MULTIPLICATION CONDITIONAL EXPECTATION TYPE OPERATORS ON ORLICZ SPACES

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Abstract. In this paper we consider a generalized conditional-type H"older-inequality and investigate some classic properties of multiplication conditional expectation type operators on Orlicz-spaces.

1. Introduction

Let $(\Omega, \Sigma, \mu)$ be a measure space and $\mathcal{A} \subseteq \Sigma$ be sub $\sigma-$algebra. For a sub-$\sigma$-finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \mapsto E^{A}f$, defined for all non-negative $f$ as well as for all $f \in L^1(\Sigma)$ and $f \in L^\infty(\Sigma)$, where $E^A f$, by the Radon-Nikodym theorem, is the unique $\mathcal{A}-$measurable function satisfying

$$\int_A f d\mu = \int_A E^A f d\mu, \quad \forall A \in \mathcal{A}.$$ 

As an operator on $L^1(\Sigma)$ and $L^\infty(\Sigma)$, $E^A$ is idempotent and $E^A(L^\infty(\Sigma)) = L^\infty(\mathcal{A})$ and $E^A(L^1(\Sigma)) = L^1(\mathcal{A})$. Thus it can be defined on all interpolation spaces of $L^1$ and $L^\infty$ such as, Orlicz spaces. If there is no possibility of confusion, we write $E(f)$ in place of $E^A(f)$. This operator will play a major role in our work and we list here some of its useful properties:

- If $g$ is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$.
- $\varphi(E(f)) \leq E(\varphi(f))$, where $\varphi$ is a convex function.
- If $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.
- For each $f \geq 0$, $\sigma(f) \subseteq \sigma(E(f))$.

A detailed discussion and verification of most of these properties may be found in [21]. We recall that an $\mathcal{A}$-atom of the measure $\mu$ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $F \in \mathcal{A}$, if $F \subseteq A$, then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space $(X, \Sigma, \mu)$ with no atoms is called non-atomic measure space.

Let $(\Omega, \Sigma, \mu)$ be a measure space and $\mathcal{A} \subseteq \Sigma$ be sub $\sigma-$algebra, such that $(\Omega, \mathcal{A}, \mu)$ has finite subset property. $E^A = E$ is conditional expectation with respect to $\mathcal{A}$. It is well-known fact that every $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ can be partitioned uniquely as $\Omega = \bigcup_{n \in \mathbb{N}} C_n \cup B$, where $\{C_n\}_{n \in \mathbb{N}}$ is a countable collection of pairwise disjoint $\Sigma$-atoms and $B$, being disjoint from each $C_n$, is non-atomic.

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Operators in function spaces defined by conditional expectations were first studied, among others, by S. T.C. Moy [19], Z. Sidak [22] and H.D. Brunk [5] in the setting of $L^p$ spaces. Conditional expectation operators on various function spaces exhibit a number of remarkable properties related to the underlying structure of the given function space or to the metric structure when the function space is equipped with a norm. P.G. Dodds, C.B. Huijsmans and B. de Pagter [7] linked these operators to averaging operators defined on abstract spaces earlier by J.L. Kelley [15], while A. Lambert [16] studied their link to classes of multiplication operators which form Hilbert $C^*$-modules. J.J. Grobler and B. de Pagter [11] showed that the classes of partial integral operators, studied by A.S. Kalitvin and others [1, 2, 3, 6, 14], were a special case of conditional expectation operators. Recently, J. Herron studied operators $E_{M_u}$ on $L^p$ spaces in [12].

Also, in [9, 10] we investigate some classic properties of multiplication conditional expectation operators $M_uE_{M_u}$ on $L^p$ spaces. In the present paper we continue the investigation of some classic properties of the operator $E_{M_u}$ on Orlicz spaces by considering Generalized conditional-type Holder inequality.

Let us now introduce the definition of convexity for functions of $n$ variables and later some particular criteria for convex functions of 2 variables.

**Definition**

Let be $f : \mathbb{R}^n \to \mathbb{R}$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we say that the function $f(x)$ is convex in $\mathbb{R}^n$ (or in a subset of $\mathbb{R}^n$) if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

for any $x = (x_1, ..., x_n)$ and $y = (x_1, ..., x_n)$ and for any $0 \leq \lambda \leq 1$.

It is well known fact that:

Given a nice function $f$ of 2 variables in a set in the plane, the function $f$ is convex if and only if both the following properties are true in such set:

1. $f_{x,x}(x, y) - f_{x,y}(x, y) - f_{y,y}(x, y) \geq 0$,
2. $f_{x,x}(x, y) \geq 0$ and $f_{y,y}(x, y) \geq 0$.

Let $\Phi : \mathbb{R} \to \mathbb{R}^+$ be a continuous convex function such that

1. $\Phi(x) = 0$ if and only if $x = 0$.
2. $\Phi(x) = \Phi(-x)$.
3. $\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty$, $\lim_{x \to \infty} \Phi(x) = \infty$.

The function $\Phi$ is called Young’s function. With each Young’s function $\Phi$, one can associate another convex function $\Psi : \mathbb{R} \to \mathbb{R}^+$ having similar properties, which is defined by

$$\Psi(y) = \sup\{x|y - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$ 

Then $\Psi$ is called complementary Young function to $\Phi$. A Young function $\Phi$ is said to satisfy the $\triangle_2$ condition (globally) if $\Phi(2x) \leq k\Phi(x), \quad x \geq x_0 \geq 0(x_0 = 0)$ for
some constant $k > 0$. Also, $\Phi$ is said to satisfy the $\triangle'(<\triangle')$ condition, in symbols $\Phi \in \triangle'(<\triangle')$, if $\exists c > 0 (b > 0)$ such that
$$\Phi(xy) \leq c\Phi(x)\Phi(y), \quad x, y \geq x_0 \geq 0$$
(and
$$\Phi(bxy) \geq \Phi(x)\Phi(y), \quad x, y \geq y_0 \geq 0).$$
If $x_0 = 0(y_0 = 0)$, then these conditions are said to hold globally. If $\Phi \in \triangle'$, then $\Phi \in \triangle_2$.

Let $\Phi_1, \Phi_2$ be two Young functions, then $\Phi_1$ is stronger than $\Phi_2$, $\Phi_1 \succ \Phi_2$ [or $\Phi_2 \prec \Phi_1$] if
$$\Phi_2(x) \leq \Phi_1(ax), \quad x \geq x_0 \geq 0$$
for some $a_0 \geq 0$ and $x_0$, if $x_0 = 0$ then this condition is said to hold globally.

Let $\Phi$ be a Young function, then the set of $\Sigma-$measurable functions

$$L^\Phi(\Sigma) = \{f : \Omega \to \mathbb{C} : \exists k > 0, \int \Omega \Phi(kf)d\mu < \infty\}$$

is a Banach space, with respect to the norm $N_\Phi(f) = \inf\{k > 0 : \int \Omega \Phi(\frac{f}{k})d\mu \leq 1\}. (L^\Phi(\Sigma), N_\Phi(.))$ is called Orlicz space.

Let $\Phi$ be a Young function and $f \in L^\Phi(\Sigma)$. Since $\Phi$ is convex, by Jensen’s inequality $\Phi(E(|f|)) \leq E(\Phi(|f|))$ so
$$\int \Omega \Phi(\frac{E(f)}{N_\Phi(f)})d\mu = \int \Omega \Phi(\frac{f}{N_\Phi(f)})d\mu \leq \int \Omega \Phi(\frac{f}{N_\Phi(f)})d\mu \leq 1.$$ This implies that $N_\Phi(E(|f|)) \leq N_\Phi(f)$ i.e, $E$ is a contraction on Orlicz spaces.

We say that $(E, \Phi)$ satisfies in Generalized conditional- type Holder inequality, if there exist some positive constant $C$ such that for all $f \in L^\Phi(\Omega, \Sigma, \mu)$ and $g \in L^\Psi(\Omega, \Sigma, \mu)$ we have
$$E(|fg|) \leq C\Phi^{-1}(E(\Phi(|f|)))\Psi^{-1}(E(\Psi(|g|))),$$
where $\Psi$ is complementary Young function to $\Phi$.

In the sequel as Lemma 1.2, Lemma 1.3 and Lemma 1.5 we give some conditions for $E$ or $\Phi$ or $\Psi$ or jointly to get Generalized conditional- type Holder inequality.

**Lemma 1.1**

Let $\Phi$ and $\Psi$ be complementary Young functions. If there exist $C_1, C_2 > 0$ such that
$$E(\Phi(\frac{f}{\Phi^{-1}(E(\Phi(|f|))})) \leq C_1, \quad E(\Psi(\frac{g}{\Psi^{-1}(E(\Psi(|g|))))) \leq C_2.$$ Then $(E, \Phi)$ satisfies in Generalized conditional-type Holder inequality.

**Proof** Let $f \in L^\Phi(\Omega, \Sigma, \mu)$ and $g \in L^\Psi(\Omega, \Sigma, \mu)$. By Young inequality we have $fg \leq \Phi(f) + \Psi(g)$. If we replace $f$ to $\frac{f}{\Phi^{-1}(E(\Phi(|f|)))}$ and $g$ to $\frac{g}{\Psi^{-1}(E(\Psi(|g|)))}$. We have
\[ \frac{fg}{\Phi^{-1}(E(\Phi(f)))\Psi^{-1}(E(\Psi(g)))} \leq \Phi\left(\frac{f}{\Phi^{-1}(E(\Phi(f)))}\right) + \Psi\left(\frac{g}{\Psi^{-1}(E(\Psi(g)))}\right). \]

By taking \( E \) we have

\[ \frac{E(fg)}{\Phi^{-1}(E(\Phi(f)))\Psi^{-1}(E(\Psi(g)))} \leq E\left(\Phi\left(\frac{f}{\Phi^{-1}(E(\Phi(f)))}\right)\right) + E\left(\Psi\left(\frac{g}{\Psi^{-1}(E(\Psi(g)))}\right)\right) \leq C_1 + C_2. \]

This implies that

\[ E(|fg|) \leq C\Phi^{-1}(E(|f|))\Psi^{-1}(E(|g|)), \]

where \( C = C_1 + C_2 \).

By using some basic facts about closed convex sets in Banach spaces in section I.2 and I.3 of [S] we have the following lemma.

**Lemma 1.2**

A mapping \( F: \mathbb{R}^n \to [0, \infty) \) can be written in the form

\[ F(x_1, x_2, ..., x_n) = \inf_{a \in A} \sum_{i=1}^{n} a_i x_i \]

for some countable set \( A \subseteq \mathbb{R}^n \) if and only if \( F \) is concave, lower semicontinuous and positive homogeneous.

Also, we recall a generalization of Jensen-inequality that is proved by Markus Haase in [17].

**Theorem 1.3**

Let \((\Omega, \Sigma, \mu)\) and \((\Omega', \Sigma', \mu')\) be measure spaces, let \( C \subseteq M(\Omega, \Sigma, \mu)_+ \) be a subcone and let \( T : C \to M(\Omega', \Sigma', \mu')_+ \) be a monotone, subadditive and positively homogeneous operator. Let \( F: \mathbb{R}^n_+ \to [0, \infty) \) be given by

\[ F(x_1, x_2, ..., x_n) = \inf_{a \in A} \sum_{i=1}^{n} a_i x_i \]

for some countable set \( A \subseteq \mathbb{R}^n_+ \). Then if \( f_1, f_2, ..., f_n \in C \) such that \( F(f_1, ..., f_n) \) is in \( C \), one has

\[ T[F(f_1, ..., f_n)] \leq F(T f_1, ..., T f_n) \]

as an inequality in \( M(\Omega', \Sigma', \mu') \).

**Corollary 1.4**

Let \( \Phi \) and \( \Psi \) be complementary Young functions such that \( \Phi''(x)\Psi''(y)\Phi(x)\Psi(y) - (\Phi'(x)\Psi'(y))^2 \geq 0 \) and the function \( F(x, y) = \Phi^{-1}(x)\Psi^{-1}(y) \) is positively homogeneous on \( \mathbb{R}^2_+ \). Then \((E, \Phi)\) satisfies in Generalized conditional-type Holder inequality, for every conditional expectation operator \( E \).

**Proof** Since \( \Phi \) and \( \Psi \) are continuous, then the map \( F(x, y) = \Phi^{-1}(x)\Psi^{-1}(y) \) is also continuous. By assumptions the map \( F \) is concave, lower semicontinuous and
positive homogeneous. By replacing $T$ with conditional expectation operator $E$ in Theorem 1.3 we have

$$E(\Phi^{-1}(f)\Psi^{-1}(g)) \leq \Phi^{-1}(E(f))\Psi^{-1}(E(g))$$

for all nonnegative measurable function on measure space $(\Omega, \Sigma, \mu)$. Direct computation shows that for all $f \in L^p(\Omega, \Sigma, \mu)$ and $g \in L^q(\Omega, \Sigma, \mu)$,

$$E(|fg|) \leq \Phi^{-1}(E(|f|))\Psi^{-1}(E(|g|)).$$

**Lemma 1.5**

Let $E$ be the conditional expectation operator and $\Phi$ and $\Psi$ be complementary Young functions. If there exists positive constant $C$ such that $E(|fg|) \leq CE(f)E(g)$, for positive measurable functions $f \in L^p(\Omega, \Sigma, \mu)$ and $g \in L^q(\Omega, \Sigma, \mu)$. Then $(E, \Phi)$ satisfies in Generalized conditional type Holder inequality.

**Proof** Let $f \in L^p(\Omega, \Sigma, \mu)$ and $g \in L^q(\Omega, \Sigma, \mu)$ such that $f > 0, g > 0$. Since $\Phi^{-1}$ and $\Psi^{-1}$ are concave, then

$$E(f) = E(\Phi^{-1}(f)) \leq \Phi^{-1}(E(f)), \quad E(g) = E(\Psi^{-1}(g)) \leq \Psi^{-1}(E(g)).$$

This implies that

$$E(fg) \leq CE(f)E(g) \leq C\Phi^{-1}(E(f))\Psi^{-1}(E(g)).$$

So for all $f \in L^p(\Omega, \Sigma, \mu)$ and $g \in L^q(\Omega, \Sigma, \mu)$ we have

$$E(|fg|) \leq C\Phi^{-1}(E(|f|))\Psi^{-1}(E(|g|)).$$

**Example 1.6**

(a) If $\Phi(x) = \frac{x^p}{p}$, $1 \leq p < \infty$. Then for all $f \in L^p(\Omega, \Sigma, \mu) = L^p(\Sigma)$ and $g \in L^q(\Omega, \Sigma, \mu) = L^q(\Sigma)$, where $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$E(|fg|) \leq (E(|f|^p))^{\frac{1}{p}}(E(|g|^q))^{\frac{1}{q}}.$$  

(b) Let $\Omega = [-1, 1]$, $d\mu = \frac{1}{2}dx$ and $A = \sigma \{(-a, a) : 0 \leq a \leq 1\}$ (Sigma algebra generated by symmetric intervals). Then

$$E^A(f)(x) = \frac{f(x) + f(-x)}{2}, \quad x \in \Omega,$$

where $E^A(f)$ is defined. Thus $E^A(|f|) \geq \frac{|f|}{2}$. Hence $|f| \leq 2E(|f|)$. Let $\Phi$ be a Young function. For each $f \in L^p(\Omega, \Sigma, \mu)$ we have $\Phi(|f|) \leq 2E(\Phi(|f|))$. Since $\Phi^{-1}$ is also increasing and concave

$$|f| = \Phi^{-1}(\Phi(|f|)) \leq \Phi^{-1}(2E(\Phi(|f|))) \leq 2\Phi^{-1}(E(\Phi(|f|))),$$

so $|f| \leq 2\Phi^{-1}(E(\Phi(|f|)))$. Similarly for $g \in L^q(\Omega, \Sigma, \mu)$ we have $|g| \leq 2\Phi^{-1}(E(\Phi(|g|)))$. Thus

$$|fg| \leq 4\Phi^{-1}(E(\Phi(|f|)))\Psi^{-1}(E(\Phi(|g|))).$$

By taking $E$ we have

$$E(|fg|) \leq 4\Phi^{-1}(E(\Phi(|f|)))\Psi^{-1}(E(\Phi(|g|))).$$
Also, by lemma 1.1 we have \( C_1 = \Phi(2) \), \( C_2 = \Psi(2) \) and \( C = \Phi(2) + \Psi(2) \).

(c) Let \( dA(z) \) be the normalized Lebesgue measure on open unit disc \( \mathbb{D} \). Recall that for \( 1 \leq p < \infty \) the Bergman space \( L^p_0(\mathbb{D}) \) is the collection of all functions \( f \in H(\mathbb{D}) \), holomorphic functions on \( \mathbb{D} \), for which \( \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty \). Let \( \mathcal{A} \) be the \( \sigma \)-algebra generated by \( \{ (z^n)^{-1}(U) : U \subseteq \mathbb{C} \text{ is open} \} \). Then

\[
E(u)(\xi) = \frac{1}{n} \sum_{\zeta^n = \xi} u(\zeta), \quad u \in H(\mathbb{D}), \quad \xi \in \mathbb{D} \setminus \{0\},
\]

(see [3]). Note that \( |u| \leq nE(|u|) \). By the same method of part(b), for every Young function \( \Phi \), \( (E, \Phi) \) satisfies in Generalized conditional-type Holder inequality.

(d) Let \( X = [0, 1], \Sigma = \text{sigma algebra of Lebesgue measurable subset of } X, \mu = \text{Lebesgue measure on } X \). Fix \( n \in \{2, 3, 4, \ldots \} \) and let \( s : [0, 1] \to [0, 1] \) be defined by \( s(x) = x + \frac{1}{n}(\text{mod } 1) \). Let \( \mathcal{B} = \{ E \in \Sigma : s^{-1}(E) = E \} \). In this case

\[
E^B(f)(x) = \sum_{j=0}^{n-1} f(s^j(x)),
\]

where \( s^j \) denotes the \( j \)-th iteration of \( s \). The functions \( f \) in the range of \( E^B \) are those for which the \( n \)-graphs of \( f \) restricted to the intervals \( \left[ \frac{j-1}{n}, \frac{j}{n} \right], 1 \leq j \leq n \), are all congruent. Also, \( |f| \leq nE^B(|f|) \) a.e. By the same method of part(b), for every Young function \( \Phi \), \( (E, \Phi) \) satisfies in Generalized conditional-type Holder inequality.

(e) Let \((\Omega, \Sigma, \mu)\) be a measure space and \( \mathcal{A} \subseteq \Sigma \) be sub \( \sigma \)-algebra. If there exists \( C_0 > 0 \) such that \( |f| \leq C_0E^A(|f|) \). Then \((E^A, \Phi)\) satisfies in Generalized conditional-type Holder inequality, for every Young function \( \Phi \).

The part (e) of the last example and proposition 2.2 of [18] show that there are many conditional expectation operators \( E \) such that the Generalized conditional-type Holder inequality holds for \((E, \Phi)\).

2. BOUNDEDNESS AND COMPACTNESS OF \( EM_u \) ON ORLICZ SPACES

**Theorem 2.1.** Let \( T = EM_u : L^p(\Omega, \Sigma, \mu) \to L^0(\Omega, \Sigma, \mu) \) such that \( T(f) = E(u,f) \) for \( f \in L^p(\Omega, \Sigma, \mu) \) is well defined, then the followings hold.

(a) If \( T \) is bounded on \( L^p(\Omega, \Sigma, \mu) \), then \( E(u) \in L^\infty(\Omega, \mathcal{A}, \mu) \).

(b) If \( \Phi \in \Delta'\text{(globally)} \) and \( T \) is bounded on \( L^p(\Omega, \Sigma, \mu) \), then \( \Psi^{-1}(E(\Psi(u))) \in L^\infty(\mathcal{A}) \).

(c) If \((E, \Phi)\) satisfies in Generalized conditional-type Holder inequality and \( \Psi^{-1}(E(\Psi(u))) \in L^\infty(\mathcal{A}) \), then \( T \) is bounded.

In this case, \( \|T\| \leq C\|\Psi^{-1}(E(\Psi(u)))\|_\infty \).

**Proof.** (a) Suppose that \( E(u) \notin L^\infty(\Omega, \mathcal{A}, \mu) \). If we set \( E_n = \{ x \in \Omega : |E(u)(x)| > n \} \), for all \( n \in \mathbb{N} \), then \( E_n \in \mathcal{A} \) and \( \mu(E_n) > 0 \).
Since \((\Omega, \mathcal{A}, \mu)\) has finite subset property, we can assume that \(0 < \mu(E_n) < \infty\), for all \(n \in \mathbb{N}\). By definition of \(E_n\) we have

\[
T(\chi_{E_n}) = E(u\chi_{E_n}) = E(u)\chi_{E_n} > n\chi_{E_n}.
\]

Since Orlicz’s norm is monotone, then

\[
\|T(\chi_{E_n})\|_\Phi > \|n\chi_{E_n}\|_\Phi = n\|\chi_{E_n}\|_\Phi.
\]

This implies that \(T\) isn’t bounded.

(b) If \(\Psi^{-1}(E(\Psi(u))) \notin L^\infty(\mathcal{A})\), then \(\mu(E_n) > 0\). Where

\[
E_n = \{x \in \Omega : \Psi^{-1}(E(\Psi(u)))(x) > n\}
\]

and so \(E_n \in \mathcal{A}\). Since \(\Phi \in \Delta_2\), then \(\Psi \in \nabla'\), i.e., \(\exists b > 0\) such that

\[
\Psi(bxy) \geq \Psi(x)\Psi(y), \quad x, y \geq 0.
\]

Also, \(\Phi \in \Delta_2\). Thus \((L^\Phi)^* = L^\Phi\) and so \(T^* = M_\Phi : L^\Phi(\mathcal{A}) \to L^\Phi(\Sigma)\), is also bounded. Hence for each \(k > 0\) we have

\[
\int_\Omega \Psi\left(\frac{k\mu\chi_{E_n}}{N\Phi(\chi_{E_n})}\right) d\mu = \int_\Omega \Psi(k\mu\chi_{E_n}\Psi^{-1}\left(\frac{1}{\mu(E_n)}\right)) d\mu = \int_\Omega \Psi(k\mu\Psi^{-1}\left(\frac{1}{\mu(E_n)}\right))\chi_{E_n} d\mu
\]

\[
\geq \int_{E_n} \Psi(u)\Psi\left(\frac{k\Psi^{-1}\left(\frac{1}{\mu(E_n)}\right)}{b}\right) d\mu \geq \left(\int_{E_n} E(\Psi(u)) d\mu\right) \left(\Psi\left(\frac{k}{b^2}\right)\Psi^{-1}\left(\frac{1}{\mu(E_n)}\right)\right)
\]

\[
\geq \Psi(n\mu(E_n))\frac{1}{\mu(E_n)} \Psi\left(\frac{k}{b^2}\right) = \Psi(n)\Psi\left(\frac{k}{b^2}\right).
\]

Thus

\[
\int_\Omega \Psi\left(\frac{k\mu\chi_{E_n}}{N\Phi(\chi_{E_n})}\right) d\mu = \int_\Omega \Psi(kM_u(f_n)) d\mu \geq \Psi(n)\Psi\left(\frac{k}{b^2}\right) \to \infty
\]

as \(n \to \infty\), where \(f_n = \frac{\chi_{E_n}}{N\Phi(\chi_{E_n})}\). Thus \(N\Phi(M_u(f_n)) \to \infty\), as \(n \to \infty\). This is a contradiction, since \(M_u\) is bounded.

(c) Put \(M = \|\Psi^{-1}(E(\Psi(u)))\|_\infty\). For \(f \in L^\Phi(\Omega, \Sigma, \mu)\) and \(g \in L^\Phi(\Omega, \Sigma, \mu)\) we have

\[
\int_\Omega \Phi\left(\frac{E(uf)}{CMN\Phi(f)}\right) d\mu = \int_\Omega \Phi\left(\frac{E(u \frac{f}{\Phi(f)})}{CM}\right) d\mu
\]

\[
\leq \int_\Omega \Phi\left(\frac{C\Phi^{-1}(E(\Phi(|\frac{f}{\Phi(f)}|)))\Psi^{-1}(E(\Psi(|u|)))}{CM}\right) d\mu
\]

\[
\leq \int_\Omega \Phi\left(\Phi^{-1}(E(\Phi(|\frac{f}{\Phi(f)}|)))\right) d\mu \leq \int_\Omega \Phi\left(\frac{f}{\Phi(f)}\right) d\mu \leq 1.
\]

So \(N\Phi(E(uf)) \leq CMN\Phi(f)\). Thus \(T = EM_u\) is bounded and \(\|T\| \leq C\|\Psi^{-1}(E(\Psi(u)))\|_\infty\).

**Corollary 2.2.**

(a) If \((E, \Phi)\) satisfies in Generalized conditional-type Holder inequality and \(\Phi \in\)
\( \Delta'(\text{globally}) \), then \( T \) is bounded if and only if \( \Psi^{-1}(E(\Psi(u))) \in L^\infty(\mathcal{A}) \).

(b) If \( \Psi \prec x \) and \( (E, \Phi) \) satisfies in Generalized conditional-type Holder inequality, then \( T \) is bounded if and only if \( \Psi^{-1}(E(\Psi(u))) \in L^\infty(\mathcal{A}) \).

**Proof.** (b) Since \( \Psi \prec x \) then \( EM_{\Psi(u)} \leq KEM_u \) for some \( K > 0 \). If \( \Psi^{-1}(E(\Psi(u))) \notin L^\infty(\mathcal{A}) \), then the operator \( EM_{\Psi(u)} \) is not bounded and so \( T = EM_u \) is not bounded.

**Theorem 2.3.** Let \( T = EM_u \) be bounded on \( L^\Phi(\Sigma) \), then the following hold.

(a) If \( T \) is compact, then
\[
N_\varepsilon(E(u)) = \{ x \in \Omega : E(u)(x) \geq \varepsilon \}
\]

consists of finitely many \( \mathcal{A} \)-atoms, for all \( \varepsilon > 0 \).

(b) If \( T \) is compact and \( \Phi \in \Delta'(\text{globally}) \), then \( N_\varepsilon(\Psi^{-1}(E(\Psi(u)))) \)

consists of finitely many \( \mathcal{A} \)-atoms, for all \( \varepsilon > 0 \), where
\[
N_\varepsilon(\Psi^{-1}(E(\Psi(u)))) = \{ x \in \Omega : \Psi^{-1}(E(\Psi(u))(x) \geq \varepsilon \}.
\]

(c) If \( (E, \Phi) \) satisfies in Generalized conditional-type Holder inequality and \( N_\varepsilon(\Psi^{-1}(E(\Psi(u)))) \)

consists of finitely many \( \mathcal{A} \)-atoms, for all \( \varepsilon > 0 \), then \( T \) is compact.

**Proof.** (a) If there exists \( \varepsilon_0 > 0 \), such that \( N_{\varepsilon_0}(E(u)) \) consists of infinitely many \( \mathcal{A} \)-atoms or a non-atomic subset of positive measure. Since \( (\Omega, \mathcal{A}, \mu) \) has finite subset property. In both cases, we can find a sequence of disjoint \( \mathcal{A} \)-measurable subsets \( \{A_n\}_{n \in \mathbb{N}} \) of \( N_{\varepsilon_0}(E(u)) \) with \( 0 < \mu(A_n) < \infty \). Let \( f_n = \frac{\chi_{A_n}}{N_{\Phi}(\chi_{A_n})} \). Hence
\[
|f_n - f_m| = |f_n + f_m| = |f_n| + |f_m|
\]

for \( n \neq m \). Also, \( E(uf_n) = E(u)f_n \geq \varepsilon_0 f_n \). By monotonicity of \( N_{\Phi}(\cdot) \) we have
\[
N_{\Phi}(|E(uf_n) - E(uf_m)|) = N_{\Phi}(|E(u)(f_n - f_m)|)
\]

= \( N_{\Phi}(|E(u)|(|f_n| + |f_m|)) \geq N_{\Phi}(|E(u)|f_n) \geq \varepsilon_0 N_{\Phi}(f_n) = \varepsilon_0 \).

Thus \( N_{\Phi}(|E(uf_n) - E(uf_m)|) \geq \varepsilon_0 \). This implies that \( T \) cannot be compact.

(b) Suppose that there exists \( \varepsilon_0 > 0 \), such that \( N_\varepsilon(\Psi^{-1}(E(\Psi(u)))) \) doesn’t consist finitely many \( \mathcal{A} \)-atoms. Since \( T \) is compact, so \( T^* = M_u \) is compact from \( L^\Phi(\mathcal{A}) \) into \( L^\Phi(\Sigma) \). By the same method of (a) and theorem 2.1 (b), we can find the sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( L^\Phi(\mathcal{A}) \), such that \( N_{\Phi}(f_n) = 1 \), and
\[
\int \Omega \Psi(uf_n) d\mu = \int \Omega \Psi(M_u(f_n)) d\mu \geq \Psi(\varepsilon_0)\Psi\left(\frac{1}{\varepsilon^2}\right).
\]

Since \( \Psi \) is increasing and \( |f_n - f_m| = |f_n| + |f_m| \), then
\[
\int \Omega \Psi(|uf_n - uf_m|) d\mu = \int \Omega \Psi(|M_u(|f_n| + |f_m|)|) d\mu \geq \Psi(\varepsilon_0)\Psi\left(\frac{1}{\varepsilon^2}\right).
\]

Thus \( \{uf_n\}_{n \in \mathbb{N}} \) has no convergence subsequence in \( \Psi \)-mean convergence. So \( \{uf_n\}_{n \in \mathbb{N}} \) has no convergence subsequence in norm. This is a contradiction.

(c) Let \( \varepsilon > 0 \) and \( N_\varepsilon = N_\varepsilon(\Psi^{-1}(E(\Psi(u)))) \). By assumption, there exist finitely many disjoint \( \mathcal{A} \)-atoms \( \{A_i\}_{i=1}^n \), such that \( N_\varepsilon = \cup_{i=1}^n A_i \). Define the operator \( T_\varepsilon \).
on $L^\Phi(\Sigma)$, such that $T_\varepsilon(f) = E(uf\chi_{N_\varepsilon})$, for $f \in L^\Phi(\Sigma)$. Since $N_\varepsilon \in \mathcal{A}$, $E(uf)$ is $\mathcal{A}$–measurable and $\mathcal{A}$–measurable functions are constant on $\mathcal{A}$–atoms, we have

$$T_\varepsilon(f) = E(uf)\chi_{N_\varepsilon} = \sum_{i=1}^n E(uf)(A_i)\chi_{A_i} \in L^\Phi(N_\varepsilon).$$

So $T_\varepsilon$ is finite rank.

For $f \in L^\Phi(\Sigma)$,

$$T(f) - T_\varepsilon(f) = E(uf) - E(uf)\chi_{N_\varepsilon} = E(uf)\chi_{\Omega \setminus N_\varepsilon}.$$

Thus

$$\int_{\Omega} \Phi\left(\frac{T(f) - T_\varepsilon(f)}{C\varepsilon N_\Phi(f)}\right) d\mu = \int_{\Omega} \Phi\left(\frac{E(uf)\chi_{\Omega \setminus N_\varepsilon}}{C\varepsilon N_\Phi(f)}\right) d\mu = \int_{\Omega} \Phi\left(\frac{C\Phi^{-1}(E(\Phi(\frac{f}{N_\Phi(f)})))\Psi^{-1}(E(\Psi(u)))}{C\varepsilon}\right) d\mu \leq \int_{\Omega} \Phi\left(\frac{f}{N_\Phi(f)}\right) d\mu \leq \int_{\Omega} \Phi\left(\frac{f}{N_\Phi(f)}\right) d\mu = \int_{\Omega} \Phi\left(\frac{f}{N_\Phi(f)}\right) d\mu = 1.$$

This implies that $N_\Phi(T(f) - T_\varepsilon(f)) \leq C\varepsilon N_\Phi(f)$, and so $\|T - T_\varepsilon\| < C\varepsilon$. This means that, $T$ is limit of a sequence of finite rank operators. So $T$ is compact.

**Corollary 2.4.**

(a) If $(E, \Phi)$ satisfies in Generalized conditional-type Holder inequality and $\Phi \in \Delta'(\text{globally})$, then $T$ is compact if and only if $N_\varepsilon(\Psi^{-1}(E(\Psi(u))))$ consists of finitely many $\mathcal{A}$–atoms, for all $\varepsilon > 0$.

(b) If $\Psi \prec x(\text{globally})$ and $(E, \Phi)$ satisfies in Generalized conditional-type Holder inequality, then then $T$ is compact if and only if $N_\varepsilon(\Psi^{-1}(E(\Psi(u))))$ consists of finitely many $\mathcal{A}$–atoms, for all $\varepsilon > 0$.

(c) If $(\Omega, \mathcal{A}, \mu)$ is non-atomic measure space, $(E, \Phi)$ satisfies in Generalized conditional-type Holder inequality and $\Phi \in \Delta'(\text{globally})$. Then $T = EM_u$ is a compact operator on $L^\Phi(\Sigma)$ if and only if $T = 0$.

**Proof (b)** Since $\Psi \prec x$ then $\Psi(u) \leq Ku$ for some $K > 0$. Suppose that, there exists $\varepsilon_0 > 0$, such that $N_\varepsilon(\Psi^{-1}(E(\Psi(u))))$ doesn’t consist finitely many $\mathcal{A}$–atoms. By the same method of (a) and theorem 2.1 (b), we can find the sequence $\{f_n\}_{n \in \mathbb{N}}$ in $L^\Phi(\mathcal{A})$, such that $N_\Phi(f_n) = 1$, and

$$\int_{\Omega} \Phi(Kuf_n) d\mu \geq \int_{\Omega} \Phi(\Psi(u)Kf_n) d\mu = \int_{\Omega} E(\Phi(\Psi(u)Kf_n)) d\mu \geq \int_{\Omega} E(\Phi(\Psi(u))) d\mu \geq \varepsilon_0 K,$$

where $K\varepsilon_0 > 1$. Since $\Phi$ is increasing and $|f_n - f_m| = |f_n| + |f_m|$, then
\[
\int_{\Omega} \Phi(|Ku_n - Kf_n|) d\mu = \int_{\Omega} \Phi(|KM_uf_n|) d\mu \geq K\varepsilon_0.
\]

Thus \(\{KM_uf_n\}\) has no convergence subsequence in \(\Phi\)-mean convergence. So \(\{M_uf_n\}\) has no convergence subsequence in norm. This is a contradiction.

In the next theorem we use the method that is used in [12].

**Theorem 2.5.** If \(A \neq \Sigma\), then \(\sigma(T) = essrange(E(u)) \cup \{0\}\).

**Proof.** Since \(A \neq \Sigma\), \(L^b(A) \not\subset L^b(\Sigma)\). Hence \(T = EM_u\) isn’t surjective and so \(0 \in \sigma(T)\). Let \(\lambda \notin essrange(E(u)), \lambda \neq 0\). We show that \(T - \lambda I\) is invertible. If \(Tf - \lambda f = 0\), then \(Euf = \lambda f\). So \(f\) is \(A\)-measurable. Thus \((E(u) - \lambda)f = E(uf) - \lambda f = 0\). Since \(\lambda \notin essrange(E(u))\), then \(E(u) - \lambda \geq \varepsilon\ a.e\) for some \(\varepsilon > 0\). So \(f = 0\ a.e\). This implies that \(T - \lambda I\) is injective. Now we show that \(T - \lambda I\) is surjective. Let \(g \in L^b(\Sigma)\). We can write

\[
g = g - E(g) + E(g), \quad g_1 = g - E(g), \quad g_2 = E(g).
\]

Since \(N_\Phi(E(g)) \leq N_\Phi(g)\), then \(g_2 \in L^b(A)\) and \(g_1 \in L^b(\Sigma)\), \((E(g_1) = 0\). Let

\[
f_1 = \frac{\lambda g_1 + T(g_2)}{\lambda (E(u) - \lambda)}, \quad f_2 = \frac{-g_2}{\lambda}.
\]

Since \(\lambda \notin essrange(E(u))\), then \(E(u) - \lambda \geq \varepsilon\ a.e\) for some \(\varepsilon > 0\). So \(\left\|\frac{1}{E(u) - \lambda}\right\|_\infty \leq \frac{1}{\varepsilon}\). Thus \(f_2 \in L^b(A), f_1 \in L^b(\Sigma)\) and \(f = f_1 + f_2 \in L^b(\Sigma)\). Direct computation shows that \(T(f) - \lambda f = g\). This implies that \(T - \lambda I\) is invertible and so \(\lambda \notin \sigma(T)\).

Conversely, let \(\lambda \notin \sigma(T)\). Define linear transformation \(S\) on \(L^b(\Sigma)\) as follows

\[
Sf = \frac{Tf - f(E(u) - \lambda)}{\lambda (E(u) - \lambda)}, \quad f \in L^b(\Sigma).
\]

If \(\lambda \notin essrange(E(u))\), then \(\left\|\frac{1}{E(u) - \lambda}\right\|_\infty \leq \frac{1}{\varepsilon}\) for some \(\varepsilon > 0\). So

\[
N_\Phi(Sf) \leq N_\Phi\left(\frac{Tf}{\lambda (E(u) - \lambda)}\right) + N_\Phi\left(\frac{f}{\lambda}\right)
\]

\[
\leq \left(\frac{\|T\|}{\lambda \varepsilon} + \frac{1}{\varepsilon}\right)N_\Phi(f).
\]

Thus \(S\) is bounded \(L^b(\Sigma)\). If \(S\) is bounded on \(L^b(\Sigma)\), then for \(f \in L^b(A)\) \(Sf = \alpha f = M_\alpha f\), where \(\alpha = \frac{1}{\lambda \varepsilon f}\). Thus multiplication operator \(M_\alpha\) is bounded on \(L^b(A)\). This implies that \(\alpha \in L_\infty(A)\) and so there exist some \(\varepsilon > 0\) such that \(E(u) - \lambda = \frac{1}{\alpha} \geq \varepsilon\ a.e\). This mean’s that \(\lambda \notin essrange(E(u))\). Also, we have

\[
S \circ (T - \lambda I) = (T - \lambda I) \circ S = I.
\]

Thus \((T - \lambda I)^{-1} = S\) and so \(\sigma(T) = essrange(E(u)) \cup \{0\}\).

Let \(\mathfrak{B}\) be a Banach space and \(K\) be the set of all compact operators on \(\mathfrak{B}\). For \(T \in L(\mathfrak{B})\), the Banach algebra of all bounded linear operators on \(\mathfrak{B}\) into itself, the essential norm of \(T\) means the distance from \(T\) to \(K\) in the operator norm, namely

\[
\|T\|_e = \inf\{\|T - S\| : S \in K\}.
\]

Clearly, \(T\) is compact if and only if \(\|T\|_e = 0\).
Let $X$ and $Y$ be reflexive Banach spaces and $T \in L(X,Y)$. It is easy to see that $\|T\|_e = \|T^*\|_e$. As is seen in [20], the essential norm plays an interesting role in the compact problem of concrete operators.

In the sequel we present an upper bound for essential norm of $EM_u$ on Orlicz space $L^\Phi(\Sigma)$.

**Theorem 2.6.** Let $EM_u : L^\Phi(\Omega, \Sigma, \mu) \to L^\Phi(\Omega, \Sigma, \mu)$ is bounded and $(E, \Phi)$ satisfies in Generalized conditional-type Holder inequality. Let $\beta = \inf \{ \varepsilon > 0 : N_\varepsilon \text{ consists of finitely many } A\text{-atoms} \}$, where $N_\varepsilon = N_\varepsilon(E(\phi(u)))$. Then

$$\|EM_u\|_e \leq \beta.$$ 

**Proof** Let $\varepsilon > 0$. Then $N_{\varepsilon + \beta}$ consist of finitely many $A$-atoms. Put $u_{\varepsilon + \beta} = u\chi_{N_{\varepsilon + \beta}}$ and $EM_{u_{\varepsilon + \beta}}$ is finite rank and so compact. By the same method that is used in theorem 2.3(c) we have

$$\|EM_u\|_e \leq \|EM_u - EM_{u_{\varepsilon + \beta}}\| \leq \beta + \varepsilon.$$ 

This implies that $\|EM_u\|_e \leq \beta$.

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