Numerical study on the influence of initial conditions on quasi-periodic oscillation of double pendulum system

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Abstract. The double pendulum system is a common and typical model to investigate nonlinear dynamics due to its various and complex dynamical phenomenon. In this paper, the effects of several parameters on the system was studied, such as the length and the initial release angle of the pendulums. Though the Swing length - frequency diagram, the details of oscillation was researched and a non-smooth inflection point was observed on the curve.

1. Introduction

The physics problems, usually described as dynamical equations, are divided into two categories: linear problems or nonlinear problems. A plenty of linear models are applied to approximate our natural world and lots of decent results have been achieved. However, with the development of human cognition, the complex problems of nonlinearity have gradually become more important. Nonlinear dynamics were born in this context[1][2][3]. The double pendulum system has a simple structure, but its oscillation state is very complicated. The simple structure helps to build the model, and its various states of oscillation makes the content of the inquiry more abundant. Under certain conditions, the nonlinear system will exhibit the bifurcation, the movement of the periodic oscillation and the aperiodic oscillation. Eventually, the state of aperiodic orderly oscillation, that is, the state of chaos, will be achieved. Sensitive to initial conditions is a very important feature of chaotic systems[4][5][6].

The energy of the double pendulum system consists of two parts: one is kinetic energy, the other part is the potential energy. Jeo Chen mentioned the method of characterizing the chaotic state of the system using the Lyapunov exponent and the Lyapunov exponent to analyze the oscillation state of the double pendulum model under damping and forced conditions[7]. Sensitive to initial conditions is not a unique feature of the double pendulum system. Meteorologist Edward Lorenz has proposed the butterfly effect, which is also a typical example of chaos[8][9]. The traditional double pendulum system generally consists of two thin rods with lengths of l₁ and l₂ and two small balls fixed at the end of the thin rod, the masses being m₁, m₂. The angle between the inner pendulum and the vertical line is θ₁; the angle between the outer pendulum and the vertical line is θ₂. The quality of the thin rod and the shape of the ball will be different for different researchers[7]. For example, some researchers will consider the moment of inertia of the rod; some researchers will change the shape and size of the ball to study the problem[10]. Many studies are not satisfied with the ideal system. Based on this basic model, the cyclical force and damping force are added to study the properties of the dissipative system[11][12]. The double pendulum system is shown in Figure 1.
Figure 1. Schematic diagram of the double pendulum system

The time series is a commonly used research tool for studying the behavior and state evolution in dynamic systems. The abscissa is time and the ordinate is the variable in the system, such as the swing angle or angular velocity of the two pendulums. Many details are reflected by the time series, such as the amplitude, phase and frequency variations of the double pendulum system. When in some initial conditions, such as the initial release angle of both pendulums are $90^\circ$, the double pendulum system will be in a chaotic oscillation state. In order to more intuitively observe the oscillation state and dynamics of the system, it is usually possible to study with the phase diagram between the various variables in the system. For example, through the phase diagram, the results shown that the double pendulum system is doing quasi-periodic oscillation and is still in a chaotic state. This phenomenon illustrates the system's sensitive dependence on initial conditions. In some articles dealing with chaotic phenomena, the degree of chaos in the double pendulum system is often measured by the size of the Lyapunov exponent of the orbit in the phase space[13][14][15]. In this paper, the double pendulum system was studied with ideal conditions. The experimental setup is shown in Figure 1. $\nu_1$, $\nu_2$ indicates the oscillation frequency of the inner pendulum and the outer pendulum, respectively.

Defined $\Delta = \theta_1 - \theta_2$. When using quasi-periodic oscillation, the oscillation frequency of the double pendulum does not change significantly with time. Then take a long enough time interval to calculate the time interval between the two maximum points that are farthest apart. Then divide the time interval by the number of cycles to get the size of the frequency $\nu$. Think of the ball as a particle, regardless of the quality of the rod, and various friction and resistance. After that, the double pendulum system is released without initial velocity. This paper mainly studies the relationship between the length of the rod, the initial release angle and the frequency of the double pendulum oscillation. First, the Lagrange equation will be used to describe the state of the system, then choose to use matlab to simulate the oscillation state of the double pendulum system. Creative use of this paper l-$\nu$ Figure to study the relationship between the length of the pendulum and the frequency of the pendulum. By changing the different initial release angles, multiple l-$\nu$ curve was got. This clearly reflects the relationship between the three in a picture. In addition, during the research process, there are some unexpected findings, such as l-$\nu$. The mutation problem of the curve, the nonlinear distribution of the frequency mutation point, and some phenomena that occur under very extreme conditions.

2. Model analysis

In this article, the zero point is set as the fixed point of the inner pendulum. Take the fixed point on the inner side of the pendulum as the origin, the horizontal direction to the right is the positive direction of the X axis, and the vertical direction to the positive direction of the Y axis establishes a two-dimensional coordinate system. According to theoretical mechanics, the Euler-Lagrange equation of the system can be written as follows:
Therefore,

\[ L = \frac{1}{2} m_1 \ddot{\theta}_1 l_1^2 + \frac{1}{2} m_2 \left( \ddot{\theta}_1 l_1^2 + \ddot{\theta}_2 l_2^2 + 2l_1 l_2 \ddot{\theta}_1 \theta_2 \cos(\theta_1 + \theta_2) \right) + m_1 g l_1 \cos \theta_1 + m_2 g \left( l_1 \cos \theta_1 + l_2 \cos \theta_2 \right) \]  

(1)

Therefore,

\[ m_2 \left[ l_1 \dddot{\theta}_1 \cos(\theta_1 - \theta_2) + l_2 \dddot{\theta}_2 - l_1 \dddot{\theta}_1 \sin(\theta_1 - \theta_2) + g \sin \theta_2 \right] = 0 \]  

(2)

\[ l_1 \dddot{\theta}_1 (m_1 + m_2) + l_2 \dddot{\theta}_2 l_2 \cos(\theta_1 - \theta_2) + m_2 \dddot{\theta}_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin \theta_1 = 0 \]  

(3)

3. Two typical oscillation modes

Through the simulation of MATLAB, in the normal state, regardless of the initial conditions, as long as \( l_1 \) or \( l_2 \) is long enough (that is, the length of the long pendulum is more than two orders of magnitude longer than the length of the pendulum), the double pendulum system will enter \( \alpha \) oscillation state or \( \beta \) oscillation state.

3.1. \( \alpha \) oscillation state

A oscillation that may occur when the length of the inner pendulum is much larger than the length of the outer pendulum. To illustrate the problem more clearly, the time series of \( \alpha \) oscillation state were shown as follows:

![Figure 2](attachment:image.png)

**Figure 2.** The time series under the condition that the initial release angle of the pendulum is 60°, \( \Delta = 3° \), \( l_2 = 1, l_1 = 250 \). Keep the quality constant: \( m_1 = m_2 = 1 \)

3.2. \( \beta \) oscillation state

What happens when \( \Delta \) is not very large, and the length of the outer pendulum is much larger than the length of the inner pendulum. To illustrate the problem more clearly, the time series of \( \beta \) oscillation state were shown as follows:
Figure 3. time series under the condition that the initial release angle of the inner pendulum is 60°, \( \Delta = 3° \), \( l_1 = 1 \), \( l_2 = 250 \). keep the quality constant: \( m_1 = m_2 = 1 \).

By comparing the time series of the two oscillation states, each time series consists of two waves. The oscillation frequency of one wave is relatively high, called it fast wave; the frequency of the other wave is relatively low, called it slow wave. The commonality of the two types of oscillations is that in the same time series, the frequency difference between the two waves is relatively large, and the oscillation range of the fast wave moves up and down with the slow wave. The difference between the two is that the fast wave amplitude of the \( \alpha \) oscillation is much smaller than the amplitude of the fast wave of the \( \beta \) type oscillation. This is intuitively easy to understand.

4. Numerical simulation results with two different initial conditions

In the state of \( \alpha \) oscillation, as shown in Figure 4, before entering this state, as the length of the outer pendulum increases, the oscillation frequency of the two pendulums will show a decreasing trend. When the length of the inner pendulum is one to two orders of magnitude larger than the length of the outer pendulum, the oscillation frequency of the outer pendulum gradually increases as the length of the inner pendulum increases. When the outside pendulum frequency is increased to a certain extent, this increasing trend will suddenly become flat. Then enter the \( \alpha \) oscillation state. The difference between the angles of the two pendulums determines the location of the abrupt point, mainly occurring in the internal pendulum length of between 10 and 150. The oscillation frequency of the outer pendulum is much larger than the oscillation frequency of the inner pendulum, but the amplitude of the outer pendulum is much smaller than the amplitude of the inner pendulum, as shown in Figure 2. It differs by about one to two orders of magnitude. Moreover, the larger the \(| \Delta |\), the sooner the double pendulum system will enter the \( \alpha \) oscillation state. The extreme case is the state of \( \Delta = 0 \) shown in Figure 5. At this time, because \( \Delta \) is infinitely small, the double pendulum system will not enter the \( \alpha \) oscillation state, or the system enters the \( \alpha \) oscillation state only when the II is infinitely long.
Figure 4. The $l_1$-$\nu_2$ curves with different $\Delta$. The case of $\Delta < 0$ is similar with this figure. ($\theta_1=60^\circ$, $l_2=1$, $m_1=m_2=1$)

Figure 5. This picture shows the extreme situation. $l_1$-$\nu$, the initial release angle of the inner and outer pendulums: $\theta_1=\theta_2=60^\circ$ ($\Delta = 0^\circ$), keep the length of the outer pendulum constant equal to 1 ($l_2 = 1$) and keep the quality constant: $m_1=m_2=1$. The frequency of the two pendulums is finally keep decreasing until it approaches zero.

For the $\beta$ type of oscillation, the double pendulum system will appear when the length of the outer pendulum is much longer than the length of the medial pendulum (the length of the outer pendulum is approximately two orders of magnitude or more of the length of the medial pendulum). Before entering the $\beta$ type of oscillation, as the length of the outer pendulum increases, the frequency of the inner pendulum gradually transitions from the oscillating state to the stable frequency $\nu_s$. As shown in Figure 6. The frequency of the outer pendulum will suddenly drop from the steady oscillation state, as shown by the sudden change of the green circle in Figure 6. As for when this mutation occurs, it is related to the initial release angle $\theta_2$ of the outer pendulum. In the process of adding $l_2$, the larger the $\theta_2$, the faster this mutation will occur. This mutation point usually occurs when the length of the outer
The inner pendulum is from 50 to 200. After the sudden change point, the frequency of the inner pendulum is basically fixed, and the frequency of the outer pendulum is continuously reduced, eventually exhibiting a β type oscillation. In the case of the β type of oscillation, the oscillation frequency of the inner pendulum is much larger than the frequency of the outer pendulum. The oscillation frequencies of the two pendulums differ by about one to two orders of magnitude. But unlike the α oscillation, the amplitude of the two pendulums of the β type is very close. The extreme case is that the initial release angle of the outer pendulum is zero, which is θ₂=0. At this time, because θ₂ is infinitely small, the double pendulum system will not enter the β type oscillation state, or the outer pendulum length l₂ is infinitely long to enter β type oscillation state.

Figure 6. The l₂-v₂ curves with different θ₂. The case of Δ<0 is similar with this figure. (θ₁=60°, l₁=1, m₁=m₂=1)

Figure 7. This picture shows the extreme situation. l₂-v, the initial release angle of the inner and outer pendulums: θ₁=60°, θ₂=0°, keep the length of the inner pendulum constant equal to 1 (l₁ = 1) and keep the quality constant: m₁=m₂=1. The frequency of the two pendulums is finally maintained at a stable value.
5. Conclusion
In this paper, a typical double pendulum system was studied. The purpose of research was to investigate the effect of the initial state of the double pendulum system on the oscillation state of the double pendulum system. But at the beginning of the study, we found that in some cases, the oscillation state of the double pendulum system would be in a revolving state (the inner pendulum or the outer pendulum crossed the highest point). In this case, the oscillation law of the double pendulum system was more difficult to explore. Therefore, this article focused on the oscillation state instead of revolving state. The results shown that the phase diagram that the oscillation state of the double pendulum system in the oscillating state was quasi-periodic. The Runge-Kutta method was applied in the numerical simulation and l-ν diagrams were plotted. It was an effective tool to investigate the relationship between the pendulum length and the frequency at a specific initial release angle. Took one point per unit length, and each curve was drawn with at least a few hundred points. The quantitative relationship between l and ν was reflected in the image, including the abrupt points in it, as shown in the green in Figure 4 and Figure 6. Then, based on this, classification studies was performed according to different initial release angles. In this paper, the model was established and simulated by MATLAB. The time series and l-ν diagram were made. The effects of two swing lengths and initial conditions on the oscillation behavior of the system were studied. The experimental results shown that in the case of extreme pendulum length, the double pendulum system will exhibit a more stable oscillation state, α and β oscillation state. The research results of this paper have guiding significance for studying the quasi-periodic oscillation field of the double pendulum system. However, due to the lack of calculation accuracy, there were still some imperfections in the case where the length of the inner pendulum was large and the difference between the two swing angles was also relatively large. Thus, more further research are expected to be attempted.

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