Gravitational Radius in view of Existence and Uniqueness Theorem

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Abstract. Talking about a black hole, one has in mind the process of unlimited self-compression of gravitating matter with a mass greater than critical. With a mass greater than the critical one, the elasticity of neutron matter cannot withstand gravitational compression. However, compression cannot be unlimited, because with increasing pressure, neutrons turn into some other "more elementary" particles. These can be bosons of the Standard Model of elementary particles. The wave function of the condensate of neutral bosons at zero temperature is a scalar field. If instead of the constraint $\det g_{ik} < 0$ we use a weaker condition of regularity (all invariants of the metric tensor $g_{ik}$ are finite), then there is a regular static spherically symmetric solution to Klein-Gordon and Einstein equations, claiming to describe the state to which the gravitational collapse leads. With no restriction on total mass. In this solution, the metric component $g^{rr}$ changes its sign twice: $g^{rr}(r) = 0$ at $r = r_g$ and $r = r_h > r_g$. Between these two gravitational radii the signature of the metric tensor $g_{ik}$ is $(+, +, -, -)$. Gravitational radius $r_g$ inside the gravitating body ensures regularity in the center. Within the framework of the phenomenological model $\lambda \psi^4$, relying on the existence and uniqueness theorem, the main properties of a collapsed black hole are determined. At $r = r_g$ a regular solution to Klein-Gordon and Einstein equations exists, but it is not a unique one. Gravitational radius $r_g$ is the branch point at which, among all possible continuous solutions, we have to choose a proper one, corresponding to the problem under consideration. We are interested in solutions that correspond to a finite mass of a black hole. It turns out that the density value of bosons is constant at $r < r_g$. It depends only on the elasticity of a condensate, and does not depend on the total mass. The energy-momentum tensor at $r \leq r_g$ corresponds to the ultra relativistic equation of state $p = \epsilon/3$. In addition to the discrete spectrum of static solutions with a mass less than the critical one (where $g^{rr} < 0$ does not change sign), there is a continuous spectrum of equilibrium states with $g^{rr}(r)$ changing sign twice, and with no restriction on mass. Among the states of continuous spectrum, the maximum possible density of bosons depends on the mass of the condensate and on the rest mass of bosons. The rest energy of massive Standard Model bosons is about 100 GeV. In this case, for the black hole in the center of our Milky Way galaxy, the maximum possible density of particles should not exceed $3 \times 10^{81} \text{cm}^{-3}$.

Spherically symmetric static solutions to the system of Klein-Gordon and Einstein equations, claiming to describe the state of matter which the gravitational collapse leads to, were found and presented in articles [1] and [2]. Getting new solutions became possible when I applied a weaker condition of regularity (all invariants of $g_{ik}$ are finite) instead of the restriction $\det g_{ik} < 0$. In these regular static solutions, the metric component $g^{rr}$ changes its sign twice: $g^{rr}(r) = 0$ at $r = r_g$ and at $r = r_h > r_g$. The
signature of the metric tensor $g_{ik}$ is $(+,+,−,−)$ within the spherical layer $r_{g} < r < r_{h}$ (between these two gravitational radii). The presence of the gravitational radius $r_{g}$ inside a gravitating body ensures regularity in the center $r = 0$. For a distant observer, the Schwarzschild's gravitational radius $r_{h}$ is the event horizon.

In the thirties of the last century, in the classical works [3],[4],[5] it was shown that there is a critical mass $M_{cr}$ of matter (of the order of the Sun mass $M_{⊙} = 2 \times 10^{33}$g), above which the Fermi elasticity of neutrons cannot withstand gravitational self-compression. Since then, there has been the opinion that objects with more than the critical mass are subject to unlimited self-compression (collapse). It would be so (that is, the collapse would be unlimited), if neutrons remained unchanged in the process of unlimited increase of pressure. In fact, mutual transformations of particles with increasing pressure can slow down and even stop the collapse.

With a further increasing pressure, a strongly compressed matter of dominating elementary particles of the Standard Model, see Figure 1, claims to be "the next stop" after neutrons.

![Figure 1. Standard Model of elementary particles][6]

Energetically, the most favorable state of matter is a condensate of massive bosons (gauge bosons $Z$ and $W$, Higgs' scalar boson $H$, and also bosonic quasiparticles formed by pairing of fermions due to the Cooper effect [7]).

At zero temperature, the wave function of a Bose-Einstein condensate is a classical scalar field $\psi$ in a curved space-time is

$$L = g_{ik} \psi_{i} \psi_{k} - U(\psi^* \psi), \quad U(0) = 0.$$  

With a very large number of particles, the gravitational interaction is predominant. In power series of the potential

$$U(|\psi|^2) = \left(\frac{m c}{\hbar}\right)^2 |\psi|^2 + \frac{1}{2} \lambda |\psi|^4 + \ldots,$$  

the first term is related to gravity of non-interacting particles ($m$ is the rest mass of a quantum). $\frac{1}{2} \lambda |\psi|^4$ and subsequent terms are corrections including interactions of non-gravitational nature. With no account of these corrections, the wave function $\psi(r)$ diverges logarithmically at $r \to 0$ [1]. The phenomenological model $\lambda \psi^4$, which takes into account the elasticity of a condensate, eliminates the divergence at the

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[6]: Figure 1. Standard Model of elementary particles [6].
center [2]. Earlier, in the article [9], it had been noted that the equilibrium configuration in the model "\( \lambda \psi^{*} \)" differs noticeably from the case of non-interacting bosons even when \( \lambda \ll 1 \).

Being scalars, \( \psi \) and \( \psi^{*} \) satisfy the Kline-Gordon equation:

\[
\frac{1}{\sqrt{\det g_{ik}}} \left( \sqrt{\det g_{ik}} g^{ik} \psi \right)_m = -\frac{\partial U}{\partial |\psi|^2} \psi.
\]

Kline-Gordon equation (2) is invariant against changing the sign of \( \det g_{ik} : \sqrt{-1} \) in numerator and denominator cancel out. A static spherically symmetric scalar field in the state of definite energy \( E \) per particle is:

\[
\psi_{\psi}(x^i) = e^{-i\omega t^0/hc} \psi(r).
\]

\( g^{rr}(r) \) is the coefficient at the highest derivative in the Kline-Gordon equation for the radial wave function \( \psi(r) \):

\[
g^{rr} \psi'' + \left( g^{rr} \right)' + \left( \frac{1}{2} \ln(\det g_{ik}) \right) g^{rr} \psi' = \left( \frac{1}{\hbar^2 c^2} (g^{00} E^2 - m^2 c^4) - \lambda |\psi|^2 \right) \psi.
\]

Equation (3) is not defined at the gravitational radii \( r = r_g \) and \( r = r_h \) where \( g^{rr}(r) = 0 \).

Kline-Gordon equation (3) together with Einstein equations (see [10], formulae (100.4) and (100.6))

\[
g^{rr} \left( 1 - (\ln g^{00}) \right) + \frac{1}{r} = \kappa r T^0_0,
\]

\[
g^{rr} \left( 1 - (\ln g^{00}) \right) + \frac{1}{r} = \kappa r T^r_r
\]

constitute a complete system of equations determining a static state of the gravitating Bose-Einstein condensate. Gravitational constant \( \kappa = (8\pi k)/c^4 = 2 \times 10^{-48} \text{sec}^2/\text{g} \times \text{cm} \). Components of the energy-momentum tensor \( T^0_0 \) and \( T^r_r \) in the Einstein equations (4), (5) are:

\[
T^0_0 = \frac{1}{\hbar^2 c^2} \left( g^{00} E^2 + m^2 c^4 \right) |\psi|^2 + \frac{1}{2} \lambda |\psi|^4 - g^{rr} |\psi|^2
\]

\[
T^r_r = \frac{1}{\hbar^2 c^2} \left( -g^{00} E^2 + m^2 c^4 \right) |\psi|^2 + \frac{1}{2} \lambda |\psi|^4 + g^{rr} |\psi|^2
\]

Energy density (6) and pressure (7) are regular in the center \( r = 0 \). It follows from Einstein equation (4) that

\[
g^{rr}(r) = -1 + 0(r^2), \quad r \to 0.
\]

Outside a gravitating body \( T^0_0 = 0 \) we have the Schwarzschild’s solution

\[
g^{rr}(r) = -1 + \frac{r_h}{r}, \quad r \geq r_h.
\]

If \( T^0_0 = 0 \) at \( r \geq r_h \), and \( T^0_0 \neq 0 \) at \( r < r_h \), then Schwarzschild’s gravitational radius \( r_h \) is the event horizon coinciding with the surface of a gravitating body. At \( r = r_h \) the derivative

\[
\frac{dg^{rr}(r_h)}{dr} = -\frac{1}{r_h}
\]

From equation (5) it follows that on the surface \( r = r_h \), where \( g^{rr}(r_h) = 0 \), the pressure does not vanish:

\[
T^r_r(r_h) = \frac{1}{\kappa r_h^2}
\]
\( \varphi_m^\mu(r) \) is a scalar, satisfying the same Klein-Gordon equation (2), though with a different rest mass of a quantum. We can say that the longitudinal vector field is a scalar field “turned inside out”. According to the theorem of existence and uniqueness, at the interface \( r = r_g \), where \( g''(r_g) = 0 \), a solution to equation (2) exists for both fields. However, a solution is not unique. This ambiguity makes it possible to ensure a balance of pressure at the interface of a black hole and dark matter at \( r = r_g \). The balance of pressure allowed determining the speed on the plateau of a galaxy rotation curve as a function of the observed mass of a black hole (see [1], formula (68)).

Currently, dark matter is observed due to gravitational interaction only. It follows from Einstein’s equation (5) that the pressure balance at the interface \( r = r_g \) depends on the observed black hole mass \( M = \frac{c^2}{2k} r_h \), \( k = 6.67 \times 10^8 \frac{cm^3}{g \times sec^2} \), see (11), regardless of the internal structure of a black hole. It was important to show that there can be a static spherically symmetric solution to Einstein equations with no restriction on mass. As an example, such a solution was found in [1] for a gravitating condensate non-interacting bosons. However, in case of an ideal Bose-gas (with no elasticity), the condensate wave function diverges logarithmically at the center.

Unfortunately, at the present state of quantum chromodynamics, we don’t have an equation of state for ultimately compressed elementary particles of the Standard Model derived from first principles. Current state of the problem, including an interpolation model named “unified equations of state”, is considered in the review [13]. The model “\( \lambda \psi^4 \)”, (1) takes into account interactions of a non-gravitational nature phenomenologically, including elasticity of the condensate. This model does not take into account that the lifetime of a Standard Model boson falls down with decreasing pressure. Taking into account the elasticity, a necessary condition for the existence of solutions with no restriction on total mass is [2]:

\[
\frac{1}{g^00(r)} \frac{dg^00(r)}{dr} < 0 \quad \text{at} \quad r = r_g. \tag{12}
\]

Under this condition, in the close vicinity of the upper boundary, numerical solutions with a finite mass practically do not differ from the regular ones at the center. The existence and uniqueness theorem prompts how to find numerical solutions in the whole range of parameters.

To get maximum advantage from the existence and uniqueness theorem, it is necessary to convert equations (3)-(5) to canonical form. In dimensionless variables:

\[ x = \frac{mc}{k} r, \quad u(x) = \sqrt{k} \varphi(r), \quad w(x) = \frac{hu}{mc} g''(r) \frac{d\varphi}{dr}, \quad g(x) = g''(r), \quad h(x) = \frac{k^2 g^00(r)}{m^2 c^4}, \tag{13}\]

we have the following system of four first-order equations, resolved with respect to derivatives [2]:

\[
\frac{du}{dx} = \frac{w}{g}, \tag{14}
\]

\[
\frac{dw}{dx} = \left( h - 1 - Au^2 \right) u - \left( \frac{2}{x} - \frac{x}{g} \right) \left( hu^2 - \frac{w^2}{g} \right) w, \tag{15}
\]

\[
\frac{dg}{dx} = x \left( (h + 1)u^2 + \frac{1}{2}Au^4 - \frac{w^2}{g} \right) - \frac{1+g}{x}, \tag{16}
\]

\[
\frac{dh}{dx} = \left( \frac{1}{x} \right) \left( 1 + \frac{1}{g} \right) \frac{x}{g} \left( (1-h)u^2 + \frac{w^2}{g} + \frac{1}{2}Au^4 \right) h. \tag{17}
\]

\( g(x) = g''(r) \) not to be confused with \( \det g \). In the model “\( \lambda \psi^4 \)”, dimensionless parameter

\[
\Lambda = \left( \frac{h}{mc} \right)^2 \frac{\lambda}{k}, \tag{18}
\]

characterizes the elasticity of a condensate. \( x_g = \frac{mc}{h} r_g \) is the dimensionless internal gravitational radius; \( x_h = \frac{mc}{h} r_h \) is the dimensionless event horizon.

Regularity in the center dictates the following boundary conditions:

\[ u(0) = u_0, \quad w(0) = 0, \quad g(0) = -1, \quad h(0) = 0. \tag{19}\]
Analytical solution to the set of equations (14)-(17) with boundary conditions (19) is
\[ u(x) = u_0, \quad w(x) = 0, \quad g(x) = -1 + \frac{1}{3} u_0^2 x^2, \quad h(x) = \frac{1}{3} \]
provided that there is a balance of density \( u_0^2 \) and elasticity \( \Lambda \):
\[ \Lambda u_0^2 = -2/3. \] (21)
In the model "\( \Lambda \psi^4 \)" density in the center \( u_0^2 \) and internal gravitational radius \( x_g \)
\( x_g^2 = \frac{3}{u_0^2} = 4.5|\Lambda| \) are determined by elasticity \( \Lambda \) of the condensate.

Analytical solution (20) exists in the whole space \( 0 \leq x < \infty \). But it is unique only inside the sphere
\( 0 \leq x < x_g \), where \( g(x) = 0 \). Right-hand sides of equations (14)-(17) are not continuous functions at
\( x = x_g \) where \( g(x) = 0 \). In accordance with the theorem of existence and uniqueness (see [14], §3), the
density \( u^2(x) = u_0^2 \) remains constant inside the central sphere \( 0 \leq x < x_g \), regardless of the total mass.
Unambiguously
\[ u(x_g) = u_0, \quad w(x_g) = 0, \quad g(x_g) = 0, \quad h(x_g) = 1/3. \] (22)
On the central plateau the components (6) and (7) of the energy-momentum tensor
\[ T_0^0 = \frac{m^2 c^2}{\kappa} u_0^2, \quad T_r^r = \frac{m^2 c^2}{3h^2 \kappa} u_0^2, \quad 0 \leq x \leq x_g \]
correspond to ultra relativistic equation of state \( p = \varepsilon/3 \). The dimensionless metric component
\( h(x) = 1/3 \) also remains constant at \( x \leq x_g \). As far as \( h'(x_g) = 0 \), the condition (12) must be specified:
\[ \frac{1}{g^{00}(r)} \frac{dg^{00}(r_g + \Delta)}{dr} < 0. \]
Gravitational radius \( x_g \) is the branch point at which, among all possible continuous extensions, we
should choose the proper one, corresponding to the problem under consideration. With boundary
conditions (22) we would have the same analytical solution (20) at \( x > x_g \), Figure 2, describing a
condensate which is uniformly spread throughout the whole space. Its total mass would be infinite.
In order to find numerically solutions with finite total mass, coinciding with (22) at \( x = x_g \), it is
reasonable to search for boundary conditions slightly beyond the gravitational radius \( x - x_g = \Delta \rightarrow 0 \):
\[ u(x_g + \Delta) = u_0 - \alpha \Delta^p, \quad g(x_g + \Delta) = \frac{(x_g + \Delta)^2}{x_g^2} - 1 - \gamma \Delta^{1+\delta}, \quad h(x_g + \Delta) = \frac{1}{3} - \beta \Delta^q. \] (23)
Since the solutions coincide at $x = x_g$, it is sufficient to leave only leading terms at $\Delta \to +0$. In order to find unknowns $\alpha, \beta, \gamma, p > 0, q > 0$, and $s > 0$, one has to submit (23) into equations (14)-(17). With accuracy of leading terms, the equations are satisfied if
\begin{equation}
\alpha(2p^2 - p)\Delta^{p-1} = \sqrt{3}\beta\Delta^q, 
\end{equation}
\begin{equation}
(1 + s)\gamma\Delta^s = \frac{3}{3} \beta\Delta^q, 
\end{equation}
\begin{equation}
(2q - 1)\beta\Delta^q = \frac{1}{3} \gamma\Delta^{1+s}. 
\end{equation}
Due to the arbitrariness of $\Delta$, it follows from (25) that $q = s$. Then relation (26) reduces to $(2q - 1)\beta\Delta^q = 0$ at $\Delta \to 0$. Hence, $q = 1/2$ (if $\beta \neq 0$), and $\gamma = 2\beta/x_g$. From (24) we have $p = q + 1 = 3/2$, and $\alpha = \beta/\sqrt{3}$. The constant $\beta$ remains a free parameter.

In the literature, the system of Einstein and Klein-Gordon equations is widely used to describe hypothetical bosonic stars being in equilibrium with their own gravitational field [9], [15]-[17]. In these and other similar articles, authors usually use the representation $g^{rr} = -e^{-\varphi}$. It imposes the restriction $g^{rr} < 0$.

In this case there is no radius $r_0$ where $g^{rr}(r_0) = 0$. The Schwarzschild gravitational radius $r_h$ appears only in the asymptotic $g^{rr}(r) = -1 + r_h/r$ which is applicable at $r \gg r_h$. In this case, $r_h$ is neither the event horizon, nor the boundary of a gravitating body.

Under the restriction (27) the solution to the eigenvalue problem for the gravitating Bose-Einstein condensate contains a discrete spectrum of eigenvalues $h_n(u_0, x = 0)$ and conjugate eigenfunctions $u_n(x)$ [18]. The continuous parameter $u_0 \equiv u(x = 0)$, on which the star’s mass $M$ depends, remains arbitrary. The total mass $M$ of the condensate in the ground state does not exceed the critical value $M_{\text{cr}}$. For bosons with a rest mass $m \sim 100 GeV/c^2$ the critical total mass $M_{\text{cr}}$ is about $10^{12}g$ (a million ton).

With no restriction (27) there are solutions with $g^{rr}$ changing its sign twice. Gravitational radius $r_h > r_g$ where $g^{rr}(r_h) = 0$ exists. $r = r_h$ is the surface of a black hole. $r_h$ is the event horizon for a remote observer. There is no restriction on total mass. The value $h(0) = 1/3$ in (20) is strictly fixed by the condition of regularity in the center. Instead of the discrete eigenvalues $h_n(u_0, 0)$, a continuous parameter $\beta > 0$ appears due to the theorem of existence and uniqueness.

With no restriction (27), there is a one-parametric continuum of regular solutions to the system of equations (14)-(17) with the same boundary conditions (22) at the gravitational radius $x = x_g$. Case $\beta = 0$ corresponds to the analytical solution (20) with infinite total mass $M$. At $\beta > 0$ we get numerically a continuous spectrum of regular solutions with a finite total mass $M$. Small deviations of $u(x)$ and $w(x)$ in ambiguous solutions are of the order $\Delta^{3/2}$ at $\Delta \to 0$. Deviations of $g(x)$ and $h(x)$ are proportional to $\Delta^{1/2}$ at $\Delta \to 0$:

\begin{equation}
u(x_g + \Delta) = u_0 - \beta \frac{\Delta^3}{3}, \quad w(x_g + \Delta) = -\beta u_0 \Delta^{3/2}, \quad g(x_g + \Delta) = \frac{2}{x_g} \Delta (1 - \beta \sqrt{\Delta}), \quad h(x_g + \Delta) = \frac{1}{3} - \beta \sqrt{\Delta} - 2\beta^2 \Delta. \end{equation}

With account of only leading deviations $(\sim \beta \sqrt{\Delta})$, boundary conditions at $x = x_g + \Delta$, $\Delta \to +0$ are:

\begin{equation} u(x_g + \Delta) = u_0, \quad w(x_g + \Delta) = 0, \quad g(x_g + \Delta) = \frac{2\Delta}{x_g} (1 - \beta \sqrt{\Delta}), \quad h(x_g + \Delta) = \frac{1}{3} - \beta \sqrt{\Delta}. \end{equation}

The smaller $\beta \sqrt{\Delta} \ll 1$, the more accurate numerical result. Certainly, within the capabilities of a computer.

No matter how small $\beta > 0$, we obtain a solution to equations (14) – (17) with boundary conditions (28) for a black hole with a finite total mass.

Luckily, I managed to find analytical solution (20). It simplified the numerical analysis of the structure and properties of an extremely compressed black hole. Currently, the main characteristic of a black hole is its mass $M$. Within the frames of the model “$\lambda \psi^4$”, varying $\beta$ within the interval $0 < \beta < \infty$, one
can determine the range of possible values of density \( u_0^2 \) in the center, and elasticity \( A \) of the ultimately compressed Bose-Einstein condensate at a fixed mass \( M \).

In order to cover the whole range (from the De Broglie wavelength of 100 GeV bosons to the size of a black hole) on a single graph, it is convenient to use a logarithmic scale along the \( x \) axis. And along the \( y \) axis also, because of a huge range of variation of the metric component \( g^{rr} \). Using a logarithmic scale along the \( y \) axis, I raised \( g(x) \) graph two steps up (I show \( \ln(g(x) + 2) \) instead of \( \ln g(x) \) in Figures 3a and 3b), because \( g(x) \) changes sign, see Figure 2.

Total mass \( M \) is the most reliable parameter of a black hole. In dimensional units, the black hole mass \( M \) is connected with the dimensionless event horizon radius \( x_h \) by

\[
M = \frac{c^2}{2k} r_h = \frac{1}{2} \frac{M_{pl}}{m} x_h.
\]

Here \( M_{pl} = \sqrt{\frac{\hbar c}{k}} = 2.177 \times 10^{-5} \text{g} \) is the Plank mass, and \( m \) is the mass of a boson. Having in mind bosons of the Standard Model with the rest energy about 100 GeV (see Figure 1), I assume a boson mass \( m \) to be \( 1.78 \times 10^{-22} \text{g} \) for quantitative estimates. Mass of the black hole in the center of our Milky Way galaxy [19] is

\[
M_{MW} = 8.6 \times 10^{39}\text{g} \tag{30}
\]

The Milky Way black hole’s event horizon \( r_h = 1.6 \times 10^{12}\text{cm} \). In dimensionless units \( x_h = \frac{mc}{\hbar} r_h = 6.46 \times 10^{27} \), \( \ln x_h = 64 \). Metric function \( g(x) \) and dimensionless density \( u^2(x) \), corresponding to the Milky Way black hole mass \( M_{MW} \) (30), are presented in Figure 3a for a very small \( \beta = 0.000045 \), and in Figure 3b for a large \( \beta = 15.5 \). For \( \beta = 0.000045 \) the mass (30) corresponds to dimensionless density in the center \( u_0^2 = 22 \). Gravitational radius \( x_g = 0.369 \), \( \ln x_g = -0.996 \). For \( \beta = 15.5 \) the density \( u_0^2 = 36 \), \( x_g = 0.289 \), \( \ln x_g = -1.242 \). Naturally, comparing two cases of different \( \beta \) for the same black hole with mass (30), the event horizon radius is the same: \( x_h = 6.46 \times 10^{27} \), \( \ln x_h = 64 \).

![Figure 3a. \( \beta = 0.000045 \). Dimensionless density in the interval \( 0 \leq x \leq x_g \) is \( u_0^2 = 22 \), Gravitational radius \( x_g = 0.369 \), \( \ln x_g = -0.996 \).](image1)

![Figure 3b. \( \beta = 15.5 \). Dimensionless density in the interval \( 0 \leq x \leq x_g \) is \( u_0^2 = 36 \), Gravitational radius \( x_g = 0.289 \), \( \ln x_g = -1.242 \).](image2)

Red lines are \( g(x) + 2 \) in double logarithmic scales. Dashed horizontal lines show the level \( \ln 2 \) where \( g(x) \) is zero. Oblique dashed lines correspond to \( g(x) \) in Figure 2.

At a fixed mass, the density \( u_0^2 \) in the center slightly increases with growing \( \beta \). Functions \( u(x)/u_0 \) and \( h(x) \) at the fixed mass \( M_{MW} = 8.6 \times 10^{39}\text{g} \) \( (x_h = 6.46 \times 10^{27}, \ln x_h = 64) \) are shown in Figure 4a for \( \beta = 0.000045 \) and in Figure 4b for \( \beta = 15.5 \).
Figure 4a. $\beta = 0.000045$. $u_0^2 = 22$. Gravitational radius $x_g = 0.369$, $\ln x_g = -0.996$.

Figure 4b. $\beta = 15.5$, $u_0^2 = 36$. Gravitational radius $x_g = 0.289$, $\ln x_g = -1.242$.

Blue curve $u(x)/u_0$ weakly depends on $\beta$. Red curve $h(x)$ turns to a step at large $\beta$.

In the center $x \leq x_g$ density $u^2(x) = u_0^2$ and metric component $h(x) = 1/3$ remain constant (plateau at $0 \leq x < x_g$) regardless of $\beta$. This is a consequence of the theorem of existence and uniqueness. At $x > x_g$ functions $u(x)$ and $h(x)$ deviate from straight lines in Figure 2 if $\beta > 0$, see Figure 4a. The wave function $u(x)$ decreases with oscillations. The metric component $h(x)$ becomes a smoothly decreasing function. $h(x)$ turns smoothly into a step at $x = x_g$ if $\beta \to \infty$, see Figure 4b. The component $g(x)$ gets positive at $x > x_g$. It turns from a growing function to a decreasing one, and changes sign back to minus at $x > x_h$. $g(x)$ tend to $-1$ at $x \to \infty$.

It looks reasonable to classify black holes with a mass $M >> M_0$ ($M_0 = 2 \times 10^{33}$ g is the Sun mass) as “heavy” black holes, and those with a mass $M \approx M_0$ as “light” ones. I consider a black hole with the mass $M \approx M_0$ as a ‘light’ black hole for the following reason. The critical mass $M_{cr}$, above which the collapse of a neutron star is inevitable, is of the order of the Sun mass $M_0$. Black holes with masses $M < M_{cr}$ could exist [20]. However, additional elasticity of neutrons as fermions (due to the Pauli principle of exclusivity) is an obstacle for the collapse. At $M > M_{cr}$, elasticity of neutrons is unable to withstand gravitational collapse. The barrier disappears, and matter can be compressed by its own gravitational field to the static state of a Bose condensate. A possibility of super-light black holes with masses $M << M_0$ does not contradict the theory of general relativity. Super-light “primordial” black holes could appear at the stage of strong compression of the early Universe [21].

Due to the parametric freedom $0 < \beta < \infty$ the model “$\lambda \psi^4$” allows the existence of black holes with the same mass $M$, but different elasticity $\Lambda$. The dependence of the density $u_0^2(\beta)$ for ‘heavy’ black holes with masses like the Milky Way’s one (30) is shown in Figure 5 (red curve). Blue curve in Figure 5 is the dependence $u_0^2(\beta)$ for ‘light’ black holes with masses $M \approx M_0$. 

Figure 5. Dependence of the density $u_0^2(\beta)$ in the central plateau of black holes with masses $M = M_{MW}$ (red curve, $\ln x_h = 64$), and with masses $M \approx M_\odot$ (blue curve, $\ln x_h = 48.76$).

Numerical analysis shows that at $\beta < < 1$ density $u_0^2(x_h, \beta)$ in the center grows logarithmically with $x_h$ and $\beta$:

$$u_0^2 = u_0^2 = 0.53 \ln x_h + 1.2 \ln \beta, \quad \beta < < 1.$$  

At $\beta >> 1$ the density $u_0^2(x_h, \beta)$ as a function of $\beta$ reaches its maximum $u_{0,\text{max}}^2(x_h)$. Dependence $u_{0,\text{max}}^2(x_h)$, found numerically,

$$u_{0,\text{max}}^2(x_h) = 2.14 + 0.512 \ln x_h,$$

is presented in Figure 6.

Figure 6. Maximum possible density of a condensate on the central plateau as a function of the total mass in dimensionless units. Two red dots correspond to the ‘heavy’ black hole in the center of Milky Way, and to a ‘light’ black hole (with a mass like that of the Sun).

$u_{0,\text{max}}^2(x_h)$ is the maximum possible density of a condensate at a fixed mass (29). Density $\rho_{0,\text{max}}(M)$ in dimensional units is

$$\rho_{0,\text{max}}(M) = \frac{m}{\hbar^2 \kappa} u_{0,\text{max}}^2(x_h) \frac{1}{cm^3}, \quad x_h = \frac{2mM}{\mu_pc}.$$
In the model \( \lambda \psi^4 \) at a fixed total mass \( M \) the maximum density \( \rho_{\text{max}}(M) \) depends only on the rest mass \( m \) of a boson. For bosons of the Standard Model (rest mass \( m \sim 100 \text{ GeV}/c^2 \)) the maximum possible density of the black hole in the center of Milky Way Galaxy (\( \ln x_h = 64 \)) is \( \rho_{\text{MW max}} = 3.1 \times 10^{81} \text{ cm}^{-3} \).

In the model \( \lambda \psi^4 \) regularity in the center dictates a strict balance between density \( u_0^2 \) and elasticity \( \lambda \) of a condensate (21). If, at a fixed mass, the density \( u_0^2 \) of a condensate in equilibrium is bounded from above, then, consequently, the elasticity \( |\lambda| \) is restricted from below.

![Graph](image)

**Figure 7.** Possible elasticity of a condensate as a function of \( \beta \) at fixed mass. Milky Way’s black hole (\( \ln x_h = 64 \)) – red line. A ‘light’ black hole with a mass like the mass of the Sun (\( \ln x_h = 48.76 \)) – blue line.

Dimensionless elasticity \( |\lambda| \) of the Milky Way’s black hole (\( \ln x_h = 64 \)) as a function of \( \beta \) is presented in Figure 7 – red line. Blue curve is \( |\lambda(\beta)| \) of a black hole with the mass like the Sun’s one (\( \ln x_h = 48.76 \)). The minimum possible elasticity as a function of the total mass

\[
|\lambda_{\text{min}}(x_h)| = \frac{1}{3.21 + 0.768 \ln x_h} \cdot x_h = \frac{2mM}{M_{Pl}^2}
\]

follows from (31). In the model \( \lambda \psi^4 \), a static state is impossible for a gravitating condensate with elasticity weaker than \( |\lambda_{\text{min}}(x_h)| \).

Thus, in the model \( \lambda \psi^4 \), in addition to the previously known discrete spectrum of static states of the gravitating Bose-Einstein condensate [18], there is also a continuous spectrum of equilibrium states claiming to describe the ultimately compressed condensate as a result of the gravitational collapse. The existence of an internal gravitational radius, at which the uniqueness of solutions to the system of Einstein and Klein-Gordon equations is violated, made it possible to establish some features of a black hole. Within the internal gravitational radius, the condensate density is uniform, and the energymomentum tensor corresponds to ultrarelativistic equation of state. The density value at the center is determined by elasticity of the condensate. For a given total mass, the range of possible density values in the center is bounded from above. The limiting value of the density depends on the rest mass of the boson \( m \) and is proportional to the logarithm of the total mass \( M \). If we proceed from bosons of the Standard Model, then the maximum possible density in the center of the black hole in our Milky Way galaxy is approximately \( 3 \times 10^{81} \text{ cm}^{-3} \).
References

[1] Meierovich B E 2019 *Universe* **5** 198
[2] Meierovich B E 2020 *Universe* **6** 113
[3] Chandrasekhar S 1931 *Astrophys. J.* **74** 81
[4] Landau L D 1932 *Phys. Zs. Sowjet.* **1** 285
[5] Oppenheimer J R and Volkoff G 1939 *Phys. Rev.* **55** 374
[6] https://en.wikipedia.org/wiki/Elementary_particles.
[7] Cooper L N 1956 *Phys. Rev.* **104** 1189
[8] Lifshits E M, Pitaevski L P 2000 *Course of Theoretical Physics Vol. 9 Statistical Physics Part 2* (Moscow: Fizmatlit)
[9] Colpi M, Shapiro S L and Wasserman I 1986 *Phys. Rev. Lett.* **57** 2485
[10] Landau L D and Lifshitz E M 1995 *Course of Theoretical Physics, Vol. 2: The Classical Theory of Fields* (Moscow: Nauka)
[11] Landau L D and Lifshitz E M 1995 *Course of Theoretical Physics, Vol. 5: Statistical Physics* (Moscow: Nauka)
[12] Meierovich B E 2013 *Physical Review D* **10** 103510
[13] Gordon Baum, Tetsuo Hatsuda, Toru Kojo et al 2017 Preprint Arxiv: 1707.04966v1 [astro-ph.HE]
[14] Pontryagin L S 1961 *Ordinary differential equations* (Moscow: Fizmatlit)
[15] Friedberg R, Lee T D and Pang Y 1987 *Phys. Rev. D* **35** 3640
[16] Kaup D J and Klein-Gordon Geon1968 *Phys. Rev.* **172** 1331
[17] Torres D F, Capozziello S and Lambiase G 2000 *Phys. Rev. D* **62** 104012
[18] Meierovich B E 2018 *J. Exp. Theor. Phys.* **127** 889
[19] Gillessen S, Eisenhauer F and Trippe S 2009 *Astrophys. J.* **692** 1075
[20] Zel'dovich Y B 1962 *JETP* **42** 641
[21] Novikov I D, Polnarev A G, Starobinsky A A and Zel'dovich Y B 1979 *Astron. Astrophys.* **20** 104