Semiclassical Particle Spectrum of Double Sine–Gordon Model

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Abstract

We present new theoretical results on the spectrum of the quantum field theory of the Double Sine Gordon model. This non–integrable model displays different varieties of kink excitations and bound states thereof. Their mass can be obtained by using a semiclassical expression of the matrix elements of the local fields. In certain regions of the coupling–constants space the semiclassical method provides a picture which is complementary to the one of the Form Factor Perturbation Theory, since the two techniques give information about the mass of different types of excitations. In other regions the two methods are comparable, since they describe the same kind of particles. Furthermore, the semiclassical picture is particularly suited to describe the phenomenon of false vacuum decay, and it also accounts in a natural way the presence of resonance states and the occurrence of a phase transition.

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1 Introduction

As a natural development of the studies on integrable quantum field theories, there has been recently an increasing interest in studying the properties of non–integrable quantum field theories in (1 + 1) dimensions, both for theoretical reasons and their application to several condensed–matter or statistical systems. However, contrary to the integrable models, many features of these quantum field theories are still poorly understood: in most of the cases, in fact, their analysis is only qualitative and even some of their basic data, such as the mass spectrum, are often not easily available. Although one could always rely on numerical methods to shed some light on their properties, it would be obviously useful to develop some theoretical tools to control them analytically. In this respect, there has been recently some progress, thanks to two different approaches.

The first approach, called the Form Factor Perturbation Theory (FFPT) [1, 2], is best suited to deal with those non–integrable theories close to the integrable ones. It permits, in particular, to obtain quantitative predictions on their mass spectrum, scattering amplitudes and other physical quantities. As any other perturbation scheme, it works finely as far as the non–integrable theory is an adiabatic deformation of the original integrable model, i.e. when the two theories are isospectral. This happens when the field which breaks the integrability is local with respect to the operator which creates the particles. If, on the contrary, the field which moves the theory away from integrability is non–local with respect to the particles, the resulting non–integrable model generally displays confinement phenomena and, in this case, some caution has to be taken in interpreting these perturbative results.

The second approach, known as Semiclassical Method and based on the seminal work of Dashen, Hasslacher and Neveu [3], is on the other hand best suited to deal with those quantum field theories (integrable or not) having kink excitations of large mass in their semiclassical limit. Under these circumstances, in fact, once the non–perturbative classical solutions are known, it is relatively simple to determine the two–particle Form Factors on the kink states of the basic fields of the theory and to extract the spectrum of the excitations from their pole structure [4, 5, 6]. Although this method is restricted to work in a semiclassical regime, it permits however to analyze non–integrable theories in the whole coupling–constants space, even far from the integrable points.

An interesting non–integrable model where both approaches can be used is the so–called Double Sine–Gordon Model (DSG). Its Lagrangian density is given by

\[ \mathcal{L} = \frac{1}{2} (\partial_{\mu} \varphi)^2 - V(\varphi), \]  

(1.1)

with

\[ V(\varphi) = -\frac{\mu}{\beta^2} \cos \beta \varphi - \frac{\lambda}{\beta^2} \cos \left( \frac{\beta}{2} \varphi + \delta \right) + C, \]  

(1.2)
where $C$ is a constant that has be chosen such that to have a vanishing potential energy of the vacuum state. The classical dynamics of this model has been extensively studied in the past by means of both analytical and numerical techniques (see [7] for a complete list of the results), while its thermodynamics has been studied in [8] by using the transfer integral method [9].

With $\lambda$ or $\mu$ equal to zero, the DSG reduces to the ordinary integrable Sine–Gordon (SG) model with frequency $\beta$ or $\beta/2$ respectively. Hence the DSG model with a small value of one of the couplings can be regarded as a deformation of the corresponding SG model and studied, therefore, by means of the FFPT [2]. On the other hand, for $\beta \to 0$, irrespectively of the value of the coupling constants $\lambda$ and $\mu$, the DSG model reduces to its semiclassical limit. Despite the non–integrable nature of the DSG model, its classical kink solutions are – remarkably enough – explicitly known [7, 8] and therefore the Semiclassical Method can be successfully applied to recover the (semi–classical) spectrum of the theory. As we will see in the following, the two approaches turn out to be complementary in certain regions of the coupling constants, i.e. both are needed in order to get the whole mass spectrum of the theory, whereas in other regions they provide the same picture about the spectrum of the excitations.

Apart from the theoretical interest in testing the efficiency of the two methods on this specific model where both are applicable, the study of the DSG is particularly important since this model plays a relevant role in several physical contexts, either as a classical non–linear system or as a quantum field theory. At the classical level, its non–linear equation of motion can be used in fact to study ultra–short optical pulses in resonant degenerate medium or texture dynamics in He$^3$ (see, for instance, [10] and references therein). As a quantum field theory, depending on the values of the parameters $\lambda, \mu, \beta, \delta$ in its Lagrangian, it displays a variety of physical effects, such as the decay of a false vacuum or the occurrence of a phase transition, the confinement of the kinks or the presence of resonances due to unstable bound states of excited kink–antikink states. Moreover, it finds interesting applications in the study of several systems, such as the massive Schwinger field theory or the Ashkin–Teller model [2], as well as in the analysis of the $O(3)$ non–linear sigma model with $\theta$ term [11], i.e. the quantum field theory relevant for understanding the dynamics of quantum spin chains [12, 13]. The DSG model also matters in the investigation of other interesting condensed matter phenomena, such as the soliton confinement of spin–Peierls antiferromagnets [14], the dynamics of the spin chains in a staggered external field or the electron interaction in a staggered potential [15].

Motivated by the above combined theoretical and physical interests, a thorough study of the spectrum of the DSG model seems therefore to be particularly interesting and in this paper we present the results of such analysis.
The paper is organised as follows: in Section 2 we briefly recall the basic formulas of the Form Factor Perturbation Theory whereas in Section 3 we remind the basic results of the Semiclassical Method. Section 4 is devoted to the semiclassical analysis of the spectrum of the DSG model and its comparison with the results coming from FFPT. Section 5 deals with the analysis of false vacuum decay. In Section 6 we discuss the occurrence of resonance phenomena in the DSG in relation with analogous effects observed in the classical scattering of kink states. Our conclusions are in Section 7. The paper also contains several appendices. In Appendix A we compute the kink mass corrections by using the FFPT, in Appendix B we collect the relevant expressions of the semiclassical Form Factors, Appendix C is devoted to the analysis of neutral states in comparison with the Sine–Gordon model, and in Appendix D we discuss the basic results in a closely related model, i.e. the Double Sinh–Gordon model.

2 Form Factor Perturbation Theory

The method of the Form Factor Perturbation Theory (FFPT) [1, 2] permits to analyse a non–integrable quantum field theory when its action $A$ is represented by a deformation of an integrable one $A_0$ through a given operator $\Psi$:

$$A = A_0 + g \int d^2 x \, \Psi(x) .$$

(2.1)

One of the first consequences of moving away from integrability is a change in the spectrum of the theory: the first order corrections to the mass of the particle $a$ belonging to the spectrum of the unperturbed theory is in fact given by

$$\delta m_a^2 = 2 g F_{a\bar{a}}^\Psi(i\pi) + O(g^2) ,$$

(2.2)

where the particle–antiparticle Form Factor of the operator $\Psi(x)$, defined by the matrix element

$$F_{a\bar{a}}^\Psi(\theta_1 - \theta_2) = \langle 0 \mid \Psi(0) \mid a(\theta_1)\bar{a}(\theta_2) \rangle ,$$

(2.3)

is introduced. The mass correction (2.2) may be finite or divergent, depending on the locality properties of the operator $\Psi(x)$ with respect to the particle $a$. The situation was clarified in [2] and it is worth recalling the main conclusion of that analysis.

In integrable theories, the Form Factors of a generic scalar operator $O(x)$ can be determined due to the simple form assumed by the Watson equation [16, 17] and for the two–particle case, one has

$$F_{a\bar{a}}^O(\theta) = S_{a\bar{a}}^{\theta\bar{\theta}}(\theta) F_{\bar{b}b}^O(-\theta) ,$$

(2.4)

1We adopt the standard parameterization of the on–shell two–dimensional momenta given in terms of the rapidity, $p^{(0)} = m \cosh \theta$, $p^{(1)} = m \sinh \theta$. 
\[ F_{\bar{a}a}^{\mathcal{O}}(\theta + 2i\pi) = e^{-2i\pi \gamma_{\mathcal{O},a}} F_{\bar{a}a}^{\mathcal{O}}(-\theta) , \]  

where \( \theta = \theta_1 - \theta_2 \). In the first equation, expressing the discontinuity of the matrix element across the unitarity cut, \( S_{\bar{a}a}^{\mathcal{O}}(\theta) \) is the elastic two-body scattering amplitude. In the second equation, expressing the crossing symmetry of the Form Factor, the explicit phase factor \( e^{-2i\pi \gamma_{\mathcal{O},a}} \) is inserted to take into account a possible semi-locality of the operator which interpolates the particle \( a \) (i.e. any operator \( \varphi_a \) such that \( \langle 0 | \varphi_a | a \rangle \neq 0 \)) with respect to the operator \( \mathcal{O}(x) \). When \( \gamma_{\mathcal{O},a} = 0 \), there is no crossing symmetric counterpart to the unitarity cut but when \( \gamma_{\mathcal{O},a} \neq 0 \), there is instead a non-locality discontinuity in the plane of the Mandelstam variable \( s \), with \( s = 0 \) as branch point. In the rapidity parameterization there is however no cut because the different Riemann sheets of the \( s \)-plane are mapped onto different sections of the \( \theta \)-plane; the branch point \( s = 0 \) is mapped onto the points \( \theta = \pm i\pi \) which become therefore the locations of simple annihilation poles. The residues at these poles are given by [17] (see also [18])

\[ -i \text{Res}_{\theta = \pm i\pi} F_{\bar{a}a}^{\mathcal{O}}(\theta) = (1 - e^{\mp 2i\pi \gamma_{\mathcal{O},a}}) \langle 0 | \mathcal{O} | 0 \rangle . \]  

In a Sine–Gordon model with frequency \( \beta \), an exponential operator \( \Psi_\alpha = e^{i\alpha \varphi} \) has a semi-locality index with respect to the soliton \( s \) of the theory given by \( \gamma_{\alpha,s} = \alpha / \beta \) whereas it has a vanishing semi-locality index with respect to the breather particles [2]. This implies that, taking the Sine–Gordon action as the integrable \( A_0 \) and \( \Psi_\alpha \) as the perturbing operator, the formula (2.2) can be safely applied to compute the first order correction to the mass of the breathers, whereas a divergence may appear in an analogous computation of the mass correction of the solitons. This divergence has to be seen as the mathematical signal that the solitons of the original integrable model no longer survive as asymptotic particles of the perturbed theory, i.e. they are confined.

3 Semiclassical Method

The semiclassical quantization of a field theory defined by a potential \( V(\phi) \) consists in identifying a classical background \( \phi_{cl}(x) \), which satisfies the equation of motion

\[ \partial_\mu \partial^\mu \phi_{cl} + V'(\phi_{cl}) = 0 , \]  

and in applying to it various well established techniques, like the path integral formalism [19] or the solution of the field equations in classical background [3], usually called the DHN method (for a systematic review, see [20]).

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\(^2\)Consistency of eq. (2.5) requires \( \gamma_{\mathcal{O},\bar{a}} = -\gamma_{\mathcal{O},a} \).

\(^3\)The Mandelstam variable \( s \) is expressed by \( s = (p_a + p_{\bar{a}})^2 = 4m_a^2 \cosh^2(\theta/2) \).
The procedure is particularly simple and interesting if one considers classical field solutions $\phi_{cl}(x)$ in $(1+1)$ dimensions which are static "kink" configurations interpolating between degenerate minima of the potential, and whose quantization gives rise to a particle-like spectrum.

A remarkable result, due to Goldstone and Jackiw [4] (see also [5] for a non-relativistic context), is that the classical background $\phi_{cl}(x)$ has the quantum meaning of Fourier transform of the Form Factor of the basic field $\phi(x)$ between kink states. The technique to derive this result relies on the Heisenberg equation of motion satisfied by the quantum field $\phi(x)$ together with the basic hypothesis that the kink momentum is very small compared to its mass$^4$. In [6], we have refined the original argument overcoming its serious drawback of being formulated non-covariantly in terms of the kink space-momenta. This was possible thanks to the use of the rapidity variable $\theta$ of the kink states (and considering it as very small), instead of the momentum.

The final result is the expression of the semiclassical form factor between kink states as the Fourier transform of the classical kink background, with respect to the Lorentz invariant rapidity difference $\theta \equiv \theta_1 - \theta_2$:

$$< p_1 | \phi(0) | p_2 > \equiv f(\theta) \equiv M_{cl} \int da e^{i M_{cl} \theta a} \phi_{cl}(a), \quad (3.2)$$

where $M_{cl}$ is the classical energy of the kink$^5$. Having a covariant formulation, it is possible to express the crossed channel form factor through the variable transformation $\theta \to i\pi - \theta$:

$$F_2(\theta) \equiv < 0 | \phi(0) | p_1, \bar{p}_2 > = f(i\pi - \theta) . \quad (3.3)$$

The analysis of this quantity provides a direct information about the spectrum of the theory. Its dynamical poles, in fact, located at $\theta^* = i(\pi - u)$ with $0 < u < \pi$, coincide with the poles of the kink–antikink $S$-matrix, and the relative bound states masses can be then expressed as

$$m_{(b)} = 2M_{cl} \sin \frac{u}{2} . \quad (3.4)$$

It is worth stressing that this procedure for extracting the semiclassical bound states masses is remarkably simpler than the standard DHN method of quantizing the corresponding classical backgrounds, because in general these solutions depend also on time

$^4$The mass of kink state is inversely proportional to the coupling constant, considered small in the semiclassical regime, and therefore the kink is a heavy particle in this limit.

$^5$Along the same lines, it is possible to prove that the form factor of an operator expressible as a function of $\phi$ is given by the Fourier transform of the same function of $\phi_{cl}$. For instance, the form factor of the energy density operator $\varepsilon$ can be computed performing the Fourier transform of $\varepsilon_{cl}(x) = \frac{1}{2} \left( \frac{d\phi_{cl}}{dx} \right)^2 + V[\phi_{cl}]$. 

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and have a much more complicated structure than the kink ones. Moreover, in non-integrable theories these backgrounds could even not exist as exact solutions of the field equations: this happens for example in the \( \phi^4 \) theory, where the DHN quantization has been performed on some approximate backgrounds [3].

In order to compute the first quantum corrections to the masses, one has to quantize semiclassically the theory around the classical solution by splitting the field as \( \phi(x, t) = \phi_{cl}(x) + \eta(x)e^{-i\omega t} \) and finding the eigenvalues \( \omega_i \) of the stability equation [3, 20]

\[
\left(-\partial_x^2 + V''[\phi_{cl}(x)]\right) \eta_i(x) = \omega_i^2 \eta_i(x) .
\]  

(3.5)

With these, the semiclassical energy levels are build as

\[
E_{\{n_i\}} = E_{cl} + \hbar \sum_i \left(n_i + \frac{1}{2}\right) \omega_i + O(\hbar^2) ,
\]

(3.6)

and, in particular, the particles masses are given by the ground state of these levels

\[
E_0 \equiv E_{\{n_i=0\}} = E_{cl} + \frac{\hbar}{2} \sum_i \omega_i + O(\hbar^2) .
\]

(3.7)

In the following, we will not include these corrections in our results, since the analytical solution of the stability equation (3.5) in the case of the DSG model is still missing. Nevertheless, these corrections are not necessary in the cases in exam, because we consider kink particles, for which the classical energy is the term of leading order in the coupling, and their bound states, for which expression (3.4) already encodes the first semiclassical corrections (see [3]).

However, the eigenvalues \( \omega_i \) play an important role in the case of unstable particles, since the fingerprint of instability is precisely the imaginary nature of some of these frequencies. Hence, although we will obtain real values for the masses of all the considered particles, we will always keep in mind that many of these masses receive imaginary contributions coming from some of the \( \omega_i \).

4 Semiclassical analysis of DSG particle spectrum

The double Sine-Gordon model is defined by the potential

\[
V_\delta(\varphi) = -\frac{\mu}{\beta^2} \cos \beta \varphi - \frac{\lambda}{\beta^2} \cos \left(\frac{\beta}{2} \varphi + \delta\right) + C ,
\]

(4.1)

with the constant \( C \) chosen such that the vacuum state has a vanishing potential energy. We will study this theory in a regime of small \( \beta \), where the semiclassical results
are expected to give a valuable approximation of the spectrum\(^6\). At the quantum level, the different Renormalization Group trajectories originating from the gaussian fixed point described by the kinetic term \(\frac{1}{2} (\partial_\mu \varphi)^2\) of the lagrangian (1.1) are labelled by the dimensionless scaling variable \(\eta = \lambda \mu^{- (8\pi - \beta^2/4)/(8\pi - \beta^2)}\) which simply reduces to the ratio \(\eta = \frac{\lambda}{\mu}\) in the semiclassical limit. When \(\lambda\) or \(\mu\) are equal to zero, the DSG model coincides with an ordinary Sine-Gordon model with coupling \(\beta\) or \(\beta/2\), and mass scale \(\sqrt{\mu}\) or \(\sqrt{\lambda}/4\), respectively.

Since for general values of the couplings the potential (4.1) presents a \(\frac{4\pi}{\beta}\)-periodicity, it was noticed in [21] that one has an adiabatic perturbation of an integrable model only if the \(\lambda = 0\) theory is regarded as a two–folded Sine-Gordon model. This theory is a modification of the standard Sine-Gordon model, where the period of the field \(\phi\) is defined to be \(\frac{4\pi}{\beta}\), instead of \(\frac{2\pi}{\beta}\) [22]. As a consequence of this new periodicity assignment, such a theory has two different degenerate vacua \(|k\rangle\), with \(k = 0, 1\) and \(|k + 2\rangle \equiv |k\rangle\), which are defined by \(\langle k| \phi |k\rangle = \frac{2\pi}{\beta} k\). Hence it has two different kinks, related to the classical backgrounds by the formula

\[
K_{k,k+1}^{cl}(x) = \frac{2k\pi}{\beta} + \frac{4}{\beta} \arctan e^{m x} \quad k = 0, 1, \tag{4.2}
\]

and two corresponding antikinks, related to the classical solutions by the expression

\[
K_{k+1,k}^{cl}(x) = \frac{2k\pi}{\beta} + \frac{4}{\beta} \arctan e^{-m x} \tag{4.3}
\]

\[
= \frac{2(k+1)\pi}{\beta} - \frac{4}{\beta} \arctan e^{m x} \quad k = 0, 1. \tag{4.4}
\]

Finally, in the spectrum there are also two sets of kink-antikink bound states \(\delta_n^{(l)}\), with \(l = 0, 1\) and \(n = 1, \ldots, \left[\frac{\xi}{\pi}\right]\).

The flow between the two limiting Sine-Gordon models (with frequency \(\beta\) or \(\beta/2\), respectively) displays a variety of different qualitative features, including confinement and phase transition phenomena, depending on the signs of \(\lambda\) and \(\mu\), and on the value of the relative phase \(\delta\). However, it was observed in [2] that the only values of \(\delta\) which lead to inequivalent theories are those given by \(|\delta| \leq \frac{\pi}{2}\). Furthermore, in virtue of the relations

\[
V_\delta (\phi + \frac{\pi}{\beta}, \lambda, \mu) = V_{\delta + \pi/2} (\phi, \lambda, -\mu), \tag{4.5}
\]

\[
V_\delta (-\phi, \lambda, \mu) = V_{-\delta} (\phi, \lambda, \mu),
\]

we can describe all the inequivalent possibilities keeping \(\mu\) positive and the relative phase in the range \(0 \leq \delta \leq \frac{\pi}{2}\). The sign of the coupling \(\lambda\), instead, simply corresponds to a

\(^6\)By applying the stability conditions found in [2] to this model, they reduce to the condition \(\beta^2 < 8\pi\). Hence, for these values of \(\beta\) and, in particular in the semiclassical limit \(\beta \to 0\), the potential (4.1) is stable under renormalization and no counterterms have to be added.
shift or a reflection of the potential, without changing its qualitative features. As we are going to show in the following, the case $\delta = \frac{\pi}{2}$ displays peculiar features, while a common description is possible for any other value of $\delta$ in the range $0 \leq \delta < \frac{\pi}{2}$.

In closing this discussion on the general properties of the DSG model, we would like to mention that the possibility of writing exact classical solutions for all the different kinds of topological objects in this model finds a deep explanation in the relation between the trigonometric potential (4.1) and power-like potentials. In fact, defining

$$\varphi = \frac{n\pi}{\beta} \pm \frac{4}{\beta} \arctan Y , \quad n = 0, 1, 2, 3 ,$$

one can easily see that the first order equation which determines the kink solution

$$\frac{1}{2} \left( \frac{d \varphi}{dx} \right)^2 = -\frac{\mu}{\beta^2} \cos \beta \varphi - \frac{\lambda}{\beta^2} \cos \left( \frac{\beta}{2} \varphi + \delta \right) + C$$

is mapped into the equation for $Y$

$$\frac{1}{2} \left( \frac{dY}{dx} \right)^2 = U(Y) ,$$

where $U(Y)$ describes various kinds of algebraic potentials, depending on the values of $n$, $\delta$ and $C$. The $\delta = 0$ case was analyzed in [23] and its classical solutions are very simple because $U(Y)$ only contains quartic and quadratic powers of $Y$. It is easy to see that a similar situation also occurs in the $\delta = \frac{\pi}{2}$ case; for instance, choosing $n = 1$ and $C = -\frac{1}{\beta^2} \left( \mu + \frac{\lambda^2}{8\mu} \right)$, one obtains the quartic potential

$$U(Y) = \frac{(4\mu + \lambda)^2}{128\mu} \left( \frac{4\mu - \lambda}{4\mu + \lambda} Y^2 - 1 \right)^2 ,$$

which has the well known classical background

$$Y(x) = \sqrt{\frac{4\mu + \lambda}{4\mu - \lambda}} \tanh \left( \sqrt{\mu - \frac{\lambda^2}{16\mu}} \frac{x}{2} \right) .$$

For generic $\delta$, instead, also cubic and linear powers of $Y$ appear, making more complicated the analysis of the classical solutions.

**4.1 $\delta = 0$ case**

It is convenient to start our discussion with the case $\delta = 0$. This case, in fact, displays those topological features which are common to all other models with $0 < \delta < \frac{\pi}{2}$, but it admits a simpler technical analysis, due to the fact that parity invariance survives the
deformation of the original SG model. As we will see explicitly, in this case the results of
the FFPT and the Semiclassical Method are complementary, since they describe different
kinds of excitations present in the theory.

Fig.1 shows the shape of this DSG potential in the two different regimes, i.e. (i)
$0 < \lambda < 4\mu$ and (ii) $\lambda > 4\mu$. The absolute minimum persists in the position $0 \pmod{4\pi}$
for any values of the couplings, while the other minimum at $\frac{2\pi}{\beta} \pmod{4\pi}$ becomes relative
and disappears at the point $\lambda = 4\mu$. The breaking of the degeneration between the two
initial vacua in the two–folded SG causes the confinement of the original SG solitons, as it
can be explicitly checked by applying the FFPT. The linearly rising potential, responsible
for the confinement of the SG solitons, gives rise then to a discrete spectrum of bound
states whose mass is beyond $2M_{SG}$, where $M_{SG}$ is the mass of the SG solitons [2, 14].

The disappearing of the initial solitons represents, of course, a drastic change in the
topological features of the spectrum. At the same time, however, a stable new static
kink solution appears for $\lambda \neq 0$, interpolating between the new vacua at $0$ and $\frac{4\pi}{\beta}$. The
existence of this new topological solution is at the origin of the complementarity between
the FFPT and the Semiclassical Method. By the first technique, in fact, one can follow
adiabatically the deformation of the SG breathers masses: these are neutral objects that
persist in the theory although the confinement of the original kinks has taken place. It
is of course impossible to see these particle states by using the Semiclassical Method,
since the corresponding solitons, which originate these breathers as their bound states,
have disappeared. Semiclassical Method can instead estimate the masses of other neutral
particles, i.e. those which appear as bound states of the new stable kink present in the
deformed theory.

This new kink solution, interpolating between $0$ and $\frac{4\pi}{\beta}$, is given explicitly by

$$
\varphi_K(x) = \frac{2\pi}{\beta} + \frac{4}{\beta} \arctan \left[ \sqrt{\frac{\lambda}{\lambda + 4\mu}} \sinh (m x) \right],
$$

(4.11)
where
\[ m^2 = \mu + \frac{\lambda}{4} \] (4.12)
is the curvature of the absolute minimum. Interestingly enough [7], this background admits an equivalent expression in terms of the superposition of two solitons of the unperturbed Sine-Gordon model, centered at the fixed points \( \pm R \)
\[ \varphi_K(x) = \varphi_{SG}(x + R) + \varphi_{SG}(x - R), \] (4.13)
where \( \varphi_{SG}(x) = \frac{4}{\beta} \arctan [e^{m \cdot x}] \) are the usual Sine-Gordon solitons with the deformed mass parameter (4.12) whereas their distance \( 2R \) is expressed in terms of the couplings by
\[ R = \frac{1}{m} \arccosh \sqrt{\frac{4\mu}{\lambda} + 1}. \]

By looking at Fig.2, it is clear that this background, in the small \( \lambda \) limit, describes the two confined solitons of SG, which become free in the \( \lambda = 0 \) point, i.e. where \( R \rightarrow \infty \).

The classical energy of this kink is given by
\[ M_K = \frac{16 m}{\beta^2} \left\{ 1 + \frac{\lambda}{\sqrt{4\mu(\lambda + 4\mu)}} \arctanh \sqrt{\frac{4\mu}{\lambda + 4\mu}} \right\}, \] (4.14)
and in the \( \lambda \rightarrow 0 \) limit it tends to twice the classical energy of the Sine-Gordon soliton, i.e.
\[ M_K \xrightarrow{\lambda \rightarrow 0} \frac{16\sqrt{\mu}}{\beta^2}, \] (4.15)
therefore confirming the above picture. In the \( \mu \rightarrow 0 \) limit, the asymptotic value of the above expression is instead the mass of the soliton in the Sine-Gordon model with coupling \( \beta/2 \). The expansion for small \( \mu \)
\[ M_K \xrightarrow{\mu \rightarrow 0} \frac{8\sqrt{\lambda/4}}{(\beta/2)^2} + \frac{\mu}{\beta^2} \frac{32}{3\sqrt{\lambda}} + O(\mu^2), \] (4.16)
gives the first order correction which is in agreement with the result of the FFPT in the semiclassical limit (see Appendix A).

The bound states created by the kink (4.11) and its antikink can be obtained by looking at the poles of the semiclassical Form Factors of the fields \( \varphi(x) \) and \( \varepsilon(x) \), reported in Appendix B, and their mass are given by\(^7\)

\[
m^{(n)}_{(K)} = 2M_K \sin \left( n \frac{m}{2M_K} \right), \quad 0 < n < \frac{\pi M_K}{m}.
\]

For small \( \mu \) we easily recognize the perturbation of the standard breathers in Sine-Gordon with \( \beta/2 \):

\[
m^{(n)}_{(K)} \xrightarrow{\mu \to 0} \frac{64}{\beta^2} \sqrt{\frac{\lambda}{4}} \sin \left( n \frac{\beta^2}{64} \right) + \frac{2}{3} \frac{\mu}{\sqrt{\lambda}} \left[ \frac{32}{\beta^2} \sin \left( n \frac{\beta^2}{64} \right) + n \cos \left( n \frac{\beta^2}{64} \right) \right] + O(\mu^2),
\]

while the expansion of the bound states masses for small \( \lambda \)

\[
m^{(n)}_{(K)} \xrightarrow{\lambda \to 0} \frac{32\sqrt{\mu}}{\beta^2} \sin \left( n \frac{\beta^2}{32} \right) + \frac{1}{8} \frac{\lambda}{\sqrt{\mu}} \left[ \left( 1 - \ln \frac{\lambda}{16\mu} \right) \frac{32}{\beta^2} \sin \left( n \frac{\beta^2}{32} \right) + n \ln \frac{\lambda}{16\mu} \cos \left( n \frac{\beta^2}{32} \right) \right] + O(\lambda^2)
\]

deserves further comments: in fact, although the above masses have well-defined asymptotic values, they do not correspond however to any state of the unperturbed SG theory. The reason is that the classical background (4.11) in the \( \lambda \to 0 \) limit does not describe any longer a localized single particle. This implies that its Fourier transform cannot be interpreted as the two-particle Form Factor and, consequently, its poles cannot be associated to any bound states.

A technical signal of the disappearing of the above mentioned bound states in the \( \lambda \to 0 \) limit can be found by computing the three particle coupling among the kink, the antikink and the lightest bound state. The residue of the kink-antikink form factor on the pole corresponding to the lightest bound state \( b^{(1)} \) has to be proportional to the one-particle form factor \( < 0 | \varphi | b^{(1)} > \) through the semiclassical 3-particle on-shell coupling of kink, antikink and elementary boson \( g_{k\bar{k}b} \):

\[
\text{Res}_{\theta=\theta_1} F_2(\theta) = i \frac{g_{k\bar{k}b}}{2\sqrt{2M_\infty m^{(1)}_b}} < 0 | \varphi | b^{(1)} >.
\]

Since the one-particle form factor takes the constant value \( 1/\sqrt{2} \), at leading order in \( \beta \) we get

\[
g_{K K b} = \frac{1}{4\sqrt{\lambda}} \left( \frac{16m}{\beta} \right)^3 \left\{ 1 + \frac{\lambda}{\sqrt{4\mu(\lambda + 4\mu)}} \arctanh \sqrt{\frac{4\mu}{\lambda + 4\mu}} \right\}.
\]

\(^7\)Due to parity invariance, the dynamical poles of the form factor of \( \varphi \) between kink states only give the bound states with \( n \) odd. The even states can be obtained from the form factor of the energy operator \( \varepsilon(x) \).
The divergence of the coupling as $\lambda \to 0$ indicates that the considered scattering processes cannot be seen anymore as a bound state creation, i.e. the corresponding bound state disappears from the theory. A general discussion of the same qualitative phenomenon for the ordinary Sine-Gordon model can be found in [24], where the disappearing from the theory of a heavy breather at specific values of $\beta$ is explicitly related to the divergence or to the imaginary nature of the three particle coupling among this breather and two lightest ones.

Summarizing, in this model we have three kinds of neutral objects, i.e. meson particles. The first kind (a) is given by the bound states originating from the confinement potential of the original solitons. These discrete states have masses above the threshold $2M_{SG}$, where $M_{SG}$ is the mass of the SG solitons, and merge in the continuum spectrum of the non-confined solitons in the $\lambda \to 0$ limit [2, 14]. The second kind (b) is represented by the deformations of SG breathers, that can be followed by means of the FFPT and have masses, for small $\lambda$, in the range $[0, 2M_{SG}]$. Finally, the third kind (c) is given by the bound states (4.17) of the stable kink of the DSG theory and they have masses in the range $[0, 4M_{SG}]$. All these mass spectra are drawn in Fig. 3.

![Figure 3: Neutral states coming from: a) solitons confinement, b) deformations of SG breathers, c) bound states of the kink (4.11)](image)

Obviously, due to the non–integrable nature of this quantum field theory not all these particles belong to the stable part of its spectrum. Apart from a selection rule coming from the conservation of parity, decay processes are expected to be simply controlled by phase–space considerations, i.e. a heavier particle with mass $M_h$ will decay in lighter particles of masses $m_i$ satisfying the condition

$$M_h \geq \sum_i m_i \ . \quad (4.21)$$

Hence, to determine the stable particles of the theory, one has initially to identify the lightest mesons of odd and even parity with mass $m^*_-$ and $m^*_+$ ($m^*_- < m^*_+$), respectively.
Then, the stable particles of even parity are those with mass \( m \) below the threshold \( 2m^*_\star \) whereas the stable particles of odd parity are those with mass \( m < m^*_\star + m^*_\star \). For instance, in the \( \mu \to 0 \) limit we know that the only stable mesons are those given by the particles \((c)\), as confirmed by the expansion (4.18). Hence, in this limit no one of the other neutral particles is present as asymptotic states. For the mesons of type \((a)\), this can be easily understood since they are all above the threshold dictated by the lightest neutral particle. The situation is more subtle, instead, for the states \((b)\). However, their absence in the theory with \( \mu \to 0 \) clearly indicates that at some particular value of \( \lambda \) even the lightest of these objects acquires a mass above the threshold \( 2m^{(l)}_\star \), with \( m^{(l)}_\star \) given by (4.17). Analogous analysis can be done for other values of the couplings so that the general conclusion is that most of the above neutral states are nothing else but resonances of the DSG model.

In addition to the above scenario of kink states and bound state thereof, in the region \( \lambda < 4\mu \) there is another non-trivial static solution of the theory, defined over the false vacuum placed at \( \varphi = \frac{2\pi}{\beta} \). It interpolates between the two values \( \frac{2\pi}{\beta} \) and \( \frac{4\pi}{\beta} - \frac{2}{\beta} \arccos(1 - \lambda/2\mu) \), and then it comes back. Its explicit expression is given by

\[
\varphi_B(x) = \frac{4\pi}{\beta} - \frac{4}{\beta} \arctan \left( \sqrt{\frac{\lambda}{4\mu - \lambda}} \cosh (m_f x) \right),
\]

(4.22)

where

\[
m^2_f = \mu - \frac{\lambda}{4}
\]

(4.23)

is the curvature of the relative minimum. Similarly to the kink (4.11), it admits an expression in terms of a soliton and an antisoliton of the unperturbed SG model:

\[
\varphi_B(x) = \varphi_{SG}(x + R) + \varphi_{SG}(-(x - R)),
\]

(4.24)

where now \( \varphi_{SG}(x) = \frac{4}{\beta} \arctan [e^{m_f x}] \) are the Sine-Gordon solitons with the deformed mass parameter (4.23) whereas their distance \( 2R \) is now given by

\[
R = \frac{1}{m_f} \arcsinh \sqrt{\frac{4\mu}{\lambda} - 1}.
\]

(4.25)

In the small \( \lambda \) limit, it is clear that this background describes the confined soliton and antisoliton of the SG model, which become free in the \( \lambda = 0 \) point, i.e. where \( R \to \infty \).

The classical background (4.22) is not related to any stable particle in the quantum theory. This can be directly seen from equation (3.5); in fact, Lorentz invariance always implies the presence of the eigenvalue \( \omega^2_\varphi = 0 \), with corresponding eigenfunction \( \eta_0(x) = \frac{d}{dx} \varphi_d(x) \). However, in the case of the solution (4.22) the eigenfunction \( \eta_0 \) clearly
displays a node, which indicates that the corresponding eigenvalue is not the smallest in the spectrum. Hence, there must be a lower eigenvalue $\omega_{-1}^2 < 0$, with a corresponding imaginary part of the mass relative to this particle state. Furthermore, the instability of (4.22) can be related to the theory of false vacuum decay [25, 26]: due to the deep physical interest of this topic, we will discuss it separately in Section 5.

4.2 Comments on generic $\delta$ case

We have already anticipated that the qualitative features of the theory relative to $\delta = 0$ case are common to all other theories associated to the values of $\delta$ in the range $0 < \delta < \frac{\pi}{2}$. This can be clearly understood by looking at the shape of the potential, which is shown in Fig. 5 for the case $\delta = \frac{\pi}{3}$.

In contrast to the $\delta = 0$ case, parity invariance is now lost in these models, and the minima move to values depending on the couplings. Furthermore, in addition to the change in the nature of the original vacuum at $\frac{2\pi}{\beta}$, which becomes a relative minimum by switching on $\lambda$, there is also a lowering of one of the two maxima. These features
make much more complicated the explicit derivation of the classical solutions, as we have mentioned at the beginning of the Section.

However, it is clear from Fig. 5 that the excitations of these theories share the same nature of the ones in the $\delta = 0$ case. In fact, the original SG solitons undergo a confinement, while a new stable topological kink appears, interpolating between the new degenerate minima. Hence, the analysis performed for $\delta = 0$ still holds in its general aspects, i.e. also in these cases the spectrum consists of a kink, antikink, and three different kinds of neutral particles.

4.3 $\delta = \frac{\pi}{2}$ case

The value $\delta = \frac{\pi}{2}$ describes the peculiar case in which no confinement phenomenon takes place, since the two different vacua of the original two–folded SG remain degenerate also in the perturbed theory. As a consequence, the original SG solitons are also asymptotic states in the perturbed theory. By means of the Semiclassical Method we can then compute their bound states, which represent the deformations of the two sets of breathers in the original two–folded SG. Hence, in this specific case FFPT and semiclassical method describe the same objects, and their results can be compared in a regime where both $\beta$ and $\lambda$ are small.

![Figure 6: DSG potential in the $\delta = \frac{\pi}{2}$ case.](image)

Fig. 6 shows the behavior of this DSG potential. There are two regions, qualitatively different, in the space of parameters, the first given by $0 < \lambda < 4\mu$ and the second given by $\lambda > 4\mu$. They are separated by the value $\lambda = 4\mu$ which has been identified in [2] as a phase transition point. We will explain how this identification is confirmed in our formalism.

Let’s start our analysis from the coupling constant region where $\lambda < 4\mu$. Switching on $\lambda$, the original inequivalent minima of the two–folded Sine-Gordon, located at $\phi_{\text{min}} =$
0, $\frac{2\pi}{\beta}$ (mod $\frac{4\pi}{\beta}$), remain degenerate and move to $\phi_{\text{min}} = -\phi_0, \frac{2\pi}{\beta} + \phi_0$ (mod $\frac{4\pi}{\beta}$), with $\phi_0 = \frac{2}{\beta} \arcsin \frac{\lambda}{4\mu}$. The common curvature of these minima is

$$m^2 = \mu - \frac{1}{16} \frac{\lambda^2}{\mu}.$$  \hfill (4.26)

Correspondingly there are two different types of kinks, one called “large kink” and interpolating through the higher barrier between $-\phi_0$ and $\frac{2\pi}{\beta} + \phi_0$, the other called “small kink” and interpolating through the lower barrier between $\frac{2\pi}{\beta} + \phi_0$ and $\frac{4\pi}{\beta} - \phi_0$. Their classical expressions are explicitly given by

$$\varphi_L(x) = \frac{\pi}{\beta} + \frac{4}{\beta} \arctan \left[ \sqrt{\frac{4\mu + \lambda}{4\mu - \lambda}} \tanh \left( \frac{m}{2} x \right) \right] \quad \text{(mod 4\pi)} ,$$

$$\varphi_S(x) = 3\frac{\pi}{\beta} + \frac{4}{\beta} \arctan \left[ \sqrt{\frac{4\mu - \lambda}{4\mu + \lambda}} \tanh \left( \frac{m}{2} x \right) \right] \quad \text{(mod 4\pi)} .$$  \hfill (4.27)

With the notation previously introduced, the vacuum structure of the corresponding quantum field theory consists of two sets of inequivalent minima, denoted by $|0\rangle$ and $|1\rangle$, identified modulo 2, i.e. $|a + 2n\rangle \equiv |a\rangle$. The spontaneous breaking of the symmetry $T : \varphi \to 2\pi - \varphi$ selects one of these minima as the vacuum. If we choose to quantize the theory around $|0\rangle$, the admitted quantum kink states are $|L\rangle = |K_{0,1}\rangle$ and $|S\rangle = |K_{0,-1}\rangle$, with the corresponding antikink states $|\overline{L}\rangle = |K_{1,0}\rangle$ and $|\overline{S}\rangle = |K_{-1,0}\rangle$, and topological charges

$$Q_L = -Q_{\overline{L}} = 1 + \frac{\beta\phi_0}{\pi} ,$$

$$Q_S = -Q_{\overline{S}} = 1 - \frac{\beta\phi_0}{\pi}.$$  \hfill (4.29)

Multi-kink states of this theory satisfy the selection rule coming from the continuity of vacuum indices and are generically given by

$$| K_{\alpha_1\alpha_2}(\theta_1) K_{\alpha_2\alpha_3}(\theta_2) \cdots K_{\alpha_{n-2}\alpha_{n-1}}(\theta_{n-2}) K_{\alpha_{n-1}\alpha_n}(\theta_{n-1}) \rangle$$ \hfill (4.30)

The leading contributions to the masses of the large and small kink are given by their classical energies, which can be easily computed

$$M_{L,S} = \frac{8m}{\beta^2} \left\{ 1 \pm \frac{\lambda}{\sqrt{16\mu^2 - \lambda^2}} \left[ \frac{\pi}{2} \pm \arcsin \frac{\lambda}{4\mu} \right] \right\} .$$  \hfill (4.31)

The expansion of this formula for small $\lambda$ is given by

$$M_{L,S} \underset{\lambda \to 0}{\rightarrow} \frac{8\sqrt{\mu}}{\beta^2} \pm \frac{\lambda}{\beta^2} \frac{\pi}{\sqrt{\mu}} + O(\lambda^2) ,$$  \hfill (4.32)

and the first order correction in $\lambda$ coincides with the result of FFPT in the semiclassical limit (see Appendix A).
Since two different types of kink $|L\rangle$ and $|S\rangle$ are present in this theory, one must be careful in applying eq. (3.2) to recover the form factors of each kink separately. In fact, one could expect that both types of kink contribute to the expansion over intermediate states used in [4] to derive the result. For instance, starting from the vacuum $|0\rangle$ located at $\phi_{\text{min}} = -\phi_0$ there might be the intermediate matrix elements $\langle 0 | \mathcal{O} | L \rangle_0$ and $\langle 0 | L | \mathcal{O} | S \rangle_0$. However, if $\mathcal{O}$ is a non–charged local operator, it is easy to see that these off-diagonal elements have to vanish for the different topological charges of $|L\rangle$ and $|S\rangle$. Hence, the expansion over intermediate states diagonalizes and one recovers again eq. (3.2).

Therefore, from the dynamical poles of the form factor of $\varphi$ on the large and small kink-antikink states, reported in Appendix B, we can extract the semiclassical masses of two sets of bound states:

$$m_{(L)}^{(n)} = 2 M_L \sin \left( n_L \frac{m}{2 M_L} \right), \quad 0 < n_L < \frac{\pi M_L}{m},$$

$$m_{(S)}^{(n)} = 2 M_S \sin \left( n_S \frac{m}{2 M_S} \right), \quad 0 < n_S < \frac{\pi M_S}{m}.$$  (4.33, 4.34)

Expanding for small $\lambda$, we can see that these states represent the perturbation of the two sets of breathers in the original two–folded Sine-Gordon model:

$$m_{(L,S)}^{(n)} \to \frac{16 \sqrt{\mu}}{\beta^2} \sin \left( n \frac{\beta^2}{16} \right) \pm 2\pi \frac{\lambda}{\sqrt{\mu}} \left[ \frac{1}{\beta^2} \sin \left( n \frac{\beta^2}{16} \right) - n \frac{\pi}{16} \cos \left( n \frac{\beta^2}{16} \right) \right] + O(\lambda^2) \quad (4.35)$$

A discussion of these results, in comparison with previous studies of this model [21], is reported in Appendix C.

Concerning the stability of the above spectrum, for $\lambda < 4\mu$ the only stable bound states are the ones with $m_{(L,S)}^{(n)} < 2m_{(S)}^{(1)}$; for $\lambda$ close enough to $4\mu$, however, the small kink creates no bound states, hence the stability condition becomes $m_{(L)}^{(n)} < 2m_{(L)}^{(1)}$.

In the limit $\lambda \to 4\mu$, $\phi_0$ tends to $\frac{\pi}{\beta}$, the two minima at $\frac{2\pi}{\beta} + \phi_0$ and $\frac{4\pi}{\beta} - \phi_0$ coincide and the small kink disappears, becoming a constant solution with zero classical energy. All the large kink bound states masses collapse to zero, and in this limit all dynamical poles of the large kink form factor disappear. This is nothing else but the semiclassical manifestation of the occurrence of the phase transition present in the DSG model (see [2]).

In the second coupling constant region, parameterized by $\lambda > 4\mu$, there is only one minimum at fixed position $-\frac{\pi}{\beta}$ (mod $\frac{4\pi}{\beta}$), with curvature

$$m^2 = \frac{\lambda}{4} - \mu.$$  (4.36)
There is now only one type of kink, given by
\[ \varphi_K(x) = \frac{\pi}{\beta} + \frac{4}{\beta} \arctan \left[ \sqrt{\frac{\lambda}{\lambda - 4\mu}} \sinh (mx) \right]. \] (4.37)

Its classical mass, expanded for small \( \mu \), is again in agreement with FFPT (see Appendix A):
\[ M_K = \frac{16 m}{\beta^2} \left\{ 1 + \frac{\lambda}{4\sqrt{\mu}(\lambda - 4\mu)} \left( \frac{\pi}{2} - \arcsin \frac{\lambda - 8\mu}{\lambda} \right) \right\} \rightarrow \lim_{\lambda \to 0} \frac{8 \sqrt{\lambda/4}}{(\beta/2)^2} - \frac{\mu}{\beta^2} \frac{32}{3\sqrt{\lambda}} + O(\mu^2). \] (4.38)

The bound states of this kink (see Appendix B for the explicit Form Factor) have masses
\[ m_{(n)}^I = 2M_K \sin \left( n \frac{m}{2M_K} \right), \quad 0 < n < \pi \frac{M_K}{m}. \] (4.40)

For small \( \mu \), these states are nothing else but the perturbed breathers of the Sine-Gordon model with coupling \( \beta/2 \):
\[ m_{(n)}^I \rightarrow \lim_{\mu \to 0} 64 \frac{\lambda}{\beta^2} \sin \left( \frac{n \beta^2}{64} \right) - \frac{2}{3} \frac{\mu}{\beta^2} \left[ \frac{32}{\beta^2} \sin \left( \frac{n \beta^2}{64} \right) + n \cos \left( \frac{n \beta^2}{64} \right) \right] + O(\mu^2). \]

In closing the discussion of the \( \delta = \pi/2 \) case, it is interesting to mention another model which presents a similar phase transition phenomenon, although in a reverse order. This is the Double Sinh–Gordon Model (DShG), discussed in Appendix D. The similarity is due to the fact that also in this case a topological excitation of the theory becomes massless at the phase transition point, but the phenomenon is reversed, because in DSG the small kink disappears when \( \lambda \) reaches the critical value, while in DShG a topological excitation appears at some value of the perturbing coupling.

5 False vacuum decay

The semiclassical study of false vacuum decay in quantum field theory has been performed by Callan and Coleman [25], in close analogy with the work of Langer [26]. The phenomenon occurs when the field theoretical potential \( U(\varphi) \) displays a relative minimum at \( \varphi_+ \); this classical point corresponds to the false vacuum in the quantum theory, which decays through tunnelling effects into the true vacuum, associated with the absolute minimum \( \varphi_-(\text{see Fig. 7}) \).
Figure 7: Generic potential for a theory with a false vacuum

The main result of [25] is the following expression for the decay width per unit time and unit volume:

$$\frac{\Gamma}{V} = \left( \frac{B}{2\pi \hbar} \right) e^{-B/\hbar} \left| \frac{\det[-\partial^2 + U''(\varphi)]}{\det[-\partial^2 + U''(\varphi_+)]} \right|^{1/2} \left[ 1 + O(\hbar) \right], \quad (5.1)$$

specialized here to the case of two-dimensional space–time. Omitting any discussion of the determinant, about which we refer to the original papers [25], we will present here an explicit analysis of the coefficient $B$.

It has been shown that $B$ coincides with the Euclidean action of the so-called “bounce” background $\varphi_B$:

$$B = S_E = 2\pi \int_0^\infty d\rho \rho \left[ \frac{1}{2} \left( \frac{d\varphi_B}{d\rho} \right)^2 + U(\varphi_B) \right]. \quad (5.2)$$

This classical solution is the field-theoretical generalization of the path of least resistance in quantum mechanical tunnelling; it only depends on the Euclidean radius $\rho = \sqrt{r^2 + x^2}$ and satisfies the equation

$$\frac{d^2\varphi_B}{d\rho^2} + \frac{1}{\rho} \frac{d\varphi_B}{d\rho} = U'[\varphi_B], \quad (5.3)$$

with boundary conditions

$$\lim_{\rho \to \infty} \varphi_B(\rho) = \varphi_+ , \quad \frac{d\varphi_B}{d\rho}(0) = 0. \quad (5.4)$$

Although in general one does not know explicitly the bounce solution, it is possible to set up some approximation to extract a closed expression for the coefficient $B$. The so-called “thin wall” approximation consists in viewing the potential $U(\varphi)$ as a perturbation of another potential $U_+(\varphi)$, which displays degenerate vacua at $\varphi_\pm$ and a kink $\varphi_K(x)$ interpolating between them. The small parameter for the approximation is the energy difference $\varepsilon = U(\varphi_+) - U(\varphi_-)$.
In this framework, one can qualitatively guess that the bounce has a value $\varphi(0)$ very close to $\varphi_-$, then it remains in this position until some vary large $\rho = R$ and finally it moves quickly towards the final value $\varphi_+$. For $\rho$ near $R$, the first–derivative term in eq. (5.3) can be neglected; if in addition one also approximates $U$ with $U_+$, then one can express the unknown bounce solution as [25]

$$
\varphi_B(\rho) = \begin{cases} 
\varphi_- & \rho \ll R \\
\varphi_K(\rho - R) & \rho \approx R \\
\varphi_+ & \rho \gg R .
\end{cases}
$$

(5.5)

Since the bounce has to represent the path of least resistance, the parameter $R$, free up to this point, can be fixed by minimizing the action

$$
S_E = -\pi R^2 \epsilon + 2\pi R M_K ,
$$

(5.6)

which is given by the sum of a volume term and a surface term. Hence, the condition $\frac{dS_E}{dR} = 0$ is realized by the balance of these two different terms in competition, and it finally gives

$$
R = \frac{M_K}{\epsilon} \implies B = \pi \frac{M_K^2}{\epsilon} .
$$

(5.7)

In the DSG model, however, we know explicitly the bounce background in the thin wall regime (here we have $\epsilon = \frac{2\lambda}{\beta^2}$), without any approximation on the potential. This is given by the solution (4.22) with $x$ replaced by $\rho$, that can be directly used to estimate the decay width. Unfortunately the integral in (5.2) does not admit a simple expression to be expanded for small $\lambda$, but it is clear from eq. (4.24) and Fig. 4 that the leading contribution is given by

$$
S_E \simeq 2\pi R \int_{R-\Delta r}^{R+\Delta r} dx \left[ \frac{d\varphi_{SG}}{dx}(R-x) \right]^2 \simeq \frac{8\pi}{\beta^2} \log \left( \frac{16\mu}{\lambda} \right) ,
$$

(5.8)

with $R$ given by (4.25). This behavior in $\lambda$ does not agree with the general prediction (5.7). The reason can be traced out in the fact that eq. (4.24) explicitly realizes the relation between the bounce and the kink of the unperturbed theory, but in a more sophisticated way than (5.5). In fact, the mass parameter $m_f$ of the SG kink $\varphi_{SG}$ is dressed to be the one of the DSG theory, and the parameter $R$ is not free, since (4.22) is already the result of a minimization process, being a solution of the Euler–Lagrange equations. The thin wall approximation can be still consistently used because $R$ is very big for small $\lambda$, while the crucial difference is that the volume term is now missing from the action, since the value $\varphi_B(0) = \varphi_1$ is the so–called classical turning point (see Fig. 7), degenerate with the
false vacuum. It is worth noting that the path of least resistance in quantum mechanics precisely interpolates between the false vacuum and the turning point.

Up to the determinant factor, our result for the leading term in the decay width is then
\[ \frac{\Gamma}{V} \simeq \frac{4}{\beta^2} \left( \frac{\lambda}{16\mu} \right)^{8\pi/\beta^2} \log \left( \frac{16\mu}{\lambda} \right). \] (5.9)

It will be interesting to investigate whether the above mentioned difference with the prediction (5.7) is a particular feature of the DSG model or it appears for a generic potential if one improves the approximate description of the bounce along the lines discussed here.

6 Other kind of resonances

The appearance of resonances in the classical scattering of the Double Sine-Gordon kinks has been extensively studied with numerical techniques, and a complete picture of this phenomenon can be found in [7]. In this work, the key ingredient for the presence of resonances was identified in the presence of a discrete eigenvalue, besides the zero mode, of the small oscillations around the kink background. This eigenvalue, called “shape mode”, represents an internal excitation of the kink [3, 20].

This mechanism can be easily interpreted also in our formalism, but unfortunately in the case of the Double Sine-Gordon model we were not able to solve analytically the stability equation around the kink backgrounds. Hence, we will limit ourselves to the discussion of the same phenomenon in a simpler theory, the $\phi^4$ field theory in the broken symmetry phase:
\[ V(\phi) = \frac{g}{4} \phi^4 - \frac{m^2}{2} \phi^2 + \frac{m^4}{4g}. \] (6.1)

The standard kink background of this theory is given by
\[ \phi_{cl}(x) = \frac{m}{\sqrt{g}} \tanh \left( \frac{mx}{\sqrt{2}} \right), \] (6.2)

with classical energy $M = \frac{2\sqrt{2} m^3}{g}$. The small oscillations (3.5) around this solution have, in addition to the usual translational mode $\omega_0 = 0$, another discrete eigenvalue
\[ \omega_1^2 = \frac{3}{2} m^2, \] (6.3)

which represents an internal excitation of the kink [3, 20]. This feature, quite crucial for the analysis performed in [27], has a counterpart in our formalism. In fact, it was shown

The main features of the numerical analysis performed in [7] for the DSG model were indeed previously recognized in this simpler theory [27].
by Goldstone and Jackiw [4] that, performing the Fourier transform of the corresponding eigenfunction \( \eta_1(a) \), one can write the form factor of the field \( \phi \) between asymptotic states containing a simple and an excited kink. Furthermore, also this result, as the previous one relative to the form factors of the elementary kinks, can be refined in terms of the rapidity variable, so that one obtains a covariant expression that can be analytically continued in the crossed channel. Since in this case the eigenfunction is

\[
\eta_1(x) = -\frac{\sinh \left( \frac{mx}{\sqrt{2}} \right)}{2 \cosh^2 \left( \frac{mx}{\sqrt{2}} \right)},
\]

(6.4)

for the corresponding form factor we have

\[
\langle 0 | \phi(0) | \bar{p}_2 p_1^* \rangle = -i \frac{M \pi}{6^{1/4} m^{5/2}} \frac{M (i\pi - \theta)}{\cosh \left( \frac{\pi}{\sqrt{2} m} M (i\pi - \theta) \right)},
\]

(6.5)

where the \( p_1^* \) denotes the momentum of the excited kink state. The dynamical poles of this object correspond to bound states with masses

\[
\left( m_{b_1^*}^{(n)} \right)^2 = 4M(M + \omega_1) \sin^2 \left( \frac{1}{3} \frac{g}{m^2} (2n + 1) \right) + \omega_1^2.
\]

(6.6)

The states with

\[
\frac{8}{3} m^2 \frac{m^2}{g} \arcsin \sqrt{\frac{4M^2 - \omega_1^2}{4M(M + \omega_1)}} < 2n + 1 < \frac{4}{3} m^2 \frac{m^2}{g} \pi
\]

(6.7)

have masses in the range

\[
2M < m_{b_1^*}^{(n)} < 2M + \omega_1,
\]

(6.8)

and, therefore, they can be seen as resonances in the kink-antikink scattering.

Since the numerical analysis done in [27] is independent of the coupling constant\(^9\), a quantitative comparison with our semiclassical result is rather difficult, due to the dependence on \( g \) of (6.7). However, the presence of many resonance states seen at classical level is qualitatively confirmed to persist also in the quantum field theory at small \( g \), i.e. in its semiclassical regime, according to (6.7).

Back to the Double Sine-Gordon model, the shape mode with the relative resonances has been numerically observed for the small kink (4.28) in the \( \delta = \frac{\pi}{2} \) case, and for the kink (4.11) in the case \( \delta = 0 \). Our analysis is in agreement with these results, and it adds another possibility for the small kink case. In fact, since in this regime it is also present the large kink, which has higher mass, the resonances seen in the small kink-antikink scattering are related both to their excited bound states with masses \( m_{b_1^*}^{(n)} \) in the range

\[
2M < m_{b_1^*}^{(n)} < 2M + \omega_1,
\]

(6.9)

\(^9\)Classically, in fact, one can always rescale the field and eliminate the coupling constant \( g \).
and to the large kink-antikink bound states with masses in the range

\[ 2M_S < m_L^{(n)} < 2M_L, \]  

(6.10)

where \( m_L^{(n)} \) are given by (4.33).

7 Conclusions

When available, the Semiclassical Method is an efficient tool for studying the mass spectrum of an integrable or a non–integrable theory. In the last case, it may be complementary to the Form Factor Perturbation Theory or it may provide results comparable with this method. We have applied both techniques for analysing the mass spectrum of the non–integrable quantum field theory given by the Double Sine–Gordon model, for few qualitatively different regions of its coupling–constants space. This model appears to be an ideal theoretical playground for understanding some of the relevant features of non–integrable models. By moving its coupling constants, it shows, in fact, different types of kink excitations and confinement phenomena, a rich spectrum of meson particles, resonance states, false vacuum decay and the occurrence of a phase transition. In light of the many applications it finds in condensed matter systems, it would be interesting to investigate further its properties.

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A Kink mass corrections in the FFPT

In this Appendix we compute by means of the FFPT the corrections to the kink masses in the semiclassical limit, which is relevant for a comparison with our results.

For small $\lambda$, we have to consider the DSG model as a perturbation of the two–folded Sine-Gordon [21]. In the $\delta = \pi/2$ case, the perturbing operator is $\Psi = \sin \frac{\beta}{2} \phi$. Its form factors between the vacuum and the two possible kink-antikink asymptotic states are obtained at the semiclassical level by performing the Fourier transform of $\sin \left[ \frac{\beta}{2} K_{k,k+1}^{cl}(x) \right]$ [6], with $K_{k,k+1}^{cl}(x)$ given by eq. (4.2). Hence we obtain

$$ F_{\Psi K_{k,k+1},K_{k,k+1}}(\theta) = \frac{8\pi}{\beta^2} (-1)^k \frac{1}{\cosh \frac{4\pi}{\beta^2}(\theta - i\pi)} . $$  \hspace{1cm} (A.1)

The first order correction in $\lambda$ to the kink masses is then

$$ \delta M_{k,k+1} = \frac{\lambda}{\beta^2} \frac{1}{M_K} F_{\Psi K_{k,k+1},K_{k,k+1}}(i\pi) = (-1)^k \frac{\lambda}{\beta^2} \frac{\pi}{\sqrt{\mu}} , $$  \hspace{1cm} (A.2)

in agreement with the correction to the classical masses (4.32), since $K_{0,1}$ is associated with the large kink, and $K_{1,2}$ with the small one.

In the $\delta = 0$ case, instead, we can explicitly see how the solitons disappear from the spectrum as soon as $\lambda$ is switched on. The form factor of the operator $\Psi = \cos \frac{\beta}{2} \phi$ has, in fact, a divergence at $\theta = i\pi$

$$ F_{\Psi K_{k,k+1},K_{k,k+1}}(\theta) = -\frac{8\pi}{\beta^2} (-1)^k \frac{1}{\sinh \frac{4\pi}{\beta^2}(i\pi - \theta)} . $$  \hspace{1cm} (A.3)

The other interesting regime to explore is the small $\mu$ limit. In the case $\delta = 0$, this can be seen as the perturbation of the SG model at coupling $\tilde{\beta} = \beta/2$ by means of the operator $\Psi = \cos 2\tilde{\beta} \varphi$. The semiclassical form factor is

$$ F_{\Psi K,K}(\theta) = \frac{16}{3} \frac{32}{\beta^2} \frac{i\pi y}{\sin i\pi y} (1 - 2y^2) , $$  \hspace{1cm} (A.4)

where we have defined $y = \frac{16}{\beta^2} (i\pi - \theta)$. The corresponding mass correction is given by

$$ \delta M_K = \frac{\mu}{\beta^2} \frac{1}{M_K} F_{\Psi K,K}(i\pi) = \frac{\mu}{\beta^2} \frac{16}{3} \frac{1}{\sqrt{\lambda/4}} , $$  \hspace{1cm} (A.5)

in agreement with (4.16).

The case $\delta = \frac{\pi}{2}$ can be described by shifting the original SG field as $\varphi \rightarrow \varphi + \frac{\pi}{\beta}$. In this way the perturbing operator becomes $-\Psi$ and we finally obtain the same mass correction but with opposite sign, as in (4.38).
B Semiclassical form factors

In this Appendix we explicitly present the expressions of the two–particle form factors, on the asymptotic states given by the different kinks appearing in the DSG theory, of the operators \( \varphi(x) \) and \( \varepsilon(x) \), the last one defined by

\[
\varepsilon(x) \equiv \frac{1}{2} \left( \frac{d\varphi}{dx} \right)^2 + V[\varphi(x)].
\]

These matrix elements are obtained by performing the Fourier transforms of the corresponding classical backgrounds, as indicated in (3.2) and (3.3). We use the notation:

\[
F_{\Psi}^{\Psi}(\theta) = \langle 0 | \Psi(0) | K(\theta_1) \bar{K}(\theta_2) \rangle,
\]

with \( \theta = \theta_1 - \theta_2 \).

For the kink (4.11) in the \( \delta = 0 \) case we have

\[
F_{\varphi}^{\varphi}(\theta) = \frac{4\pi^2}{\beta} M_K \delta [M_K(i\pi - \theta)] + i \frac{4\pi}{\beta} \frac{1}{i\pi - \theta} \cosh \left[ \frac{\alpha M_K}{M_K} (i\pi - \theta) \right] \cosh \left[ \frac{\pi M_K}{i\pi - \theta} \right], \tag{B.1}
\]

where

\[
\alpha = \text{arccosh} \sqrt{\frac{\lambda + 4\mu}{\lambda}},
\]

while \( m \) and \( M_K \) are given by (4.12) and (4.14), respectively, and

\[
F_{\varepsilon}^{\varepsilon}(\theta) = -\frac{128\pi}{\beta^2} m^3 M_K \frac{1}{\lambda} \left\{ \frac{1}{\sinh \left[ \frac{\pi M_K}{2m} (i\pi - \theta) \right]} \right\} d \frac{d}{dc} \left[ \sinh \left( \text{arccosh} \frac{c}{\sqrt{c^2 - 1}} \right) \right] + \frac{2\sinh \pi}{\cosh \left[ \frac{\pi M_K}{2m} (i\pi - \theta) \right] - 1} \frac{d}{dc} \left[ c \sinh \left( \text{arccosh} \frac{c}{\sqrt{c^2 - 1}} \right) \right], \tag{B.2}
\]

where \( c = 1 + \frac{8\mu}{\lambda} \).

For the large kink (4.27) in the \( \delta = \frac{\pi}{2} \) case (with \( \lambda < 4\mu \)) we have

\[
F_{\varepsilon}^{\varepsilon}(\theta) = \frac{2\pi^2}{\beta} M_L \delta [M_L(i\pi - \theta)] + i \frac{4\pi}{\beta} \frac{1}{i\pi - \theta} \sinh \left[ \frac{\alpha M_L}{M_L} (i\pi - \theta) \right] \cosh \left[ \frac{\pi M_L}{i\pi - \theta} \right], \tag{B.3}
\]

where

\[
\alpha = 2 \text{arctan} \sqrt{\frac{4\mu + \lambda}{4\mu - \lambda}},
\]

while \( m \) and \( M_L \) are given by (4.26) and (4.31), respectively, and

\[
F_{\varepsilon}^{\varepsilon}(\theta) = \frac{8\pi}{\beta^2} \frac{m^3 M_L}{\mu} \frac{1}{\sinh \left[ \frac{\pi M_L}{m} (i\pi - \theta) \right]} \frac{d}{dc} \left\{ \sinh \left( \text{arccos} \frac{\mu}{\sqrt{1 - c^2}} \right) \right\}, \tag{B.4}
\]

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where \( c = -\frac{\lambda}{4\mu} \).

For the small kink (4.28) in the \( \delta = \frac{\pi}{2} \) case (with \( \lambda < 4\mu \)) we have
\[
F^\varphi_{SS}(\theta) = \frac{6\pi^2}{\beta} M_S \delta [M_S(i\pi - \theta)] + i \frac{4\pi}{\beta} \frac{1}{i\pi - \theta} \frac{\sinh \left[ \lambda M_S(i\pi - \theta) \right]}{\sinh \left[ \frac{\lambda M_S}{\mu} (i\pi - \theta) \right]},
\]
(B.5)
where
\[
\alpha = 2 \arctan \sqrt{\frac{4\mu - \lambda}{4\mu + \lambda}},
\]
while \( m \) and \( M_S \) are given by (4.26) and (4.31), respectively, and
\[
F^\varphi_{SS}(\theta) = \frac{8\pi}{\beta^2} \frac{m^3 M_S}{\mu} \frac{1}{\sinh \left[ \frac{\lambda M_S}{\mu} (i\pi - \theta) \right]} \frac{d}{dc} \left\{ \frac{\sinh \left[ (\arccos c) \frac{M_S}{\mu} (i\pi - \theta) \right]}{\sqrt{1 - c^2}} \right\},
\]
(B.6)
where \( c = \frac{1}{4\mu} \).

Finally, for the kink (4.37) in the \( \delta = \frac{\pi}{2} \) case (with \( \lambda > 4\mu \)) we have
\[
F^\varphi_{KK}(\theta) = \frac{2\pi^2}{\beta} M_K \delta [M_K(i\pi - \theta)] + i \frac{4\pi}{\beta} \frac{1}{i\pi - \theta} \frac{\cos \left[ \frac{\lambda M_K}{\mu} (i\pi - \theta) \right]}{\cosh \left[ \frac{\lambda M_K}{2\mu} (i\pi - \theta) \right]},
\]
(B.7)
where
\[
\alpha = \arccosh \sqrt{\frac{\lambda - 4\mu}{\lambda}},
\]
while \( m \) and \( M_L \) are given by (4.36) and (4.38), respectively, and
\[
F^\varphi_{KK}(\theta) = -\frac{128\pi}{\beta^2} \frac{m^3 M_K}{\lambda} \frac{1}{\sinh \left[ \frac{\lambda M_K}{2\mu} (i\pi - \theta) \right]} \frac{d}{dc} \left\{ \frac{\sinh \left[ (\arccos c) \frac{M_K}{\mu} (i\pi - \theta) \right]}{\sqrt{1 - c^2}} \right\} + \frac{2 \sinh \pi}{\cosh \left[ \frac{\lambda M_K}{\mu} (i\pi - \theta) \right]} - 1 \frac{d}{dc} \left\{ c \sinh \left[ (\arccos c) \frac{M_K}{\mu} (i\pi - \theta) \right] \right\},
\]
(B.8)
where \( c = 1 - \frac{8\mu}{\lambda} \).

C Neutral states in the \( \delta = \frac{\pi}{2} \) case

The semiclassical results reported in the text, i.e. eqs. (4.33), (4.34) and (4.35), pose an interesting question about the nature of neutral states in the DSG model at \( \delta = \frac{\pi}{2} \). It should be noticed, in fact, that the first order correction in \( \lambda \) obtained by the Semiclassical Method does not match with the results reported in [21] where, by using the FFPT and
an extrapolation of numerical data, the authors concluded that this correction was instead identically zero\textsuperscript{10}. It is worth discussing this problem in more detail.

In the standard Sine–Gordon model, the breathers $|b_n\rangle$, with $n$ odd (or even), are defined as the bound states of odd (or even) combinations of $|K\bar{K}\rangle$ and $|\bar{K}K\rangle$, where $K$ represents the soliton and $\bar{K}$ the antisoliton. The combinations $|K\bar{K}\pm\bar{K}K\rangle$ are eigenstates of the parity operator $P: \phi \to -\phi$, which commutes with the hamiltonian and acts on the soliton transforming it into the antisoliton. The above mentioned identification of the bound states relies on a very peculiar feature of the Sine–Gordon $S$–matrix in the soliton sector \[28\], whose elements are defined as

\begin{align}
K(\theta_1)\bar{K}(\theta_2) &\ = \ S_T(\theta_{12})\bar{K}(\theta_2)K(\theta_1) + S_R(\theta_{12})K(\theta_2)\bar{K}(\theta_1), \quad (C.1) \\
K(\theta_1)K(\theta_2) &\ = \ S(\theta_{12})K(\theta_2)K(\theta_1), \quad (C.2) \\
\bar{K}(\theta_1)\bar{K}(\theta_2) &\ = \ S(\theta_{12})\bar{K}(\theta_2)\bar{K}(\theta_1). \quad (C.3)
\end{align}

In fact, both the transmission and the reflection amplitudes $S_T(\theta)$ and $S_R(\theta)$ display poles at $\theta^*_n = i(\pi - n\xi)$, with residua which are equal or opposite in sign depending whether $n$ is odd or even. Hence, the diagonal elements

\begin{align}
S_-(\theta) &\ = \ \frac{1}{2} [S_T(\theta) - S_R(\theta)], \quad (C.4) \\
S_+(\theta) &\ = \ \frac{1}{2} [S_T(\theta) + S_R(\theta)] \quad (C.5)
\end{align}

have only the poles with odd or even $n$, respectively, and for each $n$ there is only one bound state with definite parity.

However, this is a special feature of the Sine–Gordon model which finds no counterpart, for instance, in other problems with a similar structure. As an explicit example, one can consider the \((\text{RSOS})_3\) scattering theory, which displays a 3-fold degenerate vacuum and two types of kink and antikink with the same mass. The central vacuum is surrounded by two other minima, as in the Sine–Gordon case, and this gives the possibility to define both a kink-antikink state and an antikink-kink state around it. The minimal scattering matrix, given in \[29\], can be dressed with a CDD factor to generate bound states. It is easy to check that the common poles in the transmission and reflection amplitudes have in this case different residua, giving rise to two distinct bound states, degenerate in mass, over the central vacuum.

Hence, if we call $|\hat{b}_n^{(0)}\rangle$ the bound states of kink-antikink and $|\hat{b}_n^{(1)}\rangle$ the bound states of antikink-kink, in general we have to consider them as two distinct excitations, and if\textsuperscript{10}It is worth stressing that the linear correction (4.35) in $\lambda$ is very small even for finite values of $\beta$ (it is easy to check, indeed, that the first term of its expansion is $\frac{\pi}{24} \left(\frac{\beta^2}{16}\right)^2$ and somehow compatible with the numerical data given in [21].

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they have the same mass we can build two other states from their linear combinations

\[ |b_n^{(\pm)}\rangle = \frac{|b_n^{(0)}\rangle \pm |b_n^{(1)}\rangle}{\sqrt{2}} . \]  

(C.6)

The peculiarity of the Sine–Gordon model is the removal of this double multiplicity due to the fact that the states \( |b_{2n+1}^{(+)}\rangle \) and \( |b_{2n}^{(-)}\rangle \) decouple from the theory. This feature is shared also by the two-folded version of the model, since the kink scattering amplitudes have the same analytical form as in SG [22].

In the two–folded SG there are two different kink states \( |K_{-1,0}\rangle \) and \( |K_{0,1}\rangle \) (see Sect. 4 and ref. [22] for the notation), and the parity \( P \), which is still an exact symmetry of the theory, acts on them transforming the kink of one type into the antikink of the other type:

\[ P : |K_{0,1}\rangle \rightarrow |K_{-1,0}\rangle , \quad |K_{-1,0}\rangle \rightarrow |K_{1,0}\rangle . \]  

(C.7)

If we quantize the theory around the vacuum \( |0\rangle \), we can define \( |b_n^{(0)}\rangle \) as the bound states of \( |K_{1,1}K_{-1,0}\rangle \), and \( |b_n^{(1)}\rangle \) as the bound states of \( |K_{0,-1}K_{-1,0}\rangle \). These degenerate states, which transform under \( P \) as

\[ P : |b_n^{(0)}\rangle \rightarrow |b_n^{(1)}\rangle , \quad |b_n^{(1)}\rangle \rightarrow |b_n^{(0)}\rangle , \]  

(C.8)

can be still organized in parity eigenstates \( |b_n^{(\pm)}\rangle \), and the particular dynamics of the problem causes the decoupling of half of them from the theory. Furthermore, it is easy to see that the form factors of an odd operator between two of these states has to vanish in virtue of the relation

\[ \langle 0 | \sin \frac{\beta}{2} \phi | b_n^{(\pm)} b_n^{(\pm)} \rangle = \langle 0 | P^{-1} P \left( \sin \frac{\beta}{2} \phi \right) P^{-1} P | b_n^{(\pm)} b_n^{(\pm)} \rangle = \]  

\[ = - \langle 0 | \sin \frac{\beta}{2} \phi | b_n^{(\pm)} b_n^{(\pm)} \rangle , \]

leading to the FFPT result that the breathers receive a zero mass correction at first order in \( \lambda \), as it is claimed in [21].

However, FFPT can be applied by taking into account the nature of neutral states in the DSG model, where the addition to the Lagrangian of the term \( -\lambda \frac{\beta}{\beta_x} \sin \frac{\beta}{2} \phi \) spoils the invariance under \( P \). The kinks \( |K_{-1,0}\rangle \) and \( |K_{0,1}\rangle \) are deformed into the small and large kinks \( |S\rangle \) and \( |L\rangle \), respectively, which are not anymore degenerate in mass and cannot be superposed in linear combinations. Hence, the neutral states present in the theory are \( |b_n^{(L)}\rangle \) and \( |b_n^{(S)}\rangle \), deformations of \( |b_n^{(0)}\rangle \) and \( |b_n^{(1)}\rangle \) respectively. In virtue of the general considerations presented above, one can see that this interpretation does not lead to any drastic change in the spectrum. In fact, the states \( |b_{2n+1}^{(+)}\rangle \) and \( |b_{2n}^{(-)}\rangle \) have no reason to decouple in the DSG theory, but they have to carry a coupling which is a function of \( \lambda \) adiabatically going to zero in the two–folded SG limit.
A proper use of the FFPT on $|b_n^{(0)}\rangle$ and $|b_n^{(1)}\rangle$ reproduces indeed the situation described by (4.35), in which the two sets of breathers receive mass corrections including also odd terms in $\lambda$, but with opposite signs. This is easily seen by considering the $P$ transformations in the two–folded SG model:

$$\langle 0 | \sin \frac{\beta}{2} \phi | b_n^{(0)} b_n^{(0)} \rangle = \langle 0 | P^{-1} P \left( \sin \frac{\beta}{2} \phi \right) P^{-1} P | b_n^{(0)} b_n^{(0)} \rangle = - \langle 0 | \sin \frac{\beta}{2} \phi | b_n^{(1)} b_n^{(1)} \rangle ,$$

which gives, at first order in $\lambda$,

$$\delta m_{(L)} = - \delta m_{(S)}, \quad \text{(C.9)}$$

in agreement with our semiclassical result (4.35). It is worth noting that also with this interpretation the total spectrum of the DSG model remains unchanged under the action of $P$, which corresponds to the transformation $\lambda \rightarrow -\lambda$. In fact, the two types of kinks and breathers are mapped one into the other. This is consistent with the observation that $P$, although it is not anymore a symmetry of the perturbed theory, simply realizes a reflection of the potential, hence the total spectrum should be invariant under it.

Presently the above symmetry considerations seem to us the correct criterion to define the neutral states, and find confirmation in our semiclassical result (4.35). However, the available numerical data presented in [21] pose a challenge to this interpretation and further studies are needed to solve this interesting and delicate problem. In fact, although $\delta m_{(L)}$ and $\delta m_{(S)}$ are not forced to vanish by symmetry arguments, there is in principle the possibility that both of them are identically zero in the complete quantum computation. This could follow from the use of the exact kink masses entering eqs. (4.33) and (4.34), together with a proper shift of the semiclassical pole in the form factors, due to higher order contributions. The exact cancellation of the linear corrections is a very strong requirement, in support of which we have presently no indication in the theory, but a careful analysis of this point is nevertheless an interesting open problem.

### D Double Sinh–Gordon model

Among the different qualitative features taking place in perturbing integrable models, a situation particularly interesting is the one in which the perturbation is adiabatic for small values of the parameters but nevertheless a qualitative changes in the spectrum occurs by increasing its intensity.

This is indeed the situation in the $\delta = \frac{\pi}{2}$ case of DSG model, where we have two types of kinks for small $\lambda$, but at $\lambda = 4\mu$ one of them disappears from the spectrum. This
phenomenon is obviously unaccessible by means of FFPT, hence the semiclassical method is the best tool to describe it.

Here we consider another interesting example of this kind, realized by the Double Sinh-Gordon Model (DShG). In this case the phenomenon is even more evident, because in the unperturbed Sinh-Gordon model there are no kinks at all, but just one scalar particle, while perturbing it, at some critical value of the coupling a kink and antikink appear, i.e. there is a deconfinement phase transition of these particles.

The DShG potential, shown in Fig. 8, is expressed as

\[ V(\varphi) = \frac{\mu}{\beta^2} \cosh \beta \varphi - \frac{\lambda}{\beta^2} \cosh \left( \frac{\beta}{2} \varphi \right). \]  \hspace{1cm} (D.1)

![Figure 8: DShG potential](image)

In the regime \( \lambda < 4\mu \) the qualitative features are the same as in the unperturbed Sinh-Gordon model. At \( \lambda = 4\mu \), however, the single minimum splits in two degenerate minima, which for \( \lambda > 4\mu \) are located at \( \varphi_\pm = \pm \frac{2}{\beta} \) \( \arccosh \frac{\lambda}{4\mu} \). A study of the classical thermodynamical properties of the theory in this regime has been performed in [30] with the transfer integral method.

The kink interpolating between the two degenerate vacua is

\[ \varphi_K(x) = \frac{4}{\beta} \arctanh \left[ \sqrt{\frac{\lambda - 4\mu}{\lambda + 4\mu}} \tanh \left( \frac{m}{2} x \right) \right], \]  \hspace{1cm} (D.2)

with

\[ m^2 = \frac{\lambda^2 - 16\mu^2}{16\mu}. \]

Its classical mass is given by

\[ M_K = \frac{8m}{\beta^2} \left\{ -1 + \frac{2\lambda}{\sqrt{\lambda^2 - 16\mu^2}} \arctanh \sqrt{\frac{\lambda - 4\mu}{\lambda + 4\mu}} \right\}. \]  \hspace{1cm} (D.3)

From the form factor of \( \varphi \) on the kink-antikink asymptotic state, expressed as

\[ F_2(\theta) = -i \frac{\pi}{\beta} \frac{1}{i\pi - \theta} \sin \left[ \arccosh \frac{\lambda}{4\mu} \frac{M_K}{m} (i\pi - \theta) \right] \frac{\sinh \left[ \frac{\lambda}{4\mu} \frac{M_K}{m} (i\pi - \theta) \right]}{\sinh \left[ \frac{\lambda}{4\mu} \frac{M_K}{m} (i\pi - \theta) \right]}, \]  \hspace{1cm} (D.4)
we derive the bound states spectrum

\[ m_{(K)}^{(n)} = 2M_K \sin \left( n \frac{m}{2M_K} \right), \quad 0 < n < \pi \frac{M_K}{m} \]  

(D.5)

All the kink–antikink bound states disappear from the theory at a certain value \( \lambda^* > 4\mu \) such that \( \pi \frac{M_K}{m} \big|_{\lambda^*} = 1 \), and the kink becomes a constant solution with zero classical energy when \( \lambda \to 4\mu \). This is the semiclassical manifestation of a phase transition, analogous to the one observed in DSG with \( \delta = \frac{\pi}{2} \). As we have already anticipated, here the phenomenon occurs in a reverse order, since in this case a kink appears in the theory by increasing the coupling \( \lambda \).

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