On trapped surface formation in gravitational collapse

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Abstract
A class of perfect fluid collapse models is constructed to examine the final fate of a continual gravitational collapse. While the pressure could be negative in the interior of the cloud, the weak-energy condition is satisfied. The collapsing star radiates away its matter as the process of gravitational collapse evolves, so as to avoid the formation of trapped surfaces. The collapsing interior is matched to an exterior, which is a generalized Vaidya spacetime, to complete the model. The implications of such a result towards the resolution of the singularity issue and various black hole paradoxes at classical and quantum levels are indicated.

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1. Introduction

While the Einstein gravity has been a highly successful theory of gravitation, it is well known that it generically admits the existence of spacetime singularities. These are extreme regions in the spacetime where densities and spacetime curvatures typically blow up and the theory must breakdown. Such singularities may develop in cosmology, indicating a beginning for the universe, or in a gravitational collapse, which ensues when a massive star exhausts its nuclear fuel and undergoes the process of continual collapse. It is expected that a possible future theory of quantum gravity may resolve these singularities where all known laws of physics breakdown. While there have been many attempts in this direction over the past few decades, we still wait for a consistent quantum gravity theory to be developed, before such a possibility of singularity resolution can be realized.

The singularity theorems predicting the occurrence of the spacetime singularities, however, contain three main assumptions under which the existence of a singularity is predicted in the form of geodesic incompleteness in the spacetime (see, e.g., [1]). These are in the form of a typical causality condition which ensures a suitable and physically reasonable global structure of the spacetime, an energy condition which requires the positivity of energy density at the classical level as seen by a local observer, and finally a condition demanding that
trapped surfaces must exist in the dynamical evolution of the universe, or in the later stages of a continual gravitational collapse. A trapped region in the spacetime consists of trapped surfaces, which are 2-surfaces such that both ingoing and outgoing wavefronts normal to them must converge. Such trapped surfaces then necessarily give rise to a spacetime singularity either in gravitational collapse or in cosmology.

For the same reason, the process of trapped surface formation in gravitational collapse is also central to black hole physics. The role of such a trapping within the framework of Einstein’s theory of gravitation was highlighted by Oppenheimer, Snyder and Dutt (OSD) [2] within the context of continual collapse of a massive matter cloud. They studied the collapse of a pressureless dust model using general relativity, and showed that it leads to the formation of an event horizon and a black hole as the collapse end state, assuming that the spatial density distribution within the star was strictly homogeneous, that is, $\rho = \rho(t)$ only. At the later stages of collapse an apparent horizon develops, which is the boundary of the trapped region, thus giving rise to an event horizon. Once the collapsing star has entered the event horizon, the causal structure of spacetime there would then imply that no non-spacelike curves from that region would escape away, which is cut off from the faraway observers, thus giving rise to a black hole in the spacetime.

It is necessary, however, to study more realistic models of gravitational collapse in order to put the black hole physics on a sound footing. This is because the OSD scenario is rather idealized and pressures would play an important role in the dynamics of any realistic collapsing star. Also density distribution could not be completely homogeneous in any physically realistic model, but would be higher at the centre of the cloud, with a typical negative gradient as one moved away from the centre. The study of dynamical collapse within Einstein’s gravity is, nevertheless, a difficult subject because of the nonlinearity of the Einstein equations. It was hence proposed in 1969 by Penrose [3] that any physically realistic continual gravitational collapse must necessarily end in a black hole final state only. This is known as the cosmic censorship conjecture (CCC). Though CCC has played a rather crucial role as a basic assumption in all the physics and astrophysical applications of black holes so far, no proof or any mathematically rigorous formulation for the same is available as yet despite many attempts. Hence this has been widely recognized as the single most important problem in the theory of black holes and the gravitation physics today. What is really necessary to make any progress on CCC, or to understand the final endstate of a massive collapsing star is to study realistic gravitational collapse scenarios within the framework of Einstein’s gravity [4].

Our purpose here is to investigate gravitational collapse from such a perspective and we construct a class of perfect fluids collapsing solutions for the Einstein equations, where the trapped surface formation could be delayed or avoided during the collapse. In the models that we develop, the matter of the star escapes away and would be thrown out during the final stages of gravitational collapse. Towards realizing such a scenario, we impose various physical reasonability conditions. While we require regularity of the initial data as well as a weak-energy condition, we allow pressure to be negative. We show that in the class of perfect fluid collapse solutions we develop, no trapped surfaces form as the collapse evolves in time, thus allowing the matter to escape away at the later stages of gravitational collapse. The perfect fluid form of matter is chosen as it could admit several physically realistic equations of state, and it is a form of matter which has been studied extensively within astrophysical contexts.

Section 2 specifies the basic regularity conditions on the class of models and the Einstein equations to be solved. In section 3, the perfect fluid solutions are given where we show how the trapped surface formation is naturally avoided as the collapse progresses. To complete the model, we match this interior collapsing spacetime to an exterior Vaidya solution in section 4.
Section 5 gives some concluding remarks and we indicate possible implications of such a result towards the resolution of singularity and black hole paradoxes at the classical and quantum levels.

2. Einstein equations and regularity conditions

For the class of solutions to the Einstein field equations that we construct here, at the initial epoch we take the matter to be a regular isentropic perfect fluid with a linear equation of state \( p = k \rho \). Then the energy conditions imply that at the initial surface, the fluid has a positive pressure. As the collapse evolves, we no longer impose the condition on the perfect fluid to remain necessarily isentropic, but allow the evolution of equation of state as determined by the collapse itself and the Einstein equations, as we shall discuss. The pressure decreases monotonically as the collapse proceeds, and eventually and closer to the singularity it necessarily becomes negative in this class of solutions. It is seen that there is avoidance of any trapped surface formation in the spacetime. The weak-energy condition is, however, preserved throughout the collapse. The interior collapsing sphere is matched with a generalized Vaidya exterior spacetime, and the energy of the collapsing configuration is radiated away, when the model is smoothly matched to a flat Minkowski spacetime.

The spherical metric in a general form can be written in a comoving frame as
\[
d s^2 = -e^{2\nu(t,r)} \, dt^2 + e^{2\psi(t,r)} \, dr^2 + R^2(t,r) \, d\Omega^2,
\]
where \( d\Omega^2 \) is the line element on a 2-sphere. The energy–momentum tensor for a perfect fluid is given as
\[
T_{ij} = \begin{cases} 
-\rho(t,r) & \text{for } i=j = t,r,
\rho + p & \text{for } i = \theta, \phi,
\end{cases}
\]
where \( \rho \) and \( p \) are the energy density and pressure respectively. We assume that the matter field satisfies the weak-energy condition, that is, the energy density as measured by any local observer is non-negative, and for any timelike vector \( V^i \), we have
\[
T_{ik}V^iV^k \geq 0
\]
which amounts to \( \rho \geq 0, \rho + p \geq 0 \). The dynamic evolution of the initial data is then determined by the Einstein equations, which for metric (1) become (8\( \pi \) \( G \) = \( c \) = 1),
\[
\rho = \frac{F'}{R^2 R'}; \quad p = -\frac{\dot{F}}{R^2 R}
\]
\[
-p' = \dot{\nu}(p + \rho)
\]
\[
R' \dot{G} - 2R \dot{\nu} G = 0
\]
\[
G - H = 1 - \frac{F}{R}
\]
The functions \( G \) and \( H \) above are defined as
\[
G(t,r) = e^{-2\psi} (R')^2, \quad H(t,r) = e^{-2\nu} (\dot{R})^2.
\]
The arbitrary function \( F = F(t,r) \) here has an interpretation of the mass function for the cloud, and it gives the total mass in a shell of comoving radius \( r \) on any spacelike slice \( t = \text{const} \). The energy conditions imply \( F \geq 0 \). To preserve regularity at the initial epoch, we must have \( F(t_i,0) = 0 \), that is, the mass function should vanish at the centre of the cloud.

We would like to consider only the collapsing class of solutions to the Einstein equations because we are interested in the issues related to the spacetime singularity. This implies that...
we have $R < 0$, that is, the physical radius $R$ of the cloud decreases in time. In a continual collapse scenario this ultimately reaches $R = 0$, which is the singularity where all matter shells collapse to a zero physical radius. We use the scaling freedom for the radial coordinate $r$ to write $R = r$ at the initial epoch $t = t_i$, and introduce a function $a(t, r)$ as defined by

$$a(t, r) \equiv R/r.$$  

We then have

$$R(t, r) = ra(t, r), \quad a(t_i, r) = 1, \quad a(t_s(r), r) = 0$$  

with $\dot{a} < 0$. The time $t = t_s(r)$ corresponds to the shell-focusing singularity at $R = 0$, where all the matter shells collapse to a vanishing physical radius. Now, from equation (5), we see that the equation of state is given implicitly as

$$p = -\rho + e^{-\nu} \int \rho' e^\nu \, dr + \beta(t) e^{-\nu}.$$  

(11)

We note that there are a total of five field equations with six unknowns, $\rho, p, \psi, \nu, R$ and $F$, giving us the freedom of choice of one free function. Selection of this function, subject to the given initial data and the weak-energy condition, determines the matter distribution and metric of the spacetime, and thus leads to a particular dynamical collapse evolution of the initial data. Also, the regularity conditions (e.g. density should not blow up at the regular centre of the cloud at any epoch before the singularity) imply that at any non-singular epoch we necessarily have $F \approx r^3$ near the centre, which fixes the general form of $F$ as

$$F(t, r) = r^3 M(r, a)$$  

(12)

where $M(r, a)$ would be any general function. It is to be noted that this is not a special choice, but the general form as determined by the regularity conditions.

3. Collapsing perfect fluid models

As pointed out above, towards constructing any class of perfect fluid solutions to the Einstein equations, we have the freedom to fix one free function. We choose this to be the mass function $F(t, r)$ for the collapsing cloud. At any given epoch, the cloud has a compact support on a spacelike surface of $t = \text{const}$, outside which the collapsing ball is matched to a suitable exterior spacetime as we shall show in the next section.

Towards this purpose, we first give a specific model where a definite choice of the mass function is being made as under,

$$M = M_0[5a^2 - 4a^3].$$  

(13)

Clearly, this is a mass function with the required differentiability as needed by Einstein equations and we have $M(r, a) \geq 0$. Also, in the limit as $a \to 0$, the mass function goes to a vanishing value as a power of $a(r, t)$, and finally there is a definite value $a = a^*$ where the derivative $M_{a}$ changes sign which corresponds to $a^* = 5/6$.

With the choice of the mass function as above, let us define an arbitrary function $A(r, a)$ as $A(r, a)_{a} \equiv \nu'/R'$. Now from equation (5) we can easily see that

$$A(r, a)_{a} = -\frac{2a'}{a^2}.$$  

(14)

With the form of $A$, we can now integrate equation (6) to get

$$G(r, a) = [1 + r^2 b_0(r)] e^{2A}$$  

(15)
where $b_0(r)$ is an arbitrary function of integration. It can now be checked by direct substitution that any $A$ and $b_0(r)$ of the form

$$A = rf(r); \quad b_0(r) = \frac{(1 + \alpha r^2) e^{-2rA} - 1}{r^2}$$

(16)

where $\alpha$ is a constant, consistently solves the system of Einstein equations, with $a = a(t)$, which is given in the integral form by the equation of motion as

$$t = \int_a^1 \frac{\sqrt{a} \, da}{\sqrt{a^2 + M_0[5a^2 - 4a^2]}}.$$  

(17)

Now from equation (4), we get the density and pressure as

$$\rho = 3M_0 \left[\frac{5}{a} - 4\right]$$

(18)

and

$$p = 2M_0 \left[6 - \frac{5}{a}\right].$$

(19)

It can be easily seen now that the weak-energy condition is satisfied throughout the collapse as we have $\rho > 0$ and $\rho + p > 0$. Furthermore, we have an equation of state of the form $p = k(t)\rho$ where the function $k(t)$ is given as

$$k(t) = \frac{12a(t) - 10}{15 - 12a(t)}.$$  

(20)

Thus we see that at the initial hypersurface the pressure is positive, but as the collapse proceeds the pressure decreases and ultimately becomes negative near the singularity.

The above analysis thus provides a complete solution to the Einstein equations. In order to see that this represents a continual collapse model, it is important to note that at the initial epoch we impose the collapse condition, i.e. $\dot{R} < 0$. However, in principle, it could so happen that the collapsing system bounces back at a later epoch before actually reaching the singularity. A necessary condition for such a bounce to take place is $\dot{a} = 0$ for some value $a \in (0, 1)$. But from equation (17) it is evident that if we take the constant $\alpha$ to be positive, which is a free parameter, then $\dot{a} < 0$ holds throughout the dynamical evolution of the model. That is, in this case collapse would be continual without any bounce. Thus in this case, the suitable choice of this constant specifies continual collapse models.

Generalizing the above specific model now, let us consider the following class of mass functions $F(t, r)$ for a collapsing cloud, where $M(r, a)$ is taken to be an arbitrary function with the following properties,

(1) $M(r, a)$ is a $C^2$ function and $M(r, a) \geq 0$

(2) $\lim_{a \to 0} M$ goes to zero as $a^\alpha$, where $1 < \alpha < 3$

(3) There exists a value $a^*$, in the interval $(0, 1)$ such that, $M_{a\mid a > a^*} < 0$, $M_{a\mid a < a^*} > 0$ and $M_{a\mid a = a^*} = 0$.

Then equation (4) gives

$$\rho = \frac{3M + r[M_f + M_{a^*}]}{a^2(a + ra^*)}; \quad p = \frac{-M_{a^*}}{a^2}.$$  

(21)

Apart from motivating our discussion, the specific model given above shows that the solution set of Einstein equations for this class of mass functions with properties as above
is non-empty. Clearly, for $a > a^*$ the pressure is positive, and it changes sign for $a < a^*$. At $a = a^*$ the matter is dust-like with zero pressure. The energy conditions put some further restrictions on $M$, namely,

$$3M + r[M_r + M_\alpha a'] \geq 0, \quad 3M + rM_r - aM_\alpha \geq 0.$$  \hspace{1cm} (22)

Furthermore, if at the initial epoch the perfect fluid is isentropic with a linear equation of state $p = k\rho$, then an initial condition on the mass function is

$$3kM(r, 1) + krM(r, 1)_r + [M_\alpha]_{r=1} = 0.$$  \hspace{1cm} (23)

Now using the mass function $M$ equation (5) becomes

$$\frac{\nu'}{R'} = \frac{a[M_\alpha + a'[M_\alpha a - 2M_\alpha]]}{a[3M + rM_r - aM_\alpha]}.$$  \hspace{1cm} (24)

As earlier, let us define

$$\nu'/R' \equiv A(r, a).$$  \hspace{1cm} (25)

From the regularity conditions, and the requirement that the gradients of pressure vanish at the centre of the cloud, one can see that $A \approx rq(r, a)$ near the centre, where $q$ is any regular function. Our main interest here is in studying the shell-focusing at $R = 0$, which is the physical termination of collapse. Hence we assume that there are no shell-crosses where $R' = 0$, and so the function $A(r, v)$ is well-defined. Now using the above form of $\nu'$ we can integrate (6) to get

$$G(r, a) = b(r) e^{2A}.$$  \hspace{1cm} (26)

Here $b(r)$ is another arbitrary function of comoving coordinate $r$. A comparison with dust collapse models interprets $b(r)$ as the velocity function for the collapsing shells [7]. Following this parallel, we can write

$$b(r) = 1 + r^2 b_0(r).$$  \hspace{1cm} (27)

Thus we see that metric (1) becomes

$$ds^2 = -e^{2A} dr^2 + \frac{R^2 e^{-2A}}{[1 + r^2 b_0(r)]} dr^2 + R^2(t, r) d\Omega^2.$$  \hspace{1cm} (28)

Finally, from equation (7) we get

$$\sqrt{a} \dot{a} = -e^v \sqrt{e^{2A} ab_0(r) + ah(r, a) + M(r, a)}$$  \hspace{1cm} (29)

where the negative sign depicts a collapse scenario, and,

$$h(r, a) = \frac{e^{2vA} - 1}{r^2}.$$  \hspace{1cm} (30)

Integrating the above we get

$$t(a, r) = \int_a^1 \frac{\sqrt{a} da}{e^v \sqrt{e^{2A} ab_0 + ah + M}}.$$  \hspace{1cm} (31)

Note that the variable $r$ is treated as a constant in the above equation. Now putting $a = 0$ in the above equation would give us the time for termination of collapse $t = t_s(r)$, i.e. the time taken for the shell labelled $r$ to reach the vanishing physical radius $R = 0$,

$$t_s(r) = \int_0^1 \frac{\sqrt{a} da}{e^v \sqrt{e^{2A} ab_0 + ah + M}}.$$  \hspace{1cm} (32)
If the mass function is explicitly specified, from equations (24) and (29), we get the expressions of $a'$ and $\dot{a}$ in terms of $r$, $a$, and the function $A(r, a)$. Thus, to get a solution of $a(t, r)$ we need the Pfaffian differential equation, $a' \, dr + \dot{a} \, dt - da = 0$, to be integrable. This integrability condition gives us the required differential equation for the function $A(r, a)$. To show that solutions of this equation exist, it is sufficient to find one solution which we have done in (16). Since we know that, given one integrating factor for a Pfaffian differential equation, we can find infinity of them, this shows that there exists classes of functions $A(r, a)$ which simultaneously solve equations (24) and (29) and hence the whole collapsing system.

In general of course, given a rather general choice of the mass function with the properties as given above, it may not always be possible to find, in an explicit manner, the complete solution to the system of Einstein equations, with the area radius function $a = a(t, r)$ having a general form. We can, however, find an approximate solution near the centre ($r = 0$) by Taylor-expanding the right-hand side of equation (31) around $r = 0$ (see, for example, [7]), by using equation (24) and the form of $\nu$ as above. By doing so we can see how the collapse terminates and behaves at its later stages depending upon the initial profiles and the allowed dynamic evolutions.

The key factor that would now decide the final outcome of the collapse in terms of a spacetime singularity and black hole formation or otherwise, is the geometry of trapped surfaces which may develop as the collapse evolves. These are 2-surfaces in the spacetime geometry from which both outgoing and ingoing wavefronts necessarily converge [1]. If the trapped surfaces necessarily formed as the collapse develops, then a region arises in the spacetime where any particle or light ray entering it will not be able to escape away, and must necessarily fall into a spacetime singularity. In general, the existence of trapped surfaces, together with reasonable causality assumptions and an energy condition ensures that a spacetime singularity must develop where the spacetime curvatures and other physical quantities blow up, and where the known laws of physics must break down. This is the main content of the singularity theorems which apply to both gravitational collapse and cosmology situations. As we discussed above, the CCC would demand that any spacetime singularities forming in gravitational collapse must necessarily be hidden within a black hole, that is, the extreme strong curvature regions should not be visible to any external observers in the spacetime. Of course, if any of the basic assumptions of the singularity theorems do not hold, then there is a possibility in principle that the singularity can be avoided. While the causality and a reasonable global structure of the spacetime would be necessary for any physical model, and the energy conditions may also be required to hold, the formation or otherwise of the trapped surfaces during the collapse would depend on the nature of dynamical evolution of the collapsing cloud.

The boundary of the trapped region is the apparent horizon, which signals the black hole formation in gravitational collapse. The apparent horizon in a spherically symmetric spacetime is given by

$$F = R.$$  \hspace{1cm} (33)

The spacetime region where the mass function $F$ satisfies $F < R$ is not trapped, while $F > R$ describes a trapped region (see, e.g., [8]). Considering again the continual collapse example that we discussed in the beginning of this section, regularity would demand that there are no trapped surfaces at the initial epoch. If $r = r_b$ denotes the boundary of the cloud, then imposing the condition $M_0(r_b)^2 < 1$ ensures that there will be no trapped surfaces for any of the shells at $r \leq r_b$ because $F/R < 1$ is preserved at the initial epoch. In fact, we see that imposing a somewhat stronger condition, namely $M_0(r_b)^2 < 16/25$ is sufficient to ensure that $F/R < 1$ is preserved throughout, until the termination of collapse at $a = 0$. 

In the general case, in the limit of the termination of collapse at \( a = 0 \), we see that for the class of perfect fluid collapse models we have considered here

\[
\frac{F}{R} \approx r^2 a^{a-1} = 0. \tag{34}
\]

Thus we see that even as the collapse comes to an end as the physical radii \( a \to 0 \) for the collapsing shells, there are no trapped surfaces forming in the spacetime. The reason why this happens is that as the collapse progresses the pressures turn negative in the class of models constructed here. As seen from the Einstein equation (3), a negative pressure necessarily means \( \dot{F} \) must be negative, which implies that the mass function has to be decreasing in time for each corresponding shell at a constant value of comoving radius \( r \). In other words, as the collapse progresses, the mass is being radiated away at the same time, and as a result there is never a sufficient mass within a given radius to allow for the formation of the trapped surfaces. Finally, as the collapse terminates at \( a = 0 \), we get \( F = 0 \) for all values of \( r \) within the cloud, whereby all the mass of the cloud has been radiated away. A discussion on the effect of negative pressures within the framework of gravitational collapse models was also made in [9].

We have thus provided a class of perfect fluid solutions to the Einstein equation which represents a continual gravitational collapse. The allowed mass function and other parameters of the models are such that the trapped surface formation is avoided throughout till the termination of collapse. The matter satisfies the weak-energy condition always; however, pressures turn negative as the collapse progresses. The important point is that we have here an explicit construction of a class of models of gravitational collapse from regular initial data, with a reasonable form of matter and energy condition satisfied, such that trapping is avoided.

4. Exterior spacetime and matching conditions

To complete the model, we now need to match this class of perfect fluid interiors to a suitable exterior spacetime. In the following, we match this collapsing ball of perfect fluid to a generalized Vaidya exterior [10] at the boundary hypersurface \( \Sigma \) given by \( r = r_v \). Then the metric just inside \( \Sigma \) is

\[
\text{d}s^2 = -e^{2\nu(t,r_v)} \text{d}t^2 + e^{2\psi(t,r_v)} \text{d}r^2 + R^2(t, r) \text{d}\Omega^2_2 \tag{35}
\]

while the metric in the exterior of \( \Sigma \) is

\[
\text{d}s^2_+ = -\left(1 - \frac{2M(r_v,v)}{r_v}\right) \text{d}v^2 - 2\text{d}v \text{d}r_v + r_v^2 \text{d}\Omega^2_2 \tag{36}
\]

where \( v \) is the retarded (exploding) null coordinate and \( r_v \) is the Vaidya radius. Matching the area radius at the boundary we get

\[
R(r_v, t) = r_v(v). \tag{37}
\]

Then on the hypersurface \( \Sigma \), the interior and exterior metrics are given by

\[
\text{d}s^2_{\Sigma} = -e^{2\nu(t,r_v)} \text{d}t^2 + R^2(t, r_v) \text{d}\Omega^2_2 \tag{38}
\]

and

\[
\text{d}s^2_{\Sigma+} = -\left(1 - \frac{2M(r_v,v)}{r_v} + 2 \frac{\text{d}r_v}{\text{d}v}\right) \text{d}v^2 + r_v^2 \text{d}\Omega^2_2. \tag{39}
\]

Matching the first fundamental form gives

\[
\frac{\text{d}v}{\text{d}r} = \frac{e^{\nu(t,r_v)}}{\sqrt{1 - \frac{2M(r_v,v)}{r_v} + 2 \frac{\text{d}r_v}{\text{d}v}}}; \quad (r_v)_{\Sigma} = R(t, r_v). \tag{40}
\]
Next, to match the second fundamental form (extrinsic curvature) for interior and exterior metrics, we note that the normal to the hypersurface $\Sigma$, as calculated from the interior metric, is given as

$$n^i_v = [0, e^{-\psi(r_v)}, 0, 0]$$  (41)

and the non-vanishing components of the normal as derived from the generalized Vaidya spacetime are

$$n^v_v = -\frac{1}{\sqrt{1 - \frac{2M(r_v)}{r_v} + \frac{dr_v}{dv}}}$$  (42)

$$n^r_v = \frac{1}{\sqrt{1 - \frac{2M(r_v)}{r_v} + \frac{dr_v}{dv}}}.$$  (43)

Here the extrinsic curvature is defined as

$$K_{ab} = \frac{1}{2} L_n g_{ab}.  (44)$$

That is, the second fundamental form is the Lie derivative of the metric with respect to the normal vector $n$. The above equation is equivalent to

$$K_{ab} = \frac{1}{2} [g_{ab,c} n^c_n + g_{cb} n^c_n, a + g_{ac} n^c_n, b].  (45)$$

Now setting $[K^- - K^+]_{\Sigma} = 0$ on the hypersurface $\Sigma$ we get

$$RR' e^{-\psi} = r_v \frac{1}{\sqrt{1 - \frac{2M(r_v)}{r_v} + \frac{dr_v}{dv}}}.$$  (46)

Simplifying the above equation using equation (40) and the Einstein equations, we get

$$F(t, r_b) = 2M(r_v, v).  (47)$$

Using the above equation and (40) we now get

$$\left( \frac{dv}{dr} \right)_\Sigma = \frac{e^\nu (R' e^{-\psi} + R e^{-\psi})}{1 - \frac{2M(r_v)}{r_v} + \frac{dr_v}{dv}}.$$  (48)

Finally, setting $[K^- - K^+]_{\Sigma} = 0$, where $\tau$ is the proper time on $\Sigma$, we get

$$M(r_v, v) = \frac{F}{2R} \frac{R e^{-\nu}}{\sqrt{G}} \sqrt{H_{\tau}} + R e^{2\nu} e^{-\psi}.$$  (49)

Any generalized Vaidya mass function $M(v, r_v)$ which satisfies equation (49) will then give a unique exterior spacetime with required equations of motion given by other matching conditions, (47), (48) and (37). To give some examples of such functions $M(v, r_v)$, we can have charged Vaidya spacetime as an exterior in which $M = M(v) + Q(v)/r_v$, or one can have an anisotropic de Sitter exterior where $M = M(r_v)$, which are two different solutions of equation (49) (see, for example, [10, 11]). This gives examples of two unique exterior spacetimes, both of which are subclasses of the generalized Vaidya model considered here.

We note that the junction condition (47) specifies the value (and not the functional form) of $M(v, r_v)$ on the boundary, while (49) gives the value of partial derivative with respect to $r_v$ on the boundary. It is important to note that the value of one partial derivative (with respect to $v$) is still free and hence equations (47)–(49) actually describe a class of exterior generalized Vaidya mass functions.
Now we can easily see that along the singularity curve, \( t = t_s \),

\[
\lim_{r_v \to 0} \frac{2M(r_v, v)}{r_v} \to 0. \tag{50}
\]

Thus the exterior metric along with the singularity smoothly transform to

\[
ds = -dv^2 - 2dv \, dr_v + r_v^2 \, d\Omega^2. \tag{51}
\]

The above metric describes a Minkowski spacetime in the retarded null coordinate. Hence we see that the exterior generalized Vaidya metric, together with the singularity, can be smoothly extended to the Minkowski spacetime as the collapse completes. Figure 1 shows the schematic diagram of the complete spacetime.

5. Concluding remarks

We make several concluding remarks in this final section.

1. We have given here a class of perfect fluid collapse models, where beginning from an isentropic equation of state, and initially positive pressures, the collapse evolution is such that the pressures become negative in the later stages of evolution. This causes the mass of the cloud to be radiated away, resulting in non-formation of trapped surfaces all the way until the collapse ends. The generalized Vaidya spacetime represents the exterior geometry of such a collapsing matter cloud. In such a case, as we have seen, there are no trapped surfaces as there is never enough mass in a given spacetime radius to generate trapping of light.

2. The physical picture then would be that of a cloud radiating away most of its mass in the later stages of collapse. Eventually, as end state of collapse approaches, the entire star evaporates and the geometry is smoothly matched to a Minkowski spacetime.

3. Such a study of gravitational collapse is also warranted because of the deep paradoxes that are associated with the black holes, and which have been widely discussed. Firstly, all matter entering a black hole, must collapse into a spacetime singularity of infinite density and curvatures at the centre of the cloud, where all known laws of physics must break down. It is not clear how such a model can be stable at the classical level. Secondly, we
need to formulate mathematically and prove CCC, that a generic gravitational collapse gives rise to a black hole only. However, there have been several detailed collapse studies so far which show that the final fate of a collapsing star could be either a black hole or a naked singularity, depending on the nature of the initial data from which the collapse evolves, and the possible evolutions as allowed by the Einstein equations (see, e.g., [6] for some recent reviews). As opposed to a black hole final state, naked singularity is a scenario where ultra-strong density and curvature regions of spacetime, forming as a result of collapse, would be visible to faraway observers in the spacetime, in violation to CCC. Finally, it is well known that a black hole would create information loss, violating unitarity principle, thus creating contradiction with the basic principles of quantum theory.

(4) In such a case, one may investigate the possibility that the singularity problem as well as black hole paradoxes may be resolved by possibly avoiding the trapped surface formation in the spacetime during the process of a dynamical gravitational collapse.

Of course, in this situation, the black hole itself does not form as gravitational collapse end state. This is because as the trapped surface formation is avoided here, there is no event horizon developing in the interior of the cloud. The point, however, is that the Einstein equations readily admit such a possibility, as seen by the class of solutions given here.

(5) Such a scenario may deserve serious consideration because of the serious problems and paradoxes present at the very heart of black hole physics. What we have constructed here is a class of perfect fluid solutions to general relativity, showing that in the Einstein theory such a possibility is very much present, emerging as gravitational collapse end state, within the framework of reasonable physical conditions as outlined here. It would be relevant to note that in the context of cosmology, the avoidance of trapped surface formation has been used earlier, for a class of cylindrically symmetric perfect fluid cosmological models, to resolve the spacetime singularity [12].

In the present case, as the singularity can be matched to a flat spacetime in a continuous manner, physically it would be reasonable to think of these singularities as being in some sense ‘extendable’, even if there is a curvature singularity within the centre of the cloud.

(6) One could ask what would be the physical mechanism which could possibly give rise to a negative pressure as considered here. As such, in cosmology today, the existence of negative pressures is regarded as a fairly common phenomenon, especially if a positive cosmological constant supplied the dark energy of the universe. In fact, such a form of dark energy may also violate some of the energy conditions, as opposed to the weak-energy condition holding in our case. However, within the framework of gravitational collapse, while dealing with finite massive matter clouds such as collapsing stars, an energy condition may be supposed to hold, at least at the classical level till a fairly advanced stage of collapse is reached. On the other hand, there is no problem having negative pressures, even when the weak-energy condition is satisfied, as we have seen above. As the collapse progresses the quantum corrections may become important, and such quantum effects may effectively give rise to negative pressures, at least in a certain approximation. These issues will be discussed elsewhere.

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