STABILITY CONDITIONS ON THREEFOLDS WITH NEF TANGENT BUNDLES

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Abstract. In this paper, we prove the Bogomolov-Gieseker type inequality conjecture for threefolds with nef tangent bundles. As a corollary, there exist Bridgeland stability conditions on these threefolds.

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1. Introduction

1.1. Motivation and results. The construction of Bridgeland stability conditions on an algebraic variety \(X\) is an important problem. When \(X\) is a surface, the existence of Bridgeland stability conditions on \(X\) is proved by Bridgeland (cf. [11]) and Arcara-Bertram (cf. [1]). It has found many applications to classical problems in algebraic geometry, especially in the study of birational geometry of the moduli spaces of Gieseker stable sheaves (see e.g. [2, 4, 5, 9, 13, 14, 15, 16, 22, 23]).

When \(X\) is a threefold, the existence of Bridgeland stability conditions is an open problem in general. In the paper [7], Bayer, Macrì, and Toda reduced the problem to the so-called Bogomolov-Gieseker (BG) type inequality conjecture. The BG type inequality conjecture is known to be true for Abelian threefolds (cf. [6, 24, 25]), Fano threefolds of Picard rank one (cf. [20]), some toric threefolds (cf. [8]), product threefolds of projective spaces and Abelian varieties (cf. [19]), and quintic threefolds (cf. [21]). However, counter-examples of the original BG type inequality conjecture are constructed (see e.g. [28]). The failure of the conjecture is related to the existence of a kind of negative effective divisors on a threefold ([28], see Lemma 2.10). The modification of the conjecture is discussed in the paper [8], and they prove that the modified version of the BG type inequality holds when \(X\) is a Fano threefold of arbitrary Picard rank.

On the other hand, we can still expect that the original BG type inequality conjecture will be true if every effective divisor on \(X\) satisfies a certain positivity condition, e.g. if the pseudo-effective cone agrees with the nef cone. Actually, in this paper, we prove that the original conjecture is true for one class of threefolds satisfying this property, namely those with nef tangent bundles:

**Theorem 1.1.** Let \(X\) be a smooth projective threefold with nef tangent bundle. Then the original BG type inequality conjecture holds for \(X\).
In particular, the above theorem implies the existence of Bridgeland stability conditions on these threefolds:

**Theorem 1.2.** Let $X$ be as in Theorem 1.1. Then there exist Bridgeland stability conditions on $X$.

See Theorem 2.17, Corollary 2.18 and Theorem 2.19 for the precise statements.

### 1.2. Relation to existing works.

First recall that threefolds with nef tangent bundles are classified by F. Campana and T. Peternell.

**Theorem 1.3.** Let $X$ be a smooth projective threefold with nef tangent bundle. Then up to taking finite étale coverings, $X$ is one of the following:

1. $\mathbb{P}^3$.
2. A three dimensional smooth quadric.
3. $\mathbb{P}^1 \times \mathbb{P}^2$.
4. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
5. $\mathbb{P}(\mathbb{T}_{\mathbb{P}^2})$.
6. $\mathbb{P}(A(E))$, where $A$ is an Abelian surface and $E$ is a rank two vector bundle obtained as an extension of two line bundles in $\text{Pic}^0(A)$.
7. $\mathbb{P}(C(E))$, where $C$ is an elliptic curve and $E$ is a rank three vector bundle obtained as extensions of three line bundles of degree zero.
8. $\mathbb{P}(C(E_1)) \times \mathbb{C}P(C(E_2))$, where $C$ is an elliptic curve and $E_i$ are rank two vector bundles obtained as extensions of degree zero line bundles.
9. An Abelian threefold.

Among the above threefolds, the existence of Bridgeland stability conditions is known in the following cases:

- $\mathbb{P}^3$ by [7, 27].
- A three dimensional smooth quadric by [33].
- (3) – (5) in Theorem 1.3 by [8].
- An Abelian threefold by [6, 24, 25].

In this paper, we treat the remaining cases, i.e. (6) – (8) in Theorem 1.3. Note that $\mathbb{P}^1 \times A, \mathbb{P}^2 \times C$, and $\mathbb{P}^1 \times \mathbb{P}^1 \times C$ are treated in the author’s previous paper [19], which are the special cases of (6) – (8) in Theorem 1.3.

Furthermore, on $\mathbb{P}(\mathbb{T}_{\mathbb{P}^2})$, we will construct new Bridgeland stability conditions which were not obtained in [8].

### 1.3. Outline of the proof.

As mentioned in the last subsection, we treat the cases (6) – (8) in Theorem 1.3 in the first part of this paper. Recall that, if the bundle is a trivial bundle, then the BG type inequality conjecture is known to be true by the author’s previous paper [19]. In our situation, using this technique, we can reduce to the cases of the projectivizations of split vector bundles (see Proposition 3.3). Then for split cases, we can argue as in [19] using good finite morphisms.

In the second part, we will treat the case when $X = \mathbb{P}(\mathbb{T}_{\mathbb{P}^2})$. In [8], they used the fact that $\mathbb{P}(\mathbb{T}_{\mathbb{P}^2})$ is a Fano variety to construct Bridgeland stability conditions. On the other hand, in this paper, we regard it as a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ and use a full exceptional collection on the derived category.
1.4. Open problems.

(1) As we will see in Conjecture 2.7, the conjectural BG type inequality depends on a class $B + i\omega \in \text{NS}(X)_C$ with $\omega$ ample. For threefolds in Theorem 1.3 except for (5), we can prove the inequality for any choice of a class $B + i\omega \in \text{NS}(X)_C$ with $\omega$ ample.

On the other hand, for $\mathbb{P}(\mathcal{T}_{P_2})$, we can prove it only when $B$ and $\omega$ are proportional so far. We can hope the inequality also holds for any choice of $B + i\omega \in \text{NS}(\mathbb{P}(\mathcal{T}_{P_2}))_C$. At this moment, the author doesn’t know how to solve this problem.

(2) It is expected that the space of Bridgeland stability conditions has complex dimension equal to the rank of the algebraic cohomology (In fact, it is true in the surface case by the works [1, 11, 36]). As proven in the paper [6], the BG type inequality in Conjecture 2.7 implies the existence of a four dimensional subset in the space of Bridgeland stability conditions.

In [31, Theorem 3.21], the full dimensional family of Bridgeland stability conditions on Abelian threefolds was constructed. Proving the same statement for threefolds treated in this paper is an interesting open problem, which requires the stronger BG type inequality.

1.5. Plan of the paper. In Section 2, we recall about the theory of Bridgeland stability conditions and about threefolds with nef tangent bundles. In Section 3 we treat varieties in Theorem 1.3 (6) – (8). In Section 4, we will discuss about the stability conditions on $\mathbb{P}(\mathcal{T}_{P_2})$.

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Notation and Convention. In this paper we always work over $\mathbb{C}$. We use the following notations:

- $\text{ch}^B = (\text{ch}_0^B, \ldots , \text{ch}_n^B) := e^{-B}. \text{ch}$, where $\text{ch}$ denotes the Chern character and $B \in \text{NS}(X)_\mathbb{R}$.
- $v^B := \omega. \text{ch}^B := (\omega^n, \text{ch}_0^B, \ldots , \omega. \text{ch}_n^B - 1, \text{ch}_n^B)$, where $B, \omega \in \text{NS}(X)_\mathbb{R}$.
- $K(A)$ : the Grothendieck group of an abelian category $A$.
- $\text{hom}(E, F) := \dim \text{Hom}(E, F)$.
- $\text{ext}^i(E, F) := \dim \text{Ext}^i(E, F)$.
- $D^b(X) := D^b(\text{Coh}(X))$ : the bounded derived category of coherent sheaves on a smooth projective variety $X$.

2. Preliminaries

2.1. Bridgeland stability condition. In this subsection, we recall the notion of Bridgeland stability conditions on a triangulated category. The reference for this subsection is Bridgeland’s original paper [10]. First, we define the notion of stability functions:

**Definition 2.1.** Let $A$ be an Abelian category.
A stability function on $\mathcal{A}$ is a group homomorphism $Z: K(\mathcal{A}) \to \mathbb{C}$ satisfying the condition

$$Z(\mathcal{A} \setminus \{0\}) \subset \mathcal{H} \cup \mathbb{R}_{<0},$$

where $\mathcal{H}$ is the upper half plane.

(2) Let $Z$ be a stability function on $\mathcal{A}$. An object $E \in \mathcal{A}$ is called $Z$-stable (resp. semistable) if for every non zero proper subobject $0 \neq F \subset E$, we have an inequality

$$-\frac{\Re Z(F)}{\Im Z(F)} < (\text{resp.} \leq) -\frac{\Re Z(E)}{\Im Z(E)}.$$

Here, we define $-\frac{\Re Z(E)}{\Im Z(E)} := +\infty$ if $\Im Z(E) = 0$.

(3) A stability function $Z$ on $\mathcal{A}$ satisfies the Harder-Narasimhan (HN) property if the following holds: for every object $E \in \mathcal{A}$, there exists a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$$

such that $F_i := E_i/E_{i-1}$ are $Z$-semistable and

$$-\frac{\Re Z(F_1)}{\Im Z(F_1)} > \cdots > -\frac{\Re Z(F_m)}{\Im Z(F_m)}.$$

We now define the notion of stability conditions on a triangulated category:

**Definition 2.2.** Let $\mathcal{D}$ be a triangulated category. A stability condition on $\mathcal{D}$ is a pair consisting of the heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{D}$ and a stability function $Z$ on $\mathcal{A}$ satisfying the HN property: A stability function $Z$ is called a central charge.

### 2.2. Bogomolov-Gieseker type inequality conjecture

In this subsection, we recall the conjectural approach for the construction of stability conditions on threefolds. Let $X$ be a smooth projective threefold. Fix a class $B + i\omega \in NS(X)_{\mathbb{Q}}$ with $\omega$ ample. Conjecturally, there exists a stability condition on $D^b(X)$ with its central charge given as follows (cf. [7, Conjecture 2.1.2]):

$$Z_{\omega,B} := \int_X e^{-i\omega \cdot \text{ch}^B}.$$

It is easy to see that the pair $(Z_{\omega,B}, \text{Coh}(X))$ does not define a stability condition when $X$ is a threefold. Hence we need to introduce new hearts. Our hearts are obtained by the double-tilting construction [7] which we explain below, see the paper [17] for the general theory of torsion pairs and tilting. In the following, we assume that $B \in NS(X)_{\mathbb{Q}}$ and $\omega = mH$ for some ample divisor $H$ and $m \in \mathbb{R}_{>0}$ with $m^2 \in \mathbb{Q}$. As in the introduction, we use the following notation:

$$v^B = (v_0^B, v_1^B, v_2^B, v_3^B) := (\omega^3, \text{ch}_0^B, \omega^2, \text{ch}_1^B, \omega, \text{ch}_2^B, \omega, \text{ch}_3^B).$$

**First tilting:** We define the slope function on $\text{Coh}(X)$ as follows:

$$\mu_{\omega,B} := \frac{v_0^B}{v_0^3} : \text{Coh}(X) \to (-\infty, +\infty).$$

Then define the full subcategories $\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B} \subset \text{Coh}(X)$ as follows:

$$\mathcal{T}_{\omega,B} := \{ T \in \text{Coh}(X) : T \text{ is } \mu_{\omega,B}\text{-semistable with } \mu_{\omega,B}(T) > 0 \},$$

$$\mathcal{F}_{\omega,B} := \{ F \in \text{Coh}(X) : F \text{ is } \mu_{\omega,B}\text{-semistable with } \mu_{\omega,B}(F) \leq 0 \}.$$

Here, $\mu_{\omega,B}$-stability for coherent sheaves is defined in a standard manner, and we denote by $\langle S \rangle$ the extension closure of a set of objects $S \subset \text{Coh}(X)$. Now we define a new heart, called tilted heart by

$$\text{Coh}^{\omega,B}(X) := \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle.$$
Second tilting: As in the first tilting, we introduce a new slope function and tilting of \( \text{Coh}^{\omega,B}(X) \): A slope function \( \nu_{\omega,B} \) on \( \text{Coh}^{\omega,B}(X) \) is defined to be

\[
\nu_{\omega,B} : = \frac{v_B^1 - \frac{1}{6} v_B^0}{v_B^1} : \text{Coh}^{\omega,B}(X) \to (-\infty, +\infty],
\]

and the notion of \( \nu_{\omega,B} \)-stability for objects in \( \text{Coh}^{\omega,B}(X) \) is defined similarly as \( \mu_{\omega,B} \)-stability for coherent sheaves. We also refer to \( \nu_{\omega,B} \)-stability as tilt stability. Note that the existence of Harder-Narasimhan filtration with respect to \( \nu_{\omega,B} \)-stability is shown in the paper [7]. We define full subcategories of \( \text{Coh}^{\omega,B}(X) \) as

\[
T'_{\omega,B} : = \{ T \in \text{Coh}^{\omega,B}(X) : T \text{ is } \nu_{\omega,B} \text{-semistable with } \nu_{\omega,B}(T) > 0 \},
\]

\[
F'_{\omega,B} : = \{ F \in \text{Coh}^{\omega,B}(X) : F \text{ is } \nu_{\omega,B} \text{-semistable with } \nu_{\omega,B}(F) \leq 0 \}.
\]

Now we reach the definition of the double-tilted heart:

\[
A_{\omega,B} : = \langle F'_{\omega,B}[, T'_{\omega,B} \rangle.
\]

In the paper [7], Bayer, Macrì, and Toda conjectured the following:

**Conjecture 2.3 ([7]).** The pair \((Z_{\omega,B}, A_{\omega,B})\) is a stability condition on \( D^b(X) \).

Let us denote

\[
\Delta_{\omega,B}(E) : = v_B^1(E)^2 - 2v_B^0(E)v_B^2(E)
\]

and

\[
\nabla_{\omega,B}(E) : = 2(v_B^2(E))^2 - 3v_B^1(E)v_B^3(E).
\]

The following is the so-called Bogomolov-Gieseker (BG) type inequality conjecture ([7, 6, 32]).

**Conjecture 2.4 ([32, Conjecture 3.8]).** For any \( \nu_{\omega,B} \)-stable object \( E \), we have the inequality

\[
\Delta_{\omega,B}(E) + 6\nabla_{\omega,B}(E) \geq 0.
\]

The BG type inequality conjecture implies the existence of a stability condition:

**Proposition 2.5 ([32]).** Assume that Conjecture 2.4 holds. Then Conjecture 2.3 also holds.

2.3. Reduction theorems. In this subsection, we recall two reduction theorems of the BG type inequality conjecture due to [6, 21, 32].

First we recall the following notion.

**Definition 2.6.** Fix real numbers \( \alpha_0 > 0 \) and \( \beta_0 \). Let \( E \in \text{Coh}^{\alpha_0 \omega,B + \beta_0 \omega}(X) \) be a \( \nu_{\alpha_0 \omega,B + \beta_0 \omega} \)-semistable object.

1. We define a real number \( \bar{\beta}(E) \) as

\[
\bar{\beta}(E) : = \frac{2v_B^2(E)}{v_B^1(E) + \sqrt{\Delta_{\omega,B}(E)}}.
\]

2. \( E \) is \( \bar{\beta} \)-semistable (resp. stable) if there exists an open neighborhood \( V \) of \((0, \bar{\beta}(E))\) in the \((\alpha, \beta)\)-plane such that for every \((\alpha, \beta) \in V \) with \( \alpha > 0 \), \( E \) is \( \nu_{\alpha \omega,B + \beta \omega} \)-semistable (resp. stable).

The first reduction is of the following form.

**Conjecture 2.7 ([32, Conjecture 3.17]).** Let \( E \) be a \( \bar{\beta} \)-stable object. Then we have

\[
\text{ch}_3^{B+\bar{\beta}(E)}(E) \leq 0.
\]

**Theorem 2.8 ([32, Theorem 3.20]).** Conjectures 2.4 and 2.7 are equivalent.
Using the same technique, the following result was proved in [21].

**Theorem 2.9** ([21 Theorem 3.2]). Let $H$ be an ample divisor on $X$. Assume that there exists a real number $\alpha_{0} > 0$ such that for every real number $0 < \alpha < \alpha_{0}$, Conjecture 2.3 is true for $(X, \alpha H, B = 0)$. Then it also holds for $(X, \alpha H, \beta H)$ with any choice of $\alpha \geq \frac{1}{2\sqrt{3}}$ and $\beta \in \mathbb{R}$.

2.4. Counter-examples. Counter-examples to Conjecture 2.3 are constructed in the papers [19, 28, 34]. In particular, we have the following result:

**Lemma 2.10** ([28 Lemma 3.1]). Let $H$ be an ample divisor. Assume that there exists an effective divisor $D$ such that

$$D^3 > \frac{(H^2.D)^3}{4(H^3)^2} + \frac{3(H.D^2)^2}{4 H^2.D}.$$  

Then there exists a pair $(\alpha, \beta)$ of real numbers such that the pair $(Z_{\alpha H, \beta H}, A_{\alpha H, \beta H})$ does not define a stability condition.

**Remark 2.11.** Let $D$ be a nef divisor. We claim that $D$ does not satisfy the inequality (2.1). By the Hodge index theorem for nef divisors, we have the following inequalities:

$$H^2.D^3 \geq (H^3)^2 \cdot D^3$$  

(2.2)  

$$H.D^2)^3 \geq H^3 \cdot (D^3)^2.$$  

(2.3)  

On the other hand, by replacing $H$ with its sufficiently large multiple and taking a smooth member, the Hodge index theorem on $H$ leads the inequality

$$(H^2.D)^2 = (H.H.D)^2 \geq (H.H)^2 \cdot (D.H)^2 = H^3 \cdot H.D^2.$$  

(2.4)  

The inequality (2.2) is equivalent to the inequality

$$D^3 \leq \frac{(H^2.D)^3}{(H^3)^2}.$$  

(2.5)  

Furthermore, by the inequalities (2.2), (2.3), and (2.5), we have

$$\frac{(H.D^2)^2}{H^2.D} \geq \frac{H^3 \cdot (D^3)^2}{H^2.D \cdot H.D^2} \quad \text{(by 2.3)}$$  

(2.6)  

$$\geq \frac{(H.D^2)^2}{H.D^2 \cdot H.D^2} \quad \text{(by 2.5)}$$  

$$\geq D^3 \quad \text{(by 2.4)}.$$  

By combining the inequalities (2.5) and (2.6), we conclude that $D$ satisfies the opposite inequality to that in (2.1). Hence we can think the inequality (2.1) as a kind of negativity conditions on an effective divisor. We can still expect that Conjecture 2.3 and Conjecture 2.4 are true if all effective divisors satisfy some positivity conditions.

2.5. Threefolds with nef tangent bundles. In this subsection, we recall results on threefolds with nef tangent bundles, which we will need in this paper.

**Proposition 2.12** ([12 Proposition 2.12]). Let $X$ be a smooth projective variety with nef tangent bundle. Then every effective divisor on $X$ is nef.

The above proposition, together with Remark 2.11 shows that there does not exist an effective divisor on a threefold with nef tangent bundle satisfying the inequality (2.1) in Lemma 2.10. Furthermore, the above proposition also ensures the tilt-stability of line bundles.
Lemma 2.13 ([6, Corollary 3.11]). Let $X$ be a smooth projective threefold, $\omega$ an ample $\mathbb{R}$-divisor on $X$. Assume that for every effective divisor $D$ on $X$, we have $\omega.D^2 \geq 0$. Then for every line bundle $L$ on $X$ and $B \in \text{NS}(X)_{\mathbb{R}}$, $L$ or $L[1]$ is $\nu_{\omega,B}$-stable.

Next we recall the classification theorem of threefolds with nef tangent bundles due to the paper [12].

Theorem 2.14 ([12, Theorem 10.1]). Let $X$ be a smooth projective threefold with nef tangent bundle. Then there exists an étale covering $f: \tilde{X} \to X$ such that $\tilde{X}$ is one of the following:

1. $\mathbb{P}^3$.
2. a three dimensional smooth quadric.
3. $\mathbb{P}^1 \times \mathbb{P}^2$.
4. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
5. $\mathbb{P}(\mathcal{T}_X)$.
6. $\mathbb{P}(A)(E)$, where $A$ is an Abelian surface and $E$ is a rank two vector bundle obtained as an extension of two line bundles in $\text{Pic}^0(A)$.
7. $\mathbb{P}(C)(E)$, where $C$ is an elliptic curve and $E$ is a rank three vector bundle obtained as extensions of three line bundles of degree zero.
8. $\mathbb{P}(E_1) \times \mathbb{P}(E_2)$, where $C$ is an elliptic curve and $E_i$ are rank two vector bundles obtained as extensions of degree zero line bundles.
9. an Abelian threefold.

For our purpose, we need the following observation:

Lemma 2.15. In Theorem 2.14 we can choose an étale covering $f$ to be a Galois covering.

Proof. Let $X$ be a smooth projective threefold with nef tangent bundle. In the proof of [12, Theorem 10.1], they actually show the existence of the following diagram of smooth projective varieties:

\[
\begin{array}{ccc}
\tilde{X} := \tilde{Y} \times_X X & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\tilde{Y} & \xrightarrow{\psi} & Y' \xrightarrow{\phi} Y,
\end{array}
\]

where $Y'$ is an Abelian variety (possibly of dimension zero), $\psi$ and $\phi$ are étale coverings. Note that the morphism $\tilde{X} \to \tilde{Y}$ is same as (1) – (9) in Theorem 2.14, i.e., $\tilde{Y}$ is Spec $C$, $A$, $C$, or an Abelian threefold in the notation of Theorem 2.14.

Put $g := \phi \circ \psi$. Let us take the Galois closure of $g$, i.e. an étale covering $h: \tilde{Y} \to \tilde{Y}$ such that the morphism $h \circ g: \tilde{Y} \to Y$ is an étale Galois covering. Note that since $\tilde{Y}$ is an Abelian variety, so is $\tilde{Y}$. Hence the base change $\tilde{X} := \tilde{Y} \times_X X$ is again one of the threefolds in Theorem 2.14 (1) – (9), and is an étale Galois covering of $X$. This completes the proof. □

Remark 2.16. Among threefolds in Theorem 2.14, Conjecture 2.7 is known to be true in the following cases:

- $\mathbb{P}^3$ by [7, 27].
- a three dimensional smooth quadric by [33].
- $\mathbb{P}^1 \times \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with any choice of a class $B + i\omega$ by [8] (In the paper [8], they only treat the case when $B$ and $\omega$ are proportional. Even when they are not proportional, the same proof works according to the formulation given in Conjecture 2.7).
• \( \mathbb{P}(T_{\mathbb{P}^2}) \) with \( B \) and \( \omega \) are proportional to the anti-canonical class by [8].

• an Abelian threefold with any choice of a class \( B + i\omega \) by [8, 24, 25].

The following is our first main result, which completely solve Conjecture 2.7 for threefolds as in Theorem 2.14 (6) – (8):

**Theorem 2.17.** Let \( X \) be a threefold as in Theorem 2.14 (6), (7), or (8). Then for every class \( B + i\omega \in \text{NS}(X)_C \) with \( \omega \) ample, Conjecture 2.7 holds.

As a corollary, we obtain:

**Corollary 2.18.** Let \( X \) be a smooth projective threefold with nef tangent bundle.

**Proof.** By [6, Proposition 6.1], we may replace \( X \) by an étale Galois covering, thus we can assume it is one of the threefolds in Theorem 2.14 (see Lemma 2.15). Then Theorem 2.17, together with the previous works [6, 8, 24, 25, 27, 33], proves the required statement. □

We will also have the following result for \( X = \mathbb{P}(T_{\mathbb{P}^2}) \):

**Theorem 2.19.** Let \( X = \mathbb{P}(T_{\mathbb{P}^2}) \), \( H \) be an ample divisor on \( X \). Let \( \alpha > \frac{1}{2\sqrt{3}} \) and \( \beta \in \mathbb{R} \) be real numbers. Then Conjecture 2.4 holds for \( (X, \alpha H, \beta H) \).

3. **Proof of Theorem 2.17**

In this section, we prove Theorem 2.17. We use the following terminology.

**Definition 3.1.** Let \( X \) be as in Theorem 2.14 (6) – (8). Then \( X \) is split if the vector bundles defining \( X \) are direct sums of line bundles.

3.1. **Reduction to split cases.** In this subsection, we reduce Theorem 2.17 to the split cases. The key method is the following result announced by Bayer et al [3].

**Proposition 3.2.** Let \( f: X \to D \) be a smooth projective family of threefolds over a smooth curve \( D \) and fix a point \( 0 \in D \). Suppose that \( f \) is a trivial family over \( U := D \setminus \{0\} \), i.e. \( f^{-1}(U) \cong X \times U \) for some threefold \( X \). Take an \( f \)-ample \( \mathbb{Q} \)-divisor \( H \) and an arbitrary \( \mathbb{Q} \)-divisor \( B \) on \( X \). Let \( H_0, B_0 \) (resp. \( H, B \)) be restriction of \( H, B \) to the special fiber \( f^{-1}(0) \) (resp. the general fiber \( X \)). If Conjecture 2.7 is true for \( (f^{-1}(0), H_0, B_0) \), then it also holds for \( (X, H, B) \).

The above result follows from the existence of the relative moduli spaces of tilt-stable objects over the base \( D \), satisfying the valuative criterion for universal closedness.

**Proposition 3.3.** Assume that Theorem 2.17 holds for every split \( X \). Then it also holds for every non-split \( X \).

**Proof.** First we consider the case (6) in Theorem 2.14. Let \( A \) be an Abelian surface, \( E \) be a rank two vector bundle which fits into the non-split short exact sequence

\[
0 \to \mathcal{O}_A \to E \to 
\]

then it also holds for \( (X, H, B) \).
Indeed, the family is constructed as a $\mathbb{P}^1$-bundle $\sigma: \mathcal{X} = \mathbb{P}_{A \times \mathbb{A}^1}(\mathcal{U}) \to A \times \mathbb{A}^1$, where $\mathcal{U}$ fits into the exact sequence

$$0 \to \mathcal{U} \to q^*\mathcal{E} \to i_*\mathcal{E} \to 0.$$ 

Here, $q: A \times \mathbb{A}^1 \to A$ is a projection and $i: A \times \{0\} \to A \times \mathbb{A}^1$ is an inclusion. Let $p := q \circ \sigma: \mathcal{X} \to A$ be a projection. We also have to prove that, for a given ample divisor $H$ on $X$, there exists an $f$-ample divisor $\mathcal{H}$ on $X$ such that its restriction to $X$ coincides with $H$. Write $H = \mathcal{O}_X(a) \oplus p^*N$, where $\pi: X \to A$ is a structure morphism. We put $\mathcal{H} := \mathcal{O}_X(a) \oplus p^*N$. Then by the ampleness criterion given in Lemma 3.10 we can see that $\mathcal{H}$ is $f$-ample. Hence the result holds by Proposition 3.2.

Next let $C$ be an elliptic curve and $L_i$ be degree zero line bundles on $C$ ($i = 1, 2, 3$). Consider the case (7) in Theorem 2.14 i.e. $X = \mathbb{P}_C(\mathcal{E})$, where $\mathcal{E}$ is a rank three vector bundle obtained as follows:

$$0 \to L_1 \to \mathcal{E}' \to L_2 \to 0,$$

$$0 \to \mathcal{E}' \to \mathcal{E} \to L_3 \to 0.$$ 

As above, by considering a family over the affine line $\mathbb{A}^1 \subset \text{Ext}^1(L_1, \mathcal{E}')$ passing through the origin and a class $[\mathcal{E}]$, we may assume that $\mathcal{E} = \mathcal{E}' \oplus L_3$. Then by applying the same argument for $[\mathcal{E}'] \in \text{Ext}^1(L_2, L_1)$, we can reduce to the split case.

Finally, consider the case (8) in Theorem 2.14. For $i = 1, 2$, let $\pi_i: Y_i := \mathbb{P}_C(\mathcal{E}_i) \to C$, where $\mathcal{E}_i$ are rank two vector bundles fitting into the short exact sequences

$$0 \to \mathcal{O}_C \to \mathcal{E}_i \to L_i \to 0.$$ 

Let $X := Y_1 \times_C Y_2$. Noting that $X = \mathbb{P}_{Y_1}(\pi_1^*\mathcal{E}_2)$, we can first reduce to the case when $\mathcal{E}_2 = \mathcal{O}_C \oplus L_2$. Then by regarding as $X = \mathbb{P}_{Y_2}(\pi_2^*\mathcal{E}_1)$, we can reduce to the case when $X$ is split. 

3.2. Conclusion. In this subsection, we explain how to prove Theorem 2.14 in the split cases. We use the following notations:

- $A$ is an Abelian surface, $C$ is an elliptic curve.
- $L \in \text{Pic}^0(A)$ and $L_1, L_2 \in \text{Pic}^0(C)$.
- For $m \in \mathbb{Z}_{>0}$, $L^m$ is a line bundle such that $(L^\pm)^m \cong L$. $L_i^\pm \in \text{Pic}^0(C)$ are similarly defined.
- For $i = 1, 2, Y_i := \mathbb{P}_C(\mathcal{O}_C \oplus L_i)$.
- $X$ is $\mathbb{P}_A(\mathcal{O}_A \oplus L)$, $\mathbb{P}_C(\mathcal{O}_C \oplus L_1 \oplus L_2)$, or $Y_1 \times_C Y_2$.
- For $m \in \mathbb{Z}_{>0}$, $Y_i^{(m)} := \mathbb{P}_C(\mathcal{O}_C \oplus L_i^m)$, and $Y_i^{(\pm)} := \mathbb{P}_C(\mathcal{O}_C \oplus L_i^\pm)$. $X^{(m)}$, $X^{(\pm)}$ are defined similarly.

We start with the following easy lemma:

Lemma 3.4. Let $X$ be as in Theorem 2.14 (6) – (8) which is split, let $m \in \mathbb{Z}_{>0}$ be an positive integer. Then by identifying the tautological classes, we have a ring isomorphism

$$\Phi: H^{2*}(X^{(\pm)}), \mathbb{Q}) \to H^{2*}(X, \mathbb{Q})$$

between the even cohomology rings.

Proof. We only treat the case when $X = \mathbb{P}_A(\mathcal{O}_A \oplus L)$. Let $h \in H^2(X, \mathbb{Q})$ (resp. $h^\pm \in H^2(X^{(\pm)}, \mathbb{Q})$) be a divisor such that $\mathcal{O}_X(h) = \mathcal{O}_X(1)$ (resp. $\mathcal{O}_X(\pi^*(h^\pm)) = \mathcal{O}_X(\pi_1^*(h^\pm))$). Since $L \in \text{Pic}^0(A)$, we have ring isomorphisms

$$\Phi: H^{2*}(X^{(\pm)}), \mathbb{Q}) \cong H^{2*}(A, \mathbb{Q})[t]/(t^2) \cong H^{2*}(X, \mathbb{Q}).$$
Here, the isomorphism $H^2_\ast(A, \mathbb{Q})[t]/(t^2) \cong H^2_\ast(X, \mathbb{Q})$ sends $t$ to $[h]$ and the same is true for $X(\frac{1}{h})$. Hence $\Phi([h(\frac{1}{h})]) = [h]$. □

Next we construct finite morphisms which play important roles for our purpose.

**Proposition 3.5** (cf. [20] Proposition 5). Let $X$ be a threefold as in Theorem 2.14 which is split. Then, for every positive integer $m \in \mathbb{Z}_{>0}$, we have the following commutative diagram

\[
\begin{array}{ccc}
X(\frac{1}{m}) & \xrightarrow{g_m} & X \\
\downarrow \sigma(\frac{1}{m}) & & \downarrow \tau \\
X(m) & \xrightarrow{h_m} & X \\
\downarrow q(m) & & \downarrow q \\
Z & \xrightarrow{\varpi} & Z,
\end{array}
\]

where $Z$ is an Abelian surface $A$ or an elliptic curve $C$.

Furthermore, the pull-back via the morphism $F_m : X(\frac{1}{m}) \to X$ acts on the even cohomology as follows.

\[
\Phi \circ F_m^\ast : H^2_\ast(X, \mathbb{Q}) \ni (x, y, z, w) \mapsto (x, m^2y, m^4z, m^6w) \in H^2_\ast(X, \mathbb{Q}).
\]

**Proof.** First consider the case (6) in Theorem 2.14: $X := \mathbb{P}_A(\mathcal{O}_A \oplus L)$. Consider the multiplication map $\varpi : A \to A$. By [29] p. 71 (iii), we have $\varpi^*L \cong L^m$. Hence by base change, we have the morphism $h_m : X(m) \to X$. On the other hand, the natural inclusion

\[
O_A \oplus L^m \subset \text{Sym}^m(\mathcal{O}_A \oplus L^m) = O_A \oplus L^m \oplus \cdots \oplus (L^m)^m
\]

induces a morphism $g_m : X(\frac{1}{m}) \to X(m)$. Now we get a commutative diagram

\[
\begin{array}{ccc}
O_C \oplus L^m_1 \oplus L^m_2 \subset \text{Sym}^m(\mathcal{O}_C \oplus L_1^m \oplus L_2^m),
\end{array}
\]

we get the diagram as in (3.1). Locally over $A$, the morphism $g_m$ is nothing but the toric Frobenius morphism $\varpi_m^2 : \mathbb{P}^1 \to \mathbb{P}^1$. Hence the pull back $F_m^\ast$ acts on the cohomology as stated.

Next consider the case (7) in Theorem 2.14 i.e., $X = \mathbb{P}_C(\mathcal{O}_C \oplus L_1 \oplus L_2)$. Replacing (3.3) by the inclusion

\[
O_C \oplus L^m_1 \oplus L^m_2 \subset \text{Sym}^m(\mathcal{O}_C \oplus L_1^m \oplus L_2^m),
\]

we get the diagram as in (3.1).

Finally, consider the case (8) in Theorem 2.14 $X = Y_1 \times_C Y_2$. As above, we can construct the morphisms $Y_1(\frac{1}{m}) \to Y_1(m)$, which induce the morphism $g_m : X(\frac{1}{m}) \to X(m)$. Hence we get the diagram as in (3.1) □

**Remark 3.6.** By using the inclusion

\[
O_A \oplus L^m \subset \text{Sym}^m(O_A \oplus L)
\]

instead of (3.3), we get an endomorphism $F_m^\ast : X \to X$ which is the multiplication map $\varpi : A \to A$ on the base, and the toric Frobenius morphism $\varpi^2_1 : \mathbb{P}^1 \to \mathbb{P}^1$ on the fiber. It seems natural to use the endomorphism $F_m^\ast$ rather than $F_m$. The issue is that the endomorphism $F_m^\ast$ is not polarized. On the other hand, the morphism $F_m$ behaves like a polarized endomorphism, although it is not an endomorphism (see the formula (3.2)).

According to the description (3.2) of the pull back $F_m^\ast$, we can prove the following two results:

**Proposition 3.7** (H). Let $X$ be as in Theorem 2.14 (6), (7), or (8) which is split, let $F_m$ be the morphism constructed in Proposition 3.5. Let $E \in D^b(X)$ be a two term complex concentrated in degree $-1$ and $0$. 

(1) If there exists an ample divisor $H$ on $X^{(\frac{1}{m})}$ such that
\[ \text{hom}(O(H), F_m^*E) = 0, \]
then we have
\[ \text{hom}(O, F_m^*E) = O(m^4). \]

(2) If there exists an ample divisor $H$ on $X^{(\frac{1}{m})}$ such that
\[ \text{ext}^2(O(-H), F_m^*E) = 0, \]
then
\[ \text{ext}^2(O, F_m^*E) = O(m^4). \]

Proof. Since we know that the pull back $F_m^*$ acts on the cohomology as in (3.2), the arguments of Section 7 in [6] prove the result. \qed

Lemma 3.8. Let $X$ be as in Theorem 2.14 (6) - (8) which is split, $m,q \in \mathbb{Z}_{>0}$ be positive integers. Take a divisor $D$ on $X$ and let $D^{\frac{1}{m}}$ be a divisor on $X^{(\frac{1}{m})}$ such that $D^{\frac{1}{m}} = \Phi^{-1}(D)$ in the cohomology ring. Then for every object $E \in D^b(X)$, we have the equality
\[ \text{ch}_j \left( F_m^*E \otimes O(-m^2qD^{\frac{1}{m}}) \right) = m^6q^6 \text{ch}_3^D(E) \in \mathbb{Q} \]
as rational numbers.

Proof. Note that $\Phi(D^{\frac{1}{m}}) = D$ by definition. Hence by the formula (3.2), we have
\[
\begin{align*}
\text{ch}_3 \left( F_m^*E \otimes O(-m^2qD^{\frac{1}{m}}) \right) \\
= \Phi \left( \text{ch}_3 \left( F_m^*E \otimes O(-m^2qD^{\frac{1}{m}}) \right) \right) \\
= -\frac{1}{6}m^6q^3D^3 \text{ch}_0(E) + \frac{1}{2} \left( m^4q^2D^2 \left( m^2q^2 \text{ch}_1(E) \right) - (m^2qD) \left( m^4q^4 \text{ch}_2(E) \right) \right) \\
+ m^6q^6 \text{ch}_3(E) \\
= m^6q^6 \text{ch}_3^D(E)
\end{align*}
\]
as required. \qed

Next we prove a variant of the toric Frobenius splitting of line bundles.

Proposition 3.9 (cf. [35]). Let $X$ and $g_m$ be as in Proposition 3.5. Let $M$ be a line bundle on $X^{(\frac{1}{m})}$. Then the vector bundle $g_m^*M$ decomposes into a direct sum of line bundles. Furthermore, the direct summands are explicitly described as follows:

(1) When $X = P_A(O_A \oplus L)$ is as in Theorem 2.14 (6) and $M = O_{\pi^{(\frac{1}{m})},(\alpha \oplus \pi^{(\frac{1}{m})})^*N}$, then each direct summand of $g_m^*M$ is of the following form:
\[ O_{\pi^{(\frac{1}{m})},(\alpha \oplus \pi^{(\frac{1}{m})})^*} \left( L^j \otimes N \right), \]
where $i = \left[ \frac{a}{m^2} \right] - 1, \left[ \frac{a}{m^2} \right], 0 \leq j \leq m^2$.

(2) When $X = P_C(O_C \oplus L_1 \oplus L_2)$ is as in Theorem 2.14 (7) and $M = O_{\pi^{(\frac{1}{m}),}(\alpha \oplus \pi^{(\frac{1}{m})})^*N}$, then each direct summand of $g_m^*M$ is of the following form:
\[ O_{\pi^{(\frac{1}{m})},(\alpha \oplus \pi^{(\frac{1}{m})})^*} \left( L_1^j \otimes L_2^j \otimes N \right), \]
where $i = \left[ \frac{a}{m^2} \right] - 2, \left[ \frac{a}{m^2} \right] - 1, \left[ \frac{a}{m^2} \right], 0 \leq j_1, j_2 \leq m^2$.
When \( X = \mathbb{P}_C(\mathcal{O}_C \oplus L_1) \times_C \mathbb{P}_C(\mathcal{O}_C \oplus L_2) \) is as in Theorem 2.14 (8) and \( M = \mathcal{O}_{\frac{1}{a}, \frac{1}{b}}(a, b) \otimes \pi^{(m)}(m) \), then each direct summand of \( g_mM \) is of the following form:

\[
\mathcal{O}_{\pi^{(m)}}(i, j) \otimes \pi^{(m)}(m) \left( L_1^k \otimes L_2^k \otimes N \right),
\]

where \( i = \left\lfloor \frac{a}{m} \right\rfloor - 1, \left\lceil \frac{a}{m} \right\rceil, \left\lfloor \frac{b}{m} \right\rfloor - 1, \left\lceil \frac{b}{m} \right\rceil \), and \( 0 \leq k_1, k_2 \leq m^2 \).

Proof. (1) Let \( X = \mathbb{P}_A(\mathcal{O}_A \oplus L) \) be as in Theorem 2.14 (6). Since \( g_mM \cong g_m\mathcal{O}_{\frac{m}{a}, \frac{m}{b}}(a) \otimes \pi^{(m)}(m) \), we may assume that \( M = \mathcal{O}_{\frac{m}{a}, \frac{m}{b}}(a) \). Furthermore, since we have \( g_m\mathcal{O}_{\pi^{(m)}}(1) \cong \mathcal{O}_{\pi^{(m)}}(m^2) \), we may assume \( 0 \leq a < m^2 \). Now let \( F := g_mM \), and consider the adjoint map \( \alpha : \pi^{(m)}(m)^* \pi^{(m)}_* F \to F \). On the fiber of \( \pi^{(m)}(m) \), the map \( \alpha \) is nothing but the natural inclusion

\[
\mathcal{O}_{\pi^{(m)}}^{\oplus a+1} \subset \mathcal{O}_{\pi^{(m)}}^{\oplus a+1} \oplus \mathcal{O}_{\pi^{(m)}}(-1)^{\oplus (m^2-a-1)},
\]

Indeed by (3.4), on the fiber of \( \pi^{(m)}(m) \), we have an isomorphism

\[
F|_{\pi^{(m)}} \cong \mathcal{O}_{\pi^{(m)}}(a) \cong \mathcal{O}_{\pi^{(m)}}^{\oplus a+1} \oplus \mathcal{O}_{\pi^{(m)}}(-1)^{\oplus (m^2-a-1)},
\]

where \( m^2 \) denotes the toric Frobenius morphism on \( \mathbb{P}^1 \). Moreover, the adjoint map \( \alpha \) restricted to the fiber is nothing but the evaluation map

\[
\alpha|_{\pi^{(m)}} : H^0(\mathbb{P}^1, F|_{\pi^{(m)}}) \otimes \mathcal{O}_{\pi^{(m)}} \hookrightarrow F|_{\pi^{(m)}}.
\]

Hence globally, the map \( \alpha \) is injective and we get the short exact sequence

\[
0 \to \pi^{(m)}(m)^* \pi^{(m)}_* F \to F \to \pi^{(m)}(m)^* \mathcal{G} \otimes \mathcal{O}_{\pi^{(m)}}(-1) \to 0
\]

for some coherent sheaf \( \mathcal{G} \in \text{Coh}(A) \). First of all, we have

\[
\pi^{(m)}_* F = \pi^{(m)}_* \mathcal{O}_{\frac{m}{a}, \frac{m}{b}}(a) = \text{Sym}^{(m)}(\mathcal{O}_A \oplus L^m) = \mathcal{O}_A \oplus L^a \oplus \cdots \oplus L^m.
\]

Next we will show that \( \mathcal{G} \) is a direct sum of line bundles. Applying the functor \( \pi^{(m)}_* (- \otimes \mathcal{O}_{\pi^{(m)}}(1)) \) to the exact sequence (3.4), we have

\[
0 \to \text{Sym}^a(\mathcal{O}_A \oplus L^a) \otimes (\mathcal{O} \oplus L^m) \xrightarrow{\beta} \text{Sym}^{a+m^2}(\mathcal{O}_A \oplus L^m) \to \mathcal{G} \to 0.
\]

Note that the vector bundles \( \text{Sym}^a(\mathcal{O}_A \oplus L^a) \otimes (\mathcal{O} \oplus L^m) \) and \( \text{Sym}^{a+m^2}(\mathcal{O}_A \oplus L^m) \) are the direct sums of line bundles. By the definition of the morphism \( g_m \), the map \( \beta \) is the natural inclusion as the direct summand. Hence \( \mathcal{G} \) is isomorphic to the vector bundle

\[
L_a^{ \frac{a+1}{2} } \oplus L_a^{ \frac{a+2}{2} } \oplus \cdots \oplus L_a^{ \frac{m^2-a-1}{2} } \oplus L_m^{ \frac{m^2-1}{2} }.
\]

It remains to show that the exact sequence (3.4) splits. Let us first compute the Ext-group:

\[
\text{Ext}^1 \left( \pi^{(m)}_* \mathcal{G} \otimes \mathcal{O}_{\pi^{(m)}}(-1), \pi^{(m)}(m)^* \pi^{(m)}_* F \right)
\]

\[
\cong H^1 \left( X^{(m)}, \pi^{(m)}_* \mathcal{G} \otimes \pi^{(m)}_* F \otimes \mathcal{O}_{\pi^{(m)}}(1) \right)
\]

\[
\cong H^1 \left( A, \mathcal{G} \otimes \pi^{(m)}_* F \otimes \mathcal{O}_{\pi^{(m)}}(1) \right)
\]

\[
\cong \bigoplus_{\eta} H^1(A, L^{\frac{\eta}{2}}).
\]

Here, the last isomorphism follows from the descriptions of \( \mathcal{G}, \pi^{(m)}_* F \) given above, together with the equality \( \pi^{(m)}_* \mathcal{O}_{\pi^{(m)}}(1) = \mathcal{O}_A \oplus L^m \). Furthermore, by these descriptions, we can see that \( \eta \neq 0 \). Hence if \( L \) is not a torsion line bundle, we have the vanishing \( \text{Ext}^1 \left( \pi^{(m)}_* \mathcal{G} \otimes \mathcal{O}_{\pi^{(m)}}(-1), \pi^{(m)}(m)^* \pi^{(m)}_* F \right) = 0 \) and thus the sequence
(3.4) splits. Assume that $L^l$ is $l$-torsion, i.e., $(L^l)^l \cong \mathcal{O}_A$. Assume also that 
$\text{Ext}^1(\pi^*(\mathcal{G}) \otimes \mathcal{O}_{π^*(1)}, \pi^*(\mathcal{F}))$ contains $H^1(A, (L^l)^l) \cong H^1(A, \mathcal{O}_A)$ as a

direct summand. Suppose for a contradiction that the sequence (3.4) does not split. Consider the following Cartesian diagram:

$$
P^1 \times A \xrightarrow{\pi^2 \times \text{id}_A} P^1 \times A \xrightarrow{\pi_0} A \xrightarrow{\psi} X^{(m)} \xrightarrow{g_m} X^{(π)} \xrightarrow{l} A.$$

Since the morphisms $u$ and $v$ are flat, we have an isomorphism

$$v^*g_m, \mathcal{O}_{π^*(1)}(a) \cong (\pi^2 \times \text{id}_A), u^* \mathcal{O}_{π^*(1)}(a) \cong (\pi^2 \times \text{id}_A), \mathcal{O}_{π^*(a)}$$

and hence it is a direct sum of line bundles by the usual toric Frobenius splitting on

$P^1$. On the other hand, the pull back of the sequence (3.4) via $v$ cannot split since

$l^* : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A)$ is an isomorphism. Hence we get a contradiction.

(2) Let $X = P_C(\mathcal{O}_C \oplus L_1 \oplus L_2)$ be as in Theorem 2.14 (7). As in (1), we may assume $M = \mathcal{O}_{π^*(1)}(a)$ and $0 ≤ a < m^2$. Let $F := g_m, M$. By the same argument as

in (1), we have the following exact sequences:

$$0 \to \pi^*(\mathcal{F}) \to F \to \mathcal{O}_{π^*(1)}(−1) \to 0,$$

$$0 \to \pi^*(\mathcal{G}) \to \mathcal{F} \to \pi^*(\mathcal{G}) \otimes \mathcal{O}_{π^*(1)}(−1) \to 0,$$

which correspond to the toric Frobenius splitting

$$\pi^2 \mathcal{O}_{π^*(1)} \cong \mathcal{O}_{π^2}(−1)^{\oplus k_1} \oplus \mathcal{O}_{π^2}(−2)^{\oplus k_2}.$$

on the fiber of $\pi^*(m)$. Furthermore, we can show that the vector bundles $\mathcal{F}$ and $\mathcal{G}$

are direct sums of line bundles, and that exact sequences (3.5) split again by the

same argument as (1).

(3) Let $X = P_C(\mathcal{O}_C \oplus L_1) \times_C P_C(\mathcal{O}_C \oplus L_2)$ be as in Theorem 2.14 (8), let

$Y := P_C(\mathcal{O}_C \oplus L_1)$. Then the problem is reduced to showing the corresponding

statement for $Y^{(π)} \to Y^{(m)}$. The latter follows from the argument as in (1).

We also need the following lemma.

Lemma 3.10. Let $X$ be as in Theorem 2.14 (6), (7), or (8) which is not necessarily

split. Then by identifying the tautological classes, the Néron-Severi group of $X$

is canonically isomorphic to that of $P^1 \times A$, $P^2 \times C$, or $P^1 \times P^1 \times C$. Furthermore,

the following statements hold:

(1) Under the isomorphism, their nef cones are preserved.

(2) Under the isomorphism, their classes of the canonical divisors are preserved.

(3) If $X$ is split, then the isomorphism is compatible with the formula given in

Proposition 3.5.

Proof. Let $π : X = P_A(\mathcal{E}) \to A$ be as in Theorem 2.14 (6), where $\mathcal{E}$ is a rank two

vector bundle fitting into an exact sequence

$$0 \to \mathcal{O}_A \to \mathcal{E} \to L \to 0.$$

We only treat this case. We have $\text{NS}(X) = \mathbb{Z}[h] \oplus \text{NS}(A)$, where $h$ is a divisor

such that $\mathcal{O}(h) = \mathcal{O}_X(1)$. Hence by identifying a class $[h]$, $\text{NS}(X)$ is isomorphic to

$\text{NS}(P^1 \times A)$.

(1) We claim that the line bundle $M = \mathcal{O}_X(a) \otimes π^*N$ on $X$ is nef if and only

if $a ≥ 0$ and $N$ is nef. The ‘if’ direction is clear since $\mathcal{O}_X(1)$ is nef. Let us prove
the converse. Let \( h \in |O_x(1)| \) be a section of \( \pi \) and \( f \cong \mathbb{P}^1 \) be a fiber of \( \pi \). Then we have \( M|h \cong N \), \( M|f \cong O_{\mathbb{P}^2}(a) \) and they are nef, which proves the claim. This description of the nef cone is independent on the choice of \( L \in \text{Pic}^0(A) \) and on the choice of the extension class \( [\mathcal{E}] \in \text{Ext}^1(L, O_A) \).

(2) The canonical line bundle on \( X \) is given as \( \mathcal{O}(K_X) = \mathcal{O}(-2h) \otimes \pi^*L \). Since \( L \in \text{Pic}^0(A) \), we have \([K_X] = -2[h] \in \text{NS}(X)\) in the Néron-Severi group which is independent on the choice of \( L \in \text{Pic}^0(A) \).

(3) The statement is trivial from the proof of Proposition 5.3 again by noting that \( \text{ch}_1(L) = 0 \) for \( L \in \text{Pic}^0(A) \). \( \square \)

Now we can prove our main theorem:

Proof of Theorem 2.17. We only give an outline of the proof since the argument is same as [19]. Let \( X \) be as in Theorem 2.17. By Proposition 3.3, we may assume \( X \) is split. Take a \( \beta \)-stable object \( E \) and let \( \mathcal{B} := B + \beta(E)\omega \).

First assume that \( \mathcal{B} \) is a \( \mathbb{Q} \)-divisor. Take an integer \( q \in \mathbb{Z}_{>0} \) and an integral divisor \( D \) such that \( \mathcal{B} = \frac{1}{q}D \). For each integer \( m \in \mathbb{Z}_{>0} \), let us consider the morphism \( F_{mq} \) constructed in Proposition 5.3. Let \( D(\frac{1}{mq}) \) be the divisor on \( X(\frac{1}{mq}) \) such that \( D(\frac{1}{mq}) = \Phi^{-1}(D) \) in the cohomology. Then the Riemann-Roch theorem and Lemma 6.3 imply the inequality

\[
m^6q^6 \text{ch}_3(\mathcal{B}) + \mathcal{O}(m^4) = \chi\left( \mathcal{O}, F_{mq}^*\mathcal{E} \otimes \mathcal{O}\left(-m^2qD(\frac{1}{mq})\right) \right) \leq \text{hom}\left( \mathcal{O}, F_{mq}^*\mathcal{E} \otimes \mathcal{O}\left(-m^2qD(\frac{1}{mq})\right) \right) + \text{ext}^2\left( \mathcal{O}, F_{mq}^*\mathcal{E} \otimes \mathcal{O}\left(-m^2qD(\frac{1}{mq})\right) \right).
\]

We need to prove that the right hand side of the above inequality is of order \( m^4 \). By Proposition 3.7, to prove \( \text{Hom}\left( \mathcal{O}, F_{mq}^*\mathcal{E} \otimes \mathcal{O}\left(-m^2qD(\frac{1}{mq})\right) \right) = \mathcal{O}(m^4) \), it is enough to find an ample divisor \( H \) such that

\[
\text{Hom}\left( \mathcal{O}(H), F_{mq}^*\mathcal{E} \otimes \mathcal{O}\left(-m^2qD(\frac{1}{mq})\right) \right) = 0.
\]

By using the Serre duality and the projection formula, we have an isomorphism

\[
\text{Hom}\left( \mathcal{O}(H), F_{mq}^*\mathcal{E} \otimes \mathcal{O}\left(-m^2qD(\frac{1}{mq})\right) \right) \cong \text{Hom}\left( \mathcal{O}(-K_{X(m)} \otimes g_{mq}^*\mathcal{O}\left(H + m^2qD(\frac{1}{mq}) + K_{X(\frac{1}{mq})}\right), h_{mq}^*\mathcal{E} \right).
\]

By Proposition 5.3, the vector bundle \( \mathcal{O}(-K_{X(m)} \otimes g_{mq}^*\mathcal{O}\left(H + m^2qD(\frac{1}{mq}) + K_{X(\frac{1}{mq})}\right) \) splits into a direct sum of line bundles \( M_j \). Hence it is enough to show the vanishing \( \text{Hom}(M_j, h_{mq}^*\mathcal{E}) = 0 \) for all \( j \). Since we know the tilt stability of \( M_j \) (resp. \( h_{mq}^*\mathcal{E} \)) by Lemma 2.13 (resp. [19] Proposition 6.1), it is enough to show the inequality \( \nu(0, h_{mq}^*\mathcal{F}(M_j)) > \nu(0, h_{mq}^*\mathcal{F}(h_{mq}^*\mathcal{E})) = 0 \). The required inequality on the tilt-slope will follow if we can show that \( \text{ch}_1(h_{mq}^*\mathcal{F}(M_j)) \) is ample. By Lemma 3.10 the problem is now reduced to the case when \( X \) is \( \mathbb{P}^1 \times A, \mathbb{P}^2 \times C \), or \( \mathbb{P}^1 \times \mathbb{P}^1 \times C \), which is treated in [19] Lemma 4.6. The estimate of \( \text{ext}^2 \) will also be reduced to [19] Lemma 4.7.

When \( \mathcal{B} \) is not a \( \mathbb{Q} \)-divisor but an \( \mathbb{R} \)-divisor, we can argue as in [19] Subsection 4.3 by using Dirichlet approximation theorem. \( \square \)
4. Proof of Theorem 2.19

In this section, we will treat $\pi: X := \mathbb{P}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2}) \rightarrow \mathbb{P}^2$. Recall that $X$ is isomorphic to a $(1,1)$-divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ and hence has two projections to $\mathbb{P}^2$:

\[
\begin{array}{c}
X \\
\downarrow \pi \\
\downarrow \sigma \\
\mathbb{P}^2 \\
\downarrow \\
\mathbb{P}^2.
\end{array}
\]

Let $h_1, h_2$ be nef divisors on $X$ such that $\mathcal{O}(h_1) = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, $\mathcal{O}(h_2) = \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$. Then any line bundle on $X$ can be written as $\mathcal{O}(a,b) := \mathcal{O}(ah_1) \otimes \mathcal{O}(bh_2)$ with $a, b \in \mathbb{Z}$. In this notation, we have $\mathcal{O}_s(1) = \mathcal{O}(1, 1)$.

Fix an ample divisor $H = ah_1 + bh_2$ with $a, b \in \mathbb{Z}_{> 0}$. For a positive real number $\alpha > 0$, let $\omega := \alpha H$. We will mainly consider the following central charge and heart:

\[
\omega := \alpha H.
\]

First recall the following result due to [6, Theorem 4.2].

**Theorem 4.1.** Fix a positive real number $\alpha > 0$. Conjecture [2, 4] holds for $(X, \alpha H, B = 0)$ if and only if for every $s > \frac{1}{18}$, the pair $(Z_{\alpha,0,s}, \mathcal{A}_\alpha)$ is a stability condition on $D^b(X)$.

**Proof.** By [7, Corollary 5.2.4], the pair $(Z_{\alpha,0,s}, \mathcal{A}_\alpha)$ is a stability condition for every $s > \frac{1}{18}$ if and only if for every $\nu_{\alpha, 0}$-stable object $E \in \text{Coh}^{\alpha H \beta}(X)$ with $\nu_{\alpha, 0}(E) = 0$, we have $\text{ch}_2 \leq \frac{1}{3\alpha^2 H^2}$. Then the latter is equivalent to Conjecture 2.4 by [6, Theorem 4.2].

**Definition 4.2.** For a fixed ample divisor $H = ah_1 + bh_2$, we define a real number $\alpha_0 > 0$ as

\[
\alpha_0 := \min \left\{ \sqrt{\frac{2}{a^2 + 6ab + b^2}} : \right\}.
\]

The goal of this subsection is to prove the following:

**Proposition 4.3.** Let $H = ah_1 + bh_2$ be an ample divisor on $X$ with $b > a$. Then for every $0 < \alpha < \alpha_0$ and $s > \frac{1}{18}$, the pair $(Z_{\alpha,0,s}, \mathcal{A}_\alpha)$ is a stability condition on $X$. In particular, Conjecture 2.4 holds for $(X, \alpha H, B = 0)$.

First we prove that the above proposition implies Theorem 2.19.

**Proof of Theorem 2.19.** Let $H = ah_1 + bh_2$ be an ample divisor. By the symmetry of the diagram (4.1), we may assume that $b \geq a$. Furthermore, if $a = b$, then the result is already known due to [8]. Now we can assume that $b > a$ and then by Theorem 2.9, Proposition 4.3 implies Theorem 2.19.

To prove Proposition 4.3, we use the following result due to the paper [7], and follow the arguments in [27, 33].

**Proposition 4.4 (2 Proposition 8.1.1).** Assume there exists a heart $\mathcal{C}$ in $D^b(X)$ with the following properties:

1. There exist $\phi_0 \in (0, 1)$ and $s_0 \in \mathbb{Q}$ such that

\[
Z_{\alpha,0,s_0}(\mathcal{C}) \subset \{ r \exp(\pi \phi) : r \geq 0, \phi_0 \leq \phi \leq \phi_0 + 1 \}.
\]

2. $\mathcal{C} \subset \langle \mathcal{A}_0, 0, \mathcal{A}_0, 0(1) \rangle$.

3. For any $x \in X$, we have $\mathcal{O}_x \in \mathcal{C}$ and, for all non-zero proper subobjects $\mathcal{C} \subset \mathcal{O}_x$ in $\mathcal{C}$, we have $\exists Z_{\alpha,0,s_0}(\mathcal{C}) > 0$.
Then for all $s > s_0$, the pair $(Z_{α, 0, s}, A_{α, 0})$ is a stability condition on $D^b(X)$.

Our heart $C$ is constructed by using an Ext-exceptional collection in the sense of Definition 3.10.

**Definition 4.5.** An exceptional collection $E_1, \cdots, E_n$ on a triangulated category $D$ is Ext-exceptional if for all $i \neq j$, we have $\text{Ext}^{\geq 1}(E_i, E_j) = 0$.

**Lemma 4.6** (26, Lemma 3.14). Let $E_1, \cdots, E_n$ be a full Ext-exceptional collection on a triangulated category $D$. Then the extension closure $(E_1, \cdots, E_n)_{\text{ex}}$ is the heart of a bounded t-structure on $D$.

**Lemma 4.7.** A collection

\begin{equation}
\mathcal{O}(-1, -1)[3], \mathcal{O}(0, -1)[2], \mathcal{O}(1, -1)[1], \mathcal{O}(0, -1)[2], \mathcal{O}[1], \mathcal{O}(1, 0)
\end{equation}

is a full Ext-exceptional collection on $D^b(X)$.

**Proof.** Using the equality $\mathcal{O}_π(1) = \mathcal{O}(1, 1)$, the collection \((4.2)\) can be also written as

\begin{align*}
\pi^*\mathcal{O}_{P^2} \otimes \mathcal{O}_π(-1)[3], \pi^*\mathcal{O}_{P^2}(1) \otimes \mathcal{O}_π(-1)[2], \pi^*\mathcal{O}_{P^2}(2) \otimes \mathcal{O}_π(-1)[1], \\
\pi^*\mathcal{O}_{P^2}(-1)[2], \pi^*\mathcal{O}_{P^2}[1], \pi^*\mathcal{O}_{P^2}(1).
\end{align*}

Since we have $D^b(X) = \langle Lπ^*D^b(\mathbb{P}^2) \otimes \mathcal{O}_π(-1), Lπ^*D^b(\mathbb{P}^2) \rangle$, we can see that the collection \((4.2)\) is a full exceptional collection. To prove it is Ext-exceptional, we can use the formula

\[
RΓ(X, π^*O_{P^2}(k) \otimes O_π(l)) = \begin{cases} 
0 & (l = -1) \\
RΓ(P^2, O(k)) & (l = 0) \\
RΓ(P^2, T_{P^2}(k)) & (l = 1).
\end{cases}
\]

□

Now we can define the following heart:

**Definition 4.8.** We define a heart $C \subset D^b(X)$ as

\[
C := \langle \mathcal{O}(-1, -1)[3], \mathcal{O}(0, -1)[2], \mathcal{O}(1, -1)[1], \mathcal{O}(0, -1)[2], \mathcal{O}[1], \mathcal{O}(1, 0) \rangle_{\text{ex}}.
\]

The following will be useful in the rest of the arguments:

**Lemma 4.9.** For integers $k, l \in \mathbb{Z}$, we have the following equations.

1. $H^2.ch_1(O(k, l)) = la^2 + 2(k + l)ab + kb^2$.
2. $H.ch_2(O(k, l)) = (2k + l)la + (k + 2l)kb$.
3. $ch_3(O(k, l)) = \frac{1}{3}kl(k + l)$.

**Proof.** By using the equations $h_1^3 = h_2^2 = 0$ and $h_2^3, h_2 = h_1, h_2^2 = 1$, the straightforward computation yields the result. □

**Lemma 4.10.** For $0 < α < α_0$, we have $C \subset \langle A_{α, 0}, A_{α, 0}[1] \rangle_{\text{ex}}$.

**Proof.** By Lemma 4.9 we have $H.ch_1(O(1, 0)) > 0$ and hence $O(1, 0) \in \text{Coh}^{α, 0}(X)$. By assumption on $α$, we also have

\[
H.ch_2(O(1, 0)) - \frac{1}{6}α^2H^3.ch_0(O(1, 0)) = b - \frac{1}{2}α^2ab(a + b) > 0,
\]

i.e., $ν_{α, 0}(O(1, 0)) > 0$. Since $O(1, 0)$ is tilt stable by Lemma 2.13, we conclude that $O(1, 0) \in A_{α, 0}$.

Similar computations yield that

\[
O[1], O(-1, 0)[1], O(1, -1), O(0, -1)[1], O(-1, -1)[1] \in \text{Coh}^{αH, 0}(X)
\]
and
\[ O[1], O(-1, 0)[2], O(1, -1)[1], O(0, -1)[2], O(-1, -1)[2] \in A_{0,0}. \]

Lemma 4.11. Let \( 0 < \alpha < \alpha_0 \) and let \( \phi_0 \in (0, 1) \) be a real number such that
\[ Z_{\alpha,0}(O(1,0)) = r_0 \exp(\pi \phi_0 i) \] for some positive real number \( r_0 > 0 \). Then we have
\[ Z_{\alpha,0}(C) \subset \{ r \exp(\pi \phi i) : r \geq 0, \phi_0 \leq \phi \leq \phi_0 + 1 \}. \]

Proof. Recall that our central charge is written as
\[ Z_\alpha := Z_{\alpha,0,1/4} = -\text{ch}_3 + \frac{1}{18} \alpha^2 H^2 \cdot \text{ch}_1 + i \left( \alpha H \cdot \text{ch}_2 - \frac{1}{6} \alpha^3 H^3 \cdot \text{ch}_0 \right). \]

By Lemma 4.9 and the proof of Lemma 4.10, we can see that \( Z_\alpha(O(-1, -1)[3]) \) is in the third quadrant, \( Z_\alpha(O(1,0)) \) is in the first quadrant, and \( Z_\alpha(M) \) is in the second quadrant for other generators \( M \) of the heart \( C \). Now it is enough to check the inequality
\[ \frac{\Re Z_\alpha(O(1,0))}{\Im Z_\alpha(O(1,0))} + \frac{\Re Z_\alpha(O(-1, -1)[3])}{\Im Z_\alpha(O(-1, -1)[3])} > 0. \]

We can estimate the left hand side of the above required inequality as follows:
\[ \frac{-\frac{1}{18} \alpha^2(2a + b)b + 1 - \frac{1}{18} \alpha^2(a^2 + 4ab + b^2) - \frac{1}{18} \alpha^2(2a + b)b + 1 - \frac{1}{18} \alpha^2(a^2 + 4ab + b^2) \alpha(3a + 3b - \frac{1}{2} \alpha^2 ab(a + b))}{\alpha(3a + 3b - \frac{1}{2} \alpha^2 ab(a + b))} > 0. \]

Hence the statement holds.

Lemma 4.12. Let \( 0 < \alpha < \alpha_0 \) and \( x \in X \). Then we have \( O_x \in C \). Moreover, for any non-zero proper subobject \( C \subset O_x \) in the category \( C \), we have \( \Im Z_{\alpha,0,1/4}(C) > 0 \).

Proof. Consider the subcategories
\[ C_1 := \pi^* \left( O_{P^2}(-1)[2], O_{P^2}[1], O_{P^2}(1) \right) \text{ex}, \]
\[ C_2 := \pi^* \left( O_{P^2}[2], O_{P^2}(1)[1], O_{P^2}(2) \right) \text{ex} \oplus O_x(-1)[1] \]

of \( C \). Both of these subcategories \( C_i \) are equivalent to the category \( \text{rep}(Q, I) \) of \( Q \)-representations with certain relations \( I \). Here \( Q \) is the following quiver:

\[ \begin{array}{ccc}
0 & 1 & 2 \\
\end{array} \]

Let \( y := \pi(x) \) and denote \( L_y := \pi^{-1}(y) \cong \mathbb{P}^1 \). Then we have the following exact triangle in \( D^b(X) \)
\[ O_{L_y} \rightarrow O_x \rightarrow O_{L_y}(-1)[1] \]
with \( O_{L_y} \in C_1 \) and \( O_{L_y}(-1)[1] \in C_2 \). This proves that \( O_x \in C \). Note that the \( Q \)-representations corresponding to \( O_{L_y} \in C_1 \) and \( O_{L_y}(-1)[1] \in C_2 \) are the same representation, which has dimension vector \((1, 2, 1)\) and is generated by the vertex \( 0 \). We say that \( O_x \) has dimension vector \((1, 2, 1, 1, 2, 1)\).

To prove the second statement, recall that for an object \( M \) in \( D^b(C) \), we have \( \Im Z_{\alpha,0,1/4}(M) < 0 \) if and only if \( M = O(-1, -1)[3] = \pi^* O_{P^2} \oplus O_x(-1)[3] \). Hence it is enough to consider a subobject \( C \subset O_x \) with dimension vector \((1, a, b, c, d, e)\). We must prove that \( C = O_x \) for such a subobject \( C \). There exists an exact sequence
\[ 0 \rightarrow T_1 \rightarrow C \rightarrow T_2 \rightarrow 0 \]
in $C$ with some objects $T_i \in C_i$. Using the definition of the Ext-exceptional collection, we can see that $T_1 \subset \mathcal{O}_{L_y}$ (resp. $T_2 \subset \mathcal{O}_{L_y}(-1)[1]$) is the subobject in $C_1$ (resp. $C_2$). Since we assume that the dimension vector of $C$ is $(1, a, b, c, d, e)$, and since $\mathcal{O}_{L_y}(-1)[1]$ is generated by vertex 0 as a quiver representation, we must have $T_2 = \mathcal{O}_{L_y}(-1)[1]$. Now we get the commutative diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
0 & \to & T_1 \to C \\
0 & \downarrow & \downarrow \\
0 & \to & \mathcal{O}_{L_y} \to \mathcal{O}_x \to \mathcal{O}_{L_y}(-1)[1] \to 0 \\
K & \to & K \\
0 & \to & 0
\end{array}
\]

for some $K \in C_1$. However, since $\text{Hom}(\mathcal{O}_x, C_1) = 0$, we must have $K = 0$, i.e., $C = \mathcal{O}_x$ as required. \hfill \Box

Now we can prove Proposition 4.3.

**Proof of Proposition 4.3.** By Lemma 4.10, Lemma 4.11, and Lemma 4.12 we can apply Proposition 4.4 to get the result. \hfill \Box

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