Electric circuits for non-Hermitian Chern insulators

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We analyze the non-Hermitian Haldane model where the spin-orbit interaction is made non-Hermitian. The Dirac mass becomes complex. We propose to realize it by an LC circuit with operational amplifiers. A topological phase transition is found to occur at a critical point where the real part of the bulk spectrum is closed. The Chern number changes its value when the real part of the mass becomes zero. In the topological phase of a nanoribbon, two non-Hermitian chiral edges emerge connecting well separated conduction and valence bands. The emergence of the chiral edge states is signaled by a strong enhancement in impedance. Remarkably it is possible to observe either the left-going or right-going chiral edge by measuring one-point impedance. Furthermore, it is also possible to distinguish them by the phase of two-point impedance. Namely, the phase of the impedance acquires a dynamical degree of freedom in the non-Hermitian system.

Introduction: Non-Hermitian topological systems open a new field of topological physics. They are experimentally realized in photonic systems, microwave resonator, wave guide, quantum walks and cavity systems. The winding number and the Chern number are generalized to non-Hermitian systems. Interestingly, there are properties not shared by the Hermitian topological systems. For instance, the Hall conductance and the chiral edge conductance are not quantized in the non-Hermitian Chern insulators although the Chern number is quantized. The typical model of the Chern insulator is the Haldane model, but there is so far no extension to the non-Hermitian model in literatures. Furthermore, although there are several studies on non-Hermitian Chern insulators, there are so far no reports on how to realize them physically.

In this paper, we propose to realize the non-Hermitian Haldane model by an electric circuit. Electric circuits have proven to provide us with an ideal playground to study various topological phases. Indeed, the Hall conductance and the chiral edge conductance are not quantized in the non-Hermitian systems. We analyze the non-Hermitian Haldane model where the spin-orbit interaction is made non-Hermitian. The typical model of the Chern insulator is the Haldane model, but there is so far no extension to the non-Hermitian model in literatures. Furthermore, although there are several studies on non-Hermitian Chern insulators, there are so far no reports on how to realize them physically.

We study a non-Hermitian Haldane model on the honeycomb lattice, where the spin-orbit interaction is made non-Hermitian. It is constructed by an LC circuit together with operational amplifiers as shown in Fig.1. The spin-orbit term yields complex Dirac masses at the $K$ and $K'$ points. The chiral edge states emerge in the topological phase. They have imaginary energy though the bulk spectrum is almost real. We show a topological phase transition to occur when the real part of the bulk-energy spectrum is closed. The Chern number is determined only by the real part of the Dirac mass, while the absolute value of the bulk energy does not close at the transition point. Non-Hermitian chiral edges emerge in nanoribbon geometry, which acquire pure imaginary energy at the high symmetry points. A prominent feature is that the left-going and right-going chiral edge modes are distinguished by the phase of two-point impedance. Furthermore, it is possible to observe only one of them by a measurement of one-point impedance. These are merits of the electric-circuit realization of the non-Hermitian Haldane model.

Non-Hermitian Haldane model: We investigate a non-Hermitian Haldane model on the honeycomb lattice, where the Haldane interaction is non-Hermitian. As described later, it is realized by an electric circuit of the configuration illustrated in Fig.1. The honeycomb lattice is a bipartite lattice, which consists of the $A$ and $B$ sites. In the basis of the $A$ and $B$ sites, the Hamiltonian is given by

$$H = \begin{pmatrix} g(k) + U & f(k) \\ f^*(k) & -g(k) - U \end{pmatrix},$$

where

$$f(k) = t \left( e^{-iak_y/\sqrt{3}} + 2e^{iak_y/2\sqrt{3}} \cos \frac{ak_x}{2} \right)$$

describes hopping on the honeycomb lattice, and

$$g(k) = \frac{i}{3\sqrt{3}} \left| \lambda^r \left( e^{iak_x} + e^{-iak_x + \sqrt{3}k_y} + e^{-iak_x - \sqrt{3}k_y} \right) \right|$$

$$- \lambda^l \left( e^{-iak_x} + e^{iak_x + \sqrt{3}k_y} + e^{iak_x - \sqrt{3}k_y} \right)$$

(3)
describes the non-Hermitian Haldane interaction. It is reduced to the original Haldane interaction for $\lambda' = \lambda$. We have also added one-site staggered potential $\pm U$, which alternates between $A$ and $B$ sites. The original Haldane model has the inversion symmetry $P^{-1} H (k) P = H (-k)$ with $P = \tau_x$. It is broken in the present model where $\lambda' \neq \lambda$. Let us define $\lambda = (\lambda' + \lambda^*) / 2$ and $\gamma = (\lambda' - \lambda^*) / 2$, where $\gamma$ represents the non-Hermiticity.

A characteristic feature of the non-Hermitian theory is that the energy becomes complex. In the present model it is given by $E = \pm \sqrt{|f(k)|^2 + (g(k) + U)^2}$, which we illustrate in Fig. 2. The real part $\text{Re} [E]$ is exactly identical to the energy of the original Haldane model, while the imaginary part $\text{Im} [E]$ has strong peaks at the $K$ and $K'$ points, i.e., at $K_\xi = \frac{1}{2} (\xi \pm \sqrt{\frac{a}{3}}, 0)$ with $\xi = \pm$. The positions of these points do not shift by the non-Hermitian Haldane interaction.

A Taylor expansion of $E$ around the $K$ and $K'$ points. Hereafter, let us use $k$ as the momentum measured from each of these points. By making the Taylor expansion, we obtain $f(k + K_\xi) = \xi k_x - i k_y$ for $|k| \ll a^{-1}$. Hence, the low-energy physics near the Fermi energy is described by the Dirac theory,

$$H\xi(k) = \hbar v_F (\xi k_x \tau_x + k_y \tau_y) + m_\xi \tau_z,$$

where $v_F = \frac{\sqrt{3}}{2} a \nu t$ is the Fermi velocity. The Dirac masses at the $K$ and $K'$ points are given by

$$m_\xi = U - \xi \lambda - i \gamma / \sqrt{3}.$$

We show them as a function of $U$ in Fig. 3. As we shall soon see, the system is a Chern insulator for $|\lambda| > |U|$ and a trivial insulator for $|\lambda| < |U|$ irrespective to $\gamma$. The system undergoes a topological phase transition as a function of $U$ when $\text{Re}[\lambda]$ vanishes.

**Non-Hermitian Chern number:** The bulk spectrum looks complicated due to the imaginary part as in Fig. 2. Nevertheless, it is well decomposed into the conduction and valence bands, because $\text{Re}[E]$ has this property as in Fig. 2(a1)–(c1) and $|E| \neq 0$ anywhere as in Fig. 2(a3)–(c3). Then, the Chern number is well defined and characterizes the phase even in the non-Hermitian theory.

The non-Hermitian Chern number is defined by $\int \frac{d^n x}{(2\pi)^n} \frac{\langle \psi^k | \psi^R \rangle}{\langle \psi^k | \psi^R \rangle}$, which satisfies the biorthogonal condition, $\langle \psi^k | \psi^R \rangle = 1$.

The non-Hermitian Berry connection is defined by $A(k) = \frac{-i}{\hbar} \langle \psi^k | \partial_k | \psi^R \rangle$, which is calculated as

$$A_\alpha = \frac{h_x \partial_k h_y - h_y \partial_k h_x}{E(h - E)}.$$

The non-Hermitian Berry curvature reads

$$F(k) = \nabla \times A(k) = \frac{1}{2E^3/2\xi_{\mu \nu \rho}} h_\mu \partial_k h_\nu \partial_k h_\rho.$$

The non-Hermitian Chern number is defined by $\frac{\partial^2}{\partial U^2} \int F(k) d^2 k$, where the integration is over the Brillouin zone. It is also understood as the Pontryagin number. For the $K$ and $K'$ points we explicitly find

$$F_\xi(k) = -\xi v^2 \xi m + U \frac{2}{(\xi m + U)^2 + v^2 k^2} \epsilon^{1/2},$$

and

$$C_\xi = (\xi m + U) / (2\sqrt{(\xi m + U)^2}).$$
A complex value. We find the Berry curvature and the Chern number although the forms are exactly the same as those of the Hermitian phase. The topological phase (\(U = 0\)) and the non-Hermitian Haldane model. Namely, the bulk spectrum consists of well separated conduction and valence bands as in the Hamiltonian and the circuit Laplacian.

**Non-Hermitian chiral edge states:** According to the bulk-edge correspondence in nanoribbon geometry, the topological phase is characterized by the emergence of the left-going and right-going chiral edge states which cross each other at \(E = 0\). An intuitive reason is that the topological number cannot change its value without gap closing. We ask how it is affected by the non-Hermitian Haldane term. We calculate the bulk structures of a zigzag nanoribbon and show them in Fig. 4. The real part of the energy spectrum is identical to that of the original Haldane model as in the case of the bulk spectrum. We have already argued that the bulk spectrum consists of well separated conduction and valence bands as in the Hermitian system. The chiral edge states emerge to connect these two bands in the topological phase. In the original Haldane model (\(\gamma = 0\)), the two chiral edges along the two edges cross at \(E = 0\), where \(k = \pi\). As far as \(\text{Re}[E_{\text{edge}}]\) concerns, it has the same behavior in the non-Hermitian Haldane model (\(\gamma \neq 0\)): See Fig. 4(a2). However, these two chiral edges do not cross because \(\text{Im}[E_{\text{edge}}] \neq 0\) at \(\text{Re}[E_{\text{edge}}] = 0\) as in Fig. 4(a3), and furthermore, \(|E_{\text{edge}}| \neq 0\) anywhere as in Fig. 4(a1).

It is possible to analytically obtain \(E_{\text{edge}}(k)\) at \(k = \pi\) by the perturbation theory in \(\gamma\), which is given by \(E_{\text{edge}}(\pi) = -2i\lambda\gamma\) for the A sites and \(E_{\text{edge}}(\pi) = 2i\lambda\gamma\) for the B sites. The edge states are almost localized at the outermost edge at \(k = \pi\) for \(|\gamma| \ll t\).

**Electric circuit realization:** The Haldane model is constructed as in Fig. 1. First, a capacitor \(C\) is placed on each neighboring link of the honeycomb lattice. Second, we bridge a pair of the same type of sites by an operational amplifier, which acts as a negative impedance converter with current inversion and as the spin-orbit interaction. Then, \(A\) and \(B\) sites are grounded by inductors \(L_A\) and \(L_B\), respectively.

With the AC voltage \(V(t) = V(0)e^{i\omega t}\) applied, the Kirchhoff’s current law reads.

\[
J_{ab}(\omega) = i\omega\delta_{ab}[-\frac{1}{2\omega^2 L_A} - \frac{1}{2\omega^2 L_B} + \sum_{\epsilon \neq a} H_{\epsilon a}] - i\omega H_{ab},
\]

(12)

The voltage-current relation for the operational amplifier circuit is given by.

\[
\begin{pmatrix}
I_{ij}

\end{pmatrix}
= \frac{1}{R}
\begin{pmatrix}
-\nu & \nu \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
V_i \\
V_j
\end{pmatrix},
\]

(13)

with \(\nu = R_b/R_a\), where \(R_A\) and \(R_B\) are the resistances in the operational amplifier circuit. See Fig 1(b). The resistors in the operational amplifier circuit are tuned to be \(\nu = 1\) in the literature, so that the system becomes Hermitian. However, the system is non-Hermitian for a general value of the resistors, leading to \(\nu \neq 1\), which is the main theme of this paper.

There is a correspondence between the system parameters in the Hamiltonian and the circuit Laplacian.

\[
t = C, \quad \lambda^r = \frac{R_b}{R_A}, \quad \lambda^l = \frac{1}{R}, \quad U = \frac{1}{2\omega^2} \left(\frac{1}{L_A} - \frac{1}{L_B}\right),
\]

(14)

by way of which the physics of the two systems is identified.

**Admittance spectrum and impedance peaks:** The admittance spectrum consists of the eigenvalues of the circuit Laplacian. It corresponds to the band structure in condensed-matter physics.

Two-point impedance gives an information of the admittance band structure.

\[
Z_{ab} = \frac{V_a}{I_{ab}} = G_{aa} + G_{ib} - G_{ab} - G_{ba}, \quad G = \text{green function defined by the inverse of the Laplacian} \quad J \equiv J^{-1}.
\]

It diverges at the frequency satisfying \(J = 0\). Namely, the peaks of impedance are well described by the zeros of the admittance. After the diagonalization, the circuit Laplacian reads.

\[
J_n(\omega) = i\omega(-2\omega^2 L_A)^{-1} - (2\omega^2 L_B)^{-1} + 3C) - i\omega\varepsilon_n(\omega),
\]

(15)

where \(\varepsilon_n\) is the eigenvalue of the circuit Laplacian. It is pure imaginary for the Hermitian system but becomes complex for the non-Hermitian system. By solving \(J_n(\omega) = 0\), we obtain.

\[
\omega_{\text{Re}}(\varepsilon_n) = 1/\sqrt{(-\varepsilon_n + 3C)/(1/2L_A + 1/2L_B)},
\]

(16)

where the admittance is zero. The eigenvalue \(\varepsilon_n\) should be solved numerically for the bulk spectrum or for the nanoribbon spectrum.
FIG. 5: (a1)-(c1) Zero-admittance curves in the $k$-$\omega$ plane given by (16), along which the impedance peaks appear. (a2)-(c2) Real part of the two-point impedance $Z_{AB}$, which is quite similar to the impedance peaks in (a1)–(c1). (a3)-(c3) Imaginary part of the two-point impedance $Z_{AB}$. (a4)-(c4) Real part of the one-point impedance $G_{AA}$ at the edge at the A sites. (a5)-(c5) Real part of the one-point impedance $G_{BB}$ at the edge at the B sites. (a) $U = 0$, (b) $U = \lambda$ and (c) $U = 2\lambda$. The red (green) regions indicate positive (negative) values of the impedance. Parameters $\lambda$ and $\gamma$ are the same as in Fig.2.

We consider a nanoribbon, where it is a function of the momentum $k$. We show $\omega(k)$ as a function of $k$ in Fig.5(a1)-(c1), which are to be compared with Fig.4(a2)–(c2). The impedance diverges at these frequencies. The chiral edge states cross the $\omega = 0$ line at $k = \pi$ when $U = 0$ in Fig.5(a1). The crossing point moves towards the $K$ point as $U$ increases and reaches it when $U = \lambda$ as in Fig.5(b1). It is the signal of the topological phase. It disappears for $U > 0$ as in Fig.5(c1), indicating that the system is in the trivial phase. Thus the topological phase transition induced by the potential $U$ is clearly observed in the change of the impedance peak.

We have calculated the impedance with the use of Green functions as a function of $k$ and $\omega$, which we show in Fig.5 from (a2)–(c2) to (a5)–(c5), where the impedance strength is represented by darkness. Note that the $k$-dependent impedance is an experimentally detectable quantity.

The impedance is pure imaginary in the Hermitian model. On the other hand, there emerges the real part of the impedance in the non-Hermitian model, since the energy becomes complex. Let us focus on the two-point impedance $\text{Re}Z_{AB}$ between the two outermost edges. It is found that the chiral edge modes become prominent compared with the bulk band modes as in Fig.5(a2). This is because the imaginary part of the energy is enhanced at the $K$ and $K'$ points as in Fig.4(a2). Furthermore, it is a characteristic feature of the non-Hermitian system that the left-going and right-going chiral edges are differentiated by the sign of $\text{Re}Z_{AB}$ as in Fig.5(a2), where the sign corresponds to red or green. Equivalently, they are distinguished by the phase of the impedance. This is because the imaginary part of the energy is opposite between them.

The one-point impedance defined by $G_{ab} = V_a/I_b$ is also a measurable quantity. The one-point impedance $G_{AA}$ at the outermost edge site $A$ is shown in Fig.5(a4)-(c4), where only the left-going chiral edge has a strong peak. It is because there is only the left-going chiral edge at the $A$ edge. On the contrary, there is a strong resonance only for the right-going chiral edge when we measure $G_{BB}$ as in Fig.5(a5)-(c5). This selective detection of the chiral edge is possible only in electric circuits.

Discussion: Topological monolayer systems such as silicene, germanene and stanene provides us with rich topological phase transitions. However, their experimental realizations are yet to be made. In addition, it is hard to observe the edge states of the two dimensional topological insulators by ARPES since the intensity is very weak. On the other hand, almost all of these physics are realizable by employing electric circuits in a relatively easy manner. In particular, topological phase transitions are detectable by measuring the change of impedance. Furthermore, it is possible to observe the left-going and right-going edge states separately by measuring one-point impedance.

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