Hadronic light-by-light contribution to the muon $g - 2$

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Abstract. We have computed the hadronic light-by-light (LbL) contribution to the muon anomalous magnetic moment $a_\mu$ in the frame of Chiral Perturbation Theory with the inclusion of the lightest resonance multiplets as dynamical fields ($R\chi T$). It is essential to give a more accurate prediction of this hadronic contribution due to the future projects of J-Parc and FNAL on reducing the uncertainty in this observable. We, therefore, computed the pseudoscalar transition form factor and proposed the measurement of the $e^+e^- \rightarrow \mu^+\mu^-\pi^0$ cross section and dimuon invariant mass spectrum to determine more accurately its parameters. Then, we evaluated the pion exchange contribution to $a_\mu$, obtaining $(6.66 \pm 0.21) \cdot 10^{-10}$. By comparing the pion exchange contribution and the pion-pole approximation to the corresponding transition form factor ($\pi TFF$) we recalled that the latter underestimates the complete $\pi TFF$ by (15-20)\%.

Then, we obtained the $\eta(\prime)$ TFF, obtaining a total contribution of the lightest pseudoscalar exchanges of $(10.47 \pm 0.54) \cdot 10^{-10}$, in agreement with previous results and with smaller error.

1. Introduction
Ever since the measurement of the electron magnetic moment in the splitting of the ground states of deuterium and molecular hydrogen [1], the anomalous magnetic moment has been an ever more stringent test of the underlying theory governing the interactions among elemental particles; giving us the lead from a way of renormalizing QED [2] to an outstanding confirmation of QFT with QED contributions [3] up to order $(\frac{\alpha}{\pi})^6$. In this spirit, the $a_\mu$ has been seen as a very stringent test of beyond standard model physics (BSM). With the most recent measurements [4], a deviation from the Standard Model (SM) results would imply a contribution from BSM with a scale $\sim 100$ TeV (assuming an interaction $\sim 1$). The current discrepancy [5] of 3.6\% and future plans on measuring more accurately this observable force theorists to make more precise predictions of SM contributions to the $\mu$ anomalous magnetic moment.

Within the SM, the contributions to the $a_\mu$ that have a greater uncertainty are the hadronic ones [5]. This is due to the fact that the underlying theory cannot be taken perturbatively in the whole energy range of the quark loop integrals, forcing theorists to compute these contributions using effective field theories (EFT) based on symmetries of Quantum Chromodynamics (QCD). This hadronic contribution can be splitted into two sub-contributions, the Hadron Vacuum Polarization (HVP) and the Hadronic Light-by-Light (HLbL), shown in Fig. 1. We analize the latter one by studying the $P\gamma^*\gamma^*$ interaction through its form factor (also called P transition form factor, PTFF), which gives the leading contribution to the HLbL through a pseudoscalar exchange diagram shown in Fig. 2. At low energies (i.e. in the chiral limit), the prediction for the $\pi TFF$ has been confirmed by the measured rate of $\pi^0 \rightarrow \gamma\gamma$ decays [5]. On the other hand,
the prediction for a nearly on-shell photon and one with very large virtuality seems to be at odds with measurements at B-factories [6, 7]. These two limits have ruled the way of constructing the form factor to describe interactions in the intermediate energy region, where hadronic degrees of freedom play a crucial role. The EFT we use to compute the TFF is Resonance Chiral Theory (RχT) [8, 9], which makes use of short-distance QCD predictions to obtain the parameters of the theory in terms of known constants. In this work, we fit one of the parameters in the πTFF with the B-factories data and, using this information, we predict the η(′)TFF and then obtain the contribution to the $a_\mu$ using these form factors.

2. Theoretical Framework
Chiral Perturbation Theory ($\chi$PT) [10] is the EFT dual to QCD at low energies [11]. It is based on an expansion in powers of momenta and masses of the lightest pseudoscalar mesons over the chiral symmetry breaking scale ($\Lambda \sim 1$ GeV). Thus, the theory fails to be reliable at energies around 1 GeV; furthermore, when other mesons become relevant degrees of freedom ($\Lambda \gtrsim M_\rho$) the theory is no longer applicable. A generalization of $\chi$PT is obtained using $1/N_C$ as an expansion parameter [12] to include resonances as dynamical degrees of freedom. The theory that incorporates these elements is Resonance Chiral Theory (RχT) [8, 9], which requires unitary symmetry for the resonance multiplets. No a priori assumptions are made with respect to the role of resonances in this theory, therefore one obtains naturally Vector Meson Dominance [13] as a dynamical result of the theory [8]. The final ingredient of the theory comes from QCD behavior at short distances, which constraints a great amount of free parameters in the theory.

3. The $\pi\gamma^*\gamma^*$ form factor in RχT
In this framework, the form factor we obtain [14] is
\[ F_{\pi \gamma \gamma^*}(p^2, q^2, r^2) = \frac{2r^2}{3F} \left[ -\frac{N_C}{8\pi^2 r^2} + 4F_V^2 \frac{d_3(p^2 + q^2)}{(M_V^2 - p^2)(M_V^2 - q^2)r^2} + \frac{4F_V^2d_{123}}{(M_V^2 - p^2)(M_V^2 - q^2)} \right] + \frac{16F_V^2P_3}{(M_V^2 - p^2)(M_V^2 - q^2)(M_V^2 - r^2)} - \frac{2\sqrt{2}}{M_V^2 - p^2} \left( \frac{F_V}{M_V} \frac{r^2c_{1235} - p^2c_{1256} + q^2c_{125}}{r^2} + \frac{8P_2F_V}{(M_V^2 - r^2)} \right) + (q^2 \leftrightarrow p^2). \] (1)

It contains contributions from the pseudoscalar resonances (as can be seen through the couplings \( P_2 \) and \( P_3 \)) which need to be taken into account to obtain consistent short distance constraints [9, 15]. All the parameters, but one, can be obtained through these constraints. \( P_3 \) cannot be obtained requiring high energy constraints, therefore it is fitted using the combined analyses of \( \pi(1300) \rightarrow \gamma \gamma \) and \( \pi(1300) \rightarrow \rho \gamma \) decays as given in references [9, 14]. The consistent short distance constraints on the resonance couplings in the odd-intrinsic parity sector can be seen in refs [14, 15]. Thus, we obtain

\[ P_3 = (-1.2 \pm 0.3) \cdot 10^{-2} \text{ GeV}^2. \] (2)

Figure 3. Our best fit compared to CELLO, CLEO, BaBar and Belle data for the \( \pi \)TFF.

On the other hand, the \( \pi \)TFF does not fit very well experimental data [6, 7] when \( P_2 \) is constrained by the short distance prediction. Therefore we allowed for it a small variation in a fit to BaBar and Belle data of this form factor, where they measure the \( \pi \)TFF spectrum in a kinematical configuration that ensures that one of the photons is on-shell and the other is virtual. The form factor for such a configuration is given by taking \( ^1 p^2 \rightarrow 0 \) and \( Q^2 = -q^2 \) in eq. (1)

\[ F_{\pi \gamma \gamma^*}(Q^2) = -\frac{F Q^2(1 + 32\sqrt{2}P_2F_V)}{3} \frac{2}{M_V^4(M_V^2 + Q^2)}. \] (3)

We keep a very conservative 10\% uncertainty from the asymptotic value of \( F_V \) around its predicted value [15] of \( \sqrt{3}F \). Fig. 3 shows our best fit, with which we obtain

\[ P_2 = (-1.13 \pm 0.12) \cdot 10^{-3} \text{ GeV}, \quad \chi^2/dof = 1.01. \] (4)

1 By the kinematical configuration in which the process is chosen to be measured, the momenta of both photons are space-like.
4. The pseudoscalar exchange contribution to the $a^{HLbL}_{\mu}$

Once all the parameters in the $\pi^{TFF}$ are determined, we insert the full off-shell TFF in the relations given in [16] obtaining thus

$$a^{\pi^{-0}LbL}_{\mu} = (5.75\pm0.06)\cdot10^{-10} \text{ on-shell } \pi^0$$

$$a^{\pi^{-0}LbL}_{\mu} = (6.66\pm0.21)\cdot10^{-10} \text{ whole } \pi^0 \text{TFF.} \quad (5)$$

Table 1. Our result compared with other results obtained through different methods.

| $a^{\pi^{-0}LbL}_{\mu} \cdot 10$ | Model and Reference |
|---------------------------------|---------------------|
| 5.58 ± 0.05                    | Extended Nambu-Jona-Lasinio [17] |
| 5.56 ± 0.01                    | Naive VMD [18] |
| 5.8 ± 1.0                      | Large $N_C$ with two vector multiplets $\pi$-pole [16] |
| 7.2 ± 1.2                      | $\pi$-exchange contribution [19] |
| 6.54 ± 0.25                    | Holographic models of QCD [20] |
| 6.58 ± 0.12                    | Lightest pseudoscalar and vector resonance saturation [9] |
| 6.49 ± 0.56                    | Rational approximants [21] |
| 5.0 ± 0.4                      | Non-local chiral quark model [22] |
| 5.75 ± 0.06                    | Our result with on-shell $\pi^0$ [14] |
| 6.66 ± 0.21                    | Our result whole $\pi^0 \text{TFF}$ [14] |

This clearly shows that assuming an on-shell pion in the $a^{HLbL}_{\mu}$ underestimates the contribution in $\sim 15\%$, and the error by a factor of 4. The uncertainty comes mainly from the error in $F_V$, $P_3$ and in a chiral correction from very-low energy physics. We compare our result with previous results in table 1. The form factor for the $\eta$ and $\eta'$ can be obtained with the $\pi \text{TFF}$ through eq. (6) with the minus sign for the case of the $\eta$.

$$F_{\eta^{(s)}\gamma^*\gamma^*} = \left( \frac{5}{3} C_q^{(t)} \mp \frac{\sqrt{2}}{3} C_s^{(t)} \right) F_{\pi^{0\gamma^*\gamma^*} \text{TFF.}} \quad (6)$$

With this, we can compute the whole pseudoscalar exchange contribution to the $a^{HLbL}_{\mu}$, shown in table 2 which includes other sub-leading HLbL contributions.
Table 2. Comparison of our contributions of the full $a_{\mu}^{HLbL}$ to previous determinations.

| $a_{\mu}^{HLbL} \cdot 10^{10}$ | Contributions |
|-------------------------------|---------------|
| 11.6 ± 4.0                   | F. Jegerlehner and A. Nyffeler [19] |
| 10.5 ± 2.6                   | Prades, De Rafael and Vainshtein² [23] |
| 11.8 ± 2.0                   | Our contribution [14] |

5. Genuine probe of $\pi TFF$

All the experimental observables available to fit the parameters in the $\pi TFF$ so far need an on shell photon and/or have photons with space-like momenta, while the HLbL contribution to the $a_{\mu}$ has both photons with time-like momenta. The photons in the process we study in this section, namely $\sigma(e^+e^- \rightarrow \gamma^{*} \rightarrow \pi^0 \gamma^{*} \rightarrow \mu^+ \mu^- \pi^0)$, both have time-like momenta and can be measured at very high photon virtualities by KLOE collab. for $q^2 \sim 1 \text{ GeV}^2$ and Belle-II collab. for $q^2 \sim 10.5 \text{ GeV}^2$. With the $\pi TFF$ parameters fully determined, the prediction we obtain is shown in Fig. 5

![Figure 5](image.png)

Figure 5. Our prediction for $\sigma(s)$ (left) and for $\frac{d\sigma}{ds_1}$ with $s = 1.02 \text{ GeV}^2$, the error bands cannot be appreciated in these plots.

6. Conclusion

We found the pseudoscalar exchange contribution to the $a_{\mu}^{HLbL}$ with a very competitive uncertainty and consistent with other theoretical models; improving the analysis by including high energy constraints not realized in the reference [9] and also using Belle data released after the reference was published. Our error estimate is also more robust, since in addition to the errors of the resonance couplings, we have also included the uncertainty due to the value of the $\pi TFF$ at very low energies.

We also obtained the first prediction for the cross section $\sigma(e^+e^- \rightarrow \mu^+ \mu^- \pi^0)$, which might be measured in KLOE-2 and Belle-II. The measurement of this observable would be an interesting way of trying to reduce the error in the parameters of the $\pi TFF$. This, may also help to reduce the uncertainty on the mixing parameters between the $\eta$ and $\eta'$ mesons.

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