DIRAC OPERATORS IN TENSOR CATEGORIES AND THE MOTIVE OF ODD WEIGHT MODULAR FORMS

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Abstract. We define a motive whose realizations afford modular forms (of arbitrary weight) on an indefinite division quaternion algebra. This generalizes work of Iovita–Spiess to odd weights in the spirit of Jordan–Livné. It also generalizes a construction of Scholl to indefinite division quaternion algebras.

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1. Introduction

The paper [Sc] offers the construction of a motive whose realizations affords modular forms of even or odd weight on the indefinite split quaternion algebra over \( \mathbb{Q} \). In [IS, §10] the authors construct a motive of even weight modular forms on a quaternion division algebra (see also [Wo]). Based on ideas of Jordan and Livné (see [JL]), this motive is constructed as the kernel of an appropriate Laplace operator. More precisely, let \( h(A) \) be the motive of an abelian scheme \( A \) of dimension \( d \) (see [DM] and [Ku]). It decomposes as the direct sum

\[
h(A) = h^0(A) \oplus h^1(A) \oplus \ldots \oplus h^g(A)
\]

where \( g = 2d \) and there are canonical identifications

\[
h^i(A) = \vee^i h^1(A), \quad h^i(A) \simeq h^{2d-i}(A)^\vee (-d) \quad \text{and} \quad h^{2d}(A) \simeq \mathbb{I}(-d),
\]

where \( \vee V \) denotes the symmetric algebra of the object \( V \). It follows that the multiplication morphisms

\[
\varphi_{i,2d-i} : \vee^i h^1(A) \otimes \vee^{2d-i} h^1(A) \to \mathbb{I}(-d)
\]

are perfect. In particular, taking \( i = d \), one gets an associated Laplace operator

\[
\Delta^n : \text{Sym}^n(\vee^2 h^1(A)) \to \text{Sym}^{n-2}(\vee^2 h^1(A)(-d)), \quad n \geq 2
\]

and it is possible to show that the kernel exists. The following remark has been employed in [IS, §10]. When \( A \) is an abelian scheme of dimension 4 with multiplication by the quaternion algebra \( B \), we have that \( B \otimes B \) acts on \( \vee^2 h^1(A) \) and there is a canonical direct sum decomposition

\[
\vee^2 h^1(A) = \left( \vee^2 h^1(A) \right)_+ \oplus \left( \vee^2 h^1(A) \right)_-
\]

\[\text{For a symmetric or alternating power } M \text{ we will write } \text{Sym}^n(M) \text{ and } \text{Alt}^n(M) \text{ when considering its symmetric or alternating powers once again.}\]
is such a way that \( B^\times \subset B \otimes B \) (diagonally) acts via the reduced norm on \( (\sqrt{2}h^1 (A))_\_ \). Furthermore, since the idempotents giving rise to this decomposition are self-adjoint with respect to \( \varphi_{2,2} \), it follows that the induced pairing

\[
(\sqrt{2}h^1 (A))_\_ \otimes (\sqrt{2}h^1 (A))_\_ \rightarrow \sqrt{2}h^1 (A) \otimes \sqrt{2}h^1 (A) \rightarrow 1 (-2)
\]

is still non-degenerate and the kernel of the induced Laplace operator

\[
\Delta^n : \text{Sym}^n \left( (\sqrt{2}h^1 (A))_\_ \right) \rightarrow \text{Sym}^{n-2} \left( (\sqrt{2}h^1 (A))_\_ \right) (-2), \ n \geq 2
\]

exists. When \( A \) is taken to be the universal abelian surface, setting

\[
M_{2n} := \text{ker} (\Delta^n)
\]

gives a motive whose realizations gives incarnations of weight \( k = 2n \) modular forms.

The aim of this paper is to define a motive whose realizations afford modular forms (of arbitrary weight) on an indefinite division quaternion algebra. The idea of the construction, once again, is due to Jordan and Livnè. However a couple of remarks are in order. First, following their construction in this indefinite setting and working at the level of realizations gives the various incarnations of two copies of odd weight modular forms, rather than just one copy. It is not possible to canonically split them in a single copy: this is possible only including a splitting field for the quaternion algebra in the coefficients, but the resulting splitting depends on the choice of an identification of the base changed algebra with the split quaternion algebra. Indeed, we will construct a motive whose realizations afford two copies of odd weight modular forms.

Second, the idea of Jordan and Livnè is to construct square roots of the Laplace operators after appropriately splitting depends on the choice of an identification of the base changed algebra with the split quaternion algebra. The idea of the construction, once again, is due to Jordan and Livnè. However a couple of remarks are in order. First, following their construction in this indefinite setting and working at the level of realizations gives the various incarnations of two copies of odd weight modular forms, rather than just one copy. It is not possible to canonically split them in a single copy: this is possible only including a splitting field for the quaternion algebra in the coefficients, but the resulting splitting depends on the choice of an identification of the base changed algebra with the split quaternion algebra. Indeed, we will construct a motive whose realizations afford two copies of odd weight modular forms.

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It is in this form that we will be able to define $\partial^n$ and another $\overline{\partial}^{n-1}$ in such a way that the construction makes sense for rigid $\mathbb{Q}$-linear and pseudo-abelian $ACU$ categories and prove the generalization of the equality $\overline{\partial}^{n-1} \circ \partial^n = \Delta^n \otimes 1_B$ in this setting. Then one shows that $\mathcal{L}(\partial^n)$ computes two copies of weight $k = 2n + 1$ modular forms.

The abstract framework we work with in this paper is the following. Suppose that $\mathcal{C}$ is a rigid pseudo-abelian and $\mathbb{Q}$-linear $ACU$ tensor category with identity object $1$; if $X \in \mathcal{C}$ we write $r_X := \text{rank}(X)$. We recall from [MS] that $V$ has alternating (resp. symmetric) rank $g \in \mathbb{N}_{\geq 1}$ if $L := \wedge^n V$ (resp. $L := \vee^n V$) is invertible and if $(^{k+1}_g)$ (resp. $(^{k+1}_g)$) is invertible in $\text{End}(1)$ for every $0 \leq i \leq g$. Here, for an integer $k \geq 1$,

$$
\binom{\mathcal{T}}{k} := \frac{1}{k!} \mathcal{T}(T - 1)...(T - k + 1) \in \mathbb{Q}[T] \text{ and } \binom{\mathcal{T}}{0} = 1.
$$

Suppose first that $V$ has alternating rank $g$. We will prove that, when $g = 2i$ and $i$ is even (resp. odd), $L \simeq \mathbb{L}^{\otimes 2}$ for some invertible object and $r_{\wedge_1 V} > 0$ (resp. $r_{\wedge_1 V} < 0$) (see definition 5.3), then there is an operator

$$
\partial^n_{g-1} : \text{Sym}^n(\wedge^i V) \otimes \wedge^{i-1} V \rightarrow \text{Sym}^{n-2}(\wedge^i V) \otimes \wedge^i V \otimes L, n \geq 1
$$

(resp. $\partial^n_{g-1} : \text{Alt}^n(\wedge^i V) \otimes \wedge^{i-1} V \rightarrow \text{Alt}^{n-2}(\wedge^i V) \otimes \wedge^i V \otimes L, n \geq 1$)

such that $\ker(\partial^n_{g-1})$ exists (see Theorems 4.3 and 4.3).

Suppose now that $V$ has symmetric rank $g$. Then we prove that, when $g > 2i$, $L \simeq \mathbb{L}^{\otimes 2}$ for some invertible object and $r_{\vee_1 V} > 0$, then there is an operator

$$
\partial^n_{g-1} : \text{Sym}^n(\vee^i V) \otimes \vee^{i-1} V \rightarrow \text{Sym}^{n-2}(\vee^i V) \otimes \vee^i V \otimes L, n \geq 1
$$

such that $\ker(\partial^n_{g-1})$ exists (see Theorem 5.3).

These operators are indeed square roots of the Laplace operators induced by the multiplication pairings in the involved alternating or symmetric algebras and the existence of these kernels follows from this fact and the existence of the kernels of the Laplace operators.

Some remarks are in order about the range of applicability of our results. First of all we note that, in general, the alternating or the symmetric rank may be not uniquely determined. Suppose, however, that we know that there is a field $K$ such that $r \in K \subset \text{End}(1)$ admitting an embedding $\iota : K \hookrightarrow \mathbb{R}$. Then it follows from the formulas $\text{rank}(\wedge^k V) = \binom{k}{g}(g)$ and $\text{rank}(\vee^k V) = \binom{k}{g+1}(g)$ (see [AKh] 7.2.4 Proposition) or [De3] (7.1.2)] that we have $r \in \{-1, g\}$ (resp. $r \in \{-g, 1\}$) when $V$ has alternating (resp. symmetric) rank $g$. In particular, when $r > 0$ (resp. $r < 0$) with respect to the ordering induced by $\iota$, we deduce that $g = r$ (resp. $g = -r$), so that $g$ is a uniquely determined and $V$ has alternating (resp. symmetric) rank $g = r$ (resp. $g = -r$).

We recall that $V$ is Kimura positive (resp. negative) when $\text{rank}(\wedge^N V) = 0$ (resp. $\text{rank}(\vee^N V) = 0$) for $N \geq 0$ large enough. In this case, the formula $\text{rank}(\wedge^k V) = \binom{k}{g}$ (resp. $\text{rank}(\vee^k V) = \binom{k}{g+1}$) implies that $r \in \mathbb{Z}_{\geq 0}$ (resp. $r \in \mathbb{Z}_{<0}$) and the smallest integer $N$ such that $\wedge^N V = 0$ (resp. $\vee^N V = 0$) is $r$ (resp. $-r$).

Furthermore, it is known that $\wedge^V$ (resp. $\vee^V$) is invertible in this case (see [Kh] 11.2 Lemma]: in other words $V$ has alternating (resp. symmetric) rank $g = r$ (resp. $g = -r$). Suppose in particular that $V$ is Kimura positive (resp. negative); then $r_{\wedge_1 V} > 0$ (resp. $r_{\vee_1 V} > 0$) for $i$ even and Theorem 4.3 (resp. Theorem 4.4) (resp. Theorem 5.3) applies. On the other hand, when $i$ is odd, the condition $r_{\wedge_1 V} < 0$ (resp. $r_{\vee_1 V} < 0$) required by Theorem 4.4 (resp. Theorem 5.3) is not satisfied and we cannot apply our results.

It is known that the motive $h^1(A)$ of an abelian scheme of dimension $d$ is Kimura negative of Kimura rank $-d$ (see [Kh]). Suppose that $d = 2i \equiv 0 \mod 4$, so that $i$ is even and $r_{\vee_1 V} > 0$. Since $d$ is even, $\vee^{2d} h^1(A) \simeq h^{2d}(A) \simeq \mathbb{L}(-d)$ is the square of an invertible object. Theorem 5.3 applied to $V = h^1(A)$ implies the existence of canonical pieces

$$
\ker\left(\partial^n_{d/2-1}\right) \subset \text{Sym}^n\left(\vee^{d/2} h^1(A)\right) \otimes \vee^{d/2-1} h^1(A) \simeq \text{Sym}^n\left(h^{d/2}(A)\right) \otimes h^{d/2-11}(A)
$$

for every $n \geq 1$.

The paper is organized as follows. In §2 we recall the needed results from [MS]. In §3 we discuss generalities on Laplace and Dirac operators in rigid and pseudo-abelian tensor categories, giving condition for the existence of kernel of Laplace operators and for the Dirac operators to be square roots of Laplace
operators. We remark that the existence of kernels of Laplace operators is stated in [IS] \[\S\]10 for the category of Chow motives; the authors are indebted with M. Spiess for providing them some notes on the topics. In \[\S\]4 and \[\S\]5 we use the Poincaré morphisms from \[\S\]2 to define our Dirac operators on the alternating and symmetric powers and prove that they are indeed square roots of the Laplace operators; together with the result from \[\S\]3 we deduce Theorems (4.3) and (5.3). In \[\S\]6 we discuss how the constructions behaves with respect to additive AU tensor functors which may not respect the associativity constraint, as needed for the realization functor \(R\) (see [K]). We also apply the results to the specific case of a quaternionic object, as needed for the construction of the motives of modular forms. The subsequent section is devoted to the computation of the realization of the motives of modular forms: the reader is strongly suggested to first give a look to this section as a motivation for the abstract constructions. We work with variations of Hodge structures as a target category, following ideas of [IS], but the same computations could be worked out for other realizations following the same pattern.

Throughout this paper we will always work in a \(\mathbb{Q}\)-linear rigid and pseudo-abelian \(ACU\) category \(\mathcal{C}\) with unit object \(\mathbb{I}\) and internal homs. We let \(ev_X : X^\vee \otimes X \to \mathbb{I}\) be the evaluation and \(ev_X^\vee := ev_X \circ \tau_{X,X} : X \otimes X^\vee \to \mathbb{I}\) be the opposite evaluation.

2. Poincaré Duality Isomorphism

Given an object \(V \in \mathcal{C}\), we may consider the associated alternating and symmetric algebras, denoted by \(\wedge V\) and, respectively, \(\vee V\). If \(A\) denotes one of these algebras, we have multiplication morphisms

\[\varphi_{i,j} : A_i \otimes A_j \to A_{i+j},\]

a data which is equivalent to

\[f_{i,j} : A_i \to \text{hom}(A_j, A_{i+j}).\]

When \(g \geq i\), we may consider the composite

\[D^{i,g} : A_i \xrightarrow{f_{i,g}^{-1}} \text{hom}(A_{g-i}, A_g) \xrightarrow{d} \text{hom}(A^\vee_g, A^\vee_{g-i}) \xrightarrow{\alpha^{-1}} A^\vee_{g-i} \otimes A^\vee_g,
\]

where \(d : \text{hom}(X,Y) \to \text{hom}(Y^\vee, X^\vee)\) is the internal duality morphism and \(\alpha : \text{hom}(X,Y) \to Y \otimes X^\vee\) is the canonical morphism (see [MS] \[\S\]2). Working with the alternating or symmetric algebra of the dual \(V^\vee\) yields a morphism

\[D^{i,g} : A_i^\vee \xrightarrow{f_{i,g}^{-1}} \text{hom}(A_{g-i}, A_g^\vee) \xrightarrow{d} \text{hom}(A^\vee_g, A^\vee_{g-i}) \xrightarrow{\alpha^{-1}} A^\vee_{g-i} \otimes A^\vee_g.
\]

Employing the reflexivity morphism \(i : X \to X^{\wedge\vee}\) we can define (see [MS] \(\text{(20)}\)):

\[D_{i,g} : A_i^\vee \xrightarrow{D^{i,g}} A_i \otimes A_i^\vee \xrightarrow{\tau_{i,i}^{-1}} A_{g-i} \otimes A_g^\vee.
\]

The following results have been proved in [MS] \(\S\)5 and \(\S\)6. In order to state them, we first need to define the following morphisms:

\[\varphi^{i,j}_{13} : A_i \otimes B \otimes A_j \otimes C \xrightarrow{1 \otimes \tau_{B,A_j} \otimes 1} A_i \otimes A_j \otimes B \otimes C \xrightarrow{\varphi_{i,j} \otimes 1} A_{i+j} \otimes B \otimes C,
\]

\[\varphi^{i,j}_{13} : A_i^\vee \otimes B \otimes A_j^\vee \otimes C \xrightarrow{1 \otimes \tau_{B,A_j} \otimes 1} A_i^\vee \otimes A_j^\vee \otimes B \otimes C \xrightarrow{\varphi_{i,j} \otimes 1} A_{i+j} \otimes B \otimes C^\vee
\]

and then

\[\varphi^{13}_{g-i,j} : A_{g-i} \otimes A_g^\vee \otimes A_i \otimes A_g \xrightarrow{\varphi_{g-i,j} \otimes 1} A_g \otimes A_g \otimes A_g^\vee \xrightarrow{ev_{g,i} \otimes 1} A_g^\vee,
\]

\[\varphi^{13}_{g-i,j} : A_{g-i} \otimes A_g^\vee \otimes A_i \otimes A_g \xrightarrow{\varphi_{g-i,j} \otimes 1} A_g \otimes A_g \otimes A_g^\vee \xrightarrow{ev_{g,i} \otimes 1} A_g^\vee.
\]

In the following discussion we let \(r := \text{rank}(V) \in \text{End}(1)\).

**Theorem 2.1.** The following diagrams are commutative, for every \(g \geq i \geq 0\).
The following diagrams are commutative, for every $g \in \mathbb{N}_{\geq 1}$, if $\Lambda^g V$ is an invertible object and $(\tau^{-i})$ and $(\tau^{i+g})$ are invertible for every $0 \leq i \leq g$. For example, when $\text{End}(\mathbb{I})$ is a field or $r \in \mathbb{Q}$, the second condition means that $r$ is not a root of the polynomials $(T^{-i}) \in \mathbb{Q}[T]$ and $(T^{i+g}) \in \mathbb{Q}[T]$ for every $0 \leq i \leq g$, i.e. that $r \neq i, i + 1, \ldots, g - i - 1$ and $r \neq g - i, g - i + 1, \ldots, g - 1$ for every $1 \leq i \leq g$.

We say that $V$ has strong alternating rank $g \in \mathbb{N}_{\geq 1}$, if $\Lambda^g V$ is an invertible object and $r = g$ (hence $V$ has alternating rank $g$).

**Corollary 2.2.** If $V$ has alternating rank $g \in \mathbb{N}_{\geq 1}$, then for every $0 \leq i \leq g$, the morphisms $D^i g, D_{g-i} g, D^{g-i} g$ and $D_i g$ are isomorphisms and the multiplication maps $\varphi_{1, g-i}, \varphi_{g-i, i}, \varphi_{g-v, i}$ and $\varphi_{g-i, i}$ are perfect pairings (meaning that the associated hom valued morphisms are isomorphisms). Furthermore, when $V$ has strong alternating rank $g$, we have $(\tau^{-i}) = (\tau^{i+g}) = 1$ in the commutative diagrams of Theorem 2.7.

**Proposition 2.3.** The following diagrams are commutative, when $\Lambda^g V$ is invertible of rank $r_{\Lambda^g V}$ (hence $r_{\Lambda^g V} \in \{-1, 1\}$):

$$
\begin{align*}
\Lambda^i V \otimes \Lambda^g V \otimes V &\xrightarrow{\ (1_{\Lambda^i V} \otimes \varphi_{i-1, 1}(1_{\Lambda^g V} \otimes \varphi_{g-i, 1}) (r_{\Lambda^i V, \Lambda^g V} \otimes 1_{V}) \) } V \otimes \Lambda^i V \otimes \Lambda^g V \\
\Lambda^i V \otimes \Lambda^g i+1 V \otimes \Lambda^g V &\xrightarrow{\ (1_{\Lambda^i V} \otimes \varphi_{i-1, 1}(1_{\Lambda^g V} \otimes \varphi_{g-i, 1}) (r_{\Lambda^i V, \Lambda^g V} \otimes 1_{V}) \) } V \otimes \Lambda^i V \otimes \Lambda^g V
\end{align*}
$$

and

$$
\begin{align*}
\Lambda^i V \otimes \Lambda^g i+1 V \otimes V &\xrightarrow{\ (1_{\Lambda^i V} \otimes \varphi_{i-1, 1}(1_{\Lambda^g V} \otimes \varphi_{g-i, 1}) (r_{\Lambda^i V, \Lambda^g V} \otimes 1_{V}) \) } V \otimes \Lambda^i V \otimes \Lambda^g V \\
\Lambda^i V \otimes \Lambda^g i+1 V \otimes \Lambda^g V &\xrightarrow{\ (1_{\Lambda^i V} \otimes \varphi_{i-1, 1}(1_{\Lambda^g V} \otimes \varphi_{g-i, 1}) (r_{\Lambda^i V, \Lambda^g V} \otimes 1_{V}) \) } V \otimes \Lambda^i V \otimes \Lambda^g V
\end{align*}
$$

Here are the analogue of the above results for the symmetric algebras.

**Theorem 2.4.** The following diagrams are commutative, for every $g \geq i \geq 0$. 

pairings (meaning that the associate hom $D^{g-i}$ $g$ $V$ and $V$ $V$ are isomorphisms and the multiplication maps $ϕ_{i,g}$ and $ϕ_{g,i}$ are perfect pairings (meaning that the associate hom valued morphisms are isomorphisms). Furthermore, when $V$ has strong symmetric rank $g$, we have $\binom{r+g-1}{g-i} = (-1)^{g-i}$ and $\binom{r+g-1}{g-i} = (-1)^{i}$ in the commutative diagrams of Theorem 2.4.

**Proposition 2.6.** The following diagrams are commutative when $∀V$ is invertible of rank $r_{∀V}$ (hence $r_{∀V} ∈ \{±1\}$):

\[
\begin{array}{ccc}
V^iV ⊗ V^{−1}V ⊗ V & \xrightarrow{r_{∀V}V ⊗ V^{−1}V} & (\iota_{i[V,−1]})(ϕ_{g-i-1,i}) (\iota_{g-i-1,i}) V^iV ⊗ V^{−1}V \\
\downarrow & & \downarrow \iota_{i[V,−1]} \circ D^{g-i} \circ \iota_{g-i-1,i} \\
V^iV ⊗ V^{−1}V ⊗ V^{−1}V & \xrightarrow{D^{g-i} \circ r_{∀V}V} & V^iV ⊗ V^{−1}V
\end{array}
\]

and

\[
\begin{array}{ccc}
V^iV ⊗ V^{−1}V ⊗ V^{−1}V & \xrightarrow{r_{∀V}V ⊗ V^{−1}V} & V^iV ⊗ V^{−1}V \\
\downarrow & & \downarrow \iota_{i[V,−1]} \circ D^{g-i} \circ \iota_{g-i-1,i} \\
V^iV ⊗ V^{−1}V ⊗ V^{−1}V & \xrightarrow{D^{g-i} \circ r_{∀V}V} & V^iV ⊗ V^{−1}V
\end{array}
\]

We say that $V$ has symmetric rank $g ∈ \mathbb{N}_≥1$, if $∀V$ is an invertible object and $\binom{r+g-1}{g-i}$ and $\binom{r+g-1}{g-i}$ are invertible for every $0 ≤ i ≤ g$. For example, when $r_{∀V}$ is a field, $r_{∀V}$ is a field, and $r_{∀V}$ is a field, the second condition means that $r_{∀V}$ is a field, $r_{∀V}$ is a field, and $r_{∀V}$ is a field, i.e. that $r_{∀V} ≠ 1 - g, 2 - g, ..., i - g$ for every $1 ≤ i ≤ g$.

We say that $V$ has strong symmetric rank $g ∈ \mathbb{N}_≥1$, if $∀V$ is an invertible object and $r_{∀V} = -g$ (hence $V$ has symmetric rank $g$).

**Corollary 2.5.** If $V$ has symmetric rank $g ∈ \mathbb{N}_≥1$ then, for every $0 ≤ i ≤ g$, the morphisms $D^{i,g}$, $D^{g-i,g}$, $D^{g-i,g}$ and $D_{i,g}$ are isomorphisms and the multiplication maps $ϕ_{i,g-1}, ϕ_{g-i,i}, ϕ_{i,g-1}$ and $ϕ_{g-i,i}$ are perfect pairings (meaning that the associate hom valued morphisms are isomorphisms). Furthermore, when $V$ has strong symmetric rank $g$, we have $\binom{r+g-1}{g-i} = (-1)^{g-i}$ and $\binom{r+g-1}{g-i} = (-1)^{i}$ in the commutative diagrams of Theorem 2.4.
3. Dirac and Laplace Operators

If we have given \( \psi : X \otimes Y \to Z \), we may consider

\[
\partial_n^\psi := 1_{\otimes n-1}X \otimes \psi : \otimes^n X \otimes Y \to \otimes^{n-1} X \otimes Z
\]

for \( n \geq 1 \) and then define

\[
\partial_{p,a}^n : \wedge^n X \otimes Y \xrightarrow{i_X^{'n,\otimes 1Y}} \otimes^n X \otimes Y \xrightarrow{\partial_n^\psi} \otimes^{n-1} X \otimes Z \quad \text{and} \quad \partial_{p,a}^n : \vee^n X \otimes Y \xrightarrow{j_X^{'n,\otimes 1Y}} \otimes^n X \otimes Y \xrightarrow{\partial_n^\psi} \vee^{n-1} X \otimes Z.
\]

Here we write \( i_X^{'n,\otimes 1Y} \) and \( p_{X,a}^{n-1} \) for the canonical injective and, respectively, surjective morphisms arizing from the idempotent defining the alternating when \( * = a \) and the symmetric when \( * = s \) powers.

In particular, when \( X = Y \), we have

\[
\Delta_n^\psi = \partial_n^\psi = 1_{\mathbb{C}^{-2}} X \otimes \psi : \otimes^n X \to \otimes^{n-2} X \otimes Z \quad \text{for} \ n \geq 2
\]

inducing

\[
\Delta_{\psi,a}^n = \wedge^n X \xrightarrow{i_X^{'n,\otimes 1Y}} \otimes^n X \xrightarrow{\Delta_{\psi,a}^n} \otimes^{n-2} X \otimes Z \quad \text{and} \quad \Delta_{\psi,s}^n = \vee^n X \xrightarrow{j_X^{'n,\otimes 1Y}} \otimes^n X \xrightarrow{\Delta_{\psi,s}^n} \vee^{n-2} X \otimes Z.
\]

We may lift these morphisms to the tensor products as follows. Let \( \varepsilon \) (resp. \( 1 \)) be the sign character (resp. trivial) character of the symmetric group and, if \( X \in \{ \varepsilon, 1 \} \) and \( R \subset S_k \) is any subset, define

\[
e^R_\chi := \frac{1}{\#R} \sum_{\delta \in R} \chi^{-1} (\delta) \in \mathbb{Q}[S_k].
\]

In particular, taking \( R = S_k \) gives the idempotents \( e^R_{X,a} := e^R_{\varepsilon,1} \) and \( e^R_{X,s} := e^R_{1,1} \) defining the alternating and symmetric \( k \)-powers of any object \( X \). We have that \( \partial_n^\psi \) (resp. \( \Delta_n^\psi \)) is equivariant for the action of \( S_{n-1} = S_{\{1, \ldots, n-1\}} \subset S_n \) (resp. \( S_{n-2} = S_{\{1, \ldots, n-2\}} \subset S_n \)). Furthermore, if we choose, for every \( p \in \{1, \ldots, n\} =: I_n \) (resp. \( (p,q) \in I_n \times I_n \) with \( p \neq q \)), elements \( \delta_p^n \in S_n \) (resp. \( \delta_{p,q}^n \in S_n \)) such that \( \delta_p^n(p) = n \) (resp. \( \delta_{p,q}^{n-1,n}(p,q) = (n-1,n) \)), then \( R_{S_{n-1}} \setminus s_n := \{ \delta_p^n : p \in I_n \} \) (resp. \( R_{S_{n-2}} \setminus s_n := \{ \delta_{p,q}^{n-1,n} : (p,q) \in I_n \times I_n, p \neq q \} \)) is a set of coset representatives for \( S_{n-1} \setminus s_n \) (resp. \( S_{n-2} \setminus s_n \)). Using these facts it is not difficult to check that, setting

\[
\tilde{\partial}_{n,\psi,a}^n := \partial_n^\psi \circ e^{R_{S_{n-1}} \setminus s_n} = \frac{1}{n} \sum_{p=1}^n \chi^{-1} (\delta_p^n \cdot (1_{\otimes n-1}X \otimes \psi) \circ (\delta_p^n \otimes 1_Y)),
\]

\[
\tilde{\Delta}_{n,\psi,a}^n := \Delta_n^\psi \circ e^{R_{S_{n-2}} \setminus s_n} = \frac{1}{n (n-1)} \sum_{p,q \in I_n : p \neq q} \chi^{-1} (\delta_{p,q}^{n-1,n} \cdot (1_{\otimes n-2}X \otimes \psi) \circ (\delta_{p,q}^{n-1,n})),
\]

where \( * = a \) if \( \chi = \varepsilon \) and \( * = s \) if \( \chi = 1 \), gives morphisms making the following diagrams commutative:

\[
\begin{array}{ccc}
\otimes^n X \otimes Y & \xrightarrow{\tilde{\partial}_{n,\psi,a}^n} & \otimes^{n-1} X \otimes Z \\
\downarrow p_{X,a}^{n,\otimes 1Y} & & \downarrow p_{X,a}^{n-1,\otimes 1Z} \\
\wedge^n X \otimes Y & \xrightarrow{\tilde{\partial}_{n,\psi,a}^n} & \wedge^{n-1} X \otimes Z ,
\end{array}
\]

\[
\begin{array}{ccc}
\otimes^n X \otimes Y & \xrightarrow{\tilde{\Delta}_{n,\psi,a}^n} & \otimes^{n-2} X \otimes Z \\
\downarrow p_{X,a}^{n,\otimes 1Y} & & \downarrow p_{X,a}^{n-1,\otimes 1Z} \\
\wedge^n X \otimes Y & \xrightarrow{\tilde{\Delta}_{n,\psi,a}^n} & \wedge^{n-2} X \otimes Z,
\end{array}
\]

\[
\begin{array}{ccc}
\otimes^n X & \xrightarrow{\tilde{\partial}_{n,\psi,a}^n} & \otimes^{n-1} X \otimes Z \\
\downarrow p_{X,a}^{n,\otimes 1Y} & & \downarrow p_{X,a}^{n-1,\otimes 1Z} \\
\wedge^n X & \xrightarrow{\tilde{\partial}_{n,\psi,a}^n} & \wedge^{n-2} X \otimes Z,
\end{array}
\]

\[
\begin{array}{ccc}
\otimes^n X & \xrightarrow{\tilde{\Delta}_{n,\psi,a}^n} & \otimes^{n-2} X \otimes Z \\
\downarrow p_{X,a}^{n,\otimes 1Y} & & \downarrow p_{X,a}^{n-1,\otimes 1Z} \\
\wedge^n X & \xrightarrow{\tilde{\Delta}_{n,\psi,a}^n} & \wedge^{n-2} X \otimes Z.
\end{array}
\]

When \( \psi \) is alternating or symmetric, we can refine (12) as follows.
Lemma 3.1. Suppose that \( \psi : X \otimes X \to Z \) is such that \( \psi \circ \tau_{X,X} = -\psi \) (resp. \( \psi \circ \tau_{X,X} = \psi \)). Then \( \Delta^n_{\psi, s} = 0 \) (resp. \( \Delta^n_{\psi, a} = 0 \)) and \( \Delta^n_{\psi, a} \) (resp. \( \Delta^n_{\psi, s} \)) is induced by

\[
\tilde{\Delta}^n_{\psi, \ast} := \frac{2}{n(n-1)} \sum_{p,q \in I_n : p < q} \chi^{-1} (\delta_{p,q}^{n-1,n}) \cdot (1_{\otimes^n X} \otimes \psi) \circ \delta_{p,q}^{n-1,n}
\]

where \( \chi = \varepsilon \) (resp. \( \chi = 1 \)), \( \ast = a \) (resp. \( \ast = s \)) and \( \delta_{p,q}^{n-1,n} (p, q) = (n - 1, n) \).

Proof. The proof, based on (6) and the subsequent Remark 3.2, is left to the reader. 

Remark 3.2. Suppose that we are given actions of \( S_n \) on \( A \) and of \( S_{n-2} \) on \( B \) and that \( f : A \to B \) is an \( S_{n-2} \)-equivariant map, for an integer \( n \geq 2 \). Then we have, setting \( \tau_{n-1,n} := (n-1,n) \),

\[
e_{\chi, n-2} \circ f \circ e_{\chi, n-2, \chi, n} := e_{\chi, n-2} \circ \frac{1}{n(n-1)} \sum_{p,q \in I_n : p \neq q} \chi^{-1} (\delta_{p,q}^{n-1,n}) : f \circ \delta_{p,q}^{n-1,n}
\]

Suppose now that we have given \( \psi_1 : X \otimes Y \to Z \) and \( \psi_2 : X \otimes Z \to Y \otimes W \) and \( \psi : X \otimes X \to W \). They induce

\[
\Lambda^n X \otimes Y \xrightarrow{\rho^a_{\psi_1, a}} \Lambda^{n-1} X \otimes Z \xrightarrow{\rho^a_{\psi_2, a}} \Lambda^n X \otimes Y \otimes W,
\]

\[
\vee^n X \otimes Y \xrightarrow{\rho^s_{\psi_1, s}} \vee^{n-1} X \otimes Z \xrightarrow{\rho^s_{\psi_2, s}} \vee^n X \otimes Y \otimes W
\]

and

\[
\Delta^n_{\psi, a} : \Lambda^n X \to \Lambda^{n-2} X \otimes W,
\]

\[
\Delta^n_{\psi, s} : \vee^n X \to \vee^{n-2} X \otimes W.
\]

Lemma 3.3. Suppose that \( \psi : X \otimes X \to W \) is such that \( \psi \circ \tau_{X,X} = \nu_\ast \cdot \psi \), where \( \nu_a := -1 \), \( \nu_s := 1 \) and \( \ast \in \{ a, s \} \), and that, for some \( \rho \in \text{End}(I) \), the following diagram is commutative:

\[
X \otimes X \otimes Y \xrightarrow{(1_X \otimes \psi_1, (1_X \otimes \psi_1) \circ (\tau_{X,X} \otimes 1_Y))} X \otimes Z \otimes X \otimes Z \xrightarrow{\psi_2 \otimes \nu_\ast \cdot \psi_2} Y \otimes W.
\]

Then, when \( \ast = a \), the following diagram is commutative

\[
\Lambda^n X \otimes Y \xrightarrow{\rho^a_{\psi_1, a}} \Lambda^{n-1} X \otimes Z \xrightarrow{\rho^a_{\psi_2, a}} \Lambda^n X \otimes Y \otimes W
\]

and, when \( \ast = s \), the following diagram is commutative:

\[
\vee^n X \otimes Y \xrightarrow{\rho^s_{\psi_1, s}} \vee^{n-1} X \otimes Z \xrightarrow{\rho^s_{\psi_2, s}} \vee^n X \otimes Y \otimes W.
\]
Proof. We compute, using the notations in (3),
\[
\tilde{\delta}_{\psi_{2},*x}^{n-1} \circ \tilde{\delta}_{\psi_{1},*x}^{n} = \frac{1}{n(n-1)} \sum_{\substack{q=1,\ldots,n \backslash p \in \{1,\ldots,n-1\}}} \chi^{-1} \left( \delta_{p}^{n-1} \delta_{q}^{n} \right) \cdot (\psi_{2} \circ (1_{X} \otimes \psi_{2})) \circ (\delta_{p}^{n-1} \otimes 1_{Y}) \\
\circ (1_{X} \otimes \psi_{1}) \circ (\delta_{q}^{n} \otimes 1_{Y}) \\
= \frac{1}{n(n-1)} \sum_{\substack{q=1,\ldots,n \backslash p \in \{1,\ldots,n-1\}}} \chi^{-1} \left( \delta_{p}^{n-1} \delta_{q}^{n} \right) \cdot (\psi_{2} \circ (1_{X} \otimes \psi_{2})) \circ (\delta_{p}^{n-1} \otimes 1_{Y}) \\
\circ (\delta_{q}^{n} \otimes 1_{X} \otimes \psi_{1}) \circ (\delta_{q}^{n} \otimes 1_{Y}).
\]
(8)
Here $\delta_{p}^{n-1} \otimes 1_{X} \otimes Y$ acts on $\otimes X \otimes Y$ as $\delta_{p}^{n-1} \otimes 1_{Y}$, where now $\delta_{p}^{n-1} \in S_{n-1} = S_{\{1,\ldots,n-1\}} \subset S_{n}$ is viewed in $S_{n}$, so that $(\delta_{p}^{n-1} \otimes 1_{X} \otimes Y) \circ (\delta_{q}^{n} \otimes 1_{Y}) = \delta_{p}^{n} \otimes 1_{Y} \circ (\delta_{q}^{n} \otimes 1_{Y})$. We now remark that we may choose $\delta_{q}^{n}$ so that $\delta_{q}^{n}(p) = p$ if $p \in \{1,\ldots,n-1\} - \{q\}$ and then we find
\[
\delta_{p}^{n-1} \delta_{q}^{n}(p, q) = \delta_{p}^{n-1}(p, n) = (n-1, n), \text{ if } p \in \{1,\ldots,n-1\} - \{q\}.
\]
On the other hand, we may further assume that $\delta_{q}^{n}(n) = q$ (with $\delta_{q}^{n} = (q, n)$ both the imposed conditions are indeed satisfied). Then we find
\[
\delta_{q}^{n-1} \delta_{q}^{n}(n, q) = \delta_{q}^{n-1}(q, n) = (n-1, n), \text{ if } q \in \{1,\ldots,n-1\}.
\]
Summarizing, setting $\delta_{p,q}^{n-1,n} := \delta_{p}^{n-1} \delta_{q}^{n}$ if $p \in \{1,\ldots,n-1\} - \{q\}$ and $\delta_{p,q}^{n-1,n} := \delta_{p}^{n-1} \delta_{q}^{n}$ if $q \in \{1,\ldots,n-1\}$, we see that
\[
\{(p, q) \in I_{n} \times I_{n}, p \neq q\} = \{(p, q) : p \in I_{n-1} - \{q\}\} \sqcup \{(n, q) : q \in I_{n-1}\}
\]
and then, since $\delta_{p,q}^{n-1,n}(p, q) = (n-1, n)$, we have
\[
R_{S_{n-2} \setminus S_{n}} = \{\delta_{p,q}^{n-1,n} : (p, q) \in I_{n} \times I_{n}, p \neq q\}.
\]
Setting $f := (1_{X} \otimes \psi_{2}) \circ (1_{X} \otimes \psi_{1})$ it follows from (8) and the above discussion that we have
\[
e_{S_{n-2}}^{X} \circ \tilde{\delta}_{\psi_{2},*x}^{n-1} \circ \tilde{\delta}_{\psi_{1},*x}^{n} = \frac{1}{n(n-1)} \sum_{p,q \in I_{n}, p \neq q} \chi^{-1} \left( \delta_{p,q}^{n-1,n} \right) \cdot f \circ (\delta_{p,q}^{n-1,n} \otimes 1_{Y}).
\]
(9)
Noticing that $f$ is $S_{n-2}$-equivariant we may apply Remark 3.2 to get
\[
e_{S_{n-2}}^{X} \circ \tilde{\delta}_{\psi_{2},*x}^{n-1} \circ \tilde{\delta}_{\psi_{1},*x}^{n} = \frac{1}{n(n-1)} \sum_{p,q \in I_{n}, p \neq q} \chi^{-1} \left( \delta_{p,q}^{n-1,n} \right) \cdot \left( f + \chi^{-1} (\tau_{n-1,n}) \cdot f \circ \tau_{n-1,n} \right) \circ (\delta_{p,q}^{n-1,n} \otimes 1_{Y}).
\]
We now remark that the relation
\[
\psi_{2} \circ (1_{X} \otimes \psi_{2}) + \nu_{x} \cdot \psi_{2} \circ (1_{X} \otimes \psi_{1}) \circ (\tau_{X} \otimes 1_{Y}) = \rho \cdot (\tau_{W} \otimes \psi_{2}) \circ (\psi_{2} \otimes 1_{Y})
\]
gives, thanks to $\nu_{x} = \chi^{-1} (\tau_{n-1,n})$,
\[
f + \chi^{-1} (\tau_{n-1,n}) \cdot f \circ \tau_{n-1,n} = \left(1_{X} \otimes \psi_{2}\right) \circ (1_{X} \otimes \psi_{1}) \\
+ \nu_{x} \cdot (1_{X} \otimes \psi_{2}) \circ (1_{X} \otimes \psi_{1}) \circ (\tau_{n-1,n} \otimes 1_{Y}) \\
= \rho \cdot (1_{X} \otimes \tau_{X} \otimes \psi_{2}) \circ (1_{X} \otimes \psi_{2} \otimes 1_{Y}).
\]
Hence \ref{3.2} gives
\[
e_{S_{n-2}}^{X} \circ \tilde{\delta}_{\psi_{2},*x}^{n-1} \circ \tilde{\delta}_{\psi_{1},*x}^{n} = \frac{1}{n(n-1)} \sum_{p,q \in I_{n}, p \neq q} \chi^{-1} \left( \delta_{p,q}^{n-1,n} \right) \cdot \left( f + \chi^{-1} (\tau_{n-1,n}) \cdot f \circ \tau_{n-1,n} \right) \circ (\delta_{p,q}^{n-1,n} \otimes 1_{Y}).
\]
(10)
We have $e_{X_{n-2}}^{\psi_{n-2}} = e_{X_{n-2}}^{\psi_{n-2}} \circ 1_{T} = (p_{X_{n-2}}^{n-2} \circ 1_{T}) \circ (i_{X_{n-2}}^{n-2} \circ 1_{T}) \circ (i_{X_{n-2}}^{n-2} \circ 1_{T})$ is a monomorphism \ref{10}, where $T = Y \otimes W$ on the left hand side while $T = W \otimes Y$ on the right hand side of \ref{10}. Hence \ref{10} gives, with the notations of Lemma 3.1
\[
2 \cdot (p_{X_{n-2}}^{n-2} \circ 1_{W} \otimes W) \circ \tilde{\delta}_{\psi_{2},*x}^{n-1} \circ \tilde{\delta}_{\psi_{1},*x}^{n} = \rho \cdot (p_{X_{n-2}}^{n-2} \circ 1_{W} \otimes Y) \circ (1_{n-2} \otimes \tau_{W} \otimes Y) \circ \left( \tilde{\delta}_{\psi_{2},*x}^{n-1} \circ 1_{Y}\right)
\]
\[
= \rho \cdot (1_{n-2} \otimes \tau_{W} \otimes Y) \circ \left( p_{X_{n-2}}^{n-2} \otimes 1_{W} \otimes Y \right) \circ \left( \tilde{\delta}_{\psi_{1},*x}^{n-1} \circ 1_{Y}\right).
\]
where $1_{n-2} = 1_{X^\wedge}X$ when $*_{\chi} = a$ or, respectively, $1_{n-2} = 1_{X_X}$ when $*_{\chi} = s$. Now the claim follows from (7), which gives

$$
(p_{X,*_{\chi}}^{n-2} \otimes 1_Y \otimes W) \circ \tilde{\partial}_{\psi_{2,*_{\chi}}}^{n-1} \circ \tilde{\partial}_{\psi_{1,*_{\chi}}}^{n} = \partial_{\psi_{2,*_{\chi}}}^{n-1} \circ \tilde{\partial}_{\psi_{1,*_{\chi}}}^{n} \circ \left( p_{X,*_{\chi}}^{n-1} \otimes 1_Z \right) \circ \tilde{\partial}_{\psi_{2,*_{\chi}}}^{n-1} \circ \tilde{\partial}_{\psi_{1,*_{\chi}}}^{n} \circ \left( p_{X,*_{\chi}}^{n} \otimes 1_Y \right),
$$

and Lemma 3.1, which gives

$$
\left( p_{X,*_{\chi}}^{n-2} \otimes 1_W \otimes Y \right) \circ \left( \Delta_{\psi_{2,*_{\chi}}}^{n} \otimes 1_Y \right) = \left( \Delta_{\psi_{2,*_{\chi}}}^{n} \otimes 1_Y \right) \circ \left( p_{X,*_{\chi}}^{n} \otimes 1_Y \right),
$$

because $p_{X,*_{\chi}}^{n} \otimes 1_Y$ is an epimorphism. \hfill \Box

Suppose now that we have given a perfect pairing $\psi : X \otimes X \to \I$, meaning that the associated hom valued morphism $f_{\psi} : X \to X^\vee$ is an isomorphism. Then $(X, \psi)$ is a dual pair for $X$ and we have the Casimir element $C_\psi : \I \to X \otimes X$. It follows from well known properties of the Casimir element that we have the following commutative diagrams:

$$
1_X : X \overset{C_\psi \otimes 1_X}{\longrightarrow} X \otimes X \otimes X \overset{1_X \otimes \psi}{\longrightarrow} X,
$$

$$
1_X : X \overset{1_X \otimes C_\psi}{\longrightarrow} X \otimes X \otimes X \overset{\psi \otimes 1_X}{\longrightarrow} X.
$$

Suppose that we have $\psi \circ \tau_{X,X} = \chi (\tau_{X,X}) \cdot \psi$, where $\chi \in \{1, \phi \}$. Then we have, by definition, $r_X = \psi \circ \tau_{X,X} \circ C_\psi = \chi (\tau_{X,X}) \cdot \psi \circ C_\psi$, implying that the following diagram is commutative:

$$
\chi (\tau_{X,X}) r_X : \I \to X \otimes X \overset{\psi}{\to} \I.
$$

(13)

We may consider

$$
C^n_\psi := 1 \otimes \cdots \otimes X \otimes \cdots \otimes X \to \otimes^{n+2} X \text{ for } n \geq 0
$$

and then we define

$$
C^n_{\psi,a} : \wedge^n X = \Delta_{X,a}^n \otimes X \overset{C^n_\psi}{\longrightarrow} \otimes^{n+2} X \overset{\Delta_{X,a}^{n+2}}{\longrightarrow} \wedge^{n+2} X,
$$

$$
C^n_{\psi,s} : \vee^n X \overset{\Delta_{X,s}^n}{\longrightarrow} \otimes X \overset{C^n_\psi}{\longrightarrow} \otimes^{n+2} X \overset{\Delta_{X,s}^{n+2}}{\longrightarrow} \vee^{n+2} X.
$$

Since $C^n_\psi$ is $S_n$-equivariant, the following diagrams are commutative:

$$
\begin{array}{ccc}
\otimes^n X & \xrightarrow{C^n_\psi} & \otimes^{n+2} X \\
\downarrow_{\Delta_{\psi,a}} & & \downarrow_{\Delta_{\psi,s}^{n+2}} \\
\wedge^n X & \xrightarrow{C^n_{\psi,a}} & \wedge^{n+2} X,
\end{array}
\quad
\begin{array}{ccc}
\otimes^n X & \xrightarrow{C^n_\psi} & \otimes^{n+2} X \\
\downarrow_{\Delta_{\psi,s}} & & \downarrow_{\Delta_{\psi,s}^{n+2}} \\
\vee^n X & \xrightarrow{C^n_{\psi,s}} & \vee^{n+2} X.
\end{array}
$$

(14)

Lemma 3.4. Suppose that $\psi : X \otimes X \to \I$ is a perfect pairing such that $\psi \circ \tau_{X,X} = \nu_{*_{\chi}} \cdot \psi$, where $\nu_a := -1, \nu_s := 1$ and $*_{\chi} \in \{a, s\}$. Then we have the formulas $\Delta_{\psi,a}^2 \circ C_{\psi,a}^0 = \nu_{*_{\chi}} r_X, 3 \Delta_{\psi,s}^3 \circ C_{\psi,s}^1 = (2 + \nu_{*_{\chi}} r_X) \cdot 1_X$ and, for every $n \geq 2$,

$$
\frac{(n+2)(n+1)}{2} \cdot \Delta_{\psi,a}^{n+2} \circ C_{\psi,a}^n - \frac{n(n-1)}{2} \cdot \Delta_{\psi,s}^{n+2} \circ \Delta_{\psi,s}^n = (2n + \nu_{*_{\chi}} r_X) \cdot 1_X,
$$

where $\wedge^n X$ for $*_{\chi} = a, \vee^n X := \vee^n X$ for $*_{\chi} = s$ and $r_X := \text{rank}(X)$.

\[\text{We remark that, assuming 2 is invertible in } \text{Hom}(X \otimes X, X), \text{ we may always write } \psi \text{ as the direct sum of its alternating and symmetric part, defined respectively by the formulas } \psi_a := \psi_{-\psi_{X,X}} \text{ and } \psi_s := \psi_{\psi_{X,X}}. \text{ This means that } \psi = \psi_a \oplus \psi_s, \text{ up to the identification } \text{Hom}(X \otimes X, X) = \text{Hom}(\wedge^2 X, X) \oplus \text{Hom}(\vee^2 X, X) \text{ and the above assumption is always achieved by } \psi_a \text{ and } \psi_s.\]
Proof. We have, employing the notations in Lemma 3.1

\[
\frac{(n+2)(n+1)}{2} \cdot \Delta^2 \phi \ast x \ast \circ C^n = \frac{n(n-1)}{2} \cdot C^n \ast x \ast \circ \Delta^2 \phi \ast x \ast = \frac{n(n-1)}{2} \cdot C^n \ast x \ast \circ \Delta^2 \phi \ast x \ast
\]

We claim that we have \( \Delta^2 \phi \ast x \ast \circ C^n = \nu_x \ast r_X \), \( 3 \Delta^3 \phi \ast x \ast \circ C^1 = (2 + \nu_x \ast r_X) \cdot 1_X \) and, for every \( n \geq 2 \),

\[
\frac{(n+2)(n+1)}{2} \cdot \Delta^2 \phi \ast x \ast \circ C^n = \frac{n(n-1)}{2} \cdot C^n \ast x \ast \circ \Delta^2 \phi \ast x \ast = 2n \cdot e_{R_{n-1} \ast s} + \nu_x \ast r_X \cdot 1 \ast x.X.
\]

It will follow from Lemma 3.1 and (14) that we have

\[
\Delta^2 \phi \ast x \ast \circ C^n = \nu_x \ast r_X, \quad 3 \Delta^3 \phi \ast x \ast \circ C^1 = (2 + \nu_x \ast r_X) \cdot 1_X.
\]

and, for every \( n \geq 2 \),

\[
\frac{(n+2)(n+1)}{2} \cdot \Delta^2 \phi \ast x \ast \circ C^n = \frac{n(n-1)}{2} \cdot C^n \ast x \ast \circ \Delta^2 \phi \ast x \ast = 2n \cdot e_{R_{n-1} \ast s} + \nu_x \ast r_X \cdot p^n \ast x \ast.
\]

Here in the last equality we have employed the relation \( p^n \ast x \ast \circ e_{R_{n-1} \ast s} = p^n \ast x \ast \), which is proved noticing that, since \( i^X \ast x \ast \) is a monomorphism, it is equivalent to \( e_{R_{n-1} \ast s} \circ e_{R_{n-1} \ast s} = e_{X \ast x} \); this last relation follows from the relations \( e_G = e_{X \ast x} \) and \( e_{X \ast x} = e_{G \ast x} \), implying \( e_G e_{X \ast x} = e_G e_{X \ast x} = e_G e_{X \ast x} = e_{X \ast x} \), which can be easily checked in the group algebras. Then the claim will follow from the fact that \( p^n \ast x \ast \) is an epimorphism.

When \( n = 0 \) in (15), we have \( C^0 = C_\psi \), \( 3 \Delta^3 \phi \ast x \ast = \psi \) and the equality \( \Delta^2 \phi \ast x \ast \circ C^0 = \nu_x \ast r_X \) follows from (13). When \( n = 1 \) in (15), we have

\[
3 \Delta^3 \phi \ast x \ast \circ C^1 = \chi^{-1} (\tau_{(123)}) \cdot (1_X \otimes \psi) \circ \tau_{(123)} \circ (1_X \otimes C_\psi)
\]

\[
+ \chi^{-1} (\tau_{(12)}) \cdot (1_X \otimes \psi) \circ \tau_{(12)} \circ (1_X \otimes C_\psi)
\]

\[
+ (1_X \otimes \psi) \circ (1_X \otimes C_\psi),
\]

because we may take \( \delta_{2,1}^2 = \tau_{(123)} \), \( \delta_{1,1}^2 = \tau_{(12)} \) and \( \delta_{2,3}^2 = 1 \), where \( \tau_\sigma \) denotes the morphism attached to the permutation \( \sigma \). We have \( \tau_{(123)} = \tau_{X \otimes X, X} \) and \( (\psi \otimes 1_X) \circ (1_X \otimes C_\psi) = 1_X \) by (14). Hence we deduce the equality

\[
\chi^{-1} (\tau_{(12)}) \cdot (1_X \otimes \psi) \circ \tau_{(12)} \circ (1_X \otimes C_\psi) = (1_X \otimes \psi) \circ \tau_{X \otimes X, X} \circ (1_X \otimes C_\psi)
\]

\[
= (\psi \otimes 1_X) \circ (1_X \otimes C_\psi) = 1_X.
\]

Consider the following diagram:
The region (A) is commutative thanks to our assumption $\psi \circ \tau_{X,X} = \nu_{\ast_X} \cdot \psi$. Noticing that $\tau_{(12)} = (1_X \otimes \tau_{X,X}) \circ \tau_{X,X} \otimes X$ and that $(1_X \otimes \psi) \circ (C_\psi \otimes 1_X) = 1_X$ by (11), we deduce the equality:

\[
\chi^{-1} \left( \tau_{(12)} \right) \cdot (1_X \otimes \psi) \circ \tau_{(12)} \cdot (1_X \otimes C_\psi) = \chi^{-1} \left( \tau_{(12)} \right) \cdot (1_X \otimes \psi) \circ (1 \otimes \tau_{X,X}) \circ \tau_{X,X} \otimes X \circ (1_X \otimes C_\psi)
\]

\[
= \chi^{-1} \left( \tau_{(12)} \right) \nu_{\ast_X} \cdot (1_X \otimes \psi) \circ (C_\psi \otimes 1_X) = 1_X.
\]

Inserting (18), (19) and (13) in (17) gives $3 \Delta^2 \nu_{\ast_X} \circ C_\psi = (2 + \nu_{\ast_X} r_X) \cdot 1_X$.

Suppose now that $n \geq 2$. We remark that we have

\[
\{(p, q) \in I_{n+2} \times I_{n+2} : p < q\} = \{(p, q) \in I_n \times I_n : p < q\} \cup (I_n \times \{n+1\}) \cup (I_n \times \{n+2\}) \cup \{(n+1, n+2)\}.
\]

We may assume that we have given our choice of $\delta^1_{p,q} \in S_n$ and we choose the elements $\delta^1_{p,q} \in S_n$ as follows. First of all we view $S_n \subset S_{n+2}$ in the natural way, via $I_n \subset I_{n+2}$, and we choose elements $\delta^1_{p,q} \in S_n$ such that $\delta^1_{p,q} = n$. If $(p, q) \in I_n \times I_n$ and $p < q$, we set $\delta^1_{p,q} := \tau_{(n-1,n+1)(n,n+2)} \circ \delta^1_{p,q}$ and then we define $\delta^{n+1}_{n+2} := \tau_{(n,n+1,n+2)} \circ \delta^1_{p,q} \circ \delta^{n+1}_{p,q}$ and $\delta^{n+2}_{n+2} := \tau_{(n,n+1)} \circ \delta^1_{p,q}$ and noticing that, in every case, we have the required $\delta^{n+2}_{n+2} (p, q) = (n+1, n+2)$ satisfied. Thanks to (20), we may rewrite (15) as follows:

\[
\frac{(n+2)(n+1)}{2} \Delta^2 \nu_{\ast_X} \circ C_\psi = \sum_{p,q \in I_n : p < q} \chi^{-1} \left( \delta^2_{p,q}; \delta^1_{p,q} \right) \cdot (1 \otimes \nu_{\ast_X} \circ \psi) \circ \delta^2_{p,q} \circ (1 \otimes \nu_{\ast_X} \circ C_\psi)
\]

\[
+ \sum_{p \in I_n} \chi^{-1} \left( \delta^2_{p,n+2} \right) \cdot (1 \otimes \nu_{\ast_X} \circ \psi) \circ \delta^2_{p,n+2} \circ (1 \otimes \nu_{\ast_X} \circ C_\psi)
\]

\[
+ \sum_{p \in I_n} \chi^{-1} \left( \delta^2_{n+2,p+2} \right) \cdot (1 \otimes \nu_{\ast_X} \circ \psi) \circ \delta^2_{n+2,p+2} \circ (1 \otimes \nu_{\ast_X} \circ C_\psi)
\]

\[
+ \chi^{-1} \left( \delta^2_{n+2,n+2} \right) \cdot (1 \otimes \nu_{\ast_X} \circ \psi) \circ \delta^2_{n+2,n+2} \circ (1 \otimes \nu_{\ast_X} \circ C_\psi)
\]

\[
= \sum_{p,q \in I_n, p < q} \chi^{-1} \left( \delta^1_{p,q} \right) \cdot (1 \otimes \nu_{\ast_X} \circ \psi) \circ \tau_{(n-1,n+1)(n,n+2)} \circ (1 \otimes \nu_{\ast_X} \circ C_\psi) \circ \delta^1_{p,q}
\]

\[
+ \sum_{p \in I_n} \chi^{-1} \left( \delta^1_{p,n+2} \right) \cdot (1 \otimes \nu_{\ast_X} \circ \psi) \circ \tau_{(n,n+1,n+2)} \circ (1 \otimes \nu_{\ast_X} \circ C_\psi) \circ \delta^1_{p,n+2}
\]

\[
+ \sum_{p \in I_n} \chi^{-1} \left( \delta^1_{n+2,p+2} \right) \cdot (1 \otimes \nu_{\ast_X} \circ \psi) \circ \tau_{(n,n+1)} \circ (1 \otimes \nu_{\ast_X} \circ C_\psi) \circ \delta^1_{n+2,p}
\]

\[
+ (1 \otimes \nu_{\ast_X} \circ \psi) \circ (1 \otimes \nu_{\ast_X} \circ C_\psi)
\]

Making the substitution $(n-1, n, n+1, n+2) = (1, 2, 3, 4)$, we may write $\tau_{(n-1,n+1)(n,n+2)} = 1 \otimes \nu_{\ast_X} \otimes \tau_{(12)}$, where $\tau_{(12)}$ is acting on the last four factors $X \otimes X \otimes X \otimes X$ of $\otimes^{n+2} X$. Then the relation

\[
(1 \otimes \nu_{\ast_X} \circ \psi) \circ \tau_{X \otimes X \otimes X \otimes X} \circ (1 \otimes \nu_{\ast_X} \circ C_\psi) = (1 \otimes \nu_{\ast_X} \circ \psi) \circ (C_\psi \otimes 1 \otimes \nu_{\ast_X}) = (C_\psi \otimes \psi) = C_\psi \circ \psi
\]

implies that we have

\[
(1 \otimes \nu_{\ast_X} \circ \psi) \circ \tau_{(n-1,n+1)(n,n+2)} \circ (1 \otimes \nu_{\ast_X} \circ C_\psi) = (1 \otimes \nu_{\ast_X} \circ \psi) \circ (C_\psi \otimes 1 \otimes \nu_{\ast_X}) \circ (1 \otimes \nu_{\ast_X} \circ \psi).
\]

Hence it follows from (16) that we have:

\[
\frac{n(n-1)}{2} C_{\psi \ast_X} \nu_{\ast_X} = \frac{n(n-1)}{2} C_{\psi \ast_X} \nu_{\ast_X} \cdot \Delta^2 \nu_{\ast_X}.
\]

We are now going to compute the sums (22) and (23). Making the substitution $(n-1, n, n+1, n+2) = (1, 2, 3)$, we may write $\tau_{(n-1,n+1,n+2)} = 1 \otimes \nu_{\ast_X} \otimes \tau_{(12)}$ (resp. $\tau_{(n,n+2)} = 1 \otimes \nu_{\ast_X} \otimes \tau_{(12)}$), where $\tau_{(12)}$ (resp. $\tau_{(12)}$) is acting on the last three factors $X \otimes X \otimes X$ of $\otimes^{n+2} X$. It follows from (13) and (19) that we have, respectively,

\[
\chi^{-1} \left( \tau_{(n,n+1,n+2)} \right) \cdot (1 \otimes \nu_{\ast_X} \circ \psi) \circ \tau_{(n,n+1,n+2)} \circ (1 \otimes \nu_{\ast_X} \circ C_\psi) = 1 \otimes \nu_{\ast_X},
\]

\[
\chi^{-1} \left( \tau_{(n,n+1)} \right) \cdot (1 \otimes \nu_{\ast_X} \circ \psi) \circ \tau_{(n,n+1)} \circ (1 \otimes \nu_{\ast_X} \circ C_\psi) = 1 \otimes \nu_{\ast_X}.
\]

Hence we find

\[
\frac{n(n-1)}{2} C_{\psi \ast_X} \nu_{\ast_X} = \frac{n(n-1)}{2} C_{\psi \ast_X} \nu_{\ast_X} \cdot \Delta^2 \nu_{\ast_X}.
\]

Finally, it follows from (13) that we have

\[
\frac{n(n-1)}{2} C_{\psi \ast_X} \nu_{\ast_X} \cdot \Delta^2 \nu_{\ast_X}.
\]
It now follows from (25), (26) and (27) that we have, as claimed,
\[
\frac{(n + 2)(n + 1)}{2} \cdot \Delta_{\psi,x}^{n+2} \circ C_{\psi,a}^n = (21) + (22) + (23) + (24)
\]
\[
= \frac{n(n - 1)}{2} \cdot C_{\psi}^{n-2} \circ \Delta_{\psi,x}^n + 2n \cdot e_{K_{S_1,S_2}}^{\nu_{S_1}} + \nu_{S_1} \cdot r_X \cdot 1_{\psi,a}.
\]

The following definition will be useful in the following subsections.

**Definition 3.5.** We say the a morphism \( f : M \to M \) is diagonalizable if there is an isomorphism \( M \cong \bigoplus_{\lambda \in \text{End}(I)} M_{f,\lambda} \) such that \( M_{f,\lambda} = 0 \) for almost every \( \lambda \) and \( f \cong \bigoplus_{\lambda \in \text{End}(I)} f_{\lambda} \) via this isomorphism, with \( f_{\lambda} = \lambda : M_{f,\lambda} \to M_{f,\lambda} \) the multiplication by \( \lambda \in \text{End}(I) \). In this case, we call the set
\[
\sigma(f) := \{ \lambda : M_{f,\lambda} \neq 0 \} \subset \text{End}(I)
\]
the spectrum of \( f \).

It will be also convenient to introduce the following definition.

**Definition 3.6.** If \( S \subset \text{End}(I) \) we say that \( S \) is strictly positive (resp. positive, strictly negative or negative) and we write \( S > 0 \) (resp. \( S \geq 0, S < 0 \) or \( S \leq 0 \)) to mean that there exists an ordered field \((K,\geq)\) such that \( S \subset K \subset \text{End}(I) \) and \( s > 0 \) (resp. \( s \geq 0, s < 0 \) or \( s \leq 0 \)) in \( K \) for every \( s \in S \). If \( s \in \text{End}(I) \), we write \( s > 0 \) (resp. \( s \geq 0, s < 0 \) or \( s \leq 0 \)) to mean that \( S > 0 \) (resp. \( S \geq 0, S < 0 \) or \( S \leq 0 \)) with \( S = \{s\} \).

3.1. Laplace operators attached to \( I \)-valued perfect alternating pairings. We suppose in this subsection that we have given \( \psi : X \otimes X \to I \) which is perfect, i.e. such that the associated hom valued morphism is an isomorphism, and alternating, i.e. \( \psi \circ \tau_{X,X} = -\psi \). It follows from Lemma 3.1 that we have \( \Delta_{\psi,s}^n = 0 \) and, hence, we concentrate on \( \Delta_{\psi,a}^n \). We set \( r_X := \text{rank}(X) \) in the subsequent discussion.

**Proposition 3.7.** When \( r_X < 0 \) we have that \( \Delta_{\psi,a}^{n+2} \circ C_{\psi,a}^n \) when \( n \geq 0 \) (resp. \( C_{\psi,a}^{n-2} \circ \Delta_{\psi,a}^n \) when \( n \geq 2 \)) is diagonalizable, with spectrum
\[
\sigma\left(\Delta_{\psi,a}^{n+2} \circ C_{\psi,a}^n\right) > 0 \quad \text{(resp. } \sigma\left(C_{\psi,a}^{n-2} \circ \Delta_{\psi,a}^n\right) \geq 0\text{)}.
\]

**Proof.** It will be convenient to set \( \delta_{\psi,a}^n := \frac{m(n+1)}{2} \cdot \Delta_{\psi,a}^n \), so that Lemma 3.4 gives \( \delta_{\psi,a}^2 \circ C_{\psi,a}^0 = -r_X \), \( \delta_{\psi,a}^3 \circ C_{\psi,a}^1 = (2 - r_X) \cdot 1_X \) and, for every \( n \geq 2 \),
\[
\delta_{\psi,a}^{n+2} \circ C_{\psi,a}^n - C_{\psi,a}^{n-2} \circ \delta_{\psi,a}^n = (2n - r_X) \cdot 1_{\psi,a}.
\]

In particular, we see that \( \delta_{\psi,a}^{n+2} \circ C_{\psi,a}^n \) is diagonalizable for \( n = 0, 1 \) with \( \sigma\left(\delta_{\psi,a}^0 \circ C_{\psi,a}^0\right) = \{-r_X\} > 0 \) and \( \sigma\left(\delta_{\psi,a}^1 \circ C_{\psi,a}^1\right) = \{2 - r_X\} > 0 \). We can now assume that \( n \geq 2 \) and that, by induction, \( \delta_{\psi,a}^n \circ C_{\psi,a}^{n-2} \) is diagonalizable with spectrum \( \sigma\left(\delta_{\psi,a}^n \circ C_{\psi,a}^{n-2}\right) \geq 0 \) and we claim that this implies both that \( \Delta_{\psi,a}^{n+2} \circ C_{\psi,a}^n \) is diagonalizable with spectrum \( \geq 0 \) and that \( C_{\psi,a}^{n-2} \circ \Delta_{\psi,a}^n \) is diagonalizable with spectrum \( \geq 0 \). Here and in the following, the ordered field \((K,\geq)\) in the definition of being positive is always taken to be the one appearing in the definition of \(-r_X > 0\).

Since \( \delta_{\psi,a}^n \circ C_{\psi,a}^{n-2} \) is diagonalizable with spectrum \( > 0 \), we have that \( \delta_{\psi,a}^n \circ C_{\psi,a}^{n-2} \) is an isomorphism. It now follows from an abstract non-sense that there is a biproduct decomposition
\[
\wedge^n X \cong \ker\left(C_{\psi,a}^{n-2} \circ \delta_{\psi,a}^n\right) \oplus \wedge^{n-2} X
\]
such that
\[
C_{\psi,a}^{n-2} \circ \delta_{\psi,a}^n \simeq 0 \oplus \left(\delta_{\psi,a}^n \circ C_{\psi,a}^{n-2}\right).
\]
Since $\delta^n_{\psi,s} \circ C^n_{\psi,s}^{-2}$ is diagonalizable with spectrum $\sigma \left( \delta^n_{\psi,s} \circ C^n_{\psi,s}^{-2} \right) > 0$, it follows from (29) that $\sigma \left( C^n_{\psi,s} \circ \delta^n_{\psi,s} \right)$ is diagonalizable with spectrum

$$\sigma \left( C^n_{\psi,s} \circ \delta^n_{\psi,s} \right) \subset \{0\} \cup \sigma \left( \delta^n_{\psi,s} \circ C^n_{\psi,s} \right) \geq 0.$$ 

It now follows from (28) that $\delta^{n+2}_{\psi,s} \circ C^n_{\psi,s}^{-2}$ is diagonalizable with spectrum

$$\sigma \left( \Delta^{n+2}_{\psi,s} \circ C^n_{\psi,s} \right) \subset \left\{ \lambda + (2n - r_X) : \lambda \in \sigma \left( C^n_{\psi,s} \circ \delta^n_{\psi,s} \right) \right\} > 0.$$

□

**Corollary 3.8.** When $r_X < 0$ we have that, for every $n \geq 2$, the Laplace operator $\Delta^n_{\psi,s}$ has a section $s^n_{\psi,s} : \land^n X \to \land^n X$ such that $\Delta^n_{\psi,s} \circ s^n_{\psi,s} = 1_{\land^n X}$ and, in particular, $\ker \left( \Delta^n_{\psi,s} \right)$ exists.

**Proof.** Indeed $\Delta^n_{\psi,s} \circ C^n_{\psi,s}^{-2}$ is diagonalizable with spectrum $\sigma \left( \Delta^n_{\psi,s} \circ C^n_{\psi,s}^{-2} \right) > 0$ by Proposition 3.7, and, in particular, it is an isomorphism.

□

### 3.2. Laplace operators attached to $L$-valued perfect symmetric pairings

We suppose in this subsection that we have given a perfect pairing $\chi : X \otimes X \to \mathbb{I}$ which is perfect, i.e. such that the associated hom valued morphism is an isomorphism, and symmetric, i.e. $\psi \circ \tau_{X,X} = \psi$. It follows from Lemma 3.1 that we have $\Delta^n_{\psi,s} = 0$ and, hence, we concentrate on $\Delta^n_{\psi,s}$. We set $r_X := \text{rank}(X)$ in the subsequent discussion.

**Proposition 3.9.** When $r_X > 0$ we have that $\Delta^{n+2}_{\psi,s} \circ C^n_{\psi,s}$ when $n \geq 0$ (resp. $C^n_{\psi,s} \circ \Delta^n_{\psi,s}$ when $n \geq 2$) is diagonalizable, with spectrum

$$\sigma \left( \Delta^{n+2}_{\psi,s} \circ C^n_{\psi,s} \right) > 0 \text{ (resp. } \sigma \left( C^n_{\psi,s} \circ \Delta^n_{\psi,s} \right) \geq 0).$$

**Proof.** Setting $\delta^n_{\psi,s} := \frac{n(n-1)}{2} \Delta^n_{\psi,s}$, Lemma 3.4 gives the equalities $\delta^n_{\psi,s} \circ C^0_{\psi,s} = r_X, \delta^3_{\psi,s} \circ C^1_{\psi,s} = (2 + r_X) \cdot 1_X$ and, for every $n \geq 2$,

$$\delta^{n+2}_{\psi,s} \circ C^n_{\psi,s} - C^n_{\psi,s} \circ \delta^n_{\psi,s} = (2n + r_X) \cdot 1_{\land^n X}.$$ 

Then the proof is just a copy of those of Proposition 3.7. □

The following corollary may be deduced from Proposition 3.9 in the same way as Corollary 3.8 was deduced from Proposition 3.7.

**Corollary 3.10.** When $r_X > 0$ we have that, for every $n \geq 2$, the Laplace operator $\Delta^n_{\psi,s}$ has a section $s^n_{\psi,s} : \land^n X \to \land^n X$ such that $\Delta^n_{\psi,s} \circ s^n_{\psi,s} = 1_{\land^n X}$ and, in particular, $\ker \left( \Delta^n_{\psi,s} \right)$ exists.

### 3.3. Laplace operators attached to perfect pairings valued in squares of invertible objects

We suppose in this subsection that we have given a perfect pairing $\psi : X \otimes X \to Z$, i.e. such that $f_\psi : X \to \text{hom}(X,Z)$ is an isomorphism, and that we have $Z \simeq \mathbb{L}^{\otimes 2}$, where $\mathbb{L}$ an invertible object. We assume that $\psi$ is alternating or symmetric, i.e. $\psi \circ \tau_{X,X} = \chi (\tau_{X,X}) \cdot \psi$, where $\chi \in \{\varepsilon, 1\}$. As above, we define $s_X := a$ when $\chi = \varepsilon$ and $s_X := s$ when $\chi = 1$ and we write $s_X^k X := \land^k X$ when $\chi = \varepsilon$ and $s_X^k X := \land^k X$ when $\chi = 1$. It follows from Lemma 3.1 that we have $\Delta^n_{\psi,s} = 0$ if $\{s\} = \{a,s\} \setminus \{s_X\}$ and, hence, we concentrate on $\Delta^n_{\psi,s}$. We set $r_X := \text{rank}(X)$ and $r_L := \text{rank}(\mathbb{L})$ in the subsequent discussion, so that $r_L \in \{\pm 1\}$.

Let $\tau_{\delta_k} : \otimes^k (X \otimes \mathbb{L}) \xrightarrow{\sim} (\otimes^k X) \otimes \mathbb{L}^{\otimes k}$ be the isomorphism induced by the permutation $\delta_k \in S_{2k}$ such that $\delta_k(2i - 1) = i$ and $\delta_k(2i) = k + i$ for every $i \in I_k$. It is not difficult to show, using [De3] 7.2 Lemme, that one has

$$e^k_{X \otimes \mathbb{L},a} \simeq e^k_{X,a} \otimes 1_{\mathbb{L}^{\otimes k}} \text{ and } e^k_{X \otimes \mathbb{L},s} \simeq e^k_{X,s} \otimes 1_{\mathbb{L}^{\otimes k}} \text{ if } r_L = 1,$$

$$e^k_{X \otimes \mathbb{L},s} \simeq e^k_{X,a} \otimes 1_{\mathbb{L}^{\otimes k}} \text{ and } e^k_{X \otimes \mathbb{L},a} \simeq e^k_{X,s} \otimes 1_{\mathbb{L}^{\otimes k}} \text{ if } r_L = -1.$$
Lemma 3.11. Suppose that \( \varphi : X \otimes X \to \mathbb{L}\otimes^2 \) is alternating (resp. symmetric) and consider the composite \( \varphi_{L^{-1}} : (X \otimes L^{-1}) \to (X \otimes \mathbb{L}\otimes^{-2}) \). Suppose that \( L \) is alternating (resp. symmetric) and the following diagrams are commutative:

\[
\begin{align*}
\Lambda^n (X \otimes L^{-1}) & \xrightarrow{\Delta^n_{\varphi_{L^{-1}}}} \Lambda^n (X \otimes L^{-1}) \\
\Lambda^n (X \otimes L^{-1}) & \xrightarrow{\tau_{\delta_{n-2}}^{\varphi_{L^{-1}}}} \Lambda^n (X \otimes L^{-1})
\end{align*}
\]

where \( \tau_{\delta_{n-2}} : \Lambda^k (X \otimes \mathbb{L}) \to \Lambda^k (X \otimes \mathbb{L}) \) is the morphism induced by \( \tau_{\delta_{n-2}}^{(33)} \).

(b) If \( L \) is symmetric (resp. alternating) and the following diagrams are commutative:

\[
\begin{align*}
\Lambda^n (X \otimes L^{-1}) & \xrightarrow{\Delta^n_{\varphi_{L^{-1}}}} \Lambda^n (X \otimes L^{-1}) \\
\Lambda^n (X \otimes L^{-1}) & \xrightarrow{\tau_{\delta_{n-2}}^{\varphi_{L^{-1}}}} \Lambda^n (X \otimes L^{-1})
\end{align*}
\]

where \( \tau_{\delta_{n-2}} : \Lambda^k (X \otimes \mathbb{L}) \to \Lambda^k (X \otimes \mathbb{L}) \) is the morphism induced by \( \tau_{\delta_{n-2}}^{(33)} \).

(c) Writing \( f_{\varphi} : X \to \text{hom} (X, \mathbb{L}^\otimes^2) \) and \( f_{\varphi_{L^{-1}}} : X \otimes L^{-1} \to (X \otimes L^{-1})^\vee \) for the associated morphisms we have that \( f_{\varphi} \) is an isomorphism if and only if \( f_{\varphi_{L^{-1}}} \) is an isomorphism.

Proof. (a-b) We first claim that the following diagram is commutative:

\[
\begin{align*}
\otimes^n (X \otimes \mathbb{L}) & \xrightarrow{\Delta^n_{\varphi_{L^{-1}}}} \otimes^n (X \otimes \mathbb{L}) \\
\tau_{\delta_{n-2}} & \xrightarrow{\tau_{\delta_{n-2}}^{\varphi_{L^{-1}}}} \tau_{\delta_{n-2}}^{\varphi_{L^{-1}}}
\end{align*}
\]

A tedious computation reveals that:

\[
(\tau_{\delta_{n-2}} \circ 1_X \otimes \tau_{L^{-1},X} \otimes 1_{L^{-1}}) = (1 \otimes_{n-2} (X \otimes \varphi_{L^{-1}}) \otimes 1_{L^{-1}}) \circ \tau_{\delta_{n-2}}
\]

Hence we have:

\[
\tau_{\delta_{n-2}} \circ \Delta^n_{\varphi_{L^{-1}}} = \tau_{\delta_{n-2}} \circ (1 \otimes_{n-2} (X \otimes \varphi_{L^{-1}}) \otimes 1_{L^{-1}}) = (1 \otimes_{n-2} (X \otimes \varphi_{L^{-1}}) \otimes 1_{L^{-1}}) \circ \tau_{\delta_{n-2}}
\]

showing that \( \Delta^n_{\varphi_{L^{-1}}} \) is commutative. The claimed commutative diagrams in (a) and (b) now follows from (30), (31) and the commutativity of (32).

We view \( 1_X \otimes \tau_{L^{-1},X} \otimes 1_{L^{-1}} = \tau_{(23)} \) and \( 1 \otimes_{n-2} (X \otimes L^{-1}) \otimes 1_{L^{-1}} = \tau_{(13)(24)} \) as induced by permutations in \( S_4 \) and then, noticing that \( (23)(13)(24) = (1243) = (12)(34)(23) \) and that we have \( \varphi \circ \tau_{X,X} = \chi (\tau_{X,X}) \cdot \psi \) with
\( \chi = \varepsilon \) (resp. \( \chi = 1 \)), we find
\[
\varphi_{L^{-1}} \circ \tau_{X \otimes L^{-1}, X \otimes L^{-1}} = ev_{L \otimes 2} \circ (\varphi \otimes 1_{L \otimes 2}) \circ (1_{X} \otimes \tau_{L^{-1}, X} \otimes 1_{L^{-1}}) \circ \tau_{X \otimes L^{-1}, X \otimes L^{-1}}
\]
\[
= ev_{L \otimes 2} \circ (\varphi \otimes 1_{L \otimes 2}) \circ \tau_{(23)} \circ \tau_{(13)(24)} = ev_{L \otimes 2} \circ (\varphi \otimes 1_{L \otimes 2}) \circ \tau_{(12)} \circ \tau_{(34)} \circ \tau_{(23)}
\]
\[
= ev_{L \otimes 2} \circ (\varphi \otimes 1_{L \otimes 2}) \circ (\tau_{X,X} \otimes \tau_{L^{-1}, L^{-1}}) \circ (1_{X} \otimes \tau_{L^{-1}, X} \otimes 1_{L^{-1}})
\]
\[
= \chi (\tau_{X,X}) \cdot ev_{L \otimes 2} \circ (\varphi \otimes \tau_{L^{-1}, L^{-1}}) \circ (1_{X} \otimes \tau_{L^{-1}, X} \otimes 1_{L^{-1}}).
\]

It follows from [De3 7.2 Lemme] that we have \( \tau_{L^{-1}, L^{-1}} = r_{L} \), so that we find
\[
\varphi_{L^{-1}} \circ \tau_{X \otimes L^{-1}, X \otimes L^{-1}} = r_{L} \chi (\tau_{X,X}) \cdot \varphi_{L^{-1}}.
\]

(c) This is left to the reader. \( \square \)

**Proposition 3.12.** When \( \chi (\tau_{X,X}) r_{X} > 0 \) we have that, for every \( n \geq 2 \), the Laplace operator
\[
\Delta_{\psi, \ast, \chi}^{n} : \ast^{n-2} X \rightarrow \ast^{n-2} X
\]
has a section \( \ast^{n-2} X \rightarrow \ast^{n} X \) such that \( \Delta_{\psi, \ast, \chi}^{n} \circ \ast^{n-2} X = 1 \) and, in particular, \( \ker (\Delta_{\psi, \ast, \chi}^{n}) \) exists.

**Proof.** If \( \sigma : Z \rightarrow L \otimes Z \) is our given isomorphism, we have that \( \Delta_{\sigma \circ \psi, \ast, \chi}^{n} = (1_{\ast^{n-2} X} \otimes \sigma) \circ \Delta_{\psi, \ast, \chi}^{n} \) has a section if and only if \( \Delta_{\psi, \ast, \chi}^{n} \) has a section: hence we may assume that \( Z = L \otimes L \). We can now consider the composite:
\[
\psi_{L^{-1}} : (X \otimes L^{-1}) \otimes (X \otimes L^{-1}) \xrightarrow{1 \otimes \tau_{X \otimes L^{-1}, X \otimes L^{-1}}} X \otimes X \otimes L^{-2} \otimes L_{L^{-2}} \otimes L \rightarrow L.
\]

When \( r_{L} = 1 \) (resp. \( r_{L} = -1 \)), Lemma 3.11(a) (resp. (b)) shows that \( \Delta_{\psi, \ast, \chi}^{n} \) has a section if and only if \( \Delta_{\psi, \ast, \chi}^{n} \) (resp. \( \Delta_{\psi, \ast, \chi}^{n} \)) has a section and that \( \psi_{L^{-1}} \) satisfies \( \varphi_{L^{-1}} \circ \tau_{X \otimes L^{-1}, X \otimes L^{-1}} = \chi (\tau_{X,X}) \cdot \varphi_{L^{-1}} \) (resp. \( \varphi_{L^{-1}} \circ \tau_{X \otimes L^{-1}, X \otimes L^{-1}} = -\chi (\tau_{X,X}) \cdot \varphi_{L^{-1}} \)). It follows from this last relation that, if we define \( \varepsilon_{X \otimes L^{-1}, X \otimes L^{-1}} \) by the rule \( \varphi_{L^{-1}} \circ \tau_{X \otimes L^{-1}, X \otimes L^{-1}} = \varepsilon_{X \otimes L^{-1}, X \otimes L^{-1}} \cdot \varphi_{L^{-1}} \), then we have \( \varepsilon_{X \otimes L^{-1}, X \otimes L^{-1}} = \chi (\tau_{X,X}) \) (resp. \( \varepsilon_{X \otimes L^{-1}, X \otimes L^{-1}} = -\chi (\tau_{X,X}) \)) and
\[
\varepsilon_{X \otimes L^{-1}, X \otimes L^{-1}} \tau_{X \otimes L^{-1}} = \varepsilon_{X \otimes L^{-1}, X \otimes L^{-1}} \tau_{X \otimes L^{-1}} = \chi (\tau_{X,X}) r_{X} > 0.
\]

It follows that we may apply to \( \psi_{L^{-1}} \) Corollary 3.18 when \( \varepsilon_{X \otimes L^{-1}, X \otimes L^{-1}} = -1 \), or Corollary 3.10 when \( \varepsilon_{X \otimes L^{-1}, X \otimes L^{-1}} = 1 \), to deduce that \( \Delta_{\psi, \ast, \chi}^{n} \) has a section. \( \square \)

4. LAPLACE AND DIRAC OPERATORS FOR THE ALTERNATING ALGEBRAS

In this section we assume that we have given an object \( V \in C \) such that \( \wedge^{g} V \otimes \wedge^{g} V \) is invertible. If \( X \) is an object we set \( r_{X} := \text{rank}(X) \), so that \( r_{X \otimes V} \in \{ \pm 1 \} \), and we use the shorthand \( r := r_{V} \).

4.1. Preliminary lemmas. We define
\[
\psi_{i,1}^{V} : \wedge^{i} V \otimes V \overset{\varphi_{i,1}^{V}}{\rightarrow} \wedge^{i+1} V \overset{D_{g,i-1}^{i+1}}{\rightarrow} \wedge^{g-i+1} V \otimes \wedge^{g} V \wedge^{V},
\]
and
\[
\overline{\psi}_{g-i-1}^{V} : \wedge^{g-i} V \otimes \wedge^{g-i-1} V \overset{D_{g,i}^{g-i}}{\rightarrow} \wedge^{g-i+1} V \otimes \wedge^{g-i} V \otimes \wedge^{g} V \otimes \wedge^{V},
\]
\[
D_{g-i}^{g-i} \overline{\psi}_{g-i}^{V} : \wedge^{g-i} V \otimes \wedge^{g-i} V \otimes \wedge^{g-i} V \otimes \wedge^{g-i} V \otimes \wedge^{g} V \otimes \wedge^{V} \overset{1_{V} \otimes \overline{ev}_{\phi}^{g-i}}{\rightarrow} \wedge^{V}.
\]

We may also consider
\[
\psi_{g-i,1}^{V} : \wedge^{g-i} V \otimes \wedge^{g-i+1} V \overset{\varphi_{g-i,1}^{V}}{\rightarrow} \wedge^{g-i+1} V \otimes \wedge^{g-i} V \otimes \wedge^{V},
\]
and
\[
\overline{\psi}_{i-1}^{V} : \wedge^{i-1} V \otimes \wedge^{i} V \overset{D_{g-i-1}^{i}}{\rightarrow} \wedge^{i} V \otimes \wedge^{i-1} V \otimes \wedge^{g-i} V \otimes \wedge^{g-i-1} V \otimes \wedge^{g-i} V \otimes \wedge^{g-i} V.
\]
Lemma 4.1. Setting 
\[
\rho^{g-i}_i := (-1)^{(g-1)} r_{\mathcal{A}^gV} \left( \binom{g}{g-1} \right)^{-1} \left( \frac{g}{g-i} \right)^{-1} \left( \frac{r-1}{g-i} \right) g,
\]
\[
\nu^{g-i}_i := (-1)^{g-i} \text{ and } \nu^{1}_i := (-1)^{(g-i-1)} (g-i)
\]
the following diagram is commutative:
\[
\begin{array}{c}
\lambda^i V \otimes \lambda^g V \otimes V \left( (1_{\lambda V} \otimes \psi_{g-i-1} \otimes (1_{\lambda i V} \otimes \psi_i)) \circ (\tau_{\lambda V, \lambda i V} \otimes 1_{1V}) \right) \\
\varphi_{i,g} \otimes 1_V
\end{array}
\]
\[
\begin{array}{c}
\lambda^i V \otimes \lambda^g V \otimes V \left( (1_{\lambda V} \otimes \psi_{g-i-1} \otimes (1_{\lambda i V} \otimes \psi_i)) \circ (\tau_{\lambda V, \lambda i V} \otimes 1_{1V}) \right)
\end{array}
\]
\[
\begin{array}{c}
\rho^{g-i}_i \tau_{\lambda V, \lambda^g V} \circ (i_{\lambda V} \otimes 1_V)
\end{array}
\]
\[
\begin{array}{c}
\nu^{g-i}_i \tau_{\lambda V, \lambda^g V} \circ (i_{\lambda V} \otimes 1_V)
\end{array}
\]
\[
\begin{array}{c}
V \otimes \lambda^g V
\end{array}
\]
\[
\begin{array}{c}
V \otimes \lambda^g V \otimes V \left( (1_{\lambda V} \otimes \psi_{g-i-1} \otimes (1_{\lambda i V} \otimes \psi_i)) \circ (\tau_{\lambda V, \lambda i V} \otimes 1_{1V}) \right)
\end{array}
\]
\[
\begin{array}{c}
V \otimes \lambda^g V
\end{array}
\]

Proof. Set
\[
a := (-1)^{g-i} \cdot (\varphi_{g-i, g-i-1} \otimes 1_{\lambda^g V}) \circ (D^{i,g} \otimes 1_{\lambda i-1 V} \otimes \lambda^g V) \circ \left( 1_{\lambda^i V} \otimes \varphi_{g-i-1} \right)
\]
\[
b := (-1)^{i(g-i-1)} (g-i) \cdot (\varphi_{g-i, g-i-1} \otimes 1_{\lambda^g V}) \circ (D^{g-i,g} \otimes 1_{\lambda g-i V} \otimes \lambda^g V) \circ \left( 1_{\lambda^g V} \otimes D^{i,g+1} \right)
\]
\[
\phi := \lambda V \circ \psi_{g-i} \otimes 1_{\lambda^g V} \circ (D^{g-i,g} \otimes 1_{\lambda^g V}) \circ \left( \tau_{\lambda V, \lambda^g V} \otimes 1 V \right)
\]

Then we have \((\varphi_{i,g} \otimes 1_{\lambda^g V}) \circ \left( 1_{\lambda^i V} \otimes \psi_{g-i} \right) = \phi \circ a\) and \((\psi_{g, g-i} \otimes 1_{\lambda^g V}) \circ \left( 1_{\lambda^g V} \otimes \psi_{g-i} \right) = \phi \circ b\). With these notations, it follows from Proposition 2.3 that we have, setting
\[
\rho := \lambda_{\lambda^g V} g^{g-i} \left( \frac{g}{g-i} \right)^{-1} \left( \frac{g}{g-i} \right)
\]
\[
a + b = \rho \cdot \left( 1_{\lambda^g V} \otimes \lambda^g V \otimes V \right) \circ (D^{i,g} \otimes \lambda^g V) \circ (\psi_{i,g-1} \circ 1_{\lambda^g V})
\]
Hence we find
\[
(\psi_{i,g} \otimes 1_{\lambda^g V}) \circ \left( 1_{\lambda^i V} \otimes \psi_{g-i} \right) + (\psi_{g, g-i} \otimes 1_{\lambda^g V}) \circ \left( 1_{\lambda^g V} \otimes \psi_{g-i} \right) \circ (\tau_{\lambda V, \lambda^g V} \otimes 1 V)
\]
\[
= \rho \cdot (1 V \otimes \psi_{i,g}) \circ (\lambda V \otimes \lambda^g V) \circ \left( 1_{\lambda^g V} \otimes \psi_{g-i} \right) \circ (\tau_{\lambda V, \lambda^g V} \otimes 1 V)
\]
\[
= \rho \cdot \left( 1_{\lambda^i V} \otimes \psi_{i,g-1} \otimes 1_{\lambda^g V} \otimes V \right) \circ (D^{i,g} \otimes \psi_{i,g-1} \otimes 1_{\lambda^g V}) \circ \left( 1_{\lambda^g V} \otimes \lambda^g V \otimes 1 V \right)
\]
\[
= \rho \cdot (1 V \otimes \psi_{i,g}) \circ (\lambda V \otimes \lambda^g V) \circ (\tau_{\lambda V, \lambda^g V} \otimes 1 V)
\]
\[
= \rho \cdot \left( 1_{\lambda^i V} \otimes \psi_{i,g-1} \otimes 1_{\lambda^g V} \otimes V \right) \circ (D^{i,g} \otimes \psi_{i,g-1} \otimes 1_{\lambda^g V}) \circ \left( 1_{\lambda^g V} \otimes \lambda^g V \otimes 1 V \right)
\]
\[
= \rho \cdot (1 V \otimes 1_{\lambda^g V}) \circ (\psi_{i,g}) \circ (\tau_{\lambda V, \lambda^g V} \otimes 1 V)
\]
\[
= \rho \cdot (1 V \otimes 1_{\lambda^g V}) \circ (\psi_{i,g}) \circ (\tau_{\lambda V, \lambda^g V} \otimes 1 V)
\]
\[
= \rho \cdot (1 V \otimes 1_{\lambda^g V}) \circ (\psi_{i,g}) \circ (\tau_{\lambda V, \lambda^g V} \otimes 1 V)
\]
\[
= \rho \cdot (1 V \otimes 1_{\lambda^g V}) \circ (\psi_{i,g}) \circ (\tau_{\lambda V, \lambda^g V} \otimes 1 V)
\]
\[
= \rho \cdot (1 V \otimes 1_{\lambda^g V}) \circ (\psi_{i,g}) \circ (\tau_{\lambda V, \lambda^g V} \otimes 1 V)
\]

\[
\begin{array}{c}
\text{3The morphism denoted by } \varphi_{g-i, g-i-1} \text{ (resp. } \varphi_{i,g-i-1} \text{) in Proposition 2.3 is the one here denoted by } \varphi_{g-i, g-i-1} \otimes 1_{\lambda^g V} \text{ (resp. } \varphi_{i,g-i-1} \otimes 1_{\lambda^g V} \text{).}
\end{array}
\]

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It follows from Theorem 2.1 (1) that we have, setting $\mu$, the following diagram is commutative:

$$
(1_V \otimes ev_{V,V \cdot a}) \circ (D_{g-1,g} \otimes 1_{\Lambda^g V \cdot V}) \circ D^{1,g} = (-1)^{(g-1)} \begin{pmatrix} g \\ g-1 \end{pmatrix} (r-1) \begin{pmatrix} r-i \\ i \end{pmatrix} g,
$$

from Theorem 2.1 (1).

We now consider the following morphisms. We have

$$
\psi_{g,i-1}^V : \Lambda^g V \otimes (\Lambda^{g-i-1} V \otimes V)^{\varphi_{g,i-1}-1} \Lambda^{g-1} V \xrightarrow{\varphi_{g,i-1}} \Lambda^i V \otimes \Lambda^g V \otimes V
$$

and

$$
\bar{\psi}_{g,i-1}^V : \Lambda^g V \otimes (\Lambda^{g-i-1} V \otimes V)^{\varphi_{g,i-1}-1} \Lambda^{g-1} V \xrightarrow{\varphi_{g,i-1}} \Lambda^i V \otimes \Lambda^g V \otimes V
$$

On the other hand we have

$$
\phi_{g,i}^V := \Lambda^g i \otimes (\Lambda^{g-i} V \otimes V)^{\varphi_{g,i}-1} \Lambda^{g-1} V \xrightarrow{\varphi_{g,i}} \Lambda^i V \otimes \Lambda^g V \otimes V
$$

The proof of the following result is a bit more involved and we will leave some of the details to the reader.

**Lemma 4.2.** Setting

$$
\rho_{g,i}^V := (-1)^{(g-1)} \begin{pmatrix} g \\ g-1 \end{pmatrix} (r-1) \begin{pmatrix} r-i \\ i \end{pmatrix} \begin{pmatrix} r+i-g \\ g \end{pmatrix} g,
$$

$$
\nu_{g,i}^V := (-1)^{(i+1)(g-i)} r_{\Lambda^g V} \begin{pmatrix} g \\ i \end{pmatrix} (r+i-g) i
$$

and

$$
\nu_{g,i}^{g-i} := (-1)^i \begin{pmatrix} g \\ g-i \end{pmatrix} (r+i-g) (g-i),
$$

the following diagram is commutative:

$$
\Lambda^g V \otimes \Lambda^i V \otimes V \xrightarrow{\varphi_{g,i}^V = (\Lambda_g V \otimes \Lambda_{g-i} V \otimes V \otimes V)^{\varphi_{g,i}}(\tau_{g-i, V, \Lambda} \otimes 1_V)} \Lambda^g V \otimes \Lambda^i V \otimes \Lambda^g V \otimes V \xrightarrow{\varphi_{g,i-1}^V = (\Lambda^g V \otimes \Lambda_{g-i} V \otimes 1_V \otimes V)^{\varphi_{g,i}}(\tau_{g-i, V, \Lambda} \otimes 1_V)} \Lambda^g V \otimes \Lambda^i V \otimes \Lambda^g V \otimes V
$$

**Proof.** By definition

$$
\psi_{g,i-1}^V \circ (\Lambda^g V \otimes \varphi_{g,i-1}) = D^{g-1,g} \circ \varphi_{g,i-1} \circ (\Lambda_g V \otimes \Lambda_{g-i} V \otimes \Lambda^g V \otimes 1_V \otimes V)^{\varphi_{g,i-1}}
$$

and

$$
\psi_{g,i-1}^V \circ (\Lambda^g V \otimes (D_{g-1,g} \otimes 1_{\Lambda^g V \otimes V})) = D^{g-1,g} \circ \varphi_{g,i-1} \circ (\Lambda_g V \otimes \Lambda_{g-i} V \otimes \Lambda^g V \otimes 1_V \otimes V)^{\varphi_{g,i-1}}
$$

It follows from Theorem 2.1 (1) that we have, setting $\mu_{g,i} := (-1)^{(g-i)} (r+i-g)^{1-i}$ and $\mu_{i,g} := (-1)^{(g-i)} (r+i-g)^{1-i}$:

$$
(-1)^{(g-i)} \mu_{g,i} \cdot 1_{\Lambda^g V} = (1_{\Lambda^g V} \otimes ev_{V,V \cdot a}^{g,\tau}) \circ (D_{g-1,g} \otimes 1_{\Lambda^g V \otimes V}) \circ D^{g-1,g},
$$

$$
(-1)^{(g-i)} \mu_{i,g} \cdot 1_{\Lambda^g V} = (1_{\Lambda^g V} \otimes ev_{V,V \cdot a}^{g,\tau}) \circ (D_{g-1,g} \otimes 1_{\Lambda^g V \otimes V}) \circ D^{1,g}.
$$

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Inserting (38) in the definition (39), one checks that:

\[
(\mu_{g,i} \cdot \psi_{g-i,1} \circ (1_{\lambda^g V^V} \otimes \bar{\psi}_{g,i}) = \mu_{g,i} \cdot \psi_{g-i,1} \circ D^{g-1} \circ D^{i,g} \circ 1_{V^V},
\]

where

\[
a := (-1)^{g-i} \cdot \left( \varphi_{g,i-1} \otimes 1_{\lambda^g V^V} \otimes \lambda^g V^V \otimes \lambda^g V^V \right) \circ (D_{g-1} \otimes D_{i,g} \otimes 1_{\lambda^g V^V}) \circ \left( \varphi_{g,i-1} \otimes 1_{\lambda^g V^V} \otimes \lambda^g V^V \otimes \lambda^g V^V \right)
\]

Similarly, inserting (39) in the definition (37), one finds:

\[
(\mu_{g,i} \cdot \psi_{g-i,1} \circ (1_{\lambda^g V^V} \otimes \bar{\psi}_{g,i}) = \mu_{g,i} \cdot \psi_{g-i,1} \circ D^{g-1} \circ D^{i,g} \circ 1_{V^V},
\]

where

\[
b := (-1)^{g-i} \cdot \left( \varphi_{g,i-1} \otimes 1_{\lambda^g V^V} \otimes \lambda^g V^V \otimes \lambda^g V^V \right) \circ (D_{g-1} \otimes D_{i,g} \otimes 1_{\lambda^g V^V}) \circ \left( \varphi_{g,i-1} \otimes 1_{\lambda^g V^V} \otimes \lambda^g V^V \otimes \lambda^g V^V \right)
\]

and we have used a similar commutative diagram in the last equality.

By Proposition 2.3 in (39) we have, setting \( \rho := r_{\lambda^g V^V} \circ (g_{g-1})^{-1} (r_{g-1}) \),

\[
\rho \cdot (D_{g-1} \otimes \varphi_{g,i-1} \circ \lambda^g V^V \otimes \lambda^g V^V) = a_{0} + b_{0},
\]

where

\[
a_{0} := (-1)^{g-i} \cdot \left( \varphi_{g,i-1} \otimes 1_{\lambda^g V^V} \otimes \lambda^g V^V \right) \circ (1_{\lambda^g V^V} \otimes \lambda^g V^V \otimes \lambda^g V^V)
\]

Consider the morphism

\[
\tau_{(235)} : \Lambda^g V^V \otimes \Lambda^{g-i} V^V \otimes V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \rightarrow \Lambda^i V^V \otimes \Lambda^{g-i} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes V^V
\]

attached to the permutation (235). After a tedious computation one can verify the following relations:

\[
\tau(35) \circ a \circ \tau(235) = \tau(34) \circ (a_{0} \otimes 1_{\lambda^g V^V} \otimes \lambda^g V^V),
\]

\[
b \circ \tau(235) = \tau(34) \circ (b_{0} \otimes 1_{\lambda^g V^V} \otimes \lambda^g V^V),
\]

\[
\tau(34) \circ (D_{g-1} \otimes \varphi_{g,i-1}^{13}) \circ \lambda^g V^V \otimes \Lambda^{g-i} V^V \otimes \lambda^g V^V \otimes \Lambda^g V^V \otimes V^V \circ \tau(235) = \tau(34) \circ (c_{0} \otimes 1_{\lambda^g V^V} \otimes \lambda^g V^V)
\]

Thanks to (13), (14) and (15), the equality (12) gives

\[
\rho \cdot \tau(345) \circ (D_{g-1} \otimes \varphi_{g,i-1}^{13}) \circ \lambda^g V^V \otimes \Lambda^{g-i} V^V \otimes \lambda^g V^V \otimes \Lambda^g V^V \otimes V^V = \tau(35) \circ a + b.
\]

Finally, we need to remark that we have the following commutative diagram (by a computation of the involved permutations):

\[
\begin{array}{ccc}
\Lambda^{g-1} V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V & \xrightarrow{\tau(235)} & \Lambda^{g-1} V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \\
1_{\Lambda^{g-1} V} \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V & \xrightarrow{\tau(34)} & 1_{\Lambda^{g-1} V} \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \\
\Lambda^{g-1} V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V & \xrightarrow{\tau(35)} & \Lambda^{g-1} V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \otimes \Lambda^{g} V^V \\
\end{array}
\]
Since $\wedge^g V^\vee$ is invertible, it follows from [De3, 7.2 Lemma] that we have $\tau_{\wedge^g V^\vee, \wedge^g V^\vee} = r_{\wedge^g V}$ in the above diagram, implying that $\tau_{(35)} = r_{\wedge^g V}$ as well. Hence (40) becomes

$$\rho \cdot \tau_{(345)} \circ (D_{1,g} \otimes \varphi_{1,g-1}^{13}) \circ \tau_{\wedge^g V^\vee \otimes \wedge^g V^\vee} = r_{\wedge^g V} \cdot a + b. \quad (47)$$

We can now compute:

$$(-1)^{i(g-i)} (-1)^{g-i_i} r_{\wedge^g V} \mu_{i,g} \cdot \psi_{g-i,i-1} \circ \left(1_{\wedge^g V} \otimes \psi_{i-1}^V\right)$$

$$+ (-1)^{i(g-i)} (-1)^{(g-i-1)} \mu_{g-i,g} \cdot \psi_{V,g}^V \circ 1_{\wedge^g V} \otimes \psi_{g-i-1}^V \quad (48)$$

$$\circ \left(\tau_{\wedge^g V^\vee \otimes \wedge^g V} \otimes 1_{V^\vee}\right) \quad \text{(by 40 and 41)}$$

$$= D^{g-1,g} \circ \left(1_{\wedge^g V^\vee} \otimes ev_{V,\wedge^g V}^{g,\tau} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ r_{\wedge^g V} \circ a \circ (D^{g-1,g} \otimes D^{i,g} \otimes 1_{V^\vee})$$

$$+ D^{g-1,g} \circ \left(1_{\wedge^g V^\vee} \otimes ev_{V,\wedge^g V}^{g,\tau} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ b \circ (D^{g-1,g} \otimes D^{i,g} \otimes 1_{V^\vee})$$

$$= D^{g-1,g} \circ \left(1_{\wedge^g V^\vee} \otimes ev_{V,\wedge^g V}^{g,\tau} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ (r_{\wedge^g V} \cdot a + b) \circ (D^{g-1,g} \otimes D^{i,g} \otimes 1_{V^\vee}) \quad \text{(by 47)}$$

$$= \rho \cdot D^{g-1,g} \circ \left(1_{\wedge^g V^\vee} \otimes ev_{V,\wedge^g V}^{g,\tau} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ \tau_{(345)} \circ (D_{1,g} \otimes \varphi_{1,g-1}^{13})$$

$$\circ \left(\tau_{\wedge^g V^\vee \otimes \wedge^g V^\vee} \otimes \wedge^g V^\vee \otimes \wedge^g V^\vee\right) \circ (D^{g-1,g} \otimes D^{i,g} \otimes 1_{V^\vee}) \quad \text{(by a formal computation)}$$

$$= \rho \cdot D^{g-1,g} \circ \left(1_{\wedge^g V^\vee} \otimes ev_{V,\wedge^g V}^{g,\tau} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ (D_{1,g} \otimes 1_{\wedge^g V^\vee} \otimes 1_{\wedge^g V^\vee}) \circ \tau_{(234)}$$

$$\circ (1_{V^\vee} \otimes \varphi_{1,g-1}^{13}) \circ (1_{V^\vee} \otimes D^{g-1,g} \otimes D^{i,g}) \circ \tau_{\wedge^g V^\vee \otimes \wedge^g V^\vee} \quad (48)$$

We remark that we have, by definition, $r_{\wedge^g V} = ev_{V,\wedge^g V}^{g,\tau} \circ \tau_{\wedge^g V^\vee, \wedge^g V^\vee} \circ C_{\wedge^g V^\vee}$ and, since $\wedge^g V^\vee$ is invertible, $r_{\wedge^g V} = r_{\wedge^g V}^{-1}$ and we deduce $\left(ev_{V,\wedge^g V}^{g,\tau}\right)^{-1} = r_{\wedge^g V} \cdot \tau_{\wedge^g V^\vee, \wedge^g V^\vee} \circ C_{\wedge^g V^\vee}$. This gives the first of the subsequent equalities, while the second follows from a standard property of the Casimir element:

$$\left(1_{\wedge^g V^\vee} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ \left(ev_{V,\wedge^g V}^{g,\tau}\right)^{-1} \otimes 1_{\wedge^g V^\vee}$$

$$= r_{\wedge^g V} \cdot \left(1_{\wedge^g V^\vee} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ (\tau_{\wedge^g V^\vee, \wedge^g V^\vee} \otimes 1_{\wedge^g V^\vee}) \circ (C_{\wedge^g V^\vee} \otimes 1_{\wedge^g V^\vee})$$

$$= r_{\wedge^g V} \cdot 1_{\wedge^g V^\vee} \quad (49)$$

Thanks to Theorem 2.1, we know that $(1_{V^\vee} \otimes ev_{V,\wedge^g V}^{g,\tau}) \circ (D^{g-1,g} \otimes 1_{\wedge^g V^\vee}) \circ D_{1,g} = (-1)^{(g-1)} \mu_{g-1,g}$ with $\mu_{g-1,g} := \left(\frac{1}{g-1}\right)^{-1} \left(\frac{1}{g-1}\right)$. Employing this relation in the second of the subsequent equalities, we find

$$D^{g-1,g} \circ \left(1_{\wedge^g V^\vee} \otimes ev_{V,\wedge^g V}^{g,\tau} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ (D_{1,g} \otimes 1_{\wedge^g V^\vee} \otimes 1_{\wedge^g V^\vee})$$

$$= \left(1_{V^\vee} \otimes \wedge^g V^\vee \otimes ev_{V,\wedge^g V}^{g,\tau} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ (D^{g-1,g} \otimes 1_{\wedge^g V^\vee} \otimes 1_{\wedge^g V^\vee} \otimes 1_{\wedge^g V^\vee})$$

$$\circ (D_{1,g} \otimes 1_{\wedge^g V^\vee} \otimes 1_{\wedge^g V^\vee} \otimes 1_{\wedge^g V^\vee})$$

$$= \left(-1\right)^{(g-1)} \mu_{g-1,g} \cdot \left(1_{V^\vee} \otimes \wedge^g V^\vee \otimes ev_{V,\wedge^g V}^{g,\tau} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ (1_{V^\vee} \otimes \left(ev_{V,\wedge^g V}^{g,\tau}\right)^{-1} \otimes 1_{\wedge^g V^\vee} \otimes 1_{\wedge^g V^\vee})$$

$$= \left(-1\right)^{(g-1)} \mu_{g-1,g} \cdot \left(1_{V^\vee} \otimes \wedge^g V^\vee \otimes ev_{V,\wedge^g V}^{g,\tau} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ (1_{V^\vee} \otimes \left(ev_{V,\wedge^g V}^{g,\tau}\right)^{-1} \otimes 1_{\wedge^g V^\vee} \otimes ev_{V,\wedge^g V}^{g,\tau})$$

$$= \left(-1\right)^{(g-1)} \mu_{g-1,g} \cdot \left(1_{V^\vee} \otimes \wedge^g V^\vee \otimes ev_{V,\wedge^g V}^{g,\tau} \otimes ev_{V,\wedge^g V}^{g,\tau}\right) \circ (1_{V^\vee} \otimes \left(ev_{V,\wedge^g V}^{g,\tau}\right)^{-1} \otimes 1_{\wedge^g V^\vee} \otimes ev_{V,\wedge^g V}^{g,\tau})$$

$$\circ (1_{V^\vee} \otimes \wedge^g V^\vee \otimes ev_{V,\wedge^g V}^{g,\tau}) \quad \text{(by 49)}$$

$$= \left(-1\right)^{(g-1)} \mu_{g-1,g} \cdot r_{\wedge^g V} \cdot 1_{V^\vee} \otimes \wedge^g V^\vee \otimes ev_{V,\wedge^g V}^{g,\tau}.$$
We also have, thanks to the relation $\varphi_{13}^{\Lambda^g V \wedge V} \circ (D^{g-i,g} \otimes D^{i,g}) = \mu_{i,g} \cdot i_{\Lambda^g V} \circ \varphi_{g-i,i}$ with $\mu_{i,g} := (g)^{-1} (r+i-g)$ arising from Theorem 2.1 (2):

\[
\left(1_{V^\otimes \Lambda^g V \wedge V} \otimes ev_{V^\otimes \Lambda^g V \wedge V} \right) \circ \tau(234) \circ \left(1_{V^\otimes \varphi_{i,g-i}^{13}} \circ (1_{V^\otimes D^{g-i,g} \otimes D^{i,g}})
\right)
\]

\[
\left(1_{V^\otimes ev_{V^\otimes \Lambda^g V \wedge V}} \otimes 1_{\Lambda^g V \wedge V} \right) \circ \left(1_{V^\otimes \varphi_{i,g-i}^{13}} \circ (1_{V^\otimes D^{g-i,g} \otimes D^{i,g}})
\right)
\]

\[
\left(1_{V^\otimes \varphi_{13}^{\Lambda^g V \wedge V}} \circ (1_{V^\otimes D^{g-i,g} \otimes D^{i,g}}) = \mu_{i,g} \cdot (1_{V^\otimes i_{\Lambda^g V}} \circ (1_{V^\otimes \varphi_{g-i,i}}) \right).
\]

Inserting (50) and (51) in (48) gives the claim after a small computation. □

4.2. Laplace and Dirac operators.

We now specialize the above discussion to the case $g = 2i$, i.e. $i = g - i$, and we simply write $L$ for the invertible object $\Lambda^g V \wedge V$ and set $L^{-1} := \Lambda^g V \wedge V$. We write $\text{Alt}^n (M) := \wedge^n M$ and $\text{Sym}^n (M) := \vee^n M$ when $M$ is an alternating power of $V$. Attached to the multiplication map $\wedge^i V \otimes \wedge^j V \xrightarrow{\varphi_{i,j}} \wedge^{i+j} V$ there are the Laplace operators

\[
\Delta^n_{i,g^V \varphi_{i,i}^{i,g}} : \text{Alt}^n (\wedge^i V) \to \text{Alt}^{n-2} (\wedge^i V) \otimes L,
\]

\[
\Delta^n_{i,g^V \varphi_{i,i}^{i,g}} : \text{Sym}^n (\wedge^i V) \to \text{Sym}^{n-2} (\wedge^i V) \otimes L
\]

and, since $\varphi_{i,i} \circ \tau\wedge V \wedge V = (-1)^i \varphi_{i,i}$, by Lemma 3.1 we have $\Delta^n_{i,g^V \varphi_{i,i}^{i,g}} = 0$ (resp. $\Delta^n_{i,g^V \varphi_{i,i}^{i,g}} = 0$) when $i$ is odd (resp. is even). Hence, we set $\Delta^n := \Delta^n_{i,g^V \varphi_{i,i}^{i,g}}$ (resp. $\Delta^n := \Delta^n_{i,g^V \varphi_{i,i}^{i,g}}$) when $i$ is odd (resp. even).

Looking at the pairings defined before Lemma 4.1 we note that we have $\psi_{V,i} = \psi_{g-i,1}^V$ and $\phi_{g-i,1}^{V} = \phi_{i,i-1}^{V}$, while looking at the pairings defined before 4.2 we remark the equalities $\psi_{i,g-i-1}^V = \psi_{V,g-i-1}^V$ and $\phi_{g-i,1}^{V} = \phi_{i,i-1}^{V}$.

Suppose first that $i$ is odd. Then we define the following Dirac operators, for every integer $n \geq 1$:

\[
\partial_i^n := \partial^n_{\psi_{k,i-1}^{V}} : \text{Alt}^n (\wedge^i V) \otimes V \rightarrow \text{Alt}^{n-1} (\wedge^i V) \otimes \wedge^{i-1} V \wedge V \otimes L,
\]

\[
\partial_{i-1}^n := \partial^n_{\phi_{V,i-1}^{V}} : \text{Alt}^n (\wedge^i V) \otimes \wedge^{i-1} V \rightarrow \text{Alt}^{n-1} (\wedge^i V) \otimes V,
\]

\[
\partial_i^n := \partial^n_{\phi_{V,i-1}^{V}} : \text{Alt}^n (\wedge^i V) \otimes \wedge^{i-1} V \rightarrow \text{Alt}^{n-1} (\wedge^i V) \otimes V \wedge V \otimes L,
\]

\[
\partial_{i-1}^n := \partial^n_{\phi_{V,i-1}^{V}} : \text{Alt}^n (\wedge^i V) \otimes V \rightarrow \text{Alt}^{n-1} (\wedge^i V) \otimes \wedge^{i-1} V.
\]

Theorem 4.3. Suppose that $i$ is odd, so that $\Delta^n : \text{Alt}^n (\wedge^i V) \to \text{Alt}^{n-2} (\wedge^i V) \otimes L$ and set

\[
\rho^i := (-1)^{i+1} r_L \left( \begin{array}{c} g \\ g-1 \end{array} \right)^{-1} \left( \begin{array}{c} g \\ i \end{array} \right)^{-1} \left( \begin{array}{c} r+1 \\ g-1 \end{array} \right) \left( \begin{array}{c} r-i \\ i \end{array} \right) \frac{g}{i}.
\]

(1) The following diagram is commutative:

\[
\text{Sym}^n (\wedge^i V) \otimes V \rightarrow \text{Sym}^{n-1} (\wedge^i V) \otimes \wedge^{i-1} V \wedge V \otimes L
\]

\[
\Delta^n \otimes V \rightarrow \text{Sym}^{n-2} (\wedge^i V) \otimes V \otimes L.
\]
(2) When \( r_L = 1 \), the first of the following diagrams is commutative and it becomes equivalent to the second diagram when we further assume that \( (r_i^{-1}) \in \text{End}(I) \) is a non-zero divisor:

\[
\begin{tikzcd}
\Sym^n (\Lambda^i V) \otimes V^\vee 
\arrow{r}{\sigma_i} & \Sym^{n-1} (\Lambda^i V) \otimes \Lambda^{i-1} V \\
\Delta^n \otimes V^\vee
\end{tikzcd}
\]

\[
\begin{tikzcd}
\Sym^{n-2} (\Lambda^i V) \otimes L \otimes V^\vee 
\arrow{r}{\phi^{n-1}_{i-1}} & \Sym^{n-2} (\Lambda^i V) \otimes V^\vee \otimes L, \\
\Delta^n \otimes V^\vee
\end{tikzcd}
\]

\[
\begin{tikzcd}
\Sym^n (\Lambda^i V) \otimes V^\vee 
\arrow{r}{\sigma_i} & \Sym^{n-1} (\Lambda^i V) \otimes \Lambda^{i-1} V \\
\Delta^n \otimes V^\vee
\end{tikzcd}
\]

\[
\begin{tikzcd}
\Sym^{n-2} (\Lambda^i V) \otimes L \otimes V^\vee 
\arrow{r}{\phi^{n-1}_{i-1}} & \Sym^{n-2} (\Lambda^i V) \otimes V^\vee \otimes L. \\
\Delta^n \otimes V^\vee
\end{tikzcd}
\]

(3) Suppose that \( L \simeq L^{\otimes 2} \) for some invertible object \( L \), that \( r_{\Lambda^i V} < 0 \) (see definition 3.6) and that \( V \) has alternating rank \( g \). Then there are morphisms

\[
s_{i-1}^{n-2} : \text{Alt}^{n-2} (\Lambda^i V) \otimes L \to \text{Alt}^n (\Lambda^i V) \quad \text{for } n \geq 2,
\]

\[
s_{i-1}^{n-1} : \text{Alt}^{n-1} (\Lambda^i V) \otimes V \to \text{Alt}^n (\Lambda^i V) \otimes \Lambda^{i-1} V^\vee \quad \text{for } n \geq 1,
\]

\[
s_{i-1} : \text{Alt}^{n-1} (\Lambda^i V) \otimes V^\vee \otimes L \to \text{Alt}^n (\Lambda^i V) \otimes \Lambda^{i-1} V \quad \text{for } n \geq 1
\]

such that

\[
\Delta^n \circ s_{i-1}^{n-2} = 1_{\text{Alt}^{n-2}(\Lambda^i V) \otimes L}, \quad \sigma_i \circ s_{i-1}^{n-1} = 1_{\text{Alt}^{n-1}(\Lambda^i V) \otimes V^\vee}.
\]

In particular, the following objects exist:

\[
\ker (\Delta^n) \subset \text{Alt}^n (\Lambda^i V),
\]

\[
\ker (\phi^{n-1}_{i-1}) \subset \text{Alt}^n (\Lambda^i V) \otimes \Lambda^{i-1} V^\vee,
\]

\[
\ker (\bar{\phi}^{n-1}_{i-1}) \subset \text{Alt}^n (\Lambda^i V) \otimes \Lambda^{i-1} V.
\]

**Proof.** (1-2) Looking at the quantities \( \nu^g_{i-1} \) and \( \nu^{i-1}_{i} \) from Lemma 4.2 when \( i = g - i \), we see that \( \nu^g_{i-1} = (-1)^i \nu^{i-1}_{i} \) (resp. \( \nu^{i-1}_{i} = (-1)^{i-1} r_{\Lambda^i V} \cdot \nu^{g-1}_{g-1} \)). Since \( i \) is odd, it follows that \( \nu^g_{i-1} = -\nu^{i-1}_{i} \) (resp. \( \nu^{i-1}_{i} = -\nu^{g-1}_{g-1} \)). We have that \( i_{\Lambda^i V} \circ \nu_{i-1} \) is alternating, so that we may apply Lemma 3.3 to deduce the claimed commutativity in (1) (resp. the first commutative diagram in (2) when \( r_L = r_{\Lambda^i V} = 1 \)): we have indeed \( \nu^{g-1}_{g-1} \in \mathbb{Q}^\times \) and

\[
\rho^i_{\Lambda^i V} / \nu^{i-1}_{i} = \rho^i \quad \text{resp. } \rho^i_{\Lambda^i V} = -r_{\Lambda^i V} \left( g \overline{\Lambda}^{-1} \left( r - i \overline{i} \right) \right) i \nu^{i-1}_{i} \quad \text{and } \nu^{i-1}_{i} = r_{\Lambda^i V} \left( g \overline{\Lambda}^{-1} \left( r - i \overline{i} \right) \right) i
\]

and the commutativity of (2) is deduced simplifying by \( r_{\Lambda^i V} \left( g \overline{\Lambda}^{-1} \right) i \). If \( (r_i^{-1}) \in \text{End}(I) \) is a non-zero divisor we may further simplify to get the second commutative diagram in (2).

(3) Indeed \( L \simeq L^{\otimes 2} \) implies \( r_L = 1 \) and, since \( V \) has alternating rank \( g \), by Corollary 2.2 \( 1_{\Lambda^i V} \circ \nu_{i-1} \) is a perfect alternating pairing. Since \( r_{\Lambda^i V} < 0 \), Lemma 3.12 gives the existence of \( s_{i-1}^{n-2} \) and ker(\( \Delta^n \)). We also remark that, since \( (r_i^{-1}) = (r_i^{-1}) \in \text{End}(I) \) and \( (r_i^{-1}) \in \text{End}(I) \) is invertible (once again because \( V \) has alternating rank \( g \)), it follows that \( \pm \nu^{g-1}_{g-1} \) is invertible, that \( \alpha := \frac{g}{2} \cdot \left( 1_{\text{Alt}^{n-2}(\Lambda^i V) \otimes \Lambda^i V} \right) \) is an isomorphism and, hence, that \( f := \alpha \circ (\Delta^n \otimes 1_V) \) has a section \( s := (s_{i-1}^{n-2} \otimes 1_V) \circ \alpha^{-1} \) such that

\[
f \circ s = \alpha \circ (\Delta^n \otimes 1_V) \circ s_{i-1}^{n-2} \otimes 1_V \circ \alpha^{-1} = 1_{\text{Alt}^{n-2}(\Lambda^i V) \otimes \Lambda^i V}.
\]

Similarly \( f := \left( -\frac{g}{2} \cdot 1_{\text{Alt}^{n-2}(\Lambda^i V) \otimes V^\vee \otimes L} \right) \circ (\Delta^n \otimes 1_V) \) has a section. We can now apply the following simple remark to the commutative diagram in (1) (resp. the second commutative diagram in (2)). Suppose that we have given

\[
f : X \xrightarrow{f_1} Y \xrightarrow{f_2} Z
\]

and that \( s : Z \to X \) is a morphism such that \( f \circ s = 1_Z \). Then, setting \( s_2 := f_1 \circ s \), we see that

\[
f_2 \circ s_2 = f_2 \circ f_1 \circ s = f \circ s = 1_Z,
\]
implying that \( f_2 \) has a section. But then there is an associated idempotent \( e_2 := s_2 \circ f_2 \) and \( \ker(f_2) = \ker(e_2) \) exists because \( V \) is pseudo-abelian. This gives the existence of a section of \( \partial^n_{i-1} \otimes 1_L \), hence of \( \partial^n_{i-1} \) and \( \ker(\partial^n_{i-1}) \) because \( L \) is invertible, and of \( s^{n-1}_{\partial_{i-1}} \) and \( \ker(\partial_{i-1}) \).

Suppose now that \( i \) is even. Then we define the following Dirac operators, for every integer \( n \geq 1 \):

\[
\begin{align*}
\partial^n_i := \partial^n_{\psi_{V,i+1,s}}: & \quad \text{Sym}^n(\Lambda^i V) \otimes V \to \text{Sym}^{n-1}(\Lambda^i V) \otimes \Lambda^{i-1} V^\vee \otimes L, \\
\partial^n_{i-1} := \partial^n_{\psi_{V,i-1,s}}: & \quad \text{Sym}^n(\Lambda^i V) \otimes \Lambda^{i-1} V^\vee \to \text{Sym}^{n-1}(\Lambda^i V) \otimes V, \\
\partial^n_{n-1} := \partial^n_{\psi_{V,i-1,s}}: & \quad \text{Sym}^n(\Lambda^i V) \otimes \Lambda^{i-1} V \to \text{Sym}^{n-1}(\Lambda^i V) \otimes V^\vee \otimes L, \\
\partial^n_1 := \partial^n_{\psi_{V,i+1,s}}: & \quad \text{Sym}^n(\Lambda^i V) \otimes V^\vee \to \text{Sym}^{n-1}(\Lambda^i V) \otimes \Lambda^{i-1} V.
\end{align*}
\]

**Theorem 4.4.** Suppose that \( i \) is even, so that \( \Delta^n : \text{Sym}^n(\Lambda^i V) \to \text{Sym}^{n-2}(\Lambda^i V) \otimes L \) and set

\[
\rho_i := (-1)^{i+1} r_L \left( \frac{g}{g-1} \right)^{-1} \left( \frac{g}{i} \right)^{-1} \left( \frac{r-1}{g-1} \right)^{-1} \left( \frac{r-i}{i} \right) g.
\]

(1) The following diagram is commutative:

\[
\begin{array}{ccc}
\text{Sym}^n(\Lambda^i V) \otimes V & \xrightarrow{\partial^n_i} & \text{Sym}^{n-1}(\Lambda^i V) \otimes \Lambda^{i-1} V^\vee \otimes L \\
\Delta^n \otimes 1_V & \downarrow & \delta^n_{\partial_{i-1}} \\
\text{Sym}^{n-2}(\Lambda^i V) \otimes L \otimes V & \xrightarrow{\delta^n_{\partial_{i-1}}} & \text{Sym}^{n-2}(\Lambda^i V) \otimes V \otimes L.
\end{array}
\]

(2) When \( r_L = 1 \), the first of the following diagrams is commutative and it becomes equivalent to the second diagram when we further assume that \((\tau_{i-1}^\tau i) \in \text{End}(1)\) is a non-zero divisor:

\[
\begin{array}{ccc}
\text{Sym}^n(\Lambda^i V) \otimes V^\vee & \xrightarrow{\nu_i} & \text{Sym}^{n-1}(\Lambda^i V) \otimes \Lambda^{i-1} V \otimes L \\
\Delta^n \otimes 1_{V^\vee} & \downarrow & \nu_i \\
\text{Sym}^{n-2}(\Lambda^i V) \otimes L \otimes V^\vee & \xrightarrow{\nu_i} & \text{Sym}^{n-2}(\Lambda^i V) \otimes V^\vee \otimes L. \\
\end{array}
\]

(3) Suppose that \( L \simeq L \otimes 2 \) for some invertible object \( L \), that \( r_{\Lambda^i V} > 0 \) (see definition 3.6) and that \( V \) has alternating rank \( g \). Then there are morphisms

\[
\begin{align*}
s^n_{\partial_{i-1}} & : \text{Sym}^n(\Lambda^i V) \otimes L \to \text{Sym}^n(\Lambda^i V) \quad \text{for } n \geq 2, \\
s^{n-1}_{\partial_{i-1}} & : \text{Sym}^{n-1}(\Lambda^i V) \otimes V \to \text{Sym}^n(\Lambda^i V) \otimes \Lambda^{i-1} V^\vee \quad \text{for } n \geq 1, \\
s^{n-1}_{\partial_{i-1}} & : \text{Sym}^{n-1}(\Lambda^i V) \otimes V^\vee \otimes L \to \text{Sym}^n(\Lambda^i V) \otimes \Lambda^{i-1} V \quad \text{for } n \geq 1
\end{align*}
\]

such that

\[
\Delta^n \circ s^{n-2}_{\Delta^n} = 1_{\text{Sym}^{n-2}(\Lambda^i V) \otimes L}, \quad \partial^n_{\partial_{i-1}} \circ s^{n-1}_{\partial_{i-1}} = 1_{\text{Sym}^{n-1}(\Lambda^i V) \otimes V^\vee \otimes L}.
\]

In particular, the following objects exist:

\[
\begin{align*}
\ker(\Delta^n) & \subset \text{Sym}^n(\Lambda^i V), \\
\ker(\Delta^n_{\partial_{i-1}}) & \subset \text{Sym}^n(\Lambda^i V) \otimes \Lambda^{i-1} V^\vee, \\
\ker(\partial^n_{\partial_{i-1}}) & \subset \text{Sym}^n(\Lambda^i V) \otimes \Lambda^{i-1} V.
\end{align*}
\]
Proof. (1-2) As in the proof of Theorem 4.3 we have $\nu_{V}^{g-i,1} = (-1)^{i} \cdot \nu_{V}^{1,1}$ (resp. $\nu_{V}^{i,1} = (-1)^{i} r_{\lambda} V \cdot \nu_{V}^{g-i,1}$). Since $i$ is even, it follows that $\nu_{V}^{g-i,1} = \nu_{V}^{1,1}$ (resp. $\nu_{V}^{i,1} = r_{\lambda} V \cdot \nu_{V}^{g-i,1}$). Then the proof is identical to the proof of Theorem 4.3, noticing that we have once again $\rho_{g-i} / \nu_{V}^{g-i,1} = \rho^{1}$ and $\nu_{V}^{1,1} = r_{\lambda} V_{i}^{-1} (\lambda)^{-1} i \rho^{1}$, justifying the change of sign in the second commutative diagram of (2) with respect to that of Theorem 4.3.

(3) The proof is identical to the proof of Theorem 4.3, noticing that here we need to assume $r_{\lambda} V > 0$ in order to apply Lemma 4.12 because now $i_{\lambda} V \circ \varphi_{i,1}$ is a perfect symmetric pairing. □

5. LAPLACE AND DIRAC OPERATORS FOR THE SYMMETRIC ALGEBRAS

In this section we assume that we have given an object $V \in \mathcal{C}$ such that $\nu_{V}$ is invertible. If $X$ is an object we set $r_X := \text{rank}(X)$, so that $r_{\nu_{V}} \in \{\pm 1\}$, and we use the shorthand $r := r_{V}$.

5.1. Preliminary lemmas. We define

\[
\psi_{i,1}^{V} : i^{V} \otimes V \rightarrow i^{i+1} V D_{g-i}^{1,1} \otimes \nu_{V}^{g-i,1} \otimes \nu_{V}^{g-i,1} \otimes \nu_{V}^{1,1},
\]

and

\[
\tilde{\psi}_{i,g-i-1}^{V} : \nu_{V}^{g-i} \otimes \nu_{V}^{g-i-1} V D_{g-i}^{1,1} \otimes \nu_{V}^{g-i} \otimes \nu_{V}^{g-i} \otimes \nu_{V}^{g-i-1} \otimes \nu_{V}^{1,1} \rightarrow i^{V} \otimes V \nu_{V}^{g-i} \otimes \nu_{V}^{g-i} \otimes \nu_{V}^{1,1}.
\]

We may also consider

\[
\psi_{i-1}^{V} : i^{V} \otimes i^{i-1} V D_{g-i}^{1,1} \otimes \nu_{V}^{g-i} \otimes \nu_{V}^{g-i} \otimes \nu_{V}^{1,1} \rightarrow \nu_{V}^{g-i} \otimes \nu_{V}^{g-i} \otimes \nu_{V}^{1,1} V.
\]

Lemma 5.1. Setting

\[
\rho_{V}^{i,g-i} := r_{\nu_{V}} \frac{g}{g-1} \left( \frac{g}{g-i} \right)^{-1} \left( \frac{r + g - 1}{g - 1} \right) \left( \frac{r + g - 1}{g - i} \right) g,
\]

\[
\nu_{V}^{g-i,1} := i v_{V}^{i,1} := g - i
\]

the following diagram is commutative:

We now consider the following morphisms. We have

\[
\psi_{i,g-i-1}^{V} : i^{V} \otimes i^{i-1} V D_{g-i}^{1,1} \otimes i^{g-i-1} \rightarrow \nu_{V}^{g-i} \otimes \nu_{V}^{g-i} \otimes \nu_{V}^{1,1} \otimes \nu_{V}^{g-i} \otimes \nu_{V}^{1,1} \rightarrow \nu_{V}^{1,1} \otimes \nu_{V}^{1,1} \rightarrow V.
\]

Proof. The proof is just a copy of that of Lemma 4.11 replacing the use of Theorem 2.1 (resp. Proposition 2.3) with Theorem 2.4 (resp. Proposition 2.6). □
On the other hand we have
\[ \psi_{g-i,i-1}^V : \land^g V \otimes \land^{i-1} V \stackrel{g-i}{\to} \land^{g-1} V \to V^V \otimes \land^g V^V \]
and
\[ \overline{\psi}_{i,1} : \land^i V \otimes V^V \stackrel{\overline{g}-i}{\to} \land^{i-1} V \otimes \land^g V^V \otimes V^V \to \land^{g-i-1} V \otimes \land^g V^V \]

**Lemma 5.2.** Setting
\[
\rho_{g,i}^{g-i,i} := \left( \frac{g}{g-1} \right)^{-1} \left( \frac{g}{g-i} \right)^{-1} \left( \frac{g+1}{g-1} \right)^{-1} \left( \frac{r+g-1}{g-1} \right) \left( \frac{r+g-1}{i} \right) g,
\]
\[
\nu_{g,i}^{1,1} := r_{gV} \left( \frac{g}{g-i} \right)^{-1} \left( \frac{r+g-1}{i} \right) i \quad \text{and}
\]
\[
\nu_{g,i}^{g-i,i} := \left( \frac{g}{g-i} \right)^{-1} \left( \frac{r+g-1}{g-i} \right) (g-i),
\]
the following diagram is commutative:
\[
\xymatrix{ \land^{g-i} V \otimes \land^i V \otimes V^V & \land^{g-i} V \otimes \land^{i-1} V \oplus \land^i V \otimes \land^{g-i-1} V \ar[l]_{\varphi_{g-i,i-1} \otimes 1_{V^V}} \ar[d]_{\nu_{g,i}^{g-i,i} \otimes 1_{V^V} \otimes \tau_{\varphi_{g-i,i-1} \otimes 1_{V^V}}} & V^V \otimes \land^g V^V \ar[ll]_{\rho_{g,i}^{g-i,i} \otimes 1_{V^V} \otimes \tau_{\varphi_{g-i,i-1} \otimes 1_{V^V}}} \ar[d]_{\nu_{g,i}^{1,1} \cdot \varphi_{g-i,i-1} \otimes \nu_{g,i}^{g-i,i} \cdot \varphi_{g,i-1}} } \]

**Proof.** Again the proof is a copy of that of Lemma 4.2.

---

### 5.2. Laplace and Dirac operators.

We now specialize the above discussion to the case \( g = 2i \), i.e. \( i = g-i \), and we simply write \( L \) for the invertible object \( \land^g V^V \) and set \( L^{-1} := \land^g V^V \). We write \( \text{Alt}^n(M) := \wedge^n M \) and \( \text{Sym}^n(M) := \land^n M \) when \( M \) is a symmetric power of \( V \). Attached to the multiplication map \( \land^i V \otimes \land^i V \rightarrow \land^i V \otimes \land^i V \otimes \land^i V \otimes \land^i V \) \( L \) there are the Laplace operators
\[
\Delta^n_{i,\nu^iV \circ \varphi_{i,i},a} : \text{Alt}^n(\land^i V) \to \text{Alt}^{n-2}(\land^i V) \otimes L,
\]
\[
\Delta^n_{i,\nu^iV \circ \varphi_{i,i},a} : \text{Sym}^n(\land^i V) \to \text{Sym}^{n-2}(\land^i V) \otimes L
\]
and, since \( \varphi_{i,i} \circ \tau_{\nu^iV \circ \nu^iV} = \varphi_{i,i} \), by Lemma 3.1 we have \( \Delta^n_{i,\nu^iV \circ \varphi_{i,i},a} = 0 \). Hence we will only consider \( \Delta^n := \Delta^n_{i,\nu^iV \circ \varphi_{i,i},a} \).

Looking at the pairings defined before Lemma 5.1, we note that we have \( \psi_{i,1}^V = \psi_{g-i,1}^V \) and \( \overline{\psi}_{g-i,g-i-1} = \overline{\psi}_{i,1-1}^V \), while looking at the pairings defined before 5.2, we remark the equalities \( \psi_{i,g-i-1}^V = \psi_{g-i,i-1}^V \) and \( \overline{\psi}_{g-i,1} = \overline{\psi}_{i,1}^V \).

Then we define the following Dirac operators, for every integer \( n \geq 1 \):
\[
\overline{\delta}_n^i : \text{Sym}^n(\land^i V) \otimes V^V \to \text{Sym}^{n-1}(\land^i V) \otimes \land^{i-1} V^V \otimes L,
\]
\[
\delta_1^i = \delta_{i,1-1}^i : \text{Sym}^n(\land^i V) \otimes \land^{i-1} V \to \text{Sym}^{n-1}(\land^i V) \otimes V^V,
\]
\[
\delta_{i,1}^1 = \delta_{g-i-1,i-1}^i : \text{Sym}^n(\land^i V) \otimes \land^{i-1} V \to \text{Sym}^{n-1}(\land^i V) \otimes \land^{i-1} V^V \otimes L,
\]
\[
\delta_{i,1}^i = \delta_{g-i-1,i-1}^i : \text{Sym}^n(\land^i V) \otimes \land^{i-1} V \to \text{Sym}^{n-1}(\land^i V) \otimes \land^{i-1} V.
\]

In the same way as we have deduced Theorem 4.3 from Lemmas 4.1 and 4.2, the following result can be deduced from Lemmas 5.1 and 5.2

**Theorem 5.3.** Set
\[
\rho_i := r_{\nu^iV} \left( \frac{g}{g-1} \right)^{-1} \left( \frac{g}{g-i} \right)^{-1} \left( \frac{g+1}{g-1} \right) \left( \frac{r+g-1}{g-1} \right) \left( \frac{r+g-1}{i} \right) g.
\]
(1) The following diagram is commutative:

\[
\begin{array}{ccc}
\text{Sym}^n (v^i V) \otimes V & \xrightarrow{\partial^n_1} & \text{Sym}^{n-1} (v^i V) \otimes \text{Sym}^{i-1} V \otimes L \\
\Delta^n \otimes 1_V & & \Delta^{n-1} \otimes 1_L \\
\text{Sym}^{n-2} (v^i V) \otimes L \otimes V & \xrightarrow{\partial^{n-1}_1} & \text{Sym}^{n-2} (v^i V) \otimes V \otimes L.
\end{array}
\]

(2) When \( r_L = 1 \), the first of the following diagrams is commutative and it becomes equivalent to the second diagram when we further assume that \((r+g)_i \in \text{End} (\mathbb{I})\) is a non-zero divisor:

\[
\begin{array}{ccc}
\text{Sym}^n (v^i V) \otimes V^\vee & \xrightarrow{\partial^n_1} & \text{Sym}^{n-1} (v^i V) \otimes v^{i-1} V \\
\Delta^n \otimes 1_{V^\vee} & & (r+g)_i \partial^{n-1}_1 \\
\text{Sym}^{n-2} (v^i V) \otimes L \otimes V^\vee & \xrightarrow{\partial^{n-1}_1} & \text{Sym}^{n-2} (v^i V) \otimes V^\vee \otimes L, \\
(\Delta^{n-2} \otimes 1_{V^\vee}) (r+g)_i \partial^{n-2}_1 & & (r+g)_i \partial^{n-1}_1.
\end{array}
\]

(3) Suppose that \( L \simeq L^\otimes 2 \) for some invertible object \( E \), that \( r_{v^i V} > 0 \) (see definition \[\text{[3.6]}\]) and that \( V \) has symmetric rank \( g \). Then there are morphisms

\[
s^n_{\Delta} : \text{Sym}^n (v^i V) \otimes L \to \text{Sym}^n (v^i V) \quad \text{for} \quad n \geq 2,
\]

\[
s^{n-1}_{\Delta} : \text{Sym}^{n-1} (v^i V) \otimes V \to \text{Sym}^n (v^i V) \otimes \text{Sym}^{i-1} V \otimes V \quad \text{for} \quad n \geq 1,
\]

\[
s^{n-1}_{\partial_1} : \text{Sym}^{n-1} (v^i V) \otimes V^\vee \otimes L \to \text{Sym}^n (v^i V) \otimes \text{Sym}^{i-1} V \otimes V
\]

such that

\[
\Delta^n \circ s^n_{\Delta} = 1_{\text{Sym}^n (v^i V) \otimes L}, \quad \partial^n_1 \circ s^{n-1}_{\Delta} = 1_{\text{Sym}^{n-1} (v^i V) \otimes V}
\]

and

\[
\partial^{n-1}_1 \circ s^{n-1}_{\Delta} = 1_{\text{Sym}^{n-1} (v^i V) \otimes V^\vee \otimes L}.
\]

In particular, the following objects exist:

\[
\text{ker} (\Delta^n) \subset \text{Sym}^n (v^i V),
\]

\[
\text{ker} (\partial^{n-1}_1) \subset \text{Sym}^n (v^i V) \otimes \text{Sym}^{i-1} V^\vee,
\]

\[
\text{ker} (\partial^{n-1}_1) \subset \text{Sym}^n (v^i V) \otimes \text{Sym}^{i-1} V.
\]

6. Some remarks about the functoriality of the Dirac operators

We will assume, from now on, that we have given an object \( V \in \mathcal{C} \) and that \( \mathcal{C} \) and \( \mathcal{D} \) are \( \mathbb{Q} \)-linear rigid and pseudo-abelian \( \mathcal{ACU} \) tensor categories. Once again, if \( X \) is an object, we set \( \tau_X := \text{rank} (X) \) and we use the shorthand \( r := r_V \). As usual, we write \( e_{X,i}^g \), \( i_{X,i}^g \), and \( p_{X,i}^g \) for the idempotent \( e_{X,i}^g \) in \( \text{End} (\otimes^g X) \) giving rise to \( \wedge^g X \) when \( ? = a \) and \( \vee^g X \) when \( ? = s \) and the associated canonical injective and surjective morphisms. We denote by \( D_{ij}^g \) and \( D_{ij}^g \) the Poincare duality morphisms in the algebra \( \otimes V \) when \( ? = t \), \( \wedge V \) when \( ? = a \) and \( \vee V \) when \( ? = s \). Then it easily follows from \[\text{[MS]}\] Lemma 2.3, §5 and §6 that, for every \( g \geq i \) and \( ? = a \) or \( s \),

\[
D_{ij}^g : A_i \xrightarrow{D_{ij}^g} \otimes V^i \otimes \text{Sym}^g \otimes (\otimes^g V^\vee) \xrightarrow{p_{ij}^g \otimes p_{ij}^g \vee \vee} A_{ij}^g \otimes A_{ij}^g \quad \text{and}
\]

\[
D_{ij}^g : A_i \xrightarrow{D_{ij}^g} \otimes V^i \otimes \text{Sym}^g \otimes (\otimes^g V^\vee) \xrightarrow{p_{ij}^g \otimes p_{ij}^g \vee \vee} A_{ij}^g \otimes A_{ij}^g \quad \text{and}
\]

Suppose that we have given a (covariant) additive \( \mathcal{AU} \) tensor functor \( F : \mathcal{C} \to \mathcal{D} \); it preserves internal homs and dualities. We suppose that \( F \) has the following further properties:

- \( F (\tau_{V^\vee}) = \varepsilon \cdot \tau_{F(V^\vee)} \) and \( F (\tau_{V^\vee}) = \varepsilon \cdot \tau_{F(V^\vee)} \), where \( \varepsilon \in \{ \pm 1 \} \);
• \( F(\tau_{V,V}) = \eta \cdot \tau_{F(V)^\vee,F(V)} \) (so that \( F(\tau_{V,V}) = \eta \cdot \tau_{F(V),F(V)} \)), where \( \eta \in \{ \pm 1 \} \).

We remark that, if \( \varepsilon = 1 \) (resp. \( \varepsilon = -1 \)) and \( X \in \{ V, V^\vee, V^\vee V \} \), we have \( F(e^n_{X,a}) = e^n_{F(X),a} \) (resp. \( F(e^n_{X,a}) = e^n_{F(X),a} \)) and the same for the associated injective and surjective morphisms. The following result is now an easy consequence of this remark, (52), (53) and an explicit computation showing that \( F(D^{i,g}_{V,t}) = \eta^{2+1} \cdot F(D^{i,g}_{V,t}) \) and \( F\left(D^{i,g}_{V,t}\right) = \eta^{2+1} \cdot D^{i,g}_{F(V),t} \) (see [MS] §5 for the explicit of \( D^{i,g}_{V,t} \) and \( D^{i,g}_{V,t} \)).

**Lemma 6.1.** Suppose that we have given a (covariant) additive \( AU \) tensor functor \( F : C \to D \) as above.

1. If \( \varepsilon = 1 \) then we have
   \[
   F\left(D^{i,g}_{V,t}\right) = \eta^{2+1} \cdot D^{i,g}_{F(V),t}, \quad F\left(D^{i,g}_{V,s}\right) = \eta^{2+1} \cdot D^{i,g}_{F(V),s}.
   \]

2. If \( \varepsilon = -1 \) then we have
   \[
   F\left(D^{i,g}_{V,t}\right) = \eta^{2+1} \cdot D^{i,g}_{F(V),t}, \quad F\left(D^{i,g}_{V,s}\right) = \eta^{2+1} \cdot D^{i,g}_{F(V),s}.
   \]

Fix \( g \geq i \) such that \( g = 2i \) and set \( L_a := \wedge^g V^\vee \vee \) and \( L_s := \vee^g V^\vee \). Write \( \psi_{a,1} : G_{g-i,g-i} \) and \( \psi_{g-i,g-i-1} = \psi_{i-1} \) for the pairings defined before Lemma 4.1. Write \( \psi_{a,1} = \psi_{g-i,1} \) and \( \psi_{g-i,g-i-1} = \psi_{i-1} \) for those defined before 4.2, \( \psi_{i-1} = \psi_{g-i,1} \) and \( \psi_{g-i,g-i-1} = \psi_{i-1} \) for the ones considered before Lemma 5.1 and \( \psi_{g-i,1} = \psi_{g-i,1} \) and \( \psi_{g-i,g-i-1} = \psi_{i-1} \) for those defined before 5.2.

Consider the following operators from \( \Lambda^1 V \wedge \Lambda^1 V \).

\[
\begin{align*}
\Delta^\Lambda (\Lambda^1 V) & := \Delta^a_{i,g+i,g+i} : \Lambda^1 V \wedge \Lambda^1 V \to \Lambda^{a-1} V \wedge \Lambda^{a-1} V \vee \wedge \Lambda^a L_a, \\
\Delta^\Lambda (\Lambda^1 V) & := \Delta^a_{i,g+i,g+i} : \Lambda^1 V \wedge \Lambda^1 V \to \Lambda^{a-1} V \wedge \Lambda^{a-1} V \vee \wedge \Lambda^a L_a,
\end{align*}
\]

\[
\begin{align*}
\partial^\Lambda (\Lambda^1 V) & := \partial^a_{i,g+i,g+i} : \Lambda^1 V \wedge \Lambda^1 V \to \Lambda^{a-1} V \wedge \Lambda^a V \vee \Lambda^a L_a, \\
\partial^\Lambda (\Lambda^1 V) & := \partial^a_{i,g+i,g+i} : \Lambda^1 V \wedge \Lambda^1 V \to \Lambda^{a-1} V \wedge \Lambda^a V \vee \Lambda^a L_a.
\end{align*}
\]

---

4Indeed we have \( \tau_{V,V} = \tau_{V,V}^{-1} \), so that

\[
F(\tau_{V,V}) = F(\tau_{V,V})^{-1} = \eta \cdot \tau_{F(V),F(V)}^{-1} = \eta \cdot \tau_{F(V),F(V)}.
\]

5Of course some of them will be zero, but it will be convenient to consider all of them, in order to state the result in a symmetric way.

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Similarly, in order to symmetrically state the results, we will need to consider the operators from \( \mathbb{Q} \) together with the analogous operators induced on the alternating powers of \( \mathbb{Q} \).

\[
\Delta^{\text{Alt}}(\mathbb{Q}^i V) := \Delta^n(\mathbb{Q}^i V) \to \text{Alt}^{n-2}(\mathbb{Q}^i V) \otimes L_s,
\]

\[
\Delta^{\text{Sym}}(\mathbb{Q}^i V) := \Delta^n(\mathbb{Q}^i V) \to \text{Sym}^{n-2}(\mathbb{Q}^i V) \otimes L_s,
\]

\[
\partial_{i-1}^{\text{Alt}}(\mathbb{Q}^i V) := \partial^n_{i-1}^{\text{Alt}}(\mathbb{Q}^i V) \to \text{Alt}^{n-1}(\mathbb{Q}^i V) \otimes V^\vee,
\]

\[
\partial_{i-1}^{\text{Sym}}(\mathbb{Q}^i V) := \partial^n_{i-1}^{\text{Sym}}(\mathbb{Q}^i V) \to \text{Sym}^{n-1}(\mathbb{Q}^i V) \otimes V^\vee \otimes L_s.
\]

The following result, whose proof is left to the reader, follows from Lemma 6.1 and a small computation.

**Proposition 6.2.** Suppose that we have given a (covariant) additive \( A \) \( U \) tensor functor \( F : C \to D \) as above.

(1) If \( \varepsilon = 1 \) then we have

\[
F(\Delta^{\text{Alt}}(\mathbb{Q}^i V)) = \Delta^{\text{Alt}}(\mathbb{Q}^i F(V)), \quad F(\Delta^{\text{Sym}}(\mathbb{Q}^i V)) = \Delta^{\text{Sym}}(\mathbb{Q}^i F(V)),
\]

(2) If \( \varepsilon = -1 \) and \( i \) is even, then we have

\[
F(\Delta^{\text{Alt}}(\mathbb{Q}^i V)) = \Delta^{\text{Alt}}(\mathbb{Q}^i F(V)), \quad F(\Delta^{\text{Sym}}(\mathbb{Q}^i V)) = \Delta^{\text{Sym}}(\mathbb{Q}^i F(V)),
\]

Application to quaternionic objects. We will now focus on the case \( i = 2 \) and \( g = 2i = 4 \) and we let \( B \) be a quaternion \( \mathbb{Q} \)-algebra, whose main involution we denote by \( b \mapsto b^* \). An alternating (resp. symmetric) quaternionic object in \( C \) is a couple \((V,\theta)\) where \( V \) has alternating (resp. symmetric) rank 4 and \( \theta : B \to \text{End}(V) \) is a unitary ring homomorphism. We will assume that such a \((V,\theta)\) has been given in the following discussion.

We have \( \sqrt{2}B \subset B \otimes B \), the \( \mathbb{Q} \)-vector space generated by the elements \( b_1 \lor b_2 = \frac{1}{2}(b_1 \otimes b_2 + b_2 \otimes b_1) \). Noticing that

\[
(b_1 + b_2) \otimes (b_1 + b_2) = b_1 \otimes b_1 + b_2 \otimes b_2 + b_1 \otimes b_2 + b_2 \otimes b_1
\]
and that \( b \vee b = b \otimes b \), we see that
\[
(b_1 \vee b_2) = \frac{(b_1 + b_2) \vee (b_1 + b_2)}{2} - \frac{b_1 \vee b_1}{2} - \frac{b_2 \vee b_2}{2},
\]
so that \( \mathcal{A} \) is the \( \mathbb{Q} \)-vector space generated by the elements \( b \vee b \). Considering \( B \otimes B \) as a \( \mathbb{Q} \)-algebra in the natural way, it follows that \( \mathcal{A} \) is a subalgebra, because the product of elements of the form \( b \vee b \) is again of this form. Let \( \text{Tr}: B \to \mathbb{Q} \) and \( \text{Nr}: B \to \mathbb{Q} \) be the reduced trace and norm and set \( B_0 := \ker (\text{Tr}) \).

Write \( W \) for the \( \mathbb{Q} \)-vector space \( B \otimes B \), endowed with the action of \( B \otimes B \) defined by the rule \( b_1 \otimes b_2 \cdot x := b_1 x b_2 \). It gives rise to a unitary ring homomorphism \( f: B \otimes B \to \text{End}_\mathbb{Q} (W) \cong \mathbb{M}_4 (\mathbb{Q}) \) which is injective because \( B \otimes B \) is simple, hence an isomorphism by counting dimensions. As a \( \mathcal{A} \)-module \( W = B_0 \oplus \mathbb{Q} \) and it easily follows that the resulting homomorphism
\[
\mathcal{A} \to \text{End}_\mathbb{Q} (B_0) \oplus \text{End}_\mathbb{Q} (\mathbb{Q}) \cong \mathbb{M}_3 (\mathbb{Q}) \oplus \mathbb{Q}
\]
is an isomorphism: it is injective because \( \text{End}_\mathbb{Q} (B_0) \oplus \text{End}_\mathbb{Q} (\mathbb{Q}) \subset \text{End}_\mathbb{Q} (W) \) and \( f \) is injective, hence an isomorphism again by counting dimensions. Furthermore, the action of \( \mathcal{A} \) on \( \mathbb{Q} \) is given by the \( \mathbb{Q} \)-algebra homomorphism
\[
\chi: \mathcal{A} \to \mathbb{Q}, \chi (b_1 \vee b_2) = \frac{\text{Tr} (b_1 b_2)}{2}.
\]
It follows that there is an idempotent \( e_- \in \mathcal{A} \) characterized by \( se_- = \chi (s) e_- \) for every \( s \in \mathcal{A} \).

We have a natural \( B \otimes B \)-action on \( V \otimes V \) by \( \theta \otimes \theta := \theta \otimes \theta \) and, since \( b \otimes b \otimes c_{\lambda \lambda} := e_{\lambda \lambda} \) \( b \otimes b \) for \( \lambda \in \{a, s\} \), \( B \subset B \otimes B \) (diagonally) operates on \( \wedge^2 V \) and \( \vee^2 V \). But \( \mathcal{A} \) is generated by the elements of the form \( b \otimes b \) as a \( \mathbb{Q} \)-algebra (indeed as a \( \mathbb{Q} \)-vector space, as already noticed): hence \( \mathcal{A} \subset B \otimes B \) operates on \( \wedge^2 V \) and \( \vee^2 V \). The above discussion shows that we may write
\[
\wedge^2 V = \left( \wedge^2 V \right)_+ \oplus \left( \wedge^2 V \right)_- \quad \text{and} \quad \vee^2 V = \left( \vee^2 V \right)_+ \oplus \left( \vee^2 V \right)_-
\]
where \( \left( \wedge^2 V \right)_- := \text{Im} (\theta \otimes \theta (e_-)) \) and \( \left( \wedge^2 V \right)_+ := \text{Im} (\theta \otimes \theta (e_-)) \) are characterized by the property that \( \mathcal{A} \) acts on them via \( \chi \). Indeed we remark that, since \( \mathcal{A} \) is generated by the diagonal image of \( B \subset B \otimes B \) and \( \chi (b \otimes b) = \text{Nr} (b) \), \( \left( \wedge^2 V \right)_- \) (resp. \( \left( \wedge^2 V \right)_+ \)) is the unique maximal subobject of \( \wedge^2 V \) (resp. \( \vee^2 V \)) on which \( \mathcal{A} \) acts via the reduced norm.

Associated with \( (V, \theta) \) is the dual quaternionic object \( (V \vee, \theta^\vee) \) where \( \theta^\vee (b) := \theta (b)^\vee \). We will simply write \( \left( \wedge^2 V \right)^\vee \) (resp. \( \left( \vee^2 V \right)^\vee \)) for the \( \pm \) components attached to \( (V \vee, \theta^\vee) \) obtained in this way. Since by definition \( \theta \otimes \theta (e_-) := (\theta (e_-)^\vee)^{\otimes 2} = \left( \theta (e_-)^\otimes 2 \right)^\vee \), we have \( \left( \wedge^2 V \right)_\pm = \left( \wedge^2 V \right)^\vee \) (resp. \( \left( \vee^2 V \right)_\pm = \left( \vee^2 V \right)^\vee \)).

We summarize the above discussion in the first part of following lemma, while the second follows from the remark before Lemma 6.1.

**Lemma 6.3.** If \( (V, \theta) \) is an alternating (resp. symmetric) quaternionic object in \( \mathcal{C} \), there is a canonical decomposition [54] (in the category of quaternionic objects), where \( \left( \wedge^2 V \right)_- \) (resp. \( \left( \vee^2 V \right)_- \)) is characterized by the fact that it is the unique maximal subobject \( X \) of \( \wedge^2 V \) (resp. \( \vee^2 V \)) such that the action of \( \mathcal{A} \) acting diagonally on \( \wedge^2 V \) (resp. \( \vee^2 V \)) is given by the reduced norm on \( X \). We have \( \left( \wedge^2 V \right)_\pm = \left( \wedge^2 V \right)^\vee \) (resp. \( \left( \vee^2 V \right)_\pm = \left( \vee^2 V \right)^\vee \)).

Suppose that we have given a (covariant) additive \( AU \) tensor functor \( F : \mathcal{C} \to \mathcal{D} \) as above and define \( F (\theta) (b) := F (\theta (b)) \). Then \( (F (V), F (\theta)) \) is an alternating (resp. symmetric) quaternionic object in \( \mathcal{D} \) when \( \varepsilon = 1 \), \( (F (V), F (\theta)) \) is a symmetric (resp. alternating) quaternionic object in \( \mathcal{D} \) when \( \varepsilon = -1 \) and we have
\[
F \left( \left( \wedge^2 V \right)_\pm \right) = \left( \wedge^2 F (V) \right)_\pm \quad \text{and} \quad F \left( \left( \vee^2 V \right)_\pm \right) = \left( \vee^2 F (V) \right)_\pm \quad \text{when} \quad \varepsilon = 1
\]
and
\[
F \left( \left( \wedge^2 V \right)_\pm \right) = \left( \wedge^2 F (V) \right)_\pm \quad \text{and} \quad F \left( \left( \vee^2 V \right)_\pm \right) = \left( \vee^2 F (V) \right)_\pm \quad \text{when} \quad \varepsilon = -1.
\]
Then the Laplace and the Dirac operators induced by these pairings \( \psi^V.euclideanSpace i,1 \) and \( \psi^V.euclideanSpace g-i,1 \) are related by a commutative diagram involving injections. Hence the analogue of Theorem 4.4 (resp. Theorem 5.3) (3) is true. Finally, the statement about \( F \) follows from Lemma 6.3 and Proposition 6.2.

**Definition 6.5.** If we have given an alternating (resp. symmetric) quaternionic object in \( C \), we define \( M_2(V,\theta) := (\wedge^2V)_- \) (resp. \( M_2(V,\theta) := (\vee^2V)_- \)),

\[
M_{2n}(V,\theta) := \ker \left( \Delta^- \circ \psi^V.euclideanSpace g-i,1 \right) \subseteq \text{Sym}^n (\wedge^2V)_- \text{, where } n \geq 2
\]

**(resp. \( M_{2n}(V,\theta) := \ker \left( \Delta^- \circ \psi^V.euclideanSpace g-i,1 \right) \subseteq \text{Sym}^n (\vee^2V)_- \)).**

---

As explained in the introduction, this latter condition on the rank of the 2-powers is always fulfilled when \( V \) is Kimura positive (resp. negative).
and $M_1(V, \theta) := V$ (resp. $M_1(V, \theta) := V$)

$$M_{2n+1}(V, \theta) := \ker \left( \partial_{-}^{\text{Sym}^n(\nabla^2 V)} \right) \subset \text{Sym}^n \left( (\nabla^2 V)_- \right) \otimes V, \text{ where } n \geq 1$$

(resp. $M_{2n+1}(V, \theta) := \ker \left( \partial_{-}^{\text{Sym}^n(\nabla^2 V)} \right) \subset \text{Sym}^n \left( (\nabla^2 V)_- \right) \otimes V$)

It follows from Lemma 6.3 and Corollary 6.4 that these objects are canonical in the category of quaternionic objects and that, if we have given a (covariant) AU

and

$\text{M}$

It follows that, setting $F$ an analytic functions and $O$

analytic functions and $O$

$\text{O}$

It classifies all the possible filtrations on $O$.

7.1. The quaternionic Poincaré upper half plane. We write

$$\mathcal{P} := \mathcal{P}^1_{\mathbb{C}} - \mathcal{P}^1_{\mathbb{R}} := \mathcal{H}^+ \sqcup \mathcal{H}^- \text{ and } \mathcal{H} := \mathcal{H}^+,$$

where $\mathcal{H}^\pm$ is the connected component of $\mathcal{P}$ such that $+ i \in \mathcal{H}^\pm$. We recall that $\mathcal{P}$ has a natural moduli interpretation as follows. Set $L_1 := \mathbb{Z}^2$, $P_1 := (\mathbb{Z}^2)^\vee$ (the $\mathbb{Z}$-dual) and, for a positive integer $k$, $L_k := S_k^1(L_1)$ (the $k$-symmetric power of $L_1$) and $P_k := S_k^1(P_1) = L_k^\vee$, the space of homogeneous polynomials of degree $k$ in two variables $(X, Y)$. Then we have $O_{\mathcal{P}^1_{\mathbb{C}}}(k)(P_k^1) = P_{k, \mathbb{C}}$ and $O_{\mathcal{P}^1_{\mathbb{C}}}(-k)(P_k^1) = L_k, \mathbb{C}$. To give an $S$-point $x : S \to P_k^1$ from a locally analytic space $S$ is to give an epimorphism $O_S(P_1) \to \mathcal{H}^+$ up to isomorphism, where $\mathcal{L}$ is an invertible sheaf on $S$ and, taking $x = 1_{P_k^1}$, gives the universal quotient

$$O_{\mathcal{P}^1_{\mathbb{C}}}(P_1) \to O_{\mathcal{P}^1_{\mathbb{C}}}(1)$$

mapping the global sections $1 \otimes X, 1 \otimes Y \in O_{\mathcal{P}^1_{\mathbb{C}}}(P_1, \mathbb{C})(P_k^1)$ respectively to the global sections $X, Y \in O_{\mathcal{P}^1_{\mathbb{C}}}(1)(P_k^1)$. Taking duals we see that to give $x : S \to P_k^1$ is the same as to give a monomorphism $\mathcal{L} \hookrightarrow O_S(L_1, \mathbb{C})$ up to isomorphism, where $\mathcal{L}$ is an invertible sheaf on $S$ and the cokernel of the inclusion is locally free too; taking $x = 1_{P_k^1}$ gives the universal object

$$O_{\mathcal{P}^1_{\mathbb{C}}}(1) \hookrightarrow O_{\mathcal{P}^1_{\mathbb{C}}}(L_1)$$

mapping the global section $\partial_X, \partial_Y \in O_{\mathcal{P}^1_{\mathbb{C}}}(1)(P_k^1)$ (duals of $X$ and $Y$) to $1 \otimes \partial_X, 1 \otimes \partial_Y \in O_{\mathcal{P}^1_{\mathbb{C}}}(P_1, \mathbb{C})(P_k^1)$. It follows that, setting $F_k^{-1} \left( O_{\mathcal{P}^1_{\mathbb{C}}}(L_1) \right) := O_{\mathcal{P}^1_{\mathbb{C}}}(L_1)$ and $F_k^+ \left( O_{\mathcal{P}^1_{\mathbb{C}}}(L_1) \right) := \text{Im}(\mathcal{L} \hookrightarrow O_S(L_1))$, the space $P_k^1$ classifies all the possible filtrations on $O_S(L_1)$ by an invertible $O_S$-module having locally free cokernel.

It is easy to realize that a necessary and sufficient condition for a point $x : S \to P_k^1$ to factor through $\mathcal{P}$ is that the filtration $F_k \left( O_{\mathcal{P}^1_{\mathbb{C}}}(L_1) \right)$ on $O_{\mathcal{P}^1_{\mathbb{C}}}(L_1) = O_{\mathcal{P}^1_{\mathbb{C}}}(L_1, \mathbb{R})$ gives $L_1, \mathbb{R}, \mathcal{P}$ the structure of a variation of Hodge structures of type $\{(-1, 0), (0, -1)\}$. Hence $\mathcal{P}$ classifies variations of Hodge structures on $S$ of Hodge type $\{(-1, 0), (0, -1)\}$ with fibers in the constant coherent sheaf $L_1, \mathbb{R}, \mathcal{P}$. The universal object is

$$L_1 := \left( L_1, \mathbb{R}, O_{\mathcal{P}^1_{\mathbb{C}}}(1) \right).$$

Let us fix $B$, an indefinite quaternion algebra, an identification $B_\infty \simeq M_2(\mathbb{R})$ and a maximal order $R \subset B$. Then $R \subset M_2(\mathbb{R})$ and $\mathbb{R} \otimes \mathbb{Z} R \simeq M_2(\mathbb{R})$ are identified as $\mathbb{R}$-algebras and we also have $O_S(R) \simeq O_S(M_2(\mathbb{R}))$ for every analytic space $S$.

Definition 7.1. A quaternionic variation of Hodge structures on $S$ is a variation of Hodge structures of type $\{(-1, 0), (0, -1)\}$ with fibers in the constant coherent sheaf $O_S(M_2(\mathbb{R}))_S$ such that the action of $M_2(\mathbb{R})$ induces on $O_S(M_2(\mathbb{R}))_S$ by right multiplication preserves the filtration $F$ ($O_S(M_2(\mathbb{R}))_S$) on $O_S(M_2(\mathbb{R}))_S$.

A rigidified quaternionic variation of Hodge structures on $S$ is a variation of Hodge structures with fibers in the sheaf $R_S$ such that $(\mathbb{R} \otimes \mathbb{Z} R)_S \simeq M_2(\mathbb{R})_S$ gives by transport to the right hand side the structure of a quaternionic variation of Hodge structures on $S$.

If $M$ is an $A$-module over a ring $A \subset B$, we write $M_B := B \otimes_A M$.

If $M$ is an $A$-module over a ring $A \subset C$ and $S$ is a locally analytic space, we write $M_S$ for the associated sheaf of locally analytic functions and $O_S(M) := O_S \otimes_A M_S$. 

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We note that, having fixed \( R \subset M_2(\mathbb{R}) \), to give a quaternionic variation of Hodge structures is the same thing as to give a rigidified quaternionic variation of Hodge structures.

**Definition 7.2.** A fake (analytic) elliptic curve over \( S \) is \((A/S, i)\) where \( A/S \) is an analytic abelian surface and \( i: A \to \text{End}_S(A) \) is an injective morphism such that \( R^1{\pi}_*\mathbb{Z}^\vee_A \cong R_S \) as a right sheaves of \( R \)-modules (the action on the left hand side is by functoriality).

A rigidified fake (analytic) elliptic curve over \( S \) is a \((A/S, i, \rho)\) where \((A/S, i, \rho)\) is a fake (analytic) elliptic curve over \( S \) and \( \rho: R^1{\pi}_*\mathbb{Z}^\vee_A \cong R_S \) is an isomorphism as right sheaves of \( R \)-modules.

If we have given a rigidified fake (analytic) elliptic curve \((\pi: A \to S, i, \rho)\) over \( S \), the exponential map gives an exact sequence of sheaves on \( S \),

\[
0 \to R^1{\pi}_*\mathbb{Z}^\vee_A \to T_{A/S} \to A \to 0.
\]

Then we may define \( F^0(\mathcal{O}_S(R^1{\pi}_*\mathbb{Z}^\vee_A)) \) by means of the exact sequence

\[
0 \to F^0(\mathcal{O}_S(R^1{\pi}_*\mathbb{Z}^\vee_A)) \to \mathcal{O}_S(R^1{\pi}_*\mathbb{Z}^\vee_A) \to T_{A/S} \to 0.
\]

By means of \( i \) the ring \( R \) acts on this sequence (say from the right). The rigidification yields \( \rho: R^1{\pi}_*\mathbb{Z}^\vee_A \cong R_S \) compatible with the right action of \( R \) on \( R_S \). It follows that \( F^0(\mathcal{O}_S(R^1{\pi}_*\mathbb{Z}^\vee_A)) \cong F^0(\mathcal{O}_S(R)) \cong F^0(\mathcal{O}_S(M_2(\mathbb{R}))) \) (the right hand sides defined by transport) gives \( R_S \) a rigidified quaternionic variation of Hodge structures on \( S \) that we denote \( R^1{\pi}_*\mathbb{Z}^\vee_A \). The correspondence is indeed an equivalence of categories (see [Mi] Theorem 7.13 and [De2] 4.4.3).

The following result yields a quaternionic moduli description of \( \mathcal{P} \), which depends on the fixed identification \( B_\infty = M_2(\mathbb{R}) \) and the choice of a maximal order \( R \subset B \).

**Proposition 7.3.** The analytic space \( \mathcal{P} \) classifies rigidified fake (analytic) elliptic curve over \( S \). If \( \pi: A \to \mathcal{P} \) is the universal elliptic curve we have that \( R^1{\pi}_*\mathbb{Z}^\vee_A \cong R_\mathcal{P} \) and the associated variation of Hodge structures on \( \mathcal{O}_S(R) \cong \mathcal{O}_S(M_2(\mathbb{R})) \) is given by \( \mathcal{L}_1 \oplus \mathcal{L}_1 \).

Proof. Thanks to the above discussion we have to classify quaternionic variation of Hodge structures, rather than rigidified fake (analytic) elliptic curves. But it is easy to see that the association \( \mathcal{L} \mapsto \mathcal{L} \oplus \mathcal{L} \) realizes an identification between variations of Hodge structures on \( S \) of Hodge type \( \{(-1,0),(0,-1)\} \) with fibers in the constant coherent sheaf \( L_{1,\mathbb{R}} \) and quaternionic variation of Hodge structures. The claim follows from the above description of \( \mathcal{P} \). \( \Box \)

### 7.2. Linear algebra in the category of \( B^\times \)-representations.

We write \( x \mapsto x' \) to denote the main involution of \( B \), so that \( x + x' = \text{Tr}(x) \) and \( xx' = \text{Nr}(x) \). We let \( B^\times \) acts on \( B \) by left multiplication, while we write \( B^\times \) to denote \( B \) on which \( B^\times \) acts from the left by the rule \( b \cdot x := bxb' \). We write \( B^0 := \ker(\text{Tr}) \) to denote the trace zero elements, viewed as a \( B^\times \)-subrepresentation of \( B^* \) (indeed \( \text{Tr}(bxb') = \text{Nr}(b) \text{Tr}(x) \)). If \( V \in \text{Rep}(B^\times) \) and \( r \in \mathbb{Z} \), we let \( V(r) \) be \( V \) on which \( B^\times \) acts by \( b \cdot v = \text{Nr}^{-r}(b) bv \), so that \( V(r) = V \otimes \mathbb{Q}(r) \) (canonically).

In \( M_2(\mathbb{R}) \) certain Laplace and Dirac operators has been defined with source and target those of the subsequent Lemma 7.4. While their definition is completely explicit, it is only the definition of the Laplace operator that readily generalizes to arbitrary tensor categories; on the other hand, the definition of the Dirac operator requires the theory we have developed in order to provide good models for their kernels which have a general meaning for tensor categories. Indeed, we have the following key remark, that allows us to replace the Jordan-Livné models with ours, whose proof is left to the reader.

**Lemma 7.4.** Let

\[
f_n : \text{Sym}^n(B_0) \to \text{Sym}^{n-2}(B_0)(-2) \quad \text{and} \quad g_n : \text{Sym}^n(B_0) \otimes B \to \text{Sym}^{n-1}(B_0)(-1)
\]

be any epimorphism in \( \text{Rep}_0(B^\times) \). Once we fix \( B \otimes F \simeq M_2(F) \), where \( F \) is any splitting field of \( B \), there are canonical isomorphisms

\[
\ker(f_n) \otimes F \simeq L_{2n} \otimes F, \quad \ker(g_n) \otimes F \simeq L_{2n+1}^2 \otimes F
\]
which are compatible with the \((B \otimes \mathbb{F})^\times\)-action on the left hand side, the \(\text{GL}_2(\mathbb{F})\)-action on the right side and the induced identification \((B \otimes \mathbb{F})^\times \simeq \text{GL}_2(\mathbb{F})\).

The following Lemma will be useful. Recall that, if \(M\) is an object in a pseudo abelian \(\mathbb{Q}\)-linear category on which \(B\)-acts, we may write \(\bigwedge^2 M = (\bigwedge^2 M)_+ \oplus (\bigwedge^2 M)_-\) canonically, where \(B\) operates on \((\bigwedge^2 M)_-\) via the reduced norm. The following

**Lemma 7.5.** Let \(\theta : B \xrightarrow{\sim} \text{End}_{\text{Rep}(B^\times)}(B)\) be the isomorphism provided by the right multiplication. Then we have, in \(\text{Rep}_Q(B^\times)\),

\[
\bigwedge^2 B = (\bigwedge^2 B)_+ \oplus (\bigwedge^2 B)_- \quad \text{with} \quad (\bigwedge^2 B)_+ \simeq \mathbb{Q}(-1)^3 \quad \text{and} \quad (\bigwedge^2 B)_- \simeq B_0.
\]

Since \((B, \theta)\) is an alternating quaternionic object, we may define

\[
L^B_{2n} := M_{2n}(B, \theta) \quad \text{for } n \geq 1 \quad \text{and} \quad L^{B(2)}_{2n+1} := M_{2n+1}(B, \theta) \quad \text{for } n \geq 0
\]

and it is a consequence of Lemmas 7.3 and 7.5 that, when \(B \otimes \mathbb{F} \simeq M_2(\mathbb{F})\),

\[
L^B_{2n,\mathbb{F}} \simeq L_{2n,\mathbb{F}} \quad \text{and} \quad L^{B(2)}_{2n+1,\mathbb{F}} \simeq L^2_{2n+1,\mathbb{F}}.
\]

**7.3. Variations of Hodge structures attached to \(B^\times\)-representations.** In this subsection we define a \(\mathbb{Q}\)-additive \(ACU\) tensor functor, depending on the choice of an identification \(B_\infty \simeq M_2(\mathbb{R})\),

\[
\mathcal{L} : \text{Rep}(B^\times) \to \text{VHS}_\mathbb{P}(\mathbb{Q}),
\]

where \(\text{VHS}_\mathbb{S}(\mathbb{F})\) denotes the category of variations of Hodge structures on \(\mathbb{S}\) with coefficients in the field \(\mathbb{F} \subset \mathbb{R}\). The identification \(B_\infty \simeq M_2(\mathbb{R})\) induces \(\text{Rep}(B^\times)_\mathbb{Q} \simeq \text{Rep}(\text{GL}_2, \mathbb{R})\) and it follows from [Ha, Corollary 3.2 and its proof for uniqueness] that we may define a (unique up to equivalence) faithful and exact \(\mathbb{Q}\)-additive \(ACU\) tensor functor

\[
\mathcal{L}_\mathbb{R} : \text{Rep}(B^\times)_\mathbb{R} \to \text{VHS}_\mathbb{P}(\mathbb{R})
\]

requiring \(\mathcal{L}_\mathbb{R}(L_{1,\mathbb{R}}) := \mathcal{L}_1\). Since \(\mathcal{O}_\mathbb{P}(V) = (V_\mathbb{R})\) for every \(V \in \text{Rep}(B^\times)_\mathbb{F}\), we deduce that the restriction of \(\mathcal{L}_\mathbb{R}\) to \(\text{Rep}(B^\times)_\mathbb{F} \to \text{Rep}(B^\times)_\mathbb{R}\) via \(V \mapsto V_\mathbb{R}\) factors through \(\text{VHS}_\mathbb{P}(\mathbb{Q}) \to \text{VHS}_\mathbb{P}(\mathbb{R})\) (again via scalar extension) and gives our \(\mathcal{L}\). It follows from this description and Proposition 7.3 that we have

\[
\mathcal{L}(B) = R^1\pi_0\mathbb{Q}^{\mathcal{L},}_+,
\]

if \(B\) denotes the left \(B^\times\)-representation whose underlying vector space is \(B\) with the action given by left multiplication.

Since \((\mathcal{L}(B), \mathcal{L}(\theta))\) is an alternating quaternionic, we may define

\[
L^B_{2n} := M_{2n}(\mathcal{L}(B), \mathcal{L}(\theta)) \quad \text{for } n \geq 1 \quad \text{and} \quad L^{B(2)}_{2n+1} := M_{2n+1}(\mathcal{L}(B), \mathcal{L}(\theta)) \quad \text{for } n \geq 0.
\]

If \(K \subset \mathbb{B}^\times\) is an open and compact subgroup, we may consider the Shimura curve

\[
S_K(\mathbb{C}) := B^\times \backslash \left( \mathcal{P} \times \mathbb{B}^\times \right) / K = B^\times_+ \backslash \left( \mathcal{H} \times \mathbb{B}^\times \right) / K
\]

where:

- \(B^\times\) acts diagonally on \(\mathcal{P} \times \mathbb{B}^\times\) (via \(B^\times \subset B^\times_+\) and the diagonal embedding \(B^\times \subset B^\times\) on the second component)
- The action of \(K\) is trivial on \(\mathcal{H}\) and by right multiplication on \(\mathbb{B}^\times\).

When \(B \neq M_2(\mathbb{Q})\), \(X_K(\mathbb{C}) := S_K(\mathbb{C})\) is compact and otherwise we set \(X_K(\mathbb{C}) := \overline{S_K(\mathbb{C})}\), compactified by "adding cusps". Then \(\mathcal{L}(V)\) (for any \(V \in \text{Rep}(B^\times)\)) descend to a variation of Hodge structures \(\mathcal{L}_K(V)\) on \(S_K(\mathbb{C})\), which extends to \(X_K(\mathbb{C})\).

Setting \(\pi_0(X_K(\mathbb{C})) := B^\times_+ \backslash \mathbb{B}^\times / K\) we have

\[
\pi_0 : X_K(\mathbb{C}) \to \pi_0(X_K(\mathbb{C})) \quad \text{and} \quad \pi_K : \mathbb{B}^\times \to \pi_0(X_K(\mathbb{C})).
\]

\footnote{\text{Suffices indeed to check that } \mathcal{L}_1 \text{ descend to } S_K(\mathbb{C}) \text{ in order to get a functor } \mathcal{L}_{\mathbb{R},K} \text{ (from } \text{[Ha]} \text{) valued in } \text{VHS}_{S_K(\mathbb{C})}(\mathbb{R}) \text{ and then appeal to the uniqueness to deduce that } \mathcal{L}_{\mathbb{R},K}(V) \text{ is obtained from } \mathcal{L}_{\mathbb{R}}(V) \text{ descend for every } V. \text{ Then one can promote the restriction of } \mathcal{L}_{\mathbb{R},K} \text{ to } \text{Rep}(B^\times) \text{ to take values in } \text{VHS}_{S_K(\mathbb{C})}(\mathbb{Q}), \text{ exactly as above.}
If \( x \in \hat{B}^\times \) define \( \Gamma_K (x) := xKx^{-1} \cap B_+^\times \), where \( B^\times \subset \hat{B}^\times \) is diagonally embedded, that we view as a subgroup \( \Gamma_K (x) \subset B_+^\times \subset B_+^\times = \text{GL}_2 (\mathbb{R}) \). Then we have
\[
\pi_0^{-1} (\pi_K (x)) = B_+^\times \backslash B_+^\times (\mathcal{H} \times xK/K) \sim \Gamma_K (x) \backslash \mathcal{H}
\]
by the rule
\[
[z, xk] \mapsto z
\]
and
\[
S_K (\mathbb{C}) = \bigsqcup_{\pi_K (x) \in \pi_0(X_K (\mathbb{C})), \pi_K (x) \neq 0} \Gamma_K (x) \backslash \mathcal{H}.
\]
It follows from the Eichler-Shimura isomorphism (see [Hi, Ch. 6] and [RS, §2.5] for the statement in the quaternionic setting) and (55) that the cohomology groups (let \( (?) = \phi \) when \( k \) is even and \( (?) = (2) \) when \( k \) is odd)
\[
H^1 \left( S_K (\mathbb{C}), \mathcal{L}^{B(\gamma)}_{k, K} \right) \simeq \bigoplus_{\pi_K (x) \in \pi_0(X_K (\mathbb{C})))} \Gamma_K (x, L_k^{B(\gamma)}) ,
\]
afford weight \( k + 2 \) modular forms of level \( K \) when \( k \) is even and two copies of them when \( k \) is odd. Indeed, it is not difficult to define Hecke operators on the family \( \mathcal{L}^{B(\gamma)}_{k, K} \) by means of correspondences, which are given by double cosets on the right hand side; we also remark that the left hand side has a natural Hodge structure endowed with Hecke multiplication. The case \( B = \mathbb{M}_2 (\mathbb{Q}) \) being covered by [Sc], we will assume from now on that \( B \neq \mathbb{M}_2 (\mathbb{Q}) \), implying that the Hecke action on (57) is purely cuspidal.

7.4. The motives of quaternionic modular forms and their realizations. Let \( K \subset \hat{B}^\times \) be an open and compact subgroup which is small enough so that \( S_K \) is a fine moduli space and let \( \pi_K : A_K \to S_K \) be the universal level \( K \) fake elliptic curve over \( S_K \). Consider the relative motive \( h (A_K) \) as an object of \( \text{Mot}_+^0 (S_K, \mathbb{F}) \), where \( h \) is the contravariant functor
\[
h : \text{Sch} (S_K) \to \text{Mot}_+^0 (S_K, \mathbb{F})
\]
from the category of smooth and projective schemes over \( S \) to the category of Chow motives with coefficients in a field \( \mathbb{F} \). By functoriality of the motivic decomposition, there is \( \theta : B \to \text{End} (h^1 (A_K)) \) making \( (h^1 (A_K), \theta) \) a symmetric quaternionic object, and we may define
\[
M_{2n,K}^B := M_{2n} (h^1 (A_K), \theta) \quad \text{for} \quad n \geq 1 \quad \text{and} \quad M_{2n+1,K}^B := M_{2n+1} (h^1 (A_K), \theta) \quad \text{for} \quad n \geq 0.
\]
There is a realization functor
\[
R_{S_K} : \text{Mot}_+^0 (S_K, \mathbb{F}) \to D^b (\text{VMHS} (S_K, \mathbb{F}))
\]
extending the correspondence mapping \( \pi : X \to S_K \) to \( R_{S_K} \mathbb{F}_X \). Here \( \text{VMHS} (S_K, \mathbb{F}) \) denotes the abelian category of variations of mixed Hodge structures over \( S_K \) with coefficients in \( \mathbb{F} \). See [PS] 14.4 for details.

Theorem 7.6. Taking \( F = \mathbb{Q} \) we have the following realizations.

1. Suppose that \( 2n \geq 2 \) is even. Then:
\[
R (M_{2n,K}^B) = \mathcal{L}^B_{2n,K} [-2n].
\]

2. Suppose that \( 2n + 1 \geq 3 \) is odd. Then:
\[
R (M_{2n+1,K}^B) = \mathcal{L}^{B(2)}_{2n+1,K} [-2n+1].
\]

Proof. As in [DM] Remarks 2) after Corollary 3.2) one has \( R (h^1 (A_K)) = R^1 \pi_* \mathbb{Q}_K^\vee [-1] \), so that (55) implies \( R (h^1 (A_K)) = \mathcal{L}_K (B) [-1] \). Since \( R_{S_K} \) is a \( AU \) tensor functor (indeed anti-commutative, see [Ku] Remark (2.6.1)), we deduce from the remark after Definition (55) that (let \( (?) = \phi \) when \( k \) is even and \( (?) = (2) \) when \( k \) is odd)
\[
R (M_{k,K}^{B(\gamma)}) = M_k (\mathcal{L}_K (B) [-1], \mathcal{L}_K (\theta)) = M_k (\mathcal{L}_K (B), \mathcal{L}_K (\theta)) [-k] = \mathcal{L}^{B(\gamma)}_{k,K} [-k].
\]

Together with (57) and recalling that the group cohomology of \( L_k \) is concentrated in degree 1, we deduce the following result.
Corollary 7.7. Let $H$ be the Betti realization functor, valued in $\text{VHS}(\mathbb{Q})$, and let view $M_{2n,K}^B$ and $M_{2n+1,K}^{B(2)}$ as motives defined over $\mathbb{Q}$.

1. Suppose that $2n \geq 2$ is even. Then:

$$H^i(M_{2n,K}^B) = \begin{cases} H^1(S_{K}(\mathbb{C}), L_{k,K}^{B(2)}) & \text{if } i = 2n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

2. Suppose that $2n + 1 \geq 3$ is odd. Then:

$$H^i(M_{2n+1,K}^{B(2)}) = \begin{cases} H^1(S_{K}(\mathbb{C}), L_{k,K}^{B(2)}) & \text{if } i = 2n + 2 \\ 0 & \text{otherwise.} \end{cases}$$

As explained after [57], this motivates our designation of $M_{2n,K}^B$ (resp. $M_{2n+1,K}^{B(2)}$) as the motive of level $K$ and weight $2n + 2$ (resp. $2n + 3$) modular forms: indeed the functoriality of our construction implies that the Hecke correspondences induces a Hecke multiplication on $M_{2n,K}^B$ (resp. $M_{2n+1,K}^{B(2)}$), which is compatible with that on the realizations. The abstract approach employed here for computing the realizations, inspired by [IS], easily adapts to the other realizations: one has only to appropriately replace [50] (which is, for example, the deeper [IS] Lemma 5.10) in the $p$-adic realm considered there.

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