Abstract

In the lowest nonlinear approximation I compare two gravitational wave equations, those of Weinberg and Papapetrou. The first one is simply a form of Einstein equation and the second is claimed to be yet another field theoretical form in which the energy-momentum tensor is obtained by Belinfante or Rosenfeld method. I show that for interacting gravitational field these methods lead to different energy-momentum tensors. Both these tensors need to be complemented ”by hand” with some interaction energy-momentum tensors in order that the conservation laws of the total energy-momentum tensor give equation of motion for particles in agreement with general relativity. In approximation considered by Thirring, the Papapetrou wave equation must coincide with that of Thirring. But they differ because Thirring inserted the necessary interaction term. I show that Thirring wave equation is equivalent to Weinberg’s one. Hence the Papapetrou equation is not yet another form of Einstein equation.

1 Introduction

The Einstein equation can be written straightforwardly in a field theoretical form, see equations. (7.6.3) and (7.6.4) in Weinberg’s book [1]. Another form of wave equation was given by Papapetrou [2]. He assumed that $g_{\mu \nu}$ are simply gravitational potentials in flat space and both flat space metric $\eta_{\mu \nu}$ and $g_{\mu \nu}$ are present in his theory. It is not clear from his paper whether his wave equation is simply another form of Einstein equation or this is another theory similar to that of Rosen [3] and others.

Later, Thirring [4] showed how one can build up general relativity in the lowest nonlinear approximation starting from flat space and using field theoretical methods. It is important to note that Thirring chooses one specific way to obtain the nonlinear corrections. Namely, after considering the linear approximation, he switches from $g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}$ to $\sqrt{-g} g^{\mu \nu} \equiv \eta^{\mu \nu} - \varphi^{\mu \nu}$. Only in the linear approximation $\varphi^{\mu \nu}$ is equal $\bar{h}^{\mu \nu} = h^{\mu \nu} - \frac{1}{2} \eta^{\mu \nu} h$. I did not pay
attention to this fact and erroneously concluded that Thirring’s wave equation do not lead to Schwarzschild solution \[5\].

Later, Gupta \[6\], Halpern \[7\] and others refer to Papapetrou equation as a form of Einstein equation. Moreover, Deser \[8\], following his predecessors, claims that he deduces Einstein equation in just a few steps by field theoretical means. Yet doubts remain. We note that Halpern \[7\], following Papapetrou, obtains the the gravitational energy-momentum tensor from general relativity Lagrangian by Belinfante method. In the approximation, considered by Thirring, Halpern energy-momentum tensor must coincide with Thirring one. It does not where the matter is present. Moreover, my calculation shows that for interacting fields the Belinfante method leads to an asymmetric energy-momentum tensor. This is a good thing for Thirring because to insure the equation of motion for particles in agreement with general relativity, he introduces ”by hand” the asymmetric interaction energy-momentum tensor. Together with gravitational energy-momentum tensor it restores the necessary symmetry.

On the other hand, Deser \[8\] uses the Rosenfeld method to obtain the gravitational energy-momentum tensor. He is sure that the result must be the same as in Belinfante method. But Rosefeld method gives the symmetrical tensor. One might assume that in this method there is no need to insert ”by hand” some interaction terms to get the desired equation of motion for particles. My calculations show that this is not the case; using the Rosenfeld method one is forced again to insert some interaction terms.

In view of these doubts it is desirable to obtain an answer to the question: what theory is in agreement with general relativity, Thirring theory or Papapetrou one? My calculations show that Thirring wave equation is equivalent to Weinberg’s one. Hence, Thirring wave equation is in agreement with general relativity, not the Papapetrou one.

2 Conservation laws of total energy-momentum tensor and the equation of motion for particles

In this Section we show that the conservation laws, obtained by Thirring in a linear approximation, continue to hold also in \(h^2\) approximation. We use the notation

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1), \quad u^\mu = \frac{dx^\mu}{ds} = \dot{x}^\mu, \quad \phi_\mu = \frac{\partial}{\partial x^\mu} \phi. \tag{1}\]

According to Thirring the total energy-momentum tensor consists of three parts:

- matter part

\[
T^{\mu\nu} = \sum_a m_a u^\mu u^\nu \frac{ds}{dt} \delta(\bar{x} - \bar{x}_a(t)), \quad u^\mu = dx^\mu / ds, \tag{2}\]

- interaction part

\[
T^{\mu\nu}_{\text{int}} = T^{\mu\alpha} h_{\alpha\nu} \tag{3}\]

and gravitational part, consisting of canonical and spin parts

\[
t^{\mu\nu} = t^{\mu\nu}_{\text{can}} + t^{\mu\nu}_{s}. \tag{4}\]
The spin part $\tilde{\mu}^{\alpha\beta}$ do not contribute to the conservation laws. Calculating the divergence of the canonical tensor and using the linearized Einstein equation, we find

$$c_{\text{an}}^{\tilde{\nu}_{\nu} \mu} = -\frac{1}{2} h_{\alpha\beta}^{\nu} M_{\alpha\beta}. \tag{5}$$

Corrections to the linearized Einstein equation would lead to corrections of order $h^3$ in (5) and we neglect them.

The divergence of matter tensor is

$$M_{\nu}^{\mu} \nu_{\mu} = \sum_a m_a \dot{u}^\nu d\nu \delta(\vec{x} - \vec{x}_a(t)), \tag{6}$$

see eq.(2.8.6) in [1]. We use here the particle equation of motion in $h^2$ approximation:

$$\ddot{u}^\nu = -\Gamma_{\alpha\beta}^{\nu} u^\alpha u^\beta \approx [-h_{\alpha\beta}^{\nu} + \frac{1}{2} h_{\alpha\beta}^{\nu} + h^{\nu\sigma}(h_{\alpha\beta}^{\sigma} - \frac{1}{2} h_{\alpha\beta}^{\sigma})] u^\alpha u^\beta. \tag{7}$$

Then we get

$$M_{\nu}^{\mu} \nu_{\mu} = \sum_a m_a \frac{d\nu}{dt}(\vec{x} - \vec{x}_a(t)) u^\alpha u^\beta[-h_{\alpha\beta}^{\nu} + \frac{1}{2} h_{\alpha\beta}^{\nu} + h^{\nu\sigma}(h_{\alpha\beta}^{\sigma} - \frac{1}{2} h_{\alpha\beta}^{\sigma})]. \tag{8}$$

For the interaction tensor we have

$$int_{\nu}^{\sigma} \nu_{\mu} = M_{\nu}^{\mu} \nu_{\mu} h_{\sigma}^{\nu} + M_{\nu}^{\mu} h_{\sigma}^{\nu} \nu_{\mu}. \tag{9}$$

Using (8), we find with our accuracy

$$int_{\nu}^{\sigma} \nu_{\mu} = M_{\nu}^{\mu} \nu_{\mu} h_{\sigma}^{\nu} - h^{\nu\sigma}(h_{\sigma\beta}^{\nu} - \frac{1}{2} h_{\sigma\beta}^{\nu}) M_{\nu}^{\alpha\beta}. \tag{10}$$

From (8) and (10) we have

$$\left( M_{\nu}^{\mu} \nu_{\mu} + int_{\nu}^{\sigma} \nu_{\mu} \right)_{\nu} = \left( M_{\nu}^{\mu} \nu_{\mu} \right)_{\nu} = \frac{1}{2} h_{\alpha\beta}^{\nu} M_{\nu}^{\alpha\beta} + O(h^3). \tag{11}$$

Adding (5), we have the conservation laws for the total energy-momentum tensor

$$\left( M_{\nu}^{\mu} \nu_{\mu} + int_{\nu}^{\sigma} \nu_{\mu} + t^{\nu}_{\mu} \right)_{\nu} = 0. \tag{12}$$

It is interesting to note that lowering superscript $\nu$ in (11) with the help of $\eta$, we obtain

$$\left( M_{\nu}^{\mu} + int_{\nu}^{\sigma} \right)_{\nu} = \left( M_{\nu}^{\mu} \eta_{\sigma\nu} + h_{\sigma\nu} \right)_{\nu} = \frac{1}{2} h_{\alpha\beta}^{\nu} M_{\nu}^{\alpha\beta}. \tag{13}$$

The last equation here is an exact relation of general relativity, see eqs. (96.1) and (106.4) in [9]. So in general relativity $M_{\nu}^{\mu} \eta_{\sigma\nu} + h_{\sigma\nu}$ is simply a mixed tensor. Only in field theoretical approach $int_{\nu}^{\sigma}$ lives in a contravariant form. In general relativity we have only $M_{\nu}^{\mu}$ and it is a tensor density there, see eq.(106.4) in [9].
3 Gravitational energy-momentum tensors

In this Section we compare gravitational energy-momentum tensors computed by Belinfante and Rosenfeld methods. Later we compare also some other tensors. To facilitate the comparison, we introduce special notation for the building blocks of these tensors:

\[ T^{\mu\nu} = \eta^{\mu\nu} h_{\alpha\beta\gamma} \bar{h}^{\gamma\beta\alpha}; \quad T^{\mu}_{\nu} = \eta^{\mu\nu} h_{\alpha\beta} \bar{h}^{\alpha\beta}; \quad T^{\sigma}_{\mu\nu} = \eta^{\mu\nu} h_{\alpha\bar{h}^{\alpha}} \bar{h}^{\sigma}; \quad T^{\mu\nu\sigma} = \eta^{\mu\nu} h_{\alpha\beta} \bar{h}^{\alpha\beta}; \]

\[ T^{\mu}_{\nu\sigma} = \eta^{\mu\nu} h_{\alpha\beta\sigma} \bar{h}^{\beta\alpha}; \quad T^{\mu}_{\nu\sigma\tau} = \eta^{\mu\nu} h_{\alpha\beta\sigma\tau} \bar{h}^{\beta\alpha}; \quad T^{\mu\nu\sigma\tau} = \eta^{\mu\nu} h_{\alpha\beta\sigma\tau} \bar{h}^{\beta\alpha}; \quad T^{\mu\nu\sigma\tau\delta} = \eta^{\mu\nu} h_{\alpha\beta\sigma\tau\delta} \bar{h}^{\beta\alpha}; \]

Similarly for terms with second derivatives

\[ T^{\alpha}_{\mu\nu} = \eta^{\mu\nu} h_{\sigma\alpha} \bar{h}^{\sigma}, \quad T^{\beta}_{\mu\nu} = \eta^{\mu\nu} h_{\alpha\beta} \bar{h}^{\alpha\beta}; \quad T^{\gamma}_{\mu\nu} = \eta^{\mu\nu} h_{\sigma\alpha} \bar{h}^{\sigma\alpha}; \quad T^{\delta}_{\mu\nu} = \eta^{\mu\nu} h_{\sigma\alpha} \bar{h}^{\sigma\alpha}; \]

\[ T^{\epsilon}_{\mu\nu} = \eta^{\mu\nu} h_{\alpha\beta\sigma} \bar{h}^{\alpha\beta\sigma}, \quad T^{\eta}_{\mu\nu} = \eta^{\mu\nu} h_{\alpha\beta\sigma\tau} \bar{h}^{\alpha\beta\sigma\tau}; \quad T^{\eta}_{\mu\nu\sigma} = \eta^{\mu\nu} h_{\alpha\beta\sigma} \bar{h}^{\alpha\beta\sigma}; \quad T^{\eta}_{\mu\nu\sigma\tau} = \eta^{\mu\nu} h_{\alpha\beta\sigma\tau} \bar{h}^{\alpha\beta\sigma}; \]

Starting from the Lagrangian

\[ L = \frac{1}{32\pi G} \left[ h_{\alpha\beta\gamma} \bar{h}^{\gamma\beta\alpha} - \frac{1}{2} h_{\alpha\beta\gamma} \bar{h}^{\alpha\beta\gamma} + \frac{1}{4} h_{\alpha\beta} \bar{h}^{\alpha\beta} \right], \]

we find by Belinfante method the following expression for the sum of the canonical and spin parts

\[ t^{\mu\nu} = t^{\mu\nu}_{\text{can}} + t^{\mu\nu}_{\text{spin}} = \frac{1}{16\pi G} \left[ \frac{1}{2} T^{\mu\nu} - \frac{1}{4} T^{\mu\nu} + \frac{1}{8} T^{\mu\nu} + \frac{1}{2} T^{\mu\nu} + 6 T^{\mu\nu} + 7 T^{\mu\nu} - \frac{1}{4} T^{\mu\nu} \right. \]

\[ - \frac{1}{2} T^{\mu\nu} - 2 \frac{1}{2} T^{\mu\nu} + 2 T^{\mu\nu} + 2 \frac{1}{2} T^{\mu\nu} + 2 T^{\mu\nu} - 2 T^{\mu\nu} + \frac{1}{2} T^{\mu\nu} \]

\[ + \frac{1}{2} T^{\mu\nu} \left( h_{\alpha\beta\sigma} - h_{\alpha\beta\sigma} - h_{\alpha\beta\sigma} - \frac{1}{2} h_{\mu\alpha\beta} \bar{h}_{\alpha\beta} \right) - \frac{1}{2} h_{\alpha\beta\sigma} \left( h_{\alpha\beta\sigma} - h_{\alpha\beta\sigma} - h_{\alpha\beta\sigma} - \frac{1}{2} h_{\mu\alpha\beta} \bar{h}_{\alpha\beta} \right) \]

Here the antisymmetric part is written down explicitly. Using linearized Einstein equation, we can rewrite it as follows

\[ \frac{1}{2} \left( T^{\mu\nu} h_{\alpha\beta} - T^{\mu\alpha} h_{\alpha\nu} \right) \]
In this expression we can drop bars over \( h \) and \( M \). Then we see that (17), where the antisymmetric part is approximately equal (18), together with interaction term (3) yields the symmetric tensor.

At this stage it is convenient to compare (17) with the corresponding Halpern result. Barring a few misprints, the symmetric part in (17) agrees with eq. (3.5a) in [7]. Instead of our antisymmetric part, Halpern gives (without comments) the following asymmetric part

\[
\frac{1}{32\pi G} \bar{h}^\mu_{\alpha}(\bar{h}^{\nu\alpha,\sigma} - \bar{h}^{\nu\sigma,\alpha} - \bar{h}^{\nu\alpha,\sigma} - \frac{1}{2}\eta^{\nu\alpha}\bar{h},_{\sigma})
\]

see eq. (3.5b) in [7]. Turning back to (17), we note that using linearized Einstein equation, we find

\[
\frac{1}{16\pi G} \bar{p}^{\mu\nu} = -\frac{1}{2}(M^{\nu\alpha}\bar{h}_{\alpha}^{\mu} + M^{\mu\alpha}\bar{h}_{\alpha}^{\nu}) + \frac{1}{16\pi G}(\frac{1}{2} \bar{T}^{\mu\nu} + \bar{T}^{\mu\nu} + \bar{T}^{\mu\nu}).
\]

(19)

Similarly we get

\[
\frac{1}{16\pi G} \bar{t}^{\mu\nu} = -\frac{1}{2} \bar{h}^{\mu\nu} \frac{M}{T}, \quad \bar{T} = \bar{T}_{\alpha}^{\alpha}.
\]

(20)

So we have

\[
\frac{1}{16\pi G} \bar{p}^{\mu\nu} = -\frac{1}{2}(M^{\nu\alpha}\bar{h}_{\alpha}^{\mu} + M^{\mu\alpha}\bar{h}_{\alpha}^{\nu}) + \frac{1}{16\pi G}(-\bar{T}^{\mu\nu} + \bar{T}^{\mu\nu} + \bar{T}^{\mu\nu}) - \frac{1}{2} \bar{h}^{\mu\nu} \frac{M}{T}.
\]

(21)

Now the sum of the r.h.s. of (21) and (18) yields

\[
\bar{T}^{\mu\nu} + \bar{t}^{\mu\nu} = -\frac{1}{2} \eta^{\nu\alpha} \bar{h}_{\alpha}^{\mu} + \frac{1}{16\pi G}(\bar{T}^{\mu\nu} + \bar{T}^{\mu\nu} - \bar{T}^{\mu\nu}) - \frac{1}{2} \bar{h}^{\mu\nu} \frac{M}{T}.
\]

(22)

The first term here is

\[
-\bar{h}^{\nu}_{\alpha}(\bar{T}^{\mu\alpha} - \frac{1}{2}\eta^{\mu\alpha} \frac{M}{T}) = \bar{h}^{\nu}_{\alpha} \frac{M}{T} + \frac{1}{2} \bar{h}^{\mu\nu} \frac{M}{T}.
\]

(23)

So the last term in (22) is cancelled by the last term in (23). The first term in (23) has the form

\[
-(h^{\nu}_{\alpha} - \frac{1}{2}\eta^{\nu}_{\alpha} h) \frac{M}{T} = -\bar{T}^{\mu\nu} + \frac{1}{2} \bar{h}^{\mu\nu} \frac{M}{T}.
\]

(24)

Collecting all these results, we finally obtain

\[
\bar{T}^{\mu\nu} + \bar{t}^{\mu\nu} = \frac{1}{16\pi G} \bar{T}^{\mu\nu} + \bar{T}^{\mu\nu} + \bar{T}^{\mu\nu} - \frac{1}{4} \bar{T}^{\mu\nu}
\]

\[
- \frac{1}{2} \bar{T}^{\mu\nu} - \bar{T}^{\mu\nu} - \bar{T}^{\mu\nu} + \bar{T}^{\mu\nu} + \bar{T}^{\mu\nu} - \frac{1}{4} \bar{T}^{\mu\nu}.
\]

(25)

Together with \( M^{\mu\nu} \) it gives the total energy-momentum tensor. We shall see later that it leads to the agreement of Thirring wave equation with general relativity.

Now we give the energy-momentum tensor, obtained from the same Lagrangian (16) by Rosenfeld method:

\[
\bar{t}^{\mu\nu}_{\text{Ros}} = \frac{1}{16\pi G} \bar{T}^{\mu\nu} - \frac{1}{4} \bar{T}^{\mu\nu} + \bar{T}^{\mu\nu} + \frac{5}{2} \bar{T}^{\mu\nu} + \bar{T}^{\mu\nu} - \frac{1}{4} \bar{T}^{\mu\nu}.
\]
\[-\frac{12}{2} T^\mu\nu - 2 T_{\mu\nu} - i T^\mu\nu + \frac{1}{2} T^\mu\nu + 2 \Theta^\mu\nu]. \quad (26)\]

Using (20) we find for the difference of (25) and (26):

\[\int T^\mu\nu + t^\mu\nu - t^{(2)}_{\text{Ros}} = \frac{1}{2} h T^\mu\nu + \frac{1}{2} h^{\mu\nu} T. \quad (27)\]

Its divergence is not zero. Thus, \(t^{(2)}_{\text{Ros}}\) also needs some interaction terms.

Deser uses the method in which both \(\sqrt{-g} g^{\mu\nu}\) and \(\Gamma^\sigma_{\mu\nu}\) are a priori independent. He calculates energy-momentum tensor by Rosefeld method. It seems that his method must be equivalent the usual method in which \(\Gamma^\sigma_{\mu\nu}\) are functions of metric. If so, his wave equation is not the Einstein one already in the considered here approximation.

\section{Equivalence of Thirring and Weinberg wave equations}

In terms of \(h_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h\) where \(g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}\), the Weinberg form of the Einstein equations is

\[\bar{h}_{\mu\nu,\sigma} - (\bar{h}_{\mu\nu,\sigma} + \bar{h}_{\nu\sigma,\mu}) + \eta_{\mu\nu} \bar{h}_{\alpha\beta} = -16\pi G (T_{\mu\nu} + t_{\mu\nu}^{(2)}). \quad (28)\]

Here \(t_{\mu\nu}^{(2)}\) is given below, see (34). Thirring deals with \(\varphi^{\mu\nu}\), defined by

\[\sqrt{-g} g^{\mu\nu} \equiv \eta^{\mu\nu} - \varphi^{\mu\nu}. \quad (29)\]

It seems more natural to remain with \(h_{\mu\nu}\). Then his approach would lead to disagreement with general relativity. Thirring succeeded in obtaining the correct expression for the Schwarzschild solution in the considered approximation. To be sure that there is a complete agreement with general relativity, we have to show that his wave equation is equivalent to Weinberg’s one. To that end we rewrite (28) in terms of \(\varphi_{\mu\nu} = \eta_{\mu\alpha} \eta_{\nu\beta} \varphi^{\alpha\beta}\). With our accuracy from definition (29) we have

\[\varphi_{\mu\nu} = \bar{h}_{\mu\nu} - \eta_{\mu\nu}(\frac{1}{8} \bar{h}^2 - \frac{1}{4} \bar{h}_{\alpha\beta} \bar{h}^{\alpha\beta}) + \frac{1}{2} \bar{h}_{\mu\nu} \bar{h} - \bar{h}_{\mu\nu} \bar{h}_{\tau\nu}. \quad (30)\]

In linear approximation \(\varphi_{\mu\nu} = \bar{h}_{\mu\nu}\) and in quadratic terms we can substitute \(\varphi_{\mu\nu} \leftrightarrow \bar{h}_{\mu\nu}\). So we get

\[\bar{h}_{\mu\nu} = \varphi_{\mu\nu} + \eta_{\mu\nu}(\frac{1}{8} \varphi^2 - \frac{1}{4} \varphi_{\alpha\beta} \varphi^{\alpha\beta}) - \frac{1}{2} \varphi_{\mu\nu} \varphi + \varphi_{\mu\tau} \varphi_{\tau\nu}. \quad (31)\]

Lowering here the superscripts \(\mu\) and \(\nu\) with the help of \(\eta\) and inserting into (28), we find

\[\varphi_{\mu\nu,\sigma} - (\varphi_{\mu\sigma,\nu} + \varphi_{\nu\sigma,\mu}) + \eta_{\mu\nu} \varphi_{\alpha\beta} = -16\pi G (T_{\mu\nu} + t_{\mu\nu}^{(2)}) + [-\frac{1}{2} T_{\mu\nu} + \frac{3}{2} T_{\mu\nu} + \frac{3}{2} T_{\mu\nu} -
\frac{1}{2} \frac{4}{4} T_{\mu\nu} - \frac{5}{5} T_{\mu\nu} - \frac{7}{7} T_{\mu\nu} - \frac{8}{8} T_{\mu\nu} + \frac{9}{9} T_{\mu\nu} - \frac{10}{10} T_{\mu\nu} + \frac{11}{11} T_{\mu\nu} + \frac{13}{13} T_{\mu\nu} + \frac{14}{14} T_{\mu\nu} - \frac{16}{16} T_{\mu\nu} +
\{ - \frac{1}{2} T_{\mu\nu} + \frac{1}{2} \frac{b}{b} T_{\mu\nu} + \frac{1}{2} \frac{c}{c} T_{\mu\nu} + \frac{1}{2} \frac{d}{d} T_{\mu\nu} - \frac{1}{2} \frac{e}{e} T_{\mu\nu} + \frac{1}{2} \frac{f}{f} T_{\mu\nu} + \frac{1}{2} \frac{g}{g} T_{\mu\nu} - \frac{h}{h} T_{\mu\nu} + \frac{1}{2} \frac{i}{i} T_{\mu\nu} + \frac{1}{2} \frac{j}{j} T_{\mu\nu} + \frac{1}{2} \frac{k}{k} T_{\mu\nu} + \frac{1}{2} \frac{l}{l} T_{\mu\nu} -
\]
From (32) and (33) we have

\[ T_{\mu
u} + 2 T_{\mu
u} + 2 \bar{T}_{\mu
u} \].

(32)

Now we write down the Weinberg tensor \( T^{(2)}_{\mu\nu} \) in our notation

\[-16\pi G T^{(2)}_{\mu\nu} = \left\{ \frac{1}{2} T_{\mu\nu} - \frac{3}{4} T_{\mu\nu} - \frac{3}{4} T_{\mu\nu} + \frac{3}{4} T_{\mu\nu} \right\}.\]

(33)

Next we have to express \( T_{\mu\nu} \) in terms of \( \bar{T}_{\mu\nu} \). From definition of \( T_{\mu\nu} \) we have

\[ T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} (-g)^{-1/2} \bar{T}_{\mu\nu} \bar{T}_{\mu\nu} \approx (1 - \frac{1}{2} h) \bar{T}_{\mu\nu} \bar{T}_{\mu\nu} + \bar{T}_{\mu\nu} \bar{T}_{\mu\nu} + \bar{T}_{\mu\nu} \bar{T}_{\mu\nu} + \frac{1}{2} \bar{T}_{\mu\nu} \bar{T}_{\mu\nu} \].

(34)

In terms of \( \bar{T}_{\mu\nu} \) we have

\[ T_{\mu\nu} = T_{\mu\nu} \bar{T}_{\mu\nu} + \bar{T}_{\mu\nu} \bar{T}_{\mu\nu} + \frac{1}{2} \bar{T}_{\mu\nu} \bar{T}_{\mu\nu} \bar{T}_{\mu\nu} \].

(35)

Using linearized Einstein equation

\[ \bar{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{2} T_{\mu\nu} \bar{T}_{\mu\nu} + \frac{1}{4} T_{\mu\nu} \bar{T}_{\mu\nu} + \frac{1}{8} T_{\mu\nu} \bar{T}_{\mu\nu} + \frac{1}{16} T_{\mu\nu} \bar{T}_{\mu\nu} \]

(36)

From (32) and (33) we have

\[ \varphi_{\mu\nu} - (\varphi_{\mu\nu} + \varphi_{\nu\mu}) + \eta_{\mu\nu} \varphi_{\alpha\beta} = -16\pi G T_{\mu\nu} + \left\{ \frac{1}{2} T_{\mu\nu} - \frac{3}{4} T_{\mu\nu} - \frac{3}{4} T_{\mu\nu} + \frac{3}{4} T_{\mu\nu} \right\} \]

(37)

and

\[ \bar{T}_{\mu\nu} - \frac{1}{2} T_{\mu\nu} \bar{T}_{\mu\nu} = \frac{1}{16\pi G} \left[ 2 M_{\mu\nu} - 2 \bar{T}_{\mu\nu} - 2 \bar{T}_{\mu\nu} - 2 \bar{T}_{\mu\nu} \right]. \]

(38)

So eq.(34) takes the form

\[ \varphi_{\mu\nu} - (\varphi_{\mu\nu} + \varphi_{\nu\mu}) + \eta_{\mu\nu} \varphi_{\alpha\beta} = -16\pi G T_{\mu\nu} + \left\{ \frac{1}{2} T_{\mu\nu} - \frac{3}{4} T_{\mu\nu} - \frac{3}{4} T_{\mu\nu} + \frac{3}{4} T_{\mu\nu} \right\} \]

(39)
This is Weinberg equation in terms of $\varphi_{\mu\nu}$. As mentioned earlier, we may use $\varphi_{\mu\nu} = \bar{h}_{\mu\nu}$ in quadratic terms. Raising in (39) $\mu$ and $\nu$ with the help of $\eta$, we obtain the wave equation in Thirring form:

$$\varphi^{\mu\nu,\sigma}_{\quad \sigma} - (\varphi^{\mu\sigma,\nu}_{\quad \sigma} + \varphi^{\nu\sigma,\mu}_{\quad \sigma}) + \eta^\mu\nu \varphi_{\alpha\beta}^{\alpha\beta} = -16\pi G (\bar{T}^\mu{}_{\nu} + \bar{t}^\mu{}_{\nu} + t^\mu{}_{\nu}), \quad (40)$$

see eqs, (76) and (77) in [4]. Here use has been made of eqs. (25) and (37). Thus Thirring wave equation agrees with general relativity.

5 Conclusion

It is shown that Thirring’s field theoretical method to reproduce general relativity in the lowest nonlinear approximation is successful. Yet his approach is specific. More natural way leads to failure. There are no rigorous proofs that field theoretical derivations of gravitational wave equation lead to exact Einstein equation. These facts should stimulate the search for alternative theories of gravity, hopefully without black holes. The latter are undesirable on energy grounds [6,10]. One particular version of gravitational theory without black holes is elaborated by Logunov and his colleagues [11]. Another possibility is considered in [12].

For me it seems natural to assume that in three graviton vertex each graviton interacts with energy-momentum tensor formed by other two gravitons. One promising way to build up a gravitation theory is to deal directly with $S$—matrix formalism and use vertices and free propagators without any recourse to a wave equation.

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References

1. S.Weinberg, Gravitation and Cosmology, New York (1972).
2. A.Papapetrou, Proc. Roy. Irish. Acad. 52A, 11 (1948).
3. N.Rosen, Phys.Rev. 57, 147 (1940).
4. W.E.Thirring, Ann. Phys. (N.Y.) 16, 96 (1961).
5. A.Nikishov, gr-qc/9912034 Part. and Nucl. 32, 5 (2001).
6. S.Gupta, Proc. Phys. Soc. A 65, 608 (1952).
7. L.Halpern, Bulletin de l’Academie royal de Belgique, Classe de Sciences, 49, 256 (1963).
8 S.Deser, Gen. Rel. and Grav., 1, 9 (1970).
9. L.D.Landau and E.M.Lifshitz, The classical theory of fields, Moscow, (1973) (in Russian).
10. A.Nikishov, gr-qc/0310072
11. A.A.Logunov, Part. and Nucl. 29 (1) Jan.-Feb 1998, p 1; The Theory of Gravitational Field, Moscow, Nauka (2000), (in Russian).
12. Yu.Baryshev, gr-qc/9912003