Replica symmetric spin glass field theory

T. Temesvári

Research Group for Theoretical Physics of the Hungarian Academy of Sciences, 
Eötvös University, Pázmány Péter sétány 1/A, H-1117 Budapest, Hungary

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A new powerful method to test the stability of the replica symmetric spin glass phase is proposed by introducing a replicon generator function $g(v)$. Exact symmetry arguments are used to prove that its extremum is proportional to the inverse spin glass susceptibility. By the idea of independent droplet excitations a scaling form for $g(v)$ can be derived, whereas it can be exactly computed in the mean field Sherrington-Kirkpatrick model. It is shown by a first order perturbative treatment that the replica symmetric phase is unstable down to dimensions $d \lesssim 6$, and the mean field scaling function proves to be very robust. Although replica symmetry breaking is escalating for decreasing dimensionality, a mechanism caused by the infrared divergent replicon propagator may destroy the mean field picture at some low enough dimension.

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I. INTRODUCTION

The theory of spin glasses has by now become more than three decades old, even if one considers the archetypal model by Edwards and Anderson as its beginning. Nowadays spin glasses are regarded as the prototypes of complex systems showing highly nontrivial equilibrium properties (massless glassy phase without long-range order, nonergodic Gibbs state with a complicated free energy surface, lack of self-averaging of some quantities, etc. . . . ) in the one hand, and a very slow aging dynamics with diverging relaxation times on the other hand. The state of affairs in the understanding of basic issues of the equilibrium glassy phase is still controversial, as two rather different theories have emerged for the low-temperature phase of the short-range Ising spin glass: One of these two schools extends the idea of replica symmetry breaking (RSB), invented by Parisi (see and references therein) for the mean field version of the model — i.e. for the Sherrington-Kirkpatrick (SK) model —, to the short-range case in the physical dimensions. The other one,

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1 For reviews at different stages of the story, see [1, 2, 3, 4] and the collection of papers in [5] about different aspects of spin glass theory.
the so-called scaling or droplet picture [8, 9, 10, 11], claims — at least in its most radical version — that RSB is restricted to the SK model which, with its infinite coordination number, can be relevant only in infinite dimension \((d = \infty)\), or for long-range models in finite \(d\). Before displaying a list of the most important conflicting statements, we present the model and give some definitions.

The Hamiltonian of the Edwards-Anderson (EA) model on a \(d\)-dimensional hypercubic lattice with the nearest neighbour quenched interactions \(J_{ij}\)’s taken independently from a symmetric Gaussian distribution is [6]

\[
\mathcal{H}_J = - \sum_{ij} J_{ij} S_i S_j - H \sum_i S_i,
\]

with \(N\) Ising spins sitting on the lattice sites, and \(N \to \infty\) in the thermodynamic limit. (The second term contains the external magnetic field \(H\), although it is zero throughout this paper.) The distribution \(P_J(q)\) of the site overlaps \(q_{SS'} = \frac{1}{N} \sum_i S_i S'_i\), while picking out the spin configurations \(S\) and \(S'\) independently by the Gibbs-measures with the same \(\mathcal{H}_J\)’s, is usually considered as the order parameter function. The maximum value of the \(P_J(q)\)’s support is the EA order parameter \(q_{EA}\), although originally \(q_J \equiv \frac{1}{N} \sum_i \langle S_i \rangle^2 = \int dq q P_J(q)\) was proposed as \(q_{EA}\), which generally depends on the realization \(J\). Here and in subsequent definitions \(\langle \ldots \rangle\) means thermal average by the Gibbs measure \(\sim e^{-\mathcal{H}_J/kT}\), whereas \(\cdots\) stands for averaging over the quenched disorder. We have \(P(q) \equiv P_J(q)\). The following three 2-site (zero momentum) correlation functions are useful to define:

\[
\begin{align*}
G_1 &= \frac{1}{N} \sum_{ij} (\langle S_i S_j \rangle)^2 - N \overline{q^2} \\
G_2 &= \frac{1}{N} \sum_{ij} (\langle S_i S_j \rangle \langle S_i \rangle \langle S_j \rangle) - N \overline{q^2} \\
G_3 &= \frac{1}{N} \sum_{ij} (\langle S_i \rangle^2 \langle S_j \rangle^2 - N \overline{q^2}.
\end{align*}
\]

Two linear combinations of these are especially important to characterize the spin glass phase [8]:

\[
G_R = G_1 - 2G_2 + G_3 = \frac{1}{N} \sum_{ij} (\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle)^2 \equiv \chi_{EA}
\]

(2)

and

\[
G_L = G_1 - 4G_2 + 3G_3.
\]

The subscripts R and L are for “replicon” and “longitudinal”, and \(G_R = \chi_{EA}\) is the famous Edwards-Anderson (or spin glass) susceptibility whose inverse is the so-called replicon mass \(2m_1 = \chi_{EA}^{-1}\), which is used throughout this paper to test the stability of the spin glass phase.
In the following list, it is attempted to review the most important conflicting statements provided by the two rival theories:

1. The Gibbs measure is decomposed into an infinite number of pure states, not related by any symmetry, in RSB. The droplet picture is like the ferromagnet: there is one pair of pure states related by spin inversion. As a result, a trivial probability distribution of the overlaps between equilibrium spin configurations arises, with a pair of delta functions at $\pm q_{EA}$. A nontrivial and non-self-averaging [i.e. $P_J(q) \neq P(q)$] overlap distribution follows from the Parisi theory [2], with a continuous part for a range of overlaps smaller than $q_{EA}$ plus the delta function for self-overlap. $q_{EA}$ is, however, self-averaging in both theories.

2. The ergodic components, or pure states, in the Parisi theory are ultrametrically organized [2].

3. The average system-size low-energy excitations, when modelling in a finite size system, are proportional to $N^{\Theta/d}$, $\Theta > 0$ being the stiffness exponent, and the surface of the compact cluster of flipped spins, i.e. of the “droplet”, is a fractal with $d_s < d$ in the droplet picture [9, 11]. On the other hand, the same type of excitations are thought to be of order unity, and they are space filling ($d_s = d$) in RSB.

4. The external magnetic field $H$ destroys the spin glass transition in the droplet picture, whereas the spin glass phase is bounded by the Almeida-Thouless (AT) line $H_{AT}(T)$ in RSB [12]. Along the AT line $\chi_{EA}$ of (2) is infinite, that is the replicon mass is zero and starts to become negative, signalling the end of the replica symmetric (RS) paramagnetic phase, and the onset of RSB.

5. There is one feature both theories share, namely the spin glass susceptibility $\chi_{EA}$ is infinite, in the thermodynamic limit, in the whole glassy phase, i.e. it is a marginally stable massless phase. The details, however, already differ, as the divergence of $\chi_{EA}$ with $N$ is

$$\chi_{EA} \sim \begin{cases} N & \text{for RSB}, \\ N^{1-\Theta/d} & \text{droplet picture, see [8, 9]}. \end{cases}$$

The discussion of the above points may be started by considering “replica equivalence,” a set of identities between joint distributions of the site (or standard) overlaps, and rigorously proved for the SK model [13, 14, 15]. Its first derivation was based on the ultrametric property of the
Parisi solution of the SK model \[16\], it is, however, more generic than ultrametricity \[17\]. Applying replica equivalence to the susceptibilities in \[(1)\], we can easily derive for the leading $O(N)$ term [see also \[(2)\)]:

$$G_2 = \frac{1}{2} G_1, \quad G_3 = \frac{1}{3} G_1 \quad \text{and} \quad \chi_{\text{EA}} = \frac{1}{3} G_1 = N \frac{1}{3} \left[ \int dq q^2 P(q) - \left( \int dq q P(q) \right)^2 \right].$$

This formula is in complete accordance with point 5 above for both scenarios. It is quite interesting, and not completely understood, that the ratio $1 : \frac{1}{2} : \frac{1}{3}$ for the leading terms of the $G$’s, which are $O(N^{1-\Theta/d})$, in the droplet case remains true, although not because of replica equivalence.

Replica equivalence has recently been rigorously proven for a large class of models, including the EA model, for the link overlap $d_{SS'} = \frac{1}{d N} \sum_{(ij)} S_i S_j S'_i S'_j \ [18, 19, 20]$. A very recent numerical study \[21\] for the 3-dimensional model provided strong support that the two overlaps (site and link) are equivalent in the description of the spin glass phase of the EA model. Considering also the fact that replica equivalence is a very plausible property when deriving it in the replicated system (with the replica number $n$ going to zero) \[17\], it can now be believed that the identities of replica equivalence are also true for the standard site overlap in the finite dimensional Ising spin glass. As both scenarios, RSB and droplet, fulfill replica equivalence, it can not, however, be used to distinguish them.\(^2\)

To review the huge amount of numerical work accumulated in the last decades is impossible, the reader is referred to Refs. \[24, 25\] for extensive lists of papers. Numerical simulations work on finite systems and try to extrapolate to the infinite $N$ limit. Due to the extremely long relaxation times, notwithstanding the huge evolution of computer capacities, only relatively small systems have been simulated. That resulted in a fluctuating support for the two scenarios. Ref. \[24\] considered almost all topics of the above list, finding evidences for replica equivalence and RSB. A very recent paper \[26\] for the three-dimensional $\pm J$ model in zero external field found excellent agreement with ultrametricity — the lattice size was relatively large: $20^3$, which is a unique characteristic of RSB. The issue of the low-energy excitations (point 3) is rather controversial: Refs. \[27, 28\] found system size $O(1)$ energy excitations, but with $d_s < d$ suggesting that the link overlap distribution is trivial, whereas the standard one is not (TNT picture). This scenario, however, was ruled out by \[21\]. Zero temperature methods were also used to test the survival of the spin glass state in

\(^2\) It has emerged as an exact statement in the literature \[22, 23\] that the distribution of the site overlaps for any finite-dimensional EA model must be self-averaging — $P(q) = P_J(q)$ — in case of the infinite system. Putting together this thing with replica equivalence, one can deduce a trivial overlap distribution, i.e. a droplet scenario in any $d$. Marinari et al in Ref. \[24\] argued that the $P_J(q)$ defined by Newman and Stein is different from that appearing in the replica equivalence identities. Nevertheless, one might have the feeling that this issue has not been completely settled by now.
a nonzero external field \( \Theta \); the somewhat positive answer obtained is in favour of RSB. On the other hand, the completely different treatment in \cite{30} concluded the lack of a transition in field; though the magnetic field was not homogenous in that investigation, but chosen from a random Gaussian distribution. Finally point 5 is rather difficult to check numerically as the exponents in \cite{3} are very close to each other, due to the smallness of \( \Theta/d \).

The replica method introduced by Edwards and Anderson \cite{6}, is a powerful, though admittedly unphysical, tool to handle the quenched state by eliminating the inhomogenities caused by randomness. The effective (replicated) Hamiltonian thus obtained can be represented by a field theory using common procedures known from the study of ordinary systems. The advantage of field theory comes from its flexibility of application in any space dimension \( d \) and for a variable range of the interaction. Standard methods, like systematic perturbation expansions and the renormalization group (RG), are also available. In the first calculation by replica field theory of Harris et al \cite{31} the fixed point value and the critical exponents were obtained in leading order in \( \epsilon = 6 - d \). In two subsequent papers \cite{32,33} the ordered phase below \( T_c \), and especially the stability of the RS state in finite \( d \) was studied: the mean field, AT-like, instability was found to survive perturbatively. After Parisi’s solution of the SK model appeared and proved to be (marginally) stable \cite{34}, one-loop studies based on the ultrametric RSB ansatz were published (see \cite{35}, and references therein), the severe infrared problems were handled in the framework of a scaling theory \cite{36}. In the meantime, a search for finding the (presumably zero-temperature) fixed-point of a finite-dimensional AT line took place \cite{37,38}.

In field theory, detection of a glassy phase consistent with the droplet picture would mean finding a marginally stable RS state in zero external field. The developments enumerated in the previous paragraph were against droplet theory in finite dimensions. In Ref. \cite{32} a procedure with temporarily breaking the replica symmetry by a special — non-Parisi — type of RSB (the “two-packet” RSB), and eventually restoring RS led to a massless RS phase. We will comment this result in Section VII. The issue has been revived in two recent papers \cite{39,40}.

Alike the solution of the fully connected SK model, field theory provides results in the thermodynamic limit, i.e. for \( N \to \infty \). A direct comparison with the lattice EA model is, however, impossible due to the use of the concept of the minimum length \cite{41}, i.e. averaging out short-range fluctuations. As a result, the precise dependence of the parameters of the field theoretical Lagrangian, the bare couplings, on the control parameters (temperature, external magnetic field, etc.) of the original model is lost. This fact may be extremely important when trying to distinguish an AT line from a droplet-like phase \cite{42}: both have the generic replica symmetry with a
nonzero EA order parameter, and even an infinite spin glass susceptibility (a zero replicon mass). For that reason, we follow the ideas, introduced in Ref. [42], of using exact symmetry arguments to parametrize the bare couplings in zero external field. Our arguments are based on the only assumption that the Legendre transformed free energy is an invariant of the extra symmetry of the high-temperature phase, see Eq. (7) below. An important result of the present paper is the summation of the infinite terms which were argued in [42] to contribute to the replicon mass $2m_1$: there is a complete cancellation of these terms leading to a negative mass, i.e. to an unstable RS phase for $6 < d < 8$ and $d \lesssim 6$.

The outline of the paper is as follows: In Section II the generic replica symmetric model is introduced, and the extra symmetry of the replicated paramagnet of the high-temperature phase presented. Our basic object, the replicon generator $g(v)$, is defined, and its properties are derived by exact symmetry considerations. The full $g(v)$ is computed for the SK case in Section III. The picture of independent droplets is used to derive the scaling form of higher order susceptibilities in Section IV. A somewhat heuristic argument makes it possible to produce a scaling form of $g(v)$ for the droplet scenario when the replica number $n$ is nonzero. Exact Ward identities, which are necessary to construct the free propagators, are displayed in Section V whereas the one-loop calculation of $g(v)$ is presented in Section VI. The qualitatively different regimes — i.e. $d > 8$, $6 < d < 8$ and $d \lesssim 6$ — are discussed in separate subsections. Section VII is for a summary of the results, and it also contains our conclusions. Notations, definitions and computational details are left to the series of appendices, amongst them the block diagonalization of a generic matrix with the “two-packet” RSB is in Appendix E.

II. THE SYMMETRY OF THE SPIN GLASS FIELD THEORY IN ZERO MAGNETIC FIELD

The Ising spin glass on a hypercubic $d$-dimensional lattice — with the quenched exchange interactions taken independently from a Gaussian distribution with zero mean — can be represented (see [43] and references therein) by a replica field theory where the classical fields $\phi^{\alpha \beta}$ have the double replica indices $\alpha, \beta = 1 \ldots n$, and $\phi^{\alpha \beta} = \phi^{\beta \alpha}$ with zero diagonal $\phi^{\alpha \alpha}$. In the spirit of the replica trick [6], the special limit $n \to 0$ of this $n(n-1)/2$ component field theory provides the physical results. Unlike in the Parisi theory, the spin glass phase is almost ergodic in the droplet scenario: the two minimal free-energy phases are related by the global inversion of the spins — just like in a ferromagnet. In a replicated picture this gives rise to an RS theory with the generic
Lagrangian (displayed here in momentum space):
\[
\mathcal{L} = \frac{1}{2} \sum_{\mathbf{p}} \left( \frac{1}{2} \mathbf{p}^2 + \bar{m}_1 \right) \sum_{\alpha\beta} \phi_{\mathbf{p}}^{\alpha\beta} \phi_{-\mathbf{p}}^{\alpha\beta} + \bar{m}_2 \sum_{\alpha\beta\gamma} \phi_{\mathbf{p}}^{\alpha\beta} \phi_{\mathbf{p}}^{\beta\gamma} + \bar{m}_3 \sum_{\alpha\beta\gamma\delta} \phi_{\mathbf{p}}^{\alpha\beta} \phi_{\mathbf{p}}^{\beta\gamma} \phi_{\mathbf{p}}^{\gamma\delta} \right] + \mathcal{L}^I, \tag{4}
\]
with the interaction term containing all the higher order invariants \(I^{(k)}_j\) together with the single first order one, i.e.
\[
\mathcal{L}^I = -N^\frac{3}{2} \bar{h} \sum_{\alpha\beta} \phi_{\mathbf{p}=0}^{\alpha\beta} - \sum_{k=3}^{\infty} \frac{1}{k! N^\frac{k}{2} - 1} \sum_{\mathbf{p}_1} \sum_{j} \tilde{g}^{(k)}_j I^{(k)}_j. \tag{5}
\]

The number of spins \(N\) goes to infinity in the thermodynamic limit, its explicit indication ensures that the bare couplings, \(\bar{h}\) and the \(\tilde{g}^{(k)}_j\)'s, are independent of \(N\). Momentum conservation is understood in the primed sum over the momentums \(\mathbf{p}_1 \ldots \mathbf{p}_k\). The \(I^{(k)}_j\)'s are invariant under the global transformation \(\phi_{\mathbf{p}}^{\alpha\beta} = \phi_{\mathbf{p}}^{P_{\alpha} P_{\beta}}\), where \(P\) is any permutation of the \(n\) replicas. The three invariants of the kinetic term (\(k = 2\)) and that of the linear one are explicitly displayed in (4) and (5), while the 8 cubic and 23 quartic ones are summarized in Appendix A.3 Besides using \(\tilde{g}^{(k)}_j\) for a generic \(k\), we retain the commonly used notations for the low order couplings \(\bar{h}, -\bar{m}\)'s, \(\bar{w}\)'s (for cubic) and \(\bar{u}\)'s (for quartic). As it was pointed out in [42], an ambiguity in assigning the bare coupling constants to a given physical state occurs due to the freedom to offset the zero-momentum fields, \(\phi_{\mathbf{p}=0}^{\alpha\beta} \rightarrow \phi_{\mathbf{p}=0}^{\alpha\beta} - \sqrt{N} \Phi\), leaving all the irreducible vertices unaltered while the bare couplings changing. By this way, we can always tune the one-point function to zero, hence avoiding the appearance of any “tadpole” insertions in a perturbative treatment. We adopt this definition of the bare coupling constants in what follows. (In this case, \(\bar{h}\) is generally nonzero even if the “physical” magnetic field is switched off.)

The generic replica symmetric theory presented in the previous paragraph, however, is not confined to the description of the droplet-like spin glass phase: the nonglassy phase in an external magnetic field (even above the transition temperature) is also represented by the generic RS Lagrangian of Eqs. (4,5). Still, as a spin glass is always an infinitely correlated state, the criterium of a massless mode (the so called replicon one) seemingly may distinguish it from the magnetized paramagnet. At that point, however, we must realize that the boundary of a possible RSB phase fits the same criterium: along the AT line we have replica symmetric massless thermodynamic states.

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3 The barred notation for the bare coupling constants is used throughout the paper to distinguish them from their exact counterparts. The choice of the set of coupling constants of the generic replica symmetric Lagrangian is obviously not unique: couplings with the lowercase notation defined above are the coefficients in front of the invariants. To make the presentation as clear as possible, it will be necessary later to introduce another set with the capital letter notation (\(m_1\) vs. \(M_1\), for instance); the linear relationships between these two sets are displayed — for the quadratic, cubic and quartic vertices — in Appendix B.
The underlying problem we face when trying to distinguish an AT line from the droplet-like spin glass phase is the indirect relationship between the bare couplings of the field theory and the physical control parameters (temperature and magnetic field, for instance). To find the thermal route of the system, i.e. the dependence of the bare couplings on the reduced temperature $T_{c}-T$ while keeping the external magnetic field zero, we turn to exact symmetry arguments. The “true” paramagnet, i.e. the high temperature phase in zero magnetic field, has the extra symmetry that the Edwards-Anderson order parameter $q$ is zero. (The magnetic field makes $q$ nonzero even in the nonglassy phase.) As a result, those couplings in (4) and (5) which belong to invariants with at least one replica occuring an odd number of times (like $\bar{m}_2$, $\bar{m}_3$ and $\bar{h}$) are exactly zero. In addition to the permutational invariance, this results in the extra symmetry published in [42], and for the sake of self-containness, it is explained here again: The $n$ replicas are divided arbitrarily into two groups with $p$ and $n-p$ replicas, $p$ being a free parameter between 0 and $n$. The Lagrangian is invariant under the global transformation

$$
\phi'_{\alpha\beta}^p = \begin{cases} 
\phi_{\alpha\beta}^p & \text{for } \alpha \text{ and } \beta \text{ in the same group,} \\
-\phi_{\alpha\beta}^p & \text{for } \alpha \text{ and } \beta \text{ in different groups.} 
\end{cases} \tag{6}
$$

This transformation is obviously orthogonal in the $n(n-1)/2$-dimensional space of the field components, with the diagonal transformation matrix $O_{\alpha\beta,\gamma\delta} = (-1)^{\alpha\cap\beta+1} \delta_{\alpha\beta,\gamma\delta}^{Kr}$. We introduced here the overlap $\alpha \cap \beta$ defined as $\alpha \cap \beta = 0$ ($\alpha \cap \beta = 1$) if $\alpha$ and $\beta$ are in different groups (if $\alpha$ and $\beta$ are in the same group).

The symmetry of the Lagrangian is reflected in the invariance of the Legendre-transformed free energy $F$:

$$
F(q'_{\alpha\beta}) = F(q_{\alpha\beta}). \tag{7}
$$

The stationary condition $\frac{\delta F}{\delta q_{\alpha\beta}} = 0$ provides the symmetrical solution $q'_{\alpha\beta} = q_{\alpha\beta} \equiv 0$ for the paramagnet, while in the droplet-like phase the extra symmetry of (6) is lost leading to $q_{\alpha\beta} \equiv q \neq 0$ represented by the generic RS theory in (4) and (5). In this way, we can understand the paramagnet to spin glass transition as a symmetry breaking one.

In [42] the invariance property of (7) was combined with the freedom to choose the continuous parameter $p$ of the transformation to derive a set of Ward-like identities between exact irreducible vertex functions (derivatives of $F$ evaluated at stationarity). Here we focus on the stability of the low-temperature RS phase in zero magnetic field, and for that reason we define the generator
function $g(v)$ of replicon-like vertices as follows:

$$g(v) = \sum_{k=2}^{\infty} \frac{1}{k!} G^{(k+1)} v^k,$$

where

$$G^{(k+1)} = \sum_{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)}' G^{(k+1)}_{\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_k \beta_k}; \quad k = 1, 2, \ldots; \quad \text{and by definition } G^{(1)} \equiv G^{(1)}_{\alpha \beta}. \quad (9)$$

The prime above is to indicate the restriction $\alpha_i \cap \beta_i = 0$ for all $i = 1, \ldots, k$ in the summations, and we choose $\alpha \cap \beta = 0$ too, hence $G^{(k+1)}$ no more depends on $\alpha$ and $\beta$ in an RS state (it does, however, on $n$ and $p$). The generic definition of the exact vertices is

$$-G^{(k)}_{\alpha_1 \beta_1, \ldots, \alpha_k \beta_k} = \frac{\partial^k (F/N)}{\partial q_{\alpha_1 \beta_1} \ldots \partial q_{\alpha_k \beta_k}} \quad (10)$$

evaluated at an RS stationary point. For $k = 2, 3$ and 4 we insist — for traditional reasons — to use the notations $-M, W$ and $U$, respectively, similarly to their lowercase counterparts, see the note below Eqs. 4(5). The vertices defined in (10) take only a limited number of different values, expressing in this way the RS symmetry; in fact they may be numbered in accord with the numbering of the set with the lowercase notation. All of the relevant definitions are left to Appendix A while the linear relationships between the two sets are displayed in Appendix B for the quadratic ($k = 2$), cubic ($k = 3$) and quartic ($k = 4$) cases. (See also the footnote 3.)

Differentiating both sides of Eq. (7), and exploiting the orthogonality and diagonality of the transformation in (6), the following rule is easily derived for the transformation of the vertices:

$$G'^{(k)}_{\alpha_1 \beta_1, \ldots, \alpha_k \beta_k} = O_{\alpha_1 \beta_1, \alpha_1 \beta_1} \ldots O_{\alpha_k \beta_k, \alpha_k \beta_k} G^{(k)}_{\alpha_1 \beta_1, \ldots, \alpha_k \beta_k}, \quad (11)$$

inferring immediately

$$G^{(k+1)}' = (-1)^{k+1} G^{(k+1)}.$$

(12)

As it was explained in [42], the transformation $q_{\alpha \beta} \to q'_{\alpha \beta}$ becomes infinitesimal for $q, n, p \ll 1,$ hence the left-hand side of (12) can be Taylor-expanded around the RS state $q_{\alpha \beta} \equiv q$. From (9) and (10), and using $q'_{\alpha \beta} - q_{\alpha \beta} = 0$ unless $\alpha \cap \beta = 0$, when it is $-2q$, we can easily derive:

$$G^{(k+1)}' = \sum_{l=0}^{\infty} \frac{1}{l!} G^{(k+1+l)} (-2q)^l. \quad (13)$$

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4 Throughout the paper, we use different notations for summations over distinct pairs of replicas, $\sum_{(\alpha \beta)} = \sum_{\alpha < \beta}$, and unrestricted sums, $\sum_{\alpha \beta} = \sum_\alpha \sum_\beta$, as in the invariants in Eq. (4).
Comparing (12) and (13),

\[
\left[ (-1)^{k+1} - 1 \right] G^{(k+1)} = \sum_{l=1}^{\infty} \frac{1}{l!} G^{(k+1+l)} (-2q)^l
\]

is concluded, valid for \( k = 0, 1, \ldots \). It is useful to analyze the following cases separately:

- For \( k = 0 \), using (8), \( -2 G^{(1)} = G^{(2)} \times (-2q) + g(-2q) \) is obtained. Stationarity renders the left-hand side zero. Here we give some details of the derivation of \( G^{(2)} \) in terms of the masses, intending in this way to facilitate the reader to check more complicated formulae for \( G^{(3)} \) and \( G^{(4)} \) displayed in Appendix C:

\[
-G^{(2)} = \sum_{(\gamma\delta)} M_{\alpha\beta,\gamma\delta} = M_1 + [(p - 1) + (n - p - 1)] M_2 + (p - 1)(n - p - 1) M_3; \quad (14)
\]

for the notations used here see Appendix A. This formula and the similar ones displayed in Appendix C are much simpler when expressed in terms of the lower case vertices, though the derivation uses evidently the upper case set. By Eq. (B1) of Appendix B, we arrive to the important result

\[
2m_1 + nm_2 + 4p(n-p)m_3 = \frac{g(v)}{v} \quad \text{evaluated at} \quad v = -2q. \quad (15)
\]

- For \( k = 1 \) the right-hand side is just \( g'(v) = g(-2q) \), i.e.

\[
g'(v) = 0 \quad \text{at} \quad v = -2q. \quad (16)
\]

- The generic case, \( k \geq 2 \), is easily understood by observing that the right-hand side is \( g^{(k)}(-2q) - G^{(k+1)} \), while \( G^{(k+1)} = g^{(k)}(0) \) providing

\[
(-1)^{k+1} g^{(k)}(v = 0) = g^{(k)}(v = -2q). \quad (17)
\]

The above conditions are serious restrictions for the RS field theory representing a spin glass phase in zero magnetic field. For a given \( q \) (which is used here — instead of the reduced temperature — to parametrize the spin glass phase below \( T_c \)), the generator \( g(v) \) must have an extremum at \( v = -2q \), Eq. (16), while the higher derivatives satisfy (17). As \( 2m_1 \) is the dangerous (so called “replicon”) mass responsible for the instability of the RS state [12], stability requires \( g(v = -2q) \leq 0 \) in the limit \( n \to 0 \). (Equality means marginal stability, i.e. a massless spin glass phase, just what droplet theory suggests [8, 9].) These results were derived by generic symmetry arguments, hence they must be equally valid for systems described by mean field theory and for short-range spin glasses in physical dimensions. All these cases are reviewed in subsequent sections.
III. CALCULATION OF THE REPLICON GENERATOR IN MEAN FIELD THEORY

An explicit expression for $F(q_{\alpha\beta})$ is available in mean field theory, working it out either on the fully connected lattice (Sherrington-Kirkpatrick model [7]) or as the long-range limit on the $d$-dimensional hypercubic lattice [43]:

$$
\frac{1}{N} F = \frac{1}{2\Theta} \sum_{(\alpha\beta)} q_{\alpha\beta}^2 - \ln \text{Tr} \left( \sum_{(\alpha\beta)} q_{\alpha\beta} S^\alpha S^\beta \right).
$$ (18)

As we are in zero magnetic field, the only control parameter is $\Theta \sim (kT)^{-2}$. It is now useful to introduce the modified Legendre-transformed free energy $\tilde{F}$ by

$$
\tilde{F}(v_{\alpha\beta}) = F(q + v_{\alpha\beta}),
$$ (19)

$q$ being the RS stationary point of $F$. Obviously $\tilde{F}$ is stationary for $v_{\alpha\beta} \equiv 0$, and all the vertices in Eq. (10) can be computed equally well substituting $F$ by $\tilde{F}$ and the $q$’s by $v$’s. (The invariance property of (7) is, however, lost for $\tilde{F}$.) After Taylor-expanding $\tilde{F}$ and making the restriction $v_{\alpha\beta} = v$ for $\alpha \cap \beta = 0$, otherwise it is zero, we can express $g(v)$ by the now one-variable function $\tilde{F}(v)$:

$$
g(v) = -vG^{(2)} - \frac{1}{p(n - p)} \frac{d}{dv} (\tilde{F}/N).
$$

The second term above is the harder to compute. As it is common in mean field replica calculations, the Gaussian integral representation method can eliminate the spin traces in (18) providing for $p(n - p) \to 0$:

$$
g(v) = \int \mathcal{D}y \int \mathcal{D}x \int \mathcal{D}x' \frac{\tanh(\sqrt{q + v} y + \sqrt{-v} x) \tanh(\sqrt{q + v} y + \sqrt{-v} x') \cosh^n(\sqrt{q + v} y + \sqrt{-v} x)}{\int \mathcal{D}y \int \mathcal{D}x \int \mathcal{D}x' \cosh^n(\sqrt{q + v} y + \sqrt{-v} x)} - \tanh^2(\sqrt{q} y) - \left[ 1 + (n - 2) \tanh^2(\sqrt{q} y) - (n - 1) \tanh^4(\sqrt{q} y) \right] v.
$$ (20)

The shorthand notation

$$
\tanh^k(\sqrt{q} y) = \int \mathcal{D}y \tanh^k(\sqrt{q} y) \cosh^n(\sqrt{q} y) \quad \text{with} \quad \int \mathcal{D}y = \int \frac{dy}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2}
$$

was used here. The explicit temperature dependence has been eliminated by the stationary condition $\Theta^{-1} q = \tanh^2(\sqrt{q} y)$, and $g(v)$ is thus parametrized entirely by the order parameter $q$. 
We are now in the position to check the generic properties of \( g(v) \) deduced in the preceding section. It is relatively easy to see that \( g(v) \) is quadratic for small \( v \). Although it is somewhat harder, a straightforward application of the following two lemmas\(^5\) can prove Eqs. (16) and (17):

1. \[
\frac{d}{dv} \int \mathcal{D}y \int \mathcal{D}x \int \mathcal{D}x' f_1(z) f_2(z') = \int \mathcal{D}y \int \mathcal{D}x \int \mathcal{D}x' f'_1(z) f'_2(z')
\]

and

2. \[
\int \mathcal{D}y \int \mathcal{D}x \int \mathcal{D}x' f_1(z) f_2(z') = \int \mathcal{D}z f_1(\sqrt{q} z) f_2(-\sqrt{q} z) \quad \text{for} \quad v = -2q.
\]

\( f_1 \) and \( f_2 \) above are two arbitrary functions, while \( z = \sqrt{q + v} y + \sqrt{-v} x \) and \( z' = \sqrt{q + v} y + \sqrt{-v} x' \).

Near \( T_c \), \( g(v) \) gets into the simple scaling form \( g(v) = q^3 \hat{g}(v/q) \); the scaling function results from expanding the integrands of (20) for \( q \ll 1 \) and \( v/q = O(1) \):

\[
\hat{g}(x) = -\frac{3n-2}{3} x^2 (3 + x), \quad p(n-p) \to 0.
\]

The following remarks are appropriate here:

- \( \hat{g}(-2) > 0 \), showing by (15) the instability of the RS mean field spin glass phase just below \( T_c \).

- The leading, or scaling, term of \( g(v) \) — and only that — satisfies the following condition:

\[
g(v = -2q) = \frac{2}{3} q^2 \hat{g}''(v = 0).
\]

We shall see later that this property of the replicon generator is very robust, as it persists for the short-range system below the upper critical dimension \( 6 \).\(^6\) On the other hand, a simple one term Ward-identity follows from (22) for the replicon mass, as one can see from (15) and (22):

\[
2m_1 = \frac{g(v = -2q)}{(-2q)} = -\frac{1}{3} q G^{(3)} = -\frac{2}{3} q w_2, \quad p, n \to 0.
\]

---

\(^5\) The first one can be easily proven by a sequence of partial integration, while a change of variables to \( z, z' \) and \( y' = y/\sqrt{-v} \) in the second lemma and setting \( v = -2q \) renders the integral over \( y' \) proportional to \( \delta(z + z') \), leading us immediately to the end of the proof.

\(^6\) To be more precise, the one-loop calculations in Section \( \text{Y} \) will show that Eq. (22) is true for any \( n \) when \( d > 8 \), whereas for \( 6 < d < 8 \) and \( d \gtrsim 6 \) it is restricted to the case \( n = 0 \).
[The expression for $G^{(3)}$ in terms of the $w$'s is displayed in Appendix C, see Eq. (C1).] This formula is valid whenever the leading contribution to $g(v)$ for $n \to 0$ and $q \to 0$ satisfies (22). As it connects the replicon mass ($2m_1$) and the replicon-like cubic vertex ($w_2$), it is obviously important in testing stability. We shall therefore return to it later.

- The leading term of $g(v)$ in Eq. (21) can be deduced directly from the truncation of the infinite series (8) as

$$g(v) = \frac{1}{2} G^{(3)} v^2 + \frac{1}{6} G^{(4)} v^3.$$  

(24)

This can be seen by a direct evaluation of $G^{(3)}$ and $G^{(4)}$ from the definitions (9), (10) and using the mean field free energy (18). This truncated form of the scaling part of the replicon generator is, however, more general, as it remains valid for the finite-dimensional system when $d > 8$ (see later). Starting from (24), Eq. (22) trivially follows, together with all of its consequences.

From the generic properties of $g$ derived in the preceding section and from the mean field analysis above, one may get the feeling that the replicon mass [which is proportional to $g(v = -2q)$] is either negative or positive, but it is difficult to imagine how marginal stability may emerge. This, however, belongs to the essence of the droplet picture, and in the next subsection we try to construct $g(v)$ — or rather its scaling term — using the ideas of the theory of free droplets.

IV. THE REPLICON GENERATOR IN THE DROPLET PICTURE

The droplet picture is a low-temperature scaling theory developed for the original $d$-dimensional lattice system [8, 9, 10, 11]. As a basic inference of this theory, the spin glass state is now restricted to zero external magnetic field, and it is in fact a critical surface attracted by a zero-temperature fixed point, immediately suggesting that temperature is now a (dangerously) irrelevant parameter. The whole phase below $T_c$ is massless, as at least one correlation length, namely that of the replicon mode, is infinite. Somewhat surprisingly, there is a single independent exponent belonging to the zero-temperature fixed point — the stiffness exponent $\theta$ —, and scaling of relevant and irrelevant variables, and also of several observables (correlation functions, for instance), can be described by means of it.

Translating the results of the droplet theory to the language of replicated field theory is straightforward by using the Gaussian integral representation, called Hubbard-Stratonovich transforma-
tion, and then eliminating short ranged fluctuations by the introduction of effective coupling constants and truncation of the momentum-dependent mass term (see Ref. [43]); a Lagrangian like
that in Eqs. (4) and (5) is obtained. As a short cut to translate different quantities, we propose that whenever a pair of replicated Ising spins $S^\alpha_i S^\beta_i$ occurs ($i$ being a lattice site), it should be replaced by the field $\phi^\alpha\beta_i$; in this way, Ising averages are turned to averages with the measure $e^{-L}$. As a pedagogical example, let us consider the momentum-dependent Edwards-Anderson susceptibility defined by

$$\chi_{\text{EA}}(p) = \frac{1}{N} \sum_{ij} e^{ip(r_j-r_i)} \frac{\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle}{\langle S_i S_j \rangle}^2,$$

where $\langle \ldots \rangle$ and $\overline{\ldots}$ mean thermal average with the Ising measure and quenched average over the disorder, respectively. Applying the replica trick and introducing the fields, we get

$$\chi_{\text{EA}}(p) = \frac{1}{N} \sum_{ij} e^{ip(r_j-r_i)} \langle (S^\alpha_i S^\beta_j S^\gamma_j + S^\alpha_i S^\beta_j S^\gamma_j - 2S^\alpha_i S^\beta_j S^\gamma_j) \rangle$$

$$\rightarrow (kT)^b \left( \langle \phi^\alpha\beta \phi^\alpha\beta \rangle - 2 \langle \phi^\alpha\gamma \phi^\beta\gamma \rangle \right) \equiv (kT)^b C^{(2)}(p);$$

$C^{(2)}(p)$ being the replicon component of the 2-point connected correlation function in the replicated field theory. (To avoid complicated notations, we stick to $\langle \ldots \rangle$ for thermal averages with different measures: in the first line it is taken with the effective replicated Ising Hamiltonian resulting from the replica trick, whereas a field-theoretic average with the weight $e^{-L}$ is understood in the second one.) The emergence of the factor $(kT)^b$ above results from the transformation. Similar temperature-dependent factors with some simple exponents always appear whenever spin averages are represented by field averages. Near $T_c$ they are absolutely irrelevant, we must, however, keep them at low temperature to ensure the correct temperature scaling. (Nevertheless, we shall see that the specific values of $b$ and also $a$, which is introduced later, will not influence our basic conclusion.) We know from Refs. [8, 9, 11] that $\chi_{\text{EA}}(p) \sim (kT)^p$ yielding

$$C^{(2)}(p) \sim (kT)^{1-b} p^{d+\theta}.$$ (25)

We can apply the scaling formula

$$C^{(2)}(p) \approx \mu_1^{\frac{-2+\eta_R}{\lambda_1}} \tilde{C}^{(2)} \left( \frac{\mu_2}{\mu_1^{\lambda_2/\lambda_1}}, \frac{p}{\mu_1^{1/\lambda_1}} \right)$$

around the zero-temperature fixed point. $\mu_1$ is a relevant scaling field vanishing in the spin glass phase (e.g. the external magnetic field) with $\lambda_1 > 0$ exponent, while $\mu_2 \sim (kT)^a$ is the dangerously irrelevant one with $\lambda_2 < 0$. The scaling index of the replicon correlation function is $-2 + \eta_R$,
and the reader is reminded that, in principle, there are two different $\eta$'s ($\eta_R$ and $\eta_L$) at a generic RS fixed point when $n=0$. On the one hand, we can now exploit the fact that $C^{(2)}$ must be independent of $\mu_1$ when it goes to zero, and the correct momentum-dependence must be ensured. On the other hand, droplet theory suggests $\lambda_2 = -a \theta$. From these, we can conclude:

$$C^{(2)}(p) \cong (kT)^{2-\eta_R+\theta-d} p^{-d+\theta}. \quad (26)$$

Comparing (25) and (26), we get

$$\eta_R = 2 - d + b \theta. \quad (27)$$

The replicon mass [$2m_1$; see the table for the masses in Appendix A and (B1)] is — by Dyson’s equation [44] — just the zero momentum limit of $C^{(2)}(p)^{-1}$, and therefore obviously zero for any $T$ in the spin glass phase.

The momentum-dependence of the higher order correlation functions $C^{(k)}(p)$, $k > 2$, can be deduced by applying the single droplet excitation picture of Fisher and Huse [9, 11] for generalized Edwards-Anderson susceptibilities. As an example, for $k = 3$ we have:

$$\frac{1}{N} \sum_{ijk} e^{i(p_1 r_i + p_2 r_j + p_3 r_k)} \frac{1}{2} \left[ (S_i S_j S_k) - (S_i) (S_j S_k) - (S_j) (S_i S_k) - (S_k) (S_i S_j) + 2 (S_i) (S_j) (S_k) \right]^2 \rightarrow \delta_{p_1+p_2+p_3} (kT)^{b'} C^{(3)}(p), \quad b' = \frac{3}{2} b,$$

where momentum conservation is explicitly shown, and only a single $p$ scale is considered. $C^{(3)}(p)$ has the structure of $2w_2^2$, see (B1) and the table for the cubic vertices in Appendix A and it is fully replicon-like [43]. The lengthy formula for $C^{(3)}(p)$ is omitted here; the reader can reproduce it by replacing a cubic vertex $W_{\alpha\beta,\gamma\delta,\mu\nu}$ in $2w_2$ by the average $\sqrt{N} \langle \phi_{p_1}^{\alpha\beta} \phi_{p_2}^{\gamma\delta} \phi_{p_3}^{\mu\nu} \rangle$, and also symmetrizing with respect to the momentums. In the spirit of the droplet theory, we can conclude the momentum-dependence of the cubic and even higher order replicon correlators as

$$C^{(k)}(p) \sim p^{\theta-(k-1)d}, \quad k = 2, 3, \ldots.$$  

It follows from simple scaling arguments that the replicon correlators, which are by definition independent of $N$ (see footnote 7), have the scaling index $d + \frac{k}{2}(\eta_R - 2 - d)$, and for $\mu_2 \ll \mu_1^{\lambda_2/\lambda_1}$ we have

$$C^{(k)}(p) \cong \mu_1^{[2d+k(\eta_R-2-d)]/2\lambda_1} \left( \mu_2/\mu_1^{\lambda_2/\lambda_1} \right)^{\kappa_k} \tilde{C}^{(k)}(p/\mu_1^{1/\lambda_1}), \quad (28)$$

A The $\sqrt{N}$ factor here is to make $C^{(3)}(p)$ become independent of $N$, i.e. of order unity. For the same reason, we define $C^{(k)}(p)$ with the prefactor $N^{\frac{k}{2}-1}$. 


where the one-variable scaling function \( \tilde{C}^{(k)}(x) \sim x^{\theta-(k-1)d} \) for \( x \to \infty \), whereas — as the replicon correlation length is finite for \( \mu_1 \neq 0 \) — it goes to a constant in the opposite limit. \( \kappa_k \) is nonzero, expressing the dangerously invariant nature of \( \mu_2 \), and it can be computed by taking the limit \( \mu_1 \to 0 \), while \( \mu_2 \) and \( p \) finite, of \( C^{(k)}(p) \) in (28):

\[
\kappa_k \lambda_2 = k \left( \frac{n\tau}{2} - 1 + \frac{d}{2} \right) - \theta = \left( \frac{k}{2} - 1 \right) \theta,
\]

where the last equality follows from (27). We shall need the scaling of the zero-momentum correlation functions when approaching the spin glass state for a given temperature. It can be easily deduced from (28) and (29):

\[
C^{(k)}(p = 0) \sim \mu_1^{[\theta-(k-1)d]/\lambda_1}, \quad kT > 0 \quad \text{fixed.}
\]

(30)

It is pointed out in Appendix C that \( G^{(k)} \), Eqs. (9) and (10), is a fully replicon vertex in the spin glass limit, and therefore — applying the usual connection between vertices and connected correlation functions — it scales like \( C^{(k)}(p = 0) \times C^{(2)}(p = 0)^{-k} \). Using (30), this gives

\[
G^{(k)} \sim \mu_1^{[d-(k-1)\theta]/\lambda_1}, \quad kT > 0 \quad \text{fixed.}
\]

(31)

This equation, which is valid for \( n = 0 \), shows that \( g(v) \) of the spin glass in the droplet theory is highly singular for \( \mu_1 \to 0 \), i.e. for zero magnetic field. This is in full contrast with mean field theory, see Eq. (21). Nevertheless, (31) can be used to guess the \( n \)-dependence of \( g(v) \) in the droplet picture in the following way: Although Eq. (31) was deduced by renormalization group arguments, it merely expresses the assumption that an infinitesimally magnetic field drastically destroys the droplet picture by introducing a finite correlation length. Regarding the common observations (see [43] for instance) that the role of the replica number is alike, i.e. it makes the replicon mode more massive, we can assume tentatively the following form for the \( n \)-dependence of \( g(v) \) in zero magnetic field:

\[
g(v) = n^{d\rho/\theta} \tilde{g}(v/n^\rho).
\]

(32)

This equation is obtained by the formal replacement of \( \mu_1 \) by \( n \) in (31), and then substituting \( G^{(k)} \) into the definition of \( g(v) \) in Eq. (8). A new independent exponent \( \rho \) was introduced here, and — although not explicitly indicated — the function \( \tilde{g} \) depends on \( T \) too. We shall return to this formula later, extending it to the critical scaling region around \( T_c \).
V. WARD IDENTITIES AND THE FREE PROPAGATOR FOR $T \lesssim T_c$

Some of the identities between exact vertices presented in Ref. [42] follow directly from the properties of $g(v)$. Eqs. (8) and (15) provide:

$$2m_1 + nm_2 + 4p(n-p)m_3 = -\left(G^{(3)} - \frac{2}{3}G^{(4)} q + \ldots \right) q,$$

whereas it follows from both (16) and (17):

$$G^{(3)} = G^{(4)} q + \ldots.$$  (34)

One more expression arises from the transformation property of a mixed mass component

$$M'_{\alpha\beta,\gamma\delta} = -M_{\alpha\beta,\gamma\delta}, \quad \alpha \cap \beta = 1 \quad \text{and} \quad \gamma \cap \delta = 0$$  [see (11)], summing over $\gamma \cap \delta = 0$ and expanding the left-hand side. We get for $p \to 0$

$$m_2 = -\left[(w_1 + \frac{1}{3}w_3) + n\left(\frac{1}{3}w_5 + w_6\right)\right] q + \ldots.$$  (35)

This equation has also been displayed in Ref. [42]. Everywhere in the above formulae, the dots have the obvious meaning of higher order vertices multiplied by the appropriate power of $q$.

Eqs. (33), (34) and (35) are now used to construct the bare propagators for a one-loop calculation of the leading scaling term of $g(v)$. The following two regimes are separately treated:

- In the perturbative regime ($d > 6$) the system is defined by the set of bare couplings compatible with the symmetry of the high-temperature phase: $\tilde{w}_1; \tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4; \ldots$. The zero-loop limit of (33) and (34), using also (C2) of Appendix C, gives

$$2\tilde{m}_1 + n\tilde{m}_2 = -\frac{2}{3}\tilde{u}_2 q^2; \quad 2\tilde{m}_3 = -\frac{1}{3}(\tilde{u}_1 + 2\tilde{u}_4) q^2;$$

whereas (35) provides

$$\tilde{m}_2 = -\tilde{w}_1 q.$$  

Neglecting the $q^2$ term in the scaling limit, the free propagators assume the diagonalized form [43]

$$\frac{1}{p^2 + r_s}, \quad \text{with} \quad r_R = n(\tilde{w}_1 q), \quad r_A = 2(\tilde{w}_1 q) \quad \text{and} \quad r_L = (2 - n)(\tilde{w}_1 q).$$  (36)

[The multiplicities are: $n(n-3)/2$ for R (replicon), $n-1$ for A (anomalous) and 1 for L (longitudinal); for the details, see [43].]
• For $d < 6$ we enter the nonperturbative regime. Nevertheless, for $6 - d = \epsilon \ll 1$ the fixed point $\tilde{w}_1^*$ is small, and the perturbative renormalization group results of Ref. [45] are usable. By the scaling analysis in [42] we have shown that $G^{(k)} q^{k-2} \sim \tilde{w}_1^{*2} q^2$, with $\tilde{w}_1^{*2} = O(\epsilon)$ and $\gamma/\beta = 1 + O(\epsilon)$. The right-hand side of (33) and (35) are of order $\epsilon$, with the only exception of $-w_1 q$ in Eq. (35). This results in the bare masses

$$2\tilde{m}_1 + n \tilde{m}_2 = O(\epsilon (\tilde{w}_1^{*2} q)), \quad \tilde{m}_3 = O(\epsilon (\tilde{w}_1^{*2} q)) \quad \text{and} \quad \tilde{m}_2 = -(\tilde{w}_1^{*2} q) [1 + O(\epsilon)].$$

After replacing $\tilde{w}_1$ by $\tilde{w}_1^{*}$, the free propagators remain the same as in (36).

In conclusion, in the one-loop calculation of the next section we can use as bare masses:

$$2\tilde{m}_1 + n \tilde{m}_2 = 0, \quad \tilde{m}_3 = 0 \quad \text{and} \quad \tilde{m}_2 = \begin{cases} -\tilde{w}_1 q & \text{for } d > 6 \\ -\tilde{w}_1^{*2} q & \text{for } d < 6. \end{cases} \quad (37)$$

VI. THE ONE-LOOP CALCULATION OF $g(v)$

Our basic formula for $g(v)$ can be derived by a Legendre-transform method; leaving the technical details to Appendix D we only quote the result here:

$$g(v) = \bar{g}(v) + \bar{H} + v \sum_{(\gamma\delta)} (M_{\alpha\beta,\gamma\delta} - M_{\alpha\beta,\gamma\delta}^{(0)}) + \frac{1}{p(n-p)} \frac{d}{dv} \left[ \frac{1}{N} \ln \tilde{Z} \right], \quad \text{with}$$

$$\bar{g}(v) = \frac{1}{2} v^2 \sum_{(\gamma\delta),(\mu\nu)} W_{\alpha\beta,\gamma\delta,\mu\nu} + \frac{1}{6} v^3 \sum_{(\gamma\delta),(\mu\nu),(\rho\omega)} U_{\alpha\beta,\gamma\delta,\mu\nu,\rho\omega} + \ldots, \quad \text{and} \quad \alpha \cap \beta = 0. \quad (38)$$

[Alike in (39), the primed summation means the constraint that the replicas in a summing pair belong to different groups.] The second and third terms ensure that $g(0) = g'(0) = 0$, as it must be by the definition in (38), whereas $\bar{g}(v)$ is the replicon generator built up from the bare couplings. Due to the choice $v_{\alpha\beta} = v$ for $\alpha \cap \beta = 0$, and otherwise zero, the partition function $\tilde{Z}$ of the system with the couplings in (D4) depends now on the single variable $v$. Obviously $\ln \tilde{Z}$ has the “two-packet” RSB structure, nevertheless it serves as the generator of the replicon-like vertices of the RS system. Although Eq. (38) will be used in a first order perturbative calculation in this section, it is quite generic, and may be useful even in a nonperturbative treatment.

The one-loop contribution to the last term of (38) is

$$g_1(v) = -\frac{1}{2p(n-p)} \frac{1}{N} \sum_p \sum_j \frac{1}{p^2 + \lambda_j} \frac{\partial \lambda_j}{\partial v}, \quad (39)$$
where $\lambda_j$ is an eigenvalue of $\tilde{M}$, the bare mass of $\tilde{\mathcal{L}}$ displayed in (D4). The restriction $v_{\alpha\beta} = v$ when $\alpha \cap \beta = 0$, while otherwise it is zero, means that $\tilde{M}$ is a matrix with the “two-packet” RSB whose block diagonalization is worked out in Appendix E for the most generic case. In the scaling limit ($q \to 0$ and $v \sim q$), $\tilde{M}$ reduces to the first two terms in (D4), and only $\bar{W}_1$ is kept, as it is the only cubic coupling which is finite at criticality. With the notations of Appendix E we then have

$$
\tilde{M}_1^{(1)} = \tilde{M}_1^{(2)} = \tilde{M}_1' = \bar{M}_1,
$$
$$
\tilde{M}_2^{(1)} = \tilde{M}_2^{(2)} = \tilde{M}_2' = \bar{M}_2
$$
$$
\tilde{M}_3^{(1)} = \tilde{M}_3^{(2)} = \tilde{M}_3' = \bar{M}_3
$$

As it turns out from Appendix E, the two replicon classes are independent of $v$ in this approximation, and they do not contribute to $g_1$. To simplify the following algebra, and exploiting the arbitrariness of $p$, the symmetric choice $p = n - p = n/2$ in computing the longitudinal (L) and anomalous (A) eigenvalues will be applied. After substituting the mass elements from Eqs. (40) and (37) into the L and A blocks of Appendix E, we get

$$
\begin{align*}
\lambda_0^0_\text{L} &= 2\bar{w}_1 q \\
\lambda_{\pm}^\pm_\text{L} &= \bar{w}_1 q \left\{ 1 \pm \left[ 1 + n(n - 2) \left( 1 + \frac{v}{q} \right)^2 \right]^{\frac{1}{2}} \right\} \\
\lambda_{\pm}^\pm_\text{A} &= \bar{w}_1 q \left\{ \frac{n + 2}{2} \pm \left[ 1 + \frac{1}{4} n(n - 4)(1 + \frac{v}{q})^2 \right]^{\frac{1}{2}} \right\}
\end{align*}
$$

Putting these expressions into (39), with the multiplicities taken from Appendix E, the following result is obtained for the one-loop scaling contribution:

$$
g_1(v) = 4 \frac{n - 2}{n} q \left( 1 + \frac{v}{q} \right) \bar{w}_1^2 \frac{1}{N} \sum_p \left[ \frac{1}{(p^2 + \lambda_0^0_\text{L})(p^2 + \lambda_{\pm}^\pm_\text{L})} + \frac{n - 4}{4} \frac{1}{(p^2 + \lambda_{\pm}^\pm_\text{A})(p^2 + \lambda_{\pm}^\pm_\text{A})} \right].
$$

(For $d < 6$, $\bar{w}_1$ must be replaced by $w_1^*$. It is obvious from (41) and (42) that $g_1$ satisfies

$$
g_1(-v - 2q) = -g_1(v).
$$

By a direct calculation of the second and third terms of (38), it was checked that $g_1(v)$ in (42) ensures that the generic property $g(v = 0) = g'(v = 0) = 0$ is valid up to one-loop order.

By straightforward, though somewhat lengthy, manipulation of Eqs. (38) and (42) — whilst using the definitions of the $v$-independent masses in (36) and the $v$-dependent ones in (41) —, the
following convenient form for the one-loop leading scaling term of \( g(v) \) can be concluded:

\[
\begin{align*}
g(v) &= \tilde{g}(v) + 4(n-2) \bar{w}_1 (\bar{w}_1 q)^3 (v/q)^2 (3 + v/q) \\
&+ \frac{n-2}{N} \sum_p \left[ \frac{(n-4)^2/16}{(p^2 + \lambda_+^2)(p^2 + \lambda^-_L)(p^2 + r_L)(p^2 + r_R)} + \frac{(n-4)^3/64}{(p^2 + \lambda_+^2)(p^2 + \lambda^-_L)(p^2 + r_L)^2(p^2 + r_R)^2} \right].
\end{align*}
\]

(43)

This formula was derived for \( p = n/2 \) in the scaling limit \( q \to 0 \) and \( v \sim q \). In what follows, Eq. (43) is used to track the behaviour of \( g(v) \), and along with it the question of stability of the RS spin glass phase, starting from high dimension \( (d > 8) \) and ending below the upper critical dimension \( (d \lesssim 6) \), where the nontrivial fixed point is small, and one-loop results can be translated to the leading behaviour of \( g(v) \) in an \( \epsilon = 6 - d \) expansion.

A. The mean field like high dimensional regime: \( d > 8 \)

In this case, the bare replicon generator \( \tilde{g}(v) \) as well possesses the properties of Eq. (17) as the exact one. From this and the definition in (38), the leading term for \( q \to 0 \) and \( q \sim v \) follows as

\[
\tilde{g}(v) = \frac{1}{6} \left[ 2\bar{u}_2 + n\bar{u}_3 + \frac{1}{2} n^2(\bar{u}_1 + 2\bar{u}_4) \right] q^3 (v/q)^2 (3 + v/q),
\]

(44)

where (C2) of Appendix C has been used for \( p = n/2 \), and only the four quartic couplings remaining finite at \( T_c \) are kept. [To reproduce the mean field results of the Sherrington-Kirkpatrick model \( \tilde{g} \), we must substitute the values \( \bar{u}_1 = 3, \bar{u}_2 = 2, \bar{u}_3 = -6 \) and \( \bar{u}_4 = 0 \) into (44). We must be careful, however, if a direct comparison with Eq. (21) is to be made, as it is valid in the limit \( p \to 0 \), whereas \( p = n/2 \) was chosen in this section. Obviously, disregarding the \( n^2 \) term in (44), which comes from \( p(n - p) \), leads to complete agreement.] Putting together this result and the high momentum leading term of (43), \( g(v) \) assumes the scaling form \( q^3 \hat{g}(v/q) \), with

\[
\hat{g}(x) = \left\{ \frac{1}{6} \left[ 2\bar{u}_2 + n\bar{u}_3 + \frac{1}{2} n^2(\bar{u}_1 + 2\bar{u}_4) \right] + \frac{1}{4} (n-2)(n^2 + 8n - 16) \bar{w}_1^4 \frac{1}{N} \sum_p \frac{1}{p^3} \right\} \times x^2(3 + x)
\]

A comparison with (21) reveals the surprising fact that the scaling function \( \hat{g}(x) \sim x^2(3 + x) \) remains very robust with respect to mean field theory \( (d = \infty) \), apart from a normalization factor, at least up to one-loop order.
The replicon mass for finite $n$ can now be easily derived from (15) and (35):

$$2m_1 \cong n w_1 q - \frac{g(-2q)}{2q}. \quad (45)$$

For $n > 0$, there is a temperature domain below $T_c$ where the RS phase is stable ($2m_1 > 0$). The condition $2m_1 = 0$ signals the end of the stable RS phase, defining an Almeida-Thouless line in the $T$-$n$ plane [46], or equivalently in the $q$-$n$ plane (instead of the original AT line in the temperature-magnetic field plane):

$$q_{AT} \cong C_d n, \quad (46)$$

where the dimension-dependent slope has a finite limit for $d \to \infty$ (the zero-loop mean field limit), whereas it goes monotonically to zero while $d \to 8$; see Fig. 1. A first order perturbative correction leads to a shrinking RS phase (or growing RSB phase) when decreasing the space dimension.

**B. The transitional regime: $6 < d < 8$**

Although critical behaviour is still governed by the Gaussian fixed point, and standard perturbation expansion remains valid (no accumulating infrared divergences), scaling of $g(v)$ is now modified to

$$g(v) = \bar{w}_1 (\bar{w}_1 q)^{2-\rho/2} \hat{g}(v/q), \quad \rho = 6 - d.$$  

The scaling function $\hat{g}$ follows from (43) after sending the cutoff to infinity and rescaling the momentum $p^2 \to p^2/(\bar{w}_1 q)$. (The bare contribution $\bar{g}(v)$ is subleading for $d < 8$, and it can be neglected in what follows.) The following properties are easily concluded:

1. For $n = 0$, the mean field form $\hat{g} \sim x^2(3+x)$ is again recovered, together with its immediate aftermaths in Eqs. (22) and (23). The truncated form of (24) then follows as well. This is, however, not at all trivial when $d < 8$ — as it was pointed out in [42] —, since all the terms in (8) are now the same order, and a cancellation of the terms higher than cubic order in $v$ must occur. This solves the problem raised in Ref. [42]: the infinite sum of its Eq. (11) is exactly zero, leading to instability. This conclusion is now reached by the evaluation of the $g(v)$ function, and the previous statements are valid, of course, in this first order perturbative framework.

2. A nonanalytic $n$-dependence develops for $d < 8$, which is proportional to $n^{1-\rho/2}$. Although this nonanalyticity is innocuous with respect to the zero $n$ limit, the truncated form (24) is no longer valid for finite $n$. 
The boundary of the RS state follows again from (45) and (43), the zero \( n \) limit of the propagators taken from (36) and (41). The AT line in the \( q-n \) plane is now nonanalytic, and the tendency of a shrinking RS phase persists when approaching the upper critical dimension 6 (see Fig. 1):

\[ n \approx 16 \bar{w}_1^2 \frac{1}{N} \sum_p \frac{1}{p^4(p^2 + 2)^2} (\bar{w}_1 q_{\text{AT}})^{\varepsilon/2}, \quad -2 < \varepsilon = 6 - d < 0. \]  

(47)

C. Below the upper critical dimension: \( d < 6 \)

The nontrivial fixed point \[ 31 \] \( \bar{w}_1^2 = -\varepsilon/(n - 2) + \ldots \) governs critical behaviour below six dimensions. A scaling analysis in Ref. \[ 42 \] showed that \( G^{(k+1)} q^{k-1} \sim \bar{w}_1^2 (\bar{w}_1^* q)^{\gamma/\beta} \), yielding by Eq. (8) in the scaling limit

\[ g(v) = \bar{w}_1^* (\bar{w}_1^* q)^{\gamma/\beta + 1} \hat{g}(v/q), \quad d < 6. \]  

(48)

With this definition above, \( \hat{g} \) has a finite limit for \( \varepsilon \to 0 \), and it obviously depends on \( d \) and \( n \). Renormalization group calculations provide \( \gamma/\beta + 1 = 2 + \varepsilon/2 + \ldots \), see \[ 31, 45 \]. Our result is consistent with this form at the leading order in \( \varepsilon \), the scaling function is the same as that of the transitional regime in the previous subsection, except that the momentum integrals must be performed at \( d = 6 \). The first property of the previous subsection, concerning the \( n = 0 \) behaviour of the scaling function, can be entirely taken over, whereas the \( n \)-dependence of \( \hat{g} \), which is harmless around \( d = 6 \), will be discussed in the conclusions.

Both terms in Eq. (45) are now proportional to \( (\bar{w}_1^* q)^{\gamma/\beta} \), and we get, in the limit \( n \to 0 \), by means of (43) the replicon mass

\[ 2m_1 \approx (\bar{w}_1^* q)^{\gamma/\beta} \left[ n - 8 \varepsilon \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^4(p^2 + 2)^2} + O(\varepsilon^2) \right], \quad \varepsilon = 6 - d \gtrsim 0 \quad \text{and} \quad n \gtrsim 0. \]  

(49)

We can see from this formula that the stable RS spin glass phase for \( n \) small but finite — which does exist in mean field theory and even for \( d > 6 \) — completely disappears below six dimensions, continuing in this way the trend of a shrinking RS phase (see Fig. 1).

VII. SUMMARY AND CONCLUSIONS

The Ising spin glass in zero magnetic field has been studied in this paper, using the field theoretic representation of the replicated model and concentrating on the question of stability of the replica symmetric glassy phase. Any preconception concerning the RS spin glass phase has been avoided, except that the Legendre-transformed free energy preserves the symmetry of the
replicated paramagnet, Eqs. (6) and (7), just as in the case of common phase transitions with spontaneous symmetry breaking. As a convinient way to study stability, the replicon generator function $g(v)$ was introduced. On the one hand, it was exactly computed for the Sherrington-Kirkpatrick model, whereas droplet theory ideas were applied to guess its behaviour in case of a droplet picture scenario. In this later case, a highly singular behaviour in the spin glass limit ($n \to 0$) was found. We followed the fate of the mean field ($d = \infty$) scenario by a first order perturbative calculation down below the upper critical dimension 6, where the fixed point is small and the perturbative renormalization group is valid. The following conclusions have been reached:

- The scaling limit of $g(v)$, $q \to 0$ and $v \sim q$, has the common form for $n = 0$:

$$g(v) \sim q^\lambda \tilde{g}(v/q), \quad \text{with} \quad \lambda = \begin{cases} 3, & d > 8 \text{ and in mean field}, \\ 2 - \epsilon/2, & 6 < d < 8, \\ 1 + \gamma/\beta = 2 + \epsilon/2 + O(\epsilon^2), & d \lesssim 6; \end{cases}$$  \hspace{1cm} (50)

$\epsilon = 6 - d$. The scaling function has proved to be very robust, as it is

$$\tilde{g}(x) \sim x^2(3 + x)$$  \hspace{1cm} (51)

for all three cases.

This result is not unexpected for $d > 8$, as it must be true for the full $g(v)$ (even for any finite $n$) built up from the exact vertices. This can be proven by applying a simple scaling analysis around the Gaussian fixed point. Using (5), (C1) and (C2), the exact $g(v)$ can be separated into two parts:

$$g(v) = \left(w_2 v^2 + \frac{1}{3} w_2 v^3 + \ldots\right) + \sum_{k>\frac{d}{2}-1} \frac{1}{k!} G^{(k+1)} v^k, \quad d > 8, \quad n = 0.$$

The first term represents the analytical part, the ellipsis dots stand for its subdominant contributions with $3 < k \leq \frac{d}{2} - 1$, whereas the second one is nonanalytical with all of its subterms being the same order for $q \to 0$ and $v \sim q$:

$$\sum_{k>\frac{d}{2}-1} \frac{1}{k!} G^{(k+1)} v^k \cong \sum_{k>\frac{d}{2}-1} c_k q^{2-\frac{\epsilon}{2}} (v/q)^k \equiv q^{2-\frac{\epsilon}{2}} \hat{h}(v/q).$$  \hspace{1cm} (52)

The analytical part obviously dominates for $d > 8$, and the mean field like truncated form in Eq. (24) is valid for the exact $g(v)$ (and even for a generic $n$ too).
In the transitional regime, $6 < d < 8$, the analytical part disappears, and $g(v)$ takes the form as in (52). There is now no reason to think that the exact scaling function $\hat{h}$ has the mean field functional form $x^2(3 + x)$, since all the higher order vertices equally contribute. Nevertheless, the first order calculation provides — somewhat surprisingly — Eqs. (50) and (51), suggesting that a cancellation of terms with vertices other than the cubic and quartic ones must occur in one-loop order. A similar remark is valid for the regime below the upper critical dimension too. It must be realized, however, that the truncated form (24) or (51) holds only for the spin glass limit $n = 0$, as opposed to the $d > 8$ regime, and an infinite number of terms contribute for $n$ finite.

- The mean field scenario is very robust for decreasing $d$, see Eqs. (50) and (51), leading to an unstable RS phase for $n = 0$ even below 6 dimensions. By computing the boundary of stability, $2m_1 = 0$, in finite $n$ (a kind of AT line in zero magnetic field), even a shrinking RS phase has been found with respect to the mean field picture (Fig. 1); at least in this first order perturbative approximation.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Phase boundary of the RS-RSB transition (Almeida-Thouless line) in the $q - n$ plane. This one-loop result shows a shrinking RS phase with decreasing $d$. (a) and (b) represent (46), while the nonanalytic feature of the AT line in the transitional regime ($6 < d < 8$), Eq. (47), is displayed in (c). The stable RS domain for small but finite $n$ disappears for $d < 6$ and $n < n_c \approx \epsilon = 6 - d$, as shown in (d).}
\end{figure}

- The mean field formula Eq. (23), which connects the replicon mass $(2m_1)$ and the replicon 3-point vertex $(w_2)$ for $n = 0$, remains valid below 6 dimensions, at least in the leading $\epsilon$ order. This follows from the robustness of $\hat{g}$, as shown in (50) and (51). From the independent
calculation of the one-loop 3-point vertices \[45\] we can quote the relatively simple result for \(w_2\) (keeping \(n\) finite here):

\[
w_2 = \frac{12}{(n-2)^2} w_1^* (w_1^*)^{\gamma/\beta - 1} \int \frac{d^6p}{(2\pi)^6} \left[ \frac{4-n}{(p^2+n)^2(p^2+2)^2} - \frac{n}{(p^2+n)^3(p^2+2)} \right] \times \epsilon + O(\epsilon^2).
\]

Eq. (49) follows immediately from (23) in the spin glass limit, providing us a good check of the theory.

- As it can be observed from the above discussion, stability of the RS spin glass state in zero magnetic field, and thus the possibility of the droplet scenario itself, can not occur in those high dimensions where direct perturbation expansion (loop expansion) or the perturbative renormalization group is applicable \((d \lesssim 6)\). There is even an obvious tendency that RSB is escalating with decreasing space dimension. Although common objections against this statement might argue that a droplet-like RS state is intrinsically nonperturbative, it must be stressed that, at least for \(d > 8\), exact considerations are in accordance with the (first order) perturbative result.

- The possibility for the emergence of a marginally stable droplet phase — notwithstanding the ever increasing RSB domain — in low enough dimensions will be put forward in this remark. The \(n\)-dependent scaling form of \(g(v)\) for the droplet picture, Eq. (32), proposed in Section [IV] clearly shows that \(\hat{g}\) for the droplet model should be a nontrivial generalized homogeneous function of \(v/q\) and \(n\) near \(T_c\). Just below 6 dimensions, \(\hat{g}\) of (48) is not such, as it can be decomposed into the form \(\hat{g} = \hat{g}_{\text{an}} + \hat{g}_{\text{nonan}}\), where \(\hat{g}_{\text{an}}\) is analytical \((\hat{g}_{\text{nonan}}\) is nonanalytical) for \(n \to 0\), respectively, and \(\hat{g}_{\text{nonan}} \ll \hat{g}_{\text{an}}\) for any \(v/q\). \((\hat{g}_{\text{nonan}} \sim n^{1-\epsilon/2}\) when \(6 < d < 8\).\) As a matter of fact, \(n\) in \(\hat{g}\) plays the role of a small mass, the “replicon” mass, as contrasted to the large mass, the “longitudinal” one. This small mass is, however, innocuous around \(d = 6\), since it does not cause infrared problems. (See as an example Eq. (53): an infrared divergence would be expected at \(d = 4\), if the collapse of the perturbative approach were neglected.) We can postulate a dimension \(d_R\), which is only naively believed to be 4, where \(\hat{g}_{\text{nonan}}\) stops being negligible, and \(\hat{g}\) may become a nontrivial generalized homogeneous function of \(v/q\) and \(n\). Although a proof of this scenario and an estimate of \(d_R\) seem to be a rather hard task — due to the nonperturbative nature of the problem in these low dimensions —, these considerations could help the understanding how a marginally
stable replica symmetric phase may emerge in low enough dimensions.\footnote{\(d_R\), the lower critical dimension of RSB, should not be missed with \(d_l\), the lower critical dimension of the spin glass state. Of course, the transition from RSB to the droplet scenario can occur only if \(d_l < d_R\).}

Our final comment concerns the work of Bray and Moore \cite{32}: In this paper the authors use the same “two-packet” RSB for temporarily breaking the replica symmetry, and eventually restoring the RS state by sending \(p\) (which they call \(m\)) to infinity. What is more, this RS state is marginally stable! It must be stressed, however, that the two approaches differ substantially: \(p < n\) is always assumed here, and the “two-packet” RSB calculation used in this paper is only a tool to compute the replicon generator of the RS state. The marginally stable RS phase found in Ref. \cite{32} is obviously in conflict with our results, since the replica symmetric spin glass state has proved to be unstable in the regime \(6 < d < 8\), assuming only the invariance of the Legendre-transformed free energy, Eq. (\ref{eq:7}), as a prerequisite.

**APPENDIX A: BASIC DEFINITIONS**

In this appendix the definitions of the lower and upper case vertices are displayed in a tabular form for the generic RS state. Although not all of them are used in this paper, completeness was favoured up to the quartic order. Due to the RS symmetry, the number of different vertices is limited to 3 for the masses (\(k = 2\)), 8 for the cubic (\(k = 3\)) and 23 for the quartic (\(k = 4\)) ones. The first column provides an obvious graphical representation, while in the second one the lower case bare couplings are enumerated together with their corresponding invariants; see Eqs. (\ref{eq:4}) and (\ref{eq:5}). The alternative set of vertices with the upper case notation is best defined by the derivatives of the exact Legendre-transformed free energy \(F\), Eq. (\ref{eq:10}), and they are listed in the last column. The corresponding bare couplings enter the Lagrangian in the following way (see footnote 4):

\[
\mathcal{L} = \frac{1}{2} \sum_{(\alpha\beta)} \sum_{p=0}^{(\gamma\delta)} (p^2 \delta_{\alpha\beta,\gamma\delta} + \tilde{M}_{\alpha\beta,\gamma\delta}) \phi^{\alpha\beta}_{p} \phi^{\gamma\delta}_{-p} + \mathcal{L}^I, \quad (A1)
\]

with the interaction part

\[
\mathcal{L}^I = -N^{\frac{1}{2}} \bar{H} \sum_{(\alpha\beta)} \phi^{\alpha\beta}_{p=0} - \sum_{k=3}^{\infty} \frac{1}{k! N^{\frac{k-1}{2}}} \sum_{p_1}^{\prime} \sum_{(\alpha_1,\beta_1),\ldots,(\alpha_k,\beta_k)} \tilde{G}^{(k)}_{\alpha_1,\beta_1,\ldots,\alpha_k,\beta_k} \phi^{\alpha_1\beta_1}_{p_1} \ldots \phi^{\alpha_k\beta_k}_{p_k}. \quad (A2)
\]

\([\bar{H} = 2\tilde{h}\) follows trivially by comparing (\ref{eq:5}) and (\ref{eq:A2}).]

As already declared in the main text, the common notations are used in what follows for the quadratic (\(-\tilde{m}_j\) and \(-M_j\)), cubic (\(\tilde{w}_j\) and \(W_j\)) and quartic (\(\tilde{u}_j\) and \(U_j\)) vertices (instead of \(\tilde{g}^{(k)}_j\) and \(G^{(k)}_j, k = 2, 3\) and 4).
- The masses, $k = 2$:

| $j$ | $g_j^{(k)} I_j^{(k)}$ | $G_j^{(k)}$ |
|-----|------------------|-------------|
| 1   | $-\bar{m}_1 \sum_{\alpha\beta} \phi_{\mathbf{p}_1}^{\alpha\beta} \phi_{\mathbf{p}_2}^{\alpha\beta}$ | $-M_1 = -M_{\alpha\beta,\alpha\beta}$ |
| 2   | $-\bar{m}_2 \sum_{\alpha\beta\gamma} \phi_{\mathbf{p}_1}^{\alpha\beta} \phi_{\mathbf{p}_2}^{\gamma\beta}$ | $-M_2 = -M_{\alpha\gamma,\beta\gamma}$ |
| 3   | $-\bar{m}_3 \sum_{\alpha\beta\gamma\delta} \phi_{\mathbf{p}_1}^{\alpha\beta} \phi_{\mathbf{p}_2}^{\gamma\delta}$ | $-M_3 = -M_{\alpha\beta,\gamma\delta}$ |

- Cubic vertices, $k = 3$:

| $j$ | $g_j^{(k)} I_j^{(k)}$ | $G_j^{(k)}$ |
|-----|------------------|-------------|
| 1   | $\bar{w}_1 \sum_{\alpha\beta\gamma} \phi_{\mathbf{p}_1}^{\alpha\beta} \phi_{\mathbf{p}_2}^{\beta\gamma} \phi_{\mathbf{p}_3}^{\gamma\alpha}$ | $W_1 = W_{\alpha\beta,\beta\gamma,\gamma\alpha}$ |
| 2   | $\bar{w}_2 \sum_{\alpha\beta} \phi_{\mathbf{p}_1}^{\alpha\beta} \phi_{\mathbf{p}_2}^{\alpha\beta}$ | $W_2 = W_{\alpha\beta,\alpha\beta,\alpha\beta}$ |
| 3   | $\bar{w}_3 \sum_{\alpha\beta\gamma} \phi_{\mathbf{p}_1}^{\alpha\beta} \phi_{\mathbf{p}_2}^{\alpha\beta} \phi_{\mathbf{p}_3}^{\beta\gamma}$ | $W_3 = W_{\alpha\beta,\alpha\beta,\beta\gamma}$ |
| 4   | $\bar{w}_4 \sum_{\alpha\beta\gamma\delta} \phi_{\mathbf{p}_1}^{\alpha\beta} \phi_{\mathbf{p}_2}^{\alpha\beta} \phi_{\mathbf{p}_3}^{\gamma\delta}$ | $W_4 = W_{\alpha\beta,\alpha\beta,\gamma\delta}$ |
| 5   | $\bar{w}_5 \sum_{\alpha\beta\gamma\delta} \phi_{\mathbf{p}_1}^{\alpha\beta} \phi_{\mathbf{p}_2}^{\alpha\beta} \phi_{\mathbf{p}_3}^{\gamma\delta}$ | $W_5 = W_{\alpha\beta,\alpha\gamma,\delta\delta}$ |
| 6   | $\bar{w}_6 \sum_{\alpha\beta\gamma\delta} \phi_{\mathbf{p}_1}^{\alpha\beta} \phi_{\mathbf{p}_2}^{\alpha\beta} \phi_{\mathbf{p}_3}^{\beta\gamma}$ | $W_6 = W_{\alpha\beta,\alpha\gamma,\alpha\delta}$ |
| 7   | $\bar{w}_7 \sum_{\alpha\beta\gamma\delta\mu} \phi_{\mathbf{p}_1}^{\alpha\gamma} \phi_{\mathbf{p}_2}^{\beta\gamma} \phi_{\mathbf{p}_3}^{\delta\mu}$ | $W_7 = W_{\alpha\gamma,\beta\gamma,\delta\mu}$ |
| 8   | $\bar{w}_8 \sum_{\alpha\beta\gamma\delta\mu} \phi_{\mathbf{p}_1}^{\alpha\gamma} \phi_{\mathbf{p}_2}^{\beta\gamma} \phi_{\mathbf{p}_3}^{\delta\mu}$ | $W_8 = W_{\alpha\beta,\gamma\delta,\mu\mu}$ |

- Quartic vertices, $k = 4$:
| $j$ | $g_j^{(k)} I_j^{(k)}$ | $C_j^{(k)}$ |
|-----|-------------------|----------------|
| 1   | $\bar{u}_1 \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha} \phi_{p_2}^{\beta} \phi_{p_3}^{\gamma} \phi_{p_4}^{\delta}$ | $U_1 = U_{\alpha, \beta, \gamma, \delta}$ |
| 2   | $\bar{u}_2 \sum_{\alpha, \beta} \phi_{p_1}^{\alpha, \beta} \phi_{p_2}^{\alpha, \beta}$ | $U_2 = U_{\alpha, \beta, \alpha, \beta}$ |
| 3   | $\bar{u}_3 \sum_{\alpha, \beta} \phi_{p_1}^{\alpha, \beta} \phi_{p_2}^{\alpha, \beta}$ | $U_3 = U_{\alpha, \beta, \alpha, \beta}$ |
| 4   | $\bar{u}_4 \sum_{\alpha, \beta, \gamma} \phi_{p_1}^{\alpha, \beta, \gamma} \phi_{p_2}^{\alpha, \beta, \gamma}$ | $U_4 = U_{\alpha, \beta, \alpha, \beta}$ |
| 5   | $\bar{u}_5 \sum_{\alpha, \beta} \phi_{p_1}^{\alpha, \beta} \phi_{p_2}^{\alpha, \beta}$ | $U_5 = U_{\alpha, \beta, \alpha, \beta}$ |
| 6   | $\bar{u}_6 \sum_{\alpha, \beta, \gamma} \phi_{p_1}^{\alpha, \beta, \gamma} \phi_{p_2}^{\alpha, \beta, \gamma}$ | $U_6 = U_{\alpha, \beta, \alpha, \beta} \phi_{p_1}^{\alpha, \beta, \gamma}$ |
| 7   | $\bar{u}_7 \sum_{\alpha, \beta, \gamma} \phi_{p_1}^{\alpha, \beta, \gamma} \phi_{p_2}^{\alpha, \beta, \gamma}$ | $U_7 = U_{\alpha, \beta, \alpha, \beta}$ |
| 8   | $\bar{u}_8 \sum_{\alpha, \beta} \phi_{p_1}^{\alpha, \beta} \phi_{p_2}^{\alpha, \beta}$ | $U_8 = U_{\alpha, \beta, \alpha, \beta}$ |
| 9   | $\bar{u}_9 \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma} \phi_{p_2}^{\alpha, \beta, \gamma}$ | $U_9 = U_{\alpha, \beta, \alpha, \beta}$ |
| 10  | $\bar{u}_{10} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{10} = U_{\alpha, \beta, \alpha, \beta}$ |
| 11  | $\bar{u}_{11} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{11} = U_{\alpha, \beta, \alpha, \beta}$ |
| 12  | $\bar{u}_{12} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{12} = U_{\alpha, \beta, \alpha, \beta}$ |
| 13  | $\bar{u}_{13} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{13} = U_{\alpha, \beta, \alpha, \beta}$ |
| 14  | $\bar{u}_{14} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{14} = U_{\alpha, \beta, \alpha, \beta}$ |
| 15  | $\bar{u}_{15} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{15} = U_{\alpha, \beta, \alpha, \beta}$ |
| 16  | $\bar{u}_{16} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{16} = U_{\alpha, \beta, \alpha, \beta}$ |
| 17  | $\bar{u}_{17} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{17} = U_{\alpha, \beta, \alpha, \beta}$ |
| 18  | $\bar{u}_{18} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{18} = U_{\alpha, \beta, \alpha, \beta}$ |
| 19  | $\bar{u}_{19} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{19} = U_{\alpha, \beta, \alpha, \beta}$ |
| 20  | $\bar{u}_{20} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{20} = U_{\alpha, \beta, \alpha, \beta}$ |
| 21  | $\bar{u}_{21} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{21} = U_{\alpha, \beta, \alpha, \beta}$ |
| 22  | $\bar{u}_{22} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{22} = U_{\alpha, \beta, \alpha, \beta}$ |
| 23  | $\bar{u}_{23} \sum_{\alpha, \beta, \gamma, \delta} \phi_{p_1}^{\alpha, \beta, \gamma, \delta}$ | $U_{23} = U_{\alpha, \beta, \alpha, \beta}$ |
APPENDIX B: THE LINEAR RELATIONS BETWEEN THE TWO SETS OF VERTICES

Decomposing the restricted sums in Eq. (A2) and comparison with (4) and (5) provide the linear relations between the upper and lower case bare couplings. (Although this is straightforward combinatorics, it becomes quite time consuming for highly disconnected quartic vertices.) In what follows, these formulae are displayed by omitting the bars in the notations; in this form they can be considered as the definition of the exact lower case vertices. The results for the masses and cubic vertices are taken over from [43], and they are shown here for easing to refer them:

\[ m_1 = \frac{1}{2}(M_1 - 2M_2 + M_3) \]
\[ m_2 = M_2 - M_3 \]
\[ m_3 = \frac{1}{4}M_3; \]

\[
\begin{align*}
w_1 &= W_1 - 3W_5 + 3W_7 - W_8 \\
w_2 &= \frac{1}{2}W_2 - 3W_3 + \frac{3}{2}W_4 + 3W_5 + 2W_6 - 6W_7 + 2W_8 \\
w_3 &= 3W_3 - 3W_4 - 6W_5 - 3W_6 + 15W_7 - 6W_8 \\
w_4 &= \frac{3}{4}W_4 - \frac{3}{2}W_7 + \frac{3}{4}W_8 \\
w_5 &= 3W_5 - 6W_7 + 3W_8 \\
w_6 &= W_6 - 3W_7 + 2W_8 \\
w_7 &= \frac{3}{2}W_7 - \frac{3}{2}W_8 \\
w_8 &= \frac{1}{8}W_8. \tag{B1}
\end{align*}
\]

The set of equations for the quartic case:

\[
\begin{align*}
u_1 &= 3U_1 - 12U_{16} + 12U_{17} + 6U_{21} - 12U_{22} + 3U_{23} \\
u_2 &= 3U_1 + \frac{1}{2}U_2 - 3U_3 + \frac{3}{2}U_4 - 4U_8 + 2U_9 + 12U_{10} + 12U_{11} - 24U_{12} + 6U_{13} - 12U_{14} + 12U_{15} - 24U_{16} + 48U_{17} - 24U_{18} - 6U_{19} + 24U_{20} + 30U_{21} - 72U_{22} + 18U_{23} \\
u_3 &= -6U_1 + 3U_3 - 3U_4 - 6U_{10} - 12U_{11} + 12U_{12} + 18U_{14} - 12U_{15} + 36U_{16} - 48U_{17} + 12U_{18} + 3U_{19} - 12U_{20} - 39U_{21} + 72U_{22} - 18U_{23} \\
u_4 &= \frac{3}{4}U_4 - 3U_{14} + \frac{3}{2}U_{15} + 3U_{21} - 3U_{22} + \frac{3}{4}U_{23} \\
u_5 &= 6U_5 - 24U_6 + 12U_7 - 12U_{11} + 12U_{12} - 6U_{13} + 6U_{14} - 6U_{15} + 36U_{16} - 84U_{17} + 48U_{18} - 24U_{20} - 36U_{21} + 96U_{22} - 24U_{23} \\
u_6 &= 12U_6 - 12U_7 - 24U_{16} + 72U_{17} - 24U_{18} + 12U_{20} + 24U_{21} - 84U_{22} + 24U_{23} \\
u_7 &= 2U_7 - 6U_{17} + 6U_{22} - 2U_{23} \\
u_8 &= 4U_8 - 4U_9 - 12U_{10} - 12U_{11} + 48U_{12} - 12U_{13} + 12U_{14} - 24U_{15} + 24U_{16} - 96U_{17} + 48U_{18} + 8U_{19} - 56U_{20} - 48U_{21} + 168U_{22} - 48U_{23}
\end{align*}
\]
\[ u_9 = U_9 - 6U_{12} + 3U_{15} + 6U_{17} + 4U_{20} - 12U_{22} + 4U_{23} \]
\[ u_{10} = 6U_{10} - 12U_{12} - 6U_{14} + 12U_{15} - 12U_{16} + 48U_{17} - 12U_{18} - 6U_{19} + 24U_{20} + 30U_{21} - 108U_{22} + 36U_{23} \]
\[ u_{11} = 12U_{11} - 12U_{12} - 12U_{14} + 12U_{15} - 24U_{16} + 48U_{17} - 12U_{18} + 12U_{20} + 36U_{21} - 84U_{22} + 24U_{23} \]
\[ u_{12} = 6U_{12} - 6U_{15} - 12U_{17} - 6U_{20} + 30U_{22} - 12U_{23} \]
\[ u_{13} = -12U_{12} + 6U_{13} + 6U_{15} + 36U_{17} - 24U_{18} + 24U_{20} + 12U_{21} - 72U_{22} + 24U_{23} \]
\[ u_{14} = 3U_{14} - 3U_{15} - 6U_{21} + 9U_{22} - 3U_{23} \]
\[ u_{15} = \frac{3}{4}U_{15} - 6U_{22} + \frac{9}{4}U_{23} \]
\[ u_{16} = 12U_{16} - 24U_{17} - 12U_{21} + 36U_{22} - 12U_{23} \]
\[ u_{17} = 6U_{17} - 12U_{22} + 6U_{23} \]
\[ u_{18} = -24U_{17} + 12U_{18} - 12U_{20} - 12U_{21} + 60U_{22} - 24U_{23} \]
\[ u_{19} = U_{19} - 4U_{20} - 3U_{21} + 12U_{22} - 6U_{23} \]
\[ u_{20} = 2U_{20} - 6U_{22} + 4U_{23} \]
\[ u_{21} = 3U_{21} - 6U_{22} + 3U_{23} \]
\[ u_{22} = \frac{3}{2}U_{22} - \frac{3}{2}U_{23} \]
\[ u_{23} = \frac{1}{16}U_{23}. \]

**APPENDIX C: THE G\(^{(k)}\) COEFFICIENTS, k = 2, 3 AND 4, IN TERMS OF THE VERTICES**

Eqs. (9) and (10) define the G\(^{(k)}\)’s, which build up \(g(v)\). [See also the paragraph below (10).]

For \(k = 2\), it was worked out in Section II, see (14) and (15):

\[-G^{(2)} = \left[2m_1 + n m_2\right] + p(n - p) 4m_3.\]

We only quote the results for \(k = 3\) and \(k = 4\):

\[G^{(3)} = \left[2w_2 + n w_3 + n^2 w_6\right] + p(n - p) [(4w_4 + 2w_5 - 2w_6) + n 2w_7] + p^2(n - p)^2 8w_8 \quad \text{(C1)}\]
and

\[
G^{(4)} = [2u_2 + n(u_3 + u_8) + n^2 u_10 + n^3 u_{19}] + p(n - p) [(2u_1 + 4u_4 + 4u_9 - 2u_{10} + 2u_{11} + 2u_{13}) \\
+ n(2u_{12} + 2u_{14} + u_{16} + u_{18} - 3u_{19}) + n^2 (2u_{20} + u_{21})] \\
+ p^2(n - p)^2 [(8u_{15} + 4u_{17} - 4u_{20} + 24u_{22} + 384u_{23}) + n(4u_{22}) + \alpha^2(n - p)^3 16u_{23}].
\] 

(C2)

From these examples, we can observe the following properties valid also for generic \(k\):

- \(G^{(k)}\) is a double polinomial of \(p(n - p)\) and \(n\). We must, however, notice that the vertices themselves depend on \(n\). When constructing \(g(v)\) by its definition \((8)\), we are particularly interested in the limit \(p(n - p) \to 0\) while \(n\) fixed, and then taking the spin glass limit \(n \to 0\).

- This limit results the fully replicon vertices \(2m_1, 2w_2, 2u_2, \ldots\) etc. (These are represented by simple two-node graphs connected by \(k\) edges. See the graphical representations in the second column of the tables in Appendix [A1].) This justifies, at least in the spin glass limit, the name replicon generator for \(g(v)\).

**APPENDIX D: THE LEGENDRE-TRANSFORM METHOD TO DERIVE EQ. (38)**

The Legendre-transformed free energy \(\tilde{F}(v_{\alpha\beta})\) for the model of Eqs. (A1) and (A2), or equivalently (4) and (5), is defined by the usual rules

\[
- \tilde{F} = \ln Z - N \sum_{(\alpha\beta)} H_{\alpha\beta} v_{\alpha\beta} \quad \text{and} \quad \frac{\partial}{\partial H_{\alpha\beta}} \ln Z = N v_{\alpha\beta},
\] 

(D1)

where the partition function \(Z = \int D\phi e^{-\mathcal{L}}\) acquires its dependence on the \(H_{\alpha\beta}\)'s by adding the source term \(-N^2 \sum_{(\alpha\beta)} H_{\alpha\beta} \phi^{\alpha\beta}_p\) to the Lagrangian. By the second equation of (D1) \(\sqrt{N} v_{\alpha\beta} = \langle \phi^{\alpha\beta}_p \rangle \) follows (the average is taken by the measure \(e^{-\mathcal{L}}\)), and by the following redefinition of the fields we can make the 1-point function disappear:

\[
\varphi^{\alpha\beta}_p = \phi^{\alpha\beta}_p - \sqrt{N} v_{\alpha\beta} \delta^{K_r}_p, \quad \text{resulting} \quad \langle \varphi^{\alpha\beta}_p \rangle = 0.
\] 

(D2)

The reader is now reminded that the bare couplings of the original sourceless — i.e. \(H_{\alpha\beta} = 0\) — system are defined likewise, \(\langle \phi^{\alpha\beta}_p \rangle = 0\) [see the remark in the paragraph below (5)]. Hence we can see that \(v_{\alpha\beta} = 0\) if \(H_{\alpha\beta} = 0\), and it follows from the inverse expression \(\frac{\partial}{\partial v_{\alpha\beta}} \tilde{F} = NH_{\alpha\beta}\) that stationarity of \(\tilde{F}\) occurs for \(v_{\alpha\beta} \equiv 0\).
For computing $\tilde{F}$, $\phi$ is replaced by $\varphi$, according to Eq. (D2), in the Lagrangian \[A1\] and \[A2\], supplemented by the source to give from \[D1\]

$$-\frac{1}{N} \tilde{F} = \frac{1}{N} \ln Z - \sum_{(\alpha\beta)} H_{\alpha\beta} v_{\alpha\beta} = \frac{1}{N} \ln \tilde{Z}$$

$$+ \tilde{H} \sum_{(\alpha\beta)} v_{\alpha\beta} - \frac{1}{2} \sum_{(\alpha\beta),(\gamma\delta)} \tilde{M}_{\alpha\beta,\gamma\delta} v_{\alpha\beta} v_{\gamma\delta} + \sum_{k=3} \frac{1}{k!} \sum_{(\alpha_1\beta_1),..., (\alpha_k\beta_k)} \tilde{G}^{(k)}_{\alpha_1\beta_1,...,\alpha_k\beta_k} v_{\alpha_1\beta_1}...v_{\alpha_k\beta_k}. \quad (D3)$$

The new Lagrangian $\tilde{\mathcal{L}}$ of the shifted system gives $\ln \tilde{Z}$, its coupling parameters are listed below up to cubic order:

$$\tilde{H}_{\alpha\beta} = (\tilde{H} + H_{\alpha\beta}) - \sum_{(\gamma\delta)} \tilde{M}_{\alpha\beta,\gamma\delta} v_{\gamma\delta} + \frac{1}{2} \sum_{(\gamma\delta),(\mu\nu)} \tilde{W}_{\alpha\beta,\gamma\delta,\mu\nu} v_{\gamma\delta}v_{\mu\nu}$$

$$+ \frac{1}{6} \sum_{(\gamma\delta),(\mu\nu),(\rho\omega)} \tilde{U}_{\alpha\beta,\gamma\delta,\mu\nu,\rho\omega} v_{\gamma\delta}v_{\mu\nu}v_{\rho\omega} + \ldots,$$

$$\tilde{M}_{\alpha\beta,\gamma\delta} = \tilde{M}_{\alpha\beta,\gamma\delta} - \sum_{(\mu\nu)} \tilde{W}_{\alpha\beta,\gamma\delta,\mu\nu} v_{\mu\nu} - \frac{1}{2} \sum_{(\mu\nu),(\rho\omega)} \tilde{U}_{\alpha\beta,\gamma\delta,\mu\nu,\rho\omega} v_{\mu\nu}v_{\rho\omega} + \ldots,$$

$$\tilde{W}_{\alpha\beta,\gamma\delta,\mu\nu} = \tilde{W}_{\alpha\beta,\gamma\delta,\mu\nu} + \sum_{(\rho\omega)} \tilde{U}_{\alpha\beta,\gamma\delta,\mu\nu,\rho\omega} v_{\rho\omega} + \ldots \quad (D4)$$

Since $H_{\alpha\beta}$ enters $\tilde{\mathcal{L}}$ solely via $\tilde{H}$ and the condition in \[D2\] guarantees the lack of 1-point insertions, $\tilde{F}$ depends on the $v_{\alpha\beta}$'s alone.\(^9\)

As it is well known from the literature (see [44], [47], for instance, and references therein), $\tilde{F}$ is the generator function for the exact vertices, and since

$$\tilde{F}(v_{\alpha\beta}) = F(q + v_{\alpha\beta}) + [q\text{-dependent term}],$$

[cf. with \[19\]], we have by Eq. \[11\]:

$$-\frac{1}{N} \tilde{F} = \frac{1}{N} \ln Z(H_{\alpha\beta} \equiv 0)$$

$$- \frac{1}{2} \sum_{(\alpha\beta),(\gamma\delta)} M_{\alpha\beta,\gamma\delta} v_{\alpha\beta} v_{\gamma\delta} + \sum_{k=3} \frac{1}{k!} \sum_{(\alpha_1\beta_1),..., (\alpha_k\beta_k)} G^{(k)}_{\alpha_1\beta_1,...,\alpha_k\beta_k} v_{\alpha_1\beta_1}...v_{\alpha_k\beta_k}. \quad (D5)$$

Proceeding further along the same lines with Section \[III\], the one-variable function $\tilde{F}(v)$ is defined by taking $v_{\alpha\beta} = v$ for $\alpha \cap \beta = 0$, otherwise $v_{\alpha\beta}$ is 0. From the two alternative forms for $\tilde{F}$ in Eqs. \[D3\] and \[D5\], and using the generic definitions \[8\] and \[9\], our basic result for $g(v)$, displayed in Eq. \[38\], follows.

\(^9\) Although the derivation of $\tilde{F}$ presented here was for the RS spin glass field theory, the method is entirely general: instead of $(\alpha\beta)$, a generic index set for the field components could have been used, and the symmetry of the model does not effectively matter.
APPENDIX E: BLOCK DIAGONALIZING A MATRIX WITH THE “TWO-PACKET” REPLICA SYMMETRY BREAKING

In this Appendix the greek-roman notation of Bray and Moore [32] is used to ease the classification of matrix elements when replica symmetry is broken by dividing the $n$ replicas into two groups: the $p$ replicas in the first group are denoted by $\alpha, \beta, \ldots$, while the other $n - p$ by $a, b, \ldots$. For the 15 different matrix elements the following concise notations will be used:

- First class (diagonal elements):

  $$M^{(1)}_1 = M_{\alpha\alpha,\alpha\alpha} \quad M^{(2)}_1 = M_{\alpha\beta,\alpha\beta} \quad M^{(2)}_1 = M_{ab,ab}$$

- Second class (one common replica):

  $$M^{(1)}_2 = M_{\alpha\gamma,b\gamma} \quad M^{(2)}_2 = M_{\alpha\gamma,\beta\gamma} \quad M^{(3)}_2 = M_{a\gamma,b\gamma}$$

  $$M^{(1)}_2 = M_{\alpha c,\beta c} \quad M^{(2)}_2 = M_{ac,bc} \quad M^{(3)}_2 = M_{ac,bc}$$

- Third class (all replicas are different):

  $$M^{(1)}_3 = M_{\alpha\beta,ab} \quad M^{(2)}_3 = M_{\alpha\beta,\gamma\delta} \quad M^{(3)}_3 = M_{\alpha\beta,\gamma d} \quad M^{(4)}_3 = M_{ab,\gamma d}$$

  $$M^{(2)}_3 = M_{ab,cd} \quad M^{(3)}_3 = M_{ab,cd}$$

The procedure of Ref. [48] can be applied to find the block diagonalized matrix $\hat{M}$ by a similarity transformation (see also [43]). The basis vectors$^{10}$ of the invariant subspaces, matrix blocks and multiplicities are listed below:

1. The 3-dimensional longitudinal subspace:

   $$\phi^{\alpha\alpha}_{(L0)} = 1 \quad \phi^{\alpha\beta}_{(L1)} = 1 \quad \phi^{ab}_{(L1')} = 1$$

$^{10}$ Any elements of the basis vectors not indicated explicitly are zero.
\[ \hat{M}_{(L0),(L0)} = M_1^{(1)} + (n - p - 1)M_2^{(1)} + (p - 1)M_2'^{(1)} + (p - 1)(n - p - 1)M_3^{(4)} \]
\[ \hat{M}_{(L0),(L1)} = \frac{1}{2}(p - 1) \left[ (p - 2)M_3^{(3)} + 2M_2^{(3)} \right] \]
\[ \hat{M}_{(L0),(L1')} = \frac{1}{2}(n - p - 1) \left[ (n - p - 2)M_3^{(3)} + 2M_2^{(3)} \right] \]
\[ \hat{M}_{(L1),(L0)} = (n - p) \left[ 2M_2^{(3)} + (p - 2)M_3^{(3)} \right] \]
\[ \hat{M}_{(L1),(L1)} = M_1^{(2)} + 2(p - 2)M_2^{(2)} + \frac{1}{2}(p - 2)(p - 3)M_3^{(2)} \]
\[ \hat{M}_{(L1),(L1')} = \frac{1}{2}(n - p)(n - p - 1)M_3^{(1)} \]
\[ \hat{M}_{(L1'),(L0)} = p \left[ 2M_2'^{(3)} + (n - p - 2)M_3'^{(3)} \right] \]
\[ \hat{M}_{(L1'),(L1)} = \frac{1}{2}p(p - 1)M_3^{(1)} \]
\[ \hat{M}_{(L1'),(L1')} = M_1'^{(2)} + 2(n - p - 2)M_2'^{(2)} + \frac{1}{2}(n - p - 2)(n - p - 3)M_3'^{(2)} \]

Multiplicity: 3

2. The 2-dimensional anomalous (A) subspaces:

\[ \phi^\alpha_{(\mu_0)} = \begin{cases} 
1 & \text{if } \alpha \neq \mu \\
-(p - 1) & \text{if } \alpha = \mu 
\end{cases} \]
\[ \phi^\alpha_{(\mu_1)} = \begin{cases} 
1 & \text{if } \alpha, \beta \neq \mu \\
-\frac{1}{2}(p - 2) & \text{if } \alpha = \mu \text{ or } \beta = \mu 
\end{cases} \]
\[ \phi^\alpha_{(m_0)} = \begin{cases} 
1 & \text{if } a \neq m \\
-(n - p - 1) & \text{if } a = m 
\end{cases} \]
\[ \phi^{ab}_{(m_1)} = \begin{cases} 
1 & \text{if } a, b \neq m \\
-\frac{1}{2}(n - p - 2) & \text{if } a = m \text{ or } b = m 
\end{cases} \]
\[ \hat{\mathbf{M}}_{(\mu_0), (\mu_0)} = M_1^{(1)} + (n - p - 1)M_2^{(1)} - M_2^{(1)} - (n - p - 1)M_3^{(4)} \]
\[ \hat{\mathbf{M}}_{(\mu_0), (\mu_1)} = \frac{1}{2} (p - 2) \left( M_2^{(3)} - M_3^{(3)} \right) \]
\[ \hat{\mathbf{M}}_{(\mu_1), (\mu_0)} = 2(n - p) \left( M_2^{(3)} - M_3^{(3)} \right) \]
\[ \hat{\mathbf{M}}_{(\mu_1), (\mu_1)} = M_1^{(2)} + (p - 4)M_2^{(2)} - (p - 3)M_3^{(2)} \]

\[ \hat{\mathbf{M}}_{(m_0), (m_0)} = M_1^{(1)} + (p - 1)M_2^{(1)} - M_2^{(1)} - (p - 1)M_3^{(4)} \]
\[ \hat{\mathbf{M}}_{(m_0), (m_1)} = \frac{1}{2} (n - p - 2) \left( M_2^{(3)} - M_3^{(3)} \right) \]
\[ \hat{\mathbf{M}}_{(m_1), (m_0)} = 2p \left( M_2^{(3)} - M_3^{(3)} \right) \]
\[ \hat{\mathbf{M}}_{(m_1), (m_1)} = M_1^{(2)} + (n - p - 4)M_2^{(2)} - (n - p - 3)M_3^{(2)} \]

Multiplicity: \( 2 \left[ (p - 1) + (n - p - 1) \right] = 2(n - 2) \)

3. The 1-dimensional R0 replicon subspaces:

\[ \phi^\alpha_{(\mu m)} = \begin{cases} 
1 & \text{if } \alpha \neq \mu \text{ and } a \neq m \\
-(p - 1) & \text{if } \alpha = \mu \text{ and } a \neq m \\
-(n - p - 1) & \text{if } \alpha \neq \mu \text{ and } a = m \\
(p - 1)(n - p - 1) & \text{if } \alpha = \mu \text{ and } a = m 
\end{cases} \]

\[ \hat{\mathbf{M}}_{(\mu m), (\mu m)} = M_1^{(1)} - M_2^{(1)} - M_2^{(1)} + M_3^{(4)} \]

Multiplicity: \( (p - 1)(n - p - 1) = p(n - p) - (n - 1) \)

4. The 1-dimensional R1 replicon subspaces:
• $(\mu\nu)$ eigenvector:

\[
\phi^{\alpha\beta}_{(\mu\nu)} = \begin{cases} 
1 & \text{if } \alpha, \beta \neq \mu, \nu \\
-\frac{1}{2}(p-3) & \text{if one of } \alpha \text{ and } \beta \text{ coincides with one of } \mu \text{ and } \nu \\
\frac{1}{2}(p-2)(p-3) & \text{if } \alpha = \mu \text{ and } \beta = \nu, \text{ or } \alpha = \nu \text{ and } \beta = \mu
\end{cases}
\]

\[
\hat{M}_{(\mu\nu),(\mu\nu)} = M_1^{(2)} - 2M_2^{(2)} + M_3^{(2)}
\]

Multiplicity: \(\frac{1}{2}p(p-3)\)

• $(mn)$ eigenvector:

\[
\phi^{ab}_{(mn)} = \begin{cases} 
1 & \text{if } a, b \neq m, n \\
-\frac{1}{2}(n-p-3) & \text{if one of } a \text{ and } b \text{ coincides with one of } m \text{ and } n \\
\frac{1}{2}(n-p-2)(n-p-3) & \text{if } a = m \text{ and } b = n, \text{ or } a = n \text{ and } b = m
\end{cases}
\]

\[
\hat{M}_{(mn),(mn)} = M_1^{(2)} - 2M_2^{(2)} + M_3^{(2)}
\]

Multiplicity: \(\frac{1}{2}(n-p)(n-p-3)\)

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