QUANTUM COHOMOLOGY AND MORSE THEORY ON THE
LOOP SPACE OF TORIC VARIETIES

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Abstract. On a symplectic manifold $M$, the quantum product defines a complex, one parameter family of flat connections called the A-model or Dubrovin connections. Let $h$ denote the parameter. Associated to them is the quantum $\mathcal{D}$- module $\mathcal{D}/I$ over the Heisenberg algebra of first order differential operators on a complex torus. An element of $I$ gives a relation in the quantum cohomology of $M$ by taking the limit as $h \to 0$. Givental [10], discovered that there should be a structure of a $\mathcal{D}$- module on the (as yet not rigorously defined) $S^1$ equivariant Floer cohomology of the loop space of $M$ and conjectured that the two modules should be equal. Based on that, we formulate a conjecture about how to compute the quantum cohomology $\mathcal{D}$- module in terms of Morse theoretic data for the symplectic action functional. The conjecture is proven in the case of toric manifolds with $\int c_1 > 0$ for all nonzero classes $d$ of rational curves in $M$.

1. Introduction

In this paper we will study the quantum cohomology and more generally the quantum $\mathcal{D}$- module structure, of symplectic toric manifolds by relating it to Morse theory of the unperturbed symplectic action functional on the loop space. In particular we will use $S^1$ equivariant cohomology of the action functional.

This program was initiated by Givental in [10] and provided the inspiration for the methodology applied later in [11] in the context of Kontsevich’s space of stable maps.

In order to describe the main theorem let us briefly recall a few things about quantum cohomology. We follow mainly Givental [11] in this introductory exposition.

Let $(M, \omega)$ be a symplectic manifold and choose also a compatible almost complex structure $J$. A pseudo-holomorphic curve is a map $f : \mathbb{P}^1 \to M$ whose derivative is complex linear. Let $d \in H_2(M, \mathbb{Z})$ be a homology class. Kontsevich [14] invented the correct space parameterizing pseudo-holomorphic curves in $M$ with $k$ marked points. It is called the space of stable maps and we’ll denote it by $M_{k,d}$. Let us consider $M_{3,d}$. Its elements are equivalence classes of 4-tuples, $(f, x_1, x_2, x_3)$ where $f$ is the map and the $x_i$’s are the marked points. The 4-tuple must satisfy the stability condition that it has at most a discrete group of automorphisms. Two 4-tuples are equivalent if there is an automorphism of $\mathbb{P}^1$ that takes one to the other. The space $M_{3,d}$ is at worst an orbifold (Kontsevich [14]) if $M$ is convex.

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1To avoid any confusion, let us point out that in this paper by a pseudo-holomorphic curve we will always mean one that has arithmetic genus 0.
Moreover it comes equipped with three evaluation maps \( ev_i : M_{3,d} \to M \) for \( i = 1, 2, 3 \) given by \( ev_i(f, x_1, x_2, x_3) = f(x_i) \). Let \( a, b \) be classes in \( H^2(M, \mathbb{C}) \). Let \(( , , )\) denote the intersection pairing. Let also \( p_1, \ldots, p_r \) be classes in the Kähler cone \( \mathcal{K} \) of \( M \) which form a basis of \( H^2(M, \mathbb{Z}) \). The Kähler cone is the cone in \( H^2(M, \mathbb{R}) \) which consists of all classes whose integral over any (pseudo-) holomorphic curve is non-negative. Finally let \( d_i = \int_d p_i \) and \( q_i = e^{d_i} \) be complex variables (which can be thought of as coordinates on a complex torus). The quantum product \( a \ast b \) is defined by the property that

\[
(a \ast b, c) = \sum_d q^d \int_{M_{3,d}} ev_1^*(a) \wedge ev_2^*(b) \wedge ev_3^*(c),
\]

where \( q^d = \prod_{i=1}^r q_i^{d_i} \) and the sum is over all homology classes \( d \) of pseudo-holomorphic curves. The number \( (a \ast b, c)_d = \int_{M_{3,d}} ev_1^*(a) \wedge ev_2^*(b) \wedge ev_3^*(c) \) is called a Gromov-Witten invariant (of the symplectic structure). It should be thought of geometrically, as counting the number of curves in homology class \( d \), meeting classes dual to \( a, b \) and \( c \), when the number of such curves is finite. Otherwise it is 0. The (small) quantum cohomology ring of \( M \) is the ring \( SQH^*(M) = H^2(M, \mathbb{Z}) \otimes \mathbb{C}[[q_1, \ldots, q_r]] \) equipped with the quantum product.

The quantum product is commutative and associative. This last property is highly nontrivial and makes for many interesting consequences by itself. It turns out that the associativity can be reformulated as the flatness of the following complex one parameter family of connections

\[
\nabla_h = \hbar d - \sum_{i=1}^r \frac{dq_i}{q_i} \wedge p_i^*\nabla.
\]

acting on elements of \( SQH^*(M) \), where \( \hbar \) denotes the complex parameter. This is called the Dubrovin or A-model connection and as we’ll see shortly, it is a more fundamental object than the quantum product. Givental in his remarkable paper \[\text{1}\] found a formula for flat sections of \( \nabla_h \). It’s clear that for flat sections of \( \nabla_h \), quantum multiplication by \( p_i \) is translated to differentiation and therefore we may expect that relations in the quantum ring may be translated to differential equations. This was formulated explicitly by Givental in the following fashion: We may associate to \( \nabla_h \) a certain \( \mathcal{D} \)-module \( \mathcal{D}/I \) over the algebra of Heisenberg differential operators. This has the property that if the operator \( D(hq_i \frac{\partial}{\partial q_i}, q_i, \hbar) \) is in the ideal \( I \), then the relation \( D(p_i^*, q_i, 0) = 0 \) holds in the quantum cohomology ring \( SQH^*(M) \). Therefore the \( \mathcal{D} \)-module appears to be the real quantum object while the quantum ring arises as its “semi-classical approximation” when \( \hbar \to 0 \).

To describe the \( \mathcal{D} \)-module \( \mathcal{D}/I \) we need to introduce a new ingredient. This is the line bundle \( L_{3,d} \) which is the universal cotangent line at the second marked point, i.e., the line bundle whose fiber over \( [S, (x_1, x_2), f] \) is the cotangent line to \( S \) at the second marked point.

Let \( c \) denote the first Chern class of \( L \). Now choose bases \( T_0, \ldots, T_m \) and \( T^0, \ldots, T^m \) of \( H^{2n}(M, \mathbb{Z}) \) such that \( (T^i, T_j) = \delta_{i,j} \). We arrange that \( T_0 = 1 \in H^0(M, \mathbb{Z}) \) and \( T_i = p_i \) for \( i = 1, \ldots, r \) where \( p_i \) as before. Givental’s result \[\text{1}\] is

\[\text{2}\]Main examples of convex spaces are homogeneous spaces.

\[\text{3}\]Because of the meaning it acquires in string theory, \( c \) is called a gravitational descendant.
Let $G$ be the $H^{2*}(M,\mathbb{C})$ valued function defined as:

$$G = e^{plq/\hbar}(1 + \sum_d q^d ev_1(\frac{1}{\hbar - c})),$$

where $d$ ranges over all non-zero homology classes of pseudo-holomorphic curves and $ev_1 : M_{2,d} \to M$ is evaluation at the first marked point. Then the ideal I is generated by all polynomial differential operator that annihilate the components of $G$.

The object of this paper is to compute the quantum $D$- module and specifically the function $G$, not via the space of stable maps, but rather by working on the loop space of $M$. This is desirable for several reasons, one of them being that the setup seems to be more natural (at least as far as the $D$- module is concerned) and therefore we may get a better geometric understanding of the formulas. Another one is that we have a built in $S^1$ symmetry which, if correctly understood, should simplify the problem. (By definition the Kontsevich space of stable maps in $M$ doesn’t have that symmetry and Givental in [11] uses instead the space of stable maps $f : \mathbb{P}^1 \to M \times \mathbb{P}^1$ and the $S^1$ action on the second factor.) Both of these observations are materialized here to some extent and we expect they will even more in the future. Yet another reason to insist with the loop space is that since its $S^1$ equivariant cohomology is computed by (a version of) the cyclic bar complex we expect fruitful interaction and more powerful calculational tools to emerge when the problem is properly formulated in that setting.

The connection with the loop space was first explained by Givental in [10]. The connection between pseudo-holomorphic curves and the loop space was already present in the work of Floer and his celebrated proof of the Arnold conjecture which resulted in the definition of Floer homology. To explain the appearance of the loop space let us start with the space $\mathcal{L}M$ of free contractible loops in $M$. We can define the action functional $H$ by

$$H(\gamma) = \int_{D_\gamma} \omega,$$

where $\gamma$ is a contractible loop and $D_\gamma$ a disc contracting it. It is multi-valued if there are homologically non-trivial spheres. To resolve the ambiguity we lift it to the covering space $\tilde{\mathcal{L}}M$ of $\mathcal{L}M$, with covering group the group of spherical classes in $M$. Assume for simplicity that $M$ is simply connected, then $H_2(M,\mathbb{Z})$ is generated by spherical classes. Now the key is, that $H$ has the remarkable property that its flow lines are pseudo-holomorphic cylinders! Moreover $H$ is a Hamiltonian function with respect to the obvious circle action on $\tilde{\mathcal{L}}M$ and the symplectic form induced from the symplectic form on $M$. The critical manifolds correspond to trivial loops and are copies of $M$, one for every degree $d \in H_2(M,\mathbb{Z})$, i.e., for every floor of the cover. Denote by $M_0$ the copy on which $H$ has value 0 and by $M_d$ its translation by $d$.

A formal application of the $S^1$ equivariant localization theorem suggests that the (Floer) $S^1$ equivariant cohomology of $\tilde{\mathcal{L}}M$ should be simply $FH^{*}_{S^1}(\tilde{\mathcal{L}}M) = H^{*}(M,\mathbb{C}[\tilde{q},\tilde{q}^{-1}])$, where $\mathbb{C}[\tilde{q},\tilde{q}^{-1}]$ is the group ring of the covering group.

Givental’s observation is that $FH^{*}_{S^1}(\tilde{\mathcal{L}}M)$ bears the structure of a $D$- module over the Heisenberg algebra of differential operators. This is shown by extending the classes $\{p_1,\ldots,p_r\}$ to equivariant classes $\{P_1,\ldots,P_r\}$ (see (24)). Then if we think of the $P_k$ acting by multiplication and the $\tilde{q}_k^{d}$ by pullback it is easy to show
that

\[[P_j, \tilde{q}_k] = \delta_{j,k} \hbar \tilde{q}_k\]

Givental conjectures that this \(D\) module is the quantum \(\mathcal{D}\) module.

Now let \(\mathcal{N}_d\) denote the normal bundle to \(M_d\). Let also \(\mathcal{N}_d^+\) and \(\mathcal{N}_d^-\) denote the positive and negative normal bundles to \(M_d\), with respect to \(H\). They are both infinite dimensional. In section four we argue that if we can make sense of the ratio of \(S^1\) equivariant Euler classes \(\frac{\varepsilon_{S^1}(\mathcal{N}_d^+)}{\varepsilon_{S^1}(\mathcal{N}_0^+)}\), then the cohomology valued function

\[
F = \sum_d e^{\text{pln} q/\hbar} q^d \frac{\varepsilon_{S^1}(\mathcal{N}_d^+)}{\varepsilon_{S^1}(\mathcal{N}_0^+)}
\]

where \(d\) ranges over all homology classes of pseudo-holomorphic curves, should generate the quantum \(\mathcal{D}\) module, in the sense that the ideal \(I\) should be generated by operators annihilating the components of \(F\) in some basis. This is formulated as conjecture (1) in section (4).

Now if \(M\) is a toric variety, we construct approximations of \(\tilde{L}_M\) by finite dimensional spaces of loops of finite but arbitrarily large modes (in the Fourier expansion). In that case we are able to prove (Proposition 3, section 5) that the ratios of Euler classes stabilize and therefore we may define a “stable ratio of Euler classes” (Definition 2, section 5). Using this stable ratio in the formula for \(F\) we are then able to prove the main theorem (Theorem 4, section 5) which says that if \(M\) is a positive Kähler toric manifold in the sense that \(\int d c_1 > 0\) for all nonzero classes \(d\) of arithmetic genus 0 curves in \(M\) then

\[
F = G
\]

and therefore indeed generates the quantum \(\mathcal{D}\) module. This is the content of section five.

The structure of the paper is as follows: In the next section we gather the elements of the theory of quantum cohomology that are needed and which are contained mainly in [11]. In the third section we explain the original idea of Givental [10] relating the quantum \(\mathcal{D}\) module and \(S^1\) equivariant Floer homology of the loop space, via a sort of “Fourier” transform of cycles. Then a heuristic conjecture is presented on how to compute the “Fourier” transform of cycles arising in Givental’s work. In the fifth section the conjecture is formulated rigorously for the case of toric manifolds and then proven. Finally in the last section we make some observations and indicate what seem to be interesting directions for extension of this work.

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2. The Quantum \(\mathcal{D}\) Module

We gather here the basic facts about the quantum \(\mathcal{D}\) module that will be needed later on. Let \(M\) be a Kähler manifold and let \(T_0, T_1, \ldots, T_r, \ldots, T_m\) be a basis of \(H^2(M, \mathbb{Z})\) where \(T_0\) is the identity and \(T_1, \ldots, T_r\) are classes in the Kähler cone \(\mathcal{K}\) that generate \(H^2(M, \mathbb{Z})\). Recall that \(\mathcal{K}\) is the cone of classes whose integral over
holomorphic curves is non-negative. Set $p_i = T_i$ for $i = 1, \ldots, r$. The small quantum ring is defined to be the vector space

$$SQH^*(M) = H^{2s}(M, \mathbb{Z}) \bigotimes \mathbb{C}[q_1, \ldots, q_r]$$

equipped with the quantum product $*$ already defined in the introduction. That is, if $(\ , \ , \ , \ )$ denotes the intersection pairing and $a, b, c \in H^{2s}(M, \mathbb{Z})$ we have

$$(a * b, c) = \sum_d q^d \int_{\overline{M}_{3,d}} ev_1^*(a) \wedge ev_2^*(b) \wedge ev_3^*(c),$$

where $q^d = \prod_{i=1}^r q_i^{d_i}$, $d_i = \int q_i p_i$ and the sum is over all homology classes $d \in H_2(M, \mathbb{Z})$ of holomorphic curves. $\overline{M}_{3,d}$ is the Kontsevich space of stable maps with three marked points, whose image has arithmetic genus 0 and degree $d$ in $H_2(M, \mathbb{Z})$. $SQH^*(M)$ is graded if we assign cohomology classes their usual degree and declare $deg(q_1^{d_1} \ldots q_r^{d_r}) = 2 \int c_1(TM)$. The reason for this grading is that $\int_{\overline{M}_{3,d}} ev_1^*(a) \wedge ev_2^*(b) \wedge ev_3^*(c)$ is 0 unless the sum of the degrees of $a, b, c$ is equal to the dimension of $\overline{M}_{3,d}$ which, as computed in [12], is: $dim_{\mathbb{C}} \overline{M}_{3,d} = \frac{1}{2} \int c_1(TM)$. Recall that the quantum product is commutative and associative (see [11]).

Introduce now a one parameter family of connections with regular singular points, depending on the complex parameter $\hbar$ and defined by

$$\nabla_\hbar = \hbar \partial - \sum_{i=1}^r \frac{dq_i}{q_i} \wedge p_i *.$$

$\nabla_\hbar$ are called Dubrovin or A model connections, and act on power series in the $q_i$ with coefficients in $H^{2s}(M, \mathbb{Z})$, in other words on elements of $SQH^*(M)$. As already mentioned, one reason for considering these connections is that flat sections, if they exist, will provide a passage from quantum product to differential equations. Moreover, motivation for introducing the connections comes from mirror symmetry. In fact in the Calabi-Yau case they are the counterpart of the Gauss-Manin connection corresponding to the mirror family. On the other hand, the reason for having a whole pencil of connections is best understood from the point of view of the loop space and will be explained in the next section.

Now in Givental [11] it is proven that:

**Proposition 1.** The connection $\nabla_\hbar$ is flat for any value of $\hbar$.

The fact that $\nabla_\hbar$ is flat means that we can find flat sections. That is, sections $s$ such that $\nabla_\hbar s = 0$.

One of the remarkable results of [11] is the explicit computation of the flat sections. To describe them we introduce first the line bundle $L$ over $M_{2,d}$ which is the universal cotangent line at the second marked point, i.e., the line bundle whose fiber over $[S, (x_1, x_2), f]$ is the cotangent line to $S$ at the second marked point. Let $c$ denote the first Chern class of $L$. Choose now basis $T^0, \ldots, T^m$ of $H^{2s}(M, \mathbb{Z})$ such that $(T^i, \overline{T^i}) = \delta_{i,j}$. We still have that $T_0 = 1 \in H^0(M, \mathbb{Z})$ and that $p_i = T_i$ for $i = 1, \ldots, r$ where $p_i \in K$ are chosen to be a basis of $H^2(M, \mathbb{Z})$. Givental’s result [11] is:

**Theorem 1.** The sections

$$s_\beta = e^{\frac{plna}{\hbar}} T_\beta + \sum_{\alpha} \sum_{d \in K, d \neq 0} q^d \langle T_\alpha, e^{\frac{plna}{\hbar}} \frac{T_\beta}{\hbar - c} \rangle_d$$
for β = 0, . . . , m are flat and they provide a basis of the space of flat sections.

Here q^d is notation for q_1^{a_1} . . . q_r^{a_r}, plnq is notation for p_1lnq_1 + . . . + p_rlnq_r and

\[ <T_\alpha, e^{plnq/h} T_\beta >_{plnq/h} = \int_{M_{2,d}} e^{plnq/h} T_\alpha \wedge \frac{e^{plnq/h} T_\beta}{h-c}, \]

where M_{2,d} is the space of arithmetic genus 0 and degree d stable maps with two marked points. Finally \( \tilde{K} \) is the cone in \( H_2(M, \mathbb{Z}) \) consisting of classes of holomorphic, arithmetic genus 0 curves. It is dual to the Kähler cone \( K \). Note also that the matrix:

\[ s_{\alpha,\beta} = (T_\alpha, s_\beta) = (T_\alpha, e^{plnq/h} T_\beta) + \sum_{d_{K,d} \neq 0} q^d <T_\alpha, e^{plnq/h} \frac{T_\beta}{h-c} >_{plnq/h}. \]

is the fundamental solution matrix of the flat section equation.

Let us now explain the relation of the \( A \)-connection to the small quantum ring. Let \( G \) be the following function with values in \( H^{2*}(M, \mathbb{C}) \):

\[ G = e^{plnq/h}(1 + \sum_{d_{K,d} \neq 0} q^d e^{plnq/h}(\frac{1}{h-c})), \]

where \( e^{1} : M_{2,d} \to M \) is evaluation at the first marked point. Then \( G \) has the property (and is determined by it): \( (G, T_\beta) = (1, s_\beta) \). Indeed,

\[ (G, T_\beta) = (e^{plnq/h}, T_\beta) + \sum_{d_{K,d} \neq 0} q^d \int_{M} e^{plnq/h} e^{plnq/h}(\frac{1}{h-c}) \wedge T_\beta = \]

\[ = (1, e^{plnq/h} T_\beta) + \sum_{d_{K,d} \neq 0} q^d \int_{M_{2,d}} \frac{1}{h-c} \wedge e^{plnq/h} T_\beta = (1, s_\beta) \]

Therefore we have that

\[ G = \sum_{\beta} (1, s_\beta) T_\beta. \]

Recall that \( T_0 = 1 \) and therefore the components of \( J \) form the first row of the solution matrix \( (s_{\alpha,\beta}) \).

The following proposition is due to Givental [1]

**Proposition 2.** Let \( D(hq_i \frac{\partial}{\partial q_i}, B_i) \) be a polynomial differential operator that annihilates the components of \( G \). Then the relation \( D(p_i*, q_i, 0) = 0 \) holds in \( \text{SQH}^{*}(M) \).

Let \( D \) denote the Heisenberg algebra of differential operators on holomorphic function on a torus with coordinates \( q_i = e^{t_i} \). It is by definition generated by the operators \( hq_i \frac{\partial}{\partial q_i} = \frac{\partial}{\partial t_i} \) and multiplication by \( q_i = e^{t_i} \). Let \( I \) be the ideal of all polynomial differential operators \( D(hq_i \frac{\partial}{\partial q_i}, q_i, h) \) that annihilate the components of \( G \).

**Definition 1.** The \( \mathcal{D} \)-module \( \mathcal{D}/I \) is called the quantum cohomology \( \mathcal{D} \)-module of \( M \).

The proposition above shows that the real quantum object is the \( \mathcal{D} \)-module or equivalently the \( A \) model connection, while the quantum ring should be considered as the semi-classical limit where \( h \to 0 \). Our objective is to compute the \( \mathcal{D} \)-module in terms of the loop space of \( M \). We shall turn to this next.
3. Equivariant Floer theory

Let’s start first by considering the $S^1$ equivariant Floer homology of the unperturbed action functional $H$ in the case of a general symplectic manifold which is not necessarily toric.

Let $(M, \omega)$ be a compact symplectic manifold. Let $J$ be a compatible or calibrated almost structure on $M$. By this we mean that $\omega(v, Jv) \geq 0$ for all nonzero $v \in TM$ and $\omega(Jv, Jw) = \omega(v, w)$. The symplectic form $\omega$ along with $J$ define an invariant metric $g$ on $TM$ by $g(v, w) = \omega(v, Jw)$. Let $LM$ be the space of smooth maps $\gamma : S^1 \to M$ such that $\gamma(S^1)$ is contractible. We call $LM$ the loop space of $M$. The loop space inherits a symplectic structure $\Omega$ and an almost complex structure which we shall denote also by $J$. To describe them lets first consider the tangent bundle $TLM$. The tangent space of $LM$ at a loop $\gamma$ is $T\gamma L\gamma = C(\gamma^*TM)$, where $C$ denotes the space of sections. In other words an element of $T\gamma L\gamma$ is a vector field along the loop $\gamma$.

Consider now the Kähler cone $K \subset H^2(M, \mathbb{R})$ of $M$. We have defined $K$ to be the cone of classes in $H^2(M, \mathbb{R})$ whose integral over any pseudo-holomorphic curve is greater than or equal to zero. Assume that $K$ is spanned by the classes of symplectic two forms $\omega_1, \ldots, \omega_l$. Let $v$ and $w$ be elements of $T\gamma L\gamma$ then we define:

$$ \Omega_k|_{\gamma}(v, w) = \int_{S^1} \omega_k(v(t), w(t))dt. $$

(9)

It is not hard to show that the $\Omega_k$ are also symplectic. Moreover $J$ induces an almost complex structure by $(Jv)|_t = J(v|_t)$. Finally $T\gamma L\gamma$ becomes pre-Hilbert with the inner product

$$ g_\gamma(v, w) = \Omega_\gamma(v, Jw), $$

(10)

where

$$ \Omega_\gamma(v, w) = \int_{S^1} \omega(v(t), w(t))dt. $$

Introduce now action functionals

$$ H_k(\gamma) = \int_{D_\gamma} \omega_k, $$

(11)

for $k = 1 \ldots l$ and

$$ H(\gamma) = \int_{D_\gamma} \omega, $$

(12)

where $D_\gamma$ is a disk contracting the loop $\gamma$. These are in general not well defined since different disks contracting the same loop will not have the same symplectic areas. The ambiguity in $H_k$ is clearly given by the periods

$$ \int_S \omega_k, $$

where $S$ is a sphere obtained by gluing two different disks contracting $\gamma$, along their common boundary. The functions $H_k$ become well defined only on the covering of $LM$ with group of deck transformations the group of spherical periods of the symplectic forms $\omega_1, \ldots, \omega_l$. We shall denote this space by $\\hat{LM}$. We can describe $\\hat{LM}$ explicitly as equivalence classes of pairs $(\gamma, g)$ where $\gamma : S^1 \to M$ is a loop and $g : D \to M$ is such that $g|_{\partial D} = \gamma$. Define $(\gamma, g_1) \sim (\gamma, g_2)$ if and only if $g_1#(-g_2)$ represents a class $A \in H_2(M, \mathbb{Z})$ such that $\int_A \omega_k = 0$ for all $k = 1 \ldots l$. Observe
that by definition $\widetilde{LM}$ carries an action, denoted by $\cdot$ of the group $\Gamma$ of spherical classes in $H_2(M, \mathbb{Z})$ such that $(A \cdot (\gamma, g)) \# (\gamma, -g) = -A$ for all $A \in \Gamma$. Notice that we could have chosen a positive instead of a negative sign in the definition of the action of $\Gamma$. The reason for our choice will become apparent later (see footnote (8), page 13). Note also that since $\omega$ is a linear combination of $\omega_1, \ldots, \omega_l$ it follows that $H$ is the same linear combination of $H_1, \ldots, H_k$ and therefore also becomes a well defined function on $\widetilde{LM}$. Now it is not hard to compute that

$$dH_k|_\gamma(v) = -\int_{S^1} \omega_k(\dot{\gamma}, v(t)) dt,$$

where $\dot{\gamma}$ denotes the vector field tangent to $\gamma$. To see this, let $v(t)$ be a vector field along $\gamma$ and let $\gamma_s(e^{it})$, $s \in \mathbb{R}$ be a curve representing $v$. This means that $\gamma_0 = \gamma$ and $\frac{d}{ds}|_{s=0}\gamma_s = v(t)$. Moreover we can arrange that $\gamma_s(e^{it}) = a$ for all $s \leq -1$ where $a$ is a point in $M$. Now consider the family of maps on the disc $D$, contracting the family of loops, given by $u_s(e^{it}) = \gamma_s(e^{it})$ for $r \leq 0$. Notice that $u_s(e^{it}) = \gamma_s(e^{it})$ so that $u_s$ maps the boundary of $D$ to the image of $\gamma_s$. Moreover $u_0(e^{it}) = \gamma_0(e^{it})$. Finally $u_s = a$ for $s \leq -1$. Now

$$dH_k|_\gamma(v) = \frac{\partial}{\partial s}|_{s=0} H_k(\gamma_s) = \frac{\partial}{\partial s}|_{s=0} \int_D u_s^* \omega_k =$$

$$\frac{\partial}{\partial s}|_{s=0} \int_D \omega_k(\frac{\partial u_s}{\partial r}, \frac{\partial u_s}{\partial t}) dr \wedge dt.$$

Finally, applying Stokes' theorem gives (13).

Notice further that $LM$ and therefore $\widetilde{LM}$, support an obvious $S^1$ action $(e^{i\theta}, \gamma(e^{i\phi})) \mapsto \gamma(e^{i(\theta+\phi)})$. If we let $X$ denote the vector generating the Lie algebra of $S^1$ and $\Xi$ the induced vector field on $\widetilde{LM}$ then we have

$$\Xi(\gamma) = \dot{\gamma}.$$

Equations (9),(13) and (14) reveal the remarkable fact that

$$i_X \Omega_k = -dH_k,$$

for $k = 1 \ldots l$ and

$$i_X \Omega = -dH.$$

In that case $H_k$ is called a Hamiltonian function for $\Omega_k$ and $H$ a Hamiltonian for $\Omega$. $\Xi$ can be thought of as a symplectic gradient of $H$.

Consider now the flow of $H$. Let $u(s, t) : \mathbb{R} \times S^1 \to M$ be a flow line. Specifically this means that

$$\frac{\partial u}{\partial s} = \nabla H_{u_s(t)},$$

where $u_s(t)$ is simply $u(s, t)$. On the other hand since $\Omega$ and $J$ are compatible and the metric on $\widetilde{LM}$ is given by (10) we have that

$$\nabla H = -J \Xi.$$

Equations (11) and (12) imply then

$$\frac{\partial u}{\partial s} = -J \Xi(u_s(t)) = -J \frac{\partial u}{\partial t}.$$
Therefore
\[ \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0. \] (20)

Now recall that a map between almost complex manifolds is called (pseudo-) holomorphic if its differential respects the almost complex structures. It is easy to see that equation (20) means exactly that \( u \) is holomorphic with respect to \( J \) on \( M \) and the standard complex structure \( j \) on the infinite cylinder \( \mathbb{R} \times S^1 \). To verify that, recall that \( j \) is defined by
\[ j \left( \frac{\partial}{\partial s} \right) = \frac{\partial}{\partial t} \quad \text{and} \quad j \left( \frac{\partial}{\partial t} \right) = -\frac{\partial}{\partial s}. \]

Now \( u \) is holomorphic if \( du \circ j = J \circ du \). Evaluating at \( \frac{\partial}{\partial t} \) gives
\[ du_j \left( \frac{\partial}{\partial t} \right) = J du \left( \frac{\partial}{\partial t} \right) \iff -\frac{\partial u}{\partial s} = J \frac{\partial u}{\partial t}. \]

So indeed \( u(s,t) : \mathbb{R} \times S^1 \to M \) is a (pseudo-) holomorphic cylinder!\(^4\) This is the key reason why quantum cohomology is related to the loop space.

Floer theory is Morse theory for the action functional \( H \) on \( \tilde{LM} \). Notice that the critical manifolds are copies of \( M \), one of them corresponding to trivial loops and the rest translations by the action of the group of deck transformations, i.e., the group of spherical classes in \( H_2(M,\mathbb{Z}) \). This is easy to see using for example (16) which identifies the critical manifolds as the fixed manifolds of the circle action.

Now the fact that \( \tilde{LM} \) is infinite dimensional pauses several hard problems one needs to overcome in order to get a well defined theory. For example, for any critical manifold both the negative and positive normal bundles of \( H \) are infinite dimensional. Therefore the usual notion of index doesn’t make sense. Moreover the standard Morse theoretic method of analyzing the topology of a space simply doesn’t work. This is because we cannot describe the change in topology when going through a critical manifold, by a gluing of the negative normal sphere bundle since this is trivial!\(^5\) It was Floer’s idea to overcome this problem by constructing a Witten type Morse theory where the index is defined by counting orbits connecting critical manifolds. The key point to doing this in this case, is to use orbits of bounded energy. In other words if \( u(s,t) : \mathbb{R} \times S^1 \to M \) is a flow line, ie satisfies (20), then define the energy of \( u \) by:
\[ E(u) = \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2. \] (21)

We say that \( u \) has bounded energy if \( E(u) \) is finite. In fact it is easy to compute that when \( u \) is (pseudo-)holomorphic as is the case for flow lines, then
\[ E(u) = \int_0^1 \int_{-\infty}^{\infty} u^* \omega. \]

Still to get a well behaved theory we have to perturb the flow equation by an extra term using a periodic Hamiltonian. The critical manifolds then become points and

\(^4\) In fact we also see that the flow of \(-H\) gives antiholomorphic cylinders.

\(^5\) The negative normal bundle is trivial since any infinite dimensional bundle is trivial. Moreover the infinite dimensional sphere is homotopically trivial.
the theory can be used to prove the well known Arnold conjecture for periodic Hamiltonians.

Floer\(^6\) was able to rigorously construct a homology theory, now called Floer homology\(^\ast\), using these perturbed holomorphic cylinders connecting periodic orbits. He then showed that Floer homology is isomorphic, with respect to additive structure, to the singular homology of \(M\) with coefficients in an appropriate ring of Laurent series. It should be mentioned at this point that in \([19]\), \([16]\) and \([18]\) it is proved (in each work with different methods) that in fact the isomorphism between Floer and singular homology, respects the ring structures if \(H^2(M, \mathbb{Z})\) is equipped with the Quantum product, and Floer cohomology with the so called, pair of pants product.

The unperturbed Morse-Bott-Floer theory has been worked out to a certain extent by Ruan and Tian in \([19]\).

Following their paper, the space of connecting orbits between two critical levels that differ by a class \(d \in H_2(M, \mathbb{Z})\) should be taken to consist of maps

\[ u : \mathbb{R} \times S^1 \to M \]

that
1. are \(J\)- holomorphic
2. \[ E(u) = \int_{\mathbb{R} \times S^1} u^* \omega < \infty \]
3. \[ \lim_{s \to -\infty} u(s, t) = \text{point} \]

and

\[ \lim_{s \to \infty} u(s, t) = \text{point}, \]

in other words the infinite cylinder closes up at the ends to give a sphere with two point removed.

4. The homology class of the image of \(u\) is \(d\).

If we denote the set of such maps \(u\) by \(\mathcal{M}_d\) then we expect this space to have the same dimension as the space of holomorphic spheres of degree \(d\), the expected dimension of which, is

\[ \dim M + \int_d c_1(TM). \]

Indeed the calculation of Morrison \([17]\) (p. 277) shows that if \(\phi : S^2 \to M\) is holomorphic and \(M\) is a complex manifold then

\[ \chi(\phi^*TM) = h^0(S^2, \phi^*TM) - h^1(S^2, \phi^*TM) = \dim M + \int_d c_1(TM). \] \(22\)

In any case we wish to consider Floer \(S^1\) equivariant cohomology of \(\tilde{LM}\) so following Givental \([11]\) we bypass all that and try to use localization technics instead. The localization theorem relates the equivariant cohomology of a space with a torus acting on it, to that of the fixed components of the action. One way

\[^6\] Floer constructed the theory and proved the Arnold conjecture for so called monotone manifolds (this means that the first Chern class of the manifold is a positive multiple of the symplectic form). For general symplectic manifolds the theory was constructed in \([15]\).

\[^7\] In fact in \([13]\) the authors study a more general situation where the action functional is perturbed but without necessarily insisting that the critical manifolds be points.
of proving this theorem rests on an analysis of the $H^*(\text{point}) = H^*(\mathbb{P}^\infty) = \mathbb{C}[\hbar]$ module structure of the equivariant cohomology ring. We refer to this paper or \textcircled{3} for the more general statement which refers to a torus action. For our purposes we only need the $S^1$ case. The result \textcircled{1} then is that:

\textbf{Theorem 2.} Let $X$ be an $S^1$ (finite dimensional) compact manifold. Let $F$ denote the (possibly disconnected) fixed manifold of the action and let $i : F \to X$ be the inclusion map. Then

$$i^* : H_{S^1}^*(M) \to H_{S^1}^*(F)$$

induces an (additive) isomorphism after localization to the field of rational functions $\mathbb{C}(h)$.

Notice that since $F$ is fixed it follows that $H_{S^1}^*(F) = H^*(F) \otimes \mathbb{C}[\hbar]$. So the meaning of this theorem is that, if the fixed manifolds are $\{F_\alpha\}$ then there is an isomorphism

$$\Phi : H_{S^1}^*(M) \to \bigoplus_\alpha H^*(F_\alpha, \mathbb{C}(h))$$

and so

$$\Phi(a) = \sum_\alpha \lambda_\alpha C_\alpha$$

where $C_\alpha \in H^*(F_\alpha)$ and $\lambda_\alpha$ is a rational function in $\hbar$.

The more precise statement described in \textcircled{3} shows that we don’t really need to invert $\hbar$ to get an isomorphism but inverting some multiple of it (determined as can be expected by the equivariant Euler class of the normal bundle of $F$) is enough. In the general torus case the theorem describes precisely the localization needed in order to get an isomorphism. We refer to \textcircled{1} for the proof.

We would like now to apply this to the space $\mathcal{L}M$. Since it is infinite dimensional this is only a formal application, not a rigorous one.

With this qualification, since the fixed manifolds are copies of $M$, we expect, after Givental\textsuperscript{10}, the $S^1$ equivariant Floer cohomology of $\mathcal{L}M$ to be as an additive object,

$$\text{FH}^*_{S^1}(\mathcal{L}M) = H^*(M, \mathbb{C}[\tilde{q}, \tilde{q}^{-1}](\hbar))$$

where $\mathbb{C}[\tilde{q}, \tilde{q}^{-1}]$ is notation for the group ring $\Lambda$ of the group $\Gamma$ of spherical classes in $H^2(M, \mathbb{Z})$. In other words, instead of using a direct sum notation, we have used the group ring to enumerate the fixed components of the action.

To be more specific, elements of the ring are formal series

$$\lambda = \sum_{d \in \Gamma} \lambda_d \tilde{q}^d$$

where $\lambda_d \in \mathbb{C}$, $\tilde{q}^d = e^{2\pi i d}$ and we declare $\deg \tilde{q}^d = 2 \int_A c_1(TM)$.

Assume $M$ is Kähler and simply connected. $M$ being simply connected implies that $\Gamma = H^2(M, \mathbb{Z})$. Choose a basis $\{p_1, \ldots, p_r\}$ of the Kähler cone and let $\{A_1, \ldots, A_r\}$ be the dual basis of $\Gamma = H^2(M, \mathbb{Z})$ in the sense that $\int_{A_j} p_k = \delta_{j,k}$. Then if $d = \sum_{i=1}^r d_i A_i$ we let $\tilde{q}_k = e^{2\pi i A_k}$ and $\tilde{q}^d = e^{2\pi i d} = \prod_{k=1}^r \tilde{q}_k^{d_k}$. In this fashion the group ring $\lambda$ can be identified with the ring of formal Laurent series $\Lambda = \mathbb{C}[[\tilde{q}, \tilde{q}^{-1}, \ldots, \tilde{q}_1, \tilde{q}^{-1}]]$.

Now let $p_k$ be represented by an $S^1$ invariant kähler form $\omega_k$. Recall we have defined associated kähler classes $\Omega_k$ on $\mathcal{L}M$. Denote by the same name the pullbacks
on \( \mathcal{L}\mathcal{M} \). Let \( d_\hbar = d + \hbar \delta_{\mathcal{L}\mathcal{M}} \) be the Cartan differential, where \( \delta_{\mathcal{L}\mathcal{M}} \) is defined by (14).

Introduce now the equivariant differential forms
\[
P_k = \Omega_k + \hbar H_k,
\]
and
\[
P = \Omega + \hbar H.
\]
Then equations (15) and (16) imply that
\[
d_\hbar P_k = d_\hbar P = 0
\]
for \( k = 1 \ldots r \).

By definition of the \( A_j \) we have:
\[
\int_{A_j} \omega_k = \delta_{j,k}.
\]
Recall we have defined
\[
H_k(\gamma, g) = \int_D g^* \omega_k.
\]
Recall also that \( \Gamma \) acts on \( \mathcal{L}\mathcal{M} \) as the group of covering transformations (so in fact \( \mathcal{L}\mathcal{M}/\Gamma = \mathcal{L}\mathcal{M} \)). Now if we identify \( \mathcal{L}\mathcal{M}/\Gamma = \mathcal{L}\mathcal{M} \) then we have that
\[
\mathcal{L}\mathcal{M}/\Gamma = \mathcal{L}\mathcal{M}.
\]

Moreover, if we denote by \( P_k \) wedge product by the equivariantly closed form \( P_k \) and also denote simply by \( \mathcal{L}\mathcal{M}/\Gamma = \mathcal{L}\mathcal{M} \) then we claim that
\[
\mathcal{L}\mathcal{M}/\Gamma = \mathcal{L}\mathcal{M}.
\]

Besides (27) the operators \( P_j \) and \( \mathcal{L}\mathcal{M}/\Gamma = \mathcal{L}\mathcal{M} \) for \( j, k = 1 \ldots r \) satisfy the relations
\[
[P_j, P_k] = [\mathcal{L}\mathcal{M}/\Gamma, \mathcal{L}\mathcal{M}/\Gamma] = 0.
\]

Now let \( t_1, \ldots, t_r \) be coordinates on \( \mathbb{C}^r \). Then we have
\[
[h \frac{\partial}{\partial t_j}, e^{t_k}] = \delta_{j,k} e^{t_k},
\]
where \( e^{t_k} \) is thought of as an operator acting by multiplication on functions of \( e^{t_1}, \ldots, e^{t_r} \) and \( h \frac{\partial}{\partial t_j} \) for \( j = 1 \ldots r \) also act on such functions. The algebra \( \mathcal{D} \) of operators generated by \( e^{t_1}, \ldots, e^{t_r} \) and \( h \frac{\partial}{\partial t_1}, \ldots, h \frac{\partial}{\partial t_r} \) is called the Heisenberg algebra of differential operators. Relations (27) and (28) say that the \( S^1 \) equivariant Floer cohomology \( F_{H_{S^1}}^\ast (\mathcal{L}\mathcal{M}) \) carries the structure of a module \( \mathcal{D} \) over the Heisenberg algebra \( \mathcal{D} \)!

---

\(^8\)The reason for the choice of sign in the action of \( \Gamma \) on \( \mathcal{L}\mathcal{M} \) is precisely so that we end up with \( e^{t_j} \) instead of \( e^{-t_j} \).
In our discussion of the $A$ model connection, in the previous section, we also encountered a $\mathcal{D}$-module. That one consisted of operators which kill the first row of the solution matrix of the flat section equation for the $A$ connection. Givental’s conjecture is that the two $\mathcal{D}$-modules are in fact the same!

Of course there is no chance of proving this unless a rigorous $S^1$ equivariant Floer theory of the unperturbed action functional is constructed. In case $M$ is a toric variety though, we will construct a model for the space $\hat{\mathcal{L}}M$ in section (5). If $M$ is also positive, then we shall be able to prove that the $\mathcal{D}$-module of our model is indeed the same as the quantum $\mathcal{D}$-module.

Notice than in the previous section, we used coordinates $q_j$ which are related to the $t_j$ by $q_j = e^{t_j}$. It is also clear, that series in the $q_j$ with values in $H^{2*}(M, \mathbb{C})$ can be thought of as sections of a trivial bundle with fiber $H^{2*}(M, \mathbb{C})$, over the (algebraic) torus obtained by the lattice $H^2(M, \mathbb{Z})$ (via complexification and exponentiation). In particular this torus can be thought of as the (affine) toric variety associated to a fan consisting of a single cone, namely the Kähler cone of $M$. The $q_j$ are then identified with the toric coordinates.

Now having a $\mathcal{D}$-module how can we associate a flat connection?

Recall that from (23) we have:

$$\text{(30)} \quad FH^*_{S^1}(\hat{\mathcal{L}}M) = H^*(M, \mathbb{C}[\tilde{q}_1, \tilde{q}_1^{-1}, \ldots, \tilde{q}_r, \tilde{q}_r^{-1}](\hbar)).$$

Therefore $H^{2*}(M, \mathbb{Z})$ is embedded in $FH^*_{S^1}(\hat{\mathcal{L}}M)$. Consider again the basis $T_0, \ldots, T_m$ of $H^{2*}(M, \mathbb{Z})$. We then have

$$\text{(31)} \quad P_k \wedge (T_0, \ldots, T_m) = \tilde{A}^k(T_0, \ldots, T_m),$$

where $\tilde{A}^k$ is a matrix with coefficients functions of $\tilde{q}_k$ and $\hbar$. The coefficient functions are expected to be holomorphic so they will not contain any of the $\tilde{q}_k^{-1}$. Define now a pencil of connections $\tilde{\nabla}_h$ acting on series in the $\tilde{q}_k$ with values in $H^{2*}(M, \mathbb{C})$ by:

$$\text{(32)} \quad \tilde{\nabla}_h = \hbar d - \sum_{k=1}^r \frac{d\tilde{q}_k}{\tilde{q}_k} P_k \wedge.$$

The connection $\tilde{\nabla}_h$ is expected to be equal to the $A$ model connection $\nabla_h$ considered in our discussion of quantum cohomology based on stable maps. This means for example that $\tilde{\nabla}_h$ should be flat, i.e., that flat sections should exist. If $\sigma = \sum_{j=0}^m f_j T_j$ is a section then $\tilde{\nabla}_h \sigma = 0$ is equivalent to the system

$$\text{(33)} \quad \hbar \frac{d}{d\tilde{q}_k}(f_0, \ldots, f_m)^t = \tilde{A}^k(f_0, \ldots, f_m)^t \quad \text{for } k = 1 \ldots r.$$

Flatness of course means that (33) is integrable. To shed some more light we note that (33) can equally be written as

$$\text{(34)} \quad P_k \wedge (T_0, \ldots, T_m)(f_0, \ldots, f_m)^t = (T_0, \ldots, T_m) \hbar \frac{d}{d\tilde{q}_k}(f_0, \ldots, f_m)^t.$$

In other words the $(m + 1)$-tuple $(f_0, \ldots, f_m)^t$ defines a $\mathcal{D}$-module homomorphism between $FH^*_{S^1}(\hat{\mathcal{L}}M)$ and the sheaf $\mathcal{O}$ of holomorphic functions on the torus. Therefore we can reformulate our discussion in an invariant fashion by saying that
an element in $\text{Hom}_D(\text{FH}^*S^1(\widetilde{\mathcal{LM}}), \mathcal{O})$ defines a locally constant sheaf $\mathcal{V}$ over the torus. This sheaf defines in turn by the standard procedure a flat connection on the sheaf $\mathcal{U} = \mathcal{V} \otimes \mathcal{O}$. As an aside we note that this may remind the reader of the construction of the Gauss-Manin connection associated to a family of varieties. The locally constant sheaf is there, the one associated to the integral cohomology of the fiber. This is no accident since mirror symmetry identifies, in the case $M$ is Calabi-Yau, the $A$-model connection with the Gauss-Manin connection of a certain family of Calabi-Yau manifolds.

Instead of concentrating on the connection lets look now at the $\mathcal{D}$-module itself and try to find a presentation or at least some relations. We have seen up to now that $\text{FH}^*S^1(\widetilde{\mathcal{LM}})$ is generated by $H^{2*}(M, \mathbb{Z})$ over the ring $\Lambda$. Geometrically an equivariant Floer cycle associated to an element $T \in H^{2*}(M, \mathbb{Z})$ can be constructed as the boundary loops of all holomorphic discs whose center lies in a cycle representing the Poincaré dual of $T$. Now notice that the standard way to go between the Heisenberg algebra and its presentation in terms of the $P_k$ and $\tilde{q}_k$ is via the Fourier transform. This way relations that involve the later can be transformed to differential equations that involve the former. Indeed if $\Gamma \in \text{FH}^*S^1(\widetilde{\mathcal{LM}})$ and we denote $t_1P_1 + \cdots + t_rP_r$ by $tP$ then consider the pairing

\[
(e^{tP/\hbar}, \Gamma) = \int_{\widetilde{\mathcal{LM}}} e^{tP/\hbar} \Gamma.
\]

We claim that

\[
(e^{tP/\hbar}, R(P, \tilde{q}, \hbar)\Gamma) = R(\hbar \frac{\partial}{\partial t}, e^t, \hbar)(e^{tP/\hbar}, \Gamma).
\]

In other words that the map

\[
\mathcal{F} : \text{FH}^*S^1(\widetilde{\mathcal{LM}}) \rightarrow \mathcal{O}
\]

given by

\[
\mathcal{F}(\Gamma) = (e^{tP/\hbar}, \Gamma)
\]

is an element of $\text{Hom}_D(\text{FH}^*S^1(\widetilde{\mathcal{LM}}), \mathcal{O})$.

Indeed we can do a bit better than that. If $C_T \in \text{FH}^*S^1(\widetilde{\mathcal{LM}})$ is such that it has the same localization $T \in H^{2*}(M, \mathbb{Z})$ on every critical manifold, then

\[
(e^{tP/\hbar}C_T, R(P, \tilde{q}, \hbar)\Gamma) = R(\hbar \frac{\partial}{\partial t}, e^t, \hbar)(e^{tP/\hbar}C_T, \Gamma),
\]

and therefore the map

\[
\mathcal{F}_T : \text{FH}^*S^1(\widetilde{\mathcal{LM}}) \rightarrow \mathcal{O}
\]

given by

\[
\mathcal{F}_T(\Gamma) = (e^{tP/\hbar}C_T, \Gamma),
\]

is an element of $\text{Hom}_D(\text{FH}^*S^1(\widetilde{\mathcal{LM}}), \mathcal{O})$.

The reason is that

\[
(e^{tP/\hbar}C_T, \tilde{q}_k \Gamma) = (\tilde{q}_k^{-1}e^{tP/\hbar}C_T, \Gamma) = e^{tk}(e^{tP/\hbar}C_T, \Gamma)
\]

and

\[
(e^{tP/\hbar}C_T, P_k \Gamma) = (P_k e^{tP/\hbar}C_T, \Gamma) = (\hbar \frac{\partial}{\partial t_k} e^{tP/\hbar}C_T, \Gamma) = \hbar \frac{\partial}{\partial t_k} (e^{tP/\hbar}C_T, \Gamma).
\]
Thus we can find the differential operators and compute solution by computing $(e^{tP/h}, \Gamma)$ if we can write down $\Gamma$ and compute the integral.

In fact if $\Gamma = \Delta$ where $\Delta$ is the fundamental Floer cycle corresponding to the fundamental cycle of $M$, i.e., the cycle of all boundary loops of holomorphic discs in $M$, and the cohomology of $M$ is generated by classes in $H^2(M, \mathbb{Z})$ then polynomials $R(P, \tilde{q}, h)$ such that $R(P, \tilde{q}, h)\Delta = 0$ generate all relations. So if $I_0$ is the ideal generated by such polynomials then $FH^*_{S^1}(\hat{LM}) = \mathbb{C}[P, \tilde{q}, h]/I_0$. The reason for this is that if $f(p_1, \ldots, p_r)$ is a polynomial in the generators $\{p_1, \ldots, p_r\}$ of $H^2(M, \mathbb{Z})$ then the corresponding Floer cycle is $\Delta_f = f(p_1, \ldots, p_r)\Delta$ since out of all loops (boundaries of holomorphic discs) that have their center in $M$, this picks the ones that are in the cycle Poincaré dual to $f(p_1, \ldots, p_r)$. It is clear now that any polynomial $R(P, \tilde{q}, h)$ such that $R(P, \tilde{q}, h)\Delta_f = 0$ induces a relation $R(P, \tilde{q}, h)f(p_1, \ldots, p_r)\Delta = 0$. Therefore relations stemming form $\Delta$ generate all relations.

Up to this point our discussion of $S^1$ equivariant Floer theory of the unperturbed action functional has followed Givental’s paper [10]. We would like now to propose a conjecture about how to regularize the integral in (38). In the last section we shall prove a version of it for toric manifolds.

4. A CONJECTURE ON THE REGULARIZATION OF THE FOURIER TRANSFORM OF THE FLOER FUNDAMENTAL CYCLE

Recall first that we have chosen a basis $\{T_0, \ldots, T_m\}$ of $H^2(M, \mathbb{R})$. We arrange that $T_0 = 1$. Choose also a dual basis $\{T^0, \ldots, T^m\}$ of $H^2(M, \mathbb{Z})$ such that $(T_i, T^j) = \delta_{i,j}$, where the pairing is the Poincaré pairing. Now to compute the integral in (38) for $T = T_j$ and since the integrand is an equivariantly closed form, we could attempt to formally use a localization theorem in equivariant cohomology. The theorem we need is a stronger version of theorem (2) mentioned before and it is due independently to Berline-Vergne [1] and Atiyah-Bott [3].

**Theorem 3.** Let $\mathbb{T}$ be a torus acting on a (finite dimensional) compact manifold $M$ and let $\alpha$ be an equivariantly closed form in the Cartan model. Then

$$
\int_M \alpha = \sum_F \int_F \frac{\alpha|_F}{e_{\mathbb{T}}(N_F)}
$$

where the sum is over all the fixed components $F$ of the action and $e_{\mathbb{T}}(N_F)$ indicates the $\mathbb{T}$ equivariant Euler class of the normal bundle, $N_F$ to the fixed component $F$. By $\alpha|_F$ we denote the pullback of $\alpha$ to $F$ by the inclusion of $F$ into $M$.

Since $\hat{LM}$ is infinite dimensional an application of this theorem in our case can only be done in a formal fashion. This formal application gives:

$$
\mathcal{F}_{T_j}(\Delta) = \int_{\hat{LM}} e^{tP/h} C_{T_j} \Delta = \sum_{d \in H_2(M, \mathbb{Z})} \int_{M_d} T_j e^{\sum_{k=1}^n t_k (\omega_k/h + \frac{1}{2} \omega_k)} \frac{\Delta|_{M_d}}{e_{S^1}(N_d)},
$$

where we have used the following notation. First recall that the action functional $H$ is a function on $\hat{LM}$ whose critical manifolds are the fixed components of the $S^1$ action and therefore are just copies of $M$. Denote the copy of $M$ such that $H|_M = 0$ by $M_0$. The action of $\tilde{q}^2$ maps $M_0$ to another copy of $M$ which we denote by $M_d$. $N_d$ denotes the normal bundle to $M_d$. This is of course an infinite dimensional bundle. Notice now that $N_d$ carries a representation of $S^1$ (as a sub-bundle of
\[ T\mathcal{L}M_{(M_d)} \] and splits to the direct sum of line bundles according to the weights of this representation. The Euler class \( e_{S^1}(N_d) \) is therefore some infinite product which in general, will be divergent. Moreover recall that \( P_h = \Omega_k + hH_k \) and \( \Omega_{k_i|M_d} = \omega_k \) and finally \( H_{k_i|M_d} = \int_{M_d} \omega_k \).

Now we would like to understand better the equivariant Floer fundamental cycle \( \Delta \). Geometrically it is supposed to be the Poincaré dual of the cycle of loops which are boundary values of holomorphic discs in \( M \). We are interested in the restriction of \( \Delta \) to \( M_d \). We have already noticed (20) that flowlines of the action functional are (pseudo-) holomorphic cylinders. Therefore flowlines departing from \( M_d \) are precisely (pseudo-) holomorphic discs in \( M \). It follows that geometrically the fundamental cycle should be represented as a Morse-Witten cycle of \( H \), by the formal sum of the unstable manifolds \( K \) of the fixed components \( M_d \). In finite dimensional Morse-Bott-Witten theory an unstable manifold is fibred over the corresponding critical manifold by the obvious flow map. Moreover a neighborhood of the zero section in the positive normal bundle over a critical manifold is diffeomorphic to a neighborhood of the critical manifold in the unstable manifold \( F \). Therefore the cycle defined by the restriction of the class of the unstable manifold, to the critical manifold should be the Euler class of the positive normal bundle. This implies that the restriction of \( \Delta \) to \( M_d \) should be the equivariant Euler class \( e_{S^1}(N_d^+) \) of the positive normal bundle \( N_d^+ \) to \( M_d \).

Moreover the restriction of \( \Delta \) to \( M_d \) for \( d \) that cannot be represented by a (pseudo-) holomorphic curve should be zero. In other words the non-zero contributions come only from \( deK \).

This informal analysis suggests that (39) becomes:

\[ \mathcal{F}_{T_\beta}(\Delta) = \int_{\mathcal{L}M} e^{tP/h} C_{T_\beta} \Delta = \sum_{d \in \mathcal{K}} \int_{M_d} T_\beta e^{\sum_{k=1}^{r} t_k (\omega_k/h + f_d \omega_k)} \frac{e_{S^1}(N_d^+)}{e_{S^1}(N_d^0)}. \]

Moreover, we have

\[ e_{S^1}(N_d) = e_{S^1}(N_0) = e_{S^1}(N_0^-) e_{S^1}(N_0^+), \]

where by \( N_0^+ \) we denote of course the positive normal bundle to \( M_0 \) and by \( N_0^- \) the negative normal bundle. Therefore we may modify (40) by an overall (infinite!) constant and redefine it as:

\[ \mathcal{F}_{T_\beta}(\Delta) = \int_{\mathcal{L}M} e^{tP/h} C_{T_\beta} \Delta = \sum_{d \in \mathcal{K}} \int_{M} T_\beta e^{\sum_{k=1}^{r} t_k (\omega_k/h + f_d \omega_k)} \frac{e_{S^1}(N_d^+)}{e_{S^1}(N_0^+)}. \]

This step may seem heretical and arbitrary to the reader (and it certainly is) but will be justified in the next section where an honest mathematical proof that this "regularization" works, will be given for the toric case. A similar "regularization" is used by Givental in \[ 10 \].

\[ ^9 \text{It is quite interesting that Atiyah in} \quad 11 \text{shows that the inverse of the Euler class of the normal bundle to} \quad M \text{sitting as the space of trivial loops inside the loop space} \quad \mathcal{L}M, \text{can be normalized by} \quad \zeta \text{function regularization and turns out to be equal to} \quad A(M). \text{Another version of this computation can be found in Jones and Petrack} \quad 12. \]

\[ ^{10} \text{The unstable manifold corresponding to a critical manifold is the manifold of points on the flow lines departing from that critical manifold.} \]

\[ ^{11} \text{See for example appendix 3 of} \quad 13 \text{for a proof of this in the finite dimensional case.} \]

\[ ^{12} \text{Classes not in} \quad \mathcal{K} \text{have negative} \quad f_d \omega_k \text{for some} \quad k. \text{This is not possible for classes of (pseudo-) holomorphic curves since from equation (21) it follows that this integral is equal (up to a factor of} \quad \frac{1}{h} \text{) to the energy of the curve.} \]
Now as was explained in the previous section, in the case where the cohomology of $M$ is generated by $H^2(M, \mathbb{Z})$, the quantum $\mathcal{D}$-module is expected to be $\mathcal{D}/I_0$ where $I_0$ is the ideal of operators annihilating $\mathcal{F}_T(\Delta)$ for $\beta = 0 \ldots m$. It is more convenient to consider them all at once as follows: Consider the following $H^2_*(M, \mathbb{C})$-valued function:

$$F = \sum_{d \in \mathcal{K}} e^{\sum_{k=1}^r t_k(\omega_k/h + \int_d \omega_k)} \frac{e_{S_1}(N^d_+)}{e_{S_1}(N^d_0)}$$

Then

$$F = \sum_{\beta=1}^m \mathcal{F}_T(\Delta) T^\beta.$$

It is important to notice that the Euler class $e_{S_1}(N^d_+)$ used in (41) and (42), is not well defined, since it is an infinite divergent product (of the Euler classes of the line bundles in the -possibly virtual- splitting of $N^d_+$ according to weights of the representation of the circle action on the fibers). What we claim though, is that, certainly in the toric case (see next section) and conjecturally for general symplectic manifolds there is a way to define the ratio of the two Euler classes.

Bearing this in mind, we may now formulate a conjecture for the regularization of the “Fourier transform” of the Floer fundamental cycle $\Delta$ as follows:

**Conjecture 1.** If the cohomology of $M$ is generated by classes in $H^2(M, \mathbb{Z})$ then, the quantum cohomology $\mathcal{D}$-module of $M$ is generated by the $H^2_*(M, \mathbb{C})$-valued function:

$$F = \sum_{d \in \mathcal{K}} e^{\sum_{k=1}^r t_k(\omega_k/h + \int_d \omega_k)} \frac{e_{S_1}(N^d_+)}{e_{S_1}(N^d_0)}$$

or equivalently

$$F = e^{(t_1 \omega_1 + \cdots + t_r \omega_r)/h} \sum_{d \in \mathcal{K}} q^d \frac{e_{S_1}(N^d_+)}{e_{S_1}(N^d_0)},$$

which has components appropriate regularizations of $\mathcal{F}_T(\Delta)$. Here $q_k = e^{t_k}, d_k = \int_d \omega_k$ and $q^d = q_1^{d_1} \ldots q_r^{d_r}$.

Moreover for $d = 0$ we let $\frac{e_{S_1}(N^d_+)}{e_{S_1}(N^d_0)} = 1$.

By “generate the quantum cohomology $\mathcal{D}$-module” we mean that the quantum cohomology $\mathcal{D}$-module is equal to $\mathcal{D}/I_0$ where $I_0$ is the ideal of operators in $\mathcal{D}$ that annihilate the components of $F$.

Moreover we conjecture that:

$$F = G,$$

where $G$ is the function defined, in terms of the space of stable maps, by equation (7) and by definition has the property of generating the quantum $\mathcal{D}$-module of $M$.

Our next task will be to explain how to define the ratio of the Euler classes in the case of toric manifolds and then prove that $F = G$ and thus indeed generates the quantum $\mathcal{D}$-module.

The strategy in that case will be to reformulate everything in terms of a sequence of finite dimensional approximations of the loop space, by spaces parameterizing...
loops of arbitrarily large but finitely many modes (in their Fourier expansion). We can then consider the corresponding sequence of ratios of Euler classes. These ratios stabilize for large modes and we use this “stable ratio” as our definition. We can then calculate $F$ explicitly. Finally, we invoke the calculation of $G$ by Givental [11] to show that $F = G$.

As a final observation we note that $F_{T \beta}$ for all $\beta = 0, \ldots, m$ are elements of $\text{Hom}_D(F_{H^*} S^1(\tilde{L}_M), O)$ and therefore define flat section of the A-connection. Denote by $\Delta_\alpha$ the Floer-Witten cycle corresponding to $T_\alpha$. Then $\Delta_0 = \Delta$. For a fixed $T_\beta$, the functions $F_{T \beta}(\Delta_\alpha)$ simply give $F_{T \beta}$ in a basis. They should be

$$F_{T \beta}(\Delta_\alpha) = s_{\alpha, \beta},$$

where $(s_{\alpha, \beta})$ is the fundamental solution matrix (6) of the A-model flat section equation. In this fashion we may identify $F_{T \beta}$ with $s_\beta$, namely, the flat section found in theorem (1).

5. The Toric Case

Our goal in this section is to formulate and prove rigorously a version of Conjecture (1) formulated in the previous section. First we need to describe the set up. Let $M$ be a compact, smooth, Kähler toric variety. We choose to think of it as a symplectic quotient. To that end, in order to define $M$ we start with an exact sequence of lattices as in:

$$0 \to \mathbb{Z}^l \to \mathbb{Z}^n \to \mathbb{Z}^d \to 0, \tag{46}$$

where the first map is called $m$ and the second $\pi$. Tensoring the sequence with $\mathbb{C}$ and exponentiating gives a sequence of algebraic tori:

$$1 \to \mathbb{C}^* \to \mathbb{C}^* \to \mathbb{C}^* \to 1. \tag{47}$$

Now tensoring (46) with $i\mathbb{R}$ and exponentiating gives a sequence of real tori

$$1 \to T^l \to T^n \to T^d \to 1. \tag{48}$$

These sequences define an embedding of $\mathbb{C}^* \to \mathbb{C}^*$ and of $T^l \to T^n$. Composing this with the diagonal action of $\mathbb{C}^*$ on itself defines the action

$$(x_1, \ldots, x_n) \mapsto (\prod_{j=1}^l \lambda_j^{m_{1,j}} x_1, \ldots, \prod_{j=1}^l \lambda_j^{m_{l,j}} x_n). \tag{49}$$

Associated to this, there is the moment map

$$\mu : \mathbb{C}^n \to \mathbb{R}^l$$

given by

$$\mu = \mu_l \circ \mu_n = \frac{1}{2} \left( \sum_{k=1}^n m_{1,k} | x_k |^2 , \ldots, \sum_{k=1}^n m_{l,k} | x_k |^2 \right). \tag{50}$$

If $\lambda \in \mathbb{R}^l$ is a regular value of $\mu$ then $M$ is constructed by symplectic reduction as

$$M = M_\lambda = \mu^{-1}(\lambda)/T^l. \tag{51}$$

$M$ comes equipped with the reduced symplectic form $\omega_\lambda$. For simplicity we shall just denote it by $\omega$.\footnote{Excellent references for toric varieties are Fulton [5] for the algebraic geometric point of view and Audin [6] for the symplectic side.}
$M$ is a Kähler (and at worst) orbifold. The Kähler form is the reduction of the standard Kähler form on $\mathbb{C}^n$. Notice further that there is a cone in $\mathbb{R}^d$ defined by the conditions that it contains $\lambda$ and that the differential of $\mu$ drops rank along its walls. Reducing at any point in the cone gives a space topologically equivalent but with a different Kähler form. In fact we may identify this cone with the Kähler cone $K$ of $M$.

Now recall that if $(x_1, \ldots, x_n)$ are coordinates on $\mathbb{C}^n$ then they can be thought of as sections of corresponding line bundles $L_k$ over $M$ for $k = 1, \ldots, n$. The divisor $(x_k)$ is denoted by $D_k$. We call these divisors the toric divisors. Let $v_k = \pi(w_k)$ for $k = 1, \ldots, n$ where $\{w_1, \ldots, w_n\}$ is the standard basis of $\mathbb{Z}^n$, then rational equivalences among the $D_k$ are given by the relations

$$\sum_{k=1}^n <e_{\nu^*}, v_k> D_k = 0 \quad \text{for} \quad \nu = 1, \ldots, (n-l), \quad (52)$$

where $\{e_{1^*}, \ldots, e_{n-l^*}\}$ is the dual of the standard basis of $\mathbb{Z}^{n-l}$. Let $\alpha_k = c_1(L_k)$. The $\alpha_k$ are Poincaré dual to the $D_k$ and (52) gives the additive relations among them.

It is known (see eg Fulton [9]) that $H^2(M, \mathbb{R})$ is spanned by the $\alpha_k$ and that $H^*(M, \mathbb{R})$ is generated by classes in $H^2(M, \mathbb{R})$.

Now we would like to model somehow the space $\widetilde{LM}$ defined in section (3) as a covering of the space of free contractible loops in $M$. To this end lets consider loops in $\mathbb{C}^n$ with a finite but large number of modes $2N$. To be specific we shall consider loops which are in general of the form:

$$\gamma : S^1 \to \mathbb{C}^n \quad \text{with} \quad \gamma(e^{i\theta}) = (\gamma_1(e^{i\theta}), \ldots, \gamma_n(e^{i\theta})), \quad (53)$$

where, if we let $z = e^{i\theta}$, then

$$\gamma_k : S^1 \to \mathbb{C}, \quad (54)$$

has Fourier expansion:

$$\gamma_k(z) = \sum_{\nu = -N}^{N} a_{\nu^k} z^\nu. \quad (55)$$

Our model $\widetilde{L_N M}$ for $\widetilde{LM}$ will be defined as follows: The space $\widetilde{L_N C^n}$ of loops of finite modes in $\mathbb{C}^n$ is parametrized by the Fourier coefficients $a_{\nu^k}$ and therefore is just $\mathbb{C}^{n2N}$. Consider the $\mathbb{C}^{*1}$ (or $T^1$) action on $\widetilde{L_N C^n}$ induced by the action (49) on $\mathbb{C}^n$ defining $M$. By this we mean that the action on all the coefficients of $\gamma_k$ is the same as the action on $x_k$. The moment map attached to this action is:

$$\mu_N = \frac{1}{2} \left( \sum_{\nu = -N}^{N} \sum_{k=1}^n m_{1,k} |a_{\nu^k}|^2, \ldots, \sum_{\nu = -N}^{N} \sum_{k=1}^n m_{l,k} |a_{\nu^k}|^2 \right). \quad (56)$$

Define $\widetilde{L_N M}$ as

$$\widetilde{L_N M} = \mu_N^{-1}(\lambda)/T^1, \quad (57)$$

where we have identified the Kähler cones of $M$ and $\widetilde{L_N M}$. We can do this since the subsets of $\mathbb{R}^d$ where $\mu$ and $\mu_N$ drop rank are clearly the same. This is just because $\mu$ drops rank at some value if and only if some homogeneous coordinates are forced to be zero. At the same value $\mu_N$ drops rank since the corresponding
sums of squares are forced to be zero which in turn forces each of the squares to be zero.

Now in general the Kähler cone $K$ will not necessarily be simplicial, but it can of course be subdivided, to simplicial cones. Pick such a subdivision and consider the simplicial cone containing the value $\lambda$.

Let $\{\omega_1, \ldots, \omega_l\}$ be the basis of that cone such that

\begin{equation}
\omega = \sum_{j=1}^{l} \lambda_j \omega_j.
\end{equation}

The fact that the cone is simplicial means that $\{\omega_1, \ldots, \omega_l\}$ is a basis of $H^2(M, \mathbb{Z})$. Moreover we have that

\begin{equation}
\alpha_k = \sum_{j=1}^{l} m_{j,k} \omega_j.
\end{equation}

If $d$ is an element of $H_2(M, \mathbb{Z})$ then we let

\begin{equation}
d_j = \int_d \omega_j.
\end{equation}

In that case we may identify $d$ with the vector $(d_1, \ldots, d_l)$. Next we need to consider the action functional $H_N$ associated to $\tilde{L}_N\tilde{M}$. Recall that the action functional assigns to a pair (loop, contracting disc) the symplectic area of the contracting disc. Recall also that we have fixed the standard Kähler form on $\mathbb{C}^n$ which is

\begin{equation}
\omega_0 = \sum_{k=1}^{n} ds_k \wedge dt_k = \frac{i}{2} \sum_{k=1}^{n} dx_k \wedge d\overline{x_k},
\end{equation}

where $x_k = s_k + it_k$. Define first

\begin{equation}
H_N(\gamma) = \frac{1}{2\pi} \int_{\gamma(S^1)} \sum_{k=1}^{n} s_k dt_k = \frac{1}{2\pi} \sum_{k=1}^{n} \int_{S^1} \gamma^*(s_k dt_k) = \frac{1}{2\pi} \sum_{k=1}^{n} \int_D u^*(\frac{i}{2} dx_k \wedge d\overline{x_k}),
\end{equation}

where $\gamma$ is given by (53). In other words $H_N$ is the (normalized) action functional for loops in $\mathbb{C}^n$ or rather $\tilde{L}_N\tilde{M}$ to be exact.

An elementary calculation shows that

\begin{equation}
H_N(\gamma) = \frac{1}{2} \sum_{\nu=-N}^{N} \nu(|a_{\nu}^1|^2 + \cdots + |a_{\nu}^n|^2).
\end{equation}

To see this, it’s enough to notice that if a loop $\gamma : S^1 \to \mathbb{C}$ is given by $\gamma(e^{i\theta}) = e^{ik\theta}$ then $u : D \to \mathbb{C}$ such that $u(re^{i\theta}) = r^k e^{ik\theta}$ contracts that loop. Moreover if $x = re^{i\theta}$ is a coordinate on $\mathbb{C}$ then

\[
\int_D u^*(dx \wedge d\overline{x}) = \int_D (du \wedge d\overline{u}) = \int_0^{2\pi} \int_0^1 -2ik^2 r^{2k-1} dr \wedge d\theta = -2\pi ik.
\]

Now $H_N$ is thus far defined on $\mathbb{C}^{n2N}$, but since it is invariant under the $T^l$ action, it actually drops to a function on $\tilde{L}_N\tilde{M}$. We will still call that function by the same name $H_N$, and it is our action functional.
Consider next the $S^1$ action on $\hat{L}_N M$. It is induced by rotation on the source circle, namely by the action $e^{iθ} \mapsto e^{i(θ + φ)}$. This action induces an action on the Fourier coefficients of a loop $γ$ by
\[ a_ν^k \mapsto e^{ikφ}a_ν^k. \]
(64)
It’s clear that $H_N$ is the Hamiltonian function corresponding to this action on $\hat{L}_N C^n$ and consequently on $\hat{L}_N M$. This is in accordance with the general theory. As we saw in equation (16), the action functional is indeed the Hamiltonian of the circle action.

As a consequence the fixed components of the circle action on $\hat{L}_N M$ coincide with the critical manifolds of $H_N$. We expect those to be copies of $M$ and to correspond to homology classes $d ∈ H_2(M, \mathbb{Z})$. Recall that we may identify the class $d$ with its period vector $(d_1, \ldots, d_l)$ as in (60). Now the fixed components of the circle action can be identified as follows: The action of $T^l$ on $μ^{-1}(λ) ∈ \hat{L}_NH_N = C^{n2N}$ that defines $\hat{L}_N M$ is induced by the action in (49). When we take $λ_1 = z^{d_1}, \ldots, λ_l = z^{d_l}$, this becomes an $S^1$ action. Components in $μ^{-1}(λ)$ where the $S^1$ action from (64) coincides with the one appearing as a one parameter subgroup of the $T^l$ action as above, will lead to fixed components in $\hat{L}_N M = μ^{-1}(λ)/T^l$. Thus making the substitution
\[ λ_1 = z^{d_1}, \ldots, λ_l = z^{d_l} \]
in (49) shows immediately that loops of the form
\[ γ(z) = (a_1^{\sum_{j=1}^l m_{j1}d_j} z^{\sum_{j=1}^l m_{j1}d_j}, \ldots, a_n^{\sum_{j=1}^l m_{jn}d_j} z^{\sum_{j=1}^l m_{jn}d_j}), \]
(65)
form a fixed component of the circle action on $\hat{L}_N M$. The $T^l$ action on $C^{n2N}$ restricted to loops in $C^n$ of the form (65), restricts to the action (49) defining $M$. Therefore the reduction of the space of loops of the form (65) will indeed be exactly a copy of $M$. We shall name this component $M_d$. Notice that there is a more illuminating way to write (65). According to (59) and (60) we have
\[ \sum_{j=1}^l m_{j,k}d_j = \int_d \alpha_k, \]
where $α_k$, as we said earlier, is Poincaré dual to the toric divisor $D_k$. Therefore we see that $M_d$ consists of loops of the form:
\[ γ(z) = (a_1^{f_1α_1}z^{f_1α_1}, \ldots, a_n^{f_nα_n}z^{f_nα_n}). \]
(67)
This immediately tells us that in order to be able to study $M_d$ we must take $N ≥ \max\{f_1α_1, \ldots, f_nα_n\}$ which we will assume from now on whenever discussing $M_d$. To recapitulate the set up, up to now we have defined the spaces $\hat{L}_N M$ which as $N → ∞$ approximate $L\hat{M}$ and action functionals
\[ H_N : \hat{L}_N M → \mathbb{R}. \]
(68)
We have also described the critical manifolds $M_d$ of $H_N$, which are copies of $M$ and of course coincide with the fixed components of the circle action on $\hat{L}_N M$.

Moreover notice that since $H_N$ is the Hamiltonian of an $S^1$ action it follows from general theory that it is a perfect Morse-Bott function. This is explained for example in Audin [3]. In our case it also obvious from (63) which shows that indices of $H_N$ are even numbers. The so called lacunary principle (see e.g. Bott [3]) then
guarantees that \( H_N \) is perfect. It’s also clear that \( H_N \) is non-degenerate in the normal directions.

Now let \( \mathcal{N}_{d,N} \) denote that normal bundle to \( M_d \subset \mathcal{L}_N M \). Let \( \mathcal{N}_{d,N}^+ \) and \( \mathcal{N}_{d,N}^- \) denote the positive and negative normal bundles of \( M_d \subset \mathcal{L}_N M \). Obviously we have \( \mathcal{N}_{d,N} = \mathcal{N}_{d,N}^- \oplus \mathcal{N}_{d,N}^+ \). We shall first prove the following proposition.

**Proposition 3.** Let \( M \) be a smooth toric manifold. For every class \( d \in H_2(M, \mathbb{Z}) \), there is an integer \( N(d) \) such that the ratio of equivariant Euler classes,

\[
\frac{e_{S^1}(\mathcal{N}_{d,N}^+)}{e_{S^1}(\mathcal{N}_{d,N}^+)_{0}},
\]

remains constant for all \( N \geq N(d) \).

In other words this ratio of Euler classes stabilizes. This allows us to define the stable ratio.

**Definition 2.** Define the stable ratio \( \frac{e_{S^1}(\mathcal{N}_{d,N}^+)}{e_{S^1}(\mathcal{N}_{d,N}^+)_{0}} \) to be the common ratio \( \frac{e_{S^1}(\mathcal{N}_{d,N}^+)}{e_{S^1}(\mathcal{N}_{d,N}^+)_{0}} \) for all \( N \geq N(d) \)

After we have this and keeping the notation from above we will prove the following version of Conjecture 1.

**Theorem 4.** Let \( M \) be a smooth toric variety of Picard number \( 1 \), and assume that for every \( d \in \mathcal{N} - \{0\} \) we have \( \int_d c_1(T_M) > 0 \). Let

\[
(70) \quad F = e^{(t_1 \omega_1 + \cdots + t_l \omega_l)/h} \sum_{d \in \mathcal{N}} q^d \frac{e_{S^1}(\mathcal{N}_{d,N}^+)}{e_{S^1}(\mathcal{N}_{d,N}^+)_{0}},
\]

then

\[
(71) \quad F = G,
\]

where

\[
G = e^{(t_1 \omega_1 + \cdots + t_l \omega_l)/h} (1 + \sum_{d \in \mathcal{N}, d \neq 0} q^d e_{S^1}(1/\hbar - c))
\]

is the function defined in (7) (since \( p_i = \omega_i \) and \( q_i = e^{t_i} \)). Here \( e_{S^1} : M_{2,d} \rightarrow M \) is evaluation at the first marked point and \( c \) is the Chern class of the line bundle over \( M_{2,d} \) which is the universal cotangent line at the second marked point. Moreover, since \( G \) is the function that generates the Quantum cohomology \( \mathcal{D} \)-module of \( M \), it follows that \( F \) does too.

We begin with the proof of proposition (3). We need to analyze in detail the normal bundle \( \mathcal{N}_{d,N} \). One way to do this, would be to analyze the normal bundle by considering the affine patches that cover \( \mathcal{L}_N M \) coming from toric geometry. This is feasible but not so convenient so we take a different approach. Recall that \( \mathcal{L}_N M = \mu_N^{-1}(\lambda)/T^d \), where \( \lambda \in \mathbb{R}^l \) has components \( \lambda = (\lambda_1, \ldots, \lambda_l) \). Therefore according to (56) we have

\[
(72) \quad \frac{1}{2} \sum_{\nu=-N}^{N} \sum_{k=1}^{n} m_{\nu,k} |a_{\nu,k}|^2 = \lambda_1, \ldots, \frac{1}{2} \sum_{\nu=-N}^{N} \sum_{k=1}^{n} m_{\nu,k} |a_{\nu,k}|^2 = \lambda_l.
\]

Now the action functional \( H_N \) can be restricted to \( \mu_N^{-1}(\lambda) \) by using the relations (72). Once we do this, then we have the function on \( \mathcal{L}_N M \) since \( H_N \) is invariant
under the $T_l$ action. In order to look in the normal directions of $M_d$ and using the description of $M_d$ found in (67), we work as follows:

\[(73) \quad 2H_N(\gamma) = \sum_{\nu=-N}^{N} \nu(|a_{\nu}|^2 + \cdots + |a_{\nu}|^2) = \]

\[= \sum_{\nu \neq f_d \alpha_1} \nu |a_{\nu}|^2 + \cdots + \sum_{\nu \neq f_d \alpha_n} \nu |a_{\nu}|^2 + (\int_{f_d} a_1 |f_d \alpha_1|^2 + \cdots + (\int_{f_d} a_n |f_d \alpha_n|^2). \]

Now substituting (66) in (73) we find:

\[(74) \quad 2H_N = \sum_{\nu \neq f_d \alpha_1} \nu |a_{\nu}|^2 + \cdots + \sum_{\nu \neq f_d \alpha_n} \nu |a_{\nu}|^2 + \frac{1}{2} \sum_{j=1}^{l} m_{j,1} |f_d \alpha_1|^2 + \cdots + \frac{1}{2} \sum_{j=1}^{l} m_{j,n} |f_d \alpha_n|^2. \]

Rearranging this sum gives:

\[(75) \quad 2H_N = \sum_{\nu \neq f_d \alpha_1} \nu |a_{\nu}|^2 + \cdots + \sum_{\nu \neq f_d \alpha_n} \nu |a_{\nu}|^2 + \frac{1}{2} \sum_{k=1}^{n} |f_d \alpha_1|^2 + \cdots + \frac{1}{2} \sum_{k=1}^{n} |f_d \alpha_n|^2. \]

Now we may use (72) to obtain

\[2H_N = 2 \sum_{j=1}^{l} d_j \lambda_j + \sum_{\nu \neq f_d \alpha_1} \nu |a_{\nu}|^2 + \cdots + \sum_{\nu \neq f_d \alpha_n} \nu |a_{\nu}|^2 - \sum_{k=1}^{n} \sum_{\nu \neq f_d \alpha_k} |f_d \alpha_k|^2 - \sum_{k=1}^{n} \sum_{\nu \neq f_d \alpha_k} |f_d \alpha_k|^2. \]

Rearranging the sum once more we find:

\[H_N = \sum_{j=1}^{l} d_j \lambda_j + \frac{1}{2} \sum_{\nu \neq f_d \alpha_1} (\nu - \int_{f_d} a_1) |a_1|^2 + \cdots + \frac{1}{2} \sum_{\nu \neq f_d \alpha_n} (\nu - \int_{f_d} a_n) |a_n|^2 \]

and finally

\[(76) \quad H_N = \int_{\omega} \omega + \frac{1}{2} \sum_{\nu \neq f_d \alpha_1} (\nu - \int_{f_d} a_1) |a_1|^2 + \cdots + \frac{1}{2} \sum_{\nu \neq f_d \alpha_n} (\nu - \int_{f_d} a_n) |a_n|^2. \]

Notice that the fact that $\sum_{j=1}^{l} d_j \lambda_j = f_d \omega$, follows from (58) and (60). Moreover since $H_N$ is quadratic, computing the Hessian is immediate. Recall also from (67) that

\[(77) \quad M_d = (a_1^{f_d \alpha_1}, \ldots, a_n^{f_d \alpha_1}) / T^l, \]

where it is implied that all the other coordinates are zero. The normal bundle of the subset of $\mathbb{C}^{2N}$ whose quotient by the $T^l$ action is $M_d$, is trivial of course and has fiber coordinates given by all the variables in $\mathbb{C}^{2N}$ except for $(a_1^{f_d \alpha_1}, \ldots, a_n^{f_d \alpha_1})$. The normal bundle of $M_d$ is the quotient of the normal bundle to the subset
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(a_{f_a \alpha_1}, \ldots, a_{f_a \alpha_n}). Therefore if we let \hat{L}_{k,\nu} be the bundle over \hat{\mathcal{L}}_N M associated to \nu^k and define

\begin{align}
E_d &= \bigoplus_{\nu \neq f_a \alpha_1} \hat{L}_{1,\nu} \oplus \cdots \oplus \bigoplus_{\nu \neq f_a \alpha_n} \hat{L}_{n,\nu},
\end{align}

then

\begin{align}
N_{d,N} &= E_d|_{M_d}.
\end{align}

Of course \hat{L}_{k,\nu} is isomorphic to \hat{L}_{k,\nu'} for all \nu_1 and \nu_2 if we don’t take into account the S^1 action but they are different as S^1 equivariant bundles according to (64).

Now let

\begin{align}
L_{k,\nu} = \hat{L}_{k,\nu}|_{M_d},
\end{align}

then we have that

\begin{align}
L_{k,\nu} \text{ isomorphic to } L_k \text{ for all } \nu,
\end{align}

where again the isomorphism is taken in the usual sense, not the S^1 equivariant.

As a consequence of (79) we have that

\begin{align}
N_{d,N} = \bigoplus_{\nu \neq f_a \alpha_1} L_{1,\nu} \oplus \cdots \oplus \bigoplus_{\nu \neq f_a \alpha_n} L_{n,\nu}.
\end{align}

It is now straight forward to compute the S^1 equivariant Euler class of each of the line bundles. The key point for us, is to read off from the expression of the action functional in (76) the positive normal bundle. Clearly we have that if

\begin{align}
E^+_d &= \bigoplus_{\nu > f_a \alpha_1} \hat{L}_{1,\nu} \oplus \cdots \oplus \bigoplus_{\nu > f_a \alpha_n} \hat{L}_{n,\nu},
\end{align}

then

\begin{align}
N^+_{d,N} = E^+_d|_{M_d} = \bigoplus_{\nu > f_a \alpha_1} L_{1,\nu} \oplus \cdots \oplus \bigoplus_{\nu > f_a \alpha_n} L_{n,\nu}.
\end{align}

Finally since the S^1 action is given by (64) and using (80) we find that

\begin{align}
\epsilon_{S^1}(L_{k,\nu}) = c_1(L_{k,\nu}) + \nu \hbar = \alpha_k + \nu \hbar,
\end{align}

and therefore

\begin{align}
\epsilon_{S^1}(N^+_{d,N}) = \prod_{\nu > f_a \alpha_1} (\alpha_1 + \nu \hbar) \ldots \prod_{\nu > f_a \alpha_n} (\alpha_n + \nu \hbar).
\end{align}

As a special case it follows that

\begin{align}
\epsilon_{S^1}(N^+_{0,N}) = \prod_{\nu > 0} (\alpha_1 + \nu \hbar) \ldots \prod_{\nu > 0} (\alpha_n + \nu \hbar).
\end{align}

We may now compute the ratio

\begin{align}
\frac{\epsilon_{S^1}(N^+_{d,N})}{\epsilon_{S^1}(N^+_{0,N})} &= \frac{1}{\prod_{\nu = 1}^{f_a \alpha_1} (\alpha_1 + \nu \hbar) \ldots \prod_{\nu = 1}^{f_a \alpha_n} (\alpha_n + \nu \hbar)}.
\end{align}
It follows that indeed the ratio \( \frac{e_{S_1}(N^+_d)}{e_{S_1}(N^+_0)} \) is independent of \( N \) as long as \( N \) is greater than \( N(d) = \max \{ f_0, \ldots, f_n \} \). This concludes the proof of Proposition (3).

The proof of Theorem (4) follows now easily from the proof of Proposition (3). First note that the stable ratio is

\[
\frac{e_{S_1}(N^+_d)}{e_{S_1}(N^+_0)} = \frac{1}{\prod_{\nu=1}^{d} (\alpha_1 + \nu \hbar) \ldots \prod_{\nu=1}^{d} (\alpha_n + \nu \hbar)}.
\]

We may now compute the function \( F \) of (70). We find that

\[
F = e^{(t_1 \omega_1 + \cdots + t_n \omega_n) / \hbar} \sum_{d \in \mathcal{K}} q^d \frac{e_{S_1}(N^+_d)}{e_{S_1}(N^+_0)} = e^{(t_1 \omega_1 + \cdots + t_n \omega_n) / \hbar} \sum_{d \in \mathcal{K}} q^d \frac{1}{\prod_{\nu=1}^{d} (\alpha_1 + \nu \hbar) \ldots \prod_{\nu=1}^{d} (\alpha_n + \nu \hbar)}
\]

where \( q^d \) stands for \( q_1^{d_1} \ldots q_n^{d_n} = e^{t_1 d_1 + \cdots + t_n d_n} \) as usual.

Finally, according to Givental’s computation of the function

\[
G = e^{\frac{\ln q}{\hbar}}(1 + \sum_{d \in \mathcal{K}, d \neq 0} q^d e v_1, (\frac{1}{\hbar - c}))
\]

in [12] (Theorem (0.1), page (3) and its corollary: Example (a) page (4)) we have that, if \( \int_d c_1(TM) > 0 \) for all \( d \in \mathcal{K} \) and \( d \neq 0 \), then the function \( F \) as computed above is indeed equal to the function \( G \) and therefore it generates the Quantum \( \mathcal{D} \) module. This concludes the proof of theorem (4). It may finally be useful to consider a simple example in order to clarify things a bit more.

**Example 1.** Let us consider the simplest example which is the complex projective space \( \mathbb{P}^n \). Let \( \omega \) be the class dual to a hyperplane. The Kähler cone is a half line and is generated by \( \omega \). The toric divisor classes \( \alpha_d \) are also all equal to the class dual to a hyperplane. For \( d \) in \( H_2(\mathbb{P}^n, \mathbb{Z}) \) let \( d_1 = \int_{\omega} d = \int_{\omega} \alpha_1 \). Let \( q_1 = e^{t_1} \). Then the function \( F \) of (70) becomes

\[
F = e^{t_1 \omega / \hbar} \sum_{d \in \mathcal{K}} q^d \frac{e_{S_1}(N^+_d)}{e_{S_1}(N^+_0)} = e^{t_1 \omega / \hbar} \sum_{d_1=0}^{\infty} q_1^{d_1} \frac{1}{\prod_{\nu=1}^{d_1} (\omega + \nu \hbar)^{n+1}}
\]

We may now expand \( F \) in the basis \( \{ 1, \omega, \omega^2, \ldots, \omega^n \} \) :

\[
F = \sum_{i=0}^{n} f_i(e^{t_1}, \hbar) \omega^i.
\]

Let \( < a, b > = \int_{\mathbb{P}^n} a \wedge b \) where \( a \) and \( b \) are cohomology classes in \( \mathbb{P}^n \). Clearly we have \( f_i = < F, \omega^{n-i} > \). Moreover notice that \( < a, b > = \text{Res}_{0 \hbar} \frac{d\omega}{\omega^{n+1}} \). Therefore

\[
f_i = \text{Res}_{0 \hbar} \sum_{d_1=0}^{\infty} q_1^{d_1} \frac{\omega^{n-i} e^{t_1 \omega / \hbar}}{\prod_{\nu=1}^{d_1} (\omega + \nu \hbar)^{n+1}} \frac{d\omega}{\omega^{n+1}}.
\]

The easiest one to compute is \( f_0 \) :

\[
f_0(e^{t_1}, \hbar) = \text{Res}_{0 \hbar} \sum_{d_1=0}^{\infty} q_1^{d_1} \frac{\omega^n e^{t_1 \omega / \hbar}}{\prod_{\nu=1}^{d_1} (\omega + \nu \hbar)^{n+1}} \frac{d\omega}{\omega^{n+1}}.
\]
Therefore

\[
f_0(e^{t_1}, \hbar) = \sum_{d_1=0}^{\infty} \frac{e^{d_1} \hbar^{d_1(n+1)} (d_1!)^{n+1}}{d_1!}.
\]

The function \( f_0 \) is annihilated by the differential operator \( R(h^{\frac{\partial}{\partial t_1}}, e^{t_1}, \hbar) = (h^{\frac{\partial}{\partial t_1}})^{n+1} - e^{t_1}. \) The quantum \( D \)-module of \( \mathbb{P}^n \) is the Heisenberg algebra modulo the ideal generated by \( R. \) Finally the corresponding relation in the quantum ring of \( \mathbb{P}^n \) is \( R(p, q, 0) = 0 \) i.e., \( p^{n+1} = q. \) Indeed, the quantum cohomology of \( \mathbb{P}^n \) is \( \mathbb{C}[p, q]/(p^{n+1} = q) \) where \( p^{n+1} \) is computed by the quantum multiplication and \( p \) is the class of the hyperplane. For a computation of the quantum cohomology in terms of the space of stable maps see for example [11]. Notice also that the rest of the \( f_i \) are also annihilated by \( R. \) In fact we get a complete basis of solutions of the equation \( R = 0. \)

It is always true that the components of \( F \) satisfy the same differential equation since they all come from the Fourier transform of the same cycle, i.e., the Floer semi-infinite cycle \( \Delta. \)

6. Comments and further problems

Let us note first, that Jones and Petrack in [13] have actually constructed an extension of equivariant cohomology that works well in infinite dimensional settings and in particular for the \( S^1 \) action on the loop space. It would be very interesting to try and use it in this case.

Another observation follows directly from looking at the formulas for \( F \) (70) and \( G \) (7). It is clear that at least in the toric case

\[
\frac{e_{S^1}(\mathbb{P}^n)}{e_{S^1}(\mathbb{P}^n_0)} = ev_{1*} \left\{ \frac{1}{\hbar - e} \right\}.
\]

This kind of formula needs to be understood in more general symplectic manifolds and will possibly lead to a way to regularize the ratio of Euler classes in general. We have made some progress in this direction and plan to report it in a subsequent paper.

Finally we should mention that another interesting point of view is that of Givental in [10]. There, he thinks of the integrals (39) as integrals over the Poincaré dual of \( \Delta. \) As has already been observed this is geometrically the cycle of all loops which are boundaries of holomorphic discs. Now we have also observed that the Cauchy-Riemann equation is the flow equation (18) for the action functional. The equation remains unchanged if we change \( H \) with \( -H \) and \( J \) with \( -J. \) Now the change of the sign of \( H, \) in the \( S^1 \) equivariant theory, is equivalent to changing the sign of \( \hbar. \) Moreover the change in the sign of \( J \) is equivalent to changing the symplectic form \( \omega \) to \( -\omega. \) Notice finally that Morse theory with \( -H \) switches accenting with descending cells and the intersection of accenting and descending cells gives the spaces of flow lines connecting fixed manifolds. In our case these are spaces of holomorphic spheres. This kind of reasoning can be beautifully incarnated in concrete formulas. For example in the case of \( \mathbb{P}^n, \) Givental [10] proves that:

\[
\sum_{d=0}^{\infty} e^{d\tau} \int_{M_d} e^{(t-\tau)(H+\omega/\hbar)} = \hbar^{1-n} (F(t, \hbar), \sigma F(\tau, -\hbar)),
\]

where \( \langle, \rangle \) denotes the intersection pairing in the cohomology ring \( \mathbb{C}[\mathbb{P}]/(\mathbb{P}^n) \) of \( \mathbb{P}^n \) and \( \sigma \) is the automorphism \( \sigma(P) = -P. \) Here \( M_d \) is the projective space of degree \( d \) polynomial maps of \( \mathbb{P}^1 \) to \( \mathbb{P}^n, \) \( \omega \) is the standard Fubini Kähler form on \( M_d \) and...
$H$ is the Hamiltonian corresponding to the $S^1$ action induced on $M_d$ by rotation of $P^1$. Similar theorems exist for the function $G$ when the space of stable maps is used (see Givental [11], [12]).

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