The scalar curvature in formal deformation quantization. I

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Abstract

In the framework of formal deformation quantization, we apply our formal moment map construction on the space of almost complex structures to recover the Donaldson-Fujiki moment map picture of the Hermitian scalar curvature. In the integrable case, it yields a formal moment map deforming the scalar curvature moment map.

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1 Introduction

The Donaldson-Fujiki moment map picture [4, 7] states the Hermitian scalar curvature is a moment map on the space $J(M, \omega)$ of positive almost-complex structures on a symplectic manifold $(M, \omega)$. This famous picture motivates the use of GIT stability to treat the constant scalar curvature Kähler metric problem.

In our approach of formal moment maps [12, 13], we propose a general picture using formal deformation quantization [2] to recover and deform moment map pictures on infinite dimensional spaces. This paper proposes to apply this procedure to the space $J(M, \omega)$.

A natural Fedosov star product algebra bundle is defined above $J(M, \omega)$. Using a canonical formal connection [1] on that bundle, we show the star product trace of its curvature is a deformation of the symplectic form involved in the Donaldson-Fujiki picture. Considering the action of Hamiltonian diffeomorphisms on $J(M, \omega)$, we show this action preserves the deformed symplectic form. In the almost-Kähler situation, we show the star product trace satisfies the formal moment map equation at order 1 in $\nu$, and we show it coincides with the Donaldson-Fujiki picture. In the Kähler case, we show the star product trace of a deformed Hamiltonian gives a formal moment map on $J(M, \omega)$ which deforms the scalar curvature.

An alternative approach was proposed by Foth-Uribe [6] using the operators from geometric quantization.

2 Three connections to play with

Throughout this paper, we consider a closed symplectic manifold $(M, \omega)$ of dimension $2m$. We also deal with infinite dimensional manifolds and Lie groups, we will follow the theory from [14].

We consider the space of almost-complex structures on $(M, \omega)$:

$$J(M, \omega) := \{ J \in \Gamma \text{End}(TM) \mid J^2 = -Id, \, \omega(J\cdot, J\cdot) = \omega(\cdot, \cdot), \, \omega(J\cdot, J\cdot) > 0 \}$$

It is a Fréchet manifold. At any point $J \in J(M, \omega)$, its tangent space is

$$T_JJ(M, \omega) := \{ A \in \Gamma \text{End}(TM) \mid \omega(\cdot, A\cdot) \text{ is symmetric and } AJ = -JA \}$$

The symplectic form on $J(M, \omega)$ we will be interested in writes as :

$$\Omega^J_J(A, B) := \int_M \text{Tr}(JAB) \frac{\omega^m}{m!}, \text{ for any } A, B \in T_JJ(M, \omega).$$

(1)

Also, $J(M, \omega)$ admits a complex structure compatible with $\Omega^J$.

$$JA := JA \text{ for } J \in T_JJ(M, \omega).$$
When, there is an integrable \( J_0 \in \mathcal{J}(M, \omega) \) turning \((M, \omega, J_0)\) into a Kähler manifold, the subspace of integrable complex structures \( \mathcal{J}_{\text{int}}(M, \Omega) \subseteq \mathcal{J}(M, \omega) \) is a complex subspace so that \( \Omega^{\mathcal{J}} \) restricts to a symplectic structure
\[
\Omega^{\mathcal{J}_{\text{int}}} := \Omega^{\mathcal{J}} \bigg|_{\mathcal{J}_{\text{int}}}. 
\]

To any \( J \in \mathcal{J}(M, \omega) \), one attaches a Riemannian metric
\[
g_J(\cdot, \cdot) := \omega(\cdot, J \cdot).
\]

Then, one can consider three connections:

- the Levi-Civita connection \( \nabla^{g_J} \),
- a symplectic connection \( \nabla^{J} \) build out of \( \nabla^{g_J} \) as in \([13]\),
- the Chern connection, we will denote by \( \nabla^{\mathcal{J}} \).

## 2.1 The Levi-Civita connection \( \nabla^{g_J} \)

The Levi-Civita connection \( \nabla^{g_J} \) is the unique torsion-free connection leaving \( g_J \) parallel.

For \( \varphi \in \text{Ham}(M, \omega) \) a Hamiltonian diffeomorphism, one has a natural action of it on \( J \in \mathcal{J}(M, \Omega) \) by
\[
\varphi \cdot J := \varphi_* \circ J \circ \varphi^{-1}.
\]

One also has a natural action on a linear connection \( \nabla \) on \( TM \) by
\[
(\varphi \cdot \nabla)_X Y := \varphi_* \nabla_{\varphi^{-1}_* X} \varphi^{-1}_* Y \text{ for all } X, Y \in \mathfrak{X}(M).
\]

The next proposition follows from straightforward computations.

**Proposition 2.1.** For \( J \in \mathcal{J}(M, \Omega) \) and \( \varphi \in \text{Ham}(M, \omega) \),
\[
\nabla^{g_{\varphi \cdot J}} = \varphi_* \nabla^{g_J}.
\]

Later, we will need a formula for the first order variation of \( \nabla^{g_J} \).

**Lemma 2.2.** Let \( t \mapsto J_t \in \mathcal{J}(M, \omega) \) with \( \frac{d}{dt}\bigg|_0 J_t = A \), then
\[
g_J\left( \frac{d}{dt}\bigg|_0 \nabla^{g_{\mathcal{J}}}_{X} Y, Z \right) = \frac{1}{2} \left( (\nabla^{g_J}_{X} a)(X, Z) + (\nabla^{g_J}_{Y} a)(Y, Z) - (\nabla^{g_J}_{Z} a)(X, Y) \right),
\]
for \( X, Y, Z \in \mathfrak{X}(M) \) and \( a(X, Y) := \frac{d}{dt}\bigg|_0 g_{J_t}(X, Y) = \omega(X, AY) \).

**Proof.** A short proof can be found in P. Topping’s book \([15]\). \(\square\)
The curvature of $\nabla^g J$ is the tensor:

$$R^g J(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X,Y,Z \in \mathfrak{X}(M)$. The Ricci curvature is the symmetric 2-tensor

$$\text{Ric}(U,V) := \sum_{k=1}^{2m} g_J(R^g J(e_k, U)V, e_k),$$

for $U,V \in T_x M$ and $\{e_k | k = 1, \ldots, 2m\}$ is an orthonormal frame at point $x \in M$.

In the Kähler case, when $J$ is integrable:

$$\text{Ric}(JU, JV) = \text{Ric}(U, V)$$

and one defines the Ricci form by

$$\text{ric}(U, V) := \text{Ric}(JU, V).$$

Also, one has

$$\text{ric}(U, V) = -\frac{1}{2} \sum_{k=1}^{2m} g_J(R^g J(e_k, Je_k)U, V).$$

2.2 The symplectic connection $\nabla^J$

In the sequel, we will need to attach a symplectic connection to any almost complex structure $J \in \mathcal{J}(M, \omega)$. It is similar to what we used in [13].

First, recall that a symplectic connection on $(M, \omega)$ is a torsion-free linear connection leaving $\omega$ parallel. A symplectic connection can be built out of any torsion-free linear connection, so we build one out of $\nabla^g$, for any $J \in \mathcal{J}(M, \omega)$.

We define a 2-tensor $K^J(X,Y)$ on $M$ by

$$\omega(K^J(X,Y), Z) := (\nabla^g_X \omega)(Y, Z) \text{ for all } X, Y, Z \in TM.$$ 

Then, a symplectic connection $\nabla^J$ is obtained through the formula

$$\nabla^J_X Y := \nabla^g_X Y + \frac{1}{3} K^J(X,Y) + \frac{1}{3} K^J(Y, X),$$

for $X, Y \in \mathfrak{X}(M)$.

**Proposition 2.3.** For all $X, Y \in TM$, one has

$$K^J(X,Y) = -J(\nabla^g_X J)(Y).$$

So that,

$$\nabla^J_X Y = \nabla^g_X Y - \frac{1}{3} J(\nabla^g_J)(Y) - \frac{1}{3} J(\nabla^g_Y J)(X).$$

Moreover, $\nabla^{\varphi J} = \varphi \cdot \nabla^J$ for any $\varphi \in \text{Ham}(M, \omega)$.

**Proof.** The formula for $K^J$ follows from

$$g_J((\nabla^g_X J)(Y, Z),$$

The equivariance with respect to the action of $\varphi$ is a consequence of Proposition 2.1. □
2.3 The Chern connection $\nabla^J$ and the Hermitian scalar curvature

For any $J \in \mathcal{J}(M, \omega)$, the Chern connection $\nabla^J$ is a canonical $J$-linear connection on the complex vector bundle $(TM, J)$. It is defined by

$$\nabla^J_X Y := \nabla^g_X Y - \frac{1}{2} J (\nabla^g_X J) (Y),$$

for any $X, Y \in \mathfrak{X}(M)$. The Chern connection $\nabla^J$ preserves $g_J, J$ and then $\omega$, but has torsion. It also preserves the Hermitian metric $h^J(X, Y) := g_J(X, Y) - i \omega(X, Y)$, for any $X, Y \in T M$

which is $J$-linear in the first entry and $J$-anti-linear in the second one.

**Proposition 2.4.** $\nabla^\varphi^J = \varphi \cdot \nabla^J$ for any $\varphi \in \text{Ham}(M, \omega)$ and $J \in \mathcal{J}(M, \omega)$.

The proof again follows from Proposition 2.1.

Let us now introduce the main characters of this paper: the Hermitian Ricci form and the Hermitian scalar curvature.

The connection $\nabla^J$ induces a connection on the complex line bundle $\Lambda^m(TM, J)$ still denoted $\nabla^J$. Consider a local complex basis $\mathcal{Z} := \{Z_1, \ldots, Z_m\}$ of $(TM, J)$. This basis induces a non-zero local section $\zeta := Z_1 \wedge \ldots \wedge Z_m$ of $(TM, J)$. One computes

$$\nabla^J_X \zeta := \theta^J_{\mathcal{Z}}(X) \zeta$$

for $\theta^J_{\mathcal{Z}}(X) := (h^J)^{ki} h^J(\nabla^J_X Z_i, Z_k)$, with $(h^J)^{ki}$ denoting the inverse of the matrix of $h^J$ in the basis $\mathcal{Z}$, and we use from now on the summation convention on repeated indices. The 1-form $\theta^J_{\mathcal{Z}}$ is only locally defined, it depends on the choice of the local complex basis $\mathcal{Z}$ but its differential is globally defined.

The *Hermitian Ricci form* of $(M, \omega, J)$ is the real form

$$\rho^J := i d \theta^J_{\mathcal{Z}}.$$

In the Kähler case, it coincides with the Ricci form in Equation (2), but not in general. The *Hermitian scalar curvature* is the function $S^J$ such that:

$$\rho^J \wedge \frac{\omega^{m-1}}{(m-1)!} = \frac{1}{2} S^J \omega^m,$$

or $S^J := - \Lambda^J \rho^J$, for $\Lambda$ being the inverse matrix of the (real) coordinate matrix of $\omega$.

We will need the first order variation of $\rho^J$ which writes in term of the first order variation of $\nabla^J$. Actually, varying $J$ makes the complex structure on $(TM, J)$ vary. So we need to compensate that, we follow the ideas from the book [9].

We consider a path of almost complex structures

$$J_t := \gamma_t \circ J \circ \gamma_t^{-1}$$

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for $\gamma_t := \exp(ta)$ and $a = \frac{1}{2}JA$ for $A \in T_J(M, \omega)$, so that $\frac{d}{d|_0} J_t = A$. We consider the path of $J$-linear connections

$$\tilde{\nabla}^t := \gamma_t^{-1} \circ \nabla^h \circ \gamma_t.$$  

Consider the local unitary complex basis $Z_t := \{\gamma_t Z_1, \ldots, \gamma_t Z_m\}$ of $(TM, J_t)$ starting from a chosen local unitary complex basis $Z := \{Z_1, \ldots, Z_m\}$ of $(TM, J)$. One build the local non zero section $\zeta_t := \gamma_t Z_1 \wedge \ldots \wedge \gamma_t Z_m$. Then, for any $X \in TM$,

$$\nabla^h_{X} \zeta_t = \theta^h_{Z_t}(X) \zeta_t.$$  

On the other hand,

$$\tilde{\nabla}^t_{X} \zeta = \theta^h_{Z_t}(X) \zeta,$$

which means the 1-form $\kappa := \frac{d}{d|_0} \theta^h_{Z_t}$ is globally defined. Moreover,

$$\frac{d}{dt} \bigg|_0 \rho^h = i d\kappa.$$  

The 1-form $\kappa$ is called the first order variation of the Chern connection.

**Proposition 2.5.** For $X \in TM$,

$$\kappa(X) = \frac{i}{2} \delta^J A^b(X),$$

where $\frac{d}{d|_0} J_t = A$, $Y^b$ is the 1-form $g_J(Y, \cdot)$ and $\delta^J T(X_1, \ldots, X_n) := -(\nabla^{e_i}_T)(e_i, X_1, \ldots, X_n)$ for $T$ a n-tensor on $M$, $\{e_i | i = 1, \ldots, 2m\}$ a $g_J$-orthonormal frame and $X_1, \ldots, X_n \in TM$.

For a proof of the above Proposition, see Gauduchon’s book [7].

**Corollary 2.6.** For a path $t \mapsto J_t \in \mathcal{J}(M, \omega)$, with $\frac{d}{d|_0} J_t = A$, one computes

$$\frac{d}{dt} \bigg|_0 \rho^h = -\frac{1}{2} d\delta^J A^b \text{ and } \frac{d}{dt} \bigg|_0 S^J = \frac{1}{2} \Lambda^q (d\delta^J A^b)_q.$$  

**Remark 2.7.** We will keep the superscript $J$ in $\delta^J$ all along to emphasize its dependence in $J$ through $g_J$ but also to avoid confusion with the $\delta_F$ from Fedosov construction.

The musical isomorphism $b$ also depends on $J$, when this dependence will be investigated we will write $b_J$.

Finally, the equivariance of the Chern connection translates into the equivariance of the Hermitian Ricci form.

**Lemma 2.8.** For $J \in \mathcal{J}(M, \omega)$ and $\varphi \in \text{Ham}(M, \omega)$,

$$\rho^{\varphi^{-1}J} = \varphi^* \rho^J.$$  

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3 Formal connections and curvature

3.1 Fedosov construction

On $(M, \omega)$, consider a basis $\{e_1, \ldots, e_{2n}\}$ of $T_xM$ at $x \in M$ and its dual basis $\{y^1, \ldots, y^{2n}\}$ of $T^*_xM$. The algebra of formal symmetric forms on $T_xM$ of the kind:

$$a(y, \nu) := \sum_{2r+k=0}^{\infty} \nu^k a_{k,i_1 \ldots i_r} y^{i_1} \ldots y^{i_r},$$

where $a_{k,i_1 \ldots i_r}$ symmetric in $i_1 \ldots i_r$ and $2k + r$ is the total degree, with product,

$$(a \circ b)(y, \nu) := \left( \exp \left( \frac{\nu}{2} \Lambda^i_{ij} \partial y^i \partial z^j \right) a(y, \nu) b(z, \nu) \right) \bigg|_{y=z},$$

for two formal symmetric tensors $a(y, \nu)$ and $b(y, \nu)$, is called the formal Weyl algebra $W_x$.

The formal Weyl algebra bundle is the bundle $W := \bigsqcup_{x \in M} W_x$ over $M$. Denote by $\Gamma_W \otimes \Lambda^* M$ the space of differential forms with values in sections of $W$. Such a differential form writes locally as:

$$\sum_{2k+l \geq 0, k \geq 0, p \geq 0} \nu^k a_{k,i_1 \ldots i_l,j_1 \ldots j_p}(x) y^{i_1} \ldots y^{i_l} dx^{j_1} \wedge \ldots \wedge dx^{j_p}, \quad (4)$$

with $a_{k,i_1 \ldots i_l,j_1 \ldots j_p}(x)$ are symmetric in the $i$’s and antisymmetric in the $j$’s. The space $\Gamma_W \otimes \Lambda^* M$ is filtered with respect to the total degree

$$\Gamma_W \otimes \Lambda^* M \supset \Gamma_W \otimes \Lambda^* M \supset \Gamma_W \otimes \Lambda^* M \supset \ldots .$$

The $\circ$-product extends fiberwisely to $\Gamma_W \otimes \Lambda^* M$ making it an algebra. That is, for $a, b \in \Gamma_W$ and $\alpha, \beta \in \Omega^*(M)$, we define $(a \otimes \alpha) \circ (b \otimes \beta) := a \otimes b \otimes \alpha \wedge \beta$. It is a graded Lie algebra for the graded commutator $[s, s^\prime] := s \circ s^\prime - (-1)^{q_1 q_2} s^\prime \circ s$ where $s$, resp. $s^\prime$ are of anti-symmetric degree $q_1$, resp. $q_2$ makes $W$-valued forms.

From a symplectic connection $\nabla$ on $(M, \omega)$, one defines a derivation $\partial$ of anti-symmetric degree $+1$ on $W$-valued forms by :

$$\partial a := da + \frac{1}{\nu}[\Gamma, a] \text{ for } a \in \Gamma_W \otimes \Lambda^M,$$

where $\Gamma := \frac{1}{2} \omega_{ik} \Gamma^k_{ij} y^j dx^i$, for $\Gamma^k_{ij}$ the Christoffel symbols of $\nabla$ on a Darboux chart.

Setting $\mathcal{R} := \frac{1}{2} \omega_{ir} R^r_{jkl} y^j y^k dx^l \wedge dx^r$, for $R^r_{jkl} := (R(\partial_k, \partial_l) \partial_j)^r$ the components of the curvature tensor of $\nabla$, the curvature of $\partial$ is

$$\partial \circ \partial a := \frac{1}{\nu}[\mathcal{R}, a].$$

We look for flat connections on $\Gamma_W$ of the form

$$Da := \partial a - \delta_F a + \frac{1}{\nu}[r, a],$$

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for $r$ a $W$-valued 1-form and $\delta_F$ is defined by

$$\delta_F(a) := dx_k \wedge \partial_y a = -\frac{1}{\nu}[\omega_{ij}y^i dx^j, a],$$

the $F$ subscript is there to avoid confusion with $\delta^J$, see remark 2.7. The curvature of $D$ is

$$D^2 a = \frac{1}{\nu} \left[ \overline{R} + \partial r - \delta_F r + \frac{1}{2\nu} [r, r] - \omega, a \right].$$

Define

$$\delta^{-1}_F a_{pq} := \frac{1}{p+q} y^k i(\partial x^k) a_{pq} \text{ if } p + q > 0 \text{ and } \delta^{-1}_F a_{00} = 0,$$

where $a_{pq}$ is a $q$-form with $p y$’s and $p + q > 0$. For any given closed central 2-form $\Omega$, there exists a unique solution $r \in \Gamma W \otimes \Omega^1 M$ with $W$-degree at least 3 of equation:

$$\overline{R} + \partial r - \delta_F r + \frac{1}{\nu} r \circ r = \Omega,$$

and satisfying $\delta^{-1}_F r = 0$, see Fedosov [5]. Because $\Omega$ is central for the $\circ$-product, it makes $D$ flat.

To the flat connection $D$, one attaches the space of flat sections $\Gamma W_D := \{a \in \Gamma W|Da = 0\}$. Flat sections form an algebra for the $\circ$-product as $D$ is a derivation. The symbol map is defined by $\sigma : a \in \Gamma W_D \mapsto a|_y=0 \in C^\infty(M)[[\nu]].$ The map $\sigma$ is a bijection with inverse $Q$ (Fedosov [5]) defined by

$$Q := \sum_{k \geq 0} \left( \delta^{-1}_F (\partial + \frac{1}{\nu} [r, \cdot]) \right)^k.$$

The Fedosov star product $\star$ build with the data of $\Omega$ a formal closed 2-form and $\nabla$ a symplectic connection is, for all $F, G \in C^\infty(M)[[\nu]]$:

$$F \star G := (Q(F) \circ Q(G))|_{y=0}.$$

**Definition 3.1.** To $J \in J(M, \omega)$, we attach the star product $\star_J$ which is the Fedosov star product build with $\Omega = \nu \rho^J$ and symplectic connection $\nabla^J$.

In the sequel, when dealing with the star product $\star_J$, we may emphasize the dependence in $J$ by writing $\Gamma^J, r^J, D^J, Q^J, \ldots$ for the corresponding ingredients of the Fedosov construction performed with the symplectic connection $\nabla^J$ and $\Omega = \nu \rho^J$.

**3.2 The star products $\{\star_J\}_{J \in J(M, \omega)}$ and a formal connection**

**Definition 3.2.** Define the star product algebra bundle $\mathcal{V}$ over $J(M, \omega)$ by

$$\mathcal{V} := J(M, \omega) \times C^\infty(M)[[\nu]] \xrightarrow{\mathcal{P}} J(M, \omega),$$

where the fiber $J \in J(M, \omega)$ is equipped with the star product $\star_J$ and $\mathcal{P}$ is the projection.
A formal connection $\mathcal{D}$ on sections of $\mathcal{V}$ is an operator of the form

$$d^J + \beta,$$

with a formal series $\beta = \sum_{k \geq 1} \nu^k \beta_k$ of 1-forms on $\mathcal{J}(M,\omega)$ with values in differential operators on functions of $M$. We say the connection is compatible with the family of star products $\{*_J\}_{J \in \mathcal{J}(M,\omega)}$ when, for all sections $F, G$ of $\mathcal{V}$:

$$\mathcal{D}(F *_J G) = \mathcal{D}(F) *_J G + F *_J \mathcal{D}(G).$$

Such a connection exists \cite{1}. To define it one needs two technical lemmas about Fedosov construction of star products.

**Lemma 3.3.** Suppose $b \in \Gamma \mathcal{W} \otimes \Lambda^1 M$ satisfies $Db = 0$. Then the equation $Da = b$ admits a unique solution $a \in \Gamma \mathcal{W}$, such that $a|_{y=0} = 0$, it is given by

$$b = D^{-1} a := -Q(\delta_F^{-1} a).$$

As in \cite{12, 13}, we make use of a canonical lift of smooth path on the base manifold to isomorphisms of Fedosov star products algebra.

Define sections of the extended bundle $\mathcal{W}^+ \supset \mathcal{W}$ as locally of the form

$$\sum_{2k+l \geq 0, l \geq 0} \nu^k a_{k,1 \ldots l}(x) y_i^1 \ldots y_i^l,$$

similar to (4), with $p = 0$, but we allow $k$ to take negative values, the total degree $2k + l$ of any term must remain nonnegative and in each given nonnegative total degree there is a finite number of terms.

Given a smooth path $t \mapsto J_t \in \mathcal{J}(M,\omega)$ with $\frac{d}{dt} J_t := A_t$. Hence, by Corollary 2.6

$$\frac{d}{dt} \rho^{J_t} = -\frac{1}{2} d(\delta^{J_t} A_{b^{J_t}}).$$

The following Theorem comes from \cite{5} and is adapted to the particular case of Fedosov star products of the form of $*_J$.

**Theorem 3.4.** Consider smooth paths $t \in [0,1] \mapsto J_t \in \mathcal{J}(M,\omega)$. Then there exists maps $B_t : \Gamma \mathcal{W} \to \Gamma \mathcal{W}$ defined by

$$B_t a := v_t \circ a \circ v_t^{-1}$$

for $v_t \in \Gamma \mathcal{W}^+$ being the unique solution of the initial value problem:

$$\left\{ \begin{array}{l}
\frac{d}{dt} v_t = \frac{1}{\nu} h_t \circ v_t \\

v_0 = 1
\end{array} \right.$$ 

with

$$h_t := -(D^{J_t})^{-1} \left( \frac{d}{dt} \mathcal{I}_{J_t} + \frac{d}{dt} \nu^{J_t} + \frac{\nu}{2} \delta^{J_t} A_{b^{J_t}} \right).$$

Moreover, $B_t(D^{J_0} a) = D^{J_t}(B_t a)$ for all $a \in \Gamma \mathcal{W}$ so that

$$B_t|_{\Gamma \mathcal{W}_{D^{J_0}}} : \Gamma \mathcal{W}_{D^{J_0}} \to \Gamma \mathcal{W}_{D^{J_t}}.$$
is an isomorphism of flat sections algebras and hence
\[ \sigma \circ B_t \circ Q^{J_0} : (C^\infty(M)[[\nu]], \ast_{J_0}) \to (C^\infty(M)[[\nu]], \ast_{J_t}) \] is an equivalence of star product algebras.

The dependence of \( h_t \) in \( J_t \) and its covariant derivatives is polynomial which makes the paths \( t \mapsto h_t \) and \( t \mapsto v_t \) smooth.

Following [1], one defines a compatible formal connection \( \mathcal{D} \) by interpreting the above Theorem as a parallel lift of the path \( t \mapsto J_t \).

**Definition 3.5.** For \( A \in T_J(J(M,\omega)), \) with \( t \mapsto J_t \) so that \( \frac{d}{dt} \bigg|_0 J_t = A \), define :
- the connection 1-form \( \alpha \in \Omega^1(J(M,\omega), \Gamma W^3) \) by
  \[ \alpha_{J}(A) := (D^J)^{-1} \left( \frac{d}{dt} \bigg|_0 \Gamma^J + \frac{d}{dt} \bigg|_0 v^J + \frac{\nu}{2} \delta^J A^J \right), \]
- the 1-form \( \beta \) with values in formal differential operators:
  \[ \beta_{J}(A)(F) := \frac{1}{2} [\alpha_{J}(A), Q^J(F)] \bigg|_{y=0}, \quad \text{for } F \in C^\infty(M)[[\nu]], \]
- the formal connection \( \mathcal{D} := d^J + \beta \).

**Proposition 3.6.** \( \mathcal{D} \) is a formal connection on \( V \) compatible with the family of Fedosov star products \( \{ \ast_J \}_{J \in J(M,\omega)} \). Moreover, the parallel transport for \( \mathcal{D} \) along the path \( t \mapsto J_t \in J(M,\omega) \) is given by the equivalence of star product algebra obtained from Theorem 3.4.

The compatibility of \( \mathcal{D} \) is proved in [1] and the link with parallel transport can be proved similarly to the corresponding statement in [13].

**3.3 The curvature of \( \mathcal{D} \)**

The curvature of \( \mathcal{D} \) evaluated at vector fields \( X, Y \) on \( J \) acting on a section \( F \) of \( V \) is:
\[ (R(Y, Z)F)(J) := (D_Y(D_ZF) - D_Z(D_YF) - D_{[Y,Z]}F)(J), \]

To properly compute the curvature tensor on vectors \( A, B \in T_J(J(M,\omega)) \), we use extensions of \( A \) and \( B \) as vector fields on \( J(M,\omega) \).

For any \( a \in \text{End}(TM,\omega) \), one defines a vector field \( \hat{a} \) by
\[ \hat{a}_J := \frac{d}{dt} \bigg|_0 \exp(ta) \circ J_J \circ \exp(-ta) \in T_J(J(M,\omega)). \]

In such a way, for \( a = \frac{1}{2}JA \) with \( A \in T_J(J(M,\omega)) \), one has an extension of \( A \) as \( \hat{a}_J = A \).
One also computes the Lie bracket of two such vector fields obtained from $a, b \in \text{End}(TM, \omega)$:

$$[\hat{a}, \hat{b}]_J := -\overline{[a, b]}_J$$

In the particular case of $a := \frac{1}{2} JA$ and $b := \frac{1}{2} JB$ with $A, B \in T_J \mathcal{J}(M, \omega)$, the above Lie bracket evaluated at $J$ vanishes:

$$[\hat{a}, \hat{b}]_J = 0.$$ 

Hence, we have the following formula for the curvature on $A, B \in T_J \mathcal{J}(M, \omega)$ acting on a section $F$ of $\mathcal{J}$:

$$(\mathcal{R}(A, B)F)(J) := (D_{\hat{a}}(D_{\hat{b}}F) - D_{\hat{b}}(D_{\hat{a}}F))(J),$$

using the natural extensions $\hat{a}, \hat{b}$ defined above with $a = \frac{1}{2} JA$ and $b := \frac{1}{2} JB$.

**Theorem 3.7.** For $A, B \in T_J \mathcal{J}(M, \omega)$ and a section $F$ of $\mathcal{V}$, the curvature of $\mathcal{D}$ is given by

$$(\mathcal{R}(A, B)F)(J) = \frac{1}{\nu} [R_J(A, B), Q^J(F(J))]_{y=0}$$

(6)

for $R_J(A, B)$ being the 2-form with values in $\Gamma \mathcal{W}$ defined by

$$R_J(A, B) := \frac{\nu}{4} \text{Tr}(JAB) + d^J \alpha_J(A, B) + \frac{1}{\nu} [\alpha_J(A), \alpha_J(B)],$$

(7)

Moreover,

- $R_J(A, B) \in \Gamma \mathcal{W}_{DJ}$,
- $R_J(A, B)|_{y=0} = \frac{\nu}{4} \text{Tr}(JAB) + O(\nu^2)$.

**Proof.** The terms containing $\alpha$ in Equation (7) come from standard computations of $\mathcal{R}$. The term in $\nu$ from Equation (6) doesn’t contribute in Equation (6) but will make $R_J(A, B)$ a flat section.

To check $R_J(A, B) \in \Gamma \mathcal{W}_{DJ}$, we compute $D^J$ applied to the RHS of (7). First, because $\frac{\nu}{4} \text{Tr}(JAB)$ is a function on $M$,

$$D^J \text{Tr}(JAB) = d(\text{Tr}(JAB)).$$

Now, we detail the terms of $D^J \left( d^J \alpha_J(A, B) + \frac{1}{\nu} [\alpha_J(A), \alpha_J(B)] \right)$. To do that we use the extensions $\hat{a}$ and $\hat{b}$ defined earlier for $a = \frac{1}{2} JA$ and $b := \frac{1}{2} JB$ and we start with $D^J(\hat{a}(\alpha(\hat{b})))$. Consider the 2-parameter family of almost complex structures

$$J_{st} := \exp(sb) \exp(ta)J \exp(-ta) \exp(-sb),$$

so that

$$\left. \frac{d}{ds} \right|_{s=0} J_{st} = [b, \exp(ta)J \exp(-ta)].$$
We compute
\[ D^J(\hat{a}(\hat{b})) = D^J(\frac{d}{dt} \bigg|_0 \alpha_{J_0}(\frac{d}{ds} \bigg|_0 J_{st})), \]  
\[ = \frac{d}{dt} \bigg|_0 D^{J_{st}} \alpha_{J_0}(\frac{d}{ds} \bigg|_0 J_{st}) - \left( \frac{d}{dt} \bigg|_0 D^{J_0}(\alpha_J(\frac{d}{ds} \bigg|_0 J_{st}))). \]  
\[ (8) \]

Similarly, for \( D^J(\hat{b}(\hat{a})) \), consider the 2-parameter family of almost complex structures
\[ \tilde{J}_{st} := \exp(sa) \exp(tb)J \exp(-tb) \exp(-sa), \]
so that
\[ \frac{d}{ds} \bigg|_0 \tilde{J}_{st} = [a, \exp(tb)J \exp(-tb)]. \]

Then,
\[ D^J(\hat{b}(\hat{a})) = D^J(\frac{d}{dt} \bigg|_0 \alpha_{J_0}(\frac{d}{ds} \bigg|_0 \tilde{J}_{st})), \]
\[ = \frac{d}{dt} \bigg|_0 D^{\tilde{J}_{st}} \alpha_{J_0}(\frac{d}{ds} \bigg|_0 \tilde{J}_{st}) - \left( \frac{d}{dt} \bigg|_0 D^{\tilde{J}_0}(\alpha_J(\frac{d}{ds} \bigg|_0 \tilde{J}_{st}))). \]  
\[ (10) \]

We have no contribution from \([\hat{a}, \hat{b}]_J \) at the point \( J \).

It follows from standard computation, as in \([12, 13]\), that in
\[ D^J \left( d^{\mathcal{J}} \alpha_J(\hat{a}, \hat{b}) + \frac{1}{\nu}[\alpha_J(\hat{a}_J), \alpha_J(\hat{b}_J)] \right) \]
all the terms involving \( F^J, r^J \) and \( \alpha_J \) cancel with each other. So that all it remains is
\[ D^J \left( d^{\mathcal{J}} \alpha_J(\hat{a}, \hat{b}) + \frac{1}{\nu}[\alpha_J(\hat{a}_J), \alpha_J(\hat{b}_J)] \right) = \frac{\nu}{2} \left( \frac{d}{dt} \bigg|_0 [\delta^{J_0}(B)]^{b_{J_0}} - \frac{d}{dt} \bigg|_0 [\delta^{\tilde{J}_0}(A)]^{b_{\tilde{J}_0}} \right), \]  
\[ (12) \]

and again we have no contribution from \([\hat{a}, \hat{b}]_J \) at point \( J \).

Using Lemma 3.8, we get
\[ D^J \left( d^{\mathcal{J}} \alpha_J(\hat{a}, \hat{b}) + \frac{1}{\nu}[\alpha_J(\hat{a}_J), \alpha_J(\hat{b}_J)] \right) = -\frac{\nu}{4} d(\text{Tr}(JAB)), \]
which shows that \( R_J(A, B) \) is a \( D^J \)-flat section.

Finally, because \( \alpha(\cdot) \) is of degree at least 3, we have
\[ R_J(A, B)|_{\nu=0} = \frac{\nu}{4} \text{Tr}(JAB) + O(\nu^2). \]

\[ \square \]

Lemma 3.8. For \( A, B \in T_J \mathcal{J}(M, \omega) \) and \( J_{ts}, \tilde{J}_{ts} \) the 2-parameters families defined in the proof of the above Theorem 3.7, we have
\[ \frac{d}{dt} \bigg|_0 [\delta^{J_0}(B)]^{b_{J_0}} - \frac{d}{dt} \bigg|_0 [\delta^{\tilde{J}_0}(A)]^{b_{\tilde{J}_0}} = -\frac{1}{2} d(\text{Tr}(JAB)). \]

The proof is postponed to the Appendix.
4 Formal symplectic form and formal moment map

4.1 A formal symplectic form on $\mathcal{J}(M, \omega)$

A formal symplectic form on a manifold $F$ is a formal deformation of a symplectic form $\sigma_0$ of the form:

$$\sigma := \sigma_0 + \nu \sigma_1 + \ldots \in \Omega^2(F)[[\nu]],$$

with closed 2-forms $\sigma_i$ for all $i$.

As in our previous works [12, 13], the $\ast$-product trace and the curvature element $R$ will produce our formal moment map picture.

Consider a star product $\ast$ on a symplectic manifold, a trace for $\ast$ on a symplectic manifold $(M, \omega)$ is a character

$$\text{tr} : (C^\infty_c(M)[[\nu]], [\cdot, \cdot]_\ast) \rightarrow \mathbb{R}[[\nu^{-1}, \nu]].$$

A trace always exits for a given star product on $(M, \omega)$. It is unique if one asks for the normalisation condition

$$\text{tr}(F) = \frac{1}{(2\pi \nu)^m} \int_M BF \frac{\omega^m}{m!}, \text{ for all } F \in C^\infty_c(U)[[\nu]].$$

for all $U$ contractible Darboux chart and $B$ being local equivalences of $\ast|_{C^\infty(U)[[\nu]]}$ with the Moyal star product $\ast_{\text{Moyal}}$. The trace is given by the $L^2$-product with a formal function $\rho \in C^\infty(M)[[\nu^{-1}, \nu]]$, called the trace density

$$\text{tr}(F) = \frac{1}{(2\pi \nu)^m} \int_M F \rho \frac{\omega^m}{m!}.$$

For $J \in \mathcal{J}(M, \omega)$, we denote by $\text{tr}^{\ast_J}$ the normalised trace of the Fedosov star product $\ast_J$ and by $\rho^J$ its trace density.

Definition 4.1. Let $\tilde{\Omega}^J$ be the formal 2-form on $\mathcal{J}(M, \omega)$ defined by

$$\tilde{\Omega}^J_J(A, B) := 4(2\pi)^m \nu^{m-1} \text{tr}^{\ast_J}(R_J(A, B)|_{y=0}),$$

for $J \in \mathcal{J}(M, \omega)$ and $A, B \in T_J \mathcal{J}(M, \omega)$.

Theorem 4.2. $\tilde{\Omega}^J$ is a formal symplectic form on $\mathcal{J}(M, \omega)$ deforming $\Omega^J$ and invariant under the action of $\text{Ham}(M, \omega)$ on $\mathcal{J}(M, \omega)$.

Proof. The result follows from direct adaptation of the corresponding results from [12, 13].

The fact that $dJ^\ast \tilde{\Omega}^J = 0$ at all $J \in \mathcal{J}(M, \omega)$ is computed on vector fields of the form $\hat{a}, \hat{b}$ and $\hat{c}$ extending tangent elements $A, B$ and $C$ at $J$ and using the following Lemma.

Lemma 4.3 ([8]). Let $t \mapsto J_t$ be a smooth path in $\mathcal{J}(M, \omega)$. Then

$$\left. \frac{d}{dt} \bigg|_0 \text{tr}^{\ast_{J_t}}(F) = \text{tr}^{\ast_{J_0}} \left( \frac{1}{\nu} [\alpha_{J_0}(\frac{d}{dt}) J_t), Q^{J_0}(F)] \right|_{y=0} \right).$$
The invariance of $\tilde{\Omega}^J$ with respect to the action of $\text{Ham}(M, \omega)$, comes from the naturality of Fedosov construction and the equivariance of the ingredients we used: the Hermitian Ricci form, see Lemma 2.8 and the symplectic connection $\nabla^J$, see Proposition 2.3.

Finally, to see $\tilde{\Omega}^J$ deforms $\Omega^J$, notice the trace starts with a multiple of the integral, the first order term of $R$ in Theorem 3.7 is precisely $\frac{\nu}{4}\text{Tr}(JAB)$ and compare with Equation (1).

4.2 Deforming the Donaldson-Fujiki picture

Consider an action $\cdot$ of a regular Lie group $G$ on $(X, \sigma)$ a manifold equipped with a formal symplectic form $\sigma$ so that the action preserves $\sigma$. We define an equivariant formal moment map to be a map

$$\theta : g \rightarrow C^\infty(\mathcal{J}(M, \omega))[[\nu]],$$

for $g$ the Lie algebra of $G$, such that for all $g \in G$, $Y \in g$ and $x \in X$

$$\text{(formal moment map) } \left. \frac{d}{dt} \right|_{t=0} \exp(tY) \cdot x = d^X\theta(Y) \quad (13)$$

$$\text{(equivariance) } \theta(\text{Ad}(g)Y) = (g^{-1})^*\theta(Y).$$

Theorem 1.

1. The map

$$\mu : C^\infty_0(M) \rightarrow C^\infty(\mathcal{J}(M, \omega))[[\nu]] : H \mapsto [J \mapsto 4(2\pi)^m\nu^{m-1}\text{tr}^{*J}(H)],$$

satisfies the equivariant formal moment map at first order in $\nu$ for the action of $\text{Ham}(M, \omega)$ on $(\mathcal{J}(M, \omega), \tilde{\Omega}^J)$.

2. In the Kähler case, denote by $\Delta^JH := -\frac{1}{2}\Lambda^ks(d(dH \circ J))_{ks}$ the Laplacian for $J \in \mathcal{J}_{\text{int}}(M, \omega)$, then the map

$$\bar{\mu} : C^\infty_0(M) \rightarrow C^\infty(\mathcal{J}_{\text{int}}(M, \omega))[[\nu]] : H \mapsto [J \mapsto 4(2\pi)^m\nu^{m-1}\text{tr}^{*J}(H - \frac{\nu}{2}\Delta^JH)],$$

is a formal moment map on $(\mathcal{J}_{\text{int}}(M, \omega), \tilde{\Omega}^J_{\text{int}})$.

Moreover, at first order in $\nu$, both maps $\mu$ and $\bar{\mu}$ coincide with the Donaldson-Fujiki moment map.

We will use the next two Lemmas.

Lemma 4.4. [10] Consider $H \in C^\infty(M)$, then the derivative of the action of $\varphi_t^H$ on $\Gamma W \otimes \Lambda^M$ is given by the formula:

$$\frac{d}{dt}(\varphi_t^H)^* = (\varphi_t^H)^* \left( \left( i(X_H)D + D i(X_H) \right) + \frac{1}{\nu} \left[ -\omega_{ij}y^i X^j_H + \frac{1}{2}(\nabla^2_{\text{F}}H)_{y^i y^j} - i(X_H)r^i \right] \right),$$

where $D$ is the Fedosov flat connection obtained with symplectic connection $\nabla$ and a choice of a series of closed 2-forms.
Lemma 4.5. Let $H \in C^\infty(M)$, $J \in \mathcal{J}(M, \omega)$ inducing the symplectic connection $\nabla^J$, we have:

$$Q^J(H) = H - \omega_{ij} y^i X^j_H + \frac{1}{2}(\nabla^J)^2_{kq} H) y^k y^q - \iota(X_H) r^J + \alpha_J(\mathcal{L}_{X_H} J)$$

$$-\nu(D_J)^{-1}\left(\iota(X_H) \rho^J + \frac{1}{2}(\delta^J \mathcal{L}_{X_H} J)^{b_J}\right).$$

Moreover, if $J$ is integrable,

$$Q^J(H - \frac{\nu}{2} \Delta^J H) = H - \frac{\nu}{2} \Delta^J H - \omega_{ij} y^i X^j_H + \frac{1}{2}(\nabla^J)^2_{kq} H) y^k y^q - \iota(X_H) r^J + \alpha_J(\mathcal{L}_{X_H} J)$$

Proof. In [13], we obtained (adapted to the notations of the present paper)

$$Q^J(H) = H - \omega_{ij} y^i X^j_H + \frac{1}{2}(\nabla^J)^2_{kq} H) y^k y^q - \iota(X_H) r^J$$

$$+(D_J)^{-1}\left(\frac{d}{dt}\bigg|_0 \Gamma^{a_i}_{\mu} - \rho^{\mathcal{L}_{X_H} J}\right),$$

where the last term is what is hidden in the connection form in [13].

In Equation (14), we add what is needed to make appear $\alpha_J(\mathcal{L}_{X_H} J)$ provided that $\iota(X_H) \rho^J + \frac{1}{2}(\delta^J \mathcal{L}_{X_H} J)^{b_J}$ is a closed 1-form. But that follows from the equivariance of the Hermitian Ricci form (Lemma 2.8)

$$d\iota(X_H) \rho^J = \frac{d}{dt}\bigg|_0 \rho^{\mathcal{L}_{X_H} J},$$

and from Corollary 2.6

$$d\left(\frac{1}{2}(\delta^J \mathcal{L}_{X_H} J)^{b_J}\right) = -\frac{d}{dt}\bigg|_0 \rho^{\mathcal{L}_{X_H} J}.$$

In the Kähler case, the formula follows from Lemma 4.6 below.

Lemma 4.6. If $(M, \omega, J)$ is Kähler, then

$$\iota(X_H) \rho^J + \frac{1}{2}(\delta^J \mathcal{L}_{X_H} J)^{b_J} = d(\frac{1}{2} \Delta^J H).$$

The proof of Lemma 4.6 is postponed to the appendix.

We now prove the main Theorem.

Proof of Theorem The proof is similar to the cases studied in [12] and [13]. To shorten the proof we work directly with $\tilde{\mu}$ whose expression makes sense in the almost-Kähler case and, at first order in $\nu$, coincides with $\mu$.

The equivariance is immediate from the naturality of the Fedosov construction and the equivariance of all of its ingredients from Proposition 2.3 and Lemma 2.8.

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We check the formal moment map equation in the Kähler case. For \( J \in \mathcal{J}(M, \omega) \), and the path \( t \mapsto J_t \in \mathcal{J}(M, \omega) \) through \( J \) such that \( \frac{d}{dt}J_t = A \in \mathbb{T}_J\mathcal{J}(M, \omega) \)

\[
(\bar{d}^\mathbb{J}\bar{\mu}(H)) (A) = 4(2\pi)^m \nu^{m-1} \frac{d}{dt} \bigg|_0 \text{tr}^{*_{J_t}}(H - \frac{\nu}{2} \Delta^J H)
\]

Using Lemma 4.3 and after the formulas from Lemmas 4.4 and 4.5

\[
\frac{d}{dt} \bigg|_0 \text{tr}^{*_{J_t}}(H - \frac{\nu}{2} \Delta^J H) = \text{tr}^{*_{J_t}} \left( \frac{1}{\nu}[\alpha_J(A), Q^J(H - \frac{\nu}{2} \Delta^J H)\bigg|_{y=0} \right) - \frac{\nu}{2} \text{tr}^{*_{J_t}}(\frac{d}{dt} \bigg|_0 \Delta^J H),
\]

\[
= \text{tr}^{*_{J_t}} \left( \frac{1}{\nu}[\alpha_J(A), \alpha_J(\mathcal{L}_H J)\bigg|_{y=0} \right) + \text{tr}^{*_{J_t}} \left( -\frac{d}{dt} \bigg|_0 (\bar{\nu} H)^* \alpha_J(A)\bigg|_{y=0} \right) \]

\[
+ \text{tr}^{*_{J_t}} \left( (\mathfrak{t}(X_H)D^J + D^J \mathfrak{t}(X_H)) \alpha_J(A)\bigg|_{y=0} \right) - \frac{\nu}{2} \text{tr}^{*_{J_t}}(\frac{d}{dt} \bigg|_0 \Delta^J H).
\]

Now, \( \alpha_J(A) \) is a 0-form, all of its terms contain \( y \)'s. So, at \( y = 0 \) it remains,

\[
\frac{d}{dt} \bigg|_0 \text{tr}^{*_{J_t}}(H - \frac{\nu}{2} \Delta^J H) = \text{tr}^{*_{J_t}} \left( \frac{1}{\nu}[\alpha_J(A), \alpha_J(\mathcal{L}_H J)\bigg|_{y=0} \right) + \mathfrak{t}(X_H)D^J \alpha_J(A)\bigg|_{y=0} \right)
\]

\[
- \frac{\nu}{2} \text{tr}^{*_{J_t}}(\frac{d}{dt} \bigg|_0 \Delta^J H).
\]

By the definition of \( \alpha \) and \( \text{R}_J \), we have

\[
\frac{d}{dt} \bigg|_0 \text{tr}^{*_{J_t}}(H - \frac{\nu}{2} \Delta^J H) = - \text{tr}^{*_{J_t}} \left( \text{R}(\mathcal{L}_H J, J, A)\bigg|_{y=0} \right) - \frac{\nu}{4} \text{tr}^{*_{J_t}}(\text{Tr}(J \mathcal{L}_H J)) \quad (15)
\]

\[
+ \frac{\nu}{2} \text{tr}^{*_{J_t}} ((\bar{\delta}^J A)^{b_J}(X_H)) - \frac{\nu}{2} \text{tr}^{*_{J_t}}(\frac{d}{dt} \bigg|_0 \Delta^J H)
\]

Finally, in the Kähler case, the Lemma 4.7 below implies

\[
- \frac{\nu}{4} \text{tr}^{*_{J_t}}(\text{Tr}(J \mathcal{L}_H J)) + \frac{\nu}{2} \text{tr}^{*_{J_t}} ((\bar{\delta}^J A)^{b_J}(X_H)) - \frac{\nu}{2} \text{tr}^{*_{J_t}}(\frac{d}{dt} \bigg|_0 \Delta^J H) = 0.
\]

So that, one obtains the formal moment map equation

\[
\frac{d}{dt} \bigg|_0 4(2\pi)^m \nu^{m-1} \text{tr}^{*_{J_t}}(H - \frac{\nu}{2} \Delta^J H) = - \left( i(\mathcal{L}_H J) \bar{\Omega}^J \right)(A).
\]

In the almost-Kähler case, Equation (15) is still valid at order 1 in \( \nu \) as there is no contribution of the Laplacian because the trace starts with the integral functional. Hence, using Lemma 4.8 one get

\[
\frac{d}{dt} \bigg|_0 4(2\pi)^m \nu^{m-1} \text{tr}^{*_{J_t}}(H) = - \left( i(\mathcal{L}_H J) \bar{\Omega}^J \right)(A) + O(\nu).
\]
At first order in $\nu$, one knows (see [8] for example) the first terms of the normalised trace:

$$4(2\pi)^m \nu^{m-1} \text{tr}^{*J}(H) = -\int_M H S^J \omega^{m}/m! + O(\nu),$$

which is the Donaldson-Fujiki moment map. \hfill \square

**Lemma 4.7.** For $J \in \mathcal{J}_{\text{int}}(M, \omega)$ and $H \in C^\infty(M)$, one compute

$$\frac{d}{dt} \bigg|_0 \Delta J \cdot H = (\delta^J A)^{b_j}(X_H) - \text{Tr}(J A L X_H J)$$

**Lemma 4.8.** For $J \in \mathcal{J}(M, \omega)$ and $H \in C^\infty(M)$, we have

$$\int_M (\delta^J A)^{b_j}(X_H) \omega^{m}/m! = \int_M \text{Tr}(J A L X_H J) \omega^{m}/m!$$

The proofs of the above two Lemmas is contained in the Appendix.

**Remark 4.9.** We suspect that the Lemmas 4.6 and 4.7 are valid in the general almost-Kähler case. We postpone this task to a future work.

Our last corollary, contains at first order in $\nu$ the link that was presented in [11]. Namely, that closedness of $*J$ translates into the vanishing of a (formal) moment map. We say $*J$ is closed up to order $n$ if the integral is a trace for $*J$ modulo terms in $\nu^{n+1}$.

**Corollary 4.10.** For $J \in T J_{\text{int}}(M, \omega)$, the star product $*J$ is closed up to order $n$ if and only if $\tilde{\mu}(J) = 0 + O(\nu^{n+1})$.

**Remark 4.11.** In the almost-Kähler case, one can state the same corollary involving $\mu$, but the (formal) moment map interpretation is only valid at order 1 in $\nu$.

**Appendix**

In this appendix, we prove the key identities from Lemmas 3.8, 4.6, 4.7 and 4.8.

**Proof of Lemma 3.8.** We prove that

$$\frac{d}{dt} \bigg|_0 \left[ \delta^J_0 (A) \right]^{b_{ja}} - \frac{d}{dt} \bigg|_0 \left[ \delta^J_0 (A) \right]^{b_{ja}} = -\frac{1}{2} d(\text{Tr}(JAB)),$$

with the notations introduced in Theorem 3.7.

To identify the above LHS, we compute for all $Y \in \mathfrak{g}(M)$, the $L^2$-product of $\frac{d}{dt} \bigg|_0 \left[ \delta^J_0 (A) \right]^{b_{ja}}$ with the 1-form $g_J(Y, \cdot)$. With a frame $\{ e_k | k = 1, \ldots, 2m \}$, we obtain

$$\int_M \frac{d}{dt} \bigg|_0 \left[ \delta^J_0 (A) \right]^{b_{ja}} (e_k) g_J^j g_J(Y, e_l) \omega^m/m! = \frac{d}{dt} \bigg|_0 \int_M \left[ \delta^J_0 (A) \right]^{b_{ja}} (e_k) (g_J^j)_{k_j} (g_J_0)_{j\ell} (Y, e_l)$$

$$= \frac{d}{dt} \bigg|_0 \int_M g_J^j (e_k, Ae_q) (g_J^p)_{j\ell} (g_J_0)_{j\ell} (\nabla_{e_p} Y, e_l)$$

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Making use of Lemma 2.2, one obtains
\[
\int_M \frac{d}{dt} \left. \left[ \delta \tilde{J}^{\text{in}}(A) \right]^{b_{\text{in}}}_{\text{in}} (e_k) g^{kl} g(Y, e_l) \right|_{0}^{\omega^m_m} = - \int_M g^{kl} g(\nabla_{e_k}^g Y, J(AB + BA) e_l) \frac{\omega^m_m}{m!} \\
- \frac{1}{2} \int_M g^{pq} g(e_k, A e_q) [\nabla_{e_k}^g g(\cdot, JB \cdot)] (e_k, e_l) \frac{\omega^m_m}{m!}.
\]

Because the first term on the RHS above is symmetric in \(A, B\), denoting by \((\cdot, \cdot)^J\) the \(L^2\) product of tensors induced by \(g^J\), we have for all \(Y \in \mathfrak{X}(M)\)
\[
\left( \frac{d}{dt} \left[ \delta \tilde{J}^{\text{in}}(B) \right]^{b_{\text{in}}}_{\text{in}} - \frac{d}{dt} \left[ \delta \tilde{J}^{\text{in}}(A) \right]^{b_{\text{in}}}_{\text{in}}, g(Y, \cdot) \right)_J = - \frac{1}{2} (g(\cdot, B \cdot), [\nabla_{e_k}^g g(\cdot, JA \cdot)]_J + \frac{1}{2} (g(\cdot, A \cdot), [\nabla_{e_k}^g g(\cdot, JB \cdot)]_J, \\
= - \frac{1}{2} (d(\text{Tr}(JAB)), g(Y, \cdot))_J,
\]
which concludes the proof of the Lemma 3.8.

**Proof of Lemma 4.6.** Considering \((M, \omega, J)\) is Kähler, we will show
\[
i(X_H) \rho^J + \frac{1}{2} (\delta^J \mathcal{L}_{X_H} J)^{b_{\text{in}}} = d(\frac{1}{2} \Delta^J H).
\]

In the Kähler case,
\[
\mathcal{L}_{X_H} J(Y) = -\nabla_{JY}^g X_H + J \nabla_{Y}^g X_H. \quad (16)
\]
So that, denoting by \(\beta = g(X_H, \cdot)\) and using an unitary frame \(\{e_k \mid k = 1, \ldots, 2m\}\),
\[
\delta^J (\mathcal{L}_{X_H} J)^{b} = \sum_i (\nabla^g g)^2_{(e_i, J e_i) \beta} - \delta^J \nabla^g (\beta \circ J). \quad (17)
\]

Now, since \(\rho^J\) coincides with the Ricci form when \((M, \omega, J)\) is Kähler, Equation (3) leads to
\[
\sum_i (\nabla^g g)^2_{(e_i, J e_i) \beta} = \frac{1}{2} \sum_i R^{g^J(e_i, J e_i) \beta} = -i(X_H) \rho^J.
\]

Using the Weitzenbock formula and \(\beta \circ J = -dH\), the second term of Equation (17) becomes
\[
-\delta^J \nabla^g (\beta \circ J) = (\delta^J d + d \delta^J) dH + \sum_{i,j} e_i^* \wedge \iota(e_j) R(e_i, e_j) dH, \\
= d(\Delta^J H) - i(X_H) \rho^J.
\]
So,
\[
\delta^J (\mathcal{L}_{X_H} J)^{b} = -2i(X_H) \rho^J + d(\Delta^J H),
\]
which finishes the proof.
Proof of Lemma 4.7. In the Kähler setting, we will prove
\[
\frac{d}{dt} \left| \Delta^h H = (\delta^j A)^{b_j}(X_H) - \text{Tr}(J \mathcal{L}_{X_H} J) \right. \]

The LHS is
\[
\frac{d}{dt} \left| \Delta^h H = \frac{1}{2} \frac{d}{dt} \Lambda^{ks} [d(-dH \circ J_t)]_{ks} \right.
\]
So that in a frame \(\{e_k \mid k = 1, \ldots, 2m\}\), we have
\[
\frac{d}{dt} \left| \Delta^h H = \frac{1}{2} \frac{d}{dt} \Lambda^{ks} [d(-dH \circ A)]_{ks} \right.
\]
By Equation (16), we get
\[
\frac{d}{dt} \left| \Delta^h H = -\frac{1}{2} \text{Tr}(J \mathcal{L}_{X_H} J) + (\delta^j A)^{b_j}(X_H) \right.
\]
which concludes the proof.

Proof of Lemma 4.8. Let us prove finally that
\[
\int_M (\delta^j A)^{b_j}(X_H) \frac{\omega^m}{m!} = \int_M \text{Tr}(J \mathcal{L}_{X_H} J) \frac{\omega^m}{m!}
\]
Using the notation \((\cdot, \cdot)_J\) for the \(L^2\)-product of tensors, then :
\[
\int_M \text{Tr}(J \mathcal{L}_{X_H} J) \frac{\omega^m}{m!} = (g_J(J A \cdot, \cdot), g_J(L_{X_H} J \cdot, \cdot))_J
\]
From \(\mathcal{L}_{X_H} \omega = 0\), we obtain
\[
g_J((L_{X_H} J) U, V) = -g_J(\nabla^{q_j}_{J U} X_H, V) - g_J(J U, \nabla^{q_j}_{v} X_H).
\]
So that,
\[
(g_J(J A \cdot, \cdot), g_J(L_{X_H} J \cdot, \cdot))_J = \int_M (\delta^j A)^{b_j}(X_H) \frac{\omega^m}{m!}
\]
The proof is over. 

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Bibliography

[1] J. E. Andersen, P. Masulli, F. Schätz, Formal connections for families of star products, *Comm. Math. Physics* **342** (2), 739–768 (2016).

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnérowicz, D. Sternheimer, Deformation theory and quantization, *Annals of Physics* **111**, part I : 61–110, part II : 111–151 (1978).

[3] S.K. Donaldson, Moment maps and diffeomorphisms, *Asian J. Math.* **3** (1), 1–16 (1999).

[4] S. K. Donaldson. Remarks on gauge theory, complex geometry and 4-manifold topology. In *Fields Medallists’ lectures*, volume 5 of *World Sci. Ser. 20th Century Math.*, pages 384–403. World Sci. Publ., River Edge, NJ (1997)

[5] B.V. Fedosov, A simple geometrical construction of deformation quantization. *Journal of Differential Geometry* **40**, 213-238 (1994).

[6] T. Foth, A. Uribe, The manifold of compatible almost complex structures and geometric quantization, *Comm. Math. Phys.* **274** (2), 357–379 (2007).

[7] A. Fujiki. Moduli space of polarized algebraic manifolds and Kähler metrics [translation of Sugaku 42 (1990), no. 3, 231–243; MR1073369 (92b:32032)]. *Sugaku Expositions*, **5** (2):173–191 (1992).

[8] A. Futaki, L. La Fuente-Gravy, Quantum moment map and obstructions to the existence of closed Fedosov star products, *Journ. of Geom. and Phys.* **163**, Article 104118 (2021).

[9] P. Gauduchon, Calabi’s extremal Kähler metrics: An elementary introduction, Lecture Notes available upon request.

[10] S. Gutt, J. Rawnsley, Natural star products on symplectic manifolds and quantum moment maps, *Lett. in Math. Phys.* **66**, 123–139 (2003).

[11] L. La Fuente-Gravy, Infinite dimensional moment map geometry and closed Fedosov’s star products, *Ann. of Glob. Anal. and Geom.* **49** (1), 1–22 (2015).

[12] L. La Fuente-Gravy, The formal moment map geometry of the space of symplectic connections, [arXiv:2106.13608](https://arxiv.org/abs/2106.13608) (2021).

[13] L. La Fuente-Gravy, A formal moment map on Diff\(_0(M)\), [arXiv:2203.12287](https://arxiv.org/abs/2203.12287) (2022)

[14] K.-H. Neeb, Towards a Lie theory for locally convex groups, *Japanese Journal of Math.* **1**, 291–468 (2006).

[15] P. Topping, *Lectures on the Ricci flow*, L.M.S. Lecture note series **325** C.U.P. (2006).