The Casimir force between rough metallic plates

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Abstract. – The Casimir force between two metallic plates is affected by their roughness state. This effect is usually calculated through the so-called 'proximity force approximation' which is only valid for small enough wavevectors in the spectrum of the roughness profile. We introduce here a more general description with a wavevector-dependent roughness sensitivity of the Casimir effect. Since the proximity force approximation underestimates the effect, a measurement of the roughness spectrum is needed to achieve the desired level of accuracy in the theory-experiment comparison.

Introduction. – The Casimir force [1] has recently been measured with a good experimental precision which allows for an accurate comparison between measured values and theoretical predictions [2,3]. This comparison plays an important role in the searches of new weak forces with nanometric to millimetric ranges motivated by theoretical unification models [4,5,6]. These forces would appear as experiment/theory differences in precise measurements of the Casimir force. As far as an accurate theory-experiment comparison is aimed at, the accuracy of theory is as crucial as the precision of experiments. If the target is a given accuracy, say at the 1% level, the theoretical prediction has to be mastered at this level as well as the experimental measurement. Clearly, this requires to take into account the real conditions of the experiments which differ from the ideal situation often assumed in the theory of the Casimir effect.

Casimir initially studied the case of a pair of perfectly smooth, flat and parallel plates in the limit of zero temperature and perfect reflection. He found a force which depends only on geometrical properties, the distance \( L \) between the plates and their area \( A \) supposed to be much larger than \( L^2 \), and two fundamental constants, \( c \) and \( h \). This remarkable fact is related to the assumption of perfect reflection while the most precise experiments [7,8] are performed with metallic mirrors which are nearly perfect reflectors only at frequencies smaller than a characteristic plasma frequency \( \omega_p \) so that the force is affected by imperfect reflection.

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at distances smaller than or of the order of the plasma wavelength $\lambda_P = \frac{2\pi c}{\omega_P}$. The same experiments are performed at room temperature, which implies that the force also depends on the scattering of blackbody radiation, in a geometry different from the ideal geometry considered by Casimir: the force is measured between a plane and a spherical reflectors which show some surface roughness.

The aim of the present letter is to raise questions about the validity of the current evaluation of the effect of surface roughness. For the sake of comparison with forthcoming discussions, we first recall the principles of this evaluation which is based on the proximity force approximation [9], denoted PFA hereafter. We then introduce an improved method where the sensitivity to roughness depends on the wavevector associated with surface deformations. We get information on this sensitivity by considering metallic plates in the two limits of long or short distances, where the roughness sensitivity can be deduced from earlier calculations devoted respectively to perfect reflectors [10] and to mirrors described by the surface plasmon approach [11,12]. In the present letter, we focus our attention on questions of interest for the comparison of theoretical predictions and experimental measurements of the Casimir force in the plane-sphere geometry. We disregard the temperature effect which is significant at large distances whereas roughness corrections are more important at the smallest distances explored in the experiments. We give only the main results of our calculations and refer the reader interested by more detailed developments to forthcoming publications.

The proximity force approximation. – When using the proximity force approximation (PFA), the force $F_{PS}$ in the plane-sphere geometry is obtained as the sum of the contributions corresponding to an effective inter-plate distance $z$ which varies from the distance of closest approach $L$ to infinity. As the area element $dA = 2\pi R \, dz$ corresponding to an interval $dz$ is proportional to this interval and to the radius $R$ of the sphere, $F_{PS}$ is given by geometric arguments and by the Casimir energy $E_{PP}$ calculated between two planes

$$ F_{PS} (L) = 2\pi R \frac{E_{PP} (L)}{A} , \quad E_{PP} (L) = \int_L^\infty dz \, F_{PP} (z) $$

(1)

We emphasize that the PFA amounts to the addition of contributions corresponding to different distances $z$, assuming these contributions to be independent. But the Casimir force is not additive, so that the PFA cannot be exact, although it is often improperly called a theorem. The results available for the plane-sphere geometry [13,14,15] show that the PFA leads to correct results when the radius $R$ of the sphere is much larger than the distance $L$ of closest approach.

The roughness correction is also often evaluated by using the PFA. We suppose that the two mirrors ($i = 1,2$) have roughness profile functions $h_i (\mathbf{r})$ where $\mathbf{r}$ collects the two transverse coordinates $(x,y)$ orthogonal to the direction $z$ of the cavity. Both deformations $h_1 (\mathbf{r})$ and $h_2 (\mathbf{r})$ are counted as positive when they correspond to length increases above the mean. Using calligraphic letters for the energy between rough plates and normal letters for the energy between smooth plates, we obtain with the PFA

$$ E_{PP} (L) = \langle E_{PP} (L + h) \rangle \simeq E_{PP} (L) + \frac{E''_{PP} (L)}{2} \langle h^2 \rangle , \quad h = h_1 + h_2 $$

(2)

The symbol $\langle \ldots \rangle$ denotes an average over the transverse coordinates. We have assumed that the profiles have a null average value $\langle h_{1,2} \rangle = 0$ and restricted the evaluation at the second order in the deformations. Hence, the roughness correction depends on the second order derivative $E''_{PP} (L)$ of the energy and on the variance of the length deformation $h$. This
expression is equivalent to the procedure used for analyzing the effect of roughness in recent experiments [7, 16].

It can be simplified one step further when the area \(A\) of the plates contains a large number of correlation areas. We denote \(\ell_C\) the correlation length of the roughness profiles and suppose \(A \gg \pi \ell_C^2\). We also suppose that the profiles of the two plates have no special relation to each other so that the integration over the surface is equivalent to a statistical averaging over a number of different realisations of these profiles. It follows that the cross correlation \(\langle h_1 h_2 \rangle\) between the two profiles can be ignored, which leads to the expression

\[
\mathcal{E}_{PP}(L) \simeq \frac{E_{PP}(L)}{2} a^2, \quad a^2 = \langle h_1^2 \rangle + \langle h_2^2 \rangle, \quad A \gg \pi \ell_C^2
\] (3)

Note that the last simplification would not be applicable to the case of corrugated plates [17] which is not considered in the present letter.

The preceding equation can be translated into expressions of the Casimir forces between rough plates, denoted with similar conventions. In the plane-plane geometry, the force \(\mathcal{F}_{PP}\) between rough plates would be obtained by differentiating (3) with respect to \(L\). Here we consider the plane-sphere geometry with the radius of the sphere large enough so that the PFA [11] is valid in the absence of roughness. In the presence of roughness, we define \(R\) as the radius of the sphere realizing the best fit of the real surface of the curved mirror. We then define the roughness profile as the deviation of the real surface from the best-fit sphere and suppose the statistical properties of this profile to be uncorrelated with the mean spherical geometry. We also assume that the number of correlation areas \(\pi \ell_C^2\) contributing significantly to the whole force is very large, so that the integration over the surface is still equivalent to a statistical averaging over a number of different realisations of these profiles. This assumption is consistent when the condition \(\pi RL \gg \pi \ell_C^2\) holds (see a similar discussion in [16]) and it allows us to ignore the cross correlation between the profiles of the two mirrors. When these conditions are met, the force \(\mathcal{F}_{PS}\) is given by an expression which generalizes (1) to the case of rough plates

\[
\mathcal{F}_{PS}(L) = 2\pi R \frac{E_{PP}(L)}{A}, \quad R \gg L, \quad RL \gg \ell_C^2
\] (4)

**The spectral sensitivity to roughness.** – Clearly, the PFA can only be valid for long-wavelength deformations. In particular, it gives a correct evaluation of the effect of curvature of the spherical mirror when \(R \gg L\). As far as roughness is concerned, the PFA holds for small enough values of the transverse wavevector \(\mathbf{k} = (k_x, k_y)\) but not in the general case of an arbitrary wavevector.

In order to deal quantitatively with this problem, we introduce the spectral densities \(\sigma_{ij}\) which describe the correlation properties of the profiles \(h_i\)

\[
\sigma_{ij} [k] = \int d^2 r \ e^{-i\mathbf{k} \cdot \mathbf{r}} \langle h_i (\mathbf{r}) h_j (\mathbf{0}) \rangle
\] (5)

We write the Casimir effect at the second order in roughness profiles as

\[
\mathcal{E}_{PP}(L) \simeq E_{PP}(L) + \sum_{i,j} \int \frac{d^2 k}{4\pi^2} G_{ij} [k] \ \sigma_{ij} [k]
\] (6)

The functions \(G_{ij}\) reduce to \(E''_{PP}/2\) in the PFA so that the preceding expression reduces to (2). In the general case, the functions \(G_{ij}\) depend on the transverse wavevector \(\mathbf{k}\) and thus measure the spectral sensitivity to roughness of the Casimir effect. As previously, we disregard the
crossed correlation terms $\sigma_{ij}$ with $i \neq j$ which tend to be averaged to zero in the integration over the surface. For the sake of simplicity, we also suppose that the two mirrors are made with the same metal so that the roughness sensitivity is described by a single function

$$E_{PP} (L) \simeq E'_{PP} (L) + \int \frac{d^2k}{4\pi^2} G [k] \sigma [k] \quad , \quad \sigma [k] = \sigma_{11} [k] + \sigma_{22} [k]$$

$$G [k] = G_{11} [k] = G_{22} [k] = \frac{E''_{PP} (L)}{2} \rho [k]$$

We have introduced a reduced sensitivity $\rho$ which goes to unity in the PFA sector. Due to the cylindrical symmetry with respect to rotations in the transverse plane, $G$ and $\rho$ are functions of $|k|$ only.

The derivation of equation (4) presented above remains valid outside the PFA sector. Hence, the roughness correction to the Casimir force in the plane-sphere geometry is read

$$\frac{F_{PS} (L) - F_{PS} (L)}{F_{PS} (L)} = \frac{E''_{PP} (L)}{2E'_{PP} (L)} \int \frac{d^2k}{4\pi^2} \rho [k] \sigma [k]$$

Incidentally, we notice that the Casimir force $F_{PP}$ in the plane-plane geometry could be obtained by differentiating $E_{PP}$ with respect to $L$, taking into account that $\rho$ depends on the distance $L$. For the sake of illustrating the significance of these results, we can consider the example of a Gaussian roughness spectrum

$$\sigma [k] = a^2 \pi \ell_C^2 \exp \left( - \frac{k^2 \ell_C^2}{4} \right)$$

where $a$ measures the roughness amplitude and $\ell_C$ the roughness correlation length. This example allows one to specify the definition of the correlation length $\ell_C$ but it cannot be used as a general description of roughness profiles. In order to discuss the experimental situations, the roughness spectra must be deduced from images of the real plates.

**Metallic mirrors in the limit of long distances.** – When the inter-plate distance $L$ is much larger than the plasma length $\lambda_P$, metallic mirrors behave as perfect reflectors. It follows that the roughness sensitivity can be obtained as a by-product of recent computations by Emig et al of the Casimir force between a corrugated mirror and a flat one, both being perfect reflectors [10]. Emig et al have evaluated the effect on the Casimir force of a periodic perturbation with a wavelength $\lambda$ of one of the two plates. Equation (4) of [10] can be interpreted as giving the function $\rho$ at the wavevector $|k| = 2\pi/\lambda$

$$\rho [k] = \frac{G_{TM} (s) + G_{TE} (s)}{G_{TM} (0) + G_{TE} (0)} \quad , \quad s \equiv \frac{L}{\lambda} = \frac{|k| L}{2\pi}$$

The functions $G_{TM}$ and $G_{TE}$ correspond to the contributions of TM and TE modes and are given by equations (5,6) of [10]. They reduce to the PFA limit when $s \to 0$, a property here included in the very definition of $\rho [k]$. The variation of $\rho$ versus the dimensionless factor $|k| L$ is shown as the solid line on Figure 1.

The PFA limit $\rho \simeq 1$ is recovered for $|k| L$ smaller than or of the order of unity. This means that the PFA [9] leads to reliable evaluations when the wavevectors which contribute significantly to surface roughness lie in the PFA sector $|k| L \lesssim 1$. When this is not the case, the factor $\rho$ is almost everywhere larger than unity, which means that the effect of roughness
is underestimated in the PFA analysis. In the limiting case of large wavevectors, \( \rho \) is found to reach large values proportional to \( |k| \)

\[
\rho |k| \simeq \beta |k| L , \quad \beta = \frac{1}{3} , \quad |k| L \gtrsim 1 \quad (11)
\]

This means that the error in the PFA estimation of the roughness correction may be important.

**Metallic mirrors in the limit of short distances.** We now consider the limit \( L \ll \lambda_P \) of short distances, where the Casimir energy can be written in terms of the dispersion relation characterizing the surface plasmons [18, 19]. The change of the dispersion relation and of the Casimir energy by the surface roughness has been studied by Maradudin and Mazur [11, 12]. The roughness sensitivity can be deduced from these results through a chain of reinterpretations and substitutions.

Equation (25) of [12] can be rewritten as giving the second order correction to the Casimir energy between two rough planes

\[
\mathcal{E}_{PP}(L) \simeq E_{PP}(L) - \hbar A \int |q-p| \frac{d^2q}{4\pi^2} \frac{d^2p}{4\pi^2} \sigma(q-p) \int_0^{\infty} \frac{d\xi}{2\pi} T(q,p,i\xi) \quad (12)
\]

This is the contribution of TM modes, with TE modes having a negligible contribution at short distances. As previously, we consider the case of uncorrelated profiles on the two plates and deduce the kernel \( T \) from the \( T_2 \) of equations (26c,28) in [12] through the subtraction \( T = T_2[L] - T_2[L \to \infty] \). We write this kernel in terms of the reflexion amplitudes \( r_1 = r_2 = r \) for TM modes near grazing incidence, assuming the mirrors to be made with the same metal.
described by the plasma model. We obtain after a number of manipulations

\[
T[q, p, i\xi] = \frac{|q|}{1 - r^2 e^{-2|q|L}} \frac{|p|}{1 - r^2 e^{-2|p|L}} \\
\times \left\{ 2 \left( 1 - r^2 \right) (1 - c)^2 r^4 e^{-2|q|L} e^{-2|p|L} \\
+ \left( 2r^2 (1 - c)^2 - 2r (1 - c^2) + 4c \right) r^2 \left( e^{-2|q|L} + e^{-2|p|L} \right) \right\}
\]

\[c = \frac{q \cdot p}{|q||p|}, \quad \rho [i\xi] = \frac{1 - \epsilon |i\xi|}{1 + \epsilon |i\xi|} = -\frac{\omega_p^2}{\omega_p^2 + 2\xi^2}
\]

The denominators in \( T \) describe the resonances of the Fabry-Perot cavity formed by the two mirrors and \( c \) is the cosine of the angle between the two transverse wavevectors \( q \) and \( p \). We finally rewrite (12) under the general form (7) with \( G \) obtained through a convolution on wavevectors and \( \rho [k] \) deduced as the ratio of \( G [k] \) to \( G [0] \)

\[G[k] = -\hbar A \int \frac{d^2q}{4\pi^2} \int_0^{\infty} \frac{d\xi}{2\pi} T[q, k - q, i\xi], \quad \rho [k] = G[k] / G[0]
\]

We have computed numerically the variation of \( \rho \) versus the dimensionless factor \( |k| L \) and shown the result as the dashed line on Figure 1. The PFA sector still corresponds to \( |k| L \lesssim 1 \). Outside this sector, the factor \( \rho \) is not only larger than unity, but also larger than in the limit of long distances. The behaviour of \( \rho \) for large wavevectors is found to obey the same law as in (13) with a larger value for the coefficient \( \beta \)

\[\rho [k] \simeq \beta |k| L, \quad \beta \simeq 0.45, \quad |k| L \gtrsim 1
\]

**Discussion.** – We discuss the significance of these results for theory-experiment comparison, and focus our attention on the relative effect of roughness measured by (3) when the two conditions \( R \gg L \) and \( RL \gg \ell_C^2 \) are met. If the stronger condition \( L \lesssim \ell_C \) is true, this effect is given by the PFA

\[
\frac{F_{PS} - F_{PS}}{F_{PS}} \simeq \frac{L^2 E_{PP}^2 a^2}{2E_{PP} L^2}
\]

It is proportional to two dimensionless factors: the first one \( \frac{L^2 E_{PP}^2}{2E_{PP} a^2} \) varies from 6 at long distances (where \( E_{PP} \propto 1/L^3 \)) to 3 at short distances (where \( E_{PP} \propto 1/L^2 \)) and the second one is the square \( a^2/L^2 \) of the ratio of roughness amplitude to cavity length. This expression is equivalent to the procedure which has been used for analyzing recent experiments \( \#7 \) and led to a relative effect equal to a fraction of a percent. It is now clear that this procedure underestimates the effect as soon as the roughness spectrum contains wavevectors \( |k| L \gtrsim 1 \). This case is not excluded by a preliminary inspection of available images of the rough plates used in the experiments.

We then have to use the expression (3) which includes a third factor \( \overline{\varphi} \) representing the modification of the correction with respect to the PFA

\[
\frac{F_{PS} - F_{PS}}{F_{PS}} = \frac{L^2 E_{PP}^2 a^2}{2E_{PP} L^2} \varphi, \quad \overline{\varphi} = \int \frac{d^2k}{4\pi^2} \rho [k] \frac{\sigma [k]}{a^2}
\]

\( \overline{\varphi} \) is the mean value of the roughness sensitivity \( \rho [k] \) in the distribution given by the normalized roughness spectrum \( \sigma [k] / a^2 \). To give an illustration, it takes the simple form \( \overline{\varphi} \simeq \beta \sqrt{\pi L/\ell_C} \)
for a Gaussian roughness spectrum having an important weight outside the PFA sector. It follows that the roughness correction has completely different scaling behaviours with respect to \( L \) inside and outside the PFA sector: \( \propto \frac{a^2}{L^2} \) when \( L \lesssim \ell_C \) whereas \( \propto \beta \sqrt{\frac{a^2}{L \ell_C}} \) when \( L \gtrsim \ell_C \) (see similar discussions in [10, 20]).

We stress again that the proximity force approximation underestimates the sensitivity of the Casimir effect to roughness. In order to ensure that the informations deduced from accurate theory/experiments comparisons are not biased by this approximation, it is certainly necessary to evaluate the factor \( \rho \). This requires the measurement of the normalized roughness spectrum \( \sigma[k] / a^2 \) for the plates used in the experiments as well as the computation of the normalized roughness sensitivity function \( \rho[k] \). Here, this function \( \rho[k] \) has been obtained in the two limiting cases of long or short distances by developing previous calculations. It has been found to be different in the two cases, with a deviation from PFA more pronounced in the short distance limit where the Casimir effect is also more sensitive to roughness. This means that a more general evaluation of \( \rho[k] \), covering the experimentally explored range of values of \( L/\lambda_P \), is required for a reliable estimation of the effect of roughness.

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