CROSSED PRODUCTS BY SPECTRALLY FREE ACTIONS

CORNEL PASNICU AND N. CHRISTOPHER PHILLIPS

Abstract. We define spectral freeness for actions of discrete groups on C*-algebras. We relate spectral freeness to other freeness conditions; an example result is that for an action \( \alpha \) of a finite group \( G \), spectral freeness is equivalent to strong pointwise outerness, and also to the condition that \( \bar{\Gamma}(\alpha_g) \neq \{1\} \) for every \( g \in G \setminus \{1\} \).

We then prove permanence results for reduced crossed products by exact spectrally free actions, for crossed products by arbitrary actions of \( \mathbb{Z}/2\mathbb{Z} \), and for extensions, direct limits, stable isomorphism, and several related constructions, for the following properties:

- The combination of pure infiniteness and the ideal property.
- Residual hereditary infiniteness (closely related to pure infiniteness).
- Residual (SP) (a strengthening of Property (SP) suitable for nonsimple C*-algebras).
- The weak ideal property (closely related to the ideal property).

For the weak ideal property, we can allow arbitrary crossed products by any finite abelian group.

These properties of C*-algebras are shown to have formulations of the same general type, allowing them all to be handled using a common set of theorems.

We prove permanence results for crossed products by exact spectrally free actions. (We say more about spectral freeness below.) We also give similar results for completely arbitrary actions of the group \( \mathbb{Z}/2 \). (In fact, if one of the properties we consider holds for the fixed point algebra of an action of \( \mathbb{Z}/2 \) on a C*-algebra \( A \), then it holds for \( A \).) For the most part, we consider the following properties, either already known or related to known properties, about which we say more below:

- Residual (SP).
- Residual hereditary (proper) infiniteness.
- The combination of pure infiniteness (for nonsimple C*-algebras) and the ideal property.
- The weak ideal property.

In fact, we show that if \( \alpha : G \to \text{Aut}(A) \) is an action of an arbitrary finite group, and if the fixed point algebra has the weak ideal property, then \( A \) has the weak ideal property.

The best plausible related permanence results are that crossed products by arbitrary discrete groups should preserve pure infiniteness (not just residual hereditary infiniteness), while crossed products by exact actions of discrete groups should preserve residual (SP) and the ideal property when, except for a finite normal subgroup, the action satisfies a suitable outerness condition. We do not give theorems.

Date: 22 August 2013.
1991 Mathematics Subject Classification. Primary 46L55; Secondary 46L35, 46L40.
Some of this material is based upon work of the second author supported by the US National Science Foundation under Grants DMS-0302401, DMS-0701076, and DMS-1101742.
in anything like this generality. For example, it remains unknown whether crossed products by arbitrary actions of finite groups preserve the ideal property or pure infiniteness.

Our methods also yield easy proofs of some other permanence results, using a general scheme given in Section 5.

The outerness condition we use is spectral freeness. It seems to be an appropriate version for nonsimple C*-algebras of pointwise outerness; pointwise outerness is too weak a hypothesis to be able to prove much. It is based on the notion of a freely acting automorphism from Section 2 of [15]. It seems to be a better hypothesis for theorems than strong pointwise outerness ([28]; see Definition 1.1 below.) For finite groups, but not in general, spectral freeness is equivalent to strong pointwise outerness (Theorem 1.15, Proposition 1.11).

A C*-algebra is hereditarily infinite if every nonzero hereditary subalgebra contains an infinite positive element (in the sense of Definition 3.2 of [11]), and residually hereditarily infinite if every quotient is hereditarily infinite. This property appears (without a name) in [11], where it is shown to follow from pure infiniteness (for nonsimple C*-algebras). Question 4.8 of [11] asks whether the converse holds; this question seems to be still open. We have few permanence results for pure infiniteness of crossed products; permanence results for residual hereditary infiniteness are what we can prove instead, and are suggestive about permanence results for pure infiniteness.

The combination of pure infiniteness and the ideal property is an interesting condition in its own right, and permanence results for it are suggestive of permanence results for both properties separately.

Recall that a C*-algebra has Property (SP) if every nonzero hereditary subalgebra has a nonzero projection. This property has mostly been used for simple C*-algebras, and seems not to be the right property for nonsimple C*-algebras. In Example 7.2, we give a C*-algebra which has Property (SP) but which has quotients which do not have Property (SP). Residual (SP) is the requirement that every quotient algebra has Property (SP), and seems better for nonsimple algebras.

The weak ideal property is a weakening of the ideal property. Instead of requiring that every ideal be generated by its projections, we require that every nonzero subquotient of the stabilization contain a nonzero projection. This condition admits better permanence results (unrelated to crossed products) than the ideal property does. For example, it is preserved by extensions; it is known (Theorem 5.1 of [21]) that the ideal property is not. Permanence results for the weak ideal property for crossed products are suggestive of permanence results for the ideal property, although some results for the weak ideal property are known to fail for the ideal property.

This paper is conceptually related to [24], but has different emphasis. The paper [24] considered crossed products and fixed point algebras of actions of finite groups (with or without freeness conditions) and a different, but related, collection of properties. Here we consider crossed products by actions of infinite groups, with freeness conditions, and also give results for crossed products by actions of $\mathbb{Z}_2$ (and a few results for more general finite groups), proved since [24] was written. We presume our results for $\mathbb{Z}_2$ generalize to arbitrary finite groups, but the generalizations mostly seem to be much harder. (See the discussion in Section 4.)
After the work for this paper was done (but while we were still trying to improve the results for $\mathbb{Z}_2$ to more general finite groups), the paper [7] was posted on the arXiv. It has some overlap with our material on exact spectrally free actions and crossed products specifically by actions on purely infinite C*-algebras with the ideal property. It does everything in the more general context of partial actions, but does not address the other conditions for which we consider permanence results, and does not address general actions of $\mathbb{Z}_2$. Residual topological freeness, as defined in [7], is, when restricted to actions on commutative C*-algebras, the same as our spectral freeness. (See Proposition 1.10 below.) Our Lemma 3.2 is related to Lemma 3.11 of [7] and has a similar proof, but has differently stated hypotheses and conclusion. Our Theorem 6.8(1) is Theorem 4.2 of [7] for actions.

This paper is organized as follows. In Section 1, we define spectral freeness and relate it to other properties which have previously been considered. For example, for finite groups but not in general, spectral freeness of an action $\alpha: G \to \text{Aut}(A)$ is equivalent to strong pointwise outerness and to the condition that $\tilde{\Gamma}(\alpha_g) \neq \{1\}$ for every $g \in G \setminus \{1\}$.

In Section 2, we consider actions of $\mathbb{Z}$ with the Rokhlin property. We prove an averaging lemma, and, as consequences, we show that such actions are spectrally free, their crossed products preserve the projection property (Definition 1 of [22]) and the ideal property, and that every ideal in the crossed product is a crossed product by an invariant ideal (a result attributed to us in a 2006 preprint [17]).

Section 3 contains the main technical result on spectrally free actions: every nonzero hereditary subalgebra of the reduced crossed product contains an isomorphic image of a nonzero hereditary subalgebra of the original algebra. In Section 4, we prove this for arbitrary actions of $\mathbb{Z}_2$. In fact, for actions of $\mathbb{Z}_2$, every nonzero hereditary subalgebra of the given algebra contains an isomorphic image of a nonzero hereditary subalgebra of the fixed point algebra.

Section 5 gives permanence results for a class of properties of C*-algebras defined using hereditary subalgebras. In the remaining three sections, we use these results to treat the properties discussed earlier in the introduction. Residual hereditary (proper) infiniteness and the combination of pure infiniteness and the ideal property are treated in Section 6. Residual (SP) is treated in Section 7. The weak ideal property is treated in Section 8 along with the intermediate condition that every nonzero subquotient of $A$ contain a nonzero projection. This last property is not of the form required in Section 5, but we can still prove some permanence results for it. For both this property and the weak ideal property, we are able to prove preservation by crossed products by arbitrary finite abelian groups (not just $\mathbb{Z}_2$).

The crossed product results are derived from results involving fixed point algebras. What we actually show is that if $\alpha: \mathbb{Z}_2 \to \text{Aut}(A)$ is any action, and if the fixed point algebra has the given property, then so does $A$. For the weak ideal property and the intermediate condition above, we can allow an arbitrary finite group in place of $\mathbb{Z}_2$. The crossed product results follow by duality. This is why we require the group to be abelian when considering crossed products of algebras with the weak ideal property.

We use the following conventions and notation, some standard (some of them recalled here for reference) and some less so.
If $A$ is a C*-algebra, then $A_+$ denotes the set of positive elements of $A$. As usual, we let $K$ denote the C*-algebra of compact operators on a separable infinite dimensional Hilbert space. We set $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. (The $p$-adic integers will not appear.)

We write $a \sim b$ for Cuntz equivalence and $a \preceq b$ for Cuntz subequivalence. See Definition 2.1 of [11] (except that $a \sim b$ is written $a \approx b$ there), and see Section 2 of [11] for much more on these relations.

For a C*-algebra $A$, an automorphism $\varphi \in \text{Aut}(A)$, and a subset $S \subset A$, when we say that $S$ is $\varphi$-invariant, we mean that $\varphi(S) = S$, not merely that $\varphi(S) \subset S$. A $\varphi$-invariant quotient of $A$ means a quotient $A/I$ for a $\varphi$-invariant ideal $I \subset A$, and a $\varphi$-invariant subquotient of $A$ means a quotient $J/I$ for $\varphi$-invariant ideals $I, J \subset A$ with $I \subset J$. The automorphism $\varphi$ induces an automorphism of any invariant subalgebra, quotient, or subquotient.

For an action $\alpha : G \to \text{Aut}(A)$ of a group $G$ on a C*-algebra $A$, we give the obvious analogous meaning to the terms "$\alpha$-invariant quotient" and "$\alpha$-invariant subquotient", and note that $\alpha$ induces actions on such quotients and subquotients. Thus, for example, a $\varphi$-invariant quotient is a quotient which is invariant for the action of $\mathbb{Z}$ on $A$ generated by $\varphi$.

We sometimes use the same symbol $\alpha$ for the action $g \mapsto \alpha_g |_I$ induced by $\alpha$ on an invariant ideal $I \subset A$, and similarly with invariant subalgebras and subquotients, as well as $M_n(A)$ and similar constructions. We denote the fixed point algebra by $A^{\alpha}$.

We use the Connes spectrum and Borchers spectrum of an action $\alpha : G \to \text{Aut}(A)$ of a locally compact abelian group $G$ on a C*-algebra $A$, denoted $\Gamma(\alpha)$ and $\Gamma_B(\alpha)$. They are defined, for example, at the beginning of Section 1 of [15] (not the original source). We also use the strong Connes spectrum $\Gamma(\alpha)$, as defined in [13]. If $\varphi$ is an automorphism of a C*-algebra $B$, and $\beta : \mathbb{Z} \to \text{Aut}(B)$ is the action generated by $\varphi$, we write $\Gamma(\varphi)$ for $\Gamma(\beta)$, and similarly $\Gamma_B(\varphi)$ for $\Gamma_B(\beta)$ and $\Gamma(\varphi)$ for $\Gamma(\varphi)$.

The following definition is taken from the introduction of [34], where it is applied to the situation in which $B$ is the reduced crossed product by an action on $A$.

**Definition 0.1.** Let $B$ be a C*-algebra, and let $A \subset B$ be a subalgebra. We say that $A$ *separates the ideals of $B$* if whenever $I, J \subset B$ are ideals such that $I \cap A = J \cap A$, then $I = J$.

Some of this work was done during visits by the second author to Københavns Universitet during March–May 2012 and to the University of Texas at San Antonio and Tokyo University during November and December 2012. He is grateful to those institutions for their hospitality.

1. Spectrally free actions

In this section, we motivate and define spectral freeness, the outerness condition we use, and relate it to other conditions considered previously, particularly when the group is finite, the algebra is commutative, or the algebra is simple. Consequences of spectral freeness will be given in Section 3.

For many purposes, when $A$ is simple, the right version of freeness for an action $\alpha : G \to \text{Aut}(A)$ of a discrete group $G$ is pointwise outerness: for all $g \in G \setminus \{1\}$, the automorphism $\alpha_g$ is not implemented by a unitary in the multiplier algebra $M(A)$. For example, $C^*_r(G, A, \alpha)$ is again simple (Theorem 3.1 of [14]). For nonsimple algebras, this condition is far too weak to be useful. (Consider an action on a
direct sum which is inner on one summand and pointwise outer on the other.) In Section 4 of \[28\], the condition in the following definition was implicitly advocated as a substitute, at least for finite groups.

**Definition 1.1** (Definition 4.11 of \[28\]). Let $A$ be a C*-algebra and let $G$ be a group. An action $\alpha : G \to \text{Aut}(A)$ is said to be **strongly pointwise outer** if, for every $g \in G \setminus \{1\}$ and any two $\alpha_g$-invariant ideals $I \subset J \subset A$ with $I \neq J$, the automorphism of $J/I$ induced by $\alpha_g$ is outer, that is, not of the form $a \mapsto \text{Ad}(u)(a) = uau^*$ for any unitary $u$ in the multiplier algebra $M(J/I)$.

For finite group actions, some justification for this condition is given in Theorem 4.12 of \[28\], and Examples 4.13 and 4.14 of \[28\] show that several obvious weaker versions are not suitable. Proposition 1.11 below, however, suggests that strong pointwise outerness is too strong.

We give a preliminary definition, from the beginning of Section 2 of \[15\], where $\varphi \in \text{Aut}(A)$ is said to be a “freely acting automorphism” if the condition is satisfied. We don’t use that term, because it becomes awkward when applied to group actions.

**Definition 1.2.** Let $A$ be a C*-algebra, and let $\varphi \in \text{Aut}(A)$. Then $\varphi$ is said to be **spectrally nontrivial** if for every nonzero ideal $I \subset A$ such that $\varphi(I) = I$, we have $\Gamma_B(\varphi|_I) \neq \{1\}$ (as a subset of $\hat{Z} = S^1$). Otherwise, we say that $\varphi$ is **spectrally trivial**.

We generalize this definition as follows.

**Definition 1.3.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a C*-algebra $A$. We say that $\alpha$ is **pointwise spectrally nontrivial** if for every $g \in G \setminus \{1\}$, the automorphism $\alpha_g$ is spectrally nontrivial in the sense of Definition 1.2.

We further say that $\alpha$ is **spectrally free** if for any $\alpha$-invariant ideal $I$ of $A$ such that $I \neq A$, the induced action on $A/I$ is pointwise spectrally nontrivial.

In the rest of this section, we give various results which support the idea that spectral freeness is a good form of noncommutative freeness.

Spectral freeness should be thought of as a substitute for strong pointwise outerness. When the algebra is simple, spectral freeness and strong pointwise outerness reduce to pointwise spectral nontriviality and pointwise outerness. The following fact is essentially immediate from what is already known.

**Proposition 1.4.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a simple C*-algebra $A$. Then $\alpha$ is pointwise spectrally nontrivial if and only if $\alpha$ is pointwise outer.

**Proof.** When $A$ is simple, for every $\varphi \in \text{Aut}(A)$ the definitions immediately imply that $\Gamma_B(\varphi) = \Gamma(\varphi)$. But according to Corollary 8.9.10 of \[26\], outerness of $\varphi$ is equivalent to $\Gamma(\varphi) \neq \{1\}$. □

Although it is not directly related to spectral freeness, one direction of Proposition 1.4 is true in general.

**Proposition 1.5.** Let $\alpha : G \to \text{Aut}(A)$ be a pointwise spectrally nontrivial action of a discrete group $G$ on a C*-algebra $A$. Then $\alpha$ is pointwise outer.

**Proof.** It suffices to prove that if $\varphi \in \text{Aut}(A)$ is spectrally nontrivial, then $\varphi$ is outer. This follows from Theorem 2.1 of \[15\]. □
We can also relate spectral freeness to the strong Connes spectrum. We need a preliminary result on the strong Connes spectrum itself.

**Proposition 1.6.** Let $\alpha: G \to \text{Aut}(A)$ be an action of a locally compact abelian group $G$ on a C*-algebra $A$, and let $I_1 \subset I_2 \subset A$ be $\alpha$-invariant ideals. Let $\overline{\alpha}$ be the action of $G$ on $I_2/I_1$ determined by $\alpha$. Then $\Gamma(\alpha) \subset \overline{\Gamma(\alpha)}$.

**Proof.** Set $B = C^*(G, A, \alpha)$, $J_1 = C^*(G, I_1, \alpha)$, and $J_2 = C^*(G, I_2, \alpha)$. Lemma 2.8.2 of [2] allows us to identify $C^*(G, I_2/I_1, \overline{\alpha})$ with $J_2/J_1$, and this identification clearly respects the dual actions $\beta: \hat{G} \to \text{Aut}(B)$ and $\overline{\beta}: \hat{G} \to \text{Aut}(J_2/J_1)$. Lemma 3.4 of [1] gives

$$\overline{\Gamma(\alpha)} = \{ \tau \in \hat{G}: \tau(I) \subset J \text{ for every ideal } J \subset B \}$$

and

$$\Gamma(\overline{\alpha}) = \{ \tau \in \hat{G}: \overline{\tau}(I) \subset J \text{ for every ideal } J \subset J_2/J_1 \}.$$ 

It is clear from the compatibility of the actions $\beta$ and $\overline{\beta}$ that $\Gamma(\alpha) \subset \overline{\Gamma(\alpha)}$. □

**Proposition 1.7.** Let $\alpha: G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a C*-algebra $A$. Suppose that $\overline{\Gamma(\alpha_g)} \neq \{1\}$ for every $g \not\in G \setminus \{1\}$. Then $\alpha$ is spectrally free.

**Proof.** Let $g \in G \setminus \{1\}$, let $I \subset A$ be an $\alpha$-invariant ideal, let $\overline{\alpha}: G \to A/I$ be the action determined by $\alpha$, and let $J \subset A/I$ be an $\overline{\alpha_g}$-invariant ideal. We have to show that $\Gamma_B(\overline{\alpha_g}|J) \neq \{1\}$. By hypothesis, $\overline{\Gamma(\alpha_g)} \neq \{1\}$. Applying Proposition 1.6 with $\overline{\alpha}$ in place of $\alpha$, we obtain $\overline{\Gamma(\overline{\alpha_g}|J)} \neq \{1\}$. The result now follows from $\overline{\Gamma(\overline{\alpha_g}|J)} \subset \Gamma_B(\overline{\alpha_g}|J)$. □

In general, spectral freeness implies neither $\overline{\Gamma(\alpha_g)} \neq \{1\}$ for every $g \not\in G \setminus \{1\}$ nor strong pointwise outerness. There are counterexamples in the commutative case, which we consider next. (These conditions are all equivalent for finite groups. See Theorem 1.12 below.)

**Lemma 1.8.** Let $X$ be a locally compact Hausdorff space, and let $h: X \to X$ be a homeomorphism. Let $\varphi \in \text{Aut}(C_0(X))$ be the automorphism given by $\varphi(f) = f \circ h^{-1}$ for $f \in C_0(X)$. Then $\varphi$ is spectrally nontrivial if and only if

$$\text{int}\{x \in X: h(x) = x\} = \emptyset.$$ 

**Proof.** Set $F = \{x \in X: h(x) = x\}$. Assume that $\text{int}(F) = \emptyset$. We use Theorem 2.1 of [1] to show that $\varphi$ is spectrally nontrivial. Thus, let $B$ be a nonzero hereditary subalgebra of $C(X)$, and let $f_0 \in C(X)$. We have to show that for every $\varepsilon > 0$ there is $f \in B_+ \setminus \{0\}$ such that $\|f\| = 1$ and $\|f_0 \varphi(f)\| < \varepsilon$. There is a nonempty open set $U \subset X$ such that

$$B = \{ f \in C(X): f(y) = 0 \text{ for all } y \not\in U \}.$$ 

Then $U \not\subset F$, so there is $x \in U$ with $h(x) \neq x$. Choose an open set $V \subset U$ containing $x$ such that $h(V) \cap V = \emptyset$. Choose a continuous function $f: X \to [0, 1]$ such that $f(x) = 1$ and $\text{supp}(f) \subset V$. Then $f \varphi(f) = 0$, so $\|f_0 \varphi(f)\| = 0 < \varepsilon$. This completes the proof of spectral nontriviality.

Suppose now that $\text{int}(F) \neq \emptyset$. Set

$$I = \{ f \in C(X): f(y) = 0 \text{ for all } y \not\in \text{int}(F) \}.$$ 

Then $I$ is a $\varphi$-invariant ideal of $C_0(X)$, and $\varphi|_I = \text{id}_I$, so $\Gamma_B(\varphi) = \{1\}$. □
Proposition 1.9. Let $X$ be a locally compact Hausdorff space, and let $(g, x) \mapsto gx$ be an action of a discrete group $G$ on $X$. Let $\alpha: G \to \text{Aut}(C_0(X))$ be the action $\alpha_g(f)(x) = f(g^{-1}x)$ for $x \in X$ and $f \in C_0(X)$. Then:

1. $\alpha$ is pointwise spectrally nontrivial if and only if for every $g \in G \setminus \{1\}$, the set $\{x \in X: gx = x\}$ has empty interior.
2. $\alpha$ is spectrally free if and only if for every closed $G$-invariant subset $L \subset X$ and every $g \in G \setminus \{1\}$, the set $\{x \in L: gx = x\}$ has empty interior in $L$.
3. If $G$ is abelian, then $\alpha$ is spectrally free if and only if the action of $G$ on $X$ is free.

Proof. Part (1) is immediate from Lemma 1.8. Part (2) now follows from the fact that the $G$-invariant quotients of $C_0(X)$ are exactly the algebras $C_0(L)$ for closed $G$-invariant subsets $L \subset X$.

We prove (3). If the action is free, then $\alpha$ is spectrally free by part (2).

Now suppose that the action is not free. Choose $g_0 \in G \setminus \{1\}$ and $x_0 \in X$ such that $g_0x_0 = x_0$. Set $L = \overline{Gx_0}$. We claim that $g_0x = x$ for all $x \in L$. It suffices to consider $x = hx_0$ with $h \in G$, and we have $g_0hx_0 = hg_0x_0 = hx_0$ because $G$ is abelian. The claim follows. Apply part (2) with this choice of $L$ to see that $\alpha$ is not spectrally free.

Residual topological freeness is introduced in Definition 3.4(ii) of [7]. An action of a group $G$ on a Hausdorff topological space $X$ is essentially free if for $g \in G \setminus \{1\}$, the set $\{x \in X: gx = x\}$ has empty interior. On a commutative C*-algebra, one can check that residual topological freeness is equivalent to essential freeness of the action on every $G$-invariant closed set in the corresponding topological space.

Proposition 1.10. Let $A$ be a commutative C*-algebra, let $G$ be a discrete group, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$. Then $\alpha$ is spectrally free if and only if $\alpha$ is residually topologically free.

Proof. Apply Proposition 1.9(2) and the discussion above.

Proposition 1.11. Let $G$ be a discrete group, and let $X$ be a compact metric space with an action of $G$ which is minimal, essentially free but not free. Then the corresponding action $\alpha: G \to \text{Aut}(C(X))$ is spectrally free, but is not strongly pointwise outer and does not satisfy $\Gamma(\alpha_g) \neq \{1\}$ for every $g \in G \setminus \{1\}$.

There are many such actions (although none when $G$ is abelian). Here is an easy example. Let $G = \mathbb{Z} \rtimes \mathbb{Z}_2$ be the semidirect product of $\mathbb{Z}$ by the automorphism $n \mapsto -n$, acting on $S^1$ as follows. The generator of $\mathbb{Z}$ acts by an irrational rotation, and the generator of $\mathbb{Z}_2$ acts by $\zeta \mapsto \zeta^{-1}$.

Proof of Proposition 1.11. Spectral freeness follows from Proposition 1.9(2).

We now prove that $\alpha$ is not strongly pointwise outer and that there is $g \in G \setminus \{1\}$ such that $\Gamma(\alpha_g) = \{1\}$. By hypothesis, there exist $g \in G \setminus \{1\}$ and $x \in X$ such that $gx = x$. Then

$$C(X)/\{f \in C(X): f(x) = 0\}$$

is a nonzero $\alpha_g$-invariant subquotient on which the automorphism $\beta$ induced by $\alpha_g$ is trivial, hence not outer. Clearly also $\Gamma(\beta) = \{1\}$. So $\Gamma(\alpha_g) = \{1\}$ by Proposition 1.6.

\[\square\]
We now show the equivalence, for actions of finite groups, of spectral freeness, strong pointwise outerness, and \( \bar{\Gamma}(\alpha_g) \neq \{1\} \) for every \( g \in G \setminus \{1\} \).

**Proposition 1.12.** Let \( \alpha : G \to \text{Aut}(A) \) be a strongly pointwise outer action of a finite group \( G \) on a \( C^* \)-algebra \( A \). Then \( \bar{\Gamma}(\alpha_g) \neq \{1\} \) for every \( g \in G \setminus \{1\} \).

**Proof.** Let \( g \in G \). It is obvious that the restriction of any strongly pointwise outer action to any subgroup is again strongly pointwise outer. Therefore we may assume that \( g \) generates \( G \). Corollary 2.4 of [24] implies that \( \bar{\Gamma}(\alpha) = \hat{G} \). We must show that this implies \( \bar{\Gamma}(\alpha_g) \neq \{1\} \). The group \( \hat{G} \) acts on the primitive ideal space \( \text{Prim}(C^*(G,A,\alpha)) \) via \( \tau \cdot P = \hat{\alpha}_\tau(P) \) for \( \tau \in \hat{G} \) and \( P \in \text{Prim}(C^*(G,A,\alpha)) \).

Lemma 3.4 of [14] implies that \( \hat{\alpha}_\tau(L) \subset L \) for every ideal \( L \subset C^*(G,A,\alpha) \) and every \( \tau \in \hat{G} \). Since \( \hat{G} \) is finite, this is equivalent to \( \hat{\alpha}_\tau(L) = L \) for every ideal \( L \subset C^*(G,A,\alpha) \) and every \( \tau \in \hat{G} \). So the action of \( \hat{G} \) on \( \text{Prim}(C^*(G,A,\alpha)) \) is trivial.

Let \( n \) be the order of \( g \). The surjection \( \mathbb{Z} \to G \), sending \( 1 \in \mathbb{Z} \) to \( g \), allows us to identify \( \hat{G} \) with the subgroup of \( \hat{\mathbb{Z}} = S^1 \) consisting of all \( \exp(2\pi ik/n) \) for \( k = 0, 1, \ldots, n \). We now use Corollary 2.5 of [19], which identifies \( C^*(\mathbb{Z},A,\alpha_g) \) with the quotient of \( \hat{G} \to \hat{\mathbb{Z}} \) of the action \( \hat{\alpha} : G \to \text{Aut}(C^*(G,A,\alpha)) \), described before Theorem 2.4 in [19]. Thus, we identify \( C^*(\mathbb{Z},A,\alpha_g) \) with the set of all \( f \in C(\hat{\mathbb{Z}},C^*(G,A,\alpha)) \) which are invariant under the action \( \beta \) of \( \hat{\mathbb{Z}} \) defined by \( \beta(f)(\zeta) = \hat{\alpha}_\tau(f(\zeta) \tau) \) for \( \zeta \in \hat{\mathbb{Z}} \) and \( \tau \in \hat{\mathbb{Z}} \). The dual action \( \gamma \) of \( \hat{\mathbb{Z}} \) on \( C^*(\mathbb{Z},A,\alpha_g) \) then becomes \( \gamma(f)(\zeta) = f(\lambda^{-1}\zeta) \) for \( \lambda, \zeta \in \hat{\mathbb{Z}} \) and \( f \in C(\hat{\mathbb{Z}},C^*(G,A,\alpha)) \).

Clearly \( \text{Prim}(C(\hat{\mathbb{Z}},C^*(G,A,\alpha))) \cong \hat{\mathbb{Z}} \times \text{Prim}(C^*(G,A,\alpha)) \), and the action on it determined by \( \beta \) is \( \tau \cdot (\zeta,P) = (\zeta\tau^{-1}, \tau \cdot P) \) for \( \tau \in \hat{\mathbb{Z}} \subset \hat{\mathbb{Z}} \), \( \zeta \in \hat{\mathbb{Z}} \), and \( P \in \text{Prim}(C^*(G,A,\alpha)) \). One easily checks that the primitive ideal space of the fixed point algebra under \( \beta \) is the quotient of \( \hat{\mathbb{Z}} \times \text{Prim}(C^*(G,A,\alpha)) \) by this action. We write the image of \( (\zeta,P) \) as \( [\zeta,P] \). One further checks that the action on it determined by \( \gamma \) is \( \lambda \cdot [\zeta,P] = [\lambda \zeta,P] \) for \( \lambda, \zeta \in \hat{\mathbb{Z}} \) and \( P \in \text{Prim}(C^*(G,A,\alpha)) \).

Let \( \tau \in \hat{G} \subset \hat{\mathbb{Z}} \). Let \( \zeta \in \hat{\mathbb{Z}} \) and let \( P \in \text{Prim}(C^*(G,A,\alpha)) \). We calculate, using commutativity of \( \hat{\mathbb{Z}} \) at the first step, the formula for the action of \( \hat{G} \) on \( \hat{\mathbb{Z}} \times \text{Prim}(C^*(G,A,\alpha)) \) at the second step, and triviality of the action of \( \hat{G} \) on \( \text{Prim}(C^*(G,A,\alpha)) \) at the third step,

\[
\tau \cdot [\zeta,P] = [\zeta \tau,P] = [\zeta,P] \cdot \tau.
\]

So \( \tau \in \bar{\Gamma}(\alpha_g) \) by Lemma 3.4 of [14]. This completes the proof. \( \square \)

The main part of the proof that spectral freeness implies strong pointwise outerness is contained in Lemma 1.14 whose proof is based on Lemma 5.3.3 of [27]. We describe some notation and give one preliminary lemma. For a finite group \( G \), let \( S_G \) be the set of all subsets \( S \subset G \) such that \( 1 \in S \). Now let \( A \) be a \( C^* \)-algebra, let \( \alpha : G \to \text{Aut}(A) \) be an action of \( G \) on \( A \), and let \( I \subset A \) be an ideal. For \( S \in S_G \), we then define (following Lemma 5.3.3 of [27]) \( \alpha \)-invariant ideals \( I_S, I^-_S \subset A \) by

\[
I_S = \sum_{g \in G} \alpha_g \left( \bigcap_{h \in S} \alpha_h(I) \right) \quad \text{and} \quad I^-_S = \sum_{g \in G \setminus S} I_{S \cup \{g\}} \subset I_S.
\]

When \( S = G \), we take \( I^-_S = \{0\} \).
Lemma 1.13. Let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on a C*-algebra $A$. Let $I, L, M \subset A$ be ideals with $L \subset M$. Assume that $(I_S \cap L) + I_S^- = (I_S \cap M) + I_S^-$ for all $S \in \mathcal{S}_G$. Then $I_{\{1\}} \cap L = I_{\{1\}} \cap M$.

The hypothesis says that $L$ and $M$ have the same image in $I_S/ I_S^-$ for all $S \in \mathcal{S}_G$.

Proof of Lemma 1.13. We prove that $I_S \cap L = I_S \cap M$ by downwards induction on $S \in \mathcal{S}_G$. When $S = G$, since $I_S^- = \{0\}$, the hypothesis implies immediately that $I_G \cap L = I_G \cap M$.

Now let $S \in \mathcal{S}_G$ with $S \neq G$, and suppose that $I_{S \cup \{g\}} \cap L = I_{S \cup \{g\}} \cap M$ for every $g \in G \setminus S$. Then

\[
I_S \cap L = \left( \sum_{g \in G \setminus S} I_{S \cup \{g\}} \cap L \right) \cap L = \sum_{g \in G \setminus S} (I_{S \cup \{g\}} \cap M) = \left( \sum_{g \in G \setminus S} I_{S \cup \{g\}} \cap L \right) \cap M = I_S^- \cap M.
\]

Using $I_S^- \subset I_S$ at the first and fifth step, the assumption $(I_S \cap L) + I_S^- = (I_S \cap M) + I_S^-$ at the second step, $L \subset M$ at the third step, and $\mathfrak{L}1$ at the fourth step, we get

\[
I_S \cap L = [(I_S \cap L) + I_S^-] \cap L = [(I_S \cap M) + I_S^-] \cap L = (I_S \cap M) + (I_S^- \cap L) = I_S \cap M.
\]

This completes the induction step.

The lemma follows by taking $S = \{1\}$. \hfill \Box

Lemma 1.14. Let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on a C*-algebra $A$. Let $g_0 \in G$. Let $I, J \subset A$ be ideals such that

$I \subset J, \quad \alpha_{g_0}(I) = I, \quad \text{and} \quad \alpha_{g_0}(J) = J.$

Then there exist ideals $L, M \subset A$ such that

$L \subset M, \quad \alpha_g(L) = L \text{ for all } g \in G, \quad \text{and} \quad \alpha_{g_0}(M) = M,$

and such that there is an isomorphism $\varphi$ from $M/L$ to an $\alpha_{g_0}$-invariant subquotient of $J/I$ which intertwines the automorphisms of $M/L$ and $J/I$ induced by $\alpha_{g_0}$.

Proof. Let the notation be as before Lemma 1.13. We first assume that

\[
(I_S \cap J) + I_S^- = (I_S \cap I) + I_S^-
\]

for all $S \in \mathcal{S}_G$. Then $I_{\{1\}} \cap J = I_{\{1\}} \cap I$ by Lemma 1.13. Since $I \subset I_{\{1\}}$ (Lemma 5.3.3(3) of [27]), it follows that $I_{\{1\}} \cap J = I$. Take $M = I_{\{1\}} + J$ and $L = I_{\{1\}}$. The canonical isomorphism $M/L \to J/(I_{\{1\}} \cap J) = J/I$ clearly intertwines the automorphisms induced by $\alpha_{g_0}$. So the lemma is proved in this case.

We may therefore assume that there is $S \in \mathcal{S}_G$ such that

\[
(I_S \cap J) + I_S^- \neq (I_S \cap I) + I_S^-.
\]

Since $I_S^-$ is $\alpha$-invariant (Lemma 5.3.3(1) of [27]), we simplify the notation by defining $B = A/I_S^-$, $P = (I + I_S^-)/I_S^-$, and $Q = (J + I_S^-)/I_S^-$, and further letting $\beta: G \to \text{Aut}(B)$ be the action induced by $\alpha$. Then

\[
P_S^- = \{0\}, \quad P_S \cap P \subset P_S \cap Q, \quad \text{and} \quad P_S \cap P \neq P_S \cap Q.
\]
The quotient
\[ Q/P = (J + I + I_S^+) / \left( I + I_S^- \right) = J / \left( [I + I_S^-] \cap J \right) \]
is an \( \alpha_{g_0} \)-invariant subquotient of \( J/I \), so it suffices to use \( Q/P \) in place of \( J/I \) in the statement to be proved.

By Lemma 5.3.3(5) of [27], there is a subgroup \( H \subset G \), an \( H \)-invariant ideal \( N \subset P_S \), a system \( R \) of left coset representatives for \( H \) in \( G \), and a subset \( R_0 \subset R \), such that we have internal direct sum decompositions
\[ P_S = \bigoplus_{g \in R} \beta_g(N) \quad \text{and} \quad P \cap P_S = \bigoplus_{g \in R_0} \beta_g(N). \]

If \( g, h \in G \), then \( \beta_g(N) = \beta_h(N) \) if \( gH = hH \), and \( \beta_g(N) \cap \beta_h(N) = \varnothing \) otherwise. The relation \( \beta_{g_0}(P) = P \) therefore implies that \( g_0R_0H = R_0H \). So \( g_0(R \setminus R_0)H = (R \setminus R_0)H \), from which it follows that the ideal \( E = \sum_{g \in R \setminus R_0} \beta_g(N) \) satisfies \( \beta_{g_0}(E) = E \). Also \( P_S = (P \cap P_S) \oplus E \). Since \( Q \cap P_S \) is an ideal in \( P_S = \bigoplus_{g \in R} \beta_g(N) \), we have \( Q \cap Q \cap P_S = (Q \cap P \cap P_S) \oplus (Q \cap P) \). Since \( Q \cap P_S \) properly contains \( P \cap P_S \), we have \( Q \cap E \neq \{0\} \). Therefore \( Q \cap E \) is a nonzero \( \beta_{g_0} \)-invariant ideal in \( B \). It is \( \beta_{g_0} \)-equivariantly isomorphic to \( (Q \cap P_S) / (P \cap Q \cap P_S) \), which in turn is \( \beta_{g_0} \)-equivariantly isomorphic to the subquotient \( [(Q \cap P_S) + P] / P \) of \( Q/P \). This completes the proof. \( \square \)

**Theorem 1.15.** Let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a C*-algebra \( A \). Then the following are equivalent:

1. \( \alpha \) is spectrally free.
2. \( \alpha \) is strongly pointwise outer.
3. \( \Gamma(\alpha_g) \neq \{1\} \) for every \( g \in G \setminus \{1\} \).

**Proof.** That (3) implies (1) is Proposition 1.17 (even when \( G \) is not finite). That (2) implies (3) is Proposition 1.12.

We prove that (1) implies (2). Suppose that \( \alpha \) is not strongly pointwise outer. Then there exist \( g \in G \setminus \{1\} \) and \( \alpha_g \)-invariant ideals \( I, J \subset A \) such that \( I \subset J \), \( I \neq J \), and the automorphism \( \beta \) of \( J/I \) induced by \( \alpha_g \) is inner. By Lemma 1.14 we may assume that \( I \) is \( \alpha \)-invariant. Now \( \Gamma_B(\beta) = \{1\} \). Therefore the automorphism of \( A/I \) induced by \( \alpha_g \) is spectrally trivial, so \( \alpha \) is not spectrally free. \( \square \)

**Question 1.16.** Does strong pointwise outerness imply spectral freeness for groups which are not finite?

We give a result on spectral freeness for actions on tensor products (Proposition 1.18 below). One hopes to get the conclusion under much weaker hypotheses: simplicity of \( B \) should not be necessary, and one hopes that the trivial action on \( B \) can be replaced by an arbitrary action. We have not investigated such results. The statement we give is easy to prove, and is sufficient for a later application (in the proof of Corollary 6.9).

We start with a lemma.

**Lemma 1.17.** Let \( \alpha : G \to \text{Aut}(A) \) be an action of a locally compact abelian group \( G \) on a C*-algebra \( A \). Let \( B \) be any simple nuclear C*-algebra, and let \( \gamma : G \to \text{Aut}(B \otimes A) \) be the action \( \gamma_g = \text{id}_B \otimes \alpha_g \) for \( g \in G \). Then \( \Gamma_B(\gamma) = \Gamma_B(\alpha) \).
Proof. We use the criterion of Proposition 2.1 of [15]. We write $\hat{G}$ multiplicatively rather than additively, and for $\Omega \subset \hat{G}$ and $k \in \mathbb{Z}_{>0}$, we let

$$\Omega^k = \{\tau_1 \tau_2 \cdots \tau_k : \tau_1, \tau_2, \ldots, \tau_k \in \Omega\}.$$

We further denote the action of $\hat{G}$ on $\text{Prim}(C^*(G, A, \alpha))$ coming from $\tilde{\alpha}$ by $(\tau, x) \mapsto \tau x$.

Translated from ideals in $C^*(G, A, \alpha)$ to open subsets of $\text{Prim}(C^*(G, A, \alpha))$, Proposition 2.1 of [15] states that an element $\tau \in \hat{G}$ is in $\Gamma_B(\alpha)$ if and only if for every open set $\Omega \subset \hat{G}$ containing $\tau$, every $n \in \mathbb{Z}_{>0}$, and every open set $U \subset \text{Prim}(C^*(G, A, \alpha))$ such that $\hat{G}U$ is dense in $\text{Prim}(C^*(G, A, \alpha))$, there exist $\sigma_1 \in \Omega$, $\sigma_2 \in \Omega^2$, $\ldots$, $\sigma_n \in \Omega^n$ such that $U \cap \sigma_1 U \cap \sigma_2 U \cap \cdots \cap \sigma_n U \neq \emptyset$. Since $B$ is simple and nuclear, and since $C^*(G, B \otimes \alpha, \gamma) \cong B \otimes C^*(G, A, \alpha)$, there is an isomorphism $\text{Prim}(C^*(G, B \otimes \alpha, \gamma)) \cong \text{Prim}(C^*(G, A, \alpha))$ which is equivariant for the dual actions. Therefore the criterion for $\tau$ to be in $\Gamma_B(\alpha)$ holds if and only if the corresponding criterion for $\tau$ to be in $\Gamma_B(\gamma)$ holds. \hfill \square

Proposition 1.18. Let $\alpha : G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a C*-algebra $A$. Let $B$ be any simple nuclear C*-algebra, and let $\gamma : G \to \text{Aut}(B \otimes A)$ be the action $\gamma_g = \text{id}_B \otimes \alpha_g$ for $g \in G$. Let

1. If $\alpha$ is pointwise spectrally nontrivial, then so is $\gamma$.
2. If $\alpha$ is spectrally free, then so is $\gamma$.

Proof. We prove (1). Let $\gamma \in G \setminus \{1\}$. Since $B$ is simple and nuclear, the map $J \mapsto B \otimes J$ defines a bijection from the $\alpha_g$-invariant ideals of $A$ to the $\gamma_g$-invariant ideals of $B \otimes A$. Moreover, Lemma 1.17 implies that for each such $J$, we have $\Gamma_B(\gamma_g | J) = \Gamma_B(\alpha_g | J)$. The desired conclusion follows.

Part (2) follows from part (1) because $J \mapsto B \otimes J$ defines a bijection from the $\alpha$-invariant ideals of $A$ to the $\gamma$-invariant ideals of $B \otimes A$, and because $(B \otimes A)/(B \otimes J) = B \otimes A/J$. \hfill \square

2. Rokhlin Actions of $\mathbb{Z}$

Let $A$ be a unital C*-algebra, and let $\alpha : \mathbb{Z} \to \text{Aut}(A)$ be an action with the Rokhlin property. In this section, we show directly that all ideals in $C^*(\mathbb{Z}, A, \alpha)$ are crossed product ideals. In particular, $A$ separates the ideals in $C^*(\mathbb{Z}, A, \alpha)$. The main step is a kind of averaging result which seems of independent interest (Lemma 2.1). This is a somewhat old result; Lemma 2.1 and Theorem 2.2 are attributed to us in [17]. (See Lemma 4.1 of [17].)

We use this result to prove that such crossed products preserve the ideal and projection properties. The Rokhlin property is much stronger than needed for this. Some condition is needed, though, since crossed products by the trivial action of $\mathbb{Z}$ never have either the ideal or projection properties. We also use this result to show that actions of $\mathbb{Z}$ with the Rokhlin property satisfy the hypotheses of some of our later theorems.

Recall (Definition 2.5 of the survey article [9]) that an automorphism $\alpha$ of a unital C*-algebra $A$ (strictly speaking, the action of $\mathbb{Z}$ generated by $\alpha$) is said to have the Rokhlin property if for every $\varepsilon > 0$, every $n \in \mathbb{Z}_{>0}$, and every finite set $F \subset A$, there exist orthogonal projections $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_{n+1} \in A$
such that $\sum_{j=1}^{n} p_j + \sum_{j=1}^{n+1} q_j = 1$, such that $\|\alpha(p_j) - p_{j+1}\| < \varepsilon$ for $j = 1, 2, \ldots, n-1$ and $\|\alpha(q_j) - q_{j+1}\| < \varepsilon$ for $j = 1, 2, \ldots, n$, and such that $\|p_j a - ap_j\| < \varepsilon$ for $j = 1, 2, \ldots, n$ and $a \in F$ and $\|q_j a - aq_j\| < \varepsilon$ for $j = 1, 2, \ldots, n+1$ and $a \in F$.

**Lemma 2.1.** Let $A$ be a unital $C^*$-algebra, and let $\alpha \in \text{Aut}(A)$ have the Rokhlin property. Regard $A$ as a subalgebra of $C^*(\mathbb{Z}, A, \alpha)$ in the usual way, and let $E : C^*(\mathbb{Z}, A, \alpha) \to A$ be the standard conditional expectation. Then for every finite set $F \subset C^*(\mathbb{Z}, A, \alpha)$ and every $\varepsilon > 0$, there exist $m \in \mathbb{Z}_{\geq 0}$ and mutually orthogonal projections $e_1, e_2, \ldots, e_m \in A$ such that $\sum_{j=1}^{m} e_j = 1$ and

$$\left\| E(a) - \sum_{j=1}^{m} e_j a e_j \right\| < \varepsilon$$

for all $a \in F$.

**Proof.** Recall that $u$ denotes the standard unitary in $C^*(\mathbb{Z}, A, \alpha)$ which implements $\alpha$. Choose a finite set $S \subset A$ and $N \in \mathbb{Z}$ such that for every $b \in F$ there exist $b_{-N}, b_{-N+1}, \ldots, b_{N-1}, b_N \in S$

such that

$$\left\| b - \sum_{k=-N}^{N} b_k u^k \right\| < \frac{\varepsilon}{3}.$$

Set

$$M = \sup_{a \in S} \|a\|, \quad n = 2N + 1, \quad \text{and} \quad \delta = \frac{\varepsilon}{3n(2n+1)(6nM+1)}.$$

Apply the Rokhlin property with $S$ in place of $F$ and $\delta$ in place of $\varepsilon$, and let

$$p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_{n+1} \in A$$

be the resulting projections.

We claim that for $k = -N, -N+1, \ldots, N-1, N$, except for $k = 0$, we have $\|p_j\alpha^k(p_j)\| < 6n\delta$ for $j = 1, 2, \ldots, n$ and $\|q_j\alpha^k(q_j)\| < 6n\delta$ for $j = 1, 2, \ldots, n+1$.

We first observe that an induction argument gives $\|\alpha^l(p_j) - p_{j+l}\| < l\delta$ whenever $1 \leq j \leq n$ and $0 \leq l \leq n - j$, and $\|\alpha^l(q_j) - q_{j+l}\| < l\delta$ whenever $1 \leq j \leq n + 1$ and $0 \leq l \leq n + 1 - j$. In particular, under the given conditions, we always have

$$\|\alpha^l(p_j) - p_{j+l}\| < n\delta \quad \text{and} \quad \|\alpha^l(q_j) - q_{j+l}\| < n\delta.$$

Next,

$$\|\alpha(p_n + q_{n+1}) - (p_1 + q_1)\| = \|\alpha(1 - p_n - q_{n+1}) - (1 - p_1 - q_1)\|$$

$$= \left\| \alpha \left( \sum_{j=1}^{n-1} p_j + \sum_{j=1}^{n} q_j \right) - \left( \sum_{j=1}^{n-1} p_{j+1} + \sum_{j=1}^{n} q_{j+1} \right) \right\|$$

$$< (2n-1)\delta.$$

Now consider $\|p_j\alpha^k(p_j)\|$ when $k \in \{1, \ldots, N-1, N\}$ and $j + k \leq n$. Since $p_j p_{j+k} = 0$ and $\|\alpha^k(p_j) - p_{j+k}\| < n\delta$, we get $\|p_j\alpha^k(p_j)\| < n\delta \leq 6n\delta$. Similarly,
if \( j + k \leq n + 1 \), then \( \|q_j a^k(q_j)\| < n\delta \leq 6n\delta \). Next, suppose \( j + k > n \). Set \( r = j + k - n \). We have
\[
\|a^k(p_j + q_{j+1}) - (p_r + q_r)\| \leq \|a^{n-j}(p_j + q_{j+1}) - (p_n + q_{n+1})\| + \|a(p_n + q_{n+1}) - (p_1 + q_1)\| + \|a^{r-1}(p_1 + q_1) - (p_r + q_r)\|
\]
\[
< 2n\delta + (2n - 1)\delta + 2n\delta < 6n\delta.
\]
Since \( a^k(p_j) \leq a^k(p_j + q_{j+1}) \) and \( p_j (p_r + q_r) = 0 \) (because \( k \leq N \) and \( n > N \) imply \( r \neq j \)), we get \( \|p_j a^k(p_j)\| < 6n\delta \). A similar argument using \( r = j + k - n - 1 \) shows that if \( j + k > n + 1 \), then \( \|q_j a^k(q_j)\| < 6n\delta \).

Finally, suppose \( k \in \{-N, -N + 1, \ldots, -1\} \). Then
\[
\|p_j a^k(p_j)\| = \|a^k(p_j)p_j\| = \|p_j a^{-k}(p_j)\| < 6n\delta,
\]
and similarly \( \|q_j a^k(q_j)\| < 6n\delta \). This proves the claim.

Now let \( c = \sum_{k=-N}^{N} c_k u^k \) with \( c_{-N}, c_{-N+1}, \ldots, c_{N-1}, c_N \in S \). We claim that
\[
\left\| \sum_{j=1}^{n} p_j c p_j + \sum_{j=1}^{n+1} q_j c q_j - E(c) \right\| < \frac{\varepsilon}{3}.
\]
Set
\[
R = \{-N, -N + 1, \ldots, N - 1, N\} \setminus \{0\}.
\]
First, let \( a \in S \) and \( k \in R \). Then for \( j = 1, 2, \ldots, n \) we have
\[
\|p_j a u^k q_j\| = \|p_j a a^k(p_j) u^k\|
\]
\[
\leq \|p_j a - ap_j\| + \|a\| \cdot \|p_j a^k(p_j)\| \cdot \|u^k\| < \delta + 6nM \delta = (6nM + 1)\delta.
\]
Also, for \( k = 0 \), we get
\[
\|p_j a p_j - p_j a\| < \|p_j\| \cdot \|ap_j - p_j a\| < \delta.
\]
Similarly, for \( j = 1, 2, \ldots, n + 1 \) we have
\[
\|q_j a u^k q_j\| < (6nM + 1)\delta \quad \text{and} \quad \|q_j a q_j - q_j a\| < \delta.
\]
For \( c \) as above, we now have
\[
\left\| \sum_{j=1}^{n} p_j c p_j + \sum_{j=1}^{n+1} q_j c q_j - E(c) \right\| = \left\| \sum_{j=1}^{n} p_j c p_j + \sum_{j=1}^{n+1} q_j c q_j - \sum_{j=1}^{n} p_j c_0 - \sum_{j=1}^{n+1} q_j c_0 \right\|
\]
\[
\leq \sum_{k \in R} \left( \sum_{j=1}^{n} \|p_j c_k u^k p_j\| + \sum_{j=1}^{n+1} \|q_j c_k u^k q_j\| \right)
\]
\[
+ \sum_{j=1}^{n} \|p_j c_0 p_j - p_j c_0\| + \sum_{j=1}^{n+1} \|q_j c_0 q_j - q_j c_0\|
\]
\[
< (2N)(2n + 1)(6nM + 1)\delta + (2n + 1)\delta
\]
\[
\leq (2N + 1)(2n + 1)(6nM + 1)\delta
\]
\[
= n(2n + 1)(6nM + 1)\delta \leq \frac{\varepsilon}{3}.
\]
This proves the claim.
To prove the lemma, with \( e_1, e_2, \ldots, e_m \) being
\[
p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_{n+1},
\]
let \( a \in F \). By hypothesis, there is \( c \) of the form above such that \( \|c - a\| < \frac{1}{3} \varepsilon \). Then
\[
\left\| E(a) - \sum_{j=1}^{m} e_j a e_j \right\| \leq \|E(a) - E(c)\| + \left\| E(c) - \sum_{j=1}^{m} e_j c e_j \right\| + \|c - a\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]
This completes the proof. \( \square \)

As a corollary, we obtain the following result.

**Theorem 2.2.** Let \( A \) be a unital C*-algebra, and let \( \alpha \in \text{Aut}(A) \) have the Rohklin property. Then for every \( n \neq 0 \), the algebra \( A \) separates the ideals in \( C^*(\mathbb{Z}, A, \alpha^n) \).

The proof uses the following standard lemma. We recall that if \( \alpha : G \to \text{Aut}(A) \) is an action of a group \( G \) on a C*-algebra \( A \), and if \( I \subset A \) is a \( G \)-invariant ideal, then \( C^*(G, I, \alpha) \) is injective (Lemma 2.3.2 of [26]).

**Lemma 2.3.** Let \( A \) be a C*-algebra, let \( I \) be a discrete group, and let \( \alpha : G \to \text{Aut}(A) \) be an action of \( G \) on \( A \). Regard \( A \) as a subalgebra of \( C^*(G, A, \alpha) \) in the usual way. Let \( J \subset C^*(G, A, \alpha) \) be an ideal. Then \( I = J \cap A \) is a \( \alpha \)-invariant ideal in \( A \), and \( C^*(G, I, \alpha) \subset J \). The analogous statement holds for ideals in \( C^*_I(G, A, \alpha) \).

**Proof.** We prove the statement involving \( C^*(G, A, \alpha) \); the proof for the statement involving \( C^*_I(G, A, \alpha) \) is the same.

For \( \alpha \)-invariance, let \( a \in I \) and \( g \in G \). Then \( \alpha_g(a) = u_g a u_g^* \) is in both \( J \) and \( A \).

We have \( C^*(G, I, \alpha) \subset J \) because, if \( a \in I \) and \( g \in G \), then \( a \in J \), so \( a u_g \in J \). \( \square \)

**Proof of Theorem 2.2.** Let \( J \subset C^*(\mathbb{Z}, A, \alpha^n) \) be an ideal. Set \( I = J \cap A \), which, by Lemma 2.3, is an \( \alpha^n \)-invariant ideal in \( A \) such that \( C^*(\mathbb{Z}, I, \alpha^n) \subset J \).

Let \( E : C^*(\mathbb{Z}, A, \alpha^n) \to A \) be the standard conditional expectation. We claim that \( E(J) \subset I \). So let \( a \in J \), and let \( \varepsilon > 0 \). Think of \( a \) as an element of \( C^*(\mathbb{Z}, A, \alpha) \supset C^*(\mathbb{Z}, A, \alpha^n) \), and note that \( E \) is the restriction to \( C^*(\mathbb{Z}, A, \alpha^n) \) of the standard conditional expectation from \( C^*(\mathbb{Z}, A, \alpha) \) to \( A \). Lemma 2.1 provides \( m \in \mathbb{Z}_{>0} \) and elements \( e_1, e_2, \ldots, e_m \in A \) such that
\[
\left\| E(a) - \sum_{j=1}^{m} e_j a e_j \right\| < \varepsilon.
\]
Since \( \sum_{j=1}^{m} e_j a e_j \in J \) and \( \varepsilon > 0 \) is arbitrary, it follows that \( E(a) \in J \). The claim follows.

We finish the proof by showing that \( J \subset C^*(\mathbb{Z}, I, \alpha^n) \). Let \( a \in J \). Let \( u \in C^*(\mathbb{Z}, A, \alpha^n) \) be the standard unitary, that is, the one implementing the automorphism \( \alpha^n \). For \( k \in \mathbb{Z} \), define \( a_k = E(u a u^{-k}) \in I \). Then for every \( m \in \mathbb{Z} \), the Cesàro sum
\[
b_m = \sum_{k=-m}^{m} \left( 1 - \frac{|k|}{m+1} \right) a_k u^k
\]
is in \( C^*(\mathbb{Z}, I, \alpha^n) \), and Theorem VIII.2.2 of [6] implies that \( \lim_{m \to \infty} b_m = a \). So \( a \in C^*(\mathbb{Z}, I, \alpha^n) \). \( \square \)
Recall (Definition 1 of [22]) that a C*-algebra $A$ has the projection property if every ideal in $A$ has an approximate identity consisting of projections.

**Corollary 2.4.** Let $A$ be a unital C*-algebra, and let $\alpha \in \text{Aut}(A)$ have the Rokhlin property.

1. If $A$ has the ideal property, then $C^*(\mathbb{Z}, A, \alpha)$ has the ideal property.
2. If $A$ has the projection property, then $C^*(\mathbb{Z}, A, \alpha)$ has the projection property.

**Proof.** The first part follows from Theorem 2.2 and the fact that if $B$ is any C*-algebra which is generated as an ideal by its projections, and $\beta \in \text{Aut}(B)$ is arbitrary, then $C^*(\mathbb{Z}, B, \beta)$ is generated as an ideal by its projections. The proof of the second part is similar. □

Finally, we prove that, for actions of $\mathbb{Z}$, the Rokhlin property implies spectral freeness. The same is true for actions of finite groups, but we omit the proof.

**Proposition 2.5.** Let $A$ be a unital C*-algebra, and let $\alpha \in \text{Aut}(A)$ have the Rokhlin property. Then $\alpha$ generates an exact spectrally free action of $\mathbb{Z}$ on $A$.

**Proof.** It is clear that the Rokhlin property passes to the induced automorphism on a quotient by an $\alpha$-invariant ideal. So we need only prove that the action is pointwise spectrally nontrivial.

Fix $n \in \mathbb{Z}\setminus\{0\}$ and let $I \subset A$ be an $\alpha^n$-invariant ideal. Let $\beta : S^1 \to \text{Aut}(C^*(\mathbb{Z}, A, \alpha^n))$ be the dual action. Theorem 2.2 implies that all ideals in $C^*(\mathbb{Z}, A, \alpha^n)$ are $\beta$-invariant. Since $C^*(\mathbb{Z}, I, \alpha^n)$ is an ideal in $C^*(\mathbb{Z}, A, \alpha^n)$, it follows that all ideals in $C^*(\mathbb{Z}, I, \alpha^n)$ are invariant under the dual of the action of $\alpha^n$ on $I$. It now follows from Lemma 3.4 of [13] that $\tilde{\Gamma}(\alpha^n|_I) = S^1$. In particular, $\tilde{\Gamma}(\alpha^n|_I) \neq \{1\}$, so the larger set $\Gamma_B(\alpha^n|_I)$ is nontrivial. □

3. Using spectral freeness

In this section, we give the technical results (Lemma 3.2 and Corollary 3.10) which we use to prove that crossed products by spectrally free actions preserve some interesting properties of C*-algebras.

The proof of Lemma 3.2 is a modification of part of the proof of Lemma 10 of [16]. (Also see [14] and Lemma 3.2 of [32].) As preparation, we formally state the improvement of Lemma 3.2 of [14] mentioned in Remark 2.3 of [15]. It is essentially Lemma 7.1 of [20], but without the separability condition in [20].

**Lemma 3.1** (Remark 2.3 of [15]). Let $A$ be a C*-algebra, let $a \in A_+$, let $n \in \mathbb{Z}_{>0}$, let $a_1, a_2, \ldots, a_n \in A$, and let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \text{Aut}(A)$ be spectrally nontrivial. Then for every $\varepsilon > 0$ there is $x \in A_+$ with $\|x\| = 1$ such that $\|xa_1x\| > \|a\| - \varepsilon$ and $\|xa_kx\| < \varepsilon$ for $k = 1, 2, \ldots, n$.

After we proved the following lemma, the papers [8] and [7] were posted on the arXiv. Lemma 5.1 of [8] is a weaker version of our lemma. (It assumes simplicity of the algebra.) Lemma 3.11 of [7] is more general (applying to partial actions), but is stated differently. The proofs of both are similar to our proof.

**Lemma 3.2.** Let $\alpha : G \to \text{Aut}(A)$ be a pointwise spectrally nontrivial action of a discrete group $G$ on a C*-algebra $A$. Identify $A$ with a subalgebra of $C^*_r(G, A, \alpha)$ in the usual way. Let $a \in C^*_r(G, A, \alpha)_+ \setminus \{0\}$. Then there is $z \in C^*_r(G, A, \alpha)$ such that $za^*z$ is a nonzero element of $A$. 
Proof. Let $E: C^*_r(G, A, \alpha) \to A$ and $u_g$ for $g \in G$ be the standard conditional expectation and unitaries, as in the introduction. For $g \in G$, define $a_g = E(u_g^*)$. Since $E$ is faithful, $a_1 \neq 0$. By scaling, we may therefore assume that $\|a_1\| = 1$. Then $\|a\| \geq 1$.

Set $\varepsilon = \frac{1}{7}$.

We now follow the third paragraph of the proof of Lemma 10 of [16], with, for convenience, an extra normalization. Set

$$\delta = \min \left( \|a\|^{1/2}, \frac{1}{6\|a\|^{1/2}}, \frac{\varepsilon}{27\|a\|^{3/2}} \right).$$

Choose a finite set $S_0 \subset G$ and an element $b \in C^*_r(G, A, \alpha)$ of the form $b = \sum_{g \in S_0} b_g u_g$ with $b_g \in A$ for $g \in S_0$, such that $\|b - a^{1/2}\| < \delta$. Then

$$\|b^* b - a\| \leq \|b^*\| \cdot \|b - a^{1/2}\| + \|b^* - a^{1/2}\| \cdot \|a^{1/2}\|$$

$$< (\|a\|^{1/2} + \delta)\|a\|^{1/2} \leq 3\|a\|^{1/2}.\delta.$$

Therefore $\|E(b^* b) - a_1\| < 3\|a\|^{1/2}.\delta$. Since $3\|a\|^{1/2} < 1$, we have $E(b^* b) \neq 0$. Set $c = ||E(b^* b)||^{-1} b^* b$. Then there exist a finite set $S \subset G \setminus \{1\}$ and elements $c_1 \in A$ and $c_g \in A$ for $g \in S$ such that $c = c_1 + \sum_{g \in S} c_g u_g$. Moreover, $c_1 = ||E(b^* b)||^{-1} E(b^* b)$, so $\|c_1\| = 1$. We also have, using $\|b^* b\| < \|a\| + 3\|a\|^{1/2} \delta$,

$$\|c - a\| \leq \|c - b^* b\| + \|b^* b - a\| < \left( \frac{1}{1 - 3\|a\|^{1/2} \delta} - 1 \right) \|b^* b\| + 3\|a\|^{1/2} \delta$$

$$\leq \left( \frac{3\|a\|^{1/2} \delta}{1 - 3\|a\|^{1/2} \delta} \right) (\|a\| + 3\|a\|^{1/2} \delta) + 3\|a\|^{1/2} \delta$$

$$< 6\|a\|^{1/2} (4\|a\|) + 3\|a\|^{1/2} \delta \leq 27\|a\|^{3/2} \delta \leq \varepsilon.$$

Lemma 3.1 now provides $x \in A_+$ such that

$$\|x\| = 1, \quad \|xcx\| > 1 - \varepsilon,$$

and

$$\max_{g \in S} \|xc_g a_g(x)\| < \frac{\varepsilon}{\text{card}(S)}.$$

We estimate

$$\|xax - xa_1 x\| \leq 2\|a - c\| + \|xxc - xc_1 x\| \leq 2\|a - c\| + \sum_{g \in S} \|xc_g a_g(x)u_g\|$$

$$= 2\|a - c\| + \sum_{g \in S} \|xc_g a_g(x)u_g\| < 2\varepsilon + \varepsilon = 3\varepsilon$$

and

$$\|xa_1 x\| \geq \|xc_1 x\| - \|a - c\| > 1 - \varepsilon - \varepsilon = 1 - 2\varepsilon.$$

Recall that for $y \in A_+$ and $\rho \geq 0$, the expression $(y - \rho)_+$ is defined to be the result of applying functional calculus to $y$ with the function $t \mapsto \max(0, t - \rho)$. Using the Lemma 2.2 of [12] and (3.1), we find $d \in C^*_r(G, A, \alpha)$ such that

$$dxaxd^* = (xa_1 x - 4\varepsilon)_+.$$

Since $\varepsilon = \frac{1}{7}$ and $\|a\| = 1$, the estimate (3.2) implies that $(xa_1 x - 4\varepsilon)_+ \neq 0$. Clearly $(xa_1 x - 4\varepsilon)_+ \in A$, so we complete the proof by taking $z = dx$. \[\square\]
The following definition from [34] provides convenient terminology for an important corollary.

**Definition 3.3** (Definition 1.9 of [34]). Let $\alpha: G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a C*-algebra $A$. We say that $\alpha$ has the **intersection property** if for every nonzero ideal $I \subset C^*_r(G, A, \alpha)$, we have $I \cap A \neq 0$. We further say that $\alpha$ has the **residual intersection property** if for every $\alpha$-invariant ideal $J \subset A$, the induced action of $G$ on $A/J$ has the intersection property.

In the separable case, since spectral nontriviality implies proper outerness (Remark 2.5 of [15]), Corollary 3.4 (implicitly) and Proposition 3.5 below are contained in Remark 2.23 of [34]. Our proof uses less machinery. (There is a misprint in [34]: the reference to Theorem 1.10 of [34] there should be to Theorem 1.13 of [34].) These results were also independently obtained, in close to the form in which we give them but generalized to partial actions, as Theorem 3.12 and Corollary 3.13 of [7].

**Corollary 3.4.** Let $\alpha: G \to \text{Aut}(A)$ be a pointwise spectrally nontrivial action of a discrete group $G$ on a C*-algebra $A$. Then $\alpha$ has the intersection property.

**Proof.** Let $I \subset C^*_r(G, A, \alpha)$ be a nonzero ideal. Let $a \in I$ be a nonzero positive element. Lemma 3.2 provides $z \in C^*_r(G, A, \alpha)$ such that $zaz^*$ is a nonzero element of $A$. Since $zaz^* \in I$, we have $I \cap A \neq 0$. \qed

**Proposition 3.5.** Let $\alpha: G \to \text{Aut}(A)$ be a spectrally free action of a discrete group $G$ on a C*-algebra $A$. Suppose that $\alpha$ is exact (in the sense of Definition 1.5 of [34]). Then $A$ separates the ideals of $C^*_r(G, A, \alpha)$ (in the sense of Definition 0.1).\hfill \square

**Proof.** By Theorem 1.13 of [34], it is enough to prove that $\alpha$ has the residual intersection property. From the definitions, it is enough to prove that if $\alpha$ is spectrally nontrivial, then $\alpha$ has the intersection property. This is Corollary 3.4. \hfill \square

Any converse to Proposition 3.5 must have very restrictive hypotheses. According to Theorem 1.13 of [34], the algebra $A$ separates the ideals of $C^*_r(G, A, \alpha)$ if and only if the action is exact and has the residual intersection property. The trivial action of a nonabelian free group on $\mathbb{C}$ gives a simple reduced crossed product. Example 4.2.3 of [27] contains an action $\alpha: \mathbb{Z}_2 \times \mathbb{Z}_2 \to M_2$ such that $C^*(\mathbb{Z}_2 \times \mathbb{Z}_2, M_2, \alpha)$ is simple but $\alpha_g$ is inner for every $g \in G$. In both cases, the action is exact and has the residual intersection property, but is very far from being spectrally free.

**Corollary 3.6.** Let $\alpha: G \to \text{Aut}(A)$ be a pointwise spectrally nontrivial action of a discrete group $G$ on a C*-algebra $A$. For any nonzero ideal $I \subset C^*_r(G, A, \alpha)$ there exists a nonzero ideal $J \subset A$ such that $J \subset I$.

**Proof.** Take $J = I \cap A$ in Corollary 3.4. \hfill \square

**Corollary 3.7.** Let $\alpha: G \to \text{Aut}(A)$ be a pointwise spectrally nontrivial action of a discrete group $G$ on a C*-algebra $A$. Suppose that every nonzero ideal in $A$ contains a nonzero projection. Then every nonzero ideal in $C^*_r(G, A, \alpha)$ contains a nonzero projection.

The main part of the following lemma, namely $B \cong C$, is in 1.4 of [5], with a slightly different proof. Primarily for use in Section 4 we need additional information.
**Proof.** This is immediate from Corollary 3.3.

**Lemma 3.8.** Let $A$ be a C*-algebra, and let $a \in A$. In $A^{**}$, let $a = v(a^*a)^{1/2}$ be the polar decomposition of $a$. Set $B = (a^*a)A(a^*a)$ and $C = (aa^*)A(aa^*)$. Let $x \in A$. Then:

1. $x^*x \in B$ implies $xv^* \in A$ and $xx^* \in B$ implies $vx \in A$.
2. $x \in B$ implies $vxv^* \in C$.
3. $x^*x \in B$ implies $xv^*v = x$ and $xx^* \in B$ implies $v^*vx = x$.

Moreover, the formula $\varphi(x) = vxv^*$ defines an isomorphism $\varphi : B \to C$ such that $\varphi(a^*a) = aa^*$ and such that for every $x \in B_+$, we have $x \sim \varphi(x)$ in $A$.

**Proof.** It follows from Proposition 1.3 of [5] that for every continuous function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$, we have $vf(a^*a) \in A$. Therefore also $f(a^*a)v^* \in A$.

We claim that $x^*x \in B$ implies $\lim_{n \to \infty} x(a^*a)^{1/n} = x$. To see this, we note that $y \in B$ implies $\lim_{n \to \infty} y(a^*a)^{1/n} = \lim_{n \to \infty} (a^*a)^{1/n} = y$. We now estimate:

$$
\|x(a^*a)^{1/n} - x\|^2 = \|x(a^*a)^{1/n} - a^*a\|^2 = \|[(a^*a)^{1/n} - a^*a]a^*a\| = \|[(a^*a)^{1/n} - a^*a]a^*a\| = \|[(a^*a)^{1/n} - a^*a]a^*a\| = \|[(a^*a)^{1/n} - a^*a]a^*a\|,
$$

which converges to zero as $n \to \infty$. This proves the claim. Similarly $x^*x \in C$ implies $\lim_{n \to \infty} x(aa^*)^{1/n} = x$.

We now prove [4]. Suppose $x^*x \in B$. Then $xv^* = \lim_{n \to \infty} x(a^*a)^{1/n}v^*$ and $x(a^*a)^{1/n}v^* \in A$ for all $n \in \mathbb{Z}_{>0}$. If $xx^* \in B$, we get $xv^* \in A$, so $vx \in A$.

We next consider [3]. By standard properties of the polar decomposition, we have

$$
v^*a = (a^*a)^{1/2}.
$$

It follows that

$$
v^*va^*a = v^*v(a^*a)^{1/2}(a^*a)^{1/2} = v^*a(a^*a)^{1/2} = a^*a.
$$

Taking adjoints, we get $a^*avv^* = a^*a$. For any continuous function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$, one now gets $f(a^*a)v^*v = f(a^*a)$ by polynomial approximation. If now $x \in A$ satisfies $x^*x \in B$, we use the claim in the second paragraph to get

$$
xv^*v = \lim_{n \to \infty} x(a^*a)^{1/n}v^*v = \lim_{n \to \infty} x(a^*a)^{1/n} = x.
$$

This is one half of [3]. The other half follows by applying this to $x^*$ and taking adjoints.

We next claim that

$$
v(a^*a) = (aa^*)v.
$$

Taking adjoints in [3.3], we get $a^*v = (a^*a)^{1/2}$. So

$$
(aa^*)v = (a^*a)^{1/2}v = (a^*a)^{1/2}(a^*a)^{1/2} = va^*a,
$$

as desired.

We prove [2]. Elements of $B$ of the form $x = (a^*a)^{3/2}y(a^*a)^{3/2}$, with $y \in A$, are dense in $B$, so it suffices to consider such elements. We have, using (3.4) and its
adjoin at the second step,
\[ v xv^* = v(a^*a)(a^*a)^{1/2}y(a^*a)^{1/2}(a^*a)v^* \]
\[ = (aa^*)v(a^*a)^{1/2}y(a^*a)^{1/2}v^*(aa^*) = (aa^*)aya^*(aa^*) \in C, \]
as desired.

We now know that \( \varphi \) as in the statement is a map from \( B \) to \( C \). That it is a homomorphism follows from the relations \( v^*vx = xv^*v = x \) for \( x \in B \), which are a consequence of (3).

Define \( \psi: C \to A^{**} \) by \( \psi(x) = v^*xv \) for \( x \in C \). The same relations imply that \( (\psi \circ \varphi)(x) = x \) for all \( x \in B \).

We now claim that \( \psi(C) \subset B \). By continuity, it suffices to show that for \( y \in A \) we have \( v^*(aa^*)y(aa^*)v \in B \). Using (3.3) and its adjoint at the first step, we get
\[ v^*(aa^*)y(aa^*)v = (a^*a)^{1/2}(aa^*)^{1/2}v^*yv(a^*a)^{1/2}(aa^*)^{1/2} \]
\[ = (a^*a)^{1/2}a^*ya(a^*a)^{1/2} \in B, \]
as desired. Thus, we can treat \( \psi \) as a function from \( C \) to \( B \).

We saw above that \( \psi \circ \varphi = id_C \). It follows from (3.3) that \( vv^*aa^* = aa^* \). Using the same reasoning as in the proof of (3), by taking adjoints we get \( aa^*vv^* = aa^* \), and by polynomial approximation we get \( (aa^*)^{1/2}vv^* = (aa^*)^{1/2} \) for all \( n \in \mathbb{Z}_{>0} \).

For \( x \in C \) we then use the last statement in the second paragraph to get \( xv^* = x \) for all \( x \in C \), and take adjoints to get \( vv^*x = x \) for all \( x \in C \). These relations imply that \( \varphi \circ \psi = id_C \).

It remains only to prove that \( \varphi(x) \sim x \) for \( x \in B_+ \). Set \( c = vx^{1/2} \). We have \( c \in A \) by (1), \( c^*c = x \) by (3), and \( cc^* = \varphi(x) \), so the result follows from the discussion after Definition 2.5 of (11). \( \square \)

We need only one of the implications of the following proposition, but it seems informative to include the whole result.

**Proposition 3.9.** Let \( A \) be a C*-algebra and let \( D \subset A \) be a subalgebra. The following are equivalent:

1. For every \( a \in A_+ \setminus \{0\} \) there is \( b \in D_+ \setminus \{0\} \) such that \( b \preceq a \).
2. For every \( a \in A_+ \setminus \{0\} \) there is \( z \in A \) such that \( zaz^* \) is a nonzero element of \( D \).
3. For every nonzero hereditary subalgebra \( B \subset A \) there is a nonzero hereditary subalgebra \( E \subset D \) and an injective homomorphism \( \varphi: E \to B \) such that for all \( x \in E_+ \) we have \( \varphi(x) \sim x \).

**Proof.** We show that (1) implies (2). Let \( a \in A_+ \setminus \{0\} \). Choose \( b \in D_+ \setminus \{0\} \) such that \( b \preceq a \). Without loss of generality \( \|b\| = 1 \). The definition of Cuntz subequivalence provides \( v \in A \) such that \( \|vav^* - b\| < \frac{1}{4} \). Lemma 2.5(ii) of (11) provides \( w \in A \) such that \( (b - \frac{3}{4})^+ = wvaw^*w^* \). The element \( (b - \frac{3}{4})^+ \) is nonzero and in \( D \). Take \( z = wv \).

Now assume (2); we prove (3). Choose any nonzero element \( a \in B_+ \). Choose \( z \in A \) such that \( zaz^* \) is a nonzero element of \( D \). Set
\[ d = zaz^*, \quad E = dDd, \quad \text{and} \quad b = a^{1/2}z^*za^{1/2} \in B. \]
The last part of Lemma (3.8) provides an isomorphism \( \varphi: dD \to bD \) in \( B \) such that \( x \sim \varphi(x) \) for all \( x \in dD \). The conclusion of the lemma follows by restricting to \( dD \).
Finally, we prove that (3) implies (1). Let \( a \in A_+ \setminus \{0\} \). Set \( B = \overline{aAa} \). Let \( E \subset D \) and \( \varphi : E \to B \) as in (3). Choose any \( b \in D_+ \setminus \{0\} \). Then, using Proposition 2.7(i) of [11] for the second step, we have \( b \sim \varphi(b) \gtrsim a \).

**Corollary 3.10.** Let \( \alpha : G \to \text{Aut}(A) \) be a pointwise spectrally nontrivial action of a discrete group \( G \) on a C*-algebra \( A \). Identify \( A \) with a subalgebra of \( C^*_\alpha(G, A, \alpha) \) in the usual way. Let \( B \subset C^*_\alpha(G, A, \alpha) \) be a nonzero hereditary subalgebra. Then there exists a nonzero hereditary subalgebra \( E \subset A \) and an injective homomorphism \( \varphi : E \to B \) such that, for every \( x \in E_+ \), we have \( x \sim \varphi(x) \) in \( C^*_\alpha(G, A, \alpha) \).

**Proof.** We apply Proposition 2.7 with \( C^*_\alpha(G, A, \alpha) \) in place of \( A \) and with \( A \) in place of \( D \). Lemma 3.2 states that (2) holds, and the conclusion is that (3) holds.

The fact that we get \( x \sim \varphi(x) \) is special to the case in which we have some freeness condition. In Proposition 4.11 for an arbitrary action of \( \mathbb{Z}_2 \), we get all of the conclusion of Corollary 3.10 except for this Cuntz equivalence. For our applications, it does not matter, but it seems potentially useful.

4. Actions of \( \mathbb{Z}_2 \)

We conjecture the following analog of Corollary 3.10 for finite groups, but with no condition on the action. We omit the condition that for every \( x \in D_+ \), we have the Cuntz equivalence \( x \sim \varphi(x) \) in \( C^*(G, A, \alpha) \). The example of \( \mathbb{Z}_2 \) acting trivially on \( \mathbb{C} \) shows that, in general, this is not possible.

**Conjecture 4.1.** Let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a C*-algebra \( A \). Let \( B \subset C^*(G, A, \alpha) \) be a nonzero hereditary subalgebra. Then there exists a nonzero hereditary subalgebra \( D \subset A \) and an injective homomorphism \( \varphi : D \to B \).

We prove this conjecture for \( G = \mathbb{Z}_2 \). In fact, we prove a stronger result. If \( \alpha : \mathbb{Z}_2 \to \text{Aut}(A) \) is any action on any C*-algebra \( A \), and if \( D \subset A \) is a nonzero hereditary subalgebra, then there is a nonzero hereditary subalgebra \( B \) of the fixed point algebra of \( \alpha \) which is isomorphic to a (not necessarily hereditary) subalgebra of \( D \).

There are two obstructions to generalization. First, we use equivariant semiprojectivity of the cone over \( C(\mathbb{Z}_2) \) with the translation action of \( \mathbb{Z}_2 \) on \( \mathbb{Z}_2 \). (See Lemma 4.3.) The analogous statement for other groups is known only for cyclic groups whose order is a power of 2. (See Proposition 2.10 of [30], where equivariant projectivity is proved.) Second, the combinatorics in proofs such as that of Proposition 4.11 would need to be much more complicated, and may not give strong enough inequalities to use as hypotheses in an analog of Lemma 4.3.

**Notation 4.2.** We systematically use the same letter for an automorphism \( \alpha \) of a C*-algebra \( A \) such that \( \alpha^2 = \text{id}_A \) and the corresponding action \( \alpha : \mathbb{Z}_2 \to \text{Aut}(A) \). In particular, when appropriate, \( \hat{\alpha} \in \text{Aut}(C^*(\mathbb{Z}_2, A, \alpha)) \) is the automorphism which generates the dual action. We write the fixed point algebra as \( A^\alpha \). We also identify \( A \) with its image in \( C^*(\mathbb{Z}_2, A, \alpha) \) in the usual way.

**Lemma 4.3.** Let \( A \) be a C*-algebra, let \( \varepsilon > 0 \), and let \( a, b \in A_+ \) satisfy \( \|a - b\| < \varepsilon \). Then \( (b - \varepsilon)_+, A(b - \varepsilon)_+ \) is isomorphic to a hereditary subalgebra of \( aAa \).
Proof. Lemma 2.5(ii) of [11] provides \( v \in A \) such that \( vav^* = (b - \varepsilon)_+ \). Set \( c = a^{1/2}v^*ava^{1/2} \). Then \( cAc \cong (b - \varepsilon)_+, A(b - \varepsilon)_+ \) by the last part of Lemma 3.8 and \( cAc \) is a hereditary subalgebra of \( AAa \). \( \square \)

**Lemma 4.4.** Let \( A \) be a C*-algebra, let \( \alpha : \mathbb{Z}_2 \to \text{Aut}(A) \) be an action of \( \mathbb{Z}_2 \) on \( A \), and let \( x \in A \) satisfy

\[
0 \leq x \leq 1, \quad \|x\| = 1, \quad \text{and} \quad \|x - \alpha(x)\| < 1.
\]

Then there exists a \( \alpha \)-invariant hereditary subalgebra \( E \subset A \) which is isomorphic to a nonzero hereditary subalgebra of \( xAx \).

**Proof.** Set \( a = \frac{1}{2}[x + \alpha(x)] \). Then

\[
\|a - x\| = \frac{1}{2}\|x - \alpha(x)\| < \frac{1}{2}.
\]

Therefore \( \|a\| > \frac{1}{2} \). Set \( b = (a - \frac{1}{2})_+ \) and \( E = bab \). Then \( E \neq 0 \), and isomorphic to a hereditary subalgebra of \( xAx \) by Lemma 4.3. \( \square \)

**Definition 4.5.** Let \( A \) be a C*-algebra and let \( S, T \subset A \) be selfadjoint subsets. We say that \( S \) and \( T \) are **orthogonal** if \( ab = 0 \) for all \( a \in S \) and \( b \in T \).

The definition is symmetric in \( S \) and \( T \) because of the requirement that \( S \) and \( T \) be selfadjoint.

**Lemma 4.6.** Let \( A \) be a C*-algebra, let \( \alpha : \mathbb{Z}_2 \to \text{Aut}(A) \) be an action of \( \mathbb{Z}_2 \) on \( A \), and let \( x \in A \) be a nonzero element such that \( \alpha(x) = x^* \). Let \( D \subset A \) be the hereditary subalgebra generated by \( x^*x \). Suppose that \( D \) and \( \alpha(D) \) are orthogonal. Let \( F \subset A \) be the hereditary subalgebra generated by \( D \) and \( \alpha(D) \). Then \( F \) is \( \alpha \)-invariant, and there is an automorphism \( \psi \in \text{Aut}(D) \) such that \( \psi^2 = \text{id}_D \) and such that \( D^\psi \) is isomorphic to a corner of \( F \cap A^\alpha \).

There is surely a generalization to actions of arbitrary finite groups. If \( G \) is finite and \( \alpha : G \to \text{Aut}(A) \) is an action, then one should require that the hereditary subalgebras \( \alpha_g(D) \) be pairwise orthogonal and that there exist an equivariant homomorphism as follows. Define an action of \( G \) on the cone \( C_0((0, 1]) \otimes C(G) \) by letting \( G \) act on \( C(G) \) by translation and letting \( G \) act trivially on \( C_0((0, 1]) \). For \( g \in G \), let \( p_g \in C(G) \) be the projection \( p_g = \chi_{\{g\}} \). Let \( c \in C_0((0, 1]) \) be the function \( c(t) = t \) for \( t \in (0, 1] \). Then there should be an equivariant homomorphism \( \gamma : C_0((0, 1]) \otimes C(G) \to A \) such that \( \alpha_g(D) \) is the hereditary subalgebra of \( A \) generated by \( \gamma(c \otimes p_g) \) for all \( g \in G \). (Actually, the last part follows if we merely assume that \( D \) is the hereditary subalgebra of \( A \) generated by \( \gamma(c \otimes p_1) \).)

**Proof of Lemma 4.6.** Let \( \alpha^{**} \) be the automorphism of \( A^{**} \) induced by \( \alpha \in \text{Aut}(A) \). Apply Lemma 3.8 with \( x \) in place of \( a \), and let \( v \in A^{**} \) be as there, that is, the polar decomposition of \( x \) is \( x = v(x^*)^{1/2} \), and \( a \mapsto vav^* \) is an isomorphism from \( D \) to \( (xx^*)A(xx^*) = \alpha(D) \). Then the polar decomposition of \( x^* \) is \( x^* = v^*(xx^*)^{1/2} \). Therefore \( \alpha^{**}(v) = v^* \). Orthogonality of \( x^*x \) and \( xx^* \) implies that \( v^2 = 0 \).

Define \( \mu : M_2(D) \to A \) by

\[
\mu\left(\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\right) = a_{1,1} + a_{1,2}v^* + va_{2,1} + va_{2,2}v^*
\]

for \( a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2} \in D \). It follows from Lemma 3.8 that the range of \( \mu \), which a priori is in \( A^{**} \), really is contained in \( A \). A calculation using orthogonality of \( D \) and
\( \alpha(D) \), using \( a \in D \) implies \( vav^* \in \alpha(D) \) (Lemma 3.8[2]), and using Lemma 3.8[3] multiple times to both insert and remove factors of \( v^*v \), shows that \( \mu \) is a homomorphism.

Identify \( M_2(D) = M_2 \otimes D \), and let \((e_{j,k})_{j,k=1,2}\) be the standard system of matrix units for \( M_2 \).

The restriction of \( \mu \) to \( \mathbb{C}e_{1,1} \otimes D \) is injective, so \( \mu \) is injective.

We claim that the range \( \text{ran}(\mu) \) of \( \mu \) is equal to \( F \). We first show that \( \text{ran}(\mu) \subset F \).

Obviously \( \mu(\mathbb{C}e_{1,1} \otimes D) \subset F \) and \( \mu(\mathbb{C}e_{2,2} \otimes D) = \alpha(D) \subset F \). For \( a \in D \), we have

\[
\mu(e_{1,2} \otimes a)^* \mu(e_{1,2} \otimes a) = va^* va^* \in \alpha(D) \subset F
\]

and, using Lemma 3.8[3],

\[
\mu(e_{1,2} \otimes a)^* \mu(e_{1,2} \otimes a) = va^* va^* = aa^* \in D \subset F,
\]

so \( \mu(e_{1,2} \otimes a) \in F \). Also, \( \mu(e_{2,1} \otimes a) = \mu(e_{1,2} \otimes a)^* \in F \). Thus \( \text{ran}(\mu) \subset F \).

We now show that \( F \subset \text{ran}(\mu) \). Let \( b \in F \). Let \( (e_{\lambda})_{\lambda \in \Lambda} \) be an approximate identity for \( D \). Using orthogonality of \( D \) and \( \alpha(D) \), we see that \( D + \alpha(D) \) is a subalgebra of \( A \), and that \( (e_{\lambda} + \alpha(e_{\lambda}))_{\lambda \in \Lambda} \) is an approximate identity for \( D + \alpha(D) \) and therefore also for \( F \). So it suffices to prove that for \( \lambda \in \Lambda \) we have

\[
e_{\lambda}b_{e_{\lambda}}, e_{\lambda}b_{\alpha(e_{\lambda})}, \alpha(e_{\lambda})b_{e_{\lambda}}, \alpha(e_{\lambda})b_{\alpha(e_{\lambda})} \in \text{ran}(\mu).
\]

We have \( e_{\lambda} b_{e_{\lambda}} \in D \subset \text{ran}(\mu) \). Also

\[
\alpha(e_{\lambda})b_{\alpha(e_{\lambda})} \in \alpha(D) = vDv^* \subset \mu(\mathbb{C}e_{2,2} \otimes D) \subset \text{ran}(\mu).
\]

Next, set \( a = e_{\lambda} b_{\alpha(e_{\lambda})} v \). Then \( a \in A \) because \( \alpha(e_{\lambda})v = \alpha(e_{\lambda} v^*) \) and \( e_{\lambda} v^* \in A \) by Lemma 3.8[1]. Next,

\[
a a^* = e_{\lambda} b_{\alpha(e_{\lambda})} v^* ve_{\lambda} b_{\alpha(e_{\lambda})} e_{\lambda} \in D \quad \text{and} \quad a^* a = \alpha(ve_{\lambda} \alpha(b \epsilon_{2} b) e_{\lambda} v^*) \in \alpha(vDv^*) = D.
\]

Therefore \( a \in D \). So, using Lemma 3.8[3] at the first step,

\[
e_{\lambda} b_{\alpha(e_{\lambda})} = e_{\lambda} b_{\alpha(e_{\lambda})} v^* v = va^* = \mu(e_{1,2} \otimes a) \in \text{ran}(\mu).
\]

Finally, applying the case just done to \( b^* \), we get

\[
\alpha(e_{\lambda}) b_{e_{\lambda}} = (e_{\lambda} b_{\alpha(e_{\lambda})})^* \in \text{ran}(\mu).
\]

This completes the proof that \( \text{ran}(\mu) = F \).

Using Lemma 3.8[2], define \( \psi: D \rightarrow D \) by \( \psi(a) = \alpha(vav^*) \in \alpha^2(D) = D \). We claim that \( \psi^2 = \text{id}_D \). For \( a \in D \), we compute, using Lemma 3.8[3] at the last step,

\[
\psi^2(a) = \alpha(v\alpha(vav^*)v^*) = v^* \alpha^2(vav^*)v = v^* vav^* v = a.
\]

This proves the claim.

By Takai duality (Theorem 7.9.3 of [26]), there is an isomorphism

\[
\rho: C^*(\hat{\mathbb{Z}}_2, C^*(\mathbb{Z}_2, D, \psi), \hat{\psi}) \rightarrow M_2(D)
\]

which is equivariant for the second dual action on the domain and the \( \sigma \) action on \( M_2 \otimes D \) given by conjugation by the right regular representation on \( M_2 \), tensored with \( \psi \). The description of \( \sigma \) means that

\[
\sigma \left( \begin{array}{cc} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{array} \right) = \left( \begin{array}{cc} \psi(a_{2,2}) & \psi(a_{2,1}) \\ \psi(a_{1,2}) & \psi(a_{1,1}) \end{array} \right)
\]

for \( a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2} \in D \).
We now claim that $\mu$ is equivariant for this action. It is enough to check equivariance on a generating set for $M_2(D)$. We choose
\[
\{e_{1,1} \otimes a, e_{1,2} \otimes a : a \in D\}.
\]
We check, using the definition of $\psi$ at the second step in both calculations, and using Lemma 3.8, $\alpha^*v = v^*$, and $\alpha^*v = v$ several times: for $a \in D$,
\[
(\mu \circ \sigma)(e_{1,1} \otimes a) = \mu(e_{2,2} \otimes (\psi(a)))
= v\alpha(vav^*)v^* = \alpha((v^*va(v^*v)) = \alpha(a) = (\alpha \circ \mu)(e_{1,1} \otimes a)
\]
and
\[
(\mu \circ \sigma)(e_{1,2} \otimes a) = \mu(e_{2,1} \otimes (\psi(a)))
= v\alpha(vav^*) = \alpha((v^*va^v) = \alpha(av^*) = (\alpha \circ \mu)(e_{2,1} \otimes a).
\]
This proves the claim.

It follows that $F \cap A^\alpha = \mu(M_2(D)^\sigma)$. (We have $M_2(D)^\sigma \cong C^*(Z_2, D, \psi)$, but we won’t use this fact.) It is easy to see that
\[
M_2(D)^\sigma = \left\{ \begin{pmatrix} a & b \\ \psi(b) & \psi(a) \end{pmatrix} : a, b \in D \right\}
\]
Define a projection $e \in M_2$ by
\[
e = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]
Then $e$ is in the $\sigma$-invariant part of the multiplier algebra of $M_2(D)$, so that $eM_2(D)^\sigma e$ is a corner of $M_2(D)^\sigma$. Computations show that there is an injective homomorphism $\nu : D^\psi \to M_2(D)^\sigma$ given by
\[
\nu(c) = \frac{1}{2} \begin{pmatrix} c & c \\ c & c \end{pmatrix}
\]
for $c \in D^\psi$, that $e\nu(c)e = \nu(c)$ for $c \in D^\psi$, and that if $a, b \in D$ and we define
\[
c = \frac{1}{4}(a + b + \psi(a) + \psi(b)),
\]
then
\[
e \begin{pmatrix} a & b \\ \psi(b) & \psi(a) \end{pmatrix} = \nu(c).
\]
Therefore $\mu \circ \nu$ is an isomorphism from $D^\psi$ to the corner $\mu(eM_2(D)^\sigma e)$ of $F \cap A^\alpha$. □

**Lemma 4.7.** Let $A$ be a C*-algebra, let $\alpha : Z_2 \to \text{Aut}(A)$ be an action of $Z_2$ on $A$, and let $D \subset A$ be a hereditary subalgebra. Suppose that there is a nonzero element $x \in A$ such that $x^*x \in D$ and $xx^* \in \alpha(D)$. Then there is a nonzero element $y \in A$ such that $y^*y \in D$, $yy^* \in \alpha(D)$, and $\alpha(y) = y^*$.

**Proof.** Suppose $\alpha(x) = -x^*$. Set $y = ix$. Then $y^*y = x^*x$, $yy^* = xx^*$, and $\alpha(y) = i(-x^*) = (ix)^* = y^*$.

So suppose $\alpha(x) \neq -x^*$, and set $y = x + \alpha(x^*)$. Then $y \neq 0$. We need the fact that if $B$ is a hereditary subalgebra of $A$, and $a, b \in B$ satisfy $a^*a, b^*b \in B$, then $a^*b \in B$. (To see this, use Proposition 1.4.5 of [20] to find $v, w \in A$ such that $a = v(a^*a)^{1/4}$ and $b = w(b^*b)^{1/4}$. We have $(a^*a)^{1/4}, (b^*b)^{1/4} \in B$, so $a^*b = (a^*a)^{1/4}v^*w(a^*b)^{1/4} = B$.) Now expand:
\[
y^*y = x^*x + x^*\alpha(x^*) + \alpha(x)x + \alpha(x)\alpha(x^*).
\]
We have $\alpha(x)\alpha(x^*) = \alpha(xx^*) \in \alpha^2(D) = D$. So all four terms have the form $a^*b$ with $a^*a, b^*b \in D$. Therefore $y^*y \in D$. Essentially the same argument shows that $yy^* \in \alpha(D)$. \hfill \Box

**Lemma 4.8.** Let $A$ be a C*-algebra, and let $a, b \in A$ satisfy

$$0 \leq a \leq 1, \quad 0 \leq b \leq 1, \quad \text{and} \quad \|a - b\| = 1.$$ 

Then at least one of the following is true:

1. For every $\varepsilon > 0$ there is $x \in \overline{aAa}$ such that
   $$0 \leq x \leq 1, \quad \|x\| = 1, \quad \|xa - x\| < \varepsilon, \quad \text{and} \quad \|xb\| < \varepsilon.$$

2. For every $\varepsilon > 0$ there is $x \in \overline{bAb}$ such that
   $$0 \leq x \leq 1, \quad \|x\| < 1, \quad \|xa\| < \varepsilon, \quad \text{and} \quad \|xb - x\| < \varepsilon.$$

**Proof.** Without loss of generality $A$ is unital. For any C*-algebra $A$, let $S(A)$ denote the state space of $A$, and let $P(A)$ denote the pure state space of $A$. For any normal element $c \in A$, by using the Hahn-Banach theorem to extend from $C^*(1, c) \subset A$, we see that $\|c\| = \sup_{\omega \in S(A)} |\omega(c)|$. In particular, $\|a - b\| = \sup_{\omega \in S(A)} |\omega(a) - \omega(b)|$. Since we always have $\omega(a), \omega(b) \in [0, 1]$, it follows that at least one of the following is true:

3. For all $\delta > 0$ there is $\omega \in S(A)$ such that $\omega(a) > 1 - \delta$ and $\omega(b) < \delta$.

4. For all $\delta > 0$ there is $\omega \in S(A)$ such that $\omega(a) < \delta$ and $\omega(b) > 1 - \delta$.

We will prove that (3) implies conclusion (1) in the statement of the lemma; the same proof shows that (4) implies conclusion (2) in the statement of the lemma.

First, (3) clearly implies that $\|a\| = 1$. Set

$$\delta = \min \left( \frac{1}{32}, \frac{\varepsilon^2}{242} \right).$$

By linearity and the Krein-Milman Theorem, (3) implies that there is a pure state $\omega$ on $A$ such that

$$\omega(a) > 1 - \delta \quad \text{and} \quad \omega(b) < \delta.$$

Proposition 2.3 of [1] implies that the state $\omega$ can be excised in the sense of Definition 2.1 of [1]. In particular, there is $y \in A$ such that

$$0 \leq y \leq 1, \quad \|y\| = 1, \quad \|yay - \omega(a)y^2\| < \delta, \quad \text{and} \quad \|yby - \omega(b)y^2\| < \delta.$$

It follows that

$$\|y(1-a)y\| = \|yay-y^2\| < 2\delta \quad \text{and} \quad \|yby\| < 2\delta.$$

Since $(1-a)^2 \leq 1 - a$, we get

$$y(1-a)^2y \leq y(1-a)y,$$

whence $\|y(1-a)^2y\| < 2\delta$. Therefore

$$\|y(1-a)\| = \|(1-a)y\| < \sqrt{2\delta}.$$

Similarly $\|yb\| < \sqrt{2\delta}$. So

$$\|y(1-a)y\| \leq \|yay\| + \|y(a-y)\| < \sqrt{2\delta} + \sqrt{2\delta} = 2\sqrt{2\delta}.$$ 

Therefore

$$\|yab\| \leq \|yay\| \cdot \|b\| + \|yb\| < 2\sqrt{2\delta} + \sqrt{2\delta} = 3\sqrt{2\delta}.$$
Using (4.2) at the second step and $\delta \leq \frac{1}{32}$ at the fourth step, we get

$$1 \geq \|aya\| > \|y\| - 2\sqrt{2}\delta = 1 - 2\sqrt{2}\delta \geq \frac{1}{2}.$$ 

Therefore

$$1 \leq \frac{1}{\|aya\|} \leq 1 + 2(1 - \|aya\|) < 1 + 4\sqrt{2}\delta.$$ 

Setting $x = \|aya\|^{-1}aya$, we then get

(4.4) $$\|x - y\| \leq \|x - aya\| + \|aya - y\| < 4\sqrt{2}\delta,$$

so

$$\|y - x\| \leq \|y - aya\| + \|aya - x\| < \sqrt{2}\delta + 4\sqrt{2}\delta = 5\sqrt{2}\delta.$$ 

Combining the last two inequalities and using $\|a\| \leq 1$, we get

$$\|xy - x\| \leq \|x - y\| + \|ya - x\| < 6\sqrt{2}\delta + 5\sqrt{2}\delta = 11\sqrt{2}\delta \leq \varepsilon.$$ 

Combining (4.4) with (4.3) and using $\|b\| \leq 1$, we get

$$\|xb\| \leq \|x - aya\| + \|ayab\| < 4\sqrt{2}\delta + 3\sqrt{2}\delta = 7\sqrt{2}\delta \leq \varepsilon.$$ 

This completes the proof. \(\square\)

**Lemma 4.9.** Let $m \in \mathbb{Z}_{>0}$ and let $n = 2^m$. For every $\varepsilon > 0$ there is $\delta > 0$ such that the following holds. Let $A$ be a $C^*$-algebra, let $\alpha \in \text{Aut}(A)$ satisfy $\alpha^n = \text{id}_A$, and let $a_1, a_2, \ldots, a_n \in A$ satisfy $0 \leq a_k \leq 1$ for $k = 1, 2, \ldots, n$, $\|\alpha(a_k) - a_{k+1}\| < \delta$ for $k = 1, 2, \ldots, n$ (with $a_{n+1} = a_1$), and $\|a_ja_k\| < \delta$ for $j, k \in \{1, 2, \ldots, n\}$ with $j \neq k$. Then there exist $b_1, b_2, \ldots, b_n \in A$ such that for $k = 1, 2, \ldots, n$ we have (taking $b_{n+1} = b_1$)

$$0 \leq b_k \leq 1, \quad \alpha(b_k) = b_{k+1}, \quad \text{and} \quad \|b_k - a_k\| < \varepsilon,$$

and such that $b_jb_k = 0$ for $j, k \in \{1, 2, \ldots, n\}$ with $j \neq k$.

The case $n = 2$ (which is what we need) has a fairly easy direct proof. Set $x = a_1 - a_2$. Then $\|x + \alpha(x)\| < 2\delta$. The element $y = \frac{1}{2}(x - \alpha(x))$ satisfies $\alpha(y) = -y$ and $\|y - x\| < \delta$. Take $b_1$ to be the positive part of $y$ and take $b_2$ to be the negative part of $y$.

**Proof of Lemma 4.9.** We write $\alpha : \mathbb{Z}_n \to \text{Aut}(A)$ for the action of $\mathbb{Z}_n$ generated by $\alpha$. Replacing $A$ by the smallest $\alpha$-invariant subalgebra containing $a_1, a_2, \ldots, a_n$, we may assume that $A$ is separable.

Let $C = C_0((0, 1]) \otimes C(\mathbb{Z}_n)$ be the cone over $C(\mathbb{Z}_n)$, with the action of $\mathbb{Z}_n$ obtained by letting $\mathbb{Z}_n$ act on $C(\mathbb{Z}_n)$ by translation and letting $\mathbb{Z}_n$ act trivially on $C_0((0, 1])$. Then $C$ is equivariantly projective by Proposition 2.10 of [30]. Therefore $C$ is equivariantly semiprojective. It follows from Lemma 1.5 of [30] that the unitization $C^+$ of $C$ is equivariantly semiprojective in the unital category.

The algebra $C^+$, with the specified action of $\mathbb{Z}_n$, is given by the following $\mathbb{Z}_n$-equivariant set $(S, \sigma, R)$ of generators and relations in the sense of Definition 5.8 of [29]. We take $S = \{x_1, x_2, \ldots, x_n\}$, we take $\sigma$ to be the action of $\mathbb{Z}_n$ on $S$ generated by the cyclic permutation $x_k \mapsto x_{k+1}$ for $k = 1, 2, \ldots, n$ (with $x_{n+1} = x_1$), and we take $R$ to consist of the relations $0 \leq x_k \leq 1$ for $k = 1, 2, \ldots, n$ and $x_jx_k = 0$ for $j, k \in \{1, 2, \ldots, n\}$ with $j \neq k$. (See Remark 5.6 of [29] for the
justification of these as relations.) The pair \((S, R)\) is finite, admissible, and bounded (Definition 5.2 of [29]). Apply Theorem 5.22 of [29] to conclude that \((S, \sigma, R)\) is stable in the sense of Definition 5.20 of [29]. It is immediate from this definition that for every \(\varepsilon > 0\) there is \(\delta > 0\) such that for any C*-algebra \(A\) the conclusion of the lemma holds in the unitization \(A^+\) in place of \(A\).

So assume that \(a_1, a_2, \ldots, a_n \in A\), and let \(b_1, b_2, \ldots, b_n \in A^+\) be the resulting elements. Let \(\kappa: A^+ \to \mathbb{C}\) be the map coming from the unitization. Since \(b_j b_k = 0\) for \(j \neq k\), we have \(\kappa(b_j)\kappa(b_k) = 0\) for \(j \neq k\). Since the induced action of \(\mathbb{Z}_n\) fixes \(1 \in A^+\), we have \(\kappa(b_{k+1}) = \kappa(\alpha(b_k)) = \kappa(b_k)\) for \(k = 1, 2, \ldots, n - 1\). These two collections of relations are compatible only if \(\kappa(b_k) = 0\) for \(k = 1, 2, \ldots, n\). Therefore \(b_1, b_2, \ldots, b_n\) are in fact in \(A\).

**Proposition 4.10.** Let \(A\) be a C*-algebra, let \(\alpha: \mathbb{Z}_2 \to \text{Aut}(A)\) be an action of \(\mathbb{Z}_2\) on \(A\), and let \(D \subset A\) be a nonzero hereditary subalgebra. Then there exists a nonzero hereditary subalgebra \(B \subset A^\alpha\) which is isomorphic to a (not necessarily hereditary) subalgebra of \(D\).

**Proof.** We divide the proof into several cases. At any stage, we may replace \(D\) by any nonzero hereditary subalgebra of \(D\), and then perhaps appeal to a case already done.

**Case 1:** \(D\) and \(\alpha(D)\) are orthogonal, and there is no nonzero \(x \in A\) such that \(x^* x \in D\) and \(xx^* \in \alpha(D)\).

Orthogonality implies that \(D + \alpha(D)\) is a subalgebra of \(A\) which is equivariantly isomorphic to \(D \oplus D\) with the action generated by \(\beta(a, b) = (b, a)\) for \(a, b \in D\). We claim that \(D + \alpha(D)\) is a hereditary subalgebra of \(A\). So let \(y \in A\) satisfy \(y^* y, yy^* \in D + \alpha(D)\). Let \((e_\lambda)_{\lambda \in \Lambda}\) be an approximate identity for \(D\). Then \((e_\lambda + \alpha(e_\lambda))_{\lambda \in \Lambda}\) is an approximate identity for \(D + \alpha(D)\) and therefore also for the hereditary subalgebra generated by \(D + \alpha(D)\). For \(\lambda \in \Lambda\), the element \(x = \alpha(e_\lambda)y e_\lambda\) satisfies \(x^* x \in D\) and \(xx^* \in \alpha(D)\). Therefore \(\alpha(e_\lambda)y e_\lambda = 0\). Similarly \(\alpha(e_\lambda)y^* e_\lambda = 0\), so \(e_\lambda\alpha(e_\lambda) = 0\).

Therefore

\[
y = \lim_{\lambda \in \Lambda} [e_\lambda + \alpha(e_\lambda)]y[e_\lambda + \alpha(e_\lambda)] = \lim_{\lambda \in \Lambda} (e_\lambda y e_\lambda + \alpha(e_\lambda) \alpha(e_\lambda)) \in D + \alpha(D).
\]

The claim is proved.

Now \([D + \alpha(D)]^\alpha\) is a hereditary subalgebra of \(A^\alpha\), and \([D + \alpha(D)]^\alpha \cong D\). This completes the proof of Case 1.

**Case 2:** \(D\) and \(\alpha(D)\) are orthogonal, and there is a nonzero element \(x \in A\) such that \(x^* x \in D\) and \(xx^* \in \alpha(D)\).

By Lemma 4.7, we may assume that \(\alpha(x) = x^*\). Therefore \(\alpha(x^* x) = xx^*\). Since we can replace \(D\) by a nonzero hereditary subalgebra of \(D\), we may assume that \(D = (x^* x)A(x^* x)\). So \(\alpha(D) = (xx^*)A(xx^*)\). Let \(F \subset A\) be the hereditary subalgebra generated by \(D\) and \(\alpha(D)\). We apply Lemma 4.6 to get an automorphism \(\psi \in \text{Aut}(D)\) such that \(\psi^2 = \text{id}_D\) and such that \(D\psi\) is isomorphic to a corner of \(F \cap A^\alpha\). Then \(D\psi\) is a nonzero subalgebra of \(D\) which is isomorphic to a hereditary subalgebra of \(A^\alpha\).

**Case 3:** \(D\) and \(\alpha(D)\) are not orthogonal. Then there exist \(a \in D\) and \(b \in \alpha(D)\) such that \(ba \neq 0\). Set \(c = ba\). Then \(c^* c \in D\) and \(cc^* \in \alpha(D)\). By Lemma 4.7, there is \(x \in A\) such that

\[
x \neq 0, \quad x^* x \in D, \quad xx^* \in \alpha(D), \quad \text{and} \quad \alpha(x) = x^*.
\]
Then $\alpha(x^*x) = xx^*$. We may clearly assume that $\|x\| = 1$. Since we can replace $D$ by a nonzero hereditary subalgebra of $D$, we may assume that $D = (x^*x)A(x^*x)$. So $\alpha(D) = (x^*x)A(x^*x)$.

Suppose now that $\|x^*x - xx^*\| < 1$. Apply Lemma 4.4 with $x^*x$ in place of $x$. We obtain a nonzero $\alpha$-invariant hereditary subalgebra $E \subset A$ which is isomorphic to a nonzero hereditary subalgebra of $D$. Then $E^\alpha = E \cap A^\alpha$ is a nonzero hereditary subalgebra of $A^\alpha$ which is isomorphic to a subalgebra of $D$.

Otherwise, we have $\|x^*x - xx^*\| \geq 1$. In fact, we must have $\|x^*x - xx^*\| = 1$.

Use Lemma 4.9 to choose $\delta > 0$ such that whenever $y \in A$ satisfies $0 \leq y \leq 1$ and $\|y\alpha(y)\| < \delta$, then there exists $z \in A$ such that

\[
0 \leq z \leq 1, \quad z\alpha(z) = 0, \quad \text{and} \quad \|z - y\| < \frac{\delta}{2}.
\]

Apply Lemma 4.8 with $a = x^*x$ and $b = xx^*$. We may exchange $x$ and $x^*$ if desired. (This exchanges $D$ and $\alpha(D)$. But finding a nonzero hereditary subalgebra of $A^\alpha$ isomorphic to a subalgebra of $D$ is equivalent to finding a nonzero hereditary subalgebra of $A^\alpha$ isomorphic to a subalgebra of $\alpha(D)$.) We may therefore assume that from Lemma 4.8 we obtain $y \in D$ such that

\[
y \in D, \quad 0 \leq y \leq 1, \quad \|y\| = 1, \quad \|y(x^*x) - y\| < \frac{\delta}{2}, \quad \text{and} \quad \|y(xx^*)\| < \frac{\delta}{2}.
\]

Then also $\|(x^*x)y - y\| < \frac{\delta}{2}$.

Now

\[
\|y\alpha(y)\| \leq \|y\| \cdot \|\alpha(y) - (xx^*)\alpha(y)\| + \|y(xx^*)\| \cdot \|\alpha(y)\|
\]

\[
= \|\alpha(y - (x^*x)y)\| + \|y(xx^*)\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\]

By the choice of $\delta$, there exists $z \in A$ satisfying (4.5). Since $\|y\| = 1$, we have $\|z\| > \frac{1}{2}$, and therefore $c = (z - \frac{1}{2})_+$ is nonzero. Clearly $\alpha(c) = 0$. Let $E = cA^\alpha$. Then $E$ and $\alpha(E)$ are orthogonal hereditary subalgebras of $A$. By Case 1 or Case 2, as appropriate, there exists a nonzero hereditary subalgebra $B \subset A^\alpha$ which is isomorphic to a subalgebra of $E$. Lemma 4.3 implies that $E$, and hence $B$, is isomorphic to a subalgebra of $yA^\alpha \subset D$.

The following lemma is surely well known.

**Lemma 4.11.** Let $A$ and $B$ be C*-algebras and let $\pi : A \to B$ be a surjective homomorphism. Let $D \subset A$ be a hereditary subalgebra. Then $\pi(D)$ is a hereditary subalgebra of $B$.

**Proof.** Let $x \in \pi(D)$ and $y \in B$ satisfy $0 \leq y \leq x$. Then $\lim_{n \to \infty} x^{1/n} y x^{1/n} = y$. Choose $a \in D_+$ and $b \in A_+$ such that $\pi(a) = x$ and $\pi(b) = y$. Then the elements $a^{1/n} b a^{1/n}$ are in $D$ and $\lim_{n \to \infty} \pi(a^{1/n} b a^{1/n}) = y$. Therefore $y \in \pi(D) = \pi(D)$.

**Lemma 4.12.** Let $A$ be a C*-algebra and let $\alpha$ be an action of $\mathbb{Z}_2$ on $A$. Let $I \subset A$ be an $\alpha$-invariant ideal. Then $I^\alpha$ is an ideal in $A^\alpha$, and $A^\alpha/I^\alpha$ is isomorphic to the fixed point algebra of the induced action of $\mathbb{Z}_2$ on $A/I$.

**Proof.** This is a special case of Lemma 1.6 of [29].

**Proposition 4.13.** Let $A$ be a C*-algebra, let $\alpha$ be an action of $\mathbb{Z}_2$ on $A$, let $I \subset A$ be an ideal (not necessarily $\alpha$-invariant), and let $D \subset A/I$ be a nonzero hereditary
subalgebra. Then there exists an ideal \( J \subset A^\alpha \) and a nonzero hereditary subalgebra \( B \subset A^\alpha /J \) which is isomorphic to a (not necessarily hereditary) subalgebra of \( D \).

**Proof.** We divide the proof into several cases, some of which will be done by reduction to previous cases, possibly for a different choice of \( A \) and \( I \). We let \( \pi: A \to A/I \) be the quotient map, and we set \( E = \pi^{-1}(D) \), which is a hereditary subalgebra of \( A \) such that \( I \subset E \).

**Case 1:** The ideal \( I \) is \( \alpha \)-invariant.

There is an induced action \( \pi \) of \( \mathbb{Z}_2 \) on \( A/I \). By Proposition 4.10 there is a subalgebra \( B \subset D \) which is isomorphic to a nonzero hereditary subalgebra of \( (A/I)^{\alpha} \). Set \( J = I^\alpha \). Then \( (A/I)^{\alpha} \cong A^\alpha /J \) by Lemma 4.12. This proves Case 1.

**Case 2:** There is a \( C^* \)-algebra \( C \) such that \( A = C \oplus C \) with \( \alpha(x,y) = (y,x) \) for \( x,y \in C \), and \( I = \{0\} \oplus C \). We claim that in this case, we can take \( J = \{0\} \).

We have \( A/I \cong C \), and there is an isomorphism \( \varphi: C \to A^\alpha \) given by \( \varphi(x) = (x,x) \) for all \( x \in C \). So \( D \) is isomorphic to a hereditary subalgebra of \( C \), and thus \( D \) is isomorphic to a hereditary subalgebra of \( A^\alpha \). This proves Case 2 with \( J = \{0\} \).

**Case 3:** \( I \cap \alpha(I) = \{0\} \) and \( I \) is a proper subset of \( E \cap (I + \alpha(I)) \).

Set \( F = E \cap (I + \alpha(I)) \). Set \( C = (I + \alpha(I))/I \), let \( \kappa: I + \alpha(I) \to C \) be the quotient map, and define \( \varphi: I + \alpha(I) \to C \oplus C \) by \( \varphi(a) = (\kappa(a), (\kappa \circ \alpha)(a)) \) for \( a \in I + \alpha(I) \). Then \( \varphi \) is bijective since \( I \cap \alpha(I) = \{0\} \). Let \( \gamma: C \oplus C \to C \oplus C \) be the automorphism given by \( \gamma(x,y) = (y,x) \) for \( x,y \in C \). Using the action of \( \mathbb{Z}_2 \) that \( \gamma \) generates, \( \varphi \) becomes equivariant.

We have \( \varphi(I) = \{0\} \oplus C \) and \( \varphi(F) = \kappa(F) \oplus C \). The hypotheses of this case imply that \( \kappa(F) \neq \{0\} \). Lemma 4.11 implies that \( \kappa(F) \) is a hereditary subalgebra of \( C \subset A/I \), and clearly \( \kappa(F) \subset D \). Case 2 implies that \( \kappa(F) \) contains a subalgebra isomorphic to a nonzero hereditary subalgebra of \( I + \alpha(I) \), and hence isomorphic to a nonzero hereditary subalgebra of \( A \). This proves Case 3.

**Case 4:** \( I \cap \alpha(I) = \{0\} \) and \( I \) is not a proper subset of \( E \cap (I + \alpha(I)) \).

Since \( I \subset E \cap (I + \alpha(I)) \), the hypotheses of this case imply that \( E \cap (I + \alpha(I)) = I \). Let \( \psi: A/I \to A/(I + \alpha(I)) \) be the quotient map. Then \( \psi|_D \) is injective. Therefore \( D \) is isomorphic to a nonzero hereditary subalgebra of \( A/(I + \alpha(I)) \). Since \( I + \alpha(I) \) is \( \alpha \)-invariant, the result in Case 4 follows from Case 1.

**Case 5:** None of the previous cases applies. Thus \( I \) is not \( \alpha \)-invariant and \( I \cap \alpha(I) \neq \{0\} \). Let \( \eta: A \to A/(I \cap \alpha(I)) \) and \( \mu: A/(I \cap \alpha(I)) \to A/I \) be the quotient maps. Set \( L = \eta(I) \). Let \( \pi \) be the induced action of \( \mathbb{Z}_2 \) on \( A/(I \cap \alpha(I)) \). Then \( L \cap \pi(L) = \{0\} \) and \( \mu \) induces an isomorphism \( [A/(I \cap \alpha(I))]/L \to A/I \).

Therefore Case 3 or Case 4 applies with \( A/(I \cap \alpha(I)) \) in place of \( A \) and \( L \) in place of \( I \). Thus there is an ideal \( J_0 \subset A/(I \cap \alpha(I)) \) and a nonzero hereditary subalgebra \( B \subset A/(I \cap \alpha(I)) \) which is isomorphic to a (not necessarily hereditary) subalgebra of \( D \). We have \( [A/(I \cap \alpha(I))]/\pi \cong A^\alpha/(I \cap \alpha(I))^{\alpha} \) by Lemma 4.12. Let \( J \) be the inverse image of \( J_0 \) in \( A^\alpha \). Then \( A^\alpha /J \cong [A/(I \cap \alpha(I))]/\pi J_0 \), so we can identify \( B \) with a hereditary subalgebra of \( A^\alpha /J \). This completes the proof of Case 5.\( \square \)
5. **Permanence for properties defined in terms of hereditary subalgebras**

To avoid repetition, we present an abstract theory which gives permanence results for the properties we consider. Most of Sections 6, 7, and 8 consist of applications of this theory.

**Definition 5.1.** Let \( \mathcal{C} \) be a class of C*-algebras. We say that \( \mathcal{C} \) is *upwards directed* if whenever \( A \) is a C*-algebra which contains a subalgebra isomorphic to an algebra in \( \mathcal{C} \), then \( A \in \mathcal{C} \).

**Definition 5.2.** Let \( \mathcal{C} \) be an upwards directed class of C*-algebras, and let \( A \) be a C*-algebra.

1. We say that \( A \) is *hereditarily in* \( \mathcal{C} \) if every nonzero hereditary subalgebra of \( A \) is in \( \mathcal{C} \).

2. We say that \( A \) is *residually hereditarily in* \( \mathcal{C} \) if \( A/I \) is hereditarily in \( \mathcal{C} \) for every ideal \( I \subset A \) with \( I \neq A \).

To show the usefulness of this concept now, we point out that a C*-algebra hereditarily contains a nonzero projection if and only if it has Property (SP), and that a C*-algebra residually hereditarily contains an infinite projection if and only if it is purely infinite and has the ideal property (see Proposition 6.3 below). In this section, we give permanence results for the classes of algebras which are (residually) hereditarily in such a class \( \mathcal{C} \).

**Theorem 5.3.** Let \( \mathcal{C} \) be an upwards directed class of C*-algebras. Let \( \alpha : G \to \text{Aut}(A) \) be a pointwise spectrally nontrivial action of a discrete group \( G \) on a C*-algebra \( A \).

1. If \( \alpha \) is pointwise spectrally nontrivial and \( A \) is hereditarily in \( \mathcal{C} \), then \( \mathcal{C}^\ast_{\text{r}}(G, A, \alpha) \) is hereditarily in \( \mathcal{C} \).

2. If \( \alpha \) is exact and spectrally free, and \( A \) is residually hereditarily in \( \mathcal{C} \), then \( \mathcal{C}^\ast_{\text{r}}(G, A, \alpha) \) is residually hereditarily in \( \mathcal{C} \).

**Proof.** Part (1) is immediate from Corollary 3.10.

We prove (2). Let \( J \) be an ideal in \( \mathcal{C}^\ast_{\text{r}}(G, A, \alpha) \). By Proposition 3.5, there is an \( \alpha \)-invariant ideal \( I \) in \( A \) such that \( J = \mathcal{C}^\ast_{\text{r}}(G, I, \alpha) \). Exactness of the action implies that \( \mathcal{C}^\ast_{\text{r}}(G, A, \alpha)/J \cong \mathcal{C}^\ast_{\text{r}}(G, A/I, \alpha) \). By hypothesis, \( A/I \) is hereditarily in \( \mathcal{C} \) and the action of \( G \) on \( A/I \) is spectrally nontrivial, so \( \mathcal{C}^\ast_{\text{r}}(G, A, \alpha)/J \) is hereditarily in \( \mathcal{C} \) by part (1). \( \square \)

**Corollary 5.4.** Let \( \mathcal{C} \) be an upwards directed class of C*-algebras. Let \( A \) be a unital C*-algebra, and let \( \alpha \in \text{Aut}(A) \) have the Rokhlin property.

1. If \( A \) is hereditarily in \( \mathcal{C} \), then \( \mathcal{C}^\ast(Z, A, \alpha) \) is hereditarily in \( \mathcal{C} \).

2. If \( A \) is residually hereditarily in \( \mathcal{C} \), then \( \mathcal{C}^\ast(Z, A, \alpha) \) is residually hereditarily in \( \mathcal{C} \).

**Proof.** In view of Proposition 2.5, this follows from Theorem 5.3. \( \square \)

**Theorem 5.5.** Let \( \mathcal{C} \) be an upwards directed class of C*-algebras. Let \( \alpha : \mathbb{Z}_2 \to \text{Aut}(A) \) be an arbitrary action of \( \mathbb{Z}_2 \) on a C*-algebra \( A \).

1. If \( A^\alpha \) is hereditarily in \( \mathcal{C} \), then \( A \) is hereditarily in \( \mathcal{C} \).

2. If \( A^\alpha \) is residually hereditarily in \( \mathcal{C} \), then \( A \) is residually hereditarily in \( \mathcal{C} \).
Proof. Part (1) is immediate from Proposition 4.10. Part (2) is immediate from Proposition 4.13. □

Corollary 5.6. Let $\mathcal{C}$ be an upwards directed class of $C^*$-algebras. Let $\alpha : \mathbb{Z}_2 \to \text{Aut}(A)$ be an arbitrary action of $\mathbb{Z}_2$ on a $C^*$-algebra $A$.

(1) If $A$ is hereditarily in $\mathcal{C}$, then $C^*(\mathbb{Z}_2, A, \alpha)$ is hereditarily in $\mathcal{C}$.

(2) If $A$ is residually hereditarily in $\mathcal{C}$, then $C^*(\mathbb{Z}_2, A, \alpha)$ is residually hereditarily in $\mathcal{C}$.

Proof. Apply Theorem 5.5 with $C^*(\mathbb{Z}_2, A, \alpha)$ in place of $A$ and the dual action $\hat{\alpha}$ in place of $\alpha$. □

The same theory gives some permanence results not involving crossed products.

Proposition 5.7. Let $\mathcal{C}$ be an upwards directed class of $C^*$-algebras. Let $A$ be a $C^*$-algebra, and let $J \subset A$ be an ideal. Then $A$ is residually hereditarily in $\mathcal{C}$ if and only if $J$ and $A/J$ are both residually hereditarily in $\mathcal{C}$.

Proof. It is obvious that if $A$ is residually hereditarily in $\mathcal{C}$ then $J$ and $A/J$ are both residually hereditarily in $\mathcal{C}$. So assume that $J$ and $A/J$ are residually hereditarily in $\mathcal{C}$, let $I \subset A$ be an ideal, and let $B \subset A/I$ be a nonzero hereditary subalgebra. Set $C = B \cap (I + J)/I$, which is a hereditary subalgebra of $A/I$.

Suppose first that $C \neq \{0\}$. Then $C$ is a nonzero hereditary subalgebra of $(I + J)/I \cong J/(J \cap I)$. Since $J$ is residually hereditarily in $\mathcal{C}$, we get $C \in \mathcal{C}$, whence $B \in \mathcal{C}$ because $\mathcal{C}$ is upwards directed.

Now suppose that $C = \{0\}$. Let $\pi : A/I \to A/(I + J)$ be the quotient map. Then $\pi|_B$ is injective. Therefore $B$ is isomorphic to a nonzero hereditary subalgebra of $[A/J]/[(I + J)/J]$. Since $A/J$ is residually hereditarily in $\mathcal{C}$, we get $B \in \mathcal{C}$. □

Proposition 5.8. Let $\mathcal{C}$ be an upwards directed class of $C^*$-algebras. Let $(A_\lambda)_{\lambda \in \Lambda}$ be a direct system of $C^*$-algebras with maps $\varphi_{\mu, \lambda} : A_\lambda \to A_\mu$ for $\lambda, \mu \in \Lambda$ satisfying $\lambda \leq \mu$, with direct limit $A = \varprojlim A_\lambda$, and with canonical maps $\varphi_\lambda : A_\lambda \to A$ for $\lambda \in \Lambda$.

(1) Suppose that $A_\lambda$ is hereditarily in $\mathcal{C}$ for all $\lambda \in \Lambda$ and that $\varphi_{\mu, \lambda}$ is injective for all $\lambda, \mu \in \Lambda$ satisfying $\lambda \leq \mu$. Then $A$ is hereditarily in $\mathcal{C}$.

(2) Suppose that $A_\lambda$ is residually hereditarily in $\mathcal{C}$ for all $\lambda \in \Lambda$. Then $A$ is residually hereditarily in $\mathcal{C}$.

Proof. We prove (1). We may assume that the algebras $A_\lambda$ are all subalgebras of $A$, with $A_\lambda \subset A_\mu$ for $\lambda, \mu \in \Lambda$ satisfying $\lambda \leq \mu$, and that $A = \bigcup_{\lambda \in \Lambda} A_\lambda$.

Let $B \subset A$ be a nonzero hereditary subalgebra. Choose $z \in B_+$ such that $\|z\| = 1$. Choose $\lambda \in \Lambda$ and $y \in (A_\lambda)_+$ such that $\|z - y\| < \frac{1}{4}$. Set $x = (y - \frac{1}{4})_+ \in (A_\lambda)_+$. Then $x \neq 0$ and Lemma 4.3 implies that $x A x$ is isomorphic to a hereditary subalgebra of $\overline{z A z}$. Therefore $x A x$ is isomorphic to a subalgebra of $B$. The hypotheses imply that $x A x \in \mathcal{C}$. So $B \in \mathcal{C}$ because $\mathcal{C}$ is upwards directed. This proves (1).

Now we prove (2). Let $I \subset A$ be an ideal. For $\lambda \in \Lambda$, define $I_\lambda = A_\lambda/\varphi_\lambda^{-1}(I)$. Then $(A_\lambda/I_\lambda)_{\lambda \in \Lambda}$ is a direct system of $C^*$-algebras with injective maps and whose direct limit is $A/I$. The algebras $A_\lambda/I_\lambda$ are hereditarily in $\mathcal{C}$ by definition, so $A/I$ is hereditarily in $\mathcal{C}$ by (1). Since $I$ is arbitrary, this proves that $A$ is residually hereditarily in $\mathcal{C}$. □
Proposition 5.9. Let $\mathcal{C}$ be an upwards directed class of C*-algebras. Let $A$ be a C*-algebra, and let $B \subset A$ be a hereditary subalgebra.

(1) If $A$ is hereditarily in $\mathcal{C}$, then $B$ is hereditarily in $\mathcal{C}$.

(2) If $A$ is residually hereditarily in $\mathcal{C}$, then $B$ is residually hereditarily in $\mathcal{C}$.

Proof. Part (1) is immediate from the fact that hereditary subalgebras of hereditary subalgebras are hereditary.

We prove (2). So let $J \subset B$ be an ideal. Theorem 3.2.7 of [18] provides an ideal $I \subset A$ such that $I \cap B = J$. Let $\pi: A \to A/I$ be the quotient map. Then the restriction to $B$ of $\pi$ induces an injective homomorphism $\varphi: B/J \to A/I$. To finish the proof, by part (1) it suffices to show that $\varphi(B/J)$ is a hereditary subalgebra of $A/I$. Since $\varphi(B/J) = \pi(B)$, this follows from Lemma 4.11.

Proposition 5.10. Let $A$ be a C*-algebra, and let $n \in \mathbb{Z}_{>0}$. Then:

(1) $A$ is hereditarily in $\mathcal{C}$ if and only if $M_n(A)$ is hereditarily in $\mathcal{C}$.

(2) $A$ is residually hereditarily in $\mathcal{C}$ if and only if $M_n(A)$ is residually hereditarily in $\mathcal{C}$.

Proof. If $M_n(A)$ is (residually) hereditarily in $\mathcal{C}$, then Proposition 5.9 implies that $A$ is (residually) hereditarily in $\mathcal{C}$.

Now assume that $A$ is hereditarily in $\mathcal{C}$. Let $B \subset M_n(A)$ be a nonzero hereditary subalgebra. Choose $z \in B_+ \setminus \{0\}$. Let $(e_{j,k})_{j,k=1}^n$ be the standard system of matrix units in $M_n$. For $k = 1, 2, \ldots, n$, define $f_k = e_{k,k} \otimes 1 \in M_n(M(A))$. Since $\sum_{k=1}^n f_k = 1$ and $z^{1/2} \neq 0$, there is $k$ such that $f_k z^{1/2} \neq 0$. Set

$$x = f_k z f_k \in f_k M_n(A) f_k \cong A \quad \text{and} \quad y = z^{1/2} f_k z^{1/2} \in B.$$ 

Then $x M_n(A) x$ is isomorphic to a hereditary subalgebra of $A$, whence $x M_n(A) x \in \mathcal{C}$. The last part of Lemma 5.8 implies that $y M_n(A) y \cong x M_n(A) x$. Since $y M_n(A) y \subset B$, we get $B \in \mathcal{C}$.

Finally, assume that $A$ is residually hereditarily in $\mathcal{C}$. Let $J \subset M_n(A)$ be an ideal. Then there is an ideal $I \subset A$ such that $J = M_n(I)$. The hypotheses imply that $A/I$ is hereditarily in $\mathcal{C}$, so $M_n(A)/J$ is hereditarily in $\mathcal{C}$ by the previous paragraph.

Proposition 5.11. Let $\mathcal{C}$ be an upwards directed class of C*-algebras and let $A$ be a C*-algebra. Then:

(1) $A$ is hereditarily in $\mathcal{C}$ if and only if $K \otimes A$ is hereditarily in $\mathcal{C}$.

(2) $A$ is residually hereditarily in $\mathcal{C}$ if and only if $K \otimes A$ is residually hereditarily in $\mathcal{C}$.

Proof. If $K \otimes A$ is (residually) hereditarily in $\mathcal{C}$, then Proposition 5.9 implies that $A$ is (residually) hereditarily in $\mathcal{C}$. If $A$ is (residually) hereditarily in $\mathcal{C}$, then apply Proposition 5.10 and Proposition 5.8 to the relation $K \otimes A = \varinjlim_n M_n(A)$ to see that $K \otimes A$ is (residually) hereditarily in $\mathcal{C}$.

Corollary 5.12. Let $\mathcal{C}$ be an upwards directed class of C*-algebras Let $A$ and $B$ be Morita equivalent separable C*-algebras. Then:

(1) $A$ is hereditarily in $\mathcal{C}$ if and only if $B$ is hereditarily in $\mathcal{C}$.

(2) $A$ is residually hereditarily in $\mathcal{C}$ if and only if $B$ is residually hereditarily in $\mathcal{C}$.

Proof. Since Morita equivalence for separable C*-algebras implies stable isomorphism (Theorem 1.2 of [3]), the result is immediate from Proposition 5.11.
6. Hereditary infiniteness

We do not know whether the crossed product of a purely infinite C*-algebra by a discrete group is again purely infinite, even under extra conditions such as finiteness of the group or spectral freeness of the action. (We do have results under the additional assumption of the ideal property; see Theorem 6.8.) We therefore consider two formally weaker conditions, namely residual hereditary infiniteness and residual hereditary proper infiniteness, for which we can use the methods of Section 5 to obtain permanence results, in particular for crossed products. It is not known whether our properties are equivalent to pure infiniteness. For the weaker one, this is Question 4.8 of [11], which is one of the motivations for these properties. It is also not known whether they are equivalent to each other.

No freeness condition should be necessary for permanence results for crossed products. After all, the crossed product of a purely infinite C*-algebra by a trivial action is expected to be again purely infinite.

**Definition 6.1.** Let $A$ be a C*-algebra. We say that $A$ is **hereditarily infinite** if for every nonzero hereditary subalgebra $B \subset A$, there is an infinite positive element $a \in B$ in the sense of Definition 3.2 of [11], that is, there is $b \in A_+ \setminus \{0\}$ such that $a \oplus b \preceq a$. We say that $A$ is **residually hereditarily infinite** if $A/I$ is hereditarily infinite for every ideal $I$ in $A$.

There is a possible alternate definition.

**Definition 6.2.** Let $A$ be a C*-algebra. We say that $A$ is **hereditarily properly infinite** if for every nonzero hereditary subalgebra $B \subset A$, there is a properly infinite positive element $a \in B$ in the sense of Definition 3.2 of [11], that is, such that $a \neq 0$ and $a \oplus a \preceq a$. We say that $A$ is **residually hereditarily properly infinite** if $A/I$ is hereditarily properly infinite for every ideal $I$ in $A$.

The zero C*-algebra satisfies all the conditions in Definition 6.1 and Definition 6.2. This seems to be the convenient choice.

Pure infiniteness implies residual hereditary proper infiniteness, by Theorems 4.16 and 4.19 of [11]. Clearly residual hereditary proper infiniteness implies residual hereditary infiniteness. Question 4.8 of [11] asks whether a residually hereditarily infinite C*-algebra is necessarily purely infinite. As far as we know, this question is still open.

As a further motivation, we cite the following result, which is the equivalence of conditions (ii) and (iv) of Proposition 2.11 of [25] (valid, as shown there, even when $A$ is not separable).

**Proposition 6.3** (Proposition 2.11 of [25]). Let $A$ be a C*-algebra. Then $A$ is purely infinite and has the ideal property if and only if for every ideal $I \subset A$, every nonzero hereditary subalgebra of $A/I$ contains an infinite projection.

**Lemma 6.4.** Let $A$ be a C*-algebra, let $B \subset A$ be a hereditary subalgebra, and let $a \in B_+$. Suppose that there is $x \in A_+ \setminus \{0\}$ such that $a \oplus x \preceq a$ in $M_2(A)$. Then there is $y \in B_+ \setminus \{0\}$ such that $a \oplus y \preceq a$ in $M_2(B)$.

**Proof.** Choose $\varepsilon > 0$ such that $(x - \varepsilon)_+ \neq 0$. Lemma 2.5(ii) of [11] provides $v \in A$ such that $(x - \varepsilon)_+ = v^*av$. Set $d = a^{1/2}vv^*a^{1/2} \in B$. Then the last part of Lemma 3.8 provides an isomorphism $\varphi: (x - \varepsilon)_+A(x - \varepsilon)_+ \to dAd \subset B$ such that
SPECTRALLY FREE ACTIONS

\[ \varphi(z) \sim z \text{ for all } z \in (x - \varepsilon)_+ A (x - \varepsilon)_+. \]

Set \( y = \varphi((x - \varepsilon)_+) \). Then, in \( M_2(A) \), we have

\[ a \oplus y \sim a \oplus (x - \varepsilon)_+ \leq a \oplus x \preceq a. \]

Since \( y \in B \), it follows from Lemma 2.2(iii) of [1] that \( a \oplus y \preceq a \) in \( M_2(B) \).

**Corollary 6.5.** Let \( A \) be a C*-algebra. Let \( C \) be the class of all C*-algebras which contain an infinite element. Then \( C \) is upwards directed and \( A \) is (residually) hereditarily infinite if and only if \( A \) is (residually) hereditarily in \( C \).

**Proof.** It is obvious that \( C \) is upwards directed, and the second part follows from Lemma 6.4.

We can now give the permanence theorems. In both of them, we have listed the crossed product results first.

**Theorem 6.6.** The following operations preserve hereditary (proper) infiniteness:

1. Reduced crossed products by spectrally nontrivial actions of discrete groups.
2. Crossed products by Rokhlin actions of \( Z \).
3. Passage to a C*-algebra \( A \) from the fixed point algebra under an action of \( Z_2 \): if \( \alpha: Z_2 \to \text{Aut}(A) \) is an arbitrary action of \( Z_2 \) on \( A \), and \( A^\alpha \) is hereditarily (properly) infinite, then so is \( A \).
4. Crossed products by arbitrary actions of \( Z_2 \).
5. Passage to hereditary subalgebras.
6. Direct limits (over arbitrary index sets) of systems in which all the maps are injective.

**Theorem 6.7.** The following operations preserve residual hereditary (proper) infiniteness:

1. Reduced crossed products by exact spectrally free actions of discrete groups.
2. Crossed products by Rokhlin actions of \( Z \).
3. Passage to a C*-algebra \( A \) from the fixed point algebra under an action of \( Z_2 \): if \( \alpha: Z_2 \to \text{Aut}(A) \) is an arbitrary action of \( Z_2 \) on \( A \), and \( A^\alpha \) is residually hereditarily (properly) infinite, then so is \( A \).
4. Crossed products by arbitrary actions of \( Z_2 \).
5. Passage to hereditary subalgebras.
6. Direct limits (over arbitrary index sets). (The maps of the system need not be injective.)
7. Two out of three in short exact sequences: if \( A \) is a C*-algebra, and \( J \subset A \) is an ideal, then \( A \) is residually hereditarily (properly) infinite if and only if \( J \) and \( A/J \) are both residually hereditarily (properly) infinite.
8. Stable isomorphism, equivalently, \( A \) is residually hereditarily (properly) infinite if and only if \( K \otimes A \) is residually hereditarily (properly) infinite.
9. Morita equivalence for separable C*-algebras.

**Proofs of Theorem 6.6 and Theorem 6.7.** The statements for (residual) hereditary infiniteness follow from Corollary 6.5 and the results of Section 5.

For the statements for (residual) hereditary proper infiniteness, let \( C \) be the class of all C*-algebras which contain a properly infinite element. This class is obviously upwards directed, and a C*-algebra is (residually) hereditarily properly infinite if and only if it is (residually) hereditarily in \( C \) by definition. These statements therefore also follow from the results of Section 5.
Proposition 6.3 shows that purely infinite C*-algebras with the ideal property are also covered by our methods. We thus have (Theorem 6.8 below) the analog of Theorem 6.7. Before stating and proving it, we make some comments on the parts.

Theorem 6.8(1) has been independently proved for partial actions in Theorem 4.2 of [7]. For separable $A$, it is essentially already in the literature. First, apply Remark 2.5 of [15] to deduce that spectral freeness implies that, for every $\alpha$-invariant ideal $I \subset A$ and every $g \in G \setminus \{1\}$, the induced automorphism of $A/I$ is properly outer. Then use Remark 2.23 of [34] (relying on Theorem 1.13 of [34]; the reference to Theorem 1.10 of [34] is a misprint) to conclude that $A$ separates the ideals in $C^*_r(G, A, \alpha)$. This is enough to apply the proof of Lemma 3.1 of [32], and thus show that $C^*_r(G, A, \alpha)$ has the ideal property. Proper outerness in place of essentially free action of $G$ on $\hat{A}$ is also enough for the proof of Lemma 3.2 of [32] to be valid. Combining this with the fact that $A$ separates the ideals in $C^*_r(G, A, \alpha)$, the proof of (i) implies (iii) in Theorem 3.3 of [32] goes through, and shows that $C^*_r(G, A, \alpha)$ is purely infinite. Separability enters because the proof of Remark 2.23 of [34] relies on Lemma 7.1 of [20], which is only stated for separable C*-algebras. We do not need separability for our version, namely Lemma 3.2. Otherwise, our proof is fairly close.

On the other hand, the results for actions of $\mathbb{Z}_2$, Theorem 6.8(3) and Theorem 6.8(4), seem to be completely new.

There are situations (such as for minimal actions on infinite compact Hausdorff spaces) in which one gets the ideal property for a crossed product even without assuming it for the original algebra. No such result can be expected here, as can be seen by considering the tensor product of a pointwise outer action on a purely infinite simple C*-algebra with the trivial action on $C([0, 1])$.

The permanence results in Theorem 6.8 which don’t involve group actions seem not to have been previously published (except that part of Theorem 6.8(7) is Corollary 4.4(ii) of [23], but presumably could easily have been proved earlier. With the additional assumptions of separability and nuclearity, many of them are in Proposition 3.7(2) of [23]; see Definition 3.6 and Proposition 3.7(1) of [23]. Theorem 6.8(8) isn’t valid for the ideal property by itself, as one sees by using the algebra in Example 8.3. Also, extensions need not preserve the ideal property; see Theorem 5.1 of [21].

Theorem 6.8. The following operations preserve the class of purely infinite C*-algebras with the ideal property:

1. Reduced crossed products by exact spectrally free actions of discrete groups.
2. Crossed products by Rokhlin actions of $\mathbb{Z}$.
3. Passage to a C*-algebra $A$ from the fixed point algebra under an action of $\mathbb{Z}_2$: if $\alpha: \mathbb{Z}_2 \to \text{Aut}(A)$ is an arbitrary action of $\mathbb{Z}_2$ on $A$, and $A^\alpha$ is purely infinite and has the ideal property, then the same is true of $A$.
4. Crossed products by arbitrary actions of $\mathbb{Z}_2$.
5. Passage to hereditary subalgebras.
6. Direct limits (over arbitrary index sets). (The maps of the system need not be injective.)
7. Two out of three in short exact sequences: if $A$ is a C*-algebra, and $J \subset A$ is an ideal, then $A$ is purely infinite and has the ideal property if and only if $J$ and $A/J$ are both purely infinite and both have the ideal property.
(8) Stable isomorphism, equivalently, $A$ is purely infinite and has the ideal property if and only if $K \otimes A$ is purely infinite and has the ideal property.

(9) Morita equivalence for separable C*-algebras.

Proof. Let $\mathcal{C}$ be the class of C*-algebras $A$ which contain an infinite projection. Then $\mathcal{C}$ is clearly upwards directed. The result therefore follows from the results of Section 5. □

Recall (Remark 2.5(vi) of [4]) that a C*-algebra $A$ is said to have topological dimension zero if the topology of $\text{Prim}(A)$ has a base consisting of compact open sets. (We do not require that these sets be closed.)

Corollary 6.9. Let $\alpha: G \to \text{Aut}(A)$ be an exact and spectrally free action of a discrete group $G$ on a separable C*-algebra $A$. If $A$ has topological dimension zero, then the same is true of $C^*_r(G,A,\alpha)$.

Proof. It follows from Proposition 4.5 of [11] that $O_2 \otimes A$ is purely infinite. Now $\text{Prim}(O_2 \otimes A) \cong \text{Prim}(A)$, so $O_2 \otimes A$ has topological dimension zero. Using (i) implies (ii) in Theorem 2.11 of [25], we deduce that $O_2 \otimes A$ has the ideal property. Let $\beta: G \to \text{Aut}(O_2 \otimes A)$ be the action $\beta_g = \text{id}_{O_2} \otimes \alpha_g$ for $g \in G$. Proposition 1.18(2) implies that $\beta$ is spectrally free. Theorem 6.8(1) now implies that $O_2 \otimes C^*_r(G,A,\alpha)$ is purely infinite and has the ideal property.

Let $\beta: G \to \text{Aut}(O_2 \otimes A)$ be the action $\beta_g = \text{id}_{O_2} \otimes \alpha_g$ for $g \in G$. Proposition 1.18(2) implies that $\beta$ is spectrally free. Theorem 6.8(1) now implies that $O_2 \otimes C^*_r(G,A,\alpha)$ is purely infinite and has the ideal property.

Using (i) implies (ii) in Theorem 2.11 of [25], we deduce that $O_2 \otimes C^*_r(G,A,\alpha)$ has topological dimension zero. Since $\text{Prim}(O_2 \otimes C^*_r(G,A,\alpha)) \cong \text{Prim}(C^*_r(G,A,\alpha))$, it follows that $C^*_r(G,A,\alpha)$ has topological dimension zero. □

In Corollary 6.9, separability should not be necessary.

7. Property (SP) and residual Property (SP)

Recall that a C*-algebra has Property (SP) if every nonzero hereditary subalgebra contains a nonzero projection. This property is commonly used for simple C*-algebras. For nonsimple C*-algebras, it seems more appropriate to ask for the following strengthened version.

Definition 7.1. Let $A$ be a C*-algebra. We say that $A$ has residual (SP) if $A/I$ has Property (SP) for every ideal $I \subset A$.

This property has already implicitly appeared as a hypothesis in the literature; see Proposition 6.3.

We prove permanence results for Property (SP) and residual (SP). The results for crossed products by infinite groups definitely require some sort of freeness condition, as one can see by considering the trivial action of $\mathbb{Z}$ on $\mathbb{C}$.

Property (SP) does not imply residual (SP).

Example 7.2. Let $H = l^2(\mathbb{Z})$ and let $D \subset L(H)$ be the C*-algebra generated by $K = K(H)$ and the bilateral shift. We claim that $D$ has Property (SP) but not residual (SP).

That $D$ does not have residual (SP) follows from the fact that $D/K \cong C(S^1)$.

We verify that $D$ has Property (SP). Let $B \subset D$ be a nonzero hereditary subalgebra. We first claim that $B \cap K \neq \{0\}$. Choose $b \in B_+ \setminus \{0\}$. Choose $\xi \in H$ such that $b \xi \neq 0$. Let $p \in K$ be the projection on $\mathbb{C} \xi$. Then $bp \xi \neq 0$. Therefore $bpb = (bp)(bp)^*$ is a nonzero element of $B \cap K$. This proves the claim.
Now $B \cap K$ is a nonzero hereditary subalgebra of $K$, so contains a nonzero projection, as desired.

It follows from Theorem 4.2 of [10] that if $\alpha : G \to \text{Aut}(A)$ is a pointwise outer action of a discrete group $G$ on a simple C*-algebra $A$ with Property (SP), then $C^*_r(G, A, \alpha)$ has Property (SP). (The actual statement has slightly weaker hypotheses: one only requires that $\alpha_g$ be outer for $g$ outside some finite normal subgroup of $G$.) Theorem 7.3(1) and Theorem 7.4(1) below give generalizations to nonsimple C*-algebras.

**Theorem 7.3.** The following operations preserve Property (SP):

1. Reduced crossed products by spectrally nontrivial actions of discrete groups.
2. Crossed products by Rokhlin actions of $\mathbb{Z}$.
3. Passage to a C*-algebra $A$ from the fixed point algebra under an action of $\mathbb{Z}_2$: if $\alpha : \mathbb{Z}_2 \to \text{Aut}(A)$ is an arbitrary action of $\mathbb{Z}_2$ on $A$, and $A^\alpha$ has Property (SP), then so does $A$.
4. Crossed products by arbitrary actions of $\mathbb{Z}_2$.
5. Passage to hereditary subalgebras.
6. Direct limits (over arbitrary index sets) of systems in which all the maps are injective.

**Theorem 7.4.** The following operations preserve residual (SP):

1. Reduced crossed products by exact spectrally free actions of discrete groups.
2. Crossed products by Rokhlin actions of $\mathbb{Z}$.
3. Passage to a C*-algebra $A$ from the fixed point algebra under an action of $\mathbb{Z}_2$: if $\alpha : \mathbb{Z}_2 \to \text{Aut}(A)$ is an arbitrary action of $\mathbb{Z}_2$ on $A$, and $A^\alpha$ has residual (SP), then so does $A$.
4. Crossed products by arbitrary actions of $\mathbb{Z}_2$.
5. Passage to hereditary subalgebras.
6. Direct limits (over arbitrary index sets). (The maps of the system need not be injective.)
7. Two out of three in short exact sequences: if $A$ is a C*-algebra, and $J \subset A$ is an ideal, then $A$ has residual (SP) if and only if $J$ and $A/J$ both have residual (SP).
8. Stable isomorphism, equivalently, $A$ has residual (SP) if and only if $K \otimes A$ has residual (SP).
9. Morita equivalence for separable C*-algebras.

**Proofs of Theorem 7.3 and Theorem 7.4** Let $C$ be the class of all C*-algebras which contain a nonzero projection. This class is obviously upwards directed, and by definition a C*-algebra has (residual) (SP) if and only if it is (residually) hereditarily in $C$. All parts therefore follow from the results of Section 5.

Some condition on the action is needed to be able to prove that a crossed product has Property (SP), as one can see by considering the trivial action of $\mathbb{Z}$ on $C$.

The following example shows that pointwise spectral nontriviality (without requiring anything about the action on quotients) is not a strong enough condition to prove preservation of residual (SP).

**Example 7.5.** Let $G = \mathbb{Z}$, let $X = \mathbb{Z} \cup \{\infty\}$ be the one point compactification of $\mathbb{Z}$, and let $\mathbb{Z}$ act on $X$ by translation on $\mathbb{Z}$ and with $\infty$ fixed. Set $A = C(X)$,
and let $\alpha: \mathbb{Z} \to \text{Aut}(A)$ be the corresponding action. The algebra $A$ clearly has residual (SP). The crossed product $C^*(\mathbb{Z}, A, \alpha)$ is isomorphic to the algebra $D$ of Example 7.2 and thus does not have residual (SP).

For $n \in \mathbb{Z} \setminus \{0\}$, the set of fixed points for the action of $n$ on $X$ is $\{\infty\}$, which has empty interior. So $\alpha$ is pointwise spectrally nontrivial by Lemma 1.8.

Of course, it is easy to see that $\alpha$ is not spectrally free, by considering the quotient by the invariant ideal $C_0(\mathbb{Z})$.

8. The weak ideal property

We do not know whether the crossed product of a C*-algebra with the ideal property by an arbitrary action of a finite group again has the ideal property, and this may be false. However, a related but weaker property, which we call the weak ideal property, can be treated by the methods of this paper. In particular, the weak ideal property is preserved by crossed products by exact spectrally free actions. It is also preserved by arbitrary actions of finite abelian groups. (We don’t yet know about crossed products by finite nonabelian groups.) This weaker property also admits better permanence results of other kinds. Here, like for (SP), the trivial action of $\mathbb{Z}$ on $C$ shows that permanence results for crossed products by infinite groups require a freeness condition.

**Definition 8.1.** Let $A$ be a C*-algebra. We say that $A$ has the weak ideal property if every nonzero subquotient of $K \otimes A$ contains a nonzero projection.

An intermediate requirement is also possible: one can require that every nonzero subquotient of $A$ contain a nonzero projection. We also consider this property, although we do not give it a name.

**Proposition 8.2.** Let $A$ be a C*-algebra with the ideal property. Then $A$ has the weak ideal property; in fact, every nonzero subquotient of $A$ contains a nonzero projection.

**Proof.** Let $M \subset L \subset A$ be ideals with $M \neq L$. Since the projections in $L$ generate $L$ as an ideal in $A$, there is a projection $p \in L \setminus M$. Then $p + M$ is a nonzero projection in $L/M$. □

**Example 8.3.** Let $D$ be any infinite dimensional simple separable unital C*-algebra with no projections other than 0 and 1. (Example: The Jiang-Su algebra.) Let $A \subset D$ be any proper nonzero hereditary subalgebra. Then $A$ has the weak ideal property, because $K \otimes A \cong K \otimes D$ (by Theorem 2.8 of [2]). However, $A$ has no nonzero projections, so it is certainly not true that every nonzero subquotient of $A$ contain a nonzero projection.

**Example 8.4.** There is a separable C*-algebra $A$ with the property that that every nonzero subquotient of $A$ contain a nonzero projection, but which does not have the ideal property. Our example depends on the fact that extensions of algebras with the ideal property need not have the ideal property. Let $A$ be the C*-algebra constructed in the proof of Theorem 5.1 of [21]. There is a short exact sequence

$$0 \to I \to A \to C \to 0,$$

in which $I$ is a stabilized Bunce-Deddens algebra, and which does not split. (In particular, $A$ is not unital.) The only nonzero subquotients of $A$ are $A$, $I$, and
$A/I \cong \mathbb{C}$. All contain nonzero projections. Thus $A$ has the weak ideal property. However, by Theorem 5.1 of [21], the algebra $A$ does not have the ideal property.

The methods of Section 5 give the following permanence results for the weak ideal property. We omit the results involving actions of $\mathbb{Z}_2$, since we obtain better versions with a separate argument.

**Theorem 8.5.** The following operations preserve the weak ideal property:

1. Reduced crossed products by exact spectrally free actions of discrete groups.
2. Crossed products by Rokhlin actions of $\mathbb{Z}$.
3. Passage to hereditary subalgebras.
4. Direct limits (over arbitrary index sets). (The maps of the system need not be injective.)
5. Two out of three in short exact sequences: if $A$ is a C*-algebra, and $J \subset A$ is an ideal, then $A$ has the weak ideal property if and only if $J$ and $A/J$ both have the weak ideal property.
6. Stable isomorphism, equivalently, $A$ has the weak ideal property if and only if $K \otimes A$ has the weak ideal property.
7. Morita equivalence for separable C*-algebras.

**Proof.** Let $\mathcal{C}$ be the class of all C*-algebras $A$ such that $K \otimes A$ contains a nonzero projection. Clearly $\mathcal{C}$ is upwards directed.

Let $A$ be a C*-algebra. We claim that $A$ has the weak ideal property if and only if $A$ is residually hereditarily in $\mathcal{C}$. It suffices to show that $A$ has the property that every nonzero ideal in $K \otimes A$ contains a nonzero projection if and only if $A$ is hereditarily in $\mathcal{C}$.

Assume that $A$ is hereditarily in $\mathcal{C}$, and let $J \subset K \otimes A$ be a nonzero ideal. Then there is a nonzero ideal $I \subset A$ such that $J = K \otimes I$. Since ideals are hereditary subalgebras, it follows from the definition of $\mathcal{C}$ that $J$ contains a nonzero projection.

Now assume that every nonzero ideal in $K \otimes A$ contains a nonzero projection. Let $B \subset A$ be a nonzero hereditary subalgebra. Let $I \subset A$ be the ideal generated by $B$, and let $p \in K \otimes I$ be a nonzero projection. Then $p$ can be approximated arbitrarily well in norm by finite sums of elementary tensors of the form $k \otimes a_1ba_2$, with $k \in K$, $a_1, a_2 \in A$, and $b \in B$. Therefore there are countable subsets $S \subset A$ and $T \subset B$ such that

\[
\text{span} \left\{ k \otimes a_1ba_2 : k \in K, a_1, a_2 \in S, \text{ and } b \in T \right\}.
\]

Let $A_0 \subset A$ be the C*-subalgebra of $A$ generated by $S \cup T$, which is a separable C*-algebra. Let $B_0 \subset A_0 \cap B$ be the hereditary subalgebra of $A_0$ generated by $T$. Let $I_0 \subset A_0 \cap I$ be the ideal in $A_0$ generated by $B_0$. It follows from (8.1) that $p \in K \otimes I_0$. Since $B_0$ is full in $I_0$, we have $K \otimes B_0 \cong K \otimes I_0$ by Theorem 2.8 of [2]. So $K \otimes B_0$ contains a nonzero projection. Since $K \otimes B_0 \subset K \otimes B$, it follows that $K \otimes B$ contains a nonzero projection. This completes the proof of the claim.

Given the claim, the theorem follows immediately from the results in Section 5.

Some of these permanence results also hold for the property that every nonzero subquotient of $A$ contains a nonzero projection. It follows from Example 8.3 that hereditary subalgebras, stable isomorphism, and Morita equivalence do not preserve this condition. It is not hard to prove directly that the analogs of Theorem 8.5 hold.

\[ \Box \]
and Theorem 8.5[3] are valid. We state explicitly the analog of Theorem 8.5[1]. The analog of Theorem 8.5[2] then follows from Proposition 2.5.

**Proposition 8.6.** Let \( \alpha : G \to \text{Aut}(A) \) be an exact spectrally free action of a discrete group \( G \) on a C*-algebra \( A \). Suppose that every nonzero subquotient of \( A \) contains a nonzero projection. Then the same is true of \( C^*_r(G, A, \alpha) \).

**Proof.** In view of Proposition 8.5 this follows from Corollary 8.7. \( \square \)

We now consider actions of finite groups. We need several lemmas.

**Lemma 8.7.** Let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a C*-algebra \( A \), and let the notation related to Lemma 5.3.3 of [27] be as before Lemma 1.13. Let \( S \in S_G \), and let \( M \subseteq I_S/I_S^c \) be a nonzero ideal. (In particular, \( I_S^c \neq I_S \)) Then there is a nonzero subquotient of \( L \) of \( A^\alpha \) and an injective homomorphism from \( L \) to \( M \).

**Proof.** Since \( I_S^c \) is \( \alpha \)-invariant (Lemma 5.3.3(1) of [27]) and \( I_S^c \subseteq I_S \) (Lemma 5.3.3(2) of [27]), we can define \( B = A/I_S^c \) and \( J = (I + I_S)/I_S^c \) and further let \( \beta : G \to \text{Aut}(B) \) be the action induced by \( \alpha \). Then \( J_S^c = \{0\} \) and \( J_S = (I_S + I_S^c)/I_S^c = I_S/I_S^c \), so

\[
(J_S \cap J)/(J_S^c \cap J) = J_S \cap J = [(I_S + I_S^c) \cap (I + I_S^c)]/I_S^c
= [(I_S \cap I) + I_S^c]/I_S^c \cong (I_S \cap I)/(I_S^c \cap I).
\]

Moreover, \( B^\alpha \cong A^\alpha/(I_{11} \cap A^\alpha) \) (Lemma 1.6 of [29]). So it suffices to prove that there is an ideal in \( B^\beta \) which is isomorphic to a subalgebra of \( M \).

By Lemma 5.3.3(5) of [27], there is a subgroup \( H \subseteq G \), an \( H \)-invariant ideal \( N \subseteq J_S \), a system \( R \) of left coset representatives for \( H \) in \( G \), and a subset \( R_0 \subseteq R \), such that we have internal direct sum decompositions

\[
J_S = \bigoplus_{g \in R_0} \beta_g(N) \quad \text{and} \quad J \cap J_S = \bigoplus_{g \in R_0} \beta_g(N).
\]

Since \( J_S \neq \{0\} \), we have \( \bigcap_{h \in S} \beta_h(J) \neq \{0\} \). Therefore \( R_0 \neq \emptyset \) and \( N \neq \{0\} \).

We claim that \( (J_S)^\beta \cong \bigoplus N^{\beta|H} \). Define an injective homomorphism \( \psi : N \to J_S \) by \( \psi(x) = (\beta_k(x))_{k \in R} \). Temporarily fix \( g \in G \). There is a bijection \( \sigma : R \to R \) and a function \( \eta : R \to H \) such that \( g\kappa = \sigma(k)\eta(k) \) for all \( k \in R \). For \( k \in R \), we then get

\[
\beta_g((\beta_k(x))_{k \in R})_{\sigma(i)} = \beta_{\sigma(i)\eta(i)}(x).
\]

(8.2)

\[
\beta_g((x_k)_{k \in R}) = (\beta_{\sigma(k)(\sigma^{-1}(k))}(x))_{k \in R}.
\]

It is now clear that if \( x \in N^{\beta|H} \), then \( \beta_g(\psi(x)) = \psi(x) \). Now suppose that \( x \in N \) and \( \beta_g(\psi(x)) = \psi(x) \) for all \( g \in G \). Let \( h \in H \); we show \( \beta_h(x) = x \). There is a unique element \( \kappa_0 \in R \cap H \). Set \( g = h\kappa_0^{-1} \in H \). Let \( \sigma \) and \( \eta \) be as above. Then \( \sigma(\kappa_0) = \kappa_0 \), so \( \sigma^{-1}(\kappa_0) = \kappa_0 \), and \( \eta(\kappa_0) = \kappa_0^{-1}h \). Taking \( k = \kappa_0 \) in (8.2) gives \( \beta_h(x) = x \). So \( \psi(x) \in (J_S)^\beta \) if and only if \( x \in N^{\beta|H} \), and the claim follows.

Since \( M \) is an ideal in \( \bigoplus_{g \in R} \beta_g(N) \), there are ideals \( M_g \subseteq N \) for \( g \in R \) such that \( M = \bigoplus_{g \in R} \beta_g(M_g) \). Since \( M \neq \{0\} \), there is \( g \in R \) such that \( M_g \neq \{0\} \).

Then \( L = M_g^{\beta|H} \) is a nonzero ideal in \( N \), hence in \( B^\beta \), and \( \beta_g|_L \) is an injective homomorphism from \( L \) to \( M \). This completes the proof. \( \square \)
**Theorem 8.9.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a C*-algebra $A$, and let $C$ be a nonzero subquotient of $A$. Then there exists a nonzero subquotient of $A^a$ which is isomorphic to a subalgebra of $C$.

**Proof.** Choose ideals $I, J \subset A$ such that $I \subset J$ and $J/I = C$. We use Lemma 5.3.3 of [27], and we follow the notation before Lemma 8.7.

First assume $J \not\subset I_{(1)}$. Then $J/(J \cap I_{(1)})$ is a nonzero subquotient of $A$. Since $I \subset I_{(1)}$ (Lemma 5.3.3(3) of [27]), it suffices to prove that there is a nonzero subquotient of $A^a$ which is isomorphic to a subquotient of $J/(J \cap I_{(1)})$. Since $I_{(1)}$ is $\alpha$-invariant (Lemma 5.3.3(1) of [27]), we simplify the notation by defining $B = A/I_{(1)}$ and $M = (J + I_{(1)})/I_{(1)}$, and letting $\beta : G \to \text{Aut}(B)$ be the action induced by $\alpha$. Then $M \cong J/(J \cap I_{(1)})$, and $B^\beta \cong A^a/(I_{(1)} \cap A^a)$ by Lemma 1.6 of [27]. Since $M \subset M_{(1)}$ (Lemma 5.3.3(5) of [27]) and $M \neq \{0\}$, Lemma 1.13 provides $S \in \mathcal{S}_G$ such that $(MS \cap \{0\}) + M_S \not\subset (MS \cap M) + M_S$. Then $M_S \not\subset (MS \cap M) + M_S$, so $M_S \cap M \not\subset MS \cap M$. Lemma 8.7 provides a nonzero subquotient of $B^\beta$ which is isomorphic to a subalgebra of the subquotient $(MS \cap M)/(MS \cap M)$ of $M$. The conclusion follows in this case.

Now assume $J \subset I_{(1)}$. Since $I \not\subset J$, Lemma 1.13 provides $S \in \mathcal{S}_G$ such that $(IS \cap I) + I_S \not\subset (IS \cap J) + I_S$. By Lemma 5.3.3(5) of [27], there is a subgroup $H \subset G$, an $H$-invariant ideal $N \subset J_S$, a system $R$ of left coset representatives for $H$ in $G$, and a subset $R_0 \subset R$, such that we have internal direct sum decompositions

$$I_S/I_S = \bigoplus_{g \in R} \beta_g(N) \quad \text{and} \quad [(I \cap I_S) + I_S]/I_S = \bigoplus_{g \in R_0} \beta_g(N).$$

Since $[(I_S \cap J) + I_S]/I_S$ is an ideal in $I_S/I_S$, which strictly contains $[(I \cap I_S) + I_S]/I_S$, there is a nonzero ideal $M \subset \bigoplus_{g \in R \setminus R_0} \beta_g(N) \subset I_S/I_S$ such that

$$[(I_S \cap I) + I_S]/I_S \oplus M = [(I_S \cap J) + I_S]/I_S.$$

Then Lemma 8.7 provides a nonzero subquotient of $B^\beta$ which is isomorphic to a subalgebra of $M$. Moreover, $M$ is isomorphic to an ideal in

$$(I_S \cap J)/[(I_S \cap I) + I_S]/I_S \cong [(I_S \cap J) + I_S]/[(I_S \cap I) + I_S] \\ \cong (I_S \cap J)/[(I_S \cap I) + I_S] \cap I_S \cap J$$

which is a subquotient of $J/I$. This completes the proof. \qed

**Theorem 8.9.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a C*-algebra $A$. If $A^a$ has the weak ideal property, or if every nonzero subquotient of $A^a$ contains a nonzero projection, then the same is true of $A$.

**Proof.** For the condition that every nonzero subquotient of $A$ contains a nonzero projection, the result is immediate from Lemma 8.8. The result for the weak ideal property follows by tensoring with $K$. \qed

**Corollary 8.10.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite abelian group $G$ on a C*-algebra $A$. If $A$ has the weak ideal property, or if every nonzero subquotient of $A$ contains a nonzero projection, then the same is true of $C^*(G, A, \alpha)$. \qed

**Question 8.11.** Does Corollary 8.10 hold for finite nonabelian groups?
References

[1] C. A. Akemann, J. Anderson, and G. K. Pedersen, Excising states of C*-algebras, Canad. J. Math. 38 (1986), 1239–1260.
[2] L. G. Brown, Stable isomorphism of hereditary subalgebras of C*-algebras, Pacific J. Math. 71 (1977), 335–348.
[3] L. G. Brown, P. Green, and M. A. Rieffel, Stable isomorphism and strong Morita equivalence of C*-algebras, Pacific J. Math. 71 (1977), 349–363.
[4] L. G. Brown and G. K. Pedersen, Limits and C*-algebras of low rank or dimension, J. Operator Theory 61 (2009), 381–417.
[5] J. Cuntz, The structure of multiplication and addition in simple C*-algebras, Math. Scand. 40 (1977), 215–233.
[6] K. R. Davidson, C*-Algebras by Example, Fields Institute Monographs no. 6, Amer. Math. Soc., Providence RI, 1996.
[7] T. Giordano and A. Sierakowski, Purely infinite partial crossed products, preprint (arXiv:1303.4483v2 [math.OA]).
[8] I. Hirshberg and J. Orovitz, Tracially Z-absorbing C*-algebras, preprint (arXiv:1208.2444v1 [math.OA]).
[9] M. Izumi, The Rohlin property for automorphisms of C*-algebras, pages 191–206 in: Mathematical Physics in Mathematics and Physics (Siena, 2000), Fields Inst. Commun. vol. 30, Amer. Math. Soc., Providence RI, 2001.
[10] J. A. Jeong and H. Osaka, Extremally rich C*-crossed products and the cancellation property, J. Austral. Math. Soc. (Series A) 64 (1998), 285–301.
[11] E. Kirchberg and M. Rørdam, Non-simple purely infinite C*-algebras, Amer. J. Math. 122 (2000), 637–666.
[12] E. Kirchberg and M. Rørdam, Infinite non-simple C*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$, Adv. Math. 167 (2002), 195–264.
[13] A. Kishimoto, Simple crossed products of C*-algebras by locally compact abelian groups, Yokohama Math. J. 28 (1980), 69–85.
[14] A. Kishimoto, Outer automorphisms and reduced crossed products of simple C*-algebras, Commun. Math. Phys. 81 (1981), 429–435.
[15] A. Kishimoto, Freely acting automorphisms of C*-algebras, Yokohama Math. J. 30 (1982), 39–47.
[16] A. Kishimoto and A. Kumjian, Crossed products of Cuntz algebras by quasi-free automorphisms, pages 173–192 in: Operator Algebras and their Applications (Waterloo, ON, 1994/1995), Fields Inst. Commun. vol. 13, Amer. Math. Soc., Providence RI, 1997.
[17] H. Lin, Embedding crossed products into a unital simple AF-algebra, preprint (arXiv:math/0604047v1 [math.OA]).
[18] G. J. Murphy, C*-Algebras and Operator Theory, Academic Press, Boston, San Diego, New York, London, Sydney, Tokyo, Toronto, 1990.
[19] D. Olesen and G. K. Pedersen, Partially inner C*-dynamical systems, J. Funct. Anal. 66 (1986), 262–281.
[20] D. Olesen and G. K. Pedersen, Applications of the Connes spectrum to C*-dynamical systems, III, J. Funct. Anal. 45 (1982), 357–390.
[21] C. Pasnicu, On the AH algebras with the ideal property, J. Operator Theory 43 (2000), 389–407.
[22] C. Pasnicu, The projection property, Glasgow Math. J. 44 (2002), 293–300.
[23] C. Pasnicu, D-stable C*-algebras, the ideal property and real rank zero, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 52 (2009), 177–192.
[24] C. Pasnicu and N. C. Phillips, Permanence properties for crossed products and fixed point algebras of finite groups, Trans. Amer. Math. Soc., to appear.
[25] C. Pasnicu and M. Rørdam, Purely infinite C*-algebras of real rank zero, J. reine angew. Math. 613 (2007), 51–73.
[26] G. K. Pedersen, C*-Algebras and their Automorphism Groups, Academic Press, London, New York, San Francisco, 1979.
[27] N. C. Phillips, Equivariant K-Theory and Freeness of Group Actions on C*-Algebras, Springer-Verlag Lecture Notes in Math. no. 1274, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987.
[28] N. C. Phillips, *Freeness of actions of finite groups on C*-algebras*, pages 217–257 in: *Operator structures and dynamical systems*, M. de Jeu, S. Silvestrov, C. Skau, and J. Tomiyama (eds.), Contemporary Mathematics vol. 503, Amer. Math. Soc., Providence RI, 2009.

[29] N. C. Phillips, *Equivariant semiprojectivity*, preprint (arXiv:1112.4584v1 [math.OA]).

[30] N. C. Phillips, A. P. W. Sørensen, and H. Thiel, *Semiprojectivity with and without a group action*, in preparation (version of 15 Oct. 2012).

[31] M. Rørdam, *On the structure of simple C*-algebras tensored with a UHF-algebra. II*, J. Funct. Anal. 107(1992), 255–269.

[32] M. Rørdam and A. Sierakowski, *Purely infinite C*-algebras arising from crossed products*, Ergod. Th. Dynam. Sys. 32(2012), 273–293.

[33] J. Rosenberg, *Appendix to O. Bratteli’s paper on “Crossed products of UHF algebras”*, Duke Math. J. 46(1979), 25–26.

[34] A. Sierakowski, *The ideal structure of reduced crossed products*, Münster J. Math. 3(2010), 237–261.

**Department of Mathematics, The University of Texas at San Antonio, San Antonio TX 78249, USA.**

*E-mail address: Cornel.Pasnicu@utsa.edu*

**Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA.**

*E-mail address: ncp@darkwing.uoregon.edu*