Robust Factor Analysis without Moment Constraint

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In large-dimensional factor analysis, existing methods, such as principal component analysis (PCA), assumed finite fourth moment of the idiosyncratic components, in order to derive the convergence rates of the estimated factor loadings and scores. However, in many areas, such as finance and macroeconomics, many variables are heavy-tailed. In this case, PCA-based estimators and their variations are not theoretically underpinned. In this paper, we investigate into the $L_1$ minimization on the factor loadings and scores, which amounts to assuming a temporal and cross-sectional median structure for panel observations instead of the mean pattern in $L_2$ minimization. Without any moment constraint on the idiosyncratic errors, we correctly identify the common components for each variable. We obtained the convergence rates of a computationally feasible $L_1$ minimization estimators via iteratively alternating the median regression cross-sectionally and serially. Bahardur representations for the estimated factor loadings and scores are provided under some mild conditions. Simulation experiments checked the validity of the theory. In addition, a Robust Information Criterion (RIC) is proposed to select the factor number. Our analysis on a financial asset returns data set shows the superiority of the proposed method over other state-of-the-art methods.

Key Word: $L_1$ minimization; Large-dimensional factor analysis; Principal component analysis; Robust Information Criterion.

1 Introduction

Factor models are widely used in practice such as biology, computer science, economics and finance. The mathematical expression of a large dimensional static approximate factor model is

$$(Y_t)_{p \times 1} = L_{p \times r} (f_t)_{r \times 1} + (\epsilon_t)_{p \times 1}, \quad t = 1, ..., T,$$

where $Y_t$ is a $p$-dimensional vector observed at time $t$, $L$ is the factor loading matrix, $f_t$ is a vector of factors at time $t$, and $\epsilon_t$ is the idiosyncratic component that can be cross-sectionally weakly dependent. Recent years have seen increasing interest in statistical inference on model (1). The approximate factor structure instead of the strict factor structure was introduced and studied in Chamberlain and Rothschild (1998). Bai and Ng (2002) presented information criterions to determining the number of factors under the framework of the static approximate factor model. Bai (2003) further gave the asymptotic theory on the estimated factor loadings and scores. Stock and Watson (2002a)(2002b) incorporated the factors into the autoregressive model
to predict macroeconomic variables and detect structural changes of macroeconomics. Ahn and Horenstein (2013) proposed the eigenvalue-space-ratio estimators of the number of factors. Trapani (2017) sequentially tested the divergence of eigenvalues and found a consistent estimate of the number of factors. Onatski (2009) provided a hypothesis testing procedure to a prefixed number of factors. Kong et al. (2019) established the theory of empirical processes of the series of estimated common components and idiosyncratic components. With high-frequency data, Pelger (2018), Chen et al. (2020), Kim and Wang (2016), Kong (2017)(2018) and Kong and Liu (2018) extensively studied the continuous-time version of model (1). In the seminal paper by Forni et al. (2000), the authors proposed a generalized dynamic factor model that can accommodate a factor space of infinite dimension and the factors are loaded via linear filters. Adapting to the dynamic feature, Hallin and Liska (2007) developed an information criterion to estimate the number of factors. In this paper, we only consider robust estimation of the factors and loadings under the static model and leave extensions to the generalized dynamic factor model to our future work.

Most existing works start from decomposing the covariance matrix of the observed vector into a low-rank matrix contributed by the factors and a covariance matrix due to the idiosyncratic error vector. This decomposition is essentially the factor structure of the dispersion of high-dimensional data vectors. A basic requisite of the factor structure is the finiteness of second moment of idiosyncratic errors. For the \( \sqrt{T} \wedge \sqrt{p} \) convergence rate of the estimated factor loadings and scores, typically the finiteness of fourth moment of idiosyncratic errors is assumed in the literature mentioned above. Theoretically, a natural question is “how to do factor analysis if the fourth moment or even the second moment does not exist”. In practice, many macroeconomic variables have heavy-tailed distributions, and thus the assumption in most recent PCA-based factor analysis papers is violated. In finance, a stylized empirical fact of asset returns is leptokurtosis, c.f., Chapter 1 of Tankov and Cont (2004) and Kong et al. (2015). This motivates us to find a way to do factor analysis under model (1) without any moment constraint on the idiosyncratic errors and with theoretical guarantee and computational feasibility.

To the best of our knowledge, few papers considered robust factor analysis except for He et al. (2020) and Calzolari and Halbleib (2018). The former paper provided robust consistent estimates of the factor loadings and scores using an eigen-analysis of the spatial Kendall’s tau matrix. However, it assumed a joint elliptical distribution for the large cross-section of idiosyncratic components, which rules out the typical family of stable distributions. The latter paper assumed the stable distribution for independent factors and idiosyncratic noises and did factor analysis with indirect inference, but no asymptotic econometric theory was established. The \( L_1 \) minimization in (3) below is completely nonparametric and we are aimed at giving reliable asymptotic results for separating common components from each variable and for estimating the factor loadings and scores.

Our methodology is inspired by the equivalence of PCA and double least square estimation when there aren’t missing values. That is the PCA-based estimators of the factor loadings and scores are identical to

\[
(L, F) = \arg \min_{L,F} \{ \sum_{i=1}^{p} \sum_{t=1}^{T} (y_{it} - l_i' f_t)^2 \},
\]

up to some orthogonal transformations, where \( L = (l_1, ..., l_p)' \) and \( F = (f_1, ..., f_T) \). As in robust regression,
we simply replace the quadratic loss function by the absolute loss function. That being said,

\[
(\hat{L}, \hat{F}) = \arg \min_{L,F} \left\{ \sum_{i=1}^{p} \sum_{t=1}^{T} |y_{it} - L_i^t f_t| \right\},
\]

(3)

where \( \hat{L} = (\hat{l}_1, ..., \hat{l}_p)' \) and \( \hat{F} = (\hat{f}_1, ..., \hat{f}_T) \). The optimization solutions to (2) and (3) have the advantage that they are not much affected by the missing values in \( Y \)'s compared with the PCA solution. This is because the PCA solution relies on the input of a sample covariance matrix. To calculate the sample covariance matrix, one needs to delete the \( t \)-th column of \( Y \) if \( y_{it} \) is missing for some \( i \) or impute \( y_{it} \) with some extra effort. For optimizing the loss functions in (2) or (3), only the single loss term containing \( y_{it} \) needs to be deleted when \( y_{it} \) is missing. This advantage is advocated in machine learning area, such as image processing, c.f., Ke and Canade (2005) and Aanæs et al. (2002). However, no econometric or statistical theory had ever been presented in machine learning field. The major difficulty in deriving the asymptotic theory of the estimated factors and loadings via optimizing the double \( L_1 \) loss in (3) lies in three aspects. First, the minimizers of (3) have no closed form expression compared with the PCA solution (or equivalently the \( L_2 \) minimizer of (2)). Second, the double \( L_1 \) loss function in (3) is not a jointly convex function of \( L \) and \( F \), which is totally different from the least absolute deviation (LAD) setting in median regressions. Third, there are a large number of parameters to be optimized in (3) as \( p, T \to \infty \) simultaneously. This makes it hard to construct a small ball containing the true parameters in the parameter space, in contrary to the typical derivation of the consistency of the LAD estimators in regressions, c.f., Pollard (1991).

The \( L_1 \) minimization in (3) amounts to saying that the medians of a large cross-section of asset returns are driven by the common factor vector \( f_t \) and the corresponding exposures are measured by the loading matrix \( L \). And dynamically the medians of the return series are modeled by \( \{l_i^t f_t\} \) given latent \( f_t \). This implies the identifiability condition for the idiosyncratic components, \( Q(\epsilon_{it}|f_1, ..., f_t) = 0 \) for all \( i = 1, ..., p \) and \( t = 1, ..., T \), where \( Q(X|Y) \) refers to the median of \( X \) given \( Y \). Compared with the mean zero condition for \( \epsilon_{it} \)'s in \( L_2 \) estimation, the above identifiability condition allows for non-zero mean return (or risk premium) for the idiosyncratic part. In this paper, we derive the convergence rates of a computational feasible \( L_1 \) estimators of the common components, factor loadings and scores. We show that up to some orthogonal transformations, the \( L_1 \) estimators of the factors and loadings converge at rate \( \frac{1}{\sqrt{T}} \wedge \frac{\log p}{\sqrt{T}} \), where the \( \log p \) term stems from the aggregation of the estimation errors along the cross-sectional dimension in the solution path. Our results do not need any moment constraint on the idiosyncratic components. Under some mild conditions, we obtained the Barhadur representations of the estimated factor loadings and factors.

The present paper is arranged as follows. In Section 2, we present some setup assumptions and provide the main results of computational feasible \( L_1 \) estimators, realized by an iterative algorithm to solve the non-convex objective function in (3). Extensive simulation studies and an empirical study are given in Section 3 and Section 4, respectively. A brief conclusion and discussions on future work are given in Section 5. All the technical proofs are relegated to the Appendix.

2 Assumptions and main results

It is well known that the factor loadings and factors are only identifiable up to some orthogonal transformations. This gives the freedom to restrict the columns of the factor loading matrix to be orthogonal vectors spanning the same factor space. Notice also that the factor space spanned by the columns of \( L \) is the same
as that spanned by the $r$ principal components of $L\text{Cov}(f_t)L'$, without loss of generality and as in Fan et al. (2013), we assume that $L$ and $F$ have the canonical form in (4) below.

**Assumption 1.** (1) 

\[ L'L/p \text{ is diagonal and } \Sigma_f = \text{Cov}(f_t) = I_r, \]  

where $I_r$ stands for the $r \times r$ identity matrix and the diagonal elements of $L'L/p$ are bounded away from zero and infinity;

(2) \{f^0_t\} is a stationary and $\alpha$-mixing sequence of random vectors satisfying $E\|f^0_t\|^4 \leq C$ for some constant $C > 0$. 

Given Assumption 1(1), the $L_1$ minimization (3) can be done subject to 

\[ L'L/p \text{ is diagonal and } \frac{1}{T} \sum_{t=1}^{T} f_t'f_t' = I_r. \]  

Assumption 1(1) also assumed a strong factor condition saying that the signal strength of the common components grows at rate $p$. This condition is mainly used to derive the second-order property of the estimators. For only the consistency, this might be relaxed to the weak factor condition that $L'L/p^\alpha$ has bounded eigenvalues for some $0 < \alpha < 1$ as long as the common and idiosyncratic components are separable asymptotically. Assumption 1(2) is a standard assumption on the factor series, c.f., Fan et al. (2013) and the references therein.

**Assumption 2.** 

\[ Q(\epsilon_{it}|f_1, ..., f_t) = 0. \]  

Assumption 2 is an identifiability condition for $L_1$ optimization. When the factors are observable, it is simply the identifiability condition used in median regression. It is equivalent to stating $Q(y_{it}|f_1, ..., f_t) = L'_i f_t$ which means the medians of a large cross-section of asset returns are driven by the common factor vector $f_t$ and the corresponding exposures are measured by the loading matrix $L$. This is not in accordance with the classic CAPM theory which explains the mean cross-section excess returns via exposure to the value of the market portfolio. But the focus of the present paper is not on the finance theory but an econometric or statistical investigation into the $L_1$ estimators of the factor loadings, scores, and the common and idiosyncratic components under (3), (4) and (6).

Before presenting the next assumption on temporal and cross-sectional weak dependence on bounded functionals of $\{\epsilon_{it}\}$’s, we introduce two sums of bounded functionals of $\epsilon_{it}$. Let 

\[ H_1(\{\epsilon_{it}\}, p, T) = \sum_{i=1}^{p} \sum_{t=1}^{T} \{ (\sigma_{it} - \tau_{it}) \text{sign}(\sigma_{it}) I(0 \wedge \sigma_{it} < \tau_{it} < 0 \vee \sigma_{it}) \} \]  

\[-E_f(\sigma_{it} - \tau_{it}) \text{sign}(\sigma_{it}) I(0 \wedge \sigma_{it} < \tau_{it} < 0 \vee \sigma_{it}) \} \]

where $E_f$ stands for conditional expectation on $\{f^0_t\}$ (the true factor vector), $\sigma_{it}$’s are bounded variables, $\tau_{it} = \epsilon_{it} - \mu_{it}$ with $\mu_{it}$’s being fixed parameters. Let 

\[ H_2(\{\epsilon_{it}\}, p, T) = \sum_{i=1}^{p} \sum_{t=1}^{T} c_i t \{ I(\tau_{it} < 0) - I(\tau_{it} > 0) - E_f[I(\tau_{it} < 0) - I(\tau_{it} > 0)] \}, \]
where $c_{i\ell}$'s are bounded coefficients irrelevant to $\epsilon_{i\ell}$'s.

**Assumption 3.** (1) $\epsilon_{i\ell}$ has probability density function $h_i(x)$ satisfying $\min_i h_i(x) > 0$ for all $x \in \mathbb{R}$. The derivative function $\hat{h}_i(x)$ of $h_i(x)$ is bounded uniformly in $i$. For $M$ large enough, $\min_i \inf_{|x| \leq M} h_i(x) > c > 0$ for some constant $c$ and $h_i(x)$ does not increase as $|x| \to \infty$ for $|x| > M$;

(2) $E\{H_1(\{\epsilon_{i\ell}\}, p, T)/\sqrt{pT\max_{i\ell} |\sigma_{i\ell}|^3}\} \leq C$ and $E\{H_2(\{\epsilon_{i\ell}\}, p, T)/\sqrt{pT\max_{i\ell} \epsilon_{i\ell}^2}\} \leq C$.

Assumption 3(1) is a regular condition on the distribution functions of the idiosyncratic components. It assumes that the probability density functions of $\epsilon_{i\ell}$'s have uniform support. The assumption does not impose any moment constraint on $\epsilon_{i\ell}$'s. The moment condition in Assumption 3(2) assumes that a series of bounded functions of $\epsilon_{i\ell}$'s are weakly correlated temporally and cross-sectionally, under which the $\sqrt{pT\max_{i\ell} |\sigma_{i\ell}|^3}$ and $\sqrt{pT}$ give the scales of $H_1(\{\epsilon_{i\ell}\}, p, T)$ and $H_2(\{\epsilon_{i\ell}\}, p, T)$, respectively.

Different from the optimization problem (2), problem (3) has no explicit closed form solution. Yet the SVD algorithm designed for (2) with no missing values is not applicable to solving (3). To be computationally feasible, we introduce an alternating iterative algorithm to solve the optimization problem in (3). Although the optimization problem (3) is in general non-convex jointly in all parameters, it is indeed convex in $L$ (or $F$) when $F$ (or $L$) is fixed in advance. The above fact motivates to minimize the loss function alternatively over $L$ and $F$, each time optimizing one argument while keeping the other fixed. The alternative optimization step can be solved by linear programming or many other gradient descent schemes. The detailed algorithm is presented in Algorithm 1.

**Algorithm 1 Iterative Algorithm for Robust Factor Analysis**

**Input:** $\mathcal{D} = \{y_{it}, t = 1, \ldots, T\}$

**Output:** Alternating Iterative Estimates of the factor loadings and scores, i.e., $\hat{L}$, $\hat{F}$

1. **Initialization:** $k = 0$; Set $L^{(0)} = (l^{(0)}_{ij})$ so that (5) is satisfied.

2. $\hat{F}^{(k)} = \arg \min_F \|Y - \hat{L}^{(k-1)}F\|_1$, where $\hat{L}^{(0)} = L^{(0)}$, and then transform $\hat{F}^{(k)}$ so that (5) is satisfied.

3. $\hat{L}^{(k)} = \arg \min_L \|Y - L\hat{F}^{(k)}\|_1$ and then transform $\hat{L}^{(k)}$ so that (5) is satisfied.

4. Repeat Steps 2-3 until convergence.

5. Output $\hat{L} = \hat{L}^{(K)}$ and $\hat{F} = \hat{F}^{(K)}$ as the final estimates of the factor loading and score matrices when the convergence condition is met.

Algorithm 1 amounts to alternatively carrying out cross-sectional median regression on factors and serial median regression on loadings, starting from some initial guess of $L$. One could also start from an initial guess of $F$ and alternating the serial and cross-section median regression iteratively. To reduce the sensitivity in the initial parameter values, we can try to set a different initial parameters and choose the solution resulting in lower $L_1$ loss. As for the convergence criterion, denote the factor loading and score matrices at the $k$-th step as $\hat{L}^{(k)} = (\hat{l}^{(k)}_{ij}), \ldots, \hat{l}^{(k)}_p \rangle, \hat{F}^{(k)} = (\hat{f}^{(k)}_{ij}) = (\hat{f}^{(k)}_1, \ldots, \hat{f}^{(k)}_p)$ and let $\mathbf{C}^{(k)} = \hat{L}^{(k)}\hat{F}^{(k)} = (c^{(k)}_{ij})$. In our simulation studies, the iteration is terminated with a prefixed finite number of alternating steps or when

$$\sum_{i} \sum_{j} |c_{ij}^{(K)} - c_{ij}^{(K-1)}|/(pT|c_{ij}^{(K-1)}|) = o\left(\frac{\log p}{\sqrt{T}} + \frac{1}{\sqrt{p}}\right), \quad (7)$$

which means that the average relative iteration error for computing the common components are small enough compared with the estimation error theoretically obtained in Theorem 1 below. Our simulation
experience shows that the above accuracy condition is always met within a finite number of iterations. \( \tilde{l}_i^{(k)} \)'s and \( \tilde{f}_i^{(k)} \)'s form a solution path of the algorithm. The alternating iterative estimators are simply the ending-step solutions of the path. Notice that \( \tilde{L} \) and \( \tilde{F} \) are generally different from \( \hat{L} \) and \( \hat{F} \). \( \hat{L} \) and \( \hat{F} \) are computationally feasible while \( \tilde{L} \) and \( \tilde{F} \) are only theoretical minimizers. Therefore \( \tilde{l}_i^{(k)} \)'s and \( \tilde{f}_i^{(k)} \)'s incur two sources of errors, the computing error for a fixed sample measured by the discrepancy between \( (\tilde{l}_i^{(k)}, \tilde{f}_i^{(k)}) \) and \( (\hat{l}_i^{(K)}, \hat{f}_i^{(K)}) \), and the statistical estimation error due to the sampling randomness. Thus instead of investigating into the asymptotics of the theoretical minimizers having unknown computing error, we are concerned with the asymptotics of the feasible alternating iterative estimators. Our theory below shows that the \( \tilde{l}_i^{(k)} \)'s and \( \tilde{f}_i^{(k)} \)'s correctly identifies the realized factors and true loadings up to orthogonal transformations, and that \( C_{ij}^{(K)} \)'s consistently match the true common components.

To successfully implement the alternating iterative algorithm, we need a slightly stronger version of Assumption 1 to regularize the initial parameters that the alternating iterative algorithm starts from.

**Assumption 1'** Assumption 1 holds and

1. the eigenvalues of \( L'HL/p \) and \( L'HL^0/p \) are bounded away from zero and infinity, where \( H = \text{diag}(h_i(t_i f_i - \sigma_i^0 t_{i0})) \) is a \( p \times p \) diagonal matrix with \( h_i(x) \) being the probability density function of \( \epsilon_{it} \);
2. \( \max_i \| l_i \| \leq C \) for some generic constant \( C \).

Assumption 1' (1) demonstrates that the initial values of the loadings span a full rank-\( r \) space after being normalized by the probability density of \( \epsilon_{it} \)'s, and the spaces spanned by \( L \) and \( L^0 \) are close enough after the normalization. Assumption 1' (2) restricts that the loadings for each variable are not explosive. We remark that this assumption is not minimal. For example, the bounded loading condition could be relaxed with a more complex constraint on the increasing orders of \( p \) and \( T \). As a first attempt to establish the asymptotic theory for the \( L_1 \) robust factor analysis and for simplicity, we assume this condition in the present paper. We leave extending the theory in more general setup to our future work.

**Assumption 4.**

\[
E \exp\{p^{1/4} H_1(\{\epsilon_{it}\}, p, 1)\} \leq C, \quad E \exp\{\sigma_1^{-1} H_1(\{\epsilon_{it}\}, p, 1)\} \leq C, \\
E \exp\{H_2(\{\epsilon_{it}\}, p, 1) / \sqrt{p}\} \leq C, \quad E \exp\{H_2(\{\epsilon_{it}\}, 1, T) / \sqrt{T}\} \leq C,
\]

where for any \( c > 0 \), \( \sigma_7^2 = \frac{(\log p + c \sqrt{T/p})^3}{T^{p/2}} + \frac{T}{p^{p/2}}. \)

\( \sigma_7^2 \) is an upper bound of the variance of \( H_1(\{\epsilon_{it}\}, p, 1) \) when \( \{\epsilon_{it}\} \) is cross-sectionally and temporally weakly dependent in some sense. Assumption 4 is satisfied if \( \{\epsilon_{it}|f_i^0\} \) are independent arrays. Next assumption provides the conditions on the increasing orders of \( p \) and \( T \).

**Assumption 5.**

\[
\frac{\log p}{\sqrt{T}} + \frac{\log T}{p^{1/4} \log p} + \frac{\log \log p}{\sqrt{p}} = o(1).
\]

Assumption 5 assumes that \( p \) (or \( T \)) can not be exponentially large relative to \( T \) (or \( p \)). The reason is that \( f_i^{(k)} \)'s (or \( l_i^{(k)} \)'s) are required to converge uniformly in \( t \) (or \( t_i \)) to guarantee the convergence of \( l_i^{(k)} \) (or \( f_i^{(k)} \)) in Algorithm 1.

Now we state our theoretical results on the solution path estimators of Algorithm 1. Our first result shows that \( \tilde{f}_i \)'s and \( \tilde{l}_i \)'s have similar asymptotic results as those given in Bai and Ng (2002) and Fan et al. (2013).
Theorem 1. Under Assumptions 1-5, for $2 \leq K < \infty$ in Algorithm 1,

$$\tilde{f}_t = \tilde{W}_0 f_t^0 + O_p\left(\log\frac{p}{\sqrt{T}} + \frac{1}{\sqrt{p}}\right),$$

$$\tilde{l}_i = \tilde{W}_0^{-1} l_i^0 + O_p\left(\log\frac{p}{\sqrt{T}} + \frac{1}{\sqrt{p}}\right),$$

$$\tilde{l}_i \tilde{f}_t = l_i f_t^0 + O_p\left(\log\frac{p}{\sqrt{T}} + \frac{1}{\sqrt{p}}\right),$$

where $\tilde{W}_0 = \left\{\sum_{i=1}^{p} h_i(0) \tilde{l}_i \tilde{f}_t\right\}^{-1} \sum_{i=1}^{p} h_i(0) \tilde{l}_i l_i^0$ satisfying $\tilde{W}_0 \tilde{W}_0^* = I_r$ with probability approaching one.

If further $\frac{p \log p}{T} = o(1)$,

$$\tilde{f}_t = \tilde{W}_0 f_t^0 + \frac{1}{2} \left(\sum_{i=1}^{p} h_i(0) \tilde{l}_i \tilde{f}_t\right)^{-1} \sum_{i=1}^{p} \tilde{l}_i D_{it} + o_p\left(\frac{1}{\sqrt{T}}\right).$$

If $\frac{T \log^2 p}{p} + \log^2 p \sqrt{T} = o(1)$,

$$\tilde{l}_i = \tilde{W}_0^{-1} l_i^0 + \frac{1}{\log^2 p} \left(\sum_{i=1}^{T} \tilde{f}_t \tilde{f}_t^\top\right)^{-1} \sum_{i=1}^{T} \tilde{l}_i D_{it} + o_p\left(\frac{1}{\sqrt{T}}\right).$$

Theorem 1 demonstrates that the computationally feasible factor and loading estimates match the realized factor and true loadings up to some orthogonal transformations, and recover the common components (factor returns) consistently for each variable. Theorem 1 shows that the alternating iterative estimators share similar asymptotics with the PCA-based estimators but have lower consistency rate compared with those given in Bai and Ng (2002). One reason is that in computing iteration, the results in the current step depends on the uniform consistency rate in $i$ or $t$ of the preceding step. The other reason is the absence of an explicit decomposition of $\tilde{l}_i - \tilde{W}_0^{-1} l_i^0$ (or $\tilde{f}_t - \tilde{W}_0 f_t$) in contrast to the eigen-decomposition of the PCA-based estimators. Indeed, the Bahadur representations present the principal correction terms of orders $p^{-1/2}$ and $T^{-1/2}$, but there aren’t closed form expression for the $o_p(p^{-1/2})$ and $o_p(T^{-1/2})$ terms.

Remark 2. Lemma 6 in the technical proof demonstrates that the asymptotic results for $\tilde{l}_i$ and $\tilde{f}_t$ in Theorem 1 can be strengthened to

$$\max_i \|\tilde{f}_t - W(K) f_t^0\| = O_p\left(\frac{\log p}{\sqrt{T}} + \frac{1}{\sqrt{p}}\right),$$

$$\max_i \|\tilde{l}_i - (W(K))^{-1} l_i^0\| = O_p\left(\frac{\log p}{\sqrt{T}} + \frac{1}{\sqrt{p}}\right),$$

where $W(K)$ is defined before Lemma 5. However, the rate for the common components are not valid uniformly in $i$ and $t$. 

7
3 Numerical experiments

3.1 Data generating procedure

In this section, we introduce the general Data Generating Procedures (DGPs), which are similar as those in the simulation studies of He et al. (2020). In detail,

\[
y_{it} = \sum_{j=1}^{r} l_{ij} f_{jt} + \sqrt{\theta} u_{it}, \quad u_{it} = \sqrt{\frac{1 - \rho^2}{1 + 2J\beta^2}} e_{it},
\]

\[
e_{it} = \rho e_{i,t-1} + (1 - \beta) w_{it} + \sum_{l=\max\{1-J,1\}}^{\min\{i+j,J,p\}} \beta w_{lt}, \quad i = 1, \ldots, p, \quad t = 1, \ldots, T,
\]

where \( w_t = (w_{1t}, \ldots, w_{pt})^T \) are generated from different distributions, the loadings \( l_{ij} \)'s are independently drawn from the standard normal distribution. In model (8), \( \rho \) controls the serial correlations of idiosyncratic errors, \( \theta \) controls the signal to noise ratio (SNR), and the parameters \( \beta \) and \( J \) jointly control the cross-sectional correlations.

3.2 Estimation of loading spaces, factor spaces and common components

In this section, we assess the finite sample performances of the RIP method in terms of estimating loading spaces, factor spaces and common components. To this end, we compare the Robust Iterative Estimation Procedure (RIP) with the Robust Two-Step (RTS) method proposed by He et al. (2020) and the conventional PCA method. It is worth pointing out that the RTS method assumed that the common factors and the idiosyncratic errors are jointly elliptically distributed.

We consider the following two scenarios.

**Scenario A** Set \( r = 3, \theta = 1, \rho = \beta = J = 0 \). We consider three cases on the joint distribution of \( (f_t', w_t')' \): (i) multivariate Gaussian distribution \( \mathcal{N}(0, I_{p+r}) \); (ii) multivariate centralized t distributions \( t_{\nu}(0, I_{p+r}) \) with degree \( \nu = 3 \); (iii) \( f_t \)'s are generated from multivariate Gaussian distribution \( \mathcal{N}(0, I_r) \) while all elements of \( w_t \) are i.i.d. samples from symmetric \( \alpha \)-Stable distribution \( S_\alpha(\beta, \gamma, \delta) \) with skewness parameter \( \beta = 0 \), scale parameter \( \gamma = 1 \) and location parameter \( \delta = 0 \), \( \alpha = 1, 1.5, (p, T) = \{(150, 100), (250, 100), (250, 150), (250, 200)\} \).

**Scenario B** Set \( r = 3, \theta = 0.5, \rho = 0.2, \beta = 0.2, J = 3 \). We consider three cases on the joint distribution of \( (f_t', w_t')' \): (i) multivariate Gaussian distribution \( \mathcal{N}(0, I_{p+r}) \); (ii) multivariate centralized t distribution \( t_{\nu}(0, I_{p+r}) \) with degree \( \nu = 3 \); (iii) \( f_t \)'s are generated from multivariate Gaussian distributions \( \mathcal{N}(0, I_m) \) while all elements of \( w_t \) are i.i.d. samples from symmetric \( \alpha \)-Stable distribution \( S_\alpha(\beta, \gamma, \delta) \) with skewness parameter \( \beta = 0 \), scale parameter \( \gamma = 1 \) and location parameter \( \delta = 0 \), \( \alpha = 1, 1.5, (p, T) = \{(150, 100), (250, 100), (250, 150), (250, 200)\} \).

In **Scenario A** (i) and (ii), the settings correspond to simple cases without any serial correlations of idiosyncratic errors and \( (f_t^T, v_t^T)^T \) are jointly from elliptical distributions. \( \mathcal{N}(0, I_{p+m}) \) and heavy-tailed \( t_3(0, I_{p+m}) \) perfectly satisfy the assumptions in He et al. (2020). In **Scenario A** (iii), the factor scores are generated from multivariate Gaussian \( \mathcal{N}(0, I_r) \) while the idiosyncratic errors are generated from \( \alpha \)-stable distributions. In **Scenario B**, \( (f_t', w_t')' \) are generated in the same three ways parallel to **Scenario A**, but
the errors are now serially and cross-sectionally correlated by setting $\rho = 0.2, \beta = 0.2, J = 3$

To evaluate the empirical performances of different methods, we consider the same measurement indices in He et al. (2020), that is, the Median of the normalized estimation Errors for Common Components in terms of the matrix Frobenius norm, denoted as MEE-CC; the Average estimation Error for the Factor Loading matrices, denoted as AVE-FL; and the Average estimation Error for the Factor Score matrices, denoted as AVE-FS. In detail, the MEE-CC, AVE-FL and AVE-FS are defined as

\[
\text{MEE-CC} = \text{median}\left\{ \| \hat{L}_m\hat{F}_m^\top - LI^\top_p \|_F^2 / \| LI^\top_p \|_F^2, m = 1, \ldots, R \right\},
\]

\[
\text{AVE-FL} = \frac{\sum_{m=1}^R D(\hat{L}_m, L)}{R},
\]

\[
\text{AVE-FS} = \frac{\sum_{m=1}^R D(\hat{F}_m, F)}{R},
\]

where $R$ is the number of replicates, $\hat{L}_m$ and $\hat{F}_m$ are respectively the estimators of the factor loading matrix and factor score matrix from the $m$-th replicate, and for two orthogonal matrices $Q_1$ and $Q_2$ of sizes $p \times q_1$ and $p \times q_2$,

\[
D(Q_1, Q_2) = \left( 1 - \frac{1}{\max(q_1, q_2)} \text{Tr}(Q_1^\top Q_2 Q_2^\top Q_2) \right)^{1/2}.
\]

From the definition of $D(Q_1, Q_2)$, we can easily deduce that it is a quantity between 0 and 1, which measures the distance between the column spaces of $Q_1$ and $Q_2$. $D(Q_1, Q_2) = 0$ indicates the column spaces of $Q_1$ and $Q_2$ are the same while $D(Q_1, Q_2) = 1$ indicates the column spaces of $Q_1$ and $Q_2$ are orthogonal. In fact, $D(\cdot, \cdot)$ particularly fits to quantify the accuracy of estimated factor loading/score matrices as the factor loading and score matrices are not separately identifiable. All the simulation results are based on $R = 500$ replicates.

![Figure 1: Boxplots of the estimation errors of the estimated factor loadings and scores by RIP RTS and PCA methods under symmetric $\alpha$-Stable distributions in Scenario A (iii) with $\alpha = 1, 1.5$. $p = 250, T = 200$.](image)

We analyze the simulation results in the rest of this section. The detailed simulation results for Scenario A and Scenario B are reported in Table 1 and Table 2, respectively. From Table 1, we can see that for multivariate Gaussian case in Scenario A (i), all three methods perform very well while the RIP method
Table 1: Simulation results for Scenario A, the values in the parentheses are the interquartile ranges for MEE-CC and standard deviations for AVE-FL, AVE-FS.

| Type          | Method | $(p,T) = (150,100)$ |          | $(p,T) = (250,100)$ |          |
|---------------|--------|--------------------|----------|--------------------|----------|
| $\mathcal{N}(0, I_{p+m})$ | RIP    | 0.03(0.00)         | 0.13(0.01) | 0.10(0.01)         | 0.02(0.00) |
|               | RTS    | 0.02(0.00)         | 0.11(0.01) | 0.08(0.01)         | 0.01(0.00) |
|               | PCA    | 0.02(0.00)         | 0.10(0.01) | 0.08(0.01)         | 0.01(0.00) |
| $t_3(0, I_{p+m})$   | RIP    | 0.03(0.00)         | 0.16(0.03) | 0.11(0.02)         | 0.03(0.01) |
|               | RTS    | 0.02(0.00)         | 0.12(0.01) | 0.08(0.01)         | 0.02(0.00) |
|               | PCA    | 0.04(0.03)         | 0.20(0.06) | 0.10(0.03)         | 0.04(0.03) |
| $S_1(0,1,0)$     | RIP    | 0.05(0.01)         | 0.18(0.01) | 0.14(0.01)         | 0.04(0.01) |
|               | RTS    | 661.3(3510.92)     | 0.98(0.01) | 0.98(0.01)         | 894.64(3855.27) |
|               | PCA    | 6404.27(43169.89)  | 0.99(0.00) | 0.99(0.00)         | 11876.97(64758.55) |
| $S_{1.5}(0,1,0)$ | RIP    | 0.03(0.01)         | 0.18(0.01) | 0.15(0.01)         | 0.04(0.01) |
|               | RTS    | 8.21(15.96)        | 0.92(0.08) | 0.91(0.09)         | 9.19(20.44)  |
|               | PCA    | 8.21(15.96)        | 0.92(0.08) | 0.91(0.09)         | 9.19(20.44)  |

seems a bit worse. This is expected since least absolute regression is less efficient than least square regression when errors are normal. For multivariate $t$ distribution with degree 3 in Scenario A (ii), RTS method performs the best as the elliptical assumption is satisfied. The RIP method performs satisfactorily though not as well as RTS. The PCA method is the worst, which reflects the effect of the non-existence of the forth moment. The advantages of the proposed RIP method are well illustrated in Scenario A (iii), where the errors are from symmetric $\alpha$-stable distribution. Figure 1 shows the boxplots of the estimation errors of the estimated factor loadings and scores by RIP, RTS and PCA methods over 500 replications, with $\alpha = 1, \alpha = 1.5$ and $p = 250, T = 200$. From Figure 1, we can obviously see that the RIP method still performs very well while the RTS and PCA method totally lose power. In fact, the estimated factor loading space and factor score space by RTS and PCA method are almost orthogonal to the true spaces and the median of the normalized estimation errors of common components even reach the magnitude of order $10^3 \sim 10^4$ for symmetric $\alpha$-stable distribution with $\alpha = 1$. This is expected since the elliptical assumption required by RTS is violated. From Table 1, it can also be concluded that the performances of the RIP method tend to be better as $T$ and/or $p$ increase which is consistent with the theoretical results. For symmetric $\alpha$-stable distribution, as $\alpha$ decreases, RIP is quite stable, but RTS and PCA become worse substantially as the tail becomes thicker.

Now we turn to look at the simulation results for Scenario B in Table 2. In Scenario B, there exist
Table 2: Simulation results for Scenario B, the values in the parentheses are the interquartile ranges for MEE-CC and standard deviations for AVE-FL, AVE-FS.

| Type   | Method | \((p, T) = (150, 100)\)                                      | \((p, T) = (250, 150)\)                                      | \((p, T) = (250, 200)\)                                      |
|--------|--------|-------------------------------------------------------------|-------------------------------------------------------------|-------------------------------------------------------------|
| \(N(0, I_{p+m})\) | RIP    | 0.01(0.00) 0.09(0.01) 0.07(0.01)                            | 0.01(0.00) 0.09(0.01) 0.06(0.00)                            | 0.01(0.00) 0.09(0.01) 0.06(0.00)                            |
|        | RTS    | 0.01(0.00) 0.08(0.01) 0.06(0.00)                            | 0.01(0.00) 0.08(0.01) 0.06(0.00)                            | 0.01(0.00) 0.08(0.01) 0.06(0.00)                            |
|        | PCA    | 0.01(0.00) 0.07(0.01) 0.06(0.00)                            | 0.01(0.00) 0.07(0.01) 0.06(0.00)                            | 0.01(0.00) 0.07(0.01) 0.06(0.00)                            |
| \(t_3(0, I_{p+m})\) | RIP    | 0.02(0.00) 0.11(0.02) 0.08(0.01)                            | 0.01(0.00) 0.11(0.02) 0.06(0.01)                            | 0.01(0.00) 0.11(0.02) 0.06(0.01)                            |
|        | RTS    | 0.01(0.00) 0.08(0.01) 0.06(0.01)                            | 0.01(0.00) 0.08(0.01) 0.05(0.01)                            | 0.01(0.00) 0.08(0.01) 0.05(0.01)                            |
|        | PCA    | 0.02(0.01) 0.14(0.05) 0.07(0.03)                            | 0.02(0.01) 0.14(0.05) 0.05(0.03)                            | 0.02(0.01) 0.14(0.05) 0.05(0.03)                            |
| \(S_1(0, 1, 0)\) | RIP    | 1.45(181.64) 0.51(0.15) 0.48(0.17)                         | 1.17(0.05) 0.35(0.09) 0.26(0.11)                           | 1.17(0.05) 0.35(0.09) 0.26(0.11)                           |
|        | RTS    | 2293.52(18769.22) 0.99(0.01) 0.98(0.01)                    | 4349.65(28488.78) 0.99(0.00) 0.98(0.01)                    | 4349.65(28488.78) 0.99(0.00) 0.98(0.01)                    |
|        | PCA    | 3202.13(21579.74) 0.99(0.00) 0.98(0.01)                    | 5950.76(32357.06) 0.99(0.00) 0.99(0.01)                    | 5950.76(32357.06) 0.99(0.00) 0.99(0.01)                    |
| \(S_1(0, 1, 0)\) | RIP    | 0.04(0.01) 0.16(0.01) 0.13(0.01)                            | 0.03(0.01) 0.16(0.01) 0.10(0.01)                           | 0.03(0.01) 0.16(0.01) 0.10(0.01)                           |
|        | RTS    | 0.34(0.54) 0.28(0.05) 0.45(0.13)                            | 0.26(0.51) 0.28(0.04) 0.41(0.15)                           | 0.26(0.51) 0.28(0.04) 0.41(0.15)                           |
|        | PCA    | 4.40(8.39) 0.83(0.14) 0.82(0.14)                            | 4.92(10.51) 0.86(0.14) 0.84(0.15)                          | 4.92(10.51) 0.86(0.14) 0.84(0.15)                          |
| \(S_1(0, 1, 0)\) | RIP    | 0.11(0.01) 0.26(0.05) 0.22(0.05)                            | 0.09(0.02) 0.22(0.03) 0.21(0.03)                           | 0.09(0.02) 0.22(0.03) 0.21(0.03)                           |
|        | RTS    | 5256.14(31351.96) 0.99(0.00) 0.99(0.00)                    | 6594.49(41148.65) 0.99(0.00) 0.99(0.00)                    | 6594.49(41148.65) 0.99(0.00) 0.99(0.00)                    |
|        | PCA    | 7429.45(37392.76) 0.99(0.00) 0.99(0.00)                    | 10413.65(51591.05) 0.99(0.00) 0.99(0.00)                   | 10413.65(51591.05) 0.99(0.00) 0.99(0.00)                   |

both cross-sectional and serial correlations, but similar conclusions can be drawn as for Scenario A. The superiority of the RIP method over the RTS method is clearly illustrated when the idiosyncratic errors are from \(\alpha\)-stable distribution. For \(\alpha = 1\), when \(T, p\) are small, the RIP method does not perform well, though far much better than the RTS and PCA method. As \(T, p\) grow large, the performances of the RIP method boost quickly, while the RTS and PCA still does not work. When \(\alpha = 1.5\), \((T, p) = (100, 150)\) is enough to guarantee the good performance of RIP, while even when \(T = 200\) and \(p = 250\) the RTS and PCA still far behind.

In summary, the proposed RIP method performs robustly in both light-tailed and heavy-tailed settings, while the RTS method performs well only if the elliptical assumption is satisfied and does not work for the \(\alpha\)-stable settings. Both the RIP and RTS method perform better than the traditional PCA method in the heavy-tailed setting.
3.3 Selection of the factor numbers

To select the factor number, we propose a “Robust Information Criteria” (RIC) method, inspired by the ‘Information Criteria’ (IC) method in Bai and Ng (2002). That is,

\[
\hat{r} = \arg \min_{1 \leq k \leq k_{\text{max}}} \left\{ \log \left( \frac{1}{pT} \sum_{t=1}^{pT} ||y_{it} - (\hat{t}_t^k)^\top \hat{f}_t^k|| \right) + k \cdot l(p,T) \right\},
\]

where \(k_{\text{max}}\) is a predetermined constant no smaller than \(r\); \(\{\hat{t}_t^k\}\) and \(\{\hat{f}_t^k\}\) are the estimated factor loadings and scores by the iterative algorithm if we assume the number of factors is \(k\); \(l(p,T)\) is a penalty function of \(p\) and \(T\), which is of the form

\[
l(p,T) = \log\left( \frac{pT}{p + T} \right) \cdot \frac{p + T}{pT}.
\]

In this section, we assess the finite sample performance of the proposed “Robust Information Criteria” (RIC) for factor number selection. To this end, we compare our RIC with the “Eigenvalue-Ratio” (ER) method in Ahn and Horenstein (2013), the “Multivariate-Kendall’s tau-Eigenvalue-Ratio” (MKER) method in Yu et al. (2019) and the classical “Information Criteria” (IC) method in Bai and Ng (2002). To evaluate the empirical performance of different methods, we consider the following scenario.

| Type | \(p\) | \(T\) | \(r\) | RIC  | IC   | ER   | MKER |
|------|------|------|------|------|------|------|------|
| \(N(0, I_{p+m})\) | 50   | 50   | 3    | 3.005(0.1) | 3.000(0.0) | 3.000(0.0) | 3.000(0.0) |
|      | 100  | 100  | 3    | 3.005(0.1) | 3.000(0.0) | 3.000(0.0) | 3.000(0.0) |
|      | 150  | 150  | 3    | 3.010(0.2) | 3.000(0.0) | 3.000(0.0) | 3.000(0.0) |
|      | 200  | 200  | 3    | 3.000(0.0) | 3.000(0.0) | 3.000(0.0) | 3.000(0.0) |
| \(t_3(0, I_{p+m})\) | 50   | 50   | 3    | 3.055(1.12) | 5.570(0.186) | 2.745(38,10) | 2.990(2,0) |
|      | 100  | 100  | 3    | 3.085(0.15) | 5.815(0.190) | 2.895(15,5) | 3.000(0.0) |
|      | 150  | 150  | 3    | 3.140(0.27) | 6.145(0.197) | 3.005(9,15) | 3.000(0.0) |
|      | 200  | 200  | 3    | 3.155(0.29) | 6.485(0.194) | 3.000(5,8) | 3.000(0.0) |
| \(t_2\) | 50   | 50   | 3    | 2.485(77,0) | 4.815(9,163) | 2.335(131,45) | 2.615(57,4) |
|      | 100  | 100  | 3    | 3.000(0.0) | 5.435(0.187) | 3.020(91,80) | 2.985(3,0) |
|      | 150  | 150  | 3    | 3.005(0.1) | 6.060(0.0,19) | 3.215(81,90) | 3.000(0.0) |
|      | 200  | 200  | 3    | 3.000(0.0) | 6.420(0.196) | 3.400(74,103) | 3.000(0.0) |
| \(S_{1.5}(0,1,0)\) | 50   | 50   | 3    | 1.475(180,0) | 6.280(4,186) | 1.790(157,20) | 2.325(106,16) |
|      | 100  | 100  | 3    | 2.950(1,0) | 7.410(11,17) | 1.720(169,18) | 2.960(11,5) |
|      | 150  | 150  | 3    | 2.995(1,0) | 7.805(0,200) | 1.790(160,18) | 3.005(1,2) |
|      | 200  | 200  | 3    | 3.010(0.2) | 7.985(0,200) | 1.875(158,26) | 3.000(0.0) |

**Scenario C** Set \(r = 3, \theta = 1, \rho = \beta = J = 0\). We consider three cases on the joint distribution of \((f_t', w_t')'\): (i) multivariate Gaussian distribution \(N(0, I_{p+r})\); (ii) multivariate centralized \(t\) distributions \(t_\nu(0, I_{p+r})\) with degree \(\nu = 3\); (iii) \(f_t\)'s are generated from multivariate Gaussian distribution \(N(0, I_r)\) while all elements of \(w_t\) are i.i.d. samples from \(t_2\) distribution. (iv) \(f_t\)'s are generated from multivariate Gaussian distribution.
while all elements of \( \mathbf{w}_t \) are i.i.d. samples from symmetric \( \alpha \)-Stable distribution \( S_\alpha(\beta, \gamma, \delta) \) with skewness parameter \( \beta = 0 \), scale parameter \( \gamma = 1 \) and location parameter \( \delta = 0 \), \( \alpha = 1.5 \). \((p, T) = \{(50, 50), (100, 100), (150, 150), (200, 200)\}\). In Table 3, we show the simulation results in the form \( x(y | z) \), where \( x \) is the sample mean of the estimated factor numbers over 200 replications, while \( y \) and \( z \) are the numbers of underestimation and overestimation respectively. For the light-tailed Gaussian case in Scenario C (i), we can see from Table 3 that all the methods perform very well and \( T = p = 50 \) is sufficient for guaranteeing a satisfactory performance. For the heavy-tailed cases in Scenario C (ii), (iii) and (iv), the classical IC method always overestimate the factor number by a large margin. The performances of ER method is barely satisfactory, especially for \( \alpha \)-stable idiosyncratic errors in Scenario C (iv), it underestimate the factor number by a margin even \( p = T = 200 \). It seems that the MKER and RIC methods are comparable and they perform the best in the heavy-tailed cases. For Scenario C (ii), the factors and the idiosyncratic errors are jointly \( t_3 \) distribution, thus MKER performs the best as it’s specifically designed for this setting. For Scenario C (iii) and (iv), the factors are from Gaussian and the idiosyncratic errors are either from \( t_2 \) distribution or \( \alpha \)-stable distribution, the RIC performs satisfactorily and it can be seen that as \( p, T \) grow, the estimate by RIC converges to the true factor number.

4 An empirical study

In this empirical study, we focus on a financial asset returns dataset. We collected the weekly share returns of Standard & Poor 100 (SP100) companies in the period between January 1st, 2018 and December 31st, 2019. The data set is available at https://github.com/heyongstat/RIP, including details such as the symbol list and names of the component companies of SP100. The dataset is a large panel with dimensions \( T = 105 \) and \( p = 100 \). By preliminary time series analysis techniques such as the Augmented Dickey-Fuller tests and sample auto-correlation functions, we found that all the weekly returns series are stationary and there exists no significant serial correlations for most series. In addition, the data are heavy-tailed in the sense that the kurtosis of some series are much larger than 9, which is the theoretical kurtosis of \( t_5 \) distribution. In the following, we centralized the log returns for further analysis.

To compare the performance of various factor analysis methods, we first need to specify the factor number \( r \) for this panel data. The ER method in Ahn and Horenstein (2013), the MKER method in Yu et al. (2019) and the RIC method all estimate the factor number as \( r = 1 \), i.e., just 1 common factor while the IC method in Bai and Ng (2002) estimates the factor number as \( r = 3 \). This is consistent with the conclusion in the simulation study that the IC method tends to overestimate the factor number when data are heavy-tailed. Taking the famous Fama-French 3 factor model into account, we consider both \( r = 1 \) and \( r = 3 \) in this example for a comparison.

We first compare the empirical performance of different factor analysis methods in terms of the annual return of the year 2019 by constructing risk-minimization portfolios. To this end, we design a rolling scheme to construct time-varying portfolios. In detail, denote the true covariance matrix of the share returns by \( \Sigma \), the optimal risk-minimization portfolio weights are constructed as \( \mathbf{w}_{opt} = \Sigma^{-1} \mathbf{1} / (\mathbf{1}^T \Sigma^{-1} \mathbf{1}) \), where \( \mathbf{1} \) is a vector with all elements 1. In practice, \( \Sigma \) is unavailable and needs to be estimated from the data. We assume an Approximate Factor Model for the panel and thus \( \Sigma \) has a low-rank plus sparse structure. At each time point (week) \( t \) of the year 2019, the data of the past 52 time points (weeks), i.e., a \( 52 \times 100 \) panel,
are recursively used to train the AFM and we denote the estimated common components and idiosyncratic errors as $\hat{X}_t$ and $\hat{E}_t$, respectively, both of dimension $52 \times 100$. In the empirical study, we train the AFM by PCA, RTS and RIP methods one by one. Further, the cross-sectional correlations of the idiosyncratic errors are neglected and the covariance matrix at week $t$ is estimated as

$$\hat{\Sigma}_t = \frac{1}{52} \hat{X}_t^\top \hat{X}_t + \text{diag}\left(\frac{1}{52} \hat{E}_t^\top \hat{E}_t\right).$$

The portfolio weights are then specified as $\hat{\omega} = \hat{\Sigma}^{-1} 1 / (1^\top \hat{\Sigma}^{-1} 1)$ and the return of the risk-minimization portfolio strategy at week $t$ is $\hat{\omega}^\top x_t$, where $x_t$ is composed of the corresponding raw returns at week $t$. In Figure 2, we illustrate the net value curves of the risk-minimization strategy during the year 2019 without considering transaction cost and liquidity risk. We can clearly see from the Figure 2 that the strategy with the factor model trained by RIP method leads to the highest annual return, regardless of taking $r = 1$ or $r = 3$. The strategy with the factor model trained by RTS method take second place while the strategy with the factor model trained by PCA method is at the bottom of the list in terms of the annual return in year 2019. It also can be seen that the return of RIP dominates the returns of RTS and PCA at any time point $t$ when $r = 1$.

To further investigate the robustness of various methods, we then assess their sensitivity to outliers in the empirical study. In real replication, it’s often the case that outliers exist due to a variety of reasons such as erroneous records. We randomly select a proportion of the demeaned log returns of the two years (i.e., a $105 \times 100$ panel) and multiply their values by 5, and evaluate the sensitivity by the variation of the estimated loading space compared with the original estimated space for each method. The random contamination procedure above were repeated 100 times to reduce randomness. We report the mean variation by RIP, RTS and PCA for a variety of contamination levels in Figure 3. It can be clearly seen that the RIP almost always has a smaller variation value for both $r = 1$ and $r = 3$. The robustness of the RIP method over the other two methods is more clearly illustrated when $r = 1$, as the variations of both RTS and PCA are higher than those of RIP by a large margin at various contamination levels. It can also seen that the RTS also shows
robustness compared with the PCA method.

5 Conclusion and Discussion

In this paper, we presented a way to do robust factor analysis without any moment constraint. The method relies on alternating the $L_1$ regression in the factors cross-sectionally and in the loadings temporally. We show that after several step iteration, the terminated solution can not only identify the common components but also estimate the factors and scores consistently up to some orthogonal transformations. This provides at least a safe replacement of the PCA-based factor analysis when there are heavy-tailed idiosyncratic errors and missing values. There are still some problems that are eager to be solved in the future research. First, is there theoretical guarantee that the $\tilde{l}_i$'s and $\tilde{f}_t$'s will converge to $\hat{l}_i$'s and $\hat{f}_t$'s as $K \to \infty$, thus the computing error for $C_{ij}^{(K)}$ can be theoretically controlled. Technically, this relies on a stronger version (inclusion relation holds uniform in $k$) of Lemma 6. This is so far difficult to achieve or prove, and we leave it to our future research work. Second, one can extend the current work to a general class of loss functions beyond the absolute loss, for example, the check function used in quantile regressions.

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Appendix: technical proofs

Let $C$ be a generic constant that will be used in deriving upper bounds, and it may take different values in different places. $E_f$ stands for the expectation conditional on $f_t^0$’s. Define

$$W_0 = \{ \sum_{i=1}^{p} h_i(0)l_i l_i' \}^{-1} \sum_{i=1}^{p} h_i(0)l_i l_i'^0.$$

Assumption 1 and Assumption 3 imply that $\|W_0\| \leq C$. Reparameterize $l_i$’s and $f_t$’s with $u_i = W_0 l_i - l_i^0$ and $v_t = f_t - W_0 f_t^0$. One easily deduces the decomposition as follows,

$$\tau_{it} =: l_i' f_t - l_i'^0 f_t^0 = l_i' v_t + u_i' f_t^0.$$

Notice here that $l_i$ is still related to $u_i$ and $u_i$ simply serves as a measure of distance from $l_i$ to $l_i^0$. By model (1),

$$\sum_{i=1}^{p} \sum_{t=1}^{T} |y_{it} - l_i' f_t| = \sum_{i=1}^{p} \sum_{t=1}^{T} |\epsilon_{it} - (l_i' f_t - l_i'^0 f_t^0)|$$

Notice that

$$\langle \hat{L}, \hat{F} \rangle = \arg \min_{L,F} \sum_{i=1}^{p} \sum_{t=1}^{T} |\epsilon_{it} - \tau_{it}|$$

$$= \arg \min_{L,F} \sum_{i=1}^{p} \sum_{t=1}^{T} (|\epsilon_{it} - (l_i' f_t - l_i'^0 f_t^0)| - |\epsilon_{it}|)$$

$$= \arg \min_{L,F} \sum_{i=1}^{p} \sum_{t=1}^{T} (|\epsilon_{it} - (l_i' v_t - u_i' f_t^0)| - |\epsilon_{it}|)$$

$$=: \arg \min_{L,F} \sum_{i=1}^{p} \sum_{t=1}^{T} g_i(u_i, v_t)$$

$$= \arg \min_{L,F} \sum_{i=1}^{p} \sum_{t=1}^{T} (|\epsilon_{it} - (u_i' W_0^{-1} f_t + l_i'^0 W_0^{-1} v_t)| - |\epsilon_{it}|)$$

$$=: \arg \min_{L,F} \sum_{i=1}^{p} \sum_{t=1}^{T} g_f(u_i, v_t),$$

where $L = (l_1, ..., l_p)'$, $F = (f_1, ..., f_T)$.
One easily deduces the equality that

\[ |x - \tau| - |x| = \tau \{ I(x < 0) - I(x > 0) \} + 2(\tau - x) \text{sign}(\tau) I(0 \land \tau < x < 0 \lor \tau). \]  

(10)

This shows that

\[ g_l(u_i, v_t) =: |\epsilon_{it} - \tau_{it}| - |\epsilon_{it}| = \tau_{it}D_{it} + R_{it}, \]

(11)

where \( D_{it} = I(\epsilon_{it} < 0) - I(\epsilon_{it} > 0) \) and \( R_{it} = 2(\tau_{it} - \epsilon_{it})\text{sign}(\tau_{it})I(0 \land \tau_{it} < \epsilon_{it} < 0 \lor \tau_{it}). \) \( g_l(u_i, v_t) \) can also be expressed as

\[ g_l(u_i, v_t) = \bar{\tau}_{it}D_{it} + \bar{R}_{it} + |\epsilon_{it} - u_i'\hat{f}_t^0| - |\epsilon_{it}|, \]

(12)

where \( \bar{D}_{it} = I(\epsilon_{it} < 0) - I(\epsilon_{it} > 0) \) with \( \epsilon_{it} = \epsilon_{it} - u_i'\hat{f}_t^0, \) \( \bar{\tau}_{it} = \lambda_p^{-1/2}L_v \) and \( \bar{R}_{it} = 2(\epsilon_{it} - \epsilon_{it})\text{sign}(\epsilon_{it})I(0 \land \epsilon_{it} < \epsilon_{it} < 0 \lor \epsilon_{it}). \) There are some facts on \( \bar{R}_{it}. \) It is nonnegative, increasing in \( |\epsilon_{it}|, \) and bounded when \( \|\epsilon_{it}\| \leq C \) due to Assumption 1'.

Now we prove that for arbitrarily fixed \( l_i^{(0)} \)'s (initial guess of \( l_i \)'s in the alternating iterative algorithm) lying in the parameter space under Assumption 1' and for given \( f_t^0, \hat{v}_t^{(1)} \) satisfies \( \|\hat{v}_t^{(1)}\| =: \|\hat{f}_t^{(1)} - W(0) f_t^0\| \leq C \lambda_p^{-1/2} \) for some \( r \times r \) matrix \( W(0) \) dependent only on \( l_i^{(0)} \) and \( f_t^0, \) where \( \hat{f}_t^{(1)} \) is the optimal solution of \( f_t \) to minimizing \( \sum_{i=1}^p g_l(u_i^{(0)}, v_t) \) where \( u_i^{(0)} =: W(0)l_i^{(0)} - l_t^0 \) is fixed. Let \( \bar{\tau}_{it}^{(0)} \) and \( \bar{R}_{it}^{(0)} \) be similarly defined as \( \bar{\tau}_{it} \) and \( \bar{R}_{it} \) except for replacing \( l_i \) and \( W_0 \) by \( l_i^{(0)} \) and \( W(0) \) defined below, respectively. That being said, our first lemma gives the theoretical property of the optimal solution to the cross-sectional regression in the distance from \( f_t \) to \( f_t^0, \) for initially given design matrix \( L^{(0)} = (l_1^{(0)}, ..., l_p^{(0)}). \)

**Lemma 3.** For fixed \( l_i^{(0)} \)'s and given \( f_t^0, \) under Assumptions 1-3,

\[ P \left( \|\sqrt{\bar{D})}_{it}^{(1)} - \bar{\tau}_{it}^{(1)}\| > \delta \right) \rightarrow 0, \]

for any \( \delta > 0, \) where

\[ \bar{v}_t^{(1)} = -\frac{1}{2} \left\{ 1 \sum_{i=1}^p h_i(u_i^{(0)'}, f_t^0)L_i^{(0)} \right\}^{-1} \sum_{i=1}^p (\bar{D}_{it}^{(0)} - E_f \bar{D}_{it}^{(0)}) \frac{l_i^{(0)}}{\sqrt{p}}, \]

where \( \bar{D}_{it}^{(0)} \) is similarly defined as \( \bar{D}_{it} \) except for replacing \( u_i \) by its initial value \( u_i^{(0)}. \)
Moreover, if Assumptions 1-5 are satisfied

\[ P \left( \max_{t \leq T} \| \sqrt{p} \nu_1^{(0)} - \nu_1^{(1)} \| > \delta \right) \rightarrow 0. \]

**Proof.** First we give an expansion of \( \sum_{i=1}^{p} g_i(u_i^{(0)}, v_t) \) in \( \{ v_t : \| \nu_t \| \leq M \} \) for fixed \( l_i^{(0)} \)'s and given \( f_i^0 \) satisfying Assumption 1. (12) shows that

\[
\sum_{i=1}^{p} g_i(u_i^{(0)}, v_t) = \sum_{i=1}^{p} D_i^{(0)} l_i^{(0)} p^{-1/2} \sqrt{v_t} + \sum_{i=1}^{p} E_f \tilde{R}_i^{(0)} + \sum_{i=1}^{p} (\tilde{R}_i^{(0)} - E_f \tilde{R}_i^{(0)}) + \sum_{i=1}^{p} (|\epsilon_{it} - u_i^0 f_t^0| - |\epsilon_{it}|),
\]

where \( \bar{v}_t = \sqrt{p} v_t \) and \( \| \bar{v}_t \| \leq M \). Because \( \sum_{i=1}^{p} |(\epsilon_{it} - u_i^0 f_t^0| - |\epsilon_{it}|) \) is irrelevant to optimization in \( v_t \), we ignore this term below. Now we analyze (13) term by term. For the first term of (13),

\[
\sum_{i=1}^{p} E_f \tilde{D}_i^{(0)} l_i^{(0)} p^{-1/2} \sqrt{v_t} = 2 \sum_{i=1}^{p} l_i^{(0)} p^{-1/2} \sqrt{v_t} \left( P_f(\epsilon_{it} < u_i^{(0)} f_t^0) - 1/2 \right)
\]

\[ = 2 \sum_{i=1}^{p} l_i^{(0)} p^{-1/2} \sqrt{v_t} P_f \left( 0 < \epsilon_{it} < u_i^{(0)} f_t^0 \right) = 2 \sum_{i=1}^{p} l_i^{(0)} p^{-1/2} \sqrt{v_t} \int_{0}^{u_i^{(0)} f_t^0} h_i(x) dx \]

\[ = 2 p^{-1/2} \sqrt{v_t} \sum_{i=1}^{p} h_i(\xi_{it}^{(0)}) l_i^{(0)} (l_i^{(0)} f_t^0) - f_t^0 = 0, \]

where \( \xi_{it}^{(0)} \) is some variable in \( (0, u_i^{(0)} f_t^0) \) and in the last equality we have used the definition that

\[ W^{(0)} = \sum_{i=1}^{p} h_i(\xi_{it}^{(0)}) l_i^{(0)} - \sum_{i=1}^{p} h_i(\xi_{it}^{(0)}) l_i^{(0)} f_t^0. \]

Notice that \( W^{(0)} \) depends on \( t \), but for simplicity of notation and easy comparing with \( W_0 \), we suppress the subscript \( t \) and simply write \( W_t^{(0)} = W^{(0)} \). Assumption 1' and Assumption 3 and the restriction \( \{ \| \nu_t \| \leq M \} \) guarantee that \( P(\| W^{(0)} \| + \| (W^{(0)})^{-1} \| \leq C) \rightarrow 1 \) as \( p, T \rightarrow \infty \). In the sequel, we restrict that

\[ \| W^{(0)} \| + \| (W^{(0)})^{-1} \| \leq C. \]

For the second term of (13),

\[
\sum_{i=1}^{p} (E_f \tilde{R}_i^{(0)} - h_i((l_i^{(0)} f_t^0) - f_t^0) (l_i^{(0)} p^{-1/2} \sqrt{v_t})^2)
\]

\[ \leq C \int_{0}^{u_i^{(0)} f_t^0} (l_i^{(0)} p^{-1/2} \sqrt{v_t} - x) dx \leq C \sum_{i=1}^{p} (l_i^{(0)} p^{-1/2})^3 = o_p(1), \]
by Assumption 1'. Here \( a_p(1) \) holds uniformly in \( t \leq T \) for \( \| \tilde{v}_t \| \leq M \). For the third term of (13), we are going to prove that

\[
\left| \sum_{i=1}^{p} (\tilde{R}_{it}^{(0)} - E_f \tilde{R}_{it}^{(0)}) \right| = a_p(1),
\]

where \( a_p(1) \) holds uniformly in \( \| \tilde{v}_t \| \leq M \) for fixed \( \{l_i^{(0)}\} \)'s. Since \( v_t \) is of fixed dimension, with out of loss of generality and for simplicity of notation we assume here \( r = 1 \) in proving (16). To this end, we split the range of \( \tilde{v}_t \), \((-M, M]\), into non-overlapping intervals \((C_k, C_{k+1}]\) so that \( C_{k+1} - C_k = \delta'/M \). Then the number of subintervals is \( 2M^2/\delta' \). For convenience, we rewrite \( \tilde{R}_{it}^{(0)} \) as \( R(u_i^{(0)}, \tilde{v}_t, \epsilon_{it}) \). Then

\[
\sup_{\| \tilde{v}_t \| \leq C} \left| \sum_{i=1}^{p} (\tilde{R}_{it}^{(0)} - E_f \tilde{R}_{it}^{(0)}) \right| = \max_{k} \sup_{\tilde{v}_t \in (C_k, C_{k+1})} \left| \sum_{i=1}^{p} \{R(u_i^{(0)}, \tilde{v}_t, \epsilon_{it}) - E_f R(u_i^{(0)}, \tilde{v}_t, \epsilon_{it})\} \right|. \tag{17}
\]

Notice that \( R(u, \tilde{v}_t, \epsilon) \) is monotone in \( \tilde{v}_t \) when \( u \) is fixed,

\[
0 < R(u, C_k, \epsilon) \land R(u, C_{k+1}, \epsilon) \leq \max_{\tilde{v}_t \in (C_k, C_{k+1})} R(u, \tilde{v}_t, \epsilon) \leq R(u, C_k, \epsilon) \lor R(u, C_{k+1}, \epsilon).
\]

Let

\[
G_{bk}(u, \epsilon) = R(u, C_k, \epsilon) \lor R(u, C_{k+1}, \epsilon)
\]

and

\[
G_{sk}(u, \epsilon) = R(u, C_k, \epsilon) \land R(u, C_{k+1}, \epsilon),
\]

which are bivariate functions bounded by \( C \) for \( \| \tilde{v}_t \| \leq M \). \( G_{bk}(u_i^{(0)}, \epsilon_{it}) \) and \( G_{sk}(u_i^{(0)}, \epsilon_{it}) \) are simply the
values of $G_{bk}(u, \epsilon)$ and $G_{sk}(u, \epsilon)$ realized at $(u, \epsilon) = (u_i^{(0)}, \epsilon_{it})$. Then the right hand side of (17) is less than

$$
\max_k \left| \sum_{i=1}^p \left( G_{bk}(u_i^{(0)}, \epsilon_{it}) - E_f G_{sk}(u_i^{(0)}, \epsilon_{it}) \right) \right|
$$

$$
+ \max_k \left| \sum_{i=1}^p \left( G_{sk}(u_i^{(0)}, \epsilon_{it}) - E_f G_{bk}(u_i^{(0)}, \epsilon_{it}) \right) \right|
$$

$$
\leq \max_k \left| \sum_{i=1}^p \left( G_{bk}(u_i^{(0)}, \epsilon_{it}) - E_f G_{bk}(u_i^{(0)}, \epsilon_{it}) \right) \right|
$$

$$
+ \max_k \left| \sum_{i=1}^p \left( E_f G_{bk}(u_i^{(0)}, \epsilon_{it}) - E_f G_{sk}(u_i^{(0)}, \epsilon_{it}) \right) \right|
$$

$$
+ \max_k \left| \sum_{i=1}^p \left( E_f G_{sk}(u_i^{(0)}, \epsilon_{it}) - E_f G_{sk}(u_i^{(0)}, \epsilon_{it}) \right) \right|
$$

$$
+ \max_k \left| \sum_{i=1}^p \left( E_f G_{sk}(u_i^{(0)}, \epsilon_{it}) - G_{sk}(u_i^{(0)}, \epsilon_{it}) \right) \right| =: I_{1t} + I_{2t} + II_{1t} + II_{2t}. \tag{18}
$$

Assumption 1' and (15) show that

$$
[I_{2t} + II_{1t}] \leq C\delta' p^{-1} \sum_{i=1}^p h_i(u_i^{(0)}) f_i^{(0)} f_i^{(0)} \| t_i^{(0)} \|
$$

$$
= C\delta' p^{-1} \sum_{i=1}^p h_i(t_i^{(0)}) f_i - f_i^{(0)} + O(\frac{M}{\sqrt{p}}) \| t_i^{(0)} \|.
$$

(19)

where $O(\frac{M}{\sqrt{p}})$ holds uniformly in $i \leq p$. Assumptions 1' and 3 yield

$$
I_{1t} + II_{2t} = O_p(B) \delta' \| \frac{t_i^{(0)}}{\sqrt{p}} \| \sum_{i=1}^p \frac{h_i(u_i^{(0)}) f_i^{(0)} f_i^{(0)}}{p} = o_p(1). \tag{20}
$$

This proves (16) by letting $p \to \infty$ first and then $\delta' \to 0$. Summarizing the results for all three terms of (13), we have, by ignoring $\sum_{i=1}^p (|\epsilon_{it} - u_i^{'f_i}| - |\epsilon_{it}|),$

$$
\sum_{i=1}^p g_t(u_i^{(0)}, v_t) =: \sum_{i=1}^p G_t(u_i^{(0)}, \tilde{v}_t)
$$

$$
= \sum_{i=1}^p (\tilde{D}_{it}^{(0)} - E_f \tilde{D}_{it}^{(0)}) \frac{t_i^{(0)}}{\sqrt{p}} \tilde{v}_t + \sum_{i=1}^p h_i((W_i^{(0)} - l_i^{(0)}) f_i^{(0)}) \frac{t_i^{(0)}}{\sqrt{p}} \tilde{v}_t + o_p(1)
$$

$$
= (\tilde{v}_t - \tilde{v}_t^{(1)}) - \frac{p}{p} \sum_{i=1}^p h_i(u_i^{(0)} f_i^{(0)} f_i^{(0)}) \frac{t_i^{(0)}}{p} (\tilde{v}_t - \tilde{v}_t^{(1)}) - \frac{1}{4} \sum_{i=1}^p (\tilde{D}_{it}^{(0)} - E_f \tilde{D}_{it}^{(0)}) \frac{t_i^{(0)}}{\sqrt{p}}
$$

$$
\times \left\{ \sum_{i=1}^p h_i(u_i^{(0)} f_i^{(0)} f_i^{(0)}) \frac{t_i^{(0)}}{p} \right\}^{-1} \sum_{i=1}^p (\tilde{D}_{it}^{(0)} - E_f \tilde{D}_{it}^{(0)}) \frac{t_i^{(0)}}{\sqrt{p}} + o_p(1), \tag{21}
$$

where $o_p(1)$ holds uniformly in $\|\tilde{v}_t\| \leq M$. (21) demonstrates that $\tilde{v}_t$ achieves the minimum $\tilde{v}_t^{(1)}$ asymptot-
ically whenever $\|\tilde{v}_t\| \leq M$. Let $\sqrt{p_t}\hat{v}^{(1)}_t$ be the minimizer of $\sum_{i=1}^{p} G_i(u_i^{(0)}, \tilde{v}_t)$ over $\{\tilde{v}_t \in \mathbb{R}^r\}$ for fixed $u_i^{(0)}$ and $l_i^{(0)}$ and given $f_t^0$. Let $\tilde{v}_t = \tilde{v}^{(1)}_t + \beta_t e_t$, where $e_t$ is a vector of unit length, and let $v^*_t = \tilde{v}^{(1)}_t + \delta e_t$ so that $v^*_t$ lies in the line segment from $\tilde{v}^{(1)}_t$ to $\tilde{v}_t$. Notice that $\sum_{i=1}^{p} G_i(u_i^{(0)}, \tilde{v}_t)$ is a convex function in $\tilde{v}_t$ given $l_i^{(0)}$'s and $f_0^t$. For $\beta_t > \delta$, by convexity, (15), (16) and (21), and restricted on $\{\|\tilde{v}^{(1)}_t\| \leq M - \delta\}$,

$$\sum_{i=1}^{p} [G_i(u_i^{(0)}, \tilde{v}_t) - G_i(u_i^{(0)}, \tilde{v}^{(1)}_t)]$$

$$> \frac{\beta_t}{\delta} \sum_{i=1}^{p} [G_i(u_i^{(0)}, v^*_t) - G_i(u_i^{(0)}, \tilde{v}^{(1)}_t)]$$

$$> \delta \beta_t e_t \left( \frac{1}{p} \sum_{i=1}^{p} h_i(u_i^{(0)t}) f_t^0 l_i^{(0)t} - \|e_t - \delta^{-1} |o_p(1)\| \right), \quad (22)$$

as $p, T \to \infty$, where $o_p(1)$ holds uniformly in $\{\|\tilde{v}^{(1)}_t\| \leq M - \delta\}$. Then by Assumption 1' and 3(2) and (22), for any $\delta > 0$,

$$P\{\|\sqrt{p_t}\hat{v}^{(1)}_t - \tilde{v}^{(1)}_t\| > \delta\} \leq P\{\sum_{i=1}^{p} [G_i(u_i^{(0)}, \tilde{v}_t) - G_i(u_i^{(0)}, \tilde{v}^{(1)}_t)] < 0, \|\tilde{v}^{(1)}_t\| \leq M - \delta\}$$

$$+ P\{\|\tilde{v}^{(1)}_t\| > M - \delta\} \leq \epsilon,$$  \quad (23)

for arbitrarily small $\epsilon > 0$, where we have used the Chebyshev inequality

$$P\{\|\tilde{v}^{(1)}_t\| > M - \delta\} \leq (M - \delta)^{-2} E(\tilde{v}^{(1)}_t)^2 \leq \epsilon/2,$$ \quad (24)

by choosing $M$ large enough and Assumption 3(2).

Next, we prove the uniform result in $t \leq T$. Taking $M = C \log T$ for $C$ large enough, by the Markov inequality,

$$P\{\max_{t \leq T} \|\tilde{v}^{(1)}_t\| > C \log T\} \leq T e^{-C \log T} \max_{t} E\{E_f \exp\{\tilde{v}^{(1)}_t\}\} = o(1),$$ \quad (25)

due to Assumption 4. Hence, in the sequel, we restrict on the set $\{\max_{t} \|\tilde{v}^{(1)}_t\| \leq C \log T\}$. Repeating the steps of for the non-uniform results, we find that (19) still holds uniformly in $t \leq T$, i.e.,

$$\max_{t \leq T} (I_{2t} + II_{1t}) \leq C \delta'.$$ \quad (26)
Parallel to (20), the Markov inequality and Assumptions 4 and 5 show that

\[ P\{\max_{t \leq T}(I_{it} + H_{it}) > \epsilon\} \leq CT(\log T)^{2r} e^{-\epsilon^2/\sigma^2} = o(1), \]

where \( \sigma^2_p = \frac{1}{\sqrt{p}}. \) This proves that under the more stringent condition on \( p \) and \( T, \) (16) holds uniformly in \( t \leq T, \) and hence the \( o_p(1) \) and \( \epsilon \) terms in (21) and (23) hold uniformly in \( t \leq T. \) Then parallelizing to (23) proves the uniform (in \( t \)) results.

\[ \square \]

Now, we alternate to fix \( f_t^{(1)}, W^{(0)}, f_t^0, \) and thus \( \tilde{u}_t^{(1)}, \) i.e., the optimal solution to the cross-sectional regression in \( \tilde{u}_t \) done in Lemma 3, and run time series regression in \( u_t^{(1)} = W^{(0)}I_t - p^0. \) We write \( g_f(u_t^{(1)}, v_t^{(1)}) =: |\epsilon_t - (u_t^{(1)}(W^{(0)})^{-1} f_t^{(1)} + l_t^{(1)}(W^{(0)})^{-1} v_t^{(1)})| - |\epsilon_t| \) and set \( \tau_u^{(1)} = u_t^{(1)}(W^{(0)})^{-1} f_t^{(1)} + l_t^{(0)}(W^{(0)})^{-1} v_t^{(1)}. \) Let \( R_t^{(1)} \) be similarly defined as \( R_t \) except for replacing \( \tau_u \) by \( \tau_t^{(1)} \). Let \( \tilde{u}_t^{(1)} \) (and correspondingly \( \tilde{r}_t^{(1)} \) be the optimal solution to minimizing \( \sum_{t=1}^T g_f(u_t^{(1)}, v_t^{(1)}) \) in \( u_t^{(1)}. \)

**Lemma 4.** Given \( f_t^0, W^{(0)} \) and \( \tilde{u}_t^{(1)} \)’s, under Assumptions 1-5,

\[ \max_{i} \| \tilde{u}_t^{(1)} \| = O_p(\frac{\log p}{\sqrt{T}}) + o_p(\frac{1}{\sqrt{p}}). \]

Moreover, if further \( \frac{T}{p} (\log p)^2 + \frac{(\log p)^3}{\sqrt{T}} = o(1), \)

\[ P \left( \max_{t \leq p} \| \sqrt{T} \tilde{u}_t^{(1)} - \tilde{u}_t^{(1)} \| > \delta \right) \rightarrow 0, \]

for any constant \( \delta > 0, \) where

\[ \tilde{u}_t^{(1)} = -\frac{1}{2\tilde{u}_t^{(1)}} \left( \sum_{t=1}^T \frac{f_t^0 f_t^{(0)}}{T} \right)^{-1} \sum_{t=1}^T D_t \frac{f_t^0}{\sqrt{T}}. \]

**Proof.** Now, \( g_f(u_t^{(1)}, v_t^{(1)}) \) can be rewritten as

\[ g_f(u_t^{(1)}, v_t^{(1)}) = D_t \tau_t^{(1)} + E_f R_t^{(1)} + R_t^{(1)} - E_f R_t^{(1)} =: G_f(\tilde{u}_t^{(1)}, \tilde{v}_t^{(1)}), \]

where \( \tilde{u}_t^{(1)} = \sqrt{T} u_t^{(1)}. \) Parallel to the proof of Lemma 3 and restricted on \( \{ \max_i \| \tilde{u}_t^{(1)} \| \leq M \}, \)

\[ \sum_{t=1}^T G_f(\tilde{u}_t^{(1)}, u_t^{(1)}) = \sum_{t=1}^T D_t \tau_t^{(1)} + \sum_{t=1}^T E_f R_t^{(1)} + \sum_{t=1}^T (R_t^{(1)} - E_f R_t^{(1)}). \]
For the second term in the right hand side of (28), similar to (15) and by Lemma 3 and Assumptions 1’ and 4,

\[ E \sup_{\|v_t(1)\| \leq C \log T / \sqrt{p}} \max_i \left| \sum_{t=1}^{T} E_j R_{it}^{(1)} - \sum_{t=1}^{T} h_i(0)( \frac{\tilde{u}_i^{(1)}(W^{(0)})^{-1} f_t^{(1)}}{\sqrt{T}} + l_t^{(0)}(W^{(0)})^{-1} v_t^{(1)})^2 \right| \leq C E \sup_{\|v_t(1)\| \leq C \log T / \sqrt{p}} \max_i \left| \sum_{t=1}^{T} \max_{\tilde{u}_i^{(1)}} \left| \frac{\tilde{u}_i^{(1)} f_t^{(1)}}{\sqrt{T}} + l_t^{(0)}(W^{(0)})^{-1} v_t^{(1)} \right|^3 \right| \leq C \left( \frac{M^3}{\sqrt{T}} + \frac{T}{p^{3/2}} \right). \]

(29)

For the third term of (28), as in the proof of Lemma 3, we assume \( r = 1 \) and split the range of \( \tilde{u}_i^{(1)} \), \((-M, M]\), into \( 2M^2/\delta' \) non-overlapping subintervals \( (C_k, C_{k+1}] \) with \( C_{k+1} - C_k = \delta'/M \). Rewrite \( R_{it}^{(1)} = R(\tilde{u}_i^{(1)}, v_t^{(1)}, \epsilon_{it}) \). By the monotonicity of \( R(\tilde{u}_i^{(1)}, v_t^{(1)}, \epsilon_{it}) \) in \( \tilde{u}_i^{(1)} \), we have

\[ 0 < R(C_k, v, \epsilon) \wedge R(C_{k+1}, v, \epsilon) \leq \sup_{\tilde{u}_i^{(1)} \in (C_k, C_{k+1}]} R(\tilde{u}_i^{(1)}, v, \epsilon) \leq R(C_k, v, \epsilon) \vee R(C_{k+1}, v, \epsilon), \]

(30)

where the lower and upper bounds are irrelevant to \( \tilde{u}_i^{(1)} \). Let

\[ G_{bk}(v, \epsilon) = R(C_k, v, \epsilon) \vee R(C_{k+1}, v, \epsilon) \]

and

\[ G_{sk}(v, \epsilon) = R(C_k, v, \epsilon) \wedge R(C_{k+1}, v, \epsilon), \]

which are two bivariate functions bounded by \( C \) when \( \|\tilde{u}_i^{(1)}\| \leq M \) and \( \max_i \|v_t^{(1)}\| \leq C \log T / \sqrt{p} \). \( G_{bk}(v_t^{(1)}, \epsilon_{it}) \) and \( G_{sk}(v_t^{(1)}, \epsilon_{it}) \) are simply the values of \( G_{bk}(v, \epsilon) \) and \( G_{sk}(v, \epsilon) \) at \((v, \epsilon) = (v_t^{(1)}, \epsilon_{it}) \). Then (30) implies
that

\[
\sup_{\tilde{u}_t \in (-M,M)} \left| \sum_{t=1}^{T} (R_{it}^{(1)} - E_f R_{it}^{(1)}) \right| \leq \max_k \left| \sum_{t=1}^{T} \{G_{bk}(v_t^{(1)}, \epsilon_{it}) - E_f G_{sk}(v_t^{(1)}, \epsilon_{it})\} \right| \\
+ \max_k \left| \sum_{t=1}^{T} \{E_f G_{bk}(v_t^{(1)}, \epsilon_{it}) - G_{sk}(v_t^{(1)}, \epsilon_{it})\} \right| \\
\leq \max_k \left| \sum_{t=1}^{T} \{G_{bk}(v_t^{(1)}, \epsilon_{it}) - E_f G_{bk}(v_t^{(1)}, \epsilon_{it})\} \right| \\
+ \max_k \left| \sum_{t=1}^{T} \{E_f G_{bk}(v_t^{(1)}, \epsilon_{it}) - E_f G_{sk}(v_t^{(1)}, \epsilon_{it})\} \right| \\
+ \max_k \left| \sum_{t=1}^{T} \{E_f G_{sk}(v_t^{(1)}, \epsilon_{it}) - G_{sk}(v_t^{(1)}, \epsilon_{it})\} \right| \\
+ \max_k \left| \sum_{t=1}^{T} \{E_f G_{sk}(v_t^{(1)}, \epsilon_{it}) - E_f G_{bk}(v_t^{(1)}, \epsilon_{it})\} \right| \\
:= I_{1t}(\{v_t^{(1)}\}, \{\epsilon_{it}\}) + I_{2t}(\{v_t^{(1)}\}, \{\epsilon_{it}\}) + II_{1t}(\{v_t^{(1)}\}, \{\epsilon_{it}\}) + II_{2t}(\{v_t^{(1)}\}, \{\epsilon_{it}\}).
\]

(31)

A closer look at \(R(C_k, v_t^{(1)}, \epsilon_{it})\) shows that \(R(C_k, v_t^{(1)}, \epsilon_{it})\) is a piecewise linear function in \(v_t^{(1)}\) with turning points \(\{v_t^{(1)}; \tau_t^{(1)} = \pm \epsilon_{it}\}\), and the principal term of \(E_f R(C_k, v_t^{(1)}, \epsilon_{it})\) by (29) (i.e. \(\sum_{t=1}^{T} h_i(0)(u_t^{(1)} + v_t^{(1)} f^{(1)})^2\)) is a quadratic function in \(v_t^{(1)}\). Therefore

\[
\sup_{\max_{t \leq T} \|v_t^{(1)}\| \leq C \log T/\sqrt{p}} V(\{v_t^{(1)}\}, \{\epsilon_{it}\}) \leq |V(\{v_t^{(1)}\}, \{\epsilon_{it}\}) I(\{\tau_t^{(1)}\} \neq \pm \epsilon_{it})| + |V(\{v_t^{(1)}\}, \{\epsilon_{it}\}) I(\{\tau_t^{(1)}\} = \{\pm \epsilon_{it}\})| \\
+ O_p\left(\frac{M^3}{\sqrt{T}} + \frac{T}{p^{3/2}}\right),
\]

(32)

where the \(O_p\) term holds uniformly in \(\max_{t \leq T} \|v_t^{(1)}\| \leq C \log T/\sqrt{p}, V(\{v_t^{(1)}\}, \{\epsilon_{it}\}) = \sum_{t=1}^{T} (R(C_k, v_t^{(1)}, \epsilon_{it}) - E_f R(C_k, v_t^{(1)}, \epsilon_{it}))\) and \(v_t^{(1)}\) is the solution to \(\|v_t^{(1)}\| = C \log T\). Notice that \(v_t^{(1)}\) is independent of the index \(t\). Because \(\epsilon_{it}\) has probability density function \(h_i(x), E[|V(\{v_t^{(1)}\}, \{\epsilon_{it}\}) I(\{\tau_t^{(1)}\} = \{\pm \epsilon_{it}\})| = 0\). Then it suffices to consider \(|V(\{v_t^{(1)}\}, \{\epsilon_{it}\})|\).

By Assumption 4 with \(\mu_{it} = 0\) and the Markov inequality,

\[
P\{\max_i I_{1i}(\{v_t^{(1)}\}, \{\epsilon_{it}\}) + I_{2i}(\{v_t^{(1)}\}, \{\epsilon_{it}\}) > \epsilon_{p,T}\} \leq C p M^2 e^{\epsilon_{p,T} / \sigma_T},
\]

(33)

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where \( \sigma_T = (\frac{M^3}{T} + \frac{T}{p^{3/2}})^{1/2} \). By (29), Lemma 3 and Assumption 3,

\[
E \sup_{\max_i \|v_i^{(1)}\| \leq C \log T / \sqrt{T}} \max_i (I_2((v_i^{(1)}), \{e_{it}\}) + II_2((v_i^{(1)}), \{e_{it}\})) \leq C' \beta (1 + \frac{\sqrt{T}}{\sqrt{p}M}).
\]  

Let

\[
\tilde{u}_i^{(1)} = -\frac{1}{2h_i(0)} \sum_{t=1}^{T} (W(0)^{-1}f_{it}^{(1)}(W(0)^{1/2} + l_{it}^p(W(0)^{-1})^2 + O_p(\epsilon_{p,T} + \frac{M^3}{\sqrt{T}} + \frac{T}{p^{3/2}} + o_p(1 + \frac{\sqrt{T}}{\sqrt{p}M}))
\]

(28)-(34) and the uniform results of Lemma 3 show that

\[
\sum_{t=1}^{T} G_i(\tilde{u}_i, \tilde{v}_i^{(1)})
= \sum_{t=1}^{T} D_{it} \tilde{r}_i^{(1)} + \sum_{t=1}^{T} h_i(0)(\tilde{u}_i(W(0)^{-1}f_{it}^{(1)}))^2 + \sum_{t=1}^{T} D_{it} l_{it}^p(W(0)^{-1}v_i^{(1)}) + O_p(\epsilon_{p,T}) + o_p(1 + \frac{\sqrt{T}}{\sqrt{p}M})
\]

where the \( o_p \) and \( O_p \) terms hold uniformly in \( \{\max_i \|\tilde{u}_i\| \leq M, \max_i \|\tilde{v}_i^{(1)}\| \leq C \log T / \sqrt{T}\}. Without affecting the asymptotics below, we restrict that \( |o_p(1 + \sqrt{T/pM})| \leq \epsilon(1 + \sqrt{T/pM}) \) for some arbitrarily small \( \epsilon > 0 \).

Let

\[
\sigma_{p,T} = \frac{M^3}{\sqrt{T}} + \frac{T}{p^{3/2}} + \frac{M}{\sqrt{p}}.
\]

Next, we show that \( \tilde{u}_i \) is around \( \tilde{u}_i^{(1)} \). We first restrict our study on the set \( \{\max_i \|\tilde{u}_i\| \leq M\} \). Let \( \tilde{u}_i = \beta_i e_i + \tilde{u}_i^{(1)} \) for \( \beta_i > \delta \) and \( u_i^* = \delta e_i + \tilde{u}_i^{(1)} \). By the convexity of \( G_f(\tilde{u}_i, \tilde{v}_i^{(1)}) \) in \( \tilde{u}_i \) for fixed \( \tilde{v}_i^{(1)} \), \( f_i^0 \) and \( W^{(0)} \),

\[
\sum_{t=1}^{T} (G_f(\tilde{u}_i, \tilde{v}_i^{(1)}) - G_f(\tilde{u}_i, \tilde{v}_i^{(1)})) > \frac{\beta_i}{\delta} \sum_{t=1}^{T} (G_f(u_i^*, \tilde{v}_i^{(1)}) - G_f(\tilde{u}_i^*, \tilde{v}_i^{(1)})
\]

where the \( o_p \) and \( O_p \) terms hold uniformly in \( \{\max_i \|\tilde{u}_i\| \leq M\} \). Now, we prove the first equation by setting

\[
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\]
\[ M = C\{\log p + (\frac{\epsilon T}{p})^{1/2}\}, \quad \delta = C_1\{\log p + (\frac{\epsilon T}{p})^{1/2}\} \text{ and } \epsilon_{p,T} = \delta^2/M^* \text{ for large enough } M^*, C \text{ and } C_1. \]

By the Chebyshev inequality,

\[
P\{\max_{1 \leq i \leq p} \|\sqrt{T}\tilde{u}_i^{(1)} - \tilde{u}_i^{(1)}\| > \delta\} \\
= P\{\max_{1 \leq i \leq p} \|\sqrt{T}\tilde{u}_i^{(1)} - \tilde{u}_i^{(1)}\| > \delta, \max_i \|\tilde{u}_i^{(1)}\| \leq M, \max_i \|\tilde{u}_i^{(1)}\| \leq C \log T/\sqrt{p}\} \\
+ P\{\max_i \|\tilde{u}_i^{(1)}\| > M\} + P\{\max_i \|\tilde{u}_i^{(1)}\| > C \log T/\sqrt{p}\}. \quad (36)
\]

Assumption 4 and the Markov inequality show that

\[
P\{\max_i \|\tilde{u}_i^{(1)}\| > M\} \leq Cpe^{-M} = o(1). \quad (37)
\]

Lemma 3, Assumption 4-5, the Bonferroni inequality and the Markov inequality prove that

\[
P\{\max_i \|\tilde{u}_i^{(1)}\| > C \log T/\sqrt{p}\} = o(1). \quad (38)
\]

Assumptions 4-5 and (35) yield

\[
P\{\max_{1 \leq i \leq p} \|\sqrt{T}\tilde{u}_i^{(1)} - \tilde{u}_i^{(1)}\| > \delta, \max_i \|\tilde{u}_i^{(1)}\| \leq M, \max_i \|\tilde{u}_i^{(1)}\| \leq C \log T/\sqrt{p}\} \\
\leq P\{\beta_i \delta e_i' \left(\sum_{t=1}^{T} h_t(0) \frac{\hat{f}_i^{(1)}}{T} \hat{f}_i^{(1)'}\right) e_i + \{O_p(\epsilon_{p,T}) + O_p(\sigma_{p,T}) - \epsilon(1 + \sqrt{T/p})\} \frac{\beta_i}{\delta} < 0\} \\
= P\{\epsilon_i' \left(\sum_{t=1}^{T} h_t(0) \frac{\hat{f}_i^{(1)}}{T} \hat{f}_i^{(1)'}\right) e_i + \delta^{-2} \{O_p(\epsilon_{p,T}) + O_p(\sigma_{p,T}) - \epsilon(1 + \sqrt{T/pM})\} < 0\} \\
= P\{\epsilon_i' \left(\sum_{t=1}^{T} h_t(0) \frac{\hat{f}_i^{(1)}}{T} \hat{f}_i^{(1)'}\right) e_i - \epsilon < 0\} = o(1), \quad (39)
\]

by letting \( p, T \to \infty \) first and then \( \epsilon \to 0 \), where the last equality is due to (5) and Assumption 3. (36)-(39) prove that

\[
\tilde{u}_i^{(1)} = \frac{1}{\sqrt{T}} \tilde{u}_i^{(1)} + O_p\left(\frac{\log p + (\sqrt{T/p})}{\sqrt{T}}\right) = O_p\left(\frac{\log p}{\sqrt{T}}\right) + o_p\left(\frac{1}{\sqrt{p}}\right),
\]

where the \( O_p \) and \( o_p \) terms hold uniformly in \( i \leq p \). This together with the condition on \( D_{ui} \) in Assumption 4 proves the first equation of Lemma 4.

To prove the Barhadur representation result for \( \tilde{u}_i^{(1)} \) in Lemma 4, let \( \delta \) and \( \epsilon_{p,T} \) be two arbitrarily small constants, and \( M = C \log p \). Because we further have the condition that \( ((\log p)^2 + (\log T)^3)^{\frac{T}{p}} + \frac{\log^2 p}{\sqrt{T}} = o(1) \)
and the condition on $\epsilon_i$’s in Assumption 4, the probability in (33) is $o(1)$ and (35)-(39) are still true, which proves

$$P\{\max_{i \leq p} \|\sqrt{T} u_i^{(1)} - u_i^{(1)} \| > \delta \} \to 0.$$ 

To complete the proof of the second equation of Lemma 4, it suffices to prove

$$P\{\max_{i} \|\tilde{u}_i^{(1)} - u_i^{(1)} \| > \delta \} \to 0. \quad (40)$$

By Assumptions 2 and 3,

$$E_f\{\frac{1}{\sqrt{T}} \sum_{t=1}^{T} D_{it}(W^{(0)})^{-1}(\tilde{f}_t^{(1)} - W^{(0)} f_t^0) \} = 0, \quad (41)$$

By Lemma 3 and Assumption 3,

$$E_f\{\frac{1}{\sqrt{T}} \sum_{t=1}^{T} D_{it}(W^{(0)})^{-1}(\tilde{f}_t^{(1)} - W^{(0)} f_t^0)^2 \}
\leq \frac{C}{T} \sum_{t=1}^{T} \| (W^{(0)})^{-1}(\tilde{f}_t^{(1)} - W^{(0)} f_t^0) \|^2
\leq 2 \max_t \| (\tilde{f}_t^{(1)} - W^{(0)} f_t^0) \|^2 \frac{1}{T} \sum_{t=1}^{T} \| (W^{(0)})^{-1} \|^2 = O_p(\frac{\log^2 T}{p}), \quad (42)$$

and

$$\frac{1}{T} \sum_{t=1}^{T} \{(W^{(0)})^{-1} \tilde{\tilde{f}}_t^{(1)} - f_t^0 \} \tilde{f}_t^{(1)'} (W^{(0)'})^{-1} \tilde{f}_t^{(1)'}
\leq C \max_t \| \tilde{\tilde{f}}_t^{(1)} - W^{(0)} f_t^0 \| \frac{1}{T} \sum_{t=1}^{T} \| \tilde{f}_t^{(1)'} \| \| (W^{(0)'})^{-1} \| = O_p(\frac{\log T}{\sqrt{p}}),$$

$$\frac{1}{T} \sum_{t=1}^{T} f_t^0 \{ \tilde{\tilde{f}}_t^{(1)'} (W^{(0)'})^{-1} - f_t^{0'} \}
\leq C \max_t \| \tilde{\tilde{f}}_t^{(1)} - W^{(0)} f_t^0 \| \frac{1}{T} \sum_{t=1}^{T} \| f_t^{0'} \| \| (W^{(0)'})^{-1} \| = O_p(\frac{\log T}{\sqrt{p}}). \quad (43)$$

Assumptions 4-5 and (41)-(43) prove (40).

Next, we turn to the $k$-th ($k \geq 2$) update of $f_t$ and its corresponding distance to $W^{(k-1)} f_t^0$ with updated
$l_{i}^{(k-1)}$'s. Similar to $W^{(0)}$ defined in the proof of Lemma 3, here we define

$$W^{(k)} = \left\{ \sum_{i=1}^{p} h_{i} 1_{i}^{(k)} \tilde{t}_{i}^{(k)} \tilde{t}_{i}^{(k)'} - \sum_{i=1}^{p} h_{i} 1_{i}^{(k)} \tilde{t}_{i}^{(k)} t_{i}^{0} \right\},$$

where for some $\theta_{it}^{(k)} \in [0, 1]$, $\xi_{it}^{(k)} = \theta_{it}^{(k)} (\tilde{t}_{i}^{(k)} W^{(k)} - t_{i}^{0}) f_{i}^{0}$ is some variable between 0 and $(\tilde{t}_{i}^{(k)} W^{(k)} - t_{i}^{0}) f_{i}^{0}$.

Notice that $W^{(k)}$ depends only on $\tilde{t}_{i}^{(k)}$'s, $t_{i}^{0}$'s, and $f_{i}^{0}$. From now on, we define $\tilde{v}^{(k)} = \tilde{f}_{t}^{(k)} - W^{(k-1)} t_{i}^{0}$, and $u^{(k)} = f_{i}^{0} - W^{(k-1)} t_{i}^{0}$ and $u^{(k)} = W^{(k-1)} t_{i}^{0} - t_{i}^{0}$. Let $A_{k}$ and $B_{k}$ be the sets of samples so that $\max_{i} |\tilde{v}^{(k)}| \leq C \log \frac{T}{\sqrt{p}}$ and $\max_{i} |\tilde{u}^{(k)}| \leq C(\frac{\log p}{\sqrt{p}} + \frac{1}{\sqrt{p}})$, respectively.

**Lemma 5.** For fixed $\tilde{t}_{i}^{(k-1)}$'s and given $t_{i}^{0}$, under Assumptions 1-5,

$$P \left( \max_{t} \|\sqrt{p} \tilde{v}_{i}^{(k)} - \tilde{v}_{i}^{(k)}\| > \delta \right) \rightarrow 0,$$

for any $\delta > 0$, where

$$\tilde{v}_{i}^{(k)} = \frac{1}{2} \left( \frac{1}{p} \sum_{i=1}^{p} h_{i} 0 \tilde{t}_{i}^{(k-1)} \tilde{t}_{i}^{(k-1)'} - \sum_{i=1}^{p} D_{it} \tilde{t}_{i}^{(k-1)} \right) .$$

**Proof.** The proof of Lemma 5 is similar to that of Lemma 3 except for updating $l_{i}^{(0)}$ and $W^{(0)}$ by $\tilde{t}_{i}^{(k-1)}$ and $W^{(k-1)}$, respectively, and noting that $\tilde{u}^{(k-1)}$'s are in $B_{k-1}$. Let $\tilde{c}^{(k)}_{it} = c_{it}^{(k)} - u_{i}^{(k)} r_{it}^{0}$ and $\tilde{\theta}_{it}^{(k)} = p^{-1/2} l_{i}^{(k)} \tilde{v}_{i}^{(k)}$. Indeed, we show that the expansion of $\sum_{i=1}^{p} g_{i}(u_{i}^{(k-1)} - v_{i}^{(k)})$ in (21) with $(u_{i}^{(0)}, v_{i})$ there replaced by $(u_{i}^{(k-1)}, v_{i})$ holds uniformly in $\{\max_{i} \|v_{i}^{(k)}\| \leq C \log T / \sqrt{p}, \max_{i} \|u_{i}^{(k-1)}\| \leq C(\log p / \sqrt{T} + \epsilon / \sqrt{p}) \}$. First, (15) with $\tilde{v}_{i}^{(k)}$ there replaced by $\tilde{v}_{i}^{(k)} = \sqrt{p} v_{i}^{(k)}$ holds uniformly in $\max_{i} \|u_{i}^{(k-1)}\| \leq C(\log p / \sqrt{T} + \epsilon / \sqrt{p})$. Then it suffices to prove that

$$\sup_{\max_{i} \|u_{i}^{(k-1)}\| \leq C(\log p / \sqrt{T} + \epsilon / \sqrt{p})} \sum_{i=1}^{p} (\tilde{R}_{it}^{(k)} - E_{f} \tilde{R}_{it}^{(k)}) = o_{p}(1),$$

with $\tilde{R}_{it}^{(0)}$ in (16) replaced by $\tilde{R}_{it}^{(k)}$ which is similarly defined as $\tilde{R}_{it}^{(k)}$ except for replacing $l_{i}$ and $W_{0}$ by $l_{i}^{(k)}$ and $W^{(k)}$, respectively. To this end, replace $(u_{i}^{(0)}, t_{i}^{(0)})$ in (19) by $(u_{i}^{(k)}, t_{i}^{(k)})$, one easily shows that

$$\sup_{\max_{i} \|u_{i}^{(k-1)}\| \leq C(\log p / \sqrt{T} + \epsilon / \sqrt{p})} [I_{2it} + II_{1it}] \leq C \delta'. $$

Define

$$U(\{u_{i}^{(k)}\}, \{c_{it}\}) = \sum_{i=1}^{p} (R(u_{i}^{(k)}, c_{it}) - E_{f} R(u_{i}^{(k)}, c_{it})).$$

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We see that
\[ \sum_{i=1}^{p} \mathbb{E} F(u_i^{(k)}, C_k, \epsilon_{it}) = \sum_{i=1}^{p} h_i(0)(\tilde{\tau}_{it}^{(k)})^2 + o_p(1), \]
where \( o_p(1) \) holds uniformly in \( \max_i \|v_i^{(k)}\| \leq C \log T/\sqrt{p}, \max_i \|u_i^{(k-1)}\| \leq C(\log p/\sqrt{T} + \epsilon/\sqrt{p}) \). Notice that \( R(u_i^{(k)}, C_k, \epsilon_{it}) \) is a piecewise linear function in \( \tilde{\tau}_{it}^{(k)} \) with turning points \( \pm \epsilon_{it}^{(k)} \) while the principal term of \( \mathbb{E} F(u_i^{(k)}, C_k, \epsilon_{it}) \) is a quadratic function in \( \tilde{\tau}_{it}^{(k)} \). Then \( \sup_{\max_i \|u_i^{(k)}\| \leq C(\log p/\sqrt{T} + \epsilon/\sqrt{p})} U_{\{u_i^{(k)}\}, \{\epsilon_{it}\}} \) is achieved when \( \tilde{\tau}_{it}^{(k)} = \epsilon_{it}^{(k)} \) or \( \max_i \|\tilde{\tau}_{it}^{(k)}\| = C \log T/\sqrt{p} \). Because the probability density function exists for \( \epsilon_{it}, \max_i \|\tilde{\tau}_{it}^{(k)}\| = C \log T/\sqrt{p} \), \( \max_i \|\tilde{\tau}_{it}^{(k)}\| = C \log T/\sqrt{p} \), \( R(u_i^{(k)}, C_k, \epsilon_{it}) \) is a piecewise linear function in \( u_i^{(k)} \), and then
\[ \sup_{\max_i \|u_i^{(k)}\| \leq C(\log p/\sqrt{T} + \epsilon/\sqrt{p})} U_{\{u_i^{(k)}\}, \{\epsilon_{it}\}} \]
is achieved when \( u_i^{(k)} = \pm \epsilon_{it} - C \log T/\sqrt{p}, u_i^{(k)} = \pm \epsilon_{it}, \) or \( \max_i \|u_i^{(k)}\| = C(\log p/\sqrt{T} + \epsilon/\sqrt{p}) \) whose solution is denoted by \( u_i^{(k)} \) which is independent of \( i \). For the first the two cases, the probability of the two events are zero, hence it is enough to consider \( U_{\{u_i^{(k)}\}, \{\epsilon_{it}\}} \) which, similar to (20), is \( o_p(1) \) due to Assumptions 1’ and 3. This completes the proof of (44) and hence the expansion of \( \sum_{i=1}^{p} g_i(u_i^{(k-1)}, v_i^{(k)}) \) in (21) with \( (u_i^{(0)}, v_i, \tilde{D}_{it}^{(0)}, W_i^{(0)}, I_i^{(0)}) \) there replaced by \( (u_i^{(k-1)}, v_i^{(k)}, \tilde{D}_{it}^{(k)}, W_i^{(k)}, I_i^{(k)}) \) holds uniformly in \( \max_i \|v_i^{(k)}\| \leq C \log T/\sqrt{p}, \max_i \|u_i^{(k-1)}\| \leq C(\log p/\sqrt{T} + \epsilon/\sqrt{p}) \).

Following exactly the same lines as in the remaining proof of Lemma 3 (the lines below (21)), we have
\begin{align*}
P\{\max_t \|\tilde{\nu}_t^{(k)} - \nu_t^{(k)}\| > \delta \} &\leq P\{\sum_{i=1}^{p} G_i(\tilde{u}_i^{(k-1)}, \tilde{v}_i) - G_i(u^{(k-1)}, \nu_t^{(k)}) < 0, \max_t \|\nu_t^{(k)}\| \leq C \log T - \delta, \\
&\max_i \|\tilde{u}_i^{(k-1)}\| \leq C(\log p/\sqrt{T} + \epsilon/\sqrt{p}) \} + P\{\max_t \|\nu_t^{(k)}\| > C \log T - \delta \} \\
&+ P\{\max_i \|\tilde{u}_i^{(k-1)}\| > C(\log p/\sqrt{T} + \epsilon/\sqrt{p}) \} \leq C\epsilon. \tag{46} \end{align*}

Notice that \( \max_i |\tilde{D}_{it}^{(k)} - D_{it}| = o_p(1) \) due to the restriction that
\[ \max_i \|\tilde{u}_i^{(k-1)}\| \leq C(\log p/\sqrt{T} + \epsilon/\sqrt{p}). \]
This shows that \( \tilde{D}_{it}^{(k)} - E\tilde{D}_{it}^{(k)} \) can be replaced by \( D_{it} \) in the definition of \( \nu_t^{(1)} \). This completes the proof of
Lemma 6. Under Assumptions 1-5, $A_k \subseteq B_k \subseteq A_{k+1}$ with probability approaching one.

Proof. The proof of Lemma 3 shows that once $\hat{u}_i^{(k)}$’s enter $B_k$ and satisfy Assumptions 1-5, $\hat{v}_t^{(k+1)}$’s will satisfy the condition of $A_{k+1}$ with probability approaching one. The proof of Lemma 4 shows that once $\hat{v}_t^{(k)}$’s enter $A_k$ and satisfy Assumptions 1-5, $\hat{u}_i^{(k)}$’s will satisfy the condition of $B_k$ with probability approaching one.

Next, we show that $W^{(k)}$ is close to $W^{(k-1)}$ when $k \geq 1$ and hence as implied by Lemma 4, $\hat{l}_i^{(k)} W^{(k-1)} - l_0^p$ is close to zero, which further shows that $W^{(k)}$ and $W_0$ are close enough.

Lemma 7. Under Assumptions 1-5,

\[
\|W^{(k)} - W^{(k-1)}\| = O_p\left(\frac{\log p}{\sqrt{T}}\right) + o_p\left(\frac{1}{\sqrt{p}}\right),
\]

\[
\max_i \|W^{(k)}\hat{\ell}_i^{(k)} - l_0^p\| = O_p\left(\frac{\log p}{\sqrt{T}}\right) + o_p\left(\frac{1}{\sqrt{p}}\right),
\]

\[
\|W^{(k)} - W_0\| = O_p\left(\frac{\log p}{\sqrt{T}}\right) + o_p\left(\frac{1}{\sqrt{p}}\right).
\]

If further $(\log p)^2 \frac{T}{p} + \frac{\log^3 p}{\sqrt{T}} = o(1)$,

\[
\|W^{(k)} - W^{(k-1)}\| = o_p\left(\frac{1}{\sqrt{T}}\right),
\]

\[
\max_i \|W^{(k)}\hat{\ell}_i^{(k)} - l_0^p - \frac{1}{\sqrt{T}} \pi_i^{(k)}\| = o_p\left(\frac{1}{\sqrt{T}}\right),
\]

\[
\|W^{(k)} - W_0\| = o_p\left(\frac{1}{\sqrt{T}}\right).
\]

Proof. The proof of Lemma 4 and Lemma 5 show that

\[
\frac{1}{p} \sum_i \hat{l}_i^{(k)} \hat{\ell}_i^{(k)} h_i(\zeta_{it}) = [W^{(k-1)}]^{-1} \frac{1}{p} \sum_i l_0^p \hat{l}_i^{(k)} h_i(\zeta_{it}) [W^{(k-1)}]^{-1} + O_p\left(\frac{\log p}{\sqrt{T}}\right) + o_p\left(\frac{1}{\sqrt{p}}\right),
\]

and

\[
\frac{1}{p} \sum_i \hat{l}_i^{(k)} \hat{\ell}_i^{(k)} h_i(\zeta_{it}) = [W^{(k-1)}]^{-1} \frac{1}{p} \sum_i l_0^p \hat{l}_i^{(k)} h_i(\zeta_{it}) + O_p\left(\frac{\log p}{\sqrt{T}}\right) + o_p\left(\frac{1}{\sqrt{p}}\right).
\]
Combining (53) and (54) proves (47). (47) and Lemma 4 prove that

\[ W^{(k)} \hat{T}^{(k)}_i - l^0_i = W^{(k-1)} \hat{T}^{(k)}_i - l^0_i + \{ W^{(k)} - W^{(k-1)} \} (\hat{T}^{(k)}_i - (W^{(k-1)})^{-1} l^0_i + (W^{(k-1)})^{-1} l^0_i) \]

(55)

where the \( O_p \) and \( o_p \) terms hold uniformly in \( i \leq p \). The only difference between \( W^{(k)} \) and \( W_0 \) is the difference between \( h_i(\xi^{(k)}_{iit}) \) and \( h_i(0) \), then (49) is a straightforward result of (48) and the property \( \max_i \| \xi^{(k)}_{iit} \| \leq \max_i \| W^{(k)} \hat{T}^{(k)}_i - l^0_i \| = O_p(\sqrt{p}) + o_p(\frac{1}{\sqrt{p}}) \).

If further \( (\log p)^2 T + \log p \sqrt{T} = o(1) \), Lemma 4 and Lemma 6 show that

\[ \hat{u}^{(k)}_i = \frac{1}{\sqrt{p}} u^{(k)}_i + o_p(\frac{1}{\sqrt{T}}) \]

(56)

where the \( o_p \) term holds uniformly in \( i \leq p \). Notice that \( \xi^{(k)}_{iit} \) does not depend on \( \epsilon_{iit} \)'s, (56) together with the temporal and cross-section weak dependence condition on \( D_{iit} \)'s in Assumption 4 proves that

\[ \frac{1}{p} \sum_{i=1}^{p} h_i(\xi^{(k)}_{iit}) \{ \hat{T}^{(k)}_i - (W^{(k-1)})^{-1} l^0_i \} \]

\[ = (W^{(k-1)})^{-1} \frac{1}{p \sqrt{T}} \sum_{i=1}^{p} h_i(\xi^{(k)}_{iit}) u^{(k)}_i l^0_i + o_p(\frac{1}{\sqrt{T}}) = O_p(\frac{1}{\sqrt{pT}}) + o_p(\frac{1}{\sqrt{T}}), \]

(57)

\[ \frac{1}{p} \sum_{i=1}^{p} h_i(\xi^{(k)}_{iit}) (\hat{T}^{(k)}_i - (W^{(k-1)})^{-1} l^0_i) \hat{T}^{(k)}_i \]

\[ = (W^{(k-1)})^{-1} \frac{1}{p \sqrt{T}} \sum_{i=1}^{p} h_i(\xi^{(k)}_{iit}) u^{(k)}_i l^0_i \hat{T}^{(k)}_i + o_p(\frac{1}{\sqrt{T}}) = O_p(\frac{1}{\sqrt{pT}}) + o_p(\frac{1}{\sqrt{T}}), \]

(58)

and

\[ \frac{1}{p} \sum_{i=1}^{p} h_i(\xi^{(k)}_{iit}) (\hat{T}^{(k)}_i - (W^{(k-1)})^{-1} l^0_i) \]

\[ = (W^{(k-1)})^{-1} \frac{1}{p \sqrt{T}} \sum_{i=1}^{p} h_i(\xi^{(k)}_{iit}) u^{(k)}_i l^0_i + o_p(\frac{1}{\sqrt{T}}) = O_p(\frac{1}{\sqrt{pT}}) + o_p(\frac{1}{\sqrt{T}}), \]

(59)

where the \( O_p(\frac{1}{\sqrt{pT}}) \) term is due to \( \frac{1}{\sqrt{pT}} u^{(k)}_i \) and Assumption 3, and \( o_p(\frac{1}{\sqrt{T}}) \) is due to (56). (57)-(59) prove
(50), (55) and (50) prove (51). For \( \tilde{\theta}_i \in [0, 1] \),

\[
h_i(\xi_{it}^{(k)}) - h_i(0) = h'_i(\tilde{\theta}_i \xi_{it}^{(k)}) \theta_{it} (\hat{l}_i^{(k)})' W^{(k)} - l^0_i \implies f_i^0.  
\]  

(60)

This together with (51), the boundedness of \( h'_i(x) \), and the temporal and cross-section weak dependence condition on \( D_{it} \)'s in Assumption 4 proves that

\[
\frac{1}{p} \sum_{i=1}^{p} \{ h_i(\xi_{it}^{(k)}) - h_i(0) \} \hat{\theta}_i^{(k)} l_i^{(k)} = O_p(\frac{1}{\sqrt{pT}}) + o_p(\frac{1}{\sqrt{T}}),
\]

\[
\frac{1}{p} \sum_{i=1}^{p} \{ h_i(\xi_{it}^{(k)}) - h_i(0) \} \hat{\theta}_i^{(k)} l_i^{(k)} W_{it} = O_p(\frac{1}{\sqrt{pT}}) + o_p(\frac{1}{\sqrt{T}}),
\]

where the \( O_p(\frac{1}{\sqrt{pT}}) \) term is due to \( \frac{1}{\sqrt{T}} \) and Assumption 3, and \( o_p(\frac{1}{\sqrt{pT}}) \) is due to the \( o_p(\frac{1}{\sqrt{T}}) \) term in (51).

This completes the proof of (52).

\[\square\]

**Corollary 1.** Under the conditions in Lemma 5,

\[
\max_i \|W_{it} - l_i^0\| = O_p(\frac{\log p}{\sqrt{T}} + o_p(\frac{1}{\sqrt{pT}})).
\]

Moreover, if further \( (\log p)^2 \frac{T}{p} + \frac{\log^3 p}{\sqrt{T}} = o(1) \),

\[
\max_i \|W_{it} - l_i^0 - \frac{1}{\sqrt{T}} \tilde{w}_i^{(k)}\| = o_p(\frac{1}{\sqrt{T}}).
\]

**Proof.** Corollary 1 is a direct result of (49) and (52) in Lemma 7 by simply replacing \( W^{(k)} \) in (48) and (51) by \( W_{0} \). \[\square\]

**Corollary 2.** Under Assumptions 1-5,

\[
\hat{f}_t^{(k+1)} - W_0 f_i^0 = O_p(\frac{\log p}{\sqrt{T}}) + O_p(\frac{1}{\sqrt{p}}).
\]

Moreover, if further \( \frac{p \log^2 p}{T} = o(1) \),

\[
\hat{f}_t^{(k+1)} - W_0 f_i^0 = \frac{1}{\sqrt{p}} \tilde{v}_i^{(k+1)} + o_p(\frac{1}{\sqrt{p}}).
\]

**Proof.** By Lemmas 5-7,

\[
\hat{f}_t^{(k+1)} - W_0 f_i^0 = \hat{f}_t^{(k+1)} - W^{(k)} f_i^0 + (W^{(k)} - W_0) f_i^0 = O_p(\frac{1}{\sqrt{p}}) + O_p(\frac{\log p}{\sqrt{T}}) + o_p(\frac{1}{\sqrt{p}}).
\]

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If $\frac{\log^2 p}{T} = o(1)$, the above equation demonstrates that

$$f_t^{(k+1)} - W_0 f_t^{0} = f_t^{(k+1)} - W f_t^{0} + o_p\left(\frac{1}{\sqrt{p}}\right),$$

and hence by Lemma 6, $f_t^{(k+1)} - W_0 f_t^{0} = \frac{1}{\sqrt{p}} v_t^{(k+1)} + o_p\left(\frac{1}{\sqrt{p}}\right)$.

Proof of Theorem 1 Theorem 1 is a direct consequence of Corollaries 1 and 2. (5) and the first equation of Theorem 1 show that

$$I_r = \frac{1}{T} \sum_{t=1}^{T} \tilde{f}_t \tilde{f}_t' = \frac{1}{T} \sum_{t=1}^{T} \tilde{W}_0 f_t^{0} f_t^{0'} \tilde{W}_0' + o_p(1) = \tilde{W}_0 \tilde{W}_0' + o_p(1).$$