Quantum Games and Quantum Discord

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Abstract

We quantize prisoners dilemma and chicken game by our generalized quantization scheme to explore the role of quantum discord in quantum games. In order to establish this connection we use Werner-like state as an initial state of the game. In this quantization scheme measurement can be performed in entangled as well as in product basis. For the measurement in entangled basis the dilemma in both the games can be resolved by separable states with non-zero quantum discord. Similarly for product basis measurement the payoffs are quantum mechanical only for nonzero values of quantum discord.

1 Introduction

Entanglement is a key resource in quantum information theory. If used as a resource it can perform numerous tasks which seem rather impossible for classical resources and shared randomness. Quantum game theory is no exception in this regard where entanglement plays vital role in the solution of games. The first step to this direction was by Eisert et al. [1] who introduced an elegant scheme for the quantization of prisoner dilemma (PD). In this scheme the strategy space of the players is a two parameter set of $2 \times 2$ unitary matrices. Starting with maximally entangled initial state the authors showed that for a suitable quantum strategy the dilemma disappeared. They also pointed out a quantum strategy which always won over all the classical strategies. Later on Marinatto and Weber [2] introduced another interesting and simple scheme for the analysis of non-zero sum classical games in quantum domain. They gave Hilbert structure to the strategic spaces of the players. They also used maximally entangled

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initial state and allowed the players to play their tactics by applying probabilistic choice of unitary operators. They applied their scheme to an interesting game of Battle of Sexes and found out the strategy for which both the players can achieve equal payoffs. Both these quantization schemes gave interesting results for various quantum analogues of classical games [3, 4, 5, 6, 7, 8, 9]. In all these cases entanglement played a crucial role. However, it has recently been shown that all the non classical properties of quantum correlations can not be analyzed by entanglement only [10, 11]. Instead of entanglement quantum discord (QD) is introduced as a feature of quantumness of correlations that remains there even for separable states. Since then various aspects of QD has been studied [12, 13, 14, 15].

In this paper we explore the role of QD in quantization of games. To elaborate this role we start our game with Werner-like states which have very interesting proprieties. These states are linear combination of a maximally entangled and a maximally mixed state [16, 17, 18, 19]. Their entanglement and nonlocality depends upon a parameter \(0 \leq p \leq 1\) that parameterizes them. For \(0 < p \leq \frac{1}{3}\) they are separable, for \(\frac{1}{3} < p \leq \frac{1}{\sqrt{2}}\) entangled but not nonlocal and for the range \(\frac{1}{\sqrt{2}} < p < 1\) they become inseparable and nonlocal [10]. In the recent years these states have been investigated from different perspective like entanglement teleportation via Werner states [20], entanglement of Werner derivatives [21], Bell violation and entanglement of Werner states of two qubits in independent decay channels [22] and their application in ancilla assisted process tomography [23]. It has also been reported that despite being nonlocal, for certain range of parameter \(p\), when shared between two parties these states are a powerful resource in comparison to classical randomness [24]. In Ref. [11] the QD for these states has been found and shown positive for all \(p > 0\). This behavior of these states is in contrast with their well known separability at \(p \leq \frac{1}{3}\). Here we quantize PD and chicken game (CG) using our generalized quantization scheme taking Werner-like state as an initial quantum state [25]. In this scheme measurement can be performed in both entangled and product basis. For the measurement in entangled basis our results show that the strategy pair \((Q, Q)\) remains Nash equilibrium for both these games as in the quantization scheme of Eisert et al. [1] for all values of \(p > 0\). It is interesting to note that for \(p \leq \frac{1}{4}\) unentangled quantum state with non zero QD is capable of resolving dilemmas in PD and CG. This shows that QD also has a crucial role in quantum games. For \(p = 0\) when QD becomes zero then the payoffs become constant and independent of the players strategies. For the second case when the measurement is performed in product basis the payoffs remain quantum mechanical as in Marinatto and Weber scheme only for \(p > 0\) i.e. for nonzero values of quantum discord [2, 26].

This paper is organized as follows; in sections (2) and (3) we give a brief introduction to the classical versions of PD and CG respectively, section (4) deals with quantum discord [11] and its role in quantum games and section (5) concludes the results.
2 Prisoners’ Dilemma

This game starts with a story of two suspects, say Alice and Bob, who have committed a crime together. Now they are being interrogated in a separate cell. The two possible moves for each player are to cooperate (C) or to defect (D) without any communication between them according to the following payoff matrix

\[
\begin{array}{cc}
\text{Bob} & C & D \\
\text{Alice} & C & (3,3) & (0,5) \\
& D & (5,0) & (1,1)
\end{array}
\]

(1)

It is clear from the above payoff matrix that D is the dominant strategy for both players. Therefore, rational reasoning forces each player to play D. Thus (D, D) results as the Nash equilibrium of this game with payoffs (1, 1), which is not Pareto Optimal. However, it was possible for the players to get higher payoffs if they would have played C instead of D. This is the origin of dilemma in this game.

3 Chicken Game

The payoff matrix for this game is

\[
\begin{array}{cc}
\text{Bob} & C & D \\
\text{Alice} & C & (3,3) & (1,4) \\
& D & (4,1) & (0,0)
\end{array}
\]

(2)

In this game two players drove their cars straight towards each other. The first to swerve to avoid a collision is the loser (chicken) and the one who keeps on driving straight is the winner. There is no dominant strategy in this game. There are two Nash equilibria (C, D) and (D, C), the former is preferred by Bob and the latter is preferred by Alice. The dilemma of this game is that the Pareto Optimal strategy (C, C) is not NE.

4 Quantum Discord and Quantum Games

The Shannon entropy of a discrete variable \(X\) with discrete probability distribution \(p_x\) is defined as

\[
H(X) = -\sum_x p_x \log p_x.
\]

(3)

The conditional entropy of \(X\) given \(Y\) is the measure of the amount of uncertainty about \(X\) given the value of \(Y\). Mathematically it is written as

\[
H(X \mid Y) = -\sum_{x,y} p(x, y) \log p(x \mid y)
\]

(4)
where \( p(x, y) \) is the joint probability distribution of the random variable \( X \) and \( Y \) and \( p(x \mid y) \) is the conditional probability of the occurrence of \( X \) when \( Y \) has already occurred. The correlation between two random variables \( X, Y \) with probability distributions \( p_x \) and \( p_y \) respectively is called mutual information. Mathematically it takes the form

\[
I(X : Y) = H(X) + H(Y) - H(X, Y).
\]

(5)

where \( H(X, Y) \) is joint entropy that measures the average uncertainty of the pair \((X, Y)\).

Quantum analogue of Shannon entropy for a quantum system in state \( \rho \) is von Neumann entropy which is given as

\[
S(\rho) = -\text{Tr}(\rho \log \rho)
\]

(7)

that leads to the mutual information relation for state \( \rho_{XY} \) to be written as

\[
I(\rho_{XY}) = S(\rho_X) + S(\rho_Y) - S(\rho_{XY})
\]

(8)

But similar generalization for Eq. (6) in quantum domain is not straightforward. This is because that quantum conditional entropy \( S(\rho_X \mid \rho_Y) \) requires to specify the state of system \( \rho_X \) given the state of \( \rho_Y \). This statement in quantum mechanics is ambiguous until measurement operators \( \Pi^Y_i \) for state \( \rho_Y \) are defined. If the measurement is performed using operators \( \Pi^Y_i \) then Eq. (6) in quantum domain takes the form

\[
J(\rho_{XY}) = S(\rho_X) - S(\rho_X \mid \Pi^Y_i)
\]

(9)

where

\[
S(\rho_X \mid \Pi^Y_i) = \sum_i p_i S(\rho_{X \mid \Pi^Y_i})
\]

(10)

Quantum discord is defined as [11]

\[
D(X : Y) = I(\rho_{XY}) - J(\rho_{XY})
\]

(11)

that with the help of Eqs. (8) and (9) becomes

\[
D(X : Y) = S(\rho_X) - S(\rho_{XY}) + S(\rho_X \mid \Pi^Y_i).
\]

(12)

For two qubit Werner like state [16] of the form

\[
\rho_{\text{in}} = p |\phi^+\rangle \langle \phi^+| + \frac{(1 - p)}{4} I \otimes I
\]

(13)

where \( |\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \) is standard Bell state, the quantum discord is shown below [11]. It is clear from this graph that QD is greater than zero even for
Entangled
Separable

Dilemma is resolved in Eisert et a. case

Figure 1: Quantum Discord for Werner states

separable states i.e. for $p < \frac{1}{3}$.

Next we quantize PD and CG taking Werner like state (13) as an initial quantum state of the game with a payoff matrix of the form

$$
\begin{array}{cc}
\text{Bob} & B_1 \\
\text{Alice} & A_1 \left[ \begin{array}{cc}
(A_0^A, B_1^B) & (A_1^A, B_0^B) \\
(A_1^A, B_1^B) & (A_0^A, B_0^B) \\
\end{array} \right] \\
& A_2 \\
\end{array}
$$

(14)

The strategy of each of the players is represented by the unitary operator $U_i$ given as

$$
U_i = \cos \frac{\theta_i}{2} R_i + \sin \frac{\theta_i}{2} C_i,
$$

(15)

where $i = 1$ or $2$ and $R_i, C_i$ are the unitary operators defined as

$$
\begin{align*}
R_i |0\rangle &= e^{i\phi_i} |0\rangle, & R_i |1\rangle &= e^{-i\phi_i} |1\rangle, \\
C_i |0\rangle &= -|1\rangle, & C_i |1\rangle &= |0\rangle.
\end{align*}
$$

(16)

Here we restrict our treatment to two parameter set of strategies for mathematical simplicity in accordance with Ref. [1]. After the application of the strategies, the initial state given by Eq. (13) transforms into

$$
\rho_f = (U_1 \otimes U_2) \rho_{in} (U_1 \otimes U_2)^\dagger.
$$

(17)

The payoff operators for Alice and Bob are

$$
\begin{align*}
P^A &= A_{00}^A P_{00} + A_{11}^A P_{11} + A_{01}^A P_{01} + A_{10}^A P_{10}, \\
P^B &= B_{00}^B P_{00} + B_{11}^B P_{11} + B_{01}^B P_{01} + B_{10}^B P_{10},
\end{align*}
$$

(18)
where

\[ P_{00} = |\psi_{00}\rangle \langle \psi_{00}|, \quad |\psi_{00}\rangle = \cos \frac{\delta}{2} |00\rangle + i \sin \frac{\delta}{2} |11\rangle, \quad (19a) \]

\[ P_{11} = |\psi_{11}\rangle \langle \psi_{11}|, \quad |\psi_{11}\rangle = \cos \frac{\delta}{2} |11\rangle + i \sin \frac{\delta}{2} |00\rangle, \quad (19b) \]

\[ P_{10} = |\psi_{10}\rangle \langle \psi_{10}|, \quad |\psi_{10}\rangle = \cos \frac{\delta}{2} |10\rangle - i \sin \frac{\delta}{2} |01\rangle, \quad (19c) \]

\[ P_{01} = |\psi_{01}\rangle \langle \psi_{01}|, \quad |\psi_{01}\rangle = \cos \frac{\delta}{2} |01\rangle - i \sin \frac{\delta}{2} |10\rangle, \quad (19d) \]

with \( \delta \in [0, \frac{\pi}{2}] \) being the entanglement of the measurement basis. Above payoff operators reduce to that of Eisert’s scheme for \( \delta \) equal to \( \gamma \), which represents the entanglement of the initial state [1]. For \( \delta = 0 \) above operators transform into that of Marinatto and Weber’s scheme [2]. The payoffs for the players are calculated as

\[
\begin{align*}
A(\theta_1, \phi_1, \theta_2, \phi_2) &= \text{Tr}(P_A \rho_f), \\
B(\theta_1, \phi_1, \theta_2, \phi_2) &= \text{Tr}(P_B \rho_f),
\end{align*}
\]

where \( \text{Tr} \) represents the trace of a matrix. Using Eqs. (17), (18) and (20) the payoffs for players \( j = A, B \) are obtained as

\[
\begin{align*}
\mathcal{A}(\theta_1, \phi_1, \theta_2, \phi_2) &= \mathcal{A}_{00} \text{Tr}(P_{00} \rho_f) + \mathcal{A}_{01} \text{Tr}(P_{01} \rho_f) + \mathcal{A}_{10} \text{Tr}(P_{10} \rho_f) + \mathcal{A}_{11} \text{Tr}(P_{11} \rho_f) \\
\mathcal{B}(\theta_1, \phi_1, \theta_2, \phi_2) &= \mathcal{B}_{00} \text{Tr}(P_{00} \rho_f) + \mathcal{B}_{01} \text{Tr}(P_{01} \rho_f) + \mathcal{B}_{10} \text{Tr}(P_{10} \rho_f) + \mathcal{B}_{11} \text{Tr}(P_{11} \rho_f)
\end{align*}
\]

where we have defined

\[
\begin{align*}
\text{Tr}(P_{00} \rho_f) &= p \left\{ 1 - \sin^2 (\phi_1 + \phi_2) \sin \delta \right\} \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \\
&\quad \left\{ \cos^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta_2}{2} - \frac{1}{2} \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) \right\} - \\
&\quad \frac{(\sin \delta - 1)}{2} \left\{ \cos^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta_2}{2} - \frac{1}{2} \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) \right\} - \\
&\quad \frac{\sin \delta}{2} + \frac{1 + p}{4},
\end{align*}
\]

\[
\begin{align*}
\text{Tr}(P_{01} \rho_f) &= p \left\{ 1 + \cos 2 \phi_1 \sin \delta \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + 1 - \cos 2 \phi_2 \sin \delta \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \\
&\quad (1 + \sin \delta) \sin \phi_1 \cos \phi_2 - (1 + \sin \delta) \sin \phi_1 \sin \phi_2 \sin \theta_1 \sin \theta_2 \right\} + \frac{1 - p}{4} \\
\text{Tr}(P_{10} \rho_f) &= p \left\{ 1 - \cos 2 \phi_1 \sin \delta \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + 1 + \cos 2 \phi_2 \sin \delta \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \\
&\quad (1 + \sin \delta) \sin \phi_1 \cos \phi_2 + (1 - \sin \delta) \sin \phi_1 \sin \phi_2 \sin \theta_1 \sin \theta_2 \right\} + \frac{1 - p}{4}
\end{align*}
\]
\[
\text{Tr}(P_{11}\rho_f) = p \left[ \{1 - \cos^2(\phi_1 + \phi_2)\sin\delta\} \cos^2\frac{\theta_1}{2} \cos^2\frac{\theta_2}{2} + \\
\frac{(\sin\delta + 1)}{2} \left\{ \sin^2\frac{\theta_1}{2} \sin^2\frac{\theta_2}{2} + \frac{1}{2} \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) \right\} \right] + \\
+ \frac{1 - p}{4}
\]

(25)

In the framework of our generalized quantization scheme [25] measurement can be performed either using entangled basis \((\delta = \frac{\pi}{2})\) or product basis \((\delta = 0)\). For the measurement in entangled basis with the help of Eq. (21) the payoffs for players become

\[
\$_j(\theta_1, \phi_1, \theta_2, \phi_2) = p \left[ \$^i_{00} \left( \cos^2(\phi_1 + \phi_2) \cos^2\frac{\theta_1}{2} \cos^2\frac{\theta_2}{2} \right) + \\
\$^i_{01} \left( \cos \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \sin \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + \\
\$^i_{10} \left( \sin \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \cos \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + \\
\$^i_{11} \left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin (\phi_1 + \phi_2) + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^2 \\
+ \frac{(1 - p)}{4} (\$^i_{00} + \$^i_{01} + \$^i_{10} + \$^i_{11}) \right]
\]

(26)

For PD with the payoff matrix (1) the above equation reduce to

\[
\$^A(\theta_1, \phi_1, \theta_2, \phi_2) = p \left[ 3 \left( \cos^2(\phi_1 + \phi_2) \cos^2\frac{\theta_1}{2} \cos^2\frac{\theta_2}{2} \right) + \\
5 \left( \sin \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \cos \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + \\
\left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin (\phi_1 + \phi_2) + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^2 \right] + \\
\frac{9}{4} (1 - p)
\]

(27)

\[
\$^B(\theta_1, \phi_1, \theta_2, \phi_2) = p \left[ 3 \left( \cos^2(\phi_1 + \phi_2) \cos^2\frac{\theta_1}{2} \cos^2\frac{\theta_2}{2} \right) + \\
5 \left( \cos \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \sin \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + \\
\left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin (\phi_1 + \phi_2) + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right)^2 \right] + \\
\frac{9}{4} (1 - p)
\]

(28)
For $p = 1$ the above results reduce to that of Eisert et al. \[1\] and the dilemma in game is resolved for players strategies $U(\theta_1, \phi_1, \theta_2, \phi_2) = U(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) = Q$ with $A(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) = B(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) = (3, 3)$. Next we see whether the strategy $Q$ is NE for $p \neq 1$. In this case the NE conditions

$$
A(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) - A(\theta_1, \phi_1, 0, \frac{\pi}{2}) \geq 0
$$

$$
B(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) - B(0, \frac{\pi}{2}, \theta_2, \phi_2) \geq 0
$$

(29)
give

$$
p \left(3 \sin^2 \frac{\theta_1}{2} + 2 \cos^2 \frac{\theta_1}{2} \cos^2 \phi_1 \right) \geq 0.
$$

(30)

The above inequality is satisfied for all values of $p \geq 0$ showing that the strategy pair $(Q, Q)$ continues to be Nash equilibrium for all values of $p > 0$. It shows that although state \[13\] is not entangled for $p \leq \frac{1}{3}$ yet when shared between two players it is proved to be a better resource as compared to classical randomness. On the other hand at $p = 0$ when the initial state becomes maximally mixed the payoffs become $\frac{9}{4}$ irrespective of players strategies.

For CG with payoff matrix given by payoff matrix \[2\] the payoffs given in Eq. \[26\] become

$$
A(\theta_1, \phi_1, \theta_2, \phi_2) = p \left[ 3 \left( \cos^2 (\phi_1 + \phi_2) \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) + 
\left( \cos \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \sin \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + 
4 \left( \sin \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \cos \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 \right] + 
2 (1 - p)
$$

(31)

$$
B(\theta_1, \phi_1, \theta_2, \phi_2) = p \left[ 3 \left( \cos^2 (\phi_1 + \phi_2) \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) + 
\left( \sin \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \cos \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 + 
4 \left( \cos \phi_1 \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \sin \phi_2 \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)^2 \right] + 
2 (1 - p)
$$

(32)

With the help of Eqs. \[26\] the strategy pair $U(\theta_1, \phi_1, \theta_2, \phi_2) = U(0, \frac{\pi}{2}, 0, \frac{\pi}{2})$ will be NE of this game if

$$
p \left[ 2 + \cos^2 \frac{\theta_1}{2} (3 \cos^2 \phi_1 - 2) \right] \geq 0.
$$

(33)
The above condition is satisfied for all values of $p \geq 0$. It means that dilemma can be resolved in CG when the players share the state (13) with $p > 0$. Furthermore it can be investigated by Eqs. (31, 32) that for $p = 0$ the payoffs of the players become 2, independent of players decisions.

Comparing our results with Fig. (1) we see that for all values of quantum discord greater than zero there is no dilemma in both PD and CG. Therefore it may be safe to conclude that when Werner states are used as initial state for a quantum game it is the quantum discord the helps resolve the dilemmas.

For the measurement performed in product basis (i.e., $\delta = 0$ in Eqs. (19a to 19d) ) the Eq. (21) reduces to

$$
S_j(\theta_1, \phi_1, \theta_2, \phi_2) = \frac{p}{2} \left( \left( S_{00}^j + S_{11}^j \right) \left\{ \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1}{2} \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) \right\} + \left( S_{01}^j + S_{10}^j \right) \left\{ \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1}{2} \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) \right\} \right) + (1 - p) \frac{1}{4} \left( S_{00}^j + S_{01}^j + S_{10}^j + S_{11}^j \right)
$$

(34)

For $p > 0$ the above payoffs remain equivalent to the payoffs obtained by Marinatto and Weber’s quantization scheme where the players also have the option to manipulate the phase $\phi$ of the given qubit [2, 26]. However at $p = 0$ when the quantum discord disappears the payoffs given by Eq. (34) become average value of the entries of payoff matrix (14).

5 Conclusion

In this paper we quantized PD and CG by our generalized quantization scheme taking a Werner-like state as an initial quantum state [25] to explore the role of QD in quantum games. Generalized quantization scheme allows measurements in both entangled and product basis. For the entangled basis measurement we showed that the dilemma in both PD and CG can be resolved for all non-zero values of QD. For the case of product basis measurement the payoffs remain quantum mechanical only for nonzero quantum discord i.e. for $p > 0$. However at $p = 0$ where quantum discord disappears from the initial quantum state the payoffs become constant and independent of players strategies.

References

[1] J. Eisert, M. Wilkens, M. Lewenstein, Quantum games and quantum strategies, Phys. Rev. Lett. 83, 3077 (1999).

[2] L. Marinatto and T. Weber, A quantum approach to static games of complete information, Phys. Lett. A 272, 291 (2000) or quant-ph/0004081.
[3] Flitney and Abbott, Advantage of quantum player against a classical in 2 × 2 quantum games, Proc. Royal Soc. (London) A, 459, 2463 (2003).

[4] A. Iqbal and A. H. Toor, Quantum mechanics gives stability to a Nash equilibrium, Phys. Rev. A 65, 022306 (2002) or quant-ph/0104091.

[5] A. Iqbal and A. H. Toor, Backwards-induction outcome in a quantum game, Phys. Rev. A 65, 052328 (2002) or quant-ph/0111090.

[6] A. Iqbal and A. H. Toor, Evolutionarily stable strategies in quantum games, Phys. Lett. A 280, 249 (2001).

[7] J. Du, H. Li, X. Xu, X. Zhou and R. Han, Phase transition-like behavior of quantum games, J. Phys. A 36, 6551 (2003) or quant-ph/0111138.

[8] A. F. H. Rosero, Ph.D. thesis “Classification of quantum symmetric non zero-sum 2 × 2 games in the Eisert scheme” quant-ph/0402117.

[9] Ahmad Nawaz and A. H. Toor, Dilemma and quantum battle of sexes, J. Phys. A: Math. Gen. 37, 4437 (2004).

[10] L. Henderson and V. Vedral, J. Phys. A: Math. Gen., 34, 6899 (2001).

[11] H. Ollivier and W. Zurek, Phys. Rev. Lett., 88, 017901 (2001).

[12] M. Piani, P. Horodecki, and R. Horodecki, Phys. Rev.Lett., 100, 090502 (2008); C. A. RodrigueRosario, K. Modi, A.-M. Kuah, A. Shaji, and E. C. G. Sudarshan, J. Phys. A: Math. Theor., 41, 205301 (2008); K. Modi, T. Paterek, W. Son, V. Vedral, and M. Williamson, Phys. Rev. Lett., 104, 080501 (2010); P. Giorda and M. G. A. Paris, ibid., 105, 020503 (2010); G. Adesso and A. Datta, ibid., 105, 030501 (2010); B. Daki c, V. Vedral, and C. Brukner, arXiv:1004.0190 (2010); A. Datta, arXiv:1003.5256 (2010).

[13] A. Ferraro, L. Aolita, D. Cavalcanti, F. M. Cucchietti, and A. Acin, Phys. Rev. A, 81, 052318 (2010).

[14] T. Werlang, S. Souza, F. F. Fanchini, and C. J. Villas Boas, Phys. Rev. A, 80, 024103 (2009); J. Maziero, L. C. Celeri, R. M. Serra, and V. Vedral, ibid., 80, 044102 (2009); L. Mazzola, J. Piilo, and S. Maniscalco, Phys. Rev. Lett., 104, 200401 (2010); F. F. Fanchini, T. Werlang, C. A. Brasil, L. G. E. Arruda, and A. O. Caldeira, Phys. Rev. A, 81, 052107 (2010); J.-S. Xu et al., Nat. Comm., 1, 7 (2010).

[15] A. Datta, A. Shaji, and C. Caves, Phys. Rev. Lett. 100, 050502 (2008); B. P. Lanyon, M. Barbieri, M. P. Almeida, and A. G. White, ibid., 101, 200501 (2008).

[16] R. F. Werner, Phys. Rev. A 40, 4277 (1989).

[17] W. J. Munro, D.F.V. James. A.G. White, P.G. Kwiat, Phys. Rev. A 64, 030302(R) (2001).
[18] S. Ghosh, G. Kar1, A. Sen (De) and U. Sen, Phys. Rev. A 64, 044301 (2001).

[19] T. C. Wei1, K. Nemoto, P. M. Goldbart, P. G. Kwiat, W. J. Munro and F. Verstraete, Phys. Rev. A 67, 022110 (2003).

[20] J. Lee and M. S. Kim, Phys. Rev. Lett. 84, 4236 (2000).

[21] T. Hiroshima and S. Ishizaka, Phys. Rev. A 62, 044302 (2000).

[22] A. Miranowicz, Phys. Lett. A 327, 272 (2004).

[23] J. B. Altepeter, Phys. Rev. Lett. 90, 193601 (2003).

[24] Preskil, Lecture notes, www.theory.caltech.edu/people/preskill/ph229/

[25] Ahmad Nawaz and A. H. Toor, J. Phys. A: Math. Gen. 37, 11457-11463 (2004)

[26] Y. J. Ma, G. L. Long, F. G. Deng, F. Lee and S. X. Zhang, Cooperative three and four levels games, Phys. Lett. A, 301, 117 (2002).