Coloring translates and homothets of a convex body

Adrian Dumitrescu∗ Minghui Jiang†

August 10, 2010

Abstract

We obtain improved upper bounds and new lower bounds on the chromatic number as a linear function of the clique number, for the intersection graphs (and their complements) of finite families of translates and homothets of a convex body in \( \mathbb{R}^n \).

Keywords: graph coloring, geometric intersection graph.

1 Introduction

Let us recall the following well-known hypergraph invariants for a family \( \mathcal{F} \) of sets:

- **clique number** \( \omega(\mathcal{F}) \) is the maximum number of pairwise intersecting sets in \( \mathcal{F} \).
- **packing number** \( \nu(\mathcal{F}) \) is the maximum number of pairwise disjoint sets in \( \mathcal{F} \).
- **clique-partition number** \( \vartheta(\mathcal{F}) \) is the minimum number of classes in a partition of \( \mathcal{F} \) into subfamilies of pairwise intersecting sets.
- **coloring number** \( q(\mathcal{F}) \) is the minimum number of classes in a partition of \( \mathcal{F} \) into subfamilies of pairwise disjoint sets.

Let \( G \) be the intersection graph of \( \mathcal{F} \) such that the vertices in \( G \) correspond to the sets in \( \mathcal{F} \), one vertex for each set, and an edge connects two vertices in \( G \) if and only if the corresponding two sets in \( \mathcal{F} \) intersect. Then the four hypergraph invariants for \( \mathcal{F} \) are respectively the same as the following four graph invariants for \( G \):

- **clique number** \( \omega(G) \) is the maximum number of pairwise adjacent vertices (i.e., the maximum size of a clique) in \( G \).
- **independence number** (or **stability number** \( \alpha(G) \) is the maximum number of pairwise non-adjacent vertices (i.e., the maximum size of an independent set) in \( G \).
- **clique-partition number** \( \vartheta(G) \) is the minimum number of classes in a partition of the vertices of \( G \) into subsets of pairwise adjacent vertices.

∗Department of Computer Science, University of Wisconsin–Milwaukee, WI 53201-0784, USA. Email: dumitres@uwm.edu. Supported in part by NSF CAREER grant CCF-0444188.

†Department of Computer Science, Utah State University, Logan, UT 84322-4205, USA. Email: mjiang@cc.usu.edu. Supported in part by NSF grant DBI-0743670.
chromatic number $\chi(G)$ is the minimum number of classes in a partition of the vertices of $G$ into subsets of pairwise non-adjacent vertices.

Let $\overline{G}$ be the complement graph of $G$ with the same vertices as $G$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. Then $\alpha(G) = \omega(\overline{G})$ and $\vartheta(G) = \chi(\overline{G})$.

For any family $\mathcal{F}$ of sets, we always have the following two obvious inequalities

$$
\omega(\mathcal{F}) \leq q(\mathcal{F}), \quad \nu(\mathcal{F}) \leq \vartheta(\mathcal{F}).
$$

In graph invariants, the two inequalities become

$$
\omega(G) \leq \chi(G), \quad \omega(\overline{G}) \leq \chi(\overline{G}).
$$

Inequalities in the opposite directions, if any, are less obvious. That is, we have only limited knowledge about possible upper bounds on the chromatic number as a function of the clique number for various classes of graphs. In this paper, we focus on finite families $\mathcal{F}$ of translates or homothets of a convex body in $\mathbb{R}^n$, and study upper bounds on the chromatic number in terms of the clique number in the intersection graphs of such families $\mathcal{F}$ and in the complement graphs. Recall that a convex body is a compact convex set with non-empty interior. Many similar bounds have been studied for various geometric intersection graphs and their complements since the pioneering work of Asplund and Grünbaum [3], Gyárfás [10], and Gyárfás and Lehel [11]. We refer to Kostochka [15] for a more recent survey.

Definitions. For two convex bodies $A$ and $B$ in $\mathbb{R}^n$, denote by $A + B = \{a + b \mid a \in A, b \in B\}$ the Minkowski sum of $A$ and $B$. For a convex body $C$ in $\mathbb{R}^n$, denote by $\lambda C = \{\lambda c \mid c \in C\}$ the scaled copy of $C$ by a factor of $\lambda \in \mathbb{R}$, denote by $C + p = \{c + p \mid c \in C\}$ the translate of $C$ by a vector from the origin to a point $p \in \mathbb{R}^n$, and denote by $\lambda C + p = \{\lambda c + p \mid c \in C\}$ the homothet of $C$ obtained by first scaling $C$ by a factor of $\lambda$ then translating the scaled copy by a vector from the origin to $p$. Also denote by $-C = \{-c \mid c \in C\}$ the reflexion of $C$ about the origin, and write $C - C$ for $C + (-C)$.

We review some standard definitions concerning packing densities; see [4, Section 1.1]. A family $\mathcal{F}$ of convex bodies is a packing in a domain $Y \subseteq \mathbb{R}^n$ if $\bigcup_{C \in \mathcal{F}} C \subseteq Y$ and the convex bodies in $\mathcal{F}$ are pairwise interior-disjoint. Denote by $\mu(S)$ the Lebesgue measure of a compact set $S$ in $\mathbb{R}^n$, i.e., area in the plane, or volume in the space. Define the density of a packing $\mathcal{F}$ relative to a bounded domain $Y$ as

$$
\rho(\mathcal{F}, Y) := \frac{\sum_{C \in \mathcal{F}} \mu(C \cap Y)}{\mu(Y)}.
$$

When $Y = \mathbb{R}^n$ is the whole space, define the upper density of $\mathcal{F}$ as

$$
\overline{\rho}(\mathcal{F}, \mathbb{R}^n) := \limsup_{r \to \infty} \rho(\mathcal{F}, B^n(r)),
$$

where $B^n(r)$ denote a ball of radius $r$ centered at the origin (since we are taking the limit as $r \to \infty$, a hypercube of side length $r$ can be used instead of a ball of radius $r$). For a convex body $C$ in $\mathbb{R}^n$, define the packing density of $C$ as

$$
\delta(C) := \sup_{\mathcal{F} \text{ packing}} \overline{\rho}(\mathcal{F}, \mathbb{R}^n),
$$

where $\mathcal{F}$ ranges over all packings in $\mathbb{R}^n$ with congruent copies of $C$. If the members of $\mathcal{F}$ are restricted to translates of $C$, then we have the translative packing density $\delta_T(C)$, which is invariant under any non-singular affine transformation of $C$. 


**Translates and homothets of a convex body.** For \( n = 1 \), a convex body in \( \mathbb{R}^n \) is an interval, and the intersection graph of a finite family \( \mathcal{F} \) of translates or homothets of an interval is an interval graph. Since interval graphs and their complements are perfect graphs \([9]\), we always have perfect equalities \( \omega(\mathcal{F}) = q(\mathcal{F}) \) and \( \nu(\mathcal{F}) = \vartheta(\mathcal{F}) \).

Henceforth let \( n \geq 2 \). Let \( \mathcal{T} \) be a finite family of translates of a convex body in \( \mathbb{R}^n \). Let \( \mathcal{H} \) be a finite family of homothets of a convex body in \( \mathbb{R}^n \). Kostochka \([15]\) proved that

1. if \( \omega(\mathcal{T}) = k \), then \( q(\mathcal{T}) \leq n(2n)^{n-1}(k-1) + 1 \), and
2. if \( \omega(\mathcal{H}) = k \), then \( q(\mathcal{H}) \leq (2n)^n(k-1) + 1 \).

Kim and Nakprasit \([14]\) proved the complementary result\(^1\) that

1. if \( \nu(\mathcal{T}) = k \), then \( \vartheta(\mathcal{T}) \leq n(2n)^{n-1}(k-1) + 1 \), and
2. if \( \nu(\mathcal{H}) = k \), then \( \vartheta(\mathcal{H}) \leq (2n)^n(k-1) + 1 \).

For the planar case \( n = 2 \), there exist better bounds \( q(\mathcal{T}) \leq 3\omega(\mathcal{T}) - 2 \) and \( q(\mathcal{H}) \leq 6\omega(\mathcal{T}) - 6 \) by Kim, Kostochka, and Nakprasit \([13]\), and \( \vartheta(\mathcal{T}) \leq 3\nu(\mathcal{T}) - 2 \) and \( \vartheta(\mathcal{H}) \leq 6\nu(\mathcal{H}) - 5 \) by Kim and Nakprasit \([14]\).

For translates, we obtain the following improved bounds:

**Theorem 1.** Let \( \mathcal{T} \) be a finite family of translates of a convex body in \( \mathbb{R}^n \), \( n \geq 2 \). Let \( t_n = (n+1)^{n-1}\lceil \frac{n+1}{2} \rceil \). Then \( q(\mathcal{T}) \leq t_n \omega(\mathcal{T}) \) and \( \vartheta(\mathcal{T}) \leq t_n \nu(\mathcal{T}) \).

Note that for all \( n \geq 2 \), the multiplicative factors \( t_n = (n+1)^{n-1}\lceil \frac{n+1}{2} \rceil \) in Theorem 1 are exponentially smaller than the corresponding factors \( n(2n)^{n-1} \) in the previous bounds \([15, 14]\).

For two convex bodies \( A \) and \( B \) in \( \mathbb{R}^n \), denote by \( \kappa(A, B) \) the smallest number \( \kappa \) such that \( A \) can be covered by \( \kappa \) translates of \( B \). For homothets, we obtain the following bounds:

**Theorem 2.** Let \( \mathcal{H} \) be a finite family of homothets of a convex body \( C \) in \( \mathbb{R}^n \), \( n \geq 2 \). Let \( h(C) = \kappa(C - C, C) \). Then \( q(\mathcal{H}) \leq h(C)(\omega(\mathcal{H}) - 1) + 1 \) and \( \vartheta(\mathcal{H}) \leq h(C)(\nu(\mathcal{H}) - 1) + 1 \).

It remains to bound \( \kappa(C - C, C) \). For a convex body \( C \) in \( \mathbb{R}^n \), denote by \( \theta_T(C) \) the infimum of the covering density of \( \mathbb{R}^n \) by translates of \( C \). According to a result of Rogers \([17]\), \( \theta_T(C) \leq n \ln n + n \ln \ln n + 5n = O(n \log n) \) for any convex body \( C \) in \( \mathbb{R}^n \). The following lemma collects the previously known upper bounds on \( \kappa(C - C, C) \) from \([7]\):

**Lemma 1** (Danzer and Rogers, 1963). Let \( C \) be a convex body in \( \mathbb{R}^n \), \( n \geq 2 \). Then \( \kappa(C - C, C) \leq 3^{n+1}2^n(n+1)^{-1}\theta_T(C) = O(6^n \log n) \). Moreover, if \( C \) is centrally symmetric, then \( \kappa(C - C, C) = \kappa(2C, C) \leq \min\{5^n, 3^n\theta_T(C)\} = O(3^n n \log n) \).

Note that by Lemma 1, the multiplicative factors \( h(C) = O(6^n \log n) \) in Theorem 2 are exponentially smaller than the corresponding factors \( (2n)^n \) in the previous bounds \([15, 14]\).

For the coloring problem on finite families \( \mathcal{T} \) of translates of a convex body \( C \) in \( \mathbb{R}^n \), Kostochka \([15]\) noted that, by the following old result of Minkowski, we can assume that \( C \) is centrally symmetric:

**Lemma 2** (Minkowski, 1902). Let \( a \) and \( b \) be two points and let \( C \) be a convex body in \( \mathbb{R}^n \), \( n \geq 2 \). Then \( (C + a) \cap (C + b) \neq \emptyset \) if and only if \( \left( \frac{1}{2}(C - C) + a \right) \cap \left( \frac{1}{2}(C - C) + b \right) \neq \emptyset \).

---

\(^1\)Kim and Nakprasit \([14]\) stated their result as \( \vartheta(\mathcal{T}) \leq \lfloor n_- \rfloor [2n_-]^{n-1}(k-1) + 1 \) and \( \vartheta(\mathcal{H}) \leq [2n_-]^{n}(k-1) + 1 \), where \( n_- = (n^2 - n + 1)^{1/2} \). But since \( n - 1/2 < n_- \leq n \) for all \( n \geq 1 \), we indeed have \( \lfloor n_- \rfloor = n \) and \( [2n_-] = 2n \).
Note that if $C$ is a convex body, then $\frac{1}{2}(C - C)$ is a centrally symmetric convex body, and $\frac{1}{2}(C - C) - \frac{1}{2}(C - C) = C - C$. Thus, by Theorem 2 and Lemma 2 we have the following corollary:

**Corollary 1.** Let $T$ be a finite family of translates of a convex body $C$ in $\mathbb{R}^n$, $n \geq 2$. Let $t(C) = \kappa(C-C, \frac{1}{2}(C-C))$. Then $q(T) \leq t(C)(\omega(T) - 1) + 1$ and $\vartheta(T) \leq t(C)(\nu(T) - 1) + 1$.

Note that by Lemma 1 we have $t(C) = O(3^n n \log n)$. Thus, for sufficiently large $n$, the upper bounds in Corollary 1 are better than those in Theorem 1.

For a convex body $C$ in $\mathbb{R}^n$, define

$$r_T(C) = \sup_T \frac{q(T)}{\omega(T)}, \quad \tau_T(C) = \sup_T \frac{\vartheta(T)}{\nu(T)}, \quad r_H(C) = \sup_H \frac{q(H)}{\omega(H)}, \quad \tau_H(C) = \sup_H \frac{\vartheta(H)}{\nu(H)},$$

where $T$ ranges over all finite families of translates of $C$, and $H$ ranges over all finite families of homothets of $C$. Clearly, $r_T(C) \leq r_H(C)$ and $\tau_T(C) \leq \tau_H(C)$. Our results in Theorem 1, Theorem 2, and Corollary 1 can be summarized as follows:

$$r_T(C), \tau_T(C) \leq \min \left\{ (n+1)^{n-1} \left[ \frac{n+1}{2} \right], 5^n, 3^n \theta_T \left( \frac{1}{2}(C-C) \right) \right\} \quad (3)$$

$$r_H(C), \tau_H(C) \leq 3^{n+1} 2^n (n+1)^{-1} \theta_T(C) \quad (4)$$

A natural question is whether the four ratios $r_T(C), \tau_T(C), r_H(C),$ and $\tau_H(C)$ need to be exponential in $n$. The following theorem gives a positive answer:

**Theorem 3.** Let $C$ be a convex body in $\mathbb{R}^n$, $n \geq 2$. Then $r_H(C) \geq r_T(C) \geq 1/\delta_T(C)$ and $\tau_H(C) \geq \tau_T(C) \geq 1/\delta_T(C)$, where $\delta_T(C)$ is the translatable packing density of $C$. In particular, if $C$ is the unit ball $B^n$ in $\mathbb{R}^n$, then $r_H(C) \geq r_T(C) \geq 2^{(0.599 \pm o(1))n}$ and $\tau_H(C) \geq \tau_T(C) \geq 2^{(0.599 \pm o(1))n}$ as $n \to \infty$.

Note that our Theorem 8 gives the first general lower bounds for any convex body $C$ in $\mathbb{R}^n$, $n \geq 2$. Moreover, it gives the first lower bounds on these ratios that are exponential in the dimension $n$. Only a constant lower bound on $r_T(C)$ was previously known for the special case that $C$ is an axis-parallel square [15, 1]. We discuss this case next.

**Axis-parallel unit squares.** An interesting special case of the coloring problem is for finite families $F$ of axis-parallel unit squares in the plane. Akiyama, Hosono, and Urabe [2] proved that if $\omega(F) = 2$, then $q(F) \leq 3$, and conjectured that, in general, if $\omega(F) = k$, then $q(F) \leq k + 1$. Ahlswede and Karapetyan [1] recently gave a construction that disproves this conjecture. Their construction consists of a family $F_k$ of squares for each $k \geq 1$, which corresponds to an intersection graph that can be obtained by “replacing each vertex of a pentagon ($C_5$) by a $k$-clique”. Ahlswede and Karapetyan claimed that the family $F_k$ satisfies $q(F_k) = 3k$ and $\omega(F_k) = 2k$, and hence gives a lower bound of $3/2$ on the multiplicative factor in the linear upper bound. On the other hand, Kostochka [15, p. 132] mentioned a lower bound of only $5/4$ (for translates of any convex body in the plane), but gave no details and no references. The following theorem resolves this discrepancy by showing that the family $F_k$ in the construction by Ahlswede and Karapetyan indeed disproves the conjecture of Akiyama, Hosono, and Urabe, although it only satisfies $q(F_k) = \left\lceil \frac{3}{2} k \right\rceil$ and $\omega(F_k) = 2k$:

**Theorem 4.** For every positive integer $k$, there is a family $F_k$ of axis-parallel unit squares in the plane such that $\omega(F_k) = 2k$ and $q(F_k) = \left\lceil \frac{3}{2} k \right\rceil$, and there is a family $F_k$ of axis-parallel unit squares in the plane such that $\nu(F_k) = 2k$ and $\vartheta(F_k) = 3k$. 

4
For any finite family $F$ of axis-parallel unit hypercubes in $\mathbb{R}^n$, Perepelitsa \cite{perepelitsa1967} showed that if $\omega(F) = k$, then $q(F) \leq 2^{n-1}(k-1) + 1$. Since $\kappa(C - C, C) = \kappa(2C, C) = 2^n$ for a hypercube $C$ in $\mathbb{R}^n$, Theorem \cite{perepelitsa1967} implies that if $\nu(F) = k$, then $\vartheta(F) \leq 2^{n-1}(k-1) + 1$ too. In particular, for any finite family $F$ of axis-parallel unit squares in the plane, we have $q(F) \leq 2\omega(F) - 1$ and $\vartheta(F) \leq 2\nu(F) - 1$. By Theorem \cite{perepelitsa1967}, the multiplicative factors of 2 in these two inequalities cannot be improved to below $\frac{5}{4}$ and $\frac{3}{2}$, respectively. It is interesting that the current best lower bounds for the two factors are different.

## 2 Upper bounds for translates of a convex body in $\mathbb{R}^n$

In this section we prove Theorem \cite{perepelitsa1967}. Let $T$ be a finite family of translates of a convex body $C$ in $\mathbb{R}^n$, $n \geq 2$. Let $P$ and $Q$ be two homothetic parallelepipeds with ratio $n$ such that $P \subseteq C \subseteq Q$, as guaranteed by the following result of Chakerian and Stein \cite{chakerian1967}:

**Lemma 3** (Chakerian and Stein, 1967). Let $C$ be a convex body in $\mathbb{R}^n$. Then $C$ contains a parallelepiped $P$ such that some translate of $nP$ contains $C$.

Since the intersection graph of $T$ is invariant under any affine transformation of $\mathbb{R}^n$, we can assume without loss of generality that $P$ is an axis-parallel unit hypercube centered at the origin, and that $Q$ is an axis-parallel hypercube of side length $n$. Then each $C$-translate $C_p = C + p$ in $T$ is specified by a reference point $p$ that is the center of the corresponding $P$-translate. We first consider a special case of the coloring problem in the following lemma:

**Lemma 4.** Let $\mathcal{T}_\ell$ be a subfamily of $C$-translates in $T$ whose corresponding $P$-translates intersect a common line $\ell$ parallel to the axis $x_n$. Let $c_n = \lceil \frac{n+1}{2} \rceil$. Then $q(\mathcal{T}_\ell) \leq c_n \omega(\mathcal{T}_\ell)$ and $\vartheta(\mathcal{T}_\ell) \leq c_n \nu(\mathcal{T}_\ell)$.

**Proof.** For each integer $j$, let $U_j$ be the axis-parallel unit cube whose center is on the line $\ell$ and has $x_n$-coordinate $j$. Note that the reference point of each $C$-translate in $\mathcal{T}_\ell$ is contained in some unit cube $U_j$. Let $\mathcal{T}_c$ be the subfamily of $C$-translates in $\mathcal{T}_\ell$ whose reference points are in the unit cubes $U_j$ with $j \mod c_n = c$. We will show that the complement of the intersection graph of each subfamily $\mathcal{T}_c$, $0 \leq c \leq c_n - 1$, is a comparability graph.

Define a relation $\prec$ on the $C$-translates in $\mathcal{T}_c$ such that $C_1 \prec C_2$ if and only if (i) $C_1$ and $C_2$ are disjoint, and (ii) the reference point of $C_1$ has a smaller $x_n$-coordinate than the reference point of $C_2$. Then the complement of the intersection graph of $\mathcal{T}_c$ has an edge between two vertices $C_1$ and $C_2$ if and only if either $C_1 \prec C_2$ or $C_2 \prec C_1$. It is clear that the relation $\prec$ is irreflexive and asymmetric. We next show that $\prec$ is also transitive, and is thus a strict partial order.

Let $C_1, C_2, C_3$ be any three $C$-translates in $\mathcal{T}_c$ such that $C_1 \prec C_2$ and $C_2 \prec C_3$. Refer to Figure \cite{example} for an example in the plane. We will show that $C_1 \prec C_3$. Let $U_{j_1}, U_{j_2}, U_{j_3}$ be three unit cubes containing the reference points of $C_1, C_2, C_3$, respectively. Since any two $C$-translates with reference points in the same unit cube $U_j$ must intersect each other, the condition $C_1 \prec C_2$ implies that $j_1 < j_2$. Moreover we must have $j_1 \leq j_2 - c_n$ since $j_1 \equiv j_2 \pmod{c_n}$. Similarly, the condition $C_2 \prec C_3$ implies that $j_2 \leq j_3 - c_n$. It follows that $j_3 - j_1 \geq 2c_n \geq n + 1$. The distance between the references points of $C_1$ and $C_3$ is at least the distance between the centers of $U_{j_1}$ and $U_{j_3}$ minus 1, which is at least $n$. This implies that $C_1$ and $C_3$ are disjoint, since each $C$-translate is contained in an axis-parallel hypercube of side length $n$. Thus $C_1 \prec C_3$ because (i) $C_1$ and $C_3$ are disjoint, and (ii) the reference point of $C_1$ has smaller $x_n$-coordinate than the reference point of $C_3$. We have shown that $\prec$ is a strict partial order. Consequently, the complement of the intersection graph of each subfamily $\mathcal{T}_c$, $0 \leq c \leq c_n - 1$, is a comparability graph.
It is well-known that comparability graphs and their complements are perfect graphs [9]. So we have \( q(T_c) = \omega(T_c) \) and \( \vartheta(T_c) = \nu(T_c) \) for all \( 0 \leq c \leq c_n - 1 \). Therefore,

\[
q(T_\ell) \leq \sum_c q(T_c) = \sum_c \omega(T_c) \leq \sum_c \omega(T_\ell) = c_n \omega(T_\ell).
\]

\[
\vartheta(T_\ell) \leq \sum_c \vartheta(T_c) = \sum_c \nu(T_c) \leq \sum_c \nu(T_\ell) = c_n \nu(T_\ell).
\]

For each point \((a_1, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}\), denote by \(\langle a_1, \ldots, a_{n-1} \rangle\) the following line in \(\mathbb{R}^n\) that is parallel to the axis \(x_n\):

\[
\{ (x_1, \ldots, x_n) \mid (x_1, \ldots, x_{n-1}) = (a_1, \ldots, a_{n-1}) \}.
\]

Now consider the following (infinite) set \(\mathcal{L}\) of (periodical) parallel lines:

\[
\mathcal{L} = \{ \langle j_1 + b_1, \ldots, j_{n-1} + b_{n-1} \rangle \mid (j_1, \ldots, j_{n-1}) \in \mathbb{Z}^{n-1} \},
\]

where the offset \((b_1, \ldots, b_{n-1}) \in \mathbb{R}^{n-1}\) is chosen such that no line in \(\mathcal{L}\) is tangent to the \(P\)-translate of any \(C\)-translate in \(\mathcal{T}\). Recall that \(P\) and \(Q\) are axis-parallel hypercubes of side lengths 1 and \(n\), respectively. Thus we have the following two properties:

1. For any \(C\)-translate in \(\mathcal{T}\), the corresponding \(P\)-translate intersects exactly one line in \(\mathcal{L}\).

2. For any two \(C\)-translates in \(\mathcal{T}\), if the two corresponding \(P\)-translates intersect two different lines in \(\mathcal{L}\) at distance at least \(n+1\) along some axis \(x_i\), \(1 \leq i \leq n-1\), then the two \(C\)-translates are disjoint.
Partition $\mathcal{T}$ into subfamilies $\mathcal{T}[j_1, \ldots, j_{n-1}]$ of $C$-translates whose corresponding $P$-translates intersect a common line $(j_1 + b_1, \ldots, j_{n-1} + b_{n-1})$. By Lemma 4 the coloring number and the clique-partition number of each subfamily $\mathcal{T}[j_1, \ldots, j_{n-1}]$ are at most $c_n$ times its clique number and its packing number, respectively. For each $(k_1, \ldots, k_{n-1}) \in \{0, 1, \ldots, n\}^{n-1}$, let $\mathcal{T}_0[k_1, \ldots, k_{n-1}]$ be the union of the (pairwise-disjoint) subfamilies $\mathcal{T}[j_1, \ldots, j_{n-1}]$ with $j_i \equiv k_i \ (\text{mod} \ n + 1)$ for all $1 \leq i \leq n - 1$. Again refer to Figure 1 for an example in the plane. Then,

$$q(\mathcal{T}_0[k_1, \ldots, k_{n-1}]) = \max_{j_i \equiv k_i} q(\mathcal{T}[j_1, \ldots, j_{n-1}]) \leq \max_{j_i \equiv k_i} c_n \omega(\mathcal{T}[j_1, \ldots, j_{n-1}]) = c_n \omega(\mathcal{T}_0[k_1, \ldots, k_{n-1}]) \leq c_n \omega(\mathcal{T})$$

and

$$\vartheta(\mathcal{T}_0[k_1, \ldots, k_{n-1}]) = \sum_{j_i \equiv k_i} \vartheta(\mathcal{T}[j_1, \ldots, j_{n-1}]) \leq \sum_{j_i \equiv k_i} c_n \nu(\mathcal{T}[j_1, \ldots, j_{n-1}]) = c_n \nu(\mathcal{T}_0[k_1, \ldots, k_{n-1}]) \leq c_n \nu(\mathcal{T}).$$

Consequently,

$$q(\mathcal{T}) \leq \sum_{k_1, \ldots, k_{n-1}} q(\mathcal{T}_0[k_1, \ldots, k_{n-1}]) \leq \sum_{k_1, \ldots, k_{n-1}} c_n \omega(\mathcal{T}) = (n + 1)^{n-1} c_n \omega(\mathcal{T}) = t_n \omega(\mathcal{T}),$$

$$\vartheta(\mathcal{T}) \leq \sum_{k_1, \ldots, k_{n-1}} \vartheta(\mathcal{T}_0[k_1, \ldots, k_{n-1}]) \leq \sum_{k_1, \ldots, k_{n-1}} c_n \nu(\mathcal{T}) = (n + 1)^{n-1} c_n \nu(\mathcal{T}) = t_n \nu(\mathcal{T}).$$

This completes the proof of Theorem 1.

3 Upper bounds for homothets of a convex body in $\mathbb{R}^n$

In this section we prove Theorem 2. Let us define one more hypergraph invariant for a family $\mathcal{F}$ of sets:

transversal number $\tau(\mathcal{F})$ is the minimum cardinality of a set of elements that intersects all sets in $\mathcal{F}$.

Since any subfamily of $\mathcal{F}$ that share a common element corresponds to a clique the intersection graph of $\mathcal{F}$, we have the following inequality in addition to (4):

$$\vartheta(\mathcal{F}) \leq \tau(\mathcal{F}). \quad (5)$$

For the special case that $\mathcal{F}$ is a family of axis-parallel boxes in $\mathbb{R}^n$, we indeed have $\vartheta(\mathcal{F}) = \tau(\mathcal{F})$ since any subfamily of pairwise-intersecting axis-parallel boxes must share a common point. We will use the following lemma from a related work of ours on transversal numbers [8]:

7
Lemma 5 (Dumitrescu and Jiang, 2009). Let \( \mathcal{H} \) be a finite family of homothets of a convex body \( C \) in \( \mathbb{R}^n, n \geq 2 \). Let \( C_1 \) be the smallest homothet in \( \mathcal{H} \), and let \( \mathcal{H}_1 \) be the subfamily of homothets in \( \mathcal{H} \) that intersect \( C_1 \) (\( \mathcal{H}_1 \) includes \( C_1 \) itself). Then \( \tau(\mathcal{H}_1) \leq \kappa(C - C, C) \).

By inequality (5), we immediately have the following corollary:

**Corollary 2.** Let \( \mathcal{H} \) be a finite family of homothets of a convex body \( C \) in \( \mathbb{R}^n, n \geq 2 \). Let \( C_1 \) be the smallest homothet in \( \mathcal{H} \), and let \( \mathcal{H}_1 \) be the subfamily of homothets in \( \mathcal{H} \) that intersect \( C_1 \) (\( \mathcal{H}_1 \) includes \( C_1 \) itself). Then \( \vartheta(\mathcal{H}_1) \leq \kappa(C - C, C) \).

We first bound \( q(\mathcal{H}) \) in terms of \( \omega(\mathcal{H}) \). As in Corollary 2 let \( C_1 \) be the smallest homothet in \( \mathcal{H} \), and let \( \mathcal{H}_1 \) be the subfamily of homothets in \( \mathcal{H} \) that intersect \( C_1 \). Consider any partition of \( \mathcal{H}_1 \) into at most \( \vartheta(\mathcal{H}_1) \) classes of pairwise-intersecting homothets. Add \( C_1 \) to each class if it is not already there. Then in each class the homothets are pairwise-intersecting, and the number of homothets except \( C_1 \) is at most \( \omega(\mathcal{H}_1) - 1 \). Thus \( C_1 \) intersects a total of at most \( \vartheta(\mathcal{H}_1)(\omega(\mathcal{H}_1) - 1) \leq \kappa(C - C, C)(\omega(\mathcal{H}) - 1) \) other homothets in \( \mathcal{H} \). By a standard recursive argument, it follows that

\[
q(\mathcal{H}) \leq \kappa(C - C, C)(\omega(\mathcal{H}) - 1) + 1.
\]

We next bound \( \vartheta(\mathcal{H}) \) in terms of \( \nu(\mathcal{H}) \). Consider the following greedy partition of \( \mathcal{H} \): first find in \( \mathcal{H} \) the smallest homothet \( C_1 \) and the subfamily \( \mathcal{H}_1 \) of homothets that intersect \( C_1 \), next find in \( \mathcal{H} \setminus \mathcal{H}_1 \) the smallest homothet \( C_2 \) and the subfamily \( \mathcal{H}_2 \) of homothets that intersect \( C_2 \), and so on. Let \( \mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_k \) be the resulting partition. Then \( k \leq \nu(\mathcal{H}) \) since the homothets \( C_i \) are pairwise-disjoint. By Corollary 2 \( \vartheta(\mathcal{H}_i) \leq \kappa(C - C, C) \) for each \( \mathcal{H}_i \) in the partition. Moreover, if \( k = \nu(\mathcal{H}) \), then we must have \( \vartheta(\mathcal{H}_k) = 1 \) since otherwise there would be more than \( k \) pairwise-disjoint homothets in \( \mathcal{H} \). Thus

\[
\vartheta(\mathcal{H}) \leq \sum_{i=1}^{k} \vartheta(\mathcal{H}_i) \leq \left( \sum_{i=1}^{\nu(\mathcal{H})-1} \kappa(C - C, C) \right) + 1 = \kappa(C - C, C)(\nu(\mathcal{H}) - 1) + 1.
\]

This completes the proof of Theorem 2.

4 Lower bounds for translates of a convex body in \( \mathbb{R}^n \)

In this section we prove Theorem 3. Let \( C \) be a convex body in \( \mathbb{R}^n \) and \( m \) be a positive integer. We will show that \( \tau_T(C) \geq 1/\delta_T(C) \) and \( \tau_U(C) \geq 1/\delta_T(C) \) by constructing a finite family \( \mathcal{F}_m \) of \( m^{2n} \) translates of \( C \), such that

\[
\lim_{m \to \infty} \frac{\omega(\mathcal{F}_m)}{\omega(\mathcal{F}_m)} \geq \frac{1}{\delta_T(C)}, \tag{6}
\]

and

\[
\lim_{m \to \infty} \frac{\vartheta(\mathcal{F}_m)}{\vartheta(\mathcal{F}_m)} \geq \frac{1}{\delta_T(C)}. \tag{7}
\]

By Lemma 2 we can assume that \( C \) is centrally symmetric and is centered at the origin. We will use the following isodiametric inequality due to Busemann [4, p. 241, (2.2)]:

**Lemma 6** (Busemann, 1947). Let \( C \) be a centrally symmetric convex body in \( \mathbb{R}^n \). Let \( \mathbb{M}^n \) be the Minkowski space in which \( C \) is a ball of unit radius. For any measurable set \( S \) in \( \mathbb{R}^n \) of Minkowski diameter at most 2 in \( \mathbb{M}^n \), the Lebesgue measure of \( S \) in \( \mathbb{R}^n \) is at most the Lebesgue measure of \( C \) in \( \mathbb{R}^n \).
Let $\mathcal{F}_m$ be a family of translates of $C$

$$\mathcal{F}_m := \{ C + t \mid t \in T_m \}$$

corresponding to a set $T_m$ of $m^{2n}$ regularly placed reference points

$$T_m := \{(t_1/m, \ldots, t_n/m) \mid (t_1, \ldots, t_n) \in \mathbb{Z}^n, 1 \leq t_1, \ldots, t_n \leq m^2\}.$$

Let $U_m$ be an axis-parallel hypercube of side length $1/m$ that is centered at the origin. Observe that $U_m + T_m$ is an axis-parallel hypercube of side length $m$.

We first obtain a lower bound on $\vartheta(\mathcal{F}_m)$. Note that any two translates of $C$ in $\mathbb{R}^n$ intersect if and only if the Minkowski distance between their centers is at most 2 in $\mathbb{R}^n$. Thus any subset of pairwise intersecting translates of $C$ in $\mathcal{F}_m$ corresponds to a subset of points of Minkowski diameter at most 2 in $T_m$, and reciprocally. Consider a partition of $\mathcal{F}_m$ into $\vartheta(\mathcal{F}_m)$ subsets of pairwise intersecting translates of $C$, and let $T_{m,i} \subseteq T_m, 1 \leq i \leq \vartheta(\mathcal{F}_m)$, be the corresponding subsets of Minkowski diameter at most 2. Then the hypercube $U_m + T_m$ is covered by the union of the subsets $U_m + T_{m,i}, 1 \leq i \leq \vartheta(\mathcal{F}_m)$. Let $S_m \subseteq T_m$ be a maximum-cardinality subset of points of Minkowski diameter at most 2. Then, by a volume argument, we have

$$\vartheta(\mathcal{F}_m) \geq \frac{\mu(U_m + T_m)}{\mu(U_m + S_m)}.$$  \hspace{1cm} (8)

We next obtain an upper bound on $\nu(\mathcal{F}_m)$. Let $B$ be the smallest axis-parallel box containing $C$. For each point $t \in T_m$, the corresponding translate $C + t \in \mathcal{F}_m$ satisfies $C + t \subseteq C + T_m \subseteq B + T_m$. Recall our definition (2) that $\rho(\mathcal{F}, Y)$ is the density of a family $\mathcal{F}$ of convex bodies relative to a bounded domain $Y \subseteq \mathbb{R}^n$. Let $I_m \subseteq \mathcal{F}_m$ be a maximum-cardinality packing in $B + T_m$. Again, by a volume argument, we have

$$\nu(\mathcal{F}_m) \leq \rho(I_m, B + T_m) \cdot \frac{\mu(B + T_m)}{\mu(C)}.$$  \hspace{1cm} (9)

From (8) and (9), it follows that

$$\frac{\vartheta(\mathcal{F}_m)}{\nu(\mathcal{F}_m)} \geq \frac{1}{\rho(I_m, B + T_m)} \cdot \frac{\mu(U_m + T_m)}{\mu(B + T_m)} \cdot \frac{\mu(C)}{\mu(U_m + S_m)}.$$  \hspace{1cm} (10)

Now, taking the limit as $m \to \infty$, we clearly have $\rho(I_m, B + T_m) \to \delta_T(C)$ and $\mu(U_m + T_m)/\mu(B + T_m) \to 1$. Also, as $m \to \infty$, the Minkowski diameter of $U_m + S_m$ tends to the Minkowski diameter of $S_m$, which is at most 2. It then follows by Lemma (10) that $\lim_{m \to \infty} \mu(U_m + S_m) \leq \mu(C)$. This yields (7) as desired.

To show (6) we now obtain bounds on $q(\mathcal{F}_m)$ and $\omega(\mathcal{F}_m)$. Since $q(\mathcal{F}_m) \nu(\mathcal{F}_m) \geq |\mathcal{F}_m| = |T_m|$, it follows immediately from (9) that

$$q(\mathcal{F}_m) \geq \frac{|T_m|}{\rho(I_m, B + T_m)} \cdot \frac{\mu(C)}{\mu(B + T_m)}.$$  \hspace{1cm} (11)

Recall the definition of $S_m$ before (8). Clearly,

$$\omega(\mathcal{F}_m) = |S_m|.$$  \hspace{1cm} (12)

From (11) and (12), it follows that

$$\frac{q(\mathcal{F}_m)}{\omega(\mathcal{F}_m)} \geq \frac{1}{\rho(I_m, B + T_m)} \cdot \frac{\mu(U_m + T_m)}{\mu(B + T_m)} \cdot \frac{\mu(C)}{\mu(U_m + S_m)} \cdot \frac{\mu(U_m + S_m)}{\mu(U_m + T_m)} \cdot \frac{|T_m|}{|S_m|}.$$  \hspace{1cm} (13)
Note that \( \mu(U_m + S_m) = \mu(U_m) \cdot |S_m| \) and \( \mu(U_m + T_m) = \mu(U_m) \cdot |T_m| \). Hence the two inequalities (10) and (13) have the same the right-hand side. Taking the limit as \( m \to \infty \) in (13) yields (6).

We have shown that \( r_T(C) \geq 1/\delta_T(C) \) and \( T_T(C) \geq 1/\delta_T(C) \) for any convex body \( C \) in \( \mathbb{R}^n \). For the special case that \( C \) is the \( n \)-dimensional unit ball \( B^n \) in \( \mathbb{R}^n \), Kabatjanski˘ı and Levenštěın [12] showed that \( \delta_T(B^n) = \delta(B^n) \leq 2^{-1.599 \pm o(1)n} \) and hence \( 1/\delta_T(B^n) \geq 2^{0.599 \pm o(1)n} \) as \( n \to \infty \); see also [4, p. 50]. This completes the proof of Theorem 3.

5 Lower bounds for axis-parallel unit squares

In this section we prove Theorem 4. Refer to Figure 2(a) for the construction of the family \( \mathcal{F}_k \) given by Ahlswede and Karapetyan [1], \( k \geq 1 \).

![Figure 2: Lower bound construction for axis-parallel squares. (a) The family \( \mathcal{F}_k \) consists of 5k squares, \( k \) duplicates (or sufficiently close translates) of each of the five squares arranged into a 5-cycle. (b) A 5-coloring of the intersection graph of \( \mathcal{F}_2 \).](image)

Let \( A, B, C, D, E \) be the five groups of squares in \( \mathcal{F}_k \), \( k \) squares in each group. It is clear that \( \omega(\mathcal{F}_k) = 2k \), which is realized by any two adjacent groups of squares, for example, \( A \) and \( B \). It is also clear that \( q(\mathcal{F}_k) \leq 3k \). Let \( Q_1, Q_2, Q_3 \) be the three classes in any partition of \( 3k \) distinct colors, \( k \) colors in each class. Then we can use \( Q_1 \) for \( A \) and \( C \), \( Q_2 \) for \( B \) and \( E \), and \( Q_3 \) for \( D \). Ahlswede and Karapetyan [1] mistakenly assumed that \( q(\mathcal{F}_k) = 3k \). We next derive the correct value of \( q(\mathcal{F}_k) \).

Observe that \( \nu(\mathcal{F}_k) = 2k \). Thus we clearly have the lower bound \( q(\mathcal{F}_k) \geq |\mathcal{F}_k|/\nu(\mathcal{F}_k) = \frac{5}{2}k \); moreover \( q(\mathcal{F}_k) \geq \lceil \frac{5}{2}k \rceil \) since \( q(\mathcal{F}_k) \) is an integer. To derive the matching upper bound \( q(\mathcal{F}_k) \leq \lceil \frac{5}{2}k \rceil = k + k + \lceil k/2 \rceil \), we construct a coloring of \( \mathcal{F}_k \) with \( k \) colors from \( Q_1 \), \( k \) colors from \( Q_2 \), and \( \lceil k/2 \rceil \) colors from \( Q_3 \). Partition each color class \( Q_i \), \( 1 \leq i \leq 3 \), into two sub-classes of \( Q_{i,1} \) and \( Q_{i,2} \) of sizes \( \lceil k/2 \rceil \) and \( \lfloor k/2 \rfloor \), respectively. The coloring is as follows:

\[
A : Q_{1,1} \cup Q_{1,2} \quad B : Q_{2,1} \cup Q_{2,2} \quad C : Q_{1,2} \cup Q_{3,1} \quad D : Q_{1,1} \cup Q_{2,1} \quad E : Q_{2,2} \cup Q_{3,1}
\]

For coloring \( D \) we use any \( k \) colors from \( Q_{1,1} \cup Q_{2,1} \). Observe that \( D \) does not use any color in \( Q_3 \), and that \( C \) and \( E \) share the colors in \( Q_{3,1} \). Refer to Figure 2(b) for the case \( k = 2 \).

For the second part of the theorem, let \( \mathcal{F}_k' \) be \( k \) disjoint groups of five squares each, repeating the intersection pattern in Figure 2(a). It is easy to see that \( \nu(\mathcal{F}_k') = 2k \) and \( \delta(\mathcal{F}_k') = 3k \). This completes the proof of Theorem 4.
References

[1] R. Ahlswede and I. Karapetyan. Intersection graphs of rectangles and segments. In General Theory of Information Transfer and Combinatorics (R. Ahlswede et al., editors), volume 4123 of Lecture Notes in Computer Science, pages 1064–1065. Springer, 2006.

[2] J. Akiyama, K. Hosono, and M. Urabe. Some combinatorial problems. Discrete Mathematics, 116:291–298, 1993.

[3] E. Asplund and B. Grünbaum. On a coloring problem. Mathematica Scandinavica, 8:181–188, 1960.

[4] P. Braß, W. Moser, and J. Pach. Research Problems in Discrete Geometry. Springer, New York, 2005.

[5] H. Busemann. Intrinsic area. Annals of Mathematics, 48:234–267, 1947.

[6] G. D. Chakerian and S. K. Stein. Some intersection properties of convex bodies. Proceedings of the American Mathematical Society, 18:109–112, 1967.

[7] L. Danzer, B. Grünbaum, and V. Klee. Helly’s theorem and its relatives. In Convexity, volume 7 of Proceedings of Symposia in Pure Mathematics, pages 101–181. American Mathematical Society, 1963.

[8] A. Dumitrescu and M. Jiang. Piercing translates and homothets of a convex body. Algorithmica, doi:10.1007/s00453-010-9410-4, online first.

[9] M. Golumbic. Algorithmic Graph Theory and Perfect Graphs, 2nd edition, volume 57 of Annals of Discrete Mathematics, Elsevier, 2004.

[10] A. Gyárfás. On the chromatic number of multiple interval graphs and overlap graphs. Discrete Mathematics, 55:161–166, 1985.

[11] A. Gyárfás and J. Lehel. Covering and coloring problems for relatives of intervals. Discrete Mathematics, 55:167–180, 1985.

[12] G. A. Kabatjanskii and V. I. Levenštein. Bounds for packings on a sphere and in space (in Russian). Problemy Peredači Informacii, 14:3–25, 1978. English translation: Problems of Information Transmission, 14:1–17, 1978.

[13] S.-J. Kim, A. Kostochka, and K. Nakprasit. On the chromatic number of intersection graphs of convex sets in the plane. Electronic Journal of Combinatorics, 11:#R52, 2004.

[14] S.-J. Kim and K. Nakprasit. Coloring the complements of intersection graphs of geometric figures. Discrete Mathematics, 308:4589-4594, 2008.

[15] A. Kostochka. Coloring intersection graphs of geometric figures with a given clique number. In Towards a Theory of Geometric Graphs (J. Pach, editor), volume 342 of Contemporary Mathematics, pages 127–138. American Mathematical Society, 2004.

[16] I. G. Perepelitsa. Bounds on the chromatic number of intersection graphs of sets in the plane. Discrete Mathematics, 262:221–227, 2003.

[17] C. A. Rogers. A note on coverings. Mathematika, 4:1–6, 1957.