THE DIFFERENTIAL SEMANTICS OF ŁUKASIEWICZ SYNTACTIC CONSEQUENCE

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Abstract. The classical condition "ϕ is a semantic consequence of Θ" in infinite-valued propositional Łukasiewicz logic $L_\infty$ is refined using enriched valuations that take into account the effect on ϕ of the stability of the truth-value of all $\theta \in \Theta$ under small perturbations (or, measurement errors) of the models of Θ. The differential properties of the functions represented by ϕ and by all $\theta \in \Theta$ naturally lead to a new notion of semantic consequence $\models_\delta$ that turns out to coincide with syntactic consequence $\vdash$.

1. Prelude: semantics for Hájek propositional basic logic

Basic logic (BL) was invented by Hájek to formalize continuous t-norms. Certain axioms satisfied by any such t-norm were singled out in [10, 2.2.4]; provability of a formula $\phi$, as well as provability of $\phi$ from a set $\Theta$ of premises, were defined via Modus Ponens, in the usual way, [10, 2.2.17]. BL-algebras, BL-evaluations of formulas, and satisfiability, were then defined in [10, 2.3.3] and [10, 2.3.8], and the following completeness theorem was proved in [10, 2.3.19]:

1.1. A formula $\phi$ is provable iff every BL-evaluation satisfies $\phi$.

The following strong completeness theorem directly follows from [10, 2.4.3]:

1.2. For any formula $\phi$ and set $\Theta$ of formulas, $\phi$ is provable from $\Theta$ iff every BL-evaluation satisfying all $\theta \in \Theta$ also satisfies $\phi$, in symbols, $\Theta \models_{BL} \phi$.

Yet in [10, 2.3.23] Hájek champions a different semantics for BL. Let us agree to say that $\phi$ is a t-tautology if $\phi$ is satisfied by every evaluation of $\phi$ into a BL-algebra arising from a t-norm. The resulting t-tautology semantics is more adherent to the original motivation of BL-logic: for, Hájek’s BL-axioms in [10, Definition 2.2.4] are the result of his contemplation of continuous t-norms. The question arises: do the BL-axioms prove all t-tautologies? The problem whether BL is the logic of continuous t-norms is again posed in a final section ([10, 9.4.6]).

In the same pages [10, 9.4.1], it is noted that the traditional semantic consequence relation $\models$ in $L_\infty$ fails to be strongly complete. A counterexample is given in [10, 3.2.14]; stated otherwise, $\models$ is not compact, despite model-sets Mod($\psi$) of $L_\infty$-formulas $\psi(X_1, \ldots, X_n)$ are compact subsets of the unit n-cube $[0,1]^n$, and compactness has a pervasive role in MV-algebra theory, [6],[15].

One is then left with two rather similar problems involving the mutual role of syntax vs. semantics in BL and in $L_\infty$:

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Date: May 2, 2014.

Key words and phrases. Basic logic, t-norm, t-tautology, Łukasiewicz logic, consequence relation, syntactic and semantic consequence, MV-algebra, strongly semisimple, Bouligand-Severi tangent.

2000 Mathematics Subject Classification. Primary: 06D35 Secondary: 03B50, 03B52, 47N10, 49J52, 94D05.
(A) *Fixed semantics, amendable axioms.* In case BL were not complete for tautology semantics, how to strengthen the BL-axioms to obtain a strongly complete logic for continuous t-norms?

(B) *Fixed axioms, amendable semantics.* It being ascertained that \([0, 1]\)-valuations fail to yield a strongly complete semantics for \(L_\infty\), what new notion of “model” of a set of \(L_\infty\)-formulas, should be devised to get a strongly complete semantics?

In [11] Hájek himself gave the first substantial contribution to Problem (A), by adding to BL two (admittedly not too simple) axioms which, at the time of [10, 2.3.23] and [11] were not guaranteed to follow from the BL-axioms. The redundancy of these two axioms was finally proved in [7, 5.2], thus solving Problem (A) in the best possible way: the logic originally invented by Hájek is indeed strongly complete for valuations in t-algebras, the subset of BL-algebras directly given by continuous t-norms.

Since the strong completeness of \([0, 1]\)-valuations has been settled in the negative, and the Łukasiewicz axioms are here to stay, in order to solve Problem (B) we are left with no other choice but to modify the semantics of \(L_\infty\), looking for a novel, genuinely semantical notion of \([0, 1]\)-valuation. This is our aim in this paper.

2. Tangents, differentials and semantic consequence relations in \(L_\infty\)

We refer to [6] and [15] for notation and background on MV-algebras and infinite-valued Łukasiewicz propositional logic \(L_\infty\). The set \(\text{FORM}_n\) of \(L_\infty\)-formulas in the variables \(X_1, \ldots, X_n\) has the same definition as its boolean counterpart. The Łukasiewicz connectives \(\circ, \oplus\) of conjunction and disjunction are definable in terms of negation \(\neg\) and implication \(\rightarrow\). While in boolean logic formulas take their values in the set \([0, 1]\), \(L_\infty\)-formulas are evaluated in the unit real interval \([0, 1]\). Let \(\text{VAL}_n \subseteq [0, 1]^{\text{FORM}_n}\) denote the set of valuations (also known as evaluations, assignments, models, interpretations, possible worlds, . . . ). The *truth-functionality* property of \(L_\infty\) yields the following crucial identification:

2.1. The set \(\text{VAL}_n\) can be identified with the unit \(n\)-cube \([0, 1]^n\) via the restriction map \(V \in \text{VAL}_n \mapsto v = V \mid \{X_1, \ldots, X_n\} \in [0, 1]^{\{X_1, \ldots, X_n\}} = [0, 1]^n\).

For any fixed formula \(\phi \in \text{FORM}_n\), the map \(V \in \text{VAL}_n \mapsto V(\phi) \in [0, 1]\) defines the function \(\hat{\phi}: [0, 1]^n \rightarrow [0, 1]\) by \(\hat{\phi}(v) = V(\phi)\). The continuity and piecewise linearity of \(\hat{\phi}\) easily follow by induction on the number of connectives in \(\phi\).

2.2. Following Bolzano and Tarski (see [18, footnote on page 417]), \(L_\infty\) is now equipped with the relation \(\models\) of semantic consequence by stipulating that for all \(\Theta \subseteq \text{FORM}_n\) and \(\phi \in \text{FORM}_n\), \(\Theta \models \phi\) iff \(\forall v \in [0, 1]^n, \ (\hat{\phi}(v) = 1 \text{ for all } \theta \in \Theta \Rightarrow \hat{\phi}(v) = 1)\).

Mutatis mutandis, this notion of consequence is gratified by a completeness theorem in classical logic and in many nonclassical logics having totally disconnected valuation spaces. However,

2.3. The space \([0, 1]^n\) of valuations in \(L_\infty\) is connected. For every \(\phi \in \text{FORM}_n\), valuation \(v \in [0, 1]^n\) and unit vector \(u \in \mathbb{R}^n\) such that \(\text{conv}(v, v + \epsilon u) \in [0, 1]^n\) for all small \(\epsilon > 0\), the directional derivative \(\partial \hat{\phi}(v)/\partial u\) exists and varies continuously with \(u\), once \(v\) is kept fixed.

The following simple example involving formulas of one variable already shows that the differential properties of \(\hat{\phi}\) for all \(\theta \in \Theta\) are ignored by the semantic consequence relation \(\models\) of 2.2, although they have no less semantical content than the truth-value \(\hat{\theta}(v)\) :
2.4. Suppose $\Theta \subseteq \text{FORM}_1$ is satisfied by a unique valuation $v \in [0, 1]$, and $1 > v \in \mathbb{Q}$. Suppose $\partial \theta(v)/\partial x^+ = 0$ for all $\theta \in \Theta$. Let $\phi = \phi(X)$ be a formula with $\partial \phi(v) = 1$ and $\partial \phi(w) < 1$ for all $w > v$. Then $\Theta \models \phi$, although $\partial \phi(v)/\partial x^+ < 0$.

Intuitively, the hypothesis means that each $\theta \in \Theta$ is not only true at $v$, but is also true for all $w > v$ sufficiently close to $v$; in other words, $\theta$ is “stably” true at $v$, even if the value of $v$ were known up to a certain small error (depending on $\theta$). Although $\phi$ misses this (fault-tolerant) stability property of all $\theta \in \Theta$, $\phi$ is a semantic consequence of $\Theta$, $\Theta \models \phi$. It should be noted that $\Theta \not\models \phi$. Similarly, when $n > 1$ and $\Theta \subseteq \text{FORM}_n$, the higher-order stability properties common to all $\theta \in \Theta$ may be missing in some semantic consequence $\phi$ of $\Theta$. And again, $\Theta \not\models \phi$.

While directional derivatives make no sense in boolean logic, by 2.3 they do make sense in $L_\infty$. Accordingly, in 3.7 we will give a precise definition of “stable” consequence relation $\models_\vartheta$ which is sensitive to all higher order differentiability properties of formulas and their associated piecewise linear functions. In Section 7 this will be generalized to arbitrary (possibly uncountable) sets $\Theta$ of formulas. In 3.9 we prove that $L_\infty$ is “strongly complete” with respect to $\models_\vartheta$: indeed, $\Theta \models_\vartheta \phi$ coincides with the syntactical consequence relation $\Theta \vdash \phi$.

We then focus on the relative status of $\models_\vartheta$ with respect to $\models$. As noted in [6, p.100 and 4.6.6], from Chang completeness theorem we have

2.5. The two sets $\Theta^{=\vartheta}$ and $\Theta^=\vartheta$ of semantic and syntactic consequences of a set $\Theta$ of formulas coincide iff the Lindenbaum algebra $\text{LIND}(\Theta)$ is semisimple.

2.6. Following Dubuc and Poveda [9], we say that an MV-algebra is strongly semisimple if all its principal quotients are semisimple.

Let $\Theta \subseteq \text{FORM}_n$. Building on [5], in 4.3 we observe that $\text{LIND}_\Theta$ is strongly semisimple if $(\Theta \cup \{\psi\})^{=\vartheta} = (\Theta \cup \{\psi\})^{\models_\vartheta}$ for all $\psi \in \text{FORM}_n$. Further, when $\Theta \subseteq \text{FORM}_1$, $\text{LIND}_\Theta$ is strongly semisimple iff it is semisimple. Now suppose $\text{LIND}_\Theta$ is semisimple, with $\Theta \subseteq \text{FORM}_2$. Then $\text{LIND}_\Theta$ is strongly semisimple iff the set Mod$(\Theta) \subseteq [0, 1]^2$ of valuations satisfying $\Theta$ has no Bouligand-Severi [2, 17] outgoing rational tangent vector at any rational point $v \in \text{Mod}(\Theta)$. See 5.4. As shown in 5.5, the existence of a Bouligand-Severi rational outgoing tangent at some rational point $v$ of Mod$(\Theta)$ entails failure of strong semisimplicity in the semisimple MV-algebra $\text{LIND}(\text{Th}(\text{Mod}(\Theta)))$.

In a final section Problems (A) and (B) are retrospectively considered in the light of the results of the previous sections.

3. Semantic consequence $\models$ and stable consequence $\models_\vartheta$

The following corollary of Chang’s completeness theorem is proved in [6, 3.1.4]:

3.1. For each $n = 1, 2, \ldots$, the free $n$-generator MV-algebra $\mathcal{M}([0, 1]^n)$ consists of all functions $f : [0, 1]^n \to [0, 1]$ that are obtainable from the coordinate functions $\pi_i(x_1, \ldots, x_n) = x_i$ by pointwise application of the MV-algebraic operations of negation $\neg x = 1 - x$ and truncated addition $x \oplus y = \min(1, x + y)$. As already noted in 2.1, any such function $f$ is continuous and piecewise linear.

For any nonempty closed set $X \subseteq [0, 1]^n$ we let $\mathcal{M}(X)$ denote the MV-algebra of restrictions to $X$ of the functions in $\mathcal{M}([0, 1]^n)$, in symbols, $\mathcal{M}(X) = \{f \upharpoonright X \mid f \in \mathcal{M}([0, 1]^n)\}$. McNaughton’s characterization [6, 9.1.5] of the free MV-algebra $\mathcal{M}([0, 1]^n)$ will find no use in this paper.

In [6, 3.6.7] one can find a proof of the following result, which follows from the proof of Chang’s completeness theorem:
3.2. $\mathcal{M}(X)$ is a semisimple MV-algebra—actually, up to isomorphism, $\mathcal{M}(X)$ is the most general possible $n$-generator semisimple MV-algebra.

For every subset $Y$ of $[0, 1]^n$, conv($Y$) denotes the convex hull of $Y$. To solve Problem (B) we modify the classical notion of valuation as follows:

3.3. For $n = 1, 2, \ldots$ and $0 \leq t \leq n$ let $U = (u_0, u_1, \ldots, u_t)$ be a $(t+1)$-tuple of elements of $\mathbb{R}^n$ where $u_1, \ldots, u_t$ are pairwise orthogonal unit vectors. For each $m = 1, 2, \ldots$ let the $t$-simplex $T_{U,m} \subseteq \mathbb{R}^n$ be defined by

$$T_{U,m} = \text{conv}(u_0, u_0 + u_1/m, u_0 + u_1/m + u_2/m^2, \ldots, u_0 + u_1/m + \cdots + u_t/m^t).$$

We say that $U$ is a differential valuation (of order $t$, in $\mathbb{R}^n$) if for all large $m$ the $n$-cube $[0, 1]^n$ contains $T_{U,m}$. When this is the case, the set $p_U \subseteq \mathcal{M}([0, 1]^n)$ is defined by $p_U = \{ f \in \mathcal{M}([0, 1]^n) \mid f^{-1}(0) \supseteq T_{U,m} \text{ for some } m \}.$

Traditional valuations coincide with differential valuations of order 0.

3.4. Let $U = (u_0, u_1, \ldots, u_t)$ be a differential valuation in $\mathbb{R}^n$.

(i) For all $m = 1, 2, \ldots$, $T_{U,m} \supseteq T_{U,m+1}$.

(ii) For every $\epsilon_1, \ldots, \epsilon_t > 0$ there is $m = 1, 2, \ldots$ such that the simplex

$$S = \text{conv}(u_0, u_0 + \epsilon_1 u_1, u_0 + \epsilon_1 u_1 + \epsilon_2 u_2, \ldots, u_0 + \epsilon_1 u_1 + \cdots + \epsilon_t u_t)$$

contains $T_{U,m}$.

(iii) $p_U$ is a prime ideal of $\mathcal{M}([0, 1]^n)$.

(iv) Every prime ideal $p$ of $\mathcal{M}([0, 1]^n)$ has the form $p = p_V$ for some differential valuation $V$.

Proof. (i)-(ii) are easily verified by induction. For (iii)-(iv) use (ii) and see [4, 2.8, 2.18].

For every convex set $E \subseteq [0, 1]^n$ we let relint($E$) denote its relative interior. The prime ideals $p_{U,m}$ of $\mathcal{M}([0, 1]^n)$ are conveniently visualized as follows:

3.5. Let $U = (u_0, u_1, \ldots, u_t)$ be a differential valuation in $\mathbb{R}^n$. We then have:

0. $p_{(u_0)}$ is the maximal ideal of $\mathcal{M}([0, 1]^n)$ given by all functions of $\mathcal{M}([0, 1]^n)$ that vanish at $u_0$.

1. $p_{(u_0, u_1)}$ is the prime ideal of $\mathcal{M}([0, 1]^n)$ given by all functions $f \in \mathcal{M}([0, 1]^n)$ vanishing on an interval of the form $\text{conv}(u_0, u_0 + u_1/m)$ for some integer $m > 0$. Equivalently, $f(u_0) = 0$ and $\partial f(u_0)/\partial u_1 = 0$.

2. $p_{(u_0, u_1, u_2)}$ is the prime ideal of $\mathcal{M}([0, 1]^n)$ given by those $f \in \mathcal{M}([0, 1]^n)$ such that for some integer $m > 0$, $f$ vanishes on the segment $\text{conv}(u_0, u_0 + u_1/m)$, and $\partial f(y)/\partial u_2 = 0$ for all $y \in \text{relint}(\text{conv}(u_0, u_0 + u_1/m))$.

And inductively,

$t$. $p_{(u_0, u_1, \ldots, u_t)}$ is the prime ideal of $\mathcal{M}([0, 1]^n)$ consisting of all $f \in \mathcal{M}([0, 1]^n)$ such that for some integer $m > 0$, $f$ vanishes on the $(t-1)$-simplex

$$S = \text{conv} \left( u_0, u_0 + u_1/m, u_0 + u_1/m + u_2/m^2, \ldots, u_0 + u_1/m + \cdots + u_{t-1}/m^{t-1}, \right) ,$$

and $\partial f(y)/\partial u_{t} = 0$ for all $y \in \text{relint}(S)$.

Observe that $p_{(u_0)} \supseteq p_{(u_0, u_2)} \supseteq \cdots \supseteq p_{(u_0, u_1, \ldots, u_{t-1})} \supseteq p_{(u_0, u_1, \ldots, u_t)}$.

Generalizing the classical definitions we can now write:
3.6. Let \( U = (u_0, u_1, \ldots, u_t) \) be a differential valuation in \( \mathbb{R}^n \). Let \( \psi(X_1, \ldots, X_n) \) be a formula. We then say that \( U \) satisfies \( \psi \) if \( 1 - \hat{\psi} \in p_U \). Thus
\[
\hat{\psi}(u_0) = 1, \quad \frac{\partial \hat{\psi}(u_0)}{\partial u_i} = 0, \ldots, \quad \text{and } \hat{\psi} \text{ satisfies Conditions (2) through (t) in 3.5.}
\]

3.7. For \( \Theta \subseteq \text{FORM}_n \) and \( \psi \in \text{FORM}_n \) we say that \( \psi \) is a **stable consequence** of \( \Theta \) and we write
\[
\Theta \models_{\partial} \psi
\]
if \( \psi \) is satisfied by every differential valuation \( (u_0, u_1, \ldots, u_t) \) that satisfies every \( \theta \in \Theta \).

Observe that \( \Theta \models_{\partial} \psi \) in the sense of 2.2 iff \( \psi \) is satisfied by every differential valuation of order 0 satisfying \( \Theta \). Therefore,

3.8. Let \( \Theta \subseteq \text{FORM}_n \) and \( \psi \in \text{FORM}_n \). If \( \Theta \models_{\partial} \psi \then \Theta \models \psi \).

The strong completeness property of the stable consequence relation \( \models_{\partial} \) amounts to the following:

3.9. \( \Theta \models_{\partial} \psi \iff \Theta \vdash \psi \).

Proof. Following [6, 4.2.7], let \( j_\Theta = \langle \{ 1 - \hat{\theta} \mid \theta \in \Theta \} \rangle \) be the ideal of \( \mathcal{M}([0,1]^n) \) generated by the functions given by all negations of formulas in \( \Theta \). Equivalently, \( j_\Theta \) is the ideal generated by the congruence \( \equiv_\Theta \) of [15, 1.11]. Then

\[
\Theta \vdash \psi \iff 1 - \hat{\psi} \in j_\Theta, \quad [6, 4.2.9] \text{ or [15, 1.9]}
\]

\[
\iff 1 - \hat{\psi} \text{ belongs to every prime ideal } p \supseteq j_\Theta, \quad \text{by subdirect representation, [6, 1.2.14]}
\]

\[
\iff 1 - \hat{\psi} \text{ belongs to every prime } p \text{ such that } 1 - \hat{\theta} \in p \text{ for all } \theta \in \Theta, \quad \text{by definition of } j_\Theta
\]

\[
\iff \text{for every differential valuation } U \text{ in } \mathbb{R}^n, \text{ if } 1 - \hat{\theta} \in p_U \text{ for all } \theta \in \Theta \text{ then } 1 - \hat{\psi} \in p_U, \text{ by 3.4 (iii)-(iv)}
\]

\[
\iff \psi \text{ is satisfied by all differential valuations } U \text{ satisfying all } \theta \in \Theta, \quad \text{by 3.6}
\]

\[
\iff \Theta \models_{\partial} \psi, \quad \text{i.e., } \psi \text{ is a stable consequence of } \Theta, \text{ by 3.7.}
\]

The “finitary” character of \( \models_{\partial} \), as opposed to the non-compactness of \( \models \), is made precise by the following corollary of 3.9:

3.10. Let \( \Theta \subseteq \text{FORM}_n \) and \( \psi \in \text{FORM}_n \). Then \( \Theta \models_{\partial} \psi \iff \{ \theta_1, \ldots, \theta_k \} \models_{\partial} \psi \text{ for some finite subset } \{ \theta_1, \ldots, \theta_k \} \text{ of } \Theta. \)

Since \( \text{FORM}_n \subseteq \text{FORM}_{n+1} \), one might ask if \( \Theta \models_{\partial} \psi \) depends on \( n \), so that a more accurate notation would be \( \Theta \models_{(n,\partial)} \psi \). The following immediate corollary of 3.9 shows that such extra notation is unnecessary:

3.11. Let \( \Theta \subseteq \text{FORM}_n \) and \( \psi \in \text{FORM}_n \). Then for any \( m \geq n \), \( \Theta \models_{(n,\partial)} \psi \iff \Theta \models_{(m,\partial)} \psi \).
4. Strong semisimplicity and $\models_0$

Recall from 2.6 the definition of strongly semisimple MV-algebra. Since $\{0\}$ is a principal ideal of $A$, every strongly semisimple MV-algebra is semisimple.

4.1. All boolean algebras are strongly semisimple, and so are all simple and all finite MV-algebras.

Proof. Boolean algebras are hyperarchimedean [6, 6.3]. The second statement follows from [6, 3.5 and 3.6.5].

The set $\Theta^{=_0} \subseteq \text{FORM}_n$ is defined by $\Theta^{=_0} = \{ \psi \in \text{FORM}_n \mid \Theta \models_0 \psi \}$.

4.2. Let $\Theta \subseteq \text{FORM}_n$. Then $\text{LIND}(\Theta)$ is semisimple iff $\Theta^{=_0} = \Theta^\perp$. Thus $\text{LIND}(\Theta)$ is not semisimple iff there is $\psi \in \text{FORM}_n$ such that every differential valuation of order 0 satisfying $\Theta$ satisfies $\psi$, and there is a differential valuation $U$ satisfying $\Theta$ but not $\psi$.

Proof. [6, p.100] and 3.9 above.

4.3. Let $\Theta \subseteq \text{FORM}_n$. Then $\text{LIND}(\Theta)$ is strongly semisimple iff for all $\psi \in \text{FORM}_n$, $(\Theta \cup \{ \psi \})^{=_0} = (\Theta \cup \{ \psi \})^\perp$.

Proof. For any MV-algebra $A$ and ideal $j$ of $A$, the quotient map

$$i \mapsto i/j = \{b/i \mid b \in i\}$$

determines a 1-1 correspondence between ideals of $A$ containing $j$ and ideals of $A/j$, [6, 1.2.10]. A well known result in universal algebra, [8, 3.11], yields an isomorphism

$$\frac{a}{i} \in A \mapsto \frac{a/j}{i/j} \in A/j$$

(2)

For any $S \subseteq A$ let $\langle S \rangle$ denote the (possibly not proper) ideal of $A$ generated by $S$. When $S$ is a singleton $\{a\}$ we write $\langle a \rangle$ instead of $\{ \langle a \rangle \}$. For $j$ an ideal of $A$ we use the self-explanatory notation $S/j$ for $\{b/j \mid b \in S\}$. For any $a \in A$ we have the trivial identity

$$\langle a/j \rangle = \langle \{ a \} \rangle = \langle a \rangle$$

(3)

For any element $a/j \in A/j$, letting $\langle a/j \rangle$ be the ideal generated in $A/j$ by $a/j$, a routine exercise shows

$$\langle a/j \rangle = \langle a \rangle/j = \{b/j \mid b \leq m \cdot a \text{ for some } m = 1, 2, \ldots \}.$$ 

(4)

Here are using the notation $m \cdot a$ of [6, p.33] or [15, p.21] for $m$-fold truncated addition.

To complete the proof, for any $\Theta'$ with $\Theta \subseteq \Theta' \subseteq \Theta^\perp$ we have $\text{LIND}(\Theta) = \text{LIND}(\Theta') = \text{LIND}(\Theta^\perp)$, whence it is no loss of generality to assume $\Theta = \Theta^\perp$. The set $\{1 - \hat{\theta} \mid \theta \in \Theta\}$ is automatically an ideal $1_\Theta$ of $\mathcal{M}([0,1]^n)$ and we have the isomorphism

$$\iota : \frac{\psi}{\equiv_\Theta} \in \text{LIND}(\Theta) \cong \frac{1 - \hat{\psi}}{1_\Theta} \in \mathcal{M}([0,1]^n)/1_\Theta.$$ 

It follows that the principal ideal $\langle \psi/\equiv_\Theta \rangle$ of $\text{LIND}(\Theta)$ generated by the element $\psi/\equiv_\Theta \in \text{LIND}(\Theta)$ corresponds via $\iota$ to the principal ideal $\langle (1 - \hat{\psi})/1_\Theta \rangle$ generated by the element $\iota(\psi/\equiv_\Theta) = (1 - \hat{\psi})/1_\Theta \in \mathcal{M}([0,1]^n)/1_\Theta$. By (3)-(4) we have the identities

$$\langle 1 - \hat{\psi} \rangle_{1_\Theta} = \langle 1 - \hat{\psi} \rangle = \langle 1_\Theta \cup \{ 1 - \hat{\psi} \} \rangle_{1_\Theta}.$$
Therefore, $\text{LIND}(\Theta)$ is strongly semisimple iff so is $\mathcal{M}([0,1]^n)/j_\Theta$ iff for any principal ideal $(j_\Theta \cup \{1 - \psi\})/j_\Theta$ of $\mathcal{M}([0,1]^n)/j_\Theta$, the quotient

$$\frac{\mathcal{M}([0,1]^n)/j_\Theta}{(j_\Theta \cup \{1 - \psi\})/j_\Theta} \cong \frac{\mathcal{M}([0,1]^n)}{(j_\Theta \cup \{1 - \psi\})}$$

is semisimple. We are using (2). This is the same as saying that $\text{LIND}(\Theta \cup \{\psi\})$ is semisimple for every $\psi \in \text{FORM}_n$. Now apply 4.2. \qed

4.4. For every finite set of $L_\infty$-formulas $\Phi$, the Lindenbaum algebra $\text{LIND}_\Phi$ is strongly semisimple.

Proof. In view of 4.3, this is a reformulation of a result by Hay [12] and Wójcicki [19] (also see [6, 4.6.7] and [15, 1.6]), stating that every finitely presented MV-algebra is strongly semisimple. \qed

By a quirk of fate, when $n = 1$ strong semisimplicity boils down to semisimplicity (see ([5]) for a proof):

4.5. Let $\Theta \subseteq \text{FORM}_1$. Then $\text{LIND}(\Theta)$ is strongly semisimple iff it is semisimple.

5. Strong semisimplicity, $\models_\Theta$ and Bouligand-Severi tangents

While the strong semisimplicity of $\text{LIND}(\Theta)$ is formulated in purely algebraic terms, a deeper understanding of this property follows from an exploration of the tangent space of $\text{Mod}(\Theta)$ as a compact subset of euclidean space $\mathbb{R}^n$.

A point $x \in \mathbb{R}^n$ is said to be rational if so are all its coordinates. By a rational vector we mean a nonzero vector $w \in \mathbb{R}^n$ such that the line $\mathbb{R}w = \{\lambda w \in \mathbb{R}^n \mid \lambda \in \mathbb{R}\} \subseteq \mathbb{R}^n$ contains a rational point of $\mathbb{R}^n$ other than the origin. Any nonzero scalar multiple of a rational vector is a rational vector.

As usual, $||v||$ is the length of vector $v \in \mathbb{R}^n$.

The following definitions go back to the late twenties and early thirties of the past century, and prove very useful to understand the geometry of strong semisimplicity, and its relationship with stable consequence:

5.1. ([16, §53, p.59 and p.392], [17, §1, p.99], [2, p.32]) A half-line $H \subseteq \mathbb{R}^n$ is tangent to a set $X \subseteq \mathbb{R}^n$ at an accumulation point $x$ of $X$ if for all $\epsilon, \delta > 0$ there is $y \in X$ other than $x$ such that $||y - x|| < \epsilon$, and the angle between $H$ and the half-line through $y$ originating at $x$ is $< \delta$.

5.2. ([3, p.16]) Let $x$ be an element of a closed subset $X$ of $\mathbb{R}^n$, and $u$ a unit vector in $\mathbb{R}^n$. We then say that $u$ is a Bouligand-Severi tangent (unit) vector to $X$ at $x$ if $X$ contains a sequence $x_0, x_1, \ldots$ of elements, all different from $x$, such that

$$\lim_{i \to \infty} x_i = x \quad \text{and} \quad \lim_{i \to \infty} (x_i - x)/||x_i - x|| = u.$$ 

We further say that $u$ is outgoing if the open interval $\text{relin}(\text{conv}(x, x + \lambda u))$ is disjoint from $X$ for some $\lambda > 0$.

5.3. ([17, §5, p.103]). For any nonempty closed subset $X$ of $\mathbb{R}^n$, point $x \in X$, and unit vector $u \in \mathbb{R}^n$ the following conditions are equivalent:

(i) For all $m = 1, 2, \ldots$, the cone

$$C_{x,u,1/m,1/m^2}$$

with apex $x$, axis parallel to $u$, height $1/m$ and vertex angle $1/m^2$ contains infinitely many points of $X$. 


(ii) $u$ is a Bouligand-Severi tangent vector to $X$ at $x$.

(iii) The half-line $x + \mathbb{R}_{\geq 0}u$ is tangent to $X$.

5.4. ([5]) Let $\Theta \subseteq \text{FORM}_2$. Suppose $\text{LIND}(\Theta)$ is semisimple. Then $\text{LIND}(\Theta)$ is strongly semisimple iff $\text{Mod}(\Theta)$ does not have any Bouligand-Severi outgoing rational tangent vector at any of its rational points.

Combining [5] with our characterization 4.2 we get

5.5. Let $\Theta \subseteq \text{FORM}_n$. Suppose $\text{LIND}(\Theta)$ is semisimple and $\text{Mod}(\Theta)$ has some Bouligand-Severi outgoing rational tangent vector $u$ at some rational point $v \in \text{Mod}(\Theta)$. Then $\text{LIND}(\Theta)$ is not strongly semisimple. There are formulas $\gamma, \lambda \in \text{FORM}_n$ such that $\Theta \cup \{\gamma\} \models \lambda$ but it is not the case that $\Theta \cup \{\gamma\} \models \lambda$. Specifically, while every stable consequence $\psi$ of $\Theta \cup \{\gamma\}$ satisfies $\hat{\psi}(v) = 1$ and $\partial \hat{\psi}(v)/\partial u = 0$, for $\lambda$ we have $\hat{\lambda}(v) = 1$ and $\partial \hat{\lambda}(v)/\partial u < 0$.

As in [15, 1.3, 1.4], the operator $\text{Th}: X \subseteq [0, 1]^n \mapsto \text{Th}X \subseteq \text{FORM}_n$ is defined by

$$\text{Th}X = \{ \psi \in \text{FORM}_n \mid \hat{\psi}(w) = 1 \text{ for all } w \in X \}.$$ 

5.6. If there exists a Bouligand-Severi rational outgoing tangent vector at some rational point $v$ of $\text{Mod}(\Theta)$ then $\text{LIND}(\text{Th}(\text{Mod}(\Theta)))$ is semisimple but not strongly semisimple.

Proof. The MV-algebra $\text{LIND}(\text{Th}(\text{Mod}(\Theta)))$ is semisimple because $\text{Th}(\text{Mod}(\Theta)) = \Theta^{\text{free}}$. It is not strongly semisimple by [5].

Thus the strong semisimplicity of $\text{LIND}(\text{Th}(\text{Mod}(\Theta)))$, and more generally, of every $\Phi \subseteq \text{FORM}_n$ with $\Phi^{\models \text{free}}$, only depends on the (tangent space of the) set $\text{Mod}(\Theta) \subseteq [0, 1]^n$.

6. Concluding remarks

As shown by the examples of BL and $L_\infty$, in the beginning we are given a syntactic consequence relation $\vdash$ based on a set $R$ of axioms and rules. Then variously defined “semantic” consequence relations are tailored around $\vdash$, until a strongly complete semantic consequence relation is obtained in terms of a certain set $V^*$ of valuations: in the case of BL, $V^*$ turns out to be the subset of BL-valuations given by $t$-algebraic valuations; in the case of $L_\infty$, $V^*$ is the set of differential valuations, which contains the set of $[0, 1]$-valuations as the special 0-order case.

Historically, the emergence of semantical notions in first-order logic followed a similar path. Here a long distillation process culminated in a definitive consequence relation $\vdash$. At a later stage, motivation/confirmation of the definitive nature of $\vdash$ would be provided by suitably defined “models” (interpretations, substitutions, evaluations, possible worlds,...). Without them one cannot even speak of the correctness of the set $R$ of rules of first order logic. The completeness problem had a long gestation period. The notions of categoricity and completeness of theories were often confused with the completeness of the set $R$ of rules. Before the appearance of Tarskian models over arbitrary universes the set of arithmetical models over the fixed universe $\mathbb{N}$ was used to evaluate formulas.

Turning retrospectively to Problems (A) and (B), in the introduction we didn’t mention the following well known fact ([15, 20.7]):

6.1. For each $i = 1, 2$ and any (possibly uncountable) set $\Theta$ of formulas, let $\Theta \models_{\text{MV}_i} \phi$ be given by the following stipulation:

(I) $\Theta \models_{\text{MV}_1} \phi$ iff every $A$-valuation satisfying every $\theta \in \Theta$ also satisfies $\phi$, where $A$ ranges over arbitrary MV-algebras.
(II) $\Theta \models_{MV2} \phi$ if every $C$-valuation satisfying every $\theta \in \Theta$ also satisfies $\phi$, where $C$ ranges over arbitrary MV-chains.

Then $\models_{MV1} = \models_{MV2}$ is the syntactic consequence relation $\vdash$ of $L_\infty$.

Each consequence relation $\models_{MVi}$, while endowing $L_\infty$ with a strongly complete semantics has the same drawbacks as the consequence relation $\models_{BL}$ arising from all BL-valuations in 1.1-1.2: since $\models_{MVi}$ does not directly reflect the intuition behind the original axioms, its applicability is limited.

Consider, for instance, the complexity of the problem whether $\alpha \vdash \beta$, for $\alpha, \beta \in \bigcup_n \text{FORM}_n$. The binary relation

$$\vdash_{\text{fin}} = \left( \bigcup_n \text{FORM}_n \times \bigcup_n \text{FORM}_n \right) \cap \vdash$$

turns out to be decidable for BL and for $L_\infty$, no less than for boolean logic. However, the proper class of all BL and all MV algebras, which is needed to check $\models_{BL}$ and $\models_{MV}$, has no role in the proof of these decidability results. Actually, the proof depends on subdirect representation and completeness theorems, which, combined with results like the Hay-Wójcicki theorem, yield a dramatic restriction of the set of evaluations needed to check semantic consequence. Suitably small finite chains turn out to be sufficient to decide if $\beta$ is a consequence of $\alpha$. In this way we get polytime verifiable certificates for $\alpha \not\vdash \beta$ whence the coNP-completeness of $\vdash_{\text{fin}}$ follows. See [1] and [14]. Also see [13] for a general discussion of strong completeness in various logics, including BL and $L_\infty$.

The evolving semantical notions of valuation (model, interpretation, possible world,...), strongly impinge on the evolution of the proof theory of $\vdash$. While $\vdash$ is immutable, the recipe $R$ to check $\alpha \vdash \beta$ is not: we do not even know if “proofs”, as we understand them today in boolean logic (let alone $L_\infty$ and BL) will one day be replaced by revolutionary polytime decision procedures.

Hájek’s intuition of the BL-axioms was confirmed by a definitive strong completeness result for valuations over t-algebras rather than over arbitrary BL-algebras. Similarly, the Łukasiewicz axioms for $L_\infty$, as well as Chang’s MV-axiomatics of $L_\infty$, are now gratified by a strongly complete (genuinely semantic) consequence relation $\models_\beta$ that does not resort to valuations over exoteric MV-algebras and their “infinitesimal truth-values”. Rather, $\Theta \models_\beta \psi$ depends on (real-valued) differential valuations that check if $\psi$ has the stability properties common to all $\theta \in \Theta$.

Closing a circle of logic-algebraic-geometric ideas, our results in this paper show that the traditional semantic consequence relation $\Theta \models \phi$ fails to be strongly complete because of its total insensitivity to the Bouligand-Severi tangent space of $\text{Mod}(\Theta)$. Strong completeness is retrieved by differential valuations, which take into account the directional derivatives of formulas along the tangent space of $\text{Mod} \Theta$.

7. **Appendix: Stable Consequence for Arbitrary Sets of Sentences**

Since MV-algebras are Lindenbaum algebras of set of formulas in $L_\infty$, we have to consider arbitrarily large sets of formulas on unlimited supplies of variables. So let $\mathcal{X} = \{X_1, X_2, \ldots, X_\alpha, \ldots \mid \alpha < \kappa\}$ be a set of variables of infinite, possibly uncountable cardinality $\kappa$, indexed by all ordinals $0 < \alpha < \kappa$. We let $\text{FORM}_{\mathcal{X}}$ be the set of formulas $\psi(X_{\alpha_1}, \ldots, X_{\alpha_t})$ whose variables are contained in $\mathcal{X}$. In this appendix we routinely extend Definition 3.7 to arbitrary subsets $\Theta$ of $\text{FORM}_{\mathcal{X}}$ and formulas $\psi \in \text{FORM}_{\mathcal{X}}$.

7.1. The free MV-algebra over $\kappa$ free generators is the MV-algebra $M([0,1]^\kappa)$ of all functions on the Tychonov cube $[0,1]^\kappa = [0,1]^\kappa$ generated by the coordinate
functions \( \pi_\beta(x) = x_\beta \), \( x \in [0,1]^\kappa \), \( 0 < \beta < \kappa \) by pointwise application of the \( \neg, \oplus \) operations, \([6, 9.1.5]\).

### 7.2.
For any finite set \( \mathcal{K} = \{X_{\alpha_1}, \ldots, X_{\alpha_d}\} \subseteq \mathcal{X} \) we identify \([0,1]^\mathcal{K}\) with the set of all \( x \in [0,1]^\kappa \) such that all coordinates \( x_\beta \) of \( x \) vanish, with the possible exception of \( \beta \in \{\alpha_1, \ldots, \alpha_d\} \). For any formula \( \phi \in \text{FORM}_\mathcal{X} \), we let \( \text{var}(\phi) \) be the set of variables occurring in \( \phi \). Identifying the function \( \hat{\phi} \) with an element of \( \mathcal{M}([0,1]^\text{var}(\phi)) \) we will tacitly identify \( \mathcal{M}([0,1]^\mathcal{K}) \) with the subalgebra of \( \mathcal{M}([0,1]^\kappa) \) consisting of all \( \hat{\phi} \) for \( \phi \in \text{FORM}_\mathcal{K} \).

### 7.3.
Suppose now we are given two finite subsets \( \mathcal{H} \subseteq \mathcal{K} \subseteq \mathcal{X} \) and two differential valuations \( U = (u_0, u_1, \ldots, u_d) \in \mathbb{R}^\mathcal{H} \) and \( V = (v_0, v_1, \ldots, v_e) \in \mathbb{R}^\mathcal{K} \). We then have two prime ideals \( p_U \) of \( \mathcal{M}([0,1]^\mathcal{H}) \) and \( p_V \) of \( \mathcal{M}([0,1]^\mathcal{K}) \supseteq \mathcal{M}([0,1]^\mathcal{H}) \). Recalling 3.3, we say that \( V \) dominates \( U \), in symbols, \( V \succeq U \), if \( p_U = p_V \cap \mathcal{M}([0,1]^\mathcal{H}) \). Whenever \( V \succeq U \), the point \( u_0 \) of \([0,1]^\mathcal{H} \subseteq [0,1]^\mathcal{K}\) is obtained by forgetting all coordinates of \( v_0 \) other than those in \( \mathcal{H} \). Further information on the relationship between \( U \) and \( V \) can be found in \([4, \S 4]\).

The following definition is a straightforward generalization of 3.3:

### 7.4.
A differential valuation in \( \mathbb{R}^\kappa \) is a \( \succeq \)-direct system

\[
W = \{U_\mathcal{H} \mid \mathcal{H} \subseteq \mathcal{X}, \mathcal{H} \text{ finite}\} \tag{6}
\]
of differential valuations \( U_\mathcal{H} \in \mathbb{R}^\mathcal{H} \), indexed by all finite subsets \( \mathcal{H} \) of \( \mathcal{X} \). As usual, directedness means that, for any finite \( \mathcal{H}', \mathcal{H}'' \subseteq \mathcal{X} \), \( U_{\mathcal{H}' \cup \mathcal{H}''} \) dominates both \( U_{\mathcal{H}'} \) and \( U_{\mathcal{H}''} \). We say that \( W \) satisfies a formula \( \phi \in \text{FORM}_\mathcal{X} \) if \( U_{\text{var}(\phi)} \) satisfies \( \phi \) in the sense of 3.6, i.e., \( 1 - \phi \) belongs to \( p_{U_{\text{var}(\phi)}} \).

### 7.5.
For \( \Theta \subseteq \text{FORM}_\mathcal{X} \) and \( \psi \in \text{FORM}_\mathcal{X} \) we say that \( \psi \) is a stable consequence of \( \Theta \) and we write \( \Theta \models_\beta \psi \), if \( \psi \) is satisfied by every differential valuation \( W \) in \( \mathbb{R}^\kappa \) that satisfies every \( \theta \in \Theta \).

Recalling 3.11, it is not hard to see that \( \models_\beta \) is an extension of the stable consequence relations defined for \( \Theta \subseteq \text{FORM}_n \) and \( \psi \in \text{FORM}_n \), \( (n = 1, 2, \ldots) \). The “strong completeness” theorem for this general consequence relation \( \models_\beta \) now states:

### 7.6.
For any (possibly uncountable) set \( \mathcal{X} \) of variables, \( \Theta \subseteq \text{FORM}_\mathcal{X} \) and \( \psi \in \text{FORM}_\mathcal{X} \), \( \Theta \models_\beta \psi \) iff \( \Theta \vdash \psi \).

**Proof.** Every prime ideal \( p \) of \( \mathcal{M}([0,1]^\kappa) \) is uniquely determined by its intersections \( p \cap \mathcal{M}([0,1]^\mathcal{H}) \) letting \( \mathcal{H} \) range over finite subsets of \( \mathcal{X} \). Any such intersection is a prime ideal of \( \mathcal{M}([0,1]^\mathcal{H}) \). By \([4, 2.18]\), for every finite \( \mathcal{H} \subseteq \mathcal{X} \) there is a differential valuation \( U_{\mathcal{H}} \in \mathbb{R}^\mathcal{H} \) such that \( p \cap \mathcal{M}([0,1]^\mathcal{H}) = p_{U_{\mathcal{H}}} \). Letting now \( \mathcal{H} \) range over all finite subsets of \( \mathcal{X} \), the \( p_{U_{\mathcal{H}}} \) make a \( \succeq \)-direct system with union \( p \). Correspondingly the differential valuations \( U_{\mathcal{H}} \in \mathbb{R}^\mathcal{H} \) make a \( \succeq \)-direct system, i.e., a differential valuation \( W = W_p \) in \( \mathbb{R}^\kappa \) of the form (6). Every prime ideal \( \mathcal{M}([0,1]^\mathcal{H}) \) arises in this way from a differential valuation \( W_p \) in \( \mathbb{R}^\kappa \). Now argue as in the proof of 3.9 using the subdirect representation theorem for \( \mathcal{M}([0,1]^\mathcal{H}) \). \(\square\)

### References

[1] M. Baaz, P. Hájek, F. Montagna, H. Veith, Complexity of t-tautologies, Annals of Pure and Applied Logic, 113 (2002) 3-11.

[2] H. Bouligand, Sur les surfaces dépourvues de points hyperlimites, Ann. Soc. Polonaise Math., 9 (1930) 32–41.

[3] R.I.Bot, S.M. Grad, G.Wanka, Duality in vector optimization, Springer-Verlag, NY, 2009.

[4] M.Busaniche, D.Mundici, Geometry of Robinson consistency in Lukasiewicz logic, Annals of Pure and Applied Logic, 147 (2007) 1–22.
[5] M. Busaniche, D. Mundici, Bouligand-Severi tangents in MV-algebras, arXiv, 1204.2147v1, April 2012.
[6] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, Algebraic Foundations of many-valued Reasoning, Trends in Logic, vol. 7, Kluwer Academic Publishers, Dordrecht, (2000).
[7] R. Cignoli, F. Esteva, L. Godo, A. Torrens, Basic logic is the logic of continuous t-norms and their residua, Soft Computing, 4 (2000) 106–112.
[8] P. M. Cohn, Universal Algebra, D. Reidel Publishing Company, Dordrecht, Holland, 1980.
[9] E. Dubuc, Y. Poveda, Representation theory of MV-algebras, Annals of Pure and Applied Logic, 161 (2010) 1024–1046.
[10] P. Hájek, Metamathematics of fuzzy logic, Kluwer, Dordrecht, 1998.
[11] P. Hájek, Basic fuzzy logic and BL-algebras, Soft Computing, 2 (1998) 124-128.
[12] L. S. Hay, Axiomatization of the infinite-valued predicate calculus, Journal of Symbolic Logic, 28 (1963) 77-86.
[13] F. Montagna, Notes on the strong completeness in Łukasiewicz, product and BL logics and their first-order extensions, Lecture Notes in Artificial Intelligence, vol. 4460 (2007) 247–274.
[14] D. Mundici, Satisfiability in many-valued sentential logic is NP-complete, Theoretical Computer Science, 52 (1987) 145–153.
[15] D. Mundici, Advanced Łukasiewicz calculus and MV-algebras, Trends in Logic, Vol. 35, Springer-Verlag, Berlin, NY, 2011.
[16] F. Severi, Conferenze di geometria algebrica (Collected by B. Segre), Stabilimento tipografico del Genio Civile, Roma, 1927, and Zanichelli, Bologna, 1927–1930.
[17] F. Severi, Su alcune questioni di topologia infinitesimale, Annales Soc. Polonaise Math., 9 (1931) 97–108.
[18] A. Tarski, On the concept of logical consequence, Chapter XVI in: A. Tarski, Logic, Semantics, Metamathematics, Clarendon Press, Oxford, (1956). Reprinted: Hackett, Indianapolis, (1983).
[19] R. Wójcicki, On matrix representations of consequence operations of Łukasiewicz sentential calculi, Zeitschrift für math. Logik und Grundlagen der Mathematik, 19 (1973) 239-247. Reprinted, In: R. Wójcicki, G. Malinowski (Eds.), Selected Papers on Łukasiewicz Sentential Calculi, Ossolineum, Wrocław, 1977, pp. 101-111.

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