PARABOLIC HIGGS BUNDLES ON THE PROJECTIVE LINE, QUIVER VARIETIES AND DELIGNE-SIMPSON PROBLEM

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Abstract. We establish an isomorphism between the moduli space of parabolic Higgs bundles on the projective line and quiver variety of star-shaped quiver. As applications, we can solve the nilpotent case of the Deligne-Simpson problem in a geometrical way; and we can show that there is an algebraically completely integrable system structure on the moduli space as well as quiver variety we concerned.

1. Introduction

Parabolic bundles over curve was introduced by Metha and Seshadri in [10] as objects corresponding to unitary representations of fundamental group of punctured Riemann surface. The moduli theory of parabolic bundles is specially useful in the finite dimensional proof (in the sense of Beauville in [2]) of Verlinde formula. Verlinde formula can be seen as the dimension of nonabelian theta functions on curve. In [17], [23], [24], [25] the authors reduces the computation of Verlinde formula to computation of nonabelian parabolic theta functions on the projective line \( \mathbb{P}^1 \), using the method of degeneration of moduli spaces.

Grothendieck proved that any vector bundle on \( \mathbb{P}^1 \) is a direct sum of line bundles. So the classification of vector bundles on \( \mathbb{P}^1 \) is clear. However, if we consider parabolic bundles (parabolic Higgs bundles) on \( \mathbb{P}^1 \), things would be not so easy and become interesting. In this paper, we use \( \mathcal{M}_P \) to denote the moduli spaces of rank \( r \), degree 0 semistable parabolic bundles on \( \mathbb{P}^1 \) and use \( \text{Higgs}_P \) to denote the moduli space of homologically trivial semistable parabolic Higgs bundles. Despite the use in study of Verlinde formula, the moduli space of parabolic bundles (parabolic Higgs bundles) on \( \mathbb{P}^1 \) can be related with certain quiver variety.

A quiver is a finite oriented graph. One can define representations of quiver and the moduli spaces of quiver representations are called quiver varieties. Quiver varieties is well studied and useful in representation theory. The relation between quiver varieties and moduli spaces of parabolic Higgs bundles was firstly stated by Godinho and Mandini in [9]. They establish an isomorphism between the moduli space of rank two homologically trivial parabolic Higgs bundles on \( \mathbb{P}^1 \) and quiver variety of certain star-shaped quiver. Later in [7], Fisher and Rayan give an similar isomorphism in rank \( r \) case, but all parabolic structures are given by a one dimensional subspace and they did not fix weights of the parabolic bundles.

One main result in our paper is that we establish an isomorphism between the moduli space of parabolic bundles on \( \mathbb{P}^1 \) and quiver variety of certain star-shaped quiver, in any rank and any parabolic structure case. For a star-shaped quiver \( Q \), we use \( \mathcal{R}_\chi(\mathbf{v}) \) and \( \mathcal{M}_\chi(\mathbf{v}) \) to denote quiver varieties associated to \( Q \) (please refer to Section 4 for details). Then we have

**Theorem 1.1** (Theorem 4.6, Theorem 4.8). There are isomorphisms \( \mathcal{M}_P \cong \mathcal{R}_\chi(\mathbf{v}) \) and \( \text{Higgs}_P^r \cong \mathcal{M}_\chi(\mathbf{v}) \), where the quiver \( Q \), character \( \chi \) and dimension vector \( \mathbf{v} \) are chosen related to the parabolic data of parabolic (Higgs) bundles.

As for the parabolic bundle case, the relation between moduli stack of parabolic bundles and moduli stack of quiver representations has been mentioned by Soibelman in [21]. He study the very good property of these two moduli stacks.

The third topic in our paper is (additive) Deligne-Simpson problem. Deligne-Simpson problem can be formulated as follows: Given \( n \) conjugacy classes \( c_i \subset \mathfrak{g} \), can we find \( A_i \in c_i \) so that \( \sum_{i=1}^n A_i = 0 \)? This problem is related to the existence of certain Fuchsian system on the \( \mathbb{P}^1 \). Deligne-Simpson problem was studied by Simpson, Kostov, Crawley-Boevey and
so on in [20], [13], [14], [15], [6] and [21]. Especially, in [6] Crawley-Boevey relates solutions of Deligne-Simpson with certain star-quiver representations and gives a complete criterion to the existence of solution of Deligne-Simpson problem.

Inspired by Crawley-Boevey’s work and the isomorphism in Theorem 1.1 we relate the solutions of Deligne-Simpson problem to parabolic Higgs bundles on \( \mathbb{P}^1 \). In terms of parabolic Higgs bundle, one can study its characteristic polynomial and associated spectral curve (see subsection 2.3). In this way, we can solve the nilpotent case of Deligne-Simpson problem in a geometrical way.

To be precise, if each conjugacy class \( c_i \) is given by the conjugacy class of a nilpotent matrix \( N_i \) with rank \( \gamma_i \) (we call it the nilpotent case of Deligne-Simpson problem in this case), then we have

**Theorem 1.2** (Theorem 3.1). If \( 2r \geq \sum_{i=1}^r \gamma_i \) and \( r \geq 4 \), then the nilpotent case of Deligne-Simpson problem has irreducible resolutions.

**Remark 1.3.** Our result coincide with those in [14], [6] and [21] but with a different method. Moreover, in the proof of Theorem 1.2 we see that we can construct solutions of nilpotent case of Deligne-Simpson problem from line bundles on some smooth projective curve. So the benefit of our method seems enable us to construct the solutions explicitly, we have been working on this project recently.

The isomorphism in Theorem 1.1 enables us translate properties of two moduli spaces interchangeably. For example, we have a parabolic Hitchin map on the moduli space of parabolic Higgs bundle, then we can define a parabolic Hitchin map on the quiver variety. As a result, we have

**Theorem 1.4.** There is an algebraically completely integrable system structure on \( \text{Higgs}^\mathfrak{g}_r \cong \mathcal{M}_X(v) \).

Here we take the definition of algebraically completely integrable system from [10]. In the proof of Theorem 1.4 we use properties from both \( \text{Higgs}^\mathfrak{g}_r \) and \( \mathcal{M}_X(v) \).

This paper is organized as follows:

In Section 2, we recall the definition and some properties of parabolic bundles, homologically trivial parabolic Higgs bundles. Then under the condition of choice of weights (condition (2.1)), we construct these moduli spaces explicitly. We also study the parabolic Hitchin map in this case, recall some results in [22].

In Section 3, we establish the connection between homologically trivial parabolic Higgs bundles and solutions of nilpotent case of Deligne-Simpson problem, then use the parabolic Hitchin map and parabolic BNR correspondence (Theorem 2.13) to solve the nilpotent case of Deligne-Simpson problem.

In Section 4 we firstly recall the definition of quiver varieties and then in the case of star-shaped quiver, we construct the quiver varieties explicitly. Compare with results in Section 2 we can prove Theorem 1.1. Next we recall the definition of Poisson varieties, algebraically completely integrable system and then prove Theorem 1.4.

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## 2. Moduli space of parabolic Higgs bundles over projective line

### 2.1. Parabolic bundles on \( \mathbb{P}^1 \)

We work over an algebraically closed field \( k \) with characteristic zero. We consider the projective line \( \mathbb{P}^1 \) and let \( z \) be a local coordinate of \( \mathbb{P}^1 \). \( I = \{x_1, \cdots, x_n\} \subset \mathbb{P}^1 \) be a finite subset with \( n \geq 4 \) and \( K \) be a positive integer. We assume that \( \infty \notin I \). Consider a vector bundle \( E \) of rank \( r \) on \( \mathbb{P}^1 \), a parabolic structure on \( E \) is given by the following:
(1) Choice of flag at each $x \in I$:

$$E|_x = F^0(E_x) \supsetneq F^1(E_x) \supsetneq \cdots \supsetneq F^{\sigma_x}(E_x) = 0$$

We put $n_i(x) = \dim F^{i-1}(E_x) - \dim F^i(E_x), \ 1 \leq i \leq \sigma_x$;

(2) Choice of a sequence of integers at each $x \in D$:

$$0 \leq a_1(x) < a_2(x) < \cdots < a_{\sigma_x}(x) < K$$

We call these numbers weights.

With a parabolic structure given, we say that $E$ is a parabolic vector bundle of type $\Sigma := \{I, K, \{n_i(x)\}, \{a_i(x)\}\}$. If the parabolic type $\Sigma$ is known, we simply say that $E$ is a parabolic vector bundle.

The parabolic degree of $E$ is defined by

$$\text{pardeg}E = \deg E + \frac{1}{K} \sum_{x \in D} \sum_{i=1}^{\sigma_x} a_i(x)n_i(x)$$

and $E$ is said to be semistable if for any nontrivial subbundle $F$ of $E$, consider the induced parabolic structure on $F$, one has

$$\mu_{\text{par}}(F) := \frac{\text{pardeg}F}{\text{rk} F} \leq \mu_{\text{par}}(E) := \frac{\text{pardeg}E}{\text{rk} E}.$$ 

$E$ is said to be stable if the inequality is always strict.

The construction of moduli spaces of semistable parabolic bundles can be found in [16], [21]. What we do here is to consider the moduli space of rank $r$ degree 0 semistable parabolic bundles on $\mathbb{P}^1$ with parabolic degree being sufficiently small. We will construct the moduli space explicitly in this case.

Before going further, we introduce a condition on weights which we will use in the following:

$$\frac{1}{K} \sum_{x \in I} a_{\sigma_x}(x) < \frac{1}{r} \quad (2.1)$$

**Lemma 2.1.** Under the condition (2.1), if $E$ is semistable as parabolic vector bundle with rank $r$ degree 0, then $E$ is homologically trivial, i.e. $E \cong \mathcal{O}_{\mathbb{P}^1}^r$ as a vector bundle.

**Proof.** For any proper subbundle $F \subset E$, we have

$$\text{pardeg}F = \deg F + \frac{1}{K} \sum_{x \in D} \sum_{i=1}^{\sigma_x} a_i(x)n_i^F(x)$$

where $n_i^F(x) = \dim(F^{i-1}(E_x) \cap F|_x) - \dim(F^i(E_x) \cap F|_x)$. The condition $E$ being semistable as parabolic vector bundle says:

$$\frac{\deg F}{\text{rk} F} - \frac{\deg E}{\text{rk} E} \leq \frac{1}{K} \sum_{x \in I} \sum_{i=1}^{\sigma_x} a_i(x)(\frac{n_i(x)}{\text{rk} E} - \frac{n_i^F(x)}{\text{rk} F})$$

which means

$$\frac{\deg F}{\text{rk} F} \leq \frac{1}{K} \sum_{x \in I} \sum_{i=1}^{\sigma_x} a_i(x)(\frac{n_i(x)}{\text{rk} E} - \frac{n_i^F(x)}{\text{rk} F}).$$

Now, condition (2.1) tells the right hand side of above inequality is less than $1/r$, which shows $\deg(F) \leq 0$. By Grothendieck’s classification of vector bundles on $\mathbb{P}^1$, we see that $E \cong \mathcal{O}_{\mathbb{P}^1}^r$. \(\square\)

**Example 2.2.** Here we give a counterexample of above lemma when condition (2.1) is not satisfied. Let $I = \{x_1, \cdots, x_4\}$, $K = 2$. Consider a vector bundle $E = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, we give a parabolic structure on $E$ in the following way:

(1) At each $x_i \in I$, the choice of flag is given by

$$\mathcal{O}_{\mathbb{P}^1}(-1)|_{x_i} \oplus \mathcal{O}_{\mathbb{P}^1}(1)|_{x_i} \supset \mathcal{O}_{\mathbb{P}^1}(-1)|_{x_i} \supset 0;$$
(2) Weights are given by $a_1(x_i) = 0$, $a_2(x_i) = 1$.

Clearly the condition (2.1) is not satisfied in this case. Now we are going to explain the semistability of $E$ briefly.

Firstly we see that $\mu_{par}(E) = 1$. Next we consider all sub line bundles of $E$. When $O_{\mathbb{P}1}(1)$ is a subbundle of $E$, we see that $\mu_{par}(O_{\mathbb{P}1}(1)) = 1$; $O_{\mathbb{P}1}$ can not be a subbundle of $E$; When $O_{\mathbb{P}1}(-1)$ is a subbundle of $E$, we see that it can only be the direct summand of $E$, so $\mu_{par}(O_{\mathbb{P}1}(-1)) = 1$; For $O_{\mathbb{P}1}(-n)$, $n \geq 2$, it is easy to see that $\mu_{par}(O_{\mathbb{P}1}(-n)) < 1$. Thus $E$ is a semistable parabolic bundle of degree 0 on $\mathbb{P}1$, but it is not homologically trivial.

Lemma 2.3. Under condition (2.1), a homologically trivial parabolic vector bundle $E$ is semistable if and only if for any homologically trivial subbundle $F$ of $E$, we have

$$\frac{\text{pardeg } F}{\text{rk } F} \leq \frac{\text{pardeg } E}{\text{rk } E}.$$ 

Proof. One direction is obvious. To show the another direction, we choose any subbundle $F$ of $E$ which is not homologically trivial, thus $\text{deg } F < 0$. Then

$$\frac{\text{pardeg } F}{\text{rk } F} - \frac{\text{pardeg } E}{\text{rk } E} = \frac{\text{deg } F}{\text{rk } F} - \frac{\text{deg } E}{\text{rk } E} + \frac{1}{K} \sum_{x \in I} \sum_{i=1}^{\sigma_x} a_i(x) (n_i(x)/\text{rk } E - n_i^F(x)/\text{rk } F^F)$$

$$\leq \frac{\text{deg } F}{\text{rk } F} + \frac{1}{r} < 0.$$ 

So we only need to test homologically trivial subbundles. □

Example 2.4. As before, we are going to give a counterexample of above lemma when condition (2.1) is not satisfied.

Let $V$ be a two dimensional vector space. We consider $\mathbb{P}(V) \cong \mathbb{P}^1$ and a homologically trivial bundle $E = \mathbb{P}(V) \times V$. Notice that $E$ has a sub-line bundle $L$ so that for any one dimensional subspace $l$ of $V$, if we use $[l]$ to denote the corresponding point in $\mathbb{P}(V)$, then $L_{[l]} = l \subset V = E_{[l]}$.

Now we choose 4 different points $[l_1], \ldots, [l_4]$ of $\mathbb{P}(V)$, an integer $K = 4$, and give a parabolic structure on $E$ in the following way:

1. At each point $[l_i]$, the choice of flag is given by

$$E_{[l_i]} = V \supset l_i \supset 0;$$

2. Weights are given by $a_1([l_i]) = 0$, $a_2([l_i]) = 3$.

Again here the weights do not satisfy condition (2.1). What we are going to explain is that, for any homologically trivial subbundle $F$ of $E$, we have $\mu_{par}(F) < \mu_{par}(E)$, but $\mu_{par}(L) > \mu_{par}(E)$, which says that condition (2.1) is necessary in above lemma.

Firstly, $\mu_{par}(E) = 3/2$. A homologically trivial subbundle $F$ of $E$ is determined by an one dimensional subspace $W$ of $V$. A discussion about whether $W$ coincides with $l_i$ or not will enables us to see that $\mu_{par}(F) < \mu_{par}(E)$. On the other hand, $\mu_{par}(L) = -1 + 3/4 \times 4 = 2 > 3/2$, i.e. $\mu_{par}(L) > \mu_{par}(E)$.

Now we can construct the moduli space of semistable parabolic vector bundles with rank $r$ degree 0 under the condition (2.1).

Firstly by Lemma 2.1 every semistable parabolic vector we are about considering is isomorphic to $O_{\mathbb{P}1}^{\oplus r}$ as vector bundle. Then we put $V = H^0(\mathbb{P}^1, O_{\mathbb{P}1}^{\oplus r})$, we see that all possible parabolic structures on $O_{\mathbb{P}1}^{\oplus r}$ are parametrized by

$$F := \prod_{x \in I} \text{Flag}(V, \vec{\gamma}(x))$$

where $\text{Flag}(V, \vec{\gamma}(x))$ is the partial flag variety of flags in $V$ with dimension vector

$$\vec{\gamma}(x) = (\gamma_1(x), \gamma_2(x), \ldots, \gamma_{\sigma_x-1}(x))$$
and $\gamma_i(x) = \sum_{j=i+1}^{s_x} n_j(x)$. Moreover, notice that the group $SL(V)$ acts diagonally on $F$ so that two points in $F$ represent isomorphic parabolic vector bundle if and only if they are in a same $SL(V)$ orbit.

Next if we put a polarization of the $SL(V)$ action on $F$ by

$$\prod_{x \in I} (d_1(x), \cdots, d_{s_x-1}(x))$$

where $d_i(x) = a_{i+1}(x) - a_i(x)$. By Hilbert-Mumford criterion, a point

$$q = \prod_{x \in I} (q_1(x) \supset \cdots \supset q_{s_x-1}(x)) \in F$$

is GIT semistable if and only if for any subspace $W \subset V$, we have

$$\frac{\sum_{x \in I} \sum_{i=1}^{s_x-1} d_i(x) \dim(W \cap q_i(x))}{\sum_{x \in I} \sum_{i=1}^{s_x-1} d_i(x) \dim(q_i(x))} \leq \frac{\dim W}{r}$$

Rearranging and assuming the corresponding parabolic vector bundle of $q$ is $E$, we will see that the inequality above is equivalent to

$$\frac{\text{pardeg} W \otimes O_{P^1}}{\dim W} \leq \frac{\text{pardeg} E}{r}$$

by Lemma 2.3 we conclude that the GIT semistability coincides with parabolically semistability. Thus we have:

**Proposition 2.5.** Under condition (2.1), the moduli space of semistable parabolic vector bundle with rank $r$ and degree $0$ on $P^1$ is isomorphic to $F//SL(V)$, with polarization given above.

**2.2. Parabolic Higgs bundles on $P^1$.** In the following we will consider parabolic Higgs bundles on $P^1$ and their moduli spaces. First we define $D_I = \sum_{x \in I} x$ to be a reduced effective divisor on $P^1$. A parabolic Higgs bundle is a parabolic vector bundle $E$ together with a parabolic Higgs field: $\phi : E \to E \otimes \omega_{P^1}(D_I)$, where $\phi$ maps $F^i(E_x)$ into $F^{i+1}(E \otimes \omega_{P^1}(D_I))$ for any $x \in I$ (in this case we say that $\phi$ preserves filtration strongly). One defines a parabolic Higgs bundle $(E, \phi)$ to be semistable if for every proper sub-Higgs bundle $(F, \phi') \subset (E, \phi)$, the inequality

$$\frac{\text{pardeg} F}{\text{rk} F} \leq \frac{\text{pardeg} E}{\text{rk} E}$$

holds. Similarly if the inequality is always strict, we say that $(E, \phi)$ is stable.

Before going further to the construction of moduli space of parabolic Higgs bundles, we shall consider the following example first.

**Example 2.6.** Consider $I = \{x_1, \cdots, x_4\} \subset P^1$, $K = 16$ and a vector bundle $E = O_{P^1}(1) \oplus O_{P^1}(-1)$. We give a parabolic structure on $E$ in the following way:

1. At each point $x_i$, the choice of flag is given by

$$O_{P^1}(-1)|_{x_i} \oplus O_{P^1}(1)|_{x_i} \supset O_{P^1}(-1)|_{x_i} \supset 0;$$

2. Weights are given by $a_1(x_i) = 0, a_2(x_i) = 1$.

Clearly the choice of weights satisfies condition (2.1), and hence $E$ is not semistable as a parabolic vector bundle by Lemma 2.1. Now we fix a nonzero morphism $\phi_1 : O_{P^1}(1) \to O_{P^1}(-1) \otimes \omega_{P^1}(D_I)$ and consider a morphism on $E$: $\phi = \begin{bmatrix} 0 & 0 \\ \phi_1 & 0 \end{bmatrix} : E \to E \otimes \omega_{P^1}(D_I)$. Notice that $\phi$ is actually a parabolic Higgs field. Since $O_{P^1}(1)$ is not a sub-Higgs bundle of $(E, \phi)$, we can see that $(E, \phi)$ is a stable parabolic Higgs bundle easily.
So unlike the case of parabolic vector bundle, even if we assume that the condition (2.1) holds, a semistable parabolic Higgs bundle would have underlying vector bundle not being homologically trivial. Instead, here we only consider those parabolic Higgs bundle with underlying vector bundle being homologically trivial.

**Remark 2.7.** The moduli spaces of semistable parabolic Higgs bundles on \( X \) are constructed in [26], the homologically trivial locus forms an open subset of this moduli space.

**Lemma 2.8.** Under the condition (2.1), a homologically trivial parabolic Higgs bundle \((E, \phi)\) is semistable if and only if for all proper homologically trivial sub-Higgs bundles \((F, \phi')\), one has

\[
\frac{\text{pardeg} F}{\text{rk} F} \leq \frac{\text{pardeg} E}{\text{rk} E}.
\]

**Proof.** Similar as the proof of Lemma 2.3. \( \square \)

Before construct the moduli space of semistable homologically trivial parabolic Higgs bundle, we firstly take a closer look at parabolic Higgs fields \( \phi \). The space of all possible parabolic Higgs field on a parabolic bundle \( E \) is denoted by \( \text{Hom}_{\text{spar}}(E, E \otimes \omega_{p1}(D_I)) \). We fix an isomorphism \( H^0(P^1, E) \cong V \). So for each \( x \in I \), the filtration at \( E|_x \) would give a filtration on \( V \). Then we recall that

\[
\phi \in \text{Hom}(E, E \otimes \omega_{p1}(D_I)) \\
\cong \text{Hom}_k(V, V \otimes H^0(P^1, \omega_{p1}(D_I)))
\]

Here for any \( x \in I \), we have an residue map \( \text{Res}_x : H^0(P^1, \omega_{p1}(D_I)) \rightarrow \omega_{p1}(D_I)|_x \), hence a morphism \( \phi \in \text{Hom}_q(V, V \otimes H^0(P^1, \omega_{p1}(D_I))) \) is a parabolic Higgs field if and only if for any \( x \in I \), the compositd map:

\[
\text{Res}_x \circ \theta : V \rightarrow V \otimes \omega_{p1}(D_I)|_x
\]

preserve the filtration on \( V \)(induced by filtration on \( E|_x \)) strongly. Let \( \text{Hom}^{s.f.}(V, V \otimes \omega_{p1}(D_I)|_x) \) be the space of such morphisms. Notice that \( \text{deg}_{\omega_{p1}(D_I)} = n - 2 \), one has the following exact sequence:

\[
0 \rightarrow H^0(P^1, \omega_{p1}(D_I)) \rightarrow \bigoplus_{x \in I} \omega_{p1}(D_I)|_x \rightarrow k \rightarrow 0
\]

Now we choose a basis \( dz/(z - x) \) for \( \omega_{p1}(D_I)|_x \), then we have

\[
0 \rightarrow \text{Hom}_{\text{spar}}(E, E \otimes \omega_{p1}(D_I)) \rightarrow \bigoplus_{x \in I} \text{Hom}^{s.f.}(V, V) \rightarrow \text{Hom}(V, V) \rightarrow 0. \tag{2.2}
\]

**Remark 2.9.** Exact sequence (2.2) tells us that to give a parabolic Higgs field on a homologically trivial parabolic bundle \( E \), it is equivalent to give \( n \) linear maps \( A_x : V \rightarrow V, x \in I \), satisfying certain nilpotent conditions, so that \( \sum_{x \in I} A_x = 0 \). Moreover, if we are simply given \( n \) linear maps \( \{A_i\} \), such that \( \sum A_i = 0 \), then we have a weak parabolic Higgs field \( \phi : \mathcal{O}_{\mathbb{P}^1}^{\text{br}} \rightarrow \mathcal{O}_{\mathbb{P}^1}^{\text{br}} \otimes \omega_{p1}(D_I) \) in the sense as in [22].

Now we are going to construct the moduli space. Since we consider homologically trivial bundles only, as before, all parabolic structures are parametrized by \( F \), notice that the middle term of exact sequence (2.2) is the cotangent space of a point in \( F \). By arguments above, we have a morphism between vector bundles over \( F \): \( \mu_P : T^*F \rightarrow \text{Hom}(V, V) \) and the kernel \( \mathfrak{F} = \ker \mu_P \) parametrizes all parabolic Higgs bundles we are about to consider. Thus we have:

**Proposition 2.10.** Under condition (2.1), the moduli space of homologically trivial semistable parabolic Higgs bundle \( \text{Higgs}^s_p \) is isomorphic to \( \mathfrak{F}/\text{SL}(V) \), with polarization chosen before.

**Remark 2.11.**
(1) Consider the parabolic bundle $E$ in example 2.4. We put a parabolic Higgs field $\phi = 0$ on $E$. Thus for any homologically trivial sub-Higgs bundle $F$ of $E$, we have $\mu_{par}(F) < \mu_{par}(E)$, but $E$ is not a semistable parabolic Higgs bundle. So condition (2.1) is also necessary in parabolic Higgs bundle case.

(2) About how to do GIT quotient on $\mathfrak{g}$, please refer to [18].

2.3. Parabolic Hitchin map. Let $(E, \phi)$ be a parabolic Higgs bundle, we define its characteristic polynomial to be

$$\text{char}(E, \phi) = \lambda^r + \alpha_1 \lambda^{r-1} + \cdots + \alpha_{r-1} \lambda + \alpha_r$$

where $\alpha_i = (-1)^i \text{Tr}(\wedge^i \phi) \in H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(D_I)^{\otimes i})$. If we use coordinates to denote the characteristic polynomial $\text{char}(E, \phi)$, we may think that

$$\text{char}(E, \phi) = \alpha = (\alpha_i)_{1 \leq i \leq r} \in \mathbb{H} := \prod_{i=1}^r H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(D_I)^{\otimes i}).$$

Globally, if we use $\text{Higgs}_p$ to denote the moduli space of semistable parabolic Higgs bundles, then we have an algebraic morphism:

$$h_P : \text{Higgs}_p \rightarrow \mathbb{H}$$

which maps the $S$-equivalent class of a semistable parabolic Higgs bundle $(E, \phi)$ to $\text{char}(E, \phi)$. For detailed discussion, please refer to [22].

Notice that the parabolic Higgs field satisfies $\phi(F^i(E_x)) \subseteq F^{i+1}((E \otimes \omega_{\mathbb{P}^1}(D_I))_x)$, so at each $x \in I$, $\phi$ is nilpotent, which implies that the morphism $h_P$ above is not surjective.

To determine the image of $h_P$, for any $1 \leq j \leq r$, $x \in I$ and a fixed parabolic type

$$\Sigma = \{I, K, \{n_i(x)\}, \{u_i(x)\}\}$$

we define the following numbers:

$$\mu_j(x) = \#\{l | n_l \geq j, 1 \leq l \leq \sigma_x\}$$

$$\varepsilon_j(x) = l \iff \sum_{t \leq l-1} \mu_t(x) < j \leq \sum_{t \leq l} \mu_t(x)$$

notice that $\varepsilon_r(x) = \max\{n_i(x)\}$. Then we have

**Proposition 2.12** ([22], Theorem 3.4). The image of $h_P$ lies in the following subspace of $\mathbb{H}$:

$$\mathbb{H}_P := \prod_{j=1}^r H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes j} \otimes O_{\mathbb{P}^1}(\sum_{x \in I} (j - \varepsilon_j(x))x))$$

and we define the morphism $h_P : \text{Higgs}_p \rightarrow \mathbb{H}_P$ to be parabolic Hitchin map, $\mathbb{H}_P$ to be the parabolic Hitchin base.

If we use $|\omega_{\mathbb{P}^1}(D_I)^{-1}| = \text{Spec}(\text{Sym}(\omega_{\mathbb{P}^1}(D_I)^{-1}))$ to denote the total space of $\omega_{\mathbb{P}^1}(D_I)^{-1}$, then for any $\alpha = (\alpha_i)_{1 \leq i \leq r} \in \mathbb{H} := \prod_{i=1}^r H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(D_I)^{\otimes i})$, we can define the spectral curve $C_{\alpha}$ associated to $\alpha$ in the following sense:

For each $1 \leq i \leq r$, one has a morphism $\alpha_i : \omega_{\mathbb{P}^1}(D_I)^{-r} \rightarrow \omega_{\mathbb{P}^1}(D_I)^{-(r-i)}$. The sum of these morphisms gives a morphism $u : \omega_{\mathbb{P}^1}(D_I)^{-r} \rightarrow \text{Sym}(\omega_{\mathbb{P}^1}(D_I)^{-1})$ and we denote the ideal generated by the image of $u$ by $\mathcal{J}$. Finally, one defines the spectral curve associated to $\alpha$ by

$$C_{\alpha} = \text{Spec}(\text{Sym}(\omega_{\mathbb{P}^1}(D_I)^{-1})/\mathcal{J}).$$

There is a natural morphism $\pi_\alpha : C_{\alpha} \rightarrow \mathbb{P}^1$. For a parabolic Higgs bundle $(E, \phi)$, if $\text{char}(E, \phi) = \alpha$, we also say that $C_{\alpha}$ is the spectral curve associated to $(E, \phi)$. Loosely speaking, if we regard a parabolic Higgs bundle as a family of linear maps parametrized by $\mathbb{P}^1$, then $C_{\alpha}$ parametrizes all the eigenvalues.

In [22], we investigate the generic fibre of parabolic Hitchin map $h_P$, using the following parabolic BNR correspondence:
Theorem 2.13 ([22], Theorem 4.9). For a generic point $\alpha \in H_P$, assume that the corresponding spectral curve $C_\alpha$ is integral, then we have the a one-to-one correspondence between the following two sets:

1. Parabolic Higgs bundle $(E, \phi)$ over $\mathbb{P}^1$, such that $\text{char}(E, \phi) = \alpha$;
2. Line bundle over the normalization $\tilde{C}_\alpha$ of $C_\alpha$.

Remark 2.14.

1. The original BNR correspondence in [1] states that for Higgs bundles, there is a one-to-one correspondence between Higgs bundle over a smooth projective curve and torsion-free rank 1 sheaves on the corresponding integral spectral curve ([1], Proposition 3.6). Notice that if the spectral curve is smooth, then one can replace “torsion-free rank 1 sheaves” by “line bundles”. However, since we consider parabolic Higgs bundles, the spectral curve we encounter is barely smooth. In fact, if $\exists n_i(x) \neq 1$, then the corresponding spectral is not smooth. This is why we consider the normalization of spectral curve in [22].
2. Actually the base curves we considered in [22] has been assumed to have genus greater than 1. But what we did were mostly local computations, so conclusions hold in genus zero case.

As corollaries, we have

Corollary 2.15. Parabolic Hitchin map $h_P : \text{Higgs}_P \to H_P$ is a flat proper surjective morphism.

Corollary 2.16. For a generic point $\alpha \in H_P$, assume that the corresponding spectral curve $C_\alpha$ is integral, then the fibre $h_P^{-1}(\alpha)$ is isomorphic to a connected component of Picard group of $\tilde{C}_\alpha$. Moreover, the dimension of $h_P^{-1}(\alpha)$ is exactly half dimension of $\text{Higgs}_P$.

3. Deligne-Simpson Problem

For given $n$ conjugacy classes $\mathbf{c}_i \subset \mathfrak{gl}_r$, can we find $A_i \in \mathbf{c}_i$ so that $\sum_i A_i = 0$? This problem is called the (additive version of) Deligne-Simpson problem. If the matrices $\{A_i\}$ have no common proper invariant subspaces, then we say that the solution $\{A_i\}$ is irreducible. One of the source of Deligne-Simpson is the linear system of differential equations defined on the Riemann’s sphere:

$$dX/dt = A(t)X$$

where the $n \times n$ matrix $A(t)$ is meromorphic on $\mathbb{C}P^1$, with poles at $x_1, \cdots, x_n$. We say the system is Fuchsian if its poles are logarithmic. A Fuchsian system admits the following presentation

$$dX/dt = \left( \sum_{i=1}^{n} \frac{A_i}{z - x_i} \right)X$$

where $A_i \in \mathfrak{gl}_r$ and $\sum_i A_i = 0$. So the existence of a Fuchsian on $\mathbb{C}P^1$ is equivalent to the existence of a solution of Deligne-Simpson problem. Please refer to [15] for details.

The (additive) Deligne-Simpson problem was studied by Kostov in [13], [14] and he gives criterion for the existence of the solution for some special case, including nilpotent case. Later in [6], Crawley-Boevey gives a criterion for each case via quiver representations. In [21], Soibelman gives a sufficient condition for the moduli stack of parabolic vector bundles over $\mathbb{P}^1$ to be very good and use this property to study the space of solutions to Deligne-Simpson problem.

Here we use a more geometrical way to study Deligne-Simpson problem in nilpotent case, using parabolic Higgs bundles over $\mathbb{P}^1$ and the parabolic Hitchin map.

Now we assume that the conjugacy class $\mathbf{c}_i$ given by the conjugacy class of a nilpotent matrix $N_i$. We assume that $\text{rk}(N_i) = \gamma_i$, then $\{\gamma_i\}$ determines the conjugacy class $\mathbf{c}_i$. 
Notice that we have $\gamma_i^j - \gamma_i^{j+1} \geq \gamma_j^k - \gamma_i^{k+1}$ for any $j \leq k$. We consider rank $r$, degree 0 homologically trivial parabolic Higgs bundles with type

$$\Sigma = \{I, K, \{a_j(x_i)\}, \{a_j(x_i)\}\}$$

over $\mathbb{P}^1$, where the weights satisfy condition \[2.1\] and $\gamma_j(x_i) = \sum_{k=j+1}^{n} n_k(x_i) = \gamma_i^j$. The moduli space is denoted by $\text{Higgs}_{\pi}^0$.

For any closed point in $\text{Higgs}_{\pi}^0$, assume that one of the corresponding homologically trivial parabolic Higgs bundle is $(E, \phi)$. We choose a basis $dz/(z - x_i)$ for $\omega_{\mathbb{P}1}(D_i)|_{x_i}$ and fix an isomorphism $H^0(\mathbb{P}^1, E) \cong k^{\oplus r}$. Then by Remark \[2.9\] $(E, \phi)$ is equivalent to $n$ linear transformations $A_i : k^{\oplus r} \rightarrow k^{\oplus r}$, so that $A_i$ preserves the filtration on $k^{\oplus r}$ induced by the parabolic structure of $E$ strongly and $\sum_i A_i = 0$. However, at first sight, we can not control the conjugacy class of $A_i$, for example, we do not know whether $rkA_i = \gamma_i^1$.

So we consider the parabolic Hitchin map $h_P : \text{Higgs}_{\pi}^0 \rightarrow H_P$. Recall that

$$H_P := \bigoplus_{j=1}^{r} H^0(\mathbb{P}^1, \omega_{\mathbb{P}1}^{\oplus j} \otimes \mathcal{O}_{\mathbb{P}1}(\sum_{i=1}^{n} (j - \varepsilon_j(x_i)) x_i))$$

The image of $(E, \phi)$ under $h_P$ is given by sections $\alpha_j = \text{Tr}(\wedge^j \phi) \in H^0(\mathbb{P}^1, (\omega_{\mathbb{P}1}(D_i))^{\otimes j})$. Now the zero orders of $\{\alpha_j\}$ at $x_i$ determine the conjugacy class of $A_i$ and $rk(A_i^j) = \gamma_j^i$ if and only if the zero order of $\alpha_j$ at $x_i$ is exactly $\varepsilon_j(x_i)$.

To find such sections $\{\alpha_j\}$ in $H_P$, we need conditions

$$-2j + \sum_{i=1}^{n} (j - \varepsilon_j(x_i)) \geq 0$$

for every $j$. By the definition of $\varepsilon_j(x_i)$, we find out that these conditions is equivalent to

$$-2r + \sum_{i=1}^{n} (r - \varepsilon_r(x_i)) \geq 0$$

Recall that $\varepsilon_r(x_i) = \max\{n_j(x_i)\}$ then this condition is equivalent to

$$2r \leq \sum_{i=1}^{n} \gamma_i^1$$

(3.1)

Now we assume that condition \[3.1\] holds, then

$$H^0(\mathbb{P}^1, \omega_{\mathbb{P}1}^{\oplus r} \otimes \mathcal{O}_{\mathbb{P}1}(\sum_{i=1}^{n} (r - \varepsilon_r(x_i)) x_i)) \neq 0$$

By the argument in the Appendix of \[22\], in this case, there exists $\alpha = \{\alpha_j\} \in H_P$, so that the spectral curve $C_{\alpha}$ is integral. Notice that the condition that $C_{\alpha}$ is integral and the condition $\alpha_j$ has zero order $\varepsilon_j(x_i)$ at $x_i$ are both open conditions. So if we assume that condition \[3.1\] holds, we can choose $\alpha \in H_P$, so that $\alpha$ satisfies conditions in Theorem \[2.13\] and the $\alpha_j$ has zero order $\varepsilon_j(x_i)$ at $x_i$. We choose $(E, \phi) \in (h_P)^{-1}(\alpha)$, if $E$ is homologically trivial, we have a solution for the Delingne-Simpson problem has a solution. Moreover, this solution is actually irreducible, otherwise we would have a proper parabolic sub-Higgs bundle of $(E, \phi)$, which will give a factor of the characteristic polynomial of $(E, \phi)$ and makes $C_{\alpha}$ not integral.

If we use $\tilde{C}_{\alpha}$ to denote the normalization of $C_{\alpha}$, and $\tilde{\pi}_{\alpha} : \tilde{C}_{\alpha} \rightarrow \mathbb{P}^1$ to be the projection. By Theorem \[2.13\] we have

$$(h_P)^{-1}(\alpha) = \{\text{line bundles } \mathcal{L} \text{ over } \tilde{C}_{\alpha}, \text{ so that } \tilde{\pi}_{\alpha*} \mathcal{L} \text{ has degree 0}\}$$

Thus the homologically trivial parabolic Higgs bundles forms a nonempty affine open subset scheme of $(h_P)^{-1}(\alpha)$. In summary, we have

**Theorem 3.1.** If $2r \leq \sum_{i=1}^{n} \gamma_i^1$ and $r \geq 4$, then the nilpotent case of Deligne-Simpson problem has irreducible solutions.
Remark 3.2.

(1) In [14] and [6], they also considered the case $n \leq 3$, they call this case to be “special”. In our case, if $n \leq 3$ the moduli space of parabolic Higgs bundles on $\mathbb{P}^1$ sometimes reduces to a single point.

(2) In the following section, we will construction an isomorphism between the moduli space of homologically trivial parabolic Higgs bundles and quiver variety of “star-shaped” quiver. Our method to solve Deligne-Simpson problem is inspired by [9] and the isomorphism.

(3) The benefit of our method is that we may actually construct a solution for the Deligne-Simpson problem. By the discussion above, if we choose a line bundle $\mathcal{L}$ on the normalized curve $\tilde{C}_\alpha$, so that $\tilde{\pi}_\alpha*\mathcal{L} \cong \mathcal{O}^{\oplus r}$, then we have a solution for the Deligne-Simpson problem. The equation defining the spectral curve $C_\alpha$ is known and the local construction of $\tilde{C}_\alpha$ near $x_i \in I$ can be found in [22]. We have been recently working on this problem.

4. Quiver varieties and main theorem

In this section we work over the field $\mathbb{C}$. A quiver is a finite oriented graph. Let $Q = (I, E)$ be a quiver where $I$ is the vertex set, $E$ is the set of oriented edges. Given a dimension vector $\nu = (v_i) \in \mathbb{Z}_{\geq 0}^I$, a representation of $Q$ with dimension $\nu$ is a collection of vector spaces $\{V_i\}_{i \in I}$ with $\dim V_i = v_i$, and a collection of linear maps $\{\phi_{ij} : V_i \to V_j\}_{(i,j) \in E}$. All such representations can be gathered in a linear space:

$$R = \text{Rep}(Q, \nu) = \bigoplus_{(i,j) \in E} \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j})$$

with an action of a group $GL(\nu) := \prod_{i \in I} GL(v_i)$. Notice that the diagonal $\Delta : \mathbb{G}_m \to GL(\nu)$ acts trivially. Two representations in $\text{Rep}(Q, \nu)$ are isomorphic if and only if they are in same orbit.

One can construct the moduli space of representations of $Q$ by taking quotient space $\text{Rep}(Q, \nu)/GL(\nu)$. But this space is usually not Hausdorff. Instead one can do the GIT quotient $\mathcal{M}_0(\nu) := R/GL(\nu) = \text{Spec} \mathbb{C}[R]^{GL(\nu)}$, where $\mathbb{C}[R]$ is the coordinate ring of $R$. We have a description for this ring:

**Proposition 4.1** ([S], Proposition 2.1.1). The ring $\mathbb{C}[R]^{GL(\nu)}$ is generated by following functions:

$$\text{Tr}(\rho, -) : V \mapsto \text{Tr}(\rho, V)$$

where $V \in R$ is a representation, $\rho$ is an oriented cycle in $Q$ and $\text{Tr}(\rho, V)$ is the trace of composition of the morphisms in $V$ along cycle $\rho$.

As a corollary, if $Q$ has no oriented cycles, then $\mathcal{M}_0(\nu)$ is a point. To separate more orbits, King introduced the following definition([11]):

**Definition 4.2.** Given a character $\chi : GL(\nu) \to \mathbb{G}_m$, one defines the GIT quotient of $R$ by $GL(\nu)$ respect to $\chi$ by

$$\mathcal{M}_\chi(\nu) = \frac{R}{\chi GL(\nu)} = \text{Proj} \left( \bigoplus_{n \geq 0} \mathbb{C}[R]^{GL(\nu), \chi^n} \right)$$

where

$$\mathbb{C}[R]^{GL(\nu), \chi^n} = \{ f \in \mathbb{C}[R] \mid \forall g \in GL(\nu), x \in R, f(g^{-1}(x)) = \chi^g(n) f(x) \}$$

Clearly the natural morphism $\mathcal{M}_\chi(\nu) \to \mathcal{M}_0(\nu)$ is a projective morphism.

King also analyse stability conditions in this case:

**Lemma 4.3** ([11], Proposition 2.5). Consider a linear algebraic group $G$ acts linearly on a linear space $R$. Assume the kernel of this action is $\Delta$ and $\chi : G \to \mathbb{G}_m$ is a character.
Then a point \( x \in \mathbb{R} \) is \( \chi \)-semistable if and only if \( \chi(\Delta) = \{1\} \) and for every one parameter subgroup \( \lambda : \mathbb{G}_m \to G \), such that \( \lim_{t \to 0} \lambda(t)x \) exists, one has \( \lambda \circ \chi(t) = t^a \) for some \( a \geq 0 \).

A point \( x \) is stable if and only if the only one parameter subgroup \( \lambda \) of \( G \) which makes \( \lim_{t \to 0} \lambda(t)x \) exists and \( \lambda \circ \chi(t) = 1 \), is a subgroup of \( \Delta \).

In the following, we mainly consider the “star-shaped” quiver in Figure 1 (here I replace the vertices by dimension vector for convenience).

\[
\begin{align*}
&\gamma_1^1 \leftarrow \gamma_2^1 \leftarrow \cdots \leftarrow \gamma_{\sigma_1}^1 \\
&\gamma_1^2 \leftarrow \gamma_2^2 \leftarrow \cdots \leftarrow \gamma_{\sigma_2}^2 \\
&\vdots \quad \vdots \quad \vdots \\
&\gamma_1^n \leftarrow \gamma_2^n \leftarrow \cdots \leftarrow \gamma_{\sigma_n}^n
\end{align*}
\]

**Figure 1.** “star-shaped” quiver \( Q \)

In this case we want to describe \( \mathcal{R}_\chi(\mathcal{V}) \) explicitly. Firstly we begin with a lemma analysing semistable points in \( \mathbb{R} \) respect to a subgroup of \( GL(\mathcal{V}) \):

**Lemma 4.4.** Consider an action of \( G = GL(\gamma_1) \times \cdots \times GL(\gamma_\sigma) \) on

\[
\mathbf{R}_A = \text{Hom}(\mathbb{C}^{\gamma_1}, \mathbb{C}^r) \oplus \cdots \oplus \text{Hom}(\mathbb{C}^{\gamma_\sigma}, \mathbb{C}^{\gamma_\sigma})
\]

with character \( \chi: G \to \mathbb{G}_m : (g_1, \ldots, g_\sigma) \mapsto \prod_{i=1}^\sigma \det(g_i)^{a_i}, \) \( a_i > 0 \). Then a point \( f = (f_1, \ldots, f_\sigma) \in \mathbf{R}_A^n \) is semistable if and only if stable if and only if all \( f_i \) has rank \( \gamma_i \). As a consequence, \( G \) act freely on the semistable locus \( \mathbf{R}_A^s \) and \( \mathbf{R}_A/\chi G \cong \text{Flag}(\mathbb{C}^r, \mathbb{C}^\gamma) \).

**Proof.** Given \( f = (f_1, \ldots, f_\sigma) \in \mathbf{R}_A^n \), assume that, for example, \( f_\sigma \) has rank less than \( \gamma_\sigma \). Then one can choose a basis so that \( f_\sigma = (a_{ik}^\sigma)^{1 \leq i \leq \gamma_\sigma - 1, 1 \leq j \leq \gamma_\sigma} \) with a zero column, say \( a_{ik}^\sigma = 0 \) for \( 1 \leq i \leq \gamma_\sigma - 1 \). Now we choose a one parameter subgroup \( \lambda \) of \( G \) by

\[
\chi(t) = (I, I, \cdots, I, D)
\]

where \( I \) is identity and \( D = \text{diag}(t^{m_1}, \ldots, t^{m_\sigma}) \), \( m_j = 0 \) unless \( j = k \) and \( m_k = -1 \). In this case \( \lim_{t \to 0} \lambda(t)f = f \) exists and \( \lambda \circ \chi(t) = t^{-a_ik^\sigma}. \) This contradicts with Lemma 4.3. Similarly one can show that all \( f_i \) must have rank \( \gamma_i \).

Next assume that we are given \( f = (f_1, \ldots, f_\sigma) \in \mathbf{R}_A^n \) with rank \( f_i = \gamma_i \), we want to show \( f \) is in fact stable. For any one parameter subgroup \( \lambda \) of \( G \), we may choose a coordinate so that

\[
\lambda(t) = (D_1, \cdots, D_\sigma)
\]

where \( D_i \) are diagonal matrix. We take \( f_\sigma \) as example again, since \( f_\sigma \) has rank \( \gamma_\sigma \), under the coordinate we chosen, each column of \( f_\sigma \) must has a nonzero element. Assuming that \( \lim_{t \to 0} \lambda(t)f \) exists, then each term in \( D_\sigma \) would has power in \( t \) greater or equal than 0. Similar arguments holds for \( D_i \). By the choice of \( \chi \), we see that in order to make \( \lambda \circ \chi(t) = 1 \), \( \lambda \) must be the trivial one parameter subgroup, which is exactly the one parameter acts trivially. Hence by Lemma 4.3 \( f \) is stable. \( \square \)

**Lemma 4.5.** Let \( G = G_1 \times G_2 \) acting linearly on a linear space \( \mathbb{R} \). \( \chi : G \to \mathbb{G}_m \) is a character, which is a product of two characters \( \chi_1 \) and \( \chi_2 \) of \( G_1 \) and \( G_2 \) respectively. Then

\[
\mathbf{R}/\chi G \cong (\mathbf{R}/\chi_1 G_1)/\chi_2 G_2.
\]

**Proof.** One need to show \( \mathbb{C}[\mathbf{R}]^{G, \chi_a} = (\mathbb{C}[\mathbf{R}]^{G_1, \chi_1})^{G_2, \chi_2} \), which is straightforward. \( \square \)
Now we are going to establish an isomorphism between “star-shaped” quiver variety and moduli space of semistable parabolic bundles over \( \mathbb{P}^1 \) constructed in Section 2. Recall that \( I = \{x_1, \ldots, x_n\} \) and our parabolic type is \( \Sigma \). We consider a quiver \( Q \) as in Figure 1 and choose a character for \( GL(v) \). Let \( \gamma_i \in \mathbb{C}^* \) and \( \chi : GL(v) \rightarrow \mathbb{G}_m, (g_0, g_j) \mapsto (\det g_0)^{-N} \prod (\det g_j^d_i)^{d_i} \)

where \( g_0 \in GL(r) \); \( g_i^d \in GL(\gamma_i) \) and \( d_i = d_i(x_j) = a_{i+1}(x_j) - a_i(x_j) \); if necessary one can replace \( \chi \) by its multiple, then put \( N = (\sum_{1 \leq j \leq n} \sum_{1 \leq i \leq \sigma_j} \gamma_i^d_i) / r \) to be an integer. The reason why we choose \( N \) in this form is to make sure that \( \chi \circ \Delta = 1 \).

**Theorem 4.6.** Under the condition (2.1), and we take \( v, \chi \) as above. Then the moduli space \( M_P \) of rank \( r \), degree 0 semistable parabolic bundles with type \( \Sigma \) on \( \mathbb{P}^1 \) is isomorphic to \( \mathcal{R}_\chi(v) \)

**Proof.** We fix an isomorphism \( V \cong \mathbb{C}^{\oplus r} \). Let \( G_1 = GL(r) \), \( G_2 = GL(v) / GL(r) \). By Lemma 4.4, we see that \( R / \chi G = F \), where \( F \) is the product of flag variety defined in last section. Now by applying Lemma 4.5, we see that \( R / \chi G \cong F / GL(r) \) with polarization we given before, which gives moduli space of semistable parabolic vector bundles on \( \mathbb{P}^1 \) under condition (2.1).

In the following we will show that how to realize moduli space of homologically trivial semistable parabolic Higgs bundles on \( X \) as certain quiver variety.

Firstly we introduce the doubled quiver \( \bar{Q} \), defined by \( \bar{Q} = (I, E \cup E^{op}) \), where \( E^{op} \) is the set of oriented edges reversing to \( E \). For example, the doubled quiver of Figure 1 is as in the Figure 2.

\[
\begin{align*}
\gamma_1^1 & \leftrightarrow \gamma_2^1 \leftrightarrow \cdots \leftrightarrow \gamma_{\sigma_1}^1 \\
\gamma_1^2 & \leftrightarrow \gamma_2^2 \leftrightarrow \cdots \leftrightarrow \gamma_{\sigma_2}^2 \\
\vdots & \quad \quad \cdots \quad \quad \cdots \\
\gamma_1^n & \leftrightarrow \gamma_2^n \leftrightarrow \cdots \leftrightarrow \gamma_{\sigma_1}^n
\end{align*}
\]

**Figure 2.** Doubled quiver \( \bar{Q} \)

The representation space \( Rep(\bar{Q}, v) \) of \( \bar{Q} \) is canonically identified to cotangent bundle of \( R \) thus we have the following moment map:

\[
\mu : Rep(\bar{Q}, v) = T^*R \rightarrow g_v
\]

sending \((f_{ij}, g_{ji})\) to \( \sum(f_{ij} \circ g_{ji} - g_{ji} \circ f_{ij}) \), where \( g_v \cong g_v^* \) is the Lie algebra of \( GL(v) \). What we are going to consider is the following variety

\[
\mathfrak{M}_\chi(v) = \mu^{-1}(0) / \chi GL(v).
\]

Similarly, we want to give a description of \( \mathfrak{M}_\chi(v) \) when \( \bar{Q} \) and \( v \) are as in Figure 2. Firstly we shall analyse the action of \( G_2 = GL(v) / GL(r) \) on \( \mu^{-1}(0) \).

**Lemma 4.7.** Let \( G, R, A, \chi, \gamma \) be as in Lemma 4.4. We consider moment map as above: \( \tilde{\mu} : T^*R \rightarrow g \) and the action of \( G \) on \( \tilde{\mu}^{-1}(0) \), then

\[
\tilde{\mu}^{-1}(0) / \chi G \cong T^*\text{Flag}(\mathbb{C}^{\oplus r}, \gamma)
\]
Proof. One can find the proof in [12], Theorem 10.43.

As before, we decompose \( GL(v) \) as \( G_1 \times G_2 \), then we have

\[
\begin{array}{c}
\mathbf{T^*R} \\
\mu_1 \\
\mu_2 \\
\mu_3
\end{array} 
\begin{array}{c}
\mathfrak{g}_1 \\
\mathfrak{g}_2
\end{array}
\]

Notice that \( \mu_1 : \mu_2^{-1}(0) \to \mathfrak{g}_1 \) is \( G_2 \) equivalent, so we have a morphism \( \mu_Q : \mu_2^{-1}(0)//_{\chi_2} G_2 \to \mathfrak{g}_1 \).

**Theorem 4.8.** Assume that we have same conditions as in Theorem 4.6. Then we have an isomorphism \( \Psi : \text{Higgs}^\circ_P \to \mathcal{M}_\chi(v) \), where \( \text{Higgs}^\circ_P \) is the moduli space of rank \( r \), degree 0 homologically trivial parabolic Higgs bundles with type \( \Sigma \) on \( \mathbb{P}^1 \).

**Proof.** With the isomorphism \( V \cong \mathbb{C}^{\oplus r} \), we have the following diagram:

\[
\begin{array}{c}
\mathbf{T^*F} \\
\mu_P \downarrow \cong \downarrow \cong \\
\mu_2^{-1}(0)//_{\chi_2} G_2 \\
\mu_Q \downarrow \\
\mathfrak{g}_1
\end{array} \begin{array}{c}
\mathcal{H}om(V,V) \\
\mathfrak{g}_1
\end{array}
\]

where \( \mu_P : \mathbf{T^*F} \to \mathcal{H}om(V,V) \) is given before Proposition 4.7 and the isomorphism \( \mathbf{T^*F} \) follows from Lemma 4.7. Now one should note that \( \mu^{-1}(0) = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \), then the theorem follows as in the proof of Theorem 4.6. \( \square \)

**Remark 4.9.**

(1) If we assume that the weight of parabolic Higgs bundles are generic, or equivalently, the choice of \( \chi \) in \( \mathcal{M}_\chi(v) \) is generic, then the isomorphism \( \Psi \) is actually a symplectic isomorphism. Firstly, in this case, \( \text{Higgs}^\circ_P \) is isomorphic to the cotangent bundle of \( \mathcal{M}_P \). Secondly we know that the isomorphism \( \mathbf{T^*F} \cong \mu_2^{-1}(0)//_{\chi_2} G_2 \) is a symplectic isomorphism and \( \mu_P, \mu_Q \) are corresponding moment maps. Finally by Proposition 4.1.3 and Corollary 4.1.5 in [8], one sees that \( \Psi \) can be seen as the isomorphism of cotangent bundles of \( \mathcal{M}_P \) and \( \mathcal{M}_\chi(v) \), induced by \( \mathcal{M}_P \cong \mathcal{M}_\chi(v) \) in Theorem 4.6. Thus when we consider the natural symplectic structure on cotangent bundles, \( \Psi \) is a symplectic isomorphism.

(2) Some special cases of Theorem 4.8 have been considered before. In [9], Godinho and Mandini consider rank two case, and construct the isomorphism above. They use this isomorphism to give a description of cohomology ring of \( \text{Higgs}^\circ_P \). However, they did not assume condition (2.1), and from Remark 2.11 we know that condition (2.1) is somehow necessary in the construction of the isomorphism. Later in [4] they and their another two collaborators showed the isomorphism they constructed in [9] is actually a symplectic isomorphism. In [7], Fisher and Rayan considered rank r case and the flags at every \( x \in D \) is given by choosing a one dimensional subspace. They also showed a similar isomorphism, without considering the weights. They use this isomorphism to show that their is a parabolic Hitchin map on \( \mathcal{M}_\chi(v) \), and when \( r = 2, 3 \), they proved that the parabolic Hitchin maps are completely integrable systems.

Now we want to give a description of \( \Psi \) in the level of sets. Given a homologically trivial semistable parabolic Higgs bundle \( (E, \phi) \), notice that \( E \cong \mathcal{O}^{\oplus r}_{\mathbb{P}^1} \) and \( H^0(\mathbb{P}^1, E) = V \cong \mathbb{C}^{\oplus r} \). By exact sequence (2.29), \( \phi \) is equivalent to n linear maps \( \phi_i \in \text{Hom}^s_i(\mathbb{C}^{\oplus r}, \mathbb{C}^{\oplus r}) \) such that \( \sum_{i=1}^n \phi_i = 0 \). For every \( 1 \leq i \leq n \), there is a filtration \( F^*(\mathbb{C}^{\oplus r}) \) induced by filtration on
Now we restrict \( \phi_i \) to get morphisms \( F^i(\mathbb{C}^r) \to F^{i+1}(\mathbb{C}^r) \) and consider inclusions \( F^{i+1}(\mathbb{C}^r) \hookrightarrow F^i(\mathbb{C}^r) \). Together we have a representation \( \mathcal{V} \) for \( Q \), moreover, \( \mathcal{V} \) lies in \( \mu^{-1}(0) \) and is semistable by the semistability of \((E, \phi)\). So we get a point in \( \mathfrak{M}_\chi(\mathcal{V}) \).

Conversely, given a semistable representation \( \mathcal{V} \in \mu^{-1}(0) \), we know that there are the following maps

\[ \mathbb{C}^r \xrightarrow{f^i_1} \mathbb{C}^{\gamma^i_1} \xrightarrow{g^i_1} \mathbb{C}^r \]

and \( \sum_{i=1}^n f^i_1 \circ f^i_1 = 0 \). Moreover, the semistability of \( \mathcal{V} \) implies that \( \mathbb{C}^{\gamma^i_1} \) injects into \( \mathbb{C}^r \) so there are \( n \) filtrations on \( \mathbb{C}^r \); the condition \( \mathcal{V} \in \mu^{-1}(0) \) implies \( g^i_1 \circ f^i_1 \) preserves the filtration on \( \mathbb{C}^r \) strongly. So, by exact sequence \([22]\), we have a parabolic Higgs field on \( \mathcal{O}_{\mathbb{C}^r} \), which gives a semistable parabolic Higgs bundle.

The parabolic Hitchin map \( h_P: \text{Higgs}_P \to \mathcal{H}_P \) induces a morphism on \( \mathfrak{M}_\chi(\mathcal{V}) \):

\[ h_Q = \Psi^{-1} \circ h_P : \mathfrak{M}_\chi(\mathcal{V}) \to \mathcal{H}_P \]

which can be seen as parabolic Hitchin map on \( \mathfrak{M}_\chi(\mathcal{V}) \). Actually, \( h_Q \) is the descendent of a morphism on \( \mu^{-1}(0) \):

**Proposition 4.10.** There is a \( GL(\mathcal{V}) \) equivalent morphism

\[ \hat{h}_Q : \mu^{-1}(0) \to \mathcal{H}_P \]

such that \( h_Q \) is induced from this morphism.

**Proof.** For any representation \( \mathcal{V} \in \mu^{-1}(0) \), as before, we have morphisms

\[ \mathbb{C}^r \xrightarrow{f^i_1} \mathbb{C}^{\gamma^i_1} \xrightarrow{g^i_1} \mathbb{C}^r \]

and \( \sum_{i=1}^n g^i_1 \circ f^i_1 = 0 \). By Remark \([2,9]\) we know that \( \mathcal{V} \) gives a homologically trivial weak parabolic Higgs bundle \((E^W, \phi)\) on \( \mathbb{P}^1 \). Associate \( \mathcal{V} \) to the characteristic polynomial of \((E^W, \phi)\), so we have a morphism

\[ \hat{h}_Q : \mu^{-1}(0) \to \mathcal{H}_P \]

Now we show that the image of \( \hat{h}_Q \) lies in \( \mathcal{H}_P \). For \( g^i_1 \circ f^i_1 \), we consider the following diagram of linear maps in \( \mathcal{V} \):

\[
\begin{array}{cccccccc}
\mathbb{C}^r & \xrightarrow{f^i_1} & \mathbb{C}^{\gamma^i_1} & \xrightarrow{f^i_2} & \cdots & \xrightarrow{f^i_{j-1}} & \mathbb{C}^{\gamma^i_j} & \xrightarrow{f^i_{j+1}} & \cdots & \xrightarrow{f^i_s} & \mathbb{C}^{\gamma^i_s} \\
\downarrow{g^i_1} & & \downarrow{g^i_2} & & \cdots & & \downarrow{g^i_j} & & \cdots & & \downarrow{g^i_s} \\
\mathbb{C}^r & & \mathbb{C}^{\gamma^i_1} & & \cdots & & \mathbb{C}^{\gamma^i_j} & & \cdots & & \mathbb{C}^{\gamma^i_s} \\
\end{array}
\]

\( \mathcal{V} \in \mu^{-1}(0) \) implies \( g^i_j \circ f^i_j = f^i_{j+1} \circ g^i_{j+1} \) and \( g^i_j \circ f^i_j = 0 \).

Let \( \bar{F}^{i,j}(k^{\mathbb{C}^r}) = \text{Im}(g^i_1 \circ \cdots \circ g^i_j) \), then \( \bar{F}^{i,*} \) forms a filtration on \( k^{\mathbb{C}^r} \). For any \( v \in \bar{F}^{i,j}(k^{\mathbb{C}^r}) \), we have \( x \in k^{\mathbb{C}^r} \), so that \( v = g^i_1 \circ \cdots \circ g^i_j(x) \), then

\[
g^i_1 \circ f^i_1(v) = g^i_1 \circ f^i_1(g^i_1 \circ \cdots \circ g^i_j(x)) = g^i_1 \circ (f^i_1 \circ g^i_1) \circ g^i_2 \circ \cdots \circ g^i_j(x) = g^i_1 \circ (g^i_2 \circ f^i_2) \circ g^i_3 \circ \cdots \circ g^i_j(x) = \cdots = g^i_1 \circ \cdots \circ g^i_s \circ g^i_{j+1} \circ f^i_{j+1}(x) \in \bar{F}^{i,j+1}(k^{\mathbb{C}^r})
\]

So \( g^i_1 \circ f^i_1 \) preserves the filtration \( \bar{F}^{i,*} \) strongly. Notice that \( \dim \bar{F}^{i,j}(k^{\mathbb{C}^r}) \leq \gamma^i_j \) then by similar arguments as in the proof of Theorem 3.4 in \([22]\), we see that \( \hat{h}_Q \in \mathcal{H}_P \). Clearly \( \hat{h}_Q \) is \( GL(\mathcal{V}) \) equivalent and \( h_Q \) is the descendent of it. \( \square \)
Corollary 4.11. The parabolic Hitchin map $h_Q$ factors through $\mathcal{M}_0(v)$:

$$h_Q : \mathcal{M}_X(v) \xrightarrow{\pi} \mathcal{M}_0(v) \xrightarrow{h_Q^0} H_P$$

where $h_Q^0$ is the descendent of $\hat{h}_Q$.

4.1. Algebraically completely integrable system. In this subsection we are going to prove that, under certain condition, the parabolic Hitchin map $h_Q$ on $\mathcal{M}_X(v)$ is an algebraically completely integrable system, in the sense of Hitchin in [10].

Firstly we need to introduce Poisson variety.

Definition 4.12. A Poisson structure on a variety $X$ is a $k$-bilinear morphism on the structure sheaf $\mathcal{O}_X$:

$$\{,\} : \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X$$

(which is called Poisson bracket) satisfying the following conditions:

1. Skew-symmetry: $\{f, g\} = -\{g, f\}$;
2. Jacobi identity: $\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$;
3. Leibniz property: $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

We say that $(X, \{,\})$, or simply $X$ is a Poisson variety, if the Poisson structure is given. For any $f, g \in \mathcal{O}_X$, if $\{f, g\} = 0$, we say that $f$ and $g$ are Poisson commutative. Consider a morphism between two Poisson varieties: $\varphi : X \rightarrow Y$, if for any $f, g \in \mathcal{O}_Y$, we have $\varphi^*\{f, g\} = \{\varphi^*f, \varphi^*g\}$, then we say $\varphi$ is a Poisson morphism. For $f \in \mathcal{O}_X$, we define $X_f = \{f, -\}$, which is a vector field on $X$.

An important example of Poisson variety is symplectic variety. Let $X$ be a smooth symplectic variety, i.e. there is a non-degenerated closed two form $\omega \in \Omega^2(X)$. Then for any $f \in \mathcal{O}_X$, there is a unique vector field $X_f$ so that $\omega(-, X_f) = df$. We can define a Poisson bracket on $X$ by $\{f, g\} = \omega(X_g, X_f)$, which makes $X$ a Poisson variety.

Example 4.13. Let $V$ be a finite dimensional vector space, and $V^*$ be its dual space. Then $V \oplus V^* = T^*V$ has a Poisson structure defined as follows:

$$\{f, g\} = \sum_i \left( \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

where $p$ is a coordinate of $V$ and $q$ is the dual coordinate.

Let us go further with this example. Assume $V_1, V_2$ are two finite dimensional vector spaces, and then consider the space $V = \text{Hom}(V_1, V_2)$. Then $V^*$ can be realised as $\text{Hom}(V_2, V_1)$. By the above example, we have a Poisson structure on $V \oplus V^*$. Now consider the following functions:

$$\alpha_t = \text{Tr}(\wedge^t) : V \oplus V^* \rightarrow \mathbb{C} \quad (f, g) \mapsto \text{Tr}(\wedge^t (f \circ g))$$

and we have:

Proposition 4.14. $\{\alpha_t\}$ are pairwise Poisson commutative.

Proof. We define functions $\beta_t(f \circ g) = \text{Tr}((f \circ g)^t)$, then to prove the proposition is equivalent to prove that $\{\beta_t\}$ are pairwise Poisson commutative. We fix basis for $V_1$ and $V_2$ so that $f = (f_{ij}), g = (g_{ij})$, then $f \circ g = (\sum_k f_{ik}g_{kj}) := (h_{ij})$ and

$$\beta_t(f \circ g) = \sum_{i_1, \ldots, i_t} h_{i_1i_2}h_{i_2i_3} \cdots h_{i_ti}$$

Therefore

$$\{\beta_t, \beta_{t'}\} = \sum_{a, i_1, \ldots, i_t, b, j_1, \ldots, j_{t'}} h_{i_1i_2} \cdots h_{aia+a+1} \cdots h_{i_iti}h_{j_1j_2} \cdots h_{jbj+b+1} \cdots h_{j_{t'}j_{t'}i_1} \{h_{aia+1}, h_{jbj+b+1}\}$$
A simple calculation tells \( \{h_ia_{a+1}, h_jb_{j+1}\} = \delta_{a+1,j}h_{ia}h_{ja} - \delta_{ia,j+1}h_{ib}h_{ja+1} \). So
\[
\{\beta_t, \beta_{t'}\} = \sum_{a,b} \left( \text{Tr}((f \circ g)^{t-1}f \circ g(f \circ g)^{t'-1}) - \text{Tr}((f \circ g)^{t'-1}f \circ g(f \circ g)^{t-1}) \right) = 0
\]
So \( \{\beta_t\} \) are pairwise Poisson commutative.

Another kind of examples of Poisson varieties are quiver varieties. Notice that \( \text{Rep}(\hat{Q}, v) \cong T^* R \), so there is a Poisson structure on \( \text{Rep}(\hat{Q}, v) \). Actually this Poisson structure induces a Poisson map on \( M_\chi(v) \):

**Proposition 4.15** ([12] Theorem 9.53). For any character \( \chi \), the quiver variety \( M_\chi(v) \) has a Poisson structure. Moreover, the natural morphism \( \pi : M_\chi(v) \to M_0(v) \) is a Poisson map.

Now we state the definition of algebraically completely integrable system in [10].

**Definition 4.16.** Let \( X \) be a Poisson variety, and assume that its dimension is \( 2N \). If there are \( N \) algebraically independent, Poisson commutative functions \( f_i \in \Gamma(X, \mathcal{O}_X) \), then we say \( (X, \{f_i\}) \) is a completely integrable system. Moreover, we can consider the morphism \( F : X \to \mathbb{A}^N \) given by \( \{f_i\} \). If the generic fibre of \( F \) is an open subvariety of certain abelian variety, and the vector fields \( \{X_{f_i}\} \) are linear over generic fibres, then we say that \( (X, \{f_i\}) \), or equivalently, the morphism \( F \) is an algebraically completely integrable system.

Firstly we show that the components of \( \tilde{h}_Q : \mu^{-1}(0) \to \mathbf{H}_P \) is Poisson commutative. The proof in this part we follow [7].

Let \( V \in \mu^{-1}(0) \) be a representation, as before, we have \( n \) linear maps \( \phi^m = g^m_i \circ f^m_i \) and the value of \( \tilde{h}_Q \) on \( V \) is the trace of wedges of the matrix
\[
\phi_V := \sum_m \frac{\phi^m dz}{z - x_m} = \phi_V(z) dz
\]
Now let us consider \( \text{Tr}(\phi^m_V) = I_t(z)(dz)^t \) (please do not mix up the symbol \( \phi^m \) and the power \( \phi_V \)). If we regard \( I_t(z) \) as meromorphic function in \( z \), then its coordinates can be viewed as coordinates of \( H_P \). As in the proof of Proposition 4.14, we shall prove that for any \( t \) and \( t' \), the coordinates of \( I_t(z) \) and \( I_{t'}(z) \) are pairwise Poisson commutative.

Now we extend the Poisson structure on the coordinate ring \( \mathbb{C}[\mu^{-1}(0)] \) trivially to the formal power series ring
\[
\mathbb{C}[\mu^{-1}(0)][[z, w]]
\]
i.e. the Poisson bracket of \( z \) or \( w \) with any elements are zero. So we only need to prove that
\[
\{I_t(z), I_{t'}(w)\} = 0
\]
We define the following matrix-valued power series
\[
\Delta(z, w) = \frac{\phi_V(z) - \phi_V(w)}{w - z}
\]
then we have

**Lemma 4.17.** Let \( \phi_V(z) = (\phi_{ij}(z)) \), \( \phi^m = ((\phi^m)_{ij}) \) and \( \Delta(z, w) = (\Delta_{ij}(z, w)) \), then we have:
\[
\{\phi_{ij}(z), \phi_{kl}(w)\} = \delta_{jk} \Delta_{il}(z, w) - \delta_{li} \Delta_{kj}(z, w)
\]

**Proof.**
\[
\{\phi_{ij}(z), \phi_{kl}(w)\} = \sum_{m, m'} \left( \frac{((\phi^m)_{ij}, (\phi^m)_{kl})}{z - x_m)(w - x_{m'})} \right) = \sum_{m, m'} \frac{\delta_{mm'} \delta_{jk}(\phi^m)_{il} - \delta_{mm} \delta_{li}(\phi^m)_{kj}}{(z - x_m)(w - x_{m'})}.
\]
variety of the normalization of spectral curve $C$ is a surjective morphism. For generic of parabolic Higgs bundles on $P$ where the definition of parabolic type $\Sigma$ is generic, equivalently, the choice of we need to study its generic fibres. From now on, we assume the choice of weights in the

Assume that Condition (4.1) holds and consider the moduli space $\mathcal{M}$. Theorem 4.19.

need a condition on the parabolic type:

From the first row to the second row, we use similar argument in Proposition 4.14 from the third row to the forth row, we use the fact

$$\frac{1}{(z-x_m)(w-x_m)} = \frac{1}{w-z}(\frac{1}{z-x_m} - \frac{1}{w-x_m}).$$

Proposition 4.18. $\{I_t(z), I_t(w)\} = 0$. Thus the components of maps $\tilde{h}_Q : \mu^{-1}(0) \rightarrow H_P$, $h_Q : \mathcal{M}_\chi(v) \rightarrow H_P$ and $h_Q^0 : \mathcal{M}_0(v) \rightarrow H_P$ are Poisson commutative.

Proof. Firstly we notice that

$$I_t(z) = \sum_{i_1, \ldots, i_t} \phi_{i_1} \phi_{i_2} \cdots \phi_{i_t}$$

then we have

$$\{I_t(z), I_t(w)\} = \sum_{a, i_1, \ldots, i_t} \phi_{i_1} \cdots \phi_{i_a} \phi_{i_a+1} \cdots \phi_{i_t} \times \{\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_t}\}$$

$$= \sum_{a, i_1, \ldots, i_t} \phi_{i_1} \cdots \phi_{i_a} \phi_{i_a+1} \cdots \phi_{i_t} \times \delta_{i_1+i_a} \Delta_{i_a+1}(z, w) - \delta_{i_1+i_a} \Delta_{i_a+1}(z, w)$$

$$= \sum_{a, b} (\text{Tr}(\phi(z)^{t-1} \phi(w)^{t-1} \Delta(z, w)) - \text{Tr}(\phi(z)^{t'-1} \phi(w)^{t'-1} \Delta(z, w)))$$

$$= t \text{Tr}([\phi(z)^{t-1}, \phi(w)^{t'-1}] \Delta(z, w))$$

$\Delta(z, w)$ can be wrote into sum to two terms: one commutes with $\phi(z)^{t-1}$ and another commutes with $\phi(w)^{t'-1}$. So the final result of above calculation is 0.

In order to prove that $h_Q : \mathcal{M}_\chi(v) \rightarrow H_P$ is an algebraically completely integrable system, we need to study its generic fibres. From now on, we assume the choice of weights in the parabolic type $\Sigma$ is generic, equivalently, the choice of $\chi$ in $\mathcal{M}_\chi(v)$ is generic. By Remark 2.14 we see that in order to make sure Theorem 2.13 holds on $P^1$, we shall find at least one $\alpha \in H_P$, so that the corresponding spectral curve is integral. By arguments in Section 3 we need a condition on the parabolic type:

$$-2r + \sum_{i=1}^n (r - \varepsilon_r(x_i)) \geq 0 \quad (4.1)$$

where the definition of $\varepsilon_r(x_i)$ can be found in the definition of $H_P$.

Theorem 4.19. Assume that Condition (4.1) holds and consider the moduli space $\text{Higgs}_P$ of parabolic Higgs bundles on $P^1$ with rank $r$, degree 0, type $\Sigma$, if the choice of weights is generic, then the parabolic Hitchin map

$$h_P : \text{Higgs}_P \rightarrow H_P$$

is a surjective morphism. For generic $\alpha \in H_P$, the fibre $h_P^{-1}(\alpha)$ is isomorphic to the Picard variety of the normalization of spectral curve $C_\alpha$, and the dimension of $h_P^{-1}(\alpha)$ is equal to

$$\dim H_P = \frac{1}{2} \dim \text{Higgs}_P.$$

What we consider is the morphism $h_P^0 : \text{Higgs}_P^0 \rightarrow H_P$. For generic $\alpha \in H_P$, let $\tilde{C}_\alpha$ be the normalization of spectral curve $C_\alpha$, and $\tilde{\pi}_\alpha : \tilde{C}_\alpha \rightarrow P^1$ be the projection to $P^1$, thus

$$(h_P^0)^{-1}(\alpha) = \{\text{line bundles } \mathcal{L} \text{ on } \tilde{C}_\alpha \text{ so that } \tilde{\pi}_\alpha \mathcal{L} \cong \mathcal{O}_{P^1}^{\oplus r}\}$$
Proposition 4.23. Assume that Condition \([4.1]\) holds and the choice of weights are generic, then the maps \(h_Q : \mathfrak{M}_\chi(v) \to \mathbf{H}_P\) and \(h_P^\circ : \text{Higgs}_P^\circ \to \mathbf{H}_P\) are algebraically completely integrable systems.

Proof. We examine the definition of algebraically completely integrable systems one by one. Assume that the map \(h_Q\) is given by functions \(f_i \in \Gamma(\mathfrak{M}_\chi(v), \mathcal{O}_{\mathfrak{M}_\chi(v)}), 1 \leq i \leq \dim \mathbf{H}_P\).

Firstly, the generic fibre of \(h_Q\) has dimension \(\frac{1}{2} \dim \mathfrak{M}_\chi(v) = \dim \mathbf{H}_P\), thus \(\{f_i\}\) are algebraically independent.

Secondly, we already proved that \(\{f_i\}\) are pairwise Poisson commutative in Proposition \([4.18]\) and by the argument after Theorem \([4.19]\) the generic fibre of \(h_Q\) is open set of an abelian variety.

Lastly, in order to prove that vector fields \(X_{\bar{f}_i}\) are linear over generic fibre, we go back to the map \(h_P^\circ : \text{Higgs}_P^\circ \to \mathbf{H}_P\), which is a restriction of the map \(h_P : \text{Higgs}_P \to \mathbf{H}_P\). We assume that \(h_P\) is given by functions \(\{f_i\}\) and \(f_i\) is the restriction of \(\bar{f}_i\). Notice that by \([3]\), \(\text{Higgs}_P\) is a symplectic variety and its symplectic structure is compatible with the symplectic structure on \(\text{Higgs}_P^\circ\) we described in Remark \([4.9]\). Now, the generic fibres of \(h_P\) are abelian varieties, and the restriction of \(X_{\bar{f}_i}\) on generic fibres are linear, hence the restriction of \(X_{\bar{f}_i}\) on the generic fibres of \(h_P^\circ\) are linear.

\[Q.E.D.\]

In the following we want to prove that under certain condition, the maps \(h_Q^0 : \mathfrak{M}_0(v) \to \mathbf{H}_P\) is a completely integrable system. We already know that the components of \(h_Q^0\) are Poisson commutative, thus this can be done if we can show that the natural morphism \(\pi : \mathfrak{M}_\chi(v) \to \mathfrak{M}_0(v)\) is a birational morphism. For which we need the following assumption on the dimension vector \(v\):

\[
r - \gamma_i^j \geq \cdots \geq \gamma_j \geq \gamma_j^0 - \gamma_j^{j-1} \geq \gamma_j^0 > 0 \quad (4.2)
\]

If we assume condition \([4.1]\) and condition \([4.2]\) both holds, we can choose \(\alpha \in \mathbf{H}_P\) so that the spectral curve \(C_\alpha\) is integral, then for any \((E, \phi) \in (h_P^\circ)^{-1}(\alpha)\), it corresponds a representation \(W \in \mu^{-1}(0)\). We now argue that \(W\) is actually a simple representation.

Notice that the stability of \(W\) and condition \([4.2]\) implies the following morphisms in \(W\):

\[
\mathbb{C}^r \to \mathbb{C}^\gamma \to \cdots \to \mathbb{C}^\gamma
\]

are surjections. So a sub-representation of \(W\) would give a parabolic sub-Higgs bundle of \((E, \phi)\), which is impossible since the spectral curve \(C_\alpha\) is integral.

Definition 4.21. Assume that we have a linearly reductive group \(G\) acting on an affine variety \(X\), a point \(x \in X\) is called regular if

(1) the orbit of \(x\) is closed;

(2) the stabilizer of \(x\) is the trivial subgroup of \(G\).

We denote the set of regular points in \(X\) (considering the action of \(G\)) as \(X^{\text{reg}}\), which is an open subvariety of \(X\) (possibly empty).

Proposition 4.22 (\[12\], Theorem 9.29). Assume that \(X\) is a smooth affine variety and a linearly reductive group \(G\) acts on \(X\). We use \((X//G)^{\text{reg}}\) to denote the image of regular points in \(X//G\). Then \((X//G)^{\text{reg}}\) is a smooth open subscheme of \(X//G\). For any character \(\chi\) of \(G\), the natural morphism \((X//\chi G)^{\text{reg}} \to (X//G)^{\text{reg}}\) is an isomorphism. As corollary, if \(X//\chi G\) is a smooth variety and \((X//G)^{\text{reg}}\) is nonempty, then the morphism \(X//\chi G \to X//G\) is a resolution of singularity.

Proposition 4.23. Let \(W \in \mu^{-1}(0)\) be a simple representation, then under the action of group \(G = GL(v)/\mathbb{G}_m\), \(W\) is a regular point.
Proof. By [12] Theorem 2.10, we know that the orbit of \( W \) is closed. Now we choose \( g \in G \) be an element in the stabilizer of \( W \), then \( \{ v \in W | gv = v \} \) is a sub-representation of \( W \). We choose the representative of \( g \in GL(v) \) properly, we can assume this sub-representation is not zero, so it must be \( W \) itself, which implies that the stabilizer is trivial. 

\[ \square \]

**Definition 4.24.** Let \( X \) be a Poisson variety. A symplectic resolution of \( X \) is a smooth symplectic variety \( \bar{X} \), together with a resolution of singularity: \( \pi : \bar{X} \to X \), so that \( \pi \) is also a Poisson map.

**Theorem 4.25.** Assume that condition (4.1) and condition (4.2) both hold and the choice of \( \chi \) is generic, then the morphism \( \pi : \mathfrak{M}_X(v) \to \mathfrak{M}_0(v) \) is a symplectic resolution. Moreover, the map \( h_Q^0 : \mathfrak{M}_0(v) \to H_P \) is a completely integrable system.

Proof. From Proposition 4.22 and 4.28 we know that the morphism \( \pi \) is a resolution of singularity. Remark 4.9 tells that there is a symplectic structure on \( \mathfrak{M}_X(v) \) compatible with the Poisson structure. Recall that \( \pi \) is a Poisson map by Proposition 4.15. So \( \pi \) is a symplectic resolution. Since \( \pi \) is a birational morphism, and the morphism \( h_Q^0 : \mathfrak{M}_X(v) \to H_P \) factor through \( h_Q^0 \), by argument before, \( h_Q^0 \) is a completely integrable system. 

\[ \square \]

**Remark 4.26.** From the Theorem 9.53 in [12] we know that there are Poisson structure on quiver varieties, but we only know that for few quivers, the corresponding quiver variety has a structure of completely integrable system. The result in [7] can be seen as there is a completely integrable system structure on the quiver variety of some special “star-shaped” quiver (Notice that the dimension vector they chosen is also special). In [5] and [19], Chalykh and Silantyev show that the quiver variety of loop quiver has a structure of completely integrable system.

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