Λ-Ultrametric spaces and lattices of equivalence relations

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Dedicated to Ralph Freese, Bill Lampe, and J.B. Nation.

Abstract. For a finite lattice Λ, Λ-ultrametric spaces have, among other reasons, appeared as a means of constructing structures with lattices of equivalence relations embedding Λ. This makes use of an isomorphism of categories between Λ-ultrametric spaces and structures equipped with certain families of equivalence relations. We extend this isomorphism to the case of infinite lattices. We also pose questions about representing a given finite lattice as the lattice of ∅-definable equivalence relations of structures with model-theoretic symmetry properties.

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1. Introduction

Our goal is to present an isomorphism between two categories, which arose in simplified form in the course of constructing certain homogeneous (Definition 4.1) structures in [3].

The first category has objects consisting of a set equipped with a family of equivalence relations forming a lattice Λ, or substructures thereof. However, we initially define its objects as structures consisting of of a set equipped with a family of equivalence relations, closed under taking meets in the lattice of all equivalence relations on the set, and labeled by the elements of a fixed lattice Λ in such a way that the map from Λ to the lattice of equivalence relations is a homomorphism.

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The objects of the second category are lattice-valued metric spaces (also known as generalized ultrametric spaces) with metric valued in a certain complete lattice \( \Phi(\Lambda) \) containing \( \Lambda \).

In [3], the structures of interest were the families of equivalence relations, but the metric spaces were technically easier to work with. As \( \Lambda \) was assumed to be finite, almost everything trivializes; in particular \( \Phi(\Lambda) = \Lambda \). The correspondence is then roughly as follows. Starting with a structure equipped with a suitable family of equivalence relations, one defines \( d(x,y) \) to be the finest equivalence relation that holds between \( x \) and \( y \). From this distance, one can easily recover all the equivalence relation information.

The general correspondence is largely a matter of taking care in setting up the correct categories and in the choice of the map \( \Phi \)—we want the lattice of filters on \( \Lambda \), rather than the usual Dedekind–MacNeille completion.

Generalized ultrametric spaces were first investigated by Priess-Crampe and Ribenboim [6], who seem primarily interested in applications to valued fields. In [5,6], the correspondence with certain structures of equivalence relations is developed to some extent, but not to the point where passing to \( \Phi(\Lambda) \) becomes necessary. The correspondence also appears in restricted contexts in [1, §6.2] and [7], in order to produce embeddings of a given lattice into the lattice of all equivalence relations on a given set.

The paper is divided into three further sections. First, we present some preliminary material on the filter lattice \( \Phi(\Lambda) \), along with an example motivating the passage to it. As this lattice has appeared several times before, these results are likely standard. The next section presents the isomorphism of categories that is our main result. Finally, we close by mentioning the motivating model-theoretic context and some related open questions.

2. The filter lattice

**Definition 2.1.** For \( \Lambda \) a lattice, let \( \Phi(\Lambda) \) denote the filter lattice, defined as follows.

1. Elements are filters in \( \Lambda \) (i.e., nonempty subsets, closed under meet and closed upwards).
2. The partial order on \( \Phi(\Lambda) \) is defined as reverse inclusion.

For any \( L \subseteq \Lambda \), we write \( \langle L \rangle \) for the filter generated by \( L \), which is the upward closure of the set of elements of the form \( \bigwedge L_0 \) for \( L_0 \subseteq L \) finite.

For \( \lambda \in \Lambda \), let \( \hat{\lambda} \) denote the principal filter \( \langle \lambda \rangle = \{ x \in \Lambda \mid x \geq \lambda \} \).

The following example shows how the correspondence may fail without the passage to \( \Phi(\Lambda) \).

**Example 2.2.** Let \( \Lambda \) be the interval \([0,1] \) in \( \mathbb{R} \) with the usual ordering. Suppose we have a structure equipped with a suitable family of equivalence relations labeled by elements of \( \Lambda \). If \( xE_\lambda y \) for \( \lambda \in (0,1] \), then there is no finest equivalence relation that holds between \( x \) and \( y \), so we cannot assign a distance as in the case where \( \Lambda \) is finite.
For the given $\Lambda$, $\Phi(\Lambda)$ is isomorphic to
\[
([0, 1] \times \{0, 1\}) \setminus \{(1, 1)\}
\]
with the lexicographic ordering, with $\hat{\lambda}$ corresponding to $(\lambda, 0)$ and with $\hat{\lambda} \setminus \{\lambda\}$ corresponding to $(\lambda, 1)$. As we do not allow empty filters, we leave off $(1, 1)$.

For the $x, y$ given in the first paragraph, we can now assign $d(x, y) = (0, 1) \in \Phi(\Lambda)$.

**Lemma 2.3.** Let $\Lambda$ be a bounded lattice. Then the following hold.

1. $\Phi(\Lambda)$ is a complete lattice with join given by intersection, and meet by taking the filter generated by the union of a given set of filters.
2. The map $\phi: \Lambda \to \Phi(\Lambda)$ defined by $\lambda \mapsto \hat{\lambda}$
   is an isomorphic embedding with respect to the lattice operations.
3. If $\lambda \in \Lambda$, $L \subseteq \Lambda$, and $\hat{\lambda} = \bigwedge \{\ell \mid \ell \in L\}$ in $\Phi(\Lambda)$, then $\lambda$ is the meet of some finite subset of $L$, in $\Lambda$.

**Proof.** (1) The order is reverse inclusion, so the rules for join and meet are clear. For completeness, the only point to check is that the intersection of arbitrarily many filters is always non-empty. As $\Lambda$ has a maximum element, this holds.

(2) It suffices to check that the map $\phi$ preserves meet and join, or in other words
\[
\hat{\lambda}_1 \lor \hat{\lambda}_2 = \lambda_1 \lor \lambda_2 \\
\hat{\lambda}_1 \land \hat{\lambda}_2 = \lambda_1 \land \lambda_2
\]

More explicitly, this reads as follows.
\[
x \geq \lambda_1, \lambda_2 \iff x \geq \lambda_1 \lor \lambda_2 \\
x \in \langle \hat{\lambda}_1, \hat{\lambda}_2 \rangle \iff x \geq \lambda_1 \land \lambda_2
\]

Both points are clear.

(3) If $\hat{\lambda} = \bigwedge \{\ell \mid \ell \in L\}$ then $\lambda \in \langle L \rangle$ and therefore $\lambda \in \langle L_0 \rangle$ for some finite set $L_0$. The claim follows.

The third point in the lemma above seems to capture the key abstract property of the construction. However, it is not directly used in what follows, as we prefer to work with the explicit definition of $\Phi(\Lambda)$.

### 3. The correspondence

**Definition 3.1.** Suppose $\Lambda$ is a bounded lattice with minimal element $\emptyset$ and maximal element $\mathbb{1}$.

1. Let $\mathcal{EQ}_\Lambda$ denote the category whose objects are the models in the language
\[
\{E_\lambda \mid \lambda \in \Lambda\}
\]
for which the following hold.
• All $E_{\lambda}$ are equivalence relations.
• The map from $\lambda \rightarrow E_{\lambda}$ preserves meets; in particular, the set of relations \( \{E_{\lambda} \mid \lambda \in \Lambda\} \) is closed under intersection.
• $E_{0}$ is equality, and $E_{1}$ is the trivial relation.

We take embeddings as morphisms.

(2) A $\Lambda$-ultrametric space is a structure of the form $(X, d)$ for which
• $d: X^2 \rightarrow \Lambda$ is a symmetric function.
• $d(x, y) = 0$ iff $x = y$.
• $d(x, y) \leq d(x, z) \lor d(y, z)$ for all $x, y, z$ (Triangle Inequality).

(3) Let $\mathcal{M}_{\Lambda}$ be the category of $\Lambda$-ultrametric spaces, with isometric embeddings as morphisms.

In the model-theoretic context, we are primarily concerned with $\emptyset$-definable equivalence relations. While these form a lattice in an $\omega$-categorical structure (Definition 4.2), in general they only need form a lower semi-lattice. It would thus seem natural to define the category $\mathcal{E} Q_{\Lambda}$ for any lower semi-lattice $\Lambda$ and to define the category $\mathcal{M}_{\Lambda}$ for any upper semi-lattice $\Lambda$. But it is not clear what could replace the correspondence that we aim at below.

**Example 3.2.** If $\Lambda$ is a chain, then $\Lambda$-ultrametric spaces are ultrametric spaces in the usual sense.

**Example 3.3.** Let $\Lambda$ be a lattice and let $A$ be a set with at least two elements. Define $E_{\lambda}$ to be equality for $\lambda \in \Lambda \setminus \{1\}$, and let $E_{1}$ be trivial. Then $(A, \{E_{\lambda} \mid \lambda \in \Lambda\})$ belongs to $\mathcal{E} Q_{\Lambda}$. The map sending $\lambda$ to $E_{\lambda}$ is a semi-lattice homomorphism, but will not be a lattice homomorphism unless $1$ is join-irreducible.

This example arises naturally as a substructure of any structure in $\mathcal{E} Q_{\Lambda}$ having a pair of elements that are not related by any of the relations $E_{\lambda}$ for $\lambda < 1$. For example, there is a natural representation of the Boolean algebra with $n$ atoms by equivalence relations on the set $\{0, 1\}^n$, in which the co-atoms correspond to the relations $E_i(x, y) \iff x_i = y_i$. Intersections of co-atoms in the Boolean algebra correspond to intersections of these relations, and so specify agreement in multiple coordinates. We may then consider the substructure induced on the constant sequences, in which two elements will be related by a relation other than $E_{1}$ only if they agree in some coordinate and thus must be equal, i.e. $E_{0}$-related.

Now the main point is to consider the relationship between the categories $\mathcal{E} Q_{\Lambda}$ and $\mathcal{M}_{\Phi(\Lambda)}$. The former is the category that interests us, while the latter is easier to work with in the context of the Fraïssé theory of amalgamation classes.

**Theorem 3.4.** Let $\Lambda$ be a bounded lattice. Then the categories $\mathcal{E} Q_{\Lambda}$ and $\mathcal{M}_{\Phi(\Lambda)}$ are canonically isomorphic.

**Proof.** Given $\mathcal{A} = (A, \{E_{\lambda} \mid \lambda \in \Lambda\})$ in $\mathcal{E} Q_{\Lambda}$, let $m(\mathcal{A}) = (A, d)$, where
\[
d: A \times A \rightarrow \Phi(\Lambda)
\]
is defined by
\[ d(x, y) = \bigwedge \{ \hat{\lambda} \mid \lambda \in \Lambda \text{ and } E_\lambda(x, y) \text{ in } A \} \]

In the reverse direction, for \( M = (M, d) \) a \( \Phi(\Lambda) \)-ultrametric space, let
\[ e(M) = (M, \{ E_\lambda \mid \lambda \in \Lambda \}) \]
where \( E_\lambda \) is defined by
\[ E_\lambda(x, y) \iff \lambda \in d(x, y) \]

We will show that \( m \) and \( e \) give a pair of mutually inverse bijections between the objects, and that these bijections give rise to an isomorphism between the categories.

There are a number of points to be checked. Let us begin by listing them all.

(1) \( m : E\mathcal{Q}_\Lambda \rightarrow \mathcal{M}_{\Phi(\Lambda)} \)
(2) \( e : \mathcal{M}_{\Phi(\Lambda)} \rightarrow E\mathcal{Q}_\Lambda \)
(3) At the level of objects, the maps \( m, e \) are mutually inverse.
(4) If \( f : A \rightarrow B \) is a morphism in one category, then \( f : A' \rightarrow B' \) is also a morphism between the corresponding objects in the other category.

Claim 3.5. If \( A \in E\mathcal{Q}_\Lambda \), then \( m(A) \in \mathcal{M}_{\Phi(\Lambda)} \).

Proof of Claim. As \( \Phi(\Lambda) \) is complete, the definition of \( d(x, y) \) makes sense, and
\[ d : A \times A \rightarrow \Phi(\Lambda) \]

We first make the definition of \( d \) more explicit. Note that for \( x, y \in A \) the set \( \{ \lambda \in \Lambda \mid E_\lambda(x, y) \} \) is a filter, since the map from \( \lambda \) to \( E_\lambda \) preserves meets. We have
\[ d(x, y) = \bigwedge \{ \hat{\lambda} \mid E_\lambda(x, y) \} = \{ \lambda \mid E_\lambda(x, y) \} \]

Symmetry of \( d \) is clear. For the triangle inequality, take \( x, y, z \in A \) and note that
\[ d(x, z) \vee d(y, z) = d(x, z) \cap d(y, z) = \{ \lambda \mid E_\lambda(x, z), E_\lambda(y, z) \} \]
\[ \subseteq \{ \lambda \mid E_\lambda(x, y) \} = d(x, y) \]

Our last point is that \( d(x, y) = 0 \) iff \( x = y \). For \( x \in A \) we have \( E_\lambda(x, x) = \Lambda = \hat{0} = 0 \).

For the converse, if \( d(x, y) = 0 \) in \( \Phi(\Lambda) \) this means
\[ \{ \lambda \mid E_\lambda(x, y) \} = \Lambda \]
and thus \( E_0(x, y) \) holds. As we assume \( E_0 \) is equality, we find \( x = y \). \( \diamond \)

Claim 3.6. If \( \mathcal{M} \in \mathcal{M}_{\Phi(\Lambda)} \) then \( e(\mathcal{M}) \in E\mathcal{Q}_\Lambda \).

Proof of Claim. We check first that the relations \( E_\lambda \) are equivalence relations.

- Reflexivity: In \( \Phi(\Lambda) \), \( 0 = \Lambda \).
- Symmetry: \( d \) is symmetric.
Transitivity: The triangle inequality may be written more explicitly in the following form.

\[ d(x, y) \supseteq d(x, z) \cap d(y, z) \]

It is then clear that each relation \( E_\lambda \) is transitive.

The second point to be checked is preservation of meets:

\[ E_{\lambda \land \lambda'} = E_\lambda \cap E_{\lambda'} \]

Now

\[ E_{\lambda \land \lambda'}(x, y) \iff \lambda \land \lambda' \in d(x, y) \iff \lambda, \lambda' \in d(x, y) \]

since \( d(x, y) \) is a filter in \( \Lambda \).

The last point is that \( E_0 \) and \( E_\Pi \) are as intended.

It is easy to see that \( E_0(x, y) \) holds iff \( d(x, y) = 0 \) (in \( \Phi(\Lambda) \)), and this is equivalent to \( x = y \).

And \( E_\Pi(x, y) \) holds iff \( 1 \in d(x, y) \) which is always the case.

Claim 3.7. For \( A \in E Q_\Lambda \) and \( M \in M_{\Phi(\Lambda)} \), we have

\[ e(m(A)) = A \]
\[ m(e(M)) = M \]

Proof of Claim. We know

\[ \lambda \in d(x, y) \iff E_\lambda(x, y) \text{ when } M = m(A) \text{ and } x, y \in A \]
\[ E_\lambda(x, y) \iff \lambda \in d(x, y) \text{ when } A = e(M) \text{ and } x, y, \in M \]

Two applications of these rules will clearly bring us back where we started.

Claim 3.8. If \( M_i = e(A_i) \) for \( i = 1, 2 \), then a map \( f: A_1 \to A_2 \) will be an embedding if and only if it is a \( \Phi(\Lambda) \)-isometry.

Proof of Claim. Let us compare the two properties.

\[ E_\lambda(x, y) \iff E_\lambda(f(x), f(y)) \text{ for } x, y \in A_1 \text{ and } \lambda \in \Lambda \]
\[ d(x, y) = d(f(x), f(y)) \]

Recalling that \( \lambda \in d(x, y) \) iff \( E_\lambda(x, y) \) holds, and the same for \( f(x), f(y) \), the claim follows.

This concludes the proof of the theorem.

4. Model-theoretic context

In this section, we outline the model-theoretic context, namely homogeneous structures in the sense of Fraïssé theory, in which this correspondence arose, and mention some related open problems.

Definition 4.1. A countable structure \( M \) is homogeneous if every partial isomorphism between finitely-generated substructures extends to an automorphism of \( M \).
Definition 4.2. A countable structure $M$ is $\omega$-categorical if it is the unique countable model of its first-order theory, or equivalently if $\text{Aut}(M)$ has only finitely many orbits in its diagonal action on $M^n$ for each $n \in \mathbb{N}$.

Under the assumption of a finite relational language, homogeneity implies $\omega$-categoricity. Lying at the intersection of fields such as model theory, permutation groups, and combinatorics, these properties interact with a wide range of subjects. For example, in constraint satisfaction problems, they are crucial in allowing the universal algebraic approach from finite domains to be generalized to infinite domains [2].

The correspondence was used in the course of constructing homogeneous structures from $\mathcal{EQ}_\Lambda$, which would serve as the lattice of $\emptyset$-definable equivalence relations for the homogeneous structures of interest. This proceeds via combinatorial analysis of the finite substructures, and as the passage to substructures is simpler in $\mathcal{M}_{\Phi(\Lambda)}$, it was preferable to work in that category.

The next corollaries could be proven directly, but are immediate with the correspondence in hand.

**Corollary 4.3.** The categories $\mathcal{EQ}_\Lambda$ and $\mathcal{M}_\Lambda$ are closed under passage to substructures.

**Proof.** This is clear in the case of $\mathcal{M}_\Lambda$, and if we apply it to $\mathcal{M}_{\Phi(\Lambda)}$ and use Claim 4 of Theorem 3.4, in the case of inclusion maps, the claim follows. $\square$

**Corollary 4.4.** Let $\mathcal{A} \in \mathcal{EQ}_\Lambda$ and $\mathcal{M} = m(\mathcal{A})$. Then $\mathcal{A}$ is a homogeneous structure iff $\mathcal{M}$ is a homogeneous $\Phi(\Lambda)$-ultrametric space.

**Proof.** Substructure and isomorphism correspond between these classes by Theorem 3.4.

Suppose $\mathcal{A}$ is homogeneous. Let $X, Y \subset \mathcal{M}$ be finite such that $X \cong Y$. Then $e(X), e(Y) \subset \mathcal{A}$ are finite and isomorphic, and as $\mathcal{A}$ is homogeneous, there is an automorphism $\sigma$ of $\mathcal{A}$ sending $e(X)$ to $e(Y)$. Then $m \circ \sigma \circ e$ is an automorphism of $\mathcal{M}$ sending $X$ to $Y$.

The proof of the other direction is essentially the same, swapping the roles of $e$ and $m$. $\square$

We now turn to open questions, prompted by the following result of [3].

**Proposition 4.5.** Let $\Lambda$ be a finite distributive lattice. Then there is a homogeneous $\Lambda$-ultrametric space with lattice of $\emptyset$-definable equivalence relations isomorphic to $\Lambda$.

**Question 4.6.** What finite lattices are realizable as the lattice of $\emptyset$-definable equivalence relations of some homogeneous (resp. $\omega$-categorical) structure?

**Question 4.7.** Classify the homogeneous $\Lambda$-ultrametric spaces, for finite $\Lambda$.

In [3], it was also shown that distributivity is necessary under additional hypotheses.
Definition 4.8. Let \( \Lambda \) be the lattice of \( \emptyset \)-definable equivalence relations on \( M \). Then \( \Lambda \) has the infinite index property if whenever \( E < F \), each \( F \)-class splits into infinitely many \( E \)-classes.

Proposition 4.9. Let \( M \) be a homogeneous structure with finite lattice of \( \emptyset \)-definable equivalence relations \( \Lambda \). Suppose \( \Lambda \) has the infinite index property and the reduct of \( M \) to the language of \( \emptyset \)-definable equivalence relations is homogeneous. Then \( \Lambda \) is distributive.

However, [4, §5.2] contains an example without the infinite index property whose lattice of \( \emptyset \)-definable equivalence relations is the non-distributive lattice \( M_3 \). This example may be viewed as taking the equivalence relations to be parallel classes of lines in the affine plane over \( \mathbb{F}_2 \).

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