Bonus-Malus System with the Claim Frequency Distribution is Geometric and the Severity Distribution is Truncated Weibull

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Abstract. Bonus-Malus system is said to be optimal if it is financially balanced for insurance companies and fair for policyholders. Previous research about Bonus-Malus system concern with the determination of the risk premium which applied to all of the severity that guaranteed by the insurance company. In fact, not all of the severity that proposed by policyholder may be covered by insurance company. When the insurance company sets a maximum bound of the severity incurred, so it is necessary to modify the model of the severity distribution into the severity bound distribution. In this paper, optimal Bonus-Malus system is compound of claim frequency component has geometric distribution and severity component has truncated Weibull distribution is discussed. The number of claims considered to follow a Poisson distribution, and the expected number λ is exponentially distributed, so the number of claims has a geometric distribution. The severity with a given parameter θ is considered to have a truncated exponential distribution is modelled using the Levy distribution, so the severity have a truncated Weibull distribution.

1. Introduction
Bonus-Malus system is a system where in the next payment, policyholders who submitted one or more claim will be penalized by premium raise (malus), while policyholders who submitted no claim will be rewarded by premium reduction (bonus). Bonus-Malus system is said to be optimal if it is financially balanced for the insurance company (the total amount of bonus is equal to the total amount of malus) and fair to all policyholders, that is every policyholder pays premium proportionally with the risk [4].

In [1], it is discussed Bonus-Malus system based on the claim frequency having negative binomial distribution for modelling risk premium. While, [2] using claim frequency following Poisson inverse Gaussian distribution to minimize the risk of the insurer. Furthermore, Bonus-Malus system is not only based on the claim frequency but also based on the claim severity, were developed in [4] and [5]. This system uses a negative binomial distribution for claim frequency and Pareto distribution for claim severity. The other researchers, e.g. [3] used geometric distribution for claim frequency and Pareto distribution for claim severity on Bonus-Malus system. In the other hand, [7] is using negative binomial distribution in claim frequency and Weibull distribution for claim severity.
Previous research concern with the determination of the risk premium which applied to all of the severity that guaranteed by the insurance company. In fact, not all of the severity that proposed by policyholder may be covered by insurance company. When the insurance company sets a maximum bound of the severity incurred, so it is necessary to modify the model of the severity distribution into the severity bound distribution. Hence, in this paper, it is discussed Bonus-Malus system with the claim frequency distribution is geometric and the claim severity distribution is truncated Weibull, in this paper also compared the risk premium between the full severity and the severity with the maximum bound.

In Section 2, the process of deriving the risk premium for our models is discussed. The number of claims considered to follow a Poisson distribution, and the expected number $\lambda$ is exponentially distributed, so the number of claims has a geometric distribution. The severity with a given parameter $\theta$ is considered to have a truncated exponential distribution is modelled using the Levy distribution, so the severity have a truncated Weibull distribution. The parameters are estimated using the quadratic error loss function [6]. The risk premium based on compound of geometric distribution and truncated Weibull distribution. In Section 3, as an application, the risk premium is calculated based on the claim frequency based on geometric distribution and the severity based on Weibull distribution and truncated Weibull distribution. In this section, we compared the risk premium between the full severity based on Weibull distribution and the severity with the maximum bound based on truncated Weibull. The conclusion of this paper is presented in Section 4.

2. Optimal Bonus-Malus System

In this section we discuss how to format the title, authors and affiliations. Please follow these instructions as carefully as possible so all articles within a conference have the same style to the title page. This paragraph follows a section title so it should not be indented.

The modelling of the claim frequency is the same as the one in [3]. The number of claims for given considered to follow a Poisson distribution, and the expected number of claims $\lambda$ is exponentially distributed with parameter $\theta$, so the number of claims has a geometric distribution with parameter $\theta(\theta + 1)^{-1}$. That is,

$$ P(k) = \int_0^\infty P(k | \lambda) u(\lambda) d\lambda = \frac{\theta}{\theta+1} \left( \frac{1}{\theta+1} \right)^k, \quad k = 0, 1, 2, \ldots \text{ and } 0 < \frac{1}{\theta+1} < 1. \quad (1) $$

Furthermore, by applying the Bayesian approach, the posterior distribution is given by,

$$ u(\lambda | k_1, k_2, \ldots, k_t) = \frac{\lambda^k (\theta + t)^{K+1} e^{-\lambda(\theta + t)}}{\Gamma(K + 1)}, \lambda > 0 \quad (2) $$

where $K = \sum_{i=1}^t k_i$ represents the total of claim frequency over $t$ period with $k_i$ present the number of claims in each period respectively. Using the quadratic loss function, the following choice for $\lambda_{t+1}$, the expected number of claims of a policyholder with a claim history $k_1, k_2, \ldots, k_t$, is :

$$ \hat{\lambda}_{t+1} = \int_0^\infty \lambda u(\lambda | k_1, k_2, \ldots, k_t) d\lambda = \frac{K + 1}{\theta + t}. \quad (3) $$

Equation (3) shows that, the risk premium payable at time $t + 1$ depends on the claim history of the policyholder ($K$), time period ($t$) and the parameter of the exponential distribution ($\theta$).

The modelling of the severity in [3] is using Weibull distribution. The expected number of severity for the next period is
\[
\hat{\theta}_{t+1} = \frac{B}{K^{3/2}} \left(\frac{c\sqrt{M}}{c\sqrt{M}}\right) \left(\frac{2\sqrt{M}}{c}\right),
\]

where \(M\) is the total severity, \(K\) is the total of claim, and \(c\) is the parameter of Weibull distribution.

So that the risk premium based on compound refer to (3) and (4) is given by,

\[
\text{Premium}_{t+1} = \frac{K+1}{\theta+t} \left[ \frac{B}{K^{3/2}} \left(\frac{c\sqrt{M}}{c\sqrt{M}}\right) \left(\frac{2\sqrt{M}}{c}\right) \right].
\]

When the insurance company sets a maximum bound of the severity incurred, so it is necessary to modify the model of the severity distribution into the severity bound distribution. Now, suppose that the amount of claim \(x\) is distributed according to the truncated exponential distribution with a given parameter \(\theta\). The cumulative distribution function is given by

\[
F(X = x | \theta) = \begin{cases} 1 - e^{-\delta x}, & 0 < x < u \\ 1, & x \geq u \end{cases}
\]

The parameter \(\theta\) is referred as stable Levy distribution. Then the probability density function can be described as follows:

\[
\pi(\Theta = \theta) = \frac{c}{2\sqrt{\pi}\theta^3} e^{\frac{\theta^2}{4}} \theta > 0.
\]

So the unconditional cumulative distribution function for \(0 < x < u\) is,

\[
F(x) = 1 - \frac{2}{\sqrt{\pi}} e^{\left(\frac{\delta^2 + \frac{a^2}{4}}{\sigma^2}\right)} d\delta, \quad \frac{e^2 x}{4} = a \geq 0.
\]

The next task is solving refer to (8). First assume that,

\[
I = \int_0^\infty e^{\left(\frac{\delta - \sqrt{a}}{\sigma}\right)^2} d\delta = \int_0^\infty e^{\left(\frac{\delta}{\sigma}\right)^2} d\delta + \int_0^\infty e^{\left(\frac{\delta + \sqrt{a}}{\sigma}\right)^2} d\delta.
\]

Using substitution, let \(g = \delta - \frac{\sqrt{a}}{\sigma} \in (-\infty, \infty)\) and \(\frac{dg}{d\delta} = 1 + \frac{\sqrt{a}}{\sigma^2} > 0\), then \(g\) is a monotonically increasing function from \(-\infty\) to \(\infty\). So that refer to (9) becomes

\[
I = \int_{-\infty}^\infty e^{-\frac{1}{\sigma^2} \left(1 + \frac{\sqrt{a}}{\sigma^2}\right)} \left(1 + \frac{\sqrt{a}}{\sigma^2}\right) \frac{1}{\left(1 + \frac{\sqrt{a}}{\sigma^2}\right)} dg = \sqrt{\pi}
\]

The form obtained from solving refer to (9) is needed to have \(\int_0^\infty e^{\left(\frac{\delta - \sqrt{a}}{\sigma}\right)^2} d\delta\).

Let \(\varepsilon = -\frac{\sqrt{a}}{\sigma^2}\) and \(d\varepsilon = \frac{\sqrt{a}}{\sigma^2} d\delta\). Since \(\delta = 0\) implies \(\varepsilon = \infty\) and \(\delta = \infty\) implies \(\varepsilon = 0\), refer to (9) can be written as
Because $I = 2 \int_{0}^{2} e^{-\frac{\sqrt{\theta}}{\theta}} \, d\theta$ and $I = \sqrt{\pi}$ then we have $\int_{0}^{\infty} e^{-\frac{\sqrt{\theta}}{\theta}} \, d\theta = \frac{\sqrt{\pi}}{2}$. Equation (6) can be described as follows:

$$F(x) = 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\left(\theta \frac{\sqrt{\theta}}{\theta} + \frac{\theta}{\pi}\right)} \, d\theta$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\left(\frac{\sqrt{\theta}}{\theta} - 2\sqrt{\theta}\right)} \, d\theta$$

$$= 1 - e^{-\left(\frac{x}{\sqrt{\pi}}\right)^2}$$

(12)

So we have,

$$F(x) = \begin{cases} 1 - e^{-\left(\frac{x}{\sqrt{\pi}}\right)^2}, & 0 < x < u \\ 1, & x \geq u \end{cases}$$

(13)

Thus, the cumulative distribution function above is the truncated Weibull distribution with parameter $(c^2, 0.5)$.

If $x_i, i = 1, 2, ..., K$ denotes the amount of $i$-th claim, then the total amount of claimed for a policyholder over $t$ period will be equal to $\sum_{i=1}^{K} x_i$. We assumed that there is $n$ claim with $0 < x_i < u$, and the amount of claim for $x_i \geq u$ is $K - n$ claim. Suppose $N$ is the total of amount claimed for $0 < x_i < u$ that is $N = \sum_{i=1}^{n} x_i$. The total amount of $K$ claim becomes $\sum_{i=1}^{K} x_i = \sum_{i=1}^{n} x_i + u(K - n) = N + u(K - n)$. Hence, the posterior distribution for $\theta$ can be obtained by using Bayesian approach as follows

$$\pi(\theta | x_1, x_2, \ldots, x_k) = \frac{\theta^{K-3} e^{-\frac{1}{4\theta} \left[\theta(N+u(K-n))\right]}}{\int_{0}^{\infty} \theta^{K-3} e^{-\frac{1}{4\theta} \left[\theta(N+u(K-n))\right]} \, d\theta}.$$  

(14)
By slightly modifying the variables refer to (14), it can be rewritten as,

\[
\pi(\theta \mid x_1, x_2, \ldots, x_K) = \frac{\left(\frac{c}{2\sqrt{N+u(K-n)}}\right)^{K-\frac{1}{2}} \theta^{-\frac{3}{2}} e^{-\frac{c^2}{4\theta} + \vartheta(N+u(K-n))}}{\int_0^\infty \left(\frac{2\vartheta\sqrt{N+u(K-n)}}{c}\right)^{-\frac{3}{2}} e^{-\frac{2\vartheta\sqrt{N+u(K-n)}}{c} + \vartheta(N+u(K-n))} d\left(\frac{2\sqrt{N+u(K-n)}}{c}\right)\theta^{K-\frac{1}{2}}}. \quad (15)
\]

The integral on the denominator can be transformed to a modified Bessel function, whose integral representation is given as follow

\[B_v(x) = \frac{1}{2} \int_0^\infty y^{v-1} e^{-\frac{1}{2}y} dy, \quad B_v(x) > 0, \forall x > 0, v \in \mathbb{R}. \quad (16)\]

Then the posterior distribution refer to (15) can be written below,

\[
\pi(\theta \mid x_1, x_2, \ldots, x_K) = \frac{\left(\frac{c}{2\sqrt{N+u(K-n)}}\right)^{K-\frac{1}{2}} \theta^{-\frac{3}{2}} e^{-\frac{c^2}{4\theta} + \vartheta(N+u(K-n))}}{2B_{K-\frac{1}{2}} \left(\frac{c\sqrt{N+u(K-n)}}{2}\right)}. \quad (17)
\]

Using the quadratic loss function, the optimal choice for \(\theta_{t+1}\) for a policyholder having amount of claim reports \(x_i, i = 1, 2, \ldots, K\) over \(t\) period is estimated as

\[
\hat{\theta}_{t+1} = \frac{B_{K-\frac{1}{2}} \left(\frac{c\sqrt{N+u(K-n)}}{2}\right)}{B_{K-\frac{1}{2}} \left(\frac{c\sqrt{N+u(K-n)}}{2}\right)} \left(\frac{2\sqrt{N+u(K-n)}}{c}\right). \quad (18)
\]

The risk premium to be paid at time \(t + 1\) for a policyholder whose number of claims history is \(k_1, k_2, \ldots, k_t\) and whose amount claim history is \(x_1, x_2, \ldots, x_k\) can be calculated according to the net premium as

\[
\text{Premium}_{t+1} = \frac{K+1}{\theta + t} \left(\frac{B_{K-\frac{1}{2}} \left(\frac{c\sqrt{N+u(K-n)}}{2}\right)}{B_{K-\frac{1}{2}} \left(\frac{c\sqrt{N+u(K-n)}}{2}\right)} \left(\frac{2\sqrt{N+u(K-n)}}{c}\right)\right). \quad (19)
\]

The risk premium in (19) is compound of refer to (3) and (18). It is shows that the risk premium depends on the claim frequency, time period, and total severity.

The calculation of risk premium with Weibull distribution is refer to (5) and the risk premium based on truncated Weibull distribution is show in (19). It is give varying results according to the proposed huge losses. At the beginning and when the policyholders have no claim, the risk premium paid are equal. In other words, when the total severity is \(M \leq N + u(K-n)\), the risk premium based on Weibull distribution will be equal to the one based on truncated Weibull distribution. Whereas, when the total severity is \(M > N + u(K-n)\), the risk premium based on the Weibull distribution is more expensive than the one based on the truncated Weibull distribution.

Let suppose the ratio of two Bessel functions is
When $K = 1$ and $B_v(x) = B_{v-1}(x)$, then we have

$$ Q_1\left(c\sqrt{N+u(K-n)}\right) = \frac{B_{v-1}\left(c\sqrt{N+u(K-n)}\right)}{B_v\left(c\sqrt{N+u(K-n)}\right)} = 1. $$

Because $B_{v+1}(x) = \frac{2v}{x} B_v(x) + B_{v-1}(x)$, so the recursive function $Q_k$ is

$$ Q_{k+1}\left(c\sqrt{N+u(K-n)}\right) = \frac{2\left(K - \frac{1}{2}\right)}{c\sqrt{N+u(K-n)}} + Q_k\left(c\sqrt{N+u(K-n)}\right). $$

In (19), when the total severity is $N + u(K-n) = 0$, it means that no claim filed ($K = 0$), it shows that the risk premium is undefined. Hence, we redefine the risk premium when $K = 0$. Then the risk premium for the following period if no claim filed will be given by,

$$ \text{Premium}_{t+1 | N+u(K-n)=0} = \frac{1}{\theta + \frac{2}{c^2}}. $$

This means that when there are no claim reported, the greater of the period time will cause the smaller of the risk premium must be paid.

### 3. Numerical Illustration

#### 3.1. Bonus-Malus System without Truncated Distribution

In this set up the number of claims is assumed to follow a geometric distribution with parameter $\theta$ ($\theta + 1$)$^{-1}$, $\theta = 1.5$. Total amount of claim is assumed to follow a truncated Weibull distribution with parameter $c^2$, $c = 0.052$. The calculation of the risk premium to be paid by each policyholder if no claim filed refer to (23), whereas if there is a claim filed refer to (5). Total of claim severity used are 8000 and 10000. The risk premium are calculated for $t$-th period, $t = 0, 1, 2, \ldots, 7$ and for the number of claims $K = 0, 1, 2, \ldots, 5$.

When there is no change in the number of claims filed, the risk premium gets cheaper along with increasing time. For example, if a policyholder submits one claim with total loss is 8000 in the first period of observation, the amount of risk premiums have to be paid is 2752, it can be seen in Table 1. Furthermore, in the second period, the policyholder submits another claim with loss is 2000. It makes the number of claims during the two periods are 2 claims and the total loss is 10000, so the policyholder have to pay for 2765 that can be seen in Table 2.
Table 1. The result of risk premium with total severity is 8000

| Period | Number of claim (K) |
|--------|--------------------|
|        | 0                  | 1       | 2       | 3       | 4       | 5   |
| 0      | 493                |         |         |         |         |     |
| 1      | 296                | 2752    | 3398    | 3749    | 3918    | 3980|
| 2      | 211                | 1966    | 2427    | 2678    | 2798    | 2843|
| 3      | 164                | 1529    | 1888    | 2083    | 2176    | 2211|
| 4      | 134                | 1251    | 1544    | 1704    | 1781    | 1809|
| 5      | 114                | 1058    | 1307    | 1442    | 1507    | 1531|
| 6      | 99                 | 917     | 1133    | 1250    | 1306    | 1327|
| 7      | 87                 | 809     | 999     | 1103    | 1152    | 1171|

Table 2. The result of risk premium with total severity is 10000

| Period | Number of claim (K) |
|--------|--------------------|
|        | 0                  | 1       | 2       | 3       | 4       | 5   |
| 0      | 493                |         |         |         |         |     |
| 1      | 296                | 3077    | 3871    | 4347    | 4612    | 4744|
| 2      | 211                | 2198    | 2765    | 3105    | 3294    | 3389|
| 3      | 164                | 1709    | 2151    | 2415    | 2562    | 2636|
| 4      | 134                | 1399    | 1760    | 1976    | 2096    | 2156|
| 5      | 114                | 1183    | 1489    | 1672    | 1774    | 1825|
| 6      | 99                 | 1026    | 1290    | 1449    | 1537    | 1581|
| 7      | 87                 | 905     | 1139    | 1279    | 1356    | 1395|

3.2. Bonus-Malus System with Truncated Distribution

In order to see the effect of the truncated distribution on claim severity, the calculation of the risk premium refer to (23) if no claim filed, whereas if there is a claim filed refer to (19). The maximum bound of premiums that can be covered by insurance companies is \( u = 2300 \). Claim severity \( x_i \) is assumed to be equal to the average of the claim size, that is 1000.

Table 3 shows that the risk premium prices depend on the claim frequency variation. For example, in the beginning the policyholder pays the premium for 493. If in the first period there is no accident, there will be a premium reduction to the level 296. But, if a policyholder have one accident, the policyholder have to pay 1476 if the amount of claim is more than 2300, otherwise have to pay 973.
4. CONCLUSION

This paper consider Bonus-Malus system with the claim frequency distribution is geometric that compound distribution of Poisson distribution and exponentially distribution and the claim severity distribution is truncated Weibull which compound distribution of truncated exponentially distribution and Levy distribution. Based on the results, it can be shown that the risk premium depend on the level of risk. When the total severity are less than the maximum bound of severity or \( M \leq N + u (K - n) \), the risk premium based on Weibull distribution will be equal to the one based on truncated Weibull distribution. Where \( M \) is the total severity, \( u \) is the maximum bound that covered by insurance company, \( N \) is the total severity which the severity are lower than bound \( u \), \( K \) is the total claim frequency, and \( n \) is the claim frequency which the frequency are lower than bound \( u \). In other hand, when the total of severity are greater than the maximum bound of the severity or \( M > N + u (K - n) \), the risk premium based on the Weibull distribution is more expensive than the one based on truncated Weibull distribution.
5. References

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