On volumes of hyperbolic 3-manifolds

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Abstract

The main thrust of present note is a volume formula for hyperbolic surface bundle with the fundamental group $G$. The novelty consists in a purely algebraic approach to the above problem. Initially, we concentrate on the Baum-Connes morphism $\mu^G : K_\bullet(BG) \to K_\bullet(C^*_{red}G)$ for our class of manifolds, and then classify $\mu^G$ in terms of the ideals in the ring of integers of a quadratic number field $K$. Next, we extract the topological data (e.g. volume of the orbifold $\mathbb{H}^3/G$) from the arithmetic of field $K$.

Key words and phrases: noncommutative geometry, geometric topology

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Introduction

The topological classification of 3-dimensional manifolds is a challenge of enormous interest and difficulty. Thanks to knot theory and topological quantum field theory we understand better the nature of such manifolds. The lack of good invariants is usually blamed for a slow progress in the field. Main expectations are connected with Thurston’s idea to substitute topology by geometry, since most of the 3-manifolds admit a hyperbolic metric.

It is amazing that geometry of 3-dimensional manifolds appears to be essentially “asymptotic” or “noncommutative”. Let us explain what we mean by this. Heegaard splitting of a 3-manifold $M$ uncovers one important detail about geometry of $M$. There exist infinitely many (a countable set of) distinct Heegaard diagrams which represent $M$. Every such diagram is a closed line on surface of genus $g \geq 1$, so that the set of equivalent Heegaard diagrams can be
seen as a “spectrum” of $M$. It is clear that any reasonable topological invariant of $M$ must depend exclusively on the infinite, i.e. asymptotic, part of the spectrum. Note that for 2-dimensional manifolds (surfaces) this construction always yields finite spectra.

Let $G = \pi_1 M$, $BG$ the classifying space of $G$ and $C^*_\text{red}G$ the reduced group $C^*$-algebra of $G$. It was conjectured by Novikov, Baum and Connes that the assembly map

$$\mu^G: K_\bullet(BG) \to K_\bullet(C^*_\text{red}G)$$

is a morphism. Thus any $K$-theory invariant of $C^*_\text{red}G$ is an index, i.e. topological invariant of $M$.

Our knowledge of morphism (1) is remarkably scarce. Amenable groups, Gromov's hyperbolic groups, and some other special groups are known to satisfy the Baum-Connes conjecture. The general problem seems to be extremely hard. In this context, the aim of present note is to calculate the Baum-Connes morphism in case $G = \pi_1 M$, where $M$ is a 3-dimensional manifold fibering over the circle with surface $S$ as a fiber (surface bundle).

This class of manifolds has been studied by Stallings, Jørgensen and Thurston. Stallings described the structure of fundamental group of surface bundles. Jørgensen showed that for one punctured torus bundle with aperiodic (pseudo-Anosov) monodromy, there exists a hyperbolic metric on $M$. Thurston proved that this is true for every surface bundle with the pseudo-Anosov monodromy. On the other hand, surface bundles seem to finitely cover any hyperbolic 3-manifold.

Let $G$ be the Kleinian group of surface bundle $M$. The assembly map $\mu^G$ admits in this case an explicit construction because of the highly elaborated geometric theory of “quasi-Fuchsian deformations” of the group $\pi_1 S$ (Thurston and others). Roughly speaking, representations of $G$ and topology of $M$ are “controlled” by “geometry” of surface $S$. More precisely, certain collections of disjoint curves on $S$, called geodesic lamination, are responsible for the limit representation of $\pi_1 S$ in $\text{SL}(2,\mathbb{C})$. In turn, this limit representation generates the Kleinian group $G$ according to Stallings’ result for $\pi_1 M$ (see Appendix). On the other hand, geodesic laminations are intimately connected with the combinatorics of $AF C^*$-algebras and through it with $C^*_\text{red}G$.

It can be shown by a routine calculation that $K$-theory of $C^*_\text{red}G$ is Bott 2-periodic. We shall focus on the topological data stored in the ordered $K_0$-group $\mathcal{E} = (K_0, K_0^+, [u])$, known as dimension group of $C^*_\text{red}G$. For the surface bundles, $\mathcal{E}$ has a remarkable property of being stationary, i.e. self-similar for an automorphism $A$ of lattice $\mathbb{Z}^n$. The isomorphism class of stationary dimension group is known to intrinsically depend on the ring $\text{End} \mathcal{E}$ of the order-endomorphisms of $\mathcal{E}$. By a crucial calculation (Lemma 2) we show that

\[1\] In the context of holomorphic dynamics, this property of surface bundles has been singled out by McMullen as a “renormalization” property, see ([14]).
$E \simeq O_K$, where $O_K$ is the ring of integers in a real quadratic number field $K = \mathbb{Q}(\sqrt{d})$. Thus, $O_K$ is a Morita invariant of $K_0(C^*_\text{red}(G))$.

We wish to explain the nature of ring $O_K$ in more geometric terms. Geodesic lamination $\Lambda$ comes with definite “slope” $\theta$ on surface $S$. Real number $\theta$ equals to average inclination of the geodesic leaves in $\Lambda$, and can be thought of as extension of the well-known Poincaré rotation numbers to the case $S$ having genus $g \geq 2$ (see Appendix). Since $\Lambda$ is fixed by a pseudo-Anosov map on $S$, its $\theta$ unfolds into a periodic continued fraction, and therefore $\theta \in K = \mathbb{Q}(\sqrt{d})$ for a positive square-free integer $d$. In fact, $\theta \in O_K$ and all cyclic covers $\Lambda'$ of $\Lambda$, i.e. $\Lambda'$ such that $\psi(\Lambda') = \Lambda'$, $\psi(\Lambda) \neq \Lambda$ and $\psi^m(\Lambda) = \Lambda$, will have slopes $\theta' \in O_K$.

Thus, the ring $O_K$ is generated by the slopes of cyclic covers of lamination $\Lambda$.

Volumes of hyperbolic 3-manifolds. The volume of surface bundle $M$ is the homotopy invariant of $M$. Integration of hyperbolic volumes consists in an “ideal triangulation” of $M$, so that the sides of ideal tetrahedra satisfy certain “compatibility equations”. It is remarkable that compatibility equations are intrinsically diophantine and have clear links to number theory (Neumann-Zagier [15]). Other methods include numerical integration (SNAPPEA, Weeks [20]), and conjectures about coloured Jones polynomials (Kashaev [13]) and Mahler measures (Boyd [4]). An estimate of hyperbolic volumes through the Weil-Petersson metric can be given (Brock [5]). For a class of “arithmetic 3-manifolds” the classical Humbert’s volume formula is known (Humbert [12]). Note that our approach indicates that surface bundles are “arithmetic”, although it was proved that there are only a finite number of surface bundles among “classical” arithmetic manifolds (Bowditch-Maclachlan-Reid [3]). In general, the problem of integration seems to be extremely hard.

Scope of present note. In this note we introduce an integration technique for hyperbolic surface bundles, inspired by the ideas and methods of noncommutative geometry. The integration runs as follows. Let $N = N(p_1, q_1), \ldots, (p_n, q_n)$ be hyperbolic surface bundle obtained from a manifold $M$ with $n$ cusps by $(p_i, q_i)$-surgery on the $i$-th cusp. For brevity, consider the coprime pairs $p_i > 0, q_i = 1$ and take an infinite periodic continued fraction:

$$\theta = p_1 + \frac{1}{p_2 + \cdots + \frac{1}{p_n + \cdots}} = \text{Per } [p_1, \ldots, p_n]. \quad (2)$$

Eventually, it can be proved that $\theta$ equals to the slope of geodesic lamination $\Lambda$ on the fibre $S$ of surface bundle $N$. Take the quadratic extension $K = K(\theta)$ of field $\mathbb{Q}$. Finally, the last step of the integration process is contained in the following theorem.

**Theorem** Let $M$ be a hyperbolic surface bundle with $n$ cusps. Then the volume of manifold $N = N(p_1, 1), \ldots, (p_n, 1)$ obtained from $M$ by the $(p_i, 1)$-filling of the $i$-th
cusp is given by the formula:

\[ \text{Vol}(N) = C(M) \log \varepsilon \frac{d}{\sqrt{d}}, \quad (3) \]

where \( \varepsilon > 1 \) is the fundamental unit, \( d \) the discriminant of the field \( K = K(p_1, \ldots, p_n) \) and \( C(M) > 0 \) a real constant depending only on manifold \( M \).

Let \( \zeta_K(s) \) be the Dedekind zeta-function of field \( K \). Then \( \zeta_K(s) \) has a pole in the point \( s = 1 \). It is not hard to see that formula (3) can be written as \( \text{Vol}(N) = C(M) \text{Res } \zeta_K(1) \) in this case. The latter expression is an analogue of Humbert’s formula \( \text{Vol}(N) = \frac{1}{12} |d| \zeta_K(2) \) for the volume of “arithmetic” 3-manifold \( N \), attached to the quadratic field \( K \) with discriminant \( d < 0 \).

1 Basic lemmas

This section contains calculation of the group \( K_0(C^*_{\text{red}}G) \) (Lemma 1) and its invariants (Lemma 2). For the sake of brevity, we assume that the reader is familiar with the 3-manifolds, assembly maps, \( K \)-theory of \( AF C^* \)-algebras and algebraic number theory. For the reference we would recommend the unpublished lecture notes of Thurston ([19]) survey article of Higson ([11]), books of Rørdam et al ([17]) and Hecke ([8]). Some very brief review can be found in the Appendix to this article. Let us introduce the following notation:

- \( \mathbb{H}^3 \) hyperbolic 3-dimensional space;
- \( G \) discrete (Kleinian) group on \( \mathbb{H}^3 \);
- \( \mathcal{A} \) \( AF C^* \)-algebra;
- \( \hat{\mathcal{A}} \) representation space of algebra \( \mathcal{A} \);
- \( \mathcal{E} \) dimension (Elliott) group;
- \( \theta_\mathcal{E} \) rotation number associated to \( \mathcal{E} \);
- \( \mathcal{E}_A \) stationary dimension group given by matrix \( A \);
- \( B(\mathcal{A}) \) Bratteli diagram of \( \mathcal{A} \);
- \( K_0 \) Elliott functor \( \mathcal{A} \to \mathcal{E} \);
- \( \Lambda \) geodesic lamination on surface \( S \);
- \( C^*_{\text{red}}G \) reduced group \( C^* \)-algebra.
1.1 Calculation of the group $K_0(C^*_\mathrm{red}G)$

**Lemma 1** If $G$ is the fundamental group of 3-dimensional manifold fibering over the circle with a pseudo-Anosov monodromy $\varphi$, then

$$K_0(C^*_\mathrm{red}G) \cong E_A.$$  \hspace{1cm} (4)

**Proof.** We split the proof into several parts (Propositions 1-3), which might be of independent interest to the reader. We apologize for a fairly sketchy argument, the reader can find details in the original articles and preprints.

**Proposition 1 (Bers-Thurston)** Given pseudo-Anosov diffeomorphism $\varphi: S \to S$, the Kleinian group $G$ of Lemma 1 is determined by a geodesic lamination $\Lambda \subset S$ such that

$$\varphi(\Lambda) = \Lambda.$$  \hspace{1cm} (5)

**Proof.** The original proof of this fact is due to Thurston ([18]). Let $M$ be the mapping torus of a surface homeomorphism $\varphi: S \to S$. Thurston proved ([18], Theorem 5.5) that $M$ is either a Seifert fibration ($\varphi$ is periodic) or splits into simpler manifolds ($\varphi$ is reducible) or else a hyperbolic 3-manifold ($\varphi$ is pseudo-Anosov). In the latter case, there exists a Kleinian group $G$, which is a “geometric limit” of quasi-fuchsian deformations of the group representations $\pi_1 S \to SL(2, \mathbb{C})$. This limit has a remarkable description in terms of a $\varphi$-invariant geodesic lamination $\Lambda$ on surface $S$. (In other words, geometry on $S$ “controls” topology of $M$.) By the Mostow-Prasad rigidity of group representations, $\Lambda$’s are in a 1-1 correspondence with groups $G$ ([18]). Since $G \cong \pi_1(S) \rtimes \varphi \mathbb{Z}$, Proposition 1 follows. □

**Proposition 2 ([16])** For any $\mathcal{A}$ there exists $\Lambda$ such that $\hat{\mathcal{A}} \cong \Lambda$. Conversely, given $\Lambda$ there exists a dual $\Lambda \cong \mathcal{A}$.

**Proof.** The idea of proof is based on bijection between Bratteli diagrams of AF $C^*$-algebras and geodesic laminations on $S$, which follows, in turn, from the Koebe-Morse theory of coding of the geodesic lines. Let $\mathcal{A}$ be a simple AF $C^*$-algebra and $B(\mathcal{A})$ its Bratteli diagram of rank $2g$. Consider the set $Spec B(\mathcal{A})$ consisting of all infinite paths of the diagram $B(\mathcal{A})$. Marking the vertices (of same level) of $B(\mathcal{A})$ by the “symbols” $\mathfrak{A} = \{a_1, \ldots, a_{2g}\}$, one gets an infinite word $\{\omega_1, \omega_2, \ldots | \omega_i \in \mathfrak{A}\}$ for every element of $Spec B(\mathcal{A})$. Since each word is a Morse symbolic geodesic, one gets a correspondence between $Spec B(\mathcal{A})$ and a geodesic lamination $\Lambda$ on surface $S$. Note that $\Lambda$ is minimal since any two geodesics lie in the closure of each other by the simplicity of $\mathcal{A}$.

On the other hand, every infinite path of $B(\mathcal{A})$ defines a pure state on the $C^*$-algebra $\mathcal{A}$. In terms of the representation theory, this means that $\hat{\mathcal{A}} \cong Spec B(\mathcal{A})$. Proposition 2 follows. □
Proposition 3 Let $\Lambda$ be geodesic lamination satisfying equation (5) and let $\hat{\Lambda} \cong \mathcal{A}$ be its dual (Proposition 2). Then
\[ K_0(\mathcal{A}) = \mathcal{E}_A, \] (6)
for a stationary dimension group $\mathcal{E}_A$.

Proof. We have to show that any geodesic lamination on $S$, which is invariant under a pseudo-Anosov diffeomorphism $\varphi$, must be given by Bratteli diagram of stationary type. Every Bratteli diagram $B(\mathcal{A})$ admits a natural order $\geq$ on its set of infinite paths $Spec B(\mathcal{A})$, see ([10]). Based on this order, one can define a Putnam-Vershik homeomorphism on the set $Spec B(\mathcal{A})$, which is a Cantor set in a natural topology. If the diagram is simple, the homeomorphism is minimal.

Consider a restriction $\phi = \varphi |_{\Lambda}$ of $\varphi$ to geodesic lamination $\Lambda \cong Spec B(\mathcal{A})$. It is not hard to see that $\phi$ is conjugate to a Putnam-Vershik homeomorphism on the Cantor set $\Lambda$. Moreover, the condition $\varphi(\Lambda) = \Lambda$ would imply that $\phi$ must be order-preserving with respect to the ordered diagram $(B(\mathcal{A}), \geq)$. The latter requirement can be met if and only if the diagram admits strong structural “symmetry” which is actually a periodicity with some constant incidence matrix $A$. Proposition 3 follows. □

To finish the proof of Lemma 1, let us notice that $C^*G \cong \hat{G}$, where $\hat{G}$ is a “noncommutative dual” of the Kleinian group $G$. Using Propositions 1 and 2 we get the following chain of isomorphisms:
\[ C_{red}^*G \cong \hat{G} \cong \hat{\Lambda} \cong \mathcal{A}, \] (7)
and $K_0(C_{red}^*G) = K_0(\mathcal{A}) = \mathcal{E}_A$ by Proposition 3. Lemma 1 follows. □

1.2 Classification of groups $K_0(C_{red}^*G)$

Let $\mathcal{E} = (E, E^+, [u])$ be ordered abelian group of rank $E < \infty$. Consider the ring $End E$ of endomorphisms of $E$ under “pointwise” addition and multiplication of mappings $E \to E$. The subring of $End E$, made of endomorphisms which “preserve” positive cone $E^+$, we shall denote by $End \mathcal{E}$.

Let $\mathcal{E}_A$ be stationary ordered abelian group. In this case, the rotation number $\theta_{\mathcal{E}_A}$ is known to be real quadratic (see Appendix). Denote by $K = \mathbb{Q}(\theta_{\mathcal{E}_A})$ the field of algebraic numbers, which is an extension of $\mathbb{Q}$ by quadratic irrationality $\theta_{\mathcal{E}_A}$. Clearly, $K = \mathbb{Q}(\sqrt{d})$ for a square-free positive integer $d$.

Finally, consider the embedding
\[ End \mathcal{E}_A \to K, \] (8)
given by the formula $\mathcal{E}_i \to \theta_{\mathcal{E}_i}$, where $\mathcal{E}_i = Im f_i$ and $f_i \in End \mathcal{E}$. It is not hard to see that full image of $End \mathcal{E}_A$ is an integral ideal in $K$, which we denote by $I_{\mathcal{E}_A}$.
**Definition 1** The equivalence class \(^2\) of ideals \([I_{E_A}]\) in the ring of integers \(O_K\) we call associated to the stationary dimension group \(E_A\).

**Lemma 2** Let \([E_A]\) be the Morita equivalence class of stationary dimension group \(E_A\). The correspondence \([E_A] \leftrightarrow [I_{E_A}]\) is a bijection which classifies stationary dimension groups.

*Proof.* The idea of the proof belongs to Handelman, cf. §5 of ([7]). We split the proof into two propositions. Since \(E_A\) is stationary, its ring of order-preserving endomorphisms \(\text{End} \ E\) is non-trivial, i.e. distinct from \(\mathbb{Z}\). Moreover, in this case \(\text{End} \ E\) completely defines, and is defined by, \(E_A\) (Proposition 4). We pull back the embedding (8) and prove that Morita equivalent dimension groups generate equivalent ideals in \(O_K\) (Proposition 5).

**Proposition 4** Stationary dimension group \(E_A\) is defined by its endomorphism ring \(\text{End} \ E\).

*Proof.* Let \(E_A\) be stationary dimension group of rank \(n\). Consider the set

\[ X = \bigcap_{f \in \text{End} \ E_A} f(E_A). \tag{9} \]

\(X\) is a non-empty set, since \(\text{End} \ E_A \neq \mathbb{Z}\). It is also an additive abelian group of rank \(n\). Clearly \(f \in \text{End} \ E_A\) coincide with \(f \in \text{Aut} X\). It is not hard to see that \(\text{Aut} X\) is infinite cyclic group generated by a single automorphism \(f : \mathbb{Z}^n \to \mathbb{Z}^n\). Then our stationary group \(E_A\) is representable as limit of the simplicial dimension groups under the automorphism \(f\). \(\square\)

**Proposition 5** Let \(E_A\) and \(E'_A\) be two (stationary) dimension groups, which are Morita equivalent. Then their associated ideals \(I_{E_A}\) and \(I_{E'_A}\) are equivalent, i.e. differ by a principal ideal multiple \((\omega)\).

*Proof.* Let \(E_A = (\mathbb{Z}^n, \mathbb{Z}^*_n, [u])\) and \(E'_A = (\mathbb{Z}^n, (\mathbb{Z}^*_n)', [u'])\) be two stationary dimension groups of rank \(n\). Then \(E_A \sim E'_A\) are Morita equivalent if and only if there exists \(f \in \text{End} E_A\) such that \(f(\mathbb{Z}^*_n) = (\mathbb{Z}^*_n)\). Since every \(f\) extends to a linear map \(\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n\), one can consider the \(n\)-torus \(T^n = \mathbb{R}^n / \mathbb{Z}^n\) and the mapping:

\[ \tilde{f} : T^n \to T^n. \tag{10}\]

Let now \(I_{E_A}\) be the ideal associated to \(E_A\). Fix an algebraic integer \(\omega \in O_K\) such that

\[ N(\omega) = \deg \tilde{f}, \tag{11}\]

\(^2\)Two integral ideals \(I_1, I_2\) are equivalent if and only if there exists a principal ideal \((\omega), \omega \in O_K\) such that \(I_2 = (\omega)I_1\), see Hecke ([8]).
where \( N(\omega) \) is the norm of \( \omega \) and \( \deg \overline{f} \) is the degree of continuous map \((10)\). Since the norm is a multiplicative function on the ideals and the norm of principal ideal \((\omega)\) coincides with the \( N(\omega) \), one gets that \( I_{\mathcal{E}_A} = (\omega)I_{\mathcal{E}_A} \). □

To finish the proof of Lemma 2, it remains to apply Propositions 4 and 5, and recall the definition of the equivalence class of the ideal. □

2 Main result

In this section we state main result which relates hyperbolic volumes of surface bundles with the arithmetic of field \( K \) (volume inequalities). Our proof of these inequalities is based on an observation that “commensurability class” of hyperbolic manifold has a natural “ideal structure”, which is isomorphic to the ideal structure (“arithmetic”) of field \( K \).

Before we go to the statement of results, some general remarks on the analogy between number theory and 3-dimensional topology might be helpful. Hyperbolic surface bundles might be good example of “noncommutative varieties”. Indeed, think of \( \mathcal{E}_A \) as a “coordinate ring” of the noncommutative variety \( V(\mathcal{E}_A) \).

Then “divisor” on \( V(\mathcal{E}_A) \) coincides with an ideal in ring \( \mathcal{O}_K \) (Lemma 2). The group of equivalence classes of divisors \( \text{Pic} V(\mathcal{E}_A) = \text{Cl} K \), where \( \text{Cl} K \) is the class group of field \( K \). The role of the Chern class of the line bundle \( L_{V(\mathcal{E}_A)} \) is played by the “volume formula” \( \text{Ch} L_{V(\mathcal{E}_A)} \simeq \text{Res} \zeta_K(1) \), where \( \zeta_K \) is the Dedekind zeta-function of field \( K \).

Definition 2 Manifolds \( M_1 \) and \( M_2 \) are said to be commensurable, if there exist finite coverings \( \tilde{M}_1 \) and \( \tilde{M}_2 \), such that \( \tilde{M}_1 \simeq \tilde{M}_2 \) are homeomorphic. The equivalence class of manifolds commensurable with \( M \) is denoted by \( \mathfrak{M} \). Manifold \( M \in \mathfrak{M} \) is called prime, if it cannot be covered by any other manifold from \( \mathfrak{M} \) except \( M \) itself.

Let \( \mathfrak{M} \) be commensurability class of a hyperbolic surface bundle. Lemma 2 indicates on an isomorphism between “arithmetic” of set \( \mathfrak{M} \) and the ideal structure of ring \( \mathcal{O}_K \). This observation is by no means new in the context of “arithmetic” 3-manifolds, cf. Helling ([9]). Borel ([2]) showed that there exists infinitely many prime manifolds in every commensurability class of such manifolds. Thurston observed that one can study invariants of commensurability classes of 3-dimensional manifolds using certain algebraic number fields (trace-fields), cf §6.7 of ([19]). The following theorem is an attempt to extend Borel-Helling-Thurston theory to hyperbolic surface bundles.

Theorem 1 Let \( M \) be hyperbolic surface bundle, which has minimal volume in its commensurability class \( \mathfrak{M} \). Then there exist real constants \( k \) and \( K \) depending exclusively on \( \mathfrak{M} \), such that

\[
\frac{k \log \varepsilon}{\sqrt{d}} \leq \text{Vol} M \leq K \frac{\log \varepsilon}{\sqrt{d}},
\]

(12)
where \( \varepsilon \) is the fundamental unit of the number field \( K = \mathbb{Q}(\sqrt{d}) \) associated to \( M \). Moreover, if \( \{N_i\} \) is the family of surface bundles obtained from \( M \) by the Dehn surgery (cusp-filling), then in formula (12) \( k = K = C(M) > 0 \) for a positive constant \( C(M) \) defined for the whole family \( \{N_i\} \) and depending only on manifold \( M \).

**Corollary 1** Function \( M \to Vol M \) has degree \( h_K \) at \( M \), where \( h_K \) is the class number of field \( K \).

### 3 Proof

#### 3.1 Proof of Theorem 1

**Part 1.** We wish to prove inequality (12). Let us outline the main idea. If \( M \) is a surface bundle with the pseudo-Anosov monodromy, then it can be described as an ideal in the equivalence class of ideals \([I_{E_A}]\). Other ideals in this class correspond to manifolds commensurable with \( M \). Thus, one can extend the notion of “divisibility” from the ideals to commensurable manifolds. In this sense, prime ideals correspond to prime manifolds in the commensurability class \( \mathfrak{M} \). By the Borel-Helling lemma the number of prime manifolds is infinite (Lemma 3). Note also that since \( O_K \) is the Dedekind domain, any \( M \in \mathfrak{M} \) can be “uniquely decomposed into primes”. On the other hand, the “volume growth” of manifolds in \( \mathfrak{M} \) is discrete, in sense that it is bounded by the uniform constants \( k \) and \( K \) (Lemma 4). Thus, knowing the Dirichlet density of ideals in \([I_{E_A}]\) (Lemma 5) allows us to estimate the volume of minimal prime manifold in \( \mathfrak{M} \) in terms of the arithmetic of field \( K \).

**Division algorithm for commensurable manifolds.** Let \( \mathfrak{M} \) be commensurability class of hyperbolic surface bundles. If \( M_1, M_2 \in \mathfrak{M} \) are such that \( M_1 \) covers \( M_2 \), one can define their “ratio” \( M_1/M_2 \in \mathfrak{M} \) in the following way. Let \( E_{A_1} \) and \( E_{A_2} \) be the stationary dimension groups corresponding to \( M_1 \) and \( M_2 \) (Lemma 1). Since \( M_1 \) covers \( M_2 \), there exists a order-preserving homomorphism \( f : E_{A_1} \to E_{A_2} \). Then the normal (order) subgroup \( E_{A_1}' = Ker f \subseteq E_{A_1} \) is a stationary dimension group. Thus if \( M_1' \) is the hyperbolic surface bundle corresponding to \( E_{A_1}' \), we can define the ratio:

\[
\frac{M_1}{M_2} = M_1'.
\]

(13)

Clearly, \( M_1' \in \mathfrak{M} \) and either \( M_1' \) covers \( M_2 \) or “relatively prime” with \( M_2 \). In the first case, we repeat the division algorithm until the newly obtained ratio becomes “relatively prime” to \( M_2 \). By the construction, the algorithm stops on the \( p \)-th step, if and only if \( M_1 \) is \( p \)-fold cover to \( M_2 \).

**Lemma 3 (Borel-Helling)** The number of prime manifolds in the commensurability class \( \mathfrak{M} \) is infinite.
Proof. For hyperbolic manifolds, which are factor spaces of $\mathbb{H}^3$ by the discrete subgroup of $SL(2, \mathbb{C})$ generated by the algebraic integers of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, this lemma was proved by Borel ([2]) and Helling ([9]). We wish to prove it for hyperbolic surface bundles using the above mentioned “division algorithm”.

Let $M \in \mathfrak{M}$ be a hyperbolic surface bundle of commensurability class $\mathfrak{M}$. By the division algorithm, $M$ unfolds into a product of powers of prime manifolds: $M = (M_{\pi_1})^{p_1} (M_{\pi_2})^{p_2} \ldots (M_{\pi_k})^{p_k}$. Let $P_{\pi_i}$ be the ideals in $K$, which correspond to prime manifolds $M_{\pi_i}$. It is not hard to see that $P_{\pi_i}$ are prime ideals in $K$.

Let $\omega \in \mathcal{O}_K$ be prime algebraic integer which does not belong to any of the ideals $P_{\pi_i}, i = 1, \ldots, p$. Consider the principal ideal $(\omega)$ and take the manifold $M_\omega \in \mathfrak{M}$, which correspond to $(\omega)$. Clearly, $M_\omega$ is prime manifold, which does not belong to finite list of prime manifolds $M_{\pi_1}, M_{\pi_2}, \ldots, M_{\pi_k}$. □

**Lemma 4** Let $M_0 < M_1 < M_2 < \ldots$ be sequence of commensurable hyperbolic 3-manifolds of growing volume. Then there exists positive constants $k$ and $K$ such that for every $i = 1, \ldots, \infty$ it holds:

$$k \leq Vol M_i - Vol M_{i-1} \leq K. \quad (14)$$

Proof. Step 1. Lower bound. To prove existence of $k > 0$, it is sufficient to show that monotone sequence $Vol M_0, Vol M_1, \ldots$ cannot be a Cauchy sequence. (Indeed, for otherwise, $k = 0$.) Suppose to the contrary, that monotone growing sequence $Vol M_0, Vol M_1, \ldots$ is Cauchy. Then volume function $Vol : \mathfrak{M} \rightarrow \mathbb{R}$ is bounded by a constant $C$. This gives us a contradiction, since any $M \in \mathfrak{M}$ has an $m$-fold covering, such that $m \ Vol M > C$ for $m$ sufficiently large.

Step 2. Upper bound. Let $K$ be a constant, such that $Vol M_0 < K$. Suppose to the contrary, that there exist $i$ so that $Vol M_{i+1} - Vol M_i > K$. Let $m$ be the maximal integer which satisfy the equality $mVol M_0 \leq Vol M_i$. (In other words, one takes the maximal cover of $M_0$, whose volume does not exceed volume of $M_i$.) It is not hard to see that in view of condition $Vol M_0 < K$ we have

$$Vol M_i < (m + 1)Vol M_0 < Vol M_{i+1}. \quad (15)$$

Thus, $(m + 1)$ covering of $M_0$ lies “between” $M_i$ and $M_{i+1}$, what contradicts our assumptions. Lemma 4 is proved. □

**Lemma 5 (Dirichlet density)** Let $A$ be an equivalence class of ideals in the real quadratic number field $K$ with the discriminant $d$ and fundamental unit $\varepsilon$. Denote by $N(t, A)$ the number of ideals $a \in A$ such that $Na \leq t$, where $t$ is a positive integer. Then

$$\lim_{t \rightarrow \infty} \frac{N(t, A)}{t} = \frac{2\log \varepsilon}{\sqrt{d}}. \quad (16)$$

Moreover, the above limit exists and is the same for all equivalence classes of ideals in $K$. 

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Proof. For the proof of this lemma we refer the reader to Hecke ([8]). □

Let us complete the proof of Theorem 1. Denote by \( N(t) = N(t, A) \) the number of ideals in the ideal class \( A \) whose norm doesn’t exceed \( t \) and consider a chain of manifolds \( \{M_0, M_1, \ldots, M_{N(t)}\} \in \mathfrak{M} \) of growing volume. The telescoping sequence

\[
Vol M_{N(t)} - Vol M_{N(t) - 1} + Vol M_{N(t) - 1} - Vol M_{N(t) - 2} + \ldots + Vol M_2 - Vol M_1 + Vol M_1 - Vol M_0 = M_{N(t)} - M_0,
\]

can be evaluated using Lemma 4 as:

\[
k(N(t) - 1) < Vol M_{N(t)} - Vol M_0 < K(N(t) - 1).
\]

(17)

On the other hand, one has a prime decomposition of manifold \( M_{N(t)} \) in its commensurability class \( \mathfrak{M} \):

\[
M_{N(t)} = (M_{\pi_1})^{p_1} (M_{\pi_2})^{p_2} \ldots (M_{\pi_k})^{p_k},
\]

(18)

where \( M_{\pi_1} = M_0 \) in view of Lemma 3. Since we didn’t make any assumption on the positive integer \( t \) so far, we can now make one. Namely, take \( t \) be such that \( t = p_1 \). One can now evaluate that

\[
Vol M_{N(t)} = tVol M_0.
\]

(19)

Together with equation (17), the latter will give us that

\[
k(N(t) - 1) < (t - 1)Vol M_0 < K(N(t) - 1).
\]

(20)

Since

\[
\lim_{t \to \infty} \frac{N(t) - 1}{t - 1} = \lim_{t \to \infty} \frac{N(t)}{t} = \frac{2 \log \varepsilon}{\sqrt{d}},
\]

(21)

we get inequality (12). Part I of Theorem 1 is proved.

Part II. By Thurston’s result, we have \( Vol (N_i) \to Vol (M) \) and \( Vol (N_i) < Vol (M) \) as \( i \to \infty \) by the cusp-filling process, see e.g. ([15],[19]). On the other hand, constants \( k_i \) and \( K_i \) for \( N_i \) in (12) are exact and therefore \( k_i = K_i = C(N_i) \) for \( N_i \) whose volume is infinitely close to \( Vol (M) \). Thus, \( Vol (N_i) = C(N_i)(\log \varepsilon_i/\sqrt{d_i}) \).

In fact, \( C(N_i) = Const = C(M) \) for the whole family \( \{N_i\} \). To see this, one should take a sequence of discriminants \( \{d_i\} \), such that the Dirichlet density of the fields \( K_i = \mathbb{Q}(\sqrt{d_i}) \) is monotone growing and bounded. Since there is a subsequence in \( \{N_i\} \) corresponding to \( \{K_i\} \), then by Thurston’s formula \( Vol (N_i) \to Vol (M) \), we get \( C(N_i) = Const \). Part II of Theorem 1 is proved. □
3.2 Proof of Corollary 1

For the imaginary quadratic number fields this fact was established by Bianchi ([1]). To establish similar result for real quadratic number fields, let us notice that the Dirichlet density does not depend on the choice of the equivalence class of ideals in $O_K$ (Lemma 5). On the other hand, we have the following lemma.

**Lemma 6** Let $[E_A]$ be the Morita equivalence class of stationary dimension group $E_A$. Let $K = \mathbb{Q}(\sqrt{d})$ be the associated number field to $E_A$. If the class number $h_K$ of the field $K$ is bigger than 1, there exits $h_K$ distinct Morita equivalence classes of stationary dimension groups with the same associated number field $K$.

**Proof.** Since $h_K$ is the number of equivalence classes of ideals in the ring of integers $O_K$, one can apply Lemma 2. Then there exists exactly $h_K$ stationary dimension groups $E_A^{(1)}, E_A^{(2)}, \ldots, E_A^{(h_K)}$, which are pairwise Morita non-isomorphic, but have the same associated number field $K$. Therefore, there exists the same number of Morita equivalence classes related to the given field $K$. □

To finish the proof of corollary, let $M \to \text{Vol}M$ be the Gromov-Thurston function on the set of hyperbolic surface bundles. From the construction of $M$ it follows that stationary dimension groups $E_A^{(1)}, E_A^{(2)}, \ldots, E_A^{(h_K)}$ generate surface bundles $M^{(1)}, M^{(2)}, \ldots, M^{(h_K)}$, which are topologically distinct. However, the arithmetic of their commensurability classes is the same for all of these manifolds. By the discussion above, the arithmetic determines volume of the hyperbolic manifolds. Corollary 1 follows. □

4 Appendix

4.1 Kleinian groups and hyperbolic 3-manifolds

We briefly review Thurston’s theory of Kleinian groups for surface bundles with the pseudo-Anosov monodromy. Excellent source of original information is Thurston’s paper ([18]).

Mapping tori. Let $\varphi$ be a diffeomorphism of surface $S$. One can produce a 3-manifold $M_{\varphi}$ depending on diffeomorphism $\varphi$ by identification of points $(x, 1)$ and $(\varphi(x), 0)$ in the product $S \times I$ ($I$ - unit interval). By Thurston’s theory of surface diffeomorphisms ([18]), any $\varphi$ falls into one of three classes: (i) periodic, (ii) non-periodic (pseudo-Anosov) and (iii) reducible (combination of two preceding types). We shall be interested in $M_{\varphi}$, where $\varphi$ is pseudo-Anosov. Such 3-manifolds are known to be fibre bundles over the circle, with fibre $S$, and pseudo-Anosov monodromy $\varphi$. The interest to this case lies in the fact that $M = M_{\varphi}$ can be presented as $M = \mathbb{H}^3/G$, where $G = \pi_1M$ is a Kleinian group. In other words, $M$ is a hyperbolic manifold of finite volume. In fact, the
topology of $M$ and representation theory of group $G$ is perfectly “controlled” by geometry of surface $S$, hidden in the notion of “geodesic lamination” on $S$.

**Geodesic laminations.** Let $S = \mathbb{H}^2/\Gamma$ be a hyperbolic surface given by the action of Fuchsian group $\Gamma$ on the hyperbolic plane $\mathbb{H}^2$. By a geodesic line on $S$ one understands a line consisting of geodesic (locally shortest) arcs in the given hyperbolic metric on $S$. The geodesic line is called simple if it has no transversal self-intersection on $S$. Geodesic lamination is a disjoint union of simple geodesic lines on surface $S$. If $\gamma$ is non-periodic geodesic, then the topological closure $\text{Clos} \, \gamma$ on $S$, contains continuum of non-periodic geodesic lines, having the same closure as $\gamma$. The set of all simple non-periodic geodesic lines that are everywhere dense in the set $\text{Clos} \, \gamma$ we call minimal geodesic lamination on $S$.

**Representation theory of Fuchsian groups.** Let $\Gamma$ be Fuchsian (surface) group. Let us consider the set $\text{Rep} \, \Gamma$ of all faithful representations $\rho : \Gamma \to SL_2(\mathbb{C})$ into the group of isometries of hyperbolic space $\mathbb{H}^3$. According to Thurston’s theory for $\text{Rep} \, \Gamma$, up to conjugacy by isometries of $\mathbb{H}^3$ there exist three kinds of $\rho(\Gamma)$: (i) Fuchsian, i.e. when $\rho(\Gamma)$ preserves the boundary of unit disc $D \subset \partial \mathbb{H}^3$; (ii) quasi-Fuchsian, i.e. when $\rho(\Gamma)$ “deforms” $D$ but its boundary $\partial D$ is still a Jordan curve (topological circle) in $\partial \mathbb{H}^3$; and (iii) discontinuous, i.e. when $\rho(\Gamma)$ “breaks” the boundary $\partial D$ to a plane-filling curve, homeomorphic to $\mathbb{R}$. The invariants of the type (i) and (ii) representations are so-called (pair of) conformal structures on $S$, while the main invariant of type (iii) representation is an ending (geodesic) lamination on $S$.

**Connection to hyperbolic geometry of mapping tori.** The connection of $\text{Rep} \, \Gamma$ to the mapping tori with pseudo-Anosov monodromy is based on the Stallings’ theorem about structure of the fundamental group of manifolds which fibre over the circle. Namely, the fundamental group of $M_\varphi$ ($\varphi$ pseudo-Anosov) has a representation:

$$\pi_1 M_\varphi = \langle \pi_1 S, t \mid tgt^{-1} = \varphi_\ast(g), \forall g \in \pi_1 S \rangle,$$

where $\varphi_\ast$ is the action of $\varphi$ on $\pi_1 S$. Thus, to construct a representation $\pi_1 M_\varphi \to PSL_2(\mathbb{C})$ (Kleinian group), is equivalent to construction of such a $\rho^\ast : \pi_1 S \to PSL_2(\mathbb{C})$, which is “invariant” (up to an isometry of $\mathbb{H}^3$) under the action of $\varphi_\ast$. Note that there is no chance $\rho^\ast$ to be representation of type (i) or (ii). In fact, it is a basic observation of W. P. Thurston that $\rho^\ast$ must be of type (iii) (Thurston [18]). Thus, ending geodesic laminations on $S$ “controls” topology of manifold $M_\varphi$. The following general result is true.

**Theorem 2 (W. P. Thurston)** Let $M$ be surface bundle with a pseudo-Anosov monodromy $\varphi : S \to S$. Then $M$ is a hyperbolic 3-manifold, whose Kleinian group $G \cong \pi_1 M$ is presented by a $\varphi$-invariant (measured) geodesic lamination on surface $S$.

**Proof.** See ([18]). □
4.2 Baum-Connes morphism and assembly map

In this section we review main aspects of Baum-Connes theory connecting analytical and topological K-theory of discrete groups. For an excellent introductory material we recommend the survey of Higson ([11]).

Baum-Connes morphism for discrete groups. Let $G$ be a countable group. The analytical part of the Baum-Connes morphism involves algebraic $K$-groups of reduced $C^*$-algebra $C^*_{\text{red}} G$ of group $G$. One can think of $C^*_{\text{red}} G$ as $C^*$-algebra generated by representation of $G$ on the Hilbert space $l^2(G)$, see Higson ([11]). The $K$-theory of $C^*$-algebras is reviewed in the next section. The geometrical-topological part of the Baum-Connes morphism involves the $K$-homology of a classifying space $BG$ for group $G$. In particular, when $G$ has no torsion elements (which is always true in the case of Kleinian groups), then $BG = K(G,1)$ the Eilenberg-Mac Lane space of group $G$. The Baum-Connes morphism is a mapping

$$\mu^G_i : K^G_i (BG) \to K^*_i (C^*_{\text{red}} G), \quad i = 0, 1. \quad (23)$$

(Roughly speaking, injectivity of $\mu^G_i$ allows to obtain topological invariants of $BG$ by looking at $C^*$-algebras associated to $G$. Similarly, the surjectivity of $\mu^G_i$ gives an opportunity to classify certain $C^*$-algebras using topological data carried by the space $BG$.)

Assembly (index) map. The pointwise correspondence $\mu^G_i$ between elements of $K$-groups to both sides of morphism (24) is called an assembly map. There exists no universal recipe of how to construct such a map for the concrete countable groups. However, for many topologically significant groups (e.g. amenable, hyperbolic and fundamental groups of Haken 3-manifolds) it was done explicitly. We conclude this section by the following lemma, which illustrates how theory of geodesic laminations and Kleinian groups helps to construct the assembly map for the fundamental group of mapping tori with a pseudo-Anosov monodromy.

Lemma 7 Let $\pi_1 M$ be fundamental group of the surface bundle with pseudo-Anosov monodromy $\varphi$, such that its representation as Kleinian group $G$ is given by a $\varphi$-invariant minimal geodesic lamination $\Lambda_G = BG$. Then there exists an injective (pre-) assembly map

$$\mu^G : \Lambda_G \to C^*_{\text{SAF}} G, \quad (24)$$

where $C^*_{\text{SAF}}$ is the category of $AF C^*$-algebras of stationary type (see Section 4.3).

Proof. The statement follows our results in ([16]). The proof is based on combinatorics of $AF C^*$-algebras (Bratteli diagrams) and consists in the identification of infinite paths of Bratteli diagrams with leaves of minimal lamination $\Lambda_G$. The main technical idea comes from symbolic dynamics and uses the Koebe-Morse coding of geodesic lines. □
4.3 K-theory of AF C*-algebras

This section is reserved for the basic facts of K-theory of the AF C*-algebras. Most references can be found in the monograph of Rordam, Lautsens ([]). Stationary K0-groups and their classification in terms of the subshifts of finite type are discussed by Effros in ([6]). Hyperbolic representation and parametrization of dimension groups by continued fractions can be found in ([16]).

Dimension groups. Let $A$ be a unital C*-algebra and $V(A)$ a matrix C*-algebra with entries in $A$. Projections $p, q \in V(A)$ are equivalent if there exists a partial isometry $u$ such that $p = u^*u$ and $q = uu^*$. The equivalence class of projection $p$ is denoted by $[p]$. Equivalence classes of orthogonal projections can be made to a semigroup by putting $[p] + [q] = [p + q]$. The Grothendieck completion of this semigroup to an abelian group is called a $K_0$-group of algebra $A$. Functor $A \to K_0(A)$ maps a category of unital C*-algebras into the category of abelian groups so that the positive elements $A^+ \subset A$ correspond to a “positive cone” $K^+_0 \subset K_0(A)$ and the unit element $1 \in A$ corresponds to an “order unit” $[1] \in K_0(A)$. The ordered abelian group $(K_0, K^+_0, [1])$ with an order unit is called a dimension (Elliott) group of C*-algebra $A$.

Representation of dimension groups by geodesic lines. For $k = 1, 2, \ldots$, let us consider dimension groups of rank $2k$, i.e. dimension groups based on the lattice $\mathbb{Z}^{2k}$. Such dimension groups can be represented (and classified) by geodesic lines on the Riemann surface of genus $k$. Let us show first that any simple (with no self-intersection) non-periodic geodesic gives rise to a dimension group of rank $2k$. Fix a Riemann surface $S$ of genus $k$ together with a point $p \in S$ and simple geodesic $\gamma$ through $p$. Consider the set

$$Sp(\gamma) = \{\gamma_0, \gamma_1, \gamma_2, \ldots\},$$

(25)

of periodic geodesics $\gamma_i$ based in $p$, which monotonically approximate $\gamma$ in terms of “length” and “direction”. The set $Sp(\gamma)$ is known as spectrum of $\gamma$ and is defined uniquely upon $\gamma$. Let $H_1(S; \mathbb{Z}) = \mathbb{Z}^{2k}$ be the integral homology group of surface $S$. Since each $\gamma_i$ is a 1-cycle, there is an injective map $f : Sp(\gamma) \to H_1(S; \mathbb{Z})$, which relates every closed geodesic its homology class. Note that $f(\gamma_i) = p_i \in \mathbb{Z}^{2k}$ is “prime” in the sense that it is not an integer multiple of some other point of lattice $\mathbb{Z}^{2k}$. Denote by $Sp_f(\gamma)$ the image of $Sp(\gamma)$ under mapping $f$. Finally, let $SL(2k, \mathbb{Z})$ be the group of $2k \times 2k$ integral matrices of determinant 1 and $SL(2k, \mathbb{Z}^+)$ its semigroup consisting of matrices with strictly positive entries. It is not hard to show, that in appropriate basis in $H_1(S; \mathbb{Z})$ the following is true: (i) the coordinates of vectors $p_i$ are non-negative; (ii) there exists a matrix $A_i \in SL(2k, \mathbb{Z}^+)$ such that $p_i = A_i(p_{i-1})$ for any pair of vectors $p_{i-1}, p_i$ in $Sp_f(\gamma)$. The dimension group $(\mathbb{Z}^{2k}, (\mathbb{Z}^{2k})^+, [1])$ defined as inductive limit of simplicial dimension groups:

$$\mathbb{Z}^{2k} \xrightarrow{A_1} \mathbb{Z}^{2k} \xrightarrow{A_2} \mathbb{Z}^{2k} \xrightarrow{A_3} \ldots,$$

(26)
is called associated to geodesic \( \gamma \). In fact, every dimension group of rank \( 2k \) can be obtained in such a way (we omit the proof of this fact here).

Parametrization of dimension groups by continued fractions. It is known that dimension groups of rank 2 are parametrized by the infinite continued fraction converging to a “slope” of non-closed geodesic on 2-torus, which characterizes such groups, see e.g. ([6]). Similar description is available for the dimension groups of rank \( k \geq 2 \). Let \((S, \gamma)\) be the pair consisting of the Riemann surface \( S \) and non-periodic geodesic \( \gamma \) which represents dimension group \((\mathbb{Z}^{2k},(\mathbb{Z}^{2k})+, [1])\) as described in previous paragraph. Intuitively, every homotopy class of non-periodic simple lines has a “slope” on \( S \) which is given by a real irrational number \( \theta \). (Existence of such a number was suggested by André Weil in 1933.) This number was defined in ([16]). We assume \( G = \pi_1(S) \) is the principal congruence subgroup \( \Gamma(n) \) of group \( PSL_2(\mathbb{Z}) \).

Recall that the geodesic lines on surface \( S = \mathbb{H}/G \) fall into (i) quasi-ergodic, (ii) periodic and (iii) quasi-periodic classes (Artin, Hedlund, Hopf). Component (i) consists of geodesics (with self-intersections) which approximate any geodesic on \( S \) issued from any point in any direction. Such geodesics are “typical” in the sense of measure at the boundary \( \partial \mathbb{H} \). To the contrary, geodesics of class (ii) and (iii) aren’t quasi-ergodic and can be thought of as having specific “direction” on \( S \). Non-quasi-ergodic geodesics have measure zero and cardinality of continuum at \( \partial \mathbb{H} \). Class (ii) is countable and class (iii) is an uncountable set of simple (without self-intersections) geodesics and they are a substitute of global straight lines on \( S \). Class (iii) will be basic for our definition of “slope” since it consists of simple self-approximating geodesics with definite direction on \( S \).

Let \( \Delta \) be the fundamental domain of \( G \). Then every geodesic \( \gamma \) of class (iii) intersects \( \Delta \) infinitely often, and let us enumerate the corresponding arcs on intersection by \( \{[\gamma_0], [\gamma_1], \ldots \} \) in the order they cut \( \Delta \). Since each \([\gamma_i]\) is the arc of the geodesic half-circle on \( \mathbb{H} \), they can be moved to each other by transformations from \( G \). Put then \( g_i \in G \) be such that \([\gamma_i] = g_i([\gamma_{i-1}]) \). It can be shown that \( g_i \in \Gamma^+(n) \), where \( \Gamma^+(n) \) is the principal congruence semi-group consisting of \( 2 \times 2 \) matrices with positive integer entries. Finally, suppose that 
\[
g_i = \left( \begin{array}{cc} a_{0}^{(i)} & 0 \\ 1 & 1 \end{array} \right) \ldots \left( \begin{array}{cc} a_{n}^{(i)} & 1 \\ 1 & 0 \end{array} \right)
\]
is the Minkowski decomposition of matrix \( g_i \in G \). Then regular continued fraction
\[
\theta = a_0^{(0)} + \cfrac{1}{a_1^{(0)} + \cfrac{1}{a_2^{(0)} + \cdots}}, \tag{27}
\]
is called associated to the dimension group defined by pair \((G, \gamma)\). By analogy with the case of noncommutative torus, real number \( \theta \) is called a rotation number. Two dimension groups are order-isomorphic if and only if their continued fractions coincide, except possibly in the finite number of terms.
In other words, the corresponding rotation numbers are modular equivalent:

$$\theta' = \frac{a\theta + b}{c\theta + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = \pm 1.$$  

Stationary dimension groups and their classification. A dimension group $(\mathbb{Z}^k, (\mathbb{Z}^k)^+, [1])$ is called stationary if in formula (26) $A_i = Const = A$. If continued fraction (27) is periodic, then the corresponding dimension group is stationary. Indeed, in this case there exists a transformation $g \in G$ such that $g_i \ldots g_{i+k} = Const = g$, where $k$ is the minimal period of continued fraction. It isn’t hard to see that $g$ is a hyperbolic transformation with two fixed points at the boundary $\partial \mathcal{H}$. The geodesic $\gamma$ through these points is invariant under the action of $g$ and coincides with the geodesic corresponding to the stationary dimension group generated by the automorphism $A$. Conversely, each stationary dimension group gives rise to a geodesic, whose slope (rotation number) unfolds into a periodic continued fraction.

Since periodic continued fractions converge to quadratic irrationals, one can classify stationary dimension groups using the arithmetic of the real quadratic number fields. The corresponding classification scheme was suggested in Section 1.

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