Unitarized pion-nucleon scattering amplitude from Inverse Amplitude Method

Isabela P. Cavalcante
Universidade Federal de Mato Grosso do Sul, CCET, Departamento de Física,
Cidade Universitária, s/n, Campo Grande, MS, Brazil.

J. Sá Borges†
Universidade de Estado do Rio de Janeiro, Instituto de Física,
Rua São Francisco Xavier, 524, Maracanã, Rio de Janeiro, RJ, Brazil.

In a recent work on low energy pion-nucleon scattering, instead of using chiral perturbation theory (ChPT) amplitude, we started from a pion-nucleon soft-pion result and used elastic unitarity directly as a dynamical constraint to construct first-order unitarity corrected amplitudes. The resulting amplitudes are crossing symmetric but, as the ChPT ones, satisfy only approximate unitarity relation. In the present work, we reconsider our approach and we apply the inverse amplitude method (IAM) in order to access the energy resonance region. We present the resulting S- and P-wave phase shifts that are shown to be in qualitative agreement with experimental data.

I. INTRODUCTION

Even though Quantum Chromodynamics (QCD) is considered nowadays the theory of strong interactions, its application to low energy hadron physics is still far from a solved problem in physics. A great theoretical improvement was made by means of the method of Chiral Perturbation Theory (ChPT) [1], which is an effective theory derived from the basis of QCD. The method consists of writing down chiral Lagrangians for the physical processes and uses the conventional technique of the field theory for the calculations. It is a quite successful method when applied to meson processes.

In order to deal with baryons in the ChPT approach, the theory faced problems related to fixing the scale for momenta and quark mass expansion. This led eventually to a method known as Heavy Baryon Chiral Perturbation Theory (HBChPT) [2], which has been applied to describe pion-nucleon scattering from a complete effective Lagrangian calculated up to third order in small momenta [3]. More recently the explicit degrees of freedom corresponding to the Δ and N* resonances have also been considered within HBChPT [4]. One peculiar feature of ChPT is that it leads to partial waves satisfying only approximate elastic unitarity relation, since, for instance, the leading amplitude is a real function of energy in the physical region.

Unitarization methods have been applied to pion-nucleon scattering in the literature for a long time [5]. An interesting approach to implement unitarity in HBChPT amplitude is the Inverse Amplitude Method (IAM) [6]. This is a sort of N/D method that takes the leading contribution to the partial wave amplitude as the numerator N and includes the imaginary part that comes from loop corrections in the denominator D. The expansion of D yields the same structure of the original amplitude up to the order considered, plus corrections of higher orders, but is fully unitary. IAM partial wave amplitudes for pion-nucleon scattering have been constructed and fitted to the experimental phase shifts by fixing nine free parameters [7].

On the other hand, instead of working in an effective theory framework, one can use the hard-meson method of current algebra. In fact, current algebra approach was abandoned by several reasons but, as it is based on chiral symmetry, one expects that it gives equivalent results for meson processes. In fact, we have compared these two methods and concluded that quasi-unitarized current algebra amplitude is equivalent to ChPT one-loop calculation in the case of kaon-pion scattering and equivalent to one- and two-loop ChPT calculations for pion-pion scattering [8]. In particular, both methods lead to amplitudes satisfying the same approximate unitarity relations.

Two years ago, we developed an alternative procedure to unitarize a current algebra pion-nucleon scattering amplitude [9]. The unitarity correction was built as follows. We started from a soft-pion amplitude [10] reproducing Weinberg prediction for S-wave scattering lengths [11] and constructed auxiliary functions in order to respect approximate unitarity. Considering known the imaginary parts of partial wave amplitudes, we used the dispersion relation

---

*E-mail: ipc@dfi.ufms.br
†E-mail: saborges@uerj.br
technique to arrive at a quasi-unitarized amplitude written in terms of known functions and a two parameter polynomial part. Furthermore, we imposed exact crossing relations to Dirac amplitudes resulting very constrained partial waves. By this method, even in the absence of free parameters, the violation of unitarity of the resulting amplitudes seems to be very important mainly in the resonance region.

The motivation of the present work is to go beyond threshold by constructing partial wave amplitudes that respect the unitarity relation exactly. In the present paper, we reconsider the unitarity corrected amplitudes described above in order to construct IAM modified amplitudes. In the approximation used here, each partial wave requires a maximum of two parameters.

In the next section we present a summary of the basic formalism. In section III we present the procedure of Ref. [9] for obtaining quasi-unitary corrected amplitudes. In section IV we present the IAM unitarization procedure and the results.

II. BASIC FORMALISM

We consider the reaction \( \pi^a(k) + N(p) \rightarrow \pi^b(k') + N(p') \), which is described by the T-matrix amplitude

\[
T^{ab}(p', k'; p, k) = -A^{ab}(s, t, u) + \frac{i}{2} \left( \hat{k} + \hat{k}' \right) B^{ab}(s, t, u),
\]

with \( s = (p + k)^2 \), \( t = (k' - k)^2 \) and \( u = (p - k')^2 \).

In order to specify the various charge states, the invariant amplitudes \( A \) and \( B \) are decomposed as

\[
A^{ab} = \delta^{ab} A^+ + \frac{1}{2} [\tau^b, \tau^a] A^-,
\]

\[
B^{ab} = \delta^{ab} B^+ + \frac{1}{2} [\tau^b, \tau^a] B^-.
\]

They exhibit the following symmetry properties under crossing:

\[
A^\pm(s, t, u) = \pm A^\pm(u, t, s), \quad B^\pm(s, t, u) = \mp B^\pm(u, t, s);
\]

the amplitudes corresponding to definite isospin \( 1/2 \) and \( 3/2 \) are given by

\[
A_{1/2} = A^+ + 2A^-; \quad A_{3/2} = A^+ - A^-,
\]

and similarly for \( B_{1/2} \), \( B_{3/2} \).

We work in the center of mass system, so that the four momenta are defined as

\[
k = (\vec{k}, w), \quad k' = (\vec{k}', w), \quad p = (-\vec{k}, E), \quad \text{and} \quad p' = (-\vec{k}', E),
\]

with \( |k| = |k'| \), \( w = \sqrt{E^2 + m^2} \) and \( E = \sqrt{E^2 + M^2} \), \( M = .938 \) GeV and \( m = .140 \) GeV being the nucleon and the pion mass, respectively. The total energy and scattering angle are given, respectively, by \( W = E + w \) and \( |k|^2 \cos \theta = \vec{k} \cdot \vec{k}' \), thus, in terms of these quantities, one has \( s = W^2, t = -2 |k|^2 (1 - \cos \theta) \) and \( u = 2M^2 + 2m^2 - s - t \).

For each isospin \( I \) the Pauli amplitudes are

\[
F_{1I}(s, \cos \theta) = \frac{E + M}{8\pi W} \left[ A_I(s, \cos \theta) + (W - M)B_I(s, \cos \theta) \right],
\]

\[
F_{2I}(s, \cos \theta) = \frac{E - M}{8\pi W} \left[ -A_I(s, \cos \theta) + (W + M)B_I(s, \cos \theta) \right].
\]

Partial wave amplitudes \( f_{I\ell}^\pm \) are defined as

\[
f_{I\ell}^+(s) = F_{1I\ell}(s) + F_{2I\ell}(s), \quad \text{where} \quad F_{1I\ell}(s) = \frac{1}{2} \int_{-1}^{+1} F_{I\ell}(s, x) P_\ell(x) \, dx, \quad \text{for} \quad I = 1, 2 \quad \text{and} \quad \ell = \frac{1}{2}, \frac{3}{2}.
\]

For elastic scattering we have \( \text{Im} f_{I\ell}^+(s) = |k| \left| f_{I\ell}(s) \right|^2 \), which may be solved yielding

\[
f_{I\ell}^\pm(s) = \frac{1}{|k|} e^{i\delta_{I\ell}(s)} \sin \delta_{I\ell}(s),
\]

where \( \delta_{I\ell}(s) \) are real phase shifts.
III. UNITARIZATION PROCEDURE

Let us recall the main points in our unitarization procedure. First, we consider an amplitude reproducing S-wave scattering lengths predicted by Weinberg. The low energy amplitude we start from is the soft-pion limit obtained by Osypowsky [10] using the Ward identity technique. The final expression for the four point function, related to pion-nucleon scattering, is written in terms of form factors and propagators. By estimating the contribution of each term at threshold one is led to assume that the low energy amplitudes are

\[ A^{ca-} = \frac{\mu_v}{8Mr^2}(u - s), \quad B^{ca-} = \frac{1 + \mu_v}{2f^2}, \quad A^{ca+} = B^{ca+} = 0, \quad (7) \]

where \( f = 0.94 \) GeV is the pion decay constant, \( \mu_v \simeq 3.7 \) and the superscript \( ca \) stands for current algebra. These amplitudes lead to the well known Weinberg prediction for S-wave scattering lengths, namely \( a_0^+ = 0 \) and \( a_0 = 0.077 \) m\(^{-1}\), to be compared with the experimental values \( a_0^+ = -0.015 \pm 0.015 \) m\(^{-1}\) and \( a_0 = 0.097 \pm 0.003 \) m\(^{-1}\).

We implemented unitarity for the low-energy pion-nucleon amplitude \( A^{ca-} \) and \( B^{ca-} \). From our previous analysis on meson scattering, we conjecture that the corrected amplitudes must satisfy

\[ A_\ell(s) \simeq A^{ca}_\ell(s) + A^{(1)}_\ell(s) + O(\epsilon^2), \quad \text{for} \quad s \simeq (m_\pi + m_N)^2, \quad \text{and the same for} \ B_\ell; \]

\( A^{(1)} \) is a complex function and \( \epsilon \) is a small parameter characterizing the corrections.

For each isospin channel, soft-pion amplitudes are obtained from (5) as

\[ f^{ca}_\ell(s) = F^{ca}_\ell(s) + F^{ca}_\ell(s), \quad f^{ca-}_\ell(s) = F^{ca-}_\ell(s), \quad f^{ca+}_\ell(s) = F^{ca+}_\ell(s), \]

where \( f^{ca}_\ell \) follow from \( A^{ca} \) and \( B^{ca} \). It is evident that \( \text{Im} f^{ca\pm}_\ell(s) = 0 \), but this situation is inherent to soft-pion calculation.

We construct quasi-unitarized amplitudes by requiring that the corrections \( F^{(1)}_\ell \), for \( s \geq (M + m)^2 \), have the following imaginary parts:

\[ \text{Im} F^{(1)}_\ell(s) = |\ell| \left[ F^{ca\ell}_\ell(s) + 2F^{ca\ell}_\ell(s)F^{ca\ell}_\ell(s) \right], \]

\[ \text{Im} F^{(1)}_\ell(s) = |\ell| \left[ F^{ca\ell}_\ell(s) + 2F^{ca\ell}_\ell(s)F^{ca\ell}_\ell(s) \right], \]

\[ \text{Im} F^{(1)}_\ell(s) = |\ell| \left[ F^{ca\ell}_\ell(s) + 2F^{ca\ell}_\ell(s)F^{ca\ell}_\ell(s) \right], \]

It guarantees that the quasi-unitarized partial waves \( f^{\pm}_\ell = f^{ca\pm}_\ell + f^{(1)}\ell \) satisfy

\[ \text{Im} f^{\pm}_\ell = |\ell| \left| f^{ca\pm}_\ell \right|^2. \quad (8) \]

Expressing functions \( A \) and \( B \) in terms of Pauli amplitudes, we construct the auxiliary functions

\[ A(s, \cos \theta_s) = \frac{1}{4} \left[ a_1(s)S(s) + a_2(s)D(s) + 3 \cos \theta_s a_3(s)Q(s) \right], \]

\[ B(s, \cos \theta_s) = \frac{1}{4} \left[ b_1(s)S(s) + b_2(s)D(s) + 3 \cos \theta_s b_3(s)Q(s) \right], \quad (9) \]

where

\[ \text{Im} S(s) = \frac{2|\ell|}{W} A^{ca}_\ell(s), \quad \text{Im} D(s) = \frac{2|\ell|}{W} B^{ca}_\ell(s), \quad \text{Im} Q(s) = \frac{2|\ell|}{W} A^{ca}_\ell(s), \]

\[ a_1(s) = (W + M)(F^{ca\ell}_\ell(s) + 2F^{ca\ell}_\ell(s) + (W - M)(F^{ca\ell}_\ell(s) + 2F^{ca\ell}_\ell(s)), \]

\[ a_2(s) = (W^2 - M^2)(F^{ca\ell}_\ell(s) + 2F^{ca\ell}_\ell(s) - F^{ca\ell}_\ell(s) - F^{ca\ell}_\ell(s)), \]

\[ a_3(s) = (W + M)(F^{ca\ell}_\ell(s) + 2F^{ca\ell}_\ell(s)), \]

\[ b_1(s) = F^{ca\ell}_\ell(s) + 2F^{ca\ell}_\ell(s)_{\ell} + F^{ca\ell}_\ell(s)_{\ell}, \]

\[ b_2(s) = (W - M)(F^{ca\ell}_\ell(s) + 2F^{ca\ell}_\ell(s)_{\ell}) + (W + M)(F^{ca\ell}_\ell(s) + 2F^{ca\ell}_\ell(s)_{\ell}), \]

\[ b_3(s) = F^{ca\ell}_\ell(s) + F^{ca\ell}_\ell(s)_{\ell}. \]
In order to avoid kinematical singularities, we write subtracted dispersion relations for $S, D$ and $Q$ by introducing free parameters. At this point the present procedure must be handled in a different manner than done in Ref. [9]. In the construction of the quasi-unitarized amplitudes we introduced only two parameters. On the other hand, in the present work, one must use up to two parameters for each partial wave, chosen from the set of parameters $(\lambda_1, \lambda_2, \lambda_3)$ in the expressions

$$S(s) = s^2 \lambda_1 + A_0^a(s) G(s), \quad D(s) = s^2 \lambda_2 + B_0^a G(s), \quad Q(s) = s^2 \lambda_3 + A_1^a(s) G(s),$$

with

$$G(s) = \frac{s^3}{\pi} \int_{(M+m)^2}^{\infty} \frac{\sqrt{[x-(M+m)^2][x-(M-m)^2]}}{x^4(x-s)} \text{d}x.$$ 

One may argue that in the definition of the subtracted dispersion relations above, terms in powers of $s$ smaller than two are missing. This restricted choice of the number of subtraction constants is based on the fact that in the fitting process those extra free parameters do not provide qualitative improvement on the fits.

As done in our previous work, crossing properties and partial wave total isospin dependence are imposed to the corrected amplitudes, by taking

$$A^{(1)\pi}(s, t, u) = 2A(s, t) + (s \leftrightarrow u), \quad A^{(1)}(s, t, u) = A(s, t) - (s \leftrightarrow u),$$

$$B^{(1)\pi}(s, t, u) = 2B(s, t) - (s \leftrightarrow u), \quad B^{(1)}(s, t, u) = B(s, t) + (s \leftrightarrow u).$$

The corrections to partial wave amplitudes are calculated from (5) using (4) and are indicated by $f_{\ell I}^{(1)}(s)$. This was the final step of the procedure introduced in Ref. [9].

IV. INVERSE AMPLITUDE METHOD AND RESULTS

The quasi-unitarized amplitudes from last section do not obey exact unitarity relation. In order to restore it, we apply the IAM to the corrected partial waves $f_{\ell I} = f_{\ell I}^{ca} + f_{\ell I}^{(1)}$, by writing

$$\tilde{f}_{\ell I}(s) = \frac{f_{\ell I}^{ca}(s)}{1 - f_{\ell I}^{(1)}(s)/f_{\ell I}^{ca}(s)}.$$

In order to fit IAM amplitudes to the experimental results of phase shifts one has to be cautious. As the inelasticity for some partial waves becomes important for energies less than 1.5 GeV, one has to introduce some criteria to define the validity region of the IAM. In the present paper we decided to fit our results to experimental phase shifts corresponding to dimensionless experimental amplitudes $f_{\ell I}^{exp}$ satisfying $\text{Im} f_{\ell I}^{exp} - |f_{\ell I}^{exp}|^2 < 0.1$.

Restricting ourselves to these ranges in energy, we performed the fits of $S_{11}, S_{31}, P_{11}, P_{31}, P_{13}$ and $P_{33}$ partial waves from the form $f_{\ell I}$ above to experimental data, thus fixing the free parameters $\lambda_1, \lambda_2$ and $\lambda_3$ independently, but only a maximum of two of them for each wave. Their values are presented in the Table.

|     | S11 | P11 | P13 | S31 | P31 | P33 |
|-----|-----|-----|-----|-----|-----|-----|
| $\lambda_1$ | -15.5 | - | - | -2.3 | -48.0 | -3.8 |
| $\lambda_2$ | 24.3 | 39.8 | -6.9 | -8.5 | -27.0 | -8.3 |
| $\lambda_3$ | - | 16.3 | - | - | - | - |

TABLE I. Results from fits of IAM modified quasi-unitarized partial waves to experimental data; $\lambda_1$ and $\lambda_3$ are given in units of $(\text{GeV})^{-5}$, while $\lambda_2$, in units of $(\text{GeV})^{-6}$. 

4
The phase shifts thus obtained are shown in the figures, as functions of the cms energy, in GeV. In the same figures we plot the phase shifts obtained with the original quasi-unitarized amplitudes [9]. In that case only $\lambda_1$ and $\lambda_3$ were considered, and their values were the same for all partial waves. Concerning our present results, we observe that it is possible to fit the phase shifts of our model to the experimental data (Ref. [12]). We also observe that, using the IAM, it is not possible to simultaneously fit two waves with the same values of parameters. The qualitative agreement with experimental data is an indication that the IAM allows one to access the resonance region of pion-nucleon scattering since some low energy chiral amplitude is given.

We would like to emphasize that in the construction of the quasi-unitarized amplitude there is no commitment on the existence of any resonance. They emerge as dynamical consequences of the unitarization procedure. Notice also that we do not include the nucleon pole in the low energy amplitude we started from.

In order to further explore the results that we have obtained, one should construct corrections in the next level of unitarity approximation for the partial waves, as outlined in the last equation of Ref. [9]. We believe that the IAM applied to this new result will allow one to access the energy resonance region for higher angular momenta partial waves.

[1] J. Gasser and H. Leutwyler, Nucl. Phys. B 250 (1985) 465.
[2] For a review, see V. Bernard, N. Kaiser and U.-G. Meissner, Int. J. Mod. Phys. E4 (1995) 193.
[3] N. Fettes, U.-G. Meissner and S. Steininger, Nucl. Phys. A640 (1998) 199; M. Mojzis, Eur. Phys. J. C 2 (1998) 181.
[4] A. Datta and S. Pakvasa, Phys. Rev. D 56 (1997) 4322; P. J. Ellis and H.-B. Tang, Phys. Rev. C 57 (1998) 3356.
[5] J. L. Basdevant, Fort. der Phys. 20 (1972) 283.
[6] T. N. Truong, Phys. Rev. Lett. 61 (1988) 2526; Phys. Rev. Lett. 67 (1991) 2260; A. Dobado and J. R. Peláez, Phys. Rev. D 56 (1997) 3057.
[7] A. Gómez Nicola, J. R. Peláez, Phys. Rev. D 62 (2000) 017502; A. Gómez Nicola, J. Nieves, J. R. Peláez, E. Ruiz Arriola, Phys. Lett. B 486 (2000) 77;
[8] J. Sá Borges, Phys. Lett. B 149 (1984) 357; J. Sá Borges and F. R. A. Simão, Phys. Rev. D 53 (1996) 4806; J. Sá Borges, J. Soares Barbosa and M. Tonasse, Phys. Rev. D 57 (1998) 4108.
[9] J. Sá Borges, Nucl. Phys. A 662 (2000) 362.
[10] H. J. Schnitzer, Phys. Rev. D 2 (1970) 1621; P. Ford, Nucl. Phys. B57 (1975) 25; E. T. Osypowski, Nucl. Phys. B 21 (1970) 615.
[11] S. Weinberg, Phys. Rev. Lett. 17 (1966) 616.
[12] SAID on-line program, R. A. Arndt, R. L. Workmann et al., see http://gwdac.phys.gwu.edu.
FIG. 1. S-wave phase shifts (in degrees) as functions of cms energy (in GeV); present results (solid) and previous quasi-unitary ones (dashed).

FIG. 2. Isospin $1/2$ P-wave phase shifts (in degrees) as functions of cms energy (in GeV); present results (solid) and previous quasi-unitary ones (dashed).

FIG. 3. Isospin $3/2$ P-wave phase shifts (in degrees) as functions of cms energy (in GeV); present results (solid) and previous quasi-unitary ones (dashed).