Canonical Statistical Model and hadron production in $e^+e^-$ annihilations

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Abstract. We discuss the production of hadrons in $e^+e^-$ collisions at $\sqrt{s} = 91$ GeV. We address the question whether the particle yields measured in the final states are consistent with the statistical model predictions. In the model formulation we account for exact conservation of all relevant quantum numbers using the canonical description of the partition function. Within our model the validity of the thermodynamical approach to quantify particle production in $e^+e^-$ annihilations is not obvious.

1. Introduction

One of the essential results in heavy ion collisions was the observation that particle yields measured in a final state closely resemble a thermal equilibrium population [1, 2, 3]. The natural question was whether this statistical behavior is a unique feature of high energy nucleus-nucleus collisions or whether it is also applicable in elementary collisions like, e.g., $e^+e^-$. Previous publications [4] indicated that indeed hadron production in $e^+e^-$ collisions can be well described within a thermal model provided that local quantum number conservation is properly implemented. In view of the most complete and extended data summarized e.g. by the Particle Data Group (PDG) [5] the above question has been recently addressed independently in [6] and [7].

In this contribution we discuss hadron production in $e^+e^-$ annihilations at $\sqrt{s} = 91$ GeV based on the thermal model analysis of [6]. We also focus on different implementations of the statistical model and discuss the importance of quantum statistic effects and the mass–cut in the hadron mass spectrum. We quantify the production of heavy flavors and compare the model predictions with available data.

2. The statistical model and charge conservation

The usual form of the statistical model in the grand canonical ensemble formalism cannot be used when the number of produced charged particles is small. This is the case if either
Figure 1. The (left–hand figure) shows deviations of pion and kaon yields from their exact quantum statistics values (see text). The $k = 1$ term corresponds to the Boltzmann approximation. The (right–hand figure): the relative change of the particle yields calculated in the hadron resonance gas model with a mass spectrum cut at the mass $M = 1.7$ and $M = 3.0$ GeV.

The temperature $T$ or the volume $V$ or both are small. As a rule of thumb one needs $VT^3 >> 1$ for a grand canonical description to be applicable \[8\]. In $e^+e^-$ annihilations where modelling the particle production within a thermal approach requires an exact formulation of conservation laws. In such a system one needs to account for an exact conservation of five quantum numbers: the baryon number $N$, strangeness $S$, electric charge $Q$, charm $C$ and bottom $B$.

The appropriate tool to deal in a statistical mechanics framework with a system of quantum numbers $\vec{X} = (N, S, Q, C, B)$ is the canonical partition function \[8, 9\]

$$Z_{N,S,Q,C,B} = \frac{1}{(2\pi)^5} \int_{-\pi}^{\pi} d^5\vec{\phi} \ e^{i\vec{\phi}\vec{X}} \exp \left( \sum_j z_j \right), \quad (1)$$

where

$$z_j = g_j \frac{V}{(2\pi)^3} \int d^3p \ \ln(1 \pm \exp \left( -\sqrt{p^2 + m_j^2}/T - i\vec{x}_j \vec{\phi} \right))^{\pm 1}, \quad (2)$$

and $x_j$ is a five component vector $x_j = (N_j, S_j, Q_j, C_j, B_j)$ containing the quantum numbers of the particle species $j$. The quantity $\phi = (\phi_N, \phi_S, \phi_Q, \phi_C, \phi_B)$ is an element of the symmetry group $[U(1)]^5$ related with additive conservation laws. In this expression, each $\phi_X$ corresponds to the conservation of the corresponding quantum number $X$ and $z_j$ is the single particle partition function for particle with mass $m_j$, spin-isospin degeneracy.
factor $g_j$, and a system with volume $V$ and temperature $T$. The sum in Eq. (1) runs over all particle species in the hadronic gas.

The integral representation of the partition function in Eq. (1) is not convenient for numerical analysis as the integrand is a strongly oscillating function. Thus, we first expand the logarithm

$$\ln(1 \pm x)^{\pm 1} = \sum_{k=1}^{\infty} (\pm 1)^{k+1} \frac{x^k}{k} \quad (3)$$

and then, using the method described in [10, 11, 12], we express the partition function $Z$ in a series of Bessel functions to obtain a result that is free from oscillations. Furthermore, from Eq. (1) we obtain the multiplicity $\langle n_j \rangle$ for particle species $j$ by introducing a fugacity parameter $\lambda_j$ which multiplies the particle partition function $z_j$ and by differentiating

$$\langle n_j \rangle = \left. \frac{\partial \ln Z}{\partial \lambda_j} \right|_{\lambda_j=1} \quad (4)$$

The first term in the expansion of the logarithm in Eq. (3) corresponds to the Boltzmann approximation which is suited only for $m_j >> T$. Such a condition is satisfied for baryons since their masses are larger than a typical temperature of the hadron resonance gas which never exceeds a critical value $T_c \approx 200$ MeV required for deconfinement. However, for light bosons like pions or kaons the quantum statistics is of importance as the temperature is comparable to their masses. Fig. (1–left) shows relative deviations of pion and kaon multiplicities from their quantum statistics values with increasing numbers of terms $k$ in the expansion (3). The calculations were performed for $T = 157$ and $V = 32$ fm$^3$. It is clear that the Boltzmann approximation is by far not sufficient to reproduce the quantum statistics results. The pion yield under Boltzmann approximation deviates by more than 7% from the exact quantum statistics result. For kaons this difference is only 1% due to the larger mass. For pions, five terms are needed in the expansion (3) to get quantum statistics value with deviations below $10^{-3}$. For heavier particles, for instance for protons, the quantum and Boltzmann statistics differs by less than 0.1%. Thus, in the model comparison with $e^+e^-$ data we apply quantum statistics for light mesons and we use Boltzmann statistics for all baryons.

3. Modelling the $e^+e^-$ events within the statistical approach

The hadron multiplicity calculation within the statistical model basically proceeds in two steps. First, a primary hadron yield $N_k^{th}$, is calculated using (1) and (4). A crucial assumption of the model is that the final yields of all particles are fixed at a common temperature, the chemical decoupling point. As a second step, all resonances in the gas which are unstable against strong decays are allowed to decay into lighter stable hadrons, using appropriate branching ratios ($B$) and multiplicities ($M$) for the decay
j → h published by the PDG [13]. The abundances in the final state are thus determined by

\[ N_h = N_h^{th} + \sum_j N_j \cdot B(j \rightarrow h)M(j \rightarrow h) \]  

(5)

where the sum runs over all resonance species.

From Eq. (5) it is clear, that the final multiplicity of stable hadrons depends on the number of resonances used in the sum. In general one should include contributions of all known resonances as listed by the PDG. Fig. (1-right) shows the relative change of different particle yields when applying the mass cut \( M = 1.7 \) and \( M = 3.0 \) GeV in Eq. (5). For mesons this difference amounts to 5% and is as large as 15% for baryons. Thus, it is clear that restricting the mass spectrum only to the resonances with the mass \( M < 1.7 \) GeV might be not sufficient at the level of accuracy of data of a few percent as is the case in \( e^+e^- \) collisions. However, our knowledge of decay properties of heavier resonances is by far not complete, which causes systematic uncertainties of the
statistical model. The importance of heavier resonances originating from the Hagedorn mass spectrum in the analysis of particle production in heavy ion collisions has been recently analyzed in [14]. In the following we account for resonance contributions up to the mass $M < 3$ GeV.

In general, the resonance gas model formulated in the canonical ensemble is described by only two basic thermal parameters: the temperature $T$ and the volume $V$ of the system. To explore a possible strangeness undersaturation we introduce an additional parameter $\gamma_s$ into the partition function to account for a possible deviation of strange particle yields from their chemical equilibrium values [4]. When applying the statistical model to particle production in $e^+e^-$ annihilations we have to take into account that most hadronic events in such collisions are two-jet events, originating from quark-antiquark pairs of the five lightest flavors. Since we would like to address the issue of overall equilibration in these systems, we have to specify how the initial quantum numbers are distributed between the two-jets. We will consider two scenarios: an uncorrelated and correlated jet scheme.

In an uncorrelated jet scheme each jet is treated as a fireball with vanishing quantum numbers as fixed by the entrance channel. It is clear at this point that hadrons from jets with heavy quarks (c and b) will be greatly underestimated by the model because of the large Boltzmann suppression factors. In this approach the issue of equilibration is effectively addressed only for hadrons with light quarks (u, d, s). It is important to recognize that the measured yields of these hadrons contain the contribution from the $e^+e^-$ annihilation events into $c\bar{c}$ and $b\bar{b}$. Heavy-quark production is indeed significant and is very precisely measured, in particular at the $Z_0$ mass, where the measurements are very well described by the standard model. Hence, heavy-quark production is manifestly non-thermal in origin. We therefore consider two cases: i) we fit the data as measured and ii) we subtract from the yields of hadrons carrying light quarks the contribution originating from charm and bottom decays based on available data for the charmed and bottom hadron production and their branching ratios.

In a correlated jet scheme (case (iii) in Fig. (2) ) the initial quantum numbers are distributed such that each jet carries quantum numbers of either $S = \pm 1, C = \pm 1, B = \pm 1$, or vanishing quantum numbers in the case of $u\bar{u}$ and $d\bar{d}$ jets. The fractions of the quark flavors in hadronic events [5] are external input values, unrelated to the thermal model (see also Table II in ref. [7]).

3.1. Model comparison with $e^+e^-$ data at $\sqrt{s} = 91$ GeV

For the fit procedure we use the complete set of all measured hadron yields with the exception of those containing charm or bottom quarks. A $\chi^2$ fit is performed by minimizing

$$
\chi^2 = \sum_h \frac{(N_h^{exp} - N_h)^2}{\sigma_h^2}
$$

(6)
as a function of the three parameters $T$, $V$ and $\gamma_s$, taking account of the experimental uncertainties $\sigma_h$. The resulting best fit to the data for a correlated and uncorrelated jet scheme is shown in Fig. (2). The two cases of the fit, without and with the subtraction of the contribution from heavy quarks in an uncorrelated jet scheme are also shown in this figure. We first note the overall behavior of the data, namely an approximately exponential decrease of the particle yield with an increasing particle mass. Such a behavior is expected in the hadron resonance gas model due to the Boltzmann factors, thus indicating the presence of statistical features of hadron production in elementary collisions. The quantitative description of the data with the statistical model is, however, rather poor and certainly no improvement is visible for the case of subtracting charm and bottom contributions. The low fit quality is reflected through the large $\chi^2$ values per degree of freedom. In addition, discrepancies between individual data points and fit values larger than 5 standard deviations are not rare. There are also problems in determination of the fit parameters: $T$, $V$ and $\gamma_s$. In $\chi^2$ contour plots, both in $(T,V)$- and $(T,\gamma_s)$-plane, one notices strong anticorrelations between fit parameters [6]. In addition there is a series of local minima which makes it difficult to uniquely determine the model parameters. Such local minima are typical for poor fits and imply large uncertainties in the determinations of the fit parameters.

3.2. Model description of heavy quark hadron production

In the canonical formulation of the statistical model with exact charge conservation the abundances of charged particles depend crucially on the overall charge in a system. In order to illustrate this let us consider a model where only one charge, e.g. charm, is conserved exactly. In such a case the multiplicity $<N_i>_{C_i=\pm 1}$ of particle $i$ with mass $m_i$ that carries charge $C_i = \pm 1$ in a system of the total charge $C$, volume $V$ and temperature $T$ is obtained under the Boltzmann approximation from [3]:

$$<N_i>_{C_i=\pm 1} = V \frac{z_i^{C_i} Z_{\pm 1} I_{C\pm 1}(2Vx)}{x I_C(2Vx)}$$

(7)

where

$$z_i^{C_i} = \frac{d_i}{2\pi^2} m_i^2 TK_2(m_i/T), \quad Z_{\pm 1} = \sum_i z_i^{C_i=\pm 1}, \quad x = \sqrt{Z_1 Z_{-1}},$$

(8)

and where $I_k$ and $K_2$ are Bessel functions and the argument of $I_k$ quantifies the total number of charged particle pairs.

From Eq. (7) one recognizes an essential difference in particle yields if the total charge $C = 0$ or $C = \pm 1$. Indeed, assuming that $Vx \leq 1$ and applying an asymptotic expansion of the Bessel functions one finds e.g. that multiplicities of particles with charge $C_i = +1$ is

$$<N_i> \simeq V^2 z_i Z_{-1},$$

(9)

‡ The charge is consider here to be any quantum number related with U(1) symmetry.
if the total charge of a system $C = 0$, and

$$< N_i > \simeq \frac{z_i}{Z_{+1}},$$

(10)

if the total charge of a system $C = +1$.

For a charge neutral system the charged particles yield (9) are strongly suppressed due to canonical effects, since a particle has to be produced in a pair with an antiparticle in order to fulfill charge neutrality. This is a well known "canonical suppression effect" which is crucial e.g. to quantify "strangeness enhancement" and production in hadron-hadron and heavy ion collisions at lower energies [2]. For the total charge $C = \pm 1$ the canonical effect results in enhancement of particle yields as shown in Eq. (10). The charge $C = \pm 1$ of a system is here redistributed between all particles that carry $C_i = \pm 1$ with weights given by the ratio of the thermal phase–space of particle $i$ to the phase–space of all negatively or positively charged particles. Here, even the Boltzmann suppression is to a large extent cancelled out in the ratio (see Eq. (10)).

The above examples imply a qualitative difference in open charm and bottom production in $e^+e^-$ collisions when using an uncorrelated and correlated jet scheme scenario. If $C = B = 0$ in a jet then the thermal phase–space of charm and bottom is strongly suppressed as in Eq. (9), leading to large discrepancies between model results and data. One needs to check whether in the correlated jet scheme, where the open charm and bottom are statistically enhanced as in Eq. (9), the measured yields are comparable with model predictions.

In quantitative analysis, one calculates the heavy quark yields with the partition function (1) rather than with the approximate Eq. (10). Fig. (3) shows the statistical model results for charmed and bottomed hadrons obtained in the correlated jet scheme. One recognizes a good description of data by the canonical statistical model. Thus, if the fireballs related with each jet are charged then the model description of the distribution

![Figure 3. Comparison between the thermal model calculations in a correlated jet scheme and experimental data on charmed and bottomed hadrons for $e^+e^-$ collisions at $\sqrt{s}=91$ GeV. Also shown are the model parameters used in the calculations.](image-url)
of these charges between different particles agrees with data. The value of $\chi^2/N_{df} \simeq 2$ indicates a good quality of the model description of data. We have to stress that, in the canonical model, yields of hidden bottom and charm meson are still strongly suppressed by the Boltzmann factors, thus their multiplicities deviate from data. Indeed, in Fig. (3), the yield of $Y$ is larger by several orders of magnitudes than the model results (not shown in Fig. (3)), indicating a non-thermal origin of this particle. This is not the case for $\psi$ since in $e^+e^-$ annihilations at $\sqrt{s} = 91$ GeV as they are almost entirely originating from decays of bottom [15].

4. Summary and conclusions

We have analyzed the experimental data on hadron yields with light and heavy quarks in $e^+e^-$ collisions at $\sqrt{s} = 91$ GeV within the statistical model. The conservation of five quantum numbers was included in the framework of the canonical partition function. Our results qualitatively confirmed the previous finding [4] that statistical features are present in hadron production in $e^+e^-$ annihilation. The resulting temperature of $160 - 170$ MeV lies in the bulk expected at the chemical freezeout in heavy ion collisions at high energy [1, 2] and agrees with first results obtained in $e^+e^-$ systems [4]. Our results on open charm and bottom hadrons, using the measured $c$ and $b$ fractions of jets, showing good agreement with data also confirmed earlier observation [4]. However, in view of a rather poor fit to measured yields of hadrons with light quarks and a clearly non-thermal origin of the hidden charm and bottom particles as well as essentially different characteristics of the collision fireball in $e^+e^-$ and in heavy ion collisions [6], the general validity of the thermodynamical approach to the particle production in $e^+e^-$ annihilation at LEP energies is not obvious.

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§ For a more complete discussion on charm and bottom production see Ref. [15].