On quadratic optimization problems and canonical duality theory

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September 25, 2018

Canonical duality theory (CDT) is advertised by its author DY Gao as “a breakthrough methodological theory that can be used not only for modeling complex systems within a unified framework, but also for solving a large class of challenging problems in multidisciplinary fields of engineering, mathematics, and sciences.”

DY Gao solely or together with some of his collaborators applied CDT for solving some quadratic optimization problems with quadratic constraints. Unfortunately, in almost all papers we read on CDT there are unclear definitions, non convincing arguments in the proofs, and even false results.

The aim of this paper is to treat rigorously quadratic optimization problems by the method suggested by CDT and to compare what we get with the results obtained by DY Gao and his collaborators on this topic in several papers.

1 Notations and preliminary results

Let us consider the quadratic functions $q_k : \mathbb{R}^n \rightarrow \mathbb{R}$ for $k \in \overline{0,m}$, that is $q_k(x) := \frac{1}{2} \langle x, A_k x \rangle - \langle b_k, x \rangle + c_k$ for $x \in \mathbb{R}^n$ with given $A_k \in \mathcal{S}_n$, $b_k \in \mathbb{R}^n$ (seen as column vector) and $c_k \in \mathbb{R}$ for $k \in \overline{0,m}$, where $\mathcal{S}_n$ denotes the class of symmetric matrices from $\mathcal{M}_n := \mathbb{R}^{n \times n}$, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^n$. For $k \in \mathbb{N}^*$ we set

$$ \mathbb{R}_k^+ := \{ \eta \in \mathbb{R}^k \mid \eta_i \geq 0 \ \forall i \in \overline{1,k} \}, \quad \mathbb{R}_k^- := -\mathbb{R}_k^+, \quad \mathbb{R}_k^{++} := \text{int} \mathbb{R}_k^+, \quad \mathbb{R}_k^{--} := -\mathbb{R}_k^{++}. $$

The fact that $A \in \mathcal{S}_n$ is positive (semi) definite is denoted by $A > 0$ ($A \succeq 0$) and we set $\mathcal{S}_n^+ := \{ A \in \mathcal{S}_n \mid A > 0 \}$, $\mathcal{S}_n^{++} := \{ A \in \mathcal{S}_n \mid A > 0 \}$; it is well known that $\mathcal{S}_n^{++} = \text{int} \mathcal{S}_n^+$. In this paper we consider quadratic minimization problems with (quadratic) equality and inequality constraints. With this aim, we fix a set $J \subset \overline{1,m}$ corresponding to the equality constraints; the set $J^c := \overline{1,m} \setminus J$ will correspond to the inequality constraints. So, the general problem is

$$ (P_J) \quad \min q_0(x) \text{ s.t. } x \in X_J, $$

where

$$ X_J := \{ x \in \mathbb{R}^n \mid [\forall j \in J : q_j(x) = 0] \land [\forall j \in J^c : q_j(x) \leq 0] \}. $$

For later use we introduce also the set

$$ \Gamma_J := \{ (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m \mid \lambda_j \geq 0 \ \forall j \in J^c \}. $$
Clearly, for $J = \overline{1,m}$ $(P_J)$ becomes the quadratic minimization problem with (quadratic) equality constraints denoted $(P_e)$ with $X_e := X_{\overline{1,m}}$ its feasible set, while for $J = \emptyset$ $(P_J)$ becomes the quadratic minimization problem with inequality constraints denoted $(P_i)$ with $X_i := X_\emptyset$ its feasible set. Clearly $X_e \subset X_J \subset X_i$, the inclusions being strict in general when $\emptyset \neq J \neq \overline{1,m}$. Observe that any optimization problem with equality constraints can be seen as a problem with inequality constraints because the equality constraint $h(x) = 0$ can be replaced by the inequality constraints $g_1(x) := h(x) \leq 0$ and $g_2(x) := -h(x) \leq 0$. Excepting linear programming, such a procedure is not used in general because the constraints qualification conditions are very different for problems with equality constraints and those with inequality constraints.

To the family $(q_k)_{k \in \overline{0,m}}$ we associate the Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$L(x, \lambda) := q_0(x) + \sum_{j=1}^{m} \lambda_j q_j(x) = \frac{1}{2} \langle x, A(\lambda) x \rangle - \langle x, b(\lambda) \rangle + c(\lambda),$$

where $A(\lambda)x := [A(\lambda)] \cdot x$ and

$$A(\lambda) := \sum_{k=0}^{m} \lambda_k A_k, \quad b(\lambda) := \sum_{k=0}^{m} \lambda_k b_k, \quad c(\lambda) := \sum_{k=0}^{m} \lambda_k c_k,$$

with $\lambda_0 := 1$ and $\lambda := (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m$. Clearly, $A : \mathbb{R}^m \to \mathcal{S}_n$, $b : \mathbb{R}^m \to \mathbb{R}^n$, $c : \mathbb{R}^m \to \mathbb{R}$ defined by the above formulas are affine mappings.

Moreover, one considers the sets

$$Y_0 := \{ \lambda \in \mathbb{R}^m \mid \det A(\lambda) \neq 0 \}, \quad (1)$$

$$Y^+ := \{ \lambda \in \mathbb{R}^m \mid A(\lambda) > 0 \}, \quad Y^- := \{ \lambda \in \mathbb{R}^m \mid A(\lambda) < 0 \}. \quad (2)$$

Observe that $Y_0$ is a (possibly empty) open set, while $Y^+$ and $Y^-$ are (possibly empty) open and convex sets. Sometimes one uses also the sets

$$Y_{\text{col}} := \{ \lambda \in \mathbb{R}^m \mid b(\lambda) \in \text{Im} A(\lambda) \}, \quad (3)$$

$$Y^+_{\text{col}} := \{ \lambda \in Y_{\text{col}} \mid A(\lambda) \succeq 0 \}, \quad Y^-_{\text{col}} := \{ \lambda \in Y_{\text{col}} \mid A(\lambda) \preceq 0 \}, \quad (4)$$

where for $F \in \mathbb{R}^{m \times n}$ we set $\text{Im} F := \{ Fx \mid x \in \mathbb{R}^n \}$ and $\ker F := \{ x \in \mathbb{R}^n \mid Fx = 0 \}$. Clearly, $Y_0 \subset Y_{\text{col}}$, $Y^+ \subset Y^+_{\text{col}}$, $Y^- \subset Y^-_{\text{col}}$, and $Y_{\text{col}}$ is neither open, nor closed (in general). Unlike for $Y^+$, the convexity of $Y^+_{\text{col}}$ is less obvious. In fact the next (probably known) result holds.

**Lemma 1** (i) Let $A, B \in \mathcal{S}_n^+$. Then $\text{Im}(A + B) = \text{Im} A + \text{Im} B$.

(ii) Let $A \in \mathcal{S}_n$ and $a \in \mathbb{R}^n$, and set $q(x) := \frac{1}{2} \langle x, Ax \rangle - \langle a, x \rangle$. Then $q(x_1) = q(x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$ such that $Ax_1 = Ax_2 = a$.

Proof. (i) It is known that $\text{Im} F = (\ker F)^\perp$, and so $\mathbb{R}^n = \text{Im} F + \ker F$, provided $F \in \mathcal{S}_n$. Moreover, using Schwarz’ inequality for positive semi-definite matrices (operators) we have that $\ker F = \{ x \in \mathbb{R}^n \mid (x, Fx) = 0 \}$ whenever $F \in \mathcal{S}_n^+$. Since $A + B \in \mathcal{S}_n^+$ we get

$$(\text{Im}(A + B))^\perp = \ker(A + B) = \{ x \in \mathbb{R}^n \mid (x, (A + B)x) = 0 \}$$

$$= \ker A \cap \ker B = (\text{Im} A)^\perp \cap (\text{Im} B)^\perp = (\text{Im} A + \text{Im} B)^\perp,$$

whence the conclusion.
(ii) Take \( x_1, x_2 \in \mathbb{R}^n \) such that \( Ax_1 = Ax_2 = a; \) setting \( x := x_1 \) and \( u := x_2 - x_1, \) we have that \( x_2 = x + u \) and \( Au = 0. \) It follows that \( \langle a, u \rangle = \langle Ax, u \rangle = \langle x, Au \rangle = 0, \) and so
\[
q(x + u) = \frac{1}{2} \langle x + u, A(x + u) \rangle - \langle a, x + u \rangle = \frac{1}{2} \langle x, Ax \rangle - \langle a, x \rangle = q(x),
\]
whence \( q(x_2) = q(x_1). \)

\[\square\]

**Corollary 2** With the previous notations and assumptions, \( Y_+^\text{col} \) and \( Y_-^\text{col} \) are convex. Moreover, if \( Y_+ \) (resp. \( Y_- \)) is nonempty, then \( Y_+ = \text{int} \ Y_+^\text{col} \) (resp. \( Y_- = \text{int} \ Y_-^\text{col} \)).

**Proof.** Take \( \lambda, \lambda' \in Y_+^\text{col} \) and \( \alpha \in (0, 1). \) From the definition of \( Y_+^\text{col} \) and Lemma 1 (i), taking into account that \( A \) and \( b \) are affine, we get
\[
b(\alpha \lambda + (1 - \alpha) \lambda') = \alpha b(\lambda) + (1 - \alpha) b(\lambda') \in \alpha \text{Im} A(\lambda) + (1 - \alpha) \text{Im} A(\lambda')
\]
\[
= \text{Im}[(\alpha A(\lambda)] + \text{Im}[(1 - \alpha)A(\lambda')] = \text{Im}[\alpha A(\lambda) + (1 - \alpha)A(\lambda)]
\]
\[
= \text{Im} A(\alpha \lambda + (1 - \alpha) \lambda'),
\]
and so \( \alpha \lambda + (1 - \alpha) \lambda' \in Y_+^\text{col}. \) The proof of the convexity of \( Y_-^\text{col} \) is similar.

Assume now that \( Y_+ \neq \emptyset \) and take \( \lambda_0 \in Y_+, \lambda \in Y_+^\text{col} \) and \( \alpha \in (0, 1). \) Then \( A(\alpha \lambda_0 + (1 - \alpha) \lambda) = \alpha A(\lambda_0) + (1 - \alpha)A(\lambda) \succ 0, \) and so \( \alpha \lambda_0 + (1 - \alpha) \lambda \in Y_+. \) Taking the limit for \( \alpha \to 0 \) we obtain that \( \lambda \in \col Y_+. \) Hence \( Y_+ \subset Y_+^\text{col} \subset \col Y_+, \) and so
\[
Y_+ = \text{int} \ Y_+ \subset \text{int} \ Y_+^\text{col} \subset \text{int} (\col Y_+) = Y_+.
\]

The proof is complete. \[\square\]

Of course, for every \( (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \) we have that
\[
\nabla_x L(x, \lambda) = A(\lambda) \cdot x - b(\lambda), \quad \nabla^2_x L(x, \lambda) = A(\lambda), \quad \nabla_\lambda L(x, \lambda) = (q_j(x))_{j \in 1, m}.
\]

Hence \( L(\cdot, \lambda) \) is (strictly) convex for \( \lambda \in Y_+^\text{col} \) (\( \lambda \in Y_+ \)) and (strictly) concave for \( \lambda \in Y_-^\text{col} \) (\( \lambda \in Y_- \)). Moreover, for \( \lambda \in Y_0 \) we have that \( \nabla_x L(x, \lambda) = 0 \iff x = [A(\lambda)]^{-1} \cdot b(\lambda), \) written \( A(\lambda)^{-1} b(\lambda) \) in the sequel.

Let us consider now the (dual objective) function
\[
D : Y_\text{col} \to \mathbb{R}, \quad D(\lambda) := L(x, \lambda) \text{ with } A(\lambda)x = b(\lambda);
\]
\[\text{(6)}\]

\( D \) is well defined because for \( x_1, x_2 \in \mathbb{R}^n \) with \( A(\lambda)x_1 = A(\lambda)x_2 = b(\lambda), \) by Lemma 1 (ii), we have that \( L(x_2, \lambda) = L(x_1, \lambda). \) In particular,
\[
[\lambda \in Y_0 \land x = (A(\lambda))^{-1} \cdot b(\lambda)] \implies L(x, \lambda) = D(\lambda).
\]

Of course
\[
D(\lambda) = L(A(\lambda)^{-1} b(\lambda), \lambda) = -\frac{1}{2} \langle b(\lambda), A(\lambda)^{-1} b(\lambda) \rangle + c(\lambda) \quad \forall \lambda \in Y_0.
\]

\[\text{(7)}\]

**Lemma 3** Let \((\overline{\pi}, \overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m\) be such that \( \nabla_x L(\overline{\pi}, \overline{\lambda}) = 0 \) and \( \langle \overline{\lambda}, \nabla_\lambda L(\overline{\pi}, \overline{\lambda}) \rangle = 0. \) Then \( \overline{\lambda} \in Y_\text{col} \) and
\[
g_0(\overline{\pi}) = L(\overline{\pi}, \overline{\lambda}) = D(\overline{\lambda}).
\]

\[\text{(8)}\]

In particular, \( \overline{\pi} \in \mathbf{X}_e \) and \( \boxed{} \) hold if \((\overline{\pi}, \overline{\lambda})\) is a critical point of \( L, \) that is \( \nabla L(\overline{\pi}, \overline{\lambda}) = 0. \)

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In particular, (10) holds if $L(\bar{x}, \bar{\lambda}) = D(\bar{\lambda})$ by the definition of $D$. On the other hand,

$$L(\bar{x}, \bar{\lambda}) = q_0(\bar{x}) + \sum_{j=1}^{m} \lambda_j q_j(\bar{x}) = q_0(\bar{x}) + \langle \bar{\lambda}, \nabla L(\bar{x}, \bar{\lambda}) \rangle = q_0(\bar{x}).$$

The last assertion follows from the expression of $\nabla \lambda L(\bar{x}, \bar{\lambda})$ in (5). \hfill \Box

Formula (5) is related to the so-called “complementary-dual principle” (see [15] p. NP11, [16] p. 13]) and sometimes is called the “perfect duality formula”.

**Proposition 4**

(i) The following representation of $D$ holds:

$$D(\lambda) = \begin{cases} 
\min_{x \in \mathbb{R}^n} L(x, \lambda) & \text{if } \lambda \in Y^+_\text{col}, \\
\max_{x \in \mathbb{R}^n} L(x, \lambda) & \text{if } \lambda \in Y^-_\text{col},
\end{cases}$$

the value of $D(\lambda)$ being attained at any $x \in \mathbb{R}^n$ such that $A(\lambda)x = b(\lambda)$ whenever $\lambda \in Y^+_\text{col}\cup Y^-_\text{col}$; in particular, $D(\lambda)$ is attained uniquely at $x := A(\lambda)^{-1}b(\lambda)$ for $\lambda \in Y^+ \cup Y^-$. 

(ii) $D$ is concave and upper semicontinuous on $Y^+_\text{col}$, and convex and lower semicontinuous on $Y^-_\text{col}$.

(iii) Let $J \subset \Gamma, m$ and $(\bar{x}, \bar{\lambda}) \in X_J \times \mathbb{R}^m$ be such that $\nabla x L(\bar{x}, \bar{\lambda}) = 0$ and $\langle \bar{\lambda}, \nabla \lambda L(\bar{x}, \bar{\lambda}) \rangle = 0$. Then $\bar{\lambda} \in Y^\circ$; moreover

$$\bar{\lambda} \in \Gamma_J \cap Y^+_\text{col} \implies D(\bar{\lambda}) = \max \{ D(\lambda) \mid \lambda \in \Gamma_J \cap Y^+_\text{col} \},$$

$$\bar{\lambda} \in (-\Gamma_J) \cap Y^-_\text{col} \implies D(\bar{\lambda}) = \min \{ D(\lambda) \mid \lambda \in (-\Gamma_J) \cap Y^-_\text{col} \}.$$

(iv) Assume that $(\bar{x}, \bar{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$ is such that $\nabla L(\bar{x}, \bar{\lambda}) = 0$. Then

$$D(\bar{\lambda}) = \begin{cases} 
\max_{\lambda \in Y^+\text{col}} D(\lambda) & \text{if } \bar{\lambda} \in Y^+_\text{col}, \\
\min_{\lambda \in Y^-\text{col}} D(\lambda) & \text{if } \bar{\lambda} \in Y^-_\text{col},
\end{cases}$$

In particular, (10) holds if $\bar{\lambda} \in Y^+ \cup Y^-$ is a critical point of $D$ and $\bar{x} := x(\bar{\lambda})$.

Proof. (i) Consider $\lambda \in Y^+_\text{col}$; then there exists $u \in \mathbb{R}^n$ such that $A(\lambda)u = b(\lambda)$, and so $\nabla x L(u, \lambda) = A(\lambda)u - b(\lambda) = 0$. Because $L(\cdot, \lambda)$ is convex we obtain that $L(u, \lambda) \leq L(u', \lambda)$ for every $u' \in \mathbb{R}^n$, whence $D(\lambda) = L(u, \lambda) = \min_{u' \in \mathbb{R}^n} L(u', \lambda)$. Of course, if $\lambda \in Y^+$ then $L(\cdot, \lambda)$ is strictly convex and $u = A(\lambda)^{-1}b(\lambda)$, and so $A(\lambda)^{-1}b(\lambda)$ is the unique minimizer of $L(\cdot, \lambda)$ on $\mathbb{R}^n$. The case $\bar{\lambda} \in Y^-$ is solved similarly.

(ii) Because $L(x, \cdot)$ is linear (hence concave and convex) for every $x \in \mathbb{R}^n$, from (2) we obtain that $D$ is concave and u.s.c. on $Y^+_\text{col}$ as an infimum of concave continuous functions. The argument is similar for the other situation.

(iii) Assume that $\bar{\lambda} \in Y^+_\text{col}$ (hence $\bar{\lambda} \in \Gamma_J \cap Y^+_\text{col}$), and take $\lambda \in \Gamma_J \cap Y^+_\text{col}$. Using (2) and the fact that $\bar{x} \in X_J$, we have that

$$D(\lambda) \leq L(\bar{x}, \lambda) = q_0(\bar{x}) + \sum_{j \in J^c} \lambda_j q_j(\bar{x}) \leq q_0(\bar{x}) + \langle \bar{\lambda}, \nabla \lambda L(\bar{x}, \bar{\lambda}) \rangle = L(\bar{x}, \bar{\lambda}) = D(\bar{\lambda}),$$

and so $D(\bar{\lambda}) = \sup_{\lambda \in \Gamma_J \cap Y^+_\text{col}} D(\lambda)$. The proof for $\bar{\lambda} \in (-\Gamma_J) \cap Y^-_\text{col}$ is similar.

(iv) One applies (iii) for $J := \Gamma, m$. \hfill \Box
Observe that $D$ is a $C^\infty$ function on the open set $Y$ (assumed to be nonempty). Indeed, the operator $\varphi : \{U \in \mathcal{M}_n \mid U \text{ invertible}\} \to \mathcal{M}_n$ defined by $\varphi(U) = U^{-1}$ is Fréchet differentiable and $d\varphi(U)(S) = -U^{-1}SU^{-1}$ for $U, S \in \mathcal{M}_n$ with $U$ invertible. It follows that

$$
\frac{\partial D(\lambda)}{\partial \lambda_j} = \frac{1}{2} \langle b(\lambda), A(\lambda)^{-1}A_jA(\lambda)^{-1}b(\lambda) \rangle - \langle b_j, A(\lambda)^{-1}b(\lambda) \rangle + c_j
$$

for $\lambda \in Y_0$, where

$$
x(\lambda) := A(\lambda)^{-1}b(\lambda) \quad (\lambda \in Y_0) ;
$$

hence

$$
\nabla D(\lambda') = \nabla \lambda L(x(\lambda'), \lambda') \quad \forall \lambda' \in Y_0.
$$

Consequently,

$$
\forall \lambda' \in Y_0 : [\nabla D(\lambda') = 0 \iff \nabla \lambda L(x(\lambda'), \lambda') = 0 \iff \nabla L(x(\lambda'), \lambda') = 0].
$$

A similar computation gives

$$
\frac{\partial^2 D(\lambda)}{\partial \lambda_j \partial \lambda_k} = -\langle A_jA(\lambda)^{-1}b(\lambda), A(\lambda)^{-1}A_kA(\lambda)^{-1}b(\lambda) \rangle + \langle A_jA(\lambda)^{-1}b_k + A_kA(\lambda)^{-1}b_j, A(\lambda)^{-1}b(\lambda) \rangle - \langle b_j, A(\lambda)^{-1}b_k \rangle
$$

for $\lambda \in Y_0$. Omitting $\lambda \in Y_0$, for $v \in \mathbb{R}^m$ and $A_v := \sum_{j=1}^m v_jA_j$, $b_v := \sum_{j=1}^m v_jb_j$, we get

$$
\langle v, \nabla^2 Dv \rangle = \sum_{j,k=1}^m \frac{\partial^2 D}{\partial \lambda_j \partial \lambda_k} v_jv_k = -\langle A_vA^{-1}b - b_v, A^{-1}(A_vA^{-1}b - b_v) \rangle.
$$

Therefore, $\nabla^2 D(\lambda) \preceq 0$ if $\lambda \in Y^+$ and $\nabla^2 D(\lambda) \succeq 0$ if $\lambda \in Y^-$, confirming that $D$ is concave on $Y^+$ and convex on $Y^-$. 

## 2 Quadratic minimization problems with equality constraints

As mentioned above, for $J := \overline{1,m}$, $(P_f)$ becomes the quadratic minimization problem

$$(P_e) \quad \min q_0(x) \quad \text{s.t.} \quad x \in X_e := X_{\overline{1,m}}.$$

Using the previous facts we are in a position to state and prove the following result.

**Proposition 5** Let $(\overline{x}, \overline{x}) \in \mathbb{R}^n \times \mathbb{R}^m$.

(i) Assume that $(\overline{x}, \overline{x})$ is a critical point of $L$. Then $\overline{x} \in X_e$, $\overline{x} \in Y_{\text{col}}$, and (8) holds; moreover, for $\overline{x} \in Y_{\text{col}}^+$ we have that

$$
q_0(\overline{x}) = \inf_{x \in X_e} q_0(x) = L(\overline{x}, \overline{x}) = \sup_{\lambda \in Y_{\text{col}}^+} D(\lambda) = D(\overline{x}),
$$

while for $\overline{x} \in Y_{\text{col}}^-$ we have that

$$
q_0(\overline{x}) = \sup_{x \in X_e} q_0(x) = L(\overline{x}, \overline{x}) = \inf_{\lambda \in Y_{\text{col}}^-} D(\lambda) = D(\overline{x}).
$$
(ii) Assume that \((x, \lambda)\) is a critical point of \(L\) with \(\lambda \in Y_0\). Then \(\nabla D(\lambda) = 0\) and \(x = A(\lambda)^{-1}b(\lambda)\); moreover, \(x\) is the unique global minimizer of \(q_0\) on \(X_e\) when \(\lambda \in Y^+\), and \(x\) is the unique global maximizer of \(q_0\) on \(X_e\) when \(\lambda \in Y^-\).

Conversely, assume that \(\lambda \in Y_0\) is a critical point of \(D\). Then \((x, \lambda)\) is a critical point of \(L\), where \(x = A(\lambda)^{-1}b(\lambda)\); consequently (i) and (ii) apply.

Proof. (i) Assume that \((x, \lambda)\) is a critical point of \(L\); hence \(\nabla_x L(x, \lambda) = 0\) and \(\nabla_\lambda L(x, \lambda) = 0\). Using Lemma 3 we obtain that \(\lambda \in Y_{col, e}\) and \(x \in X_{e}\), and (3) holds.

Assume moreover that \(\lambda \in Y^+_{col}\). Because \(L(\cdot, \lambda)\) is convex, its infimum is attained at \(x\). Therefore, for \(x \in X_e\) we have that \(q_0(x) = L(x, \lambda) \leq L(x, \lambda) = q_0(x)\), and so \(q_0(x) = \inf_{x \in X_e} q_0(x)\). Using Proposition 4 (iii) for \(J := \Gamma, m\) (hence \(\Gamma_J = \mathbb{R}^m\)), we get the last equality in (14). Hence (14) holds.

The proof of (15) in the case \(\lambda \in Y^-_{col}\) is similar; an alternative proof to apply the previous case for \(q_j\) replaced by \(-q_j\) and \(\lambda_j\) by \(-\lambda_j\) for \(j \in \Gamma, m\).

(ii) Assume that \((x, \lambda)\) is a critical point of \(L\) with \(\lambda \in Y_0\). Since \(A(\lambda)^{-1}b(\lambda) = \nabla_x L(x, \lambda) = 0\), clearly \(x = A(\lambda)^{-1}b(\lambda)\). Using (12) we obtain that \(\nabla D(\lambda) = \nabla_\lambda L(x, \lambda) = 0\).

Moreover, suppose that \(\lambda \in Y^+\). Then \(L(\cdot, \lambda)\) is strictly convex, and so \(q_0(x) = L(x, \lambda) < L(x, \lambda) = q_0(x)\) for \(x \in X_e \setminus \{x\}\). Hence \(x\) is the unique global minimizer of \(q_0\) on \(X_e\). The proof in the case \(\lambda \in Y^-\) is similar.

Conversely, let \(\lambda \in Y_0\) be a critical point of \(D\) and take \(x := \lambda \in Y_{col}\); then \(\nabla_x L(x, \lambda) = 0\) by (5). Using (11) we obtain that \(x \in X_{e}\), and so \(\nabla_\lambda L(x, \lambda) = 0\). Therefore, \((x, \lambda)\) is a critical point of \(L\).

The next example shows that \((P_e)\) might have several solutions when \(\lambda \in Y^+_{col}\).

**Example 6** Take \(q_0(x, y) := xy, q_1(x, y) := \frac{1}{2}(x^2 + y^2 - 1)\) for \(x, y \in \mathbb{R}\). Then \(L(x, y, \lambda) = xy + \frac{\lambda}{2}(x^2 + y^2 - 1)\). It follows that \(A(\lambda) = \begin{pmatrix} \lambda & 1 \\ 1 & \lambda \end{pmatrix}, b(\lambda) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, c(\lambda) = -\frac{1}{2}\lambda\), \(Y_0 = \mathbb{R} \setminus \{-1, 1\}, Y^+ = -Y^- = (1, \infty), Y_{col} = \mathbb{R}, Y^+_{col} = Y^-_{col} = [1, \infty), D(\lambda) = -\frac{1}{2}\lambda\).

Clearly, \(D\) has no critical points, and the only critical points of \(L\) are \((\pm 2^{-1/2}, \pm 2^{-1/2}, 1)\) and \((\pm 2^{-1/2}, \pm 2^{-1/2}, -1)\). For \((\pm 2^{-1/2}, \pm 2^{-1/2}, 1)\) we can apply Proposition 5 (i) with \(\lambda := 1 \in Y^+_{col}\), and so both \(\pm 2^{-1/2}(1, -1)\) are solutions for problem \((P_e)\), while for \((\pm 2^{-1/2}, \pm 2^{-1/2}, -1)\) we can apply Proposition 6 (i) with \(\lambda := -1 \in Y^-_{col}\), and so \(\pm 2^{-1/2}(1, 1)\) are global maximizers of \(q_0\) on \(X_e\).

### 3 Quadratic minimization problems with equality and inequality constraints

Let us consider now the general quadratic minimization problem \((P_j)\) considered at the beginning of Section 1. To \((P_j)\) we associate the sets

\[
Y^j := \Gamma_j \cap Y_0, \quad Y^j+ := \Gamma_j \cap Y^+, \quad Y^j- := (-\Gamma_j) \cap Y^-, \\
Y^j_{col} := \Gamma_j \cap Y_{col}, \quad Y^j_{col}+ := \Gamma_j \cap Y^+_{col}, \quad Y^j_{col}- := (-\Gamma_j) \cap Y^-_{col},
\]

where \(Y_0, Y^+, Y^-, Y_{col}, Y^+_{col}\) and \(Y^-_{col}\) are defined in \(\mathbb{I}, \mathbb{P}, \mathbb{R}\) and \(\mathbb{M}\), respectively. Unlike \(Y_0, Y^+, Y^-\), the sets \(Y^j, Y^j+\) and \(Y^j-\) are (generally) not open. Because \(Y^+, Y^+_{col}\) and
$Y_{\text{col}}^-$ are convex, so are $Y_{\text{col}}^{J+}$, $Y_{\text{col}}^{J+}$, and so $L(\cdot, \lambda)$ is (strictly) convex on $Y_{\text{col}}^{J+}$ ($Y_{\text{col}}^{J+}$) and (strictly) concave on $Y_{\text{col}}^{J-}$ ($Y_{\text{col}}^{J-}$); moreover, $\text{int} Y_{\text{col}}^{J+} = \text{int} Y_{\text{col}}^{J-}$ provided $Y_{\text{col}}^{J+} \neq \emptyset$ (int $Y_{\text{col}}^{J-} \neq \emptyset$).

As observed already, for $J = \overline{1, m}$ we have that $\Gamma_J = \mathbb{R}^m$, and so $Y^+, Y^{J+}, Y^{J-}$, $Y_{\text{col}}^{J+}$, $Y_{\text{col}}^{J-}$ reduce to $Y_0$, $Y^+$, $Y_0$, $Y_{\text{col}}^+$ and $Y_{\text{col}}^-$, respectively.

Suggested by the well known necessary optimality conditions for minimization problems with equality and inequality constraints, we say that $(\overline{\pi}, \overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$ is a $J$-LKKK point of $L$ (that is a Lagrange–Karush–Kuhn–Tucker point of $L$ with respect to $J$) if $\nabla_x L(\overline{\pi}, \overline{\lambda}) = 0$ and

$$\left[ \forall j \in J^c : \overline{\lambda}_j \geq 0 \land \frac{\partial L}{\partial \overline{\lambda}_j}(\overline{\pi}, \overline{\lambda}) \leq 0 \land \overline{\lambda}_j \cdot \frac{\partial L}{\partial \overline{\lambda}_j}(\overline{\pi}, \overline{\lambda}) = 0 \right] \lor \left[ \forall j \in J^c : \overline{\lambda}_j = 0 \right],$$

or, equivalently,

$$\overline{\pi} \in X_J \land \overline{\lambda} \in \Gamma_J \lor \left[ \forall j \in J^c : \overline{x}_jq_j(\overline{\pi}) = 0 \right]; \tag{16}$$

we say that $\overline{\pi} \in \mathbb{R}^m$ is a $J$-LKKK point for $(P_J)$ if there exists $\overline{\lambda} \in \mathbb{R}^m$ such that $(\overline{\pi}, \overline{\lambda})$ verifies (16); moreover, for $D$ defined in (6), we say that $\overline{\lambda} \in Y_0$ is a $J$-LKKK point for $D$ if

$$\left[ \forall j \in J^c : \overline{\lambda}_j \geq 0 \land \frac{\partial D}{\partial \overline{\lambda}_j}(\overline{\pi}, \overline{\lambda}) \leq 0 \land \overline{\lambda}_j : \frac{\partial D}{\partial \overline{\lambda}_j}(\overline{\pi}, \overline{\lambda}) = 0 \right] \lor \left[ \forall j \in J^c : \overline{\lambda}_j = 0 \right]. \tag{17}$$

Of course, when $J = \overline{1, m}$, $(\overline{\pi}, \overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$ is a $J$-LKKK point of $L$ iff $\nabla L(\overline{\pi}, \overline{\lambda}) = 0$, while $\overline{\lambda} \in Y_0$ is a $J$-LKKK point for $D$ iff $\nabla D(\overline{\lambda}) = 0$.

**Remark 7** Notice that $\overline{\lambda} \in Y_0$ is a $J$-LKKK point of $D$ if and only if $(x(\overline{\lambda}), \overline{\lambda})$ is a $J$-LKKK point of $L$; for this just take into account (14). Moreover, taking into account (13), if $\overline{\lambda} \in Y_0$ is a critical point of $D$ then $\overline{\lambda}$ is a $J$-LKKK point of $D$ and $(x(\overline{\lambda}), \overline{\lambda})$ is a $J$-LKKK point of $L$ (being a critical point of $L$).

In general, for distinct $J$ and $J'$, the sets of $J$-LKKK and $J'$-LKKK points of $L$ (resp. $D$) are not comparable. For comparable $J$ and $J'$ we have the following result whose simple proof is omitted; its second part follows from the first one and the previous remark.

**Lemma 8** Let $J \subset J' \subset \overline{1, m}$ and $(\overline{\pi}, \overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$.

(i) If $(\overline{\pi}, \overline{\lambda})$ is a $J'$-LKKK point of $L$ and $\overline{\lambda}_j \geq 0$ for all $j \in J' \setminus J$, then $(\overline{\pi}, \overline{\lambda})$ is a $J$-LKKK point of $L$. Conversely, if $(\overline{\pi}, \overline{\lambda})$ is a $J$-LKKK point of $L$ and $\overline{\lambda}_j > 0$ for all $j \in J' \setminus J$, then $(\overline{\pi}, \overline{\lambda})$ is a $J'$-LKKK point of $L$.

(ii) If $\overline{\lambda} \in Y_0$ is a $J'$-LKKK point of $D$ and $\overline{\lambda}_j \geq 0$ for all $j \in J' \setminus J$, then $\overline{\lambda}$ is a $J$-LKKK point of $D$. Conversely, if $\overline{\lambda}$ is a $J$-LKKK point of $D$ and and $\overline{\lambda}_j > 0$ for all $j \in J' \setminus J$, then $\overline{\lambda}$ is a $J'$-LKKK point of $D$.

The result below corresponds to Proposition [5].

**Proposition 9** Let $(\overline{\pi}, \overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$.

(i) Assume that $(\overline{\pi}, \overline{\lambda})$ is a $J$-LKKK point of $L$. Then $\overline{\pi}$ is a $J$-LKKK point of $(P_J)$, $\overline{\pi} \in X_J$, $\overline{\lambda} \in Y_{\text{col}}^{J+}$, and (5) holds; moreover, if $\overline{\lambda} \in Y_{\text{col}}^{J+}$ then

$$q_0(\overline{\pi}) = \inf_{x \in X_J} q_0(x) = L(\overline{\pi}, \overline{\lambda}) = \sup_{\lambda \in Y_{\text{col}}^{J+}} D(\lambda) = D(\overline{\lambda}). \tag{18}$$

---

1It seems that the term Lagrange–Karush–Kuhn–Tucker multiplier was introduced by J.-P. Penot in [23].
(ii) Assume that $(\vec{\pi}, \vec{\lambda})$ is a J-LKKT point of $L$ with $\vec{\lambda} \in Y_0$ (or, equivalently, $\vec{\lambda} \in Y^J$). Then $\vec{x} = A(\vec{\lambda})^{-1}b(\vec{\lambda})$, and $\vec{\lambda}$ is a J-LKKT point of $D$; moreover, $\vec{x}$ is the unique global minimizer of $q_0$ on $X_J$ if $\vec{\lambda} \in Y^{J+}$.

Conversely, assume that $\vec{\lambda} \in Y_0$ is a J-LKKT point of $D$. Then $(\vec{\pi}, \vec{\lambda})$ is a J-LKKT point of $L$, where $\vec{\pi} := A(\vec{\lambda})^{-1}b(\vec{\lambda})$. Consequently, (i) and (ii) apply.

(iii) Assume that $\vec{\lambda} \in Y^{J+}$. Then

$$D(\vec{\lambda}) = \sup_{\lambda \in Y^{J+}} D(\lambda) \iff D(\vec{\lambda}) = \sup_{\lambda \in Y^{J+}} D(\lambda) \iff \vec{\lambda}$$

is a J-LKKT point of $D$.

Proof. (i) By hypothesis, (16) holds. The fact that $\vec{x}$ is a J-LKKT point of $(P_J)$ is obvious from its very definition; hence $\vec{x} \in X_J$. On the other hand, because $(\vec{\pi}, \vec{\lambda})$ is a J-LKKT point of $L$ we have that $\vec{\lambda} \in Y^{J+}$ and (8) holds by Lemma 3.

Assume that $\vec{\lambda} \in Y^{J+}$ (or, equivalently, $\vec{\lambda} \in Y^{J+}$). The last equality in (18) follows from Proposition 4(iii). Because $L(\cdot, \vec{\lambda})$ is convex, its infimum is attained at $\vec{x}$. Therefore, for $x \in X_J$ we have that

$$q_0(\vec{x}) = L(\vec{x}, \vec{\lambda}) \leq L(x, \vec{\lambda}) = q_0(x) + \sum_{j=1}^n \vec{\lambda}_j q_j(x) \leq q_0(x),$$

whence $q_0(\vec{x}) = \inf_{x \in X_J} q_0(x)$. Hence (18) holds.

(ii) Because $(\vec{\pi}, \vec{\lambda})$ is a J-LKKT point of $L$ with $\vec{\lambda} \in Y_0$, we have that $A(\vec{\lambda})\vec{\pi} - b(\vec{\lambda}) = \nabla_x L(\vec{x}, \vec{\lambda}) = 0$, and so $\vec{\pi} = x(\vec{\lambda})$. As observed in Remark 7, (18) is verified.

Suppose now that moreover that $\vec{\lambda} \in Y^{J+}$ (and so then $L(\cdot, \vec{\lambda})$ is strictly convex, and so $q_0(\vec{x}) = L(\vec{x}, \vec{\lambda}) < L(x, \vec{\lambda}) \leq q_0(x)$ for $x \in X_J \setminus \{\vec{x}\}$. Hence $\vec{x}$ is the unique global minimizer of $q_0$ on $X_J$.

Conversely, let $\vec{\lambda} \in Y_0$ be a J-LKKT point of $D$, and take $\vec{\pi} := x(\vec{\lambda})$; then $(\vec{\pi}, \vec{\lambda})$ is a J-LKKT point of $L$ by Remark 7.

(iii) If $\vec{\lambda}$ is a J-LKKT point of $D$, we have that $D(\vec{\lambda}) = \sup_{\lambda \in Y^{J+}} D(\lambda)$ by Remark 7 and (i), while $D(\vec{\lambda}) = \sup_{\lambda \in Y^{J+}} D(\lambda)$ implies $D(\vec{\lambda}) = \sup_{\lambda \in Y^{J+}} D(\lambda)$ because $Y^{J+} \subset Y^{J+}$.

Assume that $D(\vec{\lambda}) = \sup_{\lambda \in Y^{J+}} D(\lambda)$. Setting $Q := -D$, we have that $Q$ is convex and $\vec{\lambda}$ is a global minimizer of $Q$ on (the convex set) $Y^{J+}$. Using [31] Prop. 4) we have that

$$0 \leq Q'(\vec{\lambda}, \lambda - \vec{\lambda}) := \lim_{t \to 0^+} \frac{Q(\vec{\lambda} + t(\lambda - \vec{\lambda})) - Q(\vec{\lambda})}{t} = \langle \lambda - \vec{\lambda}, \nabla Q(\vec{\lambda}) \rangle \quad \forall \lambda \in Y^{J+}.$$

It follows that $\langle y, v \rangle \leq 0$ for all $y \in R^+(Y^{J+} - \vec{\lambda})$, where $v := \nabla D(\vec{\lambda})$. Because $\Gamma_J$ and $Y^{J+}$ are convex sets, $\Gamma_J \cap Y^{J+}$ and $\vec{\lambda} \in \text{int } Y^{J+} = Y^{J+}$, we have that

$$\mathbb{R}^+(Y^{J+} - \vec{\lambda}) = \mathbb{R}^+ \left[ (\Gamma_J - \vec{\lambda}) \cap (Y^{J+} - \vec{\lambda}) \right] = \mathbb{R}^+(\Gamma_J - \vec{\lambda}) = \left\{ \mu \in \mathbb{R}^m \mid \forall j \in J^c : \mu_j = 0 \Rightarrow \mu_j \geq 0 \right\}.$$

Therefore, $\frac{\partial D}{\partial \lambda_j}(\vec{\lambda}) = v_j = 0$ for $j \in J \cup \{ j \in J^c \mid \vec{\lambda}_j > 0 \}$ and $\frac{\partial D}{\partial \lambda_j}(\vec{\lambda}) = v_j \leq 0$ for $j \in \{ j' \in J \mid \vec{\lambda}_{j'} = 0 \}$. This shows that condition (17) is verified.

Corollary 10 Let $\emptyset \neq J \subset \mathbb{R}^m$ and let $(\vec{\pi}, \vec{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$ be a J-LKKT point of $L$ such that $A(\vec{\lambda}) \succeq 0$; hence $\vec{\pi} \in X_J$, $\vec{\lambda} \in Y^{J+}$ and (18) holds. If $J_{\geq} := \{ j \in J \mid \vec{\lambda}_j \geq 0 \}$ is nonempty, then $(\vec{\pi}, \vec{\lambda})$ is a $(J \setminus J_{\geq})$-LKKT point of $L$, and so $\vec{x}$ is a global minimizer of $q_0$ on $X_{J \setminus J_{\geq}} \subset X_J$. 

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Proof. The first assertion holds by Proposition 9 (i) because $\lambda \in Y^{J^+}$. In what concerns
the second assertion, it is sufficient to observe that for $j \in J^c \cup J^\geq = (J \setminus J^\geq)^c$ we have
that $\lambda_j \geq 0$, and $\lambda_j \cdot \nabla_{x_j}(\lambda, \bar{x}) = 0$ by the definition of a $J$-LKKT point of $L$, then to apply
Proposition 9 (i) for $J$ replaced by $J \setminus J^\geq$. □

Corollary 11 If $(\lambda, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^m$ is a critical point of $L$ (in particular if $\lambda \in Y_0$ is a critical
point of $D$ and $\bar{x} := x(\lambda)$), then $(\lambda, \bar{x})$ is a $J^\geq$-LKKT point of $L$, where $J := \{j \in \mathbb{N}_{\geq 1} | \lambda_j \geq 0\}$. Consequently, if moreover $A(\lambda) \geq 0$, then $\bar{x} \in X_e$ is a global minimizer of $q_0$ on $X_{J^\geq} \cap X_e$.

Proof. Apply Corollary 10 for $J := \mathbb{N}_{\geq 1}$. □

The next result is the variant of Proposition 9 for maximizing $q_0$ on $X_J$.

Proposition 12 Let $(\lambda, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^m$.

(i) Assume that $\nabla_x L(\lambda, \bar{x}) = 0$ and the condition

$$\forall j \in J^c : \lambda_j \leq 0 \land \nabla_{\lambda_j} L(\lambda, \bar{x}) \leq 0 \land \lambda_j \cdot \nabla_{x_j}(\lambda, \bar{x}) = 0 \land \forall j \in J : \nabla_{\lambda_j} L(\lambda, \bar{x}) = 0$$

is verified. Then $\bar{x} \in X_J$, $\lambda \in Y_{col}$, and

$$\forall j \in J^c : \lambda_j \leq 0 \land \nabla_{\lambda_j} L(\lambda, \bar{x}) \leq 0 \land \lambda_j \cdot \nabla_{x_j}(\lambda, \bar{x}) = 0 \land \forall j \in J : \nabla_{\lambda_j} L(\lambda, \bar{x}) = 0$$

(20)

moreover, if $\lambda \in Y_{J^-}$ (or equivalently $\lambda \in Y_{col}^-$), then

$$q_0(\bar{x}) = \sup_{x \in X_J} q_0(x) = L(\lambda, \bar{x}) = \inf_{\lambda \in Y_{col}^-} D(\lambda) = D(\lambda).$$

(ii) Assume that $\lambda \in Y_0$, $\nabla_x L(\lambda, \bar{x}) = 0$ and $(\lambda, \bar{x})$ verifies (19). Then $\bar{x} = x(\lambda)$, and $\lambda$

verifies condition (20); moreover, $\bar{x}$ is the unique global maximizer of $q_0$ on $X_J$ if $\lambda \in Y_{J^-}$.

(iii) Assume that $\lambda \in Y_{J^-}$. Then

$$D(\lambda) = \inf_{\lambda \in Y_{J^-}} D(\lambda) \iff D(\lambda) = \inf_{\lambda \in Y_{col}^-} D(\lambda) \iff \lambda \text{ verifies condition (20).}$$

The proof of the above result is an easy adaptation of the proof of Proposition 9 so we omit it.

4 Quadratic minimization problems with inequality constraints

We consider now the particular case of $(P_J)$ in which $J = \emptyset$; the problem is denoted by $(P_i)$ and
the set of its feasible solutions by $X_i$. In this case $\Gamma_J = \mathbb{R}^m_{\geq}$, and the sets $Y^i, Y^i+, Y^i-, Y^i_{col}, Y^i_{col}^+, Y^i_{col}^-$ are denoted by $Y_i, Y_i+, Y_i-, Y_i^i, Y_i^i+, Y_i^i-$, respectively. Moreover, in this situation we shall use KKT instead of $J$-LKKT. So, we say that $(\lambda, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^m$ is a Karush–Kuhn–Tucker point of $L$ if $\nabla_x L(\lambda, \bar{x}) = 0$ and

$$\lambda \in \mathbb{R}^m_{\geq} \land \nabla_\lambda L(\lambda, \bar{x}) = \mathbb{R}^m \land \quad \langle \lambda, \nabla_\lambda L(\lambda, \bar{x}) \rangle = 0,$$

or, equivalently,

$$\bar{x} \in X_i \land \lambda \in \mathbb{R}^m_{\geq} \land \quad \forall j \in \mathbb{N}_{\geq 1} : \lambda_j g_j(\bar{x}) = 0; \quad (21)$$
we say that $\pi$ is a KKT point for $(P_i)$ if there exists $\lambda \in \mathbb{R}^m$ such that (21) holds; we say that $\lambda \in Y_0$ is a KKT point for $D$ if
\[
\lambda \in \mathbb{R}^m_+ \land D(\lambda) \in \mathbb{R}^m_+ \land \langle \lambda, \nabla D(\lambda) \rangle = 0.
\]

Proposition 9 becomes the next result when $J = \emptyset$.

**Proposition 13** Let $(\pi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m$.

(i) Assume that $(\pi, \lambda)$ is a KKT point of $L$. Then $\pi$ is a KKT point of $(P_i)$, and so $\pi \in X_i, \lambda \in Y_i$ holds; moreover, for $\lambda \in Y_0$ we have that
\[
q_0(\pi) = \inf_{x \in X_i} q_0(x) = L(\pi, \lambda) = \sup_{\lambda \in Y_0} D(\lambda) = D(\lambda).
\]

(ii) Assume that $(\pi, \lambda)$ is a KKT point of $L$ with $\lambda \in Y_0$. Then $\pi = x(\lambda)$ and $\lambda$ is a KKT point of $D$; moreover, $\pi$ is the unique global minimizer of $q_0$ on $X_i$ provided $\lambda \in Y_i$.

Conversely, assume that $\lambda \in Y_0$ is a KKT point of $D$. Then $(\pi, \lambda)$ is a KKT point of $L$, where $\pi := A(\lambda)^{-1}b(\lambda)$.

(iii) Assume that $\lambda \in Y_i$. Then
\[
D(\lambda) = \sup_{\lambda \in Y_i} D(\lambda) \iff D(\lambda) = \sup_{\lambda \in Y_i} D(\lambda) \iff \lambda \text{ is a KKT point of } D.
\]

**Remark 14** Jeyakumar, Rubinov and Wu (see [21, Prop. 3.2]) proved that $\pi$ is a (global) solution of $(P_i)$ when there exists $\lambda \in Y_0$ is a KKT point of $L$; this result was established previously by Hiriart-Urruty in [20, Th. 4.6] when $m = 2$.

**Remark 15** Having in view Propositions 3, 9, 13, it is more advantageous to use their versions (i) than the second part of (ii) with $\lambda \in Y_0$ because in versions (i) one must know only the Lagrangian (hence only the data of the problems), and this provides both $\pi$ and $\lambda$, without needing to calculate effectively $D$, then to determine $\lambda$ (and after that, $\pi$). Using $D$ could be useful, maybe, if the number of constraints is much smaller than $n$. As seen in the proofs, the consideration of the dual function is not essential in finding the optimal solutions of the primal problem(s).

### 5 Comparisons with results on quadratic optimization problems obtained by using CDT

In this section we analyze results obtained by DY Gao and his collaborators in papers dedicated to quadratic optimization problems, or as particular cases of more general results. The main tool to identify the papers where quadratic problems are considered was to look in the survey papers like [2], [7] (which is almost the same as [6], both of them being cited in Gao’s papers), [19] (which is very similar to [8]), as well as in the recent book [12].

We present the results in chronological order using our notations (when possible) and with equivalent formulations; however, sometimes we quote the original formulations to feel also the flavor of those papers. When we have not notations for some sets we introduce them, often as in the respective papers; similarly for some notions. Because $c_0$ in the definition of $q_0$ may be taken always to be 0, we shall not mention it in the sequel.
Before beginning our analysis we consider it is worth having in view the following remark from the very recent paper [26] and to observe that there is not an assumption that some multiplier $X_j$ be non null in Propositions 5, 9 and 13.

"Remark 1. As we have demonstrated that by the generalized canonical duality (32), all KKT conditions can be recovered for both equality and inequality constraints. Generally speaking, the nonzero Lagrange multiplier condition for the linear equality constraint is usually ignored in optimization textbooks. But it cannot be ignored for nonlinear constraints. It is proved recently [26] that the popular augmented Lagrange multiplier method can be used mainly for linear constrained problems. Since the inequality constraint $\mu \neq 0$ produces a nonconvex feasible set $E_\mu^*$, this constraint can be replaced by either $\mu < 0$ or $\mu > 0$. But the condition $\mu < 0$ is corresponding to $y \circ (y - e_K) \geq 0$, this leads to a nonconvex open feasible set for the primal problem. By the fact that the integer constraints $y_1(y_1 - 1) = 0$ are actually a special case (boundary) of the boxed constraints $0 \leq y_1 \leq 1$, which is corresponding to $y \circ (y - e_K) \geq 0$, we should have $\mu > 0$ (see [8] and [12, 16]). In this case, the KKT condition (43) should be replaced by

$$\mu > 0, \quad y \circ (y - e_K) \leq 0, \quad \mu^T[y \circ (y - e_K)] = 0. \quad (47)$$

Therefore, as long as $\mu \neq 0$ is satisfied, the complementarity condition in (47) leads to the integer condition $y \circ (y - e_K) = 0$. Similarly, the inequality $\tau \neq 0$ can be replaced by $\tau > 0$.”

Notice that many papers (co-) authored by DY Gao, mostly in those made public in the last five years, the multipliers corresponding to nonlinear constraints (but not only) are assumed to be positive. So, in most cases Eq. (10) is true. Moreover, it is worth observing that $\bar{\pi} \in X_f$ is a local minimizer as well as a local maximizer of $q_0$ on $X_f$ whenever $X_f$ is a finite set; this is the case in many optimization problems mentioned in this section.

The quadratic problem considered by Gao in [2, Sect. 5.1] is of type (P$_1$) in which $A_1 := I_n := \text{diag} e$ with $e := (1, \ldots, 1)^T \in \mathbb{R}^n$, $b_1 = 0$, $c_1 < 0$, $A_j = 0$ for $j \in \mathbb{Z}, m$. Below, $X_{i1} := \{x \in X_i \mid q_1(x) = 0\}$ and $Y_{i1} := \{y \in Y_1 \mid \lambda_1 \geq 0\}$.

Theorem 4 in [2] (attributed to [3]) asserts: Let $\bar{\lambda} \in Y^1$ be a KKT point of $D$ and $\bar{\pi} := x(\bar{\lambda})$. Then $\bar{\pi}$ is a KKT point of (P$_1$) and $q_0(\bar{\pi}) = D(\bar{\lambda})$.

Theorem 6 in [2] asserts: Assume that $A_0$ has at least one negative eigenvalue and $(\bar{\pi}, \bar{\lambda})$ is a KKT point of $L$. If $\bar{\lambda} \in Y_1^{+}$, then $\bar{\pi} \in X^{+}_{q1}$ and $q_0(\bar{\pi}) = \min_{x \in X_{i1}} q_0(x) = \max_{\lambda \in Y_1^{+}} D(\lambda) = D(\bar{\lambda})$. If $\bar{\lambda} \in \mathbb{R}_+^m \cap Y^-$ then $q_0(\bar{\pi}) = \max_{x \in X} q_0(x) = \max_{\lambda \in \mathbb{R}_+^m \cap Y^-} D(\lambda) = D(\bar{\lambda})$.

Clearly, the conclusion of [2, Th. 4] follows from Proposition [13] (ii) and (i).

Let us look at [2, Th. 6]. Because $(\bar{\pi}, \bar{\lambda})$ is a KKT point of $L$ with $\bar{\lambda} \in Y_1^{+} \subset Y_1^{+}$, [22] holds. Moreover, because $\bar{\lambda}_i > 0$, it follows that $q_1(\bar{\pi}) = 0$, and so $\bar{\pi} \in X_i^{+}$, and so the first assertion of [2, Th. 6] holds, but [22] is stronger.

Consider now the particular case in which $b_j = 0$ and $c_j = 0$ for $j \in \mathbb{Z}, m$ (or, equivalently, $m = 1$); in this case the preceding problem becomes a “quadratic programming problem over a sphere”, considered in [2, Sect. 6]. Assume that $Y^- \supset \bar{\lambda} = \bar{\lambda}_1 > 0$. Then $\nabla D(\bar{\lambda}) = 0$, and so $\bar{\pi} \in X_i$. Using Proposition [5] we get

$$\max_{x \in X_i} q_0(x) \geq q_0(\bar{\pi}) = \max_{x \in X_c} q_0(x) = \min_{\lambda \in Y^-} D(\lambda) = \min_{\lambda \in \mathbb{R}_+^m} D(\lambda) = D(\bar{\lambda}),$$

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which does not agree with the second assertion of [23, Th. 6] because $\mathbb{R}_+ \cap Y^- \subset Y^- \subset Y_{col}$.

**Example 16** Let $n = 1$, $q_0(x) = -\frac{1}{2}(x^2 + x)$ and $q_1(x) = \frac{1}{2}(x^2 - 1)$. It follows that $X_e = \{-1, 1\}$, $X_i = [-1, 1]$, $Y^+ = (1, \infty) = Y^{i+}$ and $Y^- = (-\infty, 1) \cap [0, 1) = \mathbb{R}_+ \cap Y^-$. In this case we have that $A(\lambda) = \lambda - 1$, $b(\lambda) = \frac{1}{2}$, $c(\lambda) = -\frac{1}{2}$, $L(x, \lambda) = \frac{1}{2} - \frac{1}{2}x - \frac{1}{2}$, $\nabla L(x, \lambda) = ((\lambda - 1)x - \frac{1}{2}, \frac{1}{2}x^2 - \frac{1}{2})$, $\nabla L(x, \lambda) = 0 \iff (x, \lambda) \in \{(1, \frac{1}{2}), (1, \frac{3}{2})\}$. $D(\lambda) = \frac{1}{8(1-\lambda)} - \frac{1}{2}$. For $(\overline{\tau}, \overline{\lambda}) = (1, \frac{3}{2})$ we have that

$$q_0(\overline{\tau}) = \min_{x \in X_i} q_0(x) = \max_{\lambda \in Y_{col}^+} D(\lambda) = D(\overline{\lambda}),$$

which confirms the second assertion of [23, Th. 6], while for $(\overline{\tau}, \overline{\lambda}) = (-1, \frac{1}{2})$ we have that

$$\frac{1}{8} = \max_{x \in [-1, 1]} q_0(x) > 0 = q_0(-1) = \max_{x \in [-1, 1]} q_0(x) = \min_{\lambda \in [0, 1)} D(\lambda) = D(\frac{1}{2}) < \sup_{\lambda \in [0, 1)} D(\lambda) = \infty.$$

This shows that the third assertion of [23, Th. 6] is false.

Of course, in [23, Th. 6] there is no need to assume $A$ (i.e. our $A_0$) “has at least one negative eigenvalue”; probably this hypothesis was added in order problem $(\mathcal{P}_\lambda)$ be not a convex one.

The problems considered by DY Gao in his survey papers [6, Sect. 4] and [7, Sect. 4] (which are almost the same) refer to “box constrained problem” ([5, 14]), “integer programming” ([1, 5, 14, 28]), “mixed integer programming with fixed charge” ([18]) and “quadratic constraints” ([17]). In these survey papers the results are stated without proofs and their statements are generally different from the corresponding ones in the papers mentioned above; even more, for some results, the statements are different in the two survey papers, even if the wording (text) is almost the same. We shall mention those results from [6, Sect. 4] and/or [7, Sect. 4] which have not equivalent statements in other papers.

It seems that the first paper dedicated completely to quadratic problems with quadratic equality constraints using CDT is [1], even if [5] was published earlier; note that [1] is cited in [5] as Ref. 6 with a slightly different title (see also Ref. Fang SC, Gao DY, Sheu RL, Wu SY (2007a) in [13]).

The problems considered by Fang, Gao, Sheu and Wu in [1] are of type $(P_e)$ with $m = n$. Setting $e_j := (\delta_{jk})_{k \in 1, n} \in \mathbb{R}^n$, one has $A_j := 2 \text{diag} e_j$, $b_j := e_j$, $c_j := 0$ for $j \in 1, n$. Of course, $X_e = \{0, 1\}^n$.

**Theorem 1** in [1, Th. 1] asserts: Let $\overline{x} \in Y_0 \cap \mathbb{R}_+^n$ be a critical point of $D$ and $\overline{x} := x(\overline{\lambda})$. Then $\overline{x}$ is a KKT point for problem $(P_e)$ and $q_0(\overline{x}) = D(\overline{\lambda})$.

**Theorem 2** in [1, Th. 1] asserts: Let $\overline{x} \in Y_0 \cap \mathbb{R}_-^n$ be a critical point of $D$ and $\overline{x} := x(\overline{\lambda})$. Then $\overline{x}$ is a KKT point for the problem $(\mathcal{P}_{\max})$ of maximizing $q_0$ on $X_e$ and $q_0(\overline{x}) = D(\overline{\lambda})$.

**Theorem 3** in [1, Th. 1] asserts: Let $\overline{x} \in Y_0$ be a critical point of $D$ and $\overline{x} := x(\overline{\lambda})$.

(a) If $\overline{x} \in S_{2+} := Y^+ \cap \mathbb{R}_+^n$, then $q_0(\overline{x}) = \min_{x \in X_e} q_0(x) = \max_{\lambda \in S_{2+}} D(\lambda) = D(\overline{\lambda})$.

(b) If $\overline{x} \in S_{2-} := Y^- \cap \mathbb{R}_-^n$, then in a neighborhood $X_0 \times S_0 \subset X_e \times S_{2-}$ of $(\overline{x}, \overline{\lambda})$, $q_0(\overline{x}) = \min_{x \in X_0} q_0(x) = \max_{\lambda \in S_0} D(\lambda) = D(\overline{\lambda})$.

(c) If $\overline{x} \in S_{2+} := Y^+ \cap \mathbb{R}_-^n$, then $q_0(\overline{x}) = \max_{x \in X_e} q_0(x) = \min_{\lambda \in S_{2-}} D(\lambda) = D(\overline{\lambda})$.

(d) If $\overline{x} \in S_{2+} := Y^- \cap \mathbb{R}_-^n$, then in a neighborhood $X_0 \times S_0 \subset X_e \times S_{2+}$ of $(\overline{x}, \overline{\lambda})$, $q_0(\overline{x}) = \max_{x \in X_0} q_0(x) = \min_{\lambda \in S_0} D(\lambda) = D(\overline{\lambda})$.  

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Using Proposition 4 for \( \lambda \in Y_0 \) with \( \nabla D(\lambda) = 0 \) we have: (i) \( q_0(\pi) = D(\lambda) \) without supplementary conditions on \( \lambda \); (ii) because \( S^+_2 \subset Y^+ \), Eq. (14) is stronger than the minmax relation in (a); (iii) because \( S^-_2 \subset Y^- \), Eq. (15) is stronger than the maxmin relation in (c); (iii) because \( q_0 \) is locally constant on \( X_e \), (b) and (d) are true but their conclusions are much weaker than those provided by Eq. (15) and Eq. (14), respectively.

The quadratic problems \( (P_b) \) considered by Gao in [28, Th. 4] is of type \( (P_e) \) in which \( m \geq n \), \( q_0(x) := -\frac{1}{2} \|Ax - c\|^2 \) for some \( A \in \mathbb{R}^{p \times n} \) and \( c \in \mathbb{R}^p \), \( A_j := \text{diag} e_j \), \( b_j := 0 \), \( c_j := -\frac{1}{2} \) for \( j \in 1, n \), \( A_j = 0 \) for \( j \in n + 1, m \); hence \( X_e \subset \{-1, 1\}^n \); problem \( (P_{bo}) \) is \( (P_b) \) in the case \( m = n \). The problem of maximizing \( D \) on \( S_0 := Y_0 \cap (\mathbb{R}^n_+ \times \mathbb{R}^{m-n}) \) is denoted by \( (P^d_{bo}) \) in the general case, and by \( (P^d_b) \) for \( m = n \) (when \( S_0 := Y_0 \cap \mathbb{R}^n_+ ) \).

Theorem 4 in [5] asserts: Let \( \lambda \in S_0 \) be “a critical point of \( (P^d_b) \)” and \( \pi := x(\lambda) \). Then \( \pi \) “is a critical point of \( (P_b) \)” and \( q_0(\pi) = D(\lambda) \). Moreover, if \( \lambda \in S^+_b := Y^+ \cap (\mathbb{R}^n_+ \times \mathbb{R}^{m-n}) \), then \( q_0(\pi) = \min_{x \in X_e} q_0(x) = \max_{x \in S^+_b \cup \{0\}} D(\lambda) = D(\lambda) \).

Corollary 2 in [5] asserts: Let \( \lambda \in S_0 \) be “a KKT point the canonical dual problem \( (P^d_{bo}) \)” and \( \pi := x(\lambda) \). Then \( \pi \) “is a KKT point of the Boolean least squares problem \( (P_{bo}) \)”.

Unfortunately, it is not defined what is meant by critical points of problems \( (P^d_b) \) and \( (P_b) \), respectively. However, because \( S_0 \) and \( S^+_b \) are open sets, by “critical point of \( (P^d_b) \)” one must mean “critical point of \( D^* \)”; in this situation the conclusions of [5, Th. 4], less \( \pi \) “is a critical point of \( (P_b) \)” are true, but are much weaker than those provided by Proposition 5. Similarly, in [5, Cor. 2], \( \lambda \in S_0 \) is “a KKT point the canonical dual problem \( (P^d_{bo}) \)” equivalent to \( \lambda \) is a “critical point of \( D^* \)”.

The difference between problems \( (P_b) \) considered by Wang, Fang, Gao and Xing in [28, p. 215] and [5, Th. 4] is that in the former \( q_0 \) is a general quadratic function (hence \( X_e \subset \{-1, 1\}^n \)).

Theorem 2.2 in [28] asserts: Let \( \lambda \in S_0 := Y_0 \cap (\mathbb{R}^n_+ \times \mathbb{R}^{m-n}) \) be a critical point of \( D \) and \( \pi := x(\lambda) \). Then \( \pi \) is a KKT point of \( (P_e) \) and \( q_0(\pi) = D(\lambda) \).

Theorem 2.3 in [28] asserts: Let \( \lambda \in S^+_b := Y^+ \cap (\mathbb{R}^n_+ \times \mathbb{R}^{m-n}) \) be a critical point of \( D \) and \( \pi := x(\lambda) \). Then \( q_0(\pi) = \min_{x \in X_e} q_0(x) = \max_{x \in S^+_b \cup \{0\}} D(\lambda) = D(\lambda) \).

Theorems 3.2 and 3.3 from [28] are the versions of Theorems 2.2 and 2.3 for \( n = m \), respectively. Of course, the conclusions of Theorems 2.2 and 2.3 are valid replacing \( S_0 \) and \( S^+_b \) by \( Y_0 \) and \( Y^+ \), respectively.

The general quadratic problem with inequality constraints \( (P_i) \) is considered by Gao in [6] and [12]. In the sequel, the Moore–Penrose generalized inverse of \( F \in \mathcal{M}_n \) is denoted by \( F^+ \) or \( F^\dagger \), as in the corresponding cited papers authored by Gao and his collaborators.

Theorem 7 in [6] and Theorem 10 in [12] assert: let \( \lambda \in Y^{1+i}_{col} \) be a solution of problem \( (P^d_q) \) of maximizing \( D \) on \( Y^i_{col} \) and \( \pi := [A(\lambda)]^\dagger (\lambda) \). Then \( \pi \) is a KKT point of \( (P_i) \) and \( q_0(\pi) = D(\lambda) \). If \( A(\lambda) \succeq 0 \) then \( \lambda \) is a global maximizer of the problem \( (P^d_q) \) and \( \pi \) is a global minimizer of \( (P_i) \). If \( A(\lambda) \prec 0 \), then \( \pi \) is a local minimizer (or maximizer) of \( (P_i) \) if and only if \( \lambda \) is a local minimizer (or maximizer) of \( D \) on \( Y^{1+i}_{col} \).

The “box constrained problem” \( (P_b) \) considered by Gao in [6, Th. 3] and [12] is of type \( (P_i) \) in which \( m = n \), \( A_j := \text{diag} e_j \), \( b_j := 0 \), \( c_j := -1 \) for \( j \in 1, n \); hence \( X_i = [-1, 1]^n \).
Theorem 3 in [9] asserts: Let \( \bar{\lambda} \in Y_{\text{col}}^{i+} \) be a critical point of \( D \) and \( \bar{x} := [A(\bar{\lambda})]^\dagger b(\bar{\lambda}). \) Then \( \bar{x} \) is a KKT point of \( (P_i) \) and \( q_0(\bar{x}) = D(\bar{\lambda}). \) Moreover, if \( A(\bar{\lambda}) \succeq 0 \) then \( q_0(\bar{x}) = \min_{x \in X_i} q_0(x) = \max_{\lambda \in Y_{\text{col}}^{i+}} D(\lambda) = D(\bar{\lambda}). \) If \( A(\bar{\lambda}) \prec 0 \), then on a neighborhood \( X_o \times S_o \) of \( (\bar{x}, \bar{\lambda}) \) we have either \( q_0(\bar{x}) = \min_{x \in X_o} q_0(x) = \min_{\lambda \in S_o} D(\lambda) = D(\bar{\lambda}) \), or \( q_0(\bar{x}) = \max_{x \in X_o} q_0(x) = \max_{\lambda \in S_o} D(\lambda) = D(\bar{\lambda}). \)

The only difference between [9] Th. 3 and [7] Th. 5 is that in the latter the case \( A(\bar{\lambda}) \prec 0 \) is missing.

Probably, the intention was to take \( \bar{\lambda} \in Y_{\text{col}}^i \) instead of \( \bar{\lambda} \in Y_{\text{col}}^{i+} \) in the first assertions of [9] Ths. 3, 7 and [7] Ths. 5, 10; in fact, there is not \( \bar{\lambda} \in Y_{\text{col}}^{i+} \) such that \( A(\bar{\lambda}) \prec 0! \)

It is not clear how the criticality of \( D \) at \( \lambda \in Y_{\text{col}} \setminus Y_0 \) is defined in [9] Th. 3 and [7] Th. 5.

Let us assume that \( \bar{\lambda} \in Y_0 \) is a critical point of \( D \) in the mentioned results from [9] and [7]; in this situation [9] Th. 3 is a particular case of [9] Th. 7. Then \( \bar{x} \) is a KKT point of \( (P_i) \) iff \( \bar{\lambda} \in \mathbb{R}^n_+; \) assuming moreover that \( A(\bar{\lambda}) \succeq 0 \), the conclusion of the second assertion of [9] Th. 7 is true. However, in the case \( A(\bar{\lambda}) \prec 0 \) the conclusions of [9] Ths. 3, 7 are false, as the next example shows.

**Example 17** Consider \( n := m := 2, \ A_0 := \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix} \), \( A_1 := \text{diag} e_1, \ A_2 := \text{diag} e_2, \ b_0 := (0, -1)^T, \ b_1 := b_2 := 0, \ c_1 := c_2 := -\frac{1}{2}. \) Then \( A(\lambda) = A_0 + \lambda_1 A_1 + \lambda_2 A_2, \ b(\lambda) = b_0, \ c(\lambda) = -\frac{1}{2}(\lambda_1 + \lambda_2). \) We have that \( Y_{\text{col}} = Y_0 \) \( \{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid (\lambda_1 - 1)(\lambda_2 - 3) \neq 1 \}. \) The critical points \( (\bar{x}, \bar{\lambda}) \) of \( L \) are: \( (1, -1)^T, (0, 3)^T, \) \( (0, 1), (1, -1)^T, (2, 3)^T, \) \( (1, 1)^T, (2, 5)^T, \) \( (1, 1)^T, (0, 1)^T. \) Applying Proposition 13 we obtain that \( \bar{x} := (1, -1)^T \) is the global minimizer of \( q_0 \) on \( X_i \) \( [0, 1]^2 \) and \( \bar{\lambda} := (2, 5)^T \) is the global maximizer of \( D \) on \( Y_{\text{col}}^i = Y_i = \{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 > 2, \ (\lambda_1 - 2)(\lambda_2 - 3) > 1 \}. \)

Take now \( (\bar{x}, \bar{\lambda}) := ((1, 1)^T, (0, 1)^T); \) we have that \( \bar{\lambda} \in \mathbb{R}^2_+ \) and \( A(\bar{\lambda}) \prec 0. \) From Proposition 4 (iv), we have that \( \bar{\lambda} \) is a global minimizer of \( D \) on \( Y_{\text{col}} = Y^- = \{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 < 2, \ (\lambda_1 - 2)(\lambda_2 - 3) > 1 \}. \) Assuming that \( \bar{\lambda} \) is a local minimizer of \( D, \) because \( D \) is convex on \( Y_{\text{col}} = Y^-, \) \( D \) is constant on an open neighborhood \( U \subset Y^- \) of \( \bar{\lambda}, \) and so \( \nabla D(\lambda) = 0 \) for \( \lambda \in U; \) taking into account (13), this is a contradiction. Observe that \( \bar{x} = (1, 1) \) is not a local minimizer of \( q_0 \) on \( X_i. \) Indeed, take \( x := (1 - u, 1) \in X_i \) for \( u \in (0, 2); \) then \( q_0(x) = -\frac{1}{2}u^2 < 0 = q_0(\bar{x}), \) proving that \( \bar{x} \) is not a local minimum of \( q_0 \) on \( X_i. \)

Gao and Sherali in [19] Th. 8.16 (attributed to [4]) assert: Suppose that \( m = 1, \ A_1 > 0, \ b_1 = 0, \ c_1 < 0. \) Let \( \bar{\lambda} \in Y^i \) be a critical point of \( D \) and \( \bar{x} := x(\bar{\lambda}). \) If \( \bar{\lambda} \in Y^{i+}, \) then \( \bar{x} \) is a global minimizer of \( q_0 \) on \( X_i. \) If \( \bar{\lambda} \in \mathbb{R}_+ \cap Y^- \) then \( \bar{\lambda} \) is a local minimizer of \( q_0 \) on \( X_i. \)

As in the case of [2] Th. 6 above, the first assertion of [19] Th. 8.16 follows from Proposition 13. However, the second assertion of [19] Th. 8.16 is false as the next example shows.

**Example 18** (see [27] Ex. 1)) Consider \( n := 2, \ m := 1, \ A_0 := \begin{bmatrix} -2 & -1 \\ -1 & -3 \end{bmatrix} \), \( A_1 := I_2, \ b_0 := (-1, -1)^T, \ b_1 := 0, \ c_1 := -\frac{1}{2}. \) Then \( D(\lambda) = -\frac{1}{2}\lambda - \frac{1}{2} \frac{2\lambda - 3}{\lambda^2 - 5\lambda + 5} \) and \( D'(\lambda) = -\frac{1}{2} \frac{(\lambda - 2)^2}{(\lambda^2 - 5\lambda + 5)^2} (\lambda - 1)(\lambda - 5). \) Hence the set of critical points of \( D \) is \( \{ 1, 2, 5 \} \subset \mathbb{R}_+. \) For \( \bar{\lambda} = 1 \) we have that \( A(\bar{\lambda}) = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} \prec 0 \) and \( \bar{x} = x(\bar{\lambda}) = (1, 0)^T. \) Since \( X_i = \{ (\cos t, \sin t)^T \mid t \in (-\pi, \pi) \} \) and

\[
q_0((\cos t, \sin t)^T) = -(3 + \cos t - 2 \sin t) \sin^2 \frac{t}{2} \leq (\sqrt{5} - 3) \sin^2 \frac{t}{2} < 0 = q_0(\bar{x})
\]
for all $t \in (-\pi, \pi] \setminus \{0\}$, we have that $\overline{x}$ is the unique global maximizer of $q_0$ on $X_i$, in contradiction with the second assertion of $[19, \text{Th. 8.16}].$

The problem considered by Zhang, Zhu and Gao in $[34]$ is of type $(P_i)$ in which $m \geq n$, $A_j := \text{diag } e_j$, $b_j := 0$, $c_j \leq 0$ for $j \in \overline{1,n}$, $A_j = 0$ for $j \in \overline{n+1,m}$.

Theorem 1 in $[34]$ asserts: Let $\overline{\lambda} \in Y^i$ be a KKT point of $D$ and $\overline{x} := x(\overline{\lambda})$. Then $\overline{x}$ is a KKT point of $(P_i)$ and $q_0(\overline{x}) = D(\overline{x}).$

Theorem 2 in $[34]$ asserts: Let $\overline{\lambda} \in Y^i$ be a KKT point of $D$ and $\overline{x} := x(\overline{\lambda})$. If $\overline{\lambda} \in Y^{i+}$, then $\overline{\lambda}$ “is a global maximizer of” $D$ on $Y^{i+}$ “if and only if the vector” $\overline{x}$ “is a global minimizer of” $(P_i)$ on $X_i$, and $q_0(\overline{x}) = \min_{x \in X_i} q_0(x) = \max_{\alpha \in Y_+^{i+}} D(\lambda) = D(\overline{x})$. If $\overline{\lambda} \in \mathbb{R}_+^m \cap Y^-$, “then in a neighborhood $X_0 \times S_0 \subset \cap X_i \times (\mathbb{R}_+^m \cap Y^-)$ of $(\overline{x}, \overline{\lambda})$, “we have that either” $q_0(\overline{x}) = \min_{x \in X_0} q_0(x) = \min_{\lambda \in S_0} D(\lambda) = D(\overline{x})$, or $q_0(\overline{x}) = \max_{x \in X_0} q_0(x) = \max_{\lambda \in S_0} D(\lambda) = D(\overline{x})$.

Clearly, $[34]$ Th. 1] and the conclusion of $[34]$ Th. 2] in the case $\overline{\lambda} \in Y^{i+}$ follow from Proposition $[13]$ As shown in $[32]$ Ex. 2] and Example $[17]$ each of the alternative conclusions of $[34]$ Th. 2] in the case $\overline{\lambda} \in \mathbb{R}_+^m \cap Y^-$ is false. Observe that $[17]$ is cited in $[34]$ as a paper to appear, but not in connection with the previous result.

The problem $(P_i)$ is considered also by Gao, Ruan and Sherali in $[17]$ p. 486); the problem of maximizing $D$ on $Y^i$ is denoted by $(P_d^i)$.

Theorem 4 in $[17]$ asserts: Let $\overline{\lambda} \in Y^i$ be a critical point of $(P_d^i)$ and $\overline{x} := [A(\overline{\lambda})]^{+} b(\overline{\lambda})$. Then $\overline{x}$ is a KKT point of $(P_i)$ and $q_0(\overline{x}) = D(\overline{x})$. If $\overline{\lambda} \in Y^{i+}$, then $q_0(\overline{x}) = \min_{x \in X_i} q_0(x) = \max_{\lambda \in Y_+^{i+}} D(\lambda) = D(\overline{x})$. If $\overline{\lambda} \in Y^i$ then $\overline{\lambda}$ “is a unique global maximizer of $(P_d^i)$ and the vector $\overline{x}$ is a unique global minimizer of $(P_i)$”. If $\overline{\lambda} \in \mathbb{R}_+^m \cap Y^-$, then $\overline{\lambda}$ “is a local minimizer of” $D$ “on the neighborhood $S_0 \subset \cap \mathbb{R}_+^m \cap Y^-$ “if and only if $\overline{x}$ is a local minimizer of” $q_0$ “on the neighborhood $X_0 \subset \cap X_i$ , i.e., $q_0(\overline{x}) = \min_{x \in X_0} q_0(x) = \min_{\lambda \in S_0} D(\lambda) = D(\overline{x})$.

As noticed before Lemma $[11]$ $Y_{\text{col}}$ is not open in general, so it is not possible to speak about the differentiability of $D$ at $\lambda \in Y_{\text{col}} \setminus Y_0$. As in $[3]$ Th. 4], it is not explained what is meant by critical point of $(P_d^i)$; we interpret it as being a critical point of $D$. With the above interpretation for “critical point of $(P_d^i)$”, we agree with the first two assertions of $[17]$ Th. 4]. However, the third assertion of $[17]$ Th. 4] that $\overline{\lambda}$ is the unique global maximizer of $(P_d^i)$ provided that $\overline{\lambda} \in Y^{i+}$ is false, as seen in Example $[19]$ below. The same example shows that the fourth assertion of $[17]$ Th. 4] is false, too; another counterexample is provided by Example $[17]$.

Example 19 Let us take $n = m = 2$, $q_0(x,y) := xy - x$, and $q_1(x,y) := -q_2(x,y) := \frac{1}{2} (x^2 + y^2 - 1)$ for $(x,y) \in \mathbb{R}_2$. Clearly, the problems $(P_e)$ for $(q_0,q_1)$ and $(P_i)$ for $(q_0,q_1,q_2)$ are equivalent in the sense that they have the same objective functions and the same feasible sets (hence the same solutions). Denoting by $L^e$, $A^e$, $b^e$, $c^e$, $D^e$ and $L^i$, $A^i$, $b^i$, $c^i$, $D^i$ the functions associated to problems $(P_e)$ and $(P_i)$ mentioned above, we get: $L^e(x,y,\lambda) = xy - x + \frac{1}{2} (x^2 + y^2 - 1)$, $A^e(\lambda) = \begin{pmatrix} \lambda & 1 \\ 1 & \lambda \end{pmatrix}$, $b^e(\lambda) = (1,0)^T$, $c^e(\lambda) = -\frac{1}{2}\lambda$, $Y_{\text{col}} = Y_0 = \mathbb{R} \setminus \{-1,1\}$, $Y_+ = Y_- = Y^- = (1,\infty)$, $D^e(\lambda) = \frac{\lambda^2}{\lambda^2 - 1} - \frac{1}{2}\lambda$ for the problem $(P_e)$] and $L^i(x,y,\lambda_1,\lambda_2) = L^e(x,y,\lambda_1 - \lambda_2)$, $A^i(\lambda_1,\lambda_2) = A^e(\lambda_1 - \lambda_2)$, $b^i(\lambda_1,\lambda_2) = b^e(\lambda_1 - \lambda_2)$, $c^i(\lambda_1,\lambda_2) = c^e(\lambda_1 - \lambda_2)$, $Y_{\text{col}} = Y^i = \{\lambda_1,\lambda_2 \in \mathbb{R}_+^2 \mid \lambda_1 - \lambda_2 \neq \pm 1\}$, $Y_+ = Y_+ = \{\lambda_1,\lambda_2 \in \mathbb{R}_+^2 \mid \lambda_1 - \lambda_2 > 1\}$, $\mathbb{R}_+^2 \cap Y^- = \{\lambda_1,\lambda_2 \in \mathbb{R}_+^2 \mid \lambda_1 - \lambda_2 < 1\}$, $D^i(\lambda_1,\lambda_2) = D^e(\lambda_1 - \lambda_2)$. \]
The critical points of $L^e$ are $(0,1,0)$ and $(\pm \sqrt{3}/2, -1/2, \pm \sqrt{3})$. Using Proposition 13, it follows that $(\sqrt{3}/2, -1/2)$ is the unique global minimizer of $q_0$ on $X_e$ and $\sqrt{3}$ is a global maximizer of $D^e$ on $Y^+_{\text{col}}$ (=$Y^+$), while $(-\sqrt{3}/2, -1/2)$ is the unique global maximizer of $q_0$ on $X_e$ and $-\sqrt{3}$ is a global minimizer of $D^e$ on $Y^-_{\text{col}}$ (=$Y^-$).

Note that $(x, y, x_1, x_2)$ is a KKT point of $L^i$ iff $(x, y, x_1, x_2)$ is a critical point of $L^i$ with $(x_1, x_2) \in \mathbb{R}^2_+$. Using Proposition 13 (ii) we obtain that $(-\sqrt{3}/2, -1/2)$ is the unique global minimizer of $q_0$ on $X_i$ and any $(x_1, x_2) \in \mathbb{R}^2_+$ with $x_1 - x_2 = \sqrt{3}$ is a global maximizer of $D^i$ on $Y^{i+}$ (=$Y^+_{\text{col}}$), the latter assertion contradicting the third assertion of [17, Th. 4]. On the other hand, as seen above, $(-\sqrt{3}/2, -1/2)$ is the unique global maximizer of $q_0$ on $X_e$ and $(\sqrt{3}, 2\sqrt{3}) \in \mathbb{R}^2_+ \cap Y^-$ is a global minimizer of $(P_d^q)$, contradicting the fourth assertion of [17, Th. 4].

The problem considered by Lu, Wang, Xin and Fang in [22] is of type $(P_e)$ with $m = n$. More precisely, $A_j = 2 \text{diag } e_j, \ b_j := e_j, \ c_j := 0$ for $j \in 1, n$; hence $X_e = \{0, 1\}^n$. One must emphasize the fact that the authors use the usual Lagrangian, even if CDT is invoked.

Theorem 2.2 (resp. Theorem 2.3) of [22] asserts: If $\bar{\lambda} \in Y_0$ (resp. $\bar{\lambda} \in Y^+$) is such that $\nabla D(\bar{\lambda}) = 0$ and $x := x(\bar{\lambda})$, then $q_0(x) = D(\bar{\lambda})$ (resp. $q_0(x) = \min_{x \in X_e} q_0(x)$).

Gao and Ruan [14] considered problems $(P_e)$ and $(P_l)$ when $m = n$ and $A_j := \text{diag } e_j, \ b_j := 0, c_j := -\frac{1}{j}$ for $j \in 1, n$. Of course, $X_e = \{-1, 1\}^n$ and $X_i = [-1, 1]^n$. The problem of maximizing $D$ on $Y^{i+}$ is denoted by $(P^d)$.

Theorem 1 in [14] (attributed to [5]) asserts: “If $\bar{\sigma}$ is a critical point of” $D$, “the vector $\bar{x}(\bar{\sigma})$ “is a KKT point of” $(P_l)$ and $q_0(\bar{\sigma}) = D(\bar{\sigma})$. “If the critical point $\bar{\sigma} > 0$, then the vector $\bar{x}(\bar{\sigma}) \in X_e$ “is a local optimal solution of the integer programming problem” $(P_e)$. If $\bar{\sigma} \in Y^{i+}, \ then \ q_0(\bar{\sigma}) = \min_{x \in X_e} q_0(x) = \max_{x \in Y^{i+}} D(\bar{\sigma}) = D(\bar{\sigma})$. “If the critical point $\bar{\sigma} \in Y^{i+}$ “and $\bar{\sigma} > 0$, then the vector $\bar{x}(\bar{\sigma}) \in X_e$ “is a global minimizer to the integer programming problem” $(P_e)$. If $\bar{\sigma} \in \mathbb{R}^n_+ \cap Y^-$, “then $\bar{\sigma}$ is a local minimizer of $(P^d)$, the vector $\bar{x}(\bar{\sigma})$ is a local minimizer of” $(P_l)$, “and on the neighborhood $X_e \times S_0 \ of \ (\bar{x}(\bar{\sigma}), q_0(\bar{\sigma})) = \min_{x \in X_e} q_0(x) = \min_{\sigma \in S_0} D(\bar{\sigma}) = D(\bar{\sigma})$.

Concerning [14, Th. 1] we observe the following: In the first assertion it is not clear if $\bar{\sigma}$ belongs to $\mathbb{R}^n_+$ or not; of course, $\bar{x}$ is not a KKT point of $(P_l)$ if $\bar{\sigma} \not\in \mathbb{R}^n_+$. The second assertion is true because $X_e$ is finite (without any condition on $\bar{\sigma}$). The third assertion is false without assuming that $\bar{\sigma}$ is at least a KKT point of $D$. The fourth assertion is true without assuming $\bar{\sigma} > 0$. The fifth assertion is false if $\bar{\sigma} > 0$ and $\nabla D(\bar{\sigma}) \neq 0$.

The main difference between [14, Th. 1] and the conjunction of [14] Th. 2 & Th. 3] is that in the latter $Y_0$ is replaced by $Y_{\text{col}}$, but their statements are not more clear. This is the reason for not analyzing them here.

The problem considered by Gao, Ruan and Sherali in [18] is of type $(P_j)$ with $n = m = 2k \ (k \in \mathbb{N}^*)$ and $J := k + 1, n$. In [18] $A_0$ is such that $(A_0)_{j,j} = 0$ if $\max\{i, j\} > k, A_j := 2 \text{diag } e_j$ and $c_j := 0$ for $j \in \overline{1, m}, b_j := e_{j+k}$ for $j \in J = (\overline{1, k})$ and $c_j := e_j$ for $j \in J$; moreover, $S^+_i := Y_{\text{col}} \cap (\mathbb{R}^k_+ \times \mathbb{R}^k_{-}) \ (\subset Y^+_{\text{col}} \subset Y_{\text{col}}^J)$, $S^+_i := Y^+ \cap S^+_i (\subset Y^{i+} \subset Y_{\text{col}}^{i+})$, $S^+_i := Y_{\text{col}} \cap (\mathbb{R}^k_+ \times \mathbb{R}^k_{-})$, $S^+_i := Y^+ \cap S^+_i (\subset Y^{i+} \subset Y_{\text{col}}^{i+})$.

Theorem 1 of [18] asserts: Let $\bar{\lambda} \in S^+_i be a KKT point of $D$ and $x := x(\bar{\lambda})$. Then $\bar{x}$ is feasible to the primal problem $(P_j)$ and $q_0(\bar{x}) = L(\bar{x}, \bar{\lambda}) = D(\bar{\lambda})$. 

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Theorem 2 in [14] asserts: Let \( \lambda \in S^+_D \cup S^-_D \) be a critical point of \( D \) and \( \alpha := x(\lambda) \). If \( \lambda \in S^+_D \) then \( q_0(\alpha) = \min_{x \in X} q_0(x) = \max_{\lambda \in S^+_D} D(\lambda) = D(\lambda) \). If \( \lambda \in S^-_D \) then \( q_0(\alpha) = \max_{x \in X} q_0(x) = \min_{\lambda \in S^-_D} D(\lambda) = D(\lambda) \).

In [14] Th. 1 it is not clear what is meant by KKT point of \( D \) because \( D \) is not differentiable for \( \lambda \in S_D \setminus Y_0 \). Propositions 9 and 12 confirm [14] Th. 2, but the conclusions of the latter are much weaker than those of the former.

The quadratic problems \((P_b)\) and \((P_{bo})\) considered by Ruan and Gao in [25] (and [24]) are those from [7]. The statement of [25] Th. 5 is that of [5] Cor. 2 in which \( S_2 \) is now \( Y_0 \cap \{ \lambda \in \mathbb{R}^n \mid \lambda_j \neq 0 \ \forall j \in \{1,n\} \} \). \( S^+_2 \) being the same, that is \( Y \cap \mathbb{R}_{+}^{n} \). The statement of [25] Th. 6 is that of [5] Th. 4 in which “a critical point of \((P_{b})\)" is replaced by “a KKT point of \((P_{bo})\)."

The quadratic problem considered by Ruan and Gao in [26] is of type \((P)\) in which \( m > n \) and \( X = \{1,n\} \subset J \). In [26] \( A_j := 2 \text{ diag } e_j, b_j := e_j, c_j := 0 \) for \( j \in \{1,n\} \), \( A_j := 0 \) for \( j \in \{n+1,m\} \); hence \( X_J \subset \{0,1\}^n \). One considers \( S_a := \{ \lambda \in Y^J \mid \lambda_j \neq 0 \ \forall j \in J \} \) and \( S^+_a := \{ \lambda \in Y^{J^+} \mid \lambda_j > 0 \ \forall j \in J \} \).

Theorem 3 of [26] asserts: Let \( \lambda \in S_a \) be a J-LKKT point of \( D \) and \( \alpha = x(\lambda) \). Then \( \alpha \) is a J-LKKT point of \((P)\) and \( q(\alpha) = D(\lambda) \).

Theorem 4 in [26] asserts: Let \( \lambda \in S^+_a \) be a J-LKKT point of \( D \) and \( \alpha = x(\lambda) \). Then \( q(\alpha) = \min_{x \in X} q(x) = \max_{\lambda \in S^+_a} D(\lambda) = D(\lambda) \).

Clearly, [26] Th. 3] is an immediate consequence of Lemma 3 while [26] Th. 4] is a very particular case of Proposition 9.

The quadratic problem considered by Gao in [9], [10] and [11] is of type \((P)\) in which \( m = n + 1 \) and \( J := \{1,n\} \). In these papers \( A_0 := 0, A_j := 2 \text{ diag } e_j, b_j := e_j, c_j := 0 \) for \( j \in J \), and \( A_{n+1} := 0 \); hence \( X_J \subset \{0,1\}^n \).

Theorem 2 of [9] asserts: Let \( \lambda \in Y^{J^+} \) be a global maximizer of \( D \) on \( Y^{J^+} \). Then \( \alpha = x(\lambda) \in X_J \) and \( q(\alpha) = \min_{x \in X} q(x) = \max_{\lambda \in Y^{J^+}} D(\lambda) = D(\lambda) \).

The differences between [9] Th. 2] and [10] Th. 2] are: in the latter \( b_{n+1} := -c(u) \in \mathbb{R}^n \), \( c_{n+1} := -V_c < 0 \), and \( Y^{J^+} \) is replaced by \( \{ \lambda \in Y^{J^+} \mid \lambda_{n+1} > 0 \} \). The differences between [9] Th. 2] and [11] Th. 1] are: in the latter \( c_{n+1} := -V_c < 0 \), and \( \min_{\rho \in \mathbb{R}^n} p_u(\rho) \) is replaced by \( \min_{\rho \in \mathbb{R}^n} p_u(\rho) \); of course, \( \min_{\rho \in \mathbb{R}^n} p_u(\rho) = -\infty \) if \( c_u \neq 0 \). In all 3 papers there are provided proofs of the mentioned results.

Using Proposition 9(iii) in the context of [9] Th. 2] we have that \( \lambda \) is a J-LKKT point of \( D \); using Proposition 9(ii) and (i) we get the conclusion of [11] Th. 2].

Yuan [30] (the same as [29]) considers problem \((P)\) in its general form.

In [30] p. 340 one asserts: “One hard restriction is given” by \( b_0 \neq 0 \). “The restriction is very important to guarantee the uniqueness of a globally optimal solution of” \((P)\). Theorem 1 of [30] asserts: Let \( Y := \{ \sigma \in Y^i \mid x(\sigma) \in X_i \} \neq \emptyset \), and let \((P^d)\) be the problem of maximizing \( D \) on \( Y \). If \( \overline{\sigma} \) is a solution of \((P^d)\), then \( \overline{x} := x(\overline{\sigma}) \) is a solution of \((P)\) and \( q(\overline{\rho}) = D(\overline{\sigma}) \).

Theorem 2 of [30] asserts: Assume that \((C^1) \sum_{k=0}^{m} A_k > 0 \), and \((C^2)\) there exists \( k \in \{1,m \} \) such that \( A_k > 0 \), \( A_0 + A_k > 0 \), and \( \| D_k A_k^{-1} b_0 \| > \| b_k^T D_k^{-1} \| + \sqrt{\| b_k^T D_k^{-1} \|^2 + 2 |c_k|} \), where
\[ A_k = D_k^T D_k \text{ and } \| \cdot \| \text{ is some vector norm. Then problem } (P^d) \text{ has a unique non-zero solution } \sigma \text{ in the space } Y^{i+i} \text{.} \]

Counterexamples to both theorems of \[30\] as well as for the assertion on the “hard restriction” \( b_0 \neq 0 \) from \[30, \text{p. } 340\] are provided in \[33\].

6 Conclusions

– We made a complete study of quadratic minimization problems with quadratic equality and/or inequality constraints using the method suggested by the canonical duality theory (CDT) introduced by DY Gao. This method is based on the introduction of a dual function. Our study uses only the usual Lagrangian associated to minimization problems with equality and/or inequality constraints, without any reference to CDT; CDT is presented (or, at least, referred) in all the papers cited in Section 5.

– As observed in Remark 15, it is more advantageous to use the assertions (i) of Propositions \[5, 9, 13\] than the second part of (ii) with \( \lambda \in Y_0 \) because in versions (i) one must know only the Lagrangian (hence only the data of the problems), and this provides both \( \overline{\sigma} \) and \( \overline{\lambda} \). Using \( D \) could be useful, possibly, if the number of constraints is much smaller than \( n \).

– As seen in Section 5, many results obtained by DY Gao and his collaborators on quadratic optimization problems are not stated clearly, and some of them are even false; some statements were made more clear in subsequent papers, but we didn’t observe some warning about the false assertions. For the great majority of the correct assertions the use of the usual direct method provides stronger versions.

– Asking the strict positivity of the multipliers corresponding to nonlinear constraints (but not only, as in \[26\]), is very demanding, even for inequality constraints. Just observe that for \( k \) equality constraints one has \( 2^k \) distinct possibilities to get the feasible set, but at most one could produce strictly positive multipliers.

Acknowledgement We thank prof. Marius Durea for reading a previous version of the paper and for his useful remarks.

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