General Theory of Image Normalization

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ABSTRACT

We give a systematic, abstract formulation of the image normalization method as applied to a general group of image transformations, and then illustrate the abstract analysis by applying it to the hierarchy of viewing transformations of a planar object.
1. Introduction and Brief Review of Viewing

Transformations of a Planar Object

A central issue in pattern recognition is the efficient incorporation of invariances with respect to geometric viewing transformations. We focus in this article on a particular method for handling invariances, called “image normalization”, which has the capability of extracting all of the invariant features from an image using only a small amount of information about the image (such as a few low order moments). The great appeal of normalization is that it isolates the problem of finding the image modulo the effect of viewing transformations, from the higher order problem of deciding which features of the image are needed for a specific classification decision. Intuitively, normalization is simply a systematic method for transforming from observer–based to image–based coordinates; in the former the image depends on the view, whereas in the latter the image is viewing transformation independent. From a mathematical viewpoint, our method consists of placing a set of constraints on the transformed image equal in number to the number of viewing transformation parameters, permitting one to solve either algebraically or numerically for the parameters of a normalizing transformation. Since the constraints are necessarily viewing transformation noninvariants, their construction is in general simpler than the direct construction of viewing transformation invariants.

Let us begin our discussion with a quick review of the viewing transformations of a planar object, since these transformations will be used as illustrations of our general methods. (For further details, and a bibliography, see the excellent recent book of Reiss [15].) Under rigid 3D motions the image \( I(\vec{x}) \), with \( \vec{x} = (x_1, x_2) \) the two dimensional coordinate in the image plane, is transformed to \( I(\vec{x}') \), with \( \vec{x}' \) related to \( \vec{x} \) by the planar projective
transformation

\[
x'_n = \sum_{m=1}^{2} \frac{G_{nm} x_m + t_n}{1 + \sum_{m=1}^{2} p_m x_m}, \quad n = 1, 2.
\]  

(1)

When the depth of the object is much less than its distance from the lens, then the parameter \(p_n\) in Eq. (1) can be neglected, and Eq. (1) reduces to the linear affine transformation

\[
x'_n = \sum_{m=1}^{2} G_{nm} x_m + t_n.
\]  

(2)

[An affine transformation, with \(G_{nm}\) replaced by \(G_{nm} - t_n p_m\), also results when Eq. (1) is expanded in a power series in \(x_m\) and second and higher order terms are neglected.]

Additionally, when the viewed object is constrained to lie in the plane normal to the viewing or 3 axis, Eq. (2) specializes further to the similarity transformation group of scalings, rotations, and translations, in which \(G_{nm}\) is simply a multiple (the scale factor) of a two dimensional rotation matrix. The projective transformations, the affine transformations, and the similarity transformations all form groups, and this will be the characterizing feature of the viewing transformations studied in our general analysis.

In applications, it will be convenient to use subgroup factorizations, which are readily obtained from the group multiplication rule for the transformations of Eqs. (1) and (2). For example, a general planar projective transformation can be written as the result of composing what we will term a restricted projective transformation

\[
x''_n = \frac{x'_n}{1 + \sum_{m=1}^{2} p_m x'_m}
\]  

(3)

with the general affine transformation of Eq. (2). Another subgroup factorization expresses the general affine transformation of Eq. (2) as the result of the composition of a pure translation

\[
x''_n = x'_n + t_n
\]  

(4a)
with a homogeneous affine transformation

\[ x_n' = \sum_{m=1}^{2} G_{nm} x_m . \]  \hspace{1cm} (4b)

Yet a third subgroup factorization expresses a general homogeneous affine transformation as the result of composing what we will term a restricted affine transformation, which has vanishing upper right diagonal matrix element,

\[ x_n'' = \sum_{m=1}^{2} g_{nm} x_m' , \quad g_{12} = 0 , \]  \hspace{1cm} (5a)
with a pure rotation

\[ x_n' = \sum_{m=1}^{2} R_{nm} x_m , \quad R_{11} = R_{22} = \cos \theta , \quad R_{12} = -R_{21} = -\sin \theta . \]  \hspace{1cm} (5b)

A variant of Eqs. (5a–b) is obtained by requiring that the matrix \( g \) have unit determinant, so that it has the two–parameter form \( g_{11} = u, \ g_{12} = 0, \ g_{21} = w, \ g_{22} = w^{-1} \), and then including a scale factor \( \lambda \) in Eq. (5b), which now reads

\[ x_n' = \lambda \sum_{m=1}^{2} R_{nm} x_m . \]  \hspace{1cm} (5c)

2. General Theory of Image Normalization

We proceed now to formulate a general framework for image normalization, with the aim of understanding the common elements of the various normalization methods which appear in the literature and of generalizing them to new applications. As a preliminary to the mathematical discussion of Subsecs. 2A–E, we specify our notation for viewing transformations. Let \( \mathcal{G} = \{ S \} \) be a group of symmetry or viewing transformations \( S \), which act on the image \( I(\vec{x}) \) according to

\[ I(\vec{x}) \rightarrow I_S(\vec{x}) = I(\vec{S}(\vec{x})) . \]  \hspace{1cm} (6a)
Our notational convention, that we shall adhere to throughout, is that $\vec{x}' = \vec{S}(\vec{x})$ is the concrete image coordinate mapping induced by the abstract group element $S$. [A specific example of such a transformation would be the planar projection transformation of Eq. (1), in which $S$ would be the abstract element of the planar projective group characterized by the parameters $G_{mn}$, $t_n$, $p_m$ specifying the concrete coordinate mapping.] In this notation, the result of successive transformations with $S_1$ followed by $S_2$ is given by

$$I(\vec{x}) \rightarrow I_{S_2S_1}(\vec{x}) = I(\vec{S}_2(\vec{S}_1(\vec{x}))) \ .$$ \hspace{1cm} (6b)

The transformation groups of interest to us are in general ones with continuous parameters, in other words, Lie groups, and the reader interested in more background on Lie group theory may wish to consult the texts of Gilmore [8] and Sattinger and Weaver [13]. However, very little of the formal apparatus of Lie group theory is required in what follows; basically, all we use is the group closure property and the enumeration of the number of group parameters. In particular, no knowledge of the representation theory of Lie groups is needed.

A. The normalization recipe. We begin by giving the general prescription for an image normalization transformation. Let $\vec{N}_I(\vec{x})$ be a transformation of $\vec{x}$ which depends on the image $I$, and which is constructed so that under the image transformation of Eq. (6a), it behaves as

$$\vec{N}_{I_S}(\vec{x}) = \vec{S}^{-1}(\vec{N}_I(\vec{x})) \ ,$$ \hspace{1cm} (7a)

with $\vec{S}^{-1}$ the inverse transformation to $\vec{S}$ of Eq. (6a),

$$\vec{S}(\vec{S}^{-1}(\vec{x})) = \vec{x} \ .$$ \hspace{1cm} (7b)
Also, let \( \tilde{M}_I(\vec{x}) \) be an optional second transformation of \( \vec{x} \) which depends on the image \( I \) only through invariants under the group of transformations \( G \), that is,

\[
\tilde{M}_{I_S}(\vec{x}) = \tilde{M}_I(\vec{x}), \quad \text{all } S \in \mathcal{G} .
\] (7c)

Then

\[
\tilde{I}(\vec{x}) = I(\tilde{N}_I(\tilde{M}_I(\vec{x})))
\] (8)

is a normalized image which is invariant under all transformations of the group \( \mathcal{G} \). This is an immediate consequence of Eq. (6a) and Eqs. (7a–c), from which we have

\[
\tilde{I}_S(\vec{x}) = I_S(\tilde{N}_{I_S}(\tilde{M}_{I_S}(\vec{x})))
\]

\[
= I(\tilde{S}(\tilde{S}^{-1}(\tilde{N}_I(\tilde{M}_I(\vec{x}))))))
\]

\[
= I(\tilde{N}_I(\tilde{M}_I(\vec{x}))) = \tilde{I}(\vec{x}) .
\] (9)

B. Uniqueness. Before specifying how to actually construct a map \( \tilde{N}_I \) obeying Eq. (7a), let us address the issue of uniqueness. That is, given \( \textit{two} \) maps \( \tilde{N}_{1I}(\vec{x}) \) and \( \tilde{N}_{2I}(\vec{x}) \), both of which obey Eq. (7a), how are they related? By hypothesis, we have

\[
\tilde{N}_{1I_S}(\vec{x}) = \tilde{S}^{-1}(\tilde{N}_{1I}(\vec{x})),
\]

\[
\tilde{N}_{2I_S}(\vec{x}) = \tilde{S}^{-1}(\tilde{N}_{2I}(\vec{x})).
\] (10)

Since for any \( \tilde{f}(\vec{x}) \) and \( \tilde{g}(\vec{x}) \) we have

\[
\tilde{f}(\tilde{g}(\vec{x}))^{-1} = \tilde{g}^{-1}(\tilde{f}^{-1}(\vec{x})) ,
\] (11a)

we can rewrite the first line of Eq. (10) as

\[
\tilde{N}_{1I_S}^{-1}(\vec{x}) = \tilde{N}_{1I}^{-1}(\tilde{S}(\vec{x})).
\] (11b)

Let us now define a new map \( \tilde{M}_I(\vec{x}) \) by

\[
\tilde{M}_I(\vec{x}) \equiv \tilde{N}_{1I}^{-1}(\tilde{N}_{2I}(\vec{x})) ,
\] (12)
which reduces to the identity map when $\vec{N}_{1I} = \vec{N}_{2I}$; then by Eq. (11b) and the first line of Eq. (10), we have

$$M_{Is}(\vec{x}) = N_{1Is}(\vec{N}_{2Is}(\vec{x}))$$

$$= N_{1I}^{-1}(S(S^{-1}(\vec{N}_{2I}(\vec{x}))))$$

$$= N_{1I}^{-1}(\vec{N}_{2I}(\vec{x})) = M_I(\vec{x}) \quad (13a)$$

In other words, $M_I(\vec{x})$ depends on the image $I$ only through invariants under transformations of the group $G$, and from Eq. (12), the normalizing map $\vec{N}_{2I}$ is related to the normalizing map $\vec{N}_{1I}$ by

$$N_{2I}(\vec{x}) = N_{1I}(M_I(\vec{x})) \quad (13b)$$

This is why in writing the general normalized image corresponding to a particular normalizing map in Eq. (8), we have included in the $\vec{x}$ dependence the possible appearance of a map $M_I$ which depends on the image only through invariants under transformation by elements of $G$.

C. Construction of $\vec{N}_I$ by imposing constraints, and demonstration that normalization yields a complete set of invariants. We next show that one can construct an image normalization transformation obeying Eq. (7a) by imposing a suitable set of constraints. We shall assume now that $G$ is a $K$–parameter Lie group which is continuously connected to the identity. Let $C_k[I] = C_k[I(\vec{x})]$, $k = 1, ..., K$ (where $\vec{x}$ is a dummy variable) be a set of functionals of the image $I(\vec{x})$ with the property that the $K$ constraints

$$C_k[I_{Sr}] = C_k[I(S'(\vec{x}))] = 0, \quad k = 1, ..., K \quad (14a)$$

are satisfied for a unique element $S' = N_I$ of $G$, so that

$$C_k[I(\vec{N}_I(\vec{x}))] = 0, \quad k = 1, ..., K \quad (14b)$$

Then, as we shall now show, $\vec{N}_I(\vec{x})$ is the desired normalizing transformation.
We remark that the condition that Eqs. (14a, b) should have a unique solution can be relaxed in applications to the condition that there be only one solution in the range of relevant viewing transformation parameters. Clearly, either form of the uniqueness condition requires that the constraint functionals not be invariants under $G$, and thus their structure will in general be simpler than that of directly constructed viewing transformation invariants.

In many cases, as we will see in Sec. 3 below, the constraints can be constructed from viewing transformation covariants, which have simple algebraic properties under the transformations of $G$, permitting closed form algebraic solution for the parameters of the normalizing transformation. In more complicated cases, as discussed in Sec. 4, the constraints must be solved numerically for the normalizing transformation.

To see that the construction of Eqs. (14a,b) gives a transformation $\vec{N}_I(\vec{x})$ that obeys Eq. (7a), let us consider the effect of replacing $I$ by $I_s$ in Eqs. (14a, b). By hypothesis, the constraints

$$C_k[I_s(\vec{S}(\vec{x}))] = 0, \quad k = 1, \ldots, K \quad (15a)$$

are uniquely satisfied by a group element $S' = N_{I_s}$ of $G$, so that

$$C_k[I_s(\vec{N}_{I_s}(\vec{x}))] = 0, \quad k = 1, \ldots, K, \quad (15b)$$

with $\vec{N}_{I_s}(\vec{x})$ the proposed normalizing transformation corresponding to $I_s$. But using Eq. (6a), we can also write Eq. (15b) as

$$C_k[I(\vec{S}(\vec{N}_{I_s}(\vec{x})))]] = 0, \quad k = 1, \ldots, K, \quad (15c)$$

which has the same structure as Eq. (14b). Therefore, by uniqueness of the solution $N_I$ of Eq. (14b) we must have

$$\vec{S}(\vec{N}_{I_s}(\vec{x})) = \vec{N}_I(\vec{x}), \quad (16a)$$
which by Eq. (7b) is equivalent to

\[
\vec{N}_{I_S}(\vec{x}) = \vec{S}^{-1}(\vec{N}_I(\vec{x})) ,
\]

(16b)

showing that the \( N_I \) produced by solving the constraints does indeed obey Eq. (7a). Hence the imposition of constraints gives a constructive procedure for generating image normalization transformations.

We note that this construction makes the normalizing transformation \( \vec{N}_I \) an element of the group \( \mathcal{G} \), and the quotient \( \vec{M}_I(\vec{x}) \equiv \vec{N}_I^{-1}(\vec{N}_{2I}(\vec{x})) \) of two normalizing maps constructed by imposing different sets of constraints will likewise be an element of \( \mathcal{G} \). When both \( \vec{N}_I \) and \( \vec{M}_I \) in Eq. (8) belong to \( \mathcal{G} \), we can invert Eq. (8) to express the original image \( I \) in terms of the invariant, normalized image \( \tilde{I} \) according to

\[
I(\vec{x}) = \tilde{I}(\vec{M}_I^{-1}(\vec{N}_I^{-1}(\vec{x}))) .
\]

(16c)

This equation shows that normalization leads to a complete set of invariants, in the sense that the information in the normalized image, plus the \( K \) parameters determining the viewing transformation \( \vec{M}_I^{-1}(\vec{N}_I^{-1}(\vec{x})) \), suffice to completely reconstruct the original image. By way of contrast, the representation–theoretic methods discussed in Sec. 6.5 of Lenz [12], and the integral transform methods of Ferraro [7], although attacking the same problem as is discussed here, yield only a small fraction of the complete set of invariants. Moreover, normalization has the further advantage of requiring only a minimal knowledge of the kinematic structure of the group; the full irreducible representation structure is not needed, and the methods described here are applicable to noncompact as well as to compact groups. We note finally that the discussion of this section is slightly less general than that of Secs. 3A and 3B, where we did not require either \( \vec{N}_I \) or \( \vec{M}_I \) to belong to \( \mathcal{G} \); the most general normalizing
map $\tilde{N}_I$ is obtained from one generated by constraints by using as its argument a map $\tilde{M}_I$ which does not belong to $G$ but that is invariant under transformations of the image $I$ by $G$.

D. Extension to reflections and contrast invariance. We consider next two simple extensions of the constraint method for constructing the normalizing transformation. The first involves relaxing the requirement that $G$ be simply connected to the identity, as is needed if $G$ contains improper transformations such as reflections. Reflections are said to be independent if they do not differ solely by an element of the connected component of the group; for each independent discrete reflection $R$ in $G$, the set of constraints of Eq. (14a) must be augmented by an additional constraint $D[I(\tilde{S}^R(\bar{x}))] > 0$, where $D[I(\bar{x})]$ is a functional of the image which changes sign under the reflection operation $R$,

$$D[I(\tilde{R}(\bar{x}))] = -D[I(\bar{x})].$$

(17)

The second extension involves incorporating invariance under changes of image contrast, that is, under image transformations of the form

$$I(\bar{x}) \rightarrow cI(\bar{x}), \quad c > 0.$$  \hspace{1cm} (18a)

To the extent that illumination is sufficiently slowly varying that it can be treated as constant over a viewed object, changes in illumination level as the object is moved to different views take the form of changes in the constant $c$ in Eq. (18a), which is why incorporating contrast invariance can be important. If we require that the constraint functionals $C_k$ [and $D$ if needed] should be invariant under the change of contrast of Eq. (18a), then the image normalization transformation $\tilde{N}_I(\bar{x})$ and the auxiliary transformation $\tilde{M}_I(\bar{x})$ can be taken to be contrast invariant. A contrast invariant normalized image $\tilde{I}_c(\bar{x})$ is then obtained by
the obvious recipe
\[ \widehat{I}_c(x) = \frac{\widehat{I}(x)}{\int d^2x \, \widehat{I}(x)} . \]  
(18b)

E. Use of subgroup decompositions. Suppose that for a general element \( S \) of the group \( \mathcal{G} \), there is a subgroup decomposition of the form
\[ S = S_2 S_1 , \]  
(19a)
with \( S_2 \) belonging to a subgroup \( \mathcal{G}_2 \) of \( \mathcal{G} \), \( S_1 \) belonging to a subgroup \( \mathcal{G}_1 \) of \( \mathcal{G} \), and with the respective parameter counts \( K, K_1, \) and \( K_2 \) of \( \mathcal{G}, \mathcal{G}_1, \) and \( \mathcal{G}_2 \) obeying
\[ K = K_1 + K_2 . \]  
(19b)
(Such subgroup compositions for a general Lie group are obtained by constructing a composition series for the group, but we will not need this formal apparatus in the relatively simple applications that follow.) Let us suppose further that we can solve the problem of image normalization with respect to the group \( \mathcal{G}_1 \), and that we wish to extend this solution to the full invariance group \( \mathcal{G} \). The subgroup decomposition allows this to be done by imposing \( K_2 \) additional constraints to deal with the \( \mathcal{G}_2 \) subgroup, as follows. Let \( C_{2k}[I(x)] \), with \( k = 1, \ldots, K_2 \), be a set of functionals of the image chosen so that the constraints
\[ C_{2k}[I(N_{2f}(S_1(x)))] = 0 , \quad k = 1, \ldots, K_2 \]  
(20a)
are independent of \( S_1 \in \mathcal{G}_1 \). In particular, taking \( S_1 \) as the identity transformation, Eq. (20a) simplifies to
\[ C_{2k}[I(N_{2f}(\bar{x}))] = 0 , \quad k = 1, \ldots, K_2 , \]  
(20b)
which if we impose the requirement of a unique solution over transformations \( N_{2f} \in \mathcal{G}_2 \) determines a “partial normalization” transformation \( \bar{N}_{2f} \). Note that a sufficient condition
for the constraints of Eq. (20a) to be independent of $S_1$ is for the functionals $C_{2k}$ to be $S_1$-independent, but this is not a necessary condition; we will see examples in which, as $S_1$ traverses $G_1$, the functionals are merely covariant in some simple way that guarantees invariance of the constraints obtained by equating all the functionals to zero. To see how $\tilde{N}_{2I}$ transforms under the action of the group $\mathcal{G}$, we replace $I$ by $I_S$ in Eq. (20b), giving

$$C_{2k}[I_S(\tilde{N}_{2I_S}(\vec{x}))] = 0, \quad k = 1, ..., K_2; \quad (21a)$$

again making use of Eq. (6a) this becomes

$$C_{2k}[I(\tilde{S}(\tilde{N}_{2I_S}(\vec{x})))] = 0, \quad k = 1, ..., K_2. \quad (21b)$$

Since the argument $\tilde{S}(\tilde{N}_{2I_S}(\vec{x}))$ appearing in Eq. (21b) is no longer a member of the $G_2$ subgroup, we cannot conclude that it is equal to the argument $\tilde{N}_{2I}(\vec{x})$ appearing in Eq. (20b), but the arguments can differ at most by a transformation of $\vec{x}$ by some member $S_1'$ of the subgroup $\mathcal{G}_1$ which leaves the constraints invariant, giving

$$\tilde{N}_{2I_S}(\vec{x}) = \tilde{S}^{-1}(\tilde{N}_{2I}(\tilde{S}_1'(\vec{x}))) \quad (22a)$$

as the subgroup analog of Eq. (7a). Corresponding to this, the partially normalized image defined by

$$\tilde{I}(\vec{x}) = I(\tilde{N}_{2I}(\vec{x})) \quad (22b)$$

transforms under the group $\mathcal{G}$ as

$$\tilde{I}(\vec{x}) \rightarrow \tilde{I}_S(\vec{x}) = I_S(\tilde{N}_{2I_S}(\vec{x})) = I(\tilde{S}(\tilde{S}^{-1}(\tilde{N}_{2I}(\tilde{S}_1'(\vec{x})))))) = I(\tilde{N}_{2I}(\tilde{S}_1'(\vec{x}))) = \tilde{I}(\tilde{S}_1'(\vec{x})) \quad (22c)$$
and thus changes only by a transformation lying in the \( G_1 \) subgroup.

Further image normalization of \( \tilde{I} \) using the constraints appropriate to \( G_1 \) then gives a final normalized image

\[
\hat{I}(\bar{x}) = I(\tilde{N}_2 I(\tilde{N}_1 I(\tilde{M}_I(\bar{x})))),
\]

(23)

which is invariant with respect to the full group of transformations \( G \), where as before \( \tilde{M}_I \) is any transformation which is constructed solely using \( G \) invariants of the image.

3. Viewing Transformations of a Planar Object

With Algebraically Solvable Constraints

We proceed now to apply the general image normalization methods of Sec. 2 to the viewing transformations of a planar object. In this section we focus on cases, corresponding to linear viewing transformations, in which suitable constraints can be formed using simple viewing transformation covariants, leading to algebraically solvable constraints. In the next section we will discuss more complicated cases, several of which use the transformations of this section as building blocks, in which some of the constraints must be solved by iterative methods.

A. Translations. The translation subgroup of Eq. (1) is given by

\[
\tilde{S}(\bar{x}) = \bar{x} + \bar{t},
\]

(24a)

corresponding to which \( I_S = I(\bar{x} + \bar{t}) \) describes an image translated by the vector \(-\bar{t}\). We take as the constraint functionals

\[
C_k[I_S] = \int d^2 x \ x_k \ I(\bar{x} + \bar{t}), \quad k = 1, 2.
\]

(24b)
The constraints \( C_k = 0, \ k = 1, 2 \) can be solved explicitly for \( \vec{t} \) by making the change of integration variable \( \vec{y} = \vec{x} + \vec{t} \), giving the unique solution \( \vec{t} = \vec{t}_I \), with \( \vec{t}_I \) the image “center of mass”

\[
\vec{t}_I = \frac{\int d^2x \ \vec{x} \ I(\vec{x})}{\int d^2x \ I(\vec{x})}, \tag{25a}
\]

and the corresponding normalizing transformation is

\[
\vec{N}_I(\vec{x}) = \vec{x} + \vec{t}_I. \tag{25b}
\]

Under the action of the translation \( S \), Eq. (25a) becomes

\[
\vec{t}_{Is} = \frac{\int d^2x \ \vec{x} \ I(\vec{x} + \vec{t})}{\int d^2x \ I(\vec{x} + \vec{t})}, \tag{26a}
\]

which by a change of integration variable yields

\[
\vec{t}_{Is} = \vec{t}_I - \vec{t}. \tag{26b}
\]

Thus the normalizing transformation of Eq. (25b) behaves as

\[
\vec{N}_{Is}(\vec{x}) = \vec{x} + \vec{t}_I - \vec{t} = \vec{S}^{-1}(\vec{N}_I(\vec{x})), \tag{26c}
\]

in agreement with the general result of Eq. (7a). In accordance with Eq. (8), the translation invariant image is

\[
\bar{I}(\vec{x}) = I(\vec{N}_I(\vec{M}_I(\vec{x}))) = I(\vec{M}_I(\vec{x}) + \vec{t}_I), \tag{27a}
\]

with \( \vec{M}_I(\vec{x}) \) any transformation of \( \vec{x} \) which depends only on translation invariant image features. Usually, one makes the choice \( \vec{M}_I(\vec{x}) = \vec{x} - \vec{t}_0 \), with \( \vec{t}_0 \) a constant vector which is independent of the image \( I \). This constant vector can of course be taken to be zero, corresponding to the choice

\[
\vec{M}_I(\vec{x}) = \vec{x}, \tag{27b}
\]
or it can be adjusted to center the translation invariant form of one particular image $I_0$ at any desired point.

Once we have the translation normalized image $\tilde{I}(\tilde{x})$, all features extracted from it, such as all Fourier transform or wavelet transform amplitudes, are translation invariant. We illustrate this explicitly in the case of the Fourier transform, by showing how a translation invariant Fourier transform $\tilde{I}(\tilde{k})$ is related to the Fourier transform $I(\tilde{k})$ of the original image $I(\tilde{x})$,

$$\tilde{I}(\tilde{k}) = \int d^2 x e^{-i\tilde{k} \cdot \tilde{x}} \tilde{I}(\tilde{x})$$

$$= \int d^2 x e^{-i\tilde{k} \cdot \tilde{x}} I(\tilde{x} + \tilde{t})$$

$$= e^{i\tilde{k} \cdot \tilde{t}} I(\tilde{k}), \quad (28a)$$

where we have taken $\tilde{M}_I$ to be the identity map as in Eq. (27b), and where

$$I(\tilde{k}) = \int d^2 x e^{-i\tilde{k} \cdot \tilde{x}} I(\tilde{x}) . \quad (28b)$$

Under translation, the Fourier transform of the original image $I(\tilde{k})$ behaves as

$$I(\tilde{k}) \to I_S(\tilde{k}) = \int d^2 x e^{-i\tilde{k} \cdot \tilde{x}} I(\tilde{x} + \tilde{t}) = e^{i\tilde{k} \cdot \tilde{t}} I(\tilde{k}) , \quad (28c)$$

and is not invariant, but by Eq. (28b), the factor $e^{i\tilde{k} \cdot \tilde{t}}$ behaves as

$$e^{i\tilde{k} \cdot \tilde{t}} \to e^{i\tilde{k} \cdot \tilde{t}_S} = e^{-i\tilde{k} \cdot \tilde{t}} e^{i\tilde{k} \cdot \tilde{t}_I} , \quad (28d)$$

and has a compensating noninvariance, making the product $\tilde{I}(\tilde{k})$ appearing in Eq. (28a) invariant under image translations.

B. Separation of affine normalization into translational and homogeneous affine normalization. Since rotations and scalings are special cases of homogeneous affine transformations, before discussing them we use the subgroup decomposition method to give the general
procedure for separating the affine normalization problem into a translational normalization followed by a homogeneous affine normalization. We follow the general procedure of Eqs. (19a)–(23), taking the subgroup $G_2$ to be the translations, and the subgroup $G_1$ to be the homogeneous affine transformations, as in Eqs. (4a, b), so that

$$\vec{S}_2(\vec{x}) = \vec{x} + \vec{t}, \quad \vec{S}_1(\vec{x}) = G \cdot \vec{x},$$   \hspace{1cm} (29a)$$

$$\vec{S}(\vec{x}) = \vec{S}_2(\vec{S}_1(\vec{x})) = G \cdot \vec{x} + \vec{t},$$

where the notation $G \cdot \vec{x}$ denotes the vector with components $\sum_{m=1}^{2} G_{nm} x_m$. Applying the same translational constraint functionals $C_k$ of Eq. (24b) to the general affine transformation $\vec{S}$ of Eq. (29a), we have

$$C_k[I_S] = \int d^2x \ x_k \ I(G \cdot \vec{x} + \vec{t}),$$ \hspace{1cm} (29b)$$

which on making the change of integration variable

$$\vec{x} \rightarrow \vec{S}_1^{-1}(\vec{x}) = G^{-1} \cdot \vec{x},$$ \hspace{1cm} (29c)$$
gives

$$C_k[I_S] = \sum_{\ell} (G^{-1})_{k\ell} \int d^2x \ x_\ell \ I(\vec{x} + \vec{t}) = \sum_{\ell} (G^{-1})_{k\ell} C_\ell[I_{S_2}] .$$ \hspace{1cm} (29d)$$

Thus, although the translational constraint functionals $C_k$ are not independent of $S_1$ (or $G$), they simply mix linearly when $S_1$ is changed, and consequently the translational constraints $C_k = 0, \ k = 1, 2$ are $S_1$–independent. This permits us to normalize out the translational part of a general affine transformation independently of the homogeneous affine transformations, leading to a partially normalized image

$$\tilde{I}(\vec{x}) = I(\vec{x} + \vec{t}_f)$$ \hspace{1cm} (30a)$$
which is translation invariant. Under the full affine group, \( \vec{t}_I \) transforms as
\[
\vec{t}_I \rightarrow \vec{t}_{I_S} = \frac{\int d^2x \vec{x} I(G \cdot \vec{x} + \vec{t})}{\int d^2x I(G \cdot \vec{x} + \vec{t})},
\]
which by the same changes of integration variable used before reduces to
\[
\vec{t}_{I_S} = G^{-1} \cdot (\vec{t}_I - \vec{t}).
\]
Hence the partially normalized image of Eq. (30a) transforms under the full affine group as
\[
\vec{I}(\vec{x}) \rightarrow \vec{I}_{S}(\vec{x}) = I_S(\vec{x} + \vec{t}_{I_S}) = I(G \cdot [x + G^{-1} \cdot (\vec{t}_I - \vec{t})] + \vec{t}) = I(G \cdot \vec{x} + \vec{t}_I) = \vec{I}(G \cdot \vec{x}),
\]
in agreement with the general result of Eq. (22c). In other words, the partially normalized image \( \vec{I} \) is translation invariant, and is acted on only by the homogeneous part of the affine transformation. In discussing similarity and affine transformations in Subsecs. C–F which follow, we will assume that we are always dealing with a partially normalized image which is translation invariant, but for simplicity of notation we will drop the tilde and simply call this image \( I \). However, keeping the tilde for the moment, we note that the moments \( \mu_{pq} \) of this partially normalized image, which are called central moments, are defined by
\[
\mu_{pq} = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \ x_1^p \ x_2^q \ \vec{I}(\vec{x}).
\]
C. Rotations. We begin the discussion of homogeneous affine transformations by considering pure rotations, with the group action
\[
\vec{S}(\vec{x}) = \vec{x}_\theta \equiv (x_1 \cos \theta - x_2 \sin \theta, x_1 \sin \theta + x_2 \cos \theta),
\]
corresponding to which \( I_S(\vec{x}) = I(\vec{x}_\theta) \) describes an image rotated by the angle \( -\theta \). We shall assume, for the moment, that the image to be normalized has no rotational symmetry, in
which case we can take as the constraint functional

$$C[I_S] = \text{Phase} \left[ \int d^2 x \ e^{i\Phi(\vec{x})} f(|\vec{x}|) \ I(\vec{x}_\theta) \right] - 1 \ .$$

Here we have used the notation

$$\text{Phase}[z] = z/|z|$$

for the complex number $z$; the function $f$ is arbitrary [6], and the functions $\Phi(\vec{x})$ and $|\vec{x}|$ are defined by

$$\Phi(\vec{x}) \equiv \arctan(x_2/x_1) \ , \quad |\vec{x}| \equiv \sqrt{x_1^2 + x_2^2} \ .$$

The constraint $C[I_S] = 0$ now uniquely determines an angle $\theta = \theta_I$, which can be calculated explicitly by making a change of variable $\vec{x} \to \vec{x}_{-\theta}$ in Eq. (33a) and using the trigonometric formula

$$\Phi(\vec{x}_{-\theta}) = \arctan\left(\frac{-x_1 \sin \theta + x_2 \cos \theta}{x_1 \cos \theta + x_2 \sin \theta}\right) = \Phi(\vec{x}) - \theta \ ,$$

thus giving

$$e^{i\theta_I} = \text{Phase} \left[ \int d^2 x \ e^{i\Phi(\vec{x})} f(|\vec{x}|) \ I(\vec{x}) \right] \ ,$$

with the corresponding normalization transformation

$$\vec{N}_I(\vec{x}) = \vec{x}_{\theta_I} = (x_1 \cos \theta_I - x_2 \sin \theta_I, x_1 \sin \theta_I + x_2 \cos \theta_I) \ .$$

Under the action of the rotation $S$, Eq. (34b) is transformed to

$$e^{i\theta_{IS}} = \text{Phase} \left[ \int d^2 x \ e^{i\Phi(\vec{x})} f(|\vec{x}|) \ I(\vec{x}_\theta) \right] \ ,$$

which making the change of variable $\vec{x} \to \vec{x}_{-\theta}$ and using Eq. (34a) gives

$$\theta_{IS} = \theta_I - \theta \ .$$
Thus the normalization transformation \( \vec{N}_I \) becomes, under the action of \( S \),

\[
\vec{N}_I S(\vec{x}) = \vec{x}_{\theta_I} = \vec{x}_{\theta_I - \theta} = S^{-1}(\vec{N}_I(\vec{x})) ,
\]

(35c)
in agreement with Eq. (7a). Following the prescription of Eq. (8), the rotationally normalized image is

\[
\tilde{I}(\vec{x}) = I(\vec{N}_I(\vec{M}_I(\vec{x}))) = I(\vec{M}_I(\vec{x})_{\theta_I}) ,
\]

(36)

with \( \vec{M}_I(\vec{x}) \) any transformation of \( \vec{x} \) which depends only on rotationally invariant image features. Usually, one makes the choice \( \vec{M}_I(\vec{x}) = \vec{x}_{\theta_0} \), with \( \theta_0 \) a constant angle which is independent of the image \( I \). This angle can of course be taken to be zero, corresponding to the \( \vec{M}_I \) of Eq. (27b), or it can be used to give the rotationally invariant form of one particular image a specified orientation. Note that the angle \( \theta_I \) used to construct the normalizing transformation contains useful information about the orientation of the image in the observer–centered coordinate system, which can be used to disambiguate images which have the same invariant form, but a different classification depending on their absolute orientation. For example, in some typefaces a 6 and a 9 have the same rotationally normalized form, but their \( \theta_I \) values will differ by \( \pi \), and so the value of \( \theta_I \) modulo \( 2\pi \) can be used to resolve the six–nine ambiguity.

Up to this point we have assumed that the image to be normalized has no special rotational symmetries. Suppose now that \( I \) has an \( N \)-fold rotational symmetry, so that \( I(\vec{x}) = I(\vec{x}_{2\pi/N}) \), and let us consider the integral

\[
\int d^2 x \, e^{iM\Phi(\vec{x})} f(|\vec{x}|) I(\vec{x}) = \int d^2 x \, e^{iM\Phi(\vec{x})} f(|\vec{x}|) I(\vec{x}_{2\pi/N})
\]

\[
= \int d^2 x \, e^{iM\Phi(\vec{x}_{2\pi/N})} f(|\vec{x}|) I(\vec{x}) = e^{-iM2\pi/N} \int d^2 x \, e^{iM\Phi(\vec{x})} f(|\vec{x}|) I(\vec{x}) ,
\]

(37a)
vanishes unless $M/N$ is an integer (Hu [11]; Abu–Mostafa and Psaltis [2]). Consequently, when there is a rotational symmetry, the constraint of Eq. (33a) can no longer be used; instead, we must find the smallest value $M = N$ for which the integral of Eq. (37b) is nonvanishing, and then normalize using the constraint functional $C_N[I_S]$ defined by

$$C_N[I_S] = \text{Phase} \left[ \int d^2x \ e^{iN\Phi(x)} \ f(|x|) \ I(x) \right] - 1 . \quad (38a)$$

Solving the constraint $C_N[I_S] = 0$ now determines an angle $\theta = \theta_I$, unique up to an integer multiple of $2\pi/N$, which can be explicitly calculated from

$$e^{iN\theta_I} = \text{Phase} \left[ \int d^2x \ e^{iN\Phi(x)} \ f(|x|) \ I(x) \right] , \quad (38b)$$

with the corresponding normalizing transformation and normalized image still given by Eqs. (34c) and (36). In practice, one does not know a priori the rotational symmetry of the image being normalized; one then deals with the possibility of rotational symmetry by taking the constraint functional to have the form of Eq. (38a) and including a loop over $N = 1, 2, ...$ which terminates at the smallest value of $N$ for which the integral used to construct the constraint is nonvanishing.

D. Scaling. We turn next to scaling, with the group action

$$\tilde{S}(x) = \lambda x , \quad \lambda > 0$$

corresponding to which $I_S(\lambda x) = I(\lambda x)$ describes an image scaled in size by a factor $\lambda^{-1}$. We take as the constraint functional $C_{\mu\nu}[I_S], \ \mu \neq \nu$, given by

$$C_{\mu\nu}[I_S] = \frac{\int d^2x |x|^\mu g(\Phi(x)) I(\lambda x)}{\int d^2x |x|^\nu g(\Phi(x)) I(\lambda x)} - 1 . \quad (40a)$$
with $|\vec{x}|$ and $\Phi(\vec{x})$ as in Eq. (33c), and with $g$ an arbitrary function. The constraint $C_{\mu\nu} = 0$ determines a unique solution $\lambda = \lambda_I$, which by making the change of integration variable $\vec{x} \to \vec{x}/\lambda$ is readily found to be

$$\lambda_I = \left[ \frac{\int d^2x |\vec{x}|^{\mu} g(\Phi(\vec{x})) I(\vec{x})^{\mu}}{\int d^2x |\vec{x}|^{\nu} g(\Phi(\vec{x})) I(\vec{x})^{\nu}} \right]^{\mu-\nu}. \tag{40b}$$

The scale normalization transformation $\vec{N}_I(\vec{x})$ is then constructed as

$$\vec{N}_I(\vec{x}) = \lambda_I \vec{x}, \tag{40c}$$

and under the action of the scaling $S$, is transformed to

$$\vec{N}_{I_S}(\vec{x}) = \lambda_{I_S} \vec{x}. \tag{40d}$$

Writing the analog of Eq. (40b) for $\lambda_{I_S}$, substituting Eq. (39) and scaling $\lambda$ out of the integration variable as above, a simple calculation gives

$$\lambda_{I_S} = \lambda^{-1} \lambda_I, \tag{41a}$$

and so the normalization transformation $N_I$ becomes, under the action of $S$,

$$N_{I_S}(\vec{x}) = \lambda^{-1} \vec{N}_I(\vec{x}) = \vec{S}^{-1}(\vec{N}_I(\vec{x})), \tag{41b}$$

again in agreement with Eq. (7a). Following the recipe of Eq. (8), the image normalized with respect to scaling is

$$\vec{I}(\vec{x}) = I(\vec{N}_I(\vec{M}_I(\vec{x}))), \tag{42a}$$

with $\vec{M}_I(\vec{x})$ any transformation of $\vec{x}$ which depends only on scaling invariant image features. The customary choice of $\vec{M}_I$ is $\vec{M}_I(\vec{x}) = \lambda_0 \vec{x}$, with $\lambda_0$ an image-independent scale factor, which can be used to make the normalized form of one particular image have a specified
size. With this choice of $\vec{M}_I$, the normalizing integral $\int d^2x \vec{I}(\vec{x})$ appearing in the contrast normalized image of Eq. (18b) can be explicitly calculated in terms of the central moment $\mu_{00}$, giving

$$\int d^2x \vec{I}(\vec{x}) = \frac{\mu_{00}}{\lambda_0^2}.$$ (42b)

A useful specialization of the general method for scaling normalization is to take as the constraint

$$0 = \frac{\partial}{\partial \mu} C_{\mu \nu} [I_S] \bigg|_{\mu=\nu=0} = \int d^2x \frac{\log |\vec{x}| g(\Phi(\vec{x})) I(\vec{x})}{\int d^2x g(\Phi(\vec{x})) I(\lambda \vec{x})}.$$ (43a)

By a change of integration variable this can be solved to give a unique value $\lambda = \lambda_I$ given by

$$\log \lambda_I = \frac{\int d^2x \log |\vec{x}| g(\Phi(\vec{x})) I(\vec{x})}{\int d^2x g(\Phi(\vec{x})) I(\lambda \vec{x})},$$ (43b)

and which transforms under $S$ as

$$\log \lambda_{IS} = \log(\lambda^{-1} \lambda_I),$$ (43c)

in agreement with Eq. (41a). A potential advantage of the logarithmic weighting factor in Eq. (43b), as compared with the power law weighting factors in Eq. (40a), is that the logarithm does not suppress the contribution arising from either the center or the periphery of the image.

E. **General similarity transformations.** The general similarity transformation consists of a translation, a rotation, and a scaling, and so can be normalized using the methods of Subsecs. 2A–D. The rotational and scaling normalizations can evidently be combined into a single step, in which the arbitrary functions $f(|\vec{x}|)$ and $g(\Phi(\vec{x}))$ no longer appear, having been
replaced by the specific weightings appropriate to scaling and rotational normalizations. In
the image normalized with respect to general similarity transformations, the undetermined
map $\tilde{M}(\vec{x})$ can now depend only on general similarity invariants of the image, and can of
course be taken to be an image–independent map, including the identity transformation.

F. Homogeneous affine transformations. We turn next to the general case of the homogeneous
affine transformation, with the group action

$$\tilde{S}(\vec{x}) = G \cdot \vec{x} \equiv (G_{11}x_1 + G_{12}x_2, G_{21}x_1 + G_{22}x_2).$$  \hspace{1cm} (44a)$$

We again follow the subgroup decomposition method of Eqs. (19a)–(23), now taking (Dirilten
and Newman [6])

the subgroup $G_2$ to be the group of affine transformations with vanishing upper diagonal
matrix element, and the subgroup $G_1$ to be a rotation $R$, as in Eqs. (5a,b), so that

$$\tilde{S}_2(\vec{x}) = g \cdot \vec{x}, \quad \tilde{S}_1(\vec{x}) = \vec{x}_\theta,$$

$$\tilde{S}(\vec{x}) = \tilde{S}_2(\tilde{S}_1(\vec{x})) = g \cdot \vec{x}_\theta = G \cdot \vec{x}.$$

(44b)

To normalize the image with respect to the three–parameter group $G_2$, we will need three
constraints, which following [15] we take as

$$C_k[I_S] = 0, \quad k = 1, 2, 3,$$

(45a)

with the constraint functionals $C_{1,2,3}[I_S]$ given by

$$C_1[I_S] = \int \frac{d^2x \ x_1^2 \ I(G \cdot \vec{x})}{\int d^2x \ I(G \cdot \vec{x})} - 1,$$

$$C_2[I_S] = \int \frac{d^2x \ x_2^2 \ I(G \cdot \vec{x})}{\int d^2x \ I(G \cdot \vec{x})} - 1,$$

$$C_3[I_S] = \int d^2x \ x_1 \ x_2 \ I(G \cdot \vec{x}).$$

(45b)

Although the constraint functionals of Eq. (45b) are not independent of the element $S_1$ of
the subgroup $G_1$, that is, they are not $\theta$–independent, they are easily seen to simply mix
into $\theta$–dependent linear combinations of themselves as $\theta$ is varied, and so the constraints of Eq. (45a) are $\theta$–independent. Taking $\theta = 0$ and making a change of variable $\vec{x} \rightarrow g^{-1} \cdot \vec{x}$, one can explicitly solve the constraints to give a unique solution $g = g_I$,

$$g_{I11} = \left(\frac{\mu_{20}}{\mu_{00}}\right)^{1/2}, \quad g_{I12} = 0 ,$$

$$g_{I21} = \frac{\mu_{11}}{(\mu_{20}/\mu_{00})^{1/2}}, \quad g_{I22} = \left(\frac{\mu_{02} \mu_{20} - \mu_{11}^2}{\mu_{20}/\mu_{00}}\right)^{1/2}.$$  \(46\)

Since the Schwartz inequality implies that $\mu_{11}^2 \leq \mu_{02} \mu_{20}$, the matrix element $g_{I22}$ is always a real number. [Equation (46) assumes that $\mu_{20}$ is nonzero; if $\mu_{20}$ vanishes and if $I$ is not identically zero, then the moment $\mu_{02}$ will be nonvanishing, and so one can apply Eq. (46) after first rotating the image by 90 degrees.] The normalizing transformation for the subgroup $S_2$ is now constructed as

$$\vec{N}_{2I}(\vec{x}) = g_I \cdot \vec{x} .$$  \(47\)

By a lengthy algebraic calculation, one can verify that under a general proper (i.e., positive determinant) affine transformation $S$, the normalizing matrix $g_I$ transforms as

$$g_I \rightarrow g_{IS} = G^{-1} g_I R' ,$$  \(48a\)

with $R'$ a rotation matrix which is a complicated function of the matrix elements of $g_I$ and of $G$. Hence under the action of $S$, the normalizing transformation for $G_2$ becomes

$$\vec{N}_{2IS}(\vec{x}) = g_{IS} \cdot \vec{x}$$

$$= (G^{-1} g_I R') \cdot \vec{x} = \vec{S}^{-1}(\vec{N}_{2I}(R' \cdot \vec{x})) ,$$  \(48b\)

in agreement with Eq. (22c). To obtain an affine normalized image, one of course does not need the explicit form of $R'$; one first forms the partially normalized image

$$\tilde{I}(\vec{x}) = I(\vec{N}_{2I}(\vec{x})) ,$$  \(49a\)
and then normalizes with respect to rotations as in Eq. (36) of Subsec. 3C, to get the final normalized image

$$\hat{I}(\vec{x}) = I(N_{2I}(\vec{M}_I(\vec{x}_{\theta_1}))).$$  \(49b\)

The map $\vec{M}_I$ is constructed only from affine invariants of the image; the simplest choice is $\vec{M}_I(\vec{x}) = G_0 \cdot \vec{x}$, with $G_0$ a fixed affine transformation which can be chosen to give the affine normalized version of one particular pattern a specified form. With this choice of $\vec{M}_I$, the normalization integral $\int d^2x \hat{I}(\vec{x})$ required for contrast normalization is explicitly given in terms of central moments by

$$\int d^2x \hat{I}(\vec{x}) = \frac{\mu_{00}}{\det G_0 \det g_I} = \frac{\mu_{00}^2}{\det G_0 (\mu_{02} \mu_{20} - \mu_{11}^2)^{1/2}}. \quad (50)$$

An alternative way to normalize the homogeneous affine transformations is to use the subgroup factorization [c.f. Eq. (5c)]

$$\vec{S}_2(\vec{x}) = g' \cdot \vec{x}, \quad \vec{S}_1(\vec{x}) = \lambda \vec{x}_\theta,$$

$$\vec{S}(\vec{x}) = \vec{S}_2(\vec{S}_1(\vec{x})) = g' \cdot \lambda \vec{x}_\theta = G \cdot \vec{x}, \quad (51a)$$

with $g'$ restricted to have both zero upper right diagonal matrix element and unit determinant. Since $g'$ and $G_2$ now contain only two parameters, and since $G_1$ now includes both rotations and scalings, partial normalization with respect to $G_2$ requires two constraints which must be both rotation and scaling invariant. Inspecting Eq. (45b), we see that an obvious choice of constraint functionals is now

$$C'_1[I_S] = \int d^2x \ (x_2^2 - x_1^2) \ I(G \cdot \vec{x}),$$

$$C'_2[I_S] = \int d^2x \ x_1 x_2 \ I(G \cdot \vec{x}). \quad (51b)$$

Again, although these functionals are not rotation and (in the case of $C'_2$) scale invariant, the constraints $C'_1 = 0$, $C'_2 = 0$ are invariant, and solving them gives the not surprising result

$$g'_I = (\det g_I)^{1/2} g_I. \quad (51c)$$
with $g_I$ as given in Eq. (46). The normalizing transformation for the subgroup $S_2$ is now constructed as in Eq. (47), with $g'_I$ replacing $g_I$, and the partially normalized image is again given by Eq. (49a), but now the final step leading to a fully affine normalized image consists of a further combined normalization with respect to rotation and scaling of the type described in Subsec. 3E.

4. Viewing Transformations With Numerically Solvable Constraints

In this section we continue with the application of the general image normalization methods of Sec. 2 to the viewing transformations of a planar object, focusing on cases in which the constraints are not all algebraically solvable, so that iterative numerical methods are needed. We then go on to consider some other normalization problems of interest, that can also be solved by iterative methods.

A. Projective transformations. So far we have discussed linear transformations $\vec{S}(\vec{x})$, which within the general normalization framework of Sec. 2 lead to algebraically solvable constraints. We turn now to nonlinear transformations, to which the general analysis also applies, beginning with the planar projective transformation for which $\vec{S}(\vec{x})$ is given by

$$\vec{S}(\vec{x}) = \frac{\sum_{m=1}^{2} G_{nm} x_m + t_n}{1 + \sum_{m=1}^{2} p_m x_m}. \quad (52a)$$

We again use the subgroup decomposition method, writing

$$\vec{S}(\vec{x}) = \vec{S}_2(\vec{S}_1(\vec{x})), \quad (52b)$$

with $S_2 \in G_2$ a restricted projective transformation and $S_1 \in G_1$ an affine transformation, as in Eq. (3) and Eq. (2) respectively. Since $G_2$ is a two–parameter Lie group, we need two constraints, which must be invariant under the action of the affine transformations of $G_1$, to
partially normalize the image. We have not been able to find two simple constraint functionals which yield algebraically solvable affine invariant constraints when equated to zero, which would be the analog of our previous two applications of the subgroup method. Instead, we work with constraint functionals which are fully affine and contrast invariant, as obtained by the algebraic methods of Hu [11] and Reiss [14], which because of their complexity must be solved numerically. (Alternatively, one could formulate the projective constraints using two independent affine invariants constructed by the affine normalization procedure of the preceding section, again solving the constraints numerically. We emphasize that in either case, the constraints used for projective normalization are not projective invariants, but only invariants under the much simpler affine subgroup of the full projective group.) Using only third and lower central moments, one can form the following three functionals of the image which are affine and contrast invariant, and which are non-singular (in fact vanishing) for images with both $x_1$ and $x_2$ reflection symmetry,

$$
\Psi_1[I] = \frac{\mu_{20}^2 I_2}{I_1^3}, \quad \Psi_2[I] = \frac{\mu_{00} I_3}{I_1^2}, \quad \Psi_3[I] = \frac{\mu_{00} I_4}{I_1^3},
$$

$$
I_1 = AC - B^2,
$$

$$
I_2 = (ad - bc)^2 - 4(ac - b^2)(bd - c^2),
$$

$$
I_3 = A(bd - c^2) - B(ad - bc) + C(ac - b^2),
$$

$$
I_4 = a^2C^3 - 6abBC^2 + 6acC(2B^2 - AC) + ad(6ABC - 8B^3) + 9b^2AC^2
$$

$$
- 18bcABC + 6bdA(2B^2 - AC) + 9c^2A^2C - 6cdA^2B + d^2A^3 ,
$$

$$
A = \mu_{20} , \quad B = \mu_{11} , \quad C = \mu_{02} ,
$$

$$
a = \mu_{30} , \quad b = \mu_{21} , \quad c = \mu_{12} , \quad d = \mu_{03} .
$$
For example, \( C_1[I] = C_2[I] = 0 \) could be used as constraints, with

\[
C_1[I] = \Psi_1[I] - \Psi_0^1, \quad C_2[I] = \Psi_2[I] - \Psi_0^2, \quad (53b)
\]

provided the numerical target values \( \Psi_0^1, \Psi_0^2 \) fall within the ranges taken by \( \Psi_1, \Psi_2 \) for the image \( I \) being normalized. Since the restricted projective transformation

\[
\vec{S}_2(\vec{x}) = \frac{\vec{x}}{1 + \vec{p} \cdot \vec{x}} \quad (54a)
\]

depends nonlinearly on the parameter \( \vec{p} \), we cannot algebraically solve the constraints to find the normalizing parameter \( \vec{p}_I \), but this can be readily done numerically by an iterative method. The partial normalization transformation and partially normalized image are now given by

\[
\vec{N}_{2I}(\vec{x}) = \frac{\vec{x}}{1 + \vec{p}_I \cdot \vec{x}}, \quad (54b)
\]

\[
\tilde{I}(\vec{x}) = I(\vec{N}_{2I}(\vec{x})) .
\]

Finally, one must do a further affine normalization, as in Subsecs. 3B and 3F, to get an image normalized with respect to the full planar projection group. If the initial image is not well-centered on the raster, it may be advantageous to also do an affine normalization before the restricted projective partial normalization; this does not affect the results provided a second affine normalization is still done as the final normalization step.

The fact that one must know the range of \( \Psi_1, \Psi_2 \) to pick target numerical values for normalization may prove a significant limitation, since it is likely that there is no universal pair of target values which is guaranteed to be attainable for any image. (For a discussion of related problems with projective normalization, see Åström [1].) Consequently, it may be necessary to have a preliminary classification of the viewed object before attempting projective normalization. However, this may not be a problem in some applications, as for
example when an approaching object is tracked and can be classified when it is still far enough away for the affine approximation to the general projective transformation to be accurate. As the object gets closer, knowledge of its class can be used to determine the constraints to be used in projective normalization, and the values of $\vec{p}_I$ and the affine parameters obtained from projective normalization can then be used to deduce information about the object’s absolute orientation. Another option, in applications where preliminary classification by an affine normalizing classifier is feasible, is to use optimization of the match $M$ through the classifier to supply the projective constraints; that is, the constraints are taken as

$$\frac{\partial M}{\partial p_1} = \frac{\partial M}{\partial p_2} = 0,$$

which are solved by iteration on the restricted projective parameter $\vec{p}$ to determine $\vec{p}_I$.

B. Similarity and Affine Normalization of Partially Occluded Planar Curves. So far we have discussed only the normalization of non-occluded images, but the general methods formulated in Sec. 2 have been extended by Adler and Krishnan [3] to the more realistic problem [4], [5] of the similarity and affine normalization of a partially occluded planar curve, such as that characterizing the boundary of a partially occluded planar object. Since full details and illustrative numerical results are given in [3], we give here only a sketch of the strategy.

Consider, for simplicity, the special case in which one has a curve segment, with an identifying point $P$, distorted by an affine transformation. One can construct an affine normalization by the method of Sec. 3, by forming constraints using second moments of the curve integrated along a finite segment from $P$ to some neighboring point $P'$. To specify this integration segment in an affine invariant way, reference [3] requires that the normalized image of the segment have some specified reparameterization invariant arc-length, giving one additional constraint that must be solved by numerical iteration. The resulting normalization procedure
for partially occluded planar curves normalizes against affine transformations using as input only first and second parametric derivatives, i.e., only information about the tangent vector and the curvature of the curve. This example shows how the general methods of this paper can be used as modules in iterative procedures
to solve new, previously unsolved, classes of normalization problems.

C. Flexible template normalization. The nonlinear projective image transformations which we have just discussed are only one example of much more general nonlinear distortions which can make an observed image differ in form from the standard prototype for its class. Examples of such distortions include non–planar geometric effects when a character to be recognized is printed on a curved surface, variations among hand lettered characters produced by different individuals, and variations in facial geometry as a result of changes in facial expression. An attractive proposal for dealing with such distortions is the use of “flexible” or “deformable” templates (see, e.g. [9]), and the general normalization methods of Sec. 2 give a possible means for their implementation. We consider briefly in this subsection the case of distortions which can be modeled as an image transformation

\[ I(\vec{x}) \rightarrow I(\vec{T}(\vec{x})) , \]  

(56a)

where \( \vec{x}' = \vec{T}(\vec{x}) \) is a general nonlinear remapping or diffeomorphism of the image coordinate \( \vec{x} \). Such diffeomorphisms form a group, and so the general analysis of Sec. 2 formally applies, but since the general diffeomorphism group has an infinite number of parameters, this observation is of little practical use without making further assumptions. Let us now suppose that the predominant nonlinear distortions can be treated as small in magnitude, and are well represented by a few terms in an appropriate complete expansion basis. A
concrete example would be nonlinear distortions described by the transformation

$$\vec{T}(\vec{x}) = \vec{t} + G \cdot \vec{x} + H \cdot \vec{x}^2 + J \cdot \vec{x}^3,$$

with the $H$ term shorthand for the vector with components $\sum_{mp} H_{nmp} x_m x_p$, and similarly for the $J$ term. When $H, J$ are effectively of order unity, the transformations of Eq. (56b) do not form a group, since iteration of the transformation leads to fourth and higher order terms in $\vec{x}$. However, if $H, J$ are small enough for terms quadratic and higher order in $H, J$ to be neglected, the transformations of Eq. (56b) do, within the first order approximation, form a group, and the normalization methods of Sec. 2 become applicable. One could then proceed by first constructing an affine preprocessing classifier by the methods of Subsecs. 3A–F, thus normalizing for the linear transformation given by the first two terms of Eq. (56b). One would then normalize with respect to all of Eq. (56b) by iterating on the coefficients $H, J$ to try to get an optimal unique classification through this classifier, using the cost function

$$C = ||\text{classifier mismatch}|| + ||H|| + ||J||,$$

with the terms in Eq. (56c) giving respectively measures of the magnitude of the classifier mismatch for the class being considered, the magnitude of the coefficients $H$, and the magnitude of the coefficients $J$. Clearly, a similar method could be applied to the expansions of $\vec{T}$ on any polynomial (and perhaps more general) basis, provided the truncated basis is left invariant in form within a suitable first order approximation, for which the affine transformations form the zeroth order approximation.

D. Normalization of an Image on a Sphere. As a final application of the methods of Sec. 2, we briefly discuss the normalization of an image $I(\Omega)$ defined on the surface of a sphere of radius $R$ and angular variables $\Omega$, with respect to the group of rotations $S$ of the sphere. We
shall confine ourselves to the simplest case, in which the image $I$ has no special symmetries which make the relevant constraint integrals vanish. The normalization can be carried out in two steps. The first step imposes the constraint

$$\bar{C}[I_S] = \frac{\int d\Omega \ I_S(\Omega) \ \hat{n}_\Omega}{\int d\Omega \ I_S(\Omega) \ \hat{n}_\Omega} - (0,0,1) = 0 ,$$

(57)

with $\hat{n}_\Omega$ the outward pointing three dimensional unit normal to the sphere at $\Omega$. This rotates the sphere so that the positive $x_3$ axis (the north polar axis of the sphere) passes through the center of mass (calculated in spherical geometry) of the image. The second step consists of a rotational normalization with respect to azimuth (or longitude) using the formulas of Subsec. 3C, in which dependences on $|\vec{x}|$ are replaced by dependences on the spherical polar angle (or latitude). In the group contraction limit in which the sphere radius $R$ approaches infinity while the dimension of the region of support of the image remains bounded, this normalization recipe reduces to that of Subsecs. 3A, C for combined translational and rotational normalization of a planar image.

5. Summary and Discussion

We have given a general normalization method for viewing transformations of planar images, based on imposing a set of constraints equal in number to the parameters of the viewing transformation group, the solution of which gives the parameters of the normalizing transformation. In Sec. 3 we discussed linear viewing transformations, for which algebraically solvable constraints can be given. In Sec. 4 we discussed more complex situations, in which some of the constraints cannot be solved algebraically, but can be solved by numerical iterative methods. Although the normalization methods of Subsecs. 3A–F and 4A-B were all based on the use of moments or other weighted integrals of the image to construct the
constraints, the general analysis of Sec. 2 does not require this. Alternative methods include setting the scale normalization by a determination of the outer boundary of the image (as in [10], [16]), and setting the rotational normalization angle after scale normalization by using the maximum of $I$ in an annular ring of given radius, both of which are methods that use local image features instead of weighted integrals over the image.

The most convenient set of constraints will, in practice, depend on the specifics of the invariance problem being analyzed. In general, the larger the number of constraints that can be solved algebraically, and the smaller the number that require numerical solution, the more computationally efficient will be the resulting normalization method. For this reason, we have given particular emphasis to subgroup methods, that express some of the constraints needed for more complex normalization problems in terms of those already constructed for simpler normalization problems, for which algebraic solution methods are available.

In conclusion, we emphasize that according to the general theory established in Sec. 2 and illustrated in Secs. 3 and 4, any set of constraints that uniquely breaks the viewing transformation group invariance suffices to construct a normalization, and thereby to yield all viewing transformation invariants. The difference between normalizations constructed using alternative sets of constraints will always be representable by a residual mapping of the image, depending on the image only through viewing transformation invariants.

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