The Reinhardt Conjecture as an Optimal Control Problem

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Abstract

In 1934, Reinhardt conjectured that the shape of the centrally symmetric convex body in the plane whose densest lattice packing has the smallest density is a smoothed octagon. This conjecture is still open. We formulate the Reinhardt Conjecture as a problem in optimal control theory.

The smoothed octagon is a Pontryagin extremal trajectory with bang-bang control. More generally, the smoothed regular $6k + 2$-gon is a Pontryagin extremal with bang-bang control. The smoothed octagon is a strict (micro) local minimum to the optimal control problem.

The optimal solution to the Reinhardt problem is a trajectory without singular arcs. The extremal trajectories that do not meet the singular locus have bang-bang controls with finitely many switching times.

Finally, we reduce the Reinhardt problem to an optimization problem on a five-dimensional manifold. (Each point on the manifold is an initial condition for a potential Pontryagin extremal lifted trajectory.) We suggest that the Reinhardt conjecture might eventually be fully resolved through optimal control theory.

Some proofs are computer-assisted using a computer algebra system.

1 Introduction

In 1934, Reinhardt conjectured that the shape of centrally symmetric body in the plane whose densest lattice packing has the smallest density is a smoothed octagon (Figure 1). The corners of the octagon are rounded by hyperbolic arcs. For popular accounts of the Reinhardt conjecture, including some spectacular animated graphics by Greg Egan, see [BE14, Bae14].

This article is a continuation of an article from 2011, which formulates the Reinhardt conjecture as a problem in the calculus of variations [Hal11]. This article reformulates the Reinhardt conjecture as an optimal control problem.

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Bang-bang controls of an optimal control problem are controls that switch between extreme points of a convex control set (often with a finite number of switches). A major theme of optimal control is the study of bang-bang controls, and the extremal trajectories of many control problems have bang-bang controls. Intuitively, bang-bang controls switch from one extreme position to another: navigating a craft by flooring the accelerator pedal then slamming on the brakes; or steering a vehicle by making the sharpest possible turns to the left and to the right; or maximizing wealth by investing all resources in a single financial asset for a time, then suddenly moving all resources elsewhere.

The basic insight of this article, from which everything else follows, is that the smoothed octagon can be described by a bang-bang control with finitely many switches. The smoothed octagon exhibits the extreme behavior that is characteristic of a bang-bang control: each arc of the smoothed octagon is flattened out as much as possible (the straight edges) or is as highly curved as possible (the hyperbolic arcs), with finitely many switches between these extremes. Because of this bang-bang behavior, the natural context for the Reinhardt conjecture is optimal control theory. Viewed in this context, the Reinhardt conjecture is transformed from a puzzling problem in discrete geometry to a rather typical problem in optimal control. In fact in many ways, this is a textbook example of optimal control, by embodying significant aspects of the general theory in a single problem.

The original and guiding inspiration for this research was the visual similarity between the solutions to the Dubins car problem and segments of smoothed polygons (Figure 2). Recall that the Dubins car problem is the optimal control problem that asks for the shortest path in the plane from an initial position (and direction) to a terminal position and direction, subject to a given bound on the absolute value of the curvature at each point of the path. Roughly speaking, the Reinhardt problem is a modification of the Dubins problem that imposes hexagonal symmetry and a steering wheel that turns only to the left. In both cases, curvature constraints force the (conjectural) solution to consist of finitely many straight segments and arcs of maximal curvature. The relationship becomes more than a visual similarity when the Dubins problem is formulated as a left-invariant control problem on the group $SE(2)$ of orientation preserving
isometries of the plane or when extended to hyperbolic space \cite{Mit98, MP98}.

Figure 2: This research was motivated by the visual similarity between solutions to the Dubins car problem with its circular arcs (left) and smoothed polygons using hyperbolic arcs (right).

The main results of this article are Theorem 4.4.1 which asserts that the smoothed $6k + 2$-gon is given by a Pontryagin extremal trajectory; Theorem 4.5.1 which gives the strict local optimality of the smoothed octagon; and Theorem 5.4.2 which proves that extremal trajectories that avoid the singular locus have bang-bang controls with finitely many switches.

The hyperbolic plane plays an important role in this article. The connection with planar geometry comes through the group $\text{SL}_2(\mathbb{R})$, which acts on the plane by affine transformations and on the hyperbolic plane by isometries.

Many of the calculations are computer assisted, using Mathematica. The computer code (about 1000 lines of source) has been posted to our github repository (github.com/flyspeck). Many explicit formulas that are too long to print here can be found in the computer code that accompanies this article.

It is with some regret that I publish this article prematurely before completing a full solution to the Reinhardt conjecture. I have not encountered any obstacles to major further advances along these lines, and I believe that optimal control theory should eventually lead to a solution to the Reinhardt conjecture. The final section proposes possible end-games for this problem.

1.1 review of earlier results

We briefly review some of the main conclusions of \cite{Rei34} and \cite{Hal11}. A convex body in Euclidean space is a compact convex set with nonempty interior. A centrally symmetric convex body $D$ is a convex body such that $-D$ is a translate of $D$. In this article, a disk means a convex body in the plane and is not necessarily circular.

Reinhardt proved the existence of a convex centrally symmetric disk $D_{\text{min}}$ in the plane with the property that the density of its densest lattice packing minimizes the density of the densest lattice packing among all convex centrally symmetric disks in the plane. The Reinhardt problem is to determine the shape of $D_{\text{min}}$. Reinhardt conjectured that $D_{\text{min}}$ is a smoothed octagon.
The density is not changed by affine transformations of the plane. Thus if $D_{\text{min}}$ is a solution to the Reinhardt problem, then every affine transformation of $D_{\text{min}}$ is also a solution.

Reinhardt showed that $D_{\text{min}}$ has no corners; that is, every point on the boundary of $D_{\text{min}}$ has a unique tangent. He showed that the densest lattice packing is obtained by placing $D_{\text{min}}$ in a centrally symmetric hexagon of smallest area containing $D_{\text{min}}$, then tiling the plane with copies of the hexagon. Moreover, there is a centrally symmetric hexagon of the same minimal area passing through each point on the boundary of $D_{\text{min}}$. These hexagons never degenerate to a quadrilateral. By rescaling, we may assume without loss of generality that the centrally symmetric hexagons all have area $\sqrt{12}$ (the area of the circumscribing hexagon of circle of radius 1) and that the centrally symmetric disks are centered at the origin. This puts a structure on $D_{\text{min}}$ that we call a hexagonally symmetric disk. (This was called a hexameral domain in [Hal11].)

Let $e_0^*, e_1^*, \ldots, e_5^* \in \mathbb{R}^2$ be the vertices of a regular hexagon on a unit circle:

$$e_j^* = (\cos(2\pi j/6), \sin(2\pi j/6)).$$

Let $\text{SL}_2(\mathbb{R})$ be the group of $2 \times 2$ matrices with real coefficients with determinant 1, and let $\mathfrak{sl}_2(\mathbb{R})$ be its Lie algebra, consisting of all $2 \times 2$ matrices with real coefficients and trace 0. We write $t_f$ for the free terminal time (to be determined as part of the solution). After centering $D_{\text{min}}$ at the origin, there exists a continuously differentiable path $g : [0, t_f] \to \text{SL}_2(\mathbb{R})$ such that the boundary of $D_{\text{min}}$ is given by the six arcs

$$t \mapsto \sigma_j(t) := g(t)e_j^*, \quad t \in [0, t_f].$$

(1)

For each $t$, we may draw the tangents to the boundary of $D_{\text{min}}$ at the six points $g(t)e_j^*$, for $j = 0, \ldots, 5$. The six tangents form the six edges of an area-minimizing centrally symmetric hexagon as above. The six points are the midpoints of the six edges of the hexagon.

Remark 2. We work with the group $\text{SL}_2(\mathbb{R})$, but because of the central symmetry, we have $\sigma_{j+3} = -\sigma_j$, and we might pass to the quotient $\text{PSL}_2(\mathbb{R})$ if desired.

Each choice of path $g$ (subject to the convexity and endpoint constraints of Section 2.2) leads to a centrally symmetric disk that possesses such a family of area-minimizing centrally symmetric hexagons. We call a path $g$ in $\text{SL}_2(\mathbb{R})$ hexagonally symmetric (see Section 2.2) if it satisfies the convexity and endpoint conditions to define a convex centrally symmetric disk $D(g)$ in the plane, with boundary arcs $\mathbf{1}$. Each hexagonally symmetric disk $D$ (that is, a centrally symmetric convex disk in the plane, centered at the origin) has the form $D = D(g)$ for some $g : [0, t_f] \to \text{SL}_2(\mathbb{R})$. 


With fixed area $\sqrt{12}$ for each hexagon, the Reinhardt problem becomes equivalent to minimizing the area of a hexagonally-symmetric disk. This can be formulated as a problem in the calculus of variations, as was done in an earlier article, but the convexity constraints on the disk lead to some awkwardness. We turn to optimal control theory as a natural framework for the Reinhardt problem.

2 State

2.1 ODE

We consider the following control problem. Let

$$U = \{ (u_0, u_1, u_2) \in \mathbb{R}^3 \mid \sum_i u_i = 1, \ u_i \geq 0 \}$$

be the set of controls, a 2-simplex. We define an affine map

$$Z_0 : U \to \mathfrak{sl}_2(\mathbb{R})$$

as the inverse of the linear map $\mathfrak{sl}_2(\mathbb{R}) \to \mathbb{R}^3$, $Z \mapsto e^*_{2i} \wedge Ze^*_{2i}$; that is given $u \in \mathbb{R}^3$, by solving equations for $Z_0$:

$$e^*_{2i} \wedge Z_0(u)e^*_{2i} = u_i, \quad i = 0, 1, 2,$$

where $v \wedge v'$ is the $2 \times 2$ determinant with columns $v$ and $v'$. (This system of linear equations for $Z_0$ is nonsingular.) We refer to $Z_0(u)$ as the control matrix.

Let $g : [0, t_f] \to \text{SL}_2(\mathbb{R})$ be a $C^1$ path. We write

$$g' = gX,$$

with $X : [0, t_f] \to \mathfrak{sl}_2(\mathbb{R})$, (4)

We use a prime throughout the article to indicate the derivative with respect to $t$. We assume that

$$\det(X(t)) = 1,$$

for all $t \in [0, t_f]$. (5)

We assume that $X : [0, t_f] \to \mathfrak{sl}_2(\mathbb{R})$ is Lipschitz continuous and that

$$X' = X(\delta(u, X)Z_0(u) - X),$$

where

$$\delta = \delta(u, X) = -2/\text{trace}(Z_0(u)X).$$

As a rough guide our intuition, we can view the ODE as a Frenet-Serret type formula that determines a planar curve up to congruence by its
planar curvature. In our setting, the control \( u = (u_0, u_1, u_2) \) gives the planar curvatures of the various branches \( \sigma_{2i} \) of the hexagonally symmetric curve up to a normalization factor that has been included to make \( U \) a standard simplex. More precisely, the curvature is given as

\[
\kappa_{2i} = \left( \frac{dt}{ds_{2i}} \right)^3 \delta(u, X) u_i, \tag{8}
\]

where \( \kappa_{2i} \) is the curvature of the \( 2i \)-th curve \( t \mapsto g(t)e_{2i}^* \), and \( s_{2i} \) is its arclength parameter. The non-negativity conditions \( u_i \geq 0 \) are the local convexity conditions on the hexagonally symmetric curve.

Remark 9. Under natural disk constraints on \( X \) that can be made without loss of generality, we show in Section 2.4 that the denominator of Equation (7) is nonzero.

**Theorem 2.1.1.** Let \( g : [0, t_f] \to SL_2(\mathbb{R}) \) be related by (1) to the solution of the Reinhardt problem: \( D_{\min} = D(g) \). After a suitable reparametrization, the path \( g \) satisfies the equations (3)–(7) for some measurable control \( u : [0, t_f] \to U \).

**Proof.** We briefly indicate why an optimal solution to the Reinhardt problem gives a trajectory of this state equation. The path \( g \) is continuously differentiable. Define \( X : [0, t_f] \to \mathfrak{sl}_2(\mathbb{R}) \) by \( g' = gX \). By [Hal11, §3.3], \( X \) is Lipschitz continuous, so that \( X \) is differentiable almost everywhere. By [Hal11, §3.5], \( \det(X) > 0 \). The cost (that is, the area of a disk \( D \)) described in Section 2.3 is invariant under reparametrizations of the path \( g \). By appropriate choice of time parameter for the path \( g \), we may assume that \( g \) has unit speed in the sense that Equation (5) holds. Define a “curvature matrix” \( Z : [0, t_f] \to gl_2(\mathbb{R}) \)

\[
Z = X + X^{-1}X'. \tag{10}
\]

By Lemma 2.1.2 \( Z \) takes values in \( \mathfrak{sl}_2(\mathbb{R}) \). By [Hal11, Eqn.19], we have

\[
e^*_{2j} \wedge Z e^*_{2j} = \delta u_j \geq 0, \tag{11}
\]

for some \( \delta > 0 \) and some measurable \( u : [0, t_f] \to U \). Define \( Z_0 = Z_0(u) \) by Equation (3), so that \( Z = \delta Z_0 \). Solving Equation (10) for \( X' \), we obtain the differential equation (6). The scalar \( \delta \) is uniquely determined by the condition that \( \text{trace}(X') = 0 \):

\[
0 = \text{trace}(X') = \text{trace}(X \delta Z_0 - X^2) = \delta \text{trace}(X Z_0) + 2.
\]

\[\square\]

**Lemma 2.1.2.** If \( X : [0, t_f] \to \mathfrak{sl}_2(\mathbb{R}) \) is a Lipschitz path such that Equation (4) holds, then

\[
\text{trace}(X + X^{-1}X') = 0.
\]
Proof. The characteristic polynomial of $X$ is
\[ \lambda^2 - \text{trace}(X)\lambda + \text{det}(X) = \lambda^2 + 1. \]
By Cayley-Hamilton, $X^2 = -I$ and $X^{-1} = -X$. Differentiation gives $XX' + X'X = 0$. Then
\[ 0 = \text{trace}(-XX') = \text{trace}(X + X^{-1}X'). \]

2.2 Poincaré upper half-plane

If $X$ is any matrix, we write $c_{ij}(X)$ for the $ij$ matrix coefficient of $X$. In particular, we have linear functions $c_{ij} : \mathfrak{sl}_2(\mathbb{R}) \to \mathbb{R}$:
\[ X = \begin{pmatrix} c_{11}(X) & c_{12}(X) \\ c_{21}(X) & -c_{11}(X) \end{pmatrix}. \]
We say that $X \in \mathfrak{sl}_2(\mathbb{R})$ is positively oriented if $c_{21}(X) > 0$. By [Hal11 §3.5], a solution $X(t)$ to the Reinhardt problem has positive orientation for each $t$. (This corresponds to a counterclockwise traversal of the boundary of $D_{\text{min}}$.)

Set
\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}). \]
We have rotation matrices
\[ \exp(Jt) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \]

Lemma 2.2.1. The set of matrices $X \in \mathfrak{sl}_2(\mathbb{R})$ such that $\det(X) = 1$, trace($X$) = 0 and $c_{21}(X) > 0$ is the adjoint orbit of $J$.

Proof. Let $y = 1/c_{21}$ and $x = c_{11}/c_{21}$. Then $X$ has the form
\[ X = X(x, y) = \begin{pmatrix} x/y & -x^2/y - y \\ 1/y & -x/y \end{pmatrix} = \hat{z}J\hat{z}^{-1}, \quad \text{where} \quad \hat{z} = \hat{z}(x, y) = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}. \]

The centralizer of $J$ in $\text{SL}_2(\mathbb{R})$ is $\text{SO}_2(\mathbb{R})$. By the Iwasawa decomposition, the orbit of $J$ under $\text{SL}_2(\mathbb{R})$ is the same as the orbit under the upper triangular matrices $\hat{z}(x, y)$ with positive determinant. The result follows. (We remark that the condition $c_{21} > 0$ picks out a conjugacy class within the stable semisimple conjugacy class determined by the characteristic polynomial $\lambda^2 + 1$ of $X$.)

\[ 7 \]
The adjoint orbit of $J$ can be identified with the homogeneous space $SL_2(\mathbb{R})/SO_2(\mathbb{R})$, which can be identified with the upper half-plane $\mathfrak{h}$. This identification comes via $\hat{z} = z(x, y)$ as above:

$$\hat{z}J\hat{z}^{-1} \mapsto x + iy \in \mathfrak{h}, \quad y > 0, \quad i = \sqrt{-1}. \quad (13)$$

A calculation shows that the ODE in Equation (6) expressed in terms of coordinates $x, y$ is

$$x' = f_1(x, y; u) := \frac{y(b + 2ax - cx^2 + cy^2)}{b + 2ax - cx^2 - cy^2},$$

$$y' = f_2(x, y; u) := \frac{2(a - cx)y^2}{b + 2ax - cx^2 - cy^2}. \quad (14)$$

The dependence on the control $u = (u_0, u_1, u_2) \in U$ comes through the coefficients $a, b, c$. Specifically, $Z_0 = Z_0(u) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ is the control matrix where

$$a = c_{11}(Z_0) = \frac{u_1 - u_2}{\sqrt{3}},$$

$$b = c_{12}(Z_0) = \frac{u_0}{3} - \frac{2u_1}{3} - \frac{2u_2}{3},$$

$$c = c_{21}(Z_0) = u_0.$$

In summary, we have state equations:

$$x' = f_1(x, y; u),$$

$$y' = f_2(x, y; u),$$

$$g' = gX,$$

$$x : [0, t_f] \to \mathbb{R}; \quad y : [0, t_f] \to \mathbb{R}; \quad g : [0, t_f] \to SL_2(\mathbb{R});$$

where $X = \hat{z}(x, y)J\hat{z}(x, y)^{-1}$, and where $u : [0, t_f] \to U$ is a measurable control.

We define an admissible trajectory $(g, z) : [0, t_f] \to M := SL_2(\mathbb{R}) \times \mathfrak{h}$ to be a solution of this ODE for some measurable control $u$, with $z = x + iy$, and that satisfies the following additional conditions:

1. The image of $z$ lies in $\mathfrak{h}^* \subset \mathfrak{h}$ (see Section 2.4).
2. The endpoints of the trajectory are $(g(0), z(0)) = (I, z_0)$ and $(g(t_f), z(t_f)) = (R, R^{-1}.z_0)$, for some $z_0 \in \mathfrak{h}$, where $R := \exp(J\pi/3)$.
3. the path $g : [0, t_f] \to SL_2(\mathbb{R})$ is homotopic in $SL_2(\mathbb{R})$ to the path given by rotation:

$$t \mapsto \exp(J\pi t/(3t_f)), \quad t \in [0, t_f].$$
To each admissible trajectory we may associate a hexagonally symmetric disk $D(g, z)$. The second condition enforces that the union of the boundary arcs (Equation 1) of $D(g, z)$ is a closed curve with no corners. The third condition enforces the condition that the boundary arcs must define a simple closed curve traversed in the counterclockwise direction.

The path $g$ determines $z$ by the equations (4) and (12). Conversely $z$ determines $g$ by the same equations and the initial condition $g(0) = I$.

Thus, we sometimes abbreviate the admissible trajectory $(g, z)$ to $g$ or $z$, and write the corresponding hexagonally symmetric disk $D(g, z)$ as $D(g)$ or $D(z)$.

The condition $g(0) = I$ can be imposed without loss of generality. The group of affine transformations of the plane acts on the set of solutions to the Reinhardt problem. We have reduced the affine group of symmetries by fixing the center of $D_{	ext{min}}$ at the origin, and the fixing the area $\sqrt{12}$ of the hexagon tile. This leaves the group action of $\text{SL}_2(\mathbb{R})$ on the set of solutions to the Reinhardt problem, which we rigidify with the initial condition $g(0) = I$.

We call a link the full segment (between switching times) of a trajectory that has a constant control at a vertex of the simplex $U$:

\[ u \in \{e_1, e_2, e_3\} \subset U. \]

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1) \in U$. See Sections 4.1 and 4.2.

2.3 cost functional

The cost functional in [Hal11, §5.1] (correcting the formula there with a missing factor of 2) is

\[ g \mapsto \text{area}(D(g)) = -\frac{3}{2} \int_0^{t_f} \text{trace}(Jg^{-1}g')dt \to \text{min}. \tag{16} \]

The interpretation of the cost is the area of the hexagonally symmetric disk $D(g)$. Using $g' = gX$, the cost (16) depends only on $X$ and simplifies to

\[ -\frac{3}{2} \int_0^{t_f} \text{trace}(JX)dt \to \text{min}. \tag{17} \]

Assume now that $X$ has unit speed. Expressed in terms of coordinates $x + iy$ in the Poincaré upper half-plane, the cost takes the form

\[ \frac{3}{2} \int_0^{t_f} \frac{x^2 + y^2 + 1}{y} dt \to \text{min}. \tag{18} \]
This cost is rotationally symmetric with respect to the action of \( \text{SO}_2(\mathbb{R}) \leq \text{SL}_2(\mathbb{R}) \) on the upper half-plane. In fact, the level sets of \((x^2 + y^2 + 1)/y\) are concentric circles (centered at \(i\) in hyperbolic geometry). The cost satisfies

\[
(x^2 + y^2 + 1)/y \geq y + 1/y \geq 2,
\]

attaining its minimum at \(i = \sqrt{-1} \in \mathbb{H}\).

We may also express the cost in the Poincaré disk model \(\mathbb{D}\). Let

\[
\mathbb{D} = \{w \in \mathbb{C} \mid |w| < 1\}, \quad w = \frac{z - i}{z + i}, \quad z = \frac{-i(1 + w)}{-1 + w}, \quad z \in \mathbb{H}
\]

The cost of a path \(w : [0, t_f] \to \mathbb{D}\) in the Poincaré disk becomes

\[
3 \int_0^{t_f} \frac{1 + |w|^2}{1 - |w|^2} dt \to \min .
\]

In this model, the rotational symmetry about 0 is evident.

**Remark 21.** One model of hyperbolic geometry is the upper sheet of a hyperboloid of two sheets. We recognize \((1 + |w|^2)/(1 - |w|^2)\) as the height on the hyperboloid

\[
\{(u, v, h) \in \mathbb{R}^3 \mid h^2 = 1 + u^2 + v^2, \ h > 0\}.
\]

In more detail, we map a point \(w = (u, v, 0) \in \mathbb{R}^3\) in the unit disk \(\mathbb{D} \subset \mathbb{R}^3\) to the point \(p\) in the upper sheet whenever \(w, p,\) and \((0, 0, -1)\) are collinear (Figure 3). It is a curiosity that the area of a convex disk in the Euclidean plane in Reinhardt’s packing problem equals the integral of the height function in hyperbolic geometry.

**Figure 3:** The cost is the integral of the height \(h\) in the hyperboloid model of hyperbolic geometry. The line through \((0, -1)\) and \((|w|, 0)\) meets the hyperbola \(h^2 = 1 + x^2\) at a point \((x, h)\) with height \(h = (1 + |w|^2)/(1 - |w|^2)\).
2.4 star inequalities

We define the star inequalities on $\mathfrak{sl}_2(\mathbb{R})$ to be the following:

$$\sqrt{3} |c_{11}(X)| < c_{21}(X), \quad 3c_{12}(X) + c_{21}(X) < 0.$$  

(These conditions were obtained in [Hal11, §3.5] for the tangent $X$ at $t = 0$, but also hold for all $t$, by symmetry.) There is no loss of generality in imposing these conditions at each point of a trajectory; they are necessary conditions for the convexity of the corresponding hexagonally symmetric disk (Remark 23). Translating into the upper-half plane coordinates, the star inequalities define an open region:

$$\mathfrak{h}^* := \{(x, y) \in \mathfrak{h} \mid -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}, \quad \frac{1}{3} < x^2 + y^2\}. \quad (22)$$

The inequalities define the interior of an ideal hyperbolic triangle with vertices at $z = \pm 1/\sqrt{3}$ and $z = \infty$ on the boundary of $\mathfrak{h}$. We set $M^* = \text{SL}_2(\mathbb{R}) \times \mathfrak{h}^*$.

In disk coordinates, the star inequalities imply that $w$ lies in the interior of the ideal hyperbolic triangle with vertices $w = 1, \zeta, \zeta^2$ on the boundary of $\mathbb{D}$, where $\zeta = e^{2\pi i/3}$ (Figure 4).

![Figure 4](image-url)

Figure 4: The star inequalities define an ideal triangle in gray, shown here in the upper half-plane and disk models of hyperbolic geometry.

**Remark 23.** Every hexagonally symmetric disk that passes through the six points $e^*_i$, for $i = 0, \ldots, 5$ must be contained in the six triangular petals of the hexagram (Figure 5). The star inequalities can be interpreted geometrically as asserting that the tangent vectors at $e^*_i$ are nonzero and point into the open cones over the six triangular petals. (When some tangent vector at some $e^*_i$ points into the boundary of the open cone, the centrally symmetric hexagon $H$ degenerates to a parallelogram, which is never optimal.)

**Lemma 2.4.1.** Let $Z_0(u) \in \mathfrak{sl}_2(\mathbb{R})$ be the control matrix of $u \in U$. If $X \in \mathfrak{sl}_2(\mathbb{R})$ satisfies the star inequalities, then $\text{trace}(Z_0(u)X) < 0$.

**Proof.** The control simplex $U$ is convex, and $Z_0(U) \subset \mathfrak{sl}_2(\mathbb{R})$ is an affine image of the control simplex. Thus, the image $Z_0(U)$ is a convex set. It is enough to
Figure 5: After an affine transformation, every hexagonally symmetric disk remains confined to the six petals of a hexagram on the left. The six tangents to the disk are confined to the open cones over the six petals. These are the star inequalities.

check that \( \text{trace}(Z_0(u)X) < 0 \) at the three vertices of the control simplex. The conditions at vertices are precisely the star inequalities on \( X \).

\[ \square \]

3 Costate

3.1 Filippov’s lemma

The existence of lifted trajectories in the cotangent bundle (as discussed later in this section) is generally based on Filippov’s compactness lemma \cite[Th.10.1]{ATB3}, \cite[§4.5]{Lib12}. In this subsection, we show that the assumptions of Filippov’s lemma are fulfilled. Filippov’s lemma requires (1) that the control set \( U \) is compact, which is certainly true in our situation.

Filippov’s lemma requires (2) that for each \( x + iy \in \mathfrak{h}^* \) the velocity set (see Equation \([14]\))

\[
\{ f(x, y; u) = (f_1(x, y; u), f_2(x, y; u)) \in \mathbb{R}^2 \mid u \in U \}
\]

is convex. We prove that the velocity set is in fact the convex hull of

\[
\{ f(x, y; e_k) \mid k = 1, 2, 3 \}.
\]

Fix \( x + iy \in \mathfrak{h}^* \) and pick two vertices \( e_i, e_j \in U \). By explicit calculation, the two vertices map to distinct points in the velocity set. Let \( L : \mathbb{R}^2 \to \mathbb{R} \) be
the nonzero affine function that vanishes at \( f(x, y; e_i) \) and \( f(x, y; e_j) \). From the explicit form of Equation 14, we have

\[
L(f(x, y; u)) = \frac{\ell_1(u)}{\ell_2(u)},
\]

for some affine functions \( \ell_1, \ell_2 : U \rightarrow \mathbb{R} \) (depending on \( x, y \)), where \( \ell_2(u) \) is nonvanishing (and fixed sign) on \( U \). By direct calculation, we obtain \( \ell_1(u) = 0 \) along the segment \([e_i, e_j]) \subset U\) and that \( \ell_1 \) has fixed sign on \( U \). We conclude that the velocity set is a convex hull as claimed.

Finally, Filippov’s lemma requires (3) the compact support in \((x, y)\) of the velocity sets. This is a serious issue in our setting because the star inequalities are open conditions and the vector fields become unbounded near the boundary.

By Reinhardt, an optimal centrally symmetric \( D_{\text{min}} \) exists and its boundary has no corner. The corresponding unit-speed trajectory \( z : [0, t_f] \rightarrow \mathfrak{h}^* \) remains in the interior of some compact set \( K \subset \mathfrak{h}^* \). A standard argument using a smooth compactly-supported support function \( f : \mathfrak{h}^* \rightarrow \mathbb{R} \) with \( f|_K = 1 \) allows us to replace each vector field \( F \) on \( \mathfrak{h} \) with a vector field \( F^\epsilon \) of compact support \([\text{AS13}, \text{Remark } 10.5]\). Thus, by choosing a suitable support function, we may assume that all three of Filippov’s assumptions hold. Moreover, if desired, we can exhaust \( \mathfrak{h}^* \) by a sequence of compact sets \( K \) whose union is \( \mathfrak{h}^* \).

Pontryagin’s conditions, which are discussed below, are local around the trajectory \( z \), and so are not affected by the support function.

### 3.2 Hamiltonian

We use the formulation of the Hamiltonian for invariant problems on a Lie group from \([\text{AS13}, \text{Ch.18}]\). We use invariant vector fields to trivialize the tangent bundle of \( \text{SL}_2(\mathbb{R}) \) and use an invariant inner product to identify the cotangent space with the tangent space. We fix the invariant inner product \( \langle A, B \rangle = \text{trace}(AB) \) on \( \mathfrak{sl}_2(\mathbb{R}) \). In this formulation, according to the standard definitions, the optimal control problem has a Hamiltonian

\[
H(\lambda; u) := H_{\text{Lie}}(\Lambda, X) + H_\nu(\nu, x, y; u), \quad \text{where}
\]

\[
H_{\text{Lie}}(\Lambda, X) := \langle \Lambda, X \rangle - \frac{3}{2} \lambda_{\text{cost}} \langle J, X \rangle, \quad \text{(24)}
\]

\[
H_\nu(\nu, x, y; u) := \nu_1 f_1(x, y; u) + \nu_2 f_2(x, y; u),
\]

for costate variables \( \lambda_{\text{cost}} \in \mathbb{R} \), \( \Lambda \in \mathfrak{sl}_2(\mathbb{R}) \), \( \nu = (\nu_1, \nu_2) \in \mathbb{R}^2 \), and where \( X = \hat{\hat{z}} J \hat{\hat{z}}^{-1} \), \( \hat{\hat{z}} = \hat{z}(x, y) \). We have broken the Hamiltonian into two terms: \( H_\nu \) coming from the upper-half plane, and \( H_{\text{Lie}} \) coming from the Lie algebra and cost functional combined.
The state space $M^* := \text{SL}_2(\mathbb{R}) \times h^*$ is five-dimensional, and the costate space $\mathfrak{sl}_2(\mathbb{R}) \times \mathbb{R}^2$ is five-dimensional, viewed as the cotangent space of $M^*$ at a point under the trivialization of the cotangent bundle. We write $T^* M$ for the cotangent bundle of $M$, identified with

$$T^* M \times T^* h = (\text{SL}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})) \times (h \times \mathbb{R}^2) \ni ((g, \Lambda), (z, \nu)).$$

We write $T^*_f M$ for the subspace of $T^* M$ on which the $\text{SL}_2(\mathbb{R})$ component is $g = I$.

### 3.3 Pontryagin maximum principle

Specialized to our setting, the conditions of Pontryagin maximum principle (PMP) for our optimal control problem with free terminal time $t_f$ are the following:

1. The trajectory $(g, z)$ satisfies the ODE (15) for some measurable control $u : [0, t_f] \to U$.

2. The Hamiltonian $H(\lambda, u)$ vanishes identically along the lifted controlled trajectory $(\lambda, u)$.

3. The lifted trajectory $\lambda : [0, t_f] \to T^* M$ is Lipschitz continuous and satisfies the following ODE:

   $$\begin{align*}
   \Lambda' &= [\Lambda, X], \\
   \nu'_1 &= -\frac{\partial H^+}{\partial x}, \\
   \nu'_2 &= -\frac{\partial H^+}{\partial y},
   \end{align*}$$

   Here $H^+$ is the pointwise maximum over the control simplex $U$:

   $$H^+(\lambda_{\text{cost}}, \lambda) = \max_{u \in U} H(\lambda_{\text{cost}}, \lambda; u), \quad (\lambda_{\text{cost}}, \lambda) \in \mathbb{R} \times T^* M^*. \quad (26)$$

4. The projectivized covector is well-defined: for each $t$, the vector $(\lambda_{\text{cost}}, \lambda(t)) \in \mathbb{R} \times T^* M$ is nonzero.

5. $\lambda_{\text{cost}}$ is constant and $\lambda_{\text{cost}} \leq 0$.

6. Transversality holds at the endpoints (as described in Section 3.5).

By a lifted trajectory $\lambda : [0, t_f] \to T^* M$ of an admissible trajectory $(g, z)$ we mean a solution of the ODE (25) such that the image $(g_\lambda, z_\lambda)$ of $\lambda$ in $M$ is $(g, z)$.
A lifted trajectory satisfying the PMP conditions is called a Pontryagin extremal trajectory. The PMP gives necessary but not sufficient conditions for local optimality.

Because $\lambda_{\text{cost}} \leq 0$ is a constant, and since the PMP conditions are invariant under rescaling the costate by a positive scalar, we may take $\lambda_{\text{cost}} = 0$ (abnormal multiplier) or $\lambda_{\text{cost}} = -1$ (normal multiplier).

We define a Reinhardt trajectory to be a trajectory $(g, z)$ such that its disk $D(g, z) = D_{\text{min}}$ is a globally optimal solution to the Reinhardt problem.

**Lemma 3.3.1.** Let $(g, z)$ be a Reinhardt trajectory. Then the trajectory has a lifting

$$\lambda : [0, t_f] \to T^*M^*, \quad \lambda_{\text{cost}} \in \{0, -1\}$$

to the cotangent space. The lifted trajectory is a Pontryagin extremal trajectory.

**Proof.** Filippov’s lemma gives a Lipschitz continuous path $\lambda : [0, t_f] \to T^*M$, with components $\lambda = (\Lambda, \nu)$ satisfying the adjoint equations (25): (The first equation in (25), for $\Lambda'$, comes from a trivialization of the cotangent bundle of $\text{SL}_2(\mathbb{R})$, as stated in [AS13, Eqn.18.18].)

By the work of Pontryagin and general control theory, the PMP are necessary conditions for optimality.

### 3.4 rotational symmetry

We write $(\rho, z) \mapsto \rho \cdot z$ for the action of $\text{SL}_2(\mathbb{R})$ on $\mathfrak{sl}_2$ by linear fractional transformations.

Let $\rho \in \text{SO}_2(\mathbb{R})$ be any rotation. The symmetry acts on trajectories and related data as follows. Let $\lambda = (\Lambda, \nu, \ldots)$ be a lifted trajectory. We map the path $z = x + iy$ in the upper-half plane to the path $\bar{z} = \rho \cdot z$, where we use bars to denote transformed quantities. Then short calculations show that we obtain another lifted trajectory $\bar{\lambda} = (\bar{\Lambda}, \bar{\nu}, \ldots)$ (with different boundary values) and
associated parameters:

\[
\bar{g} = \rho g \rho^{-1}; \\
\bar{X} = \rho X \rho^{-1}; \\
\bar{z} = \rho z; \\
\bar{\Lambda} = \rho \Lambda \rho^{-1}; \\
\bar{Z}_0 = \rho Z_0 \rho^{-1}; \\
\bar{\delta} = \delta;
\]

The cost is invariant: \( \bar{\text{cost}} = \text{cost} \). The transformation rule for \( \nu \) is as follows.

The value of \( \bar{\nu} \) at \( \rho z \) is

\[
\bar{\nu}_{\rho z} = (F^t)^{-1} \nu_z, \quad \nu_z \in T_z^* \mathfrak{h},
\]

where the linear map \( F = d\rho_z \) of tangent spaces and transpose \( F^t \)

\[
F : T_z \mathfrak{h} \to T_{\rho z} \mathfrak{h}, \quad F^t : T^*_{\rho z} \mathfrak{h} \to T^*_z \mathfrak{h}.
\]

are induced from \( z \mapsto \rho z \).

It is remarkable that the entire Hamiltonian is invariant under the full rotation group \( \text{SO}_2(\mathbb{R}) \):

\[
\bar{H} = H; \quad \langle \bar{\Lambda}, \bar{X} \rangle = \langle \Lambda, X \rangle; \quad (\bar{\nu}, \bar{f}) = (\nu, f); \quad \text{trace}(J \bar{X}) = \text{trace}(J X).
\]

Moreover, assume that \( \rho \in \langle R \rangle \). Then there exists a permutation \( \pi = \pi_\rho \) of \( \{0, 1, 2\} \) such that

\[
\rho^{-1} e^*_2 = e^*_{2\pi_1}, \quad \text{for } i = 0, 1, 2.
\]

Then

\[
\bar{u} = (\bar{u}_0, \bar{u}_1, \bar{u}_2) = (u_{\pi_0}, u_{\pi_1}, u_{\pi 2}). \quad (27)
\]

We write \( \bar{u} = \rho \cdot u \) for this action.

### 3.5 terminal conditions

We define periodic boundary conditions (modulo rotation by \( R \)):

\[
g(0) = I, \quad g(t_f) = R := \exp(J\pi/3),
\]

and

\[
X(0) \in \mathfrak{sl}_2(\mathbb{R}), \quad X(t_f) = R^{-1} X(0) R \in \mathfrak{sl}_2(\mathbb{R}). \quad (28)
\]
The terminal condition $g(t_f) = R$ is necessary because the six paths of the hexagonally symmetric disk must join together to give a closed curve:

$$g(t_f)e_j^* = g(0)e_j^{*+1} \iff g(t_f) = R. \tag{29}$$

By the unit speed positive orientation conditions,

$$g(t + t_f)e_j^* = g(t)e_j^{*+1} = g(t)Re_j^*, \quad g(t + t_f) = g(t)R,$$

and taking derivatives:

$$X(t + t_f) = R^{-1}X(t)R, \quad z(t + t_f) = R^{-1}z(t). \tag{30}$$

Evaluating (30) at $t = 0$ gives the terminal condition on $X(t_f)$ in Equation 28.

Expressed in terms of coordinates on the upper half-plane, the terminal condition becomes

$$z(t_f) = R^{-1}z(0), \quad z(0), z(t_f) \in \mathfrak{h}, \tag{31}$$

a rotation about $i$ by angle $2\pi/3$. Expressed in terms of a complex variable in the Poincaré disk model, the terminal condition becomes a counterclockwise rotation by angle $2\pi/3$:

$$w(t_f) = \zeta w(0), \quad w(0), w(t_f) \in \mathbb{D}, \quad \zeta = e^{2\pi i/3}.$$

In optimal control problems such as this with free terminal time $t_f$, Pontryagin’s maximum principle (PMP) includes a transversality condition at time $t = t_f$. For periodic systems such as ours, the transversality condition can be found in Liberzon [Lib12, p134]. In our setting, the system is periodic up to rotation by $R$. Our transversality conditions can be expressed as follows:

$$\Lambda(t_f) = R^{-1}\Lambda(0)R,$$

$$F^t\lambda(t_f) = \lambda(0), \tag{32}$$

where $F = dR^{-1}_{z(0)} : T_{z(0)}\mathfrak{h} \to T_{z(t_f)}\mathfrak{h}$ is the linear map of tangent spaces induced from $z \mapsto R^{-1}z$ and its transpose is

$$F^t : T_{z(t_f)}^*\mathfrak{h} \to T_{z(0)}^*\mathfrak{h}.$$

4 Explicit trajectories with bang-bang controls

By a bang-bang control we mean a measurable control function $u$ that takes values in the set $\{e_1, e_2, e_3\}$ of vertices of the control simplex $U$. 

17
4.1 constant control at the vertex $e_2$ or $e_3$

Throughout this section, we assume that the control $u \in U$ is constant, fixed at a vertex $u = e_3$ or $u = e_2$ of $U$. Under this assumption, we give the general explicit formula for the state and costate. With such controls, the control matrix $Z_0$ in Equation (3) simplifies to the form

$$c_{11}(Z_0) = \pm 1/\sqrt{3}, \quad c_{12}(Z_0) = -2/3, \quad c_{21}(Z_0) = 0, \quad m = c_{12}/(2c_{11}) = \pm 1/\sqrt{3}.$$  

Specifically, $m = m_3 = 1/\sqrt{3}$ (control $u = e_3$) and $m_2 = -1/\sqrt{3}$, for these two controls.

In this context, the ODE (14) reduces to

$$x' = y; \quad y' = \frac{y^2}{m + x}. \quad (33)$$

The star inequalities imply that $c_0 := x(0) + m \neq 0$. Set $\alpha = y(0)/c_0$, which is also nonzero by the star inequalities. The general solution to ODE (33) is

$$x(t) = -m + c_0 e^{\alpha t}, \quad y(t) = c_0 \alpha e^{\alpha t}. \quad (34)$$

(We also write this curve as $t \mapsto z(z_0, t) \in \mathfrak{h}$, where $z_0 = x(0) + iy(0)$.) In particular, each trajectory traces out a line $y = \alpha (x + m)$ through the fixed point $(-m, 0)$. See Figure 6. The motion is away from the fixed point when $m = m_3$ and towards the fixed point when $m = m_2$. That is, $\alpha_3 > 0$ and $\alpha_2 < 0$.

Set $s = e^{\alpha t}$. Expressed in terms of the independent variable $s$, the differential equation (4) takes the form

$$\alpha s \frac{dg}{ds} = gX, \quad g(1) = I,$$

which has the explicit solution

$$g(s) = I + \frac{s - 1}{\alpha^2 c_0 s} \begin{pmatrix} c_0 - m & \ast \\ 1 & m - c_0 s \end{pmatrix},$$

where the missing entry ($\ast$) is determined by the condition $\det(g(s)) = 1$.

The adjoint equation also has an explicit exact general solution, which appears in the accompanying computer algebra calculations. Although it is entirely explicit, the solution is a bit too long to print here. The function $\nu$ is a pair of polynomials in $t$, $e^{\alpha t}$ and $e^{-\alpha t}$, and there are five constants of integration (beyond $z_0$). These constants are determined by the initial vector

$$\lambda(0) \in T^*_z(I, z_0) M.$$
Figure 6: The trajectories with constant control $u = e_1$ are circles or lines, shown here in the star region of the upper half-plane.

4.2 constant control $u = e_1$

The three extremal controls $u = e_1$, $e_2$, and $e_3$ are related by rotational symmetry of the upper-half plane. These symmetries are more visually evident in the disk model of hyperbolic space, but the solutions to the ODE take a simpler form in the upper-half plane.

Equation (27) implies that we obtain the general explicit solutions to the state and costate equations for control $u = e_1$ by rotating solutions with control $e_2$ or $e_3$, as described in Section 3.4. Details are found in the computer code.

The trajectories with constant control $e_3$ move along Euclidean lines through $(-1/\sqrt{3}, 0)$, which we view as circles through $(-1/\sqrt{3}, 0)$ and $\infty$. Under linear fractional transformations, circles map to circles. From this, we conclude that trajectories with constant control $e_1$ must move along Euclidean circles through the two fixed points $(\pm 1/\sqrt{3}, 0)$.

4.3 bang-bang controls

Lemma 4.3.1. For every $\lambda \in T^*M$, the set of maximizers of the Hamiltonian:

$$U_\lambda := \{ u \in U \mid H(\lambda, u) = \max_{u \in U} H(\lambda, u) = H^+(\lambda) \}$$

is a face of the convex set $U$; that is, $U_\lambda$ is a vertex, an edge, or all of $U$.

Proof. Fix $\lambda \in T^*M$. The only term of the Hamiltonian that depends on the control is $H_\delta$. This term is the ratio of two linear functions on $U$. For any $u_1, u_2 \in U$, let $u(s) = su_1 + (1 - s)u_2$ for $s \in [0, 1]$ be a segment in the convex control set. Then the dependence of the Hamiltonian along the segment has the
form of a linear fractional transformation
\[ s \mapsto H(\lambda, u(s)) = \frac{as + b}{cs + d}, \]
with derivative
\[ \frac{ad - bc}{(cs + d)^2} \]
of fixed sign. (The denominator is nonzero by Lemma 2.4.1.) Thus the Hamiltonian is monotonic along every segment in the control simplex \( U \). The Hamiltonian therefore assumes its maximum along a face.

If \( X \in \mathfrak{sl}_2(\mathbb{R}) \) lies in the orbit of \( J \), let \( X_h \in \mathfrak{h} \) be the corresponding element of the upper-half plane under the bijection (13). For \( t \geq 0 \) and \( z \in \mathfrak{h} \), let \( \gamma_0(z, t) \in \text{SL}_2(\mathbb{R}) \) be the trajectory with constant control \( u = e_3 \) and initial conditions
\[ \gamma_0(z, 0) = I, \quad \gamma_0'(z, 0)_\mathfrak{h} = z. \]
(As always, prime denotes the \( t \) derivative.) Let \( \gamma_i(z, t) \in \text{SL}_2(\mathbb{R}) \), for \( t \geq 0 \), \( i \in \mathbb{Z} \), and \( z \in \mathfrak{h} \) be the trajectory \( \gamma_i(z, t) := R^i \gamma_0(z, t) R^{-i} \).

We have
\[ \gamma_i(z, 0) = I, \quad (\gamma_i'(z, 0))_\mathfrak{h} = R^i.z, \]
with constant control \( u = R^i \cdot e_3 \), using the action (27) of the cyclic group \( \langle R \rangle \) on the control simplex \( U \).

We define a continuous (shifted) extension of \( \gamma_i \) that is non-constant only for \( t \in [T_1, T_2] \):
\[ \gamma_i(z, T_1, T_2, t) := \begin{cases} I, & \text{if } t \leq T_1; \\ \gamma_i(z, t - T_1), & \text{if } T_1 \leq t \leq T_2; \\ \gamma_i(z, T_2 - T_1), & \text{if } T_2 \leq t. \end{cases} \]
The derivative \( \gamma_i' \) has jump discontinuities at \( T_1 \) and \( T_2 \). Let \( z(z_0, t) \) be the solution to the ODE (33) with constant control \( u = e_3 \) and initial condition \( z_0 \). For any tuple \( \kappa = ((k_1, t_1), (k_2, t_2), \ldots, (k_n, t_n)) \) with \( k_i \in \mathbb{Z} \) and \( t_i \geq 0 \), and for any \( z_0 \in \mathfrak{h}^* \), let
\[ T_0 = 0; \]
\[ T_{i+1} = T_i + t_{i+1}; \]
\[ z_i = R^{k_i - k_{i+1}} \cdot z(t_{i-1}, t_i); \]
(35)
\[ (35) \]
\[ \gamma(\kappa, z_0, t) = \gamma_{k_1}(z_0, T_0, T_1, t) \gamma_{k_2}(z_1, T_1, T_2, t) \cdots \gamma_{k_n}(z_{n-1}, T_{n-1}, T_n, t). \]
Note that on the right-hand side of the last equation, only one factor at a time is non-constant. Then $\gamma(\kappa, z, t)$ is continuous in $t$ and has unit speed parametrization. Set $X(\kappa, z, t) := \gamma(\kappa, z, t)^{-1}\gamma'(\kappa, z, t)$. Note that for $t \in [T_{i-1}, T_i]$, when the $i$th factor is active, we have

$$X(\kappa, z_0, t) = \gamma_k(z_{i-1}, T_{i-1}, T_i, t)^{-1}\gamma'_k(z_{i-1}, T_{i-1}, T_i, t) = R^{k_i} X(z_{i-1}, t - T_{i-1}) R^{-k_i},$$

where $X(z, t) = \gamma_0(z, t)^{-1}\gamma'_0(z, t)$. Comparing left and right limits of $X(\kappa, z_0, t)$ at the boundary value $t = T_i$, we find that $X(\kappa, z_0)$ is continuous in $t$:

$$X(\kappa, z_0, T_i^-)_h = R^{k_i} z(z_{i-1}, t_i) = R^{k_{i+1}} z_i;$$

$$X(\kappa, z_0, T_i^+)_h = R^{k_{i+1}} z_i.$$

From this, it is easy to see that $\gamma(\kappa, z_0)$ is the general bang-bang trajectory with finitely many switches (at times $T_0, \ldots, T_n$), as we vary $\kappa$ and $z_0$. The control on the interval $[T_{i-1}, T_i]$ is $u = R^{k_i} \cdot e_3 \in U$.

The total cost($z_0, [0, t]$) of the trajectory (34) with initial condition $z_0$ up to time $t$ is an easy (freshman calculus) integral to compute from Equation (18), which we do not display here. The total cost of $\gamma(\kappa, z_0, t)$ from time 0 to $T_n$ is

$$\sum_{i=0}^{n-1} \text{cost}(z_i, [0, t_{i+1}]).$$

(36)

### 4.4 the smoothed regular polygon

Reinhardt conjectured that the smoothed octagon is the solution to his problem.

The smoothed octagon comes from a periodic bang-bang control to the state equations with four links (and four switching times). The control switches four times in a cyclic order around the extreme points of the control simplex $U$. The smoothed octagon itself can be visualized as being made of 24 segments: 8 smoothed corners and 16 half-edges. These 24 segments are arranged into four links, each consisting of 6 arcs. The four links are congruent, under the rotational symmetry $R$.

We generalize the smoothed octagon to a smoothed regular polygon as follows. Let $k$ be a positive integer. We consider a trajectory with $3k + 1$-links of the same length of the form $t \mapsto \gamma(\kappa, z_k, t)$, with

$$\kappa = ((0, t_k), (-1, t_k), (-2, t_k), \ldots, (-3k, t_k)),$$

(37)
where \( t_k > 0 \) and \( z_k \in \mathfrak{h} \) are to be determined as functions of \( k \geq 1 \). (Note that the meaning of \( t_k, z_k \) has changed from \( t_i, z_i \) in the previous section.)

Let \( g_k = \gamma(z_k, t_k) \in \text{SL}_2(\mathbb{R}) \) be the position at the end of a single link. The endpoint condition \((29)\) for \((37)\) is

\[
R = g_k(R^{-1}g_kR^1)(R^{-2}g_kR^2)\cdots(R^{-3k}g_kR^{3k}),
\]

or equivalently,

\[
(R^{-1}g_k)^{3k+1} = R^{-3k} = (-I)^k.
\]

Let \( \mu, \mu^{-1} \) be the eigenvalues of \( R^{-1}g_k \in \text{SL}_2(\mathbb{R}) \). Comparing eigenvalues on the two sides of \((38)\), we obtain \( \mu^{3k+1} = (-1)^k \), and

\[
\mu = e^{\pi ik/(3k+1)+2\pi i\ell/(3k+1)}, \quad \ell \in \mathbb{Z}.
\]

We pick the eigenvalues \( \mu^{\pm 1} \) that place \( g_k \) in the smallest neighborhood of \( \text{of } 1; \) that is, we take \( \ell = 0, -k \). Then

\[
\text{trace}(R^{-1}g_k) = \mu + \mu^{-1} = 2 \cos \theta_k, \quad \text{where } \theta_k = \frac{\pi k}{3k+1}.
\]

For example, for the smoothed octagon \( k = 1 \), the trace is \( \sqrt{2} \).

We impose the strong boundary condition

\[
z(z_k, t_k) = R^{-1}.z_k, \quad \text{where } z_k = 0 + iy_k.
\]

It follows that \((31)\) holds with \( t_f = (3k+1)t_k \):

\[
z(z_k, t_f) = R^{-(3k+1)}.z_k = R^{-1}.z_k.
\]

Solving \((40)\) for \( t_k \) (the time spent in each link), we obtain

\[
t_k = \frac{\ln((1 + 3y_k^2)/4)}{\sqrt{3y_k}}
\]

We have solved the nonlinear equations \((39)\) and \((41)\) explicitly for \( t_k \) and \( y_k \) in the accompanying code, but we do not display the solution here. For each positive integer \( k \), the trajectory for the smoothed \( 6k+2 \)-gon is now completely determined by these values of \( t_k \) and \( y_k \).

These formulas for \( t_k \) and \( y_k \) can be interpolated to functions that are analytic in \( k \in \mathbb{R} \). Figure 9 graphs the area of the smoothed \( 6k+2 \)-gon as a function of \( k \). It appears that the area function is increasing in \( k \) and tends to the area \( \pi \) of a circular disk.

We show that we can lift each trajectory to a Pontryagin extremal. The following is one of the main conclusions of this article. It implies in particular that the smoothed octagon \( k = 1 \) is a Pontryagin extremal.
Theorem 4.4.1. The smoothed regular $6k+2$-gon lifts to a Pontryagin extremal trajectory. The trajectory has a normal multiplier.

Proof. We show that there exists a choice of initial conditions for $\Lambda, \nu$ for which the PMP conditions hold.

We start with the endpoint condition (32) for $\Lambda$. Again, we prove a stronger form of transversality by showing

$$\Lambda(t_k) = R^{-1} \Lambda(0) R,$$

which implies (32). This is a system of three homogeneous equations for $\Lambda(0) \in \mathfrak{sl}_2(\mathbb{R})$ (three unknowns). We indicate why a nontrivial solution to this homogeneous system for $\Lambda(0)$ must exist. By the form of the differential equation it satisfies (25), as $\Lambda$ evolves in time, its determinant remains constant. We find that $\Lambda(t)$ remains in a fixed conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$. We can therefore write

$$\Lambda(t) = h(t) \Lambda(0) h(t)^{-1}$$

for some $h(t) \in \text{SL}_2(\mathbb{R})$. Equation (42) asserts that $\Lambda(0)$ lies in the centralizer of $Rh(t_k)$ in $\mathfrak{sl}_2(\mathbb{R})$. A centralizer has minimal dimension 1 (which occurs when $Rh(t_k)$ is regular, which occurs here). Thus, solutions exist and are unique up to a scalar.

We have a linear system of five equations and five unknowns. The five unknowns are $\Lambda(0) \in \mathfrak{sl}_2(\mathbb{R})$ and $\nu(0) \in \mathbb{R}^2$. Two independent equations come from (42), one from the vanishing of the Hamiltonian, and two from the endpoint condition (32) on $\nu$ (for reduced period $t_k$ instead of $t_f$). Explicit calculations give a unique solution to this linear system of equations (as homogeneous functions of $\lambda_{\text{cost}}$) for each $k$. This forces the multiplier $\lambda_{\text{cost}}$ to be normal, and we take $\lambda_{\text{cost}} = -1$.

Further explicit symbolic computer-algebra calculations show that $t = 0$ and $t = t_k$ are switching times. The following lemma completes the proof, which shows that the maximum property of Pontryagin is met for the Hamiltonian.

Lemma 4.4.2. Let $k \geq 1$ be an integer. Let $\lambda(k, t)$ be the lifted trajectory for the smoothed $6k+2$-gon along a single link with control $u = e_3$ as constructed above. Let $H_{k,u}(t)$ be the Hamiltonian restricted to the lifted trajectory, with arbitrary control function $u : [0, t_k] \rightarrow U$. Then

$$H_{k,e_3}(t) \geq H_{k,u(t)}(t), \quad t \in [0, t_k].$$

If $u \in \{e_1, e_2\}$ is a constant control at one of the first two vertices of $U$, then equality occurs only at the endpoints of the interval $[0, t_k]$.

Proof. A monotonicity result (Section 4.3) shows that the maximum of $H_{k,u}(t)$ is attained at a corner of the control simplex. It is enough to show that $\chi_{k,u} = H_{k,e_3} - H_{k,u}$, for the two constant controls $u = e_1$ and $u = e_2$. 23
An easy substitution using the explicit formulas for \( \lambda(k, t) \) gives
\[
\chi_{k,e_1}(t_k - t) = \chi_{k,e_2}(t), \quad t \in [0, t_k].
\]
Thus, it is enough to show that \( \chi_{k,e_2}(t) \geq 0 \). The function \( \chi_{k,e_2}(t) \) is equal to \( \nu(k, t)_2 \) up to a positive nonzero factor. Thus, the lemma reduces to proving that \( \nu(k, t)_2 \geq 0 \) for \( k \geq 1 \). (Here \( \nu(k, t)_2 \) is the component \( \nu_2 \) of Section 3.2 of the lifted trajectory \( t \mapsto \lambda(k, t) \).

We define new variables \((y, v)\):
\[
y = 1 + 3y_k^2, \quad v = ye^{\sqrt{3}y_k},
\]
and replace \( k \) with a continuous parameter. The region defined by \( k \geq 1 \) and \( t \in [0, t_k] \) transforms to the triangle
\[
T = \{(y, v) \in [2\sqrt{2}, 4]^2 \mid y \leq v\}.
\]
Note that \( t = 0 \) is transformed to the diagonal \( y = v \) of \( T \). We define
\[
f(y, v) = 3\sqrt{3}y_k^2v^2\nu(k, t)_2.
\]
We show that \( f \) is nonnegative on the triangle \( T \) as follows. (These calculations appear in the accompanying computer code.) First, an easy substitution gives \( f(y, y) = 0 \). (This was already verified above in a different manner, when we showed that \( t = 0 \) is a switching time.) Second, the derivative is negative on the diagonal:
\[
\frac{\partial f}{\partial y}\bigg|_{v=y} = y((y - 4) + y \ln(4/y)) \leq 0.
\]
Finally, the second derivative is positive on \( T \):
\[
\frac{\partial^2 f}{\partial y^2} = -10 - 5v + 2v/y + 7y - 2v \ln 4 + 6y \ln 4 + 4v \ln v - 2(v + 3y) \ln y \geq 0.
\]
(We leave this last inequality as a tedious but elementary exercise for the reader.) Positivity follows.

Looking more closely at the cases of equality, we see that the only zero of the switching function on \([0, t_k]\) occurs at \( t = 0 \), and that the derivative is strictly positive at \( t = 0 \). (The derivative is zero in (43) at the corner \( v = y = 4 \) of the disk, but this corresponds to the unrealizable limiting case as \( k \to \infty \).

Remark 44. A related construction gives a trajectory with \( 3k - 1 \)-links – the smoothed \( 6k - 2 \)-gon \( D_{6k-2} \), for \( k \geq 2 \). The changes are minor. We replace equation (45) with
\[
\kappa = ((1, t_k), (2, t_k), (3, t_k), \ldots, (3k - 1, t_k)).
\]
The trajectory is
\[ \gamma(\kappa, R^{-1}.z_k, t). \]

Equation 39 becomes
\[ \text{trace}(Rg_k) = 2 \cos \theta_k, \quad \text{where} \ \theta_k = \frac{\pi k}{3k - 2}. \quad (46) \]

Equation (41) is unchanged. The initial link of the smoothed octagon now has constant control \( u = e_2 \).

Remark 47. It seems that the smoothed 6k – 2-gon \( D_{6k-2} \) is not a Pontryagin extremal trajectory. Specifically, all of the conditions seem to hold, except that the Pontryagin multiplier \( \lambda_{\text{cost}} > 0 \) has the wrong sign. This suggests that these smoothed polygons are Pontryagin extremal trajectories for the problem of maximizing the area.

Remark 48. When \( k = 1 \), the smoothed polygon \( D_4 \) degenerates to a rectangle with corners (Figure 7) and area \( \sqrt{12} \). Allowing \( k \) to be non-integral, for small values of \( k > 1 \), we obtained smoothed rectangles (that do not quite satisfy the boundary conditions).

![Figure 7](image_url)

Figure 7: By taking a smoothed 6k – 2-gon and interpolating formulas to a fractional number of sides (here \( k = 1.03 \)), we see that the shape appears to be tending to a rectangle of area \( \sqrt{12} \) as \( k \to 1 \).

The trajectory in \( \mathfrak{h} \) for \( D_{6k+2} \) follows a triangle (with edges following the arcs of Figure 6) centered at \( z = i \in \mathfrak{h} \). It moves counterclockwise around \( i \), traversing one edge for each link (Figure 8). The trajectory in \( \mathfrak{h} \) for \( D_{6k-2} \) also follows an inverted triangle centered at \( z = i \in \mathfrak{h} \). It moves clockwise.

The cost increases with \( k \) for \( D_{6k+2} \) and decreases with \( k \) for \( D_{6k-2} \). In both cases, the limit of the cost is \( \pi \) as \( k \to \infty \). We show a graph of the costs of the smoothed polygons as a function of the number \( n = 6k \pm 2 \) of sides (Figure 9).
Figure 8: The trajectory in the upper-half plane of a smoothed $6k + 2$-gon follows $3k + 1$ edges moving counterclockwise on a triangular path centered at $i \in \mathfrak{h}$ (left). The trajectory for the smoothed $6k - 2$-gon follows $3k - 1$ edges moving clockwise on an inverted triangle centered at $i \in \mathfrak{h}$ (right).

Figure 9: The graph interpolates the cost $c$ of known critical points as a function of the number $n = 6k \pm 2$ of straight edge segments in the corresponding smoothed polygon. The cost tends to $\pi$ as $n$ increases. The data is consistent with Reinhardt's conjecture.

4.5 (micro) local optimality of the smoothed octagon

Nazarov has proved that the smoothed octagon is a local minimum of the Reinhardt problem [Naz88]. The following theorem should be viewed as a control-theory analogue of Nazarov's theorem. Our result gives micro-local optimality in the sense that we consider a neighborhood $V$ of the lifted extremal trajectory in the cotangent space. The following is one of the main results of this article.

**Theorem 4.5.1.** Let $\lambda_{oct} : [0, t_f] \to T^*M$ be the Pontryagin extremal lifted trajectory constructed in the previous section for the smoothed octagon ($k = 1$). Then

1. there exists a punctured neighborhood $V^*$ of $\lambda_{oct}(0) \in T^*_f M^*$ such that no initial condition in $V^*$ gives a Pontryagin extremal lifted trajectory.
2. Moreover, if any initial condition in $V^*$ gives a hexagonally symmetric...
disk $D$, then the area of $D$ is greater than that of the smoothed octagon.

**Remark 49.** I have not checked whether the other smoothed $6k + 2$-gons are local minima in the same sense.

**Proof (sketch).** From the explicit form of the lifted trajectory $\lambda_{\text{oct}}$ that was constructed in the previous section, we see that the switching functions meet the $x$-axis transversally at 0 and $t_1$ and have no other zeros on the interval $[0, t_1]$. Thus, any sufficiently small perturbation of the initial conditions will produce a small perturbation of the switching times. In particular, the trajectory will continue to consist of four links of approximately the same size and with the same controls as before on each link. We can assume without loss of generality that $t = 0$ is a switching time.

Thus, we can write the perturbed state in the form $t \mapsto \gamma(\kappa, z, t)$, where $\kappa = ((0, t_1+\eta_1), (-1, t_1+\eta_2), (-2, t_1+\eta_3), (-3, t_1+\eta_4))$, $\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6)$

and where $z = iy_1 + \eta_5 + i\eta_6 \in \mathfrak{h}$ lies in a small neighborhood of $0 + iy_1 \in \mathfrak{h}$, and $\eta_i \in \mathbb{R}$ are near 0. Here, $(t_1, y_1) = (t_k, y_k)$, constructed in Section 4.2 with $k = 1$.

We prove the second claim of the theorem first. We have a six-dimensional parameter space of initial conditions $\eta \in \mathbb{R}^6$ and five endpoint equations (29) and (31) (counting three equations from $\text{SL}_2(\mathbb{R})$ and two from $\mathfrak{h}$). These equations define a one-dimensional curve $N \subset \mathbb{R}^6$ through $p = 0$. The curve represents a 1-dimension family of deformations of the smoothed octagon that satisfies the endpoint conditions. These equations satisfy the conditions of the analytic implicit function theorem, allowing us to use $\eta_1$ as an analytic coordinate on $N$ near $p$. We write the other coordinates $\eta_2, \ldots, \eta_6$ as power series in $\eta_1$ on $N$ near $p$:

$$\tilde{\eta}_j := \eta_j|_N = \tilde{\eta}_1 a^1_j + \tilde{\eta}_2^2 a^2_j + O(\tilde{\eta}_1^3).$$

(50)

for some coefficients $a^1_j, a^2_j \in \mathbb{R}$ to be calculated. Initial calculations show that the constants $a^1_j$ have the form

$$a^1_1 = 1, \quad a^1_2 = -1, \quad a^1_3 = 1, \quad a^1_4 = -1.$$

The choice of local parameter $\tilde{\eta}_1$ on $N$ gives $a^1_7 = 0$. The terminal time for the deformation is $t_f(\tilde{\eta}_1) = t_f + \tilde{\eta}_1 + \tilde{\eta}_2 + \tilde{\eta}_3 + \tilde{\eta}_4$, where $t_f = 4t_1$ is the terminal time for the smoothed octagon. We write the periodic endpoint conditions (29) (31) in the form

$$g(\tilde{\eta}, t_f(\tilde{\eta}_1)) = R;$$
$$z(\tilde{\eta}, t_f(\tilde{\eta}_1)) = R^{-1}.(iy_1 + (\tilde{\eta}_5 + i\tilde{\eta}_6)).$$

(51)
A long computer algebra calculation (using interval arithmetic, automatic differentiation, these endpoint conditions, and the explicit formulas for the solutions to our ODEs) gives us the power series expansion of the left-hand side of (51) to second order in terms of the unknown coefficients $a_j^i$. This is a delicate calculation, which explicitly propagates the unknown coefficients along the trajectory to the endpoint. Comparing with the right-hand side of (51), we obtain explicit interval arithmetic bounds on $a_1^1, a_2^2$. Still using computer algebra calculations, we write the cost over the time interval $[0, t_f(\bar{\eta}_1)]$ as a function of $\bar{\eta}_1$ on $N$, and expand the cost in a power series in $\bar{\eta}_1$ using these interval bounds on $a_1^1, a_2^2$. These interval arithmetic calculations give the explicit bounds

$$
\text{cost}(\bar{\eta}_1) = \text{cost}(0) + b_1 \bar{\eta}_1 + b_2 \bar{\eta}_1^2 + O(\bar{\eta}_1^3), \tag{52}
$$

where $|b_1| < 10^{-9}$ and $b_2 = 4.7976 \ldots$. In particular $\text{cost}''(0) > 0$.

We know that the smoothed octagon is a Pontryagin extremal. This implies that no needle perturbations of the smoothed octagon can give a first-order improvement to the cost. In particular, $b_1 = 0$. Thus, by (52), cost has a strict local minimum at $\bar{\eta}_1 = 0$. This completes the proof of the final claim of the theorem.

Finally, we give a proof of the first claim of the theorem: there exists $V^*$ such that no initial condition in $V^*$ gives a Pontryagin extremal lifted trajectory. Any such lifted trajectory satisfies the endpoint conditions, and must therefore have an initial condition of the form (50) and must lie in $N$. On some punctured neighborhood of the smoothed octagon, along $N$, the cost (52) has nonzero derivative. This is inconsistent with PMP. This completes the proof. 

$\blacksquare$

5 The singular locus

5.1 the circle as singular arc

The circular disk is a hexagonally symmetric disk $D$ defined by the trajectory

$$
g(t) = \exp(Jt), \quad g' = gJ, \quad t \in [0, t_f], \quad t_f = \pi/3. \tag{53}
$$

(The six paths $t \mapsto \sigma_j(t) = g(t)e_j^*$, for $t \in [0, \pi/3]$ are six arcs that fill out the unit circle.) Thus, $X \equiv J$ is a constant path, and $x + iy = i$ is also constant. In the Poincaré disk, the constant path is $w \equiv 0$. The cost from Equation (20) is

$$
3 \int_0^{\pi/3} dt = \pi,
$$

which has the expected interpretation of the area of a circular disk of radius 1.
The control for the circle is constant: \( u = (1/3, 1/3, 1/3) \in U \). The ODE (14) becomes
\[
\begin{align*}
x' &= y(1 + x^2 - y^2) \\
y' &= -\frac{2xy^2}{1 + x^2 + y^2}
\end{align*}
\]
which indeed has the constant solution \( x \equiv 0, y \equiv 1 \). This constant solution determines the constant path \( X \equiv J \in \mathfrak{sl}_2(\mathbb{R}) \).

In our context, a lifted trajectory \( \lambda \) is a singular arc on \([t_1, t_2]\) if for all \( t \in [t_1, t_2] \), the face \( U_{\lambda(t)} \subseteq U \) has positive dimension. See [Lib12, §4.4.3].

**Lemma 5.1.1.** The circle is an extremal singular arc. The multiplier is normal.

**Proof.** The solution to the adjoint equations is also constant:
\[
\Lambda \equiv \Lambda_0 = \frac{3}{2} J \lambda_{\text{cost}}, \quad \nu \equiv 0.
\]
where \( \lambda_{\text{cost}} = -1 \) (for a normal multiplier). A simple calculation based on this explicit data shows the circle is an extremal. Along the lifted trajectory, the Hamiltonian is independent of the control:
\[
H(\lambda(t); u) \equiv 0.
\]
Thus, \( U_{\lambda(t)} = U \) and the lifted trajectory is a singular arc.

**Remark 55.** Second order conditions show that circular arc is not a local minimizer on any time interval \([t_1, t_2]\) so that the solution to the Reinhardt problem contains no circular arcs [Hal11, §5.2]. We recall the argument. We consider a deformation of a circular arc of the form
\[
g_\epsilon(t) = \exp \left( \epsilon \begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{12}(t) & -c_{11}(t) \end{pmatrix} \right) e^{-t}.
\]
for sufficiently small \( \epsilon > 0 \) and compactly supported \( C^\infty \) functions \( c_{11}, c_{12} \) to be determined on the interval \([t_1, t_2]\). We emphasize that \( t \) is not a unit speed parameter. Computing the cost of \( g_\epsilon \) on \([t_1, t_2]\) by (17), we find that
\[
\text{cost}(g_\epsilon) = \text{cost}(g_0) + 6\epsilon^2 \int_{t_1}^{t_2} c_{11}(t)c_{12}'(t) \, dt + O(\epsilon^3).
\]
Note that this is a second variation that is not detected by PMP. Choose \( c_{11}(t) \geq 0 \) (with positive integral \( \int c_{11} \, dt > 0 \)) with support on an interval where \( c_{12}'(t) < 0 \). Then for all sufficiently small \( \epsilon > 0 \), we have
\[
\text{cost}(g_\epsilon) < \text{cost}(g_0) = \pi.
\]
We may pick \( \epsilon > 0 \) sufficiently small so that the curvatures of the curves \( t \mapsto g_\epsilon(t)e_1^t \) are positive. Then there exists a control function \( u : [t_1, t_2] \to U \) with controlled trajectory \( g_\epsilon \).
5.2 no singular arcs

Recall that a Reinhardt trajectory is a trajectory \((g, z)\) such that its disk \(D(g, z) = D_{\text{min}}\) is a globally optimal solution to the Reinhardt problem.

**Lemma 5.2.1.** A Reinhardt trajectory contains no singular arcs.

**Proof.** Along a singular arc, the set \(U_\lambda\) of controls maximizing the Hamiltonian has positive dimension. The set can be an edge of \(U\) or all of \(U\). We first assume that \(U_\lambda(t)\) is an edge on a set of positive measure. By the continuity of the lifted singular arc, \(U_\lambda(t)\) is a fixed edge on an open set in \([t_1, t_2]\). We show that this leads to a contradiction.

By symmetry, without loss of generality, we may assume that the endpoints of the edge are \(e_2, e_3 \in U\). Thus, the first component of the control \(u\) is identically zero along the edge. That is, \(u = (0, s, s)\) along the singular arc. Interpreting the vanishing of the first component of \(u\) geometrically as a zero planar-curvature constraint \([3]\), the equation \([1]\) implies that the path \(\sigma_0\) traces out a line in \(\mathbb{R}^2\). After applying an affine transformation to make this line horizontal, we may assume that \(\sigma_0\) has the form

\[
\sigma_0(t) = (\xi(t), -1) = \begin{pmatrix} 1 & -\xi(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

for some function \(\xi : [t_0, t_1] \to \mathbb{R}\). Recall that \(\sigma_0(t) \wedge \sigma_2(t) = \sqrt{3}/2\) (see [Hal11 §3.2]). This implies that \(\sigma_2\) has the form

\[
\sigma_2(t) = \begin{pmatrix} 1 & -\xi(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{3}/2 \\ (1 + s(t))/2 \end{pmatrix}
\]

for some function \(s : [t_0, t_1] \to \mathbb{R}\).

Rather than using the unit speed normalization from \([3]\), it is more convenient to choose a linear parameter such that \(\xi(t) = a_0 t + b_0\). This requires us to make a few minor adjustments to the optimal control problem that are adapted to the singular arc. By picking the parameters \(a_0, b_0\) suitably, we can assume that \(\sigma_0\) starts at time \(t = 0\) and reaches its terminal position on the singular arc at time \(t = 1\). We optimize among trajectories with fixed initial and terminal positions: \((r(0), s(0)) = (r_0, s_0), (r(1), s(1)) = (r_1, s_1)\), where \(r := s'\).

There is a unique \(g : [0, 1] \to \text{SL}_2(\mathbb{R})\) such that \([1]\) holds. We compute

\[
g(t) = \begin{pmatrix} b_0 + a_0 t & (\sqrt{3} - (a_0 t + b_0)s)/\sqrt{3} \\ -1 & s/\sqrt{3} \end{pmatrix}.
\]

Defining \(X\) by \([1]\), we describe the state by the pair of functions \((r, s)\). (Crucially, unlike the treatment above, the function \(g\) is not included in the state.
The terminal condition for \( g \) is already determined by the terminal condition \( s_1 \) of the functions \( s \). The state equations are

\[
\begin{align*}
\dot{s} &= r, \\
\dot{r} &= \frac{2(-1 + 2u)r^2}{-1 - s + 2us}
\end{align*}
\]

The control is now \( u \in [0, 1] \) (representing an edge of the earlier control simplex \( U \)).

Without normalizing to unit speed, the star inequality gives

\[
\det(X) = \frac{a_0 r}{\sqrt{3}} > 0, \quad c_{21}(X) = a_0 > 0.
\]

The cost functional is

\[
-\frac{3}{2} \int_0^1 \text{trace}(JX) dt = \frac{1}{2} \int_0^1 (3a_0 + \sqrt{3}s' + a_0s^2) dt \\
= \frac{1}{2}(3a_0 + \sqrt{3}(s_1 - s_0)) + \frac{a_0}{2} \int_0^1 s^2 dt \\
= C_1 + \frac{a_0}{2} \int_0^1 s^2 dt.
\]

We drop the useless constant \( C_1 \) from the cost and form the Hamiltonian

\[
\nu_1 r + \nu_2 \frac{2(-1 + 2u)r^2}{-1 - s + 2us} + \frac{a_0 \lambda_{\text{cost}} s^2}{2}.
\]

(The Lie group term is no longer present.) The condition for the Hamiltonian to be independent of \( u \) is \( \nu_2 = 0 \). Thus, \( \nu_2 \equiv 0 \) along the singular arc. Solving the adjoint equations, we get \( \nu \equiv 0 \) along the singular arc. The nonvanishing of the costate gives \( \lambda_{\text{cost}} \neq 0 \). The Hamiltonian reduces to \( \frac{a_0 \lambda_{\text{cost}} s^2}{2} / 2 \), which must be constant. Hence \( s \) is constant, and \( \dot{s}' = \dot{r} = 0 \), which contradicts the star inequality (56). Hence, no singular arc exists in this case. We have completed the proof when \( U_{\lambda(t)} \) is an edge for \( t \) in some time interval.

In the remaining case, \( U_{\lambda} = U \) on some time interval. This implies that \( \nu \equiv 0 \) along the singular arc. The adjoint equations and PMP imply that for all \( t \) along the singular arc,

\[
H = -\frac{\partial H}{\partial x} = -\frac{\partial H}{\partial y} = 0.
\]

Solving these equations for \( \Lambda \), we find a unique solution

\[
\Lambda(t) \equiv \frac{3}{2} j_{\lambda_{\text{cost}}},
\]

31
The adjoint equation $\Lambda' = [\Lambda, X] \equiv 0$ implies that $x \equiv 0$, $y \equiv 1$. This is the equation of a circle, which we have seen is a singular arc. As remarked above, a circular arc is not second-order optimal and does not occur in the optimal solution to the Reinhardt problem. This completes the proof.

5.3 switching functions

We define switching functions $\chi_{ij} : T^* M \to \mathbb{R}$ by

$$
\chi_{ij}(\lambda) = H(\lambda, e_i) - H(\lambda, e_j).
$$

The optimal control is constant $u = e_i$ (that is, $U_\lambda = \{e_i\}$), on parts of the cotangent space where $\chi_{ij} > 0$ for all $j \neq i$. For example,

$$
\chi_{32} = 2\sqrt{3} \nu_2 y^2 / (1 - 3x^2),
$$

which equals $\nu_2$, up to a positive factor.

Let $\lambda_{\text{cost}} = -1$ and $\lambda_{\text{sing}, g} \in T^* M$ be the initial conditions matching the circle:

$$
\lambda_{\text{sing}, g} = (\Lambda_{\text{sing}}, \nu_{\text{sing}}) = \left( \frac{3}{2} J \lambda_{\text{cost}}, 0 \right) \in T^*_g z_0 M,
$$

$$
z_0 = (x_0, y_0) = (0, 1) \in \mathfrak{h}, \quad g \in \text{SL}_2(\mathbb{R}).
$$

We define the singular locus $\Lambda_{\text{sing}}$ of $\{-1\} \times T^* M$ by

$$
\Lambda_{\text{sing}} = \{\lambda_{\text{cost}}\} \times \{\lambda_{\text{sing}, g} \mid g \in \text{SL}_2(\mathbb{R})\} \subset T^* M.
$$

We write $\lambda_{\text{sing}} = \lambda_{\text{sing}, 1} \in T^*_I M$. We note that up to the affine transformation $g \in \text{SL}_2(\mathbb{R})$, the singular locus is the initial condition $\lambda_{\text{sing}} \in T^*_I M$ defining the singular circular arc; that is, $g_{\lambda_{\text{sing}}}(0) = I$.

We show that no transition is possible between a Pontryagin extremal link and a circular arc.

**Lemma 5.3.1.** There does not exist a Pontryagin extremal link with constant control $u \in \{e_1, e_2, e_3\}$ with initial conditions (or terminal conditions) $\lambda_{\text{sing}, g}$ and $\lambda_{\text{cost}} = -1$.

(This lemma does not rule out the possibility of a chattering arc meeting a singular arc [AS13, Fig.20.1]. See below.)

**Proof.** By symmetry, we may assume that the link (if it exists) has constant control $u = e_3$. If we take the general solution to the adjoint equation with
control $u = e_3$ and match it with the given initial conditions with normal multiplier $\lambda_{\text{cost}} = -1$, we compute that

$$
\nu_2(t) = \frac{-1 + e^{-\sqrt{3}t} + 2\sqrt{3}te^{-\sqrt{3}t}}{\sqrt{3}}
$$

This function is easily checked to be negative for all $t > 0$. Recall that $\nu_2$ has the same sign as the switching function between controls $u = e_2$ and $u = e_3$. PMP requires $\nu_2(t)$ to be positive when the control is $u = e_3$. Thus, a link that matches initial conditions with the circle cannot be a Pontryagin extremal. \(\square\)

### 5.4 finiteness of switching

We need the following simple lemma in preparation for the main theorem (5.4.2) of this section.

**Lemma 5.4.1.** Let $X : \mathfrak{h} \to \mathfrak{sl}_2(\mathbb{R})$ be given by Equation (13). For all $x + iy \in \mathfrak{h}$,

$$
X, \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y}
$$

(58)

gives a basis of $\mathfrak{sl}_2(\mathbb{R})$.

**Proof.** Let $L$ be a linear transformation that sends the standard basis:

$$
\begin{pmatrix}
1 & 0 \\
0 & \frac{1}{\sqrt{3}}
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix},
$$

to the three vectors (58). The absolute value of the determinant of the linear transformation $L$ is $2/\sqrt{3} \neq 0$. \(\square\)

**Theorem 5.4.2.** Let $\lambda : [0, t_f] \to T^*M$ be a Pontryagin extremal that does not meet the singular locus $\Lambda_{\text{sing}}$. Then $\lambda$ has a bang-bang control with finitely many switches.

**Remark 59.** In terms of Reinhardt’s problem, the theorem implies that an extremal trajectory $\lambda$ that does not meet the singular locus $\Lambda_{\text{sing}}$ defines a hexagonally-symmetric disk $D(g_\lambda, z_\lambda)$ whose boundary is a smoothed polygon, consisting of finitely many straight edges and hyperbolic arcs. A working hypothesis (6.1.1) in the final section describes what would be needed in order to remove the unwanted assumption that $\lambda$ does not meet the singular locus $\Lambda_{\text{sing}}$, and to prove unconditionally that the Reinhardt trajectory is a smoothed polygon.
Proof. Fix a Pontryagin extremal trajectory $\lambda$. By the compactness of the interval $[0, t_f]$, it is enough to show that there are finitely many switches in a neighborhood of each $t \in [0, t_f]$. By reparametrization, we may assume that $t = 0$.

Here, we give the proof when there are at least two independent switching functions $\chi_{ij}$ such that $t = 0$ is a limit point of the zero set of $\chi_{ij}$. Theorem 5.4.3 gives the proof when $t = 0$ is a limit point of the zero set of only one independent switching function.

We define a canonical coordinate system $(\xi_1, \xi_2, \mu_1, \mu_2)$ on the symplectic manifold $T^*\mathfrak{h}$ as follows. Let

$$\begin{align*}
\xi_1 &= x, \\
\xi_2 &= \frac{3y^2}{\sqrt{3x}} + \sqrt{3x} = \frac{3(x^2 + y^2) + \sqrt{3x}}{1 + \sqrt{3x}}.
\end{align*}$$

A short calculation shows that with respect to these coordinates, $\mathfrak{h}^*$ is given by a semi-infinite rectangle:

$$-\frac{1}{\sqrt{3}} < \xi_1 < \frac{1}{\sqrt{3}}, \quad 1 < \xi_2.$$

Let $\mu_i : T^*\mathfrak{h} \to \mathbb{R}$ be the usual canonical coordinates:

$$\lambda = \mu_1(\lambda)d\xi_1 + \mu_2(\lambda)d\xi_2, \quad \lambda \in T^*\mathfrak{h}^*. $$

These canonical coordinates have been chosen to be adapted to the switching functions:

$$\begin{align*}
\chi_{13} &= \mu_1 \frac{6y^3}{1 - 3x^2 - 3y^2}; \\
\chi_{23} &= \mu_2 \frac{12\sqrt{3y^3}}{(1 - \sqrt{3x})(1 + \sqrt{3x})^2}.
\end{align*}$$

Thus, up to irrelevant displayed positive factors, we may take $\mu_1$ and $\mu_2$ to be the switching functions.

Using the rotational symmetries of $U$, we may assume without loss of generality that $t = 0$ is a limit point of the zero sets of both $\mu_1$ and $\mu_2$.

We set $B := \Lambda - 3J\lambda_{\text{cost}}/2$. Then Equation (25) becomes

$$B' = \frac{3}{2}\lambda_{\text{cost}}[J, X] + [B, X]. \tag{60}$$

The Hamiltonian expressed in canonical coordinates takes the form

$$H = g_1\mu_1 + g_2\mu_2 + \langle B, X \rangle \tag{61}$$
for some vector field \((g_1, g_2)\) depending on the control \(U\). Recall that \(H(\lambda(t), u(t)) \equiv 0\) along an extremal \(\lambda\). The adjoint equation for \(\mu_i\) is

\[
\begin{align*}
\mu'_1 &= -\frac{\partial g_1}{\partial \xi_1} \mu_1 - \frac{\partial g_2}{\partial \xi_1} \mu_2 - \left\langle B, \frac{\partial X}{\partial \xi_1} \right\rangle \\
\mu'_2 &= -\frac{\partial g_1}{\partial \xi_2} \mu_1 - \frac{\partial g_2}{\partial \xi_2} \mu_2 - \left\langle B, \frac{\partial X}{\partial \xi_2} \right\rangle
\end{align*}
\]

(62)

We know that \(\mu_1, \mu_2,\) and \(B\) are absolutely continuous by the general properties of optimal control. By the form of the right-hand side of Equation (60), we see that \(B\) is continuously differentiable.

We claim that \(\mu_i\) are continuously differentiable along a Pontryagin extremal trajectory. At issue are the jumps in the functions \(\frac{\partial g_i}{\partial \xi_j}\) for arbitrary control functions \(u\) on the right-hand side of (62). These partial derivatives are bounded, so that the form of Equation (62) implies continuity of \(\mu_i'\) at \(\mu_1 = \mu_2 = 0\). Near a point where exactly one switching function is zero, the control is confined to an edge of \(U\). We argue from the form of Equation (62) (or from general facts about switching functions) that at \(\mu_1 = 0, \mu_2 = 0, \chi_{12} = 0\) respectively, the right-hand side of Equation (62) does not depend on the control restricted to the corresponding edge. This proves the continuity claim.

We assume that \(\mu_1\) and \(\mu_2\) have infinitely many zeros that accumulate at \(t = 0\). By continuity and Rolle’s theorem,

\[
\mu_1(0) = \mu'_1(0) = 0, \quad \mu_2(0) = \mu'_2(0) = 0.
\]

By Equations (61), (62) and Lemma 5.4.1, we have \(B(0) = 0\).

The lifted trajectory is not abnormal, for otherwise

\[
\lambda_{cost} = \Lambda(0) = B(0) = \mu_1(0) = \mu_2(0) = 0
\]

contrary to the PMP-projectivity condition. We set \(\lambda_{cost} = -1\).

We claim that \(B'(0) = 0\). We have

\[
\langle B'(0), X(0) \rangle = (3/2)\lambda_{cost} \langle [J, X(0)], X(0) \rangle = 0.
\]

In light of Lemma 5.4.1 to prove the claim, we assume for a contradiction that

\[
\epsilon \left\langle B'(0), \frac{\partial X}{\partial \xi_i}(0) \right\rangle > 0
\]

for some \(i\) and choice of sign \(\epsilon \in \{\pm 1\}\). By the mean-value theorem \(\mu_i(t) = t\mu'_i(\tau_i) = o(t)\) for some \(\tau_i \in (0, t)\). By Equation (63), the forcing term \(\langle B, \frac{\partial X}{\partial \xi_i} \rangle\) in Equation (62) dominates near \(t = 0\), and we have \(\epsilon \mu'_i(t) > Ct > 0\) for some
$C > 0$ and for all sufficiently small $t > 0$. This contradicts our assumption that $t = 0$ is a limit point of the zero set of $\mu_i$.

Note

$$0 = B'(0) = (3/2)\lambda \cos[J, X(0)].$$

So $[J, X(0)] = 0$, which implies $X(0) = J$ so that $x_0 = 0$ and $y_0 = 1$. This completes the proof, except for the missing piece supplied by Theorem 5.4.3.

**Theorem 5.4.3.** Let $\lambda : [0, t_f] \to T^*M$ be a Pontryagin extremal that does not meet the singular locus $\Lambda_{\text{sing}}$. Assume that two of the switching functions (say $\chi_{12}$ and $\chi_{13}$) only have finitely many zeros in some sufficiently small neighborhood of $t = 0$. Then the third switching function $\chi_{23}$ also has only finitely many zeros in some sufficiently small neighborhood of $t = 0$.

**Proof.** We may take the switching function $\chi_{23}$ to be $\nu_2$ (up to a positive factor). We assume for a contradiction that $t = 0$ is a limit point of the zero set of $\nu_2$. We recall that $\nu_2$ is continuous by the PMP. These observations imply that $\nu_2(0) = 0$.

We claim that the control function $u$ takes values in the edge $U_{23} = \{(0, *, *)\} \subset U$ (up to a set of measure zero). In fact, $\chi_{12} < 0$ and $\chi_{13} < 0$ except at finitely many points in a small neighborhood of $t = 0$. So $U_{\lambda(t)} \subseteq U_{23}$, and the claim follows.

We claim that $\nu_2$ is continuously differentiable. This follows by examining the ODE it satisfies:

$$\nu_2' = -f_{1y}\nu_1 - f_{2y}\nu_2 - \left\langle B, \frac{\partial X}{\partial y} \right\rangle.$$ 

The term $f_{1y}$ is independent of the control along the edge $U_{23}$ and is therefore continuous. The term $f_{2y}$ is bounded and has jumps only when $\nu_2 = 0$. The terms $\nu_1, \nu_2, B, \partial X/\partial y$ are continuous. This gives the claim.

We claim that $\nu_2'(0) = 0$. In fact, by Rolle, $\nu_2'(t)$ has infinitely many zeros that accumulate at $0$. The claim follows.

We define $\lambda^{(i)}(\lambda_0, t)$ to be the lifted trajectory with constant control $u = e_i \in U_{23}$ and initial condition $\lambda^{(i)}(\lambda_0, 0) = \lambda_0 \in T^*M$, for $i = 2, 3$. The lifted trajectories $\lambda^{(i)}(\lambda_0, t)$ are real analytic in $\lambda_0$ and $t$. Let $\nu_2^{(i)}(\lambda_0, t)$ be the $\nu_2$-component of $\lambda^{(i)}(\lambda_0, t)$. We have a leading term

$$\nu_2^{(i)}(\lambda_0, t) = t^d a_d + O(t^{d+1}), \quad d = d^{(i)}(\lambda_0), \quad a_d = a_d^{(i)}(\lambda_0) \neq 0.$$

We restrict to parameters $\lambda_0$ near $\lambda(0)$ such that $\nu_2^{(i)}(\lambda_0, 0) = 0$ so that $d^{(i)}(\lambda_0) > 0$. 

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We claim that \( d(i) = d(\lambda_0) = a_d(i) = a_d(\lambda_0) \) are independent of \( i \) and that \( 1 \leq d(\lambda_0) \leq 3 \). To prove the claim, we compute the power series expansion of \( \nu_2(i)(\lambda_0, t) \) at \( t = 0 \) using the explicit solutions to the ODE; and we compare the coefficients for \( i = 2, 3 \). Explicit formulas are found in the computer code. When \( d(i) > 2 \), we compute that \( a_3(i) = -1 \neq 0 \), which is independent of both \( i \) and \( \lambda_0 \). In particular \( d \leq 3 \). This proves the claim.

The theorem follows more or less from this last claim. By the Weierstrass preparation theorem, the zero set of \( \nu_2(i)(\lambda_0, t) \) coincides with that of a polynomial of degree \( d(\lambda_0) \) in \( t \) for all small \( t \) and \( \lambda_0 \) in a neighborhood of \( \lambda(0) \).

The idea is that switching function \( \nu_2 \) is closely approximated by both of the analytic functions \( \nu_2(i) \), for \( i = 2, 3 \), so that \( \nu_2 \) can have at most \( d \leq 3 \) zeros near \( t = 0 \).

Note that \( d(\lambda_0) \leq d(\lambda(0)) \) for \( \lambda_0 \) near \( \lambda(0) \).

Assume first that \( d(\lambda(0)) = 1 \). Because we are restricting to parameters \( \lambda_0 \) such that \( d(\lambda_0) > 0 \), we have \( d(\lambda_0) = 1 \) for all \( \lambda_0 \) near \( \lambda(0) \). The paths \( \nu_2(i)(\lambda_0, t) \) meet the switching hypersurface transversely. The continuous differentiability of \( \nu_2 \) implies a single switch from control \( u = e_i \) to \( u = e_j \) at the switching hypersurface.

In the remaining case, \( d(\lambda(0)) \in \{2, 3\} \). Pick small \( t_0 \) that is not a switching time. Then \( U_{\lambda(t_0)} = \{e_i\} \) for some \( i \in \{2, 3\} \). Then \( \nu_2(t_0) = \nu_2(i)(\lambda_0, t_0) \) for some \( \lambda_0 \) near \( \lambda(0) \), where \( d(\lambda_0) \leq d(\lambda(0)) \). Because \( d \leq 3 \), the Weierstrass polynomials for \( \nu_2(i)(\lambda_0, t) \) at \( t = 0 \) have at most one zero \( t_1 \) of multiplicity greater than 1. The time \( t_1 \), when it exists, is independent of \( i \). Thus, every time \( t \neq t_1 \) lies on a semi-infinite interval \((t_1, \infty)\) or \((-\infty, t_1)\) on which \( \nu_2 \) meets the switching surface transversely, with isolated switchings between controls \( u = e_2 \) and \( u = e_3 \) before leaving the small neighborhood of \( t = 0 \). This implies that \( \nu_2 \) does not have a limit point at \( t = 0 \).

6 Discussion of proposed endgames

In this section we offer some speculations about how the proof of the Reinhardt conjecture might be completed.

6.1 smoothed polygons

We have proved that a Pontryagin extremal trajectory \( \lambda \) that does not meet the singular set \( \Lambda_{sing} \) gives a smoothed polygon \( D(g_\lambda) \). Suppose that \( \lambda \) meets
λ_{sing}. We have proved that λ does not remain in Λ_{sing} for any time interval and that the only way to approach Λ_{sing} is through chattering. These are very restrictive conditions.

We suggest a working hypothesis that would complete the proof that the solution to the Reinhardt problem is a smoothed polygon. Our restrictive conditions reduce the analysis to a small neighborhood of a single point λ_{sing} in the cotangent space.

If a lifted trajectory λ meets Λ_{sing}, we may assume that the meeting occurs at \( t = 0 \) and that \( λ(t) \notin Λ_{sing} \) for some sufficiently small time interval \( t \in (0, t_0] \).

To be concrete, we may assume after applying an affine transformation that \( λ(0) = λ_{sing} \in Λ_{sing} \). Then the lifted trajectory on this interval has a bang-bang control with infinitely many switching times

\[
 t_1 > t_2 > \cdots > t_k > \cdots > 0, \tag{64}
\]

where \( t_0 \geq t_1 \) and \( \lim_{k \to \infty} t_k = 0 \). If we could show that such a trajectory is not globally optimal among trajectories in \( M \) with the same endpoints, then chattering is nonoptimal, and we would conclude established that the solution to the Reinhardt problem is a smoothed polygon.

**Working Hypothesis 6.1.1.** Let \( λ \) be a chattering extremal trajectory with bang-bang control starting at \( λ(0) = λ_{sing} \) as just described. Then there exists \( t^* \in (0, t_0) \) and a competing lifted trajectory \( λ^* \) on \([0, t^*]\) with lower cost

\[
 \text{cost}(λ^*) < \text{cost}(λ)
\]

over the interval \([0, t^*]\), and having the same endpoints in \( M \) as \( λ: \)

\[
 (g_{λ}(0), z_{λ}(0)) = (g_{λ^*}(0), z_{λ^*}(0)) = (I, i) \in SL_2(\mathbb{R}) \times \mathfrak{h},
\]

\[
 (g_{λ}(t^*), z_{λ}(t^*)) = (g_{λ^*}(t^*), z_{λ^*}(t^*)).
\]

To prove this working hypothesis, various standard methods for the treatment of chattering controls might be helpful: blowing-up along the singular locus, the Poincaré map, scaling, and self-similarity. See [ZB12].

### 6.2 a neighborhood of the circle

We have constructed extremal trajectories with bang-bang controls that have an arbitrarily large number of switches. Each neighborhood of \( V \) of Λ_{sing} contains all but finitely many of these extremal trajectories. We expect that extremal lifted trajectories λ that remain close to Λ_{sing} to give hexagonally symmetric disks \( D(g_{λ}) \) that are approximately circles. In particular, they should have cost higher than that of the smoothed octagon.

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Working Hypothesis 6.2.1. There exists (an explicit) neighborhood $V = V_{\text{sing}}$ of $\Lambda_{\text{sing}}$ and a natural number $N_V$ such that any extremal trajectory that does not meet $V$ has at most $N_V$ switches. Every extremal trajectory meeting $V$ has cost greater than that of $\lambda_{\text{oct}}$.

6.3 heuristics near the boundary of $\mathfrak{h}^*$

Recall that $\mathfrak{h}^*$ is the open subset of the upper-half plane that satisfies the star inequalities. We present some heuristics that suggest that trajectories that come close to the boundary of $\mathfrak{h}^*$ necessarily give hexagonally symmetric disks whose area is greater than that of the smoothed octagon. We formulate this as a working hypothesis.

Working Hypothesis 6.3.1. There exists (an explicit) neighborhood $V = V_*$ of the boundary of $\mathfrak{h}^*$ such that if (the projection $z_\lambda$ to $\mathfrak{h}^*$ of) an extremal admissible trajectory $\lambda$ meets $V$, then the cost of $z_\lambda$ is greater than the cost of the smoothed octagon.

Remark 65. The point of the working hypothesis is to allow us to replace $\mathfrak{h}^*$ with a slightly smaller compact set $K = \mathfrak{h}^* \setminus V$. The star inequalities (Remark 23) are strict because when equality is obtained the smallest centrally symmetric hexagon $H$ containing $D$ degenerates to a parallelogram. However, a parallelogram is the smallest centrally symmetric hexagon $H$ containing $D$ only if $D = H$ is itself a parallelogram of area $\sqrt{12}$. This suggests that extremal trajectories that come sufficiently close to the boundary of $\mathfrak{h}^*$ are approximately parallelograms of approximate area $\sqrt{12}$. Such $D$ are far from optimal. By making this intuition rigorous, a proof of the working hypothesis might be obtained.

6.4 direct computer search

We suggest two different ways that the Reinhardt conjecture might be completed from here: direct computer search or geometric methods. We can summarize the previous subsections by saying that we expect the only interesting lifted trajectories in the cotangent space to pass

1. near the boundary of $\mathfrak{h}^*$ (where $D$ is a near parallelogram);
2. near $\Lambda_{\text{sing}}$ (where $D$ is a near circular disk, including smoothed polygons $D_{6k\pm 2}$ with many sides);
3. near $\lambda_{\text{oct}}$ (where $D$ is a smoothed octagon).
Let us assume that we have a version of Theorem 4.5.1 that gives an explicit neighborhood $V_{oct}$ of $\lambda_{oct}(0)$ on which the local optimality of $\lambda_{oct}$ holds.

Each Pontryagin extremal trajectory is determined by an initial condition in $\mathbb{R}_{\text{cost}} \times T^*_\star M$. It is convenient to consider the projectivized variant:

$$\mathbb{P}(\mathbb{R}_{\text{cost}} \times T^*_\star M^*),$$

This is an explicit 7-dimensional manifold. It can be reduced by two dimensions to a 5-dimensional manifold by the vanishing of the maximized Hamiltonian [3.3] and setting the start time $t = 0$ at a switching time between controls $u = e_3$ and $u = e_2$, which gives $\nu^0_2 = 0$.

We might try to make a direct computer search through this space (say using interval arithmetic) and show that there is nothing better than the smoothed octagon. We might find for example by explicit search that the smoothed polygons of Section 4.4 are the only Pontryagin extremal trajectories (away from the singular arc).

Using our working hypotheses, by excluding a neighborhood $V_*$ of the boundary of $h^*$, a neighborhood $V_{oct}$ of $\lambda_{oct}$, and a neighborhood $V_{sing}$ of $\Lambda_{sing}$, we expect numerically stable lifted trajectories with a uniformly bounded number of switches. Given an initial condition $\lambda_0$ in the 5-dimensional manifold, we extend the trajectory until it enters one of these excluded neighborhoods $V$ (in which case we reject $\lambda_0$), until $\lambda$ meets the terminal conditions (in which case we compare the trajectory’s cost to $\lambda_{oct}$), or until $t \geq \pi/3$ (in which case we reject the trajectory if it has higher cost than the circle by Equation 19).

### 6.5 geometric methods

The transversality conditions of PMP imply that a Pontryagin extremal lifted trajectory is a closed loop $\lambda$ in

$$\mathbb{P}(\mathbb{R}_{\text{cost}} \times T^* M \setminus \Lambda_{sing})/\langle R \rangle.$$

(We remove a neighborhood of the singular locus $\Lambda_{sing}$.)

We can consider an optimization over each homology class. We have homotopy group $\pi_1(\text{SL}_2(\mathbb{R})) = \mathbb{Z}$ and the canonical map $\pi_1(\text{SL}_2(\mathbb{R})) \to \pi_1(\text{SL}_2(\mathbb{R})/R) = \mathbb{Z}$ is multiplication by 3. The Reinhardt lifted trajectory gives a generator of $\pi_1(\text{SL}_2(\mathbb{R})/R)$. We may restrict to such trajectories.

We might try to adapt the arguments of [AS13, Chapter 17]: the Poincaré-Cartan integral invariant, Hamilton-Jacobi-Bellman, etc.
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