Finiteness of multi-loop superstring amplitudes

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Abstract

Superstring amplitudes of an arbitrary genus are calculated through super-Schottky parameters by a summation over the fermion strings. For a calculation of divergent multi-loop fermion string amplitudes a supermodular invariant regularization procedure is used. A cancellation of divergences in the superstring amplitudes is established. Grassmann variables are integrated, the superstring amplitudes are obtained to be explicitly finite and modular invariant.

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1 Introduction

During years, a great deal of efforts in superstring theory \cite{1} has been invested \cite{2—10} in a construction of a perturbation series for interaction amplitudes. Especially, difficulties arose in a calculation of partition functions and of field vacuum correlators for Ramond strings where the desired values can not be derived by an obvious extension of boson string results \cite{11}. Finally, by a method given in \cite{7, 9} the partition functions and the superfield vacuum correlators were calculated \cite{9, 10} in terms of super-Schottky parameters for all the fermion strings. Multi-loop superstring amplitudes could be obtained by a summation of the fermion strings, but every fermion string amplitude is divergent. Though the divergences are expected \cite{2, 4, 12} to be canceled in the full superstring amplitude, up to now they kept the desired superstring amplitudes from being calculated. In this paper a supermodular invariant calculation \cite{13} of divergent fermion string amplitudes is proposed. So, superstring amplitudes calculated by a summation over the fermion string ones are surely invariant under supermodular group. We establish a cancellation of divergences in the above superstring amplitudes. Moreover, on integration over Grassmann moduli we obtain expressions for the superstring amplitudes that are explicitly finite and supermodular invariant. Details of this construction are need yet to be clarify, but the paper mainly completes a building of the superstring perturbation series that, in turn, opens opportunities for a wide investigation of superstrings. A topic of an essential interest that can be advanced in the next future is a summation of the superstring perturbation series in an infrared energy region of interacted states. Amplitudes of genus going to infinity dominate in this case, the discussed infrared asymptotics are expected to be quite different from each of multi-loop amplitudes taken in the infrared limit. Perhaps, they can be applied to particle interactions below the Plank mass. Another goal could be a summation of the superstring perturbation series for massless string states provided that both a number thereof and their energies tend to infinity. It might be applied to a creating of the Universe.
As it is usually, a genus-$n$ closed superstring amplitude $A_{n,m}$ with $m$ legs is given by

$$A_{n,m} = \int \prod_N dq_N dq_N' \prod_{r=1}^m dt_r dt_r' \sum_{L,L'} Z^{(n)}_{L,L'}(\{q_N, q_N'\}) E_{L,L'}^{(n,m)}(\{t_r, t_r'\}, \{q_N, q_N'; \{p_j\}\})$$

where $\{p_j\}$ are particle momenta, the overline denotes the complex conjugation and $L (L')$ labels superspin structures of right (left) superfields defined on the complex $(1|1)$ supermanifolds $[14]$. Every superspin structure $L = (l_1, l_2)$ presents a superconformal extension of the $(l_1, l_2) = \bigcup_{s=1}^n (l_{1s}, l_{2s})$ ordinary spin one $[15]$. The genus-$n$ theta function characteristics $(l_1, l_2)$ can be restricted by $l_{is} \in (0, 1/2)$. The prime denotes a product over those $(3n - 3|2n - 2)$ super-Schottky parameters $\{q_N\}$ that are chosen as moduli, $(3|2)$ thereof being fixed common to all the genus-$n$ supermanifolds by a super-Möbius transformation. Partition functions $Z^{(n)}_{L,L'}(\{q_N, \overline{q}_N\})$ are calculated from equations $[7, 11]$ expressing that the superstring amplitudes are independent of a choice of two-dim. metrics and of a gravitino field. The vacuum expectation $E_{L,L'}^{(n,m)}(\{t_r, \overline{t}_r\}, \{q_N, \overline{q}_N\}; \{p_j\})$ of the vertex product is integrated over supermanifolds $t_r = (z_r|\theta_r)$ where $z_r$ is a local complex coordinate and $\theta_r$ is its odd partner. Moduli are integrated over the fundamental domain $[16]$. Among other things, this domain depends on $L$ through terms proportional to Grassmann super-Schottky parameters since, generally, the supermodular changes of moduli and of supercoordinates depend on superspin structure $[16]$.

To calculate $A_{n,m}$ a regularization procedure is necessary because every term in (1) is divergent $[17]$ due to degeneration of Riemann surfaces. If a cutoff $[18]$ of modular integrals is used, it is necessary yet to verify the supermodular invariance of the calculated amplitudes because the cutoff $[18]$ violates the supermodular group. So we use a supermodular invariant regularization procedure given in this paper. Also, we regularize the integrals over $z_j$ that are ill defined too.

A construction of supermodular covariant functions needed for a regularization of the modular integrals is complicated by a dependence on the superspin structure of supermodular changes of super-Schottky parameters $[16]$. As example, the sum over $(L, L')$ in (1) calculated with $\{q_N\}$ common to all the superspin structures is non-covariant under the supermodular group though each of the $(L, L')$ terms is covariant $[16]$ under the group considered. Hence we perform a singular transformation $[19]$ to a new parameterization $P_{\text{split}}$ where transition groups are split,
the supermodular group being reduced to the ordinary modular one. While the superstring in not invariant under the above transformation, it is useful to integrate over Grassmann variables because in this case the integration region is independent of the above Grassmann ones.

The $P_{\text{split}}$ parameterization is considered in Sec.2. In Sec.3 regularized expressions for superstring amplitudes are given. It is argued that amplitudes of an emission of a longitudinal polarized gauge boson vanish in our scheme as it is required by the gauge symmetry. In Sec.4 the cancellation of divergences in superstring amplitudes and non-renormalization theorems [4] are verified. On integration over the $P_{\text{split}}$ Grassmann variables the expressions for the amplitudes are derived that are free from divergences and supermodular invariant as well.

2 Regularization of modular integrals

As it was mentioned above, we perform a singular $t \to \hat{t} = (\hat{z}|\hat{\theta})$ superholomorphic transformation [13, 19] to a $P_{\text{split}}$ parameterization where transition groups contain no Grassmann parameters:

$$z = f_L(\hat{z}) + f'_L(\hat{z})\hat{\theta} \xi_L(\hat{z}), \quad \theta = \sqrt{f'_L(\hat{z})} \left[ 1 + \frac{1}{2} \xi_L(\hat{z}) \xi'_L(\hat{z}) \right] \hat{\theta} + \xi_L(\hat{z}) \right], \quad f_L(\hat{z}) = \hat{z} + y_L(\hat{z}). \quad (2)$$

Here the ”prime” symbolizes $\hat{z}$-derivative, $\xi_L(\hat{z})$ is a Grassmann function and $y_L(\hat{z})$ is proportional to Grassmann modular parameters. On the $t$ supermanifold the rounds about $(A_s, B_s)$-cycles are associated with super-Schottky transformations $(\Gamma_{a,s}(l_{1s}), \Gamma_{b,s}(l_{2s}))$ where every $A_s$-cycle is a Schottky circle. In this case [8, 9] $\Gamma_{a,s}(l_{1s}) = 0 = I$, $\Gamma_{a,s}^2(l_{1s} = 1/2) = I$, but $\Gamma_{a,s}(l_{1s} = 1/2) \neq I$. So a square root cut on the $z$-plane appears for every $l_{1s} \neq 0$ with endcut points to be inside corresponding Schottky circles. For a handle $s$, the super-Schottky transformation is determined by a multiplier $k_s$ and two unmoved points $(u_s|\mu_s)$ and $(v_s|\nu_s)$ where $\mu_s$ and $\nu_s$ are Grassmann partners of $u_s$ and, respectively, of $v_s$. In the $P_{\text{split}}$ parameterization the same $(A_s, B_s)$ rounds are associated with transformations $(\hat{\Gamma}_{a,s}(l_{1s}), \hat{\Gamma}_{b,s}(l_{2s}))$. Hence

$$\Gamma_{b,s}(l_{2s})(t) = t \left( \hat{\Gamma}_{b,s}(l_{2s})(\hat{t}) \right), \quad \Gamma_{a,s}(l_{1s})(t) = t^{(s)} \left( \hat{\Gamma}_{a,s}(l_{1s})(\hat{t}) \right) \quad (3)$$
where \( t^{(s)}(\hat{t}) \) is obtained by \( 2\pi \text{-twist of } t(\hat{t}) \) on the complex \( \hat{z} \)-plane about the Schottky circle assigned to a particular handle \( s \). In this case \( \hat{\Gamma}_{a,s}(l_{1s}) \) may only give a sign of fermion fields and \( \hat{\Gamma}_{b,s}(l_{2s}) \) is a Schottky transformation with a multiplier \( \hat{k}_s \) and two unmoved local points \( \hat{u}_s \) and \( \hat{v}_s \). So both \( \hat{\Gamma}_{a,s}(l_{1s}) \) and \( \hat{\Gamma}_{b,s}(l_{2s}) \) do not contain Grassmann modular parameters. Since every super-Schottky group depends, among other things, on \( (2n - 2) \) Grassmann moduli, the transition functions in (2) necessary depend on \( (2n - 2) \) Grassmann parameters \( (\lambda^{(1)}_j, \lambda^{(2)}_j) \) where \( j = 1 \ldots n - 1 \). The equations similar to (3) were already used \([16]\) in a calculation of the acting of supermodular transformations on supercoordinates and on modular parameters. Unlike \([16]\), eqs.(3) are satisfied only if the transition functions in (2) have poles in a fundamental region of \( \hat{z} \)-plane, singular parts being proportional to Grassmann parameters. We take them possessing \((n - 1) \) poles \( \hat{z}_j \) of an order 2. For even superspin structures we choose the above poles among \( n \) zeros of the fermion Green function \( R^f_L(\hat{z}, \hat{z}_0) \) calculated for zero Grassmann moduli.\(^1\) For odd superspin structures the poles can be chosen by a similar way \([19]\). We take \( \hat{z}_0 \) common to all superspin structures. In this case supermodular changes of \( (\lambda^{(1)}_j, \lambda^{(2)}_j) \) are independent of the superspin structure and the supermodular group in the \( P_{\text{split}} \) representation is mainly reduced to the ordinary modular one. The singular parts of (2) are determined by a condition that above modular group is isomorphic to the supermodular one in the super-Schottky parameterization. From this condition, it is follows \([19]\) that near every pole \( \hat{z}_j(\hat{z}_0; L) \)

\[
\xi_L(\hat{z}) \approx \frac{1 + \xi_L(\hat{z}) \xi'_L(\hat{z})}{R^f_L(\hat{z}, \hat{z}_0)} \left[ \frac{\lambda^{(2)}_j}{R^f_L(\hat{z}, \hat{z}_0)} \frac{\partial R^f_L(\hat{z}, \hat{z}_0)}{\partial \hat{z}_0} + \lambda^{(1)}_j \right] + \frac{\lambda^{(1)}_j \lambda^{(2)}_j \xi_L(\hat{z}) \partial^2 \ln[R^f_L(\hat{z}, \hat{z}_0)]}{2[R^f_L(\hat{z}, \hat{z}_0)]^2 \frac{\partial}{\partial \hat{z}_0}} ,
\]

\[
f_L(\hat{z}) \approx \frac{\lambda^{(2)}_j \xi_L(\hat{z}) f'_L(\hat{z})}{[R^f_L(\hat{z}, \hat{z}_0)]^2} \frac{\partial R^f_L(\hat{z}, \hat{z}_0)}{\partial \hat{z}_0} + \frac{\lambda^{(1)}_j \xi_L(\hat{z}) f'_L(\hat{z})}{R^f_L(\hat{z}, \hat{z}_0)}
\]

(4)

where the "prime" symbol denotes \( \partial \hat{z} \). The calculation \([13, 19]\) of both \( y_L(\hat{z}) \) and \( \xi_L(\hat{z}) \) is quite similar to that in Sec. 3 of \([16]\). The set of (2) and of (3) determines both \( t \) and \( q_N \) in terms of \( \hat{t} \) and of \( \{\hat{q}_N\} \) up to \( SL_2 \) transformations of \( t \) where \( \{\hat{q}_N\} = \{\hat{q}_{ev}, \lambda^{(1)}_j, \lambda^{(2)}_j\} \) and

\(^1\) An another choice of the poles is discussed in \([13]\).

\(^2\) See Sec. 4 of \([\phantom{1}]\) where \( R^f_L(\hat{z}, \hat{z}_0) \) is denoted as \( R_f(z, z') \).
\{\hat{q}_{ev}\} = \{\hat{k}_s, \hat{u}_s, \hat{v}_s\}. We consider the \{r, j\} set of the solutions fixed by

\[ t(\hat{t}; \{\hat{q}_N\}; L; r, j); \quad q_N(\{\hat{q}_N\}; L; r, j) : \quad \mu_r = \nu_r = 0, \; u_r = \hat{u}_r, \; v_r = \hat{v}_r, \; u_j = \hat{u}_j. \quad (5) \]

Every solution is obtained by a \(SL_2\) transformation \(M(r, j; r_0, j_0)\) of the \((r = r_0, j = j_0)\) one as

\[ t(\hat{t}; L; r, j) = M(r, j; r_0, j_0)t(\hat{t}; L; r_0, j_0), \quad \{P(r, j)\} = M(r, j; r_0, j_0)\{P(r_0, j_0)\}. \quad (6) \]

where \(\{P(r, j)\} = \{(u_s | \mu_s), (v_s | \nu_s)\}\). The \(\{k_s\}\) multipliers are the same for all \(r, j\). The \(P_{\text{split}}\) partition functions \(\hat{Z}^{(n)}_{L,L'}(\{\hat{q}_N, \overline{q}_N\})\) can be derived by a going to the \(P_{\text{split}}\) variables in (5) as

\[ \hat{Z}^{(n)}_{L,L'}(\{\hat{q}_N, \overline{q}_N\}) = F_L(\{\hat{q}_N\}; r, j)F_{L'}(\{\hat{q}_N\}; r, j)\hat{Z}^{(n)}_{L,L'}(\{q_N, \overline{q}_N\})|(\hat{u}_r - \hat{u}_j)(\hat{v}_r - \hat{v}_j)|^2 \quad (7) \]

where \(F_L(\{\hat{q}_N\}; r, j)\) is the Jacobian of the transformation and \(q_N = q_N(\{\hat{q}_N\}; L; r, j)\). Furthermore, \(\hat{Z}^{(n)}_{L,L'}(\{q_N, \overline{q}_N\})\) being multiplied by the factor behind it, is just the partition function in (5), if \((u_r, v_r, u_j, \mu_r, \nu_r)\) are fixed in (5) as in (3) to be common to all genus-\(n\) supermanifolds (for details, see eq.(132) in [9]). Under the \(SL_2\) transformations (6) this factor is re-defined by a factor that arises in the Jacobian due to parameters of these transformations depend on \(\{q_N\}\). As the result, (5) appears invariant under the transformations (6). In the same way the partition functions in (5) being multiplied by the product of the moduli differentials, are invariant under the discussed transformations. Supermodular invariant function \(Y(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0)\) used in a regularization scheme is constructed as

\[ Y(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0) = \frac{Y_1(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0)Y_2(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0)}{Y_2(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0)} \quad (8) \]

with \(Y_1(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0) \equiv Y_1\) and \(Y_2(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0) \equiv Y_2\) defined to be

\[ Y_1 = \sum_{L \in \{L_{ev}\}} \hat{Z}^{(n)}_{L,L'}(\{\hat{q}_N, \overline{q}_N\}) \quad \text{and} \quad Y_2 = \prod_{L \in \{L_{ev}\}} \hat{Z}^{(n)}_{L,L'}(\{\hat{q}_N, \overline{q}_N\}) \quad (9) \]

where \(\{L_{ev}\}\) is the set of \(2^{n-1}(2^n+1)\) even spin structures, \(\hat{Z}^{(n)}_{L,L'}(\{\hat{q}_N, \overline{q}_N\})\) is defined by (7) and \(\{\hat{q}_N\}\)-set is common to all superspin structures. Since both \(Y_1(\{\hat{q}_N, \overline{q}_N\})\) and \(Y_2(\{\hat{q}_N, \overline{q}_N\})\) receive the same factor under modular transformation of \(\hat{q}_N\)-parameters, the right side of (8)
is invariant under supermodular transformations. In addition, it tends to infinity, if Riemann surfaces are degenerated. Indeed, if a particular handle, say $s$, become degenerated, the corresponding Schottky multiplier $k_s$ tends to zero. In this case both the nominator and the denominator in (8) tend to infinity [9], but terms associated with $l_{1s} = 0$ have an additional factor $|k_s|^{-1} \to \infty$ in a comparison with those associated with non-zero $l_{1s}$. So $Y(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0) \to \infty$. If an even spin structure of a genus-$n > 1$ is degenerated into odd spin structures, the partition functions tend to zero [10] while not vanishing, if it is degenerated into even spin ones. So again $Y(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0) \to \infty$. Hence to regularize the desired integrals we introduce in the integrand (1) a multiplier

$$B_{\text{mod}}^{(n)}(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0; \delta_0) = \{\exp[-\delta_0 Y(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0)]\}_{\text{sym}}$$

(10)

where $\delta_0 > 0$ is a parameter and the right side of (10) is symmetrized over all the sets of $(n - 1)$ zeros of the fermion Green function $R^L_{\hat{t}}(\hat{z}, \overline{z}_0)$. By the above reasons, (10) vanishes, if Riemann surfaces become degenerated that provides the finiteness of the modular integrals in (1). The right side of (10) is invariant under the $SL_2$ transformations (6) of the $\{(u_s|\mu_s), (v_s|\nu_s)\}$ set. In addition, it is invariant under those $L_2$ transformation of $\{\hat{u}_s, \hat{v}_s\}$ accompanied by a corresponding $L_2$-transformation of $\hat{z}_0$ and of $(\lambda^{(1)}_j, \lambda^{(2)}_j)$, which reduce three $(\hat{u}_r, \hat{v}_r, \hat{u}_j)$ values for particular $(r, j)$ to the fixed ones $\hat{u}_r = \hat{u}_r^{(0)}$, $\hat{v}_r = \hat{v}_r^{(0)}$ and $\hat{u}_j = \hat{u}_j^{(0)}$ common to all spin structures. For a given $\hat{t}_0 = (\hat{z}_0|\hat{\theta} = 0)$ one can calculate its image $\hat{t} = (z_0|\hat{\theta}(z_0))$ under the mapping (2). It is evidently that $\hat{t}$ is defined modulo $L_2$-transformations. In the considered case the transition functions have no poles because zeros $\hat{z}_j(\hat{z}_0; L)$ of $R^L_{\hat{t}}(\hat{z}, \overline{z}_0)$ are always different from $\hat{z}_0$. Simultaneously, so far as $\hat{z}_j(\hat{z}_0; L)$ is changed under fundamental group transformations, eqs.(3) are satisfied only if every this transformation is accompanied by an appropriate change of the $(\lambda^{(1)}_j, \lambda^{(2)}_j)$ parameters that is calculated from (4). Because the above change of $(\lambda^{(1)}_j, \lambda^{(2)}_j)$ does not depend on the superspin structure, (10) is invariant under the super-Schottky transformations of $\hat{t}$. 

6
 supermarkets

3 Superstring amplitudes

The integrals over $z_j$ in \( \text{(1)} \) are ill defined when all the vertices tend to coincide, or, alternatively, all they are moved away from each other. In addition, there is no a region in the \( \{p_j B_l\} \) space where all the nodal domain integrations giving raise to poles and to threshold singularities of $A_{n,m}$ would be finite together. As it is usual \[20\], each of the above integrals is calculated at $\text{Re } E_j^2 < 0$ where $E_j$ is a center mass energy in the channel considered. Then it is extended to $\text{Re } E_j^2 > 0$ by an analytical continuation in $E_j^2$. To regularize the $t_j$ integrals we need functions depending on two more supermanifold points $t_a = (t_{-1}, t_0)$ in addition to \( \{t_j\} \). One receives in hands the above $t_a$ points multiplying \( \text{(1)} \) by the unity arranged to be a square in the same integrals, every integral $I_{LL'}^{(n)} = 1$ being

\[
I_{LL'}^{(n)} = \frac{1}{n} \int \frac{dtd\tilde{t}}{2\pi i} I_{LL'}^{(n)}(t, \tilde{t}), \quad I_{LL'}^{(n)}(t, \tilde{t}) = D(t)[J_s(t; L) + J_s(t; L')][2\pi i \omega(L) - 2\pi i \omega(L')]^{-1}
\]

\[
\times D(t)[J_r(t; L) + J_r(t; L')], \quad D(t) = \theta \partial_{\theta} + \partial_{\theta}\,. \quad (11)
\]

Here $J_s(t; L)$ are the genus-$n$ superholomorphic functions \[9\] having periods, $D(t)$ is the spinor derivative and $\omega_{ar}(L)$ is a supermanifold period matrix dependent on the superspin structure \[3, 4\]. Due to $D(t) J_r(t; L) = 0$, both $J_s(t; L')$ and $J_r(t; L)$ could be omitted, but they are remained to provide the integrand to have no cuts on the supermanifold. Integrating \( \text{(1)} \) by parts one obtains that $I_{LL'}^{(n)} = 1$ as it was announced. With \( \text{(1)} \), we define a regularized superstring amplitude $A_{n,m}(\{\delta\})$ with $m > 3$ as

\[
A_{n,m}(\{\delta\}) = \int \left( \prod_{\gamma} d\gamma_N d\overline{\gamma}_N \right) \left( \prod_{r=1}^{m} dt_r d\tilde{t}_r \right) \sum_{L,L'} Z_{L,L'}^{(n)} E_{L,L'}^{(n,m)} \cdot \left( \prod_{a=1}^{0} dt_a d\tilde{t}_a I_{LL'}^{(n)}(t_a, \tilde{t}_a) \right) \times B_{mod}^{(n)}(\{q_N, \overline{q}_N\}; \tilde{z}_0, \overline{z}_0; \delta_0) \prod_{(jl)} B_{jl}^{(n)}(\{t_a, \tilde{t}_a\}; \{q_N, \overline{q}_N\}; \{\delta_{jl}\}; L, L') \quad (12)
\]

where $t_0 = (z_0 | \theta)$. Both $Z_{L,L'}^{(n)}$ and $E_{L,L'}^{(n,m)}$ are the same as in \( \text{(1)} \), the arguments being omitted for brevity. The $(jl)$ symbol labels pairs of the vertices, $\delta_{jl} > 0$ are parameters and $\{\delta\} = (\delta_0, \{\delta_{jl}\})$. Further, $\tilde{z}_0 = \tilde{z}_0(z_0)$ is calculated together with its Grassmann partner $\theta(z_0)$ from \( \text{(2)} \) taken at $\hat{\theta} = 0$, $z = z_0$ and $\theta = \theta(z_0)$. At $\{\delta_{jl} > 0\}$ every factor in the $(jl)$ product tends to zero at $|z_j - z_l| \to 0$ and at $|z_j - z_l| \to \infty$. Explicitly they are given in \[13\]. The superstring amplitude $A_{n,m}$
is defined as $A_{n,m}({\delta \to 0})$ calculated in line with the usual analytical continuation procedure for the integrals over nodal domains giving rise to poles and threshold singularities of $A_{n,m}$.

The (3|2) super-Schottky parameters are no moduli, say, they are $\mu_{r_0} = \nu_{r_0} = 0$, \( u_{r_0} = u_{r_0}^{(0)} \), \( v_{r_0} = v_{r_0}^{(0)} \) and $u_{j_0} = u_{j_0}^{(0)}$ common to all the supermanifolds. So, \( \{(u_s|\mu_s), (v_s|\nu_s)\} = \{P(r_0, j_0)\} \).

Due to the previous Section, \( \{\hat{q}_N\} \) for every superspin structure $L$ can be calculated as $\hat{q}_N = \hat{q}_N(Q_N(r, j); L; r, j)$ through any $\{q_N(r, j)\} = \{\{k_s\}, \{P(r, j)\}\}$ where \( \{P(r, j)\} \) is obtained by a transformation \( E \) of \( \{P(r_0, j_0)\} \). The result is independent of the choice of \( (r, j) \).

The integrations being well defined, \( \{12\} \) can be rearranged by a suitable $SL_2$-transformation $\tilde{M}$ to the integral over all (3n|2n) super-Schottky parameters and over \( (m - 3|m - 2) \) values among \( \{(z_j|\theta_j)\} \), the rest being fixed as \( \{z_b\} = (z_1 = z_1^{(0)}, z_2 = z_2^{(0)}, z_3 = z_3^{(0)}) \), $\theta_1 = \theta_2 = 0$ as

$$A_{n,m}({\delta}) = \sum_{L, L'} \int \left( \prod_N dq_N d\overline{q}_N \right) dt_0 dt_{\overline{0}} \tilde{Z}^{(n)}_{L, L'}(\{q_N, \overline{q}_N\}) K^{(n,m)}_{L, L'}(\{z_b\}, \{q_N, \overline{q}_N\}, \hat{z}_0, \overline{z}_0, \{p_j\})$$

$$\times B^{(n)}_{mod}(\{\hat{q}_N, \overline{q}_N\}; \hat{z}_0, \overline{z}_0; \delta_0) I^{(n)}_{L, L'}(t_0, \overline{t}_0) \quad (13)$$

where the $\tilde{Z}^{(n)}_{L, L'}(\{q_N, \overline{q}_N\})$ partition function is symmetrical in the super-Schottky parameters and the factor just behind it is given by

$$K^{(n,m)}_{L, L'}(\{z_b\}, \{q_N, \overline{q}_N\}, \hat{z}_0, \overline{z}_0, \{p_j\}) = \int \left( \prod_{r=4}^{m} dz_r d\overline{\sigma}_r \right) \left( \prod_{r=3}^{m} d\theta_r d\overline{\theta}_r \right) dt_{-1, \overline{t}_{-1}} E^{(n,m)}_{L, L'}$$

$$\times |(z_1^{(0)} - z_2^{(0)})(z_2^{(0)} - z_3^{(0)})(z_3^{(0)} - z_1^{(0)})|^2 I^{(n)}_{L, L'}(t_{-1, \overline{t}_{-1}}) \prod_{(j)} B^{(n)}_{j l} \{\{t_0, \overline{t}_0\}; \{\delta_{j l}\}; L, L' \} \quad (14)$$

where $E^{(n,m)}_{L, L'}$ is the same as in \( \{12\} \) and the factor between $E^{(n,m)}_{L, L'}(\{t_j, \overline{t}_j\})$ and $I^{(n)}_{L, L'}(t_{-1, \overline{t}_{-1}})$ is due to the fixing of the \( \{z_b\} \) set. The modular parameters in \( \{13\} \) are integrated over the fundamental domain \( \{16\} \) that is invariant under $SL_2$ transformations. In addition, they are restricted by both $z_1^{(0)}$, $z_2^{(0)}$ and $z_3^{(0)}$ to be outside all the Schottky circles. In \( \{13\} \) the $k_s$ multipliers are the same as in \( \{12\} \) and the \( \{(u_s|\mu_s), (v_s|\nu_s)\} = \{P\} \) set is related with \( \{P(r_0, j_0)\} \) in \( \{12\} \) by the $SL_2$-transformation $\tilde{M}$ as \( \{P(r_0, j_0)\} = \tilde{M}\{P\} \). Just as in \( \{12\} \), the \( \{\hat{q}_N\} \) set is calculated in term of \( \{P(r, j)\} \) given through \( \{P\} \) by

$$\{P(r, j)\} = M(r, j; r_0, j_0)\{P(r_0, j_0)\} = M(r, j; r_0, j_0)\tilde{M}\{P\} \quad (15)$$
where $M(r, j; r_0, j_0)$ is defined in (1). Parameters of the transition matrix in (13) depend on the super-Schottky parameters assigned to the $r$ handle and on $(\hat{u}_r, \hat{v}_r)$ in (3), but (13) is independent of $(\hat{u}_r, \hat{v}_r)$ due to $L_2$ symmetry discussed just below eq.(10).

In (13), after a suitable rewriting of the integrals over the nodal domains the regularization factors $B_{jl}^{(n)}(\{t_a, \bar{t}_a\}\{\delta_{jl}\}; L, L')$ can be removed from the integral. Hence the gauge symmetry inherent to massless modes presents though in $A_{n,m}(\{\delta\})$ it is violated due to these factors.

4 Finiteness of the superstring amplitudes

Divergences due to a degeneration of a handle are already known [5,16] to be canceled in the superstring amplitudes. Additional divergences could be when clusters $Cl$ of handles arise, the sizes being small compared with distances to vertexes (except may be to a solely dilaton-vacuum transition vertex). In this case, however, leading divergences in $A_{n,m}$ disappear due to integrations in (13) over Grassmann modular parameters associated with the $Cl$ cluster above. Indeed, here a dependence on modular parameters $\{q_{N_1}\}$ of the $Cl$ cluster is factorized in the partition functions. Besides, when all the vertexes are separated from the $Cl$ cluster the integrand (14) ceases to depend on $\{q_{N_1}\}$ except only on the limiting point $u_0$, which the $Cl$ cluster is contracted to. This $u_0$ dependence in (14) is removed by a boost of the vertex co-ordinates and of the modular parameters of the remainder. If in $u_0$ the dilaton-vacuum transition vertex is situated, an additional $\{q_{N_1}\}$ dependence arises in (14) solely as the $I_{Lr}^{(n)}(t, \bar{t})$ factor (14). Owing to the above structure of the integrand (14), two Grassmann parameters associated with the $Cl$ cluster, say, $(\mu_r, \nu_r) \in \{q_{N_1}\}$, are removed from the integrand (13) by an $SL_2$ transformation $M_r$ of $\{P_{N_1}\} = \{(u_s|\mu_s), (v_s|\nu_s)\} \in \{q_{N_1}\}$ as $\{\bar{P}_{N_1}\} = M_r\{\bar{P}_{N_1}\}$ where $\{\bar{P}_{N_1}\}\{(\bar{u}_s|\bar{\mu}_s), (\bar{v}_s|\bar{\nu}_s)\} \in \{q_{N_1}\}$. The desired $M_r$ has a form (2) with transition functions $f_r(z)$ and $\xi_r(z)$ instead of $f_L$ and $\xi_L$ where

$$M_r : \quad f_r(z) = z + \mu_r \nu_r \frac{(z - \bar{u}_r)}{(\bar{u}_r - \bar{v}_r)}, \quad \xi_r(z) = \frac{\mu_r(z - \bar{v}_r) - \nu_r(z - \bar{u}_r)}{\bar{u}_r - \bar{v}_r}. \quad (16)$$
So, \( \tilde{\mu}_r = \tilde{\nu}_r = 0, u_r = \tilde{u}_r \) and \( v_r = v_r + \mu_r \nu_r \). By (16) we go to the integration over \( \{ \tilde{\mu}_s, \tilde{\nu}_s, \tilde{\nu}_s, \tilde{\nu}_s \} \in \{ q_{N_1} \} \) and over \( (\mu_r, \nu_r) \) as well. The partition function in (13) has the form
\[
\tilde{Z}_{L,L'}^{(n)}(\{ q_N, \tilde{q}_N \}) = \tilde{Z}^{(n)}_{inv}(\{ q_{N_1}, \tilde{q}_{N_1} \}; L, L') \prod_{s=1}^{n} (u_s - v_s - \mu_s \nu_s)^{-1} \tag{17}
\]
with \( \tilde{Z}^{(n)}_{inv}(\{ q_{N_1}, \tilde{q}_{N_1} \}; L, L') \) to be \( SL_2 \) invariant \([3, 16]\). So in the new variables the \( (\mu_r, \nu_r) \) dependence in (17) is canceled by that in the Jacobian of the transformation (16). Due to (15), the \( \{ \tilde{q}_{N_1} \} \) set associated with the cluster, can be calculated through the corresponding super-Schottky multipliers and the corresponding \( \{ P(r, j) \} \)-variables, which, in turn, are calculated through \( \{ \tilde{P}_{N_1} \} \) as \( M(r, j; r_0, j_0) M M_r \{ \tilde{P}_{N_1} \} \). Since \( (\mu_r = 0, \nu_r = 0) \) in both \( \{ P(r, j) \} \) and \( \{ \tilde{P}_{N_1} \} \), the \( M(r, j; r_0, j_0) M M_r \) transformation is a properly Möbius one, its parameters depend only on \( (u_r, v_r, \tilde{u}_r, \tilde{v}_r) \), the regularization factor becomes independent of \( (\mu_r, \nu_r) \). Corrections to the partition functions and to (14) are found to be quadratic in a size of the \( Cl \) cluster that is sufficient to provide a finiteness of the \( A_{n,m} \) superstring amplitudes. So, the discussed divergences are cancelled in every superspin structure unlike those \([3, 16]\) due to a degeneration of a handle.

In the above consideration only the invariance of (11) under the \( SL_2 \) transformations (16) and (13) is used and the supermodular invariance of (10) only provides the supermodular invariance of \( A_{n,m} \). Otherwise the particular form (10) of the regularization factor is unessential.

The 0-, 1-, 2- and 3- point functions are calculated by a factorization of the amplitudes in suitable regions of the integration variables. The above finiteness of the amplitudes means that 0-, 1-, 2- and 3-point functions of massless superstring modes vanish as it is expected \([4]\).

If zeros of certain of the \( R_f^1(\hat{\zeta}, \hat{\zeta}_0) \) fermion Green functions go closely each to other, the Jacobian in (7) go to infinity that may origin in (10) singular terms proportional to powers of \( \delta \). Generally, it might give rise additional divergences in \( A_{n,m} \). We found that the discussed terms absent in two-loop amplitudes, but for arbitrary genus this matter remains to be seen. The above terms would be supermodular covariant in themselves because the \( P_{split} \) modular group is split and, so, every term in an expansion of an exponent (10) in powers of \( (\lambda_j^{(1)}, \lambda_j^{(2)}) \) is modular invariant. So, may be, these terms do not appear at all, but, contrary, may be, they are necessary for the unitarity conditions. In this case the above terms are naturally expected.
to give a finite contribution to $A_{n,m}$ because they correct finite contributions to the unitarity.

Once in (13) the integration over the $P_{\text{split}}$ Grassmann variables is performed, the regularization factor can be removed from the integrand whereas the obtained expressions are explicitly finite and modular invariant. The $P_{\text{split}}$ description is convenient for Grassmann integrations because in this case the integration region does not depend on the Grassmann variables. The transition from $(t, q_N)$ to the $P_{\text{split}}$ variables $(\hat{t}, \hat{q}_N)$ was discussed in Section 2. By $L_2$-transformation of $\hat{t}$ one can fix $(\hat{z}_1, \hat{z}_2, \hat{z}_3)$ to be the same as the \{z_b\} set. Being proportional to \{λ_j^{(1)}, λ_j^{(2)}\}, a difference between $t$ and $\hat{t}$ can not contribute to the integrals over supermanifold. Indeed, the above integrals surely do not depend on the above Grassmann parameters, if they are considered as functions of $q_N$. So, the transition to $P_{\text{split}}$ in (13) lies in substitution $q_N$ through $\hat{q}_N$ and in initiation of the Jacobian. The integration region over the supermanifold is determined now by the \{k_s, u_s, v_s\} parameters of the Schottky circles. The modular domain is determined by the ordinary modular group under the condition \{z_b\} to be outside all the Schottky circles. So only those terms contribute to the integral, which contain the product of all the $P_{\text{split}}$ Grassmann parameters. Among these terms we distinguish terms calculated at all $(\lambda_j^{(1)}, \lambda_j^{(2)})$ in $B_{\text{mod}}^{(n)}(\{q_N, \overline{q_N}\}; \overline{z_0}; \delta_0)$ to be zeros. Of the terms to be distinguished, we consider that calculated at all $(\lambda_j^{(1)}, \lambda_j^{(2)})$ in the Jacobian to be zeros. Since the product of all the super-Schottky modular parameters is the same as the inverse Jacobian multiplied by the product of the $P_{\text{split}}$ Grassmann modular ones, this term is obtained by the $(\mu_s, \nu_s)$ differentiation when \{k_s, u_s, v_s\} to be unchanged and $B_{\text{mod}}^{(n)}(\{q_N, \overline{q_N}\}; \overline{z_0}; \delta_0)$ is not differentiated. The term of interest is divided in a sum of the term obtained by the $(\mu_s, \nu_s)$ differentiation under fixed \{k_s, u_s, v_s + \mu_s \nu_s\} and of the remainder that presents total derivatives in respect to every $v_s$. The above remainder we refer to the group of terms that are due to the \{λ_j^{(1)}, λ_j^{(2)}\} dependence of the Jacobian and of \{k_s, u_s, v_s\} when they are expressed through the $P_{\text{split}}$ variables. All they are total derivatives in respect to \{k_s, \hat{u}_s, \hat{v}_s\}. Indeed, since these terms did not contain the product of all the Grassmann super-Schottky modular parameters, they can contribute to the integral only as surface terms due to the integration region to de-
pend on the above Grassmann modular ones. Hence, if in the new variables the above terms begin to depend on all the Grassmann variables, they necessary contribute to the integral as total derivatives. So, the integration over $P_{\text{split}}$ Grassmann variables being preformed, every the $(L, L')$ term in the integrand of (13) appears to be

$$B_{\text{mod}}^{(n)} \left[ \left( \prod_s \partial_{\mu_s} \partial_{\bar{\mu}_s} \partial_{\nu_s} \partial_{\bar{\nu}_s} \right) Z_{L,L'}^{(n)} K_{L,L'}^{(n,m)} A_{L,L'}^{(n)} \right]_{\{k_s, u_s, v_s + \mu_s \nu_s\}} + B_{\text{mod}}^{(n)} \sum_q \partial_q H_q^{(n,m)} + R$$

(18)

where for brevity we omit arguments in the values forming the integrand in (13). The derivatives in the first term are calculated under fixed the \(\{k_s, u_s, v_s + \mu_s \nu_s\}\) variables, which next can be replaced by \(\{\hat{k}_s, \hat{u}_s, \hat{v}_s\}\). The second term was discussed just above and $R$ is formed by terms proportional to powers of $\delta$ due to the \(\{\lambda_j^{(1)}, \lambda_j^{(2)}\}\) dependence of the regularization factor (10).

When the first term in (18) is calculated, the product behind $Z_{\text{inv}}^{(n)}(\{q_N, \bar{q}_N\}; L, L')$ in (17) can be taken at \(\{\mu_s = 0, \nu_s = 0\}\) since only the above $SL_2$ invariant part of (17) is differentiated. Hence, if the above $Cl$ clusters of handles arise, the function associated with the first term of (18) ceases to depend on certain Grassmann super-Schottky parameters, the first term in (18) vanishes in the cases of interest and, therefore, its contribution to $A_{n,m}$ is finite. This term is not, however, modular covariant because, generally, the Jacobian of a supermodular transformation depends on \(\{\mu_s, \nu_s\}\) owing to the non-split property (16) of the supermodular group. Only the sum of the first term and of the second one in (18) is supermodular invariant, the third term being supermodular invariant, as it was noted already. The integration by parts reduces the second term in (13) to an integral $\tilde{R}$ due to differentiation of $B_{\text{mod}}^{(n)}$ and to an integral $S$ over that part $\tilde{b}$ of the moduli region boundary $b$, which does not mapped oneself under modular transformations and $\tilde{b}$ can be obtained by modular changes of $\tilde{b}$. Up to terms that are supermodular invariant in themselves (if they exist at all) the integrand of $S$ can be calculated without using an explicit form of the second term in (18), only from a condition that changes of $S$ under supermodular transformations cancel the corresponding changes of the first term. In this case supermodular changes of the first term are calculated in terms of changes of super-Schottky parameters under supermodular transformations preserving the $\{z_b\}$ set in (13). The above supermodular changes of the super-Schottky parameters are obtained by the
transformation (15) of those changes of above parameters, which were calculated in [16] (see Sections 3 and 4 of [16]). The resulted boundary integral is obtained finite due to the finiteness of the first term contribution to $A_{n,m}$. Hence the regularization factors can be removed from both the discussed integrals. For genus-2 amplitudes, we found that any additions to the boundary integral are absent and $\tilde{R}$ can be neglected in the $\delta \to 0$ limit, as well as $R$. It is quite plausible that the same is true in a general case too. In this case the desired $A_{n,m}$ amplitude appears to be

$$A_{n,m} = \sum_{L,L'} \int \left( \prod_r dq_r dq\bar{r}_r \right) \left[ \left( \prod_s \partial_{\mu_s} \partial_{\nu_s} \partial_{\bar{\mu}_s} \partial_{\bar{\nu}_s} \right) Z_{L,L'}^{(n)} \mathcal{K}_{L,L'}^{(n,m)} \right]_{\{k_s,u_s,v_s+\mu_s+\nu_s\}} + S$$

where the boundary integral $S$ is calculated as it was discussed just above. The obtained $A_{n,m}$ amplitude is finite and supermodular invariant as well. Since superstrings are non-invariant under the $P_{\text{split}}$ transformation, (19) differs essentially from the expressions in [2] where a split property of the supermanifolds has been assumed. Unlike [2], our amplitudes do not depend on the choice of a basis of the gravitino zero modes. As it was discussed above, for genus-$n > 2$ amplitudes we can not at present exclude additional terms in $S$ that are supermodular invariant in themselves and additional supermodular invariant terms due to contribution of $R$ and of $\tilde{R}$. If they present or not, it can be clarify by a direct examination of the second and third terms in (18) that is in progress now. It seems, however, that the discussed terms may appear only, if it is dictated by the unitarity conditions. In this case they expected to be finite because they correct finite contributions to the unitarity.

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