THE ENTROPY OF RICCI FLOWS WITH TYPE-I SCALAR CURVATURE BOUNDS

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ABSTRACT. In this paper, we extend the theory of Ricci flows satisfying a Type-I scalar curvature bound at a finite-time singularity. In [Bam16], Bamler showed that a Type-I rescaling procedure will produce a singular shrinking gradient Ricci soliton with singularities of codimension 4. We prove that the entropy of a conjugate heat kernel based at the singular time converges to the soliton entropy of the singular soliton, and use this to characterize the singular set of the Ricci flow solution in terms of a heat kernel density function. This generalizes results previously only known with the stronger assumption of a Type-I curvature bound. We also show that in dimension 4, the singular Ricci soliton is smooth away from finitely many points, which are conical smooth orbifold singularities.

1. Introduction

This paper is concerned with the finite-time singularities of solutions \((M^n, (g_t)_{t \in [0,T)})\) to the Ricci flow
\[
\frac{\partial}{\partial t} g_t = -2 \text{Rc}(g_t)
\]
on a closed manifold which satisfy the Type-I scalar curvature condition
\[
\limsup_{t \to T} \max_M R(t, t)(T - t) < \infty,
\]
where \(T < \infty\) is the maximal existence time. In particular, we generalize some of the theory of Ricci flow solutions which satisfy the more stringent Type-I curvature assumption
\[
\limsup_{t \to T} \max_M |Rm|(t, t)(T - t) < \infty.
\]
Ricci flow solutions satisfying (1.2) have been studied in [EMT11; MM15; Nab10], where it was shown that, for any fixed \(q \in M\) and sequence of times \(t_i \nearrow T\), a subsequence of \((M^n, (T-t_i)^{-1} g_{t_i}, q)\) converges in the pointed Cheeger-Gromov sense to a complete Riemannian manifold \((M_\infty, g_\infty, q_\infty)\) equipped with a function \(f_\infty \in C^\infty(M_\infty)\) which satisfies the shrinking gradient Ricci soliton (GRS) equation
\[
\text{Rc}_{g_\infty} + \nabla^2 f_\infty = \frac{1}{2} g_\infty.
\]
While it is unknown whether this limiting soliton is uniquely determined by the basepoint \(q\), in [MM15] it is shown that all such solitons share a numerical invariant, called the shrinker entropy \(W(g_\infty, f_\infty)\), (see Section 4), which is determined by \(q\). They also show that \(W(g_\infty, f_\infty) = 0\) if and only if \((M_\infty, g_\infty, f_\infty)\) is the Gaussian shrinking soliton on flat Euclidean space.

While interesting questions about solutions satisfying (1.2) condition remain open, this condition is often too restrictive for useful applications of Ricci flow to geometry and topology. Condition (1.1), on the other hand, is satisfied by Kähler-Ricci flow on a Fano manifold with initial metric in the canonical Kähler class by the work of Perelman (see [ST08]), and is conjectured to be satisfied for a much larger class of Kähler-Ricci flow solutions (Conjecture 7.7 of [SW12]). One of the main technical difficulties that arises when studying Ricci flows satisfying (1.1) is that one cannot expect Type-I blowups to result in a smooth limiting space. In fact, most results about Ricci flows satisfying (1.2), including [CZ11; EMT11; Nab10; MM15], depend crucially on applying
the Cheeger-Gromov compactness theorems to rescaled solutions. However, in [Bam17], [Bam16], Bamler develops an extensive theory for taking weak limits of Ricci flows with uniformly bounded scalar curvature, modeled on the Cheeger-Colding-Naber-Tian theory of noncollapsed Riemannian manifolds with bounded Ricci curvature. In particular, Theorem 1.2 of [Bam16] shows that any Ricci flow satisfying (1.1) has a dilation limit which is a singular space (see section 2), and which possesses the structure of a smooth but incomplete shrinking Ricci soliton outside of a subset of Minkowski codimension 4.

The main goal of this paper is to extend Bamler’s analysis of the singular limits of dilated Ricci flows satisfying (1.1), and to relate some of their properties to the original Ricci flow. The first main theorem generalizes the aforementioned results in [MM15].

In order to state this theorem, we first recall a result in [Bam16]. Assume $(M^n, (g_t)_{t \in [0, T)})$ is a closed, pointed solution of Ricci flow satisfying (1.1), and fix any sequence $t_i \nearrow T$. According to Theorem 1.2 of [Bam16], we can pass to a subsequence to get pointed Gromov-Hausdorff convergence of $(M^n, (T - t_i)^{-1}g_{t_i}, q)$ to a pointed singular space $(\mathcal{X}, q_{\infty}) = (X, d, \mathcal{R}, g_{\infty}, q_{\infty})$. Moreover, there exists $f_{\infty} \in C^\infty(\mathcal{R})$ obtained as a limit of rescalings of a conjugate heat kernel based at the singular time, which satisfies the Ricci soliton equation on $\mathcal{R}$. The Ricci soliton $(\mathcal{R}, g_{\infty}, f_{\infty})$ has a well-defined entropy $W(g_{\infty}, f_{\infty})$, defined in Section 4, and there is a heat kernel density function (defined in Section 3) $\Theta(q) \in (-\infty, 0]$ associated to the basepoint $q$.

**Theorem 1.1.** $\Theta(q) = W(g_{\infty}, f_{\infty})$, with $\Theta(q) = 0$ if and only if $(\mathcal{R}, g_{\infty}, f_{\infty})$ is the Gaussian shrinker on flat $\mathbb{R}^n$, in which case there is a neighborhood $U$ of $q$ in $M$ such that $\sup_{U \times [-2, 0]} |Rm| < \infty$.

In particular, all singular shrinking GRS which arise as Type-I dilation limits at a fixed point in $M$ possess the same shrinker entropy. We recall the definition of the singular set of $(M, (g_t)_{t \in [0, T)})$, defined in [EMT11] as

$$\Sigma := \left\{ x \in M; \sup_{U \times [0, T]} |Rm| = \infty \text{ for every neighborhood } U \text{ of } q \text{ in } M \right\}.$$

In the general Riemannian case, little is known about the regularity or structure of $\Sigma$. In the case where $(M, g_0)$ is Kähler, it is known that $\Sigma$ is actually an analytic subvariety of $M$, even without the Type-I assumption (see [CT15]). With a Type-I curvature assumption, it was shown in [MM15] that $\Sigma$ is characterized by the density function: $\Sigma = \Theta^{-1}(0)$. We are able to generalize this result to the case of Type-I scalar curvature bounds.

**Corollary 1.2.** Suppose $(M^n, (g_t)_{t \in [0, T]}, q)$ is a closed, pointed solution of Ricci flow satisfying (1.1). Then $\Sigma = \Theta^{-1}(0)$.

Finally, in dimension 4, we extend Bamler’s results on the structure of singular shrinking GRS by giving a more precise description of the singular part of the shrinking soliton. We let $(X, d, \mathcal{R}, g_{\infty})$ be the singular space of Theorem 1.1 and the discussion preceding it.

**Theorem 1.3.** If $n = 4$, then $X \setminus \mathcal{R}$ consists of finitely many points, and $X$ has the structure of a $C^\infty$ Riemannian orbifold.

In particular, if $\mathcal{X} = (X, d, \mathcal{R}, g)$ is the singular space in Theorem 1.3, then there exists $f \in C^\infty(\mathcal{R})$ such that $(\mathcal{R}, g, f)$ is an incomplete but smooth shrinking GRS, and each $x \in X \setminus \mathcal{R}$ admits a finite group $\Gamma_x \subseteq \mathbb{R}^4$ acting linearly and freely away from the origin, along with a homeomorphism $\varphi_x : \mathbb{R}^4/\Gamma_x \supseteq B(0^4, r_0) \to B^X(x, r_0)$ such that, if $\pi_x : \mathbb{R}^4 \to \mathbb{R}^4/\Gamma_x$ is the quotient map, then $\varphi_x \circ \pi_x$ is a smooth map on $B(0^4, r_0) \setminus \{0\}$, and $(\varphi_x \circ \pi_x)^*g_i (\varphi_x \circ \pi_x)^*f$ extend smoothly to a Riemannian metric and function on $B(0^4, r_0)$.

Theorem 1.3 was proved in the setting of Fano Kähler-Ricci flow in [CW12], where it was essential that the $L^2$ norm of the curvature tensor is uniformly bounded along the flow. This fails in the general Riemannian setting (even if we assume (1.2)), so our proof must rely on different arguments.
In Section 2, we collect definitions and results related to Ricci flows satisfying certain scalar curvature bounds. In Section 3, we establish Gaussian-type estimates for conjugate heat kernels based at the singular time, largely along the lines of Bamler and Zhang’s heat kernel estimates. In Section 4, we define shrinker entropy, and prove an important integration-by-parts lemma for singular shrinking GRS. In Section 5, we prove the convergence of entropy and the heat kernel measure. In Section 6, we show that the shrinker entropy of a normalized singular GRS only depends on the underlying manifold, and use this to complete the proof of Theorem 1.1 and Corollary 1.2. Finally, in Section 7, we specialize to the case of dimension 4, and prove Theorem 1.3.

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2. Preliminaries and Notation

Given a solution \((M^n, (g_t)_{t \in [0, T)})\) of Ricci flow, we let \(d_t : M \times M \rightarrow [0, \infty)\) be the length metric induced by \(g_t\), and define

\[
B(x, t, r) := B_{g_t}(x, r) := \{y \in M; d_t(x, y) < r\},
\]

\[
Q^+(x, t, r) := \{(y, s) \in M \times [t, t + r^2]; d_s(y, x) < r\},
\]

\[
r^g_{Rm}(x, t) := t_{Rm}(x, t) := \sup\{r > 0; |Rm| \leq r^{-2} \text{ on } B(x, t, r)\}.
\]

for all \((x, t) \in M \times [0, T)\) and \(r > 0\). For measurable \(S \subseteq M\), we set \(|S|_t := \text{Vol}_{g_t}(S)\). We denote the Lebesgue measure on a Riemannian manifold \((M, g)\) as \(dg\). If we consider a rescaled flow, for example \(\tilde{g}_t = \lambda g_{\lambda^{-1}t}\), we let \(\tilde{d}_t\) be the length metric induced by \(\tilde{g}_t\), \(\tilde{B}(x, t, r) := B_{\tilde{g}_t}(x, r)\) the corresponding geodesic ball, and so on. If \((X, d)\) is a metric space, we also set

\[
B^X(x, r) := \{y \in X; d(x, y) < r\}.
\]

If in addition \(\text{diam}(X) \leq \pi\), then we denote by \((C(X), d_{C(X)}, c_0)\) the corresponding metric cone, with vertex \(c_0\).

We recall Perelman’s \(W\) functional, defined by

\[
W(g, f, \tau) := (4\pi \tau)^{-\frac{n}{2}} \int_M (\tau (R + |\nabla f|^2) + f - n) e^{-f} dg
\]

for any Riemannian metric \(g\) on \(M\), and any \(f \in C^\infty(M), \tau > 0\). For any compact Riemannian manifold, Perelman’s invariants

\[
\mu[g, \tau] := \inf \left\{W(g, f, \tau); f \in C^\infty(M) \text{ and } \int_M (4\pi \tau)^{-\frac{n}{2}} e^{-f} dg = 1 \right\},
\]

\[
\nu[g, \tau] := \inf_{s \in [0, \tau]} \mu[g, s].
\]

Note that this definition of \(\nu\) is not completely standard.

We now define the class of weak limit spaces we will be considering, following the definitions in [Bam17], [Bam16].

**Definition 2.1.** A singular space is a tuple \(X = (X, d, \mathcal{R}, g)\), where \((X, d)\) is a complete, locally compact metric length space, and \((\mathcal{R}, g)\) is a \(C^\infty\) Riemannian manifold satisfying the following:

(i) \(d((\mathcal{R} \times \mathcal{R}))\) is the length metric of \((\mathcal{R}, g)\).

(ii) \(\mathcal{R}\) is an open, dense subset of \(X\).

(iii) for any compact subset \(K \subseteq X\) and \(D \in (0, \infty)\), there exists \(\kappa = \kappa(K, D) > 0\) such that, for all \(x \in K\) and \(r \in (0, D)\), we have

\[
\kappa r^n \leq |B^X(x, r) \cap \mathcal{R}| \leq \kappa^{-1} r^n.
\]
\( X \) is said to have singularities of codimension \( p_0 > 0 \) if, for all \( p \in (0, p_0) \), \( x \in X \) and \( r_0 > 0 \), there exists \( E_{p, x, r} < \infty \) such that
\[
| \{ r_{RM} < rs \} \cap B^X(x, r) \cap R | \leq E_{p, x, r} r^n s^p
\]
for all \( r \in (0, r_0) \), \( s \in (0, 1) \). \( X \) is said to have mild singularities if, for any \( p \in \mathcal{R} \), there exists a closed subset \( Q_p \subseteq \mathcal{R} \) of measure zero such that, for any \( x \in Q_p \), there exists a minimizing geodesic from \( p \) to \( x \) lying entirely in \( \mathcal{R} \). \( X \) is \( Y \)-regular at scale \( a \) if, for any \( x \in X \) and \( r \in (0, a) \) satisfying \( |B(x, r) \cap \mathcal{R}| > (\omega_n - Y^{-1}) r^n \), we have \( r_{RM}(x) > Y^{-1}r \).

**Definition 2.2.** If \((M_i, g_i, q_i)\) is a sequence of complete, pointed Riemannian manifolds and \((X, q_\infty) = (X, d, \mathcal{R}, g, q_\infty)\) is a pointed singular space, a convergence scheme \((U_i, V_i, \phi_i)\) for the convergence \((M_i, g_i, q_i) \rightarrow (X, q_\infty)\) consists of open subsets \(V_i \subseteq M_i, U_i \subseteq \mathcal{R}\), and diffeomorphisms \(\phi_i : U_i \rightarrow V_i\) such that the following hold:

(i) \((U_i)\) is an increasing sequence with \(\bigcup_i U_i = \mathcal{R}\),
(ii) \(\phi_i^* g_i \rightarrow g\) in \(C^\infty_c(\mathcal{R})\),
(iii) there exist \(q'_i \in U_i\) with \(q'_i \rightarrow q_\infty\) and \(d_{M_i}(q_i, \phi_i(q'_i)) \rightarrow 0\),
(iv) for any \(D < \infty\) and \(\epsilon > 0\), there exists \(i_0 = i_0(D, \epsilon) \in \mathbb{N}\) such that for all \(i \geq i_0\) and \(x_1, x_2 \in B^{X_i}(q_\infty, D) \cap U_i\), we have
\[
|d_X(x_1, x_2) - d_{M_i}(\phi_i(x_1), \phi_i(x_2))| < \epsilon,
\]
and such that, for any \(y \in B^{M_i}(q_i, D)\), there exists \(x \in U_i\) such that \(d_{M_i}(\phi_i(x), y) < \epsilon\).

Note that conditions (iii), (iv) imply that \(\phi_i\) are Gromov-Hausdorff approximations. If a convergence scheme exists, we say that \((M_i, g_i, q_i)\) converges to \((X, q_\infty)\).

We will commonly rely on the main theorem of Bamler in [Bam16], which establishes weak uniqueness properties and integral curvature bounds for Ricci flows with bounded scalar curvature.

**Theorem 2.3.** (Theorems 1.4, 1.7 in [Bam16]) Suppose \((M^n_i, (g^i_t)_{t \in [-2, 0]}, q_i)\) is a sequence of closed solutions of Ricci flow satisfying the following:

i. \(\nu[g_{-2, 4}] \geq -A\),
ii. \(|R|_t \leq \rho_i \leq A\) on \(M_i \times [-2, 0]\),

where \(A < \infty\). Then some subsequence of \((M_i, g^i_0, x_i)\) converges to a pointed singular space \((X, q_\infty)\) with singularities of codimension 4, which is \(Y\)-regular at the scale 1, where \(Y = Y(n, A) < \infty\). Also, for any \(\epsilon > 0\), there exists \(C = C(A, \epsilon, n) < \infty\) such that
\[
\int_{B(x, t, r)} (r_{RM}(x, t))^{-4+2\epsilon} dg^i_t < C r^{n-4+2\epsilon}.
\]
for any \(r \in (0, 1)\). If in addition \(\rho_i \rightarrow 0\), then \(X\) is Ricci flat and has mild singularities.

The following estimate is a consequence of Perelman’s no-local-collapsing theorem [Per02], Qi Zhang’s non-inflating theorem (Theorem 1.1 of [Zha12]), and a basic covering argument (Lemma 2.1 of [BZ15]).

**Proposition 2.4.** For any \(A < \infty\), there exists \(C = C(A) < \infty\) such that, for any closed solution \((M^n, (g_t)_{t \in [-2, 0]}\) of Ricci flow satisfying:

(i) \(\nu[g_{-2, 4}] \geq -A\),
(ii) \(|R| \leq A\) on \(M \times [-2, 0]\),

then for any \((x, t) \in M \times [-1, 0], r > 0\), we have
\[
C^{-1}(\min\{1, r\})^n \leq |B(x, t, r)|_t \leq C r^n e^{Cr}.
\]
If instead of (ii) we have \(|R| \leq A |t|^{-1}\) on \(M\) for all \(t \in [-2, 0]\), then
\[
|B(x, t, r)|_t \leq C r^n
\]
for all \( r \in (0, 1] \).

We will also need the following distortion estimate for Ricci flows with bounded scalar curvature.

**Theorem 2.5** (Theorem 1.1 in [BZ15]). Given \( A < \infty \), there exists \( B = B(A, n) < \infty \) such that the following holds. Suppose \((M^n, (g_t)_{t \in [-2, 0]})\) is a closed Ricci flow satisfying:

1. \( \nu_{[g_{-2}, 4]} \geq A \),
2. \(|R| \leq A \) on \( M \times [-2, 0] \),

then for all \( x, y \in M \) and \( s, t \in [-1, 0] \), we have

\[
\frac{1}{B} d_s(x, y) - B \sqrt{|t - s|} \leq d_t(x, y) \leq Bd_s(x, y) + B \sqrt{|t - s|}.
\]

Let \((M^n, (g_t)_{t \in [-2, 0]})\) be a closed solution of Ricci flow. Standard theory (see, for example, Chapter 24, Section 2 of [Cho+10]) guarantees that, for any \((x, t) \in M \times (-2, 0)\), there exists a unique fundamental solution \( K(x, t; \cdot, \cdot) : M \times [-2, t) \to (0, \infty) \) of the conjugate heat equation based at \((x, t)\). That is, \( K(x, t; \cdot, \cdot) \) is the unique smooth function on \( M \times [-2, t) \) such that

\[
(-\partial_s - \Delta_{g_s} + R_{g_s}) K(x, t; \cdot, \cdot) = 0 \quad \text{on } M \times [-2, t),
\]

\[
\int_M K(x, t; y, s) f(y) dg_s(y) \to f(x) \quad \text{as } s \nearrow t
\]

for any continuous \( f : M \to \mathbb{R} \). Moreover, \( K \) is smooth on its domain, and if \((y, s)\) are fixed, then \( K(\cdot, \cdot; y, s) \) is the fundamental solution of the heat equation:

\[
(\partial_t - \Delta_{g_t}) K(\cdot, \cdot; y, s) = 0 \quad \text{on } M \times (s, 0),
\]

\[
\int_M K(x, t; y, s) f(x) dg_t(x) \to f(y) \quad \text{as } t \searrow s.
\]

Given \((q, t) \in M \times (-2, 0)\), let \( u_{q,t} : M \times [-2, t) \to (0, \infty) \) be the conjugate heat kernel based at \((q, t)\), and write \( u_{q,t}(y, s) = (4\pi(t-s))^{-\frac{n}{2}} e^{-f_{q,t}(y,s)} \). The corresponding pointed entropy is defined to be \( \mathcal{W}_{q,t}(\tau) := \mathcal{W}(g_{t-\tau}, f_{q,t}(t-\tau), \tau) \). Note that, if \((M^n, g_t) = (\mathbb{R}^n, g_{\text{eucl}})\) is the static, flat Euclidean space, then \( u_{x,t}(y, s) = (4\pi(t-s))^{-\frac{n}{2}} e^{-\frac{(y-x)^2}{4(t-s)}} \), and \( \mathcal{W}_{x,t}(\tau) = 0 \) for all \( \tau > 0 \). Perelman’s differential Harnack inequality guarantees that \( \mathcal{W}_{x,t}(\tau) \leq 0 \) in general, and the following \( \epsilon \)-regularity theorem demonstrates that, wherever the pointed entropy is almost-Euclidean, the space-time geometry nearby is almost-Euclidean as well.

**Theorem 2.6.** (Theorem 1.16 of [HN14]) For any \( A < \infty \), there exists \( \epsilon = \epsilon(n, A) > 0 \) such that the following holds. Let \((M^n, (g_t)_{t \in [-2, 0]})\) be a closed Ricci flow satisfying \( \nu_{[g_{-2}, 4]} \geq -A \) and \(|R| \leq A|t|^{-1} \) on \( M \) for all \( t \in [-2, 0) \). If \((q, t) \in M \times [-1, 0)\) satisfies \( \mathcal{W}_{q,t}(\tau) \geq -\epsilon \), then \((r_{Rm}(q, t))^2 \geq \epsilon \tau \).

### 3. Estimates for Conjugate Heat Kernels Based at the Singular Time

The following lemma is mostly a combination of the proofs of Theorem 1.2 in [Bam16] and Theorem 1.4 in [BZ17].

**Lemma 3.1.** For any \( A < \infty \), there exists \( C^* = C^*(A, n) < \infty \) such that the following holds. Let \((M^n, (g_t)_{t \in [-2, 0]})\) be a closed solution of Ricci flow satisfying \( \nu_{[g_{-2}, 4]} \geq -A \) and \(|R| \leq A|t|^{-1} \) for all \( t \in [-2, 0) \). Then, for any \( x, y \in M \) and \( -\frac{1}{2} \leq s < t < 0 \), we have

\[
\frac{1}{C^*(t-s)^{\frac{n}{2}}} \exp\left(-\frac{C^*d_s^2(x, y)}{t-s}\right) \leq K(x, t; y, s) \leq \frac{C^*}{(t-s)^{\frac{n}{2}}} \exp\left(-\frac{d_t^2(x, y)}{C^*(t-s)}\right).
\]
Proof. First note the reduced distance bound
\[
\ell_{(x,t)}(x,s) \leq \frac{1}{2\sqrt{t-s}} \int_0^{t-s} \frac{A}{\sqrt{t+\tau}} d\tau \leq \frac{1}{2\sqrt{t-s}} \int_0^{t-s} \frac{A}{\sqrt{t}} d\tau = A,
\]
so by Perelman’s differential Harnack inequality, \(K(x,t;x,s) \geq (4\pi(t-s))^{-\frac{n}{2}} e^{-A}\) for all \(x \in M\) and \(-2 \leq s < t < 0\).

Claim: There exists \(C' = C'(A,n) < \infty\) such that, for \(-\frac{1}{2} \leq s < 0\) and \(t \in (s,\frac{1}{2}s]\), \(x,y \in M\), we have
\[
\frac{1}{C'(t-s)^{\frac{n}{2}}} \exp\left(-\frac{C'd^2(x,y)}{t-s}\right) \leq K(x,t;y,s) \leq \frac{C'}{(t-s)^{\frac{n}{2}}} \exp\left(-\frac{d^2(x,y)}{C'(t-s)}\right),
\]
where \(\tau \in \{s,t\}\).
This will just follow from an appropriate rescaling and the corresponding Gaussian bounds for Ricci flow with bounded scalar curvature. In fact, consider the rescaled flow \(\tau = \frac{t}{t+s}g_{t+|\tau|}, r \in [-|\tau|^{-1}(2+t),0]\). Then \(|\tilde{R}| \leq A\) on \(M \times [-2,0]\), and
\[
\nu[\tilde{g}_{-2}, 4] = \nu[\tilde{g}_{3}, 4|t|] \geq \nu[\tilde{g}_{-2}, 4|t| + (2 - 3|t|)] \geq \nu[\tilde{g}_{-2}, 2 + |t|] \geq -A,
\]
so the Bamberl-Zhang heat kernel estimates [BZ17] give \(C' = C'(A,n) < \infty\) such that for all \(x,y \in M\) and \(r \in [-1,0]\) we have
\[
\frac{1}{C'|r|^{\frac{n}{2}}} \exp\left(-\frac{C'd^2(x,y)}{|r|}\right) \leq \tilde{K}(x,0;y,r) \leq \frac{C'}{|r|^{\frac{n}{2}}} \exp\left(-\frac{d^2(x,y)}{C'|r|}\right),
\]
where \(\tau \in \{0, r\}\). Also, we know the behavior of the heat kernel under rescaling: \(\tilde{K}(x,0;y,r) = |t|^\frac{n}{2}K(x,t;y,t+|t|r),\) so taking \(r := -|t|^{-1}(t-s)\) gives \(K(x,t;y,s) = |t|^{-\frac{n}{2}}\tilde{K}(x,0;y,-|t|^{-1}(t-s)).\)

Note that \(|t|^{-1}(t-s) \leq (|s|/2)^{-1} \cdot (s/2 - s) = 1\) because \(t \in (s, s/2]\), so the claim follows. □

Now consider the case where \(s/2 < t < 0\). A special case of the reproduction formula for the heat kernel is
\[
K(x,t;y,s) = \int_M K(x,t;z,\frac{1}{2}(t+s))K(z,\frac{1}{2}(t+s);y,s)dg_{\frac{1}{2}(t+s)}(z)
\]
for all \(x,y \in M\) and \(-2 \leq s < 0\) and \(t \in (s/2,0]\). Also, the above claim implies
\[
K(z,\frac{1}{2}(t+s);y,s) \leq \frac{2^nC'}{(t-s)^\frac{n}{2}},
\]
so combining this with the reproduction formula gives
\[
K(x,t;y,s) \leq \frac{2^nC'}{(t-s)^\frac{n}{2}} \int_M K(x,t;z,\frac{1}{2}(t+s))dg_{\frac{1}{2}(t+s)}(z) = \frac{2^nC'}{(t-s)^\frac{n}{2}}.
\]
Set \(c_0 := (4\pi)^{-\frac{n}{2}} e^{-A}\), so that the above claim gives \(D = D(A,n) < \infty\) such that, for any \(x,y \in M\) with \(d_{\frac{1}{2}(t+s)}(x,y) \geq D\sqrt{t-s}\), we have
\[
K(x,\frac{1}{2}(t+s);y,s) \leq \frac{c_0}{2(t-s)^\frac{n}{2}},
\]
Thus, for any \(x \in M\), we have
\[
\frac{c_0}{(t-s)^\frac{n}{2}} \leq K(x,t;x,s) = \int_M K(x,t;y,\frac{1}{2}(t+s))K(y,\frac{1}{2}(t+s);x,s)dg_{\frac{1}{2}(t+s)}(y) \leq \frac{2^nC'}{(t-s)^\frac{n}{2}} \int_{B(x,\frac{1}{2}(t+s),D\sqrt{t-s})} K(x,t;y,\frac{1}{2}(t+s))dg_{\frac{1}{2}(t+s)}(y) + \frac{c_0}{2(t-s)^\frac{n}{2}},
\]
so applying the upper bound of (3.1) with $\tau = \frac{1}{2}(t+s)$ gives

$$\int_{B(x,\frac{1}{2}(t+s),D\sqrt{t-s})} K(x,t;y,\frac{1}{2}(t+s))dg_{\frac{1}{2}(t+s)}(y) \geq c_0 \frac{\nu_{\frac{1}{2}(s+t),\frac{1}{2}(t+s)}}{2} (C')^{-1} =: c_0.'$$

Now consider the rescaled flow $\tilde{g}_r := (t-s)^{-1}g_{(t-s)r+\frac{1}{2}(s+t)}$, which satisfies

$$|\tilde{R}| \leq \frac{2A}{|s+t|}(t-s) \leq 2A$$

for $r \in [-2,0]$, and

$$\nu[\tilde{g},\nu] = \nu[g_{\frac{1}{2}(s+t)-2(t-s)},4(t-s)] \geq \nu[g_{\frac{1}{2}(s+t)-2(t-s)},4(t-s)] + \frac{1}{2}(s+t) - 2(t-s) + 2] \geq \nu[g_{\frac{1}{2}(s+t)-2(t-s)},4(t-s)]$$

since $-\frac{1}{2} \leq s \leq t < 0$. We can thus apply Theorem 2.5 to obtain $B = B(A,n) < \infty$ such that

$$\frac{1}{B} \tilde{d}_{t_1}(x_1,x_2) - B \sqrt{t_2 - t_1} \leq \tilde{d}_{t_2}(x_1,x_2) \leq B \tilde{d}_{t_1}(x_1,x_2) + B \sqrt{t_2 - t_1}$$

for all $r_1, r_2 \in [-\frac{1}{2},0]$ and $x_1, x_2 \in M$. In terms of the unrescaled flow, this gives

$$\frac{1}{B} dt_1(x_1,x_2) - B \sqrt{t_2 - t_1} \leq dt_2(x_1,x_2) \leq B dt_1(x_1,x_2) + B \sqrt{t_2 - t_1}$$

for all $t_1, t_2 \in [\frac{1}{2}(s+t)]$ and $x_1, x_2 \in M$. For any $x,y \in M$, we combine (3.1),(3.2),(3.3) to obtain

$$K(x,t;y,s) \geq \int_{B(x,\frac{1}{2}(t+s),D\sqrt{t-s})} K(x,t;z,\frac{1}{2}(t+s))K(z,\frac{1}{2}(t+s);y,s)dg_{\frac{1}{2}(t+s)}(z)$$

$$\geq \int_{B(x,\frac{1}{2}(t+s),D\sqrt{t-s})} K(x,t;z,\frac{1}{2}(t+s))$$

$$\times \frac{1}{C'(t-s)^{\frac{1}{2}}} \exp\left(-\frac{2C'}{t-s} (d_{\frac{1}{2}(s+t)}(x,y) + D \sqrt{t-s})^2\right) dg_{\frac{1}{2}(t+s)}(z)$$

$$\geq \frac{c_0}{C'(t-s)^{\frac{1}{2}}} e^{-4C'd^2} \exp\left(-\frac{4C'}{t-s} (Bd_s(x,y) + B \sqrt{|t-s|})^2\right)$$

$$\geq \frac{c_0}{C'(t-s)^{\frac{1}{2}}} e^{-4C'd^2-8C'B^2} \exp\left(-\frac{8C'B^2d_s^2(x,y)}{t-s}\right)$$

This and (3.1) give $C^*(A,n) < 0$ such that, for all $x,y \in M$ and $-\frac{1}{2} \leq s < t < 0$, we have

$$K(x,t;y,s) \geq \frac{1}{C^*(t-s)^{\frac{1}{2}}} \exp\left(-\frac{C^*_s d_s^2(x,y)}{t-s}\right).$$

In particular, for any $r \in [s,\frac{1}{2}(s+t)]$, we have

$$\int_{B(x,r,\sqrt{t-s})} K(x,t;y,r)dg_r(y) \geq \frac{e^{-C^*}}{C^*(t-s)^{\frac{1}{2}}} |B(x,r,\sqrt{t-r})|.$$
where $\tau$ is defined by $\frac{1}{2}(s + t) + (t - s)\tau = r$, so that $\tau \in [-\frac{1}{2}, 0]$. Combining estimates gives

$$\int_{B(x, r, \sqrt{t - r})} K(x, t; y, r)dg_r(y) \geq b(C^*)^{-1}e^{-C^*} =: c_*$$

for all $r \in [s, \frac{1}{2}(s + t)]$.

**Case 1:** For any $y \in M$ with $d_s(x, y) \geq 4B^2\sqrt{t - s}$, the distortion estimate (3.3) gives

$$d_r(x, y) \geq \frac{1}{B}d_s(x, y) - B\sqrt{t - s} \geq \frac{1}{2B}d_s(x, y)$$

for all $r \in [s, \frac{1}{2}(s + t)]$. Then the Hein-Naber concentration inequality (Theorem 1.30 of [HN14]) gives

$$\left(\int_{B(x, r, \sqrt{t - r})} K(x, t; z, r)dg_r(z)\right) \left(\int_{B(y, r, \sqrt{t - r})} K(x, t; z, r)dg_r(z)\right) \leq \exp\left(-\frac{(d_r(x, y) - 2\sqrt{t - r})^2}{8(t - r)}\right) \leq \exp\left(-\frac{1}{8(t - r)}\left(\frac{1}{B}d_s(x, y) - 2\sqrt{t - s}\right)^2\right) \leq \exp\left(-\frac{d_s^2(x, y)}{32B^2(t - s)}\right).$$

Combining this with 3.4 gives

$$\int_{B(y, r, \sqrt{t - r})} K(x, t; z, r)dg_r(z) \leq c_*^{-1}\exp\left(-\frac{d_s^2(x, y)}{32B^2(t - s)}\right).$$

We integrate from $r = s$ to $r = \frac{1}{2}(t + s)$ to get

$$\int_{Q^+(y, s, \sqrt{\frac{1}{2}(t - s))}} K(x, t; z, r)dg_r(z)dr \leq c_*^{-1}(t - s)\exp\left(-\frac{d_s^2(x, y)}{32B^2(t - s)}\right).$$

We now combine this with the on-diagonal upper bound, obtaining $\overline{C} = \overline{C}(A, n) < \infty$ such that

$$\int_{Q^+(y, s, \sqrt{\frac{1}{2}(t - s))}} K^2(x, t; z, r)dg_r(z)dr \leq \frac{\overline{C}}{(t - s)^{n+1}}\exp\left(-\frac{d_s^2(x, y)}{\overline{C}(t - s)}\right).$$

In terms of the rescaled flow, this is

$$\int_{\hat{Q}^+(y, \frac{1}{2}; \hat{s}, \frac{1}{2})} \hat{K}^2(x, \frac{1}{2}; z, r)d\hat{g}_r(z)dr \leq \overline{C}\exp\left(-\frac{d_s^2(x, y)}{\overline{C}(t - s)}\right),$$

The Bamler-Zhang parabolic mean value inequality for solutions to the conjugate heat equation (Lemma 4.2 in [BZ17]) applied to the rescaled flow (on the time interval $[-\frac{1}{2}, 0]$) gives

$$\hat{K}^2(x, \frac{1}{2}; y, \frac{1}{2}) \leq C'' \int_{\hat{Q}^+(y, \frac{1}{2}; \hat{s}, \frac{1}{2})} \hat{K}^2(x, \frac{1}{2}; z, r)d\hat{g}_u(z)du \leq C''\overline{C}\exp\left(-\frac{d_s^2(x, y)}{\overline{C}(t - s)}\right),$$

for some $C'' = C''(A, n) < \infty$, so rescaling back gives

$$K^2(x, t; y, s) \leq \frac{C''\overline{C}}{(t - s)^n}\exp\left(-\frac{d_s^2(x, y)}{\overline{C}(t - s)}\right).$$
Case 2: If instead $d_s(x, y) \leq 4B^2 \sqrt{t - s}$, then
\[
K(x, t; y, s) \leq \frac{2 \pi C'}{(t - s)^{\frac{3}{2}}} \leq \frac{e 2 \pi C'}{(t - s)^{\frac{3}{2}}} \exp \left( -\frac{d_s^2(x, y)}{16B^2(t - s)} \right).
\]

Throughout this section, let $u_{q,t}$ be the conjugate heat kernel based at $(q, t)$, and write $u_{q,t}(x, s) = (4\pi (t - s))^{-\frac{n}{2}} e^{-h_t(x, s)}$. The following lemma is essentially obtained by passing Lemma 3.1 to the limit as $t \to 0$, and extends Propositions 2.7 and 2.8 of [MM15].

**Lemma 3.2.** Let $(M^n, (g_t)_{t \in [0, -2]}, q)$ be a closed, pointed Ricci flow with $\nu[g_{-2}, 4] \geq -A$ and $|R(x, t)| \leq A|t|^{-1}$ for all $(x, t) \in M \times [-2, 0)$. Also suppose $t_i \to 0$ and $q_i \to q$ in $M$. Then there is some subsequence of $(u_{q_i, t_i})_{i \in \mathbb{N}}$ which converges in $C^\infty_{\text{loc}}(M \times (-1, 0))$ to some $u_{q,0} \in C^\infty(M \times [-1, 0))$ satisfying $\int_M u_{q,0}(x, t) dq_0(x) = 1$ for $t \in [-1, 0)$, as well as the conjugate heat equation $(-\partial_t - \Delta + R) u_{q,0} = 0$. In addition, there exists $C = C(A, n) < \infty$ such that
\[
\frac{1}{C|s|^{\frac{3}{2}}} \exp \left( -\frac{C d_s^2(y, q)}{|s|} \right) \leq u_{q,0}(y, s) \leq \frac{C}{|s|^{\frac{3}{2}}} \exp \left( -\frac{d_s^2(y, q)}{C|s|} \right)
\]
for all $(y, s) \in M \times [-1, 0)$.

**Proof.** For any closed solution of Ricci flow, a subsequence of $u_{q_i, t_i}$ must converge in $C^\infty_{\text{loc}}(M \times [-2, 0))$ to some $u_{q,0}$ solving the conjugate heat equation on $M \times [-2, 0)$, as shown in [MM15]. Since $M$ is closed, $\int_M u_{q,0}(x, t) dq_0(x) = 1$ is immediate, so it suffices to prove the Gaussian bounds for any limit $u_{q,0}$. Fix $\alpha \in (0, 1]$, and let $i_0 \in \mathbb{N}$ be sufficiently large so that $t_i - \alpha \geq \frac{1}{2} \alpha$ for all $i \geq i_0$. By the previously established heat kernel bounds, there exists $C^* = C^*(A, n) < \infty$ such that, for all $(y, s) \in M \times [-1, -\alpha]$ and $i \geq i_0$, we have
\[
u q_{i, t_i}(y, s) \geq \frac{1}{C^* (t_i - s)^{\frac{3}{2}}} \exp \left( -\frac{2C^*(d_s^2(q_i, q) + d_s^2(q, y))}{t_i - s} \right)
\geq \frac{1}{C^*|s|^{\frac{3}{2}}} \exp \left( -\frac{2C^* d_s^2(q, q_i)}{\frac{1}{2} \alpha} \right) \exp \left( -\frac{2C^* d_s^2(q, y)}{\frac{1}{2} |s|} \right).
\]
Note that $d_s(q_i, q) \to 0$ uniformly in $s \in [-1, -\alpha]$ as $i \to \infty$, so for any $(y, s) \in M \times [-1, -\alpha],$ we have
\[
u u_{q,0}(y, s) = \lim_{i \to \infty} u_{q_i, t_i}(y, s) \geq \frac{1}{C^*|s|^{\frac{3}{2}}} \exp \left( -\frac{4C^* d_s^2(q, y)}{|s|} \right).
\]
Similarly, for any $(y, s) \in M \times [-1, -\alpha]$ and $i \geq i_0$, we have (since $(a - b)^2 \geq \frac{1}{2} a^2 - b^2$ for $a, b > 0$)
\[
u u_{q,0}(y, s) \leq \frac{C^*}{(t_i - s)^{\frac{3}{2}}} \exp \left( -\frac{1}{2} d_s^2(q, y) + \frac{d_s^2(q, q_i)}{C^*(t_i - s)} \right)
\leq \frac{2 \pi C^*}{|s|^{\frac{3}{2}}} \exp \left( \frac{d_s^2(q, q_i)}{2C^*|s|} \right) \exp \left( -\frac{d_s^2(q, y)}{2C^*|s|} \right),
\]
so the claim follows as for the lower bound.

**Definition 3.3.** Any limit $u_{q,0}$ as in the statement of Lemma 10 is called a conjugate heat kernel based at the singular time. The set of such functions $u_{q,0}$ is denoted $\mathcal{U}_q$, as in [MM15].
Note that we are not able to establish the uniqueness of \( u_{q,0} \) given a point \( q \in M \) (in fact, this is not even known under assumption \( (1.2) \)), but the collection of such functions satisfies strong compactness properties. By the uniform Gaussian estimates and parabolic regularity on compact subsets of \( M \times (-1,0) \), \( \mathcal{U}_q \) is compact in \( C^\infty_{\text{loc}} \). Let \( \mathcal{F}_q \) be the set of \( f_{q,0} \in C^\infty(M \times (-1,0)) \), where \( u_{q,0}(x,t) = (4\pi|t|)^{-\frac{n}{2}}e^{-f_{q,0}(x,t)} \). By the locally uniform bounds on \( u_q \in \mathcal{U}_q \) and their derivatives, we observe that \( \mathcal{F}_q \) is also compact in \( C^\infty_{\text{loc}} \). Thus Perelman’s differential Harnack inequality passes to the limit to give

\[
\tau(R + 2\Delta f_q - |\nabla f_q|^2) + f_q - n \leq 0 \quad \text{on} \quad M \times (-1,0)
\]

for any \( f_q \in \mathcal{F}_q \), where \( \tau := |t| \). As in [MM15], we also define

\[
\theta_q(t) := \inf_{f \in \mathcal{F}_q} W(g_t, f(t), \tau(t)).
\]

Because \( \mathcal{F}_q \) is compact in \( C^\infty_{\text{loc}} \), and because \( \theta_q(t) \geq \mu[g_t, \tau(t)] > -\infty \), this infimum is actually achieved at any \( t \in (-1,0) \) by some \( f_t \in \mathcal{F}_q \). For \(-1 < s < t < 0\), Perelman’s entropy monotonicity gives

\[
\theta_q(s) \leq W(g_s, f_t(s), \tau(s)) \leq W(g_t, f_t(t), \tau(t)) = \theta_q(t),
\]

so \( \theta_q \) is nondecreasing. Similar reasoning gives

\[
0 \leq \theta_q(t) - \theta_q(s) \leq \int_s^t 2\rho |r| \left| Rg_r + \nabla^2 f_s(r) - \frac{g_r}{2|r|} \right|^2 \frac{e^{-f_s(r)}}{(4\pi|r|)^{\frac{n}{2}}} dgr dr,
\]

but the integrand is bounded on any compact subset of \( M \times (-1,0) \), by the uniform estimates for \( f \in \mathcal{F}_q \). Thus \( \theta_q \) is locally Lipschitz. Moreover, \( \theta_q(t) \leq 0 \) for all \( t \in (-1,0) \) by Perelman’s Harnack inequality, so we can define the heat kernel density function

\[
\Theta(q) := \lim_{t \uparrow 0} \theta_q(t).
\]

Fix a sequence \( t_i \uparrow 0 \), and consider the rescaled flows \( \tilde{g}_i(t) := |t_i|^{-1}g_{t_i + |t_i|t}, \ t_i \in [-2,0] \), and \( \tilde{f}_i(t) := f_{t_i}(t_i + |t_i|t) \). By the monotonicity of \( \theta_q \), we have \( \lim_{i \to \infty} (\theta(q(t_i)) - \theta(q(t_i - \rho |t_i|)) = 0 \) for any fixed \( \rho > 0 \). Since

\[
0 \leq W(\tilde{g}_0, \tilde{f}_i(0), 1) - W(\tilde{g}_{-\rho}, \tilde{f}_i(-\rho), \tau(-\rho)) = W(g_0, f_t(t_i), |t_i|) - W(g_{-\rho |t_i|}, f_{t_i}(t_i - \rho |t_i|), |t_i| + \rho |t_i|) \leq \theta_q(t_i) - \theta_q(t_0 - \rho |t_i|),
\]

we may conclude that

\[
(3.5) \quad 0 = \lim_{i \to \infty} W(\tilde{g}_0, \tilde{f}_i(0), 1) - W(\tilde{g}_{-\rho}, \tilde{f}_i(-\rho), 1 + \rho)
\]

\[
(3.6) \quad = \lim_{i \to \infty} \int_{-\rho}^0 2(1 + |t|) \int_M \left| Rg_t + \nabla^2 \tilde{f}_i(t) - \frac{\tilde{g}_t}{2(1 + |t|)} \right|^2 \frac{e^{-\tilde{g}_t}}{(4\pi(1 + |t|))^{-\frac{n}{2}}} d\tilde{g}_i dt.
\]

By the argument of Theorem 1.2 of [Bam16], we know that, after passing to a subsequence, \( (M, \tilde{g}_0, q) \) converge to a singular shrinking GRS, and \( \tilde{f}_i(0) \) converge to the corresponding potential function. The only difference is that, in [Bam16], the soliton potential function is obtained from limiting a fixed conjugate heat kernel based at the singular time, whereas we are obtaining a soliton potential function from a sequence in \( \mathcal{F}_q \). The proof is almost exactly the same, since the estimates for elements of \( \mathcal{F}_q \) are uniform, but we rewrite the relevant parts of the argument in [Bam16] here for completeness, and because we would like to pass the heat kernel bounds of Lemma 3.2 to the limit.

By Bamler’s compactness theorem (Theorem 2.3), we can pass to a subsequence so that \( (M, \tilde{g}_0, q) \) converge to a pointed singular space \( (X, q_\infty) = (X, d, \mathcal{R}, g, q_\infty) \), with associated convergence scheme \( \Phi_i : U_i \to V_i \). For any \( x \in \mathcal{R} \), we have \( r := r^{X}_{\text{Rom}}(x) > 0 \), so by Proposition 4.1 in [Bam16],
for sufficiently large $i \in \mathbb{N}$. Because $|R_{g^i_0}| \leq A$ and $\nu[g^i_{-2},4] \geq -A$, backwards pseudolocality (Theorem 1.5 in [BZ17]) gives $\alpha = c(n,A) > 0$ such that $r_{R_{g_0}}(y,s) > \alpha r$ for all $(y,s) \in B_{g^i}(\Phi_i(x),0,\alpha r) \times [-\alpha^2 r^2,0]$ for all $i \in \mathbb{N}$. By 9 3.2, we have the uniform bounds

\[
\frac{1}{C} \exp \left( -\frac{C_d^2(y,q)}{|s|} \right) \leq e^{-f_{i,s}(y,q)} \leq C \exp \left( -\frac{C}{C|s|} \right)
\]

for all $(y,s) \in [-1,0)$. This implies

\[
- \log C + \frac{d^2_{q_\infty,x}}{C} \leq \bar{f}_{i}(t) \leq - \log C + \frac{d^2_{q_\infty,x}}{C(1-t)}
\]

for all $t \in [-2,0]$, hence (because $\Phi_i$ is a $\epsilon_i$-Gromov-Hausdorff map for some sequence $\epsilon_i \to 0$)

\[
- \log C^* + \frac{d^2_{q_\infty,x}}{C^*} \leq \bar{f}_{i}(t) \leq - \log C^* + \frac{d^2_{q_\infty,x}}{C^*(1-t)}
\]

for all $x \in U_i \cap B^X(q_\infty,D_i)$, where $D_i \to \infty$. By parabolic regularity theory applied to $\bar{u}_i(t) := (4\pi(1-t))^{\frac{1}{2}} e^{-\bar{f}_{i}(t)}$ on $B_{g^i}(\Phi_i(x),0,\alpha r) \times [-\alpha^2 r^2,0]$, we find that

\[
\limsup_{i \to \infty} \sup_{B_{g^i}(\Phi_i(x),0,\frac{1}{2}\alpha r) \times [-\frac{1}{2}\alpha^2 r^2,0]} |\nabla^k \bar{u}_i|_{g^i} > 0
\]

for all $k \in \mathbb{N}$. Along with the locally uniform upper bound for $\bar{f}_i$, we get similar bounds for $\bar{f}_i$, so that we can pass to a subsequence such that $\bar{f}_i(0)$ converges in $C^\infty_{\text{loc}}$ to some $f_\infty \in C^\infty(\mathcal{R})$. Suppose by way of contradiction that there exists $x^* \in \mathcal{R}$ such that

\[
|\text{Rc}_{g_\infty} + \nabla^2 f_\infty - \frac{g_\infty}{2} |(x^*)^2 \geq c_0 > 0.
\]

Then this quantity is at least $\frac{1}{2} c_0$ on some ball $B^X(x^*,r) \subseteq \mathcal{R}$, so for $x \in B_{g^i_0}(\Phi_i(x^*),\frac{1}{2} r)$ and sufficiently large $i$, we have

\[
|\text{Rc}_{g^i} + \nabla^2 \bar{f}_i(0) - \frac{\bar{g}_i}{2} |(x)^2 \geq \frac{c_0}{4}.
\]

However, this along with backwards pseudolocality and parabolic regularity give

\[
|\text{Rc}_{g^i} + \nabla^2 \bar{f}_i(t) - \frac{\bar{g}_i}{2} |(x) \geq \delta
\]

for $(x,t) \in B_{g^i}(\phi_i(x^*),0,\delta) \times [-\delta^2,0]$, where $\delta > 0$ is small, (depending on $x^*$ but not on $i$) contradicting (3.5). The estimate (3) passes to the limit to give

\[
- \log C^* + \frac{d^2_{q_\infty,x}}{C^*} \leq f_\infty(x) \leq \log C^* + C^* d^2_{q_\infty,x}
\]

for all $x \in \mathcal{R}$.

4. INTEGRATION BY PARTS ON THE SINGULAR RICCI SOLITON

Now let $(X,d,\mathcal{R},g_\infty,f_\infty)$ be a singular shrinking GRS as obtained in the previous section.

**Lemma 4.1.** There exists $T = T(A,n) < \infty$ such that, for all $r > 0$, we have

\[
|B^X(q_\infty,r) \cap \mathcal{R}| \leq Tr^n.
\]
Proof. By Proposition 6, we obtain \( C = C(A) < \infty \) such that \( |B(x, t, r)|_t \leq Cr^n \) for all \( r \in (0, 1] \) and \( (x, t) \in M \times [-1, 0] \). For the rescaled flows \( \tilde{g}_i^t := |t_i|^{-1} g_{i + |t_i|t} \), this means that \( |B_{\tilde{g}_i}(x, t, r)|_{\tilde{g}_i} \leq Cr^n \) for all \( r \in (0, |t_i|^{-\frac{1}{2}}) \) and \( (x, t) \in M \times [-2|t_i|^{-1}, 0] \). Now let \((U_i, V_i, \Phi_i)\) be a convergence scheme for the convergence \((M, \tilde{g}_i, q) \to (X, q_\infty)\). Let \( K \) be any compact subset of \( B_X(q_\infty, r) \cap R \). Then, for sufficiently large \( i \in \mathbb{N} \), we have \( K \subseteq U_i \) and
\[
|K| \leq 2|\varphi_i(K)|_{\tilde{g}_i} \leq 2|B_{\tilde{g}_i}(q, 0, 2r)|_{\tilde{g}_i} \leq 2^{n+1}Cr^n.
\]
Since \( K \) was arbitrary, this means \( |B_X(q_\infty, r) \cap R| \leq 2^{n+1}Cr^n \). \( \square \)

**Definition 4.2.** The shrinker entropy of the singular shrinking GRS \((X, d, R, g_\infty, f_\infty)\) is
\[
W(g_\infty, f_\infty) := \int_R (R_{g_\infty} + |\nabla f_\infty|^2 + f_\infty - n)(4\pi)^{-\frac{n}{2}} e^{-f_\infty} dg_\infty.
\]

This integral is finite by the previous lemma, since \( |R_\infty| \) is bounded, \( f_\infty \) has quadratic growth, and \( |\nabla f_\infty|^2 \leq R + |\nabla f_\infty|^2 = f_\infty - C \) for some constant \( C \in \mathbb{R} \).

In order to prove convergence of entropy, it is essential to use Perelman’s differential Harnack inequality, so that the entropy can be rewritten as the integral of a nonpositive quantity. However, it is then necessary to prove that the integration by parts formula
\[
\int_R \Delta f_\infty e^{-f_\infty} dg_\infty = \int_R |\nabla f_\infty|^2 e^{-f_\infty} dg_\infty,
\]
holds in the singular case. This is equivalent to showing that
\[
\int_R \text{div}(\nabla e^{-f_\infty}) dg_\infty = 0.
\]

To this end, we recall the following integration by parts formula

**Lemma 4.3.** ([Bam17] Prop 5.2) Let \( X = (X, d, R, g) \) be a singular space with singularities of codimension \( p_0 > 2 \), and \( Z \) a \( C^1 \) vector field on \( R \) that vanishes on \( R \setminus B(x, r) \) for some large \( r > 0 \). Assume there is a constant \( C < \infty \) such that
\[
|Z| < Cr_{Rm}^{-1} \quad \text{and} \quad |\text{div}(Z)| < Cr_{Rm}^{-2} \quad \text{on} \quad B(x, r) \cap R.
\]
Then
\[
\int_R (\text{div} Z) dg = 0.
\]

The hypotheses of this lemma will follow from various identities for soliton potential functions.

**Lemma 4.4.** \( \int_R \Delta f_\infty e^{-f_\infty} dg_\infty = \int_R |\nabla f_\infty|^2 e^{-f_\infty} dg_\infty. \)

**Proof.** Now fix \( r > 0 \), and let \( \phi \in C^\infty(R) \) be a smoothing of a radial function, chosen such that \( \phi|B(q_\infty, r) = 1 \), \( 0 \leq \phi \leq 1 \), \( |\nabla \phi| \leq 4 \), and \( \text{supp}(\phi) \subseteq B_X(q_\infty, r + 1) \). We want to apply the previous lemma to \( Z := \phi \nabla e^{-f_\infty} \). Note that \( R_{g_\infty} \geq 0 \) since \( R_{g_\infty} \) is uniformly bounded below. Also, the bound \( |R| \leq A|t|^{-1} \) passes to the limit to give \( R_{g_\infty} \leq A \). We know that \( R_{g_\infty} + |\nabla f_\infty|^2 - f_\infty = C \) for some constant \( C \in \mathbb{R} \). For the purpose of this section, we may assume that \( C = 0 \), so that \( f_\infty \geq 0 \) and \( |\nabla f_\infty|^2 \leq f_\infty \). Also, \( R_{g_\infty} + \Delta f_\infty = \frac{\delta}{2} \) implies that \( |\Delta f_\infty| \leq A + \frac{\delta}{2} \). The quadratic growth estimates (3.7) give
\[
|Z(x)| \leq |\nabla f_\infty|e^{-f_\infty} \leq (\log C^* + C^*d^2(x, q_\infty))\frac{1}{2} \exp \left( \log C^* - \frac{1}{C^*} d^2(q_\infty, x) \right),
\]
\[
|\text{div}(Z(x))| \leq 2(|\nabla f_\infty| + |\nabla f_\infty|^2 + |\Delta f_\infty|)e^{-f_\infty} \leq 4(\log C^* + C^*d^2(x, q_\infty) + A + \frac{n}{2}) \exp \left( \log C^* - \frac{1}{C^*} d^2(q_\infty, x) \right),
\]
where \( C^* \) is a constant. The boundary term \( \int_{\partial R} \phi \Delta f_\infty \) is bounded by \( \frac{\delta}{2} \), and the remainder of the boundary term \( \int_{\partial R} \phi \Delta f_\infty \) is bounded by \( C^*d^2(q_\infty, x) \). By the divergence theorem, we have
\[
\int_R (\text{div} Z) dg = 0.
\]
for \( x \in \mathcal{R} \). Both of these terms are locally bounded on \( \mathcal{R} \), so we may apply the previous lemma to \( Z \) to obtain \( 0 = \int_{\mathcal{R}} \text{div}(\phi \nabla e^{-f_\infty}) d\sigma_\infty \). Using the volume upper bound, we can conclude

\[
\int_{\mathcal{R}} |\nabla \phi| \cdot |\nabla f_\infty| e^{-f_\infty} d\sigma_\infty \leq C(n) \int_{\mathcal{R} \cap (B_X(q_\infty, r+1) \setminus B_X(q_\infty, r))} |\nabla f_\infty| e^{-f_\infty} d\sigma_\infty
\]

\[
\leq C(n) \int_{\mathcal{R} \cap (B_X(q_\infty, r+1) \setminus B_X(q_\infty, r))} e^{-\frac{1}{2}f_\infty} d\sigma_\infty
\]

\[
\leq C(n, A) r^n \exp \left(-\frac{r^2}{C(n, A)}\right).
\]

The claim then follows by taking \( r \to \infty \), and using the dominated convergence theorem. \( \square \)

**Corollary 4.5.** The soliton entropy can also be expressed as

\[
W(g_\infty, f_\infty) = (4\pi)^{-\frac{n}{2}} \int_{\mathcal{R}} (R_{g_\infty} + 2\Delta f_\infty - |\nabla f_\infty|^2 + f_\infty - n) d\sigma_\infty,
\]

which has nonpositive integrand by passing Perelman’s differential Harnack inequality to the limit.

5. Proof of Entropy Convergence

**Theorem 5.1.** Suppose \((M^n, (g_t)_{t \in [-2, 0]}, q)\) is a closed, pointed solution of Ricci flow satisfying \( \nu'_{g_{-2}, 4} \geq -A \) and

\[
|R(\cdot, t)| \leq \frac{A}{|t|}
\]

for all \( t \in [-2, 0] \). Let \((\mathcal{X}, q_\infty) = (X, d, \mathcal{R}, g_\infty, q_\infty)\) be a singular space obtained as a pointed limit of \((M, g_0^i, q)\), where \( t_i \nearrow 0 \), and \( g_0^i := |t_i|^{-1} g_{t_i+|t_i|t} \). Also assume \( f_\infty \in C^\infty(\mathcal{R}) \) is obtained by limiting \( \tilde{f}_i(0) \) as in Section 3, where \( f_i(t) := f_i(t_i + |t_i|t) \), and \( f_i \in \mathcal{F}_q \) satisfy \( \theta_q(t_i) = W(g_{t_i}, f_i, |t_i|) \).

Then

\[
\Theta(q) = \lim_{i \to \infty} W(g_0^i, \tilde{f}_i(0), 1) = W(g_\infty, f_\infty).
\]

**Proof.** The first equality is by definition. Let \((U_i, V_i, \Phi_i)\) be the convergence scheme for \((M, g_0^i, q) \to (\mathcal{X}, q_\infty)\). Then, for any compact subset \( K \subseteq \mathcal{R} \), we have for large enough \( i \in \mathbb{N} \) that

\[
\int_K (R_{g_\infty} + 2\Delta f_\infty - |\nabla f_\infty|^2 + f_\infty - n)(4\pi)^{-\frac{n}{2}} d\sigma_\infty
\]

\[
= \lim_{i \to \infty} \int_{\Phi_i(K)} (R_{g_0^i} + 2\Delta \tilde{f}_i(0) - |\nabla \tilde{f}_i(0)|^2 + \tilde{f}_i(0) - n)(4\pi)^{-\frac{n}{2}} d\tilde{\sigma}_0^i
\]

\[
\geq \limsup_{i \to \infty} W(g_0^i, \tilde{f}_i(0), 1).
\]

Taking the infimum over all compact subsets \( K \subseteq \mathcal{R} \) gives \( W(g_\infty, f_\infty) \geq \Theta(q) \). Now fix \( \epsilon > 0 \), and choose \( K \subseteq \mathcal{R} \) compact such that

\[
(4\pi)^{-\frac{n}{2}} \int_{\mathcal{R} \setminus K} |R_{g_\infty} + |\nabla f_\infty|^2 + f_\infty - n| e^{-f_\infty} d\sigma_\infty < \epsilon.
\]

Then, for any \( K' \subseteq \mathcal{R} \) compact with \( K \subseteq K' \), we have

\[
W(g_\infty, f_\infty) \leq \int_{K'} (R_{g_\infty} + |\nabla f_\infty|^2 + f_\infty - n)(4\pi)^{-\frac{n}{2}} e^{-f_\infty} d\sigma_\infty + \epsilon
\]

\[
= \lim_{i \to \infty} \int_{\Phi_i(K')} (R_{g_0^i} + |\nabla \tilde{f}_i(0)|^2 + \tilde{f}_i(0) - n)(4\pi)^{-\frac{n}{2}} e^{-\tilde{f}_i(0)} d\tilde{\sigma}_0^i + \epsilon.
\]
In order to show $\mathcal{W}(g_\infty, f_\infty) \leq \Theta(g)$, it therefore suffices to find some $K' \subseteq \mathcal{R}$ compact (possibly depending on $\epsilon$) with $K \subseteq K'$ and
\[
\liminf_{i \to \infty} \int_{M \setminus \Phi_i(K')} (R_{\bar{g}_i} + |\nabla \tilde{f}_i(0)|^2 + \tilde{f}_i(0) - n)(4\pi)^{-\frac{n}{2}} e^{-\tilde{f}_i(0)} d\bar{g}_i > -\epsilon.
\]
Since $\tilde{f}_i(0)$ have uniform quadratic growth, and because $|R_{\bar{g}_i}| \leq A$, we can find $D = D(A, n) < \infty$ uniform such that
\[
R_{\bar{g}_i} + |\nabla \tilde{f}_i(0)|^2 + \tilde{f}_i(0) - n \geq 0 \quad \text{on} \quad M \setminus B_{\bar{g}_i}(q, 0, D)
\]
for all $i \in \mathbb{N}$. Moreover, Bamler's upper bound (Theorem 2.3) on the size of the quantitative singular set gives us $E = E(A, n) < \infty$ such that
\[
|\{r_{\bar{g}_i}^3(\cdot, 0) > s\} \cap B_{\bar{g}_i}(q, 2D)|_{\bar{g}_i} \leq E s^3
\]
for all $s \in (0, 1]$.

We also know that the entropy integrand is bounded uniformly from below on $B_{\bar{g}_i}(q, 0, D)$, and that for $i \in \mathbb{N}$ sufficiently large we have
\[
\{r_{\bar{g}_i}^3(\cdot, 0) \geq s\} \cap B_{\bar{g}_i}(q, 0, 2D) \subseteq V_i.
\]
Thus we can choose $s = s(A, n, \epsilon) > 0$ sufficiently small so that
\[
\int_{\{r_{\bar{g}_i}^3(\cdot, 0) < s\} \cap B_{\bar{g}_i}(q, 0, 2D)} (R_{\bar{g}_i} + |\nabla \tilde{f}_i(0)|^2 + \tilde{f}_i(0) - n)(4\pi)^{-\frac{n}{2}} e^{-\tilde{f}_i(0)} d\mu_{\bar{g}_i} > -\epsilon.
\]
Finally, by the definition of a convergence scheme, we can choose $K' \subseteq \mathcal{R}$ such that $K \subseteq K'$ and $\Phi_i(K) \supseteq \{r_{\bar{g}_i}^3(\cdot, 0) \geq s\} \cap B_{\bar{g}_i}(q, 0, 2D)$ (in fact, this will follow by taking $K' = \overline{U}_i$ for some large $i \in \mathbb{N}$).

**Definition 5.2.** A singular GRS $(\mathcal{R}, g, f)$ is called normalized if
\[
\int_{\mathcal{R}} (4\pi)^{-\frac{n}{2}} e^{-f} dg = 1.
\]

Recall that $R + |\nabla f|^2 - f$ is some constant $c \in \mathbb{R}$ and $R + \Delta f = \frac{n}{2}$, we can write
\[
\mathcal{W}(g, f) = (4\pi)^{-\frac{n}{2}} \int_{\mathcal{R}} (R + 2\Delta f - |\nabla f|^2 + f - n)e^{-f} dg = (4\pi)^{-\frac{n}{2}} \int_{\mathcal{R}} (-R - |\nabla f|^2 + f)e^{-f} dg = -c \int_{\mathcal{R}} (4\pi)^{-\frac{n}{2}} e^{-f} dg.
\]
That is, for a normalized soliton, we know $R + |\nabla f|^2 = f - \mathcal{W}(g, f)$.

**Proposition 5.3.** The singular shrinking GRS $(\mathcal{R}, g_\infty, f_\infty)$ of Theorem 5.1 is normalized:
\[
\int_{\mathcal{R}} (4\pi)^{-\frac{n}{2}} e^{-f_\infty} dg_\infty = 1.
\]

**Proof.** For any compact subset $K \subseteq \mathcal{R}$, we have
\[
\int_{K} e^{-f_\infty} dg_\infty = \lim_{i \to \infty} \int_{\Phi_i(K)} e^{-\tilde{f}_i(0)} d\bar{g}_i \leq (4\pi)^{\frac{n}{2}},
\]
so it suffices to prove that $\int_{\mathcal{R}} (4\pi)^{-\frac{n}{2}} e^{-f_\infty} dg_\infty \geq 1$. In fact, fix $\epsilon > 0$. By the uniform volume upper bound (Proposition 2.4) and heat kernel lower bound (Lemma 3.2), we have some $D = D(\epsilon) < \infty$
such that
\[ \int_{M \setminus B_{q,i}(q,0,D)} (4\pi)^{-\frac{n}{2}} e^{-\tilde{f}_i(0)} d\bar{g}_0 \leq C(n, A) \int_{M \setminus B_{q,i}(q,0,D)} \exp \left( -\frac{1}{C} \bar{g}_0^2(q, x) \right) d\bar{g}_0 \]
\[ = C(n, A) \int_D \text{Area}_{\bar{g}_i}(\partial B_{q,i}(q,0,r)) e^{-\frac{r^2}{2}} dr \]
\[ = C(n, A) \int_D e^{-\frac{r^2}{2}} d\bar{g}_0 |B_{\bar{g}_i}(q,0,r)|_{\bar{g}_0} dr \]
\[ \leq C(n, A) \int_D r |B_{\bar{g}_i}(q,0,r)|_{\bar{g}_0} e^{-\frac{r^2}{2}} dr \]
\[ \leq C(n, A) \int_D r^{n+1} e^{-\frac{r^2}{2}} dr < C(n, A) \frac{\epsilon}{2} \]

for all \( i \in \mathbb{N} \). Moreover, since \( e^{-f_\infty} \) is uniformly bounded on \( B_{\bar{g}_i}(q,0,2D) \) (independently of \( i \)) we also have
\[ \int_{B_{\bar{g}_i}(q,0,2D) \setminus V_i} (4\pi)^{-\frac{n}{2}} e^{-\tilde{f}_i(0)} d\mu_{\bar{g}_0} \leq C \left| \{ r_{\bar{g}_i}(\cdot,0) > s \} \cap B_{\bar{g}_i}(q,0,2D) \right|_{\bar{g}_0} \]

for any \( s > 0 \), when sufficiently large \( i \). By taking \( s > 0 \) sufficiently small, the upper bound on the size of the quantitative singular set (as in the previous section) tells us that the right hand side is less than \( \frac{1}{2} \epsilon \). This means that
\[ \int_{B_{\bar{g}_i}(q,0,2D) \cap V_i} (4\pi)^{-\frac{n}{2}} e^{-\tilde{f}_i(0)} d\mu_{\bar{g}_0} \geq 1 - \epsilon \]

for \( i \) sufficiently large, hence
\[ \int_{\mathcal{R} \cap B(q,2D)} (4\pi)^{-\frac{n}{2}} e^{-f_\infty} d\mu_{g_\infty} \geq 1 - 2 \epsilon, \]

and the claim follows. \( \square \)

**Remark 5.4.** As in the Type-I curvature case [MM15], we note that Proposition 5.3 and the entropy convergence part of Theorem 5.1 also hold if the sequence \( \tilde{f}_i \) is replaced by \( f(\cdot, t_i + |t_i|t) \) for some fixed \( f \in \mathcal{F}_{\bar{g}_i} \). The equality \( W(g_\infty, f_\infty) = \Theta(q) \) could fail a priori in that setting, though equality will follow from the results of Section 6.

### 6. Entropy Rigidity of the Gaussian Soliton

The following result extends Lemma 2.1 of [Nab10] to the setting of singular shrinking GRS. The proof of that lemma used essentially the fact that the underlying Riemannian manifold is complete, which in our setting is only true if the singular set \( X \setminus \mathcal{R} \) is empty. However, we will see that the proof can be modified to work when \( X \setminus \mathcal{R} \) has singularities of codimension strictly greater than 3, using the arguments of Claim 2.32 of [CW14]. In fact, the part of the following proof establishing the flow properties of a function \( f \in C^\infty(\mathcal{R}) \) with \( \nabla^2 f = 0 \) is taken from this claim, but since the setting of [CW14] is somewhat different, we rewrite the part of this claim we need.

**Proposition 6.1.** Suppose \( X = (X, d, \mathcal{R}, g, f_i), i = 1, 2 \) are normalized singular shrinking GRS with singularities of codimension 4. Then
\[ W(g, f_1) = W(g, f_2). \]

**Proof.** We can assume that \( f_1 - f_2 \) is not constant, otherwise the normalization condition gives the claim. Set \( f := |\nabla(f_1 - f_2)|^{-1}(f_1 - f_2) \), so that \( |\nabla f| = 1 \) and \( \nabla^2 f = 0 \) on \( \mathcal{R} \). Let \( \varphi_t(x) \) be the flow
of $\nabla f$ starting at $x \in \mathcal{R}$ for $t \in \mathbb{R}$ such that this is defined. Fix $p \in (2,4)$, $s \in (0,1]$. We first show that, for any $q \in X$, $s \in (0,1]$, and $D < \infty$, the set

$$S_{D,s} := \{ x \in \mathcal{R} \cap B^X(q,D) \mid r_{Rm}^X(\varphi_t(x)) < \frac{1}{2} s \text{ for some } t \in [-D,D] \}$$

has Minkowski codimension at least $p - 1$. We denote by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure on $\mathcal{R}$, which coincides with the Lebesgue measure on any hypersurface.

Because $r_{Rm}^X$ is 1-Lipschitz, we can find $h \in C^\infty(\mathcal{R})$ such that $|\nabla h| \leq 2$ and

$$\frac{1}{2} r_{Rm}^X < h < 2 r_{Rm}^X \quad \text{on } \mathcal{R}.$$

Using the coarea formula and the fact that singularities are of codimension 4, we have

$$\int_s^{2s} \mathcal{H}^{n-1}(h^{-1}(t) \cap B^X(q,3D))dt = \int_{\{s \leq h \leq 2s\} \cap B^X(q,3D)} |\nabla h|dg,$$

$$\leq 2 |\{ r_{Rm}^X \leq 4s \} \cap B^X(q,3D) \cap \mathcal{R} | \leq 2P_{p,3D,q} s^{p-1}.$$

By Sard’s theorem, we may therefore find $t = t(s) \in (s, 2s)$ such that $\Sigma_s := h^{-1}(t) \cap B^X(q,3D)$ is smooth and satisfies $\mathcal{H}^{n-1}(\Sigma_s) \leq 2^{2p+1} P_{p,3D,q,s^{p-1}}$. Next, write $S_{D,s} = I_s \cup II_s$, where

$$I_s := \{ x \in S_{D,s} \mid r_{Rm}^X(x) \leq 4s \},$$

$$II_s := \{ x \in S_{D,s} \mid r_{Rm}^X(x) > 4s \}.$$

Since the singularities of $\mathcal{X}$ are codimension 4, we have

$$|I_s| \leq \{|r_{Rm}^X \leq 4s\} \cap B^X(q,3D) \cap \mathcal{R} | \leq 4P_{p,3D,q} s^{p}.$$

For any $y \in II_s$, there exists $t \in (-D,D)$ such that $\varphi_t(y) \in \Sigma_s$. Moreover, $|\nabla f| = 1$ implies $d(\varphi_t(y), q) \leq 2D < 3D$. Now set

$$\Omega_s := \{(t,x) \in (-D,D) \times \Sigma_s \mid \varphi_t(x) \text{ is well defined} \},$$

which is open in $(-D,D) \times \Sigma_s$.

**Claim 1:** The Jacobian of $\eta : (\Omega_s, dt^2 + g_{\Sigma_s}) \to (\mathcal{R},g), (t,x) \mapsto \varphi_t(x)$ is $\leq 1$ everywhere.

In fact, since each $\varphi_t$ is a local isometry, we have that

$$d\eta_{(t,x)} |_{T_x \Sigma_s} : T_x \Sigma_s \to T_{\varphi_t(x)}(\varphi_t(\Sigma_s))$$

is a linear isometric embedding for all $(t,x) \in (-D,D) \times \Sigma_s$. Moreover, $d\eta_{(t,x)}(\partial/\partial t) = \nabla f(\varphi_t(x))$, and so the Jacobian of $\eta$ at $(t,x) \in \Sigma_s \times (-D,D)$ is $|\nabla f(\varphi_t(x))|^2 \leq 1$, where $\perp$ denotes the projection $T_{\varphi_t(x)} \mathcal{R} \to (T_{\varphi_t(x)} \Sigma_s)$. $\square$

Note that $II_s \subseteq \eta(\Omega_s)$ and the claim gives $|\eta(\Omega_s)| \leq \mathcal{H}^{n-1}(\Sigma_s) \cdot 2D \leq 4^{p+2} E_{q,3D,p} D s^{p-1}$, so we may conclude

$$|S_{D,s}| = |I_s| + |II_s| \leq 4^{p+3} E_{q,3D,p} s^{p-1}.$$

**Claim 2:** $\{ x \in \mathcal{R} \mid d(x, S_{D,s}) < s \} \subseteq S_{2D,4s}$.

In fact, suppose $x \in \mathcal{R}$ satisfies $d(x, S_{D,s}) < s$. If $r_{Rm}^X(x) < 2s$, then $x \in S_{2D,4s}$ by definition. If $r_{Rm}^X(x) > 2s$, then there is a minimal geodesic from $x$ to some point in $S_{2D,4s}$, and this geodesic lies entirely in $\{|r_{Rm}^X > s\} \subseteq \mathcal{R}$. By construction, there is some $t \in (-D,D)$ such that $\varphi_t(\gamma) \cap \{|r_{Rm}^X \leq \frac{1}{2} s\} \neq \emptyset$. Let $t_0 \in (-D,D)$ be such that $|t_0|$ is minimal among such $t$. We can assume, by replacing $f$ with $-f$, that $t_0 > 0$. Then, since $\varphi_{t_0}$ is a local isometry, and $r_{Rm}^X \geq \frac{1}{2} s$ along $\varphi_t(\gamma)$ for $0 \leq t \leq t_0$, we know that $\varphi_{t_0}$ is defined on $\gamma$ and that $L_g(\varphi_{t_0}(\gamma)) = L_g(\gamma) < s$.

Also, by construction we have $\varphi_{t_0}(\gamma) \cap \{|r_{Rm}^X = \frac{1}{2} s\} \neq \emptyset$. Since $r_{Rm}^X$ is 1-Lipschitz, this implies
This along with $|S_{2D,2s}| \leq 4^{p+10}E_{0,6D,p,s^{p-1}}$ implies the Minkowski dimension claim. In particular, the set $S$ of $x \in \mathcal{R}$ such that $\phi_t(x)$ does not exist for all time satisfies $|S| = 0$ and $\mathcal{H}^{n-1}(S \cap f^{-1}(0)) = 0$. Define $N := f^{-1}(0) \cap \mathcal{R}$, and let $U \subseteq \mathcal{R} \times N$ be the (open) maximal subset where $\psi(t,x) := \phi_t(x)$ is defined. Then $\mathcal{R} \setminus S \subseteq \psi(U)$, since for any $x \in \mathcal{R} \setminus S$, we have $x = \psi(f(x), \phi_{-f(x)}(x))$. In particular, $|\mathcal{R} \setminus \psi(U)| = 0$. By an computation similar to that in Claim 1, and noting that now $(\nabla f(\phi_t(x))) = 1 = (\nabla f(\phi_t(x)))$, where $\perp$ denotes the projection $T_{\phi_t(x)} \mathcal{R} \to (T_{\phi_t(x)} \phi_t(N))$, we get that $\psi(U)$ is a Riemannian isometry $(U, dt^2 + \tilde{g}) \to (\psi(U), g)$, where $\tilde{g}$ is the Riemannian metric $g$ on $N := f^{-1}(0) \cap \mathcal{R}$ induced from $g$. In particular, $\tilde{f}_i := f_i \circ \psi \in C^\infty(U)$ are soliton functions, and $(f \circ \psi)(t,x) = t$.

Claim 3: There are $a_i \in \mathbb{R}$ such that

$$\tilde{f}_i(t,x) = \tilde{f}_i(0,x) + a_i t + \frac{1}{4} t^2,$$

for all $(t,x) \in U$.

In fact, the pulled back soliton equation gives $\partial^2_t \tilde{f}_i = \frac{1}{2}$ everywhere, so

$$\tilde{f}_i(t,x) = \tilde{f}_i(0,x) + \partial_t \tilde{f}_i(0,x)t + \frac{1}{4} t^2$$

for $(t,x) \in U$. Moreover, for any $X \in \mathcal{X}(N)$, we have $\nabla X \partial_t = 0$, so the Riemannian product structure and the soliton equation give

$$X(\partial_t \tilde{f}_i) = \nabla^2 \tilde{f}_i(\partial_t, X) = \frac{1}{2} g(\partial_t, X) - Rc(\partial_t, X) = 0.$$

This means that $\nabla(\partial_t \tilde{f}_i - \frac{1}{2} t) = 0$ on $U$, hence $\nabla(\nabla f, \nabla \tilde{f}_i - \frac{1}{2} f) = 0$ on the dense open subset $\psi(U)$ of $\mathcal{R}$. Because $f, f_i$ are smooth and $\mathcal{R}$ is connected, we get that $\nabla f, \nabla \tilde{f}_i - \frac{1}{2} f$ is constant on $\psi(U)$, hence $\partial_t \tilde{f}_i - \frac{1}{2} t$ is constant on $U$. In particular, $\partial_t \tilde{f}_i$ is constant on $\{0\} \times N$, and the claim follows. □

Now we use the normalization conditions on $\tilde{f}_i$. Since $|\mathcal{R} \setminus \psi(U)| = 0$ and $\partial_t (\tilde{f}_1 - \tilde{f}_2) = 1$, we have

$$(4\pi)^{-\frac{n}{2}} \left( \int_N e^{-\tilde{f}_i(0,x)} d\tilde{g}(x) \right) \left( \int_{\mathbb{R}} e^{-\frac{t^2}{4}} dt \right) = 1,$$

$$(4\pi)^{-\frac{n}{2}} \left( \int_N e^{-\tilde{f}_i(0,x)} d\tilde{g}(x) \right) \left( \int_{\mathbb{R}} e^{-\frac{t^2}{4}} dt \right) = 1,$$

since $\tilde{f}_1 = \tilde{f}_2$ on $\{0\} \times N$. Thus

$$e^{a_1} \int_{\mathbb{R}} e^{-\frac{t^2}{4}} dt = e^{a_2} \int_{\mathbb{R}} e^{-\frac{(t-2a_2)^2}{4}} dt = e^{a_2} \int_{\mathbb{R}} e^{-\frac{(t-2a_1)}{4}} dt = e^{a_1} \int_{\mathbb{R}} e^{-\frac{t^2}{4}} dt,$$

which implies $a_1^2 = a_2^2$. Noting that $\nabla_X \tilde{f}_1(0,x) = \nabla_X \tilde{f}_2(0,x)$ for all $x \in N$ and $X \in T_x N$, we thus have

$$|\nabla \tilde{f}_1(0,x)|^2 - |\nabla \tilde{f}_2(0,x)|^2 = a_1^2 - a_2^2 = 0.$$

In particular, on $\{0\} \times N$,

$$R + |\nabla \tilde{f}_1|^2 - \tilde{f}_1 = R + |\nabla \tilde{f}_2|^2 - \tilde{f}_2.$$  

Since $f_i$ are normalized, we have $R + |\nabla f_i|^2 - f_i = -\mathcal{W}(g, f_i)$, so the proposition follows. □

Next, we address the rigidity statement of Theorem 1.1.
Proposition 6.2. Suppose \((M, (g_t)_{t \in [-2,0]}, q)\) is a closed, pointed solution of Ricci flow with
\[
\sup_{t \in [-2,0]} |R(\cdot, t)|(T - t) < \infty,
\]
and let \((\mathcal{X}, q_\infty)\) be a singular shrinking GRS obtained as a Type-I limit. If \(W(g_\infty, f_\infty) = 0\), then \(\mathcal{X}\) is the Gaussian shrinker. If this occurs, there is a neighborhood \(U\) of \(q\) in \(M\) such that
\[
\sup_{U \times [-2,0]} |Rm| < \infty.
\]

Proof. Fix \(x \in \mathcal{R}\), and let \((U_i, V_i, \Phi_i)\) be the convergence scheme for \((M, \overline{g}_0, q) \to (\mathcal{X}, q_\infty)\). Note that
\[
d^X(x, q_\infty) = \lim_{i \to \infty} d_{\overline{g}_0}^i(\Phi_i(x), q) = \lim_{i \to \infty} |t_i|^{-\frac{2}{3}} d_{g_i}(\phi_i(x), q),
\]
so \(\phi_i(x) \to q\) in \(M\). Thus, after passing to a subsequence, \(u_{\Phi_i(x), t_i}\) converges to a conjugate heat kernel at the singular time \(u \in U_q\) in \(C^\infty(M \times (-1,0))\). Writing \(u(y, s) = (4\pi s)^{-\frac{1}{2}} \exp^{-f(y, s)}\), we know from previous sections that, if \(f_i(s) := f(t_i + |t_i|s)\), then \(f_i(0) \circ \Phi_i\) converges in \(C^\infty_{\text{loc}}(\mathcal{R})\) to a normalized soliton function \(\overline{f}_\infty\), which must satisfy \(W_{\text{loc}}(g_\infty, \overline{f}_\infty) = W(g_\infty, f_\infty) = 0\) by Remark 5.4 and Proposition 6.1, hence (again using Remark 5.4)
\[
0 = \lim_{i \to \infty} W(g_i^0, f_i(0), 1) = \lim_{i \to \infty} W(|t_i|^{-1} g_i, f(t_i), 1)
\]
by Theorem 5.1. Now let \(\epsilon = \epsilon(n, C) > 0\) be the constant from Theorem 2.6. Then there exists \(\delta > 0\) such that \(W(g_i, f(t), |t|) \geq -\frac{1}{4}\epsilon\) for all \(t \in [-\delta, 0]\). Because \(f_{\Phi_i(x), t_i} \to f\) in \(C^\infty_{\text{loc}}(M \times (-1,0))\), we know that for any fixed \(t \in (-1,0)\), we have
\[
W(g_t, f(t), |t|) = \lim_{i \to \infty} W(g_{t + t_i}, f_{\Phi_i(x), t_i}(t + t_i), |t|).
\]
In particular, \(W(g_{-\delta + t_i}, f_{\Phi_i(x), t_i}(-\delta + t_i), \delta) \geq -\epsilon\) for sufficiently large \(i \in \mathbb{N}\). By Theorem 2.6, we conclude \(\left(\frac{r^2_{Rm}(\Phi_i(x), t_i)}{|t_i|}\right)^2 \geq \epsilon\delta\). This means \(\left(\frac{r^2_{Rm}(\Phi_i(x), 0)}{|t_i|}\right)^2 > \epsilon\delta\), so by backwards Pseudolocality, it follows that \((M, |t_i|^{-1} g_i, q)\) actually converges in the \(C^\infty\) Cheeger-Gromov sense to the Gaussian shrinker on flat \(\mathbb{R}^n\).

Now apply a version of Perelman’s pseudolocality theorem (Theorem 1.2 of [Lu10]) to the ball \(B(q, t_i, D\sqrt{|t_i|})\), with \(D < \infty\) and \(i \in \mathbb{N}\) sufficiently large, to conclude that \(|Rm|(x, t) \leq C\) for all \(x \in B(q, t_i, \sqrt{|t_i|})\), \(t \in (t_i, 0)\), (see also Lemma 2.4 of [EMT11]). \(\square\)

Proof of Theorem 1. By Section 3, we can pass to a further subsequence in order to assume that \(\overline{f}_i(0)\) converge to another smooth soliton potential function \(f'_\infty \in C^\infty(\mathcal{R})\), which satisfies \(W(g_\infty, f'_\infty) = \Theta(q)\). By Proposition 6.1, we have \(W(g_\infty, f'_\infty) = W(g_\infty, f_\infty)\). The remaining claim is Proposition 6.2. \(\square\)

7. Removable Singularities

In this section, we specialize to the four-dimensional case, where we first sharpen Bamler’s Minkowski dimension estimates for the singular set, obtaining that the limiting singular GRS is actually smooth outside of a discrete set of points. Using this, we are able to show the singularities are conical \(C^0\) orbifold singularities, without knowing that the global \(L^2\) norm of the curvature tensor on the regular set is finite (this is not true in general, even if we assume (1.2) so that \(\mathcal{X}\) is smooth). In fact, it is not clear how one can prove local \(L^2\) estimates for the curvature on the rescaled Ricci flow. This is because the \(L^2\) curvature bound in dimension 4 is usually proved using the Chern-Gauss-Bonnet formula, but the argument relies crucially on the (rescaled) flow having...
uniformly bounded diameter. Moreover, it is not clear how to effectively localize the Chern-Gauss-Bonnet formula in this situation: applying the formula on a subdomain results in boundary terms which depend on the principal curvatures of the boundary. In [HM11], this difficulty was overcome by using properties of level sets of a shrinking GRS, which suggests that it may be easier to prove the $L^2$ curvature estimate on the limiting singular space rather than on the Ricci flow itself.

Therefore, we aim to prove a local $L^2$ bound for $|Rm|$ near the singular points of $\mathcal{X}$, and then apply the removable singularity techniques of [Tia90],[CS07], [Uhl82]. We achieve this by estimating separately the traceless Ricci and the Weyl parts of the curvature tensor, using ideas of Haslhofer-Muller [HM11] and Donaldson-Sun [DS14], respectively. After overcoming this difficulty, the proof is fairly standard, and Uhlenbeck's theory [Uhl82] of removable singularities along with the $\epsilon$-regularity theorem proved in [Hua20], and later [GJ17], let us conclude that in fact the singular GRS has a $C^\infty$ orbifold structure.

Throughout this section, we suppose that $(M^4, (g_t)_{t \in [-2,0)})$ is a closed solution of Ricci Flow satisfying $\nu [g_{t,-2}, 4] \geq -A$ and
\[
|R(x,t)| \leq \frac{A}{|t|}
\]
for all $(x,t) \in M \times [-2,0)$. Fix a basepoint $q \in M$ and a sequence of times $t_i \to 0$. Define the rescaled sequence $	ilde{g_i} := [t_i]^{-1} g_{t_i + |t_i|}$ for $t \in [-2,0)$. Then the rescaled solutions satisfy
\[
\sup_{M \times [-2,0]} |R_{\tilde{g_i}}| \leq A \quad \text{and} \quad \nu [\tilde{g_i}, 2], 4] \geq -A \quad \text{for all} \quad i \in \mathbb{N}.
\]
By Theorem 1.2 of [Bam16], we may pass to a subsequence so that $(M, \tilde{g}_i, q)$ converges to a pointed singular space $(\mathcal{X}, q_{\infty}) = (X, d, \mathcal{R}, g, q_{\infty})$ with singularities of codimension 4, that is $\mathcal{Y} = Y(A) < \infty$, and satisfies the shrinking soliton equation $Rc + \nabla^2 f = \frac{1}{2} g$ on the regular part $\mathcal{R}$, where $f \in C^\infty(\mathcal{R})$ is the obtained from a sequence of rescaled conjugate heat kernels based at the singular time. We recall that $|R| \leq A$ on $\mathcal{R}$, and that $f$ satisfies quadratic growth estimates (3.7), which combine with the equation $R + |\nabla f|^2 = f - \mathcal{W}(g, f)$ to give a locally uniform gradient estimate for $f$.

Lemma 7.1. $X \setminus \mathcal{R}$ is discrete, and every tangent cone at $x \in X \setminus \mathcal{R}$ is isometric to $\mathbb{R}^4 / \Gamma$ for some finite subgroup $\Gamma \leq O(4, \mathbb{R})$ (which may depend on $x$ and the choice of rescalings). Moreover, there exists $N = N(A) > 0$ such that $|\Gamma| \leq N$.

Proof. Fix $x_0 \in X \setminus \mathcal{R}$, and let $(Z, d_Z, cy)$ be a tangent cone at $x_0$, with $\lambda_i \to \infty$ such that $(X, \lambda_i d_X, x_0) \to (Z, d_Z, c_z)$ in the pointed Gromov-Hausdorff sense. By Corollary 1.5 of [Bam16], $Z$ is a metric cone. Choose $x_i \in M$ such that $x_i \to x_0$ as $i \to \infty$. By definition of the convergence $(M, \tilde{g}_i, q) \to (\mathcal{X}, q_{\infty})$, for each $i \in \mathbb{N}$, we can choose $j = j(i) \geq i$ such that $(M, \lambda_i^2 g_{j(i)}, x_{i,j(i)})$ is $\lambda_i^{-1}$-close in the pointed Gromov-Hausdorff topology to $(X, \lambda_i d_X, x_0)$. Setting $\tilde{g}_i := \lambda_i^2 g_{j(i)}$, we get that $(M, (\tilde{g}_i)_{t \in [-2,0], x_{i,j(i)}}, i \in \mathbb{N})$ is a sequence of pointed Ricci flows with $\sup_{M \times [-2,0]} |R_{\tilde{g}_i}| \to 0$ and $\nu [\tilde{g}_i, 2], 4] \geq -A$, which converges in the pointed Gromov-Hausdorff sense to $(Z, d_Z, c_Z)$. In particular, $(Z, d_Z, c_Z)$ has the structure of a singular space $\Sigma = (Z, d_Z, \mathcal{R}_Z, g_Z, c_Z)$ with mild singularities of codimension 4, such that $Rc_{g_Z} = 0$ on $\mathcal{R}_Z$. However, $Z = C(\Sigma)$ is a metric cone, so the link $\Sigma$ of $Z$ is a smooth 3-dimensional Riemannian manifold. That is, $\{c_Z\}$ is a smooth metric cone $g_Z = d^2 + r^2 g_Z$ for some smooth Riemannian metric $g_Z$ on $\Sigma$. However, $Rc_{g_Z} = 0$ implies $Rc_{g_Z} = (n-1) g_Z$, and since $\text{dim}(\Sigma) = 3$, $(\Sigma, g_Z)$ must be a disjoint union of spherical space forms. Because $\mathcal{R}_Z = Z \setminus \{c_Z\}$ is connected, $\Sigma$ must be connected. Thus, $Z = C(\mathbb{S}^3 / \Gamma) = \mathbb{R}^4 / \Gamma$ for some finite subgroup $\Gamma \leq O(4, \mathbb{R})$. Moreover, because $Z$ is $Y$-tame for some $Y = Y(A) < \infty$ (by Proposition 4.2 of [Bam17]), we have
\[
c(A) < |B^Z(c_Z, 1) \setminus \{c_Z\}|_{g_Z} = \omega_n / |\Gamma|.
\]

It remains to show that $x_0$ is an isolated point of $X \setminus \mathcal{R}$. Suppose by way of contradiction that there exist $y_i \in X \setminus (\mathcal{R} \cup \{x_0\})$ such that $y_i \to x_0$. Set $\lambda_i := 1/d(x_0, y_i)$. By passing to a subsequence, we can assume $(X, \lambda_i d_X, x_0)$ converges in the pointed Gromov-Hausdorff sense to a
tangent cone $(Z, d_Z, c_Z)$ as above. For any $\alpha \in (0, 1)$, we can pass to a further subsequence so that $(B^X(y_i, \alpha_\lambda^{-1}), \lambda_i d(y_i))$ converges in the pointed Gromov-Hausdorff sense to $(B^Z(y_\infty, \alpha), d_Z, y_\infty)$ for some $y_\infty \in Z$ with $d(c_Z, y_\infty) = 1$. By possibly shrinking $\alpha > 0$, we can assume that $B^Z(y_\infty, \alpha)$ is isometric to a ball in $\mathbb{R}^n$. Applying Theorem 2.37 of [TZ16] (see the appendix of this paper), we have
\[ |B^X(y_i, \alpha_\lambda^{-1}) \cap \mathcal{R}|_g \geq (\omega_n - \epsilon_i)(\alpha_\lambda^{-1})^4 \]
for some sequence $\epsilon_i \to 0$. However, the $Y(A)$-regularity of $\mathcal{X}$ then implies $r_{Rm}(y_i) > 0$, contradicting $y_i \in X \setminus \mathcal{R}$. \hfill \Box

**Theorem 7.2.** $\mathcal{X}$ has the structure of a $C^\infty$ Riemannian orbifold with finitely many conical orbifold singularities, such that in orbifold charts around the singular points, $f$ extends smoothly across the singular points, and satisfies the gradient Ricci soliton equation everywhere.

**Proof.** Fix $x^* \in X \setminus \mathcal{R}$. Suppose by way of contradiction that there exists a sequence $x_i \to x^*$ such that $\liminf_{i \to \infty} |Rm(x_i)|d_X^2(x_i, x^*) > 0$ (since $x^*$ is an isolated point of $X \setminus \mathcal{R}$, we can assume $x_i \in \mathcal{R}$). Set $r_i := d_X(x_i, x^*)$, so that by passing to a subsequence, we may assume that $(X, r_i^{-1}d_X, x^*)$ converge in the pointed Gromov-Hausdorff sense to $(C(S^3/\Gamma), d_{C(S^3/\Gamma)}, c_0)$ for some finite subgroup $\Gamma \leq O(4, \mathbb{R})$, where $c_0 \in C(S^3/\Gamma)$ is the cone point.

We claim that there are $\epsilon_i \to 0$ such that, for all $x \in B^X(x^*, 2r_i) \setminus B^X(x^*, \frac{1}{2}r_i)$, we have the Gromov-Hausdorff distance estimate
\[ d_{GH}\left((B^X(x, \alpha r_i), r_i^{-1}d_X, x), (B^{C(S^3/\Gamma)}(\mathcal{F}, \alpha), d_{C(S^3/\Gamma)}, \mathcal{F})\right) < \epsilon_i \]
for some $\mathcal{F} \in C(S^3/\Gamma)$ with $\frac{1}{4} \leq d(c_0, \mathcal{F}) \leq 4$. Suppose by way of contradiction that there exist $\epsilon > 0$ and $y_i \in B^X(x^*, 2r_i) \setminus B^X(x^*, \frac{1}{2}r_i)$ where
\[ d_{GH}\left((B^X(y_i, \alpha r_i), r_i^{-1}d_X, y_i), (B^{C(S^3/\Gamma)}(\mathcal{F}, \alpha), d_{C(S^3/\Gamma)}, \mathcal{F})\right) > \epsilon \]
for every $\mathcal{F} \in C(S^3/\Gamma)$ with $\frac{1}{4} \leq d(c_0, \mathcal{F}) \leq 4$. Let $\psi_i : B^X(x^*, 100r_i) \to C(S^3/\Gamma)$ be $\delta_i$-Gromov-Hausdorff approximations
\[ (B^X(x^*, 100r_i), r_i^{-1}d_X, x^*) \to (B^{C(S^3/\Gamma)}(c_0, 100), d_{C(S^3/\Gamma)}, c_0), \]
where $\delta_i \to 0$. Then $\psi_i | B^X(y_i, \alpha r_i)$ is a $\frac{1}{2}\delta_i$-Gromov-Hausdorff approximation from
\[ (B^X(y_i, \alpha r_i), r_i^{-1}d_X, y_i) \to (B^{C(S^3/\Gamma)}(\psi_i(y_i), \alpha), d_{C(S^3/\Gamma)}, \psi_i(y_i)), \]
where $\frac{1}{4} \leq d(c_0, \psi_i(y_i)) \leq 3$. Passing to a subsequence, we may assume that $\psi_i(y_i)$ converges in some $y_\infty \in C(S^3/\Gamma)$ with $\frac{1}{4} \leq d(c_0, y_\infty) \leq 3$. Then $(B^X(y_i, \alpha r_i), r_i^{-1}d_X, y_i)$ converges in the pointed Gromov-Hausdorff sense to $(B^{C(S^3/\Gamma)}(y_\infty, \alpha), d_{C(S^3/\Gamma)}, y_\infty)$, a contradiction.

Because $|\Gamma| \leq N(A)$, we can choose $\alpha = \alpha(A) > 0$ sufficiently small so that $|B^{C(S^3/\Gamma)}(y, \alpha)| = \omega_n \alpha^n$ for all $y \in C(S^3/\Gamma)$ with $\frac{1}{4} \leq d(y, c_0) \leq 4$. Another application of Theorem 2.37 of [TZ16] tells us that the volume ratios $(\alpha r_i)^{-n} |B^X(x, \alpha r_i)|$ converge to the Euclidean volume ratio, locally uniformly for $x \in X$ with $2r_i > d_X(x, x^*) > \frac{1}{4}r_i$. Note that we could also perform a conformal change using the potential function $f$ as in [Zha10], and appeal to the volume convergence theorem for manifolds with Ricci curvature bounded below. By the $Y(A)$-regularity of the singular space $\mathcal{X}$, we may conclude that $r_{Rm}(x_i) > Y r_i$ for some $Y = Y(A) < \infty$. In particular, also using the lower volume bound, we may assume by passing to a subsequence and applying the Cheeger-Gromov compactness theorem, that the convergence of the rescaled metrics $(B^X(x_i, \frac{1}{2}\alpha r_i), r_i^{-2}g)$ to a ball in $B^{C(S^3/\Gamma)}(c_0, 4) \setminus B^{C(S^3/\Gamma)}(c_0, \frac{1}{4})$ is smooth. However, $C(S^3/\Gamma)$ is flat away from the cone point, so actually $|Rm(x_i)|d_X^2(x_i, x^*) \to 0$ as $i \to \infty$, a contradiction. We may therefore conclude that $|Rm|(x) d_X^2(x, x^*) \leq \epsilon(x)$ for $x \in B^X(x^*, \delta) \setminus \{x^*\}$, where $\lim_{x \to x^*} \epsilon(x) = 0$. By local estimates for shrinking GRS (c.f. the proof of Theorem 2.5 in [HM11]), we even have
Claim: $C(S^3/\Gamma)$ is the unique tangent cone of $X$ at $x^*$.

We proceed as in the proof of Lemma 5.13 in [BKN98]. The above arguments give that, for any tangent cone $C(S/\Gamma')$ of $X$ at $x^*$, there is a sequence $s_i \downarrow 0$ such that $B^X(x^*, 2s_i) \setminus B^X(x^*, s_i)$ is diffeomorphic to $(1, 2) \times S^3/\Gamma'$. It therefore suffices to find $\tau > 0$ such that $B^X(x^*, 2r) \setminus \overline{B}^X(x^*, r)$ and $B^X(x^*, r)$ are diffeomorphic for all $s, r \in (0, \tau)$. Set

$$\theta(r) := \sup\{\angle(\gamma(r), \eta(r)) ; \gamma, \eta : [0, r] \to B^X(x, r)$$

are unit-speed minimizing geodesics from $x^*$ to some $x \in \partial B^X(x, r)$.

We claim that $\lim_{r \to 0} \theta(r) = 0$. Otherwise, there exists $\tau > 0$ and a sequence of unit-speed geodesics $\gamma_i, \eta_i : [0, r_i] \to X$ from $x$ to some $x_i \in \partial B^X(x^*, r_i)$, such that $\angle(\gamma_i(r_i), \eta_i(r_i)) \geq \tau$. After passing to a subsequence, we have pointed Cheeger-Gromov convergence

$$(B^X(x^*, \alpha_i), r_i^{-2} g, x_i) \to (C^{(S^3/\Gamma')}(x, \alpha), g_{C(S^3/\Gamma')}, x)$$

for some $\alpha \in \partial C^{(S^3/\Gamma')}(c_0, 1)$,

$$r_i^{-1}\gamma_i(r_i) \to v \in T_{x_\infty} C(S^3/\Gamma'),$$

$$r_i^{-1}\eta_i(r_i) \to w \in T_{x_\infty} C(S^3/\Gamma'),$$

pointed Gromov-Hausdorff convergence

$$(B^X(x^*, \delta), r_i^{-1} d_x, x^*) \to (C(S^3/\Gamma'), g_{C(S^3/\Gamma')}, c_0)$$

and (after constant-speed reparametrization) $\gamma_i, \eta_i$ converge (smoothly on $(0, 1]$) to unit-speed geodesics $\gamma_\infty, \eta_\infty : [0, 1] \to X$ from $c_0$ to $x_\infty$ with

$$\angle(\gamma_\infty(1), \eta_\infty(1)) = \angle(v, w) \geq \tau,$$

contradicting the fact that there is a unique minimizing geodesic from $c_0$ to any $x \in C(S^3/\Gamma')$. Therefore $\lim_{r \to 0} \theta(r) = 0$, so for $r_0 > 0$ sufficiently small, we can construct a smooth vector field whose flow can be used to construct a homeomorphism (see Proposition 12.1.2 and Lemma 12.1.3 of [Pet16]) between $B^X(x^*, 2s_1) \setminus B^X(x^*, s_1)$, $i = 1, 2$, for any $s_1, s_2 \in (0, r_0]$. However, we know $B^X(x^*, 2s_1) \setminus B^X(x^*, s_1)$ is homeomorphic to $(1, 2) \times S^3/\Gamma$ for some finite subgroups, so we must have $S_3/\Gamma_1 = S_3/\Gamma_2$. □

We can now apply the argument in Step 1 of [DS14] (see also [BZ17], [BKN98], [Tia90]) verbatim to our situation to conclude that there is a diffeomorphism $F : (B(0^4, r_0) \setminus \{0^4\})/\Gamma \to B^X(x^*, r_0) \setminus \{x^*\}$ such that $(F \circ \pi)^*g$ extends to a $C^0$ Riemannian metric on $B(0^4, r_0)$, where $\pi : \mathbb{R}^4 \to \mathbb{R}^3/\Gamma$ is the quotient map. By replacing $g$ with $(F \circ \pi)^*g$, we may as well assume $\Gamma$ is trivial. Define $A(r_1, r_2) := B^X(x^*, r_2) \setminus B^X(x^*, r_1)$ and $B^* := B^X(x^*, r_0) \setminus \{x^*\}$.

Claim: $\int_{B^*} |Rm|^2d\sigma < \infty$.

In four dimensions, the curvature tensor $Rm$ admits the orthogonal decomposition

$$Rm = \frac{R}{24}g \otimes g + \frac{1}{2} \left( Rc - \frac{R}{4}g \right) \otimes g + W.$$

Because $|R| \leq A$ on $\mathcal{R}$, the first term of (7.1) is bounded pointwise. We use the method of [HM11] to estimate the second term of (7.1). Fix $\beta \in (0, 1)$, and let $\phi \in C^\infty_c(\alpha(\beta r_0, r_0))$ be a cutoff function with $|\nabla \phi| \leq C(n)(\beta r_0)^{-1}$ on $A(\beta r_0, 2\beta r_0)$, $|\nabla \phi| \leq C(n) r_0^{-1}$ on $A(1/2 r_0, 2r_0)$, and $\phi = 1$ on
Then, setting $E := \sup_B (e^{-f} + |\nabla f|)$, we get
\[
\int_{B^*} |Rc|^2 \phi^2 e^{-f} dg = \int_{B^*} \left\langle \frac{1}{2} g - \nabla^2 f, Rc \right\rangle \phi^2 e^{-f} dg
\]
\[
= C(A, E) + \int_{B^*} \langle \nabla f, \text{div}_f (\phi^2 Rc) \rangle e^{-f} dg
\]
\[
\leq C(A, E) + \int_{B^*} 2|\nabla f| \cdot |\nabla \phi| |Rc| e^{-f} dg
\]
\[
\leq C(A, E) + \frac{1}{2} \int_{B^*} |Rc|^2 \phi^2 e^{-f} dg + 2 \int_{B^*} |\nabla f|^2 |\nabla \phi|^2 e^{-f} dg,
\]

since $\text{div}_f Rc = 0$. Rearranging, we conclude
\[
\int_{B^*} |Rc|^2 \phi^2 e^{-f} dg \leq C(A, E) + C(A, E, r_0) \beta^{-2} \text{Vol}_g(A(0, 2\beta r_0))
\]
\[
\leq C(A, E) + C(A, E, r_0) \beta^2.
\]

Taking $\beta \to 0$, and recalling that $f$ is locally bounded above, we obtain $\int_{B^*} |Rc|^2 dg < \infty$. Finally, to estimate the third term of (7.1), we further decompose $W$ into the self-dual and anti-self-dual parts $W_{\pm}$, and then employ the strategy of [DS14]. Let $A_+$ be the connection on the bundle $\Lambda_+$ of self-dual forms on $\mathcal{R}$ induced by the Levi-Civita connection of $(\mathcal{R}, g)$ (see section 6.D of [Bes08] for definitions). Then, because $W_+ = 0$, $(Rc - \frac{R}{4} g) \otimes g$ is anti-self-dual, and $A_{\pm}, A_-$ are orthogonal, we have
\[
\int_{A(s, r)} \left( |W_+|^2 + \frac{R^2}{12} - |Rc - \frac{R}{4} g|^2 \right) dg = \int_{A(s, r)} tr(F_{A_+} \wedge F_{A_+})
\]

but also (by Example 2.5 of [CS85])
\[
\int_{A(s, r)} tr(F_{A_+} \wedge F_{A_+}) = CS(A_+, \partial B(x^*, r)) - CS(A_+, \partial B(x^*, s)) \quad (\text{mod } \mathbb{Z}),
\]

where
\[
CS(B, \Sigma) := \frac{1}{8\pi^2} \int_{\Sigma} tr \left( dB \wedge B + \frac{2}{3} B \wedge B \wedge B \right) \in \mathbb{R}/\mathbb{Z}.
\]

is the Chern-Simons invariant (associated to the first Pontryagin class) of a connection $\nabla = d + B$ on a trivial bundle over a 3-manifold $\Sigma$, once we have chosen an arbitrary global section of the bundle. However, by the Cheeger-Gromov convergence of $r^{-2} g|\partial B(x^*, r) \to$ a flat bundle metric on $(Tr^4) |S^3$ as $r \to \infty$, we may conclude that $A_+|\partial B(x^*, r)$ converge (after pulling back by diffeomorphisms $\psi_i : S^3 \to \partial B(x^*, r)$) to the Euclidean connection $D$ on the trivial bundle of self-dual 2-forms of $(\mathbb{R}^4 \setminus \{0\})$ restricted to $S^3$, which has Chern-Simons invariant $0$. This means
\[
CS(A_+, \partial B(x^*, r)) = CS(\psi_i^* A_+, S^3) \to CS(D, S^3) = 0
\]
as $r \downarrow 0$, so we can choose $r \in (0, r_0]$ sufficiently small such that $|CS(A_+, \partial B(x^*, s))| \leq \frac{1}{8}$ mod $\mathbb{Z}$ for all $s \in [0, r]$. Because
\[
s \mapsto \int_{A(s, r)} tr(F_{A_+} \wedge F_{A_+}) \in [-\frac{1}{4}, \frac{1}{4}] \quad (\text{mod } \mathbb{Z})
\]
is continuous, we conclude that the integral is bounded uniformly (in $\mathbb{R}$) for all $s < r$. In particular, we can take $s \downarrow 0$ to obtain $\int_{B^*} |W_+|^2 dg < \infty$, and the proof of $\int_{B^*} |W_-|^2 dg < \infty$ is similar. □

We can now argue as in [CS07], [Tia90], to conclude that in fact $B^*$ has the structure of a $C^\infty$ orbifold at $x^*$. Note that, because we have bounds on $f, |\nabla f|$ on $B^*$, the only difference in our setting is that we must use the $\epsilon$-regularity theorem that is Theorem 1.1 of [GJ17] or Theorem 1.2 of [GJ17] (note that the completeness condition can be replaced with the condition that a larger
geodesic ball is locally compact). Also, \( R + |\nabla f|^2 = f - \mathcal{W}(g, f) \), \( |R| \leq A \), and the quadratic growth of \( f \) imply that all critical points of \( f \) must occur in some bounded set. On the other hand, any orbifold point of \( X \) must be a critical point: if \( \varphi : \mathbb{R}^4/\Gamma \supseteq U \to B^X(x, \delta) \) is an orbifold chart, and \( \pi : \mathbb{R}^4 \to \mathbb{R}^4/\Gamma \) is the quotient map, then \( \nabla(\pi \circ \varphi)^*\tilde{g}(\pi \circ \varphi)^*f \) must be fixed by all of \( \Gamma \), so must be the zero vector. Since \( X \setminus \mathcal{R} \) is discrete and bounded, it must be finite. \( \square \)

**Proof of Theorem 3.** This is immediate from Theorem 7.2. \( \square \)

## 8. Appendix

In this section, we give further details for the claim in Lemma 7.1 that

\[
|B^X(y_i, \alpha \lambda_i^{-1}) \cap \mathcal{R}|_g \geq (\omega_n - \epsilon_i)(\alpha \lambda_i^{-1})^4
\]

for some sequence \( \epsilon_i \to 0 \). The main idea is to use the fact that, for \( i \in \mathbb{N} \) sufficiently large, \((B^X(y_i, \alpha \lambda_i^{-1}), \lambda_i d, y_i)\) is arbitrarily close to a Euclidean ball in the pointed Gromov-Hausdorff sense, and to then appeal to a volume convergence theorem for Riemannian manifolds with integral Ricci lower bounds.

Observe that, by Lemma 6.1 of [BZ17], we have

\[
|Rc|_{\mathcal{R}}^2(\cdot, 0) \leq C(A)(r_{\mathcal{R}}^g)^{-1}(\cdot, 0),
\]

so combining this with the integral estimate for the curvature scale (Theorem 1.7 of [Bam16]) gives

\[
\int_{B_{g_i}(x, 0, 1)} |Rc|^3(\cdot, 0) dg_i^3 \leq \int_{B_{g_i}(x, 0, 1)} (r_{\mathcal{R}}^g(\cdot, 0))^{-3} dg_i \leq C(A).
\]

Note that we actually have a local \( L^p \) bound for \( Rc \) for any \( p < 4 \), and the following arguments will work for any \( p \in (2, 4) \), but we choose \( p = 3 \) for convenience.

Let \( \mathcal{H}_d^4 = \mathcal{H}^4 \) be the 4-dimensional Hausdorff measure on the metric space \((X, d)\). Because \( \mathcal{H}^4(X \setminus \mathcal{R}) = 0 \), and because \( \mathcal{H}^4 \) agrees with the Riemannian volume measure on any 4-dimensional Riemannian manifold (in particular, on \( \mathcal{R} \)), we have \( \mathcal{H}^4(S) = |S \cap \mathcal{R}| \) for any subset \( S \subseteq X \). Thus

\[
\mathcal{H}_d^4(B^X(y_i, \alpha \lambda_i^{-1})) = \lambda_i^4|B^X(y_i, \alpha \lambda_i^{-1}) \cap \mathcal{R}|_g,
\]

\[
\mathcal{H}_d^4(B^Z(y_\infty, \alpha)) = \omega_n \alpha^n.
\]

We now restate the modification of Theorem 2.37 of [TZ16] that we will be using. Denote by \( |Rc_.|(x) \) the absolute value of the smallest negative eigenvalue of \( Rc(x) \) (if \( Rc(x) \geq 0 \), then \( |Rc_-.| = 0 \)).

**Lemma 8.1.** For any \( \kappa > 0 \), \( \Lambda < \infty \), \( n \in \mathbb{N} \), and \( p > n \), there exist \( r_0 = r_0(n, p, \kappa, \Lambda, \epsilon) > 0 \) such that the following holds. Suppose \((M^n_i, g_i, x_i)\) is a sequence of complete Riemannian manifolds satisfying:

(i) \( \int_{B(x, 1)} |Rc_-.|^p dg \leq \Lambda \) for all \( x \in M_i \),

(ii) \( |B(x, r)| \geq \kappa r^n \) for all \( r \in (0, 1] \), \( x \in M \).

Assume that \((M^n_i, g_i, x_i)\) converge in the pointed Gromov-Hausdorff sense to the complete metric length space \((X, d, p)\). Then, for any \( r \in (0, r_0] \), we have

\[
\mathcal{H}_d^4(B(x, r)) = \lim_{i \to \infty} |B(x_i, r_i)|.
\]

The difference between this lemma and Theorem 2.37 of [TZ16] is that we only require a local integral Ricci bound (i) rather than the global bound

\[
\int_M |Rc_.|^p dg \leq \Lambda
\]

assumed in [TZ16]. However, in [TZ16], the objects under consideration are time slices of a normalized Ricci flow on a Fano threefold, which have uniformly bounded diameter. The proof of
Theorem 2.37 is stated to be a modification of volume convergence for noncollapsed Riemannian manifolds with Ricci curvature bounded below, given in [Col97; CC96]. A careful examination of the proof shows that only the conditions (i), (ii) are used, essentially due to the fact that the involved arguments are all local.

The following elementary lemma is essentially a consequence of Lemma 22 and a diagonal argument.

**Lemma 8.2.** Let \((X_k, d_k, p_k)\) be a sequence of limit spaces as in Lemma 22, converging in the pointed Gromov-Hausdorff sense to \((X, d, p)\), and suppose \(r \leq r_0(n, p, \kappa, \Lambda)\). Then

\[
\mathcal{H}^n(B(p_k, r)) \to \mathcal{H}^n(B(p, r)).
\]

**Proof.** For each \(i \in \mathbb{N}\), let \((M_{k,i}, g_{k,i}, x_{k,i})\) be a sequence of complete, pointed Riemannian manifolds satisfying (i), (ii) of Lemma 8.1, which converge in the pointed Gromov-Hausdorff sense to \((X_k, d_k, x_k)\) as \(i \to \infty\). Also let \((M_i, g_i, x_i)\) be a sequence of such manifolds converging to \((X, d, p)\) in the pointed Gromov-Hausdorff sense. By Lemma 8.1, we know that

\[
\lim_{i \to \infty} |B(x_{k,i}, r)|_{g_{k,i}} = \mathcal{H}^n(B(x, r))
\]

for each \(k \in \mathbb{N}\). Thus, for each \(k \in \mathbb{N}\), we can find \(i(k) \in \mathbb{N}\) such that

\[
|B(x_{k,i(k)}, r)|_{g_{k,i(k)}} - \mathcal{H}^n(B(x, r)) \leq 2^{-k},
\]

\[
d_{GH}(B(x_{k,i(k)}, \alpha_k r), d_{g_{k,i(k)}}, x_{k,i(k)}), (B(x_k, \alpha_k r), d_k, x_k)) \leq r 2^{-k},
\]

where \(\alpha_k \to \infty\). In particular, \((M_{k,i(k)}, g_{k,i(k)}, x_{k,i(k)})\) converge in the pointed Gromov-Hausdorff sense to \((X, d, p)\), so

\[
|B(x_{k,i(k)}, r)|_{g_{k,i(k)}} - \mathcal{H}^n(B(x, r)) \to 0.
\]

Combining expressions gives the claim.

After possibly shrinking \(\alpha\) so that \(\alpha < r_0\), we can apply the previous lemma to

\[
(B^X(y_i, \alpha \lambda_i^{-1}), \lambda_i d, y_i) \to (B^Z(y_\infty, \alpha), d_Z, y_\infty),
\]

we conclude that

\[
\mathcal{H}^4(B^X(y_i, \alpha \lambda_i^{-1}) \to \mathcal{H}^4(B^Z(y_\infty, \alpha))
\]

as \(i \to \infty\). That is,

\[
\lim_{i \to \infty} \lambda_i^4 |B^X(y_i, \alpha \lambda_i^{-1}) \cap \mathcal{R}| = \omega_n \alpha^4,
\]

so the claim follows.

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