Cosmic string formation and the power spectrum of field configurations

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Abstract

We examine the statistical properties of defects formed by the breaking of a U(1) symmetry when the Higgs field has a power spectrum \( P(k) \propto k^n \). We find a marked dependence of the amount of infinite string on the spectral index \( n \) and empirically identify an analytic form for this quantity. We also confirm that this result is robust to changes in the definition of infinite string. It is possible that this result could account for the apparent absence of infinite string in recent lattice-free simulations.

I. INTRODUCTION

Cosmic strings formed at a primordial phase transition are strong candidates as seeds for large-scale structure in the universe and anisotropies in the microwave background [1,2]. Firm predictions are difficult to extract from string models, due to both the uncertainty in the statistical properties of the string network at formation and the non-linearity of network evolution. Information from numerical simulations has been the main tool for calculations to date. The observational consequences of the string scenario may be strongly dependent on whether, in addition to a scale-invariant distribution of loops, any infinite string exists. The presence or absence of infinite string may lead the string network into significantly different scaling solutions [3], and in particular to solutions that studies of large-scale structure will be able to distinguish between.

Lattice-based simulations have traditionally been employed to predict the initial fraction as infinite string, although recent work has used a lattice-free method to simulate string formation in a first order phase transition [4]. The simplest approach, however, is to assume that the string-forming field takes up random values at each site on a regular lattice, each representing a causally disconnected volume, and look for strings at each face of the lattice. Vachaspati and Vilenkin [5] first performed this on a cubic lattice and found that

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approximately 80% of string was in ‘infinite’ form. Further work on a tetrahedral lattice by Hindmarsh and Strobl demonstrated a slightly lower figure \[6\]. Furthermore, Vachaspati \[7\] has shown that for theories without an exact U(1) symmetry (i.e. one field orientation is statistically favoured) it is possible to reduce the fraction as infinite string to zero. Another approach \[8\], retaining the symmetry of the theory but introducing a variation in the ranges over which the field is correlated by means of a simple domain-laying algorithm, showed a dependence of the fraction as infinite string on the variance of domain volume. Here we present a new single-parameter method for varying the scale over which the string field is correlated to investigate the behaviour of the amount of infinite string present. It is a controlled means of varying the string-forming field configuration and allows us to employ analytic results concerning defect densities. In addition, the method makes it possible to explore a range of physically-motivated theories in which string-like defects arise \[9\], and in particular the role of causality in determining the properties of a cosmic string network on very large (super-horizon) scales.

In section 2 we discuss our method for generating string configurations and present our basic results regarding the fraction as infinite string. In section 3 we compare the total density of string to a theoretical prediction of the density of zeroes of the string-forming field. In section 4, we suggest a functional form for the relation between the power spectrum of the string field and the density of infinite string. In section 5 we confirm that our results are insensitive to changes in the definition of infinite string, and section 6 presents our conclusions.

II. THE METHOD

We construct our Higgs field \(\phi(x)\), on a lattice with periodic boundary conditions, as a realisation of gaussian random field with a simple power spectrum \(P(k) \propto k^n\), such that

\[
< \tilde{\phi}(k) \tilde{\phi}^*(k') > = (2\pi)^3 P(k) \delta^{(3)}(k - k'),
\]

where

\[
\tilde{\phi}(k) = \frac{1}{(2\pi)^{3/2}} \int d^3x e^{-ik \cdot x} \phi(x).
\]

The real and imaginary parts of each Fourier component \(\tilde{\phi}(k)\) are assigned values chosen from a gaussian distribution with mean zero and variance \(P(k)/2\). The phase of the field \(\phi(x)\) is extracted at each lattice site and windings are located. Throwing away the modulus of the field only affects the power spectrum on scales larger than the correlation length by a multiplicative factor. \[1\] Strings are paired up at random within each lattice site and the

\[1\] This can be understood qualitatively as follows. The moduli of the \(\phi(x)\) will be Rayleigh distributed with a finite mean. If we rebuild \(\tilde{\phi}(k)\) from just the phase of \(\phi(x)\), it is analogous to performing a walk on the complex plane with a fixed step length. This will give a result for \(|\tilde{\phi}(k)|\) that differs from that obtained by a walk with a variance in the step length by a multiplicative factor, independent of \(k\).
lengths of the string loops determined. The boundary conditions ensure that all string is closed.

We find that, as expected, there is a distribution of loops with a characteristic power-law density,

\[ \rho_{\text{loop}}(l) \propto l^{-5/2}, \]

where \( l \) is the length of a loop, which can be derived on dimensional grounds, assuming that the statistical properties of the loop distribution are independent of scale [5].

In addition we find a number of much longer loops, winding around the lattice several times. On a periodic lattice, it can be shown [10] that these strings belong to a distribution whose density falls off as \( l^{-1} \) and are an artifact of the boundary conditions. It is commonly believed that these represent a separate population of ‘infinite’ strings — henceforth we use the word ‘infinite’ to refer to strings which would never self-intersect in the infinite-volume limit.

The standard method for distinguishing between the loop distribution and the infinite string has been to introduce a cut-off in length that is of order \( N^2 \), where \( N \) is the size of the box in lattice units. The reason for choosing such a cut-off is that string greater than this size is likely to be “topological”, that is, to traverse opposite boundaries of the simulation. All loops longer than the cut-off are then classified as infinite. Given this standard definition of infinite string, figure 1 shows the variation of the infinite string fraction \( f_\infty \) with spectral index \( n \).
III. DEPENDENCE OF TOTAL STRING DENSITY ON SPECTRAL INDEX

We have presented the basic findings regarding the variation of the amount of infinite string with spectral index in figure 1. Without any analytical explanation of the form of the curves, we can however gain insight into some of their features by considering an analytical result for the density of zeroes $\rho_t$ on a two dimensional slice through our field. Halperin [11] shows that

$$\rho_t = \left| \frac{W(0)''}{W(0)} \right|,$$

where $W(x)$ is the correlation function, defined as

$$W(x) = \langle \phi(x)\phi^*(0) \rangle.$$

This result gives us the number density of string crossings in a plane. Naively, by extension to three dimensions and knowing that our string segments lie only in the three principal directions, we can infer that the total length of string in the box is proportional to $\rho_t$.

The relationship between the correlation function and the power spectrum is easily derived:
\[ W(x) = \left< \frac{1}{(2\pi)^3} \int d^3k \, d^3k' \, \bar{\phi}(k) \phi^*(k') e^{i k \cdot x} \right> \]

\[ = \int d^3k \, e^{i k \cdot x} P(k) \]

\[ = 4\pi \int dk \, k^2 \frac{\sin kx}{kx} P(k). \]

For power law spectra \( P(k) = k^n \), we have

\[ W(x) = 4\pi \int d\kappa \, \frac{\sin \kappa x}{\kappa} \kappa^{n+2}. \]

We can extract the spatial dependence with the substitution \( \kappa = kx \). Then

\[ W(x) = 4\pi \int d\kappa \, \frac{\kappa^{n+2} \sin \kappa}{\kappa^{n+3}} = f(n)x^{-(n+3)}. \]

We can see that a spectral index of \(-3\) implies a spatially homogeneous correlation function and thus a totally uniform field. Any value less than \(-3\) gives rise to a correlation function divergent at large \( x \), which is unphysical.

Increasing \( n \) decreases the long-range correlation with respect to short-range. The \( n = 0 \) case corresponds to white noise — \( \phi(x) \) is totally uncorrelated and is random at each lattice point. This corresponds to the original scenario considered by Vachaspati and Vilenkin, and leads to approximately 80\% 'infinite' string, which is confirmed in figure 1.

In practice, we construct our field with a number of momentum modes ranging from \( k_{\text{min}} = 2\pi/N \) to \( k_{\text{max}} = 2\pi \). From our form for the correlation function \( W \), we obtain an expression for the defect density

\[ \rho_t \simeq \frac{\int_{k_{\text{min}}}^{k_{\text{max}}} dk \, k^{n+4}}{\int_{k_{\text{min}}}^{k_{\text{max}}} dk \, k^{n+2}} \]

\[ = \left( \frac{k_{\text{max}}^{n+5} - k_{\text{min}}^{n+5}}{k_{n+3}^{n+3} - k_{n+3}^{n+3}} \right) \left( \frac{n+3}{n+5} \right). \] (2)

This function is plotted against \( n \) for various values of \( k_{\text{max}}/k_{\text{min}} \) in figure 2. It can be seen that in the infinite-volume limit, the total defect density drops to zero for values of \( n \) lower than \(-3\), as expected. The presence of string at this value of \( n \) and below is a finite size effect, and consequently the form of every plot presented in this paper for \( n < -3 \) is not significant.
FIG. 2. The dependence of $\rho_t$ on $n$, where $\rho_t$ is plotted in units of $(2\pi \times \text{lattice spacing})^{-2}$.

When we normalise the expression above to the data from the simulations, we obtain the forms shown in figure 3, which are in reasonable agreement. We note, however, that there is no simple relationship between Halperin’s result for the density of zeroes in the continuum limit and our estimation of the total length of string, assumed to lie on the dual lattice. It is not surprising that there are small discrepancies between the prediction and the data.
FIG. 3. The points show the total density of string, again averaged over 10 realisations in the \( N = 64 \) case and 4 in the \( N = 128 \). The curves show the analytic prediction, normalised by hand. Density is in arbitrary units. Errorbars are too small to appear in the plot.

IV. DEPENDENCE OF INFINITE STRING DENSITY ON SPECTRAL INDEX

In figure 4 we plot the amount of string in infinite form and in loops. We then make the observation that the former curve can be fitted extremely well with a modified version of the Halperin result for total defect density, plotted in figure 2, by replacing \( n \) with \( n - 1 \) in equation 2. That is,

\[
\rho_{\infty} = \left( \frac{k_{\text{max}}^{n+4} - k_{\text{min}}^{n+4}}{k_{\text{max}}^{n+2} - k_{\text{min}}^{n+2}} \right) \left( \frac{n + 2}{n + 4} \right).
\]

Figure 4 shows that the expression gives the correct dependence of \( \rho_{\infty} \) on \( n \). The volume-dependence of \( \rho_{\infty} \) at a given value of \( n \) is confirmed for the cases \( n = -3 \) and \( n = -1.5 \) in figure 5. In the \( n = -3 \) case, we can clearly see the density of infinite string vanishing in the infinite volume limit. Note that we introduce the volume of the lattice through the relation

\[
\frac{k_{\text{max}}}{k_{\text{min}}} = N.
\]

In the context of our definition of the correlation function, equation 2 may be rewritten as

\[
\rho_{\infty} = \frac{\int dk \ k^3 P(k)}{\int dk \ k P(k)} = \left| \frac{C''(0)}{C(0)} \right|,
\]
here cast in the same form as the Halperin result (equation 1), with $C(x)$ satisfying

$$\frac{dC}{dx} = W(x).$$

There is no direct evidence that equation 4 holds for a general $P(k)$ — this is the subject of work in progress [12]. It has two interesting properties, however, that support this suggestion. We would expect that any expression relating to infinite string would be non-local, in that it would have to take account of global properties of the field correlation function. This indeed is the case — the denominator involves the integral of the correlation function $W(x)$. Secondly, it is straightforward to confirm that for any power spectrum $P(k) \geq 0$ for all $k$, and using the expression above for $\rho_\infty$,

$$0 \leq \frac{\rho_\infty}{\rho_t} \leq 1.$$

We can then calculate the amount of string that we expect to find in loops by subtracting this modified Halperin result from the original expression for the total string density;

$$\rho_{\text{loops}} = \rho_t - \rho_\infty.$$

We see that the basic features of the loop distribution are reproduced by this expression. The major discrepancy exists for positive values of $n$, where the observed loop density tends to a finite value whilst the predicted density falls to zero. This is not entirely surprising: one artifact of working on a cubic lattice is that a given distribution of phases does not uniquely determine the length distribution of strings. In cells where four or six ends of string meet, we must make some choice about how to pair these ends up, and different choices will give rise to different amounts of infinite string and loops. This non-uniqueness, which we reiterate is purely a consequence of the cubic lattice, will become more important at high values of $n$. There the lattice becomes virtually filled with string, and cells within which the pairings are undetermined are more frequent. There will always be a non-zero probability of forming a loop of string in a full lattice when a random pairing scheme is implemented.
FIG. 4. The data points show numerical results for the density of infinite string $\rho_\infty$ (crosses) and loops $\rho_{\text{loop}}$ (solid squares). The solid and short-dashed curves show empirical fits to this data. The long-dashed line shows our analytic prediction for $\rho_\infty$ in the infinite-volume limit.
FIG. 5. The lower set of points show the density of infinite string at $n = -3$ for varying lattice sizes $N$. The upper set correspond to $n = -1.5$. The fitted curves, which are normalised individually by hand, show the prediction from equation 3.

If we are to believe the empirically-identified analytic form for $\rho_\infty$, we now see that the existence of infinite string for values of $n$ below $-2$ is purely a finite-volume effect. In the infinite-volume limit, at $n = -2$ and below there will be no infinite string, as shown clearly by the long-dashed curve in figure 4. The loop density will develop a discontinuity in its derivative with respect to $n$ at $n = -2$.

As we reduce the string density we find that the infinite string density $\rho_\infty$ also decreases, and will go to zero at some non-zero value of the total string density $\rho_t$. This may well be analogous to the findings of Vachaspati [7], in which he observes a similar decrease in $\rho_\infty$ with $\rho_t$, and a phase transition in which the infinite string vanishes at a critical and non-zero value of the string-forming probability. In addition he finds that the loop density also peaks sharply around the critical density. This similarity is particularly interesting as the means of varying the string density is fundamentally different to the present case — we preserve the underlying symmetry of the string-forming field, whereas Vachaspati introduces an explicit breaking of the U(1) symmetry.

Further, we can now explain the upturn in the plots of the variation of $f_\infty$ with $n$ that occurs below $n = -3$, as shown in figure 1. Using

$$f_\infty = \frac{\rho_\infty}{\rho_t},$$

the result is shown in figure 6.
FIG. 6. Prediction for the infinite string fraction using the empirical forms for the infinite string and loop densities, assuming a box size of $N = 128$.

V. THE DEFINITION OF INFINITE STRING

We have found that the fraction $f_\infty$ of infinite string, defined as loops of length $\gtrsim O(N^2)$, is strongly dependent on the spectral index $n$. This relationship could be a genuine indication of variation of $f_\infty$ with $n$. However, we have also shown that $f_\infty$ is a monotonically increasing function of string density for $n$ greater than $-3$. We must therefore investigate the robustness of our definition of infinite string to variations in string density before concluding that there is a fundamental dependence of $f_\infty$ on $n$.

In the Vachaspati-Vilenkin ($n = 0$) case, the strings perform approximately Brownian random walks with a step length $\xi$ of one lattice unit. A topological string, crossing a lattice of side $N$, will thus have a typical minimum length $\sim N^2$. As we decrease the spectral index, introducing more correlations on larger scales, we smooth the field and thus the string network over more lattice spacings. The step length $\xi$ is effectively increased and length of a string which crosses the box is reduced. If we define infinite string by means of a cut-off $L$, where $L$ is the length of a string that is likely to traverse the box, then we have

$$L = \xi \left( \frac{N^2}{\xi^2} \right) \propto \frac{1}{\xi}.\)$$

This suggests that the cut-off should be varied to take account of the defect density and thus $\xi$. In figure 7 we plot the fraction as string present in loops less than a certain size $L$ for various values of the spectral index. Each of these curves exhibits a distinct turnover, which corresponds to the onset of topological string. We see that the value of $L$ at this point
does indeed decrease with \( n \), and so with total string density. Consequently, by choosing our cut-off in the naive density-independent way we are introducing a systematic error which will undercount long string for lower values of \( n \).

![Diagram](image.png)

**FIG. 7.** The cumulative loop distribution (jagged lines) for three values of the spectral index, for an \( N = 64 \) lattice. The curves show the contribution to the total amount of string we would expect from a scale-invariant distribution of loops. They were fitted to the data for loops of length less than a limit \( N \), below which we expected the \( l^{-5/2} \) behaviour to hold true. The final results were insensitive to this limit — a value of \( N^2 \) resulted in an almost identical plot.

An alternative scheme [4] for estimating the fraction as infinite string is as follows. Let the total length of string in the box be \( T \). We know that a scale-invariant distribution of loops exists at small loop length \( l \). We can then calculate the total length \( T_{\text{loop}}(L) \) we would expect purely from this distribution extrapolated to an arbitrary length \( L \). We have

\[
T_{\text{loop}}(L) = \int_{l_{\min}}^{L} l \rho_{\text{loop}} \, dl
= \int_{l_{\min}}^{L} Al^{-3/2} \, dl
= 2A(l_{\min}^{-1/2} - L^{-1/2}).
\]

\( A \) and \( l_{\min} \) are obtained by a least-squares fit to data at small \( L \). The fitted curves in figure 7 show the contribution of \( T_{\text{loop}}(L) \) to the total amount of string. The fraction as infinite string is then defined as

\[
f_{\infty} = 1 - \frac{T_{\text{loop}}(\infty)}{T}.
\]
VI. CONCLUSION

We have demonstrated a strong dependence of the fraction as infinite string formed during a $U(1)$ symmetry breaking phase transition on the power spectrum of the Higgs-field configuration. For a power spectrum $P(k) \propto k^n$, we find agreement with an analytic formula for variation of the total density of string with $n$. We find a strikingly simple closed-form expression which fits the density of infinite string and is closely related to the analytic expression for the total string density. We find that the fraction as infinite string falls with $n$, and vanishes at $n = -2$ if the formula for the density of infinite string is to be believed. Between $n = -2$ and $n = -3$, string only exists in loop form, and below $n = -3$ the total string density is zero. We have also shown that the dependence of the fraction as infinite string with $n$ is robust to reasonable changes in our definition of infinite string.

For $n = 0$, the value of the Higgs field at each lattice site is independent, and as expected we reproduce the Vachaspati-Vilenkin result of approximately 80% infinite string [5]. In a cosmological phase transition, we expect values of the Higgs field to be completely independent on scales larger than the causal horizon [13]. Such a field configuration has a spectral index $n = 0$ on the largest scales. Provided string we classify as infinite in our lattice
simulation does indeed correspond to infinite string in the continuum limit, we can confirm that a non-zero fraction as infinite string will be formed in a cosmological phase transition. The existence of this population of infinite string may have important consequences for the scaling solution of the evolving network, and hence for the observational consequences of cosmic string.

Our findings show that if we want to compute the fraction as infinite string in the universe correctly using numerical simulations, the field configuration must have the correct \( n = 0 \) form on the largest scales. Failure to fulfil this condition could severely under- or over-estimate the fraction as infinite string. In particular, the results of recent lattice-free simulations of a first-order phase transition [4] appear to be consistent with an infinite string fraction equal to zero. It is important to ensure that on large scales the power spectrum of the field configuration does have a spectral index \( n = 0 \), before applying this result in a cosmological context.

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