The topology of positive scalar curvature

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Abstract. Given a smooth closed manifold $M$ we study the space of Riemannian metrics of positive scalar curvature on $M$. A long-standing question is: when is this space non-empty (i.e. when does $M$ admit a metric of positive scalar curvature)? More generally: what is the topology of this space? For example, what are its homotopy groups?

Higher index theory of the Dirac operator is the basic tool to address these questions. This has seen tremendous development in recent years, and in this survey we will discuss some of the most pertinent examples.

In particular, we will show how advancements of large scale index theory (also called coarse index theory) give rise to new types of obstructions, and provide the tools for a systematic study of the existence and classification problem via the K-theory of $C^*$-algebras. This is part of a program “mapping the topology of positive scalar curvature to analysis”.

In addition, we will show how advanced surgery theory and smoothing theory can be used to construct the first elements of infinite order in the $k$-th homotopy groups of the space of metrics of positive scalar curvature for arbitrarily large $k$. Moreover, these examples are the first ones which remain non-trivial in the moduli space of such metrics.

Mathematics Subject Classification (2010). Primary 53C21; Secondary 53C27, 58D17, 53C20, 58D27, 58B05, 53C23, 19K56, 58J22,19K33, 46L80, 57R15, 57N16, 57R65.

Keywords. Positive scalar curvature, higher index theory, large scale index theory, coarse index theory, coarse geometry, $C^*$-index theory.

1. Introduction

One of the fundamental questions at the interface of geometry and topology concerns the relation between local geometry and global topology.

More specifically, given a compact smooth manifold $M$ without boundary, what are the possibilities for Riemannian metrics on $M$? Even more specifically, can we find a metric of positive scalar curvature on $M$ and if yes, what does the space of such metrics look like?

Recall the following definition of the scalar curvature function.

*supported by the Courant Research Center “Higher order structures in Mathematics” of Georg-August-Universität Göttingen
Definition 1.1. Given an $n$-dimensional smooth Riemannian manifold $(M,g)$, the \textit{scalar curvature} at $x$ describes the volume expansion of small balls around $x$ via

$$\frac{\text{vol}(B_\epsilon(M,x))}{\text{vol}(B_\epsilon(\mathbb{R}^n,0))} = 1 - \frac{\text{scal}(x)}{6(n+2)} \epsilon^2 + O(\epsilon^4),$$

compare [3, 0.60]. In particular, this means that if $\text{scal}(x) > 0$ then geodesic balls around $x$ for small radius have smaller volume than the comparison balls in Euclidean space. Of course, alternatively, $\text{scal}(x)$ can be defined as a second contraction of the Riemannian curvature operator at $x$.

The most important tool to investigate these questions goes back to Erwin Schrödinger [29], rediscovered by André Lichnerowicz [21]: If $M$ has positive scalar curvature and a spin structure then the Dirac operator on $M$ is invertible. This forces its index (which is the super-dimension of the null space) to vanish.

Recall that a spin structure is a (global) differential geometric datum for a Riemannian manifold $M$ which allows us to construct a specific Riemannian vector bundle $S$, the spinor bundle, together with a specific differential operator of order 1, the Dirac operator $D$ (compare e.g. [20] for a nice introduction).

On the other hand non-vanishing of the index follows from index theorems, giving rise to powerful obstructions to positive scalar curvature. For example, the Atiyah-Singer index theorem says that $\text{ind}(D) = \hat{A}(M)$, where the $\hat{A}$-genus is a fundamental differential topological invariant (not depending on the metric!).

The most intriguing question around this method to rule out positive scalar curvature asks to what extent a sophisticated refinement of $\text{ind}(D)$, the \textit{Rosenberg index} $\alpha^R(M)$ which takes values in the K-theory of the (real) $C^*$-algebra of the fundamental group $\Gamma$ of $M$, is the only obstruction. This is the content of the (stable) Gromov-Lawson-Rosenberg conjecture.

\textbf{Conjecture 1.2.} Let $M$ be a connected closed spin manifold of dimension $\geq 5$. The Gromov-Lawson-Rosenberg conjecture asserts that $M$ admits a metric with positive scalar curvature if and only if $\alpha^R(M) = 0 \in KO_*(\mathbb{C}^*_\pi \pi_1(M))$.

The stable Gromov-Lawson-Rosenberg conjecture claims that $\alpha^R(M) = 0$ if and only if there is $k \in \mathbb{N}$ such that $M \times B^k$ admits a metric with positive scalar curvature. Here, $B$ is any so-called Bott manifold i.e. a simply connected 8-dimensional spin manifold with $\hat{A}(M) = 1$.

To put the stable version in context: given two closed manifolds $M, B$ such that $M$ admits a metric of positive scalar curvature, so does $M \times B$, simply using the product of a sufficiently scaled metric on $M$ with any metric on $B$. Therefore the unstable Gromov-Lawson-Rosenberg conjecture implies the stable one.

Recall here that for a discrete group $\Gamma$ the \textit{maximal group} $C^*$-\textit{algebra} $C^*_\text{max} \Gamma$ is defined as the completion of the group ring $\mathbb{C}[\Gamma]$ with respect to the maximal possible $C^*$-norm on $\mathbb{C}[\Gamma]$, and the \textit{reduced group} $C^*$-\textit{algebra} $C^*_\text{r} \Gamma$ is defined as the norm closure of $\mathbb{C}[\Gamma]$, embedded in $\mathcal{B}(l^2(\Gamma))$ via the regular representation. The \textit{real} group $C^*$-algebras $C^*_\mathbb{R} \Gamma$ and $C^*_\mathbb{R,\text{max}} \Gamma$ replace $\mathbb{C}$ by $\mathbb{R}$ throughout. Using them gives more information, necessary in the Gromov-Lawson-Rosenberg conjecture.
In this survey, for simplicity we will not discuss them but concentrate on the complex versions. For the Rosenberg index one can use them all, where a priori $\alpha_{\max}(M) \in K_*(C^*_r \pi_1(M))$ is stronger than $\alpha_r(M) \in K_*(C^*_r \pi_1(M))$. The Baum-Connes isomorphism conjecture, compare [2], predicts the calculation of $K_*(C^*_r \Gamma)$ in terms of the equivariant K-homology of a suitable classifying space. The strong Novikov conjecture predicts that this equivariant K-homology at least embeds.

In celebrated work Stephan Stolz [31, 32] has established the following two partial positive results.

**Theorem 1.3.** The Gromov-Lawson-Rosenberg conjecture is true for manifolds with trivial fundamental group. In other words, if $M$ is a closed connected spin manifold of dimension $\geq 5$ with trivial fundamental group, then $M$ admits a Riemannian metric with positive scalar curvature if and only if $\alpha^R(M) = 0$.

More generally, if $\pi_1(M)$ satisfies the strong Novikov conjecture then the stable Gromov-Lawson-Rosenberg conjecture is true for $M$.

On the other hand, recall the counterexamples of [27, 5] which show that the unstable Gromov-Lawson-Rosenberg conjecture is not always true.

**Theorem 1.4.** For $5 \leq n \leq 8$ there exist closed spin manifolds $M^n$ of dimension $n$ such that $\alpha(M^n) = 0$, but such that $M^n$ does not admit a metric with positive scalar curvature.

The manifolds $M_n$ can be constructed with fundamental groups $\mathbb{Z}^{n-1} \times \mathbb{Z}/3\mathbb{Z}$ or with appropriately chosen torsion-free fundamental group [18]. It remains one of the most intriguing open questions whether the Gromov-Lawson-Rosenberg conjecture is true for all $n$-dimensional manifolds with fundamental group $(\mathbb{Z}/3\mathbb{Z})^n$.

The obstructions used in the counterexamples of Theorem 1.4 are not based on index theory, but on the minimal hypersurface method of Richard Schoen and Shing-Tung Yau [28] which we will not discuss further in this survey.

As a companion to the Gromov-Lawson-Rosenberg conjecture we suggest a slightly weaker conjecture about the strength of the Rosenberg index:

**Conjecture 1.5.** Let $M$ be a closed spin manifold. Every obstruction to positive scalar curvature for manifolds of dimension $\geq 5$ which is based on index theory of Dirac operators can be read off the Rosenberg index $\alpha^R(M) \in KO_*(C^*_R \pi_1(M))$.

This is vague because the statement “based on index theory of Dirac operators” certainly leaves room for interpretation.

By Stolz’ Theorem 1.3, Conjecture 1.5 follows from the strong Novikov conjecture. On the other hand, every index theoretic obstruction which is not (yet) understood in terms of the Rosenberg index is particularly interesting. After all, it is a potential starting point to obtain counterexamples to the strong Novikov conjecture.

Around this question we discuss the following results [9, 10, 7].
Theorem 1.6. Let $M$ be an area-enlargeable spin manifold (which implies by the work of Mikhail Gromov and Blaine Lawson [6] that $M$ does not admit a metric of positive scalar curvature).

Then $\alpha_{\text{max}}(M) \neq 0 \in K_*(C^*\pi_1(M))$.

If $M$ is even (length)-enlargeable, then $\alpha_r(M) \neq 0 \in K_*(C^r\pi_1(M))$.

Recall that a closed $n$-dimensional manifold $M$ is called enlargeable if it admits a sequence of coverings $M_i \to M$ which come with compactly supported maps $f_i : M_i \to S^n$ of non-zero degree but such that $\sup_{x \in M_i} \|D_x f_i\|$ tends to 0 as $i \to \infty$. It is area-enlargeable if the same holds with $\|\Lambda^2 D_x f_i\|$ (a weaker condition). For the definition of the norms, we use a fixed metric on $M$ and its pull-backs to $M_i$ and a fixed metric on $S^n$.

As a potential counterexample to Conjecture 1.5 we describe a codimension-2 obstruction to positive scalar curvature (in a special form introduced by Mikhail Gromov and Blaine Lawson in [6, Theorem 7.5]) which is based on index theory of the Dirac operator, but which so far is not known to be encompassed by the Rosenberg index.

Theorem 1.7. (compare [8, Section 4]). Let $M$ be a closed connected spin manifold with vanishing second homotopy group. Assume that $N \subset M$ is a smooth submanifold of codimension 2 with trivial normal bundle and such that the inclusion induces an injection on the level of fundamental groups $\pi_1(N) \to \pi_1(M)$. Finally, assume that the Rosenberg index of the Dirac operator on the submanifold $N$ does not vanish: $0 \neq \alpha(N) \in K_*(C^*\pi_1 N)$.

Then $M$ does not admit a Riemannian metric with positive scalar curvature.

(Secondary) index invariants of the Dirac operator can be used in the classification of metrics of positive scalar curvature, if applied to appropriately constructed examples. “Classification” means in particular to understand how many deformation classes of metrics of positive scalar curvature a given manifold carries, or more generally what the topology of the space of such metrics looks like.

A promising tool to systematically study the existence and classification problem is the “Stolz positive scalar curvature long exact sequence”. It has the form

$$
\cdots \to R_{n+1}(\pi_1(M)) \to \text{Pos}_n(M) \to \Omega_n^{\text{spin}}(M) \to R_n(\pi_1(M)) \to \cdots
$$

Here the group we would like to understand is $\text{Pos}_n(M)$, the structure group of metrics of positive scalar curvature (on the spin manifold $M$ and related manifolds, and modulo a suitable bordism relation). The group $\Omega_n^{\text{spin}}(M)$ is the usual spin bordism group from algebraic topology, which is very well understood. Finally, $R_n(\pi_1(M))$ indeed depends only on the fundamental group of the manifold in question. Note that this positive scalar curvature sequence is very similar in spirit to the surgery exact sequence coming up in the classification of manifolds.

Unfortunately we have not yet been able to fully compute all the terms in this exact sequence, even for the simplest possible case of trivial fundamental group. However, a lot of information can be gained using index theory by mapping out to more manageable targets. Here, we refer in particular to [22], joint with Paolo
Piazza, where we construct a commutative diagram of maps, using large scale index theory, to the K-theory sequence of associated $C^*$-algebras

$$
\cdots \to K_{n+1}(C^*_r\Gamma) \to K_{n+1}(D^r\tilde{M}^\Gamma) \to K_n(M) \xrightarrow{\alpha} K_n(C^*_r\Gamma) \to \cdots
$$

We again abbreviate $\Gamma = \pi_1(M)$. This sequence was introduced by Nigel Higson and John Roe \cite{HR} and called there the analytic surgery exact sequence. A lot is known about this K-theory sequence: $K_n(M)$ is just the usual topological K-homology of $M$ (an important generalized homology theory). Moreover, $\alpha$ is the Baum-Connes assembly map.

A good deal of this survey will discuss the large scale index theory underlying the constructions. In particular, we will explain two primary and secondary index theorems which play key roles in the application of this theory:

- A vanishing theorem for the large scale index of the Dirac operator under partial positivity of the scalar curvature, Theorem \cite{6.1}.

- A higher secondary index theorem, that shows how the large scale index of a manifold with boundary (which has positive scalar curvature near the boundary) determines a structure invariant of the boundary’s metric of positive scalar curvature.

A fundamental problem in the use of (higher) index theory centers around the question whether there is a difference between topological information (which typically can be computed much more systematically) and analytical information (which has the desired geometric consequences but often is hard to compute). This is answered (conjecturally) by the strong Novikov conjecture. This explains why these conjectures play such a central role in higher index theory. The appropriate version for large scale index theory is the coarse Baum-Connes conjecture (here, “with coefficients”, for details compare Section \cite{4}).

**Conjecture 1.8.** Given a locally compact metric space $X$ of bounded geometry and an auxiliary coefficient $C^*$-algebra $A$, then in the composition

$$K_{*}^H(X; A) \to KX_*(X; A) \to K_*(C^*(X; A))$$

the second map is an isomorphism. Here, $K_{*}^H(X)$ is the topologists' locally finite K-homology (a generalized homology theory) of the space $X$, and $K_{*}^H(X; A)$ is a version with coefficients, still a generalized cohomology theory. Finally $KX_*(X; A)$, the coarse K-homology, is a variant which depends only on the large scale geometry of $X$.

This conjecture has many concrete applications. It implies the strong Novikov conjecture. However, I expect that counterexamples to these conjectures eventually will be found.

Concerning the classification question mentioned above, in the last part of the survey we will discuss a new construction method, based on advanced surgery theory and smoothing theory in the topology of manifolds.
Definition 1.9. We define $\text{Riem}^+(M)$ to be the space of Riemannian metrics of positive scalar curvature on $M$, an infinite dimensional manifold.

The main result is that $\pi_k(\text{Riem}^+(M))$ is very often non-trivial, even its image in the moduli space of such metrics remains non-trivial.

More precisely, we have the following theorem, derived in joint work with Bernhard Hanke and Wolfgang Steimle (compare [11, Theorem 1.1]).

Theorem 1.10. For every $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that, whenever $M$ is a connected closed spin manifold with a metric $g_0$ of positive scalar curvature and with $\dim(M) > n_k$ and $k + \dim(M) + 1 \equiv 0 \pmod{8}$, then $\pi_k(\text{Riem}^+(M), g_0)$ contains an element of infinite order.

If $M$ is a homology sphere, then the image of this element in $\pi_k(\text{Riem}^+(M)/\text{Diffeo}_{x_0}(M))$ also has infinite order, where the diffeomorphism group acts by pullback.

The second part of the theorem implies that the examples constructed do not rely on the homotopy properties of the diffeomorphism group of $M$. This is in contrast to all previous known cases, compare in particular [16, 4].

Here, $\text{Diffeo}_{x_0}(M)$ is the subgroup of the full diffeomorphism group consisting of diffeomorphisms of $M$ which fix the point $x_0 \in M$ and whose differential at $x_0$ is the identity. It is much more reasonable to use this subgroup instead of the full diffeomorphism group, because it ensures that the moduli space $\text{Riem}^+(M)/\text{Diffeo}_{x_0}(M)$ remains an infinite dimensional manifold, instead of producing a very singular space.

Remark 1.11. Most of the results mentioned so far display also how poorly the topology of positive scalar curvature is understood: the method relies on the index theory of the Dirac operator and the Schrödinger-Lichnerowicz formula. This is quite a miraculous relation which certainly is very helpful. But it requires the presence of a spin structure. Manifolds without spin structure (and where not even the universal covering admits a spin structure) a priori shouldn’t be very different from manifolds with spin structure, i.e. one would expect that many of them do not admit a metric of positive scalar curvature. But until now we have almost no tools to decide this (apart from the minimal surface method, which is only established in small dimensions).

Almost any progress in this direction would be a real breakthrough.

2. Index theory and obstructions to positive scalar curvature

The underlying principle how scalar curvature is coupled to the Dirac operator comes from a formula of Schrödinger [29], rediscovered and first applied by Lichnerowicz [21]. The starting point is a spin manifold $(M, g)$, with spinor bundle $S$ and Dirac operator $D$. Schrödinger’s formula says

$$D^2 = \nabla^* \nabla + \frac{\text{scal}}{4}.$$
The first term on the right is the “rough Laplacian”, by definition a non-negative unbounded operator on the $L^2$-sections of $S$. The second term stands for pointwise multiplication with the scalar curvature function. If the scalar curvature is bounded below by a positive number (called “uniformly positive” later), this is a positive operator and hence $D$ is invertible.

Let us recall the basics of the index theory of the Dirac operator, formulated in the language of operator algebras and K-theory. This is the most convenient setup for the generalizations we have in mind.

We start with a very brief introduction to the K-theory of $C^*$-algebras.

1. The assignment $A \mapsto K_*(A)$ is a functor from the category of $C^*$-algebras to the category of graded abelian groups.

2. We can (for a unital $C^*$-algebra $A$) define $K_0(A)$ as the group of equivalence classes of projectors in $A$ and the matrix algebras $M_n(A)$.

3. We can (for a unital $C^*$-algebra $A$) define $K_1(A)$ as the group of equivalence classes of invertible elements in $A$ and $M_n(A)$.

4. There is a natural Bott periodicity isomorphism $K_n(A) \to K_{n+2}(A)$.

5. For each short exact sequence of $C^*$-algebras $0 \to I \to A \to Q \to 0$ there is naturally associated a long exact sequence in K-theory

$$\cdots \to K_{n+1}(Q) \xrightarrow{\delta} K_n(I) \to K_n(A) \to K_n(Q) \to \cdots.$$ (2.1)

6. One can generalize K-theory for real and graded $C^*$-algebras. In the former case Bott periodicity has period 8.

7. One can use extra symmetries based on Clifford algebras to give descriptions of $K_n(A)$ which are adapted to the treatment of $n$-dimensional spin manifolds.

On an even dimensional manifold, the spinor bundle canonically splits into $S = S^+ \oplus S^-$ and the Dirac operator is odd, i.e. has the form $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$. Let $\chi: \mathbb{R} \to \mathbb{R}$ be any continuous function which is odd (i.e. $\chi(-x) = \chi(x)$ for $x \in \mathbb{R}$) and with $\lim_{x \to \infty} \chi(x) = 1$. Functional calculus allows to define $\chi(D)$, which is an odd bounded operator acting on $L^2(S)$. Choosing an isometry $U: L^2(S^-) \to L^2(S^+)$ we form $U\chi(D)^+ \in B := B(L^2(S^+))$, the $C^*$-algebra of all bounded operators on $L^2(S^+)$. If $M$ is compact, ellipticity of $D$ implies that $U\chi(D)^+$ is invertible modulo the ideal $\mathcal{K} := \mathcal{K}(L^2(S^+))$ of compact operators. The short exact sequence of $C^*$-algebras $0 \to \mathcal{K} \to B \to B/\mathcal{K} \to 0$ gives rise to a long exact K-theory sequence and the relevant piece of this exact sequence for us is

$$\cdots \to K_1(B) \to K_1(B/\mathcal{K}) \xrightarrow{\delta} K_0(\mathcal{K}) \to \cdots$$ (2.1)

As invertible elements in $A$ represent classes in $K_1(A)$, the above spectral considerations yield a class $[U\chi(D)^+] \in K_1(B/\mathcal{K})$ and we define the index to be $\text{ind}(D) := \delta(U\chi(D)^+) \in K_0(\mathcal{K})$. 

Of course, for the compact operators, $K_0(\mathcal{K})$ is isomorphic to $\mathbb{Z}$, generated by any rank 1 projector in $\mathcal{K}$. In our case $\delta(U\chi(D)^+)\in K_1(\mathcal{B}/\mathcal{K})$ in the sequence (2.1) has a lift to $K_1(\mathcal{B})$. Exactness implies $\text{ind}(D) = 0$.

A second main ingredient concerning the index of the Dirac operator is the Atiyah-Singer index theorem \[1\]. A priori $\text{ind}(D)$ (like the operator $D$) depends on the Riemannian metric on $M$. However, the index theorem expresses it in terms which are independent of the metric. More specifically, $\text{ind}(D) = \hat{A}(M)$, the $\hat{A}$-genus of $M$.

This result has vast generalizations in many directions. A very important one (introduced by Jonathan Rosenberg \[26\]) modifies the Dirac operator by “twisting” with a smooth flat bundle $E$ of (finitely generated projective) modules over an auxiliary $C^*$-algebra $A$. One then obtains an index in $K_*(A)$. Indeed, the construction is pretty much the same as above, with the important innovation that one replaces the scalars $\mathbb{C}$ by the more interesting $C^*$-algebra $A$, as detailed in Section 3.

The second generalization works for a non-compact manifold $X$. In this case the classical Fredholm property of the Dirac operator fails. To overcome this, large scale index theory, synonymously called coarse index theory is developing. Again this is based on $C^*$-techniques and pioneered by John Roe \[23\]. It is tailor-made for the non-compact setting. One obtains an index in the $K$-theory of the Roe algebra $C^*(X; A)$. In $C^*$-algebras, positivity implies invertibility, which finally implies that all the generalized indices vanish if one starts with a metric of uniformly positive scalar curvature.

The general pattern (from the point of view adopted in this article) of index theory is the following:

1. The geometry of the manifold $M$ produces an interesting operator $D$.
2. This operator defines an element in an operator algebra $A$, which depends on the precise context.
3. The operator satisfies a Fredholm condition, which means it is invertible modulo an ideal $I$ of the algebra $A$, again depending on the context.
4. The algebras in question are $C^*$-algebras. This implies that “positivity” of elements is defined, and moreover “positivity” implies invertibility.
5. A very special additional geometric input implies positivity and hence honest invertibility of our operator. For us, this special context will be the fact that we deal with a metric of uniformly positive scalar curvature.
6. Indeed, any element which is invertible in $A$ modulo an ideal $I$ defines an element in $K_{n+1}(A/I)$, where $n = \dim(M)$. (Instead of getting $K_1$, the fact
that we deal with the Dirac operator of an \( n \)-dimensional manifold produces additional symmetries (related to actions of the Clifford algebra \( Cl_n \)) which give rise to the element in \( K_{n+1}(A/I) \).

7. We interpret the class defined by the Dirac operator as a fundamental class \([M] \in K_{n+1}(A/I)\). Homotopy invariance of K-theory implies that \([M]\) does not depend on the full geometric data which goes in the construction of the operator \( D \), but only on the topology of \( M \).

8. The K-theory exact sequence of the extension \( 0 \to I \to A \to A/I \to 0 \) contains the boundary map \( \delta \). We call the image of \([M]\) under \( \delta \) the index

\[
\delta: K_{n+1}(A/I) \to K_n(I); \quad [M] \mapsto \text{ind}(D).
\]

Note that the degree arises from additional dimension-dependent symmetries which we do not discuss in this survey.

9. The additional geometric positivity assumption (uniformly positive scalar curvature) which implies invertibility already in \( A \), gives rise to a canonical lift of \([M]\) to an element \( \rho(M, g) \in K_{n+1}(A) \). Because of this, we think of \( K_{n+1}(A) \) as a structure group and \( \rho(M, g) \) is a structure class. It contains information about the underlying geometry.

Indeed, we want to advocate here the idea that the setup just described has quite a number of different manifestations, depending on the situation at hand. It can be adapted in rather flexible ways. The next section treats one example.

3. Large scale index theory

We describe “large scale index theory” for a complete Riemannian manifold of positive dimension.

Therefore, let \((M, g)\) be such a complete Riemannian manifold. Fix a Hermitian vector bundle \( E \to M \) of positive dimension. We first describe the operator algebras which are relevant. They are all defined as norm-closed subalgebras of \( B(L^2(M; E)) \).

**Definition 3.1.** We need the following concepts.

- An operator \( T: L^2(M; E) \to L^2(M; E) \) has **finite propagation** (namely \( \leq R \)) if \( \phi T\psi = 0 \) whenever \( \phi, \psi \in C_c(M) \) are compactly supported continuous functions whose supports have distance at least \( R \).

Here, we think of \( \phi \) also as bounded operator on \( L^2(M; E) \), acting by pointwise multiplication.

- \( T \) as above is called **locally compact** if \( \phi T \) and \( T \phi \) are compact operators whenever \( \phi \in C_c(M) \).
• $T$ is called pseudolocal if $\phi T \psi$ is compact whenever $\phi, \psi \in C_c(M)$ with disjoint supports, i.e. such that $\phi \psi = 0$.

• The Roe algebra $C^*(M)$ is defined as the norm closure of the algebra of all bounded finite propagation operators which are locally compact. It is an ideal in the structure algebra $D^*(M)$ which is defined as the closure of the algebra of finite propagation pseudolocal operators.

• Assume that a discrete group $\Gamma$ acts by isometries on $M$. Requiring in the above definitions that the finite propagation operators are in addition $\Gamma$-equivariant and then completing, we obtain the pair $C^*(M)^\Gamma \subset D^*(M)^\Gamma$.

Remark 3.2. For technical reasons, one actually should replace the bundle $E$ by the bundle $E \otimes I^2(\mathbb{N})$ whose fibers are separable Hilbert spaces (or in the equivariant case by $E \otimes I^2(\mathbb{N}) \otimes I^2(\Gamma)$). Via the embedding $B(L^2(M;E)) \to B(L^2(M;E \otimes I^2(\mathbb{N})))$ implicitly we think of operators on $L^2(M;E)$ as operators in the bigger algebra without mentioning this. Using the larger bundle guarantees functoriality and independence on $E$, implicit in our notion $D^*(M)$.

Let $A$ be an auxiliary $C^*$-algebra. Typical examples arise from a discrete group $\Gamma$, namely $C^*_{\text{max}} \Gamma$ or $C^*_r \Gamma$. An important role is then played by smooth bundles $E$ over $M$ with fibers finitely generated projective $A$-modules. These inherit fiberwise $A$-valued inner products (or more precisely Hilbert $A$-module structures in the sense of [19]). Integrating the fiberwise inner product then also defines a Hilbert $A$-module structure on the space of continuous compactly supported sections of $E$. By completion, we obtain the Hilbert $A$-module $L^2(M;E)$. The bounded, adjointable, $A$-linear operators on $L^2(M;E)$ form the Banach algebra $B(L^2(M;A))$. The ideal of $A$-compact operators is defined as the norm closure of the ideal generated by operators of the form $s \mapsto v \cdot (s,w); L^2(M;E) \to L^2(M;E)$, with $v,w \in L^2(M;E)$.

We now define $C^*(M;A)$ and $D^*(M;A)$ mimicking Definition 3.1 but replacing the Hilbert space concepts by the Hilbert $A$-module concepts throughout. In particular, we use $A$-compact operators instead of compact operators. Also, the stabilization as discussed in Remark 3.2 is replaced suitably.

Example 3.3. Given a connected manifold $M$ with fundamental group $\Gamma$, there is a canonical $C^*\Gamma$-module bundle, the Mishchenko bundle $\mathcal{L}$. It is the flat bundle associated to the left multiplication action of $\Gamma$ on $C^*\Gamma$, where we treat $C^*\Gamma$ as the free right $C^*\Gamma$-module of rank 1, i.e. $\mathcal{L} = \hat{M} \times_\Gamma C^*\Gamma$.

The construction of the large scale index is now based on two principles.

Proposition 3.4. Let $M$ be a complete Riemannian spin manifold, $A$ an auxiliary $C^*$-algebra and $E \to M$ a smooth bundle of (finitely generated projective) $A$-modules with compatible connection. We form the twisted Dirac operator $D_E$ as an unbounded operator on the Hilbert $A$-module of $L^2$-spinors on $M$ with values in $E$.

For the operator $D_E$, there exists a functional calculus. In particular, we can form $f(D_E)$ for $f: \mathbb{R} \to \mathbb{R}$ a continuous function which vanishes at $\infty$ or which
has limits as \( t \to \pm \infty \). Moreover, \( f(D_E) \) depends only on the restriction of \( f \) to the spectrum of \( D_E \). Then

1. if \( f \) has a compactly supported (distributional) Fourier transform then \( f(D_E) \) has finite propagation.

2. if \( f \) vanishes at infinity, then \( f(D_E) \) is locally compact; if \( f(t) \) converges as \( t \to \pm \infty \) then \( f(D_E) \) is at least pseudolocal.

The first property is a rather direct consequence of unit propagation speed for the fundamental solution of the heat equation. The second one is an incarnation of ellipticity and local elliptic regularity.

These results are well known and have been used a lot in the literature (compare in particular [23]), indeed they form the basis of “large scale index theory”. For the very general case needed in the proposition (with coefficients, arbitrary complete \( M \)), a complete proof is given in [8].

The construction of the large scale index is now rather straight-forward:

1. Take any continuous function \( \psi: \mathbb{R} \to \mathbb{R} \) (later assumed to be odd) with \( \lim_{t \to \infty} \psi(t) = 1 \). Then Proposition 3.4 implies that \( \psi(D_E) \) belongs to \( D^*(M;A) \). Even better, \( 1 - \psi^2 \) vanishes at infinity, so that \( 1 - \psi(D_E)^2 \) belongs to \( C^*(M;A) \).

2. By the principles listed at the end of Section 2, \( \psi(D_E) \) gives rise to an element \([M;E]\) in \( K_{n+1}(D^*(M;A)/C^*(M;A)) \). (Here, we again avoid the discussion of the additional symmetries which raise the index by \( n \)). Homotopy invariance of \( C^*\)-algebra K-theory implies that this element is independent of the choice of \( \psi \) and depends only on the large scale features of the metric on \( M \).

3. We define the large scale index (or synonymously “coarse index”) \( \text{ind}(D_E) \in K_{n}(C^*(M;A)) \) as the image of \([M;E]\) under the boundary map of the long exact K-theory sequence.

4. If \( M \) has uniformly positive scalar curvature and \( E \) is flat, the Lichnerowicz-Weitzenböck formula implies that \( 0 \) is not in the spectrum of \( D_E \). Then we can choose a function \( \psi \) which is equal to \(-1\) on the negative part of the spectrum of \( D_E \) and equal to \(+1\) on the positive part of the spectrum of \( D_E \), so that \( 1 - \psi^2 \) vanishes on the spectrum of \( D_E \), i.e. \( \psi^2(D_E) = 1 \).

This means that \([M;E]\) lifts in a canonical way to \( \rho(D_E) \in K_{n+1}(D^*(M;A)) \) (this class depends on the metric of positive scalar curvature) and it implies that \( \text{ind}(D_E) = 0 \).

5. A special feature is that \( K_{n+1}(D^*(M;A)/C^*(M;A)) \) indeed is homological in nature: it is canonically isomorphic to the locally finite K-homology \( K_{f,loc}^*(M;A) \), satisfying the Eilenberg-Steenrod axioms of a (locally finite) generalized homology theory.
Example 3.5. If we apply this construction to a closed $n$-dimensional spin manifold $M$ and the Mishchenko bundle $L$ on $M$, we obtain $\text{ind}(D_L) \in K_n(C^*(M;C^*\Gamma))$.

However, there is a canonical isomorphism $K_*(C^*(M;C^*\Gamma)) \cong K_*(C^*\Gamma)$. Using this isomorphism, the Rosenberg index mentioned above is exactly $\text{ind}(D_L)$:

$$\alpha(M) = \text{ind}(D_L) \in K_n(C^*\Gamma) \cong K_n(C^*(M;C^*\Gamma)).$$

The reduced $C^*$-algebra $C^*_r\Gamma$ of a discrete group is a canonical construction which captures many features of the group $\Gamma$, e.g. concerning its representation theory. However, it is very rigid. In particular, it is not functorial: a homomorphism $\Gamma_1 \to \Gamma_2$ will in general not induce a homomorphism $C^*_r\Gamma_1 \to C^*_r\Gamma_2$. As a consequence it is very hard to find homomorphisms out of $C^*_r\Gamma$ and also out of $K_*(C^*_r\Gamma)$.

Coarse geometry, however, immediately provides such a homomorphism (which allows one to detect elements in $K_*(C^*_r\Gamma)$). This is based on a simple calculation: If a discrete group $\Gamma$ isometrically acts freely and cocompactly on a metric space $X$, then $C^*X\Gamma$ is isomorphic to $C^*_r(\Gamma) \otimes K$. “Forgetting equivariance” therefore gives the composite homomorphism

$$C^*_r\Gamma \hookrightarrow C^*_r\Gamma \otimes K \cong C^*X\Gamma \hookrightarrow C^*X.$$

The induced map in K-theory allows one to detect elements in $K_*(C^*_r\Gamma)$ using large scale index theory, as we will show in one case in Section 5.

4. The coarse Baum-Connes conjecture

Being the home of important index invariants, it is very important to be able to compute the K-theory of the Roe algebras $C^*(M;A)$ for arbitrary complete manifolds $M$ and coefficient $C^*$-algebras $A$. It turns out that there are quite a number of tools to do this. Even better, at least conjecturally there is a purely homological answer to this task.

Let us start with the three most important computational tools.

1. $K_*(C^*(M;A))$ is invariant under coarse homotopy, compare [12].

2. There are powerful vanishing theorems for $K_*(C^*(M;A))$. An important one is valid if $M$ is flasque [23 Proposition 9.4]. This means that $M$ admits a shift map $f: M \to M$ such that, on the one hand, $f$ is uniformly close to the identity (i.e. there is a constant $C$ such that $d(f(x),x) < C$ for all $x \in M$). On the other hand, $f$ moves everything to infinity in the sense that for each compact subset $K$ of $M$, $\text{im}(f^k) \cap K = \emptyset$ for all sufficiently large iterations $f^k$ of $f$.

3. The group $K_*(C^*(M;A))$ satisfies a Mayer-Vietoris principle. For this, one needs a coarsely excisive decomposition $M = M_1 \cup M_2$, which means that the intersection $M_0 := M_1 \cap M_2$ captures all the large scale features of the
relation between $M_1$ and $M_2$. The technical definition is that for each $R > 0$ there is an $S > 0$ such that the $S$-neighborhood of $M_1 \cap M_2$ contains the intersection of the $R$-neighborhoods of $M_1$ and $M_2$.

In this situation, there is a long exact Mayer-Vietoris sequence (compare \[13, 30\])

\[
\cdots \to K_i(C^*(M_1; A)) \oplus K_i(C^*(M_2; A)) \to K_i(C^*(M; A)) \to K_{i-1}(C^*(M_0; A)) \to K_i(C^*(M_1; A)) \oplus K_i(C^*(M_2; A)) \to \cdots
\]

One of the powerful principles for the K-theory of $C^*$-algebras is their close relation to purely topological quantities via isomorphism conjectures. Most prominent here is the Baum-Connes conjecture for the computation of $K_\ast(C^\Gamma r)$. The properties of $K_\ast(C^*(M; A))$ listed above indicate that a similar “topological expression” should be possible here. Indeed, we have the coarse Baum-Connes conjecture (with coefficients) [23, Conjecture 8.2], verified in many cases.

**Conjecture 4.1.** Given a metric space $X$ of bounded geometry, in the composition

\[
K^\text{lf}_\ast(X; A) \to KX_\ast(X; A) \to K_\ast(C^*(X; A))
\]

the second map is an isomorphism.

Here $K^\text{lf}_\ast(X)$ is the usual locally finite K-homology of the space $X$, defined analytically as $K_{\ast+1}(D^X/C^X)$, and as we saw above it is no problem to introduce as coefficients a $C^*$-algebra $A$. The coarse K-homology $KX_\ast$ is obtained as the limit of $K^\text{lf}_\ast(\{U_i\})$, where the $U_i$ form a sequence of coverings of $X$ which become coarser as $i \to \infty$. Here $|U_i|$ is the geometric realization of the associated Čech simplicial complex. If $X$ is uniformly locally contractible, e.g. if $X$ is the universal covering of a closed non-positively curved manifold, then the “coarsening map” $K^\text{lf}_\ast(X; A) \to KX_\ast(X; A)$ is an isomorphism.

Recall that (in the context of large scale geometry) “bounded geometry” means that $X$ contains a discrete subset $T$ such that on the one hand $T$ coarsely fills the space (i.e. there is an $R > 0$ such that the $R$-neighborhood of $T$ is all of $X$), but on the other hand $T$ is uniformly discrete (i.e. for each $R > 0$ the number of elements of $T$ contained in any $R$-ball is uniformly bounded from above).

The coarse Baum-Connes conjecture has a number of important consequences. Most notably, there is a principle of descent [23, Section 5] that uses the close relation between $C^\Gamma$ and $C^X$ for any metric space $X$ on which $\Gamma$ acts properly and cocompactly. The principle of descent asserts that if such a space satisfies the coarse Baum-Connes conjecture, then the strong Novikov conjecture for $\Gamma$ is true.

On the other hand, the “bounded geometry” condition of the coarse Baum-Connes conjecture is indispensable. Guoliang Yu has constructed a metric space which is a disjoint union of (scaled) spheres of growing dimension for which the analysis of the Dirac operator shows easily that the coarse assembly map is not injective. Despite its simplicity, this example remains intriguing. It is important to understand this better and to construct other examples. We believe that the coarse Baum-Connes conjecture will not be satisfied in full generality.
5. Enlargeability and index

Let $M$ be an (area)-enlargeable closed spin manifold. Recall that this means that $M$ comes with a sequence of (not necessarily compact) coverings $M_i$ with compactly supported maps of non-zero degree $f_i: M_i \to S^n$ which are arbitrarily (area) contracting.

Mikhail Gromov and Blaine Lawson show in [6] that an enlargeable spin manifold does not admit a metric of positive scalar curvature. Theorem 1.6 verifies Conjecture 1.5 for this “enlargeability obstruction”, i.e. shows that the Rosenberg index is non-zero in this situation. This was achieved in [10] by refining the construction of Gromov and Lawson as follows:

1. One constructs vector bundles $E_i$ on $M_i$ of small curvature which represent interesting K-theory classes.

2. Next one produces associated bundles $M(E_i)$ on $M$ (using a kind of push-down). If $M_i \to M$ is a finite covering, this is finite dimensional. If the covering $M_i \to M$ is infinite, we canonically obtain an associated “structure $C^*$-algebra” $C_i$ such that $M(E_i)$ is a Hilbert $C_i$-module bundle with finitely generated projective fiber.

3. The crucial step is the construction of a bundle $E := \prod_i M(E_i)/\bigoplus_i M(E_i)$ which becomes a flat Hilbert $A$-module bundle where $A = \prod_i C_i/\bigoplus_i C_i$. Being flat, this corresponds to a representation of $\pi_1(M)$.

4. As $E$ is flat, the Schrödinger-Lichnerowicz formula implies that $\text{ind}(D_E) \in K_*(A)$ is an obstruction to positive scalar curvature.

5. On the other hand, the universal property of $C^*_{\text{max}}\pi_1(M)$ implies that the representation of $\pi_1(M)$ which gives rise to the bundle $E$ induces a $C^*$-algebra homomorphism $C^*_{\text{max}}\pi_1(M) \to A$. Moreover, the induced map in K-theory sends $\alpha_{\text{max}}(M) \in K_*(C^*_{\text{max}}\pi_1(M))$ to $\text{ind}(D_E) \in K_*(A)$.

6. Finally, an index theorem computes $\text{ind}(D_E)$ in terms of the degrees of the maps $f_i$ and in particular shows that $\text{ind}(D_E) \neq 0$. It follows that $\alpha_{\text{max}}(D) \neq 0 \in K_*(C^*_{\text{max}}\pi_1(M))$.

A main innovation is the technically quite non-trivial construction of an associated honestly flat bundle, but with infinite dimensional fibers.

In [7] we relate enlargeability to the classical strong Novikov conjecture, which deals with $C^*_r\Gamma$ instead of $C^*_{\text{max}}\Gamma$.

The main idea here is to use the functoriality of the large scale index. The argument becomes technically easier if we assume that all the covering spaces which determine the enlargeability of $M$ are the universal covering $\tilde{M}$. In this situation, the first main point is that the geometry allows us to combine all the maps $f_i: \tilde{M} \to S^n$ into one map $F: \tilde{M} \to B_\infty$, where $B_\infty$ is the “infinite balloon space”, sketched in Figure 1. It is defined using a collection of $n$-spheres of increasing radii $i = 1, 2, 3, \ldots$, with the sphere of radius $i$ attached to the point $i \in [0, \infty)$ at the south pole of $S^n$, and is equipped with the path metric.
The topology of positive scalar curvature

Using the Mayer-Vietoris sequence and induction on the dimension, one can calculate the coarse K-homology of $B_\infty$ and the K-theory of its Roe algebra $C^*B_\infty$. In particular, we obtain $KX_n(B_\infty) \cong \prod_{i \in \mathbb{N}} \mathbb{Z}/\bigoplus_{i \in \mathbb{N}} \mathbb{Z}$ and a direct calculation allows us to establish the coarse Baum-Connes conjecture for this space. We obtain a commutative diagram of K-homology and K-theory groups as follows:

$$
\begin{array}{cccc}
K_n(M) & \cong & K_{n+1}(D^*\hat{M}/C^*\hat{M}) & \longrightarrow & K_n(C^*\hat{M}) \\
\downarrow & & \downarrow & & \downarrow \\
K_{n+1}(D^*\hat{M}/C^*\hat{M}) & \longrightarrow & K_n(C^*\hat{M}) & \longrightarrow & K_n(C^*r\Gamma) \\
\downarrow & & \downarrow & & \downarrow \\
K_{n+1}(D^*B_\infty/C^*B_\infty) & \longrightarrow & K_n(C^*B_\infty) & \cong & K_n(B_\infty) \\
\downarrow & & \downarrow & & \downarrow \\
\prod \mathbb{Z}/\bigoplus \mathbb{Z} & \cong & KX_n(B_\infty) & \cong & K_n(C^*B_\infty)
\end{array}
$$

A topological calculation allows one to work out the image of the fundamental class of $M$ in $KX_n(B_\infty) \cong \prod \mathbb{Z}/\bigoplus \mathbb{Z}$: it is the class represented by the sequence of degrees $(\deg(f_i))_{i \in \mathbb{N}}$ which by assumption is non-zero. Because of the coarse Baum-Connes conjecture, the image in the bottom right corner is also non-zero, which finally implies that also the image $\alpha(M) \in K_n(C^*_r\Gamma)$ is non-zero, as claimed by the theorem.

6. Vanishing of the index under partial positivity

The main reason why one can apply index theory to geometric and topological questions is that a special geometric situation implies vanishing results for the index. It is very important to develop further instances of such vanishing theorems, in order to widen the scope of the consequences of the index method. Here we present one of these, which is valid in the context of large scale index theory:

**Theorem 6.1.** Let $M$ be a complete non-compact connected Riemannian spin manifold. Let $E \rightarrow M$ be a flat bundle of Hilbert $A$-modules for a $C^*$-algebra $A$. Assume that the scalar curvature is uniformly positive outside a compact subset.

Then the large scale index of the Dirac operator twisted with $E$ vanishes.
For $A = \mathbb{C}$, this result has been stated by John Roe [23, 25]. A different proof, which covers the general case, is given by Bernhard Hanke, Daniel Pape and the author in [8, Theorem 3.11].

A concrete application of Theorem 6.1 to compact spin manifolds is the codimension-2 obstruction to positive scalar curvature of Theorem 1.7. In its proof in [8], a gluing and bending construction of an intermediate space gives positive scalar curvature outside of a neighborhood of the hypersurface.

7. The Stolz exact sequence

Stephan Stolz (compare [22, Proposition 1.27]) introduced a long exact sequence that makes systematic the bordism classification of metrics of positive scalar curvature. It is quite similar in spirit to the surgery exact sequence for the classification of closed manifolds.

Convention 7.1. Throughout the remainder of the article, a Riemannian metric on a manifold with boundary is assumed to have product structure near the boundary.

Definition 7.2. Fix a reference space $X$.

1. The group $\Omega_{n}^{\text{spin}}(X)$ is the usual spin bordism group, consisting of cycles $f: M \to X$, with $M$ a closed $n$-dimensional spin manifold. The equivalence relation is spin bordism.

2. The structure group $\text{Pos}_{n}^{\text{spin}}(X)$ is the group of bordism classes of metrics of positive scalar curvature on $n$-dimensional closed spin manifolds with reference map to $X$. Two such manifolds $(M_{i}, g_{i}, f_{i}: M_{i} \to X)$ are called bordant if there is a manifold $W$ with boundary, with metric $G$ of positive scalar curvature and with reference map $F: W \to X$ such that its boundary is $M_{1} \cup (-M_{2})$ and $G, F$ restrict to the given $g_{i}, f_{i}$ at the boundary.

3. Finally, the group $R_{n}(X)$ is the group of equivalence classes of compact spin manifolds $W$ with boundary, with reference map $f: W \to X$ and with a metric $g$ of positive scalar curvature on $\partial W$. Again, the equivalence relation on such cycles is bordism, where a bordism between $(W_{1}, f_{1}, g_{1})$ and $(W_{2}, f_{2}, g_{2})$ is a manifold $Y$ with boundary $\partial Y = W_{1} \cup_{\partial W_{1}} Z \cup_{\partial W_{2}} -W_{2}$ (where $Z$ is a spin bordism between $W_{1}$ and $W_{2}$) together with a continuous map $f: Y \to X$ and a positive scalar curvature metric $g$ on $Z$. Of course, the restriction of $f$ to $W_{j}$ must be $f_{j}$ and the restriction of $g$ to $\partial W_{j}$ must be $g_{j}$. It turns out that $R_{n}(X)$ only depends on $\pi_{1}(X)$.

4. The group structure in each of the three cases is given by disjoint union, and the inverse is obtained by reversing the spin structure and leaving all other data unchanged.
5. There are evident "forget structure" and "take boundary" maps between these groups. Using these, one obtains a long exact sequence, the Stolz positive scalar curvature exact sequence

\[ \cdots \to R_{n+1}(\pi_1(X)) \to Pos_{n}^{\text{spin}}(X) \to \Omega_{n}^{\text{spin}}(X) \to \cdots \] (7.3)

The most useful cases of this sequence arises if \( X = M \) and \( f = \text{id} : M \to M \) or if \( X = B\Gamma \) is the classifying space of a discrete group and \( f : M \to B\Gamma \) induces the identity on the fundamental groups.

To get information about \( Pos_{n}^{\text{spin}}(M) \) we want to use index theory systematically by mapping in a consistent way to the analytic exact sequence of Nigel Higson and John Roe. This sequence is simply the long exact K-theory sequence of the extension \( 0 \to C^*\hat{M}^\Gamma \to D^*\hat{M}^\Gamma \to D^*\hat{M}^\Gamma / C^*\hat{M}^\Gamma \to 0 \), where \( \hat{M} \) is the universal covering of \( M \), and \( \Gamma = \pi_1(M) \).

Using that \( K_*^{\text{spin}}(C^*\hat{M}^\Gamma) = K_*^{\text{spin}}(C^*_\Gamma) \) and that \( K_{n+1}(D^*\hat{M}^\Gamma / C^*\hat{M}^\Gamma) = K_*^{\text{spin}}(M) \), we obtain the following theorem, compare [22, Theorem 1.39]

**Theorem 7.4.** Let \( X \) be a topological space with \( \Gamma \)-covering \( \hat{X} \). We have a natural canonical commutative diagram (established with full proof if \( n \) is odd)

\[ \begin{array}{cccccc}
\Omega_{n+1}^{\text{spin}}(X) & \longrightarrow & R_{n+1}(X) & \longrightarrow & Pos_{n}^{\text{spin}}(X) & \longrightarrow & \Omega_{n}^{\text{spin}}(X) \\
\downarrow \beta & & \downarrow \text{ind} & & \downarrow \rho \Gamma & & \downarrow \beta \\
K_{n+1}(X) & \longrightarrow & K_{n+1}(C^*_\Gamma) & \longrightarrow & K_{n+1}(D^*\hat{X}^\Gamma) & \longrightarrow & K_n(X) \\
\end{array} \]

Here, \( \beta \) is obtained by taking the large scale (equivariant) index of the Dirac operator on the covering of a cycle \( f : N \to X \) and then using functoriality of large scale index theory to push forward via \( f_* \) from \( K_*^{\text{spin}}(C^*\hat{N}^\Gamma) \) to \( K_*^{\text{spin}}(C^*\hat{X}^\Gamma) \). It coincides with the Atiyah orientation as natural transformation from spin bordism to K-homology. Similarly, \( \rho \Gamma \) is obtained by constructing the structure invariant of the positive scalar curvature metric of the covering of \( (N, g, f : N \to X) \) and then using naturality to push forward along \( f_* \) from \( K_*^{\text{spin}}(D^*\hat{N}^\Gamma) \) to \( K_*^{\text{spin}}(D^*\hat{X}^\Gamma) \).

Finally, \( \text{ind} \) assigns to a manifold with boundary and positive scalar curvature at the boundary an Atiyah-Patodi-Singer type index.

Note that the assertion of Theorem 8.1 is that the (index based) maps all exist, that they are indeed well defined, i.e. invariant under bordism, and that the diagram is commutative. This means that we have to work (for the cycles and for the equivalence relation) throughout with manifolds with boundary. It turns out that large scale index theory can very elegantly and efficiently be used to carry out index theory on manifolds with boundary, as well.

### 8. Index theory on manifolds with boundary

Our method to do index theory on a manifold with boundary is simply to attach an infinite half-cylinder to the boundary. This produces a manifold without boundary,
of course at the expense that the resulting manifold is never compact. However, large scale index theory can deal with such spaces.

To obtain the appropriate information, the construction must take the extra information into account coming from the fact that the metric of the boundary is assumed to have positive scalar curvature. Let us review the construction:

1. We start with a smooth manifold \( W \) with boundary, with a Riemannian metric \( g \) which has positive scalar curvature near the boundary (and a product structure there, by our general convention). As a metric space, \( W \) is assumed to be complete. Moreover, we fix an auxiliary \( C^* \)-algebra \( A \) and a flat Hilbert \( A \)-module bundle \( E \) on \( W \) (again with product structure near the boundary).

2. We now attach a half-cylinder \( \partial W \times [0, \infty) \) to the boundary to obtain \( W_\infty \) and extend all the structures over \( W_\infty \). We obtain a complete manifold without boundary, with product end \( \partial W \times [0, \infty) \).

3. As in Section 3, the Dirac operator \( D_E \) produces a bounded operator \( \psi(D_E) \) in \( D^*(W_\infty; A) \).

4. Now, however, we use the invertibility of \( D_E \) on \( \partial M \times [0, \infty) \) coming from the Schrödinger-Lichnerowicz formula: for suitable \( \psi \) the element \( 1 - \psi(D_E)^2 \) does not only lie in \( C^*(W_\infty; A) \) but in the smaller ideal \( C^*(W \subset W_\infty; A) \).

5. Correspondingly, the fundamental class of \( W_\infty \) has a canonical lift to a class \( [W, g_{\partial W}] \) in \( K_{n+1}(D^*(W_\infty; A)/C^*(W \subset W_\infty; A)) \). This class does in general depend on the positive scalar curvature metric on the boundary.

6. As usual, we next define the “large scale Atiyah-Patodi-Singer index” by applying the boundary map of the long exact K-theory sequence, now for the extension

\[
0 \to C^*(W \subset W_\infty; A) \to D^*(W_\infty; A) \to D^*(W_\infty; A)/C^*(W \subset W_\infty; A) \to 0,
\]

to obtain \( \text{ind}(D_W, g_{\partial W}) \in K_n(C^*(W \subset W_\infty; A)) \cong K_n(C^*(W; A)) \). The latter isomorphism is induced by the inclusion \( C^*(W; A) \to C^*(W \subset W_\infty; A) \) which just extends the operators by zero. Note that this construction of the index required the invertibility of the operator at the boundary and indeed depends in general on the metric of positive scalar curvature at \( \partial W \).

Note that large scale index theory in the situation we just described produces two invariants which depend on the positive scalar curvature metric on \( \partial W \), namely \( \text{ind}(D_W, g_{\partial W}) \in K_n(C^*(W \subset W_\infty; A)) \), but also the secondary invariant \( \rho(\partial W, g_{\partial W}) \in K_n(D^*(\partial W; A)) \). A major result, which we consider a secondary higher Atiyah-Patodi-Singer index theorem, relates these two.
The topology of positive scalar curvature

Theorem 8.1. (compare [22, Theorem 1.22]) Let \((W, g_W)\) be an even dimensional Riemannian spin-manifold with boundary \(\partial W\) such that \(g_W\) has positive scalar curvature. Assume that a group \(\Gamma\) acts isometrically on \(M\). Then

\[
\iota_* (\text{ind}_\Gamma (D_W)) = j_* (\rho(\partial W, g_{\partial W})) \quad \text{in} \quad K_0(D^*(W)^\Gamma).
\]

Here, we use \(j: D^*(\partial W)^\Gamma \to D^*(W)^\Gamma\) induced by the inclusion \(\partial W \to W\) and \(\iota: C^*(W)^\Gamma \to D^*(W)^\Gamma\) the inclusion.

Remark 8.2. Above we apply the obvious generalization of the construction of the large scale index of Sections 3 and 8 to an equivariant situation, where subalgebras \(D^*(W)^\Gamma\) and \(C^*(W)^\Gamma\) generated by invariant operators are used. This works because the Dirac operator then itself is invariant under the group \(\Gamma\).

Remark 8.3. The heart of the proof of Theorem 8.1 is an explicit secondary index calculation in a model (product) case which is surprisingly intricate.

Remark 8.4. The assertion of Theorem 8.1 should generalize to arbitrary (non-cocompact) spin manifolds, to Hilbert C∗-algebra coefficient bundles and to the K-theory of real C∗-algebras. In a preprint of Zhizhang Xie and Guoliang Yu [34] an argument is sketched which shows how to extend the result to arbitrary dimensions and to real C∗-algebras.

9. Constructions of new classes of metrics of positive scalar curvature

The fundamental idea in the construction of the “geometrically significant” homotopy classes of the space of metrics of positive scalar curvature of Theorem 1.10 is quite old and based on index theory:

Given a closed \(n\)-dimensional spin manifold \(B\) with \(A(B) \neq 0\), we know that \(B\) does not admit a metric of positive scalar curvature.

Remove an embedded disc from \(B\). The result is a manifold \(W\) with boundary \(\partial W = S^{n-1}\). Given any metric of positive scalar curvature on \(W\) (with product structure near the boundary), the corresponding boundary metric \(g_1\) can not be homotopic to the standard metric on \(S^{n-1}\) because then one could glue in the standard disc (with positive scalar curvature) to obtain a metric of positive scalar curvature on \(B\). Now, if \(g_1\) is homotopic to \(\psi^* g_{\text{eucl}}\) for a non-identity diffeomorphism we can glue the disc back in with \(\psi\) to obtain a new manifold \(B_\psi\) which is of positive scalar curvature. Note that \(B_\psi\) is not necessarily diffeomorphic to \(B\), but using the Alexander trick there is a homeomorphism between \(B_\psi\) and \(B\).

As the rational Pontryagin classes and therefore the \(A\)-genus are homeomorphism invariant, \(A(B_\psi) \neq 0\), also \(B_\psi\) can not carry a metric of positive scalar curvature.

Observe that exactly the same argument can be applied to a family situation: Let \(Y \to S^k\) be a family (i.e. bundle) of manifolds with boundary, with boundary \(S^n \times S^k\). Assume there is a family of metrics \(g_x\) of positive scalar curvature on \(Y\)
(product near the boundary). If this family of metrics is homotopic to the constant family consisting of the standard metric (or to a pullback of that one along a family of diffeomorphisms \( \psi_x \)) we can glue in \( D^{n+1} \times S^k \) to obtain a family of closed manifolds which admits a metric of positive scalar curvature (fiberwise, and then also the total space \( X \) admits such a metric). Note that this is only interesting if each \( g_x \) is in the component of the standard metric, which we therefore assume.

Alas: if the total space \( X \) has non-trivial \( \hat{A} \)-genus, this is a contradiction (and again, by the homeomorphism invariance and the Alexander trick the argument works modulo diffeomorphism).

Note that this requires two important ingredients:

1. the topological situation with the bundle \( Y \) (and \( X \))
2. the geometric input of a family of metrics of positive scalar curvature on \( Y \).

It turns out that already the topological input is surprisingly difficult to get. It means that (after the gluing) we have a fiber bundle \( M \to X \to S^k \) of spin manifolds where \( M \) does admit a metric of positive scalar curvature, therefore \( \hat{A}(M) = 0 \), whereas \( \hat{A}(X) \neq 0 \). Note that this means that the \( \hat{A} \)-genus is not multiplicative in fiber bundles, even if the base is simply connected (in contrast to the \( L \)-genus).

In [11, Theorem 1.4] we use advanced differential topology, in particular surgery theory, Casson’s theory of prefibrations and Hatcher’s theory of concordance spaces to prove that the required fiber bundles \( X \) exist:

**Theorem 9.1.** For sufficiently large \( n \), there are 4n-dimensional smooth closed spin manifolds \( X \) with non-vanishing \( \hat{A} \)-genus fitting into a smooth fiber bundle \( F \to X \to S^k \) such that \( F \) admits a metric of positive scalar curvature, is highly connected and the bundle contains as subbundle \( D^{4n-k} \times S^k \).

How about the second ingredient, the existence of the family of metrics of positive scalar curvature on \( Y = X \setminus D^{4n-k} \times S^k \)?

The only tool known which can provide such metrics is the surgery method of Gromov and Lawson. In highly non-trivial work [33] this has been extended by Mark Walsh to families of the kind \( X \) as constructed in Theorem 9.1. To apply this, we use the high connectivity and results of Kiyoshi Igusa on Morse theory for fiber bundles [17]. As a consequence we obtain Theorem 1.10

**10. Open problems**

The geometry of positive scalar curvature and the development and application of large scale index theory is a vibrant field of research, with a host of important open problems. Many of those were already mentioned above; here we want to highlight them and add a couple of further directions of research.

**Gromov-Lawson-Rosenberg conjecture.** We should find further obstructions to positive scalar curvature on spin manifolds, in particular for finite fundamental
group \((\mathbb{Z}/p\mathbb{Z})^n\) for the so-called toral manifolds. We expect that this will require fundamentally new ideas.

On the other hand, can the class of fundamental groups for which the conjecture holds be described systematically?

**Stable Gromov-Lawson-Rosenberg conjecture.** The stable Gromov-Lawson-Rosenberg conjecture and its weaker cousin \[1.5\] which states that “the Rosenberg index sees everything about positive scalar curvature which can be seen using the Dirac operator” follow from the strong Novikov conjecture. It would be spectacular to find counterexamples to either of these (they are expected to exist, but to find them will of course be very hard). It is necessary to investigate this question further. In this context, the role of the codimension-2 obstruction as discussed in Theorem \[1.7\] should be understood.

This theorem should extend to the signature operator, which will shed new light on its meaning. Vaguely, we conjecture the following.

**Conjecture 10.1.** Let \(M_1, M_2\) be two complete non-compact connected oriented Riemannian manifolds and \(f: M_1 \to M_2\) a sufficiently well behaved map which is an “oriented homotopy equivalence near infinity”.

Let \(E \to M_2\) be a flat bundle of Hilbert \(A\)-modules for a \(C^*\)-algebra \(A\). Let \(D_E^{\text{sgn}}\) be the signature operator on \(M_1\) twisted with the flat bundle \(E\), and \(D_{f^*E}^{\text{sgn}}\) the signature operator on \(M_2\) twisted by \(f^*E\). Then the large scale indices of these two operators should coincide, i.e.

\[
f_* (\text{ind}(D_{f^*E}^{\text{sgn}})) = \text{ind}(D_E^{\text{sgn}}) \in K_*(C^*(M_1; A)).
\]

Note that, in this conjecture, one has to work out the precise concept of “sufficiently well behaved” and of “homotopy equivalence at infinity”.

**Area based large scale geometry.** Large scale geometry is based on the metric spaces and distances, viewed from a coarse perspective. Curvature, on the other hand, is a concept based on the bending of surfaces, where scalar curvature looks at the average over all possible surface curvatures through a given point.

This is reflected in the fact that area-enlargeability suffices to obstruct positive scalar curvature (Theorem \[1.6\]). So far, this is not captured well by large scale index theory.

This suggests that a program should be developed for large scale geometry based on 2-dimensional areas instead of lengths. A possible starting point would be to work on a relative of the loop space and carry out the analysis there. This is interesting in its own right, with a host of potential further applications, but seems to require new analytical tools.

Note that the axiomatic abstraction from metric spaces to coarse structures as developed by John Roe \[24\] does not seem to apply here. Of course, this generalization is interesting in its own right and applications to positive scalar curvature should be developed further.
**Coarse Baum-Connes conjecture.** If the coarse Baum-Connes conjecture holds for the classifying space of a discrete group, then also the strong Novikov conjecture is true for this group. Moreover, the validity of the coarse Baum-Connes conjecture is a powerful computational tool. On the other hand, there are enigmatic counterexamples due to Guoliang Yu if one drops the “bounded geometry” condition on the space in question.

We expect that many new classes of metric spaces can be found where the coarse Baum-Connes conjecture can be established. But we also feel that the search for counterexamples (of bounded geometry) should be intensified.

**Aspherical manifolds.** A lot of attention has been given to the special class of aspherical manifolds.

**Conjecture 10.2.** Let $\mathcal{M}$ be a closed manifold whose universal covering is contractible (i.e. $\mathcal{M}$ is aspherical). Then $\mathcal{M}$ does not admit a metric of positive scalar curvature.

Often the geometry implies that a manifold is aspherical (e.g. if it admits a metric which is non-positively curved in the sense of comparison geometry). The conjecture states that in a weak sense this is the only way a manifold can be aspherical. The strong Novikov conjecture for an aspherical spin manifold $\mathcal{M}$ implies that $\alpha(\mathcal{M}) \neq 0$ because the Dirac operator of a manifold $\mathcal{M}$ always represents a non-zero K-homology class in $K_* (\mathcal{M})$, and here $\mathcal{M} = B\pi_1 (\mathcal{M})$. Of course, we now look for ways to directly use the asphericity in proofs of (special cases) of Conjecture 10.2.

**Mapping surgery to analysis to homology.** The program to map surgery to analysis has been fully carried out in [22] only for half the dimensions, and only for complex $C^*$-algebras, based on a delicate explicit index calculation. New developments, in particular the work of Zhizhang Xie and Guoliang Yu [34] extend this to all dimensions with a modified method. It remains to develop the details of this (or an alternative) approach and to carry it over to the more powerful real $C^*$-algebras.

K-theory of $C^*$-algebras is a very powerful tool. Most useful, however is its combination with homological tools (in particular Hochschild and cyclic (co)homology). To achieve this systematically and use it for the classification of metrics of positive scalar curvature, we propose a program to not only map the positive scalar curvature sequence to analysis, as described in Section 7 but then to map further to a corresponding long exact sequence of (cyclic) homology groups. There, one would then see primary and secondary numerical invariants of higher index theory.

The primary invariants are well developed. Rather not understood, however, is the theory related to the secondary invariants. Indeed, the relevant algebra $D^* \mathcal{M}$ is very large and the usual dense and holomorphically closed subalgebras on which explicit formulas for the Connes-Karoubi Chern character would make sense seem hard to come by. Exactly because of this we feel that the development of such a theory will shed important new light on the power of the secondary invariants used in Section 7.
Apart from this general program, the theory described above needs to be applied in more concrete situations. The analytic structure set $K_*(D^*M)$, despite its evident potential, so far has only been used in a small number of concrete contexts (compare in particular [14]). This must change before we will have a definite idea of its power.

**Minimal surface obstructions to positive scalar curvature.** The minimal surface method is the only known tool to obstruct the existence of a metric of positive scalar curvature which works for non-spin manifolds of dimension $\geq 5$. So far, it is controversial how to extend the method if the dimension of the manifold in question is larger than 8 (due to singularities which develop in the minimal hypersurfaces one wants to use). Joachim Lohkamp has a program to achieve this.

The minimal surface technique has so far only been used together with the Gauss-Bonnet theorem (via an iterative approach, until the hypersurface is 2-dimensional). Are there other ways to exploit it and combine it with obstructions to positive scalar curvature (Seiberg-Witten, spin Dirac) and what are the relations?

**Enlargability and non-spin manifolds.** As just one case of the question whether the known results for spin manifolds carry over to non-spin manifolds consider the following:

**Question 10.3.** Let $M$ be an arbitrary (non-spin) closed $n$-dimensional manifold with coverings $M_\epsilon$ and with maps $f_\epsilon: M_\epsilon \to S^n$ which are constant outside a compact subset of $M_\epsilon$, which have non-zero degree and which are $\epsilon$-contracting.

Can $M$ admit a metric of positive scalar curvature?

Using the minimal surface method, Gromov and Lawson [6, Section 12] have shown that this is not the case if $n \leq 7$.

**Topology of the space of metrics of positive scalar curvature.** The stable Gromov-Lawson-Rosenberg conjecture shows that there is a stability feature in the topology of the space $\text{Riem}^+(M)$ of metrics of positive scalar curvature: if index theory suggests that it should be non-empty, this might be violated by $M$ itself, but after iterated product with $B$, eventually it is non-empty. Are there similar stability features concerning the (higher) homotopy groups of $\text{Riem}^+(M)$? It would also be important to develop estimates on the stable range.

**The space of metrics of positive scalar curvature and fundamental group.** We know well that, for a spin manifold $M$ with a complicated fundamental group $\Gamma$, the existence of a metric of positive scalar curvature is not only obstructed by $A(M)$, but by $\alpha(M) \in K_*(C^*\Gamma)$, and many elements of $K_*(C^*\Gamma)$ are indeed realized as values of $\alpha(M)$.

Similarly, we should expect that the topology of $\text{Riem}^+(M)$, if non-empty, should be governed by $K_*(C^*\Gamma)$. At the moment, a precise conjecture (e.g. about the homotopy groups) seems too far-fetched. Still, the methods of large scale and higher index theory should be developed to the point that they are available to detect new elements in $\pi_*(\text{Riem}^+(M))$, and one should systematically construct non-trivial examples.
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