A Complete Statistical Analysis for the Quadrupole Amplitude in an Ellipsoidal Universe

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A model of Universe with a small eccentricity due to the presence of a magnetic field at the decoupling time (i.e. an Ellipsoidal Universe) has been recently proposed for the solution of the low quadrupole anomaly of the angular power spectrum of cosmic microwave background anisotropies. We present a complete statistical analysis of that model showing that the probability of increasing of the amplitude of the quadrupole is larger than the probability of decreasing in the whole parameters’ space.

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I. INTRODUCTION

The three year release of WMAP data \( \cite{1} \) confirms that the amplitude of the Quadrupole of Angular Power Spectrum (APS) of Cosmic Microwave Background (CMB) anisotropies is approximately five times lower than the expected value of the ΛCDM model. Other implications of this model have been considered and proposed in \( \cite{14}, \cite{15}, \cite{16}, \cite{17}, \cite{18}).

Among these models, an Ellipsoidal Universe has been proposed as a model for the explanation of this anomaly \( \cite{19} \). It is very interesting to note that the presence of a magnetic field at decoupling time induces an eccentricity of the background metric which in turn modifies the energy of the CMB photons. Following the treatment of \( \cite{19} \), this provides a novel temperature anisotropy that gives a contribution only to the quadrupole term (once expanded over Spherical Harmonics). This effect has been considered and proposed in \( \cite{19} \) to reconcile the observed quadrupole value with theoretical expectation of the ΛCDM model. Other implications of this model can be found in \( \cite{20} \).

The aim of the present paper is to associate a probability to the decreasing possibility given in this framework of Ellipsoidal Universe. We perform a complete statistical analysis of this model without constraining it to give the most favourite case (as done in literature). We show that the observed quadrupole value is more unlikely in the considered treatment of Ellipsoidal Universe than in a standard ΛCDM model.

The paper is organized as follows: in Section \( \text{II} \) we briefly describe the Ellipsoidal Universe model, in Section \( \text{III} \) technical details of the performed simulations are given and in Section \( \text{IV} \) we draw our conclusions.

II. ELLIPSOIDAL UNIVERSE

It is shown in \( \cite{19} \) that a small eccentricity \( \epsilon_{\text{dec}} \) at the decoupling time in the space-time metric provides a contribution only to the quadrupole terms (\( \ell = 2 \)) of CMB anisotropies

\[
\begin{align*}
\alpha_{20}^e & = \frac{\sqrt{\pi}}{6\sqrt{5}} \left[ 1 + 3 \cos(2\vartheta) \right] T_{\text{cmb}} \epsilon_{\text{dec}}^2 , \\
\alpha_{21}^e & = -\left( \alpha_{2,-1}^e \right)^* = -\frac{\pi}{30} e^{i\varphi} \sin(2\vartheta) T_{\text{cmb}} \epsilon_{\text{dec}}^2 , \\
\alpha_{22}^e & = \left( \alpha_{2,-2}^e \right)^* = \frac{\pi}{30} e^{-2i\varphi} \sin^2 \vartheta T_{\text{cmb}} \epsilon_{\text{dec}}^2 ,
\end{align*}
\]

where \((\vartheta,\varphi)\) represents the direction of the axis of the magnetic field that is responsible for the deviation from the perfect sphericity, \( T_{\text{cmb}} \approx 2.725 \) K \( \cite{21} \) is the CMB temperature and the eccentricity \( \epsilon_{\text{dec}} \) is related to the magnetic field \( B_0 \) through the following equation

\[
\epsilon_{\text{dec}} \approx 10^{-2} h^{-1} \frac{B_0}{10^{-8} \text{G}} ,
\]

with \( B_0 \) being the norm of the magnetic field at the present time and \( h \approx 72 \) being the reduced (dimensionless) Hubble constant (implicitly defined by \( H = h \) 100 km/s/Mpc).

These coefficients have to be added to the \( \alpha_{2m} \) that are produced by the intrinsic, independent, Gaussian dis-

\footnote{The Quadrupole anomaly is not the unique anomaly that is present at large angular scales of CMB maps. For other anomalies see for example \( \cite{2}, \cite{3}, \cite{4}, \cite{5}, \cite{6}.}

\footnote{We measure \( \alpha_{2m} \) in \( \mu \)K.}
increasing

In this way Eq. (7) can be written as

\[ C_2^{\text{obs}} = \frac{1}{3} \sum_m a_{2m}^{(\text{obs})} (a_{2m}^{(\text{obs})})^* \] (6)

is given by the following sum

\[ C_2^{\text{obs}} = C_2 + C_2^{\text{mix}} + C_2^{\text{e}}, \] (7)

where \( C_2 \) is the intrinsic one, \( C_2^{\text{e}} \) is computed from Eqs. (13) and is given by

\[ C_2^{\text{e}} = 4\pi T_{\text{cmb}}^2 \epsilon_{\text{dec}}^4 / 225 \] (8)

and \( C_2^{\text{mix}} \) is the mixing term that is writable as

\[ C_2^{\text{mix}} = -2f(\vartheta, \varphi) (C_2^{\text{e}})^{1/2} \] (9)

with the function \( f(\vartheta, \varphi) \) defined by

\[
f(\vartheta, \varphi) = \frac{-1}{4\sqrt{6}} \left[ a_{20} (1 + 3 \cos(2\vartheta)) - 2\sqrt{6} \left( a_{21}^{(R)} \cos \varphi - a_{21}^{(I)} \sin \varphi \right) \sin(2\vartheta) - \left( a_{22}^{(R)} \cos(2\varphi) - a_{22}^{(I)} \sin(2\varphi) \right) \sin^2 \vartheta \right] \] (10)

where the labels \( ^{(R)} \) and \( ^{(I)} \) stand for the real and imaginary part of the intrinsic coefficients of the spherical harmonics respectively. In this way Eq. (7) can be written as follows (in order to underline the parabolic behaviour)

\[ y = x^2 - 2\tilde{f}x + 1, \] (12)

where \( y = C_2^{\text{obs}}/C_2, x = C_2^{\text{e}}/C_2^{\text{e}} \) and \( \tilde{f} = f/C_2^{1/2} \). Eq. (12) represents a parabolic behaviour with upward concavity. Since \( x > 0 \) it is clear that \( y < 1 \) (i.e. a decreasing is obtainable for the observed quadrupole) if and only if \( \tilde{f} > 0 \) (that is the condition to have the abscissa of the vertex \( x_{\text{vertex}} = \tilde{f} > 0 \)). This is not always the case since \( \tilde{f} \) can be positive or negative depending on the input values, as can be checked from Eq. (10).

In Fig. 1 we plot Eq. (12). The green branch represents Eq. (12) for input values such that \( \tilde{f} > 0 \) whereas the red branch represents the parameter space for which \( \tilde{f} < 0 \). The horizontal black line divides the increasing from the decreasing \( y \)-region. Both cases give a possible increasing of the quadrupole amplitude but only one case (the green branch) permits an interval of decreasing of the quadrupole amplitude. As written in [19] the minimum is reached by \( x_{\text{min}} = \tilde{f} \) that gives \( y_{\text{min}} = 1 - \tilde{f}^2 \).

III. STATISTICAL ANALYSIS

It is possible to perform two kinds of analysis: the Minimum and the Full Shape Analysis. The first one, where the parameters are arbitrarily prioritized such that the model is bounded to \( x_{\text{min}} = \tilde{f} \), is performed in [19]. This is done to maximize the effect in the direction we prefer. In the second case the analysis is faced in the full general case. This is what is performed in the next subsection.

A. Full Shape Analysis

For each fixed eccentricity at decoupling \( \epsilon_{\text{dec}} \), and for each considered direction \((\vartheta, \varphi)\), we perform \( 5 \times 10^3 \) ran-
Fig. 2: Likelihood (in terms of counts, y-axis) of $\delta C_2$ (x-axis, measured in $\mu K^2$). Panels in the same row have the same $\varphi = 0$, $\pi/3$, $2\pi/3$, $\pi$, $4\pi/3$, $5\pi/3$ and $2\pi$ (in order from up to down). Panels in the same column have the same $\vartheta = 0$, $\pi/3$, $2\pi/3$, $\pi$ (in order from left to right). The eccentricity is set to $e_{dec} = 0.67 \times 10^{-2}$.

A Gaussian distribution with zero mean and standard deviation $\sigma$ of the order of the expected quadrupole for the $\Lambda$CDM model, i.e. $\sigma \sim \sqrt{1000} \mu K$. These extractions

\footnote{For the current purpose it is sufficient an estimate of the order of magnitude for $\sigma$.}
are replaced in Eq. (11) to obtain \( C_2^{\text{obs}} \) once the intrinsic quadrupole \( C_2 \) is computed. This allows to obtain the likelihood of \( \delta C_2 = C_2^{\text{obs}} - C_2 \) for the fixed parameters \( \epsilon_{\text{dec}} \) and \( (\vartheta, \varphi) \). We consider the following values for \( \epsilon_{\text{dec}} = 10^{-2} \), 0.5 \( 10^{-2} \) and 0.3 \( 10^{-2} \). Moreover we take into account \( \epsilon_{\text{dec}} = 0.67 \ 10^{-2} \) that is the “best case” present in literature [19]. This is also the considered value for \( \epsilon_{\text{dec}} \) in all the panels of Fig. 2 where we show the likelihood of \( \delta C_2 \). The directional space of \( (\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi] \) is discretized with a step of \( \pi/3 \). Fig. 2 shows that the bell shape of the likelihood of \( \delta C_2 \) is always shifted towards positive values. This means that the increasing probability is always larger than the decreasing one. The same has been obtained for the other values of the eccentricity (that are not reported for sake of brevity).

In Fig. 3 we report the probability distribution for \( \delta C_2 \), \( C_2^{\text{obs}} \) and \( C_2 \) for \( \epsilon_{\text{dec}} = 0.67 \ 10^{-2} \), 0.5 \( 10^{-2} \), 0.3 \( 10^{-2} \) at fixed \( (\vartheta, \varphi) = (\pi/3, 2\pi/3) \). Fig. 3 shows that the probability of extracting the observed WMAP value is smaller in the considered Ellipsoidal Universe than in a standard \( \Lambda \)CDM model with no eccentricity. Considering \( C_2^{\text{obs}} \) (WMAP) \( \sim 200 \mu K^2 \) we compute that for the observed quadrupole \( (C_2^{\text{obs}}) \) the probability to obtain a smaller value is 0.7\% (with \( \epsilon_{\text{dec}} = 0.67 \ 10^{-2} \)), 2.1\% (with \( \epsilon_{\text{dec}} = 0.5 \ 10^{-2} \)) and 3.5\% (with \( \epsilon_{\text{dec}} = 0.3 \ 10^{-2} \)) whereas for the intrinsic quadrupole \( (C_2) \) the probability is 3.8\%. We end this section with the expected value for the observed quadrupole (still for \( (\vartheta, \varphi) = (\pi/3, 2\pi/3) \) and \( \epsilon_{\text{dec}} = 0.67 \ 10^{-2} \)) that is computed to be \( C_2^{\text{obs}} = 1822 \mu K^2 \) whereas our intrinsic random extractions give \( C_2 = 999 \mu K^2 \).

**IV. CONCLUSIONS**

We have statistically analyzed a model of Ellipsoidal Universe recently proposed to solve the low quadrupole anomaly of CMB anisotropies. Performing our unprioritized analysis, we find that the probability of increasing of the amplitude of the quadrupole is larger than the decreasing one. We believe that this paper shows that the considered treatment of Ellipsoidal Universe cannot reconcile current observations of the quadrupole amplitude of CMB anisotropies with theoretical predictions. On the contrary in this model the observed quadrupole is more unlikely than in a standard \( \Lambda \)CDM model with no eccentricity.

**Note added** Before this paper was public, I have been informed of a different treatment of Ellipsoidal Universe which should lead to a change in the sign of the
function $f$ in Eq. (10). This modification does not affect the results of this paper since for each point $(\theta, \varphi)$ of the considered parameters space, the number of extractions that give $f > 0$ is close to the number of extractions that give $f < 0$.

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