a}) \in L_2, (\dddot{a} - \dddot{a}) \in L_2, s \to 0\) and \(x \to x_d\)
We have thus shown that all adaptive control algorithms presented in this paper, as well as the classic algorithm of Slotine and Li, can be modified to include feedback coupling to a low-pass filtered (exponentially weighted average) version of the adaptation variables. It is well known that iterate averaging for stochastic optimization algorithms such as stochastic gradient descent can improve convergence rates via variance reduction (B. T. Polyak & Juditsky, 1992). The elastic modification is similar in spirit, but employs feedback rather than series coupling. This suggests that adding the elastic term may improve robustness of adaptation algorithms, and we leave a theoretical investigation of this conjecture for future work.

### 3.5 Exponential forgetting least squares and bounded gain forgetting

We now demonstrate how to apply the techniques of exponential forgetting and bounded gain forgetting least squares (J.-J. Slotine & Li, 1991) to the adaptation algorithms presented thus far. These techniques are useful for estimation of time-varying parameters, as they rapidly discard previous information used for parameter estimation. Exponential forgetting least squares is described by a time-dependent learning-rate matrix \( P(t) \), which, in the linearly parameterized case \( \tilde{f} = Y \tilde{a} \) takes the form

\[
\dot{P} = \begin{cases} 
\lambda P - PY^TYP & \text{if } \|P\| \leq P_0 \\
0 & \text{else}
\end{cases} \quad (73)
\]

where \( \lambda > 0 \) is a constant forgetting factor, \( P_0 \) is a maximum bound on the norm, and \( \|P\| \) is a matrix norm such as the induced matrix 2-norm. Equation (73) implies for the inverse matrix

\[
\frac{d}{dt} P^{-1} = \begin{cases} 
-\lambda P^{-1} + Y^T Y & \text{if } \|P\| \leq P_0 \\
0 & \text{else}
\end{cases} \quad (74)
\]

In the nonlinearly parameterized case described by Assumption 3.3, we will replace \( Y^T \) in (73) & (74) by \( \alpha(x, t) \). In the bounded gain forgetting technique, \( \lambda \) is a time-dependent function \( \lambda(t) = \lambda_0 \left( 1 - \frac{\|P\|}{P_0} \right) \) (75)

where \( \lambda_0 > 0 \) sets the forgetting factor when the norm of \( P \) is small. It can be shown that this choice of \( \lambda(t) \) ensures that \( \|P\| \leq P_0 \), and thus we may drop the case statement in (73) & (74). The choice of \( \lambda(t) \) in bounded gain forgetting and the case statement used in (73) & (74) are employed to prevent unboundedness of the learning rate matrix.

We focus on algorithms without the elastic modification of Sec. 3.4 extension to the elastic modification is simple. We also focus on the bounded gain forgetting technique: proofs for the exponential forgetting least squares technique are identical, with the addition of an appropriate case statement in the time derivative of the Lyapunov function. For simplicity, we include only the time-dependent gain \( P(t) \) and set the scalar gains \( \kappa = \gamma = 1 \) where applicable.

### 3.6 First-order non-filtered composite

We begin with the first-order non-filtered composite (31) with \( P \) given by (73). In this case, the composite algorithm may be implemented via the PI form (32)-(35) where now \( P = P(t) \).

**Proposition 3.12.** Consider the adaptation algorithm (31) with \( P(t) \) given by (73), \( \lambda(t) \) given by (73), and \( \kappa = \gamma = 1 \). Then all trajectories \( (x, \hat{a}) \) remain bounded, \( \tilde{f} \in \mathcal{L}_2 \cap \mathcal{L}_\infty \), \( s \in \mathcal{L}_2 \cap \mathcal{L}_\infty \), \( s \to 0 \) and \( x \to x_d \).

**Proof.** The Lyapunov-like function

\[
V = \frac{1}{2} s^2 + \frac{1}{2} \hat{a}^T P^{-1} \hat{a}
\]

has time derivative

\[
\dot{V} = -\eta s^2 - \frac{1}{2} \tilde{f}^2 - \frac{\lambda}{2} \hat{a}^T P^{-1} \hat{a}
\]

which shows that \( s \) and \( \hat{a} \) remain bounded. Because \( s \) remains bounded, \( x \) remains bounded. Integrating \( \dot{V} \) shows that \( s \in \mathcal{L}_2 \) and \( \tilde{f} \in \mathcal{L}_2 \). The proof is completed by application of Lemma A.1 or directly by Barbalat’s Lemma.
3.7 Higher-order non-filtered composite

We can state a similar result for the higher-order non-filtered composite with time-dependent \( P(t) \) given by (73),

\[
\ddot{a} + \left( \beta \dot{N} - \frac{N}{N} - \dot{P} P^{-1} \right) \dot{a} = -\beta N \left( s + \hat{f} \right) P(t) Y^T,
\]

(76)

which admits the equivalent representation as two first-order systems,

\[
\dot{\hat{v}} = -P(t) \left( s + \hat{f} \right) Y^T
\]

(77)

\[
\dot{\hat{a}} = \beta N P(t) (\hat{v} - \hat{a})
\]

(78)

Equation (77) can be implemented via the PI form \( \dot{\hat{v}} = \nabla + \xi(x, x_d, t) + \rho(x, x_d, t) \) where \( \xi, \rho, \) and \( \nabla \) are given by (32), (33), and (35) respectively with \( \gamma = \kappa = 1 \).

**Proposition 3.13.** Consider the adaptation algorithm (76) or its equivalent form (77) & (78) with \( P(t) \) given by (73), \( \lambda(t) \) given by (72), \( N(t) = 1 + \mu \|Y\|^2 \) and \( \mu > \frac{3\eta_1^2}{2\eta^2} \). Then all trajectories \( (x, \hat{v}, \hat{a}) \) remain bounded, \( \hat{f} \in L_2 \cap L_\infty, \) \( s \in L_2 \cap L_\infty, \) \( s \to 0 \) and \( x \to x_d \).

The proof is deferred to the appendix.

3.8 First-order algorithm for nonlinearly parameterized systems

We now consider Tyukin’s first-order algorithm for nonlinearly parameterized systems (43) with \( P = P(t) \) given by (73). To do so, we require an additional assumption

**Assumption 3.4.** For the same function \( \alpha(x, t) \) as in Assumption 3.3 there exists a constant \( D_2 \) such that

\[
|\hat{f}(x, \hat{a}, t)| \geq D_2 |\alpha(x, t)^T \hat{a}|.
\]

(79)

**Proposition 3.14.** Consider the adaptation algorithm (43) with \( P(t) \) given by (73) and \( \lambda(t) \) given by (72) for \( \hat{f} \) satisfying Assumptions 3.3 and 3.4. Further assume that \( D_1 < 2D_2^2 \) or that \( D_2 > \frac{1}{2} \). Then, all trajectories \( (x, \hat{a}) \) remain bounded, \( \hat{f} \in L_2, s \in L_2 \cap L_\infty, s \to 0 \) and \( x \to x_d \).

**Proof.** Consider the Lyapunov-like function

\[
V = \frac{1}{2} \hat{a}^T P^{-1} \hat{a}
\]

(80)

which has time derivative

\[
\dot{V} = -\hat{f} \alpha^T \hat{a} + \frac{1}{2} (\hat{a}^T \alpha)^2 - \frac{\lambda}{2} \hat{a}^T P^{-1} \hat{a}
\]

\[
\leq -\frac{1}{D_1} \hat{f}^2 + \frac{1}{2D_2^2} \hat{f}^2 = -\left( \frac{1}{D_1} - \frac{1}{2D_2^2} \right) \hat{f}^2
\]

For \( D_1 < 2D_2^2, \) \( \dot{V} \leq 0 \) and \( \hat{f}^2 \in L_2 \). Alternatively, using the same Lyapunov function,

\[
\dot{V} \leq - (\alpha^T \hat{a})^2 \left( D_2 - \frac{1}{2} \right)
\]

For \( D_2 > \frac{1}{2}, \) \( \dot{V} \leq 0 \) and \( \alpha^T \hat{a} \in L_2 \). By Assumption 3.3, this implies that \( \hat{f} \in L_2 \). Hence, both approaches demonstrate that \( \hat{a} \) remains bounded and that \( \hat{f} \in L_2 \). By Lemma A.1, the proposition is proved. \( \square \)
3.9 Higher-order algorithm for nonlinearily parameterized systems

Last, we consider the higher-order algorithm for nonlinearily parameterized systems

\[ \ddot{a} + \left( \beta N - \frac{N}{N} - \dot{P}P^{-1} \right) \dot{a} = -\beta \dot{f}P(t)\alpha(x,t), \tag{81} \]

which admits the equivalent representation as two first-order systems,

\[ \dot{v} = -\dot{f}P(t)\alpha(x,t) \tag{82} \]
\[ \ddot{a} = \beta N\dot{P}(t)(\dot{v} - \dot{a}) \tag{83} \]

**Proposition 3.15.** Consider the adaptation algorithm \((77)\) or its equivalent form \((82) & (83)\) with \(P(t)\) given by \((75)\). \(\lambda(t)\) given by \((73)\), \(N' = 1 + \mu\|\alpha\|^2\), and \(\mu > \frac{D_2 - 2 + (2D_1 + 1)^2}{\beta(4D_2 - 1)}\). Let \(\dot{f}\) satisfy Assumptions \((3)\) and \((4)\). Further assume that \(D_2 > \frac{1}{2}\). Then, all trajectories \((x, \dot{v}, \dot{a})\) remain bounded, \(\dot{f} \in L_2, s \in L_2 \cap L_\infty, s \to 0 \) and \(x \to x_d\).

The proof is deferred to the appendix.

4 Conclusions and future directions

In this paper, we have presented a suite of new adaptive control algorithms. The algorithms combine the speed gradient methodology (A. L. Fradkov, 1999; Fradkov, 1979) with the Bregman Lagrangian (Betancourt et al., 2018; Wibisono et al., 2016) to systematically generate higher-order speed gradient algorithms, of which the recent algorithm of Gaudio et al. (2019) is a special case. Based on analogies between isotonic regression (Goel & Klivans, 2017; Karagiannis et al., 2009) or a similar embedding technique (I. Tyukin, 2011), we extended our higher-order speed gradient algorithms to the nonlinearly parameterized setting. Using a similar parallel to distributed stochastic gradient descent algorithms (Boffi & Slotine, 2020; Zhang et al., 2018), we developed a stable modification of all of our algorithms. We subsequently fused all of our developments with time-dependent learning rate update laws based on the bounded gain forgetting formalism (J.-J. Slotine & Li, 1991).

Throughout the paper, for simplicity of exposition, we focused on the \(n\)th order system \((1)\). While this represents a general class of systems including linear systems after transformation to control canonical form, many systems cannot be put in this form. As discussed in Remark \((2.2)\) our results extend to more general systems which have an error model similar to \((5)\), in the sense that the proof technique summarized by Lemma \(A.1\) is roughly preserved. The \(n\)th order system makes the employed proportional-integral forms simple, in that they can be written down explicitly as in \((32) - (35)\). As summarized in Remark \(3.6\) in the general case, a PDE needs to be solved, and solutions to this PDE may be hard to find, or may not exist at all. Solution of the PDE can be avoided by the dynamic scaling technique (Karagiannis et al., 2009) or a similar embedding technique (I. Tyukin, 2011).

There are many possible future directions of research. A first is to test the utility of these higher-order algorithms in the robotic setting. A second is to extend the higher-order formalism to “natural” speed gradient laws, so that manifold constraints may be respected throughout adaptation without projection (Lee, Kwon, & Park, 2018; Wensing & Slotine, 2018). A third is to use our analogy with the Alphatron algorithm to create higher-order algorithms for generalized linear model regression, and to understand their computational and statistical efficiency in the p-concept model (Kearns & Schapire, 1994). A fourth is to search for time-dependent learning rate update laws tailor-made to the monotonicity setting of Assumption \(3.3\) rather than the approach based on linear least squares presented in Sec. \(3.5\). Finally, we are interested in understanding if the higher-order formalism may help with more general nonlinear parameterizations, such as multiple layers of nonlinearity or a sum of single layer functions.

A Omitted proofs

**Lemma A.1.** If \(\dot{f}(x, \dot{a}, t) \in L_2, \dot{a} \in L_\infty, \) and \(\dot{f}\) is locally bounded in \(x\) and \(\dot{a}\) uniformly in \(t\), then \(s \in L_2 \cap L_\infty, s \to 0 \) and \(x \to x_d\).

**Proof.** By \((5)\), we can write explicitly

\[ s(t) = \int_0^t e^{-\eta(t-\tau)} \dot{f}(x(\tau), \dot{a}(\tau), \tau) d\tau \tag{84} \]
By the Cauchy-Schwarz inequality,
\[
s^2(t) \leq \left( \int_0^t e^{-2\eta(t-\tau)} d\tau \right) \left( \int_0^t \tilde{f}^2(\tau) d\tau \right) \\
\leq \frac{1}{2\eta} \|\tilde{f}\|_{L^2} (1 - e^{-2\eta t}) \leq \frac{1}{2\eta} \|\tilde{f}\|_{L^2}
\]
s so that \( s \in L_\infty \). Similarly, Parseval’s theorem applied to the low-pass filter \([83]\) shows that \( s \in L_2 \). Because \( s \in L_\infty \), we conclude that \( x \in L_\infty \) \([1, J.-J. Slotine & Li, 1991]\). Because \( x \in L_\infty \) and \( \hat{a} \in L_\infty \), and because \( \tilde{f} \) is locally bounded in \( x \) and \( \hat{a} \) uniformly in \( t \), \( \tilde{f} \in L_\infty \). By \([5]\), \( \dot{s} \in L_\infty \), and hence by Barbalat’s lemma, \( s \to 0 \). By definition of \( s \), we then conclude that \( x \to x_d \). \( \square \)

**Proof of Proposition 3.4**

**Proof.** Consider the Lyapunov function
\[
V = \frac{1}{2} s^2 + \frac{1}{2\gamma} (\|\tilde{\nu}\|^2 + \|\tilde{a} - \tilde{\nu}\|^2)
\]
which has time derivative
\[
\dot{V} = -\eta s^2 + s \tilde{f} + \frac{1}{\gamma} \left[ \tilde{\nu}^T (\kappa \tilde{f} - \gamma s) Y^T + (\tilde{a} - \tilde{\nu})^T \left( \beta N (\tilde{\nu} - \tilde{a}) + \gamma \tilde{a} Y^T + \kappa \tilde{f} Y^T \right) \right]
\]
\[
= -\eta s^2 - \frac{K}{\gamma} s^2 - \beta \|\tilde{a} - \tilde{\nu}\|^2 - \frac{\beta \mu}{\gamma} \|\tilde{a} - \tilde{\nu}\| \|Y\|^2 + 2s (\tilde{a} - \tilde{\nu})^T Y^T + 2 \frac{K}{\gamma} \tilde{f} (\tilde{a} - \tilde{\nu})^T Y^T
\]
\[
\leq -\epsilon_1 s^2 - \frac{K}{\gamma} s^2 - \beta \|\tilde{a} - \tilde{\nu}\|^2 - \frac{\beta \mu}{\gamma} \|\tilde{a} - \tilde{\nu}\| \|Y\|^2 + 2s \|\tilde{a} - \tilde{\nu}\| \|Y\| + 2 \frac{K}{\gamma} \|\tilde{f}\| \|\tilde{a} - \tilde{\nu}\| \|Y\|
\]
\[
\leq -\epsilon_1 s^2 - \epsilon_2 \frac{K}{\gamma} s^2 - \left( \sqrt{(1 - \epsilon_1)\eta} s - \frac{1}{\sqrt{(1 - \epsilon_1)\eta}} \|\tilde{a} - \tilde{\nu}\| \|Y\| \right)^2
\]
where \( 0 < \epsilon_1 < 1 \) and \( 0 < \epsilon_2 < 1 \) are arbitrary and where we have taken \( \mu = \frac{\gamma}{\eta (1 - \epsilon_1)} \). Because \( \epsilon_1 \) and \( \epsilon_2 \) are arbitrary, this shows that \( \dot{V} \) is negative semi-definite for \( \mu > \frac{\gamma}{\eta (1 - \epsilon_1)} \). Hence \( \dot{V} \in L_\infty \), \( \tilde{a} \in L_\infty \), and \( s \in L_\infty \). Because \( s \in L_\infty \), we automatically have \( x \in L_\infty \), which shows that \( \dot{s} \in L_\infty \) by local boundedness of \( \tilde{f} \) in \( x \) and \( \tilde{a} \) uniformly in \( t \). Integrating \( \dot{V} \) shows that \( s \in L_2 \) and hence by Barbalat’s Lemma \( s \to 0 \) and \( x \to x_d \). \( \square \)

**Proof of Proposition 3.6**

**Proof.** Consider the Lyapunov function candidate
\[
V = \frac{1}{2\gamma} \|\tilde{\nu}\|^2 + \frac{1}{2\gamma} \|\tilde{a} - \tilde{\nu}\|^2
\]
which has time derivative
\[
\dot{V} = \frac{1}{\gamma} \tilde{\nu}^T \left( \beta \tilde{N} (\tilde{\nu} - \tilde{a}) + \gamma \tilde{a} \tilde{f} \right)
\]
\[
= - (\tilde{a}^T \tilde{f} \alpha) \tilde{f} - \beta \|\tilde{a} - \tilde{\nu}\|^2 + 2 (\tilde{\nu} - \tilde{a})^T \tilde{f} \alpha
\]
\[
\leq - \frac{\tilde{f}^2}{D_1} - \frac{\beta}{\gamma} \|\tilde{a} - \tilde{\nu}\|^2 - \frac{\beta \mu}{\gamma} \|\tilde{a} - \tilde{\nu}\| \|\alpha\| \|\tilde{a} - \tilde{\nu}\| \|\tilde{f}\|
\]
\[
\leq - \frac{\epsilon}{D_1} \tilde{f}^2 - \beta \|\tilde{a} - \tilde{\nu}\|^2 - \left( \sqrt{\frac{1 - \epsilon}{D_1}} \tilde{f} - \sqrt{\frac{D_1}{1 - \epsilon}} \|\alpha\| \|\tilde{a} - \tilde{\nu}\| \right)^2
\]
where where $0 < \epsilon < 1$ is arbitrary and we have chosen $\mu = \frac{\gamma D_1}{(1-\epsilon)\beta}$. Because $\epsilon$ is arbitrary, this shows that $\hat{v}$ and $\hat{a}$ remain bounded for $\mu > \frac{\gamma D_1}{\beta}$. By integrating $\hat{V}$, we see that $\hat{f} \in L_2$. Application of Lemma A.1 completes the proof.

**Proof of Proposition 3.9**

**Proof.** The Lyapunov-like function

$$V = \frac{1}{2\gamma} (\|\hat{v}\|^2 + \|\hat{a} - \hat{v}\|^2 + \|\hat{\alpha} - \hat{a}\|^2)$$

has time derivative

$$\dot{V} = -(\hat{a}^T \alpha) \hat{f} - \frac{\beta}{\gamma} N\|\hat{a} - \hat{v}\|^2 + \frac{k \beta}{\gamma} N (\hat{a} - \hat{v})^T (\hat{a} - \hat{a}) + 2\hat{f} (\hat{a} - \hat{v})^T \alpha - 2\frac{k \beta}{\gamma} N\|\hat{a} - \hat{a}\|^2 + \frac{\beta}{\gamma} N (\hat{a} - \hat{a})^T (\hat{v} - \hat{a})$$

$$\leq -\frac{\hat{f}^2}{D_1} - \frac{\beta}{2\gamma} (1 - k) \|\hat{a} - \hat{v}\|^2 - \frac{\beta}{2\gamma} \|\hat{a} - \hat{\alpha}\|^2 (3k - 1) + 2|\hat{f}|\|\hat{a} - \hat{v}\|\|\alpha\|$$

$$\leq -\frac{\epsilon}{D_1} \hat{f}^2 - \left(\sqrt{\frac{1}{1-\epsilon}\hat{f}} - \sqrt{\frac{D_1}{1-\epsilon}\|\hat{a} - \hat{v}\|\|\alpha\|} \right)^2 - \frac{\beta}{2\gamma} (1 - k) \|\hat{a} - \hat{v}\|^2 - \frac{\beta}{2\gamma} N (3k - 1) \|\hat{a} - \hat{\alpha}\|^2$$

where $0 < \epsilon < 1$ is arbitrary and we have chosen $\mu = \frac{\gamma D_1}{\beta(1-\epsilon)\beta}$. From above, we conclude $\hat{v}, \hat{a},$ and $\hat{\alpha}$ remain bounded for $\frac{1}{\beta} \leq k < 1$. By integrating $\hat{V}$, we see that $\hat{f} \in L_2$, $(\hat{a} - \hat{\alpha}) \in L_2,$ and $(\hat{v} - \hat{\alpha}) \in L_2$. Application of Lemma A.1 completes the proof.

**Proof of Proposition 3.10**

**Proof.** The Lyapunov-like function

$$V = \frac{1}{\gamma} (\|\hat{v}\|^2 + \|\hat{\alpha} - \hat{v}\|^2)$$

has time derivative

$$\dot{V} = -(\hat{a}^T \alpha) \hat{f} + 2\hat{f} (\hat{a} - \hat{v})^T \alpha - \frac{\beta}{\gamma} N\|\hat{a} - \hat{v}\|^2 - \frac{\rho}{\gamma} \|\hat{v} - \hat{v}\|^2 - \frac{\rho}{\gamma} (\hat{a} - \hat{v})^T (\hat{v} - \hat{v})$$

$$\leq -\frac{\hat{f}^2}{D_1} + 2|\hat{f}|\|\hat{a} - \hat{v}\|\|\alpha\| - \left(\frac{\beta}{\gamma} - \frac{\rho}{2\gamma}\right) \|\hat{a} - \hat{v}\|^2 - \frac{\rho}{2\gamma} \|\hat{v} - \hat{v}\|^2$$

$$\leq -\frac{\epsilon}{D_1} \hat{f}^2 - \left(\sqrt{\frac{1}{1-\epsilon}\hat{f}} - \sqrt{\frac{D_1}{1-\epsilon}\|\hat{a} - \hat{v}\|\|\alpha\|} \right)^2 - \frac{\rho}{2\gamma} \|\hat{v} - \hat{v}\|^2 - \frac{1}{2\gamma} (2\beta - \rho) \|\hat{v} - \hat{\alpha}\|^2$$

where $0 < \epsilon < 1$ is arbitrary and we have chosen $\mu = \frac{\gamma D_1}{\beta(1-\epsilon)}$. From above, we conclude $\hat{v}, \hat{\alpha}$, and $\hat{a}$ remain bounded for $\rho < 2\beta$. Integrating $\dot{V}$ shows that $\hat{f} \in L_2$, $(\hat{v} - \hat{\alpha}) \in L_2,$ and $(\hat{v} - \hat{a}) \in L_2$. Application of Lemma A.1 completes the proof.

**Proof of Proposition 3.11**

**Proof.** The Lyapunov-like function

$$V = \frac{1}{2\gamma} (\|\hat{a} - \hat{v}\|^2 + \|\hat{\alpha} - \hat{\alpha}\|^2 + \|\hat{v}\|^2 + \|\hat{\alpha}\|^2)$$

20
has time derivative

\[
\dot{V} = - (\hat{a}^T \alpha) \dot{f} - \frac{\beta N}{\gamma} ||\hat{a} - \hat{v}||^2 - \frac{2k\beta N}{\gamma} ||\hat{a} - \hat{a}||^2 + 2\dot{f} (\hat{a} - \hat{v})^T \alpha + \beta N (\hat{a} - \hat{a})^T (\hat{v} - \hat{a}) \\
+ \frac{k\beta N}{\gamma} (\hat{a} - \hat{v})^T (\hat{a} - \hat{a}) - \frac{\rho}{\gamma} ||\hat{v} - \hat{v}||^2 - \frac{\rho}{\gamma} (\hat{a} - \hat{v})^T (\hat{v} - \hat{v}) \\
\leq - \frac{1}{D_1} \dot{f}^2 - \frac{1}{2} \left( \beta \gamma (1 - k) \rho \right) ||\hat{a} - \hat{v}||^2 - \frac{1}{2} \left( \beta \mu (1 - k) \right) ||\hat{a} - \hat{v}||^2 ||\alpha||^2 \\
- \frac{N\beta}{2\gamma} (3k - 1) ||\hat{a} - \hat{a}||^2 - \frac{\rho}{2\gamma} ||\hat{v} - \hat{v}||^2 + 2\dot{f} ||\hat{a} - \hat{v}|| ||\hat{v}|| \\
\leq - \frac{\epsilon}{D_1} \dot{f}^2 - \left( \sqrt{\frac{1 - \epsilon}{D_1}} |\dot{f}| - \sqrt{\frac{D_1}{1 - \epsilon}} ||\hat{a} - \hat{v}|| ||\alpha|| \right)^2 - \frac{\rho}{2\gamma} ||\hat{v} - \hat{v}||^2 \\
- \frac{\beta N}{2\gamma} (3k - 1) ||\hat{a} - \hat{a}||^2 - \frac{1}{2} \left( (1 - k) \beta \rho \right) ||\hat{a} - \hat{v}||^2
\]

where \( 0 < \epsilon < 1 \) is arbitrary and we have chosen \( \mu = \frac{2N D_1}{\beta(1 - k)(1 - \rho)} \). This immediately shows that \( \hat{a}, \hat{v}, \pi, \) and \( \nabla \) remain bounded for \( \frac{1}{2} \leq k < 1 \) and \( \rho < \beta(1 - k) \). Integrating \( \dot{V} \) shows that \( \dot{f} \in L_2, (\nabla - \hat{v}) \in L_2, (\hat{a} - \pi) \in L_2, \) and \( (\hat{a} - \hat{v}) \in L_2 \). Application of Lemma A.1 completes the proof.

**Proof of Proposition 5.13**

**Proof.** Consider the Lyapunov-like function

\[
V = \frac{1}{2} \dot{s}^2 + \frac{1}{2} \dot{\hat{v}}^T P^{-1} \hat{v} + \frac{1}{2} (\hat{v} - \hat{a})^T P^{-1} (\hat{v} - \hat{a})
\]

which has time derivative

\[
\dot{V} = -\eta \dot{s}^2 + (\hat{v} - \hat{a} + \hat{a})^T (s + \dot{f}) Y^T + \frac{1}{2} \dot{\hat{v}}^T Y^T (s + \dot{f}) Y^T - \frac{\lambda(t)}{2} \dot{v}^T P^{-1} \dot{v} \\
+ (\hat{v} - \hat{a})^T \left( -\beta N (\hat{v} - \hat{a}) - (s + \dot{f}) Y^T \right) + \frac{1}{2} \left( (\hat{v} - \hat{a})^T Y^T \right)^2 - \frac{\lambda(t)}{2} (\hat{v} - \hat{a})^T P^{-1} (\hat{v} - \hat{a}) \\
= -\eta \dot{s}^2 - \dot{f}^2 - 2 (\hat{v} - \hat{a})^T (s + \dot{f}) Y^T - \beta N ||\hat{v} - \hat{a}||^2 + \frac{1}{2} (\hat{v}^T Y^T)^2 + \frac{1}{2} \left( (\hat{v} - \hat{a})^T Y^T \right)^2 \\
- \frac{\lambda(t)}{2} \left( \dot{v}^T P^{-1} \dot{v} + (\hat{v} - \hat{a})^T P^{-1} (\hat{v} - \hat{a}) \right)
\]
Now we use that $\hat{V}^T Y = (\hat{v} - \hat{a})^T Y + \hat{f}$ to say that $\frac{1}{2} (\hat{v}^T Y^T)^2 = \frac{1}{2} \left[ (\hat{v} - \hat{a})^T Y^T \right]^2 = \frac{1}{2} \left[ (\hat{v} - \hat{a})^T Y^T \right]^2 + \hat{f}^2$. Hence,

$$
\dot{V} = -\eta s^2 - \frac{1}{2} \hat{f}^2 - 2s (\hat{v} - \hat{a})^T Y^T - \hat{f} (\hat{v} - \hat{a})^T Y^T - \beta N \| \hat{v} - \hat{a} \|^2 + \| (\hat{v} - \hat{a})^T Y^T \|^2
$$

Hence,

$$
\dot{V} = -\eta s^2 - \frac{1}{2} \hat{f}^2 - 2s (\hat{v} - \hat{a})^T Y^T - \hat{f} (\hat{v} - \hat{a})^T Y^T - \beta \| \hat{v} - \hat{a} \|^2 - \beta \| Y \|^2 \| \hat{v} - \hat{a} \|^2 + \left[ (\hat{v} - \hat{a})^T Y^T \right]^2
$$

$$
\dot{V} = -\eta s^2 - \frac{1}{2} \hat{f}^2 + 2s (\hat{v} - \hat{a})^T Y^T + \hat{f} (\hat{v} - \hat{a})^T Y^T - \beta \| \hat{v} - \hat{a} \|^2 - \beta \| Y \|^2 \| \hat{v} - \hat{a} \|^2 + \left[ (\hat{v} - \hat{a})^T Y^T \right]^2
$$

$$
\dot{V} = -\eta s^2 - \frac{1}{2} \hat{f}^2 - 2s (\hat{v} - \hat{a})^T Y^T - \hat{f} (\hat{v} - \hat{a})^T Y^T - \beta \| \hat{v} - \hat{a} \|^2 - \beta \| Y \|^2 \| \hat{v} - \hat{a} \|^2
$$

where $0 < \epsilon_1 < 1$ and $0 < \epsilon_2 < 1$ are both arbitrary and where we have chosen $\mu = \frac{1}{\beta} \left( 1 + \frac{1}{\eta (1 - \epsilon_1)} + \frac{1}{2 (1 - \epsilon_2)} \right)$. This shows that $s$, $\hat{v}$, and $\hat{a}$ remain bounded. Because $s$ remains bounded, $x$ remains bounded. Integrating $V$ shows that $s \in L_2$ and $\hat{f} \in L_2$. By local boundedness of $\hat{f}$ in $x$ and $\hat{a}$ uniformly in $t$, $\hat{f}$ remains bounded and hence $s$ remains bounded. By Barbalat’s Lemma, $s \to 0$ and $x \to \bar{x}$.

**Proof of Proposition 5.15**

**Proof.** Consider the Lyapunov-like function

$$
V = \frac{1}{2} \left( \hat{v}^T P \hat{v} + (\hat{a} - \hat{v})^T P^{-1} (\hat{a} - \hat{v}) \right),
$$

which has time derivative

$$
\dot{V} = -\hat{v}^T \alpha \hat{f} + \frac{1}{2} \hat{v}^T (\alpha P^{-1} + \alpha \alpha^T) \hat{v} + (\hat{a} - \hat{v})^T \left( \beta N \hat{v} - \hat{a} \right) + \frac{1}{2} (\hat{a} - \hat{v})^T (\alpha P^{-1} + \alpha \alpha^T) (\hat{a} - \hat{v})
$$

$$
\leq -\left( \alpha^T \alpha \right) \hat{f} + \frac{1}{2} \left( \hat{v}^T \alpha \right)^2 + \frac{1}{2} \left( \hat{a} - \hat{v} \right)^2 \left( \beta N \left| \hat{a} - \hat{v} \right| \right)^2 + 2 \left( \hat{a} - \hat{v} \right)^T \alpha

$$

$$
\leq -D_2 \left( \alpha^T \hat{a} \right)^2 + \frac{1}{2} \left( \hat{v}^T \alpha \right)^2 + \frac{1}{2} \left( \hat{a} - \hat{v} \right)^2 \left( \beta N \left| \hat{a} - \hat{v} \right| \right)^2 + 2 \left( \hat{a} - \hat{v} \right)^T \alpha

$$

Now, we use the fact that $\frac{1}{2} (\hat{v}^T \alpha)^2 = \frac{1}{2} \left[ (\hat{v} - \hat{a})^T \alpha \right]^2 + (\hat{a} - \hat{v})^T \alpha (\alpha^T \hat{a}) + \frac{1}{2} \left( \alpha^T \alpha \right)^2$ to rewrite

$$
\dot{V} \leq -D_2 \left( \frac{1}{2} \right) \left( \alpha^T \hat{a} \right)^2 + \left[ (\hat{a} - \hat{v})^T \alpha \right]^2 - \beta \left( \| \hat{a} - \hat{v} \|^2 + \left| \hat{v} - \hat{a} \right| \| \alpha \| \| \alpha^T \hat{a} \| \right) (2D_1 + 1)

\leq -D_2 \left( \frac{1}{2} \right) \left( \alpha^T \hat{a} \right)^2 - \beta \| \hat{a} - \hat{v} \|^2 - (\beta N \| \hat{a} - \hat{v} \|^2) + \left| \hat{v} - \hat{a} \right| \| \alpha \| \| \alpha^T \hat{a} \| (2D_1 + 1)

\leq -\epsilon \left( D_2 - \frac{1}{2} \right) \left( \alpha^T \hat{a} \right)^2 - \left( \frac{1}{(1 - \epsilon)} \left( D_2 - \frac{1}{2} \right) \right) \| \alpha^T \hat{a} \| - \frac{2D_1 + 1}{2 \sqrt{(1 - \epsilon)} \left( D_2 - \frac{1}{2} \right)} \| \hat{v} - \hat{a} \| \| \alpha \| ^2

$$

where $0 < \epsilon < 1$ is arbitrary and where we have chosen $\mu = \frac{1}{\beta} \left( 1 + \frac{2D_1 + 1}{(1 - \epsilon) (D_2 - \frac{1}{2})} \right)$. $\dot{V}$ is clearly negative semi-definite for $D_2 < \frac{1}{2}$, which shows that $\hat{v}$ and $\hat{a}$ remain bounded. Integrating $\dot{V}$ shows that $(\alpha^T \hat{a}) \in L_2$ which implies that $\hat{f} \in L_2$ by Assumption 5.3. By Lemma A.1, the proposition is proven.
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