Completion by Derived Double Centralizer

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Abstract. Let $A$ be a commutative ring, and let $a$ be a weakly proregular ideal in $A$. (If $A$ is noetherian then any ideal in it is weakly proregular.) Suppose $M$ is a compact generator of the category of cohomologically $a$-torsion complexes. We prove that the derived double centralizer of $M$ is isomorphic to the $a$-adic completion of $A$. The proof relies on the MGM equivalence from $[PSY]$ and on derived Morita equivalence. Our result extends earlier work of Dwyer-Greenlees-Iyengar $[DGI]$ and Efimov $[E]$. 

0. Introduction

Let $A$ be a commutative ring. We denote by $D(Mod A)$ the derived category of $A$-modules. Given $M \in D(Mod A)$ we define

\begin{equation}
Ext_A(M) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D(Mod A)}(M, M[i]).
\end{equation}

This is a graded $A$-algebra with the Yoneda multiplication, which we call the Ext algebra of $M$.

Suppose we choose a K-projective resolution $P \to M$. The resulting DG $A$-algebra $B := \text{End}_A(P)$ is called a derived endomorphism DG algebra of $M$. It turns out (see Proposition 2.2) that the DG algebra $B$ is unique up to quasi-isomorphism; and of course its cohomology $H(B) := \bigoplus_{i \in \mathbb{Z}} H^i(B)$ is canonically isomorphic to $Ext_A(M)$ as graded $A$-algebra.

Consider the derived category $\tilde{D}(DGMod B)$ of left DG $B$-modules. We can view $P$ as an object of $\tilde{D}(DGMod B)$, and thus, like in (0.1), we get the graded $A$-algebra $Ext_B(P)$. This is the derived double centralizer algebra of $M$. By Corollary 2.4, the graded algebra $Ext_B(P)$ is independent of the resolution $P \to M$, up to isomorphism.

Let $a$ be an ideal in $A$. The $a$-torsion functor $\Gamma_a$ can be right derived, giving a triangulated functor $R\Gamma_a$ from $D(Mod A)$ to itself. A complex $M \in D(Mod A)$ is called a cohomologically $a$-torsion complex if the canonical morphism $R\Gamma_a(M) \to M$ is an isomorphism. The full triangulated category on the cohomologically torsion complexes is denoted by $D(Mod A)_{a-tor}$. It is known that when $a$ is finitely generated, the category $D(Mod A)_{a-tor}$ is compactly generated (for instance by the Koszul complex $K(A; a)$ associated to a finite generating sequence $a = (a_1, \ldots, a_n)$ of $a$).
Let us denote by \( \hat{a} \) the a-adic completion of \( a \). This is a commutative \( A \)-algebra, and in it there is the ideal \( \hat{a} := a \cdot \hat{A} \). If \( a \) is finitely generated, then the ring \( \hat{A} \) is \( \hat{a} \)-adically complete.

A weakly proregular sequence in \( A \) is a finite sequence \( a \) of elements of \( A \), whose Koszul cohomology satisfies certain vanishing conditions; see Definition 1.2. This concept was introduced by Alonso-Jeremias-Lipman [AJL] and Schenzel [Sc]. An ideal \( a \) in \( A \) is called weakly proregular if it can be generated by a weakly proregular sequence. It is important to note that if \( A \) is noetherian, then any finite sequence in it is weakly proregular, so that any ideal in \( A \) is weakly proregular. But there are some fairly natural non-noetherian examples (see [AJL, Example 3.0(b)] and [PSY, Example 4.35]).

Here is our main result (repeated as Theorem 4.2 in the body of the paper).

**Theorem 0.2.** Let \( A \) be a commutative ring, let \( a \) be a weakly proregular ideal in \( A \), and let \( M \) be a compact generator of \( D(\text{Mod} A)_{\text{a-tor}} \). Choose a \( K \)-projective resolution \( P \to M \), and define \( B := \text{End}_A(P) \). Then there is a unique isomorphism of graded \( A \)-algebras \( \text{Ext}_B(P) \cong \hat{A} \).

Our result extends earlier work of Dwyer-Greenlees-Iyengar [DGI] and Efimov [Ef]; see Remark 4.8 for a discussion.

Let us say a few words on the proof of Theorem 0.2. We use derived Morita theory to find an isomorphism of graded algebras between \( \text{Ext}_B(P) \) and \( \text{Ext}_A(N)^{\text{op}} \), where \( N := R\Gamma_a(A) \). The necessary facts about derived Morita theory are recalled in Section 3. We then use MGM equivalence (recalled in Section 1) to prove that \( \text{Ext}_A(N) \cong \text{Ext}_A(\hat{A}) \cong \hat{A} \).

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### 1. Weak Proregularity and MGM Equivalence

Let \( A \) be a commutative ring, and let \( a \) be an ideal in it. (We do not assume that \( A \) is noetherian or \( a \)-adically complete.) There are two operations on \( A \)-modules associated to this data: the \( a \)-adic completion and the \( a \)-torsion. For an \( A \)-module \( M \) its \( a \)-adic completion is the \( A \)-module \( \Lambda_a(M) = \hat{M} := \lim_{\to-i} M/a^iM \). An element \( m \in M \) is called an \( a \)-torsion element if \( a^i m = 0 \) for \( i \gg 0 \). The \( a \)-torsion elements form the \( a \)-torsion submodule \( \Gamma_a(M) \) of \( M \).

Let us denote by \( \text{Mod} A \) the category of \( A \)-modules. So we have additive functors \( \Lambda_a \) and \( \Gamma_a \) from \( \text{Mod} A \) to itself. The functor \( \Gamma_a \) is left exact; whereas \( \Lambda_a \) is neither left exact nor right exact. An \( A \)-module is called \( a \)-adically complete if the canonical homomorphism \( \tau_M : M \to \Lambda_a(M) \) is bijective (some texts would say that \( M \) is complete and separated); and \( M \) is \( a \)-torsion if the canonical homomorphism \( \sigma_M : \Gamma_a(M) \to M \) is bijective. If the ideal \( a \) is finitely generated, then the functor \( \Lambda_a \) is idempotent; namely for any module \( M \), its completion \( \Lambda_a(M) \) is \( a \)-adically complete. (There are counterexamples to that for infinitely generated ideals – see [Ye, Example 1.8].)

The derived category of \( \text{Mod} A \) is denoted by \( D(\text{Mod} A) \). The derived functors \( L\Lambda_a, R\Gamma_a : D(\text{Mod} A) \to D(\text{Mod} A) \)
exist. The left derived functor $L\Lambda_a$ is constructed using $K$-flat resolutions, and the right derived functor $R\Gamma_a$ is constructed using $K$-injective resolutions. This means that for any $K$-flat complex $P$, the canonical morphism $\xi_P : L\Lambda_a(P) \to \Lambda_a(P)$ is an isomorphism; and for any $K$-injective complex $I$, the canonical morphism $\xi_I : \Gamma_a(I) \to R\Gamma_a(I)$ is an isomorphism. The relationship between the derived functors $R\Gamma_a$ and $L\Lambda_a$ was first studied in [AJL], where the Greenlees-May duality was established (following the paper [GM]).

A complex $M \in D(\text{Mod } A)$ is called a cohomologically $a$-torsion complex if the canonical morphism $\sigma^R_M : R\Gamma_a(M) \to M$ is an isomorphism. The complex $M$ is called a cohomologically $a$-adically complete complex if the canonical morphism $\tau^M_a : M \to L\Lambda_a(M)$ is an isomorphism. We denote by $D(\text{Mod } A)_{a,\text{tor}}$ and $D(\text{Mod } A)_{a,\text{com}}$, the full subcategories of $D(\text{Mod } A)$ consisting of cohomologically $a$-torsion complexes and cohomologically $a$-adically complete complexes, respectively. These are triangulated subcategories.

Very little can be said about the functors $L\Lambda_a$ and $R\Gamma_a$, and about the corresponding triangulated categories $D(\text{Mod } A)_{a,\text{tor}}$ and $D(\text{Mod } A)_{a,\text{com}}$, in general. However we know a lot when the ideal $a$ is weakly proregular.

Before defining weak proregularity we have to talk about Koszul complexes. Recall that for an element $a \in A$ the Koszul complex is

$$K(A; a) := (\cdots \to 0 \to A \xrightarrow{a} A \xrightarrow{0} \cdots),$$

concentrated in degrees $-1$ and $0$. Given a finite sequence $a = (a_1, \ldots, a_n)$ of elements of $A$, the Koszul complex associated to this sequence is

$$K(A; a) := K(A; a_1) \otimes_A \cdots \otimes_A K(A; a_n).$$

This is a complex of finitely generated free $A$-modules, concentrated in degrees $-n, \ldots, 0$. There is a canonical isomorphism of $A$-modules $H^0(K(A; a)) \cong A/(a)$, where $(a)$ is the ideal generated by the sequence $a$.

For any $i \geq 1$ let $a^i := (a_1^i, \ldots, a_n^i)$. If $j \geq i$ then there is a canonical homomorphism of complexes $p_{j,i} : K(A; a^j) \to K(A; a^i)$, which in $H^0$ corresponds to the surjection $A/(a^j) \to A/(a^i)$. Thus for every $k \in \mathbb{Z}$ we get an inverse system of $A$-modules

$$\{H^k(K(A; a^i))\}_{i \in \mathbb{N}},$$

with transition homomorphisms

$$H^k(p_{j,i}) : H^k(K(A; a^j)) \to H^k(K(A; a^i)).$$

Of course for $k = 0$ the inverse limit equals the $(a)$-adic completion of $A$. What turns out to be crucial is the behavior of this inverse system for $k < 0$. For more details please see [PSY] Section 4.

An inverse system $\{M_i\}_{i \in \mathbb{N}}$ of abelian groups, with transition maps $p_{j,i} : M_j \to M_i$, is called pro-zero if for every $i$ there exists $j \geq i$ such that $p_{j,i}$ is zero.

**Definition 1.2.**

1. Let $a$ be a finite sequence in $A$. The sequence $a$ is called a weakly proregular sequence if for every $k \leq -1$ the inverse system $\{M_i\}$ is pro-zero.

2. An ideal $a$ in $A$ is called a weakly proregular ideal if it is generated by some weakly proregular sequence.

The etymology of the name “weakly proregular sequence”, and the history of related concepts, are explained in [AJL] and [Sc].
If $\mathfrak{a}$ is a regular sequence, then it is weakly proregular. More important is the following result.

**Theorem 1.3** ([AJL]). If $A$ is noetherian, then every finite sequence in $A$ is weakly proregular, so that every ideal in $A$ is weakly proregular.

Here is another useful fact.

**Theorem 1.4** ([Sc]). Let $\mathfrak{a}$ be a weakly proregular ideal in a ring $A$. Then any finite sequence that generates $\mathfrak{a}$ is weakly proregular.

These theorems are repeated (with different proofs) as [PSY] Theorem 4.34 and [PSY] Corollary 6.3 respectively.

As the next theorem shows, weak proregularity is the correct condition for the derived torsion functor to be “well-behaved”. Suppose $\mathfrak{a}$ is a finite sequence that generates the ideal $\mathfrak{a} \subset A$. Consider the infinite dual Koszul complex

$$K^\vee_\infty(A; \mathfrak{a}) := \lim_{\rightarrow i} \text{Hom}_A(K(A; \mathfrak{a}^i), A).$$

Given a complex $M$, there is a canonical morphism

$$(1.5) \quad R\Gamma_\mathfrak{a}(M) \to K^\vee_\infty(A; \mathfrak{a}) \otimes_A M$$

in $D(\text{Mod } A)$.

**Theorem 1.6** ([Sc]). The sequence $\mathfrak{a}$ is weakly proregular iff the morphism $$(1.5)$$ is an isomorphism for every $M \in D(\text{Mod } A)$.

The following theorem, which is [PSY] Theorem 1.1, plays a central role in our work.

**Theorem 1.7** (MGM Equivalence). Let $A$ be a commutative ring, and $\mathfrak{a}$ a weakly proregular ideal in it.

1. For any $M \in D(\text{Mod } A)$ one has $R\Gamma_\mathfrak{a}(M) \in D(\text{Mod } A)_{\mathfrak{a}\text{-tor}}$ and $L\Lambda_\mathfrak{a}(M) \in D(\text{Mod } A)_{\mathfrak{a}\text{-com}}$.

2. The functor $R\Gamma_\mathfrak{a} : D(\text{Mod } A)_{\mathfrak{a}\text{-com}} \to D(\text{Mod } A)_{\mathfrak{a}\text{-tor}}$ is an equivalence, with quasi-inverse $L\Lambda_\mathfrak{a}$.

**Remark 1.8.** Slightly weaker versions of Theorem 1.7 appeared previously; they are [AJL] Theorem (0.3)* and [Sc] Theorem 4.5]. The difference is that in these earlier results it was assumed that the ideal $\mathfrak{a}$ is generated by a sequence $(a_1, \ldots, a_n)$ that is weakly proregular, and moreover each $a_i$ has bounded torsion. This extra condition certainly holds when $A$ is noetherian.

For the sake of convenience, in the present paper we quote [PSY] regarding derived completion and torsion. It is tacitly understood that in the noetherian case the results of [AJL] and [Sc] suffice.

2. The Derived Double Centralizer

In this section we define the derived double centralizer of a DG module. See Remarks 2.6 and 2.7 for a discussion of this concept and related literature.

Let $K$ be a commutative ring, and let $A = \bigoplus_{i \in \mathbb{Z}} A^i$ be a DG $K$-algebra (associative and unital, but not necessarily commutative). Given left DG $A$-modules $M$
and $N$, we denote by $\text{Hom}^i_A(M, N)$ the $\mathbb{K}$-module of $A$-linear homomorphisms of degree $i$. We get a DG $\mathbb{K}$-module

$$\text{Hom}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}^i_A(M, N)$$

with the usual differential.

The object $\text{End}_A(M) := \text{Hom}_A(M, M)$ is a DG $\mathbb{K}$-algebra. Since the left actions of $A$ and $\text{End}_A(M)$ on $M$ commute, we see that $M$ is a left DG module over the DG algebra $A \otimes_{\mathbb{K}} \text{End}_A(M)$.

The category of left DG $A$-modules is denoted by $\text{DGMod} A$. The set of morphisms $\text{Hom}_{\text{DGMod} A}(M, N)$ is precisely the set of 0-cocycles in the DG $\mathbb{K}$-module $\text{Hom}_A(M, N)$. Note that $\text{DGMod} A$ is a $\mathbb{K}$-linear abelian category.

Let $\tilde{K}(\text{DGMod} A)$ be the homotopy category of $\text{DGMod} A$, so that

$$\text{Hom}_{\tilde{K}(\text{DGMod} A)}(M, N) = H^0(\text{Hom}_A(M, N)).$$

The derived category $\tilde{D}(\text{DGMod} A)$ is gotten by inverting the quasi-isomorphisms in $\tilde{K}(\text{DGMod} A)$. The categories $\tilde{K}(\text{DGMod} A)$ and $\tilde{D}(\text{DGMod} A)$ are $\mathbb{K}$-linear and triangulated. If $A$ happens to be a ring (i.e. $A^i = 0$ for $i \neq 0$) then $\text{DGMod} A = \mathbb{C}(\text{Mod} A)$, the category of complexes in $\text{Mod} A$, and $\tilde{D}(\text{DGMod} A) = \text{D}(\text{Mod} A)$, the usual derived category.

For $M, N \in \text{DGMod} A$ we define

$$\text{Ext}^i_A(M, N) := \text{Hom}_{\tilde{D}(\text{DGMod} A)}(M, N[i]),$$

and

$$\text{Ext}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{Ext}^i_A(M, N).$$

**Definition 2.1.** Let $A$ be a DG $\mathbb{K}$-algebra and $M \in \text{DGMod} A$. Define

$$\text{Ext}_A(M) := \text{Ext}_A(M, M).$$

This is a graded $\mathbb{K}$-algebra with the Yoneda multiplication (i.e. composition of morphisms in $\tilde{D}(\text{DGMod} A)$). We call $\text{Ext}_A(M)$ the *Ext algebra* of $M$.

There is a canonical homomorphism of graded $\mathbb{K}$-algebras $H(\text{End}_A(M)) \to \text{Ext}_A(M)$. If $M$ is either $\text{K}$-projective or $\text{K}$-injective, then this homomorphism is bijective.

**Definition 2.2.** Let $A$ be a DG $\mathbb{K}$-algebra and $M$ a DG $A$-module. Choose a $\text{K}$-projective resolution $P \to M$ in $\text{DGMod} A$. The DG $\mathbb{K}$-algebra $B := \text{End}_A(P)$ is called a derived endomorphism DG algebra of $M$.

Note that there are isomorphisms of graded $\mathbb{K}$-algebras $H(B) \cong \text{Ext}_A(P) \cong \text{Ext}_A(M)$. The dependence of the derived endomorphism DG algebra $B = \text{End}_A(P)$ on the resolution $P \to M$ is explained in the next proposition.

**Proposition 2.3.** Let $M$ be a DG $A$-module, and let $P \to M$ and $P' \to M$ be $\text{K}$-projective resolutions in $\text{DGMod} A$. Define $B := \text{End}_A(P)$ and $B' := \text{End}_A(P')$. Then there is a DG $\mathbb{K}$-algebra $B''$, and a DG $B''$-module $P''$, with DG $\mathbb{K}$-algebra quasi-isomorphisms $B'' \to B$ and $B'' \to B'$, and DG $B''$-module quasi-isomorphisms $P'' \to P$ and $P'' \to P'$. 


Proof. Choose a quasi-isomorphism \( \phi : P' \to P \) in \( \text{DGMod } A \) lifting the given quasi-isomorphisms to \( M \); this can be done of course. Let \( L := \text{cone}(\phi) \in \text{DGMod } A \), the mapping cone of \( \phi \). So as graded \( A \)-module \( L = P \oplus P'[1] = \begin{bmatrix} P & P'[1] \end{bmatrix} \); and the differential is \( d_L = \begin{bmatrix} d_P & \phi \\ 0 & d_{P'[1]} \end{bmatrix} \), where \( \phi \) is viewed as a degree 1 homomorphism \( P'[1] \to P \). Of course \( L \) is an acyclic DG module.

Take \( Q := \text{Hom}_A(P'[1], P) \), and let \( B'' \) be the triangular matrix graded algebra \( B'' := \begin{bmatrix} B & Q \\ 0 & B' \end{bmatrix} \) with the obvious matrix multiplication. This makes sense because there is a canonical isomorphism of DG algebras \( B' \cong \text{End}_A(P'[1]) \). Note that \( B'' \) is a subalgebra of \( \text{End}_A(L) \). We make \( B'' \) into a DG algebra with differential \( d_{B''} := d_{\text{End}_A(L)}|_{B''} \). The projections \( B'' \to B \) and \( B'' \to B' \) on the diagonal entries are DG algebra quasi-isomorphisms, because their kernels are the acyclic complexes \( \text{Hom}_A(P'[1], L) \) and \( \text{Hom}_A(L, P) \) respectively.

Now under the restriction functor \( \text{DGMod}(B) \to \text{DGMod}(B'') \) we have \( P \mapsto \begin{bmatrix} P' \\ 0 \end{bmatrix} \), and likewise \( P' \mapsto \begin{bmatrix} 0 \\ P' \end{bmatrix} \). Consider the exact sequence

\[
0 \to \begin{bmatrix} P' \\ 0 \end{bmatrix} \to L \to \begin{bmatrix} 0 \\ P'[1] \end{bmatrix} \to 0
\]

in \( \text{DGMod}(B'') \). There is an induced distinguished triangle \( \begin{bmatrix} P' \\ 0 \end{bmatrix} \xrightarrow{\Delta} L \xrightarrow{\phi} \begin{bmatrix} 0 \\ P'[1] \end{bmatrix} \) in \( \mathcal{D}(\text{DGMod}(B'')) \). But \( L \) is acyclic, so \( \phi \) is an isomorphism.

Finally let us choose a \( K \)-projective resolution \( P'' \to \begin{bmatrix} P' \\ 0 \end{bmatrix} \) in \( \text{DGMod}(B'') \). Then \( \chi \) induces a quasi-isomorphism \( P'' \to \begin{bmatrix} 0 \\ P' \end{bmatrix} \) in \( \text{DGMod}(B'') \).

\[
\text{Corollary 2.4. In the situation of Proposition 2.3, there is an isomorphism of graded } \mathbb{K} \text{-algebras}
\]

\[
\text{Ext}_B(P) \cong \text{Ext}_{B'}(P')
\]

Proof. Since \( B'' \to B' \) is a quasi-isomorphism of DG algebras, it follows that the restriction functor \( \mathcal{D}(\text{DGMod}(B)) \to \mathcal{D}(\text{DGMod}(B'')) \) is an equivalence of triangulated categories. Therefore we get an induced isomorphism of graded \( \mathbb{K} \)-algebras \( \text{Ext}_B(P) \cong \text{Ext}_{B''}(P'') \). Similarly we get a graded \( \mathbb{K} \)-algebra isomorphism \( \text{Ext}_{B'}(P') \cong \text{Ext}_{B''}(P'') \).

\[
\text{Definition 2.5. Let } M \text{ be a DG } A \text{-module, and let } P \to M \text{ be a } K \text{-projective resolution in } \text{DGMod } A. \text{ The graded } \mathbb{K} \text{-algebra } \text{Ext}_B(P) \text{ is called a derived double centralizer of } M.
\]

\[
\text{Remark 2.6. The uniqueness of the graded } \mathbb{K} \text{-algebra } \text{Ext}_B(P) \text{ provided by Corollary 2.3 is sufficient for the purposes of this paper (see Theorem 0.2).}
\]

It is possible to show by a more detailed calculation that the isomorphism provided by Corollary 2.3 is in fact canonical (it does not depend on the choices made in the proof of Proposition 2.3 e.g. the quasi-isomorphism \( \phi \)).

Let us choose a \( K \)-projective resolution \( Q \to P \) in \( \text{DGMod}(B) \), and define the DG algebra \( C := \text{End}_B(Q) \). Then \( C \) should be called a double endomorphism DG algebra of \( M \). Of course \( \mathbb{H}(C) \cong \text{Ext}_B(P) \) as graded algebras. There should be a canonical DG algebra homomorphism \( A \to C \).

We tried to work out a comprehensive treatment of derived endomorphism algebras and their iterates, using the old-fashioned methods, and did not get very far (hence it is not included in the paper). We expect that a full treatment is only possible in terms of \( \infty \)-categories.
Remark 2.7. Derived endomorphism DG algebras (and the double derived ones) were treated in several earlier papers, including [DG1], [Jo] and [Ef]. These papers do not mention any uniqueness properties of these DG algebras; indeed, as far as we can tell, they just pick a convenient resolution \( P \to M \), and work with the DG algebra \( \text{End}_A(P) \). Cf. Subsection 1.5 of [DG1] where this issue is briefly discussed.

The most detailed treatment of derived endomorphism DG algebras that we know is in Keller’s paper [Ke]. In [Ke, Section 7.3] the concept of a lift of a DG module is introduced. The pair \((B, P)\) from Definition 2.2 is called a standard lift in [Ke]. It is proved that lifts are unique up to quasi-isomorphism (this is basically what is done in our Proposition 2.3); but there is no statement regarding uniqueness of these quasi-isomorphisms. Also there is no discussion of derived double centralizers.

3. Supplementation on Derived Morita Equivalence

Derived Morita theory goes back to Rickard’s paper [Ri], which dealt with rings and two-sided tilting complexes. Further generalizations can be found in [Ke, BV, Jo]. For our purposes (in Section 4) we need to know certain precise details about derived Morita equivalence in the case of DG algebras and compact generators (specifically, formula (3.3) for the functor \( F \) appearing in Theorem 3.5); and hence we give the full proof here.

Let \( E \) be a triangulated category with infinite direct sums. Recall that an object \( M \in E \) is called compact (or small) if for any collection \( \{N_z\}_{z \in Z} \) of objects of \( E \), the canonical homomorphism

\[
\bigoplus_{z \in Z} \text{Hom}_E(M, N_z) \to \text{Hom}_E\left(M, \bigoplus_{z \in Z} N_z\right)
\]

is bijective. The object \( M \) is called generator of \( E \) if for any nonzero object \( N \in E \) there is some \( i \in \mathbb{Z} \) such that \( \text{Hom}_E(M, N[i]) \neq 0 \).

As in Section 2, we consider a commutative ring \( K \) and a DG \( K \)-algebra \( A \). The next lemma seems to be known, but we could not find a reference.

**Lemma 3.1.** Let \( E \) be a triangulated category with infinite direct sums, let \( F, G : \hat{\mathcal{D}}(\text{DGMod}A) \to E \) be triangulated functors that commute with infinite direct sums, and let \( \eta : F \to G \) be a morphism of triangulated functors. Assume that \( \eta_A : F(A) \to G(A) \) is an isomorphism. Then \( \eta \) is an isomorphism.

**Proof.** Suppose we are given a distinguished triangle \( M' \to M \to M'' \xrightarrow{\Delta} \) in \( \hat{\mathcal{D}}(\text{DGMod}A) \), such that two of the three morphisms \( \eta_{M'} \), \( \eta_M \) and \( \eta_{M''} \) are isomorphisms. Then the third is also an isomorphism.

Since both functors \( F,G \) commute with shifts and direct sums, and since \( \eta_A \) is an isomorphism, it follows that \( \eta_P \) is an isomorphism for any free DG \( A \)-module \( P \).

Next consider a semi-free DG module \( P \). Choose any semi-basis \( Z = \bigcup_{j \geq 0} Z_j \) of \( P \). This gives rise to an exhaustive ascending filtration \( \{P_j\}_{j \geq -1} \) of \( P \) by DG submodules, with \( P_{-1} = 0 \). For every \( j \) we have a distinguished triangle

\[
P_{j-1} \xrightarrow{\theta_j} P_j \to P_j/P_{j-1} \xrightarrow{\Delta} \]

in \( \hat{\mathcal{D}}(\text{DGMod}A) \), where \( \theta_j : P_{j-1} \to P_j \) is the inclusion. Since \( P_j/P_{j-1} \) is a free DG module, by induction we conclude that \( \eta_{P_j} \) is an isomorphism for every \( j \). The
telescope construction (see [BN, Remark 2.2]) gives a distinguished triangle
\[ \bigoplus_{j \in \mathbb{N}} P_j \xrightarrow{\Theta} \bigoplus_{j \in \mathbb{N}} P_j \rightarrow P \xrightarrow{\Delta}, \]

with
\[ \Theta|_{P_{j-1}} := (1, -\theta_j) : P_{j-1} \rightarrow P_{j-1} \oplus P_j. \]

This shows that \( \eta_P \) is an isomorphism.

Finally, any DG module \( M \) admits a quasi-isomorphism \( P \rightarrow M \) with \( P \) semi-free. Therefore \( \eta_M \) is an isomorphism. \( \Box \)

Let \( E \) be a be a full triangulated subcategory of \( \tilde{D}(\text{DGMod} A) \) which is closed under infinite direct sums, and let \( M \in E \). Fix a K-projective resolution \( P \rightarrow M \) in \( \text{DGMod} A \), and let \( B := \text{End}_A(P) \). So \( B \) is a derived endomorphism DG algebra of \( M \) (Definition 2.2). Since \( P \in \text{DGMod} A \otimes_k B \), there is a triangulated functor
\[ (3.2) \quad G : \tilde{D}(\text{DGMod} B^{op}) \rightarrow \tilde{D}(\text{DGMod} A), \quad G(N) := N \otimes_B^H P \]

which is calculated by K-flat resolutions in \( \text{DGMod} B^{op} \). (Warning: \( P \) is usually not K-flat over \( B \).) The functor \( G \) commutes with direct sums, and \( G(B) \cong P \cong M \) in \( \tilde{D}(\text{DGMod} A) \). Therefore \( G(N) \in E \) for every \( N \in \tilde{D}(\text{DGMod} B^{op}) \).

Because \( P \) is K-projective over \( A \), there is a triangulated functor
\[ (3.3) \quad F : \tilde{D}(\text{DGMod} A) \rightarrow \tilde{D}(\text{DGMod} B^{op}), \quad F(L) := \text{Hom}_A(P, L). \]

We have \( F(M) \cong F(P) \cong B \) in \( \tilde{D}(\text{DGMod} B^{op}) \).

**Lemma 3.4.** The functor \( F|_E : E \rightarrow \tilde{D}(\text{DGMod} B^{op}) \) commutes with infinite direct sums if and only if \( M \) is a compact object of \( E \).

**Proof.** We know that
\[ \text{Hom}_{\tilde{D}(\text{DGMod} A)}(M, L[j]) \cong H^j(R\text{Hom}_A(M, L)) \cong H^j(F(L)), \]

functorially for \( L \in \tilde{D}(\text{DGMod} A) \). So \( M \) is compact relative to \( E \) if and only if the functors \( H^j \circ F \) commute with direct sums in \( E \). But that is the same as asking \( F \) to commute with direct sums in \( E \). \( \Box \)

**Theorem 3.5.** Let \( A \) be a DG \( \mathbb{k} \)-algebra, let \( E \) be a be a full triangulated subcategory of \( \tilde{D}(\text{DGMod} A) \) which is closed under infinite direct sums, and let \( M \) be a compact generator of \( E \). Choose a K-projective resolution \( P \rightarrow M \) in \( \text{DGMod} A \), and define \( B := \text{End}_A(P) \). Then the functor
\[ F|_E : E \rightarrow \tilde{D}(\text{DGMod} B^{op}) \]

from \( (3.3) \) is an equivalence of triangulated categories, with quasi-inverse the functor \( G \) from \( (3.2) \).

**Proof.** Let us write \( D(A) := \tilde{D}(\text{DGMod} A) \) etc. We begin by proving that the functors \( F \) and \( G \) are adjoints. Take any \( L \in D(A) \) and \( N \in D(B^{op}) \). We have to construct a bijection
\[ \text{Hom}_{D(A)}(G(N), L) \cong \text{Hom}_{D(B^{op})}(N, F(L)), \]
which is bifunctorial. Choose a K-projective resolution \( Q \rightarrow N \) in \( \text{DGMod} \, B^{op} \).

Since the DG \( A \)-module \( Q \otimes_B P \) is K-projective, we have a sequence of isomorphisms (of \( K \)-modules)

\[
\text{Hom}_{D(A)}(G(N), L) \cong H^0(\text{RHom}_A(G(N), L)) \\
\cong H^0(\text{Hom}_A(Q \otimes_B P, L)) \cong H^0(\text{Hom}_{B^{op}}(Q, \text{Hom}_A(P, L))) \\
\cong H^0(\text{RHom}_{B^{op}}(N, F(L))) \cong \text{Hom}_{D(B^{op})}(N, F(L)).
\]

The only choice made was in the K-projective resolution \( Q \rightarrow N \), so all is bifunctorial. The corresponding morphisms \( 1 \rightarrow F \circ G \) and \( G \circ F \rightarrow 1 \) are denoted by \( \eta \) and \( \zeta \) respectively.

Next we will prove that \( G \) is fully faithful. We do this by showing that for every \( N \) the morphism \( \eta_N : N \rightarrow (F \circ G)(N) \) in \( D(B^{op}) \) is an isomorphism. We know that \( G \) factors via the full subcategory \( E \subset D(A) \), and therefore, using Lemma 3.4 we know that the functor \( F \circ G \) commutes with infinite direct sums. So by Lemma 3.1 it suffices to check for \( N = B \). But in this case \( \eta_B \) is the canonical homomorphism of \( D(B^{op}) \)-modules \( B \rightarrow \text{Hom}_A(P, B \otimes_B P) \), which is clearly bijective.

It remains to prove that the essential image of the functor \( G \) is \( E \). Take any \( L \in E \), and consider the distinguished triangle \( (G \circ F)(L) \xrightarrow{\zeta_L} L \rightarrow L' \xrightarrow{\Delta} \) in \( E \), in which \( L' \in E \) is the mapping cone of \( \zeta_L \). Applying \( F \) and using \( \eta \) we get a distinguished triangle \( F(L) \xrightarrow{1_{F(L)}} F(L) \rightarrow F(L') \xrightarrow{\Delta} \). Therefore \( F(L') = 0 \). But \( \text{RHom}_A(M, L') \cong F(L') \), and therefore \( \text{Hom}_{D(A)}(M, L'[i]) = 0 \) for every \( i \). Since \( M \) is a generator of \( E \) we get \( L' = 0 \). Hence \( \zeta_L \) is an isomorphism, and so \( L \) is in the essential image of \( G \).

\[ \square \]

4. The Main Theorem

This is our interpretation of the completion appearing in Efimov’s recent paper [El], that is attributed to Kontsevich; cf. Remark 4.8 below for a comparison to [El] and to similar results in recent literature. Here is the setup for this section: \( A \) is a commutative ring, and \( a \) is a weakly proregular ideal in \( A \). We do not assume that \( A \) is noetherian nor \( a \)-adically complete. Let \( \hat{A} := \Lambda_a(A) \), the \( a \)-adic completion of \( A \), and let \( \hat{a} := a \cdot \hat{A} \), which is an ideal of \( \hat{A} \). Since the ideal \( a \) is finitely generated, it follows that the \( A \)-module \( \hat{A} \) is \( a \)-adically complete, and hence as a ring \( \hat{A} \) is \( \hat{a} \)-adically complete.

The full subcategory \( D(\text{Mod} A)_{a \text{-tor}} \subset D(\text{Mod} A) \) is triangulated and closed under infinite direct sums. The results of Sections 2 and 3 are invoked with \( K := A \).

Recall the Koszul complex \( K(A; a) \) associated to a finite sequence \( a \) in \( A \); see Section 3. It is a bounded complex of free \( A \)-modules, and hence it is a \( K \)-projective DG \( A \)-module. The next result was proved by several authors (see [BN, Proposition 6.1], [LN, Corollary 5.7.1(ii)] and [RO Proposition 6.6]).

**Proposition 4.1.** Let \( a \) be a finite sequence that generates \( a \). Then the Koszul complex \( K(A; a) \) is a compact generator of \( D(\text{Mod} A)_{a \text{-tor}} \).

Of course there are other compact generators of \( D(\text{Mod} A)_{a \text{-tor}} \).

**Theorem 4.2.** Let \( A \) be a commutative ring, let \( a \) be a weakly proregular ideal in \( A \), and let \( M \) be a compact generator of \( D(\text{Mod} A)_{a \text{-tor}} \). Choose some \( K \)-projective resolution \( P \rightarrow M \) in \( C(\text{Mod} A) \), and let \( B := \text{End}_A(P) \). Then \( \text{Ext}^i_B(P) = 0 \) for all \( i \neq 0 \), and there is a unique isomorphism of \( A \)-algebras \( \text{Ext}^0_B(P) \cong \hat{A} \).
Recall that the DG $A$-algebra $B$ is a derived endomorphism DG algebra of $M$ (Definition 2.2), and the graded $A$-algebra $\text{Ext}_B(P)$ is a derived double centralizer of $M$ (Definition 2.5).

We need a few lemmas before proving the theorem.

**Lemma 4.3.** Let $M$ be a compact object of $D(\text{Mod} A)_{a-tor}$. Then $M$ is also compact in $D(\text{Mod} A)$, so it is a perfect complex of $A$-modules.

**Proof.** Choose a finite sequence $\mathfrak{a}$ that generates $\mathfrak{a}$. By [PSY, Corollary 4.26] there is an isomorphism of functors $R\Gamma_{\mathfrak{a}} \cong K^\wedge_A (A; \mathfrak{a}) \otimes_A -$; where $K^\wedge_A (A; \mathfrak{a})$ is the infinite dual Koszul complex. Therefore the functor $R\Gamma_{\mathfrak{a}}$ commutes with infinite direct sums.

Let $N \in D(\text{Mod} A)$, and consider the function

$$\text{Hom}(1, \sigma^R_N): \text{Hom}_{D(\text{Mod} A)}(M, R\Gamma_{\mathfrak{a}}(N)) \to \text{Hom}_{D(\text{Mod} A)}(M, N).$$

Given a morphism $\alpha: M \to N$ in $D(\text{Mod} A)$ define

$$\beta := R\Gamma_{\mathfrak{a}}(\alpha) \circ (\sigma^R_M)^{-1}: M \to R\Gamma_{\mathfrak{a}}(N).$$

Since the functor $R\Gamma_{\mathfrak{a}}$ is idempotent (Theorem 1.7(1)), the function $\alpha \mapsto \beta$ is an inverse to $\text{Hom}(1, \sigma^R_N)$, so the latter is bijective.

Let $\{N_z\}_{z \in Z}$ be a collection of objects of $D(\text{Mod} A)$. Due to the fact that $M$ is a compact object of $D(\text{Mod} A)_{a-tor}$, and to the observations above, we get isomorphisms

$$\bigoplus_z \text{Hom}_{D(\text{Mod} A)}(M, N_z) \cong \bigoplus_z \text{Hom}_{D(\text{Mod} A)}(M, R\Gamma_{\mathfrak{a}}(N_z))$$

$$\cong \text{Hom}_{D(\text{Mod} A)}(M, \bigoplus_z R\Gamma_{\mathfrak{a}}(N_z)) \cong \text{Hom}_{D(\text{Mod} A)}(M, R\Gamma_{\mathfrak{a}}(\bigoplus_z N_z))$$

$$\cong \text{Hom}_{D(\text{Mod} A)}(M, \bigoplus_z N_z).$$

We see that $M$ is also compact in $D(\text{Mod} A)$. $\square$

Consider the contravariant functor

$$D: D(\text{Mod} B) \to D(\text{Mod} B^{op})$$

defined by choosing an injective resolution $A \to I$ over $A$, and letting $D := \text{Hom}_A(-, I)$.

**Lemma 4.4.** The functor $D$ induces a duality (i.e. a contravariant equivalence) between the full subcategory of $D(\text{Mod} B)$ consisting of objects perfect over $A$, and the full subcategory of $D(\text{Mod} B^{op})$ consisting of objects perfect over $A$.

**Proof.** Take $M \in D(\text{Mod} B)$ which is perfect over $A$. It is enough to show that the canonical homomorphism of DG $B$-modules

$$(4.5) \quad M \to (D \circ D)(M) = \text{Hom}_A(\text{Hom}_A(M, I), I)$$

is a quasi-isomorphism. For this we can forget the $B$-module structure, and just view this as a homomorphism of DG $A$-modules. Choose a resolution $P \to M$ where $P$ is a bounded complex of finitely generated projective $A$-modules. We can replace $M$ with $P$ in equation $(4.5)$, and after that we can replace $I$ with $A$: now it is clear that this is a quasi-isomorphism. $\square$

**Lemma 4.6.** Let $M$ and $N$ be $K$-flat complexes of $A$-modules. We write $\widetilde{M} := \Lambda_\mathfrak{a}(M)$ and $\widetilde{N} := \Lambda_\mathfrak{a}(N)$. 

(1) The morphisms $\xi_M : \Lambda_\alpha(M) \to \hat{M}$ and $\tau^\perp_M : \hat{M} \to \Lambda_\alpha(\hat{M})$ are isomorphisms.

(2) The homomorphism
\[ \text{Hom}(\tau_M, 1) : \text{Hom}_D(\text{Mod}_A)(\hat{M}, \hat{N}) \to \text{Hom}_D(\text{Mod}_A)(M, \hat{N}) \]
is bijective.

Proof. (1) The morphism $\xi_M$ is an isomorphism by [PSY, Proposition 3.6]. By Theorem 1.7(1) the complex $\Lambda_\alpha(M)$ is cohomologically complete; and therefore $\hat{M}$ is also cohomologically complete. But this means that $\tau^\perp_M$ is an isomorphism.

(2) Take a morphism $\alpha : M \to \hat{N}$ in $D(\text{Mod}_A)$. By part (1) we know that $\xi_M$ and $\tau^\perp_M$ are isomorphisms, so we can define
\[ \beta := (\tau^\perp_N)^{-1} \circ \Lambda_\alpha(\alpha) \circ \xi^{-1}_M : \hat{M} \to \hat{N}. \]
The function $\alpha \mapsto \beta$ is an inverse to $\text{Hom}(\tau_M, 1)$. \hfill \Box

Proof of Theorem 1.2. We shall calculate $\text{Ext}_B(P)$ indirectly.

By Lemma 4.3 we know that $M$, and hence also $P$, is perfect over $A$. So according to Lemma 4.4 there is an isomorphism of graded $A$-algebras
\[ \text{Ext}_B(P) \cong \text{Ext}_{B^{op}}(D(P))^{op}. \]
Next we note that
\[ D(P) = \text{Hom}_A(P, I) \cong \text{Hom}_A(P, A) = F(A) \]
in $\check{D}(\text{DGMod } B^{op})$. Here $F$ is the functor from (3.3). Therefore we get an isomorphism of graded $A$-algebras
\[ \text{Ext}_{B^{op}}(D(P)) \cong \text{Ext}_{B^{op}}(F(A)). \]

Let $N := R\Gamma_\alpha(A) \in D(\text{Mod}_A)$. We claim that $F(A) \cong F(N)$ in $\check{D}(\text{DGMod } B^{op})$. To see this, we first note that the canonical morphism $\sigma^B_A : N \to A$ in $D(\text{Mod}_A)$ can be represented by an actual DG module homomorphism $N \to A$ (say by replacing $N$ with a K-projective resolution of it). Consider the induced homomorphism $\text{Hom}_A(P, N) \to \text{Hom}_A(P, A)$ of DG $B^{op}$-modules. Like in the proof of Lemma 4.4 it suffices to show that this is a quasi-isomorphism of DG $A$-modules. This is true since, by $GM$ Duality [PSY, Theorem 7.12], the canonical morphism
\[ \text{RHom}(1, \sigma^B_A) : \text{RHom}_A(M, N) \to \text{RHom}_A(M, A) \]
in $D(\text{Mod}_A)$ is an isomorphism. We conclude that there is a graded $A$-algebra isomorphism
\[ \text{Ext}_{B^{op}}(F(A)) \cong \text{Ext}_{B^{op}}(F(N)). \]

Take $E := D(\text{Mod}_A)_{\alpha-tor}$ in Theorem 4.5. Since
\[ F|_E : E \to \check{D}(\text{DGMod } B^{op}) \]
is an equivalence, and $E$ is full in $D(\text{Mod}_A)$, we see that $F$ induces an isomorphism of graded $A$-algebras
\[ \text{Ext}_A(N) \cong \text{Ext}_{B^{op}}(F(N)). \]
The next step is to use the MGM equivalence. We know that $\Lambda_\alpha(N) \cong \hat{A}$ in $D(\text{Mod}_A)$. And the functor $\Lambda_\alpha$ induces an isomorphism of graded $A$-algebras
\[ \text{Ext}_A(N) \cong \text{Ext}_A(\hat{A}). \]
It remains to analyze the graded $A$-algebra $\operatorname{Ext}_A(\hat{A})$. By Lemma 4.6 the homomorphism

$$\operatorname{Hom}(\tau_A, 1) : \operatorname{Hom}_{\operatorname{D}(\operatorname{Mod} A)}(\hat{A}, \hat{A}[i]) \to \operatorname{RHom}_{\operatorname{D}(\operatorname{Mod} A)}(A, \hat{A}[i])$$

is bijective for every $i$. Therefore $\operatorname{Ext}_A^i(\hat{A}) = 0$ for $i \neq 0$, and the $A$-algebra homomorphism $\hat{A} \to \operatorname{Ext}_A^0(\hat{A})$ is bijective.

Combining all the steps above we see that $\operatorname{Ext}_B^i(P) = 0$ for $i \neq 0$, and there is an $A$-algebra isomorphism $\operatorname{Ext}_B^0(P) \cong \hat{A}^{\text{op}}$. But $\hat{A}$ is commutative, so $\hat{A}^{\text{op}} = \hat{A}$.

Regarding the uniqueness: since the image of the ring homomorphism $A \to \hat{A}$ is dense, and $\hat{A}$ is $\hat{a}$-adically complete, it follows that the only $A$-algebra automorphism of $\hat{A}$ is the identity. Therefore the $A$-algebra isomorphism $\operatorname{Ext}_B^0(P) \cong \hat{A}$ that we produced is unique. □

Remark 4.7. To explain how surprising this theorem is, take the case $P = M := K(A; a)$, the Koszul complex associated to a sequence $a = (a_1, \ldots, a_n)$ that generates the ideal $a$.

As a free $A$-module (forgetting the grading and the differential), we have $P \cong A^{n^2}$. The grading of $P$ depends on $n$ only (it is an exterior algebra). The differential of $P$ is the only place where the sequence $a$ enters. Similarly, the DG algebra $B = \operatorname{End}_A(P)$ is a graded matrix algebra over $A$, of size $n^2 \times n^2$. The differential of $B$ is where $a$ is expressed.

Forgetting the differentials, i.e. working with the graded $A$-module $P^\natural$, classical Morita theory tells us that $\operatorname{End}_B(P^\natural) \cong A$ as graded $A$-algebras. Furthermore, $P^\natural$ is a projective $B^\natural$-module, so we even have $\operatorname{Ext}_B(P^\natural) \cong A$.

However, the theorem tells us that for the DG-module structure of $P$ we have $\operatorname{Ext}_B(P) \cong \hat{A}$. Thus we get a transcendental outcome – the completion $\hat{A}$ – by a homological operation with finite input (basically finite linear algebra over $A$ together with a differential).

Remark 4.8. Our motivation to work on completion by derived double centralizer came from looking at the recent paper [Ef] by Efimov. The main result of [Ef] is Theorem 1.1 about the completion of the category $\operatorname{D}(\operatorname{QCoh} X)$ of a noetherian scheme $X$ along a closed subscheme $Y$. This idea is attributed to Kontsevich. Corollary 1.2 of [Ef] is a special case of our Theorem 4.2: it has the extra assumptions that the ring $A$ is noetherian and regular (i.e. it has finite global cohomological dimension).

After writing the first version of our paper, we learned that a similar result was proved by Dwyer-Greenlees-Iyengar [DGI]. In that paper the authors continue the work of [DG] on derived completion and torsion. Their main result is Theorem 4.10, which is a combination of MGM equivalence and derived Morita equivalence in an abstract setup (that includes algebra and topology). The manifestation of this main result in commutative algebra is [DGI, Proposition 4.20], that is also a special case of our Theorem 4.2: the ring $A$ is noetherian, and the quotient ring $A/a$ is regular.

Recall that our Theorem 4.2 only requires the ideal $a$ to be weakly proregular, and there is no regularity condition on the rings $A$ and $A/a$ (the word “regular” has a double meaning here!). It is quite possible that the methods of [DGI] or [Ef] can be pushed further to remove the regularity conditions from the rings $A$ and
However, it is less likely that these methods can handle the non-noetherian case (i.e. assuming only that the ideal $\mathfrak{a}$ is weakly proregular).

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