The One-Loop Five-Graviton Scattering Amplitude
and Its Low-Energy Limit†

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ABSTRACT

A covariant path integral calculation of the even spin structure contribution to the one-loop N-graviton scattering amplitude in the type-II superstring theory is presented. The apparent divergence of the $N = 5$ amplitude is resolved by separating it into twelve independent terms corresponding to different orders of inserting the graviton vertex operators. Each term is well defined in an appropriate kinematic region and can be analytically continued to physical regions where it develops branch cuts required by unitarity. The zero-slope limit of the $N = 5$ amplitude is performed, and the Feynman diagram content of the low-energy field theory is examined. Both one-particle irreducible (1PI) and one-particle reducible (1PR) graphs with massless internal states are generated in this limit. One set of 1PI graphs has the same divergent dependence on the cut-off as that found in the four-graviton case, and it is proved that such graphs exist for all $N$. The 1PR graphs are contributed by the poles in the world-sheet chiral Green functions.

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1 Introduction

We present an explicit calculation of the one-loop five-graviton amplitude in the critical NSR type-II string theory using path integral methods, resolve its apparent divergence, and study its low-energy field theory content. In a previous paper [1], we resolved the divergence of the one-loop four-graviton scattering amplitude in the type-II superstring theory by separating the amplitude into terms corresponding to different orders of inserting the graviton vertex operators on the world-sheet. The scalar factor for each term contains the proper-time representation of a box type Feynman diagram with the internal lines summed over the mass levels of the superstring spectrum. This decomposition of the amplitude, however, does not directly give terms representing diagrams with single-particle intermediate states which are required by unitarity. In the case of the four-graviton amplitude such terms result from duality by summing over an infinite number of mass levels on a given internal line in the box diagram. No massless single poles appear in the limit of the sum, however, and this is consistent with the fact that the two- and three-graviton amplitudes vanish on the torus.

The one-loop five-graviton amplitude does contain a massless pole in two to three particle channels in the limit when two vertex operators come close together on the world-sheet and the sum of their respective momenta goes on the graviton mass shell. The residue factorizes into a tree level three-point amplitude times a one-loop four-point amplitude. Therefore, the integrand of the one-loop five-point function must contain more than just a simple extension of the exponential dependence on the bosonic world-sheet Green function found in the one-loop four-point function. One finds no massless pole, however, in the limit where three vertex operators come close together, and the sum of their respective momenta goes on the graviton mass shell. This is consistent once again with the fact that the three-graviton amplitude vanishes on the torus.

A calculation of the one-loop five-graviton scattering amplitude using operator methods was previously presented in [4]. The path integral approach that we follow, however, makes the modular invariance and the field theory limit more straightforward. In contrast to the four-graviton amplitude where

\[1\] We use the word graviton to include all of the massless vector bosons of the theory (graviton, dilaton, and anti-symmetric tensor field).
the massive single particle poles arise only from summing over box diagrams (i.e. from duality), we find that the one-particle reducible (1PR) diagrams with a massless single pole appear directly in the zero-slope limit of the five-graviton amplitude. These massless single-particle poles are absent in the four-graviton amplitude because the only dependence of the integrand on the bosonic world-sheet coordinates of the vertex operators comes from the exponential of the bosonic world-sheet Green function. This simple structure results from the Grassman integration over fermionic world-sheet coordinates and a Riemann identity for theta functions. We find that in the case of the five-graviton amplitude no such simplification exists, and the dependence of the integrand on chiral Green functions cannot be avoided.

In addition to the 1PR diagrams, two sets of one-particle irreducible (1PI) graphs with massless internal states also occur in the zero-slope limit of the five-graviton amplitude. In each set, all possible distinct channels contribute. These distinct channels arise from dividing the integration region of the imaginary part of the coordinates for the vertex operators into their different possible orderings in exact analogy to the four-graviton amplitude\[1\]. Once again, each ordering must be independently analytically continued in the external momenta to the physical region for scattering in order that a finite answer for the amplitude be obtained without violating momentum conservation or the graviton mass shell condition. A straightforward transformation from the world-sheet coordinates to Feynman parameters can be found in the zero-slope limit, and a convenient set of momentum invariants exists for each channel which makes the identification of the two-particle cuts trivial. We argue that higher order terms in the low-energy expansion correspond to Feynman diagrams with massive internal states from the superstring spectrum.

One set of 1PI graphs has the expected logarithmic divergence in the cut-off $\alpha'\[3\]$ for a scalar five-point amplitude in ten dimensions and contains powers of the Feynman parameters in the integrand corresponding to derivative couplings in the low-energy field theory. We also find another set of 1PI graphs appearing in the zero-slope limit which has the same degree of divergence as that found in the four-point amplitude. The term that generates these graphs has exactly the same form as in the four-graviton case (i.e. a kinematic factor times a scalar integral), but with the product over world-sheet bosonic Green functions extended to include the extra particle. Therefore, in the zero-slope limit one obtains 1PI graphs with cubic
interactions but with the wrong power of \( \text{Im} \tau \) in the denominator of the integrand for a scalar five-point amplitude in ten dimensions. One can prove this result in string theory from modular invariance. The one-loop four- and five-graviton amplitudes receive contributions only from the three even spin structures on the torus. The term generating these particular 1PI graphs in the zero-slope limit comes from the even spin structure part of the amplitude. Since the integrand of this term contains the same modular invariant function no matter how many external gravitons are inserted, its dependence on the proper-time variable \( \text{Im} \tau \) is fixed by the requirement that the entire integral be modular invariant. It may then be shown that these 1PI graphs will have the same linear divergence in the cut-off to all orders in the number of external gravitons. Therefore, modular invariance from string theory tells us that inserting additional external gravitons will not change the degree of divergence of this set of 1PI Feynman graphs in the low-energy field theory.

In sect. 2 we present an explicit outline of the calculation of the even spin structure contribution to the genus-one N-graviton scattering amplitude in the type-II superstring theory using the NSR path integral formulation. Although this calculation was given by D’Hoker and Phong [4] in the superspace approach, the component formulation presented here makes the identification of the low-energy structure more straightforward. In sect. 3 we show how the four- and five-graviton amplitudes may be obtained from the general N-graviton result by performing the necessary Grassman integrations and verify that they are modular invariant. In sect. 4 we examine the Feynman diagram content of the zero-slope limit of the five-graviton amplitude and discuss the unusual ultra-violet divergence in the cut-off contained in the 1PI part. In sect. 5 we present some concluding remarks.

## 2 The N-graviton amplitude

We begin the calculation of the even spin structure contribution to the one-loop N-graviton scattering amplitude by quoting the result for the partition function [4] on the torus with fixed even spin structure \([\nu_a, \nu_b] \) for the left movers and \([\nu'_a, \nu'_b] \) for the right movers. We suppress spacetime indices for \(2[\nu_a, \nu_b] \) can take on values \([\frac{1}{2}, 0], [0, \frac{1}{2}], \) and \([0, 0] \) for even spin structure. We denote these by the subscript \(ab = 10, 01, \) and \(00 \) respectively.
the fields when unambiguous.

\[ Z_{\nu \bar{\nu}} = \int d\mu \int [D\psi][D\bar{\psi}][Dx] e^{-S} \]

\[ S = \frac{1}{2\pi\alpha'} \int id^2z \left( \partial x \bar{\partial} x + \frac{1}{2} \bar{\psi} \partial \bar{\psi} - \frac{1}{2} \bar{\psi} \bar{\partial} \bar{\psi} \right), \tag{1} \]

where the gauged fixed measure for the integral over inequivalent world-sheet metrics on the torus is given by

\[ d\mu = \frac{d^2\tau}{\tau_2} |\eta(\tau)|^4 \left| \frac{\Theta_{ab}(0|\tau)}{\eta(\tau)} \right|^2. \tag{2} \]

The complex modular parameter for the torus is \( \tau = \tau_1 + i\tau_2 \), and the Dedekind eta function is given by \( \eta(\tau) = \exp \left[ \frac{i\pi \tau}{12} \right] \prod_{n=1}^{\infty} (1 - e^{2i\pi n\tau}) \). \( \Theta_{ab}(z|\tau) \) is the Jacobi theta function with characteristics \([\nu_a, \nu_b]\) (see appendix).

We obtain the even spin structure contribution to the N-graviton scattering amplitude by inserting N vertex operators of the following form \([3]\).

\[ V(z, \bar{z}; \theta, \bar{\theta}; p) = \kappa \epsilon_{\mu\nu} D_+ \Phi^\mu D_- \Phi^\nu e^{ip \cdot \Phi} \]

\[ \Phi^\mu(z, \bar{z}, \theta, \bar{\theta}) = x^\mu + \theta \bar{\psi}^\mu - \bar{\theta} \psi^\mu, \tag{3} \]

where the superderivatives in the superconformal gauge are given by

\[ D_+ = \frac{\partial}{\partial \theta} - 2\bar{\theta} \partial \]

\[ D_- = \frac{\partial}{\partial \bar{\theta}} + 2\theta \bar{\partial}, \tag{4} \]

and \( \kappa \) is the gravitational coupling constant with dimension \([L]^4\) for \( d = 10 \). The world-sheet matter integrals are easily performed after introducing Grassman sources for the superderivative pieces. We therefore take the expectation value of the following operator,

\[ \mathcal{O}(\eta, \bar{\eta}) = \exp \sum_{i=1}^{N} \left[ i p_i \cdot \Phi_i + i \bar{\eta}_i \cdot D_+ \Phi_i + i \eta_i \cdot D_- \Phi_i \right]. \tag{5} \]

The result can be written in terms of the world-sheet Green functions \( G_{ab} \) and \( S_{ab} \) defined by

\[ \bar{\partial} \partial G_{ab}(z, \bar{z}) = \delta^{(2)}(z, \bar{z}) + \frac{1}{2i\tau_2} \delta_{a,1} \delta_{b,1} \]

\[ S_{ab}(z) = \partial G_{ab}(z, \bar{z}). \tag{6} \]
The particular Green functions useful for our purposes are

\[ G_{11}(z, \bar{z}) = -\frac{1}{2i\pi} \log \left| e^{-\pi y^2/\tau_2} \frac{\Theta_{11}(z|\tau)}{\eta(\tau)} \right|^2 \]

\[ S_{ab}(z) = -\frac{1}{2i\pi} \frac{\Theta_{ab}(z|\tau)\Theta'_{11}(0|\tau)}{\Theta_{11}(z|\tau)}, \quad ab \neq 11. \tag{7} \]

The N-graviton scattering amplitude for fixed even spin structure in both the left and right moving sectors is given by

\[
\Gamma_{\nu\nu} = \frac{\kappa^N}{(\alpha')^N} \int_F \frac{d^2\tau}{(\tau_2)^6} \int \left[ \prod_{i=1}^N \epsilon_i^{\mu_i\nu_i} d\bar{\eta}_{\mu_i} d\eta_{\nu_i} d^2\theta_i d^2z_i \right] \\
\times \exp \left\{ -\frac{i\pi\alpha'}{2} \sum_{ij} p_i \cdot p_j \left[ G_{11}^{ij} + \frac{1}{4i\tau_2} (z_{ij} - \bar{z}_{ij})^2 \right] \right\} \\
\times \left| \Theta_{ab}(0|\tau) \right|^8 \exp \left[ K_{ab}(\eta, \bar{\eta}) + K_{\sigma\tau}(\eta, \theta) \right] \\
\times \exp \left\{ \frac{i\pi\alpha'}{\tau_2} \sum_{ij} \left[ p_i \cdot \eta_j \bar{\eta}_j (z_{ij} - \bar{z}_{ij}) + p_i \cdot \eta_j \theta_j (z_{ij} - \bar{z}_{ij}) \right. \right. \\
\left. \left. + \frac{1}{8} p_i \cdot p_j (z_{ij} - \bar{z}_{ij})^2 + 2\eta_i \cdot \eta_j \bar{\eta}_j \bar{\theta}_j + \eta_i \cdot \eta_j \bar{\theta}_j + \eta_i \cdot \eta_j \theta_j \right] \right\}, \tag{8} \]

where the chiral piece \( K_{ab}(\eta, \bar{\eta}) \) depending holomorphically on \( z \) and \( \tau \) is

\[
K_{ab}(\eta, \bar{\eta}) = -i\pi\alpha' \sum_{ij} \left\{ p_i \cdot p_j \bar{\eta}_j S_{ij}^{ab} + \eta_i \cdot \eta_j S_{ij}^{ab} - 2p_i \cdot \eta_j \bar{\eta}_j S_{ij}^{ab} \right. \\
- 2p_i \cdot \eta_j \bar{\eta}_j \left[ (\partial_j G_{11}^{ij}) - \frac{1}{2i\tau_2} (z_{ij} - \bar{z}_{ij}) \right] \\
- 2\eta_i \cdot \eta_j \bar{\eta}_j \bar{\theta}_j \left[ (\partial_i \partial_j G_{11}^{ij}) - \frac{1}{2i\tau_2} \right] \right\}. \tag{9} \]

All terms proportional to delta functions have been dropped in the above expression since they do not contribute to the integral for values of the external momenta away from intermediate particle poles. The complex modular

\[ \text{The primes on the sums indicate that the } i = j \text{ terms are excluded as a consequence of normal ordering. The superscripts on the world-sheet Green functions label their z-dependence } G_{ij}^{11} = G_{11}(z_i - z_j). \]
parameter \((\tau = \tau_1 + i\tau_2)\) is integrated over the fundamental domain,

\[
-\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, \quad \tau_2 > 0, \quad \text{and} \quad |\tau| \geq 1,
\]

and the coordinates of the vertex operators \((z_i = x_i + iy_i)\) are integrated over the torus,

\[
\frac{\tau_1}{\tau_2} y_i \leq x_i \leq \frac{\tau_1}{\tau_2} y_i + 1, \quad \text{and} \quad 0 \leq y_i \leq \tau_2.
\]

Since the spin structures must be summed independently for the left and right movers, we split the mixed chirality piece by introducing the loop momenta as follows \[1\]. First, however, we include the finite \(i = j\) terms in the sum which amounts to a finite renormalization of the vertex. We can then re-write the mixed chirality piece as

\[
\exp \left\{-\frac{\pi \alpha'}{\tau_2} \sum_i \left( \eta^\mu_i \bar{\theta}_i + \eta^\mu_i \theta_i - \frac{1}{2} p_i^\mu (z_i - \bar{z}_i) \right) \right\}^2.
\]

We now split this term into a product of holomorphic and anti-holomorphic factors with the following identity.

\[
1 = \left( \frac{\tau_2}{4\alpha'} \right)^5 \int \left[ \prod_{\mu} dq^\mu \right] \exp \left\{ -\frac{\pi \alpha'}{\tau_2} \left( q^\mu - \frac{2i\alpha'}{\tau_2} \sum_i \left( \eta^\mu_i \bar{\theta}_i + \eta^\mu_i \theta_i - \frac{1}{2} p_i^\mu (z_i - \bar{z}_i) \right) \right)^2 \right\}.
\]

This allows us to write \((\ref{12})\) as

\[
\left( \frac{\tau_2}{4\alpha'} \right)^5 \int \left[ \prod_{\mu} dq^\mu \right] \left| \exp \left[ \frac{i\pi \tau}{8\alpha'} q \cdot q + i\pi \sum_i \left( q \cdot \eta_i \bar{\theta}_i - \frac{1}{2} q \cdot p_i z_i \right) \right] \right|^2.
\]

The result for the even spin structure contribution to the N-graviton amplitude then factorizes into holomorphic and anti-holomorphic pieces, and each is summed over even spin structures with phases determined by the requirement of modular invariance. We quote here the final result written in
component form.
\[
\Gamma_N = \frac{\kappa^N}{2^{10(\alpha')12-N} \int F_{\tau_2} \int \prod_{i=1}^{N} e_{\mu i}^{\nu \mu i} d\theta_{\mu i} d\eta_{\nu i} d^2 \theta_i d^2 z_i} 
\times \exp \left[ -\frac{i\pi\alpha'}{2} \sum_{ij} p_i \cdot p_j \hat{G}_{ij}^{11} \right]^2 
\times \int \prod_{\mu} dq^\mu \exp \left[ \frac{i\pi\tau_2}{8\alpha'} q \cdot q - \frac{i\pi}{2} \sum_i q \cdot p_i z_i \right]^2 
\times \left[ \sum_{a \times b = 0} (-)^{a+b} \left\{ \Theta_{ab}(0|\tau) \right\}_3^4 \exp \left[ K_{ab}(\eta, \bar{\theta}) + i\pi \sum_i q \cdot \eta_i \bar{\theta}_i \right] \right]^2 \tag{15}
\]

where we define $\hat{G}_{ij}^{11}$ to be the holomorphic part of $G_{ij}^{11} + \frac{1}{4\tau_2} (z_{ij} - \bar{z}_{ij})^2$.

We have factorized the integrand of eq. (15) into a piece depending holomorphically on $z$ and $\tau$ times a piece depending anti-holomorphically on these variables. Since some of these factors do not depend on either the spin structure or the chiral Grassman sources, for purposes of calculation it is simpler to complete the square in the $q^\mu$ integral and then shift $q^\mu \rightarrow q^\mu - \frac{i\alpha'}{\tau_2} \sum_i p_i^\mu (z_i - \bar{z}_i)$. Using momentum conservation we then define
\[
\tilde{K}_{ab}(\eta, \bar{\theta}; q) = K_{ab}(\eta, \bar{\theta}) + i\pi \sum_j \left[ q - \frac{i\alpha'}{2} \sum_i p_i (z_{ij} - \bar{z}_{ij}) \right] \cdot \eta_j \bar{\theta}_j, \tag{16}
\]
to write
\[
\Gamma_N = \frac{\kappa^N}{2^{10(\alpha')12-N} \int F_{\tau_2} \int \prod_{i=1}^{N} e_{\mu i}^{\nu \mu i} d\theta_{\mu i} d\eta_{\nu i} d^2 \theta_i d^2 z_i} 
\times \exp \left[ -\frac{i\pi\alpha'}{2} \sum_{ij} p_i \cdot p_j \hat{G}_{ij}^{11} \right] \int \prod_{\mu} dq^\mu \exp \left[ -\frac{\pi\tau_2}{4\alpha'} q^2 \right] 
\times \left[ \sum_{a \times b = 0} (-)^{a+b} \left\{ \Theta_{ab}(0|\tau) \right\}_3^4 \exp \left[ \tilde{K}_{ab}(\eta, \bar{\theta}; q) \right] \right]^2 \tag{17}
\]

3 The five-graviton amplitude

Before looking at the five-graviton amplitude, we examine the case for $N < 5$ in order to make clear the different structure that appears at $N = 5$. Instead
of expanding the exponential in eq. (17) and then performing the Grassman integration, it is somewhat easier in practice to use the fact that this process is equivalent to Grassman differentiation. Some useful expressions are

\[
\exp \left[ \tilde{K}_{ab}(\eta = 0, \theta; q) \right] = \prod_{1 \leq i < j \leq N} \left[ 1 - 2i\pi\alpha' p_i \cdot p_j \bar{\theta}_i \bar{\theta}_j S_{ij}^{ab} \right],
\]

\[
\frac{\partial}{\partial \eta_j} \tilde{K}_{ab}(\eta = 0, \theta; q) = 2i\pi\alpha' \sum_{i \neq j} p_{\mu j} \left[ \bar{\theta}_i S_{ij}^{ab} + \bar{\theta}_j \left( \partial_i G_{11}^{ij} \right) \right] + i\pi q_{\mu j} \bar{\theta}_j
\]

\[
\frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \eta_j} \tilde{K}_{ab}(\eta = 0, \theta; q) = 2i\pi\alpha' g_{\mu i, \mu j} \left[ S_{ij}^{ab} - 2\bar{\theta}_i \bar{\theta}_j \left( \partial_i \partial_j G_{11}^{ij} - \frac{1}{2i\tau_2} \right) \right].
\]

These equations allow one to differentiate using the chain rule and to drop all those terms which would have cancelled anyway.

The contribution for \( N = 0 \) vanishes trivially due to a Riemann identity for theta functions (see appendix). In fact, for \( N \leq 3 \) the only contribution allowed by the Grassman integration over the \( \theta_i \) consists of terms with products of either no world-sheet fermion propagators \( S_{ij}^{ab} \), two propagators, or three propagators. In each of these cases the contributions can be shown to vanish as a result of the same Riemann identity \( [5] \). When \( N = 4 \), however, the Riemann identity gives instead of zero

\[
\sum_{a \times b = 0} (-)^{a+b} \frac{\Theta_{ab}(0|\tau)}{[\eta(\tau)]^2} S_{ab}^{i1} S_{ab}^{i2} S_{ab}^{i3} S_{ab}^{i4} S_{ab}^{i4} = -1,
\]

and eliminates the dependence of the integrand on the fermionic Green function \( S_{ab}^{ij} \).

Since the above sum over spin structures is independent of the world-sheet coordinates \( z_i \), the kinematic structure can be factored outside the integral leaving a single integral expression for the scattering amplitude. As a direct consequence, the only dependence of the integrand on the world-sheet coordinates comes from the exponential of the bosonic Green function \( G_{11}^{ij} \). However, in order that the amplitude be well defined for non-trivial external momenta, we must divide the integration region for the \( \text{Im} z_i \) into three kinematic regions where each piece can be analytically continued separately in the external momenta \( [1] \). In the zero-slope limit these will give only the three distinct channels of 1PI graphs and no 1PR graphs. We quote here for
completeness the well known result [6].

\[ \Gamma_4 = \frac{k^4}{\alpha'} \left[ \prod_{i=1}^{4} \epsilon_i^{\mu_i \nu_i} \right] K_{\mu_1 \mu_2 \mu_3 \mu_4} \mathcal{R}_{\nu_1 \nu_2 \nu_3 \nu_4} \times \int_{\mathcal{F}} \frac{d^2 \tau}{(\tau_2)^2} \int \left[ \prod_{i=1}^{3} \frac{d^2 z_i}{\tau_2} \right] \exp \left[ -\frac{i\pi\alpha'}{2} \sum_{ij} p_i \cdot p_j \mathcal{G}_{ij}^{11} \right] \]

\[ K_{\mu_1 \mu_2 \mu_3 \mu_4} = -\frac{1}{2} \left[ g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} st + g_{\mu_1 \mu_4} g_{\mu_2 \mu_3} su + g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} tu \right] + s \left[ g_{\mu_1 \mu_3} p_{1 \mu_4} p_{3 \mu_2} + g_{\mu_1 \mu_4} p_{1 \mu_3} p_{4 \mu_2} + g_{\mu_2 \mu_3} p_{2 \mu_4} p_{3 \mu_1} + g_{\mu_2 \mu_4} p_{2 \mu_3} p_{4 \mu_1} \right] + t \left[ g_{\mu_1 \mu_2} p_{1 \mu_3} p_{2 \mu_4} + g_{\mu_1 \mu_3} p_{1 \mu_2} p_{3 \mu_4} + g_{\mu_2 \mu_4} p_{2 \mu_3} p_{4 \mu_1} + g_{\mu_3 \mu_4} p_{3 \mu_1} p_{4 \mu_2} \right] + u \left[ g_{\mu_1 \mu_2} p_{1 \mu_3} p_{2 \mu_4} + g_{\mu_1 \mu_3} p_{1 \mu_2} p_{4 \mu_3} + g_{\mu_2 \mu_4} p_{2 \mu_3} p_{4 \mu_1} + g_{\mu_3 \mu_4} p_{3 \mu_2} p_{4 \mu_1} \right], \tag{20} \]

where \( s, t, \) and \( u \) are the usual Mandelstam invariants, \( g_{\mu_i \mu_j} \) is the flat spacetime metric, and we have used translational invariance to fix \( z_4 = \tau \). When written in this form the modular invariance of the four-graviton amplitude is trivial to verify. The bosonic Green function \( G_{11}(z, \bar{z} | \tau, \bar{\tau}) \) given in eq. (7) is invariant under the modular transformation \( \tau \rightarrow \frac{a\tau + b}{c\tau + d} \) in conjunction with the conformal change of coordinates \( z \rightarrow \frac{z}{c\tau + d} \). One recovers the original integraton region for the world-sheet coordinates \( z_i \) by making appropriate lattice translations which are allowed due to the double periodicity of the integrand.

The five-graviton amplitude is the highest order one-loop graviton scattering amplitude for which the odd spin structure contribution vanishes due to the integration over the Dirac zero modes [14]. Therefore, the full amplitude can be obtained from eq. (17) by setting \( N = 5 \) and performing all the Grassman integrations. Since there are now five vertex operators, we will necessarily generate terms with products of five fermionic Green functions \( S_{ij}^{ab} \) which must be summed over even spin structures. In fact, a large number of such terms do appear in addition to others coming from the various factors in eq. (18). However, we now have no Riemann identity to simplify the sum over spin structures and eliminate the \( z_i \) dependence. Rather than presenting the complete result including the numerous permutations of both the spacetime indices and the arguments of the \( S_{ij}^{ab} \) and \( G_{11}^{ij} \), we list only the various types of terms which occur in order to verify the modular invariance of the amplitude and examine its zero-slope limit.
After performing all the Grassman integrations, the final form of the
one-loop five-graviton amplitude can be written as

$$\Gamma_5 = \kappa^5 \left[ \prod_{i=1}^{5} \epsilon_{\mu_i} \right] \int \frac{d^2 \tau}{\tau_2} \int \left[ \prod_{i=1}^{4} \frac{d^2 z_i}{\tau_2} \right] \exp \left[ -i \pi \alpha' \sum_{ij} p_i \cdot p_j G_{11}^{ij} \right]$$

$$\times \int \prod_{\mu} dq^\mu e^{-\pi q^2} K_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} (z_i, \tau; p_i, q) K_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} (\bar{z}_i, \bar{\tau}; p_i, q),$$

where we have rescaled $q^\mu \rightarrow \sqrt{\frac{4 \alpha'}{\tau_2}} q^\mu$ and used translational invariance to
fix $z_5 = \tau$. The only terms contributing to the $K_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5}$ are those resulting
from either products of four world-sheet fermionic Green functions
$S_{ij}^{ab}$ or five of them. Each of the terms with four $S_{ij}^{ab}$ multiplies a kinematic factor
containing four powers of the external momenta and can be simplified with
the identity (19). Let $P_{\mu_1 \mu_2 \mu_3 \mu_4} \prod_{i=1}^{5} \epsilon_{\mu_i}$ stand for the possible products of external
momenta $p_{\mu_5}$, external momentum invariants $p_{ij} = -(p_i + p_j)^2$, and space-
time metrics $g_{\mu_5 \mu_j}$. The terms coming from the products of four $S_{ij}^{ab}$ have the
general form

$$P_{\mu_1 \mu_2 \mu_3 \mu_4} \left[ \sum_{j \neq i} \left( \partial_{ij} G_{11}^{ij} \right) p_{j \mu_5} + \left( \frac{1}{\alpha' \tau_2} \right)^2 q_{\mu_5} \right],$$

where the indices $(i_1, i_2, i_3, i_4, i_5)$ are some permutation of $(1, 2, 3, 4, 5)$. The sum over spin structures has already been performed thereby eliminating the dependence on the $S_{ij}^{ab}$. The terms with products of five $S_{ij}^{ab}$ can be written in the same notation where the sum over spin structures remains explicit. In
this case, however, $P_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5}$ will be fifth order in powers of the external
momenta. There are two types of terms.

$$P_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \sum_{a \times b = 0} (-)^{a+b} \left( \frac{\Theta_{ab}(0|\tau)}{[\eta(\tau)]^3} \right)^4 S_{ab}^i S_{ab}^i S_{ab}^i S_{ab}^i S_{ab}^i,$$

and

$$P_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \sum_{a \times b = 0} (-)^{a+b} \left( \frac{\Theta_{ab}(0|\tau)}{[\eta(\tau)]^3} \right)^4 S_{ab}^i S_{ab}^i S_{ab}^i S_{ab}^i S_{ab}^i.$$

(21)
The modular invariance of eq. (21) can be checked easily by noting that an even number of powers of loop momenta $q_\mu$ are needed to give a non-zero contribution to the integral. Since no cross terms present in $K_{\mu_1\mu_2\mu_3\mu_4\mu_5}K_{\nu_1\nu_2\nu_3\nu_4\nu_5}$ containing a single power of $q_\mu$ appear, we get the necessary factor $1/\tau_2$ from the $q_\mu q_\nu$ piece. We will see in the next section that this term gives 1PI Feynman graphs in the low-energy limit. The properties of the world-sheet Green functions under modular transformations are listed in the appendix, and it is a simple matter to verify the modular invariance of the remaining terms in eq. (21).

When written as a single integral expression, the five-graviton amplitude (21) diverges for non-trivial values of the external momenta for the same reason as the four-graviton amplitude [7]. Following [1] we first re-write

$$\exp \left[ -\frac{i\pi\alpha'}{2} \sum_{ij} p_i \cdot p_j G_{ij}^{11} \right] = \exp \left[ \pi\alpha' \tau_2 \Phi \right] \prod_{1 \leq i < j \leq 5} \tilde{\chi}_{ij}^{-\frac{1}{2}\alpha' p_{ij}},$$

(25)

where

$$\Phi = -\frac{1}{2} \sum_{1 \leq i < j \leq 5} p_{ij} \left[ \frac{|y_i - y_j|}{\tau_2} - \left( \frac{y_i - y_j}{\tau_2} \right)^2 \right],$$

$$\tilde{\chi}_{ij} = \left| (1 - e^{2i\pi z_{ij}}) \prod_{n=1}^{\infty} \frac{(1 - e^{2i\pi n + 2i\pi z_{ij}})(1 - e^{2i\pi n - 2i\pi z_{ij}})}{(1 - e^{2i\pi n})^2} \right|. $$

(26)

Next we divide the integration region for $y_i$ into the $4!/2$ independent orderings of the difference $y_i - y_j$. The factor of $1/2$ comes from the discrete diffeomorphism $z_i \rightarrow -z_i + \tau + 1$ as explained in ref. [1]. For each of the twelve orderings there exists a convenient set of five independent invariants $p_{ij}$ which simplify the form of the function $\Phi$. For each kinematic region, the corresponding contribution must be analytically continued independently in the external momenta in order to preserve momentum conservation while on the graviton mass shell. For example we find after rescaling $y_i \rightarrow \tau_2 y_i$

$$\Phi(p_{12}, p_{23}, p_{34}, p_{45}, p_{51}) = p_{12}y_1(y_3 - y_2) + p_{23}(y_4 - y_3)(y_2 - y_1) + p_{34}(1 - y_4)(y_3 - y_2) + p_{45}y_1(y_4 - y_3) + p_{51}(1 - y_4)(y_2 - y_1), \tag{27}$$

4Here we change the normalization of the world-sheet bosonic Green function which is allowed by momentum conservation and the graviton mass shell condition.
The zero-slope limit

The zero-slope limit of eq. (21) can now be taken in a straightforward way. The most obvious source of 1PI graphs comes from the terms proportional to the loop momenta \( q_{\mu} \) found in eq. (22). As already mentioned, the terms linear in \( q_{\mu} \) do not contribute. The integral over the quadratic terms \( q_{\mu} q_{\nu} \) must be proportional to the spacetime metric \( g_{\mu\nu} \). For a particular ordering of the \( y_i \) we get a kinematic factor times

\[
\Gamma^{\text{(1PI)}}_5 = \frac{\kappa^5}{\alpha'} \int_{\mathcal{F}} \frac{d^2 \tau}{(\tau_2)^2} \int [d^2 z_i] \exp \left[ \pi \alpha' \tau_2 \Phi \prod_{ij} \tilde{\chi}^{-\frac{1}{2}} \right], \tag{28}
\]

where we have rescaled \( y_i \rightarrow \tau_2 y_i \), and the product is understood to be ordered such that \( y_i - y_j \geq 0 \).

To perform the zero-slope limit of eq. (28) we proceed as in [1]. First divide the fundamental domain into two regions with \( \mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 \), where

\[
\mathcal{F}_1 : \quad -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, \quad \text{and} \quad \sqrt{1 - (\tau_1)^2} \leq \tau_2 \leq 1.
\]

\[
\mathcal{F}_2 : \quad -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, \quad \text{and} \quad 1 \leq \tau_2 \leq \infty. \tag{29}
\]

In region \( \mathcal{F}_1 \) we get

\[
\lim_{\alpha' \rightarrow 0} \left. \Gamma^{\text{(1PI)}}_5 \right|_{\mathcal{F}_1} = \frac{\kappa^5}{\alpha'} \left( \frac{\pi}{3} - 1 \right) + \mathcal{O}(\alpha'), \tag{30}
\]

where the term zeroth order in \( \alpha' \) vanishes due to momentum conservation and the graviton mass shell condition. For region \( \mathcal{F}_2 \) we find after rescaling \( \tau_2 \rightarrow \tau_2/(\pi \alpha') \)

\[
\lim_{\alpha' \rightarrow 0} \left. \Gamma^{\text{(1PI)}}_5 \right|_{\mathcal{F}_2} = 2\pi \kappa^5 \int_{\pi \alpha'}^{\infty} \frac{d\tau_2}{(\tau_2)^2} \int_0^1 \left[ \prod_{i=1}^5 d\beta_i \right] \delta \left( 1 - \sum_{j=1}^5 \beta_j \right)
\]
\[
\times e^{\tau_2 \Phi(p_{12}, p_{23}, p_{34}, p_{45}, p_{51})} + e^{\tau_2 \Phi(p_{12}, p_{23}, p_{35}, p_{45}, p_{51})} \]

for the ordering \( 1 \leq y_4 \leq y_3 \leq y_2 \leq y_1 \leq 0 \). This contribution to the amplitude therefore converges for \( \text{Re} \, p_{12} \leq 0, \text{Re} \, p_{23} \leq 0, \text{Re} \, p_{34} \leq 0, \text{Re} \, p_{45} \leq 0, \) and \( \text{Re} \, p_{51} \leq 0 \).
\[ + e^{\tau_2 \Phi(p_{12}, p_{24}, p_{45}, p_{53}, p_{51})} + e^{\tau_2 \Phi(p_{12}, p_{24}, p_{45}, p_{53}, p_{51})} + e^{\tau_2 \Phi(p_{12}, p_{25}, p_{45}, p_{51})} + e^{\tau_2 \Phi(p_{12}, p_{25}, p_{45}, p_{51})} + e^{\tau_2 \Phi(p_{13}, p_{24}, p_{45}, p_{51})} + e^{\tau_2 \Phi(p_{13}, p_{24}, p_{45}, p_{51})} + e^{\tau_2 \Phi(p_{14}, p_{24}, p_{45}, p_{51})} + e^{\tau_2 \Phi(p_{14}, p_{24}, p_{45}, p_{51})} \]

\[ + e^{\tau_2 \Phi(p_{14}, p_{24}, p_{45}, p_{51})} + e^{\tau_2 \Phi(p_{14}, p_{24}, p_{45}, p_{51})} \] + \( O(\alpha') \).

We have introduced Feynman parameters \( \beta_i \) \(^8\) in place of the rescaled \( y_i \) found in eq. \((27)\) in order to make the field theory interpretation of the result more obvious.

\[ \Phi(p_{12}, p_{23}, p_{34}, p_{45}, p_{51}) = p_{12}\beta_1\beta_3 + p_{23}\beta_2\beta_4 + p_{34}\beta_5\beta_5 + p_{45}\beta_5\beta_5 + p_{51}\beta_3\beta_2. \]

This is a sum of the twelve independent channels of pentagon Feynman diagrams in the proper-time representation for a scalar theory with cubic interactions.

To proceed with the zero-slope limit we integrate by parts with respect to \( \tau_2 \) twice and use momentum conservation and the graviton mass shell condition to cancel the \( \log (\pi \alpha') \) terms. The \( \pi \alpha' \log (\pi \alpha') \) terms cancel as a result of expanding the lower limit of the \( \tau_2 \) integral. Therefore, to zeroth order in \( \alpha' \) we get

\[ \lim_{\alpha' \to 0} \Gamma_5^{(1P1)} \bigg|_{F_2} = \frac{\kappa^5}{\alpha'} - 2\pi\kappa^5 \int_0^1 \prod_{i=1}^5 d\beta_i \left[ \delta \left( 1 - \sum_{j=1}^5 \beta_j \right) \right] \int_0^\infty d\tau_2 \log \tau_2 \]

\[ \times \left\{ \Phi(p_{12}, p_{23}, p_{34}, p_{45}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{12}, p_{23}, p_{34}, p_{45}, p_{51})] + \left\{ \Phi(p_{12}, p_{23}, p_{34}, p_{45}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{12}, p_{23}, p_{34}, p_{45}, p_{51})] + \left\{ \Phi(p_{12}, p_{24}, p_{43}, p_{53}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{12}, p_{24}, p_{43}, p_{53}, p_{51})] + \left\{ \Phi(p_{12}, p_{24}, p_{43}, p_{53}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{12}, p_{24}, p_{43}, p_{53}, p_{51})] + \left\{ \Phi(p_{12}, p_{25}, p_{53}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{12}, p_{25}, p_{53}, p_{51})] + \left\{ \Phi(p_{12}, p_{25}, p_{53}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{12}, p_{25}, p_{53}, p_{51})] + \left\{ \Phi(p_{13}, p_{32}, p_{24}, p_{45}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{13}, p_{32}, p_{24}, p_{45}, p_{51})] + \left\{ \Phi(p_{13}, p_{32}, p_{24}, p_{45}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{13}, p_{32}, p_{24}, p_{45}, p_{51})] + \left\{ \Phi(p_{13}, p_{34}, p_{42}, p_{52}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{13}, p_{34}, p_{42}, p_{52}, p_{51})] + \left\{ \Phi(p_{13}, p_{34}, p_{42}, p_{52}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{13}, p_{34}, p_{42}, p_{52}, p_{51})] + \left\{ \Phi(p_{13}, p_{35}, p_{52}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{13}, p_{35}, p_{52}, p_{51})] + \left\{ \Phi(p_{13}, p_{35}, p_{52}, p_{51}) \right\}^2 \exp [\tau_2 \Phi(p_{13}, p_{35}, p_{52}, p_{51})]. \]
As in the four-graviton amplitude, the contribution of the region $\mathcal{F}_2$ merely renormalizes the UV cut-off in the low-energy field theory. We can replace $\pi \alpha' \to 3 \alpha' \pi$ in the lower limit of the $\tau_2$ integral in eq. (31) to write the complete result as the proper-time representation of 1PI Feynman diagrams for a scalar theory with cubic interactions with the modular parameter $\tau_2$ playing the role of the proper-time variable.

The surprising feature of eq. (31) is the power of $\tau_2$ occurring in the proper-time integral. For a scalar $\phi^3$ theory in ten dimensions with five external particles, one would find only a single power of $\tau_2$ in the denominator of the integrand (see appendix of [1]). Instead we find the same UV degree of divergence as in the zero-slope limit of the four-graviton amplitude. The form of the function $\Phi$ found in eq. (32) shows that these 1PI Feynman graphs have five internal propagators in a theory with cubic interactions. However, the low-energy supergravity theory contains both cubic and higher order interactions with non-trivial derivative couplings. The field theory limit of superstring theory gives the result of summing all of the contributions to the one-loop five-point amplitude with the Lorentz structure already factored outside the integrals over the Feynman parameters. Complicated cancellations among diagrams must occur preserving gauge invariance which produce a term having a single transverse kinematic factor times the scalar integral found in eq. (31). Therefore, from the field theory point of view the UV degree of divergence found in this expression results directly from the underlying gauge invariance of the ten dimensional low-energy field theory.

From the string theory point of view the power of $\tau_2$ occurring in the integrand of eq. (31) is required to preserve the modular invariance of the superstring scattering amplitude. The power of $\tau_2$ and therefore the degree of UV divergence will be the same no matter how many external gravitons are inserted. One can prove that such a term does come from the even spin structure contribution to the N-graviton amplitude given in eq. (17) as follows. Initially there are six powers of $\tau_2$ in the denominator before performing the Grassman integrations. These 1PI graphs come from the terms with unexponentiated powers of the loop momenta $q_{\mu_i}$ multiplying the the $\exp\left[-\frac{\tau_2}{4\alpha'} q_{\mu_i}^2\right]$ factor. Only an even number of such powers will contribute
to the integral over the loop momenta, and each pair of them gives a factor $1/\tau_2$. Each power of $q_{\mu i}$ multiplies only a single $\theta_i$ and the sum over even spin structures requires exactly four factors of the $S^{ij}_{ab}$ in order to eliminate all of the $z_i$ dependence of the integrand not found in the factor

$$\exp \left[ -\frac{i\pi\alpha'}{2} \sum_{ij} p_i \cdot p_j G^{ij}_{11} \right].$$

(34)

Since we can always have four factors of the $S^{ij}_{ab}$ each multiplying a single $\theta_i$, terms with $N - 4$ powers of $q_{\mu i}$ will occur in each of the left and right moving sectors. Overall, we will therefore get $6 + N - 4 = N + 2$ powers of $\tau_2$ in the denominator for $N$ vertex insertions. Each modular invariant integration measure for the location of the vertex operator must get a $1/\tau^2$ leaving exactly a $1/(\tau_2)^2$ for the integral over $d^2\tau$.

The rest of the diagrams appearing in the zero-slope limit of the five-graviton amplitude (21) are generated as follows. Using the representations for the chiral Green functions found in the appendix, one finds after rescaling $y_i \rightarrow \tau_2 y_i$ and then $\tau_2 \rightarrow \tau_2/(\pi \alpha')$ that their contribution to the zero-slope limit is given by

$$\lim_{\alpha' \rightarrow 0} \partial_i G^{ij}_{11} = -(y_i - y_j) + \mathcal{O}(\alpha')$$
$$\lim_{\alpha' \rightarrow 0} S^{ij}_{01} = \mathcal{O}(\alpha')$$
$$\lim_{\alpha' \rightarrow 0} S^{ij}_{00} = \mathcal{O}(\alpha')$$
$$\lim_{\alpha' \rightarrow 0} S^{ij}_{10} = \frac{1}{2} + \mathcal{O}(\alpha').$$

(35)

Since the $\tau_2$ integral diverges logarithmically in the cut-off $\alpha'$, the terms of order $\alpha'$ in the above expansions do not contribute to the zero-slope limit for $z_i \neq z_j$. One must be careful, however, since these functions each have a single pole in $z_i - z_j$ with residue $-1/2i\pi$. The poles do contribute to the zero-slope limit as $z_i \rightarrow z_j$. We therefore find a second set of 1PI diagrams that comes from the $(y_i - y_j)$ in $\partial_i G^{ij}_{11}$ and the $1/2$ in $S^{ij}_{10}$. This set of 1PI graphs has the expected logarithmic dependence on the cut-off but contains unexponentiated Feynman parameters in the integrand corresponding to derivative couplings in the low-energy field theory. The zero-slope limit of the terms that generate these graphs can be taken as before, and we do not repeat the procedure here.
The 1PR graphs which appear in the zero-slope limit of the five-graviton amplitude come from the absolute value squared of the single poles in $z_i - z_j$. For these pole terms we cannot set $\alpha'$ to zero in the factor $\prod_{ij} \tilde{\chi}_{ij}^{-1/2}$ without introducing a logarithmic divergence in the integration over the coordinates of one of the vertex operators. No cross terms between different single poles contribute as $\alpha' \to 0$ since these do not introduce any singularities and therefore vanish. We must integrate the un-rescaled $z_i$ vertex coordinate over a small disk about $z_j$ in order to pick up the pole in $\alpha' p_{ij}$ and then take the zero-slope limit as before. For example, consider the case where $z_{i_2} \to z_{i_1}$ for some $(i_1, i_2)$. In practice, we can formally include all of the twelve orderings at once. We fix $z_{i_1}$ to be anywhere on the torus due to translational invariance (instead of $z_5 = \tau$) and integrate $z_{i_2}$ over a disk of radius $\epsilon$ about that point. For the case of a product of five fermionic Green functions, when one of them is near the pole the sum over spin structure to eliminate the other four can be performed using eq. (19). In order to include all possible channels in the zero-slope limit each contribution to the pole must be considered separately.

The double pole introduced by eq. (24) will not contribute since the sum over spin structures with only three propagators vanishes. The integral over the phase angle ensures that only a $1/r^2$ will survive from the product $K_{\mu_1\nu_2\mu_3\mu_4\mu_5} \bar{K}_{\nu_1\mu_2\mu_3\mu_4\mu_5}$ in eq. (21) where $r = |z_{i_2} - z_{i_1}|$. We then find up to an overall kinematic factor tenth order in the external momenta

$$\lim_{\alpha' \to 0} \Gamma_5^{(1PR)} = \kappa^5 \int_{\mathcal{F}} \int_{\mathcal{T}_2} \left[ \frac{d^2 z_{i_3} d^2 z_{i_4} d^2 z_{i_5}}{\tau_2 \tau_2 \tau_2} \right]$$
$$\times \frac{1}{\tau_2} \int_0^{2\pi} d\phi \int_0^\epsilon dr \, r^{-1} \exp \left\{ -i\pi \alpha' p_{i_1} \cdot p_{i_2} G_{11}^{i_1i_2} \right\}$$
$$\times \exp \left\{ -i\pi \alpha' \left[ p_{i_3} \cdot p_{i_4} G_{11}^{i_3i_4} + p_{i_5} \cdot p_{i_6} G_{11}^{i_5i_6} + p_{i_7} \cdot p_{i_8} G_{11}^{i_7i_8} + \sum_{n=3}^5 \right] \right\} + \ldots$$

We now expand the rest of the integrand in $r$ keeping only the lowest order term. The bosonic Green function satisfies $G_{11}^{i_1i_2} = -(1/i\pi) \log r + \ldots$ and
we get

\[
\lim_{\alpha' \to 0} \Gamma_5^{(1PR)} = 2\pi \kappa^5 \left[ -\frac{2}{\alpha' \mathbf{p}_{i1} \mathbf{p}_{i2}} \right] \int F \left( \frac{\tau_2}{\tau_2} \right)^2 \int \left[ \frac{d^2 z_{i4} d^2 z_{i5} d^2 z_{i5}}{\tau_2 \tau_2 \tau_2} \right] \\
\times \exp \left\{ -i\pi \alpha' \left[ \mathbf{p}_{i5} \cdot \mathbf{p}_{i4} G^{i5i4}_{11} + \mathbf{p}_{i5} \cdot \mathbf{p}_{i3} G^{i5i3}_{11} + \mathbf{p}_{i4} \cdot \mathbf{p}_{i3} G^{i4i3}_{11} + (\mathbf{p}_{i1} + \mathbf{p}_{i2}) \cdot \sum_{n=3}^{5} \mathbf{p}_{in} G_{11}(z_{in}) \right] \right\} + \ldots
\]  

(37)

The above expression contains a massless pole times the scalar part of the four graviton amplitude with one external line having momenta \((\mathbf{p}_{i1} + \mathbf{p}_{i2})\). When all of the pieces of the five-graviton amplitude are included, one would find all possible factorized channels consisting of a tree level three-graviton amplitude (just a kinematic factor) times the one-loop four-graviton amplitude. The rest of the zero-slope limit can be performed exactly as in ref. \[1\] giving 1PR Feynman graphs with a massless single pole.

## 5 Conclusion

We have presented a concise and explicit calculation of the even spin structure contribution to the one-loop \(N\)-graviton scattering amplitude in the type-II superstring theory. In this calculation we chose to use the path integral formulation of the NSR string since this approach leads directly to a covariant result expressed in terms of Jacobi theta functions whose properties under modular transformations are well known. Limited superspace formalism was used in the calculation since there is no topological obstruction to the superconformal gauge for the one-loop even spin structure contribution.

For \(N \leq 5\) the even spin structure contribution gives the complete amplitude, and the amplitudes for \(N \leq 3\) vanish due to a Riemann identity for theta functions. The \(N = 4\) result is easily reproduced, and its modular invariance is trivial to verify. It must be separated into three terms, each of which is convergent in a different kinematic region. For \(N = 5\) we find in direct analogy that the final result must be divided into twelve kinematic regions where each piece can be analytically continued separately in the external momenta. Chiral world-sheet Green functions present in the integrand of the five-graviton amplitude produce massless single-particle poles in the limit when the coordinates of two vertex operators come close together.
In sect. 4 we examined the form of the zero-slope limit of the one-loop five-graviton amplitude and identified three types of contributions. The first came from terms with two powers of loop momenta $q_\mu q_\nu$, which are proportional to the spacetime metric $g_{\mu\nu}$ after integration. All of these terms have the form of a kinematic factor times the scalar integral found in eq. (28). As $\alpha' \to 0$ these terms generate pentagon Feynman diagrams in the proper-time representation for a scalar theory with cubic interactions. The square of the proper-time variable $\tau_2$ occurring in the denominator of the integrand gives the same degree of divergence in the UV limit as that found in the 1PI graphs appearing in the zero-slope limit of the four-graviton amplitude. We proved that the modular invariance of the even spin structure contribution to the N-graviton amplitude guarantees that this set of 1PI graphs will have the same UV degree of divergence for all $N$. For $N > 5$, however, there will also be a parity violating contribution coming from the odd spin structure. It does not mix with the even spin structure pieces under modular transformations, and therefore could have some different dependence on $\tau_2$ in the zero-slope limit.

The second contribution to the zero-slope limit of the five-graviton amplitude comes from the non-singular part of $\partial_i G^{ij}_{11}$ and $S^{ij}_{10}$. These terms are proportional to the same scalar integral found in eq. (28) but with only a single power of $\tau_2$ in the denominator of the integrand. They also give pentagon diagrams and have the expected logarithmic divergence in the cutoff $\alpha'$. Some of the graphs also contain unexponentiated powers of the Feynman parameters in the integrand corresponding to derivative couplings in the low-energy field theory.

The third contribution to the low-energy limit for $N = 5$ comes from the single poles in the chiral Green functions as $z_i \to z_j$. By integrating over a small region about these poles we obtain 1PR Feynman diagrams with a massless single-particle pole in the zero-slope limit. The residue of each pole is a kinematic factor times a sum of four-particle box graphs. Although no 1PR graphs are present in the zero-slope limit of the one-loop four-graviton amplitude due to a simplification of its integrand resulting from a Riemann identity for theta functions, we expect that they will contribute for all $N \geq 5$. Since each momentum invariant must multiply an $\alpha'$ on dimensional grounds, one cannot take the zero-slope limit without introducing a single-particle pole as if the sum of two external momenta are on the graviton mass shell. Therefore, if massless poles exist in the full superstring amplitude, then 1PR
Feynman diagrams must contribute directly in the zero-slope limit and are not generated by duality.

I would like to thank W.I. Weisberger for useful discussions.

A Appendix

Here we state some properties of the world-sheet Green functions and the Jacobi theta functions from which they are defined. The world-sheet fermions $\psi(z)$ carry a spin structure $[\nu_a, \nu_b]$ which can take on values $[\frac{1}{2}, \frac{1}{2}]$, $[\frac{1}{2}, 0]$, $[0, \frac{1}{2}]$, and $[0, 0]$. Their periodicity on the torus is defined by

$$
\psi(z + 1) = e^{2i\pi (\nu_a - \frac{1}{2})} \psi(z)
$$

$$
\psi(z + \tau) = e^{2i\pi (\nu_b - \frac{1}{2})} \psi(z). \tag{38}
$$

Theta functions with characteristics corresponding to these spin structures are defined by

$$
\Theta_{[\nu_a, \nu_b]}(z|\tau) = \sum_{n=-\infty}^{\infty} \exp \left\{ 2i\pi \left[ \frac{1}{2}(n + \nu_a)^2 \tau + (n + \nu_a)(z + \nu_b) \right] \right\}. \tag{39}
$$

We also have the Dedekind eta function given by

$$
\eta(\tau) = \exp \left[ i\pi \frac{\tau}{12} \right] \prod_{n=1}^{\infty} \left( 1 - e^{2i\pi \tau n} \right). \tag{40}
$$

Superstring scattering amplitudes are invariant under the modular transformations $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$ in conjunction with the conformal change of coordinates $z \rightarrow z c\tau + d$, where $a, b, c, d$ are integers. These transformations are generated by $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$. For the properties under these transformations of the Dedekind eta function and the Jacobi theta functions from which the world-sheet Green functions are defined see ref. [9]. The bosonic world-sheet Green function $G_{11}(z|\tau)$ given in eq. (7) can then be shown to satisfy $G_{11}(\frac{a\tau + b}{c\tau + d}) = G_{11}(z|\tau)$. The even spin structure fermionic Green functions $S_{ab}(z|\tau)$ satisfy

$$
S_{10}(z|\tau + 1) = S_{10}(z|\tau)
$$

$$
S_{00}(z|\tau + 1) = S_{01}(z|\tau)
$$

$$
S_{01}(z|\tau + 1) = S_{00}(z|\tau), \tag{41}
$$
\[ S_{10} \left( \frac{z}{\tau} - \frac{1}{\tau} \right) = \tau S_{01}(z|\tau) \]
\[ S_{00} \left( \frac{z}{\tau} - \frac{1}{\tau} \right) = \tau S_{00}(z|\tau) \]
\[ S_{01} \left( \frac{z}{\tau} - \frac{1}{\tau} \right) = \tau S_{10}(z|\tau). \] (42)

One can prove a form of the Riemann identity for theta functions which can be written
\[ \sum_{\nu} (-)^{a+b} 4 \prod_{i=1}^{4} \Theta_{ab}(z_i|\tau) = 2 \prod_{i=1}^{4} \Theta_{11}(w_i|\tau), \] (43)
where
\[ w_1 = \frac{1}{2}(z_1 + z_2 + z_3 + z_4) \]
\[ w_2 = \frac{1}{2}(z_1 + z_2 - z_3 - z_4) \]
\[ w_3 = \frac{1}{2}(z_1 - z_2 + z_3 - z_4) \]
\[ w_4 = \frac{1}{2}(z_1 - z_2 - z_3 + z_4). \] (44)

Using the definition of \( S^{ij}_{ab} \) given in eq. (7) and the fact that \( \Theta_{11}(0|\tau) = 0 \) one can then show
\[ \sum_{a \times b = 0} (-)^{a+b} \left\{ \frac{\Theta_{ab}(0|\tau)}{[\eta(\tau)]^3} \right\}^4 S_{ab}^{11} S_{ab}^{12} = 0 \]
\[ \sum_{a \times b = 0} (-)^{a+b} \left\{ \frac{\Theta_{ab}(0|\tau)}{[\eta(\tau)]^3} \right\}^4 S_{ab}^{11} S_{ab}^{12} S_{ab}^{13} S_{ab}^{11} = 0 \]
\[ \sum_{a \times b = 0} (-)^{a+b} \left\{ \frac{\Theta_{ab}(0|\tau)}{[\eta(\tau)]^3} \right\}^4 S_{ab}^{11} S_{ab}^{12} S_{ab}^{13} S_{ab}^{14} S_{ab}^{11} = -1. \] (45)

We may obtain representations of the chiral Green functions as infinite sums which are useful for calculating the low-energy limit of superstring scattering amplitudes. We begin with the mode expansion for the Green
function and then perform one of the two sums using the Sommerfeld-Watson transformation.

\[
\partial \left[ G_{11} + \frac{1}{4i\tau_2} (z - \bar{z})^2 \right] = \frac{1}{2} + \frac{e^{2i\pi z}}{1 - e^{2i\pi z}} + \sum_{m=1}^{\infty} \left[ \frac{e^{2i\pi m(\tau + z)} - e^{2i\pi m(\tau - z)}}{1 - e^{2i\pi m\tau}} \right],
\]
valid for \(0 \leq \text{Im} \ z \leq \text{Im} \ \tau\).

\[
S_{01}(z) = \frac{e^{i\pi z}}{1 - e^{2i\pi z}} + \sum_{m=1}^{\infty} \left[ e^{2i\pi (m-1/2)(\tau + z)} - e^{2i\pi (m-1/2)(\tau - z)} \right] \frac{1}{1 - e^{2i\pi (m-1/2)\tau}},
\]
valid for \(0 \leq \text{Im} \ z \leq \text{Im} \ \tau\).

\[
S_{00}(z) = \frac{e^{i\pi z}}{1 - e^{2i\pi z}} - \sum_{m=1}^{\infty} \left[ e^{2i\pi (m-1/2)(\tau + z)} - e^{2i\pi (m-1/2)(\tau - z)} \right] \frac{1}{1 + e^{2i\pi (m-1/2)\tau}},
\]
valid for \(0 \leq \text{Im} \ z \leq \text{Im} \ \tau\).

\[
S_{10}(z) = \frac{1}{\tau} \frac{e^{i\pi z/\tau}}{1 - e^{2i\pi z/\tau}} - \frac{1}{\tau} \sum_{m=1}^{\infty} \left[ e^{2i\pi (m-1/2)z/\tau} - e^{-2i\pi (m-1/2)z/\tau} \right] \frac{1}{1 - e^{2i\pi (m-1/2)/\tau}},
\]
valid for \(\frac{\text{Re} \ z}{\tau_2} \leq \text{Re} \ z \leq \frac{\text{Im} \ z}{\tau_2} + 1\).
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