Seifert surgery on knots via Reidemeister torsion and Casson-Walker-Lescop invariant II

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Abstract

For a knot $K$ with $\Delta_K(t) = t^2 - 3t + 1$ in a homology 3-sphere, let $M$ be the result of $2/q$-surgery on $K$. We show that an appropriate assumption on the Reidemeister torsion of the universal abelian covering of $M$ implies $q = \pm 1$, if $M$ is a Seifert fibered space.

1 Introduction

The first author [Kd1] studied the Reidemeister torsion of Seifert fibered homology lens spaces, and showed the following:

**Theorem 1.1** ([Kd1, Theorem 1.4]) Let $K$ be a knot in a homology 3-sphere $\Sigma$ such that the Alexander polynomial of $K$ is $t^2 - 3t + 1$. The only surgeries on $K$ that may produce a Seifert fibered space with base $S^2$ and with $H_1 \neq \{0\}, \mathbb{Z}$ have coefficients $2/q$ and $3/q$, and produce Seifert fibered space with three singular fibers. Moreover (1) if the coefficient is $2/q$, then the set of multiplicities is $\{2\alpha, 2\beta, 5\}$ where $\gcd(\alpha, \beta) = 1$; and (2) if the coefficient is $3/q$, then the set of multiplicities is $\{3\alpha, 3\beta, 4\}$ where $\gcd(\alpha, \beta) = 1$.

It is conjectured that Seifert surgeries on non-trivial knots are integral (except some cases). We [KMS] have studied the $2/q$-Seifert surgery, one of the remaining cases of the above theorem, by applying the Reidemeister torsion and the Casson-Walker-Lescop invariant, and have given sufficient conditions to determine the integrality of $2/q$ ([KMS Theorems 2.1, 2.3]).

In this paper, we give another condition for the integrality of $2/q$ (Theorem 2.1). Like as in [KMS], the condition is also suggested by computations for the figure eight knot ([KMS Example 2.2]).

We note two differences of this paper from [KMS]: one is that the surgery coefficient appears in the condition instead of the Casson-Walker-Lescop invariant, and another is that we need more delicate estimation for the Dedekind sum to prove the result.

(1) Let $\Sigma$ be a homology 3-sphere, and let $K$ be a knot in $\Sigma$. Then $\Delta_K(t)$ denotes the Alexander polynomial of $K$, and $\Sigma(K; p/r)$ denotes the result of $p/r$-surgery on $K$.

(2) The first author [Kd2] introduced the norm of polynomials and homology lens spaces: Let $\zeta_d$ be a primitive $d$-th root of unity. For an element $\alpha$ of $\mathbb{Q}(\zeta_d)$, $N_d(\alpha)$ denotes the norm of $\alpha$. 

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associated to the algebraic extension $\mathbb{Q}(\zeta_d)$ over $\mathbb{Q}$. Let $f(t)$ be a Laurent polynomial over $\mathbb{Z}$. We define $|f(t)|_d$ by

$$|f(t)|_d = |N_d(f(\zeta_d))| = \prod_{i \in (\mathbb{Z}/d\mathbb{Z})^\times} f(\zeta_i^d).$$

Let $X$ be a homology lens space with $H_1(X) \cong \mathbb{Z}/p\mathbb{Z}$. Then there exists a knot $K$ in a homology 3-sphere $\Sigma$ such that $X = \Sigma(K; p/r)$ ([BL, Lemma 2.1]). We define $|X|_d$ by

$$|X|_d = |\Delta_K(t)|_d,$$

where $d$ is a divisor of $p$. Then $|X|_d$ is a topological invariant of $X$ (Refer to [Kd2] for details).

(3) Let $X$ be a closed oriented 3-manifold. Then $\lambda(X)$ denotes the Lescop invariant of $X$ ([Le]). Note that $\lambda(S^3) = 0$.

## 2 Result

Let $K$ be a knot in a homology 3-sphere $\Sigma$. Let $M$ be the result of $2/q$-surgery on $K$: $M = \Sigma(K; 2/q)$. Let $\pi : X \to M$ be the universal abelian covering of $M$ (i.e. the covering associated to $\text{Ker}(\pi_1(M) \to H_1(M))$). Since $H_1(M) \cong \mathbb{Z}/2\mathbb{Z}$, $\pi$ is the 2-fold unbranched covering.

In [KMS], we have defined $|K|_{(q,d)}$ by the following formula, if $|X|_d$ is defined:

$$|K|_{(q,d)} := |X|_d.$$

Assume that the Alexander polynomial of $K$ is $t^2 - 3t + 1$. Then, as noted in [KMS], $H_1(X) \cong \mathbb{Z}/5\mathbb{Z}$ and $|K|_{(q,5)}$ is defined.

We then have the following.

**Theorem 2.1** Let $K$ be a knot in a homology 3-sphere $\Sigma$. We assume the following.

- (2.1) $\lambda(\Sigma) = 0$,
- (2.2) $\Delta_K(t) \equiv t^2 - 3t + 1$,
- (2.3) $|q| \geq 3$,
- (2.4) $\sqrt{|K|_{(q,5)}} > 4q^2$.

Then $M = \Sigma(K; 2/q)$ is not a Seifert fibered space.

**Remark 2.2** Let $K$ be the figure eight knot in $S^3$. Note that $\Delta_K(t) \equiv t^2 - 3t + 1$. Then $|K|_{(q,5)} = (5q^2 - 1)^2$ by [KMS, Example 2.2]. Hence (2.4) holds if $|q| \geq 3$.

**Remark 2.3** Theorem 2.1 seems to suggest studying the asymptotic behavior of $|K|_{(q,d)}$ as a function of $q$. 

2
3 An inequality for the Dedekind sum

To prove Theorem 2.1, we need the following inequality for the Dedekind sum \( s(\cdot, \cdot) \) ([RG]):

**Proposition 3.1** ([Ma, Lemma 3]) For an even integer \( p \geq 8 \) and for an odd integer \( q \) such that \( 3 \leq q \leq p - 3 \) and \( \gcd(p, q) = 1 \), we have

\[
|s(q, p)| < f(2, p)
\]

where \( f(2, p) = \frac{(p - 1)(p - 5)}{24p} \).

By this proposition, we immediately have the following.

**Lemma 3.2** For an even integer \( p \geq 8 \) and for an integer \( q_* \) such that \( q_* \not\equiv \pm 1 \pmod{p} \) and \( \gcd(p, q_*) = 1 \), we have

\[
|s(q_*, p)| < \frac{p}{24}.
\]

**Proof.** By assumptions, there exists \( q \) such that \( q_* \equiv q \pmod{p} \) and \( 3 \leq q \leq p - 3 \). Hence by Proposition 3.1, we have

\[
|s(q_*, p)| = |s(q, p)| < \frac{(p - 1)(p - 5)}{24p} < \frac{p}{24}.
\]

\[\Box\]

**Remark 3.3** The estimation given in Proposition 3.1 has a natural application ([Ma]).

4 Proof of Theorem 2.1

Suppose that \( M = \Sigma(K; 2/q) \) is a Seifert fibered space. Then, as shown in [KMS], we may assume that

\[ (\ast) : M \text{ has a framed link presentation as in Figure 1,} \]

where \( 1 \leq \alpha < \beta \) and \( \gcd(\alpha, \beta) = 1 \).

\[ M = \begin{array}{c}
K_1 \quad K_2 \quad K_3 \\
\frac{2\alpha}{q_1} \quad \frac{2\beta}{q_2} \quad \frac{5}{q_3} \\
0
\end{array} \]

Figure 1: A framed link presentation of \( M = \Sigma(K; 2/q) \)
Also as shown in \([KMS]\), \(\sqrt{|K_{(q,5)}|} = (\alpha \beta)^2\). Hence by (2.4),

\[
(\alpha \beta)^2 > 4q^2
\]  \((4.1)\)

By (2.1), (2.2) and \([Le\, 1.5 \, T2]\), we have \(\lambda(M) = -q\). Hence \((\alpha \beta)^2 > 4\{\lambda(M)\}^2\), and hence

\[
|\lambda(M)| < \frac{\alpha \beta}{2}
\]  \((4.2)\)

We now consider \(e\) defined as follows:

\[
e := \frac{q_1}{2\alpha} + \frac{q_2}{2\beta} + \frac{q_3}{5}.
\]

According to the sign of \(e\), we treat two cases separately: We first consider the case \(e > 0\).

Then the order of \(H_1(M)\) is \(20\alpha \beta e\). Since \(H_1(M) \cong \mathbb{Z}/2\mathbb{Z}\), \(20\alpha \beta e = 2\), and \(e = 1/(10\alpha \beta)\).

Hence by (*) and \([Le\, Proposition \, 6.1.1]\), we have

\[
\lambda(M) = \left(-\frac{4}{5}\right) \alpha \beta + \frac{5\beta}{24\alpha} + \frac{5\alpha}{24\beta} + \frac{1}{120\alpha \beta} - \frac{1}{4} - T
\]  \((4.3)\)

where \(T = s(q_1, 2\alpha) + s(q_2, 2\beta) + s(q_3, 5)\).

By (4.2), we have

\[-\frac{\alpha \beta}{2} < \lambda(M)\].

Hence by \([Le]\),

\[-\frac{\alpha \beta}{2} < \left(-\frac{4}{5}\right) \alpha \beta + \frac{5\beta}{24\alpha} + \frac{5\alpha}{24\beta} + \frac{1}{120\alpha \beta} - \frac{1}{4} + |T|.
\]

Consequently

\[
\frac{3}{10} \alpha \beta < -\frac{1}{4} + \frac{5}{24\alpha} + \frac{5}{24\beta} + \frac{1}{120\alpha \beta} + \frac{\alpha}{\beta} + |T|
\]  \((4.4)\)

As in \([KMS]\), we show that \(\alpha \geq 2\) implies a contradiction: Suppose that \(\alpha \geq 2\). Since \(\alpha < \beta\), we have \(\beta \geq 3\) and \(\alpha/\beta < 1\). Hence

\[
\frac{3}{5} \beta < -\frac{1}{4} + \frac{5}{24 \cdot 2} \beta + \frac{5}{24} + \frac{1}{120 \cdot 2 \cdot 3} + |T|.
\]

Since \(s(q_1, 2\alpha) < \frac{2\alpha}{12} < \frac{2\beta}{12}\), \(s(q_2, 2\beta) < \frac{2\beta}{12}\), and \(s(q_3, 5) < \frac{1}{5}\) as in \([KMS]\), we have

\[
|T| \leq |s(q_1, 2\alpha)| + |s(q_2, 2\beta)| + |s(q_3, 5)| \leq \beta^2 + \frac{1}{5}.
\]

Hence

\[
\frac{3}{5} \beta < -\frac{1}{4} + \frac{5}{48} \beta + \frac{5}{24} + \frac{1}{120 \cdot 6} + \left(\frac{\beta}{3} + \frac{1}{5}\right).
\]

Thus

\[
\left(\frac{3}{5} \beta - \frac{5}{48} - \frac{1}{3}\right) \beta < -\frac{1}{4} + \frac{5}{24} + \frac{1}{120 \cdot 6} + \frac{1}{5}.
\]

Therefore

\[
\frac{39}{240} \beta < \frac{1}{240} \left(38 + \frac{1}{3}\right) < \frac{39}{240}.
\]
This contradicts $\beta \geq 3$.

We next show that $\alpha = 1$ implies a contradiction: Suppose that $\alpha = 1$. By (4.1), $\beta^2 > 4q^2$.

Since $|q| \geq 3$, $\beta^2 > 4 \cdot 3^2 = 36$. Hence $\beta > 6$. Since $\alpha = 1$, $e = \frac{1}{10\beta}$. Hence

$$\frac{q_1}{2} + \frac{q_2}{2\beta} + \frac{q_3}{5} = \frac{1}{10\beta}$$

and hence we have the following equation.

$$(5\beta)q_1 + 5q_2 + (2\beta)q_3 = 1 \quad (4.5)$$

Since $q_1$ and $q_2$ are odd (see Figure 1), $\beta$ must be even. Since $\beta > 6$, we have $\beta \geq 8$. We then have

$$(\sharp) : q_2 \not\equiv \pm1 \pmod{2\beta}.$$  

In fact, since $q_1$ is odd, $(5\beta)q_1 \equiv \beta \pmod{2\beta}$. Hence by (4.5),

$$\beta + 5q_2 \equiv 1 \pmod{2\beta}.$$  

Now suppose that $q_2 \equiv 1 \pmod{2\beta}$. Then $\beta + 5 \equiv 1 \pmod{2\beta}$. This is impossible since $\beta \geq 8$.

Next suppose that $q_2 \equiv -1 \pmod{2\beta}$. Then $\beta - 5 \equiv 1 \pmod{2\beta}$. This is also impossible since $\beta \geq 8$. Thus (\sharp) holds.

Substituting $\alpha = 1$ in (4.4),

$$\frac{3}{10}\beta < -\frac{1}{4} + \frac{5}{24}\beta + \frac{5}{24\beta} + \frac{1}{120\beta} + |T|$$

where $T = s(q_2, 2\beta) + s(q_3, 5)$ (since $s(q_1, 2) = 0$). By (\sharp) and Lemma 3.2,

$$|s(q_2, 2\beta)| < \frac{2\beta}{24} = \frac{\beta}{12}.$$  

Hence

$$|T| \leq |s(q_2, 2\beta)| + |s(q_3, 5)| < \frac{\beta}{12} + \frac{1}{5}.$$  

Since $\beta \geq 8$,

$$\frac{3}{10}\beta < -\frac{1}{4} + \frac{5}{24}\beta + \frac{5}{24\cdot 8} + \frac{1}{120\cdot 8} + \left(\frac{\beta}{12} + \frac{1}{5}\right).$$

Thus

$$\left(\frac{3}{10} - \frac{5}{24} - \frac{1}{12}\right)\beta < -\frac{1}{4} + \frac{5}{24\cdot 8} + \frac{1}{120\cdot 8} + \frac{1}{5}$$

and hence $\frac{1}{120}\beta < 0$. This is a contradiction, and ends the proof in the case $e > 0$.

We finally consider the case $e < 0$. Then $e = -\frac{1}{10\alpha\beta}$. By (\ast) and [Le, Proposition 6.1.1], we have

$$\lambda(M) = -\left\{\left(\frac{4}{5}\right)\alpha\beta + \frac{5\beta}{24\alpha} + \frac{5\alpha}{24\beta} + \frac{1}{120\alpha\beta} - \frac{1}{4} + T\right\}.$$  

Remaining part of the proof is similar to that in the case $e > 0$.

This completes the proof of Theorem 2.1.  \[\square\]


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