Type II/F-theory Superpotentials and Ooguri-Vafa Invariants of Compact Calabi-Yau Threefolds with Three Deformations

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Abstract

We calculate the D-brane superpotentials for two Calabi-Yau manifolds with three deformations by the generalized hypergeometric GKZ systems, which give rise to the flux superpotentials $W_{GV}$ of the dual F-theory compactification on the relevant Calabi-Yau fourfolds in the weak decoupling limit. We also compute the Ooguri-Vafa invariants from A-model expansion with mirror symmetry, which are related to the open Gromov-Witten invariants.

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1 Introduction

Mirror symmetry has obtained much success in the $N = 2$ supersymmetric theories. The exact non-perturbative holomorphic prepotential $[1, 2]$ in such theories can be obtained by topological string theory via mirror symmetry.

Appearance of D-brane breaks the supersymmetry to $N = 1$. When a D6-brane wrapped on a special lagrangian 3-cycle, the $N = 1$, $d = 4$ superpotential term can be computed by the open topological string amplitudes $F_{g,h}$ of the A-model as follows

$$ h \int d^4 x d^2 \theta F_{g,h}(G^2)^g(F^2)^{h-1} \quad (1.1) $$

where $G$ is the gravitational chiral superfield and $F$ is the gauge chiral superfield. The formula (1.1) at $g = 0$, $h = 1$ leads to in $F$-terms of $N = 1$ supersymmetric theories:

$$ \int d^4 x d^2 \theta W(\Phi) $$

Hence we can calculate these superpotential by topological open-string theory.

For non-compact Calabi-Yau manifolds, the refs. [3–9] studied the open-closed mirror symmetry and its applications. In particular, the work [3] constructed the classical A-brane geometry with special Lagrangian submanifold and the work [5, 6] introduced $N = 1$ spacial geometry and variation of mixed Hodge structure to calculate superpotentials. However, computing such superpotentials for compact Calabi-Yau threefold is a hard work. Recently, a progress on compact manifolds came from [10–13], which studied a class of involution branes independent of open deformation moduli. Furthermore, there appeared some related works on superpotential for compact Calabi-Yau manifolds depending on open-closed deformation moduli [14–32], where the works [28, 29] studied it by the conformal field theory and matrix factorization and the others considered with Hodge theoretic method.

Open mirror symmetry also has important application in enumerative geometry. The superpotentials of A-model at large radius region count the disk invariants [3] which are related to open Gromov-Witten invariants [33–37].

In this paper, we calculate D-brane superpotentials for compact Calabi-Yau threefolds with three deformation parameters by open-closed mirror symmetry and generalized GKZ system [5, 6, 14, 41, 42, 44, 45], which is closed related to variations of mixed Hodge structure on relative cohomology group. The compact Calabi-Yau threefolds we consider are constructed as Calabi-Yau hypersurfaces in ambient toric varieties. When
including D-branes, the polyhedron associated to the above toric varieties can be extended to the so-called enhanced polyhedron with one dimension higher which gives rise to another toric variety and Calabi-Yau fourfold. There exists a duality between the type II compactification with brane on the threefold and the M/F-theory compactification on the Calabi-Yau fourfold without any branes but with fluxes [17–19,24,27,46]. In the weak decoupling limit $g_s \to 0$, the Gukov-Vafa-Witten superpotentials [38] $W_{GVW}$ of F-theory compactification on this fourfold agrees with superpotentials $\mathcal{W}$ of Type II compactification threefold with branes at lowest order in $g_s$ [18–20,24,27,39]

$$W_{GVW} = \mathcal{W} + \mathcal{O}(g_s) + \mathcal{O}(e^{-1/g_s})$$  \hspace{1cm} (1.2)

Hence, in this limit, we can obtain the flux superpotential $W_{GVW}$ from the superpotential $\mathcal{W}$ which will be given in this paper.

In Sect. 2 we give a overview of $N = 1$ special geometry and generalized GKZ system. In sect. 3 and sect. 4, we calculate the type II/F superpotentials for two compact Calabi-Yau threefold with three deformation parameters—$X_{24}(1,1,2,8,12)$ and $X_{12}(1,1,1,3,6)$, respectively. In these manifolds, we consider superpotential, mirror symmetry and Ooguri-Vafa invariants for D-brane with a single open deformation moduli. Sect. 5 is for summary.

2 $N=1$ Special Geometry, Generalized GKZ System and Type II/F Superpotentials

2.1 $N = 1$ Special Geometry, Relative Periods and Type II/F Superpotentials

Type II compactification theory is described by an effective $N = 1$ supergravity action with non-trivial superpotentials on the deformation space $M$ when adding D-branes and background fluxes. For D6-brane wrapped the whole Calabi-Yau threefold, the holomorphic Chern-Simons theory [47]

$$\mathcal{W} = \int_X \Omega^{3,0} \wedge \text{Tr}[A \wedge \bar{\partial}A + \frac{2}{3} A \wedge A \wedge A]$$  \hspace{1cm} (2.1)

gives the brane superpotential $\mathcal{W}_{brane}$, where $A$ is the gauge field with gauge group $U(N)$ for $N$ D6-branes. When reduced dimensionally, the low dimensional brane su-
perpotentials can be obtained as \[3,48\]

\[W_{\text{brane}} = N_\nu \int_{\Gamma_\nu} \Omega^{3,0}(z, \hat{z}) = \sum_\nu N_\nu \Pi^\nu\]  

(2.2)

where \(\Gamma^\nu\) is a special Lagrangian 3-chain and \((z, \hat{z})\) are closed-string complex structure moduli and D-brane moduli from open-string sector, respectively.

The background fluxes \(H^{(3)} = H^{(3)}_{RR} + \tau H^{(3)}_{NS}\), which take values in the integer cohomology group \(H^3(X, \mathbb{Z})\), also break the supersymmetry \(N = 2\) to \(N = 1\). The \(\tau = C^{(0)} + ie^{-\phi}\) is the complexified Type IIB coupling field. Its contribution to superpotentials is \([49,50]\)

\[W_{\text{flux}}(z) = \int_X H^{(3)}_{RR} \wedge \Omega^{3,0} = \sum_\alpha N_\alpha \cdot \Pi^\alpha(z), \quad N_\alpha \in \mathbb{Z}.\]  

(2.3)

The contributions of D-brane and background flux (here the NS-flux ignored) give together the general form of superpotential as follow \([5,6]\)

\[W(z, \hat{z}) = W_{\text{brane}}(z, \hat{z}) + W_{\text{flux}}(z) = \sum_{\gamma \Sigma \in H^3(Z^*, \mathcal{H})} N_\Sigma \Pi_\Sigma(z, \hat{z})\]  

(2.4)

where \(N_\Sigma = n_\Sigma + \tau m_\sigma\), \(\tau\) is the dilaton of type II string and \(\Pi_\Sigma\) is a relative periods defined in a relative cycle \(\Gamma \in H_3(X, D)\) whose boundary is wrapped by D-branes and \(D\) is a holomorphic divisor of the Calabi-Yau space. In fact, the two-cycles wrapped by the D-branes are holomorphic cycles only if the moduli are at the critical points of the superpotentials. Thus, the two-cycles are generically not holomorphic. However, according to the arguments of \([5,6,14]\), the non-holomorphic two-cycles can be replaced by a holomorphic divisor \(D\) of the ambient Calabi-Yau space with the divisor \(D\) encompassing the two-cycles.

Geometrically speaking, when varying the complex structure of Calabi-Yau space, a generic holomorphic curve will not be holomorphic with the respect to the new complex structure, and becomes obstructed to the deformation of the bulk moduli. The requirement for the holomorphy gives rise to a relation between the closed and open string moduli. Physically speaking, it turns out that the obstruction generates a superpotential for the effective theory depending on the closed and open string moduli.

The off-shell tension of D-branes, \(T(z, \hat{z})\), is equal to the relative period \([5,6,51]\)

\[\Pi_\Sigma = \int_{\Gamma_\Sigma} \Omega(z, \hat{z})\]  

(2.5)
which measures the difference between the value of on-shell superpotentials for the two D-brane configurations
\[ T(z, \hat{z}) = \mathcal{W}(C^+) - \mathcal{W}(C^-) \tag{2.6} \]
with \( \partial \Gamma_{\Sigma} = C^+ - C^- \). The on-shell domain wall tension is [24]
\[ T(z) = T(z, \hat{z}) \mid_{\hat{z} = \text{critic points}} \tag{2.7} \]
where the critical points correspond to \( \frac{d\mathcal{W}}{dz} = 0 \) [51] and the \( C^\pm \) is the holomorphic curves at those critical points. The critical points are alternatively defined as the Nother-Lefshetz locus [52]
\[ \mathcal{N} = \{(z, \hat{z}) \mid \pi(z, \hat{z}; \partial \Gamma(z, \hat{z})) \equiv 0\} \tag{2.8} \]
where
\[ \pi(z, \hat{z}; \partial \Gamma(z, \hat{z})) = \int_{\partial \Gamma} \omega^{(2,0)}_{\hat{a}}(z, \hat{z}), \quad \hat{a} = 1, \ldots, \dim(H^{2,0}(D)) \tag{2.9} \]
and \( \omega^{(2,0)}_{\hat{a}} \) is an element of the cohomology group \( H^{(2,0)}(D) \). At those critical points, the domain wall tensions are also known as normal function giving the Abel-Jacobi invariants [11, 22, 24, 52, 53]

The relative periods \( \Pi_{\Sigma} \) are related to the variations of mixed Hodge structure on the relative cohomology group \( H^3(X, D) \). Geometrically, \( H^3(X, D) \) can be viewed as the fiber of a complex vector bundle over the deformation space \( \mathcal{M} \). The space \( \mathcal{M} \), in general, can be expressed [18, 24] as fibration \( \hat{\mathcal{M}} \rightarrow \mathcal{M} \rightarrow \mathcal{M}_{CS} \), where the base \( \mathcal{M}_{CS} \) corresponds to the complex structure deformation space of the family of Calabi-Yau threefold \( X \) and the fiber \( \hat{\mathcal{M}} \) is the deformation space of the family of divisor \( D \) specifying the embedding \( i : D(z, \hat{z}) \rightarrow X(z) \), which defines the space \( \Omega(X, D) \) of the relative differential form via the exact sequence
\[ 0 \rightarrow \Omega(X, D) \rightarrow \Omega(X) \rightarrow \Omega(D) \rightarrow 0, \tag{2.10} \]
so the relative cohomology group can be representened as
\[ H^3(X, D) \simeq \ker(H^3(X) \rightarrow H^3(D)) \oplus \coker(H^2(X) \rightarrow H^2(D)). \tag{2.11} \]

The variation of mixed Hodge structure can be expressed as follow [48]
\[
\begin{array}{cccccccc}
(\Omega^3_X, 0) & \delta_x & (\Omega^2_X, 0) & \delta_x & (\Omega^1_X, 0) & \delta_x & (\Omega^0_X, 0) \\
\delta_x & \delta_x & \delta_x & \delta_x & \delta_x & \delta_x & \delta_x \\
(0, \Omega^2_D) & \delta_x, \delta_x & (0, \Omega^1_D) & \delta_x, \delta_x & (0, \Omega^0_D) & 0 \\
\end{array}
\]
which can lead to a system of differential equation for the periods in open-closed mirror symmetry [48]:

\[ \nabla_z \cdot \Pi_\Sigma = (\partial_z - A_z) \cdot \Pi_\Sigma (z, \hat{z}) = 0 \]  

(2.12)

\[ \nabla_{\hat{z}} \cdot \Pi_\Sigma = (\partial_{\hat{z}} - A_{\hat{z}}) \cdot \Pi_\Sigma (z, \hat{z}) = 0 \]  

(2.13)

where \( A \) are Gauss-Manin connection, which can be chosen as flat connection [5, 48]

\[ \begin{bmatrix} \nabla_{z_i} \nabla_{\hat{z}_j} \end{bmatrix} = \begin{bmatrix} \nabla_{z_i}, \nabla_{\hat{z}_j} \end{bmatrix} = \begin{bmatrix} \nabla_{\hat{z}_i}, \nabla_{\hat{z}_j} \end{bmatrix} = 0. \]  

(2.14)

This approach provides a powerful framework to study relative periods and off-shell superpotentials [5, 18, 26].

In A-model interpretation, the superpotential expressed in term of flat coordinates \( (t, \hat{t}) \), which relates to complex structure parameters \( (z, \hat{z}) \) in B-model through mirror map, is the generating function of the Ooguri-Vafa invariants [3, 6, 17, 54]

\[ W(t, \hat{t}) = \sum_{\vec{k}, \vec{m}} G_{\vec{k}, \vec{m}} q^d \hat{q}^d = \sum_{\vec{k}, \vec{m}} \sum_{d} n_{\vec{k}, \vec{m}} q^d \hat{q}^d \frac{k^2}{k^2} \]  

(2.15)

where \( q = e^{2\pi i t}, \hat{q} = e^{2\pi i \hat{t}} \) and \( n_{\vec{k}, \vec{m}} \) are Ooguri-Vafa invariants [54] counting disc instantons in relative homology class \( (\vec{m}, \vec{k}) \), where \( \vec{m} \) represents the elements of \( H_1(D) \) and \( \vec{k} \) represents an element of \( H_2(X) \). \( G_{\vec{k}, \vec{m}} \) are open Gromov-Witten invariants [33–37]. From string world-sheet viewpoint, these terms in the superpotential represents the contribution from instantons of sphere and disk.

### 2.2 Generalized GKZ system and Differential Operators

The generalized hypergeometric systems originated from [40] and have been applied in mirror symmetry [41–45]. The notation is as follows: \( (X^*, X) \) is the mirror pair of compact Calabi-Yau threefold defined as hypersurfaces in toric ambient spaces \( (W^*, W) \), respectively. The generators \( l^a \) of Mori cone of the toric variety [55–58] give rise to the charge vectors of the gauged linear sigma model (GLSM) [59]. \( \triangle \) is a real four dimensional reflexive polyhedron. \( W = P_{\Sigma(\triangle)} \) is the toric variety with fan \( \Sigma(\triangle) \) being the set of cones over the faces of \( \triangle^* \). \( \triangle^* \) is the dual polyhedron and \( W^* \) is the toric variety obtained from \( \Sigma(\triangle^*) \). The enhanced polyhedron \( \triangle^* \) constructed from polyhedron \( \triangle^* \) is associated to \( X^*_4 \) on which the dual F-theory compactify. The threefold \( X \) on B-model side is defined by \( p \) integral points of \( \triangle^* \) as the zero locus of the
polynomial $P$ in the toric ambient space

$$P = \sum_{i=0}^{p-1} a_i \prod_{k=0}^{4} X_k^{\nu_{i,k}}$$

(2.16)

where the $X_k$ are coordinates on an open torus $(\mathbb{C}^*)^4 \in W$ and $a_i$ are complex parameters related to the complex structure of $X$. In terms of homogeneous coordinates $x_j$ on the toric ambient space, it can be rewritten as

$$P = \sum_{i=0}^{p-1} a_i \prod_{\nu \in \Delta} x_j^{(\nu,\nu_{i,k})}+1.$$  

(2.17)

The open-string sector from D-branes can be described by the family of hypersurfaces $\mathcal{D}$, which is defined as intersections $P = 0 = Q(D)$. In toric variety, the $Q(D)$ can be defined as [17, 24]

$$Q(D) = \sum_{i=p}^{p+p'-1} a_i X_k^{\nu_{i,k}}$$

(2.18)

where additional $p'$ vertices $v_i^*$ correspond with the monomials in $Q(D)$.

When considering the dual F-theory compactify on Four-fold $X_4$, the relevant Enhanced polyhedron consists of extended vertices

$$\overline{\mathcal{V}} = \begin{cases} 
  (\nu_{i}^*, 0) & i = 0, \ldots, p - 1 \\
  (\nu_{i}^*, 1) & i = p, \ldots, p + p' - 1.
\end{cases}$$

(2.19)

The period integrals can be written as

$$\Pi_i = \int_{\gamma_i} \frac{1}{P(a, X)} \prod_{j=1}^{n} \frac{dX_j}{X_j}.$$  

(2.20)

According to the refs. [44, 45], the period integrals can be annihilated by differential operators

$$\mathcal{L}(l) = \prod_{l_i>0} (\partial_{u_i})^{l_i} - \prod_{l_i<0} (\partial_{u_i})^{l_i}$$

$$Z_k = \sum_{i=0}^{p-1} \nu_{i,k} v_i, \quad Z_0 = \sum_{i=0}^{p-1} v_i - 1$$

(2.21)
where $\vartheta_i = a_i \partial a_i$. As noted in refs. [41], the equations $Z_k \Pi(a_i) = 0$ reflect the invariance under the torus action, defining torus invariant algebraic coordinates $z_a$ on the moduli space of complex structure of $X$:

$$z_a = (-1)^{l_0} \prod_i a_i^{l_a}$$  \hspace{1cm} (2.22)

where $l_a$, $a = 1, \ldots, h^{2,1}(X)$ is generators of the Mori cone, one can rewrite the differential operators $\mathcal{L}(l)$ as [24, 41, 45]

$$\mathcal{L}(l) = \prod_{k=1}^{l_0} (\vartheta_0 - k) \prod_{l_i > 0}^{l_i-1} (\vartheta_i - k) - (-1)^{l_0} z_a \prod_{k=1}^{-l_0} (\vartheta_0 - k) \prod_{l_i < 0}^{l_i-1} (\vartheta_i - k).$$  \hspace{1cm} (2.23)

The solution to the GKZ system can be written as [24, 41, 45]

$$B_{l^a}(z^a; \rho) = \sum_{n_1, \ldots, n_N \in \mathbb{Z}^+} \frac{\Gamma(1 - \sum_a l_a^a(n_a + \rho_a))}{\prod_{k>0} \Gamma(1 + \sum_a l_a^a(n_a + \rho_a))} \prod_a z_a^{n_a + \rho_a}.$$  \hspace{1cm} (2.24)

In this paper we consider the family of divisors $\mathcal{D}$ with a single open deformation moduli $\hat{z}$

$$x_1^{b_1} + \hat{z} x_2^{b_2} = 0$$  \hspace{1cm} (2.25)

where $b_1, b_2$ are some appropriate integers. However, in [15, 25], they considered another approach which blows up along the curve $C$ and replaces the pair $(X, C)$ with a non-Calabi-Yau manifold $\hat{X}$. The relative 3-form $\Omega := (\Omega_3^X, 0)$ and the relative periods satisfy a set of differential equations [5, 6, 14, 18, 24]

$$\mathcal{L}_a(\theta, \hat{\theta}) \Omega = d\omega^{(2,0)} \Rightarrow \mathcal{L}_a(\theta, \hat{\theta}) \mathcal{T}(z, \hat{z}) = 0.$$  \hspace{1cm} (2.26)

with some corresponding two-form $\omega^{(2,0)}$. The differential operators $\mathcal{L}_a(\theta, \hat{\theta})$ can be expressed as [24]

$$\mathcal{L}_a(\theta, \hat{\theta}) := \mathcal{L}_a^b - \mathcal{L}_a^{bd} \hat{\theta}$$  \hspace{1cm} (2.27)

for $\mathcal{L}_a^b$ acting only on bulk part from closed sector, $\mathcal{L}_a^{bd}$ on boundary part from open-closed sector and $\hat{\theta} = \hat{z} \partial \hat{z}$. The explicit form of these operators will be given in following model. From the (2.9) one can obtain

$$2\pi i \hat{\theta} \mathcal{T}(z, \hat{z}) = \pi(z, \hat{z})$$  \hspace{1cm} (2.28)
for only the family of divisors $\mathcal{D}$ depending on the $\hat{z}$. From above one can obtain differential equation with the inhomogeneous term $f_a(z)$ at the critical points

$$\mathcal{L}_a^b T(z) = f_a(z)$$

and

$$2\pi i f_a(z) = \mathcal{L}_a^b \pi(z, \hat{z})|_{\hat{z}=\text{critic points}}$$

### 3 Superpotentials of Hypersurface $X_{24}(1,1,2,8,12)$

The $X_{24}(1,1,2,8,12)$ is defined as a degree 24 hypersurface in the ambient toric variety $W = P_{\Sigma(\Delta)}$ with the vertices of the polyhedron $\Delta$

$$
\nu_1 = (1,1,1,1), \quad \nu_2 = (-23,1,1,1), \quad \nu_3 = (1,-11,1,1), \quad \nu_4 = (1,1,-2,1)
\nu_5 = (1,1,1,-1)
$$

(3.1)

The vertices of dual polyhedra $\Delta^*$ are

$$
\nu_1^* = (1,2,8,12), \quad \nu_2^* = (-1,0,0,0), \quad \nu_3^* = (0,-1,0,0), \quad \nu_4^* = (0,0,-1,0)
\nu_5^* = (0,0,0,-1), \quad \nu_6^* = (0,1,4,6), \quad \nu_7^* = (0,0,2,3).
$$

(3.2)

When considering the four-fold on which the dual F-theory compactification, the extend vertices of the enhanced polyhedron $\Delta^* \supset \Delta^*$ can be constructed according to (2.19) with two extra points as follows

$$
\nu_i^* = (\nu_i^*; 0), \quad \nu_8^* = (\nu_1^*; 1), \quad \nu_9^* = (\nu_2^*; 1), \quad i = 0, ..., 7.
$$

(3.3)

The toric hypersurface, according to (2.17), is the zero locus of polynomial $P$

$$
P = a_1 x_1^{24} + a_2 x_2^{24} + a_3 x_3^{12} + a_4 x_4^3 + a_5 x_5^2 + a_6 x_1 x_2 x_3 x_4 x_5 + a_7 x_1^{12} x_2^{12} + a_8 x_1^6 x_2^6 x_3^6
\equiv x_1^{24} + x_2^{24} + x_3^{12} + x_4^3 + x_5^2 + \psi x_1 x_2 x_3 x_4 x_5 + \phi x_1^6 x_2^6 x_3^6 + \chi x_1^{12} x_2^{12}
$$

(3.4)

where the second equation is rescaled and expressed by $\psi = z_1^{-\frac{1}{3}} z_2^{-\frac{1}{3}} z_3^{-\frac{1}{3}}, \phi = z_2^{-\frac{1}{3}} z_3^{-\frac{1}{3}}$ and $\chi = z_2^{-\frac{1}{2}}$ in terms of torus invariant algebraic coordinates (2.22). The GLSM charge vectors $l_a$ are the generators of the Mori cone as follows [41]

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $l_1$ | -6 | 0 | 0 | 0 | 2 | 3 | 0 | 1 |
| $l_2$ | 0 | 1 | 1 | 0 | 0 | 0 | -2 | 0 |
| $l_3$ | 0 | 0 | 0 | 1 | 0 | 0 | 1 | -2 |

(3.5)
The mirror manifolds can be constructed as an orbifold by the Greene-Plesser orbifold group acting as $x_i \rightarrow \lambda^{g_{k,i}} x_i$ with weights

$$Z_6 : \ g_1 = (1, -1, 0, 0, 0), \ Z_6 : \ g_2 = (1, 0, -1, 0, 0), \ Z_3 : \ g_3 = (1, 0, 0, -1, 0) \quad (3.6)$$

where we denotes $\lambda^6_{1,2} = 1$ and $\lambda^3_3 = 1$. We consider the following curves

$$C_{\alpha, \pm} = \{x_2 = \xi x_1, x_3 = 0, x_4^3 = -x_5^2 - \xi^{12}\chi x_1^{24}\}, \ \xi^{24} = -1 \quad (3.7)$$

which are on the family of divisor

$$Q(D) = x_2^{24} + \hat{z} x_1^{24} \quad (3.8)$$

at the critical points $\hat{z} = 1$. The surface defined by the intersection $P = 0 = Q(D)$ is a K3 surface with the equation:

$$P_D = (x'_2)^{12} + x'_3^{12} + x'_4^3 + x'_5^2 + \psi' x'_2 x'_3 x'_4 x'_5 + \phi'(x'_2)^6 x'_3 + \chi'(x'_2)^{12} = 0 \quad (3.9)$$

where $x'_2 = x_2^2$, $\psi' = u_1^{-\frac{1}{4}}u_2^{-\frac{1}{24}}u_3^{-\frac{1}{12}}$, $\phi' = u_2^{-\frac{1}{4}}u_3^{-\frac{1}{2}}$ and $\chi' = u_2^{-\frac{1}{2}}$ are expressed in terms of new parameters as

$$u_1 = z_1 \quad u_2 = \frac{-z_2}{\hat{z}} (1 - \hat{z})^2 \quad u_3 = z_3 \quad (3.10)$$

The GLSM charge vectors for this K3 manifold are

| \hat{l}_1 | 0 | 1 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| \hat{l}_1 | -6 | 0 | 0 | 2 | 3 | 0 | 1 |
| \hat{l}_2 | 0 | 2 | 0 | 0 | 0 | -2 | 0 |
| \hat{l}_3 | 0 | 0 | 1 | 0 | 0 | 1 | -2 |

(3.11)

By the generalized GKZ system, the period on the K3 surface has the form

$$\pi = \frac{c}{2} B \{l_1, i_2, i_3\} (u_1, u_2, u_3; \frac{1}{2}, \frac{1}{2}, 0) = -\frac{4c}{\pi i} \sqrt{u_1 u_2 u_3} + \mathcal{O}((u_1 u_2)^{3/2}) \quad (3.12)$$

which vanishes at the critical locus $u_2 = 0$. According to (2.28), the off-shell superpotentials can be obtained by integrating the $\pi$:

$$\mathcal{T}^\pm_{\alpha}(z_1, z_2, z_3) = \frac{1}{2\pi i} \int \pi(\hat{z}) \frac{d\hat{z}}{\hat{z}} \quad (3.13)$$

with the appropriate integral constants [24], the superpotentials can be chosen as $W^+ = -W^-$. In this convention, the on-shell superpotentials can be obtained as

$$2W^+ = \frac{1}{2\pi i} \int_{-\hat{z}}^{\hat{z}} \pi(\zeta) \frac{d\zeta}{\zeta}, \quad W^\pm(z_1, z_2, z_3) = W^\pm(z_1, z_2, z_3)|_{\hat{z}=1} \quad (3.14)$$
Eventually, the superpotential is

\[ W^\pm(z_1, z_2, z_3, \hat{z}) = \sum_{n_1, n_2, n_3} \frac{\mp c z_1^{\frac{1}{2} + n_1} z_2^{\frac{1}{2} + n_2} z_3^{n_3} \hat{z}^{-\frac{1}{2} - 2n_2}}{\Gamma(2 + 2n_2)\Gamma(2 + 2n_1)\Gamma(\frac{5}{2} + 3n_1)\Gamma(6n_1 + 4)} \{ (1 - 2n_2) F_1(\frac{1}{2} - n_2, -2n_2, \frac{1}{2} - n_2; \hat{z}) + \hat{z}(1 + 2n_2) F_1(\frac{1}{2} - n_2, -2n_2, \frac{3}{2} - n_2; \hat{z}) \} \]

where the \( f(z_1, z_2, z_3, \hat{z}) \) are related to the open-string parameter and vanish in the critical point, \( W^\pm \) are the on-shell superpotential as follows

\[ W^\pm = \mp \frac{c}{8} B_{l_1, l_2, l_3} (\{ z_1, z_2, z_3 \}; \frac{1}{2}, \frac{1}{2}, 0) \]

substituting the vector \( l_1, l_2, l_3 \) in this hypersurface, the on-shell superpotentials are

\[ W^\pm = \frac{c}{8} \sum_{n_1, n_2, n_3} \frac{\mp z_1^{\frac{1}{2} + n_1} z_2^{\frac{1}{2} + n_2} z_3^{n_3} \Gamma(6(n_1 + \frac{1}{2}) + 1)}{\Gamma(n_2 + \frac{1}{2} + 1)\Gamma(n_3 + 1)\Gamma(2(n_1 + \frac{1}{2}) + 1)\Gamma(3(n_1 + \frac{1}{2}) + 1)} \frac{1}{\Gamma(-2n_3 + n_1 + \frac{1}{2}) + 1)\Gamma(-2n_2 + n_3)\}

The additional GLSM charge vectors corresponding to the divisor (3.8) are

\[
\begin{array}{cccccccccc}
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
l_4 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{array}
\]

The classic A-brane in the mirror Calabi-Yau manifold \( X^* \) of \( X \) determined by the additional charge vectors \( (0, -1, 1, 0, 0, 0, 0) \) is a special Lagrangian submanifold of \( X^* \) defined as \( [3, 4, 8, 9, 17] \)

\[ -|x_1|^2 + |x_2|^2 = \eta \]

where \( x_i \) are coordinates on \( X^* \), \( \eta \) is a Kähler moduli parameter with \( \hat{z} = \epsilon e^{-\eta} \) for a phase \( \epsilon \).

The differential operators, according to (2.21), can be represented as

\[
\begin{align*}
\mathcal{L}_1 &= \theta_1(\theta_1 - 2\theta_3) - 12z_1(6\theta_1 + 5)(6\theta_1 + 1) \\
\mathcal{L}_2 &= (\theta_2 - \theta_4)(\theta_2 + \theta_4) + z_2(2\theta_2 - \theta_3 + 1)(2\theta_2 - \theta_3) \\
\mathcal{L}_3 &= \theta_3(-2\theta_2 + \theta_3) + z_3(2\theta_3 - \theta_1)(2\theta_3 - \theta_1 + 1) \\
\mathcal{L}_4 &= \theta_4(\theta_2 + \theta_4) + z_4\theta_4(\theta_4 - \theta_2)
\end{align*}
\]
where $\theta_1 = \frac{d}{dz_1}$, $i = 1, \ldots, 3$ and $\theta_4 = \frac{d}{dz_2}$. From the argument in sec. 2, one obtains

\[ L_1 = L^b_1 \Rightarrow L^b_1 = \theta_1(\theta_1 - 2\theta_3) - 12z_1(6\theta_1 + 5)(6\theta_1 + 1) \]

\[ L_2 = L^b_2 - \theta_2\theta_4^2 \Rightarrow L^b_2 = \theta_2^2 + z_2(2\theta_2 - \theta_3 + 1)(2\theta_2 - \theta_3) \]

\[ L_3 = L^b_3 \Rightarrow L^b_3 = \theta_3(-2\theta_2 + \theta_3) + z_3(2\theta_3 - \theta_1)(2\theta_3 - \theta_1 + 1). \]

Hence the inhomogeneous terms, by acting with $\theta_2\theta_4$ on the (4.12), are

\[ L^b_2 W^\pm_1 = \pm \frac{c}{2\pi^2} \sqrt{z_1z_2z_3} \]  

(3.23)

For calculation of instanton corrections, one need to know mirror map. The closed-string periods are \[41\]

\[ \Pi(z) = \left( \begin{array}{c} \omega_0(z, \rho)|_{\rho=0} \\ D^{(1)}_i|_{\rho=0} \\ D^{(2)}_i|_{\rho=0} \\ D^{(3)}_i|_{\rho=0} \end{array} \right). \]

(3.24)

where $i = 1, \ldots, h_{21}(X^*)$,

\[ w_0 = \sum c(n_i + \rho_i)z^{n_i+\rho_i} \quad i = 1, 2, 3 \]

(3.25)

\[ c(n_i + \rho_i) = \frac{\Gamma(\sum_{k=1}^3 l_k^i n_k + \rho_k) + 1}{\Pi_{i=1}^3 \Gamma(\sum_{k=1}^3 l_k^i n_k + \rho_k) + 1} \]

(3.26)

and

\[ D^{(1)}_i := \partial_{\rho_i}, \quad D^{(2)}_i := \frac{1}{2} \kappa_{ijk} \partial_{\rho_j} \partial_{\rho_k}, \quad D^{(3)}_i := -\frac{1}{6} \kappa_{ijk} \partial_{\rho_j} \partial_{\rho_k} \]

(3.27)

$\kappa_{ijk}$ is intersection number of $X$.

The flat coordinates in A-model at large radius regime are related to the flat coordinates of B-model at large complex structure regime by mirror map $t_i = \frac{w_i}{w_0}$, $\omega_i := D^{(1)}_i \omega_0(z, \rho)|_{\rho=0}$

\[ 2\pi it_1 = \log(z_1) + 312z_1 + 58932z_1^2 - (1 - 120z_1)z_3 - \frac{3}{2} z_3^2 + \mathcal{O}(z_3^3) \]

\[ 2\pi it_2 = \log(z_2) + 2z_2 + 3z_2^2 + \mathcal{O}(z_3^3) \]

\[ 2\pi it_3 = \log(z_3) + 2z_3 + 3z_3^2 + (120 - 240z_3)z_1 + 34380z_1^2 - \frac{3}{2} z_1^2 - z_2 + \mathcal{O}(z_3^3) \]

(3.28)

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Table 1: Ooguri-Vafa invariants $n_{(d_1,d_2,d_3)}$ for the on-shell superpotential $W_1^+$ on the 3-fold $\mathbb{P}_{1,1,2,8,12}[24]$. The horizontal coordinates represent $d_2$ and vertical coordinates represent $d_1$.

\[
\begin{array}{|c|cccccc|}
\hline
 d_3 & 1 & 3 & 5 & 7 & 9 \\
\hline
 d_1 \backslash d_2 &  &  &  &  &  \\
1 & 8 & 0 & 0 & 0 & 0 \\
3 & -6784 & 0 & 0 & 0 & 0 \\
5 & -2167824 & -82080 & 82080 & -82080 & 82080 \\
7 & -40065280 & 66963200 & -66963200 & 66963200 & -66963200 \\
9 & -9901094392 & 25006400960 & -25006400960 & 25006400960 & -25006400960 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|cccccc|}
\hline
 d_3 & 1 & 3 & 5 & 7 & 9 \\
\hline
 d_1 \backslash d_2 &  &  &  &  &  \\
1 & 3 & -3 & 0 & 0 & 0 \\
3 & -2104 & -2104 & -4800 & 4800 & -4800 \\
5 & 556278 & 1364046 & 1933440 & -7477440 & -62400 \\
7 & -1119704976 & -649533072 & 3072437120 & 1611134080 & 4782650240 \\
9 & -563398004013 & 1412362672931 & 646095703280 & 879759458320 & 1211151714800 \\
\hline
\end{array}
\]

and we can obtain the inverse mirror map in terms of $q_i = e^{2\pi it_i}$

\[
\begin{align*}
 z_1 &= q_1 - 312q_1^2 + 87084q_1^3 + q_1q_3 - 864q_1^2q_3 + q_1q_2q_3 + \mathcal{O}(q^4) \\
 z_2 &= q_2 - 2q_2^2 + 3q_3^3 + \mathcal{O}(q^4) \\
 z_3 &= q_3 - 2q_3^2 + 3q_3^3 - 120q_1q_3 + 10260q_1^2q_3 + q_2q_3 - 120q_1q_2q_3 + 600q_1q_3^2 - 4q_2q_3^2 + \mathcal{O}(q^4). 
\end{align*}
\]

Using the modified multi-cover formula [10,23] for this case

\[
W^\pm(z(q)) = \frac{1}{(2i\pi)^2} \sum_{k \text{ odd}} \sum_{d_3,d_1,d_2 \geq 0} n^\pm_{d_1,d_2,d_3} \frac{q_1^{kd_1/2} q_2^{kd_2/2} q_3^{kd_3}}{k^2},
\]

the superpotentials $W^\pm$, at the critical points $\dot{z} = 1$, give Ooguri-Vafa invariants $n_{d_1,d_2,d_3}$ for the normalization constants $c = 1$, which are listed in Table 1. The three integers $(d_1, d_2, d_3)$ denote homology class.
4 Superpotential of Hypersurface $X_{12}(1, 1, 1, 3, 6)$

The $X_{12}(1, 1, 1, 3, 6)$ is defined as a degree 12 hypersurface in the ambient toric variety $W = P_{\Sigma(\Delta)}$ with the vertices of the polyhedron $\Delta$

$$\nu_1 = (1, 1, 1, 1), \; \nu_2 = (-11, 1, 1, 1), \; \nu_3 = (1, -11, 1, 1), \; \nu_4 = (1, 1, -3, 1)$$
$$\nu_5 = (1, 1, 1, -1)$$

(4.1)

The vertices of the dual vertices $\Delta^*$ are

$$\nu_1^* = (1, 1, 3, 6), \; \nu_2^* = (-1, 0, 0, 0), \; \nu_3^* = (0, -1, 0, 0), \; \nu_4^* = (0, 0, -1, 0)$$
$$\nu_5^* = (0, 0, 0, -1), \; \nu_6^* = (2, 1, 0, 0).$$

(4.2)

When considering the four-fold on which the dual F-theory compactification, the extend vertices of the enhanced polyhedron $\Delta^* \supset \Delta^*$ can be constructed according to (2.19) with two extra points as follows

$$\nu_i^* = (\nu_i^*; 0), \; \nu_i^* = (\nu_i^*; 1), \; \nu_i^* = (\nu_i^*; 1), \; i = 1, ..., 6.$$ (4.3)

This threefold has three complex parameters, but only two moduli can be represented as monomial deformation [41].

The toric hypersurface is defined as zero locus of $P$:

$$P = x_1^{12} + x_2^{12} + x_3^{12} + x_4^2 + x_5^2 + \psi x_1 x_2 x_3 x_4 x_5 + \phi x_1^4 x_2^4 x_3^4$$

(4.4)

where $\psi = z_1^{-1} z_2^{-1}$, $\phi = z_2^{-1}$. The GLSM charge vectors in this case are [41]

| $l_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|------|---|---|---|---|---|---|---|
| $l_2$ | -4 | 0 | 0 | 0 | 1 | 2 | 1 |

(4.5)

On the mirror manifolds, the Greene-Plesser orbifold group acts as $x_i \rightarrow \lambda_k^{g_{k,i}} x_i$ with weights

$$Z_6 : \; g_1 = (1, -1, 0, 0, 0), \; \; Z_4 : \; g_2 = (0, 1, 2, 1, 0)$$

(4.6)

where we denotes $\lambda_1^6 = 1, \; \lambda_2^4 = 1$. We consider those curves

$$C_{\alpha, \pm} = \{x_2 = \xi_1 x_1, x_5 = \xi_2 x_3^2 - \frac{\psi}{2} x_1 x_2 x_3 x_4, x_4 = \psi^2 \alpha (x_1 x_2 x_3 x_4)^2\}$$
$$\xi_1^{12} = \xi_2^2 = -1, \; \; \alpha^3 - \frac{1}{4} \alpha^2 + \frac{\phi}{\psi^6} = 0.$$ (4.7)
here with the identifications for different choice of \((\xi_1, \xi_2, \alpha)\) but for \(\xi_1^6 = \pm i\) for fixed \((\xi_2, \alpha)\).

The family of divisor we calculated is

\[
Q(D) = x_2^{12} + \hat{z} x_1^{12}
\]  
(4.8)

where the critical points is at \(\hat{z} = 1\). Solving the intersection \(P = 0 = Q(D)\) one obtains

\[
P_D = (x_2')^{12} + x_3^{12} + x_4^4 + x_5^2 + \psi' x_2' x_3 x_4 x_5 + \phi'(x_2')^4 x_3^4
\]
(4.9)

where \(x_2' = x_2^2\), \(\psi' = u_1^{-\frac{1}{2}} u_2^{-\frac{1}{12}}\) and \(\phi' = u_2^{-\frac{1}{3}}\) are expressed in terms of new parameters as

\[
u_1 = z_1 \quad u_2 = -\frac{z_2}{\hat{z}}(1 - \hat{z})^2.
\]
(4.10)

The family of divisor \(D\) as a \(K3\) surface associated with GLSM charge vectors

\[
\begin{array}{c|cccccc}
\hat{l}_1 & 0 & 2 & 3 & 4 & 5 & 6 \\
\hat{l}_2 & -4 & 0 & 0 & 1 & 2 & 1 \\
\end{array}
\]
(4.11)

has two algebraic moduli.

Besides the regular solutions the period on this K3 surface has two extra forms

\[
\begin{align*}
\pi_1(u_1, u_2) &= \frac{c_1}{2} B_{i, j_1}(u_1, u_2; 0, \frac{1}{2}) \\
\pi_2(u_1, u_2) &= \frac{c_2}{2} B_{i, j_2}(u_1, u_2; \frac{1}{2}, \frac{1}{2})
\end{align*}
\]
(4.12)

where \(c_{1,2}\) are some normalization constants not determined by the differential operator.

The additional GLSM charge vectors corresponding to the divisor are

\[
\begin{array}{c|cccccccc}
l_3 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]
(4.13)

The classic A-brane in the mirror Calabi-Yau manifold \(X^*\) of \(X\) determined by the additional charge vectors \((0, -1, 1, 0, 0, 0)\) is a special Lagrangian submanifold of \(X^*\) defined as \([3, 4, 8, 9, 17]\)

\[-|x_1|^2 + |x_2|^2 = \eta\]
(4.14)

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where \(x_i\) are coordinates on \(X^*\), \(\eta\) is a Kähler moduli parameter with \(\tilde{z} = e^{\eta}\) for a phase \(\epsilon\).

The differential operators, according to (2.21), can be obtained as
\[
\mathcal{L}_1 = \theta_1(\theta_1 - 3\theta_2) - 4z_1(4\theta_1 + 3)(4\theta_1 + 1) \\
\mathcal{L}_2 = \theta_2(\theta_2 - \theta_3)(\theta_2 + \theta_3) + z_2(3\theta_2 - \theta_1)(3\theta_2 - \theta_1 + 1)(3\theta_2 - \theta_1 + 2) \\
\mathcal{L}_3 = \theta_3(\theta_2 + \theta_3) + z_3\theta_3(\theta_2 - \theta_3)
\]
where \(\theta_i = z_i \frac{d}{dz_i}, \ i = 1,\ldots, 3\) and \(\theta_4 = \tilde{z} \frac{d}{dz}\).

Furthermore one has
\[
\mathcal{L}_1 = \mathcal{L}_1^b \Rightarrow \mathcal{L}_1^b = \theta_1(\theta_1 - 3\theta_2) - 4z_1(4\theta_1 + 3)(4\theta_1 + 1) \\
\mathcal{L}_2 = \mathcal{L}_2^b - \theta_2\theta_4^2 \Rightarrow \mathcal{L}_2^b = \theta_2^2 + z_2(3\theta_2 - \theta_1)(3\theta_2 - \theta_1 + 1)(3\theta_2 - \theta_1 + 2).
\]

Following the section 3, The on-shell superpotentials \(W^\pm\) for this manifold can be obtained as
\[
W_\alpha^\pm (z_1, z_2) = W_\alpha^\pm |_{z=1}.
\]

The explicit form of on-shell superpotentials are
\[
W_1^\pm = \mp \frac{c_1}{8} \prod_{n_1=0, n_2=0} \frac{\Gamma(1 + 4n_1)z_1^{n_1}z_2^{n_2+\frac{1}{2}}}{\Gamma^3(n_2 + \frac{3}{2})\Gamma(n_1 + 1)\Gamma(2n_1 + 1)\Gamma(n_1 - 3n_2 - \frac{1}{2})} \\
W_2^\pm = \mp \frac{c_2}{8} \prod_{n_1=0, n_2=0} \frac{\Gamma(3 + 4n_1)z_1^{n_1+\frac{1}{2}}z_2^{n_2+\frac{1}{2}}}{\Gamma^3(n_2 + \frac{3}{2})\Gamma(n_1 + \frac{3}{2})\Gamma(2n_1 + 2)\Gamma(n_1 - 3n_2)}.
\]

The flat coordinates \(t_i\) in A-model sides for this case are
\[
2\pi i t_1 = \log(z_1) + 40z_1 + 1076z_1^2 + (2 - 36z_1)z_2 - 15z_2^2 + O(z^3) \\
2\pi i t_2 = \log(z_2) - 6z_2 + 45z_2^2 + 36z_1 + 1458z_1^2 + 108z_1z_2O(z^4),
\]
and the inverse mirror map in terms of \(q_i = e^{2\pi i t_i(z)}\) are
\[
z_1 = q_1 - 40q_1^2 + 1324q_1^3 - 2q_1q_2 + 268q_1^2q_2 + 5q_1q_2^2 + O(q^4) \\
z_2 = q_2 + 6q_2^2 + 9q_2^3 - 36q_1q_2 - 468q_1q_2^2 + 630q_2^2q_2 + O(q^4).
\]

Using the modified multi-cover formula \([10,23]\) for this case
\[
\frac{W^\pm(z(q))}{w_0(z(q))} = \left(\frac{1}{2\pi i}\right)^2 \sum_{k \ odd} \sum_{d_2 \ odd, d_1 \geq 0} n_{d_1, d_2}^{k} q_1^{k_{d_1}} q_2^{k_{d_2}/2} k^2,
\]
the on-shell superpotentials \(W^\pm\), at the critical point \(\tilde{z} = 1\), give Ooguri-Vafa invariants \(n_{d_1, d_2}\) for the normalization constants \(c_1 = c_2 = 1\), which are listed in Table. 2 and 3.
Table 2: Ooguri-Vafa invariants $\frac{1}{2} n_{(d_1,d_2)}$ for the on-shell superpotential $W_1^+$ on the three-fold $\mathbb{P}_{1,1,1,3,6}[12]$. The horizontal coordinates represent $d_2$ and vertical coordinates represent $d_1$.

| $d_1 \backslash d_2$ | 1   | 3   | 5   | 7   | 9   |
|---------------------|-----|-----|-----|-----|-----|
| 0                   | 1   | -1  | 5   | -42 | 429 |
| 1                   | -54 | 54  | -486| 5454| -116316 |
| 2                   | -1107| 17199| -293463| 7513614 |
| 3                   | -10686| 20088| 8520336| -14892288| 65638798 |
| 4                   | -71496| 43155864| -1157860257| 12330791559| -150518794344 |

Table 3: Ooguri-Vafa invariants $\frac{1}{2} n_{(d_1,d_2)}$ for the on-shell superpotential $W_2^+$ on the three-fold $\mathbb{P}_{1,1,1,3,6}[12]$. The horizontal coordinates represent $d_2$ and vertical coordinates represent $d_1$.

| $d_1 \backslash d_2$ | 1   | 3   | 5   | 7   | 9   |
|---------------------|-----|-----|-----|-----|-----|
| 1                   | 0   | 0   | 0   | 0   | 0   |
| 3                   | -320| 0   | 0   | 0   | 0   |
| 5                   | -3456| 6912| -17280| 110592| 13768704 |
| 7                   | -29376| -255744| 1081728| -10596096| 287378496 |
| 9                   | -166528| 599128640| 8281234560| 865215488| -258928821888 |

5 Summary

In this paper, we constructed the generalized hypergeometric GKZ systems for two Calabi-Yau manifolds with three parameters and D-brane wrapped on a divisor with single open-string moduli. Furthermore, we calculate the D-brane superpotentials which give rise to the flux superpotential $W_{GVW}$ of the dual F-theory compactify on the relevant Calabi-Yau fourfold in the limit of $g_s \to 0$. The superpotentials are depend on bulk and open deformation moduli. By mirror symmetry, we also compute the Ooguri-Vafa invariants from A-model expansion.

The generalized hypergeometric GKZ system are closely related to the variation of mixed Hodge structure on relative cohomology group $H^3(X,Z)$. The cohomology group $H^3(X,Z)$ can be viewed as the fiber of vector bundle over the deformation space $\mathcal{M}$. Similar to closed-string, there is a flatness and integrability of the Gauss-Manin
connection which provided a powerful approach to study the geometry of B-model and compute the superpotentials. The connection in flat coordinates displays, in fact, an quantum ring structure and predictions of corrections of the disc instantons.

For the dual F-theory superpotential \([19, 20, 24, 39]\), which also are solutions of the generalized GKZ system, is not only related to the D-brane superpotential for Type II compactification, but also to superpotential for heterotic theory compactification \([16, 60]\). In type II/F-theory compactification, the vacuum structure is determined by the superpotentials, whose second derivative gives the chiral ring structure. The quantum cohomology ring structure comes from the world-sheet instanton corrections and space-time instanton corrections \([5, 6]\). In fact, the more general vacuum structure of type II/F-theory/heterotic theory compactification can be tackled in Hodge variance approach.

We will study the extremal transition and monodromy problems as well as D-brane in general case. We also try to calculate the D-brane superpotential with the method of \(A_\infty\) structure of the derived category \(D_{\text{coh}}(X)\) and path algebras of quivers.

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