Real State Transfer

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Abstract

A continuous quantum walk on a graph $X$ with adjacency matrix $A$ is specified by the 1-parameter family of unitary matrices $U(t) = \exp(itA)$. These matrices act on the state space of a quantum system, the states of which we may represent by density matrices, positive semidefinite matrices with rows and columns indexed by $V(X)$ and with trace 1. The square of the absolute values of the entries of a column of $U(t)$ define a probability density on $V(X)$, and it is precisely these densities that predict the outcomes of measurements. There are two special cases of physical interest: when the column density is supported on a vertex, and when it is uniform. In the first case we have perfect state transfer; in the second, uniform mixing.

There are many results concerning state transfer and uniform mixing. In this paper we show that these results on perfect state transfer hold largely because at the time it occurs, the density matrix is real. We also show that the results on uniform mixing obtained so far hold because the entries of the density matrix are algebraic numbers. As a consequence of these we derive strong restrictions on the occurrence of uniform mixing on bipartite graphs and on oriented graphs.

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1 Introduction

Let \( X \) be a graph with adjacency matrix \( A \). The states of a continuous quantum walk on \( X \) are represented by positive semidefinite matrices and with trace 1. Physicists refer to such matrices as density matrices.

Let \( X \) be a graph with adjacency matrix \( A \). The states of a continuous quantum walk on \( X \) are density matrices with rows and columns indexed by \( V(X) \). The behaviour of the walk is determined by its initial state and the transition matrix \( U(t) \), defined by

\[
U(t) = \exp(itA).
\]

This is a symmetric and unitary matrix. If the initial state of our walk is given by a density matrix \( D \), then the state \( D(t) \) at time \( t \) is given by

\[
D(t) = U(t)DU(-t).
\]

A density matrix \( D \) represents a pure state if \( \text{rk}(D) = 1 \). In this case \( D = zz^* \) for some complex unit vector \( z \). If \( e_a \) denotes the standard basis vector in \( \mathbb{C}^{V(X)} \) indexed by the vertex \( a \), then

\[
D_a = e_a e_a^T
\]

is the pure state associated to the vertex \( a \).

One question of interest to physicists is whether, for two distinct vertices \( a \) and \( b \), there is a time \( t \) such that \( D_a(t) = D_b \). In this case we say that there is perfect state transfer from \( a \) to \( b \). If perfect state transfer occurs, a number of interesting consequences have been established. (See e.g. [8].)

We will summarize these, after developing some more terminology. We assume that the eigenvalues of \( A \) are \( \theta_1, \ldots, \theta_m \), and that the matrix \( E_r \) represents orthogonal projection onto the \( \theta_r \)-eigenspace. Then \( E_r E_r = \delta_{r,s} E_r \) and \( \sum_r E_r = I \) and we have the spectral decomposition

\[
A = \sum_r \theta_r E_r.
\]

(One consequence of this is that \( U(t) = \sum_r e^{it\theta_r} E_r \).) Vertices \( a \) and \( b \) of \( X \) are said to be cospectral if the graphs \( X \setminus a \) and \( X \setminus b \) are cospectral, that is, they have the same characteristic polynomial. It is known that \( a \) and \( b \) are cospectral if and only if

\[
\| E_r e_a \| = \| E_r e_b \|
\]
for all \( r \). We say that \( a \) and \( b \) are strongly cospectral if, for \( r \),

\[
E_r e_a = \pm E_r e_b.
\]

For more about cospectral and strongly cospectral vertices, see [7].

We can now list the consequences of perfect state transfer. If there is pst from \( a \) to \( b \) at time \( t \), then:

(a) There is pst from \( b \) to \( a \) at time \( t \).
(b) \( D_a(2t) = D_a \).
(c) If \( E_k P E_\ell \neq 0 \) and \( E_r P E_s \neq 0 \) and \( k \neq \ell \), then

\[
\frac{\theta_r - \theta_s}{\theta_k - \theta_\ell} \in \mathbb{Q}.
\]
(d) The vertices \( a \) and \( b \) are strongly cospectral.

We refer to (a), (b) respectively as symmetry and periodicity, (c) is related to what is known as the ratio condition.

One goal of this paper is to show that all these properties are consequences of the fact that the density matrices \( D_a \) and \( D_b \) are real. Thus if \( D_1 \) and \( D_2 \) are real density matrices and \( D_1(t) = D_2 \), then strong analogs of the above four properties hold. (In fact these properties will be easy corollaries of our more general results.)

We will also see that interesting things happen if we assume that the entries of \( D_1 \) and \( D_2 \) are algebraic numbers.

For background on state transfer and continuous quantum walks, see e.g. [8, 10].

### 2 Real State Transfer

A state is real if its density matrix is real. We recall that we have perfect state transfer at time \( t \) from a density matrix \( P \) to a density matrix \( Q \) if \( P(t) = Q \). We say that we have pretty good state transfer from a state \( P \) to \( Q \) if for each positive real number \( \epsilon \) there is a time \( t_\epsilon \) such that \( \| (U(t) PU(t) - Q) \| < \epsilon \).

We define the eigenvalue support of a density matrix \( P \) to be the set of pairs \( (\theta_r, \theta_s) \) such that \( E_r P E_s \neq 0 \). (This definition extends the one given in the introduction to density matrices not of the form \( D_a \).)
2.1 Lemma. If \( P \) is a density matrix and \( E_r, E_s \) are spectral idempotents of \( A \) such that \( E_r P E_s \neq 0 \) then neither \( E_r P E_r \) nor \( E_s P E_s \) is zero.

Proof. Since \( P \) is positive semidefinite, there is a unique positive semidefinite matrix \( Q \) such that \( P = Q^2 \). Hence if \( E_r P E_s \neq 0 \), then \( E_r Q \neq 0 \) and \( E_s Q \neq 0 \). Hence \( E_r P E_r = E_r Q^2 E_r \neq 0 \) and similarly \( E_s P E_s \neq 0 \). \( \square \)

If \( E_r P E_s = 0 \) whenever \( r \neq s \), then

\[
P = \sum_r E_r P E_r
\]

and \( PA = AP \); thus \( U(t) P U(-t) = P \) for all \( t \).

We say that the eigenvalue support of \( P \) satisfies the ratio condition if, for any two pairs of distinct eigenvalues \( (\theta_r, \theta_s) \) and \( (\theta_k, \theta_\ell) \) in the eigenvalue support of \( P \) with \( \theta_k \neq \theta_\ell \), we have

\[
\frac{\theta_r - \theta_s}{\theta_k - \theta_\ell} \in \mathbb{Q}.
\]

Note that if \( P \) is pure, that is, \( P = xx^* \) for some unit vector \( x \), then \( E_r P E_s = 0 \) if and only if \( E_r x \) or \( E_s x \) is zero. (In this case we could define the eigenvalue support to be the set of eigenvalues \( \theta_r \) such that \( E_r P E_r \neq 0 \), which what we do elsewhere.)

2.2 Theorem. Let \( U(t) \) be the transition matrix corresponding to a graph \( X \). Let \( A = \sum_r e^{i\theta_r} E_r \) be the spectral decomposition of \( A \). If \( P \) is a real density matrix, there is a positive time \( t \) such that \( U(t) P U(-t) \) is real if and only if the eigenvalue support of \( P \) satisfies the ratio condition.

Proof. We have

\[
P(t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r P E_s
\]

and therefore the imaginary part of \( P(t) \) is

\[
\sum_{r,s} \sin(t(\theta_r - \theta_s)) E_r P E_s.
\]

The non-zero matrices \( E_r P E_s \) are linearly independent, and therefore the imaginary part of \( P(t) \) is zero if and only if \( \sin(t(\theta_r - \theta_s)) = 0 \) whenever \( E_r P E_s \neq 0 \). Hence if \( (\theta_r, \theta_s) \) lies in the eigenvalue support of \( P \) then there is an integer \( m_{r,s} \) such that \( t(\theta_r - \theta_s) = m_{r,s} \pi \). \( \square \)
If $E_r PE_s = 0$ whenever $r \neq s$, then

$$P = \sum_r E_r PE_r$$

and $PA = AP$; thus $U(t)PU(-t) = P$ for all $t$.

**2.3 Lemma.** Let $P$ be a real density matrix. If $P(t)$ is real, then $P(2t) = P$ and $U(2t)$ commutes with $P$ and $Q$.

*Proof.* If $U(t)PU(-t) = Q$ where $Q$ is real, then taking complex conjugates yields

$$U(-t)PU(t) = Q$$

and consequently $P = U(t)QU(-t)$. It follows at one that $U(2t)$ commutes with $P$. \hfill $\Box$

**2.4 Lemma.** If $A$ is real and symmetric and we have state transfer at $t$ between real density matrices $P$ and $Q$, then

(a) $E_r PE_r = E_r QE_r$.

(b) If $t(\theta_r - \theta_s)$ is not a multiple of $\pi$ then $E_r PE_s = E_r QE_s = 0$.

(c) Otherwise $E_r PE_s = \pm E_r QE_s$.

*Proof.* If $A$ is real and symmetric then the idempotents $E_r$ are real and symmetric. Assume

$$Q = U(\tau)PU(-\tau) = \sum_{r,s} e^{i\tau(\theta_r - \theta_s)} E_r PE_s.$$

If we multiply this expression on the left by $E_r$ and on the right by $E_s$, then

$$E_r QE_s = e^{i\tau(\theta_r - \theta_s)} E_r PE_s$$

and, since all matrices here are real, $e^{i\tau(\theta_r - \theta_s)}$ must be real.

Since $\sum_{r,s} E_r PE_s = P$, we see that if $A$ is real and there is pst from $P$ to $Q$, there are signs $\epsilon_{r,s} = \pm 1$ such that

$$Q = \sum_{r,s} \epsilon_{r,s} E_r PE_s.$$

Consider the rank-one case. If $P = uu^T$ then $E_r uu^T E_s = 0$ if and only if $E_r u = 0$ or $E_s u = 0$. Hence the constraint in (b) gives a constraint on the eigenvalue support of $u$. (In fact this is the usual ratio condition, so (b) generalizes this.)
3 Pretty Good State Transfer

We have pretty good state transfer from a state $P$ to a state $Q$ if, for each $\epsilon > 0$, there is a time $t$ such that

$$\|U(t)PU(-t) - Q\| < \epsilon.$$ 

Since the complex conjugate of

$$U(t)PU(-t) - Q$$

is

$$U(-t)PU(t) - Q = U(-t)(P - U(t)QU(-t))U(t)$$

and since $U(t)$ is unitary,

$$\|U(t)PU(-t) - Q\| = \|P - U(t)QU(-t)\|.$$ 

Hence if we have pretty good state transfer from $P$ to $Q$, then we have pretty good state transfer from $Q$ to $P$.

3.1 Lemma. Suppose $A$ is real and we have pretty good state transfer from $P$ to $Q$. Then $E_rPE_s = \pm E_rQE_s$ (for all $r$ and $s$) and $E_rPE_r = E_rQE_r$.

Proof. By assumption there is an increasing sequence of times $(t_k)_{k \geq 0}$ such that

$$U(t_k)PU(-t_k) \to Q$$

as $t_k \to \infty$. Hence

$$e^{i(\theta_r - \theta_s)t_k} E_rPE_s \to E_rQE_s$$

as $t_k \to \infty$. Since $E_rPE_s$ and $E_rQE_s$ are real, the result follows.

3.2 Corollary. If $A$ is real and $P$ is real, there are only finitely many real density matrices $Q$ for which there is pretty good state transfer from $P$ to $Q$.

Proof. Since $Q = \sum_{r,s} E_rQE_s$ it follows that $Q = \sum_{r,s} \epsilon_{r,s}E_rPE_s$, where $\epsilon_{r,s} = \pm 1$.

Pretty good state transfer is treated in some detail in [2].
4 Algebras

Because they are trace-orthogonal, the non-zero matrices $E_rPE_s$ are linearly independent. The “off-diagonal” terms $E_rPE_s$ are nilpotent, indeed the matrices

$$E_rPE_s, \quad (r < s)$$

generate a nilpotent algebra where the product of any two elements is zero. The “diagonal” terms $E_rPE_r$ generate a commutative semisimple algebra; their sum is the orthogonal projection of $P$ onto the commutant of $A$. (See [7, Section 5] for further details).

Since $U(t)$ is a linear combination of the spectral idempotents of $A$, it is a polynomial in $A$ and therefore, for each $t$ we that $P(t) \in \langle A, P \rangle$. In consequence

$$\langle P(t), A \rangle = \langle P, A \rangle$$

for all $t$.

4.1 Lemma. If we have pretty good state transfer from $P$ to $Q$, then $\langle A, P \rangle = \langle A, Q \rangle$.

Proof. The algebra $\langle A, P \rangle$ is closed and as $Q$ is a limit of a sequence of matrices in it, it follows that $Q \in \langle A, P \rangle$. If we have pretty good state transfer from $P$ to $Q$, then as we noted st the beginning of Section 3 there is pretty good state transfer from $Q$ to $P$ and so $\langle A, P \rangle = \langle A, Q \rangle$.

If $\langle A, P \rangle$ is the full matrix algebra, we say that $P$ is controllable. If $P$ is real and $P(t)$ is real, then $U(2t)$ commutes with $A$ and $P$. If $P$ is also controllable it follows that $U(2t)$ must be a scalar matrix, say $U(2t) = \zeta I$. Since $\det(U(t)) = 1$, we have $\zeta^{\mu(X)} = 1$ and therefore $\zeta$ is a root of unity.

5 Timing

We investigate the times at which perfect state transfer can occur.

5.1 Lemma. Let $P$ be a density matrix and let $S$ be given by

$$S := \{ \sigma : U(\sigma)PU(-\sigma) = P \}.$$

Then there are three possibilities:
(a) $S = \emptyset$.

(b) There is a positive real number $\tau$ and $S$ consists of all integer multiples of $\tau$.

(c) $S$ is a dense subset of $\mathbb{R}$ and $U(t)PU(-t) = P$ for all $t$.

Proof. Suppose $S \neq \emptyset$. Then $S$ is an additive subgroup of $\mathbb{R}$ and there are two cases. First, $S$ is discrete and consists of all integer multiples of its least positive element. Second, $S$ is dense in $\mathbb{R}$ and there is sequence of positive elements $(\sigma_i)_{i \geq 0}$ with limit 0. Since for small values of $t$ we have

$$U(t) \approx I + itA$$

it follows that $AP = PA$ and $U(t)PU(-t) = P$ for all $t$. \hfill \Box

If $U(t)PU(-t) = P$ when $t = \tau > 0$, but not for all $t$, we say that $P$ is periodic with period $\tau$. If a density matrix is periodic, it has a well defined minimum period. If there is perfect state transfer from $P$ to $Q$, then $P$ and $P$ are both periodic with the same minimum period.

5.2 Lemma. Suppose $P$ and $Q$ are distinct real density matrices. If there is perfect state transfer from $P$ to $Q$, then $P$ is periodic and if the minimum period of $P$ is $\sigma$, then pst occurs at time $\sigma/2$.

Proof. Suppose we have pst from $P$ to $Q$ and define

$$T := \{ t : U(t)PU(-t) = Q \}$$

Assume that the minimum period of $P$ is $\sigma$. If $t \in T$ then $P$ is periodic with period $2t$ and so $T$ is a coset of a discrete subgroup of $\mathbb{R}$. Also $T = -T$. Let $\tau$ be the least positive element of $T$. Then $2\tau \geq \sigma$ and thus

$$\tau \geq \frac{1}{2}\sigma.$$ 

If $\tau \geq \sigma$ then $\tau - \sigma \in T$ and since $\tau$ is not a period, $\tau < \sigma$. As $\sigma$ must divide $2\tau$, it follows that $\tau = \sigma/2$. \hfill \Box

5.3 Corollary. For any real density matrix $P$, there is at most one real density matrix $Q$ such that there is perfect state transfer from $P$ to $Q$. \hfill \Box
5.4 Lemma. Suppose we have pst from $P$ to $Q$ at time $t$ and that $\theta_1, \ldots, \theta_m$ are the distinct eigenvalues of $A$ in nonincreasing order. If $\text{tr}(PQ) = 0$ then

$$t \geq \frac{\pi}{\theta_1 - \theta_m}.$$ 

Suppose $U(t)PU(-t) = Q$ and $\text{tr}(PQ) = 0$. We have

$$U(t)PU(-t) = \sum_r e^{i\theta_r t}U(t)E_rU(-t)$$

and therefore

$$\text{tr}(PQ) = \sum_r e^{i\theta_r t} \text{tr}(PU(t)E_rU(-t))$$

$$= \sum_r e^{i\theta_r t} \text{tr}(U(-t)PU(t)E_r)$$

$$= \sum_r e^{i\theta_r t} \text{tr}(QE_r)$$

Since $Q$ and the idempotents $E_r$ are positive semidefinite, $\text{tr}(QE_r) \geq 0$. Also

$$\sum_r \text{tr}(QE_r) = \text{tr}(Q) = 1.$$ 

Hence if $\text{tr}(PQ) = 0$ then we see that 0 is a convex combination of the eigenvalues $e^{i\theta_r t}$ of $U(t)$. If $A$ has $m$ distinct eigenvalues

$$\theta_1 \geq \cdots \geq \theta_m,$$

this implies that $e^{i\theta_1}, \ldots, e^{i\theta_m}$ cannot be contained in an arc on the unit circle in the complex plane of length less than $\pi$, and therefore $t(\theta_1 - \theta_m) \geq \pi$. \qed

Note that in this lemma we do not need $P$ and $Q$ to be real.

An algebraic integer is totally real if all its algebraic conjugates are real, equivalently, all zeros of its minimal polynomial are real.

5.5 Theorem. Let $P$ be a rational state with eigenvalue support $S$. If $S$ satisfies the ratio condition, then there is a square-free integer $\Delta$ such that if $E_rPE_s \neq 0$, then $\theta_r - \theta_s$ is an integer multiple of $\sqrt{\Delta}$. 

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Proof. Let $\mathbb{E}$ denote the extension field of $\mathbb{Q}$ generated by the elements of $S$. Since $P$ is rational, it follows that if $E_r P E_s \neq 0$ and $\gamma \in \Gamma$, then

$$E_r^\gamma P E_r \neq 0.$$ 

We also note that, as $A$ is an integer matrix, the spectral idempotents $E_r$ are algebraic and therefore $E_r^\gamma$ is a spectral idempotent of $A$.

The product

$$\prod_{(\theta_r, \theta_s) \in S} (\theta_r - \theta_s)$$

is fixed by $\Gamma$ and is consequently an integer. Given the ratio condition, we see that if $(\theta_k, \theta_\ell) \in S$, then

$$\prod_{(\theta_r, \theta_s) \in S} \frac{\theta_k - \theta_\ell}{\theta_r - \theta_s} \in \mathbb{Q}$$

and therefore

$$(\theta_k - \theta_\ell)^{|S|} \in \mathbb{Q}.$$ 

As the eigenvalues of $A$ are algebraic integers, this implies that $(\theta_k - \theta_\ell)^{|S|}$ is an integer. The eigenvalues of $A$ are totally real, but if the polynomial $t^s - 1$ has a real root, then it must be $\pm 1$ and in this case $s$ must even. We conclude that $(\theta_k - \theta_\ell)^2$ is an integer.

Suppose there are integers $m_{k,\ell}$ and $m_{r,s}$ and square-free integers $b$ and $c$ such that

$$\theta_k - \theta_\ell = m_{k,\ell} \sqrt{b}, \quad \theta_r - \theta_s = m_{r,s} \sqrt{c}.$$ 

If

$$\frac{\sqrt{b}}{\sqrt{c}} = \frac{\theta_k - \theta_\ell}{\theta_r - \theta_s} \in \mathbb{Q},$$

then $b = c$. \hfill $\Box$

5.6 Corollary. If $P$ is a periodic rational state, then the period of $P$ is at most $2\pi$.

Proof. If $t = 2\pi/\sqrt{\Delta}$ then $t(\theta_r - \theta_s)$ is an even multiple of $\pi$ for each pair $(\theta_r, \theta_s)$ in the eigenvalue support of $P$. Consequently

$$P(t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r P E_s = \sum_{r,s} E_r P E_s = P.$$ 

\hfill $\Box$

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6 Algebraic States

We say that a state with density matrix \( D \) is algebraic if the entries of \( D \) are algebraic numbers. Clearly the states \( D_a \) are algebraic.

Suppose \( D \) is a pure state. Then \( D(t) \) is pure for all \( t \). If \( D = zz^* \), then \( D(t) = ww^* \), where \( w = U(t)z \). We say that a matrix or vector is flat if all its entries have the same absolute value. We see that a vector \( w \) is flat if and only the diagonal entries of \( ww^* \) are all equal. (Note that these entries are non-negative and real.)

We say that a quantum walk has uniform mixing relative to a pure state \( D \) if there is a time \( t \) such that

\[
D(t) \circ I = \frac{1}{n} I.
\]

We refer to uniform mixing relative to the vertex state \( D_a \) as local uniform mixing. The walk has uniform mixing if, it at some \( t \), it admits uniform mixing relative to each vertex. In many of the cases where uniform mixing is known to occur, the underlying graph is vertex transitive, and then uniform mixing occurs if and only if uniform mixing relative to a vertex occurs. The only examples we know of graphs that are not regular and that do admit uniform mixing are the complete bipartite graph \( K_{1,3} \) and its Cartesian powers (an observation due to H. Zhan). If \( n \geq 2 \), the stars \( K_{1,n} \) admit uniform mixing relative to the vertex of degree \( n \).

Carlson et al. \[5\] observed that we have uniform mixing relative to the vertex of degree \( n \) in the star \( K_{1,n} \).

We say that the continuous quantum walk on \( X \) is periodic at \( a \) if there is a time \( \tau \) such that \( D_a(\tau) = D_a \).

6.1 Theorem. Let \( A \) be a Hermitian matrix with algebraic entries and let \( U(t) = \exp(itA) \) If the density \( D \) is algebraic and, for some \( t \), the density \( D(t) \) is algebraic, then the ratio condition holds on the eigenvalue support of \( D \).

Proof. Since the entries of \( A \) are algebraic, its eigenvalues are algebraic and therefore the spectral idempotents are algebraic.

We have

\[
D(t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r D E_s.
\]
The matrices $E_r D E_s$ are pairwise orthogonal, and so, for all $r$ and $s$,

$$\langle D(t), E_r D E_s \rangle = e^{it(\theta_r - \theta_s)} \langle E_r D E_s, E_r D E_s \rangle.$$  

The entries of the spectral idempotents are algebraic, and if the entries of $D$ and $D(t)$ are algebraic, then the values of the two inner products in the above identity are algebraic numbers.

It follows that $e^{it(\theta_r - \theta_s)}$ must be algebraic, for all $r$ and $s$. Now if $k \neq \ell$, then

$$e^{it(\theta_r - \theta_s)} = (e^{it(\theta_k - \theta_\ell)})^{\frac{\theta_r - \theta_s}{\theta_k - \theta_\ell}}.$$  

The Gelfond-Schneider theorem tells us that if $\alpha$ and $\beta$ are algebraic numbers and $\alpha \neq 0, 1$ and $\beta$ is irrational, then $\alpha \beta$ is transcendental, whence we deduce that if $D$ and $D(t)$ are algebraic, then if $k \neq \ell$ and neither $E_r D E_s$ nor $E_k D E_\ell$ is zero, the ratio

$$\frac{\theta_r - \theta_s}{\theta_k - \theta_\ell}$$

is rational.

\[
\Box
\]

The Gelfond-Schneider theorem was first used as above in \cite{1}; the technique is due to Jennifer Lin. One source for the Gelfond-Schneider theorem is Burger and Tubbs \cite{3}.

# 7 Oriented Graphs

We study state transfer on oriented graphs. In this section we consider the graph theory and linear algebra, in the next we turn to the continuous walks.

An oriented graph is a structure consisting of vertices and arcs, where an arc is an ordered pair of vertices, and any two vertices lie in at most one arc. We can construct (a large number of) oriented graphs by choosing a graph and assigning a direction to each edge. We use arcs($X$) to denote the set of arcs of $X$. The adjacency matrix $S$ of $X$ id the matrix with rows and columns indexed by $V(X)$, where

$$S_{u,v} = \begin{cases} 
1, & uv \in \text{arcs}(X); \\
-1, & vu \in \text{arcs}(X); \\
0, & \text{otherwise}.
\end{cases}$$
Thus $S$ is a skew symmetric matrix. We define the degree of a vertex $v$ in $X$ to be the number of arcs that use $v$. As we defined them, each oriented graph has an underlying graph whose adjacency matrix is $S \circ S$. The total valency of a vertex in $X$ is its valency in the undirected graph that underlies $X$.

The matrix $iS$ is Hermitian, with all eigenvalues real, and therefore the eigenvalues of $S$ are purely imaginary. They are symmetric about the real axis of the complex plane, so we will assume that the $r$-th eigenvalue is $i\lambda_r$, where $\lambda_r$ is real. We can then write the spectral decomposition of $S$ as

$$S = \sum i\lambda_r F_r$$

where the idempotents $F_r$ are Hermitian. Further, the idempotent associated to the eigenvalue $-i\lambda_r$ is $\overline{F}_r$.

7.1 Lemma. Let $X$ be an oriented graph with maximum total valency $\Delta$. If $\lambda$ is an eigenvalue of $X$, then $|\lambda| \leq \Delta$.

Proof. If $z \neq 0$ and $Sz = \lambda z$, then

$$\lambda z_k = \sum \ell S_{k,\ell} z_\ell$$

and so by the triangle inequality,

$$|\lambda||z_k| \leq \sum_{\ell: S_{k,\ell} \neq 0} |z_\ell|.$$ 

By choosing $k$ so that $|z_k|$ is maximal, we obtain the stated bound. \hfill \Box

If $Y$ is a bipartite graph and

$$A(Y) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},$$

then

$$S = \begin{pmatrix} 0 & -B \\ B^T & 0 \end{pmatrix}$$

is skew symmetric. As

$$\begin{pmatrix} -iI & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} iI & 0 \\ 0 & I \end{pmatrix} = i \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},$$

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each eigenvalue of $S$ is equal to $i\theta$, for some eigenvalue $\theta$ of $Y$. The oriented graph with adjacency matrix $S$ will be called the \textit{natural orientation} of $Y$. In a sense, the spectral theory of bipartite graphs is the intersection of the spectral theory of graphs with the spectral theory of oriented graphs.

Finally, since $iS$ is Hermitian, the eigenvalues of a principal submatrix interlace the eigenvalues of $S$, and therefore the eigenvalues of an induced subgraph of an oriented graph $X$ interlace the eigenvalues of $X$.

\section{8 Quantum Walks on Oriented Graphs}

Continuous quantum walks on oriented graphs were first studied in \cite{4,6}.

In the introduction we defined the transition matrix $U(t)$ as $\exp(itA)$, where $A$ was the adjacency matrix of a graph. Thus $A$ was symmetric and real. However all that is needed is that $A$ should be Hermitian and hence, if $S$ is the adjacency matrix of an oriented graph, we may define a transition matrix

$$U(t) = \exp(it(-iS)) = \exp(ts).$$

We note that $U(t)$ is then real and orthogonal for all $t$. We have the spectral decomposition

$$U(t) = \sum_r e^{it\lambda_r} F_r$$

where $\lambda_r \in \mathbb{R}$ and $F_r$ is Hermitian.

As we noted in the previous section, if $F$ is the spectral idempotent associated to the eigenvalue $i\lambda$, then the eigenvalue associated to $-i\lambda$ is $\overline{F}$. Hence $F_r e_a = 0$ if and only if $\overline{F} e_a = 0$. Consequently the eigenvalue support of a vertex is symmetric about the real axis of the complex plane and therefore the ratio condition on the eigenvalue support of a vertex is equivalent to the condition that $\lambda/\mu \in \mathbb{Q}$ for all choices of $\lambda$ and $\mu$.

If $S$ arises as the adjacency matrix of the natural orientation of a bipartite graph $Y$, with adjacency matrix $A$, then

$$\begin{pmatrix} -iI & 0 \\ 0 & I \end{pmatrix} \exp(tS) \begin{pmatrix} iI & 0 \\ 0 & I \end{pmatrix} = \exp(itA).$$

Accordingly we have perfect state transfer from $a$ to $b$ relative to $\exp(itA)$ if and only if it occurs from $a$ to $b$ relative to $\exp(tS)$; similarly we have local uniform mixing at $a$ in the graph if and only if we have it in the oriented graph.
8.1 Theorem. If there is perfect state transfer on an oriented graph from a vertex \( a \), then the eigenvalue support of \( a \) satisfies the ratio condition.

Proof. We simply note that \( D_a \) and \( D_b \) are algebraic, whence Theorem 6.1 implies the conclusion.

8.2 Theorem. If there is local uniform mixing at a vertex \( a \) in an oriented graph, then the eigenvalue support of \( a \) satisfies the ratio condition.

Proof. Assume \( n = |V(X)| \). If there is local uniform mixing at \( a \) at time \( t \), then \( U(t)e_a \) is flat. As \( U(t) \) is real, this implies that the entries of \( U(t)e_a \) are all equal to \( \pm n^{1/2} \) and hence they are algebraic. Now apply Theorem 6.1.

We say that an oriented graph is connected if its underlying undirected graph is connected.

8.3 Lemma. Let \( X \) be an oriented graph with adjacency matrix \( S \) and suppose \( a \in V(X) \). Then \( \exp(tS)e_a \) is periodic if and only if the ratio condition holds on the eigenvalue support of \( a \).

Proof. Suppose \( U(t)e_a = e_a \). Then

\[
e_a = U(t)e_a = \sum e^{it\lambda_r}F_re_a
\]

and since we also have

\[
e_a = \sum_r F_re_a,
\]

it follows that \( e^{it\lambda_r} = 1 \) for all \( r \) such that \( i\lambda_r \in \text{supp}(a) \). Consequently \( t\lambda_r \) is an integer multiple of \( 2\pi \) for each \( r \), and therefore the ratio of any two elements of \( \text{esupp}(a) \) is rational.

Now assume conversely that the ratio condition holds on the eigenvalue support of \( a \), and set \( m = |\text{esupp}(a)| \). Let \( \Gamma \) denote the Galois group of the extension field of \( \mathbb{Q} \) generated by \( \lambda_1, \ldots, \lambda_s \). Then \( \text{esupp}(a) \) is closed under \( \Gamma \). Therefore

\[
\prod_{r=1}^{m} \lambda_r
\]

is fixed by each element of \( \Gamma \), and is thus an integer. Hence \( \prod_r \lambda_r \in \mathbb{Z} \). Now

\[
\prod_{r=1}^{m} \frac{\lambda_r}{\lambda_r}
\]

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is rational and accordingly \( \lambda_s^n \) is rational. Since it also an algebraic integer, it must be an integer. Because it is an eigenvalue of the Hermitian matrix \( iS \), all algebraic conjugates of \( \lambda_r \) are real, and therefore we must have \( \lambda_r^2 \in \mathbb{Z} \). Therefore for each \( s \) we have \( \lambda_s = a_s \sqrt{b_s} \) where \( a_s, b_s \in \mathbb{Z} \) and \( b_s \) is square free. Since \( \lambda_s/\lambda_r \) is rational it follows that \( b_s \) is independent of \( s \). Therefore \( \text{esupp}(a) \) consists of integer multiples of \( \sqrt{b} \), for some square-free integer \( b \). This shows that \( a \) is periodic, with period \( 2\pi/\sqrt{b} \).

8.4 Theorem. There are only finitely many connected bipartite graphs with maximum valency at most \( k \) which contain a periodic vertex.

Proof. Let \( c \) be the eccentricity of \( a \). The vectors

\[
(A + I)^re_a, \quad (r = 0, \ldots, c)
\]

are linearly independent, because their supports form a strictly increasing sequence. These vectors lie in the span of the non-zero vectors \( F_re_a \), whence \( c + 1 \) is bounded above by the size of the eigenvalue support of \( a \). Since any two elements of \( \text{esupp}(a) \) differ by at least one, and since the largest eigenvalue of \( S \) is at most \( k \), we have that \( |\text{esupp}(a)| \leq 2\Delta + 1 \). Hence the eccentricity of \( a \) is bounded by a function of \( \Delta \), and therefore \( |V(X)| \) is bounded.

8.5 Corollary. There are only finitely many connected bipartite graphs with maximum valency at most \( k \) which admit local uniform mixing.

The theorem also implies that there are only finitely many connected bipartite graphs with maximum valency at most \( k \) on which perfect state transfer occurs, but this holds more generally for all graphs, bipartite or not. See [9, Corollary 6.2].

9 Prospects, Problems

We have established a surprising connection between behaviour of continuous quantum walks at certain times and the field of definition of the associated density matrix. An obvious problem is to find more examples of such behaviour.

Secondly, most work on continuous quantum walk assumes that the initial state is of the form \( e_r e_r^T \). Our results indicate that it might be fruitful to consider more general initial states. Our personal feeling is that pure states will be most interesting, because their eigenvalue support tends to be smaller.
References

[1] William Adamczak, Kevin Andrew, Leon Bergen, Dillon Ethier, Peter Hernberg, Jennifer Lin, and Christino Tamon. Non-uniform mixing of quantum walk on cycles. *International Journal of Quantum Information*, 5(06):781–793, 2007.

[2] Leonardo Banchi, Gabriel Coutinho, Chris Godsil, and Simone Severini. Pretty good state transfer in qubit chains—the Heisenberg Hamiltonian. *J. Math. Phys.*, 58(3):032202, 9, 2017.

[3] Edward B Burger and Robert Tubbs. *Making transcendence transparent: An intuitive approach to classical transcendental number theory*. Springer Science & Business Media, 2004.

[4] Stephen Cameron, Shannon Fehrenbach, Leah Granger, Oliver Hennigh, Sunrose Shrestha, and Christino Tamon. Universal state transfer on graphs. *Linear Algebra and its Applications*, 455:115–142, 2014.

[5] William Carlson, Allison Ford, Elizabeth Harris, Julian Rosen, Christino Tamon, and Kathleen Wrobel. Universal mixing of quantum walk on graphs. *Quantum Inf. Comput.*, 7(8):738–751, 2007.

[6] Erin Connelly, Nathaniel Grammel, Michael Kraut, Luis Serazo, and Christino Tamon. Universality in perfect state transfer. *Linear Algebra and its Applications*, 2017.

[7] C. Godsil and J. Smith. Strongly Cospectral Vertices. *ArXiv e-prints*, September 2017.

[8] Chris Godsil. State transfer on graphs. *Discrete Math.*, 312(1):129–147, 2012.

[9] Chris Godsil. When can perfect state transfer occur? *Electron. J. Linear Algebra*, 23:877–890, 2012.

[10] Alastair Kay. Perfect, efficient, state transfer and its application as a constructive tool. *International Journal of Quantum Information*, 8(04):641–676, 2010.