PRICING AND HEDGING OF ENERGY SPREAD OPTIONS AND VOLATILITY MODULATED VOLterra PROCESSES

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Abstract. We derive the price of a spread option based on two assets which follow a bivariate volatility modulated Volterra process dynamics. Such a price dynamics is particularly relevant in energy markets, modelling for example the spot price of power and gas. Volatility modulated Volterra processes are in general not semimartingales, but contain several special cases of interest in energy markets like for example continuous-time autoregressive moving average processes. Based on a change of measure, we obtain a pricing expression based on a univariate Fourier transform of the payoff function and the characteristic function of the price dynamics. Moreover, the spread option price can be expressed in terms of the forward prices on the underlying dynamics assets. We compute a linear system of equations for the quadratic hedge for the spread option in terms of a portfolio of underlying forward contracts.

1. Introduction

Spread options are risk management tools that are extensively traded in the energy markets. For example, the owner of a gas-fired power plant lives from the spread between power and gas prices, and may apply so-called spark spread options to manage the risk of undesirably low power prices relative to gas. Tolling agreements and virtual power plants (VPP) are other classes of derivatives which are closely linked to spread options, as they can be represented as a strip of spread options on the spot prices. Although most spread options in energy markets are traded OTC, there exist some exchange-traded spread options on NYMEX written on the price differential between refined oil products.

The spot price dynamics of power and gas are very complex and call for sophisticated stochastic models. The prices possess clear seasonal features, and the fluctuations over time are typically much more volatile than in conventional financial markets. Weather factors play a key role in price determination, and sudden imbalances in supply and/or demand may produce large price spikes. We refer to Benth, Šaltytė Benth and Koekebakker [6], Eydeland and Wolyniec [15] and Geman [17] for extensive presentation of energy markets and stochastic modelling of spot prices.

Barndorff-Nielsen, Benth and Veraart [3] argue for stationarity of deseasonalized spot prices in the German power market EEX. Moreover, they find that Lévy semistationary (LSS) processes provide a flexible class of models than can be fitted to such spot price series. LSS processes can account for stationarity, stochastic volatility and spikes in an efficient way suitable for energy markets. These processes encompass many of the traditionally used models, like for example simple Gaussian Ornstein-Uhlenbeck processes. Continuous-time autoregressive moving average processes is a special class of LSS processes that has been used succesfully to model power prices (see Bernhard, Klüppelberg and Meyer-Brandis [7]).

In this paper we consider the problem of pricing spread options in energy markets where the price dynamics of the underlying assets are given as a bivariate volatility modulated Volterra (VMV) process. VMV processes are generalizations of LSS processes, and it is worth noticing that VMV processes (and also LSS processes) are not semimartingales in general.

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We apply a change of measure technique in order to translate the problem of computing the price of a call on the spread between two energies to computing the price of a call on one asset. This is a well-known approach (see Carmona and Durrleman [8] for bivariate geometric Brownian motions), which has been developed for a rather general class of semimartingale processes by Eberlein, Papapantoleon and Shiryaev [13,14]. We extend this method to the case of VMV processes, and combine it with Fourier methods in order to express the spread option price as an integral of the Fourier transform of a univariate call payoff function and the Lévy characteristics of the bivariate VMV process (see Carr and Madan [17] and Eberlein, Glau and Papapantoleon [12] for a thorough introduction and analysis of Fourier methods in derivatives pricing). We remark that although LSS processes may be the most relevant case of models for energy markets, the extension to VMV processes comes at no mathematical cost in our analysis, which is why we consider this general class.

The price of the spread option on energy spots can in our context be represented in terms of the corresponding forward prices on the spots. We apply this connection to derive a quadratic hedging strategy for the spread option, that is, the hedge portfolio in the respective forward contracts that minimizes the quadratic hedging error.

We present our results as follows. In the next Section a bivariate VMV model is introduced for the spot price dynamics. The spread option price is derived in Section 3, while we analyse the quadratic hedging problem in Section 4.

2. A BIVARIATE VOLATILITY MODULATED VOLTERRA PROCESS FOR THE SPOT DYNAMICS

Let \( L = (U, V) \) be a bivariate (two-sided) Lévy process defined on a complete probability space \((\Omega, \mathcal{F}, P)\) equipped with the filtration \( \mathcal{F}_t \) for \( t \in (-\infty, \bar{T}) \). Here, \( \bar{T} < \infty \) is some finite time horizon for the energy markets in question. We choose to work with the RCLL version of \( L \), that is, \( L \) is right-continuous with left-limits. The cumulant function of \( L \) is defined to be

\[
\psi(x, y) = \ln \mathbb{E} \left[ \exp(i x U(1) + i y V(1)) \right],
\]

and by the Lévy-Khintchin representation,

\[
\psi(x, y) = ix\gamma_1 + iy\gamma_2 - \frac{1}{2} \left( c_1^2 x^2 + 2\rho c_1 c_2 xy + c_2^2 y^2 \right) + \int_{\mathbb{R}^2} \left( \exp(ixz_1 + iyz_2) - 1 - (ixz_1 + iyz_2)1_{\{(z_1, z_2)\leq 1\}} \right) \ell(dz_1, dz_2).
\]

Here, \( \gamma_1, \gamma_2 \in \mathbb{R} \) are the drift coefficients, \( c_1, c_2 \in \mathbb{R}_+ \), the variances associated to the Brownian component of the Lévy process, \( \rho \in (-1, 1) \) the correlation coefficient of the Brownian component, and \( \ell(dz_1, dz_2) \) is the Lévy measure of \( L \). Let \( \psi_U(x) \) and \( \psi_V(x) \) denote the cumulants of the marginals \( U \) and \( V \), respectively. It holds \( \psi_U(x) = \psi(x, 0) \) and \( \psi_V(x) = \psi(0, x) \).

Introduce the two volatility modulated Volterra (VMV) processes

\[
X(t) = \int_{-\infty}^{t} g(t, s)\sigma(s-) \, dU(s),
\]

\[
Y(t) = \int_{-\infty}^{t} h(t, s)\eta(s-) \, dV(s),
\]

where \( g \) and \( h \) are two real-valued measurable functions defined on \((-\infty, \bar{T})^2\). The stochastic volatility processes \( \sigma, \eta \) are assumed to be \( \mathcal{F}_t \)-adapted RCLL processes, both being independent of \( L \). In order for the stochastic integrals in (2.3) and (2.4) to make sense, we assume that

\[
\mathbb{E} \left[ \int_{-\infty}^{t} g^2(t, s)\sigma^2(s) \, ds \right] < \infty, \quad \mathbb{E} \left[ \int_{-\infty}^{t} h^2(t, s)\eta^2(s) \, ds \right] < \infty,
\]

for all \( t \leq \bar{T} \).

We suppose that \( S_1(t) \) and \( S_2(t) \) denote the spot price dynamics of two energies (power and gas, say), defined on a logarithmic scale by

\[
\ln S_1(t) = \ln \Lambda_1(t) + X(t),
\]

\[
\ln S_2(t) = \ln \Lambda_2(t) + Y(t),
\]

where \( \Lambda_1(t) \) and \( \Lambda_2(t) \) are deterministic functions of time that are strictly positive for \( t \geq 0 \) and slowly varying. This allows to consider the spread \( S_1(t) - S_2(t) \) as a randomly fluctuated log-normal process with mean and variance functions that are slowly varying in time.
Here, \( \Lambda_i(t) > 0 \) for \( i = 1, 2 \) are deterministic and measurable functions modelling the mean level of the spot prices.

Note that in the context of pricing derivatives, it is natural to consider the spot prices for positive times \( t \) only. When studying spread option prices, we indeed focus on \( S_i(t) \) for \( t \geq 0 \), \( i = 1, 2 \). Thus, it is sufficient to specify \( \Lambda_i \), \( i = 1, 2 \) for times \( t \geq 0 \) only. We note that we may define \( g(t, s) = \tilde{g}(t, s)1(0 \leq s \leq t) \) to restrict the process \( X \) to only positive times \( t \geq 0 \). We emphasize that defining the stochastic integration in the definition of the VMV processes \( X \) and \( Y \) to start at \(-\infty\) opens for stationary stochastic dynamics, which is highly relevant in energy and in more general commodities markets (see e.g. Benth et al. [3] and Barndorff-Nielsen et al. [2]). For example, if we let \( \sigma = \eta = 1 \) and \( g(t, s) = \tilde{g}(t - s) \), \( h(t, s) = \tilde{h}(t - s) \) for functions \( \tilde{g}, \tilde{h} : \mathbb{R} \to \mathbb{R} \) being square-integrable, then \( X \) and \( Y \) are stationary processes because their cumulants are independent of time \( t \). If further we allow for stochastic volatility processes \( \sigma \) and \( \eta \) which are stationary, \( X \) and \( Y \) in (2.3) and (2.4) are known as Lévy semistationary (LSS) processes. For example, letting \( g(t - s) = \exp(-\alpha(t - s)) \) for a constant \( \alpha > 0 \), we recover the stationary solution of a Lévy-driven Ornstein-Uhlenbeck process. In other words, \( X(t) \) is the stationary solution of the stochastic differential equation

\[
   dX(t) = -\alpha X(t) \, dt + dU(t).
\]

Ornstein-Uhlenbeck processes are frequently used in factor models for energy prices like gas and power (see Benth et al. [3]). In Barndorff-Nielsen, Benth and Veraart [3], LSS processes have been proposed for modelling electricity spot prices, and empirically investigated on data from the German EEX market. Another popular class of models is the continuous-time autoregressive moving average (CARMA) processes. These have been applied in several studies to power prices, see Bernhard et al. [7] and Benth et al. [5]. A CARMA\((p,q)\)-process, for \( p > q \) being natural numbers, is defined as follows. Let \( b \in \mathbb{R}^p \) be a vector \( b^\top = (b_0, b_1, \ldots, b_q, 1, 0, \ldots, 0) \) with the first \( q \) elements being non-zero, element \( q + 1 \) equal to one and the remaining coordinates being zero. Here \( b^\top \) is the transpose of \( b \). The vector \( e_k \in \mathbb{R}^p \) for a natural number \( k \leq p \) is the \( k \)th canonical unit vector in \( \mathbb{R}^p \). Further, define the matrix \( A \in \mathbb{R}^{p \times p} \) to be

\[
   A = \begin{bmatrix}
   0 & 1 & 0 & \cdots & 0 \\
   0 & 0 & 1 & \cdots & 0 \\
   0 & 0 & 0 & \cdots & 1 \\
   \vdots & \vdots & \vdots & \ddots & \vdots \\
   -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1
   \end{bmatrix},
\]

where \( a_i > 0 \) for \( i = 1, \ldots, p \). By choosing

\[
   g(t, s) = b^\top \exp(A(t - s))e_p,
\]

we say that \( X \) in (2.3) is a volatility modulated CARMA\((p,q)\)-process. We note that \( X \) can be expressed as \( X(t) = b^\top Z(t) \), where \( Z(t) \in \mathbb{R}^p \) is the stationary solution of the Ornstein-Uhlenbeck process

\[
   dZ(t) = AZ(t) \, dt + e_p \sigma(t-) \, dU(t).
\]

A special class of CARMA processes is the continuous-time autoregressive processes, which are obtained by choosing \( q = 0 \) and denoted by CAR\((p)\). This case corresponds to selecting \( b = e_1 \).

Typical choices for the stochastic volatility processes \( \sigma \) and \( \eta \) are provided by the Barndorff-Nielsen and Shephard (BNS) model. Here, \( \sigma^2(t) \) and \( \eta^2(t) \) are defined as the stationary solutions of Ornstein-Uhlenbeck processes driven by subordinators, that is, Lévy processes with only positive jumps and non-negative drift. This ensures positive variance processes. We refer to Barndorff-Nielsen and Shephard [2] for a comprehensive analysis of this class of stochastic volatility models.

Note in passing that Benth [4] applied the BNS model in an exponential Ornstein-Uhlenbeck process to model the dynamics of UK gas spot prices. As a final note on the VMV models \( X \) and \( Y \) in (2.3) and (2.4), we recover Gaussian processes by simply choosing the \( L \) to be a bivariate Brownian motion (possibly correlated). Further, by letting the volatilities be constant...
and choosing \( g \) and \( h \) appropriately, we can allow for Gaussian processes including fractional Brownian motion (see Alos et al. [1]).

We suppose that the spot model is defined under the pricing measure directly, that is, \( P \) is assumed to be the pricing measure. From a practical viewpoint one would first specify the dynamics of the spot under the objective market probability, and then change measure to incorporate the market price of risk. The market price of risk is modelling the risk premium in the market. We refer to Barndorff-Nielsen et al. [3] for a discussion on a class of measure changes of Esscher type for LSS processes, that can be easily extended to VMV processes. As this class of measures preserves the VMV structure of the model, we refrain from introducing it to keep notation at a minimum.

3. Pricing spread options on energy spots

Let us continue with the pricing of spread options based on the bivariate spot price model in (2.6)-(2.7). To this end, let \( 0 < T \leq T \) be the exercise time for a European call option on the spread \( S_1(t) - kS_2(t) \) where \( k > 0 \) is the heat rate and strike is zero. Hence, the payoff of the option is

\[
(S_1(T) - kS_2(T))^+, \tag{3.1}
\]

where we use the notation \((x)^+ = \max(x, 0)\). The arbitrage-free price at time \( t \leq T \) of this option will be

\[
C(t, T) = e^{-r(T-t)}E \left[ (S_1(T) - kS_2(T))^+ \mid \mathcal{F}_t \right],
\]

where \( r > 0 \) is the risk-free interest rate.

In order to have the expectation in (3.1) well-defined, we assume that the price processes \( S_1 \) and \( S_2 \) are integrable, that is, that they have finite expectation. Obviously, because \( \max(x, 0) \leq |x| \), we find

\[
E \left[ (S_1(T) - kS_2(T))^+ \right] \leq E [S_1(T) - kS_2(T)] \leq E [S_1(T)] + kE [S_2(T)] < \infty.
\]

But, \( S_i(T), i = 1, 2 \) are integrable if \( X(T) \) and \( Y(T) \) have finite exponential moment. To ensure this, we introduce the following exponential integrability condition: For any \( 0 \leq t \leq T \),

\[
E \left[ \exp \left( \int_{-\infty}^{T} \psi_U \left( -i g(T, s) \sigma(s) \right) ds \right) \right] < \infty, E \left[ \exp \left( \int_{-\infty}^{T} \psi_Y \left( -i h(T, s) \eta(s) \right) ds \right) \right] < \infty.
\]

We suppose that (3.2) holds from now on.

Our aim next is to derive a numerically tractable analytic expression for the price \( C(t, T) \). We shall conveniently achieve this by Fourier methods.

Following Folland [10], the Fourier transform of a function \( g \in L^1(\mathbb{R}) \) is defined as

\[
\hat{g}(y) = \int_{\mathbb{R}} g(x) e^{-ixy} dx.
\]

Introduce the function

\[
f_{c,T}(x) := e^{-cx} \left( e^x - k \frac{\Lambda_2(T)}{\Lambda_1(T)} \right)^+.
\]

It is simple to see that \( f_{c,T} \in L^1(\mathbb{R}) \) for any \( c > 0 \). Hence, its Fourier transform exists, and calculated explicitly in the next Lemma:

**Lemma 1.** For any \( c > 1 \), the Fourier transform of \( f_{c,T} \) is given by

\[
\hat{f}_{c,T}(y) = \frac{1}{(c + iy)(c + iy - 1)} \left( k \frac{\Lambda_2(T)}{\Lambda_1(T)} \right)^{-c-iy+1}.
\]

Moreover, \( \hat{f}_{c,T} \in L^p(\mathbb{R}) \) for any \( p \geq 1 \).
Proof. The derivation follows the same steps as in Carr and Madan [9], but we include it here for the convenience of the reader. Denote for simplicity $A := k \frac{\Lambda_2(T)}{\Lambda_1(T)}$. From the definition of the Fourier transform, we find

$$
\hat{f}_{c,T}(y) = \int_{-\infty}^{\infty} e^{-cx} (e^x - A)^+ e^{-iy} dx 
$$

$$
= \int_{\ln A}^{\infty} e^{-cx+ixy} dx - A \int_{\ln A}^{\infty} e^{-cx-ixy} dx 
$$

$$
= \left( \frac{1}{c - 1 + iy} - \frac{1}{c + iy} \right) A^{-c+1-iy}. 
$$

Moreover, as $|\hat{f}_{c,T}(y)|^p \sim 1/(k + y^2)^p$ for some strictly positive constant $k$ and $p \geq 1$, integrability of $\hat{f}_{c,T}$ on $\mathbb{R}$ follows.  

We recall from Fourier analysis (see Folland [10]), that if the Fourier transform of a function $g$ is integrable, $\hat{g} \in L^1(\mathbb{R})$, then the inverse Fourier transform admits the integral representation

$$
g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(y)e^{ixy} dy. 
$$

As $\hat{f}_{c,T} \in L^1(\mathbb{R})$, we can apply the inverse Fourier transform to obtain the representation

$$
(e^x - k \frac{\Lambda_2(T)}{\Lambda_1(T)})^+ = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}_{c,T}(y)e^{ix(y-ic)} dy.
$$

Using this, we find the following price of the spread option.

**Proposition 3.1.** For a constant $c > 1$, assume that

$$
E \left[ \exp \left( \int_{-\infty}^{T} \psi(-icg(T,s)\sigma(s), -i(1-c)h(T,s)\eta(s)) ds \right) \right] < \infty.
$$

Then, the spread option price $C(t,T)$ for $0 \leq t \leq T$ defined in (3.1) is,

$$
C(t,T) = e^{-r(T-t)} \frac{\Lambda_1(T)}{2\pi} \times \int_{\mathbb{R}} \hat{f}_{c,T}(y)e^{iy+c} \int_{-\infty}^{\infty} g(T,s)\sigma(s)ds dU(s)+(1-(iy+c)) \int_{-\infty}^{\infty} h(T,s)\eta(s)dv(s) \Psi_{c,t,T}(y) dy,
$$

where

$$
\Psi_{c,t,T}(y) = E \left[ \exp \left( \int_{t}^{T} \psi ((y-ic)g(T,s)\sigma(s), ((c-1)i - y)h(T,s)\eta(s)) ds \right) | F_t \right].
$$

Here, $\hat{f}_{c,T}$ is defined in (3.6).

**Proof.** Let $G_{t,T}$ be generated by the paths of $\sigma(s)$ and $\eta(s)$ for $s \leq T$ and $F_t$. Then, using the independence of $\sigma, \eta$ and $L$, it holds by the tower property of conditional expectation,

$$
E \left[ (S_1(T) - kS_2(T))^+ | F_t \right] = E \left[ \left( \Lambda_1(T)e^{X(T)} - k\Lambda_2(T)e^{Y(T)} \right)^+ | F_t \right]
$$

$$
= \Lambda_1(T)E \left[ e^{Y(T)} \left( e^{X(T)-Y(T)} - k\frac{\Lambda_2(T)}{\Lambda_1(T)} \right)^+ | F_t \right]
$$

$$
= \Lambda_1(T)E \left[ e^{Y(T)} \left( e^{X(T)-Y(T)} - k\frac{\Lambda_2(T)}{\Lambda_1(T)} \right)^+ | G_{t,T} \right] F_t].
$$

We concentrate on the inner expectation, and observe that as long as we condition on $G_{t,T}$, we can treat $\sigma(s)$ and $\eta(s)$ pathwise, and thus view $g(t,s)\sigma(s)$ and $h(t,s)\eta(s)$ as deterministic functions in the integrals defining $X$ and $Y$. 


Define the stochastic process $R(t)$ for $t \leq T$
\[
R(t) = \exp \left( \int_{-\infty}^{t} h(T, s)\eta(s)\, d\mathbb{V}(s) - \int_{-\infty}^{t} \psi_V(-ih(T, s)\eta(s))\, ds \right).
\]
Note that by double conditioning and Jensen’s inequality, we find from the independent increment property of $V$ that
\[
\mathbb{E} \left[ \exp \left( \int_{-\infty}^{t} h(T, s)\eta(s)\, d\mathbb{V}(s) \right) \right] \leq \mathbb{E} \left[ \exp \left( \int_{-\infty}^{T} h(T, s)\eta(s)\, d\mathbb{V}(s) \right) \right]
\]
for $t \leq T$. Hence, by the exponential integrability assumption in (3.2), $R(t)$ becomes an integrable martingale process. Let $Z(t) = R(t)/R(0)$, which becomes an integrable martingale with expectation 1 for $0 \leq t \leq T$. We introduce the probability measure $Q$ with density process $Z$, that is,
\[
\frac{dQ}{dP} \big|_{\mathcal{G}_{t,T}} = Z(t).
\]
Moreover, observe that
\[
e^{Y(T)} = R(0)Z(T)e^{\int_{-\infty}^{T} \psi_V(-ih(T, s)\eta(s))\, ds}.
\]
Hence, applying Bayes’ Formula of conditional expectations twice (see Karatzas and Shreve [18]) together with $\mathcal{G}_{t,T}$-measurability of $\eta(s)$, $s \leq T$,
\[
\mathbb{E} \left[ e^{Y(T)} \left( e^{X(T) - Y(T)} - k\frac{d_{2}(T)}{d_{1}(T)} \right)^+ \bigg| \mathcal{G}_{t,T} \right] = e^{\int_{-\infty}^{T} \psi_V(-ih(T, s)\eta(s))\, ds} R(0)Z(t)E_Q \left[ \left( e^{X(T) - Y(T)} - k\frac{d_{2}(T)}{d_{1}(T)} \right)^+ \bigg| \mathcal{G}_{t,T} \right]
\]
\[
= e^{\int_{-\infty}^{T} \psi_V(-ih(T, s)\eta(s))\, ds} R(0)Z(t) \int_{R} \tilde{F}_{e,T}(y)E_Q \left[ e^{iy(x+y)(X(T) - Y(T))} \bigg| \mathcal{G}_{t,T} \right] dy
\]
\[
= \frac{1}{2\pi} \int_{R} \tilde{F}_{e,T}(y)E \left[ e^{Y(T)}e^{iy(x+y)(X(T) - Y(T))} \bigg| \mathcal{G}_{t,T} \right] dy.
\]
For two constants $a$ and $b$ (possibly complex), we find (assuming that the involved processes are integrable) that
\[
\mathbb{E} \left[ e^{aX(T) + bY(T)} \bigg| \mathcal{G}_{t,T} \right] = e^{a \int_{-\infty}^{T} g(T, s)\eta(s)\, dU(s) + b \int_{-\infty}^{T} h(T, s)\eta(s)\, dV(s)}
\]
\[
\times \mathbb{E} \left[ e^{a \int_{t}^{T} g(T, s)\eta(s)\, dU(s) + b \int_{t}^{T} h(T, s)\eta(s)\, dV(s)} \bigg| \mathcal{G}_{t,T} \right].
\]
Here, we have applied the $\mathcal{G}_{t,T}$-measurability of $U(s), V(s)$ for $s \leq t$. Because increments of $U(s)$ and $V(s)$ are independent of $\mathcal{G}_{t,T}$ for $s \in [t, T]$, we get
\[
\mathbb{E} \left[ e^{aX(T) + bY(T)} \bigg| \mathcal{G}_{t,T} \right] = e^{a \int_{-\infty}^{T} g(T, s)\eta(s)\, dU(s) + b \int_{-\infty}^{T} h(T, s)\eta(s)\, dV(s)}
\]
\[
\times e^{\int_{t}^{T} \psi(-iag(T, s)\eta(s), -ibh(T, s)\eta(s))\, ds}.
\]
Letting $a = iy + c$ and $b = 1 - (iy + c)$ yields the result by the assumed exponential integrability condition on the processes $X$ and $Y$. \hfill \Box

The trick of changing probability measure to price spread options, as we applied in the proof above, was suggested in Carmona and Durrleman [8] in the case of underlying processes being modelled by a bivariate geometric Brownian motion. Here we extend the method to general VMV processes for the underlying assets in the spread option. Worth noticing is that in the geometric Brownian motion case normality is preserved and one can compute the spread option price without resorting to an integral expression involving Fourier transform. In our much more general context it is more natural to resort to a price $C(t, x)$ expressed in term of the characteristics of the driving processes $X$ and $Y$, which naturally leads to the application of Fourier methods. As we recall from the proof above, we apply the change of measure twice, and come back to the original probability
where

By the tower law of conditional expectations, measurable for $s \in [t, T]$ and the definition of the cumulant function of $U$ with the independent increment property of a Lévy process. The proposition follows.

Recall from the definition of $S$ in the argument above, we applied that $\sigma(s)$ is $\mathcal{G}_{t,T}$-measurable for $s \in [t, T]$ and the definition of the cumulant function of $U$ with the independent increment property of a Lévy process. The proposition follows.

Remark that the option price at time $t \leq T$ is explicitly dependent on \( \int_{-\infty}^{t} g(T, s) \sigma(s) dU(s) \) and \( \int_{-\infty}^{t} h(T, s) \eta(s) dV(s) \), which are different than $X(t)$ and $Y(t)$ except at $t = T$. If we consider the special case of an OU-process, then $g(t - s) = \exp(-\alpha(t - s))$, we find

$$
\int_{-\infty}^{t} e^{-\alpha(T-s)} \sigma(s) dU(s) = e^{-\alpha(T-t)} \int_{-\infty}^{t} e^{-\alpha(t-s)} \sigma(s) dU(s) = e^{-\alpha(T-t)} X(t).
$$

Thus, we have an explicit dependency on $X(t)$ in $C(t, T)$ as long as $g$ is the kernel function of an OU-process. As it turns out, we can in the general case relate \( \int_{-\infty}^{t} g(T, s) \sigma(s) dU(s) \) and \( \int_{-\infty}^{t} h(T, s) \eta(s) dV(s) \) to the forward price on the spots. To this end, denote by $f_i(t, T)$ the forward price at time $t$ for a contract delivering the spot $S_i$ at time $T$, $t \leq T$ and $i = 1, 2$. By definition of the arbitrage-free forward price (see Duffie [11] and Benth et al. [6]),

$$
(3.7) 
\quad f_i(t, T) = \mathbb{E} [S_i(T) \mid \mathcal{F}_t], 
\quad i = 1, 2,
$$

which is well-defined as $S_i(T) \in L^1(P)$ by condition [3.2]. We find:

**Proposition 3.2.** It holds that

$$
\begin{align*}
\quad f_1(t, T) &= \Lambda_1(T) \exp \left( \int_{-\infty}^{T} g(T, s) \sigma(s) dU(s) \right) \mathbb{E} \left[ \exp \left( \int_{t}^{T} \psi_U(-i \mathbb{E}(T) \sigma(s)) ds \right) \mid \mathcal{F}_t \right]
\end{align*}
$$

and

$$
\begin{align*}
\quad f_2(t, T) &= \Lambda_2(T) \exp \left( \int_{-\infty}^{T} h(T, s) \eta(s) dV(s) \right) \mathbb{E} \left[ \exp \left( \int_{t}^{T} \psi_V(-i h(T, s) \eta(s)) ds \right) \mid \mathcal{F}_t \right]
\end{align*}
$$

for $t \leq T$.

**Proof.** By the exponential integrability condition [3.2], $S_i(T) \in L^1(P)$ and the expectation operator applied to $S_i(T)$ makes sense. Without loss of generality, we only prove the result for $i = 1$. Recall from the definition of $S_1(t)$ in [2.6] that

$$
S_1(T) = \Lambda_1(T) \exp(X(T))
$$

where

$$
X(T) = \int_{-\infty}^{T} g(T, s) \sigma(s) dU(s) = \int_{-\infty}^{t} g(T, s) \sigma(s) dU(s) + \int_{t}^{T} g(T, s) \sigma(s) dU(s).
$$

Because the first term in this decomposition of $X(T)$ is $\mathcal{F}_t$-adapted, we have

$$
\begin{align*}
\quad f_1(t, T) &= \Lambda_1(T) \exp \left( \int_{-\infty}^{T} g(T, s) \sigma(s) dU(s) \right) \mathbb{E} \left[ \exp \left( \int_{t}^{T} g(T, s) \sigma(s) dU(s) \right) \mid \mathcal{F}_t \right]
\end{align*}
$$

By the tower law of conditional expectations,

$$
\mathbb{E} \left[ \exp \left( \int_{t}^{T} g(T, s) \sigma(s) dU(s) \right) \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( \int_{t}^{T} g(T, s) \sigma(s) dU(s) \right) \mid \mathcal{G}_{t,T} \right] \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \exp \left( \int_{t}^{T} \psi_U(-i g(T, s) \sigma(s)) ds \right) \mid \mathcal{F}_t \right],
$$

where $\mathcal{G}_{t,T}$ is defined in the proof of Prop. [4.1]. In the argument above, we applied that $\sigma(s)$ is $\mathcal{G}_{t,T}$-measurable for $s \in [t, T]$ and the definition of the cumulant function of $U$ with the independent increment property of a Lévy process. The proposition follows.

From this Proposition, we can reexpress the option price as a function of the forwards, i.e.,

$$
(3.8) 
\quad C(t, T) = \tilde{C}(t, T, f_1(t, T), f_2(t, T))
$$
where, for \( x_i > 0, i = 1, 2, \)
\[
\widetilde{C}(t, T, x_1, x_2) = e^{-r(T-t)} \frac{\Lambda_1(T)}{2\pi} \int_\mathbb{R} \hat{f}_{c,T}(y) \exp \left( (iy + c) \left( \ln \frac{x_1}{\Lambda_1(T)} - \ln \Psi_U(t, T) \right) \right) \\
\times \exp \left( (1 - (iy + c)) \left( \ln \frac{x_2}{\Lambda_2(T)} - \ln \Psi_V(t, T) \right) \right) \Psi_{c,T}(y) \, dy,
\]
(3.9)
and
\[
\Psi_U(t, T) = \mathbb{E} \left[ \exp \left( \int_t^T \psi_U(-i\nu(T, s)\sigma(s)) \, ds \right) \right. | \mathcal{F}_t \]
\[
\Psi_V(t, T) = \mathbb{E} \left[ \exp \left( \int_t^T \psi_V(-i\nu(T, s)\eta(s)) \, ds \right) \right. | \mathcal{F}_t \]
\]

Note that we have rather complicated terms \( \Psi_i(t, T), \Psi_{c,t,T}(y) \) involving the conditional expectations of functionals of the stochastic volatility processes \( \sigma(s) \) and \( \eta(s) \). In the next Section we shall employ the dependency on forwards to derive hedging strategies for the option.

We recover a generalization of the Margrabe formula in the case of \( L = (B, W) \) being a bivariate Brownian motion and volatilities \( \sigma \) and \( \eta \) being deterministic. Obviously, with loss of generality, we can in the case of deterministic volatility functions let assume that \( \sigma(s) = \eta(s) = 1 \) because we can redefine the kernel functions \( g \) and \( h \) by \( \tilde{g}(t, s) = g(t, s)\sigma(s) \) and \( \tilde{h}(t, s) = h(t, s)\eta(s) \). We further assume that \( B \) and \( W \) are correlated by \( \rho \in (-1, 1) \). Then
\[
\psi(x, y) = -\frac{1}{2}(x^2 + 2\rho xy + y^2).
\]

Hence,
\[
\ln \Psi_{c,T}(y) = -\frac{1}{2} \left( (y - i\nu)^2 \int_t^T g^2(T, s) \, ds + 2\rho(y - i\nu)((c - 1)i - y) \int_t^T g(T, s)h(T, s) \, ds \\
+ ((c - 1)i - y)^2 \int_t^T h^2(T, s) \, ds \right),
\]
\[
\ln \Psi_U(t, T) = \frac{1}{2} \int_t^T g^2(T, s) \, ds \quad \text{and} \quad \ln \Psi_V(t, T) = \frac{1}{2} \int_t^T h^2(T, s) \, ds.
\]

We recall from the Fourier transform and its inverse that if \( Z \) is a random variable with characteristic function \( \psi_Z \), then (see e.g. Folland [16]),
\[
\int_\mathbb{R} \hat{f}_{c,T}(y) \psi_Z(y) \, dy = 2\pi \mathbb{E}[f_{c,T}(Z)].
\]
(3.10)

Let now \( Z \) be normally distributed with variance \( \Sigma^2(t, T) \) given as
\[
\Sigma^2(t, T) := \int_t^T \{ g^2(T, s) - 2\rho g(T, s)h(T, s) + h^2(T, s) \} \, ds
\]
(3.11)
and mean \( \mu \) as
\[
\mu := \ln \frac{x_1}{x_2} + \ln \frac{\Lambda_2(T)}{\Lambda_1(T)} + (c - 1)\Sigma^2(t, T).
\]

Collecting appropriate terms in (3.9) yields
\[
\widetilde{C}(t, T, x_1, x_2) = e^{-r(T-t)} \frac{\Lambda_1(T)}{2\pi} \mathbb{E} \left[ \int_\mathbb{R} \hat{f}_{c,T}(y) \psi_Z(y) \, dy \right] \\
e^{-r(T-t)} \Lambda_1(T) e^{\alpha} \mathbb{E} \left[ f_{c,T}(Z) \right],
\]
with
\[
\alpha := \ln \frac{x_2}{\Lambda_2(T)} + c \ln \frac{x_1}{x_2} + c \ln \frac{\Lambda_2(T)}{\Lambda_1(T)} + \frac{1}{2}(c - 1)\Sigma^2(t, T).
\]
But a straightforward computation of the expected value of $f_c(T)(Z)$ gives us the result (after some algebra)
\begin{equation}
\tilde{C}(t, T, x_1, x_2) = e^{-r(T-t)} \{x_1 N(d_1(t, T)) - k x_2 N(d_2(t, T))\},
\end{equation}
where $N(d)$ is the cumulative standard normal probability distribution function, $d_1(t, T) = d_2(t, T) + \Sigma(t, T)$, and
\begin{equation}
d_2(t, T) = \frac{\ln \frac{x_1}{x_2} - \ln k - \frac{1}{2} \Sigma^2(t, T)}{\Sigma(t, T)},
\end{equation}
and $\Sigma(t, T)$ is defined in (3.11). Not surprisingly, we are back to the Margrabe’s Formula (see Margrabe [19]) extended to Gaussian Volterra processes. We remark that in the case of stochastic volatility models, one must in practice resort to Monte Carlo methods to find $C$. It may be more efficient to go back to the original Fourier expression in this case.

4. QUADRATIC HEDGING IN THE FORWARD MARKET

In this Section we employ the functional dependency on forward prices $f_1(t, T), f_2(t, T)$ in the spread option price $C(t, T)$ to study the question of hedging. To simplify matters considerably, we focus our attention to the non-stochastic volatility case, that is, we assume that $\eta(t) \equiv \eta$ for two positive constants $\sigma$ and $\eta$. Obviously, by scaling the kernel functions $g$ and $h$, we may without loss of generality assume $\eta = 1$. Hence, from Prop. 3.2 we find the following forward prices written on $S_i, i = 1, 2$ for $t \leq T$,
\begin{align*}
f_1(t, T) &= \Lambda_1(T) \exp \left( \int_{-\infty}^t g(T, s) dU(s) + \int_t^T \psi_U(-ig(T, s)) ds \right), \\
f_2(t, T) &= \Lambda_2(T) \exp \left( \int_{-\infty}^t h(T, s) dV(s) + \int_t^T \psi_V(-ih(T, s)) ds \right).
\end{align*}
The forward price dynamics are martingales, and by a direct application of the Itô Formula for jump processes (see e.g. Øksendal and Sulem [20]) we have the following:
\begin{align*}
\frac{df_1(t, T)}{f_1(t, T)} &= c_1 g(T, s) dW_1(t) + \int_{R^2} \left( e^{zg(T, t)} - 1 \right) \tilde{N}(dz_1, dz_2, dt), \\
\frac{df_2(t, T)}{f_2(t, T)} &= c_2 h(T, t) dW_2(t) + \int_{R^2} \left( e^{zh(T, t)} - 1 \right) \tilde{N}(dz_1, dz_2, dt).
\end{align*}
Here, $\tilde{N}(dz_1, dz_2, dt)$ is the compensated Poisson random measure of $L = (U, V)$, and $W_1$ and $W_2$ are the two Brownian motions in the Lévy–Kintchine representation of $L = (U, V)$ which are correlated by $\rho$.

We seek to find a self-financing portfolio of forwards $f_1$ and $f_2$ and a bank account such that we minimize the hedging error. The hedging error is measured in terms of the expected quadratic distance between the hedging portfolio and the payoff of the spread option. This is known as the quadratic hedge (see Cont and Tankov [10]).

Denote by $(\phi_0, \phi_1, \phi_2)$ the investment strategy where $\phi_0(t)$ is the amount of money in the bank at time $t$ yielding a risk free interest $r$ and $\phi_i(t)$ is the position in forward $f_i(t, T)$ at time $t, i = 1, 2$. We suppose $t \mapsto (\phi_0(t), \phi_1(t), \phi_2(t))$ is $\mathcal{F}_t$-adapted. As forwards are costless to enter (either short or long), the value of this portfolio at time $t$, denoted $V(t)$, is the amount of money
by the self-financing hypothesis. We assume that
\[ V_{\xi}(4.2) \]
error to be \( \hat{\epsilon}(\phi_1, \phi_2) := \hat{V}(T) - \hat{C}(T, T) \),

with \( \hat{C}(t, T) = \exp(-rt)C(t, T) \). Our aim is to find a strategy that minimizes the error, that is, find \( \phi_1, \phi_2 \) such that \( \mathbb{E}[\epsilon^2(\phi_1, \phi_2)] \) is minimized. This strategy is derived in the next Proposition:

**Proposition 4.1.** Introduce the matrix \( \mathbf{A}(t) \in \mathbb{R}^{2 \times 2} \) with the elements \( a_{ij}(t), i, j = 1, 2 \) defined as

\[
a_{11}(t) = e^{-rt}f_1^2(t, T) \left\{ c_1^2 + \int_{\mathbb{R}^2} (e^{z_1(t,T)} - 1)^2 \ell(dz_1, dz_2) \right\}
\]

\[
a_{12}(t) = a_{21}(t) = e^{-rt}f_1(t, T)f_2(t, T) \left\{ \frac{1}{2} \rho c_1 c_2 + \int_{\mathbb{R}^2} (e^{z_1(t,T)} - 1)(e^{z_2(t,T)} - 1) \ell(dz_1, dz_2) \right\}
\]

\[
a_{22}(t) = e^{-rt}f_2^2(t, T) \left\{ c_2^2 + \int_{\mathbb{R}^2} (e^{z_2(t,T)} - 1)^2 \ell(dz_1, dz_2) \right\}.
\]

Furthermore, let \( \mathbf{b}(t) \in \mathbb{R}^2 \) be the vector with elements

\[
b_1(t) = \frac{\partial \hat{C}}{\partial f_1}(t, T, f_1(t, T), f_2(t, T))f_1^2(t, T)c_1^2
\]

\[
\quad + \frac{1}{2} \rho c_1 c_2 \frac{\partial \hat{C}}{\partial f_2}(t, T, f_1(t, T), f_2(t, T))f_1(t, T)f_2(t, T)
\]

\[
\quad + \int_{\mathbb{R}^2} \left\{ \hat{C}(t, f_1(t, T))(1 + z_1), f_2(t, T)(1 + z_2) - \hat{C}(t, f_1(t, T), f_2(t, T))
\]

\[
\quad - \sum_{i=1}^2 z_i \frac{\partial \hat{C}}{\partial f_i}(t, f_1(t, T), f_2(t, T)) \right\} f_1(t, T)(e^{z_1(t,T)} - 1) \ell(dz_1, dz_2),
\]

\[
b_2(t) = \frac{\partial \hat{C}}{\partial f_2}(t, T, f_1(t, T), f_2(t, T))f_2^2(t, T)c_2^2
\]

\[
\quad + \frac{1}{2} \rho c_1 c_2 \frac{\partial \hat{C}}{\partial f_1}(t, T, f_1(t, T), f_2(t, T))f_1(t, T)f_2(t, T)
\]

\[
\quad + \int_{\mathbb{R}^2} \left\{ \hat{C}(t, f_1(t, T))(1 + z_1), f_2(t, T)(1 + z_2) - \hat{C}(t, f_1(t, T), f_2(t, T))
\]

\[
\quad - \sum_{i=1}^2 z_i \frac{\partial \hat{C}}{\partial f_i}(t, f_1(t, T), f_2(t, T)) \right\} f_2(t, T)(e^{z_2(t,T)} - 1) \ell(dz_1, dz_2).
\]

Assume that \( \mathbf{A}(t) \) is invertible for every \( t \leq T \). Then the quadratic hedging strategy \( \phi(t) = (\phi_1(t), \phi_2(t))^* \) is the unique solution to \( \mathbf{A}(t)\phi(t) = \mathbf{b}(t) \).

**Proof.** We have from the definition of \( \hat{V}(t) \) in (4.1).

\[
\hat{V}(T) = V(0) + \int_0^T \phi_1(t)e^{-rt}df_1(t, T) + \int_0^T \phi_2(t)e^{-rt}df_2(t, T)
\]

\[
= V(0) + \int_0^T \phi_1(t)e^{-rt}f_1(t, T)c_1 dW_1(t) + \int_0^T \phi_2(t)e^{-rt}f_2(t, T)c_2 dW_2(t)
\]

\[
+ \int_0^T \int_{\mathbb{R}^2} \phi_1(t)e^{-rt}f_1(t, T, - T) \left( e^{z_1(T,t)} - 1 \right) \mathcal{N}(dz_1, dz_2, dt)
\]
By the isometry formula for stochastic integrals,
\[ + \int_0^T \int_{\mathbb{R}^2} \phi_2(t)e^{-rt}f_2(t-, T) \left( e^{z(t,T)} - 1 \right) \bar{N}(dz_1, dz_2, dt). \]

We next apply the Itô Formula on the martingale process \( t \mapsto \hat{C}(t, T) := \hat{C}(t, T, f_1(t, T), f_2(t, T)), \) \( t \leq T, \) where we have emphasized the explicit dependency on \( f_1(t, T) \) and \( f_2(t, T) \) in the spread option price (recalling (3.9)). We calculate,
\[
\hat{C}(T, T, f_1(T, T), f_2(T, T)) = C(0, T, f_1(0, T), f_2(0, T)) \\
+ \int_0^T \frac{\partial \hat{C}}{\partial f_1}(t, T, f_1(t, T), f_2(t, T))f_1(t, T)c_1 dW_1(t) \\
+ \int_0^T \frac{\partial \hat{C}}{\partial f_2}(t, T, f_1(t, T), f_2(t, T))f_2(t, T)c_2 dW_2(t) \\
+ \int_0^T \int_{\mathbb{R}^2} \left( \hat{C}(t-, T, f_1(t-, T)(1 + z_1), f_2(t-, T)(1 + z_2)) - \hat{C}(t-, T, f_1(t-, T, f_2(t-, T)) \right. \\
\left. + \sum_{i=1}^2 z_i \frac{\partial \hat{C}}{\partial f_i}(t-, f_1(t-, T, f_2(t-, T))f_i(t-, T)) \right) \bar{N}(dz_1, dz_2, dt).
\]

The hedging error is thus equal to (recalling that \( V(0) = C(0, T), \))
\[
\varepsilon(\phi_1, \phi_2) = \int_0^T \left\{ \phi_1(t)e^{-rt} - \frac{\partial \hat{C}}{\partial f_1}(t, T, f_1(t, T), f_2(t, T)) \right\} f_1(t, T)c_1 dW_1(t) \\
+ \int_0^T \left\{ \phi_2(t)e^{-rt} - \frac{\partial \hat{C}}{\partial f_2}(t, T, f_1(t, T), f_2(t, T)) \right\} f_2(t, T)c_2 dW_2(t) \\
+ \int_0^T \int_{\mathbb{R}^2} \left\{ \phi_1(t-)e^{-rt}f_1(t-, T)(e^{z_1g(t,T)} - 1) + \phi_2(t-)e^{-rt}f_2(t-, T)(e^{z_2h(t,T)} - 1) \right. \\
- \left( \hat{C}(t, T, f_1(t-, T)(1 + z_1), f_2(t-, T)(1 + z_2)) - \hat{C}(t, T, f_1(t-, T), f_2(t-, T)) \right) \\
\left. - \sum_{i=1}^2 z_i \frac{\partial \hat{C}}{\partial f_i}(t-, f_1(t-, T, f_2(t-, T))f_i(t-, T)) \right\} \bar{N}(dz_1, dz_2, dt).
\]

By the isometry formula for stochastic integrals,
\[
\mathbb{E} \left[ \varepsilon^2(\phi_1, \phi_2) \right] = \mathbb{E} \left[ \int_0^T \left\{ \phi_1(t)e^{-rt} - \frac{\partial \hat{C}}{\partial f_1}(t, T, f_1(t, T), f_2(t, T)) \right\}^2 f_1^2(t, T)c_1^2 dt \right] \\
+ \mathbb{E} \left[ \int_0^T \left\{ \phi_2(t)e^{-rt} - \frac{\partial \hat{C}}{\partial f_2}(t, T, f_1(t, T), f_2(t, T)) \right\}^2 f_2^2(t, T)c_2^2 dt \right] \\
+ \rho \mathbb{E} \left[ \int_0^T \left\{ \phi_1(t)e^{-rt} - \frac{\partial \hat{C}}{\partial f_1}(t, T, f_1(t, T), f_2(t, T)) \right\} \left\{ \phi_2(t)e^{-rt} - \frac{\partial \hat{C}}{\partial f_2}(t, T, f_1(t, T), f_2(t, T)) \right\} f_1(t, T)f_2(t, T)c_1c_2 dt \right]
\]
\[
+ \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^2} \left\{ \phi_1(t-)e^{-rt}f_1(t-, T)(e^{z_1g(t,T)} - 1) + \phi_2(t-)e^{-rt}f_2(t-, T)(e^{z_2h(t,T)} - 1) \right. \\
- \left( \hat{C}(t, T, f_1(t-, T)(1 + z_1), f_2(t-, T)(1 + z_2)) - \hat{C}(t, T, f_1(t-, T), f_2(t-, T)) \right) \\
\left. - \sum_{i=1}^2 z_i \frac{\partial \hat{C}}{\partial f_i}(t-, f_1(t-, T, f_2(t-, T))f_i(t-, T)) \right\}^2 \ell(dz_1, dz_2, dt). \]
To find the optimal hedges, we derive the functional differentials of the above expression with respect to $\phi_1$ and $\phi_2$ and equate this with zero, yielding first-order conditions for a minimum:

$$0 = \left\{ \phi_1(t)e^{-rt} - \frac{\partial \hat{C}}{\partial f_1}(t, T, f_1(t, T), f_2(t, T)) \right\} f_1^2(t, T)c_1^2$$

$$+ \frac{1}{2} \rho \left\{ \phi_2(t)e^{-rt} - \frac{\partial \hat{C}}{\partial f_2}(t, T, f_1(t, T), f_2(t, T)) \right\} c_1c_2f_1(t, T)f_2(t, T)$$

$$+ \int_{\mathbb{R}^2} \left\{ \phi_1(t)e^{-rt}f_1(t, T)(e^{\varepsilon_1g(t, T)} - 1) + \phi_2(t)e^{-rt}f_2(t, T)(e^{\varepsilon_2h(t, T)} - 1) - \left( \hat{C}(t, f_1(t, T)(1 + z_1), f_2(t, T)(1 + z_2)) - \hat{C}(t, f_1(t, T), f_2(t, T)) \right) \right\}$$

$$+ \sum_{i=1}^2 \int_{\mathbb{R}^2} \frac{\partial \hat{C}}{\partial f_i}(t, T, f_1(t, T), f_2(t, T)) \right\} f_1(t, T)(e^{\varepsilon_1g(t, T)} - 1) \ell(dz_1, dz_2) \right\} f_2(t, T)(e^{\varepsilon_2h(t, T)} - 1) \ell(dz_1, dz_2) \right\}$$

and

$$0 = \left\{ \phi_2(t)e^{-rt} - \frac{\partial \hat{C}}{\partial f_2}(t, T, f_1(t, T), f_2(t, T)) \right\} f_2^2(t, T)c_2^2$$

$$+ \frac{1}{2} \rho \left\{ \phi_1(t)e^{-rt} - \frac{\partial \hat{C}}{\partial f_1}(t, T, f_1(t, T), f_2(t, T)) \right\} c_1c_2f_1(t, T)f_2(t, T)$$

$$+ \int_{\mathbb{R}^2} \left\{ \phi_1(t)e^{-rt}f_1(t, T)(e^{\varepsilon_1g(t, T)} - 1) + \phi_2(t)e^{-rt}f_2(t, T)(e^{\varepsilon_2h(t, T)} - 1) - \left( \hat{C}(t, f_1(t, T)(1 + z_1), f_2(t, T)(1 + z_2)) - \hat{C}(t, f_1(t, T), f_2(t, T)) \right) \right\}$$

$$+ \sum_{i=1}^2 \int_{\mathbb{R}^2} \frac{\partial \hat{C}}{\partial f_i}(t, T, f_1(t, T), f_2(t, T)) \right\} f_2(t, T)(e^{\varepsilon_2h(t, T)} - 1) \ell(dz_1, dz_2) \right\}$$

But this leads to a linear system of two equations in $\phi_1$ and $\phi_2$, as described in the Proposition. Hence, the proof is complete. □

Note that $\hat{A}(t)\phi(t) = \mathbf{b}(t)$ has a solution if and only if the matrix $\hat{A}(t)$ is invertible. We have that the determinant of $\hat{A}(t)$ is

$$\det(\hat{A}(t)) = e^{-2rt}f_1^2(t, T)f_2^2(t, T)$$

$$\times \left\{ c_1^2 + \int_{\mathbb{R}^2} (e^{\varepsilon_1g(t, T)} - 1)^2 \ell(dz_1, dz_2) \right\} \left\{ c_2^2 + \int_{\mathbb{R}^2} (e^{\varepsilon_2h(t, T)} - 1)^2 \ell(dz_1, dz_2) \right\}$$

$$- \left\{ \frac{1}{2} \rho c_1c_2 + \int_{\mathbb{R}^2} (e^{\varepsilon_1g(t, T)} - 1)(e^{\varepsilon_2h(t, T)} - 1) \ell(dz_1, dz_2) \right\}^2$$

Hence, if this is different that zero, we find a unique solution.

Let us consider the simple case of a bivariate Brownian motion $L = (B, W)$. In this case the matrix $\hat{A}(t)$ and the vector $\mathbf{b}(t)$ have significantly simpler forms and reduce to

$$a_{11}(t) = e^{-rt}f_1^2(t, T)c_1^2$$

$$a_{12}(t) = a_{21}(t) = e^{-rt}f_1(t, T)f_2(t, T)\frac{1}{2} \rho c_1c_2$$

$$a_{22}(t) = e^{-rt}f_2^2(t, T)c_2^2.$$

and

$$b_1(t) = \frac{\partial \hat{C}}{\partial f_1}(t, T, f_1(t, T), f_2(t, T))f_1^2(t, T)c_1^2$$
\[ b_2(t) = \frac{\partial \hat{C}}{\partial f_2}(t, T, f_1(t, T), f_2(t, T))f_2^2(t, T)c_2^2 + \frac{1}{2}\rho c_1c_2 \frac{\partial \hat{C}}{\partial f_1}(t, T, f_1(t, T), f_2(t, T))f_1(t, T)f_2(t, T). \]

Because the determinant of \( \mathcal{A}(t) \) in this case becomes

\[ \det(\mathcal{A}(t)) = e^{-2rt}f_1^2(t,T)f_2^2(t,T)c_1^2c_2^2(1 - \frac{1}{4}\rho^2), \]

the unique solution of \( \mathcal{A}(t)\hat{\phi}(t) = b(t) \) always exists. One can easily compute the hedge by finding the inverse of \( \mathcal{A}(t) \).

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