Stochastic Polyak Step-size for SGD: 
An Adaptive Learning Rate for Fast Convergence

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Abstract
We propose a stochastic variant of the classical Polyak step-size (Polyak, 1987) commonly used in the subgradient method. Although computing the Polyak step-size requires knowledge of the optimal function values, this information is readily available for typical modern machine learning applications. Consequently, the proposed stochastic Polyak step-size (SPS) is an attractive choice for setting the learning rate for stochastic gradient descent (SGD). We provide theoretical convergence guarantees for SGD equipped with SPS in different settings, including strongly convex, convex and non-convex functions. Furthermore, our analysis results in novel convergence guarantees for SGD with a constant step-size. We show that SPS is particularly effective when training over-parameterized models capable of interpolating the training data. In this setting, we prove that SPS enables SGD to converge to the true solution at a fast rate without requiring the knowledge of any problem-dependent constants or additional computational overhead. We experimentally validate our theoretical results via extensive experiments on synthetic and real datasets. We demonstrate the strong performance of SGD with SPS compared to state-of-the-art optimization methods when training over-parameterized models.

1 Introduction
We consider solving the finite-sum optimization problem:
\[
\min_{x \in \mathbb{R}^d} \left[ f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right]. \tag{1}
\]
This problem is prevalent in machine learning tasks where \( x \) corresponds to the model parameters, \( f_i(x) \) represents the loss on the training point \( i \) and the aim is to minimize the average loss \( f(x) \) across training points. We denote \( X^* \subseteq \mathbb{R}^d \) to be the set of optimal points \( x^* \) of (1) and assume that \( X^* \) is not empty. We use \( f^* \) to denote the minimum value of \( f \), obtained at a point \( x^* \in X^* \). Analogously, \( f^*_i \) denotes the (unconstrained) minimum value of the function \( f_i \) for each \( i \in \{1, \ldots, n\} \). Depending on the model under study, the function \( f \) can either be strongly-convex, convex, or non-convex.

1.1 Background and Main Contributions
Stochastic gradient descent (SGD) (Robbins & Monro, 1951; Nemirovski & Yudin, 1978; 1983; Shalev-Shwartz et al., 2007; Nemirovski et al., 2009; Hardt et al., 2016), is the workhorse for training supervised machine learning problems that have the generic form (1).

Step-size selection for SGD. The main parameter for guaranteeing the convergence of SGD is the step-size or the learning rate. In recent years, several ways of selecting the step-size have been proposed. Moulines & Bach (2011); Needell et al. (2016); Needell & Ward (2017); Nguyen et al. (2018); Gower et al. (2019) propose a non-asymptotic analysis of SGD with constant step-size for convex and strongly convex functions. For non-convex functions, such an analysis can be found in Ghadimi & Lan (2013); Bottou et al. (2018). Using a constant step-size for SGD guarantees convergence to a neighbourhoood of the solution. A common technique to guarantee convergence to the exact optimum is to use a decreasing step-size (Robbins & Monro, 1951; Ghadimi & Lan, 2013; Gower et al., 2019; Nemirovski et al., 2009; Karimi et al., 2016). More recently, adaptive methods (Duchi et al., 2011; Liu et al., 2019; Kingma & Ba, 2015; Bengio, 2015; Vaswani et al., 2019b; Li & Orabona, 2019; Ward et al., 2019) that adjust the step-size on the fly have become wide-spread and are particularly beneficial when training deep neural networks.

Contributions: Inspired by the classical Polyak step-size (Polyak, 1987) commonly used with the deterministic subgradient method (Hazan & Kakade, 2019; Boyd et al., 2003), we propose a novel adaptive learning rate for SGD. The proposed step-size is a natural extension of the Polyak step-size to the stochastic setting. We name it stochastic Polyak step-
size (SPS). Although computing SPS requires knowledge of the optimal function values \( f^* \); we argue that this information is readily available for modern machine learning applications (for example is zero for most standard losses), making SPS an attractive choice for SGD.

In Section 3, we provide theoretical guarantees for the convergence of SGD with SPS in different scenarios including strongly convex, convex and non-convex smooth functions. Although SPS is provably larger than the typically used constant step-size, we guarantee its convergence to a reasonable neighborhood around the optimum. We establish a connection between SPS and the optimal step-size used in sketch and project methods for solving linear systems. Furthermore, in Appendix C, we provide convergence guarantees for convex non-smooth functions. We also show that by progressively increasing the batch-size for computing the stochastic gradients, SGD with SPS converges to the optimum.

Technical assumptions for proving convergence. Besides smoothness and convexity, several papers (Shamir & Zhang, 2013; Recht et al., 2011; Hazan & Kale, 2014; Rakhlin et al., 2012) assume that the variance of the stochastic gradient is bounded: that is there exists a \( c \) such that \( E_i \| \nabla f_i(x) \|^2 \leq c \). However, in the unconstrained setting, this assumption contradicts the assumption of strong convexity (Nguyen et al., 2018; Gower et al., 2019). In another line of work, growth conditions on the stochastic gradients have been used to guarantee convergence. In particular, the weak growth condition has been used in (Bertsekas & Tsitsiklis, 1996; Bottou et al., 2018; Nguyen et al., 2018). It states that there exist constants \( \rho, \delta \) such that \( E_i \| \nabla f_i(x) \|^2 \leq \rho E \| \nabla f(x) \|^2 + \delta \).

Contributions: As a corollary of our theoretical results, we show that SPS is particularly effective under this interpolation setting. Specifically, we prove that SPS enables SGD to converge to the true solution at a fast rate matching the deterministic case. Moreover, SPS does not require the knowledge of any problem-dependent constants or additional computational overhead.

Over-parametrized models and interpolation condition. Modern machine learning models such as non-parametric regression or over-parametrized deep neural networks are highly expressive and can fit or interpolate the training dataset completely (Zhang et al., 2016; Ma et al., 2018). In this setting, SGD with constant step-size can been shown to converge to the exact optimum at the deterministic rate (Schmidt & Roux, 2013; Ma et al., 2018; Vaswani et al., 2019a;b; Gower et al., 2019; Berrada et al., 2019).

Contributions: As a corollary of our theoretical results, we show that SPS is particularly effective under this interpolation setting. Specifically, we prove that SPS enables SGD to converge to the true solution at a fast rate matching the deterministic case. Moreover, SPS does not require the knowledge of any problem-dependent constants or additional computational overhead.

Experimental Evaluation. In Section 4, we experimentally validate our theoretical results via experiments on synthetic datasets. We also evaluate the performance of SGD equipped with SPS relative to the state-of-the-art optimization methods when training over-parameterized models for deep matrix factorization, binary classification using kernels and multi-class classification using deep neural networks. For each of these tasks, we demonstrate the superior convergence of the proposed method.

2 SGD and the Stochastic Polyak Step-size

The optimization problem (1) can be solved using SGD:

\[ x^{k+1} = x^k - \gamma_k \nabla f_i(x^k) \]  

where example \( i \in [n] \) is chosen uniformly at random and \( \gamma_k > 0 \) is the step-size in iteration \( k \).

2.1 The Polyak step-size

Before explaining the proposed stochastic Polyak step-size, we first present the deterministic variant by Polyak (Polyak, 1987). This variant is commonly used in the analysis of deterministic subgradient methods (Boyd et al., 2003; Hazan & Kakade, 2019).

The deterministic Polyak step-size. For convex functions, the deterministic Polyak step-size at iteration \( k \) is the one that minimizes an upper-bound \( Q(\gamma) \) on the distance of the iterate \( x_{k+1} \) to the optimal solution: \( \|x_{k+1} - x^*\|_2^2 \leq Q(\gamma) \):

\[ Q(\gamma) = \|x^k - x^*\|^2 - 2\gamma [f(x^k) - f^*] + \gamma^2\|g_k\|^2. \]
That is, $\gamma_k = \text{argmin}_\gamma \{ Q(\gamma_k) \} = \frac{f(x_k) - f^*}{\|g^k\|^2}$. Here $g^k$ denotes a subgradient of function $f$ at point $x_k$ and $f^*$ the optimum function value. For more details and a convergence analysis of the deterministic subgradient method, please check Appendix A.2. Note that the above step-size can be used only when the optimal value $f^*$ is known, however Boyd et al. (2003) demonstrate that $f^* = 0$ for several applications (for example, finding a point in the intersection of convex sets, positive semidefinite matrix completion and solving convex inequalities).

**Stochastic Polyak Step-size.** It is clear that using the deterministic Polyak step-size in the update rule of SGD is impractical. It requires the computation of the function value $f$ and its full gradient in each iteration.

To avoid this, we propose the stochastic Polyak step-size (SPS) for SGD:

$$\text{SPS: } \gamma_k = \frac{f_i(x_k) - f^*_i}{c\|\nabla f_i(x_k)\|^2}$$

Note that SPS requires the evaluation of only the stochastic gradient $\nabla f_i(x_k)$ and of the function $f_i(x_k)$ at the current iterate (quantities that can be computed in the update rule of SGD without further cost). However, it requires the knowledge of $f^*_i$. As we will see in Section 4, for typical machine learning applications such as empirical risk minimization where $f_i$ is the loss on a training example, the optimal values $f^*_i = 0$. An important quantity in the step-size is the parameter $0 < c \in \mathbb{R}$ which can be set theoretically based on the properties of the function under study. For example, for strongly convex functions, one should select $c = 1/2$ for optimal convergence. Thus, if the function is known to be strongly convex, $c$ is not a hyper-parameter to be tuned.

In addition to SPS, in some of our convergence results we require its bounded variant:

$$\text{SPS}_{\text{max}} : \gamma_k = \min \left\{ \frac{f_i(x_k) - f^*_i}{c\|\nabla f_i(x_k)\|^2}, \gamma_k \right\}$$

Here $\gamma_k > 0$ is a bound that restricts SPS from being very large and is essential to ensure convergence to a small neighborhood around the solution. If $\gamma_k = \infty$ then SPS_{\text{max}} is equivalent to SPS.

**Closely related work.** We now briefly compare against the recently proposed stochastic variants of the Polyak step-size (Rolinek & Martius, 2018; Oberman & Prazeres, 2019; Berrada et al., 2019). In Section 3, we present a detailed comparison of the theoretical convergence rates.

In Rolinek & Martius (2018), the L4 algorithm has been proposed showing that a stochastic variant of the Polyak step for SGD achieves good empirical results for training neural networks. However it has no theoretical convergence guarantees. The step-size is very similar to SPS (3) but each update requires an online estimation of the $f^*_i$ which does not result in robust empirical performance and requires up to three hyper-parameters.

Oberman & Prazeres (2019) use a different variant of the stochastic Polyak step-size: $\gamma_k = \frac{2(f(x_k) - f^*)}{c\|\nabla f_i(x_k)\|^2}$. This step-size requires knowledge of the quantity $E_i[\|\nabla f_i(x_k)\|^2]$ for all iterates $x_k$ and the evaluation of $f(x_k)$ in each step, making it impractical for finite-sum problems with large $n$. Moreover, their theoretical results focus only on strongly convex smooth functions.

In the ALI-G algorithm proposed by Berrada et al. (2019), the step-size is set as: $\gamma_k = \min \left\{ \frac{f_i(x_k)}{\|\nabla f_i(x_k)\|^2}, \frac{\|f_i(x_k)\|^2}{\|\nabla f_i(x_k)\|^2 - \delta \|\nabla f_i(x_k)\|^2}, \eta \right\}$, where $\delta > 0$ is a positive constant. Unlike our setting, their theoretical analysis relies on an $\epsilon$-interpolation condition. Moreover, the values of the parameter $\delta$ and $\eta$ that guarantee convergence heavily depend on the smoothness parameter of the objective $f$, limiting the method’s practical applicability. In Section 3, we show that as compared to (Berrada et al., 2019), the proposed method results in both better rates and a smaller neighborhood of convergence. For the case of over-parameterized models, our step-size selection guarantees convergence to the exact solution while the step proposed in (Berrada et al., 2019) finds only an approximate solution that could be $\delta$ away from the optimum. In Section 4, we experimentally show that SPS_{\text{max}} results in better convergence than ALI-G in several practical scenarios.

### 2.2 Optimal Objective Difference

Unlike the typical analysis of SGD that assumes a finite gradient noise $z := E[\|\nabla f_i(x^*)\|^2]$, in all our results, we assume a finite optimal objective difference.

**Assumption 2.1 (Finite optimal objective difference).**

$$\sigma^2 := E_i[f_i(x^*) - f^*_i] = f(x^*) - E_i[f^*_i]$$

This is a very weak assumption. Moreover when (1) is the training problem of an over-parametrized model such as a deep neural network or involves solving a consistent linear system or classification on linearly separable data, each individual loss function $f_i$ attains its minimum at $x^*$, and thus $f_i(x^*) - f^*_i = 0$. In this interpolation setting, it follows that $\sigma = 0$.

### 3 Convergence Analysis

In this section, we present the main convergence results. We quantify the convergence rates of SGD with the stochastic Polyak step-size for strongly convex, convex and non-convex functions. For the formal definitions and properties of functions see Appendix A.1. Proofs of all key results can be found in the Appendix B.
3.1 Upper and Lower Bounds of SPS

If a function $g$ is $\mu$-strongly convex and $L$-smooth the following bounds hold: $\frac{1}{2\mu} \| \nabla g(x) \|^2 \leq g(x) - g(x^*) \leq \frac{1}{2\mu} \| \nabla g(x) \|^2$. Using these bounds and by assuming that the functions $f_i$ in problem (1) are $\mu_i$-strongly convex and $L_i$-smooth, it is straightforward to see that SPS can be lower and upper bounded as follows:

$$\frac{1}{2cL_{\max}} \leq \frac{1}{2cL_i} \leq \gamma_k = \frac{f_i(x_k) - f_i^*}{c\| \nabla f_i(x_k) \|^2} \leq \frac{1}{2c\mu_i},$$  \hspace{1cm} (6)

where $L_{\max} = \max\{L_i\}_{i=1}^n$.

3.2 Convex Functions

We present three main theoretical results for convex functions, the first when $f$ is a strongly convex function, the second result for weakly convex functions and the third result that establishes a connection between SPS and randomized methods for solving linear systems.

3.2.1 Sum of Strongly Convex and Convex Functions

In this section, we assume that at least one of the components $f_i$ is $\mu_i$ strongly convex function, implying that the function $f$ is $\mu$-strongly convex.

**Theorem 3.1.** Let $f_i$ be $L_i$-smooth convex functions with at least one of them being a strongly convex function. SGD with $SPS_{\max}$ with $c \geq 1/2$ converges as:

$$\mathbb{E}\|x^k - x^*\|^2 \leq (1 - \mu\rho)^k \|x^0 - x^*\|^2 + \frac{2\gamma_k \sigma^2}{\mu},$$ \hspace{1cm} (7)

where $\rho := \min\{\frac{1}{2cL_{\max}}, \gamma_k\}$, $\mu = \mathbb{E}[\mu_i]$ is the average strong-convexity of the finite sum and $L_{\max} = \max\{L_i\}_{i=1}^n$ is the maximum smoothness constant. The best convergence rate and the tightest neighborhood are obtained for $c = 1/2$.

Note that in Theorem 3.1, we do not make any assumption on the value of the upper bound $\gamma_k$. However, it is clear that for convergence to a small neighborhood of the solution $x^*$ (unique solution for strongly convex functions) $\gamma_k$ should not be very large.

Another important aspect of Theorem 3.1 is that it provides convergence guarantees without requiring strong assumptions like bounded gradients or growth conditions. We do not use these conditions because SPS provides a natural bound on the norm of the gradients. In the following corollaries we make additional assumptions to better understand the convergence of SGD with $SPS_{\max}$.

In our first corollary, we assume that our model is able to interpolate the data (each individual loss function $f_i$ attains its minimum at $x^*$). The interpolation assumption enables us to guarantee the convergence of SGD with SPS, without an upper-bound on the step-size ($\gamma_k = \infty$).

**Corollary 3.2.** Assume interpolation ($\sigma = 0$) and let all assumptions of Theorem 3.1 be satisfied. SGD with SPS with $c = 1/2$ (optimal choice) converges as:

$$\mathbb{E}\|x^k - x^*\|^2 \leq \left(1 - \frac{\bar{\mu}}{L_{\max}}\right)^k \|x^0 - x^*\|^2 + \frac{2\sigma^2 L}{\mu^2}.$$ 

We compare the convergence rate in Corollary 3.2 to that of stochastic line search (SLS) proposed in (Vaswani et al., 2019b). In the same setting as Corollary 3.2, SLS achieves a slower linear rate with a worse constant $\max\left\{1 - \frac{\mu}{L_{\max}}, 1 - \gamma_k\bar{\mu}\right\}$.

In the next corollary, in order to compare against the results for ALI-G from Berrada et al. (2019), we make the strong assumption that all functions $f_i$ have the same properties. We note that such an assumption in the interpolation setting is quite strong and reduces the finite-sum optimization to minimization of a single function in the finite sum.

**Corollary 3.3.** Let all the assumptions in Theorem 3.1 be satisfied and let all $f_i$ be $\mu$-strongly convex and $L$-smooth.

SGD with $SPS_{\max}$ with $c = 1/2$ converges as:

$$\mathbb{E}\|x^k - x^*\|^2 \leq \left(1 - \frac{\mu}{L}\right)^k \|x^0 - x^*\|^2 + \frac{2\sigma^2 L}{\mu^2}.$$ 

For the interpolated case we obtain the same convergence as Corollary 3.2 with $\bar{\mu} = \mu$ and $L_{\max} = L$.

Note that, the result of Corollary 3.3 is obtained by substituting $\gamma_k = \frac{\mu}{2\bar{\mu}} = \frac{1}{2} \frac{1}{\max\{L_i\}_{i=1}^n}$ into (7).

For the setting of Corollary 3.3, Berrada et al. (2019) show the linear convergence to a much larger neighborhood than ours and with slower rate. In particular, their rate is $1 - \frac{\mu}{2L}$ and the neighborhood is $\frac{\mu}{8\mu} \left(\frac{f_i}{L} + \frac{\delta}{L}\right)$ where $\delta > 2L\epsilon$ and $\epsilon$ is the $\epsilon$-interpolation parameter $\epsilon > \max_i \|f_i(x^*) - f_i^*\|$ which by definition is bigger than $\sigma^2$. Under interpolation where $\sigma = 0$, our method converges linearly to the $x^*$ while the algorithm proposed by Berrada et al. (2019) still converges to a neighborhood that is proportional to the parameter $\delta$.

An interesting outcome of Theorem 3.1 is a novel analysis for SGD with a constant step-size. In particular, note that if the bound in $SPS_{\max}$ is selected to be $\gamma_k \leq \frac{1}{2cL_{\max}}$, then using the lower bound of (6), it can be easily shown that our method reduces to SGD with constant step-size $\gamma_k = \gamma = \gamma_k \leq \frac{1}{2cL_{\max}}$. In this case, we obtain the following...
correlation rate.

**Corollary 3.4.** Let all assumptions of Theorem 3.1 be satisfied. SGD with \( SPS_{\max} \) with \( c = 1/2 \) and \( \gamma_{\epsilon} \leq \frac{1}{L_{\max}} \) becomes SGD with constant step-size \( \gamma \leq \frac{1}{L_{\max}} \) and converges as:

\[
E[\|x^k - x^*\|^2] \leq (1 - \bar{\mu}\gamma)^k \|x^0 - x^*\|^2 + \frac{2\sigma^2}{\bar{\mu}}.
\]

If we further assume interpolation \( (\sigma = 0) \), the iterates of SGD with constant step-size \( \gamma \leq \frac{1}{L_{\max}} \) satisfy:

\[
E[\|x^k - x^*\|^2] \leq (1 - \bar{\mu}\gamma)^k \|x^0 - x^*\|^2.
\]

To the best of our knowledge, ours is the first result that shows convergence of constant step-size SGD to a neighborhood that depends on the optimal objective difference \( \sigma^2 \) (5) and not on the variance \( \bar{z}^2 = \text{E}[\|\nabla f_i(x^*)\|^2] \). If we assume that the component functions \( f_i \) are \( \mu \)-strongly convex and \( L \)-smooth functions, then the following connection: \( \frac{1}{2\rho} \bar{z}^2 \leq \sigma^2 \leq \frac{1}{2\rho} \bar{z}^2 \). Such convergence results to the neighborhood \( \sigma^2 \) have only been suggested in two papers but for different algorithms than ours. In particular, Zhang & Zhou (2019) propose an analysis of stochastic Polyak-Lojasiewicz (PL) condition (Polyak, 1987; Karimi et al., 2016). In particular, we assume that function \( f \) satisfies the PL condition but do not assume convexity of the component functions \( f_i \). The function \( f \) satisfies the PL condition if there exists \( \mu > 0 \) such that: \( \|\nabla f(x)\|^2 \geq 2\mu (f(x) - f^*) \).

**Theorem 3.6.** Assume that function \( f \) satisfies the PL condition with parameter \( \mu \), and let \( f \) and \( f_i \) be smooth functions. SGD with \( SPS_{\max} \) with \( c > \frac{L_{\max}}{4\mu} \) and \( \gamma_{\epsilon} \geq \frac{L_{\max}}{4\mu} \) converges as:

\[
E[f(x^k) - f(x^*)] \leq \nu^k [f(x^0) - f(x^*)] + \frac{L\sigma^2\gamma_{\epsilon}}{2(1 - \nu)\alpha}.
\]

where \( \nu = \gamma_{\epsilon} \left( \frac{1}{\alpha} - 2\mu + \frac{L_{\max}}{2\rho} \right) \in (0, 1] \) and \( \alpha = \min \left\{ \frac{1}{2\rho L_{\max}}, \gamma_{\epsilon} \right\} \).

Under the interpolation setting, \( \sigma = 0 \), and \( SPS_{\max} \) converges to the optimal solution at a linear rate. If \( \gamma_{\epsilon} \leq \min \left\{ \frac{1}{2\rho L_{\max}}, \frac{2\rho}{4\mu L_{\max}} \right\} \), then the analyzed method becomes the SGD with constant step-size and we obtain the following corollary.

**Corollary 3.7.** Assume that function \( f \) satisfies the PL condition and let \( f \) and \( f_i \) be smooth functions. SGD with

\[3.3 Non-convex functions\]

In this section, we present the convergence of SGD with SPS when the component functions \( f_i \) are \( L_i \)-smooth but not necessarily convex.

**3.3.1 Sum of non-convex functions: PL Objective**

We first focus on a special class of non-convex functions that satisfy the Polyak-Lojasiewicz (PL) condition (Polyak, 1987; Karimi et al., 2016). In particular, we assume that function \( f \) satisfies the PL condition but do not assume convexity of the component functions \( f_i \). The function \( f \) satisfies the PL condition if there exists \( \mu > 0 \) such that: \( \|\nabla f(x)\|^2 \geq 2\mu (f(x) - f^*) \).

**Theorem 3.6.** Assume that function \( f \) satisfies the PL condition with parameter \( \mu \), and let \( f \) and \( f_i \) be smooth functions. SGD with \( SPS_{\max} \) with \( c > \frac{L_{\max}}{4\mu} \) and \( \gamma_{\epsilon} \geq \frac{L_{\max}}{4\mu} \) converges as:

\[
E[f(x^k) - f(x^*)] \leq \nu^k [f(x^0) - f(x^*)] + \frac{L\sigma^2\gamma_{\epsilon}}{2(1 - \nu)\alpha}.
\]

where \( \nu = \gamma_{\epsilon} \left( \frac{1}{\alpha} - 2\mu + \frac{L_{\max}}{2\rho} \right) \in (0, 1] \) and \( \alpha = \min \left\{ \frac{1}{2\rho L_{\max}}, \gamma_{\epsilon} \right\} \).

Under the interpolation setting, \( \sigma = 0 \), and \( SPS_{\max} \) converges to the optimal solution at a linear rate. If \( \gamma_{\epsilon} \leq \min \left\{ \frac{1}{2\rho L_{\max}}, \frac{2\rho}{4\mu L_{\max}} \right\} \), then the analyzed method becomes the SGD with constant step-size and we obtain the following corollary.

**Corollary 3.7.** Assume that function \( f \) satisfies the PL condition and let \( f \) and \( f_i \) be smooth functions. SGD with
constant step-size \( \gamma_k = \gamma \leq \frac{\mu}{L_{\max}} \) converges as:
\[
\mathbb{E}[f(x^k) - f(x^*)] \leq \nu^k [f(x^0) - f(x^*)] + \frac{L\sigma^2\gamma}{2(1 - \nu)c}.
\]

To the best of our knowledge this is the first result for the convergence of SGD for PL functions without assuming bounded gradient or bounded variance (for more details see results in (Karimi et al., 2016) and discussion in (Gower et al., 2019)). In the interpolation case, we obtain linear convergence to the optimum with a constant step-size equal to that used in (Vaswani et al., 2019a).

### 3.3.2 GENERAL NON-CONVEX FUNCTIONS

In this section, we assume a common condition used to prove convergence of SGD in the non-convex setting (Bottou et al., 2018).
\[
\mathbb{E}[\|\nabla f_i(x)\|^2] \leq \rho \|\nabla f(x)\|^2 + \delta
\]  
where \( \rho, \delta > 0 \) constants.

**Theorem 3.8.** Let \( f \) and \( f_i \) be smooth functions and assume that there exist \( \rho, \delta > 0 \) such that the condition (8) is satisfied. SGD with \( \text{SPS}_{\max} \) with \( c > \frac{\rho L}{4\delta_{\max}} \) and \( \gamma_k < \max \left\{ \frac{2}{L^2}, \frac{1}{\sqrt{4\rho L_{\max}}} \right\} \) converges as:
\[
\min_{k \in [K]} \mathbb{E}[\|\nabla f(x^k)\|^2] \leq \frac{f(x^0) - f(x^*)}{\alpha K} + \frac{L\sigma^2\gamma^2}{2\alpha}
\]  
where \( \beta_1 = 1 - \frac{\rho c L_{\max} \gamma^2}{2} \) and \( \beta_2 = 1 - \frac{\rho L \gamma}{\sqrt{\rho L_{\max}}} \), \( \alpha = \min \left\{ \frac{\beta_1}{2c L_{\max}}, \gamma \beta_2 \right\} \).

From the above theorem, we observe that SGD with SPS results in \( O(1/K) \) convergence to a value governed by \( \delta \). For the case that \( \delta = 0 \), condition (8) reduces to the strong growth condition (SGC) used in several recent papers (Schmidt & Roux, 2013; Vaswani et al., 2019b:a). It can be easily shown that functions that satisfy the SGC condition necessarily satisfy the interpolation property (Vaswani et al., 2019a). In the special case of interpolation, SGD with SPS is able to find a first-order stationary point as efficiently as deterministic gradient descent. Moreover, for \( c \in \left( \frac{\rho L}{4\delta_{\max}}, \frac{\rho L}{2\delta_{\max}} \right) \), SPS lies in the range \( \left[ \frac{1}{4L^2}, \frac{2}{L^2} \right] \) and thus uses a step-size larger than \( \frac{1}{L^2} \), the best constant step-size analyzed in this setting (Vaswani et al., 2019a).

### 3.4 Additional Convergence Results

In Appendix C, we present some additional convergence results of SGD with SPS. In particular, we prove a \( O(1/\sqrt{K}) \) convergence rate for non-smooth convex functions. Furthermore, similar to (Schmidt et al., 2011), we propose a way to increase the mini-batch size for evaluating the stochastic gradient and guarantee convergence to the optimal solution without interpolation.

### 4 Experimental Evaluation

We validate our theoretical results using synthetic experiments in section 4.1. In section 4.2, we evaluate the performance of SGD with SPS when training over-parametrized models. In particular, we compare against state-of-the-art optimization methods for deep matrix factorization, binary classification using kernel methods and multi-class classification using standard deep neural network models.

#### 4.1 Synthetic experiments

We use a synthetic dataset to validate our theoretical results. Following the procedure outlined in (Nutini et al., 2017), we generate a sparse dataset for binary classification with the number of examples \( n = 1k \) and dimension \( d = 100 \). We use the logistic loss with and without \( \ell_2 \) regularization. The data is generated to ensure that the function \( f \) is strongly convex in both cases. We evaluate the performance of \( \text{SPS}_{\max} \) and set \( c = 1/2 \) as suggested by theorem 3.1. We experiment with three values of \( \gamma_k = \{1, 5, 100\} \). In the regularized case, we first compute the value of \( f_i^* \) for each of the examples and use it to compute the step-size. For the unregularized case, note that the logistic loss is lower-bounded by zero and since the model can correctly classify each point individually, the optimum function value \( f_i^* = 0 \). A similar observation has been used to construct a “truncated” model for improving the robustness of gradient descent in (Asi & Duchi, 2019). In both cases, we benchmark the performance of SPS against constant step-size SGD with \( \gamma = \{0.1, 0.01\} \). From figure 1, we observe that constant step-size SGD is not robust to the step-size; it has good convergence with step-size 0.1, slow convergence when using a step-size of 0.01 and we observe divergence for larger step-sizes. In contrast, all the variants of SPS converge to a neighbourhood of the optimum and the size of the neighbourhood increases as \( \gamma_k \) increases as predicted.
by the theory.

4.2 Experiments for over-parametrized models

In this section, we consider training over-paramterized models that (approximately) satisfy the interpolation condition. Following the logic of the previous section, we evaluate the performance of both the SPS and SPS\textsubscript{max} variants with $f_i^* = 0$. Throughout our experiments, we found that SPS without an upper-bound on the step-size is not robust to the misspecification of interpolation and results in large fluctuations when interpolation is not exactly satisfied. For SPS\textsubscript{max}, the value of $\gamma_b$ that results in good convergence depends on the problem and requires careful parameter tuning. This is also evidenced by the highly variable performance of ALI-G (Berrada et al., 2019) that uses a constant upper-bound on the step-size. To alleviate this problem, we use a smoothing procedure that prevents large fluctuations in the step-size across iterations. This can be viewed as using an adaptive iteration-dependent upper-bound $\gamma_b^k$ where $\gamma_b^k = \tau b/n \gamma_b^{k-1}$. Here, $\tau$ is a tunable hyper-parameter set to 2 in all our experiments, $b$ is the batch-size and $n$ is the number of examples. We note that using an adaptive $\gamma_b$ can be easily handled by our theoretical results. A similar smoothing procedure has been used to control the magnitude of the step-sizes when using the Barzilai-Borwein step-size selection procedure for SGD (Tan et al., 2016) and is related to the “reset” option for using larger step-sizes in (Vaswani et al., 2019b). We set $c = 1/2$ for binary classification using
kernels (convex case) and deep matrix factorization (non-convex PL case). For multi-class classification using deep networks, we empirically find that any value of $c \geq 0.2$ results in convergence. In this case, we observed that across models and datasets, the fastest convergence is obtained with $c = 0.2$ and use this value.

We compare our methods against Adam (Kingma & Ba, 2015), which is the most common adaptive method, and other recent methods that report better performance than Adam: (i) stochastic line-search (SLS) in (Vaswani et al., 2019b) (ii) ALI-G (Berrada et al., 2019) (iii) rectified Adam (RADAM) (Liu et al., 2019) (iv) Look-ahead optimizer (Zhang et al., 2019). To ensure a fair comparison with SPS, we do not use momentum for the competing methods. We use the default learning rates and the publicly available code for the competing methods. All our results are averaged across 5 independent runs.

**Deep matrix factorization.** In the first experiment, we use deep matrix factorization to examine the effect of over-parametrization for the different optimizers. In particular, we solve the non-convex regression problem: $\min_{W_1, W_2} \mathbb{E}_{x \sim N(0, I)} \|W_2W_1x - Ax\|^2$ and use the experimental setup in (Rolinek & Martius, 2018; Vaswani et al., 2019b; Rahimi & Recht, 2017). We choose $A \in \mathbb{R}^{10 \times 6}$ with condition number $\kappa(A) = 10^{10}$ and generate a fixed dataset of 1000 samples. We control the degree of over-parametrization via the rank $k$ of the matrix factors $W_1 \in \mathbb{R}^{k \times 6}$ and $W_2 \in \mathbb{R}^{10 \times k}$. In figure 2, we show the training loss as we vary the rank $k \in \{4, 10\}$ (additional experiments are in Appendix D). For $k = 4$, the interpolation condition is not satisfied, whereas it is exactly satisfied for $k = 10$. We observe that (i) SPS is robust to the degree of over-parametrization and (ii) has performance equal to that of SLS. However, note that SPS does not require the expensive back-tracking procedure of SLS and is arguably simpler to implement.

**Binary classification using kernels.** Next, we compare the optimizers’ performance in the convex, interpolation regime. We consider binary classification using RBF kernels, using the logistic loss without regularization. The bandwidths for the RBF kernels are set according to the validation procedure described in (Vaswani et al., 2019b). We experiment with four standard datasets: mushrooms, rcv1, icmn, and w8a from LIBSVM (Chang & Lin, 2011). Figure 2 shows the training loss on the mushrooms and icmn for the different optimizers. Again, we observe the strong performance of SPS compared to the other optimizers.

**Multi-class classification using deep networks.** We benchmark the convergence rate and generalization performance of SPS methods on standard deep learning experiments. We consider non-convex minimization for multi-class classification using deep network models on the CIFAR10 and CIFAR100 datasets. Our experimental choices follow the setup in Luo et al. (Luo et al., 2019). For CIFAR-10 and CIFAR100, we experiment with the standard image-classification architectures: ResNet-34 (He et al., 2016) and DenseNet-121 (Huang et al., 2017). For space concerns, we report only the ResNet experiments in the main paper and relegate the DenseNet and MNIST experiments to Appendix D. From figure 2, we observe that SPS results in the best training loss across models and datasets. For CIFAR-10, SPS results in competitive generalization performance compared to the other optimizers, whereas for CIFAR-100, its generalization performance is better than all optimizers except SLS. Note that ALI-G, the closest related optimizer results in worse generalization performance in all cases. We note that SPS is able to match the performance of SLS, but does not require a expensive back-tracking line-search or additional tricks.

For this set of experiments, we plot how the step-size varies across iterations for SLS, SPS and ALI-G. Interestingly, for both CIFAR-10 and CIFAR-100, we find that step-size for both SPS and SLS follows a cyclic behaviour - a warm-up period where the step-size first increases and then decreases to a constant value. Such a step-size schedule has been empirically found to result in good training and generalization performance (Loshchilov & Hutter, 2016) and our results show that SPS is able to simulate this behaviour.

**5 Conclusion**

We proposed and theoretically analyzed a stochastic variant of the classical the Polyak step-size. We quantified the convergence rate of SPS in numerous settings and used our analysis techniques to prove new results for constant step-size SGD. Furthermore, via experiments on a variety of tasks we showed the strong performance of SGD with SPS as compared to state-of-the-art optimization methods. There are many possible interesting extensions of our work: using SPS with accelerated methods, studying the effect of mini-batching and non-uniform sampling techniques and extensions to the distributed and decentralized settings.

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Stochastic Polyak Step-size for SGD

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Supplementary Material
Stochastic Polyak Step-size for SGD:
An Adaptive Learning Rate for Fast Convergence

The Supplementary Material is organized as follows: In Section A, we provide the basic definitions mentioned in the main paper. We also present the convergence of deterministic subgradient method with the classical Polyak step-size. In Section B we present the proofs of the main Theorems and in Section C we provide additional convergence results. Finally, additional numerical experiments are presented in Section D.

A Technical Preliminaries

A.1 Basic Definitions

Let us present some basic definitions used throughout the paper.

**Definition A.1 (Strong Convexity / Convexity).** The function \( f : \mathbb{R}^n \to \mathbb{R} \), is \( \mu \)-strongly convex, if there exists a constant \( \mu > 0 \) such that \( \forall x, y \in \mathbb{R}^n \):
\[
f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \| x - y \|^2 \tag{9}
\]
for all \( x \in \mathbb{R}^d \). If inequality (9) holds with \( \mu = 0 \) the function \( f \) is convex.

**Definition A.2 (Polyak-Lojasiewicz Condition).** The function \( f : \mathbb{R}^n \to \mathbb{R} \), satisfies the Polyak-Lojasiewicz (PL) condition, if there exists a constant \( \mu > 0 \) such that \( \forall x \in \mathbb{R}^n \):
\[
\| \nabla f(x) \|^2 \geq 2\mu (f(x) - f^*) \tag{10}
\]

**Definition A.3 (L-smooth).** The function \( f : \mathbb{R}^n \to \mathbb{R} \), \( L \)-smooth, if there exists a constant \( L > 0 \) such that \( \forall x, y \in \mathbb{R}^n \):
\[
\| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \| \tag{11}
\]
or equivalently:
\[
f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2} \| x - y \|^2 \tag{12}
\]

A.2 The Deterministic Polyak step-size

In this section we describe the Polyak step-size for the subgradient method as presented in (Polyak, 1987) for solving \( \min_{x \in \mathbb{R}^d} f(x) \) where \( f \) is convex, not necessarily smooth function.

Consider the subgradient method:
\[
x^{k+1} = x^k - \gamma_k g^k,
\]
where \( \gamma_k \) is the step-size (learning rate) and \( g^k \) is any subgradient of function \( f \) at point \( x^k \).

**Theorem A.4.** Let \( f \) be convex function. Let \( \gamma_k = \frac{f(x^k) - f(x^*)}{\| g^k \|^2} \) be the step-size in the update rule of subgradient method. Here \( f(x^*) \) denotes the optimum value of function \( f \). Let \( G > 0 \) such that \( \| g^k \|^2 < G^2 \). Then,
\[
f_k^* - f(x^*) \leq \frac{G \| x^0 - x^* \|}{\sqrt{k + 1}} = O \left( \frac{1}{\sqrt{k}} \right),
\]
where \( f_k^* = \min \{ f(x^i) : i = 0, 1, \ldots, k \} \).
We use the above inequality in several parts of our proofs. It is the reason that we are able to obtain an upper bound of

\[ \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - 2\gamma_k (x^k - x^*, g^k) + \gamma_k^2 \|g^k\|^2 \]

where the last line follows from the definition of subgradient:

\[ f(x^*) \geq f(x^k) + (x^k - x^*, g^k) \]

Polyak suggested to use the step-size:

\[ \gamma_k = \frac{f(x^k) - f(x^*)}{\|g^k\|^2} \]

which is precisely the step-size that minimize the right hand side of (13). That is,

\[ \gamma_k = \frac{f(x^k) - f(x^*)}{\|g^k\|^2} = \arg\min_{\gamma_k} \left[ \|x^k - x^*\|^2 - 2\gamma_k \left( f(x^k) - f(x^*) \right) + \gamma_k^2 \|g^k\|^2 \right] \]

By using this choice of step-size in (13) we obtain:

\[ \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - 2\gamma_k \left( f(x^k) - f(x^*) \right) + \gamma_k^2 \|g^k\|^2 \]

\[ \leq \|x^k - x^*\|^2 - \frac{(f(x^k) - f(x^*))^2}{\|g^k\|^2} \]

(14)

From the above note that \( \|x^k - x^*\|^2 \) is monotonic function. Now using telescopic sum and by assuming \( \|g^k\|^2 < G^2 \) we obtain:

\[ \|x^{k+1} - x^*\|^2 \leq \|x^0 - x^*\|^2 - \frac{1}{G^2} \sum_{i=0}^{k} \left( f(x^i) - f(x^*) \right)^2 \]

(16)

Thus,

\[ \frac{1}{G^2} \sum_{i=0}^{k} \left( f(x^i) - f(x^*) \right)^2 \leq \|x^0 - x^*\|^2 - \|x^{k+1} - x^*\|^2 \leq \|x^0 - x^*\|^2 \]

Let us define \( f_k^* = \min \{ f(x^i) : i = 0, 1, \ldots, k \} \) then: \( \left( f_k^* - f(x^*) \right)^2 \leq \frac{G^2 \|x^0 - x^*\|^2}{k+1} \) and

\[ f_k^* - f(x^*) \leq \frac{G \|x^0 - x^*\|}{\sqrt{k+1}} = O \left( \frac{1}{\sqrt{k}} \right) \]

For more details and slightly different analysis check (Polyak, 1987) and (Boyd et al., 2003). In (Hazan & Kakade, 2019) similar analysis to the above have been made for the deterministic gradient descent \( (g^k = \nabla f(x^k)) \) under several assumptions. (convex, strongly convex , smooth).

### B Proofs of Main Results

In this section we present the proofs of the main theoretical results presented in the main paper. That is, the convergence analysis of SGD with SPSmax and SPS under different combinations of assumptions on functions \( f_i \) and \( f \) of Problem (1).

First note that the following inequality can be easily obtained by the definition of SPSmax (4):

\[ \gamma_k^2 \|\nabla f_i(x^k)\|^2 \leq \frac{\gamma_k}{G} \left[ f_i(x^k) - f_i^* \right] \]

(17)

We use the above inequality in several parts of our proofs. It is the reason that we are able to obtain an upper bound of \( \gamma_k^2 \|\nabla f_i(x^k)\|^2 \) without any further assumptions. For the case of SPS (3), inequality (17) becomes equality.
B.1 Proof of Theorem 3.1

Proof.

\[
\|x^{k+1} - x^*\|^2 = \|x^k - \gamma_k \nabla f_i(x^k) - x^*\|^2 \\
= \|x^k - x^*\|^2 - 2\gamma_k \langle x^k - x^*, \nabla f_i(x^k) \rangle + \gamma_k^2 \|\nabla f_i(x^k)\|^2 \\
\leq (1 - \mu_i \gamma_k) \|x^k - x^*\|^2 - 2\gamma_k [f_i(x^k) - f_i(x^*)] + \gamma_k^2 \|\nabla f_i(x^k)\|^2 \\
(17) \\
\leq (1 - \mu_i \gamma_k) \|x^k - x^*\|^2 - 2\gamma_k [f_i(x^k) - f_i(x^*)] + \gamma_k \frac{\gamma_k}{c} [f_i(x^k) - f_i^*] \\
= (1 - \mu_i \gamma_k) \|x^k - x^*\|^2 - 2\gamma_k [f_i(x^k) - f_i^* + f_i^* - f_i(x^*)] \\
+ \gamma_k \frac{\gamma_k}{c} [f_i(x^k) - f_i^*] \\
= (1 - \mu_i \gamma_k) \|x^k - x^*\|^2 + \left(-2\gamma_k + \frac{\gamma_k^2}{c}\right) [f_i(x^k) - f_i^*] \\
+ 2\gamma_k [f_i(x^*) - f_i(x^*)] \\
c \geq 1/2 \\
\leq (1 - \mu_i \gamma_k) \|x^k - x^*\|^2 + 2\gamma_k \frac{\gamma_k}{c} [f_i(x^k) - f_i^*] \\
(6), (4) \\
\leq \left(1 - \mu_i \min \left\{\frac{1}{2cL^2 \max}, \gamma_i\right\}\right) \|x^k - x^*\|^2 + 2\gamma_k [f_i(x^*) - f_i^*] \\
(18)
\]

Taking expectation condition on \(x^k\)

\[
\mathbb{E}_i \|x^{k+1} - x^*\|^2 \leq \left(1 - \mathbb{E}_i[\mu_i] \min \left\{\frac{1}{2cL^2 \max}, \gamma_i\right\}\right) \|x^k - x^*\|^2 + 2\gamma_k \mathbb{E}_i [f_i(x^*) - f_i^*] \\
(5) \\
\leq \left(1 - \bar{\mu} \min \left\{\frac{1}{2cL^2 \max}, \gamma_i\right\}\right) \|x^k - x^*\|^2 + 2\gamma_k \sigma^2 \\
(19)
\]

Taking expectations again and using the tower property:

\[
\mathbb{E}_{t+k} \|x^{k+1} - x^*\|^2 \leq \left(1 - \bar{\mu} \min \left\{\frac{1}{2cL^2 \max}, \gamma_i\right\}\right) \mathbb{E}_{t+k} \|x^k - x^*\|^2 + 2\gamma_k \sigma^2 \\
(20)
\]

Recursively applying the above and summing up the resulting geometric series gives:

\[
\mathbb{E}_t \|x^k - x^*\|^2 \leq \left(1 - \bar{\mu} \min \left\{\frac{1}{2cL^2 \max}, \gamma_i\right\}\right)^k \|x^0 - x^*\|^2 + 2\gamma_k \sigma^2 \sum_{j=0}^{k-1} \left(1 - \bar{\mu} \min \left\{\frac{1}{2cL^2 \max}, \gamma_i\right\}\right)^j \\
\leq \left(1 - \bar{\mu} \min \left\{\frac{1}{2cL^2 \max}, \gamma_i\right\}\right)^k \|x^0 - x^*\|^2 + 2\gamma_k \sigma^2 \frac{1}{\bar{\mu}\min \left\{\frac{1}{2cL^2 \max}, \gamma_i\right\}} \\
(21)
\]

Let \(\alpha = \min \left\{\frac{1}{2cL^2 \max}, \gamma_i\right\}\) then,

\[
\mathbb{E}_t \|x^k - x^*\|^2 \leq \left(1 - \bar{\mu}\alpha\right)^k \|x^0 - x^*\|^2 + \frac{2\gamma_k \sigma^2}{\mu\alpha} \\
(22)
\]

From definition of \(\alpha\) is clear that having small parameter \(c\) improves both the convergence rate \(1 - \bar{\mu}\alpha\) and the neighborhood \(\frac{2\gamma_k \sigma^2}{\mu\alpha}\). Since we have the restriction \(c \geq \frac{1}{2}\) the best selection would be \(c = \frac{1}{2}\). \(\square\)
B.2 Proof of Theorem 3.5

Proof.

\[ \|x^{k+1} - x^*\|^2 = \|x^k - \gamma_k \nabla f_i(x^k) - x^*\|^2 \]
\[ = \|x^k - x^*\|^2 - 2\gamma_k (x^k - x^*, \nabla f_i(x^k)) + \gamma_k^2 \|\nabla f_i(x^k)\|^2 \]
\[ \leq \|x^k - x^*\|^2 - 2\gamma_k \left[ f_i(x^k) - f_i(x^*) \right] + \gamma_k^2 \left[ f_i(x^k) - f_i^* \right] \]
\[ = \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] \leq -\gamma_k \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] \]
\[ \leq -\gamma_k \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] \leq -\gamma_k \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] \]
\[ \leq -\gamma_k \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] \leq -\gamma_k \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] \]

By combining the above cases we obtain:

\[ -\gamma_k \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] \leq \max \left\{ -\frac{1}{2cL_{\max}}, -\gamma_b \right\} \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] \]
\[ = -\min \left\{ \frac{1}{2cL_{\max}}, \gamma_b \right\} \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] \]
\[ \leq -\frac{1}{2cL_{\max}} \left[ f_i(x^k) - f_i(x^*) \right] \]
\[ \leq -\gamma_b \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] \]

By substituting (24) into (23) we obtain:

\[ \|x^{k+1} - x^*\|^2 = \|x^k - x^*\|^2 - \min \left\{ \frac{1}{2cL_{\max}}, \gamma_b \right\} \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] + \frac{\gamma_b}{c} \left[ f_i(x^*) - f_i^* \right] \]

Let \( \alpha = \min \left\{ \frac{1}{2cL_{\max}}, \gamma_b \right\} \). By rearranging:

\[ \alpha \left( 2 - \frac{1}{c} \right) \left[ f_i(x^k) - f_i(x^*) \right] = \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + \frac{\gamma_b}{c} \left[ f_i(x^*) - f_i^* \right] \]

For our choice of \( c \) it holds that \( \alpha \left( 2 - \frac{1}{c} \right) > 0 \). By taking expectation condition on \( x^k \) and dividing by \( \alpha \left( 2 - \frac{1}{c} \right) \):

\[ \left[ f(x^k) - f(x^*) \right] = \frac{\|x^k - x^*\|^2 - \mathbb{E}_\kappa \|x^{k+1} - x^*\|^2}{\alpha \left( 2 - \frac{1}{c} \right)} + \frac{\gamma_b \sigma^2}{\alpha \left( 2 - \frac{1}{c} \right)} \]
Taking expectation again and using the tower property:

\[
\mathbb{E} \left[ f(x^k) - f(x^*) \right] = \frac{c}{\alpha(2c-1)} \left( \mathbb{E}\|x^k - x^*\|^2 - \mathbb{E}\|x^{k+1} - x^*\|^2 \right) + \frac{\gamma_0 \sigma^2}{\alpha(2c-1)} \]  \tag{28}

Summing from \(k = 0\) to \(K - 1\) and dividing by \(K\):

\[
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} [f(x^k) - f(x^*)] = \frac{c}{\alpha(2c-1)} \frac{1}{K} \sum_{k=0}^{K-1} \left( \mathbb{E}\|x^k - x^*\|^2 - \mathbb{E}\|x^{k+1} - x^*\|^2 \right) + \frac{1}{K} \sum_{k=0}^{K-1} \frac{\gamma_0 \sigma^2}{\alpha(2c-1)} 
\leq \frac{c}{\alpha(2c-1)} \frac{1}{K} \sum_{k=0}^{K-1} \|x^0 - x^*\|^2 + \frac{\gamma_0 \sigma^2}{\alpha(2c-1)} \]  \tag{29}

Let \(\bar{x}^K = \frac{1}{K} \sum_{k=0}^{K-1} x^k\), then:

\[
\mathbb{E} [f(\bar{x}^K) - f(x^*)] \leq \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} [f(x^k) - f(x^*)] \leq \frac{c}{\alpha(2c-1)} \frac{1}{K} \|x^0 - x^*\|^2 + \frac{\gamma_0 \sigma^2}{\alpha(2c-1)} \]  \tag{30}

For \(c = 1\):

\[
\mathbb{E} [f(\bar{x}^K) - f(x^*)] \leq \frac{\|x^0 - x^*\|^2}{\alpha K} + \frac{\gamma_0 \sigma^2}{\alpha} \]  \tag{31}

and this completes the proof.

At this point we highlight that \(c = 1\) is selected to simplify the expression of the upper bound in (30). This is not the optimum choice (the one that makes the rate and the neighborhood of the upper bound smaller). In order to compute the optimum value of \(c\) one needs to follow similar procedure to (Gower et al., 2019) and (Needell et al., 2016). In this case \(c\) will depend on parameter \(\sigma\) and the desired accuracy \(\epsilon\) of convergence.

However as we show bellow having \(c = 1\) allows SGD with SPS to convergence faster than the ALI-G algorithm (Berrada et al., 2019) and the SLS algorithm (Vaswani et al., 2019b) for the case of smooth convex functions.

**Comparison with other methods** Similar to the strongly convex case let us compare the above convergence for smooth convex functions with the convergence rates proposed in (Vaswani et al., 2019b) and (Berrada et al., 2019).

For the smooth convex functions, Berrada et al. (2019) show the linear convergence to a much larger neighborhood than ours and with slower rate. In particular, their rate is \(\frac{1}{K} \left( \frac{2L}{1 - 2\epsilon} \right)\) and the neighborhood is \(\frac{\delta}{L(1 - 2\epsilon)}\) where \(\delta > 2L\epsilon\) and \(\epsilon\) is the \(\epsilon\)-interpolation parameter \(\epsilon > \max_i [f_i(x^*) - f_i^*]\) which by definition is bigger than \(\sigma^2\). Under interpolation where \(\sigma = 0\), our method converges with a \(O(1/K)\) rate to the \(x^*\) while the algorithm proposed by Berrada et al. (2019) still converges to a neighborhood that is proportional to the parameter \(\delta\).

In the interpolation setting our rate is similar to the one obtain for the stochastic line search (SLS) proposed in (Vaswani et al., 2019b). In particular in the interpolation setting, SLS achieves the following \(O(1/K)\) rate \(\mathbb{E} [f(\bar{x}^K) - f(x^*)] \leq \frac{\max_i [3L_{\max_i} (2/\gamma_0)]}{K} \|x^0 - x^*\|^2\) which has slightly worse constants than SGD with SPS.

**B.3 SPS on Methods for Solving Consistent Linear Systems**

Recently several new randomized iterative methods (sketch and project methods) for solving large-scale linear systems have been proposed (Richtářik & Takáč, 2017; Loizou & Richtářik, 2017; 2019a; Gower & Richtářik, 2015). The main algorithm in this literature is the celebrated randomized Kaczmarz (RK) method (Kaczmarz, 1937; Strohmer & Vershynin, 2009) which can be seen as special case of SGD for solving least square problems (Needell et al., 2016). In this area of research, it is well known that the theoretical best constant step-size for RK method is \(\gamma = 1\).
As we have already mentioned in Section 3.2.3, given the consistent linear system

\[ \mathbf{A} x = b, \]  

Richtárik & Takáč (2017) provide a stochastic optimization reformulation of the form (1) which is equivalent to the linear system in the sense that their solution sets are identical. That is, the set of minimizers of the stochastic optimization problem \( \mathcal{X}^* \) is equal to the set of solutions of the stochastic linear system \( \mathcal{L} := \{ x : \mathbf{A} x = b \} \).

In particular, the stochastic convex quadratic optimization problem proposed in Richtárik & Takáč (2017), can be expressed as follows:

\[ \min_{x \in \mathbb{R}^n} f(x) := \mathbb{E}_{\mathbf{S} \sim \mathcal{D}} f_{\mathbf{S}}(x). \]  

(33)

Here the expectation is over random matrices \( \mathbf{S} \) drawn from an arbitrary, user defined, distribution \( \mathcal{D} \) and \( f_{\mathbf{S}} \) is a stochastic convex quadratic function of a least-squares type, defined as

\[ f_{\mathbf{S}}(x) := \frac{1}{2} \| \mathbf{A} x - b \|_\mathbf{H}^2 = \frac{1}{2} (\mathbf{A} x - b)^\top \mathbf{H} (\mathbf{A} x - b). \]  

(34)

Function \( f_{\mathbf{S}} \) depends on the matrix \( \mathbf{A} \in \mathbb{R}^{m \times n} \) and vector \( b \in \mathbb{R}^m \) of the linear system (32) and on a random symmetric positive semidefinite matrix \( \mathbf{H} := \mathbf{S} (\mathbf{S}^\top \mathbf{A} \mathbf{A}^\top \mathbf{S}) \mathbf{S}^\top \). By \( \mathbf{H}^\dagger \) we denote the Moore-Penrose pseudoinverse.

For solving problem (33), Richtárik & Takáč (2017) analyze SGD with constant step-size:

\[ x^{k+1} = x^k - \gamma \nabla f_{S_k}(x^k), \]  

(35)

where \( \nabla f_{S_k}(x^k) \) denotes the gradient of function \( f_{S_k} \). In each step the matrix \( S_k \) is drawn from the given distribution \( \mathcal{D} \).

The above update of SGD is quite general and as explained by Richtárik & Takáč (2017) the flexibility of selecting distribution \( \mathcal{D} \) allow us to obtain different stochastic reformulations of the linear system (32) and different special cases of the SGD update. For example the celebrated randomized Kaczmarz (RK) method can be seen as special cases of the above update as follows:

**Randomized Kaczmarz Method:** Let pick in each iteration the random matrix \( \mathbf{S} = e_i \) (random coordinate vector) with probability \( p_i = \| \mathbf{A} e_i \|_2^2 / \| \mathbf{A} \|_F^2 \). In this setup the update rule of SGD (35) simplifies to

\[ x^{k+1} = x^k - \omega \frac{\mathbf{A}^\top x^k - b_i}{\| \mathbf{A} e_i \|_2^2} \mathbf{A}^\top. \]  

Many other methods like Gaussian Kacmarz, Randomized Coordinate Descent, Gaussian Decsent and their block variants can be cast as special cases of the above framework. For more details on the general framework and connections with other research areas we also suggest (Loizou & Richtárik, 2019b; Loizou, 2019).

**Lemma B.1 (Properties of stochastic reformulation (Richtárik & Takáč, 2017)).** For all \( x \in \mathbb{R}^n \) and any \( \mathbf{S} \sim \mathcal{D} \) it holds that:

\[ f_{\mathbf{S}}(x) - f_{\mathbf{S}}(x^*) \bigg|_{\mathbf{f}_{\mathbf{S}} = 0} f_{\mathbf{S}}(x) = \frac{1}{2} \| \nabla f_{\mathbf{S}}(x) \|_B^2 = \frac{1}{2} (\nabla f_{\mathbf{S}}(x), x - x^*)_B. \]  

(36)

Let \( x^* \) is the projection of vector \( x \) onto the solution set \( \mathcal{X}^* \) of the optimization problem \( \min_{x \in \mathbb{R}^n} f(x) \) (Recall that by the construction of the stochastic optimization problems we have that \( \mathcal{X}^* = \mathcal{L} \)). Then:

\[ \frac{\lambda_{\min}^+(\mathbf{W})}{2} \| x - x^* \|_B^2 \leq f(x). \]  

(37)

where \( \lambda_{\min}^+ \) denotes the minimum non-zero eigenvalue of matrix \( \mathbf{W} = \mathbb{E}[\mathbf{A}^\top \mathbf{H} \mathbf{A}] \).

As we will see in the next Theorem, using the special structure of the stochastic reformulation (33), SPS (3) with \( c = 1/2 \) takes the following form:

\[ \gamma_k = (3) \frac{2 \left[ f_{\mathbf{S}}(x^k) - f_{\mathbf{S}}^* \right]}{\| \nabla f_{\mathbf{S}}(x^k) \|_2^2} \equiv 1, \]  

\[ \gamma_k = (3) \frac{2 \left[ f_{\mathbf{S}}(x^k) - f_{\mathbf{S}}^* \right]}{\| \nabla f_{\mathbf{S}}(x^k) \|_2^2} \equiv 1, \]
which is the theoretically optimal constant step-size for SGD in this setting (Richtárik & Takáč, 2017). This reduction implies that SPS results in an optimal convergence rate when solving consistent linear systems. We provide the convergence rate for SPS in the next Theorem.

Though a straight forward verification of the optimality of SPS, we believe that this is the first time that SGD with adaptive step-size is reduced to constant step-size when is used for solving linear systems. SPS does that by obtaining the best convergence rate in this setting.

**Theorem B.2.** Let $Ax = b$ be a consistent linear system and let $x^*$ is the projection of vector $x$ onto the solution set $\mathcal{X}^* = L$. Then the SGP with SPS $(3)$ with $c = 1/2$ for solving the stochastic optimization formulation $(33)$ satisfies:

$$\mathbb{E}||x^k - x^*||^2 \leq (1 - \lambda^+_{\min}(W))^k ||x^0 - x^*||^2$$

where $\lambda^+_{\min}$ denotes the minimum non-zero eigenvalue of matrix $W = \mathbb{E}[A^T HA]$.

**Proof.**

$$||x^{k+1} - x^*||^2 \overset{\text{(35)}}{=} ||x^k - \gamma_k \nabla f_{S_k}(x^k) - x^*||^2$$

$$= ||x^k - x^*||^2 - 2\gamma_k \langle x^k - x^*, \nabla f_{S_k}(x^k) \rangle + \gamma_k^2 \||\nabla f_{S_k}(x^k)||^2$$

Let us select $\gamma_k$ such that the RHS of inequality $(39)$ is minimized. That is, let us select:

$$\gamma_k = \frac{\langle x^k - x^*, \nabla f_{S_k}(x^k) \rangle}{||\nabla f_{S_k}(x^k)||^2} \overset{\text{(36)}}{=} 2 \frac{f_S(x) - f_S(x^*)}{||\nabla f_{S_k}(x^k)||^2}$$

Substitute this step-size to $(39)$ we obtain:

$$||x^{k+1} - x^*||^2 \overset{\text{(36)}}{=} ||x^k - x^*||^2 - 2f_S(x)$$

By taking expectation with respect to $S_k$ and using quadratic growth inequality $(37)$:

$$\mathbb{E}_{S_k}[||x^{k+1} - x^*||^2] \overset{\text{(37)}}{=} ||x^k - x^*||^2 - 2f(x^k)$$

$$\leq ||x^k - x^*||^2 - \lambda^+_{\min}(W)||x^k - x^*||^2$$

$$= [1 - \lambda^+_{\min}(W)] ||x^k - x^*||^2.$$  (41)

Taking expectation again and by unrolling the recurrence we obtain $(38)$.  

We highlight that the above proof provides a different viewpoint on the analysis of the optimal constant step-size for the sketch and project methods for solving consistent linear systems. The expression of Theorem B.2 is the same with the one proposed in (Richtárik & Takáč, 2017).

**B.4 Proof of Theorem 3.6**

**Proof.** By the smoothness of function $f$ we have that

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}||x^{k+1} - x^k||^2.$$
By having

Let

and by taking expectation condition on

By rearranging:

\begin{align*}
\frac{f(x^{k+1}) - f(x^k)}{\gamma_k} & \leq - \langle \nabla f(x^k), \nabla f_i(x^k) \rangle + \frac{L\gamma_k}{2} \| \nabla f_i(x^k) \|^2 \\
& \leq - \langle \nabla f(x^k), \nabla f_i(x^k) \rangle + \frac{L}{2c} [f_i(x^k) - f_i^*] \\
& = - \langle \nabla f(x^k), \nabla f_i(x^k) \rangle + \frac{L}{2c} [f_i(x^k) - f_i(x^*)] + \frac{L}{2c} [f_i(x^*) - f_i^*] 
\end{align*}

(43)

and by taking expectation condition on \( x^k \):

\begin{align*}
E_i \left[ \frac{f(x^{k+1}) - f(x^k)}{\gamma_k} \right] & \leq - \langle \nabla f(x^k), \nabla f(x^k) \rangle + \frac{L}{2c} [f(x^k) - f(x^*)] + \frac{L}{2c} E_i [f_i(x^*) - f_i^*] \\
& \leq - \| \nabla f(x^k) \|^2 + \frac{L}{2c} [f(x^k) - f(x^*)] + \frac{L}{2c} \sigma^2 \\
& \leq -2\mu [f(x^k) - f(x^*)] + \frac{L}{2c} [f(x^k) - f(x^*)] + \frac{L}{2c} \sigma^2 
\end{align*}

(44)

Let \( \alpha = \min \left\{ \frac{1}{2L_{\max}}, \gamma_k \right\} \). Then,

\begin{align*}
E_i \left[ \frac{f(x^{k+1}) - f(x^*)}{\gamma_k} \right] & \leq E_i \left[ \frac{f(x^k) - f(x^*)}{\gamma_k} \right] - 2\mu [f(x^k) - f(x^*)] + \frac{L}{2c} [f(x^k) - f(x^*)] + \frac{L}{2c} \sigma^2 \\
& \leq \frac{1}{\alpha} [f(x^k) - f(x^*)] - 2\mu [f(x^k) - f(x^*)] + \frac{L}{2c} [f(x^k) - f(x^*)] + \frac{L}{2c} \sigma^2 \\
& = \left( \frac{1}{\alpha} - 2\mu + \frac{L_{\max}}{2c} \right) [f(x^k) - f(x^*)] + \frac{L}{2c} \sigma^2 \\
& \leq L_{\max} \left[ \frac{1}{\alpha} - 2\mu + \frac{L_{\max}}{2c} \right] [f(x^k) - f(x^*)] + \frac{L}{2c} \sigma^2 
\end{align*}

(45)

Using \( \gamma_k \leq \gamma_b \) and by taking expectations again:

\begin{align*}
E \left[ f(x^{k+1}) - f(x^*) \right] & \leq \gamma_b \left( \frac{1}{\alpha} - 2\mu + \frac{L_{\max}}{2c} \right) E \left[ f(x^k) - f(x^*) \right] + \frac{L\sigma^2\gamma_b}{2c} 
\end{align*}

(46)

By having \( \nu \in (0, 1) \) and by recursively applying the above and summing the resulting geometric series we obtain:

\begin{align*}
E \left[ f(x^k) - f(x^*) \right] & \leq \nu^k \left[ f(x^0) - f(x^*) \right] + \frac{L\sigma^2\gamma_b}{2c} \sum_{j=0}^{k-1} \nu^j \\
& \leq \nu^k \left[ f(x^0) - f(x^*) \right] + \frac{L\sigma^2\gamma_b}{2(1 - \nu)c} 
\end{align*}

(47)

In the above result we require that \( 0 < \nu = \gamma_b \left( \frac{1}{\alpha} - 2\mu + \frac{L_{\max}}{2c} \right) \leq 1 \). In order for this to hold we need to make extra assumptions on the values of \( \gamma_b \) and parameter \( c \). This is what we do next.

Let us divide the analysis into two cases based on the value of parameter \( \alpha \). That is:
• (i) If $\frac{1}{2cL_{\text{max}}} \leq \gamma_b$ then,

$$\alpha = \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_b \right\} = \frac{1}{2cL_{\text{max}}} \quad \text{and} \quad \nu = \gamma_b \left( 2cL_{\text{max}} - 2\mu + \frac{L_{\text{max}}}{2c} \right) = \gamma_b \left( \frac{(2c + \frac{1}{2c})L_{\text{max}} - 2\mu}{2c} \right).$$

By preliminary computations, it can be easily shown that $\nu > 0$ for every $c \geq 0$. However for $\nu \leq 1$ we need to require that $\frac{1}{2cL_{\text{max}}} \leq \gamma_b$ and since we are already assume that $\frac{1}{2cL_{\text{max}}} \leq \gamma_b$ we need to force

$$\frac{1}{2cL_{\text{max}}} \leq \frac{1}{\left( \frac{1}{\alpha} - 2\mu + \frac{L_{\text{max}}}{2c} \right)}$$

to avoid contradiction. This is true only if $c > \frac{L_{\text{max}}}{4\mu}$ which is the assumption of Theorem 3.6.

• (ii) If $\gamma_b \leq \frac{1}{2cL_{\text{max}}}$ then,

$$\alpha = \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_b \right\} = \gamma_b \quad \text{and} \quad \nu = \gamma_b \left( \frac{1}{\gamma_b} - 2\mu + \frac{L_{\text{max}}}{2c} \right) = 1 - 2\mu\gamma_b + \frac{L_{\text{max}}\gamma_b}{2c}.$$

Note that if we have $c > \frac{L_{\text{max}}}{4\mu}$ (an assumption of Theorem 3.6) it holds that $\nu \leq 1$. In addition, by preliminary computations, it can be shown that $\nu > 0$ if $\gamma_b < \frac{2c}{4\mu c - L_{\text{max}}}$. Finally, for $c > \frac{L_{\text{max}}}{4\mu}$ it holds that $\frac{1}{2cL_{\text{max}}} \leq \frac{1}{4\mu c - L_{\text{max}}}$, and as a result $\nu > 0$ for all $\gamma_b \leq \frac{1}{2cL_{\text{max}}}$. By presenting the above cases on bound of $\nu$ we complete the proof.

Remark B.3. The expression of Corollary 3.7 is obtained by simply use $c = \frac{L_{\text{max}}}{2\mu}$ in the case (ii) of the above proof. In this case we have $\gamma \leq \frac{\mu \nu}{\nu_{\text{max}}}$ and $\nu = 1 - \mu \gamma$.

B.5 Proof of Theorem 3.8

Proof. By the smoothness of function $f$ we have that

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \| x^{k+1} - x^k \|^2.$$  

Combining this with the update rule of SGD we obtain:

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \| x^{k+1} - x^k \|^2$$

$$\leq f(x^k) - \gamma_k \langle \nabla f(x^k), \nabla f_i(x^k) \rangle + \frac{L\gamma_k^2}{2} \| \nabla f_i(x^k) \|^2$$

$$\leq f(x^k) - \gamma_k \langle \nabla f(x^k), \nabla f_i(x^k) \rangle + \frac{L\gamma_k^2}{2} \| \nabla f_i(x^k) \|^2 \quad (48)$$

At this point, we follow similar proof to the convex case. That is, note that the quantity $\langle \nabla f(x^k), \nabla f_i(x^k) \rangle$ can be either positive or negative. Thus, we divide our analysis in two cases.

(i) If $\langle \nabla f(x^k), \nabla f_i(x^k) \rangle > 0$ then,

$$-\gamma_k \langle \nabla f(x^k), \nabla f_i(x^k) \rangle \leq -\frac{1}{2cL_{\text{max}}} \langle \nabla f(x^k), \nabla f_i(x^k) \rangle$$

(ii) If $\langle \nabla f(x^k), \nabla f_i(x^k) \rangle < 0$ then,

$$-\gamma_k \langle \nabla f(x^k), \nabla f_i(x^k) \rangle \leq -\gamma_k \langle \nabla f(x^k), \nabla f_i(x^k) \rangle$$
Let $\alpha > 0$. We divide the analysis into two cases. That is:

$$
- \gamma_k \langle \nabla f(x^k), \nabla f_i(x^k) \rangle \leq \max \left\{ -\frac{1}{2cL_{\text{max}}}, -\gamma_k \right\} \langle \nabla f(x^k), \nabla f_i(x^k) \rangle = -\min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_k \right\} \langle \nabla f(x^k), \nabla f_i(x^k) \rangle
$$

By combining the above cases we obtain:

$$
f(x^{k+1}) \leq f(x^k) - \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_k \right\} \langle \nabla f(x^k), \nabla f_i(x^k) \rangle + \frac{L\gamma_k^2}{2} \| \nabla f_i(x^k) \|^2
$$

By substituting (49) into (48) we obtain:

$$
f(x^{k+1}) \leq f(x^k) - \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_k \right\} \| \nabla f(x^k) \|^2 + \frac{L\gamma_k^2}{2} \| \nabla f_i(x^k) \|^2
$$

By taking expectation condition on $x^k$ and using (8):

$$
E_i[f(x^{k+1})] \leq f(x^k) - \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_k \right\} \| \nabla f(x^k) \|^2 + \frac{L\gamma_k^2}{2} E_i[\| \nabla f_i(x^k) \|^2]
$$

By rearranging and taking expectations again:

$$
\left( \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_k \right\} - \frac{L\gamma_k^2}{2} \rho \right) E[\| \nabla f(x^k) \|^2] \leq E[f(x^k)] - E[f(x^{k+1})] + \frac{L\gamma_k^2}{2} \delta
$$

Let $\alpha > 0$ then:

$$
E[\| \nabla f(x^k) \|^2] \leq \frac{1}{\alpha} \left( E[f(x^k)] - E[f(x^{k+1})] \right) + \frac{L\gamma_k^2 \delta}{2\alpha}
$$

By summing from $k = 0$ to $K - 1$ and dividing by $K$:

$$
\frac{1}{K} \sum_{k=0}^{K-1} E[\| \nabla f(x^k) \|^2] \leq \frac{1}{\alpha} \frac{1}{K} \sum_{k=0}^{K-1} \left( E[f(x^k)] - E[f(x^{k+1})] \right) + \frac{1}{K} \sum_{k=0}^{K-1} \frac{L\gamma_k^2 \delta}{2\alpha}
$$

$$
\leq \frac{1}{\alpha} \frac{1}{K} \left( f(x^0) - E[f(x^K)] \right) + \frac{L\gamma_k^2 \delta}{2\alpha}
$$

$$
\leq \frac{1}{\alpha} \frac{1}{K} \left( f(x^0) - f(x^*) \right) + \frac{L\gamma_k^2 \delta}{2\alpha}
$$

In the above result we require that $\alpha = \left( \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_k \right\} - \frac{L\gamma_k^2}{2} \rho \right) > 0$. In order for this to hold we need to make extra assumptions on the values of $\gamma_k$ and parameter $c$. This is what we do next.

Let us divide the analysis into two cases. That is:

- (i) If $\frac{1}{2cL_{\text{max}}} \leq \gamma_k$ then,

$$
\alpha = \left( \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_k \right\} - \frac{L\gamma_k^2}{2} \rho \right) = \left( \frac{1}{2cL_{\text{max}}} - \frac{L\gamma_k^2}{2} \rho \right)
$$

By preliminary computations, it can be easily shown that $\alpha > 0$ if $\gamma_k < \frac{1}{\sqrt{L\rho cL_{\text{max}}}}$. To avoid contraction the inequality $\frac{1}{2cL_{\text{max}}} < \frac{1}{\sqrt{L\rho cL_{\text{max}}}}$ needs to be true. This is the case of $c > \frac{L\rho}{4L_{\text{max}}}$ which is the assumptions of Theorem 3.8.
• (ii) If \( \gamma_b \leq \frac{1}{2cL_{\text{max}}} \) then,

\[
\alpha = \left( \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_b \right\} - \frac{L\gamma_b^2}{2} \right) = \gamma_b - \frac{L\gamma_b^2}{2} = \gamma_b \left( 1 - \frac{L\gamma_b}{2} \right).
\]

In this case, by preliminary computations, it can be shown that \( \alpha > 0 \) if \( \gamma_b < \frac{2}{L\rho} \). For \( c > \frac{L\rho}{4L_{\text{max}}} \) it also holds that \( \frac{1}{2cL_{\text{max}}} < \frac{2}{L\rho} \).

\[ \square \]

### C Additional Convergence Results

In this section we present some additional convergence results. We first prove a \( O(1/\sqrt{K}) \) convergence rate of stochastic subgradient method with SPS for non-smooth convex functions in the interpolated setting. Furthermore, similar to (Schmidt et al., 2011), we propose a way to increase the mini-batch size for evaluating the stochastic gradient and guarantee convergence to the optimal solution without interpolation.

#### C.1 Non-smooth Convex Functions

In all of our previous results we assume that functions \( f_i \) are smooth. As a result, in the proofs of our theorems we were able to use the lower bound (6) of SPS. In the case that functions \( f_i \) are not smooth using this lower is clearly not possible. Below we present a Theorem that handles the case of non-smooth function for the convergence of stochastic subgradient method for this result we require that a constant \( G \) to use the lower bound (6) of SPS. In the case that functions \( f_i \) are convex non-smooth functions. Let \( G \) be the subgradient counterpart of SPS (3) with \( c = 1 \). Then the iterates of the stochastic subgradient method satisfy:

\[
\mathbb{E} \left[ f(x^K) - f(x^*) \right] \leq \frac{G\|x^0 - x^*\|}{\sqrt{K}} = O \left( \frac{1}{\sqrt{K}} \right)
\]

where \( x^K = \frac{1}{K} \sum_{k=0}^{K-1} x^k \).

**Proof.** The proof is similar to the deterministic case (see Theorem A.4). That is, we select the \( \gamma_k \) that minimize the right hand side of the inequality after the use of convexity.

\[
\|x^{k+1} - x^*\|^2 \leq \|x^k - \gamma_k g_k - x^*\|^2
\]

\[
= \|x^k - x^*\|^2 - 2\gamma_k \langle x^k - x^*, g_k \rangle + \gamma_k^2 \|g_k\|^2
\]

\[
\leq \|x^k - x^*\|^2 - 2\gamma_k \left[ f_i(x^k) - f_i(x^*) \right] + \gamma_k^2 \|g_k\|^2
\]

(55)

Using the subgradient counterpart of SPS (3) with \( c = 1 \), that is, \( \gamma_k = \frac{f_i(x^k) - f_i(x^*)}{\|g_k\|^2} \) we obtain:

\[
\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - 2 \frac{f_i(x^k) - f_i(x^*)}{\|g_k\|^2} \left[ f_i(x^k) - f_i(x^*) \right] + \left[ \frac{f_i(x^k) - f_i(x^*)}{\|g_k\|^2} \right]^2 \|g_k\|^2
\]

\[
= \|x^k - x^*\|^2 - \frac{\|g_k\|^2}{\|g_k\|^2} \left[ f_i(x^k) - f_i(x^*) \right]^2
\]

\[
\|g_i(x)\|^2 < G^2 \leq \|x^k - x^*\|^2 - \frac{\left[ f_i(x^k) - f_i(x^*) \right]^2}{G^2}
\]

(56)

\[ \text{Note that for non-smooth functions, it is required to have stochastic subgradient method instead of SGD. That is, in each iteration we replace the evaluation of } \nabla f_i(x) \text{ with its subgradient counterpart } g_i(x) \]
taking expectation condition on $x^k$:

$$
\mathbb{E}_i \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \frac{\mathbb{E}_i \left[f_i(x^k) - f_i(x^*)\right]^2}{G^2}
$$

Taking expectation again and using the tower property:

$$
\mathbb{E}\|x^{k+1} - x^*\|^2 \leq \mathbb{E}\|x^k - x^*\|^2 - \frac{\mathbb{E}_i \left[f_i(x^k) - f_i(x^*)\right]^2}{G^2}
$$

(57)

By rearranging, summing from $k = 0$ to $K - 1$ and dividing by $K$:

$$
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}\left[f(x^k) - f(x^*)\right]^2 \leq \frac{1}{K} \sum_{k=0}^{K-1} \left[\mathbb{E}\|x^k - x^*\|^2 - \mathbb{E}\|x^{k+1} - x^*\|^2\right]
$$

$$
= \frac{1}{K} \left[\|x^0 - x^*\|^2 - \mathbb{E}\|x^K - x^*\|^2\right]
$$

$$
= \frac{1}{K} \left[\|x^0 - x^*\|^2\right]
$$

(59)

Taking square roots and using Jensen’s inequality:

$$
\frac{1}{GK} \sum_{k=0}^{K-1} \mathbb{E}\left[f(x^k) - f(x^*)\right]^2 \leq \frac{1}{G} \sqrt{\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}\left[f(x^k) - f(x^*)\right]^2} \leq \frac{1}{\sqrt{K}} \|x^0 - x^*\|
$$

(60)

Thus,

$$
\mathbb{E}\left[f(\bar{x}^K) - f(x^*)\right] \leq \frac{1}{G} \sum_{k=0}^{K-1} \mathbb{E}\left[f(x^k) - f(x^*)\right] \leq \frac{G\|x^0 - x^*\|}{\sqrt{K}},
$$

(61)

where $\bar{x}^K = \frac{1}{K} \sum_{k=0}^{K-1} x^k$.

**C.2 Increasing Mini-batch Size**

We propose a way to increase the mini-batch size for evaluating the stochastic gradient and guarantee convergence to the optimal solution without interpolation. We present two main Theorems. In the first Theorem we assume that functions $f_i$ of problem (1) are $\mu_i$-strongly convex functions and in the second that each function $f_i$ satisfies the PL condition (10) with $\mu_i$ parameter.

**Theorem C.2.** Let us have the same assumptions as in Theorem 3.1 and let all $f_i$ be $\mu_i$-strongly convex functions then SGD with SPS and increasing the batch-size progressively such that the batch-size $b_k$ at iteration $k$ satisfies:

$$
b_k \geq \left[\frac{1}{n} + \frac{1}{2\gamma_{\text{max}}} \frac{\mu_{\text{min}} \bar{\mu}}{L_{\text{max}}} \left(\frac{\|\nabla f(x^k)\|}{L}\right)^2\right]^{-1}
$$

converges as:

$$
\mathbb{E}\|x^k - x^*\|^2 \leq \left(1 - \frac{\bar{\mu}}{4c L_{\text{max}}} \right)^k \|x^0 - x^*\|^2.
$$
Proof. Following the proof of Theorem 3.1 for the batch $b$. From Equation 18,
\[
\|x^{k+1} - x^*\|^2 \leq \left(1 - \mu_i \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_b \right\} \right) \|x^k - x^*\|^2 + 2\gamma_b [f_i(x^*) - f_i^*]
\] (62)

Taking expectation,
\[
\mathbb{E}\|x^{k+1} - x^*\|^2 \leq \mathbb{E}\left(1 - \mu_i \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_b \right\} \right) \|x^k - x^*\|^2 + 2\gamma_b \mathbb{E}[f_i(x^*) - f_i^*]
\] (63)

By strong-convexity of $f_i$,
\[
\mathbb{E} [f_b(x^*) - f_b^*] \leq \mathbb{E} \left[ \frac{1}{2\mu_i} \|\nabla f_b(x^*)\|^2 \right] \leq \frac{1}{2\mu_{\text{min}}} \mathbb{E}\|\nabla f_b(x^*)\|^2
\] (64)

By the assumption that the gradients at the optimum have bounded variance, from (Harikandeh et al., 2015; Lohr, 2019),
\[
\mathbb{E}\|\nabla f_b(x^*)\|^2 \leq \frac{n - b}{nb} z^2
\] (65)

\[\Rightarrow \mathbb{E} [f_b(x^*) - f_b^*] \leq \frac{1}{2\mu_i} \frac{n - b}{nb} z^2
\]

\[\Rightarrow \mathbb{E}\|x^{k+1} - x^*\|^2 \leq \mathbb{E}\left(1 - \mu_i \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_b \right\} \right) \|x^k - x^*\|^2 + \frac{\gamma_b}{\mu_{\text{min}}} \frac{n - b}{nb} z^2
\] (66)

If we set the batch-size in iteration $k$ such that,
\[
\frac{\gamma_b}{\mu_{\text{min}}} \frac{n - b}{nb} z^2 \leq \frac{\mu}{4cL_{\text{max}}} \left( \frac{\|\nabla f(x^*)\|}{L} \right)^2
\]

\[\Rightarrow b \geq \left[ \frac{1}{n} + \frac{1}{\gamma_{\text{max}}} \frac{\mu_{\text{min}} \bar{\mu}}{4cL_{\text{max}}} \left( \frac{\|\nabla f(x^*)\|}{L} \right)^2 \right]^{-1}
\] (67)

\[
\mathbb{E}\|x^{k+1} - x^*\|^2 \leq \mathbb{E}\left(1 - \mu_i \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_b \right\} \right) \|x^k - x^*\|^2 + \frac{\bar{\mu}}{4cL_{\text{max}}} \left( \frac{\|\nabla f(x^*)\|}{L} \right)^2
\] (68)

\[\leq \left(1 - \bar{\mu} \min \left\{ \frac{1}{2cL_{\text{max}}}, \gamma_b \right\} \right) \|x^k - x^*\|^2 + \frac{\bar{\mu}}{4cL_{\text{max}}} \|x^k - x^*\|^2
\] (69)

\[
\mathbb{E}\|x^{k+1} - x^*\|^2 \leq \left(1 - \bar{\mu} \min \left\{ \frac{1}{4cL_{\text{max}}}, \gamma_b - \frac{1}{4cL_{\text{max}}} \right\} \right) \|x^k - x^*\|^2
\] (70)

Following the remaining proof of Theorem 3.1,
\[
Exp\|x^k - x^*\|^2 \leq \left(1 - \mu_i \min \left\{ \frac{1}{4cL_{\text{max}}}, \gamma_b - \frac{1}{4cL_{\text{max}}} \right\} \right)^k \|x^0 - x^*\|^2
\] (71)

If $\gamma_b = \infty$,
\[
\mathbb{E}\|x^k - x^*\|^2 \leq \left(1 - \frac{\bar{\mu}}{4cL_{\text{max}}} \right)^k \|x^0 - x^*\|^2
\] (72)

(73)
Theorem C.3. Assume that all functions $f_i$ satisfy the PL inequality (10) and let $f$ and $f_i$ be smooth functions. SGD with $\text{SPS}_{\text{max}}$ and increasing the batch-size progressively such that the batch-size $b_k$ at iteration $k$ satisfies:

$$b \geq \left[ \frac{1}{n} + \frac{2}{\gamma_{\text{max}}} \frac{\mu_{\text{min}} v}{cL} \right]^{-1}$$

converges as

$$\mathbb{E}[f(x^k) - f(x^*)] \leq (1 - \nu/2)^k (f(x^0) - f(x^*))$$

where $\nu = 1 - \gamma_b (1 - 2\mu + \frac{L_{\text{max}}}{2c}) \in (0, 1)$.

Proof. Following the proof of Theorem 3.6, from Equation 46,

$$\mathbb{E} [f(x^{k+1}) - f(x^*)] \leq \gamma_b \left( \frac{1}{\alpha} - 2\mu + \frac{L_{\text{max}}}{2c} \right) \mathbb{E} [f(x^k) - f(x^*)] + \frac{L_{\gamma_b} c}{2c} \mathbb{E} [f_b(x^*) - f_b^*]$$

(76)

Similar to the proof of Theorem C.2, since each function $f_i$ is PL,

$$\mathbb{E} [f_b(x^*) - f_b^*] \leq \mathbb{E} \left[ \frac{1}{2\mu_i} \| \nabla f_b(x^*) \|^2 \right] \leq \frac{1}{2\mu_{\text{min}}} \mathbb{E} \| \nabla f_b(x^*) \|^2$$

$$\mathbb{E} [f_b(x^*) - f_b^*] \leq \frac{1}{2\mu_{\text{min}}} \frac{n - b}{nb} z^2$$

(77)

From the above relations,

$$\mathbb{E} [f(x^{k+1}) - f(x^*)] \leq (1 - \nu) \mathbb{E} [f(x^k) - f(x^*)] + \frac{L_{\gamma_b} c}{2c} \frac{1}{2\mu_{\text{min}}} \frac{n - b}{nb} z^2$$

(78)

If we set the batch-size $b$ s.t.

$$\frac{L_{\gamma_b} c}{2c} \frac{1}{2\mu_{\text{min}}} \frac{n - b}{nb} z^2 \leq \frac{v}{2} [f(x^k) - f(x^*)]$$

(79)

$$\implies b \geq \left[ \frac{1}{n} + \frac{2}{\gamma_{\text{max}}} \frac{\mu_{\text{min}} v}{cL} \right]^{-1}$$

(80)

$$\implies \mathbb{E} [f(x^{k+1}) - f(x^*)] \leq (1 - \nu/2) \mathbb{E} [f(x^k) - f(x^*)]$$

(81)

Following the remaining proof in 3.6,

$$\mathbb{E} [f(x^{k+1}) - f(x^*)] \leq (1 - \nu/2)^k [f(x^0) - f(x^*)]$$

(82)
D Additional Experiments

Figure 3. Deep matrix factorization
Figure 4. Binary classification using kernels
Figure 5. Multi-class classification using deep networks