The Quantum Skyrmion in Representations of General Dimension

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Abstract

The representations of general dimension are constructed for the SU(2) Skyrme model, treated quantum mechanically ab initio. This quantum Skyrme model has a negative mass term correction, that is not present in the classical Hamiltonian. The magnitude of the quantum mechanical mass correction increases with the dimension of the representation of the SU(2) group. In the case of a 5-dimensional representation it is possible to obtain satisfactory predictions for the nucleon mass with the empirical value for the pion decay constant.
1. Introduction

The $SU(2)$ version of Skyrme’s topological soliton model for the baryons [1, 2] is conventionally described with field operators that belong to the fundamental 2-dimensional representation of the $SU(2)$ group. At the classical level the predictions for the baryon observables and phenomenology turn out to be independent of the dimension of the representation used for the group in the case of the original version of the Skyrme model [3]. This situation changes when the Lagrangian density of the Skyrme model is treated quantum mechanically $ab$ $initio$. In this case, as will be shown below, the negative purely quantum mechanical mass correction that arises in the systematic quantization [4] of the model, is representation dependent, and grows in magnitude and significance with the dimension of the representation.

In this paper the quantum mechanical treatment of the Skyrme model in a representation of arbitrary dimension will be developed. The theoretical formalism builds on that developed for the classical treatment of the Skyrme model in a general representation in ref. [3]. A quantum mechanical mass formula for the baryon states is derived. In addition the expressions for the Noether and anomalous current operators are derived. Finally we study the dependence of the predictions for the phenomenological baryon structure parameters on the dimension of the presentation numerically, and show that the quantum mechanical treatment makes it possible to obtain satisfactory predictions for the baryon masses with the empirical value of the pion decay constant if a 5-dimensional representation is employed.

This paper is divided into 5 sections. In section 2 the classical treatment of the Skyrme model in a representation of general dimension [3] is reviewed. In section 3 the quantum mechanical treatment of the Skyrme model is developed. In section 4 the Noether currents of the Lagrangian density are derived, along with the expressions for the magnetic moments of the nucleons and the $\Delta_{33}$ resonances as well as the axial coupling constant of the nucleon. Numerical results for these observables are given in section 5. Section 6 contains a concluding discussion.
2. The classical skyrmion in a general representation.

The Skyrme model is based on a Lagrangian density for a unitary field $U(\vec{r}, t)$ that belongs to an irreducible representation of the $SU(2)$ group. In a general irreducible representation it is convenient to express the unitary field $U$ in terms of three unconstrained Euler angles $\alpha = (\alpha^1, \alpha^2, \alpha^3)$ as

$$U(x, t) = D^j(\alpha(x, t)).$$

(2.1)

The elements of the matrices $D^j$ are the Wigner D-functions, where $(2j + 1)$ is the dimension of the $SU(2)$ representation. The Euler angles $\alpha$ then form the dynamical variables of the theory.

The Skyrme model is defined by the chirally symmetric Lagrangian density

$$\mathcal{L}[U(x, t)] = -\frac{f_\pi^2}{4} \text{Tr}\{R_\mu R^\mu\} + \frac{1}{32e^2} \text{Tr}\{[R_\mu, R_\nu]^2\},$$

(2.2)

where the "right" current $R_\mu$ is defined as

$$R_\mu = (\partial_\mu U)U^\dagger,$$

(2.3)

and $f_\pi$ (the pion decay constant) and $e$ are parameters. As was shown in [3] the classical Lagrangian density depends on the dimension of the representation $j$ only through the overall scalar factor

$$N = \frac{2}{3}j(j + 1)(2j + 1).$$

(2.4)

which can be incorporated in the parameters by a renormalization. As a consequence the equations of motion for the dynamical variables $\alpha$ are independent of the dimension of the representation.

The trace of a bilinear combination of two generators of the group $\hat{J}_a, \hat{J}_b$ depends on the dimension of the representation as

$$\text{Tr}\langle jm|\hat{J}_a\hat{J}_b|jm'\rangle = (-)^a \frac{1}{6} j(j + 1)(2j + 1)\delta_{a,-b}.$$  

(2.5)

The commutator relations for the generators are

$$[\hat{J}_a, \hat{J}_b] = \left[ \begin{array}{ccc} 1 & 1 & 1 \\ a & b & c \\ \end{array} \right] \hat{J}_c.$$  

(2.6)
Here the factor on the r.h.s. is the Clebsch-Gordan coefficient \( (1a1b|1c) \), in a more transparent notation. The components of the operators \( \hat{J}_a \) are defined above as \( \hat{J}_\pm = -J_{\pm}\sqrt{2} \) and \( \hat{J}_0 = -J_0/\sqrt{2} \).

The "spherically symmetric" hedgehog ansatz in a general representation is invariant under the combined spatial and isospin rotation

\[
i [\mathbf{x} \times \nabla]_a \ U(\mathbf{x}) + \sqrt{2} \ [J_a, U(\mathbf{x})] = 0,
\]

where circular components are used for both the vector and isovector. The solution of (2.7) is the generalization of the usual hedgehog ansatz

\[
e^{i \vec{r} \cdot \vec{F}(r)} \Rightarrow U_0(\mathbf{x}) = \exp [-i \sqrt{2} \hat{J}_a \cdot \hat{x}^a F(r)].
\]

Here the circular coordinates of the unit vector \( \hat{x} \) are defined as

\[
\hat{x}^+ = -\frac{1}{\sqrt{2}} (\hat{x}_1 - i \hat{x}_2) = -\frac{1}{\sqrt{2}} \sin \theta e^{-i \varphi} = -\hat{x}^-, \\
\hat{x}^0 = \hat{x}_3 = \cos \theta = \hat{x}_0, \\
\hat{x}^- = \frac{1}{\sqrt{2}} (\hat{x}_1 + i \hat{x}_2) = \frac{1}{\sqrt{2}} \sin \theta e^{i \varphi} = -\hat{x}^+.
\]

The generalized hedgehog ansatz \( U_0(\mathbf{x}) = D^j(\beta(\mathbf{x})) \) can be expressed in terms of Euler angles as

\[
\beta^1(\mathbf{x}) = \varphi - \arctan(\cos \vartheta \tan F(r)) - \pi/2, \\
\beta^2(\mathbf{x}) = -2 \arcsin(\sin \vartheta \sin F(r)), \\
\beta^3(\mathbf{x}) = -\varphi - \arctan(\cos \vartheta \tan F(r)) + \pi/2.
\]

Here the angles \( \varphi \) and \( \vartheta \) are the polar angles that define the direction of the unit vector \( \hat{x} \).

With the hedgehog ansatz (2.8) the Lagrangian density (2.2) reduces to the following simple form

\[
L(F(r)) = -\frac{4}{3} j(j+1)(2j+1) \left\{ \frac{f^2}{4} \left( F''^2 + \frac{2}{r^2} \sin^2 F \right) \\
+ \frac{1}{16 e^2} \frac{\sin^2 F}{r^2} \left( 2F'^2 + \frac{\sin^2 F}{r^2} \right) \right\}.
\]
(2.4) does not affect the solution and hence the classical soliton is independent of the dimension of the representation.

After the renormalization the hedgehog mass in any representation \( j \) has the form

\[
M(F) = \frac{f_\pi}{e} \tilde{M}(F) = 2\pi \frac{f_\pi}{e} \int d\tilde{r} \tilde{r}^2 \left[ F' \sin^2 \frac{F}{\tilde{r}} \left( 2 + 2F'' + \sin^2 \frac{F}{\tilde{r}} \right) \right], \tag{2.12}
\]

where the dimensionless variable \( \tilde{r} \) is defined as \( \tilde{r} = e f_\pi r \). Variation of the mass leads to the standard differential equation

\[
F'' + 2F'' \frac{\sin^2 F}{\tilde{r}^2} + F' \frac{\sin 2F}{\tilde{r}^2} + \frac{2}{\tilde{r}^2} F' - \frac{\sin 2F}{\tilde{r}^2} - \frac{\sin 2F \sin^2 F}{\tilde{r}^4} = 0 \tag{2.13}
\]

for the chiral angle \( F(r) \).

For the hedgehog solution the baryon density takes the form

\[
B^0 = \frac{1}{24\pi^2 N} \epsilon^{0\alpha\beta\gamma} \text{Tr} R_\nu R_\beta R_\gamma = -\frac{1}{2\pi^2} \frac{\sin^2 F}{r^2} F', \tag{2.14}
\]

The renormalization factor \( N \) ensures that the lowest nonvanishing baryon number is \( B = 1 \) for the hedgehog in all representations.

3. Quantization of skyrmion in collective coordinate approach.

The quantization of Skyrme model in a general dimension is a bit intricate [3]. Following Adkins et al. [2] we shall employ collective rotational coordinates to separate the variables which depend on the time and spatial coordinates:

\[
U(x, q(t)) = A(q(t)) U_0(x) A^\dagger(q(t)). \tag{3.1}
\]

The set of three real, independent parameters \( q(t) = (q^1(t), q^2(t), q^3(t)) \) are quantum variables (skyrmion rotation Euler angles). In a general representation the unconstrained variables \( q(t) \) are more convenient than the four
constrained Euler-Rodrigues parameters used in [2]. We shall consider the Skyrme Lagrangian (2.2) quantum mechanically \textit{ab initio}. The generalized coordinates $\mathbf{q}(t)$ and velocities $\dot{\mathbf{q}}(t)$ then satisfy the commutation relations [5]:

$$[\dot{q}^a, q^b] = -if^{ab}(\mathbf{q}). \quad (3.2)$$

Here the tensor $f^{ab}(\mathbf{q})$ is a function of generalized coordinates $\mathbf{q}$ only, the explicit form of which is determined after the quantization condition has been imposed. The tensor $f^{ab}$ is symmetric with respect to interchange of the indices $a$ and $b$ as a consequence of the commutation relation $[q^a, q^b] = 0$. The commutator relation between a generalized velocity component $\dot{q}^a$ and arbitrary function $G(\mathbf{q})$ is given by

$$[\dot{q}^a, G(\mathbf{q})] = -i \sum_r f^{ar}(\mathbf{q}) \frac{\partial}{\partial q^r} G(\mathbf{q}). \quad (3.3)$$

After making the substitution (3.1) into the Lagrangian density (2.2) the dependence of the Lagrangian on the generalized velocities can be expressed as

$$L(\dot{\mathbf{q}}, \mathbf{q}, F) = \frac{1}{N} \int L(\mathbf{x}, \mathbf{q}(t), F(r)) r^2 \sin \vartheta dr d\vartheta d\varphi =$$

$$-\frac{1}{4} a(F) \dot{q}^a g(\mathbf{q})_{\alpha \beta} \dot{q}^\beta + [(\dot{q})^0-\text{order term}] \quad (3.4)$$

Here $a(F)$ is defined as the constant.

$$a(F) = \frac{1}{e^3 f_\pi} \bar{a}(F) = \frac{1}{e^3 f_\pi} \frac{8\pi}{3} \int dr r^2 \sin^2 F \left[ 1 + Fr^2 + \frac{\sin^2 F}{r^2} \right] , \quad (3.5)$$

The $3\times3$ metric tensor $g(\mathbf{q})_{\alpha \beta}$ is defined as the scalar product of a set of functions $C^{(m)}(\mathbf{q})$ [3] as

$$g(\mathbf{q})_{\alpha \beta} = \sum_m (-)^m C^{(m)}_\alpha C^{(-m)}_\beta \sum_m (-)^m C'^{(m)}_\alpha C'^{(-m)}_\beta =$$

$$-2\delta_{\alpha \beta} - 2(\delta_{\alpha 1} \delta_{\beta 3} + \delta_{\alpha 3} \delta_{\beta 1}) \cos q^2 \cos \vartheta , \quad (3.6)$$

where the functions $C'^{(m)}_\alpha$ are defined as

$$C'^{(m)}_\alpha(\mathbf{q}) = \sum_m D^{1}_{m,m'}(\mathbf{q}) C^{m}(\mathbf{q}) . \quad (3.7)$$
The orthogonality relations for the functions $C^{(m)}_{\alpha}$ are
\[
\sum_m C_{\alpha}^{(m)} C^{(m)}_{\beta} = \sum_m C_{\alpha}^{(m)} C_{\beta}^{(m)} = \delta_{\alpha,\beta},
\]
(3.8)
\[
\sum_{\alpha} C_{\alpha}^{(m)} C_{\alpha}^{(n)} = \sum_{\alpha} C_{\alpha}^{(m)} C_{\alpha}^{(n)} = \delta_{m,n}.
\]
(3.9)

The appropriate definition for the canonical momentum $p_{\alpha}$, which is conjugate to $q^{\alpha}$, is
\[
p_{\alpha}(\dot{q}, q, F) = \frac{\partial L(\dot{q}, q, F)}{\partial \dot{q}^{\alpha}} = -\frac{1}{4} a(F) \{\dot{q}^{\beta}, g(q)_{\beta\alpha}\},
\]
(3.10)
where the curly bracket denotes an anticommutator. The canonical commutation relations
\[
[p_{\alpha}(\dot{q}, q, F), q^{\beta}] = -i\delta_{\alpha\beta},
\]
(3.11)
then yield the following explicit form for the functions (3.2) $f^{ab}(q)$
\[
f^{ab}(q) = -\frac{2}{a(F)} g^{-1}_{\alpha\beta}(q).
\]
(3.12)

Because of the nonlinearity of the Skyrme model the canonical momenta defined in this way do not necessarily satisfy the relation $[p_{\alpha}, p_{\beta}] = 0$. As shown in [5], there does however exist a local transformation of the set of variables $q$, which makes it possible to satisfy these relations. Define the angular momentum operator
\[
\hat{J}^a = -\frac{i}{2} \{p_{r}, C^r_{(-a)}(q)\} = (-)^a \frac{ia(F)}{4} \{\dot{q}^{\beta}, C^r_{\beta}(-a) (q)\},
\]
(3.13)
which satisfies the commutation relations (2.5). The operator $\hat{J}^a$ is then a "right rotation" generating matrix $D^j(q)$:
\[
[\hat{J}^a, D^{l}_{m,m'}(q)] = -\langle l, m' + a | \hat{J}^a | l, m' \rangle D^{l}_{m,m' + a}(q),
\]
(3.14)
and
\[
\hat{J} = -\frac{i}{2} \{p_{r}, C^r_{(-a)}(q)\} = (-)^a \frac{ia(F)}{4} \{\dot{q}^{\beta}, C^r_{\beta}(-a) (q)\},
\]
(3.15)
is a "left rotation" generating matrix $D^j(q)$:
\[
[\hat{J}^a, D^{l}_{m,m'}(q)] = \langle l, m | \hat{J}^a | l, m - a \rangle D^{l}_{m - a, m'}(q).
\]
(3.16)
Some lengthy manipulation yields the following explicit form for the Lagrangian:

\[ L(\dot{q}, q, F) = -M(F) - \Delta M_j(F) + \frac{1}{a(F)} \dot{J}^2 = \]

\[ -M(F) - \Delta M_j(F) + \frac{1}{a(F)} \dot{J}^2, \quad (3.17) \]

where

\[ \Delta M_j(F) = e^3 f_\pi \cdot \Delta \tilde{M}_j(F) = e^3 f_\pi \frac{-2\pi}{5a^2(F)} \int d\tilde{r} \tilde{r}^2 \sin^2 F \times \]

\[ \left[ 5 + 2(2j - 1)(2j + 3) \sin^2 F + [2j(j + 1) + 1] \frac{\sin^2 F}{\tilde{r}^2} \right. \]

\[ + [8j(j + 1) - 1] F'^2 - 2(2j - 1)(2j + 3) F'^2 \sin^2 F \right] \quad (3.18) \]

The corresponding Hamilton operator is then

\[ H_j(F) = M(F) + \Delta M_j(F) + \frac{1}{a(F)} \dot{J}^2 = M(F) + \Delta M_j(F) + \frac{1}{a(F)} \dot{J}^2. \quad (3.19) \]

The most important feature of this result is that the quantum correction \( \Delta M_j(F) \) is negative definite and that it depends explicitly on the dimension of the representation of the \( SU(2) \) group. This term is lost in the usual semiclassical treatment of the Skyrme model even in the fundamental representation of \( SU(2) \), because that ignores the commutation relations \( (3.2) \). In the numerical work reported below we shall have to treat this quantum mass correction as a perturbation, and use the classical equation of motion for the chiral angle \( F(r) \) \( (2.13) \) that is obtained by variation of the classical mass expression \( (2.12) \). This implies that the quantum skyrmion is considered as a rigid rotating classical skyrmion, where the collective variables describe the spinning mode of the model. This perturbative treatment of the quantum correction is motivated by the fact that the equation of motion that would be obtained by requiring the quantum mass expression \( (3.18) \) to be stationary has physically acceptable solutions only in a very narrow parameter window \[6\].

For the Hamiltonian \( (3.19) \) are the normalized state vectors with fixed spin and isospin \( \ell \)

\[ \left| \ell \left| m, m' \right> = \frac{\sqrt{2\ell + 1}}{4\pi} D_{m,m'}^{\ell}(q) |0> \right., \quad (3.20) \]
with the eigenvalues

\[ H(j, \ell, F) = M(F) + \Delta M_j(F) + \frac{\ell(\ell + 1)}{2a(F)}. \] (3.21)

4. The Noether currents.

The Lagrangian density of the Skyrme model is invariant under left and right transformations of the unitary field \( U \). The corresponding Noether currents can be expressed in terms of the collective coordinates (3.1). The vector and axial Noether currents that are associated with the transformations

\[ U(x) \overset{V(A)}{\rightarrow} \left( 1 - i2\sqrt{2}\omega^a \hat{J}_a \right) U(x) \left( 1 + (-)i2\sqrt{2}\omega^a \hat{J}_a \right) \] (4.1)

are nevertheless simpler and directly related to physical observables. The factor \(-2\sqrt{2}\) before the generators is introduced so that the transformation (4.1) for \( j = 1/2 \) matches the infinitesimal transformation in [2]. The Noether currents are operators in terms of the generalized collective coordinates \( q \) and the generalized angular momentum operator \( \hat{J}' \) (3.13). The explicit expression for the vector current density is

\[
\hat{V}^a_b = \frac{\partial L_V}{\partial (\nabla^b \omega_a)} = \frac{4\sqrt{2}}{3} j(j + 1)(2j + 1) \sin^2 F \left( i \left\{ \frac{f^2}{e^2} \left( F'^2 + \sin^2 F \right) \right\} \right) \frac{1}{u \, s \, b} \left[ D^1_{a,s}(q) \hat{x}_u \right.
\]

\[
- \frac{\sin^2 F}{\sqrt{2} \cdot e^2 \cdot a^2(F)} (-)^s \left\{ [\hat{j}' \times \hat{x}]_{-s} D^1_{a,s}(q) \left[ [\hat{j}' \times \hat{x}] \times \hat{x} \right]_{b} \right.
\]

\[
- \frac{\sin^2 F}{\sqrt{2} \cdot e^2 \cdot a^2(F)} (-)^s \left\{ [\hat{j}' \times \hat{x}] \times \hat{x} \right\} \left[ [\hat{j}' \times \hat{x}] \times \hat{x} \right] \left[ D^1_{a,s}(q) \left[ [\hat{j}' \times \hat{x}] \times \hat{x} \right] \right]_{-s} \right). \] (4.2)

Here \( \nabla^k \) is a circular component of the gradient operator. The indexes \( a \) and \( b \) denote isospin and spin components. The time (charge) component of the vector current density becomes

\[
\hat{V}^a_t = \frac{\partial L_V}{\partial (\partial^0 \omega_a)} = \frac{4\sqrt{2}}{3 \cdot a(F)} j(j + 1)(2j + 1) \sin^2 F \left[ f_x + \frac{1}{e^2} \left( F'^2 + \frac{\sin^2 F}{r^2} \right) \right] \times
\]
The explicit expression for the axial current density takes the form

\[
\hat{A}_a^s = \frac{\partial L_A}{\partial (\nabla_b \omega_a)} = \frac{2}{3} j(j + 1)(2j + 1) \left\{ \begin{array}{c}
\frac{f_\pi^2 \sin 2F}{r} + \frac{1}{e^2} \left( \frac{F'^2}{r} + \frac{\sin^2 F}{r^2} \right) \\
- \frac{\sin^2 F}{4 \cdot a^2(F)} \end{array} \right\} D^{1}_{a,b} (q) + \left\{ \begin{array}{c}
f_\pi^2 \left( 2F' - \sin 2F \right) - \frac{1}{e^2} \left( \frac{F'^2 \sin 2F}{r} - 4F' \sin^2 F \right) \\
+ \frac{\sin^2 F}{e^2 \cdot a^2(F) \cdot r} \end{array} \right\} (-)^s D^{1}_{a,s} (q) \hat{x}_s \hat{x}_b - \frac{2F' \sin^2 F}{r^2} \times \\
(-)^s \left\{ D^{1}_{a,s} (q) \hat{x}_s \hat{j}^{\prime 2} + \hat{j}^{\prime 2} D^{1}_{a,s} (q) \hat{x}_s - 2D^{1}_{a,s} (q) \hat{x}_s (\hat{j}^{\prime} \cdot \hat{x}) (\hat{j}^{\prime} \cdot \hat{x}) \right\} \hat{x}_b \times \\
- \frac{\sin^2 F \sin 2F}{e^2 \cdot a^2(F) \cdot r} (-)^s \left\{ \left[ \hat{j}^{\prime} \times \hat{x} \right] \times \hat{x} \right\}_{a,s} \left[ \left[ \hat{j}^{\prime} \times \hat{x} \right] \times \hat{x} \right]_{b,s} \right\} \\
+ \left[ \left[ \hat{j}^{\prime} \times \hat{x} \right] \times \hat{x} \right]_a D^{1}_{a,s} (q) \left[ \left[ \hat{j}^{\prime} \times \hat{x} \right] \times \hat{x} \right]_{-s} \right\} \right\} \\
(4.4)
\]

The operators (4.2), (4.3) and (4.4) are well defined for all representations \( j \) of the classical soliton and for fixed spin and isospin \( l \) of the quantum skyrmion. The new terms which are absent in the semiclassical case are those that have the factor \( a^2(F) \) in the denominator.

The conserved topological current density in Skyrme model is the baryon current density, the components of which are

\[
B_a (x,F(r)) = \frac{1}{\sqrt{2\pi^2 a(F)}} \frac{\sin^2 F \cdot F'}{r} \left[ \hat{j}^{\prime} \times \hat{x} \right]_a. \quad (4.5)
\]

The matrix elements of the third component of the corresponding isoscalar magnetic moment operator have the form

\[
\left\langle m_t m_s \left| [\mu_{\ell=0}]_3 \right| \frac{\ell}{m_t m_s} \right\rangle = \left\langle \frac{\ell}{m_t m_s} \left| \frac{1}{2} \int d^3 x r \left[ \hat{x} \times \hat{B} \right]_0 \right| \frac{\ell}{m_t m_s} \right\rangle = \frac{[\ell(\ell + 1)]^{1/2} e}{3 \cdot \tilde{a}(F) f_\pi} \langle \hat{r}^2 \rangle_{\ell=0} \left[ \frac{\ell}{m_s} 1 \ell \right] \left[ \frac{\ell}{m_s} 0 m_s \right], \quad (4.6)
\]
where the mean square radius is given as
\[ \langle \tilde{r}_I^2 \rangle = -\frac{2}{\pi} \int \tilde{r}^2 \sin^2 F \cdot F' d\tilde{r}, \tag{4.7} \]
and \( \tilde{a} \) is defined in eq. (3.5).

The matrix elements of the third component of the isovector part of magnetic moment operator that is obtained from the isovector current (4.3) have the form
\[ \langle \ell m_t m_s | [\mu_I = 1]_3 | \ell m_t m_s \rangle = \frac{1}{2} \int d^3 x \cdot \tilde{r} \left[ \hat{x} \times \hat{V}^3 \right]_0 | \ell m_t m_s \rangle = \left[ \tilde{a}(F) \right] e^3 \cdot f_\pi + \frac{8\pi \cdot e}{3 \cdot \tilde{a}(F) \cdot f_\pi} \int d\tilde{r} \cdot \tilde{r}^2 \sin^4 F \left( 1 - \frac{(2j - 1)(2j + 3)}{2 \cdot 5} \right) \]
\[ - \frac{l(l + 1)}{3} + \frac{(-)^{2l}}{2} \left[ \frac{5l(l + 1)(2l - 1)(2l + 1)(2l + 3)}{2 \cdot 3} \right]^{1/2} \left[ \begin{array}{ccc} 1 & 2 & 1 \\ l & l & l \end{array} \right] \left[ \begin{array}{ccc} \ell & 1 & \ell \\ m_s & 0 & m_s \\ \ell & 1 & \ell \\ m_t & 0 & m_t \end{array} \right], \tag{4.8} \]
where the symbol in the curly brackets is a \( 6\jmath \) coefficient.

From the axial current density (4.4) we obtain the axial coupling constant \( g_A \) of the nucleon as
\[ g_A = -3 \left\langle \frac{1}{2}, 1/2 \left| \frac{1}{2}, 1/2 \right| d^3 x A_0^9 \right\rangle = \frac{1}{e^2} \tilde{g}_1(F) - \frac{\pi^2 e^2}{3 \cdot \tilde{a}^2(F)} \langle \tilde{r}_I^2 = 0 \rangle, \tag{4.9} \]
where
\[ \tilde{g}_1(F) = \frac{4\pi}{3} \int d\tilde{r} \left( \tilde{r}^2 F' + \tilde{r} \sin 2F + \tilde{r} \sin 2F \cdot F'^2 \right. \]
\[ + 2 \sin^2 F \cdot F' + \frac{\sin^2 F}{\tilde{r}} \sin 2F \right). \tag{4.10} \]
5. The static properties of the nucleon and the $\Delta_{33}$ resonance

The $I = J = \ell = 1/2$ and $I = J = \ell = 3/2$ skyrmions are to be identified with the nucleons and the $\Delta_{33}$ resonances. As in ref. [2] we determine the two parameters in the Lagrangian density (2.2) so that their masses take their empirical values. The expressions for the nucleon and $\Delta_{33}$ masses are

$$m_N = \frac{f_\pi}{e} \tilde{M}(F) + e^3 f_\pi \Delta \tilde{M}_j(F) + \frac{e^3 f_\pi}{2 \cdot \tilde{a}(F)} \frac{3}{4}, \tag{5.1}$$

$$m_\Delta = \frac{f_\pi}{e} \tilde{M}(F) + e^3 f_\pi \Delta \tilde{M}_j(F) + \frac{e^3 f_\pi}{2 \cdot \tilde{a}(F)} \frac{15}{4}. \tag{5.2}$$

In the evaluation of these two masses numerically we employ the chiral angle $F(r)$, which is obtained by solving the classical equation of motion that is given by the requirement that the classical mass (2.12) by stationary. The corresponding values for the Lagrangian parameters are given in Table 1 for different values of the dimension $(2j + 1)$ of the $SU(2)$ representation.

In the table we also include the predicted values for the other static nucleon properties, as well as the original predictions obtained in ref. [2] for the classical Skyrme model. In the case of the fundamental representation $j = 1/2$ the numerical importance of the quantum correction is small, as was to be expected. For larger values of $j$ the quantum corrections become increasingly important. The key qualitative feature is that the quantum mass correction $\Delta \tilde{M}_j(F)$ is negative, and hence it becomes possible to reproduce the empirical nucleon and $\Delta_{33}$ mass values with increasingly realistic values of the pion decay constant $f_\pi$. This reaches its empirical value 93 MeV for a representation of dimension 5. There is an accompanying improvement of the numerical value for the axial coupling constant $g_A$.

In the case of the isoscalar radius $r_0$ of the baryon, there is however no reduction of the difference between the predicted and the empirical value with increasing dimension of the representation. The same is true for the magnetic moments. The predicted value for the ratio of the proton and neutron magnetic moments deteriorates slowly with increasing dimension of the representation.
6. Discussion

Once the Skyrme model is treated consistently quantum mechanically \textit{ab initio} the dimension of the representation of the $SU(2)$ group becomes a significant additional model parameter. When the dimension of the representation in increased to 5 from the value 2 for the fundamental representation, it becomes possible to obtain satisfactory values for the masses of the nucleon and the $\Delta_{33}$ resonance with a value for the pion decay constant, which is very close to the empirical value (89.4 MeV vs. 93 MeV). There is unfortunately no comparable gain in quality of the predictions for the baryon magnetic moments, which deteriorate slowly with increasing dimension of the representation. The value of the axial coupling constant does on the other hand improve, but stays below 1 for representations of reasonably low dimension. The fact that the axial coupling constant remains low is a natural consequence of the vanishing axial charge commutator in the Skyrme model [7,8].

Note that the perturbative treatment used here for the quantum skyrmion breaks down when the dimension of the representation grows so large that the negative quantum mass correction becomes of the same order of magnitude as or larger than the classical skyrmion mass. This feature is clearly related to the fact that the equation of motion for the quantum skyrmion has physically acceptable solutions only in a narrow parameter window [6].

The numerical value of the quantum mass correction $\Delta M_j(F)$ (3.18) is of the order 100 MeV in the fundamental representation, but it rapidly increases in magnitude as the dimension of the representation grows. For a representation of dimension 5 it is large enough to cancel the $\sim 500$ MeV overprediction of the nucleon mass that obtains when the empirical value for the pion decay constant is employed in the classical Skyrme model. It is interesting to note that it plays a similar role to the (negative) Casimir correction to the Skyrmion energy considered in ref. [9].

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Table 1. The predicted static baryon observables as obtained with the quantum Skyrme model for representations of different dimension. The first column (ANW) are the predictions for the classical Skyrme model given in ref. [2]. The empirical results [10,11] are listed in the last column.

|     | ANW | $j = 1/2$ | $j = 1$ | $j = 3/2$ | $j = 2$ | $j = 5/2$ | Exp. |
|-----|-----|-----------|---------|-----------|---------|-----------|------|
| $m_N$ | input | input | input | input | input | input | 939 MeV. |
| $m_\Delta$ | input | input | input | input | input | input | 1232 MeV. |
| $f_\pi$ | 64.5 | 72.1 | 76.4 | 82.2 | 89.4 | 98.0 | 93 MeV. |
| $\epsilon$ | 5.45 | 5.23 | 5.15 | 5.03 | 4.89 | 4.74 | |
| $r_0$ | 0.59 | 0.55 | 0.53 | 0.51 | 0.48 | 0.45 | 0.72 fm. |
| $\mu_p$ | 1.87 | 1.90 | 1.84 | 1.78 | 1.71 | 1.64 | 2.79 |
| $\mu_n$ | -1.31 | -1.42 | -1.40 | -1.37 | -1.35 | -1.33 | -1.91 |
| $g_A$ | 0.61 | 0.65 | 0.68 | 0.71 | 0.76 | 0.80 | 1.23 |
| $\mu_{\Delta^{++}}$ | 3.70 | 3.58 | 3.44 | 3.29 | 3.15 | 4.52 | |
| $\mu_{\Delta^+}$ | 1.71 | 1.64 | 1.55 | 1.46 | 1.37 | ? | |
| $\mu_{\Delta^0}$ | -0.28 | -0.31 | -0.34 | -0.38 | -0.42 | ? | |
| $\mu_{\Delta^-}$ | -2.27 | -2.25 | -2.23 | -2.21 | -2.20 | ? | |