Effects of chiral restoration on the behaviour of the Polyakov loop at strong coupling

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We discuss the relation between the Polyakov loop and the chiral order parameter at finite temperature. For that purpose we analyse an effective model proposed by Gocksch and Ogilvie, which is constructed by the double expansion of strong coupling and large dimensionality. We improve the model and then obtain plausible results for the behaviours of both the Polyakov loop and the chiral scalar condensate. The pseudo-critical temperature read from the Polyakov loop turns out to coincide exactly with that read from the chiral scalar condensate. Within the model study based on the strong coupling expansion, the coincidence of the pseudo-critical temperatures results from the fact that the jump of the Polyakov loop in the presence of dynamical quarks should signify the chiral phase transition rather than the deconfinement transition.

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Introduction

Quantum Chromodynamics (QCD) is commonly accepted as the fundamental theory of the strong interaction. It has been intensely argued how a thermodynamic system described by QCD goes through the phase transitions at sufficiently high temperature, namely, the colour deconfinement transition and the chiral phase transition around the critical temperature $\sim 150$ MeV [1].

The analytic study of the deconfinement transition was first offered within the framework of the strong coupling expansion on the lattice [2]. In the absence of dynamical quarks a pure gluonic system has the centre symmetry and the Polyakov loop provides a criterion of confinement [3]. For the purpose of looking into the spontaneous breaking of the centre symmetry, the effective action in terms of the Polyakov loop variables has been constructed by the strong coupling expansion, as well as by the perturbative calculation. It is found that the resultant effective action leads to a second-order phase transition [4] for the SU(2) gauge theory and a first-order phase transition [5] for the SU($N \geq 3$) gauge theories. These are consistent with the anticipation from the point of view of the universality classifications [3] as well as of the lattice observations [6]. Also from the analysis of the effective action it has been shown that the perturbative vacuum of the pure gluonic system becomes unstable beyond some critical coupling strength [7].

When dynamical quarks in the fundamental representation are present in a theory, the centre symmetry is broken explicitly and the Polyakov loop is no longer regarded as an order parameter to characterise the deconfinement transition [6]. Then, the chiral symmetry associated with light quarks plays an important role in hadronic properties. There are many effective approaches based on the chiral symmetry, such as the linear-sigma model, the Nambu–Jona-Lasinio model, the chiral random matrix model and so on [1]. In those model studies, features of colour confinement are hardly taken into account. The relation between the confinement and the chiral symmetry breaking has been discussed theoretically in a variety of contexts such as the helicity conservation [9], the anomaly matching condition [10] and so on [11]. The difficulty in studying their interrelation lies in the fact that no proper criterion of confinement is established so far, despite of great efforts [12].

In the meanwhile, it has been found in the lattice simulations that the Polyakov loop behaves like an approximate order parameter even in the presence of dynamical quarks. Then the jumps of the Polyakov loop and the chiral condensate are observed at the same critical point [13]. Precise analyses thereafter in which the pseudo-critical points are defined by respective susceptibilities have sustained this coincidence [14]. It is often said that the deconfinement and the chiral restoration should take place simultaneously, though the physical reason for the coincidence is still obscure.

The author of Ref. [15] proposed the following simple explanation for the simultaneous jumps; at low temperature where the chiral symmetry is spontaneously broken, the explicit breaking of the centre symmetry is suppressed by the heavier mass of constituent quarks rather than the lighter mass of current quarks. Consequently the expectation value of the Polyakov loop stays small in the confined or chiral broken phase at low temperature. Once the constituent quark mass drops off in the chiral symmetric phase at high temperature, the expectation value of the Polyakov loop no longer receives such suppression. It follows that the jump of the Polyakov loop signifies not the deconfinement transition but the chiral phase transition. The simultaneous jumps are observed simply because the behaviours of both the Polyakov loop and the chiral condensate indicate a single phase transition of the chiral restoration.

This scenario had been partially confirmed within an effective model constructed by the double expansion of strong coupling and large dimensionality on the lattice [16]. In Ref. [16] Gocksch and Ogilvie found that the given model

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leads to the deconfinement transition at $T_\text{d} = 175$ MeV for the pure gluonic system and the chiral restoration at $T_\chi = 471$ MeV. The authors also argued that the constituent quark mass tends to suppress the magnitude of the Polyakov loop, though the simultaneous jumps could not be reproduced there. Actually in Gocksch and Ogilvie’s results, the transition temperatures relevant to the deconfinement and the chiral restoration are too different to affect each others.

The purpose of this letter is to show that the jump of the Polyakov loop should be attributed to the chiral phase transition rather than the deconfinement transition. To that end, we reexamine the model given by Ref. [16], which we call the Gocksch-Ogilvie model hereafter. We make improvements in phenomenological approximations employed in the original work and then find the deconfinement transition at $T_\text{d} = 208$ MeV for the pure gluonic system and the chiral phase transition at $T_\chi = 162$ MeV in the absence of the Polyakov loop dynamics. As compared with the empirical values [1], these transition temperatures are more reasonable than those originally derived in Ref. [16]. Furthermore, we analyse the coupled dynamics of the chiral order parameter and the Polyakov loop in a numerical way and demonstrate that the pseudo-critical temperature in regard to the Polyakov loop coincides with the pseudo-critical temperature of the chiral restoration at $T_\text{c} = 187$ MeV.

**Gocksch-Ogilvie model** The effective action of the Gocksch-Ogilvie model is constructed by the double expansion of strong coupling, where the confining property is almost trivial, and large dimensionality, where the dynamical breaking of the chiral symmetry. Since the model is based on the staggered formalism on the lattice, the flavour contents of meson fields are resolved in the differing lattice sites. $N_c$ is the number of colours. $L(\vec{n})$ denotes the Polyakov loop defined in the $d$-dimensional space-time by

$$L(\vec{n}) = \prod_{n_4=1}^{N_c} U_d(\vec{n}, n_4)$$

with $N_c$ lattice sites in the thermal (temporal) direction. The Polyakov loop is a matrix in the colour-space and $\text{Tr}_c$ represents the trace over the colour indices. The static energy $E$ of quasi-quarks is given by

$$E = \sinh^{-1}\left(\sqrt{\frac{d-1}{2} \lambda + m}\right)$$

with the current quark mass denoted by $m$. In the above expression $(d - 1)$ is the number of spatial dimensions. The propagator $V(n, m)$ of meson fields is given by

$$V(n, m) = \frac{1}{2(d-1)} \sum_j \left( \delta_{n,m+j} + \delta_{n,m-j} \right),$$

where $\hat{j}$ runs over only the spatial directions. The strength of the nearest neighbour interaction, $J$, is determined as a function of the temperature, in other words, the temperature is specified by $J$: In the strong coupling limit $(J \ll 1)$ we can express the correlation function of the Polyakov loops in twofold ways as

$$\langle \text{Tr}_c L(\vec{n}) \text{Tr}_c L(\vec{m}) \rangle \sim J_{\vec{m}-\vec{n}} \sim e^{-\beta \sigma a |\vec{m}-\vec{n}|},$$

where $\sigma$ is the string tension and $a$ is the lattice spacing. Accordingly the temperature in the Gocksch-Ogilvie model is fixed as

$$T = \frac{1}{\beta} = \frac{\sigma a}{-\ln J}. $$

It should be emphasised that the coefficient $1/2$ in front of the last part of Eq. [4] was missing in the original work by Gocksch and Ogilvie. This factor is important in counting the number of flavours from the phenomenological...
point of view. But for the coefficient $1/2$, the scalar field $\lambda(n)$ generates too many mesons with four degenerate light flavours peculiar to the staggered formalism. This coefficient effectively reduces the model to that with only two $(u$ and $d)$ light quarks, as often adopted in lattice simulations.

The prescription of taking the square root of the Dirac determinant is reliable only in the weak coupling limit where the Dirac operator is well localised. In the present case, the prescription is also reliable, though the model is based on the strong coupling expansion. The counterpart of the Dirac operator is absolutely local because the last part of Eq. (3) comes from the integration with respect to the quark field without its kinetic term which is absorbed in that of mesons.

With the factor $1/2$, the formulae at zero temperature given in Ref. [17] should receive modifications. Certainly the expectation value of $\lambda$ at zero temperature becomes $\sqrt{1/2}$ from 1, but all the physical quantities such as the hadron masses and the chiral condensate are not affected if just the current quark mass is divided by $\sqrt{1/2}$. Actually there are two parameters inherent in the model, namely, the lattice spacing $a$ and the current quark mass $m$. According to the formulae given in Ref. [13] with modifications from the factor $1/2$, the model parameters can be fixed as follows so as to reproduce the pion mass $140$ MeV and the $\rho$ meson mass $770$ MeV:

$$a^{-1} = 432 \text{ MeV}, \quad m = 5.7 \text{ MeV}. \quad (7)$$

We would emphasise the following point: In the zero temperature analysis like Ref. [17], too many light flavours would cause no serious problem as long as the mass spectrum in only the $u$ and $d$ quark sector is concerned. In the chiral limit ($m = 0$) the mean-field approximation leads to the same answer for each quark flavour regardless of the flavour number. In the presence of small current quark mass, the difference of the flavour number can be absorbed in the difference of the current quark mass. At finite temperature, on the other hand, how many degrees of freedom are excited at that temperature should affect significantly the relevant temperature to the chiral restoration. In this case, since the number of pions (Nambu-Goldstone bosons) is too large, the resultant potential energy prevents the chiral restoration until considerably high temperature is reached.\(^1\) Thus the factor $1/2$ is important in order to estimate the chiral transition temperature quantitatively.

**Mean-field approximation** Following the treatment of Ref. [16], we adopt the mean-field approximation to deal with the effective action [11]. As to the meson field, we simply replace it by a condensate $\tilde{\lambda}$. The Polyakov loop $L(\tilde{n})$ is integrated out with the mean-field action [15],

$$S_{mf}[L] = -\frac{x}{2} \sum_n \{ \text{Tr}_c(L(\tilde{n}) + \text{Tr}_c L^\dagger(\tilde{n}) \}, \quad (8)$$

where $x$ is the variational parameter to be determined afterwards from the extremal condition on the free energy. The mean-field free energy is then given by

$$\beta f_{mf}(x, \tilde{\lambda}) = -N^{-(d-1)} \left( \langle -S_{eff}[L, \tilde{\lambda}] + S_{mf}[L] \rangle_{mf} + \ln \int \mathcal{D}L e^{-S_{mf}[L]} \right)$$

$$= \beta f_{mf}^S(x) + \beta f_{mf}^g(x, \tilde{\lambda}), \quad (9)$$

where $\langle \cdots \rangle_{mf}$ stands for the expectation value calculated by means of the mean-field action [15]. In the expression of the mean-field free energy, the physical implication of each term is plainly understood as follows: In the first line, the first term is the internal energy tending to make the system ordered and the remaining terms can be regarded as due to the entropy tending to make the system disordered. $f_{mf}^S(x)$ represents the pure gluonic part of the mean-field free energy and $f_{mf}^g(x, \tilde{\lambda})$ corresponds to the chiral part with the coupling between the Polyakov loop and the scalar condensate. They are explicitly written as

$$\beta f_{mf}^S(x) = -2(d - 1)J \left( \frac{d}{dx} \ln I(x) \right)^2 + x^2 \frac{d}{dx} \left( \frac{1}{\sqrt{x}} \ln I(x) \right), \quad (10)$$

$$\beta f_{mf}^g(x, \tilde{\lambda}) = \frac{N_c N_f}{2} \tilde{\lambda}^2 - \frac{N_f}{2} \ln \left[ \cosh(N_f E) \right] - \frac{1}{2} \frac{\tilde{\lambda}^2}{I(x)} \ln [\cosh(N_f E)] \quad (11)$$

\(^1\) This might seem contradict to the fact that the chiral transition temperature is lowered as the number of flavours increases. In the treatment of the Gocksch-Ogilvie model, however, we exploit the mean-field approximation and any fluctuation of pions, which is responsible for reducing the transition temperature, is frozen from the beginning. Thus the potential energy alone is involved in the analysis and it gives higher transition temperature with more flavours.
where $I(x)$ is defined by

$$I(x) = \int dL \exp \left[ \frac{x}{2} \text{Tr}_c(L + L^\dagger) \right] = N_c! \sum_m \det I_{m-i+j}(x),$$

(12)

and $\hat{I}(x; \alpha)$ is given by

$$\hat{I}(x; \alpha) = \sum_{a=1}^{N_c} \sum_{m=-\infty}^{\infty} \epsilon_{i_1i_2...i_{N_c}} \epsilon_{j_1j_2...j_{N_c}} I_{m-i_1+j_1}(x) ... I_{m-i_a+j_a}(x; \alpha) ... I_{m-i_{N_c}+j_{N_c}}(x)$$

$$\hat{I}_n(x; \alpha) = \int_0^{2\pi} \frac{d\phi}{2\pi} \ln \left[ 1 + \frac{\cos \phi}{\alpha} \right] e^{x \cos \phi + \ln \phi}.$$  

(13)

Here $I_n(x)$ denotes the modified Bessel function of the first kind. Once we expand $\hat{I}_n(x; \alpha)$ in terms of $1/\alpha$ and furthermore expand the above expressions in terms of $x$, we can immediately reproduce the results of Ref. [16]. These expansions in terms of $1/\alpha$ and $x$ are not always regarded as reliable in fact. Although the expansion in terms of $x$ works well for the estimate of $I(x)$, it is not a good approximation for $f^\beta_{mf}(x)$ due to significant cancellations in the right-hand-side of Eq. (11). The expansion in terms of $1/\alpha$ becomes uncertain when the expectation value of the Polyakov loop is large. Therefore we do not adopt such expansions in examining the mean-field free energies in the present work.

The deconfinement transition in the pure gluonic system is described by $f^\beta_{mf}(x)$, while $f^\beta_{mf}(x, \bar{\lambda})$ determines the fate of the chiral symmetry in the presence of the Polyakov loop. Actually $f^\beta_{mf}(x)$ results in the first-order phase transition at $2(d-1)\kappa_c = 0.806$ in the case of $N_c = 3$. The corresponding temperature read from Eq. (12) is $T_d = 208$ MeV with the numerical value of the string tension $\sigma = (425$ MeV$)^2$ substituted.

Then, we will look into the chiral dynamics somewhat closely in an analytic way in the case of the chiral limit ($m = 0$). When the phase transition is second-order, that is the case in the present analysis, the transition temperature can be deduced from the condition that the quadratic term with respect to the order parameter changes its sign, or passes across zero. Thus we can derive the analytic expression for $T_\chi$ from the expansion of $f^\beta_{mf}(x, \bar{\lambda})$ in terms of $\bar{\lambda}$.

The calculation is simplified when the Polyakov loop is put as trivial, i.e., $L = 1$ for the moment. Then we can approximate the last term of Eq. (12) by $-(N_c/2) \ln[1 + 1/\cosh(N_r E)]$ looking back at Eq. (12), which leads to

$$\beta f^\beta_{mf}(\bar{\lambda}) \approx \frac{N_c N_r}{2} \bar{\lambda}^2 - \frac{(d-1)N_c N_r^2}{16} \bar{\lambda}^2 + \text{(const.)}.$$  

(14)

From this expansion, the chiral phase transition temperature in the absence of the Polyakov loop dynamics is immediately determined as

$$T_\chi = \frac{d-1}{8} a^{-1} = 162 \text{ MeV}$$

(15)

with the numerical value of $a$ given by Eq. (11). The above expression is essentially the same as (3.23) in Ref. [11].

Another interesting limit in which the calculation is simplified is that the Polyakov loop is forced to vanish, i.e., $x = 0$. Then it follows that

$$\left\langle \text{Tr}_c \ln \left[ 1 + \frac{1}{2 \cosh(N_r E)} (L + L^\dagger) \right] \right\rangle_{mf} \approx -N_c \ln 2 + \sqrt{d-1}N_c N_r |\bar{\lambda}| - \frac{(d-1)N_c N_r^2}{4} \bar{\lambda}^2.$$  

(16)

The quadratic term proportional to $\bar{\lambda}^2$ is exactly cancelled by the contribution from $N_c \ln[\cosh(N_r E)] \approx (d-1)N_c N_r^2 \bar{\lambda}^2/4$ and only the tree-term $N_c N_r \bar{\lambda}^2/2$ remains. What is surprising is that the linear term of $\bar{\lambda}$ appears in Eq. (14). Owing to the presence of this linear term, the stationary point with respect to $\lambda$ always leaves from zero. In other words, the chiral symmetry is broken at any temperature. If the vanishing Polyakov loop, $x = 0$, has something to do with confinement even in the presence of dynamical quarks, this result suggests that the chiral symmetry must be broken in the confined vacuum, which is in agreement with the arguments in Refs. [11, 12].

Hence, the chiral dynamics is substantially affected by the Polyakov loop dynamics. It means at the same time that the behaviour of the Polyakov loop is deeply related to that of the chiral order parameter.

**Numerical results** We must search for the global minimum of the free energy $f_{mf}(x, \bar{\lambda})$ numerically in general, except for the above special cases of $m = 0$, $L \simeq 1$ and $m = 0$, $x = 0$ where analytic treatments are feasible. The
variational parameter, $x$, and the scalar condensate, $\bar{\lambda}$, are determined as functions of temperature. Then we can readily acquire the expectation value of the Polyakov loop by using (see Eq. (12))

$$\left\langle \frac{1}{2N_c} (\text{Tr} e^L + \text{Tr} e^{-L'}) \right\rangle = \frac{1}{N_c} \frac{d}{dx} I(x).$$

(17)

The results are shown in Fig. 1. It is apparent that the behaviours like phase transitions are observed around the temperature $T_c \approx 180$ MeV in Fig. 1. As compared with the pure gluonic result $T_d = 208$ MeV (shown by the dotted curve for reference) it seems that the critical point concerning the Polyakov loop dynamics is smeared by dynamical quarks and, as a result, two crossovers in terms of the Polyakov loop and the scalar condensate are observed nearly at the same point.

To make the argument more quantitative, we can define the pseudo-critical temperatures by means of the peak of susceptibilities in a similar way to Ref. [13]. Here we make use of the simplest ones, which are immediately feasible in the present analysis, that is, the temperature susceptibilities $\chi^L = \partial \langle \text{Tr} e^L \rangle / \partial T/N_c$ and $\chi^\lambda = -\partial \bar{\lambda} / \partial T$. The results are shown in Fig. 2. The pseudo-critical temperatures are found exactly at the same point $T_c = 187$ MeV. As to the moderate peak of $\chi^\lambda$ slightly above $\sim 200$ MeV, it should be regarded as an accidental artifact because this peak would vanish if the current quark mass $m$ is raised or lowered by hand.

The coincidence of the pseudo-critical temperatures completely agrees with the lattice observations. Within the Gocksch-Ogilvie model, the jump of the Polyakov loop does not mean the deconfinement transition. Actually it is caused by the last term of Eq. (1) through which the behaviour of the Polyakov loop would reflect that of the scalar condensate. In analogy with the hopping parameter expansion (see Eq. (2.11) in Ref. [20]) we can regard $1/\cosh(N,E)$ as the strength of an external field to break the centre symmetry. The behaviour of the Polyakov loop is governed effectively by the chiral dynamics through this strength of an external field.

We consider that the exact coincidence of the pseudo-critical temperatures in the present study provides a credible support for the argument of Ref. [13]: The behaviour of the Polyakov loop indicates only the chiral phase transition rather than the deconfinement transition. Thus the coincidence of the critical temperatures is only a consequence of viewing a single phenomenon, i.e., the chiral restoration.

Finally we shall comment upon the plans for further work. Since the number of the space-time dimensions is four at most, it is necessary to make sure the convergence of the large dimensional expansion and to confirm the quantitative success in describing the Polyakov loop and the chiral dynamics even with next-to-leading order contributions taken into account.

It would be intriguing to construct a similar effective model in the formulation of the Wilson fermion [21]. Then the flavour decomposition becomes straightforward at the cost of the intricate structure of the Dirac indices. Also an extension to introduce adjoint fermions, which does not break the centre symmetry, would be interesting. In contrast to the case of fundamental fermions, two distinct transitions are found in the lattice simulations [22]. It would be challenging to reproduce such results within the model study parallel to the present treatment. Such analyses would serve as a double-check for the speculation on the behaviour of the Polyakov loop discussed in this letter.

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FIG. 1: The behaviours of the order parameters as functions of the temperature. Transition-like jumps are observed simultaneously around $T_c \simeq 180$ MeV. The dotted curve is the result in the pure gluonic case for reference.

FIG. 2: The temperature susceptibilities for the Polyakov loop and the chiral scalar condensate.
Figure 1

Order Parameters

$\lambda a$

$\langle \text{Tr}_c L \rangle / N_c$

pure gluonic case

Temperature [MeV]

100 200 300

0
Figure 2

Temperature [MeV]

Susceptibilities

$\chi^L_t$

$\chi^\lambda_t$