Non-positively curved Ricci Surfaces with catenoidal ends

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Abstract. A Ricci surface is defined to be a Riemannian surface \((M, g_M)\) whose Gauss curvature \(K\) satisfies the differential equation \(K\Delta K + g_M(dK, dK) + 4K^3 = 0\). In the case where \(K < 0\), this equation is equivalent to the well-known Ricci condition for the existence of minimal immersions in \(\mathbb{R}^3\). Recently, Andrei Moroianu and Sergiu Moroianu proved that a Ricci surface with non-positive Gauss curvature admits locally an isometric minimal immersion into \(\mathbb{R}^3\). In this paper, we are interested in studying non-compact orientable Ricci surfaces with non-positive Gauss curvature. Firstly, we give a definition of catenoidal end for non-positively curved Ricci surfaces. Secondly, we develop a tool which can be regarded as an analogue of the Weierstrass data to obtain some classification results for non-positively curved Ricci surfaces of genus zero with catenoidal ends. Furthermore, we also give an existence result for non-positively curved Ricci surfaces of arbitrary positive genus which have finite catenoidal ends.

1. Introduction

A classical question in the theory of minimal surfaces is to study when a Riemannian surface \((M, g_M)\) can be locally isometrically immersed into \(\mathbb{R}^3\) as a minimal surface. In the history, the first result for this question was given by Gregorio Ricci-Curbastro. He provided a necessary and sufficient condition for the existence of such immersions near points with negative Gauss curvature (see [1]). Andrei Moroianu and Sergiu Moroianu have proven later in [2] that the assumption of negative Gauss curvature could actually be left out. The main idea of their proof is to study the differential equation

\[ K\Delta K + g_M(dK, dK) + 4K^3 = 0 \]

satisfied by the Gauss curvature appearing in Ricci’s theorem. It is often called the Ricci condition in the case \(K < 0\). A Riemannian surface whose metric satisfies this equation is called a Ricci surface.

The theory of compact Ricci surfaces is developed by Andrei Moroianu and Sergiu Moroianu as well. To be more precise, they provided some methods in constructing compact Ricci surfaces. We are therefore interested in considering...
non-compact orientable Ricci surfaces. In 1958, Huber proved that a complete non-positively curved Riemannian surfaces with finite total curvature has to be biholomorphic to a compact Riemann surface with finite number of punctured points (see [3]). Consequently, a problem which should be naturally posed is to determine all the possible Ricci metrics on a given Riemann surface. However, this project seems to be very ambitious because many cases may occur around punctured points. The goal of our work is to classify Ricci surfaces by adding an assumption on the punctured points which is called the condition for catenoidal ends.

In this paper, Sect. 2 is devoted to the recall of some basic properties of the well-known Weierstrass representation for minimal immersions in $\mathbb{R}^3$. The definition of Ricci surfaces is introduced in Sect. 3, then we focus on defining the Weierstrass data for Ricci surfaces which will be our principal tools. In Sect. 4, we plan to define catenoidal ends for Ricci surfaces, and we will prove a lemma providing a local description of the Weierstrass data near a catenoidal end.

Our first main result is Theorem 9 which will be proven in Sect. 5. This theorem offers a complete classification of non-positively curved Ricci surfaces $M \simeq \mathbb{C} \setminus \{0\}$ with two catenoidal ends. It assures that a such Ricci surface can only be isometric to one of the two classes of surfaces. We will give also an explanation for the relations between our theorem and some related results. For example, Theorem 9 is equivalent to a result of Troyanov which classified all the metrics of constant Gauss curvature 1 on $\bar{\mathbb{C}}$ with two conical singularities at 0 and $\infty$ (see [4]).

In Sect. 6, we are going to discuss the case when the Ricci surface is biholomorphic to $M \simeq \mathbb{C} \setminus \{0, 1\}$ possessing three catenoidal ends. Since things become much more complicated in this situation, it could be very hard to obtain a complete classification as in the previous case. We restrict ourselves to several classification results under some additional conditions on the total curvature and the reducibility of induced metric $(-K)_{gM}$. For concrete statements of these results, readers may consult results from Theorem 11 to Corollary 17.

In the last section of this paper, we extend our discussion to Ricci surfaces with positive genus. We have proven Theorem 20 which assures that for $k, n > 0$, there exists a Ricci surface with genus $k$ and $n$ catenoidal ends.

2. Preliminaries

In this section, we plan to recall the Weierstrass representation for minimal immersions in $\mathbb{R}^3$ which plays an essential role in our discussion.

Suppose that $X : M \to \mathbb{R}^3$ is a minimal immersion, where $M$ is a smooth manifold of dimension 2. Taking a simply connected neighbourhood $(U, (x, y))$ of $p_0 \in M$ with $(x, y)$ the isothermal coordinates, then $X$ being minimal means that $X$ is a harmonic map, or equivalently,

$$\phi = (\phi_1, \phi_2, \phi_3) := \frac{\partial X}{\partial x} - i \frac{\partial X}{\partial y} = 2 \frac{\partial X}{\partial z}$$

is holomorphic, where $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})$. It is not difficult to verify that $\phi dz$ is a globally defined vector valued holomorphic 1-form on $M$. With
the help of $\phi d\zeta$, we may construct a meromorphic function $g$ and a holomorphic
1-form $\eta$ on $M$ such that the minimal immersion $X$ can be expressed as
\[ X(p) = X(p_0) + \text{Re} \int_{p_0}^p \left( \frac{i}{2} \left( 1 - g^2 \right) \eta, \frac{i}{2} \left( 1 + g^2 \right) \eta, g \eta \right). \] (2)

Formula (2) is called the Weierstrass representation of the minimal immersion
$X : M \to \mathbb{R}^3$ and the pair $(g, \eta)$ is called the Weierstrass data of $X$. Since $\phi d\zeta$ is a
holomorphic 1-form which does not vanish, we see that whenever $g$ has a pole of
order $m$ at $p \in M$, $\eta$ must have a zero of order $2m$ at $p \in M$. The function $g$ is also
called the Gauss map. To see this, it is sufficient to realize that $g$ is just the classical
Gauss map $G : M \to S^2$ composed by the stereographic projection from the north
pole of $S^2$. Moreover, it is worth noticing that all these formulas still hold if we
multiply $\eta$ by an element $e^{i\theta} \in S^1$, thus the new Weierstrass data $(g, e^{i\theta} \eta)$ give
another minimal immersion $X_\theta : M \to \mathbb{R}^3$. This leads to the following definition:

**Definition 1.** The immersions $X_\theta : M \to \mathbb{R}^3$ with $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ are called associate
minimal immersions of $X : M \to \mathbb{R}^3$.

We will discuss some geometric data of a minimal immersion with the help of
the Weierstrass representation.

Firstly, if we denote locally $\eta$ by $\alpha d\zeta$, then the induced Riemannian metric $g_M$
on $M$ will be determined to be
\[ g_M = \frac{1}{4} |\alpha|^2 \left( 1 + |g|^2 \right)^2 |d\zeta|^2. \] (3)

The second thing is to determine the Gauss curvature $K$ of $M$. In use of the
Riemannian metric $g_M$, the Gauss curvature can be computed as
\[ K = -\left[ \frac{4|g'|}{|\alpha|(1 + |g|^2)^2} \right]^2. \] (4)

We list in the end some examples of minimal immersions in $\mathbb{R}^3$ with their
Weierstrass data:

**Example 1.** (Enneper’s Surface) Let $M = \mathbb{C}$, the Weierstrass data are given by
$g(z) = z$ and $\eta = d\zeta$.

**Example 2.** (Catenoid) Let $M = \mathbb{C}\setminus\{0\}$, the Weierstrass data are given by
$g(z) = z$ and $\eta = \frac{a}{z} d\zeta$, with $a \in \mathbb{R}\setminus\{0\}$.

3. Ricci metrics and Ricci surfaces

Andrei Moroianu and Sergiu Moroianu defined in their article [2] a class of surfaces
called Ricci surfaces which can be regarded as a generalisation of minimal surfaces
in $\mathbb{R}^3$. In this section, we are going to introduce some basic properties of Ricci
surfaces. Throughout this paper, we only consider orientable Ricci surfaces.

The following theorem is an important fact which has been proven by Gregorio
Ricci-Curbastro [1]:
Theorem 1. A Riemannian surface \((M, g_M)\) with negative Gauss curvature \(K < 0\) has local isometric immersions as minimal surface in \(\mathbb{R}^3\) if and only if one of the three equivalent conditions holds:

(i). The metric \(\sqrt{-K} g_M\) is a flat metric;
(ii). The metric \((-K) g_M\) is a metric of constant Gauss curvature 1;
(iii). The Gauss curvature satisfies

\[
K \Delta K + g_M (dK, dK) + 4K^3 = 0.
\]

The third condition in Theorem 1 can even be studied without the hypothesis \(K < 0\). This observation inspires Andrei Moroianu and Sergiu Moroianu to give the following definition.

Definition 2. A Riemannian surface \((M, g_M)\) whose Gauss curvature \(K\) satisfies the following identity

\[
K \Delta K + g_M (dK, dK) + 4K^3 = 0 \tag{5}
\]

is called a Ricci surface, the metric \(g_M\) is called a Ricci metric.

From Theorem 1, we can see that all the immersed minimal surfaces in \(\mathbb{R}^3\) are Ricci surfaces. Moreover, Andrei Moroianu and Sergiu Moroianu proved in their article the following result which shows that a Ricci surface with non-positive Gauss curvature can be locally realized as an immersed minimal surface in \(\mathbb{R}^3\).

Theorem 2. Let \((M, g_M)\) be a connected Ricci surface, then its Gauss curvature \(K\) does not change sign on \(M\). If \(K \leq 0\), then \(M\) admits locally a minimal isometric immersion into \(\mathbb{R}^3\).

These two theorems indicate that the relation between Ricci surfaces with non-positive Gauss curvature and minimal immersions in \(\mathbb{R}^3\) is very close. Therefore, we are inspired to use the tools in the theory of minimal immersions in \(\mathbb{R}^3\) to study Ricci surfaces. This is possible thanks to the next proposition.

Proposition 3. Let \(g_0 = |dz|^2\) be a flat metric on some domain \(\Omega \subset \mathbb{C}\) and \(f : \Omega \to \mathbb{R}\) a smooth function. The metric \(g_\Omega = e^{-2f} g_0\) admits locally a minimal immersion in \(\mathbb{R}^3\) if and only if near every \(x \in \Omega\) there exists a pair of holomorphic functions \((a, b)\) such that \(e^{-f} = |a|^2 + |b|^2\).

Proof. (1). \(\Rightarrow\) If a such isometric immersion exists, then from the discussion in Sect. 2, there are Weierstrass data \(\eta = adz\) and \(g\) satisfying \(e^{-f} = \frac{1}{2} |a|(|1 + |g|^2|). Since \(a\) can only have a zero of order \(2m\) where \(g\) has a pole of order \(m\), we may find two holomorphic functions \(a\) and \(b\) such that

\[
a^2 = \frac{a}{2}, \quad b = ag,
\]

thus \(e^{-f} = |a|^2 + |b|^2\).

(2). \(\Leftarrow\) If there are two holomorphic functions \(a\) et \(b\) such that \(e^{-f} = |a|^2 + |b|^2\), then \(a\) can be supposed to be non-vanishing on a disc \(D\) centered at \(x\). Let us define
\( \alpha := 2a^2, \quad g := \frac{b}{a}, \) then \( \eta := \alpha dz \) is a holomorphic 1-form and \( g \) is a meromorphic function. Additionally, zeroes of \( \eta \) are compatible with the poles of \( g \) as mentioned in Sect. 2. Taking \((g, \eta)\) as Weierstrass data, we may find an isometric minimal immersion into \( \mathbb{R}^3 \) whose metric is \( g_\Omega \).

\( \square \)

Moreover, Calabi has proven the following theorem of rigidity (see [5,6]):

**Theorem 4.** Two minimal isometric immersions from a simply connected surface \( M \) into \( \mathbb{R}^3 \) are associate, up to the action of \( Isom(\mathbb{R}^3) \) on \( \mathbb{R}^3 \).

**Remark 1.** This theorem tells us that the minimal isometric immersion from a simply connected surface \( M \) into \( \mathbb{R}^3 \) is unique up to the action of \( Isom(\mathbb{R}^3) \) and the \( S^1 \)-action (associate minimal immersions). Combining Proposition 3 and Theorem 4, we may locally identify the non-positively curved Ricci surface with the equivalent class of its minimal isometric immersions into \( \mathbb{R}^3 \). This identification will be of great significance in our further discussion.

We are now interested in studying how the identification mentioned above allows us to do with the Weierstrass data. Firstly, it is easy to be seen from (2) that the Weierstrass data remain invariant under translations in \( \mathbb{R}^3 \). Secondly, Definition 1 implies that we can multiply \( \eta \) by an element \( e^{i\theta} \in S^1 \) while keeping \( g \) fixed. Thirdly, let us make a rotation in \( \mathbb{R}^3 \), i.e., applying an element \( A \in SO(3, \mathbb{R}) \) to Weierstrass representation (2), then the new surface \( \tilde{X} \) and its Gauss map \( \tilde{G} \) are written as

\[ \tilde{X} = AX, \quad \tilde{G} = AG. \]

(6)

Owing to the fact that \( \frac{\partial f}{\partial z} = 2 \frac{\partial \text{Re} f}{\partial z} \) for every holomorphic function \( f \), we obtain

\[ \tilde{\phi} := 2 \frac{\partial \tilde{X}}{\partial z} = A\phi. \]

(7)

It is known that the stereographic projection \( \pi \) and rotations preserve angles, so the composite map

\[ \pi \circ A \circ \pi^{-1} : \tilde{\mathbb{C}} \rightarrow \tilde{\mathbb{C}} \]

is conformal, hence an element of \( Aut(\tilde{\mathbb{C}}) = PSL(2, \mathbb{C}) \). In order to find the new Weierstrass data, we need the following classical result whose proof is standard.

**Lemma 5.** The map

\[ \varphi : SO(3, \mathbb{R}) \rightarrow PSL(2, \mathbb{C}) \]

\[ A \mapsto \pi \circ A \circ \pi^{-1} \]

is an injective Lie group homomorphism with \( \text{Im}\varphi = PSU(2) \), hence it induces an isomorphism \( SO(3, \mathbb{R}) \simeq PSU(2) \).
Now we are going to determine the new Weierstrass data after the rotation. As we have already seen, \( g = \pi \circ G : M \to \mathbb{C} \) is the Gauss map, hence
\[
\tilde{g} = \pi \circ A \circ G = \pi \circ A \circ \pi^{-1} \circ \pi \circ G = \varphi(A).g,
\] (8)
where \( \varphi(A).g \) denotes the action of \( \varphi(A) \) on \( g \). Then by combining (2), (7) and (8), we may get the new Weierstrass data \((\tilde{g}, \tilde{\eta})\).

Let us consider the Hopf differential defined by \( Q := dg \otimes \eta \). It is a direct observation from the previous calculation that \( Q \) is invariant under rotations in \( \mathbb{R}^3 \). Therefore, the new Weierstrass data obtained by applying \( A \in SO(3, \mathbb{R}) \) are computed as
\[
\begin{align*}
\tilde{g} &= \varphi(A).(g), \\
\tilde{\eta} &= \frac{Q}{d\tilde{g}}.
\end{align*}
\] (9)

From the proof of Proposition 3, a non-positively curved Ricci surface can be locally equipped with the Weierstrass data. It is worthwhile noticing that the Weierstrass data may not be globally defined on Ricci surfaces in general. However, they are always well-defined on the universal cover of a Ricci surface. In fact, for a non-positively curved Ricci surface \((M, g_M)\), the metric defined by \( g_1 := (-K)g_M \) is a metric of constant Gauss curvature 1, possibly with isolated conical singularities at zeroes of \( K \) (see [2,4]). It induces on the universal cover \( \tilde{M} \) of \( M \) a metric \( d\sigma^2 \) of constant Gauss curvature 1. Then there exists a meromorphic function \( g : \tilde{M} \to \mathbb{C} \) such that
\[
d\sigma^2 = \frac{4dg d\bar{g}}{(1 + |g|^2)^2}.
\] (10)
Moreover, this function \( g \) is unique up to an action of \( PSU(2) \) as in (8). Let us take \( g \) as the Gauss map, then the Weierstrass data can be constructed with the help of Proposition 3 and Theorem 4. To be convenient, we still call it the Weierstrass data of this Ricci surface. A point that should be mentioned is that the metric \( d\sigma^2 \) is invariant under the action of the deck transformation group \( \text{Deck}(\tilde{M}/M) \). It is immediate from the uniqueness of \( g \) that every deck transformation of \( \tilde{M} \) corresponds to an element in \( \text{PSU}(2) \). More precisely, if we identify the deck transformation group \( \text{Deck}(\tilde{M}/M) \) with the fundamental group \( \pi_1(M) \) of \( M \), then there exists a monodromy representation \( \rho : \pi_1(M) \to \text{PSU}(2) \) such that
\[
g \circ \tau^{-1} = \rho(\tau).g,
\] (11)
for all \( \tau \in \pi_1(M) \). This monodromy representation will be very important in our further discussion.

Even though the Hopf differential \( Q \) is only defined on the universal cover \( \tilde{M} \) in general, its modulus \( |Q| \) is actually well-defined on the Ricci surface \( M \) itself. To see this, since \( Q \) is invariant under the action of \( \text{PSU}(2) \) and it may differ just by a possible multiplication of \( e^{i\theta} \in S^1 \), the modulus \( |Q| \) is therefore independant of all these actions. Hence it descends to a well-defined quantity on \( M \).
Thanks to the identification mentioned in Remark 1, all the Weierstrass data under these operations are regarded to be equivalent. This equivalence class of Weierstrass data is exactly what the Weierstrass data of the non-positively curved Ricci surface means. From now on, once a representative pair of Weierstrass data is fixed, we are allowed to apply all the operations discussed above.

To finish this section, we also give an example of Ricci surfaces. For CMC-1 immersions in the hyperbolic 3-space $H^3$, we may define an analogue of the Weierstrass representation (see [7,8]). We know from the Lawson’s correspondence (see [9]) that CMC-1 immersions in $H^3$ are locally isometric to minimal immersions in $\mathbb{R}^3$, hence it provides some Ricci surfaces. A typical example is the following:

**Example 3.** (Catenoid cousins) Let $M = \mathbb{C}\setminus\{0\}$, the representative Weierstrass data are $g(z) = z^\mu$ and $\eta = az^{-\mu-1}dz$, with $a, \mu \in \mathbb{R}_+^*$ and $\mu \neq 1$. In particular, if $\mu$ is an integer, then it is isometric to a $\mu$-fold cover of catenoid.

### 4. Definition of catenoidal ends

In this section we plan to define catenoidal ends for a non-positively curved Ricci surface with the help of the Weierstrass data. Before beginning, we need some preparation.

**Definition 3.** Let $(M, g_M)$ be a complete surface, then the integral

$$K(M) := \int_M K \, dA$$

is called the total curvature of $M$, where $dA$ is the area element and $K$ is the Gauss curvature.

It is well-know that every two-dimensional orientable smooth Riemannian manifold can be equipped with a compatible complex structure so that it may also be regarded as a Riemann surface. We announce here an important theorem proved by Huber [3] and Osserman [10]:

**Theorem 6.** Let $(M, g_M)$ be a complete non-positively curved orientable Riemannian surface with finite total curvature. Then there exists a compact Riemann surface $S_k$ of genus $k$ and a finite number of points $p_1, \ldots, p_r$ on $S_k$ such that $M$ is biholomorphic to $S_k \setminus \{p_1, \ldots, p_r\}$.

Theorem 6 permits us to identify a complete non-positively curved Ricci surface which has finite total curvature with a Riemann surface $S_k \setminus \{p_1, \ldots, p_r\}$. It is natural for us to study the Ricci surface with some particular assumptions at the punctured points.

Examples 2 and 3 showed some complete non-positively curved Ricci surfaces which are biholomorphic to $\mathbb{C}\setminus\{0, \infty\}$. Their universal covers are isometric to that of a classical catenoid. In addition, the Hopf differential $Q$ of these examples is meromorphic near each punctured points. We have seen in the previous section that the Hopf differential $Q$ is independent of isometries of $\mathbb{R}^3$, and it is unique up to a
multiplication by an element $e^{i\theta} \in \mathbb{S}^1$. Hence we are inspired to study the order of $Q$ at punctured points. At the punctured point 0, it is not difficult to observe that $\text{ord}_0 Q = -2$. Similarly, at the punctured point $\infty$, we replace $z$ by $\frac{1}{w}$, then an easy calculation shows that $\text{ord}_\infty Q = -2$ still holds. Moreover, it is worth remarking that $\text{ord} Q = -2$ does not depend on the choice of conformal parameters. Thus the condition $\text{ord}_0 Q = -2$ is valuable for us.

Taking advantage of (3) and (4), we may observe that the modulus $|Q|$ of $Q$ coincides with the metric $\sqrt{-K_{g_M}}$ defined in Theorem 1. Therefore, around a punctured point of a Ricci surface $(M, g_M)$ corresponding to $z = 0$ where the condition $\text{ord}_0 Q = -2$ is verified, the induced flat metric $\sqrt{-K_{g_M}}$ can be locally written as

$$\sqrt{-K_{g_M}} = \left( \frac{a}{|z|^2} + o \left( \frac{1}{|z|^2} \right) \right) |dz|^2,$$

with $a \in \mathbb{R} \setminus \{0\}$. Actually, the converse is also true. To see this, let us assume that the induced metric $\sqrt{-K_{g_M}}$ of a non-positively curved Ricci surface has a local expression as in (13) near a punctured point $z = 0$. If we consider a closed curve $\gamma$ surrounding this point, then by Calabi’s rigidity theorem 4 and the definition of $Q$, we must have

$$\left( \tau^{-1} \right)^* Q = e^{2\pi i\theta} Q$$

for some $\theta \in [0, 1)$, where $\tau = [\gamma] \in \pi_1(M)$. It can be checked that $P := z^{-\theta} Q$ is a well-defined meromorphic quadratic differential near 0. Moreover, the modulus of $P$ takes the form

$$|P| = |z|^{-\theta} |Q| = \left( \frac{a}{|z|^{2+\theta}} + o \left( \frac{1}{|z|^{2+\theta}} \right) \right) |dz|^2,$$

it implies that 0 is a pole of $P$ and $\theta = 0$. Hence $Q$ is locally well-defined near 0 and satisfies $\text{ord}_0 Q = -2$. Consequently, the condition $\text{ord}_0 Q = -2$ is equivalent to saying that the induced flat metric $\sqrt{-K_{g_M}}$ has a local expression as in (13).

For these reasons, we may give the following definition.

**Definition 4.** A catenoidal end of a non-positively curved Ricci surface $M \simeq S_k \setminus \{p_1, p_2, \ldots, p_r\}$ with finite total curvature is a punctured point $p_i$ such that one of the following equivalent conditions holds:

(i). The induced flat metric $\sqrt{-K_{g_M}}$ has a cylindrical end at $p_i$, i.e. the metric $\sqrt{-K_{g_M}}$ has a local expression as

$$\sqrt{-K_{g_M}} = \left( \frac{a}{|z|^2} + o \left( \frac{1}{|z|^2} \right) \right) |dz|^2,$$

with $a \in \mathbb{R} \setminus \{0\}$.

(ii). The Hopf differential $Q$ is locally well-defined and has a pole of order 2 at $p_i$. 


Remark 2. Even though the ends in Examples 2 and 3 satisfy this definition, it should be noticed that the condition for catenoidal ends of a Ricci surface does not mean that the minimal immersion has a catenoidal end, nor that the corresponding Bryant surface has a catenoidal cousin end.

For a Ricci surface $M \simeq \mathbb{C} \setminus \{p_1, p_2, \ldots, p_r\}$ with $r$ catenoidal ends, the Hopf differential $Q$ is a well-defined holomorphic quadratic differential on $M$. In fact, the fundamental group $\pi_1(M)$ is generated by closed curves $\gamma_1, \ldots, \gamma_r$ surrounding each punctured point. By Definition 4, $Q$ is independent of all the generators of $\pi_1(M)$, thus well-defined on $M$. However, this property fails to hold for Ricci surfaces with positive genus in general. This is because there will be other generators of the fundamental group $\pi_1(M)$ than the closed curves mentioned above, hence $Q$ may not be well-defined on $M$ itself.

By using the condition of catenoidal ends, we can say even more about the Weierstrass data of the Ricci surface. The lemma below provides locally an explicit description of the catenoidal end of Ricci surfaces.

**Lemma 7.** For a non-positively curved Ricci surface $(M, g_M)$ with finite total curvature, it has a catenoidal end at the punctured point corresponding to $z = 0$ if and only if there exists an element $A \in SO(3, \mathbb{R})$ such that after the rotation, the Weierstrass data can be locally written as

$$
\begin{align*}
\tilde{\eta} &= \tilde{a} dz = \left( \frac{a}{z^{\lambda+1}} + o \left( \frac{1}{z^{\lambda+1}} \right) \right) dz, \\
\tilde{g} &= b z^\lambda + o(z^\lambda),
\end{align*}
$$

where $a, b \in \mathbb{C}\setminus\{0\}$ and $\lambda \in \mathbb{R}_+^*$.

**Proof.** $\Leftarrow$ If there is an element $A \in SO(3, \mathbb{R})$ such that after the rotation, the Weierstrass data $\tilde{\eta}$ and $\tilde{g}$ have form of (14), then it can be easily checked that $\text{ord } Q = -2$ is verified at that point.

$\Rightarrow$ Conversely, given $\text{ord } Q = -2$ at the punctured point $z = 0$, we are obliged to consider an element $\tau = [\gamma] \in \pi_1(M)$, where $\gamma$ is a closed curve around this punctured point. Since each rotation of $\mathbb{C}$ is conjugate to a rotation fixing 0 and $\infty$, in use of Lemma 5, we may suppose that the function $g$ satisfies

$$
g \circ \tau^{-1} = e^{2\pi i \theta} g,$$

with $\theta \in [0, 1)$. One may verify that $h(z) := z^{-\theta} g$ is a well-defined meromorphic function near $z = 0$. The condition that the total curvature of $M$ is finite tells us that $h$ has no essential singularity, thus meromorphic at 0 (see the proof of Proposition 4 of [7]). Therefore, the Gauss map $g$ can be locally written as $g = z^\mu \left( \sum_{j=0}^{\infty} b_j z^j \right)$ with $\mu \in \mathbb{R}$, $b_j \in \mathbb{C}$ and $b_0 \neq 0$. 
(1). Assume that $\mu > 0$, then it is immediate from the definition of $Q$ that
\[
\begin{align*}
g &= z^\mu \left( \sum_{j=0}^{\infty} b_j z^j \right), \\
\eta &= z^{-\mu - 1} \left( \sum_{j=0}^{\infty} a_j z^j \right) \, dz,
\end{align*}
\]
with $a_0, b_0 \in \mathbb{C} \setminus \{0\}$. This is the expression (14) and we can just take $A = I$.

(2). Now suppose that $\mu \leq 0$.

(i). If $\mu = 0$, then $g = b_0 + \sum_{j=1}^{\infty} b_j z^j$. The condition $\text{ord } Q = -2$ assures that $g$ is not a constant. Thus we may take a rotation $\tilde{g} = \frac{g-b_0}{b_0 g+1}$. Since in this case $\tilde{g}$ is meromorphic near $z = 0$ and $\tilde{g}(0) = 0$, it must have the form
\[
\tilde{g} = z^n \left( \sum_{j=0}^{\infty} c_j z^j \right)
\]
for some $n \in \mathbb{N}^*$, where $c_j \in \mathbb{C}$ and $c_0 \neq 0$. The isomorphism constructed in Lemma 5 implies immediately the existence of such an element $A \in SO(3, \mathbb{R})$. In addition, the corresponding $\tilde{\eta}$ can be proved to be
\[
\tilde{\eta} = z^{-n-1} \left( \sum_{j=0}^{\infty} a_j z^j \right) \, dz,
\]
with $a_j \in \mathbb{C}$ and $a_0 \neq 0$. This satisfies the expression (14).

(ii). In the case $\mu < 0$, we will apply the transformation $\tilde{g} = -\frac{1}{g}$. It is easy to see that the new Weierstrass data have the form as in (14). The corresponding $A \in SO(3, \mathbb{R})$ is also given by Lemma 5. This completes the proof.

With the help of this lemma, the Ricci metric can be locally written as
\[
G_M = \left( \frac{b}{|z|^{2\lambda+2}} + O \left( \frac{1}{|z|^2} \right) \right) |dz|^2,
\]
where $b \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{R}_+$. Moreover, taking (13) and (15) into consideration, we obtain immediately from Ricci’s theorem 1 a local expression for $(-K)_{G_M}$ which is
\[
(-K)_{G_M} = \left( c |z|^{2\lambda - 2} + o \left( |z|^{2\lambda - 2} \right) \right) |dz|^2,
\]
with $c \in \mathbb{R} \setminus \{0\}$. Since $\lambda > 0$, the metric $(-K)_{G_M}$ has a conical singularity of order $\lambda - 1 > -1$ at the catenoidal end $z = 0$.

It should be noticed that this lemma is just a local result. In order to do a global discussion, we need to study the monodromy representation introduced in the previous section. Inspired by Umehara and Yamada in their article [11], we will give the following definitions.
Definition 5. Let $d\sigma^2$ be a metric with constant constant Gauss curvature $1$ and finite number of conical singularities on $M$,

(i). We call it an irreducible metric if the image of the corresponding monodromy representation $\rho : \pi_1(M) \to PSU(2)$ is not abelian;
(ii). We call it a non-trivially reducible metric if the image of $\rho$ is abelian but not trivial;
(iii). It is called a trivially reducible metric if the image of $\rho$ is trivial.

For a non-positively curved Ricci surface $(M, g_M)$ with finite total curvature and more than one ends, if the induced metric $g_1 := (-K)g_M$ is reducible, then we can diagonalize all the monodromies simultaneously. Therefore, by applying Lemma 7 one by one to all the catenoidal ends, we may do some global analysis. However, this method does not work if the induced metric $g_1$ is irreducible.

5. $M \simeq \mathbb{C} \setminus \{0\}$ with two catenoidal ends

We are inspired by Huber’s theorem to study non-positively curved Ricci surfaces with catenoidal ends. It is immediate from the Riemann uniformization theorem that the universal cover of a such Ricci surface can only be the whole plane $\mathbb{C}$ or the upper half plane $\mathbb{H}$. In the $\mathbb{C}$ case, the corresponding Ricci surface $M$ is biholomorphic to either $\mathbb{C} \setminus \{0\}$ or $\mathbb{C}$. The following theorem affirms that the case when $M \simeq \mathbb{C}$ cannot appear.

Theorem 8. There does not exist non-positively curved Ricci surface $M \simeq \mathbb{C}$ with exactly one catenoidal end.

Proof. Let us identify $M$ with $\mathbb{C} \setminus \{0\}$, then the catenoidal end corresponds to 0. Applying Lemma 7, we may assume that the Gauss map $g$ has a pole at 0. Since $\pi_1(M)$ is trivial, $g$ can be written as $g = z^{-n}\varphi(z)$, where $n \in \mathbb{N}^*$ and $\varphi$ is a meromorphic function on $\mathbb{C}$ satisfying $\varphi(0) \neq 0$. The condition for catenoidal ends leads to the fact that $\text{ord } \eta = -2 - (-n - 1) = n - 1 \geq 0$ at 0. Therefore, $\eta$ is a holomorphic 1-form on $\mathbb{C}$ thus $\eta \equiv 0$. This implies that $Q \equiv 0$, which is not possible. \hfill $\square$

Thanks to Theorem 8, we may focus on giving a classification of non-positively curved Ricci surfaces $M \simeq \mathbb{C} \setminus \{0\}$ with two catenoidal ends.

Theorem 9. A complete non-positively curved Ricci surface $M \simeq \mathbb{C} \setminus \{0\}$ with two catenoidal ends can only be isometric to a catenoid, a catenoid cousin given as in Example 3 or the surface determined by the Weierstrass data

$$g = z^n + a, \quad \eta = bz^{-n-1}dz,$$

where $a, b \in \mathbb{R}_+^*$ and $n \in \mathbb{N}^*$. Moreover, all these surfaces are mutually non-isometric.
Proof. On the one hand, since $\text{Deck}(\tilde{M}/M) \simeq \pi_1(M) \simeq \mathbb{Z}$ is a cyclic group, the proof of Lemma 7 tells us that $h(z) := z^{-\theta} g$ is a meromorphic function on $\bar{C}$ for some $\theta \in [0, 1)$. Therefore, we may assume that the Gauss map is $g(z) = z^\mu f(z)$, where $f$ is a meromorphic function on $\bar{C}$ and $\mu > 0$. Hence it must have the form
\[
g(z) = z^\mu \frac{\varphi(z)}{\psi(z)}, \tag{18}\]
where $\varphi$ and $\psi$ are coprime polynomials such that $\psi(0) \neq 0$ and $\varphi(0) \neq 0$. From Lemma 7 and the fact that $\eta$ can only have a zero of order $2m$ where $g$ has a pole of order $m$, we obtain immediately
\[
\eta = \lambda z^{-\mu-1} \psi(z)^2 dz, \tag{19}\]
where $\lambda \in \mathbb{C}\{0\}$ is a constant number. On the other hand, as we have explained in the previous section, the Hopf differential $Q = dg \otimes \eta$ is meromorphic on $\bar{C}$ satisfying $\text{ord} \ Q = -2$ at 0 and $\infty$, thus it can be written as
\[
Q = \nu z^{-2} dz \otimes dz, \tag{20}\]
where $\nu \in \mathbb{C}\{0\}$ is a constant number.

Taking (18), (19) and (20) into consideration, we have
\[
Q = dg \otimes \eta \\
= \lambda z^{-2}[\mu \varphi \psi + z(\varphi' \psi - \varphi \psi')]dz \otimes dz \\
= \nu z^{-2} dz \otimes dz.
\]
Comparing these two expressions, we draw the following equation
\[
\mu \varphi \psi + z(\varphi' \psi - \varphi \psi') = \frac{\nu}{\lambda}. \tag{21}\]
Differentiating the two sides of (21), it becomes
\[
\psi[z \varphi'' + (1 + \mu) \varphi'] = \varphi[z \psi'' + (1 - \mu) \psi'].
\]
Since $\varphi$ and $\psi$ are coprime, there must be a polynomial $h(z)$ such that
\[
z \varphi'' + (1 + \mu) \varphi' = h(z) \varphi. \tag{22}\]
The degree of the left-hand side of (22) is at most $\deg \varphi - 1$, while the degree of the right-hand side should be at least $\deg \varphi$ unless $h(z) = 0$. This comparison tells us that $h(z) = 0$. Therefore, (22) is actually
\[
z \varphi'' + (1 + \mu) \varphi' = 0. \tag{23}\]
By solving differential Eq. (23), we get
\[
\varphi = a + bz^{-\mu}, \tag{24}\]
where $a, b \in \mathbb{C}$ are constant numbers.
Similarly, we can get
\[ \psi = cz^\mu + d, \]  
(25)
where \( c, d \in \mathbb{C} \) are constant numbers.

Since \( \mu > 0 \) and \( \varphi \) is a polynomial, we must have \( b = 0 \).

(i). If \( \mu \notin \mathbb{N} \), then \( c = 0 \) because \( \psi \) should also be a polynomial. Thus the Weierstrass data of the Ricci surface are
\[ g = \frac{a}{d} z^\mu, \quad \eta = \lambda d^2 z^{-\mu-1} dz. \]

With a change of variables \( u = \left(\frac{a}{d}\right)^{\frac{1}{\mu}} z \), these expressions can be simplified to be
\[ g = u^\mu, \quad \eta = \lambda ad u^{-\mu-1} du. \]
Moreover, by applying an \( S^1 \)-action if necessary, we may suppose \( \lambda ad \) to be a real number \( \rho > 0 \), thus the Weierstrass data of the Ricci surface can be written as
\[ g = u^\mu, \quad \eta = \rho u^{-\mu-1} du \]  
(26)
with \( \rho \in \mathbb{R}_{+} \).

(ii). If \( \mu \in \mathbb{N} \) and \( c = 0 \), then the Weierstrass data have exactly the same form as in (26). In these two cases, the Ricci surface \( M \) is isometric to a catenoid if \( \mu = 1 \), to a catenoid cousin if \( \mu \neq 1 \).

(iii). If \( \mu \in \mathbb{N} \) and \( c \neq 0 \), the Weierstrass data will have the form
\[ g = \frac{az^\mu}{cz^\mu + d}, \quad \eta = \lambda (cz^\mu + d)^2 z^{-\mu-1} dz. \]  
(27)
Denoting
\[ r = \frac{\tilde{a}}{d} \left( \frac{|c|^2}{|a|^2} + 1 \right), \quad s = \frac{\tilde{c}}{a}, \quad t = \frac{\lambda a^2 d^2}{|a|^2 + |c|^2}, \]
then a simple computation shows that after a rotation, the Weierstrass data of the Ricci surface become
\[ \tilde{g} = rz^\mu + s, \quad \tilde{\eta} = tz^{-\mu-1} dz, \]  
(28)
where \( r, s, t \in \mathbb{C} \setminus \{0\} \). With the help of a change of variables \( w = r^{\frac{1}{\mu}} z \), (28) is simplified as
\[ \tilde{g} = w^\mu + s, \quad \tilde{\eta} = tw^{-\mu-1} dw. \]  
(29)
Applying an \( S^1 \)-action if necessary, we may assume \( tr \) to be a positive real number \( \rho \), hence the Weierstrass data can be written as
\[ \tilde{g} = w^\mu + s, \quad \tilde{\eta} = \rho w^{-\mu-1} dw, \]  
(30)
where \( s \in \mathbb{C} \setminus \{0\} \) and \( \rho \in \mathbb{R}_{+} \). Moreover, if we denote \( s = \xi e^{i\theta} \) with \( \xi \in \mathbb{R}_{+} \) and \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \), then it will be immediate by a rotation and a change of variables that the formula (17) is obtained.
From Ricci’s theorem 1, we know that a Ricci metric $g_M$ induces naturally a metric $(-K)g_M$ which is of constant Gauss curvature 1, possibly with conical singularities. Therefore, we are able to compute the total curvature with the help of Gauss-Bonnet formula with conical singularities

$$\frac{1}{2\pi} \int_M (-K) dA = \chi(\bar{M}) + \sum_{i=1}^n \beta_i,$$

where $\beta_i > -1$ are the orders of conical singularities (see [4]). Taking advantage of the two expressions (26) and (17), a direct computation shows that the metric $(-K)g_M$ has two conical singularities of the same order at 0 and $\infty$, which is $\mu - 1$ for (26) and $n - 1$ for (17). Hence the total curvature is $-4\pi \mu$ for (26) and $-4\pi n$ for (17). Since the total curvature is intrinsic, any two of such Ricci surfaces with different total curvatures cannot be isometric.

Now we are going to prove that two Ricci surfaces with the same total curvature $-4\pi n$ but different pair $(a, b)$ in (17) could not be isometric. Assuming that there is an isometry $\tau : M_1 \rightarrow M_2$, where $M_1, M_2 \simeq \mathbb{C}\{0\}$ are endowed with metrics defined as in (17) with $(a_1, b_1) \neq (a_2, b_2)$. Then $\tau$ can be lifted to a biholomorphic map $\tilde{\tau} : \mathbb{C} \rightarrow \mathbb{C}$ through the universal covering map $\exp : \mathbb{C} \rightarrow \mathbb{C}\{0\}$, hence $\tilde{\tau}$ is actually an affine function. More precisely, there exists a pair $(\alpha, \beta) \in \mathbb{C}^2$ such that

$$\begin{cases}
g = e^{n(\alpha u + \beta)} + a_1 = e^{nu} + a_2, \\
\eta = b_1 \alpha e^{-n(\alpha u + \beta)} du = b_2 e^{-nu} du.
\end{cases}$$

By comparing both sides of (32), we get $\alpha = 1, a_1 = a_2, e^{n\beta} = 1$ and $b_1 = b_2$, contradiction. This completes the proof of the theorem. \qed

The previous proof is inspired by Umehara and Yamada (see [8]). We give here another proof, which provides a sight of the method that we will use for our further discussion.

**Proof.** For the same reason, we suppose that the Weierstrass data have the forms as in (18) and (19).

(1). If $\psi$ is holomorphic at $\infty$, then it should be a constant function, thus $\eta = a z^{-\mu - 1} dz$ with $a \in \mathbb{C}$. This means that $f$ does not have poles on $\mathbb{C}\{0\}$. In fact, if $f$ has a pole on $\mathbb{C}\{0\}$, then $\eta$ should have zero at this point, thus $a = 0$ and $\eta \equiv 0$. This leads to the result $Q \equiv 0$, which is a contradiction to the condition of a catenoidal end. Therefore, $f$ is holomorphic on $\mathbb{C}$.

(i). If $f$ is also holomorphic at $\infty$, then $f$ must be a constant function, hence $g = b z^\mu$ with $b \in \mathbb{C}$. In this case, we get a catenoid or a catenoid cousin.

(ii). If $f$ has poles at $\infty$, it should be a polynomial

$$f(z) = v(z - a_1) \ldots (z - a_p)$$

(33)
with \( v, a_i \in \mathbb{C} \setminus \{0\} \) and \( p \in \mathbb{N}^* \). Let us make a substitution \( z = \frac{1}{w} \), then we get

\[
\begin{aligned}
g &= vw^{-\mu - p}(1 - a_1w)\ldots(1 - a_pw), \\
\eta &= -aw_{\mu - 1}.
\end{aligned}
\]

The condition of catenoidal end at \( \infty \) becomes

\[
\text{ord } Q = -p - 2 = -2,
\]

which results in \( p = 0 \), contradiction.

(2). If \( \psi \) has poles at \( \infty \), it must have the form

\[
\psi(z) = (z - b_1)\ldots(z - b_q)
\]

with \( b_i \in \mathbb{C} \setminus \{0\} \) and \( q \in \mathbb{N}^* \). As a consequence, \( f \) has the expression

\[
f(z) = \frac{\xi(z - c_1)\ldots(z - c_r)}{(z - b_1)\ldots(z - b_q)},
\]

where \( r \in \mathbb{N} \) and \( \xi, c_j \in \mathbb{C} \setminus \{0\} \) satisfying \( b_i \neq c_j \) for all \((i, j)\). Replacing \( z \) by \( \frac{1}{w} \), we obtain

\[
\begin{aligned}
g &= \xi w^{-\mu - r + q}(1 - c_1w)\ldots(1 - c_rw) \\
\eta &= -\lambda w^{\mu - 2q - 1}(1 - b_1w)^2\ldots(1 - b_qw)^2.
\end{aligned}
\]

(i). If \(-\mu - r + q \neq 0 \), then the condition of catenoidal end at \( \infty \) implies

\[-r - q - 2 = -2, \]

thus \( r + q = 0 \), which is not possible.

(ii). The only case left is \( \mu = q - r \in \mathbb{N}^* \). Applying \( \tilde{g} = \frac{g - \xi}{\xi g + 1} \), we get the following formula

\[
\tilde{g} = \xi \frac{(1 - c_1w)\ldots(1 - c_rw) - (1 - b_1w)\ldots(1 - b_qw)}{\xi \xi(1 - c_1w)\ldots(1 - c_rw) + (1 - b_1w)\ldots(1 - b_qw)}.
\]

On the one side, it can be seen that \( \text{ord } \tilde{g} \leq q \) because \( q > r \). On the other side, since \( \text{ord } \tilde{\eta} = \text{ord } \eta = \mu - 2q - 1 = -q - r - 1 \), we can compute

\[
\text{ord } d\tilde{g} = -2 - \text{ord } \tilde{\eta} = q + r - 1 \geq 0,
\]

hence \( \text{ord } \tilde{g} = q + r \). We deduce from this comparison that \( r = 0 \). Combining (36) with the fact that \( \text{ord } \tilde{g} = q \), we may observe that

\[
\begin{aligned}
I_1 &= \sum_{1 \leq i \leq n} c_i = 0, \\
I_2 &= \sum_{1 \leq i < j \leq n} c_i c_j = 0, \\
&\quad \ldots \\
I_{n-1} &= \sum_{1 \leq i_1 < i_2 < \ldots < i_{n-1} \leq n} c_{i_1} \ldots c_{i_{n-1}} = 0, \\
I_n &= c_1 c_2 \ldots c_n \neq 0.
\end{aligned}
\]
Consequently, the Weierstrass data take the same form as in (27), hence the conclusion can be drawn by utilizing the same argument as in the first proof.

From the proofs of this theorem, we may easily obtain the following conclusion.

**Corollary 10.** A complete non-positively curved Ricci surface $M \simeq \mathbb{C}\setminus\{0\}$ with two catenoidal ends whose total curvature is $-4\pi \mu$ with $\mu \in \mathbb{R}^*_+ \setminus \mathbb{N}$ must be isometric to a catenoid cousin.

**Remark 3.** Umehara and Yamada have proven in [8] a similar result for complete CMC-1 surfaces in the hyperbolic 3-space $\mathcal{H}^3$. Then by the Lawson correspondence, these surfaces correspond to Ricci surfaces. It is worthwhile to mention that Theorem 9 gives actually more surfaces than theirs. In fact, the main difference between our theorem and the result of Umehara and Yamada is the catenoid case and that they have for the case (17) a supplementary condition

$$n^2 + 4bn = m^2 \quad (37)$$

with $m \in \mathbb{N}^*$. Condition (37) implies that $b = \frac{m^2 - n^2}{4n} \in \mathbb{Q}$, but we just require $b$ to be a real number. Then the previous argument says that we have indeed more surfaces. Moreover, up to applying a homothety, our result implies that every complete non-positively curved Ricci surface $M \simeq \mathbb{C}\setminus\{0\}$ with two catenoidal ends admits an isometric CMC-1 immersion in $\mathcal{H}^3$.

**Remark 4.** Troyanov has classified all the metrics of constant Gauss curvature 1 on $\bar{\mathbb{C}}$ with two conical singularities at 0 and $\infty$ (see [4]). In fact, he affirms that if $d\sigma^2$ is a such metric, then there exists $\lambda \geq 0$ and $\beta > -1$, such that $d\sigma^2$ has the expression

$$d\sigma^2 = (2 + 2\beta)^2 \frac{|z|^{2\beta}|dz|^2}{(1 + \lambda z^{\beta+1})^2 + |z|^{2\beta+2})^2}, \quad (38)$$

where $\beta$ is either an integer, or $\lambda = 0$. In Theorem 9, the condition for catenoidal ends at 0 and $\infty$ implies that we should have a flat metric $|Q| = \nu \frac{|dz|^2}{|z|^2}$ with $\nu \in \mathbb{R}^*_+$. Let us fix this flat metric, then each metric $d\sigma^2$ in Troyanov’s theorem gives rise to a desired Ricci surface for us (see [2], Lemma 2.3). Conversely, for every Ricci surface in Theorem 9, since the induced flat metric $\sqrt{-K} g_M$ coincides with $|Q|$ which does not vanish on $\mathbb{C}\setminus\{0\}$, its Gauss curvature $K$ has no zeroes on $\mathbb{C}\setminus\{0\}$. Hence the metric $(-K) g_M$ is of constant Gauss curvature 1 with two conical singularities at 0 and $\infty$. It is a direct verification that $(-K) g_M$ has the same expression as in (38). Therefore, our result and Troyanov’s theorem are equivalent.
6. $M \simeq \mathbb{C} \setminus \{0, 1\}$ with three catenoidal ends and reducible induced metric $(-K)g_M$

After giving a complete classification for non-positively curved Ricci surface $M \simeq \mathbb{C} \setminus \{0\}$ with two catenoidal ends, we are naturally interested in studying the case when the Ricci surface is topologically $M \simeq \mathbb{C} \setminus \{0, 1\}$ with three catenoidal ends at $0, 1, \infty$. However, since $\pi_1(M) \simeq \mathbb{Z} \ast \mathbb{Z}$ is not abelian, we have to pay attention to the monodromy representation $\rho : \pi_1(M) \to \text{PSU}(2)$, as mentioned in the end of Sect. 4.

We will only consider the case when the induced metric $(-K)g_M$ is reducible. Let us denote $\tau_1, \tau_2$ the two generators of $\pi_1(M) \simeq \mathbb{Z} \ast \mathbb{Z}$ which are represented by two closed curves $\gamma_1, \gamma_2$ surrounding $0$ and $1$, respectively. Since the image of $\rho$ is abelian, we may choose a proper $g$ such that $g \circ \tau_j^{-1} = e^{2\pi i \theta_j} g$, $j = 1, 2$, where $\theta_j \in [0, 1)$. Imitating the proof of Lemma 7, one can verify that the function $h(z) := (z-1)^{-\theta_2}z^{-\theta_1}g$ is meromorphic on $\bar{\mathbb{C}}$, thus a rational function. Therefore, $g$ should be written as

$$g = (z-1)^{\theta_2 + p}z^{\theta_1 + q} \frac{\varphi(z)}{\psi(z)},$$

where $p, q \in \mathbb{Z}$ and $\varphi, \psi$ are coprime unitary polynomials without zeroes at $0, 1$. By interchanging the roles of $0, 1$ and applying a rotation if necessary, we may divide the Weierstrass data into two possible situations:

Case 1:

$$\begin{cases}
g = \lambda (z-1)^{\mu}z^{\nu} \frac{\varphi(z)}{\psi(z)}, \\
\eta = \kappa (z-1)^{-\mu-1}z^{-\nu-1}\psi^2(z)dz,
\end{cases}$$

where $\lambda, \kappa \in \mathbb{C} \setminus \{0\}$, $\mu > 0$, $\nu \in \mathbb{R}^*$ and $\varphi, \psi$ are coprime unitary polynomials without zeroes at $0, 1$.

Case 2:

$$\begin{cases}
g = \lambda (z-1)^{\mu} \frac{\varphi(z)}{\psi(z)}, \\
\eta = \kappa (z-1)^{-\mu-1}z^{-\nu-1}\psi^2(z)dz,
\end{cases}$$

where $\lambda, \kappa \in \mathbb{C} \setminus \{0\}$, $\mu, \nu \in \mathbb{N}^*$, $\varphi, \psi$ are coprime unitary polynomials without zeroes at $0, 1$ satisfying $\deg \psi - \deg \varphi = \mu$.

In fact, if $\theta_2 + p \neq 0$, then up to replacing $g$ by $\frac{1}{g}$, we will obtain (39) or (40). In the case when $\theta_2 + p = 0$ but $\theta_1 + q \neq 0$, we may apply a Möbius transformation $\sigma(z) = 1 - z$ to interchange the role of $0$ and $1$, then we are back to the first case. For the last case, if both $\theta_2 + p$ and $\theta_1 + q$ are zero, then we may use a rotation to get a new $g$ which has zeros at $1$. Since such a rotation does not change the orders of $dg$ and $\eta$ at $1$, we will find (39) or (40) again. Additionally, we are going to give an explanation to the relation $\deg \psi - \deg \varphi = \mu$ appeared in Case 2. Actually, if
$\mu + \deg \varphi \neq \deg \psi$, then by applying a change of variables $z = \frac{1}{w}$, we can get the expression as in Case 1.

For the convenience of our further discussion, we give the following definition.

**Definition 6.** A complete non-positively curved Ricci surface $M \simeq \mathbb{C} \setminus \{0, 1\}$ with three catenoidal ends and a reducible induced metric $(-K)g_M$ is called generic if $\theta_1, \theta_2 \in (0, 1)$ and $\theta_1 + \theta_2 \neq 1$; all the other cases are said to be non-generic.

Therefore, a Ricci surface $M \simeq \mathbb{C} \setminus \{0, 1\}$ being generic is equivalent to saying that $\mu, \nu \notin \mathbb{Z}$ and $\mu + \nu \notin \mathbb{Z}$. It is not difficult to see that the generic case can only appear in Case 1.

### 6.1. Classification in the generic case

Let us first consider the generic case. In this situation, we can give a complete classification under orientation-preserving isometries for generic Ricci surface $M \simeq \mathbb{C} \setminus \{0, 1\}$ with three catenoidal ends and a reducible induced metric $(-K)g_M$.

**Theorem 11.** Let $M \simeq \mathbb{C} \setminus \{0, 1\}$ be a generic complete non-positively curved Ricci surface with three catenoidal ends and a reducible induced metric $(-K)g_M$, then its Weierstrass data must be of the form

\[
\begin{align*}
\g &= \lambda(z - 1)^\mu z^n(z - a), \\
\eta &= \kappa(z - 1)^{-\mu - 1}z^{n-1}dz,
\end{align*}
\]

where $\lambda, \kappa \in \mathbb{R}^+$, $\mu, \nu, \mu + \nu \in \mathbb{R} \setminus \mathbb{Z}$ satisfying $\nu \geq \max\{\mu, -2\mu - 1\}$, the parameter $a$ satisfies:

(i). If $\nu > \max\{\mu, -2\mu - 1\}$, then $a \in \mathbb{C} \setminus \{0, 1\}$;

(ii). If $\nu = \mu$ but $\nu \neq 2\mu - 1$, then $a \in (\mathbb{C} \setminus \{0, 1\}) \setminus \{id, z \mapsto 1 - z\}$;

(iii). If $\nu = -2\mu - 1$ but $\nu \neq \mu$, then $a \in (\mathbb{C} \setminus \{0, 1\}) \setminus \{id, z \mapsto z^{-1}\}$;

(iv). If $\nu = \mu = -\frac{1}{3}$, then $a \in (\mathbb{C} \setminus \{0, 1\}) \setminus \text{Aut}(\mathbb{C} \setminus \{0, 1\})$.

Moreover, there is no orientation-preserving isometry between the Ricci surfaces with different $(\mu, \nu, a, \lambda, \kappa)$.

**Proof.** Taking advantage of the Weierstrass data as in (39), we may get the following expression for the Hopf differential:

\[
Q = \frac{\lambda\kappa P(z)}{z^2(z - 1)^2}dz \otimes dz,
\]

where

\[
P(z) = [(\mu + \nu)z - \nu]\varphi\psi + (z^2 - z)(\varphi'\psi - \varphi\psi')
\]

is a polynomial in $z$. We know from Sect. 4 that the Hopf differential $Q$ is holomorphic on $M$ with three poles of order 2 at 0, 1 and $\infty$, thus $P(z)$ must be of degree 2 satisfying $P(0) \neq 0$ and $P(1) \neq 0$. Denote $\deg \psi = N$ and $\deg \varphi = m$, then the
highest order term of \( P(z) \) is \((\mu + \nu + m - N)z^{m+N+1}\). Since \(\mu + \nu \notin \mathbb{Z}\), we must have \(m + N + 1 = 2\), hence \((m, N) = (0, 1)\) or \((m, N) = (1, 0)\). Therefore, the only possible Weierstrass data are either given by

\[
\begin{aligned}
g &= \lambda(z - 1)^{\mu}z^{\nu}(z - a), \\
\eta &= \kappa(z - 1)^{-\mu - 1}z^{\nu - 1}dz,
\end{aligned}
\]  

or by

\[
\begin{aligned}
g &= \lambda(z - 1)^{\mu}z^{\nu - \frac{1}{z - b}}, \\
\eta &= \kappa(z - 1)^{-\mu - 1}z^{\nu - 1}(z - b)dz,
\end{aligned}
\]  

where \(\lambda, \kappa \in \mathbb{C} \setminus \{0\}\), \(a, b \in \mathbb{C} \setminus \{0, 1\}\), \(\mu > 0\) and \(\nu \in \mathbb{R}^*\).

However, for the expression (45), we may apply the transformation \(\tilde{g} = \frac{1}{g}\) to get the new Weierstrass data

\[
\begin{aligned}
g &= \frac{1}{\lambda}(z - 1)^{-\mu}z^{\nu}(z - b), \\
\eta &= -\lambda^2\kappa(z - 1)^{-\mu - 1}z^{\nu - 1}dz,
\end{aligned}
\]  

It can be easily observed that (46) has the same expression as (44), up to replacing \((\mu, \nu)\) by \((-\mu, -\nu)\). Thus we can unify expressions (44) and (46) to get the expression (41). From the formulae (3) and (4), we can see that only the modulus of \(\lambda, \kappa\) matters. Hence we may suppose that \(\lambda, \kappa \in \mathbb{R}_+^*\). The rest of the proof will be completed in the end of Sect. 6.4.

\[\square\]

Remark 5. This result is very similar to the case of \(M \simeq \mathbb{C} \setminus \{0\}\) with two catenoidal ends. In Theorem 9, when \(\mu \notin \mathbb{Z}\), there is only one possible expression for the Weierstrass data.

6.2. Results in non-generic cases

In this subsection, we are going to study non-generic cases. We begin with such cases occur in Case 1, then we pass the discussion to those of Case 2. Throughout the rest of this section, we always denote \(\deg \varphi = m\) and \(\deg \psi = N\).

Let us suppose that the Weierstrass data take the form as in (39). Whenever \(\mu + \nu \notin \mathbb{Z}\) or \(\mu + \nu \in \mathbb{Z}\) but \(\mu + \nu + m \neq N\), we may apply the same argument as in Sect. 6.1 to obtain the same expression of (41). So the only situation that need to be consider is when \(\mu + \nu + m = N\). In this context, we must have \(m + N \geq 2\). The fact that \(P(z)\) is of degree 2 gives us \((m + N - 1)\) equations. In general, these equations are quadratic equations in the coefficients of \(\varphi\) and \(\psi\). We may take the coefficients of one of \(\varphi\) and \(\psi\) as parameters, then we solve the system of linear equations with the help of these parameters. In order to better explain this method, we will give a complete classification for \(2 \leq m + N \leq 3\).

1. Suppose that \(m + N = 2\), then \((m, N) = (0, 2), (1, 1)\) or \((2, 0)\).
(i). If \((m, N) = (0, 2)\), we will have \(\mu + \nu = 2\). Let us write \(\varphi = 1\) and \(\psi = (z-b_1)(z-b_2)\) with \(b_1, b_2 \in \mathbb{C} \setminus \{0, 1\}\). An easy computation shows that

\[
P(z) = (\mu - b_1 - b_2)z^2 + [2b_1b_2 + (1 - \mu)(b_1 + b_2)]z + (\mu - 2)b_1b_2.
\]

Hence in this case, the Ricci surface is given by the Weierstrass data

\[
\begin{align*}
g &= \lambda(z - 1)^{\mu - 1}\frac{1}{(z - b_1)(z - b_2)}, \\
\eta &= \kappa(z - 1)^{-\mu - 2}(z - b_1)^2(z - b_2)^2dz,
\end{align*}
\]

where \(\mu \in \mathbb{R}_+^* \setminus \{2\}, b_1, b_2 \in \mathbb{C} \setminus \{0, 1\}\) satisfying \(\mu - b_1 - b_2 \neq 0\).

(ii). If \((m, N) = (1, 1)\), then \(\mu + \nu = 0\). Denote \(\varphi = z - a\) and \(\psi = z - b\) with \(a, b\) two distinct complex numbers in \(\mathbb{C} \setminus \{0, 1\}\), then the Ricci surface is given by

\[
\begin{align*}
g &= \lambda(z - 1)^{\mu - 1}\frac{z - a}{z - b}, \\
\eta &= \kappa(z - 1)^{-\mu - 1}(z - b)^2dz,
\end{align*}
\]

with \(\mu \in \mathbb{R}_+^*\), \(a, b \in \mathbb{C} \setminus \{0, 1\}\) satisfying \(\mu + a - b \neq 0\).

(iii). If \((m, N) = (2, 0)\), then \(\mu + \nu = 2\). Similarly, we denote \(\varphi = (z - a_1)(z - a_2)\) and \(\psi = 1\), with \(a_1, a_2 \in \mathbb{C} \setminus \{0, 1\}\). The Weierstrass data is determined to be

\[
\begin{align*}
g &= \lambda(z - 1)^{\mu - 1}(z - a_1)(z - a_2), \\
\eta &= \kappa(z - 1)^{-\mu - 1}dz,
\end{align*}
\]

where \(\mu \in \mathbb{R}_+^*\), \(a_1, a_2 \in \mathbb{C} \setminus \{0, 1\}\) satisfying \(\mu + a_1 + a_2 \neq 0\).

(2). Now we study the case when \(m + N = 3\). In this case, we may have \((m, N) = (0, 3), (1, 2), (2, 1)\) or \((3, 0)\).

(i). If \((m, N) = (0, 3)\), then \(\mu + \nu = 3\). Let us write \(\varphi = 1\) and \(\psi = (z - b_1)(z - b_2)(z - b_3)\) with \(b_1, b_2, b_3 \in \mathbb{C} \setminus \{0, 1\}\). Then \(P(z)\) can be computed to be

\[
P(z) = (\mu + b_1 - b_2 - b_3)z^3 + [b_1(1 - \mu + 2b_2) + b_2(1 - \mu + 2b_3) \\
+ b_3(1 - \mu + 2b_1)]z^2 + [(\mu - 2)(b_1b_2 + b_1b_3 + b_2b_3) \\
- 3b_1b_2b_3]z + (3 - \mu)b_1b_2b_3.
\]

Therefore, the Ricci surface is given by the Weierstrass data

\[
\begin{align*}
g &= \lambda(z - 1)^{\mu - 1}\frac{1}{(z - b_1)(z - b_2)(z - b_3)}, \\
\eta &= \kappa(z - 1)^{-\mu - 4}(z - b_1)^2(z - b_2)^2(z - b_3)^2dz,
\end{align*}
\]

where \(\mu \in \mathbb{R}_+^* \setminus \{3\}, b_1, b_2, b_3 \in \mathbb{C} \setminus \{0, 1\}\) satisfying \(\mu + b_1 - b_2 - b_3 = 0\) and \(b_1(1 - \mu + 2b_2) + b_2(1 - \mu + 2b_3) + b_3(1 - \mu + 2b_1) \neq 0\).
(ii). If \((m, N) = (1, 2)\), then we will have \(\mu + \nu = 1\). Denote \(\varphi = z - a\) and \(\psi = (z - b_1)(z - b_2)\) with \(a, b_1, b_2 \in \mathbb{C} \setminus \{0, 1\}\).

In a similar way, the Weierstrass data of the Ricci surface can be found to be

\[
\begin{align*}
g &= \lambda(z - 1)^\mu z^{1-\mu} \frac{z - a}{(z - b_1)(z - b_2)}, \\
\eta &= \kappa(z - 1)^{-\mu - 1} z^{\mu - 2} (z - b_1)^2 (z - b_2)^2 dz,
\end{align*}
\]

where \(\mu \in \mathbb{R}_+^* \setminus \{1\}, a, b_1, b_2 \in \mathbb{C} \setminus \{0, 1\}\) satisfying \(\mu + a - b_1 - b_2 = 0\) and \((1 - \mu)(a + b_1 + b_2) + 2b_1b_2 - 2a \neq 0\).

(iii). If \((m, N) = (2, 1)\), then \(\mu + \nu = -1\). We set \(\varphi = (z - a_1)(z - a_2)\) and \(\psi = z - b\) with \(a_1, a_2, b \in \mathbb{C} \setminus \{0, 1\}\). A direct computation shows that the Ricci surface is given by

\[
\begin{align*}
g &= \lambda(z - 1)^\mu z^{-1-\mu} (z - a_1)(z - a_2) \\
\eta &= \kappa(z - 1)^{-\mu - 1} z^\mu (z - b)^2 dz,
\end{align*}
\]

where \(\mu \in \mathbb{R}_+^*, a_1, a_2, b \in \mathbb{C} \setminus \{0, 1\}\) satisfying \(\mu + a_1 + a_2 - b = 0\) and \((\mu + 1)(a_1 + a_2 + b) + 2a_1a_2 - 2b \neq 0\).

(iv). If \((m, N) = (3, 0)\), then \(\mu + \nu = -3\). We write \(\varphi = (z - a_1)(z - a_2)(z - a_3)\) and \(\psi = 1\) with \(a_1, a_2, a_3 \in \mathbb{C} \setminus \{0, 1\}\). Thus the Weierstrass data of the Ricci surface have the expression

\[
\begin{align*}
g &= \lambda(z - 1)^\mu z^{-3-\mu} (z - a_1)(z - a_2)(z - a_3), \\
\eta &= \kappa(z - 1)^{-\mu - 1} z^{\mu + 2} dz,
\end{align*}
\]

where \(\mu \in \mathbb{R}_+^*, a_1, a_2, a_3 \in \mathbb{C} \setminus \{0, 1\}\) satisfying \(\mu + a_1 + a_2 + a_3 = 0\) and \((\mu + 1)(a_1 + a_2 + a_3) + 2a_1a_2 + 2a_1a_3 + 2a_2a_3 \neq 0\).

We summarize the previous discussion as the next proposition.

**Proposition 12.** Let \(M \simeq \mathbb{C} \setminus \{0, 1\}\) be a non-generic complete Ricci surface with three catenoidal ends and a reducible induced metric \((-K)g_M\), and whose Weierstrass data have the form as in (39). Then:

1. If \(\mu, \nu \in \mathbb{R}^*\) such that \(\mu + \nu + \deg \varphi \neq \deg \psi\), then \(M\) must be isometric to the surface determined by the Weierstrass data (41).
2. If \(\mu > 0, \nu \in \mathbb{R}^*\) satisfy \(\mu + \nu + \deg \varphi = \deg \psi\) and \(\deg \varphi + \deg \psi = 2\), then the Ricci surface \(M\) should be isometric to one of (47), (48) and (49).
3. If \(\mu > 0, \nu \in \mathbb{R}^*\) satisfy \(\mu + \nu + \deg \varphi = \deg \psi\) and \(\deg \varphi + \deg \psi = 3\), then \(M\) must be isometric to one of (50), (51), (52) and (53).

Now we will study Case 2. The method that we adopt here is similar to the one of Case 1. Taking advantage of expression (40), we can determine the Hopf differential \(Q\) to be

\[
Q = \frac{\lambda \kappa R(z)}{z^2(z - 1)^2} dz \otimes dz,
\]
where
\[ R(z) = z^{-v+1} \left[ \mu \varphi \psi + (z - 1)(\varphi' \psi - \varphi \psi') \right] \] (55)
is a polynomial in \( z \) of degree 2 satisfying \( R(0) \neq 0 \) and \( R(1) \neq 0 \). The highest order term of \( R(z) \) is \((\mu + m - N)z^{m+N-v+1}\). However, the condition \( \mu + m = N \) tells us that this term vanishes, so we must have \( m + N - v + 1 \geq 3 \), thus \( m + N \geq v + 2 \geq 3 \). Moreover, the fact that \( R(z) \) has degree 2 provides us with \((m + N - v - 1)\) equations, including \( \mu + m = N \). As an example, we will discuss the situation where \( m + N = 3 \). It is easy to observe that \( v = 1 \) and the only possibilities for \((m, N)\) are \((0, 3)\) and \((1, 2)\).

(i). If \((m, N) = (0, 3)\), then \( \mu = 3 \). By setting \( \varphi = 1 \) and \( \psi = (z - b_1)(z - b_2)(z - b_3) \) with \( b_1, b_2, b_3 \in \mathbb{C} \setminus \{0, 1\} \), we may find that
\[
R(z) = (3 - b_1 - b_2 - b_3) z^2 + 2 (b_1 b_2 + b_1 b_3 + b_2 b_3 - b_1 - b_2 - b_3) z + (b_1 b_2 + b_1 b_3 + b_2 b_3 - 3 b_1 b_2 b_3).
\]
The condition that \( R(z) \) has degree 2 implies \( 3 - b_1 - b_2 - b_3 \neq 0 \). In addition, we should also have
\[ R(0) = b_1 b_2 + b_1 b_3 + b_2 b_3 - 3 b_1 b_2 b_3 \neq 0. \]
Therefore, the Ricci surface is given by the Weierstrass data
\[
\begin{cases}
g = \lambda(z - 1)^3 \frac{1}{(z - b_1)(z - b_2)(z - b_3)}, \\
\eta = \kappa(z - 1)^{-4} z^{-2} (z - b_1)^2 (z - b_2)^2 (z - b_3)^2 \, dz,
\end{cases}
\] (56)
where \( b_1, b_2, b_3 \in \mathbb{C} \setminus \{0, 1\} \) should satisfy \( 3 - b_1 - b_2 - b_3 \neq 0 \) and \( b_1 b_2 + b_1 b_3 + b_2 b_3 - 3 b_1 b_2 b_3 \neq 0 \).

(ii). If \((m, N) = (1, 2)\), then \( \mu = 1 \). Denote \( \varphi = z - a \) and \( \psi = (z - b_1)(z - b_2) \) with \( a, b_1, b_2 \in \mathbb{C} \setminus \{0, 1\} \). A straightforward computation shows
\[
R(z) = (1 + a - b_1 - b_2) z^2 + 2 (b_1 b_2 - a) z + (ab_1 + ab_2 - b_1 b_2 - ab_1 b_2).
\]
Similarly, we need to require \( 1 + a - b_1 - b_2 \neq 0 \) and
\[ R(0) = ab_1 + ab_2 - b_1 b_2 - ab_1 b_2 \neq 0. \]
Hence the Weierstrass data are
\[
\begin{cases}
g = \lambda(z - 1)^3 \frac{z - a}{(z - b_1)(z - b_2)}, \\
\eta = \kappa(z - 1)^{-2} z^{-2} (z - b_1)^2 (z - b_2)^2 \, dz,
\end{cases}
\] (57)
where \( a, b_1, b_2 \in \mathbb{C} \setminus \{0, 1\} \) satisfying \( 1 + a - b_1 - b_2 \neq 0 \) and \( ab_1 + ab_2 - b_1 b_2 - ab_1 b_2 \neq 0 \).

We conclude these cases as the following proposition.

**Proposition 13.** Suppose that \( \mu > 0 \) and \( v \in \mathbb{N}^* \). Let \( M \simeq \mathbb{C} \setminus \{0, 1\} \) be a complete non-positively curved Ricci surface with three catenoidal ends and a reducible induced metric \((-K)g_M\). If its Weierstrass data take the form as in (40) with \( \deg \varphi + \deg \psi = 3 \), then \( M \) is isometric to either (56) or (57).
6.3. Method with prescribed zeros of Hopf differential

The previous method is based on the study of Hopf differential $Q$ in terms of coefficients of $\varphi$ and $\psi$. Now we are interested in seeing this problem from another viewpoint. We plan to start from the prescribed zeros of the Hopf differential $Q$, then we try to understand the relations between the zeros of $Q$ and the roots of $\varphi$ and $\psi$. As we have known before, the Hopf differential $Q = d\varphi \otimes \kappa$ is holomorphic on $M \simeq \mathbb{C} \setminus \{0, 1\}$ with three poles of order 2 at 0, 1 and $\infty$, thus it can be written as

$$Q = \alpha \frac{(z - a_1)(z - a_2)}{z^{2}(z - 1)^{2}}dz \otimes dz,$$

(58)

where $a_1, a_2 \in \mathbb{C} \setminus \{0, 1\}$ and $\alpha \in \mathbb{C} \setminus \{0\}$. Combining (39), (40) with the definition of Hopf differential $Q = dg \otimes \eta$, we can see that no matter which case shall we take, $dg$ actually has a uniform expression

$$dg = \alpha \kappa (z - 1)^{\mu - 1}z^{\nu - 1}(z - a_1)(z - a_2)\psi^{2}(z)dz.$$

(59)

Therefore, we obtain two differential equations with respect to the two cases:

For Case 1, we have

$$[(\mu + \nu)z - \nu] \varphi \psi + z(1) (\varphi' \psi - \varphi \psi') = \frac{\alpha}{\lambda \kappa} (z - a_1)(z - a_2).$$

(60)

For Case 2, the equation is

$$\mu \varphi \psi + (z - 1)(\varphi' \psi - \varphi \psi') = \frac{\alpha}{\lambda \kappa} z^{\nu - 1}(z - a_1)(z - a_2).$$

(61)

Any solution $\varphi, \psi$ of (60) or (61) will give rise to a desired Ricci surface. However, these two equations are very complicated to solve directly, so we need to do a more general discussion.

(1). Suppose that $a_1 \neq a_2$. If $\psi$ has a zero $c$ of order higher than 1, then by analysing (60) and (61), we have $(z - c)|(z - a_1)(z - a_2)$, hence $c = a_1$ or $c = a_2$ and the order of $(z - c)$ is exactly 2.

(i). If $\psi(z) = (z - a_1)^2(z - a_2)^2\widehat{\psi}$, then $\widehat{\psi}$ should be a polynomial whose zeroes are all distinct. Let us denote $\widehat{\psi}(z) = \prod_{j=1}^{n}(z - b_j)$. The existence of $g$ requires that all the residues at poles of $dg$ (zeros of $\psi$) to be zero, whence we get the following conditions:

$$\frac{\mu - 1}{b_i - 1} + \frac{\nu - 1}{b_i} - \frac{3}{b_i - a_1} - \frac{3}{b_i - a_2} - \sum_{j \neq i} \frac{2}{b_i - b_j} = 0,$$

(62)

and

$$\frac{(\mu - 1)(\mu - 2)}{(a_1 - 1)^2} + \frac{2(\mu - 1)(\nu - 1)}{a_1(a_1 - 1)} + \frac{(\nu - 1)(\nu - 2)}{a_1^3} + \frac{12}{(a_1 - a_2)^2}$$

$$- \left( \frac{\mu - 1}{a_1 - 1} + \frac{\nu - 1}{a_1} \right) \left( \frac{6}{a_1 - a_2} + \sum_{j=1}^{n} \frac{4}{a_1 - b_j} \right)$$

(63)
\[ + \frac{6}{a_1 - a_2} \left( \sum_{j=1}^{n} \frac{2}{a_1 - b_j} \right) \left( \sum_{j=1}^{n} \frac{2}{a_1 - b_j} \right)^2 + \sum_{j=1}^{n} \frac{2}{(a_1 - b_j)^2} = 0, \]

and

\[ \frac{(\mu - 1)(\mu - 2)}{(a_2 - 1)^2} + \frac{2(\mu - 1)(v - 1)}{a_2(a_2 - 1)} \left( \sum_{j=1}^{n} \frac{2}{a_2 - a_1} \right)^2 + \sum_{j=1}^{n} \frac{2}{(a_2 - a_1)^2} = 0. \]

(ii). Let us consider the case when \( \psi(z) = (z - a_1)^2 \psi', \) where \( \psi' \) is the same as in case (i). A similar computation yields that

\[
\frac{\mu - 1}{b_1 - 1} + \frac{b - 1}{b_i} + \frac{1}{b_2 - a_1} - \sum_{j \neq i} \frac{2}{b_i - b_j} = 0,
\forall i \in \{1, 2, \ldots, n\},
\]

and

\[
\frac{(\mu - 1)(\mu - 2)}{(a_1 - 1)^2} + \frac{2(\mu - 1)(v - 1)}{a_1(a_1 - 1)} \left( \sum_{j=1}^{n} \frac{4}{a_1 - b_j} \right)^2 + \sum_{j=1}^{n} \frac{2}{(a_1 - b_j)^2} = 0.
\]

(iii). The last case is when all the zeroes of \( \psi \) are distinct. We will keep the notation as in the two previous cases, then we must have

\[
\frac{\mu - 1}{b_i - 1} + \frac{v - 1}{b_i} + \frac{1}{b_i - a_1} + \frac{1}{b_i - a_2} - \sum_{j \neq i} \frac{2}{b_i - b_j} = 0,
\forall i \in \{1, 2, \ldots, n\}.
\]

(2). Now let us assume that \( a_1 = a_2. \) In this case, if \( \psi \) has a zero with multiplicity, then it could be \( (z - a_1)^2 \) or \( (z - a_1)^3. \)
(i). For the case when $\psi$ does not have zeroes with multiplicity, we easily get
\[
\frac{\mu - 1}{b_i - 1} + \frac{v - 1}{b_i} + \frac{2}{b_i - a_1} - \sum_{j \neq i} \frac{2}{b_i - b_j} = 0, \quad \forall i \in \{1, 2, \ldots, n\}.
\]
(68)

(ii). If $\psi(z) = (z - a_1)^2 \hat{\psi}$, then the conditions are
\[
\frac{\mu - 1}{b_i - 1} + \frac{v - 1}{b_i} - \frac{2}{b_i - a_1} - \sum_{j \neq i} \frac{2}{b_i - b_j} = 0, \quad \forall i \in \{1, 2, \ldots, n\},
\]
and
\[
\frac{\mu - 1}{a_1 - 1} + \frac{v - 1}{a_1} - \sum_{j=1}^{n} \frac{2}{a_1 - b_j} = 0, \quad \forall i \in \{1, 2, \ldots, n\}.
\]
(70)

(iii). The only case left is when $\psi(z) = (z - a_1)^3 \hat{\psi}$. Similarly, the conditions can be computed to be
\[
\frac{\mu - 1}{b_i - 1} + \frac{v - 1}{b_i} - \frac{4}{b_i - a_1} - \sum_{j \neq i} \frac{2}{b_i - b_j} = 0, \quad \forall i \in \{1, 2, \ldots, n\},
\]
and
\[
2 \left( \frac{\mu - 1}{a_1 - 1} + \frac{v - 1}{a_1} \right) \left[ \frac{11}{n} \left( \sum_{j=1}^{n} \frac{1}{a_1 - b_j} \right)^2 + 5 \sum_{i \neq j} \frac{1}{(a_1 - b_i)(a_1 - b_j)} \right] \\
+ 3 \left[ \frac{(\mu - 1)(\mu - 2)}{(a_1 - 1)^2} + \frac{2(\mu - 1)(v - 1)}{a_1(a_1 - 1)} + \frac{(v - 1)(v - 2)}{a_1^3} \right] \\
\times \left( \sum_{j=1}^{n} \frac{2}{a_1 - b_j} \right) + \frac{(\mu - 1)(\mu - 2)(\mu - 3)}{(a_1 - 1)^3} + \frac{3(\mu - 1)(v - 1)(\mu - 2)}{a_1(a_1 - 1)^2} \\
+ \frac{3(\mu - 1)(v - 1)(v - 2)}{a_1^2(a_1 - 1)} + \frac{(v - 1)(v - 2)(v - 3)}{a_1^3} \\
\sum_{j=1}^{n} \frac{32}{(a_1 - b_j)^3} - \sum_{i \neq j} \frac{60}{(a_1 - b_i)^2(a_1 - b_j)} \\
- \sum_{i \neq j \neq k} \frac{16}{(a_1 - b_i)(a_1 - b_j)(a_1 - b_k)} = 0.
\]
(72)
If a Ricci surface induces a reducible metric \((-K)g_M\) of constant Gauss curvature 1, then it must satisfy Eq. (60) or (61) and one of the previous cases; the converse is also true. Therefore, so as to find all the desired Ricci surfaces, we may firstly solve algebraic Eqs. (62)–(72) of each case to completely determine the polynomial \(\psi\). Then the problem is reduced to searching solutions to first order Eqs. (60) and (61) with a known polynomial \(\psi\). We summarize these arguments in the following theorem:

**Theorem 14.** Given \(\mu > 0, \nu \in \mathbb{R}^n\) and \(a_1, a_2 \in \mathbb{C} \setminus \{0, 1\}\), a complete non-positively curved Ricci surface \(M \simeq \mathbb{C} \setminus \{0, 1\}\) with a reducible metric \((-K)g_M\) and three catenoidal ends exists if only if one of the following cases is verified:

(i). If \(a_1 \neq a_2\), there exist \(n \in \mathbb{N}\) and mutually distinct \(b_1, \ldots, b_n \in \mathbb{C} \setminus \{0, 1, a_1, a_2\}\) such that (62)–(64) admits a polynomial solution \(\varphi\) with \(\psi = (z - a_1)^2(z - a_2)^2 \prod_{j=1}^n (z - b_j)\);

(ii). If \(a_1 \neq a_2\), there exist \(n \in \mathbb{N}\) and mutually distinct \(b_1, \ldots, b_n \in \mathbb{C} \setminus \{0, 1, a_1, a_2\}\) such that (65)–(66) admits a polynomial solution \(\varphi\) with \(\psi = (z - a_1)^2 \prod_{j=1}^n (z - b_j)\);

(iii). If \(a_1 \neq a_2\), there exist \(n \in \mathbb{N}\) and mutually distinct \(b_1, \ldots, b_n \in \mathbb{C} \setminus \{0, 1, a_1, a_2\}\) such that (67) admits a polynomial solution \(\varphi\) with \(\psi = \prod_{j=1}^n (z - b_j)\);

(iv). If \(a_1 = a_2\), there exist \(n \in \mathbb{N}\) and mutually distinct \(b_1, \ldots, b_n \in \mathbb{C} \setminus \{0, 1, a_1\}\) such that (68) admits a polynomial solution \(\varphi\) with \(\psi = \prod_{j=1}^n (z - b_j)\);

(v). If \(a_1 = a_2\), there exist \(n \in \mathbb{N}\) and mutually distinct \(b_1, \ldots, b_n \in \mathbb{C} \setminus \{0, 1, a_1\}\) such that (69)–(70) admits a polynomial solution \(\varphi\) with \(\psi = (z - a_1)^2 \prod_{j=1}^n (z - b_j)\);

(vi). If \(a_1 = a_2\), there exist \(n \in \mathbb{N}\) and mutually distinct \(b_1, \ldots, b_n \in \mathbb{C} \setminus \{0, 1, a_1\}\) such that (71)–(72) admits a polynomial solution \(\varphi\) with \(\psi = (z - a_1)^3 \prod_{j=1}^n (z - b_j)\).

**Remark 6.** In the case when \(M \simeq \mathbb{C} \setminus \{0, 1\}\), we are not allowed to use the method mentioned in Remark 4 to study Ricci surfaces. This is because the flat metric \(|Q|\) vanishes at \(a_1, a_2 \in \mathbb{C} \setminus \{0, 1\}\), which should coincide with zeroes of the Gauss curvature \(K\). It means that the hypothesis \(V > 0\) of Lemma 2.3 in [2] fails to be satisfied. Therefore, we are obliged to take advantage of the method as above.

### 6.4. Results related to total curvature

In this subsection, we plan to study Ricci surfaces \(M \simeq \mathbb{C} \setminus \{0, 1\}\) with a reducible metric \((-K)g_M\) under some additional conditions on the total curvature. At first, we will investigate the Ricci surfaces with finite total curvature \(-4\pi\).

**Theorem 15.** Assuming that \(M \simeq \mathbb{C} \setminus \{0, 1\}\) is a complete Ricci surface with three catenoidal ends and a reducible induced metric \((-K)g_M\), if it has finite
total curvature $-4\pi$, then it must be isometric to the surface determined by the Weierstrass data
\[
\begin{align*}
g &= \lambda(z - 1)^\mu z^v(z - a), \\
\eta &= \kappa(z - 1)^{-\mu-1}z^{-v-1}dz,
\end{align*}
\]
where $\lambda, \kappa \in \mathbb{R}^*_+$, $\mu, v \in \mathbb{R}^*_+$ satisfying $v \geq \max\{\mu, -2\mu - 1\}$, the parameter $a$ satisfies:

(i). If $v > \max\{\mu, -2\mu - 1\}$, then $a \in \mathbb{C} \setminus \{0, 1\}$;
(ii). If $v = \mu$ but $v \neq 2\mu - 1$, then $a \in \left(\mathbb{C} \setminus \{0, 1\}\right)/\langle id, z \mapsto 1 - z\rangle$;
(iii). If $v = -2\mu - 1$ but $v \neq \mu$, then $a \in \left(\mathbb{C} \setminus \{0, 1\}\right)/\langle id, z \mapsto \frac{z}{z-1}\rangle$;
(iv). If $v = \mu = -\frac{1}{3}$, then $a \in \left(\mathbb{C} \setminus \{0, 1\}\right)/\text{Aut}(\mathbb{C} \setminus \{0, 1\})$.

Moreover, there is no orientation-preserving isometry between the Ricci surfaces with different $(\mu, v, a, \lambda, \kappa)$.

Proof. By using Gauss-Bonnet formula with conical singularities (31), we are able to compute the total curvature. The expression of total curvature should be divided into several cases:

(1). Suppose that the Weierstrass data take the form as in (39) with $\mu, v > 0$. The induced metric $(-K)g_M$ can be determined to be
\[
(-K)g_M = \frac{4|\alpha|^2}{|\kappa|^2} \cdot \frac{|z - 1|^{2(\mu-1)}|z|^{2(v-1)}|z - a_1|^2|z - a_2|^2}{(|\psi|^2 + |\lambda|^2|z - 1|^{2\mu}|z|^{2v}|\varphi|^2)^2}|dz|^2.
\]

(a). If $N > \mu + v + m$, then this metric has conical singularities of order $\nu - 1, \mu - 1, 2N - 2 - \mu - v, 1$ at $0, 1, \infty, a_1, a_2$, respectively. Hence the total curvature is $-4\pi N$. The hypothesis that the total curvature should be $-4\pi$ implies $N = 1$, thus $m = 0$ and $0 < \mu + \nu < 1$. Then by applying the transformation $\tilde{g} = \frac{1}{g}$ as in the proof of Theorem 11, we obtain the desired formula.

(b). If $N = \mu + v + m$ and $N + m \geq 2$, then the total curvature is also $-4\pi N$. Hence we must have $N = 1$ and $m = 0$, which is not possible.

(c). If $N < \mu + v + m$, then the total curvature is $-4\pi(\mu + v + m)$. The condition $\mu + v + m = 1$ tells us that $m = 0$ and $N = 0$. Therefore, $\varphi = \psi = 1$. However, this case cannot appear due to Theorem 11 and Proposition 12.

(2). Now assume that the Weierstrass data also take the form as in (39) but with $\mu > 0, v < 0$. In this case, we have
\[
(-K)g_M = \frac{4|\alpha|^2}{|\kappa|^2} \cdot \frac{|z - 1|^{2(\mu-1)}|z|^{-2(v+1)}|z - a_1|^2|z - a_2|^2}{(|z|^{-2\nu}|\psi|^2 + |\lambda|^2|z - 1|^{2\mu}|\varphi|^2)^2}|dz|^2.
\]

(d). If $N > \mu + v + m$, then the total curvature is $-4\pi(N - \nu)$, thus we get $N = 0$ and $\nu = -1$. Moreover, $1 > \mu + m$ implies $m = 0$. The same argument as in (c) shows that this is impossible.

(e). If $N < \mu + v + m$, then the total curvature is $-4\pi(\mu + m)$. Again, we will obtain $m = 0$ and $N = 0$, not possible.
(f) If \( N = \mu + \nu + m \) and \( N + m \geq 2 \), then the total curvature is also 
\(-4\pi (N - \nu)\). This leads to \( N = m = 0 \), impossible.

(3) Suppose now that the Weierstrass data have the expression as in (40), where
\( \nu \in \mathbb{N}^* \), \( \mu > 0 \) satisfying \( \mu + m = N \). A similar computation shows
\[
(-K)g_M = \frac{4|\alpha|^2}{|\kappa|^2} \cdot \frac{|z - 1|^{2(\mu - 1)}|z|^{2(\nu - 1)}|z - a_1|^2|z - a_2|^2}{(|\psi|^2 + |\lambda|^2)|z - 1|^{2\mu}|\varphi|^2}|dz|^2.
\]

In this case, the total curvature is \(-4\pi N\). Therefore, we need to have \( N = 1 \) and \( m = 0 \). Since we have seen in Sect. 6.2 that \( N + m \geq \nu + 2 \geq 3 \), contradiction.

The rest of the proof is actually the same with that of Theorem 11, hence it will be accomplished in the end of this subsection as well. \( \Box \)

We get instantly the following result from the previous proof.

**Theorem 16.** There is no complete Ricci surface \( M \simeq \mathbb{C} \setminus \{0, 1\} \) with three catenoidal ends which induces a reducible metric \((-K)g_M\) and has finite total curvature \(-4\pi l\) for \( 0 < l < 1 \).

**Remark 7.** Thanks to the discussion in the previous proof of Theorem 15, we are able to express in terms of \( \mu, \nu \in \mathbb{R} \setminus \mathbb{Z} \) the total curvature of generic Ricci surfaces whose Weierstrass data take the form as in (41). This is shown in Fig. 1. In the region \( A \), the total curvature of each point is \(-4\pi (1 + \mu + \nu)\); points in the region \( B \) have total curvature \(-4\pi (1 + \nu)\). In the region \( C \), each point has total curvature \( 4\pi \mu \) whereas the region \( D \) is composed of points with total curvature \( 4\pi (\mu + \nu) \). The region \( E \) contains points with total curvature \( 4\pi \nu \). The total curvature of points in the region \( F \) is \(-4\pi (1 + \mu)\). The last region \( G \) contains Ricci surfaces with total curvature \(-4\pi \), which are the surfaces we found in Theorem 15.

In the generic case, we should have \( \mu, \nu \notin \mathbb{Z} \) and \( \mu + \nu \notin \mathbb{Z} \), thus a direct observation gives the following result.

**Corollary 17.** Let \( M \simeq \mathbb{C} \setminus \{0, 1\} \) be a complete Ricci surface with three catenoidal ends and a reducible induced metric \((-K)g_M\). If it has finite total curvature \(-4\pi k\) with \( k \geq 2 \) an integer, then \( M \) cannot be generic.

For Ricci surfaces \( M \simeq \mathbb{C} \setminus \{0, 1\} \) with total curvature \(-4\pi l\) for \( l > 1 \), things will become much more complicated. However, it is always beneficial to mention the trinoid constructed by Jorge and Meeks in [12], which is a typical example of such Ricci surfaces whose induced metric \((-K)g_M\) is reducible.

**Example 4.** (Trinoid) The classical Weierstrass data of a trinoid are
\[
g = z^2, \quad \eta = \frac{dz}{(z^3 - 1)^2}.
\]
defined on $\hat{\mathbb{C}} \setminus \{1, \xi, \xi^2\}$, where $\xi = e^{\frac{2\pi i}{3}}$. After a Mobius transformation and a rotation, we may find a new pair of Weierstrass data defined on $\mathbb{C} \setminus \{0, 1\}$ as

$$
\begin{align*}
g &= \frac{\xi - 1}{\xi + 1} \cdot \frac{(w - 1)(w + 1)}{(w - 2 - \sqrt{3})(w - 2 + \sqrt{3})}, \\
\eta &= -\frac{1}{9(2\xi + 1)} \cdot \frac{(w - 2 - \sqrt{3})^2(w - 2 + \sqrt{3})^2}{w^2(w - 1)^2} \, dw.
\end{align*}
$$

(73)

Then a little computation shows $a_1 = \frac{1 + \sqrt{3}i}{2}$ and $a_2 = \frac{1 - \sqrt{3}i}{2}$. This situation belongs to Case 2. It can be checked that conditions (67) and Eq. (61) are satisfied. Additionally, by using Gauss-Bonnet formula (31) again, we know that its total curvature is $-8\pi$.

With all the preparations that we have done above, we are now going to complete the proofs of Theorem 11 and Theorem 15.

Proof. (of Theorems 11 and 15) In Sect. 6.1, we have shown that the Weierstrass data of any generic complete non-positively curved Ricci surface with three catenoidal ends and a reducible induced metric $(-K)_{g_M}$ must be in the form of (41), with $\lambda, \kappa \in \mathbb{R}^*_+, \mu, \nu, \mu + \nu \in \mathbb{R} \setminus \mathbb{Z}$ and $a \in \mathbb{C} \setminus \{0, 1\}$. Now we want to understand if such Ricci surfaces with different parameters $(\mu, \nu, a, \lambda, \kappa)$ could be isometric. If such an orientation-preserving isometry exists, then it should be an element of the group $Aut(\mathbb{C} \setminus \{0, 1\})$ which consists of

$$
h_0(z) = z, \quad h_1(z) = \frac{z}{z - 1}, \quad h_2(z) = \frac{1}{z},
$$

Fig. 1. Total curvature distribution for generic $(\mu, \nu)$
This group is isomorphic to the dihedral group of order 6.

(1). We first look at the parameters $\mu, \nu \in \mathbb{R} \setminus \mathbb{Z}$. As we have seen in Remark 7, we can divide the $\mu\nu$-plane into 7 regions according to the total curvature expressions. By applying the transformations in $Aut(\mathbb{C} \setminus \{0, 1\})$, one can easily check that each surface in the region $C$ or $E$ is isometric to some surface in the region $A$; each surface belonging to the region $D$ or $F$ is isometric to some surface in the region $B$. Therefore, we can only consider the surfaces in the regions $A, B$ and $G$. Moreover, up to applying the transformation $h_3(z) = 1 - z$, we may restrict ourselves to the surfaces in $A, B$ and $G$ satisfying $\nu \geq \mu$.

Then we want to know if two surfaces from different regions could be isometric. Since the total curvature is intrinsic, two Ricci surfaces with different total curvature cannot be isometric. Hence we only need to see whether a surface in the region $A$ can be isometric to a surface in the region $B$ with the same total curvature. Let us denote $(\mu_1, \nu_1) \in A$ and $(\mu_2, \nu_2) \in B$, then we should have $\mu_1 + \nu_1 = \nu_2$. If such an isometry exists, then it must permute the orders of conical singularities of the induced metric $-(K)_{GM}$ at 0, 1 and $\infty$. For $(\mu_1, \nu_1) \in A$, its orders of conical singularities at 0, 1 and $\infty$ are $(\nu_1 - 1, \mu_1 - 1, \mu_1 + \nu_1) = (\nu_1 - 1, \mu_1 - 1, \nu_2)$; for $(\mu_2, \nu_2) \in B$, its orders of conical singularities at 0, 1 and $\infty$ are $(\nu_2 - 1, -\mu_2 - 1, \mu_2 + \nu_2)$. It is clear that $v_2 \neq v_1 - 1, v_2 \neq \mu_2 + v_2$. If $v_2 = -\mu_2 - 1$, then $\mu_2 + v_2 = -1 \in \mathbb{Z}$, which is not possible. This means that a surface in the region $B$ and a surface in the region $A$ cannot be isometric.

Now we try see whether two surfaces in the same region but with different $(\mu, \nu)$ can be isometric.

(i). Let us take two points $(\mu_1, \nu_1) \neq (\mu_2, \nu_2)$ in $A$ satisfying $\mu_1 + \nu_1 = \nu_2 + \nu_2$.

By using a similar argument as above, we can prove that these two surfaces cannot be isometric.

(ii). Suppose that $(\mu_1, \nu_1) \neq (\mu_2, \nu_2)$ are two different points in $B$ with $\nu_1 = \nu_2$.

By comparing the orders of conical singularities at 0, 1 and $\infty$, it is not difficult to check that if they are isometric, then we should have $\mu_2 + \nu_2 + \mu_1 = -1$. Moreover, taking advantage of the transformation $h_1(z) = \frac{z}{z-1}$, we can see that these two surfaces are actually isometric under the condition $\mu_2 + \nu_2 + \mu_1 = -1$. As a consequence, we can reduce the region $B$ to the points satisfying $\nu \geq -2\mu - 1$.

(iii). As for two different points $(\mu_1, \nu_1) \neq (\mu_2, \nu_2)$ in $G$, they have the same total curvature $-4\pi$. Their orders of conical singularities at 0, 1 and $\infty$ are $(\mu_1 - 1, \nu_1 - 1, \mu_1 + \nu_1)$ and $(\mu_2 - 1, \nu_2 - 1, \mu_2 + \nu_2)$, respectively. By utilizing a similar argument as in (ii), we are able to show that two surfaces in $G$ are isometric if and only if they are related by a transformation in $Aut(\mathbb{C} \setminus \{0, 1\})$. In this way, we can reduce the region $G$ to the points satisfying $\nu \geq \max \{\mu, -2\mu - 1\}$.
As a result of the previous discussion, we can transform Fig. 1 into Fig. 2, where the point $P$ is the barycenter of the triangular region $G$ in Fig. 1. Moreover, two different points in this figure are not isometric.

(2). Now we will study the case when two Ricci surfaces have the same parameters $(\mu, \nu)$ but different parameter $a \in \mathbb{C} \setminus \{0, 1\}$. For the points satisfying $\nu > \max\{\mu, -2\mu - 1\}$, one may check that $(\mu, \nu, a_1)$ cannot be identified with $(\mu, \nu, a_2)$ if $a_1 \neq a_2$ under the transformations in $\text{Aut}(\mathbb{C} \setminus \{0, 1\})$. If $\nu = \mu$ but $\nu \neq 2\mu - 1$, then we can see that $(\mu, \nu, a)$ is isometric to $(\mu, \nu, 1 - a)$. In this case, we will take $a \in (\mathbb{C} \setminus \{0, 1\}) / \{h_0, h_3\}$. Similarly, if $\nu = -2\mu - 1$ but $\nu \neq \mu$, then $(\mu, \nu, a)$ is isometric to $(\mu, \nu, \frac{a}{a - 1})$. The parameter $a$ should take values in $(\mathbb{C} \setminus \{0, 1\}) / \{h_0, h_1\}$.

The last case is when $\nu = \mu = -\frac{1}{2}$. In this situation, each transformation in $\text{Aut}(\mathbb{C} \setminus \{0, 1\})$ gives rise to a such isometry, thus we restrict $a$ to be in $(\mathbb{C} \setminus \{0, 1\}) / \text{Aut}(\mathbb{C} \setminus \{0, 1\})$.

We are going to prove that for $a_1 \neq a_2$ in the same fundamental domain that we mentioned above, the Ricci surfaces given by $(\mu, \nu, a_1)$ and $(\mu, \nu, a_2)$ are not isometric. It is well-known that every isometry should preserve the zeros of the Gauss curvature $K$. Taking advantage of (4) and (41), we notice that the zeros of $K$ coincides with the roots of the equation

$$(\mu + \nu + 1)z^2 - (a\mu + av + \nu + 1)z + av = 0.$$
Let us denote $b_1, b_2$ the two roots of this equation, then we have

$$b_1 + b_2 = \frac{a\mu + a\nu + \nu + 1}{\mu + \nu + 1}, \quad b_1 \cdot b_2 = \frac{a\nu}{\mu + \nu + 1}.$$ 

Since $a_1$ and $a_2$ are in the same fundamental domain, we are not allowed to apply the transformations in $\text{Aut}(\mathbb{C} \setminus \{0, 1\})$. Therefore, if such an isometry exists, it should fix the two roots of the previous equation. Hence we must have $a_1 = a_2$, which is a contradiction. As a result, if we fix $(\mu, \nu)$, then two different values of the parameter $a$ in the domains that we described above gives us non-isometric Ricci surfaces.

(3). In the end, we want to know whether two Ricci surfaces with the same parameters $(\mu, \nu, a)$ but $(\lambda_1, \kappa_1) \neq (\lambda_2, \kappa_2)$ could be isometric. If there exists such an isometry, it should preserve the Gauss curvature. Therefore, we should have

$$\frac{\lambda_1 |g_1'|}{\kappa_1 |\alpha_1| \left(1 + \lambda_1^2 |g_1|^2\right)^{\delta_1}} = \frac{\lambda_2 |g_1'|}{\kappa_2 |\alpha_1| \left(1 + \lambda_2^2 |g_1|^2\right)^{\delta_1}},$$

where $g_1 = (z - 1)^{\mu}z^\nu(z - a)$, $\alpha_1 = (z - 1)^{\mu - 1}z^{\nu - 1}$. It is easy to see that $\frac{\lambda_1}{\kappa_1} = \frac{\lambda_2}{\kappa_2}$. This implies $\lambda_1 = \lambda_2$, thus $\kappa_1 = \kappa_2$, contradiction.

Consequently, there is no orientation-preserving isometry between the Ricci surfaces with different $(\mu, \nu, a, \lambda, \kappa)$. This completes the proofs of Theorems 11 and 15.

7. Higher genus Ricci surfaces

As we have seen in Sect. 6, it is very difficult to give a total classification of Ricci surfaces with catenoidal ends in general. However, the existence of Ricci surfaces $M \simeq \mathbb{C} \setminus \{p_1, p_2, \ldots, p_n\}$ with $n$ catenoidal ends ($n \geq 2$) is well-known, which is given by the n-noid (see [13]).

Example 5. (n-noid) Suppose $M \simeq \mathbb{C} \setminus \{1, \xi, \ldots, \xi^{n-1}\}$ with $\xi = e^{\frac{2\pi i}{n}}$, then the Weierstrass data of the n-noid are given by

$$g = z^{n-1}, \quad \eta = \frac{dz}{(z^n - 1)^\frac{1}{2}}.$$ 

Now we are interested in discussing non-positively curved Ricci surfaces with positive genus and finitely many catenoidal ends. Our goal of this section is to prove that there exist Ricci surfaces $M \simeq S_k \setminus \{p_1, p_2, \ldots, p_n\}$ with $n$ catenoidal ends for $k, n > 0$, where $S_k$ is a compact orientable surface of genus $k$. To achieve this, we will use the method introduced by Andrei Moroianu and Sergiu Moroianu as mentioned in Remark 4. The first tool that we need is the following theorem proved by Gabriele Mondello and Dmitri Panov (see [14], Theorem A).
Theorem 18. Let $\chi \leq 0$ be an even number and $v_1, \ldots, v_n \in \mathbb{R}^+_+$ be such that
\[
\chi + \sum_{j=1}^n (v_j - 1) > 0,
\]
(74)
then there exists a compact Riemann surface $S_k$ of genus $k = \frac{2-\chi}{2} \geq 1$, a set of distinct points $\{p_1, p_2, \ldots, p_n\} \subset S_k$ and a metric $g_1$ of constant Gauss curvature 1 on $S_k$ such that $g_1$ has conical singularities of order $v_j - 1$ at $p_j$.

In order to get a Ricci metric on $M \simeq S_k \setminus \{p_1, p_2, \ldots, p_n\}$, we have to construct a proper flat metric. This is possible due to a result of Andrei Moroianu and Sergiu Moroianu (see [2], Lemma 6.1).

Lemma 19. Given a compact Riemann surface $S_k$ of genus $k$, let $p_1, p_2, \ldots, p_n \in S_k$ and $\beta : \{p_1, p_2, \ldots, p_n\} \to \mathbb{R}$ be a function which satisfies
\[
\sum_{j=1}^n (\beta(p_j) - 1) = -\chi(S_k),
\]
(75)
then there exists a flat metric $g_0$ on $S_k \setminus \{p_1, p_2, \ldots, p_n\}$ compatible with the complex structure of $S_k$ such that near each $p_i$, it has the form
\[
g_0 = e^{2u}|z|^{2\beta(p_i) - 2}|dz|^2
\]
(76)
with some smooth function $u \in C^\infty(S_k, \mathbb{R})$.

Thanks to these two results, we are able to prove the existence of Ricci surfaces with arbitrary genus and arbitrary number of catenoidal ends.

Theorem 20. For $k, n > 0$, there exists a non-positively curved orientable Ricci surface $M$ of genus $k$ with $n$ catenoidal ends.

Proof. Firstly, let us take $\chi = 2 - 2k \leq 0$, $v_1, \ldots, v_n \in \mathbb{R}^+_+$ and $v_{n+1} = 2(2k - 2 + n) + 1 > 0$, then it is easy to verify that
\[
\chi + \sum_{j=1}^{n+1} (v_j - 1) = \sum_{j=1}^n v_j + 2k + n - 2 > 0.
\]
Hence by Theorem 18, we will have a compact Riemann surface $S_k$ of genus $k$, a set of points $\{p_1, p_2, \ldots, p_{n+1}\} \subset S_k$ and a metric $g_1$ of constant Gauss curvature 1 on $S_k$ with conical singularities of order $v_j - 1$ at $p_j$.

Secondly, we define a function $\beta : \{p_1, p_2, \ldots, p_{n+1}\} \to \mathbb{R}$ as $\beta(p_i) = 0$ for $i = 1, 2, \ldots, n$ and $\beta(p_{n+1}) = 2k - 1 + n$. This function $\beta$ satisfies the equality (75), thus from Lemma 19, there is a flat metric $g_0$ on $S_k \setminus \{p_1, p_2, \ldots, p_{n+1}\}$ which is conformal to $g_1$ and takes the form as in (76) near every punctured point.

Both of $g_1$ and $g_0$ do not vanish on $S_k \setminus \{p_1, p_2, \ldots, p_{n+1}\}$. Since $g_0$ is conformal to $g_1$, there is a positive function $V$ defined on $S_k \setminus \{p_1, p_2, \ldots, p_{n+1}\}$ such that $g_1 = Vg_0$. It follows that $g := V^{-1}g_0$ is a Ricci metric on $S_k \setminus \{p_1, p_2, \ldots, p_{n+1}\}$
Near the point $p_{n+1}$, it is known that the metric $g_1$ can be written as

$$g_1 = \frac{4v_{n+1}^2 |z|^{4(2k-2+n)} |dz|^2}{(1 + |z|^{4(2k-2+n)+2})^2} \quad (77)$$

for a suitable complex coordinate $z$. From Lemma 19, the flat metric $g_0$ has also a local expression around $p_{n+1}$ which is

$$g_0 = e^{2u} |z|^{2(2k-2+n)} |dz|^2 \quad (78)$$

for some smooth function $u$. A direct computation shows that the Ricci metric $g$ is of the form

$$g = \frac{1}{4} v_{n+1}^{-2} e^{4u} (1 + |z|^{4(2k-2+n)+2})^2 |dz|^2 \quad (79)$$

near $p_{n+1}$. Hence $g$ is actually well-defined and smooth at $p_{n+1}$, thus a Ricci metric on $S_k \setminus \{p_1, p_2, \ldots, p_n\}$. In addition, near each $p_i \in \{p_1, p_2, \ldots, p_n\}$, $g_0$ has the form

$$g_0 = e^{2u} |z|^{-2} |dz|^2.$$

This means $p_1, p_2, \ldots, p_n$ are catenoidal ends of this Ricci surface (see the discussion after Definition 4). Consequently, we have constructed an orientable Ricci surface of genus $k$ with exactly $n$ catenoidal ends.

**Example 6.** Wayne Rossman and Katsunori Sato have constructed a genus 1 catenoid cousin which is a CMC-1 immersion into the hyperbolic 3-space $\mathcal{H}^3$ (see [15]). This is an example of Theorem 20 in the case $k = 1$ and $n = 2$.

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**References**

[1] Ricci-Curbastro, G.: Sulla teoria intrinseca delle superficie ed in ispecie di quelle di secondo grado. atti r. Ist. Ven. di Lett. ed Arti 6, 445–488 (1895)
[2] Moroianu, A., Moroianu, S.: Ricci surfaces. Annali della Scuola Normale Superiore di Pisa. Classe di scienze 14(4), 1093–1118 (2015)
[3] Huber, A.: On subharmonic functions and differential geometry in the large. Commentarii Mathematici Helvetici 32(1), 13–72 (1958)
[4] Troyanov, M.: Metrics of constant curvature on a sphere with two conical singularities. Differential Geometry, pp. 296–306. Springer (1989)
[5] Calabi, E.: Quelques applications de l’analyse complexe aux surfaces d’aire minima. Topic in Complex Manifolds, pp. 59–81 (1968)
[6] Daniel, B.: A survey on minimal isometric immersions into $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. In Minimal Surfaces: Integrable Systems and Visualisation, pp. 51–65. Springer Proceedings in Mathematics and Statistics (2021)

[7] Bryant, R.: Surfaces of mean curvature one in hyperbolic space. Astérisque 154(155), 321–347 (1987)

[8] Umehara, M., Yamada, K.: Complete surfaces of constant mean curvature-1 in the hyperbolic 3-space. Ann. Math. 137(3), 611–638 (1993)

[9] Lawson, H.B., Jr.: Complete minimal surfaces in $\mathbb{S}^3$. Ann. Math. 92(3), 335–374 (1970)

[10] Osserman, R.: A survey of minimal surfaces. Courier Corporation, United States (2013)

[11] Umehara, M., Yamada, K.: Metrics of constant curvature 1 with three conical singularities on the 2-sphere. Ill. J. Math. 44(1), 72–94 (2000)

[12] Jorge, L.P., Meeks, W.H., III.: The topology of complete minimal surfaces of finite total gaussian curvature. Topology 22(2), 203–221 (1983)

[13] Weber, M.: Classical minimal surfaces in Euclidean space by examples: geometric and computational aspects of the Weierstrass representation. Global theory of minimal surfaces 2, 19–63 (2005)

[14] Mondello, G., Panov, D.: Spherical surfaces with conical points: systole inequality and moduli spaces with many connected components. Geom. Funct. Anal. 29(4), 1110–1193 (2019)

[15] Rossman, W., Sato, K.: Constant mean curvature surfaces with two ends in hyperbolic space. Exp. Math. 7(2), 101–119 (1998)

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