The Information Projection in Moment Inequality Models: Existence, Dual Representation, and Approximation

Rami V. Tabri
School of Economics, University of Sydney, Sydney, New South Wales, Australia 2006,
Email: rami.tabri@sydney.edu.au.

November 16, 2022

Abstract

This paper presents new existence, dual representation, and approximation results for the information projection in the infinite-dimensional setting for moment inequality models. These results are established under a general specification of the moment inequality model, nesting both conditional and unconditional models, and allowing for an infinite number of such inequalities. An essential innovation of the paper is the exhibition of the dual variable as a weak vector-valued integral to formulate an approximation scheme of the $I$-projection’s equivalent Fenchel dual problem. In particular, it is shown under suitable assumptions that the dual problem’s optimum value can be approximated by the values of finite-dimensional programs and that, in addition, every accumulation point of a sequence of optimal solutions for the approximating programs is an optimal solution for the dual problem. This paper illustrates the verification of assumptions and the construction of the approximation scheme’s parameters for the cases of unconditional and conditional first-order stochastic dominance constraints and dominance conditions that characterize selectionable distributions for a random set. The paper also includes numerical experiments based on these examples that demonstrate the simplicity of the approximation scheme in practice and its straightforward implementation using off-the-shelf optimization methods.

MSC2020 subject classifications:Primary 62B10, 60B10, 65K10, 60E15; Secondary 46N10
Keywords and phrases:I-Projection, Moment Inequality Constraints, Approximation Scheme, Convergence.

1 Introduction

Given a probability distribution (PD) $Q$ and a set of PDs $\mathcal{M}$ defined on the same measurable space $(\Omega, \mathcal{F})$, a frequently encountered question that arises in statistics and econometrics is to calculate the $P \in \mathcal{M}$ “closest” to $Q$ that minimizes the Kullback-Leibler information criterion (Kullback and Leibler, 1951). This PD is known as the $I$-projection and is formally defined as the PD $P_Q \in \mathcal{M}$ that satisfies

$$I(P_Q||Q) = \min_{P \in \mathcal{M}} I(P||Q),$$

(1.1)
where

\[
I(P\|Q) = \begin{cases} 
\int_\Omega \frac{dP}{dQ} \log \left( \frac{dP}{dQ} \right) \, dQ & \text{if } P \ll Q, \\
+\infty & \text{if } P \not\ll Q.
\end{cases}
\]  

(1.2)

\(I\)-projections have played a significant role in the information theoretic approach to statistics (e.g., Kullback, 1959; Haberman, 1984; Sheehy, 1988; Kortanek, 1993; and Schennach, 2007). \(I\)-projections also arise in other areas such as the general theory of large deviations (Sanov, 1957 and Csiszar, 1984) and entropy maximization in statistical physics (e.g., Jaynes, 1957; Rao, 1973; Rüschoff and Thomsen, 1993; and Pavon et al., 2021). They also occur in econometrics in areas such as statistical testing and model selection (e.g., Vuong, 1989; Kitamura and Stutzer, 1997; Imbens et al., 1998; Sueishi, 2013; Shi, 2015; and Komunjer and Ragusa, 2016). \(I\)-projections have also been used to measure (dis)-similarity between objects that are modeled as distributions, in what is known as the “machine learning on distributions” framework (e.g., Dhillion et al., 2003; and Kandasamy et al., 2015). \(I\)-projections also arise in areas of Bayesian statistics. For example, they arise in robust Bayesian decision theory (e.g., Berger, 1985; Topsoe, 2002; and Grünwald and Dawid, 2004), and when imposing posterior constraints in Bayesian inference (e.g., Ganchev et al., 2007; Ganchev et al., 2010; and Zhu et al., 2014). Also, Granzio et al. (2019) have shown the equivalence of \(I\)-projection problems and constrained variational Bayesian inference; thus, linking \(I\)-projections to Bayesian approximation methods.

In practice, many applications call for imposing multiple constraints that take the form of infinitely many moment inequality restrictions with known moment functions. Salient examples of such restrictions include unconditional and conditional stochastic dominance orderings, unimodality, stochastic monotonicity, measures of association such as quadrant and regression dependence, shape restrictions on multivariate regression models such as convexity/concavity. Other important examples arise in partial identification of infinite-dimensional parameters of incomplete models with known moment functions, such as distributions under missing data, the distribution of the coefficients in random coefficient models without point-identifying assumptions, and the set of selectable distributions in random set theory. This paper presents a unified treatment for the existence and uniqueness, dual representation, and approximation of the \(I\)-projection in the infinite-dimensional setting that covers a wide range of constraint sets defined by infinitely many unconditional moment inequality restrictions. Our treatment covers conditional moment inequality restrictions as well, as it exploits their equivalent characterization in terms of unconditional ones using a continuum or countably infinite number of instrument functions — a result put forward by Andrews and Shi (2013).

In this paper, we establish the existence and uniqueness of the \(I\)-projection when \(M\) is defined by an arbitrary number of unconditional moment inequalities whose moment functions satisfy a mild tail condition: the existence of \(1 + \delta\) moments for some \(\delta > 0\) for elements of \(M\) (Theorem 1 below). Constrained families of densities of this sort arise naturally in statistics; see, for example, Section 6 of Bickel et al. (1993) for a myriad of examples of such families. Tail conditions of this type are also ubiquitous in the inference literature on parameters defined by moment inequality restrictions (e.g., Andrews and Shi, 2013; and Linton et al., 2010), as they enable the use of triangular array Central Limit Theorems for establishing large-sample distributional results of test statistics that are valid with uniformity. The importance of this...
tail condition in the context of this paper is that it helps establish that $M$ is variation-closed, which is a key ingredient that yields the existence and uniqueness of the $I$-projection using Theorem 2.1 of Csiszár (1975). This result of Csiszár (1975) is quite useful because we circumvent Weierstrass’ Theorem (Luenberger, 1969, Section 2.13) in establishing the $I$-projection’s existence, as imposing compactness of $M$ in the total variation norm is a severe restriction in the infinite-dimensional setting. See, for example, Theorem 18 in Section IV.8.17 of Dunford and Schwartz (1958) for a statement of the Kolmogorov-Riesz Compactness Theorem for characterizing totally bounded subsets of Lebesgue spaces.

The $I$-projection’s dual representation is developed under mild conditions on the reference PD $Q$ using the Fenchel duality results of Bhatacharya and Dykstra (1995). In particular, we impose three conditions that beget the existence and uniqueness of the dual problem’s solution, and the exponential family representation of the $I$-projection’s $Q$-density (Theorem 2 below). The first condition is that there exists at least one moment function whose Laplace transform under the PD $Q$ is finite at some point of its domain. This condition implies the dual problem’s feasible set is non-empty. The second condition is the same tail condition imposed on the moment functions that define the constraint set, but with respect to the PD $Q$. The third condition imposes non-void intersection of $Q$’s support with the subset of $\Omega$ where the moment functions satisfy the inequality constraints with strict inequality. Together, the second and third conditions imply the lower semi-continuity and coercivity of the dual problem’s objective function, with respect to the $L_1(Q)$-norm, where $L_1(Q)$ denotes the Lebesgue space of real-valued measurable functions defined on the probability space $(\Omega, \mathcal{F}, Q)$ that are absolutely integrable. We also prove that under the precompactness of the set of moment functions in the $L_1(Q)$-norm, the dual problem’s solution belongs to the $L_1(Q)$-closure of the positive linear span of the moment functions for which the moment inequalities are binding. This precompactness condition on the set of moment functions is a mild restriction that is widely used in statistics, and particularly in the development of empirical process theory. For example, Glivenko-Cantelli and Donsker classes of moment functions satisfy precompactness conditions in terms of their metric entropy covering and/or bracketing numbers (e.g., see Theorems 2.4.1 and 2.5.2, in van der Vaart and Wellner, 1996), which end up either being equivalent to or imply precompactness of the class in the $L_1(Q)$-norm.

This paper develops an approximation scheme for the $I$-projection in moment inequality models defined by infinitely many such restrictions, under the same conditions that yield our existence and dual representation results. The class of $I$-projection problems this paper considers are infinite programs, which are difficult (if perhaps not impossible) to solve in closed-form. Consequently, one is forced to use some kind of approximation scheme in order to obtain the $I$-projection. Building upon the dual representation result of this paper, we utilize the $L_1(Q)$-precompactness of the set of moment functions to exhibit the dual variable as a Gelfand-Pettis integral (Gelfand, 1936, and Pettis, 1938): a weak vector-valued integral with respect to a positive Radon measure defined on the $L_1(Q)$-closed convex hull of the set of moment functions. This characterization of the dual variable is key in our setup because the weak integral can be approximated by “Riemann sums” in the $L_1(Q)$-norm, which forms the basis of the paper’s proposed approximation scheme. The weak integral representation of the dual variable introduces a reparametrization of the $I$-projection’s dual problem in terms of these Radon measures, which we equip with their weak-star topology. The use of this topology is advantageous because we show the objective function in the reparametrized dual problem is
weak-star lower semi-continuity and coercive, yielding the existence of a solution, which may not be unique (Theorem 2 below). However, we prove that the dual problem’s optimum value can be approximated by the values of finite-dimensional programs using the “Riemann sums” approximation, and that, in addition, every weak-star accumulation point of a sequence of optimal solutions for the approximating programs is an optimal solution for the dual problem (Theorem 4 below). Furthermore, the proposed approximation scheme is straightforward to implement numerically using off-the-shelf simulation-based methods, which is an appealing feature for practice.

Leading approximation schemes for $I$-projections focus on scenarios where $\mathcal{M}$ can be expressed as $\bigcap_{t=1}^{T} \mathcal{M}_t$ with each $\mathcal{M}_t$ satisfying various properties, and $T$ is a positive integer. In the discrete case (i.e., $\Omega$ is finite), Csiszar (1975) has shown the sequence of cyclic iterated $I$-projections onto individual $\mathcal{M}_t$ that are variation-closed and linear, converges to the $I$-projection with constraint set $\mathcal{M}$. Dykstra (1985) modified this procedure to allow for the $\mathcal{M}_t$ sets to be arbitrary variation-closed convex sets, subject to a limiting condition, which Winkler (1990) shows always to hold. Bhattacharya and Dykstra (1997) give a Fenchel duality interpretation of Dykstra’s algorithm, offering a more intuitive viewing angle of his algorithm. Only more recently, an approximation scheme covering the infinite-dimensional case has been put forward by Bhattacharya (2006), which is a modified version of the discrete case Dykstra (1985) and Bhattacharya and Dykstra (1997), where the $\mathcal{M}_t$ sets are arbitrary variation-closed and convex. An essential ingredient for implementing Bhattacharya’s algorithm is that closed-form expressions for the $I$-projections onto the individual $\mathcal{M}_t$ sets can be obtained at each iteration.

The constraint sets we consider have this finite intersection representation, but our general setup may not yield closed-form expressions for the $I$-projections onto the individual $\mathcal{M}_t$ sets. Consequently, it may be infeasible to implement Bhattacharya (2006)’s algorithm without imposing further restrictions on our $I$-projection problem. Another motivation for our approximation scheme comes from this point, where the additional restrictions for executing Bhattacharya’s algorithm in our framework may be severe in practice, and hence, would reduce the scope of applications. We elaborate on this point in the paper using examples of marginal stochastic order considered by Bhattacharya (2006), who obtains closed-form expressions for the $I$-projection by imposing square-integrability restrictions on the $I$-projection problem. Contrastingly, our approximation scheme does not require this additional structure to be feasible in practice; therefore, it is more widely applicable for moment inequality models.

The paper illustrates the verification of assumptions the construction of the approximation scheme’s parameters using unconditional and conditional restricted stochastic dominance constraints. Furthermore, we implement the proposed approximation scheme numerically using examples of these constraints. Stochastic dominance constraints arise in many areas of the social, management, biological, and physical sciences. For example, they define robust rankings of income distributions in terms of poverty orderings (Foster and Shorrocks, 1988), and have been used in statistical testing on the basis of samples from income distributions (e.g., Barrett and Donald, 2002, and Davidson and Duclos, 2013). In management sciences, such constraints have been used for characterizing decision-making under uncertainty (Hadar and Russell, 1969), for example, in devising investment strategies (Linton et al., 2010), and water-conserving irrigation strategies (Harris and Mapp, 1986). In the biological sciences, such constraints have been used for rank-
ing disease control strategies in epidemiology (Verteramo Chiu et al., 2020). In the physical sciences, they have been used to define the optimal partial transport problem (Figalli, 2010), and to derive Schrödinger inequalities from an $I$-projection problem under $L_2$-type restrictions (Bhattacharya, 2006). We also present an example of the $I$-projection in the context of random sets based on the dominance conditions that characterize selectable distributions described in Artstein (1983). We demonstrate the simplicity of the approximation scheme and its straightforward implementation using off-the-shelf optimization methods with numerical experiments based on the aforementioned examples.

This paper is organised as follows. Section 2 introduces the setup and preliminary results that are used in the derivation of the main results. Section 3 presents the main results of the paper, and Section 4 presents a discussion of the scope of our main results and implications for practice. Section 5 illustrates the theory using unconditional and conditional first-order stochastic dominance constraints, and Artstein inequalities for selectable distributions. Section 6 concludes. All proofs are relegated to the Appendix for ease of exposition.

## 2 Setup and Preliminaries

We first introduce some notation. Given a PD $Q$ supported on a set $\Omega$, an $I$-sphere with center $Q$ and radius $\rho$ is the set of PDs $S(Q, \rho) = \{P : I(P||Q) < \rho\}$ for $0 < \rho \leq +\infty$. We shall work with the topology of variation distance on the set of PDs. In the setup of this paper, it is given by $|P - W| = \int_{\Omega} |p - w| dQ$, where $P$ and $W$ are absolutely continuous with respect to the PD $Q$ with corresponding $Q$-densities $p$ and $w$.

Next we recall some useful facts from functional analysis that facilitate the paper’s exposition. These facts can be found in textbooks on the subject, such as Rudin (1991), Folland (1999), and Bogachev (2007). Let $B(\mathbb{R})$ denote the Borel sigma-algebra on $\mathbb{R}$, and recall that $\mathcal{F}$ is a sigma-algebra on $\Omega$. We define for $r \geq 1$ the Lebesgue space

$$L_r(Q) = \left\{ h : \Omega \to \mathbb{R} : h \text{ is measurable } \mathcal{F}/B(\mathbb{R}) \text{ and } \left( \int_{\Omega} |h|^r dQ \right)^{\frac{1}{r}} < +\infty \right\}.$$  

These Lebesgue spaces are Banach spaces when they are equipped with their respective norms $\|h\|_{L_r(Q)} = \left( \int_{\Omega} |h|^r dQ \right)^{\frac{1}{r}}$. The results of this paper focus on the space $L_1(Q)$. Its topological dual $L_1(Q)^*$ is the vector space whose elements are the continuous linear functionals on $L_1(Q)$, plays a central role in the definition of the Gelfand-Pettis vector-valued integral. This weak integral is defined on a compact subset of $L_1(Q)$, which we denote by $X$ in this section. Suppose that $\xi$ is a positive Radon measure on the measure space $(X, B(X))$, where $B(X)$ is the Borel sigma-algebra of $X$.

**Definition 1.** Suppose that the scalar functions defined on $X$ given by $x \mapsto \Lambda(x)$ are integrable with respect to $\xi$, for every $\Lambda \in L_1(Q)^*$. If there exists a vector $y \in X$ such that

$$\Lambda(y) = \int_X \Lambda(x) d\xi(x) \quad \forall \Lambda \in L_1(Q)^*,$$
then we define \( \int_X x \, d\xi(x) = y \) 

As \( L_1(Q)^* \) separates points on \( L_1(Q) \), there is at most one such \( y \) that satisfies Definition [1]. Thus, there is no uniqueness problem. The existence of this weak integral follows from an application of Theorem 3.27 of [Rudin 1991], because of the compactness of \( X \) in the \( L_1(Q) \)-norm. Denote by \( C(X) \) the Banach space of all real continuous functions on \( X \). The Riesz Representation Theorem identifies the topological dual space \( C(X)^* \) with the space of all real Radon measures on \( X \). The subset of \( C(X)^* \) corresponding to positive measures serves as the space that we use to model the Radon measure \( \xi \) in the definition of the Gelfand-Pettis integral.

Because \( (\Omega, \mathcal{F}, Q) \) is a probability space, \( L_1(Q)^* \) is isometrically isomorphic to the Lebesgue space \( L_\infty(Q) \) through the map \( \Lambda_h : X \rightarrow \int_\Omega x \, h \, dQ \). The Lebesgue space \( L_\infty(Q) \) is also a Banach space when it is equipped with the norm \( \| \cdot \|_{L_\infty(Q)} \).

Throughout the paper, we use the following convergence notations. For \( \{y_n\}_{n \geq 1} \subset L_1(Q) \) the notation, \( y_n \overset{L_1(Q)}{\rightarrow} y \), \( y_n \overset{Q}{\rightarrow} y \) and \( y_n \overset{w}{\rightarrow} y \), denote convergence of the sequence \( \{y_n\}_{n \geq 1} \) to \( y \) in the norm \( \| \cdot \|_{L_1(Q)} \), the topology of convergence in \( Q \) measure, and in the weak topology of \( L_1(Q) \), respectively. Additionally, for \( \{\xi_n\}_{n \geq 1} \subset C(X)^* \), the notation \( \xi_n \overset{w^*}{\rightarrow} \xi \) means the sequence \( \{\xi_n\}_{n \geq 1} \) converges to \( \xi \) in the weak-star topology of \( C(X)^* \). For any subset \( X \) of a vector space, we denote the set of extreme points of \( X \) by \( \text{ex}(X) \).

Finally, we use the symbol \( \Delta \) to denote the end of an example.

### 3 Main Results

Following [Bhatacharya and Dykstra 1995], the objective function \( I(P||Q) \) can be reformulated as

\[
m(p) = \begin{cases} 
\int_\Omega p \log(p) \, dQ & \text{if } p \geq 0, \quad \int_\Omega p \, dQ = 1 \\
+\infty & \text{elsewhere},
\end{cases}
\]

which implicitly imposes the constraints that \( p \) has to be a probability density function. In (3.1) we have followed the convention of defining \( 0 \cdot \log 0 = 0 \). We shall describe \( I \)-projections with the objective function (3.1) onto a set of PDs defined by moment inequality restrictions. Given \( K, \delta > 0 \), and a set of moment functions \( \{f_\gamma : \gamma \in \Gamma \} \), where \( f_\gamma \) is real-valued and measurable \( \mathcal{F}/\mathcal{B}(\mathbb{R}) \) for each \( \gamma \in \Gamma \), the constraint set is

\[
\mathcal{M} = \left\{ p \in L_1(Q) : m(p) < +\infty, \int_\Omega |f_\gamma|^{1+\delta} p \, dQ \leq K \quad \text{and} \quad \int_\Omega f_\gamma p \, dQ \leq 0 \ \forall \gamma \in \Gamma \right\}.
\]
The condition $m(p) < +\infty$ imposes the restrictions that set $p \in \mathcal{M}$ to be a density function. The condition $\int_\Omega f_{\gamma} p \, dQ \leq 0$ for each $\gamma \in \Gamma$, imposes the moment inequality constraints of interest, and
\[
\int_\Omega |f_{\gamma}|^{1+\delta} p \, dQ \leq K \quad \forall \gamma \in \Gamma
\] (3.3)
is the tail condition on the moment functions with respect to elements in $\mathcal{M}$. The formulation of the constraint set $\mathcal{M}$ in (3.2) also includes an arbitrary number of equality constraints as well. As stated, this can be seen by considering constraints of the form $\int_\Omega f_{\gamma} p \, dQ \leq 0$ and $\int_\Omega -f_{\gamma} p \, dQ \leq 0$, where $-f_{\gamma} = f_{\gamma'}$ for some $\gamma' \neq \gamma$.

The following result establishes the existence and uniqueness of the $I$-projection onto the set $\mathcal{M}$.

**Theorem 1.** Let the objective function be given by $m(\cdot)$ in (3.1) and the constraint set $\mathcal{M}$ be given by (3.2). Furthermore, suppose that $S(Q, +\infty) \cap \mathcal{M} \neq \emptyset$. Then the $I$-projection of $Q$ onto $\mathcal{M}$, denoted by $P_Q$, exists and is unique.

**Proof.** See Appendix B.1.

The idea of the proof is to show that the conditions of this theorem enable the application of Theorem 2.1 in Csiszár (1975) to deduce the existence and uniqueness of $P_Q$. In that direction, the technical challenge is to show that $\mathcal{M}$ is variation-closed. Lemma C.1 establishes this result in Appendix C using the tail condition (3.3).

Next, we use the Fenchel duality approach of Bhatacharya and Dykstra (1995) to develop a dual optimization problem that is equivalent to the stated $I$-projection problem. The solution of the dual optimization problem, when it exists, is helpful for deriving a representation of the $Q$-density of $P_Q$. We develop conditions for the existence and uniqueness of the dual problem, and its characterization in terms of the moment functions.

Since the primal space is $L_1(Q)$, the standard dual space is $L_\infty(Q)$. Bhatacharya and Dykstra (1995) argue that $L_\infty(Q)$ can be too restrictive as a dual space since $\{f_{\gamma} : \gamma \in \Gamma\} \not\subset L_\infty(Q)$ can hold in practice, resulting in a failure of the standard duality approach. Of course, the standard duality approach would be successful if $\{f_{\gamma} : \gamma \in \Gamma\} \subset L_\infty(Q)$ holds; however, this structure on the moment functions can be restrictive in practice. Thus, to circumvent this restrictiveness of the standard approach, they propose $L_0(Q)$ – the vector space of measurable extended-valued functions on the probability space $(\Omega, F, Q)$ – as the general dual space in their approach. This vector space is a Banach space when it’s combined with the topology of convergence in $Q$ measure. The positive conjugate cone of $\mathcal{M}$, using $L_0(Q)$ as the dual space, is thus defined as
\[
\mathcal{M}^\oplus = \left\{ y \in L_0(Q) : \int_\Omega y \, p \, dQ \geq 0 \forall p \in \mathcal{M} \right\}.
\]

In light of the form of $\mathcal{M}^\oplus$, we consider the dual optimization problem on the following domain
\[
\mathcal{D} = \left\{ y \in \mathcal{M}^\oplus : y \in \overline{co}(\mathcal{V}) - \alpha, \; \alpha \geq 0 \right\},
\] (3.4)

\footnote{See Example 4.1(a) of Bhatacharya and Dykstra (1995) for a simple illustration of this point.}
where $\mathcal{V}$ is the $L_1(Q)$-norm closure of $\mathcal{V}$, $\text{co} (\mathcal{V})$ is the closed convex hull of $\mathcal{V}$ in the $L_1(Q)$-norm, and

$$
\mathcal{V} = \{-f_\gamma, K - |f_\gamma|^{1+\delta} : \gamma \in \Gamma\}.
$$

(3.5)

In particular, the dual optimization problem is given by

$$
\inf \left\{ \int_\Omega e^y \, dQ ; \ y \in D \right\}.
$$

(3.6)

The following assumptions on $Q$ and the set of moment functions $\mathcal{V}$ are helpful for developing properties of the problem (3.6):

**Assumption 1.**

(i) there exists $\gamma \in \Gamma$ and $\alpha > 0$ such that $\int_\Omega e^{-\alpha f_\gamma} \, dQ < +\infty$, (ii) $\sup_{\gamma \in \Gamma} \int_\Omega |f_\gamma|^{1+\delta} \, dQ < \infty$, and (iii) $Q (\omega \in \Omega : y(\omega) > 0 \ \forall y \in \mathcal{V}) > 0$.

and

**Assumption 2.** $\mathcal{V}$ in (3.5) is a precompact subset of $L_1(Q)$ in the norm topology.

Part (i) of Assumption 1 ensures that $\{y \in D : \int_\Omega e^y \, dQ < +\infty\} \neq \emptyset$. The integral $\int_\Omega e^{-\alpha f_\gamma} \, dQ$ is the Laplace transform of the random variable $y = f_\gamma$ evaluated at the point $\alpha$. Therefore, Part (i) of Assumption 1 imposes the existence of the Laplace transform for at least one element of $\{f_\gamma : \gamma \in \Gamma\}$ at some point in its domain. Part (ii) of Assumption 1 implies $\{f_\gamma : \gamma \in \Gamma\}$ is a uniformly integrable subset of $L_1(Q)$, and Part (iii) implies the existence of $P \in \mathcal{M}$ such that the moment inequality constraints hold with strict inequality. Together, Parts (ii) and (iii) of Assumption 1 are sufficient conditions for the lower semi-continuity and coercivity of the objective function in (3.6) with respect to the $L_1(Q)$-norm on the domain $D$, which are the ingredients for establishing existence: $\arg\inf \left\{ \int_\Omega e^y \, dQ : y \in D \right\} \neq \emptyset$. As $D$ is a convex set and the objective function is strictly convex on $D$ (because of the strict convexity of the exponential function), the dual problem has a unique minimizer. Combining this result with Assumption 2 establishes that this unique solution depends on the set of binding moments

$$
B = \left\{ v \in \mathcal{V} : \int_\Omega v \, dP_Q = 0 \right\}.
$$

(3.7)

The following result formalizes these points.

**Theorem 2.** Let the constraint set $\mathcal{M}$ be given by (3.2), and let the sets $\mathcal{V}$ and $B$ be given by (3.5) and (3.7), respectively. The following statements hold.

1. If Assumption 1 holds, then $\arg\inf \left\{ \int_\Omega e^y \, dQ : y \in D \right\} \neq \emptyset$ and is unique (up to equivalence class). Furthermore, the $Q$-density of $P_Q$ has the following representation

$$
p_Q = \frac{e^{y_0}}{\int_\Omega e^{y_0} \, dQ} \quad \text{where} \quad y_0 \equiv \arg\inf \left\{ \int_\Omega e^y \, dQ : y \in D \right\}.
$$

(3.8)

2. Suppose that $Q \notin \mathcal{M}$. If Assumptions 1 and 2 hold, then $y_0 \in \text{span}_+(B)$, where $\text{span}_+(B)$ is the $L_1(Q)$-norm closure of the positive linear span of $B$. 

8
The first result of Theorem 2 shows the I-projection is a member of the exponential family of distributions under Assumption 1. Furthermore, the representation yields the duality relationship

$$m(pQ) = -\log \left( \int_{\Omega} e^{y_0} dQ \right).$$

(3.9)

The result of Part 2 of Theorem 2 indicates that the unique minimizer of the dual problem, $y_0$, is either a finite linear combination of the moment functions in $B$ with positive coefficients, or is a limit point of such combinations in the $L_1(Q)$-norm. Assumption 2 is critical to obtaining this result, because the norm and weak closures of $V$ in $L_1(Q)$ are not necessarily equal. Note that by the Dunford-Pettis Theorem (e.g., Bogachev, 2007, Theorem 4.7.18), Part (ii) of Assumption 1 is equivalent to $\{f_\gamma : \gamma \in \Gamma\}$ having compact closure in $L_1(Q)$ with the weak topology. This property of $\{f_\gamma : \gamma \in \Gamma\}$ does not deliver the result of Part 2 of Theorem 2 unless the norm and weak closures of $V$ in $L_1(Q)$ coincide, which does not hold in general.

The computability of $y_0$ in (3.8) is key to adopting the proposed duality approach in practice. Towards that end, we develop a reparametrization of the dual problem (3.6) that enables the computation of $y_0$ using a simple approximation scheme. The reparametrization is derived from a representation of the elements of $D$ in terms of the integrator in a weak vector-valued integral. Under Assumption 2, the set $\overline{V}$ is compact in the $L_1(Q)$-norm. Denote by $\mathcal{P} \subset C(\overline{V})^*$ the set of Radon probability measures on the set $\overline{V}$. An application of Theorem 3.28 in Rudin (1991) yields

$$y \in \overline{\text{co}}(\overline{V}) \iff \exists \mu \in \mathcal{P} \text{ such that } y = \int_{\overline{V}} v d\mu(v),$$

(3.10)

where the integral in (3.10) is to be understood as in Definition 1. The characterization (3.10) is the building block of the reparametrization, as

$$y \in D \iff \exists \mu \in \mathcal{P} \text{ and } \alpha \geq 0 \text{ such that } y = \alpha \int_{\overline{V}} v d\mu(v).$$

(3.11)

Next, define $\Xi \subset C(\overline{V})^*$ as the set of all positive Radon measures on $\overline{V}$, and consider the following set

$$\Upsilon = \{\xi \in \Xi : \xi = \alpha \cdot \mu, \alpha \geq 0 \text{ and } \mu \in \mathcal{P}\}.$$

(3.12)

The dual problem (3.6) can now be reparametrized as

$$\inf \left\{ \int_{\Omega} e^{\xi(v)} v dQ : \xi \in \Upsilon \right\}.$$

(3.13)

In the above framework, the following theorem establishes the existence of a solution to (3.13).

**Theorem 3.** Let $\Upsilon$ be given by (3.12), and let $y_0$ be as in (3.8). Furthermore, suppose that Assumptions 1

Proof. See Appendix B.2. ■

The computability of $y_0$ in (3.8) is key to adopting the proposed duality approach in practice. Towards that end, we develop a reparametrization of the dual problem (3.6) that enables the computation of $y_0$ using a simple approximation scheme. The reparametrization is derived from a representation of the elements of $D$ in terms of the integrator in a weak vector-valued integral. Under Assumption 2, the set $\overline{V}$ is compact in the $L_1(Q)$-norm. Denote by $\mathcal{P} \subset C(\overline{V})^*$ the set of Radon probability measures on the set $\overline{V}$. An application of Theorem 3.28 in Rudin (1991) yields

$$y \in \overline{\text{co}}(\overline{V}) \iff \exists \mu \in \mathcal{P} \text{ such that } y = \int_{\overline{V}} v d\mu(v),$$

(3.10)

where the integral in (3.10) is to be understood as in Definition 1. The characterization (3.10) is the building block of the reparametrization, as

$$y \in D \iff \exists \mu \in \mathcal{P} \text{ and } \alpha \geq 0 \text{ such that } y = \alpha \int_{\overline{V}} v d\mu(v).$$

(3.11)

Next, define $\Xi \subset C(\overline{V})^*$ as the set of all positive Radon measures on $\overline{V}$, and consider the following set

$$\Upsilon = \{\xi \in \Xi : \xi = \alpha \cdot \mu, \alpha \geq 0 \text{ and } \mu \in \mathcal{P}\}.$$

(3.12)

The dual problem (3.6) can now be reparametrized as

$$\inf \left\{ \int_{\Omega} e^{\xi(v)} v dQ : \xi \in \Upsilon \right\}.$$

(3.13)

In the above framework, the following theorem establishes the existence of a solution to (3.13).

**Theorem 3.** Let $\Upsilon$ be given by (3.12), and let $y_0$ be as in (3.8). Furthermore, suppose that Assumptions 1

Proof. See Appendix B.2. ■
and hold. Then \( \arg \inf \left\{ \int_{\Omega} e^{f_{\nu} v} dQ : \xi \in \Upsilon \right\} \neq \emptyset , \) and

\[
\arg \inf \left\{ \int_{\Omega} e^{f_{\nu} v} dQ : \xi \in \Upsilon \right\} = \left\{ \xi \in \Upsilon : y_0 = \int_{\Upsilon} v d\xi(v) \right\}.
\]

**Proof.** See Appendix B.3. \( \blacksquare \)

Like the proof of Part 1 of Theorem 2, we prove the existence result of Theorem 3 by establishing the lower semi-continuity and coercivity of the objective function

\[
\xi \mapsto \int_{\Omega} e^{f_{\nu} v} dQ
\]

on the domain \( \Upsilon \), but with respect to the weak-star topology of \( C(V)^* \). Utilizing this topology on \( C(V)^* \) is advantageous because the Gelfand-Pettis integral \( \int_{\Upsilon} v d\xi(v) \), when identified as a mapping \( \phi : C(V)^* \to L_1(Q) \), is continuous when \( C(V)^* \) and \( L_1(Q) \) are given their weak-star and weak topologies, respectively. The continuity of this mapping with those topologies facilitates the proof of lower semi-continuity of the objective function.

The result of Theorem 3 is that the reparametrized dual problem (3.13) has a solution under the conditions of Theorem 2. Furthermore, the solution set of this optimization problem is characterized by \( y_0 \) — the solution of the Fenchel dual problem (3.6), via the representation (3.11). It is important to note that unlike the dual problem (3.6), the solution of the reparametrized dual problem (3.13) may not be unique. This non-uniqueness arises from the non-uniqueness of the representation of \( y \in co(P) \) in (3.10), since it can be represented as the moment of different elements of \( P \). It is this feature of the representation that renders the objective function in (3.13) as not necessarily being strictly convex on \( \Upsilon \).

The advantage of the proposed reparametrization of the dual optimization problem is that it enables the computation of its optimal solutions using a simple approximation scheme. Since \( L_1(Q) \) is a Banach space, and the identity map from \( \overline{V} \) into \( L_1(Q) \) is continuous with respect to the \( L_1(Q) \)-norm on both spaces, Lemma C.5 in the Appendix shows that the Gelfand-Pettis integral \( \int_{\overline{V}} h d\mu(h) \) can be approximated as the limit of “Riemann sums” in the \( L_1(Q) \)-norm. In particular, given \( \epsilon > 0 \), let \( U = \{ y \in L_1(Q) : \|y\|_{L_1(Q)} \leq \epsilon \} \). Then, there corresponds a finite partition \( \{ E_i \}_{i=1}^n \) of \( \overline{V} \) such that

\[
\int_{\overline{V}} v d\mu(v) - \sum_{i=1}^n \mu(E_i)v_i \in U \quad \forall v_i \in E_i, \ i = 1 \ldots , n.
\]

Remarkably, the only structure on the partitions that is required for (3.15) to hold is that for each \( i : v - v' \in U \) for all \( v, v' \in E_i \). Consequently, the partitions depend only on \( \epsilon, Q \) and \( V \), and not \( \mu \). This point is key to the approximation scheme put forward by this paper.

The sequence of discretizations of the infinite program (3.13) we consider is indexed by \( \{ \epsilon_m \}_{m \geq 1} \subset \mathbb{R}_{++} \) such that \( \epsilon_m \downarrow 0 \). For each \( m \), let \( U_m = \{ y \in L_1(Q) : \|y\|_{L_1(Q)} \leq \epsilon_m \} \). Then by Lemma C.5 in
Appendix C there corresponds a finite partition \( \{ E_{i,m} \}_{i=1}^{n_m} \) of \( V \) with the property

\[
\int_V v \, d\mu(v) - \sum_{i=1}^n \mu(E_{i,m}) v_i \in U_m \quad \forall v_i \in E_{i,m}, \; i = 1 \ldots, n_m.
\]

The infinite program (3.13) can be approximated by the following sequence of finite programs as \( m \to +\infty \):

\[
\inf \left\{ \int \Omega e^{\alpha \sum_{i=1}^{n_m} \mu_i v_i} \, dQ; \; \alpha, \mu_i \geq 0 \quad \forall i, \quad \text{and} \quad \sum_{i=1}^{n_m} \mu_i = 1 \right\}, \quad (3.16)
\]

where \( v_i \in E_{i,m} \) are given for each \( i \). The following theorem formalizes this point.

**Theorem 4.** Suppose the conditions of Theorem 3 hold, and let \( \xi_0 \) be a solution of (3.13). Let \( \{ \epsilon_m, U_m \}_{m \geq 1} \) be described as above. For each \( m \), let \( \{ E_{i,m} \}_{i=1}^{n_m} \) be a partition of \( V \) such that, for each \( i \), \( v - v' \in U_m \) holds for all \( v, v' \in E_{i,m} \). Then, for each \( m \), \( \arg \inf \left\{ \int \Omega e^{\alpha \sum_{i=1}^{n_m} \mu_i v_i} \, dQ; \; \alpha, \mu_i \geq 0 \quad \forall i, \quad \text{and} \quad \sum_{i=1}^{n_m} \mu_i = 1 \right\} \neq \emptyset \), (3.17)

for any \( v_i \in E_{i,m} \) where \( i = 1, \ldots, n_m \). Furthermore, let \( \xi_m = \alpha n_m \mu_{n_m} \) be a corresponding solution of (3.16) for given \( v_i \in E_{i,m} \) where \( i = 1, \ldots, n_m \). Then the following statements hold.

1. \( \lim_{m \to +\infty} \int \Omega e^{\alpha n_m \sum_{i=1}^{n_m} \mu_{i,n_m} v_i} \, dQ = \int \Omega e^{\int \Omega v \, d\xi_0} \, dQ, \) and

2. Every accumulation point of \( \{ \xi_m \}_{m \geq 1} \), in the weak-star topology of \( C(V)^* \), is a solution of (3.13).

**Proof.** See Appendix B.4.

The first result of Theorem 4 is that the dual problem’s optimum value can be approximated by the values of the finite-dimensional programs (3.16). The second result of this theorem is that every weak-star accumulation point of a sequence of optimal solutions for the approximating programs (3.16) is an optimal solution for the dual problem. Our use of the weak-star topology of \( C(V)^* \) to characterize the approximation scheme’s limiting behavior comes from the result of Theorem 3, where we have used it to establish the weak-star lower semi-continuity and coercivity of the objective function (3.14) on \( \Upsilon \).

The most appealing feature of this approximation scheme is its simplicity, as one only needs to solve a finite program in practice, and such programs can be solved numerically using off-the-shelf computational routines. However, prior to implementing this finite program, the practitioner must decide on a level of accuracy for the Riemann sum approximation in (3.15); that is, a choice of \( \epsilon \) and a corresponding partition \( \{ E_i \}_{i=1}^n \) of \( V \). These are the ingredients/parameters for setting up the finite program, as the approximation (3.15) is uniform in the choice of \( v_i \in E_i \) for each \( i \). For each \( \epsilon > 0 \) and corresponding partition, there is thus more than one choice of \( v_i \in E_i \) for each \( i \) that delivers the accuracy (3.15). The result of Theorem 2 suggests selecting \( v_i \in \overline{B} \cap E_i \) for each \( i \). The catch is that we don’t know the set \( \overline{B} \). However,

\[
0 = \int v \, dP_Q = \frac{\int \Omega v e^{b_0} \, dQ}{\int \Omega e^{b_0} \, dQ} \geq \int \Omega v \, dQ \quad \forall v \in \overline{B},
\]
by Lemma C.6 in Appendix C implying that \( \mathcal{B} \subset \{ v \in \mathcal{V} : \int_{\Omega} v \, dQ \leq 0 \} \). Consequently, we can select \( v_i \in \{ v \in \mathcal{V} : \int_{\Omega} v \, dQ \leq 0 \} \cap E_i \) for each \( i \), provided these intersections are non-empty.

4 Discussion

This section presents a discussion of the scope of our main results and implications for practice.

4.1 Existence and Exponential Family Representation

To appreciate the scope of applications the result of Theorem 1 covers, note that there is no restriction on the moment functions \( \{ f_\gamma : \gamma \in \Gamma \} \) beyond the integrability conditions (3.3) and their measurability — i.e., \( f_\gamma \) is measurable \( \mathcal{F}/\mathcal{B}(\mathbb{R}) \) for each \( \gamma \in \Gamma \). Moment inequality models in the literature usually posit more restrictive tail conditions on \( Q \) and the model \( M \), given by

\[
\sup_{\gamma \in \Gamma} \int_{\Omega} |f_\gamma|^{2+\delta} \, dQ < +\infty \quad \text{and} \quad \sup_{\gamma \in \Gamma} \int_{\Omega} |f_\gamma|^{2+\delta} p \, dQ \leq K < +\infty, \quad \text{for given } K \text{ and } \delta > 0,
\]

respectively. Thus, the spectrum of applications includes all moment inequality models in the literature: those defined by a continuum, finite, and countably infinite number of such inequalities, without specifying any particular structure on the moment functions beyond their individual \( \mathcal{F}/\mathcal{B}(\mathbb{R}) \) measurability. This coverage also includes all conditional moment inequality models in the literature, since such inequalities can be equivalently characterized as a continuum or countably infinite number unconditional inequalities using instrument functions. Instrument functions are nonnegative functions whose supports are arbitrarily small cubes, boxes, or bounded sets with other shapes; see the definitions of \( G_{\text{box}} \) and \( G_{\text{cube}} \) in Section 3.3 of Andrews and Shi (2013) for examples of instrument sets. The crux of Theorem 1 is that the \( I \)-projection exists and is unique in this general setup.

The existence and uniqueness of the \( I \)-projection also holds for moment inequality models that do not impose the tail condition (3.3) on the moment functions, but under Assumption 1 with appropriate modifications to Parts (ii) and (iii) of the assumption. Specifically, this modified version of Assumption 1 is given by

**Assumption 3.** (i) there exists \( \gamma \in \Gamma \) and \( \alpha > 0 \) such that \( \int_{\Omega} e^{-\alpha f_\gamma} \, dQ < +\infty \), (ii) \( \sup_{\gamma \in \Gamma} \int_{\Omega} |f_\gamma| \, dQ < \infty \), and (iii) \( Q(\omega \in \Omega : f_\gamma(\omega) < 0 \quad \forall \gamma \in \Gamma) > 0 \).

The constraint set of densities would be

\[
\mathcal{M}' = \left\{ p \in L_1(Q) : m(p) < +\infty \quad \text{and} \quad \int_{\Omega} f_\gamma p \, dQ \leq 0 \quad \forall \gamma \in \Gamma \right\},
\]

and in this case we would specify \( \mathcal{V} = \{-f_\gamma : \gamma \in \Gamma \} \) in setting up the corresponding Fenchel dual problem. The proof of the \( I \)-projection’s existence follows steps identical to those in the proof of Part 1 of Theorem 1.
but with appropriate adjustments to utilize the conditions of Assumption 3 instead of those of Assumption 1. Note that the key step for existence in the proof is the use of Theorem 2.2 of Bhattacharya and Dykstra (1995), which also delivers the exponential family representation of the \( I \)-projection’s \( Q \)-density. The uniqueness of the \( I \)-projection is a consequence of the convexity of the constraint set \( \mathcal{M}' \), and the strict convexity of the function \( m(\cdot) \) in (3.1) on that constraint set.

Local approaches to solving \( I \)-projection problems have also been considered (e.g., Frigyik et al., 2008). However, such approaches require the differentiability of the objective function \( m(\cdot) \) in (3.1) on the constraint set \( \mathcal{M} \) in a meaningful sense, which may not hold without further restrictions on \( \mathcal{M} \). For example, the Gâteaux derivative of \( m(\cdot) \) at \( p \in \mathcal{M} \) in the direction \( h \in L_1(Q) \) is given by \( \int_{\Omega} h(\omega) \log p(\omega) \, dQ(\omega) \), which may not be finite \( \forall (h, p) \in L_1(Q) \times \mathcal{M} \). A sufficient condition for this type of differentiability is to restrict \( \mathcal{M} \) further by also imposing \( \| \log p \|_{L_\infty(Q)} < \infty \) as an additional constraint, because it implies the existence of \( \int_{\Omega} h(\omega) \log p(\omega) \, dQ(\omega) \) \( \forall (h, p) \in L_1(Q) \times \mathcal{M} \) via Hölder’s inequality. Consequently, this approach enables a local analysis of the primal problem using Lagrange multipliers. But it comes at the expense of restricting the shape of the \( I \)-projection — i.e., it must be an element of \( L_\infty(Q) \). This point is important because such a restriction can exclude applications where it does not hold. By contrast, the Fenchel dual approach and approximation scheme this paper proposes circumvents such restrictions.

### 4.1.1 Connection to Csiszár (1984)

Csiszár (1984) developed general results on the existence and exponential family representation of the generalized \( I \)-projection, where the probability distributions are defined on a locally convex topological vector space whose mean value is constrained to a convex set. By contrast, the results of Theorem 1 and Part 1 of Theorem 2 in this paper concern the \( I \)-projection. It should be noted that the generalized \( I \)-projection coincides with the \( I \)-projection when the latter exists; see Remark 1 of Csiszár and Matus (2003). Given the arbitrary nature of \( \Gamma \) and the moment functions \( \{ f_{\gamma} : \gamma \in \Gamma \} \) in the derivation of our results, it is instructive to compare and contrast our results on existence and exponential family representation with the results of Theorems 3 and 4 in his paper.

The general framework of Csiszár (1984) focuses on the generalized \( I \)-projection of a reference probability measure \( R \) that is defined on a locally convex topological vector space \( T \), where the constraint set imposes a convexity restriction on the expectation resultant of probability measures defined on \( T \). The expectation resultant of a probability measure \( W \) defined on \( T \) is defined as the following Gelfand-Pettis integral:

\[
E[W] = t_0 \quad \text{if} \quad \int_T \Psi(t) \, dW(t) = \Psi(t_0) \quad \forall \Psi \in T^* , \quad (4.2)
\]

if such a \( t_0 \) exists, where else \( E[W] \) is undefined. Here \( T^* \) is the topological dual of \( T \). The expectation resultant is a generalization of the notion of mean value of a measure on Euclidean spaces to topological vector spaces. The constraint set is \( \{ W : E[W] \in C \} \), where \( C \subset T \) is convex. To ensure existence of this weak integral so that his constraint set is well-defined, Csiszár assumes that the reference probability measure \( R \) is convex-tight. This property of \( R \) means that there is an increasing sequence of compact and
convex subsets of $T$, $K_1 \subset K_2 \cdots$, such that $R(K_n) \to 1$ as $n \to +\infty$. Its advantage is that it permits the use of the constraint set

$$\{W : E[W] \in C \quad \text{and} \quad W(K_n) = 1 \text{ for some } n\}$$

in analyzing of the $I$-projection problem, which ensures the existence of the weak integral $E[W]$ in (4.2). The convex-tight property of $R$ is also a crucial technical condition that facilitates the development of existence and exponential family representation results in Csiszár’s setup.

The $I$-projection problem of our paper can be formulated in terms of Csiszár’s general setup. We elaborate on this point using the $I$-projection problem with constraint set $\mathcal{M}'$ in (4.1). Our focus on this constraint set is without loss of generality, as similar arguments hold for the constraint set that includes the tail condition (3.3) with appropriate modifications, but leads to notational clutter.

In our framework, $T = \mathbb{R}^\Gamma$ is the vector space of all real-valued functions, however irregular, on the set $\Gamma$, which is an arbitrary set. This specification of $T$ is a consequence of the minimal assumptions on the set of moment functions $\{f_\gamma : \gamma \in \Gamma\}$ and their index set $\Gamma$, and arises from identifying the mapping $\Phi : \Omega \to \mathbb{R}^\Gamma$, where $\Phi(\omega)$ is defined as the function $\gamma \mapsto f_\gamma(\omega)$ on the domain $\Gamma$. Consequently, to formulate our framework within Csiszár’s general setup, one must introduce a suitable topology on $\mathbb{R}^\Gamma$ that yields

(i) $\mathbb{R}^\Gamma$ as a locally convex topological vector space;

(ii) the mapping $\Phi$ as measurable $\mathcal{F}/\mathcal{B}(\mathbb{R}^\Gamma)$, where $\mathcal{B}(\mathbb{R}^\Gamma)$ is the Borel sigma-algebra of $\mathbb{R}^\Gamma$; and

(iii) the $\Phi$-image of $Q$, which is the reference probability measure $R$ in Csiszár’s setup (i.e., $R = Q \circ \Phi^{-1}$), as convex-tight on $\mathbb{R}^\Gamma$.

With such a topology on $\mathbb{R}^\Gamma$, the constraint set consists of all probability measures $W = P \circ \Phi^{-1}$ defined on the measurable space $(\mathbb{R}^\Gamma, \mathcal{B}(\mathbb{R}^\Gamma))$, where $P$ is defined on the measurable space $(\Omega, \mathcal{F})$ and is such that $\frac{dp}{dq} \in \mathcal{M}'$.

The product topology on $\mathbb{R}^\Gamma$ fits the bill. This topology renders $\mathbb{R}^\Gamma$ as a locally convex topological vector space, which addresses point (i). Regarding (ii), if $\Omega$ has a topology defined on it and $\mathcal{F}$ is now defined as the Borel sigma-algebra with respect to this topology, then an application of Theorems 2.31 and 2.37 of Wayne Patty (1993) establishes that the map $\Phi$ is continuous, when $\mathbb{R}^\Gamma$ has the product topology. Then continuity of $\Phi$ implies that it must be measurable $\mathcal{F}/\mathcal{B}(\mathbb{R}^\Gamma)$. Concerning (iii), $\mathbb{R}^\Gamma$ with the product topology is complete, and since it is a locally convex topological vector space (i.e., (i)), it is also a Banach space. Consequently, the $\Phi$-image of $Q$ is necessarily convex-tight on $\mathbb{R}^\Gamma$ with this topology, since the map $\Phi$ also satisfies the measurability requirement (ii) in this case.

Therefore, an application of Theorems 3 and 4 of Csiszár (1984) in our setup implies that the generalized $I$-projection onto the constraint set $\mathcal{M}'$ exists and has the exponential family representation; however, it is not necessarily an element of $\mathcal{M}'$. By contrast, under Assumption 3 the $I$-projection onto $\mathcal{M}'$ also exists and has the exponential family representation, and it is an element of $\mathcal{M}'$. Furthermore, under this assumption, the generalized $I$-projection coincides with the $I$-projection.
The importance of the connection of our results to the aforementioned results of Csiszár (1984) is that we can apply them to our setup to deduce that the constraint set $\mathcal{M}'$ has the Sanov Property. Specifically, let $X_1, X_2, X_3, \ldots$ be a sequence of independent random variables with common PD $Q$, and suppose that Assumption 3 holds. Then the empirical distribution $\hat{Q}_n$ of $(X_1, X_2, \ldots, X_n)$ satisfies

$$
\lim_{n \to +\infty} n^{-1} \log \Pr(\hat{Q}_n \in \mathcal{M}') = -m(p_Q),
$$

where $m(\cdot)$ is given by (3.1) and $p_Q$ is the $I$-projection’s $Q$-density onto $\mathcal{M}'$. Additionally, his results imply that $(X_1, X_2, \ldots, X_n)$ are asymptotically quasi independent conditional on the event $\hat{Q}_n \in \mathcal{M}'$. While the above arguments are based on the constraint set $\mathcal{M}'$ under Assumption 3, the same conclusion holds for the constraint set $\mathcal{M}$ (i.e., with the tail condition) in place of $\mathcal{M}'$ but under Assumption 1. Large-sample results of this sort are relevant in for studying rare events through the theory of large deviations in areas such as statistical physics (e.g., Ellis, 1999), mathematical finance (e.g., Pham, 2010), and econometrics (e.g., Canay, 2010).

4.2 Approximation of $I$-Projections

This section compares the approximation scheme of our paper with the one put forward by Bhattacharya (2006) in the context of moment inequality constraints. Bhattacharya’s algorithm applies to $I$-projection problems whose constraint sets can be represented as the finite intersection of sets that are variation-closed and convex, provided that closed-form expressions of the $I$-projections onto the intersecting sets can be obtained. The constraint sets our paper considers have this finite intersection representation, however, without imposing further structure on the $I$-projection problem in our setup, obtaining these individual $I$-projections can be a very challenging (if not impossible). By contrast, the advantage of our approximation scheme is that it does not rely upon the use of such closed-form expressions to be applicable in practice.

We discuss two examples of marginal stochastic order that intersect the setup in Section 4 of Bhattacharya (2006). These examples illustrate how closed-form expressions of $I$-projections can be obtained by imposing a number of $L_2(Q)$ restrictions on the $I$-projection problem. While convenient and simple, such restrictions can be restrictive in practice. The first example elucidates this point where the practitioner can easily obtain the closed-form expression of the $I$-projection (i.e., approximation is not required). The second example builds upon the first example and requires Bhattacharya’s algorithm to approximate the $I$-projection under a set of $L_2(Q)$ restrictions.

The first example considers the marginal stochastic order constraint set with a given marginal distribution — see Example 1 below. As such a constraint set can be formulated as an isotonic cone restriction on $Q$-densities, results from Bhattacharya and Dykstra (1995) that connect infinite-dimensional $I$-projection problems with infinite-dimensional least-squares problems are useful for obtaining closed-form solutions of the former. However, this approach to solving the $I$-projection problem comes at the expense of imposing $L_2(Q)$ restrictions on the isotonic cone, the constraint set of $Q$-densities, and the $Q$-density of the given marginal distribution. This example is a moment inequality model, and we demonstrate the applicability of our approximation scheme without having to impose any of those $L_2(Q)$ restrictions.
Example 1. Let $\Omega = [0, 1]$, $Q$ be the continuous uniform distribution on $\Omega$, and let $G$ be a given cumulative distribution function defined on $\Omega$. Consider the constraint set

$$\mathcal{M} = \left\{ p \in L_1(Q) : m(p) < +\infty \text{ and } \int_0^\gamma p \, dQ \leq G(\gamma) \ \forall \gamma \in [0, \overline{\gamma}] \right\},$$  

(4.3)

where $0 < \underline{\gamma} < 1$ is given, so that $\Gamma = [0, \overline{\gamma}]$. Based on the discussion in Section 4.1, we can verify the conditions of Assumption 3 to deduce the result of Part 1 of Theorem 2. In this example, the moment functions are $f_\gamma(\omega) = 1[\omega \leq \gamma] - G(\gamma)$, and it is a straightforward task to show that

$$\int_{\Omega} e^{-\alpha f_\gamma} \, dQ \leq e^{2\alpha} < +\infty \ \forall \alpha \geq 0 \text{ and } \forall \gamma \in \Gamma, \quad \sup_{\gamma \in \Gamma} \int_{\Omega} |f_\gamma| \, dQ \leq 2, \quad \text{and} \quad Q(\omega \in \Omega : f_\gamma(\omega) < 0 \ \forall \gamma \in \Gamma) = Q(\omega \in (\overline{\gamma}, 1)) = 1 - \overline{\gamma}.$$

Consequently, we can invoke the result of Part 1 of Theorem 2 to solve this $I$-projection problem via its corresponding dual problem. This conclusion holds for any $\overline{\gamma} \in (0, 1)$. From the above calculations, setting $\overline{\gamma} = 1$ results in $Q(\omega \in \Omega : f_\gamma(\omega) < 0 \ \forall \gamma \in \Gamma) = 0$, which violates Part (iii) of Assumption 3. Recall that in the proof of Theorem 2, this condition was helpful for establishing the coercivity of the objective function in the dual problem via Lemma C.2 in Appendix C. We can establish the coercivity of this function without this condition by direct calculation. For any $y \in D$, the divergence $\|y\|_{L_1(Q)} \to +\infty$ only holds when $\alpha \to +\infty$, because the moment functions are uniformly bounded. For any $y \in D$, $y = \alpha y'$ where $y' \in \mathcal{C}(\Gamma)$. Then, by Lemma D.1 in Appendix D.1, the objective function satisfies the inequality

$$\int_{\Omega} e^y \, dQ \geq \alpha(e^2 - \beta k) \frac{(k + 1)\beta - e^2}{\beta k}, \quad \text{where } \frac{e^2}{\beta} - 1 < k < \frac{e^2}{\beta} \text{ and } \beta = \int_{\Omega} e^y \, dQ.$$  

(4.4)

Therefore, the objective function $\int_{\Omega} e^y \, dQ \to +\infty$ as $\alpha \to +\infty$, since $\frac{(k + 1)\beta - e^2}{\beta k} > 0$, establishing that it is coercive with respect to the $L_1(Q)$-norm on the domain $D$. Since the objective function is lower semicontinuous on $D$ with respect to the $L_1(Q)$-norm, we conclude that the dual problem for the constraint set $\mathcal{M}$ with $\overline{\gamma} = 1$ has a unique solution, under Assumption 3.

The least-squares approach to solving this $I$-projection problem uses results from Bhattacharya and Dykstra (1995), which apply only if $\mathcal{M}$ in (4.3) is a subset of $L_2(Q)$, and $\frac{dG}{dQ}$ exists as an element of $L_2(Q)$. For example, with $\overline{\gamma} = 1$, the solution of the $I$-projection’s dual problem under this additional structure is given by

$$y_0(\omega) = \ln \left( E_Q \left[ \frac{dG}{dQ} \mid C \right] \right)(\omega) - \int_{\Omega} \ln \left( E_Q \left[ \frac{dG}{dQ} \mid C \right] \right)(\omega) \, dG(\omega),$$  

(4.5)

where $C = \{ y \in L_2(Q) : y \text{ is non-decreasing} \}$ and $E_Q \left[ \frac{dG}{dQ} \mid C \right]$ is the least-squares projection of $\frac{dG}{dQ}$ onto $C$.

This additional structure on $\mathcal{M}$ and $G$ constrains the shapes of the feasible $Q$-densities and $G$, which can be restrictive in practice. An example of a distribution $G$ such that $\frac{dG}{dQ}$ exists but $\frac{dG}{dQ} \notin L_2(Q)$, is
\[ \frac{dG}{dQ}(\omega) = \omega^{-1/2}/2 \] for \( \omega \in \Omega \) and zero otherwise. For this specification of \( \frac{dG}{dQ} \), the projection \( E_Q \left[ \frac{dG}{dQ} \mid \mathcal{C} \right] \) is not uniquely defined. To see why, note that it is the solution of the following least-squares problem:

\[
\min_{h \in \mathcal{C}} \int_{\Omega} \left( \frac{dG}{dQ} - h \right)^2 dQ = \min_{h \in \mathcal{C}} \left( \int_{\Omega} h^2 dQ - 2 \int_{\Omega} \left( \frac{dG}{dQ} h \right) dQ \right) + \int_{\Omega} \left( \frac{dG}{dQ} \right)^2 dQ,
\]

where the objective function equals \( +\infty \) for each \( h \in \mathcal{C} \), since

\[
\int_{\Omega} \left( \frac{dG}{dQ} \right)^2 dQ = \frac{1}{4} \int_0^1 \frac{1}{\omega} d\omega = +\infty.
\]

By contrast, the proposed approach does not require any additional structure on the constraint set (4.3) and \( G \) to approximate the \( I \)-projection.

The proposed approximation scheme applies to the constraint set (4.3) and is valid under any specification of \( G \). It is also straightforward to implement in practice. The set of moment functions in this example is given by \( \mathcal{V} = \{ G(\gamma) - 1[\omega \leq \gamma] : \gamma \in \Gamma \} \), and Lemma [D.2] in Appendix [D.1] establishes that it satisfies Assumption 2 and \( \mathcal{V} = \overline{\mathcal{V}} \). Thus, by Theorem 4 we can approximate the \( I \)-projection using the proposed approximation scheme. In doing so, we have to construct a suitable partition of \( \mathcal{V} \) for given a level of accuracy as in (3.15). For a given \( \epsilon > 0 \), a finite collection of real numbers \( 0 = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n = \overline{\gamma} \) can be found so that \( \gamma_j - \gamma_{j-1} \leq \epsilon/2 \) and \( G(\gamma_j) - G(\gamma_{j-1}) \leq \epsilon/2 \), and for all \( 1 \leq j \leq n \). This can always be done in such a way that \( n \leq 4 + 2/\epsilon \). With such a construction, we have the following finite partition of \( \mathcal{V} \) of size \( n \) that fits into our framework: \( \mathcal{V} = \bigcup_{j=1}^n E_j \), where

\[
E_1 = \{ G(\gamma) - 1[\omega \leq \gamma] : \gamma \in [\gamma_0, \gamma_1] \}, E_2 = \{ G(\gamma) - 1[\omega \leq \gamma] : \gamma \in (\gamma_1, \gamma_2] \}, \ldots,
\]

and

\[
E_n = \{ G(\gamma) - 1[\omega \leq \gamma] : \gamma \in (\gamma_{n-1}, \gamma_n] \}.
\]

The \( I \)-projection problem of Example 1 extends in a natural way to include more than one marginal distribution \( G \) and different reference distributions \( Q \). Section 4 of [Bhattacharya, 2006] considers this extension for the case of multiple marginal distributions where he also illustrates the dual form of his iterative approximation scheme. His algorithm is an extension of the celebrated iterated proportional fitting algorithm, which has been used to solve Schrödinger equations arising in statistical physics involving \( I \)-projections on to sets of densities having fixed margins (e.g., [Ruschendorf, 1993; Ruschendorf and Thomsen, 1993]). The application of Bhattacharya’s algorithm with stochastic ordering constraints gives rise to a set of Schrödinger inequalities, which is connected to optimal partial transport problems with entropic cost arising in data science and statistical physics. The next example compares our method with it using this extension of Example 1 — see Example 2 below. It should be noted that [Bhattacharya, 2006] also imposes the \( L_2(Q) \) structure to exploit results from [Bhattacharya and Dykstra, 1995] for executing his algorithm. Like with Example 1, we demonstrate the broader applicability of our approximation scheme for moment inequality models.

**Example 2.** Let \( Q = Q_{X_1,X_2} \) is a fixed bivariate probability distribution on \( \Omega = \mathbb{R}^2 \). In this setting, \( \omega = (x_1,x_2) \), and let \( G_1 \) and \( G_2 \) be given cumulative distribution function defined on \( \mathbb{R} \). Consider the
constraint set
\[ M = \left\{ p \in L_1(Q) : m(p) < +\infty, \text{ and } \int_{\Omega} 1[x_i \leq \gamma] p dQ \leq G_i(\gamma) \forall \gamma \in \mathbb{R}, i = 1, 2 \right\} . \] (4.6)

Consequently, \( V = V_1 \cup V_2 \), where
\[ V_1 = \{(G_1(\gamma) - 1[x_1 \leq \gamma]) 1[x_2 \in \mathbb{R}] : \gamma \in \mathbb{R}\} \quad \text{and} \quad V_2 = \{(G_2(\gamma) - 1[x_2 \leq \gamma]) 1[x_1 \in \mathbb{R}] : \gamma \in \mathbb{R}\} . \]

We first establish this \( I \)-projection problem with constraint set \( M \) in (4.6) is covered by our setup. Then we discuss approximation of the \( I \)-projection using the result of Theorem 4 and contrast it with the dual form of the iterative algorithm put forward by Bhattacharya (2006).

It is a straightforward task to show that Parts (i) and (ii) of Assumption 3 hold, using steps similar to those in Example 1. For brevity, we omit those details. Part (iii) of that assumption, like with \( \gamma = 1 \) in Example 1, does not necessarily hold. However, one can establish an inequality identical to (4.4) using the same arguments as in Lemma D.1, but with appropriate modifications accounting for \( Q \) now being a bivariate probability distribution and the form of \( V \). Consequently, the \( I \)-projection with constraint set (4.6) exists and is unique as an element of \( L_1(Q) \), and its \( Q \)-density has the exponential family representation. Lemma D.3 in Appendix D.2 establishes that \( V \) satisfies Assumption 2 and that \( V = \overline{V} \); holds. Thus, we can approximate the solution of this \( I \)-projection by constructing a suitable partition of \( V \). Given \( \epsilon > 0 \), a finite collection of real numbers \(-\infty = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n = +\infty \) can be found so that \( Q_{X_i}((\gamma_{j-1}, \gamma_j)) \leq \epsilon/4 \) and \( G_i(\gamma_j) - G_i(\gamma_{j-1}) \leq \epsilon/4 \), for \( i = 1, 2 \) and for all \( 1 \leq j \leq n \). This can always be done in such a way that \( n \leq 6 + 4/\epsilon \). With such a construction, we have the following finite partition of \( V \) of size \( 2n \): \( V = (\cup_{j=1}^{n} E_{1,j}) \cup (\cup_{j=1}^{n} E_{2,j}) \), where
\[
E_{1,1} = \{(G_1(\gamma) - 1[x_1 \leq \gamma]) 1[x_2 \in \mathbb{R}] : \gamma \in (\gamma_0, \gamma_1]\}, \\
E_{1,2} = \{(G_1(\gamma) - 1[x_1 \leq \gamma]) 1[x_2 \in \mathbb{R}] : \gamma \in (\gamma_1, \gamma_2]\}, \\
\vdots \\
E_{1,n} = \{(G_1(\gamma) - 1[x_1 \leq \gamma]) 1[x_2 \in \mathbb{R}] : \gamma \in (\gamma_{n-1}, \gamma_n]\},
\]
and
\[
E_{2,1} = \{(G_2(\gamma) - 1[x_2 \leq \gamma]) 1[x_1 \in \mathbb{R}] : \gamma \in (\gamma_0, \gamma_1]\}, \\
E_{2,2} = \{(G_2(\gamma) - 1[x_2 \leq \gamma]) 1[x_1 \in \mathbb{R}] : \gamma \in (\gamma_1, \gamma_2]\}, \\
\vdots \\
E_{2,n} = \{(G_2(\gamma) - 1[x_2 \leq \gamma]) 1[x_1 \in \mathbb{R}] : \gamma \in (\gamma_{n-1}, \gamma_n]\}.
\]

Therefore, one can approximate the \( I \)-projection using the proposed approximation scheme with the above partition of \( V \).
As the constraint set (4.6) has the representation $\mathcal{M} = \cap_{i=1}^2 \mathcal{M}_i$, where

$$\mathcal{M}_i = \left\{ p \in L_1(Q) : m(p) < +\infty, \text{ and } \int_{\Omega} 1[x_i \leq \gamma]p \, dQ \leq G_i(\gamma) \ \forall \gamma \in \mathbb{R}, \ i = 1, 2 \right\},$$ (4.7)

for $i = 1, 2$, Bhattacharya (2006) proposes to solve for the $I$-projection using the following dual problem:

$$\inf \left\{ \int_{\Omega} e^y \, dQ : y \in S_1 \oplus S_2 \right\} \text{ where } y = y(u_1, u_2) = y_1(u_1) + y_2(u_2) \in S_1 \oplus S_2 \text{ and }$$

$$S_1 = \left\{ y(u_1, u_2) : y(u_1, u_2) = y_1(u_1), y_1(u_1) \text{ is nondecreasing } \int_{\mathbb{R}} y_1(u_1) \, dG_1(u_1) = 0 \right\},$$

$$S_2 = \left\{ y(u_1, u_2) : y(u_1, u_2) = y_2(u_2), y_2(u_2) \text{ is nondecreasing } \int_{\mathbb{R}} y_2(u_2) \, dG_2(u_2) = 0 \right\}.$$

His cyclical descent algorithm for approximating the solution of this dual problem successively minimizes over each $y_i$ at a time while the remaining is held fixed. In particular, it follows these steps:

- Initialization: Set $y_{0,i} = 0$ and begin with $n = 1, i = 1$.

- Implementation:
  1. Let $y_{n,1}$ denote the solution to $\inf \left\{ \int_{\Omega} e^{y+y_{n-1,2}} \, dQ : y \in S_1 \right\}$.
  2. Let $y_{n,2}$ denote the solution to $\inf \left\{ \int_{\Omega} e^{y_{n,1}+y} \, dQ : y \in S_2 \right\}$.
  3. Increase $n$ by 1 and repeat the two previous steps.

To implement this algorithm, the practitioner must be able to obtain closed-form expressions of the $I$-projections onto the individual sets $\mathcal{M}_i$ and at each iteration of the algorithm. Otherwise, it is not possible to hold fixed $y_{n,i}$ while solving for the other dual variable in this algorithm. Thus, to satisfy this requirement, he further restricts the constraint sets (4.7) to be subsets of $L_2(Q)$, and simultaneously requires the existence of $dG_1/dQ_{X_1}$ and $dG_2/dQ_{X_2}$ as elements of $L_2(Q)$. Like in Example 1, this additional structure creates a way forward through the use of Theorem 3.3 in Bhattacharya and Dykstra (1995) to obtain closed-form expressions for the dual variables at each iteration. By contrast, our approach does not impose any additional structure to approximate this $I$-projection, and hence, is more widely applicable.

**4.3 Connections to Partial Identification in Econometrics**

The partial identification literature in econometrics has been leading the development of inference procedures for parameters defined by moment inequality restrictions. It is therefore instructive to connect the results of this paper with that literature. In many applications, interest focuses on a finite-dimensional parameter that enter the moment functions (e.g., Andrews and Soares, 2010 and Andrews and Shi, 2013). That is, the moment functions have the form $f_{\gamma}(\omega; \theta)$, where $\theta \in \Theta \subset \mathbb{R}^{d_\theta}$ is the finite-dimensional parameter.
In this literature, the model of interest takes the form $\mathcal{M} = \bigcup_{\theta \in \Theta} \mathcal{M}(\theta)$, where

$$\mathcal{M}(\theta) = \left\{ p \in L_1(Q) : m(p) < +\infty, \int_{\Omega} vp dQ \geq 0 \forall v \in \mathcal{V}(\theta) \right\}$$

with

$$\mathcal{V}(\theta) = \left\{ -f_\gamma(\cdot; \theta), K - |f_\gamma(\cdot; \theta)|^{1+\delta} : \gamma \in \Gamma \right\}$$

is the set of distributions that are consistent with the moment conditions for the parameter $\theta$. We point to two important applications of $I$-projection in that literature: (i) optimal inference under correct model specification (i.e., $Q \in \mathcal{M}$), and (ii) model selection tests.

The problem of optimal inference in moment inequality models defined by a finite number of such restrictions has been considered by Canay (2010). Canay shows the empirical likelihood-ratio statistic for inference on the parameter $\theta$ yields optimal inference from a large deviations perspective. The Kullback-Leibler divergence naturally appears in the asymptotics of Canay’s empirical likelihood-ratio test, as it is a nonparametric version of a maximum likelihood testing procedure. The results of our paper can be used to extend Canay’s results to models in which there are infinitely many moment inequality restrictions. Such results would be fruitful, as inference for parameters defined by infinitely many such restrictions is a fast developing area of partial identification (e.g., Andrews and Shi, 2017, and Lok and Tabri, 2021, among others), where results on asymptotic power of tests is becoming more important. Important applications in which there are infinitely many moment inequality restrictions include, but are not limited to, stochastic dominance constraints, bounds from an interval regressor in regression models (Manski, 2005), and shape constraints on regression functions (e.g., Du et al., 2013).

The problem of model selection in moment inequality models with a finite number of such restrictions, has been considered by Shi (2015). In the setting of this paper, the estimand of interest would be the quantity

$$\arg \inf_{\theta \in \Theta} \inf_{P \in \mathcal{M}(\theta)} I(P||Q),$$

which has been used for developing statistical inference on the parameter $\theta$ on the basis of a sample. For each $\theta \in \Theta$, the $I$-projection problem $\inf_{P \in \mathcal{M}(\theta)} I(P||Q)$ has a unique solution under the conditions of Theorem 1 which can be computed using the Fenchel dual approach of this paper under Assumptions 1 and 2. As the model $\mathcal{M} = \bigcup_{\theta \in \Theta} \mathcal{M}(\theta)$ is partially identified, the map $\theta \mapsto \inf_{P \in \mathcal{M}(\theta)} I(P||Q)$ can have multiple minimizers, which define the “pseudo-true set”. This concept is generalized from the “pseudo-true parameter” concept in the literature of misspecified point-identified models. The prefix “pseudo” signifies the possibility that the model may be misspecified. Shi and Hsu (2017) extend the testing framework of Shi (2015) to include conditional moment inequalities, but introduce an average generalized empirical likelihood (GEL) pseudo-distance on the set of distributions instead of utilizing the Kullback-Leibler divergence $I(P||Q)$. Their pseudo-distance averages GEL distances over the set of instrument functions with respect to a user-defined distribution on that set – a fine-tuning parameter. Hence, the results of this paper can be used for the development of conditional moment inequality model selection tests based on the $I$-projection. This

---

2The construction of their average GEL pseudo-distance is similar to existing distances that are used in other related testing problems, such as Stinchcombe and White (1998) and Lee et al. (2013), where there are infinitely many restrictions.
is advantageous, because it bypasses the introduction of a fine-tuning parameter as in Shi and Hsu (2017), which can be difficult to select in practice.

4.4 Verification of Assumptions

The marriage of Assumptions 1 and 2 are at the heart of Theorems 2, 3, and 4. In practice, the PD $Q$ is given, so that verification of Assumption 1 can be easily carried out by direct calculation, as demonstrated in Examples 1 and 2. Additionally, the set $V$ is also given in practice, and we have verified that it satisfies Assumption 2 in the examples by directly calculating the $L_1(Q)$ bracketing number of $V$. The calculations of other measures of complexity can also be used to verify Assumption 2 in practice, such as the $L_1(Q)$, covering, or packing numbers. Such calculations are heavily used in empirical process theory to characterize classes of (moment) functions that are Glivenko-Cantelli and/or Donsker, but in the $L_r(Q)$ norm for $r \geq 1$. By the result of Lemma C.7 in Appendix C such sets of moment functions that are precompact in $L_r(Q)$ with $r > 1$, are necessarily precompact in $L_1(Q)$. Thus, the results of Theorems 2, 3, and 4 also cover precompact subsets of $L_r(Q)$ for any $r > 1$. See Section 2.7 of van der Vaart and Wellner (1996) for some examples of precompact classes in $L_r(Q)$ spaces.

In applications, the $L_1(Q)$-closure of $V$ (i.e., $\overline{V}$) must be derived to implement the proposed approximation scheme. This is not a difficult task, as demonstrated by the calculations in Examples 1 and 2. The technique used in Lemmas D.2 and D.3 generalize to other classes of moment functions. The main tool for deriving $\overline{V}$ from $V$ is as follows. By considering a $L_1(Q)$ limit point of $V$, given by $y$, there is a sequence $\{y_n\}_{n \geq 1} \subset V$ such that $y_n \xrightarrow{L_1(Q)} y$, where objective is to derive the form of $y$. As

$$y_n \xrightarrow{L_1(Q)} y \implies y_n \xrightarrow{Q} y \implies \exists \{n_i\}_{i \geq 1} \subset \{n\} \text{ (non-random)}: \lim_{i \to +\infty} y_{n_i}(\omega) = y(\omega) \text{ a.s.-} Q,$$

the a.s.-$Q$ convergence can be used to deduce the form of the limit point $y$ based on the functional forms of elements in $V$. In the examples of the next section, we find that the limit point satisfies $y \in V$, implying that $V = \overline{V}$, holds. Once one obtains $\overline{V}$, constructing a finite partition $\{E_i\}_{i=1}^n$ of it for a given $\epsilon > 0$, such that (3.15) holds, is also a simple task. The next section illustrates the verification of Assumption 2, the construction of $\overline{V}$ from $V$, and the construction of the finite partition $\{E_i\}_{i=1}^n$ of $\overline{V}$, in the context of unconditional and conditional stochastic dominance constraints.

5 Dominance Constraints

In this section, we illustrate the verification of Assumption 2 and the construction of the partition of $\overline{V}$ using the examples of first-order stochastic dominance constraints, and an example from random sets with dominance criteria that characterize selectable distributions. We treat the unconditional and conditional versions of this ordering separately, as the moment functions that define these orderings are different. It is worth noting that the verifications of constraints for higher orders of dominance follows the same line of arguments as in the case of first-order dominance.

This section also presents a numerical implementation of the proposed approximation scheme for each
example using MATLAB. The code employs the parallel computing capabilities of fmincon based on the sequential quadratic programming algorithm and includes gradient evaluation in the objective function for faster or more reliable computations. For high levels of $\epsilon$, we have implemented the approximation on a desktop computer with 10 cores and 30 gigabytes of RAM. For low levels of $\epsilon$, we have implemented the approximation scheme on a cluster using 12 cores with 100 gigabytes of RAM.

5.1 Unconditional First-Order Restricted Stochastic Dominance

Let $Q = Q_{X_1,X_2}$ be a fixed bivariate probability distribution (PD) on $\Omega = \mathbb{R}^2$. This section considers the problem of finding the $I$-projection of $Q$ onto the class of all bivariate PD’s whose $X_2$-marginal cumulative distribution function (CDF) is less than or equal to its $X_1$-marginal counterpart over the interval $[\gamma, \overline{\gamma}]$. Thus the $I$-projection problem can be stated as

$$\min_{P \in \mathcal{M}} I(P|Q_{X_1,X_2}),$$

where the constraint set $\mathcal{M}$ is defined as in (3.2) but with $\Gamma = [\gamma, \overline{\gamma}]$, $\omega = (x_1, x_2)$, and $f_{\gamma} (\omega) = 1[x_2 \leq \gamma] - 1[x_1 \leq \gamma]$. As $\sup_{(\omega, \gamma) \in \Omega \times \Gamma} |f_{\gamma} (\omega)| \leq 2$ holds in this example, there no need to impose the tail condition (3.3). In this case, the constraint set is given by

$$\mathcal{M} = \left\{ p \in L_1(Q) : m(p) < +\infty, \text{ and } \int_{\Omega} f_{\gamma} p dQ \leq 0 \forall \gamma \in \Gamma \right\}.$$

In this example,

$$\mathcal{V} = \left\{ 1[x_1 \leq \gamma] - 1[x_2 \leq \gamma] : \gamma \in [\gamma, \overline{\gamma}] \right\}$$

(5.1)

The next result describes the properties of this set.

**Proposition 1.** Let $\mathcal{V}$ be given by (5.1). Then this set satisfies $\mathcal{V} = \overline{\mathcal{V}}$ and Assumption 2.

**Proof.** See Appendix B.5  

Thus, we can approximate the solution of the $I$-projection using its dual via Theorem 4 when $Q$ satisfies Assumption 1 by constructing a suitable partition of $\mathcal{V}$. Given $\epsilon > 0$, a finite collection of real numbers $\underline{\gamma} = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n = \overline{\gamma}$ of size $n$ can be found so that $Q_{X_i} ((\gamma_{j-1}, \gamma_j)) \leq \epsilon/2$ for $i = 1, 2$ and for all $1 \leq j \leq n$. With such a construction, $\mathcal{V} = \bigcup_{i=1}^{n} E_i$, where

$$E_1 = \{-f_{\gamma}(\cdot) : \gamma \in [\gamma, \gamma_1]\}, \quad E_2 = \{-f_{\gamma}(\cdot) : \gamma \in (\gamma_1, \gamma_2]\}, \ldots, \quad E_n = \{-f_{\gamma}(\cdot) : \gamma \in (\gamma_{n-1}, \overline{\gamma}]\}.$$

Now we illustrate the proposed approximation scheme using a numerical experiment based on this example with lognormal distributions where the partition of $\mathcal{V}$ is described as above. The specification is as follows: $X_1 \sim \text{LogN}(0.5,4)$ and $X_2 \sim \text{LogN}(0.5,2)$, are statistically independent, and $\Gamma = [0, 7]$. For each $\epsilon$, we
(i) specified an equidistant grid $0 = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n = 7$ that satisfies $Q_{X_i} ((\gamma_{j-1}, \gamma_j)) \leq \epsilon/2$ for $i = 1, 2$ and for all $1 \leq j \leq n$; and

(ii) implemented the approximation scheme with $v_j = -f_{\gamma_j} \in E_j$ for $j = 1, \ldots, n$.

We do not have a closed-form expression for the objective function in (3.16) with this specification. Consequently, we approximate it numerically using its sample-analogue version based on $10^4$ paired draws from $Q$ and minimize this approximate objective function instead. Finally, we have considered the following values of $\epsilon \in \{1, 0.5, 0.25, 0.126, 0.0625\}$.

Figure 1 reports the graph of $\gamma \mapsto \int_{\Omega} f_{\gamma} dQ$ under this configuration showing that $Q \notin \mathcal{M}$ since $\int_{\Omega} f_{\gamma} dQ > 0$ for all $\gamma \in [1.65, 7]$. This figure also plots this map under the $I$-projection for $\epsilon \in \{1, 0.0625\}$ only. We used these values of $\epsilon$ in the figure only to avoid visual clutter, as the maps under other values of $\epsilon$ are quite close to one another and the reported ones.

Table 1 reports the evolution of the values of $n$, the optimal values of the finite programs, and the execution times of the optimization problem as they depend on the values of $\epsilon$ that we have considered, illustrating the impact of the accuracy $\epsilon$ has on the proposed approximation scheme. Observe that the number of inequality constraints $n$ grows rapidly with lower values $\epsilon$. However, this rapid growth rate in $n$ does not pose computational challenges for the values of $\epsilon$ we have considered. Note that our numerical findings demonstrate that the optimization problem with thousands of constraints can be solved in a matter of hours. Of course, if one desires lower levels of $\epsilon$ in practice, then it may be prohibitive to execute the approximation scheme in a reasonable time on a basic computer as the number of choice variables would be quite
| $\epsilon$ | $n$ | Optimal Value ($3.16$) | Time |
|--------|-----|------------------------|------|
| 1   | 21  | 0.9837                 | 0.18s |
| 0.5 | 70  | 0.9831                 | 0.72s |
| 0.25 | 427 | 0.9828                 | 7.9s  |
| 0.125 | 1967 | 0.9827               | 167.2s |
| 0.0625* | 7315 | 0.9827               | 287.5m |

Table 1: First Order Restricted Stochastic Dominance. Notation: "s" denotes "seconds", "m" denotes "minutes", and "*" denotes implementation on a cluster.

large. In this case, it is worthwhile exploring adjustments to the approximation scheme that could reduce computational time. We present an adjustment of this sort in Section 5.3, which concerns dominance conditions arising from Artstein (1983)’s inequalities for selectionable distribution of random sets. The next section builds upon the content of this section by considering conditional first-order stochastic dominance inequality restrictions.

5.2 Conditional First-Order Restricted Stochastic Dominance

Let $Q = Q_{X_1, X_2, Z}$ be a fixed joint probability distribution (PD) on $\Omega = \mathbb{R}^{2+d_Z}$, $Z$ is a $d_Z$-dimensional vector of covariates. This section considers the problem of finding the $I$-projection of $Q$ onto the class of all PD’s whose $X_2|Z = z$ CDF is less that or equal to its $X_1|Z = z$ counterpart over the interval $[\underline{\gamma}, \overline{\gamma}]$ and for all values of $z \in \mathbb{R}^{d_Z}$. In symbols,

$$Q_{X_2|Z = z} ((-\infty, x]) - Q_{X_1|Z = z} ((-\infty, x]) \leq 0 \hspace{1em} \forall (x, z) \in [\underline{\gamma}, \overline{\gamma}] \times \mathbb{R}^{d_Z}. \quad (5.2)$$

Without loss of generality, we assume that the support of $Z$ is $[0, 1]^{d_Z} = \prod_{i=1}^{d_Z} [0, 1]$. This assumption is not restrictive as one can always apply a transformation on $Z$ such that the support of the transformed variable is $[0, 1]^{d_Z}$.

These conditional moment inequalities can be equivalently formulated as an infinite number of unconditional inequalities using instrument functions (Andrews and Shi, 2013). In this example, we consider the
countable hypercube instrument functions

\[ G_{c\text{-cube}} = \{ g(z) = 1 [z \in C] : C \in C_{c\text{-cube}} \}, \quad \text{where} \]

\[ C_{c\text{-cube}} = \left\{ C_{a,r} = \prod_{u=1}^{d_z} \left( \frac{a_u - 1}{2r}, \frac{a_u}{2r} \right] \subset [0,1]^{d_z} : a = (a_1, a_2, \ldots, a_{d_z}), a_u \in \{1, 2, \ldots, 2r\}, \right. \]

\[ \left. \text{for } u = 1, \ldots, d_z, \text{ and } r = r_0, r_0 + 1, \ldots \right\} \quad (5.4) \]

for given \( r_0 \in \mathbb{Z}_+ \). The conditional moment inequalities (5.2) can now be formulated in terms of \( Q \) as

\[ Q (\{ X_2 \leq x \} \cap C) - Q (\{ X_1 \leq x \} \cap C) \leq 0 \quad \forall (x, C) \in \overline{\gamma, \overline{\gamma}} \times G_{c\text{-cube}}. \]

In terms of the moment functions, \( \gamma = (\gamma_1, \gamma_2) \in \Gamma = \overline{\gamma, \overline{\gamma}} \times G_{c\text{-cube}} \), so that

\[ f_\gamma(\omega) = (1[x_2 \leq \gamma_1] - 1[x_1 \leq \gamma_1])\gamma_2(z), \quad \text{where} \quad \omega = (x_1, x_2, z). \]

As with unconditional restricted stochastic dominance, \( \sup_{(\omega, \gamma) \in \Omega \times \Gamma} |f_\gamma(\omega)| \leq 2 \) holds in this example. Consequently, there no need to impose the tail condition (3.3). Whence, in this example

\[ V = \{ (1[x_1 \leq \gamma_1] - 1[x_2 \leq \gamma_1])\gamma_2(z) : \gamma \in \overline{\gamma, \overline{\gamma}} \times G_{c\text{-cube}} \}. \quad (5.5) \]

The next result describes the properties of this set.

**Proposition 2.** Let \( V \) be given by (5.5). Then this set satisfies \( V = \overline{V} \) and Assumption 2.

**Proof.** See Appendix B.6. ■

Next we discuss the construction of a suitable partition of \( V \) for a given \( \epsilon > 0 \). Let \( v, v' \in V \), where \( v = (1[x_1 \leq \gamma_1] - 1[x_2 \leq \gamma_1])\gamma_2(z) \), \( v' = (1[x_1 \leq \gamma_1'] - 1[x_2 \leq \gamma_1'])\gamma_2(z) \), \( \gamma_2(z) = 1[z \in C] \), and \( \gamma_2'(z) = 1[z \in C'] \), for \( C, C' \in C_{c\text{-cube}} \). It is a simple task to show that

\[ \int_\Omega |v - v'| dQ \leq Q_{X_1} \left( \left( \gamma_1 \land \gamma_1', \gamma_1 \lor \gamma_1' \right) \right) + Q_{X_2} \left( \left( \gamma_1 \land \gamma_1', \gamma_1 \lor \gamma_1' \right) \right) + 2Q_Z(C\Delta C'), \]

holds, where \( \Delta \) denotes the symmetric difference operator on sets. The usefulness of this inequality is that in the construction of the partition, we can treat the two components of \( \gamma \) separately. To force the right side of this inequality to be less than \( \epsilon \) on each partitioning segment, we can find a finite partition of \( V = \bigsqcup_{i=1}^n E_i \), where \( Q_{X_i} \left( \left( \gamma_1 \land \gamma_1', \gamma_1 \lor \gamma_1' \right) \right) \leq \epsilon/3 \) for \( i = 1, 2 \), and \( Q_Z(C\Delta C') \leq \epsilon/6 \), holds, on each \( E_i \). For the contribution to the partition due to \( \gamma_1 \), we can proceed identically as in the previous section by selecting a finite collection of real numbers \( \gamma_i = \gamma_{1,0} < \gamma_{1,1} < \gamma_{1,2} < \cdots < \gamma_{1,n_1} = \gamma \) can be found so that \( Q_{X_i} \left( \left( \gamma_{1,j-1}, \gamma_{1,j} \right) \right) \leq \epsilon/3 \) for \( i = 1, 2 \) and for all \( 1 \leq j \leq n_1 \).

Regarding the contribution to the partition due to \( \gamma_2 \), we can set \( r_0 \) dependent on \( \epsilon \). There is no loss of information in doing so, because the cubes for \( r < r_0 \) are captured by some smaller cubes contained in
those cubes — i.e., cubes $C_{a,r}$ for $r < r_0$ are finite disjoint unions of (smaller) cubes $C_{a,r}$ for $r \geq r_0$, $\forall a$. We suggest choosing $r_0$ to be the smallest positive integer such that $Q_Z(C_{a,r_0}) \leq \epsilon/6$ for all $a$. Next, define $A_{r_0} = \{a = (a_1, a_2, \ldots, a_{d_Z}) : a_u \in \{1, 2, \ldots, 2r_0\}\}$, and note that $|A_{r_0}| = (2r_0)^{d_Z}$. Hence, we can form a finite partition $C_{c-cube}$:

$$C_{c-cube} = \bigcup_{a \in A_{r_0}} \{C \in C_{c-cube} : C \subset C_{a,r_0}\},$$

as the collections $\{C \in C_{c-cube} : C \subset C_{a,r_0}\}$ for each $a \in A_{r_0}$ are pairwise disjoint.

To simplify the exposition let $n_2 = (2r_0)^{d_Z}$, and suppose that $a_{(1)} > a_{(2)} > \cdots > a_{(n_2)}$ are the elements of $A_{r_0}$ ranked by the lexicographic ordering. Putting together these separate contributions to the partition via the crossproduct, gives the following collection of subsets of $[\gamma, \overline{\gamma}] \times G_{c-cube}$

$$\{(\gamma_{1,j-1}, \gamma_{1,j}] : j = 1, 2, \ldots, n_1\} \times \left\{\{\gamma_2 \in G_{c-cube} : C \subset C_{a_{(i)}, r_0}\} : i = 1, 2, \ldots, n_2\right\}. \quad (5.6)$$

This collection forms a finite partition of $[\gamma, \overline{\gamma}] \times G_{c-cube}$, whose size is the product $n_1 n_2$. With such a construction, $V = \bigcup_{i=1}^{n_1} \bigcup_{j=1}^{n_2} E_{i,j}$, where

$$E_{i,j} = \left\{-f_{\gamma}(\cdot) : \gamma_1 \in (\gamma_{1,i-1}, \gamma_{1,i}] \text{ and } \gamma_2 \in \{\gamma_2 \in G_{c-cube} : C \subset C_{a_{(i)}, r_0}\}\right\},$$

is a finite partition of $V$ that can be used in the implementation of the proposed approximation scheme.

Now we illustrate the proposed approximation scheme using a numerical experiment based on this example with exponential and uniform distributions where the partition of $V$ is described as above. The specification is as follows: $Z \sim U(0, 1)$, $X_1 \sim \text{Exp}(1)$ and $X_1$ is independent of the $Z$ and $X_2$, $X_2 | Z \sim \text{Exp}(Z^{-1})$, and $[\gamma, \overline{\gamma}] = [1, 4]$. This configuration has $Q \not\in M$ as the inequalities $\{5.2\}$ do not hold for all $z \in (0, 1)$. For example, this violation occurs at values $z = 0.5, 0.95$, which can be seen graphically in the right panels of Figure 2, where the blue lines plot the contrast $Q \langle X_2 | Z = z \rangle (\langle -\infty, x \rangle) - Q \langle X_1 \rangle | Z = z \rangle (\langle -\infty, x \rangle)$ showing that it is strictly positive on the interval $[1, 4]$ for those values of $z$.

For each $\epsilon$, we

(i) specified an equidistant grid $1 = \gamma = \gamma_{1,0} < \gamma_{1,1} < \gamma_{1,2} < \cdots < \gamma_{1,n_1} = \overline{\gamma} = 4$ that satisfies $Q_{X_1}(\langle (\gamma_{1,j-1}, \gamma_{1,j}) \rangle) \leq \epsilon/3$ for $i = 1, 2$ and for all $1 \leq j \leq n_1$;

(ii) set $r_0 = \lceil 3/\epsilon \rceil$; and

(iii) implemented the approximation scheme with $v_{i,j} = -f_{\gamma} \in E_{i,j}$ such that $\gamma_1 = \gamma_{1,i}$ and $\gamma_2 = 1 \in C_{j,r_0}$ for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$.

As there isn’t a closed-form expression for the objective function in $(3.16)$ with this specification, we approximate it numerically using its sample-analogue version based on $10^3$ draws of the triple $(X_1, X_2, Z)$ from $Q$, and minimize this approximate objective function instead. Finally, we have considered the following

$^3$Simulation can be carried out using the conditional distribution method, which is usually attributed to Rosenblatt (1952).
Figure 2: Left: graphs of (i) the approximate $I$-projection, $p_Q$ for $Z \in \{0.1, 0.5, 0.95\}$ with $\epsilon = 0.0625$. Right: graphs of the maps $\gamma_1 \mapsto E_{P_Q}[1[X_2 \leq \gamma_1] - 1[X_1 \leq \gamma_1] | Z]$ and $\gamma_1 \mapsto Q_{X_2|Z=z}((\infty, \gamma_1]) - Q_{X_1|Z=z}((\infty, \gamma_1])$ for $Z \in \{0.1, 0.5, 0.95\}$ and $\epsilon \in \{1, 0.5, 0.25, 0.125, 0.0625\}$

| $\epsilon$ | $n = n_1 n_2$ | Optimal Value | Time     |
|------------|----------------|---------------|----------|
| 1          | 36             | 0.8389        | 0.26s    |
| 0.5        | 72             | 0.8362        | 0.61s    |
| 0.25       | 216            | 0.8284        | 4.8s     |
| 0.125      | 1008           | 0.8130        | 99.8s    |
| 0.0625     | 3744           | 0.7905        | 154.6m   |

Table 2: Conditional First Order Stochastic Dominance. Notation: "s" denotes "seconds", and "m" denotes "minutes".
values of \( \epsilon \in \{1, 0.5, 0.25, 0.126, 0.0625 \} \).

Table \ref{table:results} reports the output of the numerical experiment. The results that are similar to those of the previous section in terms of the evolution of \( n \) and execution times of the approximation scheme with \( \epsilon \). The left side of Figure \ref{fig:contrast} reports the approximate \( I \)-projections for \( \epsilon = 0.0625 \) and \( Z = 0.1, 0.5, 0.95 \), and the right side reports the graphs of the conditional CDF contrasts for the aforementioned values of \( Z \) and the values of \( \epsilon \) that we have considered. The approximate \( I \)-projection for \( Z = 0.1 \) with \( \epsilon = 0.0625 \) is a flat surface whose altitude is the reciprocal of the optimal minimal value \( 3.16 \). This numerical finding also holds for the other values of \( \epsilon \), but we have not reported the graphs in the figure to avoid cluttering. Note that this numerical finding is expected, as the conditional CDF contrast finding is expected, as the conditional CDF contrast

Contrastingly, the approximate \( I \)-projections are non-constant surfaces when \( Z = 0.5, 0.95 \), because \( Q_{X_2\mid Z=z}((-\infty, x]) - Q_{X_1\mid Z=z}((-\infty, x]) \) is strictly positive on \([1, 4] \) for large values of \( Z \) when \( \epsilon \) is not sufficiently small. See, for instance, the plot of this contrast under \( \epsilon = 0.25 \) and \( Z = 0.95 \) (i.e., the magenta line in the bottom right panel), which depicts it taking positive values on parts of the domain in the neighborhood of 1.5. Lowering \( \epsilon \) removes this violation, as seen in the same case but with \( \epsilon = 0.0625 \) (i.e., the black line in the bottom right panel).

### 5.3 Random Set Example

This section presents a simple example involving a random set that our results cover. For simplicity, let \( \Omega = [0, 1] \). We draw inspiration from Example 1.11 of \cite{molchanov2018stable}, which is on interval data, in developing our example. The random set is \( Y = [y_L, y_U] \), where \( Y \subset [0, 1] \), the random vector \( (y_L, y_U) \) has marginal distributions \( y_L \sim U[0, 1/2], y_U \sim U[1/2, 1] \) and they are independent. This random set has capacity functional of the form

\[
\sigma ([a, b]) = \text{Prob} [y_L < a, y_U > a] + \text{Prob} [y_L \in [a, b]]
\begin{cases}
2a + \min\{2(b - a), 1 - 2a\} & \text{if } a < 1/2 \\
2a & \text{if } a \geq 1/2,
\end{cases}
(5.7)
\]

for every \([a, b] \subset \Omega\).

Following Theorem 2.1 in \cite{artstein1983stability}, the sharp identified set for selectionable distributions with respect to \( \sigma \) having \( Q \)-density \( p \) is defined by the following system of inequality restrictions:

\[
\int_\Omega 1_{\omega \in C} \, p \, dQ \leq \sigma (C) \quad \forall C \in C(\Omega),
(5.8)
\]

where \( C(\Omega) \) denotes the collection of closed interval subsets of \( \Omega \). In terms of our notation, the index set is given by \( \Gamma = C(\Omega) \), and the moment functions are

\[
f_\gamma (\omega) = 1_{\omega \in C} - \sigma (C)
(5.9)
\]
for $C \in C(\Omega)$, so that where $\gamma = C$. Thus, the constraint set has the desired form
\[
\mathcal{M} = \left\{ p \in L_1(Q) : m(p) < +\infty, \int_{\Omega} f_{\gamma} p dQ \leq 0 \forall \gamma \in \Gamma \right\}.
\] (5.10)

Furthermore,
\[
\mathcal{V} = \{ \sigma(C) - 1 [\omega \in C] : C \in C(\Omega) \}.
\] (5.11)

This example closely resembles Example 1 in Section 4.2, where the main difference is the nature of the index set $\Gamma$. Another difference is that the CDF $G$ in Example 1 has been replaced by the capacity functional $\sigma$ in (5.7). If $\mathcal{M} \neq \emptyset$, then by Theorem 1 the $I$-projection exists and is unique. This result is based on the primal formulation of the $I$-projection problem. Like in Example 1, this example satisfies all the conditions of Assumption 3 except for Condition (iii), as $Q(\omega \in \Omega : f_{\gamma}(\omega) < 0 \forall \gamma \in \Gamma) = 0$. Despite this violation, we can still establish the coercivity of the Fenchel dual’s objective function by direct calculation using arguments identical to those in Example 1 for the case $\delta = 1$. That is, using arguments similar to those in Lemma D.1 in Appendix D.1 to obtain an inequality like (4.4) to establish coercivity. Then, combining this deduction with the lower semi-continuity of the objective function and Conditions (i) and (ii) of Assumption 3 is sufficient for establishing the existence of a unique solution in the dual problem. For brevity, we omit these technical details.

We note that the capacity functional $\sigma$ in (5.7) is Lipschitz continuous:
\[
|\sigma([a_1, b_1]) - \sigma([a_2, b_2])| \leq 2 d_H ([a_1, b_1], [a_2, b_2]) \quad \forall [a_1, b_1], [a_2, b_2] \in C(\Omega),
\] (5.12)

where $d_H ([a_1, b_1], [a_2, b_2]) = \max \{|a_1 - a_2|, |b_1 - b_2|\}$ is the Hausdorff distance between the intervals $[a_1, b_1]$ and $[a_2, b_2]$, which follows from the form of $\sigma$. Furthermore, by the triangular inequality we must have that
\[
\int_{\Omega} |f_C - f_{C'}| dQ \leq |\sigma(C) - \sigma(C')| + Q(C \Delta C'),
\] (5.13)

holds for any $C$ and $C'$ in $C(\Omega)$, where $\Delta$ denotes the symmetric difference operation on sets. These inequalities are useful for establishing the following result on the set $\mathcal{V}$ in (5.11).

**Proposition 3.** Let $\mathcal{V}$ be given by (5.11). Then this set satisfies $\mathcal{V} = \overline{\mathcal{V}}$ and Assumption 2.

**Proof.** See Appendix B.7. ■

Thus, we can approximate the solution of the $I$-projection using its dual via Theorem 4 by constructing a suitable partition of $\mathcal{V}$. The inequalities (5.12) and (5.13) present a simple avenue forward in the construction. Given $\epsilon > 0$, we can obtain a partition of the unit interval $[0, 1]$ given by $0 = d_0 < d_1 < d_2 < \cdots < d_m = 1$ such that $d_i - d_{i-1}, Q([d_{i-1}, d_i]) < \epsilon/4$ for $i = 0, 1, \ldots, m$. Now let $D_1 = [0, d_1]$, and
D_i = (d_{i-1}, d_i) \forall i = 2, 3 \ldots m. Consider the collection \mathcal{E} consisting of the following sets:

\begin{align*}
E_1^1 &= \{ C \in \mathcal{C}(\Omega) : C \subset D_1 \}, E_i^1 = \{ C \in \mathcal{C}(\Omega) : C \subset D_i \} \ \forall i = 2, 3 \ldots m; \\
E_1^2 &= \{ C \in \mathcal{C}(\Omega) : C \cap D_1 \neq \emptyset, C \cap D_2 \neq \emptyset \}, \text{ and} \\
E_i^2 &= \{ C \in \mathcal{C}(\Omega) : C \cap D_i \neq \emptyset, C \cap D_{i+1} \neq \emptyset \} \ \forall i = 2, 3 \ldots m - 1; \\
E_1^3 &= \{ C \in \mathcal{C}(\Omega) : C \cap D_1 \neq \emptyset, C \cap D_2 \neq \emptyset, C \cap D_3 \neq \emptyset \}, \text{ and} \\
E_i^3 &= \{ C \in \mathcal{C}(\Omega) : C \cap D_i \neq \emptyset, C \cap D_{i+1} \neq \emptyset, C \cap D_{i+2} \neq \emptyset \} \ \forall i = 2, 3 \ldots m - 2; \\
& \vdots \\
E_1^m &= \{ C \in \mathcal{C}(\Omega) : C \subset C \cap D_i \neq \emptyset \ \forall i = 1, 2 \ldots m \}.
\end{align*}

These sets form a finite partition of \mathcal{C}(\Omega) of size \(n = (m^2 - m)/2 + 1\), and for each \(E \in \mathcal{E}\)

\[\int_{\Omega} |f_{C} - f_{C'}| \, dQ \leq \epsilon \ \forall C, C' \in E,\]

holds. Thus, the collection \(\{ -f_{\gamma} \in \mathcal{V} : \gamma \in E \}, E \in \mathcal{E} \) forms a finite partition of \(\mathcal{V}\) to which Theorem 4 applies.

Now we illustrate the proposed approximation scheme numerically with the above example for values \(\epsilon \in \{1, 0.5, 0.25, 0.125, 0.0833\}\) and \(Q\) having a Kumaraswamy distribution. Recall that this distribution has CDF \(1 - (1 - x^a)^b\) for \(x \in [0, 1]\), where \(a, b > 0\). In the numerical example we set \(a = 1\) and \(b = 3\), yielding \(Q \not\in \mathcal{M}\). This non-inclusion arises because \(\sigma(C) < Q(C)\) for all \(C \in \mathcal{C}(\Omega)\) such that \(C = [0, \ell]\) and \(\ell < \frac{3 - \sqrt{5}}{2} \approx 0.382\). Consequently, the \(I\)-projection yields a \(Q\)-density closest to this Kumaraswamy distribution, in the Kullback-Leibler sense, that is selectable with respect to the capacity functional \(\sigma\) of the random set \(Y\). As in the previous examples, we do not have a closed-form expression for the objective function in (3.16), but we approximate it using its sample-analogue version based on \(10^4\) draws from \(Q\) and minimize this approximate objective function instead.

For each \(\epsilon\), we specified an equidistant grid \(0 = d_0 < d_1 < d_2 < \cdots < d_m = 1\) such that \(d_i - d_{i-1} = \{d_{i-1}, d_i\} \leq \epsilon/4\) for \(i = 0, 1, \ldots, m\). Letting \(c_i = (d_i + d_{i-1})/2\) for \(i = 1, \ldots, m\), we implemented the approximation scheme using the following intervals:

\[D_i \in E_i^1 \ \forall i = 1 \ldots m; \ [c_i, c_{i+1}] \in E_i^2 \ \forall i = 1 \ldots, m - 1; \\
[c_i, c_{i+2}] \in E_i^3 \ \forall i = 1 \ldots, m - 2; \ldots; \text{ and } [c_1, c_m] \in E_1^m.\]

Table 3 reports the values of \(\epsilon\) we have considered and their impact on \(n\), the optimal minimal value of the approximating finite programs (3.16), and the execution times of the numerical procedure. Figure 5 reports the approximate \(I\)-projections and their \(Q\)-density weighted versions.

The rapid growth in \(n\) for lower levels of \(\epsilon\) naturally leads to computational challenges by increasing the number of choice variables in the approximating finite program. Notice in Table 3 with \(\epsilon = 0.0833\), the finite
Figure 3: Graphs of the approximate $I$-projection, $p_Q$, and its $Q$-density weighted version, $p_Q dQ$.

| $\epsilon$ | $\tau$ | Optimal Value ($\text{Eq. } 3.16$) | Time |
|----------|-------|-----------------|------|
| $1$      | 66    | $0.9871$        | $0.14s$ |
| $0.5$    | 276   | $0.9916$        | $1.3s$ |
| $0.25$   | 1128  | $0.9897$        | $22.9s$ |
| $0.125^*$| 4560  | $0.9886$        | $86m$ |
| $0.0833^*$| 10296 | $0.9853$        | $27.3h$ |

Table 3: Random Set Example. Notation: "s" denotes "seconds", "m" denotes "minutes", "h" denotes hours, and "*" denotes implementation on a cluster.
Program had 10297 choice variables and required 27.3 hours to complete. To mitigate this computational challenge, the result in Part 2 of Theorem 2 can serve as a guide towards that end. Recall that this result establishes the moment functions corresponding to the binding inequalities, i.e., the set $\overline{B}$, are the only ones that enter the $I$-projection's representation. Consequently, having information on the set $\overline{B}$ can help reduce the computational burden with lower levels of $\epsilon$. The result of Lemma C.6 implies $\overline{B} \subset \{v \in V : \int_\Omega v \, dQ \geq 0\}$, and one implements this information in the approximation scheme by constructing a suitable partition of $\{v \in V : \int_\Omega v \, dQ \geq 0\}$ instead of $V$. We demonstrate this point by repeating the above numerical experiment with the identical setup except that now for each $\epsilon$, we have specified an equidistant grid $0 = d_0 < d_1 < d_2 < \cdots < d_m = \frac{3 - \sqrt{5}}{2}$ as the interval $[0, (3 - \sqrt{5})/2]$ because various subsets of it in $C(\Omega)$ satisfy $\sigma(C) \leq Q(C)$.

The numerical results are very encouraging: we obtain results identical to those in Table 3 and Figure 3 except that now $n$ grows less rapidly, speeding up the numerical optimization procedure with a lower number of choice variables. Table 4 reports the results of this numerical experiment where we have also considered lower values of $\epsilon$ given by $\{0.0625, 0.0313\}$ because of the improved rapid computation. Figure 4 reports the graphs of the $I$-projection and its weighted version under this modification of the proposed approximation scheme, but for $\epsilon \in \{0.0833, 0.0625, 0.0313\}$ to avoid visual clutter, as the graphs for higher values of $\epsilon$ are identical to those in Figure 3. Focusing on the output for $\epsilon = 0.0833$, observe that $n = 1485$, in contrast to $n = 10296$ under the approximation scheme in the first set of experiments. Furthermore, in this case, the execution time is about 1 minute, which is a substantial improvement over 27.3 hours of execution time under the approximation scheme that ignores the additional information. With this gain in computational speed, we also computed the approximate $I$-projection for $\epsilon \in \{0.0625, 0.0313\}$, which also took about 8 minutes and 36 hours to complete, respectively. Of course, for even lower levels of $\epsilon$, one can

| $\epsilon$ | $n$   | Optimal Value | Time   |
|-----------|-------|---------------|--------|
| 1         | 10    | 0.9871        | 0.025s |
| 0.5       | 36    | 0.9916        | 0.1526s|
| 0.25      | 153   | 0.9897        | 0.8681s|
| 0.125     | 666   | 0.9886        | 9.99s  |
| 0.0833    | 1485  | 0.9853        | 62.7s  |
| 0.0625    | 2628  | 0.9842        | 486s   |
| 0.0313*   | 10731 | 0.9826        | 35.6h  |

Table 4: Random Set Example with Modified Approximation Scheme. Notation: "s" denotes "seconds", "h" denotes hours, and "*" denotes implementation on a cluster.
expect the resulting growth in $n$ to create computational challenges for this modification of the proposed approximation scheme. It is possible to further improve the modified approximation scheme by using fewer choice variables; however, developing this approach and its generalization goes beyond the intended scope of this paper and is left for future research.

6 Concluding Remarks

This paper develops new existence, dual representation, and approximation results for the $I$-projection in the infinite-dimensional setting for models defined by an infinite number of unconditional moment inequalities, which also nest conditional moment inequality models. Our results hold under mild assumptions, which are straightforward to verify in practice, and hence, apply to a broad spectrum of such models.

A direction of future research is to study the connection between the continuous optimal partial transport problem in the so-called Kantorovich form and its entropic regularized version using our results on the $I$-projection. The approach would be along the lines of arguments in Léonard (2012), who studies the relationship between entropy problems and the classical optimal transport problem. It is based on the idea to interpret the entropy regularized optimal partial transport problem as an $I$-projection of some input Gibbs distribution onto the constraint set defined by the stochastic dominance restrictions. This approach would create a pathway for applying our approximation scheme to approximate solutions of entropic regularized optimal partial transport problems. To the best of our knowledge, the body of literature that tackles approximation of solutions to continuous optimal transport problems is nascent, consisting of a few works whose focus is exclusively on the classical case (i.e., no inequality constraints): Genevay et al. (2016), Seguy et al.
Thus, our approximation scheme would address this gap in that literature.

A second direction of future research is to consider the application of our results to scenarios where only a sample from the reference distribution $Q$ is available. In such scenarios, $Q$ is replaced by the empirical measure. Then the $I$-projection problem described in this paper becomes a semi-infinite program (Shapiro, 2009), because the choice variable is finite-dimensional with dimension being equal to the sample size, and there are infinitely many moment inequality constraints. It can be viewed as a statistical procedure that tilts the empirical measure by an amount that minimizes the Kullback-Leibler divergence subject to the moment inequality constraints. This sample-analogue of the $I$-projection problem is an example of criterion-based statistical procedures known as intentionally biased bootstrap methods, described in Hall and Presnell (1999), which are used as an aid to increase performance in a range statistical problems. The sample-analogue $I$-projection problem can be used to incorporate side information that takes the form of infinitely many unconditional moment inequality restrictions, with applications to hypothesis testing, nonparametric curve estimation, and nonparametric sensitivity analysis, for instance. As our approximation scheme can also be combined with the sample-analogue version of the $I$-projection problem, it creates a pathway for implementing this statistical procedure in practice using off-the-shelf numerical optimization routines.

A third direction of future research is to apply our results to Bayesian inference problems, where one imposes constraints on the posterior distribution of latent variables. This approach is particularly useful in models where incorporating a-priori information about latent variables is complex and intractable. An important development in this literature is Ganchev et al. (2007) who incorporate such constraints within an efficient Expectation Algorithm scheme that can learn tractable graphical models satisfying additional, otherwise intractable constraints. Salient extensions of their work include Ganchev et al. (2010) and Zhu et al. (2014) with applications to supervised learning in structured latent variable models and nonparametric models. To the best of our knowledge, the focus in this literature is on the setup that has a finite number of moment inequality constraints, which can exclude important side information, like shape constraints such as monotonicity, convexity, unimodality, and stochastic orderings among the latent variables. The method of this paper can be used within their framework to incorporate such side information.

7 Acknowledgments

I thank Christopher D. Walker for helpful comments and feedback.

References

Andrews, D. W. and X. Shi (2017). Inference based on many conditional moment inequalities. *Journal of Econometrics* 196(2), 275 – 287.

Andrews, D. W. K. and X. Shi (2013). Inference Based on Conditional Moment Inequalities. *Econometrica* 82.
Andrews, D. W. K. and G. Soares (2010). Inference for Parameters Defined by Moment Inequalities using Generalized Moment Selection. *Econometrica* 78(1), 119–157.

Artstein, Z. (1983). Distributions of random sets and random selections. *Israel Journal of Mathematics* 46, 313–324.

Barrett, G. F. and S. G. Donald (2003). Consistent Tests for Stochastic Dominance. *Econometrica* 71(1), 71–104.

Berger, J. O. (1985). *Statistical Decision Theory and Bayesian Analysis* (2 ed.). Springer.

Bhattacharya, B. and R. Dykstra (1995). A General Duality Approach to I-Projections. *Journal of Statistical Planning and Inference* 47, 203–216.

Bhattacharya, B. (2006, 04). An iterative procedure for general probability measures to obtain I-projections onto intersections of convex sets. *Ann. Statist.* 34(2), 878–902.

Bhattacharya, B. and R. L. Dykstra (1997). On Dykstra’s iterative fitting procedure. *Annals of the Institute of Statistical Mathematics* 49(3), 435–446.

Bickel, P., C. A. J. Klaassen, Y. Ritov, and J. A. Wellner (1993). *Efficient and Adaptive Estimation for Semiparametric Models*. Johns Hopkins Univ. Press.

Bogachev, V. I. (2007). *Measure Theory: Volume 1*. Springer.

Canay, I. A. (2010). EL Inference for Partially Identified Models: Large Deviations Optimality and Bootstrap Validity. *Journal of Econometrics* 156(2), 408–425.

Chi, J., B. Wang, H. Chen, L. Zhang, X. Li, and J. Ouyang (2021). Approximate continuous optimal transport with copulas. *International Journal of Intelligent Systems*.

Clason, C., D. A. Lorenz, H. Mahler, and B. Wirth (2021). Entropic regularization of continuous optimal transport problems. *Journal of Mathematical Analysis and Applications* 494(1), 124432.

Csiszár, I. (1975, 02). I-divergence geometry of probability distributions and minimization problems. *Ann. Probab.* 3(1), 146–158.

Csiszár, I. (1984). Sanov Property, Generalized I-Projection and a Conditional Limit Theorem. *The Annals of Probability* 12, 768–793.

Csiszár, I. and F. Matus (2003). Information projections revisited. *IEEE Transactions on Information Theory* 49(6), 1474–1490.

Davidson, R. and J.-Y. Duclos (2013). Testing for Restricted Stochastic Dominance. *Econometric Reviews* 32(1), 84–125.

Dhillon, I. S., S. Mallela, and R. Kumar (2003). A divisive information-theoretic feature clustering algorithm for text classification. *Journal of Machine Learning Research* 3, 1256–1287.

Du, P. J., C. Parmeter, and J. Racine (2013). Nonparametric Kernel Regression with Multiple Predictors and Multiple Shape Constraints. *Statistica Sinica* 23, 1347–1371.

Dunford, N. and J. T. Schwartz (1958). *Linear Operators. Part I: General Theory*, Volume VII of *Pure and Applied Mathematics*. Interscience Publishers, INC., New York.

Dykstra, R. L. (1985). An Iterative Procedure for Obtaining I-Projections onto the Intersection of Convex Sets. *The Annals of Probability* 13(3), 975 – 984.

Ellis, R. S. (1999). The theory of large deviations: from boltzmann’s 1877 calculation to equilibrium
macrostates in 2d turbulence. *Physica D: Nonlinear Phenomena* **133**, 106–136.

Figalli, A. (2010). The optimal partial transport problem. *Archive for Rational Mechanics and Analysis* **195**, 533–560.

Folland, G. B. (1999). *Real Analysis: Modern Techniques and their Applications* (2nd ed.). Wiley InterScience.

Foster, J. and A. Shorrocks (1988). Poverty Orderings. *Econometrica* **56**(1), 173–177.

Frigyik, B. A., S. Srivastava, and M. R. Gupta (2008). Functional bregman divergence and bayesian estimation of distributions. *IEEE Transactions on Information Theory* **54**(11), 5130–5139.

Ganchev, K., J. Graca, J. Gillenwater, and B. Taskar (2010). Posterior regularization for structured latent variable models. *Journal of Machine Learning Research* **11**, 2001 – 2049.

Ganchev, K., B. Taskar, and J. Graca (2007). Expectation maximization and posterior constraints. In J. Platt, D. Koller, Y. Singer, and S. Roweis (Eds.), *Advances in Neural Information Processing Systems*, Volume 20. Curran Associates, Inc.

Gelfand, I. M. (1936). Sur un lemme de la theorie des espaces lineaires. *Communications de l’Institut des Sciences Mathématiques et mécaniques de l’Université de Kharkoff et la Société Mathématique de Kharkoff* **13**, 35–40.

Genevay, A., M. Cuturi, G. Peyre, and F. Bach (2016). Stochastic optimization for large-scale optimal transport. *Neural Information Processing Systems*, 3440–3448.

Granziol, D., B. Ru, S. Zohren, X. Dong, M. Osborne, and S. Roberts (2019). Meme: An accurate maximum entropy method for efficient approximations in large-scale machine learning. *Entropy* **21**(6).

Grimmet, G. and D. Stirzaker (2001). *Probability and Random Processes* (3 ed.). Oxford University Press.

Grünwald, P. D. and A. P. Dawid (2004). Game theory, maximum entropy, minimum discrepancy and robust Bayesian decision theory. *The Annals of Statistics* **32**(4), 1367 – 1433.

Haberman, S. J. (1984, 09). Adjustment by minimum discriminant information. *Ann. Statist.* **12**(3), 971–988.

Hadar, J. and W. R. Russell (1969). Rules for Ordering Uncertain Prospects. *The American Economic Review* **59**(1), 25–34.

Hall, P. and B. Presnell (1999). Intentionally biased bootstrap methods. *Journal of the Royal Statistical Society. Series B (Statistical Methodology)* **61**(1), pp. 143–158.

Hanche-Olsen, H. and H. Holden (2010). The Kolmogorov-Riesz compactness theorem. *Expositiones Mathematicae* **28**(4), 385 – 394.

Harris, T. R. and H. P. Mapp (1986). A stochastic dominance comparison of water-conserving irrigation strategies. *American Journal of Agricultural Economics* **68**(2), 298–305.

Imbens, G. W., R. H. Spady, and P. Johnson (1998). Information theoretic approaches to inference in moment condition models. *Econometrica* **66**(2), 333–357.

Jaynes, E. T. (1957, May). Information theory and statistical mechanics. *Phys. Rev.* **106**, 620–630.

Kandasamy, K., A. Krishnamurthy, B. Poczos, L. Wasserman, and J. M. Robins (2015). Nonparametric von mises estimators for entropies, divergences and mutual informations. In C. Cortes, N. Lawrence, D. Lee, M. Sugiyama, and R. Garnett (Eds.), *Advances in Neural Information Processing Systems*, Volume 28,
Kitamura, Y. and M. Stutzer (1997). An information-theoretic alternative to generalized method of moments estimation. *Econometrica* 65(4), 861–874.

Komunjer, I. and G. Ragusa (2016). Existence and characterization of conditional density projections. *Econometric Theory* 32(4), 947–987.

Kortanek, K. O. (1993, Oct). Semi-infinite programming duality for order restricted statistical inference models. *Zeitschrift für Operations Research* 37(3), 285–301.

Kullback, S. (1959). *Information Theory and Statistics*. New York, U.S.A: Wiley.

Kullback, S. and R. A. Leibler (1951). On Information and Sufficiency. *The Annals of Mathematical Statistics* 22(1), 79 – 86.

Lee, S., K. Song, and Y.-J. Whang (2013). Testing functional inequalities. *Journal of Econometrics* 172(1), 14–32.

Léonard (2012). From the Schrödinger problem to the Monge-Kantorovich problem. *Journal of Functional Analysis* 262(4), 1879–1920.

Linton, O., K. Song, and Y.-J. Whang (2010). An Improved Bootstrap Test for Stochastic Dominance. *Journal Of Econometrics* 154, 186–202.

Lok, T. M. and R. V. Tabri (2021). An improved bootstrap test for restricted stochastic dominance. *Journal of Econometrics*.

Luenberger, D. (1969). *Optimization by Vector Space Methods*. Series in Decision and Control. John Wiley & Sons, Inc.

Manski, C. F. (2005). Partial identification with missing data: concepts and findings. *International Journal of Approximate Reasoning* 39(2), 151 – 165. Imprecise Probabilities and Their Applications.

Molchanov, I. and F. Molinari (2018). *Random Sets in Econometrics*. Econometric Society Monographs. Cambridge University Press.

Pavon, M., G. Trigila, and E. G. Tabak (2021). The data-driven schrödinger bridge. *Communications on Pure and Applied Mathematics* 74(7), 1545–1573.

Pettis, B. J. (1938). On integration in vector spaces. *Transactions of The American Mathematical Society* 44, 70–74.

Pham, H. (2010). Large deviations in mathematical finance *.

Pollard, D. (1990). Empirical processes: Theory and applications. *NSF-CBMS Regional Conference Series in Probability and Statistics* 2, i–86.

Rao, C. R. (1973). *Linear Statistical Inference and its Applications* (2 ed.). Wiley Series in Probability and Statistics. John Wiley & Sons, Inc.

Rosenblatt, M. (1952). Remarks on a multivariate transformation. *The Annals of Mathematical Statistics* 23(3), 470–472.

Rudin, W. (1991). *Functional Analysis* (2 ed.). International Series in Pure and Applied Mathematics. McGraw-Hill.

Ruschendorf, L. (1995). Convergence of the Iterative Proportional Fitting Procedure. *The Annals of Statistics* 23(4), 1160 – 1174.
Rüschendorf, L. and W. Thomsen (1993). Note on the schrödinger equation and i-projections. *Statistics & Probability Letters* 17(5), 369–375.

Sanov, I. N. (1957). On The Probability of Large Deviation of Random Variables. *Matematiceskii Sbornik* 42, 11–44.

Schennach, S. M. (2007). Point estimation with exponentially tilted empirical likelihood. *Annals of Statistics* 35(2), 634–672.

Seguy, V., B. B. Damodaran, R. Flamy, N. Courty, A. Rolet, and M. Blondel (2018). Large-scale optimal transport and mapping estimation. In *International Conference on Learning Representations*.

Shapiro, A. (2009). Semi-Infinite Programming, Duality, Discretization, and Optimality Conditions. *Optimization* 8(2), 133–161.

Sheehy, A. (1988). Kulback-Leibler Constrained Estimation of Probability Measures. Technical Report 132, University of Washington, Department of Statistics.

Shi, X. (2015). Model selection tests for moment inequality models. *Journal of Econometrics* 187, 1–17.

Shi, X. and Y.-C. Hsu (2017). Model-selection tests for conditional moment restriction models. *Econometrics Journal* 20, 52–85.

Stinchcombe, M. B. and H. White (1998). Consistent specification testing with nuisance parameters present only under the alternative. *Econometric Theory* 14(3), 295–325.

Sueishi, N. (2013). Identification problem of the exponential tilting estimator under misspecification. *Economics Letters* 118(3), 509 – 511.

Topsoe, F. (2002). Maximum entropy versus minimum risk and applications to some classical discrete distributions. *IEEE Transactions on Information Theory* 48(8), 2368–2376.

van der Vaart, A. W. and J. Wellner (1996). *Weak Convergence and Empirical Processes* (First ed.). Springer Series in Statistics. Springer.

Verteramo Chiu, L., L. Tauer, Y. Gröhn, and R. Smith (2020). Ranking disease control strategies with stochastic outcomes. *Preventive Veterinary Medicine* 176, 104906.

Vuong, Q. H. (1989). Likelihood ratio tests for model selection and non-nested hypotheses. *Econometrica* 57(2), 307–333.

Wayne Patty, C. (1993). *Foundations of Topology*. Wavelan Press Inc.

Winkler, W. E. (1990). On dykstra’s iterative fitting procedure. *The Annals of Probability* 18(3), 1410–1415.

Zhu, J., N. Chen, and E. P. Xing (2014). Posterior regularization for structured latent variable models. *Journal of Machine Learning Research* 15, 1799 – 1847.

A Outline

In this appendix, we provide all the proofs. It is organized as follows.

- Appendix B presents the proofs of the results in the main text. These results are Theorems 1 to 4 and Propositions 1 and 2.

- Appendix C presents technical lemmas that are used in the proofs of the results in the main text.
B Proofs of Results

B.1 Theorem 1

Proof. The proof proceeds by the direct method. Lemma C.1 shows $\mathcal{M}$ is variation-closed and convex. Now the condition $S(Q, +\infty) \cap \mathcal{M} \neq \emptyset$ implies that we can apply Theorem 2.1 of Csiszár (1975) to deduce the existence and uniqueness of $P_Q$. ■

B.2 Theorem 2

Proof. Part 1. The proof proceeds by the direct method. To show under our assumptions that $\arg \inf \{ \int_\Omega e^y dQ : y \in \mathcal{D} \} \neq \emptyset$ (i.e., existence of a minimizer), holds, it is sufficient to establish that the map $y \mapsto \int_\Omega e^y dQ$ is lower semi-continuous and coercive on $\mathcal{D}$, with respect to the $L_1(Q)$-norm. The uniqueness of the minimizer (up to equivalence class, i.e., a.s.-$Q$) is implied by the strict convexity of this map on $\mathcal{D}$, which follows from the strict convexity of the exponential function and Part (i) of Assumption [1].

Toward that end, we first establish the lower semi-continuity of the map $y \mapsto \int_\Omega e^y dQ$. Suppose that $\{y_n\}_{n \geq 1} \subset \mathcal{D}$ such that $y_n \overset{L_1(Q)}{\longrightarrow} y$, then $\exists \{n_k\}_{k \geq 1}$ such that $\lim_{k \to +\infty} \int_\Omega e^{y_{n_k}} dQ = \liminf_{n \to +\infty} \int_\Omega e^{y_n} dQ$. Now we can take a further subsequence $\{n_{k\ell}\}_{\ell \geq 1}$ such that $\lim_{\ell \to +\infty} y_{n_{k\ell}}(\omega) = y(\omega)$ a.s.-$Q$. This yields

$$\int_\Omega e^y dQ = \int_\Omega \lim_{\ell \to +\infty} e^{y_{n_{k\ell}}} dQ \leq \liminf_{\ell \to +\infty} \int_\Omega e^{y_{n_{k\ell}}} dQ \quad \text{(by Fatou's Lemma)}$$

$$= \lim_{k \to +\infty} \int_\Omega e^{y_{n_k}} dQ \quad \text{(as } \{n_{k\ell}\}_{\ell \geq 1} \subset \{n_k\}_{k \geq 1}\text{)}$$

$$= \liminf_{n \to +\infty} \int_\Omega e^{y_n} dQ,$$

establishing the desired result.

We now establish the map $y \mapsto \int_\Omega e^y dQ$ is coercive with respect to the strong norm on $L_1(Q)$ on $\mathcal{D}$. This map being coercive means it satisfies the following property:

$$\int_\Omega e^y dQ \to +\infty \quad \text{as } \|y\|_{L_1(Q)} \to +\infty,$$  \hspace{1cm} (B.1)

on the set $\mathcal{D}$. The combination of the restriction of $y \in \mathcal{D}$ and Part (ii) of Assumption [1] yields the property (B.1). In particular, $y \in \mathcal{D}$ implies that $y = \alpha \cdot y'$ for some $y' \in \overline{\mathcal{C}(\mathcal{V})}$ and $\alpha \geq 0$, and Part (ii) of Assumption [1] implies

$$\|y'\|_{L_1(Q)} \leq K + \sup_{\gamma \in \Gamma} \int_\Omega |f_\gamma|^{1+\delta} dQ < \infty \quad \forall y' \in \overline{\mathcal{C}(\mathcal{V})},$$
which is a uniform bound. Consequently, \( \|y\|_{L_1(Q)} \to +\infty \) only when \( \alpha \to +\infty \), for \( y \in \mathcal{D} \). To connect this structure to the aforementioned map, we use the tail sum representation of the integral \( \int_{\Omega} e^y dQ \):

\[
\int_{\Omega} e^y dQ = \int_0^{+\infty} Q(\omega \in \Omega : e^y > t) \, dt = \int_0^{+\infty} Q(\omega \in \Omega : y' > \frac{\ln t}{\alpha}) \, dt \\
\geq \int_1^{+\infty} Q(\omega \in \Omega : y' > \frac{\ln t}{\alpha}) \, dt. \quad \text{(B.2)}
\]

Then the desired result follows upon showing that the right side of (B.2) diverges to \(+\infty\) as \( \alpha \to +\infty \), and that this is true for any \( y' \in \overline{\mathcal{D}}(\mathcal{V}) \). Lemma C.2 establishes this technical result using Part (iii) of Assumption 1.

As Part (i) of Assumption 1 implies that \( \mathcal{D} \neq \emptyset \), the arguments above establish that the set of minimisers is nonempty, i.e., \( \arg \inf \{ \int_{\Omega} e^y dQ : y \in \mathcal{D} \} \neq \emptyset \), holds. Now we shall establish the uniqueness a.s.-\( Q \) of the minimizer. Part (i) of Assumption 1 implies that the minimizers cannot be where the objective function equals \(+\infty\). Combining this implication with the strict convexity of the map \( y \mapsto \int_{\Omega} e^y dQ \) on \( \mathcal{D} \), implies that there is a unique minimizer (up to equivalence class). Let \( \beta \in (0, 1) \) and \( y_1, y_2 \in \mathcal{D} \) such that \( y_1 \neq y_2 \) holds as equivalence classes. Additionally, let \( y_3 = \beta y_1 + (1 - \beta) y_2 \). Then

\[
\int_{\Omega} e^{y_3} dQ < \beta \int_{\Omega} e^{y_1} dQ + (1 - \beta) \int_{\Omega} e^{y_2} dQ,
\]

holds, by the strict convexity of the exponential function. This establishes the strict convexity of the map, and hence, the uniqueness of the set of minimizers, i.e., \( \arg \inf \{ \int_{\Omega} e^y dQ : y \in \mathcal{D} \} \), up to equivalence class.

Next, we develop the representation of the \( I \)-projection’s \( Q \)-density. From the above arguments let \( y_0 = \arg \inf \{ \int_{\Omega} e^y dQ : y \in \mathcal{D} \} \). The set \( \mathcal{D} \) is convex, and the objective function \( g(y) = \int_{\Omega} e^y dQ \) is Gâteaux differentiable, then by Theorem 2 on page 178 of Luenberger (1969),

\[
\frac{d}{dt} g(y_0 + t(y - y_0)) \bigg|_{t=0} \geq 0 \quad \forall y \in \mathcal{D},
\]

yielding

\[
\int_{\Omega} (y - y_0) e^{y_0} dQ \geq 0 \quad \forall y \in \mathcal{D}.
\]

By choosing \( y = cy_0 \) first with \( c > 1 \) and then with \( c < 1 \) (since \( \mathcal{D} \) is also a cone), we obtain

\[
\int_{\Omega} y_0 e^{y_0} dQ = 0, \quad \text{and} \quad \int_{\Omega} y e^{y_0} dQ \geq 0 \quad \forall y \in \mathcal{D}. \quad \text{(B.3)}
\]

Let \( x_0 = \frac{e^{y_0}}{\int_{\Omega} e^{y_0} dQ} \), and note that the second part of (B.3) implies that

\[
\int_{\Omega} v x_0 dQ \geq 0 \quad \forall v \in \mathcal{V}; \quad \text{(B.4)}
\]
hence \( x_0 \in \mathcal{M} \). Furthermore,

\[
m(x_0) + \log \left( \int_{\Omega} e^{y_0} \, dQ \right) = \int_{\Omega} y_0 e^{y_0} \, dQ = 0,
\]

(B.5)

holds, by the first part of (B.3). Hence, by Theorem 2.2 of Bhattacharya and Dykstra (1995), \( x_0 \) solves the \( L \)-projection problem.

**Part 2.** The proof proceeds by the direct method. We need to establish that \( y_0 \in \operatorname{span}_+(B) \) holds. Lemma C.3 allows us to deduce that for any \( y \in D \), either (i) \( \exists n \in \mathbb{Z}_+ \text{, } \alpha_0 > 0 \text{, } p_i > 0 \text{ for each } i = 1, \ldots, n \text{ such that } \sum_{i=1}^{n} p_i = 1 \), for which \( y_0 = \alpha_0 \sum_{i=1}^{n} p_i v_i \), and \( \{v_1, \ldots, v_n\} \subset \operatorname{ex}(\overline{\mathcal{V}}) \subset \overline{\mathcal{V}} \), or (ii) that \( y \) is a limit point, in the \( L_1(Q) \)-norm, of the linear combinations described in (i). Consequently, for case (i),

\[
\int_{\Omega} y_0 e^{y_0} \, dQ = 0 \iff \int_{\Omega} y_0 \, dP_Q = 0 \iff \sum_{i=1}^{n} \beta_i \int_{\Omega} v_i \, dP_Q = 0.
\]

(B.6)

Since \( \int_{\Omega} v_i \, dP_Q \geq 0 \) for each \( i \) by the second part of (B.3) for \( v_i \in \mathcal{V} \), which also holds for \( v_i \) that are limit points of \( \mathcal{V} \), the equality in the last part of (B.6) forces \( \int_{\Omega} v_i \, dP_Q = 0 \) for each \( i \). Hence, \( y_0 \) is an element of \( \operatorname{span}_+(B) \subset \operatorname{span}(B) \).

Next, consider the case \( y_0 \) is a limit point of \( D \). Thus,

\[
\exists \{y_n\}_{n \geq 1} \subset D \text{ such that } y_n \xrightarrow{L_1(Q)} y_0.
\]

(B.7)

Since \( y_n \xrightarrow{L_1(Q)} y_0 \implies y_n \xrightarrow{Q} y_0 \text{, and } P_Q \ll Q \text{ holds}, we must have that } y_n \xrightarrow{P_Q} y_0 \text{, holds. Now because } \\
\int_{\Omega} |y_n|^{1+\delta} p_Q \, dQ \leq K \forall n \text{, the sequence } \{y_n\}_{n \geq 1} \text{ is uniformly integrable. In consequence,}

\[
\lim_{n \to +\infty} \int_{\Omega} y_n p_Q \, dQ = \int_{\Omega} y_0 p_Q \, dQ.
\]

(B.8)

Since for each \( n \), \( y_n \) has the form \( y_n = \sum_{i=1}^{m_n} a_{n_i} v_{n_i} \) for some \( m_n \in \mathbb{Z}_+, a_{n_i} > 0 \text{ for each } i \text{ and } \{v_{n_1}, \ldots, v_{n_{m_n}}\} \subset \mathcal{V} \), the limit (B.8) and the first part of (B.3) implies

\[
\lim_{n \to +\infty} \sum_{i=1}^{m_n} a_{n_i} \int_{\Omega} v_{n_i} \, dP_Q = 0.
\]

(B.9)

As \( \int_{\Omega} v_{n_i} \, dP_Q \geq 0 \text{ for all } n_i \text{ by the second part of (B.3), and } \lim_{n \to +\infty} \sum_{i=1}^{m_n} a_{n_i} > 0 \text{ (because } Q \not\in \mathcal{M} \text{), the limit (B.9) implies that }

\[
\lim_{n \to +\infty} \int_{\Omega} v_{n_i} \, dP_Q = 0 \text{ } \forall i.
\]

Hence, \( y_0 \) is an element in the boundary of \( \operatorname{span}_+(B) \) – i.e., \( y_0 \in \partial(\operatorname{span}_+(B)) \). Finally, since \( \partial(\operatorname{span}_+(B)) \subset \operatorname{span}_+(B) \), we are done. \( \blacksquare \)
B.3 Theorem 3

Proof. The proof proceeds by the direct method. It follows steps similar to that in the proof of Part 1 of Theorem 1. The goal is to establish under our assumptions that \( \arg \inf \left\{ \int_{\Omega} e^{f(v)d\xi(v)} \, dQ : \xi \in \mathcal{Y} \right\} \neq \emptyset \) (i.e., existence of a minimizer), holds. As Part (i) of Assumption 1 implies that \( \mathcal{Y} \neq \emptyset \), it is sufficient to establish that the map \( \xi \mapsto \int_{\Omega} e^{f(v)d\xi(v)} \, dQ \) is lower semi-continuous and coercive with respect to the weak-star topology of \( C(\mathcal{V})^* \) on \( \mathcal{Y} \).

Toward that end, we first prove that this map is weak-star continuous on \( \mathcal{Y} \). Let \( \{\xi_n\}_{n \geq 1} \subset \mathcal{Y} \) be such that \( \xi_n \xrightarrow{w^*} \xi \), where \( \xi \in \mathcal{Y} \). Using the fact that the map \( \xi \mapsto \int_{\mathcal{V}} v \, d\xi \) is continuous when \( C(\mathcal{V})^* \) is given the weak-star topology and \( L_1(Q) \) has the weak topology, it follows that \( \int_{\mathcal{V}} v \, d\xi_n \xrightarrow{w} \int_{\mathcal{V}} v \, d\xi \), i.e., weak convergence of \( \{\int_{\mathcal{V}} v \, d\xi_n\}_{n \geq 1} \) in \( L_1(Q) \) to \( \int_{\mathcal{V}} v \, d\xi \). Consequently, \( \{\int_{\mathcal{V}} v \, d\xi_n\}_{n \geq 1} \) converges in distribution to \( \int_{\mathcal{V}} v \, d\xi \). Since the exponential function is continuous on \( \mathbb{R} \), it follows that \( \{e^{\int_{\mathcal{V}} v \, d\xi_n}\}_{n \geq 1} \) converges in distribution to \( e^{\int_{\mathcal{V}} v \, d\xi} \). Now, by the Skorohod Representation Theorem (e.g., Theorem 7.2.14, Grimmett and Stirzaker, 2001), there exists a probability space \( (\Omega', \mathcal{F}', Q') \) and random variables \( \{y_n\}_{n \geq 1} \) and \( y_0 \), mapping \( \Omega' \) into \( \mathbb{R} \), such that:

(a) \( \{y_n\}_{n \geq 1} \) and \( y_0 \) have the same distributions as \( \{e^{\int_{\mathcal{V}} v \, d\xi_n}\}_{n \geq 1} \) and \( e^{\int_{\mathcal{V}} v \, d\xi} \), respectively,

(b) \( \lim_{n \to +\infty} y_n(\omega') = y_0(\omega') \) a.s.-\( \mathcal{F}' \).

Consequently, \( \int_{\Omega'} y_n \, dQ' \to \int_{\Omega'} y_0 \, dQ' \) as \( n \to +\infty \). However,

\[
\int_{\Omega'} y_n \, dQ' = \int_{\Omega} e^{\int_{\mathcal{V}} v \, d\xi_n} \, dQ \quad \forall n \quad \text{and} \quad \int_{\Omega} y_0 \, dQ' = \int_{\Omega} e^{\int_{\mathcal{V}} v \, d\xi_0} \, dQ.
\]

Therefore,

\[
\lim_{n \to +\infty} \int_{\Omega} e^{\int_{\mathcal{V}} v \, d\xi_n} \, dQ = \int_{\Omega} e^{\int_{\mathcal{V}} v \, d\xi_0} \, dQ,
\]

thus showing the map \( \xi \mapsto \int_{\Omega} e^{\int_{\mathcal{V}} v \, d\xi(v)} \, dQ \) is continuous with respect to the weak-star topology. This implies that it must also be lower semi-continuous in the same topology.

Next, we show the objective function is weak-star coercive on \( \mathcal{Y} \). This means establishing

\[
\int_{\Omega} e^{\int_{\mathcal{V}} v \, d\xi(v)} \, dQ \to +\infty \quad \text{as} \quad \int_{\mathcal{V}} g(v) \, d\xi(v) \, dQ \to \{+\infty, -\infty\} \quad \forall g \in C(\mathcal{V}),
\]

where the latter divergence restricts \( \xi \in \mathcal{Y} \). Note that \( \int_{\mathcal{V}} \alpha g(v) \, d\mu(v) = \alpha \int_{\mathcal{V}} g(v) \, d\mu(v) \) holds for every \( \xi \in \mathcal{Y} \), and

\[
\left| \int_{\mathcal{V}} g(v) \, d\mu(v) \right| \leq \int_{\mathcal{V}} |g(v)| \, d\mu(v) \leq \sup_{v \in \mathcal{V}} |g(v)| < +\infty
\]

(B.13)
since $\mathcal{V}$ is compact by Assumption 2 and $\mu \in \mathcal{P}$. Therefore, we only have to consider divergences such that
\[ \int_{\mathcal{V}} g(v) d\xi(v) dQ \to +\infty \quad \forall g \in C(\mathcal{V}) \]
because $\alpha \geq 0$, and such divergence arises only when $\alpha \to +\infty$. To that end, consider the tail sum representation of the objective function:
\[ \int_{\Omega} e^{\int_{\mathcal{V}} v d\xi(v)} dQ = \int_{0}^{+\infty} Q(\omega \in \Omega : e^{\int_{\mathcal{V}} v d\xi(v)} > t) \, dt \]
\[ = \int_{0}^{+\infty} Q(\omega \in \Omega : \int_{\mathcal{V}} v d\mu(v) > \frac{\ln t}{\alpha}) \, dt \]
\[ \geq \int_{1}^{+\infty} Q(\omega \in \Omega : \int_{\mathcal{V}} v d\mu(v) > \frac{\ln t}{\alpha}) \, dt. \]

Now from the characterization (3.10), $\exists y' \in \mathcal{V}$ such that $y' = \int_{\mathcal{V}} v d\mu(v)$, so that
\[ \int_{1}^{+\infty} Q(\omega \in \Omega : \int_{\mathcal{V}} v d\mu(v) > \frac{\ln t}{\alpha}) \, dt = \int_{1}^{+\infty} Q(\omega \in \Omega : y' > \frac{\ln t}{\alpha}) \, dt, \]
and Lemma C.2 establishes that $\int_{1}^{+\infty} Q(\omega \in \Omega : y' > \frac{\ln t}{\alpha}) \, dt \to +\infty$ as $\alpha \to +\infty$. This establishes (B.12).

Now we shall establish $\arg \inf \{ \int_{\Omega} e^{\int_{\mathcal{V}} v d\xi(v)} dQ : \xi \in \mathcal{Y} \} = \{ \xi \in \mathcal{Y} : y_0 = \int_{\mathcal{V}} v d\xi(v) \}$. We first prove by contradiction the direction:
\[ \xi' \in \arg \inf \{ \int_{\Omega} e^{\int_{\mathcal{V}} v d\xi(v)} dQ : \xi \in \mathcal{Y} \} \implies \xi' \in \{ \xi \in \mathcal{Y} : y_0 = \int_{\mathcal{V}} v d\xi(v) \}. \]
Suppose that $\xi' \in \arg \inf \{ \int_{\Omega} e^{\int_{\mathcal{V}} v d\xi(v)} dQ : \xi \in \mathcal{Y} \}$ and $\xi' \notin \{ \xi \in \mathcal{Y} : y_0 = \int_{\mathcal{V}} v d\xi(v) \}$. Then, the first and second inclusions imply that
\[ \int_{\Omega} e^{\int_{\mathcal{V}} v d\xi'(v)} dQ \leq \int_{\Omega} e^{y_0} dQ \iff \int_{\Omega} e^{y'} dQ \leq \int_{\Omega} e^{y_0} dQ \quad \text{where} \quad y' = \int_{\mathcal{V}} v d\xi'(v). \quad \text{(B.14)} \]
As Theorem 2 establishes that $y_0$ is the unique minimizer of (3.6), the inequality (B.18) implies that $y' = y_0$; therefore $\xi' \notin \{ \xi \in \mathcal{Y} : y_0 = \int_{\mathcal{V}} v d\xi(v) \}$ and $\xi' \in \{ \xi \in \mathcal{Y} : y_0 = \int_{\mathcal{V}} v d\xi(v) \}$, which is a contradiction.

Now we prove by contradiction the direction
\[ \xi' \in \{ \xi \in \mathcal{Y} : y_0 = \int_{\mathcal{V}} v d\xi(v) \} \implies \xi' \in \arg \inf \{ \int_{\Omega} e^{\int_{\mathcal{V}} v d\xi(v)} dQ : \xi \in \mathcal{Y} \}. \]
Suppose that $\xi' \notin \{ \xi \in \mathcal{Y} : y_0 = \int_{\mathcal{V}} v d\xi(v) \}$ and $\xi' \notin \arg \inf \{ \int_{\Omega} e^{\int_{\mathcal{V}} v d\xi(v)} dQ : \xi \in \mathcal{Y} \}$. Then these inclusions imply that
\[ \int_{\Omega} e^{y_0} dQ = \int_{\Omega} e^{\int_{\mathcal{V}} v d\xi'(v)} dQ > \int_{\Omega} e^{\int_{\mathcal{V}} v d\xi_1(v)} dQ \]

43
for some $\xi_1 \in \arg \inf \left\{ \int_{\Omega} e^{\alpha y v} d\xi \cdot\xi \in \mathcal{Y} \right\}$. By (3.11), $\exists y_1 \in D$ such that $y_1 = \int_{\Omega} y d\xi_1$, and hence, the above strict inequality becomes

$$
\int_{\Omega} e^{y_0} dQ = \int_{\Omega} e^{\alpha y v(v)} dQ > \int_{\Omega} e^{y_1} dQ.
$$

(B.15)

The inequality (B.15) yields a contradiction as Theorem 2 implies that $\int_{\Omega} e^{y_0} dQ < \int_{\Omega} e^{y_1} dQ$ must also be true. This concludes the proof.

B.4 Theorem 4

Proof. We first prove the existence result (3.17):

$$
\arg \inf \left\{ \int_{\Omega} \alpha^{\sum_{i=1}^{n_m} \mu_i v_i} dQ : \alpha, \mu_i \geq 0 \text{ } \forall i, \text{ and } \sum_{i=1}^{n_m} \mu_i = 1 \right\} \neq \emptyset,
$$

for any $v_i \in E_{i,m}$ where $i = 1, \ldots, n_m$. The proof proceeds by the direct method. We must consider two cases: where the partition is such that the optimal value (3.16) is either infinite or finite. In the former case, existence holds trivially since it implies that every element in the domain is a solution. It is the latter case which requires further analysis. In that case, under the assumptions of the theorem, it is sufficient to establish the map $\xi_m \mapsto \int_{\Omega} e^{\alpha \sum_{i=1}^{n_m} \mu_i v_i} dQ$ is lower semi-continuous and coercive on its domain of definition, with respect to the Euclidean norm on $\mathbb{R}^{n_m+1}$. The steps for establishing these properties of the map are similar to their counterparts in the proofs of Theorems 2 and 3, but with the appropriate adjustment based on using the Euclidean norm on $\mathbb{R}^{n_m+1}$ instead of the other norms used in the preceding theorems. We omit the details for brevity.

Now, we prove the two parts of the theorem.

Part 1. The proof proceeds by the direct method. Given $m$,

$$
\int_{\Omega} e^{\alpha y v} d\xi_0(v) dQ \leq \int_{\Omega} e^{\alpha_{n_m} \sum_{i=1}^{n_m} \mu_i v_i} dQ \leq \int_{\Omega} e^{\alpha_0 \sum_{i=1}^{n_m} \mu_0(E_{i,m}) v_i} dQ,
$$

(B.16)

holds, for any choice of $v_i \in E_{i,m}$ for each $i$, where $\xi_0 \in \arg \inf \left\{ \int_{\Omega} e^{\alpha y v} d\xi : \xi \in \mathcal{Y} \right\}$. The first inequality holds because $\xi_m = \alpha_{n_m} \sum_{i=1}^{n_m} \mu_i v_i$ is a discrete measure on $v_1, \ldots, v_n$. The second inequality is due to $\alpha_{n_m}, \mu_1, \mu_2, \ldots, \mu_{n_m, n_m}$ being a solution of (3.16), as $\alpha_0, \mu_0(E_{1,nm}), \ldots, \mu_0(E_{n_m,n_m})$ are in the domain of the finite program (3.16). We can apply Lemma C.4 to the sequence $\{\alpha_0 \sum_{i=1}^{n_m} \mu_0(E_{i,m}) v_i\}_{m \geq 1}$ where $y_0 = \int_{\Omega} y d\xi_0$. Note that $\{\alpha_0 \sum_{i=1}^{n_m} \mu_0(E_{i,m}) v_i\}_{m \geq 1} \subset C(\mathcal{V}) \cdot \alpha_0$. The reason is that for each $m$, $\sum_{i=1}^{n_m} \mu_0(E_{i,m}) v_i$ is a solution of (3.16) holds because $\{v_1, \ldots, v_{n_m}\} \subset \mathcal{V}$ and $\sum_{i=1}^{n_m} \mu_0(E_{i,m}) = 1$.

Hence, applying Lemma C.4 to the sequence $\{\alpha_0 \sum_{i=1}^{n_m} \mu_0(E_{i,m}) v_i\}_{m \geq 1}$ yields

$$
\lim_{m \to +\infty} \int_{\Omega} e^{\alpha_0 \sum_{i=1}^{n_m} \mu_0(E_{i,m}) v_i} dQ = \int_{\Omega} e^{\alpha y v} d\xi_0(v) dQ.
$$

(B.17)
Therefore,

\[
\lim_{m \to +\infty} \int_{\Omega} e^{\alpha_{nm} \sum_{i=1}^{m} \mu_{i,n} v_i} dQ = \int_{\Omega} e^{L \int_{\Omega} v d\xi_0(v)} dQ
\]  

(B.18)

holds, by the inequalities in (B.16) and the limit (B.17). This concludes the proof of the first part of the theorem.

**Part 2.** The proof proceeds by the direct method. Let \( \xi^* \) be an accumulation point of \( \{\xi_m\}_{m \geq 1} \) in the weak star topology of \( C(\overline{V})^* \). Therefore, there exists a subsequence \( \{m_\ell\}_{\ell \geq 1} \) such that \( \xi_{m_\ell} \overset{w^*}{\to} \xi^* \). First, recall that

\[
\xi_{m_\ell} \overset{w^*}{\to} \xi^* \iff \int_{\overline{V}} g \, d\xi_{m_\ell} \to \int_{\overline{V}} g \, d\xi^* \quad \forall g \in C(\overline{V}).
\]

(B.19)

Using the fact that the map \( \xi \mapsto \int_{\overline{V}} v \, d\xi \) is continuous when \( C(\overline{V})^* \) is given the weak-star topology and \( L_1(Q) \) has the weak topology, it follows that \( \int_{\overline{V}} v \, d\xi_{m_\ell} \overset{w}{\to} \int_{\overline{V}} v \, d\xi^* \), i.e., weak convergence of \( \{\int_{\overline{V}} v \, d\xi_{m_\ell}\}_{\ell \geq 1} \) in \( L_1(Q) \) to \( \int_{\overline{V}} v \, d\xi^* \). Now, this weak convergence implies that \( \{\int_{\overline{V}} v \, d\xi_{m_\ell}\}_{\ell \geq 1} \) converges in distribution to \( \int_{\overline{V}} v \, d\xi^* \). Since the exponential function is continuous on \( \mathbb{R} \), it follows that \( \{e^{L \int_{\overline{V}} v \, d\xi_{m_\ell}}\}_{\ell \geq 1} \) converges in distribution to \( e^{L \int_{\overline{V}} v \, d\xi^*} \). Consequently, by the Skorohod Representation Theorem (e.g., Theorem 7.2.14, Grimmett and Stirzaker, 2001), there exists a probability space \( (\Omega', \mathcal{F}', Q') \) and random variables \( \{y_\ell\}_{\ell \geq 1} \) and \( y_0 \), mapping \( \Omega' \) into \( \mathbb{R} \), such that:

(a) \( \{y_\ell\}_{\ell \geq 1} \) and \( y_0 \) have the same distributions as \( \{e^{L \int_{\overline{V}} v \, d\xi_{m_\ell}}\}_{\ell \geq 1} \) and \( e^{L \int_{\overline{V}} v \, d\xi^*} \), respectively,

(b) \( \lim_{\ell \to +\infty} y_\ell(\omega') = y_0(\omega') \) a.s.-\( Q' \).

Consequently, \( \int_{\Omega'} y_\ell \, dQ' \to \int_{\Omega'} y_0 \, dQ' \) as \( \ell \to +\infty \). However,

\[
\int_{\Omega'} y_\ell \, dQ' = \int_{\Omega} e^{L \int_{\overline{V}} v \, d\xi_{m_\ell}} \, dQ \quad \text{and} \quad \int_{\Omega'} y_0 \, dQ' = \int_{\Omega} e^{L \int_{\overline{V}} v \, d\xi^*} \, dQ.
\]

(B.20)

Therefore, \( \int_{\Omega} e^{L \int_{\overline{V}} v \, d\xi_{m_\ell}} \, dQ \to \int_{\Omega} e^{L \int_{\overline{V}} v \, d\xi^*} \, dQ \), as \( \ell \to +\infty \). Combining this deduction with the limit (B.18) we must have

\[
\int_{\Omega} e^{L \int_{\overline{V}} v \, d\xi^*} \, dQ = \lim_{\ell \to +\infty} \int_{\Omega} e^{L \int_{\overline{V}} v \, d\xi_{m_\ell}} \, dQ = \lim_{m \to +\infty} \int_{\Omega} e^{L \int_{\overline{V}} v \, d\xi_0(v)} \, dQ.
\]

(B.21)

Hence, the equalities in (B.21) show that \( \xi^* \in \arg\inf \left\{ \int_{\Omega} e^{L \int_{\overline{V}} v \, d\xi} \, dQ : \xi \in \Upsilon \right\} \) since \( \xi_0 \) is an element of that set. This concludes the proof.

**B.5 Proposition**

**Proof.** The proof proceeds by the direct method. Recall that \( \mathcal{V} = \left\{ -f_\gamma(\cdot) : \gamma \in [\underline{\gamma}, \overline{\gamma}] \right\} \), where \( f_\gamma(\omega) = 1[x_2 \leq \gamma] - 1[x_1 \leq \gamma] \) and \( \omega = (x_1, x_2) \).

First, we prove \( \mathcal{V} = \overline{\mathcal{V}} \). Consider an arbitrary sequence \( \{y_n\}_{n \geq 1} \subset \mathcal{V} \) such that \( y_n \overset{L_1(Q)}{\to} y \). To prove the desired result, we need to establish that \( y \in \mathcal{V} \). As convergence in \( L_1(Q) \) implies convergence...
in $Q$-measure, there exists a non-random increasing sequence of integers $n_1, n_2, \ldots$, such that \( \{y_{n_i}\}_{i \geq 1} \) converges to $y$ a.s.-$Q$ (e.g., see Theorem 7.2.13, [Grimmett and Stirzaker, 2001]). That is,

\[
y(\omega) = \lim_{i \to +\infty} y_{n_i}(\omega) \quad \text{a.s. - } Q.
\]

The limit (B.22) implies \( \{\gamma_{n_i}\}_{i \geq 0} \subset [\gamma, \gamma] \) holds. Thus, by the Bolzano-Weierstrass Theorem, there exists a subsequence \( \{\gamma_{n_{i\ell}}\}_{\ell \geq 0} \) such that \( \lim_{\ell \to +\infty} \gamma_{n_{i\ell}} = \gamma^* \). Combining this conclusion with the limit (B.22) yields

\[
y(\omega) = \lim_{\ell \to +\infty} y_{n_{i\ell}}(\omega) = -f_{\gamma^*}(\omega) \quad \text{a.s. - } Q,
\]

as every subsequence of \( \{y_{n_i}\}_{i \geq 1} \) converges to $y$ a.s.-$Q$. Therefore, $y \in \mathcal{V}$. Because the sequence \( \{y_{n_i}\}_{n \geq 1} \subset \mathcal{V} \) such that $y_{n_1} \xrightarrow{L_1(Q)} y$, was arbitrary, we have $\mathcal{V} = \overline{\mathcal{V}}$.

Next, we prove that $\mathcal{V}$ satisfies Assumption 2—i.e., $\mathcal{V}$ is a precompact subset of $L_1(Q)$. We establish this desired result using the notion of entropy with bracketing in $L_1(Q)$ (e.g., Definition 2.1.6, [van der Vaart and Wellner, 1996]). Given $\epsilon > 0$, denote by $N[\epsilon](\mathcal{V}, L_1(Q))$ the minimum number of $\epsilon$-brackets in $L_1(Q)$ needed to ensure that every $y \in \mathcal{V}$ lies in at least one bracket. The precompactness of $\mathcal{V}$ can be established by showing that $N[\epsilon](\mathcal{V}, L_1(Q)) < +\infty \forall \epsilon > 0$. Towards that end, for any $\epsilon > 0$, a finite collection of real numbers $\underline{\gamma} = \gamma_1 < \gamma_2 < \cdots < \gamma_k = \overline{\gamma}$ can be found so that $Q_{X_1}((\gamma_{j-1}, \gamma_j)) \leq \epsilon/2$ for $i = 1, 2$ and for all $1 \leq j \leq k$. This can always be done in such a way that $k \leq 4 + 2/\epsilon$. Simply use the $\epsilon$-brackets

\[
U_j(\omega) = 1[x_1 < \gamma_j] - 1[x_2 \leq \gamma_{j-1}] \quad \text{and} \quad l_j(\omega) = 1[x_1 \leq \gamma_{j-1}] - 1[x_2 < \gamma_j] \quad j = 1, \ldots, k,
\]

where $l_j(\omega) \leq U_j(\omega)$ for every $\omega \in \Omega$ and for all $1 \leq j \leq k$, and $\|U_j - l_j\|_{L_1(Q)} \leq Q_{X_1}((\gamma_{j-1}, \gamma_j)) + Q_{X_2}((\gamma_{j-1}, \gamma_j)) \leq \epsilon$, for every $j$. Now each $\gamma \in [\underline{\gamma}, \overline{\gamma}]$, can be sandwiched as $(j - 1) \leq \gamma \leq j$ for some $j$, so that $l_j(\omega) \leq -f_{\gamma}(\omega) \leq U_j(\omega)$ a.s. - $Q$. This concludes the proof.

**B.6 Proposition 2**

**Proof:** The proof proceeds by the direct method. Recall that $\mathcal{V} = \{-f_{\gamma}(\cdot) : \gamma \in [\underline{\gamma}, \overline{\gamma}]\}$, where $f_{\gamma}(\omega) = (1[x_2 \leq \gamma] - 1[x_1 \leq \gamma]) \gamma(z), \omega = (x_1, x_2, z)$, and $\gamma = (\gamma_1, \gamma_2) \in [\underline{\gamma}, \overline{\gamma}] \times \mathcal{G}_{\text{cube}}$. To prove that $\mathcal{V} = \overline{\mathcal{V}}$ holds, we can follow steps similar to those in the proof of Proposition 1 but now accounting for the different forms of the moment functions $f_{\gamma}$ and $\omega$. But first, we have to reparametrize the set $\mathcal{G}_{\text{cube}}$ in (5.3), which enables this approach.

Recall that $\mathcal{G}_{\text{cube}}$ consists of elements of the form

\[
C_{a, r} = \prod_{u=1}^{d_Z} \left( \frac{a_u - 1}{2r}, \frac{a_u}{2r} \right] \subset [0, 1)^{d_Z},
\]

where $a = (a_1, a_2, \ldots, a_{d_Z}), a_u \in \{1, 2, \ldots, 2r\}$, for $u = 1, \ldots, d_Z$, and $r = r_0, r_0 + 1, \ldots$. For each $u$, consider the bijective map $(a_u, r) \mapsto (m_u, \rho)$, where $m_u = \frac{a_u}{2r} - \frac{1}{4r}$ is the midpoint of the interval
The advantage of this representation is that \( \left( \frac{a_u - 1}{2r}, \frac{a_u}{2r} \right) \) and \( \rho = \frac{1}{4r} \) is its radius. Thus, each such interval in the cross product can be represented by their midpoint and radius:

\[
\left( \frac{a_u - 1}{2r}, \frac{a_u}{2r} \right) = (m_u - \rho, m_u + \rho).
\]

The advantage of this representation is that \( m_u, \rho \in [0, 1] \) for each \( u = 1, \ldots, d_Z \). Let \( m = (m_1, m_2, \ldots, m_d) \).

For any sequence \( \{y_n\}_{n \geq 1} \subset \mathcal{V} \) such that \( y_n \xrightarrow{L_1(Q)} y \), there is a subsequence \( \{y_{n_i}\}_{i \geq 1} \) that converges to \( y \) a.s.\(-Q\). Consequently, there are sequences \( \{\gamma_{1n_i}\}_{i \geq 0} \subset [\gamma, \gamma] \), \( \{m_{ni}\}_{i \geq 0} \subset [0, 1]^{d_Z} \), and \( \{\rho_{ni}\}_{i \geq 0} \subset [0, 1] \). Thus, by the Bolzano-Weierstrass Theorem applied to each of these subsequences, there exists further subsequences \( \{\gamma_{1n_{i\ell}}\}_{\ell \geq 0}, \{m_{ni\ell}\}_{\ell \geq 0}, \{\rho_{ni\ell}\}_{\ell \geq 0} \), such that \( \lim_{\ell \to +\infty} \gamma_{1n_{i\ell}} = \gamma^*_1 \), \( \lim_{\ell \to +\infty} m_{ni\ell} = m^* \), and \( \lim_{\ell \to +\infty} \rho_{ni\ell} = \rho^* \). As the limits \( \gamma^*_1, m^* \) and \( \rho^* \) index a particular element of \( \mathcal{V} \), denoted by \( \gamma^* \), the combination of this point and the almost sure convergence of \( \{y_{n_i}\}_{i \geq 1} \) yields

\[
y(\omega) = \lim_{\ell \to +\infty} y_{ni\ell}(\omega) = -f_{\gamma^*}(\omega) \quad \text{a.s.} \quad -Q,
\]

Therefore, \( y \in \mathcal{V} \). As the convergent sequence \( \{y_n\}_{n \geq 1} \subset \mathcal{V} \) was arbitrary, the above derivations hold for all such sequences. Consequently \( \mathcal{V} = \overline{\mathcal{V}} \).

To prove that \( \mathcal{V} \) satisfies Assumption 2 we use Lemma 7.1 of \cite{Andrews and Shi, 2017}. In the proof of their result, they show that the \( L_2(Q) \) packing number of this set is finite for every positive using Lemma 4.4 of \cite{Pollard, 1990}. This result implies that the \( L_2(Q) \) covering numbers of \( \mathcal{V} \) must be finite for every positive radius. Hence, by Lemma C.7 this must also be true for the \( L_1(Q) \) covering numbers of \( \mathcal{V} \).

### B.7 Proposition 3

**Proof.** The proof proceeds by the direct method. First, using the Lipschitz continuity of the capacity functional \( \sigma \) (i.e., inequality (5.12)), and following steps similar to those in the proof of Proposition 1 but now accounting for the different forms of the moment functions \( f_\gamma \) and \( \omega \), we can establish that \( \mathcal{V} = \overline{\mathcal{V}} \) holds. Specifically, the key adjustment entails using the compactness of \( \mathcal{C}(\Omega) \) (with the Hausdorff metric) as it implies that it must have the Bolzano-Weierstrass Property, which we exploit in the proof of Proposition 1 so that we can calculate the \( L_1(Q) \) limit of sequences in \( \mathcal{V} \) and deduce that they must be elements of \( \mathcal{V} \). We omit the details for brevity. This result implies that \( \mathcal{V} \) is closed in norm topology of \( L_1(Q) \).

Next, we establish that \( \mathcal{V} \) is precompact. Denote by \( \Delta \) the symmetric difference set operation on \( \mathcal{C}(\Omega) \). Given \( \epsilon > 0 \), denote by \( N(\epsilon, \mathcal{V}, L_1(Q)) \) the minimum number of \( \epsilon \)-balls in the \( L_1(Q) \)-norm needed to ensure that every \( y \in \mathcal{V} \) lies in at least one such ball. The precompactness of \( \mathcal{V} \) can be established by showing that \( N(\epsilon, \mathcal{V}, L_1(Q)) < +\infty \) \( \forall \epsilon > 0 \). Towards that end, for any \( \rho > 0 \), \( \exists n \in \mathbb{Z}_+ \) such that \( A_i \in \mathcal{C}(\Omega) \) for \( i = 1, \ldots, n \) with \( \mathcal{C}(\Omega) \subseteq \bigcup_{i=1}^n B(A_i, \rho) \), where \( B(A_i, \rho) \) is a \( \rho \)-ball in \( \mathcal{C}(\Omega) \). Next, consider the sets \( \{-f_\gamma : \gamma \in B(A_i, \rho)\} \) and let \( -f_{\gamma_0} \) be the moment function with \( \gamma = A_i \), for \( i = 1, \ldots, n \).
Fix $i$, and let $f_\gamma \in \{-f_\gamma : \gamma \in B(A_i, \rho)\}$. Then we have

\[ \int_O |f_\gamma - f_{\gamma_i}| \, dQ \leq |\sigma(A) - \sigma(A_i)| + Q(A \Delta A_i) \quad \text{(by the form of } f_\gamma) \]

\[ \leq 2d_H(A, A_i) + Q(A \Delta A_i) \quad \text{(by (6.12))} \]

\[ \leq 2\rho + Q(A \Delta A_i) \quad \text{(by precompactness of } \mathcal{C}(\Omega)) \]

\[ \leq 2\rho + 2\rho = 4\rho. \]

Thus, the set $\{-f_\gamma : \gamma \in B(A_i, \rho)\}$ is a ball in $\mathcal{V}$ based on the $L_1(Q)$-norm which is centered at $-f_{\gamma_i}$ having radius $8\rho$. As

\[ \mathcal{V} \subseteq \bigcup_{i=1}^n \{-f_\gamma : \gamma \in B(A_i, \rho)\}, \]

we can set $\rho = \epsilon/4$, to obtain $n$ balls in $\mathcal{V}$ using the desired norm, each with radius $\epsilon$, having centers $\{-f_{\gamma_i}, i = 1, \ldots, n\}$, which cover $\mathcal{V}$. This shows $N(\epsilon, \mathcal{V}, L_1(Q)) < +\infty$ for an arbitrary $\epsilon > 0$, and hence, the above argument holds for all $\epsilon > 0$. ■

## C Intermediate Technical Results

**Lemma C.1.** Let $\mathcal{M}$ be given by (3.2). Then $\mathcal{M}$ is convex and closed in the topology of variation distance.

**Proof.** The proof proceeds by the direct method. First we shall prove that $\mathcal{M}$ is closed. Let $\{p_m\}_{m=1}^{+\infty}$ be a sequence of elements in $\mathcal{M}$ that converges in $L_1(Q)$ to some $w$. To prove that $\mathcal{M}$ is closed in $L_1(Q)$, we need to show that $w \in \mathcal{M}$. To that end, the sequence $\{p_m\}_{m=1}^{+\infty} \subset \mathcal{M}$ converges in $L_1(Q)$ to $w$ means that

\[ \lim_{m \to +\infty} \int |p_m - w| \, dQ = 0, \]

and implies that $\{p_m\}_{m=1}^{+\infty}$ converges in $Q$-measure to $w$. Consequently, there exists a non-random increasing sequence of integers $m_1, m_2, \ldots$, such that $\{p_{m_i}\}_{i=1}^{+\infty}$ converges to $w$ a.s.-$Q$ (e.g., see Theorem 7.2.13, Grimmet and Stirzaker [2001]).

Firstly, we show that $m(w) < +\infty$ holds. The sequence $\{p_{m_i}\}_{i=1}^{+\infty} \subset \mathcal{M}$ converging to $w$ a.s.-$Q$ implies that $w \geq 0$ a.e.-$Q$ since $p_{m_i} \geq 0$ a.e.-$Q$ for each $i$. Next, we show $\int_O w \, dQ = 1$. Note that

\[ \left| 1 - \int_O w \, dQ \right| = \left| \int_O p_m \, dQ - \int_O w \, dQ \right| \leq \int_O |p_m - w| \, dQ \to 0 \quad \text{(C.1)} \]

as $m \to +\infty$. Thus, $m(w) < +\infty$ holds.

Next, we show for each $\gamma \in \Gamma$ that $\int_O |f_\gamma|^{1+\delta} w \, dQ \leq K$. Fix a $\gamma \in \Gamma$, and by Fatou’s Lemma

\[ \int_O |f_\gamma|^{1+\delta} w \, dQ = \int_O |f_\gamma|^{1+\delta} \lim_{i \to +\infty} p_{m_i} \, dQ \leq \liminf_{i \to +\infty} \int_O |f_\gamma|^{1+\delta} p_{m_i} \, dQ \leq K. \quad \text{(C.2)} \]

Since $\gamma \in \Gamma$ was arbitrary, the above arguments apply to each $\gamma \in \Gamma$. A useful consequence of this result is the following: for each $\gamma \in \Gamma$, $\left| \int_O |f_\gamma|(w - p_{m_i}) \, dQ \right| \to 0$ as $i \to +\infty$. The reason is that
\[ |\int_\Omega |f_\gamma| (w - p_{m_i}) \, dQ| \text{ is less than or equal to } \]

\[
\int_\Omega |f_\gamma| |w - p_{m_i}| \, dQ \leq a_i \int_\Omega |w - p_{m_i}| \, dQ + \int_{|f_\gamma| > a_i} |f_\gamma| |w - p_{m_i}| \, dQ \\
\leq a_i \int_\Omega |w - p_{m_i}| \, dQ + \int_{|f_\gamma| > a_i} |f_\gamma| w \, dQ + \int_{|f_\gamma| > a_i} |f_\gamma| p_{m_i} \, dQ \\
\leq a_i \int_\Omega |w - p_{m_i}| \, dQ + \frac{\int_\Omega |f_\gamma|^{1+\delta} w \, dQ}{a_i^{\delta}} + \frac{\int_\Omega |f_\gamma|^{1+\delta} p_{m_i} \, dQ}{a_i^{\delta}}, \\
\leq a_i \int_\Omega |w - p_{m_i}| \, dQ + \frac{2K}{a_i^{\delta}}. 
\]

(C.3)

(C.4)

(C.5)

(C.6)

Now setting \( \{a_i\} \) so that \( \lim_{i \to +\infty} a_i = +\infty \) and \( \lim_{i \to +\infty} a_i \int_\Omega |w - p_{m_i}| \, dQ = 0 \), and taking limits as \( i \to +\infty \) on the right side of the inequality (C.6) yields the desired result.

To show for each \( \gamma \in \Gamma \) that \( \int f_\gamma w \, dQ \leq 0 \), holds, consider the following: given \( \gamma \in \Gamma \), by Fatou’s Lemma

\[
\int_\Omega (|f_\gamma| + f_\gamma) w \, dQ = \int_\Omega (|f_\gamma| + f_\gamma) \lim_{i \to +\infty} p_{m_i} \, dQ \leq \liminf_{i \to +\infty} \int_\Omega (|f_\gamma| + f_\gamma) p_{m_i} \, dQ \\
= \liminf_{i \to +\infty} \left( \int_\Omega |f_\gamma| p_{m_i} \, dQ + \int_\Omega f_\gamma p_{m_i} \, dQ \right) \\
\leq \liminf_{i \to +\infty} \int_\Omega |f_\gamma| p_{m_i} \, dQ \\
= \lim_{i \to +\infty} \int_\Omega |f_\gamma| p_{m_i} \, dQ \\
= \int_\Omega |f_\gamma| w \, dQ, 
\]

and hence,

\[
\int_\Omega (|f_\gamma| + f_\gamma) w \, dQ \leq \int_\Omega |f_\gamma| w \, dQ \iff \int_\Omega f_\gamma w \, dQ \leq 0, 
\]

because \( \int_\Omega |f_\gamma| w \, dQ < +\infty \). Note that in the above derivations, we used the fact that \( \int_\Omega f_\gamma p_{m_i} \, dQ \leq 0 \ \forall i \) as \( \{p_{m_i}\}_{i=1}^{+\infty} \subset \mathcal{M} \), and the earlier result \( |\int_\Omega |f_\gamma| (w - p_{m_i}) \, dQ| \to 0 \) as \( i \to +\infty \). Now, since \( \gamma \in \Gamma \) was arbitrary, the above arguments apply to each element of \( \Gamma \), and hence,

\[
\int_\Omega f_\gamma w \, dQ \leq 0 \ \forall \gamma \in \Gamma. 
\]

To prove that \( \mathcal{M} \) is convex, let \( \lambda \in (0, 1) \), and \( P, D \in \mathcal{M} \). Consider the mixture of two PDs \( W = \lambda P + (1 - \lambda) R \), to prove that \( \mathcal{M} \) is convex, we need to show that \( W \in \mathcal{M} \). This follows since \( \forall \gamma \in \Gamma \), we have

\[
\int f_\gamma dW = \lambda \int f_\gamma dP + (1 - \lambda) \int f_\gamma dR 
\]

and that \( \int f_\gamma dP, \int f_\gamma dR \leq 0. \)
Lemma C.2. Suppose the conditions of Part 1 of Theorem 2 hold. Then for each \( y' \in \overline{\text{co}}(\nabla) \)

\[
\lim_{\alpha \to +\infty} \int_1^{+\infty} Q \left( \omega \in \Omega : y' > \frac{\ln t}{\alpha_C} \right) dt = +\infty.
\]

Proof. The proof proceeds by the direct method. Given \( y' \in \overline{\text{co}}(\nabla) \), we shall establish the desired result by showing that \( \forall C > 0, \exists \alpha_C > 0 \) sufficiently large such that \( \int_1^{+\infty} Q \left( \omega \in \Omega : y' > \frac{\ln t}{\alpha_C} \right) \geq C \).

Given \( C > 0 \) and \( y' \in \overline{\text{co}}(\nabla) \), set \( \alpha_C \) large enough so that

\[
\left| Q \left( \omega \in \Omega : y' > \frac{\ln t}{\alpha_C} \right) - Q \left( \omega \in \Omega : y' > 0 \right) \right| \leq \frac{1}{C'} \quad \forall t \in [1, C'],
\]

where \( C' \geq \frac{C+2}{Q(\omega \in \Omega : y' > 0)} \). Note that Part (iii) of Assumption 1 implies \( Q(\omega \in \Omega : y' > 0) > 0 \). Now, consider

\[
\int_1^{+\infty} Q \left( \omega \in \Omega : y' > \frac{\ln t}{\alpha_C} \right) dt = \int_1^{C'} Q \left( \omega \in \Omega : y' > \frac{\ln t}{\alpha_C} \right) dt + \int_{C'}^{+\infty} Q \left( \omega \in \Omega : y' > \frac{\ln t}{\alpha_C} \right) dt \geq \int_1^{C'} Q \left( \omega \in \Omega : y' > \frac{\ln t}{\alpha_C} \right) dt.
\]

Consequently,

\[
\int_1^{C'} Q \left( \omega \in \Omega : y' > \frac{\ln t}{\alpha_C} \right) dt \geq \int_1^{C'} \left[ Q \left( \omega \in \Omega : y' > \frac{\ln t}{\alpha_C} \right) - Q \left( \omega \in \Omega : y' > 0 \right) \right] dt + (C' - 1)Q \left( \omega \in \Omega : y' > 0 \right) \geq \frac{1}{C'} - 1 + (C' - 1)Q \left( \omega \in \Omega : y' > 0 \right)
\]

(by (C.7))

\[
\geq \frac{1}{C'} - 1 + (C' - 1)Q \left( \omega \in \Omega : y' > 0 \right) \geq C'Q \left( \omega \in \Omega : y' > 0 \right) - 2 \geq C,
\]

where the last inequality follows from the condition \( C' \geq \frac{C+2}{Q(\omega \in \Omega : y' > 0)} \). Combining the two inequalities shows \( \int_1^{+\infty} Q \left( \omega \in \Omega : y' > \frac{\ln t}{\alpha_C} \right) dt \geq C \), as desired. This concludes the proof.

Lemma C.3. Suppose that Assumption 2 holds. Then the following hold: \( \text{ex} (\overline{\text{co}}(\nabla)) \neq \emptyset \), \( \overline{\text{co}} (\text{ex} (\overline{\text{co}}(\nabla))) = \overline{\text{co}}(\nabla) \), and \( \text{ex} (\overline{\text{co}}(\nabla)) \subset \nabla \).

Proof. The proof proceeds by the direct method. Assumption 2 the set \( \nabla \) is compact in the \( L_1(Q) \)-norm (as it is complete and totally bounded). Now, since \( L_1(Q) \) is a Banach space, it is therefore a Fréchet space. This fact allows us to apply part (c) of Theorem 3.20 in [Rudin, 1991] to the set \( \nabla \) to deduce that \( \overline{\text{co}}(\nabla) \) is also compact in the same norm. Whence, we can apply the Krein-Milman Theorem (e.g., Theo-
Consequently, we must establish Lemma C.5. Suppose the conditions of Theorem 3.5 hold. Given $\epsilon > 0$, let $U = \{y \in L_1(Q) : \|y\|_{L_1(Q)} \leq \epsilon\}$. Then, there corresponds a finite partition $\{E_i\}_{i=1}^n$ of $\overline{\Omega}$ such that

$$\int_{\overline{\Omega}} v \, d\mu(v) = \sum_{i=1}^n \mu(E_i) v_i \in U \quad \forall v_i \in E_i, \ i = 1 \ldots, n.$$  

Proof. The proof proceeds by the direct method. The proof we present is a solution to Exercise 23 of [Rudin, 1991], but in the context of this paper’s framework.

First note that the neighborhood $U$ of 0 in $L_1(Q)$ is closed, balanced, and convex. The polar of $U$ is defined as $K = \{\Lambda \in L_1(Q)^* : |\Lambda(y)| \leq 1 \forall y \in U\}$. We claim that

$$U = \{y \in L_1(Q) : |\Lambda(y)| \leq 1 \forall \Lambda \in L_1(Q)^*\}.$$  

It is clear that $U$ is a subset of the right side of (C.13) which is closed. Suppose that $y \in L_1(Q)$ but $y_0 \notin U$. Theorem 3.7 of [Rudin, 1991] (with $U$ and $L_1(Q)$ in place of $B$ and $X$, respectively) then shows...
that \( \Lambda(y_0) > 1 \) for some \( \Lambda \in L_1(Q)^* \). This establishes the equality (C.13). Consequently, to show that \( z \in L_1(Q) \) is an element of \( U \), we need to establish that \( |\Lambda(z)| \leq 1 \forall \Lambda \in L_1(Q)^* \), holds.

Next, construct the partitioning sets \( E_i \) so that \( v - v' \in U \) whenever \( v \) and \( v' \) lie the same \( E_i \). Since \( \mathcal{V} \) is compact in \( L_1(Q) \), there is a finite number of such sets, \( n \in \mathbb{Z}_+ \). Now we will show (C.12). Select any \( v_i \in E_i \) for each \( i = 1, \ldots, n \), and define

\[
    z = \int_{\mathcal{V}} v \, d\mu(v) - \sum_{i=1}^{n} \mu(E_i) v_i. \tag{C.14}
\]

Then, for any \( \Lambda \in L_1(Q)^* \),

\[
    \Lambda(z) = \Lambda \left( \int_{\mathcal{V}} v \, d\mu(v) \right) - \Lambda \left( \sum_{i=1}^{n} \mu(E_i) v_i \right) \quad \text{(by linearity of } \Lambda) \\
    = \int_{\mathcal{V}} \Lambda(v) \, d\mu(v) - \sum_{i=1}^{n} \mu(E_i) \Lambda(v_i) \quad \text{(by linearity of } \Lambda) \\
    = \sum_{i=1}^{n} \left[ \int_{E_i} \Lambda(v) \, d\mu(v) - \mu(E_i) \Lambda(v_i) \right] \quad \text{(as } \{E_i\}_{i=1}^{n} \text{ is a partition of } \mathcal{V}).
\]

Whence,

\[
    |\Lambda(z)| \leq \sum_{i=1}^{n} \left| \int_{E_i} \Lambda(v) \, d\mu(v) - \mu(E_i) \Lambda(v_i) \right| \\
    = \sum_{i=1}^{n} \left| \int_{E_i} \Lambda(v) \, d\mu(v) \int_{E_i} d\mu(v) \Lambda(v_i) \right| \quad \text{(as } \int_{E_i} d\mu(v) = \mu(E_i)). \\
    = \sum_{i=1}^{n} \left| \int_{E_i} (\Lambda(v) - \Lambda(v_i)) \, d\mu(v) \right| \quad \text{(by linearity of Lebesgue integral)} \\
    = \sum_{i=1}^{n} \left| \int_{E_i} (\Lambda(v - v_i)) \, d\mu(v) \right| \quad \text{(by linearity of } \Lambda) \\
    \leq \sum_{i=1}^{n} \int_{E_i} |\Lambda(v - v_i)| \, d\mu(v) \\
    \leq 1 \quad \text{(as } v - v_i \in E_i \implies v - v_i \in U \text{ for each } i, \text{ and by the right side of (C.13)).}
\]

As \( \Lambda \in L_1(Q)^* \) in the above derivations was arbitrary, these derivations hold for all \( \Lambda \in L_1(Q)^* \). Consequently, \( |\Lambda(z)| \leq 1 \forall \Lambda \in L_1(Q)^* \), which means that \( z \) is an element of the right side of (C.13), and hence, \( z \in U \). Finally, the derivations for this \( z \) are for an arbitrary choice of elements \( v_i \in E_i \) for each \( i \). Therefore, these derivations and deductions go through for any choice of the \( v_i \in E_i \). This concludes the proof.
Lemma C.6. Suppose the conditions of Theorem 3 hold. Then
\[
\frac{\int_{\Omega} v e^{y_0} dQ}{\int_{\Omega} e^{y_0} dQ} \geq \int_{\Omega} v dQ \quad \forall v \in \overline{B}.
\]

Proof. The proof proceeds by the direct method. Let \( v \in \overline{B} \). By definition of covariance,
\[
\int_{\Omega} v e^{y_0} dQ = \int_{\Omega} e^{y_0} dQ \int_{\Omega} v dQ + \text{COV}_Q (e^{y_0}, v), \tag{C.15}
\]
where \( \text{COV}_Q (e^{y_0}, v) \) is the covariance between \( e^{y_0} \) and \( v \) under \( Q \). The equality (C.15) is equivalent to
\[
\frac{\int_{\Omega} v e^{y_0} dQ}{\int_{\Omega} e^{y_0} dQ} = \int_{\Omega} v dQ + \frac{\text{COV}_Q (e^{y_0}, v)}{\int_{\Omega} e^{y_0} dQ}. \tag{C.16}
\]
Thus, the desired conclusion would arise if \( \text{COV}_Q (e^{y_0}, v) \geq 0 \), since \( \int_{\Omega} e^{y_0} dQ > 0 \). From Theorem 2, we have that \( y_0 \in \text{span}_+ (B) \), and since \( \overline{B} \subset \text{span}_+ (B) \), we must have \( \text{COV}_Q (e^{y_0}, v) \geq 0 \). This is because \( y_0 \) must either not depend on \( v \), or it would depend on \( v \) linearly with a positive coefficient, which is on account of the positive linear span operation. This concludes the proof.

Lemma C.7. Suppose that the class of moment \( V \) in (3.5) is precompact in \( L^r (Q) \) norm for some \( r > 1 \). Then \( V \) is precompact in the \( L^1 (Q) \) norm.

Proof. The proof proceeds by the direct method and makes use of Lemma 1 in Hanche-Olsen and Holden (2010). Simply apply their result using the identity map as the choice of the function \( \Phi \) in the statement of their result.

D Technical Results for Examples 1 and 2

D.1 Example 1

Lemma D.1. Consider the setup of Example 1 but with \( \gamma = 1 \). For each \( y \in \mathcal{D} \) the inequality (4.4) holds.

Proof. The proof proceeds by the direct method. Let \( y \in \mathcal{D} \), and consider the following derivation:
\[
\int_{\Omega} e^y dQ = \int_{0}^{+\infty} Q (\omega \in \Omega : e^y > t) dt = \int_{0}^{+\infty} Q \left( \omega \in \Omega : e^{y'} > \frac{t}{\alpha} \right) dt
\]
by the tail sum formula for the mean and because \( y = \alpha y' \) where \( y' \in \text{co} (\mathcal{V}) \). As \( e^{y'} \in [0, e^2] \) a.s.-\( Q \) holds...
for any \( y' \in \overline{\co(V)} \), it follows that

\[
\int_0^{+\infty} Q \left( \omega \in \Omega : e^{y'} > \frac{t}{\alpha} \right) dt = \int_0^{\alpha e^2} Q \left( \omega \in \Omega : e^{y'} > \frac{t}{\alpha} \right) dt \\
= \int_0^{\alpha e^2 - \delta} Q \left( \omega \in \Omega : e^{y'} > \frac{t}{\alpha} \right) dt \\
+ \int_{\alpha e^2}^{\alpha e^2 - \delta} Q \left( \omega \in \Omega : e^{y'} > \frac{t}{\alpha} \right) dt \\
\geq \int_0^{\alpha e^2 - \delta} Q \left( \omega \in \Omega : e^{y'} > \frac{t}{\alpha} \right) dt.
\]

Now let \( \beta = \int_\Omega e^{y'} dQ \) and set \( \delta = \alpha \beta k \) where \( \frac{e^2}{\beta} - 1 < k < \frac{e^2}{\beta} \), and note that \( k \) only depends on \( \beta \). Then, we have

\[
\int_0^{\alpha e^2 - \delta} Q \left( \omega \in \Omega : e^{y'} > \frac{t}{\alpha} \right) dt \geq (\alpha e^2 - \delta) Q \left( \omega \in \Omega : e^{y'} > \frac{\alpha e^2 - \delta}{\alpha} \right) \\
= \alpha (e^2 - \beta k) Q \left( \omega \in \Omega : e^{y'} > \frac{e^2 - \beta k}{e^2 - \beta k} \right) \\
\geq \alpha (e^2 - \beta k) \frac{\beta - (e^2 - \beta k)}{e^2 - \beta k} \\
= \alpha (e^2 - \beta k) \frac{(k + 1) \beta - e^2}{\beta k}. \tag{D.1}
\]

where the inequality \( \text{(D.1)} \) follows from an application of Theorem 5 on page 319 of \cite{GrimmettStirzaker}. This concludes the proof.

\[\vdash\]

**Lemma D.2.** Let \( \mathcal{V} \) be given as in Example 1. Then this set satisfies \( \mathcal{V} = \overline{\mathcal{V}} \) and Assumption 2.

**Proof.** The proof proceeds by the direct method. On establishing (i), consider an arbitrary sequence \( \{y_n\}_{n \geq 1} \subset \mathcal{V} \) such that \( y_n \overset{L_1(Q)}{\longrightarrow} y \). To prove the desired result, we need to establish that \( y \in \mathcal{V} \). As convergence in \( L_1(Q) \) implies convergence in \( Q \)-measure, there exists a non-random increasing sequence of integers \( n_1, n_2, \ldots \), such that \( \{y_{n_k}\}_{k \geq 1} \) converges to \( y \) a.s.-\( Q \) (e.g., see Theorem 7.2.13, \cite{GrimmettStirzaker}). That is,

\[
y(\omega) = \lim_{k \to +\infty} y_{n_k}(\omega) \quad \text{a.s. -} \quad Q. \tag{D.2}
\]

The limit \( \text{(D.2)} \) implies \( \{\gamma_{n_k}\}_{k \geq 0} \subset [0, \overline{\gamma}] \) holds. Thus, there exists a subsequence \( \{\gamma_{n_{k_\ell}}\}_{\ell \geq 1} \) such that \( \lim_{\ell \to +\infty} \gamma_{n_{k_\ell}} = \gamma^* \in [0, \overline{\gamma}] \), by the Bolzano-Weierstrass Theorem. Combining this conclusion with the limit \( \text{(D.2)} \) yields

\[
y(\omega) = \lim_{\ell \to +\infty} y_{n_{k_\ell}}(\omega) = G_1(\gamma^*) - 1[\omega \leq \gamma^*] \quad \text{a.s. -} \quad Q, \tag{D.3}
\]

as every subsequence of \( \{y_{n_k}\}_{k \geq 1} \) converges to \( y \) a.s.-\( Q \). Therefore, \( y \in \mathcal{V} \). Because the sequence
\( \{y_n\}_{n \geq 1} \subset \mathcal{V} \) such that \( y_n \overset{L_1(Q)}{\to} y \), was arbitrary, we have \( \mathcal{V} = \overline{\mathcal{V}} \).

On establishing (ii), we will show the \( L_1(Q) \) bracketing number of \( \mathcal{V} \) is finite. Given \( \epsilon > 0 \), we can always find a partition of \( \mathbb{R} \) of size \( k \), where \( 0 = \gamma_1 < \gamma_2 < \cdots < \gamma_k = \gamma \), such that

\[
G_1(\gamma_j -) - G_1(\gamma_{j-1}) + \gamma_j - \gamma_{j-1} \leq \epsilon \quad \forall 1 \leq j \leq k, \tag{D.4}
\]

with \( G_1(\gamma_j -) = \lim_{s \uparrow \gamma_j} G_1(s) \). Consider the collection of brackets

\[
u_j(\omega) = G_1(\gamma_j) - 1[\omega \leq \gamma_{j-1}] \quad \text{and} \quad l_j(\omega) = G_1(\gamma_{j-1}) - 1[\omega < \gamma_j] \quad \forall 1 \leq j \leq k. \tag{D.5}
\]

Now each element of \( \mathcal{V} \) is in at least one bracket, i.e., for \( \gamma_{j-1} \leq \gamma \leq \gamma_j \),

\[
l_j(\omega) \leq G_1(\gamma) - 1[\omega \leq \gamma] \leq u_j(\omega) \quad \text{a.s. - } Q, \tag{D.6}
\]

and \( \|u_j - l_j\|_{L_1(Q)} \leq \epsilon \). Thus, the \( L_1(Q) \) bracketing number of \( \mathcal{V} \) satisfies \( N_{\|\cdot\|} (\epsilon, \mathcal{V}, L_1(Q)) < +\infty \forall \epsilon > 0 \), and hence, \( \mathcal{V} \) is precompact in \( L_1(Q) \).

\section*{D.2 Example 2}

\textbf{Lemma D.3.} Let \( \mathcal{V} \) be given as in Example 2 Then this set satisfies \( \mathcal{V} = \overline{\mathcal{V}} \) and Assumption 2.

\textbf{Proof.} The proof proceeds by the direct method. On establishing (i), it is sufficient to show \( \mathcal{V}_i = \overline{\mathcal{V}}_i \), for each \( i \). Without loss of generality, our derivations focus on the case \( i = 1 \). Consider an arbitrary sequence \( \{y_n\}_{n \geq 1} \subset \mathcal{V}_1 \) such that \( y_n \overset{L_1(Q)}{\to} y \). To prove the desired result, we need to establish that \( y \in \mathcal{V}_1 \). As convergence in \( L_1(Q) \) implies convergence in \( Q \)-measure, there exists a non-random increasing sequence of integers \( n_1, n_2, \ldots \), such that \( \{y_{n_k}\}_{k \geq 1} \) converges to \( y \) a.s.-\( Q \) (e.g., see Theorem 7.2.13, Grimmet and Stirzaker, 2001). That is,

\[
y(\omega) = \lim_{k \to +\infty} y_{n_k}(\omega) \quad \text{a.s. - } Q. \tag{D.7}
\]

The limit (D.7) implies \( \{y_{n_k}\}_{k \geq 0} \subset [-\infty, +\infty] \) holds. Thus, there exists a subsequence \( \{y_{n_{k_l}}\}_{l \geq 1} \) such that \( \lim_{l \to +\infty} y_{n_{k_l}} = \gamma^* \in [-\infty, +\infty] \), because \( [-\infty, +\infty] \) is the two-point compactification of \( \mathbb{R} \). Combining this conclusion with the limit (D.7) yields

\[
y(\omega) = \lim_{l \to +\infty} y_{n_{k_l}}(\omega) = (G_1(\gamma^*) - 1[x_1 \leq \gamma^*]) 1[x_2 \in \mathbb{R}] \quad \text{a.s. - } Q, \tag{D.8}
\]

as every subsequence of \( \{y_{n_k}\}_{k \geq 1} \) converges to \( y \) a.s.-\( Q \). Therefore, \( y \in \mathcal{V}_1 \). Because the sequence \( \{y_n\}_{n \geq 1} \subset \mathcal{V}_1 \) such that \( y_n \overset{L_1(Q)}{\to} y \), was arbitrary, we have \( \mathcal{V}_1 = \overline{\mathcal{V}}_1 \). An identical argument holds for establishing that \( \mathcal{V}_2 = \overline{\mathcal{V}}_2 \), which we omit for brevity.

On establishing (ii), it is sufficient to show \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are precompact in \( L_1(Q) \), since \( \mathcal{V} \) is their union. Without loss of generality, our derivations focus on the case \( i = 1 \), and shows that this set’s \( L_1(Q) \) bracketing number of \( \mathcal{V} \) is finite. Given \( \epsilon > 0 \), we can always find a partition of \( \mathbb{R} \) of size \( k \), where \( -\infty = \gamma_1 < \gamma_2 < \cdots < \gamma_k = \gamma \).
\[ \cdots < \gamma_k = +\infty, \text{ such that} \]
\[ G_1(\gamma_j -) - G_1(\gamma_{j-1}) + Q_{X_1}(\gamma_1, \gamma_j -) - Q_{X_1}(\gamma_1, \gamma_{j-1}) \leq \epsilon \quad \forall 1 \leq j \leq k, \quad (D.9) \]

with \( G_1(\gamma_j -) = \lim_{s \uparrow \gamma_j} G_1(s) \) and \( Q_{X_1}(\gamma_1, \gamma_j -) = \lim_{s \uparrow \gamma_j} Q_{X_1}(\gamma_1, s) \). Consider the collection of brackets

\[ u_j(x_1) = G_1(\gamma_j) - 1 [1 \leq \gamma_{j-1}] \quad \text{and} \quad l_j(x_1) = G_1(\gamma_{j-1} -) - 1 [1 < \gamma_j] \quad \forall 1 \leq j \leq k. \quad (D.10) \]

Now each element of \( V_1 \) is in at least one bracket, i.e., for \( \gamma_{j-1} \leq \gamma \leq \gamma_j \),

\[ l_j(x_1) \leq G_1(\gamma) - 1 [x_1 \leq \gamma] \leq u_j(x_1), \quad \text{a.s.} - Q, \quad (D.11) \]

and \( \| u_j - l_j \|_{L_1(Q)} \leq \epsilon \). Thus, \( N(\epsilon, V_1, L_1(Q)) < +\infty \forall \epsilon > 0 \), and hence, \( V_1 \) is precompact in \( L_1(Q) \). We can follow identical steps to those above to deduce that \( V_2 \) is precompact in \( L_1(Q) \). We omit these details for brevity. \( \blacksquare \)