An Entropy-based Proof of Threshold Saturation for Nonbinary SC-LDPC Ensembles on the BEC

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Abstract

In this paper we are concerned with the asymptotic analysis of nonbinary spatially-coupled low-density parity-check (SC-LDPC) ensembles defined over $GL(2^m)$ (the general linear group of degree $m$ over GF(2)). Our purpose is to prove threshold saturation when the transmission takes place on the binary erasure channel (BEC). To this end, we establish the duality rule for entropy for nonbinary variable-node (VN) and check-node (CN) convolutional operators to accommodate the nonbinary density evolution (DE) analysis. Based on this, we construct the explicit forms of the potential functions for uncoupled and coupled DE recursions. In addition, we show that these functions exhibit similar monotonicity properties as those for binary LDPC and SC-LDPC ensembles over general binary memoryless symmetric (BMS) channels. This leads to the threshold saturation theorem and its converse for nonbinary SC-LDPC ensembles on the BEC, following the proof technique developed by S. Kumar et al.

Index Terms

Density evolution, potential functions, threshold saturation, spatial coupling, nonbinary low-density parity-check codes.

I. INTRODUCTION

Spatial coupling has been recognized as an effective way of improving the performance of low-density parity-check (LDPC) codes. This concept was first introduced in [1], but its underlying idea can be traced back to the benchmark work by Zigangirov [2] for the design of LDPC codes with convolutional structures. The resultant codes, termed as spatially-coupled LDPC (SC-LDPC)
codes, are found to have better error correction capability than the uncoupled ones in terms of decoding threshold [3][4]. This finding motivates the applications of the underlying principle behind SC-LDPC codes to a wide variety of communication systems with much success. See [5][6][7] for coded modulation systems, [8][9][10] for inter-symbol interference channels, and [11][12][13] for multiple access channels.

From a design point of view, it is of particular importance to predict the asymptotic performance gain introduced by SC-LDPC codes compared with standard LDPC block codes. This can be done by calculating the belief propagation (BP) threshold of SC-LDPC codes based on the coupled DE algorithm, but at a cost of high complexity. A more efficient way is to prove the existence of the threshold saturation effect. For example, for a binary regular SC-LDPC code, it has been shown that the BP threshold saturates to the maximum-a-posteriori (MAP) threshold of its underlying uncoupled LDPC codes on the BEC [1] and general binary memoryless symmetric (BMS) channels [14]. As the MAP threshold of the binary regular LDPC code can be (tightly) calculated based on the generalized extrinsic information transfer (GEXIT) chart [15], this theoretic result provides a simple guidance to predetermine the asymptotic BP threshold of a regular SC-LDPC code, avoiding the need of the coupled DE algorithm.

For general SC-LDPC coded systems characterized by scalar recursions (e.g., the DE recursion for a binary irregular SC-LDPC code on the BEC), Yedla et al. introduced a technique based on potential functions for the proof of threshold saturation [16][17]. The underlying idea behind this technique is to construct real-valued potential functions by taking an integral of scalar DE recursions (e.g., the areas under the transfer curves in the EXIT chart). By doing this, Yedla et al. proceeded the analysis of the DE fixed points by investigating the stationary points of the potential functions. They proved that, the threshold of a scalar coupled DE recursion asymptotically coincides with the potential threshold of the uncoupled DE recursion defined by the vanishing of the so-called energy gap (a local minimum of the underlying potential function). This technique can be directly applied to binary irregular SC-LDPC codes on the BEC, proving the existence of the threshold saturation effect in this scenario.

The work by [16][17] was proposed for scalar recursions. The main difficulty of its extension to nonscalar recursions is how to construct potential functions by taking an integral over the space of density functions. For SC-LDPC codes on the BMS channels, S. Kumar et al circumvented this difficulty by specializing the replica-symmetric (RS) free entropy functional to LDPC ensembles and derived the potential functions based on entropies [18]. It turns out that these functions are
the negative of the RS free entropies associated with the code ensembles. Their analysis shed a light on the invaluable role of the duality rule for entropy \cite{15} in the construction of potential functions. This rule reveals an entropy conservation relation involving the variable-node (VN) and the check-node (CN) convolutional operators \cite{19}, establishing the bridge between the DE fixed points and the stationary points of potential functions. Following the idea by S. Kumar \textit{et al.}, we are able to extend the entropy-based proof technique to binary irregular SC-LDPC ensembles on general BMS channels.

The performance gain introduced by employing nonbinary SC-LDPC codes has been numerically observed \cite{20}\cite{21}\cite{22}. It arises a natural question whether the threshold saturation effect also exists in such scenarios. Motivated by this, the authors in \cite{23} studied nonbinary SC-LDPC codes defined over the general linear group when the transmission takes place on the BEC. They concluded that, to apply the proof technique by Yedla \textit{et al}, one should first identify the existence of the potential functions for the nonscalar DE recursions. For this reason, the authors developed a constructive criterion that is applicable to general vector spatially-coupled recursions defined over general multivariate polynomials. Although the authors conjectured that potential functions always exist, it seems not an easy task to construct these functions except for some special cases (see Table II therein).

In this paper, we focus on the asymptotic performance of nonbinary SC-LDPC ensembles defined over the general linear group GL$(2^m)$ and prove that the threshold saturation effect indeed occurs for transmission on the BEC. Our work is a nonstraightforward extension of \cite{23} and \cite{18}. Our contribution is three-fold.

- First of all, we establish the duality rule for entropy for nonbinary DE recursions on the BEC. As in the binary case mentioned above, this rule also reveals a conservation relation between the input and the output entropy of nonbinary VN and CN convolutional operators and is the key step towards constructing potential functions in the proof of threshold saturation.
- Secondly, we propose the explicit forms of nonbinary potential functions similar to those in \cite{18} derived for binary SC-LDPC ensembles over BMS channels. This proves the conjecture proposed in \cite{23} for all code degree distributions and $m$. We further show that these potential functions exhibit similar monotonicity properties including the partial order preservation properties. This finding implies that it is possible to develop the threshold saturation theorem and its converse for nonbinary SC-LDPC ensembles on the BEC, following the idea by S. Kumar \textit{et al} \cite{18}.
Finally, we modify the definition of the energy gap that is used to calculate the potential threshold of the underlying LDPC ensemble. In specific, the energy gap in [18] is defined based on the infimum over the complementary subset of the basin of attraction to the trivial DE fixed point, while in our work we restrict the complementary subset to the set of nontrivial underlying DE fixed points (see Definition 11).

The remainder of the paper is organized as follows. In Section II, we define nonbinary LDPC and SC-LDPC ensembles concerned in this paper and briefly discuss the form of the density in the nonbinary DE analysis. In Section III, we review the definitions of the entropy function and the VN and CN convolutional operators. We establish and prove several important identities and properties including the duality rule for entropy and the partial order preservation properties. In Section IV, we construct potential functions for nonbinary uncoupled and coupled DE recursions. The monotonicity properties of these functions are also proposed and proved based on the theoretic results in Section III. We establish the threshold saturation theorem and its converse at the end of Section IV. Finally, Section V concludes the whole paper.

A. Notations

We use $\mathbb{R}$ to represent the set of all real numbers and define $\mathbb{Z} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}_+ = \mathbb{Z}\setminus\{0\}$. For any $m \in \mathbb{Z}_+$, we define $\mathbb{M} = \{0, 1, \ldots, m\}$. The two integers $N$ and $w$ denote the coupling length and the coupling width for an SC-LDPC ensemble, respectively. By defining $N_w = N + w - 1$, we introduce $\mathbb{N}_v = \{0, 1, \ldots, N - 1\}$ and $\mathbb{N}_c = \{0, 1, \ldots, N_w - 1\}$ to denote the positions of VNs and CNs, respectively. Further, define $N_{\text{mid}}^w = \lfloor (N + w - 1)/2 \rfloor$ where $\lfloor x \rfloor$ represents the maximum integer less than or equal to $x \in \mathbb{R}$.

II. Preliminaries

A. LDPC and SC-LDPC Ensembles Defined Over $\text{GL}(2^m)$

Denote by $\text{LDPC}(\lambda, \rho, m)$ the nonbinary LDPC ensemble defined over the general linear group $\text{GL}(2^m)$. Here we omit the codeword length for notational brevity, since in this paper we always restrict ourselves to the limit where the codeword length trends to infinity. Following the standard notational convention, we use $\lambda(x) = \sum_i \lambda_i x^{i-1}$ and $\rho(x) = \sum_j \rho_j x^{j-1}$ to denote the edge-perspective degree distributions of VNs and CNs, respectively, with nonnegative coefficients $\lambda_i$.
and \( \rho_j \) satisfying \( \lambda(1) = \rho(1) = 1 \). We also adopt node-perspective degree distributions denoted as \( L(x) = \sum_i L_i x^i \) and \( R(x) = \sum_j R_j x^j \), the coefficients of which are determined by \[ L_i = \frac{\lambda_i / i}{\sum_k \lambda_k / k}, \quad R_j = \frac{\rho_j / j}{\sum_k \rho_k / k}. \] (1)

A nonbinary LDPC code selected from LDPC(\( \lambda, \rho, m \)) can be described in the form of a bipartite graph termed the Tanner graph. Each VN \( i \) in the Tanner graph corresponds to a coded symbol defined over \( \text{GF}(2^m) \). When the transmission takes place on the BEC, it is convenient to write the coded symbol in the form of a binary column vector of \( m \) bits, i.e. \( \mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,m})^T \) with \( x_{i,k} \in \{0, 1\}, \forall k \in \mathbb{M} \setminus \{0\} \). With this notation, we can represent the coding constraint imposed by each CN \( a \) as follows
\[ \sum_{i \in \partial a} \mathbf{W}_{i,a} \mathbf{x}_i = \mathbf{0} \] (2)

where \( \mathbf{0} \) denotes the zero vector of length \( m \), \( \partial a \) the subset of VNs connected to CN \( a \), and \( \mathbf{W}_{i,a} \), a binary \( m \)-by-\( m \) invertible matrix uniformly selected from \( \text{GL}(2^m) \) at random, is the label of the edge from VN \( i \) to CN \( a \) in the Tanner graph.

We also consider the nonbinary SC-LDPC ensemble over \( \text{GL}(2^m) \) denoted as SC-LDPC(\( \lambda, \rho, N, w, m \)) in this paper. Such an ensemble can be constructed from the graphic perspective as follows. As illustrated in Fig. 1, we first place the Tanner graphs of LDPC(\( \lambda, \rho, m \)) along a chain, the positions of which are indexed by an integer \( k \). Next, at each position \( k \), the outgoing edges of the VNs are uniformly and randomly divided into \( w \) groups, being reconnected to those CNs at positions \( \{k, k+1, \ldots, k+w-1\} \). Likewise, the CNs at each position \( k \) are also uniformly and randomly connected to the VNs at positions \( \{k-w+1, k-w+2, \ldots, k\} \). After that, we terminate the coupling chain by removing the VNs at positions \( \{\ldots, -2, -1\} \cup \{N, N+1, \ldots\} \) and their outgoing edges. As a result, all CNs with degree less than two become invalid and thus are also removed from the coupling chain. The resultant graph is referred to as the Tanner graph of SC-LDPC(\( \lambda, \rho, N, w, m \)). The termination procedure will reduce the degrees of some CNs at the two ends of the coupling chain. A coding rate loss is introduced, but it will vanish as \( N \to \infty \) (while keeping \( w \) fixed). More importantly, the termination procedure leads to a phenomenon termed decoding wave propagation in the BP decoding algorithm, which is the fundamental mechanism behind threshold saturation.

One may equivalently define SC-LDPC(\( \lambda, \rho, N, w, m \)) from the parity-check matrix perspective. See [23] for details.
Fig. 1. The Tanner graph for SC-LDPC($\lambda, \rho, 9, 3, m$), where each square (resp., circle) represents a collection of multiple CNs (resp., VNs) of the underlying LDPC ensemble located at that position. The dashed squares, circles and edges are removed in the termination procedure.

B. Densities of Messages in BP Decoding

In the DE analysis, we are interested in tracking the distributions of messages exchanged in the BP decoding algorithm. These distributions are referred to as the densities. In general, density tracking is difficult for nonbinary LDPC ensembles since the decoding performance may depend on the transmitted codeword with $2^m$ possible values for each coded symbol $x_i$. Fortunately, in the case where the transmission takes place on the BEC and the edge labels are defined over $\text{GL}(2^m)$, the form of the density can be simplified. First of all, thanks to the symmetry of the BEC, the BP decoding performance does not depend on the specific transmitted codeword, therefore we can assume that the all-zero codeword is transmitted [15]. Under this assumption, the a posteriori probability mass function (PMF) of $x_i$ is equiprobable over a subspace $S$ of the $m$-dimensional binary vector space [24]. Consider an example where $m = 3$, $x_i = (0, 0, 0)^T$ and $y_i = (0, ?, ?)^T$ with $y_i$ being the channel observation containing $k = 2$ erased bits “?”. In this example, the a posteriori PMF is given by $p(x_i|y_i) = 1/2^k = 0.25$ if $x_i$ takes values from the subspace $S = \{(0, 0, 0)^T, (0, 0, 1)^T, (0, 1, 0)^T, (0, 1, 1)^T\}$ of dimension $k = 2$, and $p(x_i|y_i) = 0$ otherwise. Secondly, it can be shown that the subspace dimension does not change when a message is passed along an edge in the BP decoding algorithm. To see this, notice that if the a posteriori PMF of $x_i$ is equiprobable over $S$, then the a posteriori PMF of $W_{i,a}x_i$ is equiprobable over $S' = \{x'|x' = W_{i,a}x, \forall x \in S\}$. Obviously, the dimensions of $S'$ and $S$ are identical due to the fact that the binary matrix $W_{i,a}$ is invertible. Therefore, it is sufficient to keep track the subspace dimensions instead of the a posteriori PMFs of coded symbols [24].

For the above reason, in this paper, our discussions are based on the density with the following form as in [23] [24].
Definition 1: The density of a message in the BP decoding algorithm for LDPC($\lambda, \rho, m$) and SC-LDPC($\lambda, \rho, N, w, m$) on the BEC is defined as the probability vector of length $m + 1$, the $k$-th entry of which is the probability that the a posteriori PMF corresponding to the message is equiprobable over a subspace of dimension $k$, $\forall k \in \mathbb{M}$. In what follows, the set of all such densities will be denoted as $\mathcal{X}$, i.e.,

$$\mathcal{X} = \left\{ a = (a_0, a_1, \ldots, a_m) \mid \sum_{k=0}^{m} a_k = 1, a_k \geq 0, k \in \mathbb{M} \right\}. \quad (3)$$

For notational brevity, we will also use $[a]_k$ to represent the $k$-th entry of $a$, $\forall k \in \mathbb{M}$.

There are two extremal densities in $\mathcal{X}$, one of which is $\Delta_m = (0, 0, \ldots, 0, 1)$ corresponding to the case where the message offers no information about the coded symbol, and the other is $\Delta_0 = (1, 0, \ldots, 0, 0)$ corresponding to the error-free case where the coded symbol can be recovered from the message perfectly. Further, we will use $\Delta_k$ to denote the density with the $k$-th entry being 1 and others being 0, $\forall k \in \mathbb{M}$.

III. Duality Rule for Entropy and Partial Ordering

A. The Duality Rule for Entropy

In this subsection, we will establish the duality rule for entropy for nonbinary LDPC and SC-LDPC ensembles on the BEC. To this end, we first present and review the definitions of the entropy function and the basic VN and CN operators.

Definition 2: For any $a \in \mathcal{X}$, the entropy function of $a$ is defined as

$$H(a) = \sum_{k=1}^{m} k a_k. \quad (4)$$

Remark 1: The entropy function $H(a)$ can be regarded as a measure of the average uncertainty of a message, the distribution of which can be determined by $a$. As discussed in Subsection II-B, the a posteriori PMF of a message is always equiprobable over a subspace $S$. Let $k$ be the dimension of $S$. Since there are $2^k$ elements in $S$ with equal probability, the uncertainty of this message is $k$ bits. Therefore, if we treat $k$ as a random variable with $a$ being the distribution, then the average uncertainty of the message is given by $\sum_{k=1}^{m} k a_k$ bits.

In this paper, we adopt the notions $\Box$ and $\Diamond$ introduced in [24] for the VN and CN convolutional operators.
Definition 3: For any $a, b \in \mathcal{X}$, $a \boxminus b$ and $a \boxtimes b$ are two densities, the $k$-th entries of which are respectively given by

$$\begin{align*}
[a \boxminus b]_k &= \sum_{i=0}^{m} \sum_{j=0}^{m} a_i V_{i,j,k}^m b_j, \\
[a \boxtimes b]_k &= \sum_{i=0}^{m} \sum_{j=0}^{m} a_i C_{i,j,k}^m b_j
\end{align*}$$

(5)

$\forall k \in \mathbb{M}$. Here, the coefficients $V_{i,j,k}^m$ and $C_{i,j,k}^m$ are respectively given by

$$\begin{align*}
V_{i,j,k}^m &= 2^{(i-k)(j-k)} \begin{bmatrix} i \cr k \end{bmatrix} \begin{bmatrix} m - i \\
 j - k \end{bmatrix}, \\
C_{i,j,k}^m &= 2^{(k-i)(k-j)} \begin{bmatrix} m - i \\
 m - k \end{bmatrix} \begin{bmatrix} i \\
 k - j \end{bmatrix}
\end{align*}$$

(6)

with $\begin{bmatrix} m \cr k \end{bmatrix}$ being the Gaussian binomial coefficient defined as follows

$$\begin{align*}
\begin{cases} 1, & k = 0 \text{ or } k = m \\
\prod_{l=0}^{k-1} \frac{2^{m-2l}}{2^l}, & 0 < k < m \\
0, & \text{otherwise.}
\end{cases}
\end{align*}$$

(7)

In addition, we define $a \boxminus_0 = a \boxminus a \boxminus \ldots \boxminus a$ and $a \boxtimes_0 = a \boxtimes a \boxtimes \ldots \boxtimes a$ for $n \in \mathbb{Z}_+$, and we use the convention that $a \boxminus_0 = \Delta_m$ and $a \boxtimes_0 = \Delta_0$ if $a \in \mathcal{X}$.

For notational convenience, in the sequel, we will use $\ast$ to denote either $\boxminus$ or $\boxtimes$. In Appendix C we will prove the commutative, distributive and associative laws of $\ast$ and apply them to the derivative analysis of the entropy function.

Remark 2: In the sequel, we will compute the difference between the entropies of two densities involving the convolutional operator $\ast$, e.g., $H(a \ast c) - H(b \ast c)$, $\forall a, b, c \in \mathcal{X}$. For notational convenience, we will extend Definitions 2 and 3 to all real-valued vectors of length $m + 1$ (not necessarily the probability vectors). By doing this, we can rewrite $H(a \ast c) - H(b \ast c)$ as $H((a - b) \ast c)$.

We are now ready for the duality rule for entropy for nonbinary LDPC and SC-LDPC ensembles on the BEC.

Lemma 1: For any $a, b \in \mathcal{X}$,

$$H(a) + H(b) = H(a \boxminus b) + H(a \boxtimes b).$$

(8)

Proof: By Definition 2

$$H(a \boxminus b) + H(a \boxtimes b) - H(a) - H(b) = \sum_{i=0}^{m} \sum_{j=0}^{m} a_i b_j \left[ \sum_{k=1}^{m} k (V_{i,j,k}^m + C_{i,j,k}^m) - (i + j) \right].$$

(9)
Therefore, it is suffice to show that, for any \( i, j \in \mathbb{M} \),
\[
\sum_{k=1}^{m} k \left( V_{i,j,k}^m + C_{i,j,k}^m \right) = i + j. \tag{10}
\]

Although (10) can be verified for small values of \( m \), how to prove it for all \( m \in \mathbb{Z}_+ \) is the most difficult step in the proof of Lemma 1. One may consider the method by induction on \( m \). However, such an idea is perhaps not feasible since the relation between \( V_{i,j,k}^m \) and \( V_{i,j,k}^{m+1} \) is quite involved. To circumvent this difficulty, we construct two bivariate functions (see (17) and (26) below), and by taking their partial derivatives we will obtain two polynomials whose coefficients are related to the left-hand side of (10). This will lead to the desired result.

Now we proceed the proof of Lemma 1 with the following Gauss’s binomial formula [25],
\[
\prod_{\alpha=1}^{m} \left( 1 + 2^{\alpha-1} x \right) = \sum_{j=0}^{m} 2^{\frac{j}{2}(j-1)} \left( \begin{array}{c} m \\ j \end{array} \right) x^j, \quad x \in \mathbb{R}. \tag{11}
\]

For \( x, y > 0 \), define \( f(x, y) \) as follows
\[
f(x, y) = \prod_{\alpha=1}^{i} \left( 1 + 2^{\alpha-1} x \right) \prod_{\beta=1}^{m-i} \left( 1 + 2^{\beta+i-1} y \right). \tag{12}
\]

By applying (11) to (12), we obtain
\[
f(x, y) = \sum_{l_1=0}^{i} 2^{\frac{l_1}{2}(l_1-1)} \left( \begin{array}{c} i \\ l_1 \end{array} \right) x^{l_1} \sum_{l_2=0}^{m-i} 2^{\frac{l_2}{2}(l_2-1)} \left( \begin{array}{c} m-i \\ l_2 \end{array} \right) (2^i y)^{l_2} \tag{13}
\]
\[
= \sum_{k=0}^{m} 2^{\frac{j}{2}(j-1)} \left( \begin{array}{c} m \\ j \end{array} \right) \sum_{k=0}^{m} V_{i,j,k}^m x^k y^{j-k} \tag{14}
\]

where (a) is obtained by replacing \( l_1 \) and \( l_2 \) with \( k \) and \( j - k \), respectively.

Now we take the partial derivative of \( f(x, y) \) in (12) with respect to \( x \), then multiply the result by \( x \), and finally replace \( y \) with \( x \). This leads to the following result
\[
\left[ x \frac{\partial}{\partial x} f(x, y) \right]_{y=x} = \left( \sum_{\alpha=1}^{i} 2^{\alpha-1} x \frac{\partial}{\partial x} \right) \prod_{\beta=1}^{m} \left( 1 + 2^{\beta-1} x \right). \tag{16}
\]

Applying the same procedure to (15) yields
\[
\left[ x \frac{\partial}{\partial x} f(x, y) \right]_{y=x} = \sum_{j=0}^{m} 2^\frac{j}{2}(j-1) \left( \begin{array}{c} m \\ j \end{array} \right) x^j \sum_{k=1}^{m} k V_{i,j,k}^m. \tag{17}
\]
Putting the above together, we obtain
\[
\left(\sum_{\alpha=1}^{i} \frac{2^{\alpha-1} x}{1 + 2^{\alpha-1} x}\right) \prod_{\beta=1}^{m} \left(1 + 2^{\beta-1} x\right) = \sum_{j=0}^{m} 2^{\frac{1}{2} j(j-1)} \left[\frac{m}{j}\right] x^j \sum_{k=1}^{m} kV_{i,j,k}.
\] (18)

Now we consider the following identity deduced from (11) by replacing \(x\) with \(x^{-1}\),
\[
\prod_{\alpha=1}^{m} \left(x + 2^{\alpha-1}\right) = \sum_{j=0}^{m} 2^{\frac{1}{2} j(j-1)} \left[\frac{m}{m-j}\right] x^{m-j}, \quad \forall x \in \mathbb{R}.
\] (19)

Similarly, for \(x, y > 0\), define \(g(x, y)\) as follows
\[
g(x, y) = \prod_{\alpha=1}^{i} \left(x + 2^{\alpha-1}\right) \prod_{\beta=1}^{m-i} \left(y + 2^{\beta+i-1}\right).
\] (20)

Again, applying (19) to (20) yields
\[
g(x, y) = 2^{i(m-i)} \prod_{\alpha=1}^{i} \left(x + 2^{\alpha-1}\right) \prod_{\beta=1}^{m-i} \left(2^{-i} y + 2^{\beta-1}\right)
\] (21)
\[
= \sum_{l_2=0}^{i} \sum_{l_1=0}^{i} 2^{i(m-i)+\frac{1}{2} l_1(l_1-1)+\frac{1}{2} l_2(l_2-1)-i(m-i-l_2)} \left[i \atop i-l_1\right] \left[i \atop i-l_2\right] x^{i-l_1} y^{m-i-l_2}
\] (22)
\[
= \sum_{j=0}^{m} 2^{\frac{1}{2} j(j-1)} \left[\frac{m}{m-j}\right] \sum_{k=0}^{m} C_{i,j,k} x^{k-j} y^{m-k}
\] (23)
\[
= \sum_{j=0}^{m} 2^{\frac{1}{2} j(j-1)} \left[\frac{m}{m-j}\right] \sum_{k=0}^{m} C_{i,j,k} x^{k-j} y^{m-k}
\] (24)

where (a) is obtained by replacing \(l_1\) and \(l_2\) with \(i + j - k\) and \(k - i\), respectively.

Following the same procedure as in (16)-(18), we can show that
\[
\left[\frac{x}{\partial x} g(x, y)\right]_{y=x} = \left(\sum_{\alpha=1}^{i} \frac{x}{x + 2^{\alpha-1}}\right) \prod_{\beta=1}^{m} \left(x + 2^{\beta-1}\right)
\] (25)
\[
= \sum_{j=0}^{m} 2^{\frac{1}{2} j(j-1)} \left[\frac{m}{m-j}\right] x^{m-j} \sum_{k=0}^{m} (k-j) C_{i,j,k}.
\] (26)

Now, by substituting \(\left[\frac{m}{m-j}\right] = \left[\frac{m}{j}\right]\) and \(\sum_{k=0}^{m} C_{i,j,k} = 1\) (see (108) in Appendix B) into (26) and replacing \(x\) with \(x^{-1}\), we can deduce that
\[
\left(\sum_{\alpha=1}^{i} \frac{1}{1 + 2^{\alpha-1} x}\right) \prod_{\beta=1}^{m} \left(1 + 2^{\beta-1} x\right) = \sum_{j=0}^{m} 2^{\frac{1}{2} j(j-1)} \left[\frac{m}{j}\right] x^j \sum_{k=1}^{m} kC_{i,j,k} - \sum_{j=0}^{m} 2^{\frac{1}{2} j(j-1)} \left[\frac{m}{j}\right] j x^j.
\] (27)
Next, by rearranging and combining (18) and (27) as follows and substituting (11) to the term \( \prod_{\beta=1}^{m} \left( 1 + 2^{\beta-1}x \right) \), we have
\[
\sum_{j=0}^{m} 2^{\frac{1}{2}j(j-1)} \binom{m}{j} \sum_{k=1}^{m} k \left( V_{i,j,k}^{m} + C_{i,j,k}^{m} \right) x^{j}
\]
\[
= \sum_{\alpha=1}^{i} \left( \frac{2^{\alpha-1}x}{1 + 2^{\alpha-1}x} + \frac{1}{1 + 2^{\alpha-1}x} \right) \prod_{\beta=1}^{m} \left( 1 + 2^{\beta-1}x \right) + \sum_{j=0}^{m} 2^{\frac{1}{2}j(j-1)} \binom{m}{j} x^{j}
\]
\[
= i \prod_{\beta=1}^{m} \left( 1 + 2^{\beta-1}x \right) + \sum_{j=0}^{m} 2^{\frac{1}{2}j(j-1)} \binom{m}{j} x^{j} = \sum_{j=0}^{m} 2^{\frac{1}{2}j(j-1)} \binom{m}{j} (i + j) x^{j}.
\]
Therefore
\[
\sum_{k=1}^{m} k \left( V_{i,j,k}^{m} + C_{i,j,k}^{m} \right) = i + j, \quad \forall i, j \in \mathbb{M}.
\]

Finally, by substituting (30) to (9), we complete the proof of Lemma 1.

**Remark 3:** Now let us briefly discuss (30) and interpret the operational meaning of the duality rule for entropy (8). For any fixed \( i, j \in \mathbb{M} \), consider two statistically independent messages, the a posteriori PMFs of which are equiprobable over a subspace \( S \) of dimension \( i \) and a subspace \( S' \) of dimension \( j \), respectively. As discussed in Remark 1, the uncertainties of the two messages are given by \( i \) bits and \( j \) bits, and therefore, the total uncertainty is given by \( i + j \) bits. If we combine the two messages based on the VN (resp. CN) decoding algorithm, then the a posteriori PMF of the combined message is equiprobable over the intersection (resp. sum) of \( S \) and \( S' \), denoted as \( S \cap S' \) (resp. \( S + S' \)). Moreover, as interpreted in Subsection II-A in [23], \( V_{i,j,k}^{m} \) (resp. \( C_{i,j,k}^{m} \)) is the probability of the event that the dimension of \( S \cap S' \) (resp. \( S + S' \)), or equivalently, the uncertainty of the combined message, is exactly \( k \) (bits). Therefore, the identity (30) indicates that the total (average) uncertainty is invariant under the combinations of two statistically independent messages based on the VN and the CN decoding algorithms. This explains why we mentioned in the introduction that the duality rule for entropy (8) reveals a conservation relation between the input and the output entropies of VNs and CNs.

Following the same line as in [18], we extend the rule (8) to the following relations and omit the details for brevity.

**Corollary 1:** For any \( a, b, c, d \in \mathcal{X} \),
\[
H \left( a \boxtimes (b - c) \right) + H \left( a \boxbar (b - c) \right) = H \left( b - c \right),
\]
\[
H \left( (a - b) \boxtimes (c - d) \right) + H \left( (a - b) \boxbar (c - d) \right) = 0.
\]
B. Partial Ordering

An important issue in the DE analysis is comparing two densities to identify which one offers more information about the coded symbols. For this purpose, a concept termed partial ordering is established in [15] based on statistical degradation in the context of binary LDPC ensembles over BMS channels. In [23], the authors defined partial ordering based on the complementary cumulative distribution function to accommodate the analysis of nonbinary LDPC ensembles on the BEC. We will exploit the notion of partial ordering in [23] in this paper, the definition of which is reformulated as follows.

Definition 4: For any \( a, b \in X \), we say that \( a \preceq b \) or \( b \succeq a \) if the following inequality holds
\[
\sum_{n=k}^{m} a_n \leq \sum_{n=k}^{m} b_n, \quad \forall k \in \mathcal{M} \setminus \{0\}
\]
and say that \( a \prec b \) or \( b \succ a \) if \( a \preceq b \) and \( a \neq b \).

Proposition 1: For any \( a, b \in X \), the strict partial order \( a \prec b \) holds if and only if there exists a nonempty set \( I \subseteq \mathcal{M} \setminus \{0\} \) such that
\[
\sum_{n=k}^{m} a_n < \sum_{n=k}^{m} b_n \quad \text{for} \quad k \in I \quad \text{and} \quad \sum_{n=k}^{m} a_n = \sum_{n=k}^{m} b_n \quad \text{for} \quad k \notin I.
\]

Proof: The proof is straightforward and we omit it for simplicity.

It is easy to justify that \( \Delta_0 \preceq a \preceq \Delta_m, \forall a \in X \).

Proposition 2: Consider a series of densities \( \{x_l\}_{l \in \mathbb{Z}} \). The limit \( \lim_{l \to \infty} x_l \) exists if either \( x_{l+1} \preceq x_l \) or \( x_{l+1} \succeq x_l \) holds for all \( l \in \mathbb{Z} \).

Proof: By definition, we can deduce that either \( \sum_{n=k}^{m} [x_{l+1}]_n \leq \sum_{n=k}^{m} [x_l]_n \) or \( \sum_{n=k}^{m} [x_{l+1}]_n \geq \sum_{n=k}^{m} [x_l]_n \) holds for each \( k \in \mathcal{M} \setminus \{0\} \). Notice that \( \sum_{n=k}^{m} [x_l]_n \) is always bounded between 0 and 1. Therefore, the limit \( \lim_{l \to \infty} x_l \) does indeed exist.

Proposition 3: The entropy function \( H(\cdot) \) preserves partial ordering. More precisely, for any \( a, b \in X \), we have \( H(a) \leq H(b) \) if \( a \preceq b \), and \( H(a) < H(b) \) if \( a \prec b \).

Proof: By Definition 4, \( a \preceq b \) implies that \( \sum_{n=k}^{m} a_n \leq \sum_{n=k}^{m} b_n, \forall k \in \mathcal{M} \setminus \{0\} \). Therefore
\[
H(a) = \sum_{k=0}^{m} k a_k = \sum_{k=1}^{m} k \sum_{n=k}^{m} a_n \leq \sum_{k=1}^{m} k \sum_{n=k}^{m} b_n = \sum_{k=0}^{m} k b_k = H(b)
\]
where (a) is based on Proposition 4 in Appendix A.

The proof of the implication \( a \prec b \Rightarrow H(a) < H(b) \) now becomes straightforward by Proposition 1.
Lemma 2: The VN and CN convolutional operators □ and ⊠ preserve partial ordering. More precisely, for any a, b, c ∈ X with a ≤ b, we have

\[ a □ c \leq b □ c \quad \text{and} \quad a ⊠ c \leq b ⊠ c. \]  

Further, if a < b, then

\[ a □ c < b □ c, \quad \text{for} \ c \neq Δ_0, \]  

\[ a ⊠ c < b ⊠ c, \quad \text{for} \ c \neq Δ_m. \]  

Proof: We focus on the results for □. The proof for ⊠ is identical by noticing that \( C^m_{i,j,k} = V^m_{i,m-j,m-k} \).

Notice that, for \( n \in \mathbb{M} \setminus \{0\} \), \( a \leq b \) implies \( \sum_{k=n}^{m} a_k \leq \sum_{k=n}^{m} b_k \) by Definition 4. Therefore

\[ \sum_{k=n}^{m} [a □ c]_k = \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=0}^{m} a_i V^m_{i,j,k} c_j \overset{(a)}{=} \sum_{i=0}^{m} c_j \sum_{k=0}^{m} \sum_{i=0}^{m} a_i V^m_{i,j,k} \]  

\[ \overset{(b)}{=} \sum_{j=0}^{m} \sum_{k=0}^{m} \left[ \sum_{l=n}^{m} \sum_{i=0}^{m} a_l \right] \sum_{i=0}^{m} \left( V^m_{i,j,k} - V^m_{i-1,j,k} \right) \]  

\[ \overset{(c)}{=} \sum_{j=0}^{m} \sum_{i=0}^{m} \left( \sum_{l=1}^{m} \sum_{i=0}^{m} a_l \right) b_l = \sum_{k=n}^{m} [b □ c]_k \]  

where (a) and (c) are based on the fact that \( V^m_{i,j,k} = 0 \) if \( 0 \leq i < n \leq k \), (b) follows from Proposition 4 in Appendix A, and (d) is based on Claim 3) of Proposition 5 in Appendix B. Therefore, by Definition 4, we obtain the desired result \( a □ c \leq b □ c \).

Next, we show that □ preserves strict partial ordering. To this end, we rearrange the above as follows

\[ \sum_{k=n}^{m} [a □ c]_k - \sum_{k=n}^{m} [b □ c]_k = \sum_{i=0}^{m} c_j \sum_{i=0}^{m} \left( \sum_{k=n}^{m} \left( V^m_{i,j,k} - V^m_{i-1,j,k} \right) \right) \left( \sum_{l=1}^{m} a_l - \sum_{l=1}^{m} b_l \right). \]  

Let \( i_0, j_0 \in \mathbb{M} \setminus \{0\} \) be two integers satisfying \( \sum_{l=i_0}^{m} a_l - \sum_{l=i_0}^{m} b_l < 0 \) and \( c_{j_0} > 0 \). The existence of such \( i_0 \) and \( j_0 \) is guaranteed by the assumption \( a < b \) and \( c \neq Δ_0 \). By letting \( n = \min \{i_0, j_0\} \) and discarding some nonpositive terms in (42), we can obtain a strictly negative upper bound, i.e.,

\[ \sum_{k=n}^{m} [a □ c]_k - \sum_{k=n}^{m} [b □ c]_k \leq c_{j_0} \left( \sum_{k=n}^{m} \left( V^m_{i_0,j_0,k} - V^m_{i_0-1,j_0,k} \right) \right) \left( \sum_{l=i_0}^{m} a_l - \sum_{l=i_0}^{m} b_l \right) < 0. \]
Thus $\sum_{k=n}^{m} [a \boxtimes c]_k < \sum_{k=n}^{m} [b \boxtimes c]_k$ holds for at least one integer $n \in \mathbb{M} \setminus \{0\}$. This completes the proof of (36).

Remark 4: The partial order preservation property in Lemma 3 guarantees that, if an uncoupled DE recursion (see (61) in the sequel) is initialized by $\Delta_m$, i.e., the most “uncertain” density, then the densities generated by the DE recursion are always partially ordered as the iteration proceeds.

Lemma 3: For any $a, b, c, d \in \mathcal{X}$ with $a \preceq b$ and $c \preceq d$, we have

$$H((a - b) \boxtimes (c - d)) \geq 0, \quad H((a - b) \boxtimes (c - d)) \leq 0$$

where the equalities hold if and only if $a = b$ and $c = d$.

Proof: For any $i, n \in \mathbb{M} \setminus \{0\}, j \in \mathbb{M}$, we define

$$D_{i,j,n}^m = \frac{2^{(i-n)(j-n+1)} \left[\frac{i-1}{n-1} \binom{m-i}{j-n} \right]}{m \binom{j}{n}}.$$ (45)

Obviously, $D_{i,j,n}^m > 0$ if $n < i$ and $i + j \leq m + n$. Further, if $j > 0$, then

$$D_{i,j,n}^m = D_{i,j-1,n}^m \frac{(2^{i-n} - 2^{-n}) (2^{m-j+n+1} - 2^j)}{(2^n - 1) (2^{m-j+1} - 1)} > D_{i,j-1,n}^m 2^{n-1} > D_{i,j-1,n}^m.$$ (46)

On the other hand, based on (18), for $i \in \mathbb{M} \setminus \{0\}$, we have

$$\sum_{j=0}^{m} \frac{2^{\frac{j(j-1)}{2}}}{1 + 2^{\frac{j(j-1)}{2}}} \sum_{k=1}^{m} k (V_{i,j,k}^m - V_{i-1,j,k}^m)$$

$$= \frac{2^{i-1} x}{1 + 2^{i-1} x} \prod_{\beta=1}^{m} \left( 1 + 2^{\beta-1} x \right) = 2^{i-1} x \prod_{\alpha=1}^{i-1} \left( 1 + 2^{\alpha-1} x \right) \prod_{\beta=1}^{m-i} \left( 1 + 2^{\beta-1} (2^i x) \right)$$ (47)

$$= \left( \sum_{l_1=0}^{a-1} \sum_{l_2=0}^{i-1} 2^{l_1} \binom{i-1}{l_1} \binom{m-i}{l_2} x^{l_1+l_2+1} \right)$$ (a)

$$\sum_{j=0}^{m} \frac{2^{\frac{j(j-1)}{2}}}{1 + 2^{\frac{j(j-1)}{2}}} \sum_{n=1}^{m} \frac{2^{(i-n)(j-n+1)}}{2^{n-1} x} \binom{i-1}{n-1} \binom{m-i}{j-n} \sum_{j=0}^{m} 2^{\frac{j(j-1)}{2}} \binom{m}{j} x^j \sum_{n=1}^{m} D_{i,j,n}^m$$ (b)

where (a) is based on (11) and (b) is obtained by $l_1 = n - 1$ and $l_2 = j - n$. Therefore, we have

$$\sum_{k=1}^{m} k (V_{i,j,k}^m - V_{i-1,j,k}^m) = \sum_{n=1}^{m} D_{i,j,n}^m \geq 0.$$ (50)

Following (46), we have

$$\sum_{k=1}^{m} k (V_{i,j,k}^m - V_{i-1,j,k}^m - V_{i-1,j-1,k}^m + V_{i-1,j-1,k}^m) = \sum_{n=1}^{m} (D_{i,j,n}^m - D_{i,j-1,n}^m) > 0.$$ (51)
By Definition 4, \( \forall i, j \in \mathbb{M} \setminus \{0\} \), \( a \preceq b \) and \( c \succeq d \) imply \( \sum_{n=i}^{m} (a_n - b_n) \leq 0 \) and \( \sum_{l=j}^{m} (c_l - d_l) \leq 0 \), respectively. The first inequality in (44) can be deduced from (51). Specifically,

\[
H ((a - b) \boxplus (c - d)) = \sum_{k=1}^{m} k \sum_{j=k}^{m} (c_j - d_j) \sum_{i=k}^{m} (a_i - b_i) V_{i,j,l}^m
\]

(52)

\[
= \sum_{k=1}^{m} k \sum_{j=k}^{m} (c_j - d_j) \sum_{i=k}^{m} (V_{i,j,k}^m - V_{i-1,j,k}^m) \sum_{n=i}^{m} (a_n - b_n)
\]

(53)

\[
= \sum_{k=1}^{m} k \sum_{i=k}^{m} \sum_{j=k}^{m} (V_{i,j,k}^m - V_{i-1,j,k}^m - V_{i-1,j-1,k}^m + V_{i,j-1,k}^m) \sum_{n=i}^{m} (a_n - b_n) \sum_{l=j}^{m} (c_l - d_l)
\]

(54)

\[
= \sum_{i=1}^{m} \sum_{j=1}^{m} \left[ \sum_{n=i}^{m} (a_n - b_n) \right] \left[ \sum_{l=j}^{m} (c_l - d_l) \right] \sum_{k=1}^{m} k (V_{i,j,k}^m - V_{i-1,j,k}^m - V_{i,j-1,k}^m + V_{i-1,j-1,k}^m) \geq 0.
\]

(55)

Further, due to the strict positiveness of the right-hand side of (51), the equality in (55) holds if and only if \( a = b \) and \( c = d \).

Now, based on (52), the proof of the second inequality in (44) becomes straightforward. \[ \square \]

The following corollary simply follows from Lemma 3.

**Corollary 2:** For any \( a, b, c \in \mathcal{X} \), we have

\[
H ((a - b) \boxplus c) \geq 0, \quad H ((a - b) \boxtimes c) \leq 0
\]

(56)

if either \( a \preceq b \) or \( a \succeq b \) holds. Moreover, the equalities in the above hold if and only if \( a = b \).

**Lemma 4:** For any \( a, b, c, d \in \mathcal{X} \) with \( a \succeq b \), we have

\[
|H ((a - b) * (c - d))| \leq H (a - b).
\]

(57)

**Proof:** We rewrite (55) as follows.

\[
H ((a - b) \boxplus (c - d)) = \sum_{j=0}^{m} c_j \sum_{i=0}^{m} \sum_{k=1}^{m} k (V_{i,j,k}^m - V_{i-1,j,k}^m) \sum_{n=i}^{m} (a_n - b_n)
\]

\[
- \sum_{j=0}^{m} d_j \sum_{i=0}^{m} \sum_{k=1}^{m} k (V_{i,j,k}^m - V_{i-1,j,k}^m) \sum_{n=i}^{m} (a_n - b_n).
\]

(58)

By assumption \( a \succeq b \) and (50), it is obvious that the four terms \( \sum_{n=i}^{m} (a_n - b_n), \sum_{k=1}^{m} k(V_{i,j,k}^m - V_{i-1,j,k}^m), c_j \) and \( d_j \) are always nonnegative, satisfying \( \sum_{j=0}^{m} c_j = \sum_{j=0}^{m} d_j = 1 \). Further, the inequality (51) indicates that the second term \( \sum_{k=1}^{m} k(V_{i,j,k}^m - V_{i-1,j,k}^m) \) is strictly increasing with respect to \( j \in \mathbb{M} \). Therefore, \( H ((a - b) \boxplus (c - d)) \) is maximized by \( c_m = d_0 = 1 \) (i.e.,
c = Δ_m, d = Δ_0) and minimized by \( c_0 = d_m = 1 \) (i.e., \( c = \Delta_0, d = \Delta_m \)). This leads to the following desired result, i.e.,

\[
-H (a - b) \leq -H ((a - b) \boxtimes (c - d)) = H ((a - b) \boxdot (c - d)) \leq H (a - b). \tag{59}
\]

\[\]

IV. POTENTIAL FUNCTIONS AND THRESHOLD SATURATION

A. LDPC(λ, ρ, m)

**Definition 5:** For LDPC(λ, ρ, m) on the BEC with erasure probability \( \epsilon \in [0, 1] \), the uncoupled DE recursion in the \( l \)-th iteration is given by

\[
\begin{align*}
y^{(l)} &= \rho \boxtimes (x^{(l-1)}) \\
x^{(l)} &= c_{\epsilon} \boxdot \lambda \boxdot (y^{(l)})
\end{align*}
\tag{60}
\]

or equivalently,

\[
x^{(l)} = c_{\epsilon} \boxdot \lambda \boxdot (\rho \boxtimes (x^{(l-1)})), \quad \forall l \in \mathbb{Z}_+
\tag{61}
\]

where \( x^{(l)} \) and \( y^{(l)} \) are respectively the output densities of VNs and CNs, \( c_{\epsilon} \) the channel density, and the operators \( \lambda \boxdot (\cdot) \) and \( \rho \boxtimes (\cdot) \) defined as

\[
\lambda \boxdot (a) = \sum_i \lambda_i a^{i-1}, \quad \rho \boxtimes (b) = \sum_j \rho_j b^{j-1}
\tag{62}
\]

\( \forall a, b \in \mathcal{X} \). Here, the \( k \)-th entry of the channel density \( c_{\epsilon} \) is determined by

\[
[c_{\epsilon}]_k = \binom{m}{k} \epsilon^k (1 - \epsilon)^{m-k}, \quad k \in \mathbb{M}.
\tag{63}
\]

As in [18], \( \forall l \in \mathbb{Z} \), we will write \( x^{(l)} \) in the form of \( x^{(l)} = T^{(l)}_s (x^{(0)}, c_{\epsilon}) \) to emphasize that the output density at VNs in the \( l \)-th iteration is determined by the initial density \( x^{(0)} \) and the channel density \( c_{\epsilon} \). Moreover, based on those propositions and lemmas in Subsection III-B, we can show that the DE operator \( T^{(l)}_s (\cdot, \cdot) \) satisfies the same monotonicity properties as stated in Lemma 18 in [18]. For convenience, we reformulate these properties in the following lemma.

**Lemma 5:** For any \( l \in \mathbb{Z}_+ \) and \( a_1, a_2, a, c_1, c_2, c \in \mathcal{X} \), the DE update operator \( T^{(l)}_s (\cdot, \cdot) \) satisfies the following properties.

1) \( T^{(l)}_s (a_1, c) \succeq T^{(l)}_s (a_2, c) \) if \( a_1 \succeq a_2 \).

2) \( T^{(l)}_s (a, c_1) \succeq T^{(l)}_s (a, c_2) \) if \( c_1 \succeq c_2 \).
3) If \( \exists a \in \mathcal{X} \) such that \( T_s^{(1)} (a, c) \leq a \), then \( T_s^{(l+1)} (a, c) \leq T_s^{(l)} (a, c) \) and the limit \( T_s^{(\infty)} (a, c) = \lim_{l \to \infty} T_s^{(l)} (a, c) \) does indeed exist, satisfying \( T_s^{(\infty)} (a, c) \leq T_s^{(l)} (a, c) \) and

\[
T_s^{(1)} \left( T_s^{(\infty)} (a, c), c \right) = T_s^{(\infty)} (a, c).
\]

(64)

4) If \( \exists a \in \mathcal{X} \) such that \( T_s^{(1)} (a, c) \geq a \), then \( T_s^{(l+1)} (a, c) \geq T_s^{(l)} (a, c) \) and the limit \( T_s^{(\infty)} (a, c) = \lim_{l \to \infty} T_s^{(l)} (a, c) \) does indeed exist, satisfying \( T_s^{(\infty)} (a, c) \geq T_s^{(l)} (a, c) \) and

\[
T_s^{(1)} \left( T_s^{(\infty)} (a, c), c \right) = T_s^{(\infty)} (a, c).
\]

(65)

**Proof:** The proof is identical to that of Lemma 18 in [18] and we omit the details here. ■

**Definition 6:** For a fixed \( \epsilon \in [0, 1] \), a density \( x \in \mathcal{X} \) is said to be an uncoupled fixed point (UFP) of the uncoupled DE recursion \((61)\) if it satisfies \( x = T_s^{(1)} (x, c, \epsilon) \). In the sequel, we will use \( \mathcal{F}_s (\epsilon) \) to denote the set of all such UFPs.

**Definition 7:** For any \( x \in \mathcal{X} \) and \( \epsilon \in [0, 1] \), the potential function for LDPC(\( \lambda, \rho, m \)) is given by

\[
U_s (x, \epsilon) = \frac{L' (1)}{R' (1)} H \left( R^\Box (x) \right) + L' (1) H \left( \rho^\Box (x) \right) - L' (1) H \left( x \boxtimes \rho^\Box (x) \right) - H \left( c, \epsilon \right) \boxtimes L^\Box \left( \rho^\Box (x) \right).
\]

(66)

**Lemma 6:** If \( \exists a \in \mathcal{X} \) such that either \( T_s^{(1)} (a, c) \leq a \) or \( T_s^{(1)} (a, c) \geq a \) holds, then

\[
U_s \left( T_s^{(l+1)} (a, c), \epsilon \right) \leq U_s \left( T_s^{(l)} (a, c), \epsilon \right), \forall l \in \mathbb{Z}.
\]

(67)

**Proof:** For notational brevity, we define

\[
W_s (x, y, \epsilon) = \frac{1}{R' (1)} H \left( R^\Box (x) \right) + H \left( y \right) - H \left( x \boxtimes y \right) - \frac{1}{L' (1)} H \left( c, \epsilon \right) \boxtimes L^\Box \left( y \right).
\]

(68)

Following \((68)\), we can rewrite \( W_s (x, y, \epsilon) \) as follows

\[
W_s (x, y, \epsilon) = \frac{1}{R' (1)} H \left( R^\Box (x) \right) - H \left( x \right) + H \left( x \boxtimes y \right) - \frac{1}{L' (1)} H \left( c, \epsilon \right) \boxtimes L^\Box \left( y \right).
\]

(69)

Obviously, the relation between \( U_s (x, \epsilon) \) and \( W_s (x, y, \epsilon) \) is given by

\[
U_s (x, \epsilon) = \frac{1}{L' (1)} W_s (x, y, \epsilon) \bigg|_{y=\rho^\Box (x)}.
\]

(70)

Let \( x^{(0)} = a \) and \( y^{(0)} = \rho^\Box (a) \). By assumption and Claims 3) and 4) in Lemma 5 the inequality \((67)\) is equivalent to the fact that the following sequence

\[
\left\{ W_s (x^{(i)}, y^{(i)}, \epsilon) \right\}_{i \in \mathbb{Z}}
\]

(71)
is nonincreasing as $l$ increases. To prove this fact, consider
\[
W_s (x^{(l+1)}, y^{(l+1)}, \epsilon) - W_s (x^{(l)}, y^{(l)}, \epsilon) = [W_s (x^{(l+1)}, y^{(l+1)}, \epsilon) - W_s (x^{(l)}, y^{(l+1)}, \epsilon)] + [W_s (x^{(l)}, y^{(l+1)}, \epsilon) - W_s (x^{(l)}, y^{(l)}, \epsilon)] .
\] (72)

We reformulate the term in the first square bracket on the right-hand side of (72) as follows,
\[
W_s (x^{(l+1)}, y^{(l+1)}, \epsilon) - W_s (x^{(l)}, y^{(l+1)}, \epsilon) = \frac{1}{R' (1)} H \left( R^{\mathbb{Z}} (x^{(l+1)}) - R^{\mathbb{Z}} (x^{(l)}) \right) - H \left( (x^{(l+1)} - x^{(l)}) \otimes y^{(l+1)} \right) \] (73)

Similarly, the other term can be rewritten as
\[
W_s (x^{(l)}, y^{(l+1)}, \epsilon) - W_s (x^{(l)}, y^{(l)}, \epsilon) = H \left( (y^{(l+1)} - y^{(l)}) \otimes c \right) - \frac{1}{L' (1)} H \left( c \otimes L^{\mathbb{Z}} (y^{(l+1)}) - c \otimes L^{\mathbb{Z}} (y^{(l)}) \right) \] (75)

By Corollary 2, the above terms are both nonnegative and hence
\[
W_s (x^{(l+1)}, y^{(l+1)}, \epsilon) \leq W_s (x^{(l)}, y^{(l)}, \epsilon) , \forall l \in \mathbb{Z}.
\] (77)

This leads to the desired result (67).

**Definition 8:** A direction defined over $\mathcal{X}$, denoted as $\delta x = (\delta x_0, \delta x_1, \ldots, \delta x_m)$, is a vector of length $m + 1$, satisfying $\sum_{i=0}^{m} \delta x_i = 0$. For convenience, in the sequel, we will always consider $\delta x_1, \ldots, \delta x_m$ as independent variables and rewrite the first entry $\delta x_0$ as $\delta x_0 = - \sum_{i=1}^{m} \delta x_i$. As a result, whenever we speak of a direction $\delta x$, we always rewrite it in the form
\[
\delta x = \left( - \sum_{i=1}^{m} \delta x_i , \delta x_1, \ldots, \delta x_m \right) ,
\] (78)
or equivalently,
\[
\delta x = \sum_{i=1}^{m} (\Delta_i - \Delta_0) \delta x_i .
\] (79)

**Definition 9:** The directional derivative of $U_s (x, \epsilon)$ with respect to $x$ in the direction $\delta x$ is defined as
\[
d_s U_s (x, \epsilon) [\delta x] = \lim_{t \to 0} \frac{U_s (x + t \delta x, \epsilon) - U_s (x, \epsilon)}{t} .
\] (80)
**Lemma 7:** The directional derivative $d_x U_s (x, \epsilon) [\delta x]$ defined as above is determined by
\[
d_x U_s (x, \epsilon) [\delta x] = L' (1) H \left( (T_s^{(1)} (x, c_\epsilon) - x) \boxtimes \rho^{02} (x) \boxtimes \delta x \right) \tag{81}
\]
where $\rho^{02} (x)$ is given by
\[
\rho^{02} (x) = \sum_j (j - 1) \rho_j x^{02j - 2}. \tag{82}
\]

The proof of Lemma 7 is identical to that of Lemma 23 in [18]. One may also prove this lemma based on the derivative of $U_s (x, \epsilon)$ with respect to $x$. See Appendix C for details.

**Definition 10:** A density $x \in \mathcal{X}$ is a stationary point of $U_s (x, \epsilon)$ if
\[
d_x U_s (x, \epsilon) [\delta x] = 0 \tag{83}
\]
for any direction $\delta x$.

**Definition 11:** For the uncoupled DE recursion (61), we define the energy gap as
\[
\Delta E (\epsilon) = \min_{x \in \mathcal{F}_s \setminus \{\Delta_0\}} U_s (x, \epsilon) \tag{84}
\]
with the convention that the minimum over the empty set is $+\infty$.

**Remark 5:** Notice that in this paper the definition of $\Delta E (\epsilon)$ is different from that introduced in [18]. While in [18] the infimum of $U_s (x, \epsilon)$ is over the densities outside the basin of attraction to $\Delta_0$ (see Definition 25 therein), the minimization in (84) is over the UFP set $\mathcal{F}_s (\epsilon)$ excluding $\Delta_0$. This modification is based on the numerical observation that the asymptotic BP threshold of SC-LDPC($\lambda, \rho, N, w, m$) on the BEC asymptotically is closely related to the sign of $U_s (x, \epsilon)$ at a nontrivial UFP $x \succ \Delta_0$.

**Lemma 8:** For any $\epsilon_1, \epsilon_2 \in [0, 1]$ with $\epsilon_1 > \epsilon_2$, we have
1) $U_s (x, \epsilon_1) < U_s (x, \epsilon_2)$ if $x \neq \Delta_0$.
2) $\Delta E (\epsilon_1) < \Delta E (\epsilon_2)$.

**Proof:** Based on the propositions and lemmas in Subsection III-B, we can prove Claim 1) following the same line as Lemma 26 in [18].

Now we put our focus on Claim 2). For $\epsilon = \epsilon_2$, let $x_2$ be the minimizer of $U_s (x, \epsilon_2)$ over $\mathcal{F}_s \setminus \{\Delta_0\}$. Since $x_2$ is a UFP of the uncoupled DE recursion (61), following Claim 1), we have
\[
\Delta E (\epsilon_2) = U_s (x_2, \epsilon_2) > U_s (x_2, \epsilon_1). \tag{85}
\]

Further, since $c_{\epsilon_1} \succ c_{\epsilon_2}$, we have
\[
x_2 = T_s^{(1)} (x_2, c_{\epsilon_2})^{(a)} \succeq T_s^{(1)} (x_2, c_{\epsilon_1}) \tag{86}
\]
where (a) follows from Claim 2) in Lemma 5.

The inequality (86) indicates that if we increase the channel erasure probability and set the initial density of the uncoupled DE recursion (61) as \( x(0) = x_2 \), then the densities generated by this recursion are always partially ordered and finally converge to a new UFP, denoted as \( x_1 \) (see Claim 3) in Lemma 5). Further, by Lemma 6 and (84), we have

\[
U_s(x_2, \epsilon_1) \geq U_s(T_1^{(1)}(x_2, c_{\epsilon_1}), \epsilon_1) \geq U_s(T_1^{(2)}(x_2, c_{\epsilon_1}), \epsilon_1) \geq \ldots \geq U_s(x_1, \epsilon_1) \geq \Delta E(\epsilon_1). \tag{87}
\]

Now combining (85) and (87) yields \( \Delta E(\epsilon_1) < \Delta E(\epsilon_2) \).

Definition 12: We define the potential threshold for the uncoupled DE recursion (61) as

\[
\epsilon_{\text{pot}} = \sup \{ \epsilon \in [0, 1] | \Delta E(\epsilon) > 0 \}. \tag{88}
\]

B. SC-LDPC(\( \lambda, \rho, N, w, m \))

Definition 13: For SC-LDPC(\( \lambda, \rho, N, w, m \)) on the BEC with erasure probability \( \epsilon \in [0, 1] \), the coupled DE recursion in the \( l \)-th iteration is given by

\[
x_i^{(l)} = \frac{1}{w} \sum_{k=0}^{w-1} c_{\epsilon,j-k} \lambda \oplus \left( \frac{1}{w} \sum_{j=0}^{w-1} \rho \oplus \left( x_i^{(l-1)} \right) \right)
\]

\( \forall l \in \mathbb{Z}_+ \), where \( x_i^{(l)} \) denotes the input density for the CNs at position \( i \) and the respective channel density \( c_{\epsilon,i} = c \) for \( i \in \mathbb{N}_v \) and \( c_{\epsilon,i} = \Delta_0 \) otherwise.

In the sequel, \( \forall l \in \mathbb{Z} \), we will use \( \mathbf{x}^{(l)} \) to represent a density sequence of length \( N_w \)

\( \) the \( i \)-th entry of which is denoted as \( x_i^{(l)} \), \( \forall i \in \mathbb{N}_c \). The set of all such density sequences is denoted as \( \mathcal{X}^{N_w} \). In addition, we will adopt \( \Delta_0 = (\Delta_0, \Delta_0, \ldots, \Delta_0) \) and \( \Delta_m = (\Delta_m, \Delta_m, \ldots, \Delta_m) \) to denote the two extremal density sequences in \( \mathcal{X}^{N_w} \). Also, we will use the operator \( T_c^{(l)}(\cdot, \cdot) \) to denote the coupled DE recursion (89) over \( l \) iterations, i.e.,

\[
\begin{align*}
\mathbf{x}^{(1)} &= T_c^{(1)}(\mathbf{x}^{(0)}, c_\epsilon), \\
\mathbf{x}^{(2)} &= T_c^{(1)}(\mathbf{x}^{(1)}, c_\epsilon) = T_c^{(2)}(\mathbf{x}^{(0)}, c_\epsilon), \\
&\quad \ldots \\
\mathbf{x}^{(l)} &= T_c^{(1)}(\mathbf{x}^{(l-1)}, c_\epsilon) = T_c^{(2)}(\mathbf{x}^{(l-2)}, c_\epsilon) = \ldots = T_c^{(l)}(\mathbf{x}^{(0)}, c_\epsilon).
\end{align*}
\tag{90}
\]

\(^1\)Unless otherwise specified, whenever we speak of a density sequence, we always assume that its length is given by \( N_w \).
**Definition 14:** For a fixed \( \epsilon \in [0, 1] \), a density sequence \( \mathbf{x} \in \mathcal{X}^{N_w} \) is said to be a coupled fixed point (CFP) of the coupled DE recursion (89) if it satisfies \( \mathbf{x} = T^{(1)}_c (\mathbf{x}, \epsilon) \). In the sequel, we will use \( \mathcal{F}_c (\epsilon, N, w) \) to denote the set of all such CFPs.

We define partial ordering between density sequences in a pointwise manner, i.e., for any \( \mathbf{x}, \mathbf{y} \in \mathcal{X}^{N_w} \), we say that \( \mathbf{x} \preceq \mathbf{y} \) or \( \mathbf{y} \succeq \mathbf{x} \) if \( x_i \preceq y_i, \forall i \in \mathcal{N}_c \). Further, we say that \( \mathbf{x} \prec \mathbf{y} \) or \( \mathbf{y} \succ \mathbf{x} \) if \( x_i < y_i, \forall i \in \mathcal{N}_c \).

**Lemma 9:** Consider \( \mathbf{a}, \mathbf{a}_1, \mathbf{a}_2 \in \mathcal{X}^{N_w} \) and \( c, c_1, c_2 \in \mathcal{X} \). For the coupled DE recursion (89) with \( l \in \mathbb{Z}_+ \), we have

1) If \( \mathbf{a}_1 \succeq \mathbf{a}_2 \), then \( T^{(l)}_c (\mathbf{a}_1, c) \succeq T^{(l)}_c (\mathbf{a}_2, c), \forall c \in \mathcal{X} \).

2) If \( c_1 \succeq c_2 \), then \( T^{(l)}_c (\mathbf{a}, c_1) \succeq T^{(l)}_c (\mathbf{a}, c_2), \forall \mathbf{a} \in \mathcal{X}^{N_w} \).

3) If \( \exists \mathbf{a} \in \mathcal{X}^{N_w} \) such that \( T^{(1)}_c (\mathbf{a}, c) \succeq \mathbf{a} \), then \( T^{(l+1)}_c (\mathbf{a}, c) \succeq T^{(l)}_c (\mathbf{a}, c) \) and the limit \( T^{(\infty)}_c (\mathbf{a}, c) = \lim_{l \to \infty} T^{(l)}_c (\mathbf{a}, c) \) does indeed exist, satisfying \( T^{(\infty)}_c (\mathbf{a}, c) \succeq T^{(l)}_c (\mathbf{a}, c) \) and

\[
T^{(1)}_c (T^{(\infty)}_c (\mathbf{a}, c), c) = T^{(\infty)}_c (\mathbf{a}, c).
\]

4) If \( \exists \mathbf{a} \in \mathcal{X}^{N_w} \) such that \( T^{(1)}_c (\mathbf{a}, c) \preceq \mathbf{a} \), then \( T^{(l+1)}_c (\mathbf{a}, c) \succeq T^{(l)}_c (\mathbf{a}, c) \) and the limit \( T^{(\infty)}_c (\mathbf{a}, c) = \lim_{l \to \infty} T^{(l)}_c (\mathbf{a}, c) \) does indeed exist, satisfying \( T^{(\infty)}_c (\mathbf{a}, c) \preceq T^{(l)}_c (\mathbf{a}, c) \) and

\[
T^{(1)}_c (T^{(\infty)}_c (\mathbf{a}, c), c) = T^{(\infty)}_c (\mathbf{a}, c).
\]

**Proof:** See the proof of Lemma 34 in [18].

For brevity, in the sequel, unless otherwise specified, whenever we speak of a coupled DE recursion, we always assume that \( \mathbf{x}^{(0)} = \Delta_{w} \). Under this assumption, Claim 3) in Lemma 9 indicates that the density sequences generated by this recursion are always partially ordered. Further, these sequences satisfy the following symmetric constraint due to the uniform coupling weights and symmetric boundary conditions [18],

\[
x^{(l)}_i = x^{(l)}_{N_w-i}, \forall i \in \mathcal{N}_c.
\]

Due to the above constraint, we focus our discussion on the “middle point” of a CFP \( \mathbf{x} \in \mathcal{F}_c (\epsilon, N, w) \), i.e., \( x^\text{mid} \). For a fixed \( \epsilon \in [0, 1] \), we write \( x^\text{mid} \) in the form of \( x^\text{mid} = m (N, w) \) to highlight the fact that this density depends on the coupling length \( N \) and the coupling width \( w \). By doing this, we can show that \( m (N, w) \) converges to a UFP as \( N \to \infty \), as stated in the following lemma.

**Lemma 10:** For any fixed \( \epsilon \in [0, 1] \) and \( w \in \mathbb{Z}_+ \), the limit \( m (\infty, w) = \lim_{N \to \infty} m (N, w) \) exists. Further, it is a UFP of the uncoupled DE recursion (61), i.e., \( m (\infty, w) \in \mathcal{F}_s (\epsilon) \).
Proof: Now consider two coupled DE recursions sharing the same degree distribution pair \((\lambda, \rho)\), coupling width \(w\) and channel erasure probability \(\epsilon\), but with different coupling lengths \(N'\) and \(N\) where \(N' < N\). Denote by \(x'\) and \(x\) the CFPs of these two coupled DE recursions.

Following Lemma 9, it is easy to verify the following facts:
1) \(x'_i \preceq x_i\).
2) \(x'_i \preceq x'_{i+1}, \forall i \in \{0, 1, \ldots, \lfloor (N' + w - 1)/2 \rfloor - 1\}\).
3) \(x_i \preceq x_{i+1}, \forall i \in \{0, 1, \ldots, \lfloor (N + w - 1)/2 \rfloor - 1\}\).

Further, we can conclude from the above facts that
\[
x'_{\lfloor (N' + w - 1)/2 \rfloor} \preceq x_{\lfloor (N' + w - 1)/2 \rfloor} \preceq x_{\lfloor (N + w - 1)/2 \rfloor}.
\] (94)

Therefore, we have \(m(N', w) \preceq m(N, w)\), and by Proposition 2 the limit \(\lim_{N \to \infty} m(N, w)\) indeed exists.

Now by fixing \(N = N' + 4w\) and letting \(N \to \infty\) (thereby \(N' \to \infty\)), we can rewrite the inequality (94) as
\[
\lim_{N' \to \infty} x'_{\lfloor (N' + w - 1)/2 \rfloor} \preceq \lim_{N \to \infty} x_{\lfloor (N + w - 1)/2 \rfloor} \preceq \lim_{N \to \infty} x_{\lfloor (N' + w - 1)/2 \rfloor} = m(\infty, w).
\] (95)

Therefore, \(\forall k \in \{0, 1, \ldots, 2w\}\),
\[
\lim_{N \to \infty} x_{\lfloor (N + w - 1)/2 \rfloor - k} = m(\infty, w).
\] (96)

By substituting the above limit to the following CFP equation with \(i = \lfloor (N + w - 1)/2 \rfloor - w\),
\[
x_i = \frac{1}{w} \sum_{k=0}^{w-1} c_{i, i-k} \boxtimes \lambda \boxtimes \left( \frac{1}{w} \sum_{j=0}^{w-1} \rho^{\otimes} (x_{i-k+j}) \right),
\] (97)
we obtain the following UFP equation \(m(\infty, w) = c \boxtimes \lambda \boxtimes (\rho^{\otimes} (m(\infty, w)))\), i.e., \(m(\infty, w)\) is a UFP of the uncoupled DE recursion (61).

Definition 15: For any \(\bar{x} \in \mathcal{X}^{Nw}, \epsilon \in [0, 1]\), the potential function for SC-LDPC(\(\lambda, \rho, N, w, m\)) is given by
\[
U_c(\bar{x}, \epsilon) = L' (1) \sum_{i=0}^{N_{mid}} H \left( \frac{1}{R' (1)} R^{\otimes} (x_i) + \rho^{\otimes} (x_i) - x_i \boxtimes \rho^{\otimes} (x_i) - c \boxtimes L \left( \frac{1}{w} \sum_{j=0}^{w-1} \rho^{\otimes} (x_{i+j}) \right) \right).
\] (98)

In general, if a density sequence \(\bar{x}\) is a CFP of the coupled DE recursion (89), then each entry of \(\bar{x}\) implicitly depends on the coupling length \(N\).
Remark 6: Notice that the potential function $U_c(x, \epsilon)$ defined in this paper is slightly different from [18] (see Definition 37 therein). Here we restrict the sum over $i \in \{0, 1, \ldots, N_{w}^{\text{mid}}\}$ based on the symmetric constraint (93), regarding the entries of the former half of $x$ as independent variables. Due to the same reason, we define the direction over $X^{N_w}$ as follows.

Definition 16: A direction over $X^{N_w}$, denoted as $\delta x$, is a sequence of length $N_w$, the first $N_{w}^{\text{mid}} + 1$ entries of which are independent directions defined over $X$ and the others are zero vectors of length $m + 1$, i.e.,

$$\delta x = (\delta x_0, \delta x_1, \ldots, \delta x_{N_{w}^{\text{mid}}}, 0, \ldots, 0).$$

(99)

Definition 17: The directional derivative of $U_c(x, \epsilon)$ with respect to $x$ in the direction $\delta x$ is defined as

$$d_x U_c(x, \epsilon) [\delta x] = \lim_{t \to 0} \frac{U_c(x + t\delta x, \epsilon) - U_c(x, \epsilon)}{t}.$$  (100)

Lemma 11: The directional derivative of $U_c(x, \epsilon)$ defined as above is given by

$$d_x U_c(x, \epsilon) [\delta x] = \frac{L'}{1} \sum_{i=0}^{N_{w}^{\text{mid}}} H \left( \left( T_{c}^{(1)}(x, c_{\epsilon}) - x \right) _i \otimes \rho^{(2)}(x_i) \otimes \delta x_i \right).$$

(101)

Proof: The proof of Lemma 11 is almost identical to that of Lemma 38 in [18], and we omit the details for brevity.

Lemma 11 indicates that $d_x U_c(x, \epsilon) [\delta x]$ vanishes if $x$ is a CFP of the coupled DE recursion (89).

Lemma 12: Define the shift operator $S(\cdot)$ as follows [18]

$$(S(x))_i = \begin{cases} 
\Delta_0, & i = 0 \\
 x_{i-1}, & i \in \mathbb{N}_c \setminus \{0\}.
\end{cases}$$

(102)

Let $x$ be a CFP of the coupled DE recursion (89). For a fixed $\epsilon \in [0, 1]$ and an arbitrary small $\eta > 0$, there exists $N_\eta \in \mathbb{Z}_+$ such that $\forall N > N_\eta$, after applying the operator $S(\cdot)$ to $x$, the change of $U_c(x, \epsilon)$ is bounded as follows

$$U_c(S(x), \epsilon) - U_c(x, \epsilon) < -\Delta E(\epsilon) + \eta.$$  (103)

Proof: First of all, following the same line as in the proof of Lemma 41 in [18], we can show that the change of $U_c(x, \epsilon)$ is bounded by the underlying potential function at the “middle point” of $x$, i.e.,

$$U_c(S(x), \epsilon) - U_c(x, \epsilon) \leq -U_s(x_{N_{w}^{\text{mid}}}, \epsilon).$$

(104)
Next, since the “middle point” $x_{N_{\text{mid}}}$ converges to a UFP $m(\infty, w)$ (see Lemma 10), we can deduce from the continuity of $U_s(x, \epsilon)$ with respect to $x$ that for any arbitrary small $\eta > 0$ there exists an integer $N_{\eta}$ such that $\forall N > N_{\eta}, |U_s(x_{N_{\text{mid}}}, \epsilon) - U_s(m(\infty, w), \epsilon)| < \eta$. Therefore,

$$U_c(S(x), \epsilon) - U_c(x, \epsilon) \leq -U_s(x_{N_{\text{mid}}}, \epsilon) < -U_s(m(\infty, w), \epsilon) + \eta \leq -\Delta E(\epsilon) + \eta. \quad (105)$$

C. Theorems for Threshold Saturation

Based on the above propositions and lemmas, we can follow a similar procedure as in [18] to establish the following theorem. See the proof of Theorem 44 therein and we do not reproduce the details in this paper.

**Theorem 1:** Consider an SC-LDPC$(\lambda, \rho, N, w, m)$ ensemble on the BEC with erasure probability $\epsilon \in [0, \epsilon_{\text{pot}})$. For arbitrary small $\eta > 0$, there exists $N_{\eta} \in \mathbb{Z}_+$, and a positive constant independent of $N$ and $w$, denoted as $K_{\lambda, \rho}$, such that $\forall N > N_{\eta}, w > K_{\lambda, \rho}/(\Delta E(\epsilon) - \eta)$, the only CFP of the coupled DE recursion (89) is $\Delta_0$.

Likewise, the converse to Theorem 1 can be shown following almost the same line as in the proof of Theorem 47 in [18].

**Theorem 2:** Consider an SC-LDPC$(\lambda, \rho, N, w_0, m)$ ensemble on the BEC with erasure probability $\epsilon \in (\epsilon_{\text{pot}}, 1]$. There exists $N_0 \in \mathbb{Z}_+$ such that $\forall N > N_0$, the CFP of the coupled DE recursion (89) initialized with $\Delta_m$ satisfies

$$T_c^{(\infty)}(\Delta_m, c_\epsilon) > \Delta_0. \quad (106)$$

V. Conclusion

We investigated the asymptotic performance for SC-LDPC ensembles defined over GL$(2^m)$. Our purpose is to prove the existence of the threshold saturation effect for transmission on the BEC. To this end, we presented a detailed analysis of the entropy function and the VN and CN convolutional operators and discussed their properties through several propositions and lemmas. In particular, we derived a nonbinary version of the duality rule for entropy to accommodate the DE analysis of nonbinary LDPC ensembles on the BEC. Based on this, we constructed potential functions for the uncoupled and coupled DE recursions, the forms of which are very similar to those in [13]. These findings led us to establish the threshold saturation theorem and its converse following almost the same approach developed by S. Kumar et al.
APPENDIX A

The following proposition is useful in the proofs of some propositions and lemmas in this paper.

**Proposition 4:** Consider two vectors \((u_1, u_2, \ldots, u_K)\) and \((v_1, v_2, \ldots, v_K)\) with \(K \in \mathbb{Z}_+\). The following identity holds for \(n = 1, \ldots, K - 1\),

\[
\sum_{i=n}^{K} v_i u_i = v_n \sum_{k=n}^{K} u_k + \sum_{i=n+1}^{K} (v_i - v_{i-1}) \sum_{k=i}^{K} u_k.
\]  
(107)

APPENDIX B

SOME PROPERTIES OF \(V^m_{i,j,k}\) AND \(C^m_{i,j,k}\)

In this Section, we discuss and prove several useful results for \(V^m_{i,j,k}\) and \(C^m_{i,j,k}\).

**Proposition 5:** For any \(m \in \mathbb{Z}_+\), the coefficients \(V^m_{i,j,k}\) and \(C^m_{i,j,k}\) satisfy the following properties.

1) For any \(i, j \in M\), we have \(0 \leq V^m_{i,j,k} \leq 1\), \(0 \leq C^m_{i,j,k} \leq 1\) and

\[
\sum_{k=0}^{m} V^m_{i,j,k} = \sum_{k=0}^{m} C^m_{i,j,k} = 1.
\]  
(108)

2) The coefficients \(V^m_{i,j,k}\) and \(C^m_{i,j,k}\) remain invariant under a swap of \(i\) and \(j\), i.e.,

\[
V^m_{i,j,k} = V^m_{j,i,k}, \quad C^m_{i,j,k} = C^m_{j,i,k}.
\]  
(109)

3) We have \(V^m_{i-1,j,k} < V^m_{i,j,k}\) if \(0 < k \leq i \leq m\), \(0 < k \leq j \leq m\), and \(C^m_{i-1,j,k} > C^m_{i,j,k}\) if \(0 < i \leq k \leq m\), \(0 < j \leq k \leq m\).

**Proof:** 1) The proof of \(V^m_{i,j,k} \geq 0\) and \(C^m_{i,j,k} \geq 0\) is trivial by definition. The identities \(\sum_{k=0}^{m} C^m_{i,j,k} = \sum_{k=0}^{m} V^m_{i,j,k} = 1\) simply follow from the fact that \(V^m_{i,j,k}\) and \(C^m_{i,j,k}\) are probabilities (see Subsection II-A in [23]). Alternatively, one may also prove them using the following well-known Vandermonde identity for the \(q\)-binomial coefficients [25],

\[
[m]_j = \sum_{k=0}^{m} 2^{(i-k)(j-k)} \binom{i}{k} \binom{m-i}{j-k}.
\]  
(110)

2) For \(n \in \mathbb{Z}\), define \([n]\) as follows

\[
[n] = \begin{cases} 
1, & n = 0 \\
\prod_{i=1}^{n} (2^i - 1), & \text{otherwise.}
\end{cases}
\]  
(111)

The Gaussian binomial coefficient \([m]_k\) can be rewritten as

\[
[m]_k = \frac{[m]}{[m-k][k]}.
\]  
(112)
Rewrite those Gaussian binomial coefficients in (6) in the form as above,
\[ V_{i,j,k}^m = \frac{2^{(i-k)(j-k)}}{[k][m][i-k][j-k][m-i-j+k]} = V_{j,i,k}^m. \tag{113} \]

Similarly, we can show that \( C_{i,j,k}^m = C_{j,i,k}^m \).

3) We focus on the first inequality and omit the proof of the other since \( C_{i,j,k}^m = V_{m-i,m-j,m-k}^m \).

In the case of \( 0 < k = i \leq m \), the first inequality holds since \( V_{m-k,1,j}^m = 0 < V_{i,j,k}^m \). For \( 0 < k < i \leq m \), this inequality follows from the fact that \( V_{i,j,k}^m > 2^{k-1}V_{i-1,j,k}^m \) (see Appendix A in \( \text{[23]} \)).

\[ \text{APPENDIX C} \]

THE COMMUTATIVE, DISTRIBUTIVE AND ASSOCIATIVE LAWS OF \( \Box \) AND \( \mathbb{X} \)

In this section, we aim to prove three important laws of the convolutional operators \( \Box \) and \( \mathbb{X} \).

**Proposition 6:** Considering three vectors of length \( m + 1 \) denoted as \( a, b \) and \( c \), we have

1) \( a \Box b = b \Box a \).
2) \( a \Box (b \Box c) = a \Box b + a \Box c \).
3) \( (a \Box b) \Box c = a \Box (b \Box c) \).

**Proof:** Claim 1) follows from (109) and Claim 2) can be easily verified by definition. Thus, we put our focus on Claim 3) for \( \Box \). The proof for \( \mathbb{X} \) is identical.

We first compare the \( k \)-th entries of \( (a \Box b) \Box c \) and \( a \Box (b \Box c) \) for any \( k \in \mathbb{M} \). On one hand,
\[ [(a \Box b) \Box c]_k = \sum_{j=0}^{m} \sum_{n=0}^{m} [a \Box b]_j V_{j,n,k}^m c_n = \sum_{i=0}^{m} \sum_{l=0}^{m} \sum_{n=0}^{m} a_i b_{l} c_n \sum_{j=0}^{m} V_{j,n,k}^m V_{i,l,j}^m. \tag{114} \]

On the other hand,
\[ [a \Box (b \Box c)]_k = \sum_{i=0}^{m} \sum_{j=0}^{m} a_i V_{i,j,k}^m [b \Box c]_j = \sum_{i=0}^{m} \sum_{l=0}^{m} \sum_{n=0}^{m} a_i b_{l} c_n \sum_{j=0}^{m} V_{i,j,k}^m V_{n,l,j}^m. \tag{115} \]

Therefore, \( (a \Box b) \Box c = a \Box (b \Box c) \) holds if
\[ \sum_{j=0}^{m} V_{j,n,k}^m V_{i,l,j}^m = \sum_{j=0}^{m} V_{j,i,k}^m V_{n,l,j}^m. \tag{116} \]

In other words, what we need to prove is that either side of (116) remains invariant when we swap the roles of \( i \) and \( n \). To this end, we substitute (6) into (116),
\[ \sum_{j=0}^{m} V_{j,n,k}^m V_{i,l,j}^m = \frac{1}{[m][i][n]} \sum_{j=0}^{m} 2^{(j-k)(i-k)+(l-j)(n-j)} [j][m-j][l][n-j][m-l]. \tag{117} \]
Applying (110) to \[
\begin{bmatrix}
m - j \\
i - k
\end{bmatrix}
\begin{bmatrix}
l - j + m - l \\
[l - j + m - l] + j' - k + i - j'
\end{bmatrix}
\begin{bmatrix}
m - l \\
[j' - k] + i - j'
\end{bmatrix}.
\] (118)

Substituting (118) into (117), we can obtain
\[
\sum_{j=0}^{m} V_{j,n,k}^m V_{i,l,j}^m = \frac{1}{m} \sum_{j=0}^{m} \sum_{j'=0}^{m} \left[ \frac{m - l}{[j' - k + i - j' - k]} \right] \times 2^{j^2+j'j'+j'^2-(j+j')(l+k)+k^2+(i+n)l-(i'j+nj)}. \]

(119)

Obviously, swapping the roles of \(i\) and \(n\) does not change \(\sum_{j=0}^{m} V_{j,n,k}^m V_{i,l,j}^m\), which completes the proof of Claim 3).

In the remainder of this appendix, we demonstrate how to apply the above laws to the derivative analysis of the entropy function involving the convolutional operators \(\Box\) and \(\bigotimes\). For convenience, we write a density \(x \in \mathcal{X}\) in the form of \(x = (1 - \sum_{i=1}^{m} x_i, x_1, x_2, \ldots x_m)\) by regarding \(x_1, x_2, \ldots x_m\) as independent variables. As a result, \(\forall i \in \mathbb{M}\setminus\{0\}\), the partial derivative of \(x\) with respect to \(x_i\) is a vector of length \(m + 1\) given by

\[
\frac{\partial}{\partial x_i} x = \Delta_i - \Delta_0.
\] (120)

Therefore, \(\forall a \in \mathcal{X}, i \in \mathbb{M}\setminus\{0\}\), we have

\[
\frac{\partial}{\partial x_i} (a \Box x) = a \Box \frac{\partial}{\partial x_i} x \quad \text{(a)} \quad \frac{\partial}{\partial x_i} (a \bigotimes x) = a \bigotimes \frac{\partial}{\partial x_i} x \quad \text{(b)}
\] (121)

where (a) and (b) are both based on the distributive law of \(\Box\) and \(\bigotimes\).

Moreover, \(\forall n \in \mathbb{Z}_+, i \in \mathbb{M}\setminus\{0\}\),

\[
\frac{\partial}{\partial x_i} x^n = \left( \frac{\partial x}{\partial x_i} \ast x \ast \ast \ast \right) + \left( x \ast \frac{\partial x}{\partial x_i} \ast \ast \ast \right) + \ldots + \left( x \ast x \ast \ast \ast \ast \right) = n \frac{\partial x}{\partial x_i} x^{n-1} \quad \text{(a)}
\] (122)

where (a) is based on the commutative law and the associative law of \(\Box\) and \(\bigotimes\).
The identity (122) is useful in the proofs of Lemma 7 and Lemma 11. For example, one can deduce from (122) that, \( \forall i \in \mathbb{M} \setminus \{0\} \),

\[
\frac{\partial}{\partial x_i} H \left( R^\rho (x) \right) = R' \left( 1 \right) H \left( \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right) \\
\frac{\partial}{\partial x_i} H \left( \rho^\rho (x) \right) = H \left( \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right) \\
\frac{\partial}{\partial x_i} H \left( x \otimes \rho^\rho (x) \right) = H \left( \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right) + H \left( x \otimes \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right) \\
\frac{\partial}{\partial x_i} H \left( c_e \square L^\rho (\rho^\rho (x)) \right) = L' \left( 1 \right) H \left[ c_e \square \lambda \square \left( \rho^\rho (x) \right) \square \left( \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right) \right].
\]  

(123)  
(124)  
(125)  
(126)

Putting the above together, we have, \( \forall i \in \mathbb{M} \setminus \{0\} \),
\[
\frac{\partial}{\partial x_i} U_s (x, \epsilon) = L' \left( 1 \right) \left\{ H \left( \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right) - H \left( x \otimes \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right) \\
- H \left[ c_e \square \lambda \square \left( \rho^\rho (x) \right) \square \left( \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right) \right] \right\} \\
= L' \left( 1 \right) \left\{ H \left[ \left( x \square \left( \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right) \right) - H \left[ c_e \square \lambda \square \left( \rho^\rho (x) \right) \square \left( \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right) \right] \right\} \\
= L' \left( 1 \right) H \left[ (x - T_s^{(1)} (x, c_e)) \square \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right] \\
\overset{(b)}{=} L' \left( 1 \right) H \left[ (T_s^{(1)} (x, c_e) - x) \otimes \rho^\rho (x) \otimes \frac{\partial x}{\partial x_i} \right]
\]

(127)  
(128)  
(129)  
(130)

where (a) and (b) follow from (31) and (32), respectively.

By the continuity of \( U_s (x, \epsilon) \) with respect to \( x \), we have
\[
d_x U_s (x, \epsilon) \otimes \delta x = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} U_s (x, \epsilon) \delta x_i \\
= L' \left( 1 \right) H \left[ (T_s^{(1)} (x, c_e) - x) \otimes \rho^\rho (x) \otimes \sum_{i=1}^{m} \frac{\partial x}{\partial x_i} \delta x_i \right] \\
\overset{(a)}{=} L' \left( 1 \right) H \left[ (T_s^{(1)} (x, c_e) - x) \otimes \rho^\rho (x) \otimes \sum_{i=1}^{m} (\Delta_i - \Delta_0) \delta x_i \right] \\
\overset{(b)}{=} L' \left( 1 \right) H \left[ (T_s^{(1)} (x, c_e) - x) \otimes \rho^\rho (x) \otimes \delta x \right]
\]

(131)  
(132)  
(133)  
(134)

where (a) is based on (120) and (b) follows from (79).

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