On Stability and Consensus of Signed Networks: A Self-loop Compensation Perspective

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Abstract

Positive semidefinite is not an inherent property of signed Laplacians, which renders the stability and consensus of multi-agent system on undirected signed networks intricate. Inspired by the correlation between diagonal dominance and spectrum of signed Laplacians, this paper proposes a self-loop compensation mechanism in the design of interaction protocol amongst agents and examines the stability/consensus of the compensated signed networks. It turns out that self-loop compensation acts as exerting a virtual leader on these agents that are incident to negative edges, steering whom towards origin. Analytical connections between self-loop compensation and the collective behavior of the compensated signed network are established. Necessary and/or sufficient conditions for predictable cluster consensus of signed networks via self-loop compensation are provided. The optimality of self-loop compensation is discussed. Furthermore, we extend our results to directed signed networks where the symmetry of signed Laplacian is not free. Simulation examples are provided to demonstrate the theoretical results.

Keywords: Signed Laplacian, self-loop compensation, positive semidefinite, positive feedback, structural balance, virtual leader

1. Introduction

Consensus is an important paradigm in distributed algorithms on networks, where graph Laplacian plays a central role in analysis and design of the network performance Mesbahi and Egerstedt [22], Kia et al. [19], Qin et al. [28]. A neat feature of graph Laplacian lies in its positive semidefiniteness which determines the stability and consensus of multi-agent networks Olfati-Saber and Murray [23], Godsil et al. [13].

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In the last two decades, the consensus problem has been extensively examined in networks with only cooperative interactions Mesbahi and Egerstedt [22]. However, antagonistic interactions can also exist which can capture, for instance, the antagonism in social networks which can lead to opinion polarization, or critical lines in power networks which may cause small-disturbance instability Song et al. [33, 32], Ding et al. [10], Proskurnikov et al. [27]. Recently, there has been a growing interest in multi-agent systems on signed networks Meng et al. [21], Chen et al. [8], Altafini [1], Xia and Cao [35], Shi et al. [29]. In this line of works, a notable observation is that the consensus or even stability of the multi-agent network cannot be guaranteed by negating the weight of only one edge from positive to negative Zelazo and Bürger [37]. From a microscopic view, the negative edges turn out to be a mechanism of positive feedback, that is, the state deviations (or tracking errors) between neighboring agents reinforce; while the positive edges are a mechanism of negative feedback, namely, the state deviations between neighboring agents reduce Shi et al. [29]. Therefore, the stability guarantee when implementing the classical consensus protocol on signed networks turns out to be a critical prerequisite with respect to the network functionality and performance Zelazo and Bürger [36], Meng [20]. Moreover, the typical collective behaviors of signed networks can often be achieving cluster consensus whose explicit characterization is challenging Pan et al. [25, 24], Xia and Cao [35], Zelazo and Bürger [37]. In fact, the stability and cluster consensus problems of signed networks can be attributed to examining the spectral properties of signed Laplacian, namely, the Laplacian matrix on signed networks Hershkowitz [16], Pan et al. [25], Bronski and De Ville [3], Chen et al. [7].

Essentially, diagonal dominance of graph Laplacian (a global property) naturally guarantees the stability of the multi-agent system on unsigned networks Pan et al. [25], Altafini [1]. For signed networks, however, there can exist mutually cancellation amongst positive and negative weights when summing them together for the diagonal entries of signed Laplacain, whose inherent properties under unsigned networks such as diagonal dominance and positive semidefiniteness of Laplacian are no long valid. Therefore, the analysis of signed Laplacian cannot fall into the traditional analysis framework such as $M$-matrix theory or using Gershgorin disc theorem etc Horn and Johnson [17], Olfati-Saber and Murray [23].

In Zelazo and Bürger [37], it is shown that a signed Laplacian with only one negative edge is positive semidefinite if the magnitude of the negative edge weight is less than or equal to the reciprocal of the effective resistance between the nodes of the negative edge over the positive subgraph. This result was then extended therein to the network with multiple negative weights under the restriction that different negative edges are not on the same cycle. The aforementioned result has been subsequently re-examined from the perspective of geometrical and passivity-based approaches Chen et al. [9]. The spectral properties of signed Laplacians with connections to eventual positivity was recently examined in Chen et al. [8]. Optimal weight allocation in signed networks and positive semidefiniteness of signed Laplacians by employing the semidefinite...
programming was investigated in Wei et al. [34]. The stability of the Laplacian matrix of signed networks is also examined in terms of matrix signatures Pan et al. [23], Bronski and DeVille [3], and the Schur stability criterion Pirani et al. [26].

Although notable results have been proposed on the interplay between edge weights and stability/consensus of signed networks, few works appeal for measures that should be taken to stabilize an unstable signed network. Matrix stability theory plays a central role in characterizing stability criteria of dynamical systems Kaszkurewicz and Bhaya [13], Arcak [2], Hershkowitz [10]. An Hermitian diagonally dominant matrix with real non-negative diagonal entries is positive semidefinite, which is a crucial fact in the stability guarantee of Laplacian of unsigned networks. However, for signed networks, a remarkable fact is that the magnitude of weights associated with negative edges between neighboring agents may lay no influence on shifting an unstable signed network with cutset into stable Song et al. [30], Zelazo and Bürger [33]. These facts motivate us to develop a distributed paradigm to guarantee the stability and consensus of signed networks. To regain the diagonal dominance of signed Laplacian, the manipulation of its diagonal entries turns out to be effective since it can be realized by each agent properly adopting feedback gains on their state in the respective protocol. In another word, this process can be implemented in a fully distributed manner. We shall refer to the aforementioned process as self-loop compensation in the following discussions.

Statement of contributions. This paper proposes a self-loop compensation mechanism in the design of interaction protocol amongst agents and examines the interplay between self-loop compensation of signed Laplacian and stability/consensus of the multi-agent system on signed networks. Analytical connections between the self-loop compensation and the collective behaviors of the compensated signed network are established. Necessary and sufficient conditions for (cluster) consensus of compensated signed networks are provided as well as the explicit characterization of their respective steady-states. The optimality of self-loop compensation is also discussed. Both undirected and directed signed networks are investigated. It is shown that structurally imbalanced networks need less self-loop compensation to be stable than structurally balanced ones.

The remainder of this paper is organized as follows. Notions and graph theory are introduced in §2. We then provide the motivation and introduction of self-loop compensation in §3 and §4, respectively, followed by the main results concerning the magnitude of self-loop compensation and stability and (cluster) consensus of the resultant compensated network in §5. Concluding remarks are finally provided in §6.

2. Preliminaries

2.1. Notations

Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ be the set of real, complex, and natural numbers, respectively. Denote $\mathbb{S} = \{1, \ldots, s\}$ for an $s \in \mathbb{N}$. Let $\text{Re}(\cdot)$ denote the real
part of a complex number. Denote the entry of a matrix $M \in \mathbb{R}^{n \times n}$ located in $i$-th row and $j$-th column as $[M]_{ij}$. Denote the eigenvalues of the matrix $M = [m_{ij}] \in \mathbb{R}^{n \times n}$ as $\lambda_i(M)$ where $i \in \mathbb{N}$. The spectrum of the matrix $M$ is the set of all its eigenvalues, denoted by $\Lambda(M)$. The spectral radius of a square matrix $M$ is the largest absolute value of its eigenvalues, denoted by $\rho(M)$. A symmetric matrix $M \in \mathbb{R}^{n \times n}$ is positive definite ($M \succ 0$) if for all nonzero $z \in \mathbb{R}^n$ such that $z^T M z > 0$, and is positive semidefinite ($M \succeq 0$) if $z^T M z \geq 0$ for all $z \in \mathbb{R}^n$. A matrix $M = [m_{ij}] \in \mathbb{R}^{p \times q}$ is said to be positive (nonnegative), denoted by $M > 0$ ($M \geq 0$), if $m_{ij} > 0$ ($m_{ij} \geq 0$) for all $i \in p$ and $j \in q$. For a pair of matrices $M \in \mathbb{R}^{p \times q}$ and $M' \in \mathbb{R}^{p \times q}$, we write $M \succeq M'$ ($M > M'$) if $M - M' \succeq 0$ ($M - M' > 0$). For two vectors $x, y \in \mathbb{R}^q$, if there exist $i', i'' \in q$ ($i' \neq i''$) such that $[x]_{i'} > [y]_{i'}$ and $[x]_{i''} < [y]_{i''}$, then we write $x \succ y$. The absolute value of a matrix $M = [m_{ij}] \in \mathbb{R}^{p \times q}$ is denoted by $|M| = [|m_{ij}]| \in \mathbb{R}^{p \times q}$.

2.2. Graph Theory

Let $G = (V, E, W)$ denote a network with the node set $V = \{1, 2, \ldots, n\}$, the edge set $E \subseteq V \times V$, and the adjacency matrix $W = [w_{ij}] \in \mathbb{R}^{n \times n}$. Here, adjacency matrix $W$ satisfies $w_{ij} \neq 0$ if and only if $(i, j) \in E$ and $w_{ij} = 0$ otherwise. A network $G$ is referred to as signed network if there exist an edge $(i, j) \in E$ such that $w_{ij} < 0$, otherwise, $G$ is an unsigned network. A network is connected if any two distinct nodes are reachable from one another via paths. The neighbor set of an agent $i \in V$ is $N_i = \{j \in V | (i, j) \in E\}$. The neighbor set of an agent $i \in V$ can be divided by $N_i = N_i^+ \cup N_i^-$ where $N_i^+ = \{j \in V | (i, j) \in E \text{ and } w_{ij} > 0\}$ and $N_i^- = \{j \in V | (i, j) \in E \text{ and } w_{ij} < 0\}$.

3. Motivation

Consider the signed Laplacian $\mathcal{L}(G) = [l_{ij}] \in \mathbb{R}^{n \times n}$ where,

\[
l_{ij} = \begin{cases} 
\sum_{k=1}^{n} w_{ik}, & i = j; \\
-w_{ij}, & i \neq j.
\end{cases}
\]

We know that the signed Laplacian $\mathcal{L}(G)$ implies the following interaction protocol amongst agents in a local level,

\[
\dot{x}_i(t) = \sum_{j \in N_i} w_{ij}(x_j(t) - x_i(t)), \quad i \in V,
\]

where $x_i(t) \in \mathbb{R}$ denotes the state of agent $i \in V$. In a global level, the collective dynamics of the multi-agent network (2) admits,

\[
\dot{x}(t) = -\mathcal{L}(G)x(t),
\]

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$.

In contrast to unsigned networks, typical collective behaviors of multi-agent system on signed networks include achieving (cluster) consensus or even
Figure 1: A signed network $G_0$ where solid lines and dotted lines represent positive and negative edges, respectively.

being unstable Zelazo and Bürger [37], Pan et al. [25]. The latter is, for the most cases, actually not a desired state that one expects the network to have. This paper proposes a self-loop compensation approach to handle this issue, which is motivated from the following two perspectives.

3.1. Network Connectivity and Signed Laplacian’s Eigenvalues

From a local perspective, it turns out that the cutset of a network plays a critical role in determining the number of negative eigenvalues of signed Laplacian Pan et al. [25], Zelazo and Bürger [36], Song et al. [30]. A cutset is a subset of edges in a network whose deletion increases the number of connected component. A cutset is referred to as a cut edge when its cardinality is equal to one. The cutset is closely related to, for instance, the transient stability of power systems Song et al. [30]. We have the estimation of number of negative eigenvalues of signed Laplacian, denoted by $i_-(\mathcal{L}(G))$, in terms of number of negative edges in the following theorem.

**Proposition 1.** Pan et al. [25] Let $G = (V, E, W)$ be a connected signed network and $i_-(\mathcal{L}(G))$ denote the number of negative eigenvalues of $\mathcal{L}(G)$.

1) Let $E_-$ be the negative edge set with the edge weights $W_-$. If $G_+ = (V, E \setminus E_-, W \setminus W_-)$ is not connected, then $i_-(\mathcal{L}(G)) \geq 1$.

2) If there exists a cutset $C$ in $G$ such that each edge in $C$ is a cut edge, edges in $E' \subseteq C$ are all negative and edges in $E \setminus E'$ are all positive, then $i_-(\mathcal{L}(G)) = |E'|$.

According to Proposition 1 the negative cutset turns out to be an incentive for negative eigenvalues of signed Laplacian. For instance, one can conclude from the Proposition 1 that the number of negative eigenvalues is exactly the number of negative edges for tree networks. Generally, consider for instance a signed network $G_0$ in Figure 1. Compute the spectrum of signed Laplacian in this example yields $\lambda(\mathcal{L}(G_0)) = \{-2.39, -1.27, -0.53, 0, 1.32, 1.59, 2.47, 2.82\}$. On the other hand, the cutset of $G_0$ in Figure 1 is $\{(1, 6), (2, 3), (3, 4)\}$, and as predicted by the Proposition 1 one has $i_-(\mathcal{L}(G_0)) = 3$.

It is reported in Zelazo and Bürger [37] that the signed Laplacian is positive semidefinite if the deletion of the negative edges does not disconnect the network and the magnitude of the negative weight is less than the reciprocal of the effective resistance between the two incident nodes. However, according to Proposition 1 if there exists a cutset in a signed network such that edges in
this cutset are all negative, then the associated signed Laplacian has at least one negative eigenvalue regardless of the selection of the magnitude of negative weights, rendering the corresponding signed network \(3\) unstable. Under this circumstance, a reasonable and intuitive option now is to rebuild the diagonal dominance of the signed Laplacian, more preferably, in a distributed manner.

3.2. Diagonal Dominance of Signed Laplacian

From a global perspective, the diagonal dominance of the signed Laplacian plays a central role in determining the stability and consensus of multi-agent network \(2\). We know that the signed Laplacian can also be characterized by \(\mathcal{L}(\mathcal{G}) = D - W\), where \(D = \text{diag}\{d_1, d_2, \ldots, d_n\} \in \mathbb{R}^{n \times n}\) is a diagonal matrix whose diagonal entries are \(d_i = \sum_{j \in N_i} w_{ij}\). If \(\mathcal{G}\) is an unsigned network, then matrix \(D\) encodes in-degree of each agent and \(\mathcal{L}(\mathcal{G})\) is diagonally dominant, namely, \(|l_{ii}| \geq \sum_{j \neq i} |l_{ij}|\) for all \(i \in \mathbb{N}\). In this case, the graph Laplacian \(\mathcal{L}(\mathcal{G})\) is positive semidefinite with only one zero eigenvalue if, and only if, the network is connected Godsil et al. [13]. Therefore, the stability and consensus of multi-agent network \(2\) on unsigned connected networks can be naturally guaranteed. Unfortunately, this fact does not hold for signed networks. As a result, the diagonal dominance can be lost since there may exist mutually cancellation amongst positive and negative weights when summing them together to construct the diagonal entries of signed Laplacian, see \(1\).

4. Self-loop Compensation

Upon the aforementioned analysis, retrieve diagonal dominance of signed Laplacian via local-level adaptation is intuitively necessary to maintain the stability, and moreover the consensus of signed networks. To this end, an intuitive and straightforward attempt is to compensate the diagonal entries of signed Laplacian that can be fulfilled by introducing \(c_i(t) = -k_i x_i(t)\) on top of \(2\),

\[
\dot{x}_i(t) = \sum_{j \in N_i} w_{ij} (x_j(t) - x_i(t)) + c_i(t), \quad i \in \mathcal{V}, \tag{4}
\]

where \(k_i \in \mathbb{R}\) is the \(i\)-th entry in the compensation vector \(k = [k_1, k_2, \ldots, k_n]^T \in \mathbb{R}^n\). In view of this, the dynamics of signed network \(3\) after self-loop compensation (or compensated dynamics) can be subsequently characterized by

\[
\dot{x}(t) = -\mathcal{L}^k x(t), \tag{5}
\]

where \(\mathcal{L}^k = \mathcal{L} + \text{diag}\{k\}\).

In the setting of \(4\), one can view the magnitude of self-loop compensation \(k_i\) as the weight \(w_{ii}\) associated with self-loop edge in the related loopy graph Song et al. [31]. Notable physical interpretations of self-loop in networks can be conductance, loads or dissipation in the context of electrical networks Dorfler and Bullo [11], Song et al. [31].
Another notable interpretation of self-loop compensation can be exerting a virtual leader on these agents that are incident to negative edges, steering whom towards origin. To see this, we take agent 2 in the signed network in Figure 2 as an example, the state update of agent 2 reads,

\[
\dot{x}_2(t) = w_{21}(x_1(t) - x_2(t)) + w_{23}(x_3(t) - x_2(t)) + k_2(x_0 - x_2(t)),
\]

where \(x_0 = 0\). Specifically, (6) signifies the attractive tendency between states of agent 2 and its neighbors in \(\mathcal{N}_2^+ = \{1\} \ (w_{21} > 0)\), (7) signifies the repulsive tendency between states of agent 2 and its neighbors in \(\mathcal{N}_2^- = \{3\} \ (w_{23} < 0)\), and the self-loop compensation term (8) signifies the tendency of \(x_2(t)\) that evolves towards the origin \((x_0 = 0)\). Note that all negative edges in a signed network lead to the loss of diagonal dominance of the corresponding signed Laplacian.

Therefore, for each negative edge \((i', j') \in \mathcal{E}\), a self-loop compensation \(k_{i'} > 0\) is desirable where the origin \((x_0 = 0)\) acts as a virtual leader Cao et al. [4]. However, different from the unsigned networks, there does not exist a general guarantee on the stability/consensus of (3) (or equivalently, positive semidefiniteness of signed Laplacian \(L\)). Therefore, whether the compensated signed network (5) can track the virtual leader (origin) depends on the structure of the signed network as well as the compensation vector.

The self-loop compensation is plausible since it can be applied to guarantee the network stability and/or even consensus in a fully distributed fashion. Conservatively, for instance, if one chooses \(k = \delta = [\delta_1, \delta_2, \ldots, \delta_n]^T \in \mathbb{R}^n\) where

\[
\delta_i = \sum_{j=1}^{n} \left( |w_{ij}| - w_{ij} \right), i \in \mathcal{V},
\]

then the system (5) may achieve the so-called bipartite consensus or trivial consensus towards origin depending on whether the underlying signed network is structurally balanced Altafini [1]. Notably, all necessary information to construct \(\delta_i\) for each agent \(i\) is locally accessible. In this special case, one can
Figure 3: A structurally balanced signed network $\mathcal{G}_1$ (top) and a structurally imbalanced signed network $\mathcal{G}_2$ (bottom), where solid lines and dotted lines represent positive edge and negative edge, respectively.

Maintain the elegant positive semidefiniteness with one zero eigenvalue, which guarantees consensus in many applications Chen et al. [8], Song et al. [33], Chen et al. [6], Altafini [1]. Moreover, $k = \delta$ also acts as a critical boundary that renders $\mathcal{L}^k$ (weak) diagonally dominant Horn and Johnson [17].

However, it is worth noting that one cannot compensate the diagonal entries of a signed Laplacian excessively, since it may lead to a loss of the behavioral diversity of the compensated network—only has the trivial consensus towards origin. Therefore, it is intricate to select the compensation vector for desired behaviors of the resultant network. Here, it is natural to ask how far away is a signed network from being stable or achieving consensus via self-loop compensation? and how the compensation vector influences the behavior of the resultant network?

We provide an example to illustrate transition of eigenvalues associated with signed Laplacian along with the variation of compensation vector.

**Example 1.** Consider the correspondence between the compensation vector $k$ and the ordered eigenvalues of $\mathcal{L}^k$ for structurally balanced ($\mathcal{G}_1$ in Figure 3) and structurally imbalanced ($\mathcal{G}_2$ in Figure 3) signed networks, respectively.

Note that the deletion of all negative edges in a structurally balanced signed network will lead to two separate node sets $V_1$ and $V_2$ such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Therefore, according to Proposition 1 there at least exists one negative eigenvalue for the Laplacian matrix of the structurally balanced signed network. Let the compensation vector be such that $k(q) = q\delta$ where $q \in [0, 1]$. Note that $q$ quantifies the influence of magnitude of the diagonal compensation on the eigenvalues of $\mathcal{L}^{k(q)} = \mathcal{L} + \text{diag} \{k(q)\}$. The eigenvalue distributions of $\mathcal{L}^{k(q)}$ for the structurally balanced network $\mathcal{G}_1$ in Figure 3 where $q = 0.1$, $q = 0.3$, $q = 0.5$, $q = 0.7$ and $q = 0.9$, respectively, are shown in Figure 4. The
Figure 4: The eigenvalue distribution of $\mathcal{L}^{k(q)}$ for the structurally balanced network $G_1$ in Figure 3.

Figure 5: The eigenvalue distribution of $\mathcal{L}^{k(q)}$, when $q$ tends to 1, for the structurally balanced network $G_1$ in Figure 3.
smallest eigenvalue of $L^k(q)$ for the structurally balanced network $G_1$ in Figure 3 where $q \in [0.1, 0.999]$ is shown in Figure 5. The increasing rate of $\lambda_1(G_1)$ becomes more and more faster when $q \rightarrow 1$, however, the smallest eigenvalue is still less than zero in case that $q < 1$. It is notable, according to Figure 4 and Figure 5 that for structurally balanced signed network $G_1$ in Figure 3 the minimal magnitude of the compensation vector to stabilize $L^k(q)$ is realized when $q = 1$, in which case $k(q) = \delta$.

The distributions of the ordered real part of eigenvalues of $L^k(q)$ for the structurally imbalanced network $G_2$ in Figure 3 where $q = 0.1$, $q = 0.3$, $q = 0.5$, $q = 0.7$ and $q = 0.9$, respectively, are shown in Figure 6. However, $L^k(q)$ is stable when $q = 0.9$ (as shown in Figure 6) implying that the stability of $L^k(q)$ can be guaranteed when $k(q) < \delta$. Inspired by the aforementioned discussions, it turns out that the magnitude of the compensation vector $k$ is closely related to the structural balance of the underlying signed network.

In the following, we proceed to analytically reveal how the compensation vector along with structural balance of underlying signed network (a paramount graph-theoretic concept) determine the collective behavior of the compensated dynamics (5). Altafini [1], Facchetti et al. [12], Heider [15], Cartwright and Harary [3].

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Footnote 2: In this paper, the stability of system $\dot{x}(t) = -Ax(t)$ is eventually examined when referring to the stability of a matrix $A$. 

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5. Compensation Vector and Network Behavior

We first present fundamental facts related to structural balance of a signed network. A signed network \( G = (\mathcal{V}, \mathcal{E}, W) \) is structurally balanced if there is a bipartition of the node set \( \mathcal{V} \), say \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) such that \( \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \) and \( \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset \), satisfies that the weights on the edges within each subset is positive, but negative for edges between the two subsets Harary et al. [14]. A signed network is structurally imbalanced if it is not structurally balanced. A Gauge transformation is performed by the matrix \( G = \text{diag} \{ \sigma_1, \sigma_2, \ldots, \sigma_n \} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n} \), where \( \sigma_i \in \{1, -1\} \) for all \( i \). Altafini [1]. If a signed network \( G = (\mathcal{V}, \mathcal{E}, W) \) is structurally balanced, then there exists a Gauge transformation \( G \), such that \( GWG \geq 0 \). For each agent \( i \in \mathcal{V} \) and an arbitrary \( x_i(0) \in \mathbb{R} \), the multi-agent system (5) is said to admit bipartite consensus if \( \lim_{t \to \infty} |x_i(t)| = \alpha > 0 \). Recall that a network is undirected when \( (i, j) \in \mathcal{E} \) if and only if \( (j, i) \in \mathcal{E} \); otherwise, it is directed. We shall start our discussion from undirected signed networks.

5.1. Undirected Signed Networks

The stability of the Laplacian matrix of undirected signed networks reduces to examining its positive semidefiniteness. Notably, for structurally balanced, connected, undirected signed networks, Altafini has shown that bipartite consensus can be achieved if \( k = \delta \) Altafini [1]. Furthermore, we shall proceed to examining the following three categories of compensation vectors.

**Theorem 1.** Let \( G = (\mathcal{V}, \mathcal{E}, W) \) be a structurally balanced, connected, undirected signed network. Then, the following statements hold.

1) If \( k \leq \delta \) and \( k \neq \delta \), then the compensated network (5) is unstable;
2) If \( k \geq \delta \) and \( k \neq \delta \), then the compensated network (5) achieves trivial consensus;
3) If \( k \neq \delta \), then the compensated network (5) cannot achieve bipartite consensus.

**Proof.**

1) Assume that \( L^k \) is positive semidefinite, without loss of generality, let \( l_k = \sum_{j=1}^{n} |w_{ij}| - \varepsilon \), where \( \varepsilon > 0 \), and \( l_i = \sum_{j=1}^{n} |w_{ij}| \), where \( i \in \mathcal{V} \), and \( i \neq 1 \). Thus \( L^k = L^\delta - \Delta \), where \( \Delta = \text{diag} \{ \varepsilon, 0, \ldots, 0 \} \). Let \( v = [v_1, v_2, \ldots, v_n]^T \in \mathbb{R}^n \) be the eigenvector of \( L^\delta \) corresponding to the zero eigenvalue, note that \( v_i \neq 0 \) for all \( i \in \mathcal{V} \), therefore \( v^T L^\delta v = 0 \) and \( v^T \Delta v > 0 \). As a result, one has \( v^T L^\delta v - v^T \Delta v < 0 \) which is a contradiction and \( L^k \) has at least one negative eigenvalue, i.e., the multi-agent system (5) is unstable.

2) Note that \( L^k = L^\delta + \text{diag} \{ k - \delta \} \), then the proof follows from traditional treatments of leader-following consensus problem via Gauge transformation Cao et al. [4], we shall omit it for space.

3) Assume that the multi-agent system (5) achieves the bipartite consensus, then there exists a Gauge transformation \( G^* \) such that \( \text{null}(L^k) = \text{span}(G^* 1_n) \).

If \( G^* = G \), then one has \( \lim_{t \to \infty} x(t) = \frac{1}{n} G 1_n 1_n^T G x(0) \) and \( G L^k 1_n = 0 \), which contradict with the fact \( k \neq \delta \).
If $G^* \neq G$, let $L^k = L^\delta + \Delta$, then one has $G^* L^k G^* 1_n = G^* (L^\delta + \Delta) G^* 1_n = G^* L^\delta G^* 1_n + \Delta 1_n$, due to $G^* \neq G$, thus $G^* L^\delta G^* 1_n \geq 0$. In addition, $\Delta 1_n$ has at least one element being positive. Therefore, $G^* L^k G^* 1_n \neq 0$, i.e., $\text{null}(L^k) \neq \text{span}(G^* 1_n)$ and the multi-agent system (5) cannot achieve the bipartite consensus.

According to Theorem 1, one can obtain the following result regarding the bipartite consensus of the compensated multi-agent network (5).

**Corollary 1.** Let $G = (V, E, W)$ be a structurally balanced, connected, undirected signed network. Then, the compensated multi-agent network (5) achieves bipartite consensus if and only if $k = \delta$. Moreover, the bipartite consensus value is

$$\lim_{t \to \infty} x(t) = \frac{1}{n} G 1_n 1_n^T G x(0)$$

where $G$ is the Gauge transformation associated with $G$.

In fact, Theorem 1 also implies the following statement.

**Corollary 2.** Let $G = (V, E, W)$ be a structurally balanced, connected, undirected signed network. Then, the $L^k$ is positive semidefinite and has a simple zero eigenvalue with eigenvector $G 1_n$ if and only if $k = \delta$, where $G$ is the Gauge transformation associated with signed network $G$.

**Remark 1.** The proof of Theorem 1 indicates the steady-state of the multi-agent system (5) in terms of the three categories of compensation vectors, i.e., $k \leq \delta$, $k = \delta$ and $k \geq \delta$. It is shown that if $k \leq \delta$, then the multi-agent system (5) is unstable; if $k \geq \delta$, then the multi-agent system (5) achieves the trivial consensus; if $k = \delta$, then the multi-agent system (5) achieves the bipartite consensus and the bipartite consensus value is unique and determined by the Gauge transformation associated with $G$.

In fact, for structurally balanced signed networks, the compensation vector $k = \delta$ reaches a sense of optimum, as stated in the following result.

**Theorem 2.** Let $G = (V, E, W)$ be a structurally balanced, connected, undirected signed network. Let $k = [k_1, k_2, \ldots, k_n]^T \in \mathbb{R}^n$ be a compensation vector such that $L^k$ is positive semidefinite. Then $\|k\|_1 \geq \|\delta\|_1$.

**Proof.** Suppose that $\|k\|_1 < \|\delta\|_1$, namely,

$$\sum_{i=1}^{n} k_i < \sum_{i=1}^{n} \delta_i.$$ 

Note that there exists a Gauge transformation $G$ such that $1_n^T G (L + \text{diag}\{\delta\}) 1_n =$
0. Therefore,

\[ 1_n^T G(\mathcal{L} + \text{diag}\{k\}) G 1_n = - \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} |w_{ij}| + \sum_{i=1}^{n} \left( \sum_{j=1, j \neq i}^{n} w_{ij} + k_i \right) < 0 \]

which implies that \( \mathcal{L} + \text{diag}\{k\} \) has a negative eigenvalue and, therefore, is not positive semidefinite. This establishes a contradiction.

In fact, Theorem 2 implies that the selection of compensation vector stated in Theorem 1 is optimal in terms of the magnitude of the compensation vector, characterized by the 1-norm of vectors.

The above discussions are mainly concentrate on the structurally balanced signed networks, and it is shown that the sufficient and necessary condition for the multi-agent system (5) achieving bipartite consensus is \( k = \delta \); for the structurally imbalanced signed network, it is shown that the multi-agent system (5) achieves trivial consensus if \( k = \delta \) [8]. However, for a connected, structurally imbalanced signed network, if the associated signed Laplacian matrix has negative eigenvalues, a natural question is whether or not the compensation vector \( k \) has to be up to \( \delta \) so as to stabilize the system (5)? We provide the following result to address this question.

**Theorem 3.** Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, W) \) be a structurally imbalanced, connected, undirected signed network. Let \( \lambda_1(\mathcal{L}^\delta) = \cdots = \lambda_p(\mathcal{L}^\delta) \) denote \( 1 \leq p \leq n \) smallest eigenvalues of the matrix \( \mathcal{L}^\delta \) with corresponding normalized eigenvectors \( v_1, \cdots, v_p \). Then, there exists a compensation vector \( k = \delta - \lambda_1(\mathcal{L}^\delta) 1_n \) such that \( \mathcal{L}^k \) is positive semidefinite and has \( p \) zero eigenvalues with corresponding eigenvectors \( v_1, \cdots, v_p \). Moreover, the compensated multi-agent network (5) achieves cluster consensus characterized by \( \lim_{t \to \infty} x(t) = \sum_{j=1}^{p} v_j v_j^T x(0) \).

**Proof.** As \( \mathcal{G} \) is structurally imbalanced, the corresponding signed Laplacian matrix \( \mathcal{L}^\delta \) is positive definite with eigenvalues ordered as

\[ 0 < \lambda_1(\mathcal{L}^\delta) \leq \lambda_2(\mathcal{L}^\delta) \leq \cdots \leq \lambda_n(\mathcal{L}^\delta) \]

and the corresponding eigenvectors \( v_i \in \mathbb{R}^n \), where \( i \in \mathbb{N} \). Due to,

\[ \mathcal{L} + \text{diag}\{k\} = \mathcal{L}^\delta - \text{diag}\{\lambda_1(\mathcal{L}^\delta) 1_n\} \]

then for any \( i \in \mathbb{N} \), one has,

\[ (\mathcal{L}^\delta - \text{diag}\{\lambda_1(\mathcal{L}^\delta) 1_n\}) v_i = (\lambda_i(\mathcal{L}^\delta) - \lambda_1(\mathcal{L}^\delta)) v_i. \]

Therefore, the eigenvalues and its corresponding eigenvectors of the matrix \( \mathcal{L} + \text{diag}\{k\} \) are the pairs \( \{\lambda_i(\mathcal{L}^\delta) - \lambda_1(\mathcal{L}^\delta), v_i\} \). Hence, \( \mathcal{L} + \text{diag}\{k\} \) is positive semidefinite and \( \lim_{t \to \infty} x(t) = v_1 v_1^T x(0) \). \qed
Remark 2. As indicated in Theorem 3, the compensation vector $k$ for structurally imbalanced networks is not necessary to be up to $\delta$ to render the compensated signed network (5) stable, in which case, $L^k$ is not necessary diagonal dominant. In fact, different from the case of structurally balanced networks, the signed Laplacian matrix for the structurally imbalanced networks may have no negative eigenvalues. However, by employing the eigenvalue of $L^\delta$ and vector $\delta$, Theorem 3 provides an elegant manner to select the compensation vector $k$ that can predict the steady-state of the compensated signed network (5).

One can see from Theorem 3 that the multi-agent system (5) can achieve the cluster consensus when we choose 

$$k = \delta - \lambda_1(L^\delta)1_n.$$ 

Similarly, a direct question here is related to the steady-state of the multi-agent system (5) in the case of $k > \delta - \lambda_1(L^\delta)1_n$ and $k < \delta - \lambda_1(L^\delta)1_n$, respectively.

Corollary 3. Let $\mathcal{G} = (V, E, W)$ be a structurally imbalanced, connected, undirected signed network. Then,

**Case 1:** If the compensation vector $k > \delta - \lambda_1(L^\delta)1_n$, then $L + \text{diag}(k)$ is positive definite, i.e., the multi-agent system (5) achieve the trivial consensus.

**Case 2:** If the compensation vector $k < \delta - \lambda_1(L^\delta)1_n$, then the multi-agent system (5) is unstable.

Proof. Case 1: Denote by $L + \text{diag}(k) = L^\delta - \text{diag}(\lambda_1(L^\delta)1_n) + \Delta$, where $\Delta \in \mathbb{R}^{n \times n}$ is diagonal with $|\Delta|_{ii} > 0$ for all $i \in V$. For any $\eta \in \mathbb{R}^n$, one has,

$$\eta^T(L^\delta - \text{diag}(\lambda_1(L^\delta)1_n) + \Delta)\eta > 0,$$

therefore, $L + \text{diag}(k)$ is positive definite and the multi-agent system (5) achieve the trivial consensus.

Case 2: Denote by

$$L + \text{diag}(k) = L^\delta - \text{diag}(\lambda_1(L^\delta)1_n) + \Delta,$$ 

where $\Delta \in \mathbb{R}^{n \times n}$ is diagonal and $|\Delta|_{ii} < 0$ for all $i \in V$. Let $\eta \in \mathbb{R}^n$ be such that $(L^\delta - \text{diag}(\lambda_1(L^\delta)1_n))\eta = 0$, then one has,

$$\eta^T(L^\delta - \text{diag}(\lambda_1(L^\delta)1_n) + \Delta)\eta < 0,$$

therefore, there exists a negative eigenvalue for the matrix $L + \text{diag}(k)$ and the multi-agent system (5) is unstable.

According to Corollary 1, there is no gap between $k$ and $\delta$ to render the multi-agent system (5) achieving the bipartite consensus if the underlying signed network is structurally balanced. Furthermore, Theorem 2 indicates that the minimum total magnitude of the self-loop compensation is $1_n^T\delta$ for structurally balanced signed networks, whereas, the magnitude of compensation can be less for structurally imbalanced signed networks (as it is shown in Example 1) and Theorem 3 provides an approach to choose this compensation vector for predictable behavior of the resultant compensated network.
5.2. Directed Signed Networks

We now proceed to examine directed signed networks. First recall the following basic facts for directed networks. A signed network $G = (V, E, W)$ is weight balanced if $\sum_{j=1}^{n} |w_{ij}| = \sum_{j=1}^{n} |w_{ji}|$ for all $i \in V$. A matrix $M = [m_{ij}] \in \mathbb{R}^{n \times n}$ is irreducible if the indices $\{1, 2, \cdots, n\}$ cannot be decomposed into two disjoint non-empty subsets $\{i_1, i_2, \cdots, i_{n_1}\}$ and $\{j_1, j_2, \cdots, j_{n_2}\}$ where $n_1 + n_2 = n$ such that $m_{i_\alpha,j_\beta} = 0$ for all $\alpha \in \{1, 2, \cdots, n_1\}$ and $\beta \in \{1, 2, \cdots, n_2\}$. The graph of a matrix $M = [m_{ij}] \in \mathbb{R}^{n \times n}$, denoted by $G(M)$, is such that $(i,j) \in E$ if and only if $m_{ij} \neq 0$ for all $i,j \in V$. A matrix $M$ is irreducible if and only if $G(M)$ is strongly connected.

Similar to the undirected signed network, we shall first discuss the condition that the compensation vector should satisfy to render the multi-agent system (5) stable.

**Theorem 4.** Let $G = (V, E, W)$ be a structurally balanced, strongly connected, directed signed network. Then, the following statements hold:

1) If $k \leq \delta$ and $k \neq \delta$, then the compensated network (5) is unstable;

2) If $k \geq \delta$ and $k \neq \delta$, then the compensated network (5) achieves trivial consensus;

3) If $k \not\subseteq \delta$, then the compensated network (5) cannot achieve bipartite consensus.

**Proof.** 1) Without loss of generality, let

$$l_{11}^k = \sum_{j=1}^{n} |w_{1j}| - \epsilon,$$

where $\epsilon > 0$, and $l_{ii}^k = \sum_{j=1}^{n} |w_{ij}|$, where $i \in \bar{V}$ and $i \neq 1$. Then $L^k = L^\delta - \Delta$, where $\Delta = \text{diag} \{\epsilon, 0, \ldots, 0\}$.

Note that $G$ is structurally balanced, thus there exists a Gauge transformation $G$ satisfying $L^\delta = GL'C'G$, where $L' = [l'_{ij}] \in \mathbb{R}^{n \times n}$ such that $l'_{ij} = -|l_{ij}^\delta|$ for all $i \neq j \in \bar{V}$ and $l_{ii}^\delta = l_{ii}'$ for all $i \in \bar{V}$. Hence

$$L^k = L^\delta - \Delta$$

$$= GL'C' - \Delta$$

$$= G(L' - \Delta)G,$$

which implies that $L^k$ is similar to $L' - \Delta$ and as such share the same eigenvalues. Since $G$ is strongly connected, $L'$ is irreducible. Also note that $l_{ii}' > 0$ for all $i \in \bar{V}$. Let $u_0 = \max_{i \in \bar{V}} \{l_{ii}'\}$ and denote $T = -L' + u_0I$. Then $T$ is an irreducible non-negative matrix.

Let $\nu$ be the eigenvector of $L'$ corresponding to the zero eigenvalue, namely, $L' \nu = 0$. Then $Tv = -L'v + u_0Iv = u_0v$ and $v$ is the eigenvector of $T$ corresponding to the eigenvalue $u_0$. Since $[T]_{ii} = u_0 - l_{ii}'$ for all $i \in \bar{V}$ and $[T]_{ij} = -l_{ij}'$
for all $i \neq j \in \mathbb{N}$, every eigenvalue of $T$ lies within at least one of the following Gershgorin discs

$$\left\{ z \in \mathbb{C} : |z - u_0 + l''_{ii}| \leq \sum_{j \neq i} |l''_{ij}| \right\}$$

where $i \in \mathbb{N}$. Hence, if $\lambda \in \lambda(T)$, then $|\lambda| \leq u_0$. Thus, $\rho(T) = u_0$. Moreover, according to Perron–Frobenius theorem (Horn and Johnson [17, Theorem 8.4.4, p. 534]), $u_0$ is a simple eigenvalue of $T$. We now have

$$T + \Delta = -(\mathcal{L}' - \Delta) + u_0 I \geq T,$$

which implies that $\rho(T + \Delta) \geq \rho(T)$.

If $\rho(T + \Delta) = \rho(T)$, then for every $j \in \mathbb{N}$, $|T + \Delta|_{jj}$ and $|T|_{jj}$ must have the same modulus according to Wielandt’s theorem (Horn and Johnson [17, Theorem 8.4.5, p. 534]). However $|T + \Delta|_{11} > |T|_{11}$, thus $\rho(T + \Delta) > \rho(T) = u_0$. Let

$$\rho(T + \Delta) = \rho(T) + \tau = u_0 + \tau.$$ 

Since $T + \Delta$ is an irreducible non-negative matrix, $u_0 + \tau$ is an eigenvalue of $T + \Delta$; and denote its corresponding eigenvector as $w$. Then

$$(u_0 I - (\mathcal{L}' - \Delta))w = (u_0 + \tau)w,$$

implying that $(\mathcal{L}' - \Delta)w = -\tau w$. As such $-\tau$ is an eigenvalue of both $\mathcal{L}' - \Delta$ and $L^k$, which is a contradiction.

2) In spirit, the proof is similar to the case of structurally balanced, connected, undirected signed networks, we shall omit it for space.

3) The proof is similar to the Case 3 in the proof of Theorem 1. \hfill \Box

In parallel, we also have the following corollary.

**Corollary 4.** Let $G = (V, E, W)$ be a structurally balanced, strongly connected, directed signed network. Then, the multi-agent system (5) achieve the bipartite consensus for any initial state $x(0) \in \mathbb{R}^n$ if and only if $k = \delta$. Moreover, $\lim_{t \to \infty} x(t) = G1_n p^T Gx(0)$, where $G$ is the Gauge transformation associated with $G$ and $p^T G L^k G = 0$ and $p^T 1_n = 1$.

**Remark 3.** The proof of Theorem 3 implies the steady-state of the multi-agent system (5) from the respect of the choice of the compensation $k$, i.e., $k \leq \delta$, $k = \delta$ and $k \geq \delta$. Similar to the undirected case, it is shown that if $k \leq \delta$, the multi-agent system (5) is unstable; if $k \geq \delta$, the multi-agent system (5) achieve the trivial consensus; if $k = \delta$, the multi-agent system (5) achieve the bipartite consensus.

For the structurally imbalanced strongly connected signed network, similar to the undirected network case, it is shown that the multi-agent system (5) achieves trivial consensus if $k = \delta$ Altafini [1]. Similarly, the compensation vector can be less than $\delta$ in order to make the system (5) stable.
Theorem 5. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be a structurally imbalanced, strongly connected signed network. If the smallest eigenvalue of $\mathcal{L}^\delta$ (ordered by real parts) is real and simple, denoted by $\lambda_1(\mathcal{L}^\delta)$ with left and right eigenvectors $\mathbf{v}_l^1$ and $\mathbf{v}_r^1$ satisfying $(\mathbf{v}_l^1)^T \mathbf{v}_r^1 = 1$. Then, there exists a compensation vector $\mathbf{k} = \delta - \lambda_1(\mathcal{L}^\delta) \mathbf{1}_n$ such that eigenvalues of $\mathcal{L}^k$ have non-negative real parts and $\mathcal{L}^k$ has simple zero eigenvalue with left and right eigenvectors $\mathbf{v}_l^1$ and $\mathbf{v}_r^1$. Moreover, the compensated multi-agent network (5) achieve cluster consensus characterized by
\[
\lim_{t \to \infty} x(t) = \mathbf{v}_r^1 (\mathbf{v}_l^1)^T x(0).
\]

Proof. The proof is similar to the proof of Theorem 3, we omitted here for brevity. \hfill \square

We provide an example to illustrate the steady-state of compensated dynamics (9) on directed, structurally imbalanced signed networks.

Consider the directed, structurally imbalanced, signed network $\mathcal{G}_2$ in Figure 3. The steady-state can be predicted via $\mathbf{v}_l^1$ and $\mathbf{v}_r^1$, the left and right normalized eigenvector associated with $\lambda_1(\mathcal{L}^\delta(\mathcal{G}_2))$, namely,
\[
\lim_{t \to \infty} x(t) = \mathbf{v}_r^1 (\mathbf{v}_l^1)^T x(0). \tag{11}
\]

The state trajectory of the compensated dynamics which is shown in Figure 7, where the black crosses indicated the steady-state predicted by (11). In this example, the right and left normalized eigenvector associated with $\lambda_1(\mathcal{L}^\delta(\mathcal{G}_2))$ are
\[
\mathbf{v}_l^1 = (-0.45, -0.13, 0.31, 0.39, -0.45, -0.35, 0.30, 0.34)^T,
\]
and
\[ \mathbf{v}_1 = (-0.50, -0.31, 0.29, 0.38, -0.50, -0.39, 0.17, 0.26)^T, \]
respectively. The initial state of agents is
\[ \mathbf{x}(0) = (-0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3)^T. \]
Therefore, one can get that
\[ \lim_{t \to \infty} \mathbf{x}(t) = (-0.12, -0.03, 0.08, 0.11, -0.12, -0.09, 0.08, 0.09)^T. \]
Finally, we discuss whether or not the compensation vector \( \mathbf{k} \) can be less than \( \delta \) so as to render the system (5) achieving the trivial consensus.

**Corollary 5.** Let \( G = (V, E, W) \) be a structurally imbalanced, weight balanced, connected signed network. If the compensation vector satisfies
\[ \mathbf{k} > \delta - \lambda_1 \left( \frac{1}{2}(L^\delta + (L^\delta)^T) \right) \mathbf{1}_n, \]
then, the multi-agent system (5) achieve the trivial consensus.

**Proof.** Note that,
\[ \frac{1}{2}(L + \text{diag}(\mathbf{k})) + (L + \text{diag}(\mathbf{k}))^T = \frac{1}{2}(L + L^T) + \text{diag}(\mathbf{k}), \]
since \( G \) is weight balanced, then \( \frac{1}{2}(L + L^T) \) is the Laplacian matrix of an undirected network corresponding to directed network \( G \). According to Corollary 3 if
\[ \mathbf{k} > \delta - \lambda_1 \left( \frac{1}{2}(L^\delta + (L^\delta)^T) \right) \mathbf{1}_n, \]
then \( \frac{1}{2}(L + L^T) + \text{diag}(\mathbf{k}) \) is positive definite, i.e., all the eigenvalues of \( \frac{1}{2}(L + L^T) + \text{diag}(\mathbf{k}) \) are positive. Due to the fact
\[ \text{Re}(\lambda_1(L + \text{diag}(\mathbf{k}))) \geq \lambda_1 \left( \frac{1}{2}(L + L^T) + \text{diag}(\mathbf{k}) \right), \]
therefore, the multi-agent system (5) achieve the trivial consensus.

Now, one can summarize the collective behavior of compensated dynamics (5) in terms of compensation vector \( \mathbf{k} \) in Table 1.

**6. Conclusion Remarks**

The negative edges in a signed network can often lead to an unstable Laplacian matrix, essentially owing to the lost of its diagonal dominance. In this paper, the connection between self-loop compensation (for re-establishment of
Table 1: Selection of compensation vector $k$ and the collective behavior of compensated dynamics (5) ($x(\infty)$), where SB and SIB refer to structural balance and structural imbalance respectively.

| $k > \delta$ | $k = \delta$ | $k < \delta$ |
|--------------|--------------|--------------|
| trivial consensus | trivial consensus | trivial consensus |
| Undirected Networks | Directed Networks | 

The correlation between the choice of compensation vector and collective behavior of the network is shown to be closely related to the structural balance of the underlying signed network. The results in this work eventually provide a novel perspective on the stability of signed Laplacian and its correspondence with the graph-theoretic characterization of the underlying signed network.

This work also provides notable implications and potential applications. For instance, this work implies novel insight into the multi-agent system on signed networks from a design perspective. It is unveiled that how much self-loop compensation is enough for stability and consensus of signed networks. For structurally balanced signed networks, the bipartite consensus can be achieved if and only if the compensation vector is $k = \delta$, in addition, it is also implied that the bipartite consensus is unique. This is useful if one is intended to steer the signed network towards the desired state via leader-following approach.

Appendix

**Lemma 1.** Horn and Johnson [17, Theorem 8.4.5, p. 534] (Wielandt’s Theorem) Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$. If $A$ is irreducible and $|B| \leq A$, then $\rho(B) \leq \rho(A)$; if the equality holds (i.e., if $\mu = \rho(A)e^{i\phi} \in \lambda(B)$ for some $\phi \in \mathbb{R}$), then $B = e^{i\phi}PA^{-1}P^{-1}$ for some $P = \text{diag} \{e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n}\}$, and conversely.

References

[1] Altafini, C., 2013. Consensus problems on networks with antagonistic interactions. IEEE Transactions on Automatic Control 58, 935–946.

[2] Arcak, M., 2011. Diagonal stability on cactus graphs and application to network stability analysis. IEEE Transactions on Automatic Control 56, 2766–2777.
[3] Bronski, J.C., DeVille, L., 2014. Spectral theory for dynamics on graphs containing attractive and repulsive interactions. SIAM Journal on Applied Mathematics 74, 83–105.

[4] Cao, Y., Ren, W., Egerstedt, M., 2012. Distributed containment control with multiple stationary or dynamic leaders in fixed and switching directed networks. Automatica 48, 1586–1597.

[5] Cartwright, D., Harary, F., 1956. Structural balance: a generalization of heider’s theory. Psychological review 63, 277.

[6] Chen, W., Wang, D., Liu, J., Başar, T., Johansson, K.H., Qiu, L., 2016a. On semidefiniteness of signed laplacians with application to microgrids. IFAC-PapersOnLine 49, 97–102.

[7] Chen, W., Wang, D., Liu, J., Bașar, T., Qiu, L., 2017. On spectral properties of signed laplacians for undirected graphs, in: IEEE 56th Annual Conference on Decision and Control, pp. 1999–2002.

[8] Chen, W., Wang, D., Liu, J., Chen, Y., Khong, S.Z., Basar, T., Johansson, K.H., Qiu, L., 2020. On spectral properties of signed laplacians with connections to eventual positivity. IEEE Transactions on Automatic Control.

[9] Chen, Y., Khong, S.Z., Georgiou, T.T., 2016b. On the definiteness of graph laplacians with negative weights: Geometrical and passivity-based approaches, in: 2016 American Control Conference (ACC), IEEE. pp. 2488–2493.

[10] Ding, T., Li, C., Yang, Y., Bo, R., Blaabjerg, F., 2016. Negative reactance impacts on the eigenvalues of the jacobian matrix in power flow and type-1 low-voltage power-flow solutions. IEEE Transactions on Power Systems 32, 3471–3481.

[11] Dorfler, F., Bullo, F., 2012. Kron reduction of graphs with applications to electrical networks. IEEE Transactions on Circuits and Systems I: Regular Papers 60, 150–163.

[12] Facchetti, G., Iacono, G., Altafini, C., 2011. Computing global structural balance in large-scale signed social networks. Proceedings of the National Academy of Sciences 108, 20953–20958.

[13] Godsil, C.D., Royle, G., Godsil, C., 2001. Algebraic graph theory. volume 207. Springer New York.

[14] Harary, F., et al., 1953. On the notion of balance of a signed graph. The Michigan Mathematical Journal 2, 143–146.

[15] Heider, F., 1946. Attitudes and cognitive organization. The Journal of psychology 21, 107–112.
[16] Hershkowitz, D., 1992. Recent directions in matrix stability. Linear Algebra and its Applications 171, 161–186.

[17] Horn, R.A., Johnson, C.R., 1990. Matrix analysis. Cambridge University Press.

[18] Kaszkurewicz, E., Bhaya, A., 2012. Matrix diagonal stability in systems and computation. Springer Science & Business Media.

[19] Kia, S.S., Van Scoy, B., Cortes, J., Freeman, R.A., Lynch, K.M., Martinez, S., 2019. Tutorial on dynamic average consensus: The problem, its applications, and the algorithms. IEEE Control Systems Magazine 39, 40–72.

[20] Meng, D., 2018. Convergence analysis of directed signed networks via an m-matrix approach. International Journal of Control 91, 827–847.

[21] Meng, D., Meng, Z., Hong, Y., 2018. Uniform convergence for signed networks under directed switching topologies. Automatica 90, 8–15.

[22] Mesbahi, M., Egerstedt, M., 2010. Graph theoretic methods in multiagent networks. Princeton University Press.

[23] Olfati-Saber, R., Murray, R.M., 2004. Consensus problems in networks of agents with switching topology and time-delays. IEEE Transactions on Automatic Control 49, 1520–1533.

[24] Pan, L., Shao, H., Li, D., Xi, Y., 2021. Cluster consensus on matrix-weighted switching networks. arXiv preprint arXiv:2107.09292.

[25] Pan, L., Shao, H., Mesbahi, M., 2016. Laplacian dynamics on signed networks, in: 2016 IEEE 55th Conference on Decision and Control, pp. 891–896.

[26] Pirani, M., Costa, T., Sundaram, S., 2014. Stability of dynamical systems on a graph, in: IEEE 53rd Annual Conference on Decision and Control, pp. 613–618.

[27] Proskurnikov, A.V., Matveev, A.S., Cao, M., 2015. Opinion dynamics in social networks with hostile camps: Consensus vs. polarization. IEEE Transactions on Automatic Control 61, 1524–1536.

[28] Qin, J., Ma, Q., Shi, Y., Wang, L., 2016. Recent advances in consensus of multi-agent systems: A brief survey. IEEE Transactions on Industrial Electronics 64, 4972–4983.

[29] Shi, G., Altafini, C., Baras, J.S., 2019. Dynamics over signed networks. SIAM Review 61, 229–257.

[30] Song, Y., Hill, D.J., Liu, T., 2017a. Characterization of cutsets in networks with application to transient stability analysis of power systems. IEEE Transactions on Control of Network Systems 5, 1261–1274.
[31] Song, Y., Hill, D.J., Liu, T., 2017b. Local stability of dc microgrids: A perspective of graph laplacians with self-loops, in: 2017 IEEE 56th Annual Conference on Decision and Control (CDC), IEEE. pp. 2629–2634.

[32] Song, Y., Hill, D.J., Liu, T., 2017c. Network-based analysis of small-disturbance angle stability of power systems. IEEE Transactions on Control of Network Systems 5, 901–912.

[33] Song, Y., Hill, D.J., Liu, T., 2017d. On extension of effective resistance with application to graph laplacian definiteness and power network stability. IEEE Transactions on Circuits and Systems I: Regular Papers 5, 901–912.

[34] Wei, J., Johansson, A., Sandberg, H., Johansson, K.H., Chen, J., 2018. Optimal weight allocation of dynamic distribution networks and positive semi-definiteness of signed laplacians. arXiv preprint arXiv:1803.05640.

[35] Xia, W., Cao, M., 2011. Clustering in diffusively coupled networks. Automatica 47, 2395–2405.

[36] Zelazo, D., Bürger, M., 2014. On the definiteness of the weighted laplacian and its connection to effective resistance, in: IEEE 53rd Annual Conference on Decision and Control, pp. 2895–2900.

[37] Zelazo, D., Bürger, M., 2017. On the robustness of uncertain consensus networks. IEEE Transactions on Control of Network Systems 4, 170–178.