Radius, girth and minimum degree

Vojtěch Dvorák  |  Peter van Hintum  |  Amy Shaw  |  Marius Tiba

Department of Pure Maths and Mathematical Statistics, University of Cambridge, Cambridge, UK

Correspondence
Vojtěch Dvorák, Department of Pure Maths and Mathematical Statistics, University of Cambridge, Wilberforce Rd, CB3 0WA, Cambridge, UK.
Email: vd273@cam.ac.uk

Funding information
UK Research and Innovation; Cambridge Trust; University of Cambridge

Abstract
The objective of the present paper is to study the maximum radius $r$ of a connected graph of order $n$, minimum degree $\delta \geq 2$ and girth at least $g \geq 4$. Erdős, Pach, Pollack and Tuza proved that if $g = 4$, that is, the graph is triangle-free, then $r \leq \frac{n-2}{\delta} + 12$, and noted that up to the value of the additive constant, this upper bound is tight. In this paper we shall determine the exact maximum. For larger values of $g$ little is known. We settle the order of the maximum $r$ for $g = 6, 8$ and $12$, and prove an upper bound for every even $g$, which we conjecture to be tight up to a constant factor. Finally, we show that our conjecture implies the so-called Erdős girth conjecture.

KEYWORDS
extremal graph theory, girth, minimum degree, radius, triangle-free graphs

1 | INTRODUCTION

The girth of a graph $G$ is the length of the shortest cycle in $G$; we set the girth to be $\infty$ if no cycle exists. The radius $r$ of a connected graph $G$ is the smallest integer such that there exists some $v \in V(G)$ with $d(v, w) \leq r$ for every $w \in V(G)$.

Consider the following question: given a connected graph $G$ on $n$ vertices, with minimum degree $\delta \geq 2$ and girth at least $g \geq 4$, what is the maximum radius $r$ this graph can have (note that the connectedness condition is superfluous if we let $r$ be the biggest radius of a connected component)? Denote this maximum value of the radius as $r(n, \delta, g)$.
Erdős, Pach, Pollak and Tuza [2] studied \( r(n, \delta, 4) \), and proved that it is at most \( \frac{n-2}{\delta} + 12 \). They also noted that, up to the additive constant 12, this bound is tight. We improve this to a best possible bound.

**Theorem 1.1.** Fix integer \( \delta \geq 2 \).

- If \( 2\delta \leq n \leq 2\delta + 1 \), then \( r(n, \delta, 4) = 2 \).
- If \( 2\delta + 2 \leq n < 4\delta \), then \( r(n, \delta, 4) = 3 \).
- If \( n \geq 4\delta \), then
  \[
  r(n, \delta, 4) = \begin{cases} 
  \left \lfloor \frac{n}{2} \right \rfloor - 1 & \text{if } \delta \text{ is odd and } n = k \delta \text{ for } k \text{ odd,} \\
  \left \lfloor \frac{n}{2} \right \rfloor & \text{otherwise.}
  \end{cases}
  \]

Observe that every graph of order \( n \) and minimum degree greater than \( n/2 \) has a triangle, so in the study of \( r(n, \delta, 4) \), we may assume that \( n \geq 2\delta \).

Next we consider the case when the girth \( g \) is bigger than 4. We shall prove the following upper bound.

**Theorem 1.2.** Let \( n, \delta \geq 2 \) and \( g = 2k \geq 4 \). Then
\[
r(n, \delta, 2k) \leq \frac{nk(\delta - 2)}{2((\delta - 1)^k - 1)} + 3k.
\]

In the cases \( g = 6, 8, 12 \), we shall prove the following lower bound.

**Theorem 1.3.** Let \( \delta \geq 2 \) be such that \( \delta - 1 \) is a prime power. Then there exists sequences \((n_i), (n'_i), (n''_i)\) with \( n_i, n'_i, n''_i \to \infty \) such that

- \( r(n_i, \delta, 6) \geq \frac{3n_i}{2(\delta^2 - \delta + 1)} - 3 = \frac{3n_i(\delta - 2)}{2((\delta - 1)^3 - 1)} - 3 \),
- \( r(n'_i, \delta, 8) \geq \frac{2n'_i}{\delta^3 - 2\delta^2 + 2\delta} - 4 = \frac{2n'_i(\delta - 2)}{(\delta - 1)^4 - 1} - 4 \),
- \( r(n''_i, \delta, 12) \geq \frac{3n''_i}{((\delta - 1)^3 + 1)(\delta^2 - \delta + 1)} - 6 = \frac{3n''_i(\delta - 2)}{(\delta - 1)^6 - 1} - 6 \).

We note that the results for the girth 6, 8 and 12 are optimal up to the value of the additive constant, as established by Theorem 1.2. We are very grateful to the anonymous referee for pointing out that our previous approach can be optimised to obtain the correct bounds even for all lower-order terms.

It would be interesting to see whether the upper bound from Theorem 1.2 is tight, at least up to some constant factor. We believe it is and hence make the following conjecture.

**Conjecture 1.4.** Let \( g = 2k \geq 4 \). Then there exist infinitely many values \( \delta \) for which the following holds.
There exists a sequence \((n(\delta)_i)\) with \(n(\delta)_i \to \infty\) and a positive constant \(c(\delta)\) such that \(r(n(\delta)_i, \delta, 2k) \geq c(\delta) \frac{n(\delta)_i}{\delta + 1}\).

As our final result, we obtain the following theorem.

**Theorem 1.5.** Let \(r, c > 0\), \(g = 2k\) and \(n \leq c(r + 1)\delta^{k-1}\), so that \(r(n, \delta, g) \geq r\). Then there exists a connected graph of girth at least \(2k\) on at most \((2k + 1)c\delta^{k-1}\) vertices with at least \(\frac{1}{2}\delta^2(\delta - 1)^{k-2}\) edges.

This theorem is related to the following girth conjecture of Erdős.

**Conjecture 1.6** (Erdős [1]). For any positive integers \(l, n\), there exists a graph with girth \(2l + 1\), \(n\) vertices and \(\Omega(n^{1+\frac{1}{l}})\) edges.

If the upper bound from Theorem 1.2 was tight up to a constant factor for some fixed \(g = 2k\), then we could find graphs \(G_i\) with \(\delta_i \to \infty\) and \(n_i \leq c(r(n_i, \delta_i, 2k) + 1)\delta_i^{k-1}\) for some fixed \(c\). By Theorem 1.5, that would verify the girth conjecture of Erdős for \(l = k - 1\).

We note that some similar problems relating to various parameters in a graph have been studied in the literature—for instance, the analogous problem for the diameter instead of the radius [2,4], and problems involving more detailed information about the degree sequence of the graph [5].

The structure of the paper is as follows: In Lemma 2.1, we establish a general tool that gives a lower bound on \(n\) in terms of \(r\) and \(\delta\) that is tight in many cases. This lemma unfortunately is not strong enough to handle all cases, so we prove the additional Lemma 3.6. We use these two lemmas (the key ingredients of the proof) to prove Lemma 3.4 which establishes the upper bound on \(r\) in Theorem 1.1. Together with Lemmas 3.1 and 3.3, which establish the lower bound on \(r\), this completes our proof. Finally, in Section 4, we consider the case of general girth.

Throughout the paper, for a vertex \(v\) in a graph, we will denote by \(N(v)\) its open neighbourhood, and by \(N[v]\) its closed neighbourhood. The difference between closed and open neighbourhoods is that the closed one contains also the vertex \(v\) itself.

## 2 | STRATEGY

The following lemma will be used throughout our paper. It tells us that if we can find a large collection of vertices in our graph such that any two vertices are either neighbours or sufficiently far away from each other, then our graph must in fact have many vertices.

We thank the anonymous referee for pointing out to us how to improve the lemma.

**Lemma 2.1.** Assume \(G\) is a graph on \(n\) vertices of girth \(g \geq 2k\) (where \(k \geq 2\)) with minimum degree \(\delta\). Let \(T \subset V(G)\) be such that all pairs of nonadjacent vertices in \(T\) have distance at least \(2k - 1\) from each other. Then we have \(n \geq |T|\left(\frac{(\delta - 1)^{k-1}}{\delta - 2}\right)\). Moreover if \(|T|\) is odd, this inequality is strict.

Note that for \(k = 2\), that is, the triangle-free case, this means that \(n \geq |T|\delta\) and if \(|T|\) is odd, then \(n > |T|\delta\).
Proof. Since $G$ contains no triangles, we know we can label

$$T = \{x_1, y_1, \ldots, x_i, y_i, z_i, \ldots, z_j\},$$

with $d(x_l, y_l) = 1$ for $1 \leq l \leq i$, while all other pairs of vertices in $T$ have distance at least $2k - 1$.

For $v \in T$, let

$$S(v) = \{w \in V(G) \mid d(v, w) = k - 1\}$$

and

$$B(v) = \{w \in V(G) \mid d(v, w) \leq k - 1\}.$$

Consider the sets $B(x_i), S(y_i) \setminus B(x_i), \ldots, B(x_i), S(y_j) \setminus B(x_i), B(z_i), \ldots, B(z_j)$. Note that by the distance condition, all these sets are disjoint. Moreover, by the girth condition, for any $v \in T$, we have

$$|B(v)| \geq 1 + \delta(1 + (\delta - 1) + \cdots + (\delta - 1)^{k-2}) = 1 + \delta\frac{(\delta - 1)^{k-1} - 1}{\delta - 2}$$

and that for any $1 \leq l \leq i$,

$$|S(y_j) \setminus B(x_i)| \geq (\delta - 1)^{k-1}.$$

We conclude

$$n = |V(G)| \geq \left| \bigcup_{i=1}^{j} B(x_i) \cup \bigcup_{l=1}^{i} S(y_l) \setminus B(x_i) \cup \bigcup_{l=1}^{j} B(z_l) \right|$$

$$= \sum_{i=1}^{j} |B(x_i)| + \sum_{l=1}^{i} |S(y_l) \setminus B(x_i)| + \sum_{l=1}^{j} |B(z_l)|$$

$$\geq (|T| - i) \left(1 + \delta\frac{(\delta - 1)^{k-1} - 1}{\delta - 2}\right) + i(\delta - 1)^{k-1}$$

$$\geq \frac{|T|}{2} \left(1 + \delta\frac{(\delta - 1)^{k-1} - 1}{\delta - 2} + (\delta - 1)^{k-1}\right)$$

$$= \frac{|T|}{2} \left(\frac{(\delta - 1)^{k-1} - 1}{\delta - 2} + (\delta - 1)^{k-1}\right).$$

Moreover, when $|T|$ is odd, the third inequality above is strict (as $|B(v)| > |S(v)|$ for any $v$ and we cannot have $j = 0$ in the odd case). Hence, the proof is finished. \hfill \Box

To find such large collections of points $T$ with restricted distances, we shall use several observations. We formulate these observations used throughout the proof in the following general setting.

Let $G$ be a graph with $n$ vertices and radius $r$. We take $v_0$ to be some fixed centre of $G$. We let $v_r$ be a vertex such that $d(v_0, v_r) = r$, and let $v_0, v_1, \ldots, v_r$ be a path of length $r$ from $v_0$ to $v_r$. 
Fix an integer $m \in \{1, \ldots, r - 1\}$, and let $v'$ be a vertex such that $d(v_m, v') \geq r$. Then let $t \geq 0$ be such that $d(v_0, v') = r - t$, and let $v_0 = v_0', v_1', \ldots, v_{r-t}' = v'$ be a path of length $r - t$ from $v_0$ to $v' = v_{r-t}'$.

**Observation 2.2.** We have $t \leq m$.

**Proof of Observation 2.2.** Assume for contradiction that we had $t > m$. Then by a triangle inequality

$$d(v_m, v_{r-t}') \leq d(v_m, v_0) + d(v_0, v_{r-t}') = m + (r - t) < r,$$

which is a contradiction. \(\square\)

**Observation 2.3.** For any $m \leq i \leq r$ and any $0 \leq j \leq r - t$, we have

$$d(v_i, v_j') \geq d(v_m, v_{r-t}') + i + j - m - r.$$

and for any $i < m$ and any $0 \leq j \leq r - t$, we have

$$d(v_i, v_j') \geq d(v_m, v_{r-t}') + i + j - m - r.$$

Moreover, in either of these cases, we also have

$$d(v_i, v_j') \geq |i - j|.$$

**Proof of Observation 2.3.** For the case $m \leq i \leq r$, note that

$$d(v_m, v_{r-t}') \leq d(v_m, v_i) + d(v_i, v_j') + d(v_j', v_{r-t}') = (i - m) + d(v_i, v_j') + (r - t - j).$$

Rearranging gives the result.

For the case $i < m$, note that

$$d(v_m, v_{r-t}') \leq d(v_m, v_i) + d(v_i, v_j') + d(v_j', v_{r-t}') = (m - i) + d(v_i, v_j') + (r - t - j).$$

Rearranging gives the result.

For the last claim, note that by triangle inequality

$$d(v_i, v_j') \geq |d(v_i, v_0) - d(v_0, v_j')| = |i - j|.$$

\(\square\)
Observation 2.4. We cannot have \( v_i = v'_i \) for any \( i > \frac{m + r - t - d(v_0, v'_0)}{2} \), and we cannot have \( v_i = v'_j \) for any \( i \neq j \).

Proof of Observation 2.4. Assume that \( v_i = v'_i \) for some \( r - t \geq i > \frac{m + r - t - d(v_0, v'_0)}{2} \). Then we obtain contradiction, as \( d(v_i, v'_i) > 0 \) by Observation 2.3.

We cannot have \( v_i = v'_j \) for any \( i \neq j \), since

\[
d(v_0, v_i) = i 
eq j = d(v_0, v'_j).
\]

\[\square\]

Now we are ready to move on to the case of the triangle-free graphs.

3 | TRIANGLE-FREE GRAPHS

To prove Theorem 1.1, we will establish the following four lemmas.

**Lemma 3.1.** Fix integers \( n \geq 4, \delta \geq 2 \). If \( n \geq 2\delta \), there exists a connected triangle-free graph on \( n \) vertices with minimum degree \( \delta \) and radius 2. If \( n \geq 2\delta + 2 \), there exists a connected triangle-free graph on \( n \) vertices with minimum degree \( \delta \) and radius 3.

**Lemma 3.2.** Every connected triangle-free graph on \( n \) vertices with minimum degree \( \delta \geq 2 \) and radius \( r \) satisfies \( r \geq 2 \) and \( n \geq 2\delta \). Moreover, if \( r = 3 \), we have \( n \geq 2\delta + 2 \).

**Lemma 3.3.** Fix integers \( r \geq 4, \delta \geq 2, c \geq 0 \). There exists a connected triangle-free graph with \( 2\left\lceil \frac{r\delta}{2} \right\rceil + c \) vertices, minimum degree \( \delta \) and radius \( r \).

**Lemma 3.4.** Every connected triangle-free graph on \( n \) vertices with minimum degree \( \delta \geq 2 \) and radius \( r \geq 4 \) satisfies \( n \geq 2\left\lceil \frac{r\delta}{2} \right\rceil \).

Let us first see how Theorem 1.1 follows from these.

**Proof of Theorem 1.1 assuming Lemma 3.1, 3.2, 3.3, 3.4.**

Case: \( 2\delta \leq n \leq 2\delta + 1 \).

As \( 2\delta \leq n \), Lemma 3.1 shows \( r(n, \delta, 4) \geq 2 \). As \( n < 2\delta + 2 \leq 4\delta \), Lemma 3.4 shows \( r(n, \delta, 4) < 4 \) and Lemma 3.2 shows \( r(n, \delta, 4) \neq 3 \). We conclude \( r(n, \delta, 4) = 2 \).

Case: \( 2\delta + 2 \leq n \leq 4\delta - 1 \).

As \( 2\delta + 2 \leq n \), Lemma 3.1 shows that \( r(n, \delta, 4) \geq 3 \). As \( n < 4\delta \), Lemma 3.4 shows \( r < 4 \). We conclude \( r(n, \delta, 4) = 3 \).

Case: \( 4\delta \leq n \).

In this case, we consider two subcases depending on the precise form of \( n \).
Subcase: \( n = k\delta \) with \( \delta \) and \( k \) both odd.

We set \( r = \frac{n}{\delta} - 1 \), \( c = \delta \) and we show that \( r(n, \delta, 4) = r \).

By Lemma 3.3, there exists a connected triangle-free graph with \( 2\left\lfloor \frac{r\delta}{2} \right\rfloor + c = n \) vertices, minimum degree \( \delta \) and radius \( r \), and hence \( r(n, \delta, 4) \geq r \).

First consider the case \( r(n, \delta, 4) < 4 \). As \( n \geq 4\delta \), we also have \( r(n, \delta, 4) < \frac{n}{\delta} \) and hence \( r(n, \delta, 4) \leq r \), finishing this case. So further assume \( r(n, \delta, 4) \geq 4 \). By Lemma 3.4, every connected triangle-free graph on \( n \) vertices and of minimum degree \( \delta \geq 2 \) and radius \( r(n, \delta, 4) \geq 4 \) satisfies \( n \geq 2\left\lfloor \frac{r(n, \delta, 4)\delta}{2} \right\rfloor \). As \( n \) is odd integer and \( 2\left\lfloor \frac{r(n, \delta, 4)\delta}{2} \right\rfloor \) is an even integer, we must even have \( n - 1 \geq 2\left\lfloor \frac{r(n, \delta, 4)\delta}{2} \right\rfloor \). So we get \( n - 1 \geq r(n, \delta, 4)\delta \).

Therefore, \( r(n, \delta, 4) < \frac{n}{\delta} \) and hence \( r(n, \delta, 4) \leq r \).

Subcase: \( n \) is not of the form \( k\delta \) with \( \delta \) and \( k \) both odd.

We set \( r = \left\lfloor \frac{n}{\delta} \right\rfloor \) and \( c = n - 2\left\lfloor \frac{r\delta}{2} \right\rfloor \) and show that \( r(n, \delta, 4) = r \).

By Lemma 3.3, there exists a connected triangle-free graph with \( 2\left\lfloor \frac{r\delta}{2} \right\rfloor + c = n \) vertices, minimum degree \( \delta \) and radius \( r \), and hence \( r(n, \delta, 4) \geq r \).

First once again consider the case \( r(n, \delta, 4) < 4 \). As \( n \geq 4\delta \), we also have \( r(n, \delta, 4) \leq \frac{n}{\delta} \) and hence \( r(n, \delta, 4) \leq r \), completing the proof in this case. Hence further assume \( r(n, \delta, 4) \geq 4 \). By Lemma 3.4, every connected triangle-free graph on \( n \) vertices and of minimum degree \( \delta \geq 2 \) and radius \( r(n, \delta, 4) \geq 4 \) satisfies \( n \geq 2\left\lfloor \frac{r(n, \delta, 4)\delta}{2} \right\rfloor \). Therefore, \( r(n, \delta, 4) \leq \frac{n}{\delta} \) and hence \( r(n, \delta, 4) \leq r \).

\square

In the rest of the section, we will prove Lemmas 3.1–3.4 and thus prove Theorem 1.1. The section will be divided into five subsections—in Section 3.1 we prove Lemma 3.3; in Section 3.2 we prove a technical lemma and we will need to prove Lemma 3.4; in Section 3.3 we prove Lemmas 3.1 and 3.2; in Section 3.4 we prove Lemma 3.4 when \( r \in \{4k, 4k + 1, 4k + 2\} \); and in Section 3.5 we prove Lemma 3.4 when \( r = 4k + 3 \).

3.1 Proof of Lemma 3.3

It suffices to prove the lemma for \( c = 0 \). Indeed, given a triangle-free graph \( G \), we can add a vertex while preserving both the radius and the minimum degree: if \( v \in V(G) \) is such that \( d(v) = \delta \), then add a vertex \( v' \) to \( V(G) \), which is connected precisely to the same vertices as \( v \) is.

For \( c = 0 \), consider the following example.

Partition \( V(G) \) into \( 2r \) sets labelled \( B_0, ..., B_{2r-1} \) such that

\[
|B_i| = \begin{cases} \left\lfloor \frac{\delta}{2} \right\rfloor & \text{if } i \equiv 0, 1 \mod 4, \\ \left\lfloor \frac{\delta}{2} \right\rfloor & \text{if } i \equiv 2, 3 \mod 4. \end{cases}
\]
Connect all vertices in $B_i$ to all vertices in $B_j$ whenever $i - j \equiv \pm 1 \text{ mod } 2r$. Example of such a graph with $r = 5$ and $\delta = 6$ is depicted in Figure 1.

It is easy to see that this is a connected triangle-free graph with $2 \left\lfloor \frac{\delta^2}{2} \right\rfloor$ vertices, minimum degree $\delta$ and radius $r$.

### 3.2 Technical lemma

First, recall Lemma 2.1 which implies the following result for triangle-free graphs.

**Lemma 3.5.** Let $G$ be a triangle-free graph on $n$ vertices and with minimum degree $\delta$. Then for any subset $T \subset V(G)$ such that no two vertices of $T$ are at distance 2, we have $n \geq 2 \left\lceil \frac{\delta |T|}{2} \right\rceil$.

We will also need another lemma of similar flavour here.

**Lemma 3.6.** Let $G$ be a triangle-free graph on $n$ vertices and with minimum degree $\delta$. Assume for some $r \geq 4$, we have a subset $U \subset V(G)$ such that $|U| = 2r$ and $U$ is as follows:
if we consider auxiliary graph $H$ such that $V(H) = U$ and in which we connect two vertices if their distance in $G$ is precisely 2, then $H$ is a disjoint union of two cycles of length $r$. Then we have $n \geq 2 \left\lceil \frac{r \delta}{2} \right\rceil$.

**Proof:** Let $c_1, \ldots, c_r$ and $d_1, \ldots, d_r$ be our two cycles of length $r$ in $H$. Consider the open neighbourhoods $N(c_1), \ldots, N(c_r)$. On the one hand, we have $|N(c_i)| \geq \delta$ for $1 \leq i \leq r$. On the other hand, each $v \in V(G)$ can be contained in the neighbourhood of at most two vertices from $\{c_1, \ldots, c_r\}$ by our triangle-free condition.

Further, as no $c_i$ and $d_j$ are at distance 2, and $G$ is triangle-free, no vertex can be contained both in an $N(c_i)$ and in an $N(d_j)$.

Let $B = \bigcup_i N(c_i)$, so that by the above discussion

$$2|B| \geq |N(c_1)| + \cdots + |N(c_r)| \geq r \delta.$$ 

Since $|B|$ is an integer, we have $|B| \geq \left\lceil \frac{r \delta}{2} \right\rceil$.

Let $B' = \bigcup_j N(d_j)$, so that similarly we get $|B'| \geq \left\lceil \frac{r \delta}{2} \right\rceil$. As $B, B'$ are disjoint, we conclude $n \geq 2 \left\lceil \frac{r \delta}{2} \right\rceil$. \qed

### 3.3 | Proof of Lemmas 3.1 and 3.2

Here we handle the small radius cases.

**Proof of Lemma 3.1.** If $n \geq 2 \delta$, note that $K_{\delta, n-\delta}$ is a connected triangle-free graph on $n$ vertices of radius 2 and minimum degree $\delta$.

If $n \geq 2 \delta + 2$, start with a complete bipartite graph $K_{\delta+1, n-\delta-1}$ with vertex classes $\{v_1, \ldots, v_{\delta+1}\}$ and $\{w_1, \ldots, w_{n-\delta-1}\}$. Erase the edges $v_1w_1, v_2w_2, \ldots, v_{\delta+1}w_{\delta+1}, v_{\delta+1}w_{\delta+2}, \ldots, v_{\delta+1}w_{n-\delta-1}$.

The resulting graph is a connected triangle-free graph on $n$ vertices of radius 3 and minimum degree $\delta$. \qed

**Proof of Lemma 3.2.** Consider a connected triangle-free graph $G$ on $n$ vertices of radius $r$ and minimum degree $\delta \geq 2$. We must have $r \geq 2$, since the only connected triangle-free graphs of radius 1 are star graphs, which have minimum degree 1.

Consider adjacent vertices $a, b \in V(G)$. It follows from Lemma 3.5 applied to $T = \{a, b\}$ that $n \geq 2 \delta$.

If $r = 3$, then we can take $a, b$ which instead satisfy $d(a, b) \geq 3$. But then even their closed neighbourhoods are disjoint, which implies

$$|V(G)| \geq |N[a] \cup N[b]| = |N[a]| + |N[b]| \geq 2\delta + 2.$$ \qed
3.4 Proof of Lemma 3.4 for \( r = 4k, 4k + 1, 4k + 2 \)

In this subsection, we prove the following.

**Lemma 3.7.** If \( G \) is a connected triangle-free graph on \( n \) vertices with minimum degree \( \delta \geq 2 \) and radius \( r \geq 4 \) such that \( r = 4k + i \) for some \( k \) and some \( i \in \{0, 1, 2\} \), then we have \( n \geq 2^{\left\lceil \frac{r^2}{2} \right\rceil} \).

*Proof:* Let \( v_0 \) be a centre of our graph \( G \) with a minimal number of vertices at distance \( r \). Let \( v_r \) be any vertex such that \( d(v_0, v_r) = r \). Let \( v_0, v_1, ..., v_r \) be a path of length \( r \) from \( v_0 \) to \( v_r \).

Let \( v'_{r-1} \) be the following vertex: if \( v_3 \) is not a centre of \( G \), then let \( v'_{r-1} \) be any vertex such that \( d(v_3, v'_{r-1}) \geq r + 1 \). If \( v_3 \) is a centre of \( G \), then let \( v'_{r-1} \) be a vertex such that \( d(v_3, v'_{r-1}) = r \) and \( d(v_0, v'_{r-1}) < r \) (such a vertex exists by a choice of \( v_0 \)).

Let \( t \) be so that \( d(v_0, v'_{r-1}) = r - t \). Let \( v_0, v_1', v_2', ..., v'_{r-1} \) be a path of length \( r - t \) from \( v_0 \) to \( v'_{r-1} \). It follows from Observation 2.2 that \( t \leq 3 \).

**Claim 3.8.** If \( r = 4k + 1 \) and \( 0 \leq t \leq 2 \), or \( r \in \{4k, 4k + 2\} \), then \( n \geq 2^{\left\lceil \frac{r^2}{2} \right\rceil} \).

*Proof of Claim 3.8.* We show that in each of these cases, we can find a collection \( C \) of \( r \) vertices in \( G \) such that no two are at distance 2. The result then follows from Lemma 3.5.

Depending on the values of \( r \) and \( t \), choose \( C \) to be the following collection.

| \( r = 4k \) | \( t = 0 \) | \( v_3, v_4, v_5, v_6, ..., v_{4k-1}, v_{4k}, v'_3, v'_4, v'_5, v'_6, ..., v'_{4k-1}, v'_{4k} \) |
| \( r = 4k \) | \( t = 1 \) | \( v_0, v_3, v_4, v_5, v_6, ..., v_{4k-1}, v_{4k}, v'_3, v'_4, v'_5, v'_6, ..., v'_{4k-5}, v'_{4k-4}, v'_{4k-1} \) |
| \( r = 4k \) | \( t = 2 \) | \( v_3, v_4, v_5, v_6, ..., v_{4k-1}, v_{4k}, v'_3, v'_4, v'_5, v'_6, ..., v'_{4k-3}, v'_{4k-2} \) |
| \( r = 4k \) | \( t = 3 \) | \( v_0, v_3, v_4, v_5, v_6, ..., v_{4k-1}, v_{4k}, v'_1, v'_3, v'_4, v'_5, v'_6, ..., v'_{4k-4}, v'_{4k-3} \) |
| \( r = 4k + 1 \) | \( t = 0 \) | \( v_0, v_4, v_5, v_6, ..., v_{4k}, v_{4k+1}, v'_4, v'_5, v'_6, ..., v'_{4k-1}, v'_{4k} \) |
| \( r = 4k + 1 \) | \( t = 1 \) | \( v_0, v_3, v_4, v_5, v_6, ..., v_{4k}, v_{4k+1}, v'_3, v'_4, v'_5, v'_6, ..., v'_{4k-1}, v'_{4k} \) |
| \( r = 4k + 1 \) | \( t = 2 \) | \( v_3, v_4, v_5, v_6, ..., v_{4k}, v_{4k+1}, v'_3, v'_4, v'_5, v'_6, ..., v'_{4k-5}, v'_{4k-4}, v'_{4k-1} \) |
| \( r = 4k + 2 \) | \( t = 0 \) | \( v_0, v_3, v_4, v_5, v_6, v_{10}, ..., v_{4k+1}, v_{4k+2}, v'_3, v'_4, v'_5, v'_6, ..., v'_{4k-1}, v'_{4k} \) |
| \( r = 4k + 2 \) | \( t = 1 \) | \( v_0, v_3, v_4, v_5, v_6, ..., v_{4k}, v_{4k+1}, v'_3, v'_4, v'_5, v'_6, ..., v'_{4k-1}, v'_{4k} \) |
| \( r = 4k + 2 \) | \( t = 2 \) | \( v_0, v_3, v_4, v_5, v_6, v_{10}, ..., v_{4k}, v_{4k+1}, v_{4k+2}, v'_3, v'_4, v'_5, v'_6, ..., v'_{4k-2}, v'_{4k-1} \) |
| \( r = 4k + 2 \) | \( t = 3 \) | \( v_1, v_2, v_5, v_6, v_{10}, ..., v_{4k+1}, v_{4k+2}, v'_3, v'_4, v'_5, v'_6, ..., v'_{4k-2}, v'_{4k-1} \) |

**Subclaim:** \( |C| = r \), and if \( v_i, v_j, v'_i, v'_j \in C \), then \( d(v_i, v_j), d(v'_i, v'_j) \neq 2 \).

*Proof of subclaim.* None of the collections above contains both \( v_i \) and \( v'_i \). For all other pairs \( v_i, v'_j \), it follows from Observation 2.4 that \( v_i \neq v'_j \). Hence, \( C \) consists of \( r \) distinct vertices.
Note that $v_0, ..., v_r$ is a path of length $r$ and $v'_0, ..., v'_{r-t}$ is an induced path of length $r - t$. Hence, $C$ contains no two vertices of the form $v_i, v_j$ such that $d(v_i, v_j) = 2$ and no two vertices of the form $v'_i, v'_j$ such that $d(v'_i, v'_j) = 2$. \[\square\]

**Subclaim:** If $v_i, v'_j \in C$, then $d(v_i, v'_j) \neq 2$.

*Proof of subclaim.* If $|i - j| > 2$, the claim follows from Observation 2.4. Henceforth assume $|i - j| \leq 2$.

**Case:** $i = 1$.

It follows from Observation 2.3 that it suffices to ensure that if our collection contains $v_1$, then it does not contain:

- $v'_1$ in the case $v_1 \neq v'_1$,
- $v'_2$ in the case $d(v_3, v'_{r-t}) + t \leq r + 2$,
- $v'_3$ in the case $d(v_3, v'_{r-t}) + t \leq r + 1$.

Recall that if $t = 0$, then $d(v_3, v'_{r-t}) \geq r + 1$. Hence, we easily verify that $C$ satisfies these conditions.

**Case:** $i = 2$.

It follows from Observation 2.3 that it suffices to ensure that if our collection contains $v_2$, then it does not contain:

- $v'_1$ in the case $d(v_3, v'_{r-t}) + t \leq r + 2$,
- $v'_2$ in the case $d(v_3, v'_{r-t}) + t \leq r + 1$.

We again can verify easily that all of the collections above satisfy these conditions.

**Case:** $i \geq 3$.

Note that by our choice of $v_0, v'_{r-t}$, we always have either $t \geq 1$ or $d(v_3, v'_{r-t}) \geq r + 1$. If $j \geq i - 1$, it follows from Observation 2.3 that $d(v_j, v'_j) \geq 3$. If $j = i - 2$, $d(v_i, v'_j) \geq 3$ follows from Observation 2.3 under additional assumption that $d(v_3, v'_{r-t}) + t \geq r + 2$. Hence, for $i \geq 3$, it is enough if $C$ does not contain both $v_i$ and $v'_{r-t}$ in the case when we have $d(v_3, v'_{r-t}) + t \leq r + 1$. But it is easy to check this condition is satisfied for all of the collections above. \[\square\]

The claim follows.

**Claim 3.9.** If $r = 4k + 1$ and $t = 3$, we have $n \geq 2 \left\lfloor \frac{rs}{2} \right\rfloor$.

*Proof of Claim 3.9.* We let $v''_{r-s}$ be such a vertex that $d(v_i, v''_{r-s}) \geq r$, then $d(v_0, v''_{r-s}) = r - s$ for some $0 \leq s \leq 1$. We consider two cases based on the value of $d(v''_{r-s}, v'_{4k-2})$.

**Case:** $d(v''_{r-s}, v'_{4k-2}) \geq 3$. 


Let
\[ T = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}, v'_1, v'_2, v'_5, v'_6, \ldots, v'_{4k-3}, v'_{4k-2}, v''_{r-s}\}. \]

Assume for a contradiction two vertices of \( T \) have distance 2. It follows from Observation 2.3 that one of them has to be \( v''_{r-s} \). Since for any \( v, w \in V(G) \), we have
\[
d(v, w) \geq |d(v, v_0) - d(w, v_0)|
\]
and as also \( d(v''_{r-s}, v'_{4k-2}) \geq 3 \) by assumption, it further follows that the other vertex would have to be \( v_{4k-2} \) or \( v_{4k-1} \). Note that if \( d(v_i, v''_{r-s}) \leq 2 \) for some \( 1 \leq i \leq 4k - 1 \), then
\[
d(v_i, v''_{r-s}) \leq d(v_i, v'_l) + d(v_i, v''_{r-s}) \leq (4k - 2) + 2 < r
\]
yielding a desired contradiction. Hence, no two vertices of \( T \) have distance 2 while \( |T| = r \). The result then follows from Lemma 3.5.

Case: \( d(v'_s, v'_{4k-2}) < 3 \).

Since
\[
d(v''_{r-s}, v'_{4k-2}) \geq |d(v''_{r-s}, v_0) - d(v'_{4k-2}, v_0)| \geq 3 - s \geq 2,
\]
this means \( s = 1 \) and \( d(v''_{r-1}, v'_{4k-2}) = 2 \). Hence there exists a vertex \( a \), such that \( a \) is neighbour of both \( v''_{r-1} \) and \( v'_{4k-2} \). Moreover, clearly \( d(a, v_0) = r - 2 \).

Consider two cases based on the value of \( d(a, v_{4k+1}) \).

Subcase: \( d(a, v_{4k+1}) \geq 3 \).

Take
\[ T = \{v_1, v_2, v_5, v_6, \ldots, v_{4k-3}, v_{4k-2}, v_{4k+1}, v'_2, v'_3, v'_6, v'_7, \ldots, v'_{4k-6}, v'_{4k-5}, v'_{4k-2}, a\}. \]

Assume for a contradiction two vertices of \( T \) are at distance 2. It follows from Observation 2.3 one of them has to be \( a \). Since for any \( v, w \) in \( G \), we have
\[
d(v, w) \geq |d(v, v_0) - d(w, v_0)|
\]
as well as \( d(a, v_{4k+1}) \geq 3 \) and \( d(a, v'_{4k-2}) = 1 \), the other vertex has to be \( v_{4k-3} \) or \( v_{4k-2} \). Note that if \( d(a, v_i) \leq 2 \) for some \( 3 \leq i \leq 4k - 2 \), then
\[
d(v_3, v'_{4k-2}) \leq d(v_3, v_i) + d(v_i, a) + d(a, v'_{4k-2}) \leq (4k - 5) + 2 + 1 < r,
\]
a contradiction. Hence, no two vertices of \( T \) are at distance 2 while \( |T| = r \). The result follows from Lemma 3.5.

Subcase: \( d(a, v_{4k+1}) < 3 \).
By the triangle inequality, we have

\[ d(a, v_{4k+1}) \geq |d(a, v_0) - d(v_0, v_{4k+1})| = 2, \]

so that \( d(a, v_{4k+1}) = 2 \). Hence, there exists a vertex \( b \) such that \( b \) is neighbour of both \( a \) and \( v_{4k+1} \). Consider

\[ U = \{v_0, v_1, v_2, \ldots, v_{4k+1}, v'_1, v'_2, \ldots, v'_{4k-2}, a, b\}. \]

We have \( |U| = 8k + 2 = 2r \). Consider auxiliary graph \( H \) on \( V(H) = U \) in which we connect two vertices if their distance in \( G \) is precisely 2. \( H \) is union of two disjoint cycles of length \( r \), first being \( v_0, v_2, \ldots, v_{4k}, b, v'_{4k-2}, \ldots, v'_2 \), and second being \( v_1, v_3, \ldots, v_{4k+1}, a, v'_{4k-3}, \ldots, v'_1 \). The result then follows from Lemma 3.6. The only nontrivial relationships needed to prove that \( H \) is union of two disjoint cycles of length \( r \) are

\[ d(b, v_{4k-1}), d(b, v_{4k-2}), d(a, v_{4k}), d(a, v_{4k-1}), d(a, v_{4k-2}), d(a, v_{4k-3}) \geq 3. \]

If any of these distances was at most 2, we could find a path of length at most \( r - 1 \) from \( v_3 \) to \( v'_{4k-2} \). That would be a contradiction.

Putting Claims 3.8 and 3.9 together now finishes the proof of Lemma 3.7.

3.5 | **Proof of Lemma 3.4 for** \( r = 4k + 3 \)

In this subsection, we prove the following.

**Lemma 3.10.** If \( G \) is a connected triangle-free graph on \( n \) vertices with minimum degree \( \delta \geq 2 \) and radius \( r \geq 4 \) such that \( r = 4k + 3 \) for some \( k \), then we have \( n \geq 2^r \).

We use a slightly weaker and more general set-up than we did in the proof of Lemma 3.7. This will have the advantage that we have more freedom in our choice of a centre \( v_0 \) as well as in the choice of \( v'_{r-t} \).

**Proof.** Take \( v_0 \) to be any centre of our graph \( G \). Let \( v_r \) be any vertex such that \( d(v_0, v_r) = r \). Let \( v_0, v_1, \ldots, v_r \) be any path of length \( r \) from \( v_0 \) to \( v_r \). Let \( v'_{r-t} \) be any vertex such that \( d(v_3, v'_{r-t}) \geq r \). Then we have \( d(v_0, v'_{r-t}) = r - t \) for some \( t \geq 0 \). Let \( v_0 = v'_0, v'_1, \ldots, v'_{r-t} \) be a path of length \( r - t \) from \( v_0 \) to \( v'_{r-t} \). By Observation 2.2, we have \( t \leq 3 \). Moreover, consider a vertex \( v''_s \) such that \( d(v_4, v''_s) \geq r \).

As before, we have \( d(v_0, v''_s) = r - s \) for some \( s \geq 0 \). Let \( v_0 = v'_0, v'_1, \ldots, v''_s \) be a path of length \( r - s \) from \( v_0 \) to \( v''_s \). By Observation 2.2, we have \( s \leq 4 \).

We will consider four cases depending on the value of \( t \).

**Case:** \( t = s = 0 \).
Let

\[ T = \{ v_0, v_1, v_2, v_6, v_7, v_8, v_{10}, v_{11}, v_{15}, v_{16}, \ldots, v_{r-5}, v_{r-4}, v_{r-1}, v_r, v_{r}' \}. \]

By Observation 2.3, no two vertices in \( T \) have distance 2. The result follows from Lemma 3.5.

Case: \( t = 0 \), \( 1 \leq s \leq 4 \).

We claim that we can find four vertices \( z_1, z_2, z_3, z_4 \) such that no two out of \( z_1, z_2, z_3, z_4 \) have distance 2, and for \( i = 1, 2, 3, 4 \);

\[ r - 4 \leq d(v_0, z_i) \leq r - 3. \]

Set \( z_1 = v_{r-4}, z_2 = v_{r-3} \) and \( z_3 = v_{r-4}' \). By Observation 2.3, we immediately see \( d(v_{r-4}, v_{r-4}') \geq 5 \) and \( d(v_{r-3}, v_{r-4}') \geq 4 \). If we have a vertex \( x \) such that \( x \) is neighbour of \( v_{r-4}' \) and \( d(x, v_0) \geq r - 4 \), we can set \( z_4 = x \) and are done. If on the other hand there exists no such \( x \), that implies \( d(v_{r-3}, v_{r-4}') \geq 3 \). By Observation 2.3 we have \( d(v_{r-4}, v_{r-3}') \geq 4 \), \( d(v_{r-3}, v_{r-3}') \geq 3 \), so we can set \( z_4 = v_{r-3}' \). Hence, we can always find suitable \( z_1, z_2, z_3, z_4 \).

Let

\[ T = \{ v_0, v_3, v_4, v_6, v_7, v_8, v_{10}, v_{11}, v_{15}, v_{16}, \ldots, v_{r-5}, v_{r-4}, v_{r-1}, v_r, v_{r}' \}. \]

It follows from Observation 2.3 that no two vertices in \( T \) have distance 2. The result follows from Lemma 3.5.

Case: \( t = 2 \).

Let

\[ T = \{ v_0, v_3, v_4, v_5, v_6, \ldots, v_{4k-1}, v_{4k}, v_{4k+3}, v_4', v_5', v_6', v_{10}, v_{11}, v_{15}, v_{16}, \ldots, v_{4k+1}, v_{4k+4} \}. \]

It follows from Observation 2.3 that no two vertices in \( T \) have distance 2. The result follows from Lemma 3.5.

Case: \( t = 3 \).

Let \( w_{r-u} \) be so that \( d(v_1, w_{r-u}) \geq r \) and \( d(v_0, w_{r-u}) = r - u \) for some \( 0 \leq u \leq 1 \). We consider subcases based on the value of \( d(w_{r-u}, v_{4k}) \).

Subcase: \( d(w_{r-u}, v_{4k}) \geq 3 \).

Let

\[ T = \{ v_0, v_1, v_4, v_5, \ldots, v_{4k}, v_{4k+1}, v_4', v_5', v_{10}, v_{11}, v_{15}, v_{16}, \ldots, v_{4k-1}, v_{4k+4}, w_{r-u} \}. \]
Assume for a contradiction that two vertices of $T$ are at distance 2. It follows from Observation 2.3 that one of them has to be $w_{r-u}$. Since for any $v, w$ in $G$, we have
\[
d(v, w) \geq |d(v, v_0) - d(w, v_0)| \quad \text{and} \quad d\left(w_{r-u}, v'_k\right) \geq 3,
\]
it further follows that the other vertex would have to be $v_{4k}$ or $v_{4k+1}$. If we had $d(v_i, w_{r-u}) \leq 2$ for some $1 \leq i \leq 4k + 1$, then
\[
d(v_1, w_{r-u}) \leq d(v_1, v_i) + d(v_i, w_{r-u}) \leq 4k + 2 < r,
\]
a contradiction. Hence, no two vertices of $T$ are at distance 2, and $|T| = r$. The result follows from Lemma 3.5.

**Subcase:** $d(w_{r-u}, v'_{4k}) < 3$.

Since
\[
d\left(w_{r-u}, v'_{4k}\right) \geq |d\left(w_{r-u}, v_0\right) - d\left(v'_k, v_0\right)| \geq 3 - u \geq 2,
\]
we have $u = 1$ and $d(w_{r-1}, v'_{4k}) = 2$. Hence there exists a vertex $a$ such that $a$ is neighbour of both $w_{r-1}$ and $v'_{4k}$. Moreover, clearly $d(a, v_0) = r - 2$.

Consider two cases based on the value of $d(a, v_{4k+3})$.

**Subsubcase:** $d(a, v_{4k+3}) \geq 3$

Let
\[
T = \left\{v_0, v_3, v_4, v_7, v_8, \ldots, v_{4k+1}, v_{4k}, v_{4k+3}, v'_1, v'_4, v'_7, v'_8, \ldots, v'_{4k-4}, v'_{4k-3}, v'_{4k}, a\right\}.
\]

Assume for a contradiction that two vertices of $T$ are at distance 2. It follows from Observation 2.3 that one of them has to be $a$. Since for any $v, w$ in $G$, we have
\[
d(v, w) \geq |d(v, v_0) - d(w, v_0)|, \quad d(a, v_{4k+3}) \geq 3 \quad \text{and} \quad d(a, v'_k) = 1,
\]
the other has to be $v_{4k-1}$ or $v_{4k}$. Since $d(a, v_i) \leq 2$ for some $3 \leq i \leq 4k$, we find
\[
d\left(v_3, v'_k\right) \leq d(v_3, v_i) + d(v_i, a) + d(a, v_{4k}) \leq (4k - 3) + 2 + 1 < r,
\]
a contradiction. Hence, no two vertices of $T$ are at distance 2 while $|T| = r$. The result follows from Lemma 3.5.

**Subsubcase:** $d(a, v_{4k+3}) < 3$.
By the triangle inequality, we have $d(a, v_{4k+3}) \geq 2$, so that $d(a, v_{4k+3}) = 2$. Hence, there exists a vertex $b$, such that $b$ is neighbour of both $a$ and $v_r$. Consider

$$U = \{v_0, v_1, v_2, v_3, \ldots, v_{4k+3}, v'_1, v'_2, \ldots, v'_{4k}, a, b\}.$$ 

We have $|U| = 8k + 6 = 2r$. Consider the auxiliary graph $H$ on $V(H) = U$ in which two vertices are connected if their distance in $G$ is precisely 2. $H$ is the union of two disjoint cycles of length $r$, the first being $v_0, v_2, \ldots, v_{4k+2}, b, v'_{4k+3}, a, v'_{k-1}, \ldots, v'_1$. Indeed, the only nontrivial relationships needed to prove that $H$ is the union of two disjoint cycles of length $r$ are

$$d(b, v_{4k+1}), d(b, v_{4k}), d(a, v_{4k+2}), d(a, v_{4k+1}), d(a, v_{4k}), d(a, v_{4k-1}) \geq 3.$$ 

If any of these distances was at most 2, we could find a path of length at most $r - 1$ from $v_3$ to $v'_{4k}$. The result follows from Lemma 3.6. This concludes the case $t = 3$.

**Case:** $t = 1$.

We start with the following useful claim.

**Claim 3.11.** Assume $r \geq 4$, $r = 4k + 3$ and $t = 1$. Further assume there are four distinct vertices $y_1, y_2, y_3, y_4$ such that no two out of them have distance 2, and $d(v_0, y_i) \leq 3$ for $i = 1, 2, 3, 4$. Then we have $n \geq \left\lceil \frac{r^2}{2} \right\rceil$.

**Proof of Claim 3.11.** Let

$$T = \{y_1, y_2, y_3, y_4, v_0, v_7, v_{10}, v_{11}, \ldots, v_{4k+2}, v_{4k+3}, v'_7, v'_1, v'_10, v'_11, \ldots, v'_{4k-2}, v'_{4k-1}, v'_{4k+2}\}.$$ 

It follows by Observation 2.3 that no two vertices of $T$ have distance 2. Moreover, we have $|T| = r$. The result follows by Lemma 3.5.

We return to the proof of Lemma 3.10 in the case $t = 1$.

**Subcase:** $v_2$ is not a centre of $G$.

There exists a vertex $c$ such that $d(v_2, c) \geq r + 1$, and by the triangle inequality $d(v_0, c) \geq r - 1$ and $d(v_3, c) \geq r$. We consider two cases: if $d(v_0, c) = r$, we could have chosen $c$ in place of $v'_{r-1}$ (as $d(v_3, c) \geq r$) and pass to a case $t = 0$ which we already solved. If, on the other hand, $d(v_0, c) = r - 1$, then let $v_0 = v''_0, v'_{1}, \ldots, v'_{r-1} = c$ be a path of length $r - 1$ from $v_0$ to $c$. No two out of $v_3, v_2, v'', v''$ can have distance 2 by Observation 2.3, using that $d(v_2, c) \geq r + 1$. Hence, we conclude by using Claim 3.11 for $y_1 = v_3, y_2 = v_2, y_3 = v'', y_4 = v''$. 


Subcase: $v_2$ is a centre of $G$.

The vertices $v_0, v_1, v_4, v_5$ all have distance at most three to $v_2$ and no two have distance 2. Now start the proof again with $v_0^\dagger := v_2$ instead of $v_0$ (choosing some vertices $v_i^\dagger$ and $(v_{r-t}^\prime)^\dagger$ in place of $v_r$ and $v_{r-t}^\prime$). If $t^\dagger \neq 1$, then the conclusion follows as before. If $t^\dagger = 1$, then we can find four distinct vertices $y_1 = v_0, y_2 = v_1, y_3 = v_4, y_4 = v_5$ such that no two out of them have distance 2, and $d(v_2, y_i) \leq 3$ for $i = 1, 2, 3, 4$. We conclude with Claim 3.11.

This finishes the proof of Lemma 3.10. □

4 | GENERAL PROBLEM FOR GIRTH $g \geq 5$

We first establish Theorem 1.2 using Lemma 2.1.

Proof of Theorem 1.2. We will find a large enough collection of vertices $T$ such that no two nonadjacent vertices of $T$ are at distance less than $2k - 1$. The result then follows by Lemma 2.1.

Let $v_0$ be a centre of $G$, $v_r$ a vertex with $d(v_0, v_r) = r$, and $v_0, v_1, ..., v_{r-1}, v_r$ a path of length $r$ in $G$ from $v_0$ to $v_r$. If $r \leq 2k$, we know the inequality holds, so assume $r \geq 2k$. We let $v_{r-t}^\prime$ be a vertex such that $d(v_0, v_{r-t}^\prime) \geq r - t$ for some $0 \leq t \leq 2k$. Further, let $v_0 = v_0', v_1', ..., v_{r-t}^\prime$ be a path of length $r - t$ from $v_0$ to $v_{r-t}^\prime$.

Let

$$T = \{v_{2ki} \mid 0 \leq i \leq \left\lfloor \frac{r}{2k} \right\rfloor \} \cup \{v_{2ki+1} \mid 0 \leq i \leq \left\lfloor \frac{r}{2k} \right\rfloor - 1 \} \cup \{v_{2ki} \mid 1 \leq i \leq \left\lfloor \frac{r}{2k} \right\rfloor - 1 \} \cup \{v_{2ki+1} \mid 1 \leq i \leq \left\lfloor \frac{r}{2k} \right\rfloor - 2 \}.$$

It follows from Observation 2.4 that the above is a disjoint union. It follows from Observation 2.3 that no two nonadjacent vertices of $T$ are at distance less than $2k - 1$. Hence, we conclude by Lemma 2.1 that

$$n \geq |T| \left( \frac{(\delta - 1)^k - 1}{\delta - 2} \right) \geq \left( \frac{2r}{k} - 6 \right) \left( \frac{(\delta - 1)^k - 1}{\delta - 2} \right).$$

We prove the next lemma using an idea similar to one of Erdős, Pollack, Pach and Tuza [2]. Its most important corollary is Theorem 1.3.

Lemma 4.1. Denote by $f(g, \delta)$ for $\delta \geq 2$, $g \geq 3$ the minimum number of vertices in the graph of girth at least $g$ and minimum degree $\delta$. Then for any $r > \frac{g}{2}$, there exists a
connected graph $G$ on $n = \left\lceil \frac{2r}{g} \right\rceil f(g, \delta)$ vertices of girth at least $g$, minimum degree $\delta$ and radius at least $r$.

**Proof.** Let $H$ be a connected graph with $f(g, \delta)$ vertices, minimum degree $\delta$ and girth at least $g$. As $\delta > 1$, we know $H$ contains a cycle. Let $v, w$ be two neighbouring vertices of $H$ such that the edge $vw$ is part of a cycle. Let $H'$ be the (still connected) graph obtained by deleting the edge $vw$ from $H$. By the girth condition, we have $d_{H'}(v, w) \geq g - 1$.

Take $\left\lceil \frac{2r}{g} \right\rceil$ identical disjoint copies of $H'$, called $H'_1, \ldots, H'_{\left\lceil \frac{2r}{g} \right\rceil}$, with vertices $v_1, \ldots, v_{\left\lceil \frac{2r}{g} \right\rceil}$ and $w_1, \ldots, w_{\left\lceil \frac{2r}{g} \right\rceil}$, and connect $v_i$ to $w_{i+1}$, where $w_{\left\lceil \frac{2r}{g} \right\rceil + 1} = w_1$. The resulting graph has $\left\lceil \frac{2r}{g} \right\rceil f(g, \delta)$ vertices, radius at least $r$, girth at least $g$, and minimum degree $\delta$. □

Theorem 1.3 follows easily.

**Proof of Theorem 1.3.** We know (see [3]) that when $\delta - 1$ is a prime power, then
\[
\begin{align*}
    f(6, \delta) &\leq 2(\delta^2 - \delta + 1), \\
    f(8, \delta) &\leq 2(\delta^3 - 2\delta^2 + 2\delta), \\
    f(12, \delta) &\leq 2((\delta - 1)^3 + 1)(\delta^2 - \delta + 1).
\end{align*}
\]

Hence the result follows directly from Lemma 4.1 by taking
\[
    n_i = \left\lfloor \frac{i}{3} \right\rfloor f(6, \delta), \quad n_i' = \left\lfloor \frac{i}{4} \right\rfloor f(8, \delta), \quad n_i'' = \left\lfloor \frac{i}{6} \right\rfloor f(12, \delta).
\]

Finally, we prove Theorem 1.5.

**Proof of Theorem 1.5.** Let $v_0$ be a centre of our graph, $v_r$ a vertex with $d(v_0, v_r) = r$ and $v_0, \ldots, v_r$ a path of length $r$.

For $0 \leq i \leq r$, let
\[
    Q(v_i) := \{v \in V(G) : d(v, v_i) \leq k\}.
\]

Every vertex in our graph is in at most $2k + 1$ of these sets, so in particular there is an $i_0$ so that
\[
    |Q(v_{i_0})| \leq (2k + 1)\delta^{k-1}.
\]

We easily find using the girth condition that $v_{i_0}$ has at least $\delta(\delta - 1)^{k-2}$ vertices at distance at most $k - 1$ from it. Hence, as all edges adjacent to these vertices are inside $Q(v_{i_0})$, we get that for every $i$ the subgraph induced by $Q(v_{i_0})$ has at least $\frac{1}{2}\delta^2(\delta - 1)^{k-2}$ edges.

We conclude that the subgraph induced by $Q(v_{i_0})$ is a connected graph of girth at least $2k$ on at most $(2k + 1)\delta^{k-1}$ vertices with at least $\frac{1}{2}\delta^2(\delta - 1)^{k-2}$ edges. □
ACKNOWLEDGEMENTS

The authors would like to thank their Ph.D. supervisor Professor Béla Bollobás for many helpful comments. The first author would like to thank the Engineering and Physical Sciences Research Council for funding under Ph.D. award Probabilistic Combinatorics, 2260624. The second author would like to thank the Engineering and Physical Sciences Research Council for funding under Ph.D. award Random Graph Percolation, 1951104 and the Cambridge trust for the Cambridge European Scholarship, 10422838.

REFERENCES

1. P. Erdős, *Extremal problems in graph theory*, Theory of Graphs and its Applications, Proc. Sympos. Smolenice, 1964, pp. 29–36.
2. P. Erdős, J. Pach, R. Pollack, and Z. Tuza, *Radius, diameter and minimum degree*, J. Combin. Theory Ser. B. 47 (1989), 73–79.
3. G. Exoo and R. Jajcay, *Dynamic cage survey*, Electron. J. Combin. DS (2013), 16.
4. B. Kim, Y. Rho, B. Song, and W. Hwang, *The maximum radius of graphs with given order and minimum degree*, Discrete Math. 312 (2012), 207–212.
5. J. P. Mazorodze and S. Mukwembi, *Radius, diameter and the degree sequence of graph*, Math. Slovaca. 65 (2015), 1223–1236.

How to cite this article: V. Dvorák, P. van Hintum, A. Shaw, and M. Tiba, *Radius, girth and minimum degree*, J. Graph Theory. 2022;100:470–488. https://doi.org/10.1002/jgt.22790