THE THRESHOLD FOR ACKERMANNIAN RAMSEY NUMBERS

MENACHEM KOJMAN AND ERAN OMRI

Abstract. For a function $g : \mathbb{N} \to \mathbb{N}$, the \textit{g-regressive Ramsey number} of $k$ is the least $N$ so that

$$N \leq \min \{k \} g.$$

This symbol means: for every $c : [N]^2 \to \mathbb{N}$ that satisfies $c(m,n) \leq g(\min \{m,n\})$ there is a

\textit{min-homogeneous} $H \subseteq N$ of size $k$, that is, the color $c(m,n)$ of a pair $\{m,n\} \subseteq H$ depends only on $\min \{m,n\}$.

It is known (\cite{4,5}) that $\text{Id}$-regressive Ramsey numbers grow in $k$ as fast as $\text{Ack}(k)$, Ackermann’s function in $k$. On the other hand, for constant $g$, the $g$-regressive Ramsey numbers grow exponentially in $k$, and are therefore primitive recursive in $k$.

We compute below the threshold in which $g$-regressive Ramsey numbers cease to be primitive recursive and become Ackermannian, by proving:

\textbf{Theorem.} Suppose $g : \mathbb{N} \to \mathbb{N}$ is weakly increasing. Then the $g$-regressive Ramsey numbers are primitive recursive if an only if for every $t > 0$ there is some $M_t$ so that for all $n \geq M_t$ it holds that $g(n) < n^{1/t}$ and $M_t$ is bounded by a primitive recursive function in $t$.

1. Introduction

We investigate Ramsey properties of pair-colorings of natural numbers in which the set of possible colors of a pair depends on the pair. A number $n \in \mathbb{N}$ is identified with the set $\{m \in \mathbb{N} : m < n\}$. The set of all two-element subsets of a set $X$ is denoted by $[X]^2$.

\textbf{Definition 1.1.} For a function $f : [N]^2 \to \mathbb{N}$, an $f$-coloring of pairs is a function $c : [N]^2 \to \mathbb{N}$ so that $c(m,n) \leq f(m,n)$ for all $\{m,n\} \in [N]^2$.

The standard Ramsey theorems for pairs can be thought of as dealing with $f$-colorings for \textit{constant} $f$. When coloring all pairs from $N$ by $C$ colors, there will be a monochromatic subset $B \subseteq N$ of size $k$ if $C^{k-C} \leq N$, by the standard proof of Ramsey’s theorem.

On the other hand, if $f$ is sufficiently large, that is, if $f(m,n) \geq \left(\max(m,n) + 1\right)^2$, then any coloring is \textit{equivalent} to an $f$ coloring. Two colorings $c_1, c_2$ are equivalent if for all $(m,n), (m',n')$ it holds that $c_1(m,n) = c_1(m',n')$ iff $c_2(m,n) = c_2(m',n')$, that is, they induce the same partition of unordered pairs.

The Ramsey behaviour of colorings of $[N]^2$ with no limitations at all on the set of colors is governed by the \textit{Canonical Ramsey Theorem} by Erdős and Rado, which asserts that for any pair coloring $c : [N]^2 \to \mathbb{N}$ there is an infinite $B \subseteq N$ so that $c \upharpoonright [B]^2$ is \textit{canonical}, that is, is equivalent to one of the following four colorings: $c_1(m,n) = \min(m,n)$, $c_2(m,n) = \max(m,n)$, $c_3(m,n) = 0$, a constant coloring or $c_4(m,n) = (m,n)$, a 1-1 coloring.

The finite version of the canonical Ramsey theorem asserts that for every $k$ there exists $N$ so that for every $c : [N]^2 \to \mathbb{N}$ there is $B \in [N]^k$ so that $c \upharpoonright B$ is canonical. Double exponential upper and lower bounds on $N$ in terms of $k$ are known for the finite canonical Ramsey theorem [6].

We are interested here in $f$-colorings where $f(m,n)$ depends only on $\min\{m,n\}$, that is, when $f(m,n) = g(\min\{m,n\})$ for some function $g : \mathbb{N} \to \mathbb{N}$. When $g = \text{Id}$, such a coloring is called \textit{regressive}. In other words, $c$ is regressive if $c(m,n) \leq \min\{m,n\}$. More generally, we say that a coloring $c$ is $g$-regressive if $c(m,n) \leq g(\min\{m,n\})$. 

We investigate Ramsey properties of pair-colorings of natural numbers in which the set of possible
A set \( B \subseteq \mathbb{N} \) is **min-homogeneous** for a coloring \( c \) if \( c(m, n) \) depends only on \( \min\{m, n\} \) for all \((m, n) \in [B]^2\). The important feature of min-homogeneity is that no matter how large a function \( g \) is, a \( g \)-regressive min-homogeneity Ramsey number exists for every \( k \):

**Fact 1.2.** Let \( g : \mathbb{N} \to \mathbb{N} \) be arbitrary. Then

1. for every \( g \)-regressive coloring \( c : [\mathbb{N}]^2 \to \mathbb{N} \) there is an infinite \( B \subseteq \mathbb{N} \) such that \( c \upharpoonright [B]^2 \) is \( g \)-homogeneous.
2. for every \( k \) there is some \( N \) so that for every \( g \)-regressive coloring \( c : [N]^2 \to \mathbb{N} \) there is a \( g \)-homogeneous \( B \subseteq N \) of size at least \( k \).

*Proof. The first item follows from the infinite canonical Ramsey theorem, since a regressive coloring cannot be equivalent neither to \( \max\{m, n\} \) nor to a 1-1 coloring on an infinite set. The second item follows from the first via compactness.*

Let us introduce the suitable symbolic notation for discussing \( g \)-Regressive colorings.

**Definition 1.3.** Let \( g : \mathbb{N} \to \mathbb{N} \) be a function. Then:

1. The symbol \( N \longrightarrow (k)_g \) means: for every \( g \)-regressive colorings \( c : [N]^2 \to \mathbb{N} \) there is a \( g \)-homogeneous \( B \subseteq N \) of size \( k \).
2. The symbol \( N \min \longrightarrow (k)_g \) means: for every \( g \)-regressive coloring \( c : [N] \to \mathbb{N} \) there is a \( g \)-homogeneous \( B \subseteq N \) of size \( k \).

The \( g \)-regressive Ramsey theorem (for pairs) is the statement

\[
(\forall k)(\exists N)(N \min \longrightarrow (k)_g)
\]

Recall that the standard proof of Ramsey’s theorem gives, for the constant number of colors \( C \),

\[
C_k \longrightarrow (k)_C, \quad C^{k,C} \longrightarrow (k)_C
\]

For any function \( f : \mathbb{N} \to \mathbb{N} \) the function \( f^{(n)}(x) \) is defined by \( f^{(0)}(x) = x \) and \( f^{(n+1)}(x) = f(f^{(n)}(x)) \). We recall that Ackermann’s function is defined as \( \text{Ack}(n) = A_n(n) \) where each \( A_n \) is the standard \( n \)-th approximation of the Ackermann function, defined by:

\[
A_1(n) = n + 1
\]

\[
A_{i+1}(n) = A_i^{(n)}(n)
\]

It is well known (see e.g. [2]) that each approximation \( A_n \) is **primitive recursive** and that every primitive recursive function is eventually dominated by some \( A_n \). Thus Ackermann’s function eventually dominates every primitive recursive function and is *truly* rapidly growing.

Ackermannian lower and upper bounds on \( N \) in terms of \( k \) are known for the regressive Ramsey theorem for \( g = \text{Id} \). This was first proved using methods from mathematical logic and then elementarily [4] [5].

We are interested here in locating the threshold for the formidable leap from a primitive recursive upper bound to an Ackermannian lower bound in the \( g \)-regressive Ramsey theorem. This threshold obviously lies between the constant functions and \( \text{Id} \).

We shall see below that if \( g(m) \leq m^\frac{1}{\beta(m)} \) for some unbounded and increasing function \( \beta : \mathbb{N} \to \mathbb{N} \) and \( \beta^{-1} \) is bounded by a primitive recursive function \( f \), then the \( g \)-regressive Ramsey numbers are dominated by \( f \); but if \( g(m) = m^{1/\beta(m)} \) where \( \beta \) grows to infinity sufficiently slowly, that is, when \( \beta^{-1} \) is Ackermannian, then the \( g \)-regressive Ramsey number are Ackermannian.

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[1] See [4], p. 60, for an amusing attempt to “grasp the magnitude” of \( A_5(5) \).
All functions below are from \( \mathbb{N} \) to \( \mathbb{N} \) and whenever an expression \( x \) may not be integer, it is intended to be replaced by \( [x] \).

2. The Results

2.1. Min-Homogeneity. For any unbounded function \( \beta : \mathbb{N} \to \mathbb{N} \) define

\[
\beta^{-1}(t) = \min\{n : \beta(n) \geq t\}
\]

(2)

Notation 2.1. For a given \( g : \mathbb{N} \to \mathbb{N} \), let \( \nu_g(k) \) denote the least \( N \) which satisfies \( N \xrightarrow{\min} (k)g \).

Theorem 2.2. Suppose \( g : \mathbb{N} \to \mathbb{N} \) and \( \beta : \mathbb{N} \to \mathbb{N} \) are nonzero, weakly increasing and \( g(n) \leq n^{1/\beta(n)} \) for all \( n \). Then for every \( k \in \mathbb{N} \) it holds that \( \nu_g(k) \leq \beta^{-1}(k) \).

Proof. Given \( 1 < k \in \mathbb{N} \) let \( N = \beta^{-1}(k) \) and we will show that \( N \xrightarrow{\min} (k)g \). Let \( C = g(N) \). Since \( g \) is increasing, \( g(m) \leq g(N) \) for all \( m < N \). Thus it suffices to show that for every coloring \( c : [N] \to C \) there exists a min-homogeneous \( B \subseteq N \) of size \( k \). This holds if \( C^k \subseteq N \). Since \( C = g(N) \leq N^{1/\beta(N)} \) it suffices to show that \( (N^{1/\beta(N)})^k \leq N \) — which is obvious, since \( \beta(N) \geq k \).

Corollary 2.3. Suppose \( g \) and \( \beta \) are weakly increasing, \( g(n) \leq n^{1/\beta(n)} \) for all \( n \) and \( \beta^{-1} \) is bounded by a primitive recursive function. Then \( \nu_g \) is bounded by a primitive recursive function.

If, furthermore, \( g \) is primitive recursive, then \( \nu_g \) is primitive recursive.

Proof. By the previous theorem \( \nu_g \) is bounded by \( \beta^{-1} \) and thus is bounded by a primitive recursive function. Since the relation \( N \xrightarrow{\min} (k)g \) is primitive recursive when \( g \) is, the computation of \( \nu_g \) requires a bounded search for a primitive recursive relation and therefore \( \nu_g \) is primitive recursive.

We now begin working towards the proof of the converse of Corollary 2.3 to show that if \( \beta^{-1} \) is Ackermann and \( g(n) = n^{1/\beta(n)} \) then \( \nu_g \) is Ackermannian. We begin by proving the special case that \( g(n) = n^{1/\beta(n)} \) and \( \beta(n) \) is bounded.

Lemma 2.4. For every \( t > 0 \) let \( g_t(n) = n^{1/t} \). Then the function \( \nu_{g_t} \) eventually dominates every primitive recursive function for all \( t > 0 \).

Proof. The proof is by induction on \( t > 0 \).

The proof involves constructing a “bad” \( f_t \)-regressive coloring for \( t \geq 1 \) by a generalization of the method of construction of a bad \( f_1 \)-regressive coloring in [4].

Definition 2.5. For a given \( t \in \mathbb{N} \setminus \{0\} \), we define a sequence of functions \( (f_t)_i : \mathbb{N} \to \mathbb{N} \) as follows.

\[
(f_t)_1(n) = n + 1
\]

(3)

\[
(f_t)_{i+1}(n) = (f_t)_i(n)^{[n^{1/t}]}(n)
\]

(4)

Claim 2.6. For all \( 0 < t \in \mathbb{N} \) the function \( f_t(k) = (f_t)_k(k) \) eventually dominates every primitive recursive function.

Proof. By induction on \( t \).

For \( t = 1 \) the functions \( (f_t)_i = A_k \), the standard \( k \)-th approximations of Ackermann’s functions, so every primitive recursive function is eventually dominated by \( f_t(k) \) (see e.g. [2]).

Claim 2.7. Let \( t > 0 \). For all \( n > 2^i, i > 0 \) it holds that \( (f_{t+1})_{i+2} + 2(n^2) > ((f_t)_i(n))^2 \).

We prove claim 2.7 by induction on \( i \). For \( i = 1 \) we need the following:
Observation 2.8. For every $t,k,n > 0$ it holds that $(f_t)_k(n) \geq n + ([n^{1/t}])^{k-1}$  

Proof. We show observation 2.8 by induction on $k$. If $k = 1$, it follows by definition that $(f_t)_k(n) = n + 1 = n + ([n^{1/t}])^{k-1}$. Let $k > 1$. By definition $(f_t)_{k+1}(n) = (f_t)_{k}([n^{1/t}])^{(n)}$ and by applying the induction hypothesis $[n^{1/t}]$ times we get that the right hand side of the equation is larger than $n + (([n^{1/t}])^{(n)})^{k-1}$ which is $n + ([n^{1/t}])^{k}$.  

Observation 2.9. $(f_{t+1})_{2t+3}(n^2) > n^2 + 2n + 1$  

Proof. By observation 2.8 we have that $(f_{t+1})_{2t+3}(n^2) \geq n^2 + ([n^{1/t+1}])^{2t+2}$. Now

$$n^2 + ([n^{1/t+1}])^{2t+2} \geq n^2 + (n^{2/t+1} - 1)^{2(t+1)} \geq n^2 + (n^{4/t+1} - 2n^{2/t+1} + 1)^{t+1} > n^2 + (n^{2/t+1}(n^{2}) - 2))^{t+1} > n^2 > n^2 + 2n + 1$$

When $i = 1$, by observation 2.8 $(f_{t+1})_{i+2t+2}(n^2) = (f_{t+1})_{2t+3}(n^2) > n^2 + 2n + 1 = ((f_t)_i(n))^2 = ((f_t)_i(n))^2$

We now assume that claim 2.7 is true for $i$ and prove it for $i+1$.

Claim 2.10. $\forall j \in \mathbb{N} (f_{t+1})_{i+2t+2}(n^2) > ((f_{t+1})_i(j))(n))^2$  

Proof. We show claim 2.10 by induction on $j$. For $j = 1$ the claim is induced by the induction hypothesis for $i$. For $j > 1$ we have $(f_{t+1})_{i+2t+2}(n^2) = (f_{t+1})_{i+2t+2}(((f_{t+1})_i(j))(n))^2$ by monotonicity and the induction hypothesis for $j$. Now, if we denote $n' = (f_{t+1})_i(j)(n)$, we easily see, by the induction hypothesis for $j$ or for $i$, that $(f_{t+1})_{i+2t+2}(((f_{t+1})_i(j))(n))^2 > ((f_{t+1})_i((f_{t+1})_i(j)(n)))^2$ which is, in fact, $(f_{t+1})_i+1(j)(n))^2$.

We still need to show the induction step for claim 2.7 $(f_{t+1})_{i+2t+2}(n^2) = (f_{t+1})_{i+2t+2}(((f_{t+1})_i(j))(n))^2$ and by claim 2.10 the latter term is larger than $((f_{t+1})_i([n^{1/t}])^{(n)})^2 = ((f_{t+1})_i(n))^2$.

That concludes the proof of claim 2.7 and therefore also of claim 2.6.

We turn now to the construction of bad $g_k$-recessive colorings.

For a given natural number $k > 2$ and a given $g : \mathbb{N} \to \mathbb{N}$ that is monotonically increasing such that for some $t \in \mathbb{N}$ it holds that $k \leq \lfloor \frac{\sqrt{g(t)}}{2} \rfloor$, we define a sequence of functions $(f_g)_i : \mathbb{N} \to \mathbb{N}$ as follows.

Definition 2.11. Let $\mu = \mu_g(k) = \min(\{t \in \mathbb{N} : k \leq \lfloor \frac{\sqrt{g(t)}}{2} \rfloor\})$ and let

$$\begin{align*}
(f_g)_1(n) &= n + 1 \\
(f_g)_{i+1}(n) &= (f_g)_i([\frac{\sqrt{g(n)}}{2}])^{(n)}
\end{align*}$$

Define a sequence of semi-metrics $\langle (d_g)_i : i \in \mathbb{N} \rangle$ on $\{ n : n \geq \mu \}$ by setting, for $m,n \geq \mu$,

$$\begin{align*}
(d_g)_i(m,n) &= |\{ l \in \mathbb{N} : m < (f_g)_i(l)(\mu) \leq n \}|
\end{align*}$$

For $n > m \geq \mu$ let $I_g(m,n)$ be the greatest $i$ for which $(d_g)_i(m,n)$ is positive, and $D_g(m,n) = (d_g)_{I(m,n)}(m,n)$. 

4
Let us fix the following (standard) pairing function $Pr$ on $\mathbb{N}^2$

$$Pr(m,n) = \left(\frac{m+n+1}{2}\right) + n$$

$Pr$ is a bijection between $[\mathbb{N}]^2$ and $\mathbb{N}$ and is monotone in each variable. Observe that if $m,n \leq l$ then $Pr(m,n) < 4l^2$ for all $l > 2$.

Define a pair coloring $c_g$ on $\{n : n \geq \mu\}$ as follows:

$$c_g(\{m,n\}) = Pr(I_g(m,n), D_g(m,n))$$

Claim 2.12. For all $n > m \geq \mu$, $D_g(m,n) \leq \frac{\sqrt{g(m)}}{2}$.

Proof. Let $i = I_g(m,n)$. Since $(d_g)_{i+1}(m,n) = 0$, there exist $t$ and $l$ such that $t = (f_g)i+1(\mu) \leq m < n < (f_g)i+1(\mu) = (f_g)i+1(t)$. But $(f_g)i+1(t) = (f_g)i+1\left(\frac{\sqrt{g(t)}}{2}\right)(t)$ and therefore $\frac{\sqrt{g(t)}}{2} = (d_g)i(t, (f_g)i+1(t)) \geq D_g(m,n)$.

Claim 2.13. $c_g$ is $g$-regressive on the interval $[\mu, (f_g)k(\mu))$.

Proof. Clearly, $(d_g)k(m,n) = 0$ for $\mu \leq m < n < (f_g)k(\mu)$ and therefore $I_g(m,n) < k \leq \frac{\sqrt{g(m)}}{2}$. From claim 2.12 we know that $D_g(m,n) \leq \frac{\sqrt{g(m)}}{2}$. Thus, $c_g(\{m,n\}) \leq Pr(\left\lfloor \frac{\sqrt{g(m)}}{2}\right\rfloor, \left\lfloor \frac{\sqrt{g(m)}}{2}\right\rfloor)$, which is $< g(m)$, since $\frac{\sqrt{g(m)}}{2} > 2$.

Claim 2.14. For every $i \in N$, every sequence $x_0 < x_1 < \cdots < x_i$ that satisfies $(d_g)i(x_0, x_i) = 0$ is not min-homogeneous for $c_g$.

Proof. The claim is proved by induction on $i$. If $i = 1$ then there are no $x_0 < x_1$ with $(d_g)1(x_0, x_1) = 0$ at all. Let $i > 1$ and suppose to the contrary that $x_0 < x_1 < \cdots < x_i$ form a min-homogeneous sequence with respect to $c_g$ and that $(d_g)i(x_0, x_i) = 0$. Necessarily, $I_g(x_0, x_i) = j < i$. By min-homogeneity, $I_g(x_0, x_i) = j$ as well, and $(d_g)j(x_0, x_i) = (d_g)j(x_0, x_i)$. Hence, $\{x_1, x_2, \ldots, x_i\}$ is min-homogeneous with $(d_g)j(x_1, x_i) = 0$ — contrary to the induction hypothesis.

Corollary 2.15. There exists no $H \subseteq [\mu, (f_g)k(\mu))$ of size $k + 1$ that is min-homogeneous for $c_g$.

Corollary 2.16. If the function $(f_g)k(k)$ dominates every primitive recursive function (Ackermannian in terms of $k$) and $\mu_g(k)$ is bound by some primitive recursive function, then the lower bound for min-homogeneity for g-regressive colorings also dominates every primitive recursive function.

Proof. The collection of primitive recursive functions is closed under composition. Thus, the function $(f_g)k(\mu_g(k)) - \mu_g(k)$ is Ackermannian in terms of $k$. Moreover, it is Ackermannian in terms of $\mu_g(k) + k + 1$. Therefore, we may allow ourselves to set the color of every pair $(m, n)$ such that $m < \mu_g(k)$ to be 0 and by that present a $g$-regressive coloring of $[(f_g)k(\mu_g(k))]^2$ that yields no min-homogeneous $H \subseteq [(f_g)k(\mu_g(k))]$ of size $\mu_g(k) + k + 1$.

Now, to conclude the proof of theorem 2.1 we need only observe that for a given $j \in \mathbb{N}$ the function $\frac{k^{\frac{1}{j}}}{2}$ grows asymptotically faster than $k^{\frac{1}{j}}$ and therefore, by claim 2.10 for any $j \in \mathbb{N}$ $(f_g)k(k)$ for $g(m) = m^j$ dominates every primitive recursive function. On the other hand, for such $g$, $\mu_g(k) \leq 4^jk^{2j}$. Hence, by corollary 2.16 we establish that the lower bound for min-homogeneity for g-regressive colorings for $g(m) = m^j$ dominates every primitive recursive function.
**Theorem 2.17.** Let \( \beta^{-1}(n) := \text{Ack}(n + 3) \) (so \( \beta \) is basically \( \text{Ack}^{-1} \)) and let \( g(n) = n^{1/\beta(n)} \). There exists a \( g \)-regressive coloring \( c : [\mathbb{N}]^2 \to \mathbb{N} \) such that for every primitive recursive function \( f : \mathbb{N} \to \mathbb{N} \) there exists \( N_f \in \mathbb{N} \) such that for all \( m > N_f \) and \( H \subseteq m \) which is min-homogeneous for \( c \) it holds that \( f(|H|) < m \).

**Proof.** We define two increasing sequences \( \{k_t\} \) and \( \{\mu_t\} \) and then let \( \beta(n) = t + 1 \) if \( \mu_t \leq n < \mu_{t+1} \).

Using the definition of \( (f_g)_t \) given in [2.11], we define a \( g \)-regressive coloring \( c \), where \( g(n) = n^{1/\beta(n)} \), so that in the interval \([\mu_t, \mu_{t+1})\) there is no min-homogeneous set of size \( k_t \).

We denote \( g_t(n) := n^{1/t} \). Let:

\[
\begin{align*}
\mu_0 &= 0 \\
\mu_1 &= 10^4 \\
k_1 &= 18
\end{align*}
\]

And for all \( t > 1 \),

\[
\begin{align*}
k_t &= \frac{\sqrt{g_{t-1}(\mu_{t-1})}}{2} = \frac{\mu_{t-1}^{1/2(t-1)}}{2} \\
\mu_t &= \text{Ack}(t + 3)
\end{align*}
\]

On \([0, \mu_1]\) we define \( c(m, n) \) as follows: color all \( \{m, n\} \) from \([0, 43]\) regessively by the colors \( \{0, 1\} \) with no min-homogeneous set of size 12. This is possible, since the (usual) Ramsey number of 5 is \( \geq 43 \), so there is a 2-coloring of \([1, 43]\) with no homogeneous set of size 5, hence with no min-homogeneous set of size 11. For \( m, n \geq 43 \) color as follows: write out \(|n - m|\) in base 10 and let \( c(m, n) = \text{Pr}(d_1, d_2 + 1) \) where \( d_1 \in \{0, 1, 2, 3\} \) is the maximal power of 10 smaller than \(|n - m|\) and \( d_2 \) is the \( f_1 \)-th decimal digit. This coloring allows no min-homogeneous sets of size 6 in \([42, 10^4]\). So letting \( c(m, n) = 0 \) for \( m < 42 \) and \( 42 \leq m \leq 10^4 \), we get that below \( \mu_1 \) there are no min-homogeneous sets of size \( k_1 = 18 \).

Now we need to define \( c \) on \([\mu_{t-1}, \mu_t)\) for all \( t > 1 \). Let \( k_t = \frac{\mu_{t-1}^{1/2(t-1)}}{2} \). Observe that we may color pairs over the interval \([\mu_{t-1}, (f_{g_t})_{k_t}(\mu_{t-1}))\) if \( \mu \geq 4(g_t(k))^2 \) using \( c_{g_t} \) with \( \mu_t \) instead of \( \mu_{g_t}(k) \). This coloring is \( g_t \)-regressive with no min-homogeneous \( H \subseteq [\mu, ((f_{g_t})_{k_t})(\mu)) \) of size \( k + 1 \). This is true since the proofs of claims 2.18 and 2.19 made no use of the minimality of \( \mu_{g_t}(k) \).

To define \( c \) on \([\mu_{t-1}, \mu_t)\) it suffices, then, to prove:

**Claim 2.18.** \( \text{Ack}(t + 3) < (f_{g_t})_{k_t}(\mu_{t-1})) \) for all \( t > 1 \).

**Proof.** We first prove claim 2.18 for \( t = 2 \). We have, by claim 2.19, that \( (f_{g_t})_{k_2}(\mu_1)) > (A_{k_2-18}(\left\lceil k_1^1/2 \right\rceil))^{8} \).

Since \( k_2 = 50 \), the latter term is \( (A_{32}(\left\lceil 10^4^{1/2} \right\rceil))^{8} \) and thus larger than \( A_{32}(3) > A_5(5) \).

Let \( t > 2 \). We know that \( \mu_{t-1} = \text{Ack}(t + 2) \) and hence it clearly holds that \( k_t - 16t^2 + 28t - 10 = \frac{\mu_{t-1}^{1/2(t-1)}}{2} - 16t^2 + 28t - 10 = (\text{Ack}(t+2))^{1/2(t-1)} - 16t^2 + 28t - 10 > t + 3 \) and it also clearly holds that \( \mu_t > t + 3 \). Thus, by claim 2.19, we have that \( (f_{g_{t-1}})_{k_t}(\mu_{t-1})) \geq (A_{k_t-16t^2+28t-10}(\left\lceil (\mu_{t-1}^{1/2(t-1)})^{-1} \right\rceil))^{24t-5} > A_{t+3}(t+3) \).

\( \square \)

**Claim 2.19.** For all \( t > 1 \) it holds that \( (f_{g_{t-1}})_{k_t}(\mu_{t-1})) > (A_{k_t-16t^2+28t-10}(\left\lceil \mu_{t-1}^{1/2(t-1)} \right\rceil))^{24t-5} \)

**Proof.** Observe that \( (A_{k_t-16t^2+28t-10}(\left\lceil \mu_{t-1}^{1/2(t-1)} \right\rceil))^{24t-5} \) is actually \( (\left\lceil (f_1)_{k_t-16t^2+28t-10}(\left\lceil \mu_{t-1}^{1/2(t-1)} \right\rceil) \right\rceil)^{24t-5} \)

Now, by applying claim 2.7 to the latter term, we get \( (f_1)_{k_t-16t^2+28t-10}(\left\lceil \mu_{t-1}^{1/2(t-1)} \right\rceil)^{24t-5} < \)
\[(f_{2})_{k_{t}-16t^{2}+28t-10}+2+2\left(\left[\mu_{t-1}^{1/2}t^{-\frac{1}{2}}\right]\right)^{2^{4t-6}},\] since the parameter \(t\) of claim \(2.21\) is here. If we apply it now to right hand side term, the parameter \(t\) of the claim would be 2 and we would find that the latter term is smaller than \((f_{3})_{k_{t}-16t^{2}+28t-10}+2+2+4+2+8\left(\left[\mu_{t-1}^{1/2}t^{-\frac{1}{2}}\right]\right)^{2^{4t-7}}.\] Generally, if we apply the claim \(j\) times we get that \((f_{2})_{k_{t}-16t^{2}+28t-10}+2+2+8\left(\left[\mu_{t-1}^{1/2}t^{-\frac{1}{2}}\right]\right)^{2^{4t-5}} < (f_{4(t-1)})_{k_{t}}(\mu_{t-1}).\) Note that we are allowed to apply claim \(2.21\) 4 times, only if for all \(1 \leq j \leq 4t - 5\) it holds that \(\mu_{t-1}^{2^{4j-5}} > 2j\), or that \(\mu_{t-1} > 2^{2^{4j-5}}\) and that is true for all \(t > 2\) since \(\mu_{t-1}\) is clearly larger than \(2^{(4(t-1))2^{4(t-1)-1}}\). For \(t = 2\) it is also true and may be easily verified by hand.

On the other hand, it holds that \((f_{9t-1})_{k_{t}}(\mu_{t-1}) > (f_{4(t-1)})_{k_{t}}(\mu_{t-1})\) since \(\mu_{t-1}\) is larger than \(2^{4t}\) for all \(t > 1\) and therefore \(\frac{\mu_{t-1}^{2^{4t}t^{-\frac{1}{2}}}}{2} \geq n_{c,\mu}^{1/24t-16t^{2}}.\)

\[\Box\]

**Observation 2.20.** The coloring \(c\) is \(g\)-regressive.

**Proof.** For any \(m, n\) such that \(\beta(m) = \beta(n) = t\) we know that \(c(m, n) \leq \left\lfloor m^{1/t} \right\rfloor\) since \(c(m, n) = c_{g,\beta}(m, n)\) is \(g_{\beta}\)-regressive on the interval. Otherwise, \(c(m, n) = 0\) which is always smaller than \(m^{1/\beta(m)}\).

\[\Box\]

**Observation 2.21.** For any given \(N \in \mathbb{N}\) with \(\beta(N) = j\), there is no min-homogeneous \(H \subseteq [N]\) of size \((k_{j-1} + 1)^{2} + 18.\)

**Proof.** From claim \(2.19\) it is clear that for all \(t > 1\) it holds that \(k_{t} < k_{t+1}\) and that \(k_{t} > t.\) Thus, since at each interval \([\mu_{t}, \mu_{t+1})\) for ant \(t < j\) there exist no min-homogeneous subset of size \(k_{t} + 1\) and hence, no min-homogeneous subset of size \(k_{j-1} + 1\). Therefore, in the union of all those intervals there is no min-homogeneous subset of size \((k_{j} + 1)t < (k_{j-1} + 1)^{2}\). Now, in the first interval there can be no no min-homogeneous of size 18, there is no min-homogeneous \(H \subseteq [N]\) of size \((k_{j-1} + 1)^{2} + 18\) in the union of the first \(j\) intervals of which \([N]\) is a subset.

To conclude the proof we only need to observe that given a primitive recursive function \(f\), there exists a \(k_{f} \in \mathbb{N}\) such that for every \(n > 4\) it holds that \(A_{k_{f}}([n^{1/\log \log n}]) > f((n + 1)^{2} + 18)\). Now, because \(k_{t}\) grows extremely faster than \(t\), we can find a \(t\) such that \(k_{t} - 2(t-1)^{2} - 4t = 5 > k_{t}\) and \(\log \log \mu_{t} > t.\) Set \(N_{f}\) to be \(\mu_{t+1}.\) Given \(n > N_{f}\) with \(\beta(n) = j.\) We have that \(j > t+1.\) Assume to the contrary that there exists a min-homogeneous \(H \subseteq [n]\) of size \(f^{-1}(n)\) then \(f^{-1}(n) < (k_{j-1} + 1)^{2} + 18.\) Thus, \(n' = \sqrt{f^{-1}(n) - 18} < k_{j-1} < \mu_{j-2}\). Now, \(n = f((n') + 1)^{2} + 18) < A_{k_{j}}([n'/\log \log n']) < A_{k_{j-1} - 32(j-2)^{2} - 4(j-2) + 3\left(\left[\mu_{j-2}, [n'/\log \log n']\right]\right]} < \mu_{j-2}.\) Contrary to \(\beta(n) = j.\)

\[\Box\]

We can now prove the main theorem of the paper:

**Theorem 2.22.** Suppose \(g : \mathbb{N} \rightarrow \mathbb{N}\) is eventually smaller than \(n^{1/t}\) for every constant \(t > 1.\) Then \(\nu_{g}\) is bounded by a primitive recursive function if and only if the least number \(M_{t}\) which satisfies \(g(n) < n^{1/t}\) for all \(n \geq M_{t}\) is bounded by a primitive recursive function in \(t.\)

**Proof.** Suppose first that \(M_{t}\) is bounded by some primitive recursive function in \(t.\) Replacing \(g(n)\) by \(\max\{g(n) : m \leq n\}\) we may assume that \(g\) is weakly increasing and \(M_{t}\) would still be bounded by a primitive recursive function. Now apply Corollary 2.3. This takes care of the “if” part.

The “only if” part follows directly from Theorem 2.22 above.

\[\Box\]
2.2. Homogeneity. We look now at the threshold $g$ at which one can guarantee the usual Ramsey theorem for $g$-recessive colorings, that is, have homogeneous rather than just min-homogeneous sets.

**Theorem 2.23.** Suppose $f : \mathbb{N} \rightarrow \mathbb{N}^+$ satisfies $\lim_{n \to \infty} (f(n)) = \infty$ and let $g(x) = \frac{\log x}{f(x) \log \log x}$ for $x \geq 4$ and $g(x) = 0$ for $x < 4$. Then $\forall k \in \mathbb{N} \rightarrow (k)_{g}^{2}$.

**Proof.** Let $f : \mathbb{N} \rightarrow \mathbb{N}^+$ be a function such that $\lim_{n \to \infty} (f(n)) = \infty$, and $g(x) = \frac{\log x}{f(x) \log \log x}$ for $x \geq 4$ and $g(x) = 0$ for $x < 4$. Given $k \in \mathbb{N}$, find $N \geq k$ so that $f(N) > k$ and $f(N) > f(m)$ for all $m < N$. Such $N$ exists, since $\lim_{m \to \infty} (f(m)) = \infty$. Since $f(N) > f(m)$ for all $m < N$, it follows that $g(m) \leq g(N)$ for all $m < N$ as well. So given a $g$-recessive coloring $c : [N]^2 \rightarrow \mathbb{N}$ we have that $c(m,n) \leq g(N)$ for all $(m,n) \in [N]^2$. Put $C = \lfloor g(N) + 1 \rfloor$. If $C = 1$ then $N$ itself is homogeneous of size $\geq k$, so assume that $C \geq 2$. The standard proof of Ramsey’s theorem with $C$ colors gives a homogeneous $B \subseteq N$ of size $k$ in case $N > C^{k-C}$, which holds here, since $\log g(N) < \log \log N$ and therefore

$$C^{k-C} = 2^{\log C^{k-C}} = 2^{\log (N^{k-g(N)})} < 2^{\log \log N \cdot g(N)} \leq 2^{\log N} = N.$$ 

It should be noted that this is of interest when $f$ grows slowly (e.g. $f(m) = \log^* m$).

**Theorem 2.24.** For every $s \in \mathbb{N}$ and for $g(i) = \frac{\log(i)}{s}$ it holds that $\exists k \in \mathbb{N} \rightarrow (k)_{g}^{2}$

**Proof.** Let $s \in \mathbb{N}$ and $g(i) = \frac{\log(i)}{s}$. We set $k = 2s + 1$ and we show a $g$-recessive coloring $C : \mathbb{N}^2 \rightarrow \mathbb{N}$ where there exists no $S \subseteq \mathbb{N}$ of size $\geq k$ that is homogeneous for $C$. For any $n \in \mathbb{N}$, let $r_s(n)$ be the representation of $n$ in $s$ basis. For any $m,n \in \mathbb{N}$ such that $m < n$ and $\lfloor \log_s(m) \rfloor = \lfloor \log_s(n) \rfloor$, let $f(m,n)$ be the smallest index $i$ such that $r_s(m)[i] \neq r_s(n)[i]$. We define $c$ as

$$c(m,n) = \begin{cases} \lfloor \log_s(m) \rfloor & \text{if } \lfloor \log_s(m) \rfloor \neq \lfloor \log_s(n) \rfloor; \\ f(m,n) & \text{if } \lfloor \log_s(m) \rfloor = \lfloor \log_s(n) \rfloor. \end{cases}$$

**Observation 2.25.** Let $Y = \{y_1, y_2, ..., y_{s+1}\}$ where $y_1 < y_2 < ... < y_{s+1}$, be a homogeneous set for $C$. Then $\lfloor \log_s(y_1) \rfloor < \lfloor \log_s(y_{s+1}) \rfloor$.

To show Observation 2.25 let $Y$ be a homogeneous set for $C$ and suppose to the contrary that $\lfloor \log_s(y_1) \rfloor = \lfloor \log_s(y_{s+1}) \rfloor$, from the definition of $c$ we get that $f$ is constant on $Y$. Thus elements of $Y$, pairwise differ in the $i$’th value in their $s$ basis representation for some index $i$, which is impossible since there are only $s$ possible values for any index. Contradiction.

Now, Let $X = \{x_1, x_2, ..., x_{s+1}\}$ $x_1 < x_2 < ... < x_{s+1}$ and suppose to the contrary that $X$ is homogeneous for $C$. By observation 2.25 we get that $\lfloor \log_s(x_1) \rfloor < \lfloor \log_s(x_{s+1}) \rfloor < \lfloor \log_s(x_{2s+1}) \rfloor$ and therefore $C(x_1, x_{s+1}) < C(x_{s+1}, x_{2s+1})$ contrary to homogeneity.

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Department of Mathematics, Ben Gurion University of the Negev

*E-mail address: kojman@math.bgu.ac.il*

Department of Computer Science, Ben Gurion University of the Negev

*E-mail address: omrier@cs.bgu.ac.il*