Cycles for rational maps of good reduction outside a prescribed set

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Abstract

Let $K$ be a number field and $S$ a fixed finite set of places of $K$ containing all the archimedean ones. Let $R_S$ be the ring of $S$-integers of $K$. In the present paper we study the cycles in $\mathbb{P}^1(K)$ for rational maps of degree $\geq 2$ with good reduction outside $S$. We say that two ordered $n$-tuples $(P_0, P_1, \ldots, P_{n-1})$ and $(Q_0, Q_1, \ldots, Q_{n-1})$ of points of $\mathbb{P}^1(K)$ are equivalent if there exists an automorphism $A \in \text{PGL}_2(R_S)$ such that $P_i = A(Q_i)$ for every index $i \in \{0, 1, \ldots, n-1\}$. We prove that if we fix two points $P_0, P_1 \in \mathbb{P}^1(K)$, then the number of inequivalent cycles for rational maps of degree $\geq 2$ with good reduction outside $S$ which admit $P_0, P_1$ as consecutive points is finite and depends only on $S$ and $K$. We also prove that this result is in a sense best possible.

1 Introduction

Let $K$ be a number field and $R$ its ring of integers. Let $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map defined over $K$ by $\Phi([x : y]) = [F(x, y) : G(x, y)]$ where $F, G \in R[x, y]$ are homogeneous of the same degree with no common factor.

Let $\mathfrak{p}$ be a prime ideal of $R$. Using the standard notation which will be introduced at the beginning of next section, we can assume that $F, G$ have coefficients in $R_{\mathfrak{p}}$ (the local ring of $R$ at the prime ideal $\mathfrak{p}$) and at least one coefficient belonging to $R_{\mathfrak{p}}^\ast$. In this way, let $K(\mathfrak{p}) = R/\mathfrak{p}$ be the residue field, we obtain the rational map defined over $K(\mathfrak{p})$

$$\tilde{\Phi} : \mathbb{P}^1 \to \mathbb{P}^1; \quad \tilde{\Phi}([x : y]) = [\tilde{F}(x, y) : \tilde{G}(x, y)]$$

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where $\tilde{F}, \tilde{G}$ are the polynomials obtained by reducing modulo $p$ the coefficients of $F$ and $G$. Since the polynomial $F, G$ have no common factors, the rational map $\tilde{\Phi}$ has degree equal to $\deg \Phi$ if and only if $\tilde{F}$ and $\tilde{G}$ do not have common roots in $\mathbb{P}_1(K(p))$, where $K(p)$ denotes the algebraic closure of $K(p)$. If this is the case we will say that $\Phi$ has good reduction at the prime ideal $p$.

A cycle of length $n$ for a rational map $\Phi$ is a ordered $n$–tuple $(P_0, P_1, \ldots, P_{n-1})$, of distinct points of $\mathbb{P}_1(K)$, with the property that $\Phi(P_i) = P_{i+1}$ for every $i \in \{0, 1, \ldots, n - 2\}$ and such that $\Phi(P_{n-1}) = P_0$. It is easy to see that every $n$–tuple of distinct points is a cycle for a suitable rational map, but imposing some restrictions on the maps gives some considerable restrictions on the cycles. In the present paper we shall consider a number field $K$ and a fixed finite set $S$ of places containing the archimedean ones. We will study the cycles for rational maps that have good reduction at every prime ideal $p$ whose associated $p$-adic place does not belong to $S$ and we will say that these maps have good reduction outside $S$.

Morton and Silverman [11, Corollary B] have proved that if $\Phi$ is a rational map of degree $\geq 2$ which has bad reduction only at $s$ prime ideals of $K$ and $P \in \mathbb{P}_1(K)$ is a periodic point with minimal period $n$, then the following inequality holds:

$$n \leq (12(s + 2) \log(5(s + 2)))^{[K:\mathbb{Q}]}. $$

These results provide some bounds for the period-length of a periodic point for a rational map $\Phi$ depending only on the number of prime ideals of bad reduction. This generalizes and improves the result by Narkiewicz [13] who was concerned with polynomial maps, in fact if $\phi(z) \in K[z]$ is a polynomial, then the corresponding map $\phi: \mathbb{P}_1 \to \mathbb{P}_1$ has good reduction outside $S$ if and only if $\phi$ has $S$-integral coefficients and its leading coefficient is a $S$-unit. For this type of polynomials, Narkiewicz found a bound for minimal period-length $n$ which is possible to write in the following way: $n \leq C^{s^2 + s[K:\mathbb{Q}]}$, where $C$ is an absolute constant. The main tool used by Narkiewicz is the finiteness of the solutions in $S$-units of the equation $u + v = 1$, and in particular the estimate of Evertse [4]. On the other hand, Morton and Silverman used their results on multiplicity and reduction obtained in [12]. R. Benedetto has recently obtained a much stronger bound for polynomial maps. He proved in [11] that if $\phi \in K[z]$ is a polynomial of degree $d \geq 2$ which has bad reduction in $s$ primes of $K$, then the number of preperiodic points of $\phi$ is at most $O(s \log s)$. The big-$O$ constant is essentially $(d^2 - 2d + 2)/\log d$ for large $s$. Benedetto’s proof relies on a detailed analysis of $p$-adic Julia sets.

Let $R_S$ be the ring of $S$-integers of $K$; the automorphism-group $\text{PGL}_2(R_S)$ acts in a canonical way on $\mathbb{P}_1(K)$. If $(P_0, P_1, \ldots, P_{n-1})$ is a $n$-cycle for a rational map
Φ, with good reduction outside S, then for every $A \in PGL_2(R_S)$ the image-$n$-tuple $(A(P_0), A(P_1), \ldots, A(P_{n-1}))$ is an $n$-cycle for the rational map $A \circ \Phi \circ A^{-1}$, which still has good reduction outside $S$; we will call the two $n$-tuples equivalent.

In [8, Theorem 1] Halter-Koch and Narkiewicz proved the finiteness of the set of possible normalized $n$-cycles in $R_S$ for polynomial maps, where a cycle is called normalized when 0 and 1 are two consecutive elements of the $n$-tuple. In the present paper we generalize to rational maps this result, in particular we prove the following corollary as a consequence of our Theorem 1 below.

**Corollary 1.** Let $P_0, P_1 \in \mathbb{P}_1(K)$ be two fixed points. The number of inequivalent cycles for rational maps of degree $\geq 2$ with good reduction outside $S$ which admit $P_0, P_1$ as consecutive points is finite and depends only on $S$ and $K$.

Let $P_1 = [x_1 : y_1], P_2 = [x_2 : y_2] \in \mathbb{P}_1(K)$ and $p$ a prime ideal of $R$. Using the notation of [12] we will denote by

$$
\delta_p(P_1, P_2) = v_p(x_1y_2 - x_2y_1) - \min\{v_p(x_1), v_p(y_1)\} - \min\{v_p(x_2), v_p(y_2)\}
$$

(1)

the $p$-adic logarithmic distance; $\delta_p(P_1, P_2)$ is independent of the choice of the homogeneous coordinates, i.e. it is well defined.

To every pair $P, Q \in \mathbb{P}_1(K)$ we associate the ideal

$$
\mathfrak{I}(P, Q) := \prod_{p \in S} p^{\delta_p(P, Q)}.
$$

It is characterized by the property that $P \equiv Q \pmod{\mathfrak{I}(P, Q)}$ and that for every ideal $\mathfrak{I}$ such that $P \equiv Q \pmod{\mathfrak{I}}$ one has $\mathfrak{I} \mid \mathfrak{I}(P, Q)$.

To every $n$-tuple $(P_0, P_1, \ldots, P_{n-1})$ we can associate the $(n-1)$-tuple of ideals $(\mathfrak{I}_1, \mathfrak{I}_2, \ldots, \mathfrak{I}_{n-1})$ defined by

$$
\mathfrak{I}_i := \prod_{p \in S} p^{\delta_p(P_0, P_i)} = \mathfrak{I}(P_0, P_i)
$$

(2)

With the above notation we will prove the following results.

**Theorem 1.** There exists a finite set $\mathfrak{I}_S$ of ideals of $R_S$, depending only on $S$ and $K$, with the following property: for every $n$-cycle $(P_0, P_1, \ldots, P_{n-1})$, for a rational map of degree $\geq 2$ with good reduction outside $S$, let $(\mathfrak{I}_1, \mathfrak{I}_2, \ldots, \mathfrak{I}_{n-1})$ be the associated $(n-1)$-tuple of ideals; then

$$
\mathfrak{I}_i \mathfrak{I}_i^{-1} \in \mathfrak{I}_S
$$

for every index $i \in \{1, \ldots, n-1\}$.
Theorem 1 will be first proved with the particular condition that \( R_S \) is a P.I.D.. Afterwards the proof in the general case will follow.

The proof of Corollary 1 will be a direct consequence of Theorem 1 by applying the results obtained by Birch and Merriman in 1972 [3].

In the proof of Theorem 1 we will also prove

**Corollary 2.** There exists a finite set \( N \) of \( n \)-tuples depending only on \( S \) and \( K \), such that if \( \Phi \) is a rational map of degree \( \geq 2 \) with good reduction outside \( S \), then every \( n \)-cycle for \( \Phi \) in \( \mathbb{P}_1(K) \) can be transformed by an automorphism in \( \text{PGL}_2(K) \) into an \( n \)-tuple in \( N \).

Two cycles \( (x_0, x_1, \ldots, x_{n-1}) \) and \( (y_0, y_1, \ldots, y_{n-1}) \) for polynomial maps are called equivalent if and only if there exist an \( S \)-integer \( a \in R_S \) and an \( S \)-unit \( \epsilon \in R_S^* \) such that \( y_i = a + \epsilon x_i \), for every index \( i \). The definition of equivalent cycles for rational maps introduced above is the natural generalization of the one just stated.

Theorem 2 in [8] states that in \( R_S \) for every \( n \geq 2 \) there is just a finite number of inequivalent \( n \)-cycles for polynomial maps of degree \( \geq 2 \). This result cannot be extended to rational maps, since we have proved the following theorem in the case \( S \) contains the places which extend the 2-adic place of \( \mathbb{Q} \).

**Theorem 2.** Let \( R_S = \mathbb{Z}[1/2] \). There exist infinitely many ideals \( \mathfrak{I} \) for which there exists a 3-cycle \((P_0, P_1, P_2)\), for a suitable rational map of degree 4 with good reduction outside \( S \), for which \( \mathfrak{I}_1 = \mathfrak{I} \) holds, where \( \mathfrak{I}_1 \) is the ideal defined in (2).

Theorem 2 proves that the conclusion of Theorem 1 is in a sense best-possible: for every cycle one has

\[
(\mathfrak{I}_1, \ldots, \mathfrak{I}_{n-1}) = \mathfrak{I}_1(R_S, \mathfrak{I}_2 \mathfrak{I}_1^{-1}, \ldots, \mathfrak{I}_{n-1} \mathfrak{I}_1^{-1})
\]

where for the factor \( (R_S, \mathfrak{I}_2 \mathfrak{I}_1^{-1}, \ldots, \mathfrak{I}_{n-1} \mathfrak{I}_1^{-1}) \) there are only finitely many possibilities in view of Theorem 1 but not for the factor \( \mathfrak{I}_1 \), in view of Theorem 2.

Our method of proof is similar to the one used by W. Narkiewicz, F. Halter-Koch and T. Pezda (see [8], [13], [14], [15]). It provides an effective bound for the cardinality of \( I_S \) depending only on \( S \). Unfortunately, by the same method, we cannot obtain an effective solution to the problem, since we shall use the theorem on the finiteness of solutions to equations in three \( S \)-units \( u_1, u_2, u_3 \):

\[
a_1u_1 + a_2u_2 + a_3u_3 = 1,
\]

which is not effective.
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2 Good reduction for \( n \)-tuples

In all the present paper we will use the following notation:

- \( K \) a number field;
- \( R \) the ring of integers of \( K \);
- \( \mathfrak{p} \) a prime ideal of \( R \), \( \mathfrak{p} \neq 0 \);
- \( R_{\mathfrak{p}} \) the local ring of \( R \) at the prime ideal \( \mathfrak{p} \);
- \( m_{\mathfrak{p}} \) the maximal ideal of \( R_{\mathfrak{p}} \) (which is principal);
- \( K(\mathfrak{p}) = R/\mathfrak{p} \cong R_{\mathfrak{p}}/m_{\mathfrak{p}} \) the residue field of the prime ideal \( \mathfrak{p} \);
- \( v_\mathfrak{p} \) the \( \mathfrak{p} \)-adic valuation on \( R \) corresponding to the prime ideal \( \mathfrak{p} \) (we always assume \( v_\mathfrak{p} \) to be normalized so that \( v_\mathfrak{p}(K^*) = \mathbb{Z} \));
- \( S \) a fixed finite set of places of \( K \) of cardinality \( s \) including all archimedean places.

We denote the ring of \( S \)-integers by

\[
R_S := \{ x \in K \mid v_\mathfrak{p}(x) \geq 0 \text{ for every prime ideal } \mathfrak{p} \not\in S \}
\]

and the group of \( S \)-units by

\[
R_S^* := \{ x \in K^* \mid v_\mathfrak{p}(x) = 0 \text{ for every prime ideal } \mathfrak{p} \not\in S \}.
\]

The canonical (mod \( \mathfrak{p} \))-projection \( \mathbb{P}_1(K) \rightarrow \mathbb{P}_1(K(\mathfrak{p})) \) is defined in the following way: for every point \( P \in \mathbb{P}_1(K) \), choose some coordinates \( P = [x : y] \) such that \( x, y \in R_{\mathfrak{p}} \) and they do not belong simultaneously to \( m_{\mathfrak{p}} \), so the point \( [x + m_{\mathfrak{p}} : y + m_{\mathfrak{p}}] \in \mathbb{P}_1(R_{\mathfrak{p}}/m_{\mathfrak{p}}) \) is well defined. By the canonical isomorphism \( R_{\mathfrak{p}}/m_{\mathfrak{p}} \cong K(\mathfrak{p}) \), for every point \( P \in \mathbb{P}_1(K) \) it is possible to associate a point of \( \mathbb{P}_1(K(\mathfrak{p})) \) which will be called the reduction modulo \( \mathfrak{p} \) of \( P \).
Definition 1. Let \( \mathfrak{p} \neq 0 \) be a prime ideal of \( R \). We say that a \( n \)-tuple \((P_0, \ldots, P_{n-1})\) of elements of \( \mathbb{P}_1(K) \) has good reduction at \( \mathfrak{p} \) if the \( n \)-tuple formed by the reduction modulo \( \mathfrak{p} \) has \( n \) distinct elements of \( \mathbb{P}_1(K(\mathfrak{p})) \); a \( n \)-tuple has good reduction outside \( S \) if it has good reduction at every prime ideal \( \mathfrak{p} \notin S \).

The finiteness of the class number of the ring \( R \) will be used to prove

Proposition 1. There exists a finite set \( S_R \) of non-archimedean places of \( K \) and an integer \( C \), depending only on \( R \), such that every point \( P \in \mathbb{P}_1(K) \) can be represented by integral homogeneous coordinates \((x, y)\) satisfying \( \min\{v_\mathfrak{p}(x), v_\mathfrak{p}(y)\} = 0 \) for all prime ideal \( \mathfrak{p} \notin S_R \) and \( \min\{v_\mathfrak{p}(x), v_\mathfrak{p}(y)\} \leq C \) for every \( \mathfrak{p} \in S_R \).

Proof. Let \( R, (a_2R + b_2R), \ldots, (a_tR + b_tR) \) be a set of representatives for the ideal classes. Each point \( P \in \mathbb{P}_1(K) \) can be expressed by integer coordinates \( P = [x : y] \) such that \((xR + yR) = (a_iR + b_iR) \) for a suitable index \( i \in \{1, 2, \ldots, t\} \) (with \( a_1 = b_1 = 1 \)), thus

\[
\min\{v_\mathfrak{p}(x), v_\mathfrak{p}(y)\} = \min\{v_\mathfrak{p}(a_i), v_\mathfrak{p}(b_i)\}
\]

for every prime ideal \( \mathfrak{p} \).

Now it is sufficient to choose \( S_R \) as the set of non-archimedean places such that \( \min\{v_\mathfrak{p}(a_i), v_\mathfrak{p}(b_i)\} \neq 0 \) for some index \( i \in \{2, \ldots, t\} \) and

\[
C = \max_{\mathfrak{p} \in S_R} \{\min\{v_\mathfrak{p}(a_i), v_\mathfrak{p}(b_i)\} | i \in \{2, \ldots, t\}\}.
\] (3)

The constant \( C \) and the set \( S_R \) can be taken depending only on \( K \) since [10] Corollary 2 to Theorem 36, Chapter 5 □

Proposition[1] allows to adopt the following convention: writing \( P = [x : y] \) for a generic element of \( \mathbb{P}_1(K) \) we will always choose \( x, y \in R \) with the property just described and we will say that \( x \) and \( y \) are almost coprime.

Notation. In the present section every point will be represented with almost coprime coordinates, except in the cases in which it will be explicitly specified. Moreover for any \( n \)-tuple \((P_0, P_1, \ldots, P_{n-1})\) of points of \( \mathbb{P}_1(K) \), for every index \( i \), \((x_i, y_i)\) always will represent almost coprime integral homogeneous coordinate for the point \( P_i \).

The \( p \)-adic logarithmic distance \( \delta_\mathfrak{p} \) defined in [1] assumes integral values and the following properties hold:

\[
\begin{align*}
\delta_\mathfrak{p}(P, Q) &\geq 0 \quad \text{for every } P \text{ and } Q \quad \text{(\( \delta' \))} \\
\delta_\mathfrak{p}(P, Q) &\geq 1 \quad \text{if and only if } P \equiv Q \pmod{\mathfrak{p}} \quad \text{(\( \delta'' \))} \\
\delta_\mathfrak{p}(P, Q) &= \infty \quad \text{if and only if } P = Q \quad \text{(\( \delta''' \))}
\end{align*}
\]
By property \((\delta'')\) it follows that a \(n\)-tuple \((P_0, P_1, \ldots, P_{n-1}) \in \mathbb{P}_1^n(K)\) has good reduction outside \(S\) if and only if \(\delta_p(P_i, P_j) = 0\) for every prime ideal \(p\) not in \(S\) and for every distinct indexes \(i, j \in \{0, \ldots, n-1\}\).

Therefore if the \(n\)-tuple \((P_0, P_1, \ldots, P_{n-1})\) has good reduction outside \(S\), then the \((n-1)\)-tuple \((\mathfrak{I}_1, \mathfrak{I}_2, \ldots, \mathfrak{I}_{n-1})\) of ideals defined by (2) is equal to \((R_S, \ldots, R_S)\).

**Definition 2.** Two \(n\)-tuples \((P_0, P_1, \ldots, P_{n-1})\) and \((Q_0, Q_1, \ldots, Q_{n-1})\) are called equivalent if there exists a projective automorphism \(A \in \text{PGL}_2(R_S)\) such that

\[
A(P_i) = Q_i \text{ for all } i \in \{0, 1, \ldots, n-1\}.
\]

If the \(n\)-tuples \((P_0, \ldots, P_{n-1})\) and \((Q_0, \ldots, Q_{n-1})\) are equivalent, then the \((n-1)\)-tuples \((\mathfrak{I}_1, \mathfrak{I}_2, \ldots, \mathfrak{I}_{n-1})\) of ideals defined by (2) coincide. Moreover if a \(n\)-tuple \((P_0, P_1, \ldots, P_{n-1})\) has good reduction outside \(S\), then every \(n\)-tuple equivalent to \((P_0, P_1, \ldots, P_{n-1})\) has good reduction outside \(S\) as well.

If \(R_S\) is a P.I.D. (principal ideal domain), then the class number of \(R_S\) is 1 and \(R_S\) is a representative of the unique ideal class. In this case, for any point of \(\mathbb{P}_1(K)\) we can choose coprime integral homogeneous coordinates and then, from this choice, for any \(n\)-tuple \((P_0, P_1, \ldots, P_{n-1})\) with good reduction outside \(S\) and for every prime ideal \(p \notin S\) it follows that

\[
v_p(x_iy_j - x_jy_i) = \delta_p(P_i, P_j) = 0
\]

thus \(x_iy_j - x_jy_i\) is a \(S\)-unit.

In general \(R_S\) is not always a P.I.D. Enlarging \(S\) (to obtain a principal domain) we change the equivalence relation between \(n\)-tuples. So we can not change \(S\) since we investigate about finiteness of the orbits of \(n\)-tuples under the action of \(\text{PGL}_2(R_S)\). But in any case we have that

**Lemma 1.** There exists a finite set \(\mathcal{R}\) of \(S\)-integers depending only on \(S\) and \(K\) such that for any \(n\)-tuple \((P_0, P_1, \ldots, P_{n-1})\) of good reduction outside \(S\) and for every distinct indexes \(i, j \in \{0, \ldots, n-1\}\) there exist \(r_{i,j} \in \mathcal{R}\) and a unit \(u_{i,j} \in R_S^*\) such that

\[
x_iy_j - x_jy_i = r_{i,j}u_{i,j}.
\]

**Proof.** By the good reduction of \((P_0, P_1, \ldots, P_{n-1})\) and the definition of logarithmic distance we have that

\[
v_p(x_iy_j - x_jy_i) = \min\{v_p(x_i), v_p(y_i)\} + \min\{v_p(x_j), v_p(y_j)\}
\]
for every \( p \notin S \). Let \( C \) be the integer and \( S_R \) the set defined in the Proposition [1]. Having chosen almost coprime coordinates it follows that

\[ v_p (x_j y_j - x_i y_i) \leq 2C, \quad (4) \]

for every \( p \in S_R / S \), and \( v_p (x_i y_j - x_j y_i) = 0 \) for every other prime ideal not in \( S \).

For every couple of distinct points \( P_i = [x_i : y_i], P_j = [x_j : y_j] \) included in a \( n \)-tuple which has good reduction outside \( S \), it is defined the following principal ideal of \( R_S \)

\[ (x_i y_j - x_j y_i) R_S = \prod_{p \in S_R \setminus S} p^{v_p(x_i y_j - x_j y_i)}. \]

By [3] we are in the position to conclude that the set of principal ideals generated in this way has finite cardinality and therefore choosing a generator for every ideal defines the finite set \( \mathcal{R} \) which has cardinality bounded by \( (2C + 1)^{|S_R \setminus S|} \). By the remarks made at the end of the proof of Proposition [1] we deduce that the cardinality of \( \mathcal{R} \) is bounded by a constant which depends only on \( S \) and \( K \). \( \Box \)

By the finiteness of classes of binary forms with given discriminant proved by Birch and Merriman in 1972 [3] we deduce the following

**Proposition 2.** The set of equivalence classes of \( n \)-tuples with good reduction outside \( S \) is finite and depends only on the set \( S \) and \( K \).

**Proof.** Note first that for large \( n \) there are no \( n \)-tuples with good reduction outside \( S \). Indeed, let \( m = \min_{p \in S} \{|K(p)|\} \). For every \( n \geq m + 2 \), each \( n \)-tuple will not be of good reduction outside \( S \), since for every prime ideal \( p \) which realizes the minimum \( m \) the projective space \( \mathbb{P}_1(K(p)) \) has only \( m + 1 \) elements.

For \( n = 1 \) the number of equivalence classes is the order of the ideal class group. Indeed, let \( r \) be the class number of \( R_S \). We choose the representatives for every class and express them, except the trivial ideal \( R_S \), through two generators \((a_2 R_S + b_2 R_S), \ldots , (a_r R_S + b_r R_S) \). Note that these representatives define \( r \) points of \( \mathbb{P}_1(K) \) equivalent for the action of \( \text{PGL}_2(R_S) \). Now we prove that every point \( P \in \mathbb{P}_1(K) \) belongs to the orbit of \([1 : 0]\) or of a point \([a_i : b_i]\) for a suitable index \( i \in \{2, \ldots , r\} \). Let \( P \in \mathbb{P}_1(K) \). We write it with integer coordinates \( P = [\bar{x} : \bar{y}] \).

If \((\bar{x} R_S + \bar{y} R_S)\) is a principal ideal (i.e. it is equivalent to the trivial ideal \( R_S \)), then \( P \) is an element of the orbit of \([1 : 0]\) under the action of \( \text{PGL}_2(R_S) \). Indeed, let \((\bar{x} R_S + \bar{y} R_S) = a R_S \) for a suitable \( a \in R_S \); then \( x = \bar{x}/a \) and \( y = \bar{y}/a \) are elements of \( R_S \) such that \((x R_S + y R_S) = R_S \); this is equivalent to the existence of two \( S \)-integers \( r_x \) and \( r_y \) such that \( x r_x + y r_y = 1 \); therefore the matrix \( \begin{pmatrix} r_x & r_y \\ -y & x \end{pmatrix} \) belongs to \( \text{SL}_2(R_S) \).
and maps the point \([x : y]\) to \([1 : 0]\).

Otherwise, there exist \(c, d \in R_S\) such that \(c(\bar{x}R_S + \bar{y}R_S) = d(a_iR_S + b_iR_S)\) for some index \(i\); therefore, denoting \(x = c\bar{x}/d\) and \(y = c\bar{y}/d\) (elements of \(R_S\)), the following holds

\[
(xR_S + yR_S) = (a_iR_S + b_iR_S) = I \subset R_S.
\]

By definition of \(I^{-1}\), there are elements \(x', y' \in I^{-1}\) satisfying \(xy' - yx' = 1\), namely \((x' \ y') \in SL_2(K)\). Moreover there are \(a', b' \in I^{-1}\) such that \(a' \bar{b'} - b' \bar{a'} = 1\), namely \((a_i \ a'_i \ b_i \ b'_i) \in SL_2(K)\). So that the following matrix

\[
\begin{pmatrix}
  a_i & a'_i \\
  b_i & b'_i \\
\end{pmatrix}
\begin{pmatrix}
  x & x' \\
  y & y' \\
\end{pmatrix}^{-1}
\begin{pmatrix}
  a_i & a'_i \\
  b_i & b'_i \\
\end{pmatrix}
\begin{pmatrix}
  y' & -x' \\
  -y & x \\
\end{pmatrix}
\begin{pmatrix}
  a_i & a'_i \\
  b_i & b'_i \\
\end{pmatrix}
\begin{pmatrix}
  a_i y' - y a'_i & -a_i x' + x a'_i \\
  -y b_i - y b'_i & -x b_i + x b'_i \\
\end{pmatrix}
\in SL_2(R_S)
\]

maps \([x : y]\) to \([a_i : b_i]\). This concludes the case \(n = 1\).

Let \(n \geq 2\) and \((P_0, P_1, \ldots, P_{n-1})\) be a \(n\)-tuple with good reduction outside \(S\). By Lemma \([\text{I}]\) we obtain for every distinct indexes \(i, j\) the following identity

\[
x_i y_j - x_j y_i = r_{i,j} u_{i,j},
\]

where \(u_{i,j} \in R_S^*\) and \(r_{i,j} \in R\). For every choice of a unit \(\lambda \in R_S^*\), these identities are verified also after replacing the almost coprime coordinates \([x_i : y_i]\) with the coordinates \([\lambda x_i : \lambda y_i]\). Now fixing the possible values of \(r_{i,j}\), to every \(n\)-tuple of points which verifies the identities \([5]\) we associate the following binary form of degree \(n\)

\[
F(X, Y) = \prod_{0 \leq i \leq n-1} (x_i X - y_i Y),
\]

defining in this way a family of forms with discriminant

\[
D(F) = u \left( \prod_{0 \leq i < j \leq n-1} r_{i,j}^2 \right),
\]

where \(u\) is a \(S\)-unit.

The multiplicative group \(R_S^*\) is finitely generated, so there exists a finite set \(\mathcal{V} \subset R_S^*\) such that every \(S\)-unit is representable as product of a \((2n - 2)\)-power of an \(S\)-unit and an element of \(\mathcal{V}\). Then, for every \(n\)-tuple \((P_0, P_1, \ldots, P_{n-1})\) with good reduction which satisfies the identities \([5]\), the unit \(u\) which appears in the equation \([6]\) relative to the discriminant of the binary form associated to
(P₀, P₁, \ldots, Pₙ₋₁) can be written as \( u = v \lambda^{2n-2} \) with \( v \in \mathcal{V} \) and \( \lambda \in \mathcal{R}_S^* \). Thus, if we replace the coordinates \((x₀, y₀)\) of \( P₀ \) with \((x₀\lambda^{-1}, y₀\lambda^{-1})\), we obtain a new binary form with discriminant \( v \prod r_{i,j}^2 \). In other words, with an appropriate choice of coordinates, the \( n \)-tuples which satisfy the identities (5) with fixed \( r_{i,j} \) define a family of binary forms of degree \( n \) whose discriminant is of the form \( v \prod r_{i,j}^2 \) with \( v \in \mathcal{V} \).

Two binary forms \( G(X, Y) \) and \( H(X, Y) \) are called equivalent if there exists a matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{R}_S) \) such that \( G(X, Y) = H(aX+bY, cX+dY) \). The equivalence of binary forms associated with \( n \)-tuples coincide with the equivalence of the corresponding unordered \( n \)-tuples. By the results obtained by Birch and Merriman in 1972 [3] (non effective) and Evertse and Győry in 1991 [6] (effective) we know that the number of classes of binary forms of degree \( n \) with fixed discriminant \( v \prod r_{i,j}^2 \) is finite and bounded by an integer depending only on \( S \).

Since the cardinalities of the sets \( \mathcal{R}, \mathcal{V} \) are finite and depend only on \( S \) and \( K \), then the set of classes of \( n \)-tuples with good reduction outside \( S \) has finite cardinality depending only on \( S \) and \( K \).

\[ \square \]

3 Cycles for rational maps

Let \( \Phi : \mathbb{P}^1 \to \mathbb{P}^1 \) a rational map defined over \( K \) by \( \Phi([x : y]) = [F(x, y) : G(x, y)] \), where \( F, G \in \mathcal{R}[x, y] \) have no common factor and are homogeneous of the same degree.

**Definition 3.** We say that a morphism \( \Phi : \mathbb{P}^1 \to \mathbb{P}^1 \) defined over \( K \) has good reduction at a prime ideal \( \mathfrak{p} \) if there exists a morphism \( \bar{\Phi} : \mathbb{P}^1 \to \mathbb{P}^1 \) defined over \( K(\mathfrak{p}) \) with \( \deg \Phi = \deg \bar{\Phi} \) such that the following diagram

\[
\begin{array}{ccc}
\mathbb{P}^1_K & \xrightarrow{\Phi} & \mathbb{P}^1_K \\
\downarrow & & \downarrow \\
\mathbb{P}^1_{K(\mathfrak{p})} & \xrightarrow{\bar{\Phi}} & \mathbb{P}^1_{K(\mathfrak{p})}
\end{array}
\]

is commutative, where \( \sim \) denotes the reduction modulo \( \mathfrak{p} \). Furthermore if \( \Phi \) has good reduction at every prime ideal \( \mathfrak{p} \notin \mathcal{S} \), we say that it has good reduction outside \( \mathcal{S} \).
We may assume that $F, G$ have coefficients in $R_p$ and that at least one coefficient is in $R_p^*$; therefore we obtain a rational map, defined over $K(p)$, as follows

$$\tilde{\Phi} : \mathbb{P}_1 \to \mathbb{P}_1; \quad \tilde{\Phi}([x : y]) = [\tilde{F}(x, y) : \tilde{G}(x, y)],$$

where $\tilde{F}, \tilde{G}$ are the polynomials obtained by reduction modulo $p$ of the coefficients of $F$ and $G$. This is the rational map which appears in the Definition [5]. Hence the rational map $\Phi$, with its coefficients chosen as described above, has good reduction at the prime ideal $p$ if and only if $\text{Res}(\tilde{F}, \tilde{G})$ is non zero (since $F$ and $G$ have no common factors).

If $H(t_1, \ldots, t_k) \in K[t_1, \ldots, t_k]$ is a non zero polynomial, following the notation of [12], we define $v_p(H)$ as

$$v_p(H) = v_p\left(\sum_i a_i t_1^{i_1} \cdots t_k^{i_k}\right) = \min_i v_p(a_i)$$

where the minimum is taken over all multi-indexes $I = (i_1, \ldots, i_k)$. That is, $v_p(H)$ is the smallest valuation of the coefficients of $H$. For any family of polynomials $H_1, \ldots, H_m \in K[t_1, \ldots, t_k]$ we define $v_p(H_1, \ldots, H_m)$ to be the minimum of the $v_p(H_i), i \in \{1, \ldots, m\}$.

Let $\Phi$ be defined as above and let $\text{Res}(F, G)$ be the resultant of homogeneous polynomials $F$ and $G$ of degree $d$. We define $\text{Disc}(\Phi)$ to be the integral ideal of $R$ whose valuation at the prime ideal $p$ is given by

$$v_p(\text{Disc}(\Phi)) = v_p(\text{Res}(F, G)) - 2v_p(F, G).$$

The definition is a good one by the properties of the resultant. Moreover choosing the coordinates of $F, G$ in $R_p$ and such that at least one coefficient is in $R_p^*$, we obtain that $v_p(\text{Disc}(\Phi)) = v_p(\text{Res}(F, G))$. Therefore $\Phi$ has good reduction at the prime ideal $p$ if and only if $v_p(\text{Disc}(\Phi)) = 0$. We will consider only cycles for rational maps which have good reduction outside $S$ and of degree $\geq 2$. These maps form a semigroup under composition on which the group $\text{PGL}_2(R_S)$ acts by conjugation.

**Definition 4.** An ordered n-tuple of elements of $\mathbb{P}_1(K)$ which is a cycle for a rational map with good reduction outside $S$ will be called a $(S, n)$-cycle.

We will use the following elementary proposition included in the paper by Morton and Silverman [12, Proposition 6.1].
Proposition 3. Let $\Phi : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational map over $K$ which has good reduction at the prime ideal $\mathfrak{p}$ and let $P \in \mathbb{P}_1(K)$ be a periodic point for $\Phi$ with minimal period $n$. Then

(a) $\delta_\mathfrak{p}(\Phi^i(P), \Phi^j(P)) = \delta_\mathfrak{p}(\Phi^{i+k}(P), \Phi^{j+k}(P))$ for every $i, j, k \in \mathbb{Z}$

(b) Let $i, j \in \mathbb{Z}$ such that $\gcd(i - j, n) = 1$. Then

$$\delta_\mathfrak{p}(\Phi^i(P), \Phi^j(P)) = \delta_\mathfrak{p}(\Phi(P), P).$$

□

This proposition states that, for every indices $i, k$, the ideals $I(P_0, P_i)$ and $I(P_k, P_{k+i})$ defined in (2) are equal. Moreover if $k$ and $n$ are coprime, then the ideals $I(P_0, P_k)$ and $I(P_0, P_1)$ are equal.

3.1 Proof of Theorem 1 in the case that $R_S$ is a P.I.D.

Since we want to obtain a finiteness result about rational maps with good reduction outside $S$, without loss of generality in this section we can suppose that $R_S$ is a P.I.D. Indeed each rational map which has good reduction outside $S$ has good reduction outside every set of places which contains $S$; hence we can enlarge $S$ so that $R_S$ becomes a P.I.D. Note that the cardinality of a minimum enlarged set, with the above property, is bounded by $s + h_{R_S} - 1$, where $h_{R_S}$ is the class number of $R_S$. In fact: if all prime ideal are principal, then $R_S$ is a P.I.D.; otherwise such a prime ideal is contained in an ideal class which is not the trivial one, so if we add this prime ideal to $S$ (obtaining a larger set $S'$) we have that the new ring $R_{S'}$ has class number $h_{R_{S'}} < h_{R_S}$; by inductive method it results that to obtain a P.I.D. it suffices to add to $S$ a number of prime ideals $\leq h_{R_S} - 1$.

In all part of this subsection we suppose that $R_S$ is a principal domain so we can adopt the convention that any point $P_i \in \mathbb{P}_1(K)$ will be represented by coprime $S$-integral homogeneous coordinates $[x_i : y_i]$. By this convention for all prime ideals $\mathfrak{p} \notin S$ and every points $P_1, P_2 \in \mathbb{P}_1(K)$ it follows that $\delta_\mathfrak{p}(P_1, P_2) = v_\mathfrak{p}(x_1y_2 - x_2y_1)$. The next lemma states part (a) of the last proposition in a form which will be useful in the sequel:

Lemma 2. For every $(S, n)$-cycle $(P_0, P_1, \ldots, P_{n-1})$ and for every $i, j \in \mathbb{Z}$ there exist an $S$-unit $u_{j,i+1} \in R_S^*$ such that

$$x_jy_{j+i} - x_{j+i}y_j = (x_0y_i - x_iy_0)u_{j,i+1}.$$  (7)
Proof. Proposition 3 asserts that, for every prime ideal $p \not\in S$ and for every couple of indexes $i, j \in \mathbb{Z}$, we have that $\delta_p(P_j, P_{j+i}) = \delta_p(P_0, P_i)$, therefore the identity $v_p(x_jy_{j+i} - x_jy_j) = v_p(x_0y_j - x_jy_0)$ holds. So we have

$$u_{j,j+i} = \frac{x_jy_{j+i} - x_jy_j}{x_0y_i - x_iy_0} \in R_S^*.$$

\[\]

Another simple but important fact is the following:

**Lemma 3.** For every $(S, n)$-cycle $(P_0, P_1, \ldots, P_{n-1})$ and for every prime ideal $p \not\in S$ the following properties hold:

1. for all indexes $j \in \{0, 1, \ldots, n - 1\}, i \not\equiv 0 \text{ (mod } n)\) we have $\delta_p(P_j, P_{j+i}) \geq \delta_p(P_0, P_1)$, or equivalently

$$C_i := \frac{x_0y_i - x_iy_0}{x_0y_1 - x_1y_0} \in R_S$$

and

$$x_jy_{j+i} - x_jy_j = C_iu_{j,j+i}(x_0y_i - x_iy_0),$$

where $u_{j,j+i} \in R_S^*$.

2. let $P_0 = [x_0 : y_0]$ and $P_1 = [x_1 : y_1]$ be the first and the second point of the $(S, n)$-cycle. The matrix $A \in \text{GL}_2(K)$

$$A = \begin{pmatrix}
-x_0 & x_0 \\
-x_0y_1 - x_1y_0 & x_0y_1 - x_1y_0 \\
y_1 & -x_1 \\
x_0y_1 - x_1y_0 & x_0y_1 - x_1y_0
\end{pmatrix}$$

maps the vector $(x_0, y_0)$ to $(0, 1)$ and the vector $(x_1, y_1)$ to $(1, 0)$. For any index $k$, if $(\bar{x}_k, \bar{y}_k)' = A(x_k, y_k)'$, then for every indexes $j \in \{0, 1, \ldots, n-1\}, i \not\equiv 0 \text{ (mod } n)\) the following identities are verified

$$\bar{x}_j\bar{y}_{j+i} - \bar{x}_{j+i}\bar{y}_j = -C_iu_{j,j+i},$$

where $u_{j,j+i}$ is the $S$-unit defined in part 1. Furthermore for every index $k > 1$

$$(\bar{x}_k, \bar{y}_k) = (C_k, -C_{k-1}u_{1,k}).$$
3. If \( i, j \) are coprime integers, then
\[
\min\{\delta_p(P_0, P_i), \delta_p(P_0, P_j)\} = \delta_p(P_0, P_1) \tag{13}
\]
and
\[
\min\{v_p(C_i), v_p(C_j)\} = 0 \tag{14}
\]

for every prime ideal \( p \notin S \).

**Proof.** 1. Since the \( p \)-adic distance satisfies the following triangle inequality [12, Proposition 5.1]:
\[
\delta_p(P, T) \geq \min\{\delta_p(P, Q), \delta_p(Q, T)\} \quad \text{for every } P, Q, T \in \mathbb{P}_1(K),
\]
it follows that
\[
\delta_p(P_j, P_{j+i}) \geq \min\{\delta_p(P_j, P_{j+1}), \ldots, \delta_p(P_{j+i-1}, P_{j+i})\} = \delta_p(P_0, P_1).
\]

for any index \( i, j \). Thus, by the choice of coprime homogeneous coordinates for every points of \( \mathbb{P}_1(K) \) and part (a) of Proposition 3 we have that \( v_p(x_0y_i - x_iy_0) \geq v_p(x_0y_1 - x_1y_0) \), so (8) is proved. The identity (9) follows from (7) and (8).

2. Let \( A \) be the matrix defined in (10). We have that \( A(x_0, y_0)^j = (0, 1)^j \) and \( A(x_1, y_1)^j = (1, 0)^j \). Putting \( (\bar{x}_k, \bar{y}_k)^j = A(x_k, y_k)^j \) for every index \( k > 1 \), the Equation (9) implies that for every indexes \( j \in \{0, 1, \ldots, n-1\}, i \neq 0 \) (mod \( n \))
\[
\bar{x}_j\bar{y}_{j+i} - \bar{x}_{j+i}\bar{y}_j = -\frac{x_jy_{j+i} - x_{j+i}y_j}{x_0y_1 - x_1y_0} = -C_iu_{j,j+i}, \tag{15}
\]
since \( \det(A) = -(x_0y_1 - x_1y_0)^{-1} \), which proves (11).

Considering (15) with \( j = 0, i = k \) and \( j = 1, k = i + 1 \) we prove (12).

3. There exist \( c, d \in \mathbb{Z} \) such that \( ci + dj = 1 \). By part 1, the triangle inequality and Proposition 3 it is verified that
\[
\delta_p(P_0, P_1) \leq \min\{\delta_p(P_0, P_{ci}), \delta_p(P_0, P_{-dj})\} \leq \delta_p(P_{ci}, P_{-dj}) = \delta_p(P_0, P_1)
\]
and
\[
\delta_p(P_0, P_j) \leq \delta_p(P_0, P_{ci}) ; \delta_p(P_0, P_j) \leq \delta_p(P_0, P_{dj})
\]
so (13) follows.

Now suppose that \( \delta_p(P_0, P_i) = \delta_p(P_0, P_1) \),
\[
\delta_p(P_0, P_i) = v_p(x_0y_i - x_iy_0) = v_p(C_i) + v_p(x_0y_1 - x_1y_0) = v_p(C_i) + \delta_p(P_0, P_1)
\]
therefore we have that \( v_p(C_i) = 0 \) which proves (14). \( \square \)
Lemma 3 states that, for any \((S,n)\)-cycle \((P_0, P_1, \ldots, P_{n-1})\) and for every couple of indexes \(j \in \{0, 1, \ldots, n-1\}, i \neq 0 \pmod{n}\), the ideal \(\mathfrak{Z}(P_0, P_1)\) divides the ideal \(\mathfrak{Z}(P_j, P_{j+i})\). The \(S\)-integer \(C_i\) generates the ideal \(\mathfrak{Z}(P_0, P_i) \cdot \mathfrak{Z}(P_0, P_1)^{-1}\). Moreover if \(i, j\) are coprime, then the greatest common divisor of \(\mathfrak{Z}(P_0, P_i)\) and \(\mathfrak{Z}(P_0, P_j)\) is \(\mathfrak{Z}(P_0, P_1)\).

**Lemma 4.** Let \((P_0, P_1, \ldots, P_{n-1})\) be a \((S,n)\)-cycle and let \(i, j \in \mathbb{Z}\); then

\[
L_{i,j} := \frac{x_0 y_{i,j} - x_{i,j} y_0}{x_0 y_j - x_j y_0} \in R_S.
\]

(16)

Moreover, let \(i, j\) be fixed coprime integers. If for every \((S,n)\)-cycle the set of principal ideals generated by the possible values of \(L_{i,j}\) is finite, then also the set of principal ideals generated by possible values of \(C_i\) is finite, where \(C_i\) is the \(S\)-integer defined in (8).

**Proof.** Let \((P_0, P_1, \ldots, P_{n-1})\) be a cycle for a rational map \(\Phi\). In order to prove the first part we simply apply Lemma 3 to the cycle \((P_0, P_j, P_{j+i}, \ldots)\), relative to rational map \(\Phi^j\).

Now suppose that \(i, j \in \mathbb{Z}\) are coprime. By (16) and (8)

\[
x_0 y_{i,j} - x_{i,j} y_0 = L_{i,j} C_j (x_0 y_1 - x_1 y_0)
\]

and

\[
x_0 y_{i,j} - x_{i,j} y_0 = L_{j,i} C_i (x_0 y_1 - x_1 y_0),
\]

therefore it follows that \(L_{i,j} C_j = L_{j,i} C_i\).

From (14) in Lemma 3 we deduce that \(v_p(C_i) \leq v_p(L_{i,j})\), for every \(p \notin S\), so the finiteness of principal ideals generated by the possible values of \(L_{i,j}\) gives the finiteness of principal ideals generated by the possible values of \(C_i\).

In the next proofs we will frequently use the fact that \(S\)-unit equations of the type

\[
a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 1,
\]

(17)

where \(a_i \in K^*\), have only a finite number of non-degenerate solutions

\[
(x_1, x_2, \ldots, x_n) \in (R_S^*)^n.
\]

A solution is called non-degenerate if no subsum vanishes (i.e. \(\sum_{i \in I} a_i x_i \neq 0\) for every nonempty subset \(I \subseteq \{1, 2, \ldots, n\}\)).

In other words, the equation \(X_1 + X_2 + \ldots + X_n = 1\) has only a finite number of
non-degenerate solutions \((X_1, \ldots, X_n) \in K^n\) with \(v_p(X_i)\) fixed for every index \(i\) and for every \(p \notin S\). Actually we shall use this result only for \(n = 2\) and \(n = 3\).

This equation has been widely studied in the literature. For \(n = 2\), the finiteness of solutions of equation \((17)\) was proved by C.L. Siegel in a particular case. Later K. Mahler studied the case \(K = \mathbb{Q}\) and generic finite set \(S\). In 1960 S. Lang extended Mahler’s result to arbitrary fields \(K\) of characteristic 0 and solutions in any group \(\Gamma \subset K^*\) of finite rank. A. Baker obtained effective results using his bound for linear forms in logarithms. J.-H. Evertse \([4]\), studying the case where \(K\) is a number field of degree \(d\) over \(\mathbb{Q}\), found that the set of solutions has cardinality \(\leq 3 \cdot 7^{d+2s}\). This upper bound depends only on \(s = \#S\) and \(d = [K : \mathbb{Q}]\). Note that \(S\) includes all archimedean places of \(K\), hence \(s \geq d/2\). In this way we obtain the upper bound \(3 \cdot 7^{4s}\) depending only on \(s\).

For general \(n > 2\), A. J. van der Poorten and H. P. Schlickewei in \([17]\) and J. H. Evertse in \([5]\) showed that the set of solutions is finite (non effective results). The best quantitative result is due to J. H. Evertse \([7]\), who found the upper bound \(2^{35n^4s}\), depending only on \(s\) and \(n\), for the cardinality of the set of solutions.

Using in the proof of Theorem \([11]\) the bounds found by Evertse, we could obtain a quantitative result, depending explicitly on the cardinality of \(S\) and the class number of \(R_S\).

**Lemma 5.** Let \(D, E \in K^*\) be fixed. Given the equation

\[ y^2 = Du + Ev, \tag{18} \]

the set

\[ \{ [u : v : y^2] \in \mathbb{P}_2(K) \mid (u, v, y) \in R_S^* \times R_S^* \times R_S \text{ is a solution of (18)} \} \]

is finite and depends only on \(S\), \(D\) and \(E\). In particular the subset of \(R_S^*\) defined by

\[ \left\{ \frac{u}{v} \mid u, v \in R_S^* \text{ satisfy (18) for a suitable } y \in R_S \right\} \tag{19} \]

is finite and depends only on \(S\), \(D\) and \(E\). The same assertion is valid for the set of principal ideals of \(R_S\) defined by

\[ \{ yR_S \mid y \text{ satisfies (18) for suitable } u, v \in R_S^* \}. \tag{20} \]

Moreover the finiteness of the last set is valid also in the case \(DE = 0\).
Proof. Since $R^*_S$ has finite rank, there exists a finite set $W \subset R^*_S$, depending only on $S$, such that for every $u \in R^*_S$ there exist $\bar{u} \in R^*_S$ and $w \in W$ such that $u = w\bar{u}^6$. Let $y$ be an integer which satisfies (18) for suitable $u, v \in R^*_S$; then there exist $\bar{u}, \bar{v} \in R^*_S$ and $w_1, w_2 \in W$ such that $y^2 = Dw_1\bar{u}^6 + Ew_2\bar{v}^6$. Therefore, we deduce that $(\bar{u}/\bar{v})$ is an $S$-integral point on the elliptic curve defined by the equation $Y^2 = Dw_1X^3 + Ew_2$. By the finiteness of $S$-integral points of elliptic curve (Siegel’s Theorem, see for example [16, Chapter 7]) and finiteness of the set $W$ we deduce that $y^2$ can assume only a finite number of values depending only on $S$. If $DE = 0$, e.g. $E = 0$, then it is trivial that $y^2R_S = DR_S$. This concludes the proof. □

Lemma 6. Let $D, E \in K$ be fixed. Given the equation

$$y^2 = D^2u^2 + DuEv + E^2v^2,$$  (22)

the set of ideals of $R_S$ defined by

$$\{yR_S \mid (u, v, y) \in R^*_S \times R^*_S \times R_S \text{ is a solution of (22)}\}$$

is finite and depends only on $S$, $D$ and $E$.

Proof. If $DE = 0$ the lemma is trivial.

Let $DE \neq 0$. We can suppose that $D$ and $E$ are integers. If they are not, we can choose an integer $F$, depending only on $D$ and $E$, such that $FD, FE \in R_S$ and replace $y^2$ with $F^2y^2$ in (22).

If $(u, v, y) \in R^*_S \times R^*_S \times R_S$ is a solution of (22) then

$$y^2 = D^2u^2 + DuEv + E^2v^2 = (Du - \zeta Ev)(Du - \bar{\zeta}Ev),$$

where $\zeta$ and $\bar{\zeta}$ are the primitive third roots of unity. The elements $(Du - \zeta Ev)$ and $(Du - \bar{\zeta}Ev)$ of the extension $K(\zeta)/K$ are integers with the property that

$$(Du - \zeta Ev) - (Du - \bar{\zeta}Ev) = (\bar{\zeta} - \zeta)Ev.$$

Let $T$ be the ring of algebraic integers of $K(\zeta)$ and let $\bar{S}$ be the finite set defined by all places of $K(\zeta)$ which lie over any place of $K$ included in $S$. We can enlarge $\bar{S}$ to a finite set of places such that $T_{\bar{S}}$ is a unique factorization domain and such that $(\bar{\zeta} - \zeta)E \in T_{\bar{S}}^*$. The $T_{\bar{S}}$-integers $(Du - \zeta Ev)$ and $(Du - \bar{\zeta}Ev)$ are coprime.
and therefore, after multiplication by a unit, are squares in \( T_5 \) and so they can be expressed in the form \( w\bar{y}^2 \) with \( w \in T_5^* \) and \( \bar{y} \in T_5 \). Applying the last lemma to the equation \( \bar{y}^2 = Du/w - \zeta Ev/w \) we easily deduce that there exists a finite set \( \mathcal{U} \), depending only on \( T_5 \), such that for every \( u, v \in T_5^* \) which satisfy (22) we have that \( u/v \in \mathcal{U} \).

Since \( R_5^* \subset T_5^* \), the last statement is true also when we consider \( u, v \in R_5^* \) and the conclusion follows. \( \square \)

**Lemma 7.** Let \( n \geq 3 \) and \( C_2 \) be the integer defined in (3) of Lemma 3 and associated to a \((S, n)\)-cycle. The set of principal ideals of \( R_5 \)

\[ \{ C_2R_5 \mid (P_0, P_1, \ldots, P_{n-1}) \text{ is a } (S, n)\text{-cycle} \} \]

is finite and depends only on \( S \) and \( K \).

**Proof.** We use the notation of Lemma 3. Let \((P_0, P_1, \ldots, P_{n-1})\) be a \((S, n)\)-cycle.

Let us consider first the case that \( n \) is an odd number. By Proposition 3(b) it follows that \( \delta_p(P_0, P_1) \) and \( \delta_p(P_0, P_2) \) are equal for every \((S, n)\)-cycle and for every \( p \not\in S \); hence \( v_p(x_0y_2 - x_2y_0) = v_p(x_0y_1 - x_1y_0) \), so from (3) we deduce that \( C_2 = 1 \). This case is proved.

By Lemma 3, Part 2, any \((S, n)\)-cycle is mapped by the automorphism defined in (10) to the following ordered \( n \)-tuple of vectors of \( K^2 \)

\[(0, 1); (1, 0); \ldots; (C_i, -C_{i-1}u_1, i); \ldots; (C_{n-1}, -C_{n-2}u_1, i) \]

Let \( n \geq 4 \) be an even number. Suppose first that \( 3 \nmid n \). By Proposition 3(b) we have that \( \delta_p(P_0, P_3) = \delta_p(P_0, P_1) \), therefore we deduce that \( C_3 = 1 \). By the identity (11) of Lemma 3 applied with \( i = 1 \), \( j = 2 \) and considering \( C_1 = 1 \) follows that \(-C_2^2u_{1,3} + u_{1,2} = -u_{2,3} \); thus we obtain that \( C_2 \) satisfies

\[ C_2^2 = \frac{u_{1,2}}{u_{1,3}} + \frac{u_{2,3}}{u_{1,3}}. \]

Lemma 3 applied with \( u_{1,2}/u_{1,3} = u \), \( u_{2,3}/u_{1,3} = v \) and \( C_2 = y \), proves this case.

Suppose now \( n = 2 \cdot 3^k \cdot m \) with \( m > 1 \) and \( 3 \nmid m \). For every \((P_0, P_1, \ldots, P_{n-1})\), the \( n \)-tuple \((P_0, P_3, \ldots, P_{(2m-1)3^k})\) is a \((S, 2m)\)-cycle and \( 2m \geq 4 \) is coprime with 3. Let

\[ L_{2,3^k} = \frac{x_0y_{2,3^k} - x_{2,3^k}y_0}{x_0y_{3^k} - x_{3^k}y_0}. \]

Applying to the above cycle the reasoning used in the previous case we obtain that, varying the possible \((S, n)\)-cycle, the set of principal ideals of \( R_5 \) generated by \( L_{2,3^k} \) is finite. Now we simply apply Lemma 4 with \( i = 2 \) and \( j = 3^k \).
The last case is \( n = 2 \cdot 3^k \). We first reduce to the case \( k = 1 \). If \( k > 1 \) we consider the cycle \((P_0, P_{3^{k-1}}, \ldots, P_{5\cdot 3^{k-1}})\) which has length 6 and if the lemma holds in the case \( n = 6 \), then one has the finiteness of ideals generated by \( L_{2,3^k} \). Therefore, by Lemma 4 applied with \( i = 2 \) and \( j = 3^k \), this case is proved.

Now we suppose that \( n = 6 \). By Lemma 3 Part 2, any cycle \((P_0, P_1, P_2, P_3, P_4, P_5)\), by the matrix defined in (10), is sent to the following ordered 6-tuple of vector of \( K^2 \)

\[
(0, 1); (1, 0); (C_2, -u_{1,2}); (C_3, -C_2u_{1,3}); (C_4, -C_3u_{1,4}); (C_5, -C_4u_{1,5})
\]

with \( u_{1,i} \in R^*_5 \) for every \( i \in \{2, 3, 4, 5\} \).

By Lemma 2 the identities \( C_4 = C_2u_{0,4} \) and \( C_5 = u_{0,5} \) hold for suitable \( u_{0,4}, u_{0,5} \in R^*_5 \). We rewrite the above 6-tuple as

\[
(0, 1); (1, 0); (C_2, -u_{1,2}); (C_3, -C_2u_{1,3}); (C_2u_{0,4}, -C_3u_{1,4}); (u_{0,5}, -C_2u_{0,4}u_{1,5})
\]

Imposing the identity (11) of Lemma 3 (after simple simplification) we obtain that \( C_2, C_3 \) satisfy the following system of equations:

\[
\begin{align*}
C_3u_{1,4} - u_{1,2}u_{0,4} &= u_{2,4} \quad \text{(a)} \\
C_2^2u_{0,4}u_{1,5} - u_{1,2}u_{0,5} &= C_3u_{2,5} \quad \text{(b)} \\
C_2^2u_{0,4}u_{1,5} - C_3u_{1,4}u_{0,5} &= u_{4,5} \quad \text{(c)}
\end{align*}
\]

The equation (a) is obtained from (11) with \( j = 2 \) and \( i = 2 \), (b) with \( j = 2 \) and \( i = 3 \), (c) with \( j = 4 \) and \( i = 1 \). From (a) it follows that

\[
C_3 = u_{1,2}u_{0,4}u_{1,4}^{-1} + u_{2,4}u_{1,4}^{-1}.
\]

Now multiplying (b) by \(-u_{0,4}\) and adding (c) we obtain

\[
u_{0,4}u_{1,2}u_{0,5} + C_3u_{2,5}u_{0,4} - C_3u_{1,4}u_{0,5} = u_{4,5},
\]

replacing \( C_3 \) with the right term of (24) in this last identity we obtain the following \( S \)-unit equation:

\[
\frac{u_{1,2}u_{0,4}^2u_{2,5}}{u_{4,5}u_{1,4}} + \frac{u_{2,4}u_{0,4}u_{2,5}}{u_{4,5}u_{1,4}} - \frac{u_{2,4}u_{0,5}}{u_{4,5}} = 1
\]

Suppose that the equation (25) does not admit vanishing subsums. By the \( S \)-units Theorem, there exist only finitely many possible values for the ratios

\[
\frac{u_{1,2}u_{0,4}^2u_{2,5}}{u_{4,5}u_{1,4}}, \quad \frac{u_{2,4}u_{0,4}u_{2,5}}{u_{4,5}u_{1,4}}, \quad \frac{u_{2,4}u_{0,5}}{u_{4,5}}.
\]
From (24) it follows that
\[ C_3 = \frac{u_{4,5}}{u_{0,4}u_{2,5}} \left( \frac{u_{1,2}u_{0,4}^2u_{2,5}}{u_{4,5}u_{1,4}} + \frac{u_{2,4}u_{0,4}u_{2,5}}{u_{4,5}u_{1,4}} \right) \] (26)

Therefore the set of principal ideals of \( R_S \) generated by \( C_3 \) is finite. By the above equation (b) and a suitable application to Lemma 5, also the set of ideals generated by \( C_2 \) is finite.

Suppose that in (25)
\[ \frac{u_{1,2}u_{0,4}^2u_{2,5}}{u_{4,5}u_{1,4}} + \frac{u_{2,4}u_{0,4}u_{2,5}}{u_{4,5}u_{1,4}} = 0. \]

In this case, by (26), we have that \( C_3 = 0 \); this situation contradicts \( n = 6 \).

Suppose that in (25)
\[ \frac{u_{1,2}u_{0,4}^2u_{2,5}}{u_{4,5}u_{1,4}} - \frac{u_{2,4}u_{0,4}u_{2,5}}{u_{4,5}} = 0; \]

which is equivalent to
\[ \frac{u_{2,4}u_{0,4}u_{2,5}}{u_{4,5}u_{1,4}} = 1. \]

From these last two identities it follows that
\[ u_{1,2}u_{0,4}u_{0,5} = \frac{u_{2,4}^2u_{0,5}^2}{u_{4,5}}. \] (27)

Now multiplying (a) by \( u_{0,5} \) and adding (c) we obtain
\[ C_2^2 = \frac{1}{u_{0,4}u_{1,5}}(u_{1,2}u_{0,4}u_{0,5} + u_{2,4}u_{0,5} + u_{4,5}) \] (28)

Replacing \( u_{1,2}u_{0,4}u_{0,5} \) in (28) with the right term of (27) we obtain
\[ C_2^2 = \frac{1}{u_{0,4}u_{1,5}u_{4,5}}(u_{2,4}^2u_{0,5}^2 + u_{2,4}u_{0,5}u_{4,5} + u_{4,5}^2); \]

so applying in the suitable way Lemma 6 we obtain the finiteness of the set of principal ideal of \( R_S \) generated by \( C_2 \). At last we consider the case
\[ \frac{u_{2,4}u_{0,4}u_{2,5}}{u_{4,5}u_{1,4}} - \frac{u_{2,4}u_{0,5}}{u_{4,5}} = 0 \] (29)
which is equivalent to

\[ \frac{u_{1,2}u_{0,4}u_{2,5}}{u_{4,5}u_{1,4}} = 1 \]

as well as

\[ \frac{u_{0,4}u_{2,5}}{u_{4,5}u_{1,4}} = \frac{1}{u_{1,2}u_{0,4}} \quad (30) \]

Replacing in (29) \((u_{0,4}u_{2,5})/(u_{4,5}u_{1,4})\) with the right term of (30) we obtain

\[ \frac{u_{2,4}}{u_{1,2}u_{0,4}} = \frac{u_{2,4}u_{0,5}}{u_{4,5}} = 0 \]

which is equivalent to \(u_{1,2}u_{0,4}u_{0,5} = u_{4,5}\). From this last identity and (28) we obtain

\[ C_2^2 = \frac{1}{u_{0,4}^2u_{1,5}}(u_{2,4}u_{0,5} + 2u_{4,5}). \]

By Lemma 5 we have the finiteness of the set of principal ideal of \(R_S\) generated by \(C_2\). This last case concludes the proof.

**Corollary 3.** For every \(l \in \mathbb{N}\), let \(C_2^l\) be the integer defined in Lemma 3 associated to a \((S, n)\)-tuple. Then the set of principal ideals of \(R_S\)

\[ \{C_2^lR_S \mid (P_0, P_1, \ldots, P_{n-1}) \text{ is a } (S, n)\text{-cycle}\} \]

is finite and depends only on \(l, S\) and \(K\).

**Proof.** We use the same notation of Lemma 3. We prove the finiteness by induction on \(l\). The case \(l = 1\) was already proved in Lemma 7. Suppose that the statement is valid for \(l - 1\). Let

\[ L_{2,2^{l-1}} := \frac{x_0y_{2^l} - x_{2^l}y_0}{x_0y_{2^{l-1}} - x_{2^{l-1}}y_0}. \]

Applying Lemma 7 to the cycle \((P_0, P_{2^{l-1}}, P_{2^l}, \ldots)\) we get the finiteness of the set of principal ideals generated by the possible values of \(L_{2,2^{l-1}}\).

By Lemma 3

\[ C_2^l(x_0y_1 - x_1y_0) = L_{2,2^{l-1}}C_2^{l-1}(x_0y_1 - x_1y_0). \]

Therefore, by inductive assumption on \(C_2^{l-1}\), the conclusion follows. \(\Box\)
Lemma 8. Let $C_3$ be the integer defined in Lemma 3 associated to a $(S, n)$-tuple. The set of principal ideals of $R_S$

$$\{ C_3 R_S \mid (P_0, P_1, \ldots, P_{n-1}) \text{ is a } (S, n)\text{-cycle} \}$$

is finite and depends only on $S$ and $K$.

Proof. We use the same notation of Lemma 3. The statement is trivial for $n < 4$. For $n = 4$ the lemma follows from Proposition 3, since $n$ and 3 are coprime. Let $n > 4$. By Lemma 3 any $(S, n)$-cycle is sent by the automorphism defined in (10) to the following ordered $n$-tuple of vectors of $K^2$:

$$(0, 1); (1, 0); (C_2, -u_{1,2}); (C_3, -C_2 u_{1,3}); (C_4, -C_3 u_{1,4}); \ldots; (C_{n-1}, -C_{n-2} u_{1,n}).$$

Thus by the identity in (11), with $j = 3, i = 1$ it follows that

$$-C_3^2 u_{1,4} + C_4 C_2 u_{1,3} = u_{3,4}.$$ 

Note that by Corollary 3 we can choose a finite set $\mathcal{C}_2$ (the choice depends only on $S$) such that $C_2 = D_2 w_1$ and $C_4 = D_4 w_2$ with $D_2, D_4 \in \mathcal{C}_2$ and $w_1, w_2$ suitable $S$-units. Then $C_3$ satisfies one of the finitely many equations

$$C_3^2 = D_4 D_2 w_1 w_2 u_{1,3} u_{1,4} - \frac{u_{3,4}}{u_{1,4}}. \quad (31)$$

Thus applying Lemma 5 to equation (31), with $w_1 w_2 u_{1,3} u_{1,4} = u, u_{3,4} u_{1,4} = v$ and $C_3 = y$, the lemma is proved. □

Remark 1. Note that the sets of ideals defined in Lemma 5 and Lemma 6 have cardinality bounded by a constant depending only on $S$, the coefficient $D, E$ and the field $K$ (see the proof Theorem D.8.3. in [9] furthermore we can effectively determine these ideal sets). So the cardinalities of the sets of ideals defined in Lemma 7 and Lemma 8 are bounded by a constant depending only on $S$ and $K$. □

Recall that the cross-ratio of four distinct points $P_1, P_2, P_3, P_4$ of $\mathbb{P}_1(K)$ is

$$\varrho(P_1, P_2, P_3, P_4) = \frac{(x_1 y_3 - x_3 y_1)(x_2 y_4 - x_4 y_2)}{(x_1 y_2 - x_2 y_1)(x_3 y_4 - x_4 y_3)}.$$
By Lemma 3 part 1 it follows that, for every \((S, n)\)-cycle \((P_0, P_1, \ldots, P_{n-1})\), the ideal \(\mathcal{I}_1 = \prod_{p \in S} p^{\delta_p(P_0, P_1)}\) divides the ideal \(\mathcal{I}_i = \prod_{p \in S} p^{\delta_p(P_0, P_i)}\), for any index \(i\). This proves that every fractional ideal
\[ \mathcal{I}_i \mathcal{I}_i^{-1} = \prod_{p \in S} p^{\delta_p(P_0, P_i) - \delta_p(P_0, P_1)} \]
is actually an integral ideal of \(R_S\).

The theorem is equivalent to proving the finiteness of the possible values for \(\delta_p(P_0, P_i) - \delta_p(P_0, P_1)\) for any \((S, n)\)-cycle and \(p \notin S\).

Let \((P_0, P_1, \ldots, P_{n-1})\) be a \((S, n)\)-cycle. The matrix \(A\) defined in (10) defines an element of \(\text{PGL}_2(K)\) which maps the ordered \(n\)-tuple \((P_0, P_1, \ldots, P_{n-1})\) to the \(n\)-tuple \((\bar{P}_0, \bar{P}_1, \ldots, \bar{P}_{n-1})\). We represent the points of the last \(n\)-tuple with coordinates as defined in (12) such that the equations (11) are satisfied.

For every index \(i\) the following identities hold:
\[
\delta_v(\bar{P}_0, \bar{P}_i) = v_p(\bar{x}_i - \bar{x}_0, \bar{y}_i - \bar{y}_0) = v_p(x_i y_0 - x_0 y_i) - v_p(x_0 y_1 - x_1 y_0) = \delta_v(P_0, P_i) - \delta_v(P_0, P_1) \tag{32}
\]

Therefore by (32) we get that for every \(p \notin S\) the following statement is true:

*the set*
\[
\Delta(p, i) := \{\delta_v(\bar{P}_0, \bar{P}_i) \mid (P_0, P_1, \ldots, P_{n-1}) \text{ is a } (S, n)\text{-cycle}\}
\]
is finite depending only on \(S\) and \(K\) if and only if the set
\[
\{\delta_v(P_0, P_i) - \delta_v(P_0, P_1) \mid (P_0, P_1, \ldots, P_{n-1}) \text{ is a } (S, n)\text{-cycle}\}
\]
is finite depending only on \(S\) and \(K\).

Now we verify the finiteness of \(\Delta(p, i)\) for every index \(i\) and for every \(p \notin S\).

As remarked in the proof of Lemma 3 we can choose a set \(\mathcal{C}_2\) (the choice depends only on \(S\) and \(K\)) such that for any possible value of \(C_2\) we have that \(C_2 = D_2 u\) with suitable \(D_2 \in \mathcal{C}_2\) and \(u \in R_s^*\).

By (12) the point \(\bar{P}_2\) is \([C_2 : -u_{1,2}]\). Thus there exists a \(S\)-unit \(u\) such that the matrix
\[
U = \begin{pmatrix} u & 0 \\ 0 & -u_{1,2} \end{pmatrix} \in \text{PGL}_2(R_s) \tag{33}
\]
maps \(\bar{P}_2\) to \([D_2 : 1]\) with \(D_2 \in \mathcal{C}_2\). Let \(\bar{U}(\bar{P}_i) = [\bar{x}_i : \bar{y}_i]\) for every index \(i \geq 3\); therefore, the automorphism \(U\) maps the \(n\)-tuple \((\bar{P}_0, \bar{P}_1, \ldots, \bar{P}_{n-1})\) to
\[
([0 : 1], [1 : 0], [D_2 : 1], \ldots, [\bar{x}_i : \bar{y}_i], \ldots, [\bar{x}_{n-1} : \bar{y}_{n-1}]), \tag{34}
\]
where for every $p \notin S$ the $p$-adic distances are not changed. It is clear that the theorem holds if $n < 4$. Otherwise by the properties of the cross-ratio
\[ \varrho(\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3) + \varrho(\tilde{P}_0, \tilde{P}_1, \tilde{P}_3, \tilde{P}_2) = 1, \]
we have that
\[ \frac{(\tilde{x}_0\tilde{y}_2 - \tilde{x}_2\tilde{y}_0)(\tilde{x}_1\tilde{y}_3 - \tilde{x}_3\tilde{y}_1)}{(\tilde{x}_0\tilde{y}_1 - \tilde{x}_1\tilde{y}_0)(\tilde{x}_2\tilde{y}_3 - \tilde{x}_3\tilde{y}_2)} - \frac{(\tilde{x}_0\tilde{y}_3 - \tilde{x}_3\tilde{y}_0)(\tilde{x}_1\tilde{y}_2 - \tilde{x}_2\tilde{y}_1)}{(\tilde{x}_0\tilde{y}_1 - \tilde{x}_1\tilde{y}_0)(\tilde{x}_2\tilde{y}_3 - \tilde{x}_3\tilde{y}_2)} = 1. \tag{35} \]
By Lemma 8 we can choose a set $\mathcal{C}_3$ (the choice depends only on $S$ and $K$), such that for any possible value of $C_3$ we have that $C_3 = D_3w$ with suitable $D_3 \in \mathcal{C}_3$ and $w \in R^*_3$.

The following identities hold:
\[ \tilde{y}_3 = D_2v_{1,3}, \tag{36} \]
\[ \tilde{x}_3 = D_3v_{0,3}, \tag{37} \]
\[ \tilde{x}_2\tilde{y}_3 - \tilde{x}_3\tilde{y}_2 = v_{2,3}, \tag{38} \]
for suitable $v_{1,3}, v_{0,3}, v_{2,3} \in R^*_3$, $\tilde{D}_2 \in \mathcal{C}_2$ and $D_3 \in \mathcal{C}_3$.

Let us rewrite equation (35) as the $S$-unit equation in $v_{1,3}/v_{2,3}, v_{0,3}/v_{2,3}$
\[ -D_2\tilde{D}_2 \left( \frac{v_{1,3}}{v_{2,3}} \right) + D_3 \left( \frac{v_{0,3}}{v_{2,3}} \right) = 1. \tag{39} \]
Note that for any $D_2, \tilde{D}_2 \in \mathcal{C}_2$, $D_3 \in \mathcal{C}_3$ fixed, there are only finitely many solutions $(v_{1,3}/v_{2,3}, v_{0,3}/v_{2,3})$.

Hence, by (37) and (36)
\[ [\tilde{x}_3 : \tilde{y}_3] = \left[ D_3 \frac{v_{0,3}}{v_{2,3}} \frac{v_{2,3}}{v_{1,3}} : \tilde{D}_2 \right] \]
There are only finitely many equations of the type (39) since the sets $\mathcal{C}_2$ and $\mathcal{C}_3$ are finite; therefore the finiteness of the possible values of $\tilde{P}_3$ follows, for every $(S, n)$-cycle $(P_0, P_1, \ldots, P_{n-1})$. By the same argument, increasing by 1 each index in (35) we obtain that the set of possible points $\tilde{P}_i$ is finite, thus it follows the finiteness of the values of $\delta_\nu(\tilde{P}_0, \tilde{P}_i)$, for every index $i \in \{2, \ldots, n - 1\}$ and for every $p \notin S$.

The proof now follows simply applying Corollary B in [11], which states that if $(P_0, P_1, \ldots, P_{n-1})$ is a $(S, n)$-cycle for rational map of degree $\geq 2$, then
\[ n \leq [12(s + 2)\log(5s + 10)]^{\text{deg}(K; \mathbb{Q})} \leq [12(s + 2)\log(5s + 10)]^{8s}. \]
\[ \square \]
By Remark 1, the cardinality of the set $I_S$ depends only on $S$ and $K$.

### 3.2 Proofs in the general case.

**Proof of Theorem 1.** We deduce the general case of Theorem 1 from the particular case in which $R_S$ is a P.I.D., treated in subsection 3.1. For every $S$ such that $R_S$ is not a P.I.D. there exist infinitely many prime ideals of $R_S$ that are not principal (it is a direct consequence of unique factorization in prime ideals and the Chinese Remainder Theorem). Therefore if $R_S$ is not a P.I.D., then there exist two disjoint finite sets $S_1$, $S_2$ of prime ideal such that $R_{S_1} \cup S_2$ and $R_{S_2} \cup S_1$ are P.I.D.. Let $(P_0, P_1, \ldots, P_{n-1})$ be a cycle for a rational map of degree $\geq 2$ with good reduction outside $S$. For every index $i \in \{1, \ldots, n-1\}$, let $I_i$ be the ideals defined in (2). For every prime ideal $p$ let $e_p$ be the exponent such that $I_i I_{i-1}^{-1} = \prod_{p \notin S} p^{e_p}$.

Applying Theorem 1 to $R_{S \cup S_1}$, we deduce that for every prime ideal $p \notin S \cup S_1$ (in particular for all $p \in S_2$) the value $e_p$ is non negative and bounded by a constant that depend only on $S$ and $K$. Now, applying Theorem 1 to $R_{S \cup S_2}$, the same statement holds for all prime ideal $p \in S_1$. In this way we have proved that there exist a constant $c$, which depends only on $S$ and $K$, such that for every prime ideal $p \notin S$, $0 \leq e_p \leq c$. It is easy to see that for all prime ideal $p$, but finitely many (they are independent from the choice of $(S, n)$-cycle), $e_p$ is zero. This concludes the proof.

**Proof of Corollary 1.** Fixing two consecutive points $P_0, P_1 \in \mathbb{P}_1(K)$ of a $(S, n)$-cycle we set the ideal $I_1$ defined by (2) and, by Theorem 1, the set of possible ideals $I_i$ is finite and fixed. Hence the choice of two consecutive points of a $(S, n)$-cycle $(P_0, P_1, \ldots, P_{n-1})$ determines the finite set of possible values for $\delta_i(P_i, P_j)$, for any couple of points $P_i, P_j$. Applying the results of Birch and Merriman [3], with the same method used in the proof of Proposition 2, we prove the corollary.

**Proof of Corollary 2.** The proof of is contained in the proof of Theorem 1. Indeed $N$ is the set of the $n$-tuples of type (34) which are obtained from any $(S, n)$-cycle $(P_0, P_1, \ldots, P_{n-1})$ under the action of the matrix $U \cdot A \in \text{PGL}_2(K)$, where $A$ and $U$ are the matrices defined in (10) and (33), respectively. Since, for every $n$, the set of possible $n$-tuples of type (34) is finite (see the proof of Theorem 1 and $n$ is bounded, then $N$ is finite.
Proof of Theorem 2. Let \( K = \mathbb{Q} \) and \( S = \{ | \cdot |_\infty; | \cdot |_2 \} \) so that \( R_S = \mathbb{Z}[1/2] \). Let

\[
T := \{ ([u : u - 1], [u - 1 : -1], [1 : u]) \mid u \in R_S^* \}
\]

We will prove that the infinite set \( T \) is formed by \((S, 3)\)-cycles for suitable rational maps of degree equal to 4 and that the set of primes defined by

\[
\{ p \mid \delta_p(P_0, P_1) > 0 \text{ for some } (P_0, P_1, P_2) \in T \}
\]

is infinite. In particular the set of possible ideals \( \mathfrak{A}_1 \) is infinite. This will prove automatically the extension of Theorem 2 to every number field \( K \) and every choice of finite set \( S \) containing all the archimedean places of \( K \) and the 2-adic ones.

For every point \( P = [x : y] \in \mathbb{P}_1(\mathbb{Q}) \) we can choose coprime \( S \)-integral coordinates \( (x, y) \). By this choice of coordinates, for every prime \( p \) and for every couple of points \( Q_1 = [x_1 : y_1], Q_2 = [x_2 : y_2] \) of \( \mathbb{P}_1(\mathbb{Q}) \) the following identity holds

\[
\delta_p(Q_1, Q_2) = v_p(x_1 y_2 - x_2 y_1).
\]

Hence we have the following identity between ideals

\[
(x_1 y_2 - x_2 y_1) = \prod_{p \text{ prime}} p^{\delta_p(Q_1, Q_2)}.
\]

To simplify the notation, to any rational map \( \phi: \mathbb{P}_1 \to \mathbb{P}_1 \) defined over \( \mathbb{Q} \) we associate, in the canonical way, the rational function \( \phi(z) \in \mathbb{Q}(z) \) by taking the pole of \( z \) as the point at infinity \([1 : 0]\). In this way, a rational function \( \phi(z) = N(z)/D(z) \) with \( N, D \in \mathbb{Z}[z] \) coprime polynomials, has good reduction at a prime \( p \) if \( p \) does not divide the resultant of polynomials \( F, D \) and \( \phi \), the rational function obtained from \( \phi \) by reduction modulo \( p \), has the same degree of \( \phi \). Writing \( \mathbb{P}_1(\mathbb{Q}) = \mathbb{Q} \cup \{ \infty \} \), we will shift from the homogeneous to the affine notation for points in \( \mathbb{P}_1(\mathbb{Q}) \) when necessary. So the point \([1 : 0]\) will correspond to \( \infty \) and any other point \([x : y]\) will correspond to the rational number \( x/y \).

Let \( U(z) = (1 - z)^{-1} \). Then for every \( x/y \in \mathbb{Q} \) it follows that

\[
U \left( \frac{x}{y} \right) = \frac{y}{y - x}; \quad U^2 \left( \frac{x}{y} \right) = \frac{y - x}{-x} \quad \text{and} \quad U^3 \left( \frac{x}{y} \right) = \frac{x}{y}.
\]

Indeed \( U \) is the automorphism of \( \mathbb{P}_1(\mathbb{Q}) \) associated to the matrix \( \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \) of order 3 in \( \text{PGL}_2(\mathbb{Z}) \), so it has good reduction at any prime, since it is defined by a matrix...
in \( \text{SL}_2(\mathbb{Z}) \). Note that every element of \( T \) is a cycle for \( U \). Moreover \( U \) admits the following cycles:

\[
0 \mapsto 1 \mapsto \infty \mapsto 0; \quad -1 \mapsto -\frac{1}{2} \mapsto 2 \mapsto -1.
\]

Define the degree three function \( \Psi(z) \in \mathbb{Q}(z) \) by

\[
\Psi(z) = \frac{(z + 1)(2z - 1)(z - 2)}{2z(z - 1)},
\]

which has good reduction outside \( S \) and satisfies \( \Psi = \Psi \circ U \).

For every \( S \)-integer \( u \), let us define \( P_0 = u/(u - 1) \), \( P_1 = U(u/(u - 1)) = -(u - 1) \) and \( P_2 = U^2(u/(u - 1)) = 1/u \). Since

\[
\Psi(P_0) = \frac{-2u^3 + 3u^2 + 3u - 2}{2u^2 - 2u}
\]

we have that the automorphism defined by

\[
H(z) = \frac{(4u^2 - 4u)z - (-4u^3 + 6u^2 + 6u - 4)}{2uz + 4u^2 + u - 2}
\]

verifies \( H(\Psi(P_0)) = 0 \) and it belongs to \( \text{PGL}_2(\mathbb{Z}_S) \) if and only if \( u \in R^*_s \). However the rational function

\[
H \circ \Psi(z) = \frac{(2u^2 - 2u)z^3 + (2u^3 - 6u^2 + 2)z^2 + (-2u^3 + 6u - 2)z + (2u^2 - 2u)}{uz^3 + (2u^2 - u - 1)z^2 + (-2u^2 - 2u + 1)z + u}
\]

verifies

\[
H \circ \Psi(P_0) = H \circ \Psi(P_1) = H \circ \Psi(P_2) = 0.
\] (40)

Define \( \Psi_1(z) = z + H \circ \Psi(z) \).

**Lemma 9.** Let \( \phi \in \mathbb{Q}(z) \) be a rational map with good reduction outside \( S \) such that \( \phi(\infty) = \infty \), let \( \begin{pmatrix} a & b \\ u & c \end{pmatrix} \in \text{GL}_2(R_S) \) and put

\[
T(z) = \frac{az + b}{uz + c} \in \text{PGL}_2(R_S).
\]

Then the rational function \( z + T \circ \phi(z) \) has degree \( \deg(\phi) + 1 \) and has good reduction outside \( S \) if and only if \( u \in R^*_s \).
Proof. Let \( \phi(z) = N(z)/D(z) \) where \( N, D \in \mathbb{Z}[z] \) are polynomials with no common factor. Since \( \phi(\infty) = \infty \) we have that \( \deg(\phi) = \deg(N) > \deg(D) \) and since \( \phi \) has good reduction outside \( S \) one has that the leading coefficient of \( N \) is a \( S \)-unit and \( N, D \) have no common factors modulo any prime \( p \notin S \). Also the rational function

\[
T \circ \phi(z) = \frac{aN(z) + bD(z)}{uN(z) + cD(z)}
\]

has good reduction outside \( S \); therefore the polynomials \((aN(z) + bD(z)), (uN(z) + cD(z))\) have no common factor modulo any prime \( p \notin S \) and have the same degree equal to \( \deg(\phi) \); moreover the leading coefficient of \((uN(z) + cD(z))\) is a \( S \)-unit if and only if \( u \in \mathbb{R}_S^* \). Now it is immediate to see that the rational function

\[
z + T \circ \phi(z) = \frac{(uN(z) + cD(z))z + aN(z) + bD(z))}{(uN(z) + cD(z))}
\]

has degree equal to \( \deg(\phi) + 1 \) and has good reduction outside \( S \) if and only if \( u \in R_S^* \). □

Since \( \Psi(\infty) = \infty \), we can apply Lemma 9 with \( \phi = \Psi \) and \( T = H \) so \( \Psi_1 \) has good reduction outside \( S \) if and only if \( u \in R_S^* \). Moreover \( \Psi_1 \) has degree equal to 4 and by \( 40 \) follows that \( \Psi_1(P_i) = P_j \) for every index \( i \in \{0, 1, 2\} \). Thus we get that \((P_0, P_1, P_2)\) is a cycle for the rational map \( \Phi = U \circ \Psi_1 \) of degree 4 with good reduction outside \( S \) (if \( u \in R_S^* \)). The 3-cycle

\[
(P_0; P_1; P_2) = ([u : u - 1]; [u - 1 : -1]; [1 : u]) \in T,
\]

with \( u = 2^n \) and \( n \in \mathbb{N} \) gives \( \mathcal{S}_1 = \mathcal{S}_2 = (2^{2n} - 2^n + 1) \cdot \mathbb{Z}[1/2] \) proving Theorem 2. Note also that the set of prime divisors of \((2^{2n} - 2^n + 1)\) for \( n \in \mathbb{N} \) is infinite. This concludes the proof of Theorem 2. □

In the last proof we constructed a set of cycles of length 3. This is a consequence of the choice of the automorphism \( U \in \text{PGL}_2(\mathbb{Z}) \) of order 3. However, for any \( n \), with a suitable number field \( K \) and a suitable finite set \( S \) of places, it is possible to employ the same method as in the proof of Theorem 2 starting with an automorphism of \( \text{PGL}_2(R_S) \) of order \( n \). In this way it is possible to construct an infinite set of \((S, n)\)-cycles which satisfies Theorem 2.
References

[1] Benedetto R (2005) Preperiodic points of polynomials over global fields. ArXiv:math.NT/0506480

[2] Beukers F, Schlickewei HP (1996) The equation $x + y = 1$ in finitely generated groups. Acta Arith LXXVIII.2: 189-199

[3] Birch BJ, Merriman JR (1972) Finiteness theorems for binary forms with given discriminant. Proc London Math Soc (3), 24: 385-394

[4] Evertse JH, (1984) On equations in S-units and the Thue-Mahler equation. Invent Math 75: 561-584

[5] Evertse JH (1984) On sums of S-units and linear recurrences. Compositio Math 53, no 2: 225-244

[6] Evertse JH, Győry K (1991) Effective finiteness results for binary forms with given discriminant. Compositio Math 79: 169-204

[7] Evertse JH (1995) The number of solutions of decomposable form equations. Invent Math 122: 559-601

[8] Halter-Koch F, Narkiewicz W (2000) Scarcity of finite polynomial orbits. Publ Math Debrecen 56/3-4: 405-414

[9] Hindry M, Silverman JH (2000) Diophantine Geometry An Introduction. Springer-Verlag, GTM 201

[10] Marcus DA (1977) Number Fields. Springer-Verlag, Universitext

[11] Morton P, Silverman JH (1994) Rational Periodic Point of Rational Functions. Inter Math Res Notices 2: 97-110

[12] Morton P, Silverman JH, (1995) Periodic points, multiplicities, and dynamical units. J Reine Angew Math 461: 81-122

[13] Narkiewicz W (1989) Polynomial cycles in algebraic number fields. Colloq Math 58: 151-155

[14] Narkiewicz W, Pezda T (1997) Finite polynomial orbits in finitely generated domain. Monatsh. Math. 124: 309-316
[15] **Pezda T** (1994) Polynomial cycles in certain local domains. Acta Arith **LXVI:1**: 11-22

[16] **Serre JP** (1989) Lectures on the Mordell-Weil Theorem. Friedr. Vieweg & Sohn. Brauschweig

[17] **van der Poorten AJ, Schlickewei HP** (1982) The growth condition for recurrence sequences. Rep. No. 82-0041, Dept. Math., Macquarie Univ., North Ryde, Australia

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