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Peter G. Casazza  
*University of Missouri*

Matthew Fickus  
*Air Force Institute of Technology*

Dustin G. Mixon  
*Princeton University*

Janet C. Tremain  
*University of Missouri*

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**Recommended Citation**

Casazza, Peter G.; Fickus, Matthew; Mixon, Dustin G.; and Tremain, Janet C., "The Bourgain-Tzafriri Conjecture and Concrete Constructions of Non-pavable Projections" (2011). *Faculty Publications*. 299.  
https://scholar.afit.edu/facpub/299

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THE BOURGAIN-TZAFRIRI CONJECTURE AND CONCRETE CONSTRUCTIONS OF NON-PAVABLE PROJECTIONS

PETER G. CASAZZA, MATTHEW FICKUS, DUSTIN G. MIXON AND JANET C. TREMAIN

Abstract. It is known that the Kadison-Singer Problem (KS) and the Paving Conjecture (PC) are equivalent to the Bourgain-Tzafiri Conjecture (BT). Also, it is known that (PC) fails for 2-paving projections with constant diagonal $1/2$. But the proofs of this fact are existence proofs. We will use variations of the discrete Fourier Transform matrices to construct concrete examples of these projections and projections with constant diagonal $1/r$ which are not $r$-pavable in a very strong sense.

In 1989, Bourgain and Tzafriri showed that the class of zero diagonal matrices with small entries (on the order of $\leq 1/\log^{1+\epsilon} n$, for an $n$-dimensional Hilbert space) are pavable. It has always been assumed that this result also holds for the BT-Conjecture - although no one formally checked it. We will show that this is not the case. We will show that if the BT-Conjecture is true for vectors with small coefficients (on the order of $\leq C/\sqrt{n}$) then the BT-Conjecture is true and hence KS and PC are true.

Keywords. Kadison-Singer Problem, Anderson Paving Problem Discrete Fourier Transform.

AMS MSC (2000). 42C15, 46C05, 46C07.

1. Introduction

It is now known that the 1959 Kadison-Singer Problem is equivalent to fundamental unsolved problems in a dozen areas of research in pure mathematics, applied mathematics and engineering [7, 8]. In 1979, Anderson [1] showed that the Kadison-Singer Problem is equivalent to the Paving Conjecture.

Paving Conjecture (PC). For $\epsilon > 0$, there is a natural number $r$ so that for every natural number $n$ and every linear operator $T$ on $l^2_n$ whose matrix has zero diagonal, we can find a partition (i.e. a paving) $\{A_j\}_{j=1}^r$ of
\{1, \ldots, n\}, such that
\[ \|Q_{A_j} T Q_{A_j}\| \leq \epsilon \|T\| \quad \text{for all } j = 1, 2, \ldots, r, \]
where \( Q_{A_j} \) is the natural projection onto the \( A_j \) coordinates of a vector.

Operators satisfying the Paving Conjecture are called **pavable operators**. A projection \( P \) on \( \mathcal{H}_n \) is \((\epsilon, r)\)-pavable if there is a partition \( \{A_j\}_{j=1}^r \) of \( \{1, 2, \ldots, n\} \) satisfying
\[ \|Q_{A_j} P Q_{A_j}\| \leq \epsilon, \quad \text{for all } j = 1, 2, \ldots, r. \]

It was shown in [5] that projections with constant diagonal \( 1/r \) are not \((r, \epsilon)\)-pavable for any \( \epsilon > 0 \). But the argument in [5] is an existence proof and the actual matrices failing paving were not known. In this note we will construct concrete examples of these projections. As a consequence, we will obtain a stronger result than that of [5]. The main question now is whether this construction can be generalized to produce a counter-example to KS.

**Notation 1.1.** Throughout this paper, if \( \mathcal{H}_n \) is an n-dimensional Hilbert space, then \( \{e_i\}_{i=1}^n \) denotes a fixed orthonormal basis for \( \mathcal{H} \).

It was shown [7] that BT is equivalent to PC. Our construction of non-2-pavable projections starts with a construction of non-2-Riesable sequences (See Section 2 for the definitions). The vectors we will produce have very small coefficients, on the order of \( 1/\sqrt{n} \) for an n-dimensional Hilbert space. However, conventional wisdom indicates that we cannot construct a counter-example to PC out of vectors with small coefficients. So next, we will show that conventional-wisdom has been wrong for the last 20 years and a counter-example to BT exists in general if and only if it exists for matrices with coefficients on the order of \( 1/\sqrt{n} \). Conventional wisdom came from a result of Bourgain and Tzafriri [2, 3] where they showed that the Paving Conjecture has a positive solution for the class of zero diagonal matrix operators \( A = (a_{ij})_{i,j=1}^n \) on \( \mathbb{H}_n \) with small coefficients. In particular, a matrix is pavable if the coefficients satisfy for some \( \epsilon > 0 \),
\[ |a_{ij}| \leq \frac{C}{\log^{1+\epsilon} n}. \]

It has always been assumed that the corresponding result holds for BT if
\[ |T e_i(j)| \leq \frac{C}{\log^{1+\epsilon} n}, \quad \text{for all } i, j = 1, 2, \ldots, n. \]

We will show that this is not the case. This is the second main theorem of this paper (See Section 2 for the definitions).

**Theorem 1.2.** The following are equivalent:
(1) The Bourgain-Tzafriri Conjecture is true.
(2) There are constants \( \delta \) and \( r \in \mathbb{N} \) so that for every \( C > 0 \) there is an \( N_0 \) so that for every \( N \geq N_0 \) if \( \{f_i\}_{i=1}^{2N} \) is a unit norm 2-tight frame for \( \mathcal{H}_N \).
satisfying
\[ |f_i(j)| \leq \frac{C}{\sqrt{2N}}, \]
then \( \{f_i\}_{i=1}^{2N} \) is \((\delta, r)\)-Rieszable.

2. Preliminaries

We will actually work with an equivalent form of the Paving Conjecture for projections with constant diagonal. In 1989, Bourgain and Tzafriri proved one of the most celebrated theorems in analysis: The Bourgain-Tzafriri Restricted Invertibility Theorem [2]. This gave rise to a major open problem in analysis.

Bourgain-Tzafriri Conjecture (BT). There is a universal constant \( A > 0 \) so that for every \( B > 1 \) there is a natural number \( r = r(B) \) satisfying: For any natural number \( n \), if \( T : \ell_2^n \to \ell_2^n \) is a linear operator with \( \|T\| \leq B \) and \( \|Te_i\| = 1 \) for all \( i = 1, 2, \ldots, n \), then there is a partition \( \{A_j\}_{j=1}^r \) of \( \{1, 2, \ldots, n\} \) so that for all \( j = 1, 2, \ldots, r \) and all choices of scalars \( \{a_i\}_{i \in A_j} \)
we have:

\[ \| \sum_{i \in A_j} a_i Te_i \| \geq A \sum_{i \in A_j} |a_i|^2. \]

It was shown in [2] that BT is equivalent to the Paving Conjecture.

Definition 2.1. A family of vectors \( \{f_i\}_{i=1}^M \) for an \( n \)-dimensional Hilbert space \( H_n \) is \((\delta, r)\)-Rieszable if there is a partition \( \{A_j\}_{j=1}^r \) of \( \{1, 2, \ldots, M\} \) so that for all \( j = 1, 2, \ldots, r \) and all scalars \( \{a_i\}_{i \in A_j} \)
we have:

\[ \| \sum_{i \in A_j} a_i f_i \| \geq \delta \sum_{i \in A_j} |a_i|^2. \]

A projection \( P \) on \( H_n \) is \((\delta, r)\)-Rieszable if \( \{Pe_i\}_{i=1}^n \) is \((\delta, r)\)-Rieszable.

Recall that a family of vectors \( \{f_i\}_{i \in I} \) is a frame for a Hilbert space \( H \) if there are constants \( 0 < A, B < \infty \), called the lower (upper) frame bounds) respectively satisfying for all \( f \in H \):

\[ A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2. \]

If \( \|f_i\| = \|f_j\| \) for all \( i, j \), we call this an equal norm frame and if \( \|f_i\| = 1 \) for all \( i \), it is a unit norm frame. If \( A = B \) this is an A-tight frame and if \( A = B = 1 \), it is a Parseval frame. It is known [4 6 9] that \( \{f_i\}_{i \in I} \) is an A-tight frame if and only if the matrix with the \( f_i \)’s as rows has orthogonal columns and the square sums of the column coefficients equal A. It is also known [4 9] that \( \{f_i\}_{i=1}^M \) is a Parseval frame for \( H_n \) if and only if there is an orthogonal projection \( P : \ell_2^M \to H_n \) with

\[ Pe_i = f_i, \quad \text{for all } i = 1, 2, \ldots, M, \]

where \( \{e_i\}_{i=1}^M \) is the unit vector basis of \( \ell_2^M \).
The following result can be found in [5, 10].

**Proposition 2.2.** Fix a natural number \( r \in \mathbb{N} \). The following are equivalent:

1. The class of projections with constant diagonal \( 1/r \) are pavable.
2. The class of projections with constant diagonal \( 1/r \) are Rieszable.
3. The class of unit norm \( r \)-tight frames \( \{ f_m \}_{m=1}^{nr} \) for \( \mathcal{H}_n \) are Rieszable.

Moreover, the Paving Conjecture is equivalent to (1)-(3) holding for some \( r \in \mathbb{N} \).

We will construct concrete counterexamples for (4) of Proposition 2.2 for the case \( r = 2 \). These will give concrete counterexamples to 1-3 in the proposition by the following result which can be found in [5]. The point here is that the proof of this proposition gives an explicit representation of each of the equivalences in the proposition in terms of all the others.

**Proposition 2.3.** Let \( P \) be an orthogonal projection on \( \mathcal{H}_n \) with matrix \( B = (a_{ij})_{i,j=1}^n \). The following are equivalent:

1. The vectors \( \{Pe_i\}_{i=1}^n \) is \((\delta, r)\)-Rieszable.
2. There is a partition \( \{A_j\}_{j=1}^r \) of \( \{1, 2, \ldots, n\} \) so that for all \( j = 1, 2, \ldots, r \) and all scalars \( \{a_i\}_{i \in A_j} \) we have
   \[
   \| \sum_{i \in A_j} a_i (I - P) e_i \|^2 \leq (1 - \delta) \sum_{i \in I} |a_i|^2.
   \]
3. The matrix of \( I - P \) is \((\delta, r)\)-pavable.

As a fundamental tool in our work, we will work with the \( n \times n \) discrete Fourier transform matrices which we will just call DFT matrices or \( DFT_{n \times n} \). For these, we fix \( n \in \mathbb{N} \) and let \( \omega \) be a primitive \( n \)th root of unity and define

\[
DFT_{n \times n} = \left( \frac{1}{\sqrt{n}} \omega^{ij} \right)_{i,j=1}^n.
\]

The main point of these \( DFT_{n \times n} \) matrices is that they are unitary matrices for which the modulus of all of the entries of the matrix are equal to 1. We will use on the following simple observation.

**Proposition 2.4.** If \( A = (a_{ij})_{i,j=1}^n \) is a matrix with \( |a_{ij}|^2 = a \) for all \( i, j \) and orthogonal columns and we multiply the \( j \)th-column of \( A \) by a constant \( C_j \) to get a new matrix \( B \), then

1. The columns of \( B \) are orthogonal.
2. The square sums of the coefficients of any row of \( B \) all equal
   \[
   a \sum_{j=1}^n C_j^2.
   \]
3. The square sum of the coefficients of the \( j \)th column of \( B \) equal \( aC_j^2 \).
3. The Bourgain-Tzafriri Conjecture for \( r=2 \)

Let us first outline our construction. For any natural number \( n \), we will alter two \( 2n \times 2n \) DFT matrices along the lines of Proposition 2.4 and then stack them on top of one another to get a \( 4n \times 2n \) matrix with the following properties:

1. Each altered DFT has the square sums of the coefficients of any row equal to 1.
2. The top altered DFT will have the square sums of the coefficients of each column \( j \) with \( 1 \leq j \leq n-1 \) equal to 2, and the square sums of the coefficients of the remaining columns will all equal \( 2/(n+1) \).
3. The combined matrix will have the square sums of the coefficients of each column equal to 2.
4. The columns of the combined matrix are orthogonal.

It follows that this is the matrix of a unit norm 2-tight frame and hence multiplying the matrix by \( 1/\sqrt{2} \) will turn it into an equal norm Parseval frame, creating the matrix of a rank \( 2n \) projection on \( \mathbb{C}^{4n} \) with constant diagonal 1/2. We will then show that the rows of this class of matrices are not uniformly 2-Rieszable to complete the example.

So we start with a \( 2n \times 2n \) DFT and multiply the first \( n-1 \) columns by \( \sqrt{2} \) and the remaining columns by \( 2/(n+1) \) to get a new matrix \( B_1 \). Now, we take the second \( 2n \times 2n \) DFT matrix and multiply the first \( n-1 \) columns by 0 and the remaining columns by \( 2/(n+1) \) to get a matrix \( B_2 \). We form the matrix \( B \) by stacking the matrices \( B_1 \) and \( B_2 \) on top of one another to get the matrix \( B \) given below.

| (n-1)-Colmns | (n+1)-Colms. |
|--------------|--------------|
| \( \sqrt{2} \) | \( 2/(n+1) \) |
| 0            | \( 2/(n+1) \) |

Now we can prove:

**Proposition 3.1.** The matrix \( B \) satisfies:

1. The columns are orthogonal and the square sum of the coefficients of every column equals 2.
2. The square sum of the coefficients of every row equals 1.

The row vectors of the matrix \( B \) are not \((\delta,2)\)-Rieszable, for any \( \delta \) independent of \( n \).

**Proof.** Clearly the columns of \( B \) are orthogonal. To check the square sums of the column coefficients, recall that for columns \( 1 \leq \ell \leq n-1 \) the modulus of all the coefficients of \( B_1 \) are \( \frac{1}{\sqrt{n}} \), the the coefficients of \( B_2 \) are 0. So the square sum of the coefficients in column \( \ell \) are:

\[
\frac{1}{n} \cdot 2n + 0 = 2.
\]
For the columns $n \leq \ell \leq 2n$, the modulus of the coefficients of $B_1$ are $\frac{1}{\sqrt{n(n+1)}}$ and the coefficients of $B_2$ are $\frac{1}{\sqrt{n+1}}$. So the square sum of the coefficients of $B$ in column $\ell$ are:

$$2n \cdot \frac{1}{n(n+1)} + 2n \cdot \frac{1}{n+1} = \frac{2}{n+1} + \frac{2n}{n+1} = 2.$$

Now we check the row sums. For any row of $B_1$, the first $n-1$ column coefficients have modulus $\frac{1}{\sqrt{n}}$ and the modulus of the coefficients of the last $n+1$ columns of $B_1$ have modulus $\frac{1}{\sqrt{n(n+1)}}$. So the square sum of the coefficients of any row of $B_1$ are:

$$(n-1)\frac{1}{n} + (n+1)\frac{1}{n(n+1)} = 1.$$  

For any row of $B_2$, the first $n-1$ column coefficients are equal to 0 and the remaining $n+1$ column coefficients have modulus $\frac{1}{\sqrt{n+1}}$. So the square sum of the row coefficients of $B_2$ are

$$(n+1)\frac{1}{n+1} + 0 = 1.$$  

We will now show that the row vectors of $B$ are not two Rieszable. So let $\{A_1, A_2\}$ be a partition of $\{1, 2, \ldots, 4n\}$. Without loss of generality, we may assume that $|A_1 \cap \{1, 2, \ldots, 2n\}| \geq n$. Let the row vectors of the matrix $B$ be $\{f_i\}_{i=1}^{4n}$ as elements of $\mathbb{C}^{2n}$. Let $P_{n-1}$ be the orthogonal projection of $\mathbb{C}^{2n}$ onto the first $n-1$ coordinates. Since $|A_1| \geq n$, there are scalars $\{a_i\}_{i \in A_1}$ so that $\sum_{i \in A_1} |a_i|^2 = 1$ and

$$P_{n-1} \left( \sum_{i \in A_1} a_i f_i \right) = 0.$$  

Also, let $\{g_j\}_{j=1}^{2n}$ be the orthonormal basis consisting of the original rows of the $DFT_{2n \times 2n}$. We now have:

$$\| \sum_{i \in A_1} a_i f_i \|^2 = \| (I - P_{n-1}) \left( \sum_{i \in A_1} a_i f_i \right) \|^2$$

$$= \frac{2}{n+1} \| (I - P_{n-1}) \left( \sum_{i \in A_1} a_i g_i \right) \|^2$$

$$\leq \frac{2}{n+1} \| \sum_{i \in A_1} a_i g_i \|^2$$

$$= \frac{2}{n+1} \sum_{i \in A_1} |a_i|^2$$

$$= \frac{2}{n+1}.$$
Letting $n \to \infty$, we have that this class of matrices is not $(\delta, 2)$-pavable for any $\delta > 0$. □

4. The Bourgain-Tzafriri Conjecture for general $r$

In this section we will extend our construction to projections with constant diagonal $1/r$ and actually prove a stronger result.

**Proposition 4.1.** For every natural number $r \geq 2$, there is a $r^2n \times rn$ projection matrix with constant diagonal $1/r$ so that whenever we partition \( \{1, 2, \ldots, r^2n\} \) into sets $\{A_j\}_{j=1}^r$, and for all $k = 1, 2, \ldots, r$, if $D_k = \{(k-1)rn + 1, (k-1)rn + 2, \ldots, krn\}$, then for every $k = 1, 2, \ldots, r-1$, there is a $j$ so that the vectors $\{f_i\}_{i \in A_j \cap D_k}$ are not uniformly 2-Rieszable.

This time, we will take $r$ DFT matrices of size $rn \times rn$ and alter their columns by certain amounts so that when we stack them on top of one another to get a matrix $B$ of size $r^2n \times rn$ satisfying:

1. The columns of $B$ are orthogonal and the sums of the squares of the coefficients of each row of $B$ equals 1.

2. The sums of the squares of the coefficients of each column of $B$ equals $r$.

3. $B$ satisfies the requirements of the proposition.

For the first matrix $B_1$ we take the $rn \times rn$ DFT and multiply the first $n-1$ columns by $\sqrt{r}$ and the remaining columns by $\sqrt{\delta_1}$ (to be chosen later). For $B_2$ we take the $rn \times rn$ DFT and multiply the first $n-1$ columns by 0, multiply the columns $n-1 + j$, $j = 1, 2, \ldots, n-1$ by $\sqrt{r - \delta_j}$, and multiply the remaining columns by $\sqrt{\delta_2}$ (to be chosen later). And for $k = 3, \ldots, r-1$ we construct the matrix $B_k$ by taking the $rn \times rn$ DFT and multiplying the first $(k-1)(n-1)$ columns by 0, multiply the columns $(k-1)(n-1) + j$ for $j = 1, 2, \ldots, n-1$ by

$$\sqrt{r - \sum_{i=1}^{k-1} \delta_{k-1}},$$

and multiplying the remaining columns by $\sqrt{\delta_k}$ (to be chosen later). Finally, for $B_r$ we take the $rn \times rn$ DFT and multiplying the first $(r-1)(n-1)$ columns by 0 and the remaining columns by $\sqrt{\delta_r}$ (to be chosen later).

We then stack these $r$, $rn \times rn$ matrices $\{B_k\}_{k=1}^r$ on top of each other to produce the matrix $B$ for which the moduli of the coefficients of $B$ are given in figure 2 below. Now we must show that the matrix $B$ has all of the properties of Proposition [4.1]

It is clear that the columns of $B$ are orthogonal. To show that the square sums of the row coefficients of the matrix $B$ are all equal to 1, we need a lemma.
Lemma 4.2. To get the rows of the matrix $B$ to square sum to 1, we need

$$
\delta_k = \frac{r^2 n}{(r - k + 1)n + k - 1][(r - k)n + k] \quad (1)
$$

Proof. We will proceed by induction on $k$ to show Equation (1) for all $k = 1, 2, \ldots, r$. For $k = 1$, we observe that the coefficients of the first $n - 1$ columns of $B_1$ have modulus equal to $1/n$, while the coefficients of the remaining $rn - (n - 1)$ columns of $B_1$ have modulus $\sqrt{\frac{\delta_1}{r n}}$. So the sum of the squares of the coefficients of any row of $B_1$ equals

$$
\frac{1}{r n} [r(n - 1) + \delta_1 (rn - (n - 1))] = 1.
$$

Hence,

$$
\delta_1 (rn - (n - 1)) = rn - r(n - 1) = r.
$$

So,

$$
\delta_1 = \frac{r}{(r - 1)n + 1} = \frac{r^2 n}{[(r - 1)n + 1 - 1][(r - 1)n + 1]}.
$$

For $k = 2$, our matrix $B_2$ has coefficients of the first $n - 1$ columns equal to 0, coefficients of the columns $(n - 1) + j$, $j = 1, 2, \ldots, n - 1$ have modulus equal to

$$
\frac{\sqrt{r - \delta_1}}{r n},
$$

and the remaining $rn - 2(n - 1)$ columns have modulus equal to $\sqrt{\frac{\delta_2}{r n}}$. So the square sums of the coefficients of any row of $B_2$ equals

$$
\frac{1}{r n} [(n - 1)(r - \delta_1) + (rn - 2(n - 1))\delta_2] = 1.
$$

Since

$$
r - \delta_1 = r - \frac{r}{(r - 1)n + 1} = \frac{r(r - 1)n}{(r - 1)n + 1},
$$

we can solve the equation to get

$$
\delta_2 = \frac{r^2 n}{[(r - 1)n + 1][(r - 2)n + 2]}.
$$

Now assume our formula holds for any $k \leq r - 1$ and we check it for $k + 1$. The matrix $B_{k+1}$ has coefficients of the first $k(n - 1)$ columns equal to 0, coefficients of the columns $k(n - 1) + j$, $j = 1, 2, \ldots, n - 1$ of modulus

$$
\left( \frac{r - \sum_{j=1}^{k} \delta_j}{rn} \right)^{1/2},
$$

and the coefficients of the remaining columns have modulus $\sqrt{\frac{\delta_{k+1}}{r n}}$. It follows that the square sums of the row coefficients of the matrix $B_{k+1}$ must
satisfy

\[
\left( r - \sum_{j=1}^{k} \delta_j \right) (n - 1) + \delta_{k+1}[rn - (k + 1)(n - 1)] = rn.
\]

Hence, letting \( a = \frac{rn}{n - 1} \) we have

\[
\sum_{j=1}^{k} \delta_j = r^2n \sum_{j=1}^{k} \frac{1}{(r - j + 1)n + j - 1} \left( (r - j)n + j \right) = \frac{r^2n}{(n - 1)^2} \sum_{j=1}^{k} \frac{1}{a + 1 - j(a - j)} \]

\[
= \frac{r^2n}{(n - 1)^2} \left( \frac{1}{a - j} - \frac{1}{a - (j - 1)} \right) \]

\[
= \frac{r^2n}{(n - 1)^2} \left( \frac{1}{a - k} - \frac{1}{a - 0} \right) \]

\[
= \frac{r^2n}{(n - 1)^2} \frac{k}{a(a - k)} \]

\[
= \frac{r^2kn}{rn(rn - k(n - 1))} \]

\[
= \frac{rk}{(r - k)n + k}. \]

Combining this with Equation 2 we have

\[
\delta_{k+1} = \frac{r + (n - 1) \sum_{j=1}^{k} \delta_j}{rn - (k + 1)(n - 1)} \]

\[
= \frac{r + (n - 1) \left( \frac{r}{(r - k)n + k} \right)}{(r - k + 1)n + k - 1} \]

\[
= \frac{r[(r - k)n + k] + (n - 1)rk}{[(r - k + 1)n + k - 1][(r - k)n + k]} \]

\[
= \frac{r^2n - rkn + rk + rnk - kr}{[(r - k + 1)n + k - 1][(r - k)n + k]} \]

\[
= \frac{r^2n}{[(r - k + 1)n + k - 1][(r - k)n + k]}. \]

\]

By Lemma 4.2, we know that the rows of the matrix \( B \) square sum to 1. Now we need to check the column sums. Most of this is true by our definitions. We check two cases:
Case 1: For a column $\ell = k(n - 1) + j$, $k = 1, 2, \ldots, r - 1$, the column coefficients for $1 \leq j \leq k - 1$ and $i = jrn + m$, $m = 1, 2, \ldots, rn$, have modulus $\sqrt{\frac{\delta_j}{rn}}$, and for $i = krn + m$, $m = 1, 2, \ldots, rm$ the modulus of the coefficients are $\sqrt{\frac{r - \sum_{j=1}^{k-1} \delta_j}{rn}}$, and all other coefficients are 0. Hence, the square sum of the column coefficients is

$$rn \sum_{j=1}^{k-1} \frac{\delta_j}{rn} + rn \left( \frac{r - \sum_{j=1}^{k-1} \delta_j}{rn} \right) = r.$$ 

Case 2: For a column $\ell = (r - 1)(n - 1) + j$, with $j = 1, 2, \ldots, rn - (r - 1)(n - 1) = r + n - 1$, the square sum of the coefficients of column $\ell$ are (using our formula for the sum of the $\delta_k$ above):

$$\sum_{k=1}^{r} \delta_k = \frac{r^2}{(r-r)n + r} = r.$$ 

Finally, we need to show that our matrix $B$ is not paving (with paving constants independent of $n$) in the strong sense given in the proposition. This follows similarly to the $DFT_{2n \times 2n}$ case. Let $\{f_i\}_{i=1}^{2n}$ be the rows of the matrix $B$ and let $\{g_i\}_{i=1}^{2n}$ be the rows of the DFT matrix. Also, let $P_k$ be the orthogonal projection of $C_{2n}^{rn}$ onto the first $k(n-1)$ coordinates. Now let $\{A_j\}_{j=1}^{r}$ be a partition of $\{1, 2, \ldots, r^2n\}$ and fix $1 \leq k \leq r - 1$. Then there is a $j$ so that $|A_j \cap D_k| \geq n$. Since the vectors $\{f_i\}_{i \in A_j \cap D_k}$ have zero coordinates for all $j = 1, 2, \ldots, (k-1)(n-1)$, and there are scalars $\{a_i\}_{i \in A_j \cap D_k}$ satisfying

1. $\sum_{i \in A_j \cap D_k} |a_i|^2 = 1.$
2. We have

$$P_k \left( \sum_{i \in A_j \cap D_k} a_i f_i \right) = 0.$$
It follows from our construction that
\[
\| \sum_{i \in A_j \cap D_k} a_i f_i \|^2 = \|(I - P_k) \left( \sum_{i \in A_j \cap D_k} a_i g_i \right) \|^2 \\
= \delta_k \|(I - P_k) \left( \sum_{i \in A_j \cap D_k} a_i g_i \right) \|^2 \\
\leq \delta_k \sum_{i \in A_j \cap D_k} a_j g_j \\
= \delta_k \sum_{i \in A_j \cap D_k} |a_i|^2 \\
= \delta_k
\]
Since \( \lim_{n \to \infty} \delta_k = 0 \), it follows that our family of matrices are not 2-Rieszable in the strong sense of the Proposition. This argument looks pictorially as:

Each square is a \( rn \times (n - 1) \) submatrix

| \[ \sqrt{\frac{\delta_k}{rn}} \] | \[ \sqrt{\frac{\delta_k}{rn}} \] | \[ \sqrt{\frac{\delta_k}{rn}} \] | \ldots | \[ \sqrt{\frac{\delta_k}{rn}} \] |
|-----------------|-----------------|-----------------|\ldots |-----------------|
| \[ 0 \] | \[ \sqrt{\frac{\delta_k}{rn}} \] | \[ \sqrt{\frac{\delta_k}{rn}} \] | \ldots | \[ \sqrt{\frac{\delta_k}{rn}} \] |
| \[ 0 \] | \[ 0 \] | \[ \sqrt{\frac{\delta_k}{rn}} \] | \ldots | \[ \sqrt{\frac{\delta_k}{rn}} \] |
| \[ \vdots \] | \[ \vdots \] | \[ \vdots \] | \[ \ldots \] | \[ \ldots \] |
| \[ 0 \] | \[ 0 \] | \[ 0 \] | \[ \ldots \] | \[ \sqrt{\frac{\delta_k}{rn}} \] |

The main question is whether it is possible to take the concrete constructions in this paper and generalize them to give a complete counterexample to the Paving Conjecture.

5. The Proof of Theorem 1.2

Proof. (1) \( \Rightarrow \) (2): This is from Proposition 2.2

(2) \( \Rightarrow \) (1): Let \( P \) be a projection with constant diagonal \( 1/2 \) on \( \mathcal{H}_{2N} \). So \( \{ \sqrt{2} P e_i \}_{i=1}^{2N} \) is a unit norm 2-tight frame for \( \mathcal{H}_{2N} \). Let \( A \) be the \( N \times N \) matrix with row vectors \( \{ \sqrt{2} P e_i \}_{i=1}^{2N} \). Define recursively,

\[
A_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} A & A \\ A & -A \end{bmatrix}
\]

and

\[
A_{K+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} A_K & A_K \\ A_K & -A_K \end{bmatrix}
\]
Note: Each $A_K$ (their rows) is a unit norm 2-tight frame for $H_{2K N}$. Since the columns of $A_K$ are orthogonal, this implies that the columns of $A_{K+1}$ are orthogonal. Also, clearly the sums of the squares of the row elements are still one and the sums of the squares of the column elements are still one.

Also, the entries $(a_{i,j})$ of $A_K$ satisfy

$$|a_{i,j}| \leq \frac{1}{\sqrt{2K}} = \frac{\sqrt{N}}{\sqrt{2K N}}. \tag{3}$$

Letting $C = \sqrt{N}$ in (2) of the theorem, there is some $N_0$ such that for every $L \geq N_0$, if $\{f_i\}_{i=1}^{2L}$ is a unit norm 2-tight frame for $H_L$ with

$$|f_{i,j}| \leq \frac{C}{\sqrt{2L}},$$

then $\{f_i\}_{i=1}^{2L}$ is $(\delta, r)$-Rieszable. Hence, for $K$ large enough, Equation 3 has this inequality. So, $A_K$ is $(\delta, r)$-Rieszable. That is, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, 2^K N\}$ so that for every $j = 1, 2, \ldots, r$ and all scalars $\{a_i\}_{i \in A_j}$ we have

$$\|\sum_{i \in A_j} a_i f_i\|^2 \geq \delta \sum_{i \in A_j} |a_i|^2,$$

where $\{f_i\}$ are the row vectors of $A_K$. Let

$$B_j = A_j \cap \{1, 2, \ldots, N\}.$$ 

Then $\{B_j\}_{j=1}^r$ is a partition of $\{1, 2, \ldots, N\}$. Now we compute,

$$\delta \leq \left\| \sum_{i \in B_j} a_i f_i \right\|^2$$

$$= \frac{1}{2^K} \sum_{\ell=1}^{2^K} \left\| \sum_{i \in B_j} a_i \sum_{j=1}^{N} f_{i, \ell+j} \right\|^2$$

$$= \frac{1}{2^K} \cdot 2^K \left\| \sum_{i \in B_j} a_i \sqrt{2P} e_i \right\|^2$$

$$= \left\| \sum_{i \in B_j} a_i \sqrt{2P} e_i \right\|^2.$$

Hence, $A$ is $(\delta, r)$-Rieszable and hence KS holds by Proposition 2.2. \qed

Remark 5.1. The above points out that there really is a major difference between “paving” and “Rieszing”. Recall that if $\{f_i\}_{i=1}^M$ is a set of vectors, the Grammian of this family is the $M \times M$ matrix $((f_i, f_j))$. In the above construction, if $G_A$ is the Grammian of the row vectors $A$ then the
Grammian of of the row vectors of $A_K$ is

$$
\begin{bmatrix}
G_A & 0 & 0 & \cdots \\
0 & G_A & 0 & \cdots \\
0 & 0 & G_A & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

That is, the coefficients of the Grammian do not get smaller in this construction while the coefficients of the matrix do get smaller.

**Remark 5.2.** This result also says that passing results on paving from the Grammian back to the matrix and the other way do not hold in general.

**Acknowledgments**

Casazza and Tremain were supported by NSF DMS 0704216 and 1008183, Fickus was supported by AFOSR F1ATA09125G003. The views expressed in this article are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government.

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Casazza/Tremain: Department of Mathematics, University of Missouri, Columbia, MO 65211-4100, Fickus: Department of Mathematics and Statistics, Air Force Institute of Technology, Wright-Patterson Air Force Base, OH 45433, Mixon: Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544

E-mail address: casazzap@missouri.edu, Matthew.Fickus@afit.edu,
E-mail address: dmixon@princeton.eduj.tremain@mchsi.com