Enumeration of Unlabeled Outerplanar Graphs

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March 2, 2022

Abstract

We determine the exact and asymptotic number of unlabeled outerplanar graphs. The exact number $g_n$ of unlabeled outerplanar graphs on $n$ vertices can be computed in polynomial time, and $g_n$ is asymptotically $g n^{-5/2} \rho^{-n}$, where $g \approx 0.00909941$ and $\rho^{-1} \approx 7.50360$ can be approximated. Using our enumerative results we investigate several statistical properties of random unlabeled outerplanar graphs on $n$ vertices, for instance concerning connectedness, chromatic number, and the number of edges. To obtain the results we combine classical cycle index enumeration with recent results from analytic combinatorics.

Keywords: unlabeled outerplanar graphs, dissections, combinatorial enumeration, cycle index, asymptotic estimates, singularity analysis

1 Introduction and results

Singularity analysis is a highly successful tool for asymptotic enumeration of combinatorial structures \cite{11}, once we have a sufficiently good description of the corresponding exponential or ordinary generating functions. If we want to count unlabeled structures, i.e., if we count the structures up to isomorphism, the potential symmetries of the structures often require a more powerful tool than generating functions, namely cycle indices, introduced by Pólya \cite{22}. From the cycle index sums for a class of combinatorial structures we can obtain its generating function, and apply singularity analysis. However, when the cycle index sums are given only implicitly, it might be a challenging task to apply this technique. This is illustrated by the situation for planar graphs: the asymptotic number of labeled planar graphs was recently determined by Giménez and Noy \cite{14}, based on singularity analysis, whereas the enumeration of unlabeled planar graphs is a research problem left open for several decades \cite{29}.

In this paper we determine the exact and asymptotic number of unlabeled outerplanar graphs, an important subclass of the class of all planar graphs. We provide a polynomial-time algorithm to compute the exact number $g_n$ of unlabeled outerplanar graphs on $n$ vertices, and prove that $g_n$ is asymptotically $g n^{-5/2} \rho^{-n}$, where $g \approx 0.00909941$ and $\rho^{-1} \approx 7.50360$ can be approximated. Building on our enumerative results we derive typical properties of random unlabeled outerplanar...
graphs on \( n \) vertices (i.e. taken uniformly at random among all unlabeled outerplanar graphs on \( n \) vertices), for example connectedness, chromatic number, the number of components, and the number of edges.

Before we provide a more detailed exposition of the results obtained in this paper, we would like to give a brief survey on the vast literature on enumerative results for planar structures. The exact and asymptotic number of embedded planar graphs (i.e., planar maps) has been studied intensively, starting with Tutte’s seminal work on the number of rooted oriented planar maps \([26]\). The number of three-connected planar maps is related to the number of three-connected planar graphs \([19, 26]\), since a three-connected planar graph has a unique embedding on the sphere \([30]\). Bender, Gao, and Wormald used this property to count labeled two-connected planar graphs \([1]\), and Giménez and Noy recently extended this work to the enumeration of labeled planar graphs \([13]\).

The asymptotic number of general unlabeled planar graphs has not yet been determined, but has been studied for quite some time \([29]\). Moreover, no polynomial time algorithm for the computation of the exact number of unlabeled planar graphs on \( n \) vertices is known. Such an algorithm is only known for unlabeled rooted two-connected planar graphs \([5]\), and for unlabeled rooted cubic planar graphs \([6]\).

An outerplanar graph is a graph that can be embedded in the plane such that every vertex lies on the outer face. Such graphs can also be characterized in terms of forbidden minors \([9]\), namely \( K_{2,3} \) and \( K_4 \). The class of outerplanar graphs is often used as a first non-trivial test-case for results about the class of all planar graphs; apart from that, this class appears frequently in various applications of graph theory. The asymptotic number of labeled outerplanar graphs was recently determined in \([3]\). In this paper, we determine the number of unlabeled outerplanar graphs, i.e., we enumerate outerplanar graphs up to isomorphism.

Two-connected outerplanar graphs can be identified with dissections of a convex polygon \([31]\). General outerplanar graphs can be decomposed according to their degree of connectivity: an outerplanar graph is a set of connected outerplanar graphs, and a connected outerplanar graph can be decomposed into two-connected blocks. In the labeled case this decomposition yields equations that link the exponential generating functions of two-connected, connected, and general outerplanar graphs \([3]\). Once labeled dissections are enumerated, these equations yield formulas for counting outerplanar graphs. In the unlabeled case we use the same decomposition. However, the potential symmetries make it more difficult to obtain exact and asymptotic results. We have to use cycle index sums, which were introduced by Pólya for unlabeled enumeration \([22]\), to obtain implicit information about the ordinary generating functions of unlabeled outerplanar graphs. We then apply singularity analysis, a very powerful tool that is thoroughly developed in the forthcoming book of Flajolet and Sedgewick \([11]\). A similar strategy was applied by Labelle, Lamathe, and Leroux for the enumeration of unlabeled \( k \)-gonal 2-trees \([10]\). However, in the singularity analysis for outerplanar graphs we face new difficulties, which did not appear in other literature, as far as we know. The generating function for connected outerplanar graphs is defined implicitly by a multiset of connected outerplanar graphs that are rooted at a two-connected component, and moreover the number of two-connected graphs has exponential growth. By applying the singular implicit function theorem we overcome these difficulties (see Section 4.1 for the details) and estimate the asymptotic number of outerplanar graphs.

**Contributions.** Our first result is the extension of Read’s counting formulas \([23]\) for the number of unlabeled two-connected outerplanar graphs to counting formulas for the number of unlabeled outerplanar graphs.

**Theorem 1.1.** The exact numbers of unlabeled two-connected outerplanar graphs \( d_n \), unlabeled connected outerplanar graphs \( c_n \), and unlabeled outerplanar graphs \( g_n \) with \( n \) vertices can be computed in polynomial time.
See the sequences A001004, A111563, and A111564 from [25] for initial values.

**Theorem 1.2.** The numbers $d_n$, $c_n$, and $g_n$ of two-connected, connected, and general outerplanar graphs with $n$ vertices have the asymptotic estimates

\[
    d_n \sim d n^{-\frac{5}{2}} \delta^{-n}, \\
    c_n \sim c n^{-\frac{5}{2}} \rho^{-n}, \\
    g_n \sim g n^{-\frac{5}{2}} \rho^{-n},
\]

with growth rates $\delta^{-1} = 3 + 2\sqrt{2} \approx 5.82843$ and $\rho^{-1} \approx 7.50360$, and constants $d \approx 0.00596026$, $c \approx 0.00760471$, and $g \approx 0.00909941$. (See Theorems 4.1, 4.3, and 4.4.)

Having the asymptotic estimates of unlabeled connected outerplanar graphs and unlabeled outerplanar graphs, we investigate asymptotic distributions of parameters such as the number of components and the number of isolated vertices of a random outerplanar graph on $n$ vertices.

**Theorem 1.3.** (1) The probability that a random outerplanar graph is connected is asymptotically equal to a constant $\approx 0.845721$.

(2) The expected number of components in a random unlabeled outerplanar graph is asymptotically equal to a constant $\approx 1.17847$.

(3) The asymptotic distribution of the number of isolated vertices in a random outerplanar graph is a geometric law with parameter $\rho$. In particular, the expected number of isolated vertices in a random outerplanar graph is asymptotically $\rho/(1 - \rho) \approx 0.153761$.

To investigate the chromatic number of a random outerplanar graph we also study the asymptotic number of unlabeled bipartite outerplanar graphs.

**Theorem 1.4.** The number of bipartite outerplanar graphs $(g_b)_n$ on $n$ vertices has the asymptotic estimate $(g_b)_n \sim bn^{-5/2} \rho_b^{-n}$, with $\rho_b^{-1} \approx 4.57717$.

An outerplanar graph is easily shown to have a 3-colouring. The fact that the growth constant of bipartite outerplanar graphs is smaller than the growth constant of outerplanar graphs yields the following result:

**Theorem 1.5.** The probability that the chromatic number of a random unlabeled outerplanar graph is different from three decays asymptotically exponentially to zero.

If we count graphs with respect to the additional parameter that specifies the number of edges, we can study the distribution of the number of edges in a random outerplanar graph.

**Theorem 1.6.** The distribution of the number of edges in a random outerplanar graph on $n$ vertices is asymptotically Gaussian with mean $\mu n$ and variance $\sigma^2 n$, where $\mu \approx 1.54894$ and $\sigma^2 \approx 0.227504$. The same holds for random connected outerplanar graphs with the same mean and variance and for random two-connected outerplanar graphs with asymptotic mean $(1 + \sqrt{2}/2) n \approx 1.70711 n$ and asymptotic variance $\sqrt{2}/8 n \approx 0.17677 n$.

**Outline.** The paper is organized as follows. Section 2 introduces well-known techniques for the enumeration of rooted and unrooted unlabeled structures and shows how to obtain asymptotic estimates. Section 3 provides exact enumeration of two-connected, connected, and general outerplanar graphs, and also of bipartite outerplanar graphs. Section 4 provides asymptotic estimates for the number of two-connected, connected, general, and bipartite outerplanar graphs. Section 4.2 shows how to approximate the growth constant for outerplanar graphs. Finally, Section 5 investigates typical properties of a random outerplanar graph on $n$ vertices, such as the probability of connectedness, the expected number of components, the expected number of isolated vertices, the chromatic number, and the distribution of the number of edges.
2 Preliminaries

We recall some concepts and techniques that we need for the enumeration of unlabeled graphs, and some facts from singularity analysis to obtain asymptotic estimates.

2.1 Cycle index sums

To enumerate unlabeled graphs, we use cycle indices as introduced by Pólya [15, 22]. For a group of permutations $A$ on an object set $X = \{1, \ldots, n\}$ (for example, the vertex set of a graph), the cycle index $Z(A)$ of $A$ with respect to the formal variables $s_1, \ldots, s_n$ is defined by

$$Z(A) := Z(A; s_1, s_2, \ldots) := \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{n} s_k^{j_k(\alpha)},$$

where $j_k(\alpha)$ denotes the number of cycles of length $k$ in the disjoint cycle decomposition of $\alpha \in A$.

For a graph $G$ on $n$ vertices with automorphism group $\Gamma(G)$, we write $Z(G) := Z(\Gamma(G))$, and for a set of graphs $\mathcal{K}$, we write $Z(\mathcal{K})$ for the cycle index sum for $\mathcal{K}$ defined by

$$Z(\mathcal{K}) := Z(\mathcal{K}; s_1, s_2, \ldots) := \sum_{K \in \mathcal{K}} Z(K; s_1, s_2, \ldots).$$

It can be shown [2] that, if $\mathcal{K}$ is the set of graphs of $\mathcal{K}$ equipped with distinct labels, then

$$Z(\mathcal{K}) = \sum_{n \geq 0} \frac{1}{n!} \sum_{K \in \mathcal{K}, \alpha \in \Gamma(K)} \prod_{k=1}^{n} s_k^{j_k(\alpha)},$$

which coincides with the classical definition of a cycle index series and shows the close relationship of cycle index sums to exponential generating functions in labeled counting.

Indeed, cycle index sums can be used for the enumeration of unlabeled structures in a similar way as generating functions for labeled enumeration. First of all, the composition of graphs corresponds to the composition of the associated cycle indices. Consider an object set $X = \{1, \ldots, n\}$ and a permutation group $A$ on $X$. A composition of $n$ graphs from $\mathcal{K}$ is a function $f : X \to \mathcal{K}$. Two compositions $f$ and $g$ are similar, $f \sim g$, if there exists a permutation $\alpha \in A$ with $f \circ \alpha = g$. We write $\mathcal{G}$ for the set of equivalence classes of compositions of $n$ graphs from $\mathcal{K}$ (with respect to the equivalence relation $\sim$). Then

$$Z(\mathcal{G}) = Z(A) [Z(\mathcal{K})] := Z(A; Z(\mathcal{K}; s_1, s_2, \ldots), Z(\mathcal{K}; s_2, s_4, \ldots), \ldots),$$

i.e., $Z(\mathcal{G})$ is obtained from $Z(A)$ by replacing each $s_i$ by $Z(\mathcal{K}; s_1, s_2, \ldots) = \sum_{K \in \mathcal{K}} Z(K; s_1, s_2, \ldots)$ [15]. Hence, Formula (2.1) makes it possible to derive the cycle index sum for a class of graphs by decomposing the graphs into simpler structures with known cycle index sum.

In many cases, such a decomposition is only possible when, for example, one vertex is distinguished from the others in the graphs, so that there is a unique point where the decomposition is applied. Graphs with a distinguished vertex are called vertex rooted graphs. The automorphism group of a vertex rooted graph consists of all permutations of the group of the unrooted graph that fix the root vertex. Hence, one can expect a close relation between the cycle index of unrooted graphs and the cycle indices of their rooted counterparts. As shown in [15], if $\mathcal{G}$ is an unlabeled set of graphs and $\mathcal{G}$ is the set of graphs of $\mathcal{G}$ rooted at a vertex, then

$$Z(\mathcal{G}) = s_1 \frac{\partial}{\partial s_1} Z(\mathcal{G}).$$

This relationship can be inverted to express the cycle index sum for the unrooted graphs in terms of the cycle index sum for the rooted graphs,

$$Z(\mathcal{G}) = \int_0^{s_1} \frac{1}{s_1} Z(\mathcal{G}) ds_1 + Z(\mathcal{G})|_{s_1=0}. \quad (2.3)$$

Observe that permutations without fixed points are not counted by the cycle indices of the rooted graphs, so that their cycle indices are added as a boundary term to $Z(\mathcal{G})$. 

4
2.2 Ordinary generating functions

Once the cycle index sum for a class of graphs of interest is known, the corresponding ordinary generating function can be derived by replacing the formal variables $s_i$ in the cycle index sums by $x^i$ (note that $Z(G; x, x^2, \ldots) = z^{G[x]}$ for a graph $G$). More generally, for a group $A$ and an ordinary generating function $K(x)$ we define

$$Z(A; K(x)) := Z(A; K(x), K(x^2), K(x^3), \ldots)$$

as the ordinary generating function obtained by substituting each $s_i$ in $Z(A)$ by $K(x^i)$, $i \geq 1$.

2.3 The dissimilarity characteristic theorem

The dissimilarity characteristic theorem expresses the number of dissimilar vertices of a graph in terms of the numbers of dissimilar blocks and the number of dissimilar vertices of each block in the graph $[15]$. In the case of trees, the blocks of the graph are the edges. An edge whose vertices are interchanged by an automorphism of the graph is called a symmetry-edge, respectively $[15]$. To also obtain the cycle index sum for unrooted trees, the dissimilarity characteristic theorem can be extended by considering vertex rooted trees whose root vertex is incident to a symmetry-edge $[28]$. The following lemma describes the singular expansion for a common case $[11, Thm. VI.1]$.

**Lemma 2.2 [standard function scale].** Let $F(z) = (1 - z)^{-\alpha}$ with $\alpha \notin \{0, -1, -2, \ldots\}$. Then the coefficients $f_n$ of $F(z)$ have a full asymptotic development in descending powers of $n$,

$$f_n = \left( n + \alpha - 1 \right) \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left( 1 + \sum_{k=1}^{\infty} \frac{e_k(\alpha)}{n^k} \right) (2.5)$$

where $\Gamma(\alpha)$ is the Gamma-Function, $\Gamma(\alpha) := \int_0^\infty e^{-t}t^{\alpha-1} dt$ for $\alpha \notin \{0, -1, -2, \ldots\}$, and $e_k(\alpha)$ is a polynomial in $\alpha$ of degree $2k$. 

2.4 Singularity analysis

To determine asymptotic estimates of the coefficients of a generating function we use singularity analysis $[11]$. The fundamental observation is that the exponential growth of the coefficients of a function that is analytic at the origin is determined by the dominant singularities of the function, i.e., singularities at the boundary of the disc of convergence. By Pringsheim’s theorem $[11]$ Thm. IV.6, a generating function $F(z)$ with non-negative coefficients and finite radius of convergence $R$ has a singularity at the point $z = R$. If $z = R$ is the unique singularity on the disk $|z| = R$, it follows from the exponential growth formula $[11]$ Thm. IV.7 that the coefficients $f_n = [z^n] F(z)$ satisfy $f_n = \theta(n) R^{-n}$ with $\limsup_{n \to \infty} |\theta(n)|^{1/n} = 1$. A closer look at the type of the dominant singularity, for example, the order of the pole, enables the computation of subexponential factors as well. The following lemma describes the singular expansion for a common case $[11]$ Thm. VI.1. 

**Lemma 2.3.** Let $G$ be an unlabeled tree. Let $V$ be the set of vertex rooted trees that have $G$ as underlying unrooted tree. Partition the set $V$ into the set $V_1$ of rooted trees where the root vertex is not incident to a symmetry-edge and the set $V_2$ where the root vertex is incident to a symmetry-edge. Furthermore, let $E$ be the set of trees obtained from $G$ by rooting at an edge, and let $S$ be the set of trees obtained from $G$ by rooting at a symmetry-edge. Then

$$Z(G) = Z(V_1) - Z(E) + 2Z(S). (2.4)$$

The lemma can be proven by induction on the number of dissimilar edges, where the initial case of the induction determines the crucial distinction between the two kinds of vertex rooted trees, $V_1$ and $V_2$ (see $[28]$ for a full proof).
In our calculations, it will appear that a generating function \( f(x) \) is given only implicitly by an equation \( H(x, f(x)) = 0 \). Theorem VII.3 in \cite{11} describes how to derive a full singular expansion of \( f(x) \) in this case. We state it here in a slightly modified version. A generating function is called aperiodic, if it can not be written in the form \( Y(x) = x^n \tilde{Y}(x^d) \) with \( d \geq 2 \) and \( \tilde{Y} \) analytic at 0.

**Theorem 2.3** [singular implicit functions]. Let \( H(x, y) \) be a bivariate function that is analytic in a complex domain \(|x| < R, |y| < S\) and verifies \( H(0, 0) = 0, \frac{\partial}{\partial y} H(0, 0) = -1\), and whose Taylor coefficients \( h_{m,n} \) satisfy the following positivity conditions: they are nonnegative except for \( h_{0,1} = -1\) (because \( \frac{\partial}{\partial y} H(0, 0) = -1\)) and \( h_{m,n} > 0\) for at least one pair \((m, n)\) with \( n \geq 2\). Assume that there are two numbers \( r \in (0, R) \) and \( s \in (0, S) \) such that
\[
H(r, s) = 0, \quad \frac{\partial}{\partial y} H(r, s) = 0, \quad (2.6)
\]
\[
\frac{\partial^2}{\partial y^2} H(r, s) \neq 0 \text{ and } \frac{\partial^2}{\partial x \partial y} H(r, s) \neq 0.
\]
Assume further that the equation \( H(x, y) = 0 \) admits a solution \( Y(x) \) that is analytic at 0, has non-negative coefficients, and is aperiodic. Then \( r \) is the unique dominant singularity of \( Y(x) \) and \( Y(x) \) converges at \( x = r \), where it has the singular expansion
\[
Y(x) = s + \sum_{i \geq 1} Y_i \left( \sqrt{1 - \frac{x}{r}} \right)^i, \quad \text{with} \quad Y_1 = -\frac{2r \partial_y H(r, s)}{\partial^2 y^2 H(r, s)} \neq 0,
\]
and computable constants \( Y_2, Y_3, \cdots \). Hence,
\[
[x^n] Y(x) = -\frac{Y_1}{2\sqrt{\pi}n^3}r^{-n} \left( 1 + O\left( \frac{1}{n} \right) \right).
\]

The formulas that express the coefficients \( Y_i \) in terms of partial derivatives of \( H(x, y) \) at \((r, s)\) can be found in \cite{10, 21}.

When a parameter \( \xi \) of a combinatorial structure is studied, the generating function \( F(x, y) \) has to be extended to a bivariate generating function \( F(x, y) = \sum_{n,m} f_{n,m} x^n y^m \) where the second variable \( y \) marks \( \xi \). We can determine the asymptotic distribution of \( \xi \) from \( F(x, y) \) by varying \( y \) in some neighbourhood of 1. The following theorem follows from the so-called quasi-powers theorem \cite{11} Thm. IX.7.

**Theorem 2.4.** Let \( F(x, y) \) be a bivariate generating function of a family of objects \( \mathcal{F} \), where the power in \( y \) corresponds to a parameter \( \xi \) on \( \mathcal{F} \), i.e., \([x^n y^m] F(x, y) = |\{ F \in \mathcal{F} \mid |F| = n, \xi(F) = m \}|\). Assume that, in a fixed complex neighbourhood of \( y = 1 \), \( F(x, y) \) has a singular expansion of the form
\[
F(x, y) = \sum_{k \geq 0} F_k(y) \left( \sqrt{1 - \frac{x}{x_0(y)}} \right)^k \quad (2.7)
\]
where \( x_0(y) \) is the dominant singularity of \( x \mapsto F(x, y) \). Furthermore, assume that there is an odd \( k_0 \in \mathbb{N} \) such that for all \( y \) in the neighbourhood of 1, \( F_{k_0}(y) \neq 0 \) and \( F_k(y) = 0 \) for \( 0 < k < k_0 \) odd. Assume that \( x_0(y) \) and \( F_{k_0}(y) \) are analytic at \( y = 1 \), and that \( x_0(y) \) satisfies the variance condition, \( x_0'(1)x_0(1) + x_0'(1)x_0(1) - x_0'(1)^2 \neq 0 \).

Let \( X_n \) be the restriction of \( \xi \) onto all objects in \( \mathcal{F} \) of size \( n \). Under these conditions, the distribution of \( X_n \) is asymptotically Gaussian with mean
\[
\mathbb{E}[X_n] \sim \mu n \quad \text{with} \quad \mu = \frac{x_0'(1)}{x_0(1)}
\]
and variance
\[
\mathbb{V}[X_n] \sim \sigma^2 n \quad \text{with} \quad \sigma^2 = \frac{x_0''(1)}{x_0(1)} - \frac{x_0'(1)}{x_0(1)} + \left( \frac{x_0'(1)}{x_0(1)} \right)^2.
\]
3 Exact enumeration of outerplanar graphs

From now on we always consider outerplanar graphs as unlabeled objects, unless stated otherwise. In Section 3.1 we derive the cycle index sums for rooted and unrooted two-connected outerplanar graphs. Section 3.2 shows how to decompose connected graphs into two-connected components, which yields expressions of the cycle index sums for rooted and unrooted connected outerplanar graphs. In Section 3.3 we use the simple fact that an unrooted outerplanar graph is an (unordered) collection of unrooted connected outerplanar graphs to obtain the cycle index sum for outerplanar graphs. In Section 3.4 we explain how to adapt the decomposition to enumerate bipartite outerplanar graphs.

3.1 Enumeration of dissections (two-connected outerplanar graphs)

A graph is two-connected if at least two of its vertices have to be removed to disconnect it. A two-connected outerplanar graph with at least three vertices has a unique Hamiltonian cycle [17] and can therefore be embedded uniquely in the plane so that this Hamiltonian cycle lies on the outer face. This unique embedding is thus a dissection of a convex polygon. Hence the task of counting two-connected outerplanar graphs coincides with the task of counting dissections of a polygon. The generating functions for unlabeled (rooted and unrooted) dissections were derived by Read [23]. In this section, we extend Read’s work to cycle index sums.

A dissection can have two types of automorphisms: reflections and rotations. We use the terminology of an oriented dissection when an orientation is imposed on the Hamiltonian cycle of the dissection. Since reflections reverse orientations, they are excluded from the automorphism group of an oriented dissection. By rooting at an edge on the outer face, rotations are excluded from the automorphism group as well. Thus, oriented dissections that are rooted at an edge on the outer face are easy to count and are the starting point for the enumeration of different types of dissections. Having the cycle index sum for oriented and non-oriented dissections that are rooted at an edge on the outer face, we can derive the cycle index sums for dissections rooted at an edge not on the outer face, and for dissections rooted at a face, by composition. Finally, we consider the dual of a dissection, which is essentially a tree, and thereby count unrooted dissections. The dissimilarity characteristic theorem for trees (Lemma 3.3) can be applied to express the cycle index sum for unrooted dissections by a combination of the cycle index sums for several types of edge rooted and face rooted dissections.

We do not consider the graph consisting of a single vertex as a dissection. However, it is convenient to include the single edge to the sets of edge rooted, vertex rooted and unrooted dissections.

Terminology [rooting, outer-edge, inner-edge, symmetry-edge, reflective]. We say that a dissection is edge (respectively face) rooted if one edge (respectively face) is distinguished from the others in the dissection. Note that the vertices of the root edge (respectively face) might be interchanged by the automorphisms of the dissection. An edge on the outer face is called outer-edge, and inner-edge otherwise. A symmetry-edge is an inner-edge such that there exists a nontrivial automorphism that fixes this edge. It is clear that a dissection can have at most one symmetry-edge. An edge rooted dissection is called reflective if its automorphism group contains a non-trivial reflection.

In the following we present the cycle index sums for several types of dissections and only sketch the proofs here. The corresponding notation is introduced in Table 1. We denote sets of graphs by calligraphic letters, ordinary generating functions by capital letters, and counts by small letters. Details can be found in the thesis of the last author [28].

We first recall the well-known results on oriented outer-edge rooted dissections ([23] and [24, A001003]). A non-oriented outer-edge rooted dissection is counted twice by the cycle index sum for oriented outer-edge rooted dissections when the dissection is not invariant under a reflection that fixes the root-edge. Therefore, to derive counting formulas for non-oriented dissections later, we also need the cycle index sum for reflective outer-edge rooted dissections. Automorphisms of
such structures can be divided into two classes: Those that fix the vertices of the root-edge and those that interchange the vertices of the root edge.

Lemma 3.1 [outer-edge rooted dissections]. The cycle index sum for oriented outer-edge rooted dissections is given by

$$Z(E_o) = \frac{s_1}{4} \left( s_1 + 1 - \sqrt{s_1^2 - 6s_1 + 1} \right).$$

(3.1)

The cycle index sum for reflective outer-edge rooted dissections is

$$Z(E_r) = \frac{1}{2} \left( Z^+ (E_r) + Z^- (E_r) \right),$$

where $Z^+ (E_r)$ counts the mappings that fix the vertices of the root-edge (i.e., the identity mappings), and $Z^- (E_r)$ counts the mappings that interchange the vertices of the root-edge (i.e., the reflection mappings),

$$Z^+ (E_r) = \frac{1 + s_1 - 3s_1^2 + s_1^3 - (1 + s_1) \sqrt{s_1^2 - 6s_1 + 1}}{4s_1},$$

(3.2)

$$Z^- (E_r) = \frac{s_2^3 - 3s_1s_2^2 + s_2 + s_1 - (s_2 + s_1) \sqrt{s_2^2 - 6s_2 + 1}}{4s_2}.$$

(3.3)

Proof. As pointed out before, oriented outer-edge rooted dissections have a trivial automorphism group. Therefore, their numbers are closely related to the bracketing numbers \[25, A001003]. To construct such a dissection, we replace the edges of an edge rooted polygon other than the root-edge by oriented outer-edge rooted dissections, see Figure 3.1. The cycle index sum \[3.1\] is derived by an application of the composition formula (2.1).

Figure 3.1: Construction of an oriented outer-edge rooted dissection

Similarly to the construction of oriented outer-edge rooted dissections a reflective outer-edge rooted dissection can be constructed by plugging oriented outer-edge rooted dissections into the edges of an edge-rooted polygon other than the root-edge. However, to obtain a reflective dissection, edges of the polygon with the same distance to the root-edge have to receive the same
outer-edge rooted dissection. If the polygon has an even number of edges, the outer-edge rooted dissection that is plugged into the edge opposite to the root-edge has to be reflective itself, see also Figure 3.2. Corresponding formulas for the cycle index sums are derived by an application of the composition formula (2.1),

$$Z^+ (E_r^o) = \frac{s_1^4 + (s_1 - s_1^2)}{s_1^2} \cdot Z (E_o^o; s_1^2), \quad Z^- (E_r^o) = \frac{s_2^2 + (s_1 - s_2)}{s_2 - 2Z (E_o^o; s_2)}.$$

Substituting formula (3.1) for $Z (E_o^o)$ yields formulas (3.2) and (3.3).

Substituting formula (3.1) for $Z (E_o^o)$ yields formulas (3.2) and (3.3).

#### Lemma 3.2 [inner-edge and symmetry-edge rooted dissections]. The cycle index sums for inner-edge rooted dissections and symmetry-edge rooted dissections, respectively, are given by

$$Z (E_i) = \frac{(3s_1 - 1 + \sqrt{s_1^2 - 6s_1 + 1})^2}{64} + \frac{(s_1 + s_2)^2}{16s_2} \left( \frac{1 + s_2 - \sqrt{s_2^2 - 6s_2 + 1}}{1 - s_2 + \sqrt{s_2^2 - 6s_2 + 1}} \right)^2.$$

$$Z (E_s) = \frac{s_1^6 - 2s_1^4s_2^2 + s_1^2s_2^4 - s_1^4s_2^2 - s_2^4}{16s_1^2s_2^2} + \frac{3}{8} \left( 1 - s_2 + \sqrt{s_2^2 - 6s_2^4 + 1} \right),$$

$$\quad - \frac{1}{8} \left( s_1^4 - 6s_1^2 + 1 - \frac{1}{16s_2} \right) \left( 1 + \frac{s_1^2}{s_2^2} \right) \sqrt{s_1^4 - 6s_1^2 + 1} - \frac{1}{16} \left( 1 + \frac{s_1^2}{s_2^2} \right) \sqrt{s_2^4 - 6s_2^2 + 1}.$$

**Proof.** An inner-edge rooted dissection can be constructed by joining two outer-edge rooted dissections at their root-edge. We consider the possible reflections and rotations. There are four kinds of transformations of the plane that map the root-edge onto itself (also see Figure 3.3): 1. the identity mapping; 2. the reflection at the root-edge; 3. the half-turn around the root-edge; 4. the reflection at the perpendicular bisector of the root-edge.

![Figure 3.3: The four kinds of transformations that fix the root edge of an inner-edge rooted dissection.](image)

Those inner-edge rooted dissections that are invariant under the identity mapping are counted by joining two independently chosen non-empty oriented outer-edge rooted dissections at their...
root-edge. Those that are invariant under the second or third kind of permutation are constructed
by joining a non-empty oriented outer-edge rooted dissection with a copy of itself on its root-edge,
where the vertices of the root-edge remain fixed (type 2) or are interchanged (type 3). Finally,
inner-edge rooted dissections invariant under the fourth kind of mapping are composed out of two
independently chosen non-empty reflective outer-edge rooted dissections, joined at the root-edge.
Hence, we get
\[ Z(\mathcal{E}_i) = \frac{1}{4} \left( \left( \frac{Z(\mathcal{E}_0^o)}{s_1^2} - 1 \right)^2 s_1^2 + \left( \frac{Z(\mathcal{E}_0^o)}{s_2^2} - 1 \right) (s_1^2 + s_2) + \left( \frac{Z^{-}(\mathcal{E}_0^o)}{s_2} - 1 \right)^2 s_2 \right), \]
which together with Lemma 3.1 yields the statement.

Similarly, a non-empty dissection rooted at a symmetry-edge can be constructed by joining an
oriented outer-edge rooted dissection with a copy of itself at the root-edge. We have to further
distinguish dissections that are invariant under reflection and dissections that are not. A similar
discussion as for inner-edge rooted dissections leads to
\[ Z(\mathcal{E}_o) = \frac{Z(\mathcal{E}_0^o; s_1^2)}{2s_1^2} + \frac{s_1^2 + s_2}{4s_2^2} Z(\mathcal{E}_0^o; s_2) - \frac{Z^+(\mathcal{E}_0^o; s_1^2, s_2^2)}{4s_1^2} + \frac{Z^-(\mathcal{E}_0^o; s_1^2, s_2^2)}{4s_2} - \frac{s_1^2 + s_2}{2}. \]
We then obtain the statement by applying Lemma 3.1.

We proceed with dissections that are rooted at a face. In this case, the automorphism groups
might include cyclic permutations of order greater than two.

**Lemma 3.3 [face rooted dissections].** The cycle index sum for face rooted dissections is given by
\[ Z(\mathcal{F}) = -\frac{1}{2} \sum_{d \geq 1} \frac{\varphi(d)}{d} \log \left( \frac{3}{4} - \frac{1}{4} s_d + \frac{1}{4} \sqrt{s_d^2 - 6s_d + 1} \right) + \frac{s_1 + 5}{32} \sqrt{s_1^2 - 6s_1 + 1} \]
\[ - \frac{s_1^2 + 2s_1 + 2s_2 + 7}{32} + \frac{s_1^2 + 6s_1s_2 + 8s_1s_2^2 + 3s_2^3 + 4s_1s_2^3 + 5s_1^2s_2^2 + 2s_2^3 - 2s_1s_2^3 - 2s_1^2s_2^2 - s_1^2}{16s_2 \left( 1 - s_2 + \sqrt{s_2^2 - 6s_2 + 1} \right)^2} \]
\[ + \left( \frac{1}{16} - \frac{s_1^2 + 2s_1s_2 + 3s_1s_2^2 + s_2^3 - 2s_1s_2^3 - 2s_1^2s_2^2 - s_1^2}{8s_2 \left( 1 - s_2 + \sqrt{s_2^2 - 6s_2 + 1} \right)^2} \right) \sqrt{s_2^2 - 6s_2 + 1}. \]

**Proof.** A face rooted dissection can be constructed by plugging oriented outer-edge rooted dissections
into the edges of a polygon. In the oriented case (see Figure 3.4), we only have to consider
cyclic permutations of the polygon. Let \( C_k \) be the cyclic group, generated by the permutation
(1 2 3 \cdots k). The composition formula (2.4) gives the cycle index sum for oriented face rooted dissections,
\[ Z(\mathcal{F}^o) = \sum_{k \geq 3} Z(C_k) \left[ Z(\mathcal{E}_0^o) / s_1 \right]. \tag{3.4} \]

\( Z(C_k) \) can be expressed with the Euler-\( \varphi \)-function, \( Z(C_k) = \frac{1}{k} \sum_{d \mid k} \varphi(d) s_d^{k/d} \), which leads to
\[ Z(\mathcal{F}^o) = -\left\{ \sum_{d \geq 1} \frac{\varphi(d)}{d} \log \left( 1 - \frac{Z(\mathcal{E}_0^o; s_d)}{s_d} \right) \right\} - \frac{Z(\mathcal{E}_0^o)}{s_1} - \frac{1}{2} \left( \frac{Z(\mathcal{E}_0^o)}{s_1} \right)^2 + \frac{Z(\mathcal{E}_0^o; s_2)}{s_2} \). \]

In the non-oriented case, we have to take care of additional reflections. Therefore, the cyclic
group \( C_k \) has to be replaced by the dihedral group \( D_k \), which is generated by the cycle (1 2 3 \cdots k)
and the reflection (1 k) (2 (k - 1)) (3 (k - 2)) \cdots , and has the cycle index
\[ Z(D_k) = \frac{1}{2} Z(C_k) + \begin{cases} \frac{1}{s_1 s_2^m}, & k \text{ odd, } k = 2m + 1, \\ \frac{1}{s_2^{m+1}} + \frac{1}{s_1^2 s_2^m}, & k \text{ even, } k = 2m + 2. \end{cases} \]
Figure 3.4: Construction of a face rooted dissection counted by $s^2_1 [Z (E^o_o) / s_1]$

The objects with cyclic automorphisms are counted by $Z (C_k)$. For the reflections, the outer-edge rooted dissections attached to corresponding pairs of edges must be the same, while the outer-edge rooted dissection that is mapped to itself must be reflective. We also have to distinguish between polygons of odd and even size. If we identify the corresponding terms in $Z (D_k)$ with the correct cycle index sums (see Figure 3.5) we get

$$Z (F) = \frac{1}{2} Z (F^o) + \frac{1}{2s_2} \left( s_1 Z (E^o_o) + \frac{s^2_1}{2s_2} Z (E^o_o; s_2) + \frac{1}{2} Z^{-} (E^o_o)^2 \right) \frac{Z (E^o_o; s_2)}{s_2 - Z (E^o_o; s_2)},$$

which together with Lemma 3.1 yields the statement.

Figure 3.5: The extra terms $s_1 s^m_2$ ($k = 2m + 1$, left) and $s^m_2 s_2$ and $s^m_2$ ($k = 2m + 2$, right) in $Z (D_k)$ for $m = 2$.

**Lemma 3.4** [face rooted dissections with root-face incident to a symmetry-edge]. The cycle index sum for face rooted dissections where the root-face is incident to a symmetry-edge is given by

$$Z (F_s) = \frac{s^6_1 (1 - 3s^2_2 - 3s^3_2)}{8s^4_1s^3_2} + \frac{s^4_1 (s^2_2 + 5s^3_2 - 3s^4_2)}{8s^4_1s^3_2} - s^3_2 - s^2_1 s^3_2$$

$$- \frac{1}{4} \sqrt{s^4_1 - 6s^2_2 + 1} + \frac{s^2_2 + s^3_2 s^3_2}{8s^4_1s^3_2} \sqrt{s^4_1 - 6s^2_2 + 1} - \frac{s^2_2 + s^3_2}{8s^3_2} \sqrt{s^4_2 - 6s^2_2 + 1}.$$

**Proof.** As in the case of symmetry-edge rooted dissections we join two (identical) non-empty oriented outer-edge rooted dissections at their root-edge and choose one of the faces incident to the root-edge to be the root face. In contrast to Lemma 3.2 we do not have to care about the permutations of the second and third kind, and obtain

$$Z (F_s) = \frac{1}{s^3_1} Z (E^o_o; s^2_1) - \frac{1}{2s^3_1} Z^+ (E^o_o; s^2_1, s^2_2) + \frac{1}{2s^2_2} Z^- (E^o_o; s^2_1, s^2_2) - \frac{1}{2} (s^2_1 + s^2_2),$$

which together with Lemma 3.1 yields the statement.
We have now found all cycle index sums that are needed to compute the cycle index sum for unrooted dissections.

**Theorem 3.5 [dissections].** The cycle index sum for dissections is given by

\[
Z(D) = -\frac{1}{2} \sum_{d \geq 1} \varphi(d) \frac{d}{d} \log \left( \frac{3}{4} - \frac{1}{4} s_d + \frac{1}{4} \sqrt{s_d^2 - 6s_d + 1} \right) + \frac{s_2 + s_1^2 - 4s_1 - 2}{16} + \frac{s_1^2 - 3s_1 s_2 + 2s_1 s_2}{16s_2} + \frac{3 - s_1}{16} \sqrt{s_2^2 - 6s_1 + 1} - \frac{1}{16} \left( \frac{s_2^2 + 2s_1}{s_2^2} \right) \sqrt{s_2^2 - 6s_2 + 1}.
\]

**Proof.** Since the dual graph of a dissection that has at least one face is a tree, we can apply the dissimilarity characteristic theorem for trees in its cycle index version (Lemma 2.1) to derive the cycle index sum for unlabeled dissections from the ones for face rooted dissections (corresponding to vertex rooted trees), inner-edge rooted dissections (corresponding to edge rooted trees), and symmetry-edge rooted dissections (corresponding to symmetry-edge rooted trees). A vertex of the tree that is not incident to a symmetry-edge corresponds to a face of the dissection that is not incident to a symmetry-edge. Lemma 2.3 with \( \mathcal{F} \setminus \mathcal{F}_s \) instead of \( \mathcal{V}_1, \mathcal{E}_1 \) instead of \( \mathcal{E} \), and \( \mathcal{E}_s \) instead of \( \mathcal{S} \) yields

\[
Z(D) = \frac{1}{2} \left( s_1^2 + s_2 \right) + Z(\mathcal{F}) - Z(\mathcal{F}_s) - Z(\mathcal{E}_1) + 2Z(\mathcal{E}_s).
\]

The additional term \( \frac{1}{2} \left( s_1^2 + s_2 \right) \) counts the dissection that consists of one edge only. We apply Lemmas 3.2, 3.3, and 3.4 to obtain the result. \( \square \)

Replacing \( s_1 \) by \( x \), \( s_2 \) by \( x^2 \), \ldots we obtain the generating function \( D(x) \) of dissections, which was already found by Read:

\[
D(x) = -\frac{1}{2} \sum_{d \geq 1} \varphi(d) \frac{d}{d} \log \left( \frac{1}{4} \left( 3 - x^d + \sqrt{x^{2d} - 6x^d + 1} \right) \right) + \frac{x^2}{8} - \frac{1}{4} x - \frac{5}{16} + \frac{1}{8x} + \frac{1}{16x^2} + \frac{3 - x}{16} \sqrt{x^2 - 6x + 1} - \frac{1 + 2x + x^2}{16x^2} \sqrt{x^4 - 6x^2 + 1}.
\]

The coefficients of \( D(x) \), counting unlabeled dissections, can be extracted in polynomial time, \( D(x) = x^2 + x^3 + 2x^4 + 3x^5 + 9x^6 + 20x^7 + 75x^8 + 262x^9 + \ldots \), matching the values computed by Read, [25, A001004].

Finally, the cycle index sum for vertex rooted dissections, which we will need in Section 3.2, can be derived by using Formula (2.2): \( Z(\mathcal{V}) = s_1 \frac{\partial}{\partial s_1} Z(D) \).

**Corollary 3.6 [vertex rooted dissections].** The cycle index sum for vertex rooted dissections is given by

\[
Z(\mathcal{V}; s_1, s_2) = \frac{s_1}{8} \left( 1 + s_1 - \sqrt{s_1^2 - 6s_1 + 1} \right) + \frac{s_1}{8s_2^2} \left( s_1 + s_2 \right) \left( 1 - 3s_2 - \sqrt{s_2^2 - 6s_2 + 1} \right). \tag{3.6}
\]

### 3.2 Enumeration of connected outerplanar graphs

We denote the set of unrooted connected outerplanar graphs by \( \mathcal{C} \), and the set of vertex rooted connected outerplanar graphs by \( \hat{\mathcal{C}} \). All rooted graphs considered in this section are rooted at a vertex. Again, ordinary generating functions are denoted by capital letters and coefficients by small letters. Thus, \( \hat{\mathcal{C}}(x) = \sum_n \hat{c}_n x^n \) and \( C(x) = \sum_n c_n x^n \).

The cycle index sum for rooted connected outerplanar graphs is derived by decomposing the graphs into rooted two-connected outerplanar graphs, i.e., vertex rooted dissections.

**Lemma 3.7 [rooted connected outerplanar graphs].** The cycle index sum for vertex rooted connected outerplanar graphs is implicitly determined by the equation

\[
Z(\hat{\mathcal{C}}) = s_1 \exp \left( \sum_{k \geq 1} \frac{Z(\mathcal{V}; Z(\hat{\mathcal{C}}; s_k, s_{2k}, \ldots), Z(\hat{\mathcal{C}}; s_{2k}, s_{4k}, \ldots))}{k Z(\hat{\mathcal{C}}; s_k, s_{2k}, \ldots)} \right). \tag{3.7}
\]
Proof. Graphs in \(\hat{C}\) rooted at a vertex that is not a cut-vertex can be constructed by taking a rooted dissection and attaching a rooted connected outerplanar graph at each vertex of the dissection other than the root vertex. By the composition formula \((2.4)\) we obtain that

\[ s_1 \left( \frac{Z(V)}{s_1} \right) \left[ Z(\hat{C}) \right] \tag{3.8} \]

is the cycle index sum for connected outerplanar graphs rooted at a non-cut-vertex. The division (resp. multiplication) by \(s_1\) is due to the removal (resp. addition) of the root vertex before (resp. after) application of Formula \((2.1)\).

The cycle index sum for rooted connected outerplanar graphs where the root vertex is incident to exactly \(n\) blocks, \(n \geq 2\), can be obtained by another application of the composition theorem. We join \(n\) connected outerplanar graphs that are rooted at a vertex other than a cut-vertex at their root vertex. Application of the composition formula \((2.4)\) with the symmetric group \(S_n\) and Formula \((3.8)\) (divided by \(s_1\)) for the cycle index sum for non-cut-vertex rooted connected outerplanar graphs (excluding the root) yields

\[ s_1 Z(S_n) \left[ \left( \frac{Z(V)}{s_1} \right) \left[ Z(\hat{C}) \right] \right]. \]

Summing over \(n \geq 0\), we get \((Z(S_0) := 1)\)

\[ Z(\hat{C}) = s_1 \sum_{n \geq 0} Z(S_n) \left[ \left( \frac{Z(V)}{s_1} \right) \left[ Z(\hat{C}) \right] \right]. \]

With the well-known formula \(\sum_{n \geq 0} Z(S_n) = \exp \left( \sum_{k \geq 1} \frac{1}{k} s_k \right)\), the statement follows. \(\square\)

Theorem 3.8 [connected outerplanar graphs]. The cycle index sum for connected outerplanar graphs is given by

\[ Z(C) = Z(\hat{C}) + Z(D; Z(\hat{C})) - Z(V; Z(\hat{C})). \tag{3.9} \]

Proof. To derive the cycle index sum for unrooted connected outerplanar graphs, one can use Formula \((2.3)\). Hence,

\[ Z(C) = \int_0^{s_1} \frac{1}{s_1} Z(\hat{C}) ds_1 + Z(C) \bigg|_{s_1=0}. \tag{3.10} \]

The term \(Z(C) \big|_{s_1=0}\) can be further replaced by \(Z(D) \big|_{s_1=0} [Z(\hat{C})]\) because each fixed-point free permutation in a connected graph \(G\) has a unique block whose vertices are setwise fixed by the automorphisms of \(G\) [15, page 190]. Using the special structure \((3.7)\) of \(Z(\hat{C})\), a closed solution for the integral in \((3.10)\) can be found [28]. We put these facts together and obtain \((3.9)\). \(\square\)

Replacing \(s_i\) by \(x^i\) in \(Z(\hat{C})\), we obtain that the generating function \(\hat{C}(x)\) counting vertex rooted connected outerplanar graphs satisfies

\[ \hat{C}(x) = x \exp \left( \sum_{k \geq 1} \frac{Z(V; \hat{C}(x^k))}{k \hat{C}(x^k)} \right), \tag{3.11} \]

from which the coefficients \(\hat{C}_n\) counting vertex rooted connected outerplanar graphs can be extracted in polynomial time: \(\hat{C}(x) = x + x^2 + 3x^3 + 10x^4 + 40x^5 + 181x^6 + 918x^7 + \ldots\), see [27, 28] for more entries. The numbers in [27] verify the correctness of our result and were computed by the polynomial algorithm proposed in [18].

In addition, it follows from \((3.9)\) that the generating function \(C(x)\) counting connected outerplanar graphs satisfies

\[ C(x) = \hat{C}(x) + Z(D; \hat{C}(x)) - Z(V; \hat{C}(x)), \tag{3.12} \]
from which the coefficients \( c_n \) counting connected outerplanar graphs can be extracted in polynomial time: 
\[
C(x) = x + x^2 + 2x^3 + 5x^4 + 13x^5 + 46x^6 + 172x^7 + \ldots, \text{ see } \text{OEIS A111563} \text{ for more entries.}
\]

### 3.3 Enumeration of outerplanar graphs

We denote the set of outerplanar graphs by \( \mathcal{G} \), its ordinary generating function by \( G(x) \) and the number of outerplanar graphs with \( n \) vertices by \( g_n \). As an outerplanar graph is a collection of connected outerplanar graphs, it is now easy to obtain the cycle index sum for outerplanar graphs. An application of the composition formula (2.1) with the symmetric group \( S_l \) and object set \( C \) yields that the cycle index sum for \( l \) connected components. Thus, by summation over all \( l \geq 0 \) (we include here also the empty graph into \( \mathcal{G} \) for convenience), we obtain the following theorem.

**Theorem 3.9 [outerplanar graphs].** The cycle index sum for outerplanar graphs is given by

\[
Z(\mathcal{G}) = \exp \left( \sum_{k \geq 1} \frac{1}{k} Z(C; s_k, s_{2k}, \ldots) \right).
\]

Hence the generating functions \( G(x) \) and \( C(x) \) of outerplanar and connected outerplanar graphs are related by

\[
G(x) = \exp \left( \sum_{k \geq 1} \frac{1}{k} C(x^k) \right).
\]

From this, we can extract in polynomial time the coefficients counting outerplanar graphs, \( C(x) = 1 + x + 2x^2 + 4x^3 + 10x^4 + 25x^5 + 80x^6 + 277x^7 + \ldots, \text{ see } \text{OEIS A111564} \text{ for more entries.}

### 3.4 Enumeration of bipartite outerplanar graphs

To study the chromatic number of a typical outerplanar graph we enumerate bipartite outerplanar graphs. Observe that an outerplanar graph is bipartite if and only if all of its blocks are bipartite. As discussed in Section 3, blocks of an outerplanar graph are dissections, and it is clear that a dissection is bipartite when all of its inner faces have an even number of vertices. The decomposition of dissections exposed in Section 3.1 can be adapted to dissections where all faces have even degree. Once the cycle index sum for bipartite dissections is obtained, the computation of the cycle index sums for bipartite connected outerplanar graphs, and then of bipartite outerplanar graphs works in the same way as for the general case, see [28] for details. From that the coefficients of the series \( G_b(x) \) counting bipartite outerplanar graphs can be extracted in polynomial time: \( G_b(x) = 1 + x + x^2 + x^3 + 7x^4 + 12x^5 + 29x^6 + 61x^7 + \ldots, \text{ see the sequences A111757, A111758, and A111759 of } \text{OEIS A111654} \text{ for the coefficients of two-connected, connected, and general bipartite outerplanar graphs.}

### 4 Asymptotic enumeration of unlabeled outerplanar graphs

To determine the asymptotic number of two-connected, connected, and general outerplanar graphs, we use singularity analysis as introduced in Section 2.4. To compute the growth constants and subexponential factors we expand the generating functions for outerplanar graphs around their dominant singularities. For unlabeled two-connected outerplanar graphs we present an analytic expression of the growth constant. For the connected and the general case we give numerical approximations of the growth constants in Section 4.
4.1 Asymptotic estimates

We now prove the first part of Theorem 1.2 on the asymptotic number of dissections.

**Theorem 4.1 [asymptotic number of unrooted dissections].** The number \( d_n \) of unlabeled two-connected outerplanar graphs on \( n \) vertices has the asymptotic estimate \( d_n \sim d n^{-4} \delta^{-n} \) with growth rate \( \delta^{-1} = 3 + 2\sqrt{2} \approx 5.82843 \) and constant \( d \approx 0.00596026 \).

**Proof.** Let \( \delta \) be the smallest root of \( x^2 - 6x + 1, \delta = 3 - 2\sqrt{2} \). Equation (3.5) implies that \( D(x) \) can be written as

\[
D(x) = -\frac{1}{2} \log \left( 1 - \frac{\sqrt{x^2 - 6x + 1}}{x - 3} \right) + \frac{3 - x}{16} \sqrt{x^2 - 6x + 1} + A(x),
\]

where \( A(x) \) is analytic at 0 with radius of convergence \( \delta \). Since the logarithmic term is analytic for \( |x| < \delta \), we can expand it and collect ascending powers of \( \sqrt{x^2 - 6x + 1} \) in \( D(x) \). Thus,

\[
D(x) = \left( -\frac{1}{16(x-3)} + \frac{1}{6(x-3)^3} \right) \left( \sqrt{x^2 - 6x + 1} \right)^3 + \sum_{k \geq 4} \frac{1}{2k} \left( \frac{\sqrt{x^2 - 6x + 1}}{x - 3} \right)^k + \hat{A}(x),
\]

where \( \hat{A}(x) \) is again analytic at 0 with radius of convergence \( \delta \). Finally, using \( \sqrt{x^2 - 6x + 1} = \sqrt{1 - x/\delta \sqrt{1 - \delta^2}} \) for \( x \leq \delta \) and applying Lemma 2.2 we obtain

\[
d_n = \left( -\frac{1}{16(\delta - 3)} + \frac{1}{6(\delta - 3)^3} \right) \left( \sqrt{1 - \delta^2} \right)^3 \frac{1}{\Gamma(-3/2)} n^{-5/2} \delta^{-n} \left( 1 + O\left( \frac{1}{n} \right) \right)
\]

\[
\sim \frac{(3\sqrt{2} - 4)^{3/2}}{8\sqrt{2\pi}} n^{-5/2} \left( 3 + 2\sqrt{2} \right)^n.
\]

We now turn to the problem of asymptotic enumeration of connected outerplanar graphs. First we have to establish the singular development of the generating function for vertex rooted connected outerplanar graphs \( \hat{C}(x) \). Let \( \rho \) be the radius of convergence of \( \hat{C}(x) \). Observe that the coefficients \( \hat{c}_n \) are bounded from below by the number of unlabeled vertex rooted dissections \( u_n \), which have exponential growth. The coefficients are bounded from above by the number of embedded outerplanar graphs with a root edge, which also have exponential growth (this follows from classical enumerative results on planar maps; see [26]). Hence \( \rho \) is in \( (0,1) \).

To apply Theorem 2.3 for rooted connected outerplanar graphs, we consider the function

\[
H(x,y) := x \exp \left( Z \left( V; y, \hat{C}(x^2) \right) + \sum_{k \geq 2} \frac{Z \left( V; \hat{C}(x^k) ; \hat{C}(x^2 k) \right)}{k \hat{C}(x^k)} \right) - y.
\]

Observe that Equation (3.11) implies that \( H(x,\hat{C}(x)) = 0 \). The difficulty in the application of the singular implicit functions theorem (Thm. 1.1) is the verification of the requirements of this theorem. Hence, to apply Theorem 2.3 we have to check that the dominant singularity of the generating functions for the connected components is determined by its implicit definition (like (3.11)) and not by a singularity of \( H(x,y) \). This analysis is the main purpose of the next proposition. Observe that it can also be easily generalized to other classes of connected unlabeled graphs with known blocks.

**Lemma 4.2.** The generating function \( \hat{C}(x) \) satisfies the conditions of Theorem 2.3 with the function \( H(x,y) \) from Equation (4.1) and \( (r,s) = (\rho, \tau) \), where \( \rho \) is the dominant singularity of \( \hat{C}(x) \) and \( \tau := \lim_{x \to \rho^-} \hat{C}(x) \).

As a consequence, Theorem 2.3 ensures that \( \hat{C}(x) \) has a singular expansion

\[
\hat{C}(x) = \sum_{k \geq 0} \hat{C}_k X^k, \quad \text{with} \quad X := \sqrt{1 - \frac{x}{\rho}}, \quad \hat{C}_0 = \tau, \quad \hat{C}_1 = -\frac{2\rho \frac{\partial}{\partial \rho} H(\rho, \tau)}{\frac{\partial^2}{\partial \rho^2} H(\rho, \tau)}, \quad (4.2)
\]
with constants $\hat{C}_k$, $k \geq 2$, which can be computed from the derivatives of $H(x,y)$ at $(\rho, \tau)$.

Proof. The conditions $H(0,0) = 0$ and $\frac{\partial}{\partial s} H(0,0) = -1$ can be verified easily. The positivity conditions on the coefficients of $H(x,y)$ follow from the positivity of the coefficients of $Z(V)$.

The analyticity domain of $H(x,y)$ is determined by the dominant singularities of $Z(V)$; that is, $H(x,y)$ is analytic for $x$ and $y$ such that $|y| < \delta$ and $|x| < \rho$ and $|\hat{C}(x^l)| < \delta$ for each $l \geq 2$.

Since $\hat{C}(x)$ is strictly increasing for positive $x$, and since $\rho < 1$, $|\hat{C}(x^l)| \leq |\hat{C}(x^2)|$ for all $l \geq 2$ and $|x| < \rho$. Therefore, $H(x,y)$ is analytic for $|x| < R := \min(\sqrt{\rho} \sqrt{C^{-1}(\delta)})$ and $|y| < S := \delta$.

We show next that $\rho < R$ and $\tau < S$.

1. We show $\tau \leq S = \delta$. Let $\hat{H}(x,y) := H(x,y) + y$. $\hat{H}(x,y)$ satisfies $\hat{H}(x,\hat{C}(x)) = \hat{C}(x)$ and has the same domain of analyticity as $H(x,y)$. Assume $\tau > \delta$. Then there exists $x_0 < \rho$ such that $\hat{C}(x_0) = \delta$. Observe that, if $|x| < x_0$ then $|\hat{C}(x^2)| \leq |\hat{C}(x)| = \delta$. Thus $(x,\hat{C}(x))$ is in the analyticity domain of $H(x,y)$, so that $H(x,\hat{C}(x)) = \hat{C}(x)$. By continuity we obtain $H(x_0,\hat{C}(x_0)) = \hat{C}(x_0)$. We have now the contradiction that $\hat{C}(x)$ is analytic at $x_0$ since $x_0 < \rho$, whereas $H(x,\hat{C}(x))$ is singular at $x_0$ because $\hat{C}(x_0) = \delta$.

2. From 1 we know that $\tau \leq \delta$, i.e., $\rho \leq \hat{C}^{-1}(\delta)$. Hence $R = \sqrt{\rho} > \rho$.

3. It remains to prove that $\tau < S$. Assume $\tau = \delta$. Observe from (8.1) and (8.11) that

$$\hat{C}(x) = x \exp(\Psi(\hat{C}(x)) + A(x))$$

where $\Psi(y) = 1/(1+y - \sqrt{1-6y+y^2})$ has a dominant singularity at $y = \delta$, and where $A(x)$ is a generating function analytic for $|x| < \rho$ and having nonnegative coefficients. (This follows from the fact that $2A(x)$ is the generating function for reflective vertex rooted dissections.) Hence, for $0 < x < \rho$,

$$\hat{C}'(x) \geq \hat{C}'(x) \Psi'(\hat{C}(x)) \hat{C}(x),$$

so that $\Psi'(\hat{C}(x)) \leq 1/\hat{C}(x)$. Thus, $\Psi'(\hat{C}(x))$ is bounded when $x \to \rho^-$, which contradicts the fact that $\lim_{y \to \delta^-} \Psi'(y) = +\infty$.

Thus, $H(x,y)$ is analytic at $(\rho, \tau)$ and $H(\rho, \tau) = 0$ is satisfied. As pointed out before, the dominant singularity $\rho$ of $\hat{C}(x)$ is determined either by a singularity in a component of Equation (8.11), or by a non-uniqueness in the definition of $\hat{C}(x)$ by Equation (8.11). The relation $\tau < \delta$ excludes the first case, so that the singularity is caused by a non-uniqueness of the inversion. Hence, the derivative of $H(x,y)$ with respect to $y$ has to vanish at $(x,y) = (\rho, \tau)$, since otherwise the implicit function theorem ensures a (unique) analytic continuation of $\hat{C}(x)$ at $x = \rho$. Therefore, the equations from (8.10) are satisfied for $(r,s) = (\rho, \tau)$.

Furthermore, it is easily verified that

$$\frac{\partial^2}{\partial y^2} H(x,y) \bigg|_{(x,y)=(\rho,\tau)} = \frac{1}{\tau} + \frac{\partial^2}{\partial s^2} Z(V; s_1, \hat{C}(\rho^2)) \bigg|_{s_1 = \tau} = \frac{1}{\tau} + \frac{\tau}{(\tau^2 - 6\tau + 1)^{3/2}}.$$

$$\frac{\partial}{\partial x} H(x,y) \bigg|_{(x,y)=(\rho,\tau)} = \tau \left( \frac{1}{\rho} + \frac{\partial}{\partial x} Z(V; \tau, \hat{C}(x^2)) \bigg|_{x = \rho} + \sum_{k \geq 2} \frac{Z(V; \hat{C}(x^k), \hat{C}(x^{2k}))}{k \hat{C}(x^k)} \bigg|_{x = \rho} \right).$$

From $0 < \tau < \delta$ and the fact that the derivative in $\frac{\partial}{\partial x} H(\rho, \tau)$ is a derivative of a formal power series with positive coefficients evaluated at $\rho > 0$, it follows that both derivatives are strictly positive and hence do not vanish.

Finally, the aperiodicity of $\hat{C}(x)$ is easily seen from the fact that $\hat{c}_1 \neq 0$ and $\hat{c}_2 \neq 0$. \qed
Theorem 4.3 [asymptotic number of connected outerplanar graphs]. The function \( C(x) \) has a singular expansion of the form

\[
C(x) = C(\rho) + \sum_{k \geq 2} C_k x^k, \quad X := \sqrt{1 - \frac{x}{\rho}}.
\]

with constants \( C_k, k \geq 2 \), which can be computed from the constants \( \hat{C}_k \), and with \( \rho \) as in Lemma \ref{lem:k2}. Hence,

\[
c_n \sim \frac{3C_4}{4\sqrt{n}} n^{-5/2} \rho^{-n}.
\]

Proof. Recall Formula \ref{eq:outerplanar} for the ordinary generating function for connected outerplanar graphs,

\[
C(x) = \hat{C}(x) + Z(D; \hat{C}(x)) - Z(V; \hat{C}(x)).
\]

Since \( \tau < \delta \), it is clear that the dominant singularity of \( C(x) \) is the same as \( \hat{C}(x) \) \cite[Ch. VI.6]{Flajolet}. The singular expansion of \( C(x) \) around \( \rho \) can then be obtained by injecting the singular expansion of \( \hat{C}(x) \) into Formula \ref{eq:outerplanar}:

\[
C(x) = \sum_{k \geq 0} \hat{C}_k x^k + Z\left(D; \sum_{k \geq 0} \hat{C}_k x^k, \hat{C}\left(\rho^2 (1 - X^2)^2\right), \hat{C}\left(\rho^3 (1 - X^2)^3\right), \ldots\right) - Z\left(V; \sum_{k \geq 0} \hat{C}_k x^k, \hat{C}\left(\rho^2 (1 - X^2)^2\right), \hat{C}\left(\rho^3 (1 - X^2)^3\right), \ldots\right).
\]

Developing in terms of \( X \) (around \( X = 0 \)) gives a singular expansion \( C(x) = \sum_{k \geq 0} C_k x^k \).

It remains to check that \( C_1 = 0 \) and \( C_3 \neq 0 \). From \ref{eq:outerplanar} it is clear that

\[
C_1 = \hat{C}_1 + \hat{C}_1 \frac{\partial}{\partial s_1} Z(D)\bigg|_{(s_1, s_2) = (\tau, \hat{C}(\rho^2))} - \hat{C}_1 \frac{\partial}{\partial s_1} Z(V)\bigg|_{(s_1, s_2) = (\tau, \hat{C}(\rho^2))}.
\]

From \ref{eq:outerplanar} we know \( s_1 \frac{\partial}{\partial s_1} Z(D) = Z(V) \), so that

\[
C_1 = \hat{C}_1 \left(1 + \frac{Z(V)}{s_1} - \frac{\partial}{\partial s_1} Z(V)\right)\bigg|_{(s_1, s_2) = (\tau, \hat{C}(\rho^2))}.
\]

On the other hand, Equation \ref{eq:outerplanar} implies that

\[
\frac{\partial}{\partial y} H(x, y) = (H(x, y) + y) \left(\frac{1}{y} \frac{\partial}{\partial s_1} Z(V; y, \hat{C}(x^2)) - \frac{1}{y^2} Z(V; y, \hat{C}(x^2))\right) - 1.
\]

By Equation \ref{eq:outerplanar} and Lemma \ref{lem:k2} \( 0 = \frac{\partial}{\partial y} H(\rho, \tau) = \frac{\partial}{\partial s_1} Z(V) - \frac{1}{s_1} Z(V) - 1\bigg|_{(s_1, s_2) = (\tau, \hat{C}(\rho^2))}. \)

Thus, \( C_1 = 0 \). Assume \( C_3 = 0 \). Then the expansion \ref{eq:outerplanar} yields \( c_n \sim O(n^{-k/2 - 1}) \rho^{-n} \) for some odd number \( k \geq 5 \). This contradicts \( n c_n \geq c_n \sim -C_1 / (2\sqrt{\pi}) n^{-3/2} \rho^{-n} \) (by Lemma \ref{lem:k2}).

Theorem 4.4 [asymptotic number of outerplanar graphs]. The function \( G(x) \) has a singular expansion of the form

\[
G(x) = G(\rho) + \sum_{k \geq 2} G_k x^k, \quad X := \sqrt{1 - \frac{x}{\rho}}
\]

where \( \rho \) is as in Lemma \ref{lem:k2} and where the constants \( G_k, k \geq 2 \), can be computed from the constants \( C_k \), in particular \( G_3 = G(\rho) C_3 \). Furthermore, \( g_n \) has the asymptotic estimate

\[
g_n = \sum_{k \geq 1} \binom{n + k - 1}{n} G_{2k+1} \rho^{-n},
\]

where \( \rho \) is as in Lemma \ref{lem:k2} and where the constants \( G_k, k \geq 2 \), can be computed from the constants \( C_k \), in particular \( G_3 = G(\rho) C_3 \). Furthermore, \( g_n \) has the asymptotic estimate

\[
g_n = \sum_{k \geq 1} \binom{n + k - 1}{n} G_{2k+1} \rho^{-n},
\]
and in particular

\[ g_n \sim \frac{3G_3}{4\sqrt{\pi}}n^{-5/2}\rho^{-n}. \]

**Proof.** Recall Formula (3.13) for the ordinary generating function for outerplanar graphs,

\[ G(x) = \exp \left( \sum_{k \geq 1} \frac{1}{k} C(x^k) \right). \]

As the exponential function \( \exp(\cdot) \) is regular, the dominant singularity of \( G(x) \) is the same as \( C(x) \). Replacing \( C(x) \) by its singular expansion (1.3) and \( x^k \) by \( \rho^k (1 - X^2)^k \) for \( k \geq 2 \), we get

\[ G(x) = \exp \left( C(\rho) + \sum_{k \geq 2} C_k X^k + \sum_{k \geq 2} \frac{1}{k} C(\rho^k (1 - X^2)^k) \right), \]

from which the singular expansion of \( G(x) \) can be computed. Then, by Lemma 2.2 we derive the asymptotic estimate of \( g_n \).

Finally, using the same techniques as for the general case, we can compute the asymptotic estimate of bipartite outerplanar graphs, given in Theorem 1.3.

### 4.2 Numerical approximation of the growth constants

As far as we know, the computation of analytic expressions for growth constants has not been possible for some classes of unlabeled structures that are even simpler than outerplanar graphs, for example, for unembedded trees, see [11, Sec. VII.2.3] and [20]. Nevertheless, we can simplify the problem by reducing it to one variable, and provide numerical estimates of the growth constants. With Formula (4.5) for \( \frac{\partial}{\partial y} H(\rho, \tau) \) and the explicit formula for \( Z(\mathcal{V}) \) from Corollary 3.6 the equation \( \frac{\partial}{\partial y} H(\rho, \tau) = 0 \) becomes

\[ \tau \left( 1 + \hat{C}(\rho^2) \left( \hat{C}(\rho^2) - 3 \right) - \frac{\hat{C}(\rho^2)^2 (\tau - 3)}{\sqrt{\tau^2 - 6\tau + 1}} - \sqrt{\hat{C}(\rho^2)^2 - 6\hat{C}(\rho^2) + 1} \right) = 8\hat{C}(\rho^2)^2. \quad (4.6) \]

With algebraic elimination [11] App. B.1], Equation (4.6) can be reformulated as a system of polynomial equations, regarding \( \hat{C}(\rho^2) \) as a fixed value. We obtain a polynomial equation of degree 8 in \( \tau \) with coefficients \( p_i(\rho) \) (depending on \( \hat{C}(\rho^2) \)), \( i = 0, \ldots, 8 \),

\[ p_0(\rho) + p_1(\rho) \tau + p_2(\rho) \tau^2 + p_3(\rho) \tau^3 + p_4(\rho) \tau^4 + p_5(\rho) \tau^5 + p_6(\rho) \tau^6 + p_7(\rho) \tau^7 + p_8(\rho) \tau^8 = 0. \quad (4.7) \]

The solutions of (4.7) do not need to satisfy Equation (4.6), but every \( \tau \) that is a solution of (4.6) is also a solution of (4.7) (see [25] for the details). We denote the solutions of (4.7) by \( \tau_1(\rho), \ldots, \tau_8(\rho) \). It remains to solve the equations

\[ H(\rho, \tau_i(\rho)) = 0, \quad i = 1, \ldots, 8, \]

and to pick the correct solution \( \rho \). Since \( H(x, y) \) depends on \( \hat{C}(x) \), which we do not know explicitly, and since it includes also an infinite sum that we were not able to simplify, we can only approximate the solutions of \( H(\rho, \tau_i(\rho)) = 0 \) by truncating the infinite sum in \( H(x, y) \) at some index \( m \) and replacing \( \hat{C}(x) \) with \( \hat{C}[m](x) := \sum_{n=1}^m \hat{c}_n x^n \) for known coefficients \( \hat{c}_1, \ldots, \hat{c}_m \). That is, we search for roots of the functions

\[ \hat{H}_i[m](\rho) := \rho \exp \left( \frac{Z(\mathcal{V}; \tau_i(\rho), \hat{C}[m](\rho^2))}{\tau_i(\rho)} + \sum_{k=2}^m \frac{Z(\mathcal{V}; \hat{C}[m](\rho^k), \hat{C}[m](\rho^{2k}))}{k \hat{C}[m](\rho^k)} \right) - \tau_i(\rho), \]

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\( i = 1, \ldots, 8 \), in the interval \((0, 1)\). We solve the equation \( \tilde{H}^{[m]}_i(\rho) = 0 \) for \( m = 25 \) numerically, select the correct root, and obtain the estimates
\[
\rho \approx 0.1332694 \quad \text{and} \quad \tau \approx 0.1707560.
\]

The residuals in the equations \( \tilde{H}^{[m]}_i(\rho, \tau) = 0 \) and \( \frac{\partial}{\partial y} \tilde{H}^{[m]}_i(\rho, \tau) = 0 \) have an order of \( 10^{-58} \). Table 2 shows approximations of \( \rho \) for several values of \( m \).

| \( m \) | approximation of \( \rho \) |
|-------|-------------------|
| 1     | \( 0.13461876886110181369... \) |
| 4     | \( 0.13327064317786556821... \) |
| 8     | \( 0.1332694328029243729... \) |
| 16    | \( 0.1332694326674682071... \) |
| 25    | \( 0.1332694326674680944... \) |

Table 2: The accuracy is improved by increasing the order of truncation.

We can now estimate the coefficients in the singular expansions of \( \hat{C}(x) \), \( C(x) \) and \( G(x) \). In particular \( \hat{C}_1 \approx -0.0255905 \), \( C_3 \approx 0.0179720 \) and \( G_3 \approx 0.0215044 \).

The growth constant for bipartite outerplanar graphs can also be estimated in the same way as \( \rho \), and we get \( \rho_b \approx 0.218475 \) (see [28] for details).

5 Random unlabeled outerplanar graphs

This section investigates typical properties of a random unlabeled outerplanar graph with \( n \) vertices. We first discuss the probability of being connected, and the number and type of components, and then proceed with the distribution of the number of edges.

5.1 Connectedness, components, and isolated vertices

We start with the proof of Theorem 1.3 (1) on the probability that a random outerplanar graph is connected.

Proof of Theorem 1.3 (1). The probability that a random outerplanar graph on \( n \) vertices is connected is exactly \( c_n/g_n \). The asymptotic estimates for \( c_n \) and \( g_n \) from Theorem 4.3 and Theorem 4.4 yield \( c_n/g_n \approx C_3/G_3 \approx 0.845721 \).

The number of components can be studied by augmenting the generating function for outerplanar graphs with a variable that counts the number of components.

Proof of Theorem 1.3 (2). Let \( \kappa_n \) denote the number of components in a random outerplanar graph on \( n \) vertices and let \( G(x, u) := \exp \left( \sum_{k \geq 1} \frac{u^k}{k} C(x^k) \right) \) be the generating function for outerplanar graphs, where the additional variable \( u \) marks the number of components. Thus, the probability that an outerplanar graph has \( k \) components is \( P[\kappa_n = k] = [x^n u^k] G(x, u)/g_n \), and the expected number of components is
\[
\mathbb{E}[\kappa_n] = \frac{1}{g_n} \sum_{k \geq 1} k [x^n u^k] G(x, u) = \frac{1}{g_n} [x^n] \frac{\partial}{\partial u} G(x, 1) = \frac{1}{g_n} [x^n] G(x) \sum_{k \geq 1} C(x^k).
\]

By asymptotic expansion around \( x = \rho \), we obtain
\[
[x^n] G(x) \sum_{k \geq 1} C(x^k) \sim G(\rho) C_3 \left( 1 + \sum_{r \geq 1} C(\rho^r) \right) \frac{1}{\Gamma(-3/2)} n^{-3/2} \rho^{-n},
\]
which together with Theorem 4.4, more precisely \( g_n \sim G(\rho) C_3 \Gamma(-3/2) n^{-5/2} \rho^{-n} \), yields

\[
\mathbb{E}[\kappa_n] \sim 1 + \sum_{r \geq 1} C(\rho^r) \approx 1.17847. \]

Given a family \( \mathcal{A} \) of connected outerplanar graphs, we can make the following statements about the probability that a random outerplanar graph has exactly \( k \) components in \( \mathcal{A} \). Denote the number of graphs in \( \mathcal{A} \) that have exactly \( n \) vertices by \( a_n \), and let \( A(x) := \sum_n a_n x^n \).

**Theorem 5.1.** Given an outerplanar graph \( G \) with \( n \) vertices, let \( \kappa_n^A \) be the number of connected components of \( G \) belonging to \( \mathcal{A} \). If the radius of convergence \( \alpha \) of \( A(x) \) is strictly larger than \( \rho \), that is, \( a_n \) is exponentially smaller than \( c_n \), then the probability that a random outerplanar graph with \( n \) vertices has exactly \( k \geq 0 \) components belonging to \( \mathcal{A} \) converges to a discrete law:

\[
P[\kappa_n^A = k] \sim Z \left( S_k^A; A(\rho) \right) \exp \left( - \sum_{r \geq 1} \frac{1}{r} A(\rho^r) \right),
\]

and the expected number of components belonging to \( \mathcal{A} \) in a random outerplanar graph with \( n \) vertices is

\[
\mathbb{E}[\kappa_n^A] \sim \sum_{r \geq 1} A(\rho^r).
\]

**Proof.** Let \( G^A(x, u) \) be the generating function for outerplanar graphs, where the additional variable \( u \) marks the number of components belonging to \( \mathcal{A} \),

\[
G^A(x, u) := \exp \left( \sum_{k \geq 1} \frac{1}{k} \left( u^k A(x^k) + (C(x^k) - A(x^k)) \right) \right)
\]

\[
= G(x) \exp \left( \sum_{k \geq 1} \frac{u^k - 1}{k} A(x^k) \right).
\]

Hence, \( P[\kappa_n^A = k] = \frac{x^k u^k}{g_n} G^A(x, u) / g_n \). Since \( A(x) \) is analytic at \( \rho \), the dominant singularity of \( G^A(x, u) \) for fixed \( u \) is determined by \( G(x) \). Thus,

\[
[x^k u^k] G^A(x, u) \sim [u^k] \exp \left( \sum_{k \geq 1} \frac{u^k - 1}{k} A(\rho^k) \right) [x^n] G(x),
\]

i.e.,

\[
P[\kappa_n^A = k] \sim [u^k] \exp \left( \sum_{k \geq 1} \frac{u^k - 1}{k} A(\rho^k) \right) = Z \left( S_k^A; A(\rho) \right) \exp \left( - \sum_{k \geq 1} \frac{1}{k} A(\rho^k) \right).
\]

For the expectation of \( \kappa_n^A \) we again use

\[
\mathbb{E}[\kappa_n^A] = \frac{1}{g_n} [x^n] \frac{\partial}{\partial u} G^A(x, 1) = \frac{1}{g_n} [x^n] G(x) \sum_{k \geq 1} A(x^k).
\]

The statement follows from the analyticity of \( A(x) \) at \( \rho \) and Theorem 4.4.

The asymptotic distribution of the number of isolated vertices in a random outerplanar graph can now be easily computed, as stated in Theorem 1.3 (3).

---

[1] Theorem 4.4
[2] Theorem 1.3 (3)
Proof of Theorem 5.1 (β). Let $A$ be the family consisting of the graph that is a single vertex, i.e., $A(x) = x$. By Theorem 5.1, $\mathbb{P} [\kappa_n^{A} = k] \sim \rho^k / (1 - \rho)$, since $Z(S_k; A(\rho)) = \rho^k$ and $\sum_r \frac{1}{r} A(\rho^r) = \log (1 - \rho)$. Hence, the distribution of the number of isolated vertices $\kappa_n^{A}$ is asymptotically a geometric law with parameter $\rho$.

Other consequences of Theorem 5.1 concern the number of two-connected components and the number of bipartite components in a random outerplanar graph.

Corollary 5.2 [two-connected components]. In a random outerplanar graph, the expected number of connected components that are two-connected is asymptotically $\sum_{k \geq 1} D(\rho^k) \approx 0.175054$.

Proof. Let $\mathcal{A} := \mathcal{D}$ be the family of dissections, $A(x) = D(x)$. The radius of convergence of $D(\rho)$ is $\delta > \rho$ (Lemma 1.2). Hence, by Theorem 5.1, $\mathbb{E} [\kappa_n^{\mathcal{D}}] = \sum_{k \geq 1} D(\rho^k)$.

Corollary 5.3 [number of bipartite components]. In a random outerplanar graph, the expected number of connected components that are bipartite is asymptotically $\sum_{k \geq 1} C_b(\rho^k) \approx 0.175427$, where $C_b(x)$ is the generating function for bipartite connected outerplanar graphs.

Proof. We apply Theorem 5.1 with $A = \mathcal{C}_b$.

5.2 Number of edges

In this section, we analyze the distribution of the number of edges in a random outerplanar graph. To do this, we add a variable $y$ whose power (in the cycle index sums and generating functions) indicates the number of edges. For a graph $G$ on $n$ vertices and $m$ edges, and with the automorphism group $\Gamma(G)$ (acting on the vertices), we define

$$Z(G; s_1, s_2, \ldots, y) := Z(\Gamma(G); s_1, s_2, \ldots; y) := y^m \frac{1}{|\Gamma(G)|} \sum_{\alpha \in \Gamma(G)} \prod_{k=1}^n s_k^{j_k(\alpha)}.$$ 

Taking the number of edges into account in the calculations of Section 3, the cycle index sums for all encountered families of outerplanar graphs can be derived with the additional variable $y$ (see [28] for more details).

$$Z(\mathcal{C}_o^0) = \frac{s_1}{2(1 + y)} \left( s_1 y + 1 - \sqrt{(s_1 y - 1)^2 - 4 s_1 y^2} \right),$$

$$Z^+ (\mathcal{E}_o^0) = s_1 \frac{(s_1 y - 1)(s_1 y^2 - 2s_1 y^2 - 1) + (1 + s_1 y) \sqrt{(s_1 y^2 - 1)^2 - 4 s_1 y^4}}{2(s_1 y^3 + s_1 y^2 + y - 1)},$$

$$Z^- (\mathcal{E}_o^0) = \frac{s_2 (2y^2 + s_2 y^3 - y) - s_1 (s_2 y^2 + 2 s_2 y^3 - 1) - (s_1 + s_2) \sqrt{(s_2 y^2 - 1)^2 - 4 s_2 y^4}}{2(y + s_2 y^2 + s_2 y^3 - 1)},$$

$$Z (\mathcal{E}_o^0) = \frac{1}{2} (Z^+ (\mathcal{E}_o^0) + Z^- (\mathcal{E}_o^0)), $$

$$Z (\mathcal{E}_1) = \frac{1}{4} \left( Z (\mathcal{E}_o^0) - s_1 \right)^2 + \frac{Z (\mathcal{E}_o^0; s_2; y^2)}{s_2 y^2} + \left( 1 - (s_2 + s_1 y) \sqrt{(s_2 y^2 - 1)^2 - 4 s_2 y^4} \right),$$

$$Z (\mathcal{E}_o) = \frac{Z (\mathcal{E}_o^0; s_1^2; y^2)}{2 s_1 y} + \frac{s_1^2 + s_2}{4 s_2 y} Z (\mathcal{E}_o^0; s_2; y^2) + \frac{Z^+ (\mathcal{E}_o^0; s_1^2; y^2)}{4 s_2 y} + \frac{Z^- (\mathcal{E}_o^0; s_1^2; s_2; y^2)}{4 s_2 y} - \frac{s_1^2 + s_2}{2},$$

$$Z (\mathcal{F}) = \frac{1}{2} \frac{\varphi (d)}{d} \log \left( 1 - \frac{Z (\mathcal{E}_o^0; s_1; y^d)}{s_1} \right) - \frac{Z (\mathcal{E}_o^0; s_1; y^d)}{s_1} \frac{1}{2} \left( \frac{Z (\mathcal{E}_o^0)}{s_1} + \frac{Z (\mathcal{E}_o^0; s_2; y^2)}{s_2} \right),$$

$$Z (\mathcal{F}) = \frac{1}{2} \left( Z (\mathcal{F}) + \frac{1}{2 s_2} \left( s_1 Z^- (\mathcal{E}_o^0) + \frac{s_2^2}{2 s_2} Z (\mathcal{E}_o^0; s_2; y^2) + \frac{1}{2} Z^- (\mathcal{E}_o^0)^2 \right) \right).$$
Similarly as in Section 3, the coefficients counting outerplanar graphs with respect to the number
that additionally counts edges is connected outerplanar graph. The generating function for oriented outer-edge rooted dissections
Proof of Theorem 1.6. We start with the limit distribution of the number of edges in a two-connected outerplanar graph. The generating function \( \hat{C}(\chi) = \exp \sum_{k \geq 1} \frac{1}{k} \left( \frac{Z(\chi; s^k, s_{2k}; y^k)}{Z(\hat{C}; s^k, s_{2k}; y^k)} \right) \),

\[
Z(\mathcal{F}) = \frac{1}{s_1^2 y} Z(\mathcal{C}^o; s_1^2; y^2) - \frac{1}{2 s_1^2 y} Z^+ (\mathcal{C}^o; s_1^2, s_2^2; y^2) + \frac{1}{2 s_2 y} Z^- (\mathcal{C}^o; s_1^2, s_2^2; y^2) - \frac{y}{2} (s_1^2 + s_2),
\]

\[
Z(\mathcal{D}) = \frac{1}{2} (s_1^2 + s_2) y + Z(\mathcal{F}) - Z(\mathcal{F}_o) - 2Z(\hat{C}),
\]

\[
Z(\mathcal{V}) = s_1 \frac{\partial}{\partial s_1} Z(\mathcal{D}, s_1, s_2, \ldots; y),
\]

\[
Z(\hat{C}) = s_1 \exp \left( \sum_{k \geq 1} \frac{1}{k} \frac{Z(\mathcal{V}; Z(\hat{C}; s^k, s_{2k}; y^k), \ldots; y^k)}{Z(\hat{C}; s^k, s_{2k}; y^k)} \right),
\]

\[
Z(C) = Z(\hat{C}) + Z(\mathcal{D}; Z(\hat{C})) - Z(\mathcal{V}; Z(\hat{C})),
\]

\[
Z(G) = \exp \left( \sum_{k \geq 1} \frac{Z(C; s^k, s_{2k}, \ldots; y^k)}{k} \right).
\]

Similarly as in Section 3, the coefficients counting outerplanar graphs with respect to the number of vertices and the number of edges can be extracted in polynomial time from the expressions of the cycle index sums, see [28] for a table.

With the help of Theorem 2.4, we can prove Theorem 1.6 giving the limit distributions of the number of edges in a random dissection and in a random outerplanar graph, respectively.

**Proof of Theorem 1.6** We start with the limit distribution of the number of edges in a two-connected outerplanar graph. The generating function for oriented outer-edge rooted dissections that additionally counts edges is

\[
E^o(x, y) = \frac{x}{2 (y + 1)} \left( xy + 1 - \sqrt{(xy - 1)^2 - 4xy^2} \right).
\]

The singularities of \( E^o(x, y) \) are determined by the equation \( (xy - 1)^2 - 4xy^2 = 0 \). Hence, for \( y \) close to 1, the dominant singularity of \( x \mapsto E^o(x, y) \) is at \( \delta(y) = \frac{2 + 1/y - 2\sqrt{1 + 1/y}}{2} \). With the same arguments as before, \( \delta(y) \) is also the dominant singularity of the generating functions for vertex rooted and unrooted dissections. Furthermore, \( \delta'(1) = -1 + \sqrt{2}/2 \) and \( \frac{\delta''(1)}{\delta'(1)} = \frac{\delta''(1)}{\delta'(1)} + \left( \frac{\delta'(1)}{\delta'(1)} \right)^2 = \sqrt{2}/8 \neq 0 \), so that the variance condition (in Theorem 2.4) on \( \delta(y) \) is satisfied. Hence, Theorem 2.4 yields the statement for dissections.

We now determine the distribution of the number of edges in a rooted connected outerplanar graph. The generating function \( \hat{C}(x, y) \) is implicitly defined by

\[
\hat{C}(x, y) = x \exp \left( \sum_{k \geq 1} \frac{Z(\mathcal{V}; \hat{C}(x^k, y^k); y^k)}{k \hat{C}(x^k, y^k)} \right).
\]

In order to apply the singular implicit functions theorem 2.3 for the function \( x \mapsto \hat{C}(x, y) \) with a fixed \( y \) close to 1, we define

\[
H(x, y, z) := x \exp \left( \frac{Z(\mathcal{V}; \hat{C}(x^2, y^2); y)}{z} + \sum_{k \geq 2} \frac{Z(\mathcal{V}; \hat{C}(x^k, y^k), \hat{C}(x^{2k}, y^{2k}); y^k)}{k \hat{C}(x^k, y^k)} \right) - z
\]

and search for a solution \( (x, z) = (\rho(y), \tau(y)) \) of the system

\[
H(x, y, z) = 0, \quad \frac{\partial}{\partial z} H(x, y, z) = 0,
\]

such that \( (\rho(y), \tau(y)) \) is in the analyticity domain of \( (x, z) \mapsto H(x, y, z) \).
For $y = 1$, the solution is at $x = \rho, z = \tau$ by Lemma 4.2. Then the classical implicit functions theorem, applied to the system (5.1), ensures that the solution $(\rho, 1, \tau)$ can be extended into solutions $(\rho(y), y, \tau(y))$ for $y$ close to 1, where the functions $\rho(y)$ and $\tau(y)$ are analytic in a neighbourhood of 1. To apply the classical implicit function theorem on system (5.1), it remains to check that the determinant of the Jacobian of system (5.1), with respect to $x$ and $z$,

\[
\begin{pmatrix}
\frac{\partial}{\partial x} H(x, y, z) & \frac{\partial}{\partial z} H(x, y, z) \\
\frac{\partial}{\partial x} H(x, y, z) & \frac{\partial}{\partial z} H(x, y, z)
\end{pmatrix},
\]

does not vanish at $(x, y, z) = (\rho(1), 1, \tau(1))$. This is clear, since from Lemma 4.2 we have $\frac{\partial}{\partial x} H(\rho(1), 1, \tau(1)) = 0$, $\frac{\partial}{\partial z} H(\rho(1), 1, \tau(1)) \neq 0$, and $\frac{\partial^2}{\partial z^2} H(\rho(1), 1, \tau(1)) \neq 0$. Hence, there exist analytic functions $\rho(y)$ and $\tau(y)$ such that

\[
H(\rho(y), y, \tau(y)) = 0, \quad \frac{\partial}{\partial y} H(\rho(y), y, \tau(y)) = 0,
\]

$\frac{\partial^2}{\partial z^2} H(\rho(y), y, \tau(y)) \neq 0$, and $\frac{\partial}{\partial z} H(\rho(y), y, \tau(y)) \neq 0$ for $y$ close to one. In addition, these solutions are in the analyticity domain of $(x, z) \mapsto H(x, y, z)$ for $y$ close to 1, by analyticity of $(x, y, z) \mapsto H(x, y, z)$ at $(\rho, 1, \tau)$. Next, the singular implicit functions theorem 2.3 yields a singular expansion $C(x, y) = \sum_{k \geq 0} C_k(y) (\sqrt{1-x/\rho(y)})^k$ with coefficients $C_k(y)$ analytic at $y = 1$ and verifying $C_1(y) \neq 0$ for $y$ close to 1.

To find $\rho'(1)$ and $\rho''(1)$ we compute the first and second derivatives of the equations in (5.2) with respect to $y$, and express $\rho'(y)$ and $\rho''(y)$ in terms of $\rho(y)$, $\tau(y)$, and the partial derivatives of $H(x, y, z)$ at $(x, z) = (\rho(y), \tau(y))$. Using the approximated values from Section 4.2 we obtain $\rho'(1) \approx -0.206426$, $\rho''(1) \approx 0.495849$, and $-\frac{\rho''(1)}{\rho'(1)} - \frac{\rho'(1)}{\rho''(1)} + (\frac{\rho''(1)}{\rho'(1)})^2 \approx 0.227504 \neq 0$. Theorem 4.2 implies that the distribution of the number of edges in a random rooted connected outerplanar graph with $n$ vertices asymptotically follows a Gaussian law with mean $\mu n$ and variance $\sigma^2 n$, where $\mu = -\frac{\rho'(1)}{\rho''(1)} \approx 1.54894$ and $\sigma^2 \approx 0.227504$. The same holds for unrooted connected outerplanar graphs and for outerplanar graphs, since their generating functions have the same dominant singularity.

6 Conclusion

A summary of the estimated growth constants and other parameters for unlabeled outerplanar graphs is presented in Table 3. For comparison we also include the corresponding labeled quantities derived in [3]. Observe that in the two-connected case the estimated quantities for the unlabeled and labeled structures do not differ, since their dominant singularity is determined by the same equation (compare Theorem 4.1 and the formula for $B(x)$ in [3]).

References

[1] E. A. Bender, Z. Gao, and N. C. Wormald, The number of labeled 2-connected planar graphs, The Electronic Journal of Combinatorics 9 (2002), #43.

[2] F. Bergeron, G. Labelle, P. Leroux, Combinatorial Species and Tree-like Structures, Cambridge University Press, Cambridge (1998)

[3] M. Bodirsky, O. Giménez, M. Kang, and M. Noy, The asymptotic number of outerplanar graphs and series-parallel graphs, in the Proceedings of European Conference on Combinatorics, Graph Theory, and Applications (EuroComb05), DMTCS Proceedings Volume AE (2005), 383 – 388.
|                              | dissections | outerplanar graphs |
|------------------------------|-------------|--------------------|
| growth constant              | $\delta^{-1} \approx 5.82843$ | $\lambda^{-1} \approx 7.32098$ |
| $\mathbb{P}[\text{connectivity}]$ | $\rho^{-1} \approx 7.50360$ | $\lambda^{-1} \approx 8.61667$ |
| $\mathbb{E}[\text{nr. of components}]$ | $1$ | $1.17847$ |
| distrib. of nr. of isolated vertices | Dirac | Geom($\rho$) |
| $\mathbb{E}[\text{nr. of isolated vertices}]$ | $0$ | $1.136593$ |
| chromatic number             | $3$ | $3$ |
| distrib. of nr. of edges     | Gaussian | Gaussian |
| $\mathbb{E}[\text{nr. of edges}]$ | $1.70711n$ | $1.54894n$ |
| $\mathbb{V}[\text{nr. of edges}]$ | $0.176777n$ | $0.227504n$ |

Table 3: Summary of growth constants, typical properties, and limit laws for unlabeled and labeled dissections and outerplanar graphs.

[4] M. Bodirsky, C. Gröpl, D. Johannsen, and M. Kang, A direct decomposition of 3-connected planar graphs, in the Proceedings of the 17th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC’05), 2005.

[5] M. Bodirsky, C. Gröpl, and M. Kang, Decomposing, counting, and generating unlabeled cubic planar graphs, submitted (2004).

[6] M. Bodirsky, C. Gröpl, and M. Kang, Sampling unlabeled biconnected planar graphs, in the Proceedings of the 16th Annual International Symposium on Algorithms and Computation (ISAAC’05), 2005, Springer LNCS 3827, 593 – 603.

[7] M. Bodirsky, C. Gröpl, and M. Kang, Generating labeled planar graphs uniformly at random, in the Proceedings of the 13th International Colloquium on Automata, Languages and Programming (ICALP’03), 2003, Springer LNCS 2719, 1095 – 1107.

[8] M. Bodirsky and M. Kang, Generating outerplanar graphs uniformly at random, accepted for publication in Combinatorics, Probability, and Computation (2003).

[9] G. Chartrand and F. Harary, Planar permutation graphs, Ann. Inst. Henry Poincaré, Nouv. Sér., Sect. B3 (1967), 433 – 438.

[10] S. Finch, Two asymptotic series (2003), available online at http://pauillac.inria.fr/algo/csolve/asym.pdf.

[11] P. Flajolet and R. Sedgewick, Analytic combinatorics, 0th Edition (October 1, 2005), available online at http://algo.inria.fr/flajolet/Publications/book051001.pdf.

[12] É. Fusy, Quadratic exact-size and linear approximate-size random sampling of planar graphs, in the Proceedings of the International Conference on the Analysis of Algorithms (AofA’05), DMTCS Proceedings Volume AD (2005), 125 – 138.

[13] É. Fusy, D. Poulalhon, and G. Schaeffer, Dissections and trees, with applications to optimal mesh encoding and random sampling, in the Proceedings of the Symposium on Discrete Algorithms (SODA’05), 2005, 690 – 699.

[14] O. Giménez and M. Noy, Asymptotic enumeration and limit laws of planar graphs (2005), available online at http://arxiv.org/abs/math/0501269.

[15] F. Harary and E. M. Palmer, Graphical enumeration, Academic Press, New York (1973).

[16] G. Labelle, C. Lamathe, and P. Leroux, Labelled and unlabelled enumeration of $k$-gonal 2-trees, Journal of Combinatorial Theory, Series A, 106 (2004), 193 – 219.
[17] J. Leydold and P. F. Stadler, Minimal cycle bases of outerplanar graphs, *Electronic Journal of Combinatorics* 5 #R16 (1998).

[18] C. McDiarmid, A. Steger, and D. Welsh, Random planar graphs, *Journal of Combinatorial Theory, Series B*, 93 (2005), 187 – 205.

[19] R. C. Mullin and P. J. Schellenberg, The enumeration of c-nets via quadrangulations, *Journal of Combinatorial Theory* 4 (1968), 259 – 276.

[20] R. Otter, The number of trees, *Annals of Mathematics* 49 (1948), 583 – 599.

[21] J. M. Plotkin and J. W. Rosenthal, How to obtain a singular expansion of a sequence from an analytic identity satisfied by its generating function, *Journal of the Australian Mathematical Society, Series A*, Vol. 56, No. 1 (1994), 131 – 143.

[22] G. Pólya, Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen, *Acta Mathematica* 68 (1937), 145 – 254.

[23] R. C. Read, On general dissections of a polygon, *Aequationes Mathematicae* 18, University of Waterloo (1978), 370 – 88.

[24] G. Schaeffer, Random sampling of large planar maps and convex polyhedra, in the Proceedings of the 31st annual ACM symposium on theory of computing (STOC’99), 1999, 760 – 769.

[25] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, [http://www.research.att.com/~njas/sequences](http://www.research.att.com/~njas/sequences)

[26] W. T. Tutte, A census of planar maps, *Canadian Journal of Mathematics* 15 (1963), 249 – 271.

[27] Y. Tomii, *Gewurzelte Unbeschriftete Outerplanare Graphen*, Studienarbeit, Humboldt-Universität zu Berlin, 2005

[28] S. Vigerske, *Asymptotic enumeration of unlabelled outerplanar graphs*, Diploma thesis, Humboldt University Berlin, 2005, available online at [http://www.informatik.hu-berlin.de/ForschungLehre/algorithmen/en/forschung/planar/vigerske.html](http://www.informatik.hu-berlin.de/ForschungLehre/algorithmen/en/forschung/planar/vigerske.html).

[29] T. Walsh and V. A. Liskovets, Ten steps to counting planar graphs, in the Proceedings of Eighteenth Southeastern International Conference on Combinatorics, Graph Theory, and Computing, *Congr. Numer.* (1987), 269 – 277.

[30] H. Whitney, Congruent graphs and the connectivity of graphs, *American Journal of Mathematics* 54 (1932), 150 – 168.