AMBROSIO-TORTORELLI APPROXIMATION OF QUASI-STATIC EVOLUTION OF BRITTLE FRACTURES

ALESSANDRO GIACOMINI

Abstract. We define a notion of quasi-static evolution for the elliptic approximation of the Mumford-Shah functional proposed by Ambrosio and Tortorelli. Then we prove that this regular evolution converges to a quasi static growth of brittle fractures in linearly elastic bodies.

Keywords: variational models, energy minimization, free discontinuity problems, elliptic approximation, crack propagation, quasi-static evolution, brittle fracture.

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1. INTRODUCTION

In 1998 Francfort and Marigo [15] proposed a model of quasi-static growth of brittle fractures in linearly elastic bodies based on the classical Griffith criterion.

Let $\Omega \subseteq \mathbb{R}^3$ be an elastic body, $\partial_D \Omega$ a part of its boundary and let $g : \partial_D \Omega \to \mathbb{R}^3$ be the spatial displacement of $\Omega$ at the points of $\partial_D \Omega$. According to Griffith theory, given a preexisting crack $\Gamma_1 \subseteq \Omega$, the new crack $\Gamma$ and the displacement $u : \Omega \setminus \Gamma \to \mathbb{R}^3$ associated to $g$ at the equilibrium minimizes the following elastic energy

$$E(v,g,\Gamma) := \int_\Omega \mu |Ev|^2 + \lambda |tr Eu|^2 \, dx + \mathcal{H}^2(\Gamma),$$

among all cracks $\Gamma$ with $\Gamma_1 \subseteq \Gamma$ and all displacements $v : \Omega \setminus \Gamma \to \mathbb{R}^3$ with $v = g$ on $\partial_D \Omega \setminus \Gamma$. Here $Ev$ denotes the symmetric part of the gradient of $v$, $tr$ denotes the trace of the matrix, and $\mathcal{H}^2$ denotes the two dimensional Hausdorff measure. Griffith criterion thus involves a competition between the bulk energy given by $\int_\Omega \mu |Ev|^2 + \lambda |tr Eu|^2 \, dx$ and the surface energy given by $\mathcal{H}^2(\Gamma)$. The boundary condition is required only on $\partial_D \Omega \setminus \Gamma$ because the displacement in a fractured region is supposed to be not transmitted. We indicate by $E(g,\Gamma)$ the minimum value of (1.1) among all $v : \Omega \setminus \Gamma \to \mathbb{R}^3$ with $v = g$ on $\partial_D \Omega \setminus \Gamma$.

Suppose that the boundary displacement $g$ varies with the time $t \in [0,1]$. The quasi-static evolution $t \to \Gamma(t)$ proposed in [15] requires that:

1. $\Gamma(t)$ is increasing in time, i.e., $\Gamma(t_1) \subseteq \Gamma(t_2)$ for all $0 \leq t_1 \leq t_2 \leq 1$;
(2) \( \mathcal{E}(g(t), \Gamma(t)) \leq \mathcal{E}(g(t), \Gamma) \) for all cracks \( \Gamma \) such that \( \cup_{s \leq t} \Gamma(s) \subseteq \Gamma \);

(3) the elastic energy \( \mathcal{E}(g(t), \Gamma(t)) \) is absolutely continuous in time.

Condition (1) stands for the irreversibility of the evolution (fracture can only increase); condition (2) states that each time \( t \) is of equilibrium, while condition (3) requires a regularity for the total energy.

The problem of giving a precise mathematical formulation of the preceding model has been the object of several recent papers. In 2000, Dal Maso and Toader [12] dealt with the case of antiplanar shear in dimension two: the authors consider a cylindric elastic body \( \Omega = \Omega' \times \mathbb{R} \) with \( \Omega' \subseteq \mathbb{R}^2 \) subject to displacements of the form \( u(\pi x)e_3 \) where \( e_3 \) is the versor of the \( x_3 \)-axis, and \( \pi \) is the projection on \( \Omega' \). The boundary antiplanar displacement is assigned on \( \partial D' \times \mathbb{R} \) while the admissible cracks are of the form \( K \times \mathbb{R} \) with \( K \) compact connected subset of \( \Omega' \) with a prescribed number of connected components and with finite \( H^1 \)-length. A generalization to non-isotropic surface energies is contained in [10].

Recently Francfort and Larsen [13] proposed a mathematical formulation which involves the space \( SBV \) of special functions of bounded variation. Their approach permits to treat antiplanar shear in a \( N \)-dimensional setting, and allows to consider fractures with a possibly infinite number of connected components. To be precise, they consider deformations of the form \( u(x)e_{N+1} \), where \( u \in SBV(\Omega) \) and \( e_{N+1} \) denotes the unitary vector of the \((N+1)\)-axis. The crack at time \( t \) is defined as \( \Gamma(t) \times \mathbb{R} \) where \( \Gamma(t) := \cup_{s<t} \left[ S_u(s) \cup (\partial D \cap \{u(s) \neq \gamma(s)\}) \right] \), and the pair \( (u(t), \Gamma(t)) \) is such that:

(a) for all \( z \in SBV(\Omega) \)

\[
\int_\Omega |\nabla u(t)|^2 + \mathcal{H}^{N-1}(\Gamma(t)) \leq \int_\Omega |\nabla z|^2 + \mathcal{H}^{N-1}(S_z \cup (\partial D \cap \{z \neq \gamma(t)\})) \cup \Gamma(t));
\]

(b) the elastic energy \( \mathcal{E}(t) := \int_\Omega |\nabla u(t)|^2 + \mathcal{H}^{N-1}(\Gamma(t)) \) is absolutely continuous and

\[
\mathcal{E}(t) = \mathcal{E}(0) + 2 \int_0^t \int_\Omega \nabla u(\tau) \nabla \dot{\gamma}(\tau) \, dx \, d\tau.
\]

Numerical computations concerning this model of evolution (see [7]) are performed using a discretization in time procedure and an approximation of the elastic energy proposed in 1990 by Ambrosio and Tortorelli (see [4, 5]). Being the new energy elliptic, the difficulties arising in the discretization of the free discontinuity term given by the fracture are avoided. Supposing to have determined the deformation \( u_i \) and the fracture \( K_i \) at the time \( t_i \), one minimizes the Ambrosio-Tortorelli functional in the domain \( \Omega \setminus K_i \) under the boundary conditions \( g(t_{i+1}) \), and hence reconstruct the couple \( (u_{i+1}, K_{i+1}) \). In this way, errors due to the discretization in time and to the approximation of the energy are introduced. In order to study the convergence of the procedure, one is led to formulate a natural notion of quasi-static evolution for the Ambrosio-Tortorelli functional. The aim of this paper is to prove the convergence of this regular evolution to an evolution of brittle fractures in the sense of [13].

The Ambrosio-Tortorelli functional is given by

\[
F_\varepsilon(u, v) = \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla v|^2 \, dx + \frac{1}{2\varepsilon} \int_\Omega (1 - v)^2 \, dx
\]

where \( (u, v) \in H^1(\Omega) \times H^1(\Omega) \), \( 0 \leq v \leq 1 \), \( 0 < \eta_\varepsilon < < \varepsilon \). \( F_\varepsilon \) contains an elliptic part

\[
\int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 \, dx
\]

and a surface part

\[
MM_\varepsilon(v) := \frac{\varepsilon}{2} \int_\Omega |\nabla v|^2 \, dx + \frac{1}{2\varepsilon} \int_\Omega (1 - v)^2 \, dx
\]
which is a term of Modica-Mortola type (see [17]).

If a sequence \((u_\varepsilon, v_\varepsilon)\) is such that \(F_\varepsilon(u_\varepsilon, v_\varepsilon) + \|u_\varepsilon\|_\infty \leq C\), then \(v_\varepsilon \to 1\) strongly in \(L^2(\Omega)\), and it turns out that, up to a subsequence, \(u_\varepsilon \to u \in SBV(\Omega)\); roughly speaking, the gradient of \(u_\varepsilon\) becomes larger and larger in the thick regions in which \(v_\varepsilon\) approaches zero, possibly creating some jumps in the limit. We conclude that the function \(u_\varepsilon\) has to be considered as a regularization of the deformation \(u\), while the function \(v_\varepsilon\) has to be intended as a function which tends to 0 in the region where \(S_u\) will appear, and to 1 elsewhere. Moreover [13] and [14] have to be interpreted as regularizations of the bulk and surface elastic energy of \(u\).

In the regular context of the Ambrosio and Tortorelli functional, we define through a variational argument the following notion of quasi-static evolution: we find a map \(t \to (u(t), v(t))\) from \([0, 1]\) to \(H^1(\Omega) \times H^1(\Omega)\), \(0 \leq v(t) \leq 1\), \(u(t) = g(t)\), \(v(t) = 1\) on \(\partial_D \Omega\) such that:

(a) for all \(0 \leq s < t \leq 1\): \(v(t) \leq v(s)\);

(b) for all \((u, v) \in H^1(\Omega) \times H^1(\Omega)\) with \(u = g(t)\), \(v = 1\) on \(\partial_D \Omega\), \(0 \leq v \leq v(t)\):

\[
F_\varepsilon(u(t), v(t)) \leq F_\varepsilon(u, v);
\]

(c) the energy \(\mathcal{E}_\varepsilon(t) := F_\varepsilon(u(t), v(t))\) is absolutely continuous and for all \(t \in [0, 1]\)

\[
\mathcal{E}_\varepsilon(t) = \mathcal{E}_\varepsilon(0) + 2 \int_0^t \int_\Omega (\eta \varepsilon + v_\varepsilon^2(\tau))|\nabla u(\tau)|^2 \varepsilon \, dx \, d\tau;
\]

(d) there exists a constant \(C\) depending only on \(g\) such that \(\mathcal{E}_\varepsilon(t) \leq C\) for all \(t \in [0, 1]\).

Condition (a) permits to recover in this regular context the fact that the fracture is increasing in time: in fact, as \(v(t)\) determines the fracture in the regions where it is near zero, the condition \(v(t) \leq v(s)\) ensures that existing cracks are preserved at subsequent times. Condition (b) reproduces the minimality condition at each time with respect to larger fractures, while condition (c) describes the evolution in time of the total energy. Condition (d) gives the necessary compactness in order to let \(\varepsilon \to 0\). In the particular case in which \(\|g(t)\|_\infty \leq C_1\) for all \(t \in [0, 1]\), it turns out that, using truncation arguments, \(\|u_\varepsilon(t)\|_\infty \leq C_1\) for all \(t\) so that a uniform \(L^\infty\) bound is available at any time. The requirement \(v(t) = 1\) on \(\partial_D \Omega\) for all \(t \in [0, 1]\) is made in such a way that, letting \(\varepsilon \to 0\), the surface energy of the fracture in the limit is the usual one also for the part touching the boundary \(\partial_D \Omega\).

The main result of the paper is that, as \(\varepsilon \to 0\), the quasi-static evolution \(t \to (u_\varepsilon(t), v_\varepsilon(t))\) for the Ambrosio-Tortorelli functional converges to a quasi-static evolution for brittle fracture in the sense of [13]. More precisely, there exists a quasi-static evolution \(t \to u(t) \in SBV(\Omega)\) relative to the boundary data \(g\) and a sequence \(\varepsilon_n \to 0\) such that for all \(t \in [0, 1]\) which are not discontinuity points of \(H^{N-1}(\Gamma(t))\) we have

\[
v_n(t) \nabla u_{\varepsilon_n}(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega, \mathbb{R}^N),
\]

\[
\int_\Omega (\eta_{\varepsilon_n} + v_{\varepsilon_n}(t)^2)|\nabla u_{\varepsilon_n}(t)|^2 \, dx \to \int_\Omega |\nabla u(t)|^2 \, dx
\]

and

\[
MM_{\varepsilon_n}(v_{\varepsilon_n}(t)) \to H^{N-1}(\Gamma(t)).
\]

Moreover \(\mathcal{E}_{\varepsilon_n}(t) \to \mathcal{E}(t)\) for all \(t \in [0, 1]\). We thus obtain an approximation of the total energy at any time, and an approximation of the gradient of the deformation, of the bulk and the surface energy at all time up to a countable set. The main step in the proof is to derive the unilateral minimality property [12] from its regularized version [13]. Given
\( z \in SBV(\Omega), \) a natural way consists in constructing \( z_n \in H^1(\Omega) \) and \( v_n \in H^1(\Omega) \) with \( z_n = g(t), \) \( v_n = 1 \) on \( \partial_D \Omega, \) \( 0 \leq v_n \leq v_n(t) \) and such that

\[
\lim \int_\Omega (\eta \varepsilon_n + v_n^2)|\nabla z_n|^2 \, dx = \int_\Omega |\nabla z|^2 \, dx
\]

and

\[
\limsup_n [MM_{\varepsilon_n}(v_n) - MM_{\varepsilon_n}(v_n(t))] \leq \mathcal{H}^{N-1}(S_z \setminus \Gamma(t)).
\]

We thus need a recovery sequence both for the deformation and the fracture: moreover we have to take into account the boundary conditions and the constraint \( v_n \leq v_n(t) \). Density results on \( z \), such that of considering \( S_z \) polyhedral, cannot be directly applied since the set \( S_z \cap \Gamma(t) \) could increase too much; on the other hand it is not possible to work in \( \Omega \setminus \Gamma(t) \) since no regularity results are available for \( \Gamma(t) \) apart from its rectifiability. It turns out that \( S_z \cap \Gamma(t) \) is the part of the fracture more difficult to be regularized, and in fact all the problems in the construction of \( (z_n,v_n) \) are already present in the particular case \( S_z \subseteq \Gamma(t) \).

In order to fix ideas, let us suppose to be in this situation; we solve the problem in two steps. We firstly construct \( \tilde{z}_n \in SBV(\Omega) \) with \( \nabla \tilde{z}_n \to \nabla z \) strongly in \( L^2(\Omega;\mathbb{R}^N) \) and such that \( S_{\tilde{z}_n} \) is related to \( u_n(t) \) and \( v_n(t) \) with precise energy estimates: this is done following the ideas of [13, Theorem 2.1], that is using local reflections and gluing along the boundaries of suitable upper levels of \( u_n(t) \), but we have to choose the upper levels in a more accurate way. In a second time, we regularize \( S_{\tilde{z}_n} \), using not only \( v_n(t) \), which is quite natural, but also \( u_n(t) \), so that (1.6) and (1.7) hold.

The plan of the paper is the following. We introduce in Section 2 the notation and the main tools employed in the rest of the paper. In Section 3 we treat the quasi-static evolution for the Ambrosio-Tortorelli functional, while in Section 5 we prove the main approximation result. The derivation of the minimality property (1.2) is contained in Section 6.

## 2. Notation and Preliminaries

In this section we state the notations and introduce the main tools used in the rest of the paper.

**Basic notation.** In the rest of the paper, we will employ the following basic notations:
- \( \Omega \) is an open bounded subset of \( \mathbb{R}^N \) with Lipschitz boundary;
- \( \partial_D \Omega \) is a subset on \( \partial \Omega \) open in the relative topology;
- \( L^p(\Omega;\mathbb{R}^m) \) with \( 1 \leq p \leq +\infty \) and \( m \geq 1 \) is the Lebesgue space of \( p \)-summable \( \mathbb{R}^m \)-valued functions;
- \( H^1(\Omega) \) is the Sobolev spaces of functions in \( L^2(\Omega) \) with distributional derivative in \( L^2(\Omega;\mathbb{R}^N) \);
- if \( u \in H^1(\Omega) \), \( \nabla u \) is its gradient;
- \( \mathcal{H}^{N-1} \) is the \((N-1)\)-dimensional Hausdorff measure;
- \( \| \cdot \|_\infty \) is the sup-norm;
- \( 1_A \) is the characteristic function of \( A \);
- if \( \sigma \in [0, +\infty] \), \( o(\sigma) \) is such that \( \lim_{\sigma \to 0^+} o(\sigma) = 0 \).

**Special functions of bounded variation.** For the general theory of functions of bounded variation, we refer to [3]; here we recall some basic definitions and theorems we need in the sequel.

Let \( A \) be an open subset of \( \mathbb{R}^N \), and let \( u : A \to \mathbb{R}^n \). We say that \( u \in BV(A;\mathbb{R}^n) \) if \( u \in L^1(A;\mathbb{R}^n) \), and its distributional derivative is a vector-valued Radon measure on \( A \).
We say that \( u \in SBV(A; \mathbb{R}^n) \) if \( u \in BV(A; \mathbb{R}^n) \) and its distributional derivative can be represented as
\[
Du(B) = \int_B \nabla u(x) \, dx + \int_{\partial S_u} (u^+(x) - u^-(x)) \otimes \nu_x \, dH^{n-1}(x)
\]
where \( \nabla u \) denotes the approximate gradient of \( u \), \( S_u \) denotes the set of approximate jumps of \( u \), \( u^+ \) and \( u^- \) are the traces of \( u \) on \( S_u \), and \( \nu_x \) is the normal to \( S_u \) at \( x \). The space \( SBV(A; \mathbb{R}^n) \) is called the space of special functions of bounded variation. Note that if \( u \in SBV(A; \mathbb{R}^n) \), then the singular part of \( Du \) is concentrated on \( S_u \) which turns out to be countably \( H^{n-1} \)-rectifiable.

We say that \( u \in GSBV(A) \) if for every \( M \geq 0 \) we have \((u \land M) \vee (-M) \in SBV(A)\). For every \( p \in [1, \infty] \), let us pose \( SBVP(A, \mathbb{R}^n) := \{ u \in SBV(A, \mathbb{R}^n) : \nabla u \in L^p(A; M^{n \times n}) \} \), and \( GSBVP(A) := \{ u \in GSBV(A) : \nabla u \in L^p(A; \mathbb{R}^n) \} \).

The space \( SBV \) is very useful when dealing with variational problems involving volume and surface energies because of the following compactness and lower semicontinuity result due to L.Ambrosio (see [1], [2]).

**Theorem 2.1.** Let \( (u_k) \) be a sequence in \( SBV(A; \mathbb{R}^n) \) such that there exist \( q > 1 \) and \( c \geq 0 \) with
\[
\int_A |\nabla u_k|^q \, dx + \mathcal{H}^{n-1}(S_{u_k}) + ||u_k||_{\infty,A} \leq c
\]
for every \( k \in \mathbb{N} \). Then there exist a subsequence \( (u_{k_h}) \) and a function \( u \in SBV(A; \mathbb{R}^n) \) such that
\[
\begin{align*}
(2.1) & \quad u_{k_h} \rightharpoonup u \quad \text{strongly in } L^1(A; \mathbb{R}^n), \\
& \quad \nabla u_{k_h} \rightharpoonup \nabla u \quad \text{weakly in } L^1(A; M^{n \times n}), \\
& \quad \mathcal{H}^{n-1}(S_u) \leq \liminf_{h \to \infty} \mathcal{H}^{n-1}(S_{u_{k_h}}).
\end{align*}
\]

In the rest of the paper, we will say that \( u_k \rightharpoonup u \) in \( SBV(\Omega) \) if \( u_k \) and \( u \) satisfy (2.1).

**Quasi-static evolution of brittle fractures.** Let \( g : [0, 1] \to H^1(\Omega) \) be absolutely continuous; we indicate the gradient of \( g \) at time \( t \) by \( \nabla g(t) \), and the time derivative of \( g \) at time \( t \) by \( \dot{g}(t) \). The main result of [24] is the following theorem.

**Theorem 2.2.** There exists a crack \( \Gamma(t) \subseteq \overline{\Omega} \) and a field \( u(t) \in SBV(\Omega) \) such that
\begin{enumerate}[(a)]
  
  (a) \( \Gamma(t) \) increases with \( t \);
  
  (b) \( u(0) \) minimizes
  \[
  \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{n-1}(S_v \cup \{ x \in \partial \Omega : v(x) \neq g(0)(x) \})
  \]
  among all \( v \in SBV(\Omega) \) (inequalities on \( \partial \Omega \) are intended for the traces of \( v \) and \( g \);
  
  (c) for \( t > 0 \), \( u(t) \) minimizes
  \[
  \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{n-1}([S_v \cup \{ x \in \partial \Omega : v(x) \neq g(t)(x) \}] \setminus \Gamma(t))
  \]
  among all \( v \in SBV(\Omega) \);

  (d) \( S_{u(t)} \cup \{ x \in \partial \Omega : u(t)(x) \neq g(t)(x) \} \subseteq \Gamma(t) \), up to a set of \( \mathcal{H}^{n-1} \)-measure 0.
\end{enumerate}  
Furthermore, the total energy
\[
\mathcal{E}(t) := \int_\Omega |\nabla u(t)|^2 \, dx + \mathcal{H}^{n-1}(\Gamma(t))
\]
is absolutely continuous and is given by
\[ \mathcal{E}(t) = \mathcal{E}(0) + 2 \int_0^t \int_\Omega \nabla u(\tau) \nabla g(\tau) \, dx \, d\tau. \]

Finally, for any countable, dense set \( I \subseteq [0,1] \), the crack \( \Gamma(t) \) and the field \( u(t) \) can be chosen such that
\[ \Gamma(t) = \bigcup_{\tau \in I, \tau \leq t} (S_u(\tau) \cup \{x \in \partial_D \Omega : u(\tau)(x) \neq g(\tau)(x)\}) \]

The Ambrosio-Tortorelli functional. In [5] and [6], Ambrosio and Tortorelli proposed an elliptic approximation of the Mumford-Shah functional in the sense of \( \Gamma \)-convergence. Their result has been extended in the vectorial case in [14], where non-isotropic surface energies are also considered. For every \( u \in GSBV(\Omega) \) let
\[ F(u) := \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(S_u) \]
the well known Mumford-Shah functional; for every \((u,v) \in H^1(\Omega) \times H^1(\Omega)\) the Ambrosio-Tortorelli functional is defined by
\[ F_\varepsilon(u,v) := \int_\Omega (\eta_\varepsilon + \varepsilon^2)|\nabla u|^2 \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla v|^2 \, dx + \frac{1}{2\varepsilon} \int_\Omega (1-v)^2 \, dx \]
where \( \eta_\varepsilon > 0 \) and \( \eta_\varepsilon \ll \varepsilon \). Let us indicate the space of Borel functions on \( \Omega \) by \( \mathcal{B}(\Omega) \) and let us consider on \( \mathcal{B}(\Omega) \times \mathcal{B}(\Omega) \) the functionals
\[ \mathcal{F}(u,v,\Omega) := \begin{cases} F(u) & u \in GSBV(\Omega), v \equiv 1 \text{ a.e. on } \Omega \\ +\infty & \text{otherwise} \end{cases} \]
and
\[ \mathcal{F}_\varepsilon(u,v,\Omega) := \begin{cases} F_\varepsilon(u,v) & (u,v) \in H^1(\Omega), 0 \leq v \leq 1 \\ +\infty & \text{otherwise} \end{cases} \]

The Ambrosio-Tortorelli result can be expressed in the following way.

**Theorem 2.3.** The functionals \((F_\varepsilon)\) on \( \mathcal{B}(\Omega) \times \mathcal{B}(\Omega) \) \( \Gamma \)-converge to \( F \) with respect to the convergence in measure.

In particular, we will use several times the following fact: if \((u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times H^1(\Omega)\) is such that \( F_\varepsilon(u_\varepsilon, v_\varepsilon) + ||u_\varepsilon||_{L^1} \leq C \), there exists \( u \in SBV(\Omega) \) and a sequence \( \varepsilon_k \rightarrow 0 \) such that \( u_\varepsilon \rightarrow u \) a.e., and
\[ \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(S_u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, v_\varepsilon). \] (2.2)

**A density result.** Let \( A \subseteq \mathbb{R}^N \) be open. We say that \( K \subseteq A \) is polyhedral (with respect to \( A \)), if it is the intersection of \( A \) with the union of a finite number of \((N-1)\)-dimensional simplexes of \( S \).

The following density result is proved in [9].

**Theorem 2.4.** Assume that \( \partial A \) is locally Lipschitz, and let \( u \in GSBV^p(A) \). For every \( \varepsilon > 0 \), there exists a function \( \nu \in SBV^p(A) \) such that

(a) \( S_\nu \) is essentially closed, i.e., \( \mathcal{H}^{N-1}(\overline{S_\nu} \setminus S_\nu) = 0 \);

(b) \( \overline{S_\nu} \) is a polyhedral set;

(c) \( \nu \in W^{k,\infty}(A \setminus \overline{S_\nu}) \) for every \( k \in \mathbb{N} \);
Let \( \partial \Omega \cap \partial \Omega = \partial \Omega \setminus \partial \Omega \) and denote by \( \Omega' := \Omega \setminus \Omega \). Given \( g \in H^1(\Omega) \) and \( u \in SBV(\Omega') \) with \( u = g \) on \( \partial \Omega \setminus \Omega \), there exists \( u_h \in SBV(\Omega') \) such that

(a) \( u_h = g \) in \( \Omega' \setminus \Omega \) and in a neighborhood of \( \partial \Omega \);

(b) \( S_{u_h} \) is polyhedral and \( S_{u_h} \subseteq \Omega \) for all \( h \);

(c) \( \nabla u_h \rightarrow \nabla u \) strongly in \( L^2(\Omega'; \mathbb{R}^N) \);

(d) for all \( A \) open subset of \( \Omega' \) with \( H^{-1}(\partial A \cap S_u) = 0 \), we have

\[
\lim_{h \to 0} H^{-1}(A \cap S_{u_h}) = H^{-1}(A \cap S_u).
\]

Proof. Using a partition of unity, we may prove the result in the case \( \Omega' := Q \setminus [0,1]^N \). Let \( g \in C^{1,1}(Q) \), \( \varphi_h = \chi_{\{y \leq f(x) + \frac{1}{h}\}} \), \( \varphi_h = 0 \) on \( \{y \geq f(x) + \frac{1}{h}\} \), and \( \|\nabla \varphi_h\|_{L^\infty} \leq \frac{1}{h} \). Let \( u := g \) on \( \Omega' \setminus \Omega \).

Let \( w_h := u(x - h \xi) \) where \( \xi \in \mathbb{R}^N \) is the versor of the \( N \)-axis, and let \( \varphi_h \) be a cut off function with \( \varphi_h = 1 \) on \( \{y \leq f(x) + \frac{1}{h}\} \), \( \varphi_h = 0 \) on \( \{y \geq f(x) + \frac{1}{h}\} \), and \( \|\nabla \varphi_h\|_{L^\infty} \leq \frac{1}{h} \). Let us pose \( v_h := \varphi_h g + (1 - \varphi_h) w_h \). We have that \( v_h = g \) in \( \Omega' \setminus \Omega \) and in a neighborhood of \( \partial \Omega \); moreover we have

\[
\nabla v_h = \varphi_h \nabla g + (1 - \varphi_h) \nabla w_h + \nabla \varphi_h (g - w_h).
\]

Since \( \nabla \varphi_h (g - w_h) \rightarrow 0 \) strongly in \( L^2(\Omega'; \mathbb{R}^N) \), we have \( \nabla v_h \rightarrow \nabla u \) strongly in \( L^2(\Omega'; \mathbb{R}^N) \). Finally, for all \( A \) open subset of \( \Omega' \) with \( H^{-1}(\partial A \cap S_u) = 0 \), we have

\[
\lim_{h \to 0} H^{-1}(A \cap S_{v_h}) = H^{-1}(A \cap S_u).
\]

In order to conclude the proof, let us apply Theorem 2.4 obtaining \( \tilde{v}_h \) with polyhedral jumps such that \( \|v_h - \tilde{v}_h\|_{L^2(\Omega')} + \|\nabla v_h - \nabla \tilde{v}_h\|_{L^2(\Omega'; \mathbb{R}^N)} \leq h^2 \), \( |H^{-1}(S_{v_h}) - H^{-1}(S_{\tilde{v}_h})| \leq h \). If we pose \( u_h := \varphi_h g + (1 - \varphi_h) \tilde{v}_h \), we obtain the thesis. \( \square \)

3. The Main Results

If \( g \in W^{1,1}([0,1]; H^1(\Omega)) \), we indicate the gradient of \( g \) at time \( t \) by \( \nabla g(t) \), and the time derivative of \( g \) at time \( t \) by \( \dot{g}(t) \).

Concerning the Ambrosio-Tortorelli functional, the following theorem holds.

**Theorem 3.1.** Let \( g \in W^{1,1}([0,1]; H^1(\Omega)) \). Then there exists a strongly measurable map

\[
[0,1] \quad \mapsto \quad H^1(\Omega) \times H^1(\Omega)
\]

\[
t \quad \mapsto \quad (u(t), v(t))
\]

such that \( 0 \leq v(t) \leq 1 \) in \( \Omega \), \( u(t) = g(t) \), \( v(t) = 1 \) on \( \partial \Omega \) for all \( t \in [0,1] \), and:

(a) for all \( 0 \leq s \leq t \leq 1 \) : \( v(t) \leq v(s) \);
Theorem 3.2. Let $g \in W^{1,1}([0,1];H^1(\Omega))$ such that $\|g(t)\|_\infty \leq C$ for all $t \in [0,1]$, and let $g_h \in W^{1,1}([0,1];H^1(\Omega))$ be a sequence of absolutely continuous functions with $\|g_h(t)\|_\infty \leq C$, $g_h(t) \in C(\bar{\Omega})$ for all $t \in [0,1]$ and such that $g_h \to g$ strongly in $W^{1,1}([0,1];H^1(\Omega))$. Then there exists a quasi-static evolution $t \to u(t) \in SBV(\Omega)$ relative to the boundary data $g$ in the sense of Theorem 2.2, and two sequences $\varepsilon_n \to 0$ and $h_n \to +\infty$ such that, posing $u_n := u_{\varepsilon_n, h_n}$ and $v_n := v_{\varepsilon_n, h_n}$, the following hold:

(a) for all $t \in [0,1]$ we have
\[ F_{\varepsilon_n}(u_n(t), v_n(t)) \to \mathcal{E}(t); \]

(b) if $\mathcal{N}$ denotes the point of discontinuity of $\mathcal{H}^{N-1}(\Gamma(\cdot))$, for all $t \in [0,1] \setminus \mathcal{N}$ we have
\[ v_n(t)\nabla u_n(t) \to \nabla u(t) \text{ strongly in } L^2(\Omega;\mathbb{R}^N), \]
and
\[ \lim_n \frac{1}{\varepsilon_n} \int_{\Omega} (|\nabla v_n(t)|^2 + \frac{1}{2\varepsilon_n} |1 - v_n(t)|^2) dx = \mathcal{H}^{N-1}(\Gamma(t)). \]

4. QUASI-STATIC EVOLUTION FOR THE AMBROSIO-TORTORELLI FUNCTIONAL

This section is devoted to the proof of Theorem 3.2 where a suitable notion of quasi-static evolution in a regular context is proposed. The evolution will be obtained through a discretization in time procedure: each step will be performed using a variational argument which will give the minimality property stated in points (b) and (c).

Let $g \in W^{1,1}([0,1];H^1(\Omega))$. Given $\delta > 0$, let $N_\delta$ be the largest integer such that $\delta N_\delta \leq 1$; for $i \geq 0$ we pose $t_i^\delta = i\delta$ and for $0 \leq i \leq N_\delta$ we pose $g_i^\delta = g(t_i^\delta)$. Define $u_0^\delta$ and $v_0^\delta$ as a minimum for the problem
\begin{equation}
(4.1) \quad \min \{ F_e(u,v) : (u,v) \in H^1(\Omega) \times H^1(\Omega), 0 \leq v \leq 1 \text{ in } \Omega, u = g_0^\delta, v = 1 \text{ on } \partial_D \Omega \},
\end{equation}
and let $(u_{i+1}^\delta, v_{i+1}^\delta)$ be a minimum for the problem
\begin{equation}
(4.2) \quad \min \{ F_e(u,v) : (u,v) \in H^1(\Omega) \times H^1(\Omega), 0 \leq v \leq v_i^\delta \text{ in } \Omega, u = g_i^\delta, v = 1 \text{ on } \partial_D \Omega \}.
\end{equation}
Problems (11.1) and (11.2) are well posed: in fact, referring for example to problem (11.2), let \((u_n, v_n)\) be a minimizing sequence. Since \((g_{i+1}^\delta, v_i^\delta)\) is an admissible pair, we obtain that there exists a constant \(C > 0\) such that for all \(n\)
\[
F_\varepsilon(u_n, v_n) \leq C.
\]
Since \(\varepsilon, \eta_\varepsilon > 0\), we deduce that \((u_n, v_n)\) is bounded in \(H^1(\Omega) \times H^1(\Omega)\) so that up to a subsequence \(u_n \rightharpoonup u\) and \(v_n \rightharpoonup v\) weakly in \(H^1(\Omega)\). We get immediately that \(u = g_{i+1}^\delta\) and \(v = 1\) on \(\partial D\Omega\) since \(u_n = g_{i+1}^\delta\) and \(v_n = 1\) on \(\partial D\Omega\) for all \(n\); on the other hand, since \(v_n \to v\) strongly in \(L^2(\Omega)\), we obtain that \(0 \leq v \leq v_i^\delta\). By semicontinuity, we have
\[
F_\varepsilon(u, v) \leq \liminf_n F_\varepsilon(u_n, v_n)
\]
so that \((u, v)\) is a minimum point for problem (11.2).

We note that by minimality of the pair \((u_{i+1}^\delta, v_{i+1}^\delta)\), we may write
\[
(4.3) \quad F_\varepsilon(u_{i+1}^\delta, v_{i+1}^\delta) \leq F_\varepsilon(u_{i+1}^\delta + g_{i+1}^\delta - g_i^\delta, v_{i+1}^\delta) =
\]
\[
= F_\varepsilon(u_i^\delta, v_i^\delta) + 2 \int_\Omega (\eta_\varepsilon + (v_i^\delta)^2) \nabla u_i^\delta \cdot \nabla (g_{i+1}^\delta - g_i^\delta) \, dx + \int_\Omega (\eta_\varepsilon + (v_i^\delta)^2) |\nabla(g_{i+1}^\delta - g_i^\delta)|^2 \, dx \leq
\]
\[
\leq F_\varepsilon(u_i^\delta, v_i^\delta) + 2 \int_{t_i^\delta}^{t_{i+1}^\delta} \int_\Omega (\eta_\varepsilon + (v_i^\delta)^2) \nabla u_i^\delta \cdot \nabla \dot{g}(\tau) \, dx \, d\tau + \varepsilon(\delta) \int_{t_i^\delta}^{t_{i+1}^\delta} ||\nabla \dot{g}(\tau)||_{L^2(\Omega; R^N)} \, d\tau,
\]
where
\[
e(\delta) := (1 + \eta_\varepsilon) \max_{0 \leq \tau \leq N_i - 1} \int_{t_i^\delta}^{t_{i+1}^\delta} ||\nabla \dot{g}(\tau)||_{L^2(\Omega; R^N)} \, d\tau
\]
is infinitesimal as \(\delta \to 0\).

We now make a piecewise constant interpolation defining
\[
(4.4) \quad u_i^\delta(t) = u_i^\delta, \quad v_i^\delta(t) = v_i^\delta, \quad g_i^\delta(t) = g_i^\delta \quad \text{for} \quad t_i^\delta \leq t < t_{i+1}^\delta.
\]
Note that by construction the map \(t \to v_i^\delta(t)\) is decreasing from \([0, 1]\) to \(L^2(\Omega)\). Moreover, iterating the estimate (4.3), we obtain
\[
F_\varepsilon(u_i^\delta(t), v_i^\delta(t)) \leq F_\varepsilon(u_i^\delta(s), v_i^\delta(s)) + 2 \int_{s_i^\delta}^{t_i^\delta} \int_\Omega (\eta_\varepsilon + v_i^\delta(\tau)^2) \nabla u_i^\delta(\tau) \cdot \nabla \dot{g}(\tau) \, dx \, d\tau +
\]
\[
\varepsilon(\delta) \int_{s_i^\delta}^{t_i^\delta} ||\nabla \dot{g}(\tau)||_{L^2(\Omega; R^N)} \, d\tau
\]
where \(s_i^\delta := s_i^\delta\) with \(s_i^\delta \leq s < s_{i+1}^\delta\), \(t_i^\delta := t_i^\delta\) with \(t_i^\delta \leq t < t_{i+1}^\delta\).

Note that by minimality of the pair \((u_i^\delta(t), v_i^\delta(t))\), we have
\[
F_\varepsilon(u_i^\delta(t), v_i^\delta(t)) \leq F_\varepsilon(g_i^\delta(t), v_i^\delta(t))
\]
so that
\[
(4.6) \quad \int_\Omega (\eta_\varepsilon + v_i^\delta(t)^2) |\nabla u_i^\delta(t)|^2 \, dx \leq \int_\Omega (\eta_\varepsilon + v_i^\delta(t)^2) |\nabla g_i^\delta(t)|^2 \, dx \leq C_1
\]
with \(C_1 > 0\) independent of \(\delta\) and \(t\). In particular by (4.4) we have that
\[
||\nabla u_i^\delta(t)||_{L^2(\Omega; R^N)} \leq \frac{C_1}{\eta_\varepsilon}.
\]
Since \(u_i^\delta(t) = g_i^\delta(t)\) on \(\partial D\Omega\), and \(g_i^\delta(t)\) is uniformly bounded in \(H^1(\Omega)\) for all \(t\) and \(\delta\), we get by a variant of Poincaré inequality that \(u_i^\delta(t)\) is uniformly bounded in \(H^1(\Omega)\) for all \(t\) and \(\delta\).
Now we pass to $v$ in order to obtain some coerciveness in the space $H^1(\Omega)$. Notice that
\[
2 \left| \int_0^t \int_\Omega (\eta_\varepsilon + v^\delta(t)^2) \nabla u^\delta(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau \right| \leq \\
\leq 2 \int_0^t \sqrt{\eta_\varepsilon + 1} \left( \int_\Omega (\eta_\varepsilon + v^\delta(t)^2) |\nabla u^\delta(t)|^2 \, dx \right)^{\frac{1}{2}} \|\nabla \dot{g}(\tau)\|_{L^2(\Omega; \mathbb{R}^N)} \, d\tau,
\]
and by (4.8), we obtain
\[
(4.7) \quad 2 \int_0^t \int_\Omega (\eta_\varepsilon + v^\delta(t)^2) \nabla u^\delta(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau \leq C_2
\]
with $C_2 > 0$ independent of $t$ and $\delta$.

By (4.5) with $s = 0$, and (4.7), we deduce
\[
\frac{\varepsilon}{2} \int_\Omega |\nabla v^\delta(t)|^2 \, dx + \frac{1}{2\varepsilon} \int_\Omega (1 - v^\delta(t))^2 \, dx \leq \\
\leq F_\varepsilon(u^\delta(0), v^\delta(0)) + 2 \int_0^t \int_\Omega \eta_\varepsilon + v^\delta(\tau)^2) \nabla u^\delta(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau + \\
+ c(\delta) \int_{\tau^*}^t \|\nabla \dot{g}(\tau)\|_{L^2(\Omega; \mathbb{R}^N)} \, d\tau \leq \\
\leq F_\varepsilon(u^\delta(0), v^\delta(0)) + C_2 + c(\delta) \int_0^1 \|\nabla \dot{g}(\tau)\|_{L^2(\Omega)} \, d\tau.
\]
We conclude that there exists $C > 0$ independent of $t$ and $\delta$ such that for all $t \in [0, 1]$
\[
(4.8) \quad \|v^\delta(t)\|_{H^1(\Omega)} \leq C.
\]
We now want to pass to the limit in $\delta$ as $\delta \to 0$.

**Lemma 4.1.** There exists a sequence $\delta_n \to 0$ and a strongly measurable map $v : [0, 1] \to H^1(\Omega)$ such that $v^{\delta_n}(t) \to v(t)$ weakly in $H^1(\Omega)$ for all $t \in [0, 1]$. Moreover, $v$ is decreasing from $[0, 1]$ to $L^2(\Omega)$, and $0 \leq v(t) \leq 1$ in $\Omega$, $v(t) = 1$ on $\partial D \Omega$ for all $t \in [0, 1]$.

**Proof.** Since the map $t \to v^\delta(t)$ is monotone decreasing from $[0, 1]$ to $L^2(\Omega)$, and $0 \leq v^\delta(t) \leq 1$ for all $t$, we deduce by a variant of Helly’s compactness theorem for sequences of monotone real functions, that there exists a subsequence $\delta_n \to 0$ and a decreasing map $v : [0, 1] \to L^2(\Omega)$ such that for all $t \in [0, 1]$ we have $v^{\delta_n}(t) \to v(t)$ strongly in $L^2(\Omega)$. In particular we deduce $0 \leq v(t) \leq 1$ in $\Omega$. By (4.5), we have that for all $t \in [0, 1]$, up to a subsequence, $v^{\delta_n}(t) \to w$ weakly in $H^1(\Omega)$; since $v^{\delta_n}(t) \to v$ strongly in $L^2(\Omega)$, we deduce that $w = v(t)$ so that $v(t) \in H^1(\Omega)$, and $v^{\delta_n}(t) \to v(t)$ weakly in $H^1(\Omega)$. As a consequence, $v(t) = 1$ on $\partial D \Omega$ for all $t \in [0, 1]$. Finally, $v$ is strongly measurable from $[0, 1]$ to $H^1(\Omega)$ because it is weakly measurable and separably valued (see Chapter V, Section 4). \[ \square \]

Let us consider the sequence $\delta_n$, and the map $v$ given by Lemma 4.1. We indicate $u^{\delta_n}$, $v^{\delta_n}$ and $g^{\delta_n}$ simply by $u_n$, $v_n$ and $g_n$.

**Lemma 4.2.** There exists a strongly measurable map $u : [0, 1] \to H^1(\Omega)$ such that $u_n(t) \to u(t)$ strongly in $H^1(\Omega)$ for all $t \in [0, 1]$. In particular, $u(t) = g(t)$ on $\partial D \Omega$ for all $t \in [0, 1]$.

**Proof.** Let $t \in [0, 1]$. We note that $u_n(t)$ is the minimum of the following problem
\[
\min \left\{ \int_\Omega (\eta_\varepsilon + v_n^2(t))|\nabla z|^2 \, dx : z \in H^1(\Omega), z = g_n(t) \text{ on } \partial D \Omega \right\}.
\]
Since by Lemma 4.1, $v_n(t) \to v(t)$ strongly in $L^2(\Omega)$, and $g_n(t) \to g(t)$ strongly in $H^1(\Omega)$, we deduce by standard results on $\Gamma$-convergence (see Chapter V), that $u_n(t) \to u(t)$ weakly in $H^1(\Omega)$
where $u(t)$ is the solution of the problem

$$\min \left\{ \int_{\Omega} (\eta + v^2(t)) |\nabla z|^2 \, dx : z \in H^1(\Omega), z = g(t) \text{ on } \partial_D \Omega \right\}.$$

Moreover, we have also convergence of energies, that is

$$\lim_n \int_{\Omega} (\eta_n + v_n^2(t)) |\nabla u_n(t)|^2 \, dx = \int_{\Omega} (\eta + v^2(t)) |\nabla u(t)|^2 \, dx. \tag{4.9}$$

Since $v_n(t) \nabla u_n(t) \to v(t) \nabla u(t)$ weakly in $L^2(\Omega; \mathbb{R}^N)$, we obtain

$$\int v_n^2(t) |\nabla u(t)|^2 \, dx \leq \liminf_n \int v_n^2(t) |\nabla u_n(t)|^2 \, dx,$$

so that by (4.9) we deduce $\nabla u_n(t) \to \nabla u(t)$ strongly in $L^2(\Omega; \mathbb{R}^N)$. We conclude that $u_n(t) \to u(t)$ strongly in $H^1(\Omega)$ for all $t \in [0, 1]$, and so the map $t \to u(t)$ is strongly measurable from $[0, 1]$ to $H^1(\Omega)$. Finally $u(t) = g(t)$ on $\partial_D \Omega$ and the proof is complete. \hfill $\Box$

The following minimality property for the pair $(u(t), v(t))$ holds.

**Proposition 4.3.** If $t \in [0, 1]$, for every $(u, v) \in H^1(\Omega) \times H^1(\Omega)$ such that $0 \leq v \leq v(t)$ in $\Omega$, and $u = g(t)$, $v = 1$ on $\partial_D \Omega$, we have

$$F_c(u(t), v(t)) \leq F_c(u, v).$$

Moreover, for all $(u, v) \in H^1(\Omega) \times H^1(\Omega)$ such that $u = g(0)$, $v = 1$ on $\partial_D \Omega$, we have

$$F_c(u(0), v(0)) \leq F_c(u, v).$$

**Proof.** Let us pose

$$u_n := u + g_n(t) - g(t),$$

and

$$v_n := \min \{v_n(t), v\};$$

we have $u_n \to u$ strongly in $H^1(\Omega)$, and $v_n \to v$ weakly in $H^1(\Omega)$. Since $0 \leq v_n \leq v(t)$ in $\Omega$, and $u_n = g_n(t)$, $v_n = 1$ on $\partial_D \Omega$, by the minimality property of the pair $(u_n(t), v_n(t))$ we get

$$F_c(u_n(t), v_n(t)) \leq F_c(u_n, v_n),$$

that is

$$\int_{\Omega} (\eta + v_n(t)^2) |\nabla u_n(t)|^2 \, dx + \frac{\varepsilon}{2} \int_{\Omega} |\nabla v_n(t)|^2 \, dx + \frac{1}{2\varepsilon} \int_{\Omega} (1 - v_n(t))^2 \, dx \leq \int_{\Omega} (\eta + v^2(t)) |\nabla u(t)|^2 \, dx + \frac{\varepsilon}{2} \int_{\Omega} |\nabla v_n(t)|^2 \, dx + \frac{1}{2\varepsilon} \int_{\Omega} (1 - v_n)^2 \, dx. \tag{4.10}$$

Notice that

$$\frac{\varepsilon}{2} \int_{\Omega} |\nabla v_n|^2 \, dx = \frac{\varepsilon}{2} \int_{\{v_n(t) < v\}} |\nabla v_n(t)|^2 \, dx + \frac{\varepsilon}{2} \int_{\{v_n(t) \geq v\}} |\nabla v|^2 \, dx$$

so that (4.10) becomes

$$\int_{\Omega} (\eta + v_n(t)^2) |\nabla u_n(t)|^2 \, dx + \frac{\varepsilon}{2} \int_{\{v_n(t) \geq v\}} |\nabla v_n(t)|^2 \, dx + \frac{1}{2\varepsilon} \int_{\Omega} (1 - v_n(t))^2 \, dx \leq \int_{\Omega} (\eta + v^2(t)) |\nabla u(t)|^2 \, dx + \frac{\varepsilon}{2} \int_{\Omega} |\nabla v|^2 \, dx + \frac{1}{2\varepsilon} \int_{\Omega} (1 - v)^2 \, dx.$$

For $n \to \infty$, the right hand side is less than $F_c(u, v)$. Let us consider the left hand side. By semicontinuity we have

$$\liminf_n \frac{\varepsilon}{2} \int_{\{v_n(t) \geq v\}} |\nabla v_n(t)|^2 \, dx \geq \frac{\varepsilon}{2} \int_{\Omega} |\nabla v(t)|^2 \, dx,$$

and so we conclude that $F_c(u(t), v(t)) \leq F_c(u, v)$. 
For the case \( t = 0 \), by lower semicontinuity we get immediately the result. \( \square \)

In order to obtain the proof of Theorem 3.1 we need the following proposition.

**Proposition 4.4.** For all \( 0 \leq s \leq t \leq 1 \), we have that

\[
F_\varepsilon(u(t), v(t)) - F_\varepsilon(u(s), v(s)) \geq 2 \int_{\Omega} (\eta_\varepsilon + \varepsilon^2) \nabla u(t) \cdot (\nabla g(t) - \nabla g(s)) \, dx + -\sigma(t - s) \int_s^t \|\nabla \dot{g}(\tau)\|_{L^2(\Omega; \mathbb{R}^N)} \, d\tau
\]

where \( \sigma \) is an increasing positive function with \( \sigma(r) \to 0 \) as \( r \to 0^+ \).

**Proof.** By Proposition 4.3 we have

\[
F_\varepsilon(u(s), v(s)) \leq F_\varepsilon(u(t) - g(t) + g(s), v(t))
\]

so that

\[
F_\varepsilon(u(s), v(s)) \leq F_\varepsilon(u(t), v(t)) - 2 \int_{\Omega} (\eta_\varepsilon + \varepsilon^2) \nabla u(t) \cdot (\nabla g(t) - \nabla g(s)) \, dx + + \int_{\Omega} (\eta_\varepsilon + \varepsilon^2) |\nabla g(t) - \nabla g(s)|^2 \, dx.
\]

Then we conclude that

\[
F_\varepsilon(u(t), v(t)) - F_\varepsilon(u(s), v(s)) \geq 2 \int_{\Omega} (\eta_\varepsilon + \varepsilon^2) \nabla u(t) \cdot (\nabla g(t) - \nabla g(s)) \, dx + -\sigma(t - s) \int_s^t \|\nabla \dot{g}(\tau)\|_{L^2(\Omega; \mathbb{R}^N)} \, d\tau
\]

where

\[
\sigma(r) := (1 + \eta_\varepsilon) \max_{t-s=r} \int_s^t \|\nabla \dot{g}(\tau)\|_{L^2(\Omega; \mathbb{R}^N)} \, d\tau,
\]

and so the proof is complete. \( \square \)

We can now prove Theorem 3.1.

**Proof of Theorem 3.1.** Let us consider the sequence \( \delta_n \to 0 \) given by Lemma 4.1 and let us indicate the discrete evolutions \( u^{\delta_n} \) and \( v^{\delta_n} \) defined in (4.4) simply by \( u_n \) and \( v_n \). Let us denote also by \( u(t) \) and \( v(t) \) their limits at time \( t \) according to Lemma 4.1 and Proposition 4.3. We have that the maps \( t \to u(t) \) and \( t \to v(t) \) are strongly measurable from \( [0, 1] \) to \( H^1(\Omega) \); moreover for all \( t \in [0, 1] \) we have \( 0 \leq v(t) \leq 1 \) in \( \Omega \), \( u(t) = g(t) \), \( v(t) = 1 \) on \( \partial \Omega \), and \( t \to v(t) \) is decreasing from \( [0, 1] \) to \( L^2(\Omega) \) so that point (a) is proved. By construction we get point (b) and by Proposition 4.3 we get point (c).

Let us pass to condition (d). Let us fix \( t \in [0, 1] \), and let us divide the interval \([0, t]\) in \( k \) subintervals with endpoints \( s_j^k := \frac{jt}{k} \) where \( j = 0, 1, \ldots, k \). Let us define \( \tilde{u}_k(s) := u(s_{j+1}^k) \), and \( \tilde{v}_k(s) := v(s_{j+1}^k) \) for \( s_j^k < s \leq s_{j+1}^k \). Then, applying Proposition 4.4 we have

\[
(4.11) \quad F_\varepsilon(u(t), v(t)) \geq F_\varepsilon(u(0), v(0)) + 2 \int_0^t \int_{\Omega} (\eta_\varepsilon + \tilde{v}_k^2(\tau)) \nabla \tilde{u}_k(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau + -\sigma \left( \frac{t}{k} \right) \int_0^t \|\nabla \dot{g}(\tau)\|_{L^2} \, d\tau.
\]

Since \( t \to v(t) \) is monotone decreasing from \( [0, 1] \) to \( L^2(\Omega) \), we have that \( \tilde{v}_k(s) \to v(s) \) strongly in \( L^2(\Omega) \) for a.e. \( s \in [0, t] \); consequently, we have that \( \tilde{u}_k(s) \to u(s) \) strongly in \( H^1(\Omega) \) as noted in Lemma 4.2. We conclude by the Dominated Convergence Theorem that

\[
\lim_k \int_0^t \int_{\Omega} (\eta_\varepsilon + \tilde{v}_k^2(\tau)) \nabla \tilde{u}_k(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau = \int_0^t \int_{\Omega} (\eta_\varepsilon + v^2(\tau)) \nabla u(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau.
\]
By (4.11) we deduce that

\[ F_\varepsilon(u(t), v(t)) \geq F_\varepsilon(u(0), v(0)) + 2 \int_0^t \int_\Omega (\eta_\varepsilon + v^2(\tau)) \nabla u(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau. \]

On the other hand, from (4.13), and since \( F_\varepsilon(u_n(0), v_n(0)) = F_\varepsilon(u(0), v(0)) \) for all \( n \), we deduce

\[ \limsup_n F_\varepsilon(u_n(t), v(t)) \leq F_\varepsilon(u(0), v(0)) + 2 \int_0^t \int_\Omega (\eta_\varepsilon + v^2(\tau)) \nabla u(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau. \]

Since by semicontinuity we have for all \( t \in [0, 1] \)

\[ F_\varepsilon(u(t), v(t)) \leq \liminf_n F_\varepsilon(u_n(t), v_n(t)), \]

by (4.12) and (4.13), we conclude that

\[ \lim_n F_\varepsilon(u_n(t), v_n(t)) = F_\varepsilon(u(t), v(t)). \]

In particular

\[ F_\varepsilon(u(t), v(t)) = F_\varepsilon(u(0), v(0)) + 2 \int_0^t \int_\Omega (\eta_\varepsilon + v^2(\tau)) \nabla u(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau, \]

and this proves point (d).

\[ \square \]

**Remark 4.5.** The map \( \{ t \rightarrow v(t), t \in [0, 1] \} \) is decreasing from \([0, 1]\) to \( L^2(\Omega) \), so that \( v \) is continuous with respect to the strong topology of \( L^2(\Omega) \) at all points except a countable set. Since

\[ \lambda(t) := \frac{\varepsilon}{2} \int_\Omega |\nabla v(t)|^2 \, dx + \frac{1}{2\varepsilon} \int_\Omega (1 - v(t))^2 \, dx \]

is monotone increasing (see Proposition 5.9), we conclude that \( v : [0, 1] \rightarrow H^1(\Omega) \) is continuous with respect to the strong topology at all points except a countable set. Then we have \( v \in L^\infty([0, 1], H^1(\Omega)) \). Moreover, we have that \( u : [0, 1] \rightarrow H^1(\Omega) \) is continuous at the continuity points of \( v \) as observed in Lemma 4.2. We conclude that \( u \in L^\infty([0, 1], H^1(\Omega)) \).

**Remark 4.6.** The minimality property of point (c) of Theorem 4.1 holds indeed in this stronger form: if \( t \in ]0, 1] \), for all \( (u, v) \in H^1(\Omega) \times H^1(\Omega) \) with 0 ≤ \( v \leq v(s) \) on \( \Omega \) for all \( s < t \), and \( u = g(t), v = 1 \) on \( \partial D \), we have

\[ F_\varepsilon(u(t), v(t)) \leq F_\varepsilon(u, v). \]

In fact, if 0 ≤ \( v \leq v(s) \), by the minimality property of \( (u(s), v(s)) \) we have

\[ F_\varepsilon(u(s), v(s)) \leq F_\varepsilon(u + g(s) - g(t), v), \]

so that, letting \( s \rightarrow t \) and using the continuity of \( F_\varepsilon(u(\cdot), v(\cdot)) \) we get the result.

This stronger minimality property is the reformulation in the context of the Ambrosio-Tortorelli functional of the minimality of the cracks required in [15] (see the Introduction).

5. **Quasi-static growth of brittle fracture**

In this section, we prove that the evolution for the Ambrosio-Tortorelli functional converges as \( \varepsilon \rightarrow 0 \) to a quasi-static evolution of brittle fractures in linearly elastic bodies in the sense of [13].

Let \( \partial_N \Omega := \partial \Omega \setminus \partial_D \Omega \) and let \( g \in W^{1,1}([0, 1]; H^1(\Omega)) \). In order to treat in a convenient way the boundary condition as \( \varepsilon \rightarrow 0 \), let \( B \) be an open ball such that \( \overline{B} \subset \Omega \), and let us pose \( \Omega' := B \setminus \partial_N \Omega \) and \( \Omega_D := \Omega' \setminus \overline{B} \). Let \( E \) be an extension operator from \( H^1(\Omega) \) to \( H^1(B) \); we indicate \( Eg(t) \) still by \( g(t) \) for all \( t \in [0, 1] \).

In this enlarged context, the following proposition holds.
Proposition 5.1. There exists a strongly measurable map
\[ [0, 1] \longrightarrow H^1(\Omega') \times H^1(\Omega') \]
\[ t \quad \mapsto \quad (u(t), v(t)) \]
such that for all \( t \in [0, 1] \) we have \( 0 \leq v(t) \leq 1 \) in \( \Omega' \), \( u(t) = g(t) \), \( v(t) = 1 \) on \( \Omega_D \) and
(a) for all \( 0 \leq s \leq t \leq 1 : v(t) \leq v(s) \);
(b) for all \( (u, v) \in H^1(\Omega') \times H^1(\Omega') \) with \( u = g(0) \), \( v = 1 \) on \( \Omega_D \):
\[ F_\varepsilon(u(0), v(0)) \leq F_\varepsilon(u, v); \]
(c) for \( t \in [0, 1] \) and for all \( (u, v) \in H^1(\Omega') \times H^1(\Omega') \) with \( 0 \leq v \leq v(t) \) on \( \Omega' \), and \( u = g(t), v = 1 \) on \( \Omega_D \):
\[ F_\varepsilon(u(t), v(t)) \leq F_\varepsilon(u, v); \]
(d) the function \( t \to F_\varepsilon(u(t), v(t)) \) is absolutely continuous and
\[ F_\varepsilon(u(t), v(t)) = F_\varepsilon(u(0), v(0)) + 2 \int_0^t \int_{\Omega'} (\eta_\varepsilon + v^2(\tau)) \nabla u(\tau) \nabla g(\tau) \, dx \, d\tau. \]

Proof. Let us consider the map \( t \to (u_\varepsilon(t), v_\varepsilon(t)) \) from \([0, 1]\) to \( H^1(\Omega) \times H^1(\Omega) \) given by Theorem 5.1. Recall that for all \( t \in [0, 1] \) we have \( u_\varepsilon(t) = g(t), v(t) = 1 \) on \( \partial_D \Omega \), and \( 0 \leq v_\varepsilon(t) \leq 1 \) in \( \Omega \). We extend \( u_\varepsilon(t) \) and \( v_\varepsilon(t) \) to \( \Omega' \) posing \( u_\varepsilon(t) = g(t) \) and \( v_\varepsilon(t) = 1 \) on \( \Omega_D \). Then we obtain a strongly measurable map \( t \to (u_\varepsilon(t), v_\varepsilon(t)) \) from \([0, 1]\) to \( H^1(\Omega') \times H^1(\Omega') \) such that \( 0 \leq v_\varepsilon(t) \leq 1 \) in \( \Omega' \), \( u_\varepsilon(t) = g(t), v_\varepsilon(t) = 1 \) on \( \Omega_D \), and such that
\[ F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \leq F_\varepsilon(u, v) \]
for all \( (u, v) \in H^1(\Omega') \times H^1(\Omega') \) with \( 0 \leq v \leq v(t) \) on \( \Omega' \), \( u = g(t), v = 1 \) on \( \Omega_D \); note in fact that the integrations on \( \Omega_D \) which appear in both sides are the same. By the same reason, we get the minimality property at time \( t = 0 \) and deduce that the function \( t \to F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \) is absolutely continuous with
\[ F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) = F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2 \int_0^t \int_{\Omega'} (\eta_\varepsilon + v_\varepsilon^2(\tau)) \nabla u_\varepsilon(\tau) \nabla g(\tau) \, dx \, d\tau. \]

From now on, we assume that there exists a constant \( C > 0 \) such that for all \( t \in [0, 1], \|g(t)\|_\infty \leq C \), and that there exists \( g_h \in W^{1,1}([0, 1], H^1(\Omega')) \) such that \( \|g_h\|_\infty \leq C \), \( g_h \in C(\overline{\Omega}) \), and \( g_h \to g \) strongly in \( W^{1,1}([0, 1], H^1(\Omega')) \). We indicate by \( (u_{\varepsilon,h}, v_{\varepsilon,h}) \) the evolution for the Ambrosio-Tortorelli functional relative to the boundary data \( g_h \) given by Proposition 5.1. The bound on the sup-norm is made in order to apply Ambrosio’s compactness theorem in \( SBV \) when \( \varepsilon \to 0 \). Notice that we may assume by a truncation argument that \( \|u_{\varepsilon,h}(t)\|_\infty \leq \|g_h(t)\|_\infty \), that is
\[ \|u_{\varepsilon,h}(t)\|_\infty \leq C. \]
We conclude that \( u_{\varepsilon,h}(t) \) is uniformly bounded in \( L^\infty(\Omega') \) as \( \varepsilon, h \) and \( t \) vary. Moreover we have that the following holds.

Lemma 5.2. There exists a constant \( C_1 \geq 0 \) depending only on \( g \) such that for all \( t \in [0, 1], \varepsilon, h \)
\[ F_\varepsilon(u_{\varepsilon,h}(t), v_{\varepsilon,h}(t)) + \|u_{\varepsilon,h}(t)\|_\infty \leq C_1. \]
Proof. Notice that $F_\varepsilon(u_{\varepsilon,h}(0), v_{\varepsilon,h}(0)) \leq F_\varepsilon(g_h(0), 1)$ so that the term $F_\varepsilon(u_{\varepsilon,h}(0), v_{\varepsilon,h}(0))$ is bounded as $\varepsilon$ and $h$ vary. We now derive an estimate for the derivative of the total energy.

Since $0 \leq v_{\varepsilon,h}(\tau) \leq 1$ and $\eta_\varepsilon \to 0$, by Hölder inequality we get

$$\left| \int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) \nabla u_{\varepsilon,h}(\tau) \nabla \dot{y}_h(\tau) \, dx \right| \leq$$

$$\leq 2 \left( \int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) \|\nabla u_{\varepsilon,h}(\tau)\|^2 \, dx \right)^{\frac{1}{2}} \|\nabla \dot{y}_h(\tau)\|_{L^2(\Omega';\mathbb{R}^N)};$$

since by the minimality property (5.2),

$$\int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) \|\nabla u_{\varepsilon,h}(\tau)\|^2 \, dx \leq \int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) \|\nabla y_h(\tau)\|^2 \, dx,$$

we get the conclusion by (5.3) and (5.4). □

As a consequence of (5.5), we have

$$\int_{\Omega'} (1 - v_{\varepsilon,h}(t)) |\nabla v_{\varepsilon,h}(t)| \, dx \leq \frac{e}{2} \int_{\Omega} |\nabla v_{\varepsilon,h}(t)|^2 \, dx + \frac{1}{2e} \int_{\Omega} (1 - v_{\varepsilon,h}(t))^2 \, dx \leq C_1,$$

so that the functions $u_{\varepsilon,h}(t) := 1 - v_{\varepsilon,h}(t)$ have uniformly bounded variation.

By coarea formula for $BV$-functions (see [4, Theorem 3.40]), we have that

$$\int_0^1 \mathcal{H}^{N-1}(\partial^* \{v_{\varepsilon,h}(t) > s\}) \, ds = \int_{\Omega'} (1 - v_{\varepsilon,h}(t)) |\nabla v_{\varepsilon,h}(t)| \, dx$$

(\partial^* denotes the essential boundary) so that by the Mean Value theorem, for all $j \geq 1$ there exists $b_{\varepsilon,h}^j(t) \in [\frac{1}{2j+1}, \frac{1}{2j}]$ with

$$\frac{1}{2j+1} \mathcal{H}^{N-1}(\partial^* \{v_{\varepsilon,n}(t) > b_{\varepsilon,h}^j(t)\}) \leq C_1.$$ (5.6)

Let us pose

$$B_{\varepsilon,h}(t) := \left\{ b_{\varepsilon,h}^j(t) : j \geq 1 \right\}.$$ (5.7)

We now let $\varepsilon \to 0$. Let $D$ be countable and dense in $[0, 1]$ with $0 \in D$.

**Lemma 5.3.** There exists a sequence $\varepsilon_n$ such that for all $t \in D$ there exists $u_h(t) \in SBV(\Omega')$, $u_h(t) = g_h(t)$ on $\Omega_D$, with

$$u_{\varepsilon,n,h}(t) \mathbb{1}_{\{u_{\varepsilon,n,h}(t) > b_{\varepsilon,n,h}^j(t)\}} \to u_h(t) \text{ in } SBV(\Omega').$$

In particular for all $t \in D$ we have

$$\int_{\Omega'} |\nabla u_h(t)|^2 \, dx + \mathcal{H}^{N-1}(S_{u_h(t)}) + \|u_h(t)\|_{\infty} \leq C_1.$$ (5.8)

**Proof.** For all $t \in [0, 1]$ we may apply Ambrosio’s compactness Theorem 2.1 to the function $z_n(t) := u_{\varepsilon,n,h}(t) \mathbb{1}_{\{u_{\varepsilon,n,h}(t) > b_{\varepsilon,n,h}^j(t)\}}$; in fact $z_n(t)$ is bounded in $L^\infty(\Omega')$ and $\nabla z_n(t)$ is bounded in $L^2(\Omega')$ by (5.5), and $S_{z_n(t)} \subseteq \partial^* \{u_{\varepsilon,n,h}(t) > b_{\varepsilon,n,h}^j(t)\}$ so that $\mathcal{H}^{N-1}(S_{z_n(t)})$ is uniformly bounded in $n$ by (5.6). Using a diagonal argument, there exists a subsequence such that for all $t \in D$, $z_n(t) \to u_h(t)$ in $SBV(\Omega')$; in particular, we have that $u_h(t) = g_h(t)$ on $\Omega_D$, and by (5.3) and the $\Gamma$-liminf inequality for the Ambrosio-Tortorelli functional (2.2), we get □

The following lemma deals with the possibility of truncating at other levels given by the elements of $B_{\varepsilon,h}(t)$.

**Lemma 5.4.** Let $t \in D$ and $j \geq 1$. For every $b_{\varepsilon,n,h}^j(t) \in B_{\varepsilon,h}(t)$ we have that

$$u_{\varepsilon,n,h}(t) \mathbb{1}_{\{u_{\varepsilon,n,h}(t) > b_{\varepsilon,n,h}^j(t)\}} \to u_h(t) \text{ in } SBV(\Omega').$$
Proof. Note that, up to a subsequence, \( u_{\varepsilon_n,h}(t)1_{\{v_{\varepsilon_n,h}(t) > b^j_{\varepsilon_n,h}(t)\}} \to z \) in \( SBV(\Omega') \) because of Ambrosio’s Theorem 2.1. By (5.4), we have that

\[
\left\| u_{\varepsilon_n,h}(t)1_{\{v_{\varepsilon_n,h}(t) > b^j_{\varepsilon_n,h}(t)\}} - u_{\varepsilon_n,h}(t)1_{\{v_{\varepsilon_n,h}(t) > b^j_{\varepsilon_n,h}(t)\}} \right\|_{L^2(\Omega')} \leq C \left\| \{b^j_{\varepsilon_n,h}(t) \leq v_{\varepsilon_n,h}(t) \leq b^1_{\varepsilon_n,h}(t)\} \right\|.
\]

Since \( v_{\varepsilon_n,h}(t) \to 1 \) strongly in \( L^2(\Omega') \), we conclude that

\[
\left\| \{b^j_{\varepsilon_n,h}(t) \leq v_{\varepsilon_n,h}(t) \leq b^1_{\varepsilon_n,h}(t)\} \right\| \to 0,
\]

so that

\[
\left\| z - u_h(t) \right\|_{L^2(\Omega')} = \lim_n \left\| u_{\varepsilon_n,h}(t)1_{\{v_{\varepsilon_n,h}(t) > b^j_{\varepsilon_n,h}(t)\}} - u_{\varepsilon_n,h}(t)1_{\{v_{\varepsilon_n,h}(t) > b^j_{\varepsilon_n,h}(t)\}} \right\|_{L^2(\Omega')} = 0,
\]

that is \( z = u_h(t) \) and the proof is complete. \( \square \)

The following lemma deals with the possibility of truncating at time \( s \) using the function \( v(t) \) for \( t \geq s \).

**Lemma 5.5.** Let \( s,t \in D \) with \( s \leq t \), and \( j \geq 1 \). Then for every \( b^j_{\varepsilon_n,h}(t) \in B_{\varepsilon_n,h}(t) \) we have that

\[
u_{\varepsilon_n,h}(s)1_{\{v_{\varepsilon_n,h}(t) > b^j_{\varepsilon_n,h}(t)\}} \to u_h(s) \quad \text{in} \quad SBV(\Omega').
\]

**Proof.** Up to a subsequence, by Ambrosio’s Theorem, we have that

\[
u_{\varepsilon_n,h}(s)1_{\{v_{\varepsilon_n,h}(t) > b^j_{\varepsilon_n,h}(t)\}} \to z \quad \text{in} \quad SBV(\Omega').
\]

Since \( v_{\varepsilon_n,h}(t) \leq v_{\varepsilon_n,h}(s) \), we have that \( \{v_{\varepsilon_n,h}(t) > b^j_{\varepsilon_n,h}(t)\} \subseteq \{v_{\varepsilon_n,h}(s) > b^{j+1}_{\varepsilon_n,h}(s)\} \). Then we have

\[
\left\| u_{\varepsilon_n,h}(s)1_{\{v_{\varepsilon_n,h}(t) > b^j_{\varepsilon_n,h}(t)\}} - u_{\varepsilon_n,h}(s)1_{\{v_{\varepsilon_n,h}(s) > b^{j+1}_{\varepsilon_n,h}(s)\}} \right\|_{L^2(\Omega')} \leq C \left\| \{v_{\varepsilon_n,h}(t) \leq b^{j+1}_{\varepsilon_n,h}(t)\} \right\|.
\]

Since \( v_{\varepsilon_n,h}(t) \to 1 \) strongly in \( L^2(\Omega') \), we conclude that \( \left\| \{v_{\varepsilon_n,h}(t) \leq b^j_{\varepsilon_n,h}(t)\} \right\| \to 0 \). By Lemma 5.4, we have

\[
u_{\varepsilon_n,h}(s)1_{\{v_{\varepsilon_n,h}(s) > b^j_{\varepsilon_n,h}(s)\}} \to u_h(s) \quad \text{in} \quad SBV(\Omega'),
\]

so that \( z = u_h(s) \) and the proof is complete. \( \square \)

We now pass to the analysis of \( u_h(t) \) with \( t \in D \). The following minimality property for the functions \( u_h(t) \) with \( t \in D \) is crucial for the subsequent results.

**Theorem 5.6.** Let \( t \in D \). Then for every \( z \in SBV(\Omega') \) with \( z = g_h(t) \) on \( \Omega_D \), we have that

\[
\int_{\Omega'} |\nabla u_h(t)|^2 \, dx \leq \int_{\Omega'} |\nabla z|^2 \, dx + \mathcal{H}^{N-1}(S_z \setminus \bigcup_{s \leq t, s \in D} S_{u_h(s)}) \cdot S_{u_h(s)}.
\]

The proof is quite technical, and it is postponed to Section 6. We now let \( h \to \infty \).

**Proposition 5.7.** There exists \( h_n \to \infty \) such that for all \( t \in D \) there exists \( u(t) \in SBV(\Omega') \) with \( u(t) = g(t) \) on \( \Omega_D \) such that \( u_{h_n}(t) \to u(t) \) in \( SBV(\Omega') \). Moreover, \( \nabla u_{h_n}(t) \to \nabla u(t) \) strongly in \( L^2(\Omega'; \mathbb{R}^N) \) and for all \( z \in SBV(\Omega') \) with \( z = g(t) \) on \( \Omega_D \) we have

\[
\int_{\Omega'} |\nabla u(t)|^2 \, dx \leq \int_{\Omega'} |\nabla z|^2 \, dx + \mathcal{H}^{N-1}(S_z \setminus \bigcup_{s \leq t, s \in D} S_{u(s)}) \cdot S_{u(s)}.
\]
Proof. The compactness is given by Ambrosio’s Theorem in view of [13]. The strong
convergence of the gradients and the minimality property is a consequence of the minimality
property of Theorem [5.6] and of [13, Theorem 2.1].

We can now deal with $\varepsilon$ and $h$ at the same time.

**Proposition 5.8.** There exists $\varepsilon_n \to 0$ and $h_n \to +\infty$ such that for all $t \in D$ there exists
$u(t) \in SBV(\Omega')$ with $u(t) = g(t)$ on $\Omega_D$ such that for all $j \geq 1$
$$u_{\varepsilon_n, h_n}(t)1_{\{v_{\varepsilon_n, h_n}(t) > \varepsilon_n, h_n(t)\}} \to u(t) \quad \text{in } SBV(\Omega').$$
Furthermore for all $z \in SBV(\Omega')$ with $z = g(t)$ on $\Omega_D$ we have
$$\int_{\Omega'} |\nabla u(t)|^2 \, dx \leq \int_{\Omega'} |\nabla z|^2 \, dx + \mathcal{H}^{N-1}\left(S_z \setminus \bigcup_{s \leq t, s \in D} S_{u(s)}\right),$$
and we may suppose that the functions $\lambda_{\varepsilon_n, h_n}$ converge pointwise on $[0, 1]$ to an increasing
function $\lambda$ such that for all $t \in D$
$$\lambda(t) \geq \mathcal{H}^{N-1}\left(\bigcup_{s \leq t, s \in D} S_{u(s)}\right).$$

Finally, we have that for all $t \in D$
$$\int_{\Omega'} |\nabla u(t)|^2 \, dx + \mathcal{H}^{N-1}(S_u(t)) + \|u(t)\|_\infty \leq C_1.$$

Proof. We find $\varepsilon_n$ and $h_n$ combining Lemma [5.8] and Proposition [5.7] and using a diagonal
argument. Passing to the second part of the proposition, notice that the functions $\lambda_{\varepsilon_n, h_n}$
are monotone increasing. In fact if $s \leq t$, since $v_{\varepsilon_n, h_n}(t) \leq v_{\varepsilon_n, h_n}(s)$, and $v_{\varepsilon_n, h_n}(t) = 1$ on
$\Omega_D$, by the minimality property [5.2], we have that
$$F_{\varepsilon_n}(u_{\varepsilon_n, h_n}(s), v_{\varepsilon_n, h_n}(s)) \leq F_{\varepsilon_n}(u_{\varepsilon_n, h_n}(s), v_{\varepsilon_n, h_n}(t),$$
so that
$$\lambda_{\varepsilon_n, h_n}(t) - \lambda_{\varepsilon_n, h_n}(s) \geq \int_{\Omega'} (\eta_{\varepsilon_n} + v_{\varepsilon_n, h_n}(s)^2)|\nabla u_{\varepsilon_n, h_n}(s)|^2 \, dx - \int_{\Omega'} (\eta_{\varepsilon_n} + v_{\varepsilon_n, h_n}(t)^2)|\nabla u_{\varepsilon_n, h_n}(s)|^2 \, dx \geq 0.$$
Moreover by [5.8] we have $0 \leq \lambda_{\varepsilon_n, h_n} \leq C_1$. Applying Helly’s theorem, we get that there
exists an increasing function $\lambda$ up to a subsequence $\lambda_{\varepsilon_n, h_n} \to \lambda$ pointwise in $[0, 1]$. In order
to prove [5.7], let us fix $s_1, \ldots, s_m \in D \cap [0, t]$: we want to prove that
$$\lambda(t) = \lim_n \lambda_{\varepsilon_n, h_n}(t) \geq \mathcal{H}^{N-1}\left(\bigcup_{i=1}^m S_{u(s_i)}\right).$$

Then taking the sup over all possible $s_1, \ldots, s_m$, we can deduce [5.9]. Consider $z_n \in SBV(\Omega', \mathbb{R}^m)$
defined as
$$z_n(x) := (u_{\varepsilon_n, h_n}(s_1), \ldots, u_{\varepsilon_n, h_n}(s_m)).$$
Notice that by [5.5], and the fact that $t \to v_{\varepsilon_n, h_n}(t)$ is decreasing in $L^2(\Omega')$, we obtain that there
exists $C' > 0$ such that for all $n$
$$\int_{\Omega'} (\eta_{\varepsilon_n} + v_{\varepsilon_n, h_n}(t)^2)|\nabla z_n(t)|^2 \, dx + \frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla v_{\varepsilon_n, h_n}(t)|^2 \, dx + \frac{1}{2\varepsilon_n} \int_{\Omega'} (1 - v_{\varepsilon_n, h_n}(t))^2 \, dx \leq C'.$$
Then we may apply [13, Lemma 3.2] obtaining [5.11]. Finally [5.10] is a consequence of [5.9] and the lower semicontinuity
[5.2]. The proof is now concluded. \qed
Let us extend the evolution \( \{ t \to u(t), t \in D \} \) of Proposition 5.8 to the entire interval \([0, 1]\). Let us pose for every \( t \in [0, 1] \)
\[(5.12) \quad \Gamma(t) := \bigcup_{s \in D, s \leq t} S_u(s).\]

**Proposition 5.9.** For every \( t \in [0, 1] \) there exists \( u(t) \in SBV(\Omega') \) with \( u(t) = g(t) \) on \( \Omega_D \)
such that \( \nabla u \in L^\infty([0, 1], L^2(\Omega'; \mathbb{R}^N)) \), \( \nabla u \) is left continuous in \([0, 1] \setminus D \) with respect to the strong topology, and such that, if \( \Gamma \) is as in (5.12), the following hold:

(a) for all \( t \in [0, 1] \)
\[(5.13) \quad S(u(t)) \subseteq \Gamma(t) \quad \text{up to a set of } \mathcal{H}^{N-1} - \text{measure } 0,\]

and if \( \lambda \) is as in Proposition 5.5
\[(5.14) \quad \lambda(t) \geq \mathcal{H}^{N-1}(\Gamma(t));\]

(b) for all \( z \in SBV(\Omega') \) with \( z = g(0) \) on \( \Omega_D \)
\[(5.15) \quad \int_{\Omega'} |\nabla u(0)|^2 \, dx + \mathcal{H}^{N-1}(S_u) \leq \int_{\Omega'} |\nabla z|^2 \, dx + \mathcal{H}^{N-1}(S_z).\]

(c) for all \( t \in [0, 1] \) and for all \( z \in SBV(\Omega') \) with \( z = g(t) \) on \( \Omega_D \)
\[(5.16) \quad \int_{\Omega'} |\nabla u(t)|^2 \, dx \leq \int_{\Omega'} |\nabla z|^2 \, dx + \mathcal{H}^{N-1}(S_z \setminus \Gamma(t)).\]

Finally,
\[(5.17) \quad \mathcal{E}(t) \geq \mathcal{E}(0) + 2 \int_0^t \int_{\Omega'} \nabla u(\tau) \nabla g(\tau) \, dx \, d\tau\]
where
\[(5.18) \quad \mathcal{E}(t) := \int_{\Omega'} |\nabla u(t)|^2 \, dx + \mathcal{H}^{N-1}(\Gamma(t)).\]

**Proof.** Let \( t \in [0, 1] \setminus D \) and let \( t_n \in D \) with \( t_n \to t \); by (5.10) we can apply Ambrosio’s Theorem obtaining \( u \in SBV(\Omega') \) with \( u = g(t) \) on \( \Omega_D \) such that \( u(t_n) \to u \) in \( SBV(\Omega') \) up to subsequences. Let us pose \( u(t) := u \). By [13, Lemma 3.7], we have that (5.13) and (5.16) hold, and that the convergence \( \nabla u(t_n) \to \nabla u \) is strong in \( L^2(\Omega'; \mathbb{R}^N) \). Notice that \( \nabla u(t) \) is uniquely determined by (5.13) and (5.16) since the gradient of the solutions of the minimum problem
\[
\min \left\{ \int_{\Omega'} |\nabla u|^2 \, dx : u = g(t) \text{ on } \Omega_D, S_u \subseteq \Gamma(t) \text{ up to a set of } \mathcal{H}^{N-1} - \text{measure } 0 \right\}
\]
is unique by the strict convexity of the functional. We conclude that \( \nabla u(t) \) is well defined. The argument above proves that \( \nabla u \) is left continuous at all the points of \([0, 1] \setminus D \). It turns out that \( \nabla u \) is continuous in \([0, 1] \) up to a countable set. In fact let us consider \( t \in [0, 1] \setminus (D \cup \mathcal{N}) \) where \( \mathcal{N} \) is the set of discontinuities of the function \( \mathcal{H}^{N-1}(\Gamma(\cdot)) \). Let \( t_n \to t \). By Ambrosio’s Theorem, we have that there exists \( u \in SBV(\Omega') \) with \( u = g(t) \) on \( \Omega_D \) such that, up to a subsequence, \( u(t_n) \to u \) in \( SBV(\Omega') \). Since \( t \) is a continuity point of \( \mathcal{H}^{N-1}(\Gamma(\cdot)) \), we deduce that \( S_u \subseteq \Gamma(t) \) up to a set of \( \mathcal{H}^{N-1} - \text{measure } 0 \). Moreover by [13, Lemma 3.7] we have that \( u \) satisfies the minimality property (5.16), and \( \nabla u(t_n) \to \nabla u \) strongly in \( L^2(\Omega'; \mathbb{R}^N) \). We deduce that \( \nabla u = \nabla u(t) \), and so \( \nabla u(\cdot) \) is continuous in \([0, 1] \setminus (D \cup \mathcal{N}) \). We conclude that \( \nabla u(\cdot) \) is continuous in \([0, 1] \) up to a countable set, so that \( \nabla u \in L^\infty([0, 1]; L^2(\Omega'; \mathbb{R}^N)) \).

We have that (5.14) is a direct consequence of (5.5), while (5.13) is a consequence of (5.1) and the \( \Gamma \)-convergence result of Ambrosio and Tortorelli [5] and [6].
Finally, in order to prove (5.17), we can reason in the following way. Given \( t \in [0,1] \) and \( m > 0 \), let \( s^m_i := \frac{1}{m} t \) for all \( i = 0, \ldots, m \). Let us pose \( u^m(s) := u(s^m_{i+1}) \) for \( s^m_i < s \leq s^m_{i+1} \). By (5.10) we have

\[
E(t) \geq E(0) + 2 \int_0^t \int_{\Omega} \nabla u^m(\tau) \nabla \dot{g}(\tau) \, d\tau \, dx + \alpha_m
\]

where \( \alpha_m \to 0 \) for \( m \to +\infty \) because \( g \) is absolutely continuous. Since \( \nabla u \) is continuous with respect to the strong topology of \( L^2(\Omega; \mathbb{R}^N) \) in \( [0,1] \) up to a countable set, passing to the limit for \( m \to +\infty \) we deduce that (5.17) holds, and the proof is concluded. \( \square \)

We are now in position to prove our convergence result. We need the following lemma.

**Lemma 5.10.** Let \( \tilde{\mathcal{N}} \) be the set of discontinuity points of the function \( \lambda \) given by Proposition 5.9. Then for every \( t \in [0,1] \setminus \tilde{\mathcal{N}} \), and \( j \geq 1 \) we have that

\[
\nabla u_{\varepsilon,n,h_1}(t)1_{\{s,t \leq t^\prime_{\varepsilon,n,h_1}(t)\}} \rightharpoonup \nabla u(t) \text{ weakly in } L^2(\Omega; \mathbb{R}^N).
\]

**Proof.** Let \( t \in [0,1] \setminus \tilde{\mathcal{N}} \): we may suppose that \( t \notin D \), since otherwise the result has already been established. Let \( s \in D \) with \( s < t \). We pose

\[
J := \inf \left\{ \int_{\Omega} (\eta_{\varepsilon,n} + v^2_{\varepsilon,n,h_1}(t))|\nabla z|^2 \, dx : z = g_h_n(s) \text{ on } \Omega \right\},
\]

and we indicate by \( w_n(s,t) \) the minimum point of this problem. Notice that \( u_{\varepsilon,n,h_1}(t) - w_n(s,t) \) is the minimum for

\[
K := \inf \left\{ \int_{\Omega} (\eta_{\varepsilon,n} + v^2_{\varepsilon,n,h_1}(t))|\nabla z|^2 \, dx : z = g_h_n(t) - g_h_n(s) \text{ on } \Omega \right\}.
\]

Comparing \( u_{\varepsilon,n,h_1}(t) - w_n(s,t) \) with \( g_h_n(t) - g_h_n(s) \), we have

\[
\int_{\Omega} (\eta_{\varepsilon,n} + v^2_{\varepsilon,n,h_1}(t))|\nabla u_{\varepsilon,n,h_1}(t) - \nabla w_n(s,t)|^2 \, dx \leq \int_{\Omega} (\eta_{\varepsilon,n} + v^2_{\varepsilon,n,h_1}(t))|\nabla g_h_n(t) - \nabla g_h_n(s)|^2 \, dx.
\]

Since \( u_{\varepsilon,n,h_1}(s) - w_n(s,t) \) is a good test for \( J \), we have

\[
\int_{\Omega} (\eta_{\varepsilon,n} + v^2_{\varepsilon,n,h_1}(t)) \nabla w_n(s,t) \cdot (\nabla u_{\varepsilon,n,h_1}(s) - \nabla w_n(s,t)) \, dx = 0,
\]

and so the following equality holds

\[
\int_{\Omega} (\eta_{\varepsilon,n} + v^2_{\varepsilon,n,h_1}(t))(|\nabla u_{\varepsilon,n,h_1}(s)|^2 - |\nabla w_n(s,t)|^2) \, dx = \int_{\Omega} (\eta_{\varepsilon,n} + v^2_{\varepsilon,n,h_1}(t))(|\nabla u_{\varepsilon,n,h_1}(s) - \nabla w_n(s,t)|^2) \, dx.
\]

Since \( v_{\varepsilon,n,h_1}(t) \leq u_{\varepsilon,n,h_1}(s) \) and by minimality of \( u_{\varepsilon,n,h_1}(s) \) we have

\[
\int_{\Omega} (\eta_{\varepsilon,n} + v^2_{\varepsilon,n,h_1}(t))|\nabla u_{\varepsilon,n,h_1}(s)|^2 \, dx + \lambda_{\varepsilon,n,h_1}(s) \leq \int_{\Omega} (\eta_{\varepsilon,n} + v^2_{\varepsilon,n,h_1}(s))|\nabla u_{\varepsilon,n,h_1}(s)|^2 \, dx + \lambda_{\varepsilon,n,h_1}(s) \leq \int_{\Omega} (\eta_{\varepsilon,n} + v^2_{\varepsilon,n,h_1}(t))|\nabla w_n(s,t)|^2 \, dx + \lambda_{\varepsilon,n,h_1}(t).
\]
so that

\[ (5.21) \quad \int_{\Omega'} (\eta_{\varepsilon_n} + \varepsilon_{\varepsilon_n, h_n}(t))(|\nabla u_{\varepsilon_n, h_n}(s) - \nabla w_n(s, t)|^2) \, dx = \]

\[ = \int_{\Omega'} (\eta_{\varepsilon_n} + \varepsilon_{\varepsilon_n, h_n}(t))(|\nabla u_{\varepsilon_n, h_n}(s)|^2 - |\nabla w_n(s, t)|^2) \, dx \leq \lambda_{\varepsilon_n, h_n}(t) - \lambda_{\varepsilon_n, h_n}(s). \]

By (5.20) and (5.21), we conclude that there exists \( C' > 0 \) with

\[ (5.22) \quad \int_{\Omega'} (\eta_{\varepsilon_n} + \varepsilon_{\varepsilon_n, h_n}(t))(|\nabla u_{\varepsilon_n, h_n}(t) - \nabla u_{\varepsilon_n, h_n}(s)|^2) \, dx \leq C'\|\nabla g_{h_n}(t) - \nabla g_{h_n}(s)\| + (\lambda_{\varepsilon_n, h_n}(t) - \lambda_{\varepsilon_n, h_n}(s)). \]

Then we conclude that for \( b'_{\varepsilon_n, h_n}(t) \in B_{\varepsilon_n, h_n}(t) \)

\[ (5.23) \quad \|\nabla u_{\varepsilon_n, h_n}(t)1_{\{v_{\varepsilon_n, h_n}(t) > b'_{\varepsilon_n, h_n}(t)\}} - \nabla u_{\varepsilon_n, h_n}(s)1_{\{v_{\varepsilon_n, h_n}(t) > b'_{\varepsilon_n, h_n}(t)\}}\|_{L^2(\Omega'; \mathbb{R}^N)} \leq \alpha(t - s) \]

since \( \lambda_{\varepsilon_n, h_n} \to \lambda \) pointwise, and \( t \) is a continuity point for \( \lambda \). Recall that by Lemma 5.5

\[ \nabla u_{\varepsilon_n, h_n}(s)1_{\{v_{\varepsilon_n, h_n}(t) > b'_{\varepsilon_n, h_n}(t)\}} \rightharpoonup \nabla u(s) \text{ weakly in } L^2(\Omega'; \mathbb{R}^N). \]

Since

\[ \nabla u_{\varepsilon_n, h_n}(t)1_{\{v_{\varepsilon_n, h_n}(t) > b'_{\varepsilon_n, h_n}(t)\}} - \nabla u(t) = \]

\[ = (\nabla u_{\varepsilon_n, h_n}(t)1_{\{v_{\varepsilon_n, h_n}(t) > b'_{\varepsilon_n, h_n}(t)\}} - \nabla u_{\varepsilon_n, h_n}(s)1_{\{v_{\varepsilon_n, h_n}(t) > b'_{\varepsilon_n, h_n}(t)\}}) + \]

\[ + (\nabla u_{\varepsilon_n, h_n}(s)1_{\{v_{\varepsilon_n, h_n}(t) > b'_{\varepsilon_n, h_n}(t)\}} - \nabla u(s)) + (\nabla u(s) - \nabla u(t)), \]

by (5.23) and the left continuity of \( \{\tau \to \nabla u(\tau)\} \) at the points of \([0, 1] \setminus D\), we have that

\[ \nabla u_{\varepsilon_n, h_n}(t)1_{\{v_{\varepsilon_n, h_n}(t) > b'_{\varepsilon_n, h_n}(t)\}} \rightharpoonup \nabla u(t) \text{ weakly in } L^2(\Omega'; \mathbb{R}^N), \]

so that the lemma is proved.

We can now pass to the proof of the main theorem of the paper.

**Proof of Theorem 5.2** By Proposition 5.1 we may extend \((u_{\varepsilon, h}(t), v_{\varepsilon, h}(t))\) to \( \Omega' \) posing \( u_{\varepsilon, h}(t) = g_{h}(t) \) and \( v_{\varepsilon, h}(t) = 1 \) on \( \Omega_{\varepsilon, h} \), obtaining a quasi-static evolution in \( \Omega' \). In this context, the points of \( \partial_{\Omega} \) where the boundary condition is violated in the limit simply become discontinuity points of the extended function. Thus we prove the result in this equivalent setting involving \( \Omega' \).

Let \( \varepsilon_n \to 0 \) and \( h_n \to +\infty \) be the sequences determined by Proposition 5.8. Let us denote by \( u_{\varepsilon_n, h}(t), v_{\varepsilon_n, h}(t), B_{\varepsilon_n, h}(t) \) and \( F_{\varepsilon_n}(t) \) the evolution relative to the boundary data \( g \) given by Proposition 5.9 up to a subsequence, we have that \( u_n(t)1_{\{v_{\varepsilon_n}(t) > b'_{\varepsilon_n}(t)\}} \to u(t) \) in \( SBV(\Omega') \) for all \( j \geq 1 \) and for all \( t \) in a countable and dense subset \( D \subseteq [0, 1] \) with \( 0 \in D \). Moreover we have that

\[ (5.24) \quad \mathcal{E}(t) \geq \mathcal{E}(0) + 2 \int_0^t \int_{\Omega'} \nabla u(\tau) \nabla \tilde{g}(\tau) \, dx \, d\tau, \]

where \( \mathcal{E}(t) := \int_{\Omega'} |\nabla u(t)|^2 \, dx + H^{N-1}(\Gamma(t)) \) and \( \Gamma(t) \) is as in (5.12).

By point (b) of Proposition 5.1 and the Ambrosio-Tortorelli Theorem we have

\[ (5.25) \quad \lim_{n \to \infty} F_{\varepsilon_n}(u_n(0), v_n(0)) = \mathcal{E}(0). \]
For $m \geq 1$, notice that
\[
\int_{\Omega'} (\eta_n + v_n^2(\tau)) \nabla u_n(\tau) \nabla g_{h_n}(\tau) \, dx = \int_{\Omega'} (\eta_n + v_n^2(\tau)) \nabla u_n(\tau)1_{\{v_n(\tau) > b_n(\tau)\}} \nabla g_{h_n}(\tau) \, dx + \\
+ \int_{\Omega'} (\eta_n + v_n^2(\tau)) \nabla u_n(\tau)1_{\{v_n(\tau) \leq b_n(\tau)\}} \nabla g_{h_n}(\tau) \, dx.
\]

If $\tau \in [0, 1]$, we have the estimate
\[
\left| \int_{\Omega'} (\eta_n + v_n^2(\tau)) \nabla u_n(\tau)1_{\{v_n(\tau) \leq b_n(\tau)\}} \nabla g_{h_n}(\tau) \, dx \right| \leq \\
\leq \sqrt{\eta_n + \frac{1}{2m}} \left( \int_{\Omega'} (\eta_n + v_n^2(\tau)) |\nabla u_n(\tau)|^2 \, dx \right)^{\frac{1}{2}} \|\nabla g_{h_n}(\tau)\|_{L^2(\Omega' ; \mathbb{R}^N)} \leq \\
\leq \sqrt{\eta_n + \frac{1}{2m}} C \to \frac{C}{2m}.
\]

Moreover, by Lemma 5.10, we have that for a.e. $\tau \in [0, 1]$
\[
\lim_n \int_{\Omega'} (\eta_n + v_n^2(\tau)) \nabla u_n(\tau)1_{\{v_n(\tau) > b_n(\tau)\}} \nabla g_{h_n}(\tau) \, dx = \int_{\Omega'} \nabla u(\tau) \nabla \hat{g}(\tau) \, dx,
\]
and we deduce that for such $\tau$
\[
\limsup_n \left| \int_{\Omega'} (\eta_n + v_n^2(\tau)) \nabla u_n(\tau) \nabla g_{h_n}(\tau) \, dx - \int_{\Omega'} \nabla u(\tau) \nabla \hat{g}(\tau) \right| \leq \frac{C}{2m}.
\]

Since $m$ is arbitrary, we have that for a.e. $\tau \in [0, 1]$
\[
(5.26) \quad \lim_n \int_{\Omega'} (\eta_n + v_n^2(\tau)) \nabla u_n(\tau) \nabla g_{h_n}(\tau) \, dx = \int_{\Omega'} \nabla u(\tau) \nabla \hat{g}(\tau) \, dx.
\]

By 5.3, 5.20, 5.21 and the Dominated Convergence Theorem, we conclude that for all $t \in [0, 1]$
\[
(5.27) \quad \lim_n F_n(u_n(t), v_n(t)) = \mathcal{E}(0) + 2 \int_0^t \int_{\Omega'} \nabla u(\tau) \nabla \hat{g}(\tau) \, dx \, d\tau.
\]

Since $\liminf_n F_n(u_n(t), v_n(t)) \geq \mathcal{E}(t)$ by 2.29, by 7.24 we have that for all $t \in [0, 1]$
\[
\lim_n F_n(u_n(t), v_n(t)) = \mathcal{E}(t).
\]

In particular we get
\[
(5.28) \quad \mathcal{E}(t) = \mathcal{E}(0) + 2 \int_0^t \int_{\Omega'} \nabla u(\tau) \nabla \hat{g}(\tau) \, dx \, d\tau,
\]
so that, recalling all the properties stated in Proposition 5.9, we deduce that $\{t \to u(t) : t \in [0, 1]\}$ is a quasi-static evolution relative to the boundary data $g$. Point (a) is a consequence of 5.29 and 5.30.

Let us pass to point (b). By Lemma 5.10, we know that if $\tilde{\mathcal{N}}$ is the set of discontinuity points of $\lambda$, for all $t \in [0, 1] \setminus \tilde{\mathcal{N}}$ and for all $j \geq 1$ we have $\nabla u_n(t)1_{\{v_n(t) > b_j(t)\}} \to \nabla u(t)$ weakly in $L^2(\Omega', \mathbb{R}^N)$. Since
\[
v_n(t) \nabla u_n(t) = v_n(t) \nabla u_n(t)1_{\{v_n(t) > b_j(t)\}} + v_n(t) \nabla u_n(t)1_{\{v_n(t) < b_j(t)\}},
\]
we get immediately that $v_n(t) \nabla u_n(t) \to \nabla u(t)$ weakly in $L^2(\Omega', \mathbb{R}^N)$. For all such $t$, we have that
\[
\liminf_n \int_{\Omega'} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx \geq \int_{\Omega'} |\nabla u(t)|^2 \, dx,
\]
and by 5.14
\[
\liminf_n \frac{1}{2} \int_{\Omega'} |\nabla v_n(t)|^2 \, dx + \frac{1}{2n} \int_{\Omega'} (1 - v_n(t))^2 \, dx \geq H^{N-1}(\Gamma(t)).
\]
By point (a), we have that the two preceding inequalities are equalities. In particular, $\lambda$ and $H^{N-1}(\Gamma(\cdot))$ coincide up to a countable set in $[0, 1]$. We deduce that $\lambda$ and $H^{N-1}(\Gamma(\cdot))$ have the same continuity points, that is $\mathcal{N} = \mathcal{N}$. We conclude that for all $t \in [0, 1] \setminus \mathcal{N}$ we have $v_n(t) \nabla u_n(t) \to \nabla u(t)$ strongly in $L^2(\Omega', \mathbb{R}^N)$,

$$\lim_n \int_{\Omega'} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx = \int_{\Omega'} |\nabla u(t)|^2 \, dx,$$

and

$$\lim_n \frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla v_n(t)|^2 \, dx + \frac{1}{2\varepsilon_n} \int_{\Omega'} (1 - v_n(t))^2 \, dx = H^{N-1}(\Gamma(t)),$$

so that point (b) is proved, and the proof of the theorem is complete. \hfill $\Box$

6. Proof of Theorem 5.6

In this section we give the proof of Theorem 5.6 which is an essential step in the analysis of Section 5. For simplicity of notation, for all $t \in D$ we denote $u(t)$, $v_n(t)$ and $v_n$ for $u_n(t)$, $u_{\varepsilon_n,h}(t)$ and $v_{\varepsilon_n,h}(t)$ respectively. Moreover, let us write $B_n(t)$, $b_n(t)$ and $b_{\varepsilon_n,h}(t)$, where $b_{\varepsilon_n,h}(t)$ is defined as in (5.7).

Given $z \in SBV(\Omega')$ with $z = g_h(t)$ on $\Omega_D$, we want to see that

$$\int_{\Omega'} |\nabla u(t)|^2 \, dx \leq \int_{\Omega'} |\nabla z|^2 \, dx + H^{N-1}(S_z \setminus \Gamma(t)),$$

where $g_h(t) \in H^1(\Omega') \cap C(\overline{\Omega'})$ and $\Gamma(t) = \bigcup_{t \leq s \in D} S_{u(s)}$.

The plan is to use the minimality property (5.2) of the approximating evolution, so that the main point is to construct a sequence $(z_n, v_n) \in H^1(\Omega') \times H^1(\Omega')$ such that $z_n = g_h(t)$, $v_n = 1$ on $\Omega_D$, $0 \leq v_n \leq u_n(t)$, and such that

$$\lim_n \int_{\Omega'} (\eta_n + v_n^2(t)|\nabla z_n|^2 \, dx = \int_{\Omega'} |\nabla z|^2 \, dx$$

and

$$\limsup_n \left| MM_n(v_n) - MM_n(v_n(t)) \right| \leq H^{N-1}(S_z \setminus \Gamma(t)),$$

where we use the notation

$$MM_n(w) := \frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla w|^2 \, dx + \frac{1}{2\varepsilon_n} \int_{\Omega'} (1 - w)^2 \, dx.$$

If a sequence with these properties exists, then by property (5.2) we get the result. The following lemma contains the main ideas in order to prove Theorem 5.6.

**Lemma 6.1.** Let $t \in D$: given $z \in SBV(\Omega')$ with $z = g_h(t)$ on $\Omega_D$ we have that

$$\int_{\Omega'} |\nabla u(t)|^2 \, dx \leq \int_{\Omega'} |\nabla z|^2 \, dx + H^{N-1}(S_z \setminus S_{u(t)}).$$

In order to prove Lemma 6.1 we need several preliminary results. Let $z \in SBV(\Omega')$ be such that $z = g_h(t)$ on $\Omega_D$. Given $\sigma > 0$, let $U$ be a neighborhood of $S_{u(t)}$ such that $|U| \leq \sigma$, and $||\nabla z||_{L^2(U; \mathbb{R}^N)} \leq \sigma$. Let $C := \{ x \in \partial D \Omega : \partial D \Omega \text{ is not differentiable at } x \}$. We recall that there exists a countable and dense set $A \subseteq \mathbb{R}$ such that up to a set of $H^{N-1}$-measure zero

$$S_{u(t)} = \bigcup_{a,b \in A} \partial^* E_a \cap \partial^* E_b$$

where $E_a := \{ x \in \Omega' : u(t)(x) \geq a \}$ and $\partial^*$ denotes the essential boundary. Consider

$$J_j := \left\{ x \in S_{u(t)} \setminus C : [u(t)(x)] \geq \frac{1}{j} \right\},$$

with $j$ chosen in such a way that $H^{N-1}(S_{u(t)} \setminus J_j) \leq \sigma$. For $x \in J_j$, let $a_1(x), a_2(x) \in A$ be such that $u^1(x) < a_1(x) < a_2(x) < u^2(x)$ and $a_2(x) - a_1(x) \geq \frac{1}{2j}$. Following 13...
Theorem 2.1], we consider a finite disjoint collection of closed cubes \( \{Q_i\}_{i=1}^k \) with center \( x_i \in J_j \), radius \( r_i \) and with normal \( \nu(x_i) \) such that \( \bigcup_{i=1}^k Q_i \subseteq U \), \( \mathcal{H}^{N-1}(J_j \setminus \bigcup_{i=1}^k Q_i) \leq \sigma \), and for all \( i = 1, \ldots, k \), \( j = 1, 2 \)

1. \( \mathcal{H}^{N-1}(S_{u(t)} \cap \partial Q_i) = 0; \)

2. \( r_i^{-N} \leq c\mathcal{H}^{N-1}(S_{u(t)} \cap Q_i) \) for some constant \( c > 0; \)

3. \( \mathcal{H}^{N-1}\left([S_{u(t)} \setminus \partial^* E_{a_j(x_i)}] \cap Q_i\right) \leq \sigma r_i^{-N-1}; \)

4. \( \mathcal{H}^{N-1}\left(\{y \in \partial^* E_{a_j(x_i)} \cap Q_i : \text{dist}(y, H_i) \geq \frac{2}{\sigma} r_i\}\right) < \sigma r_i^{-N-1} \) where \( H_i \) denotes the intersection of \( Q_i \) with the hyperplane through \( x_i \) orthogonal to \( \nu(x_i); \)

5. \( \mathcal{H}^{N-1}(S_z \cap \partial Q_i) < \sigma r_i^{-N-1} \) and \( \mathcal{H}^{N-1}(S \cap \partial Q_i) = 0. \)

Note that we may suppose that \( Q_i \subseteq \Omega \) if \( x_i \in \Omega \). Moreover we may require that (see Theorem 2.1 and references therein) for all \( i = 1, \ldots, k \) and \( j = 1, 2 \)

\[
(6.3) \quad \|1_{E_{a_j(x_i)} \cap Q_i} - 1_{Q_i^+}\|_{L^1(\Omega)} \leq \sigma^2 r_i^{-N}.
\]

Let us indicate by \( R_i \) the rectangle given by the intersection of \( Q_i \) with the strip centered at \( H_i \) with width \( 2r_i \), and let us pose \( V_i := \{y + sv(x_i) : y \in \partial Q_i, s \in \mathbb{R}\} \cap R_i \). Note that up to changing the strip, we can suppose \( \mathcal{H}^{N-1}(\partial R_i \cap (S_u \cup S_z)) = 0 \).

If \( x_i \in \partial D \Omega \), since \( x_i \notin C \), we may require that

\[
(6.4) \quad \text{dist}(\partial \Omega \cap Q_i, H_i) < \sigma r_i;
\]

moreover, if \( (Q_i^+ \setminus R_i) \subseteq \Omega \), we can assume that \( g_i(t) < a_1(x_i) \) on \( \partial \Omega \cap Q_i \) because \( g_i(t) \) is continuous and \( g_i(t)(x_i) = u^-(x_i) < a_1(x_i) \). Similarly we may require that \( g_i(t) > a_2(x_i) \) on \( \partial \Omega \cap Q_i \) in the case \( (Q_i^+ \setminus R_i) \subseteq \Omega \).

Since we can reason up to subsequences of \( \varepsilon_n \), we may suppose that \( \sum_n \varepsilon_n \leq \frac{1}{\sigma} \). Since by \( (5.5) \) we have that \( \|u_n(t)\|_\infty < C_1 \) and \( v_n(t) \rightarrow 1 \) strongly in \( L^2(\Omega) \), by Lemma 5.3 we deduce that \( u_n(t) \rightarrow u(t) \) in measure. By \( (6.3) \), we deduce that for \( n \) large enough

\[
(6.5) \quad |Q_i^+ \setminus E^n_{a_2(x_i)}| \leq 2\sigma^2 r_i^{-N},
\]

where we use the notation \( E^n_a := \{x \in \Omega : u_n(t)(x) \geq a\} \). Let \( G_n \subseteq \frac{2}{\sigma} r_i \), \( \frac{2}{\sigma} r_i \) be the set of all \( s \) such that

\[
\int_{H_i(s)} (\eta_n + v^2_n(t))|\nabla u_n(t)|^2 \, d\mathcal{H}^{N-1} \geq \frac{C_1}{\sigma r_i \varepsilon_n};
\]

we get immediately by \( (5.5) \) that

\[
|G_n| \leq \sigma r_i \varepsilon_n,
\]

so that, posing \( G := \bigcup_n G_n \), we have \( |G| \leq \frac{2}{\sigma} r_i \) and \( |\frac{2}{\sigma} r_i \setminus G| \geq \frac{2}{\sigma} r_i \). From \( (6.5) \), applying Fubini's Theorem we obtain

\[
\int_{\frac{2}{\sigma} r_i \setminus G} \mathcal{H}^{N-1}\left(H_i(s) \setminus E^n_{a_2(x_i)}\right) \, ds \leq 2\sigma^2 r_i^{-N},
\]

so that there exists \( \varepsilon \in \frac{2}{\sigma} r_i \setminus \frac{2}{\sigma} r_i \setminus G \) such that, posing \( H^+_i := H_i(\varepsilon) \), we have

\[
(6.6) \quad \mathcal{H}^{N-1}\left(H_i^+ \setminus E^n_{a_2(x_i)}\right) \leq 16\sigma r_i^{-N-1}.
\]

Moreover we have by construction

\[
(6.7) \quad \int_{H_i^+} (\eta_n + v^2_n(t))|\nabla u_n|^2 \, d\mathcal{H}^{N-1} \leq K_n,
\]
where \( K_n \) is of the order of \( \frac{1}{\varepsilon_n} \). In a similar way, there exists \( H_i^+ := H_i(\tilde{s}) \) with \( \tilde{s} \in ] - \frac{2}{3} r_i, -\frac{2}{3} r_i[ \) and

\[
\mathcal{H}^{N-1}(H_i^- \cap E_{\varepsilon_i(x_i)} + \frac{1}{2}) \leq 16\sigma_i^{N-1},
\]

and

\[
\int_{H_i^-} (\eta_n + \nu_n^2(t))|\nabla u_n|^2 d\mathcal{H}^{N-1} \leq K_n
\]

where \( K_n \) is of the order of \( \frac{1}{\varepsilon_n} \). We indicate by \( \bar{R}_i \) the intersection of \( Q_i \) with the strip determined by \( H_i^+ \) and \( H_i^- \).

A similar argument prove that, up to reducing \( Q_i \) (preserving the estimates previously stated), we may suppose that

\[
\int_{V_i} (\eta_n + \nu_n^2(t))|\nabla u_n(t)|^2 d\mathcal{H}^{N-1} \leq K_n.
\]

where \( K_n \) is of the order of \( \frac{1}{\varepsilon_n} \).

In order to prove Lemma 6.2, we claim that we can suppose \( z = g_h(t) \) on \( \Omega_D \) and in a neighborhood \( V \) of \( \partial_D \Omega \setminus \bigcup_{i=1}^k Q_i, S_2 \setminus \bigcup_{i=1}^k R_i \) polyhedral with closure contained in \( \Omega \), and \( \mathcal{H}^{N-1}((S_2 \setminus S_\varepsilon(t)) \cap Q_i) \leq \sigma_i^{N-1} \) for all \( i = 1, \ldots, k \). In fact, by Proposition 2.5, there exists \( w_m \in SBV(\Omega') \) with \( w_m = g_h(t) \) in \( \Omega' \setminus \bar{\Omega} \) and in a neighborhood \( V_m \) of \( \partial_D \Omega \) such that \( w_m \rightarrow z \) strongly in \( L^2(\Omega') \), \( \nabla w_m \rightarrow \nabla z \) strongly in \( L^2(\Omega'; \mathbb{R}^N) \), \( S_{w_m} \subseteq \Omega \), and such that for all \( A \) open subset of \( \Omega' \) with \( \mathcal{H}^{N-1}(\partial A \cap S_z) = 0 \), we have

\[
\lim_m \mathcal{H}^{N-1}(A \cap S_{w_m}) = \mathcal{H}^{N-1}(A \cap S_z).
\]

Let us fix \( \sigma' > 0 \) and let us consider for all \( i = 1, \ldots, k \) a rectangle \( R'_i \) centered in \( x_i \), oriented as \( R_i \) and such that \( R'_i \subseteq \text{int}(R_i) \), \( \mathcal{H}^{N-1}(\partial R'_i \cap S_z) = 0 \), \( \mathcal{H}^{N-1}(S_z \cap (\text{int}(R_i) \setminus R'_i)) \leq \sigma_i^{N-1} \), where \( \text{int}(R_i) \) denotes the interior part of \( R_i \). Let \( \psi_i \) be a smooth function such that \( 0 \leq \psi_i \leq 1 \), \( \psi_i = 1 \) on \( R'_i \) and \( \psi_i = 0 \) outside \( R_i \). Posing \( \psi := \sum_{i=1}^k \psi_i \), let us consider \( z_m := \psi z + (1 - \psi) w_m \). Note that \( z_m \rightarrow z \) strongly in \( L^2(\Omega') \), \( \nabla z_m \rightarrow \nabla z \) strongly in \( L^2(\Omega'; \mathbb{R}^N) \), \( z_m = g_h(t) \) in \( \Omega_D \) and in a neighborhood \( V' \) of \( \partial_D \Omega \), and \( S_{z_m} \subseteq \Omega \) polyhedral with closure contained in \( \bar{\Omega} \). Finally, for \( m \rightarrow +\infty \), we have

\[
\mathcal{H}^{N-1}((S_{z_m} \setminus \bigcup_{i=1}^k Q_i) \rightarrow \mathcal{H}^{N-1}(S_z \setminus \bigcup_{i=1}^k Q_i)) \quad \text{and} \quad \limsup_m \mathcal{H}^{N-1}(S_{z_m} \cap (\text{int}(R_i) \setminus R'_i)) \leq 2\mathcal{H}^{N-1}(S_z \cap (\text{int}(R_i) \setminus R'_i)) \leq \sigma' \varepsilon_i^{N-1}.
\]

So, if (6.2) holds for \( z_m \), we obtain for \( m \rightarrow +\infty \) that (6.11) holds also for \( z \) since \( \sigma' \) is arbitrary, and so the claim is proved.

We begin with the following lemma.

**Lemma 6.2.** Let \( B_n(t) \) be as in (6.2), and let us consider \( b_k^2 := b_n^2(t), b_k^3 := b_n^3(t) \in B_n(t) \) with \( j_2 > j_3 > 1 \). Suppose that \( k_n := \frac{k}{n^2} > 1 \) and let \( k, b \) be such that \( 1 < k \leq k_n, b_n^3 \leq b \) for all \( n \). Then posing

\[
\left\{
\begin{array}{ll}
\frac{k}{n^2}(v_n(t) - b_n^3) + b_n^3 & \text{in } \{b_n^2 \leq v_n(t) \leq b_n^3\} \\
0 & \text{in } \{v_n(t) \leq b_n^2\} \\
v_n(t) & \text{in } \{v_n(t) \geq b_n^3\}
\end{array}
\right.
\]

we have that \( w_n \in H^1(\Omega') \) with \( w_n = 1 \) on \( \Omega_D \), \( 0 \leq w_n \leq v_n(t) \) in \( \Omega' \) and

\[
\limsup_n (M_{b_n}(w_n) - M_{b_n}(v_n(t))) \leq \frac{2C_1 k}{(k - 1)^2} + \frac{C_1}{(k - 1)(1 - b)^2} + \frac{C_1 b}{(1 - b)^2},
\]

where \( C_1 \) is given by (6.2). Moreover there exist \( b_n^1 := b_n^1(t) \in B_n(t) \) with \( j_1 > j_2 > 1 \) and a cut-off function \( \varphi_n \in H^1(\Omega') \) with \( \varphi_n = 0 \) in \( \{v_n(t) \leq b_n^1\} \), \( \varphi_n = 1 \) on \( \{v_n(t) \geq b_n^2\} \) (in
particular on $\Omega_D$) and such that

$$
(6.12) \quad \lim_{n} \eta_n \int_{\Omega'} |\nabla \varphi_n|^2 \, dx = 0
$$

Proof. $w_n$ is well defined in $H^1(\Omega')$, and by construction $w_n = 1$ on $\Omega_D$ and $0 \leq w_n \leq v_n(t)$ in $\Omega'$. Let us estimate $MM_n(w_n) - MM_n(v_n)$. Since

$$
\frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla w_n|^2 \, dx \geq \frac{\varepsilon_n}{2} \int_{\{v_n(t) \geq b_n^1\}} |\nabla v_n(t)|^2 \, dx + \frac{\varepsilon_n}{2} \int_{\{b_n^2 \leq v_n(t) \leq b_n^3\}} |\nabla w_n|^2 \, dx,
$$

and $MM_n(v_n(t)) \leq C_1$ by (5.3), we have that

$$
\frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla w_n|^2 \, dx - \frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla v_n(t)|^2 \, dx \leq C_1 \left( \frac{k_n^2}{(k_n - 1)^2} - 1 \right) |\nabla v_n(t)|^2 \, dx - \frac{\varepsilon_n}{2} \int_{\{v_n(t) \leq b_n^3\}} |\nabla v_n(t)|^2 \, dx \leq C_1 \left( \frac{k_n^2}{(k_n - 1)^2} - 1 \right) = \frac{C_1 (2k_n - 1)}{(k_n - 1)^2} \leq \frac{2C_1 k}{(k - 1)^2}.
$$

Moreover we have that

$$
\frac{1}{2\varepsilon_n} \int_{\Omega'} (1 - w_n)^2 \, dx - \frac{1}{2\varepsilon_n} \int_{\Omega'} (1 - v_n(t))^2 \, dx = \frac{1}{2\varepsilon_n} \int_{\Omega'} [(1 - w_n)^2 - (1 - v_n(t))^2] \, dx = \frac{1}{2\varepsilon_n} \int_{\Omega'} (v_n(t) - w_n)(2 - v_n(t) - w_n) \, dx = \frac{1}{2\varepsilon_n} \int_{\{b_n^2 \leq v_n(t) \leq b_n^3\}} (v_n(t) - \frac{k_n}{k_n - 1}(v_n(t) - b_n^3) - b_n^3) (2 - v_n(t) - w_n) \, dx + \frac{1}{2\varepsilon_n} \int_{\{v_n(t) \leq b_n^3\}} v_n(t)(2 - v_n(t)) \, dx = \frac{1}{2\varepsilon_n} \int_{\{b_n^2 \leq v_n(t) \leq b_n^3\}} \frac{1}{k_n - 1} (b_n^3 - v_n(t))(2 - v_n(t) - w_n) \, dx + \frac{1}{2\varepsilon_n} \int_{\{v_n(t) \leq b_n^3\}} v_n(t)(2 - v_n(t)) \, dx \leq \frac{C_1}{(k_n - 1)(1 - b_n^1)^2} + \frac{C_1 b_n^2}{(1 - b_n^2)^2} \leq \frac{C_1}{(k - 1)(1 - b)^2} + \frac{C_1 b}{(1 - b)^2}
$$

because $\frac{(v_n(t) \leq b_n^1)}{\varepsilon_n} \leq \frac{C_1}{(1 - s)^2}$. We conclude that

$$
\limsup_{n} (MM_n(w_n) - MM_n(v_n(t))) \leq \frac{2C_1 k}{(k - 1)^2} + \frac{C_1}{(k - 1)(1 - b)^2} + \frac{C_1 b}{(1 - b)^2}.
$$

Let $j_1 > j_2 + 1$: we have that $b_n^1 := b_n^{j_1}$ and $b_n^2$ are not in adjacent intervals, and so there exists $l > 0$ with $0 < l \leq b_n^2 - b_n^1$. Let us divide the interval $[b_n^1, b_n^2]$ in $h_n$ intervals of the same size $I_j, j = 1, \ldots, h_n$, with $h_n$ such that $\frac{h_n}{\varepsilon_n} h_n \to 0$. Since

$$
\sum_{j=1}^{h_n} \frac{\varepsilon_n}{2} \int_{\{v_n(t) \in I_j\}} |\nabla v_n(t)|^2 \, dx \leq \frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla v_n(t)|^2 \, dx \leq C_1,
$$

we deduce that there exists $I_n$ such that

$$
(6.13) \quad \frac{\varepsilon_n}{2} \int_{\{v_n(t) \in I_n\}} |\nabla v_n(t)|^2 \, dx \leq \frac{C_1}{h_n}.
$$
Let $\alpha_n, \beta_n$ be the extremes of $I_n$. Let us pose

\begin{equation}
\varphi_n := \frac{1}{\beta_n - \alpha_n} (v_n - \alpha_n)^+ \wedge 1.
\end{equation}

Then $\varphi_n \in H^1(\Omega)$, $\varphi_n = 0$ in $\{v_n(t) \leq b^1_n\}$, and $\varphi_n = 1$ on $\{v_n(t) \geq b^2_n\}$ (in particular on $\Omega_D$) and by (6.14) and the choice of $h_n$ we have that

$$
\eta_n \int_{\Omega'} |\nabla \varphi_n|^2 \, dx = \eta_n \int_{\{\alpha_n \leq v_n(t) \leq \beta_n\}} \frac{1}{(\beta_n - \alpha_n)^2} |\nabla v_n(t)|^2 \, dx \leq \frac{\eta_n 2C_1 h_n^2}{\varepsilon_n} \to 0,
$$

so that the proof is complete. \hfill \Box

Let $b^1_n$ be as in Lemma 6.2 and let $\delta := \frac{1}{8j}$ so that for all $i = 1, \ldots, k$

$$
a_1(x_i) < a_1(x_i) + \delta < a_2(x_i) - \delta < a_2(x_i).
$$

**Lemma 6.3.** For each $i = 1, \ldots, k$, there exists $w^{2,i}_n \in H^1(Q_i)$ and $[\gamma^i_n - \tau^i_n, \gamma^i_n + \tau^i_n] \subseteq [a_1(x_i) + \delta, a_2(x_i) - \delta]$ such that $0 \leq w^{2,i}_n \leq 1$, $w^{2,i}_n = 0$ in $[\gamma^i_n - \tau^i_n \leq u_n(t) \leq \gamma^i_n + \tau^i_n] \cap Q_i$, $w^{2,i}_n = 1$ on $\left\{(u_n(t) \leq a_1(x_i) + \frac{2}{3}\delta) \cup \{u_n(t) \geq a_2(x_i) - \frac{2}{3}\delta\}\right\} \cap Q_i$, and

\begin{equation}
\limsup_{n} \sum_{i=1}^{k} MM_n(w^{2,i}_n)_{\{u_n(t) > b^1_n\}} \leq o(\sigma).
\end{equation}

Moreover there exists $\varphi^{2,i}_n \in H^1(Q_i)$ such that $0 \leq \varphi^{2,i}_n \leq 1$, $\varphi^{2,i}_n = 0$ on $[\gamma^i_n - \frac{1}{2}\tau^i_n \leq u_n(t) \leq \gamma^i_n + \frac{1}{2}\tau^i_n] \cap Q_i$, $\varphi^{2,i}_n = 1$ on $\left\{(u_n(t) \leq \gamma^i_n - \tau^i_n) \cup \{u_n(t) \geq \gamma^i_n + \tau^i_n\}\right\} \cap Q_i$, and

\begin{equation}
\lim_{n} \eta_n \int_{Q_i \cap \{u_n(t) > b^1_n\}} |\nabla \varphi^{2,i}_n|^2 \, dx = 0.
\end{equation}

**Proof.** For each $i$ let us consider the strip

$$
S^i_n := E^n_{a_1(x_i) + \delta} \setminus E^n_{a_2(x_i) - \delta}.
$$

Let $h_n \in \mathbb{N}$ and let us divide $[a_1(x_i) + \delta, a_2(x_i) - \delta]$ in $h_n$ intervals of the same size: there exists a subinterval with extremes $\alpha^i_n$ and $\beta^i_n$ such that, positioning $\tilde{S}^i_n := \{x \in \Omega' : \alpha^i_n \leq u_n(t) \leq \beta^i_n\}$,

\begin{equation}
\int_{\tilde{S}^i_n \cap Q_i} \left[ \sigma(\eta_n + v^2_n(t))|\nabla u_n(t)|^2 + (1 - \sigma) \right] \, dx \leq \frac{1}{h_n} \int_{S^i_n \cap Q_i} \left[ \sigma(\eta_n + v^2_n(t))|\nabla u_n(t)|^2 + (1 - \sigma) \right] \, dx.
\end{equation}

Let $\gamma^i_n := \frac{\alpha^i_n + \beta^i_n}{2}$ and $\tau^i_n := \frac{a_2(x_i) - a_1(x_i) - 2\delta}{4h_n}$. We pose

$$
w^{2,i}_n := \begin{cases} 
\frac{1}{\gamma^i_n - \alpha^i_n - \tau^i_n} (u_n(t) - \gamma^i_n - \tau^i_n)^+ \wedge 1 & \text{in } \{u_n(t) \geq \gamma^i_n + \tau^i_n\} \cap Q_i \\
0 & \text{in } \{\gamma^i_n - \tau^i_n \leq u_n(t) \leq \gamma^i_n + \tau^i_n\} \cap Q_i \\
\frac{1}{\gamma^i_n - \tau^i_n - \alpha^i_n} (u_n(t) - \gamma^i_n + \tau^i_n)^- \wedge 1 & \text{in } \{u_n(t) \leq \gamma^i_n - \tau^i_n\} \cap Q_i.
\end{cases}
$$

We have that

\begin{align*}
\frac{\varepsilon_n}{2} \int_{Q_i \cap \{u_n(t) > b^1_n\}} |\nabla w^{2,i}_n|^2 \, dx + \frac{1}{2\varepsilon_n} \int_{Q_i \cap \{u_n(t) > b^1_n\}} (1 - w^{2,i}_n)^2 \, dx & \leq \frac{\varepsilon_n}{2} \left( \frac{4h_n^2}{\delta^2} \int_{S^i_n \cap (Q_i \cap \{u_n(t) > b^1_n\})} |\nabla u_n(t)|^2 \, dx \right) + \frac{1}{2\varepsilon_n} |\tilde{S}^i_n \cap (Q_i \cap \{u_n(t) > b^1_n\})|.
\end{align*}
Since by (6.17)
\[ \int_{S_n^i \cap Q_i} (\eta_n + \nu_n^2(t)) |\nabla u_n(t)|^2 \, dx \leq \frac{1}{h_n} \left[ \int_{S_n^i \cap Q_i} (\eta_n + \nu_n^2(t)) |\nabla u_n(t)|^2 \, dx + \frac{1-\sigma}{\sigma} |S_n^i \cap Q_i| \right] \]

and
\[ |\bar{S}_n^i \cap Q_i| \leq \frac{1}{h_n} \left[ \frac{\sigma}{1-\sigma} \int_{\bar{S}_n^i \cap Q_i} (\eta_n + \nu_n^2(t)) |\nabla u_n(t)|^2 \, dx + |S_n^i \cap Q_i| \right] \]

we have
\[ \text{MM}_n(\nu_n^2)_{\{v_n(t) > b_n^1\}} \leq \frac{2h_n\varepsilon_n}{\delta^2(\eta_n + (b_n^1)^2)} \left[ \int_{\bar{S}_n^i \cap Q_i} (\eta_n + \nu_n^2(t)) |\nabla u_n(t)|^2 \, dx + \frac{1-\sigma}{\sigma} |\cup Q_i| \right] + \frac{1}{2\varepsilon_n h_n} \frac{1}{\sigma} \int_{\bar{S}_n^i \cap Q_i} (\eta_n + \nu_n^2(t)) |\nabla u_n(t)|^2 \, dx + |S_n^i \cap Q_i| \right]. \]

Summing on \( i = 1, \ldots, k \), recalling \( b_n \) and letting \( d \in [0, 1] \) with \( \eta_n + (b_n^1)^2 \geq d^2 \) for all \( n \), we obtain
\[ \sum_{i=1}^{k} \text{MM}_n(\nu_n^2)_{\{v_n(t) > b_n^1\}} \leq \frac{1}{\delta^2 d^2} \left( C_1 + 1 - \sigma \right) \left( \frac{\sigma}{1-\sigma} C_1 + \sigma \right). \]

We choose \( h_n \) in such a way that the preceding quantity is less than (recall that \( |\cup Q_i| \leq |\Omega| < \sigma \))
\[ \sqrt{\frac{1}{\delta^2 d^2} (C_1 + 1 - \sigma) \left( \frac{\sigma}{1-\sigma} C_1 + \sigma \right)}. \]

Then we obtain
\[ \sum_{i=1}^{k} \text{MM}_n(\nu_n^2)_{\{v_n(t) > b_n^1\}} \leq \sqrt{\frac{1}{\delta^2 d^2} (C_1 + 1 - \sigma) \left( \frac{\sigma}{1-\sigma} C_1 + \sigma \right)} = o(\sigma). \]

This prove the first part of the lemma.

Let us define \( \varphi_n^{2,i} \) as \( \nu_n^{2,i} \) but operating with the levels \( \gamma_n^i - \tau_n^i \leq \gamma_n^i - \frac{\tau_n^i}{2} \) and \( \gamma_n^i + \frac{\tau_n^i}{2} \leq \gamma_n^i + \tau_n^i \). Reasoning as above we obtain
\[ \eta_n \sum_{i=1}^{k} \int_{Q_i \cap \{v_n(t) > b_n^1\}} |\nabla \varphi_n^{2,i}|^2 \, dx \leq \frac{16\eta_n h_n^2}{\delta^2 d^2} (C_1 + 1 - \sigma) \rightarrow 0 \]

since \( h_n \) has been chosen of the order of \( \frac{1}{\varepsilon_n} \). \( \square \)

**Lemma 6.4.** Let \( Q_i \subseteq \Omega \). Then there exists \( \nu_n^{3,i} \in H^1(Q_i) \) such that \( 0 \leq \nu_n^{3,i} \leq 1 \), \( \nu_n^{3,i} = 0 \) in a neighborhood of \( H_i^+ \setminus E_{a_2(x)} - \frac{\delta}{2} \) and of \( H_i^+ \cap E_{a_1(x)} + \frac{\delta}{4} \), \( \nu_n^{3,i} = 1 \) on \( Q_i \setminus R_i \) for \( n \) large, and
\[ \limsup_{Q_i \subseteq \Omega} \sum_{i=1}^{k} \text{MM}_n(\nu_n^{3,i})_{\{v_n(t) > b_n^1\}} \leq o(\sigma). \]

Moreover there exists a cut-off function \( \varphi_n^{3,i} \in H^1(Q_i) \) such that \( \varphi_n^{3,i} = 0 \) in a neighborhood of \( H_i^+ \setminus E_{a_2(x)} - \delta \) and of \( H_i^+ \cap E_{a_1(x)} + \delta \), \( \varphi_n^{3,i} = 1 \) on \( Q_i \setminus R_i \) for \( n \) large, sup(\( \nabla \varphi_n^{3,i} \)) \subseteq \{ \nu_n^{3,i} = 0 \}, \) and
\[ \lim_{n} \eta_n \int_{Q_i \cap \{v_n(t) > b_n^1\}} |\nabla \varphi_n^{3,i}|^2 \, dx = 0. \]
Proof. Let \( \pi^+_i \) be the planes which contain \( H^+_i \), and for \( x \in \Omega' \), let \( \pi^+_i \) be its projection on \( \pi^+_i \). Let us now consider \( (u_n(t))_{H^+_i} \): we pose

\[
\psi^{i,+}_n(y) := \frac{4}{\delta} \left( u_n(y) - a_2(x_i) + \frac{3}{4} \right)^+ \wedge 1
\]

Note that \( \psi^{i,+}_n \) is equal to zero on \( H^+_i \setminus E^a_{n_{a_2(x_i)}} \) and so on \( \{ x \in H^+_i : u_n(t)(x) = \gamma_n \} \) where \( \gamma_n \) is defined as in Lemma\textsuperscript{28}. Moreover, \( \psi^{i,+}_n \) is 1 on \( H^+_i \cap E^a_{a_2(x_i)} \). If \( d \in [0, 1] \) is such that \( \eta_n + (b^1_n)^2 \geq d^2 \), by \textsuperscript{28} we have

\[
(6.20) \quad \int_{H^+_i \cap \{ v_n(t) > b^1_n \}} |\nabla \psi^{i,+}_n|^2 d\mathcal{H}^{N-1} \leq \frac{16}{\delta^2} \int_{H^+_i \cap \{ v_n(t) > b^1_n \}} |\nabla u_n|^2 d\mathcal{H}^{N-1} \leq \frac{16K_n}{\delta^2 d^2}.
\]

Let us define

\[
\tilde{\psi}^{i,+}_n(y) := \frac{4}{\delta} (u_n(y) - a_2(x_i) + \delta)^+ \wedge 1
\]

which is null on \( H^+_i \setminus E^a_{a_2(x_i)} \).

In a similar way we construct \( \psi^{i,-}_n \) and \( \tilde{\psi}^{i,-}_n \) on \( H^-_i \) which are null on \( H^-_i \cap E^a_{a_1(x_i) + \frac{\delta}{2}} \) and on \( H^-_i \cap E^a_{a_1(x_i) + \delta} \) respectively. Let us pose

\[
w^{3,i,+}_n(x) := \left[ \psi^{i,+}_n(\pi^+_i x) + \frac{1}{\varepsilon_n} (d_{H^+_i}(x) - l^i_n)^+ \right] \wedge 1
\]

with \( \frac{l^i_n}{\varepsilon_n} \to 0 \) and \( \frac{\varepsilon_n}{(l^i_n)^2} \to 0 \). This is possible since \( \eta_n \ll \varepsilon_n \). Let \( A^i_n := (H^+_i \setminus E^a_{a_2(x_i)} + \frac{\delta}{2}) - \varepsilon_n - l^i_n, \varepsilon_n + l^i_n \cap \{ v_n(t) > b^1_n \} \). Then we have by definition of \( \psi^{i,+}_n \), by \textsuperscript{28}, \textsuperscript{6.20} and the fact that \( K_n \varepsilon_n \) is bounded in \( n \)

\[
\limsup_n M M_n(w^{3,i,+}_n)_{\{ v_n(t) > b^1_n \}} = \limsup_n \left\{ \frac{\varepsilon_n}{2} \int_{Q_i \cap \{ v_n(t) > b^1_n \}} |\nabla w^{3,i,+}_n|^2 d\mathcal{H} \leq \frac{1}{2 \varepsilon_n} \int_{Q_i \cap \{ v_n(t) > b^1_n \}} \left( 1 - w^{3,i,+}_n \right)^2 d\mathcal{H} \right\} \leq \limsup_n \left\{ \frac{\varepsilon_n}{2} \left( |\nabla \psi^{i,+}_n(\pi^+_i x)|^2 + \frac{1}{\varepsilon_n} \right) d\mathcal{H} + \frac{1}{2 \varepsilon_n} \left( 2 \mathcal{H}^{N-1}(H^+_i \setminus E^a_{a_2(x_i)} - \frac{\delta}{2}) (\varepsilon_n + l^i_n) \right) \right\}
\]

so that we get

\[
\limsup_n M M_n(w^{3,i,+}_n)_{\{ v_n(t) > b^1_n \}} = \limsup_n \left\{ \frac{\varepsilon_n}{2} \frac{16K_n}{\delta^2 d^2} (\varepsilon_n + l^i_n)^2 + \frac{\varepsilon_n}{2} (\varepsilon_n + l^i_n) \mathcal{H}^{N-1}(H^+_i \setminus E^a_{a_2(x_i)} - \frac{\delta}{2}) \right\} \leq 2 \limsup_n \mathcal{H}^{N-1}(H^+_i \setminus E^a_{a_2(x_i)} - \frac{\delta}{2}) \leq 4 \sigma r_i^{N-1}.
\]

Similar calculations hold for \( w^{3,i,-}_n \). Let us pose \( w^{3,i} := w^{3,i,+} \wedge w^{3,i,-}_n \). Then \( 0 \leq w^{3,i}_n \leq 1 \), \( w^{3,i}_n = 0 \) in a neighborhood of \( H^+_i \setminus E^a_{a_2(x_i)} - \frac{\delta}{2} \) and of \( H^-_i \cap E^a_{a_1(x_i) + \frac{\delta}{2}} \), \( w^{3,i}_n = 1 \) on \( Q_i \setminus R_i \) for \( n \) large, and we have that

\[
\limsup_n \sum_{Q_i \subseteq \Omega} M M_n(w^{3,i}_n)_{\{ v_n(t) > b^1_n \}} \leq o(\sigma),
\]

which proves the first part of the lemma.
We define
\[ \varphi_{n}^{3,i}(x) := \left[ \psi_{n}^{i,+}(\pi_{n}^{i}x) + \frac{1}{l_{n}^{i}} \left( d_{H_{n}^{i}}(x) - \frac{l_{n}^{i}}{2} \right) \right] \wedge \left[ \psi_{n}^{i,-}(\pi_{n}^{i}x) + \frac{1}{l_{n}^{i}} \left( d_{H_{n}^{i}}(x) - \frac{l_{n}^{i}}{2} \right) \right] \wedge 1. \]

The previous calculations prove that
\[ \lim_{n} \frac{\eta_{n}}{(l_{n}^{i})^{2}} \int_{Q_{i} \cap \{v_{n}(t) > b_{n}^{i}\}} |\nabla \varphi_{n}^{3,i}|^{2} \, dx = 0 \]

since \( \frac{\eta_{n}}{(l_{n}^{i})^{2}} \to 0 \). Moreover \( \varphi_{n}^{3,i} = 1 \) on \( Q_{i} \setminus R_{i} \) for \( n \) large.

**Lemma 6.5.** Suppose that \( Q_{i} \subseteq \Omega \); then there exists \( w_{n}^{4,i} \in H^{1}(\Omega') \) such that \( 0 \leq w_{n}^{4,i} \leq 1 \), \( w_{n}^{4,i} = 0 \) in a neighborhood of \( V_{i} \), \( w_{n}^{4,i} = 1 \) on \( \Omega_{D} \) for \( n \) large and

\[ \limsup_{n} \sum_{Q_{i} \subseteq \Omega} M M_{n}(w_{n}^{4,i}) \leq o(\sigma). \]

Moreover there exists a cut-off function \( \varphi_{n}^{4,i} \) such that \( \varphi_{n}^{4,i} = 0 \) in a neighborhood of \( V_{i} \), \( \varphi_{n}^{4,i} = 1 \) on \( \Omega_{D} \) for \( n \) large, \( \text{sup}(\nabla \varphi_{n}^{4,i}) \subseteq \{w_{n}^{4,i} = 0\} \), and

\[ \lim_{n} \eta_{n} \int_{\Omega'} |\nabla \varphi_{n}^{4,i}|^{2} \, dx = 0. \]

**Proof.** Let us pose
\[ w_{n}^{4,i}(x) := \frac{1}{\varepsilon_{n}}(d_{\Omega}(x) - l_{n}^{i})^{+} \wedge 1, \]

and
\[ \varphi_{n}^{4,i}(x) := \frac{1}{l_{n}^{i}} \left( d_{\Omega}(x) - \frac{l_{n}^{i}}{2} \right)^{+} \wedge 1 \]

where \( \frac{l_{n}^{i}}{\varepsilon_{n}} \to 0 \) and \( \frac{\eta_{n}}{(l_{n}^{i})^{2}} \to 0 \). We have immediately (since \( \sum_{Q_{i} \subseteq \Omega} \mathcal{H}^{N-1}(V_{i}) \leq o(\sigma) \))

\[ \limsup_{n} \sum_{Q_{i} \subseteq \Omega} M M_{n}(w_{n}^{4,i}) \leq o(\sigma) \]

while
\[ \lim_{n} \eta_{n} \int_{\Omega'} |\nabla \varphi_{n}^{4,i}|^{2} \, dx = 0 \]

since \( \frac{\eta_{n}}{(l_{n}^{i})^{2}} \to 0 \). For \( n \) large enough, \( w_{n}^{4,i} = 1 \), \( \varphi_{n}^{4,i} = 1 \) on \( \Omega_{D} \) and the proof is complete.

We recall that \( z = g_{h}(t) \) in a neighborhood \( \Omega \) of \( \partial \Omega \cup Q_{i} \).

**Lemma 6.6.** Let \( Q_{i} \cap \partial D \Omega \neq \emptyset \) with \( Q_{i}^{+} \setminus R_{i} \subseteq \Omega \). Then \( E_{a_{1}(x_{i})+\delta}^{n} \cap Q_{i} \subseteq \Omega \) for all \( n \), and there exists \( w_{n}^{b,i,+} \in H^{1}(\Omega') \) with \( 0 \leq w_{n}^{b,i,+} \leq 1 \), \( w_{n}^{b,i,+} = 1 \) on \( \Omega_{D} \), \( w_{n}^{b,i,+} = 0 \) in a neighborhood of
\[ V_{n}^{i,+} := \left[ V_{i} \cap E_{a_{1}(x_{i})+\delta}^{n} \right] \cup \left( (V_{i} \cap Q_{i}^{+}) \setminus \Omega \right), \]

and such that

\[ \limsup_{n} \sum_{Q_{i} \cap \partial D \Omega \neq \emptyset} M M_{n}(w_{n}^{b,i,+})_{\{v_{n}(t) > b_{n}^{i}\}} \leq o(\sigma). \]

Moreover there exists a cut-off function \( \varphi_{n}^{b,i,+} \) such that \( \varphi_{n}^{b,i,+} = 1 \) on \( \Omega_{D} \), \( \varphi_{n}^{b,i,+} = 0 \) in a neighborhood of \( V_{n}^{i,+} \), suppt(\nabla \varphi_{n}^{b,i,+}) \subseteq \{w_{n}^{b,i,+} = 0\} \), and

\[ \lim_{n} \eta_{n} \int_{\Omega' \cap \{v_{n}(t) > b_{n}^{i}\}} |\nabla \varphi_{n}^{b,i,+}|^{2} \, dx = 0 \]
Proof. Note that by construction, $E_{a_1(x_i)+\delta}^n \cap Q_i \subseteq \Omega$ since $u_n(t)$ is continuous and $u_n(t) = g_h(t)$ on $Q$. It is now sufficient to operate as in Lemma 6.4 and in Lemma 6.5. In fact, in view of (6.20), we may construct $\hat{w}_n^{b,i,+} \in H^1(\Omega')$ such that $0 \leq \hat{w}_n^{b,i,+} \leq 1$, $\hat{w}_n^{b,i,+} = 0$ in a neighborhood of $V_i \cap E_{a_1(x_i)+\delta}^n$, $\hat{w}_n^{b,i,+} = 1$ on $\Omega_D$ and on $V_i \setminus E_{a_1(x_i)+\delta}^n$, and such that $\limsup_n MM_n(\hat{w}_n^{b,i,+})_{\{\|v_n(t)\| > b_i\}} \leq o(\sigma)\eta_i^{N-1}$. Referring to $(V_i \cap Q_t^\tau) \setminus \mathcal{V}$, we can reason as in Lemma 6.4 and getting $w_n^{b,i,+}$, such that $0 \leq w_n^{b,i,+} \leq 1$, $w_n^{b,i,+} = 0$ in a neighborhood of $(V_i \cap Q^\tau_t) \setminus \mathcal{V}$, $\hat{w}_n^{b,i,+} = 1$ on $\Omega_D$, and such that $\limsup_n MM_n(w_n^{b,i,+}) \leq o(\sigma)\eta_i^{N-1}$.

Posing $w_n^{b,i,+} := \hat{w}_n^{b,i,+} \land w_n^{b,i,+}$, we get the first part of the thesis. Similarly, we may construct $\varphi_n^{b,i,+}$ which satisfies (6.24). 

In a similar way we can prove the following lemma.

Lemma 6.7. Let $Q_i \cap \partial_D \Omega \neq \emptyset$ with $Q_i^\tau \not\subseteq R_i \subseteq \Omega$. Then $Q_i \setminus E_{a_2(x_i)-\delta}^n \subseteq \Omega$ for all $n$, and there exists $w_n^{b,i,-} \in H^1(\Omega')$ with $0 \leq w_n^{b,i,-} \leq 1$, $w_n^{b,i,-} = 1$ on $\Omega_D$, $w_n^{b,i,-} = 0$ in a neighborhood of

$$V_i^{n,-} := \left[ V_i \setminus E_{a_2(x_i)-\delta}^n \right] \cup \left[ (V_i \cap Q_i^\tau) \setminus \mathcal{V} \right],$$

and such that

$$(6.25) \quad \limsup_n \sum_{Q_i \cap \partial_D \Omega \neq \emptyset} MM_n(w_n^{b,i,-})_{\{|v_n(t)| > b_i^+\}} \leq o(\sigma).$$

Moreover there exists a cut-off function $\varphi_n^{b,i,-}$ such that $\varphi_n^{b,i,-} = 1$ on $\Omega_D$, $\varphi_n^{b,i,-} = 0$ in a neighborhood of $V_i^{n,-}$, $\sup_{\Omega} (\nabla \varphi_n^{b,i,-}) \subseteq \{w_n^{b,i,-} = 0\}$, and

$$(6.26) \quad \lim_n \eta_n \int_{\Omega \cap \{|v_n(t)| > b_i^+\}} \|\nabla \varphi_n^{b,i,-}\|^2 dx = 0.$$

We can now prove Lemma 6.1.

Proof of Lemma 6.7. We employ the notation of the preceding lemmas. Following [12, Theorem 2.1], for each $i$ let us define $z_i^+$ on $Q_i^\tau \setminus R_i$ to be equal to $z$ on $Q_i^\tau \setminus R_i$ and to $H_1(\sigma)$ on $R_i$. Similarly we define $z_i^-$. For each $Q_i \subseteq \Omega$, let us pose $z_i^0$ to be equal to $z_i^+$ on $(Q_i^\tau \setminus R_i) \cup (E_{\gamma_i}^n \setminus R_i)$, and to $z_i^-$ in the rest of $Q_i$.

If $Q_i \cap \partial \Omega \neq \emptyset$ with $Q_i^\tau \setminus R_i \subseteq \Omega$, by Lemma 6.4 and Lemma 6.6 we have $E_{\gamma_i}^n \setminus E_{z_i^0} \subseteq Q_i \subseteq Q_i^\tau$ for all $n$, and its closure does not intersect $\partial \Omega$. We define $z_i^+ \not\subseteq Q_i^\tau \setminus R_i \cup (E_{\gamma_i}^n \setminus R_i)$, and to $g_h(t)$ in the rest of $Q_i$. If $Q_i^\tau \setminus R_i \subseteq \Omega$, by Lemma 6.4 and Lemma 6.6 we have $Q_i \setminus E_{\gamma_i}^n \subseteq \Omega$, and its closure does not intersect $\partial \Omega$. We define $z_i^+ \not\subseteq Q_i^\tau \setminus R_i \cup (E_{\gamma_i}^n \setminus R_i)$, and to $g_h(t)$ in the rest of $Q_i$.

Let us now define $z_i$ to be equal to $z$ outside $\bigcup_{i=1}^{k} R_i$, and to $z_i^+$ inside each $R_i$. We have $z_n = g_h(t)$ on $\Omega_D$. Note that if $Q_i \subseteq \Omega$, $H_i^+ \setminus E_{\gamma_i}^n \subseteq H_i^+ \cap H_i \subseteq E_{\gamma_i}^n$, $V_i^{+}$, and $\vartheta^* E_{\gamma_i}^n \setminus Q_i$ could be contained in $S_{\gamma_i}$. Similarly, if $Q_i \cap \partial \Omega \neq \emptyset$ and $Q_i^\tau \setminus R_i \subseteq \Omega$ (the other case being similar), then $H_i^+ \setminus E_{\gamma_i}^n \subseteq V_i^{\pm}$, and $\vartheta^* E_{\gamma_i}^n \setminus Q_i$ could be contained in $S_{\gamma_i}$. By assumption on $U$, we have that

$$(6.27) \quad \|z_n - z\|_{L^2(\mathcal{Y})} + \|\nabla z_n - \nabla z\|_{L^2(\mathcal{Y} ; \mathbb{R}^N)} \leq o(\sigma);$$

moreover, besides the possible jumps previously individuated, $\tilde{z}$ has in $R_i$ polyhedral jumps which are a reflected version of the polyhedral jumps of $z$ in $Q_i$. By assumption on $z$, we conclude that the union of these polyhedral sets $P(S_i)$ has $\mathcal{H}^{N-1}$-measure which is of the order of $\sigma$ that is $\mathcal{H}^{N-1}(P(S_i)) \leq o(\sigma)$ where $P(S_i) := \bigcup_{i=1}^{k} P_i(S_i)$.
Let \( \tilde{w}_n \) be optimal for the Ambrosio-Tortorelli approximation of \([S_z \setminus (\bigcup Q_i)] \cup P(S_z)\) (as we can find for example in [14] Lemma 3.3), that is \( \tilde{w}_n \) is null in a neighborhood of \([S_z \setminus (\bigcup Q_i)] \cup P(S_z)\) and

\[
\limsup_n MM_n(\tilde{w}_n) \leq \mathcal{H}^{N-1}(S_z \setminus (\bigcup Q_i)) \leq \\
\mathcal{H}^{N-1}(S_z \setminus S_{u(t)}) + o(\sigma).
\]

As in [14], let \( \tilde{\varphi}_n \) be a cut-off function associated to \( \tilde{w}_n \), such that

\[
\lim_n \eta_n \int_{\Omega} |\nabla \tilde{\varphi}_n|^2 \, dx = 0.
\]

Let us pose for all \( Q_i \subseteq \Omega \)

\[
w^i_n := \begin{cases} 
\min\{\tilde{w}_n, w^{2,i}_n, w^{3,i}_n, w^{4,i}_n\} & \text{in } \tilde{R}_i \\
\min\{\tilde{w}_n, w^{3,i}_n, w^{4,i}_n\} & \text{in } R_i \setminus \tilde{R}_i \\
\min\{\tilde{w}_n, w^{4,i}_n\} & \text{outside } R_i,
\end{cases}
\]

and

\[
\varphi^i_n := \begin{cases} 
\min\{\tilde{\varphi}_n, \varphi^{2,i}_n, \varphi^{3,i}_n, \varphi^{4,i}_n\} & \text{in } \tilde{R}_i \\
\min\{\tilde{\varphi}_n, \varphi^{3,i}_n, \varphi^{4,i}_n\} & \text{in } R_i \setminus \tilde{R}_i \\
\min\{\tilde{\varphi}_n, \varphi^{4,i}_n\} & \text{outside } R_i.
\end{cases}
\]

For all \( Q_i \) such that \( Q_i \cap \partial_D \Omega \neq \emptyset \) with \( Q_i^+ \setminus R_i \subseteq \Omega \), let us pose

\[
w^i_n := \begin{cases} 
\min\{\tilde{w}_n, w^{2,i}_n, w^{3,i}_n, b^{i,+}_n\} & \text{in } \tilde{R}_i \cap E^+_n \\
\min\{w^{2,i}_n, b^{i,+}_n\} & \text{in } (\tilde{R}_i \setminus E^+_n) \cup Q_i^- \\
\min\{\tilde{w}_n, w^{3,i}_n, b^{i,+}_n\} & \text{in } R_i \setminus (\tilde{R}_i \cup Q_i^-) \\
1 & \text{in } \Omega_D \\
\min\{\tilde{w}_n, b^{i,+}_n\} & \text{otherwise}
\end{cases}
\]

and

\[
\varphi^i_n := \begin{cases} 
\min\{\tilde{\varphi}_n, \varphi^{2,i}_n, \varphi^{3,i}_n, \varphi^{4,i}_n, \varphi^{b,i}_n\} & \text{in } \tilde{R}_i \cap E^+_n \\
\min\{\varphi^{2,i}_n, \varphi^{b,i}_n\} & \text{in } (\tilde{R}_i \setminus E^+_n) \cup Q_i^- \\
\min\{\tilde{\varphi}_n, \varphi^{3,i}_n, \varphi^{b,i}_n\} & \text{in } R_i \setminus (\tilde{R}_i \cup Q_i^-) \\
1 & \text{in } \Omega_D \\
\min\{\tilde{\varphi}_n, \varphi^{b,i}_n\} & \text{otherwise}
\end{cases}
\]

Similarly we reason for the case \( Q_i^- \setminus R_i \subseteq \Omega \). By construction, for all \( i = 1, \ldots, k \) we have that \( w^i_n, \varphi^i_n \in H^1(\Omega) \), \( 0 \leq w^i_n, \varphi^i_n \leq 1 \) and \( w^i_n, \varphi^i_n = 1 \) on \( \Omega_D \) for \( n \) large.

Note that by Lemmas 6.3, 6.4, 6.5, 6.6 and 6.7 and by (6.28) and (6.29), we have that

\[
\limsup_n \sum_{i=1}^k MM_n(w^i_n)_{\{\varphi_n(t) > b^i_n\}} \leq \mathcal{H}^{N-1}(S_z \setminus S_{u(t)}) + o(\sigma),
\]
and

\[ \lim_{n} \frac{1}{\eta_n} \sum_{i=1}^{k} \int_{\Omega \cap \{v_n(t) > b_k^i\}} |\nabla \varphi_n^i(x)|^2 \, dx = 0. \tag{6.31} \]

We are now in a position to conclude the proof. We pose

\[ v_n := \min\{w_n, w_n^i, i = 1, \ldots, k\}, \quad \varphi_n := \min\{\varphi_n, \varphi_n^i, i = 1, \ldots, k\}. \]

Note that \( \varphi_n = 0 \) in a neighborhood of \( S_{\Xi_n} \), and \( \varphi_n = 1 \) on \( \Omega_D \) for \( n \) large. Moreover \( 0 \leq v_n \leq w_n \leq v_n(t) \) in \( \Omega' \) and \( v_n = 1 \) on \( \Omega_D \). Let \( z_n := \varphi_n \tilde{z}_n \); we have \( z_n \in H^1(\Omega') \) with \( z_n = g_n(t) \) on \( \Omega_D \). By (6.27), we have that

\[ F_{z_n}(u_n(t), v_n(t)) \leq F_{z_n}(z_n, v_n), \]

and so

\[ \int_{\Omega} (\eta_n + v_n(t)^2) |\nabla u_n(t)|^2 \, dx \leq \int_{\Omega} (\eta_n + v_n^2) |\nabla (\varphi_n \tilde{z}_n)|^2 \, dx + M_{n}(v_n) - M_{n}(v_n(t)). \]

We may write

\[
\begin{align*}
\int_{\Omega'} (\eta_n + v_n(t)^2) |\nabla u_n(t)|^2 \, dx & \leq \int_{\Omega'} (\eta_n + v_n^2) |\nabla \tilde{z}_n|^2 \, dx + \int_{\Omega'} (\eta_n + v_n^2)(2|\nabla \varphi_n \nabla \tilde{z}_n + \tilde{z}_n| |\nabla \varphi_n|^2) \, dx + \\
& + M_{n}(w_n) - M_{n}(v_n(t)) + \sum_{i=1}^{k} M_{n}(w_n^i) \big|_{\{v_n(t) > b_k^i\}}.
\end{align*}
\]

Taking into account (6.27), (6.12), (6.31), (6.11), and (6.30), we have that passing to the limit

\[
\int_{\Omega} |\nabla u|^2 \, dx \leq \int_{\Omega} |\nabla z|^2 \, dx + H^{N-1}(S_z \setminus S_{u(t)}) + \\
+ \frac{2Ck}{(k-1)^2} + \frac{C}{(k-1)(1-b)^2} + \frac{Cb}{(1-b)^2} + o(\sigma),
\]

so that, letting \( \sigma \to 0 \) and then \( b \to 0 \), \( k \to \infty \) (which is permitted choosing appropriately \( j_2 \) and \( j_3 \)), we obtain the thesis. \( \square \)

We can now pass to the proof of Theorem 5.6. Given \( 0 = t_1 \leq t_2 \leq \ldots \leq t_k = t \), it is sufficient to prove that

\[ \int_{\Omega'} |\nabla u(t)|^2 \, dx \leq \int_{\Omega'} |\nabla z|^2 \, dx + H^{N-1} \left( S_z \setminus \left( \bigcup_{i=1}^{k} S_{u(t_i)} \right) \right). \tag{6.32} \]

Passing to the sup on \( t_1, \ldots, t_k \), we deduce in fact the thesis. We obtain (6.32) using the same arguments of Lemma 6.1, defining

\[ J_j := \left\{ x \in \bigcup_{m=1}^{k} \left( \bigcup_{a_1, a_2 \in A^k} [\partial^* E^k_{a_1} \cap \partial^* E^k_{a_2}] \right) : \min_{l=1, \ldots, k} \left[ u_l(x) \right] > \frac{1}{j} \right\}, \]

where \( E^k_{a} \) and \( A_k \) are defined as the corresponding sets for \( u(t) \), following (13), we cover \( J_j \) in such a way that for all \( x_i \in J_j \) there exists \( l \) with \( x_i \in S_{u(t_l)} \) and

\[ H^{N-1} \left( \bigcup_{r=1}^{k} \left( S_{u(t_r)} \setminus S_{u(t_l)} \right) \right) \cap Q_i \leq \sigma \varepsilon_i^{N-1}. \]
So in each $Q_i$ there exists $u(t_l)$ such that $igcup_{l=1}^k S_{u(t_l)} \cap Q_i$ is essentially (with respect to the measure $\mathcal{H}^{N-1}$) $S_{u(t_l)} \cap Q_i$. Recalling that $v_n(t_l) \leq v_n(t_l)$ for all $l = 1, \ldots, k$, we have

$$\int_{\Omega'} (\eta(t_l) + v_n(t_l)) |\nabla u_n(t_l)|^2 \, dx \leq \int_{\Omega'} (\eta(t_l) + v_n(t_l)) |\nabla u_n(t_l)|^2 \, dx \leq C,$$

and so it is readily seen that the arguments of Lemma 6.1 can be adapted to prove (6.32).

7. A FINAL REMARK

The previous results can be extended to recover the case of non-isotropic surface energies, i.e., energies of the form

$$(7.1) \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Gamma} \varphi(\nu_x) \, d\mathcal{H}^{N-1}(x)$$

where $\nu_x$ is the normal to $\Gamma$ at $x$, and $\varphi$ is a norm on $\mathbb{R}^N$. In fact all the previous arguments are based on Theorem 2.3 concerning the elliptic approximation and on Theorem 2.4 about the density of piecewise smooth functions with respect to the elastic energy. An elliptic approximation of Ambrosio-Tortorelli type of (7.1) has been proved in [14], while a density result of piecewise smooth functions with respect to non-isotropic surface energies has been proved in [10]. We conclude that all the previous theorems can be modified in order to treat the more general energy (7.1).

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REFERENCES

[1] Ambrosio L.: A compactness theorem for a new class of functions of bounded variations. Boll. Un. Mat. Ital. 3-B (1989), 857-881.
[2] Ambrosio L.: Existence theory for a new class of variational problems. Arch. Ration. Mech. Anal. 111 (1990) 291-322.
[3] Ambrosio L.: A new proof of the SBV compactness theorem. Calc. Var. 3 (1995) 127-137.
[4] Ambrosio L., Fusco N., Pallara D.: Functions of bounded variations and Free Discontinuity Problems. Clarendon Press, Oxford, 2000.
[5] Ambrosio L., Tortorelli V.M.: Approximation of functionals depending on jumps by elliptic functionals via $\Gamma$-convergence. Comm. Pure Appl. Math. 43 (1990), 999-1036.
[6] Dal Maso G.: An introduction to $\Gamma$-convergence. Birkhäuser, 1993.
[7] Dal Maso G., Toader R.: A density result in $SBV$ with respect to non-isotropic energies. Nonlinear Anal. 38 (1999) 585-604.
[8] Dal Maso G., Toader R.: A model for the quasi-static growth of brittle fractures: existence and approximation results. Arch. Ration. Mech. Anal. 162 (2002), 101-135.
[9] Francfort G.A., Larsen C.J.: Existence and convergence for quasi-static evolution in brittle fracture. To appear.
[10] Francfort G.A., Marigo J.J.: Numerical experiments in revisited brittle fracture. J. Mech. Phys. Solids 48-4, 797-826, 2000.
[11] Brezis, H.: Opérateurs maximaux monotones et semi-groupe de contractions dans les espaces de Hilbert. North-Holland, Amsterdam, 1973.
[12] Cortesani G.: Strong approximation of $GSBV$ functions by piecewise smooth functions. Ann. Univ. Ferrara - Sez VII - Sc. Mat. 43 (1997) 27-49.
[13] Cortesani G., Toader R.: A density result in $SBV$ with respect to non-isotropic energies. Nonlinear Anal. 38 (1999) 585-604.
[17] Modica L., Mortola S.: Un esempio di Γ-convergenza. *Boll. Un. Mat. Ital.* **14-B** (1977), 285-299.
[18] Yosida K.: *Functional Analysis.* Springer, 1965.

(Alessandro Giacomini) S.I.S.S.A., Via Beirut 2-4, 34014, Trieste, Italy

*E-mail address,* A. Giacomini: giacomin@sissa.it