NON-EXTREME WEIGHT MODULES FOR QUANTIZED UNIVERSAL 
ENVELOPING ALGEBRAS

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Abstract. For quantized universal enveloping algebras we construct weight modules by 
inducing representations of the centralizer of the Cartan subalgebra in the quantized universal 
enveloping algebra. The induced modules arising from finite-dimensional weight modules 
the centralizer algebra are studied. In particular, we study the induction of one-dimensional 
modules, and this is related to the study of commutative subalgebras of the centralizer 
algabra. For the special case of \( U_q(sl(2, \mathbb{C})) \) we show that we get the admissible unitary 
representations corresponding to the non-compact real form \( U_q(sl(1,1)) \).

1. Introduction

Large classes of representations of quantized universal enveloping algebras \( U_q(\mathfrak{g}) \) for simple 
complex Lie algebras, such as finite dimensional representations or Verma modules, see e.g. 
[4], for the classical case, are well understood, see e.g. [3], [9], [10], [11]. On the other 
hand, these representations (or modules) do not suffice for the harmonic analysis on quantum 
analogs of non-compact quantum groups. The best known example of an analytically studied 
non-compact quantum group is the quantum analog of the universal enveloping algebra of \( su(1,1) \). The irreducible \(*\)-representations have been classified by Vaksman and Korogodski˘ ı [19], by Burban and Klimyk [2] and by Masuda et al. [14]. In this case we see that the 
representation theory of \( U_q(su(1,1)) \) differs from the irreducible unitary representations of 
the Lie algebra \( su(1,1) \). The so-called strange series representations do not have a classical 
analog; they formally vanish in the limit \( q \to 1 \). It turns out that in the analytic study of 
this non-compact quantum group these representations play an important role, see [7], [12], 
[18], [19] and references given there. The representations that play a role in this example 
are non-extremal weight representations, i.e. these representations are weight representations 
that have neither a highest weight nor a lowest weight. In this paper we present another way 
to obtain these representations.

The idea is to use the centralizer of the analog \( U^0 \) of the Cartan subalgebra of \( U_q(\mathfrak{g}) = \mathfrak{u}^- \otimes U^0 \otimes \mathfrak{u}^+ \), i.e. the trivial weight space in the weight decomposition \( U_q(\mathfrak{g}) = \bigoplus_{\beta \in Q} U_q(\mathfrak{g})_{\beta} \), 
where \( Q \) is the corresponding root lattice, see Section 1.1 for notation. We then construct 
weight representations of \( U_q(\mathfrak{g}) \) by inducing a weight module of the centralizer algebra \( U_0 \). 
The construction is called Mathieu module, being inspired by the paper [15] by Mathieu on 
the study of weight modules for Lie algebras. In Mathieu’s paper [15] the parabolic induction 
is the key procedure, and in Futorny et al. [5] a quantum analogue for \( \mathfrak{sl}(n, \mathbb{C}) \) is given.

In particular, we are interested in the case of the induction of 1-dimensional modules of 
the centralizer algebra \( U_0 \). In order to do so, we look for commutative subalgebras of the 
centralizer algebra \( U_0 \), which is closely related to strongly orthogonal roots, see [4], [13]. We
show that for the case of \( g = \mathfrak{sl}(2, \mathbb{C}) \) and for the *-structure for the non-compact real form \( U_q(\mathfrak{su}(1, 1)) \) we recover the representations of \( [2, 11, 19] \).

In Section 2 we introduce and study the centralizer algebra \( U_0 \) using the PBW-basis and suitable height functions. We discuss commutative subalgebras of \( U_0 \) in relation to strongly orthogonal roots. In Section 3 we introduce the induced representations, which we call Mathieu modules. In Section 4 we focus our attention on the induction of 1-dimensional representations. We study the simplest case \( g = \mathfrak{sl}(2, \mathbb{C}) \) in Section 5. In Section 6 we discuss some aspects of this construction for \( g = \mathfrak{sl}(n + 1, \mathbb{C}) \).

We expect that the non-extremal weight modules constructed in this way can be used to improve the understanding of the harmonic analysis of non-compact quantum groups, see [18].

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1.1. Notation and conventions. We use the notation \( \mathbb{N} = \{1, 2, 3, \cdots \} \) and we use \( \mathbb{N}_0 \) for the set \( \{0, 1, 2, 3, \cdots \} \).

We use the conventions and notations for quantized universal enveloping algebras as in [11], see also e.g. [3, 9]. All statements in this section can be found in [11].

Let \( g \) be a complex semi-simple Lie algebra with a Cartan subalgebra \( h \) and \( \Phi \) be the corresponding root system. Let \( n = \text{rank} g \) and fix the simple roots \( \Pi = \{\alpha_1, \cdots, \alpha_n\} \). Let \( \Phi^+ \) be the set of positive roots and set \( r = |\Phi^+| \). By \( Q = \bigoplus_{i=1}^n \mathbb{Z} \alpha_i \subset h^* \) we denote the root lattice and \( Q^+ = \bigoplus_{i=1}^n \mathbb{N}_0 \alpha_i \) denotes the corresponding positive roots. The Cartan matrix is \( A = (a_{ij})_{i,j=1}^n \). Let \( D = \text{diag}(d_1, \cdots, d_n) \) be the diagonal matrix so that \( d_i \in \{1, 2, 3\} \) and \( DA \) is symmetric and positive definite. Let \( (\cdot, \cdot) \) be the corresponding bilinear form on \( h^* \).

We consider \( q \) as a non-zero element of \( \mathbb{C} \), and we assume \( q \) is not a root of unity. We let \( q_i = q^{d_i} \) and we use the \( q \)-binomial coefficient for \( n, k \in \mathbb{N}_0 \) with \( 0 \leq k \leq n; \)

\[
\begin{align*}
[k]_q &= \frac{[n]!}{[k]! [n-k]!}, & [k]_q! &= \prod_{j=1}^{k} [j]_q, & [j]_q &= \frac{q^j - q^{-j}}{q - q^{-1}}
\end{align*}
\]

**Definition 1.1.** The quantized enveloping algebra \( U = U_q(g) \) is the unital associative algebra generated by elements \( E_i, F_i, K_i, K_i^{-1} \), \( i \in \{1, \cdots, n\} \), subject to the relations:

\[
K_i K_i^{-1} = 1 = K_i^{-1} K_i, \quad K_i K_j = K_j K_i, \quad K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j, \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{k} \binom{1-a_{ij}-r}{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0, \quad i \neq j,
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{k} \binom{1-a_{ij}-r}{q_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0, \quad i \neq j.
\]

Note that \( q_i^{a_{ij}} = q^{(\alpha_i, \alpha_j)} \). The last two relations in Definition 1.1 are known as the \( q \)-analog of the Serre relations.
Denote by $U^+ = U_q(n^+)$ the subalgebra generated by $E_i$, $1 \leq i \leq n$, and similarly we let $U^- = U_q(n^-)$ be the subalgebra generated by $F_i$, $1 \leq i \leq n$, which are the analogues of the universal enveloping algebra for the subalgebras $n^\pm$ in the decomposition $\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+$. Put $U^0$ for the subalgebra generated by $K_i^\pm$, $1 \leq i \leq n$. Then the multiplication map

$$U^+ \otimes U^0 \otimes U^- \to U$$

is an isomorphism of vector spaces.

In order to describe the PBW (Poincaré-Birkhoff-Witt) basis of $U = U_q(\mathfrak{g})$ we fix a reduced decomposition $w_0 = s_i \cdots s_{i_r}$ of the longest Weyl group element $w_0 \in W$ in terms of the reflections $s_i$ corresponding to the simple root $\alpha_i$. Then $\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \ldots, \beta_r = s_{i_1} \cdots s_{i_{r-1}}(\alpha_{i_r})$ exhaust the positive roots $\Phi^+$. In the quantum case, there exist elements $T_i$, $1 \leq i \leq n$, satisfying the braid relations for $\mathfrak{g}$, and the root vectors $E_{\beta_i}, F_{\beta_i}$ are defined as

$$E_{\beta_i} = T_i \cdots T_{i_{r-1}}(E_{i_r}), \quad F_{\beta_i} = T_i \cdots T_{i_{r-1}}(F_{i_r}).$$

Now the PBW basis for $U$ is given by

$$\{ E_{\beta_i}^{m_1} \cdots E_{\beta_i}^{m_r} K_1^{l_1} \cdots K_n^{l_n} F_{\beta_i}^{k_1} \cdots F_{\beta_i}^{k_r}; \; m_i, k_i \in \mathbb{N}_0, \; l_i \in \mathbb{Z} \}$$

and writing $m = (m_1, \ldots, m_r) \in \mathbb{N}_0^n$, $k = (k_1, \ldots, k_r) \in \mathbb{N}_0^n$, $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$, we abbreviate such a basis element as $E^m K^l F^k$.

Next consider the adjoint action restricted to $U^0$. For $\gamma \in Q$ we consider the root subspace

$$U_\gamma = \{ X \in U \mid K_i X K_i^{-1} = q^{(\alpha_i, \gamma)} X \},$$

and similarly defined $U^\pm_\gamma = \{ X \in U^\pm \mid K_i X K_i^{-1} = q^{(\alpha_i, \gamma)} X \}$. Then we have

$$U = \bigoplus_{\gamma \in Q} U_\gamma, \quad U^+ = \bigoplus_{\gamma \in Q^+} U^\gamma, \quad U^- = \bigoplus_{\gamma \in Q^-} U^-\gamma.$$

Then $\dim U^+_\gamma = \dim U^-_\gamma = K(\gamma)$, where $K(\gamma)$ is the Kostant partition function, i.e. the number of partitions of $\gamma$ as a sum of positive roots. Note that $U_\beta U_\gamma \subset U_{\beta + \gamma}$ and $U^+_\beta U^-_\gamma \subset U^\pm_{\beta + \gamma}$. For $X \in U_\gamma$ we say $\text{root}(X) = \gamma$, so for $X \in U_\gamma, Y \in U_\beta$ we have $\text{root}(XY) = \text{root}(X) + \text{root}(Y)$.

Finally, if we write $X \in U_\gamma$ in the PBW-basis, $X = \sum_{m,k,l} \xi_{m,k,l} E^m K^l F^k$, then $\xi_{m,k,l} \neq 0$ implies $E^m K^l F^k \in U_\gamma$. The PBW-basis is a joint eigenbasis for the adjoint action of $U^0$.

2. The centralizer of the Cartan subalgebra

In this section we study the structure of the 0-root space of $U$ as well as some of its properties. So we study $U_0$, which is the centralizer of the Cartan subalgebra $U^0$ of the quantized enveloping algebra $U$. We are in particular interested in abelian subalgebras of $U_0$. These will be used later to define Mathieu modules.

We start by defining

$$U_{\gamma, \sigma} = U^+_\sigma U^0 U^-_{\gamma - \sigma}, \quad \gamma \in Q, \sigma \in Q^+.$$  

(2.1)

Note that the space is trivial unless $\gamma < \sigma$. Then the PBW-basis element $E^m K^l F^k \in U_{\gamma, \sigma}$ if and only if $\sum_i m_i \beta_i = \sigma$ and $\sum_i k_i \beta_i = \sigma - \gamma$. As a consequence, we have

$$U_\gamma = \bigoplus_{\sigma \in Q^+} U_{\gamma, \sigma}, \quad U_0 = \bigoplus_{\sigma \in Q^+} U_{0, \sigma} = \bigoplus_{\sigma \in Q^+} U^+_\sigma U^0 U^-_{\gamma - \sigma}.$$  

(2.2)
Note that the PBW-basis gives a basis for the spaces $U_{\gamma,\sigma}$. We extend the definition of root($X$) = $\gamma$ for $X \in U_\gamma$ to E-root($X$) = $\sigma$ whenever $X \in U_{\gamma,\sigma}$. In particular, the E-root of a PBW-basis element is well-defined. Similarly, the F-root($X$) can be defined, but we do not use this.

Recall that we have fixed a set $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ of simple roots, and for $i \in \{1, \cdots, n\}$ we define the $i$-the height function

$$h_i: Q^+ \to \mathbb{N}, \quad h_i(\sum_{j=1}^n m_j \alpha_j) = m_i. \quad (2.3)$$

For a PBW-basis element $E^mK^1F^k$ we define $h_i(E^mK^1F^k) = h_i(E\text{-root}(E^mK^1F^k))$.

**Definition 2.1.** For $X = \sum_{m,k,l} \xi_{m,k,l} E^mK^1F^k \in U$ define

$$h_i^-(X) = \min_{\xi_{m,k,l} \neq 0} h_i(E^mK^1F^k), \quad h_i^+(X) = \max_{\xi_{m,k,l} \neq 0} h_i(E^mK^1F^k).$$

An alternative description of the height functions is the following. Let $X \in U$, then, upon decomposing $X$ in the PBW basis, we can group the PBW basis elements that have the same $i$-height obtaining $X = \sum_{j=0}^\infty X_j$, with $h_i(X_j) = j$. Only a finite number of $X_j$ is nonzero, and $h_i^-(X) = \min_{j \neq 0} j$ and $h_i^+(X) = \max_{j \neq 0} j$.

We have that $h_i^+(X) = 0$ for all $i$ if and only if $X \in U^0$, but it is not true that $h_i^-(X) = 0$ for all $i$ implies $X \in U^0$. Furthermore, multiplying by elements of the Cartan subalgebra $U^0$ on the left or right does not alter the minimal or maximal $i$-heights; if $X \in U^0$ and $Y \in U$ then $h_i^+(XY) = h_i^+(YX) = h_i^+(Y)$.

Finally, note that $h_i^-(X + Y) \geq \min(h_i^-(X), h_i^-(Y))$ for $X, Y \in U$.

**Lemma 2.2.** Let $m, m', k, k' \in \mathbb{N}^r$. Then for every $X \in U^0U^-$, $Y \in U^+U^0$, we have $h_i^-(E^mE^{m'}X) = h_i^-(E^{m+m'}X)$ and $h_i^-(YF^kF^{k'}) = h_i^-(Y^{F+k'})$.

**Proof.** Take $\gamma = E\text{-root}(E^m) + E\text{-root}(E^{m'})$. Since $U^+$ is a subalgebra of $U$, we can decompose $E^{m+m'} = \sum_{n \in \mathbb{N}^r} \xi_n E^n \in U_{\gamma}$ with respect to the PBW-basis. All of these elements in the PBW expansion satisfy $E^n \in U_{\gamma,\gamma}$. Since $E^{m+m'} \in U_{\gamma,\gamma}$ as well, we have $h_i^-(E^mE^{m'}X) = h_i(\gamma) = h_i^-(E^{m+m'}X)$.

The proof of the other statement follows analogously. \qed

We are in particular interested in the function $h_i^-$ on the centralizer algebra $U_0$.

**Proposition 2.3.** For $X, Y \in U_0$ we have $h_i^-(XY) \geq \max(h_i^-(X), h_i^-(Y))$.

**Proof.** We start with $X$ and $Y$ elements from the PBW basis. For PBW-basis elements $E^mF^k$ and $E^{m'}F^{k'}$ elements in $U_0$, we write $F^kE^{m'} = \sum \xi_{m',k'} \xi_{m,k} E^{m'}K^{l'}F^{k'}$ in the PBW basis. Then

$$h_i^-(E^mF^kE^{m'}F^{k'}) = h_i^-\left(\sum \xi_{m',k'} \xi_{m,k} E^{m'}K^{l'}F^{k'}\right) \geq \min_{\xi_{m',k'} \neq 0} h_i^-\left(\sum \xi_{m',k'} K^{l'}F^{k'}\right) = \min_{\xi_{m',k'} \neq 0} h_i^-\left(E^{m+m'}F^{k+k'}\right)$$

using Lemma 2.2 and the fact that $h_i(E^mK^1F^k) = h_i(E^mF^k)$. Since $h_i^-\left(E^{m+m'}F^{k+k'}\right) = h_i(\sum_{j=1}^m m_j + m_j') \geq h_i(\sum_{j=1}^r m_j \beta_j) = h_i^-(E^mF^k)$ it follows that $h_i^-(E^mF^kE^{m'}F^{k'}) \geq h_i^-(E^mF^k)$. 

Since $E^{m+m'}F^{k''+k'} \in U_0$ we have $\sum_{j=1}^r (m_j + m'_j) \beta_j = \sum_{j=1}^r (k''_j + k'_j) \beta_j$. Hence
\[
h_i^- (E^{m+m''}F^{k''+k'}) = h_i (\sum_{j=1}^r (k''_j + k'_j) \beta_j) = h_i (\sum_{j=1}^r m'_j \beta_j) = h_i^- (E^{m'}F^{k'})
\]
so that we have proved the statement for $X = E^mF^k, Y = E^{m'}F^{k'}$ in $U_0$.

The proof of the case $X = E^mF^k, Y = \sum \xi_{m',v',k'} E^{m'}K^{v'}F^{k'}$ in $U_0$ uses that any non-trivial element $E^{m'}K^{v'}F^{k'}$ in the expansion for $Y$ is in $U_0$. Then we reduce to the previous case by
\[
h_i^- (E^mF^k Y) = h_i^- (E^mF^k \sum \xi_{m',v',k'} E^{m'}K^{v'}F^{k'}) \geq \min_{\xi_{m',v',k'} \neq 0} h_i^- (E^mF^k E^{m'}K^{v'}F^{k'}) = \max (h_i^- (E^mF^k), \min_{\xi_{m',v',k'} \neq 0} h_i^- (E^{m'}F^{k'})) = \max (h_i^- (E^mF^k), h_i^- (Y))
\]
using that the appearance of $K^{v'}$ is immaterial, and the value $h_i^- (E^mF^k)$ is independent of the condition for the minimalization.

The general case then follows by writing $X = \sum \xi_{m,1,k} E^mK^1F^k$ and follow
\[
h_i^- (XY) = h_i^- (\sum \xi_{m,1,k} E^mK^1F^k Y) \geq \min_{\xi_{m,1,k} \neq 0} h_i^- (E^mK^1F^k Y) = \min_{\xi_{m,1,k} \neq 0} h_i^- (E^mF^k Y) \geq \max \left( \min_{\xi_{m,1,k} \neq 0} h_i^- (E^mF^k), h_i^- (Y) \right) = \max (h_i^- (X), h_i^- (Y))
\]
again using that the appearance of $K^1$ is immaterial, and the value $h_i^- (Y)$ is independent of the condition for minimalization. \(\square\)

Consider a subset $S \subset \{1, \cdots, n\}$, then there exists an associated disjoint decomposition $Q^+ = Q^+_S \cap (Q^+_S)^c$;
\[
Q^+_S = \{ \gamma \in Q^+ \mid h_i (\gamma) = 0 \ \forall i \notin S \} = \{ \gamma = \sum_{j=1}^n b_j \alpha_j \in Q^+ \mid i \notin S \Rightarrow b_i = 0 \}
\]
are the roots that can be completely written in terms of the simple roots $\{\alpha_i \mid i \in S\}$. Then
\[
(Q^+_S)^c = \{ \gamma \in Q^+ \mid \exists i \notin S: h_i (\gamma) > 0 \} = \{ \gamma = \sum_{j=1}^n b_j \alpha_j \in Q^+ \mid \exists i \notin S: b_i > 0 \}.
\]
From (2.23) we get a decomposition for the centralizer algebra;
\[
U_0 = U_0^S \oplus I^S, \quad U_0^S = \bigoplus_{\gamma \in Q^+_S} U_{0,\gamma}, \quad I^S = \bigoplus_{\gamma \in (Q^+_S)^c} U_{0,\gamma}.
\]
Consider a PBW basis element $X \in U_0$, then $X \in I^S$ if and only if $h_i^- (X) > 0$ for some $i \notin S$ and $X \in U_0^S$ if and only if for all $i \notin S$ we have $h_i^- (X) = 0$.

Remark 2.4. Keeping in the Dynkin diagram of $\mathfrak{g}$ only the vertices from $S$ and the corresponding edges, we obtain a Dynkin diagram to which we associate the Lie algebra $\mathfrak{g}_S$. 
Restricting the diagonal matrix \( D \) to the set \( S \), we can similarly define the quantized universal enveloping algebra \( U_q(\mathfrak{g}_S) \). Then \( U_q(\mathfrak{g}_S) \subset U_q(\mathfrak{g}) \) is a Hopf subalgebra which is invariant for the adjoint action of \( U^0 = U^{00}(\mathfrak{g}) \). Then \( U^0_S \) is generated by \( U_q(\mathfrak{g}_S) = U_q(\mathfrak{g}_S) \cap U^0 \) and \( K_i \) for \( i \notin S \).

Now we look at commutative subalgebras of the centralizer of the Cartan subalgebra. Recall the notational conventions for the Lie algebra \( \mathfrak{g} \), in particular its Cartan matrix \((a_{ij})_{1 \leq i,j \leq n}\) and the quantized universal enveloping algebra \( U = U_q(\mathfrak{g}) \) as in Definition 4.1.

**Theorem 2.5.** Let \( S \subset \{1, \cdots, n\} \). Assume that \( a_{i,j} = 0 \) is for each pair \((i,j)\) with \( i \neq j \) and \( i, j \in S \). Consider the corresponding decomposition \( U_0 = U^0_0 \oplus I^S \), then \( U^0_0 \) is a commutative subalgebra of \( U^0 \) generated by

\[
E_i F_i, \ i \in S, \ K_j^\pm, \ 1 \leq j \leq n
\]

The subspace \( I^S \) is a two-sided ideal of \( U_0 \).

**Remark 2.6.** (i) Recall that two non-proportional roots \( \alpha, \beta \) are strongly orthogonal if \( \alpha \perp \beta \) and if \( \alpha \pm \beta \) are not roots, which plays an important role in determining maximal abelian subspaces in symmetric pairs, see [8, VIII, §7] for the quantum case. Note that the condition in Theorem 2.5 means that \( \{\alpha_i \mid i \in S\} \) forms a set of strongly orthogonal roots. Indeed, \( \alpha_i - \alpha_j \) is not a root, and if \( \alpha_i + \alpha_j \) would be a root, so would the reflection \( \alpha_i - \alpha_j \) in the hyperplane orthogonal to \( \alpha_j \). See e.g. [1] for classification results on maximal families of strongly orthogonal roots.

(ii) The case \( S = \emptyset \) gives \( U^0_0 = U^0 \) and \( I^0 = \bigoplus_{\gamma \in Q^+ \setminus \{0\}} U_{0,\gamma} \). Then \( I^0 \) is the kernel of the Harish-Chandra homomorphism \([11] \) §6.3.4, and \([12] \) §7.4 for the classical case.

**Corollary 2.7.** A 1-dimensional representation of the commutative subalgebra \( U^0_0 \) extends to a 1-dimensional representation of \( U_0 \).

**Proof.** Let \( \pi: U^0_0 \to \mathbb{C} \) be a 1-dimensional representation, then we extend \( \pi \) to \( U_0 \) by putting \( \pi|_{I^0} = 0 \). Since \( I^S \) is a 2-sided ideal, \( \pi: U_0 \to \mathbb{C} \) is a representation. \(\square\)

**Proof of Theorem 2.3.** In this case the Lie algebra \( \mathfrak{g}_S \) as in Remark 2.4 consists of \( |S| \) copies of \( \mathfrak{sl}(2, \mathbb{C}) \), so the positive roots for \( \mathfrak{g}_S \) are just the simple roots (corresponding to \( S \)), i.e. \( \Phi^+_S = \{\alpha_i \mid i \in S\} \). Note that \( E_i F_i, i \in S \), and \( K_j^\pm, 1 \leq j \leq n \) are in \( U^0_0 \). Also, since \( a_{i,j} = 0 \) for \( i \neq j \) and \( i, j \in S \), it follows from the Serre relations of Definition 1.1 that \( E_i E_j = E_j E_i, F_i F_j = F_j F_i \) for all \( i, j \in S \). Hence, \( [E_i F_i, E_j F_j] = 0 \) for \( i, j \in S \). And since \( E_i F_i \in U_0 \), we see that \( E_i F_i, i \in S \), and \( K_j^\pm, 1 \leq j \leq n \), generate a commutative subalgebra \( A \) of the subalgebra \( U_0 \subset U \) which only involves elements from the root lattice \( Q^+_S \), hence \( A \subset U^0_0 \). So it suffices to show \( A \supset U^0_0 \).

Now take a PBW basis element in \( U_{0,\gamma} \subset U^0_0 \) for \( \gamma \in Q^+_S \), which can be written as

\[
E_{i_1}^{k_1} \cdots E_{i_s}^{k_s} K_1^{k_1} F_{i_1} \cdots F_{i_s}^{k_s}
\]

where \( S = \{i_1, \cdots, i_s\} \) since the positive roots of \( \mathfrak{g}_S \) are the simple roots \( \alpha_{i_1}, \cdots, \alpha_{i_s} \). The \( E_{i_j} \)'s, respectively \( F_{i_j} \)'s, commute amongst each other, and we can move the \( K_1 \) around at the cost of a power of \( q \). So we can rewrite this element, up to a power of \( q \), as \( E_{i_1}^{k_1} F_{i_1}^{k_1} \cdots E_{i_s}^{k_s} F_{i_s}^{k_s} K_1^{k_1} \). It suffices to do the \( U_q(\mathfrak{sl}(2)) \)-calculation that \( E^k F^k \) is a polynomial in \( EF \) with coefficients polynomial in \( K, K^{-1} \), see Lemma 5.4. This also shows that \( U^0_0 \) is an algebra, as for the \( U_q(\mathfrak{sl}(2, \mathbb{C})) \) calculations in Section 5.

To show that \( I^S \) is an ideal, it suffices to take PBW-basis elements \( E^m F^k \in U_0 \) and \( E^m F^k \in I^S \) and show that \( E^m F^k E^m F^k \in I^S \) and \( E^m F^k E^m F^k \in I^S \). Note we can
assume $E^{m_1}F^{k_1} \in U_{0,\gamma}$ with $\gamma \in (Q^+_S)^c$. Pick $i \notin S$ with $h_i(\gamma) > 0$, then by Proposition 2.3 we have

$$h_i^-(E^{m_1}F^{k_1}E^{m_1}F^{k_1}) \geq h_i^-(E^{m_1}F^{k_1}) = h_i(\gamma) > 0$$

hence $E^{m_1}F^{k_1}E^{m_1}F^{k_1} \in I_S$. Similarly, the reversed order can be dealt with and obtain that $I_S$ is a two-sided ideal in $U_0$. 

3. Mathieu modules

We stick to the notation for the quantized enveloping algebra $U = U_q(g)$, the corresponding Cartan subalgebra $U^0$ and its centralizer $U_0$ in $U$. We view $U$ as a right $U_0$-module, and recall that $U^0 \subset U_0$.

**Definition 3.1.** Let $V$ be any (left) $U_0$-module and consider the induced $U$-module $U \otimes_{U_0} V$. If $V = \bigoplus \gamma V_\gamma$ is a weight module, i.e. decomposes in terms of finite-dimensional weight spaces for the $U^0$-action, we say that the induced module $M(V) = U \otimes_{U_0} V$ is a Mathieu module of $U$ induced by $V$. We call $\dim V$ the rank of the Mathieu module $M$. The Mathieu module $M(V)$ is called degenerate in case $X \cdot v = 0$ for all $v \in V$ and all $X \in U_{0,\gamma}$ for all $\gamma \in Q^+ \setminus \{0\}$.

Note that a weight module is a module with a direct sum decomposition with respect to the action of $U^0$. Here $\gamma : U^0 \to \mathbb{C}$ is a homomorphism, and then $V_\gamma = \{v \in V \mid Kv = \gamma(K)v\}$. For the adjoint action of $U^0$ on $U = U_q(g)$ we obtain the decomposition in weight spaces $U_\lambda$, $\lambda \in Q$, corresponding to the homomorphism $q^\lambda : U^0 \to \mathbb{C}$, $K_1 \mapsto q^{(\alpha_1, \lambda)}$.

For a Mathieu module $M(V)$, the subspace $1 \otimes V \subset M(V)$ is a sub-$U_0$-module isomorphic to $V$. Observe that the Mathieu module is a weight module;

$$M(V)_\gamma = \bigoplus_{\gamma=q^\lambda \mu} U_\lambda \otimes V_\mu. \quad (3.1)$$

Here $q^\lambda \mu : U^0 \to \mathbb{C}$ defined by $q^\lambda \mu(K) = q^\lambda(K)\mu(K)$ for $K \in U^0$, since all $K \in U^0$ are group-like elements.

Recall that the weight module $V = \bigoplus \gamma V_\gamma$ is a highest, respectively lowest, weight module if the weights occurring are of the form $q^\lambda \mu$ for some fixed $\mu$ and $\lambda \in -Q^+$, respectively $\lambda \in Q^+$. Assuming $V_\mu \neq \{0\}$, we say that $\mu$ is the highest, respectively lowest, weight of the $U$-module $V$.

Note that the construction of Definition 3.1 is functorial, i.e. if $\psi : V \to \tilde{V}$ is a $U_0$-module map between weight modules $V$ and $\tilde{V}$, then $M(\psi) = \text{Id} \otimes \psi : M(V) \to M(\tilde{V})$ is a $U$-module morphism extending $\psi$, and using (3.1) we find that $\psi$ is surjective, respectively injective, if and only if $M(\psi)$ is surjective, respectively injective. So the Mathieu module is determined by the equivalence class of the $U_0$-module $V$.

**Lemma 3.2.** Assume $W$ is $U$-module which is a weight module. Let $V \subseteq W$ be a $U_0$-submodule, and let $\tilde{V}$ be a $U_0$-module which is a weight module. Assume $\psi : \tilde{V} \to V$ is a $U_0$-module homomorphism, then there is a $U$-module homomorphism $\Psi : M(\tilde{V}) \to W$ extending $\psi$.

**Proof.** Consider the bilinear map

$$\Psi : U \times \tilde{V} \to W, \quad \Phi(X, v) = X \cdot \psi(v) \in W, \quad X \in U, \quad v \in \tilde{V}.$$

Then for $Z \in U_0$ we have $\Psi(XZ, v) = \Psi(X, Z \cdot v)$, since $XZ \cdot \psi(v) = X \cdot \psi(Z \cdot v)$ as $\psi$ is a $U_0$-intertwiner. By universality we obtain a map, also denoted $\Psi : M(\tilde{V}) = U \otimes_{U_0} \tilde{V} \to W$,
Ψ(X ⊗ v) = X · ψ(v), which by construction intertwines the U-action. Moreover, \( Ψ(1 ⊗ v) = ψ(v) \), so that Ψ extends ψ.

**Proposition 3.3.** Let \( W \) be an \( U \)-module generated by a weight vector \( w \in W \), then \( W \) is isomorphic to a quotient of a Mathieu module. In particular, an irreducible weight representation of \( U \) is isomorphic to a quotient of a Mathieu module.

**Proof.** Define the \( U_0 \)-module \( V \) generated by \( w \), i.e. \( V = U_0w \), then \( V \) is a weight module with only one weight occurring, which is the same weight as that of \( w \). The identity map is a \( U_0 \)-module homomorphism \( τ: V → V ⊂ W \). By Lemma 3.2, there is a \( U \)-module homomorphism \( Ψ: M(V) → W \) extending \( τ \). Note that Ψ is surjective, since \( w \) generates \( W \). Hence \( W \cong M(V)/\text{Ker}(Ψ) \).

**Corollary 3.4.** Let \( W \) be an irreducible highest weight \( U \)-module, or an irreducible lowest weight \( U \)-module, then \( W \) is isomorphic to a quotient of a Mathieu module of rank 1.

We say that \( W \) is an extremal weight \( U \)-module if \( W \) is an irreducible highest weight \( U \)-module or an irreducible lowest weight \( U \)-module.

**Proof.** By Proposition 3.3 it suffices to show that we can take a Mathieu module of rank 1. Let \( w \) be the highest weight vector of \( W \), then \( U_0w = Cw \) since this is the only space with the same weight as the weight of \( w \). So take the \( U_0 \)-module \( Cw \) of dimension 1 and apply the construction to obtain the corresponding Mathieu module \( M \) of rank 1.

## 4. Mathieu modules of rank 1

**Lemma 4.1.** Let \( M(V) \) be a rank 1 Mathieu module for \( U \). Then, as vector spaces \( U \cong M(V) \otimes U_0 \).

**Proof.** This follows from the associativity property of tensor products of modules over rings. Let \( V \cong C \) as \( U_0 \)-module, then

\[
M(V) \otimes_C U_0 \cong (U \otimes_{U_0} C) \otimes_C U_0 \cong U \otimes_{U_0} (C \otimes_C U_0) \cong U \otimes_{U_0} U_0 \cong U.
\]

**Proposition 4.2.** Let \( M(V) \) be a rank 1 Mathieu module, then there exists a unique maximal proper submodule \( W(V) \). So \( M(V)/W(V) \) is the unique irreducible quotient of the Mathieu module.

**Proof.** Let \( V \cong C_λ \) with weight \( λ: U^0 → C \), so that \( M(V) \cong U \otimes_{U_0} C \) has weight space decomposition \( M(V)_γ = U_γ \otimes_{U_0} V \). In particular, for \( γ = 0 \), \( M(V)_λ = U_0 \otimes_{U_0} V \) is one-dimensional. Since a proper submodule \( W \) is a weight module, \( W \) cannot contain \( M(V)_λ \) since \( M(V)_λ = 1 \otimes V \) generates \( M(V) \). So the union of all proper submodules is proper, and gives the unique maximal proper submodule \( W(V) \).

In order to construct Mathieu modules of rank 1 we consider Theorem 2.3. So take \( S = \{i_1, \ldots, i_s\}, s = |S| \), as in Theorem 2.3 and consider \( μ = (μ_{i_1}, \ldots, μ_{i_s}) ∈ C^s, μ_i ≠ 0 \) for all \( i ∈ S \) and \( λ: U^0 → C \). Define the one-dimensional module \( φ^{S}_{λ,μ}: U_0 = U_0^S + IS → C = C^S_{λ,μ} \) by

\[
\text{Ker } φ^{S}_{λ,μ} = I^S, \quad E_i F_i ↦ μ_i, \quad i ∈ S, \quad K_j ↦ λ_j, \quad 1 ≤ j ≤ n.
\]

Note that allowing \( μ_i \)'s to be zero would mean to consider a smaller subset of \( S \).

**Definition 4.3.** Define \( M^S_{λ,μ} = M(C^S_{λ,μ}) \) as the rank 1 Mathieu modules induced by the one-dimensional \( U_0 \)-representations \( φ^{S}_{λ,μ} \).
Proof. Since \( \lambda \) is well-defined by the previous observation. Then the Cauchy-Schwarz inequality \( \langle \cdot, \cdot \rangle \) is a proper subspace, which contains \( \lambda \) trivially to \( U' \). According to Proposition \( 4.3 \), the module \( V(\lambda) \) is a quotient of a Mathieu module.

Proposition 4.4. Assume the lowest weight Verma module \( V(\lambda) \) is irreducible, then \( V(\lambda) \cong M_\lambda/W \) where \( M_\lambda \) is the degenerate Mathieu module and \( W \) its maximal proper invariant subspace.

Proof. Consider the invariant space \( W_0 \) of \( M_\lambda \) generated by \( F_i \circ 1, 1 \leq i \leq n \). We first observe that \( W_0 \) is a proper subspace, and for this it suffices to show that \( 1 \circ 1 \notin W_0 \). Indeed, if it does then we have \( X_i \in U \) so that \( 1 \circ 1 = \sum_{i=1}^{n} X_i F_i \circ 1 \), and decomposing \( X_i = \sum_\beta X_i^\beta \) according to \( U = \bigoplus_{\beta \in Q} U_\beta \) we have \( 1 \circ 1 = \sum_{i=1}^{n} \sum_\beta X_i^\beta F_i \circ 1 \). Considering the weight \( \lambda \) we require that for non-zero terms in the sum we have \( X_i^\beta F_i \in U_0 \), and then \( h_i^{-1}(X_i^\beta F_i) > 0 \), so that \( X_i^\beta F_i \in I = I^0 \) which acts as zero. So \( 1 \circ 1 \notin W_0 \).

Let \( W \) be the maximal proper subspace, which contains \( W_0 \) by Proposition \( 4.2 \). Then the image \( v \) of \( 1 \circ 1 \) in \( M(\mathbb{C}_\lambda)/W \) satisfies \( F_i \cdot v = 0 \) for all \( 1 \leq i \leq n \). So it is a lowest weight vector of weight \( \lambda \). Assuming \( V(\lambda) \) is irreducible, we find \( M(\mathbb{C}_\lambda)/W \cong V(\lambda) \). \( \square \)

Next we discuss the unitarizability of the rank 1 Mathieu modules. We restrict to case of real \( q \), and we consider the \(*\)-structures as in the classification of Twietmeyer [17], see [3] §9.4. Then the \(*\)-structure is given by an involutive Dynkin diagram automorphism \( \eta \) and a set of numbers \( s_i \in \{\pm 1\}, 1 \leq i \leq n \), so that

\[
K_i^* = K_{\eta(i)}, \quad E_i^* = s_i F_{\eta(i)} K_{\eta(i)}, \quad F_i^* = s_i K_{\eta(i)}^{-1} E_{\eta(i)}
\]

with the condition that \( s_i = 1 \) if \( \eta(i) \neq i \). From (4.1) we see that \((U^0)^* = U^0\), and this gives \((U_\beta)^* = U_{-\eta(\beta)}\) extending \( \eta \) to \( Q \) by \( \eta(\beta) = \eta(\sum_{i=1}^{n} b_i \alpha_i) = \sum_{i=1}^{n} b_i \eta(\alpha_i) \).

We extend \( \phi_{\lambda,\mu}^S : U = \bigoplus_{\beta \in Q} U_\beta \to \mathbb{C} \) by first projecting on \( U_\beta \) and next applying the 1-dimensional representation \( \phi_{\lambda,\mu}^S \) of \( U_0 \).

Proposition 4.5. Let the \(*\)-structure be given by (4.1), and assume \( \phi_{\lambda,\mu}^S : U \to \mathbb{C} \) as defined above is a positive linear functional. Then \( M(\mathbb{C}_\lambda^S)/N \) is an irreducible unitary \( U \)-module, where

\[
N = \{ X \cdot (1 \otimes 1) \mid \phi_{\lambda,\mu}^S(X^* X) = 0 \}.
\]

Note that \( X \in U_\beta \) gives \( X^* X \in U_{\beta - \eta(\beta)} \) so that \( U_{\beta} \subset N \) in case \( \eta(\beta) \neq \beta \).

Proof. Since \( S, \lambda \) and \( \mu \) are fixed, we use the notation \( \phi = \phi_{\lambda,\mu}^S \) in the proof. Note that for \( X \in U \), \( Z \in U_0 \) we have \( \phi(XZ) = \phi(ZX) = \phi(Z)\phi(X) \), since this is true for \( X \in U_\beta \) for any \( \beta \in Q \) by \( U_0 U_\beta \subset U_\beta \) and \( U_0 \) being \(*\)-invariant. Define the sesquilinear form

\[
\langle \cdot, \cdot \rangle : M(\mathbb{C}_\lambda^S) \times M(\mathbb{C}_\lambda^S) \to \mathbb{C}, \quad \langle X \cdot (1 \otimes 1), Y \cdot (1 \otimes 1) \rangle = \phi(Y^* X),
\]

which is well-defined by the previous observation. Then the Cauchy-Schwarz inequality

\[
|\phi(Y^* X)|^2 \leq \phi(X^* X)\phi(Y^* Y)
\]
implies that $N$ is invariant subspace. The space $M(C^S_{\lambda,\mu})/N$ is an inner product space and the action of $U$ is unitary by construction.

The subspace $V$ generated by the action of $U$ on the image of $1 \otimes 1$ in $M(C^S_{\lambda,\mu})/N$ is an invariant subspace. Since the representation is unitary, we know that the orthocomplement is invariant as well and we show it is trivial. So assume $X \cdot (1 \otimes 1 + N)$ is perpendicular to $V$, then

$$\phi(Y^* X) = \langle X \cdot (1 \otimes 1 + N), Y \cdot (1 \otimes 1 + N) \rangle = 0 \quad \forall Y \in U.$$ 

In particular, taking $Y = X$ gives $\phi(X^* X) = 0$ and $X \cdot (1 \otimes 1 + N) \in N$, so the orthocomplement is trivial.

Since we require $\phi_{\lambda,\mu}^S$ to be a positive functional, we see that we require $\overline{\lambda}_i = \lambda_{\eta(i)}$ and $\overline{\mu}_i = \mu_{\eta(i)}$, since $(E_\ast F_i)^* = E_{\eta(i)} F_{\eta(i)}$ and $K_i^* = K_{\eta(i)}$. Assuming that $S \subset \{i \mid \eta(i) = i\}$, we have $\mu_i, \lambda_i \in \mathbb{R}$ and

$$E_i F_i = s_i K_i F_i^* F_i \implies \mu_i = s_i \lambda_i \phi_{\lambda,\mu}^S (F_i^* F_i)$$

so that $\mu_i \lambda_i > 0$ in case $s_i = 1$ and $\mu_i \lambda_i < 0$ in case $s_i = -1$.

5. Mathieu Modules for $U_q(sl(2, \mathbb{C}))$

In this section we discuss Mathieu modules for the simplest quantum algebra $U_q(sl(2, \mathbb{C}))$. The quantum algebra $U_q(sl(2, \mathbb{C}))$ is of type $A_1$ and has the $1 \times 1$ Cartan matrix (2). By Definition 1.1, $U_q(sl(2, \mathbb{C}))$ is generated by elements $E = E_1$, $F = F_1$, $K = K_1$, where the quantum Serre relations are void. The root system is $\Phi = \{\pm \alpha\}$.

We show that the Mathieu modules can be used to obtain all irreducible unitary modules for the $U_q(sl(1,1))$, i.e. the quantum algebra $U_q(sl(2, \mathbb{C}))$ equipped with the $*$-structure

$$K^* = K, \quad E^* = -FK, \quad F^* = -K^{-1}E,$$  

see (4.1).

**Definition 5.1.** A $U_q(sl(1,1))$-module $V$ is admissible if $V$ has a weight space decomposition $V = \bigoplus V_\sigma$ for the action of $K$ with finite-dimensional weight spaces $V_\sigma$. The module $V$ is of type $I$ if the eigenvalues $\sigma$ are of the form $q^r$ for $r \in \mathbb{R}$.

The unitary admissible type I representations of $U_q(sl(1,1))$ have been classified by Vaksman and Korogodski˘ı [19], Burban and Klimyk [2] and Masuda et al. [14], and they play an important role in the harmonic analysis on the quantum group analog of $SU(1,1)$. The purpose is to show that one can obtain these representations from the Mathieu modules for $U_q(sl(2, \mathbb{C}))$.

5.1. Mathieu modules for $U_q(sl(2, \mathbb{C}))$. For future reference we collect some well-known commutation relations in Lemma 5.2. The proof is a straightforward verification by induction and the relations of Definition 1.1 for the case $U_q(sl(2, \mathbb{C}))$, see e.g. [11].

**Lemma 5.2.** For $n, m \in \mathbb{N}_0$ we have

(i) $K^n E^m = q^{2mn} E^m K^n$ and $K^n F^m = q^{-2mn} F^m K^n$,

(ii) $E F^m = F^m E + \frac{q^n - q^{-n}}{q - q^{-1}} F^{m-1} q^{1-n} K - \frac{q^{-n} K^{-1}}{q - q^{-1}}$.
Lemma 5.3. For each $n \in \mathbb{N}_0$, $U_{0,\alpha}$ is one-dimensional $U^0$-module spanned by $E^n F^n$. Moreover, $\{E^n K^l F^n; \ n \in \mathbb{N}_0, l \in \mathbb{Z}\}$ is a basis for $U_0$.

Lemma 5.4. $E^n F^n = (EF)^n + \sum_{i=1}^{n-1} (EF)^i c_i(K, K^{-1})$ for some polynomial $c_i(K, K^{-1})$ in $K$ and $K^{-1}$ for all $n \in \mathbb{N}$.

Proof. The case $n = 1$ is trivial, and the induction step follows from Lemma 5.2(ii):

$$E^{n+1} F^{n+1} = E^n F^n EF = E^n F^n EF + \frac{q^n - q^{-n}}{q - q^{-1}} E^n F^{n-1} \frac{q^{1-n} K - q^{-1-n} K^{-1}}{q - q^{-1}}.$$

Moving $F$ through $K^{\pm 1}$, we can apply the induction hypothesis. Since $EF$ commutes with $K^{\pm 1}$, the result follows. $\square$

Corollary 5.5. $U^0 = \mathbb{C}[EF, K, K^{-1}]$ is a commutative algebra.

Note that Corollary 5.5 is the special case $S = \{1\}$ in the notation of Theorem 2.10.

As in Section 4, we define the 1-dimensional $U_0$-modules $\mathbb{C}_{\lambda,\mu} \cong \mathbb{C}$ by choosing $K \cdot 1 = \lambda 1$ and $EF \cdot 1 = \mu 1$, where $\lambda, \mu \in \mathbb{C}$, $\lambda \neq 0$. The case $\mu = 0$ corresponds to the degenerate case. Denote this 1-dimensional $U_0$-representation by $\phi = \phi_{\lambda,\mu}$. We then consider the Mathieu module $M(\mathbb{C}_{\lambda,\mu}) = U_q(\mathfrak{sl}(2, \mathbb{C})) \otimes \mathbb{C}_{\lambda,\mu}$ associated to this 1-dimensional $U_0$-module. We denote 1 for the element $1 \otimes 1 \in M(\mathbb{C}_{\lambda,\mu})$.

Proposition 5.6. The set $\{E^n \cdot 1\}_{n\in\mathbb{N}} \cup \{1\} \cup \{F^n \cdot 1\}_{n\in\mathbb{N}}$ is a basis of the $U_q(\mathfrak{sl}(2, \mathbb{C}))$-module $M(\mathbb{C}_{\lambda,\mu})$ and the generators act on elements of this basis as follows:

1. $K(E^n \cdot 1) = q^{2n} \lambda E^n \cdot 1$ and $K(F^n \cdot 1) = q^{-2n} \lambda F^n \cdot 1$ for $n \in \mathbb{N}_0$,
2. $E(E^n \cdot 1) = E^{n+1} \cdot 1$ and $F(F^n \cdot 1) = F^{n+1} \cdot 1$ for $n \in \mathbb{N}_0$,
3. $E(F^n \cdot 1) = \left(1 + \frac{q^n - q^{-n}}{q - q^{-1}} \right) F^{n-1} \cdot 1$ for $n \in \mathbb{N}$,
4. $F(E^n \cdot 1) = \left(1 - \frac{q^n - q^{-n}}{q - q^{-1}} \right) E^{n-1} \cdot 1$ for $n \in \mathbb{N}$.

Proof. The elements $E^n \cdot 1$, $1$ and $F^n \cdot 1$ are non-zero by Lemma 4.3 and they are linearly independent as weight vectors for different weights. To show that they span the module, we consider first the case where $E^n K^l F^k$ where $m \geq k$. We write $E^n K^l F^k \cdot 1 = E^{m-k} E^k K^l F^k \cdot 1 = \phi(E^k K^l F^k) E^{m-k} \cdot 1$. For $k \geq m$ the situation is slightly more complicated. Write $E^n K^l F^k = q^{2(m-k)} E^{m-k} K^l F^m$ and next use Lemma 5.2(ii) repeatedly to find that $E^n K^l F^k = F^{k-m} Z$ for some $Z \in U_0$. Hence $E^n K^l F^k \cdot 1 = \phi(Z) F^{k-m} \cdot 1$.

The action of the generators on these elements in (i) and (ii) follow. For (iii) we have by Lemma 5.2

$$EF^n \cdot 1 = F^{n-1} FE \cdot 1 + \frac{(q^n - q^{-n})(q^{1-n} \lambda - q^{-1-n} \lambda^{-1})}{(q - q^{-1})^2} E^{n-1} \cdot 1.$$
and using \( FE \cdot 1 = EF \cdot 1 - (q - q^{-1})^{-1}(K - K^{-1}) \cdot 1 = (\mu + (q - q^{-1})^{-1}(\lambda - \lambda^{-1})) \cdot 1 \) we find (iii) after a straightforward calculation. The proof of (iv) is similar and slightly simpler. □

From Proposition 5.6 we see that the representation space has a weight space decomposition for the action of \( K \); \( M(\mathbb{C}_{\lambda,\mu}) = \bigoplus_{k \in \mathbb{Z}} M(\mathbb{C}_{\lambda,\mu})_{\lambda q^{2k}} \), where each \( M(\mathbb{C}_{\lambda,\mu})_{\lambda q^{2k}} \) is 1-dimensional and spanned by \( E^k \cdot 1 \) if \( k > 0 \), by \( F^k \cdot 1 \) if \( k < 0 \) and by 1 if \( k = 0 \). Here we use \( \lambda q^{2k} : U_0 \to \mathbb{C} \) as the homomorphism sending \( K \mapsto \lambda q^{2k} \), which corresponds to \( \lambda q^{k \alpha} \). Proposition 5.6 shows that

\[
E : M(\mathbb{C}_{\lambda,\mu})_{\lambda q^{2k}} \to M(\mathbb{C}_{\lambda,\mu})_{\lambda q^{2(k+1)}}, \quad F : M(\mathbb{C}_{\lambda,\mu})_{\lambda q^{2k}} \to M(\mathbb{C}_{\lambda,\mu})_{\lambda q^{2(k-1)}}.
\]

Recall the Casimir element

\[
\Omega = EF + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2} = FE + \frac{Kq + K^{-1}q^{-1}}{(q - q^{-1})^2}
\]

for \( U_q(\mathfrak{sl}(2, \mathbb{C})) \), see e.g. [11, §3.1.1]. Then \( \Omega \) is central, and it generates the center of \( U_q(\mathfrak{sl}(2, \mathbb{C})) \). Using Proposition 5.6 we can calculate the action of \( \Omega \) on any basis element of the \( U_q(\mathfrak{sl}(2, \mathbb{C})) \)-module \( M(\mathbb{C}_{\lambda,\mu}) \).

**Corollary 5.7.** The Casimir operator \( \Omega \) acts as the constant \( \mu + \frac{q^{-1} + q^{2k}}{(q - q^{-1})^2} \) times the identity on the \( U_q(\mathfrak{sl}(2, \mathbb{C})) \)-module \( M(\mathbb{C}_{\lambda,\mu}) \).

### 5.2. Reducibility

The Mathieu module \( M(\mathbb{C}_{\lambda,\mu}) \) is admissible in the sense of Definition 5.1 since it has a weight space decomposition with finite-dimensional weight spaces. Hence, in case \( M(\mathbb{C}_{\lambda,\mu}) \) is reducible, a non-trivial invariant subspace has a weight space decomposition. Since the weight spaces are 1-dimensional, we can only have a non-trivial invariant subspace in case \( E \), respectively \( F \), kills a weight space. From Proposition 5.6 we see that this can only happen in cases (ii) and (iv).

In case (iii), \( E \) kills a weight space if there exists \( n_E \in \mathbb{N} \) with

\[
(q^{n_E} - q^{\lambda^{2k}})(q^{n_E} - q^{-1}) = 0,
\]

and then \( E \cdot (F^{n_E} \cdot 1) = 0 \). Note that for fixed \( \lambda \) and \( \mu \), at most one solution \( n_E \in \mathbb{N} \) for (5.4) exists. In this case the submodule \( M_{n_E}^{-} = \bigoplus_{k \geq -n_E} M(\mathbb{C}_{\lambda,\mu})_{\lambda q^{2k}} \), being the span of \( F^{n_E} \cdot 1 \), \( p \in \mathbb{N}_0 \), is invariant. The spectrum of \( K \) on the invariant subspace is \( \lambda q^{-2n_E} \cdot 1 \), so that we can consider \( M_{n_E}^{-} \) as a highest weight representation.

Similarly, in case (iv), we only get a zero action by \( F \) on \( F^{n_F} \cdot 1 \) if there exists \( n_F \in \mathbb{N} \) so that

\[
(q^{n_F} - q^{-n_F})(q^{n_F} - q^{1-n_F}) = 0,
\]

so that \( F \cdot (E^{n_F} \cdot 1) = 0 \). Again, there is at most one solution of (5.5) in \( \mathbb{N} \). The submodule \( M_{n_F}^{+} = \bigoplus_{k \geq n_F} M(\mathbb{C}_{\lambda,\mu})_{\lambda q^{2k}} \), being the span of \( E^{n_F} \cdot 1 \), \( p \in \mathbb{N}_0 \), is invariant. The spectrum of \( K \) on the invariant subspace is \( \lambda q^{2n_F} \cdot 1 \), so that we can consider \( M_{n_F}^{+} \) as a lowest weight representation.

These considerations prove the first part of Proposition 5.8.

**Proposition 5.8.** The Mathieu module \( M(\mathbb{C}_{\lambda,\mu}) \) is generically irreducible. More precisely, assume that (5.5) has no solution \( n_F \in \mathbb{N} \) and that (5.4) has no solution \( n_E \in \mathbb{N} \), then \( M(\mathbb{C}_{\lambda,\mu}) \) is irreducible. Conversely, if \( M(\mathbb{C}_{\lambda,\mu}) \) is irreducible, then (5.5) has no solution \( n_F \in \mathbb{N} \) and (5.4) has no solution \( n_E \in \mathbb{N} \).
Proof. It remains to prove the converse statement. Since $M(C_{\lambda,\mu})$ is the sum of the weight spaces, and, using the PBW basis, the only elements in $U_q(\mathfrak{sl}(2, \mathbb{C}))$ mapping $M(C_{\lambda,\mu})_\lambda$ to $M(C_{\lambda,\mu})_{\lambda q^2k}$ ($k \geq 0$) are elements from $E^k U_0$. By irreducibility, the map $E^k$ has to be non-zero, and by (5.2), we see that each $E: M(C_{\lambda,\mu})_{\lambda q^2p} \to M(C_{\lambda,\mu})_{\lambda q^{2p+1}}$ for $0 \leq p < k$ has to be non-zero. Since $k$ is arbitrary, we find that (5.4) has no solution $n_E \in \mathbb{N}$.

The statement for (5.5) is proved similarly. \hfill \Box

In case there exists a $n_E \in \mathbb{N}$ satisfying (5.4), and there exists no $n_F \in \mathbb{N}$ satisfying (5.5), the quotient $M(C_{\lambda,\mu})/M_{n_E}^-$ gives an irreducible $U_q(\mathfrak{sl}(2, \mathbb{C}))$-representation, which we can view as a lowest weight module with lowest weight $\lambda q^{2n_E}$. Similarly, in case there exists a $n_F \in \mathbb{N}$ satisfying (5.5) and there exists no $n_F \in \mathbb{N}$ satisfying (5.4), the quotient $M(C_{\lambda,\mu})/M_{n_F}^+$ gives an irreducible $U_q(\mathfrak{sl}(2, \mathbb{C}))$-representation, which we can view as a highest weight module with highest weight $\lambda q^{2n_F-2}$. In case there exists a solution $n_E \in \mathbb{N}$ to (5.4) and a solution $n_F \in \mathbb{N}$ to (5.5), then the $M(C_{\lambda,\mu})/(M_{n_E}^- + M_{n_F}^+)$ is a finite-dimensional irreducible $U_q(\mathfrak{sl}(2, \mathbb{C}))$-representation.

5.3. Equivalence. In general the equivalence question for general Mathieu modules seems to be difficult. For the case of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ and irreducible Mathieu modules, it is possible to describe it in detail.

**Proposition 5.9.** Assume that $M(C_{\lambda,\mu})$ and $M(C'_{\lambda',\mu'})$ are irreducible Mathieu modules. Then $M(C_{\lambda,\mu}) \cong M(C'_{\lambda',\mu'})$ if and only if there exists $n \in \mathbb{Z}$ with $\lambda' = \lambda q^{2n}$ and

$$\mu' = \mu - \frac{(q^n - q^{-n})(\lambda q^{n-1} - \lambda^{-1}q^{1-n})}{(q - q^{-1})^2}.$$

**Proof.** Assume first that the modules are equivalent. Since the spectrum of $K$ in both modules has to be equal, we find $\lambda q^{2n} = \lambda q^{2n'}$. Hence, there exists $n \in \mathbb{Z}$ with $\lambda' = \lambda q^{2n}$. By considering the action of the Casimir element $\Omega$, Corollary 5.7 gives the relation between $\mu$ and $\mu'$.

To prove the converse, we use Lemma 3.2. Let $W = M(C'_{\lambda',\mu'})$. Let $\tilde{V} = C_{(\lambda,\mu)} \cong \mathbb{C} \cdot 1_{(\lambda,\mu)} \subset M(C_{\lambda,\mu})$, stressing the dependence on $(\lambda,\mu)$. Then we define

$$\psi: \mathbb{C} \cdot 1_{(\lambda,\mu)} \rightarrow CF^n \cdot 1_{(\lambda',\mu')}, \quad 1_{(\lambda,\mu)} \rightarrow F^n \cdot 1_{(\lambda',\mu')}, \quad n \in \mathbb{N}_0,$$

$$\psi: \mathbb{C} \cdot 1_{(\lambda,\mu)} \rightarrow CE^{-n} \cdot 1_{(\lambda',\mu')}, \quad 1_{(\lambda,\mu)} \mapsto E^{-n} \cdot 1_{(\lambda',\mu')}, \quad -n \in \mathbb{N}.$$

By a straightforward calculation using Proposition 5.6 we see that $\psi$ intertwines the action of $K$ and $EF$. Then $V$, the image of $\psi$, is a $U_0$-submodule of $M(C_{\lambda,\mu})$. Lemma 3.2 gives an intertwiner $\Psi: M(V) = M(C_{\lambda,\mu}) \rightarrow W = M(C'_{\lambda',\mu'})$, which is non-zero, since it extends the non-zero map $\psi$. Since $M(C_{\lambda,\mu})$ and $M(C'_{\lambda',\mu'})$ are irreducible, they are equivalent. \hfill \Box

**Remark 5.10.** Introduce the map $\text{Tr}(M): (U_0^0)^* \times U_0 \rightarrow \mathbb{C}$ for a weight module $M = \bigoplus_{\lambda} M_{\lambda}$ by $\text{Tr}(M)(\lambda, X) = \text{Tr}(X|_{M_{\lambda}})$, see Mathieu [15, §2]. Using Proposition 5.6 we obtain

$$\text{Tr}(M(C_{\lambda,\mu}))(\lambda q^{2k}, (EF)^j K^l) = (\lambda q^{2k})^j \left(\mu - \frac{(q^k - q^{-k})(q^{k-1}\lambda - q^{1-k}\lambda^{-1})}{(q - q^{-1})^2}\right)^j$$

for $j \in \mathbb{N}_0$, $l \in \mathbb{Z}$, $k \in \mathbb{Z}$. By the explicit expression we see that $\text{Tr}(M(C_{\lambda,\mu})) = \text{Tr}(M(C'_{\lambda',\mu'}))$ if and only if $(\lambda, \mu)$ and $(\lambda', \mu')$ are related as in Proposition 5.9.
Remark 5.11. The proof of Proposition 5.9 is in case of irreducible Mathieu modules. It is straightforward to write down the intertwiner explicitly. E.g. in case \( n \in \mathbb{N}_0 \) we have
\[
\Psi(E^k \cdot 1_{(\lambda,\mu)}) = (E^k F^n) \cdot 1_{(\lambda',\mu')}, \quad \Psi(F^k \cdot 1_{(\lambda,\mu)}) = F^{k+n} \cdot 1_{(\lambda',\mu')},
\]
Assuming \((\lambda,\mu)\) and \((\lambda',\mu')\) related as in Proposition 5.9 we can check that \( \Psi \) intertwines the action using Proposition 5.6 directly. There are two non-trivial relations to check, namely \( \Psi(E(F^k \cdot 1_{(\lambda,\mu)})) = E \cdot \Psi(F^k \cdot 1_{(\lambda,\mu)}) \) and \( \Psi(F(E^k \cdot 1_{(\lambda,\mu)})) = F \cdot \Psi(E^k \cdot 1_{(\lambda,\mu)}) \). In the first case, Proposition 5.6 and the relation between \((\lambda,\mu)\) and \((\lambda',\mu')\) give the result. In the second case, the left hand side follows from Proposition 5.6. For the right hand side we use Lemma 5.2 ii), (iii) to write
\[
FE^k F^n = E^{k-1} E F^n F - \frac{q^k - q^{-k}}{q - q^{-1}} E^{k-1} \frac{q^{k-1} K - q^{-1} K^{-1}}{q - q^{-1}} F^n F,
\]
\[
E^{k-1} E F^n F = E^{k-1} F^n (EF) + E^{k-1} \frac{q^n - q^{-n}}{q - q^{-1}} F^{n+1} K - q^{-1} K^{-1} F.
\]
Then \( F \cdot \Psi(E^k \cdot 1_{(\lambda,\mu)}) = FE^k F^n \cdot 1_{(\lambda',\mu')} \) can be calculated directly in terms of \((\lambda',\mu')\). Using the relation between \((\lambda,\mu)\) and \((\lambda',\mu')\) then shows equality with the left hand side. Similarly, we have an explicit intertwiner for \(-n \in \mathbb{N}\).

Now the transition \((\lambda,\mu) \mapsto (\lambda',\mu')\) is invertible, and of the same type, i.e. \( \lambda = \lambda' q^{-2n} \) and \( \mu = \mu' - \frac{(q^n - q^{-n})(\lambda' q^{-n-1} - (\lambda)^{-1} q^{1+n})}{(q - q^{-1})^2} \).

So, we then similarly find an intertwiner \( \Psi' : M(C_{\lambda',\mu'}) \to M(C_{\lambda,\mu}) \). By considering the action on each of the basis vectors, we can obtain \( \Psi' \circ \Psi = \phi_{\lambda,\mu}(FE^n) \text{Id} \). We will not use this result, and we skip its proof.

5.4. Unitarizability. Next we consider which Mathieu modules for \( U_q(\mathfrak{sl}(2,\mathbb{C})) \) can be made into unitary representations for the \(*\)-structure (5.1) corresponding to the quantized universal enveloping algebra \( U_q(\mathfrak{su}(1,1)) \). Recall that we assume \( 0 < q < 1 \).

Observe that \( K^* = K \) and \( EF = -EE^* K^{-1} \), so that acting on \( 1 \in M(C_{\lambda,\mu}) \) and recalling that \( EE^* \) is a positive operator, we find the necessary conditions
\[
\lambda \in \mathbb{R} \setminus \{0\}, \quad \mu \lambda < 0, \quad (5.6)
\]
for \( M(C_{\lambda,\mu}) \) to be unitary. In particular, for type I representations, see Definition 5.1 we need \( \lambda > 0 \) and \( \mu > 0 \), see Section 3 and Proposition 4.5.

Since the basis vectors of Proposition 5.6 are eigenvectors of the (formally) self-adjoint operator \( K \) for different eigenvalues, this constitutes an orthogonal basis in case \( M(C_{\lambda,\mu}) \) is unitary. See Schmüdgen [16] Ch. 8 for the notion of a representation by unbounded operators, where in this case the common domain is the finite linear combinations of the basis vectors of Proposition 5.6.

Assume \( M(C_{\lambda,\mu}) \) is a unitary module for \( U_q(\mathfrak{su}(1,1)) \) with respect to the inner product \( \langle \cdot, \cdot \rangle \). We derive a recursive expression; take \( n > 0 \) and
\[
\langle E^n \cdot 1 | E^n \cdot 1 \rangle = \langle E^* E^n \cdot 1 | E^{n-1} \cdot 1 \rangle = -\langle FE^n \cdot 1 | E^{n-1} \cdot 1 \rangle = -q^{2n} \lambda \left( \mu - \frac{(q^n - q^{-n})(q^{n-1} - q^{-1} \lambda^{-1})}{(q - q^{-1})^2} \right) \langle E^{n-1} \cdot 1 | E^{n-1} \cdot 1 \rangle.
\]
using Proposition 5.6. This is a simple recursion, and we find, using \( \mu = M/(q - q^{-1})^2 \) and normalizing \( \langle 1|1 \rangle = 1 \),

\[
\langle E^n \cdot 1|E^n \cdot 1 \rangle = \left( \frac{q}{(q - q^{-1})^2} \right)^n \prod_{k=0}^{n-1} (1 - (\lambda^2 + q^2 + qM\lambda)q^{2k} + \lambda^2 q^2 q^{4k})
\]

so that \( \langle E^n \cdot 1|E^n \cdot 1 \rangle > 0 \) for all \( n \in \mathbb{N} \) if and only if \( 1 - (\lambda^2 + q^2 + qM\lambda)x + \lambda^2 q^2 x^2 > 0 \) for all \( x \in q^{2N_0} \).

Similarly, we find

\[
\langle F^n \cdot 1|F^n \cdot 1 \rangle = -\langle K^{-1}E F^n \cdot 1|F^{n-1} \cdot 1 \rangle = -\lambda^{-1} q^{2(n-1)} (\mu + \frac{(q^{n-1} - q^{1-n})(q^{-n}\lambda - q^n\lambda^{-1})}{(q - q^{-1})^2}) \langle F^{n-1} \cdot 1|F^{n-1} \cdot 1 \rangle.
\]

and hence

\[
\langle F^n \cdot 1|F^n \cdot 1 \rangle = \left( \frac{q^{-1}}{(q - q^{-1})^2} \right)^n \prod_{k=0}^{n-1} (1 - (q^2\lambda^{-2} + 1 + \frac{qM}{\lambda})q^{2k} + \frac{q^2}{\lambda^2} q^{4k}).
\]

The considerations for the positivity of \( \langle E^n \cdot 1|E^n \cdot 1 \rangle \) and \( \langle F^n \cdot 1|F^n \cdot 1 \rangle \) for all \( n \in \mathbb{N} \) lead to Theorem 5.12.

**Theorem 5.12.** Assume \( M(\mathbb{C}_{\lambda,\mu}) \) is an irreducible Mathieu module. Then \( M(\mathbb{C}_{\lambda,\mu}) \) is unitarizable for the *-structure \([5.11]\) for \( U_q(\mathfrak{su}(1, 1)) \) if and only if \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( \lambda \mu < 0 \) and, relabeling \( \mu = M/(q - q^{-1})^2 \),

\[
1 - (\lambda^2 + q^2 + qM\lambda)x + q^2\lambda^2 x^2 > 0, \quad \forall x \in q^{2N_0},
\]

\[
1 - (q^2\lambda^{-2} + 1 + \frac{qM}{\lambda})x + \frac{q^2}{\lambda^2} x^2 > 0, \quad \forall x \in q^{2N_0}.
\]

In this case, the basis of Proposition 5.6 is orthogonal, with squared norms given by \( \langle 1|1 \rangle = 1 \) and

\[
\langle F^n \cdot 1|F^n \cdot 1 \rangle = \left( \frac{q^{-1}}{(q - q^{-1})^2} \right)^n \prod_{k=1}^{n} (1 - (q^2\lambda^{-2} + 1 + \frac{qM}{\lambda})q^{2k} + \frac{q^2}{\lambda^2} q^{4k}),
\]

\[
\langle E^n \cdot 1|E^n \cdot 1 \rangle = \left( \frac{q}{(q - q^{-1})^2} \right)^n \prod_{k=1}^{n} (1 - (\lambda^2 + q^2 + qM\lambda)q^{2k} + q^2\lambda^2 q^{4k}).
\]

Note that by putting \( A, B, C \) and \( D \) by

\[
\begin{align*}
AB &= q^2\lambda^2, \\
A + B &= q\lambda M + q^2 + \lambda^2, \\
C &= qM\lambda^{-1} + 1 + q^2\lambda^{-2}
\end{align*}
\]

we can rewrite the positivity condition as

\[
(A, B; q^2)_n > 0, \quad (C, D; q^2)_n > 0 \quad \forall n \in \mathbb{N}
\]

using the standard notation for \( q \)-shifted factorials \([6]\).

**Proof.** It remains to check that the inner product indeed gives a unitary representation of \( U_q(\mathfrak{su}(1, 1)) \). The relation \( K = K^* \) is clear, and the relation \( E^* = -FK \) follows by construction. \( \square \)
In case the Mathieu modules are reducible, see Proposition 5.8, the analysis of unitarizability can be done similarly for the quotient space.

5.5. The irreducible admissible unitary representations of $U_q(\mathfrak{su}(1,1))$. The representations of $U_q(\mathfrak{su}(1,1))$ have been classified under certain conditions in [2], [14], [19]. We restrict to the case of the irreducible unitary representation $U_q(\mathfrak{su}(1,1))$ that play an important role in the harmonic analysis on the non-compact quantum group, in the von Neumann algebraic setting, as the analogue of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$, see [7]. We restrict to type I admissible representations of $U_q(\mathfrak{su}(1,1))$, because the eigenvalues of the action of $K$ are contained in $q^{2\varepsilon+2\mathbb{Z}}$, $\varepsilon \in \{0, \frac{1}{2}\}$. Translating the relevant representations we have the following irreducible $*$-representations of $U_q(\mathfrak{su}(1,1))$, where one should note that the representations are given by unbounded operators defined on the domain of the finite linear combinations of the basis vectors. The Hilbert space is $\ell^2(N_0)$, respectively $\ell^2(\mathbb{Z})$, equipped with orthonormal basis $\{e_k\}_{k \in \mathbb{N}_0}$, respectively $\{e_k\}_{k \in \mathbb{Z}}$. These representations are classified by the action of the Casimir and the eigenvalues of $K$, where the Casimir operator $\Omega$ acts as $(q^{2\sigma+1} + q^{-2\sigma-1})/(q^{-1} - q)^2$ with the value for $\sigma$ given below for the non-extremal unitary representations of $U_q(\mathfrak{su}(1,1))$.

(i) Principal series acts in $\ell^2(\mathbb{Z})$. Labeling $\sigma = -\frac{1}{2} + ib$, with $0 \leq b \leq -(\pi/2 \ln q)$ and $\varepsilon \in \{0, \frac{1}{2}\}$ and assume $(\sigma, \varepsilon) \neq (-\frac{1}{2}, \frac{1}{2})$. The eigenvalues of $K$ are $q^{2\varepsilon+2\mathbb{Z}}$.

(ii) Strange series acts in $\ell^2(\mathbb{Z})$. Labeling $\varepsilon \in \{0, \frac{1}{2}\}$, $\sigma = -\frac{1}{2} - (i\pi/2 \ln q) + a$, $a > 0$. The eigenvalues of $K$ are $q^{2\varepsilon+2\mathbb{Z}}$.

(iii) Complementary series acts in $\ell^2(\mathbb{Z})$. Labeling $-\frac{1}{2} < \sigma < 0$. The eigenvalues of $K$ are $q^{2\varepsilon}$.

The explicit action can be found in e.g. [12], see also [2], [14], [19] for more general representations.

Upon comparing with Proposition 5.6 and Corollary 5.7 and Proposition 5.9 we see that the principal series, strange series and complementary series can be matched by considering the suitable $(\lambda, \mu)$ such that the spectrum of $K$, i.e. $\lambda q^{2\mathbb{Z}}$, and the eigenvalue of the Casimir match, i.e.

$$\mu + \frac{q^{-1}\lambda + q^{\lambda-1}}{(q - q^{-1})^2} = \frac{q^{2\sigma+1} + q^{-2\sigma-1}}{(q^{-1} - q)^2},$$

see Corollary 5.4. Then, by Proposition 5.9, all these choices lead to equivalent representations. So we recover the principal series, strange series and complementary series representations of $U_q(\mathfrak{su}(1,1))$ as irreducible unitary Mathieu modules $M(C_{\lambda,\mu})$, where the values of $(\lambda, \mu)$ are not uniquely determined, but the corresponding Mathieu module is.

Apart from the non-extremal unitary representations, $U_q(\mathfrak{su}(1,1))$ has two sets of extremal unitary representations. These are the positive and negative discrete series representations;

(iv) Positive discrete series acts in $\ell^2(N_0)$. Labeling $\sigma = -k$, $k \in \frac{1}{2}\mathbb{N}$, and the eigenvalues of $K$ are $q^{2k+2\mathbb{N}}$.

(iv) Negative discrete series acts in $\ell^2(N_0)$. Labeling $\sigma = -k$, $k \in \frac{1}{2}\mathbb{N}$, and the eigenvalues of $K$ are $q^{-2k-2\mathbb{N}}$.

For the positive and negative series representations we need to take a quotient of the Mathieu module, see Corollary 5.4. For the positive discrete series we can take $n_F = 1$ in (5.5), so $\mu = 0$, which corresponds to the degenerate Mathieu module, cf. Proposition 4.3.
Next take $\lambda = q^{2k}$, and then the positive discrete series is equivalent to the quotient of the corresponding Mathieu module by the invariant subspace. The negative discrete series can be dealt with by taking $n_E = 1$. It is well known that these representations are unitary, as can be checked using by performing the analysis of Theorem 5.12 in case of non-irreducible Mathieu modules.

6. Rank 1 Mathieu modules for $U_q(\mathfrak{sl}(n+1, \mathbb{C}))$

The setting of Section 3 for the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ is very special, since the weight space $U_0$ is a commutative algebra. In this section we consider the case of $U_q(\mathfrak{sl}(n+1, \mathbb{C}))$ for $n \geq 2$, in which $U_0$ is not commutative.

In the setting of Theorem 2.5 we take $S = \{i_1, \cdots, i_s\}$, $s = |S|$, any subset of $\{1, \cdots, n\}$ with the condition that $|i_k - i_l| > 1$. Then $S$ is a set of strongly orthogonal roots, see Remark 2.6 and 4. Then $U_q(\mathfrak{g}_S)$ corresponds to a product of commuting copies of $U_q(\mathfrak{sl}(2, \mathbb{C}))$, see Remark 2.4. So $U_q(\mathfrak{g}_S)$ is Hopf subalgebras of $U_q(\mathfrak{sl}(n+1, \mathbb{C}))$ generated by $E_j, F_j, K_j^{\pm 1}, j \in S$.

Denote by $U_q(\mathfrak{g}_S)^+$, respectively $U_q(\mathfrak{g}_S)^0, U_q(\mathfrak{g}_S)^-$, the subalgebras of $U_q(\mathfrak{g}_S)$ generated by $E_j$, respectively $F_j, K_j^{\pm 1}$, for $j \in S$.

In case $S$ consists of one element, we write $U_q(\mathfrak{g}_{(j)}) = U_q(\mathfrak{g}_j)$, and then $U_q(\mathfrak{g}_j) \cong U_q(\mathfrak{sl}(2, \mathbb{C}))$.

Using the description of $U_0$ for $U_q(\mathfrak{sl}(2, \mathbb{C}))$ in Corollary 5.5 we obtain Lemma 6.1.

**Lemma 6.1.** $U_0^S$ is the commutative algebra generated by $U^0$ and by $E_j F_j, j \in S$.

So in particular, for $j \in S$ we have $\mathbb{C}[K_j^{\pm 1}, E_j F_j] \subset U_0^S$.

**Lemma 6.2.** If $v$ is an element of a $U_q(\mathfrak{sl}(n+1, \mathbb{C}))$-module $W$ such that $E_j F_j v = \mu_j v$ and $K_j^{\pm 1} = \lambda_j^{\pm 1} v$ for $\mu_j, \lambda_j \in \mathbb{C}$, then the $U_q(\mathfrak{g}_j)$-module generated by $v$ is isomorphic to a quotient of the $U_q(\mathfrak{sl}(2, \mathbb{C}))$-module $M(\mathbb{C}_{\lambda_j, \mu_j})$. In particular, if $M(\mathbb{C}_{\lambda_j, \mu_j})$ is irreducible as $U_q(\mathfrak{sl}(2, \mathbb{C}))$-module, the $U_q(\mathfrak{g}_j)$-module generated by $v$ is isomorphic to the $U_q(\mathfrak{sl}(2, \mathbb{C}))$-module $M(\mathbb{C}_{\lambda_j, \mu_j})$.

**Proof.** Apply Lemma 3.2 to the case $U = U_q(\mathfrak{sl}(2, \mathbb{C}))$ with $V = \mathbb{C} v$ and $\tilde{V} = \mathbb{C}_{\lambda_j, \mu_j}$ with $\psi$ mapping 1 to $v$. Then $\Psi = M(\psi): M(\mathbb{C}_{\lambda_j, \mu_j}) \rightarrow W_j \subset W, W_j = U_q(\mathfrak{g}_j) \cdot v$ gives the $U_q(\mathfrak{g}_j)$-intertwiner. Then $\Psi$ is surjective, and hence $W_j$ is a quotient of $M(\mathbb{C}_{\lambda_j, \mu_j})$. □

Note that from Lemma 0.2 and (6.4), (6.5) we can determine when $E_j^m \cdot v = 0$ or $F_j^m \cdot v = 0$ for some $m \in \mathbb{N}$ in order to study the reducibility of the corresponding Mathieu modules for $U_q(\mathfrak{sl}(n+1, \mathbb{C}))$. However, in case the Mathieu module is associated to a set $S$ of strongly orthogonal roots, the module is always reducible.

**Proposition 6.3.** Let $S \subset \{1, \cdots, n\}$ as above, and let $\lambda \in \mathbb{C}^n, \mu \in \mathbb{C}^s$. Let $\phi_{\lambda, \mu}^S$ be the corresponding 1-dimensional representation $U_0^S \rightarrow \mathbb{C}$ sending $K_i \mapsto \lambda_i, E_j F_j \mapsto \mu_j$ for $j \in S$, and we denote the extension to $U_0$ by $\phi_{\lambda, \mu}^S$ as well. Let $M(\mathbb{C}_{\lambda, \mu}^S)$ be the corresponding rank 1 Mathieu module, then $M(\mathbb{C}_{\lambda, \mu}^S)$ has a non-trivial invariant subspace.

**Proof.** Let $W$ be the invariant subspace generated by $F_j \otimes 1$ in $U \otimes U_0 \mathbb{C}_{\lambda, \mu}$ for $j \notin S$. As in the proof of Proposition 4.4, we see that $1 \otimes 1 \notin W$, so that $W$ is a proper invariant subspace. □

**Remark 6.4.** The representations constructed in Proposition 6.3 by modding out the maximal proper subspace are in general non-extremal modules of $U_q(\mathfrak{sl}(n+1, \mathbb{C}))$. 
We expect that generically the invariant subspace $W$ is the maximal proper subspace, so that $M(C^S_{\lambda,\mu})/W$ is irreducible. A further study of these representations, possibly in relation to the results of [5], is needed in order to determine the usefulness in the analytic study of the non-compact quantum group analogs of $SU(r, s)$, $r + s = n + 1$, and related homogeneous spaces.

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