On timelike and spacelike hard exclusive reactions.

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We show to next-to-leading order accuracy in the strong coupling $\alpha_s$ how the collinear factorization properties of QCD in the generalized Bjorken regime relate exclusive amplitudes for spacelike and timelike hadronic processes. This yields simple space–to–timelike relations linking the amplitudes for electroproduction of a photon or meson to those for photo- or meso-production of a lepton pair. These relations constitute a new test of the relevance of leading twist analyses of experimental data.

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In the traditional collinear factorization framework the scattering amplitude for exclusive processes has been shown \cite{1} to factorize in specific kinematical regions, provided a large scale controls the separation of short distance dominated partonic subprocesses and long distance hadronic matrix elements. This large scale may come from a spacelike momentum exchange, as in hard lepton-production processes, or from a timelike momentum as in electron-positron annihilation or lepton pair production.

The complementarity of spacelike and timelike processes has been much used in inclusive reactions to understand in detail parton distribution and parton fragmentation functions, in particular through deep inelastic lepton production and Drell–Yan processes in hadron reactions. In the realm of exclusive reaction, much work has been devoted to the electromagnetic form factors. In particular, the spacelike and timelike meson form factors were analyzed in great details in Ref. \cite{2}.

Analyticity of the factorized amplitude is the basic property that allows us to derive the new relations Eqs. \cite{1,2,3} at the heart of our paper. Analyticity, which is a consequence of causality in relativistic field theory, and factorization of short distance vs long distance properties, are common tools in many fields of theoretical physics. Our instance is to our knowledge the first case where they are put together to obtain useful relations between observables.

We shall detail two instances of direct interest to near future phenomenological studies, illustrated in Fig.1, firstly near forward deeply virtual Compton scattering (DVCS) and timelike Compton scattering (TCS), and secondly near forward deeply virtual meson leptoproduction (DVMP) and mesoproduction of a lepton pair. The momentum transfer square $t$ in these processes is taken to be small w.r.t. the large virtuality of one photon.

The DVCS and TCS amplitudes. Let us begin with near forward virtual Compton scattering

$$\gamma^{(e)}(q_{in})N(p) \to \gamma^{(e)}(q_{out})N'(p').$$

In its spacelike version the DVCS amplitude is accessible in deep electroproduction of a photon, i.e., $q_{out}^2 = 0$,

$$e(k_1)N(p) \to e'(k_2)\gamma(q_{out})N'(p')$$

with a large spacelike virtuality $q_{in}^2 = (k_1 - k_2)^2 = -Q^2$. The timelike TCS amplitude is accessible in the photoproduction, i.e., $q_{in}^2 = 0$, of a lepton pair \cite{4}:

$$\gamma(q_{in})N(p) \to l^+(k^+)l^-(k^-)N'(p')$$

with a large timelike virtuality $q_{out}^2 = (k^+ + k^-)^2 = +Q^2$. The other common variables, describing the processes of interest in this generalized Bjorken limit, are the scaling variable $\xi$ and skewness $\eta > 0$:

$$\xi = -\frac{q_{out}^2 + q_{in}^2}{q_{out}^2 - q_{in}^2}, \quad \eta = \frac{q_{out}^2 - q_{in}^2}{(p + p') \cdot (q_{out} + q_{in})}.$$ 

Hence, $\xi = +\eta > 0$ in DVCS and $\xi = -\eta < 0$ in TCS kinematics. This allows us to relate spacelike and timelike amplitudes for equal $\eta$, $t$, and $Q^2$ values by the rule:

$$F(\xi = \eta, t, Q^2) \rightarrow \frac{vL}{vL} F(\xi = -\eta, t, -Q^2),$$

where $vL$ is the helicity flip probability of the lepton pair.
where the c.o.m. energy square $s = (p + q_m)^2$ might differ.

We first study the DVCS amplitude which is usually parameterized in terms of Compton form factors (CFFs) \( \mathcal{F} \). After renormalization, a leading twist CFF read as sum over quarks \((q)\) and gluon \((g)\) in its factorized form:

\[
\mathcal{F}(\xi, t, Q^2) = \int \frac{dx}{1} \sum_{i=u,d,\cdots} ST_i(x, \xi) F_i(x, \xi, t, \mu^2),
\]

where we adopt for generalized parton distributions (GPDs) \( F_i^\ast \) the common conventions \( [1] \). The hard coefficients \( ST_i \) depend on both the virtuality and factorization scale and read to next-to-leading order (NLO) accuracy in \( \alpha_s \):

\[
ST_i^{NLO} \equiv SC_i^0 + \frac{\alpha_s(\mu^2)}{2\pi} \left[ SC_i^1 + SC_{coll}^i \ln \frac{Q^2}{\mu^2} \right].
\]

Note that the possible distinction between factorization scale \( \mu_F \) and renormalization scale \( \mu_R \) has no consequence on our arguments and we simplify notations by equating \( \mu = \mu_F = \mu_R \). The collinear coefficient \( SC_{coll}^i \) are given as convolution of the Born term \( SC_0^i \) with the GPD evolution leading order (LO) kernels. Since DVCS amplitude is symmetric under \( s \leftrightarrow u \)-channel crossing, the CFFs and \( SC_i \) coefficients have definite symmetry properties under \( \xi \)-reflection. Moreover, boost-invariance tells us that all \( SC_i \) coefficients are functions of the variable \( x/\xi \), apart from an overall scaling factor. For \( \xi \) (or \( s \leftrightarrow u \)) symmetric coefficients we write here explicitly

\[
C_0^i(x, \xi) = e_q^i \left( \frac{1}{\xi - x} - \frac{1}{x + \xi} \right), \quad C_0^\ast(x, \xi) = 0. \tag{8}
\]

Note that gluons do not contribute in LO but at NLO:

\[
C_{coll}^q(x, \xi) = -\frac{1}{2} \sum_x e_x^q \frac{\ln \xi - x}{\xi + x} + (x \to -x), \tag{9}
\]

\[
C_1^q(x, \xi) = \frac{1}{2} \sum_x e_x^q \left( \frac{3\xi - x}{\xi + x} - \frac{1}{2} \ln \frac{\xi - x}{x} - \frac{2}{2\xi} \ln \frac{\xi - x}{x} \right), \tag{10}
\]

\[
\times \ln \frac{\xi - x}{x} + (x \to -x),
\]

where the remaining quark and antisymmetric coefficients in this representation can be read off from \( [1] \). Obviously, for \( z = x/\xi \) these functions are holomorphic in the complex plane except for a \(-u\)-(and \(-u\))-channel poles at \( z = 1 \) \((z = -1)\) and \(-s\)-(and \(-u\))-channel cuts \([1, \infty] \) \([(-\infty, -1)]\) on the real axis. Their physical value on the cuts (or poles) is governed by causality, i.e., by the \(+i\epsilon\) prescription of propagators. This yields the extension of the scaling variable \( \xi_S = \xi - i\epsilon \) into the complex domain, which can be also read off from \( \xi = Q^2/(2s + Q^2) \), resulting from \( [1] \), and decorating \( s \) with \(+i\epsilon\). All hard coefficients in \( (8-10) \) can be then uniquely extended:

\[
SC_i^\ast(x, \xi) = C_i^\ast(x, \xi_S), \tag{11}
\]

consistent with the physical sheet.

For the TCS amplitude, i.e., time-like CFFs, the situation is in general more intricate due to existence of possible poles and cuts, caused by the time-like virtuality of the outgoing photon. However, in our perturbative description of TCS we require that we are away from the resonance region and we might employ the substitution rule \([3, 4]\) and causality to find the hard coefficients in the timelike region by analytic continuation, e.g., from \([6]\) \([8]\). However, from \([3]\) we immediately see that the factorization \( \ln \frac{Q^2}{\mu^2} \) goes into \( \ln \frac{-Q^2}{\mu^2} \), providing us additional \( \pm i\pi C_{coll}^i \) terms at NLO. To pick up the proper sign, we might analyze Feynman diagrams or, equivalently, we can use a convolution representation for DVCS coefficients, in which the \( \ln \frac{Q^2(\xi - ie + x)}{2\mu^2} \) appears \( [1] \). As we show below, the rule \([3]\) together with the \( ie\) prescription provides then an unique answer.

Let us verify this statement and also provide us a more usable time-like-to-space-like relation. At Born level we easily realize in accordance with a diagrammatic evaluation a rule, conveniently written with \( \xi_T = \eta + i\epsilon \):

\[
\frac{1}{\xi - i\epsilon + x} \overset{SL \to TL}{\rightarrow} \frac{1}{-\eta - i\epsilon + x} = -\frac{1}{\xi_T + x} = -\frac{1}{x_S^{\pm} \pm x}. \tag{12}
\]

This exercise exemplifies our main result, namely, the time-like \( s(u)\)-channel coefficients are given by complex conjugation of the space-like \( u(s)\) one. Utilizing Schwarz reflection principle, we write for a generic \((N)LO\) coefficient:

\[
SC(x, \xi_S) \overset{SL \to TL}{\rightarrow} T C(x, \xi_T) = \mp S C^\ast(-x, \xi_S), \tag{13}
\]

where the upper sign applies for quarks and the lower for gluons (compared to quark GPDs our gluon GPDs contain a relative \( x \) and so quark and gluon coefficients have different symmetry properties under \( \xi\) and \( x\)-reflection). From the analyticity of hard coefficients, see, e.g., \([3, 4, 7]\), and the substitution \([3]\) we also establish the rule \([3]\) at NLO. As said, there is an additional imaginary part, uniquely fixed by causality, that is associated with the factorization logarithms \((\ln\text{’}s)\). Indeed, in a diagrammatic NLO calculation \([3]\) we realize that they appear in \( \ln \frac{\xi_T}{\mu^2} \) and \( \ln \frac{i\epsilon + \xi}{\mu^2} \) terms, where \( s = \frac{Q^2}{2\xi} \) and \( \hat{u} = \frac{-Q^2}{2\xi} \) are Mandelstam variables for partonic sub-processes. In the DVCS case the \( \hat{s}\) cut is contained in:

\[
\ln \frac{-\hat{s}S - i\epsilon}{\mu^2} = \ln \frac{Q^2}{2\xi^2 \mu^2} + \ln(\xi - i\epsilon - x), \tag{14}
\]

where after applying \([3]\) goes into the TCS expression, which can then be expressed by the space-like \( u\)-channel contribution and a \(-i\pi\) addendum:

\[
\ln \frac{-\hat{s}T - i\epsilon}{\mu^2} = \ln \frac{2\eta}{2\mu^2} + \ln(-\eta - i\epsilon - x) \tag{15}
\]

\[
= \left[ \ln \frac{-\hat{u}S - i\epsilon}{\mu^2} \right]^* - i\pi.
\]
An analogous result holds for the $\hat{u}$-channel and, thus, independently from the considered channel the space–to–timelike relation \(4\) is accompanied by
\[
\ln \frac{Q^2}{\mu^2}^{\text{SL} \rightarrow \text{TL}} = \ln \frac{Q^2}{\mu^2} - i\pi . \tag{16}
\]

Employing the space–to–timelike relation \(13\) to the net NLO coefficient \(1\), we find the timelike ones:
\[
T_{T^{\text{NLO}}}^{i} = \pm S_{T^{\text{NLO}}}^{i} + i\pi \frac{\alpha_s}{2\pi} C_{s}^{\text{coll}} ; \tag{17}
\]

upper (lower) sign applies to $\xi$-(anti)symmetric CFFs.

For the symmetric case the space–to–timelike relation \(14\) has been exemplified by a diagrammatic NLO evaluation \(15\). As we have seen, \(17\) arises from general field theoretical principles and is an example of a more general result for hard NLO coefficients at twist-two accuracy.

**DVMP and exclusive Drell-Yan.** Let us now turn to a slightly different pair of reactions where amplitudes factorize in both GPDs and a meson distribution amplitude (DA). Specifically, we consider $\gamma^*_L N \rightarrow \pi N'$, a subprocess in near forward lepton production, and $\pi N \rightarrow \gamma^*_L N'$, appearing in the exclusive limit of Drell–Yan process. The factorization theorem \(13\) states that the $\gamma^*_L p \rightarrow \pi^* n$ amplitude, written in terms of $F_\pi^\pm (\xi, t, Q^2)$ transition form factors (TFFs), factorizes up to a constant factor as
\[
\tilde{F} \propto \frac{1}{Q^2} \int du \int dx \tilde{F}_{ud}(x, \xi, t)S_{T_{ud}}(u, x, \xi)\varphi_\pi(u) . \tag{18}
\]

Here, $ud$ denote the exchanged quark pair, the flavor off-diagonal GPD $\tilde{F}_{ud} = \tilde{F}_{u} - \tilde{F}_{d}$ is expressed by diagonal ones via $SU(2)$ symmetry, and the pion DA $\varphi_\pi$ is symmetric w.r.t. $u \rightarrow 1 - u$. In analogy to DVCS, we introduce $C(u, x, \xi)$ coefficients and write
\[
S_{T_{ud}} = \left[ e_u C(u, x, \xi_S) - e_d C(u, x, -\xi_S) \right] , \tag{19}
\]

where the physical sheet is picked up by $-i\epsilon$ in $\xi_S$. Note that we use here and in the following $u \rightarrow 1 - u$ symmetry and that already the LO result is proportional to $\alpha_s (\mu^2)$,
\[
C^{\text{NLO}} = \alpha_s (\mu^2) C_0 + \frac{\alpha_s^2 (\mu^2)}{2\pi} \left[ C_{\text{div}} \ln \frac{Q^2}{\mu^2} + C_1 \right] , \tag{20}
\]
\[
C_0(u, x, \xi) = \frac{1}{u(\xi - x)} , \tag{21}
\]
\[
C_{\text{div}} = -\frac{\beta_0}{2} C_0 + C_F^{\text{coll}} + C_S^{\text{coll}} . \tag{22}
\]

Here, $\beta_0 = 11 - 2n_f / 3$ controls the running of $\alpha_s$ at LO, the collinear coefficients $C_F^{\text{coll}}$ and $C_S^{\text{coll}}$ are given as convolution of LO evolution kernels with the LO coefficient \(21\). All these coefficients can be obtained from known pion form factor results \(1\). $C_{\cdot}$ are $Q^2$ independent. Moreover, analytic properties, seen in DVCS coefficients such as \(17\), hold for the coefficients in \(22\) as function of $z = x/\xi$, too, which justifies the replacement $\xi \rightarrow \xi_S$ in \(19\) \(14\).

The factorization proof \(3\) may be extended to the crossed reaction $\pi N \rightarrow \gamma^*_L N'$, appearing in $\pi^* p \rightarrow \gamma^*_L n$, might be in full analogy to the space like form factor \(13\) written as convolution of $F_{\pi u} = -F_{\pi ud}$ GPD and pion DA, where hard coefficients read to LO accuracy as \(10\):
\[
T_{du}(u, x, \xi_T) = \left[ e_u C_0(u, x, \xi_T) - e_d C_0(u, -x, \xi_T) \right] . \tag{23}
\]

Taking the physical sheet in spacelike region, the reflection \(3\) implies the space–to–timelike relation \(14\) for NLO coefficients. As in DVCS, from the explicit NLO result we can read of the rule \(16\) for the continuation of renormalization and factorization ln’s, e.g., the ln’s of the $\beta_0$ proportional part can be collected in a ln $\frac{u^2 - \mu^2}{\mu^2}$ or ln $\frac{u^2}{u^2}$ term. Hence, both rules can be employed to the net coefficient \(13\) and so we obtain with \(20\) \(22\) the NLO approximation for timelike coefficient \(23\), where
\[
T C (u, x, \xi_T) = \left[ C^{*} - i\pi \frac{\alpha_s}{2\pi} C_{\text{div}} \right] (u, -x, \xi_S) . \tag{24}
\]

Note that coefficients for spacelike [timelike] TFFs $\tilde{F}_{\pi^*}$ [$T F_{\pi^*}$] for $\pi^*$ DVMP [and exclusive Drell–Yan in $\pi^*$] off neutron follows from \(19\) \(23\) by $u \leftrightarrow d$ exchange.

This result generalizes the relation obtained in Ref. \(1\) between the timelike and spacelike pion form factors, in which only the first and third term on the r.h.s. of Eq. \(22\) appear.

**Phenomenological perspectives.** As we have seen, the space–to–timelike relation of hard coefficients is at NLO modified by $-i\pi$ proportional terms that are associated with factorization and renormalization ln’s. Since GPDs and DA are real valued, our findings imply a relation among CFFs or TFFs. In the case of $\xi$-(anti)symmetric CFFs, called $\mathcal{H}$ ($\tilde{\mathcal{H}}$) and $\mathcal{E}$ ($\tilde{\mathcal{E}}$), Eq. \(17\) yields, e.g.,
\[
T_{\mathcal{H}}^{\text{NLO}} = \mathcal{H}^* - i\pi Q^2 \frac{\partial}{\partial Q^2} \mathcal{H}^* , \tag{25}
\]
\[
T_{\tilde{\mathcal{H}}}^{\text{NLO}} = -\tilde{\mathcal{H}}^* + i\pi Q^2 \frac{\partial}{\partial Q^2} \tilde{\mathcal{H}}^* . \tag{26}
\]

Analog relations connects (up to a conventional phase) timelike $\pi^\pm$ with spacelike $\pi^\mp$ TFFs, see \(18\) \(19\) \(23\) \(24\).
\[
T_{\mathcal{H}}^{\pm} \cong \mathcal{H}^{\pm} - i\pi Q^2 \frac{\partial}{\partial Q^2} Q^{\pm} \mathcal{H}^{\mp} . \tag{27}
\]

The NLO relations \(23\) \(27\) tell us that if scaling violations are small, the timelike CFFs (TFFs) can be obtained from the spacelike ones by complex conjugations. Moreover, GPD model studies indicate that in the valence region, i.e., for $\xi \sim 0.2$, CFFs might only evolve mild. This rather generic statement, which will be quantified by model studies \(13\), might be tested in future (after 12GeV upgrade) Jefferson Lab experiments.
On the other hand it is known that the evolution of CFF $H$ in the small $\xi$ region is driven by the “pomeron” pole in the gluon evolution kernel which also interfere with the effective “pomeron” intercepts of GPDs at the input scale. The effective “pomeron” trajectory induces then that the imaginary part $\Im H$ dominates over the real one $\Re H$, which is consistent with a phenomenological analysis of HERA data [14]. Since of the $-i\pi$ proportional NLO addenda in (25), the small $\Re H$ will only mildly influence the LO prediction $\Im H^{LO} = -3mH$. On the other hand we expect huge NLO corrections to $\Re H^{LO}$ $\Re H$, induced by $3mH$. Utilizing Goloskokov-Kroll model for $H$ GPDs [13], we illustrate this effect in Fig. 2 for $10^{-4} < \xi < 10^{-2}$, accessible in a suggested Electron-Ion-Collider [1], and $t = 0$. We plot $\Re H$ vs. $\xi$, for LO DVCS or TCS (solid), NLO DVCS (dashed) and NLO TCS (dotted) at the input scale $\mu^2 = Q^2 = 4$ GeV$^2$. In the case of NLO TCS $-\Re H^{T}$ is shown, since even the sign changes. We read off that the NLO correction to $\Re H^{T}$ is of the order of $-400\%$ and so the real part in TCS becomes of similar importance as the imaginary part. This NLO prediction is testable via a lepton-pair scattering into a massive lepton pair and nucleon. Here also analyticity allows to relate NLO corrections in both processes. We shall discuss that elsewhere.

Conclusions. We have shown that the factorization property of exclusive amplitudes at leading twist together with analyticity allow to link various processes at NLO accuracy. Thereby, we specialized to near forward processes in the generalized Bjorken regime where collinear factorization holds. The space–time-like relation [13,14] helps to understand the previously published result of [1], leads to new results written in [14,15], and indicates a more general relation that might be established by a perturbative analyze of Feynman diagrams.

The extension of $q\bar{q}$ and $gg$ exchange to $qqq$ exchange in a generalized Bjorken regime, much related to the DVCS one, generalizes the GPD concept, yielding the definition of transition distribution amplitudes (TDAs) and to a factorized formula for backward DVCS and backward lepton production of a $\pi$ meson [14]. For the latter $\pi N$ TDAs factorize from the hard subprocess. The corresponding timelike processes occur in meson proton scattering into a massive lepton pair and nucleon. Here also analyticity allows to relate NLO corrections in both processes. We shall discuss that elsewhere.

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FIG. 2: The real part of CFF $H$ vs. $\xi$ with $\mu^2 = Q^2 = 4$ GeV$^2$ and $t = 0$ at LO (solid) and NLO for DVCS (dashed). For TCS at NLO its negative value is shown as dotted curve.

\[ |\Re H(\xi, Q^2, 0) = j_q Q^2| \]

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