Two Component Kaup - Kupershmidt Equation

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Abstract

The Kaup - Kupershmidt equation is generalized to the system of equations in the same manner as the Korteweg - de Vries equation is generalized to the Hirota - Satsuma equation. The Gelfand - Dikii - Lax and Hamiltonian formulation for this generalization is given. The same construction is repeated for the constrained Kadomtsev - Pietviasvili - Lax operator what leads to the four component Kaup - Kupershmidt equation. The modified version of the two component Kaup - Kupershmidt equation is presented and analysed.

Introduction.

Large classes of nonlinear partial differential equations are integrable by the inverse spectral transform method and its modifications [1, 2]. It is well known that most of the integrable partial differential equations, 

$$u_t = F(t, x, u, u_x, u_{xx}, ...)$$  \hspace{1cm} (1)

admit so called Lax representation 

$$\frac{\partial L}{\partial t} = [A, L],$$  \hspace{1cm} (2)

and hence the inverse scattering method is applicable.

We shall consider the case where the Lax operator is a differential operator 

$$L = \partial^m + u_{m-2} \partial^{m-2} + ... + u_0,$$  \hspace{1cm} (3)

where $$u_i, i = 0, 1, ... m - 2$$ are functions of $$x, t$$. Then the equation (2) gives us the Gelfand - Dikii system where $$A = L^{n/m}$$ is a pseudodifferential series of the form 

$$L^{n/m} = \sum_{i=\infty}^{n} v_i \partial^i$$ and $$L^{n/m}_{\geq 0} = \sum_{i=0}^{n} v_i \partial^i$$.

Quite different systems of equations could be obtained considering the Kadomtsev - Pietviasvili hierarchy within Sato’s approach [3, 4]. In this case the Lax operator is spanned by infinitely many fields 

$$L_{KP} = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + ...$$.  \hspace{1cm} (4)
with the following Lax pair representation

$$\frac{\partial L}{\partial t} = [(L^N)_{\geq 0}, L], \quad (5)$$

Both these hierarchies describe a large classes of nonlinear partial differential equations. In order to find some interesting equations, in these hierarchies, sometime we need to apply the reduction procedure in which some functions are described in terms of other functions used in the Lax operator. We have no the unique prescription how to carry out such procedure at the moment. Kupershmidt \[5\] has noticed, that certain invariance of the partial differential nonlinear equations, can be extracted from the Lax operator. This observation allowed him to put some constraints on the functions appearing in the Lax operator. This procedure, is called now the Kupershmidt reduction \[1\].

In this paper we would like to consider some specific reduction of the Gelfand - Dikii Lax operator in which Lax operator can be factorized as the product of two Lax operators. This idea follows from the observation that the product of two Lax operators \[6\] of the Korteweg - de Vries equations

$$L = (\partial^2 + u)(\partial^2 + v) \quad (6)$$

creates the whole hierarchy of equations with the following Lax pair representation

$$\frac{\partial L}{\partial t_n} = 8\left[(L^{(2n+1)/4})_{\geq 0}, L\right], \quad (7)$$

where $n = 0, 1, 2, \ldots$ and factor 8 was chosen in such a way that to normalize the higher term in the equation. For $n = 1$ we have Hirota - Satsuma equation \[7\]

$$\begin{align*}
\frac{\partial u}{\partial t_1} &= \left( -u_{xxx} + 3v_{xx} - 6u_x u + 6v_x + 12v_u \right), \\
\frac{\partial v}{\partial t_1} &= \left( -v_{xxx} + 3u_{xx} - 6v_x v + 6v_x + 12v_v \right),
\end{align*} \quad (8)$$

while for $n = 2$

$$\begin{align*}
\frac{\partial u}{\partial t_2} &= \left( -3u_{xxxx} - 15u_{xxx} u - 15u_{xx} u_x - 15u_x u^2 + 5v_{xxxx} + \\
&\quad 25v_{xx} u + 5v_{xxx} v + 25v_{xx} u_x + 15v_{xx} v_x + 15v_x u_{xx} + \\
&\quad 20v_x u^2 + 20v_x v u + 5v^2 u_x + 5v u_{xxx} + 30v u u \right)/4, \\
\frac{\partial v}{\partial t_2} &= \left( 5u_{xxxx} + 5u_{xxx} u + 15u_{xx} u_x - 3v_{xxxx} + 5v_{xxx} u - \\
&\quad 15v_{xxx} v + 15v_{xx} u_x - 15v_{xx} v_x + 25v_x u_{xx} + 5v_x u^2 - \\
&\quad 15v_x v^2 + 30v_x v u + 20v^2 u_x + 25v u_{xxx} + 20v u u \right)/4,
\end{align*} \quad (9)$$

Let us notice that both these equations could be rewritten in the hamiltonian form as

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_n} = J \begin{pmatrix} \frac{\delta H_n}{\delta u} \\ \frac{\delta H_n}{\delta v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \partial^3 - 2u \partial - u_x \\ 0 \\ \frac{1}{2} \partial^3 - 2v \partial - v_x \end{pmatrix} \begin{pmatrix} \frac{\delta H_n}{\delta u} \\ \frac{\delta H_n}{\delta v} \end{pmatrix}, \quad (10)$$
where \( n = 1, 2 \) and
\[
H_1 = \int dx \text{ Res}(L^{3/2}) = \int dx (u^2 + v^2 - 6uv),
\]
\[
H_2 = \int dx \text{ Res}(L^{5/2}) = \int dx ((3u_{xx} + 10v_{xx})u - u^3 - 3v_{xx}v - v^3 + 5vu(v + u))
\]
and \( \text{Res} \) denotes the coefficient standing in the \( \partial^{-1} \) term.

Recently it was showed in [8] that the similar construction could be carried out for the Harry Dym equation which leads to the system of interacting equations. However the Lax operator for the Harry Dym equation does not belong to the Gelfand - Dikii system.

Both these equations could be considered either as the extensions of known equations or as the reduction of the Lax pair representations. Indeed Lax operator (6) could be considered as the admissible reduction of the fourth-order Gelfand - Dikii - Lax operator
\[
L = \partial^4 + f_2 \partial^2 + f_1 \partial + f_0,
\]
where
\[
f_2 = u + v, \quad f_1 = 2v_x, \quad f_0 = v_{xx} + vu.
\]

We would like now to repeat the similar construction for the Boussinesq type Lax operators. We choose third order Lax operator of the form
\[
L = \partial^3 + u \partial + \lambda u_x
\]
where at the moment \( \lambda \) is a free parameter.

This Lax operator generate the whole hierarchy of equations and the first non-trivial equation starts from the fifth flow
\[
\frac{\partial L}{\partial t_5} = 9\left[ (L^{5/3})_{\geq 0}, L \right],
\]
of the form
\[
u_t = \left( -u_{4x} - 5u_{xx}u + 15\lambda(\lambda - 1)u_x^2 - \frac{5}{3}u^3 \right)_x
\]
only when \( \lambda = \frac{1}{2}, 1, 0 \). Notice that the factor 9 was chosen in such a way that to normalize the higher terms in the equation.

For \( \lambda = \frac{1}{2} \) we have Kaup - Kupershmidt hierarchy [9, 10] while for \( \lambda = 1 \) or \( \lambda = 0 \) we obtain Sawada - Kotera hierarchy [11]. Both these equations are hamiltonian equations where
\[
u_t = \left( c \partial^3 + \frac{1}{15}(\partial u + u \partial) \right) \frac{\delta H}{\delta u}
\]
where
\[
H_1 = \int dx \left( 3(3\lambda^2 - 3\lambda + 1)u_x^2 - 5u^3 \right)
\]
and \( c = \frac{2}{15} \) for \( \lambda = \frac{1}{2} \) or \( c = \frac{1}{15} \) for \( \lambda = 1 \) or \( \lambda = 0 \).

Now we are prepare to consider new Lax operator as the product of two different Lax operators of the Boussinesq type
\[
L := (\partial^3 + v \partial + \lambda v_x)(\partial^3 + (u - v) \partial + \lambda (u_x - v_x))
\]

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The consistent hierarchy could be obtained only for \( \lambda = \frac{1}{2} \) and first two nontrivial flows are

\[
\frac{\partial L}{\partial t_n} = 9 \left[ (L^{\eta/6})_{\eta \geq 0}, L \right],
\]

give us

\[
v_t_3 = \frac{9}{2} \left( u_{xxxx} - 2v_{xxx} + \frac{1}{2}v_x u - 3v_x v + vu_x \right)
\]

\[
u_t_3 = \frac{9}{2} \left( -\frac{3}{4}u^2 - 3v^2 + 3uv \right)
\]

\[
n_t_3 = \left( -5u_{xxxx} + 9v_{xxxx} - \frac{5}{2}u_{xx}u - \frac{5}{2}u_x^2 + 15v_x v + \frac{15}{4}v_x^2 + \frac{5}{2}v^3 - \frac{5}{2}vu_{xx} - \frac{5}{8}vu^2 \right) - \frac{5}{2}vu_{xxxx} - \frac{5}{4}vu_x u
\]

\[
u_t_5 = \left( -u_{xxxx} + 5u_{xx}u + \frac{35}{24}u^3 - 15v_{xx}u + 30v_{xx} v + \frac{15}{2}v_x^2 - \frac{15}{2}vu_{ux} + \frac{15}{2}u^2v - 15vu_{xx}x - \frac{15}{2}vu_x^2 \right)
\]

The last system of the equations is our two component generalized Kaup - Kupershmidt equation. This system cannot be reduced to the system of equations (9) by the linear transformation.

Interestingly this two - component generalization have been considered first time in [4] where authors investigated the so called constrained Kadomtsev - Pietviashvilli (KP) hierarchy. The constrained KP hierarchy is obtained from the usual KP hierarchy as

\[
L^{N}_{KP} = (L^{N}_{KP})_{\geq 0} + \Psi \partial^{-1} \Phi
\]

with \( L_{KP} \) defined by (4). Then the equations (21,22) could be obtained choosing

\[
L_{KP} = \partial^3 + \frac{1}{2}u \partial + \frac{1}{4}u_x + \frac{1}{16}(2v - u) \partial^{-1} (2v - u)
\]

In contrast to the usual Kaup - Kupershmidt hierarchy, which starts from the fifth flow, our hierarchy begin from the third flow. Notice that our Lax operator as well as the equations allows the reduction to the standard Kaup - Kupershmidt Lax operator or equations when \( u = 2v \).

Both these systems are hamiltonians where

\[
\left( \begin{array}{c} u \\ v \end{array} \right) _{t_n} = J \frac{\delta H_n}{\delta v} = \frac{1}{216} \left( 4\partial^3 + \partial u + u \partial \ 2\partial^3 + \partial v + v \partial \ 2\partial^3 + \partial v + v \partial \right) \left( \frac{\delta H_n}{\delta v} \right)
\]

and

\[
H_3 = \int dx \text{Res}(L^{3/6}) = 54 \int dx \left( 4uv - 4v^2 - u^2 \right)
\]

\[
H_5 = \int dx \text{Res}(L^{5/6}) = \int dx \left( 7u^3 + 24u_{xx} u - (108v_{xx} - 36vu)(v - u) \right)
\]

By straightforward calculations it is easy to show that the hamiltonian operator \( J \) satisfy the Jacobi identity.
Let us now consider the following Miura transformation

\[ u = a_x, \quad v = b_x - \frac{1}{4}b^2. \]  

(27)

where \( a, b \) are functions of \( x \) and \( t \). It is easy to show that this transforms the systems of equations

\[
\begin{align*}
\frac{a_t}{a_t^3} &= \frac{1}{16} \left( -12a_x^2 - 48b_x^2 + 48b_x a_x + 24b_x b^2 - 3b^4 - 12b^2 a_x \right) \\
\frac{b_t}{b_t^3} &= \frac{1}{4} \left( 4a_{xx} - 8b_{xx} + b^2 + 2b_{ax} \right)
\end{align*}
\]

(28)

to the systems (21) or (22) respectively.

Notice that the equations (28) describe system of two interacting fields of the modified Korteweg - de Vries type. This system of equations does not belong to the class of the interacting fields considered by Foursov [12]. Foursov has classified all integrable systems of two interacting modified KdV-type equations which could be reduced to the symmetrical form

\[
\begin{align*}
\frac{u_t}{u_t^3} &= F[u, v] \\
\frac{v_t}{v_t^3} &= F[v, u],
\end{align*}
\]

(30)

where \( F[u, v] = F[u, u_x, u_{xx}, \ldots v_x, v_{xx}, \ldots] \) denotes differential polynomial function of two variables. However our system of equations (24) cannot be reduced to the symmetrical form by the linear transformation.

Interestingly the system (28) collapses when \( u = 2v \). Indeed the condition \( u = 2v \) is equivalent with the assumption that

\[ a_x = 2b_x - \frac{1}{2}b^2 \]

(31)

and therefore we have \( a_{t3} = 0 \). The system of equation (29) reduces when \( u = 2v \) to the modified version of the Kaup - Kupershmidt equation

\[
\frac{b_t}{b_t^5} = \frac{1}{16} \left( -16b_{xxxx} - 40b_{xx}b_x + 2b_{xx} b^2 + 20b_x b^2 - b^5 \right)
\]

(32)

Our equations (28) and (29) are hamiltonians equations where

\[
\left( \begin{array}{c}
\dot{a} \\
\dot{b}
\end{array} \right)_{\text{tn}} = D \left( \begin{array}{c}
\frac{\delta H_n}{\delta a} \\
\frac{\delta H_n}{\delta b}
\end{array} \right) = \frac{1}{2} \left( \begin{array}{cc}
-4\partial - (\partial^{-1}a_x - a_x \partial^{-1}) & -2\partial - \partial^{-1}b_x + b \\
-2\partial - b - b_x \partial^{-1} & -2\partial
\end{array} \right) \left( \begin{array}{c}
\frac{\delta H_n}{\delta a} \\
\frac{\delta H_n}{\delta b}
\end{array} \right),
\]

(33)
where \( n = 3, 5 \) and
\[
H_3 = \int dx \left( \frac{1}{2} a_{xx} a - 2 b_{xx} a + 2 b_{xx} b + b_x b a - \frac{1}{8} b^4 \right) \quad (34)
\]
\[
H_5 = \int dx \left( 24 a_{xxxx} a + 14 a_{xx} a a - 108 b_{xxxx} (a - b) + 54 b_{xxx} b b + b_{xx} (234 b_x a - 36 a_x a - 108 b_x b - 18 b^2 a) + b_x (b b^3 a - 36 a_x a + 18 b a a) + 9 b^2 a_{xx} a \right) \quad (34)
\]

It is easy to check that the operator \( \mathcal{D} \) is the Hamiltonian operator. Indeed it is enough to notice that under the Miura transformation (27) this operator transforms to the \( \hat{J} = \mathcal{F} \mathcal{D} \mathcal{F}^{*} \) where \( \mathcal{F} \) is the Freche derivative of Miura transformation and \( * \) denotes the hermitian conjugation.

\[
\hat{J} = \left( \begin{array}{cc} \partial & 0 \\ 0 & -\partial - \frac{1}{2} b \end{array} \right) \quad (35)
\]

Let us apply finally the factorization procedure directly to constrained Kadomtsev - Petviashvili - Lax operator. We consider therefore two different Lax operators
\[
L_1 = \partial^3 + v \partial + \frac{1}{2} v_x + h \partial^{-1} h, \quad (36)
\]
\[
L_2 = \partial^3 + (u - v) \partial + \frac{1}{2} (u_x - v_x) + g \partial^{-1} g,
\]
and construct new Lax operator as
\[
L = L_1 L_2. \quad (37)
\]

This Lax operator generate the integrable hierarchy of four interacting fields. The first nontrivial equations are
\[
u_{t_5} = \frac{9}{2} \left( 6 g^2 + 6 h^2 - \frac{3}{2} u^2 + 6 v u - 6 v^2 \right)_x
\]
\[
u_{t_5} = \frac{9}{2} \left( 12 h h_x + 2 u_{xxx} - 4 v_{xxx} + v_x u - 6 v_x v + 2 v v_x \right)
\]
\[
u_{t_5} = \frac{9}{2} \left( 2 g_{xxx} - u_x g + u g_x + 3 v_x g \right)
\]
\[
u_{t_5} = \frac{9}{2} \left( 2 h_{xxx} + 2 u_x h + u h_x - 3 v_x h \right)
\]

\[
u_{t_5} = \left( 60 g_{xxx} g + 15 g_x g + 60 h_{xxx} h + 15 h_x h - u_{xxxx} + 5 u_{xx} u + \frac{35}{24} u^3 
\]
\[
- \frac{15}{2} u g^2 + \frac{75}{2} u h^2 - 15 v_x u u + 30 v_{xxx} v + \frac{15}{2} v^2 - \frac{15}{2} v_x u x + \frac{15}{2} v^2 u 
\]
\[
+ 45 u v^2 - 45 v^2 h - 15 v u_{xx} - \frac{15}{2} v u^2 \right)_x,
\]
\[
u_{t_5} = 30 g_{xxx} g + 90 g_x g_x + 30 h_{xxx} h - 5 u_{xxxx} - \frac{5}{2} u_{xxx} u - \frac{15}{2} u_{xx} u_x + 30 u_x h^2 +
\]

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The last system of equations could be considered as the four-component generalized Kaup-Kupershmidt equation. This equation reduces to the two-component Kaup-Kupershmidt equation when $g = h = 0$.

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