Adaptive fixed-time synchronization of Lorenz systems with application in chaotic finance systems

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Received: 12 March 2022 / Accepted: 1 June 2022 / Published online: 21 June 2022
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Abstract This paper concerns the problem of fixed-time synchronization of master–slave Lorenz systems. The adaptive control and fixed-time control strategies are successfully integrated so that not only the Lorenz systems can be synchronized within a fixed-time, but also the related controlling gains are not necessary to select beforehand. Distinguished from the conventional fixed-time control schemes, the proposed controller do not contain the signum function anymore, thereby the chattering behavior is avoided owing to its smoothness. The synchronizing condition is deduced according to the theory of fixed-time stability, and the upper bound of settling time is also estimated, which is irrelevant to the initial states of Lorenz systems. Apart from the correctness and effectiveness of theoretical analysis is validated by simulating Lorenz systems, the spirit of adaptive fixed-time control strategy is applied into synchronizing two finance systems finally.

Keywords Chaotic system · Finance system · Synchronization · Adaptive control

1 Introduction

In 1963, the butterfly effect, synonymous with chaos, was firstly observed and explored by Lorenz [1], which was ruled by a group of ordinary differential equations (ODEs). Later on, lots of interesting chaotic phenomena were found in many fields, such as permanent magnet synchronous motor (PMSM) [2], finance system [3], brushless DC motors [4], and food web system (FWS) [5]. The prominent feature of chaos is that it is extremely sensitive to the initial conditions of the system, which makes it impossible to accurately predict the long-term behavior of chaotic system. And the chaos behavior is usually an unpopular phenomenon, which may result in the instability or desynchronization of systems. Therefore, the notable OGY-type [6,7], PID [8], adaptive [9], and other control protocols have been developed for stabilizing or synchronizing the chaotic systems in the early years.

In order to acquire a rapid convergence rate of chaotic system, the finite-time stability and synchronization problems have been discussed a lot in the past years. The sliding mode and adaptive techniques have been employed for guaranteeing the finite-time synchronization (FtS) between two distinct chaotic systems [10]. Chen et al. have derived the sufficient conditions for achieving the global FtS between Lorenz–Stenflo systems [11]. For different dimensional chaotic...
systems, Zhang et al. have also discussed the problem of global FtS [12]. Recently, Wang et al. have studied the FtS of memristor chaotic systems, and the obtained results have been applied into image encryption [13]. But there exists another problem that the upper bound time estimation relates with the initial state of the system [10,12,13]. That is, the final convergence time is not deterministic. Therefore, it is necessary to design a new control strategy for realizing FtS without relevance to the initial conditions. Fortunately, this drawback has been overcome by the theory of fixed-time stability [14]. In power systems, the technique of fixed-time terminal sliding mode control (SMC) has been employed to eliminate the chaos behavior [15]. Recently, the adaptive fixed-time control method has been used to synchronize the different dimensional chaotic systems [16], complex networks [17], and PMSMs [18].

For stabilizing or synchronizing the Lorenz systems, many effective control strategies have been proposed in the past years. Xie et al. have adopted the impulsive control to achieve the stabilization and synchronization of Lorenz systems [19]. Wang and Ge have employed the adaptive backstepping control for the Lorenz system with unknown parameters [20]. After that, the backstepping method has also been used for chaotic control of Lorenz system [21]. Kose and Mhrc have found that the SMC is superior to the adaptive pole placement control method for the Lorenz system [22]. For obtaining a fast convergence rate, the finite-time and fixed-time control techniques have been further implemented for chaotic Lorenz systems in recent years [23–25]. Besides, the complexity [26], high-dimension [27], and unstable periodic orbits analysis [28] of the generalized Lorenz systems have been discussed a lot.

However, the signum function\(^1\) is usually contained in the conventional finite-time [10–12,23] and fixed-time [15,17] control techniques, which may incur the troublesome chattering phenomenon and even damage the apparatus in industrial production due to its discontinuity. Therefore, it is necessary to strip the signum function out and pick a continuous and smooth fixed-time control protocol for achieving synchronization of chaotic systems. Besides, the adaptive control strategy is an effective approach to identify the uncertain parameters. In other words, the related controlling gains are not necessarily selected in advance, which can be adjusted by the system itself according to the designed adaptive laws.

Although the FxS and control of chaotic systems have been successfully applied into many fields, such as power systems [15], PMSMs [18], memristor chaotic circuits [29], FWS [30], and so forth, there are few results for the chaotic finance systems. Zhao et al. have studied the chaotic synchronization between the drive-response finance systems [31]. Then, the SMC [32] and passive control [33] have been also used to synchronize chaotic and hyperchaotic nonlinear finance systems. But the convergence time of [31–33] tends to infinity, namely asymptotical synchronization. Recently, the FtS of chaotic finance systems has been addressed with delayed feedback control [34], resilient fault-tolerant strategy [35], and impulsive control [36]. At the present, there is no any effort devoted to investigating the fixed-time stabilization or synchronization of chaotic finance systems yet.

To a certain extent, the chaotic financial system [3] is similar to Lorenz system [1]. On the one hand, both of them are described by the nonlinear ordinary differential system with the same dimension, namely 3-D. On the other hand, the strange attractors will emerge in both chaotic systems with similar periodic orbits, which looks like a butterfly. Therefore, the control scheme exerted in Lorenz system will also work on the chaotic finance system, namely compatible. The topic of FtS of chaotic finance systems has been discussed a lot recently [34–36], but all of them cannot get rid of the dependence of initial conditions. This problem is to be resolved by the forthcoming investigation, in which the adaptive fixed-time control strategy exerted in Lorenz system is successfully applied into synchronizing two chaotic finance systems.

Since the Lorenz system is a typical representative of multitudinous chaotic systems and it is tightly related with the chaotic financial model, this paper discusses the problem of FxS between two Lorenz systems and its application in finance systems. The main contributions are stated as follows:

\(^1\) This investigation successfully integrates the adaptive control and fixed-time control strategies so that not only the master–slave Lorenz systems can be synchronized in a fixed-time, but also the related controlling gains are not necessary to be chosen beforehand.
2 This paper picks a continuous and smooth fixed-time control protocol for achieving synchronization between two Lorenz systems, in which there is no signum function contained, so that the synchronization performance of chaotic systems is improved effectively;  

3 Based on the fixed-time stability theory, the synchronizing condition is deduced for guaranteeing the FxS of Lorenz systems, and the upper bound of settling time is estimated, which is irrelevant to the initial states of the chaotic systems;  

4 The obtained results have been effectively applied into synchronizing two chaotic finance systems, which provides a kind of guidance to ensure the stable operation of financial market.

The architecture of the rest investigation is organized as follows. Section 2 gives the preliminaries and problem description. The main results are obtained in Sect. 3, which includes the design of adaptive fixed-time control protocol and a theorem(corollary) for ensuring FxS(FxS) of master–slave Lorenz systems. The correctness and effectiveness of designed control technique for achieving FxS of Lorenz systems is demonstrated by simulation in Sect. 4. The spirit of adaptive fixed-time control strategy is applied into finance systems for realizing chaotic synchronization in Sect. 5. Section 6 draws some conclusions and points out potential directions.

2 Preliminaries and problem description

Consider the master Lorenz system [1]:  
\[
\begin{align*}
\dot{x}_1 &= -ax_1 + ax_2, \\
\dot{x}_2 &= bx_1 - x_2 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - cx_3,
\end{align*}
\]

where parameters \(a = 10, b = 28, c = \frac{8}{3}\), initial states \(x_1(0) = 2, x_2(0) = -10, x_3(0) = 10\). Then, an attractor is generated by the above master system as shown in Fig. 1a.

The slave Lorenz system with controllers \(u_i(t), i = 1, 2, 3\), is ruled as  
\[
\begin{align*}
\dot{y}_1 &= -ay_1 + ay_2 + u_1(t), \\
\dot{y}_2 &= by_1 - y_2 - y_1y_3 + u_2(t), \\
\dot{y}_3 &= y_1y_2 - cy_3 + u_3(t),
\end{align*}
\]

where parameters \(a = 10, b = 28, c = \frac{8}{3}\), initial states \(y_1(0) = -10, y_2(0) = 12, y_3(0) = 5\). The chaotic attractor is depicted in Fig. 1b. And the state errors between the master–slave systems (1–2) is defined as  
\[
\begin{align*}
e_i(t) &= y_i(t) - x_i(t), \quad i = 1, 2, 3.
\end{align*}
\]

Then, the synchronization error system is  
\[
\begin{align*}
\dot{e}_1(t) &= -ae_1 + ae_2 + u_1(t), \\
\dot{e}_2(t) &= be_1 - e_2 - y_1y_3 + x_1x_3 + u_2(t), \\
\dot{e}_3(t) &= y_1y_2 - x_1x_2 - ce_3 + u_3(t).
\end{align*}
\]

Fig. 1 The chaotic attractors generated by a master and b slave Lorenz systems, respectively, without any control.
Definition 1 The fixed-time chaos synchronization between the master–slave Lorenz systems (1–2) is achieved, if there exists a settling time function \( T > 0 \) which is irrelevant to initial values \( x_i(0), y_i(0) \), such that
\[
\lim_{t \to T} |e_i(t)| = \lim_{t \to T} |\gamma_i(t) - x_i(t)| = 0, \quad i = 1, 2, 3 \}
and \( e_i(0) = 0, \forall t \geq T \).

Lemma 1 [37] If \( x_1, x_2, \ldots, x_n \geq 0 \), then
\[
\sum_{i=1}^{n} x_i^{\eta} \leq \left( \sum_{i=1}^{n} x_i \right)^{\eta}, \quad 0 < \eta \leq 1,
\]
\[
\sum_{i=1}^{n} x_i^{\zeta} \geq n^{1-\zeta} \left( \sum_{i=1}^{n} x_i \right)^{\zeta}, \quad \zeta > 1.
\]

Lemma 2 [14] If there exists a continuous radially unbounded function \( V : R^n \to R_{+} \cup \{0\} \) such that
(1) \( V(x) = 0 \iff x = 0 \);
(2) Any solution \( x(t) \) satisfies the inequality
\[
D^* V(x(t)) \leq -\alpha V^p(x(t)) + \beta V^q(x(t))^k,
\]
for some parameters \( \alpha, \beta, p, q, k > 0 \), and \( pk < 1 \), \( qk > 1 \), where \( D^* V(x(t)) \) denotes the upper right-hand derivative of the function \( V(x(t)) \). Then, the origin is globally fixed time stable and the settling time bounded by
\[
T \leq \frac{1}{\alpha k(1-pk)} + \frac{1}{\beta k(qk-1)}.
\]

3 Main results

Theorem 1 The master–slave Lorenz systems (1–2) can achieve synchronization within the fixed-time \( T \leq \frac{m^2}{\lambda \mu (m-1)^2} \) via adopting the following smooth control protocol
\[
u_1(t) = -ae_1(t) - k_1 e_1^1(t) - k_1 e_1^2(t),
\]
\[
u_2(t) = -be_2(t) + y_1 y_3 - x_1 x_3 - k_2 e_2^1(t) - k_2 e_2^2(t),
\]
\[
u_3(t) = x_1 x_2 - y_1 y_2 - k_3 e_3^1(t) - k_3 e_3^2(t),
\]
in which the control gains \( k_i \) are adjusted complying with the adaptive law
\[
\dot{k}_i = (1-k_i^{-1}) e_i^{1+\frac{1}{m}}(t) + (1-k_i^{-1}) e_i^{1+\frac{1}{r}}(t)
\]
\[-k_i^{\frac{1}{m}} - k_i^{\frac{1}{r}}, \quad i = 1, 2, 3, \}
and the positive odd integers \( l, m, r, s \) satisfy \( l < m \) and \( r > s \), the initial conditions \( k_i(0) > 0 \), \( e_i(0) = y_i(0) - x_i(0) \).

Remark 1 One can find that the control gains \( k_1, k_2 \) in [13] and \( \lambda, \mu \) in [25] must be determined in advance, which will cause waste for the large values and slow convergence rate for small cases. This drawback can be overcome by employing the above adaptive feedback law (8). That is, it is not necessary to set the related control gain \( k_i \) in advance, which can be adjusted adaptively complying with the system state. Additionally, the above control scheme (7–8) does not contain the signum function. It is continuous and differentiable, namely smooth, which can eliminate the chattering phenomenon effectively.

Proof At this point, the synchronization error system is governed by
\[
\dot{e}_1(t) = -ae_1(t) - k_1 e_1^1(t) - k_1 e_1^2(t),
\]
\[
\dot{e}_2(t) = -e_2(t) - k_2 e_2^1(t) - k_2 e_2^2(t),
\]
\[
\dot{e}_3(t) = -e_3(t) - k_3 e_3^1(t) - k_3 e_3^2(t).
\]
Pick the Lyapunov function
\[
V(t) = \frac{1}{2} \sum_{i=1}^{3} e_i^2(t) + \frac{1}{2} \sum_{i=1}^{3} k_i^2.
\]
Differentiating the above Lyapunov function (10) along the synchronization error system (9) arrives at
\[
\dot{V}(t) = \sum_{i=1}^{3} e_i(t) \dot{e}_i(t) + \sum_{i=1}^{3} k_i \dot{k}_i
\]
\[
= e_1(t)[-ae_1(t) - k_1 e_1^1(t) - k_1 e_1^2(t)]
\]
\[
+ e_2(t)[-e_2(t) - k_2 e_2^1(t) - k_2 e_2^2(t)]
\]
\[
+ e_3(t)[-e_3(t) - k_3 e_3^1(t) - k_3 e_3^2(t)]
\]
\[
+ \sum_{i=1}^{3} k_i [(1-k_i^{-1}) e_i^{1+\frac{1}{m}}(t)]
\]
\[
+(1-k_i^{-1}) e_i^{1+\frac{1}{m}}(t) - k_i^{\frac{1}{m}} - k_i^{\frac{1}{r}}
\]
\[
= -ae_1^2(t) - e_2^2(t) - e_3^2(t) - \sum_{i=1}^{3} k_i^{1+\frac{1}{m}}
\]
\[
- \sum_{i=1}^{3} e_i^{1+\frac{1}{r}}(t) - \sum_{i=1}^{3} e_i^{1+\frac{1}{s}}(t)
\]
\[
\leq - \sum_{i=1}^{3} e_i^{1+\frac{1}{r}}(t) - \sum_{i=1}^{3} k_i^{1+\frac{1}{m}}
\]
\[
- \sum_{i=1}^{3} e_i^{1+\frac{1}{s}}(t) - \sum_{i=1}^{3} k_i^{1+\frac{1}{r}}.
\]
systems (1) and (2), which is essentially distinct from in Corollary 1. The finite-time chaos synchronization to be discussed in the light of Lemma 2, the FxS of master–slave Lorenz systems (1) and (2) can be achieved, and the settling time \( T \leq m \cdot 2^{\frac{m-1}{2m}} \cdot V(e_i(0)) \cdot \frac{m-1}{m} \) via adopting the following control scheme

\[
\begin{align*}
\dot{u}_1(t) &= -ae_2(t) - k_1 e_i^l(t), \\
\dot{u}_2(t) &= -be_1(t) + y_1 y_3 - x_1 x_3 - k_2 e_2^l(t), \\
\dot{u}_3(t) &= x_1 y_2 - y_1 y_3 - k_3 e_3^l(t),
\end{align*}
\]

and the unified control gains \( k_i \) are adaptively adjusted according to

\[
\dot{k}_i = (1 - k_i^{-1}) e_i^{1+m} - k_i^m, \quad i = 1, 2, 3,
\]

where the positive odd integers \( l, m \) satisfy \( l < m \), and the initial states \( k_i(0) > 0, e_i(0) = y_i(0) - x_i(0) \).

**Proof** From (5) and (12), now the synchronization error system is

\[
\begin{align*}
\dot{e}_1(t) &= -ae_1(t) - k_1 e_i^l(t), \\
\dot{e}_2(t) &= -e_2(t) - k_2 e_2^l(t), \\
\dot{e}_3(t) &= -ce_3(t) - k_3 e_3^l(t).
\end{align*}
\]

Considering the same Lyapunov function (10) and differentiating it along the synchronization error system (14) yields

\[
\dot{V}(t) = \sum_{i=1}^{3} e_i(t) \dot{e}_i(t) + \sum_{i=1}^{3} k_i \dot{k}_i
\]

In the light of Lemma 2, the FxS of master–slave Lorenz systems (1) and (2) can be achieved, and the settling time

\[
T \leq m \cdot 2^{\frac{m-1}{2m}} \cdot \frac{m-1}{m} \cdot \frac{2^{\frac{m-1}{2m}}}{r - s} \cdot \left(\frac{2}{3}\right)^{\frac{m-1}{2m}}. \quad (11)
\]

This completes the proof. \( \square \)

**Remark 2** Looking back to Theorem 1, it is easy to observe that the estimation of upper bound settling time (11) of FxS is independent of the initial states of chaotic systems (1) and (2), which is essentially distinct from the finite-time chaos synchronization to be discussed in Corollary 1.

If all of the last terms of (7) are not implemented due to lack of control spending or other factors, the Lorenz systems just can achieve FtS.

**Corollary 1** The master–slave Lorenz systems (1) and (2) can achieve chaotic synchronization within the finite-time \( T \leq m \cdot 2^{\frac{m-1}{2m}} \cdot V(e_i(0)) \cdot \frac{m-1}{m} \) via adopting the following control scheme

\[
\begin{align*}
\dot{u}_1(t) &= -ae_2(t) - k_1 e_i^l(t), \\
\dot{u}_2(t) &= -be_1(t) + y_1 y_3 - x_1 x_3 - k_2 e_2^l(t), \\
\dot{u}_3(t) &= x_1 y_2 - y_1 y_3 - k_3 e_3^l(t),
\end{align*}
\]

and the unified control gains \( k_i \) are adaptively adjusted according to

\[
\dot{k}_i = (1 - k_i^{-1}) e_i^{1+m} - k_i^m, \quad i = 1, 2, 3,
\]

where the positive odd integers \( l, m \) satisfy \( l < m \), and the initial states \( k_i(0) > 0, e_i(0) = y_i(0) - x_i(0) \).

4 Simulation results for Lorenz systems

This section verifies the correctness and effectiveness of above theoretical analysis. The initial values of the...
master–slave Lorenz systems are given in Sect. 2. For the proposed adaptive fixed-time control scheme \((7–8)\), we take the initial values \(k_i(0) = 0.01, i = 1, 2, 3\), and the odd integers \(l = 7, m = 9, r = 9, s = 7\).

The trajectories of state variables \(x_i, y_i, i = 1, 2, 3\), without and with implementing control strategy are displayed in Fig. 2a, b, respectively. The trajectories synchronization errors \(e_i, i = 1, 2, 3\), between the master–slave Lorenz systems \((1)-(2)\) are given in Fig. 3. Figure 2a depicts that the trajectories of Lorenz systems \((1)\) and \((2)\) are chaotic, both which cannot be synchronized by themselves. From Figs. 2b and 3, the chaos synchronization of Lorenz systems is achieved within the time \(t = 1\). The trajectories of control signals \(u_i, i = 1, 2, 3\), and the corresponding controlling gains \(k_i, i = 1, 2, 3\), are illustrated in Fig. 4a, b, respectively, in which we find that the control inputs will smoothly converge to zero and the gains will also adaptively approach to zero at last.

For further validating the effectiveness of the proposed control scheme \((7–8)\), the influence of external disturbance should be considered. We assume the system suffers from unknown disturbance at time \(t = 2\) (other time is also permissible), where the time evolution of error system is shown in Fig. 5 and control signals are illustrated in Fig. 6a with the gains in Fig. 6b. One can find that the system will be synchronized quickly. That is, the proposed controller is robust to external disturbance. Moreover, we select different initial conditions to study the performance of the proposed method. Six groups of initial values \(x_i(0), y_i(0)\) are generated from the interval \([-10, 10]\) randomly (other larger intervals are also permissible). And the trajectories of total errors \(\delta(t) = \|\mathbf{e}(t)\|\) are displayed in Fig. 7, where \(\mathbf{e}(t) = [e_1, e_2, e_3]^T\). From Fig. 7, one can observe that the master–slave Lorenz system will always achieve synchronization within \(t = 1\), which is irrelevant to initial conditions, namely fixed-time synchronization.

In addition, for uncovering the effect of exponents \(l/m\) and \(r/s\) on the convergence time, several different ratio values are tested, in which we take \(l/m = 1/9, 1/3, 5/9, 7/9\), and \(r/s = r/s = 9/7, 11/9, 13/11, 15/13\), respectively. And the total errors \(\delta(t) = \|\mathbf{e}(t)\|\) are displayed in Fig. 8, which
Fig. 4 Time evolution of a controllers $u_i, i = 1, 2, 3$, and b adaptive controlling gains $k_i, i = 1, 2, 3$

Fig. 5 Synchronization errors $e_i, i = 1, 2, 3$, with external disturbance

Fig. 6 Time evolution of a controllers $u_i, i = 1, 2, 3$, and b adaptive controlling gains $k_i, i = 1, 2, 3$, with external disturbance

shows that the convergence rate increases with the ratio $l/m(r/s)$ increases(decreases).

5 Application in nonlinear finance systems

This section applies the proposed adaptive fixed-time control technique into nonlinear chaotic finance systems for achieving synchronization. The master nonlinear finance system is ruled as follows [3]:

$$\begin{align*}
\dot{x}_1 &= z_1 + (y_1 - a)x_1, \\
\dot{y}_1 &= 1 - by_1 - x_1^2, \\
\dot{z}_1 &= -x_1 - cz_1,
\end{align*}$$

where the variables $x_1, y_1, z_1$ are the interest rate, investment demand, and (normalized) price index,
exhibits chaotic behavior as illustrated in Fig. 9a.

The time evolution of state variables and parameters are the same as those in master system (16), but with different initial states $x_2(0) = -1$, $y_2(0) = 1.7$, $z_2(0) = 0.5$, $u_i(t), i = 1, 2, 3$, are controllers to be designed. The drivers are adaptively adjusted complying with

$\dot{k}_i = (1 - k_i^{-1})e_i^{1+\frac{i}{m}}(t) + (1 - k_i^{-1})e_i^{1+\frac{i}{r}}(t)$

$-k_i^m - k_i^r$, $i = 1, 2, 3$. (21)

By using the same Lyapunov function (10), we omit the similar proof here for simplicity.

The smooth control scheme is designed as

$\left\{ \begin{array}{ll}
  u_1(t) &= x_1 y_1 - x_2 y_2 - k_1 e_1^m(t) - k_1 e_1^r(t), \\
  u_2(t) &= x_2^2 - x_1^2 - k_2 e_2^m(t) - k_2 e_2^r(t), \\
  u_3(t) &= -k_3 e_3^m(t) - k_3 e_3^r(t),
\end{array} \right.$ (20)

in which the control gains $k_i$ are adaptively adjusted complying with

$\dot{k}_i = (1 - k_i^{-1})e_i^{1+\frac{i}{m}}(t) + (1 - k_i^{-1})e_i^{1+\frac{i}{r}}(t)$

$-k_i^m - k_i^r$, $i = 1, 2, 3$. (21)

Between the master–slave finance systems, we define the synchronization error

$\left\{ \begin{array}{ll}
  e_1 &= x_2 - x_1, \\
  e_2 &= y_2 - y_1, \\
  e_3 &= z_2 - z_1.
\end{array} \right.$ (18)

Then, the dynamics of error system is governed by

$\left\{ \begin{array}{ll}
  \dot{e}_1 &= e_3 - ae_1 + x_2 y_2 - x_1 y_1 + u_1(t), \\
  \dot{e}_2 &= -be_2 - x_2^2 + x_1^2 + u_2(t), \\
  \dot{e}_3 &= -2\dot{z}_1 + u_3(t) = -e_1 - ce_3 + u_3(t).
\end{array} \right.$ (19)

The smooth control scheme is designed as

Between the master–slave finance systems, we define the synchronization error

$\left\{ \begin{array}{ll}
  e_1 &= x_2 - x_1, \\
  e_2 &= y_2 - y_1, \\
  e_3 &= z_2 - z_1.
\end{array} \right.$ (18)

Then, the dynamics of error system is governed by

$\left\{ \begin{array}{ll}
  \dot{e}_1 &= e_3 - ae_1 + x_2 y_2 - x_1 y_1 + u_1(t), \\
  \dot{e}_2 &= -be_2 - x_2^2 + x_1^2 + u_2(t), \\
  \dot{e}_3 &= -2\dot{z}_1 + u_3(t) = -e_1 - ce_3 + u_3(t).
\end{array} \right.$ (19)

The smooth control scheme is designed as

$\left\{ \begin{array}{ll}
  u_1(t) &= x_1 y_1 - x_2 y_2 - k_1 e_1^m(t) - k_1 e_1^r(t), \\
  u_2(t) &= x_2^2 - x_1^2 - k_2 e_2^m(t) - k_2 e_2^r(t), \\
  u_3(t) &= -k_3 e_3^m(t) - k_3 e_3^r(t),
\end{array} \right.$ (20)

in which the control gains $k_i$ are adaptively adjusted complying with

$\dot{k}_i = (1 - k_i^{-1})e_i^{1+\frac{i}{m}}(t) + (1 - k_i^{-1})e_i^{1+\frac{i}{r}}(t)$

$-k_i^m - k_i^r$, $i = 1, 2, 3$. (21)

By using the same Lyapunov function (10), we omit the similar proof here for simplicity.

The time evolution of state variables $x_i, y_i, z_i, i = 1, 2$, without and with employing control strategy are plotted in Fig. 10a, b, respectively. The trajectories synchronization errors $e_i, i = 1, 2, 3$, between the master–slave finance systems (16–17) are given in Fig. 11. Figure. 10a shows that the trajectories of finance systems
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(a) Fig. 9 The chaotic attractors generated by a master and b slave finance systems (16) and (17), respectively, without any control

(b) Fig. 10 Time evolution of state variables $x_i, y_i, z_i, i = 1, 2$ a without and b with implementing control strategy

(16) and (17) are chaotic, both which cannot be synchronized spontaneously. From Figs. 10b and 11, the chaotic synchronization of finance systems is achieved within the time $t = 3$.

In this application example, we set the initial values $k_i(0) = 0.01, i = 1, 2, 3$, and $l = 7, m = 9, r = 9, s = 7$ for the adaptive fixed-time control scheme (20–21). The time trajectories of control signals $u_i, i = 1, 2, 3$, and the corresponding controlling gains $k_i, i = 1, 2, 3$ are displayed in Fig. 12a, b, respectively, in which we observe that the control inputs will also smoothly converge to zero and the gains will also adaptively approach to zero eventually.

Besides the control technique has been implemented at the beginning, it is also can be practiced at different times. Now, we do not employ the proposed control strategy
Fig. 12 The trajectories of \( a \) controllers \( u_i, i = 1, 2, 3 \), and \( b \) adaptive controlling gains \( k_i, i = 1, 2, 3 \)

Fig. 13 The trajectories of \( a \) state variables \( x_i, y_i, z_i, i = 1, 2 \), and \( b \) synchronization errors \( e_i, i = 1, 2, 3 \), between master and slave nonlinear finance systems (16–17)

protocol (20–21) until \( t = 5 \) (other parameters and initial values are unchanged). Figure 13a, b depicts the trajectories of state variables \( x_i, y_i, z_i, i = 1, 2 \), and synchronization errors \( e_i, i = 1, 2, 3 \), respectively, which shows that the master and slave chaotic finance systems (16) and (17) can also be synchronized within \( t = 20 \). The control signals \( u_i \) and controlling gains \( k_i, i = 1, 2, 3 \), are displayed in Fig. 14a, b accordingly, in which we observe that the control inputs \( u_i \) and the controlling gains \( k_i \) will also tend to zero when the chaotic synchronization of finance systems is realized.

6 Conclusions

This paper has discussed the problem of adaptive FxS of master–slave Lorenz systems with application in finance systems. The designed adaptive fixed-time control method is well-performed, which not only can synchronize two Lorenz systems in a fixed-time, but also the related controlling gains can be adjusted complying with the updated laws. Moreover, the adaptive fixed-time control protocol is continuous and smooth, which strips out the signum function from the conventional fixed-time control schemes so that the chattering phenomenon is eliminated successfully. The synchronizing condition has been obtained in the light of the theory of
Adaptive fixed-time synchronization of Lorenz systems has been given, which is irrelevant to the initial states of Lorenz systems. On the one hand, the simulation is tested for the master–slave Lorenz systems, which has validated the correctness of designed control scheme. On the other hand, the adaptive fixed-time control strategy has been effectively applied to the finance systems.

Besides the finance systems, the proposed adaptive fixed-time control technique can be applied into image encryption, PMSMs, FWS, and so on. But the problem existed in the FxS of Lorenz(or finance) systems is that the estimation of convergence time is conservative and not explicit. This drawback will be overcome by exploring the predefined-time synchronization of chaotic systems [39], in which the upper limit of the predefined-time is a real minimum.

Acknowledgements This work was supported in part by the Fundamental Research Funds of Suzhou University of Science and Technology [No.332114604], National Natural Science Foundation of China [No.62103166], and Jiangsu Shuangchuang (Innovation and Entrepreneurship) Talent Program [No.JSSCBS20210840].

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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