A LINK BETWEEN HARMONICITY OF 2-DISTANCE FUNCTIONS AND INCOMPRESSIBILITY OF CANONICAL VECTOR FIELDS

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Abstract. Let $M$ be a Riemannian submanifold of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field $\tilde{Z}$. Let $Z$ denote the restriction of $\tilde{Z}$ along $M$ and let $Z^T$ be the tangential component of $Z$ on $M$, called the canonical vector field of $M$. The 2-distance function $\delta_2^Z$ of $M$ (associated with $Z$) is defined by $\delta_2^Z = \langle Z, Z \rangle$.

In this article, we initiate the study of submanifolds $M$ of $\tilde{M}$ with incompressible canonical vector field $Z^T$ arisen from a concurrent vector field $\tilde{Z}$ on the ambient space $\tilde{M}$. First, we derive some necessary and sufficient conditions for such canonical vector fields to be incompressible. In particular, we prove that the 2-distance function $\delta_2^Z$ is harmonic if and only if the canonical vector field $Z^T$ on $M$ is an incompressible vector field. Then we provide some applications of our main results.

1. Incompressible vector fields

In fluid mechanics, many liquids are hard to compress (i.e., their volumes or densities don’t change much when you squeeze them), so that the density $\rho$ is essentially a constant. For such an incompressible fluid the equation of continuity simplifies to the divergence of the flow velocity $v$ is zero, i.e.,

$$\text{div}(v) = 0 \text{ (incompressible)},$$

so that the velocity field $v$ is an incompressible vector field (also known as a solenoidal vector field or a divergence-free vector field). This condition is analogous to the condition $\text{div}(B) = 0$ in electromagnetism that the magnetic field $B$ has zero divergence.

It is well-known that incompressible vector fields are important in magnetohydrodynamics. Moreover, magnetic fields are widely used throughout modern technology, particularly in electrical engineering and electromechanics (cf. e.g., [1, 15, 16]).

Based on the reasons mentioned above, one has the following.

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1All vector fields, functions, immersions and manifolds are assumed to be smooth.
Definition 1.1. A vector field \( X \) on a Riemannian manifold \( M \) is called *incompressible* if the divergence of \( X \) is zero, i.e., \( \text{div}(X) = 0 \).

Let \( \phi : M \to \tilde{M} \) be an isometric immersion of a Riemannian manifold \( M \) into another Riemannian manifold \( \tilde{M} \). Denote by \( \langle \cdot, \cdot \rangle \) the inner product of \( M \) as well as of \( \tilde{M} \). Assume that \( \tilde{Y} \) is a vector field of \( \tilde{M} \). Denote by \( Y \) the restriction of \( \tilde{Y} \) along \( M \). Then \( Y \) admits an orthogonal decomposition:

\[
Y = Y^T + Y^\perp, \tag{1.2}
\]

where \( Y^T \) and \( Y^\perp \) are the tangential and the normal components of \( Y \), respectively. The tangent vector field \( Y^T \) of \( M \) is called the *canonical vector field* of \( M \) associated with \( Y \).

For a submanifold \( M \) of a Euclidean space \( \mathbb{E}^m \), the most elementary and natural vector field on \( M \) is the position vector field \( x \). The tangential component \( x^T \) of \( x \) is simply called the *canonical vector field* of \( M \) \([11, 12]\). It is well-known that the position vector field of \( \mathbb{E}^m \) is a concurrent vector field (see Definition 2.2 and Example 2.1).

In earlier articles, we have investigated Euclidean submanifolds whose canonical vector fields are concurrent \([6, 8]\), concircular \([14]\), torse-forming \([13]\), conformal \([12]\), or incompressible \([11]\). (See also recent surveys \([9, 10]\) for several topics on position vector fields on Euclidean submanifolds.)

In this article, we initiate the investigation of submanifolds \( M \) of \( \tilde{M} \) with incompressible canonical vector field \( Z^T \) arisen from a concurrent vector field \( \tilde{Z} \) on the ambient space \( \tilde{M} \). First, we derive some necessary and sufficient conditions for such canonical vector fields to be incompressible. In particular, we prove that the 2-distance function \( \delta^2_Z \) is harmonic if and only if the canonical vector field \( Z^T \) on \( M \) is an incompressible vector field. Then we provide some applications of our main results.

2. Preliminaries

Let \( \phi : M \to \tilde{M} \) be an isometric immersion of a connected Riemannian \( n \)-manifold \( M \) into a Riemannian \( m \)-manifold \( \tilde{M} \). For each point \( p \in M \), we denote by \( T_pM \) and \( T^\perp_pM \) the tangent space and the normal space of \( M \) at \( p \), respectively. Let \( \nabla \) and \( \tilde{\nabla} \) denote the Levi–Civita connections of \( M \) and \( \mathbb{E}^m \), respectively.

The formula of Gauss and the formula of Weingarten are then given respectively by (cf. \([3, 4, 7]\))

\[
\tilde{\nabla}_XY = \nabla_XY + h(X, Y), \tag{2.1}
\]

\[
\tilde{\nabla}_X\xi = -A_\xi X + D_X\xi, \tag{2.2}
\]
for vector fields $X, Y$ tangent to $M$ and $\xi$ normal to $M$, where $h$ denotes the second fundamental form, $D$ is the normal connection and $A$ is the shape operator of $M$.

For each normal vector $\xi$ at $p$, the shape operator $A_\xi$ is a self-adjoint endomorphism of $T_p M$. The second fundamental form $h$ and the shape operator $A$ are related by

$$\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle. \quad (2.3)$$

The mean curvature vector field $H$ of an $n$-dimensional submanifold $M$ is defined by

$$H = \left( \frac{1}{n} \right) \text{trace } h. \quad (2.4)$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame on $M$, then the divergence of a vector field $X$ on $M$, denoted by $\text{div}(X)$, is defined by

$$\text{div}(X) = \sum_{j=1}^{n} \langle \nabla_{e_j} X, e_i \rangle. \quad (2.5)$$

The gradient $\nabla f$ of a function $f$ on $M$ is defined by

$$\langle \nabla f, Y \rangle = Y f$$

for any vector $Y$ tangent to $M$. Hence, in terms of an orthonormal frame $\{e_1, \ldots, e_n\}$ on $M$, we have

$$\nabla f = \sum_{i=1}^{n} (e_i f) e_i. \quad (2.6)$$

And the Laplacian $\Delta$ of $M$ acting on a function $f$ on $M$ is given by

$$\Delta f = -\sum_{i=1}^{n} \{e_i e_i(f) - \nabla_{e_i} e_i(f)\}. \quad (2.7)$$

Now, we present some basic definitions for later use.

**Definition 2.2.** A vector field $\tilde{Z}$ on a Riemannian manifold $\tilde{M}$ is called a concurrent vector field if it satisfies

$$\tilde{\nabla}_X \tilde{Z} = X \quad (2.8)$$

for all vectors $X$ tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection of $\tilde{M}$ (cf. [19, 20]).

Concurrent vector fields play some important roles in differential geometry and mathematical physics. For instance, it was proved in [19] that if the holonomy group of a Riemannian manifold $\tilde{M}$ leaves a point invariant, then $\tilde{M}$ admits a concurrent vector field. Concurrent vector fields have also been studied in Finsler geometry since the beginning of 1950s (cf. [17, 18]).

The simplest example of Riemannian manifold with a concurrent vector field is a Euclidean space.
Example 2.1. The position vector field $x$ of the Euclidean $m$-space $E^m$ is a concurrent vector field.

Definition 2.3. Let $B$ and $F$ be two Riemannian manifolds of positive dimensions equipped with metrics $g_B$ and $g_F$, respectively, and let $f$ be a positive smooth function on $B$.

The warped product $M = B \times_f F$ is the product manifold $B \times F$ equipped with the warped product metric

$$g = g_B + f^2 g_F.$$ \hspace{1cm} (2.9)

The function $f$ is called the warping function of the warped product (cf. [2, 11]).

For a warped product $B \times_f F$, $B$ is called the base and $F$ the fiber. The leaves $B \times \{q\} = \eta^{-1}(q), \, q \in F$, and the fibers $\{b\} \times F = \pi^{-1}(p), \, b \in B$ are Riemannian submanifolds of $B \times_f F$.

Example 2.2. It is direct to verify that $E^m_\ast = E^m - \{0\} \subset E^m$ can be regarded as the warped product $R^+ \times_s S^{m-1}$ equipped with the warped product metric

$$g = ds^2 + s^2 g_S,$$

where $g_S$ is the metric tensor of the unit $(m-1)$-sphere $S^{m-1}$. In this case, the position vector field $x$ of $E^m_\ast$ is given by $s \frac{\partial}{\partial s}$.

The distance function $\delta$ from the origin $o \in E^m$ to a point of $E^m$ is given by

$$\delta = \sqrt{\langle x, x \rangle}.$$ 

Example 2.3. Let $F$ be any Riemannian manifold and let $I = (a, b)$ be an open interval with $0 \notin I$. Consider the warped product $I \times_s F$ equipped with the warped product metric

$$\tilde{g} = ds^2 + s^2 g_F.$$ \hspace{1cm} (2.10)

where $g_F$ denotes the Riemannian metric of $F$. Then the vector field $\tilde{Z} = s \frac{\partial}{\partial s}$ is a concurrent vector field on $I \times_s F$ (cf. Example 3.1 of [5]). Moreover, in this case the vector field $\tilde{Z} = s \frac{\partial}{\partial s}$ can be considered as the radial vector field of $I \times_s F$.

3. Theorems

Now, we define the notion of $r$-distance function on a submanifold $M$ of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field as follows.
**Definition 3.4.** Let $M$ be a submanifold of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field $\tilde{Z}$. Denote by $Z$ the restriction of $\tilde{Z}$ on $M$. Then the function

$$\delta_{Z}^{r}(p) = |Z_{p}|^{r} = \langle Z_{p}, Z_{p} \rangle^{r/2}$$

is called the $r$-distance function (associated with $Z$) (or simply the $r$-distance function if there is no confusion arisen).

**Lemma 3.1.** Let $M$ be a submanifold of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field $\tilde{Z}$ on $\tilde{M}$. Then the corresponding canonical vector field $Z^{T}$ and the gradient of the 2-distance function $\delta_{Z}^{2}$ of $M$ are related by

$$Z^{T} = \frac{1}{2} \nabla \delta_{Z}^{2}. \quad (3.1)$$

**Proof.** Let $M$ be a submanifold of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field $\tilde{Z}$ on $\tilde{M}$. Then the 2-distance function $\delta_{Z}^{2}$ of $M$ is given by

$$\delta_{Z}^{2} = \langle Z, Z \rangle, \quad (3.2)$$

where $Z$ is the restriction of $\tilde{Z}$ on $M$.

Let $\{e_{1}, \ldots, e_{n}\}$ be an orthonormal local frame on $M$. Then it follows from (2.6), (2.8) and (3.2) that

$$\nabla \delta_{Z}^{2} = \sum_{i=1}^{n} (e_{i} \langle Z, Z \rangle) e_{i} = 2 \sum_{i=1}^{n} \langle \tilde{\nabla}_{e_{i}} Z, Z \rangle e_{i}$$

$$= 2n \sum_{i=1}^{n} \langle e_{i}, Z \rangle e_{i} = 2Z^{T},$$

which proves (3.1). \hfill \Box

The next result provides a simple characterization of an incompressible canonical vector field on a submanifold arisen from a concurrent vector field on its ambient space.

**Theorem 3.1.** Let $M$ be a submanifold of a Riemannian manifold $\tilde{M}$ with a concurrent vector field $\tilde{Z}$ on $\tilde{M}$. Then the canonical vector field $Z^{T}$ on $M$ is incompressible if and only if the mean curvature vector field $H$ of $M$ in $\tilde{M}$ satisfies

$$\langle H, Z \rangle = -1 \quad (3.3)$$

identically.

**Proof.** Let $M$ be a submanifold of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field $\tilde{Z}$ on $\tilde{M}$. Then, according to Definition 3.4, the canonical vector field $Z^{T}$ is the tangential component of the restriction $Z$ of the concurrent vector field $\tilde{Z}$ along $M$. 

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Now, let us compute the divergence $\text{div}(Z^T)$. It follows from (2.1), (2.4), (2.5) and Lemma 3.1 that

$$\text{div}(Z^T) = \frac{1}{2} \sum_{i=1}^{n} \langle \nabla e_i \nabla \delta^2 Z, e_i \rangle = \sum_{i,j=1}^{n} \langle \nabla e_i(\langle e_j, Z \rangle e_j), e_i \rangle$$

$$= \sum_{i,j=1}^{n} \left( \langle \tilde{\nabla}_e e_j, Z \rangle \langle e_j, e_i \rangle + \langle e_j, e_i \rangle^2 + \langle e_j, Z \rangle \langle \nabla e_i e_j, e_i \rangle \right)$$

$$= n(1 + \langle H, Z \rangle) + \sum_{i,j=1}^{n} \left( \langle \nabla e_i e_j, Z \rangle \langle e_j, e_i \rangle + \langle e_j, Z \rangle \langle \nabla e_i e_j, e_i \rangle \right). \quad (3.4)$$

Let us put

$$\nabla_X e_i = \sum_{k=1}^{n} \omega^k_i(X)e_k \quad (3.5)$$

for tangent vectors $X$ of $M$. Then we find from the fact that $\nabla$ is a metric connection that

$$\omega^k_i = -\omega^i_k \quad (3.6)$$

for $1 \leq i, k \leq n$.

From (3.5) and (3.6) we obtain

$$\sum_{i,j=1}^{n} \left( \langle \tilde{\nabla}_e e_j, Z \rangle \langle e_j, e_i \rangle + \langle e_j, Z \rangle \langle \nabla e_i e_j, e_i \rangle \right)$$

$$= \sum_{i,k=1}^{n} \omega^k_i(e_i) \langle e_k, X \rangle + \sum_{i,j=1}^{n} \omega^j_i(e_i) \langle e_j, Z \rangle$$

$$= 0. \quad (3.7)$$

Therefore, after combining (3.4) and (3.7) we have

$$\text{div}(Z^T) = n(1 + \langle H, Z \rangle).$$

Consequently, the canonical vector field $Z^T$ is incompressible if and only if $\langle H, Z \rangle = -1$ holds identically. \qed

**Remark 3.1.** Lemma 3.1 and Theorem (3.1) generalize statement (a) and statement (b) Theorem 3.1 of [11], respectively.

The next result is the **main theorem** of this article. This main theorem provides a very simple link between harmonicity of the 2-distance function $\delta^2_Z$ and the incompressibility of the canonical vector field $Z^T$. 

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[11] Chen, B. Y. (1999). *Harmonic Maps, Lie Groups, and Representation Theory*. World Scientific Publishing Co., Inc., River Edge, NJ.
Theorem 3.2. Let $M$ be a submanifold of a Riemannian manifold $\tilde{M}$ with a concurrent vector field $\tilde{Z}$. Then the 2-distance function $\delta_Z^2$ is harmonic if and only if the canonical vector field $Z^T$ is incompressible.

Proof. Let $M$ be a submanifold of a Riemannian manifold $\tilde{M}$. Assume that $\tilde{M}$ admits a concurrent vector field $\tilde{Z}$. Let us compute the Laplacian of the 2-distance function $\delta_Z^2$ of $M$ as follows.

$$
\Delta \delta_Z^2 = -\sum_{i=1}^{n} e_i e_i (\delta_Z^2) + \sum_{i=1}^{n} \nabla e_i e_i (\delta_Z^2)
= -2 \sum_{i=1}^{n} e_i \langle e_i, Z \rangle + 2 \sum_{i=1}^{n} \langle \nabla e_i e_i Z, Z \rangle
= -2 \sum_{i=1}^{n} \langle \nabla e_i e_i Z, Z \rangle - 2n + 2 \sum_{i=1}^{n} \langle \nabla e_i e_i Z, Z \rangle
= -2 \sum_{i=1}^{n} \langle h(e_i, e_i), Z \rangle - 2n
= -2n (\langle H, Z \rangle + 1). \quad (3.8)
$$

Now, by combining (3.8) and Theorem 3.1 we obtain the theorem.

For a Euclidean submanifold $M$, if we denote the tangential component of the position vector field $\mathbf{x}$ of $M$ by $\mathbf{x}^T$, then $\mathbf{x}^T$ is the canonical vector field of the Euclidean submanifold $M$.

For Euclidean submanifolds, Theorem 3.2 yields the following.

Theorem 3.3. Let $M$ be an arbitrary Euclidean submanifold $M$ of $\mathbb{E}^m$. Then the canonical vector field $\mathbf{x}^T$ of $M$ is incompressible if and only if the 2-distance function $\delta^2 = \langle \mathbf{x}, \mathbf{x} \rangle$ of $M$ is a harmonic function.

Proof. This is an immediate consequence of Theorem 3.2 since the position vector field $\mathbf{x}$ is a concurrent vector field on $\mathbb{E}^m$.

4. Some applications

In this section we make the following.

Assumption. Let $M$ be a submanifold of the warped product $\tilde{M} = I \times_s F$. We consider the canonical concurrent vector field $\tilde{Z} = s \frac{\partial}{\partial s}$ on $I \times_s F$.

Now, we provide the following applications of Theorems 3.1–3.3.
Corollary 4.1. Let $\tilde{M} = I \times_s F$ be a warped product with warped product metric $\tilde{g} = ds^2 + s^2 g_F$. Then, for every submanifold $B$ of $F$, the canonical vector field $Z^T$ of $I \times_s B$ is never incompressible.

Proof. Under the hypothesis, the restriction $Z$ of $\tilde{Z}$ on $I \times_s B$ is always tangent to $I \times_s B$, i.e., $Z^\perp = 0$. Therefore, condition (2.1) never holds at each point. Consequently, the canonical vector field $Z^T$ of $I \times_s B$ is never incompressible according to Theorem 3.1. \hfill $\Box$

Corollary 4.2. Let $\tilde{M} = I \times_s F$ be a warped product with warped product metric $\tilde{g} = ds^2 + s^2 g_F$. Then the canonical vector field $Z^T$ of every fiber $\{s_0\} \times_s F$ in $I \times_s F$ is always incompressible.

Proof. Let $M$ be a submanifold of the warped product $\tilde{M} = I \times_s F$ endowed with a concurrent vector field $\tilde{Z} = s \frac{\partial}{\partial s}$. Then the 2-distance function of $M$ is given by $\delta_Z^2 = s^2$. Suppose that $M$ is a fiber of $I \times_s F$ defined by $\{s_0\} \times F$. Then the 2-distance function $\delta_Z^2$ is the constant $s_0^2$. Hence it is a harmonic function trivially. Consequently, Theorem 3.2 implies that the canonical vector field $Z^T$ is always incompressible. \hfill $\Box$

Corollary 4.3. Let $\tilde{M} = I \times_s S^{m-1}$ be the warped product of $I = (0, \infty)$ and the unit $(m-1)$-sphere $S^{m-1}$ equipped with the warped product metric $\tilde{g} = ds^2 + s^2 g_S$. Consider the canonical concurrent vector field $Z = s \frac{\partial}{\partial s}$ on $I \times_s S^{m-1}$. Then, for any map $\gamma : I \rightarrow S^{m-1}$, the curve defined by

$$\psi : I \rightarrow I \times_s S^{m-1}; I \ni s \mapsto (\sqrt{1+2s}, \gamma(s)) \in I \times_s S^{m-1}$$

(4.1)

has incompressible canonical vector field $Z^T$.

Proof. Under the hypothesis, the 2-distance function $\delta_Z^2$ of the curve $\psi$ given by (4.1) is $\delta_Z^2 = 1 + 2s$, which is a harmonic function. Consequently, the canonical vector field $Z^T$ is incompressible according to Theorem 3.3. \hfill $\Box$

Example 4.1. Consider the map $\gamma : I \rightarrow S^1$, $I = (0, \infty)$, defined by

$$\gamma(s) = \left(\frac{\cos \sqrt{2s} + \sqrt{2s} \sin \sqrt{2s}}{\sqrt{1+2s}}, \frac{\sin \sqrt{2s} - \sqrt{2s} \cos \sqrt{2s}}{\sqrt{1+2s}}\right).$$

(4.2)

Then the curve $\psi$ in (4.1) of Corollary 4.3 is given by

$$\psi(s) = \left(\sqrt{1+2s}, \gamma(s)\right) \in I \times_s S^1.$$  

(4.3)

Therefore, according to Corollary 4.3, the canonical vector field $x^T = Z^T$ is an incompressible vector field.

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