Remarks on the numerical impact of potential theoretical systematics in the prediction of QCD instanton cross sections

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Abstract: We discuss the origin and size of potential uncertainties arising in the estimate of cross sections for the production of multiparticle final states induced by QCD instantons at the LHC.

1 Introduction

This note is a more detailed version of a contribution to the “QCD Instantons” discussion session at the Workshop on “Topological Effects in the Standard Model: Instantons, Sphalerons and Beyond at LHC”, https://indico.cern.ch/event/965112/. It follows the two presentations by A.Ringwald and V.Khoze, and in particular it links to the latter talk, and to the recent papers on which this was based, refs. [1, 2]. It incorporates some additional remarks emerged during the discussions at the Workshop, and further follow up.

This note will not address the underlying formalism used and developed in refs. [1, 2]. Taking the formalism and the results for granted, I focus on possible systematics that should still be attached to the results in [1, 2], helping to put in perspective the interpretation of the LHC searches for final states induced by QCD instantons.
2 Results and remarks

The amplitude for the instanton-induced production of $n_g$ gluons and $n_f$ quark-antiquark pairs is given in ref. [1] as:

$$\mathcal{A}(2 \to n_g + 2n_f) \sim \int_0^\infty d\rho^2 \left( \rho^2 \right)^{n_g + n_f - 1} e^{-\frac{2\pi}{\alpha_s(1/\rho)} - \frac{n_g(1/\rho)}{16\pi} E^2 \rho^2 \log(E^2 \rho^2)}$$

where $\rho$ is the instanton radius, and $E = \sqrt{s}$ is the partonic CM energy. As in ref. [1], we neglect overall constants, wave function normalization factors, etc. The second term in the exponent, proportional to $\rho^2 \log(E^2 \rho^2)$, reflects the Mueller’s form-factor, evaluated and discussed in refs. [3–5]. This is critical to suppress the contribution of large-size instantons, and to guarantee the convergence of the integral over instanton configurations. We note that eq. 1 makes explicit use of the relation $\rho \mu_r = 1$ between the renormalization scale $\mu_r$ (which otherwise enters in the value of the strong coupling constant, in the instanton density and in Mueller’s exponent) and the radius $\rho$ [1]. This relation enforces the renormalization group invariance of the amplitude, and reflects the intuitive notion that the inverse size of the instanton defines the proper scale at which strong interactions act in the instanton field; as discussed in the following, however, the relation should be taken only as a functional relation fulfilling scale invariance, leaving room for a possible overall numerical factor, namely $\rho \mu_r = \mathcal{O}(1)$.

**Remark 1.** From dimensional analysis, the amplitude given above scales with $E$ as follows:

$$\mathcal{A} \propto E^{-2(n_g + n_f + b_0/2)} \propto E^{-2(n_g + n_f)} \left( \frac{\Lambda}{E} \right)^{b_0},$$

where at leading order $\alpha_s(\mu) = 4\pi/[b_0 \log(\mu^2/\Lambda^2)]$ and $b_0 = 11 - 2/3 n_f$. The second expression above highlights the fact that, while $(n_f + n_g)$ powers of $1/E^2$ are matched by the energy dependence of the wave function normalization of the external states and by the final phase-space integration, leading to $\dim(\sigma) = -2$, the $b_0$ powers of $1/E$ are matched instead by the QCD scale $\Lambda$, which therefore must appear in the final amplitude expression. This is the consequence of the power suppressed nature of this non-perturbative amplitude, embodied by the contribution of the instanton action $\exp(-2\pi/\alpha_s(1/\rho))$. This means that the amplitude has an intrinsic $\Lambda^{b_0}$ dependence. More on this later.

As indicated in ref. [1], the amplitude can be evaluated in the saddle-point approximation, where, leaving out again constant numerical factors:

$$\mathcal{A} \sim \int_0^\infty d\rho^2 e^{f(\rho^2)} = e^{f(\bar{\rho})} \sqrt{-f''(\bar{\rho})},$$

where

$$f(\rho^2) = (n_g + n_f - 1 + b_0/2) \log \rho^2 - \frac{\alpha_s(1/\rho)}{16\pi} E^2 \rho^2 \log(E^2 \rho^2),$$

The saddle point $\bar{\rho}$ is defined through:

$$\partial f(\rho^2)/\partial \rho^2 = \frac{A}{\rho^2} - \frac{E^2}{16\pi} \alpha_s(1/\rho) \log(E^2 \rho^2) + \mathcal{O}(\alpha_s^2) = 0 \quad \text{at} \quad \rho = \bar{\rho}$$

where $A = n_g + n_f - 1 + b_0/2$. We used the a-posteriori knowledge that $E^2 \rho^2 \gg 1$, to neglect in the derivative a term of order 1 w.r.t. $\log(E^2 \rho^2)$.

**Remark 2.** Notice that in eq. 5 we neglected terms formally of higher powers of $\alpha_s$. Some arise by including the term proportional to $\partial \alpha_s(1/\rho)/\partial \rho^2$ ~ $b_0 \alpha_s^2(1/\rho)$ in the derivative of $f(\rho)$, others would
arise in taking the NLO beta-function rather than the LO one; others, unknown, arise from NLO corrections to the function \( f(\rho) \) itself, in particular to Mueller's form factor. Since we cannot control the exact form of these higher order terms, we stick to the strict LO expression, but must keep in mind their existence for the assessment of the systematics of the final result.

The approximate solution of the saddle-point condition leads to the relation:

\[
\frac{1}{\hat{\rho}^2} = \eta E^2 \alpha_s(\eta E^2) \log(1/\eta) + \mathcal{O}(\alpha_s^2) \tag{6}
\]

where \( \eta = 1/(16\pi A) \). This approximate solution to eq. 5 agrees to within 10% with the exact solution, estimated numerically. Equation 6 leads to a value of the inverse instanton radius of \( \epsilon = 1/\hat{\rho} = \gamma E \), with \( \gamma \) in the range of 1/20 – 1/30 for \( E \sim 100 – 3000 \) GeV. This is consistent with the findings of Fig. 4 and Table 1 of ref. [2]. The fact that the inverse instanton radius is significantly larger than the invariant mass of the system, \( E \), has important implications. On one side it sets a threshold for the creation of massive quarks \( Q \) in the instanton decay: for the instanton to resolve the heavy quark, it must be \( \rho < 1/m_Q \), and therefore \( E > m_Q/\eta \sim 30m_Q \). This means that the threshold to produce bottom quarks is at around 150 GeV, and to produce top quarks it must be \( E \gtrsim 5 \) TeV. On the other side, it implies that for \( \alpha_s \) to remain in the perturbative domain, \( 1/\rho > \Lambda \), the minimum energy should be \( E > \Lambda/\eta \sim 4 \) GeV.

Evaluating \( f''(\rho) \) at the saddle point gives

\[
f''(\hat{\rho}) = -\frac{A}{\hat{\rho}^3}(1 + \mathcal{O}(\alpha_s)) \tag{7}
\]

and, putting things back into eq. 3 and neglecting overall constant factors, we obtain:

\[
A \sim \left( \frac{\Lambda}{E} \right)^{b_0} \left( \frac{1}{E} \right)^{n_g+n_f} \left[ \frac{1}{\alpha_s(\eta E^2)[1 + \mathcal{O}(\alpha_s)]} \right]^{n_g+n_f+b_0/2}. \tag{8}
\]

3 Discussion

I discuss here in more detail the numerical impact on the final amplitude, eq. 8, of the possible sources of systematics highlighted so far.

3.1 \( \Lambda^{b_0} \)

The first point is the \( \Lambda^{b_0} \) term upfront. On one side this inherits the intrinsic 1% uncertainty on \( \alpha_s(M_Z) \). But \( \Delta A/\Lambda \sim \Delta \alpha_s/\alpha_s \times \log(M_Z/\Lambda) \sim 6\% \), leading to \( \Delta A/A \sim \pm 60\% \), which is negligible overall. On the other hand, the choice of the perturbative order at which \( \Lambda \) is estimated is not well defined, and e.g. the difference between \( \Lambda_{LO} \) and \( \Lambda_{NLO} \) is large. For example, to obtain \( \alpha_s(M_Z) = 0.12 \) from the 1-loop evolution we get \( \Lambda \sim 100 \) MeV, while at 2-loop we get \( \Lambda \sim 260 \) MeV. So, in the cross section \( \sigma \propto A^2 \) there is a potential systematics factor in the range of \( 2^{\pm b_0} \sim [4 \times 10^{-3} – 250] \).

3.2 NLO effects

There is an independent uncertainty arising from the \( O(\alpha_s) \) corrections indicated in eq. 8. This is independent of the LO vs NLO issue raised in the previous remark: there we dealt with the order at which the leading power-suppressed instanton action, \( \exp(-2\pi/\alpha_s) \), is calculated. Here we are dealing with higher-order corrections to Mueller’s form factor. It is reasonable to expect these \( O(\alpha_s) \) uncertainties to be limited to a \( \pm 20\% \), but when raised to the power of \( (n_g+n_f+b_0/2) \) this can become an overall factor of \( (1.2/0.8)^{(n_g+n_f+b_0/2)} \geq 50 \) for the amplitude, and greatly more for the cross sections. More in general, as mentioned above, the relation between renormalization scale \( \mu_r \) and the instanton radius, \( \rho \mu_r = 1 \), should be subject to the usual factor of 2 uncertainty. The legitimate choice of \( \mu_r \rho = [0.5 – 2] \) would lead to a factor of \([0.5 – 2]\) rescaling of \( \Lambda \) in the argument of \( \alpha_s \), leading again to a systematics similar to what discussed at point 1 above.
Table 1. Parton luminosity ratios evaluated at $\mu_F = E$ and $\mu_F = 1/\rho$, at different partonic CM energies $E$. For each $E$, the corresponding value of $1/\rho$ is derived by linear interpolation from Table 1 of ref. [2], subject to the additional constraint $\mu_F \geq 1.65$ GeV.

| $E$ (GeV) | $1/\rho$ (GeV) | 20 | 30 | 40 | 50 | 100 | 200 | 500 |
|-----------|----------------|----|----|----|----|-----|-----|-----|
|           | $[d\mathcal{L}/d\tau]_{\mu_F=E} / [d\mathcal{L}/d\tau]_{\mu_F=1/\rho}$ | 1.65 | 2.1 | 2.7 | 3.2 | 5.4 | 9.8 | 22 |
|           |                | 49 | 15 | 7.4 | 5.0 | 2.1 | 1.2 | 0.8 |

Table 2. Cross sections, at 13 TeV, for the production of QCD instantons with mass larger than $E_{\text{min}}$. First row: the results from Table 2 of ref. [2]. (a) My result, mimiking the prescriptions in ref. [2], including $\mu_F = 1/\rho$. (b) Same approach, but with $\mu_F = E$. In rows (c) and (d) I repeat the calculation of (a) and (b), using a local power-like interpolation for the partonic cross section, instead of the linear interpolation adopted in [2]. Notice that, while significantly reduced with respect to the result of the linear interpolation, the rate for $E > 20$ GeV and with $\mu_F = E$ is still larger than the total pp cross section.

| $E_{\text{min}}$ (GeV) | $\sigma(pp \rightarrow I)$ [2] | 20 | 30 | 40 | 50 | 100 | 200 | 500 |
|------------------------|-----------------------------|----|----|----|----|-----|-----|-----|
|                        | $\sigma(pp \rightarrow I)$ (a) | 6.3 mb | 64 bi | 58 nb | 80 nb | 105 pb | 3.5 mb |
|                        | $\sigma(pp \rightarrow I)$ (b) | 5.8 mb | 0.91 mb | 0.17 mb | 40 mb | 79 nb | 106 pb | 3.5 fb |
|                        | $\sigma(pp \rightarrow I)$ (c) | 170 mb | 9.1 mb | 0.9 mb | 0.2 mb | 150 nb | 120 pb | 2.5 fb |
|                        | $\sigma(pp \rightarrow I)$ (d) | 4.0 mb | 0.63 nb | 0.12 nb | 25 mb | 41 nb | 39 pb | 2.0 fb |
|                        | $\sigma(pp \rightarrow I)$ (d) | 110 mb | 6.5 mb | 0.71 nb | 0.11 nb | 80 nb | 46 pb | 1.4 fb |

3.3 Factorization scale systematics

When the partonic cross sections are convoluted with parton luminosities, to extract hadronic cross sections in pp collisions, a further source of systematics arises from the choice of the factorization scale, $\mu_F$. In general, this is set equal to the renormalization scale, which for the instanton case, as discussed above, is chosen around the value of $1/\rho \ll E$. Since the factorization scale is related to the removal of initial state collinear singularities, which factorize out of the details of the hard process itself, it is fair to argue however that the choice $\mu_F \sim E$, if not preferable, should at least be considered here as a possible alternative. One can anticipate that this could lead to a large uncertainty, since the partonic luminosity at the small values of $\mu_F$ probed by the choice $\mu_F = 1/\rho$ has a very strong scale dependence, as shown in Table 1.

Table 2 shows the actual impact of this systematics from the choice of $\mu_F$. The first row of the table shows the results of ref. [2]: these were obtained by convoluting the parton level cross section $\hat{\sigma}(E)$ with the partonic luminosity at $\mu_F = \max[1/\rho, Q_{\text{min}}]$, $Q_{\text{min}} = 1.65$ GeV being the minimum value of $Q$ supported by the NNPDF3.1 PDF set chosen for the calculation. For simplicity, ref. [2] obtained $\hat{\sigma}(E)$ from a linear interpolation of the values calculated for a few fixed choices of the instanton mass $E$, given in Table 1 of ref. [2].

The second row in the Table is what I get by replicating the calculation of ref. [2]. The results show good agreement, with minor differences likely due to the different implementations of the interpolation criterion (I assume also a linear interpolation for the values of $\rho$ at different $E$ values in the integration). Having established that I can reproduce the results of ref. [2], the third row shows the results I obtain by setting $\mu_F = E$, the instanton mass. The larger evolution of the PDFs in $Q^2$ leads to a drastic increase of the cross section at small $E$ values, driven by the low-$x$ behaviour of the gluon density. At larger $E$, the $Q^2$ evolution depletes the gluon at larger $x$ values, leading to a reduced rate. This implies that it is not just the absolute value of the cross sections that is affected

\[ \text{Remark that, given the rapid falloff of the cross section, which for } E \gtrsim 50 \text{ GeV behaves like } 1/E^{-9}, \text{ a bin-by-bin linear interpolation of } \log \hat{\sigma}(E), \text{ namely a power fit in } E \text{ of } \hat{\sigma}(E) \text{ rather than a linear interpolation, would give a more faithful representation.} \]
by potentially large uncertainties, but also the shape of the instanton-energy dependence seems to be uncertain, making it less useful in separating signal and backgrounds or in the interpretation of potential signals. For example, in the region of interest for the extraction of a signal, Table 2 shows that the drop in rate between $\sigma(E_{\text{min}} = 50 \text{GeV})$ and $\sigma(E_{\text{min}} = 500 \text{GeV})$ is 7 times larger with the choice $\mu_F = E$ than with $\mu_F = 1/\rho$.

We also note that, for $E_{\text{min}} = 20 \text{ GeV}$, the total cross section largely surpasses the total pp cross section, more than doubling the inelastic rate. This apparent breaking of unitarity would simply imply that the average number of QCD instantons with $E > 20 \text{ GeV}$ produced in a pp collision is larger than 1 $^2$, but would also imply that the total pp cross section is saturated by instanton production, a rather unlikely possibility. Among other effects, this would lead to the striking prediction of an average multiplicity of charm quark pairs produced per pp collisions larger than 1 (the standard charm production mechanism predicts a total NLO rate of about 10 mb, although with a large uncertainty. This corresponds to about one charm pair produced for every 8 inelastic events).

4 Conclusions

In conclusion, it appears that there could be large sources of systematics associated to the predictions for instanton-induced QCD processes at the LHC. If the analysis reported in this note is correct, these uncertainties could cover several orders of magnitude. This does not remove interest in the search for such final states, but a possible lack of evidence does not lead to the immediate conclusion that instantons “do not exist”, but simply that their actual production rate is unfortunately on the lower end of the systematics, with respect to the central baseline rates discussed in refs. [1, 2]. These uncertainties would also clearly influence the interpretation of a possible signal, and its clear identification in terms of instantons, as opposed to other possible sources, within or beyond the Standard Model.

References

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$^2$A similar phenomenon occurs when we calculate the cross section for production of minijets with a low minimum $p_T$. Notice also that for $\sqrt{s} \sim 20 \text{ GeV}$ at the LHC one does not expect gluon-shadowing and saturation effects to play a unitarization role, since there is no evidence of gluon saturation from the measurement of other hard processes at this $(x, Q)$ scale (e.g. charm and bottom production at small $p_T$).