

ABSTRACT. This is a sequel to our previous paper of oriented bivariant theory \[14\]. In 2001 M. Levine and F. Morel constructed algebraic cobordism \(\Omega^*(X)\) for schemes \(X\) over a field \(k\) in an abstract way and later M. Levine and R. Pandharipande reconstructed it more geometrically. In this paper in a similar manner we construct an algebraic cobordism \(\Omega^*(X \to S)\) for a scheme \(X\) over a fixed scheme \(S\) in such a way that if the target scheme \(S\) is the point \(pt = \text{Spec } k\), then \(\Omega^{-i}(X \to pt)\) is isomorphic to Levine–Morel’s algebraic cobordism \(\Omega_i(X)\).

1. INTRODUCTION

V. Voevodsky has introduced algebraic cobordism (now called higher algebraic cobordism), which was used in his proof of Milnor’s conjecture \[13\]. D. Quillen introduced the notion of (complex) oriented cohomology theory on the category of differential manifolds \[10\] and this notion can be formally extended to the category of smooth schemes in algebraic geometry. M. Levine and F. Morel constructed a universal oriented cohomology theory on smooth schemes, which they also call algebraic cobordism \[6\], and recently M. Levine and R. Pandharipande \[7\] gave another equivalent construction of the algebraic cobordism via what they call “double point degeneration” and they found a nice application of it in the Donaldson–Thomas theory of 3-folds, i.e. they proved what is called MNOP conjecture \[8\].

The algebraic cobordism \(\Omega_*(X)\) of a scheme \(X\) is roughly speaking constructed as follows. First they consider the Grothendieck group or the group completion, denoted by \(Z_* (X)\), of the monoid consisting of isomorphism classes \([M \rightarrow h X, L_1, \cdots, L_r]\) of a projective morphism \(h : M \rightarrow X\) from a quasi-projective smooth scheme \(M\) together with line bundles \(L_i\) over the source scheme \(M\). The functor \(Z_*\) carries the four data \((D1), (D2), (D3), (D4)\), and they satisfy the eight conditions \((A1), (A2), \cdots, (A8)\). These data and conditions are not written here, but in the following section we will write them in our context. A functor having such four data and satisfying the eight conditions is called an oriented Borel–Moore functor with products.

If \(A_*\) is an oriented Borel–Moore functor with products, then the abelian group \(A_*(pt)\) of the point \(pt\) becomes a commutative graded ring. Given a commutative ring \(R_\ast\), an oriented Borel-Moore functor with product \(A_*\) together with a graded ring homomorphism \(R_\ast \rightarrow A_*(pt)\) is called an oriented Borel–Moore \(R_\ast\)-functor with products. Let \(L_*\) be the Lazard ring. Then, an oriented Borel–Moore functor with products of geometric type is defined to be an oriented Borel–Moore \(\mathbb{L}_*\)-functor with products which satisfies the following three axioms (see \[6\], Definition 2.2.1):

- (Dim) Dimension Axiom: For any smooth scheme \(Y\) and any family \((L_1, \cdots, L_n)\) of line bundles on \(Y\) with \(n > \dim Y\), one has

\[
[Y \xrightarrow{id} Y; L_1, \cdots, L_n] = 0 \in A_*(Y).
\]

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Moore functor with products
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pt
quasi-projective
from
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Ω
pre-algebraic cobordism
of
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project
was that we observed that for any bivariant theory
braic cobordism [6] (see also [7]), we introduce an oriented bivariant theory. The starting point of this

restrict ourselves to the category of smooth schemes
if the ground field is of characteristic zero (because the resolution of singularities is used) and if we

The above group \( Z_\ast(X) \) modded out by a certain subgroup \( R_\ast(X) \) involving the above three axioms,

\[
\Omega_\ast(X) := \frac{Z_\ast(X)}{R_\ast(X)}
\]

becomes an oriented Borel–Moore functor with products of geometric type. To be a bit more precisely, they construct it step by step.

1. First they consider the subgroup \( \langle R^{\text{Dim}} \rangle(X) \subset Z_\ast(X) \) dealing with (Dim) and define the quotient

\[
\underline{Z_\ast}(X) := \frac{Z_\ast(X)}{\langle R^{\text{Dim}} \rangle(X)}.
\]

2. Secondly, they consider the subgroup \( \langle R^{\text{Sect}} \rangle(X) \subset Z_\ast(X) \) dealing with (Sect) and define the quotient

\[
\underline{Z_\ast}(X) := \frac{Z_\ast(X)}{\langle R^{\text{Sect}} \rangle(X)}.
\]

3. Finally, they consider the subgroup \( \langle R^{\text{FGL}} \rangle(X) \subset L_\ast \otimes Z_\ast(X) \) dealing with (FGL) and define the quotient

\[
\Omega_\ast(X) := \frac{L_\ast \otimes Z_\ast(X)}{\langle R^{\text{FGL}} \rangle(X)}.
\]

It turns out (via the construction of \( \Omega_\ast(X) \)) that Levine–Morel’s \( \Omega_\ast(X) \) is the universal one among such oriented Borel–Moore functors with products of geometric type. The main theorem of [6] is that if the ground field is of characteristic zero (because the resolution of singularities is used) and if we restrict ourselves to the category of smooth schemes \( X \) the theory of algebraic cobordism \( \Omega^\ast(X) := \Omega_{\dim X} X \rightarrow pt(X) \) is in fact the universal oriented cohomology theory[4]. The group \( Z_\ast(X) \) shall be called the pre-algebraic cobordism of \( X \).

In the definition of the algebraic cobordism [6] (also see [7]), they consider projective morphisms from quasi-projective smooth varieties, or equivalently proper morphisms from quasi-projective smooth varieties. Very recently J. L. González and K. Karu [5] (cf. [4]) have observed that the assumption of quasi-projectivity can be dropped, i.e., one can consider proper morphisms from smooth varieties, to get the same algebraic cobordism. So in the rest of the paper we will use González-Karu’s description.

In our previous paper [14] aiming at the construction of a bivariant version of Levine–Morel’s algebraic cobordism [6] (see also [7]), we introduce an oriented bivariant theory. The starting point of this project was that we observed that for any bivariant theory \( B \), the covariant theory \( B_\ast(X) := B^\ast X \rightarrow pt \) and the contravariant theory \( B^\ast(X) := B^\ast X \rightarrow X \) are both more or less what is called a Borel–Moore functor with products without the Chern operator defined (if the Chern operator is also defined

\[1\] In this sense Levine-Morel’s algebraic cobordism \( \Omega_\ast(X) \) is a bordism theory, thus could be called “algebraic bordism”.

\[2\] Using the same idea as in [14], in [11] we construct a bivariant version of what is called a motivic characteristic class [1].
and compatible with the pushforward, pullback and exterior product, it is called an oriented Borel–Moore functor with products.

In this paper we start with our following oriented bivariant theory

\[
Z^*(X \xrightarrow{f} Y) := \left\{ \left[ V \xrightarrow{h} X; L_1, \cdots, L_r \right] \right\}
\]

which is the graded abelian group generated by the cobordism cycle \( [V \xrightarrow{h} X; L_1, \cdots, L_r] \) (as defined in [6]) such that

1. \( h : V \to X \) is proper,
2. the composite \( f \circ h : V \to Y \) is smooth,
3. \( L_i \) is a line bundle over \( V \).

The grading is defined by

\[
[V \xrightarrow{h} X; L_1, \cdots, L_r] \in Z^i(X \xrightarrow{f} Y) \iff -i + r = \dim(f \circ h),
\]

where the dimension \( \dim(f \circ h) \) is the relative dimension of the smooth morphism, i.e., the dimension of the fiber, or \( \dim(f \circ h) = \dim V - \dim Y \).

**Remark 1.1.**

1. The reason why we consider such gradings is that eventually we want to capture \( \Omega^*(X \xrightarrow{\pi_X} S) \) as a bivariant-theoretical group in the sense of Fulton–MacPherson [3]. In their bivariant theory \( B \), the grading is such that \( B^{-i}(X \to \text{pt}) = B_i(X) \), which is the associated covariant group.

2. When \( Y \) is a point, then our \( Z^*(X \xrightarrow{f} \text{pt}) \) is the same as Levine-Morel’s pre-algebraic cobordism \( Z^*(X) \).

In this paper we restrict the above group to the category of \( S \)-schemes over a fixed scheme, namely the over category \( \text{Sch}/S \), whose objects are morphisms \( \pi_X : X \to S \) and morphisms from \( \pi_X : X \to S \) to \( \pi_Y : Y \to S \) are \( f : X \to Y \) such that the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\pi_X} & & \downarrow{\pi_Y} \\
S & \xleftarrow{\pi_Y} & S.
\end{array}
\]

Thus we consider the graded abelian group \( Z^*(X \xrightarrow{\pi_X} S) \) on the over category \( \text{Sch}/S \). Then, in an analogous manner as done in Levine-Morel’s construction, we proceed as follows:

1. First, consider the subgroup \( \langle R^{\text{Dim}} \rangle(X \xrightarrow{\pi_X} S) \subset Z_*(X \xrightarrow{\pi_X} S) \) dealing with (rel-Dim) a “relative” Dimension Axiom and define the quotient

\[
\underline{Z}^*(X \xrightarrow{\pi_X} S) := \frac{Z^*(X \xrightarrow{\pi_X} S)}{\langle R^{\text{Dim}} \rangle(X \xrightarrow{\pi_X} S)}.
\]

2. Secondly, consider the subgroup \( \langle R^{\text{Sect}} \rangle(X \xrightarrow{\pi_X} S) \subset Z_*(X \xrightarrow{\pi_X} S) \) dealing with (rel-Sect) a “relative” Section Axiom, and define the quotient

\[
\underline{Z}^*(X \xrightarrow{\pi_X} S) := \frac{Z^*(X \xrightarrow{\pi_X} S)}{\langle R^{\text{Sect}} \rangle(X \xrightarrow{\pi_X} S)}.
\]

3. Finally, consider the subgroup \( \langle R^{\text{FGL}} \rangle(X \xrightarrow{\pi_X} S) \subset L_* \otimes \underline{Z}^*(X \xrightarrow{\pi_X} S) \) dealing with (FGL) a “relative” Formal Group Law Axiom, and define the quotient

\[
\Omega^*(X \xrightarrow{\pi_X} S) := \frac{L_* \otimes \underline{Z}^*(X \xrightarrow{\pi_X} S)}{\langle R^{\text{FGL}} \rangle(X \xrightarrow{\pi_X} S)}.
\]
$\Omega^*(X \xrightarrow{\pi_X} S)$ is a $Sch/S$-version of an oriented Borel–Moore functor with products of geometric type for a scheme $X$ over a fixed scheme $S$ such that $\Omega^*(X \to pt)$ is equal to Levine–Morel’s algebraic cobordism $\Omega_*(X)$. In this sense our group $\Omega^*(X \xrightarrow{\pi_X} S)$, algebraic cobordism of an $S$-scheme $\pi_X : X \to S$, could be called a relative algebraic cobordism.

**Remark 1.2.** Motivated by the construction given in the present paper, furthermore we tried to get a bivariant version of algebraic cobordism, which is our ultimate goal. However we have been unable to obtain one, mainly because the bivariant product is not well-defined under the present construction, as we will make a remark later. We hope that by modifying the present construction slightly we would be able to get a bivariant algebraic cobordism.

2. **FULTON–MACPHERSON’S BIVARIANT THEORY AND A UNIVERSAL BIVARIANT THEORY**

We make a quick review of Fulton–MacPherson’s bivariant theory [3] (also see [2]) and a universal bivariant theory [14].

Let $V$ be a category which has a final object $pt$ and on which the fiber product or fiber square is well-defined. Also we consider a class of maps, called “confined maps” (e.g., proper maps, projective maps, in algebraic geometry), which are closed under composition and base change and contain all the identity maps, and a class of fiber squares, called “independent squares” (or “confined squares”, e.g., “Tor-independent” in algebraic geometry, a fiber square with some extra conditions required on morphisms of the square), which satisfy the following:

(i) if the two inside squares in

$$
\begin{array}{c}
X'' \xrightarrow{h'} X' \xrightarrow{g'} X \\
\downarrow f'' \quad \downarrow f' \\
Y'' \xrightarrow{h} Y' \xrightarrow{g} Y
\end{array}
$$

or

$$
\begin{array}{c}
X' \xrightarrow{h''} X \\
\downarrow f \\
Y' \xrightarrow{h'} Y
\end{array}
$$

are independent, then the outside square is also independent,

(ii) any square of the following forms are independent:

$$
\begin{array}{c}
\begin{array}{c}
X \xrightarrow{id_X} X \\
\downarrow f \\
Y
\end{array}
\end{array}
$$

where $f : X \to Y$ is any morphism.

A bivariant theory $B$ on a category $V$ with values in the category of graded abelian groups is an assignment to each morphism $X \xrightarrow{f} Y$ in the category $V$ a graded abelian group (in most cases we ignore the grading) $B(X \xrightarrow{f} Y)$ which is equipped with the following three basic operations. The $i$-th component of $B(X \xrightarrow{f} Y)$, $i \in \mathbb{Z}$, is denoted by $B^i(X \xrightarrow{f} Y)$.

1. **Product:** For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$
\bullet : B^i(X \xrightarrow{f} Y) \otimes B^j(Y \xrightarrow{g} Z) \to B^{i+j}(X \xrightarrow{gf} Z)
$$

is defined.
(2) **Pushforward**: For morphisms \( f : X \to Y \) and \( g : Y \to Z \) with \( f \) confined, the pushforward operation
\[
f_* : \mathcal{B}(X \xrightarrow{g} Z) \to \mathcal{B}(Y \xrightarrow{g} Z)
\]

is defined.

(3) **Pullback**: For an independent square
\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow & & \downarrow f \\
Y' & \xrightarrow{g} & Y,
\end{array}
\]

the pullback operation
\[
g^* : \mathcal{B}(X \xrightarrow{f} Y) \to \mathcal{B}(X' \xrightarrow{f'} Y')
\]

is defined.

These three operations are required to satisfy the following seven compatibility axioms ([3, Part I, §2.2]):

(A1) **Product is associative**: given a diagram \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \) with \( \alpha \in \mathcal{B}(X \xrightarrow{f} Y), \beta \in \mathcal{B}(Y \xrightarrow{g} Z), \gamma \in \mathcal{B}(Z \xrightarrow{h} W) \),
\[
(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).
\]

(A2) **Pushforward is functorial**: given a diagram \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \) with \( f \) and \( g \) confined and \( \alpha \in \mathcal{B}(X \xrightarrow{h \circ g \circ f} W) \),
\[
(g \circ f)_*(\alpha) = g_*(f_*(\alpha)).
\]

(A3) **Pullback is functorial**: given independent squares
\[
\begin{array}{ccc}
X'' & \xrightarrow{h'} & X' \\
\downarrow f'' & & \downarrow f' \\
Y'' & \xrightarrow{g} & Y \\
\end{array}
\]

and \( \alpha \in \mathcal{B}(X \xrightarrow{f} Y) \),
\[
(g \circ h)^*(\alpha) = h^*(g^*(\alpha)).
\]

(A12) **Product and pushforward commute**: given a diagram \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \) with \( f \) confined and \( \alpha \in \mathcal{B}(X \xrightarrow{g \circ f} Z), \beta \in \mathcal{B}(Z \xrightarrow{h} W) \),
\[
f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot \beta.
\]

(A13) **Product and pullback commute**: given independent squares
\[
\begin{array}{ccc}
X' & \xrightarrow{h''} & X \\
f' & & \downarrow f \\
Y' & \xrightarrow{h'} & Y \\
g' & & \downarrow g \\
Z' & \xrightarrow{h} & Z
\end{array}
\]
with $\alpha \in \mathcal{B}(X \xrightarrow{f} Y), \beta \in \mathcal{B}(Y \xrightarrow{g} Z), \gamma \in \mathcal{B}(Z \xrightarrow{h} W)$, 

$$h^*(\alpha \cdot \beta) = h^*(\alpha) \cdot h^*(\beta).$$

(A.23) **Pushforward and pullback commute:** given independent squares

$$X' \xrightarrow{g'} X \quad f' \downarrow \quad f \downarrow \quad Y' \xrightarrow{h'} Y \quad g' \downarrow \quad g \downarrow \quad Z' \xrightarrow{h} Z$$

with $f$ confined and $\alpha \in \mathcal{B}(X \xrightarrow{g\circ f} Z)$,

$$f^*(h^*(\alpha)) = h^*(f_*(\alpha)).$$

(A.123) **Projection formula:** given an independent square with $g$ confined and $\alpha \in \mathcal{B}(X \xrightarrow{f} Y), \beta \in \mathcal{B}(Y \xrightarrow{h} Z)$

$$X' \xrightarrow{g} X \quad f' \downarrow \quad f \quad Y' \xrightarrow{g} Y \xrightarrow{h} Z$$

and $\alpha \in \mathcal{B}(X \xrightarrow{f} Y), \beta \in \mathcal{B}(Y \xrightarrow{h} Z), \gamma \in \mathcal{B}(Z \xrightarrow{h} W)$,

$$g^*(g^*(\alpha) \cdot \beta) = \alpha \cdot g_*(\beta).$$

We also assume that $\mathcal{B}$ has units:

- **Units:** $\mathcal{B}$ has units, i.e., there is an element $1_X \in \mathcal{B}(X \xrightarrow{id} X)$ such that $\alpha \cdot 1_X = \alpha$ for all morphisms $W \to X$ and all $\alpha \in \mathcal{B}(W \to X)$, such that $1_X \cdot \beta = \beta$ for all morphisms $X \to Y$ and all $\beta \in \mathcal{B}(X \to Y)$, and such that $g^*1_X = 1_{X'}$ for all $g : X' \to X$.

- **Commutativity:** $\mathcal{B}$ is called commutative if whenever both

$$W \xrightarrow{g} X \quad W \xrightarrow{f} Y \quad f' \downarrow \quad f \quad g' \downarrow \quad g \quad Y \xrightarrow{g} Z \quad X \xrightarrow{g} Z$$

are independent squares with $\alpha \in \mathcal{B}(X \xrightarrow{f} Z)$ and $\beta \in \mathcal{B}(Y \xrightarrow{g} Z),

$$g^*(\alpha) \cdot \beta = f^*(\beta) \cdot \alpha.$$ (NOTE: If $g^*(\alpha) \cdot \beta = (-1)^{\deg(\alpha) \deg(\beta)} f^*(\beta) \cdot \alpha$ holds, it is called skew-commutative. In this paper we assume that bivariant theories are commutative.) Let $\mathcal{B}, \mathcal{B}'$ be two bivariant theories on a category $\mathcal{V}$.

A Grothendieck transformation from $\mathcal{B}$ to $\mathcal{B}'$, $\gamma : \mathcal{B} \to \mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(X \to Y) \to \mathcal{B}'(X \to Y)$ for a morphism $X \to Y$ in the category $\mathcal{V}$, which preserves the above three basic operations:

1. $\gamma(\alpha \cdot \beta) = \gamma(\alpha) \cdot \gamma(\beta)$,
2. $\gamma(f_*\alpha) = f_*\gamma(\alpha)$, and
3. $\gamma(g^*\alpha) = g^*\gamma(\alpha)$. 

7\SHIOJ YOKURA\(\mathrm{\text{\textit{t}t}}\)
A bivariant theory unifies both a covariant theory and a contravariant theory in the following sense:

\[ \mathbb{B}_*(X) := \mathbb{B}(X \to pt) \]

becomes a covariant functor for \textit{confined} morphisms and

\[ \mathbb{B}^*(X) := \mathbb{B}(X \xrightarrow{id} X) \]

becomes a contravariant functor for \textit{any} morphisms. A Grothendieck transformation \( \gamma : \mathbb{B} \to \mathbb{B}' \) induces natural transformations \( \gamma_* : \mathbb{B}_* \to \mathbb{B}'_* \) and \( \gamma^* : \mathbb{B}^* \to \mathbb{B}'^* \).

**Definition 2.2.** As to the grading, \( \mathbb{B}_i(X) := \mathbb{B}^{-i}(X \xrightarrow{id} X) \) and \( \mathbb{B}'_i(X) := \mathbb{B}'^i(X \xrightarrow{id} X) \).

**Definition 2.3.** (\textit{[3] Part I, §2.6.2 Definition}) Let \( S \) be a class of maps in \( \mathcal{V} \), which is closed under compositions and containing all identity maps. Suppose that to each \( f : X \to Y \) in \( S \) there is assigned an element \( \theta(f) \in \mathbb{B}(X \xrightarrow{f} Y) \) satisfying that

(i) \( \theta(g \circ f) = \theta(f) \circ \theta(g) \) for all \( f : X \to Y, \ g : Y \to Z \in S \) and

(ii) \( \theta(id_X) = 1_X \) for all \( X \in \mathbb{B}^*(X) := B(X \xrightarrow{id_X} X) \) the unit element.

Then \( \theta(f) \) is called an \textit{orientation} of \( f \). (In \textit{[3] Part I, §2.6.2 Definition} it is called a \textit{canonical orientation} of \( f \), but in this paper it shall be simply called an orientation.)

\textbf{Gysin homomorphisms:} Note that such an orientation makes the covariant functor \( \mathbb{B}_*(X) \) a contravariant functor for morphisms in \( S \), and also makes the contravariant functor \( \mathbb{B}^* \) a covariant functor for morphisms in \( \mathcal{C} \cap S \): Indeed,

1. As to the covariant functor \( \mathbb{B}_*(X) \): For a morphism \( f : X \to Y \in S \) and the orientation \( \theta \) on \( S \) the following \textit{Gysin homomorphism}

\[ f^* : \mathbb{B}_*(Y) \to \mathbb{B}_*(X) \quad \text{defined by} \quad f^*(\alpha) := \theta(f) \circ \alpha \]

is \textit{contravariantly functorial}.

2. As to contravariant functor \( \mathbb{B}^* \): For a fiber square (which is an independent square by hypothesis)

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{id_X} & & \downarrow{id_Y} \\
X & \xrightarrow{f} & Y,
\end{array}
\]

where \( f \in \mathcal{C} \cap S \), the following \textit{Gysin homomorphism}

\[ f_* : \mathbb{B}^*(X) \to \mathbb{B}^*(Y) \quad \text{defined by} \quad f_*(\alpha) := f_*(\alpha \circ \theta(f)) \]

is \textit{covariantly functorial}.

The notation should carry the information of \( S \) and the orientation \( \theta \), but it will be usually omitted if it is not necessary to be mentioned. Note that the above conditions (i) and (ii) of Definition \textit{2.2} are certainly necessary for the above Gysin homomorphisms to be functorial.

**Definition 2.4.** (i) Let \( S \) be another class of maps called “specialized maps” (e.g., smooth maps in algebraic geometry) in \( \mathcal{V} \), which is closed under composition, closed under base change and containing all identity maps. Let \( \mathbb{B} \) be a bivariant theory. If \( S \) has orientations in \( \mathbb{B} \), then we say that \( S \) is \( \mathbb{B} \)-oriented and an element of \( S \) is called a \( \mathbb{B} \)-oriented morphism. (Of course \( S \) is also a class of confined maps, but since we consider the above extra condition of \( \mathbb{B} \)-orientation on \( S \), we give a different name to \( S \).)

(ii) Let \( S \) be as in (i). Let \( \mathbb{B} \) be a bivariant theory and \( S \) be \( \mathbb{B} \)-oriented. Furthermore, if the orientation \( \theta \) on \( S \) satisfies that for an independent square with \( f \in S \)

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

then \( \theta(f) \) is a \textit{canonical orientation} of \( f \).

\textbf{Remark:} Since there is no consistent naming for \( \gamma \), this might be a confusing notation.
the following condition holds: \( \theta(f') = g^* \theta(f) \), (which means that the orientation \( \theta \) preserves the pullback operation), then we call \( \theta \) a stable orientation and say that \( S \) is stably \( \mathcal{B} \)-oriented and an element of \( S \) is called a stably \( \mathcal{B} \)-oriented morphism.

The following theorem is about the existence of a universal one of the bivariant theories for a given category \( V \) with a class \( C \) of confined morphisms, a class of independent squares and a class \( S \) of specialized morphisms.

**Theorem 2.4.** (\textsuperscript{[14]} Theorem 3.1) (A universal bivariant theory) Let \( V \) be a category with a class \( C \) of confined morphisms, a class of independent squares and a class \( S \) of specialized maps. We define

\[
\mathbb{M}^C_S(X \xrightarrow{f} Y)
\]

to be the free abelian group generated by the set of isomorphism classes of confined morphisms \( h : W \to X \) such that the composite of \( h \) and \( f \) is a specialized map:

\[
h \in C \quad \text{and} \quad f \circ h : W \to Y \in S.
\]

1. The association \( \mathbb{M}^C_S \) is a bivariant theory if the three bivariant operations are defined as follows:
   
   (a) **Product:** For morphisms \( f : X \to Y \) and \( g : Y \to Z \), the product operation

   \[
   \bullet : \mathbb{M}^C_S(X \xrightarrow{f} Y) \otimes \mathbb{M}^C_S(Y \xrightarrow{g} Z) \to \mathbb{M}^C_S(X \xrightarrow{gf} Z)
   \]

   is defined by

   \[
   [V \xrightarrow{h} X] \bullet [W \xrightarrow{k} Y] := [V' \xrightarrow{h \circ k'} X]
   \]

   and extended linearly, where we consider the following fiber squares

   \[
   \begin{array}{ccc}
   V' & \xrightarrow{h'} & X' \\
   \downarrow{k'} & & \downarrow{k} \\
   V & \xrightarrow{h} & X
   \end{array}
   \begin{array}{ccc}
   & & \xrightarrow{f'} & W \\
   \downarrow{f} & & \downarrow{g} \\
   & & \xrightarrow{g} & Z.
   \end{array}
   \]

   (b) **Pushforward:** For morphisms \( f : X \to Y \) and \( g : Y \to Z \) with \( f \) confined, the pushforward operation

   \[
   f_* : \mathbb{M}^C_S(X \xrightarrow{g \circ f} Z) \to \mathbb{M}^C_S(Y \xrightarrow{g} Z)
   \]

   is defined by

   \[
   f_* \left([V \xrightarrow{h} X]\right) := [V' \xrightarrow{f \circ h} Y]
   \]

   and extended linearly.

   (c) **Pullback:** For an independent square

   \[
   \begin{array}{ccc}
   X' & \xrightarrow{g'} & X \\
   \downarrow{f'} & & \downarrow{f} \\
   Y' & \xrightarrow{g} & Y,
   \end{array}
   \]

   the pullback operation

   \[
   g^* : \mathbb{M}^C_S(X \xrightarrow{f} Y) \to \mathbb{M}^C_S(X' \xrightarrow{g} Y')
   \]

   is defined by

   \[
   g^* \left([V \xrightarrow{h} X]\right) := [V' \xrightarrow{h'} X']
   \]
and extended linearly, where we consider the following fiber squares:

\[
\begin{array}{ccc}
V' & \xrightarrow{g''} & V \\
\downarrow h' & & \downarrow h \\
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y.
\end{array}
\]

(2) Let \(BT\) be a class of bivariant theories \(B\) on the same category \(V\) with a class \(C\) of confined morphisms, a class of independent squares and a class \(S\) of specialized maps. Let \(S\) be nice canonical \(B\)-orientable for any bivariant theory \(B \in BT\). Then, for each bivariant theory \(B \in BT\) there exists a unique Grothendieck transformation

\[
\gamma_B : M_C^S \rightarrow B
\]

such that for a specialized morphism \(f : X \rightarrow Y \in S\) the homomorphism \(\gamma_B : M_C^S(X \xrightarrow{f} Y) \rightarrow B(X \xrightarrow{f} Y)\) satisfies the normalization condition that

\[
\gamma_B([X \xrightarrow{id_X} X]) = \theta_B(f).
\]

3. ORIENTED BIVARIANT THEORY AND A UNIVERSAL ORIENTED BIVARIANT THEORY

Levine–Morel’s algebraic cobordism is the universal one among the so-called oriented Borel–Moore functors with products for algebraic schemes. Here “oriented” means that the given Borel–Moore functor \(H_*\) is equipped with the endomorphism \(\tilde{c}_1(L) : H_*(X) \rightarrow H_*(X)\) for a line bundle \(L\) over the scheme \(X\). Motivated by this “orientation” (which is different from the one given in Definition 2.2, but we still call this “orientation” using a different symbol so that the reader will not be confused with terminologies), in [14, §4] we introduce an orientation to bivariant theories for any category, using the notion of fibered categories in abstract category theory (e.g., see [12]) and such a bivariant theory equipped with such an orientation (Chern operator) is called an oriented bivariant theory.

**Definition 3.1.** Let \(L\) be a fibered category over \(V\). An object in the fiber \(L(X)\) over an object \(X \in V\) is called an “fiber-object over \(X\)”, abusing words, and denoted by \(L, M, \ldots\).

**Definition 3.2.** ([14, Definition 4.2]) (an oriented bivariant theory) Let \(B\) be a bivariant theory on a category \(V\).

(1) For a fiber-object \(L\) over \(X\), the “operator” on \(B\) associated to \(L\), denoted by \(\phi(L)\), is defined to be an endomorphism

\[
\phi(L) : B(X \xrightarrow{f} Y) \rightarrow B(X \xrightarrow{f} Y)
\]

which satisfies the following properties:

(O-1) **identity:** If \(L\) and \(L'\) are line bundles over \(X\) and isomorphic (i.e., if \(f : L \rightarrow X\) and \(f' : L' \rightarrow X\), then there exists an isomorphism \(i : L \rightarrow L'\) such that \(f = f' \circ i\)), then we have

\[
\phi(L) = \phi(L') : B(X \xrightarrow{f} Y) \rightarrow B(X \xrightarrow{f} Y).
\]

(O-2) **commutativity:** Let \(L\) and \(L'\) be two fiber-objects over \(X\), then we have

\[
\phi(L) \circ \phi(L') = \phi(L') \circ \phi(L) : B(X \xrightarrow{f} Y) \rightarrow B(X \xrightarrow{f} Y).
\]
(O-3) **compatibility with product:** For morphisms \( f : X \to Y \) and \( g : Y \to Z \), \( \alpha \in B(X \xrightarrow{L} Y) \) and \( \beta \in B(Y \xrightarrow{M} Z) \), a fiber-object \( L \) over \( X \) and a fiber-object \( M \) over \( Y \), we have
\[
\phi(L)(\alpha \bullet \beta) = \phi(L)(\alpha) \bullet \beta, \quad \phi(f^*M)(\alpha \bullet \beta) = \alpha \bullet \phi(M)(\beta).
\]

(0-4) **compatibility with pushforward:** For a confined morphism \( f : X \to Y \) and a fiber-object \( M \) over \( Y \), we have
\[
f_* (\phi(f^*M)(\alpha)) = \phi(M)(f_* \alpha).
\]

(0-5) **compatibility with pullback:** For an independent square and a fiber-object \( L \) over \( X \)
\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]
we have
\[
g^* (\phi(L)(\alpha)) = \phi(g'^*L)(g^* \alpha).
\]

The above operator is called an “orientation” and a bivariant theory equipped with such an orientation is called an **oriented bivariant theory**, denoted by \( \mathcal{OB} \).

(2) An **oriented Grothendieck transformation** between two oriented bivariant theories is a Grothendieck transformation which preserves or is compatible with the operator, i.e., for two oriented bivariant theories \( \mathcal{OB} \) with an orientation \( \phi \) and \( \mathcal{OB}' \) with an orientation \( \phi' \) the following diagram commutes
\[
\begin{array}{ccc}
\mathcal{OB}(X \xrightarrow{L} Y) & \xrightarrow{\phi(L)} & \mathcal{OB}(X \xrightarrow{L} Y) \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
\mathcal{OB}'(X \xrightarrow{L} Y) & \xrightarrow{\phi'(L)} & \mathcal{OB}'(X \xrightarrow{L} Y).
\end{array}
\]

**Theorem 3.3.** ([14, Theorem 4.6]) **(A universal oriented bivariant theory)** Let \( V \) be a category with a class \( C \) of confined morphisms, a class of independent squares, a class \( S \) of specialized morphisms and \( L \) a fibered category over \( V \). We define
\[
\mathcal{OM}_{C}^{S}(X \xrightarrow{L} Y)
\]
to be the free abelian group generated by the set of isomorphism classes of cobordism cycles over \( X \)
\[
[V \xrightarrow{h} X; L_1, L_2, \cdots, L_r]
\]
such that \( h \in C \), \( f \circ h : W \to Y \in S \) and \( L_i \) a fiber-object over \( V \).

1. The association \( \mathcal{OM}_{C}^{S} \) becomes an oriented bivariant theory if the four operations are defined as follows:
   a. **Orientation** \( \Phi \): For a morphism \( f : X \to Y \) and a fiber-object \( L \) over \( X \), the operator
   \[
   \phi(L) : \mathcal{OM}_{C}^{S}(X \xrightarrow{L} Y) \to \mathcal{OM}_{C}^{S}(X \xrightarrow{L} Y)
   \]
is defined by
   \[
   \phi(L)([V \xrightarrow{h} X; L_1, L_2, \cdots, L_r]) := [V \xrightarrow{h} X; L_1, L_2, \cdots, L_r, h^*L],
   \]
   and extended linearly.

(b) **Product:** For morphisms \( f : X \to Y \) and \( g : Y \to Z \), the product operation

\[ \bullet : \mathcal{OM}_S^C(X \overset{f}{\to} Y) \otimes \mathcal{OM}_S^C(Y \overset{g}{\to} Z) \to \mathcal{OM}_S^C(X \overset{gf}{\to} Z) \]

is defined as follows: The product is defined by

\[ [V \overset{h}{\to} X; L_1, \cdots, L_r] \bullet [W \overset{k}{\to} Y; M_1, \cdots, M_s] := [V' \overset{k''}{\to} X; k''^*L_1, \cdots, k''^*L_r, (f' \circ h')^*M_1, \cdots, (f' \circ h')^*M_s] \]

and extended bilinearly. Here we consider the following fiber squares

\[ V' \overset{h'}{\to} X' \overset{f'}{\to} W \]

\[ V \quad \overset{h}{\to} \quad X \quad \overset{f}{\to} \quad Y \quad \overset{g}{\to} \quad Z. \]

(c) **Pushforward:** For morphisms \( f : X \to Y \) and \( g : Y \to Z \) with \( f \) confined, the pushforward operation

\[ f_* : \mathcal{OM}_S^C(X \overset{gf}{\to} Z) \to \mathcal{OM}_S^C(Y \overset{g}{\to} Z) \]

is defined by

\[ f_* \left( [V \overset{h}{\to} X; L_1, \cdots, L_r] \right) := [V \overset{f \circ h}{\to} Y; L_1, \cdots, L_r] \]

and extended linearly.

(d) **Pullback:** For an independent square

\[ X' \overset{g'}{\to} X \]

\[ f' \downarrow \quad \quad \quad f \downarrow \]

\[ Y' \overset{g}{\to} Y; \]

the pullback operation

\[ g^* : \mathcal{OM}_S^C(X \overset{f}{\to} Y) \to \mathcal{OM}_S^C(X' \overset{f'}{\to} Y') \]

is defined by

\[ g^* \left( [V \overset{h}{\to} X; L_1, \cdots, L_r] \right) := [V' \overset{h^*}{\to} X'; g'^*L_1, \cdots, g'^*L_r] \]

and extended linearly, where we consider the following fiber squares:

\[ V' \overset{g''}{\to} V \]

\[ h' \downarrow \quad \quad \quad h \downarrow \]

\[ X' \overset{g'}{\to} X \]

\[ f' \downarrow \quad \quad \quad f \downarrow \]

\[ Y' \overset{g}{\to} Y. \]
The following proposition is a generalization of \cite[Proposition 2.3]{14} and its proof is done in the same way as in the proof of \cite[Proposition 2.3]{14}, using the axioms $A_{12}$, $A_{23}$ and $A_{123}$, so omitted, or left for the reader.

**Proposition 3.4.** Let $\mathbb{B}$ a commutative bivariant theory and we restrict $\mathbb{B}$ to the over category $\mathcal{V}/S$. Let us define the exterior product

$$\times_S : \mathbb{B}(X \overset{\pi_X}{\longrightarrow} S) \times \mathbb{B}(Y \overset{\pi_Y}{\longrightarrow} S) \to \mathbb{B}(X \times_S Y \overset{\pi_{X \times_S Y}}{\longrightarrow} S)$$

by $\alpha \times_S \beta := \pi_Y^\ast \alpha \bullet \beta = (\pi_X^\ast \beta \bullet \alpha)$ (note that $\mathbb{B}$ is commutative), where we consider the following independent square

$$\begin{array}{c}
X \times_S Y \xrightarrow{p_2} Y \\
p_1 \downarrow \quad \downarrow \pi_Y \\
X \xrightarrow{\pi_X} S.
\end{array}$$

The morphism $\pi_{X \times_S Y} : X \times_S Y \to S$ is the composite $\pi_Y \circ p_2 = \pi_X \circ p_1$.

(1) For two confined morphisms $f : X_1 \to X_2$ from $\pi_{X_1} : X_1 \to S$ to $\pi_{X_2} : X_2 \to S$ and $g : Y_1 \to Y_2$ from $\pi_{Y_1} : Y_1 \to S$ to $\pi_{Y_2} : Y_2 \to S$, we have that for $\alpha_1 \in \mathbb{B}(X_1 \overset{\pi_{X_1}}{\longrightarrow} S)$ and $\beta_1 \in \mathbb{B}(Y_1 \overset{\pi_{Y_1}}{\longrightarrow} S)$

$$f_\ast \alpha_1 \times_S g_\ast \beta_1 = (f \times_S g)_\ast (\alpha_1 \times_S \beta_1),$$

namely, the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{B}(X_1 \overset{\pi_{X_1}}{\longrightarrow} S) \times \mathbb{B}(Y_1 \overset{\pi_{Y_1}}{\longrightarrow} S) & \xrightarrow{x \times_S} & \mathbb{B}(X_1 \times_S Y_1 \overset{\pi_{X_1 \times_S Y_1}}{\longrightarrow} S) \\
\downarrow f_\ast \times_S g_\ast & & \downarrow (f \times_S g)_\ast \\
\mathbb{B}(X_2 \overset{\pi_{X_2}}{\longrightarrow} S) \times \mathbb{B}(Y_2 \overset{\pi_{Y_2}}{\longrightarrow} S) & \xrightarrow{x \times_S} & \mathbb{B}(X_2 \times_S Y_2 \overset{\pi_{X_2 \times_S Y_2}}{\longrightarrow} S).
\end{array}
\]
(2) If the above two morphisms \( f \) and \( g \) are specialized, then for \( \alpha_2 \in \mathbb{B}(X_2 \xrightarrow{\pi_{X_2}} S) \) and \( \beta_2 \in \mathbb{B}(Y_2 \xrightarrow{\pi_{Y_2}} S) \), we have

\[
f^* \alpha_2 \times_S g^* \beta_2 = (f \times_S g)^*(\alpha_2 \times_S \beta_2),
\]

namely, the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{B}(X_2 \xrightarrow{\pi_{X_2}} S) \times \mathbb{B}(Y_2 \xrightarrow{\pi_{Y_2}} S) & \xrightarrow{\times_x} & \mathbb{B}(X_2 \times_S Y_2 \xrightarrow{\pi_{X_2 \times_S Y_2}} S) \\
\downarrow{f^* \times g^*} & & \downarrow{(f \times_S g)^*} \\
\mathbb{B}(X_1 \xrightarrow{\pi_{X_1}} S) \times \mathbb{B}(Y_1 \xrightarrow{\pi_{Y_1}} S) & \xrightarrow{\times_x} & \mathbb{B}(X_1 \times_S Y_1 \xrightarrow{\pi_{X_1 \times_S Y_1}} S).
\end{array}
\]

Here we use the following independent squares:

\[
\begin{array}{ccc}
X_1 \times_S Y_1 \xrightarrow{f''} X_2 \times_S Y_1 & \xrightarrow{p_2} & Y_1 \\
\downarrow{g''} & & \downarrow{g} \\
X_1 \times_S Y_2 \xrightarrow{f'} X_2 \times_S Y_2 & \xrightarrow{p_2} & Y_2 \\
\downarrow{p_1} & & \downarrow{\pi_{Y_2}} \\
X_1 & \xrightarrow{f} & X_2 & \xrightarrow{\pi_{X_2}} & S.
\end{array}
\]

And \( \pi_{X_1 \times_S Y_1} \) is any composite of the morphisms from \( X_1 \times_S Y_1 \) to \( S \), e.g., \( \pi_{X_1 \times_S Y_1} = \pi_{Y_2} \circ g \circ p_2 \circ f'' \) and \( f \times_S g : X_1 \times_S Y_1 \to X_2 \times_S Y_2 \) is the composite \( g' \circ f'' = r' \circ g'' \).

**Remark 3.5.** If \( S \) is a terminal object \( pt \) (e.g., a point \( pt \) in the category of topological spaces, complex algebraic varieties, schemes, etc.), then the over category \( \mathcal{V}/S = \mathcal{V}/pt \) is the same as \( \mathcal{V} \) and we get [14, Proposition 2.3].

**Theorem 3.6.** Let \( \mathbb{B} \) be a commutative bivariant theory and we restrict \( \mathbb{B} \) to the over category \( \mathcal{V}/S \). Then we have the following.

(D1)' Let \( AB \) be the category of abelian groups. Then on the subcategory \( \mathcal{V}'/S \subset \mathcal{V}/S \) of confined morphisms, \( \mathbb{B}^*(\mathcal{V}'/S) : \mathcal{V}'/S \to AB \) is a covariant functor. Here, for a confined morphism \( f : X \to Y \) from \( \pi_X : X \to S \) to \( \pi_Y : Y \to S \), the pushforward

\[
f_* : \mathbb{B}^*(X \xrightarrow{\pi_X} S) \to \mathbb{B}^*(Y \xrightarrow{\pi_Y} S)
\]

is defined by the bivariant pushforward.

(D2) For a specialized morphism \( f : X \to Y \) from \( \pi_X : X \to S \) to \( \pi_Y : Y \to S \), the pullback

\[
f^* : \mathbb{B}^*(Y \xrightarrow{\pi_Y} S) \to \mathbb{B}^*(X \xrightarrow{\pi_X} S)
\]

is defined by \( f^*(\alpha) := \theta(f) \cdot \alpha \).

(D3) For a fiber-object \( L \) over \( X \), we have the operator

\[
\phi(L) : \mathbb{B}^*(X \xrightarrow{\pi_X} S) \to \mathbb{B}^*(X \xrightarrow{\pi_X} S).
\]

(D4) The above external product

\[
\times_S : \mathbb{B}^*(X \xrightarrow{\pi_X} S) \times \mathbb{B}^*(Y \xrightarrow{\pi_Y} S) \to \mathbb{B}^*(X \times_S Y \xrightarrow{\pi_{X \times_S Y}} S)
\]

is commutative, associative and admits \( 1 \in \mathbb{B}^*(S \xrightarrow{\text{id}_S} S) \).

(A1) For specialized morphisms \( f : X \to Y \) from \( \pi_X : X \to S \) to \( \pi_Y : Y \to S \) and \( g : Y \to Z \) from \( \pi_Y : Y \to S \) to \( \pi_Z : Z \to S \), we have

\[
(g \circ f)^* = f^* \circ g^* : \mathbb{B}^*(Z \xrightarrow{\pi_Z} S) \to \mathbb{B}^*(X \xrightarrow{\pi_X} S).
\]
(A2) For an independent square

\[
\begin{array}{ccc}
W & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y & \xrightarrow{g} & Z
\end{array}
\]

where \( f \in \mathcal{C} \) is confined and \( g \in \mathcal{S} \) is specialized, we have that \( g^* \circ f_* = (f')^* (g')^* \), i.e. the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{B}^*(X, \xrightarrow{\pi_X} S) & \xrightarrow{g'^*} & \mathbb{B}^*(W, \xrightarrow{\pi_W} S) \\
\downarrow{f_*} & & \downarrow{(f')^*} \\
\mathbb{B}^*(Z, \xrightarrow{\pi_Z} S) & \xrightarrow{g^*} & \mathbb{B}^*(Y, \xrightarrow{\pi_Y} S).
\end{array}
\]

(A3) For a confined morphism \( f : X \to Y \) from \( \pi_X : X \to S \) to \( \pi_Y : Y \to S \) and a fiber-object \( M \)

\[ f_\ast \circ \phi(f^* M) = \phi(M) \circ f_\ast : \mathbb{B}^*(X, \xrightarrow{\pi_X} S) \to \mathbb{B}^*(Y, \xrightarrow{\pi_Y} S). \]

(A4) For a specialized morphism \( f : X \to Y \) from \( \pi_X : X \to S \) to \( \pi_Y : Y \to S \) and a fiber-object \( M \)

\[ \phi(f^* M) \circ f^* = f^* \circ \phi(M) : \mathbb{B}^*(Y, \xrightarrow{\pi_Y} S) \to \mathbb{B}^*(X, \xrightarrow{\pi_X} S). \]

(A5) Let \( L \) and \( L' \) be two fiber-objects over \( X \), then we have

\[ \phi(L) \circ \phi(L') = \phi(L') \circ \phi(L) : \mathbb{B}^*(X, \xrightarrow{\pi_X} S) \to \mathbb{B}^*(X, \xrightarrow{\pi_X} S). \]

Moreover, if \( L \) and \( L' \) are isomorphic, then we have that \( \phi(L) = \phi(L') \).

(A6) For proper morphisms \( f : X_1 \to X_2 \) from \( \pi_{X_1} : X_1 \to S \) to \( \pi_{X_2} : X_2 \to S \) and \( g : Y_1 \to Y_2 \)

\[ f_* \circ \alpha \times_S \circ g_* \circ \beta = (f \times_S g)_* (\alpha \times_S \beta). \]

(A7) For smooth morphisms \( f : X_1 \to X_2 \) from \( \pi_{X_1} : X_1 \to S \) to \( \pi_{X_2} : X_2 \to S \) and \( g : Y_1 \to Y_2 \)

\[ f^* \alpha \times_S \circ g_* \circ \beta = (f \times_S g)^* (\alpha \times_S \beta). \]

(A8) For a fiber-object \( L \) and \( \alpha \in \mathbb{B}^*(X, \xrightarrow{\pi_X} S) \) and \( \beta \in \mathbb{B}^*(Y, \xrightarrow{\pi_Y} S) \),

\[ \phi(L) \circ \alpha \times_S \beta = \phi(p_1^* L) (\alpha \times_S \beta) \in \mathbb{B}^*(X \times_S Y, \xrightarrow{\pi_X \times_S \pi_Y} S). \]

Here we use the fiber square

\[
\begin{array}{ccc}
X \times_S Y & \xrightarrow{p_2} & Y \\
\downarrow{p_1} & & \downarrow{\pi_Y} \\
X & \xrightarrow{\pi_X} & S.
\end{array}
\]

We also point out that the contravariant functor \( \mathbb{B}^*(X) = \mathbb{B}^*(X, \xrightarrow{id_X} X) \) have similar properties as above:

**Proposition 3.7.** Let \( \mathbb{B} \) a commutative bivariant theory on the category \( \mathcal{V} \) with \( \mathcal{C}, \mathcal{S}, \mathcal{L} \) as above.
(D1) For confined and specialized morphisms in \( C \cap S \), i.e., for confined and nice canonical \( B \)-orientable morphisms \( f : X \to Y \), the Gysin (pushforward) homomorphisms 
\[ f_* : \mathbb{B}^*(X) \to \mathbb{B}^*(Y) \]

are covariantly functorial.

(D2) For any morphisms \( f : X \to Y \), we have the pullback homomorphism:
\[ f^* : \mathbb{B}^*(Y) \to \mathbb{B}^*(X) \]

(D3) For a fiber-object \( L \) over \( X \) we have the operator
\[ \phi(L) : \mathbb{B}^*(X) \to \mathbb{B}^*(X) \].

(D4) We have the exterior product
\[ \times : \mathbb{B}^*(X) \times \mathbb{B}^*(Y) \to \mathbb{B}^*(X \times Y) \]

which is commutative, associative and admits \( 1 \in \mathbb{B}^*(X) \).

(A1) For any morphisms \( f : X \to Y \) and \( g : Y \to Z \) we have
\[ (g \circ f)^* = f^* \circ g^*. \]

(A2) For an independent square
\[
\begin{array}{ccc}
X' & \to & X \\
\downarrow g' & & \downarrow f \\
Y' & \to & Y \\
\end{array}
\]

with \( g \in C \cap S \), we have \( f^* \circ g_* = g'_* \circ (f')^* \), i.e., the following diagram commutes:
\[
\begin{align*}
\mathbb{B}^*(Y') & \xrightarrow{f'^*} \mathbb{B}^*(X') \\
g_* & \downarrow \quad \downarrow g'_*
\end{align*}
\]

(A3) For a confined and specialized morphism \( f : X \to Y \) and a fiber-object \( M \) over \( Y \), we have
\[ f_* \circ \phi(f^* M) = \phi(M) \circ f_* : \mathbb{O}\mathbb{B}^*(X) \to \mathbb{O}\mathbb{B}^*(Y). \]

(A4) For any morphism \( f : X \to Y \) and fiber-object \( M \) over \( Y \), we have
\[ \phi(f^* M) \circ f^* = f^* \circ \phi(M) : \mathbb{O}\mathbb{B}^*(Y) \to \mathbb{O}\mathbb{B}^*(X). \]

(A5) Let \( L \) and \( L' \) be two fiber-objects over \( X \), then we have
\[ \phi(L) \circ \phi(L') = \phi(L') \circ \phi(L) : \mathbb{O}\mathbb{B}^*(X) \to \mathbb{O}\mathbb{B}^*(X), \]

and if \( L \) and \( L' \) are isomorphic, then we have that \( \phi(L) = \phi(L') \).

(A6) For both confined and specialized morphisms \( f : X_1 \to X_2 \) and \( g : Y_1 \to Y_2 \), and \( \alpha \in \mathbb{O}\mathbb{B}^*(X_1) \) and \( \beta \in \mathbb{O}\mathbb{B}^*(Y_1) \) we have we have
\[ f_* \alpha \times g_* \beta = (f \times g)_* (\alpha \times \beta). \]

(A7) For any morphisms \( f : X_1 \to X_2 \) and \( g : Y_1 \to Y_2 \), and \( \alpha \in \mathbb{O}\mathbb{B}^*(X_2) \) and \( \beta \in \mathbb{O}\mathbb{B}^*(Y_2) \) we have
\[ f^* \alpha \times g^* \beta = (f \times g)^* (\alpha \times \beta). \]

(A8) For fiber-object \( L \) over \( X \) and for \( \alpha \in \mathbb{O}\mathbb{B}^*(X) \) and \( \beta \in \mathbb{O}\mathbb{B}^*(Y) \), we have
\[ \phi(L)(\alpha) \times \beta = \phi(p_1^* L)(\alpha \times \beta). \]

Here \( p_1 : X \times Y \to X \) is the projection.
Remark 3.8.  
(1) Let $S = pt$. Then in the above proposition $\mathbb{B}^*(X \xrightarrow{\pi_X} S) = \mathbb{B}^*(X \xrightarrow{\pi_X} pt)$ is replaced by $\mathbb{B}_s(X)$ and all these properties exactly correspond to the properties $(D1), \cdots (D4)$ and $(A1), \cdots , (A8)$ of Levine-Morel’s oriented Borel–Moore functor with products [6], except $(D1)$ which requires that the functor $\mathbb{B}_s(X)$ is supposed to be additive, i.e., $\mathbb{B}_s(X \sqcup Y) = \mathbb{B}_s(X) \oplus \mathbb{B}_s(Y)$. In order to deal with such requirements, in [14, Remark 2.5] we introduce the notion of additive bivariant theory.

(2) Our oriented bivariant theory is a kind of bivariant-theoretic generalization of Levine-Morel’s oriented Borel–Moore functor with products. In fact, as shown in [14, Proposition 2.4], having observed that given a bivariant theory $\mathbb{B}$ both the covariant functor $\mathbb{B}_s(X)$ and the contravariant functor $\mathbb{B}^*(X)$ satisfy the properties $(D1'), (D2), (D4), (A1), (A2), (A6), (A7)$ (having nothing to do with orientation $\phi(L)$) was a motivation for introducing an oriented bivariant theory so that the oriented bivariant theory satisfies the other properties $(D3), (A3), (A4), (A5), (A8)$ involving the orientation $\phi(L)$.

(3) Even in the case of an oriented bivariant theory $\mathbb{O}B$ for a general situation, the special functors $\mathbb{O}B^*(X \xrightarrow{\pi_X} S)$ of $\mathbb{V}, \mathbb{O}B_*(X)$ and $\mathbb{O}B_s(X)$ shall be also called oriented Borel–Moore functors with products.

Corollary 3.9. The abelian group $\mathbb{OM}^C_{S*}(X) := \mathbb{OM}^C_S(X \rightarrow pt)$ is the free abelian group generated by the set of isomorphism classes of cobordism cycles

$$[V \xrightarrow{h_V} X; L_1, \cdots , L_r]$$

such that $h_V : V \rightarrow X \in C$ and $V \rightarrow pt$ is a specialized map in $S$ and $L_i$ is a fiber-object over $V$. The abelian group $\mathbb{OM}^C_{S*}(X) := \mathbb{OM}^C_S(X \xrightarrow{id_X} X)$ is the free abelian group generated by the set of isomorphism classes of cobordism cycles

$$[V \xrightarrow{h_V} X; L_1, \cdots , L_r]$$

such that $h_V : V \rightarrow X \in C \cap S$ and $L_i$ is fiber-object over $V$. Both functor $\mathbb{OM}^C_{S*}$ and $\mathbb{OM}^C_{S*}$ are oriented Borel–Moore functors with products in the sense of Levine–Morel.

Corollary 3.10. (A universal oriented Borel–Moore functor with products) Let $\mathbb{B}T$ be a class of oriented additive bivariant theories $\mathbb{B}$ on the same category $\mathbb{V}$ with a class $C$ of confined morphisms, a class of independent squares, a class $S$ of specialized maps and $\mathbb{L}$ a fibered category over $\mathbb{V}$. Let $\mathbb{S}$ be nice canonical $\mathbb{O}B$-orientable for any oriented bivariant theory $\mathbb{O}B \in \mathbb{O}BT$. Then, for each oriented bivariant theory $\mathbb{O}B \in \mathbb{O}BT$ with an orientation $\phi$,

(1) there exists a unique natural transformation of oriented Borel–Moore functors with products

$$\gamma_{\mathbb{O}B_s} : \mathbb{OM}^C_{S*} \rightarrow \mathbb{O}B_s$$

such that if $\pi_X : X \rightarrow pt$ is in $S$

$$\gamma_{\mathbb{O}B_s}[X \xrightarrow{id_X} X; L_1, \cdots , L_r] = \phi(L) \circ \cdots \circ \phi(L_r)(\pi_X^*(1pt)), \text{ and}$$

(2) there exists a unique natural transformation of oriented Borel–Moore functors with products

$$\gamma_{\mathbb{O}B}^* : \mathbb{OM}^C_{S*} \rightarrow \mathbb{O}B^*$$

such that for any object $X$

$$\gamma_{\mathbb{O}B}[X \xrightarrow{id_X} X; L_1, \cdots , L_r] = \phi(L) \circ \cdots \circ \phi(L_r)(1_X).$$
4. A UNIVERSAL ORIENTED BIVARIANT THEORY ON SCHEMES

Now, from this section on, instead of considering a general situation we consider the category $\text{Sch}$ of schemes and the category of schemes over a fixed scheme $S$ is nothing but the over category $\text{Sch}/S$. Of course, we can consider the category $\text{Var}_\mathbb{C}$ of complex algebraic varieties and the over category $\text{Var}_\mathbb{C}/S$ over a fixed variety. In this context, a confined morphism is a proper morphism, a specialized morphism is a smooth morphism, an independent square is a fiber square or fiber product, and a fiber-object $L$ over $X$ is a line bundle over $X$.

In this setup, our universal oriented bivariant theory $\Omega M^*_S(X \rightarrow Y)$ shall be denoted by $Z^*(X \rightarrow Y)$ mimicking the notation used in [6].

**Definition 4.1.** We define

$$Z^*(X \xrightarrow{\pi_X} S)$$

to be the free abelian group generated by the set of isomorphism classes of cobordism cycles $[V \xrightarrow{h} X; L_1, L_2, \ldots, L_r]$ over $X$, where $L_i$ $(1 \leq i \leq r)$ is a line bundle over $V$, such that

1. $h : V \rightarrow X$ is proper,
2. the composite $\pi_X \circ h : V \rightarrow S$ is smooth.

So far we never pay attention to the grading, so we define the grading as follows:

**Definition 4.2.** The grading $i$ of the graded group $Z^i(X \xrightarrow{\pi_X} S)$ is defined by:

$$[V \xrightarrow{h} X; L_1, \ldots, L_r] \in Z^i(X \xrightarrow{\pi_X} S) \iff -i + r = \dim(\pi_X \circ h),$$

where $\dim(\pi_X \circ h)$ is the relative dimension of the smooth morphism $\pi_X \circ h$, i.e. the dimension of the (smooth) fiber of $\pi_X \circ h$, which is equal to $\dim V - \dim S$.

**Remark 4.3.** Such a grading is due to the requirement that for $S = \text{pt}$ we want to have $Z^i(X \xrightarrow{\pi_X} \text{pt}) = Z_{-i}(X)$ (see Definition 2.1). According to the definition ([6, Definition 2.1.6]) of grading of Levine-Morel’s algebraic pre-cobordism $Z_*(X)$, the degree (or dimension) of the cobordism cycle $[V \xrightarrow{h} X; L_1, \ldots, L_r] \in Z_*(X)$ is $\dim V - r$, i.e.

$$[V \xrightarrow{h} X; L_1, \ldots, L_r] \in Z_{-1}(X) \iff -i = \dim V - r,$$ namely, $-i + r = \dim V$.

Then we have the following theorem, which is just a rewritten of Theorem 3.3 with a bit more detailed information, in particular gradings and with a different notation for the “orientation” $\phi(L)$, however we write it down again for the sake of reader.

**Theorem 4.4 (14).** (A universal oriented bivariant theory on schemes)

1. The assignment $Z^*$ becomes an oriented bivariant theory if the bivariant operations are defined as follows:

   a. **Orientation** $\widetilde{c}_1$: For a morphism $f : X \rightarrow Y$ and a line bundle $L$ over $X$, the “Chern operator”

   $$\widetilde{c}_1(L) : Z^i(Y \xrightarrow{f} X) \rightarrow Z^{i+1}(X \xrightarrow{f} Y)$$

   is defined by

   $$\widetilde{c}_1(L)([V \xrightarrow{h} X; L_1, L_2, \ldots, L_r]) := [V \xrightarrow{h \circ f} X; L_1, L_2, \ldots, L_r, h^* L].$$

   b. **Product**: For morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, the product operation

   $$\cdot : Z^i(X \xrightarrow{f} Y) \otimes Z^j(Y \xrightarrow{g} Z) \rightarrow Z^{i+j}(X \xrightarrow{gf} Z)$$
is defined as follows: The product on generators is defined by

\[ [V \xrightarrow{h} X; L_1, \ldots, L_r] \bullet [W \xrightarrow{k} Y; M_1, \ldots, M_s] = [V' \xrightarrow{h*k''} X; k'^*L_1, \ldots, (f' \circ h')^*M_1, \ldots, (f' \circ h')^*M_s], \]

and it extends bilinearly. Here we consider the following fiber squares

\[
\begin{array}{c}
V' \xrightarrow{h'} X' \xrightarrow{f'} W \\
k'' \downarrow \quad k' \downarrow \quad k \downarrow \\
V \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z.
\end{array}
\]

(c) **Pushforward:** For morphisms \( f : X \to Y \) and \( g : Y \to Z \) with \( f \) a proper morphism, the pushforward

\[ f_* : \mathcal{Z}(X \xrightarrow{gf} Z) \to \mathcal{Z}(Y \xrightarrow{g} Z) \]

is defined by

\[ f_* \left( [V \xrightarrow{h} X; L_1, \ldots, L_r] \right) := [V \xrightarrow{f\circ h} Y; L_1, \ldots, L_r]. \]

(d) **Pullback:** For a fiber square

\[
\begin{array}{ccc}
V' & \xrightarrow{g'} & X' \\
\downarrow & \downarrow & \downarrow \\
Y' & \xrightarrow{g} & Y,
\end{array}
\]

the pullback

\[ g^* : \mathcal{Z}(X \xrightarrow{f} Y) \to \mathcal{Z}(X' \xrightarrow{f'} Y') \]

is defined by

\[ g^* \left( [V \xrightarrow{h} X; L_1, \ldots, L_r] \right) := [V' \xrightarrow{h'} X'; g'^*L_1, \ldots, g'^*L_r], \]

where we consider the following fiber squares:

\[
\begin{array}{c}
V' \xrightarrow{g''} V \\
h' \downarrow \quad \downarrow h \\
X' \xrightarrow{g'} X \\
f' \downarrow \quad \downarrow f \\
Y' \xrightarrow{g} Y.
\end{array}
\]

(2) Let \( \text{OBT} \) be a class of oriented bivariant theories \( \mathfrak{OB} \) on the category \( \text{Sch} \) of schemes with the same class \( \mathcal{C} \) of proper morphisms, the class of fiber squares, the class \( \mathcal{S} \) of smooth morphisms and \( \mathcal{L} \) consisting of line bundles over \( \text{Sch} \). Let \( \mathcal{S} \) be stably \( \mathfrak{OB} \)-oriented for any oriented bivariant theory \( \mathfrak{OB} \in \text{OBT} \). Then, for each oriented bivariant theory \( \mathfrak{OB} \in \text{OBT} \) with an orientation \( \tilde{c}_1 \) there exists a unique oriented Grothendieck transformation

\[ \gamma_{\mathfrak{OB}} : \mathcal{Z} \to \mathfrak{OB} \]
such that for any \( f : X \to Y \in S \) the homomorphism \( \gamma_{OB} : \mathcal{Z}(X \xrightarrow{\Delta} Y) \to \mathcal{OB}(X \xrightarrow{\Delta} Y) \) satisfies the normalization condition that
\[
\gamma_{OB}(\mathcal{Z}(X \xrightarrow{id_S} X; L_1, \ldots, L_r)) = \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)(\theta_{OB}(f)).
\]

**Remark 4.5.** Here we point out that in the product operation \( \bullet \), the grading is correct. Indeed, \([V \xrightarrow{h} X; L_1, \ldots, L_r] \in \mathcal{Z}^j(X \xrightarrow{\Delta} Y) \) and \([W \xrightarrow{k} Y; M_1, \ldots, M_s] \in \mathcal{Z}^i(Y \xrightarrow{\Delta} Z) \) imply that we have
\[
-i + r = \dim(f \circ h) \quad \text{and} \quad -j + s = \dim(g \circ k).
\]

Hence
\[
-i + r + (-j + s) = \dim(f \circ h) + \dim(g \circ k).
\]

Thus we have
\[
-(i + j) + (r + s) = \dim(f \circ h) + \dim(g \circ k)
\]
\[
= \dim(f' \circ h') + \dim(g \circ k) \quad \text{(the relative dimension is preserved by the pullback)}
\]
\[
= \dim(g \circ f \circ h \circ k'' \circ g \circ f \circ h \circ k'')
\]
\[
= \dim((g \circ f) \circ (h \circ k'')).
\]

Thus we have
\[
[V \xrightarrow{h} X; L_1, \ldots, L_r] \bullet [W \xrightarrow{k} Y; M_1, \ldots, M_s]
\]
\[
= [V' \xrightarrow{\text{h} \circ \text{k}''} X; k''^*L_1, \ldots, k''^*L_r, (f' \circ h')^*M_1, \ldots, (f' \circ h')^*M_s] \in \mathcal{Z}^{i+j}(X \xrightarrow{g \circ f} Z)
\]

Then as in Proposition 3.4 we have the following external product on the over category \( \text{Sch}/S \):
\[
\times_S : \mathcal{Z}^i(X \xrightarrow{\pi_X} S) \times \mathcal{Z}^j(Y \xrightarrow{\pi_Y} S) \to \mathcal{Z}^{i+j}(X \times_S Y \xrightarrow{\pi_X \times \pi_Y} S),
\]

which we recall is defined by: for \( \alpha \in \mathcal{Z}^i(X \xrightarrow{\pi_X} S) \) and \( \beta \in \mathcal{Z}^j(Y \xrightarrow{\pi_Y} S) \)
\[
\pi_{\tilde{Y}}(\alpha) \bullet \beta
\]
where we consider the fiber square
\[
\begin{array}{ccc}
X \times_S Y & \xrightarrow{p_2} & Y \\
\downarrow p_1 & & \downarrow \pi_Y \\
X & \xrightarrow{\pi_X} & S.
\end{array}
\]

More precisely, we have
\[
[V \xrightarrow{h} X; L_1, \ldots, L_s] \times_S [W \xrightarrow{k} X; M_1, \ldots, M_i]
\]
\[
:= [V \times_S W \xrightarrow{\text{h} \circ \text{k}} X \times_S Y; (\tilde{p}_1 \tilde{k})^*L_1, \ldots, (\tilde{p}_1 \tilde{k})^*L_s, (\tilde{p}_2 \tilde{k})^*M_1, \ldots, (\tilde{p}_2 \tilde{k})^*M_i]
\]
\[
\begin{array}{ccc}
V \times_S W & \xrightarrow{\text{h} \circ \text{k}} & \widetilde{W} \xrightarrow{\text{p}_2} W \\
\downarrow \tilde{k} & & \downarrow \text{k} \\
\tilde{V} & \xrightarrow{\text{h}' \circ \tilde{k}} & X \times_D Y \xrightarrow{p_2} Y \\
\downarrow \tilde{p}_1 & & \downarrow p_1 \\
V & \xrightarrow{h} & X \xrightarrow{\pi_X} S.
\end{array}
\]

If we restrict the oriented bivariant theory \( \mathcal{Z}^* \) to the over category \( \text{Sch}/S \), we have the following theorem, which is just Theorem 3.6 rewritten with a bit more detailed information, in particular gradings and with a different notation for the “orientation” \( \phi(L) \), however we write it down again for the sake of reader.
Theorem 4.6. (cf. [6] Definition 2.1.2, Definition 2.1.10)]

(D1) Let $\mathcal{A}B$ be the category of abelian groups. Then on the subcategory $\text{Sch}'/S \subset \text{Sch}/S$ of proper morphisms, $Z^*(- \to S) : \text{Sch}'/S \to \mathcal{A}B$ is an additive functor. Here, for a proper morphism $f : X \to Y$ from $\pi_X : X \to S$ to $\pi_Y : Y \to S$, the pushforward

$$f_* : Z^i(X \xrightarrow{\pi_X} S) \to Z^i(Y \xrightarrow{\pi_Y} S)$$

defined by $f_*[V \xrightarrow{h} Y; L_1, \ldots, L_k] := [V \xrightarrow{f; h} X; L_1, \ldots, L_k]$ is well-defined.

(D2) For a smooth morphism $f : X \to Y$ from $\pi_X : X \to S$ to $\pi_Y : Y \to S$, the pullback

$$f^* : Z^i(Y \xrightarrow{\pi_Y} S) \to Z^{i-\dim f}(X \xrightarrow{\pi_X} S)$$

defined by $f^*[W \xrightarrow{h} Y; L_1, \ldots, L_r] := [W' \xrightarrow{k'} X; (f')^*L_1, \ldots, (f')^*L_r]$ is well-defined, where we use the following fiber square:

$$\begin{array}{ccc}
W' & \xrightarrow{f'} & W \\
\downarrow{k'} & & \downarrow{k} \\
X & \xrightarrow{f} & Y,
\end{array}$$

(D3) For a line bundle $L$ over $X$, the operator

$$\tilde{c}_1(L) : Z^i(X \xrightarrow{\pi_X} S) \to Z^{i+1}(X \xrightarrow{\pi_X} S)$$

defined by $\tilde{c}_1(L)([V \xrightarrow{h} X; L_1, \ldots, L_k]) := [V \xrightarrow{h} X; L_1, \ldots, L_k, h^*L]$ is well-defined.

(D4) The above external product

$$\times_S : Z^i(X \xrightarrow{\pi_X} S) \times Z^j(Y \xrightarrow{\pi_Y} S) \to Z^{i+j}(X \times_S Y \xrightarrow{\pi_X \times \pi_Y} S)$$

is commutative, associative and admits $1 := [S \xrightarrow{\text{id}} S] \in Z^0(S \xrightarrow{\text{id}} S)$.

(A1) For smooth morphisms $f : X \to Y$ from $\pi_X : X \to S$ to $\pi_Y : Y \to S$ and $g : Y \to Z$ from $\pi_Y : Y \to S$ to $\pi_Z : Z \to S$, we have

$$(g \circ f)^* = f^* \circ g^* : Z^i(Z \xrightarrow{\pi_Z} S) \to Z^{i-\dim f-\dim g}(X \xrightarrow{\pi_X} S).$$

(A2) For a fiber square

$$\begin{array}{ccc}
W & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y & \xrightarrow{g} & Z
\end{array}$$

where $f$ is proper and $g$ is smooth, we have that $g^* \circ f_* = (f')_*(g')^*$, i.e. the following diagram commutes:

$$\begin{array}{ccc}
Z^i(X \xrightarrow{\pi_X} S) & \xrightarrow{g^*} & Z^{i-\dim g'}(W \xrightarrow{\pi_W} S) \\
\downarrow{f_*} & & \downarrow{(f')_*} \\
Z^i(Z \xrightarrow{\pi_Z} S) & \xrightarrow{g^*} & Z^{i-\dim g}(Y \xrightarrow{\pi_Y} S)
\end{array}$$

Here we note that $\dim g = \dim g'$. 
(A3) For a proper morphism \( f : X \to Y \) from \( \pi_X : X \to S \) to \( \pi_Y : Y \to S \) and a line bundle \( M \) over \( Y \),
\[
f_* \circ c_1(f^*M) = c_1(M) \circ f_* : Z^i(X, \pi_X) \to Z^{i+1}(Y, \pi_Y).
\]

(A4) For a smooth morphism \( f : X \to Y \) from \( \pi_X : X \to S \) to \( \pi_Y : Y \to S \) and a line bundle \( M \) over \( Y \),
\[
c_1(f^*M) \circ f^* = f^* \circ c_1(M) : Z^i(Y, \pi_Y) \to Z^{i+1}(X, \pi_X).
\]

(A5) Let \( L \) and \( L' \) be two line bundles over \( X \), then we have
\[
c_1(L) \circ c_1(L') = c_1(L') \circ c_1(L) : Z^i(X, \pi_X) \to Z^{i+2}(X, \pi_X).
\]
Moreover, if \( L \) and \( L' \) are isomorphic, then we have that \( c_1(L) = c_1(L') \).

(A6) For proper morphisms \( f : X_1 \to X_2 \) from \( \pi_{X_1} : X_1 \to S \) to \( \pi_{X_2} : X_2 \to S \) and \( g : Y_1 \to Y_2 \)
from \( \pi_{Y_1} : Y_1 \to S \) to \( \pi_{Y_2} : Y_2 \to S \), and \( \alpha \in Z^*(X_1, \pi_{X_1}) \) and \( \beta \in Z^*(Y_1, \pi_{Y_1}) \), we have
\[
f_* \alpha \times_S g_* \beta = (f \times g)_* (\alpha \times \beta).
\]

(A7) For smooth morphisms \( f : X_1 \to X_2 \) from \( \pi_{X_1} : X_1 \to S \) to \( \pi_{X_2} : X_2 \to S \) and \( g : Y_1 \to Y_2 \)
from \( \pi_{Y_1} : Y_1 \to S \) to \( \pi_{Y_2} : Y_2 \to S \), and \( \alpha \in Z^*(X_1, \pi_{X_1}) \) and \( \beta \in Z^*(Y_2, \pi_{Y_2}) \), we have
\[
c_1(L) (\alpha) \times_S \beta = c_1(p_1^* L)(\alpha \times \beta) \in Z^{i+1}(X, \pi_X) \times_Y S = Z^{i+1}(X, \pi_X) \cdot S.
\]

Remark 4.7. We just make a remark on (D2). Let us consider the following commutative diagram

\[
\begin{array}{ccc}
W' & \xrightarrow{f'} & W \\
\downarrow{k'} & & \downarrow{k} \\
X & \xrightarrow{f} & Y \\
\downarrow{\pi_X} & & \downarrow{\pi_Y} \\
S. & & S. \\
\end{array}
\]

\[
\pi_X \circ k' = (\pi_Y \circ f) \circ k'
\]
\[
= \pi_Y \circ (f \circ k')
\]
\[
= \pi_Y \circ (k \circ f')
\]
\[
= (\pi_Y \circ k) \circ f'.
\]

Since \( f' \) is smooth as the pullback of the smooth map \( f \) (which is the given condition) and \( \pi_Y \circ k \)
is smooth, \( \pi_X \circ k' \) is smooth. As to the grading, we can see it as follows. Suppose that \( [W \xrightarrow{k} Y; L_1, \ldots, L_r] \in Z^i(Y, \pi_Y) \), i.e., \( -i + r = \dim(\pi_Y \circ k) \). From which we get
\[
-i + r + \dim(f) = \dim(\pi_Y \circ k) + \dim(f)
\]
\[
= \dim(\pi_Y \circ k) + \dim(f') \quad \text{ (since } \dim(f) = \dim(f'))
\]
\[
= \dim(\pi_Y \circ k \circ f')
\]
\[
= \dim(\pi_Y \circ f \circ k')
\]
\[
= \dim((\pi_Y \circ f) \circ k')
\]
\[
= \dim(\pi_X \circ k').
\]
Then the uniqueness of $\tau$ follows from its naturality and the condition that $\pi$.

**Remark 4.8.** We note that if $S$ is a point $pt = \text{Spec} k$, $Z^i(X \xrightarrow{p_X} pt)$ is Levine–Morel’s oriented Borel–Moore functor with products $\mathcal{Z}_i(X)$ on $\text{Sch}_k \ [6, \text{Definition 2.1.6}].$

**Definition 4.9.** A functor $A^*$ assigning $A^*(X \xrightarrow{\pi_X} S)$ to a $S$-scheme $\pi_X : X \to S$ satisfying all the properties in the above theorem is called an oriented Borel–Moore functor with products on the category of $S$-schemes, i.e. the over category $\text{Sch}/S$.

**Theorem 4.10.** $Z^*(\xrightarrow{-} \pi_X \to S)$ is the universal oriented Borel–Moore functor with products on $\text{Sch}_S$ in the sense that for any oriented Borel–Moore functor $A^*(\xrightarrow{-} \to S)$ with products there exists a unique natural transformation $\tau_{A^*} : Z^*(\xrightarrow{-} \to S) \to A^*(\xrightarrow{-} \to S)$ with the requirement $\tau_{A^*}(\xrightarrow{\text{id}_{\text{Sch}/S}} S) = 1_S \in A^0(S \xrightarrow{\text{id}_{\text{Sch}/S}} S)$, where $1_S$ is the unit.

**Proof.** Let $[V \xrightarrow{h_Y} X, L_1, \cdots, L_r] \in Z^r(X \xrightarrow{\pi_X} S)$. Noticing that $(\pi_X \circ h)^*(\xrightarrow{\text{id}_{\text{Sch}/S}} S) = \xrightarrow{\text{id}_{\text{Sch}/S}} V \in Z^r(\xrightarrow{-} \to S)$, we get the following equality:

$$
[V \xrightarrow{h_Y} X, L_1, \cdots, L_r] = \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r) \circ h_s([V \xrightarrow{\text{id}_{\text{Sch}/S}} V])
$$

Then we can define the homomorphism $\tau_{A^*} : Z^*(\xrightarrow{-} \pi_X \to S) \to A^*(\xrightarrow{-} \pi_X \to S)$ by

$$
\tau_{A^*}([V \xrightarrow{h_Y} X, L_1, \cdots, L_r]) = \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r) \circ h_s(\pi_X \circ h)^*(\xrightarrow{\text{id}_{\text{Sch}/S}} S).
$$

Then the uniqueness of $\tau_{A^*}$ follows from its naturality and the condition that $\tau_{A^*}(\xrightarrow{\text{id}_{\text{Sch}/S}} S) = 1_S$. For the sake of clarity we write down the following sequence of the above homomorphisms:

$$
\begin{array}{ccc}
Z^0(S \xrightarrow{\text{id}_{\text{Sch}/S}} S) & \xrightarrow{\tau_{A^*}} & A^0(S \xrightarrow{\text{id}_{\text{Sch}/S}} S) \\
(\pi_X \circ h)^* & & (\pi_X \circ h)^* \\
Z^{-\dim(\pi_X \circ h)}(V \xrightarrow{\pi_X \circ h} S) & \xrightarrow{\tau_{A^*}} & A^{-\dim(\pi_X \circ h)}(V \xrightarrow{\pi_X \circ h} S) \\
h_s & & h_s \\
Z^{-\dim(\pi_X \circ h)}(X \xrightarrow{\pi_X} S) & \xrightarrow{\tau_{A^*}} & A^{-\dim(\pi_X \circ h)}(X \xrightarrow{\pi_X} S) \\
\tilde{c}_1(L_r) & & \tilde{c}_1(L_r) \\
Z^{-\dim(\pi_X \circ h) + 1}(X \xrightarrow{\pi_X} S) & \xrightarrow{\tau_{A^*}} & A^{-\dim(\pi_X \circ h) + 1}(X \xrightarrow{\pi_X} S) \\
\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_{r-1}) & & \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_{r-1}) \\
Z^i(X \xrightarrow{\pi_X} S) & \xrightarrow{\tau_{A^*}} & A^i(X \xrightarrow{\pi_X} S) \\
\end{array}
$$

Here we use the fact that $-i + r = \dim(\pi_X \circ h)$. \qed

**Remark 4.11.** It is clear that if $S$ is a point, $A_s(X) := A^*(X \xrightarrow{\text{pt}} \text{pt})$ is Levine–Morel’s oriented Borel–Moore functor with products. $Z_s(\xrightarrow{-})$ is the universal one among all the oriented Borel–Moore functor with products in the sense that for any oriented Borel–Moore functor $A_s$ with products there exists
a unique natural transformation \( \tau_\ast : Z_\ast \to A_\ast \) with the requirement \( \tau_\ast([pt \xrightarrow{id_{pt}} pt]) = 1_{pt} \in A_\ast(pt) \), where \( 1_{pt} \) is the unit. The proof of this is adopted in the proof of the above theorem.

**Remark 4.12.** We note that in Levine–Morel’s algebraic cobordism \( \Omega_\ast(X) \), the particular cobordism cycle \([X \xrightarrow{id_X} X; L_1, \ldots, L_r] \) belongs to the algebraic pre-cobordism \( Z_\ast(X) \) if and only if \( X \) is smooth because of the definition of \( Z_\ast(X) \). In our case we have that \([X \xrightarrow{id_X} X; L_1, \ldots, L_r] \) belongs to \( Z^\ast(X \xrightarrow{\pi_X} S) \) if and only if \( \pi_X : X \to S \) is smooth. We also note that \([X \xrightarrow{id_X} X; L_1, \ldots, L_r] \) always belongs to \( Z^\ast(X \xrightarrow{id_X} X) \) whether \( X \) is smooth or singular.

### 5. Algebraic Cobordism \( \Omega^\ast(X \to S) \) of \( S \)-Schemes

Levine and Morel \([6]\) construct their algebraic cobordism \( \Omega_\ast(X) \) from the oriented Borel–Morel functor \( Z_\ast(X) \) imposing three axioms, \( \text{(Dim)} \) the dimension axiom, \( \text{(Sect)} \) the section axiom and \( \text{(FGL)} \) Formal group law axiom. In this section, from the above oriented Borel–Moore functor \( Z^\ast(X \xrightarrow{\pi_X} S) \) on the over category \( Sch/S \) we construct an “algebraic cobordism” \( \Omega^\ast(X \to S) \) on the over category \( Sch/S \) by imposing relative versions of these three axioms. In other words \( \Omega^\ast(X \to S) \) is an algebraic cobordism of \( S \)-schemes.

As we will see later, the construction in this section does not give a bivariant-theoretic analogue of the algebraic cobordism \( \Omega_\ast(X) \).

First we define the following relative analogues of Levine–Morel’s definitions \([6]\) Definition 2.1.12, Definition 2.2.1]:

**Definition 5.1.** Let \( R_\ast \) be a commutative graded ring with unit. An oriented Borel–Moore functor with \( \ast \)-theory \( A(X \xrightarrow{\pi_X} S) \) is one together with a graded ring homomorphism

\[
\Phi : R_\ast \to A(S \xrightarrow{id_S} S).
\]

**Remark 5.2.** In the above definition we should note that the external product on \( A(S \xrightarrow{id_S} S) \) gives a ring structure.

Let \( \mathbb{L}_\ast \) be the Lazard ring homologically graded and let \( F_\mathbb{L}(u, v) \in \mathbb{L}_\ast[[u, v]] \) denote the universal formal group law.

**Definition 5.3.** An oriented Borel–Moore functor with \( \ast \)-theory \( A \) is called “of geometric type” if the following three axioms are satisfied:

1. \( \text{(rel-Dim = Relative Dimension Axiom)} \) For any smooth morphism \( \pi_X : X \to S \) and any family \( \{L_1, L_2, \ldots, L_n\} \) of line bundles on \( X \) with \( n > \dim(\pi_X) \), one has
   \[
   \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)(\pi_X^\ast 1_S) = 0 \in A^\ast(X \xrightarrow{\pi_X} S).
   \]
   Here we note that \( \pi_X^\ast 1_S \in A^{-\dim(\pi_X)}(X \xrightarrow{\pi_X} S) \).

2. \( \text{(rel-Sect = Relative Section Axiom)} \) For any smooth morphism \( \pi_X : X \to S \), any line bundle \( L \) over \( X \) and any section \( s \) of \( L \) which is transverse fiberwisely (with respect to the smooth map \( \pi_X \circ h \)) to the zero section of \( L \) with \( Z := s^{-1}(0) \), i.e., \( \pi_Z := \pi_X|_Z : Z \to S \) is smooth, one has
   \[
   \tilde{c}_1(L)(\pi_X^\ast 1_S) = i_Z^\ast(\pi_Z^\ast 1_S),
   \]
   Here \( i_Z : Z \to X \) is the closed immersion (note: \( \pi_Z = \pi_X \circ i_Z \)).

3. \( \text{(rel-FGL = Relative Formal Group Law Axiom)} \) Let \( \tilde{c}_1 : L_\ast \to A(pt \to pt) \) be the ring homomorphism giving the \( L_\ast \)-structure and let \( F_A \in A(pt \to pt) \) be the image of the universal formal group law \( F_A \in \mathbb{L}_\ast[[u, v]] \) by \( \Phi \). Then for any smooth morphism \( \pi_X : X \to S \) and any pair \( \{L, M\} \) of line bundles over \( X \), one has
   \[
   F_A(\tilde{c}_1(L), \tilde{c}_1(M))(\pi_X^\ast 1_S) = \tilde{c}_1(L \otimes M)(\pi_X^\ast 1_S).
   \]
Remark 5.4. In the above definitions, if the target scheme $S$ is a point, we recover Levine–Morel’s original definitions.

First we consider imposing the (rel-Dim) on $\mathbb{Z}^\ast(X \xrightarrow{\pi_X} S)$.

Definition 5.5. We define the following subgroup of $\mathbb{Z}^\ast(X \xrightarrow{\pi_X} S)$:

$$\langle \mathcal{R}^{\dim} \rangle(X \xrightarrow{\pi_X} S)$$

is generated by the cobordism cycles of the form

$$[V \xrightarrow{h} X; \pi_X^* L_1, \pi_X^* L_2, \ldots, \pi_X^* L_r, M_1, \ldots M_s]$$

where

1. the following diagram commutes

   $\begin{array}{ccc}
   V & \xrightarrow{h} & X \\
   & \searrow \pi & \downarrow \pi_X \\
   & & S
   \end{array}$

   $\begin{array}{ccc}
   & & S' \\
   \pi & \searrow \nu & \\
   & S \\
   \end{array}$

2. $\pi : V \to S'$ and $\nu : S' \to S$ are both smooth.
3. $L_1, L_2, \ldots, L_r$ are line bundles over $S'$ and $r > \dim \nu = \dim S' - \dim S$,
4. $M_1, \ldots, M_s$ are line bundles over $V$.

Remark 5.6. Let $L_i(1 \leq i \leq r)$ be any line bundle over the base scheme $S$. Then any cobordism cycle $[V \xrightarrow{h} X; (\pi_X \circ h)^* L_1, (\pi_X \circ h)^* L_2, \ldots, (\pi_X \circ h)^* L_r, M_1, \ldots M_s]$ always belong to $\langle \mathcal{R}^{\dim} \rangle(X \xrightarrow{\pi_X} S)$. Because we can consider the following obvious commutative diagram

$\begin{array}{ccc}
V & \xrightarrow{h} & X \\
& \searrow \pi_X \circ h & \downarrow \pi_X \\
& & S \\
\end{array}$

and the condition (3) above is satisfied: $r \geq 1 > \dim(\nu) = \dim(\text{id}_S) = 0$.

Definition 5.7. We define the following quotient

$$\mathbb{Z}^\ast(X \xrightarrow{\pi_X} S) := \frac{\mathbb{Z}^\ast(X \xrightarrow{\pi_X} S)}{\langle \mathcal{R}^{\dim} \rangle(X \xrightarrow{\pi_X} S)}.$$

The equivalence class of $[V \xrightarrow{h} X; (\pi_X \circ h)^* L_1, (\pi_X \circ h)^* L_2, \ldots, (\pi_X \circ h)^* L_r, M_1, \ldots M_s]$ in the quotient group $\mathbb{Z}^\ast(X \xrightarrow{\pi_X} S)$ shall be denoted by $[[V \xrightarrow{h} X; L_1, L_2, \ldots, L_k]]$.

Remark 5.8. If the target scheme $S$ is a point, then the above $\langle \mathcal{R}^{\dim} \rangle(X \xrightarrow{\pi_X} S)$ is equal to the subgroup $\langle \mathcal{R}^{\dim} \rangle(X)$ defined in [6, Lemma 2.4.2]. Therefore we have

$$\langle \mathcal{R}^{\dim} \rangle(X \xrightarrow{\pi_X} \text{pt}) = \langle \mathcal{R}^{\dim} \rangle(X), \quad \mathbb{Z}^{-1}(X \xrightarrow{\pi_X} \text{pt}) = \mathbb{Z}(X).$$

Theorem 5.9. For the above group $\mathbb{Z}^\ast(X \xrightarrow{\pi_X} S)$, we define the following four operations:

- **External product:** The external product $\times_{S} : \mathbb{Z}^\ast(X \xrightarrow{\pi_X} S) \times \mathbb{Z}^\ast(Y \xrightarrow{\pi_Y} S) \to \mathbb{Z}^\ast(X \times_{S} Y \xrightarrow{\pi_{X \times S \pi_Y}} S)$
Proof. (1) To show that the pushforward, the pullback and the orientation are well-defined, it suffices to show that each operation preserves \( \langle R^{\dim} \rangle \), to be more precise,

(a) \( f_* (\langle R^{\dim} \rangle (X \xrightarrow{\pi_X} S)) \subset \langle R^{\dim} \rangle (Y \xrightarrow{\pi_Y} S) \): Suppose that \([V \xrightarrow{h} X; \pi^* L_1, \ldots, \pi^* L_r, M_1, \ldots, M_s] \in \langle R^{\dim} \rangle (X \xrightarrow{\pi_X} S)\) as in Definition 5.5. Then we have \( f_* (\langle R^{\dim} \rangle (X \xrightarrow{\pi_X} S)) \subset \langle R^{\dim} \rangle (Y \xrightarrow{\pi_Y} S)\) as the following commutative diagram:

(b) \( f^* (\langle R^{\dim} \rangle (Y \xrightarrow{\pi_Y} S)) \subset \langle R^{\dim} \rangle (X \xrightarrow{\pi_X} S)\): Here we should note that \( f : X \to Y \) is smooth, which is important. Suppose that \([W \xrightarrow{k} Y; \pi^* L_1, \ldots, \pi^* L_r, M_1, \ldots, M_s] \in \langle R^{\dim} \rangle (Y \xrightarrow{\pi_Y} S)\) as in Definition 5.5. Consider the following commutative diagram:
Then we have

\[ f^*([W \xrightarrow{k} Y; \pi^*L_1, \ldots, \pi^*L_r, M_1, \ldots, M_s]) = [W' \xrightarrow{k'} X; (f')^*\pi^*L_1, \ldots, (f')^*\pi^*L_r, (f')^*M_1, \ldots, (f')^*M_s] \]

which belongs to \((\mathcal{R}^\text{dim})(X \xrightarrow{\pi_x} S)\).

(c) \(\tilde{c}_1(L)((\mathcal{R}^\text{dim})(X \xrightarrow{\pi_x} S)) \subseteq \langle \mathcal{R}^\text{dim}(X \xrightarrow{\pi_y} S) \rangle\): It is clear.

(d) As to the external product, we need to show that

(i) \(\mathcal{Z}^*(X \xrightarrow{\pi_x} S) \times S (\mathcal{R}^\text{dim}(Y \xrightarrow{\pi_y} S)) \subseteq \langle \mathcal{R}^\text{dim}(X \times_S Y \xrightarrow{\pi_x \times \pi_y} S) \rangle\),

(ii) \((\mathcal{R}^\text{dim})(X \xrightarrow{\pi_x} S) \times S \mathcal{Z}^*(Y \xrightarrow{\pi_y} S) \subseteq \langle \mathcal{R}^\text{dim}(X \times_S Y \xrightarrow{\pi_x \times \pi_y} S) \rangle\),

(iii) \((\mathcal{R}^\text{dim})(X \xrightarrow{\pi_x} S) \times S (\mathcal{R}^\text{dim})(Y \xrightarrow{\pi_y} S) \subseteq \langle \mathcal{R}^\text{dim}(X \times_S Y \xrightarrow{\pi_x \times \pi_y} S) \rangle\).

Since (iii) is a special case, it suffices to show (i) and (ii). For (ii), suppose that

\([V \xrightarrow{h} X; \pi^*L_1, \ldots, \pi^*L_r, M_1, \ldots, M_s] \in \langle \mathcal{R}^\text{dim}(X \xrightarrow{\pi_x} S) \rangle\) as in Definition 5.5 and

\([W \xrightarrow{h'} N_1, \ldots, N_t] \in \mathcal{Z}^*(Y \xrightarrow{\pi_y} S)\). Then we consider the following commutative diagrams:

\[
\begin{array}{ccc}
V \times_S W & \xrightarrow{h} & \tilde{W} & \xrightarrow{\tilde{p}_2} & W \\
\downarrow{k} & & \downarrow{k'} & & \downarrow{k} \\
\tilde{V} & \xrightarrow{h'} & X \times_S Y & \xrightarrow{p_2} & Y \\
\downarrow{\tilde{p}_1} & & \downarrow{p_2} & & \downarrow{\pi_y} \\
V & \xrightarrow{h} & X & \xrightarrow{\pi_x} & S \\
\downarrow{\pi} & & \downarrow{\nu} & & \downarrow{\nu} \\
S' & & & & \\
\end{array}
\]

Here we note that \(\pi \tilde{p}_1 \tilde{k} : V \times_S W \to S'\) is smooth because \(\pi\) is smooth by hypothesis and \(\tilde{p}_1 \circ \tilde{k} : V \times_S W \to V\) is smooth since it is the pullback of the smooth morphism \(\pi \circ \tilde{k} : W \to S\). The proof of (i) is the same as this, so omitted.

(2) (D1), \ldots, (D4) are already checked above, thus it suffices to see (A1), \ldots, (A8). But they follow from the definitions of these four operations. E.g., as to (A1), we can see it as follows: for

\([x] \in \mathcal{Z}(X \xrightarrow{\pi_x} S)\), where \([x] = [V \xrightarrow{h} X; L_1, \ldots, L_s]\),
we have
\[
(g \circ f)^*([x]) = [(g \circ f)^*([x])]
\]
(by the definition)
\[
= [(f^* \circ g^*)([x])]
\]
\[
= f^*([g^*([x])])
\]
\[
= f^* \circ g^*([x])
\]
Thus we have \((g \circ f)^* = f^* \circ g^*\).

(3) In our case, since \(1_S = [S \xrightarrow{\text{id}_S} S]\) and \(\pi_X^* 1_S = [X \xrightarrow{\text{id}_X} X]\), we have that
\[
\overline{\pi}_1(L_1) \circ \overline{\pi}_2(L_2) \circ \ldots \circ \overline{\pi}_n(L_n)(\pi_X^* 1_S) = [X \xrightarrow{\text{id}_X} X; L_1, L_2, \ldots, L_n]
\]
with \(\pi_X : X \to S\) is smooth. Then we have the following obvious commutative diagram:
\[
\begin{array}{ccc}
X & \xrightarrow{\text{id}_X} & S \\
\downarrow{\pi_X} & & \downarrow{\pi_X} \\
X & \xrightarrow{\nu = \pi_X} & X
\end{array}
\]
Since \(r > \dim(\pi_X) = \dim(\nu)\), we have that
\[
[X \xrightarrow{\text{id}_X} X; L_1, L_2, \ldots, L_n] \in \langle \mathcal{R}^{\text{dim}} \rangle(X \xrightarrow{\pi_X} S).
\]
Therefore we have that
\[
[[X \xrightarrow{\text{id}_X} X; L_1, L_2, \ldots, L_n]] = 0 \in \mathcal{Z}^*(X \xrightarrow{\pi_X} S).
\]

Next we impose the axiom (rel-Sect) on the above quotient group \(\mathcal{Z}^*(X \xrightarrow{\pi_X} S)\).

**Definition 5.10.** We define the following subgroup of \(\mathcal{Z}^*(X \xrightarrow{\pi_X} S)\):
\[
\langle \mathcal{R}^\text{Sect} \rangle(X \xrightarrow{\pi_X} S)
\]
is generated by elements of the form
\[
[[V \xrightarrow{h} X; L_1, \ldots, L_r]] - [[Z \xrightarrow{h|_Z} X; i^*L_1, \ldots, i^*L_{r-1}]],
\]
where
1. \(r > 0\)
2. \(Z = s^{-1}(0)\), where \(s\) is a section of the line bundle \(L_r\) which is transverse fiberwisely (with respect to the smooth map \(\pi_X \circ h\)) to the zero section of \(L_r\) (hence \(\pi_X \circ h|_Z\) is smooth) and \(i : Z \hookrightarrow X\) is the inclusion and \(h|_Z = h \circ i\).

**Definition 5.11.** We define the following quotient group
\[
\Omega^*(X \xrightarrow{\pi_X} S) := \mathcal{Z}^*(X \xrightarrow{\pi_X} S) / \langle \mathcal{R}^\text{Sect} \rangle(X \xrightarrow{\pi_X} S).
\]
The equivalence class of \([[V \xrightarrow{h} X; L_1, L_2, \ldots, L_k]]\) in \(\mathcal{Z}^*(X \xrightarrow{\pi_X} S)\) in the quotient group \(\Omega^*(X \xrightarrow{\pi_X} S)\) shall be denoted by \([[V \xrightarrow{h} X; L_1, L_2, \ldots, L_k]]\).

**Remark 5.12.** If the target scheme \(Y\) is a point, then the above \(\langle \mathcal{R}^\text{Sect} \rangle(X \xrightarrow{\rho_X} Y)\) is equal to the subgroup \(\langle \mathcal{R}^\text{Sect} \rangle(X)\) defined in [G Lemma 2.4.7]. Therefore we have
\[
\langle \mathcal{R}^\text{Sect} \rangle(X \xrightarrow{\rho_X} pt) = \langle \mathcal{R}^\text{Sect} \rangle(X), \quad \Omega^{-1}(X \xrightarrow{\rho_X} pt) = \Omega(X).
\]
Theorem 5.13. For the above group $\Omega^\ast(X \xrightarrow{\pi_X} S)$, we define the following four operations as follows:

- (external product) The external product
  
  $$\times_S : \Omega^\ast(X \xrightarrow{\pi_X} S) \times \Omega^\ast(Y \xrightarrow{\pi_Y} S) \to \Omega^\ast(X \times_S Y \xrightarrow{\pi_X \times_S \pi_Y} S)$$

  is defined by

  $$\left[[V \xrightarrow{h} X; L_1, \cdots, L_n]\right] \times_S \left[[W \xrightarrow{k} X; M_1, \cdots, M_m]\right] := \left[[V \xrightarrow{h} X; L_1, \cdots, L_n] \times_S [W \xrightarrow{k} X; M_1, \cdots, M_m]\right].$$

- (pushforward) For a proper morphism $f : X \to Y$ from $\pi_X : X \to S$ to $\pi_Y : Y \to S$, the pushforward
  
  $$f_* : \Omega^\ast(X \xrightarrow{\pi_X} S) \to \Omega^\ast(Y \xrightarrow{\pi_Y} S)$$

  is defined by $f_*\left([[V \xrightarrow{h} X; L_1, \cdots, L_n]]\right) := \left[[\pi_Y \circ f \circ h \left|_{\pi_X^{-1}(V)}\right.; L_1, \cdots, L_n]\right]$.

- (pullback) For a smooth morphism $f : X \to Y$ from $\pi_X : X \to S$ to $\pi_Y : Y \to S$, the pullback
  
  $$f^* : \Omega^\ast(Y \xrightarrow{\pi_Y} S) \to \Omega^{\dim f}(X \xrightarrow{\pi_X} S)$$

  is defined by $f^*\left([[V \xrightarrow{h} Y; L_1, \cdots, L_n]]\right) := \left[[V \xrightarrow{h} Y; f^* L_1, \cdots, f^* L_n]\right]$.

- (orientation = the Chern operator $\tilde{c}_1(L)$) For a line bundle $L$ over $X$, the operator
  
  $$\tilde{c}_1(L) : \Omega^\ast(X \xrightarrow{\pi_X} S) \to \Omega^{\dim f}(X \xrightarrow{\pi_X} S)$$

  is defined by $\tilde{c}_1(L)\left([[V \xrightarrow{h} X; L_1, \cdots, L_n]]\right) := \left[[\tilde{c}_1(L) h \left|_{\pi_X^{-1}(V)}\right.; L_1, \cdots, L_n]\right]$.

1. The above operations are well-defined.
2. The above theory $\Omega^\ast(X \xrightarrow{\pi_X} S)$ is an oriented Borel–More functor with products, i.e. it satisfies all the properties (D1), · · · , (D4) and (A1), · · · , (A8).
3. The above theory $\Omega^\ast(X \xrightarrow{\pi_X} S)$ satisfies (rel-Dim) and (rel-Sec).

Proof. The proof is similar to that of the above Theorem 5.9. But we will write them down for the sake of completeness.

(1) To show that the pushforward, the pullback and the orientation are well-defined, it suffices to show that each operation preserves $\langle R^{\text{Sect}} \rangle$, to be more precise,

(i) $f_*\left(\langle R^{\text{Sect}} \rangle(X \xrightarrow{\pi_X} S)\right) \subset \langle R^{\text{Sect}} \rangle(Y \xrightarrow{\pi_Y} S)$: Indeed, let

$$[[V \xrightarrow{h} X; L_1, \cdots, L_n]] \in \langle R^{\text{Sect}} \rangle(X \xrightarrow{\pi_X} S),$$

where $Z$, the line bundles $L_i$’s and $i : Z \to V$ are as in Definition 5.10 above.

Then

$$f_*\left([[V \xrightarrow{h} X; L_1, \cdots, L_n]]\right) = [f_*[[V \xrightarrow{h} X; i^* L_1, \cdots, i^* L_n]]] = f_*[[Z \xrightarrow{h \circ i} X; i^*L_1, \cdots, i^* L_n]] = [[[V \xrightarrow{f \circ h \circ i} Y; L_1, \cdots, L_n]]] \in \langle R^{\text{Sect}} \rangle(Y \xrightarrow{\pi_Y} S),$$

(ii) $f^*\left(\langle R^{\text{Sect}} \rangle(Y \xrightarrow{\pi_Y} S)\right) \subset \langle R^{\text{Sect}} \rangle(X \xrightarrow{\pi_X} S)$: Let


Hence we have 

\[ k \] is the composite

\[ Z \]

Here \( i : Z \to W \) is the inclusion and thus the pullback \( i' : Z' \to W \) is also an inclusion and \( k|_Z : Z \to Y \) is the composite \( k \circ i \) and \( k|_{Z'} : Z' \to Y \) is the composite \( k' \circ i' \). Then we have

\[
(f''^*)(i^*M_j) = (i')^*((f')^*M_j).
\]

Hence we have

\[
\begin{align*}
\text{f}^*(([[W \xrightarrow{h} Y; M_1, \ldots, M_r]] - [[Z \xrightarrow{k|_Z} Y; i^*M_1, \ldots, i^*M_{r-1}]])) \\
= [f^*)(([[W \xrightarrow{h} Y; M_1, \ldots, M_r]] - [[Z \xrightarrow{k|_Z} Y; i^*M_1, \ldots, i^*M_{r-1}]])) \\
= [[[W' \xrightarrow{k'} X; (f')^*M_1, \ldots, (f')^*M_r]] - [[Z' \xrightarrow{k'|_{Z'}} X; (f'')^*(i^*M_1), \ldots, (f'')^*(i^*M_{r-1})]]] \\
= [[[W' \xrightarrow{k'} X; (f')^*M_1, \ldots, (f')^*M_r]] - [[Z' \xrightarrow{k'|_{Z'}} X; (i')^*((f')^*M_1), \ldots, (i')^*((f')^*M_{r-1})]]] \\
= [[[W' \xrightarrow{k'} X; (f')^*M_1, \ldots, (f')^*M_r]] - [[Z' \xrightarrow{k'|_{Z'}} X; (i')^*((f')^*M_1), \ldots, (i')^*((f')^*M_{r-1})]]] \\
\end{align*}
\]

Note that \( Z' \) is the zero locus of the section \( s' : W' \to (f')^*M_r \) which is the pullback of the section \( s : W \to M_r \). Hence

\[
[[W' \xrightarrow{k'} X; (f')^*M_1, \ldots, (f')^*M_r]] - [[Z' \xrightarrow{k'|_{Z'}} X; (i')^*((f')^*M_1), \ldots, (i')^*((f')^*M_{r-1})]] \\
\]

belongs to \( (\mathcal{R}_{\text{Sect}}(X \xrightarrow{\pi_S} S)) \).

(iii) \( \overline{c_1}(L)([\mathcal{R}_{\text{Sect}}(X \xrightarrow{\pi_S} S)]) \subset (\mathcal{R}_{\text{Sect}}(X \xrightarrow{\pi_S} S)) \): Let \( L \) be a line bundle over \( X \) and let

\[
[[V \xrightarrow{h} X; L_1, \ldots, L_r]] - [[Z \xrightarrow{h|_Z} X; i^*L_1, \ldots, i^*L_{r-1}]] \in (\mathcal{R}_{\text{Sect}}(X \xrightarrow{\pi_S} S)).
\]

\[
\overline{c_1}(L)([[V \xrightarrow{h} X; L_1, \ldots, L_r]] - [[Z \xrightarrow{h|_Z} X; i^*L_1, \ldots, i^*L_{r-1}]]))
\]

\[
= [[(V \xrightarrow{h} X; L_1, \ldots, L_r)] - [[(Z \xrightarrow{h|_Z} X; i^*L_1, \ldots, i^*L_{r-1})])]
\]

\[
= [[[V \xrightarrow{h} X; L_1, \ldots, L_r; h^*L]] - [[[Z \xrightarrow{h|_Z} X; i^*L_1, \ldots, i^*L_{r-1}, (h|_Z)^*L]]]
\]

\[
= [[[V \xrightarrow{h} X; L_1, \ldots, L_r; h^*L]] - [[[Z \xrightarrow{h|_Z} X; i^*L_1, \ldots, i^*L_{r-1}, h^*L]]], \quad \text{(since } h|_Z = h \circ i)\]
\]

which belongs to \( (\mathcal{R}_{\text{Sect}}(X \xrightarrow{\pi_S} S)) \).

(iv) As to the external product, we need to see that not only

\[
(\mathcal{R}_{\text{Sect}}(X \xrightarrow{\pi_S} S) \times_S (\mathcal{R}_{\text{Sect}}(Y \xrightarrow{\pi_Y} S)) \subset (\mathcal{R}_{\text{Sect}}(X \times_S Y \xrightarrow{\pi_{X \times_S Y}} S))
\]

but also

\[
(\mathcal{R}_{\text{Sect}}(X \xrightarrow{\pi_S} S) \times_S (\mathcal{R}_{\text{Sect}}(Y \xrightarrow{\pi_Y} S)) \subset (\mathcal{R}_{\text{Sect}}(X \times_S Y \xrightarrow{\pi_{X \times_S Y}} S)),
\]

\[
(\mathcal{R}_{\text{Sect}}(X \xrightarrow{\pi_S} S) \times_S (\mathcal{R}_{\text{Sect}}(Y \xrightarrow{\pi_Y} S)) \subset (\mathcal{R}_{\text{Sect}}(X \times_S Y \xrightarrow{\pi_{X \times_S Y}} S)).
\]
For this, it suffices to show the second one, because the other two are similar. So, we consider

\[ [[V \xrightarrow{h} X; L_1, \ldots, L_r]] \times_S \left( [[W \xrightarrow{k} Y; M_1, \ldots, M_q]] - [[Z \xrightarrow{k[Z]} Y; i^* M_1, \ldots, i^* M_{q-1}]] \right) \]

\[ = [[V \xrightarrow{h} X; L_1, \ldots, L_r]] \times_S [[W \xrightarrow{k} Y; M_1, \ldots, M_q]] \]

\[ - [[V \xrightarrow{h} X; L_1, \ldots, L_r]] \times_S [[Z \xrightarrow{k[Z]} Y; i^* M_1, \ldots, i^* M_{q-1}]] \]

We note that \( i : Z \to W \) is the inclusion and \( k|_Z \) is the composite \( k \circ i \), and we recall that

\[ [[V \xrightarrow{h} X; L_1, \ldots, L_r]] \times_S [[W \xrightarrow{k} X; M_1, \ldots, M_q]] \]

\[ = [[V \xrightarrow{h} X; L_1, \ldots, L_r]] \times_S [[W \xrightarrow{k} X; M_1, \ldots, M_q]] \]

\[ = [[V \times_S W \xrightarrow{k[Z]} X \times_S Y; (\tilde{p}_1 \tilde{k})^* L_1, \ldots, (\tilde{p}_1 \tilde{k})^* L_r, (\tilde{p}_2 \tilde{h})^* M_1, \ldots, (\tilde{p}_2 \tilde{h})^* M_q]] \]

\[ = [[V \xrightarrow{h} X; L_1, \ldots, L_r]] \times_S [[Z \xrightarrow{k[Z]} Y; i^* M_1, \ldots, i^* M_{q-1}]] \]

\[ = [[V \times_S Z \xrightarrow{\tilde{i}[k[Z]]} X \times_S Y; i^* (\tilde{p}_1 \tilde{k})^* L_1, \ldots, i^* (\tilde{p}_1 \tilde{k})^* L_r, (\tilde{p}_2 \tilde{h})^* M_1, \ldots, (\tilde{p}_2 \tilde{h})^* M_q]] \]

where \( V \times_S Z \) is the zero locus of the section from \( V \times_S W \) to the pullbacked line bundle \( (\tilde{p}_2 \tilde{h})^* M_q \). So the last one belongs to \( (\mathbb{R}^{Sect}) (X \times_S Y \xrightarrow{\pi_X \times_S \pi_Y} S) \). Here we use the diagram:

\[
\begin{array}{c}
V \times_S Z \xrightarrow{\bar{h}} \bar{Z} \xrightarrow{\bar{p}_2} Z \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
V \times_S W \xrightarrow{\bar{h}} \bar{W} \xrightarrow{\bar{p}_2} W \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\tilde{V} \xrightarrow{k'} X \times_B Y \xrightarrow{p_2} Y \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
V \xrightarrow{h} X \xrightarrow{\pi_X} S.
\end{array}
\]

(2) (D1), \ldots, (D4) are already checked above, thus it suffices to see (A1), \ldots, (A8). But they follow from the definitions of these four operations.

(3) \( \tilde{c}(L)(\pi_X 1_S) = (i_Z)(\pi_Z 1_S) \) is nothing but \( [X \xrightarrow{id_X} X; L] = [Z \xrightarrow{p} X] \). Since we have \( [[X \xrightarrow{id_X} X; L]] - [[Z \xrightarrow{p} X]] \in (\mathbb{R}^{Sect}) (X \xrightarrow{\pi_X} S) \), we have \( \tilde{c}(L)(\pi_X 1_S) = i_Z(\pi_Z 1_S) \).

\[ \square \]

Finally we define the following

**Definition 5.14.** We define the following subgroup of \( \mathbb{L}_* \otimes \Omega^*(X \xrightarrow{\pi_X} S) \):

\[ (\mathbb{L}_*{\mathbb{R}^{FGT}})(X \xrightarrow{\pi_X} S) \]

is generated by the elements of the form

\[ [[V \xrightarrow{h} X; L_1, \ldots L_r, F_L(L, M)]] - [[V \xrightarrow{h} X; L_1, \ldots L_r, L \otimes M]] \]
Theorem 5.16. For the above group \( \Omega^*(X \xrightarrow{\pi_X} S) \), we define the following four operations as follows:

- (external product) The external product
  \[ \times_S : \Omega^*(X \xrightarrow{\pi_X} S) \times \Omega^*(Y \xrightarrow{\pi_Y} S) \to \Omega^*(X \times_S Y \xrightarrow{\pi_{X \times_S Y}} S) \]
  is defined by
  \[
  [\left[ [V \xrightarrow{h} X; L_1, \ldots, L_k] \right]] \times_S [\left[ [W \xrightarrow{k} X; M_1, \ldots, M_l] \right]] := [\left[ [W \xrightarrow{h'} L_1, \ldots, L_k] \right] \times_S [\left[ [V \xrightarrow{k'} M_1, \ldots, M_l] \right]].
  \]

- (pullback) For a smooth morphism \( f : X \to Y \) from \( \pi_X : X \to S \) to \( \pi_Y : Y \to S \), the pullback
  \[ f^* : \Omega^*(Y \xrightarrow{\pi_Y} S) \to \Omega^{\dim f}(X \xrightarrow{\pi_X} S) \]
  is defined by
  \[ f^*([\left[ [W \xrightarrow{h} Y; L_1, \ldots, L_k] \right]]) := [\left[ f^*(W) \xrightarrow{k} X; (f^*)^*L_1, \ldots, (f^*)^*L_k \right]]. \]

- (orientation = the Chern operator \( \tilde{c}_1(L) \)) For a line bundle \( L \) over \( X \), the operator
  \[ \tilde{c}_1(L) : \Omega^j(X \xrightarrow{\pi_X} S) \to \Omega^{j+1}(X \xrightarrow{\pi_X} S) \]
  is defined by
  \[ \tilde{c}_1(L)(\left[ [V \xrightarrow{h} X; L_1, \ldots, L_k] \right]) := \tilde{c}_1(L)(\left[ [V \xrightarrow{h} X; L_1, \ldots, L_k, h^*L] \right]). \]

1. The above operations are well-defined.
2. The above theory \( \Omega^*(X \xrightarrow{\pi_X} S) \) is an oriented Borel–More functor with products, i.e. it satisfies all the properties (D1), (D4) and (A1), (A4), (A8).
3. The above theory \( \Omega^*(X \xrightarrow{\pi_X} S) \) satisfies (rel-Dim), (rel-Sec) and (rel-FGL).
Proof. It is easy to see that as above the pushforward, the pullback and the orientation are all well-defined. As to the external product, we basically deal with pulling back line bundles, and the tensor $\otimes$ and the formal group law commute with the pullback operation, therefore we can see that the external product is also well-defined. It is also clear that it satisfies (A1), $\cdots$, (A8) and (rel-Dim), (rel-Sec) and (rel-FGL).

**Remark 5.17.** If the target scheme $Y$ is a point, then the above $\langle L_\ast R^FGL \rangle(X \xrightarrow{f} Y)$ is equal to the subgroup $\langle L_\ast R^FGL \rangle(X)$ defined in Remark 2.4.11. Hence we have

$$\langle L_\ast R^FGL \rangle(X \xrightarrow{\text{pt}} pt) = \langle L_\ast R^FGL \rangle(X), \quad \Omega^{-i}(X \xrightarrow{\text{pt}} pt) = \Omega_i(X).$$

Therefore we get the following theorem

**Theorem 5.18.** The above theory $\Omega^{\ast}(X \xrightarrow{\text{pt}} S)$ is an oriented Borel–Moore functor with products of geometric type on $\text{Sch}/S$ such that if $S = \text{Spec}(k) = pt$, then $\Omega^{-\ast}(X \rightarrow pt)$ is equal to Levine–Morel’s algebraic cobordism $\Omega_\ast(X)$.

Since $\Omega_\ast(X) := \Omega^{-\ast}(X \rightarrow pt)$ is equal to Levine–Morel’s algebraic cobordism $\Omega_\ast(X)$, in this paper we call the above theory $\Omega^{\ast}(X \xrightarrow{\text{pt}} S)$ algebraic cobordism on $\text{Sch}/S$.

In a similar manner to the proof of the universality of $\mathcal{Z}^{\ast}(X \xrightarrow{\text{pt}} S)$ in Theorem 4.10, we can show the following universality of $\Omega^{\ast}(X \xrightarrow{\text{pt}} S)$:

**Corollary 5.19.** $\Omega^{\ast}(X \xrightarrow{\text{pt}} S)$ is the universal one among the oriented Borel–Moore functor with products of geometric type $A^{\ast}(X \xrightarrow{\text{pt}} S)$ for $S$-schemes.

**Remark 5.20.** Motivated by the present construction, we defined a bivariant-theoretic analogue $\langle R^{\text{Dim}} \rangle(X \xrightarrow{f} Y) \subset \mathcal{Z}^{\ast}(X \xrightarrow{f} Y)$ and we thought that for the quotient

$$\mathcal{Z}^{\ast}(X \xrightarrow{f} Y) := \frac{\mathcal{Z}^{\ast}(X \xrightarrow{f} Y)}{\langle R^{\text{Dim}} \rangle(X \xrightarrow{f} Y)}$$

the above four operations, (i) orientation $\tilde{c}_i(L)$, (ii) the bivariant product $\bullet$, (iii) the bivariant pushforward and (iv) the bivariant pullback were all well-defined. Unfortunately only biarient product was not well-defined. Thus we hope to be able to come up with a reasonable subgroup $\langle R^{\text{Dim}} \rangle(X \xrightarrow{f} Y) \subset \mathcal{Z}^{\ast}(X \xrightarrow{f} Y)$ such that the bivariant product on the quotient $\mathcal{Z}^{\ast}(X \xrightarrow{f} Y)$ is well-defined.

**Remark 5.21.** In a different paper we would like to consider whether we could construct the above algebraic cobordism of $S$-schemes analogously using “double point degeneration” of Levine–Pandharipande’s construction.

6. **Some properties of $\Omega^{\ast}(X \xrightarrow{\text{pt}} X)$**

We denote $\Omega^{\ast}(X \xrightarrow{\text{pt}} X)$ by $\widetilde{\Omega}^{\ast}(X)$ to avoid confusion with Levine–Morel’s algebraic cobordism $\Omega^{\ast}(X)$ in the case when $X$ is smooth. We emphasize that our $\widetilde{\Omega}^{\ast}(X)$ is defined for any scheme $X$.

Similarly we denote $\mathcal{Z}^{\ast}(X \xrightarrow{\text{pt}} X)$ by $\widetilde{\mathcal{Z}}^{\ast}(X)$.

Here we list basic properties of $\widetilde{\Omega}^{\ast}(X)$:

1. For any morphism $f : X \rightarrow Y$ we have the pullback homomorphism

$$f^* : \widetilde{\Omega}^{\ast}(Y) \rightarrow \widetilde{\Omega}^{\ast}(X).$$

It is clear that on the level of $\widetilde{\mathcal{Z}}^{\ast}(X)$ we have the pullback homomorphism

$$f^* : \widetilde{\mathcal{Z}}^{\ast}(Y) \rightarrow \widetilde{\mathcal{Z}}^{\ast}(X).$$
We have the following canonical cap product:
\[ \cap : \hat{\Omega}^i(X) \otimes \Omega_{\dim X}(X) \to \Omega_{\dim X - i}(X), \]
which is defined by
\[ [V \xrightarrow{h} X; L_1, \ldots, L_r] \cap [W \xrightarrow{k} X] := [V \times_X W \xrightarrow{h \times_X k} X; p_1^* L_1, \ldots, p_r^* L_r], \]
where \( p_1 : V \times_X W \to V \) is the projection and note that \( V \times_X W \) is smooth.

In particular, when \( X \) is smooth, we have the following canonical map:
\[ \mathcal{D} : \hat{\Omega}^i(X) \to \Omega_{\dim X - i}(X), \]
which is defined by
\[ \mathcal{D}([V \xrightarrow{h} X, L_1, \ldots, L_r]) := [V \xrightarrow{h} X, L_1, \ldots, L_r] \cap [X \xrightarrow{\text{id}_X} X], \]

namely
\[ \mathcal{D}([V \xrightarrow{h} X, L_1, \ldots, L_r]) := [V \xrightarrow{h} X, L_1, \ldots, L_r]. \]

Since Levine and Morel define \( \Omega^i(X) := \Omega_{\dim X - i}(X) \) in the case when \( X \) is smooth, the above canonical homomorphism \( \mathcal{D} \) is also expressed as
\[ \mathcal{D} : \hat{\Omega}^i(X) \to \Omega^i(X). \]

Suppose that \( X \) is not smooth. Then whenever we are given a resolution of singularities \( \pi : \tilde{X} \to X \), we have the corresponding homomorphism
\[ \mathcal{D}_\pi : \hat{\Omega}^i(X) \to \Omega_{\dim X - i}(X), \]
which is defined by
\[ \mathcal{D}_\pi([V \xrightarrow{h} X, L_1, \ldots, L_r]) := [V \xrightarrow{h} X, L_1, \ldots, L_r] \cap [\tilde{X} \xrightarrow{\pi} X] \]
\[ = [V \times_X \tilde{X} \xrightarrow{h \times_X \pi} X; p_1^* L_1, \ldots, p_r^* L_r], \]
where \( p_1 : V \times_X \tilde{X} \to V \) is the projection and note that \( V \times_X \tilde{X} \) is smooth. Here we note that all the resolutions of singularities make a direct system, indeed we can see this as follows. Let \( \mathcal{R}_X \) denote the set of all the resolution of singularities of \( X \). If \( X \) is nonsingular, then \( \mathcal{R}_X \) is defined to be just \( \{ \text{id}_X : X \to X \} \), the identity map. For two resolutions \( \pi_1 : \tilde{X}_1 \to X \) and \( \pi_2 : \tilde{X}_2 \to X \) we define the order \( \pi_1 \leq \pi_2 \) by
\[ \pi_1 \leq \pi_2 \iff \exists \pi_{12} : \tilde{X}_2 \to \tilde{X}_1 \text{ such that } \pi_2 = \pi_1 \circ \pi_{12}. \]

Then we have

**Lemma 6.2.** The ordered set \((\mathcal{R}_X, \leq)\) is a directed set.

**Proof.** Indeed, for any two resolutions \( \pi_1 : \tilde{X}_1 \to X \) and \( \pi_2 : \tilde{X}_2 \to X \), we consider the following fiber product:
\[ \tilde{X}_1 \times_X \tilde{X}_2 \xrightarrow{\pi_2} \tilde{X}_2 \]
\[ \pi_1 \]
\[ \tilde{X}_1 \xrightarrow{\pi_1} X. \]

Let \( \pi : \tilde{X}_1 \times_X \tilde{X}_2 \to X \) be a resolution of singularities and let \( \pi_3 : \tilde{X}_1 \times_X \tilde{X}_2 \to X \) be the composite
\[ \pi_3 = \pi_1 \circ (\pi_2 \circ \pi) = \pi_2 \circ (\pi_1 \circ \pi). \]

Which means that \( \pi_1 \leq \pi_3 \) and \( \pi_2 \leq \pi_3 \), therefore \((\mathcal{R}_X, \leq)\) is a directed set. \( \Box \)
Now, for each resolution \( \pi : \tilde{X} \to X \in \mathcal{R}_X \), we let
\[
\Omega_{\dim X - i}(X)_{\pi} := \text{Im} D_{\pi}(\tilde{\Omega}(X))
\]
\[
= \left\{ [V \xrightarrow{h} X, L_1, \cdots, L_r] \cap [\tilde{X} \xrightarrow{\pi} X] \mid [V \xrightarrow{h} X, L_1, \cdots, L_r] \in \tilde{\Omega}(X) \right\} \subset \Omega_{\dim X - i}(X).
\]
Then we have a directed system \( \{ \Omega_{\dim X - i}(X)_{\pi}, \phi_{\pi_1 \pi_2} \} \) where for \( \pi_1 \leq \pi_2 \) the morphism \( \phi_{\pi_1 \pi_2} : \Omega_{\dim X - i}(X)_{\pi_1} \to \Omega_{\dim X - i}(X)_{\pi_2} \) is defined by
\[
\phi_{\pi_1 \pi_2}([V \xrightarrow{h} X, L_1, \cdots, L_r] \cap [\tilde{X}_1 \xrightarrow{\pi_1} X]) = [V \xrightarrow{h} X, L_1, \cdots, L_r] \cap [\tilde{X}_2 \xrightarrow{\pi_2} X]
\]
Thus we can define the canonical map
\[
\tilde{D} : \tilde{\Omega}(X) \to \lim_{\pi \in \mathcal{R}_X} \Omega_{\dim X - i}(X)_{\pi} \subset \Omega_{\dim X - i}(X)
\]
by
\[
\tilde{D}([V \xrightarrow{h} X, L_1, \cdots, L_r]) := \lim_{\pi \in \mathcal{R}_X} D_{\pi}([V \xrightarrow{h} X, L_1, \cdots, L_r]).
\]

7. A REMARK ON A RELATION WITH GONZALÉZ–KARU’S OPERATIONAL BIVARIANT ALGEBRAIC COBORDISM

Finally we want to mention about a relation with González–Karu’s operational bivariant algebraic cobordism \([4]\), which shall be denoted by \( B_{op}^{GK} \Omega(X \to Y) \).

We expect that there is a canonical transformation
\[
\tau : \Omega^*(X \xrightarrow{\pi_S} S) \to B_{op}^{GK} \Omega(X \xrightarrow{\pi_S} S)
\]
defined as follows: for each element \( [V \xrightarrow{h} X; L_1, \cdots, L_r] \) and for any morphism \( g : S' \to S \)
\[
\tau([V \xrightarrow{h} X; L_1, \cdots, L_r]) := \left\{ h' \circ \tilde{c}_1((g')^* L_1) \circ \cdots \circ \tilde{c}_1((g')^* L_r) \circ (\pi_{X'} \circ h')^* : \Omega_*(S') \to \Omega_*(X') \right\}_{g : S' \to S},
\]
where we consider the following fiber squares:
\[
\begin{array}{ccc}
V' & \xrightarrow{g'} & V \\
\downarrow h' & & \downarrow h \\
X' & \xrightarrow{g'} & X \\
\downarrow \pi_{X'} & & \downarrow \pi_X \\
S' & \xrightarrow{g} & S
\end{array}
\]
We would like to treat this in a different paper.
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