We study topological open membranes of BF type in a manifest BV formalism. Our main interest is the effect of the bulk deformations on the algebra of boundary operators. This forms a homotopy Lie algebra, which can be understood in terms of a closed string field theory. The simplest models are associated to quasi-Lie bialgebras and are of Chern-Simons type. More generally, the induced structure is a Courant algebroid, or “quasi-Lie bialgebroid”, with boundary conditions related to Dirac bundles. A canonical example is the topological open membrane coupling to a closed 3-form, modeling the deformation of strings by a C-field. The Courant algebroid for this model describes a modification of deformation quantization. We propose our models as a tool to find a formal solution to the quantization problem of Courant algebroids.
1. Introduction

Topological field theories have emerged as an important tool for performing exact calculations in physics. They are also very well fit to apply field theoretical methods to mathematical problems. The topological Poisson sigma model introduced in [1] has been used in [2] to give the solution of Kontsevich [3, 4] to the problem of deformation quantization in terms of correlation functions for a topological string theory. This model captures the essence of the emergence of noncommutative geometry in open string theory in the presence of a $B$-field background [5, 6, 7]. This topological model, which is of BF-type, is most succinctly formulated in a BV language. In this formulation it can be viewed as a sigma-model with a particular symplectic superspace as target manifold. Many 2-dimensional topological field theories—such as the A- and the B-model [8, 9]—can be formulated in such a way. These topological models put on the disc describe the deformation theory of the algebra of boundary operators [10, 11]. In the case of [2] the boundary algebra was simply the algebra of functions $C^\infty(M)$ on some manifold, which was quantized to a noncommutative algebra by a Poisson bivector coupling to the bulk.

This idea has a straightforward generalization to higher dimensions. Many higher dimensional topological field theories can be formulated as gauge fixed versions of similar BF type BV sigma models, e.g. Chern-Simons theory, Rozanski-Witten theory [12], Donaldson-Witten theory [13, 14], and the membrane coupling to a 3-form [9]. This gives rise to open $p$-branes as introduced in [9], which have various applications to both physics and mathematics. In this paper we will specialize to the case $p = 2$, that is 3 dimensions, and demonstrate the algebraic and geometrical structure of these theories.

The main focus will be the deformation of the theory living on the boundary by the bulk theory. The couplings in the bulk can be viewed as elements of the cohomology of the deformation complex for the boundary theory. The path integral then calculates the corresponding deformation, as a straightforward generalization of deformation quantization. Indeed this was the basic strategy adopted in [2]. The present paper will mainly deal with the semi-classical part of the quantization, that is it will only deal with the first order deformation. In a subsequent paper [15] we will discuss how to use path integral techniques to extent this to a full quantization, at least on a formal level.

Perhaps the most interesting example, and indeed our primary motivation, is the open membrane coupling to a closed 3-form, which was called the open 2-brane in [9]. This model has many interesting relations to both physics and mathematics. This model could be used to study the effect of the $C$-field to the little string theory living on the 5-brane,
In [16, 17] a constraint canonical quantization was used to study the model. This approach however is hard to extend to all orders; the method of BV quantization is much more suitable for this. Also the solution could become singular, as it involves the inversion of a 3-form. In this paper we will show that the topological open membrane coupling to the 3-form describes what is known as an exact Courant algebroid [23]. The classic Courant algebroid is based on the space $T M \oplus T^* M$, and was used to study general constraint quantization of gauged systems. It was shown that this Courant algebroid is deformed by a closed 3-form. Quantization of this object is still unsolved, but probably has a connection to gerbes. The deformed exact Courant algebroid controls a deformed version of quantization; the 3-form deforms a Poisson structure to a quasi-Poisson structure. In principle, the path integral for the open membrane model defines a formal quantization for this object.

The simplest examples of our class of open membrane models are based on general quasi-Lie bialgebras, or Manin pairs $(g, g^*)$. These models are closely related to Chern-Simons theories. The relation between the topological open membrane and the quantization of the boundary string can be seen as a generalization of the relation between Chern-Simons and WZW models [24]. The relation of the $G/G$ quotient WZW model and a double Chern-Simons [25, 26] will explicitly appear as a special case of the topological membrane related to a particular quasi-Lie bialgebra. Quasi-Lie bialgebras are the infinitesimal objects related to (quasi-)Hopf algebras [27], a generalization of quantum groups. In our BV models the Yang-Baxter equation will be identified with part of the master equation, while scrooching/twisting of quasi-Lie bialgebras comes out simply as a canonical transformation. In a follow up paper [15] we will show that the boundary theory will have the structure of the corresponding quasi-Hopf algebras. Quantizability of the general Lie bialgebra was proven recently by Etinghof-Kazhdan [28]. The path integral of our model will give an alternative universal quantization formula for general quasi-Lie bialgebras.

More generally we will find models based on Courant algebroids, which might also be called quasi-Lie bialgebroids. Algebroids combine the structure of tangent spaces and Lie algebras. Sections of the tangent bundle have a natural Lie bracket, which involves first order derivatives. Algebroids generalize this structure to more general fiber bundles. Lie bialgebroids can be described as dual pairs $(A, A^*)$ of Lie algebroids. The basic example is $A = TM$, which is equivalent to the exact Courant algebroid mentioned above [29]. Courant algebroids correspond to the generic topological open membrane. Recently Xu asked the question whether any Lie bialgebroid is quantizable [30]. A Lie bialgebroid is the geometrical structure underlying the classical dynamical Yang-Baxter equation. The corresponding
quantum dynamical Yang-Baxter equation is relevant in quantizing Liouville theory, the Knizhnik-Zamolodchikov-Bernard equation, the Calogero-Moser model, and many related problems. Our approach gives a unified geometrical description of QDYB equations in terms of topological membrane theories. In particular, it gives a proposal for a formal universal quantization formula of Courant algebroids, based on 3-dimensional Feynman diagrams.

The topological open membrane theories we will study give an interesting class of toy closed string field theories, \[\text{[31, 32, 33, 34, 35]}\] which in some cases can be solved exactly. In general closed string field theory has the structure of a \(L_\infty\) algebra \[\text{[31]}\]. In fact this \(L_\infty\) algebra will play an important role in our discussion of the open membrane. It is this structure that will be deformed by the bulk deformations. Especially in the cases related to quasi-Lie bialgebras, the quasi-Hopf algebras will be constructed out of the closed string field theory. Other areas in physics where our model could be useful is the study of instanton effects in M-theory \[\text{[36, 37]}\] and the study of D-branes in the presence of a 3-form field strength.

This paper is organized as follows. In Section 2 we introduce a convenient BV formulation in terms of superfields which allow us to give a simple geometric construction of topological membrane theories.

In Section 3 we discuss the general algebraic structure of the master equation that follows from the semiclassical topological open membrane. In Section 4 we discuss the semi-classical structure of the algebra of boundary operators for the open membrane.

In Section 5 the simplest class of models related to quasi-Lie bialgebras are discussed in some detail.

In Section 6 we turn to topological open membranes based on exact Courant algebroid structures. These are related to membranes coupling to a closed 3-form. This is then generalized to more general Courant algebroids, combining the above situations of the tangent bundle and the quasi-Lie bialgebras. These models are the most general solutions of the master equation if one does not introduce negative ghost number superfields.

In Section 7 we review the mathematical structure of Courant algebroids, and show how our open membranes give rise to this structure.

In Section 8 we end with some conclusions and discussions on the results.

While this paper was being finished, the paper \[\text{[38]}\] appeared, which has some overlap with the present paper.
2. **BV Actions for Topological Open Membranes**

In this section we will develop a convenient description of a general class of BV actions for topological open membranes. We will only recall the main results of the detailed construction of [9] relevant for the present paper.

2.1. **Superfields and BV Structure**

The theory of topological open $p$-branes developed in [9], specialized to the case $p = 2$, involves an Euclidean open membrane living in a Euclidean target space $\mathcal{M}$. The worldvolume theory of the membrane will be a topological theory, meaning that it does not depend on the worldvolume metric. The models studied in this paper will be manifestly independent of the metric, and be of BF type. The fields are differential forms, which have an action of the form

$$S_{BF} = \int_V \eta_{ij} B^i_{(2-p)} dA^j_{(p)} + \text{interactions},$$

where the index between brackets denotes the form degree and the interactions are formed by wedge products of the fields. Note that the form degree $p$ is at most 2. These theories have a lot of gauge symmetries which have to be gauge fixed. A general procedure to find a gauge fixed action is to use the BV formalism. For each of the fields $A^i$ and $B^i$, we need to introduce a whole set of ghost and antighost fields. The ghosts (and ghost-for-ghosts) for a $p$-form field $A^i_{(p)}$ will be corresponding lower degree fields. The antighosts are fields of all higher degree. It will be convenient to combine a field with all its ghosts and antighosts into a single superfield. These superfields can then be considered as maps between superspaces. Another advantage of using this superfield language is that it automatically takes care of some extra signs that are needed in the BV formulation.

Quite generally, a topological field theory contains two operators of crucial importance: a BRST operator $Q$ and a fermionic operator $G_\mu$ transforming as a worldvolume 1-form (the current of which is usually denoted $b$ in string theory). They satisfy the crucial anti-commutation relation \{\(Q, G_\mu\)\} = $\partial_\mu$. Furthermore, there is a conserved ghost number charge called ghost, with $Q$ and $G$ having ghost numbers 1 and $-1$ respectively. Given any BRST closed worldvolume scalar operator $\mathcal{O}$ we will define a set of descendants defined by $\mathcal{O}^{(p+1)} = G\mathcal{O}^{(p)}$, where $\mathcal{O}^{(0)} = \mathcal{O}$. These operators satisfy the descent equation $Q\mathcal{O}^{(p+1)} = d\mathcal{O}^{(p)}$, due to the anti-commutation relation above. As $G$ is a 1-form, the $p$th descendant $\mathcal{O}^{(p)}$ will be a worldvolume $p$-form.
Any physical field (of ghost number zero) will be the descendant of some scalar field \( \phi^I \), generically a ghost. These scalars can be viewed as coordinates on a target superspace \( \mathcal{M} \). Equivalently, they can be seen as components of a map \( \phi : V \to \mathcal{M} \), where \( V \) is the worldvolume. The coordinates on \( V \) will be denoted \( x^\mu \). As noted above, the \( p \)-form descendants of the coordinate fields can be combined into superfie lds which will be denoted \( \phi^I \). For this purpose we introduce fermionic worldvolume coordinates \( \theta^\mu \) of ghost degree 1. Together the super coordinates \( (x^\mu|\theta^\mu) \) can be viewed as coordinates on the superspace \( \mathcal{V} = \Pi TV \), where \( \Pi \) denotes the shift of the degree by 1 (acting on the fiber). We will sometimes denote the supercoordinates collectively by \( x \). The supercoordinate fields are then functions of \( (x^\mu|\theta^\mu) \) which can be expanded as

\[
\phi^I(x, \theta) = \phi^I(x) + \theta^\mu \phi^{I(1)}_\mu(x) + \frac{1}{2} \theta^\mu \theta^\nu \phi^{I(2)}_{\mu\nu}(x) + \frac{1}{3!} \theta^\mu \theta^\nu \theta^\rho \phi^{I(3)}_{\mu\nu\rho}(x).
\]

We treat the descendant components as separate fields. The descendant operator acts on superfields simply as \( G^\mu = \frac{\partial}{\partial \theta^\mu} \). Combined together, the super coordinates can be viewed as a map between superspaces, \( \phi : \mathcal{V} \to \mathcal{M} \). Note that if the superfield \( \phi^I \) has ghost number \( g \), the \( p \)th descendant will have ghost number \( g - p \). The ghost number \( g \) therefore equals the form degree of the physical field in the superfield.

Instead of starting with the BF theory and constructing a BV action we will start right away from the BV action. This will be a rather simple matter in the language of superfields. In order to define a BV structure for the membrane, the target space \( \mathcal{M} \) must be symplectic with symplectic form \( \omega \). In this paper we will only consider constant \( \omega \), though this restriction is not essential. This induces a symplectic form on the space of superfields by

\[
\omega_{BV} = \int_{\mathcal{V}} \phi^* \omega = \frac{1}{2} \int_{\mathcal{V}} \omega_{IJ} \delta \phi^I \delta \phi^J,
\]

where \( \delta \) denotes the De Rham differential on field space. Here the integral over \( \mathcal{V} \) involves integration over \( x \) and \( \theta \). It also defines a BV antibracket as the corresponding Poisson bracket, which we shall formally denote as follows,

\[
(\cdot, \cdot) = \int_{\mathcal{V}} \omega^{IJ} \frac{\partial R}{\partial \phi^I} \wedge \frac{\partial L}{\partial \phi^J}.
\]

Here the \( L \) (\( R \)) subscript indicates the left (right) derivative. These derivatives are functional derivatives with respect to the superfields \( \phi^I \), defined in the usual way by

\[
\frac{\partial}{\partial \epsilon} f(\phi + \epsilon \xi) \bigg|_{\epsilon=0} = \int_{\mathcal{V}} \xi^I \frac{\partial L}{\partial \phi^I}(\phi) = \int_{\mathcal{V}} \frac{\partial R}{\partial \phi^I}(\phi) \xi^I.
\]
This BV bracket is derived from a BV operator, which is a second order differential operator formally given by
\[
\Delta = \frac{1}{2} \int_V \omega^{IJ} \frac{\partial^2}{\partial \phi^I \partial \phi^J},
\]  
(6)
where the derivatives are left-derivatives.

The BV bracket \((\cdot, \cdot)\) should have degree 1, or equivalently the symplectic structure \(\omega_{BV}\) should have degree \(-1\). Therefore, the symplectic structure \(\omega\) on the target space must have degree 2, since the integration over \(V\) has ghost degree \(-3\). Hence we find that the target space \(\mathcal{M}\) is a symplectic supermanifold with a symplectic structure of degree 2.

Let us recall some basic facts about BV algebras. The BV bracket is related to the BV operator by the relation
\[
(\alpha, \beta) = (-1)^{|\alpha|} \Delta (\alpha \beta) - (-1)^{|\alpha|} \Delta (\alpha) \beta - \alpha \Delta \beta.
\]  
(7)
The BV bracket is graded antisymmetric in the following shifted sense
\[
(\alpha, \beta) = -(-1)^{(|\alpha|+1)(|\beta|+1)} (\beta, \alpha),
\]  
(8)
and it satisfies the following graded Jacobi identity
\[
(\alpha, (\beta, \gamma)) = ((\alpha, \beta), \gamma) + (-1)^{(|\alpha|+1)(|\beta|+1)} (\beta, (\alpha, \gamma)).
\]  
(9)

A BV action \(S_{BV}\) determines a BRST operator by the relation \(Q = (S_{BV}, \cdot)\). It squares to zero if the BV action satisfies the classical master equation \((S_{BV}, S_{BV}) = 0\). Quantum mechanically this is not strictly necessary, but rather the BV action has to satisfy the quantum master equation \(\Delta S_{BV} + \frac{1}{2} (S_{BV}, S_{BV}) = 0\). The Jacobi identity for the BV bracket implies the derivation condition for the BRST operator
\[
Q(\alpha, \beta) = (Q \alpha, \beta) - (-1)^{|\alpha|} (\alpha, Q \beta).
\]  
(10)

Let us describe the structure of the target superspace. We will make use of the fact that for any supermanifold the the nonzero degrees form a fiber bundle over the degree zero submanifold, which we will denote \(M\). In fact, if we denote by \(M^p\) the submanifold of degree at most \(p\), we find that \(M^{p+1}\) is a fibration over \(M^p\). For this paper we will assume that the target space is symplectic, or equivalently that the BV structure is nondegenerate. This can always be accomplished by adding extra fields. Furthermore we assume that all superfields will contain a physical (i.e. ghost degree zero) component. This reduces the degrees of the superfields, and thereby in the superspace \(\mathcal{M}\), to 0, 1, or 2. The degree 1 submanifold \(M^1\)
is a graded fiber bundle over \( M = \mathcal{M}^0 \). As the BV structure is considered nondegenerate, there should be a natural (symmetric) pairing in the fiber. This implies that we can, at least locally, write the fiber bundle as \( \mathcal{M}^1 = \mathbb{A} \oplus \mathbb{A}^* \). The fiber of degree 2 must be dual to the linearization of the degree 0 base. In other words it can be described by the fiber of the twisted cotangent bundle \( T^*[2]\mathcal{M} \). Combining this with the structure of the degree 1 fiber, we can describe the total target superspace as a twisted cotangent bundle \( \mathcal{M} = T^*[2]\mathcal{A} \). Here we used that the cotangent direction of the fiber is naturally the dual fiber, and the twist of the degree by 2 maps it degree back to 1.

Locally the coordinates \( \phi^I \) split into sets of conjugate coordinates \( \phi_i \) on the base \( \mathbb{A} \) and \( \phi^+_i \) on the fiber. The shift implies that their degrees are related by \( |\phi^+_i| = 2 - |\phi^i| \). The cotangent bundle comes with the canonical symplectic structure \( \frac{1}{2}\omega_{IJ} d\phi^I d\phi^J = d\phi^+_i d\phi^i \). Due to the shift this has the required degree of 2. In the BV formulations, the conjugate superfields \( \phi^+_i \) will contain the antifields of \( \phi^i \), and vice versa.

2.2. BV Action and BRST Operator

The first part of the BV action will be given by the kinetic term, which in this paper will always be written in a first order form. Explicitly, our kinetic term will directly be determined by the BV structure and be given by\(^1\)

\[
S_0 = \frac{1}{2} \int_V \omega_{IJ} \phi^I d\phi^J,
\]

where \( d = \theta^\mu \frac{\partial}{\partial x^\mu} \) is the De Rham differential on the worldvolume in the superfield formalism. This action satisfies the classical master equation \( \{S_0, S_0\} = 0 \), and also the the quantum master equation, as \( \triangle S_{BV} = 0 \). This indeed has the BF form \( \{\}\), whith the “\( A \)” and “\( B \)” fields residing in conjugate superfields with respect to the BV structure. The induced BV-BRST operator is given by \( Q = d \). This indeed satisfies the correct anticommutation relations with the operator \( G \).

The interaction terms in the action the membrane action will be given by a function of the superfields. The total bulk action will have the form

\[
S = S_0 + \int_V \gamma,
\]

\(^1\)In general \( [p] \) denotes a shift of the (fiber) degree by \( p \).
\(^2\)Here we assumed \( \omega \) to be constant. In general the integrand is given in terms of a 1-form potential \( \tau \) satisfying \( d\tau = \omega \) as \( \phi^* \tau = \tau_1(\phi)d\phi^I \).
where $\gamma(x, \theta) = (\phi^* \gamma)(x, \theta) = \gamma(\phi(x, \theta))$ for some function $\gamma \in C^\infty(\mathcal{M})$.\footnote{Here and in the following we will denote a pullback by the superfields by a boldface character.} We will require that $\gamma$ satisfies $\Delta \gamma = 0$, so that the classical master equation will imply the quantum master equation. The master equation then takes the form $\int d\gamma + \frac{1}{2} (\int \gamma, \int \gamma) = 0$. If we can ignore boundary terms, the first term is a total derivative and therefore vanishes identically. Note that in order to get an action of ghost degree zero, $\gamma$ should be a function of degree 3. In the presence of the deformation $\gamma$, the BRST operator takes the form $Q = d + (\int \gamma, \cdot)$. The (classical) master equation is then indeed equivalent to $Q^2 = 0$. The anticommutation relation of the deformed BRST operator with $G$ is preserved by this deformation, due to the superfield structure. We can also add a boundary term of the form

$$\int_{\partial \mathcal{V}} \beta,$$

where $\beta = \phi^* \beta$ for a function $\beta \in C^\infty(\mathcal{M})$ of degree 2, and $\partial \mathcal{V} = \Pi T(\partial \mathcal{V})$ is the boundary of the super worldvolume.\footnote{Note that this is given by fixing both the even and odd normal coordinates.}

3. Observables and the Master Equation

In this section we discuss the master equation of the class of topological open membranes introduced above. We formulate this in terms of a convenient algebraic framework related to the target space algebra.

3.1. The Bulk Algebra

First we discuss the precise relation between the field theory on the closed membrane to the algebra in the target space. In the rest of this section we discuss the generalization to open membranes.

Observables for the bulk membrane can be found as functions of the superfields. They are therefore associated to functions on the target superspace. Let us denote this algebra of functions $\mathcal{A} = C^\infty(\mathcal{M})$. The basic observable in the field theory on the membrane associated to $f \in \mathcal{A}$ is the pullback to the super worldvolume $\mathcal{V}$ of the membrane, $f = \phi^* f$ where $\phi : \mathcal{V} \to \mathcal{M}$ is the map formed by the superfields. The BV symplectic structure on the superfields was inherited by pullback of $\omega$ from the target space $\mathcal{M}$. Let us denote the
dual Poisson bracket on \( \mathcal{A} \) by \([\cdot, \cdot]\). This bracket is related to the BV bracket on field space by pullback,
\[
\int V \phi^*([f, g]) = (\int V f, \int V g).
\]
(14)
The bracket \([\cdot, \cdot]\) in \( \mathcal{A} \) has degree \(-2\), and therefore has the usual graded antisymmetry and Jacobi identity, rather than the shifted ones for the BV antibracket \((\cdot, \cdot)\).

Similarly, the BRST operator \( Q \) in field space induces a nilpotent operator \( Q \) on \( \mathcal{A} \). We have to be careful here, as the action involves a derivative on the worldvolume. And in our description using function on the target space, we did not included operators involving derivatives. To define \( Q \) in the algebra \( \mathcal{A} \) we will drop total derivatives over the worldvolume. For the closed membrane this will indeed be sufficient. Below we will be more careful about these contributions when we study the open membrane. With the above form of the action, we have
\[
Q \int V f = \int V df + (\int V \gamma, \int V f).
\]
(15)
Dropping total derivatives, the operator \( Q \) in the algebra is determined by the second term, and can be written \( Qf = [\gamma, f] \).

The algebraic structures on the target space are related to correlators in the field theory. For example, the bracket in the algebra \( \mathcal{A} = C^\infty(\mathcal{M}) \) can be defined by the relation
\[
\phi^*([f, g]) = \oint_S (\phi^* f)^{(2)} \phi^* g,
\]
(16)
where \( S \) is a 2-cycle enclosing the insertion point of \( g \). In terms of the superfields this can be written in the form \( \phi^*([f, g]) = f_{\Pi TS} f g \). The integral over \( \Pi TS \) includes in integral over two fermionic coordinates tangent to the cycle, and therefore picks out the first descendant when we specialize to the zeroth descendant component.

The reason for the coincidence of the BV bracket with the above operator product is a result of the kinetic term, involving \( \omega \) and \( d \). Using the (gauge fixed) propagator, this gives
\[
\left\langle \oint_S \phi^{(2)}(x) \phi^J(y) \right\rangle \sim \omega^{IJ} \oint_S \frac{n_\mu(x - y)^\mu}{\|x - y\|^3} \sim \omega^{IJ},
\]
(17)
where \( n_\mu \) is the normal vector to the surface \( S \). This correlation function is topological, and therefore only depends on the homology class of \( S \).

This is the structure of the closed membrane algebra. If we would introduce a boundary for the membrane, the above will still be valid when we assume that the observables \( f \) all vanish on the boundary, because then the total derivatives still vanish when integrated. This can actually be achieved by restriction on the algebra \( \mathcal{A} \). We will call this restricted bulk
algebra $A_0$. This would describe the pure bulk theory. However, we are interested basically in what happens on the boundary. We will now turn to the boundary algebra, which will be treated in a similar way.

### 3.2. Including Boundary Terms

The full target space is the superspace $\mathcal{M}$. In the present paper, our main goal is the open membrane. Therefore, we have to specify boundary conditions. These will be determined by a choice of Lagrangian subspace $\mathcal{L} \subset \mathcal{M}$ (with respect to the BV structure). The boundary condition for the superfields is such that the boundary of the super-worldvolume $\partial V = \Pi T(\partial V)$ is mapped into this Lagrangian subspace $\mathcal{L}$. The bulk operators were related to functions on the target space, giving the algebra $A = C^\infty(\mathcal{M})$. The Lagrangian condition ensures that the kinetic term $S_0$ satisfies the master equation $(S_0, S_0) = 0$, including the boundary term.

As above, we consider a target space which is a twisted cotangent bundle, $\mathcal{M} = T^*[2]A$. A natural choice for the Lagrangian subspace $\mathcal{L}$ is a section of this fiber bundle. In case $\mathcal{L}$ is everywhere transverse to the fiber, we can canonically identify $\mathcal{L}$ with the base $A$.

The operators on the boundary can be interpreted as functions on the Lagrangian subspace $\mathcal{B} = C^\infty(\mathcal{L})$. Given the Lagrangian subspace, we have a map $P_{\mathcal{L}} : A \to \mathcal{B}$ mapping functions on the total target space to functions on the Lagrangian subspace, defined by restriction. Note that the restricted bulk algebra mentioned above is given by $A_0 = \ker P_{\mathcal{L}}$.

Taking into account the boundary term, the total BRST operator $Q$ acting on a bulk observable $f = \phi^*f$ can be written

$$Q \int_V f = Q \int_V \phi^*f = \int_V \phi^*(Qf) + \int_{\partial V} \phi^*f.$$  \hspace{1cm} (18)

The first term indeed generates just the BRST operator in $A$, which we used above. In general, we have also the boundary term. We could set it to zero by demanding the extra condition $P_{\mathcal{L}}f = f|_{\mathcal{L}} = 0$. Indeed, as $\phi$ restricted to the boundary maps into $\mathcal{L}$, this gives a vanishing boundary term. These functions represent the pure bulk operators. More generally, we incorporate the boundary terms into our description by extending the space of operators to $\bar{A} = A \oplus \mathcal{B}$ including both the bulk and the boundary deformations. Elements are pairs $f \oplus g \in A \oplus \mathcal{B}$, for which we define the ($\phi$-dependent) formal integral

$$\int f \oplus g \equiv \int_V f + \int_{\partial V} g.$$  \hspace{1cm} (19)
We can interpret the restriction map $P_L$ as an off-diagonal map in this extended algebra $P_L : f \oplus g \mapsto 0 \oplus P_L f$. Note that this operation trivially squares to 0. It is in fact the unperturbed BRST operator, for $\gamma = 0$. With these notations, we can write the above identity — also including a boundary term — in the form $Q \int f \oplus g = \int Q f \oplus (P_L f - Q_L g)$, where $Q_L$ denotes the restriction of $Q$ to the boundary. Here the relative minus sign in front of $Q_L$ is due to the fact that $\int_{\partial V}$ has degree $-2$ (or equivalently, it involves a degree one delta-function on the boundary). This leads to a BRST operator on the extended operator space $\bar{A}$ having the block form
\[
Q = \begin{pmatrix} Q & 0 \\ P_L & -Q_L \end{pmatrix} : A \oplus B \rightarrow A \oplus B.
\] (20)
The relation $Q_L P_L = P_L Q$ ensures that $\bar{Q}^2 = 0$.

We also need to know how the bracket extends to the total space $\bar{A}$. The bracket will be zero when restricted to the boundary, due to the Lagrangian boundary condition. So we only need to give the prescription for the bracket acting between $A$ and $B$. To find an expression for this we will use the derivation condition of the unperturbed BRST operator $P_L$,
\[
P_L[\alpha, \beta] = [P_L \alpha, \beta] + (-1)^{||\alpha||}[\alpha, P_L \beta],
\] (21)
which is a consequence of the corresponding identity in field space. To give a more explicit description, we will need an explicit embedding $i_L : B \rightarrow A$, satisfying $P_L \circ i_L = 1_B$. For $\alpha = f \oplus 0 \in A_0$ and $\beta = i_L g \oplus 0$ the above implies
\[
[f \oplus 0, 0 \oplus g] = 0 \oplus (-1)^{|f|}P_L[f, i_L g].
\] (22)
This will be independent of the choice of embedding $i_L$ due to the above identity. For $P_L f \neq 0$, the simplified description in terms of the algebra will not be sufficient anymore. We will however not need this generalization. Of course, this result can also be derived from the BV bracket on field space.

### 3.3. Deformations, BRST Cohomology and Canonical Transformations

Infinitesimal deformations of the action are controlled by the BRST cohomology. This should be the cohomology for the total BRST operator $\bar{Q}$. The total space $\bar{A}$ can be viewed as the total complex of a double complex, with differentials $P_L$ and $Q$. The total cohomology can be calculated using spectral sequence techniques. In the following calculation we will assume that $Q_L = 0$ for simplicity, although one can easily generalize.
We decompose $\bar{Q} = Q + P_L$, and first take cohomology with respect to $Q$. The first term in the spectral sequence is then $E_1 = H_Q(\mathcal{A}) \oplus \mathcal{B}$, as $Q$ acts only on $\mathcal{A}$. The term $E_1$ has differential induced by $P_L$. We denote by $P'_L : H_Q(\mathcal{A}) \to \mathcal{B}$ the induced projection $P_L$ reduced to $H_Q(\mathcal{A})$. Note that this is well defined, as $P_L = 0$ on $\text{im} \ Q$ by our assumption. Taking its cohomology restricts the bulk term to elements in the kernel of $P'_L$. In other words, the bulk deformations are $Q$-cohomology classes vanishing on the boundary. The boundary term is defined up to the image of $P'_L$. The spectral sequence terminates at the second term because there is no room for higher differentials. We conclude $H_Q(\bar{\mathcal{A}}) \cong E_2 \cong \ker P'_L \oplus (\mathcal{B} / \text{im} \ P'_L)$.

For nonzero $Q_L$, we should have replaced $\mathcal{B}$ by $H_Q(L)$. An alternative way to calculate the cohomology is to start the spectral sequence with $P_L$. Then the first term is given by $E_1 = H_{P_L}(\bar{\mathcal{A}}) = \mathcal{A}_0 \oplus 0$, as $P_L$ is surjective. Denoting $Q' = Q |_{\mathcal{A}_0}$, we have $E_2 = H_Q(\mathcal{A}_0) \oplus 0$. The spectral sequence terminates at the second term, as $E_1$ is concentrated in a single degree (in the $P_L$ direction). Therefore $H_Q(\bar{\mathcal{A}}) \cong E_2 \cong H_{Q'}(\ker P_L)$.

The two answers do not look the same. For example, the first one contains boundary terms, while the second has only bulk deformations. The two results are however equivalent. We will see below how boundary deformations can be turned into bulk terms in vice versa by canonical transformations.

The BRST cohomology is closely related to canonical transformations in the BV theory. For any function $\beta \in \mathcal{A}$ let us define the operator $\delta_{\beta} = [\cdot, \beta]$. In the following we will mainly use $\beta$ of degree 2. It is basically the Hamiltonian vector field with respect to the symplectic structure. Similarly, on superfield space we define the operator $\delta_{\beta} = (\cdot, \int \beta)$. This operator is the generator of a canonical transformation. The relation between the BRST cohomology and a canonical transformation is based on the following relation

$$e^{t\delta_{\beta}} S = S + tQ \int \beta + O(t^2).$$

In other words, to first order in $t$ a canonical transformation shifts the action by a BRST exact term. The first order shift of the action by a BRST exact term usually does not produce a solution of the master equation. It can however be turned into a solution of the master equation by adding higher order corrections, which are generated by the full canonical transformation. A canonical transformation is a true symmetry of the theory, while the BRST exact terms only give an approximation.

In terms of the algebraic language we have developed above, and in case we can ignore boundary terms, the above can be reduced to the algebra $\mathcal{A}$,

$$e^{t\delta_{\beta}} \gamma = \gamma + tQ \beta + O(t^2).$$
Even if there are boundary terms, the term $e^{\delta \beta \gamma}$ is still a solution of the bulk master equation when $\gamma$ is. It is a solution to the full master equation if in addition the boundary master equation $P_C(e^{\delta \beta \gamma}) = 0$ is satisfied. The solution however is not necessarily equivalent to $\gamma$, as the canonical transformation can produce boundary terms, which we have here ignored.

3.4. Boundary Deformations

We next consider the case where the boundary term does not vanish. Actually, we can use what we have found above for the case where there is no boundary term.

First, we have to be careful about the kinetic term in the action. In general, we write the full action as $S_0 + \Gamma$, where $S_0$ is the kinetic term and $\Gamma$ is assumed to be the integral of the pull-back of a function $\gamma$ on $\mathcal{M}$. Furthermore, we will deform the action by an extra boundary term, which is the integral of a pullback from $\mathcal{A}$. Note that for $\gamma = 0$ we have $Q = Q_{CL} = 0$.

First we note that

$$\delta \beta S_0 = (S_0, \int_V \beta) = \int_V d\beta = \int_{\partial V} \beta. \tag{25}$$

To be able to describe this in terms of the algebra $\mathcal{A} \oplus \mathcal{B}$, we adjoin to the bulk algebra $\mathcal{A}$ a formal element $\tau$ corresponding to $S_0$, i.e. formally $S_0 = \int \tau$, and satisfies

$$\delta \beta \tau = 0 \oplus P_{CL} \beta \tag{26}$$

for any $\beta$. Then we have

$$e^{\delta \beta} \int (\tau + \gamma) \oplus 0 = \int (\tau + e^{\delta \beta \gamma}) \oplus \left( \sum_{n \geq 1} \frac{1}{n!} (\delta \beta)^n P_{CL} \beta \right). \tag{27}$$

We assume that $[\beta, \beta] = 0$, so that only the $n = 1$ term survives in the boundary term. An important case where this is satisfied is when $\beta \in i_{CL} (\mathcal{B})$. If we ignore the boundary term, we find what we used before: the canonical transformation of the kinetic term is a total derivative, and therefore trivial, so we only transform the bulk deformation $\gamma$. We know that the pure bulk term $S_0 + \Gamma_{\gamma, \beta} = S_0 + e^{-\delta \beta} \int \gamma$ is a solution to the full master equation if $\gamma$ is a solution of the bulk master equation, i.e. $[\gamma, \gamma] = 0$, and the boundary term vanishes, $P_C(e^{-\delta \beta \gamma}) = 0$. However, if $P_{CL} \beta \neq 0$, this solution is not equivalent to the solution $S_0 + \int \gamma$. In fact, we have

$$e^{\delta \beta} (S_0 + \Gamma_{\gamma, \beta}) = e^{\delta \beta} \int (\tau + e^{-\delta \beta \gamma}) = \int (\tau + \gamma) \oplus P_{CL} \beta = S_0 + \int \gamma \oplus P_{CL} \beta, \tag{28}$$
where we assumed that \([\beta,\beta] = 0\) to prevent higher order terms in the boundary term. As this includes all contributions of the canonical transformation, it should be equivalent to the full action \(S + \Gamma_{\gamma,\beta}\). So we have actually written the deformation using \(\beta\) in terms of a boundary term. Therefore, if we can solve our constraint of vanishing field strength, we can add a boundary term to cancel the boundary term in the master equation. So although the action looks simple, the BV master equation is much more nontrivial due to the boundary term. In general, it can be found by writing the terms again as superfields, and the boundary term as a bulk term using \(d\), writing the master equation for the bulk and writing total derivatives again as boundary terms. It has in general two components: a bulk and a boundary term, given by

\[
Q\gamma + \frac{1}{2}[\gamma,\gamma] = 0, \quad P_L\gamma = 0. \tag{29}
\]

At first sight, this seems to be the master equation for \(\beta = 0\), rather than the one for nonzero boundary term to which it is supposed to be equivalent. We have to be very careful however with the boundary condition for the fields, as they are different in both cases. Assume that before the canonical transformation we had a boundary condition \(\psi|_{\partial V} = 0\). After the canonical transformation, we have changed the fields, which means that in the new variables the boundary condition becomes

\[
e^{-\delta_\beta}\psi^i|_{\partial V} = (\psi^i - [\psi^i,\beta])|_{\partial V} = 0. \tag{30}
\]

Thus can also be found by realizing that variation with respect to \(\chi_i\) has a boundary term \(\delta\chi_i(\psi^i - \frac{\partial \gamma}{\partial \chi_i})\). As \(\delta\chi_i\) is arbitrary on the boundary, this requires the above boundary condition for \(\psi^i\).

This implies that the projector \(P_L\) has changed due to the presence of the boundary term \(\beta\). To see how, let us call the original projector \(P^0_L\), and the projector in the presence of a boundary term \(P^\beta_L\). These two operators are then related by a canonical transformation as

\[
P^\beta_L = e^{\delta_\beta} \circ P^0_L \circ e^{-\delta_\beta}. \tag{31}
\]

The boundary master equation has to be interpreted as \(P^\beta_L\gamma = 0\). This is indeed the same as the original constraint \(P^0_L(e^{-\delta_\beta}\gamma) = 0\) we found for the equivalent pure bulk action. Expanding the exponential, this can be written in the form

\[
\sum_{n \geq 0} \frac{1}{n!} P^0_L(-\delta_\beta)^n\gamma = P^0_L\gamma - P^0_L[\gamma,\beta] + \frac{1}{2} P^0_L[[\gamma,\beta],\beta] + \cdots = 0. \tag{32}
\]

Later, we will give an interpretation of the various terms in this equation.
4. The Algebraic Structure of Open Membranes

We will now discuss the general structure of the deformed boundary algebra that arises as sketched above. We will see that in general there is a structure of $L_\infty$ algebra, which arises in a way we call a derived $L_\infty$ algebra, generalizing the notion of derived bracket.

4.1. Correlators and the Boundary Algebra

Let us first discuss the correlation functions of boundary operators in the open membrane theory in the presence of a nontrivial bulk term $\gamma$. As we discussed the basic boundary observables are determined by functions on the Lagrangian $L \subset M$.

First we write the action as the sum of a kinetic term and an interaction term, $S = S_0 + S_{int}$, where we took $S_{int} = \int \gamma$. Using a Gaussian integral in the path integral, we can write the correlation functions as

$$\langle \prod_a O_a \rangle = \int D\phi e^{i[\pi \phi]} \left( e^{iS_{int}[\phi]} \prod_a O_a \right).$$

(33)

The propagator in the above expression, seen as a bidifferential operator, can be written in the form

$$\Pi \left[ \frac{\partial}{\partial \phi} \right] = \int_V d\mathbf{x} \int_V d\mathbf{y} \Pi(x, y) \omega^{IJ} \frac{\partial}{\partial \phi^I(x)} \frac{\partial}{\partial \phi^J(y)}.$$  

(34)

Here $\Pi(x, y)$ is the integral kernel for the inverse kinetic operator $d^{-1}$ (after gauge fixing). We recognize in this expression the BV bracket structure. Because of this we will see that we can effectively describe the algebraic structure on the boundary operators in terms of the original BV bracket.

The boundary theory is basically a topological closed string theory. As discussed in [39], one of the essential operations in the algebra of observables is based on the bracket determined by the contour integral of one operator around another,

$$\{f, g\} = \oint_C f^{(1)} g,$$  

(35)

where $C$ is a 1-cycle enclosing the insertion point of $g$. Interpreting the 1-form $f^{(1)}$ as a worldsheet current, this is actually the action of the current on a scalar operator. This bracket determines the current algebra in the string theory. For example, a Ward identity implies that the supercommutator $[ff^{(1)}, fg^{(1)}] = f\{f, g\}^{(1)}$.

The bracket $\{\cdot, \cdot\}$ introduced above is an antibracket of degree $-1$. Therefore, the graded antisymmetry and Jacobi identity are similar to those of the BV antibracket $(\cdot, \cdot)$ on field
space. This is part of the reason that the closed string algebra forms has the on-shell structure of a BV algebra [31].

This bracket can again conveniently be written in terms of the superfields. We introduce the super 1-cycle \( C = \Pi TC \); the integration over the fermionic coordinates picks up the first descendant in the tangent direction. More precisely, we can define the operation in terms of the correlation function

\[
\langle \delta_{\phi_0} f \rangle_c ,
\]

where all the operators are put on the boundary and \( \delta_{\phi_0} \) is a delta function fixing the scalar fields to a fixed value \( \phi_0 \) consistent with the boundary condition. After contractions, and using the expression for the propagator above, the lowest order term can be written

\[
\int_V dz \int_C dy \Pi(z, y) \Pi(z, x) \int d\phi \delta(\phi - \phi_0) \omega^{K L} \omega^{I J} \frac{\partial^2 \gamma}{\partial \phi^K \partial \phi^I \partial \phi^J} \partial f \partial g .
\]

This is just the Feynman integral corresponding to a 2-legged tree-level diagram. The integral is a universal factor, that does no longer depend on the precise choice of operators. The dependence on the functions \( f \) and \( g \), and therefore the choice of boundary observables is expressed in terms of differential operators acting on these functions. To see that this is nontrivial, one should check that the integral indeed is a number different from zero. That this is indeed the case will be shown elsewhere [40]. In terms of the the boundary algebra of functions \( B = C^\infty(\mathcal{L}) \), the bracket can now be written (after a proper normalization and including signs)

\[
\{ f, g \} = (-1)^{|f|+1} P_L[[\gamma, f], g] + (-1)^{|f||g|+1} P_L[[\gamma, g], f] .
\]

Here the \( P_L \) results from the projection on the outgoing state \( \delta_{\phi_0} \), or the delta-function in the zero-mode integral over \( \phi \). More precisely, we should interpret the boundary operators like \( f \) as embedded in the algebra \( A \); so we should write \( i_L f \).

In the above form of the bracket, the reader can readily recognize the structure of a term in the boundary master equation we met before. This is no coincidence, and is a direct consequence of the equivalence between the deformed theories with and without a boundary term.

More general correlation functions can be found by introducing more integrated operators. The operator products related to the brackets in the boundary string are given by the operator equation

\[
\{ f_1, \ldots, f_n \} = (-1)^{|f_1|+|f_2|+\ldots+|f_{n-2}|} \int_{\partial V} f_1 \ldots \int_{\partial V} f_{n-2} \int_C f_{n-1} f_n + \text{perms} ,
\]
where $\mathcal{C} = \PiTC$ is a super 1-cycle in the boundary. In terms of the components of the superfields, this can be written

$$\{f_1, \ldots, f_n\} = (-1)^{|f_1|+|f_2|+\ldots+|f_{n-2}|} \int_{\partial V} f_1^{(2)} \cdots \int_{\partial V} f_{n-2}^{(2)} \oint_C f_{n-1}^{(1)} f_n + \text{perms.} \tag{40}$$

The corresponding correlation functions are topological. The relevance of these operations and the relation to the $L_\infty$ structure was explained in [39]. They can be interpreted as the structure constant for the bosonic closed string field theory [31] of the corresponding boundary string.

The semiclassical approximation to these brackets are calculated analogously to that of the bracket, involving the various contractions. Due to the form of the propagator the brackets in the boundary algebra are induced by the BV bracket in the bulk and the bulk term in the action. In fact, the form (38) is almost that of the well known mathematical notion of a derived bracket.

### 4.2. Derived $L_\infty$ Algebra

We can express the above results of the boundary brackets in our algebraic language in terms of the basic structure, the bracket and the BRST operator. This gives rise to a generalization of so-called derived brackets.

Let us assume a graded differential Lie algebra $\mathcal{A}$. This means that it is provided with a Lie-bracket $[\cdot, \cdot]$ of degree $p$ and a derivation $d$ of this bracket which squares to zero. Then we can define the derived bracket $\circ$, of degree $p + 1$, by

$$f \circ g = (-1)^{|f|+1}[df, g]. \tag{41}$$

In general this derived bracket is not skew symmetric. It satisfies a close analog of the Jacobi identity, making $\mathcal{A}$ into a Loday algebra. The differential $d$ is also a derivation of the derived bracket. We could have also considered the skew-symmetrization of $\circ$, which is sometimes called the derived bracket. This bracket will in general not satisfy the Jacobi identity. The derived bracket becomes important when we study an abelian subalgebra $\mathcal{B}$ of $\mathcal{A}$ (with respect to $[\cdot, \cdot]$). It can be shown that restricted to $\mathcal{B}$ the derived bracket is actually skew-symmetric. When $\mathcal{B}$ in addition is closed with respect to $d$ and $\circ$, it is a graded differential Lie algebra itself, with a bracket of degree $p + 1$.

A well known example of a derived bracket is a Poisson bracket. Consider a manifold $M$ and the algebra $\mathcal{A} = C^\infty(\PiTM) = \Gamma(\Lambda^* TM)$ is the algebra of multivector fields. This is naturally provided with the Schouten-Nijenhuis bracket $[\cdot, \cdot]$ (the generalization to

17
multivector fields of the Lie bracket). Now we choose a bivector $\pi$ satisfying $[\pi, \pi] = 0$ (Poisson structure), and consider the derivation $d_\pi = [\pi, \cdot]$. This makes $\mathcal{A}$ into a differential Lie algebra. The algebra $\mathcal{B} = C^\infty(M)$ of functions is an abelian subalgebra stable with respect to $d_\pi$. The derived bracket on $\mathcal{B}$, given by $\{f, g\} = (-1)^{|f|+1}[[\pi, f], g]$, is precisely the Poisson bracket generated by $\pi$. This example is actually the analogous boundary bracket for the open string of the Poisson-sigma model [2].

We now observe that the tree level result for the bracket (38) in the boundary algebra has the form of the skew-symmetrization of a derived bracket, with $d = [\gamma, \cdot] = Q$, apart from the projection $P_L$. This projection had to be inserted because the boundary algebra $\mathcal{B}$ is not closed under the derived bracket. Indeed, this is a natural extension of an induced derived bracket. More generally at tree level, the differential (BRST operator), bracket and trilinear bracket on the boundary algebra $\mathcal{B}$ are given by the expressions

$$Q_L f = P_L[\gamma, i_L f],$$

$$\{f, g\} = (-1)^{|f|+1}P_L[[\gamma, i_L f], i_L g] \pm \text{perms.,}$$

$$\{f, g, h\} = (-1)^{2|f|+|g|+3}P_L[[[\gamma, i_L f], i_L g], i_L h] \pm \text{perms.}$$

(42)

One could go on, but at least semi-classically, the higher brackets all vanish in the theories we study, due to the degree. This is an obvious generalization of the notion of derived bracket to higher brackets. Note that the induced derived bracket on $\mathcal{B}$ with the projection $P_L$ is not a Lie bracket in general. However as it turns out they do satisfy the relations of an $L_\infty$ algebra or homotopy Lie algebra. We will therefore call this a derived $L_\infty$ algebra.

We can now recognize the boundary master equation (32) as the Maurer-Cartan equation $Q_L \beta + \{\beta, \beta\} + \cdots = 0$ of this derived $L_\infty$ algebra. The bilinear boundary bracket should be interpreted as the BV bracket of the boundary string. We note that the BV algebra in the boundary string is of a more general homotopy type, including higher brackets. Such generalizations of the BV algebra appeared in the context of BV quantization in [41], were they were called quantum antibrackets.

4.3. Path Integral Quantization and Deformation Theory

Now that we have described the semiclassical deformation structure of our model, let us shortly discuss how to pass to the quantization. This will be a generalization of the problem of deformation quantization for the associative algebra of functions.

Topological field theories in $d$ dimensions are closely related to $d$-algebras. Indeed, $d$-algebras can be defined in terms of the homology of configuration spaces of punctured $d$-
dimensional discs \([\mathbb{R}^2] \). A 1-algebra is simply an associative algebra, while for \(d \geq 2\), a \(d\)-algebra in general is a (super)commutative associative algebra with a twisted Lie-bracket of degree \(d - 1\) \([\mathbb{R}^3] \). Particularly important examples of (super)commutative algebras are provided by the algebra of functions \(\mathcal{B} = C^\infty(\mathcal{A})\) on some (super)manifold \(\mathcal{A}\). They become \(d\)-algebras when provided with a (possibly zero) twisted Lie bracket of degree \(d - 1\). The BV sigma model canonically associates a topological membrane theory to any such 2-algebra. Our quantization can be considered as expressing the generalized Deligne conjecture, which states that the deformation of a \(d\)-algebra is a \((d + 1)\)-algebra, see for example \([\mathbb{R}^4] \). We interpret this by saying that the \((d + 1)\)-dimensional topological field theory deforms the \(d\)-dimensional topological field theory on the boundary. The Hochschild cohomology—closely related to the deformation complex—of \(\mathcal{B} = C^\infty(\mathcal{A})\) as a \(d\)-algebra is given by the algebra of functions on the twisted cotangent space, \(HH^*(\mathcal{B}) = C^\infty(T^*[d]\mathcal{A})\), c.f. \([\mathbb{R}^4] \). This is naturally reflected in our BV sigma models, where the target superspace of the bulk membrane has the form \(\mathcal{M} = T^*[2]\mathcal{A}\), with the boundary string living in \(\mathcal{B}\).

The objective of the quantization program will be to construct a map from the Hochschild cohomology \(\mathcal{A} = HH^*(\mathcal{B})\) to the Hochschild complex of the 2-algebra,

\[
Q : C^\infty(\mathcal{M}) \to C^*(\mathcal{B}, \mathcal{B}).
\]

(43)

This map should be intertwining, at least up to a quasi-isomorphism. It then gives a formality of the complex as a \(G_\infty\) or homotopy Gerstenhaber algebra. The map will be constructed using the path integral of the topological open membrane corresponding to the \(G\) algebra \(\mathcal{B}\). For this we also need an inner product, which is provided by the 2-point function. For \(\gamma \in C^\infty(\mathcal{M})\) the proposed quantization map is given by

\[
\langle f_0, Q_C(\gamma)(f_1, \cdots, f_n) \rangle = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}(S_0 + S_\gamma)} \mathcal{O}_C(f_0, \cdots, f_n),
\]

(44)

where the deformed action is given by \(S_\gamma = f_\gamma \gamma\). \(\mathcal{O}_C\) is a boundary observable composed out of the operators \(f_i \in \mathcal{B}\) and depends on an extra label, which runs over chains in the configuration space of the \(n + 1\) insertion points. They run over the labels of the maps defining the \(G_\infty\) structure. So any \(\gamma \in C^\infty(\mathcal{M})\) gives rise to a whole set of multilinear maps \(Q_C(\gamma) \in C^{nc}(\mathcal{B}, \mathcal{B}) = \text{Hom}(\mathcal{B}^{\otimes nC}, \mathcal{B})\). The brackets of the \(L_\infty\) algebra correspond to particular examples of the boundary observables \([\mathbb{R}^4] \). Indeed, this is precisely a reduction in the topological context for the \(L_\infty\) algebra in string field theory \([\mathbb{R}^4] \). We can find a quantization of the full \(G_\infty\) algebra by considering more general observables. For example, we should include an observable \(\mathcal{O}_{A,2}(f_1, f_2) = f_1 f_2\) for the product, at least to lowest order. A more detailed discussion will be given in a forthcoming paper \([\mathbb{R}^4] \).
The path integral can be perturbatively expanded as a sum over Feynman diagrams, each corresponding to a particular term in the expansion of the products, and a universal weight given by an integral involving CS propagators \( d^{-1} \). As the above quantization map is intertwining, a solution to the master equation in \( C^\infty(M) \) is mapped to a deformed \( G_\infty \) structure in the complex.

5. Topological Membranes from Quasi-Lie Bialgebras

In this section we discuss a particular class of open membrane models based on a purely fermionic target space. The models, which are of a Chern-Simons type, have a semi-classical structure of a quasi-Lie bialgebra or Manin pair. This allows us to interpret the open membrane model as a quantization of these mathematical objects, which are quasi-Hopf algebras or quantum groups.

5.1. Quasi-Lie Bialgebras and Open Membranes

Above we have described a construction for topological membranes based on a twisted cotangent bundle of a supermanifold. As the symplectic structure should have degree two, the simplest way to get this is to take a manifold which is completely of degree one. This is the class of models we will study in this section. More explicitly, the target space will be given by \( M = T^*[2](\Pi g) = \Pi(g \oplus g^*) \), where \( g \) is any vector space and \( g^* \) its dual. In other words, the base space is the graded space \( A = \Pi g \). Hence we will initially take the Lagrangian to be the zero section, \( L = \Pi g \). We will choose flat coordinates \( \chi^i \) and \( \psi^i \), on the base and the fiber respectively. The Lagrangian submanifold \( L \) is then given by the equations \( \psi^i = 0 \).

This gives two superfields of ghost degree one,

\[
\psi^i = \psi^i + \theta B^i + \cdots, \quad \chi^i = \chi^i + \theta A^i + \cdots.
\]  

The action for this model following the general description takes the form

\[
S = \int_V (\psi^i d\chi^i + \gamma(\psi^i, \chi^i)).
\]

The function \( \gamma \) is cubic, and has the general form

\[
\gamma = \frac{1}{2} c^{ijk} \psi^i \psi^j \chi^k + \frac{1}{2} f^{ijk} \psi^i \psi^j \chi^k + \frac{1}{3!} \varphi^{ijk} \psi^i \psi^j \psi^k.
\]
The \( \chi \chi \chi \) term has to vanish in order for the condition \( P_L \gamma = 0 \) to be satisfied. The condition \( \Delta \gamma = 0 \) implies the vanishing of the traces \( c_{ij} = 0 = f_{ij}^{ij} \). The master equation \([\gamma, \gamma] = 0\) is equivalent to the 4 relations

\[
\begin{align*}
  c_{ijk}^{[ij} c_{k]m} &= 0, \\
  f_{ij}^{ij} f_{i}^{k]m} + c_{m}^{i} \varphi^{jk]m} &= 0, \\
  c_{m[i} f_{j]}^{f]m} &= 0, \\
  f_{m}^{ij} \varphi^{kl]m} &= 0.
\end{align*}
\]

(48)

The first condition implies that \( c_{ij}^{k} \) are the structure constants of a Lie-algebra, based on the vector space \( g \). When the \( \varphi^{ijk} \) vanish, we see from the second equation that the \( f_{ij}^{ij} \) also are the structure constants of a Lie-algebra. In other words \( g^{*} \) is also a Lie-algebra in this case. Note that always the total space \( g \oplus g^{*} \) has the structure of a Lie algebra.\[5\]

For more concreteness, let us introducing a basis \( e_{i} \) for \( g \) and a dual basis \( e^{i} \) for \( g^{*} \). With respect to this basis, the Lie-bracket on \( g \oplus g^{*} \) can be written as

\[
\begin{align*}
  [e_{i}, e_{j}] &= c_{ij}^{k} e_{k}, \\
  [e_{i}, e^{j}] &= f_{i}^{j} k \ e_{k} - c_{ik}^{j} e_{k}, \\
  [e^{i}, e^{j}] &= f_{j}^{ij} e^{k} - c_{ij}^{k} e_{k}.
\end{align*}
\]

(49)

When \( \varphi^{ijk} \) vanishes, we see that both \( g \) and \( g^{*} \) have the structure of a Lie algebra, with some extra compatibility condition between the two structures. One also calls the triple \( (g \oplus g^{*}, g, g^{*}) \) a Manin triple in this case. It consists of a Lie algebra \( g \oplus g^{*} \) with an invariant nondegenerate inner product and two isotropic Lie subalgebras. The structure constants \( f_{i}^{jk} \) can also be interpreted as a so-called cocommutator, a map \( \delta : g \rightarrow \Lambda^{2} g \), given by

\[
\delta(e_{i}) = f_{i}^{jk} e_{j} \wedge e_{k}.
\]

(50)

The above conditions say that \( \delta \) squares to zero and is a cocycle. The Lie algebra \( g \) with the cocommutator \( \delta \) is called a Lie bialgebra. Note that this notion is dual, as also \( g^{*} \) is a Lie bialgebra. The commutator of \( g^{*} \) is dual to the cocommutator of \( g \), and vice versa. When only \( c_{ij}^{k} \) is nonzero, we find the canonical Lie bialgebra structure, consisting of the Lie bracket \( \Lambda^{2} g \rightarrow g \) and the adjoint action of \( g \) on its dual, \( g \otimes g^{*} \rightarrow g^{*} \).

More generally, consider the case that the \( \varphi^{ijk} \) do not vanish. The above equations show that \( g^{*} \) no longer is a Lie algebra. Hence we just have a Lie algebra \( g \oplus g^{*} \) with an isotropic Lie subalgebra \( g \). In this situation, the pair \( (g \oplus g^{*}, g) \) is known as a Manin pair.\[5\]

---

\[5\]We denote here the total space by \( g \oplus g^{*} \), which is only true as a vector space. The reader should be aware that as a Lie-algebra it is not simply a direct sum.
Equivalently, the Lie algebra $\mathfrak{g}$, supplied with the additional cocommutator $\delta$ and $\varphi \in \Lambda^3 \mathfrak{g}$, is said to be a quasi-Lie bialgebra. Hence, we find that the solutions of the BV master equation are given by quasi-Lie bialgebras. Reversely, any quasi-Lie bialgebra (Manin pair) gives rise to a topological open membrane model of the above form. The total Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$ is called the Drinfeld double.

Using this basis we can combine the superfields into a single $\Pi(\mathfrak{g} \oplus \mathfrak{g}^*)$-valued superfield $\Psi = \chi^i e_i + \psi_i e^i$. Using the canonical inner product $\langle e_i, e^j \rangle = \delta^j_i$, we can write the action in the form

$$S = \int_V \left( \frac{1}{2} \langle \Psi, d\Psi \rangle + \frac{1}{3} \langle \Psi, [\Psi, \Psi] \rangle \right).$$

(51)

Noting that the physical fields of ghost number 0 are the vector components, this can be identified with the Chern-Simons theory for the total Lie-algebra $\mathfrak{g} \oplus \mathfrak{g}^*$. This total Lie-algebra is also known as the Drinfeld double. Hence, our membrane theory based on the quasi-Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ reduces to Chern-Simons for the Drinfeld double.

5.2. Scrooching as a Canonical Transformation

We now consider the canonical transformation

$$\chi^i \rightarrow \chi^i + \frac{\partial \alpha}{\partial \psi_i},$$

(52)

where the generating function is the degree 2 function $\alpha = \frac{1}{2} a^{ij} \psi_i \psi_j$. If we started with a quasi-Lie bialgebra, that is $\gamma$ satisfies the master equation, we still have a solution to the master equation and hence a quasi-Lie bialgebra after this transformation. Note that also the boundary term is not affected as $P_L \beta = 0$. So in fact the solution is equivalent. The effect on the structure constants $c^i_{jk}, f_i^{jk}$ and $\varphi^{ijk}$, are precisely the transformations that Drinfeld originally dubbed as twisting of the quasi-Lie bialgebra, and is also known as scrooching.

Similarly, we can consider the canonical transformation

$$\psi_i \rightarrow \psi_i + \frac{\partial \beta}{\partial \chi^i},$$

(53)

with $\beta = \frac{1}{2} b_{ij} \chi^i \chi^j$. Now we have to be careful that the boundary term vanishes. If we consider the case where $f = h = 0$, this implies the condition

$$c^i_{jk} \chi^j \frac{\partial \beta}{\partial \chi^2} \frac{\partial \beta}{\partial \chi^k} = 0,$$

(54)

or in components

$$f_i^{jm} b_{ij} b_{mk} + f_k^{jm} b_{ki} b_{mj} + f_j^{im} b_{lk} b_{mi} = 0.$$  

(55)
This is easily identified with the classical Yang-Baxter equation for $b_{ij}$. More generally, we find that $P_L(e^{-\beta \gamma}) = 0$ can be written
\[
\frac{1}{2} c_{ij}^k \chi^i \chi^j \frac{\partial \beta}{\partial \chi^k} + \frac{1}{2} f_{ij}^k \chi^i \frac{\partial \beta}{\partial \chi^j} + \frac{1}{3!} \varphi^{ijk} \frac{\partial \beta}{\partial \chi^i} \frac{\partial \beta}{\partial \chi^j} \frac{\partial \beta}{\partial \chi^k} = 0.
\] (56)

The boundary observables are functions on the base space $\mathbb{A} = \Pi g$. The space of these functions can be identified with the exterior algebra $\mathcal{B} = \Lambda g^*$. The induced $L_\infty$ structure on these boundary observables is given by the following differential, bracket, and 3-bracket.

\[
d^* = \frac{1}{2} c_{ij}^k \chi^i \chi^j \frac{\partial}{\partial \chi^k},
\]
\[
\{\cdot, \cdot\}^* = f_{ij}^k \chi^i \frac{\partial}{\partial \chi^j} \wedge \frac{\partial}{\partial \chi^k},
\]
\[
\{\cdot, \cdot, \cdot\}^* = \varphi^{ijk} \frac{\partial}{\partial \chi^i} \wedge \frac{\partial}{\partial \chi^j} \wedge \frac{\partial}{\partial \chi^k}.
\]

Identifying the generators $\beta$ of the above canonical transformations with boundary observables, we can write the condition $P_L \beta = 0$ as
\[
d^* \beta + \frac{1}{2} \{\beta, \beta\}^* + \frac{1}{3!} \{\beta, \beta, \beta\}^* = 0.
\] (57)

This is actually a generalization of the quantum Yang-Baxter equation.

Notice that indeed the structure constants $f_{ij}^k$ determine the (Lie) bracket on $g^*$, and more generally the corresponding Schouten-Nijenhuis bracket on the exterior algebra $\Lambda g^*$. Also, the structure constants $c_{ij}^k$ of the Lie algebra $g$ induce a cocommutator $d^* : g^* \to \Lambda^2 g^*$ on the dual space $g^*$, generalizing to a differential on $\Lambda g^*$. This canonical relation between the Lie-bracket on $g$ and a cocommutator on $g^*$ is well known, and plays an important role in the theory of (quasi-)Hopf algebras.

5.3. Relation to CFT

It is well known for a long time that Chern-Simons is related to the closed WZW model for the same group $[2]$. More recently, it has been shown that also the $G/H$ quotient WZW models can be related to Chern-Simons theories. The gauge group of the CS in this case $G \times H$. The two gauge fields $A_\pm$ satisfy some nontrivial boundary condition, relating the $H$ part of the two gauge fields. For the $G/G$ model, this becomes the double CS theory $[25]$

\[
CS(A_+) - CS(A_-) = \int_V (A_+ dA_+ - A_- dA_- + \frac{2}{3} A_+ A_+ A_+ - \frac{2}{3} A_- A_- A_-),
\] (58)
with the boundary condition \( A_+ = A_- \).

We can relate this to our theory. Let \( \mathfrak{g} \) be a Lie algebra with structure constants \( c^i_{jk} \), and invariant inner product \( \eta_{ij} \). The two sets of gauge fields \( A^i \) and \( B_i \) can then be considered as taking values in the same Lie algebra. We identify these with the fields in the double CS above by take the diagonal and anti-diagonal gauge fields,

\[
A^i_\pm = \frac{1}{2} (A^i \pm \eta^{ij} B_j). \tag{59}
\]

Indeed, the above double CS action is then equivalent to

\[
\int_V \left( B_i dA^i + \frac{1}{2} c^i_{jk} B_i A^j A^k + \frac{1}{6} c^{ijk} B_i B_j B_k \right), \tag{60}
\]

and the boundary condition reduces to our boundary conditions \( B_i = 0 \). We see that \((\mathfrak{g}, [\cdot, \cdot], \delta = 0, \varphi)\), where the coassociator is related to the structure constants as \( \varphi^{ijk} = c^{ijk} \). It is well known that the \( G/G \) model indeed is related to the quasi-Hopf algebra \( U_h(\mathfrak{g}) \) with nontrivial coassociator related to the structure constants.

More generally, there are CFT’s canonically related to any Manin pair or Manin triples [45, 46]. The above \( G/G \) model is particular example of these models. It seems suggestive that these CFT’s could be dual to our open membranes relate to Manin pairs. The relation would be similar to the one [25]. Half of the currents on the boundary would be the 1-forms \( A^i \). The other currents are of the form \( g^{-1} dg \), where \( g \) is defined as a Wilson line for \( B \) ending on the boundary of the membrane. Indeed these would give rise to a current algebra reflecting the Lie algebra structure of the double \( \mathfrak{g} \oplus \mathfrak{g}^* \), as in [45, 46].

6. Open 2-Branes and Quasi-Lie Bialgebroids

The original open 2-brane model of [9] was that of a pure 3-form WZ term. This model and some generalizations are discussed in this section. They can be related to mathematical objects called Courant algebroids, which are studied by mathematicians in the context of generalized Dirac quantization. They are also known to be related as infinitesimal objects to gerbes.

6.1. The Canonical Topological Open Membrane

For our next model, we take for the target superspace \( \mathcal{M} = T^*[2](\Pi T^*M) \) for any manifold \( M \). Notice that this falls in the special class of twisted cotangent bundles we have singled
out. So we can take for the Lagrangian subspace a section \( \mathcal{L} \cong \Lambda = \Pi T^* M \) of this fiber bundle.

The open membrane theory is defined by four sets of superfields \( X^i, \chi^i, \psi^i, \) and \( F_i \) of ghost degree 0, 1, 1, and 2 respectively. \( M^i \) and \( \chi_i \) are coordinates on the base \( \Lambda = \Pi T^* M \) and \( \psi^i \) and \( F_i \) are coordinates on the fiber. The BV structure is determined by the BV bracket

\[
\int_V \left( \frac{\partial}{\partial X^i} \wedge \frac{\partial}{\partial F_i} + \frac{\partial}{\partial \chi^i} \wedge \frac{\partial}{\partial \psi^i} \right).
\]

(61)

The BV action functional will be given by

\[
S = \int_V \left( F_i dX^i + \psi^i d\chi^i + c \right).
\]

(62)

the interaction term \( c \) satisfies the master equation \([c, c] = 0 \) and \( P_L c = 0 \). The boundary term will be Dirichlet for \( \psi^i \) and \( F_i \). To write down this interaction, we introduce two separate gradings, one for \( \chi^i \) and one for \( \psi^i \) and \( F_i \) (the latter will have degree 1). Note that these will not be preserved separately. According to these gradings, we split the interaction term, \( c = c^{3,0} + c^{2,1} + c^{1,2} + c^{0,3} \). The most general expressions are

\[
c^{3,0} = \frac{1}{3!} h^{ijk}(X) \chi_i \chi_j \chi_k,
\]

\[
c^{2,1} = -b^{ij}(X) \chi_i F_j + \frac{1}{2} f^{jk}(X) \psi^i \chi_j \chi_k,
\]

\[
c^{1,2} = a_{ij}^k(X) \psi^i F_j + \frac{1}{2} g_{ij}^k(X) \psi^i \psi^j \chi_k,
\]

\[
c^{0,3} = \frac{1}{3!} c^{ijk}(X) \psi^i \psi^j \psi^k.
\]

First when \( a \) is an invertible matrix, we can always use a canonical transformation to make it equal to \( a^i_j = \delta^i_j \). In the following we will assume this is the case. In general, we can transform \( a \) to \( aU \) for any invertible matrix \( U \). Therefore, the only relevant information is the rank of \( a \). The bulk master equation \([c, c] = 0 \) then implies the following constraints

\[
f^{jk}_i = -\partial_i b^{jk} + c_{ilm} b^{lj} b^{mk}, \quad g^{k}_ij = -c_{ijkl} b^{jk}, \quad h^{ijk} = -b^{[i} \partial_{b^{jk]} - c_{lmi} b^{li} b^{mj} b^{nk}},
\]

(63)

and furthermore \( \partial_i c_{jkl} = 0 \). The boundary master equation \( P_L c = 0 \) constrains \( h^{ijk} = 0 \). In other words, the data is given by a closed 3-form \( c \) and a bivector \( b \), satisfying the above constraint. We note that for \( c = 0 \) the constraint \( h = 0 \) says that \( b^{ij} \) is a Poisson bivector. Hence we have a deformed version of the a Poisson bivector, also called a quasi-Poisson structure. We will later see that there is a gauge transformation which changes \( c \) by an
exact form, so that actually the data is a 3-form class and a quasi-Poisson structure for a representative of this class.

Combining the above, the total action can be written

\[
S = \int_V \left( F_i dX^i + \psi^i d\chi_i + F_i \psi^i - b^{ij} F_i \chi_j - \frac{1}{2} \partial_k b^{ij} \psi^k \chi_i \chi_j + \frac{1}{2} b^{ij} \partial_l b^{jk} \chi_i \chi_j \chi_k 
+ \frac{1}{6} c_{ijk} (\psi^i - b^{il} \chi_l) (\psi^j - b^{jm} \chi_m) (\psi^k - b^{kn} \chi_n) \right).
\] (64)

The above action can be derived from the deformation by \( e^{-\delta \beta \gamma} \), where

\[
\gamma = \psi^i F_i + \frac{1}{6} c_{ijk} \psi^i \psi^j \psi^k, \quad \beta = \frac{1}{2} b^{ij} \chi_i \chi_j.
\] (65)

It is therefore equivalent to the bulk/boundary action

\[
S = \int_V \left( F_i dX^i + \psi^i d\chi_i + F_i \psi^i + \frac{1}{6} c_{ijk} \psi^i \psi^j \psi^k \right) + \int_{\partial V} \frac{1}{2} b^{ij} \chi_i \chi_j.
\] (66)

We have to be careful however that in the latter case the Lagrangian embedding \( L \subset \mathcal{M} \) is not the zero section, but rather is determined by the equations

\[
\psi^i = [\beta, \psi^i] = b^{ij} \chi_j, \quad F_i = [\beta, F_i] = \frac{1}{2} \partial_k b^{ij} \psi^k \chi_i \chi_j.
\] (67)

This affects the projector \( P^\beta_L \), and therefore the boundary master equation \( P^\beta_L \gamma = 0 \).

This membrane action is classically equivalent to the membrane coupling to the \( c \)-field through the WZ term \( f, c \), and the (closed) Cataneo-Felder on the boundary. Therefore, it can be interpreted as a deformation of the Cataneo-Felder model by the 3-form.

6.2. Deformations of the 2-Algebras of Polyvector Fields

The boundary algebra \( B \) of this model has the form \( C^\infty(\Pi T^* M) = \Gamma(\wedge TM) \). In other words, it is the exterior algebra of polyvector fields. These are written as functions of \( X^i \) and \( \chi_i \). The \( \chi_i \) can indeed be seen as a basis of vector fields. The product in this algebra is the wedge product in this exterior algebra. The bracket on this algebra for \( c = 0 \) is given by

\[
\{ \cdot, \cdot \} = \frac{\partial}{\partial X^i} \wedge \frac{\partial}{\partial \chi_i}.
\] (68)

This bracket on the algebra of polyvector fields is well known, and is the Schouten-Nijenhuis bracket and is the extension to polyvector fields of the Lie bracket on vector fields. With this structure, the algebra of polyvector fields is well known to be a Gerstenhaber or 2-algebra.
When we turn on $c$ and $b$, the deformed $L_\infty$ algebra structure is given by
\[ Q = b^{ij} \chi_j \frac{\partial}{\partial X_i} + \frac{1}{2}(\partial_k b^{ij} + c_{klm} b^{li} b^{mj}) \chi_i \chi_j \frac{\partial}{\partial X_k} + O(c^2), \]
\[ \{\cdot, \cdot\} = \frac{\partial}{\partial X_i} \wedge \frac{\partial}{\partial X_i} + \frac{1}{2} c_{ijk} b^{kl} \chi_i \frac{\partial}{\partial X_i} \wedge \frac{\partial}{\partial X_j} + O(c^2), \]
\[ \{\cdot, \cdot, \cdot\} = \frac{1}{6} c_{ijk} \frac{\partial}{\partial X_i} \wedge \frac{\partial}{\partial X_j} \wedge \frac{\partial}{\partial X_k} + O(c^2). \] (69)

This forms a $G_\infty$ algebra when we take into account the (canonical) product.

6.3. Canonical Transformations and Gauge Transformations

In this subsection, we look at space-time gauge transformations in the topological open membrane. In the worldvolume, they correspond to adding BRST exact terms to the action. Actually, in the BV formalism there are corrections to this statement, as the BRST exact terms only represent the infinitesimal gauge transformations.

We are interested in the generalization of a gauge symmetry of the form $c \rightarrow c + da$, where $a$ is a 2-form. Let us start with the situation $b = 0$. Then we can add an exact term of the form $Q\alpha$, where
\[ \alpha = \frac{1}{2} a_{ij} \psi^i \psi^j. \] (70)
This term can be written $\frac{1}{6}(da)_{ijk} \psi^i \psi^j \psi^k + d\alpha$. The last term vanishes, due to the boundary condition of $\psi$. It can easily be confirmed that there are no higher order corrections. Therefore this term exactly generates the space-time gauge transformation $c \rightarrow c + da$.

When $b \neq 0$, there are corrections involving $b$. With the description of canonical transformations above they are not too difficult to write down. We noticed above that when $P_L \beta \neq 0$, the canonical transformation generated by $\beta$ is not a symmetry due to the boundary term. Such transformations generate a series of theories, which, as we saw above, are equivalent to adding a boundary term. These can be understood in terms of the Goldstone modes of the broken symmetry. The remaining (space-time) symmetries are generated by $\alpha \in A_0 = \ker P_L$.

In order to find the corrections for nonzero $b$, we have to be a bit more careful in the analysis of the canonical transformation generated by $\alpha$ above. The first step is to find two functions $\beta' \in B$ and $\alpha' \in A_0$ such that
\[ e^{\delta \alpha} e^{\delta \beta} = e^{\delta \beta'} e^{\delta \alpha'} \] (71)
The solution has the form

\[ \alpha' = \frac{1}{2} a_{ij}^\prime \psi^i \psi^j - a_{ij} \psi^i \chi^j, \quad \beta' = \frac{1}{2} b_{ij} \psi^i \psi^j, \] (72)

with \( b' = b(1 + ab)^{-1} \). This can be shown as follows. First, replace \( \alpha \) by \( t \alpha \). Note that \( \beta' \) and \( \alpha' \) depend on \( t \), therefore we denote them \( \beta'_t \) and \( \alpha'_t \) respectively. For \( t = 0 \) we clearly have \( \beta'_0 = \beta \). We still take \( \beta'_t \) in the above form, with \( b' \) now depending on \( t \). We then find

\[ \frac{d}{dt} \left( e^{-\delta \beta'_t} e^{\delta \alpha} \right) = e^{-\delta \beta'_t} \delta \alpha e^{\delta \alpha} - \delta \beta'_t e^{-\delta \beta'_t} e^{\delta \alpha} = \delta \gamma_t e^{-\delta \beta'_t} e^{\delta \alpha}, \] (73)

where

\[ \gamma_t = e^{-\delta \beta'_t} (\alpha) - \beta'_t = \frac{1}{2} a(\psi - b'_t \chi)^2 - \frac{1}{2} b'_t \chi^2. \] (74)

We need \( \gamma_t \) to be in \( \mathcal{A}_0 \), which means that the \( \chi^2 \) term vanishes. This gives a differential equation for \( b'_t \), which is solved by \( b'_t = b(1 + tab)^{-1} \). Setting \( t = 1 \) gives back our solution above. To solve for \( \alpha' \), we have to solve

\[ \frac{d}{dt} e^{\delta \alpha} = \delta \gamma_t e^{\delta \alpha}. \] (75)

As \( \alpha'_t \) now depends on \( t \), this will be a rather complicated differential equation. Luckily we will not need the explicit solution; the only relevant fact is that the solution for \( \alpha' \) is in \( \mathcal{A}_0 \), which therefore has the form given above. For this it was necessary that \( \gamma_t \) vanishes on the boundary.

Using this relation, we have the following relations between pure bulk actions

\[ \int (\tau + e^{-\delta \beta} \gamma) \sim e^{\delta \alpha'} \int (\tau + e^{-\delta \beta} \gamma) = \int (\tau + e^{-\delta \beta'} e^{\delta \alpha} \gamma). \] (76)

Note that the generator \( \alpha' \) of the canonical transformation that is used vanishes on the boundary, and therefore does not change the boundary conditions: it is a true canonical transformation on the bulk. The left hand side is equivalent to \( S_0 + \int \gamma \oplus P_L \beta \), while the last expression is equivalent to \( S_0 + \int e^{\delta \alpha} \gamma \oplus P_L \beta' \), so that we have established the equivalence of the two.

The conclusion of this is that the symmetry generated by \( \alpha \) on the total algebra \( \tilde{\mathcal{A}} \) is given by

\[ c \rightarrow c + da, \quad b \rightarrow b(1 + ab)^{-1}. \] (77)

When both \( b \) and \( 1 + ab \) are invertible, we can write the latter as \( b^{-1} \rightarrow b^{-1} + a \). As a consistency check, it can be shown that the combined transformation is a symmetry of the master equation, both \( dc = 0 \) and \( b \partial b + b^3 c = 0 \). When \( b \) is invertible, the invariance of the latter can be seen by writing it as \( d(b^{-1}) = c \). We will denote the transformation of \( b \) by \([a]b \equiv b(1 + ab)^{-1}\).
6.4. Boundary Conditions, Duality, and Large $c$

In the supergravity of decoupled open membranes ending on $M5$-branes an important role is played by a 3-vector rather than the 3-form $[16, 17, 18, 19]$. This relation between a 3-form and 3-vector is similar to the relation between the bivector and 2-form in the open string case $[3, 4, 7]$. We have build our model on the base space $A = \Pi T^* M$, parametrized by the coordinates $(X^i, \chi_i)$. For $b = 0$, the Lagrangian $\mathcal{L} \subset \mathcal{M}$ is precisely the zero section. However by turning on a boundary term $\beta$, we can change it to any section of the twisted cotangent bundle $\mathcal{M} = T^* [2] A$. It was sometimes useful to identify the section $\mathcal{L}$ with the base space $A$. Indeed, when $\mathcal{L}$ is always transverse to the fiber, the projection to the base gives a canonical identification. One can however easily convince oneself that there is however no need for the section to be transverse. In fact, it does not even have to be a section. Our description of the model then is not really appropriate and could better be arranged differently. As we will see this gives rise to some interesting dualities. As an extreme case, by changing the section $\mathcal{L}$ we can smoothly go from the undeformed situation $\psi^i = 0$ to an $\mathcal{L}$ determined by $\chi_i = 0$ (and $F_i = 0$ in both situations). This can be seen as a change of constant $b$ from 0 to $\infty$. The latter situation is an example where $\mathcal{L}$ is indeed not given as a section of the cotangent bundle. As it turns out, we can however still describe this situation as a section of some twisted cotangent bundle. To see this, we note the equivalence $T^* [2] (\Pi T^* M) = T^* [2] (\Pi TM)$. The Lagrangian determined by $\chi_i = F_i = 0$ is now given as the zero section of the latter way of writing. Geometrically, this has however a completely different interpretation. Note that the base space $A = \Pi T^* M$ is replaced by its dual $A^* = \Pi TM$. The boundary algebra $B$, which at first was the algebra of polyvector fields, now has changed into the algebra of differential forms. In terms of the fields, we roughly have interchanged $\chi_i$ and $\psi^i$. That this goes further even than the identification of the observables can be seen by looking at the algebra. In the new situation the undeformed algebra has zero bracket and 3-bracket, but has a differential which is precisely the De Rham differential.

We can spell out the duality in some more detail when $M$ is an even dimensional manifold. The bivector $b$ is not everywhere invertible, but we can always write it as the difference of two invertible bivectors, $b = \epsilon + (b - \epsilon)$. This allows us to write the action in terms of bulk and a boundary term given by

$$\gamma = \psi F + \epsilon \chi F + \frac{1}{2} \epsilon \partial \epsilon \chi^3 + \frac{1}{3!} \epsilon (\psi + \epsilon \chi)^3, \quad \beta = \frac{1}{2} (b - \epsilon) \chi^2. \quad (78)$$

The boundary condition for $\psi$ is $\psi^i = (b - \epsilon)^i j \chi_j$. As $b - \epsilon$ is invertible we can actually write
this as a dual boundary condition for $\chi$ rather than $\psi$, $\chi_i = ((b - \epsilon)^{-1})_{ij} \psi^j$, and write the boundary term as

$$\beta = \frac{1}{2} ((b - \epsilon)^{-1})_{ij} \psi^i \psi^j.$$  \hspace{1cm} (79)

Using a canonical transformation generated by this $\beta$, we can write the action in terms of a pure bulk term. The boundary conditions have now however changed to $\chi_i = 0$. This pure bulk term has the general form above, with the matrix $a$ not necessarily equal to $\delta$ anymore. In fact we will now argue that in general it is not invertible. Straightforwardly working out the canonical transformation shows that the matrix $a$ is given by $a = 1 + \epsilon^{-1} (b - \epsilon) = \epsilon^{-1} b$. The assumption that $b$ is not invertible, now is seen to be equivalent to the new $a$ being not invertible. However, as we also assumed $\epsilon$ to be invertible, there is a term $\epsilon \chi F$. Using this we can always by a canonical transformation get rid of the $\psi F$ term. This same canonical transformation will also simplify the rest of the bulk terms involving $c$. After the canonical transformation the bulk and the boundary term will have the following form

$$\gamma = \epsilon \chi F + \frac{1}{3!} \epsilon' \chi'^3, \quad \beta = \frac{1}{2} b' \psi^2,$$  \hspace{1cm} (80)

where $c'^{ijk} = \epsilon^{il} \epsilon^{jm} \epsilon^{kn} c_{lmn} + \epsilon^{[i} \partial_l \epsilon^{jk]}$, and $b' = (b - \epsilon)^{-1} + \epsilon^{-1}$. Note that using a canonical transformation we can change $\epsilon$ to almost any fixed — but invertible — form we want.

This duality could be useful for studying the large $c$ limit of the theory. We can take $\epsilon$ very small, such that $\epsilon'$ is small in all directions. Then the bulk interactions are all small, and we can apply perturbation theory. Note that the boundary term has to be treated exactly, as it will always be large. in this situation.

More generally, we can try to deform the algebra of differential forms. It turns out however that there are no nontrivial global deformations. The only nontrivial bulk deformation is still the 3-form deformation by $\gamma = \frac{1}{6} c_{ijk} \psi^i \psi^j \psi^k$. However, this does not satisfy the boundary condition $\mathcal{P} \gamma = 0$. This could be remedied by adding a boundary term, but only if $c = db$ for some 2-form $b$. The boundary term then is simply given by $\beta = \frac{1}{2} b_{ij} \psi^i \psi^j$. However, now the total deformation is BRST exact (or more precisely, it is generated by a canonical transformation). We might only get something nontrivial if $b$ is not defined globally in the target space, but only on patches.

The duality between $\mathbb{A}$ and $\mathbb{A}^*$ show that the deformed 2-algebras are equivalent. In a sense, the algebra of forms deformed by a 3-vector $c_{ijk}$ can be seen as a $c_{ijk} \to \infty$ limit of the deformed algebra of polyvector fields. As it is believed that a stack of M5-branes in the presence of a large $c$-field reduces to exactly the TOM model we studied for $c_{ijk} \to \infty$, this would lead us to study precisely this deformation of the algebra of differential forms.
Notice that the physical boundary observables are precisely given by 2-forms \( f(X, \psi) = \frac{1}{2} B_{ij}(X) \psi^i \psi^j \).

In \([9]\) it was shown how also the topological A- and B-model could be found by taking particular boundary conditions for the open membrane model when \( M \) is a Kähler manifold. In these models, one can smoothly interpolate between the boundary conditions for the A- and the B-Model. This is suggestive of mirror symmetry.

### 6.5. Generalized Topological Open Membrane

We now discuss a further generalization of the above, based on the general form of the target superspace. It combines the Lie bialgebra case and the structure of the canonical open membrane.

To construct the target space of the generalized model we start from the total space of a Grassmann bundle \( A = \Pi A \), the twist of a vector bundle \( A \to M \) over a manifold \( M \). Following the construction discussed before, we take for the target superspace the twisted cotangent space \( M = T^*[2]A \). The above model is a special case, with \( A = TM \). For the Lagrangian subspace we can again take a section \( L \simeq A \) of this fiber bundle.

The open membrane theory is defined by four sets of superfields \( X^i, \chi^a, \psi^a \), and \( F^i \) of ghost degree 0, 1, 1, and 2 respectively. \( X^i \) and \( \chi^a \) are coordinates on the base \( A = \Pi A \) and \( \psi^a \) and \( F^i \) are coordinates on the fiber. The BV structure is determined by the BV bracket

\[
\int_V \left( \frac{\partial}{\partial X^i} \wedge \frac{\partial}{\partial F_i} + \frac{\partial}{\partial \chi^a} \wedge \frac{\partial}{\partial \psi^a} \right).
\]

The BV action functional will be given by

\[
S = \int_V \left( F_i dX^i + \psi^a d\chi^a + \gamma \right).
\]

the interaction term \( \gamma \) satisfies the master equations \([\gamma, \gamma] = 0 \) and \( P_L \gamma = 0 \). The boundary term will be Dirichlet for \( \psi^a \) and \( F_i \). To write down this interaction, we split the ghost number into two separate gradings, such that \( \chi^a, \psi^a \) and \( F_i \) have degrees \((1, 0), (0, 1)\) and \((1, 1)\). Note that these will not be preserved separately. According to these gradings, we split the interaction term, \( \gamma = \gamma^{3,0} + \gamma^{2,1} + \gamma^{1,2} + \gamma^{0,3} \). The most general expressions are

\[
\gamma^{3,0} = \frac{1}{3!} h^{abc}(X) \chi_a \chi_b \chi_c,
\]

\[
\gamma^{2,1} = b^{ai}(X) \chi_a F_i + \frac{1}{2} f^{abc}(X) \psi^a \chi_b \chi_c.
\]
\[ \gamma^{1,2} = a^i_a(X)\psi^a F_i + \frac{1}{2} g_{ab}(X)\psi^a \psi^b \chi_c, \]
\[ \gamma^{0,3} = \frac{1}{3!} c_{abc}(X)\psi^a\psi^b \psi^c. \]

Without a boundary term, the boundary master equation will set \( \gamma^{3,0} = 0 \). The bulk master equation will be a combination of the two cases we studied before. In components, the equations are
\[ b^{[a} \partial_i f^{bc]} + f^{[ab} f^{c]}_d = 0, \]
\[ a_{[a} \partial_i g_{bc]} + b^{[ab} \partial_i c_{abc} + \frac{1}{2} g^{[ab} g_{cd]} e + f^{de}_{[a c} e = 0, \]
\[ a_{[a} \partial_i f^{bc]} + b^{[a} \partial_i f_{[bc]} + f^{e[a b]} g_{d]e} = 0, \]
\[ a_{[a} \partial_i c_{bcd]} + f^{e[b c]} g_{d]e} = 0, \]
\[ b^{[a} \partial_i b^{[b]} + b^{bc} f^{ab} = 0, \]
\[ b^{[a} \partial_j b_{[b]} + c_{[a} f^{bc]} + b^{bc} g_{ab} = 0, \]
\[ a_{[a} \partial_j a_{[b]} + c_{[a} g_{bc]} + b^{bc} c_{cab} = 0. \]

It can be interpreted as a local version of a quasi-Lie bialgebra. This can be called a quasi-Lie bialgebroid. The structure it gives is also known in the mathematical literature as a Courant algebroid, reviewed in the next section.

To see this more precisely, denote by \([\cdot, \cdot]_0\) the BV bracket in the fiber direction only. Note that this is exactly the same bracket as for the quasi Lie-algebra \( g \oplus g^* \) case. Correspondingly we write the function \( \gamma \) as \( \gamma_0 + \gamma_1 \), where \( \gamma_0 \) does not involve \( F \) and \( \gamma_1 \) is linear in \( F \). We write
\[ \delta = [\gamma_1, \cdot] - [\gamma_1, \cdot]_0 = (a^i_a \psi^a + b^{ia} \chi_a) \frac{\partial}{\partial X^i}. \]  
(83)
We can then write the full master equation in the form
\[ \delta \gamma_0 + [\gamma_0, \gamma_0]_0 = 0, \quad \delta^2 + [\gamma_0, \delta]_0 = 0. \]  
(84)
second equation says that $\delta$ takes it values in the center of this quasi-Lie bialgebra. When the structure constants are not constant, this will be modified as above.

7. **Courant Algebroids and Gerbes**

In this section we shortly discuss the mathematical structure of Courant algebroids and its relation to the topological open membrane.

7.1. **Courant Algebroids**

Algebroids are objects that interpolate between Lie algebras and tangent bundles. A *Lie algebroid* over a manifold $M$ is a bundle $A$ over $M$ provided with a Lie-bracket $\{\cdot,\cdot\}_A$ on the space of sections and a map $a : A \to TM$ to the tangent space called the anchor. This map should intertwine the Lie-bracket on $A$ and on vector fields, that is $a([X,Y]_A) = [a(X),a(Y)]_{TM}$. When $M$ is a point, a Lie algebroid is the same thing as a Lie algebra. Another special Lie algebroid is the tangent space itself, where we take for $a$ the identity map and for $\{\cdot,\cdot\}_A$ the canonical Lie bracket on vector fields.

Now let us turn to the case of our main interest. A *Courant algebroid* is a vector bundle $E \to M$ with a pseudo-Euclidean inner product $\langle \cdot, \cdot \rangle$ together with a bilinear operation $\circ$ on $\Gamma(E)$ and a bundle map $\rho : E \to TM$, called the anchor, satisfying the following properties

1. $e \circ (e_1 \circ e_2) = (e \circ e_1) \circ e_2 + e_1 \circ (e \circ e_1)$,
2. $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]_{TM}$,
3. $e_1 \circ (fe_2) = fe_1 \circ e_2 + (\rho(e_1) \cdot f)e_2$,
4. $\langle e, e_1 \circ e_2 + e_2 \circ e_1 \rangle = \rho(e)\langle e_1, e_2 \rangle$,
5. $\rho(e)\langle e_1, e_2 \rangle = \langle e \circ e_1, e_2 \rangle + \langle e_1, e \circ e_2 \rangle$.

Here $e, e_1, e_2 \in \Gamma(E)$ are sections of $E$ and $f$ is a function on $M$. Sections of $E$ naturally act on sections of $E$ through left multiplication $e \circ$, and on functions through $\rho$. Properties 1, 3 and 5 say that this action is a derivation of all the products and the inner product. Note that the first property becomes the Jacobi identity when the product $\circ$ is antisymmetric. This generalization of a Lie algebra is called a Loday algebra. The second and third property are similar to that for the anchor map of a Lie algebroid. Property 3 shows that the product
acts as a first order differential operator. The fourth property shows that the symmetric part of the product is in some sense “infinitesimal”. If we introduce the operator \( D : C^\infty(M) \to \Gamma(E) \) defined by \( \langle e, Df \rangle = \rho(e)f \), we can write the symmetric part as \( e_1 \circ e_2 + e_2 \circ e_1 = D(e_1, e_2) \). Equivalently, the properties above can be expressed in terms of the skew-symmetrization \( \{e_1, e_2\} = \frac{1}{2}(e_1 \circ e_2 - e_2 \circ e_1) \), as was done the original formulation of Courant algebroids \([50, 29]\). The Jacobi identity for this skew-symmetric bracket has an anomaly, as the first property becomes \( \{e_1, \{e_2, e_3\}\} + \text{cycl.} + D\{e_1, e_2, e_3\} = 0 \), where \( \{e_1, e_2, e_3\} = -\frac{1}{6}\{\{e_1, e_2\}, e_3\} + \text{cycl.} \). Together with the other identities, these structures give rise to a structure of homotopy Lie algebra on the total space of sections and functions \([29]\).

When \( M \) is a point, the definitions above reduce to that of (the double of) a quasi-Lie bialgebra. Indeed, as \( \rho = 0 \) the product is skew-symmetric, and reduces to a genuine Lie bracket on the double.

A special and canonical example is the so called exact Courant algebroid. This is an extension of the tangent bundle, which can be described locally as \( E \cong TM \oplus T^*M \). The product, or Courant bracket, is defined as an extension of the Lie-bracket of vector fields by the formula

\[
(v, \xi) \circ (w, \eta) = \left( [v, w], \mathcal{L}_v \eta - \iota_w d\xi + \iota_v \iota_w c \right),
\]

where \( c \) is a closed 3-form and \( \iota \) denotes contraction. With the canonical inner product \( \langle (v, \xi), (w, \eta) \rangle = \iota_v \eta + \iota_w \xi \) and anchor \( \rho(v, \xi) = v \) one can check that this is indeed a Courant algebroid. An exact Courant algebroid can be defined globally as an extension

\[
0 \to T^*M \to E \to TM \to 0.
\]

A particular choice of splitting \( TM \oplus T^*M \) is called a connection. The difference of two such connections can be identified locally with a 2-form. The corresponding curvature of the connection, which can be identified locally with the exterior derivative of the 2-form connection, is a globally defined closed 3-form. In fact this 3-form can be identified with the 3-form \( c \) appearing in the product above. In this way exact Courant algebroids can be classified by a 3-form class.

A more general class of Courant algebroids can be defined as extensions of Lie algebroids of the form

\[
0 \to A^* \to E \to A \to 0,
\]

where \( A \) and \( A^* \) are dual Lie algebroids, possibly with trivial anchor and bracket. We can locally split this bundle as \( E = A \oplus A^* \). The Courant algebroid structure on \( A \oplus A^* \) is a direct generalization of the exact Courant algebroid. Note that we can still define a contraction
\( \iota : A \times A^* \to \mathbb{R}_M \). Also, we can define a generalization \( d : C^\infty(M) \to \Gamma(A^*) \) of the De Rham differential, using the anchor map \( a : A \to TM \). It is given by the composition of the ordinary De Rham differential and the adjoint \( a^* : T^*M \to A^* \) of \( a \). The Lie derivative on sections of \( A^* \) can then be defined using the standard formula \( \mathcal{L}_v = \iota_v d + d \iota_v \) for \( v \in \Gamma(A) \). This allows us to write down a generalization of the Courant bracket (85) for \( E = A \oplus A^* \), with \( c \in \Gamma(\Lambda^3 A^*) \) satisfying \( dc = 0 \).

We can generally relate the closed topological membrane with target superspace \( \mathcal{M} = T^*[2](\Pi A) \) with the Courant algebroid by identifying \( E = A \oplus A^* \). Sections of \( E \) can then be identified as degree one elements of the closed membrane algebra \( \mathcal{A} \), \( \Gamma(\Pi E) = \mathcal{A}^1 \), and functions on \( M \) with degree zero elements. More explicitly, a section \( e = (v, \xi) \in \Gamma(E) \) is identified with the element \( v^\alpha(X) \chi_\alpha + \xi_\alpha(X) \psi^\alpha \in \mathcal{A} \). On this subset of \( \mathcal{A} \) the Courant algebroid structure is defined as

\[
\langle e_1, e_2 \rangle = [e_1, e_2], \quad e_1 \circ e_2 = [[\gamma, e_1], e_2], \quad \rho(e) f = [[\gamma, e], f],
\]

(88)

where \( \gamma \) is the bulk deformation of the BV action. Note that on degree one elements the bracket \( \langle \cdot, \cdot \rangle \) is symmetric. Also note that we can identify \( \mathcal{D} f = [\gamma, f] = Q f \). More generally, we can identify the \( L_\infty \) brackets in the skew-symmetric formulation with the derived higher brackets as in (12), but without the \( P_L \) and \( i_L \). One easily verifies that these satisfy the above conditions for a Courant algebroid, as a result of the master equation. We observe that the topological membranes based on \( \mathcal{M} = T^*[2](\Pi T^*M) \) give rise to an exact Courant algebroid, while the more general topological membranes with target superspace \( \mathcal{M} = T^*[2](\Pi A) \) correspond to the more general Courant algebroid \( E = A \oplus A^* \).

It was observed in [29] that the total space \( A \oplus A^* \) of the Courant algebroid is not a symplectic manifold. However it can be naturally embedded into the symplectic manifold \( T^*A \cong T^*A^* \cong A \oplus A^* \oplus T^*M \). By twisting, we find precisely the target supermanifold \( \mathcal{M} \) of the generalized topological open membrane. This symplectic supermanifold naturally appears in the mathematical construction [29]. In fact, Courant algebroids are in 1-to-1 correspondence to supermanifolds of this form supplied with the BV structure and the BRST operator [51].

### 7.2. Dirac Structures

To extent this to the open membrane, we need to discuss Dirac structures. A Dirac bundle is a maximally isotropic subbundle \( L \subset E \) (with respect to \( \langle \cdot, \cdot \rangle \)) that is closed under the action of the Courant bracket. We will concentrate on the exact case; the generalization
is straightforward. A canonical choice for $L$ is the subbundle $T^*M$. Deformations of this Dirac structures can be found in terms of a bivector $b \in \Gamma(\Lambda^2 T^*M)$. Such a bivector can be identified with a map $\tilde{b} : T^*M \to TM$. For $c = 0$, the graph of this map, spanned by elements $(\tilde{b}(\xi), \xi) \in TM \oplus T^*M$, defines a Dirac structure if $[b, b] = 0$, where the bracket is the Schouten-Nijenhuis bracket, i.e. $b$ is a Poisson structure. For nonzero $c$, the condition is replaced by $[b, b] = \tilde{b}^3 c$, where the right-hand side is defined as triple contraction. In local coordinates we have $3b^{l[i} \partial_l b^{jk]} = b^{il} b^{jm} b^{kn} \epsilon_{lmn}$. This is precisely the condition on $b$ we found for the master equation.

Another canonical Dirac bundle is given by the graph of a 2-form $b'$, spanned by $(v, \iota_v b') \in TM \oplus T^*M$. In the undeformed case, this defines a Dirac structure if $db' = 0$. This corresponds to the dual boundary conditions $\chi = 0$, but with the interaction $\gamma = \psi^i F_i$. When we deform by $\frac{1}{6} \epsilon_{ijk} \psi^i \psi^j \psi^k$, we find that the condition for a Dirac bundle is changed to $db' = c$. This is different from the above situation, for three reasons. The first one is that there is not always a global solution; only if $c$ is exact. Next the solution is fixed up to trivial terms by $c$, and $b' = 0$ is not a solution. Lastly, the situation is actually gauge equivalent to the trivial situation $b' = c = 0$. This was not appreciated in the mathematical context. But in our case the fact that this situation is more constraint and actually trivial follows from the master equation. The difference is the extra condition coming from the boundary term in the master equation. As we saw earlier, a more interesting case is to trade in the interaction $\gamma = \psi^i F_i$ by $\gamma = \epsilon^{ij} \chi_i F_j$. This does have a nontrivial deformation involving a 3-vector, namely by $\frac{1}{6} \epsilon^{ijk} \chi_i \chi_j \chi_k$.

It is clear that the choice of a Dirac bundle $L$ corresponds to the choice of a boundary condition for the open membrane. The precise relation is that for a Lagrangian $L$ describing the boundary condition the corresponding Dirac structure is perpendicular with respect to the inner product $\langle \cdot, \cdot \rangle$, that is $\Pi L = L^\perp$. Therefore the sections of $L$ are identified with the degree one elements in the kernel of $P_L$, i.e. $\Gamma(\Pi L) = A^1_0$. One can indeed see that the boundary master equation $P_L \gamma = 0$ is equivalent to the closure of the Courant bracket on $L$. We already saw how the boundary bracket, and more generally the $L_\infty$ algebra of the boundary, was related to the derived bracket. According to the identification above, the latter is precisely the Courant bracket. The boundary $L_\infty$ algebra is the projection of the $L_\infty$ algebra related to the skew-symmetrized Courant bracket $\{\cdot, \cdot\}$ mentioned above. Reversely, the Courant algebra structure may be seen as the general form of the boundary algebra irrespective of the boundary condition.

As a short aside, let us remark that originally Courant algebroids and Dirac structures were discovered in the context of general constraint quantization of the manifold $M$.\[36\]
Generically, the Dirac bundle $L$ can be split into three parts: $L \cap TM$, $L \cap T^*M$, and the rest. The first factor consists of pure vectors, and corresponds to gauge transformations. The second part corresponds to Casimirs generating constraints, while all the rest are ordinary dynamical degrees of freedom. For example, in the case of the graph of a Poisson bivector $b$, the Casimirs correspond to the kernel of the Poisson bivector. These are indeed the central elements for the Poisson bracket. Similarly, for the graph of a closed 2-form $b'$, the vectors in the kernel of $b'$ generate gauge symmetries for the system. $b'$ then becomes a symplectic structure on the quotient manifold of the corresponding foliation. More general Dirac bundles can combine both effects. This story applies to zero deformation. When we turn on $c$ we deform the quantization procedure. However, we see that we still have a well defined notion of (integrable) Dirac bundles. This will allow us to perform a quantization of $M$. What is the precise meaning of the twisting by $c$ is not completely clear yet, however. This involves the quantization of the Courant algebroid.

7.3. Gerbes and Local Star Products

Courant algebroids have are closely related to abelian gerbes. This is already suggested by the fact that both the exact Courant algebroid and the abelian gerbe is classified by a 3-form class. One can be more specific than this. An important role in the connection is played by the gauge transformations we found above. The data for the model is encoded by the closed 3-form $c$ and the boundary bivector $b$, modulo gauge transformations. A connection on abelian gerbes is a 2-form $a$. The idea is to identify the 3-form $c$ with the curvature of this connection. Thus locally we want to write $c = da$. If we could write $c = da$ globally on $M$, we can use the gauge transformations to gauge away $c$. The boundary data $B$ will be replaced by $[-a]b = b(1 - ab)^{-1}$. As $c = 0$ after the gauge transformation this is a genuine Poisson structure. In this case we actually know what quantization does: it gives a global star product through deformation quantization.

For the case that $c$ is globally not exact, let us choose a good covering $\{U_\alpha\}$ for $M$. As $c$ is closed, we can on each patch $U_\alpha$ choose a 2-form $a_\alpha$ satisfying $c = da_\alpha$. We can then locally gauge away $c$ by a gauge transformation generated by $a_\alpha$. This gives on each patch a Poisson bivector $b_\alpha = [-a_\alpha]b = b(1 - a_\alpha b)^{-1}$. On overlaps $U_{\alpha \beta} = U_\alpha \cap U_\beta$ the 2-forms $a_\alpha$ differ by exact 2-forms, $a_\alpha - a_\beta = d\lambda_{\alpha \beta}$ on $U_{\alpha \beta}$. Here we assumed that the patches are chosen such that intersections are always contractable. The transition 1-forms $\{\lambda_{\alpha \beta}\}$ automatically satisfy the cocycle condition $d\lambda_{\alpha \beta} + d\lambda_{\beta \gamma} + d\lambda_{\gamma \alpha} = 0$ on triple intersections. One easily sees that the local Poisson bivectors $\{b_\alpha\}$ are related on intersections by the gauge transformation
\[ b_{\beta} = [d\lambda_{\alpha\beta}]b_{\alpha} \text{ on } U_{\alpha\beta}. \]

We find that the original global data \((c, b)\) of a closed 3-form and a quasi-Poisson bivector can be translated into a set of local Poisson bivectors \(b_{\alpha}\) and transition 1-forms \(\lambda_{\alpha\beta}\) satisfying the cocycle condition on triple interactions. The 3-form class of \(c\) can be recovered from this local data. Given \(a_{\alpha 0}\) at one patch, the \(a_{\beta}\) in any other patch are determined by the transition 1-forms \(\lambda_{\alpha\beta}\). The uniqueness of these is guaranteed by the cocycle condition of the \(\lambda_{\alpha\beta}\). This determines a 3-form \(c = da_{\alpha}\), which is globally defined because the \(a_{\alpha}\) differ by exact forms on intersections. Note that this determines \(c\) only determined modulo an exact form, due to the fact that we had to make an initial choice \(a_{\alpha 0}\). But we indeed find that the local data is classified by the 3-form class of \(c\). In addition, from the local data we can recover the quasi-Poisson bivector \(b\), which also should be globally defined. Hence we are able to recover from the local data \(\{b_{\alpha}, \lambda_{\alpha\beta}\}\) the complete global data \(\{c, b\}\) up to global gauge transformation.

The undeformed Courant algebroid is related to (deformation) quantization. The 3-form deformation of the Courant algebroid is therefore expected to change the deformation quantization. The way in which this occurs can now be described as follows \[52\]. On each patch \(U_{\alpha}\) we have the local algebra of functions \(A_{\alpha} = C^\infty(U_{\alpha})\). Using the above construction we have also a bivector \(b_{\alpha}\). We can use \(b_{\alpha}\) and deformation quantization \[3\] to construct an associative star product \(*_{\alpha}\) on \(A_{\alpha}\) \[53\]. On each intersection \(U_{\alpha\beta}\) the Poisson bivectors \(b_{\alpha}\) and \(b_{\beta}\) are related by a gauge transformation generated by the exact 2-form \(d\lambda_{\alpha\beta}\). This implies that the two star products \(*_{\alpha}\) and \(*_{\beta}\) restricted to the subalgebras \(A_{\alpha\beta} = C^\infty(U_{\alpha} \cap U_{\beta})\) are equivalent. This equivalence is closely related to the Seiberg-Witten map \[7\]. So what we end up with is a set of of algebras \((A_{\alpha}, *_{\alpha})\), with for any pair \(\alpha, \beta\) two subalgebras \(A_{\alpha\beta} \subset A_{\alpha}\) and \(A_{\beta\alpha} \subset A_{\beta}\), which are equivalent deformation quantizations, \((A_{\alpha\beta}, *_{\alpha}) \simeq (A_{\beta\alpha}, *_{\beta})\). These “noncommutative gerbes” were recently discussed also in \[54\].

8. Discussion and Conclusion

In this paper we studied a large class of BV actions for topological open membranes. We gave a geometric construction of the algebra of bulk and boundary operators. The boundary theory has the structure of an homotopy Lie algebra, determined by the bulk deformations. This \(L_\infty\) algebra is the natural structure of a string field theory \[31\]. The main result is that the generic solutions to the master equations are given in terms of Courant algebroids, or “quasi-Lie bialgebroids”. We should stress the fact that the Courant algebroid gives the structure of the boundary \(L_\infty\) algebra irrespective of the choice of boundary conditions. This
means that the boundary algebra is given by a projection of the Courant algebroid structure.

Just as the path integral of topological open strings can give an expansion of the (deformation) quantization of function algebras as in [2], the path integral for the membrane will give a quantization of the corresponding Courant bialgebroids. These structures are much more complicated, mainly due to the fact that we are now deforming a string theory on the boundary. We only sketched the first order, semi-classical, approximation in this paper. Also, we mainly focused on the homotopy Lie algebra structure. While this is an important ingredient of the string theory [31, 39], the boundary string has a more intricate structure of homotopy Gerstenhaber algebra [34, 35, 15]. The quantization problem of Courant algebroids should be a generalization of that for the simpler subclass of quasi-Lie bialgebras. The quantization of the latter has been solved effectively by [28], and leads as expected to quasi-Hopf algebras [27]. It turns out that the full Hopf algebra structure arises naturally only after passing to the full homotopy Gerstenhaber structure [15]. Therefore we expect that we need to go beyond the $L_\infty$ structure to quantize this object.

A natural and important question that arises is whether our topological open membrane model arises as a decoupling limit in string theory. For the 3-form model this was argued in [16], where it appeared in the context of $M_2$-branes ending on $M_5$-branes. More generally, we expect non-abelian 2-form theories to arise in certain little string theories living on 5-branes. These may also be related to nonabelian generalizations of Dixmier-Douady gerbes. In that sense it is also encouraging that we found a natural extension involving non-abelian Lie algebra structures for our model, namely the general Courant algebroids. In the case of open strings, the structure of gauge theories based on nonabelian 1-form is very similar to the one deformed by the $B$-field in noncommutative geometry. One might similarly wonder if the structure of nonabelian 2-forms will be analogous to that of 2-form theories deformed by the $C$-field. In this way, the study of our model based on the exact Courant algebroid could learn us about nonabelian 2-forms.

The deformation of the open membrane by a 3-form also has its effect on the corresponding open string theory corresponding to the boundary string. We know that the bivector coupling of the string results in deformation quantization of the function algebra, as exemplified by the Cattaneo-Felder model. The 3-form will deform this quantization in a nontrivial way. This is already seen in the fact that the Poisson condition for the bivector changes into the quasi-Poisson condition. In the last subsection we used the local gauge symmetries to write the formal quantization due to such a quasi-Poisson bivector as a set of local star-products on patches. One problem with this approach is that this only makes sense for formal quantization, i.e., viewed as a formal power series in a quantization parameter $\bar{\hbar}$. For
finite $\hbar$ the star-product algebra can not be localized to a patch.

Another approach which could make more global sense uses the path integral quantization of our model. For this we have to model the open string, which is nontrivial as the boundary of a membrane itself has no boundary. This can be solved by including boundaries with corners, and allowing different boundary conditions on various regions of the boundary. Let us divide the boundary $\partial V$ into two regions. We take the usual boundary conditions corresponding to the Lagrangian submanifold $\mathcal{L} = \Pi T^*M$ for one region, but restrict the fields to only $M$ for the other region. The interface of the two regions can be viewed as the boundary for region one. On the interface live only operators corresponding to functions on $M$, as on the boundary of the Poisson-sigma model. Coupling the 3-form $c_{ijk}$ to the bulk and a quasi-Poisson bivector $b^{ij}$ to region one will induce a nontrivial quantum product on the function algebra living on the interface, defined through the path integral. For $c = 0$ we can completely forget about the bulk and we reproduce star-product of Kontsevich, as in [2].

Our models naturally have bulk and boundary deformations, which we saw can be non-trivially intertwined. We have seen that the bulk algebra $\mathcal{A}$ can be understood as the Hochschild cohomology of the boundary algebra $\mathcal{B}$. In fact, also the boundary deformations can naturally be understood in the context of the full deformation complex. In general the deformation complex of a $d$-algebra $\mathcal{B}$ fits in a short exact sequence of the form $\mathcal{B}[d - 1] \to \text{Def}^*(\mathcal{B}) \to C^*(\mathcal{B}, \mathcal{B})[d]$, c.f. [3]. This induces a long exact sequence in cohomology, $\cdots \to HH^{i+d-1}(\mathcal{B}) \to B^{i+d-1} \to H^i(\text{Def}(\mathcal{B})) \to HH^{i+d}(\mathcal{B}) \to \cdots$. Let’s take our canonical example $\mathcal{B} = C^\infty(\Pi T^*M) = \Gamma(M, \Lambda^\ast TM)$ as a 2-algebra, corresponding to the exact Courant algebroid. From the above exact sequence we learn that the space $H^1(\text{Def}(\mathcal{B}))$ encoding a deformation of this 2-algebra consists of the two pieces, $HH^3(\mathcal{B}) = H^3(M)$ and $H^2 = \Gamma(M, \Lambda^2 TM)$. Indeed we found that for the open membrane with this boundary algebra the deformation of the model was precisely encoded in a 3-form class and a bivector. We observe that the two components of the deformation complex explicitly reflect the bulk and boundary deformations $\gamma$ and $\beta$ of the open membrane.

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