LIMIT THEOREMS FOR NUMBER OF DIFFUSION PROCESSES WHICH DID NOT ABSORB BY BOUNDARIES.\textsuperscript{1}

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Abstract. We have random number of independent diffusion processes with absorption on boundaries in some region at initial time \( t = 0 \). The initial numbers and positions of processes in region is defined by Poisson random measure. It is required to estimate of number of the unabsorbed processes for the fixed time \( \tau > 0 \). The Poisson random measure depends on \( \tau \) and \( \tau \to \infty \).

Consider the set of independent random diffusion processes \( \xi_k(t) \), \( k = 1, N \), \( t \geq 0 \), \( \xi_k(0) = x_k \), \( x_k \in Q \subset R^d \). We wish to investigate of distribution of the number of the processes \( \xi_k(t) \) which was into \( Q \) for all moments of time \( t \leq \tau \).

Let domain \( Q \subset R^d \) be open connected region and it is limited by smooth surface \( \partial Q \). All processes \( \xi_k(t) \) are diffusion processes with absorption on the boundary \( \partial Q \). These processes are solutions of the following stochastic differential equations in \( Q \)

\[
d\xi(t) = a(t, \xi(t))dt + \sum_{i=1}^{d} b_i(t, \xi(t))dw_i^{(k)}(t) \quad \xi(t) \in R^d
\]

with an initial condition: \( \xi(0) = x_k \in D \).

Here the \( W^{(k)}(t) = (w_i^{(k)}(t), \quad 1 \leq i \leq d), \quad 1 \leq k \leq N \) are independent in totality \( d \)-dimensional Wiener processes.

Thus, these processes have the identical diffusion matrices and shift vectors, but they have different initial states.

Let \( Q \) is bounded and boundary \( \partial Q \) is Lyapunov surface \( C^{(1, \lambda)} \).

The initial number and positions of processes are defined by the random Poisson measure \( \mu(\cdot, \tau) \) in \( Q \):

\[
P(\mu(A, \tau) = k) = \frac{m^k(A, \tau)}{k!} e^{-m(A, \tau)}
\]

where \( m(\cdot, \tau) \) is finitely additive positive measure on \( Q \) for fixed \( \tau \).

This task was offered in [1] as the mathematical model of practice problem. The authors in article [2] investigated case when initial number and positions of diffusion

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processes are defined by determinate limited measure $N(B, \tau)$. Where the $N(B, \tau)$ is equal to number of points $x_k$ in a set $B$ and $N = N(Q, \tau) < \infty$ for fixed $\tau > 0$.

We consider the following case
\[ a(t, x) = a = (0, \ldots, 0), \quad b_i(t, x) = b_i = (b_{i1}, \ldots, b_{id}), \quad 1 \leq i \leq d; \]

We define matrix $\sigma = B^T B$, $B = (b_{ij})$, $1 \leq i, j \leq d$ and differential operator $A : \sum_{1 \leq i, j \leq d} \sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$.

Let $\sigma$ be a matrix with the following property
\[ \sum_{1 \leq i, j \leq d} \sigma_{ij} z_i z_j \geq \mu |z|^2. \]

Here $\mu$, there is fixed positive number, and $z = (z_1, \ldots, z_d)$ there is an arbitrary real vector.

This operator acts in the following space
\[ H_A = \{ u : u \in L_2(Q) \cap Au \in L_2(Q) \cap u(\partial Q) = 0 \} \]
with inner product $(u, v)_A = (Au, v).$ Here $(,)$ is inner product in $L_2(Q)$. The operator $A$ is positive operator.

It is known [3] that the following eigenvalues problem
\[ Au = -\lambda u, \quad u(\partial Q) = 0 \]
has infinity set of real eigenvalues $\lambda_i \to \infty$ and
\[ 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s < \cdots. \]

The corresponding eigenfunctions
\[ f_{11}, \ldots, f_{1n_1}, \ldots, f_{s1}, \ldots, f_{sn_s}, \ldots \]
form complete system of functions both in $H_A$ and $L_2(Q) := \{ u : u \in L_2(Q) \cap u(\partial Q) = 0 \}$. Here the number $n_k$ is equal to multiplicity of eigenvalue $\lambda_k$.

We denote by $\eta(\tau)$ the number of remaining processes in the region $D$ at time instant $\tau$.

We also assume that $\sigma$-additive measure $\nu$ is given on the $\Sigma_\nu$-algebra sets of $Q$, $\nu(Q) < \infty$. All eigenfunctions $f_{ij} : Q \to R^1$ and all measures $m(\cdot, \tau)$ are $(\Sigma_\nu, \Sigma_Y)$ measurable. Here $\Sigma_Y$ is system of Borel sets of $R^1$. Let $\Rightarrow$ denotes the weak convergence of random values.

Put
\[ g(\tau) = \exp \left( -\frac{\tau}{2} \lambda_1 \right). \]

**Theorem 1.** We suppose that $m(\cdot, \tau)$ satisfies the condition
\[ \lim_{\tau \to \infty} m(B, \tau) g(\tau) = \nu(B), \quad B \in \Sigma_\nu. \]

Then $\eta(\tau) \Rightarrow \eta$ if $\tau \to \infty$ where $\eta$ has Poisson distribution function with the parameter $a = \int_Q F(x) d\nu$ and $F(x) = \sum_{i=1}^{n_1} f_{1i}(x) c_{i1}$, $c_{i1} = \int_Q f_{1i}(\bar{x}) d\bar{x}$. 


Proof. We consider the following initial-boundary problem

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{1 \leq i, j \leq d} \sigma_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad x \in Q;
\]

\[
u(t, x) = \begin{cases} 1 & \text{if } x \in Q; \\ 0 & \text{if } x \in \partial Q, t \geq 0 \end{cases}
\]  

It is known \([4]\), that \(\nu(\tau, x)\) is equal to probability of remaining in the region \(Q\) at time instant \(\tau\) of a diffusion process from (1) which occurs at the point \((0, x)\) at the initial moment \((\xi(0) = x, x \in Q)\). We designate through \(\gamma_k = (x_{k1}, \ldots, x_{kd})\) the initial position of \(k\)-th process. We define the value of \(\nu(\tau, \gamma_k)\).

We define a particular solution of (2) in form

\[
u(t, x) = \nu_1(t) \nu_2(x).
\]

The ordinary argumentaion leads to definition of joined constant \(\lambda\):

\[
2 \frac{1}{\nu_1} \frac{\partial \nu_1}{\partial t} = \frac{A \nu_2}{\nu_2} = -\lambda.
\]

We obtain the following system of tasks due the latter one

\[
A \nu_2 = -\lambda \nu_2; \quad \nu_2(\partial Q) = 0,
\]

\[
\frac{\partial \nu_1}{\partial t} = \frac{\lambda}{2} \nu_1; \quad \nu_1(0) = 1
\]

It is clear that \(\nu_1(t, \lambda) = \exp(-\frac{t}{2} \lambda)\) is solution of (5). The solution of (3) was described above. We assume that system of functions \(\{f_{ij}(x), i \geq 1, 1 \leq j \leq n_i\}\) is orthonormalized with respect to space \(L_2^0(Q)\).

The general solution of problem (2) has the following form

\[
u(t, x) = \sum_{j=1}^{\infty} \exp(-\frac{t}{2} \lambda_j) \sum_{m=1}^{n_j} c_{jm} f_{jm}(x)
\]

where coefficients \(c_{jm}\) are equal to coefficients of decomposition of initial value (unit) by system of functions \(f_{jm}\): \(c_{jm} = \int_Q f_{jm}(x)dx\). The Parseval - Steklov equality is true for these coefficients:

\[
\sum_{j=1}^{\infty} \sum_{m=1}^{n_j} c_{jm}^2 = |Q|.
\]

Put \(F(x) = \sum_{m=1}^{n_1} c_{1m} f_{1m}(x)\). The function \(F(x)\) is continuous and bounded function on the \(\bar{Q}\). Since \(\nu(t, x)\) is probability, it is not difficult to show that \(F(x) \geq 0\) for all \(x \in Q\). Let \(M = \sup_{x \in Q} F(x)\). We introduce the following sets
Here \(0 \leq k \leq n - 1\) and \(n > 1\).

Let us denote by \(\zeta_{k,n}(\tau)\), \(1 \leq k \leq n\) the number of unabsorbed processes at time instant \(\tau\) which occur in the region \(B_{k,n}\) at the initial time. These values are independent in totality by assumption. The distribution function of \(\zeta_{k,n}(\tau)\) is defined by the following formula

\[
P(\zeta_{k,n}(\tau) = l) = \sum_{d=l}^{\infty} P(\mu(B_{k,n}, \tau) = d) \times 
\sum_{1 \leq i_1, \ldots, i_d \leq d, i_m \neq i_j, m \neq j} \prod_{k=1}^{d} u(\tau, \gamma_{i_k}) \prod_{s=l+1, i_s \notin (i_1, \ldots, i_d), i_m \neq i_j} (1 - u(\tau, \gamma_{i_s})), \quad l = 0, 1, \ldots.
\]

Here \(x_{ij} \in B_{k,n}\).

We set

\[
a_{k,n}(\tau) = \min_{x \in B_{k,n}} u(\tau, x), \quad \bar{a}_{k,n}(\tau) = 1 - a_{k,n}(\tau);
\]

\[
b_{k,n}(\tau) = \max_{x \in B_{k,n}} u(\tau, x), \quad \bar{b}_{k,n}(\tau) = 1 - b_{k,n}(\tau).
\]

Now

\[
J_{k,n}(l, \tau) := \sum_{d=l}^{\infty} \frac{m^d(B_{k,n}, \tau)}{d!} \exp(-m(B_{k,n}, \tau)) C_d^l a_{k,n}(\tau) \bar{b}_{k,n}^{d-l}(\tau) \leq \sum_{d=l}^{\infty} \frac{m^d(B_{k,n}, \tau)}{d!} \exp(-m(B_{k,n}, \tau)) C_d^l b_{k,n}(\tau) \bar{a}_{k,n}^{d-l}(\tau) =: I_{k,n}(l, \tau).
\]

Further

\[
J_{k,n}(l, \tau) = \frac{(m(B_{k,n}, \tau)a_{k,n}(\tau))^l}{l!} \exp(-m(B_{k,n}, \tau)) \sum_{d=l}^{\infty} \frac{(b_{k,n}(\tau)m(B_{k,n}, \tau))^{d-l}}{(d-l)!} = \frac{(m(B_{k,n}, \tau)a_{k,n}(\tau))^l}{l!} \exp(-b_{k,n}(\tau)m(B_{k,n}, \tau));
\]

By analogy:

\[
I_{k,n}(l, \tau) = \frac{(m(B_{k,n}, \tau)b_{k,n}(\tau))^l}{l!} \exp(-a_{k,n}(\tau)m(B_{k,n}, \tau)).
\]

We introduce the following generating functions

\[
\varphi(\tau, s) = \sum_{l \geq 0} s^l P(\eta(\tau) = l).
\]
\begin{align*}
\varphi_{k,n}(\tau, s) &= \sum_{l \geq 0} s^l P(\zeta_{k,n}(\tau) = l), \quad k = 0, n - 1, \quad 0 \leq s \leq 1.
\end{align*}

By the construction, \( \eta(\tau) \) can be represented in the form \( \eta(\tau) = \zeta_{1,n} + \cdots + \zeta_{n-1,n}(\tau) \).

Thus
\begin{align*}
\varphi(\tau, s) &= \prod_{k=0}^{n-1} \varphi_{k,n}(\tau, s). \tag{8}
\end{align*}

Combining (6)-(8), we conclude that
\begin{align*}
\exp\{ (sa_{k,n}(\tau) - b_{k,n}(\tau)) m(B_{k,n}, \tau) \} &\leq \varphi_{k,n}(\tau, s) \leq \\
&\leq \exp\{ (sb_{k,n}(\tau) - a_{k,n}(\tau)) m(B_{k,n}, \tau) \}
\end{align*}

and
\begin{align*}
\exp \left\{ \sum_{k=0}^{n-1} (sa_{k,n}(\tau) - b_{k,n}(\tau)) m(B_{k,n}, \tau) \right\} &\leq \varphi(\tau, s) \leq \\
&\leq \exp \left\{ \sum_{k=0}^{n-1} (sb_{k,n}(\tau) - a_{k,n}(\tau)) m(B_{k,n}, \tau) \right\}. \tag{9}
\end{align*}

Since function \( u(\tau, x) \) is continuous function in \( x \in Q \), there exit a points \( x_s, x^* \in B_{k,n} \) such that the following equalities have place
\begin{align*}
a_{k,n}(\tau) &= u_1(\tau, \lambda_1) F(x_s) + \sum_{k \geq 2} u_1(\tau, \lambda_k) \sum_{m=1}^{n_k} c_{km} f_{km}(x_s), \\
b_{k,n}(\tau) &= u_1(\tau, \lambda_1) F(x^*) + \sum_{k \geq 2} u_1(\tau, \lambda_k) \sum_{m=1}^{n_k} c_{km} f_{km}(x^*),
\end{align*}

here \( x_s := x_s(k, n, \tau), x^* := x^*(k, n, \tau) \).

Now, we can rewrite the sums in exponents from (9) in the following forms
\begin{align*}
\sum_{k=0}^{n-1} (sF(x_s) - F(x^*)) \exp\left(-\frac{\tau}{2}\lambda_1\right) m(B_{k,n}, \tau) + \\
\sum_{k=0}^{n-1} \exp\left(-\frac{\tau}{2}\lambda_1\right) m(B_{k,n}, \tau) \sum_{j \geq 2} \exp\left(-\frac{\tau}{2}(\lambda_j - \lambda_1)\right) \sum_{m=1}^{n_j} c_{jm} (sf_{jm}(x_s) - f_{jm}(x^*)), \tag{10}
\end{align*}

\begin{align*}
\sum_{k=0}^{n-1} (sF(x^*) - F(x_s)) \exp\left(-\frac{\tau}{2}\lambda_1\right) m(B_{k,n}, \tau) + \\
\sum_{k=0}^{n-1} \exp\left(-\frac{\tau}{2}\lambda_1\right) m(B_{k,n}, \tau) \sum_{j \geq 2} \exp\left(-\frac{\tau}{2}(\lambda_j - \lambda_1)\right) \sum_{m=1}^{n_j} c_{jm} (sf_{jm}(x^*) - f_{jm}(x_s)), \tag{11}
\end{align*}

We calculate limit of (10) if \( \tau \to \infty \). The first sum of (10) converges to the following limit under the condition of theorem
\[
\sum_{k=0}^{n-1} sF(x_\ast)\nu(B_{k,n}) - \sum_{k=0}^{n-1} F(x_\ast)\nu(B_{k,n}).
\]

This is difference of two integral sums which has the following limit under \( n \to \infty \) (see \([5]\))
\[
(s - 1) \int_Q F(x)\nu(dx).
\]

Put
\[
s_\tau(x) = \sum_{j \geq 2} \exp \left( -\frac{\tau}{2}(\lambda_j - \lambda_1) \right) \sum_{m=1}^{n_j} c_{km} f_{km}(x).
\]

We consider sums of eigenfunctions in the form
\[
e(x, \lambda) = \sum_{\lambda_j \leq \lambda} f_{jl}(x)
\]

The following result is proved in monography \([6, \text{Thm. 17.5.3}]\)
\[
\sup_{x \in Q} e(x, \lambda) \leq C\lambda^{\frac{d}{2}}.
\]

Asymptotic characteristic of eigenvalues \( \lambda_j \) under \( j \to \infty \) is defined by the following inequalities \([3, \text{sec. 18}]\)
\[
c_1 j^{\frac{d}{2}} \leq \lambda_j \leq c_2 j^{\frac{d}{2}}, \quad \text{where} \quad c_1, c_2 = \text{const}.
\]

The latter one, \((5)\) and Caushy-Bunyakovskii inequality lead to the following convergence under \( \tau \to \infty \)
\[
|s_\tau(x)| \leq \sum_{j \geq 2} \exp \left( -\frac{\tau}{2}(\lambda_j - \lambda_1) \right) \sqrt{\sum_{m=1}^{n_j} c_{jm}^2} \leq \sqrt{C} \sum_{j \geq 2} \lambda_j^{\frac{d}{2}} \exp \left( -\frac{\tau}{2}(\lambda_j - \lambda) \right) \sqrt{\sum_{m=1}^{n_j} c_{jm}^2} \leq \sqrt{C} \sum_{j \geq 2} \lambda_j^{\frac{d}{2}} \exp \left( -\tau(\lambda_j - \lambda_1) \right) \sqrt{\sum_{j \geq 2} \sum_{m=1}^{n_j} c_{jm}^2} \to 0.
\]

Thus the second sum from \((10)\) convergences to zero.

The similar considerations apply to \((11)\). Proof is complete.

**Example.** Now we apply the general approach to the particular case.

We consider the case if \( Q \) is circle \( Q = \{(x, y) : x^2 + y^2 \leq r_0^2\} \). We assume that the diffusion processes occurs at the point \((x_k, y_k) \in Q\) at the initial time.

The processes are described in \( Q \) by the following stochastic differential equations
\[ d\xi(t) = \sum_{i=1}^{2} b_i dw_i(t) \quad (12) \]
\[ \xi(0) = \xi_0 = (x_k, y_k). \]

where \( b_1 = (\sigma, 0), b_2 = (0, \sigma) \) and \( W(t) = (w_i(t), i = 1, 2) \) is 2-dimensional Wiener process.

We assume that the equation (12) defines a diffusion process with absorption on the boundary \( \partial Q = \{(x, y, z) : x^2 + y^2 = r_0^2 \} \).

It follows that the \( J_0(x), J_1(x) \) are Bessel functions zero and first order. They are defined as the solutions of the following equations

\[ \frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left( 1 - \frac{n^2}{x^2} \right) = 0, \]
\[ y(x_0) = 0, \quad (x_0 = \sqrt{\lambda r}); \quad |y(0)| < \infty; \]

under \( n = 0 \) and \( n = 1 \).

The value of \( \mu_0^{(0)} \) is equal to \( m \)-th root of the equation \( J_0(\mu) = 0 \) [7,8].

Let \( mes(\cdot) \) denotes the Lebesgue measure.

We set

\[ f(\tau) := \exp \left( -\frac{\tau}{2} \left( \frac{\sigma \mu_0^{(0)}}{r} \right)^2 \right). \]

We suppose that \( m(\cdot, \tau) \) holds the condition

\[ m(\cdot, \tau)f(\tau) \Rightarrow mes(\cdot) \quad \text{if} \quad \tau \to \infty. \]

In this case the system of tasks (3),(4) has the following form

\[ \triangle u_2 = -\mu u_2, \quad (x, y) \in C; \quad u_2(x, y) = 0 \quad \text{if} \quad x^2 + y^2 = r_0^2 \quad (13), \]
\[ \frac{\partial u_1}{\partial t} = -\frac{\sigma^2}{2} \mu u_1, \quad u_1(0) = 1. \quad (14) \]

According to general approach for construction of solution \( u(t, x, y) \) (see, for example, [7, sec.1V] we rewrite the task of (13) in polar coordinates: \( u_3(r, \varphi) := u_2(r \cos \varphi, r \sin \varphi). \)

The \( u_3 \) is solution the following problem

\[ \frac{\partial^2 u_3}{\partial r^2} + \frac{1}{r} \frac{\partial u_3}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_3}{\partial \varphi^2} + \mu u_3 = 0, \]
\[ u_3(r_0, \varphi) = 0. \]

We obtain

\[ u(t, x, y) = u(t, r) = \sum_{m=1}^{\infty} c_m J_0 \left( \frac{\mu_0^{(0)} m}{r_0} \right) \exp \left( -\frac{t}{2} \left( \frac{\sigma \mu_0^{(0)}}{r_0} \right)^2 \right), \]
where \( c_m = 2 \left( m_m^{(0)} J_1(\mu_m^{(0)}) \right)^{-1} \).

The function \( J_0\left( \frac{\mu}{\mu_0} r \right) \) is strictly decreasing function if \( 0 \leq r \leq r_0 \). Thus we can construct the partitions \( \tilde{B}_{k,n} \) by the following partitions

\[
\tilde{B}_{k,n} = \left\{ (x, y) \in C : \frac{r_0 k}{n} < \sqrt{x^2 + y^2} \leq \frac{r_0 (k + 1)}{n} \right\}, \quad 0 \leq i \leq n - 1.
\]

Now \( \text{mes}(\tilde{B}_{k,n}) = g\left( \frac{k+1}{n} \right) - g\left( \frac{k}{n} \right) \) where \( g(x) = \pi r_0^2 x \), \( 0 \leq x \leq 1 \).
Finally, the parameter of Poisson distribution is equal to

\[
a = 2 \left( m_1^{(0)} J_1(\mu_1^{(0)}) \right)^{-1} \int_0^1 J_0(\mu_1^{(0)}) x dx = \frac{2 \pi r_0^2}{\mu_1^{(0)}} \alpha J_0(\alpha) = [\alpha J_1(\alpha)]' \quad [7,p.466]
\]

for calculation of the latter integral.

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