The signal from an emitting source moving in a Schwarzschild spacetime under the influence of a radiation field

Donato Bini$^{1,2}$, Maurizio Falanga$^3$, Andrea Geralico$^{2,4}$ and Luigi Stella$^5$

$^1$ Istituto per le Applicazioni del Calcolo ‘M Picone’, CNR, I-00161 Rome, Italy
$^2$ ICRA, University of Rome ‘La Sapienza’, I-00185 Rome, Italy
$^3$ International Space Science Institute (ISSI), CH-3012 Bern, Switzerland
$^4$ Physics Department, University of Rome ‘La Sapienza’, I-00185 Rome, Italy
$^5$ Osservatorio Astronomico di Roma, via Frascati 33, I-00040 Monteporzio Catone (Roma), Italy

E-mail: binid@icra.it

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Abstract

The motion of matter immersed in a radiation field is affected by a radiation drag, as a result of scattering or absorption and re-emission. The resulting friction-like drag, also known as the Poynting–Robertson effect, has been recently studied in the general relativistic background of the Schwarzschild and Kerr metric, under the assumption that all photons in the radiation field possess the same angular momentum. We calculate here the signal produced by an emitting point-like specific source moving in a Schwarzschild spacetime under the influence of such a radiation field. We derive the flux, redshift factor and solid angle of the hot spot as a function of (coordinate) time, as well as the time-integrated image of the hot spot as seen by an observer at infinity. The results are then compared with those for a spot moving on a circular geodesic in a Schwarzschild metric.

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1. Introduction

In previous works [1, 2] we studied the motion of test particles in the gravitational background of a black hole under the influence of a superimposed radiation field. The particle–radiation interaction is assumed to take place through Thomson scattering only. Radiation exerts a drag force on the particle’s motion, a well-known effect in the Newtonian regime that is often referred to as the Poynting–Robertson effect [3, 4]. We limited ourselves to the case of particles moving in the equatorial plane of the Schwarzschild and Kerr backgrounds, immersed in an (equatorial) radiation field composed of photons having the same specific angular momentum and traveling along geodesics. In [1], the simplest case was considered of
a radiation field with zero angular momentum, i.e. photons moving in a purely radial direction with respect to the locally nonrotating frames, naturally associated with the family of zero angular momentum observers (ZAMOs). This analysis was extended in [2] to the case of a radiation field characterized by photons possessing all the same, non-null specific angular momentum. We found that those particles which do not escape to infinity are attracted to a single critical radius outside the horizon where they stay at rest with respect to ZAMOs; if the radiation field and the black hole have zero angular momentum, or move in a circular orbit, then a non-null angular momentum characterizes the radiation field and/or the black hole.

In this paper we consider an emitting source (hot spot) which moves in a Schwarzschild metric under the influence of a radiation field, and calculate the corresponding signal seen by an observer at infinity. In particular, we derive the flux, redshift factor and solid angle of the hot spot as a function of (coordinate) time, as well as the time-integrated image of the hot spot in the observer’s sky. The results are then compared with those for a spot moving on a circular geodesic in a Schwarzschild field. This analysis will then be extended to the more complex case of a Kerr field in a forthcoming paper.

2. Motion in the Schwarzschild spacetime

Consider a Schwarzschild spacetime, whose line element written in standard coordinates is given by

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \] (2.1)

where \( N = (1 - 2M/r)^{1/2} \) denotes the lapse function and introduces the usual orthonormal frame adapted to ZAMOs following the time lines

\[ n = e_t = N^{-1} \partial_t, \quad e_r = N \partial_r, \quad e_\theta = \frac{1}{r} \partial_\theta, \quad e_\phi = \frac{1}{r \sin \theta} \partial_\phi, \] (2.2)

where \{\partial_t, \partial_r, \partial_\theta, \partial_\phi\} is the coordinate frame.

2.1. Circular geodesic motion

Circular geodesic motion of a test particle in the equatorial plane \( \theta = \pi/2 \) at \( r = r_0 \) is characterized by the 4-velocity

\[ U = U_K = \gamma_K (n \pm v_K e_\phi), \] (2.3)

where the Keplerian value of speed (\( v_K \)), the associated Lorentz factor (\( \gamma_K \)) and angular velocity (\( \zeta_K \)) are given by

\[ v_K = \frac{M}{\sqrt{r_0 - 2M}}, \quad \gamma_K = \frac{r_0 - 2M}{r_0 - 3M}, \quad \zeta_K = \frac{M}{r_0^2}. \] (2.4)

The \( \pm \) signs in equation (2.3) correspond to co-rotating (+) or counter-rotating (−) orbits with respect to increasing values of the azimuthal coordinate \( \phi \) (counter-clockwise motion as seen from above).

The parametric equations of \( U_K \) are

\[ t_K = t_0 + \Gamma_K \tau, \quad r_K = r_0, \quad \theta_K = \frac{\pi}{2}, \quad \phi_K = \phi_0 \pm \Omega_K \tau, \] (2.5)

where

\[ \Gamma_K = \sqrt{\frac{r_0}{r_0 - 3M}}, \quad \Omega_K = \frac{1}{r_0} \sqrt{\frac{M}{r_0 - 3M}}. \] (2.6)
2.2. Test particle undergoing the Poynting–Robertson effect

Consider now a test particle in arbitrary motion on the equatorial plane \( \theta = \pi/2 \), i.e. with 4-velocity and 3-velocity with respect to the ZAMOs, respectively

\[
U = \gamma(U, n)[n + v(U, n)], \quad v(U, n) = \dot{v} \hat{e}_r + \dot{v} \hat{e}_\phi = v \sin \alpha \hat{e}_r + v \cos \alpha \hat{e}_\phi,
\]

(2.7)

where \( \gamma(U, n) = 1/\sqrt{1 - ||v(U, n)||^2} \) is the Lorentz factor and the abbreviated notation \( v^\alpha = v(U, n)^\alpha \) has been used. In a similar abbreviated notation, \( v = ||v(U, n)|| \geq 0 \) and \( \alpha \) are the magnitude of the spatial velocity \( v(U, n) \) and its polar angle measured clockwise from the positive \( \phi \) direction in the \( r-\phi \) tangent plane. Note that \( \sin \alpha = 0 \) (i.e. \( \alpha = 0, \pi \)) corresponds to a pure azimuthal motion of the particle with respect to the ZAMOs, while \( \cos \alpha = 0 \) (i.e. \( \alpha = \pm \pi/2 \)) corresponds to (outward/inward) purely radial motion with respect to the ZAMOs.

Let the particle be accelerated by a test radiation field propagating in a general direction on the equatorial plane. The corresponding equations for \( v \) and \( \alpha \) as derived in [2] are given by

\[
\frac{d\alpha}{d\tau} = -\frac{\nu \cos \alpha N}{v} \left( \frac{v^2 - M}{r N^2} \right) + \frac{A}{v} \left[ 1 - v \cos(\alpha - \beta) \right] \frac{1}{N^2 r^2 |\sin \beta|} \sin(\beta - \alpha),
\]

(2.8)

where \( \tau \) is the proper time parameter along \( U \), \( A \) is a positive constant for a given fixed radiation field and

\[
\cos \beta = \frac{b_{(rad)} N}{r},
\]

(2.9)

the constant \( b_{(rad)} \) being the photon impact parameter defined as the ratio between the conserved angular momentum and the energy associated with the rotational and time-like Killing vector fields, respectively. In the zero angular momentum limit, \( b_{(rad)} \rightarrow 0 \), \( \cos \beta \rightarrow 0 \), then \( \sin \beta \rightarrow \pm 1 \), \( \cos(\alpha - \beta) \rightarrow \pm \sin \alpha \), \( \sin(\beta - \alpha) \rightarrow \pm \cos \alpha \) and the two equations (2.8) reduce to those in [1].

Finally, we have in addition the remaining equations

\[
\frac{dr}{d\tau} = \frac{\gamma v}{N}, \quad \frac{d\rho}{d\tau} = \gamma N v \sin \alpha, \quad \frac{d\phi}{d\tau} = \frac{\gamma}{r} v \cos \alpha.
\]

(2.10)

The system of four differential equations for \( v, \alpha, r \) and \( \phi \) admits a critical solution at a radial equilibrium which corresponds to a circular orbit of constant radius \( r = r_0 \), constant speed \( v = v_0 \) and constant angles \( \beta = \beta_0, \alpha = \alpha_0 \). The constancy of the radius requires \( \sin \alpha_0 = 0, \cos \alpha_0 = \pm 1 \) and therefore \( \sin(\beta_0 - \alpha_0) = \cos \alpha_0 \sin \beta_0 \) and \( \cos(\alpha_0 - \beta_0) = \cos \alpha_0 \cos \beta_0 \).

The force balance equation for the critical circular orbits is then

\[
N \gamma_0 \left( 1 - \frac{v_0^2}{v_K^2} \right) = \text{sgn}(\sin \beta_0) \frac{A}{M},
\]

(2.11)

with

\[
\pm v_0 = \cos \beta_0 = \frac{b_{(rad)} N}{r_0} \quad \rightarrow \quad \gamma_0 = 1/|\sin \beta_0|.
\]

(2.12)

In the case \( b_{(rad)} = 0 \) (i.e. \( v_0 = 0, \gamma_0 = 1 \)) and \( \sin \beta_0 > 0 \) of purely radial outward photon motion, the previous equation reduces to the result of [1]

\[
\frac{A}{M} = N = \left( 1 - \frac{2M}{r_0} \right)^{1/2},
\]

(2.13)

which requires \( A/M < 1 \) for a solution to exist.
2.3. Radiation from the spot

Let the spot to radiate isotropically in its own rest frame. Consider then a (geodesic) photon connecting the emitter world line with the observer world line, i.e.

\[ K = \Gamma_{\text{ph}} \left[ \partial_t + \xi_{\text{ph}}^r \partial_r \right], \]  

(2.14)

For a general motion we have

\[ K' = \frac{dr}{d\lambda} = \frac{E}{N^2} = \Gamma_{\text{ph}}', \]

\[ K' = \frac{dr}{d\lambda} = \xi_{\text{ph}}^r \sqrt{R} = \Gamma_{\text{ph}} \xi_{\text{ph}}^r, \]

\[ K^\theta = \frac{d\theta}{d\lambda} = \frac{\epsilon_{\theta} E}{r^2} \sqrt{\Theta} = \Gamma_{\text{ph}} \xi_{\text{ph}}^\theta, \]

\[ K^\phi = \frac{d\phi}{d\lambda} = \frac{E}{r^2} \frac{b}{\sin^2 \theta} = \Gamma_{\text{ph}} \xi_{\text{ph}}^\phi, \]

(2.15)

where \( \lambda \) is an affine parameter, \( \epsilon_r \) and \( \epsilon_{\theta} \) are sign indicators, \( R = r^4 - 2r^2 N^2 (b^2 + q^2) = r [r^3 - (r - 2M) (b^2 + q^2)] \),

\( \Theta = q^2 - b^2 \cot^2 \theta, \)

and the following notation has been introduced:

\[ b = \frac{L}{E}, \quad q^2 = \frac{K}{E^2}. \]  

(2.16)

Here \( E = -K_t \) and \( L = K_\phi \) are conserved Killing quantities, whereas \( K \) is a separation constant related to Carter’s fourth constant of the motion \( Q \) by \( K = Q + L^2 \). Note that \( q^2 \) can take any value.

The geodesic equations can be formally integrated by eliminating the affine parameter as follows [5]:

\[ \epsilon_r \int^r \frac{dr}{\sqrt{R(r)}} = \epsilon_{\theta} \int^\theta \frac{d\theta}{\sqrt{\Theta(\theta)}}, \]

\[ t = \epsilon_r \int^r \frac{r^2}{N^2 \sqrt{R(r)}} \, dr, \]

\[ \phi = b \epsilon_{\theta} \int^\theta \frac{d\theta}{\sin^2 \theta \sqrt{\Theta(\theta)}}. \]

(2.18)

The integrals are along the path of motion.

3. Energy shift

We consider here the case of an emitter with 4-velocity \( U_{\text{em}} \) moving on the equatorial plane of a Schwarzschild spacetime. The observer with 4-velocity \( U_{\text{obs}} \) is at rest (very far from the origin) at a point not necessarily belonging to the equatorial plane. Let \( K \) be the 4-momentum of the photon connecting the two world lines. Hence we have

\[ U_{\text{em}} = \Gamma_{\text{em}} \left[ \partial_t + \xi_{\text{em}}^r \partial_r + \xi_{\text{em}}^\phi \partial_\phi \right], \]

\[ U_{\text{obs}} = \partial_t, \]

\[ K = \Gamma_{\text{ph}} \left[ \partial_t + \xi_{\text{ph}}^r \partial_r + \xi_{\text{ph}}^\theta \partial_\theta + \xi_{\text{ph}}^\phi \partial_\phi \right]. \]

(3.1)
The energy of the photon at the emission point \( P_{\text{em}} \), as measured by \( U_{\text{em}} \), is
\[
E_{\text{em}} \equiv E(K, U_{\text{em}}) = -K \cdot U_{\text{em}} |_{\mathcal{P}_{\text{em}}} = -\Gamma (\phi_{\text{em}}) \Gamma (\theta_{\text{em}}) \left[ -N^2 + N^{-2} \zeta_{\phi} r^2 \zeta_{\phi} \zeta_{\phi} (\phi_{\text{em}}) + r^2 \zeta_{\phi} r^2 \zeta_{\phi} (\phi_{\text{em}}) \right] |_{\mathcal{P}_{\text{em}}}.
\]
while the one observed at the point \( P_{\text{obs}} \) by \( U_{\text{obs}} \) is
\[
E_{\text{obs}} \equiv E(K, U_{\text{obs}}) = -K \cdot U_{\text{obs}} |_{\mathcal{P}_{\text{obs}}} = -E |_{\mathcal{P}_{\text{obs}}}.
\]
Therefore, the ratio \( E_{\text{obs}} / E_{\text{em}} \) is
\[
g \equiv \frac{E_{\text{obs}}}{E_{\text{em}}} = \frac{\sqrt{N^2 - N^{-2} \zeta^2 - r^2 \zeta^2}}{1 - N^{-4} \zeta^2 - b \zeta^2}
\]
with all quantities evaluated at the emission point. Usually, one also introduces the ‘redshift’ parameter \( z \)
\[
z = \frac{E_{\text{em}} - E_{\text{obs}}}{E_{\text{obs}}}, \quad g = (z + 1)^{-1}.
\]
If the emitter is in a geodesic circular orbit, we have \( \zeta_{\phi} = 0 \) and \( \zeta_{\phi} = \pm \zeta_{K} \) so that
\[
g = \frac{1}{1 + b \sqrt{\zeta_{K}^2}}.
\]

4. Ray tracing

Consider a photon emitted at the points \( r_{\text{em}}, \theta_{\text{em}} \) and \( \phi_{\text{em}} \) at the coordinate time \( t_{\text{em}} \), which reaches an observer located at \( r_{\text{obs}}, \theta_{\text{obs}} \) and \( \phi_{\text{obs}} \) at the coordinate time \( t_{\text{obs}} \). The photon trajectories originating at the emitter must satisfy the following integral equation:
\[
\epsilon_e \int_{r_{\text{em}}}^{r_{\text{obs}}} \frac{dr}{\sqrt{R(r)}} = \epsilon_\theta \int_{\theta_{\text{em}}}^{\theta_{\text{obs}}} \frac{d\theta}{\sqrt{\Theta(\theta)}},
\]
as from equation (2.18). The signs \( \epsilon_e \) and \( \epsilon_\theta \) change when a turning point is reached. Turning points in \( r \) and \( \theta \) are solutions of the equations \( R = 0 \) and \( \Theta = 0 \), respectively. To find out which photons actually reach the observer, one thus must find those pairs \( (b, q^2) \) that satisfy equation (4.1).

It is useful to introduce the new variable \( \mu = \cos \theta \), so that the null geodesic equations (2.18) become
\[
\epsilon_e \int_{r_{\text{em}}}^{r} \frac{dr}{\sqrt{R(r)}} = \epsilon_\mu \int_{\mu_{\text{em}}}^{\mu_{\text{obs}}} \frac{d\mu}{\sqrt{\Theta_{\mu}(\mu)}},
\]
\[
t = \epsilon_e \int_{r_{\text{em}}}^{r} \frac{r^2}{\sqrt{R(r)}} \frac{dr}{N^2 \sqrt{R(r)}} = \epsilon_\mu \int_{\mu_{\text{em}}}^{\mu_{\text{obs}}} \frac{d\mu}{1 - \mu^2} \frac{1}{\sqrt{\Theta_{\mu}(\mu)}},
\]
where
\[
\Theta_{\mu} = q^2 - (q^2 + b^2) \mu^2 \equiv (q^2 + b^2) (\mu^2 - \mu^2), \quad \mu^2 = \frac{q^2}{q^2 + b^2}.
\]
We will consider the case of an emitting source moving on the equatorial plane (i.e. $\theta_{em} = \pi/2$) and a distant observer located far away from the black hole (i.e. $r_{obs} \to \infty$) at the azimuthal position $\phi_{obs} = 0$. For a photon emitted by the spot, we thus have $\mu_{em} = 0$. Furthermore, for a photon crossing the equatorial plane (which is the case we are interested in), we have $q^2 > 0$ so that $\mu^2 > 0$.

Consider first equation (4.2). The integral over $\mu$ is straightforward

$$\int_{\mu}^{\mu_0} \frac{d\mu}{\sqrt{\Theta_\mu}} = -\frac{1}{\sqrt{q^2 + b^2}} \arctan \left( \frac{\mu}{\sqrt{\mu^2 - \mu^2}} \right).$$

(4.6)

The integral over $r$ can be worked out with the inverse Jacobian elliptic integrals. We will use the notations and conventions of [6] for the definition of the Jacobi functions

$$sn(u, k) = \sin \varphi, \quad cn(u, k) = \cos \varphi,$$

(4.7)

where

$$u = \int_0^\varphi \frac{d\varphi}{\sqrt{1 - k \sin^2 \varphi}}$$

(4.8)

is the incomplete elliptic integral of the first kind. Let us denote the four roots of $R(r) = 0$ by $r_1, r_2, r_3$ and $r_4$. There are two relevant cases to be considered. In fact, since $r = 0$ is a root of the equation $R(r) = 0$, the reduced cubic equation admits either three real roots or one real root and two complex conjugate roots.

**Case A:** $R(r) = 0$ has four real roots.

Let the roots be ordered so that $r_1 \geq r_2 \geq r_3 \geq r_4$, with $r_3 = 0$ and $r_4 \leq 0$.

Physically allowed regions for photons are given by $R \geq 0$, i.e. $r \geq r_1$ (region I) and $r_3 \leq r \leq r_2$ (region II) with $r > 2M$. In region I the integral over $r$ can be worked out by the following integration:

$$\int_{r_1}^{r} \frac{dr}{\sqrt{R(r)}} = \frac{2}{\sqrt{r_1(r_2 - r_3)}} sn^{-1}(\sin \varphi_{A_1}, k_A),$$

(4.9)

where

$$\sin \varphi_{A_1} = \sqrt{\frac{(r_2 - r_4)(r - r_1)}{(r_1 - r_4)(r - r_2)}}, \quad k_A = \sqrt{\frac{r_2(r_1 - r_4)}{r_1(r_2 - r_4)}},$$

(4.10)

when $r_1 \neq r_2$. The case of two equal roots $r_1 = r_2$ should be treated separately.

In region II the integral over $r$ can be worked out by the following integration:

$$\int_{r_3}^{r_2} \frac{dr}{\sqrt{R(r)}} = \frac{2}{\sqrt{r_1(r_2 - r_4)}} sn^{-1}(\sin \varphi_{A_2}, k_A),$$

(4.11)

where

$$\sin \varphi_{A_2} = \sqrt{\frac{r_1(r_2 - r)}{r_2(r_1 - r)}}, \quad r_1 \neq r_2.$$  

(4.12)

**Case B:** $R(r) = 0$ has two complex roots and two real roots.

Let us assume that $r_1$ and $r_2$ are complex, with $r_1 = \bar{r}_2$, whereas $r_3 = 0$ and $r_4$ is real such that $r_4 \leq 0$. The physically allowed region for photons is given by $r > 2M$.

The integral over $r$ can be worked out with the following integration:

$$\int_{2M}^{r} \frac{dr}{\sqrt{R(r)}} = \int_{0}^{r} \frac{dr}{\sqrt{R(r)}} = \int_{0}^{2M} \frac{dr}{\sqrt{R(r)}}.$$  

(4.13)
\[ \int_0^r \frac{dr}{\sqrt{R(r)}} = \frac{1}{\sqrt{AB}} \text{cn}^{-1}(\sin \varphi_B, k_B), \]

\[ \int_0^{2M} \frac{dr}{\sqrt{R(r)}} = \frac{1}{\sqrt{AB}} \text{cn}^{-1}(\sin \varphi_B^H, k_B), \]

where

\[ A^2 = u^2 + v^2, \quad B^2 = (r_4 - u)^2 + v^2, \] (4.15)

with

\[ u = \text{Re}(r_1) \quad \text{and} \quad v = \text{Im}(r_1), \]

\[ \sin \varphi_B = \frac{(A - B)r - r_4A}{(A + B)r - r_4A}, \quad k_B = \sqrt{\frac{(A + B)^2 - r_4^2}{4AB}}, \] (4.16)

\[ \sin \varphi_B^H = \sin \varphi_B(r = 2M). \]

5. Images

The apparent position of the image of the emitting source on the celestial sphere is represented by two impact parameters, \( \alpha \) and \( \beta \), measured on a plane centered about the observer location and perpendicular to the direction \( \theta_{\text{obs}} \). They are defined by [7]

\[ \alpha = \lim_{r_{\text{obs}} \to \infty} -r_{\text{obs}} \frac{K_\hat{\phi}}{K^i} = -\frac{b}{\sin \theta_{\text{obs}}}, \]

\[ \beta = \lim_{r_{\text{obs}} \to \infty} r_{\text{obs}} \frac{K_\hat{\theta}}{K^i} = \epsilon_{\theta_{\text{obs}}} \sqrt{q^2 - b^2 \cot^2 \theta_{\text{obs}}}, \] (5.1)

where \( K^i \) are the frame components of \( K \) with respect to the ZAMOs (its coordinate components are instead listed in equation (2.15)). Equivalent expressions can be obtained by decomposing the photon 4-velocity as follows:

\[ K = E(n) [e_t + \hat{v}^\phi e_\phi], \quad \hat{v} \cdot \hat{v} = 1 \]

so that

\[ \alpha = \lim_{r_{\text{obs}} \to \infty} -r_{\text{obs}} \hat{v}^\phi = \lim_{r_{\text{obs}} \to \infty} [\vec{r} \times \hat{v}]^\phi, \]

\[ \beta = \lim_{r_{\text{obs}} \to \infty} r_{\text{obs}} \hat{v}^\theta = \lim_{r_{\text{obs}} \to \infty} [\vec{r} \times \hat{v}]^\theta. \] (5.3)

The line of sight to the black hole center marks the origin of the coordinates, where \( \alpha = 0 = \beta \). Now imagine a source of illumination behind the black hole whose angular size is large compared with the angular size of the black hole. As seen by the distant observer, the black hole will appear as a black region in the middle of the larger bright source. No photons with impact parameters in a certain range about \( \alpha = 0 = \beta \) will reach the observer. The rim of the black hole corresponds to photon trajectories which are marginally trapped by the black hole; they spiral around many times before they reach the observer.

The image of the trajectory is thus obtained by determining all pairs \((b, q^2)\) satisfying equation (4.1) (or equivalently equation (4.2)), then substituting back into equation (5.1) in order to obtain the corresponding coordinates on the observer’s plane. Alternatively, one can solve equation (5.1) for \( b \) and \( q^2 \), i.e.

\[ b = -\alpha \sin \theta_{\text{obs}}, \quad q^2 = \beta^2 + \alpha^2 \cos^2 \theta_{\text{obs}} \quad \Rightarrow \quad b^2 + q^2 = \alpha^2 + \beta^2, \] (5.4)
then substituting back into equation (4.2) and solving for all allowed pairs of impact parameters \((\alpha, \beta)\).

The images of the source obtained in this way can be classified according to the number of times the photon trajectory crosses the equatorial plane between the emitting source and the observer. The trajectory of the ‘direct’ image does not cross the equatorial plane, while that of the ‘first-order’ image crosses it once, and so on.

We are interested in constructing direct images only and refer to appendix A for details.

6. Light curves

The observed differential flux is given by [8]

\[
\frac{dF_{\text{obs}}}{d\Omega} = I_{\text{obs}} \frac{d\alpha}{r_{\text{obs}}}.
\]  

(6.1)

where \(d\Omega\) is the solid angle subtended by the light source on the observer sky and \(I_{\text{obs}}\) is the intensity of the source integrated over its effective frequency range, i.e.

\[
I_{\text{obs}} = g^4 I_{\text{em}}.
\]  

(6.2)

The intensity \(I_{\text{em}}\) measured at the rest frame of the spot can be normalized as \(I_{\text{em}} = 1\). Furthermore, the solid angle can be expressed in terms of the observer’s plane coordinates \([\alpha, \beta]\) as

\[
d\Omega = \frac{d\alpha}{r_{\text{obs}}} d\beta.
\]  

(6.3)

Introducing then polar coordinates on the observer’s plane and switching integration over \(r_{\text{em}}\) and \(\phi_{\text{em}}\) (see appendix B for details), the observed differential flux is expressed as

\[
\frac{dF_{\text{obs}}}{r_{\text{obs}}^2} = \frac{g^4}{1 - \cos^2 \phi_{\text{em}} \sin^2 \theta_{\text{obs}}} \left| \frac{\partial b}{\partial r_{\text{em}}} + q \frac{\partial q}{\partial r_{\text{em}}} \right| dr_{\text{em}} d\phi_{\text{em}}.
\]  

(6.4)

Finally, the light curve of the emitting source is constructed by introducing the time dependence of the radiation received by the distant observer, including the time delay effects. Therefore, we also need to evaluate the coordinate time interval spent by each photon to reach the observer (see appendix C).

7. Results

Figures 1–5 show the apparent position of the direct image, light curve, redshift factor and solid angle for an emitting spot under the effect of both the gravitational and radiation fields. Different figures correspond to different properties of the radiation field. The distant observer is located at the polar angle \(\theta_{\text{obs}} = 80^\circ\) in all cases. The case of an emitting spot in circular geodesic motion on the equatorial plane of a Schwarzschild spacetime at the same initial \(r_{\text{em}}\) is also shown for comparison.

In figures 1–3 the initial radius and azimuthal angle of the emitting spot are \(r_{\text{em}}(0) = 10M\), \(\phi_{\text{em}}(0) = 0\); the initial velocity is that of a Keplerian circular orbit at that radius. The radiation field is radially outgoing, with different values of the luminosity parameter \(A/M\). For small values of \(A/M\), the emitter spirals toward the critical radius located close to the horizon, by undergoing several azimuthal cycles around the black hole (see figure 1). The trajectory is nearly circular after the first revolution so that the light curve is very close to the circular Keplerian one. The spiraling then becomes faster, and the light curve peaks occur faster and faster.
Figure 1. The apparent position (direct image only), light curve, redshift factor and solid angle of the emitting spot are shown for the orbit depicted in the upper left panel in the case of a radially outgoing radiation field. The orbital parameters and initial conditions are $A/M = 0.01$, $r_{em}(0) = 10M$, $\phi_{em}(0) = 0$, $v_{em}(0) = v_{K} \approx 0.35$ and $\alpha_{em}(0) = 0$. The distant observer is located at the polar angle $\theta_{obs} = 80^\circ$. $x = (r_{em}/M) \cos \phi_{em}$ and $y = (r_{em}/M) \sin \phi_{em}$ are the Cartesian-like coordinates expressed in units of $M$. The black circle represents the Schwarzschild horizon $r = 2M$. The critical radius approaches the horizon in this case ($r_{crit}(\approx 2.0002M)$). The flux is given in arbitrary units as a function of the coordinate time given in seconds, corresponding to the choice of $M = 1.0M_\odot$. The observed time is given by the orbital time plus the light-bending travel time delay. The relative time delay is then evaluated by using the geodesic equations in such a way that the first photon emitted at the starting point defines the reference time which is set to zero. For comparison purposes, the corresponding curves for an emitting spot in circular geodesic orbit at $r_{em} = 10M$ on the equatorial plane of a Schwarzschild spacetime are shown (dashed curves).

Figure 2. The same as in figure 1 but with $A/M = 0.1$. The critical radius approaches the horizon also in this case ($r_{crit}(\approx 2.02M)$).
Increasing the luminosity parameter while maintaining the initial conditions fixed causes the emitting spot to drift initially to larger radii and then quickly spiral inward down to the critical radius, which, as expected, is larger in this case (see figures 2 and 3 and equation (2.13)).
Figure 5. The same as in figure 4 but with an initial negative value of the azimuthal velocity, i.e. $\alpha_\text{em}(0) = \pi$. The values of the critical radius as well as critical velocity do not change.

Figures 4 and 5 show the effect of a nonzero angular momentum of the radiation field. In the case of figure 4 the emitting spot drifts away from the black hole before going back inward, ending in the circular equilibrium orbit in a few revolutions. This is also evident from the comparison of the corresponding light curve with the Keplerian one. In fact, the emitter spends a fairly long time far from the source before completing the first revolution, whereas in the same time interval, the Keplerian emitter orbits the source several times. Then the orbit soon becomes circular, at the smaller radius given by equation (2.11). Figure 5 refers to an emitter with the same initial conditions as in figure 4, except for the negative sign of the azimuthal velocity. In this case, the emitter initially moves clockwise around the black hole and is quickly dragged by the radiation field in the opposite direction, soon reaching the same equilibrium orbit as in figure 4.

8. Concluding remarks

We have calculated the signal produced by an emitting point-like source moving in the equatorial plane of a Schwarzschild spacetime under the influence of a radiation field. The latter consists of photons having the same specific angular momentum and traveling along geodesics. The interaction with the photon field leads to a friction-like drag force responsible for the so-called Poynting–Robertson effect. Previous studies have shown that, in the case of photons with zero angular momentum, i.e. propagating radially with respect to the ZAMOs, there exists an equilibrium radius representing the balance of the outward radiation force with the inward gravitational force, where the emitter can remain at rest. The location of such a critical radius depends on the luminosity parameter. If the outward photon flux possesses a nonzero angular momentum, emitting spots that do not escape end up in circular orbits.

In this paper we have derived the flux, redshift factor and solid angle as a function of the (coordinate) time, as well as the time-integrated image of the spot in the observer’s sky. The
results are clearly different from those for an emitting spot in circular geodesic motion, as shown by numerical examples where the effect of the interaction with the radiation field has been investigated by varying both the luminosity parameter and the photon angular momentum.

The treatment and results presented here hold a potential for astrophysical applications. Matter accretion toward white dwarfs, neutron stars or black holes that emit radiation at a sizeable fraction of their Eddington luminosity (i.e. $L/L_{\text{Edd}} \gtrsim 1\%$) will be influenced by general relativistic Poynting–Robertson-type effects. Departures from the unperturbed motion (i.e. in the absence of the radiation field) are substantial and may lead to observable phenomena. The range of astrophysical applications is vast; examples include quasi-periodic oscillations that are observed in accreting neutron stars and black holes [9, 10], thermonuclear flashes that occur on the surface of accreting neutron stars (the so-called type I bursts [11]) and the very broad Fe–Kα line profiles produced in the innermost regions of accretion disks around collapsed objects [12]. However, the treatment we have developed here is idealized in several respects and the impact of some approximations should be carefully assessed before detailed predictions for astrophysical systems are worked out. Treating the photon field as if all photons had the same angular momentum presents clear advantages for the analytical calculations presented in [1, 2], but would require some caution in an astrophysical context. For instance, the flux emitted from the surface of an accreting neutron star comprises photons emitted in virtually all directions (and thus possessing a range of different angular momenta). Moreover, in accreting black holes the motion of matter in the vicinity of the innermost stable circular orbit will be mainly affected by radiation coming from the outer disk regions, in turn involving a radiation field emitted in a range of different photon directions and emission radii. Finally, in an astrophysical environment one must also consider the impact of two key assumptions, which are intrinsic to any Poynting–Robertson-type theory, namely that matter is directly exposed to the radiation field (meaning that the optical depth to the source must be $< 1$) and that the radiation re-emitted or scattered by matter propagates unimpeded to infinity without undergoing other interactions.

Despite these limitations the analysis presented here captures some essential features of the motion of matter in the strong field regime under the effects of an intense source of radiation. Future work will be devoted to generalizing the present treatment and addressing specific astrophysical situations in which the general relativistic version of the Poynting–Robertson effect can be relevant.

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Appendix A.

We list below for completeness the details on the construction of the direct image, the derivation of the observed energy flux and the calculation of the coordinate time interval between the emitter and observer. This is a well-known topic addressed by many authors in the literature (see, e.g., [13–17]). However, different and sophisticated techniques are used simply to show light curves as well as images, without entering the underlying analytical framework or referring to previous related works.
A.1. Constructing the direct image

The direct image results from photons which never cross the equatorial plane. As the photon reaches the observer, on the photon orbit, we have \( \frac{d\theta}{dr} > 0 \) (i.e. \( \frac{d\mu}{dr} < 0 \)) if \( \beta > 0 \), and \( \frac{d\theta}{dr} < 0 \) (i.e. \( \frac{d\mu}{dr} > 0 \)) if \( \beta < 0 \). Therefore, when \( \beta > 0 \), the photon must encounter a turning point at \( \mu = \bar{\mu} \): \( \mu \) starts from 0, goes up to \( \bar{\mu} \) and then goes down to \( \mu_{\text{obs}} \) (which is \( \leq \bar{\mu} \)). When \( \beta < 0 \), the photon does not encounter a turning point at \( \mu = \bar{\mu} \): \( \mu \) starts from 0 and monotonically increases to \( \mu_{\text{obs}} \).

The total integration over \( \mu \) along the path of the photon from the emitting source to the observer is thus given by

\[
I_\mu = \begin{cases} 
\int_0^{\bar{\mu}} \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}} + \int_{\bar{\mu}}^{\mu_{\text{obs}}} \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}} & (\beta > 0) \\
\int_0^{\mu_{\text{obs}}} \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}} = \int_0^{\mu_{\text{obs}}} \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}} & (\beta < 0). 
\end{cases}
\]

(A.1)

By using equation (4.6) we obtain

\[
I_\mu = \begin{cases} 
\frac{\pi}{\sqrt{\alpha^2 + \beta^2}} - I_{\mu_{\text{obs}}} & (\beta > 0) \\
I_{\mu_{\text{obs}}} & (\beta < 0), 
\end{cases}
\]

(A.2)

where

\[
I_{\mu_{\text{obs}}} = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \arctan \left( \frac{\sqrt{\alpha^2 + \beta^2}}{|\beta|} \cot \theta_{\text{obs}} \right).
\]

(A.3)

Now let us consider the integration over \( r \). Since the observer is at infinity, the photon reaching him/her must have been moving in the allowed region defined by \( r \geq r_1 \) when \( R(r) = 0 \) has four real roots (case A), or the allowed region defined by \( r > 2M \) when \( R(r) = 0 \) has two complex roots and two real roots (case B). There are then two possibilities for the photon during its trip: it has encountered a turning point at \( r = r_1 \), or it has not encountered any turning point in \( r \). Define

\[
I_\infty = \int_{r_1}^\infty \frac{dr}{\sqrt{R(r)}}, \quad I_{r_{\text{em}}} = \int_{r_1}^{r_{\text{em}}} \frac{dr}{\sqrt{R(r)}}.
\]

(A.4)

Obviously, according to equation (4.2), a necessary and sufficient condition for the occurrence of a turning point in \( r \) on the path of the photon is that \( I_\infty < I_\mu \). Therefore, the total integration over \( r \) along the path of the photon from the emitting source to the observer is

\[
I_r = \begin{cases} 
I_\infty + I_{r_{\text{em}}} & (I_\infty < I_\mu) \\
I_\infty - I_{r_{\text{em}}} & (I_\infty \geq I_\mu). 
\end{cases}
\]

(A.5)

By definition, \( I_\infty, I_{r_{\text{em}}} \) and \( I_r \) are all positive. According to equation (4.2), we must have \( I_r = I_\mu \) for the orbit of a photon. The relevant cases to be considered are the following.

Case A: \( R(r) = 0 \) has four real roots.

When \( r_1 \neq r_2 \), by using equation (4.9) to evaluate integrals in equation (A.4) with \( r_i = r_1 \), we obtain

\[
I_\infty = \frac{2}{\sqrt{r_1(r_2 - r_3)}} \ \text{sn}^{-1} \left( \sin \varphi^\infty_i, k_i \right),
\]

\[
I_{r_{\text{em}}} = \frac{2}{\sqrt{r_1(r_2 - r_3)}} \ \text{sn}^{-1} \left( \sin \varphi^*_{r_{\text{em}}}, k_i \right).
\]

(A.6)
where \( \sin \varphi_{A}^{\infty} = \sin \varphi_{A}(r \to \infty) \) and \( \sin \varphi_{A}^{\text{em}} = \sin \varphi_{A}(r = r_{\text{em}}) \). Substitute these expressions into equation (A.5); let \( I_{r} = I_{r_{\text{f}}} \) and finally solve for \( r_{\text{em}} \):

\[
 r_{\text{em}} = \frac{r_{1}(r_{2} - r_{A}) - r_{2}(r_{1} - r_{A})\sin^{2}(\xi_{A}^{\infty}, k_{A})}{(r_{2} - r_{A}) - (r_{1} - r_{A})\sin^{2}(\xi_{A}, k_{A})}, \tag{A.7}
\]

where

\[
 \sin \xi_{A} = \frac{1}{2}(I_{r} - I_{\infty})\sqrt{r_{1} - r_{A}}. \tag{A.8}
\]

Since \( \sin^{2}(\xi_{A}, k_{A}) = \sin^{2}(\xi_{A}^{\infty}, k_{A}) \), the solution given by equation (A.7) applies whether \( I_{r} - I_{\infty} \) is positive or negative, i.e. no matter whether there is a turning point in \( r \) or not along the path of the photon.

Case B: \( R(r) = 0 \) has two complex roots and two real roots.

No turning points occur in this case. Therefore, we have

\[
 I_{r} = \int_{r_{\text{em}}}^{\infty} \frac{dr}{\sqrt{R(r)}} = I_{\infty} - I_{r_{\text{em}}}, \tag{A.9}
\]

with

\[
 I_{\infty} = \frac{1}{\sqrt{AB}} \text{cn}^{-1}(\sin \varphi_{B}^{\infty}, k_{B}), \quad I_{r_{\text{em}}} = \frac{1}{\sqrt{AB}} \text{cn}^{-1}(\sin \varphi_{B}^{\text{em}}, k_{B}). \tag{A.10}
\]

by using equation (4.13) to evaluate integrals in equation (A.4); also recall equation (4.15). Solving then the equation \( I_{r} = I_{r_{\text{f}}} \) for \( r_{\text{em}} \) gives

\[
 r_{\text{em}} = \frac{r_{A}[1 - \text{cn}(\sin \xi_{B}, k_{B})]}{(A - B) - (A + B)\text{cn}(\sin \xi_{B}, k_{B})}, \tag{A.11}
\]

where \( \sin \varphi_{B}^{\infty} = \sin \varphi_{B}(r \to \infty) \) and \( \sin \varphi_{B}^{\text{em}} = \sin \varphi_{B}(r = r_{\text{em}}) \) and

\[
 \sin \xi_{B} = (I_{r_{\text{f}}} - I_{\infty})\sqrt{AB}. \tag{A.12}
\]

### Appendix B. Evaluating the observed energy flux

The solid angle once expressed in terms of the observer’s plane coordinates \( \alpha, \beta \) is given by

\[
 d\Omega = \frac{d\alpha d\beta}{r_{\text{obs}}^{2}}. \tag{B.1}
\]

Introduce polar coordinates on the observer’s plane

\[
 \alpha = \rho \cos \psi, \quad \beta = \rho \sin \psi. \tag{B.2}
\]

The integration over the observer’s plane coordinates can then be switched over \( r_{\text{em}} \) and \( \phi_{\text{em}} \) by

\[
 d\alpha d\beta = \rho d\rho d\psi = \sqrt{\alpha^{2} + \beta^{2}} \left| \frac{\partial(\rho, \psi)}{\partial(r_{\text{em}}, \phi_{\text{em}})} \right| dr_{\text{em}} d\phi_{\text{em}}, \tag{B.3}
\]

where

\[
 \left| \frac{\partial(\rho, \psi)}{\partial(r_{\text{em}}, \phi_{\text{em}})} \right| = \left| \frac{\partial \rho}{\partial r_{\text{em}}} \frac{\partial \psi}{\partial \phi_{\text{em}}} - \frac{\partial \rho}{\partial \phi_{\text{em}}} \frac{\partial \psi}{\partial r_{\text{em}}} \right| \tag{B.4}
\]

is the Jacobian of the transformation \( (\rho, \psi) \rightarrow (r_{\text{em}}, \phi_{\text{em}}) \). Since

\[
 \rho = \sqrt{\alpha^{2} + \beta^{2}} = \sqrt{b^{2} + q^{2}} \tag{B.5}
\]

does not depend on \( \phi_{\text{em}} \), we have to evaluate only

\[
 \frac{\partial \rho}{\partial r_{\text{em}}} = \frac{1}{\rho} \left( b \frac{\partial b}{\partial r_{\text{em}}} + q \frac{\partial q}{\partial r_{\text{em}}} \right), \tag{B.6}
\]
where the derivatives of \( b \) and \( q \) with respect to \( r_{\text{em}} \) are obtained simply by inverting the derivatives \( \partial r_{\text{em}}/\partial q \) and \( \partial r_{\text{em}}/\partial b \) which can be evaluated from equations (A.7) and (A.11). \( \partial \psi / \partial \phi_{\text{em}} \) instead can be evaluated from equation (4.4) governing the azimuthal motion, i.e.

\[
\phi = b \epsilon_{\mu} \int_{0}^{\mu} \frac{1}{1 - \mu^2 \Theta_{\mu}(\mu)} \, d\mu,
\]

(B.7)

taking into account that \( \phi_{\text{obs}} = 0 \) and \( \tan \psi = \beta / \alpha \). The integration is straightforward

\[
\phi = - \epsilon_{\mu} \arctan \left( \frac{b}{\mu} \sqrt{\Theta_{\mu}(\mu)} \right)
\]

(B.8)

so that

\[
\phi_{\text{em}} = \begin{cases} 
\pi + \arctan \left( \frac{\alpha}{\beta} \cos \theta_{\text{obs}} \right) & (\beta > 0) \\
- \arctan \left( \frac{\alpha}{|\beta|} \cos \theta_{\text{obs}} \right) & (\beta < 0),
\end{cases}
\]

(B.9)

whence

\[
\tan \phi_{\text{em}} = \frac{\alpha}{\beta} \cos \theta_{\text{obs}} = \cot \psi \cos \theta_{\text{obs}}.
\]

(B.10)

Therefore,

\[
\sin \psi = \frac{\cos \phi_{\text{em}} \cos \theta_{\text{obs}}}{\sqrt{1 - \cos^2 \phi_{\text{em}} \sin^2 \theta_{\text{obs}}}}, \quad \cos \psi = \frac{\sin \phi_{\text{em}}}{\sqrt{1 - \cos^2 \phi_{\text{em}} \sin^2 \theta_{\text{obs}}}},
\]

implying that

\[
\frac{\partial \psi}{\partial \phi_{\text{em}}} = \frac{\cos \theta_{\text{obs}}}{1 - \cos^2 \phi_{\text{em}} \sin^2 \theta_{\text{obs}}}.
\]

(B.12)

Finally, the solid angle (B.1) turns out to be given by

\[
d\Omega = \frac{1}{r_{\text{obs}}^2} \cos \theta_{\text{obs}} \left| b \frac{\partial b}{\partial r_{\text{em}}} + q \frac{\partial q}{\partial r_{\text{em}}} \right| dr_{\text{em}} d\phi_{\text{em}},
\]

(B.13)

leading to expression (6.4) for the observed differential flux.

**Appendix C. Evaluating the coordinate time integral**

The light travel time between the emitter and observer is given by equation (4.3), i.e.

\[
t_{\text{obs}} - t_{\text{em}} = \epsilon_{\mu} \int_{r_{\text{em}}}^{r_{\text{obs}}} \frac{r^2}{N^2 \sqrt{R(r)}} \, dr,
\]

(C.1)

where the integration has to be done properly. The integral can be conveniently decomposed as follows:

\[
\int_{r_{\text{em}}}^{r_{\text{obs}}} \frac{r^2}{N^2 \sqrt{R(r)}} \, dr = \int_{r_{\text{em}}}^{r_{\text{obs}}} \frac{r^3}{(r - 2M) \sqrt{R(r)}} \, dr
\]

\[
= 8M^3 \int_{r_{\text{em}}}^{r_{\text{obs}}} \frac{dr}{(r - 2M) \sqrt{R(r)}} + \int_{r_{\text{em}}}^{r_{\text{obs}}} \sqrt{R(r)} \, dr
\]

\[
+ 2M \int_{r_{\text{em}}}^{r_{\text{obs}}} \frac{r}{\sqrt{R(r)}} \, dr + 4M^2 \int_{r_{\text{em}}}^{r_{\text{obs}}} \frac{dr}{\sqrt{R(r)}}.
\]

(C.2)

Each term can be evaluated in terms of elliptic functions, e.g., by using the table of integrals in [6].
Consider first the case A, where $R(r) = 0$ has four real roots. Let the roots be ordered so that $r_1 > r_2 > r_3 > r_4$, with $r_4 < 0$. Physically allowed regions for photons are given by $R > 0$, i.e. $r > r_1$ (region I) and $r_2 > r > r_3$ (region II). In region I the integrals entering equation (C.2) have to be worked out by the formulas nn 258.00, 258.11, 258.39 on pp 128–32 of [6]. In region II instead we refer to nn 255.00, 255.17, 255.38 on pp 116–20.

In case B the equation $R(r) = 0$ has two complex roots and two real roots. Let us assume that $r_1$ and $r_2$ are complex, $r_3$ and $r_4$ are real and $r_3 \geq r_4$. Then, we must have $r_1 = \bar{r_2}$, whereas $r_3 \geq 0$ and $r_4 \leq 0$. The physically allowed region for photons is given by $r > r_3$. The integrals entering equation (C.2) have to be worked out by the formulas nn 260.00, 260.03, 260.04 on pp 135–36 of [6].

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