Classical Theorems in Noncommutative Quantum Field Theory

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Abstract

Classical results of the axiomatic quantum field theory - Reeh and Schlösser’s theorems, irreducibility of the set of field operators and generalized Haag’s theorem are proven in $SO(1,1)$ invariant quantum field theory, of which an important example is noncommutative quantum field theory. In $SO(1,3)$ invariant theory new consequences of generalized Haag’s theorem are obtained. It has been proven that the equality of four-point Wightman functions in two theories leads to the equality of elastic scattering amplitudes and thus the total cross-sections in these theories.

1 Introduction

Quantum field theory (QFT) as a mathematically rigorous and consistent theory was formulated in the framework of the axiomatic approach in the works of Wightman, Jost, Bogoliubov, Haag and others (\cite{1} - \cite{5}).

Within the framework of this theory on the basis of most general principles such as Poincaré invariance, local commutativity and spectra, a number of fundamental physical results, for example, the CPT-theorem and the spin-statistics theorem were proven \cite{1} - \cite{3}.

Noncommutative quantum field theory (NC QFT) being one of the generalizations of standard QFT has been intensively developed during the past years (for reviews, see \cite{6, 7}). The idea of such a generalization of QFT ascends to Heisenberg and it was initially developed in Snyder’s work \cite{8}. The present development in this direction is connected with the construction of noncommutative geometry \cite{9} and new physical arguments in favour of such a generalization of QFT \cite{10}. Essential interest in NC QFT is also due the fact that in some cases it is a low-energy limit of string theory \cite{11}. The simplest and at the
same time most studied version of noncommutative field theory is based on the following Heisenberg-like commutation relations between coordinates:

\[ [\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu\nu}, \quad (1) \]

where \( \theta^{\mu\nu} \) is a constant antisymmetric matrix.

It is known that the construction of NC QFT in a general case \( (\theta^{0i} \neq 0) \) meets serious difficulties with unitarity and causality \(^{[13]} - [16] \). For this reason the version with \( \theta^{0i} = 0 \) (space-space noncommutativity), in which there do not appear such difficulties and which is a low-energy limit of the string theory, draws special attention. Then always there is a system of coordinates, in which only \( \theta^{12} = -\theta^{21} \neq 0 \). Thus, when \( \theta^{0i} = 0 \), without loss of generality it is possible to choose coordinates \( x^0 \) and \( x^3 \) as commutative and coordinates \( x^1 \) and \( x^2 \) as noncommutative.

The relation (1) breaks the Lorentz invariance of the theory, while the symmetry under the \( SO(1,1) \times SO(2) \) subgroup of the Lorentz group survives \(^{[13]} \). Translational invariance is still valid. Below we shall consider the theory to be \( SO(1,1) \) invariant with respect to coordinates \( x^0 \) and \( x^3 \). Besides these classical groups of symmetry, in the paper \(^{[17]} \) it was shown, that the noncommutative field theory with the commutation relation (1) of the coordinates, and built according to the Weyl-Moyal correspondence, has also a quantum symmetry, i.e. twisted Poincaré invariance.

In the works \(^{[18]} - [20] \) the Wightman approach was formulated for NC QFT. For scalar fields the CPT theorem and the spin-statistics theorem were proven in the case \( \theta^{0i} = 0 \).

In \(^{[18]} \) it was proposed that Wightman functions in the noncommutative case can be written down in the standard form

\[ W(x_1, \ldots, x_n) = \langle \Psi_0, \varphi(x_1) \cdots \varphi(x_n) \Psi_0 \rangle, \quad (2) \]

where \( \Psi_0 \) is the vacuum state. However, unlike the commutative case, these Wightman functions are only \( SO(1,1) \otimes SO(2) \) invariant.

In \(^{[19]} \) it was proposed that in the noncommutative case the usual product of operators in the Wightman functions be replaced by the Moyal-type product (see also \(^{[7]} \)):

\[ \varphi(x_1) \cdots \varphi(x_n) = \prod_{a < b} \exp \left( \frac{i}{2} \theta^{\alpha\beta} \frac{\partial}{\partial x_\alpha^a} \frac{\partial}{\partial x_\beta^b} \right) \varphi(x_1) \cdots \varphi(x_n), \quad a, b = 1, 2, \ldots, n. \quad (3) \]

Such a product of operators is compatible with the twisted Poincaré invariance of the theory \(^{[21]} \) and also reflects the natural physical assumption, that noncommutativity should change the product of operators not only in coinciding points, but also in different ones. This follows also from another interpretation of NC QFT in terms of a quantum shift operator \(^{[22]} \). In \(^{[20]} \) it was shown that in the derivation of some axiomatic results, the concrete type of product of operators in various points is insignificant. It is essential only that from the appropriate spectral condition (see formula (9)), the analyticity of Wightman
functions with respect to the commutative variables $x^0$ and $x^3$ follows, while $x^1$ and $x^2$ do not need to be complexified. The Wightman functions can be written down as follows [20]:

$$W_\sim(x_1,x_2,\ldots,x_n) = \langle \Psi_0, \varphi(x_1) \tilde{\star} \cdots \tilde{\star} \varphi(x_n) \Psi_0 \rangle.$$  \hfill (4)

The meaning of $\tilde{\star}$ depends on the considered case. In particular,

$$\varphi(x) \tilde{\star} \varphi(y) = \varphi(x) \varphi(y) \text{ according to } [18] \quad \text{or} \quad \varphi(x) \tilde{\star} \varphi(y) = \varphi(x) \exp \left( \frac{i}{2} \frac{\partial^\mu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right) \varphi(y) \text{ according to } [19].$$

Note that actually field operators are the smoothed operators

$$\varphi_f \equiv \int \varphi(x) f(x) \, dx,$$

where $f(x)$ are test functions. This point will be considered in detail in Section 2.

In [20] it was shown that, besides the above-mentioned theorems, in NC QFT (with $\theta^0 = 0$) a number of other classical results of the axiomatic theory remain valid. In [21] on the basis of the twisted Poincaré invariance of the theory the Haag’s theorem was obtained [23, 24] (see also [1] and references therein).

In the present work, analogues of some known results of the axiomatic approach in quantum field theory are obtained for the $SO(1,1)$ invariant field theory, of which an important example is NC QFT. In the $SO(1,3)$ invariant theory new consequences of the generalized Haag’s theorem are found, without analogues in NC QFT. At the same time it is proven that the basic physical conclusion of Haag’s theorem is valid also in the $SO(1,1)$ invariant theory, and it is sufficient that spectrality, local commutativity condition and translational invariance be fulfilled only for the transformations concerning the commuting coordinates. The analysis of Haag’s theorem reveals essential distinctions between commutative and noncommutative cases, more precisely between the $SO(1,3)$ and $SO(1,1)$ invariant theories. In the commutative case, the conditions [33] and [34], whose consequence is generalized Haag’s theorem, lead to the equality of Wightman functions in two theories up to four-point ones. In the present paper it is shown that in the $SO(1,1)$ invariant theory, unlike the commutative case, only two-point Wightman functions are equal and it is shown that from the equality of two-point Wightman functions in two theories it follows that if in one of them the current is equal to zero, it is equal to zero in the other as well and under weaker conditions than the standard ones. It is also shown that for the derivation of eq. [34] it is sufficient to assume that the vacuum vector is a unique normalized vector, invariant under translations along the axis $x^3$. It is proven that from the equality of four-point Wightman functions in two theories, the equality of their elastic scattering amplitudes follows and, owing to the optical theorem, the equality of total cross sections as
well. In derivation of this result LCC is not used. In the noncommutative theory we also prove that classical results, such as the irreducibility of the set of field operators, the theorems of Reeh and Schlieder [1] - [3] remain valid in the noncommutative case. It should be emphasized that the results obtained in this paper do not depend on the $SO(2)$ invariance of the theory in the variables $x^1$ and $x^2$ and therefore can be extended to more general cases. The first theorem of Reeh and Schlieder and the irreducibility of the set of field operators remain valid in any theory, in which the spectral condition (9) leads to the analyticity of the Wightman functions in the variables $x^0$ and $x^3$ in the primitive domains of analyticity ("tubes").

The study of Wightman functions leads still to new nontrivial consequences also in the commutative case.

The paper is arranged as follows. In section 2 the necessary properties of Wightman functions are formulated; in section 3 generalizations of the theorems of Reeh and Schlieder to NC QFT are obtained; in section 4 the irreducibility of the set of field operators is proven; section 5 is devoted to generalized Haag's theorem; in section 6 it is shown that in the commutative case, the conditions of weak local commutativity (WLCC) and of local commutativity (LCC), which are valid in the noncommutative case ((18) and (19)), appear to be equivalent to the usual WLCC and LCC, respectively.

## 2 Basic Properties of Wightman Functions

In Wightman’s approach cyclicity of the vacuum vector is assumed. In the noncommutative case this means that any vector of the space under consideration, $J$, can be approximated with arbitrary accuracy by vectors of the type

$$\varphi_{f_1} \hat{\star} \cdots \hat{\star} \varphi_{f_n} \Psi_0,$$

where $f_i$ is a proper test function $\varphi$ and the $\hat{\star}$ in (6) acts only on the test functions $f_1, \ldots, f_n$. For simplicity we consider the case of a real field, however, the results are easily extended to a complex field. As well as in the commutative case we assume that the domain of definition of operators $\varphi_f$ is dense in $J$.

We admit that the scalar product of two vectors $\Phi = \varphi_{f_1} \hat{\star} \cdots \hat{\star} \varphi_{f_n} \Psi_0$ and $\Psi = \varphi_{f_{n+1}} \hat{\star} \cdots \hat{\star} \varphi_{f_n}$ is the following

$$\langle \Phi, \Psi \rangle = \langle \Psi_0, \varphi_{f_1} \hat{\star} \cdots \hat{\star} \varphi_{f_n} \Psi_0 \rangle =$$

$$\int W(x_1, \ldots, x_n) f_1(x_1) \hat{\star} \cdots \hat{\star} f_n(x_n) \; dx_1 \ldots dx_n. \tag{7}$$

For reasons given below, it is important only that the scalar product of any two vectors in $J$ can be approximated by linear combination of the scalar products of the type (7) with arbitrary precision. As noncommutativity does not affect the commutative variables we assume that noncommutative Wightman

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1 Part of the results have been presented in the talk [25].
functions after smearing on noncommutative variables are tempered distributions with respect to \( x_0^i \) and \( x_3^i \). As to their properties with respect to noncommutative variables it is natural to assume, as it is done in various versions of nonlocal theories \[26\] - \[28\], that they belong to one of the Gel’fand-Shilov spaces \[29\]. This question will be considered in a paper which is in preparation. Let us stress that the concrete choice of Gel’fand-Shilov space is not important in the derivation of our results.

Let us point out that the results of this work are new also in the case of the standard multiplication of functions \( f_i(x_i) \).

For the results obtained below, translational invariance only in commuting coordinates is essential, therefore we write down the Wightman functions as:

\[
W(x_1, \ldots, x_n) = W(\xi_1, \ldots, \xi_{n-1}, X), \tag{8}
\]

where \( X \) designates the set of noncommutative variables \( x_1^i, x_2^i, i = 1, \ldots n \), and \( \xi_j = \{\xi^0_j, \xi^3_j\} \), where \( \xi^0_j = x^0_j - x^0_{j+1}, \xi^3_j = x^3_j - x^3_{j+1}, j = 1, \ldots, n - 1 \).

Let us formulate now the spectral condition. We assume that any vector in \( p \) space, belonging to the complete system of these vectors, is time-like with respect to momentum components \( P^0_n \) and \( P^3_n \), i.e. that

\[
P^0_n \geq |P^3_n|. \tag{9}
\]

The condition \( \text{(9)} \) is conveniently written as \( P_n \in \tilde{V}_2^+ \), where \( \tilde{V}_2^+ \) is the set of the four-dimensional vectors satisfying the condition \( P^0 \geq |P^3| \). Recall that the usual spectral condition for these vectors looks like \( P_n \in V^+ \), i.e. \( P^0_n \geq |\vec{P}_n| \).

From the condition \( \text{(9)} \) and the completeness of the system of the vectors \( \Psi_{P_n} \): \[
\langle \Phi, \Psi \rangle = \sum_n \int dP_n \langle \Phi, \Psi_{P_n} \rangle \langle \Psi_{P_n}, \Psi \rangle, \tag{10}
\]

it follows that

\[
\int d a e^{-i P a} \langle \Phi, U(a) \Psi \rangle = 0, \quad \text{if} \quad p \notin \tilde{V}_2^+, \tag{11}
\]

where \( a = \{a^0, a^3\} \) is a two-dimensional vector, \( U(a) \) is a translation in the plane \( p^0, p^3 \), and \( \Phi \) and \( \Psi \) are arbitrary vectors. The equality \( \text{(11)} \) is similar to the corresponding equality in the standard case (\cite{1}, Chap. 2.6). Let us point out that in fact the complete system of vectors \( \Psi_{P_n} \) contains vectors with infinite norms. But this point can be settled in the noncommutative case just as in commutative one \( \cite{1} \). The point is that noncommutativity does not affect the momentum operators. So in momentum space the only difference between commutative and noncommutative cases is the weaker spectral condition in the latter case.

A direct consequence of the equality \( \text{(11)} \) is the spectral property of Wightman functions:

\[
W(P_1, \ldots, P_{n-1}, X) = \frac{1}{(2\pi)^{n-1}} \int e^{i P_j \xi_j} W(\xi_1, \ldots, \xi_{n-1}, X) d\xi_1 \ldots d\xi_{n-1} = 0, \tag{12}
\]

\[\text{5}\]
if \( P_j \notin V_2^+ \). The proof of the equality (12) is similar to the proof of the spectral condition in the commutative case [1, 3]. Recall that in the latter case the equality (12) is valid, if \( P_j \notin V^+ \). Having written down \( W(\xi_1, \ldots, \xi_{n-1}, X) \) as

\[
W(\xi_1, \ldots, \xi_{n-1}, X) = \frac{1}{(2\pi)^{n-1}} \int e^{-i P_j \xi_j} W(P_1, \ldots, P_{n-1}, X) dP_1 \ldots dP_{n-1},
\]

and taking into account that Wightman functions are tempered distributions with respect to the commutative variables, we obtain that, due to the condition (12), \( W(\nu_1, \ldots, \nu_{n-1}, X) \) is analytical in the "tube" \( T_n^- \):

\[
\nu_i \in T_n^- \quad \text{if} \quad \nu_i = \xi_i - i \eta_i, \, \eta_i \in V_2^+, \, \eta_i = \{\eta_i^0, \eta_i^3\}.
\]

It should be stressed that the noncommutative coordinates \( x_1^1, x_i^2 \) remain always real.

Owing to \( SO(1,1) \) invariance and according to the Bargmann-Hall-Wightman theorem [1] - [3], \( W(\nu_1, \ldots, \nu_{n-1}, X) \) is analytical in the domain \( T_n \):

\[
T_n = \cup_{\Lambda_c} \Lambda_c T_n^-,
\]

where \( \Lambda_c \in SO_c(1,1) \) is the two-dimensional analogue of the complex Lorentz group. This expansion is similar to the transition from tubes to expanded tubes in the commutative case. Just as in the commutative case, the expanded domain of analyticity contains real points \( x_i \), which are the noncommutative Jost points, satisfying the condition \( x_i \sim x_j, \forall i, j \), which means that

\[
(x_i^0 - x_j^0)^2 - (x_i^3 - x_j^3)^2 < 0.
\]

It should be emphasized that the noncommutative Jost points are a subset of the set of Jost points of the commutative case, when

\[
(x_i - x_j)^2 < 0 \quad \forall i, j.
\]

Let us point out that WLCC and LCC, respectively have the same form as in the local theory:

\[
W(x_1, \ldots, x_n) = W(x_n, \ldots, x_1), \quad \text{if} \quad x_i \sim x_j, \forall i, j;
\]

\[
W(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = W(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n),
\]

if \( \text{supp} \, f_i \in O_i \times R^2 \), \( \text{supp} \, f_{i+1} \in O_{i+1} \times R^2 \), \( O_i \sim O_{i+1} \), which means that the condition (16) is valid for any points \( x_i \in O_i \times R^2 \) and \( x_{i+1} \in O_{i+1} \times R^2 \).

3 Theorems of Reeh and Schlieder in NC QFT

In the following we shall prove the analogues of the theorems of Reeh and Schlieder [1, 2] for the noncommutative case.
Theorem 1  Let supports of functions $f_i$ belong to $O \times R^2$, where $O$ is any open domain on variables $x_i^0$ and $x_i^3$.

Then there is no vector distinct from zero, which is orthogonal to all vectors of the type $\varphi_{f_1} \hat{\otimes} \cdots \hat{\otimes} \varphi_{f_n} \Psi_0$, supp $f_i \in O \times R^2$. Let us consider two vectors

$$
\Phi = \varphi_{f_1} \hat{\otimes} \cdots \hat{\otimes} \varphi_{f_n} \Psi_0, \quad \text{supp } f_i \in O \times R^2 \ \forall \ i,
$$

$$
\Psi = \varphi_{f_m} \hat{\otimes} \cdots \hat{\otimes} \varphi_{f_1} \Psi_0.
$$

On supp $\tilde{f}_i$ no restrictions are imposed. We shall prove that $\Psi = 0$, if for any vector $\Phi$

$$
\langle \Psi, \Phi \rangle = 0.
$$

For the proof it is sufficient to notice that the corresponding Wightman function

$$
\langle \Psi_0, \varphi(y_1) \hat{\otimes} \cdots \hat{\otimes} \varphi(y_m) \hat{\otimes} \varphi(x_1) \hat{\otimes} \cdots \hat{\otimes} \varphi(x_n) \rangle = W(y_1, \ldots, y_m, x_1, \ldots, x_n)
$$

is an analytical function in the variables $-x_i^0 - i \eta_0^i, -x_i^3 - i \eta_3^i$, $\nu_i = \xi_i - i \eta_i$, $i = 1, \ldots, n - 1$, if $\eta_i \in V_2^-$. According to the condition (21), this function is equal to zero on the border, if $x_i \in O \times R^2$. As $O$ is an open domain, $W(y_1, \ldots, y_m, x_1, \ldots, x_n) \equiv 0$. Thus the vector $\Psi$ is orthogonal to all vectors of the type (6) and, according to the cyclicity of the vacuum vector, $\Psi = 0$.

Taking into account the completeness of the system of vectors $\Psi$ we come to the statement of the Theorem. Remark that for the proof of the Theorem 1 only the analyticity of the Wightman functions in the domain $T_n$ has been used.

Theorem 2  Let the support of $\tilde{f} \in \tilde{O} \times R^2$, where $\tilde{O}$ is such a domain of commutative variables, for which domain $O \sim \tilde{O}$, satisfying the condition of the Theorem 1, exists. Then the condition

$$
\varphi_{\tilde{f}} \Psi_0 = 0
$$

implies that

$$
\varphi_{\tilde{f}} \equiv 0,
$$

if the operator $\varphi_{\tilde{f}}$ satisfies the LCC.

In accordance with LCC

$$
\varphi_{\tilde{f}} \hat{\otimes} \Phi = 0,
$$

if vector $\Phi$ is defined as in eq. (20). Hence, for any vector $\Psi$ belonging to the domain of definition of the Hermitian operator $\varphi_{\tilde{f}}$,

$$
\langle \varphi_{\tilde{f}} \hat{\otimes} \Psi, \Phi \rangle = \langle \Psi, \varphi_{\tilde{f}} \hat{\otimes} \Phi \rangle = 0.
$$

According to the Theorem 1, the condition (26) means that $\varphi_{\tilde{f}} \hat{\otimes} \Psi = 0$. As the domain of definition of the operator $\varphi_{\tilde{f}}$ is dense in $J$, this equality means the validity of the equality (23).

Remark  Theorem 2 remains true for any densely defined operator $\psi_{\tilde{f}}$, mutually local with $\varphi_{\tilde{f}}$, i.e. if

$$
\psi_{\tilde{f}} \hat{\otimes} \varphi_{\tilde{f}} \hat{\otimes} \Phi = \varphi_{\tilde{f}} \hat{\otimes} \psi_{\tilde{f}} \hat{\otimes} \Phi,
$$

if supp $f \in O \times R^2$, supp $\tilde{f} \in \tilde{O} \times R^2$, $O \sim \tilde{O}$, vector $\Phi$ belongs to the domain of definition of operators $\varphi_{\tilde{f}}$ and $\psi_{\tilde{f}}$. 

7
4 Irreducibility of the set of field operators $\varphi_f$ in NC QFT

In the noncommutative case, the irreducibility of a set of field operators $\varphi_f$ implies that, from the condition

$$A \varphi_{f_1} \star \cdots \star \varphi_{f_n}, \Psi_0 = \varphi_{f_1} \star \cdots \star \varphi_{f_n} A \Psi_0$$  \hspace{1cm} (27)$$

where $f_i$ are arbitrary test functions and $A$ is a bounded operator, follows that

$$A = C \mathbf{1} \quad C \in \mathbb{C}$$  \hspace{1cm} (28)$$

where $\mathbf{1}$ is the identity operator.

In the noncommutative case the condition of irreducibility of the set of operators $\varphi_f$ is valid as well as in commutative case. The point is that for this it is sufficient to have the translational invariance in the variable $x^0$ and the spectral condition, which can be weakened up to the condition

$$P_n^0 \geq 0.$$  \hspace{1cm} (29)$$

Using condition (27) and the invariance of the vacuum vector with respect to the translations $U(a)$ on the axis $x^0$, we obtain the following equality

$$\langle A^* \Psi_0, U(a) \varphi_{f_1} \star \cdots \star \varphi_{f_n}, \Psi_0 \rangle = \langle \varphi_{f_n} \star \cdots \star \varphi_{f_1}, U(-a) A \Psi_0 \rangle.$$  \hspace{1cm} (30)$$

In accordance with the eq. (11)

$$\int da e^{-i p^0 a} \langle A^* \Psi_0, U(a) \varphi_{f_1} \star \cdots \star \varphi_{f_n}, \Psi_0 \rangle \neq 0,$$

only if $p^0 \geq 0$. However,

$$\int da e^{-i p^0 a} \langle \varphi_{f_1} \star \cdots \star \varphi_{f_n}, U(-a) A \Psi_0 \rangle \neq 0,$$

only if $p^0 \leq 0$. Hence, the equality (30) can be fulfilled only when $p^0 = 0$ and so, in accordance with the eq. (10),

$$A \Psi_0 = C \Psi_0.$$  \hspace{1cm} (31)$$

Thus owing to (27) and (31)

$$A \varphi_{f_1} \star \cdots \star \varphi_{f_n} \Psi_0 = C \varphi_{f_1} \star \cdots \star \varphi_{f_n} \Psi_0.$$  \hspace{1cm} (32)$$

The required equality (28) follows from eq. (32) in accordance with the boundedness of the operator $A$ and ciclicity of the vacuum vector.
Recall the formulation of the generalized Haag’s theorem in the commutative case ([1], Theorem 4.17):

Let \( \varphi^1_f (t) \) and \( \varphi^2_f (t) \), \( \text{supp} \ f \in \mathbb{R}^3 \) be two irreducible sets of operators, for which the vacuum vectors \( \Psi^1_0 \) and \( \Psi^2_0 \) are cyclic. Further, let the corresponding Wightman functions be analytical in the domain \( \mathcal{T} \).

Then the two-, three- and four-point Wightman functions coincide in the two theories if there is a unitary operator \( V \), such that

1) \( \varphi^2_f (t) = V \varphi^1_f (t) V^* \), \( \quad (33) \)
2) \( \Psi^2_0 = C V \Psi^1_0, \ C \in \mathbb{C}, \ |C| = 1 \). \( \quad (34) \)

It should be emphasized that actually the condition 2) is a consequence of condition 1) with rather general assumptions (see the Statement below). In the formulation of Haag’s theorem it is assumed that the formal operators \( \varphi_i (t, \mathbf{x}) \) can be smeared only on the spatial variables. This assumption is natural also in noncommutative case if \( \theta^0 = 0 \).

Let us consider Haag’s theorem in the \( SO(1,1) \) invariant field theory and show that the corresponding equality is true only for two-point Wightman functions.

For the proof we first note that in the noncommutative case, just as in the commutative one, from conditions 1) and 2) it follows that the Wightman functions in the two theories coincide at equal times

\[
\langle \Psi^1_0, \varphi_1 (t, x_1^1) \cdots \varphi_1 (t, x_n^1) \Psi^1_0 \rangle = \langle \Psi^2_0, \varphi_2 (t, x_1^2) \cdots \varphi_2 (t, x_n^2) \Psi^2_0 \rangle. \quad (35)
\]

Having written down the two-point Wightman functions \( W_i (x_1, x_2) \), \( i = 1, 2 \) as \( W_i (u_1, v_1; u_2, v_2) \), where \( u_i = \{ x_i^0, x_i^1 \}, \ v_i = \{ x_i^1, x_i^2 \} \) we can write for them equality \( (35) \) as:

\[
W_1 (0, \xi^3; v_1, v_2) = W_2 (0, \xi^3; v_1, v_2), \quad (36)
\]

where \( \xi = u_1 - u_2 \), \( v_1 \) and \( v_2 \) are arbitrary vectors. Now we notice that, due to the \( SO(1,1) \) invariance,

\[
W_i (0, \xi^3; v_1, v_2) = W_i (\tilde{\xi}; v_1, v_2) \quad (37)
\]

hence,

\[
W_1 (\tilde{\xi}; v_1, v_2) = W_2 (\tilde{\xi}; v_1, v_2), \quad (38)
\]

where \( \tilde{\xi} \) is any Jost point. Due to the analyticity of the Wightman functions in the commuting variables they are completely determined by their values at the Jost points. Thus at any \( \xi \) from the equality \( (38) \), it follows that

\[
W_i (\xi; v_1, v_2) = W_i (\xi; v_1, v_2). \quad (39)
\]

\[\text{Remark:} \quad \text{The required analyticity of the Wightman functions follows only from the spectral condition and the } SO(1,3) \text{ invariance of the theory.}\]
As \( v_1 \) and \( v_2 \) are arbitrary, the formula (39) means the equality of two-point Wightman functions at all values of arguments.

Thus, for the equality of the two-point Wightman functions in two theories related by the conditions (33) and (34), the \( SO(1,1) \) invariance of the theory and corresponding spectral condition are sufficient.

It is impossible to extend this proof to three-point Wightman functions. Indeed, let us write down

\[
W_i(x_1,x_2,x_3) = W_i(u_1,u_2,u_3;v_1,v_2,v_3),
\]

where vectors \( u_i \) and \( v_i \) are determined as before. Equality (36) means that

\[
W_1(0,\xi_1^1,0,\xi_2^1;v_1,v_2,v_3) = W_2(0,\xi_1^1,0,\xi_2^1;v_1,v_2,v_3),
\]

(40)

\( v_1, v_2, v_3 \) are arbitrary. In order to have equality of the three-point Wightman functions in the two theories from the \( SO(1,1) \) invariance, the existence of transformations \( \Lambda \in SO(1,1) \) connecting the points \( (0,\xi_1^1) \) and \( (0,\xi_2^1) \) with an open vicinity of Jost points is necessary. That would be possible, if there existed two-dimensional vectors \( \tilde{\xi}_1 \) and \( \tilde{\xi}_2 \), \( (\tilde{\xi}_1, \tilde{\xi}_2) \), satisfying the inequalities:

\[
|\langle \tilde{\xi}_1, \tilde{\xi}_2 \rangle| < \sqrt{\left(\tilde{\xi}_1 \right)^2 \left(\tilde{\xi}_2 \right)^2}.
\]

These inequalities are similar to the corresponding inequalities in the commutative case (see equation (4.87) in [1]). However, it is easy to check that the last of these inequalities can not be fulfilled, while the first two are fulfilled.

Let us show now that the condition (34) actually is a consequence of the condition (33).

**Statement** Condition (34) is fulfilled, if the vacuum vectors \( \Psi_0^1 \) are unique, normalized, translationally invariant vectors with respect to translations \( U_i(a) \) along the axis \( x^3 \).

It is easy to see that the operator \( U_1^{-1}(a) V^{-1} U_2(a) V \) commutes with operators \( \varphi_i f(t) \) and, owing to the irreducibility of the set of these operators, it is proportional to the identity operator. Having considered the limit \( a = 0 \), we see that

\[
U_1^{-1}(a) V^{-1} U_2(a) V = I
\]

(41)

From the equality (41) it follows directly that if

\[
U_1(a) \Psi_0^1 = \Psi_0^1,
\]

(42)

then

\[
U_2(a) V \Psi_0^1 = V \Psi_0^1,
\]

(43)

i.e. the condition (34) is fulfilled. If the theory is translationally invariant in all variables, the equality (43) is true, if the vacuum vector is unique, normalized, translationally invariant in the spatial coordinates.

The most important consequence of the generalized Haag theorem is the following statement: if one of the two fields related by conditions (33) and (34) is a free field, the other is also free. In deriving this result the equality of the two-point Wightman functions in the two theories and LCC are used. In [21] it is proved that this result is valid also in the noncommutative theory, if \( \theta^{0i} = 0 \).
Here we obtain the close result in the \( SO(1,1) \) symmetric theory using the spectral conditions and translational invariance only with respect to the commutating coordinates. In this case the equality of the two-point Wightman functions in the two theories leads to the conclusion that if LCC is fulfilled and the current in one of the theories is equal to zero, for example, \( j^1_I = 0 \), then \( j^2_I = 0 \) as well; \( j^I_I = (\Box + m^2) \varphi^I_I \). Indeed as \( W_1(x_1, x_2) = W_2(x_1, x_2) \),
\[
\langle \Psi_0^1, j^1_I(x_1) j^1_I(x_2) \Psi_0^1 \rangle = \langle \Psi_0^2, j^2_I(x_1) j^2_I(x_2) \Psi_0^2 \rangle = 0, \quad (44)
\]
since \( j^I_I = 0 \). Hence,
\begin{equation}
\int \prod_{j=1}^4 \langle \Box_j + m^2 \rangle < 0 | T \varphi_i(x_1) \cdots \varphi_i(x_4) | 0 >,
\end{equation}
where \( T \varphi_i(x_1) \cdots \varphi_i(x_4) \) is the chronological product of operators. From the equality
\[
W_2(x_1, \ldots, x_4) = W_1(x_1, \ldots, x_4)
\]
it follows that
\[
\langle p_3, p_4 | p_1, p_2 \rangle_1 = \langle p_3, p_4 | p_1, p_2 \rangle_2 = \langle p_3, p_4 | p_1, p_2 \rangle_1
\]
for any \( p_i \). Having applied this equality for the forward elastic scattering amplitudes, we obtain that, according to the optical theorem, the total cross-sections for the fields \( \varphi_1(x) \) and \( \varphi_2(x) \) coincide. If now the \( S \)-matrix for the field \( \varphi_1(x) \)
is unity, then it is also unity for field $\phi_2(x)$. We stress that the equality of the four-point Wightman functions in the two theories related by the conditions (33) and (34) are valid only in the commutative field theory but not in the noncommutative case.

6 Equivalence of various conditions of local commutativity in QFT

Let us show that in the commutative case, when Wightman functions are analytical ones in the usual domain, the conditions (18) and (19) are equivalent to the standard conditions of WLC and LC, i.e. the latter remain valid if the condition (17) is fulfilled. In effect, (18) is a sufficient condition for the theory to be CPT invariant [18]. However, in the commutative case, from CPT invariance the standard condition of WLC follows. [1] - [3].

The equivalence of LCC (19) with the standard one follows from the fact that, for the validity of usual LCC its validity on arbitrary small spatially divided domains is sufficient (see [4], Proposal 9.12). Indeed, validity of "noncommutative" LCC (19) in commutative the case means validity of standard LCC in the domain $(x^0 - y^0)^2 - (x^3 - y^3)^2 < 0$, $x^k, y^k$, $k = 1, 2$ are arbitrary. This domain satisfies the requirements of the above mentioned statement.

Besides we can replace (19) with the formally weaker condition, requiring that it is valid only when

$$(x^0_i - x^0_j)^2 - (x^3_i - x^3_j)^2 < -l^2, \forall i, j,$$

(47)

where $l$ is any fixed fundamental length. Indeed, in the commutative theory, according to the results of Wightman, Petrina and Vladimirov (see [31], Chapter 5 and references therein) the condition

$$[\phi(x), \phi(y)] = 0, \quad (x - y)^2 < -l^2,$$

(48)

for any finite $l$, is equivalent to standard LCC ($l = 0$). Similarly if (19) is fulfilled at (47), then it is fulfilled also at $l = 0$.

Thus, the analysis of Wightman functions in NC QFT, carried out in this and our previous works [19], [20], [21], shows that the basic axiomatic results are valid (or have analogues) in NC QFT as well, at least in the case when $\theta^{0i} = 0$.

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