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Samy Skander Bahoura

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A COMPACTNESS RESULT FOR AN ELLIPTIC EQUATION IN DIMENSION 2.

SAMY SKANDER BAHOURA

ABSTRACT. We give a blow-up analysis for the solutions of an elliptic equation under some conditions on the prescribed curvature. Also, we derive a compactness result for this elliptic equation under a Lipschitz condition.

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Keywords: quantization, blow-up, boundary, elliptic equation, a priori estimate, Lipschitz condition.

1. INTRODUCTION AND MAIN RESULTS

We consider the following equation:

\[ (P_\epsilon) \begin{cases} -\Delta u - \epsilon(x_1 \partial_1 u + x_2 \partial_2 u) = V e^u \text{ in } \Omega \\ u = 0 \text{ on } \partial \Omega. \end{cases} \]

Here, we assume that:

\[ \Omega \text{ starshaped}, \]

and,

\[ u \in W^{1,1}_0(\Omega), \ e^u \in L^1(\Omega), \ 0 \leq V \leq b, \ \epsilon \geq 0. \]

When \( \epsilon = 0 \), the previous equation was studied by many authors, with or without the boundary condition, also for Riemann surfaces, see [1,18], we can find some existence and compactness results.

Among other results, we can see in [4] the following important Theorem,

**Theorem.** (Brezis-Merle [4]). If \((u_i)_i\) and \((V_i)_i\) are two sequences of functions relatively to the problem \((P_0)\) with, \( \epsilon = 0 \) and \( 0 < a \leq V_i \leq b < +\infty \), then, for all compact set \( K \) of \( \Omega \),

\[ \sup_K u_i \leq c = c(a, b, m, K, \Omega). \]

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We can find in [4] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of $e^{u_i}$.

If we assume $V$ with more regularity, we can have another type of estimates, a $\sup + \inf$ type inequalities. It was proved by Shafrir see [15], that, if $(u_i), (V_i)$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left( \frac{a}{b} \right) \sup_{\Omega} u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

We can see in [7] an explicit value of $C \left( \frac{a}{b} \right) = \sqrt{\frac{a}{b}}$. In his proof, Shafrir has used the Stokes formula and an isoperimetric inequality, see [2]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)$ uniformly Lipschitzian with $A$ the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see Brézis-Shafrir [5]. This result was extended for Hölderian sequences $(V_i)$ by Chen-Lin, see [7]. Also, we can see in [12], an extension of the Brezis-Shafrir result to compact Riemann surface without boundary. We can see in [13] explicit form, $(8\pi m, m \in \mathbb{N}^*)$ exactly, for the numbers in front of the Dirac masses when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, we can see in [8] and [18] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

Here we give the behavior of the blow-up points on the boundary and a new proof of Brezis-Merle conjecture with Lipschitz condition.

The Brezis-Merle Conjecture (see [4]) is:

**Conjecture.** Suppose that $V_i \to V$ in $C^0(\overline{\Omega})$ with $0 \leq V_i \leq b$ for some positive constant $b$. Also, we consider a sequence of solutions $(u_i)$ of $(P_\epsilon)$ relatively to $(V_i)$ such that,

$$\int_{\Omega} e^{u_i} dx \leq C,$$

is it possible to have:

$$\|u_i\|_{L^\infty} \leq C = C(b, \Omega, C)?$$

Here, we give a blow-up analysis on the boundary when the prescribed curvature are nonnegative and bounded and on the other hand, if we add the assumption that these curvature are uniformly Lipschitzian, we have a compactness of the solutions of the problem $(P_\epsilon)$ for $\epsilon$ small enough. (In particular we can take a sequence of $\epsilon_i$ tending to 0):

For the behavior of the blow-up points on the boundary, the following condition is enough,
\[0 \leq V_i \leq b,\]

The condition \(V_i \to V\) in \(C^0(\bar{\Omega})\) is not necessary.

But for the compactness of the solutions we add the following condition:

\[||\nabla V_i||_{L^\infty} \leq A.\]

We have the following characterization of the behavior of the blow-up points on the boundary.

**Theorem 1.1.** Assume that \(\max_{\Omega} u_i \to +\infty\), where \((u_i)\) are solutions of the problem \((P_{\epsilon_i})\) with:

\[0 \leq V_i \leq b, \quad \text{and} \quad \int_{\Omega} e^{u_i} \, dx \leq C, \quad \epsilon_i \to 0,\]

then, after passing to a subsequence, there is a function \(u\), there is a number \(N \in \mathbb{N}\) and \(N\) points \(x_1, \ldots, x_N \in \partial \Omega\), such that,

\[
\partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^N \alpha_j \delta_{x_j}, \; \alpha_j \geq 4\pi, \; \text{weakly in the sense of measure } L^1(\partial \Omega).
\]

\[u_i \to u \text{ in } C^1_{\text{loc}}(\bar{\Omega} - \{x_1, \ldots, x_N\}).\]

In the following theorem, we have a compactness result:

**Theorem 1.2.** Assume that \((u_i)\) are solutions of \((P_{\epsilon_i})\) relatively to \((V_i)\) with the following conditions:

\[||\nabla V_i||_{L^\infty} \leq A \text{ and } \int_{\Omega} e^{u_i} \, dx \leq C, \; \epsilon_i \to 0.\]

Then, we have:

\[||u_i||_{L^\infty} \leq c(b, A, C, \Omega),\]

**2. Proof of the theorems**

**Proof of theorem 1.1:**

First of all, remark that, we have for two positive constants \(C_q = C(q, \Omega)\) and \(C_1 = C_1(\Omega):\)
\[ ||\nabla u_i||_{L^q} \leq C_q ||\Delta u_i||_{L^1} \leq (C'_q + \epsilon C_1 ||\nabla u_i||_{L^1}), \forall i \text{ and } 1 < q < 2. \]

Thus, if \( \epsilon > 0 \) is small enough and by the Holder inequality, we have the following estimate:

\[ ||\nabla u_i||_{L^q} \leq C''_q, \forall i \text{ and } 1 < q < 2. \]

**Step 1: interior estimate**

First of all remark that, if we consider the following equation:

\[
\begin{cases}
-\Delta w_i = -\epsilon_i (x_1 \partial_1 u_i + x_2 \partial_2 u_i) \in L^q, & 1 < q < 2 \text{ in } \Omega \\
w_i = 0 \text{ on } \partial \Omega.
\end{cases}
\]

We have by the elliptic estimates that \( w_i \in W^{2,1+\epsilon} \subset L^\infty \), and we can write the equation of the Problem as:

\[
\begin{cases}
-\Delta (u_i - w_i) = \tilde{V}_i e^{u_i - w_i}, & \text{in } \Omega \\
u_i - w_i = 0 \text{ on } \partial \Omega.
\end{cases}
\]

with,

\[
0 \leq \tilde{V}_i = V_i e^{w_i} \leq \tilde{b}, \quad \int_{\Omega} e^{u_i - w_i} \leq \tilde{C}.
\]

We apply the Brezis-Merle theorem to \( u_i - w_i \) to have:

\[ u_i - w_i \in L^\infty_{loc}(\Omega), \]

and, thus:

\[ u_i \in L^\infty_{loc}(\Omega). \]

**Step 2: boundary estimate**

Now, we have:

\[ \int_{\partial \Omega} \partial_\nu u_i d\sigma \leq C, \]

and
Without loss of generality, we can assume that $\partial_\nu u_i > 0$. Thus, (using the weak convergence in the space of Radon measures), we have the existence of a positive Radon measure $\mu$ such that,

$$
\int_{\partial \Omega} \partial_\nu u_i \varphi d\sigma \to \mu(\varphi), \quad \forall \ \varphi \in C^0(\partial \Omega).
$$

We take an $x_0 \in \partial \Omega$ such that, $\mu(x_0) < 4\pi$. Without loss of generality, we can assume that the following curve, $B(x_0, \epsilon) \cap \partial \Omega := I_\epsilon$ is an interval. (In this case, it is more simple to construct the following test function $\eta_\epsilon$). We choose a function $\eta_\epsilon$ such that,

$$
\eta_\epsilon \equiv 1, \text{ on } I_\epsilon, \quad 0 < \epsilon < \delta/2,
$$
$$
\eta_\epsilon \equiv 0, \text{ outside } I_{2\epsilon},
$$
$$
0 \leq \eta_\epsilon \leq 1,
$$
$$
||\nabla \eta_\epsilon||_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}.
$$

We take a $\tilde{\eta}_\epsilon$ such that,

$$
\left\{ \begin{array}{l}
-\Delta \tilde{\eta}_\epsilon = 0 \text{ in } \Omega \\
\tilde{\eta}_\epsilon = \eta_\epsilon \text{ on } \partial \Omega.
\end{array} \right.
$$

We use the following estimate, see [3, 11, 17],

$$
||\nabla u_i||_{L^q} \leq C_q, \ \forall \ i \text{ and } 1 < q < 2.
$$

We deduce from the last estimate that, $(u_i)$ converge weakly in $W^{1,q}_0(\Omega)$, almost everywhere to a function $u \geq 0$ and $\int_{\Omega} e^u < +\infty$ (by Fatou lemma). Also, $V_i$ weakly converge to a nonnegative function $V$ in $L^\infty$. The function $u$ is in $W^{1,q}_0(\Omega)$ solution of :

$$
\left\{ \begin{array}{l}
-\Delta u = V e^u \in L^1(\Omega) \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega,
\end{array} \right.
$$

According to the corollary 1 of Brezis-Merle result, see [4], we have $e^{ku} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have $u \in C^1(\Omega)$.

We can write,

$$
-\Delta((u_i - u)\tilde{\eta}_\epsilon) = (V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2 < \nabla (u_i - u) \nabla \tilde{\eta}_\epsilon >.
$$

We use the interior estimate of Brezis-Merle, see [4].
**Step 1**: Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde{\eta}_\epsilon$ and $u$, we obtain,

\[
\int_{\Omega} V e^{u} \tilde{\eta}_\epsilon \, dx = \int_{\partial \Omega} \partial_{\nu} u \eta \leq 4\epsilon \| \partial_{\nu} u \|_{L^\infty} = C \epsilon \quad (2)
\]

We have,

\[
\begin{cases}
-\Delta u_i = V_i e^{u_i} \text{ in } \Omega \\
u_i = 0 \text{ on } \partial \Omega.
\end{cases}
\]

We use the Green formula between $u_i$ and $\tilde{\eta}_\epsilon$ to have:

\[
\int_{\Omega} V_i e^{u_i} \tilde{\eta}_\epsilon \, dx = \int_{\partial \Omega} \partial_{\nu} u_i \eta \, d\sigma \rightarrow \mu(\eta_\epsilon) \leq \mu(I) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0 \quad (3)
\]

From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

\[
\int_{\Omega} |(V_i e^{u_i} - V e^{u})\tilde{\eta}_\epsilon| \, dx \leq 4\pi - \epsilon_0 + C \epsilon \quad (4)
\]

**Step 2.1**: Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{ x \in \Omega, d(x, \partial \Omega) = \epsilon^2 \}$ and $\Omega_{\epsilon, 2} = \{ x \in \Omega, d(x, \partial \Omega) \geq \epsilon^2 \}$, $\epsilon > 0$. Then, for $\epsilon$ small enough, $\Sigma_\epsilon$ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon, 2}$ is $k_2 \epsilon^2 \leq \mu_L(\Omega - \Omega_{\epsilon, 2}) \leq k_1 \epsilon^2$.

**Remark**: for the unit ball $B(0, 1)$, our new manifold is $B(0, 1 - \epsilon^2)$.

We write,

\[
\int_{\Omega} | \nabla (u_i - u) \nabla \tilde{\eta}_\epsilon | \, dx = \int_{\Omega_{\epsilon, 2}} | \nabla (u_i - u) \nabla \tilde{\eta}_\epsilon | \, dx + \int_{\Omega - \Omega_{\epsilon, 2}} | \nabla (u_i - u) \nabla \tilde{\eta}_\epsilon | \, dx. \quad (5)
\]

**Step 2.1.1**: Estimate of $\int_{\Omega - \Omega_{\epsilon, 2}} | \nabla (u_i - u) \nabla \tilde{\eta}_\epsilon | \, dx$.

First, we know from the elliptic estimates that $\| \nabla \tilde{\eta}_\epsilon \|_{L^\infty} \leq C_1 / \epsilon$, $C_1$ depends on $\Omega$.
We know that \(|\nabla u_i|_i\) is bounded in \(L^q, 1 < q < 2\), we can extract from this sequence a subsequence which converge weakly to \(h \in L^q\). But, we know that we have locally the uniform convergence to \(|\nabla u|\) (by Brezis-Merle theorem), then, \(h = |\nabla u|\) a.e. Let \(q'\) be the conjugate of \(q\).

We have, \(\forall f \in L^{q'}(\Omega)\)

\[
\int_{\Omega} |\nabla u_i| f \, dx \to \int_{\Omega} |\nabla u| f \, dx
\]

If we take \(f = 1_{\Omega - \Omega_{\epsilon_2}}\), we have:

for \(\epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, \ i \geq i_1, \int_{\Omega - \Omega_{\epsilon_2}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon_2}} |\nabla u| + \epsilon^2.

Then, for \(i \geq i_1(\epsilon), \)

\[
\int_{\Omega - \Omega_{\epsilon_2}} |\nabla u_i| \leq \text{mes}(\Omega - \Omega_{\epsilon_2}) ||\nabla u||_{L^\infty} + \epsilon^2 = \epsilon^2(k_1 ||\nabla u||_{L^\infty} + 1).
\]

Thus, we obtain,

\[
\int_{\Omega - \Omega_{\epsilon_2}} |\nabla (u_i - u)| \nabla \tilde{\eta}_\epsilon > |dx \leq \epsilon C_1(2k_1 ||\nabla u||_{L^\infty} + 1) \quad (6)
\]

The constant \(C_1\) does not depend on \(\epsilon\) but on \(\Omega\).

**Step 2.1.2:** Estimate of \(\int_{\Omega_{\epsilon_2}} |\nabla (u_i - u)| \nabla \tilde{\eta}_\epsilon > |dx\).

We know that, \(\Omega_{\epsilon} \subset \subset \Omega\), and (because of Brezis-Merle’s interior estimates) \(u_i \to u\) in \(C^1(\Omega_{\epsilon_2})\). We have,

\[ ||\nabla (u_i - u)||_{L^\infty(\Omega_{\epsilon_2})} \leq \epsilon^2, \text{ for } i \geq i_3 = i_3(\epsilon). \]

We write,

\[
\int_{\Omega_{\epsilon_2}} |\nabla (u_i - u)| \nabla \tilde{\eta}_\epsilon > |dx \leq ||\nabla (u_i - u)||_{L^\infty(\Omega_{\epsilon_2})} ||\nabla \tilde{\eta}_\epsilon||_{L^\infty} \leq C_1 \epsilon \text{ for } i \geq i_3,
\]

For \(\epsilon > 0\), we have for \(i \in \mathbb{N}, \ i \geq \max\{i_1, i_2, i_3\}\),
\[
\int_{\Omega} | < \nabla (u_i - u) \nabla \tilde{\eta} > | \ dx \leq \epsilon C_1 (2k_1 \| \nabla u \|_{L^\infty} + 2) \quad (7)
\]

From (4) and (7), we have, for \( \epsilon > 0 \), there is \( i_3 = i_3(\epsilon) \in \mathbb{N} \), \( i_3 = \max \{ i_0, i_1, i_2 \} \) such that,

\[
\int_{\Omega} | \Delta [(u_i - u) \tilde{\eta}] | \ dx \leq 4\pi - \epsilon_0 + \epsilon C_1 (2k_1 \| \nabla u \|_{L^\infty} + 2 + C) \quad (8)
\]

We choose \( \epsilon > 0 \) small enough to have a good estimate of (1).

Indeed, we have:

\[
\begin{cases}
- \Delta [(u_i - u) \tilde{\eta}] = g_{i, \epsilon} \text{ in } \Omega, \\
(u_i - u) \tilde{\eta} = 0 \text{ on } \partial \Omega.
\end{cases}
\]

with \( \| g_{i, \epsilon} \|_{L^1(\Omega)} \leq 4\pi - \epsilon_0 \).

We can use Theorem 1 of [4] to conclude that there is \( q > 1 \) such that:

\[
\int_{V_\epsilon(x_0)} e^{q(u_i - u)} \ dx \leq \int_{\Omega} e^{q(u_i - u) \tilde{\eta}} \ dx \leq C(\epsilon, \Omega).
\]

where, \( V_\epsilon(x_0) \) is a neighborhood of \( x_0 \) in \( \Omega \).

Thus, for each \( x_0 \in \partial \Omega - \{ \bar{x}_1, \ldots, \bar{x}_m \} \) there is \( \epsilon_{x_0} > 0, q_{x_0} > 1 \) such that:

\[
\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0}u_i} \ dx \leq C, \ \forall \ i.
\]

Now, we consider a cutoff function \( \eta \in C^\infty(\mathbb{R}^2) \) such that:

\[
\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).
\]

We write,

\[-\Delta (u_i \eta) = V_i e^{u_i \eta} - 2 < \nabla u_i | \nabla \eta > -u_i \Delta \eta.
\]

By the elliptic estimates, \( (u_i \eta)_i \) is uniformly bounded in \( W^{2,q_1}(\Omega) \) and also, in \( C^1(\Omega) \).

Finally, we have, for some \( \epsilon > 0 \) small enough,
We have proved that, there is a finite number of points $\bar{x}_1, \ldots, \bar{x}_m$ such that the sequence $(u_i)_i$ is locally uniformly bounded in $\Omega - \{\bar{x}_1, \ldots, \bar{x}_m\}$.

**Proof of theorem 1.2:**

The Pohozaev identity gives:

$$
\int_{\partial \Omega} < x|\nu > (\partial_\nu u_i)^2 d\sigma + \epsilon \int_\Omega (\langle x|\nabla u_i \rangle)^2 dx + \int_{\partial \Omega} < x|\nu > V_i e^{u_i} d\sigma = \int_\Omega (\langle x|\nabla V_i \rangle + 2 V_i) e^{u_i} dx
$$

We use the boundary condition and the fact that $\Omega$ is starshaped and the fact that $\epsilon > 0$ to have that:

$$
\int_{\partial \Omega} (\partial_\nu u_i)^2 dx \leq c_0(b, A, C, \Omega).
$$

Thus we can use the weak convergence in $L^2(\partial \Omega)$ to have a subsequence $\partial_\nu u_i$, such that:

$$
\int_{\partial \Omega} \partial_\nu u_i \varphi dx \to \int_{\partial \Omega} \partial_\nu u \varphi dx, \quad \forall \varphi \in L^2(\partial \Omega),
$$

Thus, $\alpha_j = 0, j = 1, \ldots, N$ and $(u_i)$ is uniformly bounded.

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Département de Mathématiques, Université Pierre et Marie Curie, 2 place Jussieu, 75005, Paris, France.

E-mail address: samybahoura@yahoo.fr