BLOWUP RAMSEY NUMBERS

VICTOR SOUZA

Abstract. We study a generalisation of the bipartite Ramsey numbers to blowups of graphs. For a graph $G$, denote the $t$-blowup of $G$ by $G[t]$. We say that $G$ is $r$-Ramsey for $H$, and write $G \rightarrow H$, if every $r$-colouring of the edges of $G$ has a monochromatic copy of $H$. We show that if $G \rightarrow H$, then for all $t$, there exists $n$ such that $G[n] \rightarrow H[t]$. In fact, we provide exponential lower and upper bounds for the minimum $n$ with $G[n] \rightarrow H[t]$, and conjecture an upper bound of the form $c^t$, where $c$ depends on $H$ and $r$, but not on $G$. We also show that this conjecture holds for $G(n, p)$ with high probability, above the threshold for the event $G(n, p) \rightarrow H$.

1. Introduction

We say that a graph $G$ is $r$-Ramsey for a graph $H$, and write $G \rightarrow H$, if every $r$-colouring of the edges of $G$ contains a monochromatic copy of $H$. The classical theorem of Ramsey [30] from 1930 shows that for every $t$, there is an $n$ such that $K_n \rightarrow K_t$. The smallest $n$ with this property is called the diagonal $r$-Ramsey number and is denoted $R_r(t)$. It was proved by Erdős and Szekeres [18] and by Erdős [13] that $2^t/2 \leq R_2(t) \leq 4^t$. Currently, the best bounds are

$$(1 + o(1))(\sqrt{2/e}) t^{2t/2} \leq R_2(t) \leq t^{-\log t} 4^t,$$

for some constant $c > 0$. The lower bound, due to Spencer [37], is an application of the Lovász Local Lemma [16]. The upper bound was due to Conlon [12], improving on previous results of Rödl [22] and Thomason [38].

The first results on the bipartite analogue of the Ramsey numbers were proved by Beineke and Schwenk [3] in 1975. The bipartite $r$-Ramsey number $B_r(t)$ is defined to be the smallest $n$ such that $K_{n,n} \rightarrow K_{t,t}$. The best current bounds for these numbers, in the case $r = 2$, are

$$(1 + o(1))(\sqrt{2/e}) t^{2t/2} \leq B_2(t) \leq (1 + o(1)) \log_2(t)2^{t+1},$$

where the lower bound is due to Hattingh and Henning [23], and the upper bound to Conlon [11].

In this paper, we consider a generalisation of the bipartite Ramsey numbers to blowups of general graphs. We denote by $G[t]$ the $t$-blowup of $G$ (see Section 2 for a precise definition) and call a copy of $H[t]$ in $G[n]$ canonical if it is the $t$-blowup of a copy of $H$ in $G$. We say that $G[n]$ is canonically $r$-Ramsey for $H[t]$, and write $G[n] \rightarrow H[t]$, if every $r$-colouring of the edges of $G[n]$ has a canonical monochromatic copy of $H[t]$. Define

$$B(G \rightarrow H; t) = \min \{ n : G[n] \rightarrow H[t] \},$$

as the blowup Ramsey numbers. This generalises the bipartite Ramsey numbers, since every copy of $K_2[t]$ in $K_2[n]$ is canonical, so $B_r(t) = B(K_2 \rightarrow K_2; t)$. 


A necessary condition for these numbers to be finite is that $G \rightarrow H$. Indeed, if $G \not\rightarrow H$, consider a colouring of $G$ without monochromatic copies of $H$. Taking the $n$-blowup of this colouring, we see that $G[n] \not\rightarrow H$ for all $n$. Assuming that $G \rightarrow H$, one can obtain a bound on $B(G \rightarrow H; t)$ in the following way. Let $n$ be sufficiently large and consider an $r$-colouring of the edges $G[n]$. Repeatedly apply the bipartite Ramsey theorem between vertex classes in $G[n]$ corresponding to edges of $G$. Each time, restrict the vertex classes to contain only vertices used by the monochromatic bipartite graph we obtain. Doing this for each of the $e(G)$ pairs of vertex classes with edges between, we obtain a canonical copy of $G[t]$ in $G[n]$ for which the colouring is the blowup of a colouring of $G$. Since $G \rightarrow H$, we have a monochromatic canonical copy of $H[t]$. In fact, this shows that $B(G \rightarrow H; t) \leq B_r(B_r(\cdots B_r(t) \cdots))$, that is, an exponential tower of height $e(G)$. Our first result gives a singly exponential bound for $B(G \rightarrow H; t)$.

**Theorem 1.1.** If $G \rightarrow H$, then $G[c^r] \rightarrow H[t]$ for some constant $c = c(G, H, r)$.

In Section 3, using the Lovász Local Lemma, we prove a corresponding exponential lower bound of the form

$$B(G \rightarrow H; t) \geq (1 + o(1)) \left( r^{d(H)/2} \right)^t,$$

where $d(H) = 2e(H)/v(H)$ is the average degree of $H$. Theorem 3.1 provides the precise lower bound we obtain, which recovers the best known bound on $B_2(t)$.

Note that the lower bound we obtain does not depend on the graph $G$ asymptotically. If $G$ is large and has many copies of $H$, it should be harder for a random colouring of $G[n]$ to avoid a canonical monochromatic copy of $H[t]$. In fact, we conjecture that for all $r \geq 2$ and graphs $H$, there is a constant $c = c(H, r)$ such that if $G \rightarrow H$, then $G[c^r] \rightarrow H[t]$. See Conjecture 5.1.

Although we have not managed to settle this conjecture, not even the case $r = 2$ and $H = K_3$, we can provide some evidence to support it. More precisely, we show that above the threshold for the event $G(n, p) \rightarrow H$, the conjecture holds for $G(n, p)$ with high probability.

Let $m_2(H)$ be the 2-density of a graph $H$ (see Section 3 for a precise definition). Rödl and Ruciński 33 proved that $p = n^{-\frac{1}{m_2(H)}}$ is the correct order for the threshold of the event $G(n, p) \rightarrow H$ when $H$ has at least one component that is not a star. When $H$ is a star forest and $\Delta(H) \geq 2$, the threshold occurs at a lower value of $p$, while if $\Delta(H) = 1$ then there is a coarse threshold at $p = 1/n^2$.

**Theorem 1.2.** Let $r \geq 2$ and let $H$ be a graph with maximum degree $\Delta(H) \geq 2$. There are constants $c = c(H, r)$ and $C = C(H, r)$ such that, if $p \geq C n^{-1/m_2(H)}$ then

$$\lim_{n \to \infty} \mathbb{P} \left( G(n, p)[c^r] \rightarrow H[t] \right) = 1.$$
We prove Theorem 1.2 in Section 4, using the hypergraph container method of Balogh, Morris and Samotij [4] and Saxton and Thomason [35]. More specifically, we use a container theorem for sparse sets in \( H \)-free graphs, stated explicitly by Saxton and Thomason.

In Section 5 we propose some conjectures and open problems, including the aforementioned Conjecture 5.1. We also examine a family of minimal graphs with the property that \( G \rightarrow H \), but for which the bound given by our proof of Theorem 1.1 is not strong enough to deduce Conjecture 5.1 in this case. The construction of this family, due to Burr, Erdős and Lovász [8], uses the so called signal senders.

2. Upper Bound

In this section, we prove a quantified version of Theorem 1.1. Before stating this result precisely, we introduce some concepts and some notation.

Let \( G \) be a graph on \( n \) vertices. Given \( t_1, \ldots, t_n \) positive integers, we define the \((t_1, \ldots, t_n)\)-blowup of \( G \), denote by \( G[t_1, \ldots, t_n] \), as the graph obtained from \( G \) by replacing each vertex \( v_i \) with an independent set \( U_i \) of \( t_i \) vertices. For every edge \( v_iv_j \) in \( G \), we put a complete bipartite graph between \( U_i \) and \( U_j \). The \( t \)-blowup of \( G \) is the graph \( G[t] = G[t, \ldots, t] \).

For graphs \( G \) and \( H \), we define the \( r \)-multiplicity of \( H \) in \( G \) as the minimum number of monochromatic copies of \( H \) over all \( r \)-colourings of the edges of \( G \), and denote that quantity by \( M_r(H; G) \). Note that the statement that \( G \rightarrow H \) is then equivalent to the statement that \( M_r(H; G) \geq 1 \). Also, note that \( M_1(H; G) \) is the number of copies of \( H \) in \( G \). We call the ratio

\[
\beta_r(H; G) := \frac{M_r(H; G)}{M_1(H; G)}
\]

the Ramsey \( r \)-robustness of \( H \) in \( G \). Thus, \( \beta_r(H; G) \) is the minimum proportion of monochromatic copies of \( H \) in \( G \) that we can guarantee that appears in any \( r \)-colouring of the edges of \( G \). Note that \( \beta_r(H; G) > 0 \) if, and only if, \( G \rightarrow H \). We prove the following theorem, which implies Theorem 1.1 and which we also use to prove Theorem 1.2.

**Theorem 2.1.** If \( G \rightarrow H \), then \( G[c^t] \rightarrow H[t] \), where \( c \) is given by

\[
c = \exp \left( \frac{\beta_r(H; G) c^{r^2} - c^2}{\beta_r(H; G)} \right). \tag{2.1}
\]

Actually, we are going to prove the stronger statement that \( G[c^t] \rightarrow H[t, \ldots, t, c_0^t] \), where

\[
c_0 = c^{1 - \frac{1}{r} \beta_r(H; G)} c^{r - 1}. \tag{2.2}
\]

This is a strengthening of Theorem 1.1 since \( c_0^t \geq t \). Note that, via a relabelling of the vertices of \( H \), we can choose which vertex class receives the larger part.

The main ingredient in the proof of upper bound is a variant of the following beautiful theorem of Nikiforov [27, 28].

**Theorem 2.2.** Let \( H \) be a graph with \( k \geq 2 \) vertices. Let \( G \) a graph with \( n \) vertices and \( \rho < 1/4 \). If \( G \) contains at least \( \rho n^k \) copies of \( H \), then \( G \) contains a copy of \( H[t, \ldots, t, n^{1-\rho^2-1}] \) where \( t = \lfloor \rho^2 \log n \rfloor \). \( \square \)
We can obtain the constant $c$ in Theorem 2.1
\[ c = \exp \left( \frac{r^v(H) \nu(H)^2}{\beta_r(H; G) \nu(H)} \right). \]

We can obtain the constant $c$ given in Theorem 2.1 by applying the following variant of Nikiforov’s theorem, Theorem 2.2

**Theorem 2.3.** Let $H$ be a graph with $k \geq 2$ vertices. If $G$ is a subgraph of $H[n]$ with $\rho n^k$ canonical copies of $H$, then it has a canonical copy of $H[t, \ldots, t, n^{1-\rho-1}]$ where $t = \lfloor \rho^k 4^{-k^2+k} \log n \rfloor$.

The proof of Theorem 2.3 is very similar to the original proof of Nikiforov, and is therefore postponed to Appendix A. We now deduce Theorem 2.1

**Proof of Theorem 2.1.** Let $G$ and $H$ be graphs with $G \supseteq H$. Consider an $r$-colouring of the edges of $G[n]$. Note that there are $n^{v(G)}$ canonical copies of $G$ in $G[n]$. Each one of these copies has $M_r(H; G) \geq 1$ canonical monochromatic copies of $H$. But a canonical copy of $H$ can appear in $n^{v(G)-v(H)}$ distinct copies of $G$ in $G[n]$. Therefore the number of distinct canonical monochromatic copies of $H$ in $G[n]$ is at least
\[ \frac{n^{v(G)} M_r(H; G)}{n^{v(G)-v(H)}} = M_r(H; G) n^{v(H)}. \]

Thus, there is a colour $i \in [r]$ such that there are at least $n^{v(H)} M_r(H; G)/r$ copies of $H$ in colour $i$. Furthermore, every canonical copy of $H$ in $G[n]$ correspond to a copy of $H$ in $G$. There are $M_1(H; G)$ copies of $H$ in $G$, and hence, there are
\[ L := \left( \frac{M_r(H; G)}{r M_1(H; G)} \right) n^{v(H)} = r^{-1} \beta_r(H; G) n^{v(H)} \]
canonical copies of $H$ in $G[n]$, of colour $i$, all corresponding to the same copy of $H$ in $G$. In other words, there is some copy $H'$ of $H$ in $G$ such that there are $L$ canonical copies of $H$ of colour $i$ in $H'[n]$.

Let $H'[n]_{(i)}$ be the subgraph of $H'[n]$ of the edges of colour $i$. We apply Theorem 2.3 to $H'[n]_{(i)}$ with $\rho = r^{-1} \beta_r(H; G)$ to obtain a copy of $H[t, \ldots, t, n^{1-\rho-1}]$ of colour $i$ in $G[n]$, where
\[ t = r^{-v(H)} \beta_r(H; G)^{v(H)} 4^{-v(H)^2+v(H) \log n}. \]

Consequently, if we take $c = c(G, H, r) = \exp(r^{v(H)} \beta_r(H; G)^{-v(H)} 4^{v(H)^2-v(H)})$, we have that $G[e^c] \supseteq H[t, \ldots, t, c_0]$, for $c_0 = c^{1-(r^{-1} \beta_r(H; G))^{v(H)-1}}$. □

We point out that in whichever version of Nikiforov’s theorem is used, we actually can find a blowup of $H$ with one of the parts of size polynomial in $v(G)$, instead of logarithmic. This stronger conclusion is not needed if we just want to find a monochromatic copy of $H[t]$ in $G[n]$, but we get a larger part for free by applying this result. This asymmetric phenomenon seems to appear naturally in extremal and Ramsey questions in blowup ambient graphs.
3. Lower Bound

In this section, we set out to prove the following theorem.

**Theorem 3.1.** For $r \geq 2$ and graphs $G$ and $H$, we have

$$B(G \rightarrow H; t) \geq \left(1 + o(1)\right) \left(r^{d(H)/v(H)}e^{-1}\right) t r^{d(H)H/2}.$$  

We recall that $d(H) = 2e(H)/v(H)$ is the average degree of $H$. To obtain this lower bound, we produce a $r$-colouring of the edges of $G[n]$ randomly. For a suitable value of $n$, we check, via the Lovász Local Lemma that the probability that it has no canonical monochromatic copy of $H[t]$ is positive.

Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. A graph $D = (V, E)$ on the set of vertices $V = \{1, 2, ..., n\}$ is called a dependency graph for the events $A_1, \ldots, A_n$ if for each $i, 1 \leq i \leq n$, the event $A_i$ is mutually independent of all the events $\{A_j : \{i, j\} \notin E\}$. As we want to avoid the same graph in all the colours, the simpler symmetric version of the Lovász Local Lemma is sufficient. We use the following version of the local lemma, see [1, Corollary 5.1.2]:

**Lemma 3.2 (Lovász Local Lemma).** Suppose that $D = (V, E)$ is a dependency graph for the events $A_1, A_2, \ldots, A_n$. Suppose that $D$ has degree bounded by $d$ and that $\mathbb{P}(A_i) \leq p$ for all $1 \leq i \leq n$. If $ep(d + 1) \leq 1$, then $\mathbb{P}(\bigcap_{i=1}^{n} A_i) > 0$.

In the next proposition, we apply the local lemma to provide a condition that produces good lower bounds for blowup Ramsey numbers in a very general setting. Indeed, we allow the blowup of $H$ we avoid to have vertex classes of distinct sizes.

Before we state the proposition, we recall that a graph homomorphism $\phi : H \rightarrow G$ is a map from the vertices of $H$ to the vertices of $G$ such that if $u \sim_H v$ then $\phi(u) \sim_G \phi(v)$. We denote by $\text{inj}(H, G)$ the number of injective homomorphism from $H$ to $G$. An injective homomorphism from $H$ to $G$ can be thought as an copy of $H$ in $G$, but you also keep track of which vertex of $H$ corresponds to which vertex of $G$.

**Proposition 3.3.** Let $G$ and $H$ be graphs and $t_1, \ldots, t_{v(H)}$ positive integers. Set $\bar{H} = H[t_1, \ldots, t_{v(H)}]$ and $\Delta = \max\{t_it_j : i \sim_H j\}$. Then $G[n] \rightarrow \bar{H}$, given that

$$e \text{ inj}(H, G) e(\bar{H}) r^{1-e(\bar{H})} \frac{\Delta}{n^2} \prod_{w \in [k]} \left(\frac{n}{t_w}\right) \leq 1. \quad (3.1)$$

**Proof.** Consider the blowup $G[n]$, where for each vertex $j$, we associate a vertex class $V_i$ of size $n$. Consider a random uniform $r$-colouring of the edges of $G[n]$ and define, for each canonical copy of $\bar{H}$ in $G$, the event that such copy is monochromatic. Each one of these events have the same probability $p = r^{1-e(\bar{H})}$. Furthermore, each event is mutually independent from all other events whose corresponding copy of $\bar{H}$ is edge-disjoint with its own copy of $\bar{H}$.

Having fixed a copy of $\bar{H}$ in $G[n]$ and an edge $i \sim j$ in $H$, we bound the number of canonical copies of $H$ that has at least one edge in common between the vertex classes $V_{\phi(i)}$ and $V_{\phi(j)}$ by

$$t_it_j \text{ inj}(H, G) \binom{n-1}{t_u-1} \binom{n-1}{t_v-1} \prod_{w \neq u, v} \left(\frac{n}{t_w}\right) = \frac{\text{inj}(H, G) t_it_j t_ut_v}{n^2} \prod_{w} \left(\frac{n}{t_w}\right). \quad (3.2)$$
There are $t_i t_j$ choices for the intersecting edge. At most inj$(H,G)$ choices for a homomorphism $\varphi$ with $\varphi(u) = \phi(i)$ and $\varphi(v) = \phi(j)$ and the remaining vertices of $\tilde{H}$ are chosen without restriction. Now we sum (3.2) over the possible choices for the edge $i \sim j$:

$$d \leq \sum_{i \sim j} \frac{\text{inj}(H,G) t_i t_j t_w}{n^2} \prod_w \left( \frac{n}{t_w} \right) \leq \text{inj}(H,G) e(\tilde{H}) \frac{\Delta}{n^2} \prod_w \left( \frac{n}{t_w} \right).$$

Thus, the condition in (3.1) implies that $epd \leq 1$, so we can apply Lemma 3.2 and conclude that with positive probability, none of the events occur. \hfill \Box

Now, we get Theorem 3.1 as a consequence of Proposition 3.3. We just have take all $t_i$'s equal and work out asymptotically the best value of $n$ such that condition (3.1) holds.

**Proof of Theorem 3.1.** Set $t_i = t$ for all $i$, $\Delta = t^2$ and apply Proposition 3.3. Condition (3.1) translates to

$$e \text{inj}(H,G)e(H)r^{1-e(H)/2} \leq 1.$$  \tag{3.3}

We want to find the largest $n$, as a function of $t$, such that condition (3.3) holds, since we then have $B(G \not\rightarrow H; t) > n$. We can take $n$ to be at least exponential in $t$, so we can approximate the binomial coefficients as $\binom{n}{t} \sim n^t / t!$. Also recall Stirling’s formula $t! \sim \sqrt{2\pi t} (t/e)^t$. Thus, it suffices to show, as $t \to \infty$, that

$$e \text{inj}(H,G)e(H)r \cdot t^4 \frac{e^t n^t}{t^t} \cdot \frac{v(H)}{t^t} \leq 1.$$  \tag{3.4}

Ignoring terms that are constant in $t$ and regrouping, we have

$$\left( \frac{e^t}{\sqrt{t^t}} \right)^{v(H)/t} \ll 1.$$  \tag{3.5}

Let $P_1$ and $P_2$ be as identified above in equation (3.4). If for some $\beta > 0$, we have that $P_1 \ll \beta^{-t}$ and $P_2 \ll \beta^t$, then the condition (3.3) is satisfied for large enough $t$. Note that $P_2 \ll \beta^t$ gives us

$$n \leq \left( \beta^{1/v(H)} e^{-1} \right) t r^{d(H)/t}. $$

Therefore, if $\varepsilon > 0$, the condition $P_1 \ll \beta^{-t}$ is satisfied for $\beta = r^{d(H)} - \varepsilon$. If $n_0$ is the largest $n$ that satisfy condition (3.3), then we have

$$n_0 \geq \left( 1 + o(1) \right) \left( (r^{d(H)} - \varepsilon)^{1/v(H)} e^{-1} \right) t r^{d(H)/t},$$

for all $\varepsilon > 0$. Sending $\varepsilon$ to zero, we obtain

$$B(G \not\rightarrow H; t) \geq \left( 1 + o(1) \right) \left( r^{d(H)/v(H)} e^{-1} \right) t r^{d(H)/t}. $$  \hfill \Box

As already discussed, this lower bound does not depend on the graph $G$. Note that the condition (3.3) itself depends on $G$, but this is lost in the asymptotic behaviour. If $G$ had no copies of $H$ whatsoever, then inj$(H,G) = 0$ and condition (3.3) is trivial. Furthermore, recall that we do not assume that $G \not\rightarrow H$.

It is important to notice that a similar lower bound could also be obtained by an application of the first moment method. In fact, we would obtain the weaker bound:

$$B(G \not\rightarrow H; t) \geq \left( 1 + o(1) \right) e^{-1} t r^{d(H)/t}. $$
Like in the bipartite case, the application of the Lovász Local Lemma provides only a constant improvement, in this case, an improvement of $r^{d(H)/\nu(H)}$. This is a very minor enhancement over the first moment method, but it recaptures the lower bound by Hattingh and Henning [23].

We observe that Proposition 3.3 can be used to provide a more direct counterpart to Theorem 2.1 in the following sense. We can show that for every constant $k > 1$, we have

$$G[(t/e)(r^{d(v)}k)^t] \nrightarrow H[t, \ldots, t, k^t].$$

Indeed, this shows that any method capable of proving that $G[e^C] \nrightarrow H[t, \ldots, t, k^t]$ is bound to give a relatively weak upper bound on $c$. Take the case $K_2 \nrightarrow K_2$ for example. We know from the bounds on the bipartite Ramsey numbers that $K_2[\log_2(t)2^{t+1}] \nrightarrow K_2[t]$, but by applying Theorem 2.1 we get that $K_2[e^{64t}] \nrightarrow K_2[t]$. This bound is much weaker than the bound we already have, but what we actually prove is that $K_2[e^{64t}] \nrightarrow K_2[t, e^{32t}]$. If our target graph is $K_2[t,e^{32t}]$, then by (3.5) we have $K_2[e^{32.6t}] \nrightarrow K_2[t,e^{32t}]$. This quantifies the limitations of our method, considering the original aim of finding $n$ such that $G[n] \nrightarrow H[t]$.

4. Random Graphs

Before we prove Theorem 1.2 let us recall some results concerning Ramsey properties of random graphs. Denote by $G(n,p)$ the usual model for a graph of $n$ vertices and where each edge is present with probability $p$, independent from all other edges. See Bollobás [H] for background on random graphs. In light of Theorem 2.1 we are interested in finding $m$ such that $G(n,p)[m] \nrightarrow H[t]$. For this to be possible, it is necessary that $G(n,p) \nrightarrow H$. Thus, we first recall some of what is known about the threshold for this property.

A graph parameter that is useful in the analysis of Ramsey properties of random graphs is the 2-density $m_2(G)$ of a nonempty graph $G$. First, define $d_2(K_2) = 1/2$ and $d_2(G) = \frac{e(G)-1}{v(G)-2}$ otherwise. The 2-density of $G$ is defined as

$$m_2(G) = \max \{d_2(J) : J \subseteq G, e(J) \geq 1 \}.$$

The study of Ramsey properties of random graphs was initiated by Erdős. Answering his question, Frankl and Rödl [19] showed that if $p \geq n^{-1/2+\varepsilon}$ then $G(n,p) \nrightarrow K_3$ with high probability. Later Luczak, Ruciński, and Voigt [25] (and independently Erdős, Sós and Spencer) proved that $p = n^{-1/2}$ is the threshold for the event $G(n,p) \nrightarrow K_3$. In a series of papers, Rödl and Ruciński [31, 33] proved the following remarkable theorem.

**Theorem 4.1.** Let $r \geq 2$ and suppose that $H$ is a graph such that at least one component of $H$ is not a star, and in the case $r = 2$, also not a path of length three. Then, there exist positive constants $c$ and $C$ such that

$$\lim_{n \to \infty} \mathbb{P}(G(n,p) \nrightarrow H) = \begin{cases} 0 & \text{if } p \leq cn^{-1/m_2(H)} \\ 1 & \text{if } p \geq Cn^{-1/m_2(H)}. \end{cases}$$

Moreover, Rödl and Ruciński showed that in the case where $H$ is a star forest, the threshold for the property $G(n,p) \nrightarrow H$ is actually $n^{-1-1/((\Delta(H)-1)+1)}$, where $\Delta(H)$ is the maximum degree of $H$. This occurs before the $m_2$ threshold as above. In the case $r = 2$ and $H$ being a forest whose components are stars and $P_3$’s, with at least one $P_3$, the
1-statement of Theorem 4.1 still holds, but it is necessary to assume \( p \ll n^{-1/m_2(H)} = 1/n \) for the 0-statement.

As we want to show that \( G(n, p)[m] \xrightarrow{\varepsilon} H[t] \) if \( m \) is large enough, we only need the 1-statement. It is necessary to assume that \( p \geq C n^{-1/m_2(H)} \) when \( H \) is not a star forest, but we assume this condition for all \( H \) for simplicity. In fact, we also assume that \( \Delta(H) \geq 2 \). Under these conditions, the subgraph count \( X_H = M_1(H; G(n, p)) \) is concentrated around it’s mean, see Janson, Łuczak and Ruciński [24, Section 3.1]. This holds since \( \Delta(H) \geq 2 \) implies \( m(H) < m_2(H) \), where \( m(H) \) is the maximum density \( d(J) = e(J)/v(J) \) of a subgraph \( J \subseteq H \). In particular, \( X_H \lesssim 2 \mathbb{E}(X_H) \) holds with high probability.

The best that we can hope is that there is a canonical monochromatic copy of \( H[t] \) in every \( r \)-colouring of a \( m \)-blowup of \( G(n, p) \), where \( m = m(H, r, t) \). That is, \( m \) does not depend on the ambient graph, only on the graph that we want to find.

In view of Theorem 2.1 it is sufficient show that the multiplicity \( \beta_r(H; G(n, p)) \) is bounded away from 0 with high probability. To prove this, we adapt the proof of Nenadov and Steger [26] of the 1-statement of Theorem 4.1. As in their proof, we use a version of hypergraph container theorem of Balogh, Morris and Samotij [2] and Saxton and Thomason [35]. We differ from Nenadov and Steger by using a container for sparse sets, instead of independent sets.

We introduce some notation. Let \( \mathcal{G}(n) \) be the set of \( 2^{(\binom{n}{2})} \) graphs with vertex set \([n] \). Let \( \mathcal{F}_r(H, n, b) \) be the family defined as

\[
\mathcal{F}_r(H, n, b) = \{ G \in \mathcal{G}(n) : M_r(H; G) \leq b \}.
\]

Note that \( \mathcal{F}_1(H, n, 0) \) is the family of \( H \)-free graphs on \( n \) vertices and \( \mathcal{F}_r(H, n, 0) \) is the family of graphs \( G \) on \( n \) vertices for which \( G \not\Rightarrow H \).

The specific container theorem we use, concerning \( H \)-sparse graphs, was stated explicitly by Saxton and Thomason.

**Theorem 4.2.** Let \( H \) be a graph with \( e(H) \geq 2 \). For any \( \varepsilon > 0 \) there exists \( n_0 \) and \( k = k(H, \varepsilon) > 0 \) such that the following is true for all \( n \geq n_0 \). For \( n^{-1/m_2(H)} \leq q \leq 1/n_0 \), there exists functions

\[
f : \mathcal{G}(n) \to \mathcal{G}(n) \quad \text{and} \quad g : \mathcal{F}_1(H, n, q^{e(H)}n^{v(H)}) \to \mathcal{G}(n),
\]

such that for all \( G \in \mathcal{F}_1(H, n, q^{e(H)}n^{v(H)}) \),

(i) \( e(g(G)) \leq kqn^2 \),

(ii) \( M_1(H; f(g(G))) \leq \varepsilon n^{v(H)} \),

(iii) \( g(G) \subseteq G \subseteq f(g(G)) \).

This theorem is a consequence of Theorem 9.2 in [35] (take \( \ell = 2 \), \( \hat{G} = G(n, H) \)), and note that (b) implies (ii). To obtain (i) and (iii), use (a) and (d) together with Remark 2.2 in [36], which allows us to consider a single set as a ‘signature’, instead of a tuple \( T = (T_1, \ldots, T_s) \). The following standard lemma is also useful for us. For a proof, see Corollary 8.3 in [20], for instance.

**Lemma 4.3.** Let \( r \geq 1 \) and \( H \) be a graph. Then there are constants \( \delta, \varepsilon > 0 \) and \( n_0 \) such that the following is true for all \( n \geq n_0 \). For any graphs \( G_1, \ldots, G_r \in \mathcal{F}_1(H, n, \varepsilon n^{v(H)}) \), we have \( e(K_n \setminus (G_1 \cup \cdots \cup G_r)) \geq \delta n^2 \).

Now, we can precisely state and prove the robustness result for \( G(n, p) \).
Theorem 4.4. Let \( r \geq 2 \) and let \( H \) be a graph with \( \Delta(H) \geq 2 \). There are constants \( \eta = \eta(H, r) > 0 \), and \( C = C(H, r) \) such that, if \( p \geq C n^{-1/m_2(H)} \) then

\[
\lim_{n \to \infty} \mathbb{P}(\beta_r(H; G(n, p)) \geq \eta) = 1.
\]

Proof. Let \( \varepsilon = \varepsilon(H, r) \) and \( \delta = \delta(H, r) \) be as in Lemma 4.3 and let \( n_0 = n_0(H, r) \) and \( k = k(H, r) \) be as in Theorem 4.2 for \( H \), \( \varepsilon \) and \( q = \gamma p \), where \( \gamma \geq 0 \) is to be determined. Choose \( C = 1/\gamma \) and let \( p \geq C n^{-1/m_2(H)} \), with \( p = o(1) \), and assume that \( n \geq n_0 \). We have \( p \gg 1/n^2 \), since \( m_2(H) \geq 1 \) whenever \( \Delta(H) \geq 2 \).

Consider the event \( \mathcal{E} = \{ M_r(H; G(n, p)) \leq q^{e(H)n^v(H)} \} \). If \( \mathcal{E} \) holds, then there exists a colouring \( c: E(G(n, p)) \to \lbrack r \rbrack \) such that for all \( i \in \lbrack r \rbrack \), the subgraph of edges of colour \( i \), \( G_i := (G(n, p))_{(i)} \), have few copies of \( H \), namely, \( M_1(H; G_i) \leq q^{e(H)n^v(H)} \). By Theorem 4.2 for all \( i \in \lbrack r \rbrack \) there is a ‘signature’ graph \( S_i := g(G_i) \), such that \( S_i \subseteq G_i \subseteq f(S_i) \). Define the graph

\[
K(S_1, \ldots, S_r) := K_n \setminus (f(S_1) \cup \cdots \cup f(S_r)),
\]

and note that \( G(n, p) \) avoids all the edges of \( K(S_1, \ldots, S_r) \). Hence, by the union bound, we have

\[
\mathbb{P}(\mathcal{E}) \leq \sum_{(S_1, \ldots, S_r)} \mathbb{P}
\left(S_1, \ldots, S_r \subseteq G(n, p) \text{ and } K(S_1, \ldots, S_r) \subseteq (G(n, p))^c\right),
\]

where \((S_1, \ldots, S_r)\) runs over all the possible sequences of signatures given by Theorem 4.2. Note that \( E(S_1) \cup \cdots \cup E(S_r) \) and \( E(K(S_1, \ldots, S_r)) \) are disjoint sets of edges, and hence the events \( S_1, \ldots, S_r \subseteq G(n, p) \) and \( K(S_1, \ldots, S_r) \subseteq (G(n, p))^c \) are independent.

Since \( M_1(H; f(S_i)) \leq \varepsilon n^v(H) \), Lemma 4.3 implies that \( e(K(S_1, \ldots, S_r)) \geq \delta n^2 \). Defining \( S^+ := \bigcup_{i \in \lbrack r \rbrack} E(S_i) \), we can bound

\[
\mathbb{P}(S_1, \ldots, S_R \subseteq G(n, p)) \leq p^{|S^+|}, \quad \text{and}
\]

\[
\mathbb{P}(K(S_1, \ldots, S_r) \subseteq (G(n, p))^c) \leq (1 - p)^{\delta n^2} \leq \exp(-\delta p n^2).
\]

Now, the whole sum can be bounded via the following strategy. We sum over the possible values of \( s := |S^+| \leq rkqn^2 \). First, we choose \( s \) edges in \( K_n \) to correspond to the union \( S^+ \), so we have \( \binom{n}{s} \) choices. Next, for each edge, we choose which signatures they will appear. Since the signatures are disjoint, each edge has \( r \) choices, so we have at most \( r^s \) possibilities in total. In total, we have

\[
\mathbb{P}(\mathcal{E}) \leq \exp(-\delta p n^2) \sum_{s \leq rkqn^2} \binom{n}{s}^r p^s \leq \exp(-\delta p n^2) \sum_{s \leq rkqn^2} \left( \frac{erpn^2}{2s} \right)^s.
\]

Observing that \( x \mapsto (A/x)^x \) is increasing on the interval \((0, A/e)\), as long as \( M < A/e \), we can bound the sum \( \sum_{x \leq M} (A/x)^x \) by \( M(A/M)^M \). Recall that \( q = \gamma p \), and by choosing \( \gamma = \gamma(H, r) \) sufficiently small with respect to \( k = k(H, r) \), and consequently, choosing \( C = 1/\gamma \) large enough, we have

\[
\sum_{s \leq rkqn^2} \left( \frac{erpn^2}{2s} \right)^s \leq rk\gamma p n^2 \left( \frac{e}{2k\gamma} \right)^{rk\gamma p n^2} \leq \exp(\delta p n^2/2).
\]
In the end, we obtain \( \mathbb{P}(\mathcal{E}) \leq \exp(-\delta n^2p/2) = o(1) \), since \( p \gg 1/n^2 \). Now, consider the random variable \( X_H = M_1(H; G(n, p)) \). We have seen above that \( X_H \leq 2 \mathbb{E}(X_H) \) holds with high probability in the range we consider. Since \( \mathbb{E}(X_H) \leq n^v(H)p^\gamma(H) \), we have that

\[
\beta_r(H; G(n, p)) = \frac{M_r(H; G(n, p))}{M_1(H; G(n, p))} \geq \frac{q^{v(H)n^v(H)}}{2 \mathbb{E}[X_H]} \geq \frac{\gamma^{v(H)}}{2}
\]

holds with high probability. Thus, choosing \( \eta = \gamma^{v(H)}/2 \), we are done.

If \( \Delta(H) \leq 1 \), then \( m_2(H) = m(H) = 1/2 \), thus the threshold function for \( G(n, p) \xrightarrow{\gamma} H \) is \( p = 1/n^2 \). In this range, the number of edges in \( G(n, p) \) converges in distribution to a Poisson random variable, so in particular, there is a positive probability that \( G(n, p) \) is empty. Therefore, Theorem 4.4 cannot hold for all \( p \geq C/n^2 \) in such cases. If we instead assume \( p \gg 1/n^2 \), it is easy to see that \( G(n, p) \xrightarrow{\gamma} H \) with high probability. For completeness, we observe that Theorem 4.4 implies 1.2.

**Proof of Theorem 1.2.** By Theorem 4.4, \( \beta_r(H; G(n, p)) \geq \eta \) with high probability for some \( \eta = \eta(H, r) > 0 \). In particular, \( G(n, p) \xrightarrow{\gamma} H \), so by Theorem 2.1, \( G(n, p)[c^t] \xrightarrow{\gamma} H[t] \) for

\[
c = \exp\left( \frac{r^{v(H)}4^{v(H)^2-v(H)}}{\beta_r(H; G(n, p))^{v(H)}} \right) \leq \exp\left( \frac{r^{v(H)}4^{v(H)^2-v(H)}}{\eta^{v(H)}} \right),
\]

an upper bound that is a function of \( H \) and \( \eta = \eta(H, r) \) only.

In particular, we have \( B(G(n, p) \xrightarrow{\gamma} H; t) \leq c^t \) with high probability for a constant \( c = c(H, r) \), whenever \( p \geq Cn^{-1/m_2(H)} \).

5. **Conjectures and Open Problems**

As already noted in Section 1, the lower bound on \( B(G \xrightarrow{\gamma} H; t) \) we obtain in Theorem 3.1 does not depend on \( G \) asymptotically. Additionally, Theorem 1.2 implies an upper bound on \( B(G \xrightarrow{\gamma} H; t) \) that does not depend on \( G = G(n, p) \) and \( p \geq Cn^{-1/m_2(H)} \). Given this evidence, we conjecture that one could find exponential upper bounds that are uniform on \( G \) with \( G \xrightarrow{\gamma} H \).

**Conjecture 5.1.** Let \( r \geq 2 \) and let \( H \) be a graph. There is a constant \( c = c(H, r) \) such that if \( G \xrightarrow{\gamma} H \), then \( G[c^t] \xrightarrow{\gamma} H[t] \).

In our efforts to establish Conjecture 5.1, we obtained Theorem 2.1. Note that if \( G \xrightarrow{\gamma} H \) has a subgraph \( G' \subseteq G \) such that \( G' \xrightarrow{\gamma} H \), we could apply Theorem 2.1 to \( G' \) and obtain a constant \( c \) as a function of \( G' \). This shows that to certify Conjecture 5.1 we can consider only graphs \( G \) that are minimal with respect to the Ramsey property \( G \xrightarrow{\gamma} H \). Indeed, one may be tempted to show that the robustness \( \beta_r(H; G) \) is bounded away from zero among the class or minimal graphs \( M_r(H) = \{ G : G \text{ minimal such that } G \xrightarrow{\gamma} H \} \). Indeed, it would be enough that

\[
\inf\{ \beta_r(H; G) : G \in M_r(H) \} > 0.
\]

We will show that this is not the case in general. For some graphs \( H \), we show that there exist minimal graphs \( G \in M_r(H) \) with arbitrarily low robustness. To be able to construct such graphs, we recall the concept of signal senders.
We construct the graph we added are minimal with respect to the signal property, so if the remove an edge minimal. Finally, observe that \( M \) repeats without modifications and shows that concatenation is also a minimal positive signal sender. Furthermore, the argument above definition of the signal senders, all the edges between\( \{Q_1, \ldots, Q_t\} \) and let \( v \) be one of its vertices. Consider \( t \) disjoint cliques \( Q_1, \ldots, Q_t \) in \( F - v \), each of size \( t \). We construct the graph \( G \) as follows. Start with a copy of \( F \), together with an edge \( e \) disjoint from \( F \). For every edge \( f = \{x, y\} \) with \( x \in Q_i, y \in Q_j, i \neq j \), add a disjoint minimal positive signal sender \( S^+(r; K_{t+1}, e, f) \). We require the distance from \( e \) to \( f \) to be at least 3, in order to guarantee that there is no copy of \( K_{t+1} \) in \( G \) other than the copies completely inside \( F \) or those completely inside one of the signal senders.

Now, note that \( G \rightarrow K_{t+1} \). Indeed, suppose that we have a red-blue colouring of the edges of \( G \) without monochromatic \( K_{t+1} \). Additionally, suppose that \( e \) is red. By the definition of the signal senders, all the edges between \( Q_i \)'s are also red. This forces all \( Q_i \) to be blue cliques. Now consider the edges incident to \( v \). If \( v \) sends only blue edges to some \( Q_i \), then \( Q_i \) with \( v \) forms a blue \( K_{t+1} \). On the other hand, if \( v \) sends a red edge to each of the \( Q_i \), then \( v \) together with theses red neighbours forms a red \( K_{t+1} \), so \( G \rightarrow K_{t+1} \). Furthermore, if \( v \) sends exactly one red edge to each one of the \( Q_i \), then this colouring has precisely one monochromatic \( K_{t+1} \), so \( M_2(K_{t+1}; G) = 1 \).

It is not hard to see that \( G \) is also Ramsey minimal. Just note that the signal senders we added are minimal with respect to the signal property, so if the remove an edge from it, we can find a colouring \( c \) avoiding monochromatic \( K_{t+1} \) and with \( c(e) \neq c(f) \).

Consider now the family \( G_n \) of graphs that are constructed just as \( G \), but we replace one of the minimal positive signal senders \( S \) from \( e \) to \( f \) by a concatenation of \( n \) more such signal senders. More precisely, we add \( n \) new edges \( g_1, \ldots, g_n \) to \( G \) and then add internally disjoint copies of \( S \) from \( e \) to \( g_1 \), from \( g_1 \) to \( g_{t+1} \) and from \( g_n \) to \( f \). Such concatenation is also a minimal positive signal sender. Furthermore, the argument above repeats without modifications and shows that \( M_2(K_{t+1}; G_n) = 1 \) and that \( G_n \) is Ramsey minimal. Finally, observe that \( M_1(K_{t+1}; G_n) = M_1(K_{t+1}; G) + nM_1(K_{t+1}; S) \rightarrow \infty \).
In particular, there are graphs $G$ with arbitrarily low robustness $\beta_2(K_{t+1}; G)$. This shows that Theorem 2.1 is not enough to settle Conjecture 5.1 in general. Even so, 
\[
\inf \{ \beta_r(H; G) : G \in \mathcal{M}_r(H) \}
\]
can be positive for some graphs $H$. An easy case for that is when $H$ is $r$-Ramsey-finite, that is, the family $\mathcal{M}_r(H)$ is finite. It can be shown from Theorem 4.1, however, that $H$ is not $r$-Ramsey-finite for all $r$ whenever $H$ has a cycle. For $r = 2$, it is known for instance that $H$ Ramsey-finite when $H$ is a star with an odd number of edges [6] and when $H$ is a matching [7]. We conjecture that the only possible reason for $\beta_r(H; G)$ to be bounded away from zero is when $\mathcal{M}_r(H)$ is finite.

**Conjecture 5.4.** If $H$ is $r$-Ramsey-infinite, then 
\[
\inf \{ \beta_r(H; G) : G \in \mathcal{M}_r(H) \} = 0.
\]

The study of this quantity fits well within the framework of minimisation problems on Ramsey-minimal graphs. Given a graph parameter $F$, one can investigate the quantity 
\[
\inf \{ F(G) : G \in \mathcal{M}_r(H) \}.
\]
When $F$ is the number of vertices, we obtain the Ramsey numbers $r^r_r(H)$, and when $F$ is the number of edges, size-Ramsey numbers $\hat{r}^r_r(H)$ [15]. Other parameters such as chromatic number, minimal and maximum degree have also been studied [8].

The study of the Ramsey multiplicities is interesting in itself. Goodman [21] initiated the subject by determining $M_2(K_3; K_n)$ precisely for all $n$. A survey of Burr and Rosta [10] collect several results on multiplicities. They show that $\beta_r(H; K_n)$ is monotone nondecreasing on $n$ and bounded above by $r^{1-e(H)}$. Thus, is natural to define Ramsey multiplicity constant of $H$ as the following converging limit
\[
C_r(H) = \lim_{n \to \infty} \beta_r(H; K_n).
\]

Erdős [14] conjectured that $C_2(K_t) = 2^{1-\binom{t}{2}}$ and Goodman’s result implies that this is true for $t = 3$. Burr and Rosta further conjectured that $C_2(H) = 2^{1-e(H)}$ for all graphs $H$. Erdős conjecture was later disproved by Thomason [39] for $H = K_t$, $t \geq 4$.

Similar questions can be raised for different ambient graphs in place of the complete graph $K_n$. Erdős and Moon [17], for instance, have shown that
\[
\lim_{m \to \infty} \beta_2(K_{a,b}; K_{n,m}) = 2^{1-ab},
\]
where they considered copies of $K_{a,b}$ in $K_{n,m}$ where the part of size $a$ is sitting inside the part of size $n$. This confirms Erdős conjecture in the bipartite setting. Another natural setting to consider is a random graph. Theorem 4.1 shows that $\beta_r(H; G(n, p)) \geq \eta > 0$ with high probability, given that $p \geq Cn^{-1/m_2(H)}$.

Finally, we note that while the upper bound on the Blowup Ramsey numbers provided on Theorem 2.1 is of the form $c^t$, the constant $c$ can be quite large. It is natural to ask for more effective upper bounds on these numbers. We find the following to be specially interesting.

**Open Problem 5.5.** What is the smallest $n$ such that $K_6[n] \stackrel{2}{\to} K_3[t]$?

All we know at the moment is that Theorem 2.1 and Theorem 3.1 imply the weak bounds: $2^t \leq n \leq e^{(3.3 \times 10^7)t}$. 

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Appendix A. Proof of Theorem 2.3

In this appendix, we provide a proof of Theorem 2.3. Our proof is essentially the same as that of Nikiforov [27, 28], with some modifications. Following his strategy, we deduce it from a routine lemma, adjusted for our purposes:

Lemma A.1. Let $k \geq 2$, let $0 < \rho \leq 1$, and let $F$ be a bipartite graph with parts $A$ and $B$. If $e(F) \geq (\rho/2)|A||B|$ and $\rho|A|/4 + 1 \geq \lfloor \rho^k 4^{-k+2+k} \log |B| \rfloor \geq 1$, then $F$ contains a $K_2[s,t]$ with parts $A_0 \subseteq A$ and $B_0 \subseteq B$, such that

$$ s = |A_0| = \lfloor \rho^k 4^{-k+2+k} \log |B| \rfloor \quad \text{and} \quad t = |B_0| \geq |B|^{1 - \rho^{-k}}. $$

Proof. Let $m := |A|$ and $n := |B|$, and define

$$ t := \max \{ x : \text{there exists } K_2[s,x] \subseteq F \text{ with part of size } s \text{ in } A \}. $$

For any $X \subseteq A$, write $d(X)$ for the number of vertices that are neighbours of all vertices of $X$. For each $X$ with $|X| = s$, we have $d(X) \leq t$, thus

$$ \sum_{v \in B} \binom{d(v)}{s} = \sum_{X \subset A, |X| = s} d(X) \leq t \binom{m}{s}. \quad (A.1) $$

By convexity of function $x \mapsto \binom{x}{s} 1_{\{x \geq s-1\}}$, we obtain

$$ \sum_{v \in B} \binom{d(v)}{s} \geq n \binom{e(F)/n}{s} \geq n \binom{\rho m/2}{s}. $$
Combining this inequality with (A.1), we have
\[
t \geq n \left( \frac{\rho m/2}{s} \right) \left( \frac{m}{s} \right)^{-1} \geq n \left( \frac{\rho}{4} \right)^s \geq n^{1 + \rho^4 \cdot 4^{-k^2 + k} \log(\rho/4)},
\]
where we used that \( s \leq \rho m/4 + 1 \) on the last step. Since \( \rho \log(4/\rho) < 4^{k^2 - k} \) we have
\[
t \geq n^{1 - \rho^{k-1}}. \tag*{$\square$}
\]

Before proceeding to the proof of Theorem 22.3 we introduce some notation. For a subgraph \( G \) of \( H[n] \), denote by \( \mathcal{N}(H, G) \) the set of canonical copies of \( H \) in \( G \). Given \( \mathcal{L} \subseteq \mathcal{N}(H, G) \), and a subgraph \( H' \) of \( H \), we write \( \mathcal{N}(H', \mathcal{L}) \) to denote the set of canonical copies of \( H' \) that are contained in some member of \( \mathcal{L} \). Also, given a subgraph \( G' \subseteq G \) such that \( G' = H[t_1, \ldots, t_k] \), we say that a family \( \mathcal{L} \subseteq \mathcal{N}(H, G) \) covers \( G' \) if \( E(G') \subseteq \mathcal{N}(K_2, \mathcal{L}) \), that is, the union of the edges of elements of \( \mathcal{L} \) covers the graph \( G' \), and there are \( \min\{t_1, \ldots, t_k\} \) disjoint elements of \( \mathcal{L} \) as subgraphs of \( G' \).

Still assuming that \( G \) is a subgraph of \( H[n] \), for a vertex \( v \in V(H) \) we denote by \( G - v \) the subgraph of \( (H - v)[n] \) obtained from \( G \) by the removal of the vertex class of \( v \) in \( H[n] \). Finally, for any subset \( \mathcal{L} \subseteq \mathcal{N}(H, G) \), and \( R \in \mathcal{N}(H - v, G - v) \), we denote by \( d_{\mathcal{L}}(R) \) the number ways that we can extend \( R \), a canonical copy of \( H - v \), to a element of \( \mathcal{L} \).

**Proof of Theorem 22.3.** We are going to prove by induction on \( k \geq 2 \) the following statement: that every subset \( \mathcal{M} \subseteq \mathcal{N}(H, H[n]) \) of canonical copies of \( H \) in \( H[n] \), with \( |\mathcal{M}| \geq \rho n^k \), covers a \( H[t_1, \ldots, t_k, n^{1 - \rho^{k-1}}] \) with \( t = \rho^4 - k^2 + k \log n \).

For \( k = 2 \), let \( \mathcal{M} \subseteq \mathcal{N}(K_2, K_2[n]) \) with \( |\mathcal{M}| \geq \rho n^2 \) and apply Lemma A.1 to \( K_2[n] \). We obtain that \( \mathcal{M} \) covers a \( K_2[\rho^4 - 2\rho + 2 \log n, n^{1 - \rho}] \).

Now we proceed to the induction step, with \( k > 2 \). Let \( G \) be a subgraph of \( H[n] \) with \( |\mathcal{N}(H, G)| \geq \rho n^k \) and let \( v \in V(H) \). The first step is to show that there is a subset \( \mathcal{L} \subseteq \mathcal{N}(H, G) \), with \( |\mathcal{L}| \geq (\rho/2)n^k \) such that for all \( R \in \mathcal{L} \), \( d_{\mathcal{L}}(R - v) \geq (\rho/2)n \). We construct this subset via the following procedure:

\[
\mathcal{L} \leftarrow \mathcal{N}(H, G).
\]

While there exists an \( R \in \mathcal{L} \) with \( d_{\mathcal{L}}(R - v) < (\rho/2)n \) do

\[
\mathcal{L} \leftarrow \mathcal{L} \setminus \{R' \in \mathcal{L} : R' \text{ is an extension of } R - v\}.
\]

End while

When it ends, we have a subset \( \mathcal{L} \) with the property that \( d_{\mathcal{L}}(R - v) \geq (\rho/2)n \) for all \( R \in \mathcal{L} \). Also, we have
\[
|\mathcal{L}| > |\mathcal{N}(H, G)| - (\rho/2)n |\mathcal{N}(H - v, G - v)| \geq (\rho/2)n^k.
\]

Now, observe that \( \mathcal{N}(H - v, \mathcal{L}) \subseteq \mathcal{N}(H - v, G - v) \) with
\[
|\mathcal{N}(H - v, \mathcal{L})| \geq |\mathcal{L}|/n \geq (\rho/2)n^{k-1}.
\]

By the induction hypothesis, \( \mathcal{N}(H - v, \mathcal{L}) \) covers a copy of \( (H - v)[t'] \), where we have \( t' = [(\rho/2)^{k-1} - 4^{(k-1)^2 + k-1} \log n] \).

Now we build a bipartite graph \( F \) with parts \( A \) and \( B \), where \( A \) is a set of disjoint canonical copies of \( H - v \) in the blowup \( (H - v)[t'] \), and \( B \) is the vertex class of \( v \). This gives us \( |A| = t' \), \( |B| = n \). We put an edge between a copy of \( H - v \) and a vertex \( u \in B \) if together, they form an element of \( \mathcal{L} \). Therefore,
\[
e(F) \geq d_{\mathcal{L}}(R)|A| \geq (\rho/2)n|A| = (\rho/2)|A||B|.
\]
We will apply Lemma A.1 to the bipartite graph $F$. If $t := \lfloor \rho k 4^{-k^2+k} \log |B| \rfloor$, we have to check that $t \leq \rho |A|/4 + 1$. Indeed
\[
t \leq \rho k 4^{-k^2+k} \log n \leq (\rho/4)(\rho/2)^{k-1} 4^{-k^2+k+1+(k-1)/2} \log n \leq (\rho/4)(\rho/2)^{k-1} 4^{-(k-1)^2+k-1} \log n \leq \rho |A|/4 + 1,
\]
where we used that $-k^2 + k + 1 + (k-1)/2 \leq -(k-1)^2 + k - 1$ for $k \geq 2$. Thus, by Lemma A.1 we have $K_{2[t, n^{1-\rho^{k-1}}]} \subseteq F$, with parts $A_0 \subseteq A$ of size $t$ and $B_0 \subseteq B$ of size $n^{1-\rho^{k-1}}$. Let $H^*$ be the subgraph of $(H-v)|n|$ induced by the union of the members of $A_0$. For every copy $H'$ of $H-v$ in $A_0$, it can be joined to any vertex of $u \in B_0$ to form a copy of $H$. This implies that $H^*$ covers a copy of $H[|A_0|, \ldots, |A_0|, |B_0|] = H[t, \ldots, t, n^{1-\rho^{k-1}}]$. □