Is the Energy Balance Paradox Solved?

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Abstract

The question of what is the source of the energy carried by radiation is discussed. The case of a hyperbolic motion is analyzed, describing the solutions suggested in the past for the “energy balance paradox”. The solution to the paradox is found in considering the stress force created in the curved electric field of the accelerated charge, that acts as a reaction force. The work performed by the external force to overcome this stress force is the source of the energy carried by the radiation. This stress force is the spatial component of the four-vector called “Schott term”, that appears in Abraham four-vector. This novel approach solves the “energy balance paradox”.

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1. Introduction

The Energy Balance Paradox concerns the question of - what is the source of the energy carried by radiation. This problem is complicated and demands a careful analysis of all the factors relevant to the process. Hence, we shall concentrate here on the most simple case of creation of radiation - a charge accelerated uniformly in its own system of reference, in which case its motion is described in a four dimensional free space as a hyperbolic motion [1].

One expects that the energy carried by the radiation is created by a work done by the external force against a certain force that exists in the system, where this work should be performed in addition to the work done in creating the kinetic energy of the charged particle. In many publications this problem is treated by using the concept of “radiation reaction force”, where it is assumed that a reaction force created by the radiation is the force that the external force should overcome, and the work done against the reaction force is the source of the energy carried by the radiation. However, in the simple case of the hyperbolic motion this approach meets difficulties that were named “The Energy Balance Paradox”, as no such force exists in this motion.

The difficulties have two sources: one - when the accelerated charge moves with low (zero) velocity, its radiation is symmetric with respect to the plane which is perpendicular to the direction of motion. Thus the radiation does not impart any counter momentum to the accelerated charge, and no radiation reaction force exists. The second source of the difficulties is that usually, it is considered that in the equation of motion of the accelerated charge, a term called Abraham four vector is considered as representing the radiation reaction force. It comes out that in a hyperbolic motion, this vector vanishes. This suits the fact that in this case no radiation reaction force exists, but it leaves us with the paradox.

In the present work we shall analyze the paradox, and the solutions given to it. We shall also show that the solution to the paradox is found when a reaction force is identified, which is not the “radiation reaction force”, but rather a stress force that exists in the curved electric field of the accelerated charge. The work done by the external force to overcome the stress force, is the source of the energy carried by the radiation.

2. The Problem.

The formula for the angular distribution of the radiation power is [2]:

\[
\frac{dP}{d\Omega} = \frac{\epsilon^2 a^2}{4\pi c^3} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}
\]  

(1)

where \( \epsilon \) is the accelerated charge, \( a \) is the acceleration, \( c \) is the speed of light, and \( \theta \) is the angle measured from the direction of motion. Integrating eq. 1 over the angles, yields:

\[
P = \frac{2}{3} \frac{\epsilon^2 a^2}{c^3 (1 - \beta^2)^3} = \frac{2}{3} \frac{\epsilon^2 (\gamma^3 a)^2}{c^3}
\]  

(2)

where for \( \beta \to 0 \ (\gamma \to 1) \), yields Larmor formula for the power carried by the radiation:

\[
P = \frac{2}{3} \frac{\epsilon^2 a^2}{c^3}
\]  

(3)
In Fig. 1 we plot the angular distribution of the radiation for several values of $\beta$. It is clearly observed that for low velocities ($\beta \leq 0.01$), no counter momentum is imparted by the radiation to the charge, and no radiation reaction exists.

The Dirac-Lorentz equation (or LAD as called by Rohrlich[3]) for a charged particle is:

$$ma^\mu = F^\mu_\text{ext} + \Gamma^\mu = F^\mu_\text{ext} + \frac{2e^2}{3c^3} \left( \dot{a}^\mu - \frac{1}{c^2} a^\lambda a_\lambda v^\mu \right),$$  

(4)

where $m$ is the particle mass, $v^\mu$ and $a^\mu$ are the velocity 4-vector and the acceleration 4-vector, respectively, and $F^\mu_\text{ext}$ is the external force 4-vector that drives the particle. $\Gamma^\mu$ is Abraham 4-vector, which was usually considered as representing the radiation reaction force.

The first term in $\Gamma^\mu$, $\left( \frac{2e^2}{3c^3} \dot{a}^\mu \right)$, is called the “Schott term”. In a hyperbolic motion, $\dot{a}^\mu = \frac{1}{c^2} a^\lambda a_\lambda v^\mu$, and $\Gamma^\mu$ vanishes. This is in accord with the nonexistence of a radiation reaction force in such a motion as shown in Fig. 1 (concerning the conjecture that $\Gamma^\mu$ represents the radiation reaction force). In the present work we follow Rohrlich[3], who argues that the reaction force responsible for the creation of the radiation should be included in Schott term, thus showing that the vanishing of $\Gamma^\mu$ shows that the power carried by the radiation, which is represented in the second term of $\Gamma^\mu$ is created by the force represented in Schott term. However, this force is not a radiation reaction force, but rather a reaction force created by the stress force the exists in the curved electric field of the accelerated charge. This reaction force should be overcome by the accelerating external force, and the work performed against this force is the source of the energy carried by the radiation (see section 4).

The presence of $\dot{a}$ (the third time derivative of the position) in the equation of motion (eq.4), demands a third initial condition for the solution of the equation of motion of an electric charge (the initial acceleration). In section 4 we find that a stress force, $f_s$, which exists in the curved electric field of the accelerated charge, is proportional to the acceleration. Thus, the third initial condition is needed to complement the picture of all the factors involved in such a motion. It is also found that this force, $f_s$, is responsible for the creation of the radiation.

3. Solutions suggested
One of the solutions was [4], that there exists a charged plane, whose charge is equal and opposite in sign to the accelerated charge, and it recesses with the speed of light in a direction opposite to the direction of the acceleration. The interaction between this charged plane and the accelerated charge creates the energy carried by the radiation. Another suggestion was [5, (eq. 4.9)], that in such a case the energy radiated is supplied from the self-energy of the charge. Evidently, these suggestions are far from being satisfactory. Some people deduced that a uniformly accelerated charge does not radiate (see Singal [6]). It should be noted that the idea suggested in [4] resolves another difficulty concerned with this topic, which is the existence of a single electric charge. As we assume that the matter in the universe is neutral, the existence of a solitary charge is a local phenomenon, whose validity is limited to distance scales that are much shorter than distance scales that characterize gravitational considerations. Any treatment of this topic that carries calculations to infinity, cannot be a valid treatment. The treatment suggested by Leibovitz and Peres [4], considers a system which is neutral.

In a recently published paper, Rohrlich[3] discusses this topic, and criticizing earlier works (including his earlier work[7]), he considers a treatment that should solve the problem. Let us discuss here briefly the treatment suggested in [3].

The equation of motion given in [3] is:

\[ m_0 \dot{v}^\mu = F^\mu_{\text{ext}} + F^\mu_{\text{self}} \]  \hspace{1cm} (5)

where \( m_0 \) is the physical rest mass involved in the process \( m_0 = m + \delta m \), [3], and \( F^\mu_{\text{self}} \) actually equals Abraham 4-vector. \( \delta m \) represents the electromagnetic mass which will be discussed later. Rohrlich obtained this equation by generalizing the equation of motion, and by demanding that the constant in the generalized equation be \( \frac{2e^2}{3c^3} \). Rohrlich concludes that the vanishing of Abraham 4-vector (\( F^\mu_{\text{self}} \) in his notation) shows that the radiation energy comes from the Schott term.

This argumentation makes sense but it raises a question: The terms that appear in Abraham 4-vector in this case are the zeroth components of a 4-vector (components that represent power, and not force, see [3]). The question where is the force that does the work is still there. We are looking for a three dimensional force that acts as a reaction force, and we do not find it in this treatment. The Schott term in this case represents the power performed by the force we are looking for, but not the force itself. We discuss this point in the next section.

4. The stress force solution

When an electric charge is accelerated, its electric field is not accelerated with the charge. It is detached from the charge, and remains inertial. Hence the electric field of the charge becomes curved[8]. Fulton and Rohrlich [5] calculated the electromagnetic fields of a charge moving in a hyperbolic motion, along the \( z \) axis, using the retarded potentials method. The equation for the electric field lines were calculated by Singal[6], and they are drawn in Fig. 2.

The same results were obtained by Gupta and Padmanabhan[9], where they calculate the electric fields of an accelerated charge in the rest system of the charge, and then transform them to the inertial frame. They also show that by using the retarded coordinates \( (x_{\text{ret}}, t_{\text{ret}}) \) for the accelerated charge, one obtains the expressions for the electromagnetic fields as given in the standard textbooks[2,10].

The radius of curvature of the field lines is: \( R_c = \frac{c^2}{a \sin \theta} \), where \( \theta \) is the angle between the initial direction of the field line and the acceleration. Now, we follow in brief the calculations
Figure 2: A curved electric field of a uniformly accelerated charge.

given in [11]. In a curved electric field a stress force exists, whose density, $f_s$, is given by:

$$f_s = \frac{E^2}{4\pi R_c},$$

(6)

where $E$ is the electric field, and $R_c$ is the radius of curvature of the field lines. In the immediate vicinity of the charge, the electric field can be taken as $E = e/r^2$, and we have:

$$f_s = \frac{E^2}{4\pi R_c} = \frac{a \sin \theta e^2}{4\pi c^2 r^4}$$

(7)

The stress force is perpendicular to the field lines, so that the component of this force along the acceleration is $-f_s \sin \phi$, where $\phi$ is the angle between the local field line and the acceleration. In the immediate vicinity of the charge ($r \ll c^2/a$, where the direction of the field line did not change much from its initial direction), $\phi \sim \theta$, and we can write for the parallel component of the stress force:

$$-f_s \sin \phi \simeq -f_s \sin \theta = -\frac{a \sin^2 \theta e^2}{4\pi c^2 r^4}$$

(8)

It is interesting to note that the angular distribution of the parallel component of the stress force is similar to that of the radiation. We have to sum over the parallel component of the stress force, $f_s$, and calculate the work done against this force.

In order to sum over $f_s$, we have to integrate over a sphere whose center is located on the charge. Naturally, such an integration involves a divergence (at the center). To avoid such a divergence, we take as the lower limit of the integration a small distance from the center, $r = c\Delta t$, (where $\Delta t$ is infinitesimal). We calculate the work done by the stress force in the volume defined by $c\Delta t \leq r \leq r_{up}$, where $r_{up}$ is some large distance from the charge, satisfying the demand: $c\Delta t \ll r_{up} \ll c^2/a$. These calculations are performed in a system of reference $S$, which is a flat system that momentarily coincides with the frame of reference of the accelerated charge at time $t = 0$, at the charge location.
Since the calculations are performed in the flat system $S$, the integration can be carried
without using any terms concerning space curvature. Integration of the stress force over a
volume extending from $r = c\Delta t$ to $r_{up}$, yields the total force due to stress, $F_s$:

$$F_s = 2\pi \int_{c\Delta t}^{r_{up}} r^2 dr \int_0^\pi \sin \theta d\theta [-f_s \sin \theta] = -\frac{2}{3} \frac{a}{c^2} \left(\frac{e^2}{c}\right) \left(1 - \frac{c\Delta t}{r_{up}}\right).$$ (9)

Clearly the second term in the parenthesis can be neglected. $F_s$ is the reaction force. The
power supplied by the external force on acting against the electric stress is $P_s = -F_s v$, where $v$
is the velocity of the charge in system $S$ at time $t = \Delta t$, $v = a\Delta t$. Substituting this value for $v$
we get for $P_s$:

$$P_s = \frac{2}{3} \frac{a^2 e^2}{c^3}.$$ (10)

This is the power radiated by an accelerated charged particle at zero velocity (eq. 3).

In order to include this result in the equation of motion (eq. 5), we should add the force
$-F_s$ to the external force, which is the force needed to overcome the stress force, and we should
include $F_s$ in the spatial part of the Schott 4-vector (Schott term in the notation of [3]), whose
zeroth component is the power created by this force. The work performed by $-F_s$ is the source
of the energy carried by the radiation. This extended equation includes all the forces involved
in the process, including the force that performs the work that creates the energy carried by the
radiation. Rohrlich is right in his conclusion that the Schott term is the source of the power
of the radiation, but this definition becomes complete when the stress force ($F_s$ from eq. 9) is
included in the spatial part of the Schott term.

The expression for the force in eq. 9 (before substituting the integration limits, $F_s = -\frac{2}{3} \frac{a e^2}{c^2 r}$) equals the inertial force $(4m_e a/3)$ of the electromagnetic mass of the charge as calculated by Lorentz (see ref. [2], p. 790). However, a work performed against an inertial force cannot be the work that creates the energy carried by the radiation, because such a work creates a kinetic energy of the electromagnetic mass $m_e$, and this leaves us with the energy balance paradox. Actually, this electromagnetic mass is already included in the equation of motion by Rohrlich, (eq. 5) as $\delta m$ (see [3]), and the force needed to accelerate this mass is already included in $F_{\mu \text{ext}}$.

As an example, let us analyze the case of an oscillatory motion of a charge in a linear antenna
(length of $2D$), in the $x$ direction. the equations of motion are:

$$x = D \sin \omega t; \quad v = \omega D \cos \omega t$$ (11.a)
$$a = -\omega^2 D \sin \omega t = -\omega^2 x; \quad \dot{a} = -\omega^3 D \cos \omega t = -\omega^2 v$$ (11.b)

At time $\Delta t$ (where $\Delta t$ is infinitesimal), we use the approximations:

$\sin \omega t \simeq \omega \Delta t$, and $\cos \omega t \simeq 1$:

$$x = \omega D \Delta t; \quad v = \omega D$$ (12.a)
$$a = -\omega^2 D \Delta t; \quad \dot{a} = -\omega^3 D$$ (12.b)

and we find that when we start from a point at which $a = 0$, $a(\Delta t) = \dot{a} \Delta t$. 
The motion is linear and from eq. 9 we have the reaction force $F_{\text{reac}}$ created in a linear motion: 
$$F_{\text{reac}} = F_s = \frac{2e^2}{3c^2} \Delta t.$$ 

Substituting for $a$, we find: $F_s = \frac{-2e^2 \dot{a}}{3c^2}$, which is exactly Schott term. Jackson [2, eq. 17.8] calls this expression $F_{\text{rad}}$, relating it to the radiation reaction force. Since we know that in a linear motion, at low velocities ($\beta < 0.01$), no radiation reaction force exists, we deduce that $F_{\text{reac}}$, that expresses the stress force in the curved electric field, should replace Jackson’s $F_{\text{rad}}$, the non-existing radiation reaction force.

5. Non-Zero Velocity

Equation (10) shows the power emitted by a uniformly accelerated charge calculated at zero velocity. We can consider the case of an accelerated charge, moving with low velocity. From Fig. 1 we observe that in this case, a part of the radiation is radiated forward, and evidently, this part of the radiation imparts a backward momentum to the radiating charge, thus creating a reaction force. To calculate the power created by this reaction force, we follow the calculations given in [12]. We multiply eq. 1 by $\cos \theta/c$, (to obtain the parallel component of the momentum flux of the radiation), and integrate over the angles. The integration yields for the parallel component of the momentum flux ($p_{\text{par}}$):

$$p_{\text{par}} = \frac{2e^2 (\gamma^3 a)^2}{3c^4} \beta,$$

while the total absolute value of the momentum flux, $p$, is found by dividing eq. 2 by $c$. (The perpendicular momentum flux vanishes because of the symmetry in the plane perpendicular to the direction of motion, but we still can compare the parallel component of the momentum flux to the total absolute value of the momentum flux of the radiation). By dividing $p_{\text{par}}$ by $p$, we find the weight of the parallel component of the momentum flux in the total absolute value of the momentum flux:

$$\frac{p_{\text{par}}}{p} = \beta$$

This fraction is the weight of the parallel component of the momentum flux of the radiation, and this fraction creates a reaction force - a radiation reaction force. To get the weight of the work done by the radiation reaction force we should multiply this fraction by $\beta c$ (the velocity of the charge in the rest frame), and we find that the weight of the work done in overcoming the radiation reaction force (the reaction force created by the radiation) in the total power radiated is $\beta^2$. The other part of the energy ($1 - \beta^2 = 1/\gamma^2$), is created by the stress force that exists in the curved electric field. We find that the relative weight of the work done by the stress force in the total work done by the total reaction force decreases when $\gamma$ increases.

5. Conclusions

The solution to the energy balance paradox is found when we consider the stress force that exists in the curved electric field of an accelerated charge. At low (zero) velocities, the work done by the external (accelerating) force in overcoming the stress force is the source of the energy carried by the radiation. The stress force is the spatial component of the Schott 4-vector, which is usually called the Schott term in Abraham 4-vector. Thus, the radiation energy comes from the Schott term, as suggested by Rohrlich[3].
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