Minority game with arbitrary cutoffs

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We study a model of a competing population of $N$ adaptive agents, with similar capabilities, repeatedly deciding whether to attend a bar with an arbitrary cutoff $L$. Decisions are based upon past outcomes. The agents are only told whether the actual attendance is above or below $L$. For $L \sim N/2$, the game reproduces the main features of Challet and Zhang’s minority game. As $L$ is lowered, however, the mean attendances in different runs tend to divide into two groups. The corresponding standard deviations for these two groups are very different. This grouping effect results from the dynamical feedback governing the game’s time-evolution, and is not reproduced if the agents are fed a random history.

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I. INTRODUCTION

Complex adaptive systems \cite{1} have been the subject of much recent attention. These systems typically exhibit rich global behaviour which cannot be straightforwardly deduced from the microscopic details of the constituent objects (agents). Fascinating self-organized phenomena \cite{2} have been observed or discussed for a wide range of systems such as sandpiles, traffic flow, financial markets and other social phenomena. Within the physics community, moreover, many ‘microscopic’ economics-based models have been proposed \cite{3,4} in the growing subfield of ‘econophysics’.

Challet and Zhang \cite{5} recently introduced a simple minority game in which agents repeatedly compete to be in the minority group. These agents have similar capabilities and each of them makes decisions based on the past history of outcomes. Further work on the basic minority game is presented in Refs. \cite{6,7}. While the minority game is stated in terms of agents choosing between two rooms, the model can be stated in different ways to fit different situations; for example, the agents may as well be deciding whether to buy or sell a certain stock in a simple market model. A more general version of the minority game, namely the bar-attendance model \cite{8} in which agents decide whether to attend a bar with a certain seating capacity, has also been studied recently \cite{9}.

In this paper, we generalize the minority game to the case of arbitrary cutoff. There are $N$ agents, each of whom possesses $s$ strategies, deciding whether to attend a bar with a seating capacity, i.e. cutoff, equal to $L$. Good (bad) decisions correspond to attending an undercrowded (overcrowded) bar, or not attending an overcrowded (undercrowded) room. The only information given to the agents after each run is whether the actual attendance was above or below the cutoff. We present extensive numerical results for the mean attendances and corresponding standard deviations as a function of the agents’ capabilities and the cutoff $L$. For $L = N/2$, the game reduces to the minority game \cite{6,7}. The present model for $L \neq N/2$ thus represents an intermediate model between the minority game \cite{6,7} and the bar-attendance model \cite{5,8}; it allows for arbitrary cutoff as in the bar-attendance model, but the actual attendances are not announced as in the minority game. As $L$ is reduced below $N/2$, the mean attendances and standard deviations tend to be distributed into two groups of values for different runs of the same game. One group comprises a mean attendance which is insensitive to the agents’ capabilities, and is accompanied by a small standard deviation of attendance. The other group contains a spread in mean attendances, each with a larger corresponding standard deviation. An explanation of these features is given.

The plan of the paper is as follows. In Sec. II, the game with arbitrary cutoff is defined. Results for the mean attendances and standard deviations for different cutoff $L$, and for different number of strategies $s$ per agent, are presented and discussed in Sec. III. The relationship between games with cutoff $L$ and $L' = N - L$ is also discussed. Section IV summarizes our main findings.

II. THE GENERALIZED MINORITY GAME

Consider a game with $N$ agents deciding whether to go to a bar with a seating capacity of $L$. Let the actual attendance at the bar in the $n$-th turn be $A_n$. If $A_n \leq L$, the outcome, which is the only information made known to all agents, is the signal ‘undercrowded’. In contrast, if $A_n > L$ then the outcome is the signal ‘overcrowded’. Hence, the outcome can be represented by a string of zeros (representing, say, ‘undercrowded’).
and ones (representing ‘overcrowded’). The value of the cutoff \( L \) is not announced. The agents are not allowed to communicate among themselves. They interact with each other through the common knowledge of the past history of outcomes, and the fact that each agent’s decision is influencing the others’ chances of being right. All agents are assumed to have the same level of capability and to decide the next move based on the most recent \( m \) outcomes. The strategy space, therefore, consists of a total of \( 2^m \) strategies, analogous to the minority game \([6]\). At the beginning of the game, each agent randomly picks \( s \) strategies from the pool of strategies, with repetitions allowed during picking. Each agent uses his best strategy in making the next decision, i.e., he uses the one with the highest accumulated merit points. The merit points are assigned in the following way. After the outcome in a turn is announced, every agent assigns a point to each of his strategies which would have made the correct decision. The correct decisions are attending (not attending) the room with the outcome being ‘undercrowded’ (‘overcrowded’). The incorrect decisions are attending (not attending) the room with the outcome being ‘overcrowded’ (‘undercrowded’). The original minority game \([6]\) thus corresponds to having an odd value of \( s \) (‘undercrowded’). The incorrect decisions are attending (not attending) the room with the outcome being ‘overcrowded’ (‘undercrowded’). The value of the ‘mental’ value of a particular market.

In this case, the relevant cutoff will be larger than the price will go up and the majority can then benefit. For example, when many agents are buying the same stock, the outcome is a binary digit instead of the actual attendance is above or below the cutoff, i.e., the actual attendance is above or below the cutoff, i.e. the outcome is a binary digit instead of the actual attendance. This generalized model may be relevant to some situations arising in trading in real markets. For example, when many agents are buying the same stock, the price will go up and the majority can then benefit. In this case, the relevant cutoff will be larger than \( N/2 \). Alternatively, \( L \) may represent some underlying ‘fundamental’ value of a particular market.

### III. RESULTS AND DISCUSSION

We have performed numerical simulations for a range of values of the cutoff \( L \). The number of agents is fixed at \( N = 101 \). One of the interesting results in the minority game is that the mean standard deviation (SD) or the mean volatility, i.e., the average of SD’s over different runs, shows a minimum as a function of \( m \) for small values of \( s \) (e.g., \( s = 2, 3, \cdots \)) \([3, 6]\). The minimum value of the mean SD is smaller than for the ‘random’ game in which agents independently decide by tossing a coin \([6]\); the system has thereby managed to self-organize itself into a state which maximizes the number of satisfied agents in the community. This minimum has been explained by invoking the idea that the effects of the crowd of agents using the best strategy can become counter-balanced by the effects of the anticrowd of agents using the corresponding anti-correlated strategy \([8, 10]\).

Figure 1 shows the standard deviation (SD) and the mean attendance (inset) for \( s = 2 \) as a function of \( m \), with cutoff \( L = 48 \); this represents a small deviation from the minority game \((L = 50)\). For each value of \( m \), 32 runs were carried out using different random initial conditions. Each run corresponds to a total of 10000 turns. It is clear that there still exists a minimum in the mean SD at around \( m = 5 \), a feature analogous to the minority game result \([6, 7]\). For small \( m \), the SD is large since the strategy space is small. For large \( m \), the mean SD approaches the random coin-toss limit of \( \sigma = \sqrt{N/2} \). For small \( m \), the spread in the mean attendance is larger. For large \( m \), the mean attendance is slightly larger than the cutoff and the spread in the mean attendance is small. It should be noted that the averaged mean attendance for \( L = 48 \) is lower than that in the minority game with \( L = 50 \), indicating the adaptation of the population to the lower cutoff value: this occurs despite the fact that the cutoff \( L \) is unknown to the agents.

Figure 2 shows the SD and mean attendance for \( s = 2 \) and cutoff \( L = 40 \). In general, the SD’s in the 32 runs spread over a much larger range of values than in the case with \( L = 48 \). For the minority game, the SD’s for \( m = 2, 3 \) are all above the value corresponding to the random coin-toss limit. For \( L = 40 \), however, the SD’s at \( m = 3 \) take on values below the coin-toss limit. An interesting feature in Fig. 2 is the tendency of having small SD’s in some runs for all values of \( m \). A minimum in the averaged SD exists at a smaller value of \( m \) (i.e. \( m \approx 3 \)) than for \( L = 48 \). The values of the mean attendance at each value of \( m \) also tend to spread over a larger range than for \( L = 48 \). The averaged mean attendance is lower than for \( L = 48 \), demonstrating once more the adaptation of the population to the (unknown) cutoff.

As the cutoff \( L \) is further reduced, a new feature arises: the results for the SD and mean attendance show the formation of two separate groups. One group consists of large SD’s with magnitudes similar to the minority game, while the other group consists of small SD’s. Figure 3 shows the results for \( L = 30 \). For each value of \( m \), the upper SD branch is broad while the lower branch is narrow. The lower branch has a small mean SD (SD < 1) and does not vary much with \( m \). If we take an average over the two branches to obtain a mean SD as in the minority game, the result is that the minimum in the mean SD gradually shifts to lower values of \( m \) as the cutoff is lowered. However, such an average may be misleading as the two branches actually correspond to different types of outcomes (i.e., trajectories) of the game. Taken together with the previous results for \( L = 48 \) and 40, the gradual splitting into two branches of both the SD and mean attendance distributions is clearly shown as \( L \) decreases. From this perspective, the minority game with \( L = N/2 \) corresponds to the special case in which the two branches merge together to form a rather symmetrical distribution.

The formation of a branch with a steady mean attendance together with a corresponding small SD, for small values of \( s \), can be understood qualitatively as follows.
For small $s$, the chance for an agent to have picked $s$ strategies with the same response to a particular $m$-bit history string, is not negligible. Suppose that the recent string of outcomes is $\cdots 111111$, representing a series of ‘overcrowded’ attendances. Let $N'$ be the number of agents who only have strategies with the decision to attend the bar when faced with an $m$-bit history consisting of just 1’s. If $N' > L$, these agents will lose, but they will keep attending since they have no alternative strategies. The agents who decide not to attend, i.e. those having strategies with the decision to stay given the $m$-bit history of 1’s, will stay away and win. The mean attendance is hence approximately $N'$ and the corresponding SD is small. The series of outcomes $\cdots 11111111 \cdots$ thus corresponds to an attractor of the dynamics for $N' > L$.

If $N' < L$, the $N'$ agents will attend when faced with an $m$-bit history of 1’s. Depending on the decisions of the other $(N - N')$ agents, the next outcome can either be ‘overcrowded’ or ‘undercrowded’, with a higher chance of overcrowding for $N' \lesssim L$. Therefore, the series $\cdots 111111 \cdots$ with all 1’s is no longer an attractor: the series of outcomes now consists of 0’s and 1’s. The mean attendance will be larger in this case with a larger spread. The 0’s and 1’s in the time series lead to a larger mean SD as the outcomes make the game look more like the minority game. For $N' \lesssim L$, the mean attendance will be higher than the cutoff as there will be more ‘overcrowded’ attendances than ‘undercrowded’ attendances. For fixed $N$, the splitting into two branches appears in the regime of intermediate $L$ where the situations $N' < L$ and $N' > L$ occur with comparable probabilities. For $s = 2$, there will be about $N/4$ agents picking two strategies with the same response to a given history on the average. However, in each run there will be fluctuations around this mean number. The game with $L = 30$ and $N = 101$ represents the situation in which $N' < L$ and $N' > L$ may arise for different runs, and hence leads to the formation of the two branches in the mean attendance and SD. If we further reduce the cutoff, more runs corresponding to $N' > L$ (i.e. the lower branch) occur.

It has recently been argued that the real history time-series, and hence the memory, is irrelevant in the minority game. The real $m$-bit history fed to the agents is replaced by a random $m$-bit string, then the standard deviation of the attendance at one of the rooms, say 0, is essentially unchanged. However, replacing the real history by a random one in the present system does not reproduce the results in Fig. 3 for $L = 30$. Both the spreads and values for the upper branch of the SD and mean attendance differ between the random and real-history versions. More importantly, the lower branch is completely absent in the random-history version. This is because the lower branch depends crucially for its existence on non-local time correlations in the real history time-series (e.g. $\cdots 11111 \cdots$). Our findings therefore demonstrate the possible dangers of replacing the true history by a random one when evaluating dynamical properties of such complex adaptive systems. We note in passing that even in the basic minority game with $L = N/2$, correlations do arise in the real history time-series.

Figure 4 shows the mean attendance for $L = 70$ and $s = 2$. From the definition of the game, games with cutoff $L' = N - L$ are related to games with cutoff $L$ in that the mean attendance in the game with $L'$ can be found from the mean population of agents not attending the room in a game with $L$. It is, therefore, sufficient to study the range of values with $L \leq N/2$. Comparing the results in Fig. 3 ($L = 30$) with those in Fig. 4 ($L' = 70$), the symmetry between games with $L'$ and $L$ is clearly shown.

Figures 5 and 6 show results with a larger value of $s$ ($s = 5$) for the cases of $L = 40$ and 30, respectively. For $L = 40$, the minimum in the mean SD can clearly be seen at $m = 5$. For $L = 30$ the minimum, although not as sharp as for larger values of $L$, is at $m = 4$: for larger $m$, the SD’s start to spread out and show the tendency to distribute into two groups. Comparing the results at cutoff $L = 30$ for $s = 2$ (Fig. 3) and $s = 5$ (Fig. 6), the formation of the two branches is much more pronounced for small $s$. This is consistent with our argument above since the chance that an agent picks $s$ strategies with the same response to a particular $m$-bit history is smaller for larger values of $s$. Hence, having $L = 30$ and $s = 5$ corresponds to the case with $N' \ll L$: the resulting behavior is hence similar to a minority game, with the exception that some runs have smaller SD’s. It should also be noted from the results for both $s = 2$ and $s = 5$ that the minimum in the mean SD gradually shifts to a lower value of $m$ as the cutoff is reduced.

**IV. CONCLUSIONS**

A generalized version of the minority game with arbitrary cutoff $L$ was proposed and studied. Features in the mean attendance and standard deviation can be quite different from the basic minority game as the cutoff shifts away from $N/2$. In particular, the mean attendances in different runs tend to divide into two groups. The corresponding standard deviations for these two groups are very different. These features are not reproduced if the agents are fed a random history, thereby demonstrating the importance of dynamical feedback and hence memory in this system.

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Figure Captions

Figure 1: The standard deviation (SD) and mean attendance (inset) as a function of the memory $m$ for $N = 101$ agents, $s = 2$ strategies per agent and bar cutoff $L = 48$. For each value of $m$, data from 32 different runs are shown.

Figure 2: Same as in Figure 1 for $N = 101$, $s = 2$ and $L = 40$.

Figure 3: Same as in Figure 1 for $N = 101$, $s = 2$ and $L = 30$.

Figure 4: Same as in Figure 1 for $N = 101$, $s = 2$ and $L = 70$. Note the relationship between the games with $L = 30$ (Figure 3) and $L = 70$.

Figure 5: The standard deviation and mean attendance (inset) as a function of $m$ for $N = 101$, $s = 5$ and cutoff $L = 40$. For each value of $m$, data from 32 different runs are shown.

Figure 6: Same as in Figure 5 for $N = 101$, $s = 5$ and $L = 30$. 