Two existence results between an affine resolvable SRGD design
and a difference scheme

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(Received June 30, 2017)
(Revised June 7, 2018)

Abstract. The existence of affine resolvable block designs has been discussed since
1942 in the literature (cf. Bose (1942), Clatworthy (1973), Raghavarao (1988)). Kadowaki
and Kageyama (2009, 2010, 2012) obtained a number of results on combinatorics
for the existence of an affine resolvable SRGD design. In this paper, a new existence
result is shown as a generalization of Theorem 3.3.3 given in Kadowaki and Kageyama
(2009, 2010). Furthermore, another existence result is shown as a conditional converse
of Theorem 3.3.3 and also a generalization of Theorem 3.3.4, both theorems given in
Kadowaki and Kageyama (2009, 2010).

1. Introduction

A block design $BD(v, b, r, k)$ with $v$ points is said to be resolvable if the $b$
blocks of size $k$ each can be grouped into $r$ resolution sets of $b/r$ blocks each
such that in each resolution set every point occurs exactly once. A resolvable
BD is said to be affine resolvable if every two blocks belonging to different
resolution sets intersect in the same number, say $q$, of points. It is known that
for an affine resolvable BD $BD(v, b, r, k)$, $q = k^2/v$ holds.

A $BD(v, b, r, k)$ is called a group divisible (GD) design with parameters
$v = mn, b, r, k, \lambda_1, \lambda_2$ if the $mn$ points are divided into $m$ groups of $n$
points each such that any two points in the same group occur together in exactly
$\lambda_1$ blocks, whereas any two points from different groups occur together in
exactly $\lambda_2$ blocks. The GD designs are further classified into three subclasses:
Singular if $r - \lambda_1 = 0$; Semi-Regular (SR) if $r - \lambda_1 > 0$ and $rk - \nu \lambda_2 = 0$;
Regular if $r - \lambda_1 > 0$ and $rk - \nu \lambda_2 > 0$.

Furthermore, a special type of a difference scheme is utilized. An $sx \times sx$
matrix $A$ with entries from an abelian group $S$ of order $s(\geq 2)$ is called a
difference scheme, denoted by $DS(sx, s; x)$, if in a vector difference on any two
columns of $A$ every entry of $S$ occurs $x$ times. $DS(sx, s; x)$ is also called a

2010 Mathematics Subject Classification. 05B05, 62K10, 51E05.

Key words and phrases. Affine $\lambda$-resolvability, $\lambda$-resolvability, affine plane, difference scheme, BIB
design, PBIB design, GD design.
generalized Hadamard matrix, usually denoted by $GH(s, x)$, or a difference matrix, usually denoted by $D(m, m, s)$ in literature. It is seen that (i) all entries in the first row and first column of a $DS(sx, s; x)$ can be set 0, and further (ii) in each of columns except for the first, every entry of $S$ occurs $x$ times. Furthermore, the following properties can be derived.

(iii) In each of rows except for the first one of the $DS(sx, s; x)$, every entry of $S$ occurs $x$ times.

(iv) In a vector difference on any two rows of a $DS(sx, s; x)$, every entry of $S$ occurs $x$ times.

It is clear that a $DS(2x, 2; x)$ exists if a Hadamard matrix of order $2x$ exists. The following results are also available.

**Theorem 1** (Theorem 3.3.3 corrected in [3]). For a prime $s$, the existence of a $DS(sx, s; x)$ implies the existence of an affine resolvable SRGD design with parameters $v = b = sx^2$, $r = k = sx$, $\lambda_1 = 0$, $\lambda_2 = x$, $q = x$; $m = sx$, $n = s$ for $s \geq 2$.

**Theorem 2** (Theorem 3.3.4 in [3]). The existence of a Hadamard matrix of order $2x$ is equivalent to the existence of an affine resolvable SRGD design with parameters $v = b = 4x$, $r = k = 2x$, $\lambda_1 = 0$, $\lambda_2 = x$, $q = x$; $m = 2x$, $n = 2$.

In this paper, we derive a new existence result which provides a generalization of Theorem 1. Furthermore, we show another existence result which reveals a conditional converse of Theorem 1 and also a generalization of Theorem 2.

2. Statement

The following result will be shown as a generalization of Theorem 1.

**Theorem 3.** Let $s$ be a prime or a prime power. Then the existence of a $DS(sx, s; x)$ implies the existence of an affine resolvable SRGD design with parameters $v = b = sx^2$, $r = k = sx$, $\lambda_1 = 0$, $\lambda_2 = x$, $q = x$; $m = sx$, $n = s$.

Before the proof of Theorem 3, some preliminaries are made. Let $s = p^n$, where $p$ is a prime and $n$ is a positive integer, and $S = \{x_0, x_1, \ldots, x_{s-1}\}$. Consider $\pi I_p$ with a row-permutation $\pi$ and the identity matrix $I_p$ of order $p$. Also take the following $s \times s$ matrix as

$$\pi^{L_i} I_s = (\pi^{a_0} I_p) \otimes (\pi^{a_1} I_p) \otimes \cdots \otimes (\pi^{a_i} I_p),$$

where $L_i = a_{i_0} + a_{i_1} x + \cdots + a_{i_{n-1}} x^{n-1}$ for $a_{i_0}, a_{i_1}, \ldots, a_{i_{n-1}} \in \mathbb{Z}_p$, $i = 0, 1, \ldots, s-1$, $\otimes$ denotes the Kronecker product of matrices, and also $L_i$’s constitute $GF(s)$ ($= S$, say).
An illustration of Theorem 3 is given for $s = 4 = 2^2$ ($p = n = 2$) and $x = 1$, i.e., $S = GF(4) = \{0, 1, x, 1 + x\}$ with $x^2 = 1 + x$.

Consider a $DS(4, 2^2; 1)$ given by, for example,

$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & x & 1 + x \\
0 & 1 + x & 1 & x \\
0 & x & 1 + x & 1
\end{bmatrix}.
$$

Then take the following four matrices as

$$
\pi^{L_0} I_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \pi^{L_1} I_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
$$

$$
\pi^{L_0 x} I_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \pi^{L_1 + x} I_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
$$

with $L_0 = 0 + 0 \cdot x$, $L_1 = 1 + 0 \cdot x$, $L_x = 0 + 1 \cdot x$ and $L_{1+x} = 1 + 1 \cdot x$. By replacing elements $0, 1, x, 1 + x (\in S)$ in the above $DS(4, 2^2; 1)$ with $\pi^{L_0} I_4$, $\pi^{L_1} I_4$, $\pi^{L_0 x} I_4$, $\pi^{L_1 + x} I_4$, respectively, we get the following $16 \times 16$ matrix $D$, which can be checked to be the usual incidence matrix of an affine resolvable SRGD design with parameters $v = b = 16$, $r = k = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$, $q = 1$; $m = n = 4$:

$$
D = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}.
$$
This illustrates Theorem 3 for \( s = 4 \) and \( x = 1 \). The illustration can be generalized as the following proof shows.

**Proof.** By replacing elements \( a_0, a_1, \ldots, a_{s-1} \) (\( \in S \)) in the \( s \times s \) matrix as the existent \( DS(sx, s; x) \) with \( \pi^{L_0} I_s, \pi^{L_1} I_s, \ldots, \pi^{L_{s-1}} I_s \), respectively, we get an \( xs^2 \times xs^2 \) matrix \( D \). Now it will be shown that the matrix \( D \) itself is the incidence matrix of the required affine resolvable SRGD design. In fact, \( v = b = xs^2 \) is obvious and the resolvability is introduced as usual. The other design parameters can be obtained as follows. At first a GD association scheme of \( xs^2 \) points is here given by the \( sx^2 \) array as

\[
\begin{bmatrix}
1 & 2 & \cdots & s \\
1 & 2 & \cdots & 2s \\
\vdots & \vdots & \ddots & \vdots \\
(s(xs - 1) + 1 & s(xs - 1) + 2 & \cdots & xs^2
\end{bmatrix}
\]

Here let for each column \( m = sx \) (i.e., the number of groups in the GD association scheme) and for each row \( n = s \) (i.e., the number of points in each group in the GD association scheme). Since there is exactly one ‘1’ in every row of the matrix \( \pi^{L_i} I_s \), it is clear that \( r = sx \). Similarly, since there is exactly one ‘1’ in every column of the matrix \( \pi^{L_i} I_s \), it is seen that \( k = sx \) and \( \lambda_1 = 0 \). Furthermore it follows that \( \lambda_2 = x \), because each element of \( S \) in row vector differences of \( DS(sx, s; x) \) occurs \( x \) times and on the matrices \( \pi^{L_0} I_s, \pi^{L_1} I_s, \ldots, \pi^{L_{s-1}} I_s \), by definition, the \( \{L_i\} \) coincides with \( GF(s) = S \). Similarly, it can be seen that \( q = x \) (showing the affine resolvability).

Next, we consider a converse of Theorem 1 under some assumption.

By the definition, an affine resolvable SRGD design with parameters \( v = b = xs^2 \), \( r = k = sx \), \( \lambda_1 = 0 \), \( \lambda_2 = x \), \( q = x \); \( m = sx \), \( n = s \) has \( m(= sx) \) groups in the GD association scheme and \( r(= sx) \) resolution sets of \( s \) blocks each. In the incidence matrix, \( (xs)^2 \) submatrices \( C_{ij} \) of order \( s \) are newly introduced such that (i) \( C_{ij} \)'s are \( (0,1) \)-matrices corresponding to the \( i \)-th group of the GD association scheme and the \( j \)-th resolution set of the design for \( i, j = 1, 2, \ldots, sx \), and (ii) \( C_{ij} = I_s \) for \( i = 1 \) or \( j = 1 \). For example, the incidence matrix \( D \) in the illustration of Theorem 3 is expressed by

\[
D = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
C_{21} & C_{22} & C_{23} & C_{24} \\
C_{31} & C_{32} & C_{33} & C_{34} \\
C_{41} & C_{42} & C_{43} & C_{44}
\end{bmatrix}
\]

with \( C_{ij} = I_4 \) for \( i = 1 \) or \( j = 1 \). Some conditions on these \( C_{ij} \) are newly assumed in the following theorem.
Theorem 4. Let $s$ be a prime. Then the existence of an affine resolvable SRGD design with parameters $v = b = xs^2$, $r = k = sx$, $\lambda_1 = 0$, $\lambda_2 = x$, $q = x$; $m = sx$, $n = s$ implies the existence of a DS$(sx, s; x)$, if all $C_{ij}$'s have a structure formed by some cyclic row-permutations of $I_s$.

Before the proof of Theorem 4, we will give an illustration of Theorem 4 for $s = 3$ and $x = 1$, along with new three procedures, $T_1$, $T_2$, $T_3$, of transformation.

Now let $D$ be the incidence matrix of an affine resolvable SRGD design with parameters $v = b = 9$, $r = k = 3$, $\lambda_1 = 0$, $\lambda_2 = 1$, $q = 1$; $m = n = 3$, whose solution can be found in Table VI of [2], with the following incidence matrix

$$
D = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.
$$

The matrix $D$ can be transformed into the following $D^*$, whose first three rows and columns are the juxtaposition of $I_3$, without loss of generality, by some permutation of rows and/or columns in $D$ (let this type of transformation be called $T_1$):

$$
D \overset{T_1}{\mapsto} D^* = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}
$$

with $C_{ij} = I_3$ for $i = 1$ or $j = 1$.

Here it should be noted that $C_{ij}$'s of $D^*$ have a structure formed by some cyclic row-permutations of $I_3$ (i.e., the assumption on $C_{ij}$ is satisfied in the
illustration), and each of 3 columns displayed above corresponds to each of 3 resolution sets in the starting affine resolvable design.

Next form a new $9 \times 3$ matrix $D^{**}$, as a submatrix of the matrix $D^{*}$, of consisting only of the first column in each of 3 resolution sets in $D^{*}$ (let this type of transformation be called $T_{2}$):

$$D^{*} \xrightarrow{T_{2}} D^{**} = \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}.$$

The matrix $D^{**}$ is now partitioned into 3 groups of 3 rows each and then let $d_{ij}$ be the $j$-th column vector of size 3 in the $i$-th group for $1 \leq i \leq 3$ and $1 \leq j \leq 3$. For example, $(d_{1j}^{T}, d_{2j}^{T}, d_{3j}^{T})^{T}$ is the $j$-th column of $D^{**}$. Here, since in the starting SRGD design every block contains only one point from each group (by $k/m = 1$), $d_{ij}$'s have only one ‘1’ and other 2 ‘0’s for all $i$ and $j$. In this stage, the following procedure is now taken (this type of replacement procedure will be called $T_{3}$): For $1 \leq l \leq 3$ when the $l$-th component of $d_{ij}$ is a ‘1’, the $d_{ij}$ is replaced with a value $l - 1$, that is, $(1,0,0)^{T}$ is replaced by 0, $(0,1,0)^{T}$ by 1 and $(0,0,1)^{T}$ by 2. It is obvious that each column, except for the first column, contains all the distinct elements of $Z_{3} = \{0,1,2\}$ once. Hence the resulting matrix $D^{***}$ of order 3 is clearly a $DS(3,3;1)$ based on the additive group $S = Z_{3}$:

$$D^{**} \xrightarrow{T_{3}} D^{***} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 2 \\
0 & 2 & 1
\end{bmatrix}.$$

This illustrates Theorem 4 for $s = 3$ and $x = 1$. The illustration can be generalized as the following proof shows.

**Proof.** Take an affine resolvable SRGD design with the given parameters, having the incidence matrix $D$. The matrix $D$ can be transformed into the following $D_{1}$, whose first $s$ rows and columns are the juxtaposition of $I_{s}$, of order $sx^{2}$ without loss of generality, by some permutation of rows and/or columns in $D$ (this is done by transformation $T_{1}$):
Next form a new \( \times s^2 \times x \) matrix \( D_1^* \), which is a submatrix of the matrix \( D_1^* \), consisting only of the first column in each of \( x \) resolution sets in \( D_1^* \) (this is done by \( T_2 \)). The matrix \( D_1^* \) is now partitioned into \( x \) groups of \( s \) rows each and let \( d_{ij} \) be the \( j \)-th column vector of size \( s \) in the \( i \)-th group for \( 1 \leq i \leq x \) and \( 1 \leq j \leq x \). Here, since in the starting SRGD design every block contains only one point from each group (by \( k = m = 1 \), \( d_{ij} \)'s have only one ‘1’ and other \( s - 1 \) ‘0’s for all \( i \) and \( j \). In this stage, the following replacement procedure (called \( T_3 \)) is now taken: For \( 1 \leq l \leq s \) when the \( l \)-th component of \( d_{ij} \) is a ‘1’, the \( d_{ij} \) is replaced with a value \( l - 1 \) which will become possible elements of the required \( DS \). Then the resulting matrix \( D_1^{***} \) of order \( \times s \) can be shown to be the required \( DS_{\times s} \) on \( Z_s = \{0, 1, \ldots, s - 1\} \) as follows.

Let \( S = Z_s \). In \( D_1^* \), any column in the first resolution set has an inner product \( q \) (\( = x \)) as vectors with the first column in other resolution sets. This means that in \( D_1^{***} \) formed from \( D_1^* \) by both \( T_2 \) and \( T_3 \), any \( j \)-th column for \( 2 \leq j \leq x \) contains each of elements of \( Z_s \) \( x \) times. That is, in the vector differences between the first column and any \( j \)-th column of \( D_1^{***} \) for \( 2 \leq j \leq x \) each element of \( Z_s \) appears \( x \) times.

On the other hand, since, by the assumption, \( C_{ij} \)'s of \( D_1^* \) have a structure formed by some cyclic row-permutations of \( I_s \) for \( i, j = 1, 2, \ldots, x \), the matrix \( D_1^* \) can be transformed equivalently into the following \( D_2^{**} \), whose first \( s \) rows and \( s \) columns in the second resolution set are the juxtaposition of \( I_s \), of order \( \times s^2 \) by some cyclic row-permutations in each group (let this type of transformation be called \( T_4 \)):

\[
D_1^* \xrightarrow{T_4} D_2^{**} = \begin{bmatrix}
I_s & I_s & \cdots & I_s \\
I_s & I_s & \cdots & I_s \\
\vdots & \vdots & \ddots & \vdots \\
I_s & I_s & \cdots & I_s
\end{bmatrix} \text{ \( x \) times}
\]

As before, let \( D_2^{***} \) be an \( \times s^2 \times x \) matrix formed from \( D_2^{**} \) by \( T_2 \) and further let \( D_2^{****} \) be formed from the matrix \( D_2^{***} \) by \( T_3 \). Then \( D_2^{****} \) is of the form:

\[
D_2^{****} = \begin{bmatrix}
I_s & I_s & \cdots & I_s \\
I_s & I_s & \cdots & I_s \\
\vdots & \vdots & \ddots & \vdots \\
I_s & I_s & \cdots & I_s
\end{bmatrix} \text{ \( x \) times}
\]
Under the procedures of transforming $D_2^*$ to $D_2^{***}$, it follows that in the matrix $D_2^{***}$, any $j$-th column for $1 \leq j \neq 2 \leq xs$ contains each of elements of $Z_s$ $x$ times, because any column in the second resolution set of $D_2^*$ has an inner product $q (= x)$ as vectors with the first column in other resolution sets. It is further shown that in the vector differences between the second column and any $j$-th column of $D_2^{***}$ for $1 \leq j \neq 2 \leq xs$ each element of $Z_s$ appears $x$ times. In fact, let $d_{12}$ be an element of the $i$-th row and the second column of $D_1^{***}$ and $d_{13}$ be an element of the $i$-th row and the third column of $D_1^{***}$. Similarly, let $d_{22}$ be an element of the $i$-th row and the second column of $D_2^{***}$ and $d_{23}$ be an element of the $i$-th row and the third column of $D_2^{***}$. Furthermore in the $i$-th group of $D_1^*$ (and $D_2^*$), let $\mu_i$ be the frequency of cyclic row-permutations depending on $T_4$. Then, it holds that $d_{12} + \mu_i \equiv d_{22} (\mod s)$ and $d_{13} + \mu_i \equiv d_{23} (\mod s)$. Thus, it follows that $d_{12} - d_{13} \equiv (d_{12} + \mu_i) - (d_{13} + \mu_i) \equiv d_{22} - d_{23} (\mod s)$. Furthermore, it is remembered that in the vector differences between the first column and any $j$-th column of $D_1^{***}$ for $2 \leq j \leq xs$ each element of $Z_s$ appears $x$ times, and in the vector differences between the second column and any $j$-th column of $D_2^{***}$ for $1 \leq j \neq 2 \leq xs$ each element of $Z_s$ appears $x$ times. Therefore these mean that in the vector differences between the second and third columns of $D_1^{***}$ each element of $Z_s$ appears equally in the vector differences between the second and third columns of $D_2^{***}$.

Thus, similarly to the transformation $D_1^* \rightsquigarrow D_2^*$, if we consider the transformation $D_1^* \rightsquigarrow D_j^*$ for $3 \leq j \leq xs$, it can be seen that in the vector differences between “any two columns” of $D_1^{***}$ each element of $Z_s$ appears $x$ times. This means that the matrix $D_1^{***}$ is a $DS(xs,s,x)$ on $Z_s$.

Note that Theorem 4 shows a generalization of Theorem 2.

Remark. The affine resolvability in the proof of Theorem 3 is also shown by use of the property (Corollary 8.5.10.1 in [5]) such that a resolvable SRGD design is affine resolvable if and only if (a) $b = v - m + r$ and (b) $k^2 \div v$ is an integer, which can be easily checked in the present case.

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