A NEW MINIMAL NON $\sigma$-SCATTERED LINEAR ORDER

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Abstract. We will show it is consistent with GCH that there is a minimal non $\sigma$-scattered linear order which does not contain any real or Aronszajn type. In particular the assumption $PFA^+$ in the main result of [5] is necessary, and there are other obstructions than real and Aronszajn types to the sharpness of Laver’s theorem in [5].

1. Introduction

Fraïssé in [4], conjectured that, every descending sequence of countable order types is finite, and every antichain of countable order types is finite. That is, the class of countable linear orders is well quasi ordered. Laver confirmed the conjecture by proving a stronger statement than what Fraïssé conjectured.

Theorem 1.1. [8] The class of $\sigma$-scattered linear orders is well quasi ordered. In particular every descending chain of $\sigma$-scattered linear orders is finite.

Here the class of linear orders is considered with the quasi order of embeddability. Recall that a linear order $L$ is said to be scattered if it does not contain a copy of the rationals. $L$ is called $\sigma$-scattered if it is a countable union of scattered linear orders. At the end of his paper Laver asks about the behavior of non $\sigma$-scattered linear orders under embeddability. For instance to what extent Laver’s theorem is sharp? If the answer to this question is independent of ZFC, what are the obstructions to the sharpness of Laver’s theorem?

Not very long after Laver proved Theorem 1.1 various theorems in the direction of showing that Laver’s theorem is consistently not sharp were proved. Baumgartner in [2] showed that it is consistent that all $\aleph_1$-dense sets of the reals are isomorphic. He mentions in [2] that one can add all $\aleph_1$ sized subsets of the reals to the class of all $\sigma$-scattered linear orders in order to obtain a class $L$ of linear orders such that

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\( \mathcal{L} \) is strictly larger than the class of \( \sigma \)-scattered linear orders, \( \mathcal{L} \) is closed under taking suborders and it is consistent that \( \mathcal{L} \) is well quasi ordered. Another result in the direction of "Laver's theorem is consistently not sharp" is due to Abraham and Shelah. In [I], they showed that the proper forcing axiom implies that every two non-stationary Countryman lines are either isomorphic or reverse isomorphic. Since every Countryman line contains an uncountable non-stationary suborder, one can even considers a larger class of linear orders than what Baumgartner and Laver considered and still have a class of linear orders which is consistently well quasi ordered and which is closed under taking suborders. Later Martinez-Ranero in [9] showed that under PF \( \mathcal{A} \) the class of all Aronszajn lines is well quasi ordered.

Baumgartner seems to be the first person who considered the other side of Laver’s question, i.e, to what extent is Laver’s theorem sharp? In [3], he introduces a class of non \( \sigma \)-scattered linear orders and proves in ZFC that his examples are not minimal with respect to being non \( \sigma \)-scattered. Baumgartner’s example can be described as follows. Let \( L = \{C_\alpha : \alpha \in S\} \) ordered lexicographically, where \( S \) is a stationary subset of \( \omega_1 \) consisting of limit ordinals and \( C_\alpha \) is a cofinal sequence in \( \alpha \) that has order type \( \omega \). Note that if \( f : L \rightarrow L \) is an embedding then the set \( \{\xi \in S : f(C_\xi) \neq C_\xi\} \) is not stationary by pressing down lemma. Therefore if \( S_0 \subset S \) is such that \( S \setminus S_0 \) and \( S_0 \) are stationary then \( L \) does not embed into the linear order corresponding to \( S_0 \). In this paper Baumgartner type refers to Baumgartner’s examples or the revers of them.

The behavior of Baumgartner types inspired Ishiu and Moore to generalize the situation above, for a broader class of linear orders and prove the following theorem.

**Theorem 1.2.** [5] PF \( \mathcal{A}^+ \) implies that every minimal non \( \sigma \)-scattered linear order is either a real or a Countryman type.

In other words under PF \( \mathcal{A}^+ \), the only obstructions to the sharpness of Laver’s theorem are real and Countryman types. This breakthrough should be considered with the following result.

**Theorem 1.3.** [10] It is consistent with CH that \( \omega_1 \) and \( \omega_1^* \) are the only linear orders that are minimal with respect to being uncountable.

Later the methods in [5] and [10] was used to prove Laver’s theorem is sharp, i.e, it is impossible to improve the theorem in ZFC.

**Theorem 1.4.** [7] If there is a supercompact cardinal, then there is a forcing extension which satisfies CH in which there are no minimal non \( \sigma \)-scattered linear orders.
Note that all of the results proving that Laver’s theorem is consistently not sharp were based on the consistency of the minimality of real types or Aronszajn types. So it is natural to ask:

**does any minimal non σ-scattered linear order have to be real or Aronszajn type?**

This question is also important from the point of view of the work involved in proving Theorems 1.2, 1.3 and 1.4. An affirmative answer to this question asserts that the assumption PFA+ is not needed in order to obtain the results in [5]. Consequently, the model Moore came up with in order to prove Theorem 1.3 already satisfies ”Laver’s theorem is sharp.” Therefore the work in [7] as well as the large cardinal assumption were not needed to prove Theorem 1.4. In this paper we will provide a negative answer to this question. in particular real and Aronszajn types are not the only possible obstructions to the sharpness of Laver’s theorem.

**Theorem 1.5.** It is consistent with GCH that there is a non σ-scattered linear order L which contains no real or Aronszajn type and is minimal with respect to not being σ-scattered.

Moreover, Theorem 1.5 is related the following question which is due to Galvin.

**Question 1.6.** [3, problem 4] Is there a linear order which is minimal with respect to not being σ-scattered and which has the property that all of its uncountable suborders contain a copy of ω₁?

Note that a consistent negative answer is already given by Theorem 1.2. Theorem 1.5 does not answer Galvin’s question because the linear order we introduce, has a lot of copies of ω₁.

This paper is organized as follows. Section 2 reviews some notation, definition and facts regarding linear orders. Section 3 is devoted to constructing a specific Kurepa that is a suitable candidate for having suborders that witness Theorem 1.5. We also show that this tree has a lot of non σ-scattered linear order which become σ-scattered to get the main result. In Section 4 we introduce the posets that add isomorphisms we need. Section 5 finishes the proof of Theorem 1.5.

2. **Preliminaries**

This section is devoted to some background, notation and definitions regarding trees, linearly ordered sets, forcings and their iterations. More discussion can be found in [5], [6], [7], and [11].
To avoid ambiguity we fix some terminology and notation. An $\omega_1$-tree is a tree which is of height $\omega_1$, has countable levels and does not branch at limit heights, i.e. if $s, t$ are of the same limit height and have the same predecessors then they are equal. A branch of a tree $T$ is a chain in $T$ which intersects all levels. An $\omega_1$-tree $T$ is called Aronszajn if it has no branches. It is called Kurepa if it has at least $\omega_2$ many branches. If $T$ is a tree and $A$ is a set of ordinals, by $T \upharpoonright A$ we mean $\{ t \in T : \text{ht}(t) \in A \}$, with the order inherited from $T$. If $S, T$ are trees of height $\kappa$ and $C \subset \kappa$ is a club and $f : T \upharpoonright C \rightarrow S \upharpoonright C$ is one to one, level and order preserving then $f$ is called a club embedding from $T$ to $S$. $\mathcal{B}(T)$ refers to the collection of all branches of $T$. If $L$ is a linearly ordered set, $\hat{L}$ denotes the completion of $L$. Formally $\hat{L}$ consists of all Dedekind cuts of $L$.

The following few definitions and facts give a characterization of $\sigma$-scatteredness which we use in the proof of Theorem 1.5. They also generalize the behavior of Baumgartner types that causes them to be non minimal. We will use this to show that the generic tree that we build in section 3 has suborders that are obstructions to minimality.

**Definition 2.1.** Assume $L$ is a linear order and $Z$ is a countable set. We say $Z$ captures $x \in L$ if there is a $z \in Z \cap \hat{L}$ such that there is no element of $Z \cap L$ strictly in between $x, z$.

**Fact 2.2.** Suppose $L$ is a linear order and $\kappa$ is a regular large enough cardinal. If $M$ is a countable elementary submodel of $H_\kappa$ such that $L \in M$ and $x \in L \setminus M$, then $M$ captures $x \in L$ iff it there is a unique $z \in \hat{L} \cap M$ such that there is no element of $M \cap L$ strictly in between $x$ and $z$. In this case we say $M$ captures $x$ via $z$.

**Definition 2.3.** Assume $L$ is a linear order. $\Omega(L)$ is the set of all countable $Z \subset \hat{L}$ which capture all elements of $L$. $\Gamma(L) = [\hat{L}]^\omega \setminus \Omega(L)$.

**Proposition 2.4.** A linear order $L$ is $\sigma$-scattered iff $\Gamma(L)$ is not stationary in $[\hat{L}]^\omega$.

**Proposition 2.5.** Assume $L$ is a linear order, $x \in L$, and $M$ is a countable elementary submodel of $H_\theta$ where $\theta > 2^{|L|}$ is a large enough regular cardinal. We say $x$ is internal to $M$ if there is a club $E \subset M$ such that whenever $Z \in M \cap E$, $Z$ captures $x \in L$. We say $L$ is amenable if for all large enough regular cardinals $\theta$, for all countable elementary submodels $M$ of $H_\theta$ that contain $L$ as an element, and for all $x \in L$, $x$ is internal to $M$. 

The following proposition shows that amenability is what causes Baumgartner types and consistently more linear orders to be non minimal, see \cite{7}, discussion after the proof of Theorem 3.1.

**Proposition 2.6.** \cite{5} If $L$ is an amenable non $\sigma$-scattered linear order, then it is not minimal with respect to being non $\sigma$-scattered.

During this paper we consider the invariants $\Omega$, and $\Gamma$ for trees and linear orders with different definitions. The point is that all these definitions coincide modulo an equivalence relation that is defined here.

**Definition 2.7.** Assume $X, Y$ are two countable sets and $A, B$ are two collections of countable subsets of $X, Y$ such that $\bigcup A = X$ and $\bigcup B = Y$. We say $A, B$ are equivalent if there is a bijective function $f : X \rightarrow Y$ and a club $E \subset [X]^{\omega}$ such that for all $M \in E$, $M \in A \iff f[M] \in B$.

The invariant $\Omega$ together with the equivalence relation mentioned above was used in \cite{5}. By the work in \cite{5}, if $L_0 \subset L$ and $L$ embeds into $L_0$ then $\Omega(L)$ is equivalent to $\Omega(L_0)$. In fact the strategy in that work was to find a suborder $L_0$ of a given non $\sigma$-scattered linear order $L$ such that $\Omega(L_0)$ is stationary and not equivalent to $\Omega(L)$. This seems to be the motivation of Problem 5.10 in \cite{5}. The problem asks, assuming that $S$ is stationary, is the class of all linear orders $L$ with $\Omega(L) \equiv S$ well quasi ordered? Here we will show that even with such a restriction on the $\Omega$ of non $\sigma$-scattered linear orders it is impossible to obtain a well quasi ordered class.

Assume $S \subset \omega_1$ is a stationary set consisting of limit ordinals, and $\{S_i : i \in \omega\}$ is a partition of $S$ into infinitely many stationary pieces. Let $\langle C_\alpha : \alpha \in S \rangle$ be a collection of cofinal sequences of order type $\omega$. Let $L = \{C_\alpha : \alpha \in S\}$ and $L_i = \{C_\alpha : \alpha \in \bigcup_{j \geq i} S_i\}$ ordered with the lex order. By pressing down lemma, for any one to one function $f : L_i \rightarrow L_j$, $f(C_\xi) = C_\xi$ except for a non-stationary set. Therefore the sequence $\langle L \oplus L_i : i \in \omega \rangle$ is a strictly decreasing chain of linear orders. If $A, B$ are two linear orders $A \oplus B$ is the linear order consisting of the disjoint union of $A, B$ where every element of $A$ is less than every element of $B$. On the other hand, if $M$ is a countable elementary submodel of $H_\theta (\theta > 2^{\omega_1})$, $M$ captures all elements of $L \oplus L_i$ iff $M \cap \omega_1 \notin S$. This shows that $\Omega(L \oplus L_i) \equiv \Omega(L \oplus L_j)$.

If $T$ is an $\omega_1$-tree that is equipped with a lexicographic order, and $L = (T, \leq_{\text{lex}})$, then $\Omega(L)$ defined here is equivalent to the $\Omega(T)$ defined in \cite{6}.

**Definition 2.8.** \cite{6} Assume $T$ is an $\omega_1$-tree. $\Omega(T)$ is the set of all countable $Z \subset B(T)$ with the property that for all $t \in T_{az}$ there is a
$b \in Z$ with $t \in b$, where $\alpha_Z = \sup \{b \Delta b' : b, b' \in Z\}$. Here $\mathcal{B}(T)$ is the collection of all cofinal branches of $T$.

To see why these two definitions give $\Omega$ that are equivalent in the sense of Definition 2.7, consider a suitable countable model $M$ for $T$ which captures all $t \in T$. It is easy to see that countable chains of $T$ cannot define interesting cuts $z$, ($z$ is interesting if for some $x \notin M$, $M$ captures $x$ via $z$). So $M \cap \mathcal{B}(T)$ has to capture all elements of $T$. This motivates us to extend the definition of capturing for $\omega_1$-trees.

**Definition 2.9.** Assume $T$ is an $\omega_1$-tree, $\mathcal{B}(T)$ is the collection of cofinal branches in $T$, $Z$ is a countable set, and $t \in T$. Then we say $Z$ captures $t \in T$ if either $t \in Z$ or there is a branch $b \in Z \cap \mathcal{B}(T)$ such that $t \Delta b \geq M \cap \omega_1$. Here $t \Delta b$ is the height of the least element of $b$ that is not a predecessor of $t$.

Not that when $Z \subset \hat{L}$ is countable, and $x \in L' \subset L$, in general it might be the case that $Z$ captures $x \in L'$ but $Z$ does not capture $x \in L$. But the similar situation does not happen for the notion of capturing for trees. The following fact shows that the obstruction about linear orders and their suborders happens so rarely that we can ignore it. The proof of this fact only uses routine elementarity arguments which we leave to the reader.

**Fact 2.10.** Assume $L' \subset L$ are linear orders, $x \in L'$ and $M$ is a countable elementary submodel of $H_\theta$ that has $L, L'$ as elements, where $\theta > 2^{\aleph_1}$ is regular. Then $M$ captures $x$ as an element of $L$ iff $M$ captures $x$ as an element of $L'$.

The following fact shows that after adding embeddings and making many non $\sigma$-scattered suborders $\sigma$-scattered, the linear order to which we added embeddings has still non $\sigma$-scattered dense suborders, which are in fact the witnesses for Theorem 1.5.

**Fact 2.11.** Assume $L$ is a linear order which has size $\aleph_2$, all elements of $L$ have cofinality and coinitiality $\omega_1$, and $L' \subset L$ is dense and has cardinality $\aleph_1$. Then $L'$ is $\sigma$-scattered.

**Proof.** Since all $x \in L$ have cofinality and coinitiality $\omega_1$, there is a scattered suborder $L_0$ of $L'$ whose closure has cardinality $\aleph_2$. For $x, y \in L_0$ let $x \sim y$ if there are at most $\aleph_1$ many elements of $\bar{L}_0$ in between $x, y$. Note that there are exactly $\aleph_1$ many equivalence classes and between every two distinct equivalence classes there are infinitely many, in fact $\aleph_1$ many, equivalence classes. Now let $L_1$ be a suborder of $L_0$ which intersect each equivalence class at exactly one point. $L_1$ is in infinite dense linear order which contradicts $\sigma$-scatteredness of $L_0$. \qed
As mentioned in the previous section, we will be using forcings which are not proper to add embeddings among suborders of a specific Kurepa tree which is equipped with a lexicographic order. The rest of this section is devoted to the facts and lemmas which enable us to show that countable support iterations of these forcings are robust enough to preserve cardinals, under mild assumptions like $CH$. More discussion can be found in [6] and [11]. The notion of $S$-completeness here seems to be the same as $E$-completeness in [11], where it is proved that countable support iterations of these forcings preserve $\omega_1$. The lemmas needed for $\aleph_2$-chain conditions are from [6] where it is proved, based on the work in [11] and the notion of proper isomorphism condition, that under some additional assumption we can iterate these posets while preserving $\omega_2$.

**Definition 2.12.** Assume $X$ is uncountable and $S \subset [X]^\omega$ is stationary. A poset $\mathcal{P}$ is said to be $S$-complete, if every descending $(M, \mathcal{P})$-generic sequence, $\langle p_n : n \in \omega \rangle$ has a lower bound, for all $M$ with $M \cap X \in S$ and $M$ suitable for $X, \mathcal{P}$.

**Fact 2.13.** [11] Assume $X$ is uncountable and $S \subset [X]^\omega$ is stationary. If $\mathcal{P}$ is an $S$-complete forcing then it preserves $\omega_1$ and adds no new countable sequences of ordinals.

**Corollary 2.14.** [11] Assume $X$ is uncountable and $S \subset [X]^\omega$ is stationary. Then the class of $S$-complete forcings are closed under countable support iterations.

The following fact follows routinely from the definition above. The proof can be found in [6], which is basically a modification of the proof of the well known fact that countable support iterations of proper posets is proper, which is due to Shelah.

**Fact 2.15.** [6] Assume $T$ is an $\omega_1$-tree which has no Aronszajn subtree in the ground model $V$, $\Omega(T) \subset [\mathcal{B}(T)]^{\omega}$ is stationary, and $\mathcal{P}$ is an $\Omega(T)$-complete forcing. Then $T$ has no Aronszajn subtree in $V^\mathcal{P}$.

The following definition is a modification of the Shelah’s notion for chain condition, $\kappa$-proper isomorphism condition. We will be using it for verifying certain chain conditions.

**Definition 2.16.** Assume $S, X$ are as above. We say that $\mathcal{P}$ satisfies the $S$-closedness isomorphism condition for $\kappa$, or $\mathcal{P}$ has the $S$-cic for $\kappa$, where $\kappa$ is an ordinal, if whenever
- $M, N$ are suitable models for $\mathcal{P}$,
- both $M \cap X, N \cap X$ are in $S$, 


\( h : M \to N \) is an isomorphism such that \( h \upharpoonright (M \cap N) = id \),

- there are \( \alpha_M, \alpha_N \) in \( M \cap \kappa \) and \( N \cap \kappa \) respectively with \( h(\alpha_M) = \alpha_N \), \( sup(M \cap \kappa) < \alpha_N \), \( M \cap \alpha_M = N \cap \alpha_N \), and

- \( \langle p_n : n \in \omega \rangle \) is an \((M, \mathcal{P})\)-generic sequence,

then there is a common lower bound \( q \in \mathcal{P} \) for \( \langle p_n : n \in \omega \rangle \) and \( \langle h(p_n) : n \in \omega \rangle \).

**Lemma 2.17.** Assume \( 2^{\aleph_0} < \kappa \), \( \kappa \) is a regular cardinal and that \( S, X \) are as above. If \( \mathcal{P} \) satisfies the \( S \)-cic for \( \kappa \) then it has the \( \kappa \)-c.c.

The proof of the following fact which is useful in verifying the chain condition properties of an iteration of posets can be found in [6].

**Lemma 2.18.** Suppose \( \langle \mathcal{P}_i, \dot{\mathcal{Q}}_j : i \leq \delta, j < \delta \rangle \) is a countable support iteration of \( S \)-complete forcings, where \( S \subset [X]^{\omega} \) is stationary and \( X \) is uncountable. Assume in addition that

\[ \models_{\mathcal{P}_i} " \dot{\mathcal{Q}} \text{ has the } \vec{S} \text{-cic for } \kappa " \]

for all \( i \in \delta \). Then \( \mathcal{P}_\delta \) has the \( S \)-cic for \( \kappa \).

3. **The generic homogeneous Kurepa tree**

**Definition 3.1.** The poset \( \mathcal{H} \) consists of conditions \( q = (T_q, b_q, \Pi_q) \) for which the following statements hold.

1. \( T_q \) is a countable tree of height \( \alpha_q + 1 \) which is equipped with a lexicographic order such that for all \( t \in (T_q)_{< \alpha_q} \), the set \( t^+ \), consisting of all immediate successors of \( t \), is isomorphic to the rationals when considered with the lexicographic order.

2. \( b_q \) is a partial bijective function from a countable subset of \( \omega_2 \) to the last level of \( T_q \).

3. \( \Pi_q = \langle \pi_{t,s}^q : (t, s) \in \bigcup_{\xi \in \alpha_q} (T_q)_{\xi}^2 \rangle \) such that \( \pi_{t,s}^q \) is a tree isomorphism from the tree of all \( t' \in T_q \) that are compatible with \( t \) to the tree of all \( s' \in T_q \) that are compatible with \( s \), which preserves the lexicographic order.

4. \( \Pi_q \) is coherent, in the sense that if \( t' > t \) and \( \pi_{t,s}^q(t') = s' \) then \( \pi_{t',s'}^q \) is equal to \( \pi_{t,s}^q \) restricted to the elements that are compatible with \( t' \).

5. \( \Pi_q \) is symmetric in the sense that \( \pi_{s,t}^q = (\pi_{t,s}^q)^{-1} \).

6. \( \Pi_q \) respects the club \( C \), in the sense that if \( \alpha \in C \), \( t, s \) are in \( T_q \) and have the same height then, \( \xi < \alpha \) iff \( b_q^{-1}(\pi_{t,s}^q(b_q(\xi))) < \alpha \).

7. \( \Pi_q \) respects the composition operation, in the sense that if \( t, s, u \) are in \( (T_q)_{\xi} \) and \( \xi < \alpha_q \) then \( \pi_{s,u}^q \circ \pi_{t,s}^q = \pi_{t,u}^q \).

For \( p, q \in \mathcal{H} \) we let \( q \leq p \) if
1. \((T_q)_{\leq \alpha_p} = T_p\) and the lexicographic order on \(T_q\) is the same as the one on \(T_p\).

2. \(\text{dom}(b_p) \subset \text{dom}(b_q)\),

3. for all \(\xi \in \text{dom}(b_p)\), \(b_p(\xi) \leq b_q(\xi)\),

4. for all \((t, s) \in \bigcup_{\xi \in \alpha_q} (T_p)^2\), \(\pi_{t,s}^p\) is equal to \(\pi_{t,s}^q\) \(p\upharpoonright T_p\), and

5. for all \((t, s) \in \bigcup_{\xi \in \alpha_q} (T_p)^2\), and \(\xi, \eta \in \text{dom}(b_p)\), if \(\pi_{t,s}^p(b_p(\xi)) = b_p(\eta)\) then \(\pi_{t,s}^q(b_q(\xi)) = b_q(\eta)\).

**Notation 3.2.** Assume \(G\) is a generic filter for \(\mathcal{H}\). \(T_G = \bigcup_{q \in \mathcal{G}} T_q\), \(b_\xi\) is the branch \(\{b_q(\xi) : q \in \mathcal{G}\}\). If \(t, s\) are in \(T_G\) and have the same height then \(\pi_{t,s} = \bigcup_{q \in \mathcal{G}} \pi_{t,s}^q\).

**Lemma 3.3.** \(\mathcal{H}\) is \(\sigma\)-closed.

**Proof.** Let \(\langle p_n : n \in \omega \rangle\) be a decreasing sequence in \(\mathcal{H}\) and \(\sup(\alpha_{p_n})_{n \in \omega} = \alpha\). Let \(T = \bigcup_{n \in \omega} T_{p_n}\). Note that \(\langle b_{p_n}(\xi) : n \in \omega \rangle\) is a cofinal chain in \(T\) for all \(\xi \in \bigcup_{n \in \omega} \text{dom}(b_{p_n})\). Let \(T_q\) be a countable tree of height \(\alpha + 1\) such that

- \((T_q)_{\leq \alpha} = T\),
- for all \(\xi \in \bigcup_{n \in \omega} \text{dom}(b_{p_n})\), \(\langle b_{p_n}(\xi) : n \in \omega \rangle\) has an upper bound in \(T_q\), and
- every element of height \(\alpha\) is an upper bound for \(\langle b_{p_n}(\xi) : n \in \omega \rangle\),

for some \(\xi \in \bigcup_{n \in \omega} \text{dom}(b_{p_n})\).

Now let \(q\) be the condition with \(\alpha_q = \alpha\) and \(T_q\) as above. Let \(b_q\) be the function form \(\bigcup_{n \in \omega} \text{dom}(b_{p_n})\) to the last level of \(T_q\) such that for all \(\xi\) in the domain, \(b_q(\xi)\) is the upper bound for the chain \(\langle b_{p_n}(\xi) : n \in \omega \rangle\). Similarly \(\bigcup_{n \in \omega} \pi_{t,s}^p\), can be extended to the last level of \(T_q\), for all \(t, s\) that are of the same height and are in \(T\). It is easy to see that the condition \(q\) described above is a lower bound for the sequence \(\langle p_n : n \in \omega \rangle\).

**Lemma 3.4.** \(\text{GCH}\) implies that \(\mathcal{H}\) has the \(\aleph_2\)-cc.

**Proof.** Let \(\langle q_\xi : \xi \in \omega_2 \rangle\) be a collection of conditions in \(\mathcal{H}\). Since there are many possibilities for \(T_q\) and \(\Pi_q\), we can thin down this collection to a subset of the same cardinality so that \(T_{q_\xi}\) and \(\Pi_{q_\xi}\) do not depend on \(\xi\). Now define \(f : C \rightarrow \omega_2\) by \(f(\xi) = \sup(\text{dom}(b_{q_\xi}) \cap \xi)\), where \(C\) is the club that all elements of \(\mathcal{H}\) respect. Note that for all \(\xi \in C\) with \(cf(\xi) > \omega\), \(f\) is regressive. So there is a stationary \(S \subset C\), and \(\alpha \in \omega_2\) such that \(f \upharpoonright S\) is the constant \(\alpha\). We can thin down \(S\) to a stationary subset \(S'\) if necessary, so that in \(\langle q_\xi : \xi \in S' \rangle\), \(\text{dom}(b_{q_\xi}) \cap \alpha \) and \(b_{q_\xi} \upharpoonright \alpha\) do not depend on \(\xi\). Let \(S'' \subset S' \setminus (\alpha + 1)\) be of size \(\aleph_2\) and whenever \(\xi < \eta\) are in \(S''\), \(\sup(\text{dom}(b_{q_\xi})) < \eta\). Note that \(\langle b_{q_\xi} : \xi \in S'' \rangle\) forms a \(\Delta\)-system with root \(r\) such that the \(\text{dom}(r) \subset \alpha\). Moreover for all
\[ \xi \in S'', \min(\text{dom}(b_{q_\xi}) \setminus r) \geq \xi. \] Since \( S'' \subset C \), every two conditions in \( \langle q_\xi : \xi \in S'' \rangle \) are compatible. \( \square \)

The following can routinely be verified.

**Fact 3.5.** The following sets are dense in \( \mathcal{H} \).

- \( H_\alpha := \{ q \in \mathcal{H} | \alpha_q > \alpha \} \).
- For \( \xi \in \omega_2 \), \( I_\xi := \{ q \in \mathcal{H} : \xi \in \text{dom}(b_q) \} \).

The proof of the following lemma is the same as 3.3

**Lemma 3.6.** If \( M \) is suitable for \( \mathcal{H} \) and \( \langle p_n : n \in \omega \rangle \) is a decreasing \((M, \mathcal{H})\)-generic sequence, then there is a lower bound \( q \) for \( \langle p_n : n \in \omega \rangle \) such that \( \text{dom}(b_q) = M \cap \omega_2 \), \( \alpha_q = M \cap \omega_1 \), and \( \text{ran}(b_q) = (T_q)_{\alpha_q} \).

**Fact 3.7.**
- Assume \( G \) is a generic filter for \( \mathcal{H} \). Then the generic tree \( T := \bigcup_{q \in G} T_q \) is a Kurepa tree such that \( \langle \{ b_q(\xi) : q \in G \} : \xi \in \omega_2 \rangle \) is an enumeration of the set of all branches.
- \( T \) has no Aronszajn subtree. Moreover, any uncountable downward closed subtree of \( T \) contains a branch \( b_\xi \) for some \( \xi \in \omega_2 \).
- Assume \( L \) is the linear order consisting of all branches of \( T \), \( B(T) \), ordered by the lexicographic order of the tree \( T \). Then \( \Omega(L) \) is stationary.

**Proof.** The first two statements follow vacuously from the last lemma. For the last statement, let \( M \) be suitable for \( \mathcal{H} \) and \( p \in M \cap \mathcal{H} \). Then the \((M, \mathcal{H})\)-generic condition from the last lemma forces that \( M[\dot{G}] \cap B(L) \in \Omega(L) \).

From now on \( T \) is the generic Kurepa tree generated by \( \mathcal{H} \) unless otherwise mentioned. Also \( K \) is the linear order \( B(T) \) ordered by the lexicographic order of the tree \( T \). We fix an enumeration of of \( B(T) = \langle b_\xi : \xi \in \omega_2 \rangle \).

The rest of this section is devoted to showing that \( K \) has a lot of non \( \sigma \)-scattered suborders that are amenable. These facts are not used in the proof of the results in the next sections but show some possible obstruction for the minimality of suborders of \( K \). In the next section these non \( \sigma \)-scattered suborders are forced to be \( \sigma \)-scattered by an improper forcing. Here we say a countable sequence of conditions in \( \mathcal{H} \) forces a statement if every lower bound of that sequence forces that statement, equivalently all generic filter that contain the sequence extends the model to the one in which the statement holds.

**Definition 3.8.** Let \( T \) be the \( \mathcal{H} \)-generic tree and \( t \in T \). \( t \) is said to be simple if whenever \( M \) is a countable elementary submodel of \( H_\theta \) containing \( T \), \( M \) captures \( t \in T \). Otherwise \( t \) is said to be complex.
Lemma 3.9. Assume GCH holds in $V$, $M$ is suitable for $\mathcal{H}$, $\langle p_n : n \in \omega \rangle$ is an $(M, \mathcal{H})$-generic sequence, $t \in T_0 := \bigcup_{n \in \omega} T_{p_n}$, $\langle p_n \rangle_{n \in \omega} \models \"t is simple\"$, $b$ is a branch in $T_0$ and $ht(t) < \alpha < \delta := M \cap \omega_1$. Then there exists $s \in T_0$ such that $ht(s) = \alpha$, $t < s$, $s \notin b$ and $\langle p_n \rangle_{n \in \omega} \models \"s is simple\"$.

Proof. First note that if $G$ is $\mathcal{H}$-generic over $V$ then $H_{\omega_3}[G] = H_{\omega_3}^{V[G]}$ has a well ordering $\triangleleft$. Let $\triangleleft$ be an $\mathcal{H}$-name for $\triangleleft$. Since $\langle p_n \rangle_{n \in \omega}$ is $M$-generic it decides $\triangleleft \cap (M[\dot{G}])^2$, in the sense that, if $\tau$ and $\pi$ are two $\mathcal{H}$-names that are in $M$ then there is an $n \in \omega$ such that $p_n \models \"\tau \triangleleft \pi\"$ or $p_n \models \"\pi \triangleleft \tau\"$.

Also note that if $t$ is simple then so is every $t' \in t^+$. Now let $\sigma \in M$ be an $\mathcal{H}$-name for a branch of the $\mathcal{H}$-generic tree such that $\langle p_n \rangle_{n \in \omega} \models \sigma$ forces that

- $t \in \sigma$,
- $\sigma(ht(t) + 1) \neq b(ht(t) + 1)$, and
- $\sigma$ is the $\triangleleft$-minimum branch of $T$ with the properties above.

Let $s \in T_0$ such that $\langle p_n \rangle_{n \in \omega} \models \"s is simple\"$. Let $G$ be an $\mathcal{H}$-generic filter containing $\langle p_n \rangle_{n \in \omega}$ and in $V[G]$, $N$ be a countable elementary submodel of $H_{\omega_3}$. If $N \cap \omega_1 \leq ht(t)$, by simplicity of $t$, $N$ captures $s$. If $ht(t) < N \cap \omega_1$ then $t^+ \subset N$ so $\sigma_G$ which is $\min_{\triangleleft} \{ b \in \mathcal{B}(T) : b(ht(t) = 1) = s(ht(t) + 1) \}$. So by elementarity $\sigma_G \in N$ and $N$ captures $s$. □

Proposition 3.10. Assume GCH holds in $V$ and $G$ is $V$-generic for $\mathcal{H}$. Then $K$ has an amenable non $\sigma$-scattered suborder.

Proof. Let $L = \{ t \in T : t \text{ is minimal complex} \}$ ordered by the lexicographic order of the $\mathcal{H}$-generic tree $T$. To see $L$ is amenable, let $t \in L$ and $M$ be countable elementary submodel of $H_\theta$ with $T, L \in M$, where $\theta$ is a regular large enough cardinal. Let $E = \{ N \cap \mathcal{B}(T) : N \text{ is a countable elementary submodel of } H_{\omega_3} \text{ with } T, L \in N \}$. Since $t$ is a minimal complex element of $T$ every $N \in E \cap M$ captures $t$. So $t$ is internal to $M$ and $L$ is amenable.

In order to see $L$ is not $\sigma$-scattered we will show that $\Gamma(L)$ is stationary in $[\dot{L}]^\omega$. Assume $\dot{E}$ is an $\mathcal{H}$-name for for a club in $[\dot{L}]^\omega$ and $q \in \mathcal{H}$. In $V$, let $M$ be countable elementary submodel of $H_\theta$ and $\theta$ be a regular large enough cardinal. Let $\langle p_n : n \in \omega \rangle$ be an $M$-generic sequence such that $p_0 = q$. Also let $\langle b_n : n \in \omega \rangle$ be an enumeration of all branches of $T_0 = \bigcup_{n \in \omega} T_{p_n}$ which are downward closure of $\{ b_{p_n}(\xi) : n \in \omega \}$ for some $\xi \in M \cap \omega_2$. By the previous lemma there is a sequence $\langle t_k : k \in \omega \rangle$ of elements in $T_0$ such that for all $k \in \omega$, $\langle p_n \rangle_{n \in \omega}$ forces that $t_k$ is simple, $t_k < t_{k+1}$, $t_k \notin b_k$ and $\sup \{ ht(t_k) : k \in \omega \} = \delta := M \cap \omega_1$. 

Now let $T_p = T_0 \cup (T_p)_\delta$ where is a minimal set such that

- for each $\xi \in M \cap \omega_2$, $\{b_{p_n}(\xi) : n \in \omega\}$ has a unique upper bound in $T_\delta$,
- the sequence $t_k$ has a unique upper bound for in $T_\delta$, and
- for each $u, v \in T_0$ and $t \in T_\delta$, $(\pi^{p_n}_{u,v})[\{s \in T_0 : s < t\}]$ has a unique upper bound in $T_\delta$.

It is easy to see that there are $b_p$ which is a from a countable subset of $\omega_2$ to $T_\delta$ and $\Pi_p$ consisting of natural extensions of the maps $\pi^{p_n}_{u,v}$ where $u, v$ are in $T_0$, such that $p = (T_p, b_p, \Pi_p)$ is a lower bound for $p$.

On the other hand $p$ forces the following statement.

- There are minimal complex elements at the $\delta$'th level of the $\mathcal{H}$-generic tree $T$.
- $M[\dot{G}] \cap \tau \in \dot{E}$, where $\tau$ is an $\mathcal{H}$-name for $\dot{L}$ in $M$.
- $M[\dot{G}] \cap \tau$ does not capture all elements of $\dot{L}$.

Therefore $1_H \models " \dot{L} is not $\sigma$-scattered."$ Note that the elements of $L$ form an antichain in $T$. Let $L' \subset K$ such that for every $t \in L$ there is a unique branch $b \in L'$ with $t \in b$. Then $L'$ is isomorphic to $L$, hence $K$ has an amenable non $\sigma$-scattered suborder.

\[\square\]

4. Adding embeddings

In the previous section we introduced a forcing which generates a Kurepa tree $T$ equipped with a lexicographic order which also has some homogeneity properties. In this section we use the homogeneity of $T$ to prove the countable support iteration of some forcings that add embeddings among the $\aleph_1$-sized dense subsets of the linear order $K = (\mathcal{B}(T), <_{lex})$ do not collapse cardinals. We fix an enumeration $\langle b_\xi : \xi \in \omega_2 \rangle$ of the branches of the tree $T$ for the rest of the paper, and recall that for each $t \in T$, the set $t^+$, consisting of all immediate successors of $t$ with respect to $<_T$, is isomorphic to the rationals when considered with the lex order inherited from the tree $T$. Here homogeneity of $T$ means that there is a collection $\Pi = \langle \pi_{t,s} : t, s \in T \& h(t) = h(s)\rangle$ with the following properties.

1. for all $t, s$ in $T$ which have the same height $\pi_{t,s}$ is a tree and lex order isomorphism from the tree of all elements that are compatible with $t$ to the tree of all elements that are compatible with $s$.
2. $\Pi$ is symmetric, in the sense that $\pi_{t,s} = (\pi_{s,t})^{-1}$.
3. $\Pi$ is coherent in the sense that if $t, s, t', s'$ are in $T$, $ht(t) = ht(s)$, $ht(t') = ht(s')$, $t < t'$ and $s < s'$, then $\pi_{t,s} \upharpoonright (t' \uparrow) = \pi_{t',s'}$, where $t' \uparrow = \{u \in T : t' \succ u\}$.

**Definition 4.1.** Assume $T$ is as above and $X, Y$ are two $\aleph_1$-sized dense subsets of $\omega_2$ such that both $\langle b_\xi ; \xi \in X \rangle$ and $\langle b_\xi ; \xi \in Y \rangle$ are dense in $K$. $\mathcal{F}_{XY} (= \mathcal{F})$ is the poset consisting of all conditions $p = (f_p, \phi_p)$ for which the following holds.

1. $f_p : T \upharpoonright A_p \rightarrow T \upharpoonright A_p$ is a lex order and level preserving tree isomorphism where $A_p \subset \omega_1$ is countable and closed with $\max A_p = \alpha_p$.
2. $\phi_p$ is a countable partial injection from $\omega_2$ to $\omega_2$ such that
2.a. for all $\xi \in \text{dom}(\phi_p)$, $\xi \in X \leftarrow \phi_p(\xi) \in Y$,
2.b. for all $\xi \in \text{dom}(\phi_p) \setminus X$, $b_{\phi_p(\xi)} = \pi_{t,s}(b_\xi)$, where $t$ is an immediate successor of $b_\xi(\alpha_p)$ and $s$ is a successor of $f_p(b_\xi(\alpha_p))$, and
2.c. the map $b_\xi \mapsto b_{\phi_p(\xi)}$ is lexicographic order preserving.
3. For all $t \in T_{\alpha_p}$, there are at most finitely many $\xi \in \text{dom}(\phi_p)$ with $t \in b_\xi$.
4. For all $\xi \in \text{dom}(\phi_p)$, $f_p(b_\xi(\alpha_p)) = b_{\phi_p(\xi)}(\alpha_p)$

We let $q \preceq p$ if $f_q \subset f_p$ and $\phi_q \subset \phi_p$.

It is obvious that the sets $\{q \in \mathcal{F} : \alpha_q > \beta\}$ and $\{q \in \mathcal{F} : \xi \in \text{dom}(\phi_q)\}$ are dense for all $\beta \in \omega_1$ and $\xi \in \omega_2$. Therefore the forcing $\mathcal{F}$ adds a lexicographic order isomorphism from $X$ to $Y$ via the map $\Phi \upharpoonright X$ where $\Phi = \bigcup_{p \in G} \phi_p$ and $G$ is the generic filter for $\mathcal{F}$. We will show that countable support iterations of these forcings do not collapse cardinals.

**Lemma 4.2.** Assume $\mathcal{P}$ is an $S$-complete forcing where $S = \Omega(T)$, and $\hat{X}, \hat{Y}$ are $\mathcal{P}$-names for the indexes of the elements of $\aleph_1$-sized dense subsets of $K$. Then

1) $\Vdash \text{"} \mathcal{F}_{XY} \text{ is } S\text{-complete"}$, and
2) $\Vdash \text{"} \mathcal{F}_{XY} \text{ has the } \check{S}\text{-cic for } \check{\omega}_2\text{"}$

**Proof.** Let $G \subset \mathcal{P}$ is generic. We work in $\mathcal{V}[G]$. To see (1), assume $M$ is suitable for $\mathcal{F}$ and $M \cap K \in S$. Also let $\langle p_n = (f_n, \phi_n) : n \in \omega \rangle$ be a descending $(M, \mathcal{F})$-generic sequence and $\delta = M \cap \omega_1$. Note that $M \cap \omega_2 = \bigcup_{n \in \omega} \text{dom}(\phi_n)$ and $\bigcup_{n \in \omega} A_{p_n}$ is cofinal in $\delta$. Now let $\phi_p = \bigcup_{n \in \omega} \phi_n$, and $f_p = \bigcup_{n \in \omega} f_n \cup f$, where $f(b_\xi(\delta)) = b_{\phi_p(\xi)}(\delta)$. This makes $p$ a lower bound for $\langle p_n : n \in \omega \rangle$, since $M \cap K \in S$, and $\{b_\xi(\delta) : \xi \in M \cap \omega_2\} = T_\delta$.

For (2), still in $\mathcal{V}[G]$, let $M, N, \langle p_n = (f_n, \phi_n) : n \in \omega \rangle$, and $h$ be as in definition 2.10 with $M \cap \omega_1 = N \cap \omega_1 = \delta$. Since $h$ fixes the
intersection $h(f_n) = f_n$ and $b(\delta) = [h(b)](\delta)$, for all $b \in M \cap B(T)$. Let $\phi_p = \bigcup_{n \in \omega}(\phi_n \cup h(\phi_n))$ and $f_p = \bigcup_{n \in \omega} f_n \cup f$, where $f(b(\delta)) = b_{\phi_p(\xi)}(\delta)$. Note that by part 2.b. of the definition 4.1 hypothesis on $M, N$, and the fact that all $\pi_{t,s}$ in $\Pi$ preserve the lexicographic order of $T$, $\phi_p$ is one to one and the map $b_\xi \mapsto b_{\phi_p(\xi)}$ is lex order preserving. This shows that $p$ is a condition and a common lower bound for $\langle p_n : n \in \omega \rangle$ and its image under $h$. 

5. Proof of the main theorem

In this section we will finish the proof of Theorem 4.5. The strategy is to show that if two $\aleph_1$-sized $L, L' \subset K$ have closure of cardinality $\aleph_2$, then they are isomorphic. Note that by lemma 3.10, $K$ has non $\sigma$-scattered suborders whose closure have cardinality $\aleph_1$. So in order to use the strategy mentioned above, we need to make these suborders $\sigma$-scattered by forcings for which the analogue of Lemma 4.2 holds. We finish this section with a proof of Theorem 4.5.

**Definition 5.1.** Assume $L \subset K$, $|L| \leq \aleph_1$. $\mathcal{P}_L(=\mathcal{P})$ is the poset consisting of all conditions $p : \alpha_p + 1 \longrightarrow [L]^\omega \cap \Omega(L)$ that are $\subset$-increasing and continuous.

**Lemma 5.2.** Assume $S = \Omega(K)$, $\mathcal{Q}$ is an $S$-complete forcing, and $\hat{L}$ is a $\mathcal{Q}$-name for a suborder of $K$ whose closure has size $\leq \aleph_1$. Then

1) $\Vdash \mathcal{P}_L$ is $\hat{S}$-complete, and

2) $\Vdash \mathcal{P}_L$ has the $\hat{S}$-cic for $\hat{\omega}_2$.

**Proof.** Let $G \subset \mathcal{Q}$ is generic. We work in $V[G]$. To see (1), let $M$ be suitable for $\mathcal{P}$ and $M \cap K \subset S$. It is enough to show that $M \cap L \in \Omega(L)$. First note that $M$ does not capture any $x \in K \setminus M$ via cuts of countable cofinality or initiality. So if $M$ captures an element that is not in $M$ it has to capture it via a cut $z \in \check{K}$ of cofinality and coinitiality $\aleph_1$. But then $z$ determines a branch in $T$ which means that $z \in K$.

Now let $M$ capture $x \in L \setminus M$ via $z \in K \cap M$. Note that $K \setminus \check{L}$ is the union of a collection consisting of pairwise disjoint convex open subsets of $K$. So if $z \in (K \setminus \check{L}) \cap M$ there is a convex open set $I$ containing $z$ which is in $M$. Since $I \in M$ the endpoints of $I$ are in $M \cap \check{K}$. But $M$ captures $x$ via a unique cut, so $z$ is an endpoint of $I$ which contradicts the fact that $I$ is open.

For (2), note that if $h : M \longrightarrow N$ is an isomorphism that fixes $M \cap N$, then $h$ fixes $\check{L} \cap M$ because $|\check{L}| = \aleph_1$. So any lower bound for an $M$-generic sequence is a lower bound for an $N$-generic sequence. □

Now we are ready to prove Theorem 4.5. Assume $GCH$ holds in $V$ and $T$ is the generic Kurepa tree from the forcing $\mathcal{H}$ in $V^\mathcal{H}$. By Facts
and Lemmas \([2.15, 3.7, 4.2, 5.2]\) and the work in [6] there is countable support iteration of forcings of length \(\omega_2\) which is \(\Omega(T)\)-complete and extends \(V^H\) to a model in which the following holds.

1. \(T\) is club isomorphic to all of its everywhere Kurepa subtrees and has no Aronszajn subtree.
2. If \(X, Y\) are two dense suborders of \(K = (B(T), <_{lex})\) and \(|X| = |Y| = \aleph_1\) then they are isomorphic.
3. If \(X \subset K\) and \(|\bar{X}| \leq \aleph_1\) then \(X\) is \(\sigma\)-scattered.

Note that if \(L \subset K, |L| = \aleph_1, |\bar{L}| = \aleph_2\), then there is \(L_0 \subset L\) such that \(\bar{L}_0\) is \(\aleph_2\)-dense. To see this, for \(b, b' \in L\), let \(b \sim b'\) if there are at most \(\aleph_1\) many elements of \(L\) in between \(b, b'\). It is obvious that there are at least two distinct equivalence classes and since \(ht(T) = \omega_1\), the set of equivalences is \(\aleph_1\)-dense. Here the equivalence classes are ordered by the order of their elements and since the equivalence classes are convex subsets of \(L\) this order is well defined. Now let \(L_0\) be the suborder that intersects each equivalence class at exactly one point. \(\bar{L}_0\) is \(\aleph_2\)-dense since \(\bar{L} \setminus \bar{L}_0 \subset \{\bar{L} \cap [b] : b \in L\}, and |\{\bar{L} \cap [b] : b \in L\}| \leq \aleph_1\).

Note that for such an \(L_0\), the tree \(\bigcup \bar{L}_0\) is an everywhere Kurepa subtree of \(T\). So \(L_0\) is isomorphic to an \(\aleph_1\)-sized dense suborder of \(K\). This finishes the proof because all \(\aleph_1\)-sized dense suborders of \(K\) are isomorphic.

We will finish the paper with some remarks about the iteration of the forcings we used. The most important features of the forcings we used are \(\Omega(T)\)-completeness and \(\aleph_2\)-chain conditions. These forcings preserve the stationarity of stationary subsets of \(\Omega(T)\), but they do not need to preserve the stationarity of stationary subsets of \(\Gamma(T)\). In fact some of the iterands we considered, shoot clubs into the complement of some stationary subsets of \(\Gamma(T)\). On the other hand the set \(\Gamma(T)\) itself remains stationary in the final model we obtain, by Proposition \(2.4\). The only way to see that \(\Gamma(T)\) is stationary is that \(\omega_2\) is preserved and consequently \(K\) is not \(\sigma\)-scattered. The phenomenon that only preserving \(\omega_2\) without any control on countable structures which come from \(\Gamma(T)\) guarantees that \(\Gamma(T)\) remains stationary seems to be new and mysterious. For instance, assume \(S \subset \Gamma(T)\) is stationary and is not in the form of \(\Omega\) or \(\Gamma\) of any suborder of \(K\). Is there any way to determine whether or not \(S\) remains stationary in the extension under countable support iterations of these forcings?

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