STOCHASTIC GRADIENT DESCENT WITH POLYAK’S LEARNING RATE

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ABSTRACT. Stochastic gradient descent (SGD) for strongly convex functions converges at the rate $O(1/k)$. However, achieving the rate with the optimal constant requires knowledge of parameters of the function which are usually not available. In practice, the learning rate is tuned. In this paper we propose a generalization of the Polyak step size to SGD. We prove convergence at the rate $O(1/k)$ with a rate constant which can be better than the corresponding rate constant for optimally scheduled SGD. We demonstrate that the method is effective in practice on small and large scale problems.

1. Introduction

Stochastic Gradient Descent (SGD) [RM51, Kal60] is now a widely used optimization algorithm of current interest due to its ubiquitous use in machine learning [BCN16]. Convergence rates are available in a wide setting. In the simple but important setting of strongly convex functions, achieving the optimal convergence rate requires a scheduled learning rate (time step), which depends on knowledge of parameters of the function which are often not available. Practical implementations sometimes rely on careful tuning of the learning rate to achieve good performance.

SGD is the method of choice for large scale machine learning problems [Bot91]. When SGD is combined with momentum [Pol64, Nes13] empirical performance is improved, but there is still no theoretical justification for the improvement [KNJK18]. Other popular stochastic optimization algorithms that use different forms of averaging or variance reduction are still being studied and developed [BCN16].

In this work, we focus on SGD without momentum. We study a learning rate formulation based on a generalization of Polyak’s learning rate [Pol87, Chapter 5.3.] to the stochastic setting. Polyak’s learning rate (see also [Bec17, Page 204]) is commonly used for the subgradient method. It corresponds to

$$x_{k+1} = x_k - h(x_k)\partial f(x_k), \quad h(x_k) = \frac{f(x_k) - f^*}{||\partial f(x_k)||^2}.$$ 

In this case, the learning rate depends on an estimate of the value of $f^* = \min_x f(x)$. In some applications, the $f^*$ value is known (or zero), and the method can be applied without estimation. A principled method for estimation, which is provably convergent with the same rate, is provided in [BM13, Chapter 4.2]. The estimation involves an auxiliary sequence $\gamma_k$, leading to the learning rate

$$h^{est}(x_k) = \frac{f(x_k) - f_{k}^{best} + \gamma_k}{||\partial f(x_k)||^2},$$

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where \( f_{k}^{\text{best}} = \min_{j \leq k} f(x_j) \). Convergence follows provided

\[
\gamma_k > 0, \quad \gamma_k \to 0, \quad \sum_{k=1}^{\infty} \gamma_k = \infty.
\]

In particular, \( f_{k}^{\text{best}} \to f^\ast \) as \( k \to \infty \) in this case. Additional methods for estimating \( f^\ast \) can be found in [Sho12, BM13] and [KAC90].

The optimal convergence rate for SGD in the strongly convex case is \( O(1/k) \) [RSS11, AWBR09, RR11]. Achieving the optimal rate constant for scheduled SGD is difficult in practise, as we review below, since it depends on parameters for the function, such as the strong convexity constant, which may not be known. Hence, in practise the rate can be slower than optimal. In practise, the learning rate is tuned to the problem at hand, or adaptive methods are used [DHS11, HSS12, Zei12, KB14, WWB18]. These methods often perform well in practise, but they lack optimal rates of convergence, or in some cases, they lack convergence guarantees [RKK18]. In the deep learning setting, tuned SGD tends to outperform these methods in terms of generalization [HRS15, WRS + 17].

However, while the rate cannot be improved, the rate constant can. In this work, we show that using the stochastic version of the Polyak step size, we can obtain a convergence rate with a constant that is better (provided the initial point is not too far) compared to the constant for optimally scheduled SGD. We demonstrate that the method is effective in practise, using small scale and deep learning examples. While we do not prove any results in the non-convex case, the Polyak schedule for SGD can still be applied. It has the advantage of requiring only the parameter \( f^\ast \) which can be sometimes preferred to the parameters required for scheduled non-convex SGD.

Contents. We start this paper by introducing some mathematical background. Then, in Section 2, which is provided for background and is not needed for the sequel, we prove a convergent rate for gradient descent using Polyak’s step size. In Section 3, we recall SGD with optimally scheduled learning rate and its properties. Then we establish a convergence rate of \( O(1/k) \), although with different constants, for optimally scheduled SGD and SGD with Polyak’s step size, and compare. Finally, in Section 4, we present numerical results. We show both results for generated mini-batch noise and for an image recognition problem in deep learning.

1.1. Notation and Convex Function Inequalities. In this section, we recall some definitions and establish notation. Write,

\[
f^\ast = \min_x f(x), \quad x^\ast \in \arg\min_x f(x),
\]

when such quantities are defined. Write \( g(x) = \nabla f(x) \) and \( g_k = \nabla f(x_k) \). Write as well \( q(x) = \frac{1}{2} |x - x^\ast|^2 \) and \( q_k = \frac{1}{2} |x_k - x^\ast|^2 \).

The following definitions can be found in [Pol87] and [Bec17, Chapter 5].

**Definition 1.1.** The function \( f : \mathbb{R}^d \to \mathbb{R} \) is \( \mu \)-strongly convex if

(\( \mu \)-convex) \( f(x) - (f(y) + \nabla f(y) \cdot (x - y)) \geq \frac{\mu}{2} |x - y|^2, \quad x, y \in \mathbb{R}^d. \)

The function \( f : \mathbb{R}^d \to \mathbb{R} \) is \( L \)-smooth if

(\( L \)-smooth) \( f(x) - (f(y) - \nabla f(y) \cdot (x - y)) \geq \frac{1}{2L} |\nabla f(x) - \nabla f(y)|^2, \quad x, y \in \mathbb{R}^d. \)
We will use the following inequalities in the sequel:

1. \( f(x) - f^* \geq \mu q(x), \quad x \in \mathbb{R}^d \)
2. \( f(x^*) - (f(x_k) + g_k \cdot (x^* - x_k)) \geq \frac{\mu}{2} g_k, \quad x_k \in \mathbb{R}^d \)
3. \( 2L(f(x) - f^*) \geq \|\nabla f(x)\|^2, \quad x \in \mathbb{R}^d \)
4. \( q(x) - q(y) = (x - y) \cdot (y - x^*) + \frac{1}{2} |x - y|^2, \quad x, y \in \mathbb{R}^d. \)

**Proof.** The first three inequalities follow from (\( \mu \)-convex), with choice \( y = x^* \) and with choices \( x = x^* \) and \( y = x_k \), and from (L-smooth), with choice \( y = x^* \), respectively. To establish (4) directly, use the algebraic identity \( a^2 - b^2 = (a + b)(a - b) \) applied to \( q(x) - q(y) \) to obtain

\[
q(x) - q(y) = \frac{1}{2} (x + y - 2x^*) \cdot (x - y) \\
= (x - y) \cdot \left( \frac{x + y}{2} - x^* \right) \\
= (x - y) \cdot (y - x^*) + \frac{1}{2} |x - y|^2.
\]

2. Gradient Descent with Polyak’s step size

In this section, we prove a convergence rate for Polyak’s step size in the full gradient case. Polyak’s step size [Pol87] is most often used in the subgradient case, as discussed above. This result is not needed in the sequel, but it is included for context, and because the proof is a simplified version of the proof in the stochastic case.

Define the Polyak \( x \)-dependent learning rate by

\[
(PLR) \quad h(x) = \frac{2f(x) - f^*}{\|\nabla f(x)\|^2}.
\]

The adaptive gradient descent sequence is given by

\[
(AGD) \quad x_{k+1} = x_k - h_k \nabla f(x_k),
\]
where \( 0 < h_k = h(x_k) \). Next, we prove convergence of the iterates \( x_k \) to \( x^* \) in terms of the difference squared, \( q(x) = \frac{1}{2} |x - x^*|^2 \).

**Lemma 2.1.** Suppose that \( f(x) \) is \( \mu \)-strongly convex and \( L \)-smooth. Let \( x_k, h_k \) be the sequence given by (AGD) and (PLR). Then,

\[
\frac{1}{L} \leq h_k \leq \frac{1}{\mu}
\]

and

\[
q_k \leq (1 - \mu/L)^k q_0.
\]
Proof of Lemma 2.1. The inequality (5) follows from (3) and (1). Using the identity (4) with $x_{k+1}$ and $x_k$ we obtain,

$$q_{k+1} - q_k = (x_{k+1} - x_k)(x_k - x^*) + \frac{1}{2}(x_{k+1} - x_k)^2$$

by (4)

$$= -h_k g_k(x_k - x^*) + \frac{1}{2}h_k^2 g_k^2$$

by (AGD)

$$\leq -h_k \mu q_k - h_k(f(x_k) - f^*) + \frac{1}{2}h_k^2 g_k^2$$

by (2)

$$\leq -h_k \mu q_k$$

by (PLR)

$$\leq -\mu q_k / L$$

by (5) \qquad \Box

One outcome of the Polyak’s learning rate (PLR) is that the usual restriction $h_k \leq 1/L$ on the learning rate is relaxed to $h_k \leq 1/\mu$. For example, on a quadratic $f(x) = (\mu x_1^2 + Lx_2^2)/2$, whenever $x_2 = 0$, we have $h(x) = 1/\mu$. More generally, Figure 2 illustrates $h(x)$. Clearly from (PLR), smaller gradients allow for larger learning rates and vice-versa.

3. Stochastic Gradient Descent with Polyak’s Step

Now we consider stochastic gradient descent. For the purpose of our analysis, we use an abstract representation, where we write $\nabla f(x)$ for the full gradient and are given an approximation $\nabla_{mb} f(x)$ such that

$$\nabla_{mb} f(x) = \nabla f(x) + e,$$

with $e$ a random error term.

The notation $\nabla_{mb}$ is meant to suggest mini-batch SGD, the important special case where the loss is of the form, $f(x) = \sum f_i(x)$, and a mini-batch approximation is give by $f_{mb}(x) = \sum_{i \in mb} f_i(x)$. However, in our analysis we consider a general

**Figure 1.** Level sets of $f$ superimposed on level sets of $h(x) = 2(f(x) - f^*)/\nabla f(x)^2$. 

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The following text describes the figure and provides a detailed explanation of the mathematical concepts involved. The figure illustrates the level sets of $f$ and $h(x)$, showing how the gradient descent process adapts to the landscape of the function. The notation and mathematical identities used in the text are applied to demonstrate the behavior of the learning rate and its implications on the convergence of the optimization process.
loss function and abstract the mini-batch error into the error term. For clarity, we assume the error is random with zero mean and finite variance,

\[ E[e] = 0, \quad E[e^2] = \sigma^2. \]

More general assumptions on the error are discussed in [BCN16]. If variance reduction [JZ13] [ZMJ13] is incorporated, it is reflected in the \( \sigma^2 \) term in (6).

3.1. Scheduled and Polyak SGD. The SGD step is given by

\[ x_{k+1} = x_k - h_k \nabla_{mb}f(x_k), \]

where \( h_k > 0 \) is the learning rate.

In order to achieve the optimal convergence rate for scheduled SGD, the schedule makes use of \( \mu \) and \( q_0 = \|x_0 - x^*\|^2 / 2. \) The optimal scheduled convergence rate is achieved using the following schedule

\[ h_k = \frac{1}{\mu (k + q_0^{-1} \alpha^*_S)}, \]

where \( \alpha^*_S \) is defined below in (7). The proof (in a more general setting) can be found in [BCN16]; we provide a shorter proof below.

For Polyak SGD we define the stochastic Polyak learning rate as a function of \( x_k, \sigma^2, \) and \( f^*. \)

\[ h_k = h(x_k) = 2 \frac{f(x_k) - f^*}{E[\|\nabla_{mb}f(x_k)\|^2]}. \]

Remark 3.1. From (SPLR) we can conclude that increasing the mini-batch size will also increase the learning rate, since increased mini-batch size decreases the variance of the \( \nabla_{mb}f(x). \)

A similar learning rate schedule, replacing the expectation in (SPLR) with \( \|\nabla_{mb}f(x_k)\|^2 \) was implemented effectively in [RM18], however no convergence proof is presented.

Remark 3.2. To achieve the optimal constant in the convergence rate for scheduled SDG using (SLR) requires knowing \( q_0 \) and \( \mu. \) The stochastic Polyak learning rate (SPLR) requires knowing or estimating \( f^*, \) as well as the variance of the stochastic gradient, which can be easily approximated. To reduce the computational cost, we can evaluate (SPLR) every fixed number of number of iterations (or epochs).

Convergence rates for SGD using both the optimally scheduled and Polyak learning rates are proven in Theorem 3.4 and 3.5 below, respectively.

3.2. Basic inequality. We begin by proving an inequality which we be used for the proof of the convergence rate for both methods.

Lemma 3.3. Suppose \( f \) is \( \mu \)-strongly convex. Assume that (6) holds. Let \( x_{k+1} \) be given by (SGD) for any \( h_k > 0. \) Then,

\[ E[q_{k+1} | x_k] \leq (1 - \mu h_k)q_k - h_k(f(x_k) - f^*) + \frac{1}{2} h_k^2 (g_k^2 + E[e^2]). \]
Proof. We start by estimating the difference between consecutive iterates,
\[ q_{k+1} - q_k = (x_{k+1} - x_k)(x_k - x^*) + \frac{1}{2}(x_{k+1} - x_k)^2 \]  
by (4)
\[ = -h_k(g_k + e)(x_k - x^*) + \frac{1}{2}h_k^2(g_k + e)^2 \]  
by (SGD)
\[ \leq -h_k\mu q(x_k) - h_k(f(x_k) - f^*) - h_k e(x_k - x^*) + \frac{1}{2}h_k^2(g_k + e)^2 \]  
by (2).
Then, by taking the expectations of the last inequality and conditioning on \(x_k\) we obtain,
\[ \mathbb{E}[q_{k+1} | x_k] \leq (1 - \mu h_k)q_k - h_k(f(x_k) - f^*) - h_k\mathbb{E}[e(x_k - x^*)] \]
\[ + \frac{1}{2}h_k^2\mathbb{E}[(g_k + e)^2]. \]
Note that, by assumption (6), it follows that
\[ \mathbb{E}[e(x_k - x^*)] = 0. \]
Moreover, we may expand the last term as
\[ \mathbb{E}[(g_k + e)^2] = \mathbb{E}[g_k^2] + 2g_k\mathbb{E}[e] + \mathbb{E}[e^2] = g_k^2 + \mathbb{E}[e^2]. \]
Thus,
\[ \mathbb{E}[q_{k+1} | x_k] \leq (1 - \mu h_k)q_k - h_k(f(x_k) - f^*) + \frac{1}{2}h_k^2(g_k^2 + \mathbb{E}[e^2]), \]
which establishes the inequality. \(\square\)

3.3. Scheduled SGD Rate. Now, we prove the rate for SGD with schedule (SLR). We use this proof structure to easily draw a parallel between the different learning rate choices, (SLR) and (SPLR).

Theorem 3.4. Suppose \(f\) is \(\mu\)-strongly convex and \(L\)-smooth. Assume (6). Let \(x_k, h_k\) be the sequence given by (SGD) with the optimal learning rate schedule (SLR). Set \(M = \max_k \|g_k\|\). Then,
\[ \mathbb{E}[q_k | x_{k-1}] \leq \frac{1}{\alpha S k + q_0^{-1}}, \text{ for all } k \geq 0, \]
where
\[ \alpha_S = \frac{2\mu^2}{\sigma^2 + M^2}. \]  
(7)
Proof. We prove the rate by induction. Clearly, it holds for \(k = 0\). For the induction step, assume it the rate holds and define \(\hat{k} = k + q_0^{-1}\alpha_S^{-1}. \) Then, from Lemma 3.3,
\[ \mathbb{E}[q_{k+1}] \leq (1 - 2\mu h_k)q_k + \frac{1}{2}h_k^2(g_k^2 + \mathbb{E}[e^2]) \]
\[ \leq (1 - \frac{2}{\hat{k}}) q_k + \frac{1}{2\mu^2\hat{k}^2}(M^2 + \sigma^2) \]  
by (SLR) and assumption
\[ \leq (1 - \frac{2}{\hat{k}}) \frac{1}{\alpha S k} + \frac{1}{\alpha S k^2} \]  
by the induction hypothesis
\[ = \left(\frac{\hat{k} - 1}{\hat{k}^2}\right) \frac{1}{\alpha S} \leq \frac{1}{\alpha S} \frac{1}{k + 1} = \frac{1}{\alpha S(k + 1) + q_0^{-1}}, \]
which proves the rate. \(\square\)
3.4. Polyak SGD Rate. Now we prove the rate for Polyak SGD, (SGD) using (SPLR).

**Theorem 3.5.** Suppose $f$ is $\mu$-strongly convex and $L$-smooth. Assume the mean and variance of $e$ are given by (6). Let $x_k$, $h_k$ be the sequence given by (SGD) and (SPLR). Then,

$$\mathbb{E}[q_k | x_{k-1}] \leq \frac{1}{\alpha_p k + q_0}, \quad \text{for all } k \geq 0,$$

where

$$\alpha_p = \frac{2\mu^2}{\sigma^2 + 2\mu^2(L - \mu)q_0}.$$

The proof of the theorem requires two auxiliary Lemmas. Lemma 3.6 provides an inequality for $q_{k+1}$ in terms on the previous iterate, $q_k$. Lemma 3.7 provides a further bound. We combine both in an induction proof at the end of the section.

**Lemma 3.6.** Suppose $f$ is $\mu$-strongly convex and $L$-smooth. Let $x_k$, $h_k$ be the sequence given by (SGD) and (SPLR). Assume that (6) holds. Then,

$$\mathbb{E}[q_{k+1} | x_k] \leq (1 - \mu h_k)q_k \leq \frac{\beta + r q_k}{\beta + q_k} q_k,$$

where $r = 1 - \frac{\mu}{L}$ and $\beta = \frac{\sigma^2}{2\mu L}$.

**Proof.** Apply Lemma 3.3 with $h_k$ given by (SPLR) to obtain

$$\mathbb{E}[q_{k+1} | x_k] \leq (1 - \mu h_k)q_k.$$

We have the following estimate on the learning rate,

$$h_k = \frac{f_k - f^*}{|g_k|^2 + \sigma^2} \quad \text{by (SPLR)}$$

$$\geq \frac{f_k - f^*}{2L(f_k - f^*) + \sigma^2} \quad \text{by (3)}$$

$$= \frac{2}{2L + \frac{\sigma^2}{f_k - f^*}} \quad \text{divide by } f_k - f^*$$

$$\geq \frac{2}{2L + \frac{\sigma^2}{\mu q_k}} \quad \text{by (1)}.$$

This establishes the second inequality in (8) as follows,

$$(1 - \mu h_k)q_k \leq \left(1 - \frac{2\mu}{2L + \frac{\sigma^2}{\mu q_k}}\right)q_k \leq \frac{2Lq_k + \frac{\sigma^2}{\mu} - 2\mu q_k}{2L + \frac{\sigma^2}{\mu q_k}}$$

$$= \frac{(1 - \frac{\mu}{L})q_k + \frac{\sigma^2}{2\mu L}}{q_k + \frac{\sigma^2}{2\mu L}} q_k$$

$$= \frac{rq_k + \beta}{q_k + \beta} q_k,$$

where $r = 1 - \frac{\mu}{L}$ and $\beta = \frac{\sigma^2}{2\mu L}$. \qed
Next, we establish the following technical, but elementary Lemma, which is used to prove the rate.

**Lemma 3.7.** Define the transformation $T$, $$T(x) = \frac{\beta + rx}{\beta + x} x,$$
for $\beta > 0$ and for $r \in [0,1)$. Given $b > 0$ define
$$c = \frac{1 - r}{\beta + r/b}.$$ 
Take $x_0 > 0$, and assume the sequence $\{x_n\}$ satisfies
$$x_{n+1} \leq T(x_n).$$ 
Then, for all $k \geq 0$,
$$T(ck + b) \leq \frac{1}{c(k+1) + b}.$$ 

**Proof.** Consider the auxiliary sequence $$y_n = \frac{1}{x_n}.$$ 
Then, $y_{n+1} \geq S(y_n)$, where $S$ is the multiplicative inverse of $T$, $$S(y) = y^\beta \frac{y + 1}{\beta y + r}.$$ 
In fact,
$$y_{n+1} = \frac{1}{x_{n+1}} \geq \frac{1}{T(x_n)} = \frac{1}{x_n} \frac{\beta + x_n}{\beta + r x_n} = y_n \frac{\beta y_n + 1}{\beta y_n + r} = S(y_n).$$ 
Note that $S$ is increasing monotonically, since $r \in [0,1)$ and $y_0 > 0$.

Next, we consider the sequence $a_k := ck + b$. We start by expanding the difference:
$$S(a_k) - a_{k+1} = a_k \frac{\beta a_k + 1}{\beta a_k + r} - (a_k + c)$$
$$= \frac{a_k (1 - r - c\beta) - cr}{\beta a_k + r}$$
$$= \frac{kc(1 - r - c\beta) + (b(1 - r - c\beta) - cr)}{\beta ck + (\beta b + r)}.$$ 
This difference is positive if both the denominator and numerator are positive for $k \geq 0$. The denominator’s positivity follows from $b, c, r, \beta \geq 0$. The numerator is positive for all $k \geq 0$ provided
$$1 - r - c\beta \geq 0,$$
since the choice of $c$ guarantees $b(1 - r - c\beta) - cr = 0$. In fact, positivity of this expression also holds for our choice of $c$.

Use the fact $S(a_k) \geq a_{k+1}$, and rewrite in terms of $T$ and $a_k$ to obtain
$$T(a_k) \leq \frac{1}{a_{k+1}} = \frac{1}{c(k+1) + b},$$
and the lemma follows. \qed
Proof of Theorem 3.5. We proceed by induction. The base case corresponds to \( q_0 \leq \frac{1}{\alpha_P k + q_0} \) for \( k = 0 \), which clearly holds.

For the induction step, assume that \( q_k \leq \frac{1}{\alpha_P k + q_0} \). We will apply Lemma 3.7 with \( b = q_0^{-1} \) and the corresponding \( c \).

\[
\mathbb{E}[q_{k+1}|x_k] \leq T(q_k) \quad \text{by Lemma 3.6}
\]
\[
\leq T\left(\frac{1}{\alpha_P k + q_0^{-1}}\right) \quad \text{by monotonicity of } T \text{ and hypothesis}
\]
\[
\leq \frac{1}{\alpha_P (k + 1) + q_0^{-1}} \quad \text{by Lemma 3.7},
\]
which concludes the induction. \( \square \)

3.5. Comparison of the Polyak and Scheduled Rate Constants. The constant in the rate established in Theorem 3.5 for Polyak SGD can be smaller than the rate for scheduled SGD in Theorem 3.4. Recall the rates are given by,

\[
\alpha_S = \frac{2\mu^2}{\sigma^2 + M^2}, \quad \alpha_P = \frac{2\mu^2}{\sigma^2 + 2\mu^2(L - \mu)q_0}.
\]

Now, we note that \( \alpha_P \geq \alpha_S \) if

\[
q_0 \leq \frac{M^2}{2\mu^2(L - \mu)}.
\]

Since we can reset the algorithm, this condition will eventually hold, using the reset value for \( q_0 \).

4. Numerical Results

In this section, we implement the Polyak adaptive SGD algorithm and compare it to scheduled SGD. The Polyak SGD code is implemented in PyTorch\(^1\). For practical implementations, we capped the learning rates with a minimum and maximum value to protect against errors in the estimation of \( f^* \).

The first problem we considered was

\[
f(x) = \frac{1}{2N} \sum_{i=1}^{N} \|x - x_i\|^2,
\]
for a given data set \( x_1, \ldots, x_N \in \mathbb{R}^2 \). The stochastic gradients were obtained by taking a different random minibatch of fixed size \( M \) at each iteration. A minibatch of size \( M \) is a random subset \( I \subset [1, \ldots, N] \) with \( |I| = M \). Then,

\[
f_I = \frac{1}{2|I|} \sum_{i \in I} \|x - x_i\|^2
\]
and the stochastic gradient is the gradient of \( f_I \).

In Figure 2, we illustrate the learning rate as a function of the mini-batch size and the location of the current iterate. As expected, see (SPLR), the learning rate increases when the mini-batch noise is smaller, which happens when we have a larger mini-batch.

\(^1\)https://github.com/marianapraz/polyakSGD
We see, in Figure 3, how Polyak SGD compares to two schedules of SGD. The first schedule is a typical deep learning schedule that reduces the learning rate every hundred steps by a fixed amount (in this case, a sixth). The second is the optimal schedule (SLR).

In Figure 4, we share a special case where we start close to an optimal value, which Polyak immediately recognizes by lowering the learning rate.

4.1. Deep Learning Example. We trained an AllCNN architecture, [SDBR14], for image classification on CIFAR-10. Our baseline was trained using SGD and the schedule we used reduces the learning rate by a factor of five every 60 epochs. We used no momentum and no regularization. In Figure 5, we see that we obtain similar results in minimizing the training loss without impacting the testing error (the most relevant metric for the success of a neural net training). The schedule of SGD for the CIFAR-10 dataset has already been tuned very well, so it performs similarly to Polyak SGD, which required no tuning. The estimate of $f^*$ was obtained with one run of scheduled SGD. In practise, networks are trained many times using similar parameters, so Polyak SGD could provided some advantages in training time when averaged over many runs.

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Figure 3. Comparison of SGD with Polyak (adaptive) SGD. Top left: paths of optimally scheduled SGD, epoch scheduled SGD, and Polyak (adaptive) SGD. From one path it is not clear which algorithm is faster. Bottom left: average excess loss over 40 runs of each algorithm. Now it is clear that the Polyak algorithm is faster on average. Top right: plots of a path. Bottom right: illustration of the learning rates computed.

Figure 4. Comparison of SGD with Polyak (adaptive) SGD. In the case where the initial value is close to optimal, (non-optimally) scheduled SGD forgets the good initialization. On the other hand, the Polyak method detects the good initialization and improves the values.
Figure 5. Comparison of scheduled SGD with Polyak SGD. The first plot is the training loss and the second one is the test error.

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