A Gabriel Theorem for Coherent Twisted Sheaves

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Abstract

The aim of this work is to give a generalization of Gabriel’s theorem for twisted sheaves over smooth varieties. We start by showing that we can reconstruct a variety $X$ from the category $\text{Coh}(X, \alpha)$ of coherent $\alpha$–twisted sheaves over $X$. This follows from the bijective correspondence between closed subsets of $X$ and Serre subcategories of finite type of $\text{Coh}(X, \alpha)$. Then we show that any equivalence between $\text{Coh}(X, \alpha)$ and $\text{Coh}(Y, \beta)$, where $X$ and $Y$ are smooth varieties, induces an isomorphism between $X$ and $Y$. Here, the problem is to show that we can extend any coherent twisted sheaf on an open sub-scheme of $X$ to a coherent twisted sheaf on $X$. In order to do this, we study perfect and compact objects in $\text{D}(\text{QCoh}(X, \alpha))$. As a complement, we study the problem of saturatedness of $\text{D}^b(X, \alpha)$, which will be proved at least for smooth and proper varieties.

1 Introduction

Gabriel’s theorem is one of the main results of the use of category theory in algebraic geometry. It says that for every noetherian scheme $X$, we can construct a scheme $E_X$ from $\text{Coh}(X)$, and an isomorphism between $E_X$ and $X$, so that we can say that $\text{Coh}(X)$ carries informations about the scheme structure of $X$. Moreover, we have that two noetherian schemes have equivalent categories of coherent sheaves if and only if they are isomorphic.

What we want to do is to show that we can extend this theorem to the case of $\text{Coh}(X, \alpha)$, the category of coherent $\alpha$–twisted sheaves over $X$, where such $X$ is a (smooth) $k$–variety, that is, a separated scheme of finite type over a field $k$. More precisely, we want to show the following:

**Theorem 1.** Let $X$ be a variety over a field $k$, and $\alpha \in \text{Br}X$. Then the abelian category $\text{Coh}(X, \alpha)$ determines $X$. Moreover, if $X$ and $Y$ are two smooth varieties, $\alpha \in \text{Br}X$, $\beta \in \text{Br}Y$, any equivalence between $\text{Coh}(X, \alpha)$ and $\text{Coh}(Y, \beta)$ induces an isomorphism between $X$ and $Y$. 
Note that this theorem does not tell anything about how \( f^* \) acts on \( Br Y \), namely if \( f^* \beta = \alpha \). This would be an easy consequence of the twisted version of Orlov’s theorem which is shown in \[CS\] (see \[CS\], Remark 5.4 for the proof of this fact).

In this introduction, we would like to recall the notions of twisted sheaf and of category of (quasi) coherent twisted sheaves. The main reference for definitions and proofs will be \[Ca\]. In the following, let \( X \) be a variety over a field \( k \). We will denote by \( Br' X := H^2(X, \mathcal{O}_X^*)_{\text{tors}} \) the cohomological Brauer group of \( X \), and by \( Br X \) the Brauer group of \( X \), that is, the group of equivalence classes of Azumaya algebras over \( X \). From Theorem 1.1.8 in \[Ca\], we know that \( Br X \) is a subgroup of \( Br' X \), so that, in particular, we will think an element \( \alpha \in Br X \) as the cohomology class of a 2-cocycle \( \{\alpha_{ijk}\} \in \check{C}^2(X, U, \mathcal{O}_X^*) \), where \( U = \{U_i\}_{i \in I} \) is an open covering of \( X \). We will denote \( U_{ij} = U_i \cap U_j \) and \( U_{ijk} = U_i \cap U_j \cap U_k \), so that \( \alpha_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_X^*) \) and the 2-cocycle condition is satisfied. In this way we have the following definition.

**Definition 1.** We call sheaf twisted by \( \alpha \in \check{C}^2(X, U, \mathcal{O}_X^*) \), or simply \( \alpha \)-sheaf, a family \( F = (F_i, \varphi_{ij})_{i,j \in I} \) where \( F_i \) is an \( \mathcal{O}_{U_i} \)-module and \( \varphi_{ij} : F_j|_{U_{ij}} \rightarrow F_i|_{U_{ij}} \) is an isomorphism of \( \mathcal{O}_{U_{ij}} \)-modules such that:

1. \( \varphi_{ii} = id_{F_i} \) for every \( i \in I \);
2. \( \varphi_{ij} = \varphi_{ji}^{-1} \) for every \( i, j \in I \);
3. \( \varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot id_{F_i|_{U_{ijk}}} \).

If the sheaves \( F_i \) are quasi-coherent (coherent, locally free) \( \mathcal{O}_{U_i} \)-modules for every \( i \in I \), we say that \( F \) is a quasi-coherent (coherent, locally free) \( \alpha \)-sheaf.

**Definition 2.** A morphism between two \( \alpha \)-sheaves \( F = (F_i, \varphi_{ij}) \) and \( G = (G_i, \psi_{ij}) \) is a family \( f = (f_i)_{i \in I} \) where \( f_i : F_i \rightarrow G_i \) is a morphism of \( \mathcal{O}_{U_i} \)-modules such that \( \psi_{ij} \circ f_j = f_i \circ \varphi_{ij} \) for every \( i, j \in I \).

We will write \( \text{Mod}(X, U, \alpha) \) the category whose objects are \( \alpha \)-sheaves over \( X \) and morphisms are morphisms of \( \alpha \)-sheaves. We define also its full subcategories \( \text{QCoh}(X, U, \alpha) \) and \( \text{Coh}(X, U, \alpha) \) in the obvious way.

Actually, we can show that if we change the open covering \( U \) into \( U' \), we have that \( \text{Mod}(X, U, \alpha) \) is canonically equivalent to \( \text{Mod}(X, U', \alpha) \), so that we are allowed to write \( \text{Mod}(X, \alpha) \) instead of \( \text{Mod}(X, U, \alpha) \) (see \[Ca\].
Lemma 1.2.3 and Corollary 1.2.6). Moreover, we can show that if we change the 2-cocycle $\alpha$ to an equivalent one $\beta$, we can find a (non-canonical) equivalence between $\text{Mod}(X, \alpha)$ and $\text{Mod}(X, \beta)$, so that $\text{Mod}(X, \alpha)$ depends only on $\alpha \in Br'X$ (see [Ca], Lemma 1.2.8).

It is easy to show that $\text{Mod}(X, \alpha)$, $\text{QCoh}(X, \alpha)$ and $\text{Coh}(X, \alpha)$ are abelian categories. Moreover, $\text{Mod}(X, \alpha)$ and $\text{QCoh}(X, \alpha)$ have enough injective objects (see [Ca], Lemma 2.1.1).

**Remark 1.** If $F = (F_i, \varphi_{ij})$ is an $\alpha$–sheaf and $x \in U_i$, we write $F_x := F_{i,x}$.

Actually, this does not define a true stalk for the twisted sheaf, since it is determined only up to isomorphism. Anyway, the following definition makes sense:

**Definition 3.** Let $\mathcal{F}$ an $\alpha$–sheaf over $X$. We call support of $\mathcal{F}$ the set

$$\text{Supp} \mathcal{F} := \{x \in X \mid F_x \neq 0\}.$$ 

**Remark 2.** If $\mathcal{F}$ is a coherent $\alpha$–sheaf, then $\text{Supp} \mathcal{F}$ is a closed subset of $X$.

**Remark 3.** Let $x \in X$ be a point. Then the skyscraper sheaf $k(x)$ has a natural structure of $\alpha$–sheaf, for any $\alpha \in Br'X$.

We want also to recall some result on geometrical functors on twisted sheaves. In particular we have:

**Proposition 2.** Let $X, Y$ be two varieties, $\alpha, \alpha' \in Br'X$, $\beta \in Br'Y$ and $f : X \to Y$ a morphism. Then we can define the following functors:

- $\mathcal{H}om(.,.) : \text{Mod}(X, \alpha) \times \text{Mod}(X, \alpha') \to \text{Mod}(X, \alpha'\alpha'^{-1})$
- $\otimes : \text{Mod}(X, \alpha) \times \text{Mod}(X, \alpha') \to \text{Mod}(X, \alpha\alpha')$
- $f^* : \text{Mod}(Y, \beta) \to \text{Mod}(X, f^*\beta)$
- $f_* : \text{Mod}(X, f^*\beta) \to \text{Mod}(Y, \beta)$

**Proof.** See [Ca], Proposition 1.2.10. □

**Proposition 3.** Let $X$ be a variety. Then $\alpha \in BrX$ if and only if there is an $\alpha$–sheaf $\mathcal{E}$ that is locally free of finite rank. In this case, we have that the sheaf $A := \text{End}_{X,\alpha}(\mathcal{E})$ has the structure of Azumaya algebra on $X$, and that there is an equivalence

$$\text{Mod}(X, \alpha) \cong \text{Mod}(A), \quad \mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{E}^\vee,$$

where $\text{Mod}(A)$ is the category of $\mathcal{O}_X$–modules which have the structure of right $A$–module.

**Proof.** See [Ca], Theorem 1.3.5. and Theorem 1.3.7. □
If we pass to derived categories and functors, we can show the following proposition:

**Proposition 4.** Let $X, Y$ be two varieties, $\alpha, \alpha' \in BrX$, $\beta \in BrY$ and $f : X \to Y$ a morphism. Then we can define the following functors:

\[
\otimes^L : D^-(X, \alpha) \times D^-(X, \alpha') \to D^-(X, \alpha\alpha')
\]

\[
R\text{Hom}(.,.) : D^b(X, \alpha)^\circ \times D^+(X, \alpha') \to D^b(X, \alpha'\alpha^{-1})
\]

\[
Lf^* : D^-(Y, \beta) \to D^-(X, f^*\beta)
\]

\[
Rf_* : D(QCoh(X, f^*\beta)) \to D(QCoh(Y, \beta)).
\]

If $f$ is a proper morphism, we have

\[
Rf_* : D(X, f^*\beta) \to D(Y, \beta).
\]

Moreover, if $X$ and $Y$ are smooth of finite dimension and $f$ is a proper morphism, we have

\[
\otimes^L : D^b(X, \alpha) \times D^b(X, \alpha') \to D^b(X, \alpha\alpha')
\]

\[
R\text{Hom}(.,.) : D^b(X, \alpha)^\circ \times D^b(X, \alpha') \to D^b(X, \alpha'\alpha^{-1})
\]

\[
Lf^* : D^b(Y, \beta) \to D^b(X, f^*\beta)
\]

\[
Rf_* : D^b(X, f^*\beta) \to D^b(Y, \beta).
\]

**Proof.** See [Ca], Theorem 2.2.4 and Theorem 2.2.6.

For relations among these derived functors, see [Ca], Section 2.3.

## 2 Reconstruction of a variety

In this section we will show that any variety $X$ can be recovered from the category $Coh(X, \alpha)$ of $\alpha$-sheaves, where $\alpha \in BrX$. The idea is that we can give a ringed space structure to the set $E_{X,\alpha}$ of irreducible Serre subcategories of finite type of $Coh(X, \alpha)$, and that it is, in fact, a scheme isomorphic to $X$.

In order to do so, we will introduce the notion of Serre subcategory of an abelian category. Using the fact that skyscraper sheaves of points of $X$ are twisted, we will show that (irreducible) Serre subcategories of finite type of $Coh(X, \alpha)$ are in bijective correspondence with the (irreducible) closed subsets of $X$. This allows us to put a topology on $E_{X,\alpha}$, which recovers the topology of $X$. The problem will be to give a good definition of a structure sheaf on $E_{X,\alpha}$, such that we can get an isomorphism between $E_{X,\alpha}$ and $X$. 
2.1 Serre subcategories of an abelian category

**Definition 4.** Let $\mathcal{A}$ be an abelian category. A subcategory $\mathcal{I}$ of $\mathcal{A}$ is called a **Serre subcategory** if for every short exact sequence in $\mathcal{A}$

$$0 \to A \to B \to C \to 0$$

we have $B \in \mathcal{I}$ if and only if $A, C \in \mathcal{I}$.

We say that $\mathcal{I}$ is a **Serre subcategory of finite type** if it is a Serre subcategory of $\mathcal{A}$ generated by an element $A \in \mathcal{I}$, that is, $\mathcal{I}$ is the smallest Serre subcategory of $\mathcal{A}$ that contains $A$. Such an $A$ will be called a **generator** for $\mathcal{I}$.

We say that $\mathcal{I}$ is an **irreducible Serre subcategory** if it is not generated (as Serre subcategory) by two proper Serre subcategories.

**Example 1.** Let $\text{Coh}_Z(X, \alpha)$ be the full subcategory of $\text{Coh}(X, \alpha)$ whose objects have support contained in the closed set $Z$ of $X$. Then it is easy to show that it is a Serre subcategory of $\text{Coh}(X, \alpha)$.

**Definition 5.** If $\mathcal{I}$ is a subcategory of $\mathcal{A}$, the **quotient category** $\mathcal{A}/\mathcal{I}$ is the category which has the same objects as $\mathcal{A}$ and morphisms are defined in this way: if $A, B \in \mathcal{A}$ we have

$$\text{Hom}_{\mathcal{A}/\mathcal{I}}(A, B) = \lim_{\to} \text{Hom}_{\mathcal{A}}(A', B')$$

where $i : A' \hookrightarrow A$ is a sub-object of $A$ such that $\text{coker}(i) \in \mathcal{I}$ and $p : B \to B'$ is a quotient of $B$ such that $\text{ker}(p) \in \mathcal{I}$.

If $\mathcal{I}$ is a Serre subcategory of $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is an abelian category. We have the following lemmas, very easy to show:

**Lemma 5.** Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ an exact functor that admits a fully faithful right adjoint. Then $\ker F$ is a Serre subcategory of $\mathcal{A}$ and the induced functor $\mathcal{A}/\ker F \to \mathcal{B}$ is an equivalence.

**Lemma 6.** Let $\mathcal{A}$ be an abelian category, $\mathcal{A}'$ a full abelian subcategory of $\mathcal{A}$ and $\mathcal{I}$ a Serre subcategory of $\mathcal{A}$. Suppose that for every $M \in \mathcal{A}'$, $N \in \mathcal{I}$ a sub-object or a quotient of $M$, we have that $N \in \mathcal{I} \cap \mathcal{A}'$. Then the induced functor

$$i : \mathcal{A}'/\mathcal{I} \cap \mathcal{A}' \to \mathcal{A}/\mathcal{I}$$

is fully faithful.

Now, let $X$ be a variety over a field $k$, $\alpha \in \text{Br}X$ and $Z$ a closed subset of $X$. Let $U = X \setminus Z$ and $j_U : U \to X$ the corresponding open immersion.

We have

$$j_U^* : \text{QCoh}(X, \alpha) \to \text{QCoh}(U, \alpha|_U)$$

that is an exact functor with a fully faithful right adjoint $j_{U*}$. Using Lemma 5 we find that

$$j_U^* : \text{QCoh}(X, \alpha)/\text{QCoh}_Z(X, \alpha) \to \text{QCoh}(U, \alpha|_U)$$

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is an equivalence and, using Lemma 6, that

\[ j^*_U : \text{Coh}(X, \alpha)/\text{Coh}_Z(X, \alpha) \to \text{Coh}(U, \alpha|_U) \]

is fully faithful.

Remark 4. If \( \alpha = 1 \), so that \( \text{Coh}(X, \alpha) \approx \text{Coh}(X) \), we can actually show that the functor \( j^*_U \) above is even an equivalence. Indeed, we know that every coherent sheaf over an open subscheme \( U \) of \( X \) is restriction to \( U \) of a coherent sheaf on \( X \). This is not clear in the case of twisted sheaves: we will discuss this problem in Section 3.

### 2.2 Closed subsets and Serre subcategories

Let \( X \) be a variety. In this section, we will show that for every \( \alpha \in BrX \), there is a bijective correspondence between closed subsets of \( X \) and Serre subcategories of finite type of \( \text{Coh}(X, \alpha) \). The main point here is the following:

**Proposition 7.** Let \( X \) be a variety over a field \( k \), \( \alpha \in BrX \) and \( Z \) a closed subset of \( X \). Then \( \text{Coh}_Z(X, \alpha) \) is a Serre subcategory of finite type of \( \text{Coh}(X, \alpha) \). More precisely, it is generated, as Serre subcategory, by any \( \alpha \)-sheaf \( \mathcal{F} \) such that \( \text{Supp} \mathcal{F} = Z \).

**Proof.** First, we have to show that such an \( \alpha \)-sheaf exists: since \( \alpha \in BrX \), there is a locally free \( \alpha \)-sheaf \( E \) of finite rank over \( X \) (see Proposition 3) so that its restriction to \( Z \) (thought as an \( \alpha \)-sheaf over \( X \)) has support equal to \( Z \). Now, choose \( \mathcal{F} \in \text{Coh}(X, \alpha) \) such that \( \text{Supp} \mathcal{F} = Z \), and write \( \langle \mathcal{F} \rangle \) for the Serre subcategory of \( \text{Coh}(X, \alpha) \) generated by \( \mathcal{F} \). It is clear that \( \langle \mathcal{F} \rangle \subseteq \text{Coh}_Z(X, \alpha) \), so that it remains to show the opposite inclusion.

First, we reduce to \( Z = X \): if we note \( i : Z \to X \) the closed immersion of \( Z \), let \( J \) be the Serre subcategory of \( \text{Coh}(Z, i^* \alpha) \) generated by \( i^* \mathcal{F} \). It is easy to show that \( i_* i^* \mathcal{F} \in \langle \mathcal{F} \rangle \), so that \( i_* J \subseteq \langle \mathcal{F} \rangle \). If we have that \( J = \text{Coh}(Z, i^* \alpha) \), we get \( \text{Coh}_Z(X, \alpha) = i_* J \subseteq \langle \mathcal{F} \rangle \). Using the same kind of arguments, we can even suppose \( X \) irreducible.

We now proceed by induction over the dimension of \( X \). The case of dimension 0 is clear (here, every twist is trivial). Now, let us suppose that \( \dim X = n \) and that the proposition is true for all schemes of dimension smaller or equal to \( n - 1 \). Let \( Y \) be a proper closed subscheme of \( X \). By induction we have that \( \text{Coh}_Y(X, \alpha) \subseteq \langle \mathcal{F} \rangle \).

Now let \( \mathcal{G} \in \text{Coh}(X, \alpha) \), \( j : U \to X \) an open affine subscheme of \( X \), and let \( Y = X \setminus U \). If \( U \) is little enough, we can even suppose \( j^* \mathcal{G} \) and \( j^* \mathcal{F} \) free of ranks \( r \) and \( s \) respectively (this is possible, since \( \alpha|_U \in BrU \)). In this way we have an isomorphism

\[ \tilde{f} : j^* \mathcal{G} \xrightarrow{\sim} j^* \mathcal{F} \]
in the category $\text{Coh}(U, \alpha|_U)$. By Lemma 6, as we saw, the functor
\[ j^*_U : \text{Coh}(X, \alpha)/\text{Coh}_Y(X, \alpha) \to \text{Coh}(U, \alpha|_U) \]
is fully faithful, so that the isomorphism $\tilde{f}$ comes from an isomorphism $f : \mathcal{G} \to \mathcal{F}$ in $\text{Coh}(X, \alpha)/\text{Coh}_Y(X, \alpha)$, that is $\ker f$ and $\text{coker } f$ are in $\text{Coh}_Y(X, \alpha) \subseteq \langle \mathcal{F} \rangle$. But $\langle \mathcal{F} \rangle$ is a Serre subcategory, so $\mathcal{G} \in \langle \mathcal{F} \rangle$.

It is now easy to show the following:

**Corollary 8.** Let $X$ be a variety over a field $k$ and $\alpha \in \text{Br} X$. There is a bijective correspondence between the set $C$ of closed subsets of $X$ and the set $S$ of Serre subcategories of finite type of $\text{Coh}(X, \alpha)$. In particular, this induces a bijective correspondence between the points of $X$ and the set $E_{X, \alpha}$ of irreducible Serre subcategories of finite type of $\text{Coh}(X, \alpha)$.

**Proof.** We can define
\[ i : C \to S, \quad Z \mapsto \text{Coh}_Z(X, \alpha), \]
and
\[ j : S \to C, \quad \mathcal{I} = \langle \mathcal{F} \rangle \mapsto \text{Supp } \mathcal{F}. \]
Now, $j$ is well defined: two generators of the same Serre subcategory have the same support (this follows from definition of generator and Remark 3). In view of Proposition 7 it is straightforward to show that $i = j^{-1}$. Moreover, it is easy to show that $Z$ is irreducible if and only if $\text{Coh}_Z(X, \alpha)$ is irreducible as Serre subcategory of $\text{Coh}(X, \alpha)$. This gives the bijective correspondence between the points of $X$ (which are the generic points of irreducible closed sets of $X$) and the elements of $E_{X, \alpha}$.

**2.3 The reconstruction of a variety from $\text{Coh}(X, \alpha)$**

We are now able to describe how one can recover the variety $X$ from $\text{Coh}(X, \alpha)$. As we saw in Corollary 8, we can think the points of $X$ as the irreducible Serre subcategories of finite type of $\text{Coh}(X, \alpha)$.

Let $E = E_{X, \alpha}$ be the set of irreducible Serre subcategories of finite type of $\text{Coh}(X, \alpha)$. On $E$ we can define the following topology: let $\mathcal{J}$ be a Serre subcategory of finite type of $\text{Coh}(X, \alpha)$, and write
\[ D(\mathcal{J}) := \{ \mathcal{I} \in E | \mathcal{I} \nsubseteq \mathcal{J} \}. \]
It is easy to verify that this family of subsets forms a topology over $E$ and that the following morphism:
\[ f := f_{X, \alpha} : E \to X, \quad f(\mathcal{I} = \text{Coh}_{\{x\}}(X, \alpha)) = x \]
is a homeomorphism (use Proposition 7 and Corollary 8). More precisely, if $Z$ is a closed subset of $X$, $U = X \setminus Z$ and $\mathcal{J} = \text{Coh}_Z(X, \alpha)$, then $f$ gives a bijective correspondence between $D(\mathcal{J})$ and $U$. In this way, we have shown that we can recover the topological space underlying $X$ from $\text{Coh}(X, \alpha)$.

It remains to define a structure sheaf on $E$, in order to make $f$ an isomorphism of schemes. Let us recall the notion of center of a category.

**Definition 6.** Let $\mathcal{A}$ be a category. We call *center of* $\mathcal{A}$ the ring $Z(\mathcal{A})$ of endomorphisms of the identity functor of $\mathcal{A}$: $Z(\mathcal{A}) := \text{End}_\mathcal{A}(\text{id}_\mathcal{A})$.

It is very easy to show the following lemma:

**Lemma 9.** Let $A$ be a ring with unity, $Z(A)$ his center, and $\text{Mod}_{ft}(A)$ the category of modules of finite type over $A$. The canonical morphism

$$Z(A) \longrightarrow Z(\text{Mod}_{ft}(A)), \quad a \mapsto a$$

is an isomorphism of commutative rings.

Using the center of a category and the notations we used above, we can define the following sheaf on $E$:

$$\mathcal{O}_E(D(\mathcal{J})) = Z(\text{Coh}_U(\alpha|_U)),$$

and the morphism of sheaves $f^\sharp : \mathcal{O}_X \longrightarrow f_* \mathcal{O}_E$ which is given over every open set $U$ of $X$ by

$$f^\sharp(U) : \mathcal{O}_X(U) \longrightarrow Z(\text{Coh}_U(\alpha|_U)), \quad s \mapsto s.$$

In this way we have given to $E$ the structure of ringed space. We have now the following theorem, which shows the first part of Theorem 1:

**Theorem 10.** The morphism $(f, f^\sharp) : E \longrightarrow X$ is an isomorphism of ringed spaces over $k$. In particular, $E$ is a $k$–variety which depends only on $\text{Coh}(X, \alpha)$.

**Proof.** We only need to show that $f^\sharp$ is an isomorphism of rings. It suffices to show that it is an isomorphism on open affine subschemes of $X$. So, let’s take $U = \text{Spec} A$ an open affine subscheme of $X$.

As we want to show that $f^\sharp(U)$ is an isomorphism, we begin by studying the ring $Z(\text{Coh}_U(\alpha|_U))$. Since $\alpha|_U \in \text{Br} U$, from Proposition 8 we know that there is a locally free $\alpha$–sheaf $\mathcal{E}$ of rank $r$, and that there is an equivalence of categories

$$\text{Mod}(U, \alpha|_U) \xrightarrow{\sim} \text{Mod}(\mathcal{E}nd_{U, \alpha|_U}(\mathcal{E})), \quad \mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{E}^\vee.$$

If we note $\mathcal{A} = \mathcal{E}nd_{U, \alpha|_U}(\mathcal{E})$, this is an Azumaya algebra on $X$ (so that, in particular, it is an $\mathcal{O}_X$–module). Looking at this equivalence, we can
see that it sends any (quasi) coherent $\alpha_U$–sheaf to a (quasi) coherent sheaf which has the structure of right $A$–module, so that we get an equivalence between $\text{Coh}(U, \alpha_U)$ and the full subcategory $\mathcal{C}$ of $\text{Mod}(A)$ whose objects are coherent sheaves with the structure of right $A$–module. Since we are on an affine scheme, taking global sections we get an equivalence between $\mathcal{C}$ and $\text{Mod}_{f^*}(\text{End}_{U, \alpha_U}(\mathcal{E}))$, so that we finally have the isomorphisms

$$Z(\text{Coh}(U, \alpha_U)) \simeq Z(\text{Mod}_{f^*}(\text{End}_{U, \alpha_U}(\mathcal{E}))) \simeq Z(\text{End}_{U, \alpha_U}(\mathcal{E}))$$

where the second isomorphism follows from Lemma \[3\].

We are now reduced to study the center of the ring of endomorphisms of $\mathcal{E}$ as an $\alpha_U$–sheaf. Let’s see what this ring looks like. As $U = \text{Spec } A$, we can find $f_1, ..., f_n \in A$ such that $U = \bigcup_{i=1}^n D(f_i)$. We represent $\mathcal{E}$ over this open covering, so that $\mathcal{E} = (\mathcal{E}_i, \varphi_{ij})$, where $\mathcal{E}_i$ is an locally free $A_{f_i}$–module of rank $r$ for every $i$, and

$$\varphi_{ij} : \mathcal{E}_{ij}[D(f_i)] \xrightarrow{\sim} \mathcal{E}_{ij}[D(f_j)]$$

is an isomorphism of $A_{f_i,f_j}$–modules for every $i,j$, verifying the conditions of Definition \[4\]. We can even take $D(f_i)$ small enough such that, for every $i$, $\mathcal{E}_i \simeq (A_{f_i})^r$. With this choice, we see that $\varphi_{ij}$ can be thought as an isomorphism of $(A_{f_i,f_j})^r$, that is a matrix $B_{ij} \in GL_r(A_{f_i,f_j})$.

Now, to give an endomorphism $M$ of $\mathcal{E}$ is to give a family $(M_1, ..., M_n)$, where $M_i \in \text{End}_{A_{f_i}}((A_{f_i})^r) = M_r(A_{f_i})$ is a square matrix of rank $r$ with elements in $A_{f_i}$, such that for every $i,j$ we have $B_{ij}M_i = M_jB_{ij}$ in the ring $M_r(A_{f_i,f_j})$. In conclusion, we have

$$\text{End}_{U, \alpha_U}(\mathcal{E}) = \{(M_1, ..., M_n) \mid M_i \in M_r(A_{f_i}), B_{ij}M_i = M_jB_{ij} \in M_r(A_{f_i,f_j})\}$$

where sum and multiplication are the obvious ones. It is easy to see that the morphism $f^*(U)$ is now given by

$$A \longrightarrow \text{End}_{U, \alpha_U}(\mathcal{E}), \quad a \mapsto (\text{diag}(a), ..., \text{diag}(a)).$$

It is quite clear that this map is injective: if $a, b \in A$ have the same image, this means that $a = b$ in $A_{f_i}$ for every $i$. Since $a, b$ are global section of the sheaf associated to $A$, this means that $a = b$ in $A$.

It remains to show the surjectivity. Let $M = (M_1, ..., M_n)$ be an element in $Z(\text{End}_{U, \alpha_U}(\mathcal{E}))$. This means that for every $(N_1, ..., N_n) \in \text{End}_{U, \alpha_U}(\mathcal{E})$ we have

$$(M_1N_1, ..., M_nN_n) = (N_1M_1, ..., N_nM_n),$$

that is, for every $i$, $M_i \in Z(M_r(A_{f_i})) \simeq A_{f_i}$, so that there is $b_i \in A_{f_i}$ such that $M_i = \text{diag}(b_i)$. In particular, $M_i \in Z(M_r(A_{f_i,f_j}))$ for every $i,j$. The condition $B_{ij}M_i = M_jB_{ij}$ in the ring $M_r(A_{f_i,f_j})$ gives $\text{diag}(b_i - b_j)B_{ij} = 0$. Since $B_{ij}$ is invertible, this tells us that $b_i = b_j$ in $A_{f_i,f_j}$ for every $i,j$. Since $b_i$ is a section on $D(f_i)$ of the sheaf associated to $A$, we get an element $a \in A$ such that $b_i = a$ in $A_{f_i}$ for every $i$. This tells us that $(M_1, ..., M_n) = (\text{diag}(a), ..., \text{diag}(a))$ for a (unique) $a \in A$. \[Q.E.D.\]
3 Isomorphism induced by an equivalence

We have shown a generalization of the first part of Gabriel’s theorem to twisted coherent sheaves, as we wanted at the beginning, so, the next question is if any equivalence between $\text{Coh}(X, \alpha)$ and $\text{Coh}(Y, \beta)$ gives rise to an isomorphism between $X$ and $Y$.

First of all, we can show that this problem can be reduced to the following one: let $X$ be a variety, $\alpha \in \text{Br}_X$, $Z$ a closed subset, $U = X \setminus Z$ and $j_U : U \to X$ the open immersion. Is the functor

$$j_U^*: \text{Coh}(X, \alpha)/\text{Coh}_Z(X, \alpha) \to \text{Coh}(U, \alpha|_U)$$

an equivalence?

If we have a positive answer for every open subscheme $U$ we’ll say that $(X, \alpha)$ satisfies the restriction condition.

So, let $(Y, \beta)$ be another variety which satisfies the restriction condition, and

$$F: \text{Coh}(X, \alpha) \sim \to \text{Coh}(Y, \beta)$$

an equivalence. It is trivial to show that if $J$ is a(n irreducible) Serre subcategory of finite type of $\text{Coh}(X, \alpha)$, then $F(J)$ is a(n irreducible) Serre subcategory of finite type of $\text{Coh}(Y, \beta)$. This gives a bijective correspondence

$$f_F: X \to Y, \quad f_F(x) = f_{Y, \beta}(F(f_{X, \alpha}^{-1}(x))),$$

where $f_{X, \alpha}$ (resp. $f_{Y, \beta}$) is the isomorphism between $E_{X, \alpha}$ and $X$ (resp. between $E_{Y, \beta}$ and $Y$) we defined in the previous section. It is also easy to show that $f_F$ is an homeomorphism and even an isomorphism of schemes: $U$ is an open subscheme of $X$, we have that $f_F$ induces a bijective correspondence between $U$ and $W := f_{Y, \beta}(D(F(f_{X, \alpha}^{-1}(U))))$. Moreover, since $F$ is an equivalence, we have that

$$F: \text{Coh}(X, \alpha)/\text{Coh}_X(U)(X, \alpha) \sim \to \text{Coh}(Y, \beta)/\text{Coh}_Y(W)(Y, \beta)$$

is an equivalence (this follows easily from Lemma 5). Using this and the fact that $(X, \alpha)$ and $(Y, \beta)$ verify the restriction condition, it’s trivial to show that

$$j_W^* \circ F \circ (j_U^*)^{-1}: Z(\text{Coh}(U, \alpha|_U)) \sim \to Z(\text{Coh}(W, \beta|_W))$$

is an isomorphism, that is, we get an isomorphism $g: \mathcal{O}_{E_{X, \alpha}} \sim \to \mathcal{O}_{E_{Y, \beta}}$. Now, from Theorem 10 it’s obvious that

$$f_F^*: \mathcal{O}_{E_{X, \alpha}} \circ g^{-1} \circ f_{Y, \beta}^{-1}: \mathcal{O}_Y \sim \to f_F^* \mathcal{O}_X$$

is an isomorphism. In conclusion, we have shown the following:
Theorem 11. Let $X,Y$ be two varieties over a field $k$, $\alpha \in BrX$ and $\beta \in BrY$ which verify the restriction condition above. Then any equivalence $F : Coh(X,\alpha) \sim \rightarrow Coh(Y,\beta)$ induces an isomorphism of varieties $f : X \sim \rightarrow Y$.

We are now reduced to study when a couple $(X,\alpha)$ verifies the restriction condition. Actually, we can show it only when $X$ is a smooth variety. To approach the problem, we get into the domain of derived category, where we can use perfect and compact objects in $D(QCoh(X,\alpha))$.

3.1 Thick subcategories and compact objects of a triangulated category

In this section we introduce the notions of thick subcategory and Bousfield subcategory of a triangulated category. The main references here will be [Ro1] and [Ro2]. Let $\mathcal{T}$ be a triangulated category.

Definition 7. We say that a subcategory $I$ of $\mathcal{T}$ is thick (or épaisse) if it is a triangulated subcategory such that for every $M,N \in \mathcal{T}$, if $M \oplus N \in I$ then $M,N \in I$.

If $I$ is a thick subcategory of $\mathcal{T}$, we have that the quotient category $\mathcal{T}/I$ is again triangulated. It is clear that we have a (n essentially surjective) functor $j^* : \mathcal{T} \rightarrow \mathcal{T}/I$.

Definition 8. A thick subcategory $I$ of $\mathcal{T}$ is called Bousfield subcategory if $j^*$ admits a right adjoint, which will be noted $j_*$. Now, let $I$ be a full triangulated subcategory of $\mathcal{T}$. We define the following subcategories of $\mathcal{T}$:

1. If $\mathcal{T}$ admits infinite direct sums, $\overline{I}$ will be smallest thick subcategory of $\mathcal{T}$ which contains $I$ and which is stable for infinite direct sums;

2. $\langle I \rangle$ is the smallest thick subcategory of $\mathcal{T}$ which contains $I$;

3. $I^\perp$ is the subcategory of objects $C \in \mathcal{T}$ such that for every $D \in \langle I \rangle$ we have $Hom_{\mathcal{T}}(D,C) = 0$;

4. if $I_1,I_2$ are two full subcategories of $\mathcal{T}$ we define $I_1 \ast I_2$ as the subcategory of $\mathcal{T}$ whose objects $M$ are such that there is a distinguished triangle $M_1 \rightarrow M \rightarrow M_2 \sim\rightarrow$ where $M_i \in I_i$, and $I_1 \circ I_2 = \langle I_1 \ast I_2 \rangle$;
5. \langle \mathcal{I} \rangle_0 = 0, and by induction over \( i \) we define \( \langle \mathcal{I} \rangle_i = \langle \mathcal{I} \rangle_{i-1} \circ \langle \mathcal{I} \rangle \) and \( \langle \mathcal{I} \rangle_\infty = \bigcup_{i \geq 0} \langle \mathcal{I} \rangle_i \). In particular, if \( \mathcal{T} \) admits infinite direct sums, we have \( \langle \mathcal{I} \rangle_\infty = \overline{\langle \mathcal{I} \rangle} \).

**Lemma 12.** If \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) are two Bousfield subcategories of \( \mathcal{T} \), then \( \mathcal{I}_1 \cap \mathcal{I}_2 \) and \( \langle \mathcal{I}_1 \cup \mathcal{I}_2 \rangle_\infty \) are Bousfield subcategories of \( \mathcal{T} \).

**Proof.** See [Ro2], Lemma 5.8. \( \square \)

**Definition 9.** An object \( C \in \mathcal{T} \) is said to be **compact** in \( \mathcal{T} \) if for every family \( \mathcal{E} \) of objects of \( \mathcal{T} \), the canonical morphism

\[
\bigoplus_{E \in \mathcal{E}} \text{Hom}(C, E) \rightarrow \text{Hom}(C, \bigoplus_{E \in \mathcal{E}} E)
\]

is an isomorphism. We will note \( \mathcal{T}_c \) the full subcategory of \( \mathcal{T} \) of compact objects.

We can now recall some result which will be used later.

**Lemma 13.** Let \( \mathcal{T}, \mathcal{T}' \) be two triangulated categories and \( F : \mathcal{T} \rightarrow \mathcal{T}' \) an exact functor which admits a fully faithful right adjoint. Then \( \ker F \) is a thick subcategory of \( \mathcal{T} \) and the induced functor \( F : \mathcal{T}/\ker F \rightarrow \mathcal{T}' \) is an equivalence.

**Lemma 14.** Let \( \mathcal{T} \) be a triangulated category, \( \mathcal{I} \) a thick subcategory of \( \mathcal{T} \) and \( \mathcal{T}' \) a full triangulated subcategory of \( \mathcal{T} \). If for every \( C \in \mathcal{T}' \), \( D \in \mathcal{I} \), every morphism from \( C \) to \( D \) factorizes by an object of \( \mathcal{I} \cap \mathcal{T}' \), we have that the induced functor

\[
i : \mathcal{T}'/\mathcal{I} \cap \mathcal{T}' \rightarrow \mathcal{T}/\mathcal{I}
\]

is fully faithful.

**Lemma 15.** Let \( \mathcal{T} \) be a triangulated category that admits infinite direct sums and \( \mathcal{I} \) a thick subcategory of \( \mathcal{T}_c \). Then any morphism from a compact object of \( \mathcal{T} \) to an object of \( \mathcal{I} \) factorizes by an element of \( \mathcal{I} \). In particular \( \mathcal{T}_c \cap \mathcal{I} = \mathcal{I} \).

Moreover, \( \mathcal{I} = \mathcal{I} \) if and only if \( \mathcal{I}^\perp = 0 \).

**Proof.** This is proved in [BN]. \( \square \)

**Lemma 16.** Let \( \mathcal{I}_1, \mathcal{I}_2 \) be two Bousfield subcategories of \( \mathcal{T} \) with \( \mathcal{I}_1 \cap \mathcal{I}_2 = 0 \). Then for every object \( D \in \mathcal{T} \) there is a distinguished triangle:

\[
D \rightarrow j_1^* j_1^! D \oplus j_2^* j_2^! D \rightarrow j_\cup^* j_\cup^! D \rightarrow D[1]
\]

which is called Mayer-Vietoris triangle for \( D \), where \( j_i^* : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{I}_i \) and \( j_\cup^* : \mathcal{T} \rightarrow \mathcal{T}/(\mathcal{I}_1 \cup \mathcal{I}_2)_{\infty} \) are the projection functors.

**Proof.** See [Ro2], Proposition 5.10. \( \square \)
3.2 Perfect objects in $\mathcal{D}(QCoh(X, \alpha))$

Let $X$ be a variety over a field $k$ and $\alpha \in BrX$. If $Z$ is a closed subset of $X$, it is easy to see that $\mathcal{D}_Z(QCoh(X, \alpha))$ is a thick subcategory of $\mathcal{D}(QCoh(X, \alpha))$. Moreover, using Lemma 13 we can see that if $U = X \setminus Z$ and $j_U$ is the open immersion of $U$ in $X$, then

$$j_U^* : \mathcal{D}(QCoh(X, \alpha)) / \mathcal{D}_Z(QCoh(X, \alpha)) \xrightarrow{\sim} \mathcal{D}(QCoh(U, \alpha|_U))$$

is an equivalence, since we have the functor $Rj_U^*$ which is fully faithful and is right adjoint to $j_U^*$. In this way we have also shown that $\mathcal{D}_Z(QCoh(X, \alpha))$ is a Bousfield subcategory of $\mathcal{D}(QCoh(X, \alpha))$.

Now, let $Z_1, Z_2$ be two closed subsets of $X$ such that $Z_1 \cap Z_2 = \emptyset$, and let $U_i = X \setminus Z_i$, $U_{12} = U_1 \cap U_2$. It’s clear that

$$\mathcal{D}_{Z_1}(QCoh(X, \alpha)) \cap \mathcal{D}_{Z_2}(QCoh(X, \alpha)) = 0,$$

and it’s easy to show that

$$(\mathcal{D}_{Z_1}(QCoh(X, \alpha)) \cup \mathcal{D}_{Z_2}(QCoh(X, \alpha))) = \mathcal{D}_{Z_1 \cup Z_2}(QCoh(X, \alpha)).$$

Using Lemma 16 if $D \in \mathcal{D}(QCoh(X, \alpha))$, there is a distinguished triangle

$$D \rightarrow Rj_1^*j_1^*D \oplus Rj_2^*j_2^*D \rightarrow Rj_{12}^*j_{12}^*D \rightarrow D[1] \quad (1)$$

where $j_i : U_i \rightarrow X$ and $j_{12} : U_{12} \rightarrow X$.

We give the following definition:

**Definition 10.** An object $C \in \mathcal{D}(QCoh(X, \alpha))$ is called **perfect** if it is locally quasi-isomorphic to a bounded complex of locally free $\alpha$-sheaves of finite rank. We will denote $\text{Perf}(X, \alpha)$ the subcategory of perfect objects in $\mathcal{D}(QCoh(X, \alpha))$.

$\text{Perf}(X, \alpha)$ is a (non empty) thick subcategory of $\mathcal{D}^b(X, \alpha)$. Moreover, if $X$ is a smooth variety, we have $\text{Perf}(X, \alpha) = \mathcal{D}^b(X, \alpha)$ (see [Ca], Lemma 2.1.4 and Proposition 2.1.8).

We have the following theorem, which will be basic for what will follow.

**Theorem 17.** Let $X$ be a variety over a field $k$, $\alpha \in BrX$. Then we have $\text{Perf}(X, \alpha) = \mathcal{D}(QCoh(X, \alpha))^c$.

**Proof.** The proof will proceed by induction on the minimal number of open affine subschemes which cover $X$. We start with $X = \text{Spec} \ A$. Since we know that $\alpha \in BrX$, there is a locally free $\alpha$-sheaf $\mathcal{E}$ of finite rank over $X$.

First of all, $\mathcal{E}$ is compact: let $\{\mathcal{F}_i\}_{i \in I}$ a family of complexes of quasi-coherent $\alpha$-sheaves, and let $\mathcal{E}^\vee = R\text{Hom}(\mathcal{E}, \mathcal{O}_X)$ be the dual complex of $\mathcal{E}$. We have

$$\bigoplus_{i \in I} \text{Hom}_{X, \alpha}(\mathcal{E}, \mathcal{F}_i) = \bigoplus_{i \in I} \text{Hom}_X(\mathcal{O}_X, \mathcal{E}^\vee \otimes \mathcal{F}_i),$$
and since $\mathcal{O}_X$ is compact in $D(QCoh(X))$ (this is trivial), we have

$$\bigoplus_{i \in I} \text{Hom}_X(\mathcal{O}_X, \mathcal{E}^\vee \otimes \mathcal{I}_i) = \text{Hom}_X(\mathcal{O}_X, \bigoplus_{i \in I} \mathcal{E}^\vee \otimes \mathcal{I}_i) = \text{Hom}_X(\mathcal{E}^\vee, \bigoplus_{i \in I} \mathcal{I}_i),$$

so that $\text{Perf}(X, \alpha) \subseteq D(QCoh(X, \alpha))^c$. We have even that $\text{Perf}(X, \alpha)$ is a thick subcategory of $D(QCoh(X, \alpha))^c$.

Now we can show that $\text{Perf}(X, \alpha) = D(QCoh(X, \alpha))$: using Lemma 15, it suffices to show that $\text{Perf}(X, \alpha) \perp = 0$. So, let $C \in \text{Perf}(X, \alpha) \perp$. Since $\mathcal{E}$ is perfect, we have that

$$0 = R\text{Hom}_{X, \alpha}(\mathcal{E}, C) = R\text{Hom}_{X}(\mathcal{O}_X, \mathcal{E}^\vee \otimes C)$$

that is $\mathcal{H}^i(\mathcal{E}^\vee \otimes C) = 0$ for every $i$. This implies clearly $\mathcal{E}^\vee \otimes C = 0$, that is $C = 0$. Using again Lemma 15, we have $\text{Perf}(X, \alpha) = D(QCoh(X, \alpha))^c$.

Now suppose that $X = U_1 \cup U_2$, where $U_1$ is affine and $U_2$ verifies the theorem. Let $C, D \in D(QCoh(X, \alpha))$. Using the Mayer-Vietoris triangle (11), it is easy to show that $C$ is compact if and only if $j_1^*C$, $j_2^*C$ and $j_{12}^*C$ are.

Since $U_1$, $U_2$ and $U_{12}$ verify the theorem by induction, we have that $j_1^*C$, $j_2^*C$ and $j_{12}^*C$ are compact if and only if they are perfect, that is, there are $E_i \in D(QCoh(U_i, \alpha|_{U_i}))$ and $E_{12} \in D(QCoh(U_{12}, \alpha|_{U_{12}}))$ bounded complexes of locally free twisted sheaves of finite rank such that $j_i^*C$ is locally quasi-isomorphic to $E_i$ and $j_{12}^*C$ is locally quasi-isomorphic to $E_{12}$. This means that $E_1|_{U_{12}} \simeq E_{12} \simeq E_2|_{U_{12}}$, that is, we can glue $E_1$ and $E_2$ over $E_{12}$, obtaining a locally free $\alpha$–sheaf of finite rank on $X$, locally quasi-isomorphic to $C$. \hfill \Box

Let us denote $\text{Perf}_Z(X, \alpha) = \text{Perf}(X, \alpha) \cap D_Z(QCoh(X, \alpha))$.

**Definition 11.** We define the group $K_0(\mathcal{T})$ of a triangulated category $\mathcal{T}$ as the quotient of the free abelian group generated by the objects in $\mathcal{T}$ by the relation $[N] = [M] + [L]$ if there is a distinguished triangle

$$M \rightarrow N \rightarrow L \rightarrow M[1].$$

In particular, we will denote $K_0(X, \alpha) = K_0(\text{Perf}(X, \alpha))$ and, if $Z$ is a closed subset of $X$, $K_{0,Z}(X, \alpha) = K_0(\text{Perf}_Z(X, \alpha))$.

We have the following lemma, due to Thomason.

**Lemma 18.** Let $\mathcal{I}$ be a triangulated category. The correspondence which sends a full triangulated subcategory $\mathcal{J}$ of $\mathcal{I}$ such that $\mathcal{J} = \mathcal{I}$ to the image of $K_0(\mathcal{J})$ in $K_0(\mathcal{T})$ of the group morphism induced by the inclusion $i : \mathcal{J} \rightarrow \mathcal{I}$ is bijective.

Thanks to this lemma, and the others we stated in section 3.1, we can show the following:
Theorem 19. Let $X$ be a variety over a field $k$, $\alpha \in BrX$. Let $Y, Z$ two closed subsets of $X$, $U = X \setminus Z$ and $j_U$ the corresponding open immersion. Then the functor

$$j_U^*: \text{Perf}_Y(X, \alpha)/\text{Perf}_{Z\cap Y}(X, \alpha) \to \text{Perf}_{U\cap Y}(U, \alpha|_U)$$

is fully faithful. Moreover, an object $F \in \text{Perf}_{U\cap Y}(X, \alpha|_U)$ is restriction to $U$ of an object in $\text{Perf}_Y(X, \alpha)$ if and only if $[F] \in K_{0,U\cap Y}(U, \alpha|_U)$ is restriction of a class in $K_{0,Y}(X, \alpha)$.

Proof. Suppose that for every variety $X$, $\alpha \in BrX$ and $Y$ closed subset of $X$, we have that $\text{Perf}_Y(X, \alpha) = D_Y(\text{QCoh}(X, \alpha))$. Using this condition, Lemma 14 and Lemma 15 we can easily see that the functor $j_U^*$ is fully faithful. Now, let $I$ be the essential image of $j_U^*$. This is a full triangulated subcategory of $\text{Perf}_{U\cap Y}(X, \alpha|_U)$. Since we have supposed that $\text{Perf}_Y(X, \alpha) = D_Y(\text{QCoh}(X, \alpha))$, we have $I = D_{U\cap Y}(\text{QCoh}(U, \alpha|_U))$, and from Lemma 15, we have that $I$ generates $\text{Perf}_{U\cap Y}(X, \alpha|_U)$ as thick subcategory. Now the theorem follows from Lemma 18.

We have now to show that $\text{Perf}_Y(X, \alpha) = D_Y(\text{QCoh}(X, \alpha))$ for every variety $X$, $\alpha \in BrX$ and $Y$ a closed subset of $X$. Using Lemma 15, it suffices to show that $\text{Perf}_Y(X, \alpha)^\perp = 0$. We will proceed by induction on the minimal number of open affine subschemes that cover $X$. So, let $C \in \text{Perf}_Y(X, \alpha)^\perp$.

Let $X = \text{Spec} A$, so that $Y$ will be the closed subset corresponding to the ideal of $A$ generated by $r$ elements $f_1, ..., f_r$, or $Y = \emptyset$, that is $r = 0$. Let $E$ a locally free $\alpha$–sheaf on $X$ and

$$G_r = \bigotimes_{i=1}^r (0 \to O_X \otimes E^\vee \xrightarrow{f_i \otimes id} O_X \otimes E^\vee \to 0) \in \text{Perf}_Y(X, \alpha^{-1}).$$

It is easy to show that $C = 0$ if and only if $G_r \otimes C = 0$ (here we have that $G_r \otimes C$ is a sheaf, so we can use the same argument in the proof of Lemme 2.10 in [Ro1]). Now, since $G_r \in \text{Perf}_Y(X, \alpha^{-1})$, we have that $G_r^\vee \in \text{Perf}_Y(X, \alpha)$, so that

$$0 = R\text{Hom}_{X, \alpha}(G_r^\vee, C) = R\text{Hom}_X(O_X, G_r \otimes C)$$

that is $\mathcal{H}^i(G_r \otimes C) = 0$ for every $i$, and so $G_r \otimes C = 0$, which implies $C = 0$.

Now, let $X = U_1 \cup U_2$, where $U_1$ is affine and $U_2$ verifies the theorem. Moreover let $Z_i = X \setminus U_i$. Let $C \in \text{Perf}_Y(X, \alpha)^\perp$, $j_i$ the open immersion of $U_i$ in $X$, $j_{12}$ the open immersion of $U_{12} = U_1 \cap U_2$ in $X$. 

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First we show that the adjunction morphism $\gamma : C \rightarrow Rj_{2*}j_2^*C$ is a quasi-isomorphism. Let $D \in Perf_{Y \cap Z_2}(U_1, \alpha|_{U_1})$. Since $Y \cap Z_2 \subseteq U_1$, the functor $Rj_{1*}$ induces the two following equivalences:

$$D_{Y \cap Z_2}(QCoh(U_1, \alpha|_{U_1})) \xrightarrow{Rj_{1*}} D_{Y \cap Z_2}(QCoh(X, \alpha))$$

and

$$Perf_{Y \cap Z_2}(U_1, \alpha|_{U_1}) \xrightarrow{Rj_{1*}} Perf_{Y \cap Z_2}(X, \alpha).$$

Using this, we have that $Rj_{1*}D \in Perf_{Y \cap Z_2}(X, \alpha) \subseteq Perf_Y(X, \alpha)$, so that $Hom(Rj_{1*}D, C) = 0$. Moreover

$$Hom(Rj_{1*}D, Rj_{2*}j_2^*C) = Hom(j_2^*Rj_{1*}D, j_2^*C) = 0.$$

If $C'$ is the cocone of $\gamma$, applying the functor $Hom(Rj_{1*}D, .)$ to the distinguished triangle

$$C' \rightarrow C \xrightarrow{\gamma} Rj_{2*}j_2^*C \rightarrow C'[1]$$

we get $Hom(Rj_{1*}D, C') = 0$. Now, we know that $C' \in D_{Y \cap Z_2}(QCoh(X, \alpha))$. Using equivalence (2), we get $C'' \in D_{Y \cap Z_2}(QCoh(U_1, \alpha|_{U_1}))$ with the property that $Rj_{1*}C'' \simeq C'$. In this way we have that $Hom(D, C'') = 0$ for every $D \in Perf_{Y \cap Z_2}(U_1, \alpha|_{U_1})$. But $U_1$ is affine, so the first part of the proof says $C'' = 0$, and then $C' = 0$, that is, $\gamma$ is a quasi-isomorphism.

Now let $E' \in Perf_{Y \cap U_2}(U_2, \alpha|_{U_2})$ and $E = E' \oplus E'[1]$, so that $[E] = 0$ in $K_{0,Y \cap U_2}(U_2, \alpha|_{U_2})$. Let $G = E|_{U_12}$. By the affine part of the theorem (which we have already proven), we know that there is $F \in Perf_{Y \cap U_1}(U_1, \alpha|_{U_1})$ such that $F|_{U_12} \simeq G$. Consider $\delta : Rj_{1*}F \oplus Rj_{2*}E \rightarrow Rj_{12*}G$ the adjunction morphism, and let $D$ be the cocone of $\delta$, so that we have the distinguished triangle

$$D \rightarrow Rj_{1*}F \oplus Rj_{2*}E \xrightarrow{\delta} Rj_{12*}G \rightarrow D[1].$$

Applying to it the exact functor $j_1^*$ we find that $j_1^*D \simeq F$, while applying $j_2^*$ we get $j_2^*D \simeq E$. In this way we have shown that $D \in Perf_Y(X, \alpha)$. By the hypothesis on $C$, we have:

$$0 = Hom(D, C) = Hom(D, Rj_{2*}j_2^*C) = Hom(j_2^*D, j_2^*C) = Hom(E, j_2^*C)$$

so that $Hom(E', j_2^*C) = 0$ for every $E' \in Perf_{Y \cap U_2}(U_2, \alpha|_{U_2})$. But $U_2$ verifies the theorem by induction, so this implies $j_2^*C = 0$, that is $C = 0$. \[\square\]

Using this theorem we can show the following:

**Corollary 20.** Let $X$ be a smooth variety over a field $k$, $\alpha \in BrX$, $Z$ a closed subset of $X$, $U = X \setminus Z$. Let $\mathcal{F} \in Coh(U, \alpha|_{U})$. Then, there is $\mathcal{E} \in Coh(X, \alpha)$ such that $\mathcal{E}|_{U} \simeq \mathcal{F}$. In particular, $(X, \alpha)$ verifies the restriction condition.
Proof. Take $C = \mathcal{F} \oplus \mathcal{F}[1]$, so that $[C] = 0$ in $K_0(U, \alpha|_U)$. Since $X$ is smooth, $C \in Perf(U, \alpha|_U)$. By Theorem 19 we have that there is $C' \in Perf(X, \alpha) = D^b(X, \alpha)$ such that $C'|_U$ is quasi-isomorphic to $C$. Now, let $\mathcal{E} = \mathcal{F}^0(C') \in Coh(X, \alpha)$. Since the restriction is an exact functor, we have

$$\mathcal{E}|_U = \mathcal{F}^0(C'|_U) \simeq \mathcal{F}^0(C) = \mathcal{F}.$$

\[\square\]

4 Saturatedness of $D^b(X, \alpha)$

Using the results of the previous section, following [BdB] we can even show that if $X$ is a smooth and proper variety over a field $k$ and $\alpha \in BrX$, then $D^b(X, \alpha)$ is saturated.

In [BdB] is given the following definition

**Definition 12.** Let $\mathcal{T}$ a triangulated category such that for every $A, B$ in $\mathcal{T}$ we have $\sum_{i \in \mathbb{Z}} \dim \text{Hom}(A, B[i]) < \infty$. $\mathcal{T}$ is called saturated if every contravariant cohomological functor of finite type $H : \mathcal{T} \to \text{Vect}(k)$ is representable.

We shall need the following definitions and results. Here we use the same notations as in section 3.1.

**Definition 13.** We say that a family of objects $\mathcal{E} \subseteq \mathcal{T}$ generates (resp. strongly generates) $\mathcal{T}$ if we have $\langle \mathcal{E} \rangle_{\infty} = \mathcal{T}$ (resp. if there is an integer $n$ such that $\langle \mathcal{E} \rangle_n = \mathcal{T}$). Moreover, we say that $\mathcal{T}$ is finitely generated (resp. finitely strongly generated) if we can find a generating family given by just one object, that will be called a generator (resp. strong generator).

**Proposition 21.** Let $\mathcal{T}$ be a triangulated category which admits infinite direct sums, and let $\mathcal{T}^c$ the full triangulated subcategory of compact objects. Then $\mathcal{T}^c$ is Karoubian.

Proof. This is shown in [BN]. \[\square\]

In [BdB] it is show the following theorem:

**Theorem 22.** Let $\mathcal{T}$ be a triangulated category such that for every $A, B \in \mathcal{T}$ we have $\sum_{i \in \mathbb{Z}} \dim \text{Hom}(A, B[i]) < \infty$. If $\mathcal{T}$ is Karoubian and is strongly finitely generated, then $\mathcal{T}$ is saturated.

Proof. See proof of Theorem 1.3 in [BdB]. \[\square\]

We want to use this criterion to show that if $X$ is a smooth proper variety over a field $k$ and $\alpha \in BrX$, then $D^b(X, \alpha)$ is saturated.
The fact that $X$ is proper implies that for any $A, B \in \mathcal{D}^b(X, \alpha)$ we have $\sum_{i \in \mathbb{Z}} \dim \text{Hom}(A, B[i]) < \infty$. Moreover, the fact that $X$ is smooth and that $\alpha \in \text{Br}X$ implies, by Theorem 17, that

$$\mathcal{D}^b(X, \alpha) = \text{Perf}(X, \alpha) = \mathcal{D}^b(Q\text{Coh}(X, \alpha))^c,$$

so that Proposition 24 tells us that $\mathcal{D}^b(X, \alpha)$ is Karoubian. It remains to show that we can find a strong generator. So, we start by proving that we are able to find a generator for $\text{Perf}(X, \alpha)$.

**Proposition 23.** Let $X$ be a variety, $\alpha \in \text{Br}X$. Then $\mathcal{D}^b(\text{QCoh}(X, \alpha))$ is generated by a perfect complex. In particular, $\text{Perf}(X, \alpha)$ is finitely generated.

**Proof.** By Lemma 15, it suffices to show that there is a perfect complex whose orthogonal is zero. The proof goes by induction on the minimal number of affine open subschemes that cover $X$. We start by $X = \text{Spec}A$. This was done in Theorem 17: there, we showed that a generator is a locally free $\alpha$–sheaf of finite rank (thought as a complex concentrated in degree 0), that we will denote, as usual, $E$.

Now, let’s suppose $X = U_1 \cup U_2$, where $U_1 = \text{Spec}A$ is an open affine subscheme of $X$, and $U_2$ is an open subscheme of $X$ which verifies the proposition. We will use the same notations as in the proof of Theorem 19. In particular, we are able to find a perfect generator $\mathcal{F}$ of $\mathcal{D}(\text{QCoh}(U_2, \alpha|_{U_2}))$. Now let $\mathcal{F}' = \mathcal{F} \oplus \mathcal{F}[1]$, so that, from Theorem 19 we know that there is a perfect complex $P \in \text{Perf}(X, \alpha)$ such that $j_1^*P \simeq \mathcal{F}'$.

Moreover, let $Y = X \setminus U_2 = U_1 \setminus U_{12}$, which is a closed subscheme of $\text{Spec}A$, so that it will be given by $f_1, \ldots, f_r \in A$. In the proof of Theorem 19 we showed that the complex $Q := G'_{r'}$ associated to $Y$ is a perfect generator of $\mathcal{D}_Y(\text{QCoh}(U_1, \alpha|_{U_1}))$.

We want to show is that $C = P \oplus Rj_{1*}Q$ is a perfect generator of $\mathcal{D}(\text{QCoh}(X, \alpha))$. By Theorem 17 and Lemma 15 we know that it suffices to show that $C^\perp = 0$.

Since $\text{Supp}Q \subseteq Y$, we have that $j_1^*Rj_{1*}Q = Q$ and $j_2^*Rj_{1*}Q = 0$. In this way we see that $Rj_{1*}Q$ is in $\text{Perf}(X, \alpha)$, so that $C \in \text{Perf}(X, \alpha)$. Moreover, using Lemma 16 we can show that for every $D \in \mathcal{D}(\text{QCoh}(X, \alpha))$ we have

$$R\text{Hom}(Rj_{1*}Q, D) = R\text{Hom}(Q, j_1^*D).$$

Now, suppose $D \in C^\perp$. This gives $R\text{Hom}(Rj_{1*}Q, D) = 0$, so that $j_1^*D \in Q^\perp$. As we saw in the proof of Theorem 19 this implies that we get a canonical isomorphism $D \simeq Rj_{2*}j_2^*D$.

But we have also that $R\text{Hom}(P, D) = 0$. This means

$$0 = R\text{Hom}(P, Rj_{2*}j_2^*D) = R\text{Hom}(j_2^*P, j_2^*D)$$

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and since \( j_2^*P \simeq \mathcal{F} \), this implies that \( j_2^*D \) is orthogonal to \( \mathcal{F} \), which is a generator of \( D(\text{QCoh}(U_2, \alpha|_{U_2})) \). This tells us that \( j_2^*D \) must be 0, and so \( D = 0 \).

Now that we have shown that we can find a perfect generator for the derived category \( D(\text{QCoh}(X, \alpha)) \), we can show the following:

**Proposition 24.** Let \( X, Y \) be two varieties, \( \alpha \in BrX, \beta \in BrY, \mathcal{F} \) a perfect generator of \( D(\text{QCoh}(X, \alpha)) \), \( \mathcal{G} \) a perfect generator of \( D(\text{QCoh}(Y, \beta)) \). Then \( \mathcal{F} \otimes_{\mathcal{G}} \mathcal{G} \) is a perfect generator of \( D(\text{QCoh}(X \times Y, p^*\alpha \cdot q^*\beta)) \), where \( p, q \) are the projections of \( X \times Y \) on \( X \) and \( Y \) respectively.

**Proof.** The fact that \( \mathcal{F} \otimes_{\mathcal{G}} \mathcal{G} \) is perfect is clear. We have to show that if \( D \in \mathcal{F} \otimes_{\mathcal{G}} \mathcal{G}^\perp = 0 \), then \( D = 0 \). We have, for every \( i, j \in \mathbb{Z} \)

\[
0 = \text{Hom}(p^*\mathcal{F} \otimes q^*\mathcal{G}, D[i + j]) = \text{Hom}(p^*\mathcal{F}, \text{RHom}(q^*\mathcal{G}, D)[i][j]) = \text{Hom}(\mathcal{F}, \text{RHom}(q^*\mathcal{G}, D[i])[j]).
\]

Since \( \mathcal{F} \) is a generator for \( D(\text{QCoh}(X, \alpha)) \), we get \( \text{RHom}(q^*\mathcal{G}, D[i]) = 0 \) for every \( i \in \mathbb{Z} \). Now, take \( U \) an open affine subscheme of \( X \) and \( \mathcal{E} \) a locally free \( \alpha \)-sheaf of finite rank in \( D(\text{QCoh}(X, \alpha)) \). We have

\[
\text{Hom}(\mathcal{E}, \text{RHom}(q^*\mathcal{G}, D[i])) = 0
\]

so that

\[
0 = \text{Hom}(\mathcal{E}|_U, (\text{RHom}(q^*\mathcal{G}, D[i]))|_U).
\]

Now, let’s denote \( p_U \) and \( q_U \) the projection of \( U \times Y \) to \( U \) and \( Y \). We get

\[
0 = \text{Hom}(p_U^*\mathcal{E}, \text{RHom}((q^*\mathcal{G})|_{U \times Y}, D[i]|_{U \times Y})) = \text{Hom}(\mathcal{G}|_U, \text{RHom}(p_U^*\mathcal{E}, D[i]|_{U \times Y})).
\]

Since \( \mathcal{G} \) is a generator of \( D(\text{QCoh}(Y, \beta)) \) this implies

\[
0 = \text{RHom}(p_U^*\mathcal{E}, D[i]|_{U \times Y}) = \text{RHom}(p_U^*\mathcal{G} \otimes D[i]|_{U \times Y})
\]

the last one being a sheaf on \( Y \). Now, take \( V \) an open affine subscheme of \( Y \), so that

\[
\text{Hom}(\mathcal{O}_V, \text{RHom}(\mathcal{O}_{U \times Y}, (p^*\mathcal{E}^\vee \otimes D[i]|_{U \times Y})) = 0
\]

that is

\[
\text{RHom}(\mathcal{O}_{X \times Y}, p^*\mathcal{E} \otimes D[i]|_{U \times V} = 0
\]

for every \( i \in \mathbb{Z}, U, V \) open affine subschemes of \( X \) and \( Y \) respectively. We have found that \( \text{RHom}(\mathcal{O}_{X \times Y}, p^*\mathcal{E}^\vee \otimes D[i]) = 0 \), that implies \( p^*\mathcal{E}^\vee \otimes D = 0 \), and so \( D = 0 \).
Now we can use Propositions 23 and 24 to show the following:

**Proposition 25.** Let $X$ be a smooth variety over a field $k$, $\alpha \in BrX$. Then $D^b(X, \alpha)$ has a strong generator.

**Proof.** Let $\gamma = p^*\alpha \cdot q^*\alpha^{-1}$. Since $X$ is smooth, we know that $X \times X$ is smooth and that the structure sheaf $O_\Delta$ of the diagonal is perfect. We have that $O_\Delta$ has structure of $\gamma$-sheaf: if $\delta$ is the closed immersion of $\Delta$ in $X \times X$, it’s easy to show that $\delta^*\gamma = 1$.

Let $\mathcal{F}$ be a perfect generator of $D(QCoh(X, \alpha))$, and $\mathcal{G}$ a perfect generator of $D(QCoh(X, \alpha^{-1}))$ (we know that there are such generators from Proposition 23). From Proposition 24 we know that $\mathcal{F} \boxtimes \mathcal{G}$ is a perfect generator of $D(QCoh(X \times X, \gamma))$, so that there is $n \in \mathbb{N}$ such that $O_\Delta \in \langle \mathcal{F} \boxtimes \mathcal{G} \rangle^\oplus$.

Now, we know that the Fourier-Mukai transform
\[
\Phi_{O_\Delta} : D^b(X, \alpha) \longrightarrow D^b(X, \alpha)
\]
is the identity, so that for every $D \in D(QCoh(X, \alpha))$, we have
\[
D = Rp_*(q^*D \otimes O_\Delta) \in \langle Rp_*(q^*D \otimes (\mathcal{F} \boxtimes \mathcal{G})) \rangle^\oplus_n.
\]

Since $Rp_*(q^*D \otimes (\mathcal{F} \boxtimes \mathcal{G})) = \mathcal{F} \otimes Rp_*q^*(D \otimes \mathcal{G})$, and $D \otimes \mathcal{G}$ is a sheaf on $X$, we have that
\[
D \in \langle \mathcal{F} \otimes RT\Gamma(X, D \otimes \mathcal{G}) \rangle_n = \langle \mathcal{F} \rangle_n.
\]

Using Lemma 15 we get $D(QCoh(X, \alpha))^c = \langle \mathcal{F} \rangle_n$, and since $X$ is smooth, $D^b(X, \alpha) = \langle \mathcal{F} \rangle_n$. \hfill \qed

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