ON FLAT CONNECTIONS INDUCED OVER COVERING MAPS

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Abstract

Flat connections induced over covering maps are studied and the trivial ones among them are described. In the sequel, we deal with the resulting holonomy bundles.

Key words: Flat connection, covering manifold, holonomy bundles.

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0 Introduction.

Let \( P, M \) be connected paracompact smooth manifolds and suppose that \( P(M, G) \) is a principal fiber bundle, where the Lie group \( G \) acts smoothly from the left on the total space \( P \). Given the flat connection \( \omega \) on \( P(M, G) \) and a regular covering manifold \((\tilde{M}, q, M)\) of \( M \), a flat connection \( \tilde{\omega} \) is constructed on the bundle \( \tilde{P}(\tilde{M}, G) \), induced over \( \tilde{M} \). Furthermore, over the universal covering manifold \( \tilde{M} \) the induced connection is trivial, i.e., it is (isomorphic to) the flat connection on the product bundle \( \tilde{P} = G \times \tilde{M} \), [2].

In this work we are looking for all those coverings, where the induced connection is trivial. In this respect, we compute the holonomy group of an induced connection in terms of the holonomy morphism defined by the given connection and the fundamental group of the covering. To be more precise, in Section 1 we prove the following.

\footnote{This work is in final form and no part of this will publish elsewhere.}
Theorem 0.1 Let $h_\omega : \pi_1(M) \to G$ denote the holomony homorphism for the flat connection $\omega$ of the principal bundle $P(M,G)$ and $\pi_1(\hat{M})$ be the fundamental group of the covering $(\hat{M}, q, M)$. Then, for the induced by $q$ connection $\hat{\omega}$, we have the formula $\text{Im}(h_\hat{\omega}) \cong (h_\omega|\pi_1(\hat{M}))$.

Corollary 0.2 The induced connection is trivial if and only if $\pi_1(\hat{M}) \subseteq \text{Ker}_h\omega$.

In Section 2 we deal with certain coverings inducing non trivial connections and study the corresponding holomony bundles. We prove that if $\text{Ker}_h\omega \subseteq \pi_1(\hat{M})$, all these bundles have the same total space, namely that of $P(M,G)$ (cf. 2.2).

1 The holomony morphism for the induced bundles.

We first formulate a preparatory result concerning bundles induced over covering projections.

Lemma 1.1 Let $(E, p, B, F)$ be a (locally trivial) fiber bundle with connected fiber, $(B_1, f, B)$ a covering space and $(E_1, p, B_1, F)$ denote the bundle induced by $f$. Then the induced map $f_\xi : E_1 \to E$ is a covering map.

Proof. Using standard notation (see [1], Chapter 2), we have the following commutative diagram.

\[
\begin{array}{ccc}
E_1 & \xrightarrow{f_\xi} & E \\
p_1 \downarrow & & \downarrow p \\
B_1 & \xrightarrow{f} & B
\end{array}
\]

Let $V$ be an evenly covered neighbourhood in $B$ which is trivializable with respect to $p$, i.e., $f^{-1}(V) = \bigcup V_i$ and $p^{-1}(V) \cong F \times V$, where the union is discrete. In view of the above diagram, we have $f_\xi^{-1}(p^{-1}(V)) = p_1^{-1}(\bigcup V_i) = \bigcup p_1^{-1}(V_i)$. Therefore, the open set $W = p^{-1}(V)$ is the desired evenly covered neighbourhood for $f_\xi$.

We now come on our main object of study and fix first some notation. Given the principal bundle $P(M,G) = (P, p, M, G)$ and the covering...
it follows from Lemma 1.1 that the total space $\hat{P}$ is a covering space of $P$. Denoting by $\hat{p}$ and $\hat{q}$ the corresponding projections, we have the following commutative diagram.

$$
\begin{array}{ccc}
G & \rightarrow & \hat{P} \\
\hat{\varphi} & \downarrow & \downarrow \hat{q} \\
G & \rightarrow & P \\
\end{array}
$$

If $\omega$ is a flat connection on $P(M,G)$ and $\hat{\omega}$ its lifting on $\hat{P}(\hat{M},G)$, then $\hat{\omega}$ is also flat and $\hat{q}$ preserves the horizontality defined by these connections. Let $N(x_o)$ be the maximal horizontal leaf of $\omega$ through a basic point $x_o \in P$ and $\hat{x}_o \in q^{-1}(x_o)$. Then $N(\hat{x}_o)$ is a covering space of $N(x_o)$, the restriction $p|N(x_o) : N(x_o) \rightarrow G/P \cong M$ is a covering map and $N(x_o)$ is the total space for the holonomy bundle of $\omega$. For further terminology see [2], [3]; Chapter 4 and [4] where proofs of these facts and exact references are given. In the sequel base points, if not necessary, would be ommited.

**Proposition 1.2** Let $h_{\omega}$ (resp. $h_{\hat{\omega}}$) be the holonomy morphism of $\omega$ (resp. $\hat{\omega}$). Then, $h_{\hat{\omega}} = h_{\omega} \circ q_{\#}$, where $q_{\#} = q_{\#,x_o} : \pi_1(\hat{M},\hat{\varphi}(\hat{x}_o)) \rightarrow \pi_1(M,p(x_o))$ is the induced by $q$ morphism, at the fundamental group level.

**Proof.** Let $w$ be a path in $\hat{M}$ closed at $\hat{\varphi}(\hat{x}_o)$ and $r = q \circ w$. If $\overline{w}$ denotes the horizontal lifting of $w$ starting at $\hat{x}_o$, then there exists exactly one $g_w \in G$ such that $\overline{w}(1) = h_{\hat{\omega}}(\overline{w}) = g_w \hat{x}_o$. Using similar notation, we have $\overline{r}(1) = h_{\omega}(r) = g_r x_o$. We claim that $g_r = g_w$. Indeed, since $\hat{q}$ preserves horizontality, the path $\hat{q} \circ \overline{w}$ is also the horizontal lifting of $r$ starting at $x_o$. Therefore,

$$
\hat{q}(g_w \hat{x}_o) = (\hat{q} \circ \overline{w})(1) = \overline{r}(1) = g_r x_o.
$$

Because $\hat{q}$ also preserves the orbit parametrization and the fibration is principal, it follows that $g_w x_o = g_r x_o$ and finally $g_w = g_r$, as required.

As $q_{\#}$ is a monomorphism, the proof of the Theorem follows from Prop. 1.2, as well as the Corollary too. The further discription of $\hat{P}(\hat{M},G)$ is carried out below.
2 The holonomy bundles for certain induced connections.

We now proceed further and deal with the resulting holonomy bundles over certain coverings of $M$. Our previous notation is still in force.

**Proposition 2.1** If for the regular covering $(\hat{M}, q, M)$ the relation $\text{Ker}h_{\omega} = \pi_1(\hat{M})$ holds, then the flat connections $\omega$ and $\hat{\omega}$ have the same total space for their holonomy bundles, i.e., $N(x_o) \cong N(\hat{x}_o)$.

**Proof.** As $N(\hat{x}_o) \cong \hat{M}$ and it is a covering space of $N(x_o)$, it is enough to prove that they have isomorphic fundamental groups. Let $z$ be a horizontal path closed at $x_o$, $\tilde{z}$ its lift with respect to the covering $(N(\hat{x}_o), \hat{\omega}N(\hat{x}_o), N(x_o))$ starting from $\hat{x}_o$ and suppose that $\tilde{z}(1) \neq \tilde{z}(0)$. Since $\hat{P}(\hat{M}, G)$ is the product bundle and has trivial holonomy, the path $\hat{p} \circ \hat{z}$ is not closed in $\hat{M}$, while $(q \circ \hat{p}) \circ \hat{z} = p \circ z$ is closed in $M$. Because $(\hat{M}, q, M)$ is a covering, it follows that the homotopy class $[p \circ z]$ is not contained in $\text{Ker}h_{\omega} \cong \pi_1(M)$. This in particular implies that the horizontal lift $\hat{\omega} \circ \hat{z}$ of $p \circ z$, with respect to $p$, is not closed. As $z$ is also horizontal, we have $\hat{p} \circ \hat{z} = z$, which is closed, contradiction. Thus, for every $[\gamma] \in \pi_1(N(x_o), x_o)$ we have that $\hat{\gamma}$ is a closed path, i.e., $\pi_1(N(\hat{x}_o), \hat{x}_o) \cong \pi_1(N(x_o), x_o)$.

**Corollary 2.2** For all coverings $(\hat{M}, q, M)$ with $\text{Ker}h_{\omega} \subseteq \pi_1(\hat{M})$, we have that $N(\hat{x}_o) \cong N(x_o)$.

**Proof.** These horizontal spaces are covered by that of the bundle induced over the covering corresponding to $\text{Ker}h_{\omega}$. At the same time, they cover $N(x_o)$. This, in view of the above proved proposition, completes the proof.

**Proposition 2.3** If we deal with $\hat{P}(\hat{M}, G)$ where $\pi_1(\hat{M}) = \text{Ker}h_{\omega}$, we have $\pi_1(\hat{P}) \cong p_\#^{-1}(\text{Ker}h_{\omega})$.

**Proof.** Since $(\hat{P}, \hat{q}, \hat{P})$ is a covering (cf. Lemma 1.1), it is enough to see that $p_\#^{-1}(\text{Ker}h_{\omega}) = \hat{q}_\#(\pi_1(\hat{P}))$. Let $c$ be a path of $P$ closed at $x_o$, such that $[c] \in p_\#^{-1}(\text{Ker}h_{\omega})$. As $[p \circ c] \in \text{Ker}h_{\omega}$, the path $p \circ c$ is lifted to a closed path $\hat{p} \circ \hat{c}$ on $\hat{M}$ (note that $\pi_1(\hat{M}) = \text{Ker}h_{\omega}$). Because $\hat{P} \cong G(\hat{x}_o) \times \hat{M}$, the path $\hat{p} \circ \hat{c}$ also gives the lifting of $c$ with respect to $\hat{p}$. This implies that the lifting of $c$ with respect to $\hat{q}$ is closed, hence $p_\#^{-1}(\text{Ker}h_{\omega}) \subseteq \hat{q}_\#(\pi_1(\hat{P}))$. As the other inclusion is obvious, the proof is completed.
We finally identify $N(x_o)$ in terms of the holonomy morphism and data related to $P(M,G)$.

**Proposition 2.4** Under the above notation $\pi_1(N(x_o), x_o) \cong p^{-1}_*(\text{Ker}h_\omega)/\pi_1(G(x_o), x_o)$.

**Proof.** Let $\gamma$ be a path of $P$, closed at $x_o$, such that $p_\#([\gamma]) \in \text{Ker}h_\omega$. The horizontal lift $p \circ \gamma$ of this path with respect to $p$ is again closed at $x_o$ and $[p \circ \gamma] \in \pi_1(N(x_o), x_o)$. As $p|N(x_o)$ is a covering map, $p \circ \gamma$ is nullhomotopic if and only if $[\gamma] \in \text{Kerp}_#x_o \cong \pi_1(G(x_o), x_o)$ (cf. the exact homotopy sequence of the fibration $G \to P \to M$). Hence the correspondence $[\gamma] \to [p \circ \gamma]$ defines a homomorphism of $p^{-1}_*(\text{Ker}h_\omega)$ onto $\pi_1(N(x_o), x_o)$, whose kernel is exactly the group $\pi_1(G(x_o), x_o)$.

**Added in Proof.** In a forthcoming work we consider similar problems in the setting of associated fiber bundles and suitable flat connections defined on them.

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