LOG K-STABILITY OF GIT-STABLE DIVISORS ON FANO MANIFOLDS

CHUYU ZHOU

Abstract. For a given K-polystable Fano variety $X$ and a natural number $l$ such that 
$(X, lB)$ is log canonical for some $B \in |-|lK_X|$, we show that there exists a rational number 
$0 < c_1 < 1$ depending only on $X$ and $l$, such that $D \in |-|lK_X|$ is GIT-(semi/poly)stable under the action of $\text{Aut}(X)$ if and only if the pair $(X, \xi D)$ is K-(semi/poly)stable for some rational $0 < \epsilon < c_1$.

CONTENTS

1. Introduction 1
   Acknowledgement 2
2. Preliminaries 2
   2.1. K-stability of Q-Fano varieties 3
   2.2. Valuative criterion 3
   2.3. CM-line bundle 4
3. Q-Fano degenerations of a Q-Fano variety 5
4. Proof of the main result 7
References 8

1. INTRODUCTION

In the past few years, people have made tremendous progress on the construction of K-moduli space of Fano varieties, e.g. [Jia20, BLX22, Xu20, BX19, ABHLX20, XZ20, LXZ22]. For smoothable Fano varieties, via the help of analytic tools, e.g. [CDS15a, CDS15b, CDS15c, Tia15, TW20] etc., people could construct proper K-moduli for smoothable Fano varieties with Kähler-Einstein metrics, e.g. [LWX19, ADL19]. With properness in hand, moduli continuity method is now widely applied in the literature to construct explicit K-moduli spaces of some special kinds of Fano varieties, e.g. K-moduli for cubic 3-folds ([LX19]) and cubic 4-folds ([Liu22]). It is well known that GIT-stability and K-stability, although not the same, have close relationship via CM-line bundle. More precisely, the generalized Futaki invariant of an one parameter subgroup can be identified with the corresponding GIT-weight via the CM-line bundle on the base (up to a positive multiple), thus in many cases one can construct a morphism from a K-moduli space to a GIT-moduli space. The moduli continuity method aims to establish an isomorphism between these two spaces, which becomes a powerful way to confirm K-stability of explicit Fano varieties.

2010 Mathematics Subject Classification: 14J10, 14J45.
Keywords: Fano varieties, K-stability, GIT-stability, K-moduli.
In order to state the main theorem, we first fix some notation. Let $X$ be a $K$-polystable Fano variety and $l$ a positive natural number such that $(X, \frac{1}{l}B)$ is log canonical for some $B \in |-lK_X|$. As $\text{Aut}(X)$ is reductive (e.g. [ABHLX20, CDS15a, CDS15b, CDS15c, Tia15, Mat57]), it is natural to determine GIT-stability of elements in $|-lK_X|$ under the action of $\text{Aut}(X)$. For a rational number $0 < \epsilon < 1$, we denote by $M_{X,l,\epsilon}^{K}$ the Artin stack which parametrizes all $K$-semistable pairs of the form $(\tilde{Y}, \frac{\epsilon}{l} \tilde{D})$, where $(\tilde{Y}, \tilde{D})$ is the degeneration of $(X, |-lK_X|)$ (see Section 4). We also write $M_{GIT}^{X,l} := [|-lK_X|^{ss}/\text{Aut}(X)]$. Denote by $M_{X,l,\epsilon}^{K}$ (resp. $M_{X,l,\epsilon}^{GIT}$) the good moduli space of $M_{X,l,\epsilon}^{K}$ (resp. $M_{X,l,\epsilon}^{GIT}$). It is recognized that the GIT-stability of $D \in |-lK_X|$ is related to the K-stability of the log Fano pair $(X, \frac{\epsilon}{l}D)$ for small $\epsilon$. For example, in the case $X = \mathbb{P}^n$, the work [GMGS21] establishes such a correspondence between GIT-stability and K-stability for $n = 1, 2, 3$, and the work [ADL19, Theorem 1.4] establishes it for every $n$. In this paper, we apply moduli continuity method to prove the following theorem, which is conjectured in [GMGS21, Conjecture 1.3].

**Theorem 1.1.** Let $X$ be a $K$-polystable Fano variety and $l$ a positive integer such that $(X, \frac{1}{l}B)$ is log canonical for some $B \in |-lK_X|$. Then there exists a rational number $0 < c_1 < 1$ depending only on $X$ and $l$ such that the following two statements are equivalent.

1. $D \in |-lK_X|$ is GIT-(semi/poly)stable under the action of $\text{Aut}(X)$,
2. the pair $(X, \frac{\epsilon}{l}D)$ is K-(semi/poly)stable for any rational $0 < \epsilon < c_1$.

In general, there is a morphism $\phi_\epsilon : M_{X,l,\epsilon}^{K} \to M_{X,l}^{GIT}$ which admits an isomorphic descent $\phi_\epsilon' : M_{X,l,\epsilon}^{K} \to M_{X,l}^{GIT}$ for any rational $0 < \epsilon < c_1$.

**Remark 1.2.** The above result generalizes [ADL19, Theorem 1.4] to any $K$-polystable Fano variety.

The key point for the proof of the theorem is to confirm that, the $K$-semistable degeneration of $(X, \frac{\epsilon}{l}D)$ for small $\epsilon$ and $D \in |-lK_X|$ preserves the ambient space. This is reduced to the fact that, if a $K$-polystable $\mathbb{Q}$-Fano variety admits a $K$-semistable degeneration, then the degeneration is still $K$-polystable (see Theorem 3.5). Here we say that $Y$ is a $\mathbb{Q}$-Fano degeneration of $X$ if there is a $\mathbb{Q}$-Gorenstein flat family $X \to C$ over a smooth pointed curve $0 \in C$ such that $-K_{X/C}$ is a relatively ample $\mathbb{Q}$-Cartier divisor, $X_t \cong X$ for $t \neq 0$, and $X_0 \cong Y$ is a $\mathbb{Q}$-Fano variety. In fact, if the degeneration is obtained by a test configuration, then the fact is well known by [LWX21, Section 3] (see also Definition 2.2).

**Acknowledgement.** The author would like to thank Chen Jiang, Yuchen Liu, Ziquan Zhuang for helpful discussions and beneficial comments. The author is supported by grant European Research Council (ERC-804334).

## 2. Preliminaries

In this section, we provide necessary preliminaries. We work over complex number field $\mathbb{C}$. We say that $(X, \Delta)$ is a log pair if $X$ is a projective normal variety and $\Delta$ is an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say a log pair $(X, \Delta)$ is log Fano if it admits klt singularities and $-K_X - \Delta$ is ample. If $\Delta = 0$, we just say a log Fano pair $(X, \Delta)$ is a $\mathbb{Q}$-Fano variety. For the concepts of singularities in birational geometry such as klt singularities, we refer to [KM98, Kol13].
2.1. K-stability of $\mathbb{Q}$-Fano varieties. Let $(X, \Delta)$ be a log Fano pair of dimension $n$, we denote $L := -K_X - \Delta$, which is an ample $\mathbb{Q}$-line bundle.

**Definition 2.1.** We say that a triple $\pi : (\mathcal{X}, \Delta_{tc}; \mathcal{L}) \to \mathbb{A}^1$ is a test configuration of $(X, \Delta; L)$ if the following conditions are satisfied:

1. $\pi$ is a flat projective morphism from a normal variety $\mathcal{X}$ and $\Delta_{tc} \subset \mathcal{X}$ is a $\mathbb{Q}$-divisor flat over $\mathbb{A}^1$,
2. $\mathcal{L}$ is a relatively ample $\mathbb{Q}$-line bundle on $\mathcal{X}$ with a $\mathbb{C}^*$-action induced by the natural multiplication on $\mathbb{A}^1$,
3. $(\mathcal{X}^*, \Delta_{tc}^*; \mathcal{L}^*)$ is $\mathbb{C}^*$-equivariantly isomorphic to $(X \times \mathbb{C}^*, \Delta \times \mathbb{C}^*; L \times \mathbb{C}^*)$, where $\mathcal{X}^* := \mathcal{X} \setminus \mathcal{X}_0$.

The compactification of the test configuration is denoted by $(\bar{\mathcal{X}}, \bar{\Delta}_{tc}; \bar{\mathcal{L}}) \to \mathbb{P}^1$, which is obtained by gluing $(\mathcal{X}, \Delta_{tc})$ and $(X \times (\mathbb{P}^1 \setminus 0), \Delta \times (\mathbb{P}^1 \setminus 0))$ along $(X \times \mathbb{C}^*, \Delta \times \mathbb{C}^*)$.

**Definition 2.2.** Let $(\mathcal{X}, \Delta_{tc}; \mathcal{L}) \to \mathbb{A}^1$ be a test configuration of $(X, \Delta; L)$, then the generalized Futaki invariant of this test configuration is defined as follows:

$$\text{Fut}(\mathcal{X}, \Delta_{tc}; \mathcal{L}) := \frac{n\bar{\mathcal{L}}^{n+1}}{(n+1)(-K_X - \Delta)^n} + \frac{\bar{\mathcal{L}}^n(K_{\bar{\mathcal{X}}/\mathbb{P}^1} + \bar{\Delta}_{tc})}{(-K_X - \Delta)^n}.$$ 

We say that $(X, \Delta)$ is K-semistable if $\text{Fut}(\mathcal{X}, \Delta_{tc}; \mathcal{L}) \geq 0$ for every test configuration. We say that $(X, \Delta)$ is K-polystable if it is K-semistable and for any test configuration of $(X, \Delta; L)$ whose central fiber is K-semistable, we have an isomorphism between $(X, \Delta)$ and the central fiber.

2.2. Valuative criterion. Let $(X, \Delta)$ be a log Fano pair of dimension $n$. We say $E$ is a prime divisor over $X$ if there is a proper birational morphism from a normal variety $\phi : Y \to X$ such that $E$ is a prime divisor on $Y$. We define

$$A_{X,\Delta}(E) := \ord_E(K_Y - \phi^*(K_X + \Delta)) + 1,$$

$$S_{X,\Delta}(E) := \frac{1}{(-K_X - \Delta)^n} \int_0^\infty \text{vol}(-\phi^*(K_X + \Delta) - tE)dt.$$ 

**Definition 2.3.** The beta-invariant of a prime divisor $E$ over a log Fano pair $(X, \Delta)$ is defined as follows:

$$\beta_{X,\Delta}(E) := A_{X,\Delta}(E) - S_{X,\Delta}(E).$$

The delta invariant of a log Fano pair $(X, \Delta)$ is defined as follows:

$$\delta(X, \Delta) := \inf_E \frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)},$$

where $E$ runs through all prime divisors over $X$.

We have the following well-known theorem due to [Fuj19, Li17, FO18, BJ20].

**Theorem 2.4.** Let $(X, \Delta)$ be a log Fano pair of dimension $n$, then

1. $(X, \Delta)$ is K-semistable if and only if $\beta_{X,\Delta}(E) \geq 0$ for any prime divisor $E$ over $X$.
2. $(X, \Delta)$ is K-semistable if and only if $\delta(X, \Delta) \geq 1$.

---

1This is not the original definition of K-polystability, but equivalent to the original one by [LWX21].
2.3. CM-line bundle. Let \( \pi : (X, D; L) \to T \) be a flat family of projective normal varieties of dimension \( n \) over a normal base \( T \), where \( D \) is an effective \( \mathbb{Q} \)-divisor on \( X \) whose components are all flat over \( T \), and \( L \) is a relative ample \( \mathbb{Q} \)-line bundle. By the work of Mumford-Knudsen [KM76] there exist \( \mathbb{Q} \)-line bundles \( \lambda_i, i = 0, 1, \ldots, n + 1 \) and \( \tilde{\lambda}_i, i = 0, 1, \ldots, n \), on \( T \) such that we have the following expansions for all sufficiently large \( k \in \mathbb{N} \):

\[
\det \pi_*(L^k) = \lambda_{n+1}^{(k)} \otimes \lambda_n^{(k)} \otimes \ldots \otimes \lambda_1^{(k)} \otimes \lambda_0,
\]

\[
\det \pi_*(L^k|_D) = \tilde{\lambda}_n^{(k)} \otimes \tilde{\lambda}_{n-1}^{(k)} \otimes \ldots \otimes \tilde{\lambda}_0.
\]

By Riemann-Roch formula, cf [CP21 Appendix], we have

\[
c_1(\pi_* L^k) = \frac{\pi_*(L^{n+1})}{(n+1)!} k^{n+1} + \frac{\pi_*(-K_X/T L^n)}{2n!} k^n + \ldots,
\]

\[
c_1(\pi_* L^k|_D) = \frac{\pi_*(L^n D)}{n!} k^n + \ldots.
\]

From above formulas, it’s not hard to see

\[
\lambda_{n+1} = \pi_* (L^{n+1}), \lambda_n = \frac{n}{2} \pi_* (L^{n+1}) + \frac{1}{2} \pi_* (-K_X/T L^n) \quad \text{and} \quad \tilde{\lambda}_n = \pi_* (L^n D).
\]

By flatness of \( \pi \) and \( \pi_D \), we write

\[
h^0(X_t, kL_t) = a_0 k^n + a_1 k^{n-1} + o(k^{n-1}) \quad \text{and} \quad h^0(D_t, kL_t|_{D_t}) = \tilde{a}_0 k^{n-1} + o(k^{n-1}),
\]

which do not depend on the choice of \( t \in T \). Then we have

\[
a_0 = \frac{L_t^n}{n!}, a_1 = -\frac{K_X L_t^{n-1}}{2(n-1)!} \quad \text{and} \quad \tilde{a}_0 = \frac{L_t^{n-1} D_t}{(n-1)!}.
\]

**Definition 2.5.** We define the CM-line bundles for the family \( \pi : (X, D; L) \to T \) as follows:

\[
\lambda_{CM,(X,L;\pi)} := \lambda_{n+1}^{2\alpha_0/n+1+n(n+1)} \otimes \lambda_n^{-2(n+1)},
\]

\[
\tilde{\lambda}_{CM,(X,D;\pi)} := \lambda_{n+1}^{2\alpha_0/n+1+n(n+1)} \otimes \tilde{\lambda}_n^{-2(n+1)} \otimes \tilde{\lambda}_n^{n+1}.
\]

Now we assume \( \pi : (X, D; L) \to \mathbb{P}^1 \) to be a compactification test configuration of a log pair \((X, D; L)\) where \( L \) is an ample \( \mathbb{Q} \)-line bundle on \( X \). As \( \pi_* L^k \) is a \( \mathbb{C}^* \)-equivariant vector bundle on \( \mathbb{P}^1 \), we write the total weights to be:

\[
w(\det(\pi_* L^k)) = b_0 k^{n+1} + b_1 k^n + o(k^n) \quad \text{and} \quad w(\det(\pi_* L^k|_D)) = \tilde{b}_0 k^n + o(k^n).
\]

It is not hard to compute that

1. \( b_0 = \frac{L_t^{n+1}}{(n+1)!} = \frac{w(\lambda_{n+1})}{(n+1)!} \),
2. \( b_1 = -\frac{K_X L_t^n}{2n!} = \frac{w(\lambda_n)}{n!} - \frac{n(n+1)}{2(n+1)!} w(\lambda_{n+1}) \),
3. \( \tilde{b}_0 = \frac{L_t^n D}{n!} = \frac{w(\lambda_n)}{n!} \).

**Definition 2.6.** Notation as above, we define

1. The generalized Futaki invariant of \((X, L)\):

\[
\text{Fut}(X, L) := \frac{2(b_0 a_1 - b_1 a_0)}{a_0^2} = \frac{1}{(n+1)L^n} w(\lambda_{n+1}^{2\alpha_0/n+1+n(n+1)} \otimes \lambda_n^{-2(n+1)}),
\]
Remark 2.7. We have a few remarks for the above definition.

1. If \( \pi : (X, D; L) \to T \) is a compactification test configuration of a log Fano pair, then the generalized Futaki invariants here coincide with the definition in Section 2.1.

2. In the case of test configurations, we see that the generalized Futaki invariants coincide with the GIT-weights of CM-line bundles up to a multiple. We also note here that CM-line bundles are in fact CM-line bundles on the base.

3. Suppose \( \pi : (X, D; L) \to T \) is a flat family of log Fano varieties such that \( L \sim_{\mathbb{Q}} -K_{X/T} - D \), then

\[
\lambda_{CM,(X,D;L)} = -\pi_*(L^{n+1}) \sim_{\mathbb{Q}} -\pi_*(-K_{X/T} - D)^{n+1}.
\]

Example 2.8. Let \( X \) be a smooth Fano manifold of dimension \( n \) and \( l \) a sufficiently divisible natural number. We denote \( \mathbb{P}^N := |-lK_X| \) and \( D \subset X \times \mathbb{P}^N \) the universal divisor corresponding to the linear system \( |-lK_X| \). For a rational number \( 0 < \epsilon < 1 \), we compute the CM-line bundle for the family \( \pi : (X \times \mathbb{P}^N, \mathbb{P}^N; L) \to \mathbb{P}^N \), where \( L \sim_{\mathbb{Q}} -K_{X \times \mathbb{P}^N} - \pi^*D \).

As the base is of Picard number 1, it suffices to consider a base change for a general rational curve \( \mathbb{P}^1 \hookrightarrow \mathbb{P}^N \), thus we get a family \( (X \times \mathbb{P}^1, \mathbb{P}^1; L_{\mathbb{P}^1}) \to \mathbb{P}^1 \), still denoted by \( \pi \) for convenience. By Remark 2.7 we have

\[
\lambda_{CM,(X \times \mathbb{P}^1, \mathbb{P}^1; L_{\mathbb{P}^1})} \sim_{\mathbb{Q}} -\pi_*(-K_{X \times \mathbb{P}^1} - \epsilon \mathbb{P}^1)^{n+1} \sim_{\mathbb{Q}} (n+1)(1-\epsilon)(1-\epsilon)^n (-K_X)^n \mathcal{O}_{\mathbb{P}^1}(1).
\]

Therefore \( \lambda_{CM,(X \times \mathbb{P}^N, \mathbb{P}^N; L; \pi)} \) is an ample \( \mathbb{Q} \)-line bundle on \( \mathbb{P}^N \). Thus the GIT-stability of \( D \in |-lK_X| \) under the action of \( \text{Aut}(X) \) with respect to \( \lambda_{CM,(X \times \mathbb{P}^N, \mathbb{P}^N; L; \pi)} \) is the same as that with respect to \( \mathcal{O}_{\mathbb{P}^N}(1) \).

3. \( \mathbb{Q} \)-Fano degenerations of a \( \mathbb{Q} \)-Fano variety

In this section, we fix \( X \) to be a \( \mathbb{Q} \)-Fano variety of dimension \( n \).

Definition 3.1. We say that a variety \( Y \) is a \( \mathbb{Q} \)-Fano degeneration of \( X \) if there is a \( \mathbb{Q} \)-Gorenstein flat family \( X \to C \) over a smooth pointed curve \( 0 \in C \) such that

1. \(-K_{X/C}\) is a relative ample \( \mathbb{Q} \)-line bundle,
2. for \( t \neq 0 \), \( X_t \cong X \),
3. \( X_0 \cong Y \) is a \( \mathbb{Q} \)-Fano variety.

Fix a rational number \( 0 < \epsilon_0 < 1 \), we consider the set \( F \) of \( \mathbb{Q} \)-Fano varieties such that \( Y \in F \) if and only if

1. \( Y \) is a \( \mathbb{Q} \)-Fano degeneration of \( X \),
2. the pair \( (Y, cD) \) is K-semistable for some rational \( 0 < c < 1 - \epsilon_0 \) and \( D \sim_{\mathbb{Q}} -K_Y \).
We have the following lemma.

**Lemma 3.2.** Notation as above, $\mathcal{F}$ is contained in a bounded family, and there is a rational number $0 < \eta < 1$ depending only on $X$ such that $Y$ is $K$-semistable once $\delta(Y) \geq \eta$.

**Proof.** For $Y \in \mathcal{F}$, by valuative criterion (see Section 2.2), we have $\frac{A_{Y,tD}(E)}{S_{Y,tD}(E)} \geq 1$ for any prime divisor $E$ over $Y$. Note that

$$S_{Y,tD}(E) = (1 - c)S_Y(E),$$

thus one sees the following

$$\frac{A_Y(E)}{S_Y(E)} \geq 1 - c > \epsilon_0,$$

which implies that $\delta(Y) \geq \epsilon_0$. As $\text{vol}(-K_Y) = \text{vol}(-K_X)$, combining the work [Jia20], we know that $\mathcal{F}$ is contained in a bounded family. By [BLX22] and [Xu20], the set

$$\{\min\{\delta(Y), 1\} | Y \in \mathcal{F}\}$$

is finite. Then there exists a rational number $0 < \eta < 1$ such that $Y$ is $K$-semistable if $\delta(Y) \geq \eta$ for any $Y \in \mathcal{F}$. The proof is finished. \qed

We are ready to prove the following theorem, which is related to [ADL19] Theorem 1.2.

**Theorem 3.3.** Notation as above, assume $Y \in \mathcal{F}$, then there exists a rational number $0 < t_1 < 1$ depending only on $X$ such that if $(Y, tD)$ is $K$-semistable for some rational $t \in (0, t_1)$ and $D \sim_Q -K_Y$, then $Y$ is $K$-semistable.

**Proof.** Choose a positive rational number $t_1 < \min\{1 - \epsilon_0, 1 - \eta\}$, we show that it satisfies our requirement. By Lemma 3.2 it suffices to show that $\delta(Y) \geq \eta$. To see this, let $E$ be any prime divisor over $Y$, we have the following computation:

$$\frac{A_Y(E)}{S_Y(E)} \geq \frac{A_{Y,tD}(E)}{S_Y(E)} = (1 - t) \frac{A_{Y,tD}(E)}{S_{Y,tD}(E)} \geq 1 - t \geq \eta.$$

Thus we obtain $\delta(Y) \geq \eta$ by valuative criterion. \qed

**Remark 3.4.** If we assume $X$ is smoothable, then $\text{vol}(-K_X)$ automatically admits a positive lower bound which does not depend on $X$, thus $t_1$ only depends on the dimension of $X$.

**Theorem 3.5.** Suppose $X$ is a $K$-polystable Fano variety and $Y$ is a $Q$-Fano degeneration of $X$. If $Y$ is $K$-semistable, then $Y \cong X$.

**Proof.** Let $f : \mathcal{X} \to C$ be a $Q$-Gorenstein flat family over a smooth pointed curve $0_1 \in C$ which produces the degeneration from $X$ to $Y$. By [BX19] Theorem 1.1, we know there is a degeneration from $Y$ to $X$ via a special test configuration, denoted by $g : Y \to \mathbb{A}^1$. We denote the origin $0_2 \in \mathbb{A}^1$, then $\mathcal{Y}_{0_2} \cong X$. Choose a sufficiently large natural number $r$ such that $-rK_X$ and $-rK_Y$ are both very ample, and $\mathcal{X}$ (resp. $\mathcal{Y}$) can be embedded into $C \times \mathbb{P}^N$ (resp. $\mathbb{A}^1 \times \mathbb{P}^N$). Suppose $p(m) = \chi(-mK_X)$, we denote $\text{Hilb}$ the Hilbert scheme whose points parametrize closed sub-varieties in $\mathbb{P}^N$ with Hilbert polynomial $p(m)$. Then the morphism $f$ (resp. $g$) induces a morphism $C \to \text{Hilb}$ (resp. $\mathbb{A}^1 \to \text{Hilb}$), with $C \setminus 0_1$ and $0_2$ (resp. $\mathbb{A}^1 \setminus 0_2$ and $0_1$) being sent to $[X] \in \text{Hilb}$ (resp. $[Y] \in \text{Hilb}$). Thus we see $\text{PGL}(N + 1)[X] \subset \text{PGL}(N + 1)[Y]$ and $\text{PGL}(N + 1)[Y] \subset \text{PGL}(N + 1)[X]$. Suppose $Y$ is not isomorphic to $X$, then the containments are strict, and we directly have $\dim \text{PGL}(n + 1)[Y] < \dim \text{PGL}(n + 1)[X] < \dim \text{PGL}(n + 1)[Y]$, contradiction. \qed
4. Proof of the main result

In this section, we fix $X$ to be a $K$-polystable Fano variety of dimension $n$. Let $l$ be a positive natural number such that $(X, \frac{1}{l} B)$ is log canonical for some $B \in |−lK_X|$

**Definition 4.1.** We say that the log Fano pair $(Y, \frac{1}{l} \tilde{D})$ is a degeneration of $(X, \frac{1}{l} |−lK_X|)$ if there is a $\mathbb{Q}$-Gorenstein flat family $(\mathcal{X}, \frac{1}{l} \mathcal{D}) \to C$ over a smooth pointed curve $0 \in C$ such that

1. $−K_X − \frac{1}{l} \mathcal{D}$ is a relative ample $\mathbb{Q}$-line bundle,
2. for each $t \neq 0$, $\mathcal{X}_t \cong X$ and $\mathcal{D}_t \in |−lK_{\mathcal{X}_t}|$, 
3. $(\mathcal{X}_0, \mathcal{D}_0) \cong (Y, \tilde{D})$.

We denote $\mathcal{M}_{X,l,\epsilon}^K$ to be the Artin stack parametrizing all K-semistable log Fano pairs of the form $(Y, \frac{1}{l} \tilde{D})$, which is the degeneration of $(X, \frac{1}{l} |−lK_X|)$, and $\mathcal{M}_{X,l}^{GIT} := [|−lK_X|]^{ss}/\text{Aut}(X)$. By taking good moduli spaces, we denote $\mathcal{M}_{X,l,\epsilon}^K$ (resp. $\mathcal{M}_{X,l}^{GIT}$) to be the descent of $\mathcal{M}_{X,l,\epsilon}^K$ (resp. $\mathcal{M}_{X,l}^{GIT}$) whose closed points parametrize K-polystable (resp. GIT-polystable) elements. We first have the following lemma.

**Lemma 4.2.** Notation as above, for general $D \in |−lK_X|$, the pair $(X, \frac{1}{l} D)$ is K-semistable for any rational number $0 < \epsilon < 1$.

**Proof.** Since $(X, \frac{1}{l} B)$ is log canonical for some $B \in |−lK_X|$, we see that for general $D \in |−lK_X|$, the pair $(X, \frac{1}{l} D)$ is an lc Calabi-Yau pair, thus being K-semistable by [BHJ17 Corollary 9.4] or [Oda13 Theorem 1.5]. By the interpolation property of K-stability (e.g. [ADL19 Proposition 2.13]), we at once see that $(X, \frac{1}{l} D)$ is K-semistable for any general $D \in |−lK_X|$ and any rational $0 < \epsilon < 1$. In particular, the stack $\mathcal{M}_{X,l,\epsilon}^K$ is not empty. □

We are ready to prove the main result.

**Theorem 4.3.** (= Theorem 1.1) Let $X$ be a $K$-polystable Fano variety and $l$ a positive integer such that $(X, \frac{1}{l} B)$ is log canonical for some $B \in |−lK_X|$. Then there exists a rational number $0 < c_1 < 1$ depending only on $X$ and $l$ such that the following two statements are equivalent.

1. $D \in |−lK_X|$ is GIT-(semi/poly)stable under the action of Aut$(X)$,
2. the pair $(X, \frac{1}{l} D)$ is K-(semi/poly)stable for any rational $0 < \epsilon < c_1$.

In general, there is a morphism $\phi_\epsilon : \mathcal{M}_{X,l,\epsilon}^K \to \mathcal{M}_{X,l}^{GIT}$ which admits an isomorphic descent $\phi'_\epsilon : M_{X,l,\epsilon}^K \to M_{X,l}^{GIT}$ for any rational $0 < \epsilon < c_1$.

**Proof.** We first assume $0 < \epsilon \ll 1$. Suppose $(X, \frac{1}{l} D)$ is K-(semi/poly)stable for $D \in |−lK_X|$, then $D$ is naturally GIT-(semi/poly)stable with respect to an ample CM-line bundle by Example 2.8. For any K-semistable element $[(Y, \frac{1}{l} \tilde{D})] \in \mathcal{M}_{X,l,\epsilon}^K$, by Theorem 3.3 we know that $Y$ is K-semistable (note that we currently assume $0 < \epsilon \ll 1$). By Theorem 3.5 we have $Y \cong X$. Thus there is an open immersion $\phi_\epsilon : M_{X,l,\epsilon}^K \to \mathcal{M}_{X,l}^{GIT}$ (e.g. [BLX22, Xu20]). By taking good moduli spaces, we have an injective descent $\phi'_\epsilon : M_{X,l,\epsilon}^K \to M_{X,l}^{GIT}$. We aim to show that $\phi_\epsilon$ is an isomorphism.

To show that $\phi_\epsilon$ is an isomorphism, the key point is to show that the moduli space $M_{X,l,\epsilon}^K$ is proper, which is due to [ABHLX20, LXZ22]. Thus we see that $\phi'_\epsilon : M_{X,l,\epsilon}^K \to M_{X,l}^{GIT}$ is a proper injective morphism. By [Alp13 Proposition 6.4] we know $\phi_\epsilon$ is finite. We are
concluded by noting that $\phi_\epsilon$ is an open immersion. By now, we get the following equivalence for $D \in | - lK_X |$:

1. $D \in | - lK_X |$ is GIT-(semi/poly)stable under the action of $\text{Aut}(X)$,
2. the pair $(X, \mathcal{O}_X(D))$ is K-(semi/poly)stable for any rational $0 < \epsilon \ll 1$.

The existence of $c_1$ is given by [Zho21], where we show that $\mathcal{M}_{X,l,\epsilon}^K$ (resp. $\mathcal{M}_{X,l,\epsilon}^K$) does not change as we vary $\epsilon$ in $(0, c_1)$, and $c_1$ depends on $X$ and the coefficient data $l$. The proof is finished. □

References

[ABHLX20] Jarod Alper, Harold Blum, Daniel Halpern-Leistner, and Chenyang Xu, Reductivity of the automorphism group of K-polystable Fano varieties, Invent. Math. 222 (2020), no. 3, 995–1032. MR4169054
[ADL19] Kenneth Ascher, Kristin DeVleming, and Yuchen Liu, Wall crossing for k-moduli spaces of plane curves, 2019.
[Alp13] Jarod Alper, Good moduli spaces for Artin stacks, Ann. Inst. Fourier (Grenoble) 63 (2013), no. 6, 2349–2402. MR3237451
[BHJ17] Sébastien Boucksom, Tomoyuki Hisamoto, and Mattias Jonsson, Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs, Ann. Inst. Fourier (Grenoble) 67 (2017), no. 2, 743–841. MR3669511
[BJ20] Harold Blum and Mattias Jonsson, Thresholds, valuations, and K-stability, Adv. Math. 365 (2020), 107062, 57. MR4067358
[BLX22] Harold Blum, Yuchen Liu, and Chenyang Xu, Openness of K-semistability for Fano varieties, Duke Math. J. 171 (2022), no. 13, 2753–2797. MR4505846
[BX19] Harold Blum and Chenyang Xu, Uniqueness of K-polystable degenerations of Fano varieties, Ann. of Math. (2) 190 (2019), no. 2, 609–656. MR3997130
[CDS15a] Xiuxiong Chen, Simon Donaldson, and Song Sun, Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities, J. Amer. Math. Soc. 28 (2015), no. 1, 183–197. MR3264766
[CDS15b] , Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than $2\pi$, J. Amer. Math. Soc. 28 (2015), no. 1, 199–234. MR3264767
[CDS15c] , Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches $2\pi$ and completion of the main proof, J. Amer. Math. Soc. 28 (2015), no. 1, 235–278. MR3264768
[CP21] Giulio Codogni and Zsolt Patakfalvi, Positivity of the CM line bundle for families of K-stable klt Fano varieties, Invent. Math. 223 (2021), no. 3, 811–894. MR4213768
[FO18] Kento Fujita and Yuji Odaka, On the K-stability of Fano varieties and anticanonical divisors, Tohoku Math. J. (2) 70 (2018), no. 4, 511–521. MR3896135
[Fuj19] Kento Fujita, A valuative criterion for uniform K-stability of Q-Fano varieties, J. Reine Angew. Math. 751 (2019), 309–338.
[GMGS21] Patricio Gallardo, Jesus Martinez-Garcia, and Cristiano Spotti, Applications of the moduli continuity method to log K-stable pairs, J. Lond. Math. Soc. (2) 103 (2021), no. 2, 729–759. MR4230917
[Jia20] Chen Jiang, Boundedness of Q-Fano varieties with degrees and alpha-invariants bounded from below, Ann. Sci. Éc. Norm. Supér. (4) 53 (2020), no. 5, 1235–1248. MR4174851
[KM76] Finn Faye Knudsen and David Mumford, The projectivity of the moduli space of stable curves, I. Preliminaries on “det” and “Div”, Math. Scand. 39 (1976), no. 1, 19–55. MR375414
[KM98] János Kollár and Shigefumi Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR1658959
[Kol13] János Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács. MR3057950

[Li17] Chi Li, *K-semistability is equivariant volume minimization*, Duke Math. J. 166 (2017), no. 16, 3147–3218. MR3715806

[Liu22] Yuchen Liu, *K-stability of cubic fourfolds*, J. Reine Angew. Math. 786 (2022), 55–77. MR4434748

[LWX19] Chi Li, Xiaowei Wang, and Chenyang Xu, *On the proper moduli spaces of smoothable Kähler-Einstein Fano varieties*, Duke Math. J. 168 (2019), no. 8, 1387–1459. MR3959862

[LWX21] __________, *Algebraicity of the metric tangent cones and equivariant K-stability*, J. Amer. Math. Soc. 34 (2021), no. 4, 1175–1214. MR4301561

[LX19] Yuchen Liu and Chenyang Xu, *K-stability of cubic threefolds*, Duke Math. J. 168 (2019), no. 11, 2029–2073. MR3992032

[LXZ22] Yuchen Liu, Chenyang Xu, and Ziquan Zhuang, *Finite generation for valuations computing stability thresholds and applications to K-stability*, Ann. of Math. (2) 196 (2022), no. 2, 507–566. MR4445441

[Mat57] Yozô Matsushima, *Sur la structure du groupe d’homéomorphismes analytiques d’une certaine variété kählérienne*, Nagoya Math. J. 11 (1957), 145–150. MR94478

[Oda13] Yuji Odaka, *The GIT stability of polarized varieties via discrepancy*, Ann. of Math. (2) 177 (2013), no. 2, 645–661. MR3010808

[Tia15] Gang Tian, *K-stability and Kähler-Einstein metrics*, Comm. Pure Appl. Math. 68 (2015), no. 7, 1085–1156. MR3352459

[TW20] Gang Tian and Feng Wang, *On the existence of conic Kähler-Einstein metrics*, Adv. Math. 375 (2020), 107413, 42. MR4170229

[Xu20] Chenyang Xu, *A minimizing valuation is quasi-monomial*, Ann. of Math. (2) 191 (2020), no. 3, 1003–1030. MR4088355

[XZ20] Chenyang Xu and Ziquan Zhuang, *On positivity of the CM line bundle on K-moduli spaces*, Ann. of Math. (2) 192 (2020), no. 3, 1005–1068. MR4172625

[Zho21] Chuyu Zhou, *On wall crossing for K-stability*, arXiv, 2021.

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL), MA C3 615, STATION 8, 1015 LAUSANNE, SWITZERLAND

*Email address*: chuyu.zhou@epfl.ch