MODULI SPACES OF ULRICH BUNDLES
ON THE FANO 3-FOLD $V_5$

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Abstract. We study moduli spaces of Ulrich bundles of rank $r \geq 2$ on the Fano 3-fold $V_5$ of Picard number 1, degree 5 and index 2. We prove that the moduli space of stable rank $r$ Ulrich bundles on $V_5$ is a smooth $(r^2 + 1)$-dimensional open subset of the moduli space of stable quiver representations.

1. Introduction

Ulrich bundles form interesting classes of vector bundles on higher dimensional algebraic varieties. Recently, existence and moduli of Ulrich bundles draw lots of attention. It turns out that a large class of algebraic varieties admit Ulrich bundles (see [3] for more details) and it seems that the moduli space of Ulrich bundles of a given variety reflects many interesting features of the variety.

Kuznetsov studied instanton bundles on some Fano 3-folds via bounded derived categories of coherent sheaves in [11]. Using derived categories, Lahoz, Macrì and Stellari studied moduli spaces of Ulrich bundles on cubic 3-folds and 4-folds in [12, 13], and Cho, Kim and the first named author studied intersection of two 4-dimensional quadrics cases in [7].

In this paper, we study moduli spaces of Ulrich bundles on $V_5$ which is the unique smooth Fano 3-fold of Picard number 1, degree 5 and index 2. The Fano variety $V_5$ is a very interesting Fano 3-fold which enjoy many beautiful geometric and topological properties. Vector bundles on $V_5$ was studied by Faenzi in [9], Arrondo-Costa in [1] and Kuznetsov in [11]. The bounded derived category $D(V_5)$ of coherent sheaves on $V_5$ was studied by Orlov in [15] where he proved that there are full exceptional collections in $D(V_5)$. From one of the exceptional collection and Bondal’s work (cf. [4]), we see that the derived category of the finitely generated modules of the path algebra associated to a quiver $\Gamma$ is embedded in $D(V_5)$. Using this semiorthogonal decomposition of derived category of $V_5$ we obtain the following result (see Theorem 3.11 for more precise statement).

Theorem 1.1. For any $r \geq 2$, there is a nontrivial $(r^2 + 1)$-dimensional family of stable rank $r$ Ulrich bundles on $V_5$. Moreover, the moduli space of stable rank $r$ Ulrich bundles on $V_5$ is a smooth open subset of the moduli space of stable quiver representations. In particular, $V_5$ is of wild representation type.

Conventions. We will work over $\mathbb{C}$. For a variety $X$, we will use $D(X)$ to denote the bounded derived category of coherent sheaves on $X$. For a quiver $\Gamma$, we will consider only finite-dimensional representations of $\Gamma$.

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2. Preliminaries

In this section we briefly recall basic definitions and facts about Ulrich bundles, quiver representations and the Fano 3-fold $V_5$.

2.1. Ulrich bundles. Let us recall the definition of Ulrich bundles on smooth projective varieties.

Definition 2.1. Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension $d$. An Ulrich bundle $E$ is a vector bundle on $X$ satisfying

$$H^i(X, E(-j)) = 0$$

for all $i = 0, \cdots, d$ and $j = 1, \cdots, d$. Here, we denote the twisted bundle $E \otimes \mathcal{O}_X(-j)$ by $E(-j)$.

Ulrich bundles enjoy many nice properties and existence and moduli spaces of Ulrich bundles on a given variety tell us many things about the variety. See [3] and references therein for more details about Ulrich bundles.

2.2. Quiver representations. Let us recall some basic definitions and properties of quiver representations as follows. See [8] for more discussions about quiver representations.

Definition 2.2. A quiver $\Gamma$ is an oriented graph, i.e. $\Gamma = (\Gamma_0, \Gamma_1)$ consists of a finite set $\Gamma_0$ of vertices and a finite set $\Gamma_1$ of arrows.

Definition 2.3. Let $\Gamma$ be a quiver. A representation $V_{\Gamma}$ of the quiver $\Gamma$ is a collection of vector spaces $V_i$ for each vertex $i \in \Gamma_0$ and a collection of linear maps $f_{ij}^V : V_i \to V_j$ for each arrow $\rho_{ij} \in \Gamma_1$. A subrepresentation $W_{\Gamma}$ of $V_{\Gamma}$ is a quiver representation with injective linear maps $\iota_i : W_i \to V_i$ such that the following diagram commutes for each $\rho_{ij} \in \Gamma_1$.

$$
\begin{array}{c}
W_i \xrightarrow{f_{ij}^W} W_j \\
\downarrow \iota_i & & \downarrow \iota_j \\
V_i \xrightarrow{f_{ij}^V} V_j
\end{array}
$$

When we have a quiver $\Gamma$ and a sequence of integers $d = (d_i)_{i \in \Gamma_0}$, it is an interesting task to classify $\Gamma$-representations $V_{\Gamma}$ which have given dimension $d_{\Gamma} = (d_i)_{i \in \Gamma_0}$, i.e. each vector space $V_i$ has dimension $d_i$. Two standard ways will be constructing the moduli space of such representations using stacks or introducing certain stability conditions on the $\Gamma$-representations. Indeed, there are well-known stability conditions studied by King as follows (cf. [10], [16]).
**Definition 2.4.** Let $\Gamma$ be a quiver and $\theta : \mathbb{Z}^{|\Gamma_0|} \to \mathbb{Z}$ be a $\mathbb{Z}$-linear map. Then a $\Gamma$-representation $V_\Gamma = (V_i, f_{ij})$ is $\theta$-(semi)stable if $\theta(\underline{d}^W) < \theta(\underline{d}^V)$ (respectively, $\theta(\underline{d}^W) \leq \theta(\underline{d}^V)$) for every subrepresentation $W_\Gamma$ of $V_\Gamma$.

Using the above $\theta$-stability condition, King constructed moduli spaces of $\theta$-semistable $\Gamma$-representations via geometric invariant theory in [10].

**Theorem 2.5 (10).** Fix $\Gamma$, $\underline{d}$ and $\theta$ as above. Then there exists a coarse moduli space $M^\theta, ss(\Gamma)$ (resp. $M^\theta, ss(\Gamma)$) of $\theta$-semistable (resp. $\theta$-stable) $\Gamma$-representations which is an irreducible normal projective (resp. quasi-projective) variety.

From now on let us use $M^\theta, ss(\Gamma)$ (resp. $M^\theta, ss(\Gamma)$) to denote the moduli space of $\theta$-semistable (resp. $\theta$-stable) $\Gamma$-representations of dimension $\underline{d}$.

When we have a quiver $\Gamma$, we have an associated path algebra $\mathcal{C}\Gamma$ and it is well-known that there is a natural equivalence between the category of quiver representations and the category of $\mathcal{C}\Gamma$-modules. The following result is also well-known and see [8] for more details.

**Proposition 2.6 ([8]).** (1) Let $\mathcal{C}\Gamma$ be the path algebra associated to a quiver $\Gamma$. Then the homological dimension of $\mathcal{C}\Gamma$-mod is at most 1.

(2) Let $A_\Gamma$ (resp. $B_\Gamma$) be a finite-dimensional quiver representation of $\Gamma$, and let $\underline{a}$ (resp. $\underline{b}$) be its dimension vector. Then we have $\chi(A_\Gamma, B_\Gamma) = \dim \text{Hom}(A_\Gamma, B_\Gamma) - \dim \text{Ext}^1(A_\Gamma, B_\Gamma) = \sum_{i \in \Gamma_0} a_i b_i - \sum_{\rho, \gamma \in \Gamma_1} a_\rho b_\gamma = \chi(\underline{a}, \underline{b})$.

We can compute extensions between quiver representations using the above result. Moreover, using general arguments in homological algebra we have the following corollary.

**Corollary 2.7.** Any general element $F \in \mathsf{D}(\mathcal{C}\Gamma$-mod) in the derived category of $\mathcal{C}\Gamma$-modules is isomorphic to $\bigoplus_i H^i(F)[-i]$.

### 2.3. Equivariant bundles on Grassmannian $\text{Gr}(2, 5)$

Let $\text{Gr}(2, 5)$ be the Grassmannian of 2-dimensional linear subspaces in a 5-dimensional vector space $V$. We have the following tautological sequence on $\text{Gr}(2, 5)$

$$0 \to \mathcal{U} \to \mathcal{O}^{\otimes 5} \to \mathcal{Q} \to 0,$$

where $\mathcal{U}$ is the tautological subbundle of rank 2 and $\mathcal{Q}$ is the universal quotient bundle of rank 3.

In general, we can consider equivariant vector bundles on a rational homogeneous variety $G/P$, where $G$ is a complex semisimple algebraic group and $P$ is its parabolic subgroup. Given an integral $P$-dominant weight $\omega \in \Lambda^+_P$, we have an irreducible representation $V(\omega)$ of $P$ with the highest weight $\omega$. We denote by $\mathcal{E}_\omega$ the corresponding irreducible equivariant vector bundle $G \times_P V(\omega)$ on $G/P$. When we see $\text{Gr}(2, 5)$ as a homogeneous variety of $\text{SL}(5, \mathbb{C})$, we can write $\mathcal{U}, \mathcal{U}^*, \mathcal{Q}, \mathcal{Q}^*$ as equivariant bundles as follows (see [5] [14] [17] and references therein for details about equivariant bundles), where $\omega_1, \cdots, \omega_4$ are the fundamental weights of $\text{SL}(5)$.

$$\mathcal{U} = \mathcal{E}_{\omega_1 - \omega_2} \quad \mathcal{U}^* = \mathcal{E}_{\omega_1}$$

$$\mathcal{Q} = \mathcal{E}_{\omega_2} \quad \mathcal{Q}^* = \mathcal{E}_{-\omega_2 + \omega_3}$$
To compute the cohomology of irreducible equivariant vector bundles on rational homogeneous varieties \(G/P\), we use the famous Borel-Weil-Bott theorem from [5].

**Theorem 2.8** (Borel-Weil-Bott theorem). Let \(G\) be a simply connected complex semisimple algebraic group and \(P \subset G\) be its parabolic subgroup. Suppose that \(\omega\) is an integral weight for \(G\) which is dominant with respect to \(P\). Let \(\rho\) be the sum of fundamental weights of \(G\).

1. If a weight \(\omega + \rho\) is singular, that is, it is orthogonal to some (positive) root of \(G\), then every cohomology group \(H^i(G/P, \mathcal{E}_\omega)\) vanishes.
2. Otherwise, \(\omega + \rho\) is regular, that is, it lies in the interior of some Weyl chamber, then \(H^i(\omega)(G/P, \mathcal{E}_\omega) = V_G(\omega + \rho - \rho)\) and any other cohomology group vanishes. Here, \(\omega \in W\) is a unique element of the Weyl group of \(G\) such that \(\omega(\omega + \rho)\) is strictly dominant and \(\ell(\omega)\) denotes the length of \(\omega \in W\).

2.4. **Fano 3-fold** \(V_5\). The Fano 3-fold \(V_5\) is the unique smooth 3-fold of Picard number 1, degree 5 and index 2. It is well-known that \(V_5\) is a generic codimension 3 linear section of \(\text{Gr}(2, 5)\). The cohomology group of \(V_5\) over \(\mathbb{Z}\) is isomorphic to \(\mathbb{Z}^4\) as an abelian group as follows.

\[
H^*(V_5, \mathbb{Z}) = H^0(V_5, \mathbb{Z}) \oplus H^2(V_5, \mathbb{Z}) \oplus H^4(V_5, \mathbb{Z}) \oplus H^6(V_5, \mathbb{Z})
\]

where \(h \cdot l = p\), \(h^2 = 5l\) and \(h^3 = 5p\). See [9, 11, 15] for more details about the Fano 3-fold \(V_5\).

Because \(V_5\) is a linear section of codimension 3 in \(\text{Gr}(2, 5)\), we have the following natural exact sequence of vector bundles on \(V_5\):

\[
0 \to U \to V \otimes \mathcal{O} \to Q \to 0,
\]

where \(U\) is the restriction of the universal subbundle \(\mathcal{U}\) on \(\text{Gr}(2, 5)\) to \(V_5\) and \(Q\) is the restriction of the universal quotient bundle \(Q\) on \(\text{Gr}(2, 5)\) to \(V_5\). We can compute the Chern classes of \(U, U^*, Q, Q^*\) as follows (cf. Section 3 of [9]).

1. The Chern classes of \(U\) are \(\text{rk}(U) = 2\), \(c_1(U) = -h\) and \(c_2(U) = 2l\).
2. The Chern classes of \(U^*\) are \(\text{rk}(U^*) = 2\), \(c_1(U^*) = h\) and \(c_2(U^*) = 2l\).
3. The Chern classes of \(Q\) are \(\text{rk}(Q) = 3\), \(c_1(Q) = h\), \(c_2(Q) = 3l\) and \(c_3(Q) = p\).
4. The Chern classes of \(Q^*\) are \(\text{rk}(Q^*) = 3\), \(c_1(Q^*) = -h\), \(c_2(Q^*) = 3l\) and \(c_3(Q^*) = -p\).

Using Borel-Weil-Bott theorem, we can compute the cohomology groups \(H^i(V_5, U(j))\) and \(H^i(V_5, Q^*(j))\) for \(j = -2, -1, 0, 1\).

**Lemma 2.9.** The cohomology groups \(H^i(V_5, U(j))\) and \(H^i(V_5, Q^*(j))\) for \(j = -2, -1, 0, 1\) are as follows.

\[
H^i(V_5, U(1)) = \begin{cases} \mathbb{C}^5 & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}
\]

\[
H^i(V_5, Q^*(1)) = \begin{cases} \mathbb{C}^{10} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}
\]

\[
H^*(V_5, U) = H^*(V_5, U(-1)) = 0
\]

\[
H^*(V_5, Q^*) = H^*(V_5, Q^*(-1)) = 0
\]
\[ H^i(V_5, U(-2)) = \begin{cases} \mathbb{C}^5 & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases} \]

\[ H^i(V_5, Q^*(-2)) = \begin{cases} \mathbb{C}^5 & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** Let \( E \) be a vector bundle on \( \text{Gr}(2, 5) \). From the Koszul resolution
\[ 0 \to \mathcal{O}(-3) \to \mathcal{O}(-2)^{\oplus 3} \to \mathcal{O}(-1)^{\oplus 3} \to \mathcal{O} \to \mathcal{O}_{V_5} \to 0, \]
we have an exact sequence
\[ 0 \to E(-3) \to E(-2)^{\oplus 3} \to E(-1)^{\oplus 3} \to E \to E|_{V_5} \to 0. \]

In particular, we have the following two exact sequences:

(2.1) \[ 0 \to U(-2) \to U(-1)^{\oplus 3} \to U^{\oplus 3} \to U(1) \to U(1) \to 0 \]

and

(2.2) \[ 0 \to Q^*(-2) \to Q^*(-1)^{\oplus 3} \to Q^{*\oplus 3} \to Q^*(1) \to Q^*(1) \to 0. \]

Because \( U \) and \( Q^* \) are irreducible equivariant bundles on \( \text{Gr}(2, 5) \), we can compute cohomology groups using Borel-Weil-Bott theorem. For example, note that the weight defining \( U(-1) \) is \( (\omega_1 - \omega_2) - \omega_2 = \omega_1 - 2\omega_2 \). After adding \( \rho \), we apply the second simple reflection on the weight \( \omega_1 - 2\omega_2 + \rho = 2\omega_1 - \omega_2 + \omega_3 + \omega_4 \), then we get a singular weight \( \omega_1 + \omega_2 + \omega_4 \).

\[
\begin{array}{cccccccccc}
2 & -1 & 1 & 1 & \rightarrow & 1 & 1 & 0 & 1 \\
\sigma_2 & & & & & & & & \\
\end{array}
\]

On the other hand, since the weight defining \( U(-5) \) is \( (\omega_1 - \omega_2) - 5\omega_2 = \omega_1 - 6\omega_2 \). After adding \( \rho \), we apply the 6 simple reflections on the weight \( \omega_1 - 6\omega_2 + \rho = 2\omega_1 - 5\omega_2 + \omega_3 + \omega_4 \), then we get a strictly dominant weight \( \omega_1 + \omega_2 + \omega_3 + 2\omega_4 \).

\[
\begin{array}{cccccccccc}
2 & -5 & 1 & 1 & \rightarrow & -3 & 5 & -4 & 1 & \rightarrow & 3 & 2 & -4 & 1 \\
\sigma_2 & & & & & & & & & & & & \\
\sigma_3 & & & & & & & & & & & & \\
\sigma_4 & & & & & & & & & & & & \\
\end{array}
\]

Consequently, we see that
\[ H^i(\text{Gr}(2, 5), U(1)) = \begin{cases} V_{\text{SL}(5)}(\omega_1) \cong \mathbb{C}^5 & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases} \]

\[ H^*(\text{Gr}(2, 5), U) = 0 \]

\[ H^*(\text{Gr}(2, 5), U(-1)) = 0 \]

\[ H^*(\text{Gr}(2, 5), U(-2)) = 0 \]

\[ H^*(\text{Gr}(2, 5), U(-3)) = 0 \]

\[ H^*(\text{Gr}(2, 5), U(-4)) = 0 \]
\[ H^i(\text{Gr}(2,5), U(-5)) = \begin{cases} V_{\text{SL}(5)}(\omega_4) \cong \mathbb{C}^5 & \text{if } i = 6, \\ 0 & \text{otherwise.} \end{cases} \]

Therefore, from the exact sequence (2.1) we get
\[
\begin{align*}
H^i(V_5, U(1)) &= \begin{cases} \mathbb{C}^5 & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases} \\
H^*(V_5, U) &= H^*(V_5, U(-1)) = 0 \\
H^i(V_5, U(-2)) &= \begin{cases} \mathbb{C}^5 & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}
\end{align*}
\]

Similarly, by Borel-Weil-Bott theorem we have
\[
\begin{align*}
H^i(\text{Gr}(2,5), Q^*(1)) &= \begin{cases} V_{\text{SL}(5)}(\omega_3) = \bigwedge^3 \mathbb{C}^5 \cong \mathbb{C}^{10} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases} \\
H^*(\text{Gr}(2,5), Q^*) = 0 \\
H^*(\text{Gr}(2,5), Q^*(-1)) = 0 \\
H^*(\text{Gr}(2,5), Q^*(-2)) = 0 \\
H^*(\text{Gr}(2,5), Q^*(-3)) = 0 \\
H^*(\text{Gr}(2,5), Q^*(-4)) = 0
\end{align*}
\]

\[ H^i(\text{Gr}(2,5), Q^*(-5)) = \begin{cases} V_{\text{SL}(5)}(\omega_1) \cong \mathbb{C}^5 & \text{if } i = 6, \\ 0 & \text{otherwise.} \end{cases} \]

Therefore, from the exact sequence (2.2) we have
\[
\begin{align*}
H^i(V_5, Q^*(1)) &= \begin{cases} \mathbb{C}^{10} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases} \\
H^*(V_5, Q^*) &= H^*(V_5, Q^*(-1)) = 0 \\
H^i(V_5, Q^*(-2)) &= \begin{cases} \mathbb{C}^5 & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}
\end{align*}
\]

□

Orlov proved that there are full exceptional collections of length 4 on \( V_5 \).

**Proposition 2.10** (Theorem 1 of [15]). There are following full exceptional collections in the derived category \( \text{D}(V_5) \).

\[
\text{D}(V_5) = \langle U, Q^*, \mathcal{O}, \mathcal{O}(1) \rangle = \langle \mathcal{O}(-2), \mathcal{O}(-1), (V/U)(-1), U \rangle
\]

From Bondal’s theorem in [4], we see that the triangulated subcategory \( \langle U, Q^* \rangle \) is equivalent to the derived category \( \text{D}(\Gamma\text{-mod}) \) of \( \Gamma \)-modules, where \( \Gamma \) is the quiver with two vertices and three arrows between them. From now on \( \Gamma \) will denote this specific quiver. See [11, 15] for more details.

Let us consider the following diagram with two projections \( p \) and \( q \).

\[
\begin{diagram}
\node{V_5 \times V_5} \arrow{p}{p} \node[2]{V_5} \arrow{q}{q} \node[2]{V_5}
\end{diagram}
\]

Let us recall a resolution of the diagonal over \( V_5 \times V_5 \) obtained by Faenzi.
Proposition 2.11 (Theorem 4.1 and Corollary 5.4 of [9]). There is a resolution of
the diagonal over \( V_5 \times V_5 \) as follows.

\[
0 \to \mathcal{O}(-1,-1) \to U \supset \wedge^2 Q^* \to Q^* \not\boxtimes U \to \mathcal{O} \to \mathcal{O}_\Delta \to 0
\]

By mutation of the exceptional collection, we have another resolution

\[
0 \to U(-1) \not\boxtimes U \to Q^*(-1) \not\boxtimes Q^* \to \mathcal{O}(-1) \not\boxtimes \Omega^1_{\mathbb{P}^6}(1)|_{V_5} \to \mathcal{O} \to \mathcal{O}_\Delta \to 0.
\]

The above resolution will be very useful to compute projection functors.

3. Ulrich bundles on \( V_5 \)

In this section we discuss existence and moduli spaces of rank \( r \) Ulrich bundles on \( V_5 \) for any \( r \geq 2 \).

3.1. Generalities on Ulrich bundles on \( V_5 \). First, it is easy to see that there
is no Ulrich line bundle on \( V_5 \). Moreover, we can see that \((-1)\)-twists of Ulrich
bundles lie on the specific semiorthogonal component.

Lemma 3.1. Let \( E \) be an Ulrich bundle on \( V_5 \). Then \( E(-1) \in \langle U, Q^* \rangle \).

Proof. By the definition of Ulrich bundles, we see that

\[
H^*(V_5, E(-1)) = H^*(V_5, E(-2)) = H^*(V_5, E(-3)) = 0.
\]

Hence, we have that \( E(-1) \in \langle U, Q^* \rangle \) by Proposition 2.10. \( \square \)

Therefore, we can study Ulrich bundles on \( V_5 \) in terms of \( U, Q^* \).

3.2. Rank 2 and 3 cases. Beauville proved that every Fano 3-fold of even index
admits a rank 2 Ulrich bundle and computed its deformation via Serre correspond-
ence.

Proposition 3.2 (Proposition 6.1 and Proposition 6.4 of [3]). There are rank 2
Ulrich bundles on \( V_5 \) and the moduli space of rank 2 special Ulrich bundles is smooth
of dimension 5.

Arrondo and Costa gave explicit rank 3 Ulrich bundle on \( V_5 \) in [1]. Faenzi also
studied ACM bundles on \( V_5 \) up to rank 3 in [9].

Proposition 3.3 (Section 4 of [1], Proposition 6.8 of [9]). There is a rank 3 Ulrich
bundle on \( V_5 \).

Proof. Using Borel-Weil-Bott theorem, one can check that \( \text{Sym}^2(U^*) = \mathcal{E}_{2\omega_1} \) is an
Ulrich bundle on \( \text{Gr}(2,5) \).

From the above diagram, we can see that \( \text{Sym}^2(U^*) = \mathcal{E}_{2\omega_1} \) is an Ulrich bundle on
Gr(2, 5). Because the restriction of an Ulrich bundle to a general hyperplane section is Ulrich, its restriction Sym^2(U^*) to V_5 is also an Ulrich bundle.

3.3. Higher rank cases. Suppose that there exists a rank r Ulrich bundle on V_5. Because we know that E(-1) lies on \langle U, Q^* \rangle, we can express E(-1) in terms of U and Q^*. Let us compute the image of E(-1) in \langle U, Q^* \rangle in terms of quiver representations as follows.

**Proposition 3.4.** For any r \geq 2, an Ulrich bundle E on V_5 corresponds to the following quiver representation.

\[ E(-1) = \text{Coker}(U^{\oplus r} \to Q^{\oplus r}) \]

**Proof.** As a quiver representation, to find the image of E(-1) in \langle U, Q^* \rangle is equivalent to find the image of E(-2) in \langle U(-1), Q^*(-1) \rangle. From the following resolution of diagonal over V_5 \times V_5 (Proposition 2.11),

\[ 0 \to U(-1) \otimes U \to Q^*(-1) \otimes Q^* \to \mathcal{O}(-1) \otimes \Omega^1_{\mathbb{P}^5}(1)|_{V_5} \to \mathcal{O} \to \mathcal{O}_\Delta \to 0, \]

we have the following exact sequence

\[ 0 \to U(-1) \otimes H^*(V_5, U \otimes E(-2)) \to Q^*(-1) \otimes H^*(V_5, Q^* \otimes E(-2)) \]

\[ \to \mathcal{O}(-1) \otimes H^*(V_5, \Omega^1_{\mathbb{P}^5}(1)|_{V_5} \otimes E(-2)) \to \mathcal{O} \otimes H^*(V_5, E(-2)) \to E(-2) \to 0 \]

on V_5. From the exact sequence

\[ 0 \to Q^* \to \mathcal{O}^{\oplus 5} \to U^* \to 0 \]

and the isomorphism U^* \cong U(1) we have the following exact sequence

\[ 0 \to Q^* \otimes E(i) \to E(i)^{\oplus 5} \to U \otimes E(i+1) \to 0. \]

Because E is an Ulrich bundle, it is easy to see that \( H^0(V_5, Q^* \otimes E(-2)) = 0 \) and \( H^3(V_5, U \otimes E(-2)) = 0 \). By stability condition we see that

\[ H^0(V_5, Q^* \otimes E(-2)) = 0, \]

and similarly Serre duality and stability condition imply

\[ H^3(V_5, Q^* \otimes E(-2)) \cong H^0(V_5, Q \otimes E^*)^* = 0. \]

From the exact sequence

\[ 0 \to U \to \mathcal{O}^{\oplus 5} \to Q \to 0, \]

we have the following exact sequence

\[ 0 \to U \otimes E(i) \to E(i)^{\oplus 5} \to Q \otimes E(i) \to 0. \]

Again from stability we have

\[ H^1(V_5, U \otimes E(-2)) \cong H^0(V_5, Q \otimes E(-2)) = 0 \]

and

\[ H^1(V_5, Q^* \otimes E(-2)) \cong H^0(V_5, U \otimes E(-1)) = 0 \]

From the Riemann-Roch formula, we have

\[ h^2(V_5, U \otimes E(-2)) = \chi(V_5, U \otimes E(-2)) = r \]

and

\[ h^2(V_5, Q^* \otimes E(-2)) = \chi(V_5, Q^* \otimes E(-2)) = r. \]

Therefore, we get the conclusion. \( \square \)
Let us compute the image of the line bundle $O(2)$ on $V_5$ in $\langle U, Q^* \rangle$.

**Lemma 3.5.** The image of $O(2)$ in $\langle U, Q^* \rangle$ is $\text{Coker}(U \oplus_5 \to Q^* \oplus_{10})$.

**Proof.** As a quiver representation, to find the image of $O(2)$ in $\langle U, Q^* \rangle$ is equivalent to find the image of $O(1)$ in $\langle U(-1), Q^*(-1) \rangle$. From the resolution of diagonal over $V_5 \times V_5$ in Proposition 2.11, we have the following exact sequence
\[
0 \to U(-1) \otimes H^*(V_5, U(1)) \to Q^*(-1) \otimes H^*(V_5, Q^*(1)) \to O(-1) \otimes H^*(V_5, O(1)) \to O(1) \to 0
\]
on $V_5$. We already in Lemma 2.9 saw that $H^i(V_5, U(1)) = \begin{cases} \mathbb{C}^5 & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$ and $H^i(V_5, Q^*(1)) = \begin{cases} \mathbb{C}^{10} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$ Therefore, we see that the image of $O(2)$ in $\langle U, Q^* \rangle$ is $\text{Coker}(U \oplus_5 \to Q^* \oplus_{10})$. \qed

From now on, let us use $R_\bullet$ to denote the quiver representation of $\Gamma$ corresponding to the image of $O(2)$ in $\langle U, Q^* \rangle$.

**Proposition 3.6.** For any $r \geq 2$, the moduli space of stable rank $r$ Ulrich bundles is smooth of dimension $r^2 + 1$.

**Proof.** From the presentation of a stable rank $r$ Ulrich bundle $E$ via quiver representation (Proposition 3.4)
\[
E(-1) = \text{Coker}(U^{\oplus r} \to Q^{\oplus_{10}}),
\]
we see that $\text{ext}^2(E, E) = \text{ext}^3(E, E) = 0$ and $\text{ext}^0(E, E) - \text{ext}^1(E, E) = \chi(E, E) = (r^2 + r^2) - 3r^2 = -r^2$. Note that the path algebra associated to the quiver is hereditary, i.e. $\text{Ext}^i(A_\Gamma, B_\Gamma) = 0$ for any $i \geq 2$ and $\Gamma$-representations $A_\Gamma, B_\Gamma$. Therefore, the moduli space of stable rank $r$ Ulrich bundle is smooth of dimension $r^2 + 1$. \qed

It remains to prove the existence of rank $r$ Ulrich bundles on $V_5$ for any $r \geq 2$. From the Serre correspondence in [3] and explicit construction in [1], we know that there are rank 2 and rank 3 Ulrich bundles on $V_5$. Then we can prove it via deformation argument which was also used in [6, 7]. The argument is almost the same as that of [6, Theorem 5.7], possibly except the computation of $\text{ext}^i$. Indeed, we can easily compute $\text{ext}^i$ between two quiver representations and the computation of $V_5$ case is exactly the same as the cubic 3-fold and intersection of two 4-dimensional quadrics cases in [6, 7]. Therefore, let us skip the proof.

**Lemma 3.7.** For any $r \geq 2$, there is a stable rank $r$ Ulrich bundle on $V_5$.

Using the existence, we can prove that a general member of the above quiver representations gives an Ulrich bundle.

**Lemma 3.8.** Let $E(-1) = \text{Coker}(U^{\oplus r} \to Q^{\oplus_{10}})$ be a general member of the quiver representation. Then $E$ is an Ulrich bundle of rank $r$ on $V_5$. 

Proof. Let \( E \) be a vector bundle on \( \text{Gr}(2, 5) \) such that \( E|_{V_5} = E \) on \( V_5 \). As in the proof of Lemma 2.9 from the Koszul resolution
\[
0 \to \mathcal{O}(-3) \to \mathcal{O}(-2)^{\oplus 3} \to \mathcal{O}(-1)^{\oplus 3} \to \mathcal{O} \to \mathcal{O}_{V_5} \to 0,
\]
we have an exact sequence
\[
0 \to \mathcal{E}(-3) \to \mathcal{E}(-2)^{\oplus 3} \to \mathcal{E}(-1)^{\oplus 3} \to \mathcal{E} \to \mathcal{E}|_{V_5} \to 0.
\]
In particular, we have the following two sequences
\[
0 \to \mathcal{U}(k - 3) \to \mathcal{U}(k - 2)^{\oplus 3} \to \mathcal{U}(k - 1)^{\oplus 3} \to \mathcal{U}(k) \to \mathcal{U}(k)|_{V_5} \to 0
\]
and
\[
0 \to \mathcal{Q}^*(k - 3) \to \mathcal{Q}^*(k - 2)^{\oplus 3} \to \mathcal{Q}^*(k - 1)^{\oplus 3} \to \mathcal{Q}^*(k) \to \mathcal{Q}^*(k)|_{V_5} \to 0.
\]
From the condition \( E(-1) = \text{Coker}(U^{\oplus r} \to Q^{\oplus r}) \) and the results in Lemma 2.9, we have \( H^*(V_5, E(-1)) = H^*(V_5, E(-2)) = 0 \). Moreover, we have \( H^i(V_5, E(-3)) = 0 \) for \( i = 0, 1, 2 \) and the following sequence
\[
0 \to H^3(V_5, U(-2)^{\oplus r}) \to H^3(V_5, Q^*(-2)^{\oplus r}) \to H^3(V_5, E(-3)) \to 0.
\]
We already computed that the first and second terms are 5r-dimensional vector spaces and existence of Ulrich bundles of rank \( r \) implies that there is a map \( U^{\oplus r} \to Q^{\oplus r} \) inducing an isomorphism between first two terms. Because being injective morphism between vector bundles and being isomorphism are open conditions, we obtain the desired result. \( \square \)

Let us recall the result of Casanellas and Hartshorne in [6].

**Theorem 3.9** (Theorem 2.9 of [6]). Let \( X \) be a smooth projective variety, and \( E \) be an Ulrich bundle on \( X \). Then \( E \) is a semistable bundle.

Using similar arguments in [7], we have the following result.

**Proposition 3.10.** For any \( r \geq 2 \), a (semi-)stable Ulrich bundle corresponds to a (semi-)stable quiver representation.

**Proof.** Let \( E \) be an Ulrich bundle of rank \( r \) on \( V_5 \). Then we have the following sequence.
\[
0 \to U^{\oplus r} \to Q^{\oplus r} \to E(-1) \to 0
\]
Let \( V_\bullet \) be the quiver representation corresponding to \( E(-1) \). From the definition of Ulrich bundles, we see that \( \chi(R_\bullet, V_\bullet) = 0 \) and \( H^0(R_\bullet, V_\bullet) = 0 \).

Let \( V'_\bullet \) be a subrepresentation of \( V_\bullet \) and consider the following exact sequence.
\[
0 \to V'_\bullet \to V_\bullet \to V''_\bullet \to 0
\]
Then we have the long exact sequence
\[
0 \to H^0(R_\bullet, V'_\bullet) \to H^0(R_\bullet, V_\bullet) \to H^0(R_\bullet, V''_\bullet) \to H^1(R_\bullet, V'_\bullet) \to H^1(R_\bullet, V_\bullet) \to H^1(R_\bullet, V''_\bullet) \to 0,
\]
so this implies that
\[
H^0(R_\bullet, V'_\bullet) = 0 \text{ and } H^1(R_\bullet, V''_\bullet) = 0.
\]
Hence, \( \chi(R_\bullet, V'_\bullet) \leq 0 \) and \( \chi(R_\bullet, V''_\bullet) \geq 0 \). Let \( \theta : K_0(\Gamma\text{-mod}) \to \mathbb{Z} \) be the character defined by \( \theta = \chi(R_\bullet, -) \). From the above inequality, we see that \( V_\bullet \) is a \( \theta \)-semistable \( \Gamma \)-representation.
If $E$ is strictly semistable, then there exists the destabilizing sequence
$$0 \to E' \to E \to E'' \to 0,$$
where $\mu(E') = \mu(E)$, and from [3, 6] we see that $E'$ is also an Ulrich bundle of smaller rank. Therefore, we see that $V_\bullet$ is a strictly semistable $\Gamma$-representation.

Suppose that $E$ is a stable Ulrich bundle but the corresponding quiver representation $V_\bullet$ is strictly semistable. Let $W_\bullet$ be the proper $\Gamma$-subrepresentation of $V_\bullet$ with $\theta(W_\bullet) = 0$ and let $U^{\oplus a} \to Q^{\oplus b}$ be the corresponding element in $\langle U, Q^* \rangle$. Then we have the following commutative diagram.

From the Snake Lemma, we have the following exact sequence
$$0 \to \text{Ker} \to F \to E(-1) \to \text{Cok} \to 0.$$ 
Because $F$ is a coherent sheaf with $H^i(V_5, F(-j)) = 0$ for $i = 0, 1, 2, 3$ and $j = 1, 2, 3$, we see that $F$ is an Ulrich subbundle with $\mu(F) = \mu(E(-1))$ which gives a contradiction. Thus, we see that when $E$ is stable the corresponding $\Gamma$-representation must be also stable.

From the above discussions, we get the desired result. Let $\Gamma$, $\theta$ be as above.

**Theorem 3.11.** Let $M(r)$ (resp. $M^s(r)$) be the moduli space of $S$-equivalence classes of Ulrich bundles (resp. stable Ulrich bundles) of rank $r \geq 2$ on $V_5$. Then the natural functor $D(V_5) \to D(\text{C}^r\text{-mod})$ induces a morphism $\varphi: M(r) \to M^{\theta,\text{ss}}(r,r)(\Gamma)$ satisfying the following properties:

1. $\varphi$ is an injective map;
2. $\varphi$ maps stable (resp. strictly semistable) objects to stable (resp. strictly semistable) objects;
3. $\varphi$ induces an isomorphism of $M^s(r)$ onto

\[
\varphi(M^s(r)) = \left\{ V_\bullet \in M^{\theta,\text{ss}}(r,r)(\Gamma) : \begin{array}{l}
\text{Hom}(R_\bullet, V_\bullet) = 0 \text{ and the corresponding } \\
U^{\oplus r} \to Q^{\oplus r} \text{ is injective}
\end{array} \right\},
\]

which forms a smooth $(r^2 + 1)$-dimensional open subset of $M^{\theta,\text{ss}}(r,r)(\Gamma) \subset M^{\theta,\text{ss}}(r,r,\Gamma)$.

It is remarkable that the moduli spaces of rank $r(\geq 2)$ Ulrich bundles on cubic threefolds, intersections of two 4-dimensional quadrics, $V_5$ are all smooth quasi-projective variety of dimension $r^2 + 1$. (See [7, 12] for more details.) We conjecture that for any smooth Fano 3-fold of index 2 the moduli space of rank $r(\geq 2)$ stable Ulrich bundles is nonempty and a smooth quasi-projective variety of dimension $r^2 + 1$. Moreover, in many cases the arguments used in this paper is similar to the arguments in [7, 12], especially stability of Ulrich bundles on the Fano 3-folds is related to another stability of objects in the semiorthogonal component. Recently,
Bayer, Lahoz, Macri and Stellari introduce a method to induce Bridgeland stability conditions on semiorthogonal decompositions in [2]. We think there might be a uniform proof using their method for the results in this paper and [7, 12] which might work for more general Fano varieties.

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