ζ-function renormalization of one-loop stress tensors in curved spacetime, a check on the method in the conical manifold and other cases.

Valter Moretti

European Centre for Theoretical Nuclear Physics and Related Areas
Villa Tambosi, Strada delle Tabarelle 286 I-38050 Villazzano (TN), Italy
and
Dipartimento di Fisica, Università di Trento, Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Trento, via Sommarive 14 38050 Povo (TN), Italy

June 1997

Abstract:
A previously introduced method to renormalize the one-loop stress tensor and the one-loop vacuum fluctuations in a curved background by a direct local ζ-function approach is checked in some thermal and nonthermal cases.

First the method is checked in the case of a conformally coupled massless field in the static Einstein universe where all hypotheses initially requested by the method hold true.

Secondly, dropping the hypothesis of a closed manifold, the method is checked in the open static Einstein universe. Finally, the method is checked for a massless scalar field in the presence of a conical singularity in the Euclidean manifold (i.e. Rindler spacetimes/large mass black hole manifold/cosmic string manifold). In all cases, a complete agreement with other approaches is found.

Concerning the last case in particular, the method is proven to give rise to the stress tensor already got by the point-splitting approach for every coupling with the curvature regardless of the presence of the singular curvature. In the last case, comments on the measure employed in the path integral, the use of the optical manifold and the different approaches to renormalize the Hamiltonian are made.

PACS number(s): 04.62.+v

1e-mail: moretti@science.unitn.it
Introduction

In [1] we have introduced a direct method to compute the one-loop Euclidean renormalized stress tensor through a $\zeta$ function approach. The considered stress tensor is that obtained by varying the one-loop effective action $S_{\text{eff}}[g_{ab}]$ with respect to the background metric $g_{ab}$. Let us summarize the principal feature of that method very briefly.

In a closed (namely, compact without boundary) Euclidean manifold, the one-loop renormalized stress tensor of a scalar field $\phi(x)$ of a $\zeta$-regular theory\footnote{That is a QFT theory involving meromorphic $x$-smooth local $\zeta$ functions obtained by analytic continuation of corresponding series, such that the function $\zeta(s,x|A)$ is analytic in $s=0$ and $\zeta_{ab}(s,x|A)$ takes a simple pole at $s=1$.} (see [1]) is obtained as

$$
\langle T_{ab}(x) \rangle = \left\{ \zeta_{ab}(s+1,x|A) + \frac{1}{2} g_{ab}(x) \zeta(s,x|A) + 
+ s \left[ \zeta_{ab}(s+1,x|A) + \ln(\mu^2)\zeta_{ab}(s+1,x|A) \right] \right\}_{s=0}.
$$

(1)

$\zeta(s,x|A)$ is the usual local $\zeta$ function of the Euclidean second order motion operator $A$, namely, the motion equations are given by $A\phi = 0$. The tensorial $\zeta$ function $\zeta_{ab}(s,x|A)$ is obtained through the $s$ analytic continuation in the whole complex plane of the series (where from now on ‘ indicates that possible null eigenvalues are omitted)

$$
\zeta_{ab}(s,x|A) := \sum_n \lambda_n^{-s} T_{ab}[\phi_n^*, \phi_n](x),
$$

(2)

$T_{ab}[\phi_n^*, \phi_n](x)$ being the stress tensor evaluated on the normal modes of $A$.

We have seen that the definition above is equivalent to the simpler one

$$
\langle T_{ab}(x) \rangle = \zeta_{ab}(1,x|A) + \frac{1}{2} g_{ab}(x) \zeta(0,x|A),
$$

(3)

which can be used in the case of a super $\zeta$-regular theory, namely, when the function $\zeta_{ab}(s,x|A)$ is analytic also in $s=1$ where a simple pole appears in general. Notice that, in the considered case, the renormalization scale $\mu$ as well as the finite renormalization counterterms ([1]) disappear from the final result. We have seen that such $\mu$-dependent counterterms, whenever they exist, have a simple form related to the heat kernel coefficients, are conserved and depend locally on the geometry: they represent a finite renormalization of the coefficients which stay on the geometrical side of customary generalized Einstein’s equations.

We remark that the formulae given above arise by an opportune interpretation of the formal identity

$$
\langle T_{ab}(x) \rangle = -\frac{2}{g^{1/2}} \frac{\delta S_{\text{eff}}[g]}{\delta g^{ab}(x)}
$$

when we have computed the effective action through the $\zeta$ function

$$
S_{\text{eff}}[g_{ab}] = \frac{1}{2} \frac{d}{ds}|_{s=0} \zeta(s|A/\mu^2),
$$

(4)

$\frac{1}{2} \frac{d}{ds}|_{s=0} \zeta(s|A/\mu^2)$,
where \( \zeta(s|A) = \int d^4x \sqrt{g(x)} \zeta(s, x|A). \)

In [1], we have seen that, dealing with a scalar field (in general massive) coupled with the curvature through the parameter \( \xi \) as usually and propagating into an Euclidean compact without boundary manifold, the function \( \zeta_{ab}(s, x|A) \) is meromorphic and its poles are related to the heat kernel coefficients in a simple fashion. Hence, we have a \( \zeta \)-regular theory.

We have also proven that the definition of one-loop renormalized stress tensor given in (1), (3) produces a conserved stress tensor provided the manifold were closed and the theory \( \zeta \)-regular. Actually, one can simply prove that the hypotheses of a compact without boundary manifold is not necessary. Indeed, on noncompact manifolds, one can assume (1) by definition considering spectral measure and integrations instead of summations. Thus, by means of the same proof of the stress-tensor conservation used in [1] with a few changes, one gets that, if the involved \( \zeta \) functions are correctly meromorphic defining a \( \zeta \)-regular theory, the left hand side of (1) (or (3)) will be automatically conserved. Similarly, in the same hypotheses, one finds the anomalous trace of a massless conformally coupled field in terms of the local \( \zeta \) function evaluated at \( s = 0 \) [1] (i.e. the heat kernel expansion coefficient \( a_2(x|A) \), provided the local heat kernel expansion holds true in the considered case).

In [1] we have also considered identities as

\[
- \frac{\partial \ln Z_\beta}{\partial \beta} = - \int_\Sigma d\bar{x} \sqrt{-g_L} \langle T^0_0(\bar{x}) \rangle_\beta
\]

and

\[
- \frac{\partial \ln Z_\beta}{\partial L_i} = \frac{\beta}{L_i} \int_\Sigma d\bar{x} \sqrt{-g_L} \langle T^i_i(\bar{x}) \rangle_\beta,
\]

where we defined \( \ln Z_\beta := S_{\text{eff}} \), provided the (Euclidean and Lorentzian) manifold admit a global (Lorentzian time-like) Killing vector arising from the Euclidean temporal coordinate with a period \( \beta = 1/T \). \( \bar{x} \) represents the spatial coordinates which belong to the spatial section \( \Sigma \) and \( g_L = -g \) is the determinant of the Lorentzian metric. Obviously, \( Z_\beta \) has to be interpreted as a partition function. Notice that all quantities which appear in the formula above do not depend on the Euclidean or Lorentzian time because the manifold is stationary and thus no time dependence arises from the metric. Furthermore the analytic continuation to the Lorentzian time yields \( \langle T^0_0(\bar{x}) \rangle = \langle T^0_0(\bar{x}) \rangle \) and \( \langle T^i_i(\bar{x}) \rangle = \langle T^i_i(\bar{x}) \rangle \). Eq.(1) holds assuming both the homogeneity along \( x^0 \) and \( x^i \), \( L \) being the “period” of the manifold along \( x^i \).

We have proved both identities above requiring explicitly a compact (finite) spatial section without boundary of the manifold. It is not obvious whether or not such identities hold dropping that hypothesis and introducing some cut-off in both sides of the considered equations in order to handle finite quantities.

A further subtlety which arises dropping our simple hypotheses is related to the ambiguity in

\(^3\)The presence of a boundary involves well-known changes in the heat kernel expansion we shall not consider here.

\(^4\)Throughout the paper we shall employ the signature \((1,1,1,1)\) for the Euclidean metric and \((-1,1,1,1)\) for the Lorentzian one.
defining the right hand side of the couple of identities above. Indeed, let us consider the first identity above for example. From the statistical mechanics, one expects to find, in the right hand side, the averaged Hamiltonian instead of the spatial integral of the averaged $T_{00}$. Actually, the latter differs from the former just by total spatial derivative of objects (built up by employing the field fluctuations $\langle \phi^2(x) \rangle_\beta$) which vanish after one integrates provided the spatial manifold have no boundary or the field does vanish on this, whenever it exists. This situation could change dramatically in the presence of boundaries or infinite spatial sections and, in general, has to be analyzed case by case.

In \cite{2} D. Iellici and the author of the present paper introduced also a method to renormalize the one-loop field fluctuations $\langle \phi^2(x) \rangle$ in a curved spacetime. Also that method is based on a local $\zeta$ function and, in principle, holds when the involved Euclidean manifold is closed. Nevertheless we expect that it holds in a more general case.

We have checked the method in several manifolds obtaining reasonable results.

The field fluctuations are regularized as

$$\langle \phi^2(x) \rangle = \frac{d}{ds}|_{s=0} \frac{s}{\mu^2} \zeta(s + 1, x|A/\mu^2).$$  \hspace{1cm} (7)

Notice that, if the local $\zeta$ function of the effective action $\zeta(s + 1, x|A/\mu^2)$ is analytic at $s = 1$, the formula above reduces to the more usual expression also expected from the Green function analysis

$$\langle \phi^2(x) \rangle = \zeta(1, x|A).$$  \hspace{1cm} (8)

Notice the disappearance of the scale $\mu$.

The renormalized field fluctuations play an important role also in the relationship between the renormalized stress tensor and the renormalized Hamiltonian \cite{3, 1} as stressed above. We shall come back on this point in the final discussion.

In this paper we shall analyze our method to renormalize the stress tensor and field fluctuations in three different situations. First we shall consider the (thermal) theory of a conformally coupled massless scalar field within the closed Einstein universe. The Euclidean related manifold satisfies completely our initial hypotheses of a closed manifold.

Secondly, we shall consider the same field propagating in the open Einstein universe. The related Euclidean manifold is not compact and this is a first nontrivial ground where check our approach assumed by definition.

The third case we shall consider is the Euclidean manifold related both to the cosmic string manifold and Rindler space (which can be considered also as the manifold containing a very large mass black hole). That Euclidean manifold is not ultrastatic differently from the two manifolds considered above, moreover, it has a conical singularity which, for some aspects, could be considered as a boundary. That singularity involves a lot of difficulties dealing with $\zeta$ function approaches to renormalize the effective action. In particular, stress tensor components built up through the local $\zeta$ function of the effective action in the physical manifold have been obtained.
making direct use of mechanical-statistical laws or supposing \textit{a priori} a particular form of the stress tensor. These results disagree, at law energies, with those obtained by the point-splitting method (see Section II of [4] and the final discussion of [4] for a discussion and references on these topics). In this paper we shall see that, as for it concerns the stress tensor in the conical manifold, it is possible to get the same results arising also from the point-splitting approach, for every value of the coupling parameter $\xi$ by means of our local $\zeta$ function approach. This result will be carried out not depending on the mechanical-statistical laws and without supposing any particular form of the stress tensor \textit{a priori}. In the final discussion we shall perform some remark on the problem of the choice of the configuration-space measure in the path integral to define the partition function of the fields. This problem becomes important as far as the physics in the conical manifold is concerned. We shall see that these problems are related to the renormalization procedure involved in defining physical quantities, concerning the Hamiltonian in particular. These problems have become important after the interesting works [5, 3, 6] where the renormalized value of the vacuum fluctuations plays a central role in explaining the finite Bekenstein-Hawking entropy in the approach of the induced gravity.

1 Einstein’s closed static universe

The ultrastatic metric of the (Euclidean) Einstein closed static universe is \[ ds^2_{ECS} = d\theta^2 + g_{ij} dx^i dx^j = d\theta^2 + a^2 \left(dX^2 + \sin^2 X d\Omega^2_2\right). \]

$X$ ranges from 0 to $\pi$ and $d\Omega^2_2$ is the usual metric on $S^2$. The time coordinate $\theta$ ranges from 0 to $\beta \leq +\infty$. $\beta$ is the inverse temperature of the considered thermal state referred to the Killing vector generated by the Lorentzian time $i\theta$. The related vacuum state corresponds to the choice $\beta = +\infty$. The curvature of the space is $R = 6/a^2$ and the Ricci tensor reads $R_{ij} = 2g_{ij}/a^2$, the remaining components vanish.

This manifold is closed, namely compact without boundary. Also the spatial section at $\theta$ = constant are closed and their volume is $V = 2\pi^2a^3$.

Let us consider a conformally coupled massless scalar field propagating within this manifold. We want to compute its stress tensor referred to the thermal states pointed out above, in particular we want to get the vacuum stress tensor which is known in literature [7]. Notice that all the hypotheses required in [1] to implement the stress-tensor $\zeta$-function approach are fulfilled.

Let us build up the function $\zeta_{ab}(s,x|A)$ necessary to get $\langle T_{ab}(x)\rangle_\beta$ through (1) or (3). The general expression of $\zeta_{ab}(s,x|A)$ is [1]

\[
\zeta_{ab}(s,x|A) = \bar{\zeta}_{ab}(s,x|A) - \xi \nabla_a \nabla_b \zeta(s,x|A) + \left(\xi - \frac{1}{4}\right) g_{ab}(x) \Delta \zeta(s,x|A) \\
+ \xi R_{ab}(x) \zeta(s,x|A) - \frac{1}{2} g_{ab}(x) \zeta(s-1,x|A),
\]

where, in the sense of the analytic continuation of both sides in the whole $s$ complex plane:

\[
\bar{\zeta}_{ab}(s,x|A) = \sum_k \lambda_k^{-s} \nabla_a \phi_k^*(x) \nabla_b \phi_k(x).
\]
We are interested in the case $\xi = \xi_c := 1/6$ (conformal coupling in four dimensions). The local $\zeta$ function is similarly given by

$$\zeta(s, x|A) = \sum_k' \lambda_k^{-s} \phi_k^*(x) \phi_k(x).$$  \hspace{1cm} (11)$$

The functions $\phi_k(x)$ define a normalized complete set of eigenvectors of the Euclidean motion operator:

$$A \phi_k = \lambda_k \phi_k,$$

where, in our case

$$A = -\partial_0^2 - a^{-2} \Delta_{S^3} + \xi_c R,$$

The explicit form of the considered eigenvalues and Kronecker’s delta-normalized eigenvectors is well-known \cite{7}. In particular we have $k \equiv (n, q, l, m)$ where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, $q = 1, 2, 3, \ldots$, $l = 0, 1, 2, \ldots, q - 1$, $m = 0, \pm 1, \pm 2, \ldots, \pm l$ and

$$\lambda_k = \left( \frac{2\pi n}{\beta} \right)^2 + \left( \frac{q}{a} \right)^2.$$  \hspace{1cm} (12)$$

The following relations, which hold true for normalized eigenvectors, are also useful. We leave the proofs of these to the reader.

$$\sum_{lm} \phi_k^*(x) \phi_k(x) = \frac{q^2}{V \beta};$$  \hspace{1cm} (13)$$

notice that the right hand side of the equation above is nothing but the degeneracy of each eigenspace times $1/2 \beta V$ (or $1/\beta V$ when $n = 0$);

$$\sum_{lm} \partial_i \phi_k^*(x) \partial_j \phi_k(x) = g_{ij}(x) \frac{q^2(q^2 - 1)}{3V \beta a^2},$$  \hspace{1cm} (14)$$

and $(x^0 := \theta)$

$$\sum_{lm} \partial_0 \phi_k^*(x) \partial_0 \phi_k(x) = \frac{(2\pi n q)^2}{V \beta^3}.$$  \hspace{1cm} (15)$$

We have also, because of the homogeneity of the space

$$\zeta(s, x|A) = \frac{\zeta(s|A)}{V \beta},$$  \hspace{1cm} (16)$$

where $\zeta(s|A)$ is the global $\zeta$ function obtained by summing over $\lambda_k^{-s}$ as usually

$$\zeta(s|A) = \sum_k' \lambda_k^{-s}.$$  \hspace{1cm} (17)
It is possible to relate the function \( \zeta_{ab}(s, x|A) \) to the function \( \zeta(s, x|A) \). Indeed, we notice that

\[
\lambda_k^{-s} \left( \frac{2\pi n}{\beta} \right)^2 = \frac{\beta}{2(s-1)} \frac{\partial \lambda_k^{-(s-1)}}{\partial \beta}.
\]

The identity above inserted into the definition (10) for \( a = b = 0 \), taking (15) into account, yields

\[
\bar{\zeta}_{00}(s+1, x|A) = \frac{1}{2V} \frac{\partial}{\partial \beta} \zeta(s|A),
\]

or equivalently

\[
\zeta_{00}(s+1, x|A) = \bar{\zeta}_{00}(s+1, x|A) = -\frac{a}{2V} \frac{\partial}{\partial a} \zeta(s|A),
\]

which follows from the identity above taking account of

\[
2s\zeta(s|A) = \beta \frac{\partial}{\partial \beta} \zeta(s|A) + a \frac{\partial}{\partial a} \zeta(s|A).
\]

The last identity is a simple consequence of the expression of the eigenvalues (12).

Concerning the components \( ij \) (the remaining components vanish) we can take advantage from the identity

\[
\lambda_k^{-s} q^2 = \frac{3a^3}{2(s-1)} \frac{\partial \lambda_k^{-(s-1)}}{\partial a}.
\]

Inserting this into (10) for \( a = i, b = j \), taking (15) into account, it arises

\[
\bar{\zeta}_{ij}(s+1, x|A) = \frac{g_{ij}(x)}{3V} \left[ -\zeta(s+1|A) + \frac{a^3}{2s} \frac{\partial}{\partial a} \zeta(s|A) \right].
\]

To get the renormalized stress tensor, we have to compute \( \zeta(s|A) \) or equivalently \( \zeta(s, x|A) \) only.

The expansion of the latter over the eigenvalues reads

\[
\zeta(s, x|A) = \frac{2}{V\beta} \sum_{q=1}^{+\infty} \sum_{n=1}^{+\infty} q^2 \left[ \left( \frac{2\pi n}{\beta} \right)^2 + \left( \frac{q}{a} \right)^2 \right]^{-s} + \frac{1}{V\beta} \sum_{q=1}^{+\infty} q^2 \left[ \left( \frac{q}{a} \right)^2 \right]^{-s} + \frac{a^2 s}{V\beta} \zeta_R(2s-2).
\]

The last \( \zeta \) function is Riemann’s one.

Let us introduce the Epstein function \( E \) obtained by continuing (into a meromorphic function) the series in the variable \( s \)

\[
E(s, x, y) := \sum_{n,m=1}^{+\infty} \left( x^2 n^2 + y^2 m^2 \right)^{-s}.
\]
We get trivially
\[ \sum_{n,m=1}^{+\infty} m^2 (x^2 n^2 + y^2 m^2)^{-s} = -\frac{1}{2y(s-1)} \frac{\partial}{\partial y} E(s-1,x,y). \]

Employing such an identity, we can rewrite the expression (23) of \( \zeta(s,x|A) \) as
\[
\zeta(s,x|A) = \frac{a^{2s}}{V \beta} \zeta_R(2s - 2) + \frac{a^3}{V \beta(s-1)} \frac{\partial}{\partial a} E(s-1, \frac{2\pi}{\beta}, \frac{1}{a}). \tag{25}
\]

No expression of the Epstein function in terms of elementary functions exists in literature. Any-
more, there exist a well-know expansion in terms of MacDonal d functions \[8\]
\[
E(s,x,y) = -\frac{1}{2} y^{-2s} \zeta_R(2s) + \frac{\sqrt{\pi} \Gamma(s-1/2)}{2x \Gamma(s)} y^{1-2s} \zeta_R(2s - 1)
+ \frac{2\sqrt{\pi} x^{-2s}}{\Gamma(s)} \sum_{m,n=1}^{+\infty} \left( \frac{\pi m x}{yn} \right)^{s-1/2} K_{s-1/2} \left( \frac{2\pi y nm}{x} \right). \tag{26}
\]

Notice that, due to the negative exponential behaviour of MacDonal ds functions \( K_a(x) \) at large
arguments, the last series defines a function which is analytic on the whole \( s \) complex plane.
The structure of the poles of the Epstein function is due to the gamma and (Riemann’s) zeta
functions in the first line of the formula above. In particular there are only two simple poles at
\( s = 1/2 \) and \( s = 1 \).

Taking account of the expression above and (25), we find
\[
\zeta(s,x|A) = \frac{\sqrt{\pi}}{4\pi V} \frac{\Gamma(s-3/2)}{\Gamma(s)} (2s-3) a^{2s-1} \zeta_R(2s - 3) - \frac{a}{V \Gamma(s)} \left( \frac{\beta}{2\pi} \right)^{2s-2} \Xi(s,\beta/a), \tag{27}
\]
where the function \( \Xi(s,\beta/a) \) given by
\[
\Xi(s,z) = 2\pi \frac{d}{dz} \sum_{m,n=1}^{+\infty} \left( \frac{2\pi m z}{zn} \right)^{s-3/2} K_{s-3/2}(nmz), \tag{28}
\]
is analytic throughout the \( s \) complex plane and, due to the large argument behaviour of the
MacDonal d functions, vanishes as \( \beta \to +\infty \) like \((\beta/a)^{5/2-s} \exp -\beta/a\) when Re \( s \geq 0\).

Reminding the relation
\[
2\frac{d}{du} K_a(u) = K_{a-1}(u) + K_{a+1}(u) \tag{29}
\]
the function \( \Xi(s,z) \) and its \( z \) derivative (see below) can be evaluated numerically at the physical
values \( s = 0 \) and \( s = 1 \) (see below).

The expression (27) which is very useful as far as the low temperature thermodynamics in our
manifold is concerned. Notice that, changing the role of \( x \) an \( y \) in the expression (28), one may
get an expression for \( \zeta(s,x|A) \) useful at large temperatures.
Some remarks on (27) are in order. First notice that, due to the gamma functions into the denominators, \( \zeta(s, x | A) \to 0 \) like \( s \) when \( s \to 0 \) and thus no trace anomaly appears and neither renormalization scale \( \mu \) remains in the renormalized effective action. The found \( \zeta \) function is analytic throughout the \( s \) complex plane except for the point \( s = 2 \) where a simple pole appears. Employing (18) and (22) we find that \( \zeta_{ab}(s, x | A) \) is analytic at \( s = 1 \) and thus the theory is a super \( \zeta \)-regular theory.

Employing the definition (3), (9) and the obtained expression for \( \zeta_{ab}(s, x | A) \), a few calculations lead us to

\[
\langle T^b_a(x) \rangle_\beta = \langle T^b_a(x) \rangle_\beta \equiv T(\beta) \left(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right),
\]

where

\[
T(\beta) = -\frac{1}{2V} \frac{\partial}{\partial \beta} \frac{\zeta(s|A)}{s} \big|_{s=0} = -\frac{1}{480a^4\pi^2} + \frac{1}{a^4} \frac{d}{dz} \big|_{z=\beta/a} \Xi(0, z).
\]

Notice that the last derivative term vanishes very fast at low temperatures. Now, one can prove very simply that the obtained stress tensor is conserved, has a vanishing trace and reduces to the well-known vacuum stress tensor in the closed Einstein universe \([7]\) as \( \beta \to +\infty \)

\[
\langle T^b_a(x) \rangle_\text{vacuum} \equiv \frac{1}{480a^4\pi^2} \left(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).
\]

Taking account of \( \zeta(0|A) = 0 \), we can rewrite (31) as

\[
T(\beta) = -\frac{1}{2V} \frac{\partial}{\partial \beta} \zeta'(0|A) = -\frac{1}{V} \ln Z_\beta
\]

where the prime means the \( s \) derivative. Hence, the relation (3) holds true trivially. The general relation between the Hamiltonian density and the stress-tensor energy density in case of static coordinates reads

\[
H = -T^0_0 + \xi g^{-1/2} \partial_i [g^{1/2} (g^{ij} \partial_j \phi^2 - \phi^2 w^i)],
\]

where \( w^a = \frac{1}{2} \nabla^a \ln g_{00} \). \( w^a \) vanishes in the present case. Let us employ such a relationship to evaluate the averaged value of the quantum Hamiltonian. We have to interpret (33) as

\[
\langle H \rangle_\beta = -\langle T^0_0 \rangle_\beta + \xi g^{-1/2} \partial_i [g^{1/2} (g^{ij} \partial_j \langle \phi^2 \rangle_\beta - \langle \phi^2 \rangle_\beta w^i)].
\]

As is well-known, provided the local \( \zeta \) function were regular at \( s = 1 \), we can use the relation (8) to evaluate \( \langle \phi^2(x) \rangle \). This is the case and we find

\[
\langle \phi^2(x) \rangle_\beta = -\frac{1}{48\pi^2 a^2} - \frac{1}{2\pi^2 a^2} \Xi(1, \beta/a).
\]

5 Notice that we are writing Lorentzian relations employing the Euclidean metric. We could pass to use the more usual Lorentzian metric simply through the identities \( g = -g_L, g_{00} = -g_{L00} \) and \( g^{ij} = g^{ij}_L \).
This reduces to the known value as $\beta \to +\infty$ \([7]\). Notice that, due to the homogeneity of the space, there is not dependence on $x$ and thus all derivatives in (34) vanish yielding $\langle H \rangle_\beta = -\langle T_0^0 \rangle_\beta$. Then (5) can be rewritten in terms of the averaged Hamiltonian in the right hand side

$$\frac{-\partial \ln Z_\beta}{\partial \beta} = \langle H \rangle_\beta$$ \hspace{1cm} (35)

2 Einstein’s open static universe

The ultrastatic metric of the (Euclidean) Einstein closed static universe is \([7]\)

$$ds^2_{EOS} = d\theta^2 + g_{ij} dx^i dx^j = d\theta^2 + a^2 \left(dX^2 + \sinh^2 X d\Omega_2^2\right).$$

$X$ ranges from 0 to $+\infty$ and $d\Omega_2^2$ is the usual metric on $S_2$. The time coordinate $\theta$ ranges from 0 to $\beta \leq +\infty$. Again, $\beta$ is the inverse temperature of the considered thermal state referred to the Killing vector generated by the Lorentzian time $i\theta$ and the related vacuum state corresponds to the choice $\beta = +\infty$. The curvature of the space is $R = -6/a^2$ and the Ricci tensor reads $R_{ij} = -2g_{ij}/a^2$, the remaining components vanish.

This manifold is not closed and the spatial sections have not a finite volume.

Let us consider a conformally coupled massless scalar field propagating within this manifold. As in the previously considered case, we want to compute its stress tensor referred to the thermal states, in particular we want to get the vacuum stress tensor. Notice that not all the hypotheses required in \([4]\) to implement the stress-tensor $\zeta$-function approach are fulfilled. The manifold has no boundary but it is not compact. We expect to find a continuous spectrum as far as the Euclidean motion operator is concerned.

However, we shall find that our method does work also in this case. Notice that, now, we have to employ our definition (1) or (3) by definition and check on the obtained results finally.

The form of the eigenvalues $\lambda_k$ of the conformally coupled massless Euclidean motion operator

$$A = -\partial_\theta^2 - a^{-2}\Delta_{H_3} + \xi_c R,$$

is well-known \([3, 4]\), we have, exactly as in the previous case

$$\lambda_k = \left(\frac{2\pi n}{\beta}\right)^2 + \left(\frac{q}{a}\right)^2,$$ \hspace{1cm} (36)

where $k \equiv (n, q, l, m)$ and $n = 0, \pm 1, \pm 2, \pm 3, ...$, $q \in [0, +\infty)$, $l = 0, 1, 2, 3, ...$, $m = 0, \pm 1, \pm 2, ...$. The degeneracy depends only on the indexes $l$ and $m$.

The following relations which hold true for eigenvectors $\phi_k(x)$ (which are Dirac’s delta normalized in $q$ and Kroneker’s delta normalized in the remaining variables) are also useful. We leave the proofs of these to the reader (see also \([5]\)).

$$\sum_{l,m} \phi_k^*(x)\phi_k(x) = \frac{q^2}{2\pi^2 a^3 \beta},$$ \hspace{1cm} (37)
\[ \sum_{l,m} \partial_i \phi^*_k(x) \partial_j \phi_k(x) = g_{ij}(x) \frac{q^2(q^2 + 1)}{6\pi^2 a^5 \beta}, \]  
\hspace{1cm} (38)

and \((x^0 := \theta)\)

\[ \sum_{l,m} \partial_0 \phi^*_k(x) \partial_0 \phi_k(x) = \frac{(2\pi n q)^2}{2\pi^2 a^3 \beta^3}. \]  
\hspace{1cm} (39)

Notice that the global \(\zeta\) function simply does not exist because the infinite spatial volume of the manifold. Anyhow, we can compute the local \(\zeta\) function as

\[ \zeta(s, x|A) := \int_0^{+\infty} dq \sum_{l,m,n} \phi^*_k(x) \phi_k(x) \lambda^{-s}_k. \]  
\hspace{1cm} (40)

It is convenient to separate the contribution due to the terms with \(n = 0\) and introduce, as far as these terms are concerned, a cutoff \(\epsilon\) at low \(q\). A few trivial manipulations of the expression above yields

\[ \zeta(s, x|A) = \frac{a^{2s-3}}{2\pi^2 \beta} \int_{\epsilon}^{+\infty} dq \; q^{2s-2} + \frac{1}{4\pi^2 \beta} \left( \frac{\beta}{2\pi} \right)^{2s-3} \zeta_R(2s - 3) \frac{\Gamma(1/2)\Gamma(s - 3/2)}{\Gamma(s)}. \]  
\hspace{1cm} (41)

The apparent divergent integral as \(\epsilon \to 0^+\) can be made harmless as in [10] putting \(\epsilon \to 0^+\) after one has fixed \(\Re s\) large finite. This procedure generalize the finite volume prescription to drop the null eigenvalues in defining the \(\zeta\) function for the case of an infinite spatial volume. We have finally

\[ \zeta(s, x|A) = \frac{1}{8\pi^2 \sqrt{\pi}} \left( \frac{\beta}{2\pi} \right)^{2s-4} \zeta_R(2s - 3) \frac{\Gamma(s - 3/2)}{\Gamma(s)}. \]  
\hspace{1cm} (42)

Notice that \(\zeta(0, x|A) = 0\) and thus no renormalization scale appears in the (infinite) partition function.

Let us evaluate \(\tilde{\zeta}_{ab}(s, x|A)\). The only nonvanishing components are 00 and \(ij\). In the first case we have directly from the definitions (omitting the terms with \(n = 0\) as above)

\[ \zeta_{00}(s + 1, x|A) = \tilde{\zeta}_{00}(s + 1, x|A) = \int dq \sum_{l,m,n} \left( \frac{2\pi n}{\beta} \right)^2 \lambda^{-s}_k \phi^*_k(x) \phi_k(x) \]  
\hspace{1cm} = \frac{1}{8\pi^2 \sqrt{\pi}} \left( \frac{\beta}{2\pi} \right)^{2s-4} \zeta_R(2s - 4) \frac{\Gamma(s - 1/2)}{\Gamma(s + 1)}. \]  
\hspace{1cm} (43)

In order to compute the remaining components of \(\tilde{\zeta}_{ab}\) we can use (38) and the relation in (21) once again. We find

\[ \tilde{\zeta}_{ij}(s + 1, x|A) = \frac{1}{3a^5} g_{ij}(x) \zeta(s + 1, x|A) + \frac{1}{2s} g_{ij}(x) \zeta(s, x|A). \]  
\hspace{1cm} (44)
We have found that $\zeta_{ab}(s,x|A)$ is analytic in $s = 1$, hence the theory is a super $\zeta$-regular theory once again. We can use (3) to compute the stress tensor.

Through (4) and (3) we find finally

$$\langle T^b_{La} \rangle_\beta = \langle T^b_a \rangle_\beta \equiv T(\beta)(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}),$$

(45)

where

$$T(\beta) = \frac{\pi^2}{30\beta^4}.$$  

(46)

The stress tensor in (45) is conserved and traceless as we expected from the general theory. $\langle T^b_a \rangle_\beta$ vanishes as $\beta \to +\infty$, this agrees with the known result [7] that the stress tensor in the vacuum state of the open Einstein universe vanishes.

Notice that the found stress tensor, in the considered components, is exactly the same than in Minkowski spacetime.

Let us finally consider (5). In this case the left hand side of (5) does not exist because that simply diverges. Nevertheless, we can notice that the divergence of the partition function is due to the volume divergence only and the remaining factor does not depend on the position on the spatial section, namely

$$\ln Z_\beta = V \ln Z_\beta = V \times \left(\frac{1}{2} \zeta'(0,x|A)\right) = V \times \frac{\pi^3}{90\beta^3}$$

(47)

where $V$ diverges and, actually, $\zeta'(0,x|A)$ does not depend on $x$ due to the homogeneity of the spatial manifold. This is the same situation than arises in the Minkowski spacetime.

We expect that, although (5) does not make sense, a local version could yet make sense. Indeed, one can get very simply from (45) and (47)

$$-\frac{\partial V \ln Z_\beta}{\partial \beta} = -V \langle T^0_0 \rangle_\beta = - \int_V d\vec{x} \sqrt{g} \langle T^0_{L0} (\vec{x}) \rangle_\beta$$

(48)

on any finite volume $V$. As in the previously discussed case, $\langle \phi^2(x) \rangle_\beta$ can be obtained through (8)

$$\langle \phi^2(x) \rangle_\beta = \frac{1}{12\beta^2}.$$  

(49)

Notice that this vanishes as $\beta \to +\infty$ namely, in the vacuum state as is known [9]. Furthermore, it does not depend on $x$ and thus, through (33) and noticing that $u^a = 0$ (see the Einstein closed universe case), $\langle T^0_0 \rangle_\beta = -\langle H \rangle_\beta$. We can write finally, with an obvious meaning

$$-\frac{\partial V \ln Z_\beta}{\partial \beta} = \langle H_V \rangle_\beta.$$  

(50)
3 The conical manifold

Let us consider the Euclidean manifold $\mathbb{C}_\beta \times \mathbb{R}^2$ endowed with the metric

$$ds^2 = r^2 d\theta^2 + dr^2 + dz_1^2 + dz_2^2,$$

where $(z_1, z_2) \in \mathbb{R}^2$, $r \in [0, +\infty)$, $\theta \in [0, \beta)$ when 0 is identified with $\beta$. $\mathbb{C}_\beta \times \mathbb{R}^2$ is a cone with deficit angle given by $2\pi - \beta$. That is the Euclidean manifold corresponding to the finite temperature ($T = 1/\beta$) quantum field theory in the Rindler space. In such a case $\theta$ is the Euclidean time of the theory. This is also a good approximation of a large mass black hole near the event horizon. Equivalently, considering $z_1$ as the Euclidean time, the metric above defines the Euclidean section (at zero temperature) of a cosmic string background. In this case $(2\pi - \beta)/8\pi G$ is the mass of the string.

The metric in (51), considered as the Rindler Euclidean metric, is static but not ultrastatic. Another important point is that such a metric is not homogeneous in the spatial section.

The considered manifold is flat everywhere except for conical singularities which appear at $r = 0$ whenever $\beta \neq 2\pi$. These singularities produces well-known Dirac’s delta singularities in the curvatures of the manifolds at $r = 0$ [11]. The physics involved in such anomalous curvature it is not completely known. Actually, we shall see shortly that one can ignore completely the anomalous curvature dealing with the stress tensor renormalization also considering nonminimal coupling with the scalar curve.

As is well-known, the particular value $\beta_H = 2\pi$ defines the Hawking-Unruh temperature in the Rindler/large-mass-black-hole interpretation, the corresponding thermal state being nothing but the Minkowski vacuum/Hartle-Hawking (large mass) vacuum.

The thermal Rindler stress tensor (renormalized with respect to the Minkowski vacuum) which coincide, in the Euclidean approach, to the zero-temperature cosmic-string stress tensor (renormalized with respect to the Minkowski vacuum) has been computed by the point splitting approach [12].

Such results has been only partially reproduced by some $\zeta$-function or (local) heat kernel approach [13, 14]. This is because these approaches were employed to renormalize the effective action only, and thus the stress tensor was computed assuming further hypotheses on its form or assuming some statistical-mechanical law as holding true [13, 14].

Recently, in [16], also the massive case has been considered by employing an off-diagonal $\zeta$-function approach and a subtraction procedure similar to that is employed within the point-splitting framework.

Here, we shall consider the massless case only. We shall check our approach for every value of the curvature coupling proving that the same results got by the point-splitting approach naturally arise. The important point is that, due to the complete independence of the method from statistical mechanics, we shall be able to discuss the statistical mechanics meaning (if it exists) of our results a posteriori.

Let us consider first the case of the minimal coupling $\xi = 0$. This avoids all problems involved dealing with the singular curvature on the tip of the cone generated by the conical singularity.
The \( \zeta \) function of the effective action in conic backgrounds has been computed by several authors \[15\] also in the massive scalar case \[16\] and for photons and gravitons \[4\].

Discarding the singular curvature by posing \( \xi = 0 \), a complete normalized set of eigenvectors of the massless Euclidean motion operator

\[
A = -\Delta_{\mathbb{C}^{\mathbb{R}^2}}
\]

is \[15\]

\[
\phi_q(x) = \frac{1}{2\pi} \sqrt{\frac{\lambda}{\beta}} e^{\frac{i2\pi n}{\beta}} e^{i\frac{2\pi k\theta}{\beta}} \frac{1}{r^{2|x|=1}} (\lambda r)
\]

where \( z = (z_1, z_2) \), \( q = (n, k, \lambda) \), \( n = 0, \pm 1, \pm 2, ... \), \( k = (k_1, k_2) \in \mathbb{R}^2 \), \( \lambda \in [0, +\infty) \). The considered eigenfunctions are Kronecker’s delta normalized in the index \( n \) and Dirac’s delta normalized in the remaining indices. The corresponding eigenvalues are

\[
\lambda_q = \lambda^2 + k^2.
\]

The \( \zeta \) function of \( A \) has been computed explicitly and reads

\[
\zeta(s, x|A) = \frac{r^{2s-4}}{4\pi \beta \Gamma(s)} I_\beta(s-1).
\]

\( I_\beta(s) \) is a well-known meromorphic function \[15\] carrying a simple pole at \( s = 1 \). Known values are also

\[
I_\beta(0) = \frac{1}{6\nu}(\nu^2 - 1),
\]

\[
I_\beta(-1) = \frac{1}{90\nu}(\nu^2 - 1)(\nu^2 + 11),
\]

where we defined \( \nu := 2\pi/\beta \).

Notice that \( \zeta(0, x|A) = 0 \) and thus no scale remains into the renormalized local effective action and \( \langle \phi^2(x) \rangle \) can be computed through \[8\].

The function \( \tilde{\zeta}_{ab}(s, x|A) \) can be computed making use of intermediate results contained in \[14\]. A few calculations lead us to

\[
\tilde{\zeta}_{\theta\theta}(s, x|A) = \frac{r^{2s-4}\Gamma(s-3/2)}{4\pi \sqrt{\pi} \beta \Gamma(s)} H_\beta(s-1),
\]

\[
\tilde{\zeta}_{rr}(s, x|A) = \frac{1}{2r} \partial_r r \partial_r \zeta(s, x|A) - \frac{1}{r^2} \tilde{\zeta}_{\theta\theta}(s, x|A) + 4\pi(s-2)\zeta_{D=6}(s, x|A),
\]

\[
\tilde{\zeta}_{z_1z_1}(s, x|A) = \tilde{\zeta}_{z_2z_2}(s, x|A) = 2\pi \zeta_{D=6}(s, x|A).
\]

All remaining components vanish. The meromorphic function \( H_\beta(s) \) has been defined in \[16\], it has a simple pole at \( s = 2 \) and known values are

\[
H_\beta(0) = \frac{1}{120\nu}(\nu^4 - 1),
\]

\[
H_\beta(1) = -\frac{1}{12\nu}(\nu^2 - 1).
\]

\[\]
The function $\zeta_{D=6}(s, x|A)$ is the $\zeta$ function of the effective action in $C_\beta \times \mathbb{R}^4 [19]$, it reads

$$\zeta_{D=6}(s, x|A) = \frac{r^{2s-6}}{(4\pi)^2 \beta \Gamma(s)} I_\beta(s-2).$$

(62)

From the above equations and (9) it follows that $\bar{\zeta}_{ab}(s, x|A)$ is analytic at $s = 1$ and thus the theory is super $\zeta$-regular once again. Hence, we can use (3) to compute the stress tensor. Trivial calculations employing (9) with $\xi = 0$ and (3) produce

$$\langle T_{ab}(x)_{\xi=0} \rangle_\beta = \frac{1}{1440\pi^2 r^4} \left\{ \left[ \left( \frac{2\pi}{\beta} \right)^4 - 1 \right] \text{diag}(-3, 1, 1, 1) - 20 \left[ \left( \frac{2\pi}{\beta} \right)^2 - 1 \right] \text{diag}\left(\frac{3}{2}, -\frac{1}{2}, 1, 1\right) \right\}. \quad (63)$$

This is the correct result arising by the point-splitting approach [12] in the case of the minimal coupling. Let us prove that our method reproduce also the remaining cases.

In general, the relationship between the minimally coupled stress-tensor and the generally coupled stress tensor can be trivially obtained by varying the action containing the usual coupling with the curvature, it reads

$$T_{ab}(x)_{\xi} = T_{ab}(x)_{\xi=0} + \xi \left[ (R_{ab} - \frac{1}{2} g_{ab} R) \phi^2(x) + g_{ab} \Delta \phi^2(x) - \nabla_a \nabla_b \phi^2(x) \right]. \quad (64)$$

It is worthwhile stressing that the last $\xi$-parametrized term appears also when the manifold is flat. We can interpret quantistically this relationship as

$$\langle T_{ab}(x)_{\xi} \rangle = \langle T_{ab}(x)_{\xi=0} \rangle + \langle Q(x)_{ab} \rangle,$$

where

$$\langle Q(x)_{ab} \rangle := \left[ (R_{ab}(x) - \frac{1}{2} g_{ab}(x) R(x)) \langle \phi^2(x) \rangle + g_{ab} \Delta \langle \phi^2(x) \rangle - \nabla_a \nabla_b \langle \phi^2(x) \rangle \right]. \quad (65)$$

Now $\langle T_{ab}(x)_{\xi=0} \rangle_\beta$ is known by (13), $R_{ab}(x) = 0$, $R(x) = 0$ and thus we can compute $\langle T_{ab}(x)_{\xi} \rangle_\beta$ employing the known value of $\langle \phi^2(x) \rangle_\beta$. We have, through (14) and (8)

$$\langle \phi^2(x) \rangle_\beta = \frac{1}{48\pi^2 r^2} \left[ \left( \frac{2\pi}{\beta} \right)^2 - 1 \right]. \quad (66)$$

The final result is exactly that of the point-splitting approach:

$$\langle T^b_{La}(x)_{\xi} \rangle_\beta = \langle T^b_{La}(x)_{\xi=0} \rangle_\beta = \frac{1}{1440\pi^2 r^4} \left\{ \left[ \left( \frac{2\pi}{\beta} \right)^4 - 1 \right] \text{diag}(-3, 1, 1, 1) + 20(6\xi - 1) \left[ \left( \frac{2\pi}{\beta} \right)^2 - 1 \right] \text{diag}\left(\frac{3}{2}, -\frac{1}{2}, 1, 1\right) \right\}. \quad (67)$$
The same result arises by employing the definition of \( \zeta_{ab}(s, x|A) \) given in (9) with the chosen value of \( \xi \), provided \( \bar{\zeta}_{ab}(s, x|A) \) and \( \zeta(s, x|A) \) were those computed in the minimal coupling case. This means that, concerning the renormalization of the stress tensor, the presence of the conical singularity which determines a singular curvature on the tip of the cone is completely irrelevant. Concerning the quantum state, there is no difference between different couplings with the curvature. The \( \xi \)-parametrized term remains as a relic in the stress tensor because of the classical formula (64). This term does not come out from the quantum state once one fixed the renormalization procedure. We see that the renormalization of the stress tensor can be managed completely by our Euclidean \( \zeta \)-function approach on the physical manifold instead of the optical manifold (see the final discussion) not depending on the presence of the conical singularity in the Euclidean manifold.

The knowledge of the averaged and renormalized stress tensor makes us able to compute the averaged and renormalized Hamiltonian of the system. The Hamiltonian of the theory should not depend on the parameter \( \xi \) because that cannot appear into the Lorentzian action the manifold being flat. Notice that there is no conical singularity in the Lorentzian theory! Not depending on \( \xi \), the classical Hamiltonian density coincides with the changed sign energy component of the stress tensor in the minimal coupling. Indeed, employing (33), we can write down

\[
\langle H(x) \rangle_{\beta} = -\langle T^0_0(x)_{\xi=0} \rangle_{\beta} = \frac{3}{1440\pi^2 r^4}
\left[ \left( \frac{2\pi}{\beta} \right)^4 - 10 \left( \frac{2\pi}{\beta} \right)^2 - 11 \right].
\]

Let us finally consider the problem of the validity of the relation (5) in some sense. The spatial section is neither finite nor homogeneous, we could have problems with the use of cutoffs. It is not obvious that such a relation as (5) can hold true in our case considering cutoff smeared quantities as\[7\]

\[
\ln Z_{\beta\epsilon} := \int_{r>\epsilon} d^4x \sqrt{g} \frac{1}{2} \frac{d}{ds} |_{s=0} \zeta(s, x|A),
\]

\[
Q_{\epsilon}(\beta) := \int_{r>\epsilon} d^3x \sqrt{g} \langle Q^0_0(x) \rangle_{\beta},
\]

\[
\langle H_{\epsilon} \rangle_{\beta} := \int_{r>\epsilon} d^3x \sqrt{g} \langle H \rangle_{\beta}
\]

and finally

\[
E_{\epsilon\xi}(\beta) := -\int_{r>\epsilon} d^3x \sqrt{g} \langle T^0_0(x)_{\xi} \rangle_{\beta} = \int_{r>\epsilon} d^3x \sqrt{g} \langle H \rangle_{\beta} - \xi Q_{\epsilon}(\beta).
\]

In particular we have from (54)

\[
\ln Z_{\beta\epsilon} = \frac{A\beta}{2880\pi^2 \epsilon^2} \left[ \left( \frac{2\pi}{\beta} \right)^4 + 10 \left( \frac{2\pi}{\beta} \right)^2 - 11 \right],
\]

\[7\text{Notice that also the area } A \text{ of the horizon is a cutoff because the actual area is infinite. This cutoff is a trivial overall factor. We shall omit this cutoff as an index in the following formulae for sake of simplicity.}\]
where $A$ is the area of the event horizon, the regularized volume of the spatial section is $V_\epsilon = A/(2\epsilon^2)$. Notice that, actually, the conserved charge $Q_\epsilon(\beta)$ is a boundary integral which diverges on the conical singularity. Indeed, it can be expressed by the integration of (33) and it should discarded if the manifold were regular. Notice that the choice of values of $\xi$ determines different values of $\mathcal{E}_\xi$ due to the $\xi$-parametrized boundary term $\xi Q_\epsilon$ in the stress tensor. Conversely, $\ln Z_\beta$ does not depend on $\xi$.

If something like (5) holds true for a fixed value of $\epsilon$, it does just for a particular and unique value of $\xi$. Actually few calculation through (67) prove that, not depending on the value of $\epsilon$

$$\frac{\partial \ln Z_\beta}{\partial \beta} = \mathcal{E}_{\epsilon=1/9}(\beta) + \mathcal{E}_\epsilon = \langle H_\epsilon \rangle_\beta - \frac{1}{9} Q_\epsilon(\beta) + \mathcal{E}_\epsilon. \quad (74)$$

The last term is an opportune constant energy

$$\mathcal{E}_\epsilon = \frac{A}{120\pi^2 \epsilon^2}.$$ 

The presence of such an added constant could be expected from the fact that the energy $\mathcal{E}_\epsilon$ is renormalized to vanish at $\beta = 2\pi$ instead of $\beta = +\infty$. Conversely, there is no trivial explanation of the presence of the $\beta$-dependent term $-\frac{1}{9} Q_\epsilon(\beta)$. Then, in the considered case, in the right hand side of (5) does not appear the Hamiltonian which, at least classically, corresponds to the value $\xi = 0$ as discussed above.

One could wonder whether or not $Z_\beta$ defined in (69) can be considered a (regularized) partition function of the system. The simplest answer is obviously not because a fundamental relationship of statistical mechanics does not hold true.

In general, one could think that this negative result arises because we have dropped a contribution due to the conical singularity. This singularity produces a Dirac delta in the curvature on the tip of the cone in the Euclidean manifold. The integral of the Lagrangian get a contribution from this term in the case of a nonminimal coupling with the curvature. The problem of the contributions of these possible terms, in particular in relation to the black-hole entropy has been studied by several authors (see [22, 13, 14, 23, 24] and references therein), anyhow, in this paper we shall not explore such a possibility.

In any cases, it is worthwhile stressing that the found Euclidean effective action (73) is the correct one in order to get the thermal renormalized stress tensor by (formal) variation with respect to the background metric. We re-stress that the obtained stress tensor is exactly that obtained by the point-splitting approach.

4 Summary and discussion

In this paper we have checked the method to renormalize the one-loop stress tensor introduced in [1]. We have studied the application of the method to three different cases both considering the thermal and non thermal QFT. The first case (the closed static Einstein universe) fulfills completely the hypotheses requested in [1]. The found results coincide with known results. The
remaining two cases (open Einstein static universe and Euclidean Rindler/cosmic-string manifold) have concerned two situations where the hypotheses requested in [1] were not completely fulfilled. In particular the last case, the conical manifold, could seem quite subtle concerning \( \zeta \) function techniques due to the presence of the singular curvature. Actually, we have found that the presence of the conical singularity does not involve particular problems in renormalizing the stress tensor because the kernel of the physical information is completely contained in the minimally coupled case. Anyhow, we have found correct results in both cases. In particular, these results agree with those obtained by the point-splitting approach.

In general, we expect that the method introduced in [1] to renormalize the one-loop stress tensor could work also relaxing the hypotheses of an Euclidean closed manifold as we have found in these examples. Moreover, we expect that the results should coincide with those obtained by the point splitting approach.

We shall conclude this work with some comments on the result found in the conical manifold.

There are still some unresolved problems concerning the theory on the conical manifold interpreted as the thermal Rindler space.

The question whether or not the effective action computed by the \( \zeta \) function defines also the logarithm of the partition function (renormalized with respect to the Minkowski vacuum) is not a simple question.

The problem is interesting on a physical ground also because the partition function of the field around a black hole (we remind the reader that the Rindler metric represents a large mass black hole) is used to compute the quantum corrections to the Bekenstein-Hawking entropy as early suggested by ’t Hooft [18] or to explain the complete B-H entropy in the framework of the induced gravity considering massive fields nonconformally coupled [5, 1, 3].

On a more general ground, the considered problem is also interesting because there exist two not completely equivalent approaches to implement the statistical mechanics of a quantum field in a curved spacetime through the use of a path integral techniques and, up the knowledge of the author, there is not a definitive choice of the method. In this work, we have employed the path integral in the physical manifold instead of in the optical related manifold. We remind the reader that in the case of a static but not ultrastatic spacetime, the naive approach based on the phase-space path integral leads one to a definition of the partition function as an Euclidean path integral performed in the configuration space within the optical manifold\(^8\) instead of the physical one [13]. Other approaches [20] lead one to the definition of the partition function as a path integral in the physical manifold.

When the spatial section of the space is regular (e.g. closed) and thus the path integral regularized through the \( \zeta \)-function approach yields a finite result, formal manipulations of the path integral prove that these two different definitions lead to the same result up to the renormalization of the zero point energy [21]. In such a case these definitions are substantially equivalent.

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\(^8\)This is the ultrastatic manifold conformally related to the physical manifold by defining the optical metric through \( g_{ab} := \frac{g_{ab}}{g_{00}} \). The Euclidean action employed on the optical manifold is the physical action conformally transformed (including the matter fields) following the conformal transformation written above.
When the manifold is not regular, e.g. it has spatial sections with an infinite volume or has boundaries, in principle one may lose such an equivalence. Indeed, as far as the effective actions are concerned in our case we have

\[
\ln Z_{\beta \epsilon} = \frac{A \beta}{2880 \pi^2 \epsilon^2} \left[ \left( \frac{2\pi}{\beta} \right)^4 + 10 \left( \frac{2\pi}{\beta} \right)^2 - 11 \right],
\]

and

\[
\ln Z_{\text{opt}} = \frac{A \beta}{2880 \pi^2 \epsilon^2} \left( \frac{2\pi}{\beta} \right)^4 \tag{75}
\]

The latter result can be directly obtained noticing that the optical manifold of the Rindler space is the open Einstein static universe \( \mathbb{E} \). Hence the latter effective action above is nothing but that computed previously in the open Einstein universe (in the conformal coupling).

Further comments on the renormalization of the Hamiltonian in the conical manifold are in order. Considering the effective action computed as a path integral in the optical manifold we have

\[
- \frac{\partial \ln Z_{\beta \epsilon}}{\partial \beta} = \mathcal{E}_{\epsilon = 1/6} + \mathcal{E}' \tag{76}
\]

One could conclude that, once again, there is not the Hamiltonian in the right hand side, also discarding the constant energy. Actually, this result involves more subtle considerations. Indeed, we shall prove that this naive conclusion is not correct.

Let us suppose to implement the canonical QFT \( \mathbb{F} \) for a massless field conformally coupled directly on the optical manifold, namely in the open Einstein static universe as it were the physical manifold. Obviously, we should get exactly the effective action which appears in \( \mathbb{F} \). Furthermore, \( \mathbb{F} \) is nothing but \( \mathbb{F} \) and the right hand side of \( \mathbb{F} \) is nothing but the averaged \( \epsilon \)-regularized Hamiltonian of the QFT in the open Einstein universe. Such a Hamiltonian can be also obtained as a thermal average of the Hamiltonian operator got from the canonical QFT employing the normal order prescription.

Implementing the canonical quantization in the Rindler space for a massless scalar field, one trivially finds that an isomorphism exists between the Fock space built up on the Fulling-Rindler vacuum and the Fock space built up on the natural vacuum of the QFT in the open Einstein static universe (in the conformal coupling). Indeed, this isomorphism arises from the conformal relationship between the wavefunctions of the particles related to the quantized fields. This relation defines a one-to-one map from the one-particle Hilbert space of the Einstein open universe to the one-particle Hilbert space of the Rindler space which maintains the value of the corresponding indefinite scalar products \( \mathbb{F} \). This map defines a unitary isomorphism between the two Fock spaces provided one require that this isomorphism transform the vacuum state of the Einstein open universe into the Fulling-Rindler vacuum. In particular, also the Hamiltonian operators are unitarily identified provided one use the normal order prescription in both cases. As a result we find that the right hand side of \( \mathbb{F} \) coincides also with the averaged Hamiltonian.
operator built up in the framework of the canonical quantization in the Rindler space with respect to the Fulling-Rindler vacuum!

In this sense (76) is the usual statistical mechanical relationship between the canonical energy and the partition function in the Rindler space.

The central point is that the renormalization scheme employed is the normal order prescription with respect to the Fulling-Rindler vacuum and not the point-splitting procedure.

We can finally compare the averaged Rindler Hamiltonian of the canonical quantization \( \langle H^\text{can}_\epsilon \rangle_\beta \) which is renormalized by the normal order prescription in the the Fulling-Rindler vacuum with the averaged Rindler Hamiltonian \( \langle H_\epsilon \rangle_\beta \) obtained by integrating (68). The latter is renormalized with respect the Minkowski vacuum by the point-splitting procedure. We find

\[
\langle H_\epsilon \rangle_\beta - \langle H^\text{can}_\epsilon \rangle_\beta = -\frac{3}{2880\pi^2\epsilon^2} - \frac{30}{2880\pi^2\epsilon^2} \left( \left( \frac{2\pi}{\beta} \right)^2 - 1 \right) = -\frac{1}{960\pi^2\epsilon^2} - \frac{1}{6} Q_\epsilon(\beta).
\]  

(77)

The first term in the right hand side is trivial: it takes account of the difference of the zero-point energy. The second term is quite unexpected. It proves that the point-splitting procedure (or equivalently our \( \zeta \)-function procedure) to renormalize the stress tensor and hence the Hamiltonian is not so trivial as one could expect, this is because it involves terms which do not represent a trivial zero-point energy renormalization.

Concerning the conical manifold, the conclusion is that the theory in the optical manifold leads us naturally to an effective action which can be considered the logarithm of the partition function provided we renormalize the theory with respect to the Fulling-Rindler vacuum. Conversely, the effective action evaluated in the physical manifold is the correct effective action which produces the thermal stress tensor by formal variations with respect to the metric (this is the method exposed in [1]). This stress tensor is that obtained also by the point-splitting procedure and thus renormalizing with respect to the Minkowski vacuum.

**Acknowledgments**

This work has been financially supported by the ECT* (European Centre for Theoretical Nuclear Physics and Related Areas).

I would like to thank R. Balbinot, D.V. Fursaev, D. Iellici, L. Vanzo and A. I. Zelnikov for valuable discussions and suggestions.

I am grateful to the Dept. of Physics of the Trento University for the hospitality during a part of the time spent to produce this paper.
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