Bogomol’nyi Equations of Maxwell-Chern-Simons Vortices
from a generalized Abelian Higgs Model

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Abstract

We consider a generalization of the abelian Higgs model with a Chern-Simons term by modifying two terms of the usual Lagrangian. We multiply a dielectric function with the Maxwell kinetic energy term and incorporate nonminimal interaction by considering generalized covariant derivative. We show that for a particular choice of the dielectric function this model admits both topological as well as nontopological charged vortices satisfying Bogomol’nyi bound for which the magnetic flux, charge and angular momentum are not quantized. However the energy for the topolgical vortices is quantized and in each sector these topological vortex solutions are infinitely degenerate. In the nonrelativistic limit, this model admits static self-dual soliton solutions with nonzero finite energy configuration. For the whole class of dielectric function for which the nontopological vortices exists in the relativistic theory, the charge density satisfies the same Liouville equation in the nonrelativistic limit.

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I. INTRODUCTION

Recently, the charged vortex solutions \cite{1} in the abelian Higgs model with a Chern-Simons (CS) term have received considerable attention in the literature because of their possible relevance in context of cosmic strings as well as planar condensed matter system. More recently, relativistic model in which the gauge field is solely given by the CS term and the usual $|\phi|^4$ potential is replaced by $|\phi|^6$ -type potential have been considered \cite{2}. For a specific choice of the coupling constant, minimum energy topological \cite{2,3} as well as nontopological vortex \cite{4,4} configuration arise in this model that satisfy certain first order differential equation consistent with the second order equations of motion. These first order differential equations are known as Bogomol’nyi equations \cite{5}. Subsequently, Lee et. al. \cite{7} showed that even the usual $|\phi|^4$ -type abelin Higgs model (with a specific choice of the coupling constant) with both Maxwell and the CS (MCS) term admits minimum energy vortex solutions satisfying Bogomol’nyi bound. However this was possible by the addition of an extra neutral scalar in the theory. Besides the Gauss Law equation is still second and not first order in nature.

In the nonrelativistic (NR) limit, the pure CS theory provides a second quantized description of point particles moving(nonrelativistically) in $\delta$- function potentials and interacting via CS term \cite{8}. The matter equation of motion becomes a gauged nonlinear Schrodinger equation. For static self-dual soliton solution of this theory, the charge density solves Liouville equation. The nonrelativistic limit of MCS theory also have been discussed and shown to admit static self-dual soliton solutions \cite{9}. However unlike corresponding pure CS case, no analytical solution can be obtained in this case. In the nonrelativistic limit, self-dual solitons for both the pure CS and MCS theory are of zero energy configuration.
In this paper we consider a generalization of the abelian Higgs model with a CS term, in which we have a “dielectric function” multiplying the Maxwell term and an extra gauge invariant nonminimal contribution to the covariant derivative. Specifically we are interested in the case where dielectric function depends on the Higgs field. We show that for a class of dielectric function and for a specific choice of the coupling constant this model admits both topological and nontopological static minimum energy charged vortex solutions. The Gauss law equation for these vortex solutions are of first order in nature and not second order like the other model for self-dual MCS vortices considered by Lee et. al [7]. Remarkably enough, for a particular choice of the dielectric function the first order Bogomol’nyi equations obtained in this generalized Maxwell-Chern-Simons(GMCS) theory can be mapped into the corresponding equations of pure CS vortices [2] upto a scale transformation of the variables. Also the Bogomol’nyi equations of Torres model [10] where an anomalous magnetic moment contribution plays important role are obtained as a special case.

The general feature of topological vortices, both neutral [8] and charged, in gauge theories known to date is that the magnetic flux of the vortices are necessarily quantized. Furthermore, the vortex solution in each topological sector is nondegenerate. However, we show in this paper that for a very special choice of the dielectric function, this model admits topological charged vortex solutions obeying Bogomol’nyi bound for which the magnetic flux, and hence the charge and the angular momentum need not necessarily be quantized even though the energy is quantized. The topological vortex solutions are infinitely degenerate in energy in each sector and these degenerate vortex solutions in a particular sector differ from each other by flux, charge and the angular momentum. In particular, the topological vortex solutions in each sector are characterized by energy $E = \frac{\pi \kappa^2 n}{2e^2}$, flux $\Phi = \frac{2\pi (n-\beta)}{e}$, angular momentum $J = \frac{\pi \kappa (\beta^2 - n^2)}{e^2}$ and charge $Q = -\kappa \Phi$.
(where $n$ is the winding number, $\kappa$ is the coefficient of the $\text{CS}$ term, $\epsilon$ is a coupling constant and $\beta$ is a parameter describing the solutions). We derive the sum rules for these topological vortices using which we find that $\beta$ is restricted as, $\frac{1}{4} < \beta < n$.

We also study the nontopological vortex solutions present in this model for the whole class of the dielectric function. The energy, magnetic flux, charge and angular momentum of these nontopological vortices are not quantized and determined in terms of a constant which is the asymptotic value of the gauge field (with our choice of the ansatz) at large distances. We also derive sum rules for these nontopological vortices. Using these sum rules we show that lower bound on the energy, magnetic flux, charge and angular momentum can be obtained for some particular choices of the dielectric function.

Furthermore, we study the $NR$ limit of our model and obtain static self-dual soliton solutions. However, self-dual soliton solutions in the nonrelativistic limit is possible only when a particular relation between two parameters (one characterizing the dielectric function and the other characterizing the scalar potential), to be discussed later, is satisfied. This particular constraint, in order to have nonrelativistic self-dual soliton solutions, is not present in the relativistic theory. The interesting feature of these $NR$ soliton solutions is that unlike corresponding pure $\text{CS}$ [8] and $\text{MCS}$ [9] case they saturate the lower bound at some finite nonzero value of the static energy functional. For the whole class of the dielectric function for which nontopological vortex solutions is possible in the relativistic theory, the charge density in the nonrelativistic theory satisfies the same Liouville equation which is completely integrable.

We organize the paper as following. In subsec. II.A we set up the the relativistic model describing in detail the motivation behind considering such a Lagrangian. We also obtain second order equations of motion and the expression for the energy functional. The Bogomol'nyi equations are obtained in subsec. II.B. In subsec. II.C we
study both the topological and nontopological vortex solutions along with their physical properties. We also show that our model admits a new type of topological vortex solutions for which the magnetic flux, charge and the angular momentum is not quantized and these solutions are infinitely degenerate in each sector. In subsec. II.D we rigorously show that a bound can be put on magnetic flux, charge and angular momentum for both nontopological as well as topological vortices by deriving sum rules. We take nonrelativistic limit of the relativistic model and set up the equations of motion as well as the energy functional in subsec. III.A. In subsec. III.B we obtain static self-dual soliton solutions in the nonrelativistic model. Finally we conclude describing our results in brief and indicating future directions in sec.IV.

II. RELATIVISTIC THEORY

A. The Model

We first define our theory by writing the Lagrangian density

\[ \mathcal{L} = -\frac{1}{4}G(\phi)F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi(D^\mu\phi)^* + \frac{\kappa}{4}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} - V(\phi) \]  

(1)

where \( G(\phi) \) is the scalar field dependent dielectric function and the generalized covariant derivative is given by

\[ D_\mu\phi = (\partial_\mu - i e A_\mu - \frac{i g}{4} G(\phi) \epsilon_{\mu\nu\lambda} F^{\nu\lambda})\phi \]  

(2)

Our notation is \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \mu = (0, 1, 2), g_{\mu\nu} = \text{diag}(1, -1, -1) \). We choose \( c = \hbar = 1 \) throughout this paper except in section III where we discuss the NR limit of this model and keep \( c \) explicitly.

With a choice of symmetry breaking potential and using the covariant derivative \( (2) \), both mass term for the gauge field and the CS term can be generated via spontaneous
symmetry breaking (SSB) mechanism \[12\]. In fact the nonminimal part of the covariant derivative generates the CS term after SSB. Also it is interesting to note that for \(G(| \phi |) = 1\) the nonminimal part can be interpreted as the anomalous magnetic moment \[13\]. This is due to the fact that in 2+1 spacetime Dirac matrices obey \(SO(2,1)\) algebra and Pauli coupling can be incorporated in the generalized covariant derivative even for the scalar field without any reference to the spin degrees of freedom.

The modification to the Maxwell kinetic term can be viewed as an effective action for a system in a medium described by a suitable dielectric function. In fact, soliton bag models \[14\] of quarks and gluons are described by Lagrangian, where such a dielectric function is multiplied with the Maxwell kinetic energy term. Also in certain supersymmetric theories with a non-compact gauge group \[15\], such a nonminimal kinetic term was necessary in order to have a sensible gauge theory. In the context of vortex solutions, this non-minimal coupling is interesting because of the existence of Bogomol’nyi bounds for a more general form of the scalar potential \[11,16\]. Lee et. al. \[11\] considered the Lagrangian (1) with \(\kappa = 0\) and without the nonminimal contribution to the covariant derivative (2) and have shown that the model admits topological as well as nontopological static self-dual neutral vortex solutions. Torres \[10\] considered the Lagrangian (1) with \(G(| \phi |) = 1\) and obtained static minimum energy nontopological vortex configuration for a simple \(| \phi |^2\) potential. As a natural extension, we consider the effect of both the dielectric function and the generalized covariant derivative for arbitrary \(G(| \phi |)\) and study the Bogomol’nyi limit for topological as well as nontopological vortex solution.

The equations of motion for the Lagrangian (1) are

\[
D_\mu D^\mu \phi = -2 \frac{\partial V(| \phi |)}{\partial \phi^*} - \frac{1}{2} \frac{\partial G(| \phi |)}{\partial \phi^*} F_\mu F^{\mu \nu} - \frac{g}{2e} \frac{\partial G(| \phi |)}{\partial \phi^*} \epsilon^{\mu \nu \lambda} J_\mu F_{\nu \lambda} \tag{3}
\]
\[ \epsilon_{\mu\nu\lambda} \partial^\mu \left[ G(|\phi|) \left( F^\lambda + \frac{g}{2e} J^\lambda \right) \right] = J_\nu - \kappa F_\nu \]  

(4)

where the dual field \( F_\mu \) and the conserved current \( J_\mu \) is

\[ F_\mu = \frac{1}{2} \epsilon_{\mu\nu\alpha} F^{\nu\alpha} \]  

(5)

\[ J_\mu = - \frac{i\epsilon}{2} \left[ \phi^* D_\mu \phi - \phi (D_\mu \phi)^* \right] \]  

(6)

The energy momentum tensor \( T_{\mu\nu} \) is obtained by varying the curved space form of the action with respect to the metric

\[ T_{\mu\nu} = G(|\phi|) \left[ 1 - \frac{g^2}{4} G(|\phi|)|\phi|^2 \right] [F_\mu F_\nu - \frac{1}{2} g_{\mu\nu} F^\alpha F_\alpha] \]

\[ + \frac{1}{2} \left[ \nabla_\mu \phi (\nabla_\nu \phi)^* + \nabla_\nu \phi (\nabla_\mu \phi)^* \right] \]

\[ - g_{\mu\nu} \left[ \frac{1}{2} |\nabla_\alpha \phi|^2 - V(|\phi|) \right] \]

(7)

where \( \nabla_\alpha = \partial_\alpha - i e A_\alpha \) includes only the gauge potential contribution.

**B. Bogomoln’nyi Equation**

Now notice that the first order equation

\[ J_\nu = k F_\nu \]  

(8)

solves the gauge field equations (4) for arbitrary \( G(|\phi|) \) provided the following relation among the coupling constant holds

\[ g = - \frac{2e}{\kappa} \]  

(9)

Like pure CS Higgs theory [2], the zero component of (8), i.e, Gauss law implies that the solution with charge \( Q \) also carries magnetic flux \( \Phi = - \frac{Q}{\kappa} \). It should be noted that
for $G(|\phi|) \neq 0$ equation (8) is essentially different from that of corresponding equation for pure CS vortices as $D_\mu$ receives contribution from the nonminimal part also. In particular, using equations (6) and (9) the gauge field equation (8) can be rewritten as

$$\kappa \ F_\mu = J_\mu = (1 - \frac{e^2}{\kappa^2} G(|\phi|)|\phi|^2)^{-1} \tilde{J}_\mu$$

(10)

where $\tilde{J}_\mu$ receives only minimal contribution

$$\tilde{J}_\mu = -\frac{ie}{2} [\phi^* \nabla_\mu \phi - \phi (\nabla_\mu \phi)^*]$$

(11)

We notice that for the choice of $G(|\phi|) = \frac{e^2}{\kappa^2} (1 - C_0)|\phi|^2$, equations (10) is also the gauge field equation for the pure CS Higgs theory except the overall constant factor $1/C_0$ multiplying with $\tilde{J}_\mu$. This implies that the electric and magnetic field components of pure CS Higgs theory and our model with the above choice of the dielectric function differ by the scale factor $1/C_0$. For $C_0 = 1$, i.e, $G = 0$, our model reduces to the pure CS Higgs theory as can be seen from (1) and (2). It may therefore be worthwhile to consider the possibility of obtaining Bogomol’nyi limit, for the choice $G(|\phi|) = \frac{e^2}{\kappa^2} (1 - C_0)|\phi|^2$. However we are interested in a more general class of dielectric function and the scalar potential. So we keep $G(|\phi|)$ arbitrary unless mentioned otherwise and obtain the results for above mentioned choice of $G(|\phi|)$ as a special case. Using equation (8) and (9), we rewrite the scalar field equation of motion (3)

$$D_\mu D^\mu \phi = -2 \frac{\partial V(|\phi|)}{\partial \phi^*} + \frac{1}{2} \frac{\partial G(|\phi|)}{\partial \phi^*} F_{\mu\nu} F^{\mu\nu}$$

(12)

Now we seek vortex solutions in the system described by the equations (10) and (12), when the relation (9) is satisfied. We choose the ansatz for rotationally symmetric solution of winding number $n$

$$A^r(r) = -\theta \frac{a(r)}{er}, A_0(r) = \frac{k}{e} h(r), \phi(r) = \frac{k}{e} f(r)e^{-in\theta}$$

(13)
After substituting the ansatz (13), equations (10) and (12) can be reduced to

\[ \frac{1}{r} [1 - G(f)f^2]a' + k^2 f^2 h = 0 \] (14)

\[ r[1 - G(f)f^2]h' + af^2 = 0 \] (15)

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{k^2 f}{(1 - G(f)f^2)^2} \left[ f \partial G(f) h^2 - \frac{a^2}{k^2 r^2} \right] = \frac{e^2}{k^2} \partial V \] (16)

where prime denotes differentiation with respect to r. The energy functional that is obtained from equation (7) for the ansatz (13) is

\[ E = \frac{k^2}{2e^2} \int d^2 x \left\{ \left[ f' \pm \frac{fa}{r(1 - G(f)f^2)} \right]^2 + \frac{f^2 a^2}{r^2(1 - G(f)f^2)} \left( \frac{a'}{kr} \right)^2 \left( 1 - G(f)f^2 \right) + \frac{2e^2}{k^2} V(f) \right\} \] (17)

After eliminating \( h(r) \) and \( h'(r) \) from equation (17) using equations (14) and (15) the energy functional can be written solely in terms of \( f(r) \) and \( a(r) \)

\[ E = \frac{k^2}{2e^2} \int d^2 x \left\{ (f')^2 + \frac{f^2 a^2}{r^2(1 - G(f)f^2)} + \left( \frac{a'}{kr} \right)^2 \left( 1 - G(f)f^2 \right) + \frac{2e^2}{k^2} V(f) \right\} \] (18)

Let us define a function \( W(f) \) shortly to be fixed in terms of \( G(f) \). The energy functional (18) can be rearranged using Bogomol’nyi trick as,

\[ E = \frac{k^2}{2e^2} \int d^2 x \left\{ \left[ f' \pm \frac{fa}{r(1 - G(f)f^2)} \right]^2 + \frac{1 - G(f)f^2}{f^2} \left( \frac{a'}{kr} \right)^2 + W^2 + \frac{2e^2}{k^2} V(f) \right\} \pm \frac{2\pi \kappa^2}{\sqrt{C_0(1 + \gamma)}e^2} [R(\infty) - R(0)] \] (19)

where

\[ R(r) = \frac{\sqrt{C_0(1 + \gamma)}}{\kappa f^2} (1 - G(f)f^2)^{1/2} aW \] (20)

and \( W(f) \) is determined by
\[
\frac{d}{dr}\left[\frac{1 - G(f)f^2}{\kappa f^2}\right] W = -\frac{f' f}{(1 - G(f)f^2)^{\frac{3}{2}}} \tag{21}
\]

However for arbitrary \( G(f) \) no analytical solution can be obtained for \( W(f) \). For simplicity, we choose \( G(f) \) to be
\[
G(f) = f^{-2} - C_0 f^{-2} (1 - f^2)^{1-\gamma} \tag{22}
\]
where \( \gamma \) is a real number and \( C_0 \) is a positive constant. The function \( W(f) \) for this choice of \( G(f) \) follows from equation (21)
\[
W(f) = \frac{\kappa f^2}{C_{0}^{\frac{4}{2}}(1 + \gamma)} (1 - f^2)^{\frac{3\gamma - 1}{2}} \tag{23}
\]
For this choice of \( G(f) \), and hence \( W(f) \), the energy functional can be written as
\[
E = \frac{k^2}{2e^2} \int d^2x \left\{ \left[ f' \pm \frac{f a}{C_0^{\frac{1}{2}} r} \right] (1 - f^2)^{\frac{3\gamma - 1}{2}} \right\}^2
+ \frac{C_0 f^{-2}(1 - f^2)^{1-\gamma} \left[ \frac{a'}{\kappa r} \mp \frac{\kappa}{C_0^{\frac{3}{2}}(1 + \gamma)} f^2(1 - f^2)^{\frac{3\gamma - 1}{2}} \right]^2}
+ \frac{2e^2}{k^2} V(f) - \frac{\kappa^2}{C_0^{\frac{4}{2}}(1 + \gamma)^2} f^2(1 - f^2)^{2\gamma} \right\} \pm \frac{2\pi \kappa^2}{C_0^{\frac{1}{2}}(1 + \gamma) e^2} [R(\infty) - R(0)] \tag{24}
\]
where
\[
R(r) = a(1 - f^2)^{\frac{1 + \gamma}{\gamma}} \tag{25}
\]
When \( \gamma \) is odd integer there is a lower bound on the energy provided we choose the scalar potential
\[
V(f) = \frac{\kappa^4}{2C_0^{\frac{5}{2}} e^2} f^2(1 - f^2)^{2\gamma} \tag{26}
\]
The lower bound on the energy also exists for arbitrary \( \gamma \) (and above choice of \( V(f) \)) provided \( 0 \leq f \leq 1 \). The lower bound on energy is saturated when the following Bogomol’nyi equations are satisfied
\[ f' = \pm \frac{f a}{C_0^\frac{1}{2} r} (1 - f^2)^{\frac{\gamma - 1}{2}} \]  

(27)

\[ \frac{\alpha'}{\kappa r} = \pm \frac{\kappa}{C_0^{\frac{1}{2}} (1 + \gamma)} f^2 (1 - f^2)^{\frac{\gamma - 1}{2}} \]

(28)

One can easily check that these two first order differential equations are consistent with the second order differential equation (16). At the Bogomol’nyi limit, two diagonal elements of the energy momentum tensor other than \( T_{00} \), i.e, \( T_{rr} \) and \( T_{\theta\theta} \) vanishes. The off-diagonal element \( T_{t\theta} \) is however nonvanishing

\[ T_{t\theta} = -\frac{\kappa}{e^2 r^2} \alpha' a \]

implying that the solution to the Bogomol’nyi equations carry finite angular momentum for well behaved \( a(r) \)

\[ J = \frac{\pi \kappa}{e^2} [a(\infty)^2 - a(0)^2] \]

(30)

Note that the following scale transformation

\[ a \to C_0^{\frac{1}{2}} a, \quad r \to C_0 r, \quad f \to f \]

(31)

eliminates \( C_0 \) from the equations (27) and (28). The decoupled second order equation for the \( f(r) \) is

\[ f'' + \frac{f'}{r} + \frac{\kappa^2}{\gamma + 1} f^3 (1 - f^2)^{2\gamma - 1} - \frac{f'^2}{f} + (\gamma - 1) \frac{f'^2 f}{1 - f^2} = 0 \]

(32)

where the scale transformation (31) have been performed. Equation (32) is highly nonlinear and we do not have the analytical solution for it. However we can obtain asymptotic as well as numerical solution for this equation. We do so while discussing vortex solution for different choices of \( \gamma \).
C. Vortex Solution

In this section we discuss asymptotic vortex solutions for well-behaving scalar potential. We choose either $\gamma = 0$ or $\gamma$ odd so that the scalar potential is bounded from below and $f(r)$ is not restricted.

I. $\gamma = 0$, i.e., $G = (1 - C_0 + C_0 f^2)f^{-2}$: The Bogomol’nyi bound arises when the scalar field mass $m$ and the CS coefficient $\kappa$ are related as, $m = \frac{\kappa}{C_0}$. For $C_0 = 1$, i.e., $G = 1$; our model reduces to that of considered by Torres [10]. For arbitrary $C_0 (C_0 > 0)$, Bogomol’nyi equations of our model can be mapped into corresponding equations of Torres model up to a scale transformation of the variables. For this case only nontopological vortices exist. The flux, charge, and angular momentum of these nontopological vortices are independent of $C_0$ and same for both Torres model as well as our model. However, the energy $E = \frac{\kappa^2}{\sqrt{C_0 e}} \Phi$ is parametrized by $C_0$ and distinguishes between Torres model and our model.

II. $\gamma = 1$, i.e., $G = (1 - C_0)f^{-2}$: For $C_0 = 1$, i.e., $G = 0$, our model reduces to pure CS Higgs theory. Equations (26) to (28) reproduce the Bogomol’nyi equations as well as the potential of pure CS Higgs theory. Even for arbitrary $C_0 (C_0 > 0)$ when Maxwell term is present, these Bogomol’nyi equations of GMCS are exactly same as those of corresponding pure CS theory [3] provided the scale transformation (31) is performed. This implies that the Bogomol’nyi equations of GMCS Higgs theory can be mapped into the Bogomol’nyi equations of pure CS Higgs theory up to a scale transformation of the variables. Hence both topological as well as nontopological vortex solutions also exist in this case. The topological vortex solutions are characterized by the flux $\Phi_{\text{top}} = \frac{2\pi}{e} n$, the energy $E_{\text{top}} = \frac{\kappa^2}{\sqrt{C_0 e}} \Phi_{\text{top}}$, the charge $Q_{\text{top}} = -\kappa \Phi_{\text{top}}$, and the angular momentum $J_{\text{top}} = -\frac{\pi \kappa^2 e n^2}{e^2}$. The nontopological vortex solutions are
characterized by the flux $\Phi^{\text{nontop.}} = \frac{2\pi}{e}(n + \alpha)$, the energy $E^{\text{nontop.}} = \frac{\kappa^2}{\sqrt{\epsilon \mu_e}} \Phi^{\text{nontop.}}$, the charge $Q^{\text{nontop.}} = -\kappa \Phi^{\text{nontop.}}$ and the angular momentum $J^{\text{nontop.}} = \frac{\pi \kappa}{e}(\alpha^2 - n^2)$, where $\alpha$ is a parameter describing the nontopological solutions. We shall see in subsec. II.D that $\alpha \geq n + 2$.

III. $\gamma > 1$: The finiteness of the energy can be ensured by requiring either (i) $a(\infty) = -\alpha$, $f(\infty) = 0$ or (ii) $a(\infty) = \beta$, $f(\infty) = 1$. Here $\alpha$ and $\beta$ are two real positive constants. Further, on demanding nonsingular field variables, boundary condition at the origin gets fixed as (iii) $a(0) = n$, $f(0) = 0$ for $n \neq 0$, (iv) $a(0) = 0$, $f(0) = G_0$ for $n = 0$. Since the solutions for $n$ and $-n$ are related by the transformation $f \rightarrow f$, $a \rightarrow -a$; we consider only the case $n \geq 0$, further without any loss of generality we choose $C_0 = 1$. Notice that for $\gamma > 1$, the constant $C_0$ does not play any special role. The boundary condition (i) corresponds to nontopological vortex solution while the boundary condition (ii) corresponds to topological vortex solution which we now discuss in some detail.

(a) Nontopological vortex solutions: For nontopological vortices, at large distances $f(r) \rightarrow 0$ and $a(r) \rightarrow -\alpha$. The power series solution of equations (27), (28), (32) can be shown to be

$$f(r) = \frac{A}{(kr)^\alpha} - \frac{A^3}{4(\gamma + 1)(\alpha - 1)^2(\kappa r)^{3\alpha - 2}} + O((\frac{1}{\kappa r})^{3\alpha})$$  \hspace{1cm} (33)

$$a(r) = -\alpha + \frac{A^2}{2(\gamma + 1)(\alpha - 1)(\kappa r)^{2\alpha - 2}}$$

$$-\frac{A^4}{8(\gamma + 1)^2(\alpha - 1)^3(\kappa r)^{4\alpha - 4}} + O((\frac{1}{\kappa r})^{4\alpha - 2})$$  \hspace{1cm} (34)

The behavior at small distances when $n \neq 0$ is given by

$$f(r) = B(\kappa r)^n - \frac{\gamma - 1}{4} B^3(\kappa r)^{3n} + O((\kappa r)^{3n + 2})$$  \hspace{1cm} (35)

$$a(r) = n - \frac{B^2(\kappa r)^{2n + 2}}{2(\gamma + 1)(n + 1)} + \frac{(2\gamma - 1)B^4(\kappa r)^{4n + 2}}{2(2n + 1)(\gamma + 1)} + O((\kappa r)^{6n + 2})$$  \hspace{1cm} (36)
These solutions are characterized by magnetic flux $\Phi = \frac{2\pi(n+\alpha)}{e}$, charge $Q = -\kappa \Phi$, energy $E = \frac{\kappa^2}{(1+\gamma)e} \Phi$ and angular momentum $J = \frac{\pi \kappa (\alpha^2 - n^2)}{e^2}$.

As in Ref. [3], the constant $B$ is not determined by the behaviour of the fields near the origin, but is instead fixed by requiring proper behaviour as $r \to \infty$. In particular, $B$ is a function of the asymptotic value of the gauge field, i.e, $\alpha$. Now notice from equation (34) that $\alpha > 1$. Since the second term in (34) is subleading compare to the first term only in that case. So, all values of $B$ are not allowed in order to have nontopological vortex solutions. It turns out numerically that for each integer $n$ a continuous set of nontopological vortex solutions exists corresponding to the range $0 < B < B_0 (B_0 < 1)$, where $B_0$ is a critical value of $B$ separating nontopological vortices from topological vortices. As an illustration, we plot nontopological vortices of this kind for $\gamma = 3$ when $n = 1$ and $n = 2$ in Fig. 1 and Fig. 2 respectively. From both the figures we observe that $\text{max}(f) < 1$. In fact, the peak of $f(r)$ gradually decreases as $B$ takes comparatively lower value than $B_0$. We have checked for other values of $B$ close to $B_0$ that $\text{max}(f) < 1$. In subsec. II. D, we shall derive sum rules for these nontopological vortices and using this fact, shall show that $\alpha \geq n + 2$ (for $C_0 = 1$).

For zero vorticity, i.e, $n = 0$, $f(0)$ is not constrained. So the power series solution for both $f(r)$ and $a(r)$ near the origin is quite different from the $n \neq 0$ case. We find

\begin{align*}
  f(r) &= G_0 - \frac{G_0^3(1 - G_0^2)^{2\gamma - 1}}{4(\gamma + 1)} (\kappa r)^2 \\
  &\quad + \frac{G_0^5(1 - G_0^2)^{4\gamma - 3}}{64(\gamma + 1)^2} (4 - G_0^2 - 5\gamma G_0^2) (\kappa r)^4 + O((\kappa r)^6) \quad (37) \\
  a(r) &= -\frac{G_0^2(1 - G_0^2)^{3\gamma - 1}}{2(\gamma + 1)} (\kappa r)^2 \\
  &\quad + \frac{G_0^4(1 - G_0^2)^{7\gamma - 5}}{16(\gamma + 1)^2} (2 - G_0^2 - 3\gamma G_0^2) (\kappa r)^4 + O((\kappa r)^6) \quad (38)
\end{align*}

We plot a nontopological soliton of this kind for $\gamma = 3$ in Fig. 3.
(b) **Topological vortex solutions**: For simplicity, we restrict our discussion on the topological vortex solutions to $\gamma = 3$. The behaviour of the field variables near the origin for $n \neq 0$ can be obtained from (35) and (36) by putting $\gamma = 3$. We obtain large distance behaviour as following

$$f(r) = 1 + \frac{D_{2\beta}}{(\kappa r)^{2\beta}} + \frac{3D^2}{2(\kappa r)^{4\beta}} + O\left(\frac{1}{\kappa r}^{6\beta}\right)$$ (39)

$$a(r) = \beta + \frac{2D^4}{(4\beta - 1)(\kappa r)^{8\beta - 2}} + \frac{16D^5}{(5\beta - 1)(\kappa r)^{10\beta - 2}} + O\left(\frac{1}{\kappa r}^{12\beta - 2}\right)$$ (40)

Note that the large distance behaviour of the scalar field and the gauge field for these topological vortices are of semi-local [18] type, i.e., they fall off obeying power law.

It is remarkable to note that when the scalar field $f(r)$ attains its asymmetric vacuum value at large distances, $a(r)$ does not vanish; a feature not known so far for the topological vortices. The novel consequence is that even for topological vortices the magnetic flux, and hence the charge and the angular momentum, need not necessarily be quantized; while the energy, as evident from (24), is quantized. In particular, these topological vortex solutions are characterized by energy $E = \frac{\pi \kappa^2 n}{2e^2}$, flux $\Phi = \frac{2\pi}{e}(n - \beta)$, charge $Q = -\kappa \Phi$ and angular momentum $J = \frac{\pi \kappa e^2}{e^2}(\beta^2 - n^2)$. Since for each $n$, there is a set of solutions parameterized by $\beta$; the solutions are degenerate in each topological sector and one solution differs from another by charge, flux and angular momentum. At this point it is worthwhile to ask whether $\beta$ can take any positive value or is bounded from above and/or below. We find that $\beta$ is indeed bounded from above and below, i.e., $\frac{1}{4} < \beta < n$. The lower bound on $\beta$ is due to the fact that the second term in (40) is subleading compare to the first term only when $\beta > \frac{1}{4}$. The upper bound follows from the sum rules for these topological vortices, derived below in subsec. II.D.

As in nontopological vortex solutions, $B$ is a function of $\beta$. Since $\frac{1}{4} < \beta < n$, hence $B$ is also restricted. We find numerically that for each $n$ a continuous set of topological
vortex solutions exists corresponding to the range $B_0 < B < 1$. We plot $f(r)$ (solid line) and $a(r)$ (dashed line) in Fig. 4 for (I) $n = 1, \beta = 0.78$; (II) $n = 2, \beta = 1.91$; (III) $n = 2, \beta = 1.53$ and (IV) $n = 2, \beta = 1.23$. We have given only one plot for $n = 1$ since the profile of $f(r)$ and $a(r)$ are almost same for different $\beta$.

The Bogomol’nyi equations (27) and (28) are also satisfied by the trivial vacuum solution $f(r) = 1, a(r) = n$. This particular topological vortex solution has zero energy, charge, flux and angular momentum. Usually any model admitting topological vortex solutions has trivial vacuum solution only for $n = 0$. However in our model trivial vacuum solution exists in each topological sector, i.e, for any $n$. We have also looked for a nontrivial solution with the boundary condition $a(0) = n, f(0) = 1$ for $n \neq 0$. However no well behaved power series solution near origin is possible for this choice of boundary condition.

D. Sum Rules

The magnetic flux, and hence the charge and the angular momentum of both topological as well as nontopological vortices are determined in terms of unknown constants which are the assymptotic values of the gauge field (with our choice of ansatz) at large distances. We now show that a bound on these constants can be obtained rigorously using the first order Bogomol’nyi equations. Infact, Khare [17] obtained countable infinite number of sum rules for pure CS topological as well as nontopological vortices and using the first two he was able to put lower bound on the flux of nontopological vortices. In the same spirit, we derive and study the first few sum rules for the topological as well as nontopological GMCS vortices.

The usual technique for deriving sum rules is following. Let us consider the identity
\[
\frac{1}{l+1} \frac{d}{dr}a^{l+1} = a' a'
\] (41)

where \( l \) is any nonnegative integer, i.e, \( l = 0, 1, 2, \ldots \). On integrating both sides with respect to \( r \) from 0 to \( \infty \) we find that the left hand side of (41) gives \( \frac{1}{l+1}[a(\infty)^{l+1} - a(0)^{l+1}] \). The right hand side of (41) can be simplified step by step by making use of the Bogomol'nyi equations (27), (28) and boundary conditions (i), (ii), (iii). Since the boundary conditions for the topological vortices are different from those of the nontopological vortices, we discuss these two cases separately.

(a) **Sum rules for nontopological vortices** : The first sum rule (i.e, for \( l = 0 \)) is obtained after integrating equation (28) once with respect to \( r \) and using boundary conditions (i) and (iii)

\[
\alpha + n = \frac{\kappa^2}{(1 + \gamma)C_0^2} \int_0^\infty rdr f^2 (1 - f^2)^{\frac{\gamma - 1}{2}}
\] (42)

In order to obtain the second sum rule (i.e, for \( l = 1 \)), first we rewrite \( aa' \) solely in terms of \( f, f' \) and \( r \), by using equations (27) and (28). Then, on expanding \((1 - f^2)^\gamma\) binomially and after integrating the right hand side of equation (41) by parts and using the boundary conditions for the nontopological vortices; we have

\[
\alpha^2 - n^2 = \frac{2\kappa^2}{(1 + \gamma)\sqrt{C_0}} \int_0^\infty rdr \sum_{\rho=0}^\gamma \frac{(-1)^\rho}{\rho + 1} \gamma C_\rho f^{2(\rho + 1)}
\] (43)

where \( \gamma C_\rho = \gamma!/(\rho!(\gamma - \rho)!). \) For \( \gamma = 0 \) the second sum rule (43) is simply \( \alpha^2 - n^2 = \frac{2\kappa^2}{C_0} \int_0^\infty rdr f^2 > 0. \) This implies that \( \alpha > n \) for arbitrary \( C_0 \) since right hand side is manifestly positive definite. Not surprisingly the numerical calculation [10] for \( C_0 = 1 \) is in agreement with this exact result. Similarly for \( \gamma = 1 \), using both the sum rules (12) and (13) we find that \( \alpha \geq n + 2\sqrt{C_0} \) as in pure CS case (i.e, \( C_0 = 1 \)); but now for arbitrary \( C_0 (C_0 > 0) \), i.e, for GMCS vortices. The magnetic moment of these nontopological vortices [4,17]
\[
\mu_z = \frac{1}{2} \int d^2r (\vec{r} \times \vec{J})_z = -\frac{\pi \kappa^2}{e} \int r^2 dr h'(r)
\]  
(44)
can also be calculated for \( \gamma = 1 \) using the sum rules (12) and (13). In particular, we find
\[
\mu_{z_{\text{nontopo.}}} = -\frac{2\pi \kappa^2 \sqrt{C_0}}{e} (\alpha + n)(\alpha - n - \sqrt{C_0})
\]  
(45)
Note that \( \mu_{z_{\text{nontopo.}}} \) is always negative.

For \( \gamma > 1 \), the bound \( \alpha > 1 \) which follows from (34) can not be improved using these two sum rules alone. This can be seen as follows. Since we are considering only odd \( \gamma \), the term with highest power in \( f(r) \) in the second sum rule (13) is \( -\frac{2\kappa^2}{C_0}(\gamma + 1)^{-2}f^{2(\gamma + 1)} \), while for the first sum rule (12) it is \( \mp \frac{\kappa^2}{C_0}(1 + \gamma)^{-1}f^{3\gamma + 1} \); where negative sign is meant for \( \gamma = 4m + 1 \) and positive sign for \( \gamma = 4m - 1, m = 1, 2, 3... \). It is obvious that no manifestly positive or negative definite expression for \( f(r) \) is possible for \( \gamma = 4m + 1 \), since \( 3\gamma + 1 > 2(\gamma + 1) \) when \( \gamma > 1 \). Considering the term with second highest power in \( f(r) \) in second sum rule (13), the possibility \( \gamma = 4m - 1 \) is also ruled out. However if the numerical calculation is any guide, we can put a lower bound on the magnetic flux of the nontopological vortices at least for \( \gamma = 3 \). For \( \gamma = 3 \), the two sum rules (12) and (13) can be combined to write the following expression.
\[
\alpha^2 - n^2 - 2\sqrt{C_0}(\alpha + n) = \frac{\kappa^2}{2C_0} \int_0^\infty r dr [\frac{5}{2} f^4(1 - f^2)^2 + f^8(\frac{5}{4} - f^2)]
\]  
(46)
We know from the numerical calculation that \( \max(f) < 1 \) for \( n = 0, 1, 2 \) and we believe this is true for \( n > 2 \) also. So right hand side of (46) is manifestly positive definite and \( \alpha \geq n + 2 \) for \( C_0 = 1 \).

(b) **Sum rules for topological vortices** : In order to obtain sum rules for topological vortices we recall that the boundary conditions are, \( f(\infty) = 1, a(\infty) = \beta, f(0) = 0, a(0) = n \). Following exactly the same technique we find that the first three sum rules (i.e, for \( l = 0, l = 1 \) and \( l = 2 \)) are
\[ n - \beta = \frac{\kappa^2}{(1 + \gamma)c_0} \int_0^\infty rdr f^2 (1 - f^2)^{3\gamma - 1} \]  
(47)

\[ n^2 - \beta^2 = \frac{2\kappa^2}{(1 + \gamma)^2 c_0} \int_0^\infty rdr (1 - f^2)^{1+\gamma} \]  
(48)

\[ n^3 - \beta^3 = -\frac{3\kappa^2}{(1 + \gamma)^2 \sqrt{c_0}} \int rdr \left[ \frac{\gamma^2}{2(1 + \gamma)c_0^{\frac{3}{2}}} f^2 (1 - f^2)^{\frac{\gamma+1}{2}} + 2 \log f \right. 
- 2 \sum_{p=1}^{\gamma+3} \left. \frac{(-1)^p}{2p} \frac{\gamma+3}{2} C_p (1 - f^{2p}) \right] \]  
(49)

where \( \gamma = 1, 3 \). When \( \gamma = 1 \), the magnetic moment of these topological vortices can be calculated as in case of nontopological vortices. Using equations (44), (47) and (48) we find

\[ \mu_{\text{topo.}}^z = \frac{2\pi \kappa^2 \sqrt{c_0}}{\epsilon} n(n + \sqrt{c_0}) \]  
(50)

Note that \( \mu_{\text{topo.}}^z \) is positive definite and can be obtained from \( \mu_{\text{nontopo.}}^z \) which is always negative, by putting \( \alpha = 0 \).

When \( \gamma = 3 \), for both the first two sum rules (47) and (48) the right hand side is positive definite and this implies that \( \beta \leq n \). When \( \beta \) saturates the upper bound, the Bogomol’nyi equations are satisfied by the trivial vacuum solution \( f(r) = 1, a(r) = n \) in each topological sector. We have already seen that \( \beta > \frac{1}{4} \) from equation (34). So, the infinitely degenerate vortex solutions exists in each topological sector for \( \frac{1}{4} < \beta < n \), characterized by the same energy but with different flux, charge and angular momentum.

### III. NONRELATIVISTIC THEORY

#### A. Nonrelativistic Limit

In this section, we want to study the nonrelativistic limit of our model. We consider \( \hbar = 1 \) and keep the velocity of light \( c \) explicitly since we are interested in the nonrelativis-
tic limit $c \to \infty$. All other conventions we follow in this section are same as described in section II. A. We want to compare our model directly with that of Jackiw and Pi [8]. So, for convenience, we remove the factor $\frac{1}{2}$ multiplying with the scalar kinetic energy term in the Lagrangian (1). The subsequent effect of this change on the dielectric function as well as on the scalar potential have been taken into account. After keeping the velocity of light $c$ explicitly and making the above mentioned change in the Lagrangian (1), we have

$$\mathcal{L} = -\frac{1}{4} G(|\phi|) F_{\mu\nu} F^{\mu\nu} + D_\mu \phi (D^\mu \phi)^* + \frac{\kappa}{4} \epsilon_{\mu\nu\lambda} A_\mu F_{\nu\lambda} - \frac{4e^4 v^{4-4\gamma}}{(\gamma + 1)^2 C_0^2 \kappa^2 c^4} |\phi|^2 (v^2 - |\phi|^2)^{2\gamma}$$

(51)

where the covariant derivative is given by

$$D_\mu \phi = (\partial_\mu - \frac{ie}{c} A_\mu - \frac{ig}{4c} G(|\phi|) \epsilon_{\mu\nu\lambda} F^{\nu\lambda}) \phi$$

(52)

Nontopological vortex solution exists in this theory for any positive odd integer $\gamma$, when the dielectric function $G(|\phi|)$ assumes the following form

$$G(|\phi|) = \frac{\kappa^2 c^2}{2e^2 |\phi|^2} - \frac{\kappa^2 c^2 C_0 v^{2(\gamma - 1)}}{2e^2 |\phi|^2 (v^2 - |\phi|^2)^{\gamma - 1}}$$

(53)

and $C_0 > 0$.

Recall the fact that the mass of a scalar field is defined through the coefficient of the quadratic term in a scalar field potential, which is $m^2 c^2$ in this case. Comparing with the potential term in (51), we find that $v^2$ should have the value

$$v^2 = \frac{mc^3 \kappa |C_0(\gamma + 1)|}{2e^2}$$

(54)

Now the matter part of the Lagrangian density (51) can be written in terms of the scalar field mass $m$.
\[ \mathcal{L}_{\text{matter}} = \frac{1}{c^2} \left[ \partial_t - i e A_0 - \frac{ig}{2} F_{12} G(\phi) \right]^2 - (D_\mu \phi)(D^\mu \phi)^* - m^2 c^2 |\phi|^2 + \frac{4\gamma e^2 m}{(\gamma + 1)cC_0 |\kappa|} |\phi|^4 + \frac{4e^4}{(\gamma + 1)^2 C_0^2 \kappa^2 c^4} \sum_{p=2}^{2\gamma} (-1)^p 2\gamma C_p v^{2(2-p)} |\phi|^{2(p+1)} \]  

(55)

To consider the nonrelativistic limit, we first substitute in (53) and (55)

\[ \phi = \frac{1}{\sqrt{2m}} \left[ e^{-imc^2t} \psi + e^{imc^2t} \tilde{\psi}^* \right] \]  

(56)

and drop all terms which oscillate as \( c \to \infty \). Keeping only dominant inverse powers of \( c \) and setting \( \tilde{\psi} = 0 \) in order to work in the zero antiparticle sector (since particles and antiparticles are separately conserved), we obtain the nonrelativistic dielectric function

\[ G(\rho) = \frac{\kappa^2 e^2 m(1 - C_0)}{e^2 \rho} \]  

(57)

as well as the Lagrangian density corresponding to the relativistic Lagrangian (51),

\[ \mathcal{L} = \psi^* i(\partial_t - i e A_0 - \frac{ig}{2} \frac{\kappa^2 e^2 m(1 - C_0)}{\rho e^2} F_{12}) \psi - \frac{1}{2m} (D_\mu \psi)(D^\mu \psi)^* + \frac{\gamma e^2 \rho^2}{(\gamma + 1)mcC_0 |\kappa|} - \frac{\kappa^2 e^2 m(1 - C_0)}{4e^2 \rho} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{4} \epsilon_{\mu\nu\lambda} A^\mu F^{\nu\lambda} \]  

(58)

where \( \rho = \psi^* \psi \) is the particle density. The Lagrangian (58) describes the second quantized version of a fixed number of nonrelativistic particles moving in \( \delta \)-function potential and interacting with massive relativistic photons. Note that the effect of the scalar potential with higher values of \( \gamma \) is to change the strength of the \( \delta \)-function interaction in the nonrelativistic limit. Since \( C_0 \) is positive in order to have nontopological vortex solutions in the relativistic theory and \( m \) is defined to be positive through (54), the \( \delta \)-function interaction is attractive.

We obtain the equation of motion by varying the Lagrangian (58) with respect to \( \psi^* \) and \( A_\mu \) respectively.
\[-i(\partial_t - ieA_0) - \frac{ig}{2} \frac{\kappa^2 c^2 m(1 - C_0)}{e^2 \rho} F_{12}\psi = \frac{1}{4} \frac{\kappa^2 c^2 m(1 - C_0)}{e^2 \rho^2} [F_{\mu\nu} F^{\mu\nu} + \frac{g}{ec} \epsilon_{\mu\nu\alpha} J^{\mu} F^{\nu\alpha}] \psi + \frac{1}{2m} D_i D_i \psi + \frac{2\gamma e^2 \rho}{(\gamma + 1) mcC_0} \kappa \psi \] (59)

\[\epsilon_{\mu\nu\lambda} \partial^\mu \left[ \frac{\kappa^2 c^2 m(1 - C_0)}{e^2 \rho} \right] (F^\lambda + \frac{g}{2ec} J^\lambda) = \frac{1}{c} J_\nu - \kappa F_\nu \] (60)

where \( J_\nu \) is the current density

\[ J_\nu = (-ec\rho, J_i) \]

\[ = [-ec\psi^* \psi, -\frac{ie}{2m} \{ \psi^* D_i \psi - \psi (D_i \psi)^* \}] \] (61)

The energy is given by

\[ E = c \int d^2 x \left[ \frac{1}{2m} (\nabla_i \psi) (\nabla_i \psi)^* - \frac{\gamma e^2 \rho^2}{(\gamma + 1) mcC_0} \right] - \frac{\kappa^2 c^2 m C_0 (1 - C_0)}{2 e^2 \rho} (F_{12})^2 + \frac{\kappa^2 c^2 m C_0 (1 - C_0)}{2 e^2 \rho} (F_{0i})^2 \] (62)

where \( \nabla_i = \partial_i - \frac{ie}{c} A_i \) includes only the gauge potential. Like relativistic case notice that the solution to the equation

\[ J_\nu = \kappa c F_\nu \] (63)

also solves gauge field equation (60) provided \( g = -\frac{2e}{\kappa} \). The time component of equation (63), i.e., the CS modified Gauss law gives \( B = \frac{2e}{\kappa} \rho \), where \( B \) is the magnetic field. The immediate consequence of equation (63) is that any solution with flux \( \Phi \) also carries charge \( Q = -\kappa c \Phi \). The space component of equation (63) can be written in component form as

\[ E^i = -\partial_i A^0 - \frac{1}{c} \partial_i A^i \]

\[ = \frac{1}{c \kappa} e^{ij} J^j \] (64)
This equation can be further simplified as, $E^i = \frac{1}{c^2} \epsilon^{ij} J^j = \frac{1}{c^2 \epsilon} \epsilon^{ij} \tilde{J}^j$. Note that the expression for the electric field components is exactly same as that of nonrelativistic pure CS case [8] except the factor $1/C_0$. We rewrite equation (59) as

$$- i(\partial_t - ieA_0 - \frac{ig}{2} \kappa c^2 m (1 - C_0) F_{12}) \psi = \frac{1}{2m} D_i D_i \psi - \frac{1}{4} \frac{\kappa c^2 m (1 - C_0)}{e^2 \rho^2} F_{\mu \nu} F^{\mu \nu} \psi + \frac{2 \gamma e^2 \rho}{(\gamma + 1) mc C_0 |\kappa|} \psi$$

(65)

B. Self-dual Solution

Now we seek rotationally symmetric static soliton solutions in the system described by the equations (63) and (65). We choose the following Ansatz

$$\psi = e^{-i n \theta} \rho^4(r), \tilde{A} = - \frac{\dot{c}(a(r) - n)}{e r}; A_0 = h(r).$$

(66)

Substituting (66) into the equation (64), and (65), we have

$$h'(r) = - \frac{e a \rho}{\kappa m c r C_0}$$

(67)

$$\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r} \log \rho) = - \frac{8 \gamma e^2 \rho}{(\gamma + 1) e C_0 |\kappa|} - 4 mc A_0 - 2 m^2 e^2 (1 - C_0) + \frac{2 a^2}{r^2 C_0} - \frac{\rho^2}{2 \rho^2}$$

(68)

where use have been made of the relation $B = \frac{e}{\kappa} \rho$. The energy functional can be written as,

$$E = c \int d^2 x \left[ \frac{1}{8 m} \left( \frac{\rho'}{\sqrt{\rho}} \right)^2 + \frac{a^2 \rho}{2 m C_0 r^2} - \frac{\gamma e^2 \rho^2}{(\gamma + 1) m C_0 |\kappa|} + \frac{mc^2 (1 - C_0) \rho}{2} \right]$$

(69)

Using Bogomol’nyi trick we rewrite $E$

$$E = c \int d^2 x \left[ \frac{1}{2 m} \left( \frac{\rho'}{2 \sqrt{\rho}} \right)^2 \pm \frac{a \sqrt{\rho}}{\sqrt{C_0} r} \right] + \frac{e^2 \rho^2}{2 mc \sqrt{C_0} \kappa} - \frac{\gamma e^2 \rho^2}{(\gamma + 1) m C_0 |\kappa|} + \frac{\kappa mc^3 (1 - C_0)}{2 e} \Phi$$

(70)
There is a lower bound on the energy \( E \geq \frac{\kappa mc^3(1-C_0)}{2e} \Phi \) provided we choose

\[
\sqrt{C_0} = \frac{2\gamma}{(\gamma + 1)}
\]  

(71)

and take the upper sign in case \( \kappa \) negative while the lower sign for \( \kappa \) positive. The bound is saturated when the following first order equation holds

\[
\frac{\rho'}{2\sqrt{\rho}} = \pm a \frac{\sqrt{\rho}}{\sqrt{C_0} r}
\]  

(72)

Notice from equations (70) and (71) that when \( \gamma = 1 \), and hence \( C_0 = 1 \) (i.e, for pure CS theory in the NR limit), static self-dual soliton solution exists with zero energy. However the effect of the Maxwell term in the NR limit can be seen for \( \gamma > 1 \). As is clear from equations (70) and (71) the lower bound on the energy is saturated at nonzero finite value. Since \( C_0 > 1 \) for \( \gamma > 1 \) and the positive flux corresponds to the lower sign in (70) and vice versa; the minimum energy is always negative.

The decoupled equation for the charge density \( \rho \) is,

\[
\nabla^2 \log \rho(r) = \pm \frac{2\tau}{\sqrt{C_0}} \rho(r)
\]  

(73)

where \( \tau = \frac{e^2}{c \kappa} \). The equation (73) is the Liouville equation which is completely integrable. It is worth while to mention that for the whole class of the dielectric function for which the nontopological vortex solutions exists in the relativistic theory, the charge density solves the same Liouville equation (73) in the nonrelativistic limit. However, both the soliton energy and the Liouville equation are parametrized by \( C_0 \) which is determined in terms of \( \gamma \). The Liouville equation admits nonsingular nonnegative solutions for \( \rho(r) \) when the numerical constant on the right hand side of (73) is negative. For both the self-dual and anti self-dual solutions, the numerical constant \( \pm \frac{2\tau}{\sqrt{C_0}} \) is indeed negative according to the sign convention we fixed after the equation (70). The matter density \( \rho(r) \) that solves (73) is
\[ \rho(r) = \frac{4(n + \sqrt{C_0})^2}{\sqrt{C_0} \tau r^2} \left[ \left( \frac{r_0}{r} \right)^{\frac{n}{\sqrt{C_0}} + 1} + \left( \frac{r}{r_0} \right)^{\frac{n}{\sqrt{C_0}} + 1} \right]^{-2} \]  

(74)

where \( r_0 \) is a parameter describing the solution. After substituting (74) into equation (72), we find

\[ a(r) = \pm \left[ \sqrt{C_0} - (n + \sqrt{C_0}) \left( \frac{r_0}{r} \right)^{\frac{n}{\sqrt{C_0}} + 1} - \left( \frac{r}{r_0} \right)^{\frac{n}{\sqrt{C_0}} + 1} \right] \]  

(75)

Note that both \( f(r) \) and \( a(r) \) depends on \( C_0 \) nontrivially. The charge density \( \rho(r) \) vanishes both at the origin and at the asymptotic infinity for any \( C_0 \) satisfying the relation (71). However, for fixed \( r_0, n \) and \( \tau \), the rate of fall off at both the limit is slower for higher values of \( C_0 \), i.e, for the higher values of \( \gamma \). As \( r \to 0 \), \( a(r) \to \mp n \) ensuring the nonsingularity of the gauge field \( A_2 \). At large distances \( a(r) \) approaches the value \( \pm (n + 2\sqrt{C_0}) \) which, interestingly enough, is exactly the lower bound on \( \alpha \) in the relativistic theory as shown in subsec. II. D. In other words, \( \alpha \) saturates its lower bound in the nonrelativistic limit. These soliton solutions are characterized by the magnetic flux \( \Phi = \mp \frac{4\pi e}{c} (n + \sqrt{C_0}) \), the charge \( Q = -\kappa c \Phi \) and the angular momentum \( J = \mp \frac{2\pi c}{e} \sqrt{C_0} \Phi \). It is obvious from equation (71) that \( C_0 < 4 \) for any value of \( \gamma \). So, the magnetic flux, the charge, the angular momentum and the soliton energy are finite for any \( \gamma \) and in particular, \( (n + 1) \leq \frac{e}{4\pi c} \mid \Phi \mid < (n + 2) \).

To show that the first order Bogomol’nyi equations are consistent with the second order equation (68), note that the current density \( J_2 \) can be written in London form,

\[ J_2 = \frac{1}{C_0} \tilde{J}_2 = -\frac{e}{mC_0r} \rho(r)a(r) \]  

(76)

Since we are considering only static soliton solution, we choose \( J^2 \) to be transverse. Following Jackiw and Pi [8], \( A_0 \) can be fixed as,

\[ A_0 = -\frac{mc^2(1 - C_0)}{2e} \pm \frac{e\rho}{2mc\sqrt{C_0}\kappa} \]  

(77)
Now it is easy to see that the first order equations are consistent with the second order equation (68).

IV. CONCLUSION

We have considered a generalization of the abelian Higgs model with a CS term by multiplying a dielectric function with the Maxwell term and taking generalized covariant derivative. We have obtained Bogomol’nyi bound for this model, when certain relation among the coupling constant holds and the dielectric function assumes a particular form. The Bogomol’nyi equations of pure CS vortices and of Torres model are obtained as two special cases, up to a scale transformation of the variables. We have also obtained a novel new type of topological vortex solutions for which flux, charge and angular momentum need not be quantized, even though the energy is quantized. These topological vortex solutions are infinitely degenerate in each sector and differ from each other by flux, charge and angular momentum. We have been sucessfull to put both upper and lower bound on the magnetic flux, the charge and the angular momentum of these topological vortices by deriving sum rules. We have also studied nontopological vortex solutions in this model along with their physical properties.

Furthermore, we have considered the NR limit of our model and obtained static self-dual soliton solutions. Though both the NR pure CS and MCS self-dual soliton solutions are zero energy configuration, the self-dual soliton solutions in GMCS model saturates the lower bound at some nonzero finite value of the energy. For the whole class of scalar potential for which nontopological vortex solution exists in the relativistic GMCS theory, the charge density in the NR theory satisfies the same Liouville equation. Since the contribution of the scalar potential with higher values of \( \gamma \) to the NR theory is to change the strength of the \( \delta \)-function interaction, both the Liouville
equation and the soliton energy are parameterized by $C_0$ which is dependent on $\gamma$. In fact, the relation between $C_0$ and $\gamma$ which is not present in the relativistic theory comes as a constraint in order to have self-dual solution in the nonrelativistic limit.

For further investigations, this model raises a number of interesting questions. First of all, it is well known that for the Lagrangian considered by Torres (i.e., $G=1$), the physical photon mass does not receive one-loop radiative correction at the Bogomol’nyi Limit (i.e., $g = \frac{-2\kappa}{\gamma}$) [19]. Does the physical photon mass in GMCS model receives one loop radiative correction at the Bogomol’nyi limit, when only renormalizable potential are allowed? The bogomol’nyi equations are often manifestation of $N = 2$ supersymmetry [20]. Is it true for GMCS vortices also? If the radiative correction to photon mass vanishes at one loop level at the Bogomol’nyi limit and a $N = 2$ superfield formalism is possible for GMCS model, could one correlate these two phenomena?

The Bogomol’nyi equations we obtained are quite different in nature from the corresponding equations for vortices in other well known models. Naturally it is worthwhile to study whether the usual technique for showing the uniqueness and existence of soliton solutions goes through in this case or not. Also it would be interesting to know the total number of independent zero modes present in this model for generic dielectric function and the corresponding scalar potential. Above all, the most interesting thing it would be if this model can be realized in any planar condensed matter system where dielectric function plays major role.

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FIGURES

FIG. 1. A plot of $f(r)$ (solid line) and $a(r)$ (dashed line) for $\gamma = 3$ nontopological vortices with $n = 1$, $\alpha = 4.27$.

FIG. 2. A plot of $f(r)$ (solid line) and $a(r)$ (dashed line) for $\gamma = 3$ nontopological vortices with $n = 2$, $\alpha = 5.51$.

FIG. 3. A plot of $f(r)$ (solid line) and $a(r)$ (dashed line) for $\gamma = 3$ nontopological soliton with $\alpha = 4.22$ and $n = 0$.

FIG. 4. A plot of $f(r)$ (solid line) and $a(r)$ (dashed line) for $\gamma = 3$ topological vortices with (I) $n = 1$, $\beta = 0.78$; (II) $n = 2$, $\beta = 1.91$; (III) $n = 2$, $\beta = 1.53$ and (IV) $n = 2$, $\beta = 1.23$. 
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