Difference Sequence Spaces Derived by Using Generalized Means

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Abstract

This paper deals with new sequence spaces $X(r, s, t; Δ)$ for $X ∈ \{l_∞, c, c_0\}$ defined by using generalized means and difference operator. It is shown that these spaces are complete normed linear spaces and the spaces $X(r, s, t; Δ)$ for $X ∈ \{c, c_0\}$ have Schauder basis. Furthermore, the $α$-, $β$-, $γ$- duals of these sequence spaces are computed and also established necessary and sufficient conditions for matrix transformations from $X(r, s, t; Δ)$ to $X$.

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1 Introduction

The study of sequence spaces has importance in the several branches of analysis, namely, the structural theory of topological vector spaces, summability theory, Schauder basis theory etc. Besides this, the theory of sequence spaces is a powerful tool for obtaining some topological and geometrical results using Schauder basis.

Let $\mathbb{w}$ be the space of all real or complex sequences $x = (x_n), n ∈ \mathbb{N}_0$. For an infinite matrix $A$ and a sequence space $\lambda$, the matrix domain of $A$, which is denoted by $\lambda_A$ and defined as $\lambda_A = \{x ∈ \mathbb{w} : Ax ∈ \lambda\}$ [17]. Basic methods, which are used to determine the topologies, matrix transformations and inclusion relations on sequence spaces can also be applied to study the matrix domain $\lambda_A$. In recent times, there is an approach of forming new sequence spaces by using matrix domain of a suitable matrix and characterize the matrix mappings between these sequence spaces.

Kizmaz first introduced and studied the difference sequence spaces in [8]. Later on, several authors including Ahmad and Mursaleen [11], Çolak and Et [5], Başar and Altay [3], Orhan [13], Polat and Altay [15], Aydin and Başar [4] etc. have introduced and studied new sequence spaces defined by using difference operator.

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On the other hand, sequence spaces are also defined by using generalized weighted means. Some of them can be viewed in Malkowsky and Savas [11], Altay and Başar [2]. Mursaleen and Noman [12] introduced a sequence space of generalized means, which includes most of the earlier known sequence spaces. But till 2011, there was no such literature available in which a sequence space is generated by combining both the weighted means and the difference operator. This was firstly initiated by Polat et al. [14]. The authors in [14] have introduced the sequence spaces \( \lambda(u, v; \Delta) \) for \( \lambda \in \{l_\infty, c, c_0\} \) defined as

\[
\lambda(u, v; \Delta) = \{ x \in w : (G(u, v) \Delta) x \in \lambda \},
\]

where \( u, v \in w \) such that \( u_n, v_n \neq 0 \) for all \( n \) and the matrices \( G(u, v) = (g_{nk}) \) and \( \Delta = (\delta_{nk}) \) are defined by

\[
g_{nk} = \begin{cases} 
  u_n v_k & \text{if } 0 \leq k \leq n, \\
  0 & \text{if } k > n
\end{cases}
\]

\[
\delta_{nk} = \begin{cases} 
  0 & \text{if } 0 \leq k < n - 1, \\
  (-1)^{n-k} & \text{if } n - 1 \leq k \leq n, \\
  0 & \text{if } k > n,
\end{cases}
\]

respectively.

The aim of this article is to introduce new sequence spaces defined by using both the generalized means and the difference operator. We investigate some topological properties as well as \( \alpha-, \beta-, \gamma- \) duals and bases of the new sequence spaces are obtained. Further, we characterize some matrix transformations between these new sequence spaces.

## 2 Preliminaries

Let \( l_\infty, c \) and \( c_0 \) be the spaces of all bounded, convergent and null sequences \( x = (x_n) \) respectively, with norm \( \|x\|_\infty = \sup \|x_n\| \). Let \( bs \) and \( cs \) be the sequence spaces of all bounded and convergent series respectively. We denote by \( e = (1, 1, \cdots) \) and \( e_n \) for the sequence whose \( n \)-th term is 1 and others are zero and \( N_0 = \mathbb{N} \cup \{0\} \), where \( \mathbb{N} \) is the set of all natural numbers. A sequence \( (b_n) \) in a normed linear space \( (X, \|\cdot\|) \) is called a Schauder basis for \( X \) if for every \( x \in X \) there is a unique sequence of scalars \( (\mu_n) \) such that

\[
\|x - \sum_{n=0}^{k} \mu_n b_n\| \to 0 \text{ as } k \to \infty,
\]

i.e., \( x = \sum_{n=0}^{\infty} \mu_n b_n \). ([17], [9]).

For any subsets \( U \) and \( V \) of \( w \), the multiplier space \( M(U, V) \) of \( U \) and \( V \) is defined as

\[
M(U, V) = \{ a = (a_n) \in w : au = (a_n u_n) \in V \text{ for all } u \in U \}.
\]

In particular,

\[
U^\alpha = M(U, l_1), \quad U^\beta = M(U, cs) \text{ and } U^\gamma = M(U, bs)
\]

are called the \( \alpha-, \beta- \) and \( \gamma- \) duals of \( U \) respectively ([10]).

Let \( A = (a_{nk})_{n,k} \) be an infinite matrix with real or complex entries \( a_{nk} \). We write \( A_n \) as the sequence of the \( n \)-th row of \( A \), i.e., \( A_n = (a_{nk}) \) for every \( n \). For \( x = (x_n) \in w \), the \( A \)-transform of \( x \) is defined as the sequence \( Ax = ((Ax)_n) \), where

\[
(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k.
\]
\[ A_n(x) = (Ax)_n = \sum_{k=0}^{n} a_{nk}x_k, \]

provided the series on the right side converges for each \( n \). For any two sequence spaces \( U \) and \( V \), we denote by \( (U, V) \), the class of all infinite matrices \( A \) that map \( U \) into \( V \). Therefore \( A \in (U, V) \) if and only if \( Ax = ((Ax)_n) \in V \) for all \( x \in U \). In other words, \( A \in (U, V) \) if and only if \( A_n \in \mathcal{U}^2 \) for all \( n \) \[17\].

An infinite matrix \( T = (t_{nk})_{n,k} \) is said to be triangle if \( t_{nk} = 0 \) for \( k > n \) and \( t_{nn} \neq 0, n \in \mathbb{N}_0 \).

### 3 Sequence spaces \( X(r, s, t; \Delta) \) for \( X \in \{l_\infty, c, c_0\} \)

In this section, we first begin with the notion of generalized means given by Mursaleen et al. \[12\].

We denote the sets \( U \) and \( U_0 \) as

\[ U = \left\{ u = (u_n)_{n=0}^{\infty} \in w : u_n \neq 0 \text{ for all } n \right\} \text{ and } U_0 = \left\{ u = (u_n)_{n=0}^{\infty} \in w : u_0 \neq 0 \right\}. \]

Let \( r, t \in U \) and \( s \in U_0 \). The sequence \( y = (y_n) \) of generalized means of a sequence \( x = (x_n) \) is defined by

\[ y_n = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k}t_kx_k \quad (n \in \mathbb{N}_0). \]

The infinite matrix \( A(r, s, t) \) of generalized means is defined by

\[ (A(r, s, t))_{nk} = \begin{cases} \frac{s_{n-k}t_k}{r_n} & 0 \leq k \leq n, \\ 0 & k > n. \end{cases} \]

Since \( A(r, s, t) \) is a triangle, it has a unique inverse and the inverse is also a triangle \[7\]. Take \( D_0^{(s)} = \frac{1}{s_0} \) and

\[ D_n^{(s)} = \frac{1}{s_n} \begin{pmatrix} s_1 & s_0 & 0 & \cdots & 0 \\ s_2 & s_1 & s_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & \cdots & s_1 \end{pmatrix} \quad \text{for } n = 1, 2, 3, \ldots \]

Then the inverse of \( A(r, s, t) \) is the triangle \( B = (b_{nk})_{n,k} \) which is defined as

\[ b_{nk} = \begin{cases} (-1)^{n-k} \frac{b_{nk}^{(s)}}{s_n} r_k & 0 \leq k \leq n, \\ 0 & k > n. \end{cases} \]

We now introduce the sequence spaces \( X(r, s, t; \Delta) \) for \( X \in \{l_\infty, c, c_0\} \) as

\[ X(r, s, t; \Delta) = \left\{ x = (x_n) \in w : \left( \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k}t_kx_k \right)_n \in X \right\}, \]

which is a combination of the generalized means and the difference operator \( \Delta \) such that \( \Delta x_k = x_k - x_{k-1} \), \( x_{-1} = 0 \). By using matrix domain, we can write \( X(r, s, t; \Delta) = X_{A(r, s, t; \Delta)} = \left\{ x \in w : A(r, s, t; \Delta)x \in X \right\} \), where \( A(r, s, t; \Delta) = A(r, s, t) \Delta \), product of two triangles \( A(r, s, t) \) and \( \Delta \).
These sequence spaces include many known sequence spaces studied by several authors. For examples,

I. if \( r_n = \frac{1}{n}, t_n = v_n \) and \( s_n = 1 \) \( \forall \) \( n \), then the sequence spaces \( X(r, s, t; \Delta) \) reduce to \( X(u, v; \Delta) \) for \( X \in \{ l_\infty, c, c_0 \} \) introduced and studied by Polat et al. [14].

II. if \( t_n = 1, s_n = 1 \) \( \forall \) \( n \) and \( r_n = n + 1 \), then the sequence space \( X(r, s, t; \Delta) \) for \( X = l_\infty \) reduces to \( c_{es, \infty}(\Delta) \) studied by Orhan [13].

III. if \( r_n = \frac{1}{n}, t_n = \frac{\alpha n}{n}, s_n = \frac{(1-\alpha)n}{n!} \), where \( 0 < \alpha < 1 \), then the sequence spaces \( X(r, s, t; \Delta) \) for \( X \in \{ l_\infty, c, c_0 \} \) reduce to \( c_\infty^\alpha(\Delta), c_\infty^\alpha(\Delta) \) and \( c_\infty^\alpha(\Delta) \) respectively [15].

IV. if \( r_n = n + 1, t_n = 1 + \alpha^n \), where \( 0 < \alpha < 1 \) and \( s_n = 1 \) \( \forall n \), then the sequence spaces \( X(r, s, t; \Delta) \) for \( X \in \{ c, c_0 \} \) reduce to the spaces of sequences \( a_\infty^\alpha(\Delta) \) and \( a_\infty^\alpha(\Delta) \) studied by Aydin and Başar [4]. For \( X = l_\infty \), the sequence space \( X(r, s, t; \Delta) \) reduces to \( a_\infty^\alpha(\Delta) \) studied by Djolović [6].

4 Main results

In this section, we begin with some topological results of the newly defined sequence spaces.

**Theorem 4.1.** The sequence spaces \( X(r, s, t; \Delta) \) for \( X \in \{ l_\infty, c, c_0 \} \) are complete normed linear spaces under the norm defined by

\[
\|x\|_{X(r,s,t;\Delta)} = \sup_n \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta x_k \right| = \sup_n \left| (A(r,s,t;\Delta)x)_n \right|
\]

**Proof.** Let \( u, v \in X(r, s, t; \Delta) \) and \( \alpha, \beta \) be any two scalars. Then

\[
\sup_n \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta (\alpha u_k + \beta v_k) \right| \leq |\alpha| \sup_n \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta u_k \right| + |\beta| \sup_n \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k \Delta v_k \right|
\]

and hence \( \alpha u + \beta v \in X(r, s, t; \Delta) \). Therefore \( X(r, s, t; \Delta) \) is a linear space. It is easy to show that the functional \( \| \cdot \|_{X(r,s,t;\Delta)} \) defined above gives a norm on the linear space \( X(r, s, t; \Delta) \).

To show completeness, let \((x^m)\) be a Cauchy sequence in \( X(r, s, t; \Delta) \), where \( x^m = (x^m_k) = (x^m_0, x^m_1, x^m_2, \ldots) \in X(r, s, t; \Delta) \), for each \( m \in \mathbb{N}_0 \). Then for every \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that

\[
\|x^m - x^l\|_{X(r,s,t;\Delta)} < \epsilon \quad \text{for} \quad m, l \geq n_0.
\]

The above implies that for each \( k \in \mathbb{N}_0 \),

\[
|A(r, s, t; \Delta)(x^m_k - x^l_k)| < \epsilon \quad \text{for all} \quad m, l \geq n_0. \tag{4.1}
\]

Therefore \((A(r, s, t; \Delta)x^m_k)\) is a Cauchy sequence of scalars for each \( k \in \mathbb{N}_0 \) and hence \((A(r, s, t; \Delta)x^m_k)\) converges for each \( k \). We write

\[
\lim_{m \to \infty} ((A(r, s, t; \Delta)x^m_k)) = ((A(r, s, t; \Delta)x_k), \quad k \in \mathbb{N}_0.
\]

Letting \( l \to \infty \) in (4.1), we obtain

\[
|A(r, s, t; \Delta)(x^m_k - x^l_k)| < \epsilon \quad \text{for all} \quad m \geq n_0 \text{ and each} \quad k \in \mathbb{N}_0. \tag{4.2}
\]

Hence by definition \( \|x^m - x\|_{X(r,s,t;\Delta)} < \epsilon \) for all \( m \geq n_0 \). Next we show that \( x \in X(r, s, t; \Delta) \). Consider
\[ \|x\|_{X(r,s,t;\Delta)} \leq \|x^m\|_{X(r,s,t;\Delta)} + \|x^m - x\|_{X(r,s,t;\Delta)}, \]
which is finite for \( m \geq n_0 \) and hence \( x \in X(r,s,t;\Delta) \). This completes the proof. \( \square \)

**Theorem 4.2.** The sequence spaces \( X(r,s,t;\Delta) \), where \( X \in \{l_\infty, c, c_0\} \) are linearly isomorphic to the spaces \( X \in \{l_\infty, c, c_0\} \) respectively i.e. \( l_\infty(r,s,t;\Delta) \cong l_\infty \), \( c(r,s,t;\Delta) \cong c \) and \( c_0(r,s,t;\Delta) \cong c_0 \).

**Proof.** We prove the theorem only for the case \( X = l_\infty \). To prove this, we need to show that there exists a bijective linear map from \( l_\infty(r,s,t;\Delta) \) to \( l_\infty \).

We define a map \( T: l_\infty(r,s,t;\Delta) \to l_\infty \) by \( x \mapsto Tx = y \), where

\[ y_n = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} l_k \Delta x_k. \]

Since \( \Delta \) is a linear operator, so the linearity of \( T \) is trivial. It is clear from the definition that \( Tx = 0 \) implies \( x = 0 \). Thus \( T \) is injective. To prove \( T \) is surjective, let \( y = (y_n) \in l_\infty \). Since \( y = (A(r,s,t;\Delta)x) \), i.e.,

\[ x = (A(r,s,t;\Delta))^{-1} y = \Delta^{-1} A(r,s,t;\Delta)^{-1} y, \]

we can get a sequence \( x = (x_n) \) as

\[ x_n = \sum_{j=0}^{n-1} (-1)^j \frac{D_j(s)}{l_{k+j}} r_j y_j, \quad n \in \mathbb{N}_0. \tag{4.3} \]

Then

\[ \|x\|_{l_\infty(r,s,t;\Delta)} = \sup_n \left| \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} l_k \Delta x_k \right| = \sup_n |y_n| = \|y\|_\infty < \infty. \]

Thus \( x \in l_\infty(r,s,t;\Delta) \) and this shows that \( T \) is surjective. Hence \( T \) is a linear bijection from \( l_\infty(r,s,t;\Delta) \) to \( l_\infty \). Also \( T \) is norm preserving. So \( l_\infty(r,s,t;\Delta) \cong l_\infty \).

In the same way, we can prove that \( c_0(r,s,t;\Delta) \cong c_0 \), \( c(r,s,t;\Delta) \cong c \). This completes the proof. \( \square \)

Since \( X(r,s,t;\Delta) \cong X \) for \( X \in \{c_0, c\} \), the Schauder bases of the sequence spaces \( X(r,s,t;\Delta) \) are the inverse image of the bases of \( X \) for \( X \in \{c_0, c\} \). So, we have the following theorem without proof.

**Theorem 4.3.** Let \( \mu_k = (A(r,s,t;\Delta)x)_k \) for all \( k \in \mathbb{N}_0 \). Define the sequences \( b^{(j)}_n = (b^{(j)}_n)_j, j \in \mathbb{N}_0 \) and \( b^{(-1)}_n \) as

\[ b^{(j)}_n = \begin{cases} 
\sum_{k=0}^{n-j} (-1)^k \frac{D_j(s)}{l_{k+j}} r_j & \text{if } 0 \leq j \leq n \\
0 & \text{if } j > n.
\end{cases} \]

Then the followings are true:

(i) The sequence \( (b^{(j)})^\infty_{j=0} \) is a basis for the space \( c_0(r,s,t;\Delta) \) and any sequence \( x \in c_0(r,s,t;\Delta) \) has a unique representation of the form

\[ x = \sum_{j=0}^{\infty} \mu_j b^{(j)}. \]
(ii) The sequence \((b^{(j)})_{j=-1}^{\infty}\) is a basis for the space \(c(r,s,t;\Delta)\) and any \(x \in c(r,s,t;\Delta)\) has a unique representation of the form

\[
x = \ell b^{(-1)} + \sum_{j=0}^{\infty} (\mu_j - \ell) b^{(j)},
\]

where \(\ell = \lim_{n \to \infty} (A(r,s,t;\Delta)x)_n\).

**Remark 4.1.** In particular, if we choose \(r_n = \frac{1}{u_n}, t_n = v_n, s_n = 1, \forall n\) then the sequence spaces \(X(r,s,t;\Delta)\) reduce to \(X(u,v;\Delta)\) for \(X \in \{l_{\infty}, c, c_0\}\). With this choice of \((s_n)\), we have \(D_0^{(s)} = D_1^{(s)} = 1\) and \(D_n^{(s)} = 0\) for \(n \geq 2\). Thus the sequences \((b^{(j)}) = (b^{(j)}_n), j \in \mathbb{N}_0\) and \((b^{(-1)}) = (b^{(-1)}_n)\) reduce to

\[
b^{(j)}_n = \begin{cases} 
\frac{1}{u_j} \left( \frac{1}{v_j} - \frac{1}{v_{j+1}} \right) & \text{if } 0 \leq j < n \\
\frac{1}{u_n v_n} & \text{if } j = n \\
0 & \text{if } j > n.
\end{cases}
\]

The sequences \((b^{(j)})_{j=0}^{\infty}\) and \((b^{(-1)})_{j=-1}^{\infty}\) are the bases for the spaces \(c_0(u,v;\Delta)\) and \(c(u,v;\Delta)\) respectively [14].

Let \(F\) be the collection of all nonempty finite subsets of the set of all natural numbers and \(A = (a_{nk})_{n,k}\) be an infinite matrix satisfying the conditions:

\[
\sup_{K \in F} \sum_{n=0}^{\infty} \left| \sum_{k \in K} a_{nk} \right| < \infty \quad (4.4)
\]

\[
\sup_{n} \sum_{k=0}^{\infty} |a_{nk}| < \infty \quad (4.5)
\]

\[
\lim_{n} \sum_{k=0}^{\infty} |a_{nk}| = 0 \quad (4.6)
\]

\[
\lim_{n} a_{nk} = 0 \text{ for all } k \quad (4.7)
\]

\[
\lim_{n} \sum_{k=0}^{\infty} a_{nk} = 0 \quad (4.8)
\]

\[
\lim_{n} a_{nk} \text{ exists for all } k \quad (4.9)
\]

\[
\lim_{n} \sum_{k=0}^{\infty} |a_{nk} - \lim_{n} a_{nk}| = 0 \quad (4.10)
\]

\[
\lim_{n} \sum_{k=0}^{\infty} a_{nk} \text{ exists} \quad (4.11)
\]

We now state some results of Stieglitz and Tietz [16] which are required to obtain the duals and to characterize some matrix transformations.

**Theorem 4.4.** [16] (a) \(A \in (c_0, l_1), A \in (c, l_1), A \in (l_\infty, l_1)\) if and only if (4.4) holds.
(b) \(A \in (c_0, l_{\infty}), A \in (c, l_{\infty}), A \in (l_\infty, l_{\infty})\) if and only if (4.5) holds.
(c) \(A \in (c_0, c_0)\) if and only if (4.5) and (4.7) hold.
(d) \( A \in (l_\infty, c_0) \) if and only if (4.6) holds.
(e) \( A \in (c, c_0) \) if and only if (4.5), (4.7) and (4.8) hold.
(f) \( A \in (c_0, c) \) if and only if (4.5) and (4.9) hold.
(g) \( A \in (l_\infty, c) \) if and only if (4.5), (4.9) and (4.10) hold.
(h) \( A \in (c, c) \) if and only if (4.5), (4.9) and (4.11) hold.

4.1 The \( \alpha \)-dual, \( \gamma \)-dual of \( X(r, s, t; \Delta) \) for \( X \in \{l_\infty, c, c_0\} \)

**Theorem 4.5.** The \( \alpha \)-dual of the space \( X(r, s, t; \Delta) \) for \( X \in \{l_\infty, c, c_0\} \) is the set

\[ \Lambda = \left\{ a = (a_n) \in w : \sup_{K \in J} \frac{1}{\sum_n} \left| \sum_{j \in K} (-1)^{j-k} \frac{D_j(s)}{l_{k+j}} r_{j} a_{n} \right| < \infty \right\}. \]

*Proof.* Let \( a = (a_n) \in w, x \in X(r, s, t; \Delta) \) and \( y \in X \) for \( X \in \{l_\infty, c, c_0\} \). Then for each \( n \in \mathbb{N}_0 \), we have

\[ a_n x_n = \sum_{j=0}^{n} \sum_{k=0}^{n-j} (-1)^{k} \frac{D_j(s)}{l_{k+j}} r_{j} a_{n} y_{j} = (Cy)_n, \]

where the matrix \( C = (c_{nj}) \) is defined as

\[ c_{nj} = \begin{cases} \sum_{k=0}^{n-j} (-1)^{k} \frac{D_j(s)}{l_{k+j}} r_{j} a_{n} & \text{if } 0 \leq j \leq n \\ 0 & \text{if } j > n \end{cases} \]

and \( x_n \) is given by (4.3). Thus for each \( x \in X(r, s, t; \Delta) \), \( (a_n x_n)_n \in l_1 \) if and only if \( (Cy)_n \in l_1 \) where \( y \in X \) for \( X \in \{l_\infty, c, c_0\} \). Therefore \( a = (a_n) \in [X(r, s, t; \Delta)]^\alpha \) if and only if \( C \in (X, l_1) \). By using Theorem 4.4(a), we have

\[ [X(r, s, t; \Delta)]^\alpha = \Lambda. \]

\[ \square \]

**Theorem 4.6.** The \( \gamma \)-dual of \( X(r, s, t; \Delta) \) for \( X \in \{l_\infty, c, c_0\} \) is the set

\[ \Gamma = \left\{ a = (a_n) \in w : \sup_{n} \sum_{m=0}^{\infty} |e_{mn}| < \infty \right\}, \]

where the matrix \( E = (e_{mn}) \) is defined by

\[ E = (e_{mn}) = \begin{cases} r_n \left[ \frac{a_n}{s_0 t_n} + \frac{(D_0(s))}{l_{n+1}} \sum_{j=n+1}^{m} a_j + \sum_{j=n+2}^{m} (-1)^{j-n} \frac{D_j(s)}{l_j} \left( \sum_{k=j}^{m} a_k \right) \right] & 0 \leq n \leq m, \\ 0 & n > m. \end{cases} \]

**Note:** We mean \( \sum_{j=n}^{m} = 0 \) if \( n > m \).

*Proof.* Let \( a = (a_n) \in w, x \in X(r, s, t; \Delta) \) and \( y \in X \), where \( X \in \{l_\infty, c, c_0\} \). Then by using (4.3), we
have

\[
\sum_{n=0}^{m} a_n x_n = \sum_{n=0}^{m} \sum_{j=0}^{m-n} (-1)^j \frac{D(k)_j}{t_{k+j}} r_j y_j a_n \\
= \sum_{n=0}^{m-1} \sum_{j=0}^{m-n} (-1)^j \frac{D(k)_j}{t_{k+j}} + \sum_{j=0}^{m} \sum_{k=0}^{m-j} (-1)^j \frac{D(k)_j}{t_{k+j}} r_j y_j a_m \\
= \sum_{n=0}^{m-1} \sum_{k=0}^{m-n} (-1)^k \frac{D(k)_k}{t_{k+j}} a_n + \sum_{k=0}^{m} (-1)^k \frac{D(k)_k}{t_{k+j}} a_m \left[r_0 y_0 + \ldots + \frac{D(s)_0}{t_m} r_m y_m a_m \right] \\
= \left[ \frac{D(s)_0}{t_0} a_0 + \left( \frac{D(s)_0}{t_1} - \frac{D(s)_1}{t_2} \right) \sum_{j=1}^{m} a_j + \sum_{j=2}^{m} (-1)^j \frac{D(s)_j}{t_j} \left( \sum_{k=j}^{m} a_k \right) \right] r_0 y_0 \ldots + \frac{D(s)_m}{t_m} y_m \\
= \left( E y \right)_m ,
\]

where \( E = (e_{mn}) \) is the matrix defined above.

Thus \( a \in [X(r, s, t; \Delta)]^\gamma \) if and only if \( ax = (a_n x_n) \in bs \) for \( x \in X(r, s, t; \Delta) \) if and only if \( \left( \sum_{n=0}^{m} a_n x_n \right) \in l_s \), i.e. \( (E y)_m \in l_s \), for \( y \in X \). Hence by Theorem 4.4(b), we have

\[
[ X(r, s, t; \Delta) ]^\gamma = \Gamma.
\]

4.2 \( \beta \)-dual and Matrix transformations

We now first discuss about the \( \beta \)-dual and then characterize some matrix transformations. Let \( T \) be a triangle and \( X_T \) be the matrix domain of \( T \).

**Theorem 4.7.** ([7], Theorem 2.6) Let \( X \) be a BK space with AK property and \( R = S^t \), the transpose of \( S \), where \( S = (s_{jk}) \) is the inverse of the matrix \( T \). Then \( a \in (X_T)^3 \) if and only if \( a \in (X^2)_R \) and \( W \in (X, c_0) \), where the triangle \( W \) is defined by \( w_{nk} = \sum_{j=0}^{m} a_j s_{jk} \). Moreover if \( a \in (X_T)^3 \), then

\[
\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty} R_k(a) T_k(z) \quad \forall z \in X_T.
\]

**Remark 4.2.** ([7], Remark 2.7) The conclusion of the Theorem 4.7 is also true for \( X = l_s \).

**Remark 4.3.** ([10], [7]) We have \( a \in (c_T)^3 \) if and only if \( Ra \in l_1 \) and \( W \in (c, c) \). Moreover, if
Then we have for all $z \in c_T$

$$
\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty} R_k(a) T_k(z) - \eta \gamma,
$$

where $\eta = \lim_{k \to \infty} T_k(z)$ and $\gamma = \lim_{m \to \infty} \sum_{k=0}^{m} w_{mk}$.

To find $\beta$-duals of the sequence spaces $X(r, s, t; \Delta)$ for $X \in \{l_\infty, c, c_0\}$, we list the following sets:

- $B_1 = \{ a \in w : \sum_{k=0}^{\infty} |R_k(a)| < \infty \}$
- $B_2 = \{ a \in w : \lim_{m \to \infty} w_{mk} = 0 \text{ for all } k \}$
- $B_3 = \{ a \in w : \sup_{m} \sum_{k=0}^{\infty} |w_{mk}| < \infty \}$
- $B_4 = \{ a \in w : \lim_{m \to \infty} \sum_{k=0}^{m} |w_{mk}| = 0 \}$
- $B_5 = \{ a \in w : \lim_{m \to \infty} w_{mk} \text{ exists for all } k \}$
- $B_6 = \{ a \in w : \lim_{m \to \infty} \sum_{k=0}^{m} w_{mk} \text{ exists } \}$

where $R_k(a) = r_k \left[ \frac{a_k}{s_{nk}} + \left( \frac{B^{(r)}_{(s)}}{t_{k+1}} - \frac{B^{(r)}_{(t)}}{t_{k+1}} \right) \sum_{j=k+1}^{\infty} a_j + \sum_{l=2}^{\infty} (-1)^l \frac{B^{(r)}_{(s)}}{t_{l+k}} \sum_{j=k+l}^{\infty} a_j \right], \ R(a) = (R_k(a))_k$ and

$$
w_{mk} = r_k \left[ \sum_{l=0}^{m-k} (-1)^l \frac{B^{(r)}_{(s)}}{t_{l+k}} \sum_{j=m}^{\infty} a_j + \sum_{l=m-k+1}^{\infty} (-1)^l \frac{B^{(r)}_{(s)}}{t_{l+k}} \sum_{j=k+l}^{\infty} a_j \right].
$$

**Theorem 4.8.** We have $[c_0(r, s, t; \Delta)]^\beta = B_1 \cap B_2 \cap B_3$, $[l_\infty(r, s, t; \Delta)]^\beta = B_1 \cap B_4$ and $[c(r, s, t; \Delta)]^\beta = B_1 \cap B_2 \cap B_3 \cap B_4$.

**Proof.** Here the matrix $T = A(r, s, t) \Delta = (t'_{mk})$, where

$$
t'_{mk} = \begin{cases} 
\frac{1}{r_n} [s_{n-k} t_k - s_{n-k+1} t_{k+1}] & \text{if } 0 \leq k < n \\
\frac{1}{r_n} & \text{if } k = n \\
0 & \text{if } k > n.
\end{cases}
$$

So, $T^{-1} = (A(r, s, t) \Delta)^{-1} = \Delta^{-1} A(r, s, t)^{-1}$. Let $S = (s_{jk})$ be the inverse of $T$. Then we easily get

$$
s_{jk} = \begin{cases} 
\sum_{i=0}^{j-k} (-1)^i \frac{B^{(r)}_{(s)}}{t_{i+k}} r_k & \text{if } 0 \leq k \leq j \\
0 & \text{if } k > j.
\end{cases}
$$
To compute $\beta$-duals, we first determine $W = (w_{mk})$ and $R(a) = (R_k(a))$, where $R = S^t$.

$$R_k(a) = \sum_{j=k}^{\infty} a_j s_{jk}$$

$$= \frac{D_0^{(s)}}{l_k} r_k a_k + \sum_{j=k+1}^{\infty} \sum_{l=0}^{j-k} (-1)^l \frac{D_l^{(s)}}{l_{l+k}} r_k a_j$$

$$= \frac{D_0^{(s)}}{l_k} r_k a_k + \sum_{l=0}^{1} (-1)^l \frac{D_l^{(s)}}{l_{l+k}} r_k a_{k+1} + \sum_{l=0}^{2} (-1)^l \frac{D_l^{(s)}}{l_{l+k}} r_k a_{k+2} + \cdots$$

$$= r_k \left[ \frac{a_k}{s_{0k}} + \left( \frac{D_0^{(s)}}{l_k} - \frac{D_l^{(s)}}{l_{l+k}} \right) \sum_{j=k+1}^{\infty} a_j + \sum_{l=2}^{\infty} (-1)^l \frac{D_l^{(s)}}{l_{l+k}} \sum_{j=k+l}^{\infty} a_j \right]$$

and

$$w_{mk} = \sum_{j=m}^{\infty} a_j s_{jk}$$

$$= \sum_{j=m}^{\infty} \sum_{l=0}^{j-k} (-1)^l \frac{D_l^{(s)}}{l_{l+k}} r_k a_j$$

$$= r_k \left[ \sum_{l=0}^{m-k} (-1)^l \frac{D_l^{(s)}}{l_{l+k}} \sum_{j=m}^{\infty} a_j + \sum_{l=m-k+1}^{\infty} (-1)^l \frac{D_l^{(s)}}{l_{l+k}} \sum_{j=k+l}^{\infty} a_j \right].$$

Using Theorem 4.7 and Remarks 4.2 & 4.3 we have $[c_0(r, s; t; \Delta)]^\beta = B_1 \cap B_2 \cap B_3$, $[t_\infty(r, s; t; \Delta)]^\beta = B_1 \cap B_4$ and $[c(r, s; t; \Delta)]^\beta = B_1 \cap B_3 \cap B_5 \cap B_6$. □

**Theorem 4.9. (7)** Let $X$ be a BK space with AK property, $Y$ be an arbitrary subset of $w$ and $R = S^t$. Then $A \in (X, Y)$ if and only if $B^A \in (X, Y)$ and $W^A_n \in (X, c_0)$ for all $n = 0, 1, 2, \cdots$, where $B^A$ is the matrix with rows $B^A_n = R(A_n)$, $A_n$ are the rows of $A$ and the triangles $W^A_n$ are defined by

$$w_{mk}^{A_n} = \left\{ \begin{array}{ll} \sum_{j=m}^{\infty} a_{nj} s_{jk} & : 0 \leq k \leq m \\ 0 & : k > m. \end{array} \right.$$  

**Theorem 4.10. (7)** Let $Y$ be any linear subspace of $w$. Then $A \in (c_T, Y)$ if and only if $R_k(A_n) \in (c_0, Y)$ and $W^A_n \in (c, c)$ for all $n$ and $R_k(A_n) e - (\gamma_n) \in Y$, where $\gamma_n = \lim_{m \to \infty} \sum_{k=0}^{m} w_{mk}^{A_n}$ for $n = 0, 1, 2, \cdots$ and $e = (1, 1, 1, \cdots)$.

Moreover, if $A \in (c_T, Y)$ then we have

$$Az = R_k(A_n)(T(z)) - \eta(\gamma_n) \quad \text{for all } z \in c_T, \quad \text{where } \eta = \lim_{k \to \infty} T_k(z).$$

To characterize the matrix transformations $A \in (X(r, s; t; \Delta), Y)$ for $X, Y \in \{l_\infty, c, c_0\}$, we consider the following conditions:
Theorem 4.11. (a) $A \in (c_0(r, s, t; \Delta), c_0)$ if and only if the conditions (4.12), (4.13), (4.14) and (4.15) hold.

(b) $A \in (c_0(r, s, t; \Delta), c)$ if and only if the conditions (4.12), (4.14), (4.15) and (4.16) hold.

(c) $A \in (c_0(r, s, t; \Delta), l_\infty)$ if and only if the conditions (4.12), (4.14) and (4.15) hold.

Proof. We prove only the part (a) of this theorem. The other parts follow in a similar way. For this, we first compute the matrices $B^A = R_k(A_n)$ and $W^A_n = (w^A_{mk})$ for $n = 0, 1, 2, \cdots$ of Theorem 4.9 to determine the conditions $B^A \in (c_0, c_0)$ and $W^A_n \in (c_0, c_0)$. Using the same lines of proof as used in
Theorem 4.8, we have

\[ R_k(A_n) = \sum_{j=k}^{\infty} s_{jk} a_{nj} \]

\[ = \sum_{j=k+1}^{\infty} \sum_{l=0}^{j-k} (-1)^l \frac{D_k^{(s)}}{l_{k+1}} r_k a_{nj} + \frac{D_k^{(s)}}{l_k} r_k a_{nk} \]

\[ = r_k \left[ \frac{a_{nk}}{s_0 l_k} + \frac{D_k^{(s)}}{l_k} \sum_{j=k+1}^{\infty} a_{nj} + \sum_{l=0}^{j-m} (-1)^l \frac{D_k^{(s)}}{l_{k+1}} \sum_{j=k+1}^{\infty} a_{nj} \right] \]

and

\[ w_{A_n}^{l_{nk}} = \sum_{j=m}^{\infty} a_{nj} s_{jk} \]

\[ = r_k \left[ \sum_{l=0}^{m-k} (-1)^l \frac{D_k^{(s)}}{l_{k+1}} \sum_{j=m}^{\infty} a_{nj} + \sum_{l=m-k+1}^{\infty} (-1)^l \frac{D_k^{(s)}}{l_{k+1}} \sum_{j=k+1}^{\infty} a_{nj} \right]. \]

Using Theorem 4.9, we have \( A \in (c_0 (r, s, t; \Delta), c_0) \) if and only if the conditions (4.12), (4.13), (4.14) and (4.15) hold.

We can also obtain the following results.

Corollary 4.1. (a) \( A \in (l_\infty (r, s, t; \Delta), c_0) \) if and only if the conditions (4.17) and (4.18) hold.
(b) \( A \in (l_\infty (r, s, t; \Delta), c) \) if and only if the conditions (4.12), (4.15), (4.18) and (4.20) hold.
(c) \( A \in (l_\infty (r, s, t; \Delta), l_\infty) \) if and only if the conditions (4.12) and (4.18) hold.

Corollary 4.2. (a) \( A \in (c (r, s, t; \Delta), c_0) \) if and only if the conditions (4.12), (4.13), (4.14), (4.20), (4.21) and (4.22) hold.
(b) \( A \in (c (r, s, t; \Delta), c) \) if and only if the conditions (4.12), (4.14), (4.16), (4.20), (4.21) and (4.24) hold.
(c) \( A \in (c (r, s, t; \Delta), l_\infty) \) if and only if the conditions (4.12), (4.14), (4.20), (4.21) and (4.23) hold.

5 Some applications

In this section, we justify our results in some special cases. Also, we illustrate the results related to matrix transformations given by Djolovic [6], Polat and Altay [15], Aydin and Başar [4] etc.

(I) In particular, if we choose \( r_n = \frac{1}{n^\alpha}, t_n = \frac{n^\alpha}{n^\beta} \) and \( s_n = \frac{(1-\alpha)^n}{n^\gamma} \), where \( 0 < \alpha < 1 \) then the sequence spaces \( X(r, s, t; \Delta) \) for \( X \in \{l_\infty, c_0, c\} \) reduce to the Euler difference sequence spaces \( e_\infty(\Delta), e_0(\Delta) \) and \( e_0(\Delta) \) respectively [15]. By the above choice of \( r, s \) and \( t \), we have \( D_k^{(s)} = 1, D_k^{(s)} = (1-\alpha), D_k^{(s)} = (1-\alpha)^2, \) and so on. Therefore \( D_k^{(s)} = \frac{(1-\alpha)^k}{k!}, k \in \mathbb{N}_0 \). Thus the \( \alpha \)-dual of the Euler difference sequence spaces is the set

\[ \left\{ a \in w : \sup_{K \in \mathbb{F}} \sum_n \sum_{j \in K} \sum_{k=j}^{n} (-1)^{k-j} \frac{(1-\alpha)^{k-j}}{(k-j)!} \frac{1}{j!} \alpha^{-k} a_{nj} < \infty \right\} \]

\[ = \left\{ a \in w : \sup_{K \in \mathbb{F}} \sum_n \sum_{j \in K} \sum_{k=j}^{n} (-1)^{k-j} \frac{k^j}{j!} (1-\alpha)^{k-j} \alpha^{-k} a_{nj} < \infty \right\}. \]
Here we illustrate that how the characterization of matrix transformation $A \in (e^{\alpha}_{\infty}(\Delta), l_{\infty})$ can be obtained with the help of Corollary 4.1(c)

$$R_k(A_n) = \sum_{j=k}^{\infty} a_{nj}s_{jk}$$

$$= r_k \left[ \frac{a_{nk}}{s_{nk}} + \left( \frac{D_0^{(s)}}{t_k} - \frac{D_1^{(s)}}{t_{k+1}} \right) \sum_{j=k+1}^{\infty} a_{nj} + \sum_{l=2}^{\infty} (-1)^l \frac{D_l^{(s)}}{t_{k+l}} \sum_{j=k+l}^{\infty} a_{nj} \right]$$

$$= \frac{a_{nk}}{s_{nk}} + \left( \frac{1}{\alpha^k} - \frac{1}{\alpha^{k+1}} \right) \sum_{j=k+1}^{\infty} a_{nj} + \sum_{l=2}^{\infty} (-1)^l \left( \frac{1}{\alpha^l} \right) \sum_{j=k+l}^{\infty} a_{nj}. $$

$$w_{nk}^{A_n} = \sum_{j=m}^{\infty} a_{nj}s_{jk}$$

$$= r_k \left[ \sum_{l=0}^{m-k} (-1)^l \frac{D_l^{(s)}}{t_{l+1}} \sum_{j=m-k+1}^{\infty} a_{nj} + \sum_{l=1}^{\infty} (-1)^l \frac{D_l^{(s)}}{t_{l+1}} \sum_{j=m+k+l}^{\infty} a_{nj} \right]$$

$$= \frac{1}{k!} \left[ \sum_{l=0}^{m-k} (-1)^l \frac{(l+1)}{\alpha^{l+k+1}} \sum_{j=m-k+1}^{\infty} a_{nj} + \sum_{l=m-k+1}^{\infty} (-1)^l \frac{(l+1)}{\alpha^{l+k+1}} \sum_{j=m+k+l}^{\infty} a_{nj} \right]$$

$$= \left[ \sum_{l=0}^{m-k} (-1)^l \frac{(l+1)}{\alpha^{l+k+1}} \sum_{j=m-k+1}^{\infty} a_{nj} + \sum_{l=m-k+1}^{\infty} (-1)^l \frac{(l+1)}{\alpha^{l+k+1}} \sum_{j=m+k+l}^{\infty} a_{nj} \right].$$

So $A \in (e^{\alpha}_{\infty}(\Delta), l_{\infty})$ if and only if

$$\sup_n \sum_{k=0}^{\infty} |R_k(A_n)| < \infty$$

and

$$\lim_{m \to \infty} \sum_{k=0}^{m} |w_{nk}^{A_n}| = 0 \quad \text{for all } n.$$

\((II)\) We choose $s_n = 1 \forall n, r_n = (n+1), t_n = 1 + \alpha^n,$ where $0 < \alpha < 1$ then the sequence spaces $X(r, s, \Delta)$ for $X \in \{l_{\infty}, c, c_0\}$ reduce to $a_{\alpha}^{\infty}(\Delta), a_{\alpha}^{c}(\Delta)$ and $a_{\alpha}^{0}(\Delta)$ respectively. With this choice $D_0^{(s)} = 1 = D_1^{(s)}$ and $D_k^{(s)} = 0$ for all $k \geq 2$. Therefore the matrices $R_k(A_n)$ and $W^{A_n} = (w_{nk}^{A_n})$ become

$$R_k(A_n) = (k+1) \left[ \frac{a_{nk}}{1 + \alpha^k} + \left( \frac{1}{1 + \alpha^k} - \frac{1}{1 + \alpha^{k+1}} \right) \sum_{j=k+1}^{\infty} a_{nj} \right],$$

and

$$w_{nk}^{A_n} = r_k \left[ \left( \frac{D_l^{(s)}}{t_k} - \frac{D_l^{(s)}}{t_{k+1}} \right) \sum_{j=m}^{\infty} a_{nj} - \frac{D_l^{(s)}}{t_{m+1}} \sum_{j=m+1}^{\infty} a_{nj} \right]$$

$$= (k+1) \left[ \left( \frac{1}{1 + \alpha^k} - \frac{1}{1 + \alpha^{k+1}} \right) \sum_{j=m}^{\infty} a_{nj} - \frac{1}{1 + \alpha^{m+1}} \sum_{j=m+1}^{\infty} a_{nj} \right].$$
By evaluating, we have
\[ \sum_{k=0}^{m} |w_{mk}| = \sum_{k=0}^{m} (k+1) \left( \frac{1}{1 + \alpha^k} - \frac{1}{1 + \alpha^{k+1}} \right) \left| \sum_{j=m}^{\infty} a_{nj} \right| + |a_{nm}| \frac{m+1}{1 + \alpha^{m+1}}. \]

Therefore by Corollary 4.1(a), we have \( A \in (\sigma_n^{\infty}(\Delta), c_0) \) if and only if
\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} |R_k(A_n)| = 0
\]
and
\[
\lim_{m \to \infty} \sum_{k=0}^{m} |w_{mk}| = 0 \text{ for all } n.
\]

References

[1] Z. U. Ahmad, M. Mursaleen, Köthe-Toeplitz duals of some new sequence spaces and their matrix maps, *Publ. Inst. Math.(Beograd)*, 42(56)(1987), 57-61.

[2] B. Altay, F. Başar, Generalization of the sequence space \( \ell(p) \) derived by weighted mean, *J. Math. Anal. Appl.*, 330(2007), 174-185.

[3] B. Altay , F. Başar, The fine spectrum and matrix domain of the difference operator \( \Delta \) on the sequence space \( \ell_p \), \( (0 < p < 1) \), *Commun. Math. Anal.*, 2 (2)(2007), 1-11.

[4] C. Aydin, F. Başar, Some new difference sequence spaces, *Appl. Math. Comput.*, 157(3)(2004), 677-693.

[5] R. Çolak, M. Et, On some generalized difference sequence spaces and related matrix transformations, *Hokkaido Math. J.*, 26(3)(1997), 483-492.

[6] I. Djolović , On the space of bounded Euler difference sequences and some classes of compact operators, *Appl. Math. Comput.*, 182(2) (2006),1803-1811.

[7] A. M. Jarrah, E. Malkowsky, Ordinary, absolute and strong summability and matrix transformations, *Filomat*, 17(2003), 59-78.

[8] H. Kizmaz, On certain sequence spaces, *Canad. Math. Bull.*, 24(2)(1981), 169-176.

[9] I. J. Maddox, Elements of Functional Analysis, 2nd Edition, *The University Press, Cambridge*, 1988.

[10] E. Malkowsky, V. Rakočević, On matrix domains of triangles, *Appl. Math. Comput.*, 189(2)(2007), 1146-1163.

[11] E. Malkowsky, E. Savas, Matrix transformations between sequence spaces of generalized weighted means, *Appl. Math. Comput.*, 147(2004), 333-345.

[12] M. Mursaleen, A. K. Noman, On generalized means and some related sequence spaces, *Comput. Math. Appl.*, 61(4)(2011), 988-999.

[13] C. Orhan, Matrix transformations on Cesro difference sequence spaces, *Comm. Fac. Sci. Univ. Ankara Sr. A1 Math.*, 33(1)(1984), 1-8.
[14] H. Polat, V. Karakaya, N. Şimşek, Difference sequence spaces derived by using a generalized weighted mean, *Appl. Math. Lett.*, 24(5)(2011), 608-614.

[15] H. Polat, B. Atay, On some new Euler difference sequence spaces, *Southeast Asian Bulletin Math.*, 30(2006), 209-220.

[16] M. Stieglitz, H. Tietz, Matrix trasformationen von Folenraumen Eine Erebsisubersicht, *Mathematische Zeitschrift(Math. Z.)*, 154(1977), 1-16.

[17] A. Wilansky, Summability through Functional Analysis, *North-Holland Math. Stud., vol. 85*, Elsevier Science Publishers, Amsterdam, New York, Oxford, 1984.