Global Convergence of Deep Networks with One Wide Layer 
Followed by Pyramidal Topology

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Abstract

A recent line of research has provided convergence guarantees for gradient descent algorithms in the excessive over-parameterization regime where the widths of all the hidden layers are required to be polynomially large in the number of training samples. However, the widths of practical deep networks are often only large in the first layer(s) and then start to decrease towards the output layer. This raises an interesting open question whether similar results also hold under this empirically relevant setting. Existing theoretical insights suggest that the loss surface of this class of networks is well-behaved, but these results usually do not provide direct algorithmic guarantees for optimization. In this paper, we close the gap by showing that one wide layer followed by pyramidal deep network topology suffices for gradient descent to find a global minimum with a geometric rate. Our proof is based on a weak form of Polyak-Lojasiewicz inequality which holds for deep pyramidal networks in the manifold of full-rank weight matrices.

1. Introduction

It is well-known that training a neural network is, in the worst-case, NP-Hard (Blum & Rivest, 1992), and that the optimization problem can have exponentially many distinct local minima (Auer et al., 1996). Yet, in practice neural networks can be successfully trained with many more parameters than training samples. Indeed, gradient descent methods can find a globally optimal solution when using appropriate initialization schemes, such as Xavier’s initialization (Glorot & Bengio, 2010; He et al., 2015). We refer to (Chizat et al., 2019) for more details about the NTK regime, and to (Park et al., 2019) for a comparison between the empirical per-

Jacot et al. (2018) showed that, for the limiting case of infinitely wide neural networks, training can be described precisely by the so-called ‘Neural Tangent Kernel’ (NTK). Motivated by this, a series of recent papers studied the convergence of gradient descent for over-parameterized neural nets. Despite being conceptually interesting, most of the existing results are either restricted to one-hidden-layer networks (Du et al., 2019b; Brutzkus et al., 2018; Li & Liang, 2018; Wu et al., 2019; Oymak & Soltanolkotabi, 2019; Ji & Telgarsky, 2020; Daniely, 2019), or require all the hidden layers to be simultaneously large (Allen-Zhu et al., 2019; Du et al., 2019a; Zou et al., 2018; Zou & Gu, 2019). In contrast, neural networks used in practice are typically only wide at the first layer(s), after which the width starts to decrease toward the output layer, see e.g. Figure 1.

This leads to an interesting open theoretical question: Do similar results also hold for this kind of ‘pyramidal’ network topology? Into this direction, Nguyen (2019) showed that the loss landscape of these networks has connected sublevel sets, and consequently it is geometrically well behaved. However, no algorithmic guarantees are provided, which means that gradient descent can still get trapped in a bad local minimum (Safran & Shamir, 2018; Yun et al., 2019).

In this paper, we close the gap and answer positively the question above. We show that gradient descend converges to a global minimum at a geometric rate, when the network consists of one wide hidden layer followed by a pyramidal topology. Our analysis considers the standard parametrization (as opposed to the NTK parameterization), and it assumes the widely employed Xavier’s initialization (Glorot & Bengio, 2010; He et al., 2015). We refer to (Chizat et al., 2019) for more details about the NTK regime, and to (Park et al., 2019) for a comparison between the empirical per-
Table 1. Recent progress on convergence of gradient descent for training neural networks. \( \phi \) is the minimum Euclidean distance between any pair of data points; \( \lambda_0 \) is the smallest eigenvalue of a Gram matrix defined for 2-layer neural networks (Du et al., 2019b) which might depend on the number of samples \( N \); \( \lambda_{\text{min}}(R^{(L)}) \) is the smallest eigenvalue of a Gram matrix defined recursively for \( L \)-layer neural networks (Du et al., 2019a); \( \Gamma \) is the margin of a limiting kernel (Ji & Telgarsky, 2020) which has been shown to scale as \( 1/\text{poly}(N) \) for generic data (but can have a more favorable scaling under additional assumptions); \( R \) is defined in (Chen et al., 2019) as a constant such that the ball \( B(\theta_0, R/\sqrt{m}) \) contains a solution with training loss less than \( \epsilon \), where \( \theta_0 \) is the initial point and \( m \) is the number of neurons per layer in their notation. Depending on the data assumption, Chen et al. (2019) show that the per-layer width scales as \( N^{2L} \exp^{2\phi} \), or can have more favorable scaling as \( \text{polylog}(N, 1/\epsilon) \) under the similar separability assumption as in (Ji & Telgarsky, 2020). Most of the prior work assume that the data lies on the unit sphere. \( \lambda_* \) is defined in (13). In the last line of this table, we write \( a \lor b \lor c \) to denote \( \max(a, b, c) \).

|                     | Deep? | Multiple Outputs? | Loss Activation | Layer Width | Iteration complexity | Parameterization | Train All Layers? | # Wide Layers |
|---------------------|-------|-------------------|-----------------|-------------|----------------------|-----------------|-------------------|--------------|
| (Allen-Zhu et al., 2019) | Deep  | Yes               | General         | General     | \( N^{\frac{24 + 12}{9^2}} \) \( \frac{2^{O(L)}}{\lambda_{\text{min}}(R^{(L)})} \) \( \frac{1}{\epsilon} \) | Standard        | No               | All           |
| (Du et al., 2019a)   | Deep  | No                | Square          | Smooth      | \( N^{\frac{24 + 12}{9^2}} \) \( \frac{2^{O(L)}}{\lambda_{\text{min}}(R^{(L)})} \) \( \frac{1}{\epsilon} \) | NTK             | Yes              | All           |
| (Oymak & Soltanolkotabi, 2019) | Shallow | No                | Square          | Smooth      | \( N \parallel X \parallel^2 \) \( \frac{1}{\epsilon} \) | NTK             | No               | x             |
| (Zou & Gu, 2019)     | Deep  | No                | General         | ReLU        | \( N^{\frac{8^3}{9^2}} \) \( \frac{2^{O(L)}}{\lambda_{\text{min}}(R^{(L)})} \) \( \frac{1}{\epsilon} \) | NTK             | No               | x             |
| (Ji & Telgarsky, 2020) | Shallow | No                | Logistic        | ReLU        | \( \frac{\log(N)}{\epsilon^{2L} X^{1/4}} \) | NTK             | No               | x             |
| (Chen et al., 2019)   | Deep  | No                | Logistic        | ReLU        | \( \frac{\log(N)}{\epsilon^{2L} X^{1/4}} \) | NTK             | No               | x             |
| This paper           | Deep  | Yes               | Square          | Smooth      | \( \frac{N^\frac{24}{9^2}}{\lambda_*} \parallel X \parallel^2 \) \( \frac{1}{\epsilon} \) | Standard        | Yes              | One           |

In particular, the latter paper shows that the standard parameterization consistently outperforms the NTK one for several convolutional neural networks, especially those with a ‘bottom-heavy’ architecture, as analyzed in this paper. Our main results can be summarized as follows:

1. In Section 3.1, we show that gradient descent converges to a global optimum at a geometric rate for a class of pyramidal networks. More specifically, we provide a set of sufficient conditions on the initialization and the network topology that delivers such guarantees.

2. In Section 3.2, we specialize our results to the widely used Xavier’s initialization. In particular, we prove that the sufficient conditions for convergence are satisfied under a moderate over-parameterization condition: the width of the first layer needs to scale as

\[
\max\left(N, \frac{d \| X \|^2}{\lambda_*}, \frac{2^{cL} \| X \|^4}{d^2 \lambda_*^2}\right),
\]

where \( N \) is the number of samples, \( d \) the input dimension, \( L \) the number of layers, \( X \) the training data, \( \lambda_* \) the smallest eigenvalue of the expected feature output at the first layer, and \( c \) a constant. Let us emphasize that only the first layer is required to be wide. The other layers follow a pyramidal structure and can have a constant (i.e., independent of \( N \) and \( d \)) number of neurons. We also characterize the convergence rate, and we show that the number of iterations needed to achieve \( \epsilon \) training loss scales as

\[
\frac{2^{cL} \| X \|^3}{\lambda_* d^{3/2}} \log(1/\epsilon).
\]

3. In Section 3.3, we prove that \( \lambda_* > 0 \) under mild assumptions on the activation function and training data. Furthermore, for training data uniformly distributed on the sphere, we show that \( \lambda_* \) is lower bounded by a numerical constant (independent of \( N \) and \( d \)). Thus, in this setting, the width of the first layer scales as \( N^2 \), and the iteration complexity scales as \( N^{3/2} \log(1/\epsilon) \).

To clarify how the settings and results of our paper differ from previous works, we provide a comparison in Table 1: the key difference between this work and the prior work is that we consider an empirically relevant deep network topology with only one wide hidden layer. Our network parameterization and initialization are standard. Similarly to (Du et al., 2019a; Chen et al., 2019), we train all layers, while several other existing works train a subset of them. The recent papers (Chen et al., 2019; Ji & Telgarsky, 2020) require only a polylogarithmic layer width, but they make strong assumptions on the data separability and focus on other parameterizations (similar to the NTK (Jacot et al., 2018) or lazy regime (Chizat et al., 2019)) which are outperformed by the standard one with Xavier’s initialization.
2. Problem Setup

We consider an $L$-layer neural network with activation function $\sigma : \mathbb{R} \to \mathbb{R}$ and parameters $\theta = (W_l)_{l=1}^L$, where $W_l \in \mathbb{R}^{n_{l-1} \times n_l}$ is the weight matrix at layer $l$. Given $\theta_l = (W_l^\theta)_{l=1}^L$ and $\theta_l = (W_l^\theta)_{l=1}^L$, their $L_2$ distance is given by $||\theta_l - \theta_l||_2 = \sqrt{\sum_{l=1}^L \|W_l^\theta - W_l^\theta\|_F^2}$, where $\cdot \|F\|$ denotes the Frobenius norm. Let $X \in \mathbb{R}^{N \times d}$ and $Y \in \mathbb{R}^{N \times n_L}$ be respectively the training input and output, where $N$ is the number of training samples, $d$ is the input dimension and $n_L$ is the output dimension (for consistency, we set $n_0 = d$). Let $F_l \in \mathbb{R}^{n_L \times n_k}$ be the output of layer $l$, which is defined as

$$F_l = \begin{cases} X & l = 0, \\ \sigma(F_{l-1}W_l) & l \in [L-1], \\ F_{l-1}W_l & l = L, \end{cases}$$

where $[L-1] = \{1, \ldots, L-1\}$ and the activation function $\sigma$ is applied componentwise. Denote by $G_l \in \mathbb{R}^{n_L \times n_l}$ the pre-activation output, which is defined as $G_l = F_{l-1}W_k$ for $l \in [L-1]$ and $G_L = F_L$. Let $f_l = \text{vec}(F_l) \in \mathbb{R}^{Nn_l}$ and $y = \text{vec}(Y) \in \mathbb{R}^{Nn_L}$ be obtained by concatenating the columns of $F_l$ and $Y$, respectively.

We are interested in minimizing the square loss given by

$$\Phi(\theta) = \frac{1}{2} \|f_L(\theta) - y\|_2^2.$$ 

To do so, we consider the gradient descent (GD) update

$$\theta_{k+1} = \theta_k - \eta \nabla \Phi(\theta_k),$$

where $\eta$ is the step size and $\theta_k = (W_l^\theta)_{l=1}^L$ contains all parameters at step $k$.

In this paper, we consider neural networks with the following topological structure, see e.g. (Nguyen, 2019; Nguyen & Hein, 2017; 2018) and Figure 1.

**Assumption 2.1 (Pyramidal network topology)** Let $n_1 \geq N$ and $n_2 \geq n_3 \geq \ldots \geq n_L$.

We make the following assumptions on the activation $\sigma$.

**Assumption 2.2 (Activation function)** Fix $\gamma \in (0, 1)$ and $\beta > 0$. Let $\sigma$ satisfy that: (i) $\sigma'(x) \in [\gamma, 1]$, (ii) $|\sigma(x)| \leq |x|$ for every $x \in \mathbb{R}$, and (iii) $\sigma'$ is $\beta$-Lipschitz.

As a concrete example, we consider a family of parameterized ReLU functions, smoothed out by a Gaussian kernel:

$$\sigma(x) = -\frac{(1-\gamma)^2}{2\pi\beta} + \frac{\beta}{1 - \gamma} \int_{-\infty}^\infty \max(\gamma u, u) e^{-\frac{u^2}{(1-\gamma)^2}} du.$$  

Then, $\sigma$ fulfills the conditions of Assumption 2.2. Furthermore, $\sigma$ uniformly approximates the Leaky-ReLU function over the real line, i.e., $\lim_{\beta \to \infty, \beta \in \mathbb{R}} \sigma(x) = \max(\gamma x, x)$. These statements are proved in Lemma B.1 in the appendix.

Figure 2 shows several examples of $\sigma$ as a uniform approximation of $\max(0, 1.1x, x)$.

3. Main Results

3.1. General Framework

First, let us introduce some notation for the singular values of the weight matrices at initialization:

$$\tilde{\lambda}_l = \frac{2}{3} (1 + \|W_l^0\|_2), \quad l \in \{1, 2\},$$

$$\check{\lambda}_l = \|W_l^0\|_2, \quad l \in \{3, \ldots, L\},$$

$$\lambda_l = \sigma_{\min}(W_l^0), \quad l \in [L],$$

$$\lambda_{i \to j} = \prod_{l=i}^{j} \lambda_l, \quad \check{\lambda}_{i \to j} = \prod_{l=i}^{j} \check{\lambda}_l,$$

where $\sigma_{\min}(A)$ and $\|A\|_2$ are the smallest resp. largest singular value of the matrix $A$. We define

$$\lambda_F = \sigma_{\min}(\sigma(XW_1^0))$$

as the smallest singular value of the output of the first hidden layer at initialization.

We make the following assumptions on the initialization.

**Assumption 3.1 (Initial conditions)**

$$\lambda_F^2 \geq \frac{\gamma^4}{3} \left( \frac{6}{\gamma^2} \right)^L \|X\|_F \sqrt{2\Phi(\theta_0)} \frac{\bar{\lambda}_3 - \bar{\lambda}_L}{\bar{\lambda}_3 - \bar{\lambda}_L} \max_{\lambda \in \{3, \ldots, L\}} \frac{2\lambda_1 \lambda_2}{\min_{\lambda \in \{3, \ldots, L\}} \lambda_1 \lambda_2} \bar{\lambda}_1, \bar{\lambda}_2,$$ 

$$\lambda_2^3 \geq \frac{\gamma^2}{3} \left( \frac{6}{\gamma^2} \right)^L \|X\|_2 \|X\|_F \sqrt{2\Phi(\theta_0)} \frac{\bar{\lambda}_3 - \bar{\lambda}_L}{\bar{\lambda}_3 - \bar{\lambda}_L} \bar{\lambda}_2.$$ 

Note that (6) and (7) involve the number of neurons at all layers (and hence the full network topology). At this point, we are ready to present our most general result. Its proof is presented in Section 4.1.
Theorem 3.2 Let the network satisfy Assumption 2.1, the activation function satisfy Assumption 2.2 and the initial conditions satisfy Assumption 3.1. Define
\[
\alpha_0 = \frac{4}{\gamma^4} \left( \frac{2^2 \gamma^2}{4} \right)^L \lambda_1^2 \lambda_2^3 \cdots \lambda_L^L,
\]
\[
Q_0 = L \sqrt{L} \frac{3}{2} 2^{(L-1)} \|X\|_F^2 \frac{\lambda_1^2 \lambda_2 \cdots \lambda_L^L}{\min_i \{L \lambda_i^L \}}
\] (8) with \( R = \prod_{p=1}^{L} \max \left( 1, \frac{3}{2} \lambda_p \right) \). Let the learning rate be \( \eta < \min \left( \frac{1}{\alpha_0}, \frac{1}{Q_0} \right) \).

Then, the training loss vanishes at a geometric rate as
\[
\Phi(\theta_k) \leq (1 - \eta \alpha_0)^k \Phi(\theta_0).
\] (10)

Furthermore, define
\[
Q_1 = \frac{4}{3} \left( \frac{3}{2} \right)^L \|X\|_F^2 \alpha_0 \sum_{i=1}^{L} \frac{\lambda_1^L}{\lambda_i^i} \sqrt{2} \Phi(\theta_0).
\] (11)

Then, the network parameters converge to a global minimizer \( \theta_* \) at a geometric rate as
\[
\|\theta_k - \theta_*\|_2 \leq (1 - \eta \alpha_0)^{k/2} Q_1.
\] (12)

3.2. Xavier’s Initialization

Theorem 3.2 proves that gradient descent converges to a global minimum at a geometric rate under the rather general conditions (6)-(7). In this section, we specialize these results to the popular Xavier’s initialization. Let us define the following crucial quantity which appear in our bounds:
\[
G_* = E_{w \sim \mathcal{N}(0, \frac{1}{d^2}I_d)} \left[ \sigma(Xw)\sigma(Xw)^T \right], \quad \lambda_* = \lambda_{\min}(G_*).
\] (13)

As the entries of \( W_1 \in \mathbb{R}^{d \times n_1} \) are i.i.d. \( \mathcal{N}(0, 1/d) \), we have that \( n_1 G_* = E_{w \sim \mathcal{N}(0, 1/d)} \left[ \sigma(Xw)\sigma(Xw)^T \right] \). Thus, \( n_1 G_* \) can be interpreted as the expected Gram matrix with respect to the feature representation of the first layer.

Theorem 3.3 Let the activation function satisfy Assumption 2.2. Fix \( t > 0 \), \( t_0 \geq \sqrt{4d - 1} \ln \max \left( 1, 2\sqrt{d} \|X\|_2^2 \lambda_*^{-1} \right) \), and denote by \( c \) a large enough constant depending only on \( \gamma, \beta \). Let the widths of the neural network satisfy the following conditions:
\[
\sqrt{n_{l-1}} \geq \left( 1 + \frac{1}{100} \right) (\sqrt{n_l} + t), \quad \forall l \in \{2, \ldots, L\},
\] (14)
\[
n_1 \geq \max \left( N, \frac{2 \sqrt{2} \lambda_*}{\sqrt{d} \lambda_* \ln N} \right),
\]
\[
\frac{2cL \|X\|_F^2}{\bar{d} \lambda_*^2 \left( (\sqrt{n_{l-1}} + t) \|X\|_F + \|Y\|_F \right)^2}.
\] (15)

Assume that \( d \leq N \) and consider Xavier’s initialization:
\[
[W_i^0]_{ij} \sim \mathcal{N}(0, 1/n_{l-1}), \quad \forall l \in [L], i \in [n_{l-1}], j \in [n_i].
\]

Let the learning rate satisfy
\[
\eta < \left( \frac{2cL n_1}{d} \max(1, \|X\|_F^2) \right)^{-1} \cdot \max \left( 1, \left( \frac{\sqrt{n_{l-1}} + t}{\sqrt{d} \lambda_*} \right) (\sqrt{n_l} + t) \|X\|_F + \|Y\|_F \right)^{-1}.
\] (16)

Then, the training loss vanishes and the network parameters converge to a global minimizer \( \theta_* \) at a geometric rate as
\[
\Phi(\theta_k) \leq \left( 1 - \frac{\eta n_1 \lambda_*}{2cL} \right)^k \Phi(\theta_0),
\] (17)
\[
\|\theta_k - \theta_*\|_2 \leq \left( 1 - \frac{\eta n_1 \lambda_*}{2cL} \right)^{k/2} \cdot \left( \frac{2cL \|X\|_F}{\sqrt{n_1 d} \lambda_*} \right) \left( \frac{\sqrt{n_{l-1}} + t}{\sqrt{d} \lambda_*} \right) (\sqrt{n_l} + t) \|X\|_F + \|Y\|_F \right)^{-1},
\] (18)

with probability at least \( 1 - 3Le^{-t_0^2/2} - 2e^{-t_0^2/2} \).

Condition (14) provides a mild requirement on the speed at which the widths of the layers need to decrease. Condition (15) requires a lower bound on the size of the first layer, which is the widest. Let us emphasize that only the first layer needs to be over-parameterized: the widths of all the other layers can be set to constants that do not depend on the number of training samples or the input dimension.

Assume that (i) \( \|X\|_F \) is lower bounded by a constant, (ii) \( n_L \) is a constant and (iii) the entries of \( X \) are of the same order as the entries of \( Y \) (this implies that \( \|X\|_F / \sqrt{d} \lambda_* \) is of the same order as \( \|Y\|_F / \sqrt{d} \lambda_* \)). Then, the width \( n_1 \) of the first layer scales as
\[
\tilde{\Omega} \left( \max(N, \frac{d \|X\|_2^2}{\lambda_*}, \frac{2cL \|X\|_F^2}{d^2 \lambda_*^2}) \right),
\] (19)
Global Convergence of Deep Networks with One Wide Layer Followed by Pyramidal Topology

The family of activation functions (3) satisfies both Assumption 2.1 and, under the assumptions mentioned above, the number of iterations needed to achieve \( \epsilon \) training loss scales as

\[
\Omega \left( \frac{2^{pL} \|X\|_F^2}{\lambda_* \sigma_d^{3/2}} \log(1/\epsilon) \right). \tag{20}
\]

The proof of Theorem 3.3 is provided in Appendix C. The high level idea is as follows. First, we bound the output of the network at initialization using Gaussian concentration arguments. Then, by following an approach similar to that of Theorem 3.2 in (Oymak & Soltanolkotabi, 2019), we show that, with high probability, \( \lambda_F \geq \sqrt{n_1 \lambda_* / 4} \). Finally, we bound the quantities \( \lambda_l \) and \( \lambda_\ast \) using results on the singular values of random Gaussian matrices. By combining these bounds, we show that conditions (6)-(7) are satisfied. Then, we compute the quantities \( \alpha_0, Q_0 \) and \( Q_1 \) from Theorem 3.2 to get the corresponding statements of Theorem 3.3.

### 3.3. Lower Bound on \( \lambda_* \)

Theorem 3.3 shows that the amount of over-parameterization for the first layer depends on the quantity \( \lambda_* \) as given by (13). Note that, by definition, \( \lambda_* \) only depends on the activation function \( \sigma \), the training data \( X \), and hence \( N \) and \( d \) in this section. In this study, how we study this quantity scales with respect to these variables. First, let us show that \( \lambda_* > 0 \) under some mild conditions on \( \sigma \) and \( X \). The proof of this result is contained in Appendix D.1.

**Lemma 3.4** Assume there exists an index \( s \in [d] \) such that (i) \( X_s \neq X_j \) for all \( j \neq s \), and (ii) \( X_s = 0 \) for all \( i \in [N] \). Further, assume that \( \sigma \) is analytic and \( \sigma^{(2k)}(0) \neq 0 \) for all \( k \geq 1 \). Then, \( \lambda_* > 0 \).

The family of activation functions (3) satisfies both Assumption 2.2 and the hypotheses of Lemma 3.4. Furthermore, the set of data points that do not satisfy the hypotheses of Lemma 3.4 has Lebesgue measure zero. It has been shown that the result of Lemma 3.4 also holds when the data points are not parallel and the activation function is analytic and not polynomial, see (Du et al., 2019a).

Lemma 3.4 shows that \( \lambda_* > 0 \), but it does not provide information on how the lower bound scales with \( N \) and \( d \). If we further assume that the data points are uniform on the sphere with radius \( \sqrt{d} \), then \( \lambda_* \) is lower bounded by a constant independent of \( N \) and \( d \).

**Theorem 3.5** Let \( X \in \mathbb{R}^{N \times d} \) be a matrix whose rows are i.i.d. random vectors uniformly distributed on the sphere of radius \( \sqrt{d} \). Assume that (i) \( \sigma \) is not a linear function, (ii) \( \sigma \) is continuous and piecewise smooth, and (iii) for every \( j \in [0, 4] \), there exists a polynomial \( P_j \) of finite degree such that \( |\sigma^{(j)}(x)| \leq P_j(|x|) \). Then, there exists \( c_1 \in (0, 1) \) such that, for \( d \leq N \leq c_1 d^2 \), we have

\[
\mathbb{P}(\lambda_* \geq c_2) \geq 1 - e^{-d^{c_1}} - e^{-b_2 \sqrt{N}}, \tag{21}
\]

for some constants \( c_2, b_1, b_2 > 0 \) that do not depend on \( N, d \).

The lower bound (21) is tight (up to a constant). In fact,

\[
\lambda_* \leq \frac{\text{tr}(G_*)}{N} = \frac{\mathbb{E}\|\sigma(Xw)\|_2^2}{N} \leq \frac{\mathbb{E}\|Xw\|_2^2}{N} = \frac{\|X\|_F^2}{Nd}, \tag{22}
\]

and the RHS of (22) is equal to 1 when the rows of \( X \) have norm \( \sqrt{d} \). Note also that the requirement that \( \sigma \) is not a linear function is necessary. In fact, if \( \sigma(x) = x \), then \( G_* = \frac{1}{d} XX^T \) and \( \lambda_* = 0 \) as long as \( d < N \). The other two conditions on \( \sigma \) are purely technical, and they require smoothness as well as bounds on the growth of the derivatives of \( \sigma \). Note that the conditions of Theorem 3.5 are satisfied for the family of activation functions (3), as we verify in Appendix D.2.

If the rows of \( X \in \mathbb{R}^{N \times d} \) are uniform on the sphere of radius \( \sqrt{d} \), then \( \|X\|_F^2 \) is of order \( N \), see e.g. Theorem 5.39 in (Vershynin, 2010). Thus, an immediate consequence of Theorem 3.5 is that, in the regime \( d \leq N \leq c_1 d^2 \), the requirement of Theorem 3.3 on the over-parameterization of the first layer is moderate: \( n_1 \) scales as \( \tilde{O}(N^2) \). We also obtain an explicit bound on the convergence rate: the number of iterations needed to achieve \( \epsilon \) training loss scales as \( \tilde{O}(N^{3/2} \log(1/\epsilon)) \).

### 4. Sketch of the Proofs

#### 4.1. General Framework (Theorem 3.2)

Let \( \otimes \) denote the Kronecker product, and let \( \Sigma_t = \text{diag}[\text{vec}(\sigma'(G_t))] \in \mathbb{R}^{n_1 \times N n_1} \). Below, we frequently encounter situations where we need to evaluate the matrices \( \Sigma_t, F_t \) at specific iterations of gradient descent. To this end, we define the shorthands \( F^k_t = F_t(\theta_k), f^k = \text{vec}(F^k_t), \) and \( \Sigma^k_t = \Sigma_t(\theta_k) \). We omit the parameter \( \theta_t \) and write just \( F_i, \Sigma \) when it is clear from the context.

We recall the following results from (Nguyen & Hein, 2018; 2017), which provide a PL-type inequality for the training objective. This is one of the key components in our proofs to show convergence of gradient descent to a global minimum.

**Lemma 4.1** Let Assumption 2.1 hold. Then,

1. \( \text{vec} (\nabla_{W_i} \Phi) = (I_{n_1} \otimes F^T_{l-1}) \prod_{p=l+1}^r (W_p \otimes I_N)(f_L - y). \)
Global Convergence of Deep Networks with One Wide Layer Followed by Pyramidal Topology

2. \( \frac{\partial}{\partial \text{vec}(W_p^T)} \sum_{p=l+1}^{L} (W_p^T \otimes I_N) \sum_{p=1}^{l} (I_n \otimes F_{l-1}) \).

3. \( \| \text{vec}(\nabla W_2 \Phi) \|_2 \geq \sigma_{\min}(F_1) \sum_{p=3}^{L} \sigma_{\min}(\sum_{p=1}^{l} \sigma_{\min}(W_p)) \| f_L - y \|_2 \).

4. Let \( \sigma \) be such that \( \sigma'(x) \neq 0 \) for every \( x \in \mathbb{R} \). Then every stationary point \( \theta = (W_l^T)_{l=1}^{L} \) for which \( F_1 \) has full rank and all the weight matrices \( (W_l^T)_{l=3}^{L} \) have full rank is a global minimizer of \( \Phi \).

Note that the first two statements of Lemma 4.1 are just gradient computation which we state here for the purpose of the proof. The last two statements suggest that in order to show convergence of gradient descent to a global minimum, it suffices to (i) initialize all the matrices \( \{ F_1, W_3, \ldots, W_L \} \) to be full rank and (ii) make sure that the dynamics of gradient descent stays inside the manifold of full-rank matrices. One way to do that is to keep track of the smallest singular values of these matrices and ensure that they are kept away from zero throughout training. This is the key idea that we will use to prove our Theorem 3.2.

The next lemmas serve as key building blocks for the proof (for the proof see Appendix B.2, B.3, B.4 and B.5).

**Lemma 4.2** Let Assumption 2.2 hold. Then, for every \( \theta = (W_p^T)_{p=1}^{L} \) and \( l \in [L] \),

\[
\| F_l \|_F \leq \| X \|_F \prod_{p=1}^{l} \| W_p \|_2 ,
\]

\[
\| \nabla W_l \Phi \|_F \leq \| X \|_F \prod_{p=1}^{L} \| W_p^k \|_2 \| f_L - y \|_2 .
\]

Furthermore, let \( \theta_a = (W_p^T)_{p=1}^{L}, \theta_b = (W_p^T)_{p=1}^{L}, \) and \( \tilde{\lambda}_l \geq \max(\| W_l^T \|_2, \| W_l^0 \|_2) \) for some scalars \( \tilde{\lambda}_l \). Then, for \( l \in [L] \),

\[
\| F_l^a - F_l^b \|_F \leq \sqrt{L} \| X \|_F \frac{\prod_{p=1}^{L} \tilde{\lambda}_l}{\min_{l \in [L]} \tilde{\lambda}_l} \| \theta_a - \theta_b \|_2 ,
\]

\[
\| \frac{\partial f_L(\theta)}{\partial \text{vec}(W_l^T)} - \frac{\partial f_L(\theta)}{\partial \text{vec}(W_l^T)} \|_2 \leq \sqrt{L} \| X \|_F R \left( 1 + L \beta \| X \|_F R \right) \| \theta_a - \theta_b \|_2 ,
\]

where \( R = \prod_{p=1}^{L} \max(1, \tilde{\lambda}_p) \).

We recall the following standard result from optimization, which is proved for completeness in Appendix B.6.

**Lemma 4.3** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a twice continuously differentiable function. Let \( x, y \in \mathbb{R}^n \) be given, and assume that \( \| \nabla f(z) - \nabla f(x) \| \leq C \| z - x \|_2 \) for every \( z = x + t(y - x) \) where \( t \in [0, 1] \). Then,

\[
f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{C}{2} \| x - y \|_2^2 .
\]

**Proof of Theorem 3.2.** We show by induction that, for every \( k \geq 0 \), the following holds:

\[
\left\{ \begin{aligned}
\sigma_{\min}(W_l^T) &\geq \frac{1}{2} \lambda_l, & l &\in \{3, \ldots, L\}, \\
\| W_l^T \|_2 &\leq \frac{3}{2} \tilde{\lambda}_l, & l &\in [L], \\
\| F_l^T \|_2 &\geq \frac{1}{2} \lambda_F, & r &\in \{0, \ldots, k\}, \\
\Phi(\theta_r) &\leq (1 - \eta \alpha_0)^r \Phi(\theta_0), & r &\in \{0, \ldots, k\},
\end{aligned} \right.
\]

which holds for \( k = 0 \). Now, suppose that (27) holds for all iterates from 0 to \( k \), and let us show the claim at iteration \( k + 1 \). For every \( r \in \{0, \ldots, k\} \), we have by triangle inequality,

\[
\| W_l^{r+1} - W_0^r \|_F \leq \frac{r}{\eta} \| W_l^{r+1} - W_0^r \|_F
\]

\[
\leq \eta \| \nabla W_l \Phi(\theta_r) \|_F
\]

\[
\leq \eta \| \nabla W_l \Phi(\theta_r) \|_F \leq \eta \| X \|_F \| W_l^T \|_2 \| f_L - y \|_2 .
\]

where in the last line we have used our induction hypothesis. Set \( u := \sqrt{1 - \eta \alpha_0} \). Then, the RHS of the previous expression can be written as

\[
\frac{1}{\alpha_0} \| X \|_F \left( \frac{3}{2} \right)^{L-1} \frac{\tilde{\lambda}_1 - L}{\tilde{\lambda}_l} \sum_{s=0}^{r} (1 - \eta \alpha_0)^{s/2} \| f_L^0 - y \|_2 ,
\]

which, by using (6) and \( u \in (0, 1) \), is upper bounded by

\[
\begin{cases}
\frac{1}{2} \lambda_l, & l \in \{3, \ldots, L\}, \\
1, & l \in \{1, 2\}.
\end{cases}
\]

By Weyl’s inequality, we obtain that

\[
\left\{ \begin{aligned}
\sigma_{\min}(W_l^{r+1}) &\geq \sigma_{\min}(W_l^0) - \frac{1}{2} \lambda_l = \frac{1}{2} \lambda_l, & l &\in \{3, \ldots, L\}, \\
\| W_l^{r+1} \|_2 &\geq \| W_l^0 \|_2 + \frac{3}{2} \tilde{\lambda}_l = \frac{3}{2} \tilde{\lambda}_l, & l &\in [L], \\
\| W_l^{r+1} \|_2 &\geq \| W_l^0 \|_2 + \frac{3}{2} \tilde{\lambda}_l, & l &\in \{3, \ldots, L\}, \\
\| W_l^{r+1} \|_2 &\geq \| W_l^0 \|_2 + \frac{3}{2} \tilde{\lambda}_l, & l &\in \{3, \ldots, L\}.
\end{aligned} \right.
\]
Similarly, we have that, for every \( r \in \{0, \ldots, k\} \)
\[
\|F_{r+1} - F_r\|_F = \|\sigma(W_{r+1}) - \sigma(W_r)\|_F \\
\leq \|X\|_2 \|W_{r+1} - W_r\|_F, \\
\leq \frac{2}{\alpha_0} \|X\|_2 \|X\|_F \left( \frac{3}{2} \right)^{L-1} \lambda_{2-L} \|F_0 - y\|_2 \\
\leq \frac{1}{2} \lambda_F,
\]
where in the first inequality we use Assumption 2.2 and in the last line we use (7). It follows that
\[
\sigma_{\min}(F_{r+1})^{(r+1)} \geq \sigma_{\min}(F_r^{(r)}) - \frac{1}{2} \lambda_F = \frac{1}{2} \lambda_F.
\]

So far, we have shown that the first three statements in (27) hold for \( k + 1 \). It remains to show that \( \Phi(\theta_{k+1}) \leq (1 - \eta\alpha_0)^k \Phi(\theta_0) \). To do so, define the shorthand \( J_{FL} \) for the Jacobian of the network:
\[
J_{FL} = \left[ \frac{\partial f_l}{\partial \text{vec}(W_l)} \right] \text{, where } \frac{\partial f_l}{\partial \text{vec}(W_l)} \in \mathbb{R}^{(N_{l+1}) \times (n_{l} - 1)n_l} \text{ for } l \in [L].
\]

We first derive a Lipschitz constant of the gradient restricted to the line segment \([\theta_k, \theta_{k+1}]\). Let \( \theta_{t} = \theta_k + t(\theta_{k+1} - \theta_k) \) for \( t \in [0, 1] \). Then, by triangle inequality,
\[
\| \nabla \Phi(\theta_t) - \nabla \Phi(\theta_k) \|_2 \\
= \| J_{FL}(\theta_t)^T [f_L(\theta_t) - y] - J_{FL}(\theta_k)^T [f_L(\theta_k) - y] \|_2 \\
\leq \| f_L(\theta_t) - f_L(\theta_k) \|_2 \| J_{FL}(\theta_t) \|_2 \\
+ \| J_{FL}(\theta_t) - J_{FL}(\theta_k) \|_2 \| f_L(\theta_t) - y \|_2.
\]

In the following, we bound each term in (28). We first note that, for \( l \in [L] \) and \( t \in [0, 1] \),
\[
\| W_l(\theta_t) - W_t^l \|_F \\
\leq \| W_l(\theta_k) - W_t^l \|_F + \sum_{s=0}^{k-1} \| W_s^{l+1} - W_s^l \|_F \\
= \| t \eta \nabla W_l \Phi(\theta_k) \|_F + \sum_{s=0}^{k-1} \| \eta \nabla W_l \Phi(\theta_s) \|_F \\
\leq \| \eta \sum_{s=0}^{k} \nabla W_l \Phi(\theta_s) \|_F.
\]

By following a similar chain of inequalities as done in the beginning, we obtain that, for \( l \in [L] \),
\[
\max(\| W_l(\theta_k) \|_2, \| W_l(\theta_k) \|_2) \leq \frac{3}{2} \lambda_{L}.
\]

By using (25) and (29), we get
\[
\| f_L(\theta_t) - f_L(\theta_k) \|_2 \\
\leq \sqrt{L} \| X \|_F \left( \frac{3}{2} \right)^{L-1} \lambda_{2-L} \min_{l \in [L]} \lambda_l \| \theta_t - \theta_k \|_2.
\]

Note that, for a partitioned matrix \( A = [A_1, \ldots, A_n] \), we have that \( \| A \|_2 \leq \sum_{i=1}^n \| A_i \|_2 \). Thus,
\[
\| f_L(\theta_t) - f_L(\theta_k) \|_2 \\
\leq \sum_{l=1}^L \left\| \frac{\partial f_L(\theta_k^l)}{\partial \text{vec}(W_l)} \right\|_2 \\
\leq \sum_{l=1}^L \prod_{p=1}^L \| W_p(\theta_k^l) \|_2 \| f_{l-1}(\theta_k^l) \|_2 \text{ by Lemma 4.1} \\
\leq L \| X \|_F \sum_{l=1}^L \prod_{p=1}^L \| W_p(\theta_k^l) \|_2 \text{ by (23)} \\
\leq L \| X \|_F \left( \frac{3}{2} \right)^{L-1} \lambda_{2-L} \min_{l \in [L]} \lambda_l.
\]

We now bound the Lipschitz constant of the Jacobian restricted to the segment \([\theta_k, \theta_{k+1}]\). By using (26) and (29),
\[
\| f_L(\theta_t) - f_L(\theta_k) \|_2 \\
\leq \sum_{l=1}^L \left\| \frac{\partial f_L(\theta_k^l)}{\partial \text{vec}(W_l)} - \frac{\partial f_L(\theta_k^l)}{\partial \text{vec}(W_l)} \right\|_2 \\
\leq L \sqrt{L} \| X \|_F R \left( 1 + L \beta \| X \|_F R \right) \| \theta_t - \theta_k \|_2.
\]

Plugging all these bounds into (28) gives
\[
\| \nabla \Phi(\theta_t) - \nabla \Phi(\theta_k) \|_2 \leq Q_0 \| \theta_t - \theta_k \|_2,
\]
with \( Q_0 \) defined in (8). By using Lemma 4.3, we have that
\[
\Phi(\theta_{k+1}) \leq \Phi(\theta_k) + \langle \nabla \Phi(\theta_k), \theta_{k+1} - \theta_k \rangle + \frac{Q_0}{2} \| \theta_{k+1} - \theta_k \|_2^2 \\
= \Phi(\theta_k) - \eta \| \nabla \Phi(\theta_k) \|_2^2 + \frac{Q_0}{2} \eta^2 \| \nabla \Phi(\theta_k) \|_2^2 \\
\leq \Phi(\theta_k) - \frac{1}{2} \eta \| \nabla \Phi(\theta_k) \|_2^2 \text{ by (9)} \\
\leq \Phi(\theta_k) \left( 1 - \frac{1}{2} \eta \sigma_{\min} \left( F_k^{k-1} \right)^2 \right) \text{ by Lemma 4.1} \\
\cdot \prod_{l=3}^L \sigma_{\min} \left( \Sigma_{p=1}^{k-1} \lambda_{p-1} \right) \| f_{l-1}(\theta_k) \|_2^2 \\
\leq \Phi(\theta_k) \left( 1 - \frac{1}{2} \eta \sigma_{\min} \left( F_k^{k-1} \right)^2 \right) \| f_{L-1}(\theta_k) \|_2^2.
\]

which, by the induction hypothesis (27) and Assumption 2.2, is upper bounded by
\[
\Phi(\theta_k) - \eta \gamma(\theta_k) \left( 1 - \frac{1}{2} \lambda_{L-2} \right) \lambda_{L-2}^2 \lambda_{L-1}^2 \| f_{L-1}(\theta_k) \|_2^2 \\
= \Phi(\theta_k) \left( 1 - \eta \alpha_0 \right), \text{ by def. of } \alpha_0 \text{ in (8)}.
\]

So far, we have proven the hypothesis (27). It remains to show that the iterates \( \{ \theta_k \} \) also converge. From arguments
similar to those used at the beginning of this proof, one can prove that \( \{\theta_k\}_{k=0}^{\infty} \) is a Cauchy sequence and that (12) holds (a detailed proof is given in Appendix B.7). Thus, \( \{\theta_k\}_{k=0}^{\infty} \) is a convergent sequence and there exists some \( \theta_* \) such that \( \lim_{k \to \infty} \theta_k = \theta_* \). By continuity, \( \Phi(\theta_*) = \lim_{k \to \infty} \Phi(\theta_k) = 0 \), hence \( \theta_* \) is a global minimizer. \( \square \)

### 4.2. Lower Bound on \( \lambda_* \) (Theorem 3.5)

For two matrices \( A = [a_1, a_2, \ldots, a_p]^T \in \mathbb{R}^{p \times m} \) and \( B = [b_1, b_2, \ldots, b_p]^T \in \mathbb{R}^{p \times n} \), we define their Kratki-Rao product as \( A \ast B = [a_1 \otimes b_1, a_2 \otimes b_2, \ldots, a_p \otimes b_p]^T \in \mathbb{R}^{p \times mn} \). For \( r \in \mathbb{N} \), we denote the \( r \)-th Kratki-Rao power of \( A \) by \( A^r \in \mathbb{R}^{n \times d^r} \). Given a function \( \sigma \) with bounded Gaussian measure, let \( \mu_r(\sigma) \) denote its \( r \)-th Hermite coefficient (see Appendix D.3 for some background on Hermite expansions).

The proof of Theorem 3.5 relies on two intermediate lemmas. The first one connects \( G_* \) with the Kratki-Rao powers of \( X \). This connection is stated in Lemma H.2 of (Oymak & Soltanolkotabi, 2019) for ReLU and softplus activation functions. As a fully rigorous proof is missing in (Oymak & Soltanolkotabi, 2019), we provide it in Appendix D.4.

#### Lemma 4.4

Let \( X = [x_1, \ldots, x_N]^T \in \mathbb{R}^{N \times d} \) where \( \|x_i\|_2 = \sqrt{d} \) for all \( i \in [N] \). Assume that (i) \( \sigma \) is continuous and piecewise smooth and (ii) for every \( j \in [0, 4] \), there exists a polynomial \( P_j \) of finite degree such that \( |\sigma^{(j)}(x)| \leq P_j(|x|) \). Let \( G_* \), be defined as in (13). Then,

\[
G_* = \sum_{r=0}^\infty \int \frac{\mu^2_r(\sigma)}{d^r} (X^*)^r (X^*)^r)^T.
\]

Here, “\( \ast \)” is understood in the sense of uniform convergence, that is, for every \( \epsilon > 0 \), there exists a sufficiently large \( r_0 \geq 0 \) such that

\[
|\langle (G_*)_ij - (S_r)_ij \rangle | < \epsilon, \ \forall i, j \in [N], \ \forall r \geq r_0,
\]

where \( S_r = \sum_{k=0}^r \frac{\mu^2_k(\sigma)}{d^k} (X^*)^k (X^*)^k)^T \).

The second intermediate lemma lower bounds the smallest singular value of \( X^* \), when the rows of \( X \) are uniformly distributed on the sphere.

#### Lemma 4.5

Let \( X \in \mathbb{R}^{N \times d} \) be a matrix whose rows are i.i.d. random vectors uniformly distributed on the sphere of radius \( \sqrt{d} \). Fix an integer \( r \geq 2 \). Then, there exists \( c_1 \in (0, 1) \) such that, for \( d \leq N \leq c_1 d^2 \), we have

\[
\sigma_{\min}(X^*) \geq d^{r/2}/2
\]

with probability \( \geq 1 - 2Ne^{c_2 d^{r/2}} - (1 + 3 \log N)e^{-11\sqrt{N}} \), for some constant \( c_2 > 0 \).

Let us emphasize that the constants \( c_1, c_2 > 0 \) do not depend on \( N \) and \( d \), but they can depend on the integer \( r \). At this point, we are ready to prove Theorem 3.5.

#### Proof of Theorem 3.5

As \( \sigma \) is not linear, there exists an integer \( r \geq 2 \) such that \( \mu_r(\sigma) \neq 0 \). Thus, Lemma 4.5 implies that

\[
\lambda_{\min}((X^*)^r (X^*)^r)^T \geq \frac{d^{r/2}}{2}.
\]

Furthermore, by Lemma 4.4, there exists \( r' \geq r \) such that

\[
\|G_* - S_{r'}\|_F < \frac{\mu^2_r(\sigma)}{2d} \lambda_{\min}((X^*)^r (X^*)^r)^T = \frac{\xi}{2}.
\]

Note that \( \lambda_{\min}(S_{r'}) \geq \lambda_{\min}(S_r) \geq \xi \). Thus by Weyl’s inequality, we get

\[
\lambda_* = \lambda_{\min}(G_*) \geq \lambda_{\min}(S_{r'}) - \frac{\xi}{2} \geq \frac{\xi}{2} \geq \frac{\mu^2_r(\sigma)}{8},
\]

which completes the proof. \( \square \)

In the rest of this section, we provide a few comments on Lemma 4.5, whose proof is contained in Appendix D.5. A result similar to Lemma 4.5 is shown in Corollary 7.5 of (Soltanolkotabi et al., 2018) for the special case \( r = 1 \); it is assumed that \( X \in \mathbb{R}^{N \times d} \) is a matrix with i.i.d. \( N(0, 1) \) entries and that \( d \leq N \leq c_1 d^2 \), with \( c_1 \in (0, 1) \) a sufficiently small constant; then, it is proved that, with high probability, \( \sigma_{\min}(X^* X) \geq cd \), for some constant \( c \). Here, we consider a different data distribution (the data points are uniformly distributed on a sphere) and, most importantly, we provide a bound on higher order Kratki-Rao powers. Consequently, we give a lower bound on \( \lambda_* \) for a wider class of activation functions \( \sigma \). In fact, by showing Lemma 4.5 for \( r = 2 \), we obtain a non-trivial bound for \( \lambda_* \) only for activation functions \( \sigma \) such that \( \mu_2(\sigma) \neq 0 \).

The proof strategy is similar to that of Corollary 7.5 in (Soltanolkotabi et al., 2018): it builds on results from (Adamczak et al., 2011), in order to bound the smallest singular value of a random matrix whose rows \( \{x_i\}_{i=1}^N \) are i.i.d. and have bounded sub-exponential norm. For \( r = 2 \), the bound on the sub-exponential norm follows from a general version of the Hanson-Wright inequality proved in (Adamczak, 2015). Here, the main technical challenge is to obtain a tight bound on the sub-exponential norm for \( r > 2 \). To solve this issue, we study the tails of homogeneous tetrahedral polynomials of degree \( r \) whose variables are given by the components of \( x_i \), and we apply a concentration inequality for non-Lipschitz functions with bounded derivatives of higher order (Adamczak & Wolff, 2015; Bobkov et al., 2019). As a final remark, let us point out that, if \( d = o(N^{1/r}) \), then \( \lambda_{\min}((X^*)^r (X^*)^r)^T = 0 \). Lemma 4.5 shows that \( \sigma_{\min}(X^*) = \Omega(1) \), as long as \( d = \Omega(\sqrt{N}) \). Thus, it
remains an open problem to characterize the behavior of $\sigma_{\min}(X^r)$, and hence $\lambda_*$, in the regime $N^{1/r} \lesssim d \lesssim \sqrt{N}$.

5. Concluding Remarks

In this paper, we show that one wide layer followed by a pyramidal topology suffices for gradient descent to converge to a global minimum with a geometric rate. Our analysis is based on a PL-type inequality, which holds for a class of deep pyramidal networks in the manifold of full-rank weight matrices. The crux of the argument is to bound the smallest singular value of the feature output at the first layer. This makes our analysis to some extent simpler than existing approaches, which study the recursively-defined $L$-th layer Gram matrix (Du et al., 2019a), or the full Jacobian of the network (Oymak & Soltanolkotabi, 2019).

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A. Mathematical Tools

**Proposition A.1** Let $\int_A f \, d\mu(dx) = 0$ for a nonnegative function $f$. Then, $\mu(\{x \in A \mid f(x) \neq 0\}) = 0$.

**Proof:** Let $B_n = \{x \in A \mid f(x) \geq \frac{1}{n}\}$ and $B = \{x \in A \mid f(x) > 0\}$. Then, $B = \cup_{n=1}^{\infty} B_n$. Since $\frac{1}{n} \mathbf{1}_{B_n}(x) \leq f(x)$ for every $x$, it holds

$$0 = \int_A f \, d\mu \geq \frac{1}{n} \int_A \mathbf{1}_{B_n} \, d\mu = \frac{1}{n} \mu(B_n),$$

which implies $\mu(B_n) = 0$. Thus, $\mu(B) = \mu(\cup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mu(B_n) = 0$. \hfill \Box

**Proposition A.2** Let $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}$. Then $\|AB\|_F \leq \|A\|_F \|B\|_2$ and $\|AB\|_F \leq \|A\|_2 \|B\|_F$.

**Proof:** Let $A = [a_1^T, \ldots, a_m^T]^T, B = [b_1, \ldots, b_p]$. Then

$$\|AB\|_F^2 = \text{tr}(B^T A^T A B) = \sum_{i=1}^{p} b_i^T A^T A b_i \leq \sum_{i=1}^{p} \|b_i\|_2^2 \|A^T A\|_2 = \|A\|_2^2 \|B\|_2^2,$$

$$\|AB\|_F^2 = \text{tr}(A B B^T A^T) = \sum_{i=1}^{m} a_i^T B B^T A_i \leq \sum_{i=1}^{m} \|a_i\|_2^2 \|B B^T\|_2 = \|A\|_F^2 \|B\|_2^2.$$ \hfill \Box

**Lemma A.3** (Zeros of analytic functions, see e.g. (Mityagin, 2015)) If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a real analytic function which is not identically zero then the set $\{x \in \mathbb{R}^n \mid f(x) = 0\}$ has Lebesgue measure zero.

**Proposition A.4** (Weyl’s inequality, see e.g. (Stewart, 1990)) Let $A, B \in \mathbb{R}^{m \times n}$ with $\sigma_1(A) \geq \ldots \geq \sigma_r(A)$ and $\sigma_1(B) \geq \ldots \geq \sigma_r(B)$ where $r = \min(m, n)$. Then $|\sigma_i(A) - \sigma_i(B)| \leq \|A - B\|_2$ for every $i \in [r]$.

**Lemma A.5** (Singular values of random gaussian matrices, see e.g. (Vershynin, 2010)) Let $A \in \mathbb{R}^{m \times n}$ be a random matrix with $m \geq n$ and $A_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$. Then for every $t \geq 0$, we have that, with probability at least $1 - 2e^{-t^2/2}$

$$\sqrt{m} - \sqrt{n} - t \leq \sigma_{\min}(A) \leq \|A\|_2 \leq \sqrt{m} + \sqrt{n} + t.$$ 

**Theorem A.6** (Matrix Chernoff) Let $\{X_i\}_{i=1}^{n} \in \mathbb{R}^{d \times d}$ be a sequence of independent, random, symmetric matrices. Assume that $0 \leq \lambda_{\min}(X_i) \leq \lambda_{\max}(X_i) \leq R$. Let $S = \sum_{i=1}^{n} X_i$. Then,

$$\mathbb{P}(\lambda_{\min}(S) \leq (1 - \epsilon) \lambda_{\min}(E S)) \leq d \left[ \frac{e^{-\epsilon}}{(1 - \epsilon)^{1-\epsilon}} \right]^{\lambda_{\max}(E S) / R} \quad \forall \epsilon \in [0, 1),$$

$$\mathbb{P}(\lambda_{\max}(S) \geq (1 + \epsilon) \lambda_{\min}(E S)) \leq d \left[ \frac{e^\epsilon}{(1 + \epsilon)^{1+\epsilon}} \right]^{\lambda_{\max}(E S) / R} \quad \forall \epsilon \geq 0.$$

Supplementary Material (Appendix)

Global Convergence of Deep Networks with One Wide Layer

Followed by Pyramidal Topology
Global Convergence of Deep Networks with One Wide Layer Followed by Pyramidal Topology

B. Proofs for General Framework

In the following, we frequently use a basic inequality, namely, for every $A, B \in \mathbb{R}^{m \times n}$, $\|AB\|_F \leq \|A\|_2 \|B\|_F$ and $\|AB\|_F \leq \|A\|_F \|B\|_2$, see Proposition A.2 for the proof.

B.1. Properties of Activation Function (3)

Lemma B.1 Let $\sigma : \mathbb{R} \to \mathbb{R}$ be given as in (3). Then,

1. $\sigma$ is real analytic.
2. $\sigma'(x) \in [\gamma, 1]$ for every $x \in \mathbb{R}$.
3. $|\sigma(x)| \leq |x|$ for every $x \in \mathbb{R}$.
4. $\sigma'$ is $\beta$-Lipschitz.
5. $\lim_{\beta \to \infty} \sup_{x \in \mathbb{R}} |\sigma(x) - \max(\gamma x, x)| = 0$.

Proof: Let $\Psi$ be the CDF of the standard normal distribution. Then, after some manipulations, we have that

$$\sigma(x) = -\frac{(1 - \gamma)^2}{2\pi \beta} + \frac{(1 - \gamma)^2}{2\pi \beta} \exp \left( - \frac{\pi \beta^2 x^2}{(1 - \gamma)^2} \right) + x \Psi \left( \frac{\beta \sqrt{2\pi} x}{1 - \gamma} \right) + \gamma x \Psi \left( -\frac{\beta \sqrt{2\pi} x}{1 - \gamma} \right).$$

(31)

1. Since $\Psi$ is known as an entire function (i.e. analytic everywhere), it follows from (31) that $\sigma$ is analytic on $\mathbb{R}$.

2. Note that $\Psi'(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ and $\Psi(-z) = 1 - \Psi(z)$. Thus, after some simplifications, we have that

$$\sigma'(x) = \gamma + (1 - \gamma) \Psi \left( \frac{\beta \sqrt{2\pi} x}{1 - \gamma} \right).$$

(32)

The result follows by noting that $\Psi(\cdot) \in [0, 1]$.

3. It is easy to check that $\sigma(0) = 0$. Moreover, $\sigma$ is 1-Lipschitz and thus $|\sigma(x)| = |\sigma(x) - \sigma(0)| \leq |x|$.

4. We have that

$$\sigma''(x) = \beta \sqrt{2\pi} \Psi' \left( \frac{\beta \sqrt{2\pi} x}{1 - \gamma} \right) \leq \beta.$$

Thus $\sigma'$ is $\beta$-Lipschitz.

5. Note that

$$\frac{\beta}{1 - \gamma} \int_{-\infty}^{\infty} \exp \left( - \frac{\pi \beta^2 (x - u)^2}{(1 - \gamma)^2} \right) du = 1,$$

which implies that

$$\max(\gamma x, x) = \frac{\beta}{1 - \gamma} \int_{-\infty}^{\infty} \max(\gamma x, x) \exp \left( - \frac{\pi \beta^2 (x - u)^2}{(1 - \gamma)^2} \right) du.$$
Thus, the following chain of inequalities holds:

\[
|\sigma(x) - \max(\gamma x, x)| = \left| \frac{(1-\gamma)^2}{2\pi\beta} + \frac{\beta}{1-\gamma} \int_{-\infty}^{\infty} \max(\gamma u, u) \exp\left(-\frac{\pi\beta^2 (x-u)^2}{(1-\gamma)^2}\right) du - \frac{\beta}{1-\gamma} \int_{-\infty}^{\max(\gamma x, x)} \exp\left(-\frac{\pi\beta^2 (x-u)^2}{(1-\gamma)^2}\right) du \right|
\]

\[
\leq \frac{(1-\gamma)^2}{2\pi\beta} + \frac{\beta}{1-\gamma} \int_{-\infty}^{\infty} |\max(\gamma u, u) - \max(\gamma x, x)| \exp\left(-\frac{\pi\beta^2 (x-u)^2}{(1-\gamma)^2}\right) du
\]

\[
\leq \frac{(1-\gamma)^2}{2\pi\beta} + \frac{\beta}{1-\gamma} \int_{-\infty}^{\infty} |x-u| \exp\left(-\frac{\pi\beta^2 (x-u)^2}{(1-\gamma)^2}\right) du
\]

\[
= \frac{(1-\gamma)^2}{2\pi\beta} + \frac{2\beta}{1-\gamma} \int_{0}^{\infty} v \exp\left(-\frac{\pi\beta^2 v^2}{(1-\gamma)^2}\right) dv
\]

\[
= \frac{(1-\gamma)^2}{2\pi\beta} + \frac{1-\gamma}{\pi\beta}.
\]

Taking the supremum and the limit on both sides yields the result.

\[\square\]

**B.2. Proof of (23) in Lemma 4.2**

We prove by induction on \(l\). Note that the lemma holds for \(l = 1\) since

\[
\|F_1\|_F = \|\sigma(W_1)\|_F \leq \|W_1\|_F \leq \|X\|_F \|W_1\|_2
\]

where the second step follows from our assumption on \(\sigma\). Now suppose the lemma holds for \(l - 1\), that is,

\[
\|F_{l-1}\|_F \leq \|X\|_F \prod_{p=1}^{l-1} \|W_p\|_2.
\]

It is easy to verify that it also holds for \(l\). Indeed,

\[
\|F_l\|_F = \|\sigma(F_{l-1}W_l)\|_F \quad \text{by definition}
\]

\[
\leq \|F_{l-1}W_l\|_F \quad |\sigma(x)| \leq |x|
\]

\[
\leq \|F_{l-1}\|_F \|W_l\|_2
\]

\[
\leq \|X\|_F \prod_{p=1}^{l} \|W_p\|_2 \quad \text{by induction assump.}
\]

Note that for \(l = L\) one can skip the first equality above, as there is no activation at the output layer. \(\square\)

**B.3. Proof of (25) in Lemma 4.2**

We first prove the following intermediate result.

**Lemma B.2** Let \(\sigma\) be 1-Lipschitz and \(|\sigma(x)| \leq |x|\) for every \(x \in \mathbb{R}\). Let \(\theta_a = (W^a_l)_{l=1}^L, \theta_b = (W^b_l)_{l=1}^L\). Let \(\lambda_l \geq \max(\|W^a_l\|_2, \|W^b_l\|_2)\). Then, for every \(l \in [L]\),

\[
\|F_l^a - F_l^b\|_F \leq \|G_l^a - G_l^b\|_F \leq \|X\|_F \sum_{p=1}^{l} \lambda_p^{-1} \|W_p^a - W_p^b\|_2.
\]

Here, we denote \(\lambda_{i \rightarrow j} = \prod_{p=i}^{j} \lambda_p\).
Applying Lemma B.2 to the output layer yields:

\[\|F^a_l - F^b_l\|_F = \|\sigma(G^a_l) - \sigma(G^b_l)\|_F\]

by definition

\[\leq \|G^a_l - G^b_l\|_F\]

\[\sigma\text{ is 1-Lipschitz}\]

\[= \|XW^a_l - XW^b_l\|_F\]

\[\leq \|X\|_F \|W^a_l - W^b_l\|_2.\]

Suppose the lemma holds for \(l - 1\) and we want to prove it for \(l\). We have

\[\|F^a_l - F^b_l\|_F = \|\sigma(G^a_l) - \sigma(G^b_l)\|_F\]

definition

\[\leq \|G^a_l - G^b_l\|_F\]

\[\sigma\text{ is 1-Lipschitz}\]

\[= \|F^a_{l-1}W^a_l - F^b_{l-1}W^b_l\|_F\]

triangle inequality

\[\leq \|F^a_{l-1}W^a_l - F^b_{l-1}W^b_l\|_F + \|F^b_{l-1}W^a_l - F^b_{l-1}W^b_l\|_F\]

\[\leq \|F^a_{l-1} - F^b_{l-1}\|_F \|W^a_l\|_2 + \|F^b_{l-1}\|_F \|W^a_l - W^b_l\|_2\]

\[\leq \|F^a_{l-1} - F^b_{l-1}\|_F \|W^a_l\|_2 + \|X\|_F \left[\prod_{p=1}^{l-2} \|W^b_p\|_2\right] \|W^a_l - W^b_l\|_2\]

by (23)

\[\leq \|F^a_{l-1} - F^b_{l-1}\|_F L_{\lambda_l} + \|X\|_F \|W^a_l - W^b_l\|_2\]

by (23)

\[\leq \|X\|_F L_{\lambda_1\rightarrow l} \sum_{p=1}^{l} \lambda^{-1}_p \|W^a_p - W^b_p\|_2\]

induction assumption

\[\□\]

Applying Lemma B.2 to the output layer yields:

\[\|F^a_L - F^b_L\|_F = \|G^a_L - G^b_L\|_F\]

\[\leq \|X\|_F L_{\lambda_1\rightarrow L} \sum_{p=1}^{L} \lambda^{-1}_p \|W^a_p - W^b_p\|_2\]

\[\leq \sqrt{L} \|X\|_F \frac{L_{\lambda_1\rightarrow L}}{\min_{l\in[L]} \lambda_l} \|\theta_a - \theta_b\|_2\]

Cauchy-Schwarz

\[\□\]

B.4. Proof of (24) in Lemma 4.2

\[\|\nabla W_p \Phi\|_F = \|\text{vec}(\nabla W_p \Phi)\|_2\]

\[= \left\| (I_n \otimes F^T_{l-1}) \left[ \prod_{p=l+1}^{L} \Sigma_{p-1} (W_p \otimes I_N) \right] (f_L - y) \right\|_2\]

Lemma 4.1

\[\leq \|f_L - y\|_2 \left[ \prod_{p=l+1}^{L} \|W_p\|_2 \right] \|f_L - y\|_2\]

\[\leq \|X\|_F \left[ \prod_{p=1}^{L} \|W^k_p\|_2 \right] \|f_L - y\|_2\]

by (23).

\[\□\]
B.5. Proof of (26) in Lemma 4.2

We start by showing the following intermediate result.

**Lemma B.3** Let \( \sigma \) be 1-Lipschitz, and let \( |\sigma(x)| \leq |x| \) and \( |\sigma'(x)| \leq 1 \) hold for every \( x \in \mathbb{R} \). Let \( \theta_a = (W^a_L)_{l=1}^L, \theta_b = (W^b_L)_{l=1}^L \). Let \( \tilde{\lambda}_l = \max(\|W^a_l\|_2, \|W^b_l\|_2) \). Then, for every \( l \in [L] \),

\[
\left\| \frac{\partial f_L(\theta_a)}{\partial \text{vec}(W^a_L)} - \frac{\partial f_L(\theta_b)}{\partial \text{vec}(W^b_L)} \right\|_2 \\
\leq \|X\|_F \tilde{\lambda}_{1 \to L} \sigma^{-1} \sum_{p=1}^L \tilde{\lambda}_p^{-1} \|W^a_p - W^b_p\|_2 + \|X\|_F + \tilde{\lambda}_{1 \to L} \lambda_1 \sigma^{-1} \sum_{p=1}^L \|\Sigma^a_p - \Sigma^b_p\|_2 + \tilde{\lambda}_{t+1 \to L} \|F^a_L - F^b_L\|_2.
\]

Here, we denote \( \tilde{\lambda}_{i \to j} = \prod_{k=i}^j \tilde{\lambda}_k \).

**Proof:** For every \( t \in \{l, \ldots, L\} \), let

\[
M^a_t = \prod_{p=l+1}^t ((W^a_p)^T \otimes I_N) \sigma^{-1}_p, \quad M^b_t = \prod_{p=l+1}^t ((W^b_p)^T \otimes I_N) \sigma^{-1}_p.
\]

Note that for \( t = l \) the product terms inside brackets are inactive, and thus \( M^a_l = (I_{N_l} \otimes F^a_{l-1}) \) and \( M^b_l = (I_{N_l} \otimes F^b_{l-1}) \). It follows from Lemma 4.1 that

\[
\frac{\partial f_L(\theta_a)}{\partial \text{vec}(W^a_L)} = M^a_L, \quad \frac{\partial f_L(\theta_b)}{\partial \text{vec}(W^b_L)} = M^b_L.
\]

The following inequality holds

\[
\|M^a_t - M^b_t\|_2 \leq \prod_{p=l+1}^t \|W^a_p\|_2 \|\sigma^{-1}_p\|_2 \leq \prod_{p=l+1}^t \|W^a_p\|_2 \leq \|X\|_F \tilde{\lambda}_{1 \to L} \|X\|_F \cdot (33)
\]

where the second inequality follows from (23) and \( |\sigma'| \leq 1 \). To prove the lemma, we will prove that, for every \( t \in \{l, \ldots, L\} \),

\[
\|M^a_t - M^b_t\|_2 \\
\leq \|X\|_F \sum_{p=l+1}^t \tilde{\lambda}_{t \to p} \sigma^{-1}_p \|W^a_p - W^b_p\|_2 + \|X\|_F \tilde{\lambda}_{1 \to L} \lambda_1 \sigma^{-1} \sum_{p=1}^{l-1} \|\Sigma^a_p - \Sigma^b_p\|_2 + \tilde{\lambda}_{t \to L} \|F^a_L - F^b_L\|_2.
\]

Then setting \( t = L \) in (34) leads to the desired result. First we note that (34) holds for \( t = l \) since

\[
\|M^a_t - M^b_t\|_2 = \|(I_{N_l} \otimes F^a_{l-1}) - (I_{N_l} \otimes F^b_{l-1})\|_2 = \|F^a_{l-1} - F^b_{l-1}\|_2.
\]

Suppose that it holds for \( t - 1 \) with \( t \geq l + 1 \), and we want to show it for \( t \). Then,

\[
\|M^a_t - M^b_t\|_2 \\
= \|(W^a_t)^T \otimes I_N) \Sigma^a_{t-1} M^a_{t-1} - (W^b_t)^T \otimes I_N) \Sigma^b_{t-1} M^b_{t-1}\|_2 \\
\leq \|(W^a_t)^T \otimes I_N) \Sigma^a_{t-1} M^a_{t-1} - (W^b_t)^T \otimes I_N) \Sigma^b_{t-1} M^b_{t-1}\|_2 + \|(W^b_t)^T \otimes I_N) \Sigma^b_{t-1} M^b_{t-1} - (W^b_t)^T \otimes I_N) \Sigma^b_{t-1} M^b_{t-1}\|_2 \\
\leq \|W^a_t - W^b_t\|_2 \|\Sigma^a_{t-1}\|_2 \|\Sigma^a_{t-1} M^a_{t-1}\|_2 + \|W^b_t\|_2 \|\Sigma^a_{t-1} M^a_{t-1} - \Sigma^b_{t-1} M^b_{t-1}\|_2 \\
\leq \|W^a_t - W^b_t\|_2 \|\Sigma^a_{t-1}\|_2 \|\Sigma^a_{t-1} M^a_{t-1} - \Sigma^b_{t-1} M^b_{t-1}\|_2, \quad \text{by (33) and } |\sigma'| \leq 1
\]

\[
\leq \|W^a_t - W^b_t\|_2 \tilde{\lambda}_{1 \to t} \|X\|_F + \tilde{\lambda}_l \|\Sigma^a_{t-1} M^a_{t-1} - \Sigma^b_{t-1} M^b_{t-1}\|_2 + \|\Sigma^b_{t-1} M^b_{t-1}\|_2 \\
\leq \|W^a_t - W^b_t\|_2 \tilde{\lambda}_{1 \to t} \|X\|_F + \tilde{\lambda}_l \|\Sigma^a_{t-1} - \Sigma^b_{t-1}\|_2 \tilde{\lambda}_{1 \to t} \|X\|_F + \|\Sigma^a_{t-1} - \Sigma^b_{t-1}\|_2 \\
\leq \|X\|_F \tilde{\lambda}_{1 \to t} \|W^a_t - W^b_t\|_2 + \|X\|_F \tilde{\lambda}_{1 \to t} \|\Sigma^a_{t-1} - \Sigma^b_{t-1}\|_2 + \tilde{\lambda}_l \|M^a_t - M^b_t\|_2 \\
\leq \|X\|_F \tilde{\lambda}_{1 \to t} \|W^a_t - W^b_t\|_2 + \|X\|_F \tilde{\lambda}_{1 \to t} \|\Sigma^a_{t-1} - \Sigma^b_{t-1}\|_2 + \tilde{\lambda}_l \|M^a_t - M^b_t\|_2,
\]

\[
\leq \|X\|_F \tilde{\lambda}_{1 \to t} \sum_{p=l+1}^{t-1} \|W^a_p - W^b_p\|_2 + \|X\|_F \tilde{\lambda}_{1 \to t} \sum_{p=l+1}^{t-1} \|\Sigma^a_p - \Sigma^b_p\|_2 + \tilde{\lambda}_l \|M^a_t - M^b_t\|_2,
\]

Global Convergence of Deep Networks with One Wide Layer Followed by Pyramidal Topology
where the last line follows by plugging the bound of $\|M_{t-1}^a - M_{t-1}^b\|_2$ from the induction assumption.

**Proof of (26) in Lemma 4.2.** Let $S = \|X\|_F \overset{\lambda_1-\lambda L}{\lambda_l^1} \sum_{p=1}^{L} \lambda_p^{-1} \|W_p^a - W_p^b\|_2$. Then,

$$\left\| \frac{\partial f_{L}(\theta_a)}{\text{vec}(W^a_I)} - \frac{\partial f_{L}(\theta_b)}{\text{vec}(W^b_I)} \right\|_2 \leq S + \|X\|_F \lambda_1-\lambda L \lambda_l^1 \sum_{p=1}^{L} \|\Sigma_p^a - \Sigma_p^b\|_2 + \lambda_{l+1}\lambda L \|F_{l-1}^a - F_{l-1}^b\|_2 \leq S + \|X\|_F \lambda_1-\lambda L \lambda_l^1 \sum_{p=1}^{L} \|\Sigma_p^a - \Sigma_p^b\|_2 + \lambda_{l+1}\lambda L \|F_{l-1}^a - F_{l-1}^b\|_2$$

**Lemma B.2**

$$\leq S + \|X\|_F \lambda_1-\lambda L \lambda_l^1 \sum_{p=1}^{L} \|\Sigma_p^a - \Sigma_p^b\|_2 + \lambda_{l+1}\lambda L \|F_{l-1}^a - F_{l-1}^b\|_2 \leq \|X\|_F \lambda_1-\lambda L \lambda_l^1 \sum_{p=1}^{L} \|\Sigma_p^a - \Sigma_p^b\|_2 + \lambda_{l+1}\lambda L \|F_{l-1}^a - F_{l-1}^b\|_2$$

**Lemma B.3**

$$\leq S + \|X\|_F \lambda_1-\lambda L \lambda_l^1 \sum_{p=1}^{L} \|\Sigma_p^a - \Sigma_p^b\|_2 + \lambda_{l+1}\lambda L \|F_{l-1}^a - F_{l-1}^b\|_2$$

Then,

$$= \|X\|_F \lambda_1-\lambda L \lambda_l^1 \sum_{p=1}^{L} \|\Sigma_p^a - \Sigma_p^b\|_2 + \lambda_{l+1}\lambda L \|F_{l-1}^a - F_{l-1}^b\|_2$$

Cauchy-Schwarz

□
B.6. Proof of Lemma 4.3

Let \( g(t) = f(x + t(y - x)) \). Then

\[
f(y) - f(x) = g(1) - g(0) = \int_0^1 g'(t) dt
\]

\[
= \int_0^1 \langle \nabla f(x + t(y - x)), y - x \rangle dt
\]

\[
= \langle \nabla f(x), y - x \rangle + \int_0^1 \langle \nabla f(x + t(y - x)) - \nabla f(x), y - x \rangle dt
\]

\[
\leq \langle \nabla f(x), y - x \rangle + \int_0^1 c \| y - x \|^2 dt
\]

\[
= \langle \nabla f(x), y - x \rangle + \frac{c}{2} \| x - y \|^2.
\]

\[\square\]

B.7. Proof that \( \{\theta_k\}_{k=1}^\infty \) is a Cauchy Sequence

Let us fix any \( \epsilon > 0 \). We need to show that there exists \( r > 0 \) such that for every \( i, j \geq r \), \( \| \theta_j - \theta_i \| < \epsilon \). The case \( i = j \) is trivial, so we assume w.l.o.g. that \( i < j \). Then, the following chain of inequalities hold

\[
\| \theta_j - \theta_i \| = \sqrt{\sum_{l=1}^L \left\| W_l^j - W_l^i \right\|_F^2}
\]

\[
\leq \sum_{l=1}^L \left\| W_l^j - W_l^i \right\|_F
\]

\[
\leq \sum_{l=1}^L \sum_{s=i}^{j-1} \left\| W_s^{i+1} - W_s^i \right\|_F \quad \text{triangle inequality}
\]

\[
= \sum_{l=1}^L \sum_{s=i}^{j-1} \eta \| \nabla W_l \Phi(\theta_s) \|_F
\]

\[
\leq \sum_{l=1}^L \sum_{s=i}^{j-1} \eta \| X \|_F \| f_L^s - y \|_2 \prod_{p=1}^L \| W_p^s \|_2
\]

\[
\leq \sum_{l=1}^L \eta \| X \|_F 1.5^{L-1} \sum_{s=i}^{j-1} (1 - \eta \alpha_0)^{s/2} \| f_L^s - y \|_2
\]

\[
= (1 - \eta \alpha_0)^{i/2} \sum_{l=1}^L \eta \| X \|_F 1.5^{L-1} \sum_{s=0}^{j-i-1} (1 - \eta \alpha_0)^{s/2} \| f_L^0 - y \|_2
\]

\[
= (1 - \eta \alpha_0)^{i/2} \left[ \eta \| X \|_F 1.5^{L-1} \sum_{l=1}^L \sum_{s=0}^{L-1} \lambda_{l-1}^{-1} \lambda_{L-1} \frac{1 - \sqrt{1 - \eta \alpha_0}}{1 - \sqrt{1 - \eta \alpha_0}} \| f_L^0 - y \|_2 \right]
\]

\[
= (1 - \eta \alpha_0)^{i/2} \left[ \frac{1}{\alpha_0} \| X \|_F 1.5^{L-1} \sum_{l=1}^L \sum_{s=0}^{L-1} \lambda_{l-1}^{-1} \lambda_{L-1} (1 - u^2) \frac{1 - u^{j-i}}{1 - u} \| f_L^0 - y \|_2 \right] \quad u := \sqrt{1 - \eta \alpha_0}
\]

\[
\leq (1 - \eta \alpha_0)^{i/2} \left[ \frac{2}{\alpha_0} \| X \|_F 1.5^{L-1} \sum_{l=1}^L \sum_{s=0}^{L-1} \lambda_{l-1}^{-1} \lambda_{L-1} \| f_L^0 - y \|_2 \right] \quad u \in (0, 1)
\]

Note that \( (1 - \eta \alpha_0)^{i/2} \leq (1 - \eta \alpha_0)^{i/2} \) and thus there exists a sufficiently large \( r \) such that \( \| \theta_j - \theta_i \| < \epsilon \). This shows that \( \{\theta_k\}_{k=0}^\infty \) is a Cauchy sequence, and hence convergent to some \( \theta_* \). By continuity, \( \Phi(\theta_*) = \Phi(\lim_{k \to \infty} \theta_k) = \Phi(\theta_0) \).
\[\lim_{k \to \infty} \Phi(\theta_k) = 0, \text{ and thus } \theta_* \text{ is a global minimizer. The rate of convergence is}\]
\[
\|\theta_k - \theta_*\| = \lim_{j \to \infty} \|\theta_k - \theta_j\| \leq (1 - \eta \alpha_0)^{k/2} \left[ \frac{2}{\alpha_0} \|X\|_F 1.5^{L-1} \sum_{l=1}^L \lambda_1^{-1} \lambda_{1-L} \|f^0_l - y\|_2 \right].
\]

\[\square\]

C. Proofs for Xavier’s Initialization

Before presenting the proof of the convergence result under Xavier’s initialization in Appendix C.3, let us state two helpful lemmas. The first lemma bounds the output of the network at initialization using standard Gaussian concentration and it is proved in Appendix C.1.

**Lemma C.1** Let \( \sigma \) be 1-Lipschitz, and consider Xavier’s initialization scheme:

\[ [W_l]_{i,j} \sim \mathcal{N}(0, 1/n_{l-1}), \quad \forall l \in [L], i \in [n_{l-1}], j \in [n_l]. \]

Assume that \( \sqrt{n_l} \geq t \) for any \( l \in [L - 1] \). Then,

\[ \|F_l\|_F \leq 2^{L-1} \frac{\|X\|_F}{\sqrt{d}} (\sqrt{n_L} + t), \tag{35} \]

with probability at least \( 1 - L e^{-t^2/2} \).

The second lemma identifies sufficient conditions on \( n_1 \) so that the quantity \( \lambda_F \) defined in (5) is bounded away from zero. The proof is similar to that of Theorem 3.2 of (Oymak & Soltanolkotabi, 2019) (see Section 6.8 in their appendix), and we provide it in Appendix C.2.

**Lemma C.2** Let \( |\sigma(x)| \leq |x| \) for every \( x \in \mathbb{R} \). Define \( F_1 = \sigma(XW) \) with \( X \in \mathbb{R}^{N \times d}, W \in \mathbb{R}^{d \times n_1}, \) and \( W_{ij} \sim \mathcal{N}(0, \zeta^2) \) for all \( i \in [d], j \in [n_1] \). Define also

\[ G_* = \mathbb{E}_{w \sim \mathcal{N}(0, \zeta^2 Id)} [\sigma(Xw)\sigma(Xw)^T], \quad \lambda_* = \lambda_{\min}(G_*). \]

Then, for \( t \geq \sqrt{4\zeta^2 \ln \max \left(1, 2\sqrt{\frac{1}{6}} \|X\|_2^2 \alpha_2 \lambda_*^{-1}\right)} \) and \( n_1 \geq \max \left(N, \frac{20 \|X\|_2^2 dt^2 (t^2/2 + \ln(N)/2)}{\lambda_*} \right) \), we have

\[ \sigma_{\min}(F_1) \geq \sqrt{n_1 \lambda_* / 4}, \tag{36} \]

with probability at least \( 1 - 2e^{-t^2/2} \).

C.1. Proof of Lemma C.1

It is straightforward to show the inequality.

**Lemma C.3** Let \( |\sigma(x)| \leq |x| \) for every \( x \in \mathbb{R} \). Let \( [W_l]_{i,j} \sim \mathcal{N} \left(0, \frac{1}{n_{l-1}}\right) \) for every \( l \in [L], i \in [n_{l-1}], j \in [n_l] \). Then, for every \( l \in [L] \) we have \( \mathbb{E} \|F_l\|_F^2 \leq \frac{n_l}{n_{l-1}} \mathbb{E} \|F_{l-1}\|_F^2 \).

**Proof:**

\[
\mathbb{E} \|F_l\|_F^2 = \mathbb{E} \|\sigma(F_{l-1}W_l)\|_F^2 \leq \mathbb{E} \|F_{l-1}W_l\|_F^2 = \mathbb{E} \text{tr} \left(F_{l-1}W_lW_l^TF_{l-1}^T\right) = \frac{n_l}{n_{l-1}} \mathbb{E} \|F_{l-1}\|_F^2,
\]

where the first inequality follows from our assumption on \( \sigma \), and the last equality follows from the fact that \( W_lW_l^T = \sum_{j=1}^{n_l} (W_l)_{i,j}(W_l)_{i,j}^T \) and \( \mathbb{E}(W_l)_{i,j}(W_l)_{i,j}^T = \frac{1}{n_{l-1}} \mathbb{I}_{n_{l-1}} \) for every \( j \in [n_l] \). \( \square \)
Global Convergence of Deep Networks with One Wide Layer Followed by Pyramidal Topology

**Proof of Lemma C.1.** In the following, we write $\text{subG}(\xi^2)$ to denote a sub-gaussian random variable with mean zero and variance proxy $\xi^2$. It is well-known that if $Z \sim \text{subG}(\xi^2)$ then for every $t \geq 0$ we have $\Pr(|Z| \geq t) \leq 2 \exp(-\frac{t^2}{2\xi^2})$.

We prove by induction on $l \in [L]$ that, if $\sqrt{n_p} \geq t$ for every $p \in [l-1]$, then it holds w.p. $\geq 1 - le^{-t^2/2}$ over $(W_p)_{p=1}^l$ that

$$
\|F_l\|_F \leq \frac{\|X_F\|_F 2^{l-1} \sqrt{n_l} + t}{}.
$$

Let us check the case $l = 1$ first. We have

$$
\left|\|F_1(W_1)\|_F - \|F_1(W_1')\|_F\right| \leq \|F_1(W_1) - F_1(W_1')\|_F
$$

$$
= \|\sigma(XW_1) - \sigma(XW_1')\|_F
$$

$$
\leq \|XW_1 - XW_1'\|_F
$$

$$
\leq \|X\|_F \|W_1 - W_1'\|_F.
$$

It follows that $\|F_1\|_F - \mathbb{E}\|F_1\|_F \sim \text{subG}\left(\frac{\|X\|_F^2}{d}\right)$. By Gaussian concentration inequality, we have w.p. at least $1 - e^{-t^2/2}$,

$$
\|F_1\|_F \leq \mathbb{E}\|F_1\|_F + \frac{\|X\|_F t}{\sqrt{d}}
$$

$$
\leq \frac{\sqrt{n_1}}{\sqrt{d}} \|X\|_F + \frac{\|X\|_F t}{\sqrt{d}}
$$

$$
= \frac{\|X\|_F}{\sqrt{d}} \sqrt{n_1} + t.
$$

Thus the hypothesis holds for $l = 1$. Now suppose it holds for $l - 1$, that is, we have w.p. $\geq 1 - (l-1)e^{-t^2/2}$ over $(W_p)_{p=1}^{l-1}$,

$$
\|F_{l-1}\|_F \leq \frac{\|X\|_F 2^{l-2} \sqrt{n_{l-1}} + t}{}.
$$

Conditioned on $(W_p)_{p=1}^{l-1}$, we note that $\|F_1\|_F$ is Lipschitz w.r.t. $W_l$ because

$$
\left|\|F_l(W_l)\|_F - \|F_l(W_l')\|_F\right| \leq \|F_{l-1}\|_F \|W_l - W_l'\|_F
$$

and thus $\|F_l\|_F - \mathbb{E}\|F_l\|_F \sim \text{subG}\left(\frac{\|F_{l-1}\|_F^2}{n_{l-1}}\right)$. By Gaussian concentration inequality, we have w.p. $\geq 1 - e^{-t^2/2}$ over $W_l$,

$$
\|F_l\|_F \leq \mathbb{E}\|F_l\|_F + \frac{\|F_{l-1}\|_F t}{\sqrt{n_{l-1}}}.
$$

Thus the above events hold w.p. at least $1 - le^{-t^2/2}$ over $(W_p)_{p=1}^l$, in which case we get

$$
\|F_l\|_F \leq \mathbb{E}\|F_l\|_F + \frac{\|F_{l-1}\|_F t}{\sqrt{n_{l-1}}}
$$

$$
\leq \frac{\sqrt{n_l}}{\sqrt{n_{l-1}}} \|F_{l-1}\|_F + \frac{\|F_{l-1}\|_F t}{\sqrt{n_{l-1}}}
$$

$$
\leq \frac{\|X\|_F 2^{l-2} \sqrt{n_{l-1}} + t}{\sqrt{d}}
$$

$$
\leq \frac{\|X\|_F 2^{l-1} \sqrt{n_{l-1}} + t}{\sqrt{d}}
$$

Thus, the hypothesis also holds for $l$. 

$\square$
C.2. Proof of Lemma C.2

Let $A \in \mathbb{R}^{N \times n_1}$ be a random matrix defined as $A_j = \sigma(XW_j) \mathbb{I}_{\|W_j\|_{\infty} \leq t}$ $\forall j \in [n_1]$. Then,

$$\lambda_{\min} (F_1 F_1^T) = \lambda_{\min} \left( \sum_{j=1}^{n_1} \sigma(XW_j)\sigma(XW_j)^T \right) \geq \lambda_{\min} (AA^T).$$

Thus, by using our assumption on $\sigma$,

$$\lambda_{\max} (A_jA_j^T) = \|A_j\|^2_2 = \|\sigma(XW_j) \mathbb{I}_{\|W_j\|_{\infty} \leq t}\|^2_2 \leq \|X\|^2_2 \|W_j\|_{\infty} \leq \|X\|^2_2 dt =: R$$

Let $G = \mathbb{E}_{w \sim \mathcal{N}(0, \xi^2 I_d)} \left[ \sigma(Xw)\sigma(Xw)^T \mathbb{I}_{\|w\|_{\infty} \leq t} \right]$. Applying Matrix Chernoff bound (Theorem A.6) to the sum of random p.s.d. matrices, $AA^T = \sum_{j=1}^{n_1} A_jA_j^T$, we obtain that for every $\epsilon \in [0, 1)$

$$\mathbb{P} \left( \lambda_{\min} (AA^T) \leq (1 - \epsilon)\lambda_{\min} (\mathbb{E}AA^T) \right) \leq N \left[ \frac{e^{-\epsilon}}{(1 - \epsilon)^{1-\epsilon}} \right]^{\lambda_{\max}(\mathbb{E}AA^T)/R}$$

Substituting $\mathbb{E}[AA^T] = n_1 G$ and $R = \|X\|^2_2 dt$ and $\epsilon = 1/2$ gives

$$\mathbb{P} \left( \lambda_{\min} (AA^T) \leq n_1 \lambda_{\min} (G) / 2 \right) \leq N \left[ \sqrt{2e}^{-1/2} \right]^{n_1 \lambda_{\max}(G)/R} \leq \exp \left( - \frac{n_1 \lambda_{\min}(G)}{10 \|X\|^2_2 dt^2 + \ln N} \right)$$

Thus, as long as $n_1$ is large enough, in particular,

$$n_1 \geq \frac{10 \|X\|^2_2 dt^2 \left( t^2/2 + \ln(N/2) \right)}{\lambda_{\min}(G)}$$

we have $\lambda_{\min} (AA^T) \geq n_1 \lambda_{\min} (G) / 2$ w.p. at least $1 - 2e^{-t^2/2}$.

The idea now is to lower bound $\lambda_{\min} (G)$ in terms of $\lambda_{\min} (G_*)$ .

$$\|G - G_*\|_2 = \left\| \mathbb{E} \left[ \sigma(Xw)\sigma(Xw)^T \mathbb{I}_{\|w\|_{\infty} \leq t} \right] - \mathbb{E} \left[ \sigma(Xw)\sigma(Xw)^T \right] \right\|_2$$

\[= \mathbb{E} \left[ \|\sigma(Xw)\sigma(Xw)^T \mathbb{I}_{\|w\|_{\infty} > t} \right] - \mathbb{E} \left[ \sigma(Xw)\sigma(Xw)^T \right] \right\|_2 \quad \text{Jensen inequality} \]

\[= \mathbb{E} \left[ \|\sigma(Xw)\|_2^2 \mathbb{I}_{\|w\|_{\infty} > t} \right] \quad \text{assump. on } \sigma \]

\[\leq \mathbb{E} \left[ \|\|w\|_{\infty} > t \right] \quad \text{Cauchy-Schwarz} \]

\[\leq \mathbb{E} \left[ \left\{ \sum_{i=1}^{d} w_i^4 \right\} \mathbb{P} \left( \|w\|_{\infty} > t \right) \right] \quad \text{Cauchy-Schwarz} \]

\[\leq \mathbb{E}_{x \sim \mathcal{N}(0, 1)} \left[ \|x\|^4 \mathbb{P} \left( \|w\|_{\infty} > t \right) \right] \quad \text{union bound} \]

\[\leq \mathbb{E}_{x \sim \mathcal{N}(0, 1)} \left[ \|w\|^4 \mathbb{P} \left( \|w\|_{\infty} > t \right) \right] \quad \text{normal dist.} \]

\[\leq \lambda_* \quad \text{by assumption on } \lambda \]

This implies that $\lambda_{\min} (G) \geq \lambda_{\min} (G_*) - \lambda_* / 2 = \lambda_* / 2$. Plugging this into the above statement yields for every

$$n_1 \geq \frac{20 \|X\|^2_2 dt^2 \left( t^2/2 + \ln(N/2) \right)}{\lambda_*}.$$
it holds w.p. at least $1 - 2e^{-t^2/2}$ that

$$
\lambda_{\text{min}} \left( F_1 F_1^T \right) \geq \lambda_{\text{min}} \left( A A^T \right)
\geq n_1 \lambda_{\text{min}}(G)/2
\geq n_1 (\lambda_{\text{min}}(G_x) - \lambda_x)/2
\geq n_1 \lambda_x/4
$$

Lastly, since $n_1 \geq N$ we get $\sigma_{\text{min}}(F_1) = \sqrt{\lambda_{\text{min}} \left( F_1 F_1^T \right)} \geq \sqrt{n_1} \lambda_x/4$. \hfill \( \square \)

### C.3. Proof of Theorem 3.3

From known results on random Gaussian matrices, we have, w.p. $\geq 1 - 2e^{-t^2/2}$,

$$
\|W^0\|_2^2 \leq \frac{\sqrt{n_1} + \sqrt{d} + t}{\sqrt{d}} \leq 3^{1/4} \sqrt{n_1},
\|W^0_2\|_2^2 \leq \frac{\sqrt{n_1} + \sqrt{n_2} + t}{\sqrt{n_1}} \leq 2,
$$

where the last inequality in each line follows from (14) and from $n_1 \geq N \geq d$. From the definition (4), we get

$$
\bar{\lambda}_1 = \frac{2}{3} (1 + \|W^0\|_2) \leq \frac{8}{3} \sqrt{n_1},
\bar{\lambda}_2 = \frac{2}{3} (1 + \|W^0_2\|_2) \leq 2. \tag{37}
$$

Similarly, for any $l \in \{3, \ldots, L\}$, we have, w.p. $\geq 1 - 2e^{-t^2/2}$,

$$
\frac{1}{101} \leq \frac{\sqrt{n_{l-1}} - \sqrt{n_l} - t}{\sqrt{n_{l-1}}} \leq \lambda_l \leq \frac{\sqrt{n_{l-1}} + \sqrt{n_l} + t}{\sqrt{n_{l-1}}} \leq 2. \tag{38}
$$

Furthermore, by Lemma C.1 and C.2, we have, w.p. $\geq 1 - Le^{-t^2/2} - 2e^{-t^2/2}$,

$$
\lambda_F = \sigma_{\text{min}}(F^0_1) \geq \sqrt{n_1} \lambda_x/4,
\sqrt{2 \Phi(\theta_0)} \leq 2^{L-1} (\sqrt{n_L} + t) \frac{\|X\|_F}{\sqrt{d}} + \|Y\|_F. \tag{39,40}
$$

as long as the width of the first layer satisfies the following condition from Lemma C.2:

$$
n_1 \geq \max \left( N, \frac{c t_0^2 d \|X\|_2^2 (t_0^2 + \ln N)}{\lambda_x} \right). \tag{41}
$$

for a suitable constant $c$. From (39), we get a lower bound on the LHS of (6); and from (37), (38) and (40) we get an upper bound on the RHS of (6). Thus in order to satisfy the initial condition (6), it suffices to have (41) and

$$
n_1 \lambda_x \geq 2^{cL} \|X\|_F \sqrt{n_1 d} \left( \sqrt{n_L + t} \frac{\|X\|_F}{\sqrt{d}} + \|Y\|_F \right), \tag{42}
$$

which together leads to condition (15).

To satisfy the initial condition (7), it suffices to have in addition to (6) that $\lambda_F \geq 2 \|X\|_2$, which is fulfilled for $n_1 \geq \frac{16 \|X\|_2^2}{\lambda_x}$. One can verify that this is true under the condition (15). Indeed, by (15) we have $n_1 \geq c \frac{\|X\|_F^2}{\lambda_x}$ for some large enough constant $c$, which combined with (22) leads to $n_1 \geq c \frac{\|X\|_F^2}{\lambda_x} \geq c \frac{\|X\|_2^2}{\lambda_x}$.
As a result, the initial conditions (6)-(7) are satisfied and we can apply Theorem 3.2. Let us now bound the quantities \( \alpha_0, Q_0 \) and \( Q_1 \) defined in (8). Note that \( \lambda_F = \sigma_{\min}(\sigma(XW_0^0)) \leq \|\sigma(XW_0^0)\|_F \leq \|X\|_F \|W_0^0\|_2 \). Then,
\[
\frac{\lambda_1}{2e^L} \leq \alpha_0 \leq 2e^L \frac{\|X\|_F^3}{\lambda_1},
\]
and
\[
Q_0 \leq 2e^L \|X\|_F^2 \frac{\lambda_1}{d} + 2e^L \frac{\lambda_1}{d} \|X\|_F (1 + \|X\|_F) \sqrt{2F(\theta_0)} \leq \frac{2e^L \lambda_1}{d} \max(1, \|X\|_F^2) \max \left( 1, \frac{\sqrt{\lambda_1 + t}}{\sqrt{d}}, \|Y\|_F \right)
\]
by (40).

It is easy to see that the upper bound of \( Q_0 \) dominates that of \( \alpha_0 \). Thus to satisfy the learning rate condition (9) from Theorem 3.2, it suffices to set \( \eta \) to be smaller than the inverse of the upper bound on \( Q_0 \), which leads to condition (16).

From the lower bound of \( \alpha_0 \) in (43) and (10), we immediately get the convergence of the loss as stated in (17). Similarly, one can compute the quantity \( Q_1 \) defined in (11) to get the convergence of the parameters as stated in (18).

\[\square\]

**D. Proofs for Lower Bound on \( \lambda_* \)**

**D.1. Proof of Lemma 3.4**

We first show an intermediate result in Lemma D.1. The idea of using a Vandermonde matrix for proving full-rank features is similar to (Li et al., 2018).

**Lemma D.1** Let the training data and the activation function satisfy the conditions of Lemma 3.4. Then the set \( \{W \in \mathbb{R}^{d \times N} \mid \text{rank}(\sigma(XW)) < N\} \) has Lebesgue measure zero.

**Proof:** Let \( X = [x_1, \ldots, x_N]^T, W = [w_1, \ldots, w_N] \) and \( f : \mathbb{R}^{d \times N} \rightarrow \mathbb{R} \) be a function defined as
\[
f(W) = \det(\sigma(XW)) = \det \begin{bmatrix}
\sigma(x_1^Tw_1) & \sigma(x_1^Tw_2) & \cdots & \sigma(x_1^Tw_N) \\
\sigma(x_2^Tw_1) & \sigma(x_2^Tw_2) & \cdots & \sigma(x_2^Tw_N) \\
& \ddots & \ddots & \ddots \\
\sigma(x_N^Tw_1) & \sigma(x_N^Tw_2) & \cdots & \sigma(x_N^Tw_N)
\end{bmatrix}.
\]

By hypothesis, \( \sigma \) is real analytic, and thus \( f \) is a real analytic function. It follows from Lemma A.3 that either \( f \) is a constant zero function, or the zeros of \( f \) has Lebesgue measure zero in which case the result of the lemma follows immediately. Thus it remains to check that \( f \) is not zero everywhere. In the following, we prove this statement by induction on \( N \). First, the statement is true for \( N = 1 \) since one can pick \( w_1 = b \frac{x_1}{\|x_1\|_2} \) for some \( b \in \mathbb{R} \) with \( \sigma(b) \neq 0 \) so that \( f(W) = \sigma(x_1^Tw_1) = \sigma(b) \neq 0 \). Now, suppose that the statement holds for \( N - 1 \), and we want to prove it for \( N \).

Indeed, assume by contradiction that \( f(W) = 0 \) for every \( W \in \mathbb{R}^{d \times N} \). One can easily check that the high order derivatives of \( f \) w.r.t. \( w_1, s \) are given by
\[
\frac{\partial^k f}{\partial w_1^k} = \det \begin{bmatrix}
x_1^s \sigma^{(k)}(x_1^Tw_1) & \sigma(x_1^Tw_2) & \cdots & \sigma(x_1^Tw_N) \\
x_2^s \sigma^{(k)}(x_2^Tw_1) & \sigma(x_2^Tw_2) & \cdots & \sigma(x_2^Tw_N) \\
& \ddots & \ddots & \ddots \\
x_N^s \sigma^{(k)}(x_N^Tw_1) & \sigma(x_N^Tw_2) & \cdots & \sigma(x_N^Tw_N)
\end{bmatrix} =: \det(B_k).
\]

Since \( f \) is constant zero, it holds that \( \frac{\partial^k f}{\partial w_1^k} \bigg|_W = 0 \) for every \( W \) and \( k \geq 0 \). By induction assumption, one can pick \( N - 1 \) vectors \( \{w_2, \ldots, w_N\} \) so that the following matrix
\[
E = \begin{bmatrix}
\sigma(x_1^Tw_2) & \cdots & \sigma(x_1^Tw_N) \\
\sigma(x_2^Tw_2) & \cdots & \sigma(x_2^Tw_N) \\
& \ddots & \ddots \\
\sigma(x_{N-1}^Tw_2) & \cdots & \sigma(x_{N-1}^Tw_N)
\end{bmatrix} \in \mathbb{R}^{(N-1) \times (N-1)}
\]
Global Convergence of Deep Networks with One Wide Layer Followed by Pyramidal Topology

has full rank. One observes that $E$ is a submatrix contained in the last $N - 1$ columns of $B_k$, and so the last $N - 1$ columns of $B_k$ are linearly independent. This combined with the fact that $\det B_k = 0$ for all $W$ implies that the set of vectors associated to the 1st column of $B_k$ for all possible values of $w_1$ and $k \geq 0$ must lie on the same linear subspace spanned by the last $N - 1$ columns of $B_k$, which has been fixed by the choice of $\{w_2, \ldots, w_N\}$ as above. Pick $w_1 = 0$ and denote the first column of $B_k$ as

$$v_k = \begin{bmatrix} x_{1,s}^k \sigma^{(k)}(0) \\ x_{2,s}^k \sigma^{(k)}(0) \\ \vdots \\ x_{N,s}^k \sigma^{(k)}(0) \end{bmatrix}.$$  

Then it follows from above that $\{v_2, v_4, \ldots, v_{2N}\}$ lie on the same linear subspace of dimension $N - 1$, and thus the following matrix

$$V := [v_2, v_4, \ldots, v_{2N}] = \begin{bmatrix} x_{1,s}^2 \sigma^{(2)}(0) & x_{1,s}^4 \sigma^{(4)}(0) & \cdots & x_{1,s}^{2N} \sigma^{(2N)}(0) \\ x_{2,s}^2 \sigma^{(2)}(0) & x_{2,s}^4 \sigma^{(4)}(0) & \cdots & x_{2,s}^{2N} \sigma^{(2N)}(0) \\ \vdots & \vdots & \ddots & \vdots \\ x_{N,s}^2 \sigma^{(2)}(0) & x_{N,s}^4 \sigma^{(4)}(0) & \cdots & x_{N,s}^{2N} \sigma^{(2N)}(0) \end{bmatrix} \in \mathbb{R}^{N \times N}$$

must have rank at most $N - 1$. Let $z_i = x_{is}^2$ for every $i \in [N]$ then,

$$V = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} 1 & z_1 & z_2^2 & \cdots & z_1^{N-1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_N & z_N^2 & \cdots & z_N^{N-1} \end{bmatrix} \begin{bmatrix} \sigma^{(2)}(0) \\ \sigma^{(4)}(0) \\ \vdots \\ \sigma^{(2N)}(0) \end{bmatrix}.$$

Note that the middle one is a Vandermonde matrix and thus,

$$\det(V) = \left( \prod_{i=1}^{N} z_i \right) \left( \prod_{1 \leq i < j \leq N} (z_j - z_i) \right) \left( \prod_{k=1}^{N} \sigma^{(2k)}(0) \right).$$

By our assumption, $z_i = x_{is}^2 \neq 0$ and $z_i \neq z_j$ for all $i \neq j$. Moreover, $\sigma^{(2k)}(0) \neq 0$ for all $k \geq 1$. Thus $\det(V) \neq 0$, which is a contradiction. \hfill \Box

**Proof of Lemma 3.4.** Recall $X = [x_1, \ldots, x_N]^T \in \mathbb{R}^{N \times d}$. Let $f_i : \mathbb{R}^d \to \mathbb{R}$ be defined as $f_i(w) = \sigma((x_i, w))$ for every $i \in [N]$. Let $P$ denote the Gaussian measure in $\mathbb{R}^d$ with mean zero and covariance matrix $\frac{1}{2} I_d$. By definition, $[G_{*}]_{ij} = \int_{\mathbb{R}^d} f_i(x)f_j(x)dP(x)$. Suppose by contradiction that $\lambda_* = 0$. Then there exist coefficients $\alpha_1, \ldots, \alpha_N$ not all zero so that

$$0 = \sum_{i=1}^{N} \alpha_i \int f_i f_j dP = \int \left( \sum_{i=1}^{N} \alpha_i f_i \right) f_j dP, \quad \forall j \in [N]$$

which implies

$$0 = \sum_{j=1}^{N} \alpha_j \int \left( \sum_{i=1}^{N} \alpha_i f_i \right) f_j dP = \int \left( \sum_{i=1}^{N} \alpha_i f_i \right)^2 dP$$

and thus by Proposition A.1,

$$P\left( \sum_{i=1}^{N} \alpha_i f_i(w) \neq 0 \right) = 0.$$
Let $W = [w_1, \ldots, w_N]$ where $w_j$’s are $N$ independent random vectors drawn from $P$. Let $P_W$ denote the joint distribution over $W$. By the union bound,

$$P_W\left( \exists j \in [N]: \sum_{i=1}^{N} \alpha_i f_i(w_j) \neq 0 \right) \leq \sum_{j=1}^{N} P_W\left( \sum_{i=1}^{N} \alpha_i f_i(w_j) \neq 0 \right) = 0$$

Let $A = \sigma(XW)$. Since not all $\alpha_i$’s are zero,

$$P_W\left( \exists j \in [N]: \sum_{i=1}^{N} \alpha_i f_i(w_j) \neq 0 \right) = P_W\left( \sum_{i=1}^{N} \alpha_i A_i \neq 0 \right) \geq P_W(\det(A) \neq 0).$$

This combined with the previous inequality leads to $P_W(\det(A) \neq 0) = 0$. Moreover, by Lemma D.1, the set of $W$ for which $\det(A) = 0$ has Lebesgue measure zero. As $P_W$ has a density w.r.t. the Lebesgue measure, this implies $P_W(\det(A) = 0) = 0$ and thus $P_W(\det(A) \neq 0) = 1$, which is a contradiction. Thus $\lambda_* > 0$, as $G_*$ is p.s.d. \hfill \Box

D.2. Activation Function (3) Satisfies the Conditions of Theorem 3.5.

Clearly $\sigma$ is continuous and smooth according to Lemma B.1. Moreover, we have $|\sigma(x)| \leq |x|$ and thus,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma(x)| e^{-x^2/2} dx \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-x^2/2} dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{-x^2/2} dx = \frac{2}{\sqrt{2\pi}} < \infty.$$ 

Next, we verify the polynomial bounds on the derivatives of $\sigma$. Let $\beta = (1 - \gamma)\beta_0$. Then from Lemma B.1 we have:

$$\sigma^{(1)}(x) = \gamma + (1 - \gamma)\Psi(\beta_0 \sqrt{2\pi} x),$$

$$\sigma^{(2)}(x) = (1 - \gamma)\beta_0 \sqrt{2\pi} \Psi'(\beta_0 \sqrt{2\pi} x) = (1 - \gamma)\beta_0 e^{-\beta_0^2 \pi x^2},$$

$$\sigma^{(3)}(x) = -(1 - \gamma)2\beta_0^3 \pi x e^{-\beta_0^2 \pi x^2},$$

$$\sigma^{(4)}(x) = -(1 - \gamma)2\beta_0^3 \pi \left(e^{-\beta_0^2 \pi x^2} - 2\beta_0^2 \pi x^2 e^{-\beta_0^2 \pi x^2}\right).$$

This implies that

$$|\sigma^{(1)}(x)| \leq (1 - \gamma), \quad |\sigma^{(2)}(x)| \leq (1 - \gamma)\beta_0,$$

$$|\sigma^{(3)}(x)| \leq (1 - \gamma)2\beta_0^3 \pi |x|, \quad |\sigma^{(4)}(x)| \leq (1 - \gamma)2\beta_0^3 \pi (1 + 2\beta_0^2 \pi |x|^2).$$

\hfill \Box

D.3. Background on Hermite Expansions

Let $L^2(\mathbb{R}, w(x))$ denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} f^2(x)w(x)dx < \infty.$$ 

The normalized probabilist’s hermite polynomials are given by

$$h_r(x) = \frac{1}{\sqrt{r!}}(-1)^r e^{x^2/2} \frac{d^r}{dx^r} e^{-x^2/2}.$$ 

The functions $\{h_r(x)\}_{r=0}^{\infty}$ form a complete basis of $L^2\left(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}}\right)$. In particular, let $\sigma \in L^2\left(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}}\right)$ then

$$\sigma(x) = \sum_{r=0}^{\infty} \mu_r(\sigma) h_r(x)$$ \hspace{1cm} (44)
where “=” is understood in the sense of $L^2$ convergence, and $\mu_r(\sigma)$ is the $r$-th Hermite coefficient given by

$$\mu_r(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma(y) h_r(y) e^{-y^2/2} dy.$$ 

Lemma D.2, D.3 and D.4 state some useful properties of Hermite polynomials, see (Abramowitz, 1974).

**Lemma D.2** We have for every $r \geq 0$ and $x \in \mathbb{R}$ that

$$h_r(x) = \sqrt{\pi} \sum_{m=0}^{[\frac{r}{2}]} (-1)^m \frac{2^m}{m!(r-2m)!} x^{r-2m}.$$ 

**Lemma D.3 (Cramer’s inequality)** $|h_r(x)| \leq 2e^{x^2}/4$ for every $r \geq 0, x \in \mathbb{R}$.

Note that the constant 2 above is not optimal compared to the original version, but it suffices for the purpose of this paper.

**Lemma D.4** $h'_r(x) = \sqrt{r} h_{r-1}(x)$ for every $r \geq 0, x \in \mathbb{R}$.

**Lemma D.5** Let $k \geq 1$ be given. For every $j \in [0, 2k - 1]$, we assume that $|\sigma^{(j)}(x)| \leq P_j(|x|)$ for some polynomial $P_j$ of finite degree. Then,

$$|\mu_r(\sigma)| \leq (r+1)^{-k} \int_{-\infty}^{\infty} \left| \frac{d^{(2k)}}{d^r} \right| (\sigma(y)e^{-y^2/2}) h_{r+1}(y) dy, \quad \forall r \geq 0.$$ 

**Proof:** By Lemma D.4 we have $\int h_r(y) dy = \frac{h_{r+1}}{\sqrt{r+1}}$. Integrating by parts $\mu_r(\sigma)$ we have

$$\mu_r(\sigma) = \frac{1}{\sqrt{2\pi}} \left[ \sigma(y)e^{-y^2/2} \frac{h_{r+1}(y)}{\sqrt{r+1}} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dy} \left( \sigma(t)e^{-t^2/2} \right) \frac{h_{r+1}(y)}{\sqrt{r+1}} dy.$$ 

As $|\sigma(y)|$ does not grow faster than a polynomial, the presence of $e^{-y^2/2}$ makes sure that the first term vanishes. Thus,

$$\mu_r(\sigma) = \frac{1}{\sqrt{2\pi}} \frac{(-1)^k}{\sqrt{r+1}} \int_{-\infty}^{\infty} \frac{d}{dy} \left( \sigma(y)e^{-y^2/2} \right) h_{r+1}(y) dy$$

Similarly, we can integrate by parts the RHS to higher orders. From our assumption, it is easy to see that for every $j \in [0, 2k - 1]$, there exists a polynomial $P$ of finite degree such that $\left| \frac{d^{(2k)}}{d^r} \sigma(y)e^{-y^2/2} \right| \leq P(|y|)e^{-y^2/2}$. Thus, the presence of $e^{-y^2/2}$ always make the first term of the integration by parts vanish. Finally, we get

$$\mu_r(\sigma) = \frac{1}{\sqrt{2\pi}} \frac{(-1)^{2k}}{\sqrt{(r+1)\ldots(r+2k)}} \int_{-\infty}^{\infty} \frac{d^{(2k)}}{d^r} \left( \sigma(y)e^{-y^2/2} \right) h_{r+1}(y) dy$$

which implies the desired result. \( \square \)

**Corollary D.6** Let $|\sigma^{(j)}(x)| \leq P_j(|x|)$ for some polynomial $P_j$ of finite degree, for every $j \in [0, 4]$. Then, there exists a constant $C$ such that, for every $r \geq 0$, we have

$$|\mu_r(\sigma)| \leq \frac{C}{(r+1)^2}.$$ 

**Proof:** By assumption, there exists a polynomial $P$ of finite degree such that

$$\left| \frac{d^4}{d^4} \sigma(y)e^{-y^2/2} \right| \leq P(|y|)e^{-y^2/2}. \quad (45)$$
Suppose that $P$ has degree $n$ and is given by $P(|y|) = \sum_{k=0}^{n} a_k |y|^k$. Using Lemma D.5 for $k = 2$, we get

$$|\mu_r(\sigma)| \leq (r + 1)^{-2} \int_{-\infty}^{\infty} \left| \frac{d^4}{dy^4} \left( \sigma(y)e^{-y^2/2} \right) \right| h_{r+1}(y) \, dy$$

$$\leq (r + 1)^{-2} \int_{-\infty}^{\infty} P(|y|)e^{-y^2/2} |h_{r+1}(y)| \, dy \quad \text{by (45)}$$

$$\leq 2(r + 1)^{-2} \int_{-\infty}^{\infty} P(|y|)e^{-y^2/4} \, dy$$

$$= 4(r + 1)^{-2} \sum_{k=0}^{n} a_k \int_{0}^{\infty} y^k e^{-y^2/4} \, dy$$

$$= 8(r + 1)^{-2} \sum_{k=0}^{n} a_k 2^{k-1} \Gamma \left( \frac{k+1}{2} \right)$$

$$\leq C(r + 1)^{-2},$$

where we bounded the sum by a constant $C$. Note here that $n$ and the $a_k$’s are being treated as constants of $\sigma$. $\square$

**Lemma D.7** Let $x, y \in \mathbb{R}^d$ be such that $\|x\|_2 = \|y\|_2 = 1$. Then, for every $j, k \geq 0$,

$$\mathbb{E}_{w \sim \mathcal{N}(0, I_d)} \left[ h_j(\langle w, x \rangle)h_k(\langle w, y \rangle) \right] = \begin{cases} \langle x, y \rangle^j & j = k \\ 0 & j \neq k \end{cases}$$

**Proof:** Let $s, t \in \mathbb{R}$ be given finite variables. Then,

$$\mathbb{E} \exp \left( s \langle w, x \rangle + t \langle w, y \rangle \right) = \prod_{i=1}^{d} \mathbb{E} \exp \left( w_i (sx_i + ty_i) \right)$$

$$= \prod_{i=1}^{d} \exp \left( \frac{(sx_i + ty_i)^2}{2} \right)$$

$$= \exp \left( \frac{s^2 + t^2 + 2st \langle x, y \rangle}{2} \right).$$

Thus, it follows that

$$\mathbb{E} \exp \left( s \langle w, x \rangle - \frac{s^2}{2} \right) \exp \left( t \langle w, y \rangle - \frac{t^2}{2} \right) = \exp \left( st \langle x, y \rangle \right).$$

The generating function of the normalized probabilist’s Hermite polynomials is

$$\exp \left( su - \frac{s^2}{2} \right) = \sum_{j=0}^{\infty} h_j(u) \frac{s^j}{\sqrt{j!}}.$$

Plugging this into the above formula gives

$$\mathbb{E} \sum_{j,k=0}^{\infty} h_j(\langle w, x \rangle)h_k(\langle w, y \rangle) \frac{s^j t^k}{\sqrt{j!k!}} = \sum_{j=0}^{\infty} \langle x, y \rangle^j \frac{s^j t^j}{j! \sqrt{j!}}.$$

Note that the LHS can be rewritten as

$$\mathbb{E} \lim_{n \to \infty} \sum_{j,k=0}^{n} h_j(\langle w, x \rangle)h_k(\langle w, y \rangle) \frac{s^j t^k}{\sqrt{j!k!}} =: f_{x,y}(w).$$
By Lebesgue’s dominated convergence theorem, we get
\[
\sum_{j,k=0}^{\infty} \mathbb{E} \left( \frac{h_j(\langle w, x \rangle) h_k(\langle w, y \rangle)}{\sqrt{j!k!}} \right) s^j t^k = \sum_{j=0}^{\infty} \frac{\langle x, y \rangle^j}{j!} s^j t^j, \quad \forall s, t \in \mathbb{R}.
\]
Equating the coefficients on both sides gives the desired result.

It remains to justify the application of the dominated convergence theorem. Since \(\{f_n\}\) are obtained from the Taylor series of exponential functions in \(f\), it follows that \(\lim_{n \to \infty} f_n(w) = f(w)\) for every \(w \in \mathbb{R}^d\). By abuse of notation, let \(\mu\) denote the standard Gaussian measure in \(\mathbb{R}^d\). Then,
\[
|f_n(w)| \leq \left[ \sum_{j=0}^{n} \frac{|h_j(\langle w, x \rangle)||s|^j}{\sqrt{j!}} \right] \left[ \sum_{k=0}^{n} \frac{|h_k(\langle w, y \rangle)||t|^k}{\sqrt{k!}} \right]
\]
Let \(u = \langle w, x \rangle\). By Lemma D.2 we have
\[
\sum_{j=0}^{n} \frac{|h_j(\langle w, x \rangle)||s|^j}{\sqrt{j!}} = n \sum_{j=0}^{n} \left( \frac{1}{2} \right)^{j-2m} |s|^{j-2m} \frac{1}{m!(j-2m)!2^m} 
\leq \left( \frac{1}{2} \right)^n \sum_{m=0}^{n} \frac{|s|^{2m}}{m!2^m} 
\leq \exp \left( \|us\| + |s|^2 \right).
\]
By a similar calculation, we obtain
\[
|f_n(w)| \leq \exp \left( |s|^2 + |t|^2 \right) \exp \left( |s \langle w, x \rangle| + |t \langle w, y \rangle| \right) =: g(w)
\]
Thus \(f_n\)'s are dominated by \(g\) on \(\mathbb{R}^d\). It suffices to show that \(g\) is integrable w.r.t. the measure \(\mu\). Let
\[
R_1 := \{ w \in \mathbb{R}^d \mid \langle w, sx \rangle \geq 0, \langle w, ty \rangle \geq 0 \}, \quad R_2 := \{ w \in \mathbb{R}^d \mid \langle w, sx \rangle \geq 0, \langle w, ty \rangle \leq 0 \}, \\
R_3 := \{ w \in \mathbb{R}^d \mid \langle w, sx \rangle \leq 0, \langle w, ty \rangle \geq 0 \}, \quad R_4 := \{ w \in \mathbb{R}^d \mid \langle w, sx \rangle \leq 0, \langle w, ty \rangle \leq 0 \}.
\]
Note that \(\mathbb{R}^d = R_1 \cup R_2 \cup R_3 \cup R_4\). We have
\[
\int_{\mathbb{R}^d} |g| \, d\mu = \exp \left( |s|^2 + |t|^2 \right) (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp \left( -\frac{\|w\|^2}{2} - 2 \frac{\langle w, sx \rangle}{\|t\|} - 2 \frac{\langle w, ty \rangle}{\|t\|} \right) \, dw.
\]
The integral of \(Q(w)\) over \(R_1\) is given by
\[
\int_{R_1} \exp \left( -\frac{\|w-(sx+ty)\|^2}{2} \right) \exp \left( \frac{\|sx+ty\|^2}{2} \right) \, dw 
\leq \exp \left( \frac{\|sx+ty\|^2}{2} \right) \int_{\mathbb{R}^d} \exp \left( -\frac{\|w-(sx+ty)\|^2}{2} \right) \, dw 
< \infty.
\]
By similar calculations for \(R_2, R_3, R_4\), we get \(\int_{\mathbb{R}^d} Q(w) \, d\mu < \infty\), and thus \(g\) is integrable w.r.t. the measure \(\mu\). \qed
D.4. Proof of Lemma 4.4

Let \( \bar{x}_i = x_i / \|x_i\|_2 \) for \( i \in [N] \). From the definition of \( G_\ast \), we have

\[
(G_\ast)_{ij} = E_{\bar{w} \sim \mathcal{N}(0,1_d/d)}[\sigma(\langle w, \bar{x}_i \rangle)\sigma(\langle w, x_j \rangle)] \\
= E_{\bar{w} \sim \mathcal{N}(0,1_d)}[\sigma(\langle \bar{w}, \bar{x}_i \rangle)\sigma(\langle \bar{w}, \bar{x}_j \rangle)] \\
= E_{\bar{w} \sim \mathcal{N}(0,1_d)}\left[ \sum_{r,s=0}^\infty \mu_r(\sigma)\mu_s(\sigma)h_r(\langle \bar{w}, \bar{x}_i \rangle)h_s(\langle \bar{w}, \bar{x}_j \rangle) \right].
\]

By Fubini’s theorem, we have

\[
(G_\ast)_{ij} = \sum_{r,s=0}^\infty \mu_r(\sigma)\mu_s(\sigma)E_{\bar{w} \sim \mathcal{N}(0,1_d)}[h_r(\langle \bar{w}, \bar{x}_i \rangle)h_s(\langle \bar{w}, \bar{x}_j \rangle)] \\
= \sum_{r=0}^\infty \mu_r^2(\sigma) \langle \bar{x}_i, \bar{x}_j \rangle^r
\]

Lemma D.7

Here, we note that \( \langle \bar{x}_i, \bar{x}_j \rangle^r = \frac{1}{d^r} \langle x_i \otimes \cdots \otimes x_i, x_j \otimes \cdots \otimes x_j \rangle \). Thus,

\[
G_\ast = \sum_{r=0}^\infty \frac{\mu_r^2(\sigma)}{d^r} (X^{*r})(X^{*r})^T.
\]

It remains to justify the application of Fubini’s Theorem. It suffices to show that

\[
\sum_{r,s=0}^\infty E_{\bar{w} \sim \mathcal{N}(0,1_d)}[\mu_r(\sigma)\mu_s(\sigma)h_r(\langle \bar{w}, \bar{x}_i \rangle)h_s(\langle \bar{w}, \bar{x}_j \rangle)] < \infty.
\]

Indeed, the LHS is upper bounded by

\[
\sum_{r,s=0}^\infty |\mu_r(\sigma)||\mu_s(\sigma)| \sqrt{E_{\bar{w} \sim \mathcal{N}(0,1_d)}h_r(\langle \bar{w}, \bar{x}_i \rangle)^2} \sqrt{E_{\bar{w} \sim \mathcal{N}(0,1_d)}h_s(\langle \bar{w}, \bar{x}_j \rangle)^2} = \left[ \sum_{r=0}^\infty |\mu_r(\sigma)| \right]^2,
\]

where the equality follows from Lemma D.7. By Corollary D.6, the infinite series on the RHS can be upper bounded, up to some constant, by the series \( \sum_{r=0}^\infty \frac{1}{(r+1)^r} \), which is known to be convergent. \( \square \)

D.5. Proof of Lemma 4.5

Recall that, given a random variable \( Y \in \mathbb{R} \), its sub-exponential norm is defined as

\[
\|Y\|_{\psi_1} = \inf \{C > 0 : E[e^{\|Y\|/C}] \leq 2 \}.
\]

Furthermore, for a centered random vector \( x \in \mathbb{R}^d \), its sub-exponential norm is defined as

\[
\|x\|_{\psi_1} = \sup_{\|y\|_2 = 1} \|\langle x, y \rangle\|_{\psi_1}.
\]

We start by stating two intermediate results that will be useful for the proof.

**Lemma D.8** Consider an \( r \)-indexed matrix \( A = (a_{i_1,...,i_r})_{i_1,...,i_r=1}^d \) such that \( a_{i_1,...,i_r} = 0 \) whenever \( i_j = i_k \) for some \( j \neq k \). Let \( x = (x_1, \ldots, x_d) \) be a random vector in \( \mathbb{R}^d \) uniformly distributed on the unit sphere, and define

\[
Z = \sum_{i \in [d]^r} a_i \prod_{j=1}^r x_{i_j}.
\]

Then,

\[
E \left[ e^{Cd|Z|^2/r} \right] \leq 2,
\]

where \( C \) is a numerical constant.
If \( x \) is uniformly distributed on the unit sphere, then it satisfies the logarithmic Sobolev inequality with constant \( 2/d \), see Corollary 1.1 in (Dolbeault et al., 2014). Thus, Lemma D.8 follows from Theorem 1.14 in (Bobkov et al., 2019), where \( \alpha^2 = 1/d \) (see (1.18) in (Bobkov et al., 2019)) and the function \( f \) is a homogeneous tetrahedral polynomial of degree \( r \).

Let us also point out that, in (Adamczak & Wolff, 2015), a similar concentration result is proved for a random vector \( x \) that satisfies

\[
|g(x) - \mathbb{E}[g(x)]|_p \leq C \sqrt[p]{\|\nabla g(x)\|_2^p},
\]

for any \( p \geq 2 \), for any smooth bounded function \( g : \mathbb{R}^d \to \mathbb{R} \), and for some constant \( C \) independent of \( g \) and \( p \). It is known that if \( x \) satisfies the logarithmic Sobolev inequality with constant \( D_{LS} \), then (50) holds with \( C = \sqrt{D_{LS}/2} \), see (Aida & Stroock, 1994) or Theorem 3.4 of (Adamczak & Wolff, 2015). However, these results refer to random vectors (and test functions \( g \)) defined on \( \mathbb{R}^d \). Here, we are interested in a concentration result for vectors uniformly distributed on the sphere, and this case is analyzed in (Bobkov et al., 2019).

The second intermediate lemma is stated below and it follows from Theorem 5.1 of (Adamczak et al., 2011) (this is also basically a restatement of Lemma F.2 of (Soltanolkotabi et al., 2018)).

**Lemma D.9** Let \( u_1, u_2, \ldots, u_N \) be independent sub-exponential random vectors with \( \psi = \max_{i \in [N]} \|u_i\|_{\psi_1} \). Let \( \eta_{\max} = \max_{i \in [N]} \|u_i\|_2 \) and define

\[
B_N = \sup_{z : \|z\|_2 = 1} \left| \sum_{i \neq j} \langle z_i u_i, z_j u_j \rangle \right|^{1/2}.
\]

Then,

\[
P \left( B_N^2 \geq \max(B^2, \eta_{\max}^2 B, \eta_{\max}^2/4) \right) \leq (1 + 3 \log N) e^{-11\sqrt{N}},
\]

where

\[
B = C_0 \psi \sqrt{N},
\]

and \( C_0 \) is a numerical constant.

At this point, we are ready to provide the proof of Lemma 4.5.

**Proof of Lemma 4.5.** The first step is to drop columns from \( X^{*r} \). Define \( K = X^{*r} \) and note that, for \( i \in [N] \), the \( i \)-th row of \( K \) is given by the \( r \)-th Kronecker power of \( x_i \), namely, \( x_i^{\otimes r} = x_i \otimes x_i \otimes \cdots \otimes x_i \in \mathbb{R}^{d^r} \). Let \( x_i = (x_{i,1}, \ldots, x_{i,d}) \) and index the columns of \( K \) as \( (j_1, j_2, \ldots, j_r) \), with \( j_p \in [d] \) for all \( p \in [r] \), so that the element of \( K \) in row \( i \) and column \((j_1, j_2, \ldots, j_r)\) is given by \( \prod_{p=1}^{r} x_{i,j_p} \). Consider the matrix \( \tilde{K} \) obtained by keeping only the columns of \( K \) where the indices \( j_1, j_2, \ldots, j_r \) are all different. Note that \( \tilde{K} \) has \( \prod_{j=0}^{r-1}(d - j) \geq N \) columns as \( N \leq c_1 d^2 \). Thus, as \( \tilde{K} = X^{*r} \) is obtained by dropping columns from \( K \), then

\[
\sigma_{\min}(K) \geq \sigma_{\min}(\tilde{K}).
\]

The second step is to bound the sub-exponential norm of the rows of \( \tilde{K} \). Let \( \tilde{k}_x \) be the row of \( \tilde{K} \) corresponding to the data point \( x = (x_1, \ldots, x_d) \). Let us emphasize that, from now till the end of the proof, we denote by \( x_i \in \mathbb{R} \) the \( i \)-th element of the vector \( x \) (and not the \( i \)-th training sample, which is a vector in \( \mathbb{R}^d \)). Let \( A \) be the set of \( r \)-indexed matrices \( A = (a_{i_1, \ldots, i_r})_{i_1, \ldots, i_r=1} \) such that \( \sum_{i \in [d]} a_i^2 = 1 \) and \( a_{i_1, \ldots, i_r} = 0 \) whenever \( i_j = i_k \) for some \( j \neq k \). Then, by definition of sub-exponential norm of a vector, we have that

\[
\|\tilde{k}_x\|_{\psi_1} = \sup_{A \in A} \left\| \sum_{i \in [d]^r} a_i \prod_{j=1}^{r} x_{i,j} \right\|_{\psi_1}.
\]

Note that, for all \( A \in \mathcal{A} \),

\[
\left\| \sum_{i \in [d]^r} a_i \prod_{j=1}^{r} x_{i,j} \right\|_{\psi_1} \leq \sum_{i \in [d]} a_i^2 \left\| \prod_{j=1}^{r} x_{i,j} \right\|_{\psi_1} \approx d^{r/2},
\]

(56)
As for the lower bound, we have that \( \sum_{i \in [d]} a_i^2 = 1 \) and \( \|x\|_2 = \sqrt{d} \). Consequently,

\[
\|\tilde{k}_x\|_{\psi_1} = \sup_{A \in \mathcal{A}} \left\| \sum_{i \in [d]} a_i \prod_{j=1}^r x_{i_j} \right\|^{1-2/r} \| \sum_{i \in [d]} a_i \prod_{j=1}^r x_{i_j} \|^{2/r}_{\psi_1}
\leq d^{r/2-1} \sup_{A \in \mathcal{A}} \left\| \sum_{i \in [d]} a_i \prod_{j=1}^r x_{i_j} \right\|^{2/r}_{\psi_1}.
\]  

(57)

Note that Lemma D.8 considers a vector \( x \) uniformly distributed on the unit sphere, while in (57) \( x \) is uniformly distributed on the sphere with radius \( \sqrt{d} \). Thus, (49) can be re-written as

\[
\mathbb{E} \left[ \exp \left( C \left| \sum_{i \in [d]} a_i \prod_{j=1}^r x_{i_j} \right|^{2/r} \right) \right] \leq 2.
\]  

(58)

By definition (46) of sub-exponential norm, we obtain that

\[
\sup_{A \in \mathcal{A}} \left\| \sum_{i \in [d]} a_i \prod_{j=1}^r x_{i_j} \right\|^{2/r}_{\psi_1} = \frac{1}{C},
\]  

(59)

which, combined with (57), leads to

\[
\|\tilde{k}_x\|_{\psi_1} \leq C_1 d^{r/2-1},
\]  

(60)

where \( C_1 \) is a numerical constant.

The third step is to bound the Euclidean norm of the rows of \( \tilde{K} \). Recall that \( \tilde{k}_x \) is obtained by keeping the elements of \( x^{\otimes r} \) where the indices \( i_1, i_2, \ldots, i_r \) are all different. As for the upper bound, we have that

\[
\|\tilde{k}_x\|_2^2 \leq \|x^{\otimes r}\|_2^2 = d^r.
\]  

(61)

As for the lower bound, we have that

\[
\|\tilde{k}_x\|_2^2 \geq \|x^{\otimes r}\|_2^2 - \left( d^r - \prod_{j=0}^{r-1} (d - j) \right) \left( \max_{i \in [d]} |x_i| \right)^{2r},
\]  

(62)

since \( x^{\otimes r} \) contains \( d^r \) entries, \( \tilde{k}_x \) contains \( \prod_{j=0}^{r-1} (d - j) \) entries and each of these entries is at most \( \left( \max_{i \in [d]} |x_i| \right)^r \). Note that \( \prod_{j=0}^{r-1} (d - j) \) is a polynomial in \( d \) of degree \( r \) whose leading coefficient is 1. Thus,

\[
\prod_{j=0}^{r-1} (d - j) \geq d^r - C_2 d^{r-1},
\]  

for some constant \( C_2 \) that depends only on \( r \). Consequently,

\[
\|\tilde{k}_x\|_2^2 \geq d^r - C_2 d^{r-1} \left( \max_{i \in [d]} |x_i| \right)^{2r}.
\]  

(63)

As \( x \) is uniform on the sphere of radius \( \sqrt{d} \), we can write

\[
x = \sqrt{d} \frac{g}{\|g\|_2},
\]  

(64)

where \( g = (g_1, \ldots, g_d) \sim \mathcal{N}(0, I_d) \). Then,

\[
\left( \max_{i \in [d]} |x_i| \right)^{2r} = \left( \sqrt{d} \frac{\|g\|_2}{\|g\|_2} \right)^{2r} \left( \max_{i \in [d]} |g_i| \right)^{2r}.
\]  

(65)
Recall that the norm of a vector is a 1-Lipschitz function of the components of the vector. Thus,

\[ P \left( \|g\|_2 - E[\|g\|_2] \geq t \right) \leq 2e^{-t^2/2}. \]  

(66)

Furthermore,

\[ E[\|g\|_2] = \frac{\sqrt{2} \Gamma \left( \frac{d+1}{2} \right)}{\Gamma \left( \frac{d}{2} \right)}, \]

(67)

where \( \Gamma \) denotes Euler’s gamma function. By Gautschi’s inequality, we have the following upper and lower bounds on \( E[\|g\|_2] \):

\[ \sqrt{d-1} \leq E[\|g\|_2] \leq \sqrt{d+1}. \]

(68)

As a result,

\[ P \left( \left( \frac{\sqrt{d}}{\|g\|_2} \right)^{2r} - 1 > \frac{1}{2} \right) \leq 2e^{-C_3d}, \]

(69)

for some constant \( C_3 > 0 \) depending on \( r \) (but not on \( d \)). Consequently, with probability at least \( 1 - 2e^{-C_3d} \), we have that

\[ (\max_{i \in [d]} \|x_i\|)^{2r} \leq \frac{3}{2} (\max_{i \in [d]} |g_i|)^{2r}. \]

(70)

An application of Theorem 5.8 of (Boucheron et al., 2013) gives that, for any \( t > 0 \),

\[ P(\max_{i \in [d]} \|g_i - E[\max_{i \in [d]} \|g_i\|] \geq t) \leq e^{-t^2/2}. \]

(71)

Furthermore, we have that, for any \( \alpha > 0 \),

\[ e^{-\alpha E[\max_{i \in [d]} \|g_i\|]} \leq E\left[e^{\alpha \max_{i \in [d]} \|g_i\|}\right] = E\left[\max_{i \in [d]} e^{\alpha g_i}\right] \leq \sum_{i=1}^{d} E\left[e^{\alpha g_i}\right] = d e^{\alpha^2/2}, \]

(72)

where the first passage follows from Jensen’s inequality. By taking \( \alpha = \sqrt{2 \log d} \), we obtain

\[ E[\max_{i \in [d]} \|g_i\|] \leq \sqrt{2 \log d}, \]

(73)

which, combined with (71), leads to

\[ P\left(\max_{i \in [d]} \|g_i\| \geq \left(\frac{d}{3C_2}\right)^{\frac{1}{\alpha}}\right) \leq 2e^{-C_4d^{1/r}}, \]

(74)

where \( C_2 \) is the constant in (63) and \( C_4 > 0 \) is a constant that depends only on \( r \) (and not on \( d \)). Since the Gaussian distribution is symmetric, we also have that

\[ P\left(\max_{i \in [d]} |g_i| \geq \left(\frac{d}{3C_2}\right)^{\frac{1}{\alpha}}\right) \leq 4e^{-C_4d^{1/r}}. \]

(75)

By combining (63), (70) and (75), we obtain that, with probability at least \( 1 - 2e^{-C_5d^{1/r}} \),

\[ \|\tilde{K}_\ell\|_2^2 \geq \frac{d^{r}}{2}. \]

(76)

Hence, by doing a union bound on the rows of \( \tilde{K} \), we have that, with probability at least \( 1 - 2N e^{-C_5d^{1/r}} \),

\[ \min_{i \in [N]} \|\tilde{K}_i\|_2^2 \geq \frac{d^{r}}{2}, \]

(77)

\[ \max_{i \in [N]} \|\tilde{K}_i\|_2^2 \leq d^{r}, \]

(78)

where \( \tilde{K}_i \) denotes the \( i \)-th row of \( \tilde{K} \).
The last step is to apply the results of Lemma D.9. Let \( z = (z_1, \ldots, z_N) \in \mathbb{R}^N \) be such that \( \|z\|_2 = 1 \). Then,

\[
\|\tilde{K}^T z\|^2_2 = \sum_{i=1}^{N} z_i^2 \|\tilde{K}_i\|^2_2 + \sum_{i \neq j} \langle z_i \tilde{K}_i, z_j \tilde{K}_j \rangle,
\]

which immediately implies that

\[
\sigma^2_{\min}(\tilde{K}) \geq \min_{i \in [N]} \|\tilde{K}_i\|^2_2 - B_N^2,
\]

with

\[
B_N = \sup_{z: \|z\|_2 = 1} \left| \sum_{i \neq j} \langle z_i \tilde{K}_i, z_j \tilde{K}_j \rangle \right|^{1/2}.
\]

By applying Lemma D.9 and using the bounds (60) and (78), we have that

\[
P \left( B_N^2 \geq \max(C_6 d^r - 2 N, C_6 d^r - 1 \sqrt{N}, d^r / 4) \right) \leq (1 + 3 \log N)e^{-11\sqrt{N}},
\]

for some constant \( C_6 \) depending on \( r \). Recall that \( N \leq c_1 d^2 \) for a sufficiently small constant \( c_1 \) (which can depend on \( r \)). Thus, we have that

\[
P \left( B_N^2 \geq d^r / 4 \right) \leq (1 + 3 \log N)e^{-11\sqrt{N}}.
\]

By combining (80), (77) and (83), we obtain that

\[
P \left( \sigma^2_{\min}(\tilde{K}) \geq d^r / 4 \right) \geq 1 - 2Ne^{-C_5 d^r / r} - (1 + 3 \log N)e^{-11\sqrt{N}},
\]

which, together with (54), gives the desired result. \( \square \)