On the propagation of regularity and decay of solutions to the Benjamin equation

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Abstract

In this paper, we investigate some special regularities and decay properties of solutions to the initial value problem (IVP) of the Benjamin equation. The main result shows that: for initial datum \( u_0 \in H^s(\mathbb{R}) \) with \( s > 3/4 \), if the restriction of \( u_0 \) belongs to \( H^l((x_0, \infty)) \) for some \( l \in \mathbb{Z}^+ \) and \( x_0 \in \mathbb{R} \), then the restriction of the corresponding solution \( u(\cdot, t) \) belongs to \( H^l((\alpha, \infty)) \) for any \( \alpha \in \mathbb{R} \) and any \( t \in (0, T) \). Consequently, this type of regularity travels with infinite speed to its left as time evolves.

MSC: primary 35Q53, secondary 35B05.

Key words: Benjamin equation; Propagation of regularity; Decay

1 Introduction

In this paper, we are concerned with the IVP of the following Benjamin equation

\[
\begin{align*}
  u_t + \partial_x^3 u - H \partial_x^2 u + u \partial_x u &= 0, \quad x, t \in \mathbb{R}, \\
  u(x, 0) &= u_0(x),
\end{align*}
\]  

(1.1)

where \( H \) is the one-dimensional Hilbert transform

\[
H f(x) = \frac{1}{\pi} \text{p.v.} \left( \frac{1}{x} \ast f \right)(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|y| > \epsilon} \frac{f(x - y)}{y} dy = \left(-i \text{sgn}(\xi) \hat{f}(\xi)\right)'(x)
\]

and \( u = u(x, t) \) is a real valued function.

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We will derive some special properties including the propagation of regularity and decay of solutions to equation (1.1).

The integro-differential equation (1.1) models the unidirectional propagation of long waves in a two-fluid system, where the lower fluid with greater density is infinitely deep and the interface is subject to capillarity. It was derived by Benjamin [3] to study gravity-capillary surface waves of solitary type on deep water. He also showed that the solutions of the Benjamin equation (1.1) satisfy the following conservation laws

\[ I_1(u) = \int_{-\infty}^{+\infty} u(x,t)dx, \]
\[ I_2(u) = \int_{-\infty}^{+\infty} u^2(x,t)dx, \]
\[ I_3(u) = \int_{-\infty}^{+\infty} \left[ \frac{1}{2} (\partial_x u)^2(x,t) - \frac{1}{2} u(x,t)H \partial_x u(x,t) + \frac{1}{3} u^3(x,t) \right]dx. \]

Notice that the conservation law for solutions of (1.1)
\[ I_1(u_0) = \int_{-\infty}^{+\infty} u_0(x)dx \]
guarantees that the property \( \dot{u}(0) = 0 \) is preserved by the solution flow.

Following the definition of T. Kato [14] it is said that the IVP (1.1) is locally well-posed (LWP) in the Banach space \( X \) if given any datum \( u_0 \in X \) there exists \( T > 0 \) and a unique solution

\[ u \in C([-T,T]; X) \cap Y(T) \]  \hspace{1cm} (1.2)

with \( Y(T) \) be an auxiliary function space. Furthermore, the solution map \( u_0 \mapsto u \) is continuous from \( X \) into the class (1.2). This notion of LWP, which includes the “persistent” property, i.e., the solution describes a continuous curve on \( X \), implies that the solution of (1.1) defines a dynamic system on \( X \). If \( T \) can be taken arbitrarily large, the IVP (1.1) is said to be globally well-posed (GWP).

The problem of finding the minimal regularity property, measured in the classical Sobolev space
\[ H^s(\mathbb{R}) = (1 - \partial_x^2)^{-s/2} L^2(\mathbb{R}), \quad s \in \mathbb{R}, \]
required to guarantee that the IVP (1.1) is locally or globally well-posed in \( H^s(\mathbb{R}) \) has been extensively studied. We list some of the main results here.

Employing the Fourier restriction method introduced by Bourgain [4], Linares [18] established the LWP result for (1.1) in \( H^s(\mathbb{R}) \) with \( s \geq 0 \), which combined with the conservation law \( I_2 \), leads to the GWP for (1.1) in \( L^2 \). Guo and Huo [11] obtained the LWP result in \( H^s(\mathbb{R}) \) for \( s > -3/4 \). The best LWP results were established by Li and Wu [19] and Chen, Guo and Xiao [6]. They also asserted the GWP for (1.1) in \( H^s(\mathbb{R}) \) for \( s \geq -3/4 \). On the other hand, for the study of existence, stability and asymptotics of solitary wave solutions of equation (1.1), we can refer to [1–3, 21, 22].

The well-posedness problem has also been studied in the following weighted Sobolev spaces concerning with regularity and decay property
\[ Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^r dx), \quad s, r \in \mathbb{R} \]
and
\[ \dot{Z}_{s,r} = \{ f \in Z_{s,r} : \hat{f}(0) = 0 \}. \]

In this respect we can refer to, such as, the articles [8–10] for the Benjamin-Ono and the dispersion generalized Benjamin-Ono equations, the paper of Nahas and Ponce [20] for the nonlinear Schrödinger equation, and so on.

For the Benjamin equation (1.1), Urrea [24] established the LWP in weighted Sobolev spaces \( Z_{s,r} \) with \( s \geq 1, r \in [0, s/2] \) and \( r < 5/2 \), the GWP in \( Z_{s,r} \) with \( s \geq 1, r \in [0, s/2] \) and \( 3/2 < r < 5/2 \), and the GWP in \( \dot{Z}_{s,r} \) with \( r \in [0, s/2] \) and \( 5/2 \leq r < 7/2 \). In particular, this implies the well-posedness of the IVP (1.1) in the Schwartz space. He also established a unique continuity property for solutions of (1.1). More precisely, he showed that if \( u \in C([0, T]; Z_{7,7/2}) \) is a solution of the IVP (1.1) and there exists three different times \( t_1, t_2, t_3 \in [0, T] \) such that \( u(\cdot, t_j) \in \dot{Z}_{7,7/2} \) for \( j = 1, 2, 3 \), then \( u(x,t) \equiv 0 \).

Also, there are works concerning with special regularities and decay properties of some dispersive models.

Isaza, Linares and Ponce [12] consider these problems for the the k-generalized KdV equations
\[
\begin{align*}
    &u_t + \partial_x^3 u + u^k \partial_x u = 0, \quad x, t \in \mathbb{R}, \\
    &u(x,0) = u_0(x).
\end{align*}
\]

They mainly established two results.

The first one describes the propagation of regularity in the right hand side of the initial value for positive times. It asserts that this regularity travels with infinite speed to its left as time goes by. Note that in [13], they proved similar result for the following Benjamin-Ono equation with negative dispersion
\[
\begin{align*}
    &u_t - H \partial_x^2 u + u \partial_x u = 0, \quad x, t \in \mathbb{R}, \\
    &u(x,0) = u_0(x).
\end{align*}
\]

The difference between [12] and [13] lies in the regularity of the initial data. For the k-generalized KdV equations, the initial value \( u_0 \) belongs to \( H^{3/4+}(\mathbb{R}) \), while \( u_0 \) lies in \( H^{3/2}(\mathbb{R}) \) for the Benjamin-Ono equation.

The second conclusion in [12] is that if the initial value \( u_0 \in H^{3/4+}(\mathbb{R}) \) of the k-generalized KdV equations has polynomial decay in the positive real line, then the corresponding solution possesses some persistence properties and regularity effects for positive times.

Segata and Smith [23] extend the results of [12] to the following fifth order dispersive equation with \( a_1, a_2, a_3 \) be three constants
\[
\begin{align*}
    &u_t - \partial_x^5 u + a_1 u^2 \partial_x u + a_2 \partial_x u \partial_x^2 u + a_3 u \partial_x^3 u = 0, \quad x, t \in \mathbb{R}, \\
    &u(x,0) = u_0(x).
\end{align*}
\]

However, the regularity of the initial data need to be \( 5/2+ \) for equation (1.5).

Motivated by the above works, the objective of this paper is to extend the results of [12] to the IVP (1.1).

Before stating our results we describe the following Theorem providing us with the space of solutions where we shall be working on.
**Theorem A.** Let \( u_0 \in H^{3/4+}(\mathbb{R}) \). Then there exists a constant \( T = T(\| u_0 \|_{H^{3/4+}}) \) and a unique local solution of the IVP (1.1) such that

\[
\begin{align*}
(i) & \quad u \in C([-T, T]; H^{3/4+}(\mathbb{R})), \\
(ii) & \quad \partial_x u \in L^4([-T, T]; L^\infty(\mathbb{R})), \\
(iii) & \quad \sup_x \int_{-T}^T |J^r \partial_x u(x, t)|^2 dt < \infty, \quad \text{for} \ r \in [0, 3/4], \\
(iv) & \quad \int_{-\infty}^\infty \sup_{-T \leq t \leq T} |u(x, t)|^2 dx < \infty,
\end{align*}
\]

where \( J = (1 - \partial_x^2)^{1/2} \) denotes the Bessel potential. Moreover, the map data-solution, \( u_0 \mapsto u(x, t) \) is locally continuous (smooth) from \( H^{3/4+}(\mathbb{R}) \) into the class defined by (1.6).

**Remark 1.1.** The above well-posedness Theorem can be obtained by combining the properties of the unitary group associated to the linear part of equation (1.1) and the commutator estimate established by Kato and Ponce [15]. For the method of its proof, we refer the reader to [16] and [17], and we omit the details here.

We first describe the propagation of one-sided regularity displayed by solutions to the IVP (1.1) provided by Theorem A.

**Theorem 1.1.** Assume \( u_0 \in H^{3/4+}(\mathbb{R}) \) and for some \( l \in \mathbb{Z}^+, \ l \geq 1 \) and \( x_0 \in \mathbb{R} \) there holds

\[
\| \partial_x^j u_0 \|^2_{L^2((x_0, \infty))} = \int_{x_0}^\infty |\partial_x^j u_0(x)|^2 dx < \infty, \quad (1.7)
\]

then the solution of the IVP (1.1) provided by Theorem A satisfies that for any \( v > 0 \) and \( \epsilon > 0 \)

\[
\sup_{0 \leq t \leq T} \int_{x_0 + \epsilon - vt}^{x_0 + \epsilon + vt} (\partial_x^j u)^2(x, t) dx \leq c_0, \quad (1.8)
\]

for \( j = 0, 1, 2, \ldots, l \) with \( c_0 = c_0(\| u_0 \|_{H^{3/4+}}; \| \partial_x^j u_0 \|_{L^2((x_0, \infty))}; l; v; \epsilon; T) \).

In particular, for all \( t \in (0, T] \), the restriction of \( u(\cdot, t) \) to any interval \((x_1, \infty)\) belongs to \( H^l((x_1, \infty)) \).

Moreover, for any \( v \geq 0 \), \( \epsilon > 0 \) and \( R > 0 \)

\[
\int_0^T \int_{x_0 + \epsilon - vt}^{x_0 + R - vt} (\partial_x^j u(x, t))^2 dx dt \leq c_0, \quad (1.9)
\]

\[
\int_0^T \int_{x_0 + \epsilon - vt}^{x_0 + R - vt} (\partial_x^{j+1} u(x, t))^2 dx dt \leq c_1, \quad (1.10)
\]

where \( c_1 = c_1(l; \| u_0 \|_{H^{3/4+}}; \| \partial_x^j u_0 \|_{L^2((x_0, \infty))}; v; \epsilon; T; R) \).

**Remark 1.2.** The functions \( \eta(x; \epsilon, b) \) mentioned in Theorem 1.1 and \( \eta_j(x; \epsilon, b) \) in Theorem 1.2 will be defined in section 2. In addition, without loss of generality, we shall assume from now on \( x_0 = 0 \) in Theorem 1.1.

The persistence of decay and regularity effects established in [12] can also be extended to the IVP (1.1). In fact, we have
Theorem 1.2. Assume $u_0 \in H^{3/4^+}(\mathbb{R})$ and for some $n \in \mathbb{Z}^+$, $n \geq 1$ there holds
\[
\|x^{n/2}u_0\|^2_{L^2((0,\infty))} = \int_0^\infty |x^n| |u_0(x)|^2 \, dx < \infty, \tag{1.11}
\]
then the solution of the IVP (1.1) provided by Theorem A satisfies that
\[
\sup_{0 \leq t \leq T} \int_0^\infty |x^n| |u(x,t)|^2 \, dx \leq c_2 \tag{1.12}
\]
with $c_2 = c_2(\|u_0\|_{H^{3/4^+}}; \|x^{n/2}u_0\|_{L^2((0,\infty))}; T; n)$.

Furthermore, for any $v \geq 0$, $\epsilon, \delta > 0$, $m, j \in \mathbb{Z}^+$, $m + j \leq n$ and $m \geq 1$,
\[
\sup_{\delta \leq t \leq T} \int_{|x-\epsilon t|}^\infty (\partial_x^m u)(x,t) x_j^1 \, dx + \int_{\delta}^T \int_{|x-\epsilon t|}^\infty (\partial_x^{m+1} u)^2(x,t) x_j^{1-1} \, dx \, dt
\]
\[
+ \int_{\delta}^T \int |\partial_x^2 (\partial_x^m u(x,t) \eta_j(x+\beta t; \epsilon, \delta))|^2 \, dx \, dt \leq c_3, \tag{1.13}
\]
where $c_3 = c_3(||u_0||_{H^{3/4^+}}; ||x^{n/2}u_0||_{L^2((x_0,\infty))}; v; c; T; \delta; \epsilon), x_+ = \max\{x, 0\}$.

Simple analysis of the proof of Theorems 1.1 and 1.2 yields their validity for the “defocusing” Benjamin equation
\[
\begin{align*}
& u_t + \partial_x^3 u - Hu^2 u - u \partial_x u = 0, \quad x, t \in \mathbb{R}, \\
& u(x,0) = u_0(x).
\end{align*} \tag{1.14}
\]
Consequently, our results still hold for $u(-x,-t)$ with $u(x,t)$ be the solution of (1.1). Put another way, for datum satisfying the assumption (1.7) and (1.11) on the left hand side of the real line, respectively, Theorems 1.1 and 1.2 remain true backward in time.

On the other hand, equation (1.1) is time reversible. In fact, let $v(x,t) = u(-x,-t)$ with $u(x,t)$ be the solution of equation (1.1). Using the relation $(Hv)(x,t) = -(Hu)(-x,-t)$, one has
\[
\begin{align*}
& v_t + \partial_x^3 v - H\partial_x^2 v + v \partial_x v = 0, \quad x, t \in \mathbb{R}, \\
& v(x,0) = u_0(-x).
\end{align*} \tag{1.15}
\]

Theorems 1.1 and 1.2 combining with the above two points indicate

Corollary 1.1. Let $u \in C([-T, T]; H^{3/4^+}(\mathbb{R}))$ be a solution of the equation (1.1) provided by Theorem A such that
\[
\partial_x^m u(\cdot, \hat{t}) \notin L^2((a, \infty)) \text{ for some } \hat{t} \in (-T, T), \quad a \in \mathbb{R} \text{ and } m \in \mathbb{Z}^+.
\]
Then for any $t \in [-T, \hat{t})$ and any $\beta \in \mathbb{R}$
\[
\partial_x^m u(\cdot, t) \notin L^2((\beta, \infty)) \quad \text{and} \quad x^{m/2} u(\cdot, t) \notin L^2((0, \infty)).
\]

Next, Theorems 1.1 and 1.2 yield that the singularity of the solution corresponding to an appropriate class of initial data propagates with infinite speed to the left as time goes by. Also, since equation (1.1) is time reversible, the solution cannot have had some regularity in the past. More precisely, we have
Corollary 1.2. Let \( u \in C([-T,T]; H^{3/4^+}(\mathbb{R})) \) be a solution of the equation (1.1) provided by Theorem A. Suppose there exists \( n, m \in \mathbb{Z}^+ \) with \( m \leq n \) such that for some \( a, b \in \mathbb{R} \) with \( a < b \)
\[
\int_{b}^{\infty} |\partial_x^n u_0(x)|^2 \, dx < \infty \quad \text{but} \quad \partial_x^m u_0 \notin L^2((a, \infty)).
\] (1.16)

Then for any \( t \in (0, T) \) and any \( v, \epsilon > 0 \)
\[
\int_{b+\epsilon-\epsilon v}^{\infty} |\partial_x^n u(x,t)|^2 \, dx < \infty
\]
and for any \( t \in (-T, 0) \) and any \( \alpha \in \mathbb{R} \)
\[
\int_{\alpha}^{\infty} |\partial_x^m u(x,t)|^2 \, dx = \infty.
\]

We now discuss some of the ingredients in the proof of Theorems 1.1 and 1.2.

The first one is concerned with the proof of Theorem 1.1. As in [12], we mainly use induction. To treat the Benjamin-Ono term \(-H \partial_x^2 u\), we follow the idea in [13], where the commutator estimate for the Hilbert transform (2.12) plays a vital role. In spite of this, there is a little difference between [13] and this paper when handling the following two terms (see (3.8))
\[
\int_{0}^{T} \int_{0}^{T} (\partial_x^2 u)^2(\eta')^2 \, dx \, dt + \int_{0}^{T} \int (\partial_x^2 u \eta)^2 \, dx \, dt.
\] (1.17)

In [13] for the Benjamin-Ono equation (1.4), these two terms can be controlled by sufficient local smoothing effect. More precisely, the condition (1.6)(iii) in [13] reads
\[
\int_{-T}^{T} \int_{-R}^{R} (|\partial_x D_x u|^2 + |\partial_x^2 u|^2) \, dx \, dt \leq c_0,
\]
where \( R \) is arbitrary and finite. This combined with the boundedness of \( \eta' \) and \( \eta \) on the support of \( \eta \) immediately yields the finiteness of (1.17). However, (1.6)(iii) in this paper provides us at most \( 7/4^+ \) order local smoothing effect, which is not enough to bound (1.17). Fortunately, in the first step (the case \( l=1 \) in the proof of Theorem 1.1) in our induction process, the KdV term provides us with the finiteness of (see (3.4))
\[
\int_{0}^{T} \int (\partial_x^2 u)^2(x,t) \chi' (x+vt; \epsilon, b) \, dx \, dt.
\]
This permits us to use the properties of \( \eta' \) and \( \eta \), i.e. (2.3) and (2.4), to control (1.17).

The second one relates to the proof of Theorem 1.2. The difficulty still comes from the Benjamin-Ono term. For the term \( A_{422} \) in (4.2), note that because of the factor \( x^n \) in the definition of \( \eta_n' \) and \( \eta_n \), the support of \( \eta_n \) is not \([\epsilon, b] \) at all for the general case. As a consequence, \( \eta_n \) and \( \eta_n' \) may be unbounded. However, we notice that (2.9) and (2.10) provide us with a relation between \( \chi_n \) and \( \chi_{n-1} \), therefore, we could use induction to treat this term. (2.9) and (2.10) are also used to bound the term in (4.20).

The rest of this paper is organized as follows: in section 2 we construct our cut-off functions and state a lemma to be used in the proof of Theorem 1.1 and Theorem 1.2. The proof of Theorem 1.1 and Theorem 1.2 will be given in section 3 and section 4, respectively.
2 Preliminaries

Let us first construct our cutoff functions, the construction of this family of cutoff functions is motivated by Segata and Smith [23].

Let \( p \) be large enough and let \( \rho(x) \) be defined as follows

\[
\rho(x) = a \int_{0}^{x} y^p (1 - y)^p dy
\]

with the constant \( a = a(p) \) be chosen to satisfy \( \rho(1) = 1 \).

**Remark 2.1.** According to Lemma 2.1 below, when come across the \( L^p \) norm of the commutator related to the Hilbert transform, we want to put all derivatives to the smooth function \( \psi \), this is the reason for \( p \) in the definition of \( \rho(x) \) being large enough.

With the above definition, we have

\[
\begin{align*}
\rho(0) &= 0, \quad \rho(1) = 1, \\
\rho'(0) &= \rho''(0) = \cdots = \rho^{(p)}(0) = 0, \\
\rho'(1) &= \rho''(1) = \cdots = \rho^{(p)}(1) = 0
\end{align*}
\]

with \( 0 < \rho, \rho' \) for \( 0 < x < 1 \).

Next, for parameters \( \epsilon, b > 0 \), define \( \chi \in C^p(\mathbb{R}) \) by

\[
\chi(x; \epsilon, b) = \begin{cases} 
0, & x \leq \epsilon \\
\rho((x - \epsilon)/b), & \epsilon < x < b + \epsilon \\
1, & b + \epsilon \leq x.
\end{cases}
\]

In addition, we define \( \chi_n = x^n \chi \in C^p(\mathbb{R}) \).

By their definitions, \( \chi \) and \( \chi_n \) are both positive for \( x \in (\epsilon, \infty) \).

Computing as section 2 in [23], we can derive the following properties concerning \( \chi \) and \( \chi_n \):

\[
\begin{align*}
(1) & \quad \chi(x; \epsilon/10, \epsilon/2) = 1 \text{ on } \text{supp}(\chi; \epsilon, b) = [\epsilon, \infty); \\
(2) & \quad |\chi(x, \epsilon, b)| \leq \chi'(x, \epsilon, b); \\
(3) & \quad \left| \frac{\chi''(x; \epsilon, b)}{\chi'(x; \epsilon, b)} \right| \leq c(\epsilon, b) \chi'(x/3, b + \epsilon) \text{ on support of } \chi'; \\
(4) & \quad |\chi^{(j)}(x; \epsilon, b)| \leq c(j, \epsilon, b) \chi'(x/3, b + \epsilon) \text{ on } [\epsilon, b + \epsilon] \text{ for } j = 1, 2, \ldots, p; \\
(5) & \quad |\chi''(x, \epsilon, b)| \leq c(n, l) \chi_{n-\epsilon}(x, \epsilon, b) + c(b, v, \epsilon, T) \chi'(x/3, b + \epsilon) \text{ for } l \leq n - 2; \\
(6) & \quad |\chi''(x, \epsilon, b)| \leq c(n, l) \chi_{n-\epsilon}(x, \epsilon, b) + c(n, l, b) \chi(x/10, \epsilon/2) \text{ for } l \leq n - 3; \\
(7) & \quad n\chi_{n-1}(x, \epsilon, b) \leq \chi'(x, \epsilon, b); \\
(8) & \quad |\chi^{(j)}(x; \epsilon, b)| \leq c(j, n, b)[1 + \chi_n(x; \epsilon, b)] \text{ for } j = 1, 2, \ldots, p; \\
(9) & \quad |\chi^{(j)}(x; \epsilon, b)| \leq c(n, b, \epsilon) \chi_{n-1}(x; \epsilon/3, b + \epsilon) \text{ for } j = 1, 2, 3, \ldots, p; \\
(10) & \quad \left| \frac{\chi''(x; \epsilon, b)}{\chi'(x; \epsilon, b)} \right| \leq c(\epsilon, b, n) \chi_{n-1}(x; \epsilon/3, b + \epsilon) \text{ on support of } \chi_n'.
\end{align*}
\]
Moreover, we define
\[ \eta(x; \epsilon, b) = \sqrt{\chi'(x; \epsilon, b)}, \]
\[ \eta_n(x; \epsilon, b) = \sqrt{\chi_n'(x; \epsilon, b)}. \]  \hspace{1cm} (2.11)

Then, reasoning as section 2 in [13], we derive that \( \eta(x; \epsilon, b) \) and \( \eta_n(x; \epsilon, b) \) are both in \( C^p(\mathbb{R}) \).

The following commutator estimate is an extension of the Calderón theorem [5], it was proved by Dawson, McGahagan and Ponce [7].

**Lemma 2.1.** For any \( p \in (1, \infty) \) and \( l, m \in \mathbb{Z}^+ \cup \{0\} \), \( l + m \geq 1 \) there exists a constant \( C = C(p, l, m) > 0 \) such that
\[
\|\partial_x^l [H; \psi] \partial_x^m f\|_{L^p} \leq C\|\partial_x^{l+m} \psi\|_{L^\infty} \|f\|_{L^p} \tag{2.12}
\]
with \( H \) be the Hilbert transform.

### 3 Proof of Theorem 1.1

To prove Theorem 1.1, we follow the idea in [12] and use an induction argument. To illuminate our method, we first prove (1.8) for \( l = 1 \) and \( l = 2 \).

Let us first prove the case \( l = 1 \).

Formally, applying \( \partial_x \) to equation (1.1) and multiplying the result by \( \partial_x u(x, t) \chi(x + vt; \epsilon, b) \), after some integration by parts, one deduces
\[
\frac{1}{2} \frac{d}{dt} \int (\partial_x u)^2(x, t) \chi(x + vt) dx - \frac{1}{2} \int (\partial_x u)^2(x, t) \chi'(x + vt) dx \tag{3.1}
\]
\[
+ \frac{3}{2} v \int (\partial_x^2 u)^2(x, t) \chi'(x + vt) dx - \frac{1}{2} \int (\partial_x u)^2(x, t) \chi''(x + vt) dx \tag{3.1}
\]
\[
+ \int \partial_x(u \partial_x u) \partial_x u(x, t) \chi(x + vt) dx - \int H \partial_x^3 u \partial_x u(x, t) \chi(x + vt) dx = 0,
\]

where in \( \chi \) we omit the parameters \( \epsilon \) and \( b \).

We estimate the integrals in (3.1) term by term.

Using (1.6)(iii) with \( r = 0 \) and the support property of \( \chi'(x) \), it holds that
\[
\int_0^T |A_1(t)| dt \leq \int_0^T \int (\partial_x u)^2(x, t) \chi'(x + vt) dx dt \leq c_0
\]
and similarly
\[
\int_0^T |A_2(t)| dt \leq c_0.
\]

For the term \( A_3 \), direct computation yields
\[
A_3(t) = \int (\partial_x u)^3 \chi(x + vt) dx + \int u \partial_x^2 u \partial_x u \chi(x + vt) dx
\]
\[
= \frac{1}{2} \int (\partial_x u)^3 \chi(x + vt) dx - \frac{1}{2} \int u \partial_x u \partial_x u \chi'(x + vt) dx
\]
\[ \leq \|\partial_x u\|_{L^\infty} \int (\partial_x u)^2 \chi(x + vt) dx + \|u\|_{L^\infty} \int (\partial_x u)^2 \chi'(x + vt) dx = A_{31} + A_{32}. \]

By Sobolev embedding theorem, one obtains
\[ \int_0^T |A_{32}(t)| dt \leq \sup_{[0,T]} \|u\|_{H^{3/4}} \int (\partial_x u)^2 \chi(x + vt) dx dt. \]

The term \( A_{31} \) will be controlled by using (1.6)(ii) and the Gronwall inequality.

Finally, to estimate \( A_4 \), we follow the idea described in [13]. Integration by parts yields
\[ A_4 = -\int H \partial_x^2 u \partial_x u \chi(x + vt) dx \]
\[ = \int H \partial_x^2 u \partial_x^2 u \chi(x + vt) dx + \int H \partial_x^2 u \partial_x u \chi'(x + vt) dx = A_{41} + A_{42}. \]

Since the Hilbert transform is skew symmetric, we have
\[ A_{41} = \int H \partial_x^2 u \partial_x^2 u \chi(x + vt) dx \]
\[ = -\int \partial_x^2 u H(\partial_x^2 u \chi(x + vt)) dx \]
\[ = -\int \partial_x^2 u H \partial_x^2 u \chi(x + vt) dx - \int \partial_x^2 u [H; \chi] \partial_x^2 u dx \]
\[ = -A_{41} - \int \partial_x^2 u [H; \chi] \partial_x^2 u dx. \]

Therefore, (2.12) leads to
\[ A_{41} = -\frac{1}{2} \int \partial_x^2 u [H; \chi] \partial_x^2 u dx \]
\[ = -\frac{1}{2} \int u \partial_x^2 [H; \chi] \partial_x^2 u dx \]
\[ \leq c \|u\|_{L^2} \|\partial_x^2 u [H; \chi] \partial_x^2 u\|_{L^2} \]
\[ \leq c \|u\|_{L^2}^2 = c \|u_0\|_{L^2}^2. \]

Concerning the term \( A_{42} \), let us recall the definition of \( \eta(x; \epsilon, b) \) in (2.11), we can write \( A_{42} \) as
\[ A_{42} = \int H \partial_x^2 u \partial_x u \chi'(x + vt) dx \]
\[ = \int H \partial_x^2 u \eta \partial_x u \eta dx \]
\[ = \int H (\partial_x^2 u \eta) \partial_x u \eta dx - \int [H; \eta] \partial_x^2 u \partial_x u \eta dx \]
\[ = \int H \partial_x (\partial_x u \eta) \partial_x u \eta dx - \int H (\partial_x u \eta') \partial_x u \eta dx - \int [H; \eta] \partial_x^2 u \partial_x u \eta dx \]
\[ = A_{421} + A_{422} + A_{423}. \]
Plancherel’s identity yields

$$A_{421} = \int H\partial_x(\partial_x u\eta)\partial_x u\eta dx = \int |D_x^2(\partial_x u\eta)|^2 dx,$$

which is positive and will stay at the left hand side of (3.1).

The boundedness of the Hilbert transform in \( L^2 \) and the Young inequality produce

$$\int_0^T |A_{422}| dt = \int_0^T \left| \int H(\partial_x u\eta')\partial_x u\eta dx \right| dt \leq \int_0^T (\partial_x u)^2(\eta')^2 dx dt + c \int_0^T (\partial_x u)^2\eta^2 dx dt. \quad (3.3)$$

Employing the boundedness of \( \eta \) and \( \eta' \) on the support of \( \eta \) and using (1.6)(iii) with \( r = 0 \), we obtain

$$\int_0^T |A_{422}| dt \leq c_0.$$ 

Invoking the commutator estimate (2.12), we derive

$$|A_{423}| = \left| \int [H; \eta]\partial_x^2 u\partial_x u\eta dx \right| \leq \| [H; \eta]\partial_x^2 u \|_{L^2} \| \partial_x u\eta \|_{L^2} \leq c\|u\|_{L^2}^2 + c\|\partial_x u\eta\|_{L^2}^2 \leq c\|u_0\|_{L^2}^2 + c\|\partial_x u\eta\|_{L^2}^2.$$ 

After integration in time, the term \( \|\partial_x u\eta\|_{L^2}^2 \) can be controlled as that in (3.3).

Substituting the above information in (3.1), using the Gronwall inequality and (1.6)(ii), one obtains

$$\sup_{\epsilon \in [0,T]} \int (\partial_x u)^2(x,t)\chi(x+vt; \epsilon, b)dx + \int_0^T \int (\partial_x^2 u)^2(x,t)\chi'(x+vt; \epsilon, b)dx dt + \int_0^T \int [D_x^2(\partial_x u(x,t)\eta(x+vt; \epsilon, b))]^2 dx dt \leq c_0. \quad (3.4)$$

This completes the proof of the case \( l = 1 \).

Next, we prove (1.8) for the case \( l = 2 \).

Applying \( \partial_x^2 \) to equation (1.1), then multiplying \( \partial_x^2 u(x,t)\chi(x+vt; \epsilon, b) \) and integrating, we find

$$\frac{1}{2} \frac{d}{dt} \int (\partial_x^2 u)^2(x,t)\chi(x+vt)dx - \frac{1}{2} v \int (\partial_x^2 u)^2(x,t)\chi'(x+vt)dx \leq A_1$$

$$+ \frac{3}{2} v \int (\partial_x^2 u)^2(x,t)\chi''(x+vt)dx - \frac{1}{2} \int (\partial_x^2 u)^2(x,t)\chi''(x+vt)dx \leq A_2$$

$$+ \int \partial_x^2(u\partial_x u)\partial_x^2 u(x,t)\chi(x+vt)dx - \int H\partial_x^4 u\partial_x^2 u(x,t)\chi(x+vt)dx = 0. \quad (3.5)$$
Invoking (3.4), one has
\[ \int_0^T |A_1(t)|dt \leq |v| \int_0^T (\partial_x^2 u)^2(x, t)\chi(x + vt)dx \leq c_0. \]

Employing (2.4) with \( j = 3 \) and using (3.4) with \((\epsilon/3, b + \epsilon)\) instead of \((\epsilon, b)\), it holds that
\[ \int_0^T |A_2(t)|dt \leq \int_0^T (\partial_x^2 u)^2(x, t)\chi''(x + vt; \epsilon, b)dxdt \leq \int_0^T (\partial_x^2 u)^2(x, t)\chi'(x + vt; \epsilon/3, b + \epsilon)dxdt \leq c_0. \quad (3.6) \]

Integration by parts yields
\[
A_3(t) = 3 \int \partial_x u (\partial_x^2 u)^2 \chi(x + vt)dx + \int u \partial_x^2 u \partial_x^2 u \chi(x + vt)dx \\
= \frac{5}{2} \int \partial_x u (\partial_x^2 u)^2 \chi(x + vt)dx - \frac{1}{2} \int u (\partial_x^2 u)^2 \chi'(x + vt)dx \\
\leq \|\partial_x u\|_{L^\infty} \int (\partial_x^2 u)^2 \chi(x + vt)dx + \|u\|_{L^\infty} \int (\partial_x^2 u)^2 \chi'(x + vt)dx \\
= A_{31} + A_{32}.
\]

Again, using Sobolev embedding theorem and (3.4), one obtains
\[ \int_0^T |A_{32}(t)|dt < \sup_{[0,T]} \|u\|_{H^{3/4}+} \int_0^T (\partial_x^2 u)^2 \chi'(x + vt)dxdt \leq c_0. \]

The term \( A_{31} \) will be controlled by using (1.6)(ii) and the Gronwall inequality.

We now estimate \( A_4 \).

Integration by parts leads to
\[ A_4 = - \int H \partial_x^3 u \partial_x^2 u \chi(x + vt)dx \\
= \int H \partial_x^3 u \partial_x^2 u \chi(x + vt)dx + \int H \partial_x^3 u \partial_x^2 u \chi'(x + vt)dx \\
= A_{41} + A_{42}. \]

Invoking again the fact that the Hilbert transform is skew symmetric, we have
\[
A_{41} = \int H \partial_x^2 u \partial_x^3 u \chi(x + vt)dx \\
= - \int \partial_x^3 u H (\partial_x^3 u \chi(x + vt))dx \\
= - \int \partial_x^3 u H \partial_x^3 u \chi(x + vt)dx - \int \partial_x^3 u [H; \chi] \partial_x^3 udx \\
= -A_{41} - \int \partial_x^3 u [H; \chi] \partial_x^3 udx. \quad (3.7)
\]

Consequently, (2.12) produces
\[
A_{41} = - \frac{1}{2} \int \partial_x^3 u [H; \chi] \partial_x^3 udx \\
= \frac{1}{2} \int u \partial_x^3 [H; \chi] \partial_x^3 udx \\
\leq c\|u\|_{L^2} \|\partial_x^3 [H; \chi] \partial_x^3 u\|_{L^2} \\
\leq c\|u\|_{L^2}^2 = c\|u_0\|_{L^2}^2.
\]
Applying (2.11) yields

\[ A_{42} = \int H \partial_x^3 u \partial_x^2 u \chi'(x + vt) dx \]

\[ = \int H \partial_x^3 u \partial_x^2 u \eta dx \]

\[ = \int H(\partial_x^3 u \eta)\eta dx - \int [H; \eta] \partial_x^3 u \partial_x^2 u \eta dx \]

\[ = \int H(\partial_x^3 u \eta)\partial_x^2 u \eta dx - \int H(\partial_x^3 u \eta)\partial_x^2 u \eta dx - \int [H; \eta] \partial_x^3 u \partial_x^2 u \eta dx \]

\[ = A_{421} + A_{422} + A_{423}. \]

Similar to the treatment of (3.2), we write \( A_{421} \) as

\[ A_{421} = \int H \partial_x(\partial_x^2 u \eta) \partial_x^2 u \eta dx = \int [D_2^\frac{1}{2} (\partial_x^2 u \eta)]^2 dx. \]

The Young inequality leads to

\[
\int_0^T |A_{422}| dt \leq \int_0^T \left| \int H(\partial_x^3 u \eta)\partial_x^2 u \eta \right| dx dt
\]

\[ \leq c_0 \int_0^T \int (\partial_x^2 u)^2(\eta')^2 dx dt + c_0 \int_0^T \int (\partial_x^2 u)^2 \eta^2 dx dt. \]  (3.8)

Invoking (2.3) and using (3.4) with \((\epsilon/3, b + \epsilon)\) instead of \((\epsilon, b)\) yield

\[ \int_0^T \int (\partial_x^2 u)^2 \eta dx dt + \int_0^T \int (\partial_x^2 u)^2(\eta')^2 dx dt \]

\[ \leq \int_0^T \int (\partial_x^2 u)^2 \chi'(x + vt; \epsilon, b) dx dt + \int_0^T \int (\partial_x^2 u)^2 \chi'(x + vt; \epsilon/3, b + \epsilon) dx dt \]

\[ \leq c_0. \]

For the term \( A_{423} \), (2.12) leads to

\[ A_{423} = - \int [H; \eta] \partial_x^3 u \partial_x^2 u \eta dx \]

\[ = \int \partial_x[H; \eta] \partial_x^3 u \partial_x u \eta dx + \int [H; \eta] \partial_x^3 u \partial_x \eta' dx \]

\[ \leq \| \partial_x[H; \eta] \partial_x^3 u \|_{L^2} \| \partial_x u \eta \|_{L^2} + \|[H; \eta] \partial_x^3 u \|_{L^2} \| \partial_x \eta' \|_{L^2} \]

\[ \leq c \| u \|_{L^2} \| \partial_x \eta \|_{L^2} + c \| u \|_{L^2} \| \partial_x u \eta' \|_{L^2} \]

\[ \leq c \| u \|_{L^2}^2 + c \| \partial_x \eta \|_{L^2}^2 + c \| \partial_x u \eta' \|_{L^2}^2, \]

which, after integration in time, can be controlled by using similar method as that in (3.8).

Accordingly, gathering the above information in (3.5) and invoking the Gronwall inequality, one derives

\[
\sup_{t \in [0, T]} \int (\partial_x^2 u)^2(x, t) \chi(x + vt; \epsilon, b) dx + \int_0^T \int (\partial_x^3 u)^2(x, t) \chi'(x + vt; \epsilon, b) dx dt
\]

\[ + \int_0^T \int [D_2^\frac{1}{2} (\partial_x^2 u(x, t) \eta(x + vt; \epsilon, b))]^2 dx dt \leq c_0. \]  (3.9)
We prove the general case $l \geq 2$ by induction. In details, we assume: If $u_0$ satisfies (1.7) then (1.8) holds, that is to say
\[
\sup_{t \in [0,T]} \int (\partial_x^l u)^2 (x, t) \chi(x + vt; \epsilon, b) dx + \int_0^T \int (\partial_x^{l+1} u)^2 (x, t) \chi'(x + vt; \epsilon, b) dx dt \\
+ \int_0^T \int [D_x^2 (\partial_x^l u(x, t) \eta(x + vt; \epsilon, b))]^2 dx dt \leq c_0
\] (3.10)
for $j = 1, 2, \ldots, l, l \geq 2$, and for any $\epsilon, b, v > 0$.

Now we have that
\[
u_0|_{(0, \infty)} \in H^{l+1}((0, \infty)).
\]

Thus from the previous step (3.10) holds. And formally, we have for $\epsilon, b, v > 0$ the following identity
\[
\left. \frac{1}{2} \frac{d}{dt} \int (\partial_x^{l+1} u)^2 (x, t) \chi(x + vt) dx - \frac{1}{2} v \int (\partial_x^{l+1} u)^2 (x, t) \chi'(x + vt) dx \right|_{A_1} \\
+ \frac{3}{2} \int (\partial_x^{l+2} u)^2 (x, t) \chi'(x + vt) dx - \frac{1}{2} \int (\partial_x^{l+1} u)^2 (x, t) \chi''(x + vt) dx \right|_{A_2} \\
+ \int \partial_x^{l+1} (u \partial_x u) \partial_x^{l+1} u(x, t) \chi(x + vt) dx - \int H \partial_x^{l+3} u \partial_x^{l+1} u(x, t) \chi(x + vt) dx = 0. \tag{3.11}
\]

Invoking (3.10) with $j = l$, it holds that
\[
\int_0^T |A_1(t)| dt \leq |v| \int_0^T \int (\partial_x^{l+1} u)^2 (x, t) \chi'(x + vt) dx \leq c_0.
\]

Using similar method of treating (3.6), we find
\[
\int_0^T |A_2(t)| dt \leq \int_0^T \int (\partial_x^{l+1} u)^2 (x, t) \chi''(x + vt; \epsilon, b) dx dt \leq \frac{3}{2} \int_0^T \int (\partial_x^{l+1} u)^2 (x, t) \chi'(x + vt; \epsilon/3, b + \epsilon) dx dt \leq c_0.
\]

We estimate $A_3$ by considering two cases: The first case is when $l + 1 = 3$ and the second is $l + 1 \geq 4$.

When $l + 1 = 3$, we have after integration by parts
\[
A_3(t) = 4 \int \partial_x u (\partial_x^3 u)^2 \chi(x + vt) dx + \int u \partial_x^3 u \partial_x^3 u \chi(x + vt) dx \\
+ 3 \int (\partial_x^2 u)^2 \partial_x^3 u \chi(x + vt) dx \\
= \frac{7}{2} \int \partial_x u (\partial_x^3 u)^2 \chi(x + vt) dx - \frac{1}{2} \int u (\partial_x^3 u)^2 \chi'(x + vt) dx \\
+ 3 \int (\partial_x^2 u)^2 \partial_x^3 u \chi(x + vt) dx \\
= A_{31} + A_{32} + A_{33}.
\]
Simple computation leads to

\[ |A_{31}(t)| < \|\partial_x u\|_{L^\infty} \int (\partial_x^3 u)^2 \chi(x + vt)dx \]

with the integral be the quantity to be estimated.

Employing (3.10) with \( j = l = 2 \), one deduces

\[
\int_0^T |A_{32}(t)|dt \leq \sup_{t \in [0,T]} \|u\|_{L^\infty} \int_0^T (\partial_x^3 u)^2 \chi'(x + vt)dxdt
\]

\[
\leq \sup_{t \in [0,T]} \|u\|_{H^{3/4}} \int_0^T (\partial_x^3 u)^2 \chi'(x + vt)dxdt
\]

\[
\leq c_0.
\]

Integration by parts leads to

\[ A_{33} = 3 \int (\partial_x^2 u)^2 \partial_x^2 u \chi(x + vt)dx = -\int (\partial_x^2 u)^3 \chi'(x + vt)dx \]

Using (2.1), we have

\[ |A_{33}| \leq \|\partial_x^2 u \chi'(\cdot + vt; \epsilon, b)\|_{L^\infty} \int (\partial_x^2 u)^2 \chi(x + vt; \epsilon/10, \epsilon/2)dx \quad (3.12) \]

with the integral be bounded in \( t \in (0, T] \) by a constant \( c_0(\epsilon, b, v) \) resulting from (3.10)(j=2).

Therefore, from the boundedness of \( \chi \) and the Sobolev inequality \( \|f\|_{L^\infty} \leq \|f\|_{H^{1,1}} \), one has

\[ |A_{33}| \leq c\|\partial_x^2 u \chi'(\cdot + vt; \epsilon, b)\|^2_{L^\infty} + c \]

\[ \leq c\|\partial_x^2 u \chi'(\cdot + vt; \epsilon, b)\|_{L^\infty} + c \]

\[ \leq c\int \|\partial_x(\partial_x^2 u)^2 \chi'(x + vt; \epsilon, b)\|dx + c \]

\[ \leq c\int \partial_x^2 u \partial_x^2 u \chi'(x + vt; \epsilon, b)dx + c\int \partial_x^2 u \partial_x^2 u \chi''(x + vt; \epsilon, b)dx \]

\[ \leq c\int (\partial_x^2 u)^2 \chi'(x + vt; \epsilon, b)dx + c\int (\partial_x^2 u)^2 \chi'(x + vt; \epsilon, b)dx \]

\[ + c\int |\partial_x^2 u \partial_x^2 u \chi'(x + vt; \epsilon, b)dx + c, \quad (3.13) \]

where we have used (2.4) with \( j = 2 \).

Employing (3.10) with \( j = 1, 2 \) and integration in time, we obtain

\[ \int_0^T |A_{33}|dt \leq c_0. \quad (3.14) \]

We turn our attention to the second case \( l + 1 \geq 4 \) in \( A_3 \).

By integration by parts, one derives

\[ A_3 = d_0 \int u(\partial_x^{l+1} u)^2 \chi'(x + vt)dx + d_1 \int \partial_x u(\partial_x^{l+1} u)^2 \chi(x + vt)dx \]

\[ + d_2 \int \partial_x^2 u \partial_x^2 u(\partial_x^{l+1} u) \chi(x + vt)dx + \sum_{j=3}^{l-1} \int \partial_x^2 u \partial_x^{l+2-j} u \partial_x^{l+1} u \chi(x + vt)dx \]

\[ = A_{3,0} + A_{3,1} + A_{3,2} + \sum_{j=3}^{l-1} A_{3,j}. \]
Using (3.10) with $j = l$ and the Sobolev embedding, one obtains

$$
\int_0^T |A_{3,0}| \, dt \leq \int_0^T \|u\|_{L^\infty} \int (\partial_x^{l+1}u)^2 \chi'(x + vt) \, dx \, dt
$$

$$
\leq \sup_{0 \leq t \leq T} \|u\|_{H^{3/4}} \int_0^T (\partial_x^{l+1}u)^2 \chi'(x + vt) \, dx \, dt \leq c_0.
$$

Direct computation leads to

$$
|A_{3,1}| \leq \|\partial_x u\|_{L^\infty} \int (\partial_x^{l+1}u)^2 \chi(x + vt) \, dx,
$$

which can be handled by the Gronwall inequality and (1.6)(ii).

To estimate $A_{3,2}$ we follow the argument in the previous case. Accordingly, we need to estimate $\sum_{j=3}^{l-1} A_{3,j}$ which only appears when $l - 1 \geq 3$.

The Young inequality leads to

$$
|A_{3,j}| \leq \frac{1}{2} \int (\partial_x^ju\partial_x^{l+2-j}u)^2 \chi(x + vt) \, dx + \frac{1}{2} \int (\partial_x^{l+1}u)^2 \chi(x + vt) \, dx
$$

$$
= A_{3,j,1} + \frac{1}{2} \int (\partial_x^{l+1}u)^2 \chi(x + vt) \, dx
$$

with the last integral be the quantity to be estimated.

To handle $A_{3,j,1}$, one observes that $j, l + 2 - j \leq l - 1$ and accordingly

$$
|A_{3,j,1}| \leq \|(\partial_x^j u)^2 \chi(\cdot + vt; \epsilon/10, \epsilon/2)\|_{L^\infty} \int (\partial_x^{l+2-j}u)^2 \chi(x + vt; \epsilon, b) \, dx
$$

with the last integral be bounded by (3.10). Moreover, Sobolev embedding yields

$$
\|(\partial_x^2 u)^2 \chi(\cdot + vt; \epsilon/10, \epsilon/2)\|_{L^\infty}
$$

$$
\leq \|\partial_x[(\partial_x^2 u)^2 \chi(\cdot + vt; \epsilon/10, \epsilon/2)]\|_{L^1}
$$

$$
\leq \|\partial_x^2 u \partial_x^{l+1}u \chi(\cdot + vt; \epsilon/10, \epsilon/2)\|_{L^1} + \|\partial_x^2 u \partial_x^j u \chi' (\cdot + vt; \epsilon/10, \epsilon/2)\|_{L^1}
$$

$$
\leq c \int (\partial_x^2 u)^2 \chi(x + vt; \epsilon/10, \epsilon/2) \, dx + c \int (\partial_x^{l+1}u)^2 \chi(x + vt; \epsilon/10, \epsilon/2) \, dx
$$

$$
+ c \int (\partial_x^j u)^2 \chi'(x + vt; \epsilon/10, \epsilon/2) \, dx,
$$

which can be treated after integration in time by invoking (3.10).

Finally, we estimate $A_4$.

After integration by parts, we find

$$
A_4 = - \int H \partial_x^{l+3}u \partial_x^{l+1}u \chi(x + vt) \, dx
$$

$$
= \int H \partial_x^{l+2}u \partial_x^{l+2}u \chi(x + vt) \, dx + \int H \partial_x^{l+2}u \partial_x^{l+1}u \chi'(x + vt) \, dx
$$

$$
= A_{41} + A_{42}.
$$

Similar to (3.7), we write $A_{41}$ as

$$
A_{41} = \int H \partial_x^{l+2}u \partial_x^{l+2}u \chi(x + vt) \, dx
$$

$$
= - \int \partial_x^{l+2}u H (\partial_x^{l+2}u \chi(x + vt)) \, dx
$$

15
For the term

\[ \int \partial_t^{l+2} u \partial_t^{l+2} u \chi(x + vt) dx - \int \partial_t^{l+2} u \partial_t^{l+2} u dx \]

Consequently, there holds

\[ A_{41} = -\frac{1}{2} \int \partial_t^{l+2} u [H; \chi] \partial_t^{l+2} u dx \]

Recall \( \eta = \sqrt{\chi} \); therefore

\[ A_{42} = \int H \partial_t^{l+2} u \partial_t^{l+1} u \chi'(x + vt) dx \]

The H"older and Young inequality yield

\[ \int_0^T |A_{42}| dt \leq c \int_0^T \left( \int \partial_t^{l+1} u \right)^2 dx dt + c \int_0^T \left( \int \partial_t^{l+1} u \right)^2 dx dt. \quad (3.15) \]

Thus, we can handle this term by using a similar method as that in (3.8). Invoking (2.12), one finds

\[ A_{421} = \int H \partial_x (\partial_t^{l+1} u \eta) \partial_t^{l+1} u dx = \int \left[ D_x (\partial_t^{l+1} u \eta) \right]^2 dx. \]

For the term \( A_{421} \), one has

\[ A_{421} = \int H \partial_x (\partial_t^{l+1} u \eta) \partial_t^{l+1} u dx = \int \left[ D_x (\partial_t^{l+1} u \eta) \right]^2 dx. \]

The Hölder and Young inequality yield

\[ \int_0^T |A_{422}| dt \leq c \int_0^T \left( \int \partial_t^{l+1} u \right)^2 dx dt + c \int_0^T \left( \int \partial_t^{l+1} u \right)^2 dx dt. \]

Thus, we can handle this term by using a similar method as that in (3.8). Invoking (2.12), one finds

\[ A_{423} = \int [H; \eta] \partial_t^{l+2} u \partial_t^{l+2} u dx \]

\[ = \int \partial_x [H; \eta] \partial_t^{l+2} u \partial_t^{l+2} u dx + \int [H; \eta] \partial_t^{l+2} u \partial_t^{l+2} u dx \]

\[ \leq \| \partial_x [H; \eta] \partial_t^{l+2} u \|_L^2 \| \partial_t^{l+2} u \|_L^2 + \|[H; \eta] \partial_t^{l+2} u \|_L^2 \| \partial_t^{l+2} u \|_L^2 \]

\[ \leq c \| u \|_L^2 \| \partial_t^{l+2} u \|_L^2 + c \| \partial_t^{l+2} u \|_L^2, \]

where \( c \) is a constant depending on the specific problem.
which can also be controlled by using a similar way as that in (3.8).

As a consequence, substituting the above information into (3.11) and employing the Gronwall inequality, one deduces

\[
\sup_{t \in [0, T]} \int (\partial_x^{l+1} u)^2 \chi(x + vt; \epsilon, b) dx + \int_0^T \int (\partial_x^{l+2} u)^2 \chi'(x + vt; \epsilon, b) dx dt \\
+ \int_0^T \int [D_x^2 (\partial_x^{l+1} u)]^2 dx dt \leq c_0.
\]

(3.16)

This close our induction.

To justify the previous formal computations we refer the reader to [12] and we omit the details here.

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

We first prove (1.12) for any \( n \in \mathbb{Z}^+ \).

Note that \( x_n u_0 \in L^2(\mathbb{R}) \) implies \( \chi_n(x + vt; \epsilon, b) u_0 \in L^2(\mathbb{R}) \).

Multiplying equation (1.1) with \( u(x, t) \chi_n(x + vt; \epsilon, b) \) and integrating, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int u^2(x, t) \chi_n(x + vt; \epsilon, b) dx - \frac{1}{2} \int u^2(x, t) \chi'_n(x + vt; \epsilon, b) dx \\
+ \frac{3}{2} \int (\partial_x u)^2(x, t) \chi'_n(x + vt; \epsilon, b) dx - \frac{1}{2} \int u^2(x, t) \chi''_n(x + vt; \epsilon, b) dx \\
+ \int u \partial_x uu(x, t) \chi_n(x + vt; \epsilon, b) dx - \int H \partial_x^2 uu(x, t) \chi_n(x + vt; \epsilon, b) dx = 0.
\]

(4.1)

Employing (2.8) with \( j = 1 \), one easily deduces

\[
|A_1(t)| \leq |v| \int u^2 \chi_n(x + vt; \epsilon, b) dx + |v| c(n, b) \int u^2 dx \\
\leq |v| \int u^2 \chi_n(x + vt; \epsilon, b) dx + |v| c(n, b) \|u_0\|_{L^2}^2.
\]

Again, invoking (2.8) with \( j = 3 \), we derive

\[
|A_2(t)| \leq c(n, b) \int u^2 dx + \int u^2 \chi_n(x + vt; \epsilon, b) dx \\
\leq c(n, b) \|u_0\|_{L^2}^2 + \int u^2 \chi_n(x + vt; \epsilon, b) dx.
\]

We estimate \( A_3 \) as

\[
|A_3(t)| \leq \|\partial_x u\|_{L^\infty} \int u^2 \chi_n(x + vt; \epsilon, b) dx.
\]

17
For $A_4$, integration by parts produces

$$A_4 = - \int H \partial_x^2 uu \chi_n(x + vt; \epsilon, b) dx$$

$$= \int H \partial_x u \partial_x u \chi_n(x + vt; \epsilon, b) dx + \int H \partial_x uu \chi_n'(x + vt; \epsilon, b) dx$$

$$= A_{41} + A_{42}.$$

Reasoning as (3.7), it holds that

$$A_{41} = \int H \partial_x u \partial_x u \chi_n(x + vt; \epsilon, b) dx$$

$$= - \int \partial_x u H (\partial_x u \chi_n(x + vt; \epsilon, b)) dx$$

$$= - \int \partial_x u H \partial_x u \chi_n(x + vt; \epsilon, b) dx - \int \partial_x u[H; \chi_n] \partial_x u dx$$

$$= - A_{41} - \int \partial_x u[H; \chi_n] \partial_x u dx.$$

As a result, we find

$$A_{41} = - \frac{1}{2} \int \partial_x u[H; \chi_n] \partial_x u dx$$

$$= \frac{1}{2} \int u \partial_x [H; \chi_n] \partial_x u dx$$

$$\leq c||u||_{L^2} || \partial_x [H; \chi_n] \partial_x u||_{L^2}$$

$$\leq c||u||_{L^2}^2 = c||u_0||_{L^2}^2.$$

Integration by parts and (2.11) lead to

$$A_{42} = \int H \partial_x uu \chi_n'(x + vt) dx$$

$$= \int H \partial_x u \eta_n \eta_n dx$$

$$= \int (\partial_x u \eta_n) \eta_n dx - \int [H; \eta_n] \partial_x uu \eta_n dx$$

$$= \int H \partial_x (u \eta_n) \eta_n dx - \int H (u \eta_n') \eta_n dx - \int [H; \eta_n] \partial_x uu \eta_n dx$$

$$= A_{421} + A_{422} + A_{423}. \quad (4.2)$$

Again, using the Plancherel theorem, we write $A_{421}$ as

$$A_{421} = \int H \partial_x (u \eta_n) \eta_n dx = \int |D_x^2 (u \eta_n)|^2 dx.$$

For the term $A_{422}$, note that because of the factor $x^n$, the support of $\chi_n'$ is not $[\epsilon, b]$ at all for the general case. As a consequence, $\eta_n$ and $\eta_n'$ may be unbounded. However, we notice that (2.9) and (2.10) provide us with a relation between $\chi_n$ and $\chi_{n-1}$, therefore, we could use induction to close our proof.

Let us first consider the case $n = 0$ and thus $\eta_0 = \eta$.
At this point, we derive

\[ A_{422} = - \int H(u\eta')u\eta dx \leq c\|u\eta'\|_{L^2}\|u\eta\|_{L^2} \leq c\|u\|_{L^2}^2 \leq c\|u_0\|_{L^2}^2. \]

Invoking (2.12), we find

\[ A_{423} = - \int [H;\eta] \partial_x u\eta dx \leq \|[H;\eta] \partial_x u\|_{L^2}\|u\eta\|_{L^2} \leq c\|u\|_{L^2}^2 \leq c\|u_0\|_{L^2}^2. \]

Hence, we obtain the following inequality when \( n = 0 \):

\[
\begin{align*}
\sup_{t \in [0,T]} & \int u^2(x,t)\chi(x+vt;\epsilon,b)dx + \int_0^T \int (\partial_x u)^2(x,t)\chi'(x+vt;\epsilon,b)dx dt \\
+ & \int_0^T [D^2_x(u(x,t)\eta(x+vt;\epsilon,b))]^2 dx dt \leq c_2.
\end{align*}
\]

Let us assume the case \( n \geq 0 \) holds, i.e.,

\[
\begin{align*}
\sup_{t \in [0,T]} & \int u^2(x,t)\chi_n(x+vt;\epsilon,b)dx + \int_0^T \int (\partial_x u)^2(x,t)\chi_n'(x+vt;\epsilon,b)dx dt \\
+ & \int_0^T [D^2_x(u(x,t)\eta_n(x+vt;\epsilon,b))]^2 dx dt \leq c_2.
\end{align*}
\]

We shall prove the case \( n + 1 \).

We only need to treat the terms \( A_{422} \) and \( A_{423} \) with \( n + 1 \) instead of \( n \).

Employing (2.9) and (2.10), we find

\[
A_{422} = - \int H(u\eta_{n+1}')u\eta_{n+1} dx \\
\leq c_2\|u\eta_{n+1}'\|_{L^2}\|u\eta_{n+1}\|_{L^2} \\
\leq c_2 \int u^2(\eta_{n+1}')^2 dx + c_2 \int u^2(\eta_{n+1})^2 dx \\
\leq c_2 \int u^2\chi_n(x+vt;\epsilon/3,b+\epsilon)dx,
\]

which can be handled by using (4.3) with \((\epsilon/3, b + \epsilon)\) instead of \((\epsilon, b)\).

The term \( A_{423} \) can be controlled similarly.

Thus we completes the proof of (1.12). And for convenience, we view (4.3) as a conclusion in the following of this paper.

Next, we prove (1.13).

We first prove the case \( n = 1 \).

From (4.3) with \( n = 1 \) and (2.2), it follows that for any \( \delta > 0 \) there exists \( \hat{t} \in (0, \delta) \) such that

\[ \int (\partial_x u)^2(x,\hat{t})\chi(x;\epsilon,b)dx < \infty. \]

A smooth solution \( u \) to the IVP (1.1) satisfies the following identity:

\[
\frac{1}{2} \frac{d}{dt} \int (\partial_x u)^2(x,t)\chi(x+vt;\epsilon,b)dx - \frac{1}{2} \int (\partial_x u)^2(x,t)\chi'(x+vt;\epsilon,b)dx \\
\leq \int (\partial_x u)^2(x,t)\chi(x+vt;\epsilon,b)dx
\]

\[
A_1
\]
By Sobolev embedding theorem, one obtains

\[
\frac{3}{2} \int (\partial^2_x u)^2(x, t) \chi'(x + vt; \epsilon, b) dx - \frac{1}{2} \int (\partial_x u)^2(x, t) \chi''(x + vt; \epsilon, b) dx
\]

\[
+ \int \partial_x(u \partial_x u) \partial_x u(x, t) \chi(x + vt; \epsilon, b) dx - \int H \partial^3_x u(x, t) \partial_x u \chi(x + vt; \epsilon, b) dx = 0. \tag{4.4}
\]

Using (4.3) with \( n = 0 \), one obtains

\[
\int_0^T |A_1(t)| dt \leq c_3.
\]

Again, using (4.3) with \( n = 0 \) and \( (\epsilon/3, b + \epsilon) \) instead of \( (\epsilon, b) \), there holds

\[
\int_0^T |A_2(t)| dt \leq c_3.
\]

For the term \( A_3 \), integration by parts leads to

\[
A_3(t) = \frac{1}{2} \int (\partial_x u)^3 \chi(x + vt; \epsilon, b) dx - \frac{1}{2} \int u \partial_x u \partial_x \chi'(x + vt; \epsilon, b) dx
\]

\[
\leq \| \partial_x u \|_{L^\infty} \int (\partial_x u)^2 \chi(x + vt; \epsilon, b) dx + \| u \|_{L^\infty} \int (\partial_x u)^2 \chi'(x + vt; \epsilon, b) dx
\]

\[
= A_{31} + A_{32}.
\]

By Sobolev embedding theorem, one obtains

\[
\int_0^T |A_{32}(t)| dt < \sup_{t \in [\bar{t}, T]} \| u \|_{H^{3/4+}} \int_0^T (\partial_x u)^2 \chi'(x + vt; \epsilon, b) dx dt.
\]

The term \( A_{31} \) will be controlled by using (1.6)(ii) and the Gronwall inequality.

The term \( A_4 \) can be estimated as in the proof of Theorem 1.1, we omit it.

Substituting the above information in (4.4), using Gronwall inequality and (1.6)(ii), one obtains

\[
\sup_{t \in [\bar{t}, T]} \int (\partial_x u)^2(x, t) \chi(x + vt; \epsilon, b) dx + \int_0^T (\partial^2_x u)^2(x, t) \chi'(x + vt; \epsilon, b) dx dt
\]

\[
+ \int_0^T (D^2_x \partial_x u(x, t) \eta(x + vt; \epsilon, b))^2 dx dt \leq c_3. \tag{4.5}
\]

Next, we turn to the case \( n = 2 \) in the proof of (1.13).

Since \( x + u_0 \in L^2(\mathbb{R}) \), using (4.3) with \( n = 2 \), one finds

\[
\sup_{t \in [\bar{t}, T]} \int u_2^2(x, t) \chi_2(x + vt; \epsilon, b) dx + \int_0^T (\partial_x u)^2(x, t) \chi_2'(x + vt; \epsilon, b) dx dt
\]

\[
+ \int_0^T (D^2_x u(x, t) \eta_2(x + vt; \epsilon, b))^2 dx dt \leq c_2. \tag{4.6}
\]

Using (4.6) and (2.7), we derive that for any \( \delta > 0 \) there exists \( \bar{t} \in (0, \delta) \) such that

\[
\int (\partial_x u)^2(x, \bar{t}) \chi_1(x; \epsilon, b) dx < \infty.
\]

20
Consider the following identity

\[
\frac{1}{2} \frac{d}{dt} \int (\partial_x u)^2 \chi_1(x + vt; \epsilon, b) dx - \frac{1}{2} v \int (\partial_x u)^2 (x, t) \chi_1'(x + vt; \epsilon, b) dx \\
+ \frac{3}{2} \int (\partial^2_x u)^2 (x, t) \chi_1'(x + vt; \epsilon, b) dx - \frac{1}{2} \int (\partial_x u)^2 (x, t) \chi'_1(x + vt; \epsilon, b) dx \\
+ \int \partial_x (u \partial_x u) \partial_x u \chi_1(x + vt; \epsilon, b) dx - \int H \partial^3_x u \partial_x u (x, t) \chi_1(x + vt; \epsilon, b) dx = 0.
\]

Invoking (4.3) with \( n = 1 \), it holds that

\[
\int_t^T |A_1(t)| dt \leq |v| \int_t^T (\partial_x u)^2 (x, t) \chi_1'(x + vt; \epsilon, b) dx dt \leq c_3.
\]

For the term \( A_2 \), employing the fact that \( \chi''_1 \) is supported in \( [\epsilon, b] \), we deduce

\[
\int_t^T |A_2(t)| dt \leq c_3,
\]

where we have used (1.6)(iii) with \( r = 0 \).

Concerning the term \( A_3 \), integration by parts leads to

\[
A_3(t) = \frac{1}{2} \int (\partial_x u)^3 \chi_1(x + vt; \epsilon, b) dx - \frac{1}{2} \int u \partial_x u \partial_x u \chi_1(x + vt; \epsilon, b) dx \\
\leq \|\partial_x u\|_{L^\infty} \int (\partial_x u)^2 \chi_1(x + vt; \epsilon, b) dx + \|u\|_{L^\infty} \int (\partial_x u)^2 \chi_1'(x + vt; \epsilon, b) dx \\
= A_{31} + A_{32},
\]

which can treated as in the proof of Theorem 1.1.

Finally, we control \( A_4 \).

Integration by parts yields

\[
A_4 = - \int H \partial^3_x u \partial_x u \chi_1(x + vt; \epsilon, b) dx \\
= \int H \partial^2_x u \partial_x u \chi_1(x + vt; \epsilon, b) dx + \int H \partial^2_x u \partial_x \chi_1(x + vt; \epsilon, b) dx \\
= A_{41} + A_{42}.
\]

The term \( A_{41} \) can be handled by using integration by parts, the skew symmetry of the Hilbert transform and the commutator estimate (2.12), we omit it.

Now, we focus on the term \( A_{42} \). Recall \( (\eta_1)^2 = \chi'_1 \), one has

\[
A_{42} = \int H \partial^2_x u \partial_x u \chi_1'(x + vt) dx \\
= \int H \partial^2_x u \eta_1 \partial_x \eta_1 dx
\]

\[
\text{(4.7)}
\]
\[
\int H(\partial^2_x u_\eta)\partial_x u_\eta dx - \int [H; \eta] \partial^2_x u \partial_x u_\eta dx
\]
\[
\int H\partial_x(\partial_x u_\eta)\partial_x u_\eta dx - \int H(\partial_x u_\eta')\partial_x u_\eta dx - \int [H; \eta] \partial^2_x u \partial_x u_\eta dx
\]
\[
= A_{421} + A_{422} + A_{423}.
\]
For the term \(A_{421}\), we have
\[
A_{421} = \int H\partial_x(\partial_x u_\eta)\partial_x u_\eta dx = \int |D\frac{1}{2}(\partial_x u_\eta)|^2 dx.
\]
The Young inequality leads to
\[
\int_i^T |A_{422}| dt = \int_i^T \left| \int H(\partial_x u_\eta')\partial_x u_\eta dx \right| dt
\]
\[
\leq c_3 \int_i^T (\partial_x u)^2(\eta_1')^2 dx dt + c_3 \int_i^T (\partial_x u)^2\eta_1^2 dx dt.
\]
Note that \(\eta_1\) in unbounded in support of \(\chi_1'\). However, invoking (4.3) with \(n = 1\), we find
\[
\int_i^T (\partial_x u)^2\eta_1 dx dt \leq c_3.
\]
Now, simple computation yields \(\chi(x; \epsilon, b) \leq \chi_1'(x; \epsilon, b)\). This fact combining with (2.10) and (4.3) with \((\epsilon/3, b + \epsilon)\) instead of \((\epsilon, b)\) permits us to conclude
\[
\int_i^T (\partial_x u)^2(\eta_1')^2 dx dt \leq c_3.
\]
Thus we have controlled \(A_{422}\) after integration in time. The term \(A_{423}\) can be handled by using the above method and (2.12), we omit it.

As a result, we conclude after invoking the Gronwall inequality that
\[
\sup_{t \in [\hat{t}, T]} \int (\partial_x u)^2(x, t)\chi(x + vt; \epsilon, b) dx + \int_i^T (\partial^2_x u)^2(x, t)\chi_1'(x + vt; \epsilon, b) dx dt
\]
\[
+ \int_i^T |D\frac{\chi}{\chi}(\partial_x u(x, t)\eta_1(x + vt; \epsilon, b))|^2 dx dt \leq c_3.
\]  
(4.8)

By (4.8) for any \(\delta > 0\) there exists \(\hat{t} \in (\hat{t}, \delta)\) such that
\[
\int (\partial^2_x u)^2(x, \hat{t})\chi_1(x; \epsilon, b) dx < \infty,
\]
this produces
\[
\int (\partial^2_x u)^2(x, \hat{t})\chi(x; \epsilon, b) dx < \infty.
\]

Hence, the result of propagation of regularity (3.9) yields:
\[
\sup_{t \in [\delta, T]} \int (\partial^2_x u)^2(x, t)\chi(x + vt; \epsilon, b) dx + \int_\delta^T (\partial^3_x u)^2(x, t)\chi'(x + vt; \epsilon, b) dx dt
\]
\[
+ \int_\delta^T |D\frac{\chi}{\chi}(\partial_x^2 u(x, t)\eta(x + vt; \epsilon, b))|^2 dx dt \leq c_3.
\]
This completes the proof of the case \( n = 2 \).

For the general case, we use induction.

Given \((m, l) \in \mathbb{Z}^+ \times \mathbb{Z}^+\) we say that

\[
(m, l) > (\hat{m}, \hat{l}) \iff \begin{cases} (i) & m > \hat{m} \\ or \\ (ii) & m = \hat{m} \quad \text{and} \quad l > \hat{l}. \end{cases}
\]

(4.9)

Similarly, we say that \((m, l) \geq (\hat{m}, \hat{l})\) if \((ii)\) in the right hand side of (4.9) holds with \(\geq\) instead of \(>\).

The general case \((m, l)\) reads:

For any \(\epsilon, b, v > 0\)

\[
\sup_{t \in [\delta, T]} \int (\partial_t^l u)^2(x, t)\chi_m(x + vt; \epsilon, b)dx + \int_\delta^T \int (\partial_{x}^{l+1} u)^2(x, t)\chi_m(x + vt; \epsilon, b)dxdt \\
+ \int_\delta^T \int [D_x^2(\partial_t^l u(x, t)\eta_m(x + vt; \epsilon, b))]^2dxdt \leq c_3.
\]

(4.10)

Notice that we have already proved the following cases:

1. \((0, 1)\) and \((1, 0)\)
2. \((0, 2)\) \((1, 1)\) and \((2, 0)\)
3. Under the hypothesis \(x^n_{+}/2u_0 \in L^2(\mathbb{R})\), we proved (4.3), i.e. \((n, 0)\) for all \(n \in \mathbb{Z}^+\)
4. By Theorem 1.1 (propagation of regularity): If (4.10) holds with \((m, l) = (1, l)\) \((\delta/2\) instead of \(\delta)\), then there exists \(\tilde{t} \in (\delta/2, \delta)\) such that

\[
\int (\partial_t^{l+1} u)^2\chi_1'(x + \tilde{t}; \epsilon, b)dx < \infty
\]

which implies that

\[
\int (\partial_t^{l+1} u)^2\chi(x + \tilde{t}; \epsilon, b)dx < \infty.
\]

By the propagation of regularity (Theorem 1.1), one has the result (4.10) with \((m, l) = (0, l + 1)\), that is, \((1, l)\) implies \((0, l + 1)\) for any \(l \in \mathbb{Z}^+\).

Now we assume (4.10) holds for \((m, k)\) such that

\[
\begin{cases} (a) & (m, k) \leq (n - j, j) \quad \text{for some} \quad j = 0, 1, 2, \ldots, n \\ \text{and} \\ (b) & (m, k) = (n + 1, 0), (n, 1), \ldots, (n + 1 - l, l) \quad \text{for some} \quad l \leq n. \end{cases}
\]

We need to prove the case \((n + 1 - (l + 1), l + 1) = (n - l, l + 1)\). From (4) above, since \((1, l)\) implies \((0, l + 1)\), this case is already true for \(l = n\). Thus it suffices to consider \(l \leq n - 1\).

From the previous step \((n - l + 1, l)\) we have that for any \(\delta, v, \epsilon > 0,\)
Simple computation yields

\[ \chi'_{n+1-l}(x; \epsilon, b) \geq c \chi_{n-l}(x; \epsilon, b). \]

According to (4.11), there exists \( \hat{t} \in (\delta', 2\delta') \) such that

\[ \int (D_x^{l+1}u)^2(x, \hat{t})\chi_{n-l}(x + vt; \epsilon, b)dx < \infty. \]

For smooth solution of equation (1.1), consider

\[
\frac{1}{2} \frac{d}{dt} \int (D_x^{l+1}u)^2(x, t)\chi_{n-l}(x + vt; \epsilon, b)dx - \frac{1}{2} \int (D_x^{l+1}u)^2(x, t)\chi'_{n-l}(x + vt; \epsilon, b)dx \\
+ \frac{3}{2} \int (D_x^{l+2}u)^2(x, t)\chi''_{n-l}(x + vt; \epsilon, b)dx - \frac{1}{2} \int (D_x^{l+1}u)^2(x, t)\chi'''_{n-l}(x + vt; \epsilon, b)dx \\
+ \int D_x^{l+1}(uD_x u)D_x^{l+1}u(x, t)\chi_{n-l}(x + vt; \epsilon, b)dx \\
- \int H D_x^{l+3}uD_x^{l+1}u(x, t)\chi_{n-l}(x + vt; \epsilon, b)dx = 0. \tag{4.12}
\]

From the previous step \((n - l, l)\), we derive

\[
\int_{\hat{t}}^T |A_1(t)|dt \leq |v| \int_{\hat{t}}^T (D_x^{l+1}u)^2(x, t)\chi_{n-l}(x + vt; \epsilon, b)dxdt \leq c_3.
\]

Invoking (2.6), one obtains

\[
|A_2(t)| \leq c_3 \int (D_x^{l+1}u)^2(x, t)\chi_{n-l-3}(x + vt; \epsilon, b)dx \\
+ c_3 \int (D_x^{l+1}u)^2(x, t)\chi(x + vt; \epsilon/10, \epsilon/2)dx. \tag{4.13}
\]

According to previous steps \((n - l - 3, l + 1)\) and \((0, l + 1)\), we know that (4.13) is bounded.

Notice that the step \((0, l + 1)\) is implied by the step \((1, l) = (l + 1 - l, l) \leq (n - l, l)\).

For the term \(A_3\), Leibniz formula leads to

\[
A_3 = d_0 \int uD_x^{l+2}u D_x^{l+1}u\chi_{n-l}(x + vt; \epsilon, b)dx + d_1 \int \partial_x u (D_x^{l+1}u)^2 \chi_{n-l}(x + vt; \epsilon, b)dx
\]
After integration by parts, we deduce

\[
A_l = d_2 \int \partial_x^2 u \partial_x^{l+1} u \chi_{n-l}(x + vt; \epsilon, b) dx + \sum_{j=3}^{l-1} \int \partial_x^2 u \partial_x^{l+2-j} u \partial_x^{j+1} u \chi_{n-l}(x + vt; \epsilon, b) dx
\]

\[
= A_{3.0} + A_{3.1} + A_{3.2} + \sum_{j=3}^{l-1} A_{3.j}
\]

After integration by parts, we deduce

\[
A_{3.0} = -\frac{d_0}{2} \int \partial_x u (\partial_x^{l+1} u)^2 \chi_{n-l}(x + vt; \epsilon, b) dx
\]

\[
- \frac{d_0}{2} \int u (\partial_x^{l+1} u)^2 \chi_{n-l}(x + vt; \epsilon, b) dx
\]

\[
= A_{3.01} + A_{3.02}
\]

with \(A_{3.01}\) be similar to \(A_{3.1}\).

Sobolev embedding yields

\[
\int_\ell^T |A_{3.02}(t)| dt \leq \sup_{t \in [\ell, T]} \|u\|_{L^\infty} \int_\ell^T (\partial_x^{l+1} u)^2 \chi_{n-l}(x + vt; \epsilon, b) dx dt
\]

\[
\leq \sup_{t \in [\ell, T]} \|u\|_{H^{3/4}} \int_\ell^T (\partial_x^{l+1} u)^2 \chi_{n-l}(x + vt; \epsilon, b) dx dt,
\]

where the last integral corresponds to the case \((n - l, l)\), which is part of our hypothesis of induction.

For the term \(A_{3.1}\), we have

\[
|A_{3.1}| \leq c_3 \|\partial_x u\|_{L^\infty} \int (\partial_x^{l+1} u)^2 \chi_{n-l}(x + vt; \epsilon, b) dx
\]

which can be handled by the Gronwall inequality.

Consider now \(A_{3.2}\) which appears only if \(l \geq 2\) (we recall that \(n \geq 3\) to be proved \((n-l,l+1)\))

\[
A_{3.2} = d_2 \int \partial_x^2 u \partial_x^l u \partial_x^{l+1} u \chi_{n-l}(x + vt; \epsilon, b) dx
\]

(4.15)

Following the idea in [12], we study two cases: \(l = 2\) and \(l \geq 3\).

We first consider \(l = 2\).

Similar to the estimates of (3.12)-(3.14) in the previous section, one derives

\[
|A_{3.2}| = \left| -\frac{d_2}{3} \int \partial_x^2 u \partial_x^2 u \partial_x^2 u \chi_{n-l}(x + vt; \epsilon, b) dx \right|
\]

\[
\leq c \|\partial_x^2 u\|_L^{\infty} \chi_{n-l}(x + vt; \epsilon, b) \|L^\infty \int (\partial_x^2 u)^2 \chi(x + vt; \epsilon/10, \epsilon/2) dx
\]

\[
\leq c_3 \|\partial_x^2 u\|_L^{n-l} \chi_{n-l}(x + vt; \epsilon, b) \|L^\infty + c_3
\]

\[
\leq c_3 \int |\partial_x [(\partial_x^2 u)^2 \chi_{n-l}(x + vt; \epsilon, b)]| dx + c_3
\]

\[
\leq c_3 \int |(\partial_x^2 u)^2 \chi_{n-l}(x + vt; \epsilon, b)| dx + c_3 \int |((\partial_x^2 u)^2 \chi_{n-l}(x + vt; \epsilon, b)| dx
\]

\[
+ c_3 \int |(\partial_x^2 u)^2 \chi_{n-l}(x + vt; \epsilon, b)| dx
\]

\[
= A_{3.21} + A_{3.22} + A_{3.23} + c_3
\]

(4.16)
From our induction hypothesis we know that \( A_{3,21} \) and \( A_{3,22} \) are bounded after integration in time, since \( A_{3,21} \) corresponds to the case \((n-l,1) = (n-2,1)\) and \( A_{3,22} \) corresponds to the case \((n-l,2) = (n-2,2)\).

Moreover, invoking (2.5), one derives

\[
|A_{3,23}| \leq c_3 \int (\partial_x^2 u)^2 \chi_{n-l-2}(x+vt; \epsilon, b)dx + c_3 \int (\partial_x^2 u)^2 \chi'(x+vt; \epsilon/3, b + \epsilon)dx
= A_{3,231} + A_{3,232}. \tag{4.17}
\]

From the induction cases \((n-l-2,2)\) and \((0,1)\), we deduce that \( A_{3,231} \) is bounded in time \( t \in [\bar{t}, T] \) and \( A_{3,232} \) can be controlled after integration in time.

This completes the proof of (4.15) in the case \( l = 2 \).

Next, we turn to the case \( l \geq 3 \).

Integration by parts leads to

\[
A_{3,2} = -d_2 \int \partial_x^2 u \partial_x^{l+1} u \partial_x^{l+1} \chi_{n-l}(x+vt; \epsilon, b)dx
+ \frac{d_2}{2} \int \partial_x^2 u (\partial_x^l u)^2 \chi_{n-l}(x+vt; \epsilon, b)dx \tag{4.18}
\]

For the integrals on the right hand side of (4.18), using (4.10) and reasoning as (4.16) produce

\[
|A_{3,2}| \leq c_3 \int |(\partial_x^2 u)^2 \chi_{n-l}(x+vt; \epsilon, b)|dx + c_3 \int |(\partial_x^2 u)^2 \chi_{n-l}(x+vt; \epsilon, b)|dx
+ c_3 \int |(\partial_x^2 u)^2 \chi'(x+vt; \epsilon, b)|dx + c_3 \int |(\partial_x^2 u)^2 \chi'(x+vt; \epsilon, b)|dx
+ c_3 \int |(\partial_x^2 u)^2 \chi_{n-l}(x+vt; \epsilon, b)|dx.
\]

Since \( l \geq 3 \), after integration in time, the first two and the fourth integrals correspond to the previous cases \((n-l,1)\) , \((n-l,2)\) and \((n-l,3)\), respectively, which are all implied in the case \((n-l, l)\). The third and fifth integrals can be treated using a similar way as (4.17), where the fifth integral corresponds to the case \((n-l-2,3)\) and \((0,2)\) after using (2.5). Note that the case \((0,2)\) is implied by the case \((1,1)\), which can be deduced from the previous case \( l = 1 \).

Therefore, we only need to consider the remainder terms in (4.14), i.e.,

\[
A_{3,j} = c_j \int \partial_x^j u \partial_x^{l+2-j} u \partial_x^{l+1} u \chi_{n-l}(x+vt; \epsilon, b)dx.
\]

Without loss of generality , we can assume \( 3 \leq j \leq l/2 + 1 \). Consequently, one finds

\[
|A_{3,j}| \leq c_j \int (\partial_x^j u \partial_x^{l+2-j} u)^2 \chi_{n-l}(x+vt; \epsilon, b)dx + c_j \int (\partial_x^{l+1} u)^2 \chi_{n-l}(x+vt; \epsilon, b)dx
\]

with the second integral be the quantity to be estimated.

For the first integral, we have

\[
c_j \int (\partial_x^j u \partial_x^{l+2-j} u)^2 \chi_{n-l}(x+vt; \epsilon, b)dx
\leq \|(\partial_x^j u)^2 \chi(x+vt; \epsilon/10, \epsilon/2)\|_{L^\infty} \int (\partial_x^{l+2-j} u)^2 \chi_{n-l}(x+vt; \epsilon, b)dx. \tag{4.19}
\]
From the induction hypothesis \((n-l, l+2-j)\) with \(j \geq 3\), we deduce that the integral in (4.19) is bounded. Thus it remains to control the \(L^\infty\) norm.

For this purpose, we employ the Sobolev inequality \(\|f\|_{L^\infty} \leq \|f\|_{H^{1,1}}\) to obtain

\[
\|(\partial^j_x u)^2 \chi(x + vt; \epsilon/10, \epsilon/2)\|_{L^\infty} \\
\leq \int |\partial_x [(\partial^j_x u)^2 \chi(x + vt; \epsilon/10, \epsilon/2)]| dx \\
\leq c \int \|\partial^j_x u \partial^{j+1}_x u \chi(x + vt; \epsilon/10, \epsilon/2)\| dx + c \int \|\partial^j_x u \chi'(x + vt; \epsilon/10, \epsilon/2)\| dx \\
\leq c \int \|\partial^j_x u \chi(x + vt; \epsilon/10, \epsilon/2)\| dx + c \int \|\partial^{j+1}_x u \chi(x + vt; \epsilon/10, \epsilon/2)\| dx \\
+ c \int \|\partial^j_x u \chi'(x + vt; \epsilon/10, \epsilon/2)\| dx.
\]

Since \(j \leq l - 1\), we have \(j + 1 \leq l \leq n\). Thus, previous cases \((0, j)\) and \((0, j + 1)\) imply the boundedness of the first two integrals, respectively. The third integral corresponds to the case \((0, j - 1)\) after integration in time.

Finally, we estimate \(A_4\). As before, we write

\[
A_4 = - \int H \partial^{j+3}_x u \partial^{j+1}_x u \chi_{n-l}(x + vt; \epsilon, b) dx \\
= \int H \partial^{j+2}_x u \partial^{j+2}_x u \chi_{n-l}(x + vt; \epsilon, b) dx + \int H \partial^{j+2}_x u \partial^{j+1}_x \chi_{n-l}(x + vt; \epsilon, b) dx \\
= A_{41} + A_{42}.
\]

The term \(A_{41}\) can be treated easily, we omit it.

For the term \(A_{42}\), one has

\[
A_{42} = \int H \partial^{j+2}_x u \partial^{j+1}_x \chi_{n-l}(x + vt) dx \\
= \int H \partial^{j+2}_x u \partial^{j+1}_x u \eta_{n-l} dx \\
= \int H(\partial^{j+2}_x u \eta_{n-l}) \partial^{j+1}_x u \eta_{n-l} dx - \int [H; \eta_{n-l}] \partial^{j+2}_x u \partial^{j+1}_x u \eta_{n-l} dx \\
= \int H \partial_x (\partial^{j+2}_x \eta_{n-l}) \partial^{j+1}_x u \eta_{n-l} dx - \int H(\partial^{j+2}_x \eta_{n-l}) \partial^{j+1}_x u \eta_{n-l} dx \\
= \int [H; \eta_{n-l}] \partial^{j+2}_x u \partial^{j+1}_x u \eta_{n-l} dx \\
= A_{421} + A_{422} + A_{423},
\]

where \(A_{421}\) is positive and will stay at the left hand side of (4.12).

(2.9) and (2.10) lead to

\[
|A_{422}| \leq \left| \int H(\partial^{j+1}_x \eta_{n-l}) \partial^{j+1}_x u \eta_{n-l} dx \right| \\
\leq c \int (\partial^{j+1}_x u)^2 (\eta_{n-l})^2 dx + c \int (\partial^{j+1}_x u \eta_{n-l})^2 dx \\
\leq c \int (\partial^{j+1}_x u)^2 \chi_{n-l-1}(x; \epsilon/3, b + c) dx,
\]

which can be handled by the previous step \((n-l-1, l+1)\) since \(l + 1 \leq n\).
The term $A_{421}$ can be handled similarly, we omit it.

This basically completes the proof of Theorem 1.2.

To justify the previous formal computation, we approximate the initial data $u_0$ by Schwartz functions, say $u_0^\mu$, $\mu > 0$, which can be satisfied by convolution $u_0$ with a family of mollifiers. Using the well-posedness in the class of Schwartz functions, we obtain a family of solutions $u^\mu(\cdot,t)$ for which each step of the above argument can be justified. From our construction those estimates are uniform in the parameter $\mu > 0$, which yields the desired estimate by passing to the limit.

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