Self-similar cosmological solutions in $f(R, T)$ gravity theory

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We study the $f(R, T)$ cosmological models under the self-similarity hypothesis. We determine the exact form that each physical and geometrical quantity may take in order that the Field Equations (FE) admit exact self-similar solutions through the matter collineation approach. We study two models: the case $f(R, T) = f_1(R) + f_2(T)$ and the case $f(R, T) = f_1(R) + f_2(R) f_3(T)$. In each case, we state general theorems which determine completely the form of the unknown functions $f_i$ such that the field equations admit self-similar solutions. We also state some corollaries as limiting cases. These results are quite general and valid for any homogeneous self-similar metric. In this way, we are able to generate new cosmological scenarios. As examples, we study two cases by finding exact solutions to these particular models.

Keywords: $f(R, T)$ gravity; Self-similarity; Exact solutions.

1. Introduction
The physical and mathematical importance of the modified gravitational models has recently been pointed out by several authors (see, for instance, [1] [2] [3], and [4], respectively, since this kind of theories explain in a better way the dynamics of the very early universe, as well as its current acceleration. Obviously, although such class of theories are more general than the usual Jordan-Brans-Dicke models, they may be generalized in order to incorporate corrections to the Ricci scalar term as the $f(R)$ models (see for example [5]).

Recently, a new theory, known as $f(R, T)$ gravity, has been formulated in [6]. In the context of $f(R, T)$ modified theories of gravity, the gravitational Lagrangian
is given by an arbitrary function of the Ricci scalar $R$ and of the trace of the energy-momentum tensor $T$. Hence, $f(R, T)$ gravity theories can be interpreted as alternatives of the $f(R, L_m)$ type gravity theories [5], with the gravitational action depending not on the matter Lagrangian, but on the trace of the matter energy-momentum tensor. The gravitational field equations in the metric formalism show that the field equations explicitly depend on the nature of the matter source. The dependence on $T$ may be induced by exotic imperfect fluids, or quantum effects (conformal anomaly) [7].

This theory has received recently a great amount of attention. For extensive reviews of $f(R, T)$ gravity see [9][10][11][12]. In the following, we discuss some of the recent relevant works in the field. Houndjo [13] studied the cosmological reconstruction of $f(R, T)$ gravity describing matter-dominated and accelerated phases. Special attention was paid to the specific case of $f(R, T) = f_1(R) + f_2(T)$. In [14], the authors have shown that the dust fluid reproduces ΛCDM cosmology. In [15], the authors discussed a non-equilibrium picture of thermodynamics at the apparent horizon of an Friedmann-Lemaître-Robertson-Walker (FLRW) universe in $f(R, T)$ gravity, and the validity of the first and second law of thermodynamics in this scenario were checked. It was shown that the Friedmann equations can be expressed in the form of the first law of thermodynamics and that the second law of thermodynamics holds both in the phantom and non-phantom phases. The energy conditions have also been extensively explored in $f(R, T)$ gravity, for instance, in an FLRW universe with perfect fluid [16], and in the context of exact power-law solutions [17]. It was found that certain constraints have to be met in order to ensure that power law solutions may be stable, and match the bounds prescribed by the energy conditions. In [18], it was also shown that the energy conditions were satisfied for specific models. Furthermore, an analysis of the perturbations and stability of de Sitter solutions and power-law solutions was performed, and it was shown that for those models in which the energy conditions are satisfied, de Sitter solutions and power-law solutions may be stable.

The gravitational baryogenesis epoch in the context of $f(R, T)$ theory was studied in [19]. In particular, the following two models, $f(R, T) = R + \alpha T + \beta T^2$ and $f(R, T) = R + \mu R^2 + \lambda T$, were considered, with the assumption that the universe is filled by dark energy and perfect fluid. Moreover, it was assumed that the baryon to entropy ratio during the radiation domination era was non-zero. In [20] a particular form for $f(R, T) = R + \lambda T$ was adopted, and inflationary scenarios were investigated. By taking the slow-roll approximation, interesting models were obtained, with a Klein-Gordon potential leading to observationally consistent inflation observables. These results make it clear that, in addition to the Ricci scalar and scalar fields, the trace of the energy momentum tensor also plays a major role in driving inflationary evolution. The slow-roll approximation of cosmic inflation within the context of $f(R, T)$ gravity was studied in [21]. A gravitational model unifying both $f(R, L_m)$ and $f(R, T)$ theories into the $f(R, L_m, T)$ was developed in [22].
In all the studied models, it is necessary to impose a specific form for the function $f(R, T)$, where usually $f(R, T) = R + 2f(T) = R + \lambda T$, and the matter content is a perfect fluid. Therefore, it would be highly desirable to have a general method for determining the functional form of $f(R, T)$ in such a way that the resulting FE are integrable (in the Louville form). Such an approach would also allow us to make definite cosmological predictions, and to compare theory with observations. Moreover, it would be important to determine the form of the different physical and geometrical quantities in order that the FE could be integrated. Therefore, it would be necessary to have a fundamental method according to which the form (or forms) of the geometrical and physical quantities could be fixed, and, if possible, to calculate exact solutions for the proposed models. In this context, several geometric methods can be used, such as, the matter collineation (self-similar solutions) approach.

For the matter content in the $f(R, T)$ gravity it is generally assumed that it consists of a fluid that can be characterized by two thermodynamic parameters only, the energy density and the pressure, respectively. To solve the resulting FE it is necessary to consider particular classes of $f(R, T)$-modified gravity models, obtained by explicitly specifying the functional form of $f$. Generally, the field equations also depend on the physical nature of the matter field. Hence, in the case of $f(R, T)$ gravity, depending on the nature of the matter source, for each choice of $f$, we can obtain several theoretical models, corresponding to different matter models.

The study of self-similar (SS) models is quite important since, as it has been pointed out in [23], they correspond to equilibrium points. Therefore, a large class of orthogonal spatially homogeneous models are asymptotically self-similar at the initial singularity, and are approximated by exact perfect fluid, or vacuum self-similar power law models.

Exact self-similar power-law models can also approximate general Bianchi models at intermediate stages of their evolution [24]. From the geometrical point of view, self-similarity is defined by the existence of a homothetic vector field $\mathcal{H}$ in the spacetime, which satisfies the equation $L_{\mathcal{H}} g_{\mu\nu} = 2\alpha g_{\mu\nu}$ [25]-[26]. The geometry and physics at different points on an integral curve of a homothetic vector field (HVF) differ only by a change in the overall length scale, and in particular, any dimensionless scalar will be constant along the integral curves. Self-similarity also plays an important role in the study of stellar structure [27].

Therefore, the aim of this paper is to study $f(R, T)$ cosmological models by using geometrical methods in order to determine the form of the physical quantities, as well as the other unknown functions that appear in the field equations. In particular, we are interested in studying whether self-similar solutions do exist, and how must each physical quantity behaves in order that the FE admit such class of solutions. We also show how to use this approach in order to generate additional solutions.

The present paper is organized as follows. In Section II we introduce first the basic gravitational model, and we outline the field equations. Then we determine the exact form that each physical quantity may take in order for the FE to admit
exact self-similar solutions through the matter collineation approach. Two models are explicitly studied. In Section III, we study the case \( f(R, T) = f_1(R) + f_2(T) \), and in Section IV, we explore the case \( f(R, T) = f_1(R) + f_2(R)f_3(T) \) is studied. In each case, we state a general theorem which determines completely the form of the unknown functions \( f_i \) for the field equations to admit self-similar solutions. We also state some corollaries as limiting cases. The results are quite general and valid for any self-similar metric. Section V is devoted to the investigation of exact solutions to two particular models. In Section VI, we discuss our findings and conclude our work. In the Appendix, we show how to apply this geometrical method to obtain other types of solutions, such as the exponential one.

2. Field equations of \( f(R, T) \) gravity, and self-similarity

In this Section we briefly introduce the field equations of the \( f(R, T) \) gravity theory, and we outline the basic concepts of self-similarity.

2.1. Field equations

For a general function \( f = f(R, T) \), the FE and the matter conservation equation read (see for instance [7], and [28]-[29] for the correct conservation equation),

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{f_R} \left( (8\pi - f_T) T_{\mu\nu} - f_T \Theta_{\mu\nu} + \frac{1}{2} \left( f - f_R R \right) g_{\mu\nu} - (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R \right),
\]

\[
\nabla^\mu T_{\mu\nu} = \frac{f_T}{8\pi - f_T} \left[ \left( T_{\mu\nu} + \Theta_{\mu\nu} \right) \nabla^\nu \ln f_T + \nabla^\mu \Theta_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \nabla^\nu f_T \right],
\]

where

\[
\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu} L_m - 2g^{\alpha\beta} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}},
\]

and \( L_m \) stands for the Lagrangian of the matter field. We consider as matter model a perfect fluid, and we take \( L_m = -\rho \) [30]-[31], thus obtaining

\[
\Theta_{\mu\nu} = -2T_{\mu\nu} - \rho g_{\mu\nu}, \quad T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \quad T = -\rho + 3p, \quad p = \omega \rho.
\]

2.2. Self-similarity and \( f(R, T) \) gravity

Self-similarity is defined by the existence of a homothetic vector field \( V \) in the spacetime ([32]-[33]), which satisfies the condition

\[
L_V g_{\mu\nu} = 2\alpha g_{\mu\nu},
\]

where \( g_{\mu\nu} \) is the metric tensor, \( L_V \) denotes Lie differentiation along the vector field \( V \in \mathfrak{X}(\mathcal{M}) \) and \( \alpha \) is a constant (see for general reviews [25] and [34], respectively).
If we consider the Einstein equations in standard general relativity \( G_{\mu\nu} = 8\pi G T_{\mu\nu} \), where \( T_{\mu\nu} \) is an effective stress-energy tensor, then if the spacetime is homothetic, the energy-momentum tensor of the matter fields must satisfy \( L_V T_{\mu\nu} = 0 \).

Nevertheless, in this work, we are not interested in finding the set of vector fields \( V \in \mathfrak{X}(M) \), that satisfy the standard general relativistic condition. If the homothetic vector field is known (HVF) \( H \), that is, the vector field satisfying the condition \( L_H g_{\mu\nu} = 2g_{\mu\nu} \), (see, for example [25]), then \( H \) is also a matter collineation, i.e., it satisfies the condition \( L_H T_{\mu\nu} = 0 \). In the following, we extend this condition to the \( f(R, T) \) theory to determine the behavior of the main geometrical and physical quantities so that the field equations admit self-similar solutions (see [34]).

Therefore, in a modified gravity theory we impose generally the condition

\[
L_H T^\text{eff}_{\mu\nu} = 0,
\]

where \( H \) is a homothetic vector field (HVF), i.e. it verifies the equation: \( L_H g_{\mu\nu} = 2g_{\mu\nu} \), for a given metric tensor, and where \( T^\text{eff}_{\mu\nu} \) is the effective energy-momentum tensor of the theory. In [35] it was shown that it is enough to calculate \( L_H (i) T^\text{eff}_{\mu\nu} = 0 \), for each component of the energy-momentum tensor.

For simplicity, in the following we consider the flat, homogeneous and isotropic FLRW metric only,

\[
ds^2 = dt^2 - a^2(t) \left( dx^2 + dy^2 + dz^2 \right),
\]

where \( a \) is the scale factor. Thus the HVF yields (see for instance [36]),

\[
H = t\partial_t + (1 - a_1) \left( x\partial_x + y\partial_y + z\partial_z \right),
\]

where \( a_1 \in \mathbb{R} \), is a numerical constant.

Note at this point that the scale factor must behave as \( a(t) = t^{a_1}, \ a_1 \in \mathbb{R}^+ \).

We adopt such a simplification since, as we have shown in [35], all the physical quantities are homogeneous, that is, they only depend on the time \( t \). Hence, the unique equation of \( L_H T_{\mu\nu} = 0 \), that is interesting for us, is the one corresponding to the temporal coordinate \( t\partial_t \). Nevertheless, the results are also valid for any homogeneous self-similar metric (Bianchi models, for example) since their homothetic vector field always takes the from \( H = t\partial_t + ... \) (see for instance [37]). This kind of models are called spatially homogeneous, that is, the \( G_6 \) FLRW. The \( G_4 \) models. If the \( G_4 \) has a subgroup \( G_3 \) which acts simply transitively on the three-dimensional orbits we obtain the LRS Bianchi models. Otherwise, we obtain the Kantowski-Sachs models, and the SS Bianchi models. In addition, although the model evolves in time, the evolution is fully specified by this symmetry property, and the FE are purely algebraic when expressed in terms of expansion-normalized variables.

Then the FE of the \( f(R, T) \) theory do admit self-similar solutions if

\[
L_H (T^\text{eff}) = 0, \quad T^\text{eff} = \frac{1}{f_R} \left( (8\pi - f_R) T_{\mu\nu} - f_R \Theta_{\mu\nu} + \frac{1}{2} \left( f - f_R R \right) g_{\mu\nu} - (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R \right),
\]

where \( L_H \) is the Lie derivative along the HVF (see for instance [35] and [38] where this method was applied to other gravitational theories).
3. Case 1: \( f(R, T) = f_1(R) + f_2(T) \)

Under the hypothesis \( f(R, T) = f_1(R) + f_2(T) \), the FE (12) take the form

\[
G_{\mu\nu} = \frac{1}{f_1 R} \left[ (8\pi + f_2 T) T_{\mu\nu} + f_2 T g_{\mu\nu} + \frac{1}{2} f_2 g_{\mu\nu} + \frac{1}{2} (f_1 - f_1 R) g_{\mu\nu} - (g_{\mu\nu} \Box - \nabla_{(\mu} \nabla_{\nu)} f_1 R) \right].
\]

(10)

We state the following Theorem.

**Theorem 1.** The FE (10) admit SS solutions if and only if \( f_1 = R^n \), and \( f_2 = \lambda T \), with \( n, \lambda \in \mathbb{R} \).

**Proof.** We start by proving that \( f_1 = R^n \). Thus, from \( L \mathcal{H} (T_{\mu\nu}^R) = 0 \), that is,

\[
T_{\mu\nu}^R = \frac{1}{2} (F^{-1} f_1 - R) g_{\mu\nu} + F^{-1} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) F, \quad F = f_1 R,
\]

then

\[
L \mathcal{H} \left( \frac{1}{2} (F^{-1} f_1 - R) g_{\mu\nu} \right) = 0,
\]

(12)

we calculate only the first component, \( L \mathcal{H} (T_{11}^R) = 0 \), since with the other components we only obtain restrictions on the scale factor. Thus, this yields

\[
t (f_1' - R' F - F' f_1 F^{-1}) = -2(f_1 - FR),\]

(13)

where \( ' = d/dt \), which may be re-written in the following form

\[
t (-f_1^{-1} f_1 R R' f_1) + 2(f_1 - f_1 R) = 0,
\]

(14)

since

\[
f_1' = \frac{df_1}{dR} R' = FR', \quad F' = f_1 R R'.
\]

(15)

Now, by taking into account that \( t R' \approx -2R \), then, the equation \( L \mathcal{H} (T_{11}) = 0 \) yields

\[
f_{1RR} = \frac{f_2 R}{f} - \frac{f_1}{R} \quad \Rightarrow \quad f_1 = C_1 R^n, \quad n \in \mathbb{R},
\]

(16)

where \( C_1 \) a constant of integration.

Now, we calculate

\[
L \mathcal{H} (F^{-1} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) F) = 0,
\]

(17)

where the first component of the above equation, \( L \mathcal{H} (T_{11}) = 0 \), yields

\[
t \left( F'' H + F' H' - \frac{F'^2}{F} H \right) = -2HF',
\]

(18)

where \( H = a'/a \), but we already know that \( H = h_0 t^{-1} \) (within the self-similar approach), so we rewrite this as

\[
F'' = \frac{F'^2}{F} - \frac{F'}{t} \quad \Rightarrow \quad F = F_0 t^{C_1}, \quad C_1 \in \mathbb{R}.
\]

(19)
Therefore, if we set $C_1 = 2(1-n)$ then we arrive at the conclusion that $F = F_0 t^{2(1-n)} = n R^{-n}$. Note that $R = R_0 t^{-2}$.

Now we calculate

$$L_H \left( T^{\mu \nu}_{\mu \nu} \right) = L_H \left( F^{-1} \left( (8\pi + f_{2T}) T_{\mu \nu} + f_{2T p g_{\mu \nu}} + \frac{1}{2} f_{2g_{\mu \nu}} \right) \right) = 0,$$

where it will be sufficient to prove that

$$L_H \left( F^{-1} T_{\mu \nu} \right) = 0, \quad L_H \left( F^{-1} f_{2g_{\mu \nu}} \right) = 0,$$

so, from $L_H \left( F^{-1} T_{11} \right) = 0$, we get that

$$\frac{\rho'}{\rho} - \frac{F'}{F} = \frac{2}{t} \quad \implies \quad \rho F^{-1} = \rho_0 t^{-2},$$

where $\rho_0$ is a constant of integration.

In the same way, we obtain a similar behavior for $p$, that is, $pF^{-1} = p_0 t^{-2}$. By taking into account our previous result, $F = F_0 t^{2(1-n)}$, then

$$\rho = \rho_0 t^{-2n},$$

which implies that

$$T = -\rho + 3p = T_0 t^{-2n}.$$

Now from $L_H \left( F^{-1} f_{2T g_{\mu \nu}} \right) = 0$,

$$\frac{f_2'}{f} - \frac{F'}{F} = \frac{2}{t}, \quad f_2' = f_{2T} T', \quad \implies \quad f_2 F^{-1} = t^{-2},$$

thus we may set

$$f_2 = \lambda T, \quad \lambda \in \mathbb{R},$$

as it is required.

Note that with these results, the rest of equations, that is, $L_H \left( F^{-1} f_{2T T_{\mu \nu}} \right) = 0$, and $L_H \left( F^{-1} f_{2T p g_{\mu \nu}} \right) = 0$, are trivially satisfied.

Therefore, **Theorem 1** states that $f_1(R) = R^n, n \geq 1$, in the framework of the self-similar solution, the scale factor (or scale factors for the self-similar Bianchi models, for example) follows a power-law $a = t^{a_1}$, which implies that the Ricci curvature (scalar curvature) behaves as $R = R_0 t^{-2}$. We summarize the result below in the form

$$f(R, T) = R^n + \lambda T, \quad R = R_0 t^{-2}, \quad a = t^{a_1}, \quad a_1 \in \mathbb{R}^+, \quad p = \omega \rho = p_0 t^{-2n}, \quad T = -\rho + 3p = T_0 t^{-2n}, \quad n, \lambda \in \mathbb{R}.$$
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We define

Proof.

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The FE (29) admit self-similar solutions if and only if

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4. Case 2: \( f(R, T) = f_1(R) + f_2(R) f_3(T) \)

Under the assumption \( f(R, T) = f_1(R) + f_2(R) f_3(T) \), the FE (12) read

\[
G_{\mu\nu} = \frac{1}{K} \left( (8\pi + f_T) T_{\mu\nu} + f_T p g_{\mu\nu} + \frac{1}{2} f_2 f_3 g_{\mu\nu} + \frac{1}{2} (f_1 - K R) g_{\mu\nu} - (g_{\mu\nu} - \nabla_\mu \nabla_\nu) K \right),
\]

(29)

where \( K = f_1 R + f_2 R f_3 \) and \( f_T = f_2 f_3 T \). We state the following Theorem.

Theorem 3. The FE (29) admit self-similar solutions if and only if \( f(R, T) = f_1(R) + f_2(R) f_3(T) = R^n + R^{n(1-\ell)} T^\ell \), with \( n, \ell \in \mathbb{R} \).

Proof. We define

\[
T^{eff}_{\mu\nu} = \frac{1}{K} \left( (8\pi + f_T) T_{\mu\nu} + f_T p g_{\mu\nu} + \frac{1}{2} f_2 f_3 g_{\mu\nu} + \frac{1}{2} (f_1 - K R) g_{\mu\nu} - (g_{\mu\nu} - \nabla_\mu \nabla_\nu) K \right),
\]

(30)

\[
K = f_1 R + f_2 R f_3, \quad f_T = f_2 f_3 T,
\]

(31)

and also consider

\[
T^1_{\mu\nu} = \frac{1}{K} T_{\mu\nu}, \quad T^2_{\mu\nu} = \frac{f_T}{K} T_{\mu\nu}, \quad T^3_{\mu\nu} = \frac{f_T}{K} p g_{\mu\nu}, \quad T^4_{\mu\nu} = \frac{f_2 f_3}{K} g_{\mu\nu},
\]

(32)

\[
T^5_{\mu\nu} = \frac{f_1}{K} g_{\mu\nu}, \quad T^6_{\mu\nu} = R g_{\mu\nu}, \quad T^7_{\mu\nu} = \frac{1}{K} (g_{\mu\nu} - \nabla_\mu \nabla_\nu) K
\]

(33)

where \( R, \) by hypothesis, behaves as \( R \approx t^{-2} \).

Then, from

\[
T^1_{\mu\nu} = \frac{1}{K} T_{\mu\nu}, \quad L_H \left( \frac{1}{K} T_{11} \right) = \frac{\rho'}{\rho} - \frac{K'}{K} = -2 \frac{t}{t}, \quad \frac{\rho}{K} = t^{-2},
\]

(34)

\[
T^2_{\mu\nu} = \frac{f_T}{K} T_{\mu\nu}, \quad L_H \left( \frac{f_T}{K} T_{11} \right) = \frac{\rho'}{\rho} + \frac{f_T'}{f_T} - \frac{K'}{K} = -2 \frac{t}{t}, \quad \frac{f_T}{K} \rho = t^{-2},
\]

(35)

\[
T^3_{\mu\nu} = \frac{f_T}{K} p g_{\mu\nu}, \quad L_H \left( \frac{f_T}{K} p g_{11} \right) = \frac{\rho'}{\rho} + \frac{f_T'}{f_T} - \frac{K'}{K} = -2 \frac{t}{t}, \quad \frac{f_T}{K} \rho \approx t^{-2},
\]

(36)

\[
T^4_{\mu\nu} = \frac{f_2 f_3}{K} g_{\mu\nu}, \quad L_H \left( \frac{f_2 f_3}{K} g_{11} \right) = \frac{f_2}{f_2} + \frac{f_3}{f_3} - \frac{K'}{K} = -2 \frac{t}{t}, \quad \frac{f_2 f_3}{K} \approx t^{-2},
\]

(37)

\[
T^5_{\mu\nu} = \frac{f_1}{K} g_{\mu\nu}, \quad L_H \left( \frac{f_1}{K} g_{11} \right) = \frac{f_1}{f_1} - \frac{K'}{K} = -2 \frac{t}{t}, \quad \frac{f_1}{K} \approx t^{-2},
\]

(38)
and the trivial (by hypothesis) result
\[ T^6_{\mu\nu} = Rg_{\mu\nu}, \quad L_\mu (Rg_{11}) = \frac{R'}{R} = -\frac{2}{t} \quad R \approx t^{-2}, \] (39)
so this means that
\[ \frac{\rho}{K} \approx t^{-2}, \quad \frac{f_T}{K} \rho \approx t^{-2}, \quad \frac{f_T}{K} p \approx t^{-2}, \quad \frac{f_2 f_3}{K} \approx t^{-2}, \quad \frac{f_3}{K} \approx t^{-2}, \] (40)
then
\[ f_T = f_2 f_3 T = \text{const.}, \quad f_1 \approx f_2 f_3 \approx \rho \approx p, \quad f_{1R} \approx f_{2R} f_3. \] (41)

Note that \( \approx \) means that the quantities have the same order of magnitude. Therefore, we are only able to obtain relationships between the physical quantities, but we do not know anything about their individual behavior.

We need to assume the following hypothesis (within the self-similar framework)
\[ \rho \approx p \approx T \approx t^{-x}, \quad x \in \mathbb{R}^+ \] (42)
so that
\[ f_1(R) = R^n, \quad f_2(R) = R^m, \quad f_3(T) = T^l, \quad n, m, l \in \mathbb{R}, \] (43)
where \( R \approx t^{-2} \). Therefore
\[ f_2 f_3 T = \text{const.} : -2m - x(l - 1) = 0 \quad x = \frac{2m}{1 - l} \]
\[ f_{1R} \approx f_{2R} f_3 : \quad -2(n - 1) = -2(m - 1) - xl \quad 2n = 2m + xl \]
\[ f_1 \approx f_2 f_3 : \quad -2n = -2m - xl \quad 2n = 2m + xl \] (44)

Now, from \( f_1 \approx \rho \approx p \), we obtain
\[ f_1 \approx t^{-2n} \approx \rho \approx t^{-x}, \quad \implies \quad x = 2n \] (45)
and therefore
\[ x = \frac{2m}{1 - l} = 2n, \quad \implies \quad 2n = 2m + xl \quad \implies \quad m = n(1 - l) \quad x = 2n, \] (46)
or
\[ l = -\frac{1}{n} (m - n). \] (47)
Hence the model reduces to
\[ f(R, T) = f_1(R) + f_2(R) f_3(T) = R^n + R^{n(1 - l)} T^l, \quad n, l \in \mathbb{R}, \] (48)
\[ f(R, T) = f_1(R) + f_2(R) f_3(T) = R^n + R^m T^{1 - \frac{m}{n}}, \quad n, m \in \mathbb{R}, \] (49)
where
\[ \rho \approx p \approx T \approx t^{-2n}, \quad R \approx t^{-2}, \] (50)
as it is required.
In conclusion, the above Theorem states that if \( f(R, T) = R^n + R^{n(1-b)} T^l \), then the FE admit self-similar solutions. For cosmological reasons, we usually consider \( n \geq 1 \) and \( l \leq 1 \). Note that if we fix \( l = 1 \) then the model reduces to the \( f(R, T) = f_1(R) + f_2(T) \) case. From the above Theorem, we have the following corollaries.

**Corollary 4.** If \( n = 1 \), the model reduces to
\[
f(R, T) = f_1(R) + f_2(R) f_3(T) = R + R^m T^l, \quad / \quad m + l = 1.
\]

Note that from the proof of the above Theorem, it is easy to see that \( f_2 f_3 T = \text{const.} \), \( f_1 R \approx f_2 f_3 \approx \rho \approx p \approx T \approx t^{-2} \).

**Corollary 5.** If \( n = 0 \), the model collapses to
\[
f(R, T) = f_2(R) f_3(T) = R^m T^l,
\]
where \( \rho \approx p \approx T \approx t^{-x} \), with the constraint: \( x (1 - l) = 2m \). Thus, if we set \( x = 2 \), then \( m + l = 1 \), since \( f_2 f_3 T = \text{const.} \).

5. Some exact solutions

The above theoretical results only determine the behavior of the physical and geometrical quantities. They state the order of magnitude of the different quantities, but they do not say anything about the constants \( a_1, \rho_0, \) etc. To find an exact analytical solution that is also cosmologically relevant, it is necessary to determine them in such a way that we may make predictions about the behavior of the solution. As an example, in what follows, we find two exact solutions, and we explore their cosmological consequences.

In the first example we study the case \( f(R, T) = f_1(R) + f_2(T) = R^n + \lambda T \), while in the second example we investigate the case \( f(R, T) = f_2(R) f_3(T) = R^m T^l \).

### 5.1. Case 1: \( f(R, T) = f_1(R) + f_2(T) \)

In this case, the FE are:
\[
G_{\mu\nu} = \frac{1}{f_1 R} \left( (8\pi + f_2 T) T_{\mu\nu} + f_2 T p g_{\mu\nu} + \frac{1}{2} f_2 g_{\mu\nu} + \frac{1}{2} (f_1 - f_1 R) g_{\mu\nu} - (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) f_1 R \right),
\]
\[
\left( 1 + \frac{2f_T}{8\pi - f_T} \right) \nabla^{\lambda} T_{\lambda \mu} = \frac{f_T}{8\pi - f_T} \left[ (-T_{\mu\nu} - p g_{\mu\nu}) \nabla^{\lambda} \ln f_T - \nabla^{\lambda} (p g_{\mu\nu}) - \frac{1}{2} g_{\mu\nu} \nabla^{\lambda} T \right],
\]
and we may reformulate them as an effective Einstein field equations of the form
\[
G_{\mu\nu} = G_{\text{eff}} T_{\mu\nu} + T_{\text{eff}}^{\mu\nu}, \quad G_{\text{eff}} = \frac{1}{f_1 R} (1 + f_2 T),
\]

5.2. Case 2: \( f(R, T) = f_2(R) f_3(T) \)
where $G_{\text{eff}}$ is the effective gravitational coupling. In fact, it is a time varying coupling, while $T_{\mu\nu}^{\text{eff}}$ is defined as

$$T_{\mu\nu}^{\text{eff}} = \frac{1}{f_{1R}} \left( \frac{1}{2} (2f_2r + f_2 + (f_1 - f_{1R}R)) g_{\mu\nu} - (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) f_{1R} \right). \quad (58)$$

Under the above assumptions about the matter field, and taking into account the previous results [27], that is:

$$f(R, T) = f_1(R) + f_2(T) = R^n + \lambda T, \quad (59)$$

where $R = R_0 t^{-2}$, $a = t^{n_1}$, $a_1 \in \mathbb{R}^+$, and $p = \omega = \rho_0 t^{-2n}$, $T = -\rho + 3p = T_0 t^{-2n}$, with $n, \lambda \in \mathbb{R}$, then, the FE and the conservation equation for a flat FLRW metric read:

$$3f_{1R}H^2 = (1 + f_{2T}) \rho - f_2r p - \frac{1}{2}f_2 - 3f_{1R}H + \frac{1}{2} (f_{1R}R - f_1), \quad (60)$$

$$f_{1R} (2H' + 3H^2) = - (1 + f_{2T}) p - \frac{1}{2}f_2 - f_{1R}' - 2f_{1R}'H + \frac{1}{2} (f_{1R}R - f_1), \quad (61)$$

$$\left[ 1 + \frac{1}{2} (3 - 5\omega) f_{2T} \right] \rho' + [3(1 + f_{2T}) (\omega + 1) H + (1 - \omega) f_{2T}] \rho = 0. \quad (62)$$

Starting from the results [27], we have $f_{1R} = nR_0^{n-1}$; $f_{1R}' = -2n(n - 1)R_0^{n-1}t^{-2n+1}$ and $f_{1R}'' = n(n - 1)(2n - 1)R_0^{n-1}t^{-2n}$.

Additionally, from [27] we have the relation $T = T_0 t^{-2n}$, with $T_0 = \rho_0 (3\omega - 1)$. By inserting these expressions into Eqs. (60, 62) we obtain

$$\rho_0 \left[ 1 + \frac{\lambda}{2} (3 - 5\omega) \right] = A, \quad a_1 = \frac{n (2 - \lambda (5\omega - 3))}{3 (\omega + 1) (\lambda + 1)}, \quad (63)$$

where

$$A \equiv R_0^{n-1} \left( 3a_1^2 + \frac{1}{2} (1 - n) (R_0 + 12na_1) \right), \quad R_0 = 6a_1 (2a_1 - 1), \quad (64)$$

and therefore

$$G_{\text{eff}} = \frac{(1 + \lambda)}{n R_0^{n-1}} t^{2(n-1)}. \quad (65)$$

If $\lambda = 0$ and $n = 1$, from Eq. (63) one recovers the Standard $\Lambda$CDM Model relations

$$a_1 = \frac{2}{3 (\omega + 1)}, \quad \rho_0 = 3a_1^2, \quad G_{\text{eff}} = \text{const}. \quad (66)$$

In order to study the solution described in Eq. (63), that is, to determine the range of the parameters $(\omega, n, \lambda)$, we have performed a numerical analysis for two cases.

In Case 1, we have fixed $n = 1$ (the Standard Model) and in Case 2, we have fixed $n = 2$. It is possible to extend this analysis to any value of $n$, that is, $n = 3/2$, $n = 3$, etc.
Case 1a, \( n = 1 \). To estimate the \( \Lambda \)-effect on Standard Model, we focus to the case \( n = 1 \). The physical quantities behave as

\[
a = t^{a_1}, \quad a_1 = \frac{(2 - \lambda(5\omega - 3))}{3(\omega + 1)(\lambda + 1)}, \quad \rho = \rho_0 t^{-2}, \quad \rho_0 = \frac{3a_1^2}{1 + \frac{\lambda}{2}(3 - 5\omega)},
\]

while the effective gravitational constant becomes a constant \( G_{\text{eff}} = \text{const} \), but under the restriction \( \lambda > -1 \).

![Fig. 1. Case 1a \( n = 1 \). Plots of \( a_1 \) (blue area \( a_1 > 0 \), red color \( a_1 \leq 0 \)). Second figure, we show the region where \( a_1 > 1 \) (blue color), red color means \( a_1 \leq 1 \). Last figure, \( \rho_0 \). Blue color \( \rho_0 > 0 \), while red color \( \rho_0 \leq 0 \).](image)

In Figs. 1 we have performed a numerical analysis of the quantities \( a_1 \) and \( \rho_0 \), by plotting these quantities in the plane \( (\omega, \lambda) \). We have fixed \( \omega \in [-2, 1] \) and \( \lambda \in [-2, 2] \), noting that the analysis may be enlarged by considering \( \lambda \in [-m, m] \), \( m \in \mathbb{R} \), since, a priori, there is no any restriction on this numerical constant.

We have started the analysis by plotting the area within the region \( (\mathcal{R} = [-2, 1] \times [-2, 2]) \) where \( a_1 > 0 \) (blue color) while red color means \( a_1 \leq 0 \). In the mid sub-panel, we have calculated the region where \( a_1 > 1 \) (blue color), that is, we have calculated those values of \( \omega \) and \( \lambda \) for which the solution represents an accelerated expansion (red color means \( a_1 \leq 1 \)). In the right sub-panel, we have calculated the values of \( (\omega, \lambda) \) for which \( \rho_0 > 0 \) (blue color) while, as in the above pictures, red color means \( \rho_0 \leq 0 \).

We may conclude, roughly speaking, that the solution has physical meaning if \( \omega \in (-1, 1] \) and \( \lambda \in [-2, 2] \). Note that \( a_1 \) tends to infinity when \( \omega \to 1 \), and \( \lambda \to -1 \). If \( \omega \in (-1, 0) \), then the solution accelerates. Only by taking into account the condition \( G_{\text{eff}} = \text{const} > 0 \), we may find the following constraint on \( \lambda: \lambda > -1 \). Thus, our solution works well in the region \( \mathcal{R} = (-1, 1] \times (-1, 2] \). From the observations, the deceleration parameter \( q \) is measured as \( q = -0.18^{+0.12}_{-0.12} \), thus this means that \( a_1 \in [1.064, 1.429] \).

Case 1b, \( n = 2 \). In order to estimate the curvature-effect on the Standard
Model, we focus on the case \( n = 2 \). The physical quantities behaves as
\[
a = t^{a_1}, \quad a_1 = \frac{2(2 - \lambda (5\omega - 3))}{3(\omega + 1)(\lambda + 1)}, \quad \rho = \rho_0 t^{-4}, \quad \rho_0 = \frac{R_0 (12a_1^2 - (R_0 + 24a_1))}{2 + \lambda(3 - 5\omega)},
\]
while the effective gravitational constant behaves as
\[
G_{\text{eff}} = \frac{(1 + \lambda)}{2R_0} t^2 = \frac{3(1 + \lambda)^3 (\omega + 1)^2}{8 (\lambda (5\omega - 3) - 2)(3\omega - 5 + \lambda (23\omega - 9))} t^2,
\]
so it is a positive function, if \( \lambda > -1 \). Note that for these numerical values it is an increasing function.

\[
\begin{align*}
\text{Fig. 2. Case 1b } n = 2. \text{ Plots of } a_1 \text{ (blue area } a_1 > 0, \text{ red color } a_1 \leq 0). \text{ Second figure, we show the region where } a_1 > 1 \text{ (blue color) red color means } a_1 \leq 1. \text{ Last figure, } \rho_0. \text{ Blue color } \rho_0 > 0, \text{ while red color } \rho_0 \leq 0.
\end{align*}
\]

By carrying out a similar analysis as in the previous case, we have fixed a region \( \mathcal{R} \) in the plane \( (\omega, \lambda) \), with \( \mathcal{R} = [-2, 1] \times [-2, 2] \). Thus, in Figs. (2), we have calculated the area within the region \( \mathcal{R} \) where the exponent of the scale factor, \( a_1 \) is positive (left sub panel) and where it is \( a_1 > 1 \) (mid sub panel). In the right sub-panel of Fig. (2) we have plotted the area where \( \rho_0 > 0 \) (blue color) while, as in the above pictures, red color means \( \rho_0 \leq 0 \).

As we can see from this last picture, basically \( \omega \in (-1, 0] \) where furthermore \( a_1 > 1 \), and \( \lambda \in (-1, 2] \), in such a way that in this region, \( \mathcal{R} = (-1, 0] \times (-1, 2] \), \( G_{\text{eff}} = \text{const} > 0 \).

5.2. Case 2: \( f(R, T) = f_1(R)f_2(T) \)

For this model the FE reduce to
\[
G_{\mu\nu} = G_{\text{eff}} T_{\mu\nu} + T_{\mu\nu}^{\text{eff}},
\]
\[
G_{\mu\nu} = \frac{1}{K} \left( (8\pi + f_T) T_{\mu\nu} + f_T p g_{\mu\nu} + \frac{1}{2} (f - KR) g_{\mu\nu} - (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) K \right),
\]
with \( f_R = f_1 R f_2 = K \), and \( f_T = f_2 T f_1 \), where

\[
G_{\text{eff}} = \frac{8 \pi + f_2 T f_1}{f_1 R f_2}.
\]

(72)

The conservation equation becomes

\[
(1 + 2C) \nabla^\mu T_{\mu \nu} = -C \left[ (T_{\mu \nu} + p g_{\mu \nu}) \nabla^\mu \ln (f_1 f_2 T) + \frac{5}{2} g_{\mu \nu} \nabla^\mu p - \frac{1}{2} g_{\mu \nu} \nabla^\mu \rho \right],
\]

(73)

where \( C = \frac{f_1 f_2}{1 - f_1 f_2} \).

For a flat FLRW metric, the FE and the conservation equation read

\[
3 K H^2 = \rho (1 + f_1 f_2 T) - p f_1 f_2 T - 3 K' H + \frac{1}{2} (K R - f_1 f_2),
\]

(74)

\[
K (2 H' + 3 H^2) = -p (1 + 2 f_1 f_2 T) - K'' - 2 K' H + \frac{1}{2} (K R - f_1 f_2),
\]

(75)

\[
\left(1 + \frac{5}{2} (1 - \omega) C\right) \rho' = \left[C (\omega - 1) \left(\frac{f_1 f_2 T}{f_1 T} + \frac{f_1}{f_2}\right) - 3 (\omega + 1) H (1 + 2 C)\right] \rho.
\]

(76)

By taking into account the previous results (48),

\[
f(R, T) = f_1(R) f_2(T) = \lambda T R^m, \quad 2m = x (1 - l),
\]

(77)

where

\[
f_1(R) = R^m = R_0 t^{-2m}, \quad f_2(T) = \lambda T^l = \lambda T_0 t^{-xl}, \quad \lambda, m, l, x \in \mathbb{R},
\]

(78)

and recalling that \( \rho \approx p \approx T \approx t^{-x}, \ (T_0 = \rho_0 (3 \omega - 1)) \), we obtain

\[
a = t^{\alpha_1}, \quad \alpha_1 \in \mathbb{R}^+, \quad R = 6 \left(\frac{a'}{a} + H^2\right) = 6 a_1 (2 a_1 - 1) t^{-2} = R_0 t^{-2}.
\]

(79)

Note that if we set \( x = 2 \), then \( m + l = 1 \), since \( f_1 f_2 T = \text{const} \), the Standard Model is recovered if we set: \( m = 1 \) and \( \ell = 0 \) (\( \lambda = 1 \)).

By solving Eqs. (74-76) we find

\[
\rho_0 = [6 a_1 (2 a_1 - 1)]^{\frac{1}{2}} \left(\frac{m \lambda (3 \omega - 1) - \frac{2 m}{3}}{6 a_1 (2 a_1 - 1) (1 + \omega)} \right) \left[2 a_1 + (2 - x) (a_1 + x + 1) - \lambda (1 - \frac{2 m}{3}) \right]^{\frac{3}{2}},
\]

(80)

and

\[
m = \frac{\omega}{x - 2 m - 1} + \frac{1}{2},
\]

(81)

with

\[
B = \frac{a_1 (2 - 3 a_1 (1 + \omega)) + (2 - x) (x - 1 - a_1 (2 + 3 \omega))}{6 a_1 (2 a_1 - 1) (1 + \omega)}.
\]

(82)

We may obtain an expression for \( a_1 \) from Eq. (81). It must satisfy the following quadratic equation

\[
c_1 a_1^2 + c_2 a_1 + c_3 = 0,
\]

(83)
where
\[ c_1 = 3 \left(1 - \frac{2}{m}\right) - \frac{12\omega}{x(3\omega - 1)} \left(\frac{x}{m} - 2\right), \]
\[ c_2 = \frac{2 + (x - 2)(2 + 3\omega)}{1 + \omega} + \frac{6\omega}{x(3\omega - 1)} \left(\frac{x}{m} - 2\right) + 3 \left(\frac{1}{m} - 1\right), \]
\[ c_3 = \frac{(x - 2)(x - 1)}{1 + \omega}, \]  
so we obtain two solutions for \( a_1 = a_1 \pm \). Note that, in the Standard Model case, the above Eqs. are reduced to
\[ \rho_0 = \frac{2a_1}{1 + \omega}, \quad m = 1, \quad a_1 = \frac{2}{3(1 + \omega)}. \]

Therefore, we have solved completely our proposed model. The constants \( \rho_0 \) and \( a_1 \) depend on \((m, x, \omega)\) with the relationship \( 2m = x(1 - l) \). Since \( m \) and \( x \) are positive, then \( l < 1 \). Thus, our model is described for the function
\[ f(R, T) = f_1(R)f_2(T) = \lambda R^m T^l, \quad l < 1, \quad 2m = x(1 - l), \quad \lambda \in \mathbb{R}. \]

We may study the behavior of this solution in several ways. For example, by fixing \( m \) so the constants \( \rho_0 \) and \( a_1 \) depend on \((x, \omega)\) or by fixing \( l \).

We have explored the case \( l = -1 \) which implies that \( m = x \) \((f = R^x T^{-1})\), in such a way that Eqs. (80 and 83) are reduced to
\[ \rho_0 = \left[6a_1(2a_1 - 1)\right]^\frac{1}{2} \left(\frac{x\lambda(2a_1 - (2 - x)(a_1 + x - 1))}{6a_1(2a_1 - 1)(1 + \omega)(3\omega - 1)\omega} + \frac{\lambda}{(3\omega - 1)^2}\right)^\frac{1}{2} \]
\[ c_1 a_1^2 + c_2 a_1 + c_3 = 0 \implies a_1 = a_1 \pm \]
where
\[ c_1 = 3 \left(1 - \frac{2}{x}\right) + \frac{12\omega}{x(3\omega - 1)}, \quad c_3 = \frac{(x - 2)(x - 1)}{1 + \omega}, \]
\[ c_2 = \frac{2 + (x - 2)(2 + 3\omega)}{1 + \omega} - \frac{6\omega}{x(3\omega - 1)} + 3 \left(\frac{1}{x} - 1\right), \]  
and
\[ G_{\text{eff}} = \frac{1 + f}{1 + f} = \left(\frac{R_0^{1-x}}{x\lambda T_0} (T_0^2 - \lambda R_0^x)\right)^{t^{-2}} = G_{\text{eff}}^t t^{-2}. \]

To analyse this solution we have carried out a numerical study, as in the above cases. By fixing a region \( \mathcal{R} \) in the plane \((\omega, x)\), with \( \mathcal{R} = (-1, 1) \times [1, 3] \), then we may see where the solution has a physical meaning. Thus, in Figs. (3) we have calculated the area within the region \( \mathcal{R} \) where the exponent of the scale factor, \( a_1 \) is positive (left sub-panel) and where it is \( a_1 > 1 \) (mid sub-panel).

In the right sub-panel of Fig. (3) we have plotted the area where \( \rho_0 > 0 \) (blue color) while, as in the above pictures, red color means \( \rho_0 \leq 0 \). As we can see from this last picture (right sub-panel), basically, the parameter \( \rho_0 \) is not positive for
all \( \omega \in (-1, 0] \), thus, this solution looks restrictive. With regard to the behavior of \( G_{\text{eff}} \), we may say that it behaves as an increasing time function if \( x > 2 \), constant if \( x = 2 \) and a decreasing function if \( x < 2 \).

As we have mentioned above, it is possible to analyse the solution in several ways. This could be achieved by following a different approach. For example, fixing \( x = 2 \), leads to \( m = (1 - l) \), and this could result in a less restrictive solution.

6. Conclusions

In the present paper, we have explored whether the gravitational model \( f(R, T) \) admits self-similar solutions, or not. We have considered only two cases for the function \( f(R, T) \). The first studied case corresponds to \( f(R, T) = f_1(R) + f_2(T) \) while the second one is \( f(R, T) = f_1(R) + f_2(R)f_3(T) \). In order to determine if these models admit self similar solutions, we have used the geometrical method of the matter collineations, by considering that the matter collineation vector field is the homothetic one. This approach allows us to determine the exact form of the unknown functions \( f_1(R), f_2(T) \), as well as the rest of the physical and geometrical quantities.

For simplicity, we have considered a perfect fluid, and we have determined the behavior of the energy density \( \rho \). The obtained Theorems are quite general, and valid not only for the FLRW metric, but also valid for all the self-similar Bianchi models and the Kantowski-Sachs one.

In the first of the studied cases, with \( f(R, T) = f_1(R) + f_2(T) \), we have been able to determine that the FE admit self-similar solutions if \( f_1 = R^n \), and \( f_2 = \lambda T \), with \( n, \lambda \in \mathbb{R} \), therefore \( f(R, T) = R^n + \lambda T \) (see [4]). This new result generalizes the model proposed by Harko et al. in [7]. In the Corollary (2), we show that the model \( f(R, T) = R + \lambda T \), admits self-similar solutions as well.
In the second of the studied cases: \( f(R, T) = f_1(R) + f_2(R)f_3(T) \), we have stated in Theorem 3 that the FE admit self-similar solutions, if \( f(R, T) = R^n + R^{m(1-l)}T^l \). As we have pointed out, if we fix \( l = 1 \), then the model reduces to the first studied case, that is, the case \( f(R, T) = f_1(R) + f_2(T) \). From the Theorem 3 we have the Corollaries 4 and 5, which allow us to obtain two particular cases: \( f(R, T) = R + R^mT^l \), such that \( m + l = 1 \) and \( f(R, T) = R^mT^l \), under the constraint \( x(1-l) = 2m \). All these results are also new, leading to novel classes of cosmological models.

Furthermore, we have shown how to calculate the exact solution in two particular cases, \( f(R, T) = R^n + \lambda T \), and \( f(R, T) = R^mT^l \), respectively.

The determinations by the Planck satellite of the Cosmic Microwave Background Radiation temperature fluctuations [39,40], together with the observations of the light curves of the distant supernovae [41] have provided a powerful confirmation of the amazing fact that the Universe is in a phase experiencing a de Sitter type accelerating expansion. Moreover, other amazing observational result and that the matter composition of the Universe consists of only 5% baryonic matter, while 95% of matter-energy is represented by two mysterious forms of energy/matter, called dark energy and dark matter, respectively. To find an explanation for the cosmological observational results, the ΛCDM paradigm was introduced, which is essentially based by the introduction in the Einstein gravitational field equations of the cosmological constant \( \Lambda \). The cosmological constant was introduced in general relativity by Einstein in 1917 [42], in order to construct a static cosmological model. The ΛCDM model gives and excellent fit to the observational data, especially at low redshifts. However, its theoretical basis is questionable, since there are no convincing answers to the many problems raised by physical nature and interpretation of \( \Lambda \).

Therefore, to obtain a physically and mathematically acceptable description of the Universe, one must go beyond standard general relativity. In the present paper we have adopted the dark gravity approach, in which it is assumed that the nature of the gravitational interaction drastically changes on astrophysical (galactic) and cosmological scales. Therefore, the standard general relativistic Einstein equations, which offer a very precise description of the physics at the level of the Solar System, must be change by a new theory of gravity. In the present paper we have assumed that such a theory is represented by the \( f(R, T) \) theory, which implies a geometry-matter coupling [43]. Such an approach leads to gravitational models more complicated than standard general relativity. Cosmological models built in the framework of \( f(R, T) \) gravity represent an interesting possibility of explaining dark energy, the accelerating expansion of the Universe, and perhaps even dark matter. However, this type of theory also raises a number of extremely difficult mathematical problems, and the present paper represents a systematic approach for investigating some of the properties of the theory, as well as proposing a rigorous mathematical algorithm for obtaining cosmological solutions.

From a cosmological, as well as a physical point of view, the relevance of the
obtained results is twofold. Firstly, it gives the possibility of obtaining the functional forms of $f(R, T)$ that admit cosmological power law solutions of the form $a(t) = t^{a_1}$, which are relevant for several phases of the evolution of the Universe. The deceleration parameter corresponding to this solution is given by $q = 1/a_1 - 1$, and it can describe accelerating cosmological models for $a_1 > 1$, decelerating evolution for $a_1 < 1$, and marginally accelerating solutions for $a_1 = 1$. Hence, one could obtain severe constraints for the allowed functional form of the $f(R, T)$ function by requiring the presence in the theory of this power law form of the scale factor. Generally, for this form of the scale factor, the geometrical (Ricci scalar), and physical (energy density) parameters have a simple time dependence, being inversely proportional to the square of the time. This indicates that the considered self-similar solutions do present a singular behavior at $t = 0$. The effective gravitational coupling has a power law dependence on time, and, for the $f(R, T) = f_1(R) + f_2(T)$ it increases as $G_{eff} \propto t^2$.

The second implication of the self-similar approach is related to the possibility of determining the integration constants that appear in the theoretical models. Usually, these constants are determined from the initial conditions, which are not, or poorly known. The present approach allows an independent determination of the free parameters of the different cosmological models, leading to the possibility of the direct confrontation of the theoretical results with observations.

The self-similar approach for the investigation of the gravitational theories represents a powerful approach that can give important information on the mathematical and physical structures of the particular models. In the present study we have performed such an analysis for the $f(R, T)$ modified gravity theory, with the results of our analysis indicating that simple power law cosmological models can be obtained in the framework of this modified gravity theory. The cosmological implications of the present results will be fully investigated in a future work.

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Appendix A. Alternative approach

For the model $f(R, T) = f_1(R) + f_2(T)$ FE read (for a flat FLRW metric, and $T_{\mu\nu}$ a perfect fluid)

$$3f_{1R}H^2 = (1 + f_{2T})\rho - f_{2T}p - \frac{1}{2}f_2 - 3f_{1R}'H + \frac{1}{2}(f_{1R}R - f_1), \quad (A.1)$$

$$f_{1R}(2H' + 3H^2) = -(1 + 2f_{2T})\rho - \frac{1}{2}f_2 - f_{1R}'' - 2f_{1R}'H + \frac{1}{2}(f_{1R}R - f_1), \quad (A.2)$$
\[
\left[1 + \frac{1}{2} (3 - 5\omega) f_{2T}\right] \rho' + [3(1 + f_{2T}) (\omega + 1) H + (1 - \omega) f_{2T}'] \rho = 0.
\] (A.3)

If one fixes the geometrical background as
\[
a = e^{H_0 t},
\] (A.4)
this implies that
\[
H = H_0 = \text{const.}
\] (A.5)
and therefore
\[
R = R_0 = \text{const.}
\] (A.6)

Hence, in a naive approach (dimensional analysis point of view), \( f_1(R) = \text{const.} \) and the first of the FE give
\[
3f_1HR^2 = (1 + f_{2T}) \rho - f_{2T}\rho - \frac{1}{2}f_2 - 3f_{1R}H + \frac{1}{2}(f_{1R} - f_1) \]
(A.7)
\[
K_1 = (1 + f_{2T}) \rho - f_{2T}\rho - \frac{1}{2}f_2 - K_2 + K_3,
\] (A.8)
where \( K_i \) are constants. To keep the dimensional homogeneity of the equation, each of the remaining terms must be constant as well, that is,
\[
\rho = \rho_0, \implies p = p_0
\] (A.9)
\[
f_{2T}\rho = f_{2T}p = C_1,
\] (A.10)
\[
\frac{1}{2}f_2 = C_2.
\] (A.11)

However, \( \rho = \rho_0 \) is not a realistic physical result today. Note that if one fix \( f_1 = f_2 = 1 \), then the FE are reduced to the standard FLRW FE.

To formalize this approach, we may to develop the following strategy. We consider the following vector field (VF)
\[
X = K\partial_t - x\partial_x - y\partial_y - z\partial_z \in \mathfrak{X}(M), \quad K \in \mathbb{R},
\] (A.12)
then (for a FLRW metric) the condition
\[
L_X g_{\mu\nu} = 0,
\] (A.13)
yields
\[
K a' - a = 0 \iff a = a_0 e^{H_0 t},
\] (A.14)
that is, \( a = \exp(H_0 t) \).

In the same way, if we calculate \( L_X T_{\mu\nu} = 0 \), it yields
\[
\rho'K = 0,
\] (A.15)
\[
2Kpa' + Kfp' - 2pf = 0,
\] (A.16)
that is

\[ \rho = \text{const.} \quad \text{(A.17)} \]

\[ 2K \frac{a'}{a} + K \frac{p'}{p} = 2. \quad \text{(A.18)} \]

But, if we take into account that \( a = a_0 e^{\frac{K}{a} t} \), then

\[ 2 + K \frac{p'}{p} = 2, \quad \text{(A.19)} \]

and therefore

\[ p'K = 0. \quad \text{(A.20)} \]

Thus, \( \rho = \text{const} = p \) and \( f_2 = \text{const.} \) as \( f_1 \).

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