Existence and Exponential Stability of Almost Pseudo Automorphic Solution for Neutral Stochastic Evolution Equations Driven by G-Brownian Motion

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Abstract. This paper mainly concerns the quasi sure exponential stability of square mean almost pseudo automorphic mild solution for a class of neutral stochastic evolution equations driven by G-Brownian motion. By means of evolution operator theorem and fixed point theorem, existence and uniqueness of square mean almost pseudo automorphic mild solution is obtained. Also, a series of sufficient conditions on exponential stability and quasi sure exponential stability are established.

1. Introduction

The article aims to the quasi sure exponential stability of square mean almost pseudo automorphic mild solution for neutral stochastic evolution equations driven by G-Brownian motion (G-NSEEs for short)

\[
d[X(t) - D(t, X(t))] = [AX(t) + f(t, X(t))] dt + g(t, X(t)) dB(t) + h(t, X(t)) dB(t), \quad t \in \mathbb{R}
\]

where \(A(t) : \mathcal{D}(A(t)) \subset L^2_G(\mathcal{F}) \rightarrow L^2_G(\mathcal{F})\) is densely closed linear operator, and satisfies the well known Acquistapace-Terrani conditions (one can see [1] and [5]). \(B(t)\) is a one dimensional G-Brownian motion, the functions \(D, f, g, h : \mathbb{R} \times L^2_G(\mathcal{F}) \rightarrow L^2_G(\mathcal{F})\) are jointly continuous. Since Bochner [3] firstly introduced the results of automorphy, many authors made further study and improvement (one can see [4], [10], [17]).

Because of various applications of almost pseudo automorphy, there have been a wide range of interests on this issue. In particular, under the framework of classical Brownian motion, the stability and existence of pseudo almost automorphic solutions of stochastic differential equations have been considerably discussed. Chen and Lin [6] studied the square mean almost pseudo automorphic solution of SEEs. By means of Weyl fractional derivative, Pardoa and Lizama [19] obtained the existence and uniqueness of weighted pseudo almost automorphic mild solutions for fractional abstract differential equation. In the sense of distribution, Feng and Zong [9] discussed the square mean pseudo almost automorphic solution to stochastic differential equation driven by Lévy process. Most recently, by use of \(C_0\)-semigroup, Cui and Rong [7] established exponential stability of \(\mu\)-pseudo almost automorphic mild solutions for nonlinear SEEs.

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In order to solve some problems in finance, Peng [20] firstly established the basic theory of G-expectation. Moreover, Peng [21] introduced the G-Brownian motion and related G-Itô stochastic calculus. Since then, many scholars made further research on the the G-Brownian motion (one can see [8], [14], [25]). Importantly, based on the G-stochastic analysis theory, stochastic differential equations driven by G-Brownian motion (G-SDEs in short) have been attracting much attention (one can see [2], [16], [22], [23]). Especially, Gu et al. [12] established existence and uniqueness of square mean pseudo almost automorphic mild solutions for G-NSEEs. To close the gap, we first aim to derive the existence of square mean almost automorphic for a class of impulsive G-SDEs with the help of Lyapunov function.

As we know, stability has been one of the most interesting topics of SDEs since Mao [18] established the stability theorem. As to G-SDEs, there are a lot of interesting works including exponential stability, \( H_q \) stability and almost sure exponential stability (one can see [24], [27]). By Razumikhin theorem, Li and Yang [15] derived \( p \)-th moment exponential stability of mild solution of neutral stochastic functional differential equations driven by G-Brownian motion. Recently, Hu et al. [13] considered exponential stability of the square mean almost automorphic for a class of impulsive G-SDEs with the help of Lyapunov function.

However, to the best of our knowledge, there is no result on the existence and stability of square mean pseudo almost automorphic mild solutions for G-NSEEs. To close the gap, we first aim to derive the existence and uniqueness of square mean pseudo almost automorphic mild solutions for G-NSEEs. In section 4, we shall discuss exponential stability and quasi sure exponential stability of square mean pseudo almost automorphic mild solutions.

The structure of this article is arranged as follows. In section 2, some basic notions, preliminaries and lemmas are provided. Section 3 is devoted to studying the existence and uniqueness of square mean pseudo almost automorphic mild solutions for G-NSEEs. In section 4, we shall discuss exponential stability and quasi sure exponential stability of square mean pseudo almost automorphic mild solutions.

2. Notations and Preliminaries

Throughout the paper, we will use the following specified notation. Denote \( \mathbb{R} = (\infty, +\infty), \mathbb{R}^+ = [0, +\infty), \mathbb{N} = \{1, 2, \cdots\} \). If A is a vector or matrix, its transpose is denoted by \( A^T \) and the norm \( |A|^2 = \text{trace}(AA^T) \).

2.1. Itô integral of G-Brownian motion

In this subsection, we begin with some notations and preliminary results with respect to G-Brownian motion. \( \Omega \) is the space of all \( \mathbb{R}^n \)-valued continuous functions with \( \omega_0 = 0 \), equipped with the distance

\[
\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i} \left[ \max_{t \in [0, t]} |\omega^1_t - \omega^2_t| \right] \land 1,
\]

then \((\Omega, \rho)\) is a metric space. We suppose that \( \mathcal{H} \) satisfies \( c \in \mathcal{H} \) for each constant \( c \). If \( X \in \mathcal{H} \), then \( |X| \in \mathcal{H} \). If \( X_1, X_2, \cdots, X_n \in \mathcal{H} \), then \( \varphi(X_1, X_2, \cdots, X_n) \in \mathcal{H} \) for each \( \varphi \in C_{Lip}(\mathbb{R}^n) \), where \( C_{Lip}(\mathbb{R}^n) \) is defined as

\[
C_{Lip}(\mathbb{R}^n) = \{ \varphi : \mathbb{R}^n \to \mathbb{R} | \exists C \in \mathbb{R}^+ \text{ s.t. } |\varphi(x) - \varphi(y)| \leq C (1 + |x|^m + |y|^m) |x - y| \}.
\]

Definition 2.1. \( E : \mathcal{H} \to \mathbb{R} \) is called as a sublinear expectation, if for any \( X, Y \in \mathcal{H} \),

1. if \( X \geq Y \), then \( E(X) \geq E(Y) \).
2. \( E(c) = c \), for any \( c \in \mathbb{R} \).
3. \( E(X + Y) \leq E(X) + E(Y) \).
4. \( E(\lambda X) = \lambda E(X) \), for any \( \lambda \geq 0 \).
For any $\omega \in \Omega$, the canonical process $B_t(\omega)$ is defined by $B_t(\omega) = \omega_t$, $t \geq 0$. The filtration $\mathcal{F}_t$ generated by $(B_t)_{t \geq 0}$ is defined

$$\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t).$$

Let $C_{\phi,\text{Lip}}(\mathbb{R}^n)$ denote the set of all bounded and continuous Lipschitz functions on $\mathbb{R}^n$. For any $t > 0$, let

$$\mathcal{L}_{\phi,\text{Lip}}(\mathcal{F}_t) = \left\{ \xi(\omega) := \phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}), n \geq 1, t_1, t_2, \cdots, t_n \in [0, t], \phi \in C_{\phi,\text{Lip}}(\mathbb{R}^{\infty \times n}) \right\}.$$

Let

$$\mathcal{L}_{\phi,\text{Lip}}(\mathcal{F}) = \bigcup_{t=1}^{\infty} \mathcal{L}_{\phi,\text{Lip}}(\mathcal{F}_t).$$

If $\xi(\omega) = \phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}) \in \mathcal{L}_{\phi,\text{Lip}}(\mathcal{F})$ with $0 < t_1 < t_2 < \cdots < t_n < \infty$, we let

$$E \left[ \phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}) \right] := \phi_n,$$

where $\phi_n$ is iterative procedure defined as

$$\begin{align*}
\phi_1(x_1, x_2, \cdots, x_{n-1}) &= E \left[ \phi(x_1, x_2, \cdots, x_{n-1}, B_{t_2} - B_{t_1}) \right], \\
\phi_2(x_1, x_2, \cdots, x_{n-2}) &= E \left[ \phi_1(x_1, x_2, \cdots, x_{n-2}, B_{t_3} - B_{t_2}) \right], \\
& \vdots \\
\phi_{n-1}(x_1) &= E \left[ \phi_{n-2}(x_1, B_{t_n} - B_{t_{n-1}}) \right], \\
\phi_n(x_1) &= E \left[ \phi_{n-1}(B_{t_n}) \right].
\end{align*}$$

The conditional expectation of $\xi := \phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$ is given by

$$E \left[ \xi | \mathcal{F}_t \right] := \phi_{n-1}(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}).$$

**Definition 2.2.** (G-normal distribution) Assuming that $\underline{\sigma}$, $\overline{\sigma}$ are given nonnegative numbers satisfying $0 \leq \underline{\sigma} \leq \overline{\sigma}$. A random variable $X$ is subject to G-normal distribution, denoted by $X \sim N\left(0, \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\right)$, if for each $\phi \in \mathcal{L}_{\phi,\text{Lip}}(\mathcal{F})$, the operator is defined by

$$E \left[ \phi(B(t) + X) \right] := u(t, X),$$

$u(t, x)$ is the viscosity solution of the following nonlinear heat equation

$$\begin{cases}
\frac{\partial u}{\partial t} - G \frac{\partial^2 u}{\partial x^2} = 0, \\
u(0, x) = \phi(x).
\end{cases}$$

where $G(r) = \frac{1}{2} \left( \overline{\sigma}^2 r^+ - \underline{\sigma}^2 r^- \right)$, $r \in \mathbb{R}$.

**Definition 2.3.** (G-Brownian motion) The expectation operator $E$ on $\mathcal{H}$ defined through the above process is called the G-expectation and the canonical process $B(t)$ is called G-Brownian motion.

Next, we give the definitions of Itô Integral and quadratic variation process with respect to G-Brownian motion.
Definition 2.4. (1) For \( p \geq 1, T > 0, \mathcal{M}^{\emptyset, 0}_{G}([0, T]) \) denotes the space of simple processes by

\[
\mathcal{M}^{\emptyset, 0}_{G}([0, T]) = \{ \eta_\omega(t) = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j, t_{j+1})}(t); \; \xi_j \in L_{G}^p(\mathcal{F}_t), \forall N \geq 1, \\
0 = t_0 < t_1 < \cdots < t_N = T, \; j = 0, 1, 2, \cdots, N-1 \}.
\]

(2) For any \( \eta_\omega(t) = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j, t_{j+1})} \in \mathcal{M}^{\emptyset, 0}_{G}([0, T]) \), its Bochner integral is defined as follows

\[
\int_0^t \eta_\omega(t)dt = \sum_{j=0}^{N-1} \xi_j(\omega)(t_{j+1} - t_j).
\]

(3) Let

\[
\mathcal{E}_T = \frac{1}{T} \int_0^T E[\eta_\omega(t)]dt = \frac{1}{T} \sum_{j=0}^{N-1} E[\xi_j(\omega)](t_{j+1} - t_j),
\]

then, \( \mathcal{E}_T : \mathcal{M}^{\emptyset, 0}_{G} \rightarrow \mathbb{R} \) is also a sublinear expectation.

For each \( p \geq 1, \mathcal{M}^{\emptyset, 0}_{G}([0, T]) \) is the completion of \( \mathcal{M}^{\emptyset, 0}_{G}([0, T]) \) equipped with the form

\[
\| \eta \|_{\mathcal{M}^{\emptyset, 0}_{G}([0, T])} = \left( \frac{1}{T} \int_0^T \| \eta_\omega(t) \|^p dt \right)^{\frac{1}{p}} = \left( \frac{1}{T} \sum_{j=0}^{N-1} E[\xi_j(\omega)]^p(t_{j+1} - t_j) \right)^{\frac{1}{p}}.
\]

Definition 2.5. (Itô Integral)For \( \eta_\omega(t) = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j, t_{j+1})} \in \mathcal{M}^{\emptyset, 0}_{G}([0, T]) \), the Itô integral is defined by

\[
I(\eta) = \int_0^t \eta_\omega dB(s) := \sum_{j=0}^{N-1} \xi_j(B(t_{j+1}) - B(t_j)).
\]

\( L_{G}^p(\mathcal{F}_T) \) (\( p \geq 1 \)) is the completion of \( \mathcal{L}_{L_{G}^p}(\mathcal{F}_T) \) with norm of \( \| X \| = \{ E|X|^p \}^{\frac{1}{p}} \), as well as, \( L_{G}^p(\mathcal{F}) \) the completion of \( \mathcal{L}_{L_{G}^p}(\mathcal{F}) \). It is natural to construct the G-expectation on \((\Omega, L_{G}^p(\mathcal{F}))\) (one can see [21]).

Remark 2.6. For any \( \eta \in \mathcal{M}^{\emptyset, 0}_{G}([0, T]) \), we have \( E\left[ \int_0^T \eta_\omega dB(s) \right] = 0 \).

Remark 2.7. From [11] and [26], we conclude that the map \( I : \mathcal{M}^{\emptyset, 0}_{G}([0, T]) \rightarrow L_{G}^p(\mathcal{F}_T) \) is linear and continuous. Moreover, it can be extended as \( I : \mathcal{M}^{\emptyset, 0}_{G}([0, T]) \rightarrow L_{G}^p(\mathcal{F}) \).

Definition 2.8. When \( t > 0 \), the sequence \( \pi_t \) is a partition of \([0, t] \), \( \pi_t \) : \( 0 = t_0 < t_1 < \cdots < t_N = t \), with the mesh \( \mu(\pi_t) \rightarrow 0 \) as \( N \rightarrow \infty \). The quadratic variation process of G-Brownian motion \( B(t) \) is

\[
\langle B \rangle(t) := \lim_{N \rightarrow \infty} \sum_{j=0}^{N-1} (B(t_{j+1}) - B(t_j))^2 = \hat{B}(t) - 2 \int_0^t B(s)dB(s).
\]

In addition, the mutual variation process of \( B \) and \( \bar{B} \) is

\[
\langle B, \bar{B} \rangle(t) := \frac{1}{4} (\langle B + \bar{B} \rangle(t) - \langle B - \bar{B} \rangle(t)).
\]
Definition 2.9. (Integral w.r.t (B)) For any $\eta \in \mathcal{M}_C^1([0, T])$, the map $Q_{\eta, t}(\eta) : \mathcal{M}_C^1([0, T]) \rightarrow L_C^1(\mathcal{F}_T)$ is defined by

$$Q_{\eta, t}(\eta) = \int_0^t \eta \mathrm{d} (B)(t) := \sum_{j=0}^{N-1} \xi_j \left( \langle B \rangle (t_{j+1}) - \langle B \rangle (t_j) \right).$$

Remark 2.10. $Q_{\eta, t}(\eta)$ is linear and continuous, and can be continuously extended $Q_{\eta, t}(\eta) : \mathcal{M}_C^1([0, T]) \rightarrow L_C^1(\mathcal{F}_T)$.

In order to get the our main results, we introduce some technical lemmas which can be found in [11], [20].

Lemma 2.11. For any $0 \leq t < \infty$,

1. $\mathbb{E} \left[ \left| \int_0^t \eta \mathrm{d} (B)(t) \right|^p \right] \leq \sigma^p \mathbb{E} \left[ \int_0^t |\eta| \mathrm{d}t \right]$ for any $\eta \in \mathcal{M}_C^1([0, T])$.

2. $\mathbb{E} \left[ \left( \int_0^t \eta \mathrm{d} (B)(t) \right)^2 \right] = \mathbb{E} \left[ \int_0^t \eta^2 \mathrm{d} (B)(t) \right]$ for any $\eta \in \mathcal{M}_C^2([0, T])$.

3. $\mathbb{E} \left[ \left( \int_0^t |\eta|^p \mathrm{d}t \right)^2 \right] \leq \int_0^t \mathbb{E}[|\eta|^p] \mathrm{d}t$, for any $\eta \in \mathcal{M}_C^p([0, T]), p \geq 1$.

Lemma 2.12. Let $p \geq 2$, $\eta = \{\eta_r\} \in \mathcal{M}_C^p([0, T])$. Then,

$$\mathbb{E} \left( \sup_{t \leq s \leq t} \left| \int_s^t \eta \mathrm{d}B(r) \right|^p \right) \leq \left( \frac{p}{p - 1} \right)^p \mathbb{E} \left( \left| \int_s^t \eta \mathrm{d}B(r) \right|^p \right).$$

Lemma 2.13. For $p \geq 1$, $\eta = \{\eta_r\} \in \mathcal{M}_C^p([0, T])$. Then,

$$\mathbb{E} \left( \sup_{t \leq s \leq t} \left| \int_s^t \eta \mathrm{d} \langle B \rangle (r) \right|^p \right) \leq \sigma^p |t - s|^{p-1} \left| \int_s^t \mathbb{E}[|\eta|^p] \mathrm{d}r \right|.$$  

By Denis et al. [8] and Wei et al. [28], there exists a weakly compact family $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\mathbb{E}[X] = \sup_{\mathcal{P} \in \mathcal{P}} \mathbb{E}_\mathcal{P}[X], \forall X \in L_C^1(\mathcal{F}).$$

And the related Choquet capacities is defined by

$$\mathcal{C}(A) = \sup_{\mathcal{P} \in \mathcal{P}} \mathcal{P}(A), A \in \mathcal{B}(\Omega).$$

A set $A$ is called polar if $\mathcal{C}(A) = 0$, and a property holds quasi surely (q.s. in short) if it holds outside a polar set.

Lemma 2.14. Suppose $X \in L_C^1(\mathcal{F}_T)$ satisfies $\mathbb{E}[X]^p < \infty$ for some $p > 0$. Then

$$\mathcal{C}(\{X > M\}) \leq \frac{\mathbb{E}[X]^p}{M^p}.$$
2.2. square mean almost automorphic stochastic process

In this subsection, we introduce some concepts of square mean almost automorphic stochastic processes and related properties.

**Definition 2.15.** A stochastically continuous process \( X(t) : \mathbb{R} \to L^2_G(F) \) is square mean almost automorphic if for any real sequence \( \{r_n\}_{n \in \mathbb{N}} \) there exist a subsequence \( \{r_{n_k}\}_{k \in \mathbb{N}} \) and \( Y(t) : \mathbb{R} \to L^2_G(F) \) such that

\[
\lim_{n \to \infty} \mathbb{E}\|X(t + r_n) - Y(t)\|^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}\|Y(t - r_n) - X(t)\|^2 = 0
\]

hold. The collection of all square mean almost automorphic processes is denoted by \( \text{SAA}(\mathbb{R}, L^2_G(F)) \).

\( \text{SBC}(\mathbb{R}, L^2_G(F)) \) is served as the collection of all the stochastically bounded and continuous processes.

**Remark 2.16.** \( \text{SBC}(\mathbb{R}, L^2_G(F)) \) is a Banach space with the norm

\[
\|X\|_\infty = \sup_{t \in \mathbb{R}} \left( \mathbb{E}\|X(t)\|^2 \right)^{\frac{1}{2}}.
\]

**Definition 2.17.** A stochastic process \( X(t) \) belongs to \( \text{SBC}_0(\mathbb{R}, L^2_G(F)) \), if it is one of \( \text{SBC}(\mathbb{R}, L^2_G(F)) \) and satisfies

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \mathbb{E}\|X(t)\|^2 \, dt = 0.
\]

**Remark 2.18.** \( \text{SBC}_0(\mathbb{R}, L^2_G(F)) \) is also a Banach space with the norm \( \|X\|_\infty \).

**Remark 2.19.** If \( X(t) \in \text{SAA}(\mathbb{R}, L^2_G(F)) \), then \( X(t) \) is bounded with the norm \( \|X\|_\infty \). That is, \( \text{SAA}(\mathbb{R}, L^2_G(F)) \subset \text{SBC}(\mathbb{R}, L^2_G(F)) \).

**Definition 2.20.** A continuous stochastic process \( f(t) : \mathbb{R} \to L^2_G(F) \) is called square mean pseudo almost automorphic if it can be decomposed as \( f(t) = g(t) + \varphi(t) \), where \( g(t) \in \text{SAA}(\mathbb{R}, L^2_G(F)) \), \( \varphi(t) \in \text{SBC}_0(\mathbb{R}, L^2_G(F)) \).

We denote \( \text{SPAA}(\mathbb{R}, L^2_G(F)) \) the collection of square mean pseudo almost automorphic processes.

**Remark 2.21.** Under the norm \( \|X\|_\infty \), \( \text{SPAA}(\mathbb{R}, L^2_G(F)) \) is a Banach space.

**Definition 2.22.** A jointly continuous function \( f(t, x) : \mathbb{R} \times L^2_G(F) \to L^2_G(F) \) is square mean pseudo almost automorphic at \( t \) for any \( x \in L^2_G(F) \) if it can be decomposed as \( f = g + \varphi \), where \( g \in \text{SAA}(\mathbb{R} \times L^2_G(F), L^2_G(F)) \), \( \varphi \in \text{SBC}_0(\mathbb{R} \times L^2_G(F), L^2_G(F)) \). We denote the set of all such stochastically continuous processes by \( \text{SPAA}(\mathbb{R} \times L^2_G(F), L^2_G(F)) \).

**Lemma 2.23.** ([5]) If \( f(t, x) : \mathbb{R} \times L^2_G(F) \to L^2_G(F) \) is square mean almost automorphic and satisfies

\[
\mathbb{E}\|f(t, x) - f(t, y)\|^2 \leq C_1 \|x - y\|^2, \quad \text{for all } x, y \in L^2_G(F), \ t \in \mathbb{R},
\]

where \( C_1 \geq 0 \) is independent of \( t \). Then for each \( X(t) \in \text{SPAA}(\mathbb{R}, L^2_G(F)) \), the stochastic process \( F(\cdot) = f(\cdot, X(\cdot)) \) is also square mean almost automorphic.

**Lemma 2.24.** Suppose that \( f(t, x) \in \text{SPAA}(\mathbb{R} \times L^2_G(F), L^2_G(F)) \), and there exists nonnegative constant \( C \) such that,

\[
\mathbb{E}\|f(t, x) - f(t, y)\|^2 \leq C \|x - y\|^2, \quad \text{for any } x, y \in L^2_G(F), \ t \in \mathbb{R}.
\]

Then, \( f(t, X(t)) \in \text{SPAA}(\mathbb{R}, L^2_G(F)) \) for any \( X(t) \in \text{SPAA}(\mathbb{R}, L^2_G(F)) \).
3. Existence of square mean pseudo almost automorphic mild solution

In order to investigate the existence and uniqueness of square mean pseudo almost automorphic mild solution for G-NSEEs, we begin with definition of the mild solutions and some assumptions.

Definition 3.1. An \( \mathcal{F} \)-progressively measurable process \( \{X(t)\}_{t \in \mathbb{R}} \) is called a mild solution of the (1) if the following stochastic integral equation is satisfied

\[
X(t) - D(t, X(t)) = U(t, s) [X(s) - D(s, X(s))] + \int_{s}^{t} U(t, r)f(r, X(r)) \, dr
+ \int_{s}^{t} U(t, r)g(r, X(r)) \, dB(r) + \int_{s}^{t} U(t, r)h(r, X(r)) \, d\langle B \rangle(r)
\]  

(2)

for any \( t \geq s \) and \( s \in \mathbb{R} \).

In order to get the main results, we impose the following assumptions on evolution family and coefficients.

(H1) There exist positive constants \( M \) and \( \mu \) such that the evolution family \( U(t, s) \) generated by \( A(t) \) is exponentially stable,

\[
\|U(t, s)\| \leq Me^{-\mu(t-s)}, \quad t \geq s.
\]

(H2) The coefficients \( D(t, x), f(t, x), g(t, x) \) and \( h(t, x) \): \( \mathbb{R} \times L^2_G(\mathcal{F}) \to L^2_G(\mathcal{F}) \) are functions of SPAA \( \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right) \).

Furthermore, there exist nonnegative constants \( L_D, L_f, L_g \) and \( L_h \) such that

\[
\|D(t, x) - D(t, y)\| \leq L_D \|x - y\|, \quad \|f(t, x) - f(t, y)\| \leq L_f \|x - y\|
\]

and

\[
\|g(t, x) - g(t, y)\| \leq L_g \|x - y\|, \quad \|h(t, x) - h(t, y)\| \leq L_h \|x - y\|
\]

for \( x, y \in L^2_G(\mathcal{F}) \) and \( t \in \mathbb{R} \).

(H3) \( D = D_1 + D_2 \in \text{SPAA} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right) \), where \( D_1 \in \text{SAA} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right), \) \( D_2 \in \text{SBC} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right) \). \( f = f_1 + f_2 \in \text{SPAA} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right), \) where \( f_1 \in \text{SAA} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right), \) \( f_2 \in \text{SBC} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right) \), \( g = g_1 + g_2 \in \text{SPAA} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right), \) where \( g_1 \in \text{SAA} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right), g_2 \in \text{SBC} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right) \), \( h = h_1 + h_2 \in \text{SPAA} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right), \) where \( h_1 \in \text{SAA} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right), h_2 \in \text{SBC} \left( \mathbb{R} \times L^2_G(\mathcal{F}), L^2_G(\mathcal{F}) \right) \).

The following theorem presents the existence and uniqueness of square mean pseudo almost automorphic mild solution.

Theorem 3.2. Assuming that the conditions (H1)-(H3) are satisfied, and

\[
4L_D + 4M^2L_f \frac{\mu}{\mu^2} + 4\sigma^2M^2L_g \frac{\mu}{\mu^2} + 2\sigma^2M^2L_h \frac{\mu}{\mu} < 1.
\]

Then, the system (1) has a unique mild solution \( X \in \text{SPAA} \left( \mathbb{R}, L^2_G(\mathcal{F}) \right) \). Moreover, the solution can be expressed by

\[
X(t) = D(t, X(t)) + \int_{-\infty}^{t} U(t, r)f(r, X(r)) \, dr + \int_{-\infty}^{t} U(t, r)g(r, X(r)) \, dB(r)
+ \int_{-\infty}^{t} U(t, r)h(r, X(r)) \, d\langle B \rangle(r)
\]  

(3)
Due to the dominated convergence theorem, it shows
Because
that $(\Phi X)(t)$ is a mild solution of (1).

By means of the properties of evolution family $U(t, r)$ and elementary inequality, we get
$$
\mathbb{E} \left\| \int_{-\infty}^{t+s} U(t+s, t) f_1 (r, X(r)) \, dr \right\|^2
\leq 2\mathbb{E} \left\| \int_{-\infty}^{t} (U(t+s, t) - I) U(t, r) f_1 (r, X(r)) \, dr \right\|^2 + 2\mathbb{E} \left\| \int_{t}^{t+s} U(t+s, t) f_1 (r, X(r)) \, dr \right\|^2.
$$

Due to the dominated convergence theorem, it shows
$$
\lim_{s \to 0} \mathbb{E} \left\| \int_{-\infty}^{t+s} U(t+s, t) f_1 (r, X(r)) \, dr - \int_{-\infty}^{t} U(t, r) f_1 (r, X(r)) \, dr \right\|^2 = 0.
$$
Combining the properties of evolution family $U(t, r)$ with Lemma 2.11, we have
\[
\mathbb{E} \left\| \int_{-\infty}^{t+s} U(t+s, r)g_1 (r, X(r)) \, d\langle B \rangle (r) - \int_{-\infty}^{t} U(t, r)g_1 (r, X(r)) \, d\langle B \rangle (r) \right\|^2
\]
\[
= \mathbb{E} \left\| \int_{-\infty}^{t} \left( U(t+s, r) - U(t, r) \right) g_1 (r, X(r)) \, d\langle B \rangle (r) + \int_{t}^{t+s} U(t+s, r)g_1 (r, X(r)) \, d\langle B \rangle (r) \right\|^2
\]
\[
\leq 2\mathbb{E} \left\| \int_{-\infty}^{t} U(t+s, r) - U(t, r) \, d\langle B \rangle (r) \right\|^2 + 2\mathbb{E} \left\| \int_{t}^{t+s} U(t+s, r)g_1 (r, X(r)) \, d\langle B \rangle (r) \right\|^2
\]
\[
\leq 2\mathbb{E} \left( \left\| U(t+s, r) - U(t, r) \right\| \left\| g_1 (r, X(r)) \right\| \right)^2 \, dr + 2\mathbb{E} \left( \left\| \int_{t}^{t+s} g_1 (r, X(r)) \, d\langle B \rangle (r) \right\|^2 \right).
\]
And
\[
\mathbb{E} \left\| \int_{-\infty}^{t+s} U(t+s, r)h_1 (r, X(r)) \, dB(r) - \int_{-\infty}^{t} U(t, r)h_1 (r, X(r)) \, dB(r) \right\|^2
\]
\[
= \mathbb{E} \left\| \int_{-\infty}^{t} \left( U(t+s, r) - U(t, r) \right) h_1 (r, X(r)) \, dB(r) + \int_{t}^{t+s} U(t+s, r)h_1 (r, X(r)) \, dB(r) \right\|^2
\]
\[
\leq 2\mathbb{E} \left( \left\| U(t+s, r) - U(t, r) \right\| \left\| h_1 (r, X(r)) \right\| \right)^2 \, dr + 2\mathbb{E} \left( \left\| \int_{t}^{t+s} h_1 (r, X(r)) \, dB(r) \right\|^2 \right).
\]
So, it follows
\[
\lim_{s \to 0} \mathbb{E} \left\| (\Phi_1 X) (t+s) - (\Phi_1 X) (t) \right\|^2 = 0.
\]

Step 2. Because $D(t, x), f(t, x), g(t, x)$ and $h(t, x)$ are the functions of $SAA(\mathbb{R} \times L^2_G (F), L^2_G (F))$, thus, there exists a subsequence $\{n_k\}$ of any real numbers $\{r_k\}_{k \in \mathbb{N}}$ for some stochastic process $\overline{D_t}, \overline{f_t}, \overline{g_t}$ and $\overline{h_t} : \mathbb{R} \times L^2_G (F) \to L^2_G (F)$, such that
\[
\lim_{n \to \infty} \mathbb{E} \left\| D_t (t + r_{n}, X(t + r_{n})) - \overline{D_t} (t, X(t)) \right\|^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E} \left\| D_t (t - r_{n}, X(t - r_{n})) - D_t (t, X(t)) \right\|^2 = 0,
\]
\[
\lim_{n \to \infty} \mathbb{E} \left\| f_t (t + r_{n}, X(t + r_{n})) - \overline{f_t} (t, X(t)) \right\|^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E} \left\| f_t (t - r_{n}, X(t - r_{n})) - f_t (t, X(t)) \right\|^2 = 0,
\]
\[
\lim_{n \to \infty} \mathbb{E} \left\| g_t (t + r_{n}, X(t + r_{n})) - \overline{g_t} (t, X(t)) \right\|^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E} \left\| g_t (t - r_{n}, X(t - r_{n})) - g_t (t, X(t)) \right\|^2 = 0,
\]
\[
\lim_{n \to \infty} \mathbb{E} \left\| h_t (t + r_{n}, X(t + r_{n})) - \overline{h_t} (t, X(t)) \right\|^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E} \left\| h_t (t - r_{n}, X(t - r_{n})) - h_t (t, X(t)) \right\|^2 = 0,
\]
for each $t \in \mathbb{R}$ and $X(t) \in L^2_G (F)$.
In order to verify that $(\Phi_1 X) (t)$ is a square mean almost automorphic process, we consider the operator
\[
(\Phi_1 X) (t) = \overline{D_t} (t, X(t)) + \int_{-\infty}^{t} U(t, r) \overline{f_t} (r, X(r)) \, dr + \int_{-\infty}^{t} U(t, r) \overline{g_t} (r, X(r)) \, d\langle B \rangle (r) + \int_{-\infty}^{t} U(t, r) \overline{h_t} (r, X(r)) \, dB(r),
\]
Then, we have

\[
\mathbb{E} \left\| (\Phi_t X) t + r_n - (\tilde{\Phi}_t X) (t) \right\|^2 \\
= \mathbb{E} \left\| D_1 (t + r_n, X(t) + r_n) + \int_{-\infty}^{t+r_n} U(t + r_n, r) f_1 (r, X(r)) \, dr + \int_{-\infty}^{t+r_n} U(t + r_n, r) g_1 (r, X(r)) \, dB(r) \, (r) \\
+ \int_{-\infty}^{t+r_n} U(t + r_n, r) h_1 (r, X(r)) \, dB(r) - \tilde{D}_1 (t, X(t)) - \int_{-\infty}^{t} U(t, r) \tilde{f}_1 (r, X(r)) \, dr \\
- \int_{-\infty}^{t} U(t, r) \tilde{g}_1 (r, X(r)) \, dB(r) - \int_{-\infty}^{t} U(t, r) \tilde{h}_1 (r, X(r)) \, dB(r) \right\|^2 \\
\leq 4 \mathbb{E} \left\| D_1 (t + r_n, X(t) + r_n) - \tilde{D}_1 (t, X(t)) \right\|^2 \\
+ 4 \mathbb{E} \left\| \int_{-\infty}^{t+r_n} U(t + r_n, r) f_1 (r, X(r)) \, dr - \int_{-\infty}^{t} U(t, r) \tilde{f}_1 (r, X(r)) \, dr \right\|^2 \\
+ 4 \mathbb{E} \left\| \int_{-\infty}^{t+r_n} U(t + r_n, r) g_1 (r, X(r)) \, dB(r) - \int_{-\infty}^{t} U(t, r) \tilde{g}_1 (r, X(r)) \, dB(r) \right\|^2 \\
+ 4 \mathbb{E} \left\| \int_{-\infty}^{t+r_n} U(t + r_n, r) h_1 (r, X(r)) \, dB(r) - \int_{-\infty}^{t} U(t, r) \tilde{h}_1 (r, X(r)) \, dB(r) \right\|^2 .
\]  

(10)

By the Cauchy-Schwarz inequality, we have

\[
\mathbb{E} \left\| \int_{-\infty}^{t+r_n} U(t + r_n, r) f_1 (r, X(r)) \, dr - \int_{-\infty}^{t} U(t, r) \tilde{f}_1 (r, X(r)) \, dr \right\|^2 \\
= \mathbb{E} \left\| \int_{-\infty}^{t} U(t, r) f_1 (r + r_n, X(r + r_n)) \, dr - \int_{-\infty}^{t} U(t, r) \tilde{f}_1 (r, X(r)) \, dr \right\|^2 \\
\leq \int_{-\infty}^{t} U(t, r) \, dr \int_{-\infty}^{t} U(t, r) \mathbb{E} \left\| f_1 (r + r_n, X(r + r_n)) - \tilde{f}_1 (r, X(r)) \right\|^2 \, dr,
\]  

(11)

where the last estimate converges to zero as \( n \to \infty \).

Noting that, for any \( t \in \mathbb{R}, \langle B \rangle (t) := \langle B \rangle (t + r_n) - \langle B \rangle (r_n) \) has the same distribution with \( \langle B \rangle (t) \) and taking the Cauchy-Schwarz inequality again, we have

\[
\mathbb{E} \left\| \int_{-\infty}^{t+r_n} U(t + r_n, r) g_1 (r, X(r)) \, dB(r) - \int_{-\infty}^{t} U(t, r) \tilde{g}_1 (r, X(r)) \, dB(r) \right\|^2 \\
= \mathbb{E} \left\| \int_{-\infty}^{t} U(t, r) \left[ g_1 (r + r_n, X(r + r_n)) - \tilde{g}_1 (r, X(r)) \right] \, dB(r) \right\|^2 \\
\leq \delta^4 \mathbb{E} \left\| \int_{-\infty}^{t} U(t, r) \left[ g_1 (r + r_n, X(r + r_n)) - \tilde{g}_1 (r, X(r)) \right] \, dr \right\|^2 \\
\leq \delta^4 \int_{-\infty}^{t} U(t, r) \, dr \int_{-\infty}^{t} U(t, r) \mathbb{E} \left\| g_1 (r + r_n, X(r + r_n)) - \tilde{g}_1 (r, X(r)) \right\|^2 \, dr.
\]  

(12)

Therefor, we have

\[
\lim_{n \to \infty} \mathbb{E} \left\| \int_{-\infty}^{t+r_n} U(t + r_n, r) g_1 (r, X(r)) \, dB(r) - \int_{-\infty}^{t} U(t, r) \tilde{g}_1 (r, X(r)) \, dB(r) \right\|^2 = 0.
\]
Let \( \tilde{B}(t) = B(t + r_n) - B(r_n) \) for each \( t \in \mathbb{R} \), then \( \tilde{B}(t) \) is also a G-Brownian motion with the same distribution as \( B(t) \), we obtain

\[
\mathbb{E} \| \int_{-\infty}^{t+r_n} U(t + r_n, r) h_1(r, X(r)) \, dB(r) - \int_{-\infty}^{t} U(t, r) \tilde{h}_1(r, X(r)) \, dB(r) \|^2 \\
= \mathbb{E} \left\| \int_{-\infty}^{t} U(t, r) \left[ h_1(r + r_n, X(r + r_n)) - \tilde{h}_1(r, X(r)) \right] \, dB(r) \right\|^2 \\
\leq \sigma^2 \int_{-\infty}^{t} \| U(t, r) \|^2 \mathbb{E} \left\| h_1(r + r_n, X(r + r_n)) - \tilde{h}_1(r, X(r)) \right\|^2 \, dr, \tag{13}
\]

where the last estimate converges to zero as \( n \to \infty \).

Therefore, we can conclude that

\[
\lim_{n \to \infty} \mathbb{E} \left\| (\Phi_1X)(t + r_n) - (\tilde{\Phi}_1X)(t) \right\|^2 = 0.
\]

By an analogous arguments as above, we have

\[
\lim_{n \to \infty} \mathbb{E} \left\| (\tilde{\Phi}_1X)(t - r_n) - (\Phi_1X)(t) \right\|^2 = 0.
\]

From the Steps 1 and 2, we have \((\Phi_1X)(t) \in SAA \left( \mathbb{R}, L^2_G(F) \right)\).

**Step 3.** As the similar way as Step 1, we can prove that \((\Phi_2X)(t)\) is stochastically continuous process. According to the functions \( D_2, F_2, G_2 \) and \( H_2 \in SBC_0 \left( \mathbb{R} \times L^2_G(F), L^2_G(F) \right) \), it follows that \((\Phi_2X)(t)\) is stochastically bounded. In what follows, we aim to prove

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \mathbb{E} \| (\Phi_2X)(t) \|^2 \, dt = 0.
\]
From the definition of $(\Phi_2 X)(t)$, we have

\[
\frac{1}{2T} \int_{-T}^{T} E \|(\Phi_2 X) (t)\|^2 dt \\
\leq 4 \left\{ \frac{1}{2T} \int_{-T}^{T} E \| D_2 (t, X(t)) \|^2 dt + \frac{1}{2T} \int_{-T}^{T} E \left\| \int_{-\infty}^{t} U(t, r) f_2 (r, X(r)) dr \right\|^2 dt \\
+ \frac{1}{2T} \int_{-T}^{T} E \left\| \int_{-\infty}^{t} U(t, r) g_2 (r, X(r)) d \langle B \rangle (r) \right\|^2 dt \right\}
\]

As to the second part of the last inequality, it follows

\[
\frac{1}{2T} \int_{-T}^{T} E \| D_2 (t, X(t)) \|^2 dt \\
+ \frac{M^2}{\mu} \times \frac{1}{2T} \int_{-T}^{T} \left[ \int_{-\infty}^{t} e^{-\mu(t-r)} \| f_2 (r, X(r)) \|^2 dr \right] dt \\
+ \frac{M^2 \sigma^4}{\mu} \times \frac{1}{2T} \int_{-T}^{T} \left[ \int_{-\infty}^{t} e^{-\mu(t-r)} \| g_2 (r, X(r)) \|^2 dr \right] dt \\
+ \frac{M^2 \sigma^2}{\mu} \times \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{t} e^{-2\mu(t-r)} \| h_2 (r, X(r)) \|^2 dr dt \}
\]

As the second part of the last inequality, it follows

\[
\frac{1}{2T} \int_{-T}^{T} dt \int_{-\infty}^{t} e^{-\mu(t-r)} \| f_2 (r, X(r)) \|^2 dr \\
= \frac{1}{2T} \int_{-T}^{T} dr \int_{-\infty}^{t} e^{-\mu(t-r)} \| f_2 (r, X(r)) \|^2 dr + \frac{1}{2T} \int_{-T}^{T} dt \int_{-\infty}^{t} e^{-\mu(t-r)} \| f_2 (r, X(r)) \|^2 dr \\
= \frac{1}{2T} \int_{-T}^{T} dr \int_{-\infty}^{t} e^{-\mu(t-r)} \| f_2 (r, X(r)) \|^2 dr + \frac{1}{2T} \int_{-T}^{T} dt \int_{-\infty}^{t} e^{-\mu(t-r)} \| f_2 (r, X(r)) \|^2 dr \\
\leq \frac{1}{\mu} \frac{1}{2T} \int_{-T}^{T} \| f_2 (r, X(r)) \|^2 dr + \frac{1}{\mu^2} \frac{1}{2T} \| f_2 (r, X(r)) \|^{2}_{\infty} \rightarrow 0
\]

as $T \to \infty$.

Taking the similar method, we have

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[ \int_{-\infty}^{t} e^{-\mu(t-r)} \| g_2 (r, X(r)) \|^2 dr \right] dt = 0
\]

and

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{t} e^{-2\mu(t-r)} \| h_2 (r, X(r)) \|^2 dr dt = 0.
\]
Thus, we proved that \((\Phi X)(t) \in SBC_0(\mathbb{R}, L^2_{\mathbb{C}}(F))\). According to the above three steps, we could demonstrate \((\Phi X)(t) \in SPA\{\mathbb{R}, L^2_{\mathbb{C}}(F)\}\).

**Uniqueness** In the following parts, we will introduce that \(\Phi\) has a unique fixed point. If \(X(t)\) and \(Y(t)\) are the solutions of (1), we have

\[
E\left\| (\Phi X)(t) - (\Phi Y)(t) \right\|^2 \\
= E\left\| D(t, X(t)) + \int_{t}^{4} U(t, r)f(r, X(r))dr + \int_{t}^{4} U(t, r)g(r, X(r))d\langle B \rangle(r) \\
+ \int_{t}^{4} U(t, r)h(r, X(r))dB(r) - D(t, Y(t)) + \int_{t}^{4} U(t, r)f(r, Y(r))dr \\
- \int_{t}^{4} U(t, r)g(r, Y(r))d\langle B \rangle(r) - \int_{t}^{4} U(t, r)h(r, Y(r))dB(r) \right\|^2 \\
\leq 4E\|D(t, X(t)) - D(t, Y(t))\|^2 + 4E\left\| \int_{t}^{4} U(t, r)[f(r, X(r)) - f(r, Y(r))]dr \right\|^2 \\
+ 4E\left\| \int_{t}^{4} U(t, r)[g(r, X(r)) - g(r, Y(r))]d\langle B \rangle(r) \right\|^2 \\
+ 4E\left\| \int_{t}^{4} U(t, r)[h(r, X(r)) - h(r, Y(r))]dB(r) \right\|^2 \\
= 4 \sum_{i=1}^{4} \Pi_i(t)
\]

From the assumption (H2), we get

\[
\Pi_1(t) = E\left\| D(t, X(t)) - D(t, Y(t)) \right\|^2 \leq L_D \sup_{t \in \mathbb{R}} E\|X(t) - Y(t)\|^2.
\]

By Cauchy-Schwarz inequality, (H1) and (H2), we obtain

\[
\Pi_2(t) = E\left\| \int_{t}^{4} U(t, r)[f(r, X(r)) - f(r, Y(r))]dr \right\|^2 \\
\leq \int_{t}^{4} U(t, r)drE\left\| \int_{t}^{4} U(t, r)[f(r, X(r)) - f(r, Y(r))]dr \right\|^2 \\
\leq M^2L_f \int_{t}^{4} e^{\mu(t-r)}E\|X(r) - Y(r)\|^2dr \\
\leq \frac{M^2L_f}{\mu^2} \sup_{t \in \mathbb{R}} E\|X(t) - Y(t)\|^2.
\]

From Lemma 2.11, Cauchy-Schwarz inequality, (H1) and (H2), one can prove that

\[
\Pi_3(t) = E\left\| \int_{t}^{4} U(t, r)[g(r, X(r)) - g(r, Y(r))]d\langle B \rangle(r) \right\|^2 \\
\leq \sigma^4E\left\| \int_{t}^{4} U(t, r)[g(r, X(r)) - g(r, Y(r))]dr \right\|^2 \\
\leq \sigma^4 \int_{t}^{4} U(t, r)dr \int_{t}^{4} U(t, r)E\|g(r, X(r)) - g(r, Y(r))\|^2dr \\
\leq \frac{\sigma^4M^2L_g}{\mu^2} \sup_{t \in \mathbb{R}} E\|X(t) - Y(t)\|^2.
\]
From Lemma 2.11, (H1) and (H2), we can verify

\[
\Pi_4(t) = \mathbb{E} \left\| \int_{-\infty}^{t} U(t, r) [h(r, X(r)) - h(r, Y(r))] \, dB(r) \right\|^2 \\
= \mathbb{E} \int_{-\infty}^{t} \left\| U(t, r) [h(r, X(r)) - h(r, Y(r))] \right\|^2 \, d(B)(r) \\
\leq \sigma^2 M^2 L_h \int_{-\infty}^{t} e^{-2\bar{\sigma}(t-r)} \mathbb{E} \left\| X(r) - Y(r) \right\|^2 \, dr \\
\leq \frac{\sigma^2 M^2 L_h}{2\mu} \sup_{t \in \mathbb{R}} \mathbb{E} \left\| X(t) - Y(t) \right\|^2.
\]

(18)

It follows from (15) to (18), we deduce

\[
\mathbb{E} \left\| (\Phi X)(t) - (\Phi Y)(t) \right\|^2 \leq \left[ 4L_D + \frac{4M^2 L_f}{\mu^2} + \frac{\sigma^4 M^2 L_g}{\mu^2} + \frac{2\sigma^2 M^2 L_h}{\mu} \right] \sup_{t \in \mathbb{R}} \mathbb{E} \left\| X(t) - Y(t) \right\|^2.
\]

So,

\[
\left\| (\Phi X)(t) - (\Phi Y)(t) \right\|^2_{\text{SPAA}} \leq \left[ 4L_D + \frac{4M^2 L_f}{\mu^2} + \frac{\sigma^4 M^2 L_g}{\mu^2} + \frac{2\sigma^2 M^2 L_h}{\mu} \right] \left\| X(t) - Y(t) \right\|^2_{\text{SPAA}}.
\]

(20)

Consequently, \( \Phi \) has a unique fixed point in \( \text{SPAA} \left( \mathbb{R}, L^2_{\text{C}}(F) \right) \), which shows that (1) has unique square mean pseudo almost automorphic mild solution.

4. Stability of square mean pseudo almost automorphic solution

In this section, we firstly introduce the definitions of exponential stability. In order to obtain the main results, we let \( D(t, 0) = f(t, 0) = g(t, 0) = h(t, 0) = 0 \).

**Definition 4.1.** The square mean pseudo almost automorphic mild solution \( X(t) \) of (1) is

1. exponentially stable in mean square if for any initial value \( X(t_0) \), the solution \( X(t) \) satisfies

\[
\mathbb{E} \left\| X(t) \right\|^2 \leq C \mathbb{E} \left\| X(t_0) \right\|^2 e^{-\lambda t},
\]

where \( \lambda \) and \( C \) are positive constants independent of \( t_0 \).

2. quasi sure exponentially stable if for any initial value \( X(t_0) \), the solution \( X(t) \) satisfies

\[
\limsup_{t \to \infty} \frac{1}{t} \log \left\| X(t) \right\| \leq -\lambda, \ q.s.,
\]

where \( \lambda > 0 \).

**Theorem 4.2.** Assuming that the conditions (H1)-(H3) are satisfied and

\[
5L_D + \frac{5M^2 L_f}{\mu^2} + \frac{5\sigma^4 M^2 L_g}{\mu^2} + \frac{5\sigma^2 M^2 L_h}{2\mu} < 1.
\]

Then the square mean pseudo almost automorphic mild solution \( X(t) \) of (1) is exponentially stable.
Therefore, we get

\[ C \]

Similar to the proof of \textbf{Uniqueness}, we have

\[
E\|X(t)\|^2 \leq \left[ 5L_D + \frac{5M^4L_f}{\mu^2} + \frac{5\sigma^4M^2L_{L_2}}{\mu^2} + \frac{5\sigma^2M^2L_{L_0}}{2\mu} \right] \sup_{t \in \mathbb{R}} E\|X(t)\|^2 + 10M^2(1 + L_D)e^{-\mu(t-t_0)}E\|X(t_0)\|^2.
\]

Therefore, we get

\[
E\|X(t)\|^2 \leq C E\|X(t_0)\|^2 e^{-\mu(t-t_0)}.
\]

where \( C = 10M^2(1 + L_D) / (1 - 5L_D - \frac{5M^4L_f}{\mu^2} - \frac{5\sigma^4M^2L_{L_2}}{\mu^2} - \frac{5\sigma^2M^2L_{L_0}}{2\mu}) \). So, we can find that the square mean pseudo almost automorphic mild solution \( X(t) \) of (1) is exponentially stable.

\textbf{Theorem 4.3.} Assuming that all the conditions of 4.2 are satisfied. Then the square mean pseudo almost automorphic mild solution \( X(t) \) of (1) is said to be quasi sure exponentially stable.

\textbf{Proof:} From the Theorem 4.2, we have

\[
E\|X(t)\|^2 \leq C E\|X(t_0)\|^2 e^{-\mu(t-t_0)}.
\] (21)

By the elementary inequality, we obtain

\[
\|X(t + s)\|^2 = \|D(t + s, X(t + s)) + \int_t^{t+s} U(t + s, r) f (r, X(r)) \, dr + \int_t^{t+s} U(t + s, r) g (r, X(r)) \, d \langle B \rangle (r) + \int_t^{t+s} U(t + s, r) h (r, X(r)) \, dB(r)
\]

\[ + U(t + s, t) \left[ X(t) - D(t, X(t)) \right] \|^2 \]

\[ \leq 5\|D(t + s, X(t + s))\|^2 + 5\left\| \int_t^{t+s} U(t + s, r) f (r, X(r)) \, dr \right\|^2 + 5\left\| \int_t^{t+s} U(t + s, r) g (r, X(r)) \, d \langle B \rangle (r) \right\|^2
\]

\[ + 5\left\| \int_t^{t+s} U(t + s, r) h (r, X(r)) \, dB(r) \right\|^2 + 5\left\| U(t + s, t) \left[ X(t) - D(t, X(t)) \right] \right\|^2
\]

\[ := 5 \sum_{i=1}^{5} \upsilon_i(t). \] (22)

From assumption (H1) and (H2), we obtain

\[
E\left[ \sup_{0 \leq s \leq T} \upsilon_1(s) \right] = E\left[ \sup_{0 \leq s \leq T}\|D(t + s, X(t + s))\|^2 \right]
\]

\[ \leq L_D E\left[ \sup_{0 \leq s \leq T}\|X(t + s)\|^2 \right]. \] (23)
By Cauchy-Schwarz inequality, (H1) and (H2), we get

\[
E \left[ \sup_{0 \leq s \leq t} Y_2(s) \right] = \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| \int_t^{t+s} U(t+s,r) f(r, X(r)) \, dr \right\|^2 \right]
\leq \mathbb{E} \sup_{0 \leq s \leq t} \left\{ \int_t^{t+s} |U(t+s,r)|^2 \, dr \int_t^{t+s} \| f(r, X(r)) \|^2 \, dr \right\}
\leq \frac{M^2 L_f}{2\mu} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| X(s) \|^2 \right]
\leq \frac{M^2 L_f}{2\mu} \int_t^{t+s} \mathbb{E} \| X(r) \|^2 \, dr
\leq \frac{M^2 L_f}{2\mu} \mathbb{E} \| X(t_0) \|^2 \int_t^{t+s} e^{-\mu(t-t_0)} \, dr
\leq \frac{M^2 L_f}{2\mu^2} \mathbb{E} \| X(t_0) \|^2 e^{-\mu(t-t_0)}.
\tag{24}
\]

From (H1), (H2) and Lemma 2.13, it shows

\[
E \left[ \sup_{0 \leq s \leq t} Y_3(s) \right] \leq M^2 \sigma^2 \gamma \int_t^{t+s} e^{-2\mu(t-r)} \mathbb{E} \| g(r, X(r)) \|^2 \, dr
\leq M^2 \sigma^2 \gamma^2 L_g \int_t^{t+s} e^{-2\mu(t-r)} \mathbb{E} \| X(r) \|^2 \, dr
\leq M^2 \sigma^2 \gamma^2 L_g \mathbb{E} \| X(t_0) \|^2 \int_t^{t+s} e^{-2\mu(t-r)} e^{-\mu(t-t_0)} \, dr
\leq \frac{M^2 \sigma^2 \gamma^2 L_g}{\mu} \mathbb{E} \| X(t_0) \|^2 e^{-\mu(t-t_0)}.
\tag{25}
\]

From Lemma 2.11 and 2.12, (H1) and (H2),

\[
E \left[ \sup_{0 \leq s \leq t} Y_4(s) \right] \leq 4\alpha^2 M^2 L_h \int_t^{t+s} e^{-2\mu(t-r)} \mathbb{E} \| X(r) \|^2 \, dr
\leq 4\alpha^2 M^2 L_h \mathbb{E} \| X(t_0) \|^2 \int_t^{t+s} e^{-2\mu(t-r)} e^{-\mu(t-t_0)} \, dr
\leq \frac{4\alpha^2 M^2 L_h}{\mu} \mathbb{E} \| X(t_0) \|^2 e^{-\mu(t-t_0)}.
\tag{26}
\]

By elementary inequality, (H1) and (H2), we have

\[
5E \left[ \sup_{0 \leq s \leq t} Y_3(s) \right] = 5E \| U(t + s, t) [X(t) - D(t, X(t))] \|^2
\leq 10M^2 (1 + L_D) \mathbb{E} \| X(t_0) \|^2 e^{-\mu(t-t_0)}.
\tag{27}
\]

It follows from (23) to (27), we deduce

\[
(1 - 5L_D) E \left[ \sup_{h \leq s \leq t} \| X(s) \|^2 \right]
\leq \frac{5M^2 L_f}{2\mu^2} + \frac{4\alpha^2 M^2 L_h}{\mu} \mathbb{E} \| X(t_0) \|^2 e^{-\mu(t-t_0)}.
\tag{28}
\]
So, by virtue of \( 5L_D + \frac{5M_2L_f}{\mu} + \frac{5M_2L_h}{\mu} + \frac{5\sigma^2M_2}{2\mu} < 1 \), we get

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \| X(t + s) \|_2^2 \right] \leq M_0 e^{-\mu(t-t_0)},
\]

(29)

where \( M_0 = \left[ \frac{5M_2L_f}{2\mu} + \frac{5M_2\sigma^2}{\mu} e^\tau + \frac{20\sigma^2M_2}{\mu} e^\tau + 10M_2(1 + L_D) \right] \mathbb{E} \| X(t_0) \|_2^2 / (1 - 5L_D) \).

Consequently, for any \( \epsilon \in (0, \mu) \),

\[
\mathcal{C} \left( \omega : \sup_{0 \leq s \leq t} \| X(n\tau + s) \|_2^2 \geq e^{-n(\mu-\epsilon)\tau} \right) \leq \frac{\mathbb{E} \left[ \sup_{0 \leq s \leq t} \| X(n\tau + s) \|_2^2 \right]}{e^{-n(\mu-\epsilon)\tau}} \leq M_0 e^{-n\epsilon \tau} \mathbb{E} \| X(t_0) \|_2^2.
\]

According to Borel-Cantelli Lemma, we can conclude there exists a \( k_0(\omega) \) such that for almost all \( \omega \in \Omega, k \geq k_0(\omega) \),

\[
\sup_{0 \leq s \leq t} \| X(n\tau + s) \|_2^2 \leq e^{-n(\mu-\epsilon)\tau}.
\]

This implies

\[
\log \sup_{n \tau \leq s \leq (n+1)\tau} \| X(t) \| \leq -\frac{\mu - \epsilon}{2}, \text{ q. s.}
\]

Therefore, we can obtain

\[
\limsup_{t \to \infty} \frac{\log \| X(t) \|}{t} \leq -\frac{\mu - \epsilon}{2}, \text{ q. s.}
\]

Letting \( \epsilon \to 0 \), we obtain the desired results.

5. Conclusion

In this paper, a class of neutral stochastic evolution equations driven by G-Brownian motion has been studied. Firstly, under classical Lipchitz conditions, the existence and uniqueness of square mean pseudo almost automorphic mild solutions to the stochastic system. Next, the quasi sure exponential stability of square mean pseudo almost automorphic mild solutions to neutral stochastic evolution equations is investigated based stochastic analysis theory and Borel-Cantelli Lemma. Moreover, we obtained the exponential stability of square mean pseudo almost automorphic mild solutions.

Availability of supporting data
The data sets supporting the results are included within the article.

Competing interests
The author declare that he has no competing interests.

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References

[1] P. Acquistapace, F. Flandoli, B. Terreni. Initial boundary value problems and optimal control for nonautonomous parabolic systems. SIAM J. Control Optim. 29 (1991) 89-118.
[2] X. Bai, Y. Lin, On the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion with integral-Lipschitz coefficients. Acta. Math. Appl. Sin Engl Ser. 30 (2014) 589 - 610.
[3] S. Bochner, Uniform convergence of monotone sequences of functions, Proc. Natl. Acad. Sci. USA 47(4) (1961) 582 - 585.
[4] P. Cieutat, S. Fatajou, G. N’Guérékata, Composition of pseudo almost periodic and pseudo almost automorphic functions and applications to evolution equations. Appl. Anal. 89(1) (2010) 11 - 27.
[5] Y. Chang, Z. Zhao, G. N’Guérékata. Square mean almost automorphic mild solutions to non-autonomous stochastic differential equations in Hilbert spaces. Computer. Math. Appl. 61 (2011) 384-391.
[6] Z. Chen, W. Lin, Square mean pseudo almost automorphic process and its application to stochastic evolution equations. J. Funct. Anal. 261(2011) 69-89.
[7] J. Cui, W. Rong, Existence and stability of µ-pseudo almost automorphic solutions for stochastic evolution equations, Front. Math. China 14(2) (2019) 261-280.
[8] L. Denis, M. Hu, S. Peng, Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths. Potential Anal. 34 (2011) 139-161.
[9] X. Feng, G. Zong, Pseudo almost automorphic solution to stochastic differential equation driven by Lévy process, Front. Math. China 13(4) (2018) 779-796.
[10] M. Fu, Almost automorphic solutions for nonautonomous stochastic differential equations, J. Math Anal. Appl. 393 (2012) 231-238.
[11] F. Gao, Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion, Stoch. Proc. Appl. 119 (2009) 3356-3382.
[12] Y. Gu, Y. Ren, R. Sakthivel, Square mean pseudo almost automorphic mild solutions for stochastic evolution equations driven by G-Brownian motion, Stoch. Anal. Appl. 34(3) (2016) 528-545.
[13] L. Hu, Y. Ren, R. Sakthivel, Stability of square mean almost automorphic mild solutions to impulsive stochastic differential equations driven by G-Brownian motion, Int. J. Control. 93(12)(2020) 3016-3025.
[14] M. Hu, F. Wang, G. Zheng, Quasi-continuous random variables and processes under the G-expectation framework, Stoch. Proc. Appl. 126(8) (2016) 2367-2387.
[15] G. Li, Q. Yang, Stability of neutral stochastic functional differential equations with Markovian switching driven by G-Brownian motion, Appl. Anal. 97(15) (2018) 2555-2572.
[16] X. Li, X. Lin, Y. Lin, Lyapunov-type conditions and stochastic differential equations driven by G-Brownian motion. J Math. Anal. Appl. 439(1) (2016) 235 -255.
[17] Z. Li, L. Xu, L. Van, Stepanov-like almost automorphic solutions for stochastic differential equations with Lévy noise, Communications in Statistics - Theory and Methods, 47(6) (2018) 1350-1371.
[18] X. Mao, Stochastic stabilization and destabilization, Systems Control Lett. 23 (1994) 279-290.
[19] E. Pardoa, C. Lizama, Weighted pseudo almost automorphic mild solutions for two-term fractional order differential equations, Appl. Math. Comput. 271 (2015) 154-167.
[20] S. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Itô type. Stoch. Anal. Appl. Abel. Symp. 2 (2007) 51-56.
[21] S. Peng, Nonlinear expections and stochastic calculus under uncertainty. 2010. arXiv:1002.4546.
[22] Y. Ren, W. Yin, Asymptotical boundedness for stochastic coupled systems on networks with time-varying delay driven by G-Brownian motion, Int. J. Control 92(10) (2019) 2235-2242.
[23] W. Wei, M. Zhang, P. Luo, Asymptotic estimates for the solution of stochastic differential equations driven By G-Brownian motion, Appl. Anal. 97(12) (2018) 2025-2036.
[24] L. Xu, S. Ge, H. Hu, Boundedness and stability analysis for impulsive stochastic differential equations driven by G-Brownian motion, Int. J. Control 92(3)(2019) 642-652.
[25] L. Yan, Y. Li, K. He, Rough path analysis for local time of G-Brownian motion, Appl. Anal. 99(6)(2020) 899-921.
[26] B. Zhang, J. Xu, D. Kannan. Extention and application of Itô’s formula under G-framework. Stoch. Anal. Appl. 28 (2010) 322-349.
[27] D. Zhang, Z. Chen, Exponential stability for stochastic differential equation driven by G-Brownian motion. Appl. Math. Lett. 25(11) (2012) 1906-1910.
[28] W. Wei, M. Zhang, P. Luo, Asymptotic estimates for the solution of stochastic differential equations driven By G-Brownian motion, Appl. Anal. 97(12) (2018) 2025-2036.
[29] F. Wu, S. Hu , Y. Liu, Positive solution and its asymptotic behavior if stochastic functional Kolmogorov type system. J. Math. Anal. Appl. 364 (2010) 104-118.