RESEARCH ARTICLE

Sobolev inequality on manifolds with asymptotically nonnegative Bakry–Émery Ricci curvature

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Abstract
In this paper, inspired by Brendle (Comm. Pure Appl. Math. 76 (2023), 2192) and Johne (arXiv:2103.08496, 2021), we prove a Sobolev inequality on manifolds with density and asymptotically nonnegative Bakry–Émery Ricci curvature.

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1 | INTRODUCTION

In 2019, S. Brendle [3] proved a Sobolev inequality for submanifolds in Euclidean space. Moreover, he obtained a sharp isoperimetric inequality for compact minimal submanifolds in Euclidean space with codimension at most 2. In 2020, he generalized the results in [3] to the case of manifolds with nonnegative curvature [4]. Recently, we extended [4] to manifolds with asymptotically nonnegative curvature [6]. In 2021, F. Johne [9] generalized the results of [4] to the case of manifolds with density and nonnegative Bakry–Émery Ricci curvature. In this note, we establish a Sobolev inequality in manifolds with density and asymptotically nonnegative Bakry–Émery Ricci curvature.

Let $(M, g, w d\text{vol}_g)$ be a smooth complete noncompact $n$-dimensional Riemannian manifold with density, where $w$ is a smooth positive density function on $M$ and $d\text{vol}_g$ is the Riemannian volume measure with respect to the metric $g$. As a generalization of Ricci curvature, the
Bakry–Émery Ricci curvature [2] of \((M, g, w \, d\text{vol}_g)\) is defined by

\[
\text{Ric}_\alpha^w = \text{Ric} - D^2(\log w) - \frac{1}{\alpha} D \log w \otimes D \log w,
\]

where \(\text{Ric}\) denotes the Ricci curvature of \(M\) and \(D\) is the Levi–Civita connection with respect to the metric \(g\) and \(\alpha > 0\). If the density function \(w\) is constant, the Bakry–Émery Ricci curvature \(\text{Ric}_\alpha^w\) reduces to the Ricci curvature.

Suppose \(\lambda : [0, +\infty) \to [0, +\infty)\) is a nonnegative and nonincreasing continuous function satisfying

\[
b_0 := \int_0^{+\infty} s\lambda(s) \, ds < +\infty, \quad (1.2)
\]

which implies

\[
b_1 := \int_0^{+\infty} \lambda(s) \, ds < +\infty. \quad (1.3)
\]

A complete noncompact Riemannian manifold \((M, g, w \, d\text{vol}_g)\) with density of dimension \(n\) is said to have \textit{asymptotically nonnegative Bakry–Émery Ricci curvature} if there exists a base point \(o \in M\) such that

\[
\text{Ric}_\alpha^w(q) \geq -(n + \alpha - 1) \lambda(d(o, q)), \quad \forall q \in M, \quad (1.4)
\]

where \(d\) is the distance function of \(M\). In particular, \(\lambda \equiv 0\) in (1.4) corresponds to the case treated in [9].

Suppose \(h : [0, T) \to \mathbb{R}\) is the unique solution of the initial value problem

\[
\begin{align*}
    h''(t) &= \lambda(t)h(t), \\
    h(0) &= 0, \quad h'(0) = 1.
\end{align*} \quad (1.5)
\]

By the theory of ordinary differential equations [16], the solution exists for all time, that is, \(T = +\infty\). We remark that \(h\) reduces to the radius function, if \(\lambda = 0\). Similar to the work of F. Johne [9], we define the \(\alpha\)-\textit{asymptotic volume ratio} \(V_\alpha\) of \((M, g, w \, d\text{vol}_g)\) by

\[
V_\alpha := \lim_{r \to +\infty} \frac{\int_{B_r(o)} w}{(n + \alpha) \int_0^r h^{n+\alpha-1}(t) \, dt}, \quad (1.6)
\]

where \(o\) is the base point and \(B_r(o)\) denotes the geodesic ball of radius \(r\), that is, \(B_r(o) = \{q \in M : d(o, q) < r\}\). In Theorem 2.2, we will show a comparison theorem for weighted volumes, to be more precise we will show

\[
\frac{\int_{B_r(o)} w}{(n + \alpha) \int_0^r h^{n+\alpha-1}(t) \, dt}
\]
is a nonincreasing function for \( r \in (0, +\infty) \), so \( \mathcal{V}_\alpha \) is well defined.

By combining the ABP-method in [4, 9] with some comparison theorems for ordinary differential equations, we establish a Sobolev inequality for a compact domain in manifolds with density, under the asymptotically nonnegative Bakry–Émery Ricci curvature as follows.

**Theorem 1.1.** Let \((M, g, w\, d\text{vol}_g)\) be a smooth complete noncompact \( n \)-dimensional Riemannian manifold of smooth density \( w > 0 \) and asymptotically nonnegative Bakry–Émery Ricci curvature with respect to a base point \( o \in M \). Let \( \Omega \) be a compact domain in \( M \) with boundary \( \partial \Omega \), and \( f \) be a positive smooth function on \( \Omega \). Then,

\[
\int_{\partial \Omega} w f + \int_\Omega w |Df| + 2b_1(n + \alpha - 1) \int_\Omega w f \\
\geq (n + \alpha) \mathcal{V}_\alpha^{\frac{1}{n+\alpha}} \left( \frac{1 + b_0}{e^{2r_0b_1+b_0}} \right)^{\frac{n+\alpha-1}{n+\alpha}} \left( \int_\Omega w \right)^{\frac{1}{n+\alpha}} \left( \int_\Omega w \right)^{\frac{n+\alpha-1}{n+\alpha}},
\]

where \( r_0 = \max\{d(o, x) | x \in \Omega\} \), \( \mathcal{V}_\alpha \) is the \( \alpha \)-asymptotic volume ratio of \( M \) given by (1.6), \( b_0, b_1 \) are defined in (1.2) and (1.3), respectively.

Taking \( f = 1 \) in Theorem 1.1, we obtain an isoperimetric inequality.

**Corollary 1.2.** Let \((M, g, w\, d\text{vol}_g)\) be a smooth complete noncompact \( n \)-dimensional Riemannian manifold of smooth density \( w > 0 \) and asymptotically nonnegative Bakry–Émery Ricci curvature with respect to a base point \( o \in M \). Then,

\[
\int_{\partial \Omega} w \geq \left( (n + \alpha) \mathcal{V}_\alpha^{\frac{1}{n+\alpha}} \left( \frac{1 + b_0}{e^{2r_0b_1+b_0}} \right)^{\frac{n+\alpha-1}{n+\alpha}} - 2(n + \alpha - 1) b_1 \left( \int_\Omega w \right)^{\frac{1}{n+\alpha}} \left( \int_\Omega w \right)^{\frac{n+\alpha-1}{n+\alpha}} \right) \left( \int_\Omega w \right)^{\frac{1}{n+\alpha}},
\]

where \( r_0 = \max\{d(o, x) | x \in \Omega\} \), \( \mathcal{V}_\alpha \) is the \( \alpha \)-asymptotic volume ratio of \( M \) given by (1.6), \( b_0, b_1 \) are defined in (1.2) and (1.3).

When \( w = 1, b_0 = b_1 = \alpha = 0 \), Corollary 1.2 was first given by V. Agostiniani, M. Fogagnolo, and L. Mazziari [1] in dimension 3 and obtained by S. Brendle [4] for any dimension. In recent years, the study of the isoperimetric problem in manifolds with density attracts much attention, see [5, 7, 11, 15]. For more about manifolds with density, we refer the reader to [12, 13] and references therein.

2 | PRELIMINARIES

In this section, we give a proof of the Bishop–Gromov volume comparison theorem for complete noncompact Riemannian manifold with density and asymptotically nonnegative Bakry–Émery Ricci curvature.
The following lemma is an almost verbatim combination of Lemma 2.1 and Corollary 2.2 in [14]. We should point out, however, that it deals with a different initial value case than [14].

**Lemma 2.1.** Let \( G \) be a continuous nonnegative function on \([0, +\infty)\) and let \( g, \psi \) be solutions to the following problems

\[
\begin{align*}
g' + g^2 & \leqslant G, \quad t \in (0, +\infty), \\
g(t) & = \frac{\beta}{t} + O(1), \text{ as } t \to 0^+,
\end{align*}
\]

\[
\begin{align*}
\psi'' & = G\psi, \quad t \in (0, +\infty), \\
\psi(0) & = 0, \psi'(0) = 1,
\end{align*}
\]

where \( 0 < \beta \leqslant 1 \). Then, we have

\[
g \leqslant \frac{\psi'}{\psi} \text{ on } (0, +\infty).
\]

**Proof.** Observe that the initial conditions imply \( \psi \geqslant 0 \). Then, the Fundamental Theorem of Calculus implies \( \psi(t) \geqslant t \) and \( \psi'(t) \geqslant 1 \). Let \( \phi(t) = t^\beta e^{\int_0^t (g - \frac{\beta}{\tau}) \, d\tau} > 0 \) for \( t \in (0, +\infty) \). Similar to the proof of Lemma 2.1 and Corollary 2.2 in [14], it is easy to show that

\[
g = \frac{\phi'}{\phi}, \quad \phi'' \leqslant G\phi,
\]

\[
\phi(t) = t^\beta(1 + O(1)), \text{ as } t \to 0^+,
\]

\[
\lim_{t \to 0^+} \frac{\psi(t)}{t} = \frac{\psi'(0)}{1} = 1,
\]

\[
\lim_{t \to 0^+} (\phi'\psi - \phi\psi') = \lim_{t \to 0^+} (g\phi\psi - \phi\psi') = 0.
\]

Using (2.1) and (2.2), we conclude that

\[
(\phi'\psi - \psi'\phi)' = \phi''\psi - \phi\psi'' \leqslant G(t)\psi\phi - G(t)\psi\phi = 0
\]

and

\[
g = \frac{\phi'}{\phi} \leqslant \frac{\psi'}{\psi} \text{ on } (0, +\infty).
\]

Thus,

\[
g = \frac{\phi'}{\phi} \leqslant \frac{\psi'}{\psi} \text{ on } (0, +\infty).
\]

The proof of the following theorem is a close adaption of Theorem A.1 in F. Johne [9].
Theorem 2.2. Let \((M, g, w \text{dvol}_g)\) be a smooth complete noncompact \(n\)-dimensional Riemannian manifold of smooth density \(w > 0\) and asymptotically nonnegative Bakry–Émery Ricci curvature with respect to a base point \(o \in M\). Then the function

\[
r \mapsto \frac{\int_{B_r(o)} w}{(n + \alpha) \int_0^r h^{n+\alpha-1}(t) \, dt}
\]

is nonincreasing.

Proof. Fix the base point \(o \in M\) and \(r > 0\), let \(D_o = M \setminus \text{cut}(o)\) be the domain of the normal geodesic coordinates centered at \(o\). We define \(B_r(o) = \{q \in M : d(o, q) < r\}\) and its boundary by \(S_r(o) = \partial B_r(o)\). Denote the second fundamental form of the hypersurface \(S_r(o) \cap D_o\) by \(B\) and the mean curvature of the geodesic sphere with an inward pointing normal vector by \(H\).

Let \(\gamma(t) := \exp_o(tv), t \in [0, r]\) be a normal geodesic such that \(\gamma(t) \in S_t(o) \cap D_o\). We consider the variation set of hypersurfaces that have a constant distance from \(N\). By (1.6) in [10], it is easy to know that

\[
\frac{d}{dt} H = -|B|^2 - \text{Ric}(\gamma', \gamma'), \quad t \in (0, r),
\]

provided \(\gamma(t) \in S_t(o) \cap D_o\). By the definition of Bakry–Émery Ricci curvature (1.1), we deduce that

\[
\frac{d}{dt}[H + \langle D \log w, \gamma' \rangle] = -\frac{1}{\alpha} \langle D \log w, \gamma' \rangle^2 - \text{Ric}_\alpha(\gamma', \gamma') - |B|^2
\]

\[
\leq -\frac{1}{n - 1} H^2 - \frac{1}{\alpha} \langle D \log w, \gamma' \rangle^2 - \text{Ric}_\alpha(\gamma', \gamma')
\]

\[
= -\frac{1}{n + \alpha - 1}(H + \langle D \log w, \gamma' \rangle)^2 - \frac{n - 1}{\alpha(n + \alpha - 1)} \left(\frac{\alpha}{n - 1} H - \langle D \log w, \gamma' \rangle\right)^2 - \text{Ric}_\alpha(\gamma', \gamma')
\]

\[
\leq -\frac{1}{n + \alpha - 1}(H + \langle D \log w, \gamma' \rangle)^2 - \text{Ric}_\alpha(\gamma', \gamma'). \quad (2.3)
\]

Set \(g_o = \frac{1}{n + \alpha - 1}[H + \langle D \log w, \gamma' \rangle], t \in (0, r)\). Using \(\lim_{t \to 0^+} tH(t) = n - 1\) and the smoothness of the density \(w\), by (1.4) and (2.3), we can find

\[
\begin{aligned}
g_o' + g_o^2 &\leq \lambda, \quad t \in (0, r), \\
g_o(t) &= \frac{n - 1}{(n + \alpha - 1)t} + O(1), \quad \text{as } t \to 0^+.
\end{aligned}
\]

Note that \(0 < \frac{n - 1}{n + \alpha - 1} \leq 1\), from (1.5) and Lemma 2.1, it follows

\[
g_o \leq \frac{h'}{h}.
\]
that is,

\[ H + \langle D(\log w), \gamma' \rangle \leq (n + \alpha - 1) \frac{h'}{h}. \]

By the first variation formula for the manifold with density, we obtain

\[
\frac{d}{dt} \left( \int_{S_t(o)} w \right) = \frac{d}{dt} \left( \int_{S_t(o) \cap D_o} w \right) = \int_{S_t(o) \cap D_o} (H + \langle \nu, D\log w \rangle) w \\
\leq (n + \alpha - 1) \frac{h'}{h} \int_{S_t(o) \cap D_o} w = (n + \alpha - 1) \frac{h'}{h} \int_{S_t(o)} w,
\]

where \( \nu \) is the unit outward vector field along \( S_t(o) \cap D_0 \). This implies that

\[ t \mapsto \frac{\int_{S_t(o)} w}{h^{n+\alpha-1}(t)} \]

is a nonincreasing function. Following Lemma 2.2 in [17], we derive that

\[
\int_{B_r(o)} w = \int_0^r \frac{\int_{S_t(o)} w}{h^{n+\alpha-1}} h^{n+\alpha-1} dt \geq \frac{\int_{S_t(o)} w}{h^{n+\alpha-1}(r)} \int_0^r h^{n+\alpha-1} dt,
\]

which implies

\[
\frac{d}{dr} \left( \frac{\int_{B_r(o)} w}{\int_0^r h^{n+\alpha-1} dt} \right) = \frac{h^{n+\alpha-1}(r)}{\int_0^r h^{n+\alpha-1} dt} \left( \frac{\int_{S_t(o)} w}{h^{n+\alpha-1}(r)} - \frac{\int_{B_r(o)} w}{\int_0^r h^{n+\alpha-1} dt} \right) \leq 0.
\]

This proves the assertion. \( \square \)

### 3 PROOF OF THEOREM 1.1

Let \((M, g, w\, d\text{vol}_g)\) be a complete noncompact \(n\)-dimensional Riemannian manifold with smooth density \( w > 0 \) and asymptotically nonnegative Bakry-Émery Ricci curvature with respect to a base point \( o \in M \). Let \( \Omega \) be a compact and connected domain in \( M \) with smooth boundary \( \partial \Omega \), and \( f \) be a smooth positive function on \( \Omega \).

We only need to prove Theorem 1.1 in the case that \( \Omega \) is connected. By scaling, we may assume that

\[
\int_{\partial \Omega} w f + \int_{\Omega} |Df| + 2(n + \alpha - 1)b_1 \int_{\Omega} w f = (n + \alpha) \int_{\Omega} w f^{\frac{n+\alpha}{n+\alpha-1}}. \tag{3.1}
\]
Due to (3.1) and the connectedness of $\Omega$, we can find a solution to the following Neumann problem

$$\begin{cases}
\text{div}(w f Du) = (n + \alpha)w f^{\frac{n+\alpha}{n+\alpha-1}} - w|Df| - 2(n + \alpha - 1)b_1w f \\
\langle Du, v \rangle = 1,
\end{cases}
$$

where $v$ is the outward unit normal vector field of $\partial \Omega$. By standard elliptic regularity theory (see Theorem 6.31 in [8]), we know that $u \in C^{2,\gamma}$ for each $0 < \gamma < 1$.

Following the notions in [4], we define

$$U := \{ x \in \Omega \setminus \partial \Omega : |Du(x)| < 1 \}.$$

For any $r > 0$, we denote $A_r$ by

$$\{ \tilde{x} \in U : ru(x) + \frac{1}{2}d(x, \exp \tilde{x}(r Du(\tilde{x})))^2 \geq ru(\tilde{x}) + \frac{1}{2}r^2|Du(\tilde{x})|^2, \forall x \in \Omega \}$$

and the transport map $\Phi_r : \Omega \to M$ by

$$\Phi_r(x) = \exp_x(r Du(x)), \quad \forall x \in \Omega.$$

Using the regularity of the solution $u$ of the Neumann problem, we know that the transport map is of class $C^{1,\gamma}$, $0 < \gamma < 1$.

We obtain the following lemma similar to Lemma 2.1 in [4].

**Lemma 3.1.** Assume that $x \in U$. Then, we have

$$w\Delta u + \langle Dw, Du \rangle + 2(n + \alpha - 1)b_1w \leq (n + \alpha)w f^{\frac{1}{n+\alpha-1}}.$$

**Proof.** Using the Cauchy–Schwarz inequality and the property that $|Du| < 1$ for $x \in U$, we get

$$-\langle Df, Du \rangle \leq |Df|.$$

In terms of (3.2), we derive that

$$f(w\Delta u + \langle Dw, Du \rangle + 2(n + \alpha - 1)b_1w) = (n + \alpha)w f^{\frac{n+\alpha}{n+\alpha-1}} - w(|Df| + \langle Df, Du \rangle)$$

$$\leq (n + \alpha)w f^{\frac{n+\alpha}{n+\alpha-1}}.$$

This proves the assertion. \qed

The proofs of the following three lemmas are identical to those of Lemmas 2.2–2.4 in [4] without any change for the case of asymptotically nonnegative Bakry–Émery Ricci curvature. So, we omit them here.
Lemma 3.2 (cf. S. Brendle, Lemma 2.2 in [4]). The set
\[ \{ q \in M : d(x, q) < r, \forall x \in \Omega \} \]
is contained in \( \Phi_r(A_r) \).

Lemma 3.3 (cf. S. Brendle, Lemma 2.3 in [4]). Assume that \( \tilde{x} \in A_r \), and let \( \tilde{\gamma}(t) := \text{exp}_{\tilde{x}}(t Du(\tilde{x})) \) for all \( t \in [0, r] \). If \( Z \) is a smooth vector field along \( \tilde{\gamma} \) satisfying \( Z(r) = 0 \), then
\[
(D^2u)(Z(0), Z(0)) + \int_0^r \left( |D_t Z(t)|^2 - R(\tilde{\gamma}'(t), Z(t), \tilde{\gamma}'(t), Z(t)) \right) dt \geq 0,
\]
where \( R \) is the Riemannian curvature tensor of \( M \).

Lemma 3.4 (cf. S. Brendle, Lemma 2.2 in [4]). Assume that \( \tilde{x} \in A_r \), and let \( \tilde{\gamma}(t) := \text{exp}_{\tilde{x}}(t Du(\tilde{x})) \) for all \( t \in [0, r] \). Moreover, let \( \{ e_1, \ldots, e_n \} \) be an orthonormal basis of \( T_{\tilde{x}}M \). Suppose that \( W \) is a Jacobi field along \( \tilde{\gamma} \) satisfying
\[
\langle D_t W(0), e_j \rangle = (D^2u)(W(0), e_j), \quad 1 \leq j \leq n.
\]
If \( W(\tau) = 0 \) for some \( \tau \in (0, r) \), then \( W \) vanishes identically.

We now give the proof of Theorem 1.1. The strategy of the proof follows the work of S. Brendle [4] closely.

Proof of Theorem 1.1. For any \( r > 0 \) and \( \tilde{x} \in A_r \), let \( \{ e_1, \ldots, e_n \} \) be an orthonormal basis of the tangent space \( T_{\tilde{x}}M \). Choose the geodesic normal coordinates \( (x^1, \ldots, x^n) \) around \( \tilde{x} \), such that \( \frac{\partial}{\partial x^i} = e_i \) at \( \tilde{x} \). Let \( \tilde{\gamma}(t) := \text{exp}_{\tilde{x}}(t Du(\tilde{x})) \) for all \( t \in [0, r] \). For \( 1 \leq i \leq n \), let \( E_i(t) \) be the parallel transport of \( e_i \) along \( \tilde{\gamma} \). For \( 1 \leq i \leq n \), let \( X_i(t) \) be the Jacobi field along \( \tilde{\gamma} \) with the initial conditions of \( X_i(0) = e_i \) and
\[
\langle D_t X_i(0), e_j \rangle = (D^2u)(e_i, e_j), \quad 1 \leq j \leq n.
\]
Let \( P(t) = (P_{ij}(t)) \) be a matrix defined by
\[
P_{ij}(t) = \langle X_i(t), E_j(t) \rangle, \quad 1 \leq i, j \leq n.
\]
It follows from Lemma 3.4 that \( X_1(t), \ldots, X_n(t) \) are linearly independent for each \( t \in (0, r) \), which implies that the matrix \( P(t) \) is invertible for each \( t \in (0, r) \). It is obvious that \( \det P(t) > 0 \) if \( t \) is sufficiently small. Therefore, \( \det | \det D\Phi_1(\tilde{x}) | = \det P(t) > 0 \), for \( t \in (0, r) \). Let \( S(t) = (S_{ij}(t)) \) be a matrix defined by
\[
S_{ij}(t) = R(\tilde{\gamma}'(t), E_i(t), \tilde{\gamma}'(t), E_j(t)), \quad 1 \leq i, j \leq n,
\]
where $R$ denotes the Riemannian curvature tensor of $M$. By the Jacobi equation, we obtain
\[
\begin{cases}
    P''(t) = -P(t)S(t), & t \in [0, r], \\
    P_{ij}(0) = \delta_{ij}, & P'_{ij}(0) = (D^2 u)(e_i, e_j).
\end{cases}
\tag{3.3}
\]
Let $Q(t) = P(t)^{-1}P'(t), t \in (0, r)$, which is symmetric showed by S. Brendle [4]. By (3.3), a simple computation yields
\[
\frac{d}{dt} Q(t) = -S(t) - Q^2(t).
\]
Recalling that
\[
\text{Ric}_w^\alpha := \text{Ric} - D^2(\log w) - \frac{1}{\alpha} D \log w \otimes D \log w,
\]
we follow the computation by F. Johne [9] to derive that
\[
\frac{d}{dt} [\text{tr} Q + \langle D \log w, \bar{\gamma}' \rangle] = -\frac{1}{\alpha} \langle D \log w, \bar{\gamma}' \rangle^2 - \text{Ric}_w^\alpha (\bar{\gamma}', \bar{\gamma}') - \text{tr} [Q^2]
\leq -\frac{1}{n} [\text{tr} Q]^2 - \frac{1}{\alpha} \langle D \log w, \bar{\gamma}' \rangle^2 - \text{Ric}_w^\alpha (\bar{\gamma}', \bar{\gamma}')
= -\frac{1}{n + \alpha} (\text{tr} Q + \langle D \log w, \bar{\gamma}' \rangle)^2 - \frac{n}{\alpha(n + \alpha)} \left( \frac{\alpha}{n} \text{tr} Q - \langle D \log w, \bar{\gamma}' \rangle \right)^2 - \text{Ric}_w^\alpha (\bar{\gamma}', \bar{\gamma}')
\leq -\frac{1}{n + \alpha} (\text{tr} Q + \langle D \log w, \bar{\gamma}' \rangle)^2 - \text{Ric}_w^\alpha (\bar{\gamma}', \bar{\gamma}').
\]
Set $g = \frac{1}{n + \alpha} [\text{tr} Q + \langle D (\log w), \bar{\gamma}'(t) \rangle]$. The assumption of asymptotic nonnegative Bakry–Émery Ricci curvature gives
\[
g' + g^2 \leq \frac{n + \alpha - 1}{n + \alpha} |Du(\bar{x})|^2 \lambda(d(o, \bar{\gamma}(t))),
\tag{3.4}
\]
where $o$ is the base point. By the triangle inequality, we get
\[
d(o, \bar{\gamma}(t)) \geq |d(o, \bar{x}) - d(\bar{x}, \bar{\gamma}(t))| = |d(o, \bar{x}) - t|D u(\bar{x})||.
\tag{3.5}
\]
Set
\[
\Lambda_\bar{x}(t) = \frac{n + \alpha - 1}{n + \alpha} |D u(\bar{x})|^2 \lambda(|d(o, \bar{x}) - t|D u(\bar{x})||).
Since $\lambda$ is a nonincreasing function, it follows from (3.3), (3.4), and (3.5) that

\[
\begin{cases}
g'(t) + g(t)^2 \leq \Lambda_{\bar{x}}(t), & t \in (0, r), \\
g(0) = \frac{1}{n+\alpha}[\Delta u(\bar{x}) + \langle D(logw)(\bar{x}), Du(\bar{x}) \rangle].
\end{cases}
\]

Let $\phi = \int_0^t g(\tau) d\tau$, then

\[
\begin{cases}
\phi'' \leq \Lambda_{\bar{x}}(t)\phi, & t \in (0, r), \\
\phi(0) = 1, \phi'(0) = g(0).
\end{cases}
\] (3.6)

Set $\psi_1, \psi_2$ be solutions of the following problems:

\[
\begin{cases}
\psi_1'' = \Lambda_{\bar{x}}(t)\psi_1, & t \in (0, r), \\
\psi_1(0) = 0, \psi_1'(0) = 1,
\end{cases}
\] \quad \text{and}

\[
\begin{cases}
\psi_2'' = \Lambda_{\bar{x}}(t)\psi_2, & t \in (0, r), \\
\psi_2(0) = 1, \psi_2'(0) = 0.
\end{cases}
\] (3.7)

By the assumption of (1.2), one knows that $\int_0^\infty \Lambda_{\bar{x}}(t) \, dt < \infty$. Similar to the proof of Lemma 2.6 in [6], we have

\[
\frac{\psi_2}{\psi_1}(r) \leq \int_0^\infty \Lambda_{\bar{x}}(t) \, dt + \frac{1}{r} \leq 2\frac{n + \alpha - 1}{n + \alpha} b_1|Du(\bar{x})| + \frac{1}{r}.
\] (3.8)

Noting that $|Du(\bar{x})| < 1$, then

\[
\frac{\psi_2}{\psi_1}(r) \leq 2\frac{n + \alpha - 1}{n + \alpha} b_1 + \frac{1}{r}.
\] (3.9)

By Lemma 2.13 in [14] and (3.7), we derive that

\[
\psi_1(t) \leq \int_0^t e^{\int_0^s \tau \Lambda_{\bar{x}}(\tau) \, d\tau} \, ds \leq te^{\int_0^\infty \tau \Lambda_{\bar{x}}(\tau) \, d\tau} = te^{\frac{n+\alpha-1}{n+\alpha} \int_0^\infty \lambda(|d(o,\bar{x})-v|) \, dv} \leq te^{\frac{n+\alpha-1}{n+\alpha}(2r_0b_1+b_0)},
\] (3.10)

where $r_0 = \max\{d(o, x)|x \in \Omega\}$.

Letting $\psi(t) = \psi_2(t) + g(0)\psi_1(t)$, using (3.6), (3.7), and Lemma 2.5 in [6], we obtain

\[
\frac{1}{n+\alpha}[\text{tr}Q + \langle D \log w, \bar{y}' \rangle] = \frac{\phi'}{\phi} \leq \frac{\psi'}{\psi}, \quad \forall t \in (0, r).
\]
Consequently,
\[ \frac{d}{dt} \log[w(\tilde{\gamma}(t)) \det P(t)] = \text{tr}Q(t) + (D \log w(\tilde{\gamma}(t)), \tilde{\gamma}'(t)) \leq (n + \alpha) \frac{\psi'}{\psi}. \] (3.11)

Through (3.11), we can get
\[ w(\Phi_t(\tilde{x})) | \det D\Phi_t(\tilde{x})| = w(\Phi_t(\tilde{x})) | \det P(t) \]
\[ \leq w(\tilde{x}) \left( \psi_2(t) + \frac{1}{n + \alpha} [\Delta u(\tilde{x}) + (D \log w(\tilde{x}), Du(\tilde{x}))] \psi_1(t) \right)^{n + \alpha} \]
for all \( t \in [0, r] \). This implies
\[ w(\Phi_r(\tilde{x})) | \det D\Phi_r(\tilde{x})| \leq w(\tilde{x}) \left( \frac{\psi_2(r)}{\psi_1(r)} + g(0) \right)^{n + \alpha} \psi_1^{n + \alpha}(r) \]
for any \( \tilde{x} \in A_r \). Using (3.9), (3.10), and Lemma 3.1, it follows that
\[ w(\Phi_r(\tilde{x})) | \det D\Phi_r(\tilde{x})| \]
\[ \leq w(\tilde{x}) \left( \frac{n + \alpha - 1}{n + \alpha} 2b_1 + \frac{1}{r} + \frac{1}{n + \alpha} [\Delta u(\tilde{x}) + (D(\log w)(\tilde{x}), Du(\tilde{x}))] \right)^{n + \alpha} \]
\[ \cdot r^{n + \alpha} e^{(n + \alpha - 1)(2r_0b_1 + b_0)} \]
\[ \leq w(\tilde{x}) \left( \frac{1}{r} + f \frac{1}{n + \alpha - 1}(\tilde{x}) \right)^{n + \alpha} r^{n + \alpha} e^{(n + \alpha - 1)(2r_0b_1 + b_0)} \]
for any \( \tilde{x} \in A_r \). Moreover, by (1.5), we obtain \( h(t) \geq t \) and
\[ \lim_{t \to \infty} h'(t) = 1 + \int_0^\infty h(s)\lambda(s) \, ds \geq 1 + \int_0^\infty s\lambda(s) \, ds = 1 + b_0. \] (3.13)

Combining Lemma 3.2 and (3.12) with the formula for change of variables in multiple integrals, we conclude that
\[ \int_{\{q \in M : d(x,q) < r \text{ for all } x \in \Omega\}} w \, d\text{vol}_g(x) \]
\[ \leq \int_{A_r} | \det D\Phi_r(\tilde{x})|w(\Phi_r(\tilde{x})) \, d\text{vol}_g(\tilde{x}) \]
\[ \leq \int_{A_r} \left( \frac{1}{r} + f \frac{1}{n + \alpha - 1}(\tilde{x}) \right)^{n + \alpha} r^{n + \alpha} e^{(n + \alpha - 1)(2r_0b_1 + b_0)} w(\tilde{x}) \, d\text{vol}_g(\tilde{x}). \]
(3.14)

Let \( r > r_0 \), the triangle inequality implies that
\[ B_{r-r_0}(o) \subset \{q \in M : d(x,q) < r \text{ for all } x \in \Omega\} \subset B_{r+r_0}(o). \] (3.15)
Using \( \lim_{r \to \infty} \int_{f_0}^{r-r_0} \frac{h(t)^{n+\alpha-1}}{dt} = \lim_{r \to \infty} \frac{h(r-r_0)^{n+\alpha-1}}{h(r)^{n+\alpha-1}} \) by the L'Hospital's rule, and combining (1.6) and (3.15) with Lemma 2.7 in [6], we have

\[
\mathcal{V}_\alpha = \mathcal{V}_\alpha \lim_{r \to \infty} \frac{h(r-r_0)^{n+\alpha-1}}{h(r)^{n+\alpha-1}}
\]

\[
= \lim_{r \to \infty} \frac{\int_{B_r-r_0(\alpha)} w \, d\text{vol}_g}{(n + \alpha) \int_{0}^{r-r_0} h(t)^{n+\alpha-1} \, dt} \int_{0}^{r-r_0} h(t)^{n+\alpha-1} \, dt
\]

\[
\leq \lim_{r \to \infty} \frac{\int_{\{q \in M : d(x,q) < r \text{ for all } x \in \Omega\}} w \, d\text{vol}_g}{(n + \alpha) \int_{0}^{r} h(t)^{n+\alpha-1} \, dt} \int_{0}^{r} h(t)^{n+\alpha-1} \, dt
\]

\[
\leq \mathcal{V}_\alpha \lim_{r \to \infty} \frac{h(r+r_0)^{n+\alpha-1}}{h(r)^{n+\alpha-1}}
\]

which implies that

\[
\mathcal{V}_\alpha = \lim_{r \to \infty} \frac{\int_{\{q \in M : d(x,q) < r \text{ for all } x \in \Omega\}} w \, d\text{vol}_g}{(n + \alpha) \int_{0}^{r} h(t)^{n+\alpha-1} \, dt}. \quad (3.16)
\]

Dividing both side of (3.14) by \((n + \alpha) \int_{0}^{r} h(t)^{n+\alpha-1} \, dt\) and letting \(r \to \infty\), using (3.13) and (3.16), one can find that

\[
\mathcal{V}_\alpha \leq e^{(n+\alpha-1)(2r_0b_1+b_0)} \int_{\Omega} wf^{n+\alpha-1} \lim_{r \to \infty} \frac{r^{n+\alpha}}{(n + \alpha) \int_{0}^{r} h(t)^{n+\alpha-1} \, dt}
\]

\[
= e^{(n+\alpha-1)(2r_0b_1+b_0)} \int_{\Omega} wf^{n+\alpha-1} \lim_{r \to \infty} \frac{1}{h'(t)^{n+\alpha-1}}
\]

\[
\leq \left( \frac{e^{2r_0b_1+b_0}}{1+b_0} \right)^{n+\alpha-1} \int_{\Omega} wf^{n+\alpha-1}.
\]

Under our scaling assumption (3.1), we obtain

\[
\int_{\partial \Omega} w f + \int_{\Omega} w |Df| + (n + \alpha - 1)2b_1 \int_{\Omega} w f = (n + \alpha) \int_{\Omega} wf^{n+\alpha-1}
\]

\[
\geq (n + \alpha) \mathcal{V}_{\alpha}^{1+\alpha} \left( \frac{1+b_0}{e^{2r_0b_1+b_0}} \right)^{n+\alpha-1} \left( \int_{\Omega} wf^{n+\alpha-1} \right)^{n+\alpha-1}.
\]

\[\square\]
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