ON \( p \)-ADIC SIMPSON AND RIEMANN-HILBERT CORRESPONDENCES IN THE IMPERFECT RESIDUE FIELD CASE

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Abstract. Let \( K \) be a mixed characteristic complete discrete valuation field with residue field admitting a finite \( p \)-basis, and let \( G_K \) be the Galois group. Inspired by Liu and Zhu’s construction of \( p \)-adic Simpson and Riemann-Hilbert correspondences over rigid analytic varieties, we construct such correspondences for representations of \( G_K \). As an application, we prove a Hodge-Tate (resp. de Rham) “rigidity” theorem for \( p \)-adic representations of \( G_K \), generalizing a result of Morita.

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1. Introduction

1.1. Overview and main theorems. Let \( K \) be a CDVF (complete discrete valuation field) of characteristic zero with residue field \( k_K \) of characteristic \( p \) such that \([k_K : k^p_K] < \infty\), and let \( G_K \) be its absolute Galois group. The scope of this paper is to study \( p \)-adic representations of \( G_K \). Such representations naturally arise in the study of relative \( p \)-adic Hodge theory. For example, let \( R = \mathbb{Z}_p(x^{\pm 1}) \) be the (integral) ring corresponding to a rigid torus, and let \( \widetilde{R}_{(p)} \) be the \( p \)-adic completion of the localization \( R_{(p)} \). Then \( \widetilde{R}_{(p)}[1/p] \) is a CDVF with residue field \( \mathbb{F}_p(x) \).

It turns out that to understand \( p \)-adic representations of \( \pi_1^{\text{ét}}(R[1/p]) \) (called “the relative case” in the following), it is crucial to understand representations for these CDVF{s} with imperfect residue fields (called “the imperfect residue field case” in the following). Such examples are indeed numerous. For example, Brinon studies the Hodge-Tate, de Rham, and crystalline representations in the imperfect residue field case in [Bri06], which then pave the way for the studies in the relative case in [Bri08]. For another example, very recently, Shimizu proves a variant of \( p \)-adic local monodromy theorem in the relative case in [Shi]; the proof makes crucial use of \( p \)-adic local monodromy theorem in the imperfect residue field case established in [Mor14, Ohk13].

Contrary to the above examples where results in the imperfect residue field case are obtained first and then are used in the study of relative case, this paper goes in the opposite direction. Indeed, inspired by the work of Liu-Zhu [LZ17] where they construct a \( p \)-adic Simpson correspondence and a \( p \)-adic Riemann-Hilbert correspondence in the relative case, we construct in this paper such correspondences in the imperfect residue field case. Before we discuss such correspondences, we start with an application which

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is indeed a main motivation of this paper. This application, as will be discussed in the ensuing remark, in turn is motivated by considerations in \textit{integral p-adic Hodge theory}. We start by introducing some notations.

\textbf{Notation 1.1.1.} Fix an algebraic closure $\overline{K}$ of $K$ and let $G_K := \text{Gal}(\overline{K}/K)$. Throughout this paper, we use $L$ to denote another mixed characteristic CDVF whose residue field has a finite $p$-basis, and similarly fix some algebraic closure $\overline{L}$ and denote $G_L := \text{Gal}(\overline{L}/L)$. Furthermore, we assume there is some non-zero (hence injective) \textit{continuous} field homomorphism

$$i : K \hookrightarrow L.$$  

Fix a such $i$, and extend it to some embedding (still denoted by $i$) of their algebraic closures

$$i : \overline{K} \hookrightarrow \overline{L}.$$  

(Such extensions are not unique, but we fix one.) With respect to these fixed field embeddings, we obtain a continuous group homomorphism (still denoted by $i$)

$$i : G_L \rightarrow G_K.$$  

Given a $p$-adic Galois representation $V$ of $G_K$, we use $V|_{G_L}$ to denote the $p$-adic Galois representation of $G_L$ via the map $i$, and call it the “restriction” of $V$ to $G_L$ (note though the map $i : G_L \rightarrow G_K$ is not necessarily injective).

Given a $V$ of dimension $n$ as above, Brinon defined the Fontaine modules $D_{\text{HT},K}(V)$ resp. $D_{\text{dR},K}(V)$ (cf. §2.2): they are both $K$-vector spaces of dimension $\leq n$, and $V$ is called Hodge-Tate resp. de Rham if $\dim_K D_{\text{HT},K}(V) = n$ resp. $\dim_K D_{\text{dR},K}(V) = n$. One can similarly define $D_{\text{HT},L}, D_{\text{dR},L}$.

\textbf{Theorem 1.1.2.} (cf. §5.2 and §7.2.) Use the above notations.

(1) We have “base-change” isomorphisms

$$D_{\text{HT},K}(V) \otimes_K L \simeq D_{\text{HT},L}(V|_{G_L}),$$  

$$D_{\text{dR},K}(V) \otimes_K L \simeq D_{\text{dR},L}(V|_{G_L}).$$  

(2) The $G_K$-representation $V$ is Hodge-Tate (resp. de Rham) if and only if $V|_{G_L}$ (considered as a representation of $G_L$) is so.

\textbf{Remark 1.1.3.} We comment on the history, motivation, and future outlook of the above theorem.

(1) Note that Item (2) of Thm. 1.1.2 is a trivial consequence of Item (1). We call Item (2) the Hodge-Tate \textit{rigidity} theorem (resp. de Rham \textit{rigidity} theorem) for $p$-adic Galois representations in the imperfect residue field case. The terminology “rigidity” (sometimes also called “purity”) comes from similar results in the relative case established by Liu-Zhu, Shimizu and Petrov, cf. Rem. 1.1.5 for more details.

(2) Item (2) of Thm. 1.1.2 was proved by Morita [Mor10] for certain special type embeddings denoted as $K \hookrightarrow K^\text{pf}$ (cf. Example 3.1.2). Indeed, in Morita’s setting, $K^\text{pf}$ is the “perfection” field of $K$ (hence in particular $K^\text{pf}$ has perfect residue field); furthermore, this embedding has many advantageous features (cf. Rem. 3.1.3) that allow Morita to obtain the rigidity theorems by an \textit{explicit} method. However, even in Morita’s set-up, he did not obtain Item (1) of Thm. 1.1.2; furthermore, as far as we understand, Morita’s method can not work in the general setting.

(3) There are natural reasons to consider all possible embeddings $i : K \hookrightarrow L$. Indeed, in the study of crystalline (resp. semi-stable) representations, particularly in \textit{integral p-adic Hodge theory}, a “Frobenius-compatible” embedding (introduced by Brinon) that we denote as $i_\sigma : K \hookrightarrow \mathcal{K}_\sigma$ (cf. Example 3.1.4 for details) naturally arises, and is not covered by Morita’s setting.
(4) For integral $p$-adic Hodge theory, we have better understanding in the imperfect residue field case (cf. [BT08, Gao]) than in the relative case: a major reason being Kedlaya’s slope filtration theorem [Ked04, Ked05] is only available in the former setting. As Thm. 1.1.2 is applicable in the natural setting of integral theory in the imperfect residue field case, we speculate it would be of further use in the development of relative integral theory.

In contrast to Morita’s explicit method, our approach is completely conceptual, and is inspired by the work of Liu-Zhu [LZ17] (and subsequent works such as [DLLZ, Shi18, Pet]). Let us first quickly recall the $p$-adic Simpson and Riemann-Hilbert correspondences of Liu-Zhu. Let $k/Q_p$ be a finite extension, let $k_\infty$ be the extension by adjoining all $p$-power roots of unity, and let $\hat{k_\infty}$ be the $p$-adic completion. With respect to the perfectoid field $k_\infty$, one can construct the usual Fontaine ring $B_{dR}(\hat{k_\infty})$. (Instead of $k_\infty$ and $\hat{k_\infty}$, one can also use the algebraic closure $\bar{k}$ and its completion $\hat{k}$). Since we only use Thm. 1.1.4 and Rem. 1.1.5 for motivational reasons, we refer the readers to the references cited therein for more details.

**Theorem 1.1.4.** [LZ17] Let $X$ be a smooth rigid analytic variety over $k$. Then

1. there is a tensor functor $\mathcal{H}$ from the category of $\mathbb{Q}_p$-local systems on $X_{\text{ét}}$ to the category of nilpotent Higgs bundles on $X_{\hat{k_\infty}}$. In addition,
2. there is a tensor functor $\mathcal{R}\mathcal{H}$ from the category of $\mathbb{Q}_p$-local systems on $X_{\text{ét}}$ to the category of vector bundles on the ringed space $X = (X_{\hat{k_\infty}}, O_X \hat{\otimes} B_{dR}(\hat{k_\infty}))$, equipped with a semi-linear action of $\text{Gal}(k_\infty/k)$, and with a filtration and an integrable connection that satisfy Griffiths transversality.

The functors $\mathcal{H}$ and $\mathcal{R}\mathcal{H}$ are called the $p$-adic Simpson correspondence and the $p$-adic Riemann-Hilbert correspondence respectively. They are both compatible with pullback along arbitrary morphisms and pushforward under smooth proper morphisms.

**Remark 1.1.5.** Let us summarize some “rigidity” theorems in the relative case. Let $X$ be as in above theorem and suppose it is geometrically connected, let $L$ be an étale local system.

1. Liu-Zhu [LZ17, Thm. 1.3] shows that $L$ is de Rham if and only if any of its specializations to classical points is so. The key argument is that they can prove $D_{\text{dR}}(L)$ is a vector bundle and satisfies a base change property.
2. Shimizu [Shi18, Thm. 1.1] shows that the generalized Hodge-Tate weights of the $p$-adic Galois representation $L_x$ (the specialization of $L$ at any classical point $x$) are constant. A key ingredient in the proof is Shimizu’s construction of decompleted Simpson and Riemann-Hilbert correspondences (denote by $\mathcal{H}$ and RH respectively).
3. If in addition $X$ comes from analytification of a smooth algebraic variety over $k$, then Petrov [Pet, Prop. 1.2] shows that $L$ is Hodge-Tate if and only if any of its specializations to classical points is so. Petrov’s proof makes crucial use of the log-Simpson and log-Riemann-Hilbert correspondences constructed in [DLLZ].

Now, let us state our $p$-adic Simpson and Riemann-Hilbert correspondences in the imperfect residue field case. Let $K_\infty/K$ be the extension by adjoining all $p$-power roots of unity, and let $\tilde{K_\infty}$ be the $p$-adic completion. Let $G_{K_\infty} := \text{Gal}(\overline{K}/K_\infty)$, $\Gamma_K := \text{Gal}(K_\infty/K)$.

Let $C$ be the $p$-adic completion of $\overline{K}$, and let $B_{\text{dR}} = B_{\text{dR}}(C)$ be the “usual” Fontaine’s ring associated to the perfectoid field $C$, and let $t$ be the usual Fontaine’s element. Let $\mathcal{O}B_{\text{dR}}$ be the de Rham period ring in the imperfect residue field case (which contains $B_{\text{dR}}$). (There are some subtle issues about $\mathcal{O}B_{\text{dR}}$, cf. Rem. 2.1.5 for details). Let $\mathcal{O}L_{\text{dR}} := (\mathcal{O}B_{\text{dR}})^{G_{K_\infty}}$. See Notation 1.3.1 for notations of the form “$\text{Rep}_G(R)$” (the usual category of semi-linear representations).
Theorem 1.1.6. (cf. Thm. 5.1.2 and Thm. 7.1.2 for precise statements). There are two commutative diagrams of tensor functors where the vertical functors are tensor equivalences.

\[
\begin{array}{ccc}
\text{Rep}_{G_K}(\mathbb{Q}_p) & \longrightarrow & \text{Rep}_{G_K}(C) \\
\downarrow^\mathcal{H} & & \downarrow^\simeq \\
\text{Rep}_{T_K}(K_{\infty}) & & \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{Rep}_{G_K}(\mathbb{Q}_p) & \longrightarrow & \text{Rep}_{G_K}(B_{dR}) \\
\downarrow^\mathcal{RH} & & \downarrow^\simeq \\
\text{Rep}_{G_K}(\mathcal{O}_{L_{dR}}) & & \\
\end{array}
\]

(1) We call the functor \(\mathcal{H}\) (resp. \(\mathcal{H}\)) the Simpson correspondence (resp. the decompleted Simpson correspondence). Furthermore, we can upgrade \(\mathcal{H}\) so that it maps to a category of representations equipped with Higgs fields.

(2) We call the functor \(\mathcal{RH}\) (resp. \(\mathcal{RH}\)) the Riemann-Hilbert correspondence (resp. the decompleted Riemann-Hilbert correspondence). Furthermore, we can upgrade \(\mathcal{RH}\) so that it maps to a category of representations equipped with integrable connections.

(3) Let \(i : K \hookrightarrow L\) be the continuous embedding as in Notation 1.1.1. All the functors \(\mathcal{H}, \mathcal{H}, \mathcal{RH}, \mathcal{RH}\) (as well as the upgraded versions) satisfy natural base change properties with respect to \(i\). For a quick example (the other cases are similar), let \(W \in \text{Rep}_{G_K}(C)\), then we have

\[
\mathcal{H}(W) \otimes_{K_{\infty}} \mathcal{L}_{\infty} \simeq \mathcal{H}_L(W|_{G_L} \otimes_C C_L)
\]

where \(\mathcal{L}_{\infty}, C_L\) (resp. \(\mathcal{H}_L\)) are the analogous fields (resp. the analogous Simpson correspondence functor) for the field \(L\).

Once the above correspondences are established, it is relatively easy (compared to the results in Rem. 1.1.5) to obtain Thm. 1.1.2. Indeed, the situation is much simpler in our case: our \(D_{HT,K}(V)\) and \(D_{dr,K}(V)\) are automatically “vector bundles”; indeed they are just finite dimensional vector spaces over \(K\). It hence suffices to establish the base change property as in Thm. 1.1.2(1): this will follow from the base change property of the Simpson and Riemann-Hilbert correspondences, together with some cohomological computations.

Remark 1.1.7. We now comment on the relation between Thm. 1.1.6 and various “Sen theories”.

(1) Let \(K\) be a CDVF with perfect residue field, then classical Sen theory says that we have a diagram of tensor functors:

\[
\begin{array}{ccc}
\text{Rep}_{G_K}(\mathbb{Q}_p) & \longrightarrow & \text{Rep}_{G_K}(C) \\
\downarrow^\mathcal{H}_{K,\infty} & & \downarrow^\simeq \\
\text{Rep}_{G_K}(\mathbb{K}_{\infty}) & & \\
\end{array}
\]
(Note in contrast to the general case in Thm. 1.1.6, all the functors on the triangles are tensor equivalences). Hence our $p$-adic Simpson correspondence could be regarded as a “generalization” of the classical Sen theory. (Similarly, our Riemann-Hilbert correspondence could be regarded as a “generalization” of the classical Sen-Fontaine theory reviewed in §6.1).

(2) For general $K$, Brinon [Bri03] also developed a “Sen theory”. Let us only very briefly recall here, cf. §4.2 for more details. Indeed, it is related to the embedding $K \hookrightarrow K^{\text{pf}}$ that we mentioned in Rem. 1.1.3(2), where $K^{\text{pf}}$ is the “perfection” field of $K$. Let $\overline{K}_\infty^{\text{pf}}$ is the cyclotomic extension with $\overline{K}_\infty^{\text{pf}}$ being the $p$-adic completion, and let $\Gamma := \text{Aut}(C/K^{\text{pf}})$. Then Brinon constructs a tensor equivalence (there is also a “decompleted” version)

$$\text{Rep}_{\Gamma_K}(C) \simeq \text{Rep}_{\Gamma}(\overline{K}_\infty^{\text{pf}}).$$

Here are some general remarks:

(a) The embedding $K \hookrightarrow K^{\text{pf}}$, and hence Brinon’s theory, depend on choices of “local coordinates” $t_i$, cf. Notation 2.1.10. In addition, Brinon’s theory has no functoriality (or base change property) with respect to embeddings $i : K \hookrightarrow L$.

(b) Morita’s proof of Hodge-Tate rigidity with respect to $K \hookrightarrow K^{\text{pf}}$ (cf. Rem. 1.1.3(2)) makes use of Brinon’s theory.

(c) The field $\overline{K}_\infty$ in our Simpson correspondence is a subfield of $\overline{K}_\infty^{\text{pf}}$, hence our theory can be regarded as a refinement of Brinon’s theory. Note in particular the field $\overline{K}_\infty$ (unlike $\overline{K}^{\text{pf}}_\infty$) is independent of the “local coordinates” $t_i$.

(3) Andreatta-Brinon [AB10] then further developed a “Sen-Fontaine theory” for general $K$, cf. §6.3, which lifts Brinon’s Sen theory, and is used in Morita’s proof of de Rham rigidity with respect to $K \hookrightarrow K^{\text{pf}}$. In comparison, our Riemann-Hilbert correspondence lifts the Simpson correspondence, which however is via a very different lifting process. Indeed, Andreatta-Brinon’s theory is a “ker $\theta$-adic lifting”, whereas ours is a “$t$-adic lifting”; compare the diagrams in Rem. 6.3.3 and Rem. 7.1.5.

Let us give some brief technical remarks about the proof of Thm. 1.1.6. The equivalence $\text{Rep}_{\Gamma_K}(\overline{K}_\infty) \simeq \text{Rep}_{\Gamma}(\overline{K}_\infty)$ is proved using similar ideas as in the perfect residue field case, namely, via Tate’s normalized traces. The key part then is to construct the functor $\text{Rep}_{\Gamma_K}(C) \to \text{Rep}_{\Gamma}(\overline{K}_\infty)$, and this is where we are inspired by Liu-Zhu’s construction, which uses some argument from Grothendieck’s proof of $\ell$-adic monodromy theorem. Finally, the Riemann-Hilbert correspondence can be obtained by standard dévissage argument.

1.2. Structure of the paper. In §2, we review (Hodge-Tate and de Rham) Fontaine rings and Fontaine modules; then in §3, we introduce similar notations with respect to the embedding $K \hookrightarrow L$. In §4, we discuss Sen theory and Tate-Sen decomposition, which are used in §5 to construct the Simpson correspondence and to prove the Hodge-Tate rigidity theorem. Via dévissage argument, we discuss Sen-Fontaine theory and Tate-Sen-Fontaine decomposition in §6, which are used in §7 to construct the Riemann-Hilbert correspondence and to prove the de Rham rigidity theorem.

1.3. Notations and conventions.

Notation 1.3.1. Suppose $\mathcal{G}$ is a topological group that acts continuously on a topological ring $R$. We use $\text{Rep}_G(R)$ to denote the category where an object is a finite free $R$-module $M$ (topologized via the topology on $R$) with a continuous and semi-linear $\mathcal{G}$-action in the usual sense that

$$g(rx) = g(r)g(x), \forall g \in \mathcal{G}, r \in R, x \in M.$$ (The only case in this paper where the action is linear is when $R = \mathbb{Q}_p$). Morphisms in the category are the obvious ones.
1.3.2. *Locally analytic vectors.* Let us very quickly recall the notion of locally analytic vectors, which is particularly convenient in the study of *Sen theory,* see [BC16, §2] and [Ber16, §2] for more details. Recall the multi-index notations: if \( \mathbf{c} = (c_1, \ldots, c_d) \) and \( \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d \) (here \( \mathbb{N} = \mathbb{Z}_{\geq 0} \)), then we let \( \mathbf{c}^\mathbf{k} = c_1^{k_1} \cdot \ldots \cdot c_d^{k_d} \). Let \( G \) be a \( p \)-adic Lie group, and let \((W, \| \cdot \|)\) be a \( \mathbb{Q}_p \)-Banach representation of \( G \). Let \( H \) be an open subgroup of \( G \) such that there exist coordinates \( c_1, \ldots, c_d : H \to \mathbb{Z}_p \) giving rise to an analytic bijection \( c : H \to \mathbb{Z}_p^d \). We say that an element \( w \in W \) is an \( H \)-analytic vector if there exists a sequence \( \{w_\mathbf{k}\}_{\mathbf{k} \in \mathbb{N}^d} \) with \( w_\mathbf{k} \to 0 \) in \( W \), such that \[
g(w) = \sum_{\mathbf{k} \in \mathbb{N}^d} c(\mathbf{g})^{\mathbf{k}} w_\mathbf{k}, \quad \forall g \in H.
\]
Let \( W^{H \text{-an}} \) denote the space of \( H \)-analytic vectors. We say that a vector \( w \in W \) is locally analytic if there exists an open subgroup \( H \) as above such that \( w \in W^{H \text{-an}} \).

1.3.3. *Some other conventions.* All group cohomologies are continuous group cohomologies. We use \( \text{Mat}(*) \) to denote the set of matrices with elements in \(*\); the size of the matrices will be clear from the context.

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## 2. Fontaine rings and Fontaine modules

In this section, we review Fontaine rings and Fontaine modules in the imperfect residue field case.

**Notation 2.0.1.** Let \( K \) be a complete discrete valuation field of characteristic 0 with residue field \( k_K \) of characteristic \( p > 0 \) such that \([k_K : k_{K^p}] = p^d\) where \( d \geq 0 \). Let \( \mathcal{O}_K \) be the ring of integers. Fix an algebraic closure \( \overline{K} \) of \( K \), and let \( G_K = \text{Gal}(\overline{K}/K) \). Let \( C \) be the \( p \)-adic completion of \( \overline{K} \). Let \( \varepsilon_0 = 1 \), let \( \varepsilon_1 \in \overline{K} \) be a primitive \( p \)-th root of unity, and inductively define a sequence \( \{\varepsilon_n\}_{n \geq 0} \) in \( \overline{K} \) where \( \varepsilon_{n+1}^p = \varepsilon_n, \forall n \).

**Remark 2.0.2.** Let us remark on the various choices we make in this paper.

1. One can readily check that different choices of \( \overline{K} \) produce “equivalent results” for all constructions in this paper.
2. We will use the fixed \( \{\varepsilon_n\}_{n \geq 0} \) to define Fontaine’s usual element \( t \). A different choice of these elements also produce “equivalent results” for all constructions in this paper.
3. In Notation 2.1.10, we will introduce a system of “coordinates” \( t_i \)'s for \( K \): these coordinates make it possible to write out “explicit” expressions for the period rings such as in Prop. 2.1.11. Note that
   a. The main result in §2, namely the Fontaine module theory Prop. 2.2.2, is independent of these \( t_i \)'s. (So is the connection operator \( \nabla \), cf. the paragraph above Notation 2.3.2).
   b. The embedding \( K \hookrightarrow K^{\text{pf}} \) in Example 3.1.2 not only depends on \( t_i \), it further depends on choices of \( p \)-power roots of them. This means that Brinon’s Sen theory reviewed in §4.2 and hence Morita’s result Thm. 3.3.1 depend on these choices.
   c. In contrast, our main results on Simpson and Riemann-Hilbert correspondences are independent of these coordinates \( t_i \)'s (although we do use them for intermediate computations).

### 2.1. Fontaine rings

In this subsection, we review some Fontaine period rings in the imperfect residue field case.
Convention 2.1.1. In this paper, we choose to use the “relative”-style notations (e.g., similar to those in [LZ17]), since our main constructions are parallel to those in the relative case. Nonetheless, most of the results in this subsection are developed in [Bri06]. Here, we compare the notations:

1. Our \( C, A_{\text{inf}}(\mathcal{O}_C/\mathcal{O}_K) \) are the same as those in [Bri06]. Our notation \( \mathcal{O} \) does not appear in [Bri06] (but the ring is used there).
2. Our \( \mathcal{O}_{dR}^{+\text{nc}}, \mathcal{O}_{dR}^{\text{nc}} \) correspond to “\( \mathcal{B}_{dR}^{+}, \mathcal{B}_{dR}^{\text{nc}}, \mathcal{B}_{dR}^{\text{nc}} \)” in [Bri06] respectively. Our \( \mathcal{O}_{dR} \) is not defined in [Bri06]. (There is indeed some subtle issues about these de Rham period rings, cf. Rem. 2.1.5.)
3. Our \( \mathcal{B}_{HT}, \mathcal{O}_{HT} \) correspond to “\( \mathfrak{g}_r \mathcal{B}_{dR}^{\text{nc}}, \mathcal{B}_{HT} \)” in [Bri06] respectively.

Notation 2.1.2. Since \( \mathcal{O}_C \) is a perfectoid ring, we can define its tilt \( \mathcal{O}_C^{p} \); let \( W(\mathcal{O}_C^{p}) \) be the ring of Witt vectors. Elements in \( \mathcal{O}_C^{p} \) are in bijection with sequences \( (x(n))_{n \geq 0} \) where \( x(n) \in \mathcal{O}_C \) and \( (x(n+1)^p = x(n) \). Let \( \xi \in \mathcal{O}_C^{p} \) be the element defined by the sequence \( \{ \xi_n \}_{n \geq 0} \), and let \( [\xi] \in W(\mathcal{O}_C) \) be its Teichmüller lift. Let

\[
\theta : W(\mathcal{O}_C) \to \mathcal{O}_C
\]

be the usual Fontaine’s map, it extends to

\[
\theta : W(\mathcal{O}_C^p)[1/p] \to C.
\]

Both these maps have principle kernels generated by \( [\xi] - 1 \).

Notation 2.1.3. Let \( \mathcal{B}_{dR}^{+} \) be the \((\ker \theta)\)-adic completion of \( W(\mathcal{O}_C^p)[1/p] \), and hence the \( \theta \)-map extends to

\[
\theta : \mathcal{B}_{dR}^{+} \to C.
\]

Let \( t := \log[\xi] \in \mathcal{B}_{dR}^{+} \) be the usual element, which is also a generator of the kernel of the \( \theta \)-map on \( \mathcal{B}_{dR}^{+} \). We equip \( \mathcal{B}_{dR}^{+} \) with the \( t \)-adic filtration, and extend it to a filtration on \( \mathcal{B}_{dR} := \mathcal{B}_{dR}^{+}[1/t] \) via Fil\( i \mathcal{B}_{dR} := t^i \cdot \mathcal{B}_{dR}, \forall i \in \mathbb{Z} \). Define

\[
\mathcal{B}_{HT} := \bigoplus_{i \in \mathbb{Z}} \text{gr}^i(\mathcal{B}_{dR}).
\]

Note that the \( G_K \)-action on \( W(\mathcal{O}_C^p) \) naturally induces actions on \( \mathcal{B}_{dR}^{+} \) and \( \mathcal{B}_{HT} \).

Definition 2.1.4. By scalar extension, the \( \theta \)-map induces an \((\mathcal{O}_K\text{-linear})\) map

\[
\theta_K : \mathcal{O}_K \otimes_Z W(\mathcal{O}_C^p) \to \mathcal{O}_C.
\]

Let

\[
\text{A}_{\text{inf}}(\mathcal{O}_C/\mathcal{O}_K) := \lim_{\rightarrow} \left( \mathcal{O}_K \otimes_Z W(\mathcal{O}_C^p) \right) / \left( \theta_K^{-1}(p\mathcal{O}_C) \right)^n.
\]

Then the \( \theta_K \)-map extends to

\[
\theta_K : \text{A}_{\text{inf}}(\mathcal{O}_C/\mathcal{O}_K) \to \mathcal{O}_C \quad \text{and} \quad \theta_K : \text{A}_{\text{inf}}(\mathcal{O}_C/\mathcal{O}_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to C.
\]

Let

\[
\mathcal{O}_{dR}^{+\text{nc}} := \lim_{\rightarrow} A_{\text{inf}}(\mathcal{O}_C/\mathcal{O}_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) / (\ker \theta_K)^n.
\]

(Here, the superscript “nc” stands for “non-complete”, as will be explained in Rem. 2.1.5.) Note that there is a canonical map \( \mathcal{B}_{dR}^{+} \to \mathcal{O}_{dR}^{+\text{nc}} \), and hence we can regard \( t \) as an element in \( \mathcal{O}_{dR}^{+\text{nc}} \). The \( \theta_K \)-map extends to

\[
\theta_K : \mathcal{O}_{dR}^{+\text{nc}} \to C.
\]

Let \( m_{dR} := \ker \theta_K \) for the above map. Define a \( \mathbb{Z} \geq 0 \)-filtration on \( \mathcal{O}_{dR}^{+\text{nc}} \) where

\[
\text{fil}^i \mathcal{O}_{dR}^{+\text{nc}} := m_{dR}^i, \forall i \geq 0.
\]

Using \( \text{fil}^i \), define a \( \mathbb{Z} \)-filtration on \( \mathcal{O}_{dR}^{+\text{nc}}[1/t] \) by:

\[
\text{Fil}^0 \mathcal{O}_{dR}^{\text{nc}} := \bigoplus_{n=0}^\infty t^{-n} \cdot \text{fil}^n \mathcal{O}_{dR}^{+\text{nc}}
\]

\[
\text{Fil}^i \mathcal{O}_{dR}^{\text{nc}} := t^i \cdot \text{Fil}^0 \mathcal{O}_{dR}^{\text{nc}}, \forall i \in \mathbb{Z}.
\]
Remark 2.1.5. Given a ring $R$ with a decreasing filtration \{Fil\^i \, R\}_{i \in \mathbb{Z}}$, we say this filtration is complete if for each $i$, we have \( \lim_{j \to -\infty} \text{Fil}^{i+j} R = \text{Fil}^i R \).

1. Note that the ring \( \mathcal{O}B_{dr}^{\text{nc}+} \) is \( t \)-adically complete, hence the filtration \{Fil\^i \, \mathcal{O}B_{dr}^{\text{nc}+}\}_{i \in \mathbb{Z}} is a complete filtration on \( \mathcal{O}B_{dr}^{\text{nc}} \). However, this is not the correct filtration for Fontaine module theory; the correct one is the Fil\^i \, \mathcal{O}B_{dr}^{\text{nc}+} filtration defined above. As we will see in below, the ring \( \mathcal{O}B_{dr}^{\text{nc}+} \) together with this filtration is enough to establish the usual Fontaine module theory, cf. Rem. 2.2.3.

2. However, a subtle issue is that the filtration Fil\^i \, \mathcal{O}B_{dr}^{\text{nc}+} is not complete (unless \( k_k \) is perfect)! As observed by Shimizu and noted in [DLLZ, Rem. 2.2.11], this causes trouble in the construction of Riemann-Hilbert correspondences, which is a lift of the Simpson correspondence. Fortunately (cf. loc. cit.), this issue can be resolved by introducing the \( t \)-adic completions of the ring and its filtrations, cf. Def. 2.1.6.

Definition 2.1.6. Note that Fil\^0 \mathcal{O}B_{dr}^{\text{nc}} is a ring, and hence define its \( t \)-adic completion:

\[
\text{Fil}^0 \mathcal{O}B_{dr} := \lim_{n \to \infty} (\text{Fil}^0 \mathcal{O}B_{dr}^{\text{nc}})/t^n.
\]

Let \( \mathcal{O}B_{dr} := \text{Fil}^0 \mathcal{O}B_{dr}[1/t] \), and equip it with the \( t \)-adic filtration, namely

\[
\text{Fil}^i \mathcal{O}B_{dr} := t^i \cdot \text{Fil}^0 \mathcal{O}B_{dr}, \forall i \in \mathbb{Z}.
\]

Remark 2.1.7. Note that we do not define any “\( \mathcal{O}B_{dr}^{\text{nc}+} \)”. Indeed, given a CDVF with perfect residue field, then with notations in the classical setting (cf. Notation 3.2.2), there is a co-incidence Fil\^0 \mathcal{B}_{dr} = \mathbb{B}^{+}_{dr}. In the general imperfect residue field case (or in relative case), we simply use Fil\^0 \mathcal{O}B_{dr} as the “effective” de Rham ring.

Definition 2.1.8. For \( -\infty \leq a \leq b \leq +\infty \), define

\[
\mathcal{O}B_{dr}^{\text{nc}[a,b]} := \text{Fil}^{a} \mathcal{O}B_{dr}^{\text{nc}}/\text{Fil}^{b+1} \mathcal{O}B_{dr}^{\text{nc}},
\]
\[
\mathcal{O}B_{dr}^{[a,b]} := \text{Fil}^{a} \mathcal{O}B_{dr}/\text{Fil}^{b+1} \mathcal{O}B_{dr},
\]

where Fil\^\infty \mathcal{O}B_{dr}^{\text{nc}} = \mathcal{O}B_{dr}^{\text{nc}}, Fil\^\infty \mathcal{O}B_{dr}^{\text{nc}} = 0 and similarly for \( \mathcal{O}B_{dr} \). Note that since Fil\^i \mathcal{O}B_{dr} is the \( t \)-adic completion of Fil\^i \mathcal{O}B_{dr}^{\text{nc}} for all \( i \in \mathbb{Z} \), we have

\[
\mathcal{O}B_{dr}^{\text{nc}[a,b]} = \mathcal{O}B_{dr}^{[a,b]}, \forall a, b \neq \pm \infty.
\]

In particular, we have gr\^i \mathcal{O}B_{dr}^{\text{nc}} \simeq \text{gr} gr\^i \mathcal{O}B_{dr} , \forall i \in \mathbb{Z}. Thus, we can define

\[
\mathcal{O}C := \bigoplus_{i \in \mathbb{Z}} \text{gr}^i \mathcal{O}B_{dr}^{\text{nc}} \simeq \bigoplus_{i \in \mathbb{Z}} \text{gr}^i \mathcal{O}B_{dr}.
\]

Remark 2.1.9. The relations of all the “de Rham rings” are summarized in the following diagram where all rows are short exact.

\[
0 \to \text{Fil}^1 B^{+}_{dr} \to B^{+}_{dr} \to C \to 0
\]
\[
0 \to \text{Fil}^1 \mathcal{O}B^{+\text{nc}}_{dr} \to \mathcal{O}B^{+\text{nc}}_{dr} \to \mathcal{O}C \to 0
\]

(2.1.2)

We claim in fact the three squares in the left side are all Cartesian: namely, the embeddings of the various “de Rham rings” in the center column are in fact strict with respect to their filtrations (where we use fil\^filtration for \( \mathcal{O}B_{dr}^{+\text{nc}} \)). The only
non-trivial part of the claim is about the square in the second floor, see [Bri06, Prop. 2.20] for the proof.

To ease the mind, let us mention that in our main construction (in the Riemann-Hilbert correspondence), we only use the following small part of the above diagram:

\[ 0 \rightarrow \text{Fil}^1 B_{\text{dR}} \rightarrow B_{\text{dR}} \rightarrow C \rightarrow 0 \]

\[ 0 \rightarrow \text{Fil}^1 \mathcal{O}B_{\text{dR}} \rightarrow \text{Fil}^0 \mathcal{O}B_{\text{dR}} \rightarrow \mathcal{O}C \rightarrow 0 \]

Notation 2.1.10. Recall \([k_K : k_K^p] = p^d\). Let \( \bar{t}_i, \cdots, \bar{t}_d \) be a \( p \)-basis of \( k_K \), and for each \( i \), let \( t_i \in \mathcal{O}_K \) be some fixed lift of \( \bar{t}_i \). For each \( m \geq 0 \), fix an element \( t_{i,m} \in \overline{K} \) where \( t_{i,0} = t_i \) and \( (t_{i,m+1})^p = t_{i,m}, \forall m \). By abuse of notations, we simply denote \( t_i^m := t_{i,m} \).

Note that the sequence \( \{t_i^m\}_{m \geq 0} \) defines an element in \( \mathcal{O}_C^p \), which we denote as \( t_i^p \). Let \( \{t_i^p\} \in W(\mathcal{O}_C^p) \) be the Teichmüller lift. Let

\[ u_i = t_i \otimes 1 - 1 \otimes [t_i^p] \in \mathcal{O}_K \otimes \mathbb{Z} W(\mathcal{O}_C^p). \]

We (briefly) recall some constructions above [Bri06, Prop. 2.9]. There is a natural homomorphism \( B_{\text{dR}}^+ \rightarrow \mathcal{O}B_{\text{dR}}^{+,\text{nc}} \). Note that \( u_i \in m_{\text{dR}} \) and \( \mathcal{O}B_{\text{dR}}^{+,\text{nc}} \) is complete with respect to the \( m_{\text{dR}} \)-adic topology, hence there is a natural homomorphism

\[ f : B_{\text{dR}}^+[[u_1, \cdots, u_d]] \rightarrow \mathcal{O}B_{\text{dR}}^{+,\text{nc}}. \]

Proposition 2.1.11. [Bri06, Prop. 2.9, Prop. 2.11]

1. The above homomorphism \( f : B_{\text{dR}}^+[[u_1, \cdots, u_d]] \rightarrow \mathcal{O}B_{\text{dR}}^{+,\text{nc}} \) is an isomorphism; in addition, \( f(u_i) \in \text{Fil}^1 \mathcal{O}B_{\text{dR}}^+ \).
2. Let \( \bar{u}_i \) be the image of \( f(u_i) \) in \( \text{gr}^1 \mathcal{O}B_{\text{dR}}^{\text{nc}} \), then \( f \) induces an isomorphism of graded rings

\[ \mathcal{O}B_{\text{HT}} \cong C[[t^{\pm 1}, \bar{u}_1, \cdots, \bar{u}_d]] \]

where the right hand side is graded by total degree (where \( t^{-1} \) has degree \(-1\)). Hence, there is an isomorphism

\[ \mathcal{O}C \cong C[[\frac{\bar{u}_1}{t}, \cdots, \frac{\bar{u}_d}{t}]]. \]

Notation 2.1.12. Define

\[ V_i = \frac{1}{t} \log\left(\frac{[t_i^p]}{t_i^p}\right). \]

These are elements of \( \text{Fil}^0 \mathcal{O}B_{\text{dR}}^{\text{nc}} \), and by abuse of notation we also regard \( V_i \) as an element in \( \text{gr}^0 \mathcal{O}B_{\text{dR}}^{\text{nc}} = \mathcal{O}C \). Then by Prop. 2.1.11, we can deduce

\[ \mathcal{O}B_{\text{dR}}^{+,\text{nc}} \cong B_{\text{dR}}^+[[tV_1, \cdots, tV_d]] \]

\[ \mathcal{O}C \cong C[V_1, \cdots, V_d]. \]

2.2. Fontaine modules.

Definition 2.2.1. For \( V \in \text{Rep}_{G_K}(\mathbb{Q}_p) \), define

\[ D_{\text{dR},K}^{\text{nc}}(V) = (V \otimes_{\mathbb{Q}_p} \mathcal{O}B_{\text{dR}}^{\text{nc}})^{G_K} \]

\[ D_{\text{dR},K}(V) = (V \otimes_{\mathbb{Q}_p} \mathcal{O}B_{\text{dR}})^{G_K} \]

\[ D_{\text{HT},K}(V) = (V \otimes_{\mathbb{Q}_p} \mathcal{O}B_{\text{HT}})^{G_K}. \]

Proposition 2.2.2.

1. There are canonical isomorphisms

\[ (\mathcal{O}B_{\text{dR}}^{\text{nc}})^{G_K} = (\mathcal{O}B_{\text{dR}})^{G_K} = K, \quad (\mathcal{O}B_{\text{HT}})^{G_K} = K. \]

2. The natural injection \( D_{\text{dR}}^{\text{nc}}(V) \hookrightarrow D_{\text{dR}}(V) \) induced by the injection \( \mathcal{O}B_{\text{dR}}^{\text{nc}} \hookrightarrow \mathcal{O}B_{\text{dR}} \) is an isomorphism of \( K \)-vector spaces.
3. \( D_{\text{dR},K}(V) \), \( D_{\text{dR},K}(V) \), \( D_{\text{HT},K}(V) \) are \( K \)-vector space of dimension \( \leq \text{dim}_{\mathbb{Q}_p} V \).
Proof. The statements concerning $\mathcal{O}B^\text{nc}_{\text{dR}}$ are proved in [Bri06, Prop. 2.16, Prop. 3.22], the statements concerning $\mathcal{O}B_{\text{HT}}$ are mentioned in the proof of [Bri06, Prop. 3.35]. To finish the proof, it would suffice to show $D^\text{nc}_{\text{dR}}(V) \to D_{\text{dR}}(V)$ is an isomorphism (note that the case for $V = \mathbb{Q}_p$ implies $(\mathcal{O}B^\text{nc}_{\text{dR}})^{G_K} = (\mathcal{O}B_{\text{dR}})^{G_K}$).

Now, [Bri06, Prop. 3.22] already shows that $D^\text{nc}_{\text{dR}}(V)$ is a finite dimensional $K$-vector space, and [Bri06, Prop. 4.19] already shows that the filtration on $D^\text{nc}_{\text{dR}}(V)$, namely:

$$\text{Fil}^i D^\text{nc}_{\text{dR}}(V) := (V \otimes_{\mathbb{Q}_p} \text{Fil}^i \mathcal{O}B^\text{nc}_{\text{dR}})^{G_K}, \forall i \in \mathbb{Z}$$

is exhaustive and separated. Namely, there exists some finite numbers $c \leq d$ such that

$$D^\text{nc}_{\text{dR}}(V) = (V \otimes_{\mathbb{Q}_p} \mathcal{O}B^\text{nc}_{\text{dR}}_{[a,b]})^{G_K}, \forall -\infty \leq a \leq c \leq d \leq b \leq +\infty.$$  

Note that since $c, d$ are finite, we have

$$(V \otimes_{\mathbb{Q}_p} \mathcal{O}B^\text{nc}_{\text{dR}}_{[c,d]})^{G_K} = (V \otimes_{\mathbb{Q}_p} \mathcal{O}B_{\text{dR}}^{[c,d]})^{G_K}.$$  

An obvious inductive argument then shows

$$D_{\text{dR}}(V) = (V \otimes_{\mathbb{Q}_p} \mathcal{O}B^\text{nc}_{\text{dR}}_{[a,b]})^{G_K}, \forall -\infty \leq a \leq c \leq d \leq b \leq +\infty.$$  

Remark 2.2.3. The isomorphism $D^\text{nc}_{\text{dR}}(V) \simeq D_{\text{dR}}(V)$ shows that both $\mathcal{O}B^\text{nc}_{\text{dR}}$ and $\mathcal{O}B_{\text{dR}}$ can be used to construct the de Rham Fontaine module theory.

Definition 2.2.4. Let $V \in \text{Rep}_{G_K}(\mathbb{Q}_p)$.

(1) $V$ is a called de Rham if $D^\text{nc}_{\text{dR},K}(V)$ (equivalently $D_{\text{dR},K}(V)$) is of $K$-dimension $\dim_{\mathbb{Q}_p} V$.

(2) $V$ is called Hodge-Tate if $D_{\text{HT},K}(V)$ is of $K$-dimension $\dim_{\mathbb{Q}_p} V$.

2.3. Connection maps.

Notation 2.3.1. Let

$$\hat{\Omega}_{O_K} := \lim_{\overset{n \to 0}{\longleftarrow}} \Omega^1_{O_K/\mathbb{Z}}/p^n\Omega^1_{O_K/\mathbb{Z}}$$

be the $p$-adically continuous Kähler differentials, and let

$$\hat{\Omega}_K = \hat{\Omega}_{O_K} \otimes_{O_K} K.$$  

$\hat{\Omega}_{O_K}$ is a free $O_K$-module with a set of basis $\{d\log(t_i)\}_{1 \leq i \leq d}$, where $t_1, \cdots, t_d \in O_K$ are introduced in Notation 2.1.10 as a lift of a $p$-basis of $K$. Let

$$d : O_K \to \hat{\Omega}_{O_K}$$

be the induced differential map (extending the canonical differential map $d : O_K \to \Omega_{O_K}$).

We now construct connections on $\mathcal{O}B^\text{nc}_{\text{dR}}$ and $\mathcal{O}B_{\text{dR}}$. In [Bri06, §2.2], a connection is defined in an explicit way that seemingly to depend on various choices (such as $u_i$’s). Here, we follow the presentation of [Sch13, §6] to see that the connection is indeed a unique extension of that on $K$, and in particular is an intrinsic notion.

Construction 2.3.2. Recall we have $d : O_K \to \hat{\Omega}_{O_K}$. It extends $W(O'_C)$-linearly to a unique (continuous) connection

$$\nabla : W(O'_C) \otimes_{\mathbb{Z}} O_K \to W(O'_C) \otimes_{\mathbb{Z}} \hat{\Omega}_{O_K}.$$  

Via (2.1.1), it then extends $W(O'_C)$-linearly (but not $A_{\inf}(O_C/O_K)$-linearly, since $\nabla$ does not kill $\theta_K^{-1}(pO'_C)$) to

$$\nabla : A_{\inf}(O_C/O_K) \to A_{\inf}(O_C/O_K) \otimes_{O_K} \hat{\Omega}_{O_K},$$  

and hence $B^+_{\text{dR}}$-linearly (extending the $W(O'_C)$-linearity above, via ker $\theta$-completion) to

$$\nabla : \mathcal{O}B^+_{\text{dR},\text{nc}} \to \mathcal{O}B^+_{\text{dR},\text{nc}} \otimes_{K} \hat{\Omega}_K,$$  

and finally $B_{\text{dR}}$-linearly (which is possible since $t \in B^+_{\text{dR}}$) to

$$\nabla : \mathcal{O}B^\text{nc}_{\text{dR}} \to \mathcal{O}B^\text{nc}_{\text{dR}} \otimes_{K} \hat{\Omega}_K.$$
One can easily check that $\nabla(u_i) = dt_i$, hence indeed this *intrinsically* defined $\nabla$ is exactly the same as the one explicitly defined in [Bri06]. Namely, using the expression $\mathcal{O}B_{\text{dr}}^{+ \text{nc}} = B_{\text{dr}}^+[[u_1, \ldots, u_d]]$, and for each $i$, let $N_i$ be the unique $B_{\text{dr}}^+$-derivation of $\mathcal{O}B_{\text{dr}}^{+ \text{nc}}$ such that

$$N_i(u_j) = \delta_{i,j} t_j$$

where $\delta_{i,j}$ is the Kronecker symbol, then $\nabla$ has the following explicit formula:

$$(2.3.1) \quad \nabla : \mathcal{O}B_{\text{dr}}^{+ \text{nc}} \to \mathcal{O}B_{\text{dr}}^{+ \text{nc}} \otimes_K \widehat{\Omega}_K, \quad x \mapsto \sum_{i=1}^d N_i(x) \otimes d\log(t_i).$$

We list some basic properties of $\nabla$.

1. $\nabla$ is integrable (since $N_i$’s commute with each other).
2. We have $(\mathcal{O}B_{\text{dr}}^{+ \text{nc}})^{\nabla=0} = B_{\text{dr}}$.
3. By [Bri06, Prop. 2.23], $\nabla$ satisfies Griffiths transversality, i.e.,

$$\nabla(\text{Fil}^r \mathcal{O}B_{\text{dr}}^{+ \text{nc}}) \subset \text{Fil}^{r-1} \mathcal{O}B_{\text{dr}}^{+ \text{nc}} \otimes_K \widehat{\Omega}_K.$$ 

4. By [Bri06, Prop. 2.24], $\nabla$ commutes with $G_K$-action.
5. By [Bri06, Prop. 2.25], $\nabla|_K$ is precisely the differential $d : K \to \widehat{\Omega}_K$. (Note this is now a “vacuous” statement, as we have just shown $\nabla$ comes from a unique extension of $d$ on $K$).

**Construction 2.3.3.** Since $\nabla(t) = 0$, the connection $\nabla$ on $\mathcal{O}B_{\text{dr}}^{+ \text{nc}}$ continuously extends to

$$\nabla : \mathcal{O}B_{\text{dr}} \to \mathcal{O}B_{\text{dr}} \otimes_K \widehat{\Omega}_K.$$ 

Note $\nabla$ on $\mathcal{O}B_{\text{dr}}$ still satisfies all the listed properties above, namely:

1. $\nabla$ is integrable.
2. $(\mathcal{O}B_{\text{dr}})^{\nabla=0} = B_{\text{dr}}$.
3. $\nabla$ satisfies Griffiths transversality, i.e.,

$$\nabla(\text{Fil}^r \mathcal{O}B_{\text{dr}}) \subset \text{Fil}^{r-1} \mathcal{O}B_{\text{dr}} \otimes_K \widehat{\Omega}_K.$$ 

4. $\nabla$ commutes with $G_K$-action.
5. $\nabla|_K$ is precisely the differential $d : K \to \widehat{\Omega}_K$.

**Construction 2.3.4.** Since $\nabla$ on $\mathcal{O}B_{\text{dr}}^{+ \text{nc}}$ (and $\mathcal{O}B_{\text{dr}}$) satisfies Griffiths transversality, we can define the zero-th graded:

$$\text{gr}^0 \nabla : \mathcal{O}C \to \mathcal{O}C(-1) \otimes_K \widehat{\Omega}_K = \mathcal{O}C \otimes_K \widehat{\Omega}_K(-1).$$

Using the expression $\mathcal{O}C \simeq C[\frac{t_1}{t}, \ldots, \frac{t_d}{t}]$, one sees that $\text{gr}^0 \nabla$ is $C$-linear, and

$$\text{gr}^0 \nabla(\frac{u_i}{t}) = t_i \otimes \frac{d\log t_i}{t}.$$ 

More conveniently, using the expression $\mathcal{O}C \simeq C[V_1, \ldots, V_d]$, we have

$$\text{gr}^0 \nabla(V_i) = 1 \otimes \frac{d\log t_i}{t}.$$ 

(Let us note that $\nabla$ is not $K$-linear, but $\text{gr}^0 \nabla$ is $C$-linear hence $K$-linear.)

### 3. Change of base fields

In this section, we give some examples of the continuous embedding $i : K \hookrightarrow L$, and we set up notations for the field $L$. In the end, we recall Morita’s rigidity theorem for a special embedding.

**Notation 3.0.1.**

1. Recall in Notation 1.1.1, we fixed a continuous field embedding $i : K \hookrightarrow L$. We also fixed $i : \overline{K} \hookrightarrow \overline{L}$ and $i : G_L \to G_K$.

2. Throughout this paper, we use $\mathbb{K}$ to denote a mixed characteristic CDVF with perfect residue field ($\mathbb{K}$ may or may not be related with $K$).
3.1. Embedding of fields. We introduce three examples of \( K \hookrightarrow L \) to orient the readers.

Example 3.1.1. Let \( \mathbb{Q}_p \hookrightarrow L = W(\mathbb{F}_p(x^{1/p})) [1/p] \) be the inclusion embedding. Note that Thm. 1.1.2 implies that a representation \( V \in \text{Rep}_{\mathbb{G}_m}(\mathbb{Q}_p) \) is Hodge-Tate resp. de Rham if and only if \( V|_{G_L} \) is so. Note that although \( L/\mathbb{Q}_p \) is an unramified extension, the extension of their residue fields is not algebraic. Hence any induced inclusion \( C_{\mathbb{Q}_p} \hookrightarrow C_L \) is not an isomorphism, and hence the mentioned Hodge-Tate resp. de Rham rigidity does not follow from the “usually known argument”.

Example 3.1.2. In Notation 2.1.10, for each \( m \geq 0 \), we fixed the elements \( t_i^{1/p^m} \in \overline{K} \). Let \( K^{(pf)}/K \) be the algebraic extension by adjoining all these \( t_i^{1/p^m} \), namely:

\[
K^{(pf)} = \bigcup_{m \geq 0} K(t_1^{1/p^m}, \ldots, t_d^{1/p^m}).
\]

Let \( K^{pf} \) be its \( p \)-adic completion. Let

\[
i : K \hookrightarrow K^{pf}
\]

be the inclusion embedding.

Remark 3.1.3. Let us summarize some advantageous features of the above embedding, which are crucially exploited in [Bri03] and then used in [Mor10] to prove Thm. 3.3.1.

1. We introduced \( u_i, V_i \) in Notations 2.1.10 and 2.1.12; \( u_i \) and \( tV_i \) are fixed by \( G_{K^{pf}} \).
2. The algebraic extension \( K \to K^{(pf)} \) has an obvious Galois closure by adjoining all \( p \)-power roots of unity whose Galois group is an explicit \( p \)-adic Lie group, cf. Notation 4.2.3.

Example 3.1.4. (This example is key for the study of crystalline resp. semi-stable representations, as well as integral \( p \)-adic Hodge theory in the imperfect residue field case). Now suppose the residue field \( k_K \) is not perfect (otherwise, the following discussion is vacuous). The readers can consult [Bri06, §1] and [BT08, §2] for more details.

- Let \( k_K \) be the residue field of \( K \), which is an algebraic closure of \( k_K \). Let \( k \) be the radical closure of \( k_K \subset \overline{k_K} \); note that \( k \) is a perfect field. Let \( W(k) \) be the ring of Witt vectors for \( k \), recall there is a unique Frobenius lifting on \( W(k) \).
- Let \( C(k_K) \) be the Cohen ring of \( k_K \), and fix an embedding \( C(k_K) \hookrightarrow O_K \) (which is not unique) and denote the image as \( O_{K_0} \subset O_K \); let \( K_0 = O_{K_0}[1/p] \).
- Fix a Frobenius lifting \( \sigma : O_{K_0} \to O_{K_0} \) (such Frobenius liftings are not unique). With \( \sigma \) fixed, there then exists a unique continuous Frobenius-equivariant ring embedding

\[
i_\sigma : O_{K_0} \hookrightarrow W(k)
\]

whose reduction modulo \( p \) is the inclusion \( k_K \hookrightarrow k \).
- Let \( \mathbb{K}_0 = W(k)[1/p] \), and fix \( \mathbb{K}_0 \) an algebraic closure of \( \mathbb{K}_0 \). The map \( i_\sigma \) extends (non-uniquely) to some embedding

\[
i_\sigma : \overline{K} \hookrightarrow \overline{k}.
\]
- Finally, define \( \mathbb{K}_\sigma := i_\sigma(K)\mathbb{K}_0 \). Then we obtain an embedding

\[
i_\sigma : K \hookrightarrow \mathbb{K}_\sigma.
\]

Note that \( \mathbb{K}_\sigma \) is always isomorphic to \( K^{pf} \) in Notation 3.1.2 as they are both the “smallest” CDVF containing \( K \) whose residue field is the perfection of \( k_K \). However, it is clear the embeddings \( i_\sigma \) can be very different from the inclusion \( K \hookrightarrow K^{pf} \).

Remark 3.1.5. Clearly, the features mentioned in Rem. 3.1.3 in general do not apply in Example 3.1.4.
3.2. Fontaine rings for different base fields.

**Notation 3.2.1.** Recall in Notation 1.1.1, we fixed a continuous embedding \( i : K \rightarrow L \). We also fixed \( i : \overline{K} \rightarrow \overline{L} \) and \( i : G_L \rightarrow G_K \). Let \( C_L \) be the \( p \)-adic completion of \( \overline{L} \), then we have an induced embedding \( i : C \rightarrow C_L \) that is compatible with the Galois actions via \( i : G_L \rightarrow G_K \). As \( L \) is also a CDVF whose residue field has finite \( p \)-basis, we can carry out all the constructions in Section 2. Indeed, we can first define the rings \( \mathcal{O}_{C_L}^i, W(\mathcal{O}_{C_L}^i) \); note that \( i \) induces continuous homomorphisms \( \mathcal{O}_{C}^i \rightarrow \mathcal{O}_{C_L}^i \), \( W(\mathcal{O}_{C}^i) \rightarrow W(\mathcal{O}_{C_L}^i) \) that are compatible with Galois actions via \( i : G_K \rightarrow G_L \). Note that since \( i : K \hookrightarrow L \) is continuous, there exists some \( s \in \mathbb{Z} \) such that \( i(\mathcal{O}_K) \subset p^s\mathcal{O}_L \). This implies that \( K \hookrightarrow L \) can be extended to a continuous homomorphism

\[
i : A_{\text{inf}}(\mathcal{O}_C / \mathcal{O}_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \rightarrow A_{\text{inf}}(\mathcal{O}_{C_L} / \mathcal{O}_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p
\]

and then to

\[
i : \mathcal{O}B_{\text{dR}}^{+,-} \rightarrow \mathcal{O}B_{\text{dR},L}^{+,-}.
\]

There are similar continuous homomorphisms to \( \mathcal{O}B_{\text{dR},L}, \mathcal{O}B_{\text{HT},L}, \mathcal{O}C_L \) etc., that are always compatible with structures such as Galois actions, filtrations, connections (whenever applicable).

**Notation 3.2.2.** For \( \mathbb{K} \) a CDVF with perfect residue field (\( \mathbb{K} \) may or may not be related with \( K \)), we fix an algebraic closure \( \overline{\mathbb{K}} \) and its \( p \)-adic completion \( \mathbb{C} \). We then use notations such as

\[
\mathbb{B}_{\text{dR}}, \mathbb{B}_{\text{HT}}
\]

to denote the “usual” Fontaine rings (in the perfect residue field case).

3.3. Morita’s results. The following theorem of Morita is a special case of our Thm. 1.1.2(2). Morita’s argument relies on the features mentioned in Rem. 3.1.3, and cannot be adapted to the general case.

**Theorem 3.3.1.** [Mor10] Let \( V \in \text{Rep}_{G_K}(\mathbb{Q}_p) \). Let \( K \hookrightarrow K^{\text{pf}} \) be the embedding in Example 3.1.2. Then \( V \) is Hodge-Tate (resp. de Rham) if and only if \( V|_{G_{K^{\text{pf}}}} \) is so.

**Remark 3.3.2.** In [Mor14], Morita also proves that \( V \) is potentially semi-stable (resp. potentially crystalline) if and only if \( V|_{G_{K^{\text{pf}}}} \) is so. This, together with Thm. 3.3.1, allows him to obtain \( p \)-adic local monodromy theorem in the imperfect residue field case: namely, \( V \) is de Rham if and only it is potentially semi-stable. Alternatively, another proof which works even when \([k : k^p] = +\infty \) is supplied by Ohkubo [Ohk13].

4. Sen theory and Tate-Sen decompletion

In this section, we review Sen theory, first in the classical perfect residue field case, then in the imperfect residue field case. We also carry out a Tate-Sen decompletion in the imperfect residue field case, and use it to carry out some cohomological computations for later use.

4.1. Sen theory in the perfect residue field case. The following theorem is proved in [Sen81], with some key ingredients from [Tat67]. We also refer to [Fon04] for a very clear exposition. Let \( \mathbb{K} \) be a CDVF with perfect residue field as in Notation 3.2.2. Let \( \mathbb{K}_{\infty} \) be the extension by adjoining all \( p \)-power roots of unity, and let \( \overline{\mathbb{K}}_{\infty} \) be the \( p \)-adic completion. Let \( G_{\mathbb{K}_{\infty}} := \text{Gal}(\overline{\mathbb{K}} / \mathbb{K}_{\infty}), \Gamma_{\mathbb{K}} := \text{Gal}(\mathbb{K}_{\infty} / \mathbb{K}) \).

**Proposition 4.1.1.** The rule

\[
\text{Rep}_{G_{\mathbb{K}}}(\mathbb{C}) \ni W \mapsto \mathcal{H}_{\mathbb{K}}(W) := W^{G_{\mathbb{K}_{\infty}}}
\]

induces a tensor equivalence of categories

\[
\mathcal{H}_{\mathbb{K}} : \text{Rep}_{G_{\mathbb{K}}}(\mathbb{C}) \xrightarrow{\sim} \text{Rep}_{\Gamma_{\mathbb{K}}}(\overline{\mathbb{K}}_{\infty}).
\]

The base change map

\[
\text{Rep}_{\Gamma_{\mathbb{K}}}(\mathbb{K}_{\infty}) \ni M \mapsto M \otimes_{\mathbb{K}_{\infty}} \overline{\mathbb{K}}_{\infty}
\]
induces a tensor equivalence of categories
\[ \text{Rep}_{\Gamma}(\mathbb{K}_\infty) \xrightarrow{\cong} \text{Rep}_{\Gamma}(\hat{\mathbb{K}}_\infty). \]
These equivalences then induce a tensor equivalence, denoted as \( H_{\mathcal{K}} \):
\[ H_{\mathcal{K}} : \text{Rep}_{G_{\mathcal{K}}}(\mathbb{C}) \xrightarrow{\cong} \text{Rep}_{\Gamma}(\hat{\mathbb{K}}_\infty). \]
Let us gather some results used in the proof of above theorem.

**Proposition 4.1.2.**

1. \( H^1(G_{\mathcal{K}_\infty}, \text{GL}_n(\mathbb{C})) = 0 \) for any \( n \geq 1 \).
2. Any \( W \in \text{Rep}_{G_{\mathcal{K}_\infty}}(\mathbb{C}) \) is trivial.
3. Let \( W \in \text{Rep}_{G_{\mathcal{K}_\infty}}(\mathbb{C}) \), then \( H^1(G_{\mathcal{K}_\infty}, W) = 0 \). (As a special case, \( H^1(G_{\mathcal{K}_\infty}, \mathbb{C}) = 0 \)).

**Proof.** See [Fon04, Thm. 1.1, Cor 2.3]. \( \square \)

**Corollary 4.1.3.**

1. \( H^i(G_{\mathcal{K}_\infty}, \mathbb{C}) = 0, \forall i \geq 1 \).
2. Let \( W \in \text{Rep}_{G_{\mathcal{K}_\infty}}(\mathbb{C}) \), then \( H^i(G_{\mathcal{K}_\infty}, W) = 0, \forall i \geq 1 \).

**Proof.** Note that \( G_{\mathcal{K}_\infty} \simeq G_{\hat{\mathcal{K}}_\infty} \) where the later denotes the Galois group of the perfectoid field \( \hat{\mathcal{K}}_\infty \). Then Item (1) follows from the (almost) vanishing theorem of [Sch12, Prop. 7.13]. For Item (2), \( W \) is a trivial representation by Prop. 4.1.2, hence the statement follows from Item (1). \( \square \)

### 4.2. Brinon’s Sen theory in the imperfect residue field case.

We now recall Brinon’s Sen theory in the imperfect residue field case, cf. [Bri03].

**Notation 4.2.1.** We introduce the following diagram of field extensions.

\[
\begin{array}{cccccc}
\mathcal{K} & \xrightarrow{H_L} & \mathcal{K}^{\text{pf}} & \xrightarrow{\text{completion}} & \mathcal{K}^{\text{pf}}_{\infty} & \xrightarrow{C} \\
\downarrow^\Gamma & & & & & \\
K^{(\text{pf})} & \xrightarrow{\text{completion}} & K^{\text{pf}} & \xrightarrow{\text{completion}} & K^{\text{pf}}_{\infty} & \\
\end{array}
\]

(4.2.1)

Let us explain the notations:

1. The general pattern of the diagram is that all vertical arrows are algebraic extensions, and all horizontal arrows are dense embeddings. All fields on the right most \( (K, K^{\text{pf}}, K^{\text{pf}}_{\infty}, C) \) are \( p \)-adically complete.
2. Fields on the third row are introduced Example 3.1.2.
3. On the second row: \( K^{(\text{pf})}_{\infty} \) resp. \( K^{\text{pf}}_{\infty} \) are obtained from \( K^{(\text{pf})} \) resp. \( K^{\text{pf}} \) by adjoining all \( p \)-power roots of unity; \( K^{\text{pf}}_{\infty} \) is the \( p \)-adic completion.
4. Fields on the first row (except \( C \)) are algebraic closures of fields on the second row; \( C \) is the \( p \)-adic completion of all fields on this row.
5. We also introduce the Galois groups \( H_L := \text{Gal}(\mathcal{K}/K^{(\text{pf})}) \) and \( \Gamma := \text{Gal}(K^{(\text{pf})}/K) \).

**Proposition 4.2.2.** [Bri03, Thm. 1, Thm. 2]

1. \( C^{H_L} = \hat{K}^{\text{pf}}_{\infty} \).
2. For \( W \in \text{Rep}_{G_{\mathcal{K}}}(\mathbb{C}) \), \( W^{H_L} \) is an object in \( \text{Rep}_{\hat{\Gamma}}(\hat{K}^{\text{pf}}_{\infty}) \), and there is a \( G_{\mathcal{K}} \)-equivariant isomorphism
\[
W^{H_L} \otimes_{\hat{K}^{\text{pf}}_{\infty}} C \simeq W.
\]
The rule \( W \leadsto W^{H_L} \) induces a tensor equivalence of categories:
\[ \text{Rep}_{G_{\mathcal{K}}}(\mathbb{C}) \simeq \text{Rep}_{\hat{\Gamma}}(\hat{K}^{\text{pf}}_{\infty}). \]
(3) The rule
\[ \text{Rep}_\Gamma(K^{(pf)}_{\infty}) \ni M \leadsto M \otimes_{K^{(pf)}_{\infty}} \widehat{K}^{pf}_{\infty} \in \text{Rep}_\Gamma(\widehat{K}^{pf}_{\infty}) \]
induces a tensor equivalence of categories:
\[ \text{Rep}_\Gamma(K^{(pf)}_{\infty}) \simeq \text{Rep}_\Gamma(\widehat{K}^{pf}_{\infty}). \]

**Notation 4.2.3.** Let \( K_{\infty} = \bigcup_{n \geq 0} K(\varepsilon_n) \), then we have the following diagram of algebraic extensions, where the groups over the arrows denote Galois groups:
\[
\begin{array}{ccc}
K_{\infty} & \xrightarrow{\Gamma_{\text{geom}}} & K^{(pf)}_{\infty} \\
\downarrow & & \downarrow \\
K & \xrightarrow{\Gamma_K} & K^{(pf)}
\end{array}
\]
(4.2.2)

Note that since \( K \hookrightarrow K^{(pf)} \) is unramified, there is a natural isomorphism
\[ \Gamma_K^{(pf)} \simeq \Gamma_K. \]

There is a short exact sequence
\[ 1 \rightarrow \Gamma_{\text{geom}} \rightarrow \Gamma \rightarrow \Gamma_K \rightarrow 1. \]

Note \( \Gamma_{\text{geom}} := \text{Gal}(K^{(pf)}_{\infty}/K_{\infty}) \simeq \mathbb{Z}_p(1)^d \), where \( \mathbb{Z}_p(1)^d \) signifies that \( \Gamma_K^{(pf)} \) acts on \( \Gamma_{\text{geom}} \) via \( \chi_p^{\otimes d} \) where \( \chi_p \) denotes the \( p \)-adic cyclotomic character. Thus, we also have a semi-direct product
\[ \Gamma \simeq \Gamma_K^{(pf)} \ltimes \Gamma_{\text{geom}}. \]

Note all the three groups above are \( p \)-adic Lie groups, and we use \( \text{Lie}\Gamma \) etc. to denote their Lie algebras.

**Proposition 4.2.4.** [Bri03, Prop. 5] (cf. also the summary in [Ohk11, Lem. 4.4]). Let \( M \in \text{Rep}_\Gamma(K^{(pf)}_{\infty}) \).

(1) The action of \( \Gamma_K^{(pf)} \) on \( M \) is locally analytic (cf. §1.3.2), in the sense that there exists a \( K^{(pf)}_{\infty} \)-linear endomorphism \( \varphi \) of \( M \) such that for any \( y \in M \), there exists an open subgroup \( H(y) \subset \Gamma_K^{(pf)} \) such that
\[ g(y) = \exp(\varphi \log \chi_p(g))y, \forall g \in H(y). \]
Indeed, the \( \varphi \)-operator is nothing but the action of (the 1-dimensional Lie algebra) \( \text{Lie}(\Gamma_K^{(pf)}) \).

(2) The action of the (commutative Lie-group) \( \Gamma_{\text{geom}} \) on \( M \) is locally analytic, in the sense that there exist \( K^{(pf)}_{\infty} \)-linear endomorphisms \( \mu_1, \ldots, \mu_d \) of \( M \) such that for any \( y \in M \), there exists an open subgroup \( H(y) \subset \Gamma_{\text{geom}} \) such that
\[ g(y) = \exp\left( \sum_{i=1}^d e_i \mu_i \right)y, \forall g = \prod_{i=1}^d \tau_i^{e_i} \in H(y). \]
Indeed, these \( \mu_i \)-operators come from the action of (the \( d \)-dimensional Lie algebra) \( \text{Lie}(\Gamma_{\text{geom}}) \).

(3) Furthermore, the endomorphisms \( \mu_1, \ldots, \mu_d \) are nilpotent.

4.3. **Tate-Sen decompletion in the imperfect residue field case.**

**Notation 4.3.1.** Recall we have the valuation-compatible inclusion \( K \hookrightarrow K^{pf} \). We use \( v_p \) to denote the valuation such that \( v_p(p) = 1 \). Assume:

- \( K \) (hence \( K^{pf} \)) contains all \( p^2 \)-th roots of unity.
Hence there are isomorphisms
\[ \Gamma_{K^{pf}} \simeq \Gamma_K \simeq \mathbb{Z}_p. \]
Use \( \gamma_0 \) to denote a topological generator of these groups. For each \( s \geq 0 \), let \( K_s \subset K_\infty \) be the unique sub-extension such that \( [K_s : K] = p^s \). For \( x \in K_\infty \), define Tate’s normalized traces
\[ t_{K_s}(x) := p^{-n+s}\text{Tr}_{K_n/K_s}(x) \in K \text{ for } n \gg 0 \text{ such that } x \in K_n \]
Similarly define \( K_s^{pf} \) and \( t_{K^{pf}} \). It is clear that these trace maps are compatible (for elements in \( K_\infty \)), namely,
\[ (4.3.1) \quad t_{K_s}(x) = t_{K_s^{pf}}(x), \forall x \in K_\infty \]

**Proposition 4.3.2.** Suppose \( K \) contains all \( p^2 \)-th roots of unity. For any \( h \geq 1 \), the map
\[ (4.3.2) \quad \iota : H^1(\Gamma_K, \text{GL}_h(K_\infty)) \rightarrow H^1(\Gamma_K, \text{GL}_h(\widehat{K_\infty})) \]
induced by \( \text{GL}_h(K_\infty) \hookrightarrow \text{GL}_h(\widehat{K_\infty}) \) is a bijection.

**Proof.** When \( K \) has perfect residue field, this is a theorem of Sen (cf. [Fon04, Thm. 1.2’]). Thus, we have
\[ (4.3.3) \quad \iota_{K^{pf}} : H^1(\Gamma_{K^{pf}}, \text{GL}_h(K_s^{pf})) \rightarrow H^1(\Gamma_{K^{pf}}, \text{GL}_h(\widehat{K_\infty})) \]
is a bijection.

We now show that indeed all the ingredients leading to the proof of (4.3.3) still hold for \( K \), and hence the same strategy can be used to prove (4.3.2) is bijective. We will follow the exposition in the proof [Fon04, Thm. 1.2’].

**Ingredient 1** (cf. [Fon04, Prop. 1.13]): For \( x \in K_\infty \), we have
\[ (4.3.4) \quad v_p(t_K(x) - x) \geq v_p((\gamma_0 - 1)x) - \frac{p}{p-1} \]
Note this is proved for \( x \in K_s^{pf} \) (and using \( t_{K^{pf}} \)) in [Fon04, Prop. 1.13], hence this **automatically** holds for \( x \in K_\infty \) by the compatibility (4.3.1).

**Ingredient 2** (cf. [Fon04, Prop. 1.15]): the following statements hold.

1. \( t_K : K_\infty \rightarrow K \) is continuous.
2. Let \( \hat{t}_K : \widehat{K_\infty} \rightarrow K \) be the continuous extension of \( t_K \), and let \( L_0 \) denote the kernel. Then we have \( \widehat{K_\infty} = K \oplus L_0 \). Furthermore, \( \gamma_0 - 1 \) is bijective on \( L_0 \) with a continuous inverse \( \rho \).
3. \( v_p(\hat{t}_K(x)) \geq v_p(x) - \frac{p}{p-1} \) for all \( x \in \widehat{K_\infty} \) and \( v_p(\rho(y)) \geq v_p(y) - \frac{p}{p-1} \) for all \( y \in L_0 \).
4. For all \( x \in \widehat{K_\infty} \), we have \( \lim_{s \rightarrow \infty} \hat{t}_{K_s}(x) = x \)

Item (1), (4), as well as the first inequality of Item (3) follow by (4.3.1) and the known result for \( K^{pf} \). It suffices to construct \( \rho \) in Item (2), and prove the second inequality of Item (3). The construction of \( \rho \) does not directly follow from the \( K^{pf} \)-case, but the exact same argument works, and let us give a quick sketch. (We also find the original argument in [Tat67, p173, Prop. 7] very concise.) Indeed, note that \( L_0 = (1-\hat{t}_K)\widehat{K_\infty} \).

Now let \( K_{n,0} = K_n \cap L_0 \), and let \( K_{\infty,0} = \bigcup_{n \geq 0} K_{n,0} \), then its closure is \( L_0 \). Note \( \gamma_0 - 1 \) is injective and \( K \)-linear on the finite dimensional \( K \)-vector spaces \( K_{n,0} \), and hence is bijective. Thus \( \gamma_0 - 1 \) is bijective on \( K_{\infty,0} \); let
\[ \rho : K_{\infty,0} \rightarrow K_{\infty,0} \]
be its inverse. Let \( x \in K_{\infty,0} \) (hence \( t_K(x) = 0 \)), let \( y = (\gamma_0 - 1)x \), hence \( \rho(y) = x \).

Thus Eqn. (4.3.4) in Ingredient 1 implies that
\[ (4.3.5) \quad v_p(\rho(y)) \geq v_p(y) - \frac{p}{p-1}. \]

Thus \( \rho \) is continuous (as \( y \) runs through all of \( K_{\infty,0} \)) and extends to \( L_0 \), and the second inequality of Item (3) follows from (4.3.5).
Ingredient 3: A finite dimensional $K$-sub vector space contained in $\overline{K}_\infty$, which is stable by $\gamma_0$, is contained in $K_\infty$. The proof of [Fon04, Prop. 1.16] works verbatim here, since we have Ingredient 2.

With all three ingredients established, all the argument below [Fon04, Prop. 1.16] work through verbatim, which establishes (4.3.2).

Remark 4.3.3. Note that Ingredients 1 and 2 above show that if we use

$$G_0 = \Gamma_K, \quad \tilde{\Lambda} = \overline{K}_\infty, \quad \Lambda_n = K_n, \quad R_n = i_{K_n}$$

then these data satisfy all the axioms (TS1), (TS2), (TS3) in [BC08, Def. 3.1.3]. Note that in this case, the kernel (denoted by $H_0$ in loc. cit.) of $\Gamma_K \to \mathbb{Z}_p^\times$ is trivial, hence (TS1) of loc. cit. becomes vacuous, and hence we can choose any $c_1 > 0$. We can then choose $c_2 = c_3 = \frac{p}{p-1}$.

Proposition 4.3.4. For $X \in \text{Rep}_{\Gamma_K}(\overline{K}_\infty)$, let $X_f$ be the union of finite dimensional $\overline{K}_\infty$-sub-vector spaces of $X$ that are stable under $\Gamma_K$, then $X_f \in \text{Rep}_{\Gamma_K}(K_\infty)$, and $X_f \otimes_{K_\infty} \overline{K}_\infty \to X$ is a canonical isomorphism.

Proof. If $K$ contains all $p^2$-th roots of unity, then the argument in [Fon04, Thm. 2.4] works verbatim, as now Prop. 4.3.2 is established. The general case follows easily: indeed, simply consider $K' / K$ the extension by adjoining all $p^2$-th roots of unity. Restrict $X$ as representation of $\Gamma_{K'}$, and let $X'_f$ be the union of finite dimensional $K$-sub vector spaces of $X$ that are stable under $\Gamma_{K'}$. Now the key point is to note we must have $X_f = X'_f$ as $\Gamma_{K'} \subset \Gamma_K$ is of finite index, hence $X'_f$ is indeed automatically $\Gamma_K$-stable.

Corollary 4.3.5. For $Y \in \text{Rep}_{\Gamma_K}(K_\infty)$, the extension $Y \otimes_{K_\infty} \overline{K}_\infty$ is an object in $\text{Rep}_{\Gamma_K}(\overline{K}_\infty)$. This induces a tensor equivalence of categories:

$$\text{Rep}_{\Gamma_K}(K_\infty) \xrightarrow{\sim} \text{Rep}_{\Gamma_K}(\overline{K}_\infty).$$

4.4. On $\Gamma_K$-cohomology.

Lemma 4.4.1. Let $M \in \text{Rep}_{\Gamma_K}(K_{m_0})$ for some $m_0 \geq 0$. If $m \gg m_0$, then for all $\gamma \in \Gamma_K$ such that $v_p(\chi_p(\gamma)-1) > m, \gamma - 1$ is continuously invertible on $(M \otimes_{K_{m_0}} \overline{K}_\infty)/(M \otimes_{K_{m_0}} K_m)$. Thus, for $i \geq 0$, we have natural isomorphisms

$$H^i(\Gamma_K, M \otimes_{K_{m_0}} K_m) \simeq H^i(\Gamma_K, M \otimes_{K_{m_0}} \overline{K}_\infty)$$

Proof. The proof is exactly the same as [LZ17, Lem. 3.10], since we already constructed Tate’s normalized trace in this setting.

Lemma 4.4.2. Let $M$ be a finite dimensional vector space over a characteristic zero field $P$, and let $G$ a finite cyclic group that acts $P$-linearly on $M$. Then

$$H^i(G, M) = 0, \forall i \geq 1.$$

Proof. This is standard, cf. e.g. [Bro82, p. 58, Example 2, Exercise 1]. Indeed, one sees that for $i \geq 1$, $H^i(G, M)$ is a $P$-vector space and is killed by the order of $G$, and hence has to be zero.

Corollary 4.4.3. Let $\hat{M} \in \text{Rep}_{\Gamma_K}(\overline{K}_\infty)$, then

1. $H^i(\Gamma_K, \hat{M})$ is a finite dimensional $K$-vector space for $i \geq 0$, and vanishes when $i \geq 2$.

2. Let $K \hookrightarrow L$ be the field embedding as in Notation 1.1.1. For each $i \geq 0$, the natural map

$$H^i(\Gamma_K, \hat{M}) \otimes_K L \to H^i(\Gamma_L, \hat{M} \otimes_{\overline{K}_\infty} \overline{L}_\infty)$$

is an isomorphism.
Proof. Consider Item (1). By Cor. 4.3.5, there exists some \( M \in \text{Rep}_{\Gamma_K}(K_{\infty}) \) and hence some \( M_m \in \text{Rep}_{\Gamma_K}(K_m) \) whose base change to \( \hat{K}_{\infty} \) is \( \hat{M} \). By Lem. 4.4.1, \( H^i(\Gamma_K, \hat{M}) = H^i(\Gamma_K, M_{m_0}) \) and hence is of finite \( K \)-dimension; they vanish for \( i \geq 2 \) since \( \Gamma_K \) has cohomological dimension 1.

Consider Item (2). By Lem. 4.4.1, it suffices to show for \( m \gg m_0 \), the natural map

\[
H^i(\Gamma_K, M_m) \otimes_K L \to H^i(\Gamma_L, M_m \otimes_K L)
\]

is an isomorphism. When \( i = 1 \), \( H^1(\Gamma_K, M_m) \otimes_K L \) equals to \( H^1(\Gamma_L, M_m \otimes_K L) \) by Hochschild-Serre spectral sequence together with Lem. 4.4.2, which further equals to \( H^1(\Gamma_L, M_m \otimes_K L) \) by flat base change.

It now suffices to treat the case \( i = 0 \). First, for any \( s \geq 0 \), by Galois descent, we have

\[
H^0(\Gamma_K, M_m) \otimes_K K_s \simeq H^0(\Gamma_K, M_m), \forall m \geq s.
\]

Now, fix some \( s \gg 0 \) so that the map \( \Gamma_L \hookrightarrow \Gamma_K \) is an isomorphism. To prove Eqn. (4.4.1) is an isomorphism, it suffices to show

\[
H^0(\Gamma_K, M_m) \otimes_K K_s L \otimes L L_s \simeq H^0(\Gamma_K, M_m \otimes_K L_s) \otimes L L_s, \forall m \gg s.
\]

Using (4.4.2), it is the same to show

\[
H^0(\Gamma_K, M_m) \otimes_K K_s L_s \simeq H^0(\Gamma_K, M_m \otimes_K L_s),
\]

which is

\[
H^0(\Gamma_K, M_m) \otimes_K K_s L \simeq H^0(\Gamma_K, M_m \otimes_K L).
\]

But the above holds by flat base change. \( \square \)

5. \( p \)-adic Simpson correspondence and Hodge-Tate rigidity

In this section, we construct the \( p \)-adic Simpson correspondence together with its decompleted version, and use it to prove a Hodge-Tate rigidity theorem. The following diagram summarizes the (base) fields and Galois groups that we use in this section.

\[
\begin{array}{ccc}
\hat{K}_{\infty} & \to & K_{\infty}^{pf} \\
\uparrow & & \uparrow \\
K_{\infty} & \to & K_{\infty}^{geom} \\
\Gamma_K & \to & G_{K_{\infty}} \\
\end{array}
\]

5.1. \( p \)-adic Simpson correspondence in the imperfect residue field case.

Definition 5.1.1. Let \( \text{Higgs}_{\Gamma_K}(\hat{K}_{\infty}) \) be the category where an object is some \( M \in \text{Rep}_{\Gamma_K}(\hat{K}_{\infty}) \) equipped with a \( \Gamma_K \)-equivariant Higgs field:

\[
\theta_M : M \to M \otimes_K \hat{\Omega}_K(-1).
\]

Theorem 5.1.2. For \( W \in \text{Rep}_{G_K}(C) \), define

\[
\mathcal{H}(W) := (W \otimes_C OC)^{G_{K_{\infty}}}
\]

(1) The above rule defines a functor

\[
\mathcal{H} : \text{Rep}_{G_K}(C) \to \text{Rep}_{\Gamma_K}(\hat{K}_{\infty}).
\]

Via the equivalence in Cor. 4.3.5, we can define a functor

\[
\mathcal{H} : \text{Rep}_{G_K}(C) \to \text{Rep}_{\Gamma_K}(K_{\infty}).
\]
(2) Using the field $L$, we can similarly define functors
\[
\mathcal{H}_L : \text{Rep}_{GL}(C_L) \to \text{Rep}_{\Gamma_L}(\widehat{L}_\infty) \\
H_L : \text{Rep}_{GL}(C_L) \to \text{Rep}_{\Gamma_L}(L_\infty).
\]
Note that for $W \in \text{Rep}_{G_K}(C)$, the base change $W \otimes_C C_L$ can be regarded as an object in $\text{Rep}_{GL}(C_L)$. With these notations, we have the following canonical "base change" isomorphisms:
\[
\mathcal{H}(W) \otimes_{\widehat{K}_\infty} \widehat{L}_\infty \simeq \mathcal{H}_L(W \otimes_C C_L),
\]
\[
H(W) \otimes_{K_\infty} L_\infty \simeq H_L(W \otimes_C C_L).
\]
(3) The functor $\mathcal{H}$ can be upgraded to a functor
\[
\text{Rep}_{G_K}(C) \to \text{Higgs}_{\Gamma_K}(\widehat{K}_\infty),
\]
in the sense its composite with the forgetful functor $\text{Higgs}_{\Gamma_K}(\widehat{K}_\infty) \to \text{Rep}_{\Gamma_K}(L_\infty)$ recovers $\mathcal{H}$.
(4) The functors $\mathcal{H}$ (as well as its upgraded version) and $H$ are tensor functors and are compatible with duality.

**Remark 5.1.3.** By abuse of terminology, we call both $\mathcal{H}$ and its upgraded version the $p$-adic Simpson correspondence. (Indeed, the Higgs field on $\mathcal{H}(W)$ does not play any further role in this paper.) We call the functor $H$ the decompleted $p$-adic Simpson correspondence.

**Proof.** Item (1) will be proved in Thm. 5.1.5, where we also prove some cohomology vanishing results for later use; Items (2)-(4) are formal consequences of Item (1), and will be proved after Thm. 5.1.5.

Let us now sketch the ideas of the proof of Item (1), which mimics the strategy of [LZ17]. The main content of the proof can be summarized in the following diagram
\[
\begin{array}{ccc}
\text{Rep}_{G_K}(C) & \xrightarrow{\simeq} & \text{Rep}_{\Gamma}(\widehat{K}_\infty^{\text{pf}}) \\
& \downarrow \text{Higgs}_{\Gamma_K} & \downarrow \text{Higgs}_{\Gamma_K} \\
\text{Rep}_{\Gamma_K}(\widehat{K}_\infty) & \xrightarrow{\simeq} & \text{Rep}_{\Gamma_K}(K_\infty)
\end{array}
\]
(5.1.1)

Here, the top horizontal equivalences are due to Brinon as in Thm. 4.2.2. The vertical functors are defined by “taking unipotent part under the $\Gamma_{\text{geom}}$-action”. Note that a natural question then is if the composite functor $\text{Rep}_{G_K}(C) \to \text{Rep}_{\Gamma_K}(\widehat{K}_\infty)$ depends on the choices of $t_i$: indeed it does not, as answered in Step 2 in the proof of Thm. 5.1.5. $\square$

We start with a lemma.

**Lemma 5.1.4.** Let $W \in \text{Rep}_{G_K}(C)$.

(1) $H^i(H_L, W \otimes_C OC) = 0$ for $i \geq 1$.
(2) $H^i(G_{K_\infty}, W \otimes_C OC) = H^i(\Gamma_{\text{geom}}, (W \otimes_C OC)^{H_L})$ for $i \geq 0$.

**Proof.** For Item (1), note that $OC = C[V_1, \ldots, V_d]$ and $H_L$ acts on $V_i$ trivially, hence it suffices to show $H^i(H^L, W) = 0$. This follows from Cor. 4.1.3 since $H^L \simeq G_{K_{\text{geg}}}$ (and $K_{\text{pf}}$ has perfect residue field). Item (2) now follows by Hochschild-Serre spectral sequence. $\square$

**Theorem 5.1.5.** Let $W \in \text{Rep}_{G_K}(C)$. Then $\mathcal{H}(W) := (W \otimes_C OC)^{G_{K_\infty}}$ is an object in $\text{Rep}_{\Gamma_K}(\widehat{K}_\infty)$ and there is a $G_K$-equivariant isomorphism
\[
\mathcal{H}(W) \otimes_{\widehat{K}_\infty} OC \simeq W \otimes_C OC.
\]
In addition,
\[
H^i(G_{K_\infty}, W \otimes_C OC) = 0, \forall i \geq 1.
\]
(5.1.2)
(5.1.3)
Proof. In the proof, we omit certain “Galois equivariance” issues which will be discussed in detail in Rem. 5.1.6 (they cause no logical problem, but could be confusing). In Step 1, we first construct a $\widehat{K}_\infty$-vector space $H$ (of correct dimension). In Step 2, we show it is isomorphic to the desired $H(W)$ as a vector space; note that $H(W)$ is automatically equipped with a $\Gamma_K$-action by definition.

**Step 1:** Let $$M \in \text{Rep}_r(\widehat{K}_\infty^{pf}), \quad \text{resp.} \quad M \in \text{Rep}_r(K_\infty^{pf})$$ be the objects corresponding to $W$ via Prop. 4.2.2. Recall $K_\infty^{pf} = K(1_{\widehat{\mathbb{Q}}}^1, t_1^{\frac{1}{p^m}}, \ldots, t_d^{\frac{1}{p^m}})$. Since $\Gamma$ is finitely generated, there exists some $m \gg 0$ such that there exists some $M_m \in \text{Rep}_r(K(1_{\widehat{\mathbb{Q}}}^1, t_1^{\frac{1}{p^m}}, \ldots, t_d^{\frac{1}{p^m}}))$ such that its base change to $K_\infty^{pf}$ is $M$.

By Prop. 4.2.4(3), the $K$-linear $\Gamma_{\text{geom}}$-action on $M_m$ is quasi-unipotent. Hence by exactly the same argument as below [LZ17, Lem. 2.15], we have a decomposition

$$M_m = \bigoplus_\tau M_{m,\tau} \tag{5.1.4}$$

where $\tau$ runs through characters of $\Gamma_{\text{geom}}$ of finite order, and

$$M_{m,\tau} = \{ m \in M_m \mid \exists N \gg 0, (\gamma - \tau(\gamma))^N m = 0, \forall \gamma \in \Gamma_{\text{geom}} \}$$

is the corresponding generalized eigenspace. Now exactly the same argument as in ibid. shows that if we let $H_m := M_{m,1}$ be the unipotent part (where 1 denotes the trivial character), then $H_m$ is a finite dimensional $K_m$-vector space stable under $\Gamma$-action such that there is a $\Gamma$-equivariant isomorphism

$$H_m \otimes_{K_m} K(1_{\widehat{\mathbb{Q}}}^1, t_1^{\frac{1}{p^m}}, \ldots, t_d^{\frac{1}{p^m}}) \simeq M_m. \tag{5.1.5}$$

Define

$$H := H_m \otimes_{K_m} \widehat{K}_\infty. \tag{5.1.6}$$

By construction, there is a $\Gamma$-equivariant isomorphism

$$H \otimes_{\widehat{K}_\infty} \widehat{K}_\infty^{pf} \simeq M, \tag{5.1.7}$$

and hence there is a $G_K$-equivariant isomorphism

$$H \otimes_{\widehat{K}_\infty} C \simeq W. \tag{5.1.8}$$

Note that so far, we only consider the $\Gamma$-action on $H$, which is enough for our purpose; cf. Rem. 5.1.6 for other subtle issues.

**Step 2.** In order to finish the proof, it suffices to prove there are $\widehat{K}_\infty$-linear isomorphisms

$$H^i(G_{K_{\infty}}, W \otimes_C OC) \simeq \begin{cases} H & i = 0 \\ 0 & i > 0, \end{cases} \tag{5.1.9}$$

The proof of these isomorphisms follow the same path as [LZ17, Lem. 2.9]. (Note as mentioned in the end of Step 1, we now only consider $\Gamma$-action on $H$, hence we cannot talk about Galois equivariances in the above isomorphism.)

First note by Lem. 5.1.4,

$$H^i(G_{K_{\infty}}, W \otimes_C OC) \simeq H^i(\Gamma_{\text{geom}}, M \otimes_{\widehat{K}_\infty^{pf}} \widehat{K}_\infty^{pf}[V_1, \ldots, V_d]), \quad \forall i \geq 0. \tag{5.1.10}$$

We claim the $\Gamma$-equivariant hence $\Gamma_{\text{geom}}$-equivariant inclusion $H \hookrightarrow M$ induces an isomorphism

$$H^i(\Gamma_{\text{geom}}, H \otimes_{\widehat{K}_\infty^{pf}} \widehat{K}_\infty^{pf}[V_1, \ldots, V_d]) \simeq H^i(\Gamma_{\text{geom}}, M \otimes_{\widehat{K}_\infty^{pf}} \widehat{K}_\infty^{pf}[V_1, \ldots, V_d]), \quad \forall i \geq 0. \tag{5.1.11}$$

The claim follows from similar argument in the paragraph below [Sch13, Lem. 6.17]. Indeed, the increasing $\mathbb{Z}^{\geq 0}$-filtration on $\widehat{K}_\infty^{pf}[V_1, \ldots, V_d]$ (resp. on $\widehat{K}_\infty^{pf}[V_1, \ldots, V_d]$) defined by polynomial degree is $\Gamma_{\text{geom}}$-stable, and hence to prove (5.1.11), it suffices
to consider the graded pieces with respect to above filtrations. In both gradeds, the action of $\Gamma_{\text{geom}}$ on the $V_i$'s is trivial, and hence it reduces to show that
\begin{equation}
H^i(\Gamma_{\text{geom}}, \mathcal{H}) \simeq H^i(\Gamma_{\text{geom}}, \mathcal{M}), \quad \forall i \geq 0.
\end{equation}
Note (5.1.4) induces a $\Gamma_{\text{geom}}$-equivariant decomposition
$$\mathcal{M} = \mathcal{H} \oplus \bigoplus_{\tau \neq 1} \mathcal{M}_\tau.$$ 
Note for each $\tau \neq 1$, there exists some $i$ such that $\gamma_i - 1$ acts on $\mathcal{M}_\tau$ bijectively (by considering the eigenvalues); here $\gamma_i$ is a topological generator of a copy of $\mathbb{Z}_p(1)$ in $\Gamma_{\text{geom}} \simeq \mathbb{Z}_p(1)^d$. This gives (5.1.9) and hence (5.1.8). Finally, using the projection map (modulo the $V_i$'s)
$$\mathcal{H} \otimes_{\widehat{K}_\infty} \widehat{K}_\infty[V_1, \cdots, V_d] \to \mathcal{H},$$
and inductively applying $[\text{LZ17, Lem. 2.10}]$, we have
\begin{equation}
H^i(\Gamma_{\text{geom}}, \mathcal{H} \otimes_{\widehat{K}_\infty} \widehat{K}_\infty[V_1, \cdots, V_d]) \simeq \begin{cases} \mathcal{H} & i = 0 \\ 0 & i > 0. \end{cases}
\end{equation}
This in particular shows there is a $\Gamma$-equivariant isomorphism
\begin{equation}
\mathcal{H}(W) \otimes_{\widehat{K}_\infty} \widehat{K}_\infty[V_1, \cdots, V_d] \simeq \mathcal{H} \otimes_{\widehat{K}_\infty} \widehat{K}_\infty[V_1, \cdots, V_d].
\end{equation}
Note that $\Gamma_{\text{geom}}$-action is trivial on $\mathcal{H}(W)$ but not on $\mathcal{H}$. Via the surjection $G_K \to \Gamma$ and using Eqn. (5.1.6), we have $G_K$-equivariant isomorphisms
$$\mathcal{H}(W) \otimes_{\widehat{K}_\infty} \mathcal{O}C \simeq \mathcal{H} \otimes_{\widehat{K}_\infty} \mathcal{O}C \simeq W \otimes_\mathcal{O}C \mathcal{O}C.$$ 

\begin{remark} \textbf{5.1.6.} \textbf{We discuss and clarify some Galois equivariance issues in the proof of Thm. 5.1.5.} In Step 1 of the proof above, we constructed a $\widehat{K}_\infty$-vector space $\mathcal{H}$, which is a $\Gamma$-stable subset of $\mathcal{M}$. However, the $\Gamma_{\text{geom}}$-action on $\mathcal{H}$ is only unipotent but not trivial, hence we cannot direct talk about the “quotient action” of $\Gamma_K$ on $\mathcal{H}$. Nonetheless, $\Gamma_{K(p)}$ is a subgroup of $\Gamma_K$, and hence $\Gamma_{K(p)}$ acts on $\mathcal{H}$. Thus via $\Gamma_{K(p)} \simeq \Gamma_K$, we can still define a $\Gamma_K$-action on $\mathcal{H}$. 
In Step 2 of the proof above, we defined $\mathcal{H}(W)$ as $H^0(G_{K_{\infty}}, W \otimes_\mathcal{O}C \mathcal{O}C)$ which is naturally equipped with a $\Gamma_K$-action. Now we claim the isomorphism
$$\mathcal{H}(W) \simeq \mathcal{H}$$
in Eqn. (5.1.7) is indeed $\Gamma_K$-equivariant, using the $\Gamma_K$-action on $\mathcal{H}$ defined in last paragraph. Indeed, on the left hand side of (5.1.11), $\Gamma_{\text{geom}}$-action on $\mathcal{H}(W)$ is trivial, hence $\Gamma_{K(p)}$-action on $\mathcal{H}(W)$ is precisely the same as the \textit{quotient action} of $\Gamma_K$. Now it suffices to note that the projection map
$$\mathcal{H} \otimes_{\widehat{K}_\infty} \widehat{K}_\infty[V_1, \cdots, V_d] \to \mathcal{H},$$
is $\Gamma_{K(p)}$-equivariant: this is because for $g \in \Gamma_{K(p)}$, $g(V_i) = \chi_p(g)^{-1}V_i$. (Note this projection map is not $\Gamma_{\text{geom}}$-equivariant hence not $\Gamma$-equivariant).
\end{remark}

\begin{proof} \textbf{of Thm. 5.1.2(2-4).} We now prove remaining parts of Thm. 5.1.2.

Item (2): by Thm. 5.1.5, there is a $G_K$-equivariant isomorphism
$$W \otimes_\mathcal{O}C \mathcal{O}C \simeq \mathcal{H}(W) \otimes_{\widehat{K}_\infty} \mathcal{O}C \mathcal{O}C.$$ 
Hence we can obtain a $G_L$-equivariant isomorphism
$$W \otimes_\mathcal{O}C \mathcal{O}C \otimes_\mathcal{O}C \mathcal{O}C_L \simeq \mathcal{H}(W) \otimes_{\widehat{K}_\infty} \mathcal{O}C \otimes_\mathcal{O}C \mathcal{O}C_L,$$
which, after re-organization becomes
$$(W \otimes_\mathcal{O}C C_L) \otimes_\mathcal{O}C_L \mathcal{O}C_L \simeq (\mathcal{H}(W) \otimes_{\widehat{K}_\infty} \mathcal{L}_\infty) \otimes_{\mathcal{L}_\infty} \mathcal{O}C_L.$$ 
Taking $G_{L,\infty}$-invariant on both sides gives
$$\mathcal{H}_L(W \otimes_\mathcal{O}C C_L) \simeq \mathcal{H}(W) \otimes_{\widehat{K}_\infty} \mathcal{L}_\infty.$$ 
The decompleted base change isomorphism then follows.
\end{proof}
Item (3): The Higgs field on $\mathcal{H}(W) = (W \otimes_C \mathcal{O}C)^{G_K}$ is naturally induced from $\text{gr}^0 \nabla$ on $\mathcal{O}C$ defined in Notation 2.3.4, since $\text{gr}^0 \nabla$ commutes with $G_K$-actions.

Item (4): it suffices to prove that $\mathcal{H}$ is compatible with tensor products and duality, the upgraded version and the $H$-version immediately follow. Let $W_1, W_2 \in \text{Rep}_{G_K}(C)$, then it is easy to check that

$$(\mathcal{H}(W_1) \otimes_{\widehat{K}^p} \mathcal{H}(W_2)) \otimes_{\widehat{K}^p} \mathcal{O}C \simeq (W_1 \otimes_C W_2) \otimes_C \mathcal{O}C.$$ 

Take $G_{K^p}$-invariants on both sides gives

$$(5.1.12) \quad \mathcal{H}(W_1) \otimes_{\widehat{K}^p} \mathcal{H}(W_2) \simeq \mathcal{H}(W_1 \otimes_C W_2).$$

We now consider duality. Apply (5.1.12) to $W_1 = W, W_2 = W^\vee$, then we can obtain a homomorphism of finite dimensional $\widehat{K}$-vector spaces

$$(\mathcal{H}(W^\vee) \to \mathcal{H}(W)^\vee.$$ 

It is an isomorphism because its base change over (the field) $\widehat{K}^p$ is so, since the base change, via Item (2), recovers duality morphism in Sen theory for the field $K^p$ (with perfect residue field).

5.2. Hodge-Tate rigidity.

**Definition 5.2.1.** For $W \in \text{Rep}_{G_K}(C)$, define

$$D_{HT,K}(W) := (W \otimes_C \mathcal{O}B_{HT})^{G_K}.$$ 

**Proposition 5.2.2.** There is a base change isomorphism

$$D_{HT,K}(W) \otimes_K L \simeq D_{HT,L}(W \otimes_C C_L).$$

**Proof.** It is clear $D_{HT}(W) \simeq \bigoplus_{j \in \mathbb{Z}} (\mathcal{H}(W(j)))^{\Gamma_K}$, where $W(j)$ is the Tate twist. It suffices to note that for each $j$, $(\mathcal{H}(W(j)))^{\widehat{K}} \otimes_K L \simeq (\mathcal{H}_L(W(j)))^{\Gamma_K}$ by Cor. 4.4.3. □

**Corollary 5.2.3.** For $W \in \text{Rep}_{G_K}(C)$, $D_{HT,K}(W)$ is a $K$-vector space such that

$$\dim_K D_{HT,K}(W) \leq \dim_C W.$$ 

**Proof.** Apply above proposition to the embedding $K \hookrightarrow K^p$, then it suffices to treat the case $\widehat{K} = K^p$, which is well-known, cf. [Fon04, §2.7]. □

**Definition 5.2.4.** Say $W \in \text{Rep}_{G_K}(C)$ a Hodge-Tate $C$-representation (or simply, $W$ is Hodge-Tate) if

$$\dim_K D_{HT,K}(W) = \dim_C W.$$ 

**Theorem 5.2.5.** (1) Let $W \in \text{Rep}_{G_K}(C)$, then $W$ is Hodge-Tate if and only the $G_L$-representation $C_L \otimes_C (W|_{G_L})$ is so.

(2) Let $V \in \text{Rep}_{G_K}(\mathbb{Q}_p)$, then $V$ is Hodge-Tate if and only $V|_{G_L}$ is Hodge-Tate.

**Proof.** Apply Prop. 5.2.2. □

We also discuss some results related with “Sen operators” (which is not covered in Brinon’s Sen theory [Bri03]).

**Theorem 5.2.6.** (1) Let $H \in \text{Rep}_{\Gamma_K}(\widehat{K}_\infty)$, then the $\Gamma_K$-action on $H$ is locally analytic. As a consequence, one can define a $K_\infty$-linear endomorphism $\varphi_H : H \to H$, called the Sen operator.

(2) Let $W \in \text{Rep}_{G_K}(C)$, then $W$ is Hodge-Tate if and only if the Sen operator on $H(W)$ (defined via Thm. 5.1.2) is semi-simple with integer eigenvalues.

**Proof.** For Item (1), the base change $H \otimes_{K_\infty} K_{\infty}^p$ is an object in $\text{Rep}_{\Gamma_{K^p}}(K_{\infty}^p)$, with the $\Gamma_K$-action locally analytic by Sen theory (in the perfect residue field case). Note that $\Gamma_K$-action on $H$ is also locally analytic. Item (2) then follows immediately (from the case for $K^p$). □
6. Sen-Fontaine theory

In this section, we first review the Sen-Fontaine theory in the classical perfect residue field case. We then carry out a Tate-Sen-Fontaine decompletion in the imperfect residue field case. Finally, we also summarize Andreatta-Brinon’s “Sen-Fontaine theory” in the imperfect residue field case; it serves for comparison purposes (with our Riemann-Hilbert correspondence) only, and will not be used in our paper.

6.1. Sen-Fontaine theory in the perfect residue field case. In continuation of §4.1, we recall Sen-Fontaine theory for the CDVF $\mathbb{K}$ with perfect residue field.

Notation 6.1.1. Let

$$\mathbb{L}_{\text{dR}} = (\mathbb{B}_{\text{dR}})^{G_{\mathbb{K}}} \quad \text{and} \quad \mathbb{L}_{\text{dR}}^+ = (\mathbb{B}_{\text{dR}}^+)^{G_{\mathbb{K}}}.$$

The filtration on $\mathbb{B}_{\text{dR}}$ induces a filtration on $\mathbb{L}_{\text{dR}}$. Note $\text{Fil}^0 \mathbb{L}_{\text{dR}} = \mathbb{L}_{\text{dR}}^+$.

Proposition 6.1.2. [Fon04, Thm. 3.6] The embeddings $\mathbb{K}_\infty[[t]] \hookrightarrow \mathbb{L}_{\text{dR}}^+ \hookrightarrow \mathbb{B}_{\text{dR}}^+$ induce tensor equivalences of categories

$$\text{Rep}_{\mathbb{K}}(\mathbb{K}_\infty[[t]]) \rightarrow \text{Rep}_{\mathbb{K}}(\mathbb{L}_{\text{dR}}^+) \rightarrow \text{Rep}_{G_{\mathbb{K}}}(\mathbb{B}_{\text{dR}}^+),$$

as well as

$$\text{Rep}_{\mathbb{K}}(\mathbb{K}_\infty((t))) \rightarrow \text{Rep}_{\mathbb{K}}(\mathbb{L}_{\text{dR}}) \rightarrow \text{Rep}_{G_{\mathbb{K}}}(\mathbb{B}_{\text{dR}}).$$

Notation 6.1.3. Define the following functors:

$$\mathcal{R}H_{\mathbb{K}}^+: \text{Rep}_{G_{\mathbb{K}}}(\mathbb{B}_{\text{dR}}^+) \simeq \text{Rep}_{G_{\mathbb{K}}}(\mathbb{L}_{\text{dR}}^+),$$

$$\mathcal{R}H_{\mathbb{K}}: \text{Rep}_{G_{\mathbb{K}}}(\mathbb{B}_{\text{dR}}) \simeq \text{Rep}_{G_{\mathbb{K}}}(\mathbb{L}_{\text{dR}}),$$

$$\text{RH}_{\mathbb{K}}^+: \text{Rep}_{G_{\mathbb{K}}}(\mathbb{B}_{\text{dR}}^+) \simeq \text{Rep}_{G_{\mathbb{K}}}(\mathbb{K}_\infty[[t]]),$$

$$\text{RH}_{\mathbb{K}}: \text{Rep}_{G_{\mathbb{K}}}(\mathbb{B}_{\text{dR}}) \simeq \text{Rep}_{G_{\mathbb{K}}}(\mathbb{K}_\infty((t))).$$

Corollary 6.1.4. Let $W \in \text{Rep}_{G_{\mathbb{K}}}(\mathbb{B}_{\text{dR}}^+)$, then we have canonical isomorphisms

$$\mathcal{R}H_{\mathbb{K}}^+(W)/t \simeq \mathcal{H}_{\mathbb{K}}(W/tW),$$

$$\text{RH}_{\mathbb{K}}^+(W)/t \simeq \mathcal{H}_{\mathbb{K}}(W/tW).$$

Remark 6.1.5. (1) Let $V \in \text{Rep}_{G_{\mathbb{K}}}(\mathbb{Q}_p)$ and let $W = V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}^+ \in \text{Rep}_{G_{\mathbb{K}}}(\mathbb{B}_{\text{dR}}^+)$. Then $\text{RH}_{\mathbb{K}}^+(W)/t \simeq \mathcal{H}_{\mathbb{K}}(W/tW)$ is precisely the usually written isomorphism “$D_{\text{dR}}^+(V)/t \simeq D_{\text{Sen}}(V)$” in the literature, cf. e.g., [Ber02, §5.3].

(2) We can summarize Sen theory and Sen-Fontaine theory in the perfect residue field case as follows.

$$\begin{array}{ccc}
\text{Rep}_{G_{\mathbb{K}}}(\mathbb{C}) & \overset{\sim}{\rightarrow} & \text{Rep}_{G_{\mathbb{K}}}(\overline{\mathbb{K}}_{\infty}) \\
\text{mod } t & & \text{mod } t \\
\text{Rep}_{G_{\mathbb{K}}}(\mathbb{Q}_p) & \overset{\sim}{\rightarrow} & \text{Rep}_{G_{\mathbb{K}}}(\mathbb{B}_{\text{dR}}^+) \\
\text{mod } t & & \text{mod } t \\
\text{Rep}_{G_{\mathbb{K}}}(\mathbb{B}_{\text{dR}}^+)) & \overset{\sim}{\rightarrow} & \text{Rep}_{G_{\mathbb{K}}}(\mathbb{L}_{\text{dR}}^+) \\
\text{mod } t & & \text{mod } t \\
& & \text{Rep}_{G_{\mathbb{K}}}(\mathbb{K}_\infty[[t]])
\end{array}$$

6.2. Tate-Sen-Fontaine decompletion in the imperfect residue field case. In continuation of §4.3, we develop some Tate-Sen-Fontaine decompletion results in the imperfect residue field case.

Lemma 6.2.1. Let $U \in \text{Rep}_{G_{\mathbb{K}}}(\mathbb{B}_{\text{dR}}^+)$. Suppose $-\infty \leq a \leq b \leq +\infty$.

(1) $H^i(\mathbb{H}^e; U \otimes_{\mathbb{B}_{\text{dR}}} \mathcal{O}_{\mathbb{B}_{\text{dR}}^{a,b}}) = 0$ for $i \geq 1$.

(2) $H^i(G_{\mathbb{K}_{\infty}}; U \otimes_{\mathbb{B}_{\text{dR}}} \mathcal{O}_{\mathbb{B}_{\text{dR}}^{a,b}}) = 0$ for $i \geq 1$.

Proof. Consider Item (1), the case $a = b$ follows from Lem. 5.1.4. The case where $b, a \neq \pm\infty$ then follows from induction on the length $b-a$. The case $a = -\infty$ follows since $\mathbb{H}^e$ acts on $t$ trivially. The case $b = +\infty$ follows from [Sch13, Lem. 3.18] (note:
it is important that the filtration on $\mathcal{O}B_{\text{dr}}$ is complete in order to apply loc. cit., cf. Rem. 2.1.5.

The argument for Item (2) is similar, where the $a = b$ case follows from Thm. 5.1.5.

**Definition 6.2.2.** Let $\mathcal{O}L_{\text{dr}} := (\mathcal{O}B_{\text{dr}})^{G_{K_\infty}}$ and induce a filtration from that on $\mathcal{O}B_{\text{dr}}$.

**Remark 6.2.3.** (1) We do not define $\mathcal{O}L_{\text{dr}}^+$ as we do not define $\mathcal{O}B_{\text{dr}}^+$, cf. Rem. 2.1.7.

(2) By [Bri06, Lem. 2.31], there is an inclusion

$$K \otimes_{K^{\tau=0}} B_{\text{dr}}^{G_{K_\infty}=1} \subset \mathcal{O}L_{\text{dr}}.$$  

We do not know if this is an equality.

**Lemma 6.2.4.** For $i \in \mathbb{Z}$, we have $\text{gr}^i(\mathcal{O}L_{\text{dr}}) \simeq \widehat{K}_{\infty}(i)$ as $\Gamma_K$-representations.

**Proof.** Consider the short exact sequence

$$0 \to \mathcal{O}B_{\text{dr}}^{[-\infty,i]} \to \mathcal{O}B_{\text{dr}}^{[-\infty,i+1]} \to \text{gr}^i \mathcal{O}B_{\text{dr}} \to 0.$$  

By Lem. 6.2.1, it is still short exact after taking $G_{K_\infty}$-invariants. Now note $\text{gr}^i \mathcal{O}B_{\text{dr}} = \mathcal{O}C(i)$, and by Thm. 5.1.2, $(\mathcal{O}C(i))^{G_{K_\infty}} = \mathcal{H}(C(i)) = \widehat{K}_{\infty}(i).$ □

**Proposition 6.2.5.** Let $X \in \text{Rep}_{\Gamma_K}(\text{Fil}^0 \mathcal{O}L_{\text{dr}})$, define

$$X_f = \lim_{s \geq 1}(X/t^sX)_f,$$

where $(X/t^sX)_f$ denotes the union of $K$-sub-vector spaces of $X/t^sX$ stable under $\Gamma_K$-action. Then

(1) $X_f$ is also the union of $K_\infty[[t]]$-sub-modules of $X$ stable under $\Gamma_K$-action; furthermore, $X_f$ is an object in $\text{Rep}_{\Gamma_K}(K_\infty[[t]])$.

(2) In addition, the natural map

$$X_f \otimes_{K_\infty[[t]]} \text{Fil}^0 \mathcal{O}L_{\text{dr}} \to X$$

is an isomorphism.

**Proof.** If $K$ has perfect residue field, then this is [Fon04, Thm. 3.6]; the general case follows similar argument. Here, we briefly sketch the argument following the clear exposition of [Shi18, Prop. 2.11]. Indeed, it suffices to show:

- for each $s \geq 1$, the finite length representation $V = X/t^sX$ satisfies that $V_f \in \text{Rep}_{\Gamma_K}(K_\infty[t]/t^s)$ and $V_f \otimes_{K_\infty[t]/t^s} \text{Fil}^0 \mathcal{O}L_{\text{dr}}/t^s \simeq V$.

Note that when $s = 1$, then this is precisely Prop. 4.3.4, because $\text{Fil}^0 \mathcal{O}L_{\text{dr}}/(t) \simeq \widehat{K}_{\infty}$ by Lem. 6.2.4. Suppose the above claim is true for $s \leq n - 1$, and now consider $V = X/t^nX$. Let $V' = t^{n-1}V$ and $V'' = V/V'$, and consider the short exact sequence

$$0 \to V' \to V \to V'' \to 0.$$  

Pick any $K_\infty$-basis $\tilde{v}_1, \cdots, \tilde{v}_r$ of $V''$ (which is automatically a $\widehat{K}_{\infty}$-basis of $V''$ by induction hypothesis), and lift it to elements $v_1, \cdots, v_r \in V$, which by Nakayama Lemma, is a $\text{Fil}^0 \mathcal{O}L_{\text{dr}}/t^n$-basis of $V$. We want to modify $v_i$ so that they fall into $V_f$. Indeed, given any $\gamma \in \Gamma_K$, let $T$ be the matrix of its action with respect to the basis $v_1, \cdots, v_r$. As $\tilde{v}_1, \cdots, \tilde{v}_r$ is a basis of $V''$, the matrix $T$ is of the form

$$T = T^0 + t^{n-1}T^1, \quad \text{where } T^0 \in \text{Mat}(K_\infty[t]/t^n), \quad T^1 \in \text{Mat}(\widehat{K}_{\infty}).$$  

Let $U := T^0 \pmod{t}$ in $\text{Mat}(K_\infty)$ which has to be invertible since it is the matrix of $\gamma$ acting on $V/tV$. Choose a specific $\gamma \neq 1$ close to 1 so that the matrix satisfies $v_\mu(U - 1) > c_3 = \frac{1}{p-1}$. Then exactly the same argument of [Shi18, Claim 2.12] –the key ingredient being the normalized Tate traces, which we established in Rem. 4.3.3– shows that after changing $v_1, \cdots, v_d$ via a matrix $1 + t^{n-1}M$ with $M \in \text{Mat}(\widehat{K}_{\infty})$, then the matrix of $\gamma$ acting on them falls inside $\text{Mat}(K_\infty[t]/t^n)$, which quickly implies that these new basis elements –which we still denote as $v_i$– are elements in $V_f$ (using
the fact \( \gamma_{\mathcal{P}} \subset \Gamma_K \) is of finite index). Finally, a standard argument shows these (new) \( v_1, \ldots, v_d \) has to be a basis of \( V_f \).

\[ \square \]

**Corollary 6.2.6.** For \( Y \) an object in \( \text{Rep}_{\Gamma_K}(K_\infty[[t]]) \), the base change \( Y \otimes_{K_\infty[[t]]} \text{Fil}^0 \mathcal{O}_{\text{dR}} \) is an object in \( \text{Rep}_{\Gamma_K}(\text{Fil}^0 \mathcal{O}_{\text{dR}}) \). This induces a tensor equivalence of categories:

\[ \text{Rep}_{\Gamma_K}(K_\infty[[t]]) \cong \text{Rep}_{\Gamma_K}(\text{Fil}^0 \mathcal{O}_{\text{dR}}). \]

Similarly there is a tensor equivalence of categories:

\[ \text{Rep}_{\Gamma_K}(K_\infty((t))) \cong \text{Rep}_{\Gamma_K}(\mathcal{O}_{\text{dR}}). \]

### 6.3. Sen-Fontaine theory of Andreatta-Brinon

By Prop. 2.1.11, and note that \( H_\mathbb{Z} \) fixes the elements \([t^i]\), we have

\[ (\mathcal{O}B_{\text{dR}}^{\text{ac}})^{H_\mathbb{Z}} = (B_{\text{dR}}^{\text{ac}})^{H_\mathbb{Z}}[[u_1, \ldots, u_d]] = (B_{\text{dR}}^{\text{ac}})^{H_\mathbb{Z}}[[tV_1, \ldots, tV_d]], \]

which then contains the ring \( K_{\infty}^{(pf)}[[t]][[tV_1, \ldots, tV_d]] \).

**Proposition 6.3.1.** There is a chain of tensor equivalences of categories

\[ \text{Rep}_{\Gamma}(K_\infty^{(pf)}[[t, tV_1, \ldots, tV_d]]) \cong \text{Rep}_{\Gamma}(B_{\text{dR}}^{\text{ac}})^{H_\mathbb{Z}}[[tV_1, \ldots, tV_d]]) \cong \text{Rep}_{\Gamma}(K_\infty^{(pf)}), \]

where all the functors are defined via base change.

**Proof.** These are proved in [AB10, Cor. 3.8, Thm. 3.23] in the (affine) relative case; similar argument renders the imperfect residue field case, as reviewed in [Mor10]. \( \square \)

**Remark 6.3.2.** Let \( M \in \text{Rep}_{\Gamma}(K_\infty^{(pf)}[[t, tV_1, \ldots, tV_d]]) \), one can define Lie algebra actions and hence introduce various differential modules, cf. [AB10, §4] and [Mor10, §3.1]. Indeed, Morita’s proof of Thm. 3.3.1 makes key use of these differential modules.

**Remark 6.3.3.** The relation of Andretta-Brinon’s theory and Brinon’s theory can be summarized in the following.

\[ \begin{array}{ccc}
\text{Rep}_{\Gamma_K}(\mathbb{Q}_p) & \xrightarrow{\theta_K} & \text{Rep}_{\Gamma}(K_{\infty}^{(pf)}) \\
\text{Rep}_{\Gamma_K}(\mathcal{O}B_{\text{dR}}^{\text{ac}}) & \xrightarrow{\theta_K} & \text{Rep}_{\Gamma}(K_{\infty}^{(pf)})
\end{array} \]

### 7. p-adic RIEMANN-HILBERT CORRESPONDENCE AND DE RHAM RIGIDITY

In this section, we construct the \( p \)-adic Riemann-Hilbert correspondence together with its decompleted version, and use it to prove a de Rham rigidity theorem.

#### 7.1. p-adic Riemann-Hilbert correspondence in the imperfect residue field case

Recall in Construction 2.3.3, we have defined a connection operator over the ring \( \mathcal{O}B_{\text{dR}} \). (In particular, our situation is simpler than that in [LZ17, §3], as we do not need to deal with sheaves.)

**Definition 7.1.1.** Let \( \text{MIC}_{\Gamma_K}(\mathcal{O}B_{\text{dR}}) \) be the category where an object is some \( M \in \text{Rep}_{\Gamma_K}(\mathcal{O}B_{\text{dR}}) \) equipped with:

- a \( \Gamma_K \)-stable \( \mathbb{Z} \)-filtration \( \text{Fil}^i M \) such that \( \text{Fil}^i M = t^i \text{Fil}^0 M, \forall i \in \mathbb{Z} \)
- a \( \Gamma_K \)-equivariant integrable connection \( \nabla : M \to M \otimes_{\mathbb{Z}} \hat{\Omega}_K \) satisfying Griffiths transversality, namely \( \nabla(\text{Fil}^i M) \subset \text{Fil}^{i+1} M \otimes_{\mathbb{Z}} \hat{\Omega}_K, \forall i \).

**Theorem 7.1.2.** For \( W \in \text{Rep}_{\Gamma_K}(\mathbb{B}_{\text{dR}}) \), define

\[ \mathcal{R}H(W) := (W \otimes_{\mathbb{B}_{\text{dR}}} \mathcal{O}B_{\text{dR}})^{G_{\infty}} \]
6.2.6

By abuse of terminology, we call both

\[ \text{Remark 7.1.3.} \]

Then we have the following canonical isomorphisms:

\[ \text{Corollary 7.1.4.} \]

It defines a tensor functor

\[ \text{Corollary 7.1.4.} \]

Via the equivalence in Cor. 6.2.6, we have another tensor functor

\[ \text{Corollary 7.1.4.} \]

Then we have the following canonical isomorphisms:

\[ \text{Remark 7.1.3.} \]

(1) The above rule defines a functor

\[ \mathcal{R} \mathcal{H} : \text{Rep}_{G_K}(B_{\text{dr}}) \to \text{Rep}_{\Gamma_K}(\mathcal{O}L_{\text{dR}}) \]

(2) These functors satisfy base change properties with respect to the embedding \( K \hookrightarrow L \), in the sense that we have canonical isomorphisms

\[ \mathcal{R} \mathcal{H}(W) \otimes_{\mathcal{O}L_{\text{dR}}} \mathcal{O}L_{\text{dR}, L} \simeq \mathcal{R} \mathcal{H}_L(W|_{G_L} \otimes_{B_{\text{dR}}} B_{\text{dR}, L}), \]

\[ \mathcal{R} \mathcal{H}(W) \otimes_{K_{\infty}((t))} L_{\infty}((t)) \simeq \mathcal{R} \mathcal{H}_L(W|_{G_L} \otimes_{B_{\text{dR}}} B_{\text{dR}, L}), \]

where \( W|_{G_L} \otimes_{B_{\text{dR}}} B_{\text{dR}, L} \) is regarded as an object in \( \text{Rep}_{G_L}(B_{\text{dR}, L}) \).

(3) The functor \( \mathcal{R} \mathcal{H} \) can be upgraded to a functor

\[ \text{Corollary 7.1.4.} \]

(4) The functors \( \mathcal{R} \mathcal{H} \) (as well as its upgraded version) and \( \mathcal{R} \mathcal{H} \) are tensor functors and are compatible with duality.

**Remark 7.1.3.** By abuse of terminology, we call both \( \mathcal{R} \mathcal{H} \) and its upgraded version the \( p \)-adic Riemann-Hilbert correspondence. (Indeed, the connection on \( \mathcal{R} \mathcal{H}(W) \) does not play any further role in this paper.) We call the functor \( \mathcal{R} \mathcal{H} \) the decompleted \( p \)-adic Riemann-Hilbert correspondence.

**Proof.** Since \( G_K \) is compact, one can easily show any \( W \in \text{Rep}_{G_K}(B_{\text{dR}}) \) admits a \( G_K \)-stable \( B_{\text{dR}} \)-lattice, cf. the discussion above [Fon04, Thm. 3.9]. Hence for Item (1), it suffices to show for any \( U \in \text{Rep}_{G_K}(B^+_\text{dR}) \) and for all \(-\infty \leq a \leq b \leq +\infty\), the space

\[ \mathcal{R} \mathcal{H}^{[a,b]}(U) := H^0(G_{K_{\infty}}, U \otimes B^+_\text{dR} \mathcal{O}B_{\text{dR}}^{[a,b]}) \]

is a finite free module over \( \mathcal{O}L_{\text{dR}}^{[a,b]} := (\mathcal{O}B_{\text{dR}}^{[a,b]})^{G_{K_{\infty}}} \), and the natural map

\[ \mathcal{R} \mathcal{H}^{[a,b]}(U) \otimes_{\mathcal{O}L_{\text{dR}}^{[a,b]}} \mathcal{O}B_{\text{dR}}^{[a,b]} \to U \otimes B^+_\text{dR} \mathcal{O}B_{\text{dR}}^{[a,b]} \]

is a \( G_K \)-equivariant isomorphism. The case when \( a = b \) is precisely the (twisted) Simpson correspondence in Thm. 5.1.2. The general case then follows by standard d\évissage argument, by making use of the cohomology vanishing result of Lem. 6.2.1(2). Note that again, just as in Lem. 6.2.1, it is crucial we use the \( t \)-adically complete ring \( \mathcal{O}B_{\text{dR}} \).

Items (2)-(4) are formal consequences of the above construction, and follows similar argument as in Thm. 5.1.2.

**Corollary 7.1.4.** For \( U \in \text{Rep}_{G_K}(B^+_\text{dR}) \), denote

\[ \mathcal{R} \mathcal{H}^+(U) := \mathcal{R} \mathcal{H}^{[0, +\infty]}(U) = (U \otimes B^+_\text{dR} \text{Fil}^0 \mathcal{O}B_{\text{dR}})^{G_{K_{\infty}}}. \]

It defines a tensor functor

\[ \mathcal{R} \mathcal{H}^+ : \text{Rep}_{G_K}(B^+_\text{dR}) \to \text{Rep}_{\Gamma_K}(\text{Fil}^0 \mathcal{O}L_{\text{dR}}). \]

Via equivalence in Cor. 6.2.6, we have another tensor functor

\[ \mathcal{R} \mathcal{H}^+ : \text{Rep}_{G_K}(B^+_\text{dR}) \to \text{Rep}_{\Gamma_K}(K_{\infty}[t]). \]

Then we have the following canonical isomorphisms:

\[ \mathcal{R} \mathcal{H}^+(U)/t \simeq \mathcal{H}(U \otimes B^+_\text{dR} C), \]

\[ \mathcal{R} \mathcal{H}^+(U)/t \simeq \mathcal{H}(U \otimes B^+_\text{dR} C). \]
Remark 7.1.5. The relations of the Simpson and Riemann-Hilbert correspondences can be summarized in the following diagram.

\[
\begin{array}{ccc}
\text{Rep}_{G_K}(C) & \longrightarrow & \text{Rep}_{G_K}(K_\infty) \\
\text{mod } t & & \text{mod } t \\
\text{Rep}_{G_K}(\mathbb{B}_{\text{dr}}^+) & \longrightarrow & \text{Rep}_{G_K}(\text{Fil}^0 \mathcal{O}_{\text{dR}}) \xrightarrow{\simeq} \text{Rep}_{G_K}(K_\infty[[t]]) \\
\end{array}
\]

7.2. de Rham rigidity.

Definition 7.2.1. For \( W \in \text{Rep}_{G_K}(\mathbb{B}_{\text{dr}}) \), define
\[
D_{\text{dR},K}(W) := (W \otimes_{\mathbb{B}_{\text{dr}}} \mathcal{O}_{\text{dR}})^{G_K}.
\]

Proposition 7.2.2. Let \( W \in \text{Rep}_{G_K}(\mathbb{B}_{\text{dr}}) \).

1. For each \( i \geq 0 \), the cohomology space \( H^i(\Gamma_K, \mathcal{R} \mathcal{H}(W)) \) is a finite dimensional \( K \)-vector space, and vanishes when \( i \geq 2 \).

2. Furthermore, the natural map
\[
H^i(\Gamma_K, \mathcal{R} \mathcal{H}(W)) \otimes_K L \rightarrow H^i(\Gamma_L, \mathcal{R} \mathcal{H}_L(W|_{G_L} \otimes_{\mathbb{B}_{\text{dr}}} \mathbb{B}_{\text{dr},L}))
\]

is an isomorphism. Hence in particular
\[
D_{\text{dR},K}(W) \otimes_K L \cong D_{\text{dR},L}(W|_{G_L} \otimes_{\mathbb{B}_{\text{dr}}} \mathbb{B}_{\text{dr},L})
\]

Proof. Let \( U \in \text{Rep}_{G_K}(\mathbb{B}_{\text{dr}}^+) \) be a \( G_K \)-stable lattice of \( W \), and let \( V = U/tU \in \text{Rep}_{G_K}(C) \). Let \( V(j) \) denote the Tate twists. By standard dévissage, it suffices to prove for any \( j \in \mathbb{Z} \) we have:

- \( H^i(\Gamma_K, \mathcal{H}(V(j))) \) is a finite dimensional \( K \)-vector space, and vanishes when \( i \geq 2 \).
- Furthermore, the natural map
\[
H^i(\Gamma_K, \mathcal{H}(V(j))) \otimes_K L \rightarrow H^i(\Gamma_L, \mathcal{H}_L(V(j)|_{G_L}) \otimes_C C_L)
\]

is an isomorphism.

These follow from Cor. 4.4.3.

Corollary 7.2.3. For \( W \in \text{Rep}_{G_K}(\mathbb{B}_{\text{dr}}) \), \( D_{\text{dR},K}(W) \) is a \( K \)-vector space such that
\[
\dim_K D_{\text{dR},K}(W) \leq \dim_{\mathbb{B}_{\text{dr}}} W.
\]

Proof. Argue similarly as Cor. 5.2.3, using Prop. 7.2.2.

Definition 7.2.4. Say \( W \in \text{Rep}_{G_K}(\mathbb{B}_{\text{dr}}) \) is de Rham if
\[
\dim_K D_{\text{dR},K}(W) = \dim_{\mathbb{B}_{\text{dr}}} W.
\]

Say \( U \in \text{Rep}_{G_K}(\mathbb{B}_{\text{dr}}^+) \) is de Rham if \( U[1/t] \) is so.

Theorem 7.2.5. (1) Let \( W \in \text{Rep}_{G_K}(\mathbb{B}_{\text{dr}}) \). Then \( W \) is de Rham if and only \( W|_{G_L} \otimes_{\mathbb{B}_{\text{dr}}} \mathbb{B}_{\text{dr},L} \) is de Rham.

(2) Let \( V \in \text{Rep}_{G_K}(\mathbb{Q}_p) \), then \( V \) is de Rham if and only \( V|_{G_L} \) is de Rham.

Proof. Apply Prop. 7.2.2.

In the statement of the following theorem, we refer to the paragraph above [Ber08, Lem. III.1.3] for a quick review of concepts such as formal connections, regular connections, and trivial connections.

Theorem 7.2.6. (1) The \( \Gamma_K \)-action on \( K_\infty[[t]] \) is locally analytic, and the Lie algebra action induces a differential map \( \nabla_{\gamma} : K_\infty[[t]] \rightarrow K_\infty[[t]] \) which is \( K_\infty \)-linear and \( \nabla_{\gamma}(t) = t \).

(2) Let \( Y \in \text{Rep}_{G_K}(K_\infty[[t]]) \), then the \( \Gamma_K \)-action on \( Y \) is locally analytic. Thus the Lie algebra action induces a regular connection \( \nabla_{\gamma} : Y \rightarrow Y \).
(3) Let $W \in \text{Rep}_{G_K}(\mathcal{B}_{\text{dr}})$. Then it is de Rham if and only the regular connection $\nabla_\gamma : \text{RH}(W) \to \text{RH}(W)$ is trivial.

**Proof.** Item (1) is obvious. Item (2) and (3) follows similar argument as in Thm. 5.2.6. Indeed, for Item (2), the base change $Y \otimes_{K_{\text{pf}}} K_{\text{pf}}[[t]]$ has locally analytic action by $\Gamma_{K_{\text{pf}}}$, by the classical theory [Fon04, Prop. 3.7]. Hence $\Gamma_K$ action on $Y$ is also locally analytic since $\Gamma_{K_{\text{pf}}} \simeq \Gamma_K$.

Finally consider Item (3). First, note that for any $G_K$-stable lattice $U \in \text{Rep}_{G_K}(\mathcal{B}_{\text{dr}})$ of $W$, the connection $\nabla_\gamma : \text{RH}^+(U) \to \text{RH}^+(U)$ is regular by Item (2), and hence the induced connection on $\text{RH}(W)$ is also regular. Now for brevity, let $X = W|_{G_K}$, $\mathcal{B}_{\text{dr},K_{\text{pf}}} \simeq \mathcal{B}_{\text{dr}}$. If $\nabla_\gamma$ on $\text{RH}(W)$ is trivial, then the similar connection $\nabla_\gamma$ on $\text{RH}(X)$ is also trivial, and hence $X$ is de Rham by the classical theory in the perfect residue field case [Ber02, Prop. 5.9] (which treated the case when $W = V \otimes_{\mathbb{Q}_p} \mathcal{B}_{\text{dr}}$ for some $V = \text{Rep}_{G_K}(\mathbb{Q}_p)$, but the general case is the same). Thus $W$ is de Rham by Thm. 7.2.5. Conversely, if $W$ is de Rham, then $D_{\text{dr},K}(W)$ is already a solution of $\nabla_\gamma$ and hence the connection is trivial.

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