Dimensional regularization of a compact dimension.

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Abstract

An extension of dimensional regularization to the case of compact dimensions is presented. The procedure preserves the Kaluza-Klein tower structure, but has a regulator specific to the compact dimension. Possible 5 and 4 dimensional divergent as well as manifest finite contributions of (one-loop) Feynman graphs can easily be identified in this scheme.
1 Introduction

It is well known that dimensional regularization in 4 uncompact dimensions is a very powerful and convenient regularization scheme. The main idea is to obtain analytic expressions in the number of dimensions $D$ and then analytically continue it to 4 dimensions. This scheme is universal in the sense that it can be applied to an arbitrary loop calculation of Feynman graphs in a quantum field theory. It respects all symmetries of the classical theory, except when this symmetry develops an anomaly at the quantum level. In addition, calculations with this procedure are simple and elegant since they rely on various properties of complex function theory.

The central idea of this paper is to extend the dimensional regularization prescription such that it can also be applied to a space-time that has some compact dimensions in addition to the non-compact ones. We assume here a factorizable geometry $M^4 \times C$ where $M^4$ denotes the 4 dimensional Minkowski space and $C$ is a compact manifold.

Furthermore, we restrict our attention to the case where the compact manifold is one dimensional, i.e. typically a circle $S^1$, an orbicircle $S^1/Z_2$ or an orbifold $S^1/Z_2 \times Z_2'$. Models based on these types of orbifold have been the subject of study in the recent literature [1, 2]. However the general regularization prescription that we present could in principle also be applied to other (more complicated) cases. Because of the compactness sums – rather than integrals – are obtained in momentum space. One would expect that in the limit of a large radius (small spacing of the momentum eigenvalues), a good approximation would be to replace the sum by the corresponding integral. In the approach described in this article we take this further and rewrite all sums as integrals. Since both sum and integrals are over momenta, the question which should be performed first should be irrelevant, in the regularization prescription we present this is indeed the case.

Since the geometry of the manifold is a direct product $M^4 \times C$, we use two regulator parameters: $D_4$ denotes the complex extension of the dimension of the Minkowski space and $D_5$ denotes the complex extension of the single dimension of the compact manifold $C$. In addition we introduce two arbitrary renormalization scales $\mu_4$ and $\mu_5$. As we will see the appearance of $D_4$ and $D_5$ is very convenient because it allows us to trace whether a divergence has a 4 or 5 dimensional origin, depending on whether $D_5$, $D_4$ or $D_4 + D_5$ appear in a regulated expression.

Dimensional regularization has been used before in the connection with compact manifolds. For example, in ref. [3, 4, 5] it was combined with $\zeta$-function regularization (see ref. [6] for a general review of this method) for the compact dimension. Hence there is only one regulator parameter for both the compact and non-compact dimensions. An attempt to have a separate regulator for compact dimensions is discussed in [7]. The approach we discuss in this paper is more direct and makes a clear distinction between regulator effects and properties of the momentum spectrum due to the compact manifold.

Furthermore, the regularization prescription can in principle be applied to any effective field theory description obtained by integrating out some addi-
tional dimension. Therefore, it can be used in situations more general than those considered in this paper. Just as an example, we mention that (loop) calculations with the mass spectra obtained in [8] for a warped geometry [9, 10] can be investigated with the method described here. (In fact, the asymptotic form of the KK masses are treated in this paper.)

In the following we first explain how after turning a sum into complex integral, the complex dimension $D_5$ can be introduced. The original properties of the sum are translated into the properties of a complex function that we call the “pole function” of the momentum spectrum. After a general description of such pole functions, we determine them for compactifications on a circle $S^1$ or orbicircle $S^1/Z_2$. Here we also discuss why one should sum the complete towers as is common practice in the “Kaluza-Klein regularization”. Next we show that from the structure of those pole functions, we can identify the 4 and 5 dimensional divergences and finite contributions could be encountered in the calculation of a Feynman graph. A separate section is devoted to a discussion how fermions can be incorporated in this approach. We illustrate our method with the computation of the effective potential of the model presented in [1].

2 Sum as $D_5$ dimensional integral

In this section we describe how a typical sum-integral, that arises from a Feynman graph in the 4 dimensional effective theory with KK towers, can be dimensionally regulated. Here we treat graphs with bosons only, however a large part of the discussion can directly be applied to graphs with fermions as well. Our discussion here is far from being complete, the main focus here is to convey the ideas behind the procedure.

The Laplacian of the fifth dimension $\Delta_5 \,(=\partial_5^2 \text{ for flat manifolds})$ has eigenfunctions $\phi_n$ with real positive eigenvalues $m_n^2$. Furthermore, let $f(p_4, p_5)$ be an arbitrary function of 4 dimensional momentum vector $p_4$ and 5 dimensional momentum $p_5$, where the latter can only take values $m_n$. For convenience, we assume here that each eigenvalue $m_n$ has multiplicity 1 and that $|m_n| \to \infty$ if there are infinitely many eigenvalues. Furthermore we assume that $f(p_4, p_5)$ only depends on the combination $p_4^2 + p_5^2$. (These constrains are not really necessary as will become clear below. However, they make the discussion here more accessible.) Only the infinite case ($n \to \infty$) is discussed here, as it is easy to make a restriction to finite number of eigenvalues. We make the additional assumption that the function $f$ is meromorphic in $p_5$, but does not have poles on the real axis. This means that all poles $p_5 = X$ of $f(p_4, p_5)$ satisfy $|\text{Im} X| \geq \epsilon$ for a given $\epsilon > 0$ for any $p_4$. If the function $f$ does have poles on the real axis, then by a slight modification of this function they can be taken away from this axis by introducing an infrared regulator. In particular, this may be needed if one considers massless particles in the 5 dimensional theory.

After these generalities, we want to compute the sum-integral

$$\int d^4p_4 \sum_{n \in \mathbb{N}} f(p_4, m_n)$$

(1)
using dimensional regularization for both the four dimensional integral and for the sum. (All momentum integrals are Euclidean; the Wick rotation from Minkowski space is assumed to be performed.) To turn the sum over the eigenvalues into an integral, we define the pole function as the meromorphic function $\mathcal{P}(p_5)$ with the properties:

1. its set of poles is the set eigenvalues $\{m_n \mid n \in \mathbb{N}\}$,
2. the residue at each of these poles is 1,
3. for $p_5 \to \infty$ with $\pm \text{Im } p_5 > \epsilon$ its goes to an imaginary constant $\mathcal{P}(p_5) \to \mp i r$.

Existence and uniqueness of $\mathcal{P}$ follows from Mittag-Leffler’s theorem which is closely related to the Weierstrass’ product theorem, see for example [11, 12]. The first two conditions only determine the meromorphic function up to an arbitrary holomorphic function. The third condition fixes this function. The constant $r$ measures the size of the fifth dimension. In the examples of the circle and orbicircle discussed in the next section we obtain precise relations between this quantity $r$ and their radius $R$.

Let us first consider the situation where the infinite sum is convergent: to be more precise we assume that $C, \alpha > 0$ exist (which may be $p_4$ dependent) such that $|f(p_4, p_5)| \leq C|p_5|^{-1-\alpha}$. Around any eigenvalue $m_n$ we can consider a contour of the form of a box $\Box_{(n, \epsilon)}$ with height $2\epsilon$ symmetric around the real axis. These boxes can be assumed be infinitely close together, their union is denoted by $\equiv \equiv \cup_n \Box_{(n, \epsilon)}$. This contour goes infinitely close around the real axis. Using standard contour integration the sum is rewritten as an integral [13, 14]. This integral can be rewritten as contour integral $\ominus$ over the upper and lower half plane with opposite orientation (anti-clockwise) to the $\equiv$ contour. Here we have used that the arc contours at infinity vanish because of the bound on the function $f$ given above.

\[
\sum_{n \in \mathbb{N}} f(p_4, m_n) = \sum_{n \in \mathbb{N}} \frac{-1}{2\pi i} \int_{\Box_{(n, \epsilon)}} dp_5 \mathcal{P}(p_5)f(p_4, p_5) = \tag{2}
\]

\[
= \frac{-1}{2\pi i} \int_{\equiv} dp_5 \mathcal{P}(p_5)f(p_4, p_5) = \frac{1}{2\pi i} \int_{\ominus} dp_5 \mathcal{P}(p_5)f(p_4, p_5).
\]

The figure below gives a schematic picture of this situation in the complex $p_5$-plane:
The symbols “X” denote the positions of poles $f$, and the dots • denote the real values which are the eigenvalues $m_n$; the poles of $\mathcal{P}$. The contour orientations are described above.

Now we want to set up a method that can handle a (divergent) sum-integral following the idea of replacing the sum by the contour integral $\odot$. The dimensional regularization procedure is defined by the following replacement:

$$\int d^4p_4 \sum_{n \in \mathbb{N}} f(p_4, m_n) \rightarrow \frac{1}{2\pi i} \int_\odot d^Dp_5 \int d^4p_4 \mathcal{P}(p_5)f(p_4, p_5) \equiv (3)$$

where we have introduced the regulator functions $R_4(p_4)$ and $R_5(p_5)$ with regulators $D_4$ and $D_5$ for the 4 dimensional and 5 dimensional integrations, given by

$$R_4(p_4) = \frac{2\pi^{\frac{1}{2}(D_4)}}{\Gamma\left(\frac{1}{2}D_4\right)} p_4^\frac{1}{2}(\frac{p_4}{\mu_4})^{D_4-4}, \quad R_5(p_5) = \frac{\pi^{\frac{1}{2}(D_5)}}{\Gamma\left(\frac{1}{2}D_5\right)} (\frac{p_5}{\mu_5})^{D_5-1}. \quad (4)$$

Here we have introduced two (arbitrary) renormalization scales $\mu_4$ and $\mu_5$. The regulator function $R_4$ is the standard function for dimensional regularization of 4 non-compact dimensions [16]. The motivations for the regulator function $R_5$ are the following: it should reduce to unit when $D_5 \rightarrow 1$. The straight line $-\infty < p_5 < \infty$ becomes a sphere with infinite radius in more (integer) dimensions, therefore one expects the usual volume factor $\pi^{\frac{1}{2}(D_5)}/\Gamma\left(\frac{1}{2}D_5\right)$. To define the regulator function $R_5$ we need to introduce the complex logarithm which has a branch cut. Because the contour $\odot$ does not contain the real line, it is convenient to take it along the negative real axis.

For the convergent sum (2), that serves at the inspiration for our regularization procedure, it is irrelevant in which order the sum is performed. In more mathematical words: for any bijection $P : \mathbb{N} \rightarrow \mathbb{N}$, we have that

$$\sum_{n \in \mathbb{N}} f(p_4, m_n) = \sum_{n \in \mathbb{N}} f(p_4, m_{P(n)}). \quad (5)$$

In physics such bijections can be interpreted as symmetries of the spectrum. An important property of the regularization prescription in (3) is that it preserve such symmetries. The reason for this is that the sums are determined by the pole functions $\mathcal{P}$ and $\mathcal{P}_P$, where the latter is determined by the set of poles $\{m_{P(n)} \mid n \in \mathbb{N}\}$. But since this set and the original set $\{m_n \mid n \in \mathbb{N}\}$ that determines $\mathcal{P}$ are identical, it follows that $\mathcal{P}_P = \mathcal{P}$. Note that such bijections cannot be interpreted on the level of the complex integration variable $p_5$.

This procedure regulates the sum and the integral separately and the prescription is independent of the mass structure of the KK theory that is encoded in the pole function $\mathcal{P}$. The properties of the pole function determine the structure of possible divergences: in the next section we make a distinction between 5 and 4 dimensional divergences and finite contributions. The pole function $\mathcal{P}$ contains information about the physical momentum spectrum; the regulator functions are just to make standard manipulations with divergent sum-integrals meaningful.
3 Properties of the pole function

Up to now the discussion was general in the sense that it did not specify the structure of the compact manifold. We now restrict the attention to the circle $S^1$ and the orbicircle $S^1/\mathbb{Z}_2$, to be able to exemplify important properties of the pole function.

The mode functions are $\phi_n(x_5) = e^{inx_5/R}$, $n \in \mathbb{Z}$, for $S^1$ and $\phi^+_n(x_5) = \cos \frac{n\pi x_5}{R}$, $n \geq 0$, and $\phi^-_n(x_5) = \sin \frac{n\pi x_5}{R}$, $n > 0$ for $S^1/\mathbb{Z}_2$. The corresponding eigenvalue momentum can be read off easily. It is straightforward to check that the pole function for $S^1$ is given by $P_0(p_5)$, with

$$P_\omega(p_5) = \frac{\pi R}{\tan \pi R(p_5 - \omega)}. \quad (6)$$

This more general form of the pole function $P_\omega(p_5)$ describes a momentum spectrum that is shifted by $\omega$. This form we use in the calculation of the effective potential discussed in section 5.

As the pole function is determined by the KK momentum, it is clear that it reflects the symmetries of the KK towers and that these symmetries are not broken by this regularization procedure. In the case of a compactification on a circle, the shifts over $2\pi R$ leave the manifold invariant, hence the momentum in the 5th dimension is quantized to be elements of $\mathbb{Z}/R$. The set of integers are invariant under shifts by any integer. These shifts are symmetries of the pole function $P_0$ and hence respected by the dimensional regularization procedure applied to the compact dimension, since they define bijections $P_m(n) = n + m$ with $m$ a given integer.

For the pole functions of the orbicircle a slight complication arises because the integer $n$ is either non-negative or positive. However, since we assumed that $f$ is a function of $p_5^2$ it is symmetric under the interchange of $n \rightarrow -n$, therefore may take the sum over $n \in \mathbb{Z}$ instead and divide by 2. Only for $n = 0$ we have to be a bit more careful: if $n = 0$ is included in the sum, $\frac{1}{2}p_5$ has to be added to give the pole $\frac{1}{p_5}$ residue 1, while if $n = 0$ is not included $\frac{1}{2}p_5$ has to be subtracted. This gives

$$P^\pm(p_5) = \frac{1}{2} \left( \pm \frac{1}{p_5} + P_0(p_5) \right) = \frac{1}{2} \left( \pm \frac{1}{p_5} + \frac{\pi R}{\tan \pi R p_5} \right). \quad (7)$$

The pole function $P_\omega$ has important properties that can be used in identifying the structure of divergences that can occur. We define the remainder $\rho_\omega \pm$ of $P_\omega$ by

$$P_\omega(p_5) = \mp i\pi R + \rho_\omega \pm(p_5). \quad (8)$$

It is not difficult to show that the remainder is bounded by

$$\pi^2 R^2 \left(1 \mp \tan x\right)^2 \leq |\rho_\omega \pm(p_5)|^2 \leq \pi^2 R^2 \frac{(1 \mp \tanh x)^2}{\tanh^2 x}, \quad (9)$$

with $x = \pi R \text{Im} p_5$. From these inequalities the following important bounds can be derived. Using the assumption that there is an $\epsilon > 0$ such that $|x| = \ldots$
\[ \pi R |\text{Im} \rho_5| > \epsilon \] one obtains

\[ \pi^2 R^2 e^{-4|x|} \leq |\rho_{\omega \pm}(\rho_5)|^2 \leq \pi^2 R^2 \frac{1}{\sinh^2 \epsilon} e^{-4|x|}, \]  

where for \( x > \epsilon \) the bounds apply to \( \rho_{\omega +} \) and for \( -x > \epsilon \) the bound apply to \( \rho_{\omega -} \), respectively.

With this result we can now identify the different possible divergences that can occur. As an example, we consider the sum-integration of the function \( f \) for the even mode functions \( \phi_n^+ \) on the orbicircle

\[ I = \int d^4 p_4 \sum_{n \geq 0} f(p_4, \frac{n}{R}) \rightarrow \frac{1}{2\pi i} \int d^D p_4 \int d^D p_5 \mathcal{P}^+(p_5) f(p_4, p_5). \]  

The dimensionally regulated sum-integral now naturally splits into three parts

\[ I = I_{5D} + I_{4D} + I_{finite}. \]  

The constant parts \( \pm \frac{1}{2} i \pi R \) of the pole function \( \mathcal{P}^+ \), see (8), give the contributions

\[ I_{5D} = \frac{1}{2\pi i} \frac{2 \pi^{\frac{1}{2}(D_4+D_5)}}{\Gamma(\frac{1}{2}D_4)\Gamma(\frac{1}{2}D_5) \mu_4^{D_4-4} \mu_5^{D_5-4}} \int_0^\infty dp_4 p_4^{D_4-1} \int_{-\infty}^{\infty} dp_5 p_5^{D_5-1} \left\{ p_5^{D_5-1} f(p_4, p_5) \left( -\frac{i\pi R}{2} \right) + e^{i\pi (e^{i\pi} p_5)} f(p_4, e^{-i\pi} p_5) \left( \frac{i\pi R}{2} \right) \right\}. \]  

Here we used that it is possible to take the real part of \( D_5 \) small enough such that the arc contours at infinity do not contribute. Since in \( f \) only \( p_5^2 \) appears, the factor \( e^{i\pi} \) is irrelevant for the insertion of \( p_5 \) in \( f \). We can restrict the integration over \( p_5 \) to \( 0 < p_5 < \infty \) by including an additional factor 2. Since the function \( f \) only depends on \( p_4 \) and \( p_5 \) via the combination \( p_4^2 + p_5^2 \), we can employ a change of variables \( p_5 = pt^{\frac{1}{2}} \), \( p_4 = p(1 - t)^{\frac{1}{2}} \) with \( 0 < p < \infty \) and \( 0 < t < 1 \). Combining this with the standard result

\[ \int_0^1 dt \ t^{\alpha - 1} (1 - t)^{\beta - 1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \]  

the divergent part finally reads

\[ I_{5D} = -\frac{1}{4} + e^{i\pi (D_5-1)} \frac{R}{\mu_4^{D_4-4} \mu_5^{D_5-1}} \frac{2 \pi^{\frac{1}{2}(D_4+D_5)}}{\Gamma(\frac{1}{2}(D_4 + D_5))} \int_0^\infty dp \ p^{D_4+D_5-1} f(p). \]  

Here we denote with \( p = (p_4, p_5) \) the full 5 dimensional momentum. This part behaves identically to a purely 5 dimensional integral since depends on \( D_4 + D_5 \) only. Therefore, its possible divergence has a 5 dimensional character. Notice that this calculation shows that the sum-integral can also be regulated using either \( D_4 \) or \( D_5 \). Actually, in dimensional regularization this contribution is regulated to be finite even if the regulators are taken away. However, this is just an artifact of dimensional regularization: if it is applied to a Minkowski space-time of odd dimension, all regulated integrals are finite [3].
Next we investigate the contribution due to the term $\frac{1}{2} \frac{1}{p_5}$ in the pole function $P^+$. As this term is a pole function that just gives only a contribution at $p_5 = 0$ there is no need to regulate the sum here anymore. It is similar to a contribution of a single particle. Therefore, we set $D_5 = 1$, perform the integration over contour $\equiv$, and obtain a purely 4 dimensional integral

$$I_{4D} = \frac{1}{2} \frac{1}{\mu_4^{D_4-4}} \frac{2\pi^{D_4}}{\Gamma(D_4)} \int dp_4 p_4^{D_4-1} f(p_4,0). \tag{15}$$

Finally, due to $\frac{1}{2} \rho_0 \pm$ in the pole function (8) we obtain the contribution

$$I_{\text{finite}} = \frac{1}{2\pi i} \int d^{D_4} p_4 \int d^{D_5} p_5 \frac{1}{2} \rho_0 \pm (p_5) f(p_4, p_5), \tag{16}$$

where we sum over $\pm$. The poles $p_{5 \pm i} = \sqrt{(p_4 - k_i)^2 + m_i^2}$ of the function $f(p_4, p_5)$ in $p_5$ are labeled by the finite number of different pole pairs. Possible external 4 dimensional momenta are denoted by $k_i$ and $m_i^2$ are some masses that appear in the propagators. The contour around $\ominus$ contains all these poles. Therefore we get contributions with factors $\rho_0 \pm (p_{5 \pm})$. We can conclude that these contributions are finite: integrating the bounds on the integrants (9) give convergent results. As we only get a finite number of pole contributions, we conclude that this contribution $I_{\text{finite}}$ is indeed finite. This means we can take $D_4 = 4$ and $D_5 = 1$ to calculate this contribution.

## 4 Inclusion of fermions

The procedure we described so far applies to bosons only. The extension of this formulation with fermions is very similar to the situation in the well-known 4 dimensional case. We first review this briefly and then turn to the generalization to 5 dimensions of which one is compact.

There are three basic ways of regulating Clifford algebra properties with the extension of the momentum integrals to arbitrary complex dimensions [15]: naive dimensional regularization, dimensional regularization and dimensional reduction. The naive dimensional regularization scheme just introduces additional gamma matrices that anti-commute with each other and the 4 dimensional gamma matrices including $\gamma^5$.

In dimensional regularization the 4 dimensional Clifford algebra is extended to $D$ gamma matrices $\gamma^\mu$ of which the first 4 are the original gamma matrices in 4 dimensions. They anti-commute to the generalized metric in $D$ dimensions. However, the chirality operator $\gamma^5$ has a special role [14]: although it anti-commutes with the original gamma matrices, it commutes with the additional ones. This treatment is fully consistent [17] and produces the axial-anomaly.

In dimensional reduction the Clifford algebra and the spinor traces are worked out in 4 dimensions. The remaining momentum integrals are extended to $D$ dimensions is the standard way. This procedure may be used if only even parity fermionic loops are present [18]. Since the spinor properties remain unchanged, this regularization scheme is well suited for supersymmetric calculations [19].
We return to the situation in 5 dimensions. If the theory has 5 non-compact dimensions, we can just apply any procedure, described above. In (naive) dimensional regularization now the 5 gamma matrices are extended to $D$. However, we do not have the difficulty with the chirality operator as is trivial in odd dimensions. On the contrary, if the 5th dimension is compactified, the 5 dimensional Lorentz invariance is broken, hence $\gamma^5$ has a special role. This means that we are in a similar situation as in 4 dimensions. Therefore, using dimensional reduction the treatment is the same as in 4 dimensions.

We now investigate how dimensional regularization of the Clifford algebra can be generalized to included a compact dimension. In principle, we can now introduce both additional gamma matrices for both the 4 dimensional Minkowski space and the compact manifold. However, similar consistency arguments as those presented in [17] apply. Depending on whether the additional gamma matrices for the compact dimension $\gamma^\perp_a$ commute or anti-commute with $\gamma^5$, we find

$$ (D_5 - 1) \text{tr} \, \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5 = \text{tr} \, \gamma^\perp_a \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5 = \pm \text{tr} \, \gamma^\perp_a \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5. \quad (17) $$

In the anti-commuting case (where we pick up the minus) we see that for all dimensions $D_5 \neq 1$ we find that the trace $\text{tr} \, \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5$ has to vanish. As we certainly want to avoid this, we have to make the commuting choice. This means that there is no distinction between the additional gamma matrices for the 4 non-compact dimensions and the 1 compact dimension, so that as far as the gamma algebra in dimensional regularization is concerned it goes exactly the same as in the standard 4 dimensional case.

5 Example: an effective one-loop potential

A typical form of an effective potential with a KK tower formally reads

$$ V_\omega = \int \frac{d^4 p_4}{(2\pi)^4} \sum_{n \in \mathbb{Z}} \ln \left[ p_4^2 + \left( \frac{n}{R} + \omega \right)^2 + m^2 \right] $$

$$ = \frac{1}{2\pi i} \int_\mathbb{C} dp_5 \int d^4 \frac{d^4 p_4}{(2\pi)^4} \mathcal{P}_\omega(p_5) \ln (p_4^2 + p_5^2 + m^2), \quad (18) $$

where $R$ is the compactification radius, $\omega$ a shift parameter in the lattice of five dimensional momenta. In addition, $m$ is a mass parameter that either represents a physical mass or the IR regulator introduced to shift the poles away from the real axis, as discussed above. This type of mass spectrum $\left( \frac{n^2}{R^2} + \omega^2 \right) + m^2$ was investigated in [1, 2] for compactification of the fifth dimension on an orbicircle $S^1/\mathbb{Z}_2$ or $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2'$.

The techniques of manipulation of the pole functions like the one discussed in section 2 can be used here as well, therefore we just quote results here. The 5 dimensional divergent part of the effective potential reads

$$ V_{\omega, 5D} = -\sin^2 \frac{\pi}{2} D_5 \frac{\pi \frac{1}{2} (D_4 + D_5)}{(2\pi)^{D_4}} \Gamma(-\frac{1}{2}(D_4 + D_5)) \frac{R m^{D_4 + D_5}}{\mu_{D_4 - 4} \mu_{D_5 - 1}}. \quad (19) $$
Note that this result is independent of the momentum lattice shift parameter \( \omega \), because the part \( \mp i \pi R \) of the pole function (8) which is responsible for the 5 dimensional divergence does not depend on \( \omega \). In the limit \( D_4 \to 4 \) and \( D_5 \to 1 \) we find a finite result. As explained in our general discussion in section 3, this is just an artifact of dimensional regularization.

Apart from this 5 dimensional divergent part, we obtain a finite part

\[
V_{\omega\ finite} = -\frac{2\pi^{\frac{1}{2}(D_4+D_5)}}{(2\pi)^{D_4}} \frac{\Gamma(D_4 + D_5)}{\Gamma(\frac{1}{2}D_4 + 1)\Gamma(\frac{1}{2}D_5)} \frac{(2\pi R)^{1-D_4-D_5}}{\mu_4^{D_4-1}\mu_5^{D_5-1}} \text{K}(\omega R).
\] (20)

Here the poly-logarithm \( L_\sigma(z) = \sum_{n \geq 1} \frac{z^n}{n^\sigma} \) is introduced in the definition of the function

\[
\text{K}(\omega R) = e^{i\frac{\pi}{4}(D_5-1)}L_{D_4+D_5}(e^{-2\pi i \omega R}) + e^{-i\frac{\pi}{4}(D_5-1)}L_{D_4+D_5}(e^{2\pi i \omega R}).
\] (21)

We conclude that the effective potential \( V_\omega \) has a quintic divergence and a finite part; a quadratic divergence is absent [21, 24]. In the model of [1] the boson and fermion masses are related by \( \omega_F = \omega_B + \frac{1}{2} \). Then in the effective one loop potential the overall 5 dimensional divergent term (19) cancels and we find the finite contribution

\[
V_{\text{eff}} = V_{\omega B\ finite} - V_{\omega F\ finite}.
\] (22)

6 Conclusion

The main purpose of this paper was to extend the dimensional regularization procedure to a space-time with compact dimensions. In particular we focused on \( M^4 \times C \), with \( C \) a compact one dimensional manifold. Since the topology and geometry of the 5th dimension is different from the 4 dimensional Minkowski space, two regulators may in principle be needed to regularize such a theory. From experience with un-compact dimensions we have learned that dimensional regularization is a powerful procedure. However, it cannot be applied directly to a compact dimension because we are confronted with sums rather than integrals over the momentum. Using complex function theory and a “pole function” (that has poles at the KK momenta), the sum over the KK tower can be turned into a contour integral, which may be extended to an arbitrary complex dimension in a manner inspired by dimensional regularization. In this way we have constructed a regularization prescription that carefully treats the additional dimension separately, just as it was done in ref. [21], without any prejudice of how the regulators should be related. But at the same time our regularization prescription reflects the properties of the KK towers since they are encoded in a regularization independent pole function. The latter was not respected by putting a momentum cut-off on the KK sums in their prescription, unless the cut-off for the sum was taken much larger than the cut-off for the 4 dimensional momentum.

The extension of this method so as that fermion loops can also be treated. Just like in 4 dimensions, there are various prescriptions possible. We discussed
dimensional reduction where the gamma algebra and the traces are performed in 5 dimensions and after that the momentum sum and integrals are regularized using dimensional regularization. Secondly, we discussed dimensional regularization of the Clifford algebra. In order not to run into an inconsistency, it was necessary that the additional gamma matrices for the compact dimension commute with $\gamma^5$ just like the additional gamma matrices in 4 dimensions.

Using the dimensional regularization procedure of both compact and non-compact dimensions, the possible types of divergences can be identified: there can be true 5 dimensional divergences, 4 dimensional divergences and finite contributions. This is encoded in the asymptotic behavior of the pole function: for the orbifold $S^1/Z_2$, for example, the pole function has a constant, a single pole and an exponentially suppressed part for the 5 dimensional momentum going to $i\infty$; these correspond to the afore mentioned 5, 4 dimensional divergent and finite parts. Because this method makes transparent the structure of divergences that can occur and what the nature of these are; it also easier to understand why they may cancel in certain situations.

As an example of this the effective one-loop potential due to Yukawa interactions in the model of ref. [1] was calculated using this prescription. For both the bosonic and fermionic parts of this potential we found the same 5 dimensional divergent piece but different finite parts. The fact that there is an equal number of bosonic and fermionic towers (a left over of the $N = 2$ supersymmetry that was present before the orbifolding) the divergent pieces cancel out. We should stress that this one-loop calculation does not show that is free of divergences to all orders, or even at one-loop since only Yukawa couplings are taken into account. Higher order corrections, like the two-loop Yukawa corrections to the potential calculated in ref. [23] are divergent. However, their result seems to show that the superpotential on the branes does not renormalize as they are able to absorb the linear divergence by wave function renormalization. In ref. [22] we show that the Fayet-Iliopoulos tadpole can be quadratically divergent using methods discussed in the paper.

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