Complex projective surfaces and infinite groups

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March 19, 2022

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1 Introduction

It is well known that complex projective surfaces can have highly nontrivial fundamental groups. It is also known that not all finitely presented groups can occur as fundamental groups of projective

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*Partially supported by DMS Grant- 9500774
†Partially supported by A.P. Sloan Dissertational Fellowship
surfaces. The fundamental problem in the theory then, is to determine which groups can occur as the fundamental groups of the complex projective surfaces and to describe complex manifolds which occur as nonramified coverings of the surface. The only interesting case is when the group is infinite since finite coverings of projective surfaces are projective and every finite group occurs as a fundamental group of some projective surface.

In this paper we mainly consider a surface with a representation as a family of projective curves over a curve. It does not put much restriction on the choice of a surface since any surface has such a representation after blowing up a finite number of points. We use a base change construction combined with a finite ramified covering to associate to a given surface a collection of surfaces with infinite fundamental groups. Most of these fundamental groups were not previously known to be fundamental groups of smooth projective surfaces. Every surface we construct comes equipped with a regular map to a curve of high genus. The kernel of the corresponding map of fundamental groups is obtained from the fundamental group of a generic fiber by imposing torsion relations on some elements and a very big class of infinite groups occur in this way. We also analyze the universal coverings of these surfaces in the context of the Shafarevich’s conjecture which states that the universal covering of the smooth complex projective variety must be holomorphically convex.

Partial affirmative results [11] regarding Shafarevich’s conjecture can be applied to the fundamental groups we obtain. It leads to nontrivial purely algebraic results on the structure of this groups. On the other hand the generality of our construction indicates that not all fundamental groups we construct will satisfy the algebraic restrictions imposed by Shafarevich’s conjecture. It would appear then that the conjecture may be false in general. We describe a series of potential counterexamples in section 4.

We begin with a local version of the general construction. Namely we have a local fibration without multiple components and with only double singular points in the central fiber. As a first step we make a local base change which produces a new surface with singular points corresponding to the singular points of the central fiber. This move changes the image of the fundamental group of a generic fiber in the fundamental group of the Zariski open subset of nonsingular points. Namely the kernel of this map is generated by the $N$-th powers of the initial vanishing cycle where $N$ is the degree of the local base change. As a second step we desingularize the surface by taking a finite fiberwise covering of the singular surface which is ramified at singular points only.

The global construction follows the same pattern, but as a result we obtain a surface with a highly nontrivial fundamental group coming from the fiber even if the surface at the beginning was simply connected. First we construct a singular projective surface with a big fundamental group of the compliment to the set of singular points. Smooth projective surface is obtained at the second step as a finite covering of the singular surface ramified at singular points only. We prove a general
theorem (Theorem 2.4) which establishes a close similarity between the fundamental groups and universal coverings for two classes of surfaces: normal projective surfaces and the surfaces obtained from them by deleting a finite number of points. The above construction enlarges the image of the fundamental group of the fiber in the fundamental group of the whole surface. The resulting group can be described in purely algebraic terms. Let \( \pi_g \) be a fundamental group of a projective curve of genus \( g \) (generic fiber of the fibration). Consider a finite set of pairs of \( (s_i \in \pi_g, N_i \in \mathbb{N}) \) and a subgroup \( M \) of the automorphisms of \( \pi_g \). We assume that all \( s_i \) are vanishing cycles. These are special conjugacy classes in the fundamental group of the curve which constitute a finite number of orbits under the action of the mapping class group \( \text{Map}(g) \) (see section 2). The orbits \( Ms_i^N_i \) generate the normal subgroup \( \Xi(M, s_i, N_i) \) of \( \pi_g \).

Now we can give an algebraic version of the description of the corresponding group.

**Definition 1.1** Define a Burnside type quotient of \( \pi_g \) to be the group \( \pi_g / \Xi(M, s_i, N_i) \).

In the geometric situation \( s_i \) are the vanishing cycles of the initial fibration and \( M \) is its monodromy group. Geometric Burnside type groups constitute a proper subset among all Burnside type groups. In particular not every data \( s_i, N_i, M \) can be geometrically realized. The problem which data can appear in geometry is the most substantial problem of the above construction. It is clear that we can change \( M \) into its own subgroup of finite index but at the expense of changing the set of elements \( s_i \). The following lemma shows that we are free to vary \( N_i \).

**Lemma 1.1** Let \( s_i, N_i = 1, M \) describe the data of the smooth projective family of curves of a given genus \( g \). Then \( s_i, N_i, M \) corresponds to the geometric Burnside group for any choice \( N_i \) satisfying the condition: \( N_i = N_j \) if the singular points corresponding to the cycles \( s_i \) belong to the same singular fiber of the fibration.

The topological structure of the algebraic family of curves can be rather arbitrary if we consider its restriction on a disc inside the base curve. We summarize relevant results of the article in the following proposition.

**Proposition 1.1** Let \( \Gamma \) be any finitely presented group and \( f_g : \pi_g \to \Gamma \) be any surjective map with \( \text{Ker} f_g \) generated by a finite subset \( VS \) of the conjugacy classes in \( \pi_g \). There exists a surjective map \( p_h : \pi_h \to \pi_g \) which corresponds to the contraction of a set of nonintersecting handles \( H_i \) inside the curve of genus \( h \) with the following properties:

1) Every element \( s_j \in VS \) has a preimage \( s_j' \) in \( \pi_h \) which is realized by a smooth cycle.

2) There is a smooth holomorphic family \( X \) of curves of genus \( h \) with simple singularities over an algebraic curve which has all cycles \( s_j' \) and the generating cycles \( a_i, b_i \) of the handles \( H_i \) as vanishing cycles.
3) The family $X$ above can have as a monodromy group any subgroup of finite index in $\text{Map}(h)$ containing the elements corresponding to positive Dehn twists for all $s_j', a_i, b_i$.

Thus any topological data $s_i, M$ can be realized as part of some geometric data $s_i, s_i'M'$ where $M'$ is any subgroup of finite index in $\text{Map}(g)$ containing $M$.

The previous lemma allows to transfer all additional relations $s_i'$ into their $N$-powers. Hence we have two parameter “approximation” of any topological data by geometric data. One of the parameters corresponds to $N$ and converges to infinity. Another parameter runs through subgroups of finite index in $\text{Map}(h)$ which containing the monodromy group $M$ of the fibration over the disc. Since $\text{Map}(h)$ is residually finite the corresponding sequence of groups converges to $M$.

**Remark 1.1** These results provide with a tool to produce a big variety of fundamental groups and universal coverings using our construction. It is worth noticing that though the initial group with $N_i = 1$ can be small (even trivial) the groups which appear for other choices of $N_i$ are quite diverse.

**Definition 1.2** Let us take a free group $\mathbb{F}^l$ on $l$ generators, the normal subgroup $\Xi$ generated by $N$-th powers of all the primitive elements can be described as in definition 1.1. Namely $\Xi$ is generated by $Ms^N$ where $s$ is a primitive element of $\mathbb{F}^l$ and $M$ is the group of all automorphisms of $\mathbb{F}^l$. We will denote $\mathbb{F}^l/\Xi$ by $\text{BT}(l, N)$.

**Definition 1.3** Denote by $\pi_g/(x^N = 1)$ the quotient of the fundamental group of a Riemann surface of genus $g$ by the group generated by the $N$-th powers of all primitive elements $x$ in $\pi_g$.

**Conjecture 1.1** (Zelmanov) For big $l, g, N$ the groups $\text{BT}(l, N)$ and $\pi_g/(x^N = 1)$ are nonresidually finite groups.

If the above conjecture is true then the base change construction provides us with many new simple examples of nonresidually finite fundamental groups of smooth projective surfaces. The first example of such a group was constructed by Toledo ([21], see also [3]).

In section four we discuss the holomorphic convexity of the universal coverings of the surfaces we have constructed. Recently many new powerful methods have been developed to investigate the structure of the fundamental groups and universal coverings of complex projective surfaces. These methods lead to many new remarkable results. In particular quite a few positive results on Shafarevich’s conjecture were proved e.g. the results of F. Campana, H. Grauert, R. Gurjar, J. Kollár, B. Lasell, R. Narasimhan, T. Napier, M. Nori, M. Ramachandran, C. Simpson, S. Shasrty, K. Zuo, S.T. Yau (see e.g. [4], [11], [14], [12], [15], [16], [19], [20], [22]).
In particular holomorphic convexity is established (see [11]) for coverings of normal projective surface $X$ corresponding to the homomorphisms of $\pi_1(X)$ to $GL(n, \mathbb{C})$ such that the image of these homomorphisms is virtually not equal to $\mathbb{Z}$ (see also [13]). We suggest possible applications of the construction related to the Shafarevich’s conjecture. Holomorphic convexity of a variety implies the absence of infinite chains of compact curves. This can be expressed in our case as a restriction on the images on the fundamental groups of the components of a singular fiber (see lemma 4.1). We provide with a scheme how to control the behavior of the monodromy and vanishing cycles to construct a counterexample to the Shafarevich conjecture.

A free group with a given number of generators can be identified with a fundamental group of a Riemann surface with one or two ends. The number of generators in the free group defines the genus of the surface and the number of ends (two if the number of generators is odd and one if the latter is even).

**Definition 1.4** Denote by $\mathbb{F}_k^g$ a free group with $k$ generators realized as a fundamental group of a Riemann surface $B$ with one or two ends. Denote by $P^g(k, N)$ the quotient of $\mathbb{F}_k^g$ by a normal subgroup generated in $\mathbb{F}_k^g$ by all primitive elements in $\mathbb{F}_k^g$ which map into primitive elements (embedded curves) in $B$.

The following group theoretic question is closely related to the Shafarevich conjecture:

**Question** Are there such a $k > 1$ and $N$ that that $P^g(2k, N)$ is a finite group and $\pi_{2k}/(x^N = 1)$ is an infinite group?

If the answer to this question is affirmative then one gets a counterexample to the Shafarevich conjecture (see section 4).

We also suggest potential counterexamples by considering simplest nontrivial case $N = 3$ in which we establish the finiteness of all the groups $P^g(k, 3)$ (Appendix B). In this case we suggest a family of potential counterexamples to the conjecture which depends on a non-finiteness of some groups in the family “approximating” some infinite group. The corresponding family of groups can be described as follows. Let us take a chain or ring of curves $X_0$ containing more than two curves of genus greater than zero. A natural contraction of a generic curve of genus $g$ onto this special fiber defines a subgroup $M_D$ inside the mapping class group $Map(g)$ which commutes with the contraction. If every component $C_i$ of $X_0$ contains at least one vanishing cycle $s_i$ then the group $\pi_g/(M_D s_i^3)$ is infinite though the images of the groups corresponding to different components $C_i \subset X_0$ are finite (Appendix B). It is true even if we consider smaller subgroups $M^f \subset M_D$. Now by our “approximation” results we can realize $X_0$ as a fiber in an algebraic fibration with the corresponding fiber group $\pi_g/(M^f s_i^3, M^f s_{k(N)}^3)^{3N}$. Here $M^f_j$ is any open subgroup of finite index in $Map(g)$ containing $M^f$, $N$ is any integer and $s_{k(N)}$ runs through the set of additional vanishing
classes. We conjecture that this set of groups contains many infinite groups which will imply that there are many counterexamples to the Shafarevich’s conjecture obtained from the surfaces above.

Remark 1.2 Since the group we “approximate” surjects onto a nontrivial free product of finite groups we expect that the proof of the above conjecture can be found within the modern theory of infinite groups.

In this section we briefly discuss the symplectic version of our construction. We also formulate an arithmetic variant of the Shafarevich conjecture which is presumably easier to prove though the conjecture is formally stronger than a direct analogue of the complex case.

Acknowledgments:
The authors would like to thank M. Gromov, J. Kollár, T. Pantev, G. Tian, D. Toledo for useful conversations and comments. The second author would like to thank A. Beilinson, J. Carlson, H. Clemens, K. Corlette, P. Deligne, R. Donagi, S. Gersten, S. Ivanov, M. Kapovich, M. Newman, M. Nori, A. Olshanskii, M. Ramachandran, C. Simpson, Y. T. Siu, S. Weinberger, E. Zelmanov and S. T. Yau for useful conversations and constant attention to work. We would like also to thank the referee for pointing out an erroneous statement in the initial version and for helpful suggestions on the organization of the paper. We thank M. Fried for looking through the arithmetic part of the paper.

2 The general construction
2.1 Vanishing cycles - a local construction

In this subsection we explain the local computation with the vanishing cycles on which the whole construction is based. We begin with some classical results on degeneration of curves that can be found in [6].

Let $X_D$ be a smooth complex surface fibered over a disc $D$. We assume that fibers over a punctured disc $D^* = D - 0$ are smooth curves of genus $g$ and the projection $t : X \to D$ is a complex Morse function. In particular the fiber $X_0$ over $0 \in D$ has only quadratic singular points and it has no multiple components. Denote by $P$ the set of singular points of $X_0$ and by $T : X_t \to X_t$ the monodromy transformation acting on the fundamental group of the general fiber $X_t$. This action can be described in terms of Dehn twists. Obviously this action defines an action on the first homology group of the general fiber. The following proposition describes completely the topology of $X_D$ and the projection $t : X_D \to D$.

Proposition 2.1 1) There is a natural topological contraction $cr : X_D \to X_0$. 
2) The restriction of $cr$ to $X_t$ is an isomorphism outside singular points $P_i \in X_0$. It contracts the circle $S_i \subset X_t$ into $P_i$. The monodromy transformation $T$ is the identity outside small band $B_i$ around $S_i$ and in $B_i$ the transformation $T$ coincides with a standard Dehn twist.

**Proof** See [6].

The above contraction is an isomorphism from $X_t$ minus preimage of $P$ on $X_0 - P$. The preimage of any singular point $P_i$ is a smooth, homotopically nontrivial curve $S_i \subset X_t$.

**Definition 2.1** We will call the free homotopy class of $S_i$ in the fundamental group of $X_t$ a geometric vanishing cycle. It defines a conjugacy class in $\pi_g(\text{vanishing cycle})$.

**Remark 2.1** The direction of the standard Dehn twist is defined by the orientation of $S_i$ which in turn is defined by the complex structure of the neighborhood of the corresponding singular point of the singular fiber.

Let us denote by $De_i$ the topological Dehn transformation of $X_t$ and by $De_{i,H}$ its action on the homology of $X_t$. In a neighborhood of $X_0$ the monodromy transformation $T$ is a product of Dehn transformations $De_i$ with non-intersecting support.

**Lemma 2.1** 1) The monodromy transformation $T$ acts via unipotent transformation $T_H$ on the homology group $H_1(X_t, \mathbb{Z})$.

2) $(1 - T_H)^2 = 0$.

3) $(1 - T_H^N) = 0 \pmod{N}$ for any $N$.

**Proof** It is enough to prove (2) for $De_{i,H}$. The topological description of $De_i$ implies that $(1 - De_{i,H})x = (x, s_i)s_i$, where $s_i$ a homology class of the vanishing cycle $s_i$ and $(x, s_i)$ is the intersection number.

Since $(s_i, s_i) = 0$ we obtain $(1 - De_{i,H})^2 = 0$. The image of $(1 - De_{i,H})$ consists of the elements proportional to $s_i$. We also have $(1 - De_{i,H}^N) = (1 - NDe_{i,H}) = 0 \pmod{N}$.

The topological transformations $De_i$ commute for different $i$ since they have disjoint supports. Therefore the same holds for $De_{i,H}$. We also have $De_{i,H}s_j = 0$ for any vanishing cycle $s_j$ since the corresponding circles $S_i, S_j$ don’t intersect in $X_t$. We now see that

$$(1 - T_H)^2 = (1 - \prod De_{i,H})^2 = \prod (1 - De_{i,H})^2 = 0$$

where the last equality follows from the above formulas. Similarly we obtain 1) and 3).

$\square$
The geometric vanishing cycles consist of two different types:

1) The first type includes homologically nontrivial vanishing classes. They are all equivalent under the mapping class group $\text{Map}(g)$. The latter can be described either as group of connected components of the orientation preserving homeomorphisms of the Riemann surface of genus $g$ or as the group of exterior automorphisms of the group $\pi_g$. The vanishing cycle from this class is a primitive element of $\pi_g$ which means it can be included into a set of generators of $\pi_g$ satisfying standard relation defining the fundamental group of the curve. We shall denote the vanishing cycles of the first type type as $\text{NZ}$-cycles.

2) The second type consists of elements in $\pi_g$ which are homologous to zero. Any vanishing cycle of this type cuts the Riemann surface $X_t$ into two pieces and the number of handles in these pieces is the only invariant which distinguishes the type of a cycle under the action of $\text{Map}(g)$. We shall denote the vanishing cycles of the second type by $\text{Z}$-cycles.

Vanishing $\text{Z}$-cycles correspond to the singular points of the singular fiber which divide this fiber into two components and $\text{NZ}$-cycles to the ones which do not. Each $\text{Z}$-cycle defines a primitive element in the center of the quotient $\pi_g/[[\pi_g, \pi_g], \pi_g]$, while each $\text{NZ}$-cycle defines a primitive element in the abelian quotient $\pi_g/[\pi_g, \pi_g] = \mathbb{Z}^{2g}$. Assume now that we made a change of the variable $t = u^N$ and consider the induced family of curves over $D$ with a coordinate $u$. Denote the resulting family as $X_U$. It is a singular surface and the singular set can be identified with the set $P$ of the singular points of the fiber $X_0$. For the new monodromy transformation we have $T^u = T^N$. Therefore by lemma 2.3 it acts trivially on $H_1(X_t, \mathbb{Z}_N), t \neq 0$.

Lemma 2.2 The surface $X_U$ contracts to the central fiber $X_0$.

Indeed the fiberwise contraction of $X_D$ to $X_0$ can be lifted into a contraction of $X_U$.

Remark 2.2 The fundamental group $\pi_1(X_D - P) = \pi_1(X_D) = \pi_1(X_0)$ since the singular points of the fiber $X_0$ are nonsingular points of $X_D$. The analogous statement is not true however for $X_U$.

Theorem 2.1 The fundamental group $\pi_1(X_U - P)$ is equal to the quotient of $\pi_1(X_t) = \pi_g$ by a normal subgroup generated by the elements $s_i^N$.

Proof The fundamental group of a complex coincides with the fundamental group of any two-skeleton of the complex. There is a natural two-dimensional complex with a fundamental group as in the theorem. Namely let us take a curve $X_t$ and attach two-dimensional disks $D_i$ via the boundary maps $f_i : dD_i \to S_i$ of degree $N$. The resulting two dimensional complex $X_t^a$ evidently has a fundamental group isomorphic to the group described in the theorem. We are going to show that $X_t^a$ can be realized as a two skeleton of $X_U - P$. We prove first the following statement:
Lemma 2.3 The surface $X_U - P$ retracts onto a three dimensional complex $X^3$ which is a union of generic curve $X_t$ and a set of three-dimensional lens spaces $L^i_N$. Each lens space corresponds to the singular point $P_i$ of the singular fiber and $L^i_N$ intersects $X_t$ along the band $B_i$.

Proof Locally near $P_i$ the surface $X_U$ is described by the equation $t^N = z_1 z_2$. Hence a neighborhood $U_i$ of $P_i$ is a cone over the three-dimensional lens space $L^i_N = S^3/\mathbb{Z}_N$. Here $S^3$ is a three dimensional sphere and $\mathbb{Z}_N$ is generated by the matrix with eigenvalues $\chi, \chi^{-1}$, where $\chi$ is a primitive root of unity of order $N$. We can assume that the surface $X_t$ intersects $L^i_N$ along a two-dimensional band $B_i$ with a central circle $S_i$ defining a generator of $\pi_1(L^i_N)$. Though the band $B_i$ is a direct product of $S_i$ by the interval its embedding into $L^i_N$ is nontrivial: the boundary circles have nonzero linking number. We use a fiberwise contraction to contract $X_U - P$ to the union to $X^3$. It coincides with a standard contraction outside of the cones over $L^i_N$. Therefore we obtain a contraction of $X_U - P$ onto $X^3$.

\[ \square \]

The two-skeleton of $X^3$ can be obtained as a union of $X_t$ and a two-skeleton of $L^i_N$. The latter can be seen as a retract of a complimentary set to the point in $L^i_N$. Here is the topological picture we are looking at.

Lemma 2.4 Let $L^i_N$ be a lens space as above. Then the complimentary set to a point retracts on complex $L^2_i$ obtained by attaching a disc to the circle $S_i$ via the boundary map of degree $N$.

Proof The sphere $S^3$ can be represented as a joint of two circles $S_i$ and $S'$. In other words it consists of intervals connecting different points of $S_i, S'$. The action of $\mathbb{Z}_N$ with $L^3_N = S^3/\mathbb{Z}_N$ rotates both circle. Define the disc $D_x$ to be a cone over $S_i$. Different discs $D_x, x \in S'$, don’t intersect and the image of $D_x$ in $L^3_N$ is $L^2_i$. The fundamental domain of $\mathbb{Z}_N$-action lies between discs $D_x, D_{gx}$, where $g$ is a generator of $\mathbb{Z}_N$. Hence this domain is isomorphic to $D^3$ and coincides with complimentary of $L^2_i$ in $L^i_N$. The last proves the lemma.

\[ \square \]

Corollary 2.1 There is an embedding of $X^a_t$ into $X_U - P$ which induces an isomorphism of the fundamental groups.

Indeed we obtain the two skeleton of $X^3$ gluing $X_t$ and $L^2_i$ along $S_i$, but the resulting two-complex coincides with $X^a_t$. Since $X^3$ is a retract of $X_U - P$ we obtain the corollary and finish the proof of the theorem.

\[ \square \]
Let us denote by $G_X$ the fundamental group of $X_U - P$ and by $\tilde{X}$ its universal covering. The description of the group $G_X$ can be obtained in pure geometric terms. Namely the vanishing cycles $S_i$ don’t intersect and therefore we can first contract $Z$-cycles to obtain the union of smooth Riemann surfaces $X_i$ with normal intersections. The graph corresponding to this system of surfaces is tree since every point corresponding to $Z$-cycle splits it into two components. Remaining $NZ$-cycles lie on different surfaces $X_j$ and constitute a finite isotropic subset of primitive elements in $\pi_1(X_j)$. The vanishing cycle $S_i$ which contracts to the point of $X_j$ defines a map of a free group onto $\pi_1(X_j)$ with $S_i$ corresponding to standard relations in $\pi_1(X_j)$. We also have the following lemma.

**Lemma 2.5** If $N$ is odd or divisible by four then the group $G_X$ has a natural surjective projection on the quotient group $\pi_g/[\pi_g,\pi_g]$ with additional relation $x^N = 1$. If $N = 2$ then $G_X$ maps surjectively on a central extension of $\mathbb{Z}_2^{2g}$ by $\mathbb{Z}_2$ and the images of all $s_i$ have order 2.

**Proof** The case when $N$ is odd or divisible by four is clear since all the elements $s_i$ have order $N$. In the case $N = 2$ we use the fact that $NZ$ cycles $s_i$ contained in the component $X_j$ lie in the isotropic subspace. Hence there exists a standard $\mathbb{Z}_2$ central extension of $\mathbb{Z}_2^{2g_j}$ where the images of all $s_i \in X_j$ are exactly of order 2. Denote this group by $G_j$ and the generator of its center by $c_j$. Now consider the product of $G_j$ for all $j$ and factor it by a central subgroup generated by $c_k - c_j$ if $X_k$ and $X_j$ intersect. Since the graph is a tree we obtain that the quotient of the center by the above group is equal to $\mathbb{Z}_2$ which is identified with the zero homology with coefficients $\mathbb{Z}_2$. The image of $Z$-cycle $s_i$ coincide with $c_j$ if $s_i \in X_j$ and hence is never zero. Vanishing $NZ$-cycles project into nonzero elements of the abelian quotient of the group. □

**Remark 2.3** If $N$ is odd or divisible by four we obtain a canonical quotient of $G_X$ which is equivariant under $Aut(\pi_g)$. We denote this group by $UC_0^N$. For $N = 2$ our construction is less canonical, it depends on the choice of isotropic subspace in $H_1(X_t,\mathbb{Z}_2)$

Here we start to develop an idea that will be constantly used throughout the paper. Namely we show that we can work with open surfaces and get results concerning the universal coverings of the closed surfaces.

**Theorem 2.2** There is a natural $G_X$-invariant embedding of $\tilde{X}$ into a smooth surface $\tilde{X}_U$ with $\tilde{X}_U/G_X = X_U$. The complimentary of $\tilde{X}$ in $\tilde{X}_U$ consists of a discrete subset of points.

**Proof** Let $P_i$ is a point that corresponds to a $NZ$ vanishing cycle. The preimage of a local neighborhood $U_i$ of $P_i$ in $\tilde{X}$ consists of a number of nonramified coverings of $U_i - P_i$. Since
We also have the following:

\textbf{Lemma 2.6} Let $G^0_X$ be the kernel of projection of $G_X$ into one of the finite groups defined in lemma 1.13. Then $G^0_X$ acts freely on $\tilde{X}_U$ and the quotient is a family of compact curves without multiple fibers and with a fundamental group $G^0_X$.

\textbf{Proof} Indeed the surface $\tilde{X}_U$ was obtained from $\tilde{X}$ by adding points and since $\tilde{X}$ was simplyconnected the former is simplyconnected either. Any element of $G_X$ which has invariant point in $\tilde{X}_U$ is conjugated to the power of $s_i$, but the latter are not contained in $G^0_X$.

\hfill \Box

We are done with our description of the local computations. By taking finite coverings and taking away the singularities of the covering we were able to make all local geometric vanishing cycles to be torsion elements.

\subsection*{2.2 The global construction}

In this subsection we globalize the construction of section 2.1 to a compact surface.

Let $X$ be smooth surface with a proper map to a smooth projective curve $C$. We assume that the map $f : X \rightarrow C$ is described locally by a set of holomorphic Morse functions and hence satisfies the conditions of the previous section in the neighborhood of any fiber. The generic fiber is a smooth curve $X_t, t \in C$ of genus $g$.

We denote by $P$ the set of all singular points of the fibers and by $P_C$ the set of points in $C$ corresponding to the singular fibers, $f(P) = P_C$. The main difference of the global situation lies in the presence of the global monodromy group which is the image of $\pi_1(C - P_C)$ in the mapping class group $Map(g)$. We denote this group by $M_X$.

Let us choose an integer $N$ and consider a base change $h : R \rightarrow C$ where $R$ such that the map $h$ is $N$-ramified at all the preimages of the points from $P_C$ in $R$. Consider a surface $S$ obtained via a base change $h : R \rightarrow C$. We have the finite map $h' : S \rightarrow X$ defined via $h$ and the projection
$g : S \to R$ with a generic fiber $S_t = X_{h(t)}$. The surface $S$ is singular with the set of singular points equal to $h^{-1}(P) = Q$ and the set of singular fibers over the points of $h^{-1}P_C = P_R$. The monodromy group $M_S$ of the family $S$ is subgroup of finite index of the group $M_X$.

**Theorem 2.3** The fundamental group $\pi_1(S-Q)$ surjects onto $\pi_1(R)$. The kernel of this surjection is a quotient of $\pi_g$ by a normal subgroup generated by the orbits of $M_S(s_i^N)$.

**Proof** Indeed we have a natural surjection of $\pi_g$ on the kernel of projection onto $\pi_1(R)$ since there are no multiple fibers in the projection $g$. A standard argument reduces all relations to the local ones and the local relations where described in the previous section (see theorem 2.1).

Now we move to the second step of our construction getting out of $S - Q$ a smooth compact surface $S^N$ with almost the same fundamental group. In the next subsection we will develop some partial theory of this second step summarizing and generalizing some known results. Let us assume that $N$ is either odd or divisible by 4.

**Lemma 2.7** There exists a smooth projective surface $S^N$ with a finite map $f : S^N \to S$ such that the image of the homomorphism $f_* : \pi_1(S^N) \to \pi_1(S-Q)$ is a subgroup of finite index in $\pi_1(S-Q)$.

As it was already shown in lemma 2.5 there exists a projection of $\pi_1(X_g)$ to a finite group $UC_g^N$ which is invariant under $Aut(\pi_g)$ and factors through the fundamental group of a small neighborhood of a singular fiber in $S - Q$. Therefore, if the map $h : S \to R$ has a topological section we obtain a finite fiberwise covering of $S$ which is ramified only over the singular set $Q$. The resulting surface coincides locally with the smooth surface described in theorem 2. It is also smooth and the map $f$ is finite. If there is not a topological section we first make a base change nonramified at $P_R$ in order to obtain such a section (see theorem 5.2 Appendix A) and then apply the previous argument. Since the map $f$ is finite the preimage $f^{-1}(Q)$ consists of a finite number of smooth points and therefore $\pi_1(S^N) = \pi_1(S^N - f^{-1}(Q))$ (see theorem 5.1 Appendix A). This also proves that the image of the homomorphism $f_* : \pi_1(S^N) \to \pi_1(S-Q)$ is a subgroup of finite index in $\pi_1(S-Q)$.

As we have said in the introduction to find a counterexample to the Shafarevich’s conjecture we try to control the existence of an infinite connected chain of compact curves in the universal covering of $S^N$. The above construction shows that we can do it by controlling the image of the fundamental groups of the open irreducible components of the reducible singular fibers in the fundamental group of the open surface $S - Q$. 

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2.3 Comparison theorem

The construction discussed in the previous section can be applied to both projective and quasiprojective fibered surfaces. This indicates that the universal coverings and the fundamental groups for these two classes of surfaces have a similar structure. In this section we illustrate another flavor of the same principle by describing a procedure comparing the fundamental groups of surfaces with quotient singularities to the fundamental groups of certain smooth surfaces. For future reference we will set up this transition in a slightly bigger generality.

Let $V$ be a normal projective complex surface and $Q \subset V$ be the finite set of its singular points.

The fundamental group $\pi_1(V)$ is the quotient of $\pi_1(V - \{q\})$ by the normal subgroup generated by the images of the local fundamental groups of the points $q \in Q$. Recall that the local fundamental group $L_q$ of a point $q \in Q$ is defined as the fundamental group of a deleted neighborhood of $q$, i.e. as the group $\pi_1(U_q - \{q\})$ where $U_q$ is a small analytic neighborhood of the point $q \in V$.

The topology of the neighborhood $U_q$ is completely determined by $L_q$ as it was shown by D. Mumford [18]. The following theorem is a generalization of a theorem by J. Kollár (see [16]).

**Theorem 2.4** Let $V$ be a normal projective surface and let $Q$ be the finite set of its singular points. Consider for any $q \in Q$ a normal subgroup of finite index $K_q \triangleleft L_q$ which contains the kernel of the natural map $L_q \rightarrow \pi_1(V - Q)$. Then there exists a smooth projective surface $F$ and a surjective finite map $r : F \rightarrow V$ which induces an isomorphism between $\pi_1(F)$ and the quotient of $\pi_1(V - Q)$ by the normal subgroup generated by the images of $K_q \in \pi_1(V - Q), q \in Q$.

**Proof.** Denote by $K_Q$ the normal subgroup of $\pi_1(V - Q)$ generated by the subgroups $K_q \subset L_q$. We obtain the surface $N$ as a generic hyperplane section of a singular projective variety $W$ with the following property: $W$ contains a subvariety $S$ of codimension $\geq 3$ with $\pi_1(W - S) = \pi_1(V - Q)/K_Q$. We may assume that $F$ does not intersect $S$ since the latter has codimension at least 3 in $W$. The fundamental group $\pi_1(F) = \pi_1(W - S)$ since $F$ is a generic complete intersection in $W$. We are going to construct $W$ as a union of two quasiprojective subvarieties. Denote by $G_q$ the finite quotient $L_q/K_q$ and by $G$ the direct product of all the groups $G_q$. Denote by $g_q$ the coordinate projection of $G$ onto $G_q$ and by $i_q$ the coordinate embedding of $G_q$ into $G$. For any $q$ there is a natural finite covering $M_q$ of $U_q$ corresponding to the projection $L_q \rightarrow G_q$. The preimage of $q$ in $M_q$ consists of a single point and the projection $M_q \rightarrow U_q$ is nonramified outside $q$. In the next lemma we prove the existence of an algebraic extension of this local covering.

**Lemma 2.8** There exist an open affine subvariety $V_q \subset V$ containing $q$ and an affine variety $B_q$ which is a $G_q$-Galois covering of $V_q$ ramified only at $q$ so that $B_q \times_{V_q} U_q$ and $M_q$ are isomorphic.
Proof Let $\hat{A}_q$ be the completed local ring of $q \in V$. A local $G_q$-covering defines a finite algebraic extension $\hat{B}_q$ of $\hat{A}_q$. By Artin’s approximation theorem there exists an affine ring $A \subset \mathbb{C}(V)$ and a finite algebraic extension $B$ of $A$ which locally at $q$ corresponds to the extension $\hat{B}_q$ over $\hat{A}_q$. Explicitly the extension $\hat{B}_q$ is described by a monic polynomial $f(x)$ with coefficients in $\hat{A}_q$. If we now consider any monic polynomial $g(x)$ over the ring $A$ with $g(x) = f(x) \mod m_q^N$ for a big enough $N$ then the resulting algebraic extension $B$ will be the one we need. The ring $A$ defines an open algebraic subvariety $\text{Spec}(A) \subset V$ containing $q$. Similarly $B$ defines an affine variety $\text{Spec}(B)$ with a finite projection $p_q : \text{Spec}(B) \to \text{Spec}(A)$. This projection is unramified outside $q$ in the formal neighborhood of $q$. Since we have the freedom to impose any finite number of extra conditions on $g(x)$ we can choose $p_q$ to be unramified at any finite number of points. In particular we may assume that the projection $p_q$ is nonramified over $Q - \{q\}$. That means that the divisorial part $D \subset \text{Spec}(A)$ of the ramification of $p_q$ does not intersect $Q$. Now we can take $\text{Spec}(A)/D$ as $V_q$. Let $B_q$ denote $\text{Spec}(B) \times_{\text{Spec}(A)} V_q$. It is an affine variety with affine $G_q$-action since it extends the local nonramified Galois covering $U_q - \{q\}$ and has the same degree. 

Let $B_0$ be the product of all $B_q$’s over $V$. This is an affine variety with the action of $G$. The quotient $B_0/G = V_0'$ is an open affine subvariety of $V$ which contains $Q$. The action of $G$ on $B_0$ is free outside of the preimage of $Q$. Let $G \to GL(E)$ be a (not-necessarily irreducible) faithful linear representation of $G$ of dimension $e$ with the property that only $1 \in G$ is represented by scalar matrix. Consider the diagonal action of $G$ on the product $B_0 \times E$. There exists a natural $\mathbb{C}^*$ action on $E$ - multiplication by scalars. It extends to a $\mathbb{C}^*$-action on the product which commutes with the $G$-action.

Let $F'_0 = (B_0 \times E)/G$ be the quotient variety. It is an affine variety with induced $\mathbb{C}^*$ action which has a natural projection $\pi_0 : F'_0 \to V'_0$ and a zero section $i(V'_0) = (B_0 \times 0)/G$. For any $s \in V'_0 - Q$ the preimage $\pi_0^{-1}(s)$ is a vector space isomorphic to $E$. Moreover $F'_0$ contains a natural vector bundle $I$ over $V'_0 - Q$. Its sheaf of sections coincides with the sheaf of $G$-equivariant sections of the constant sheaf $O \otimes E$ over $B_0$. Let us choose a smaller affine variety $V_0 \subset V'_0, Q \subset V_0$ with the property that $I$ is constant on $V_0 - Q$. We define $F_0$ as the preimage of $V_0$ in $F'_0$. Let $V_1$ be an open subvariety of $V$ which does not contain $Q$ and such that the union of $V_1$ and $V_0$ is equal to $V$. Let $J$ be the trivial bundle of rank $e$ on $V_1$. Choose a linear algebraic isomorphism of $F_0$ and $J$ over the intersection of $V_0$ and $V_1$. Use this isomorphism to glue the projectivization $\mathbb{P}(J)$ with the singular variety $X = (N_0 - i(V'_0))\mathbb{C}^*$. The resulting proper variety $W$ has a natural projection $p : W \to V$ with all the fibers outside $Q$ isomorphic to projective space $\mathbb{P}^{e-1}$. Moreover the preimage of $V - Q$ in $W$ coincides with the projectivization of a vector bundle according to the construction of $W$. The fiber $W_q$ over $q$ coincides with $\mathbb{P}^{e-1}/i_q(G_q)$.

The action of $i(G_q)$ on $\mathbb{P}^{e-1}$ is effective because of our assumption on the representation of
$G \to GL(E)$. We denote by $S_q$ the singular subset of $W_q$. It lies in the image of the fixed sets $\text{Fix}_g(\mathbb{P}^{e-1}) \subset \mathbb{P}^{e-1}$ for different elements $g \in i_q(G_q), g \neq 1$. Define a subvariety $S$ as the union of the varieties $S_q, q \in Q$. The set $S$ has codimension $\geq 3$ in $W$ since the codimension of $S_q$ in $W_q$ is at least 1.

**Lemma 2.9** The variety $W - S$ has a fundamental group isomorphic to $\pi_1(V - Q)/K_Q$.

**Proof** The fundamental group of $W - S$ is the quotient of the fundamental group of $\pi_1(V - Q)$ since $W$ contains an open subvariety which is $\mathbb{P}^{e-1}$ fibration over $V - Q$ and therefore has the same fundamental group. The group $K_q$ maps into zero under the surjective map $\pi_1(V - Q) \to \pi_1(W - S)$ since the image of a neighborhood of $q$ via the zero section $i$ has $K_q$ as a local fundamental group. All the relations are local and concentrated near special fibers. A formal neighborhood of $W_q - S_q$ in $W - S$ is topologically isomorphic to a fibration over $W_q - S_q$ with $M_q$ as a fiber. Therefore all local relations follow from $K_q = 1$. It finishes the proof that $\pi_1(W - S) = \pi_1(V - Q)/K_Q$.  

Finally we prove the projectivity of $W$ by constructing an ample line bundle on it. Start with a line bundle $L$ on $W$ whose sections give an embedding of $X$ into a projective space. To see that such an $L$ exists consider first the $G$-invariant and $\mathbb{C}^*$ homogeneous sections of the trivial bundle $O \otimes E$ over $B_0$. Assume that the degree of homogeneity is big enough and divisible by the order of $G$. It is known that such sections separate the points in the quotient variety $X$ and we obtain an embedding of $X$ into a projective space. Thus the induced bundle $O(1)$ is defined on $X$. Denote by $L$ some extension of $O(1)$ to $W$. Such extension exists since the complement of $X$ in $W$ is smooth.

Next by choosing a polarization $H$ on $V$ appropriately we may assume that the global sections of $L \otimes p^*H$ on $W$ separate all the points of $X$. The restriction of $L$ on $\mathbb{P}(J)$ coincides with $O_{\mathbb{P}(J)}(m)$ for some positive integer $m$. By replacing $H$ with a high power of $H$ if necessary we can produce enough sections of $L \otimes p^*H$ to separate the points of $\mathbb{P}(J)$. Therefore $L \otimes p^*H$ gives an embedding of $W$ into a projective space.  

**Remark 2.4** J.Kollár ([15]) obtained similar result under additional assumption of existence of a surjective map of $\pi_1(V - Q)$ onto a finite group $H$ with $K_q$ as a kernel of the induced map on $L_q$ for every $q \in Q$.

The universal covering $\tilde{F}$ of the smooth projective surface $F$ is very similar to the ramified covering $\tilde{V}$ of $V$ corresponding to the quotient group $\pi_1(V - Q)/K_Q$. Namely there is a natural finite map $p : F \to V$ which induces a finite map from $\tilde{F}$ to $\tilde{V}$. Hence both $\tilde{F}$ and $\tilde{V}$ are simultaneously either holomorphically convex or not.
2.4 Fiber groups

The global construction described in section 2.2 treats separately the part of the fundamental group of the fibered surface which lies in the image of the fundamental group of the fiber. Let $V$ be normal projective complex surface and $Q$ be a set of its singular points. Suppose that there is a projection of $V$ on a smooth curve $C$ which has no multiple fibers and the generic fiber of the projection is a curve of genus $g > 1$.

**Definition 2.2** Denote by $\pi_{1,f}(V - Q)$ the image of the fundamental group $\pi_g$ of the general fiber in $\pi_1(V - Q)$. We will call this group a general fiber group.

In this article we mostly consider the case when the set of singular points in $V$ includes only singularities with finite local fundamental groups. It is well known that these are exactly the quotient singularities.

**Definition 2.3** We will call the group $\pi_{1,f}(V - Q)$ above a fiber group if $Q$ consists of the quotient singularities only.

We shall also give a special notation for the case when $Q$ is empty.

**Definition 2.4** We will call the group $\pi_{1,f}(V)$ projective fiber group if $V$ is a projective fibered surface over $C$ without multiple fibers.

**Remark 2.5** Though we don’t allow multiple fibers in the above definition we allow some multiple components in the singular fibers. We need that at least one component of each singular fiber has multiplicity one.

Thus we have defined three classes of groups. These groups are equipped with a surjective map from the group $\pi_g$. The principal difference between these three classes of groups lies in the nontriviality of the local fundamental groups of normal surface singularity.

**Remark 2.6** It follows that the general fiber group occurs also as a fiber group of a projective surface if the images of all local fundamental groups are finite. In particular this is true if all singular points have finite local fundamental groups.

As a consequences of theorem 2.4 we get:

**Corollary 2.2** The classes of fundamental groups and fiber groups are the same for projective smooth surfaces and projective surfaces minus quotient singularities.
The above results suggest that finding examples of smooth projective surfaces with pathological behavior of fundamental groups and universal coverings can be reduced to a similar problem for normal projective surfaces minus singular points. The latter seems to be an easier task.

3 Nonresidually finite groups

This section contains some material that shows opportunities to make our construction applicable to a big variety of examples. The second subsection shows how one can use our construction and a conjecture by Zelmanov to obtain a variety of potential examples of surfaces with nonresidually finite fundamental groups. It is a pleasure for the second author to thank M. Nori for many illuminating discussions concerning the related ideas.

3.1 Variety of constructions

In this subsection we analyze the groups that can be obtained as fiber groups. Recall that the fiber group depends on the genus $g$ of the generic fiber, the monodromy group and the set of geometric vanishing cycles. Thus we can define an abstract algebraic data which gives us an abstract analogue of the fiber group. Let $M$ be a subgroup of the mapping class group $\text{Map}(g)$ and $g_1, ..., g_N$ be any finite collection of elements in $\pi_g$ which are powers of vanishing cycles. Let $Mg_i$ be the orbit of $g_i$ under the action of $M$.

**Definition 3.1** An abstract fiber invariant is a set of the form $(g, M, Mg_i)$ for some $M \subset \text{Map}(g)$ and some finite set of $g_i \in \pi_g$ as above.

An abstract fiber invariant defines an abstract fiber group as the quotient of $\pi_g$ by a normal subgroup generated by $Mg_i$. We are interested in determining conditions under which this abstract fiber group is the actual fiber group of some geometric fibration which means a complex quasiprojective surface with a fibration over a curve.

**Remark 3.1** The answer is rather simple in smooth or symplectic categories because all the elements of $\text{Map}(g)$ can be realized by the automorphisms of the Riemann surface of genus $g$ which preserve a given volume form. However the question about geometric fiber invariants (smooth projective case) is substantially more delicate.

The following theorem shows that we still have a significant freedom to vary geometric fiber invariants.

**Theorem 3.1** Assume that $(g, M, Ms_i)$ is a fiber invariant of a projection $p : X \to C$ where $X$ is a smooth compact surface and $p$ has only Morse singularities and $s_i$ are the vanishing cycles
corresponding to the singular points $P_i$ of the fibers of $p$. Then $(g, M, Ms_i^{N_i})$ is also a fiber invariant associated with some other fibration provided $N_i = N_j$ if $P_i, P_j$ are contained in the same singular fiber.

**Proof** We shall construct a new fibration with the desired properties by applying the base change construction to the fibration $p : X \to C$. Let $P$ be the set of points of $C$ corresponding to the singular fibers of $p$ and $N(p)$ be a function on $P$ with positive integer values such that $N(s) = N_i = N(P_i)$ if a singular point $P_i \in X_s$ for some $s \in P$.

Let us take a base change $h : R \to C$ where $R$ is a cyclic covering of $C$ of degree $N$ - the minimal integer divisible by all $N_i$, and with ramification indices $N_i$ at $P_i$ and $N$ at some $c \notin P$.

This covering satisfies the following properties:

1) The preimage of $s \in P$ in $R$ is $N(s)$ ramified.
2) The map $\pi_1(R - h^{-1}(P)) \to \pi_1(C - M)$ is surjective.

Indeed the first property is obvious from the construction of $h$. The point $c$ has exactly one preimage in $R$. Hence any closed loop in $C - P$ containing $c$ lifts into a closed loop in $R - h^{-1}(P)$ which proves surjectivity of the corresponding map of fundamental groups.

Now we can induce the family of curves on $R$ by the map $h$. The resulting surface $Y$ is a singular surface with a finite map $f : Y \to X$. The singular points of $Y$ are the preimages of the points $P_i \in X$ with $N_i > 1$. All the singularities of $Y$ are quotient singularities. If we denote the set of singular points by $Q$ then the fiber invariant of the projection $p_h : (Y - Q) \to R$ is described by the data $g, M, Mg_i$.

Indeed the monodromy of the new family coincides with the image of the group $\pi_1(R - h^{-1}(P))$ in $Map(g)$ but the latter is equal to the image of $\pi_1(C - P)$ in $Map(g)$ as it was proved above. Hence the monodromy of the newly obtained family is $M$. All relations in the fiber group of $Y - Q$ are generated by local relations. The latter correspond to the singular points of the fibers of $p_h$. If $f(Q_1) = P_i$ then the corresponding relation is described by $s_i^{N_i}$ as it was shown in section 2.1. It finishes the proof of the theorem.

The above theorem suggests that we can construct a big class of Burnside type groups as fiber groups. We can consider the set of geometric vanishing cycles $s_i$ as a set of simple curves on one copy of the fiber.

The following construction allows to approximate any topological data by the geometric ones. Let $M^L_g$ be the compactified moduli space of curves of genus $g$ corresponding to a subgroup of finite index $M_L$ in the group $Map(g)$. It is an algebraic variety with quotient singularities only which contains a family of similar type irreducible divisors $S_f$ with normal crossings corresponding
to different type of stable degenerations. Singular points of $M^L_g$ correspond to stable curves with automorphisms and constitute a subset of codimension more than 1 if genus $g > 2$. Let $M^L_{g0}$ be an open nonsingular subvariety in $M^L_g$ which corresponds to smooth curves of genus $g$ without extra automorphisms. Then $\pi_1(M^L_{g0}) = M_L$.

The space $M^0_g$, $g > 3$ is far from being affine. If fact there is a natural map of $M^L_g$ into the Satake compactification of the moduli of abelian varieties of dimension $g$ with a principal polarization. It maps each stable curve to a point corresponding to the Jacobian of its normalization. Thus all divisors corresponding to degenerate curve have images of codimension at least 2 if $g > 2$ under this map. In particular generic hyperplane sections of Satake compactification produce complete curves which lie in $M^0_g$.

We are interested in constructing holomorphic families of curves with a given set of singularities. The following construction shows that there almost no restrictions in constructing such families over a disc.

**Lemma 3.1** Let $X_0$ be curve of genus $g > 2$ with a given set of smooth noncontractible cycles $s^k_i$ on it. Assume that for a given $k$ all cycles $s^k_i, s^k_j$ don’t intersect and correspond to different conjugated classes in the fundamental group of $X_0$. Then there is a holomorphic family of curves over a disc $D$ which contains $X_0$ as nonsingular fiber, the cycles $s^k_i$ correspond to the vanishing cycles for degenerate fibers and monodromy group is generated by the products of Dehn twists over $s^k_i$ for each $k$.

**Proof** The condition on $s^k_i$ means that each set $s^k_i$ corresponds to some type of stable degeneration $I(k)$ modulo the action of $Map(g)$. Consider a small complex disc $D_k$ around a generic point of $S_{I(k)}$ in $M_g$. There is a local family of stable curves over $D_k$ which consists of smooth curves outside the point of intersection of $D_k$ and $S_{I(k)}$. Let $p_k$ be a point on the boundary circle $dD_k$. We can find a path $t^k_0$ connecting 0 and $D_k$ inside $M^0_g$ which provides with a diffeomorphisms $X_0 \to X_{p_k}$ which maps $s^k_i$ into a family of vanishing cycles on $X_{p_k}$. Indeed different paths provide with maps which differ by the elements of $Map(g) = \pi_1(M^0_g)$.

Let us take an extension of $t^k_0$ into smooth real analytic curves which ends up transversally at the intersection point of $D_k$ and $S_{I(k)}$. We can assume that all the curves $t^k_0$ meet at 0 being tangent to some one-dimensional complex subspace. Thus we constructed a one dimensional “octopus” $W$ consisting of the extended curves $t^k_0$. After a small variation we can complexify the resulting one-dimensional real set into a complex disc $D$ which contains 0 and intersects a given set of divisors $S_{I(k)}$ with a prescribed monodromy corresponding to the connecting path $t^k_0$. It follows from the fact that a small neighborhood of $W$ is Stein and holomorphic functions on it approximate continuous functions on $W$. The family of stable curves induced on $D$ satisfies the lemma.
Lemma 3.2 For any finitely presented group $\Gamma$ we can construct a relatively projective family of curves $X_t$ over a holomorphic disc which has $\Gamma$ as a fundamental group.

Proof For any group $\Gamma$ above we can find a surjective homomorphism $r : \pi_g \to \Gamma$ for some $g$. Let $N$ be the kernel of $r$. It has a finite number of generators $k_i$ as normal subgroup of $\pi_g$. The elements $k_i$ can be realized as cycles with normal intersections (including selfintersections) only on the curve $X_g$. Let us add a handle at each intersection. We can lift $k_i$ in a new Riemann surface $X_h$ into a family of cycles $\tilde{k}_i$ without selfintersections. Let us add to this family of cycles the generating cycles $a_j, b_j$ of the additional small handles. We obtain the set of conjugated classes which generates the kernel of the projection $\pi_h \to \Gamma$ and each of the elements in this set is realized by a smooth cycle.

Introduce a complex structure on $X_h$ and consider positive Dehn twists corresponding to $\tilde{k}_i, a_j, b_j$. All these cycles correspond to the conjugation classes which belong to the kernel of the projection $\pi_h : \pi_h \to \Gamma$. Note that the Dehn twist along the cycle $s$ acts trivially on the quotient of $\pi_h$ by a normal subgroup generated by $s$. Therefore we can apply lemma 3.1 to $X_h, \tilde{k}_i, a_j, b_j$ and obtain a relatively projective family $X$ of curves of genus $h$ over a disc with $\pi_1(X) = \Gamma$.

Though the holomorphic families of curves look very different from algebraic ones we can rather easily embed a small deformation of such a family into an algebraic one.

Denote by $M$ the monodromy group of the family of curves over disc described in the lemma. Note that $D$ is a Stein subvariety in $M_g$. We can lift it into any covering of $M_g$ which is unramified along $D$. In particular we can lift $D$ into any variety $M^L_g$ for a subgroup of finite index $M^L \subset \text{Map}(g)$ containing $M$. Since $D$ lies in affine subset of $M_g$ we can find an algebraic curve $C \subset M_g$ (respectively $M^L_g$) which contains a small variation of $D$. The family over $C$ induced from $M_g$ or $M^L_g$ extends globally a small variation of the family over $D$ without changing its topological data. By taking generic $C$ we can assume that the resulting family $Y$ of curves over a normalization of $C$ is smooth and the projection $p : Y \to C$ is a Morse type map. Thus having a family over disc with arbitrary data $g, Ms_i$ where $M$ is generated by the Dehn twists over $s_i$ we obtain the following geometric data $(g, M_Ls_i, M_Ls_j)$ for any subgroup of finite index in $M_g$ containing $M$. Accordingly considering any data $g, Ms_i$ we can obtain a Burnside type approximation $(g, M_Ls_i^{N_1}, M_Ls_j^{N_2})$ for any $N_1, N_2$. By taking $N_1 = N$ and $N_2 = N^B$, increasing the integer $B$ and decreasing $M_L$ we obtain series which approximate the group given by $(g, Ms_i^N)$.

Remark 3.2 It seems plausible that most of the groups in such a series are nonresidually finite and violate any other good properties of linear groups if the group $(g, Ms_i)$ violates them.
Remark 3.3 The above construction can be applied also to algebraic manifolds parametrizing special curves instead of moduli spaces. It results in different series of groups as monodromy groups.

3.2 Potential examples

Now we suggest a construction that can lead to rather simple new examples of surfaces with nonresidually finite groups. We thank V. Alexeev, S. Keel and M.Nori, for the fruitful discussions of the construction.

Consider the map $f$ of the moduli space $\mathcal{M}^L_g$ into Satake compactification $S^g$ of the moduli space of abelian varieties of dimension $g$ with a principal polarization. The latter is a projective variety which has a representation as a union of $A^g$, $A^{g-1}$,..., where $A^g$ is a quotient of the space of positively defined hermitian matrices of rank $g$ by the action of $Sp(2g,\mathbb{Z})$. If $x \in \mathcal{M}^L_g$ corresponds to a stable curve $X$ then $f(x) \in S^g$ is a point corresponding to the Jacobian of the normalization of the curve $X$. Denote by $SM^g$ the closure of the image $f(M_g)$ in $S^g$. If $g > 3$ the map $f$ contracts analytic subvarieties corresponding to the degenerate curves and curves with nontrivial automorphisms into proper analytic subvarieties in $SM^g$ of codimension at least two. Denote by $\Delta_0$ the divisor in $\mathcal{M}^L_g$ corresponding to irreducible stable curves with one node. The image of $f(\Delta_0)$ consists of all the points of $S_{g-1} \subset S^g$ which correspond to the Jacobian varieties of dimension $g-1$. Generic jacobian is known to be a simple abelian variety. On the other hand the images of divisors corresponding to different type of stable degeneration intersect $S_{g-1}$ in proper subvarieties corresponding to nonsimple abelian subvarieties (they decompose into a product after isogeny).

Consider the surface $V$ in $SM^g$ obtained by a set of hyperplane sections.

We can assume that:

1) The fundamental group of an open part of $V$ surjects onto $Map(g)$.

2) The surface $V$ intersects the images of different divisors $S_I$ at a point only.

3) The intersection of $V$ and $f(\Delta_0)$ does not include points from the images of other divisors $S_I$ or subsets corresponding to curves with automorphisms.

Denote the latter as $R \subset V$. By resolving $V$ at $R$ only we again obtain a projective surface $V'$. General hyperplane section $C$ of $V'$ will lift into a curve $C'$ in $\mathcal{M}^L_g$ which intersects only $\Delta_0$ and $C' - \Delta_0$ is contained in $\mathcal{M}^{L,0}_g$. Thus we have a family of curves $X_g$ over $C'$ which has singular fibers of one type only and the monodromy group of it coincides with $M^L$.

Now we apply our construction to the above surface $V'$ and get a surface $V^N$ whose fundamental group $\pi_1(V^N)$ is a of a finite index in an extension of the fundamental group of a Riemann surface by $\pi^d/(x^N = 1)$. Therefore Zelmanov’s conjecture implies that the group $\pi_1(V^N)$ is nonresidually finite. Hence we obtain a series of simple potential examples of surfaces with nonresidually finite fundamental groups.
The considerations from the previous subsection allow us to get even bigger variety of examples. Let us make a:

**Definition 3.2** Let $x$ be a primitive element in the fundamental group $\pi_1(g)$ of a Riemann surface of genus $g > 1$ and $M^L$ be a subgroup of finite index in $\text{Map}(g)$. Consider the orbit of $x$ under $M^L$, $(M^Lx)$ and take the $N$-th powers of all this elements. Consider the normal closure of this powers in $\pi_1(g)$. Let us denote this normal closure by $(M^Lx)^N = 1$. We will denote the quotient group by $\pi_1(g)/(M^Lx)^N = 1$. (Observe that definition depends on the choice of $x$.)

Now the following generalization of the conjecture of Zelmanov's gives us a way of constructing more examples of surfaces with nonresidually finite fundamental groups.

**Question** (Zelmanov) For big $g$ and $N$ the groups $\pi_1(g)/(M^Lx)^N = 1$ are nonresidually finite for any primitive $x$ and $M^L$ a subgroup of finite index in $\text{Map}(g)$.

### 4 Some remarks on Shafarevich’s conjecture for fibered surfaces

#### 4.1 The case of projective surfaces

In this section we consider potential counterexamples to the Shafarevich’s conjecture based on our construction. We begin with the general setting. Let $f : X \to R$ be a Morse type fibration with $X_t$ as generic fiber. Suppose that the fiber $X_0$ is singular and has more than one component $X_0 = \bigcup C_i$. We also assume that all components $C_i$ are smooth and without selfintersection. Denote the intersection graph of $X_0$ by $\Gamma_0$. Consider the retraction $cr : X_t \to X_0$ of generic fiber on special fiber (see section 1).

The preimage $cr^{-1}(C_i)$ in $X_t$ is an open Riemann surface with a boundary consisting of geometric vanishing cycles corresponding to the intersection points of $C_i$ with other components of $X_0$. The fundamental group $\pi_1(cr^{-1}(C_i))$ is free. The natural embedding $cr^{-1}(C_i)$ into generic fiber $X_g$ defines an embedding of the fundamental groups $\pi_1(cr^{-1}(C_i)) = \mathbb{F}_i \to \pi_g$. Similar construction holds for any proper subgraph of curves in $X_0$.

**Definition 4.1** For any proper subgraph $K \subset \Gamma_0$ define a subgroup $\mathbb{F}_K \subset \pi_g$ as a fundamental group of the preimage $cr^{-1}(\bigcup C_i), i \in K$.

**Remark 4.1** If the graph $K$ is connected then its preimage in $X_t$ has only one component and vise versa.

Let $\pi_{1,f}$ be a fiber group obtained from $\pi_g$ by our base change construction for some $N$. 

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Lemma 4.1 Suppose that there is a decomposition of a connected subgraph $K \subset \Gamma_0$ into a union $K_1 \cup K_2$ so that the image of $F_K$ in $\pi_1, f$ is infinite, but the image of both $F_{K_1}, F_{K_2}$ is finite then the Shafarevich conjecture is not true.

**Proof** Indeed under the conditions of the lemma we obtain an infinite connected graph of compact curves in the universal covering of the surface $S^N$.

Now we choose $N = 3$. In this case we can apply the above lemma due to the group theoretic result which concerns the quotients of the free groups.

Definition 4.2 Let $\mathbb{F}_k^g$ be a free group with $k$ generators with a realization as a fundamental group of curve minus one or two points (depending on $k$). Define $P^g(k, 3)$ as the quotient of $\mathbb{F}_k^g$ by the set of relations $x^3 = 1$ for all primitive elements of $\mathbb{F}_k$ which can be realized by smooth nonintersecting curves in the above geometric realization $\mathbb{F}_k^g$ of $\mathbb{F}_k$.

Theorem 4.1 The group $P^g(k, 3)$ is equal to the Burnside group $B(k, 3)$ and hence finite.

**Proof** See Appendix B.

Corollary 4.1 The group $\pi_g/(x^3 = 1)$ is a quotient of $B(2g, 3)$ by one additional relation.

Let $X_0$ be a graph of smooth curves $C_i$ with each curve intersecting at most two others (chain or ring). Suppose that there exists a Morse family of curves $X \to R$ with a set of vanishing cycles $VS$ and a monodromy group $M$ which has $X_0$ as fiber. Assume that cycles from $VS$ correspond to different singular points of $R$ unless they correspond to $X_0$. Assume that $VS$ is decomposed into a union $S_0 \cup S_1 \cup S_2$. Assume that cycles from different subsets $S_i, i = 0, 1, 2$ correspond to different singular points of $R$.

Lemma 4.2 Suppose that the monodromy group $M$ and the sets of vanishing cycles $S_0, S_1, S_2$ satisfy the following properties:

1. The image of $\pi_1(C_i)$ in $\pi_g/Ms_j^3$ is finite for any $i, s_j \in S_0 \cup S_1$.
2. The quotient group $\pi_g/(Ms_j^3, Ms_k^S)$ is infinite for some integer $S$ and $s_j \in S_0 \cup S_1, s_k \in S_2$.

Then the universal covering $\tilde{X}$ is holomorphically nonconvex.

**Proof** Indeed the universal covering $\tilde{X}$ contains an infinite covering of $X_0$ which is connected and consists of compact curves. Hence $\tilde{X}$ is not holomorphically convex.
We are going to construct a family of surfaces which presumably contain an infinite number of surfaces with above property. We would like to produce such families from a standard family over an interval \( I \).

Let \( g > 3 \) and consider a family over an interval \( I = [0,1] \) which has a fiber \( X_0 \) over 0 and a singular fiber \( X_1 \). The generic fiber \( X_t \) surjects on \( X_0 \) and \( X_1 \) and vanishing cycles for both singular fibers are realized as smooth curves on \( X_t \). Denote the corresponding set of cycles as \( S_0, S_1 \) respectively. We assume that they don’t intersect and for any component \( C_i \subset X_0 \) there is a corresponding cycle \( s_i \in S_1 \) which projects into a smooth homologically nontrivial cycle in \( X_i \).

This family over interval can be complexified into an algebraic family over a complete curve \( R \). We can assume that the monodromy of the resulting family is any subgroup of finite index in the group \( \text{Map}(g) \) which contains commuting monodromy transformations \( T_0, T_1 \) defined by the fibers \( X_0, X_1 \).

Let \( M_D \) be a subgroup of \( \text{Map}(g) \) which commutes a contraction map \( \pi_g \rightarrow \pi_1(X_0) \). Since the group \( M_D \) contains \( \text{Map}(g) \) for any curve \( C_i \) we have obtain that the image \( \pi_1(\text{cr}^{-1}(C_i))/(M_D s_i^3) \) is a finite group of exponent 3 for any component \( C_i \).

**Lemma 4.3** The quotient \( \pi_g/(M_D s_i^3) \) is infinite if the number of components \( C_i, g(C_i) > 0 \) is more than 1.

**Proof** Indeed the group above maps surjectively onto a free product of nontrivial Burnside groups of exponent 3 corresponding to different components of \( X_0 \) with nonzero genus. The latter is infinite which implies the lemma. □

Let \( M^f \) be any subgroup of \( M_D \) which contains \( T_0, T_1 \) and has the property that the image of \( \pi_1(\text{cr}^{-1}C_i) \) in \( \pi_g/(M_D s_i^3) \) is finite.

The group \( M^f \) defines a set of subgroups \( M^f_j \) of finite index in \( \text{Map}(g) \) containing \( M^f \). The intersection of this set of subgroups coincides with \( M^f \) since \( \text{Map}(g) \) is residually finite.

For each \( M^f_j \) we can find a curve \( R_j \) with a Morse family of curves \( X_j \) which contains a topological family over interval constructed above and with a monodromy \( M^f_j \). Let \( S_0^j \) be a complementary set of vanishing cycles in the family \( X_j \). We can now consider any \( S \) and construct a new family \( X^S_j \) using the theorem 3.1 with \( N_0 = N_1 = 3, N_2 = 3^A \).

**Corollary 4.2** For \( M^f_j \) and integer \( A > 1 \) we obtain a group \( \pi_g/((M^f_j s_i^3, M^f_j s_k^A)) \) as a fiber group. Here \( s_i \in S_0 \subset S_1, s_k \subset S_2 \).

**Remark 4.2** The subset \( S_2 \) depends on the actual curve \( R_j \). The dependence of the fiber group on \( S_2 \) weakens with \( A \) converging to infinity. The resulting family of groups approximates the infinite group \( \pi_g/(M^f s_i^3) \) as \( M^f_j \) converges to \( M^f \) and \( A \) converges to infinity.
Conjecture 4.1 Let $X_0$ be a nontrivial chain or ring of curves of genus greater than 0. Assume that $g$ is an arithmetic genus of $X_0$ and $X_I$ is a family of curves over an interval described above. For any small enough subgroup $M_f \subset M_D$ defined above there exists a subgroup $M_f^I \subset \text{Map}(g)_I$ of finite index and an integer $A > 1$ such that $\pi_g/(M_f^I s_i^3, M_f s_k^3)$ is infinite for $s_i \subset S_0 \cup S_1$ and any finite subset $s_k \in \pi_g$

If the answer to the above conjecture is positive then the Shafarevich conjecture is not true. The fact that the family of groups parameterized by $M_f^I$ and $A$ approximates a group which has a nontrivial free product of groups as a quotient provides with a strong evidence supporting the above conjecture. On the other and the resulting fiber group does not have infinite linear representations which are equivariant with respect to the action of the monodromy group (\cite{14}).

There is also another possibility to satisfy condition 2. We can easily construct a family of curves of genus $g = 2k, k > 1$ such that there is a fiber in this family which consists of two components each of genus $k$ that give us a tree of components. Applying the base change construction for a given $N$ we obtain a surface $S - Q$ with the image of the fundamental group of every component in $\pi_1(S - Q)$ being equal to $P_g(2k, N)$. The fiber group of $S - Q$ is equal to $\pi_{2k}/(x^N = 1)$. Now as we have shown we have the same behavior on $S^N$ for the closed curves and surfaces. We can formulate the following question:

**Question** Are there such a $k$ and $N$ such that $P_g(2k, N)$ is a finite group and $\pi_{2k}/(x^N = 1)$ is an infinite group?

If the answer of the above question is affirmative for some $N, k$ we get a counterexample to the Shafarevich conjecture. We should point out that if the groups obtained from the components are finite then $\pi_{2g}/(x^N = 1)$ does not have infinite linear representation. If the Shafarevich conjecture is correct the answer of the above question is negative. It also implies the answer to many similar group theoretic questions. The most basic question seems to be the following:

**Question** Is there such an $N$ and such $2 \leq m_1 < m_2$ for which $B(m_1, N)$ is finite and $B(m_2, N)$ is infinite?

Recently we were informed by Zelmanov that he can show that there exists an integer $d(0)$ so that for every prime number $p$ and an integer $d$ the group $B(d_0, p)$ is finite if and only if the group $B(d, p)$ is finite. This result suggests the existence of an abstract group theoretic version of holomorphic convexity. The above considerations indicate a possibility for analysis of infinite groups by analytic methods.
4.2 Other applications

In this subsection we discuss symplectic and arithmetic versions of our construction. As Gompf has shown \[9\], every finitely presented group can be realized as a fundamental group of a symplectic manifold. It is reasonable to ask the if being symplectic puts any restrictions on the structure of universal coverings. Our construction easily extends to the symplectic category. It leads to interesting examples of symplectic four dimensional manifolds (see \[2\]). Using this construction we have defined in \[2\] an obstruction to a symplectic Lefschetz pencil being a Kähler Lefschetz pencil.

We begin with a symplectic fourfold $X$. Consider the corresponding Lefschetz pencil with reducible fiber and apply to it the our construction. So we get for a fix integer $N$ a symplectic fourfold $S^N$. Let $\rho$ be a generic representation $\rho: \pi_1(S^N) \to GL(n, \mathbb{C})$ whose image is not virtually equal to $\mathbb{Z}$. Denote by $Y_i$ the components of the preimage of the reducible fiber of $S$ in $S^N$ and denote by $F$ the general fiber of $S^N$. Denote by $\Gamma$ the image of $\pi_1(F)$ in $\pi_1(S^N)$ and by $\Gamma_i$ the images of the fundamental groups of $Y_i$ in $\pi_1(S^N)$.

If the restrictions of $\rho$ on $\Gamma$ and $\Gamma_i$ for all $i$ are both finite or infinite we will say that the obstruction $O(X)^{N,n}$ is equal to zero and to one otherwise.

**Proposition 4.1** If $S$ is Kähler $O(X)^{N,n}$ is trivial for every pair $N, n$.

Indeed otherwise we contradict the holomorphic convexity of the covering corresponding to infinite linear representation of the fundamental group. In \[2\] we have constructed examples of symplectic fourfolds with nontrivial $O(X)^{N,n}$.

In this article we consider only surfaces with given projection on a curve. Algebraically this means that the field of rational functions on the surface is provided with a structure of a one-dimensional field over a field of rational functions of the base curve. This picture is parallel to that of a curve defined over number field. It suggests that there should be a natural arithmetic version of the Shafarevich’s conjecture. The notion of holomorphic convexity is not well defined in the arithmetic case. Instead we can describe the analogue of the absence of infinite chains of compact curves in the universal covering in the arithmetic case. The absence of infinite chains of compact curves seems to be the only obstruction to the holomorphic convexity in the case of compact complex surfaces.

Let $C$ be a projective semi-stable curve over $K$. Since $K$ is not algebraically closed we obtain a nontrivial map $C \to Spec(O_K)$ where $O_K$ is the ring of integers. Extending $K$ if necessary we can assume that $C$ is a semistable curve. That means $C$ is a normal variety with semi-stable fibers consisting of normally intersecting divisors. The function field $K(C)$ is regular and has dimension one over $K$. Any maximal ideal $\nu$ in $O_K$ defines a subring $O_{\nu}$ of $K(C)$ which consists of the elements
of $K(C)$ which are regular at the generic point of any component of the preimage of $\rho$ in the scheme of $C$. The ring $O_\nu$ contains the ideal of elements which are trivial on the preimage of $\nu$. Denote this ideal by $M_\nu$. The quotient ring $O_\nu/M_\nu$ is a finite sum of fields of rational functions on the components of the fiber of $C$ over $\nu$. Consider the maximal nonramified extension $K(C)^{nr}$ of $K(C)$. It is a Galois extension with a profinite Galois group $Gal^{nr}(K(C))$. The field $K(C)^{nr}$ contains the maximal nonramified extension $K^{nr}$ of the field $K$ as a subfield. (Observe that the field $\mathbb{Q}^{nr} = \mathbb{Q}$ but for many other fields $K$ the field $K^{nr}$ is an infinite extension.) The group $Gal^{nr}(K(C))$ maps surjectively onto $Gal^{nr}(K)$. Denote the kernel of the corresponding projection as $Gal^{nr}_g K(C)$.

Any maximal ideal $\rho$ in the ring of integers $O_{K^{nr}}$ contains the unique maximal ideal $\nu$ of $O_K$. We can now define the subring $A_{\rho}$ as an integral algebraic closure of the subring $O_\nu$ in $K^{nr}(C)$. Let $Res_\rho = A_{\rho}/I(\rho)$ be the quotient ring by the ideal generated by $\rho$ in $A_{\rho}$. It is a semisimple ring of finite characteristics. We formulate a strong arithmetic analogue of the Shafarevich’s conjecture.

**Conjecture 4.2** Let $K(C)$ be a field as above. For some finite extension $F$ of $K$ we can find a semistable model $C'$ of the field $F(C)$ such that the ring $Res_\rho$ is a direct sum of a finite number of fields for any ideal $\rho$ of the field $F^{nr}$.

We can also formulate this conjecture in a more geometric language. Namely if $C'$ is a semistable curve over $F$ then for any finite nonramified extension $L$ of $F(C')$ we have a model $C_L$ with a finite map onto $C'$. The fibers of this new model $C_L$ are uniquely defined by $C'$. The result of the conjecture depends (at least formally) on the chosen model $C'$. If we blow up a generic point on a fiber the conjecture becomes false if the corresponding covering induces an infinite covering of the fiber. Thus the arithmetic version is sensitive to the change of the semistable model whereas the geometric conjecture is not.

**Remark 4.3** The fields $F(C')^{nr}$ correspond to the factor of $\pi_1^{fin}(C')$ that is acted trivially by all inertia groups. (Here we denote by $\pi_1^{fin}(C')$ the profinite completion of the geometric fundamental group of $C'$.) Geometrically this means that we consider coverings of $C_L$ that are unramified over a generic point of every irreducible divisor in $C_L$.

There exists some evidence for the above arithmetic conjectures. Partial results in this direction were obtained by the first author several years ago (1982). Namely he proved that the torsion group
of an abelian variety $A$ is finite for any infinite algebraic extension of $K$ which contains only finite abelian extensions of $K$.

In particular it is true for infinite nonramified extension of $K$ where $K$ is a finite extension of $\mathbb{Q}$. (This result was announced at Delange-Puiso seminar in Paris, May 1982 and later appeared in [7]). Yu. Zarhin proved that the same is true if the infinite extension of $K$ contains only finite number of roots of unity under some conditions on the algebra of endomorphisms $A$.

The above result states that the group $Gal^{nr}_gK(C)$ has a finite abelian quotient and hence the conjecture 4.2 is evidently true for the quotient of $Gal^{nr}_gK(C)$ by the commutant of $Gal^{nr}_gK(C)$. For the same reason it is true for the quotient of $Gal^{nr}_gK(C)$ by any iterated commutant.

**Remark 4.5** Any curve over arithmetic field can be obtained as a covering of $\mathbb{P}^1$ ramified at $(0, 1, \infty)$ according to the famous Bely’s theorem. As it was pointed out by Yu. Manin any arithmetic curve has a ramified covering which is a nonramified covering of a modular curve. Therefore one might reduce the above conjecture to modular curves being considered over different number fields. In the complex case we don’t have similar simple class of dominant manifolds (see the discussion in [3]).

If Conjecture 4.2 is correct for every curve over every finite extension of $\mathbb{Q}$ then we get that there isn’t any infinite chains of compact curves on the universal covering of a projective surface with a residually finite fundamental group. According to a conjecture of M. Ramachandran this is the only obstruction to holomorphic convexity for the universal coverings of a projective surfaces.

5 Appendix A - Fiber groups and monodromy

This appendix includes several technical results from the theory of surfaces which we use in our article. First theorem generalizes the result we used in order to pass from quasiprojective surface to the projective smooth surface in section 2.

**Theorem 5.1** Let $V$ be a normal projective surface with $Q$ being the set of singular points in $V$ and $f : V' \to V$ be a finite surjective map from another normal projective surface $V'$. Assume that for any $q \in f^{-1}(Q)$ the map of the local fundamental groups $f_\ast : \pi_1(U_q - q) \to \pi_1(V - Q)$ is zero, where $U_q$ is small topological neighborhood of $q$. Then there is a natural map $f_\ast : \pi_1(V') \to \pi_1(V - Q)$ which is a surjection on a subgroup of finite index bounded from above by the degree of $f$.

**Proof** Indeed $Q$ is a set of isolated singular points. Following the proof of the lemma 2.5 we obtain a map $f_\ast : \pi_1(V' - f^{-1}(Q)) \to \pi_1(V - Q)$. In order to prove the theorem it is sufficient to show that the kernel of natural surjection $i_\ast : \pi_1(V' - f^{-1}(Q)) \to \pi_1(V')$ lies in the kernel of $f_\ast$. 

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The preimage $f^{-1}(Q)$ also consists of a finite number of points and the kernel of $i_*$ is generated as a normal subgroup in $\pi_1(V' - f^{-1}(Q))$ by the local subgroups $\pi_1(U_q - q), q \in f^{-1}(Q)$. Due to the condition of the theorem the images of these groups are trivial in $\pi_1(V - Q)$ and hence we obtain the map $f_*$.

We shall also need the following general result which allows to compare the properties of fiber groups and fundamental groups of the smooth quasiprojective surfaces.

**Theorem 5.2** Let $f : X \to R$ be a quasiprojective surface which has a structure of a family of curves over a smooth curve $R$. Assume that generic fiber is an irreducible smooth projective curve and any fiber contains a component of multiplicity one. Let $h : \pi_1, f(X) \to G$ be $M_X$-invariant homomorphism into a finite group $G$ with $M_X$ action.

Let $K$ be the kernel of $h$. Then there exists a subgroup of finite index $H \in \pi_1(X)$ that the intersection of $H$ with $\pi_1, f(X)$ is equal to $K$.

**Remark 5.1** This is true if there exists a section $s : \pi_1(R) \to \pi_1(X)$, since we can define $H$ as a subgroup of $\pi_1(X)$ generated by products of the elements of $s(\pi_1(R))$ and $K$. If the curve $R$ is open then the group $\pi_1(R)$ is free and hence a section always exists.

**Proof** The group $K$ is a normal subgroup of $\pi_1(X)$ since it is invariant under the conjugations from both $\pi_1(R)$ and $\pi_1, f(X)$. Denote by $Q$ the quotient $\pi_1(X)/K$. It is an extension of $\pi_1(R)$ by $G$. Hence there is an action of $\pi_1(R)$ over $G$. Since $G$ is a finite group $\pi_1(R)$ contains a subgroup of finite index which acts trivially on $G$ via interior endomorphisms. This subgroup corresponds to a finite nonramified covering $\phi : C \to R$ and will be denoted by $\pi_1(C)$. Its preimage $Q'$ in $Q$ is a subgroup of finite index. Consider a subgroup $K_G$ of $Q'$ consisting of the elements commuting with all the elements of $G$. The group $K_G$ projects onto $\pi_1(C)$ by the definition of $\pi_1(C)$ and therefore its intersection with $G$ is a cyclic subgroup of the center of $G$. The group $\pi_1(C)$ contains a subgroup of finite index $K'$ where the corresponding central extension splits. Namely if the order of the corresponding cyclic extension is $n$ then it splits over any cyclic covering of order $n$. The preimage of $K'$ in $Q'$ splits into a direct product of $G$ and $K'$. Hence on the preimage of a subgroup $K' \subset \pi_1(C) \subset \pi_1(R)$ we have a natural extension of the projection from $\pi_1, f(X) \to G$.

### 6 Appendix B - Some group theoretic results

This appendix contains several group theoretic results. We present proofs though most of the results are presumably known to the experts in the group theory.

Recall that we denote by $BT(n, m)$ the quotient of a free group $F^n$ by a normal subgroup generated by the elements $x^m = 1$ where $x$ runs through all primitive elements of $F^n$. 

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Proposition 6.1 If \( m \) is divisible by 4 and \( n \geq 2 \) then \( BT(2, m) \) is infinite.

**Proof** The group \( BT(2, 4) \) has an infinite representation. Namely let \( Q_8 \) be the group of the unit quaternions of order 8. It acts on \( H = \mathbb{R}^4 \) by multiplications. Consider a group of the affine transformations of \( \mathbb{R}^4 \) generated by two generating rotations \( g_1, g_2 \) in \( Q_8 \) but with different invariant points. The resulting group \( G \) will be infinite. It has two generators and \( G \) has \( Q_8 \) as its quotient group. Any element of \( g \in G \) which projects nontrivially into \( G \) has order 4. Indeed \( g \) is an affine transformation of \( \mathbb{R}^4 \) with a linear part of order 4 and without 1 as eigenvalue. Hence \( g \) in the conjugacy class of its linear part and has order 4.

\( \square \)

**Remark 6.1** Similar argument can be applied to any finite subgroup of a skewfield instead of \( Q_8 \) (see the description of such finite groups in J. Amitsur, Ann. of Math., 1955, vol. 62, p. 8). The above result makes it plausible that the groups \( P(n, m) \) are infinite if \( m > 3, n > 1 \).

The case \( BT(n, 3) \) is different. The following result gives a hint on the effects which occur with the exponent 3.

**Lemma 6.1** Let \( G_i, i = 1, 2 \) be the groups generated by \( a, b \) with relations

1) \( G_1 : a^3 = b^3 = (ab)^3 = 1 \)
2) \( G_2 : a^3 = b^3 = (ab)^3 = (ab^2)^3 = 1 \).

Then \( G_1 \) is infinite and \( G_2 \) coincides with the Burnside group \( B(2, 3) \) and hence it is finite.

**Proof** The group \( G_1 \) has a natural geometric realization. Namely let us take \( \mathbb{P}^1 \) minus three points \( p_i, i = 1, 2, 3 \). The fundamental group \( \pi_1(\mathbb{P}^1 - p_i) = \mathbb{F}_2 \). If we impose relations \( x^3 \) on the elements which can be realized by smooth curves in \( \mathbb{P}^1 - p_i \) then we obtain the set 1). Let us take a \( \mathbb{Z}_3 \) character \( \chi \) of \( \mathbb{F}_2 \) which is nontrivial on \( a, b, ab \). We obtain a covering of \( \mathbb{P}^1 \) ramified over three points. Imposing the above relations corresponds to the completion of the curve. Hence the subgroup of \( G_1 \) which is the kernel of \( \chi \) coincides with the fundamental group of the corresponding complete curve. Since the above curve is a torus the group \( G_1 \) is an extension of \( \mathbb{Z}_3 \) by a free abelian group \( \mathbb{Z} + \mathbb{Z} \). The description of 2) immediately follows since it is the quotient group of the group in 1). The element \( ab^2 \) generates \( \mathbb{Z} + \mathbb{Z} \) as a \( \mathbb{Z}_3 \)-module. Hence \( (ab^2)^3 \) generates \( 3(\mathbb{Z} + \mathbb{Z}) \) and the resulting group is an extension of \( \mathbb{Z}_3 \) by the group \( \mathbb{Z}_3 + \mathbb{Z}_3 \).

\( \square \)

We shall use the following standard notations. Denote by \( (a, b) \) the commutator of the elements \( a, b \) and we put a sequence of brackets to denote an element obtained by iteration of the procedure.
The group $B(n,3)$ has a rather simple description. It is a metabelian group with a central series of length three. The elements $(a, x)$ where $x \in [B(n,3), B(n,3)]$ are in the center of $B(n,3)$. In fact $((a, b), c)$ varies under permutation according to standard $\mathbb{Z}_2$ character of the group $S_3$ for any $((a, b), c)$. In particular $((x, r), x) = 1$ for any $x$.

**Lemma 6.2** Let $G$ be a finitely generated group with a given set $S$ of generators. Assume that $((a, b), f)$ is invariant under any even permutation of $a, b, f$ for any $a, b \in S, f \in S \cup (S, S)$ where the latter denotes the set of pairwise commutators of the elements from $S$. Then $G$ is a metabelian group and it has a central series of length at most 3.

**Proof** The proof closely follows the proof from [17]. Indeed we can write $((a, b), (c, d)) = (a, (c, d)), b)$. We deduce next that the above expression is invariant under even permutations. The latter implies that it is equal to 1 and hence $((a, b), c)$ commutes with any element from $S$ and hence lies in the center of $G$. The quotient of $G$ by the center $Z \in G$ is also generated by $S$ with equality $((a, b), c) = 1$ for any $a, b, c \in S$ which means that $(a, b)$ is in the center of $G/Z$ for any $a, b \in S$. That means $G/Z$ is a central extension of abelian group which finishes the proof.

\[\square\]

**Corollary 6.1** Under the above conditions the commutant $[G, G]$ is additively generated by the elements $(a, b), ((a, b)c), a, b, c \in S$.

**Lemma 6.3** Assume that $S$ in the above lemma consists of $n$ elements , $[S], [S, S]$ consists of elements of order 3 and there is a surjective map $p : G \rightarrow B(n,3)$. Then $p$ is an isomorphism.

**Proof** The commutant $[G, G]$ is additively generated as $G^{ab}$-module by the elements $(a, b)$ under the above conditions. Hence it is of exponent 3 and the number of elements in $G$ is not greater than the number of elements in the commutant of $B(n,3)$. The abelian quotient $G^{ab}$ is isomorphic to $B(n,3)^{ab}$. Therefore the number of elements in $G$ is not greater than in $B(n,3)$ and a surjective map $p : G \rightarrow B(n,3)$ is an isomorphism.

\[\square\]

**Proposition 6.2** The group $BT(n,3)$ is finite and coincides with the Burnside group $B(n,3)$ if $BT(3,3) = B(3,3)$.

**Proof** Indeed this is true for $n = 2$ as it was shown above ( see e.g. [17]). The group $BT(n,3)$ has a natural surjective map onto $B(n,3)$ and hence we have to check the condition on $(a, b), c)$. The latter is enough to check for the group with three generators.

\[\square\]
Lemma 6.4 \( BT(3, 3) = B(3, 3) \).

Proof

The group \( BT(n, 3) \) is obtained as an extension of \( B(3, 2) \) by \( c \) and since \( cx \) is a generator for any \( x \in B(3, 2) \) we obtain that \( (cx)^3 = 1 \). Therefore \( x^{-1}cx, c \) commute for any \( x \in B(3, 2) \) and the kernel \( K \) of the projection \( p_c : BT(3, 3) \to B(3, 2) \) is an abelian group of exponent 3. The sum \( x^{-1}cx + c + xcx^{-1} = 0 \) and we can easily deduce that \( K \) has 4 generators as a \( \mathbb{Z}_3 \)-space. Therefore the group \( BT(3, 3) \) has the same number of elements as \( B(3, 3) \) and since there is a natural surjection \( BT(3, 3) \to B(3, 3) \) they are isomorphic.

\( \square \)

Lemma 6.5 Let \( S \) be a finite set of generators of the group \( G \). Assume that any subset of 4 elements in \( S \) generates a subgroup of exponent 3. Then \( G \) is of exponent 3.

Proof The assumption implies that any three-commutator of \( a, b, c \in S \) lies in the center \( C \) of the group \( G \) and the group satisfies lemmas 6.2, 6.3.

\( \square \)

Lemma 6.6 Assume that \( G \) has four generators \( a, b, c, d \) that \( G \) contains a set of subgroups of exponent 3 which includes groups generated by triples of generators \( a, b, c, d \) and also the groups \( a, b, (cd), c, d, (a, b) \). Then \( G \) is of exponent 3.

Proof It follows from the above results that \( ((a, b)c) \) and similar combinations are invariant under cyclic permutations. We also have \( ((a, b)(c, d)) = (((c, d)a)b = (((d, c)b)a) = (((a, b)c)d) = (((a, b)d)c) \). Thus the value of the above commutator does not depend on any even permutation of the symbols. On the other hand it transforms into opposite if we permute \( (c, d)(a, b) \). Hence all the above elements are equal to 1. This implies the lemma.

\( \square \)

We are interested in the groups which occur as the quotients of the fundamental groups of curves. Namely we will study the groups \( P^g(n, m) \) defined in section four. We are going to use a representation of the fundamental group of a Riemann surface as a subgroup of index 2 in the group generated by involutions which was successfully used by J.Birman, M.Nori, W.Thurston , B.Wainrieb and many others. Let \( X \) be a curve of genus \( g \). It can be represented as a double covering of \( \mathbb{P}^1 \) ramified over \( 2g + 2 \) points and \( \pi_g \) is a subgroup of index two in the group generated by \( 2g + 2 \) involutions \( x_i \) with additional relation that the product of all involutions is an involution again. Similarly the group \( \pi_1(X_g - pt) \) is realized as subgroup of index two in the group generated
by \(2g+1\) involutions. The fundamental group of a curve minus two points is realized as a subgroup of index two in the group generated by \(2g+2\) involutions without any additional relations.

The groups above are free groups, but they are provided with a special realization as the fundamental groups of open curves.

**Definition 6.1** We shall denote by \(F^n_g\) a free group of \(n\) generators provided with a realization as a fundamental group of a curve of genus \(g\) minus one or two points. In case \(n\) is odd \(F^n_g\) is realized as a fundamental group of a curve of genus \(g\) minus one point. In case \(n\) is even \(F^n_g\) is realized as a fundamental group of a curve of genus \(g\) minus two points.

Recall that \(P^g(n, m)\) is a quotient of \(F^n_g\) by the relations \(x^m = 1\) where \(x\) runs through the primitive elements of \(F^n_g\) which can be realized as a smooth loops in the above geometric realization.

The following lemma reduces a general case to \(n \leq 4\)

**Lemma 6.7** If \(P^g(4, 3) = B(4, 3)\) then \(P^g(n, 3) = B(n, 3)\) for any \(n \geq 4\).

**Proof**

The group \(P^g\) is represented as subgroup of index two generated by involutions \(x_1, \ldots, x_{n+1}\). The set of standard generators of \(F^n_g\) can be taken as \(x_1x_i, i \neq 1\). Any four elements \(x_1x_j\) generate a subgroup of \(P^g(n, 3)\) which is a quotient of \(P^g(4, 3)\) and by assumption of the lemma the latter is of exponent 3. Hence by lemma 6.5 \(P^g(n, 3)\) is of exponent 3. Since the set of generators includes only primitive elements \(F^n\) the group \(P^g(n, 3)\) coincides with \(B(n, 3)\).

\(\Box\)

**Lemma 6.8** If \(P^g(3, 3) = B(3, 3)\) then \(P^g(4, 3) = B(4, 3)\).

**Proof** The group \(P^g(4, 3)\) is obtained from the curve of genus 2 minus a point. Consider a standard decomposition of \(X^2\) into a union of two handles corresponding to pairs of generators \(a, b\) and \(c, d\) of \(F^4_4\). There are topological embeddings of tori with two discs removed corresponding to the subgroups generated by any three symbols from the set \((a, b, c, d)\). The group \(P^g(3, 3)\) is realized as the quotient of the fundamental group of torus minus two discs. The above tori are obtained from one of the handles by adjoining a neighborhood of a generator in another handle. By assumption of the lemma \(((a, b), c)\) is transformed into itself or the opposite element under the permutation of symbols. Similar groups correspond to the triples \((a, b(cd)), (c, d, (ab))\). Namely we can consider a corresponding handle minus a point. Thus we can apply lemma 6.6 and obtain that \(P^g(4, 3)\) is of exponent 3 if the group \(P^g(3, 3)\) is.

\(\Box\)
Theorem 6.1 \( P^g(3,3) = B(3,3) \).

Let us first describe the geometric picture. Consider the torus \( T^2 \) with a small embedded interval \( I \). Let \( p_1, p_2 \) be two different points in the interval \( I \) and \( I_1 \) be the interval between \( p_1, p_2 \) inside \( I \). We assume that \( p_0 \) is a point in \( I - I_1 \) and identify \( p_0 \) as an initial point for the fundamental group \( \pi_1(T^2 - p_1 - p_2) = F_3 \). Assume that \( T^2, I \) are given with orientation. We consider smooth oriented loops through \( p_0 \) which are transversal to \( I \).

Lemma 6.9 Any element of \( P^g(2,3) \) with \( p_0 \) as an initial point is represented by an oriented curve \( A \) without selfintersection which does not intersect \( I \) and such that \( A, I \) defines a standard orientation of \( T \) at \( p_0 \).

Proof Let \( a, b \) be standard generators of \( F_2 \) with a given orientation. The group \( \text{Out}(F_2) = SL(2, \mathbb{Z}) \) can be realized by linear periodic map of torus. In particular we can represent topologically the elements of \( SL(2, \mathbb{Z}) \) by maps which stabilize the points of \( I \). In this way we obtain any map of \( F_2 \) into itself which transforms \( aba^{-1}b^{-1} = C \) into itself. Any homomorphism of the free group with above property is induced by this action of \( SL(2, \mathbb{Z}) \). Thus \( g(a), g \in SL(2, \mathbb{Z}), a \in B(3,2) \) can be any element such that there exists \( b' \in B(3,2) \) with \( g(a)b'g(a)^{-1}b'^{-1} = C \) where \( C \) is a given generator of the center. But we can find such \( b' \) for any \( g(a) \) which is not in the center. The curve \( g(a) = A \) will be the image of a map which is linear outside a neighborhood of \( I \) and keeps orientation intact.

\[ \Box \]

Remark 6.2 Since \( A^{-1}I \) represents the opposite orientation the lemma actually shows that the element \( A^{-1} \) can be represented by another simple closed curve \( B \) with orientation \( BI \) opposite to \( A^{-1}I \).

The group \( P^g(3,3) \) is generated by \( B(2,3) \) realized as above and an element \( r \) realized by a curve \( R \) with one selfintersection. We have a natural representation of \( r \) as \( x_1x_2^{-1} \) where \( x_1, x_2 \) are simple curves with the same orientation which move around \( p_1, p_2 \) respectively. The element \( c = x_1x_2 \) is a natural central loop in \( P^g(2,3) = B(2,3) \).

Lemma 6.10 Any element \( bx_i \) is realized by a simple loop if \( b \) is not in the center of \( B(2,3) \).

Proof We have to find a simple representative of \( b \) through \( p_0 \) with an appropriate orientation. The latter exists due to the previous lemma.

\[ \Box \]
Lemma 6.11 Any element \( br \in P^g(3,3), b \in B(2,3) \) can be represented by a simple curve in its conjugation class unless \( b \) is in the center of \( B(2,3) \).

We have \( br = bx_1x_2 \). The element \( bx_1 \) is represented by a simple curve \( B \). Let \( S \) be curve which contains \( I_1 \) and intersects \( B \) transversally at exactly one point inside \( I_1 \). The complimentary \( T - (S - I_1) \) defines another group \( B(3,2) \). We assume that \( p_0 \) is not in \( S \) and hence \( bx_1 \) is equivalent to a simple curve with a desired orientation with respect to \( x_2 \). That means \( (bx_1)x_2 \) can be realized by a class of simple curve in \( P^g(3,3) \). The classes \( x_i \) are also realized by simple curves.

\[ \square \]

Corollary 6.2 The elements \( (br)^3 = 1 \) for any \( b \neq c = x_1x_2 \).

Indeed if \( b \) is not in the center then \( br \) is realized by a simple curve and we get the result. The element \( c^{-1}r = x_2^{-1}x^{-1}x_1^{-1}x_2^{-1} = x_2^{-2} = x_2 \) in \( P^g(3,3) \).

In dealing with the extension of \( B(2,3) \) by an element \( r \) we will be using the following general argument. Let \( G \) be a finite group of exponent 3 and \( G' \) is obtained from \( G \) by adding \( r \) and some relations of type \( (br)^3 = 1 \). Then the kernel of a natural projection \( p : G' \rightarrow G \) is generated by the elements \( ra = ara^{-1}, a \in G \). The group \( G \) acts on this set of elements by left translation \( g : ra \rightarrow r^{ga} \). Any relation \( (br)^3 = 1 \) implies the relation : \( 1 = brbrbr = brb^{-1}b^{-1}br = rbr^{-1}r = 1 \) and similar relation for left translations of the orbits of the cyclic group \( B = (1,b,b^{-1}) \). If in addition \( (b^{-1}r)^3 = 1 \) all the elements \( r^b, r^{b^{-1}}, r \) commute and any pair of them generate the same abelian group.

Lemma 6.12 The group \( P^g(3,3) \) is an abelian extension of \( B(2,3) \).

Proof The kernel of the projection \( pr : P^g(3,3) \rightarrow B(2,3) \) which maps \( r \) to 1 is generated by the elements \( r^b = brb^{-1} \). All these elements commute with \( r \) unless \( b = c \). We also have \( r^brr^{-1}b^{-1} = 1 \) and they all commute if \( b \) is not in the center. Therefore \( r^a \) commutes with \( r \) if it commutes with both \( r^b, r^{b^{-1}} \). On the other hand \( r^{ab}, r^a, r^{ab^{-1}} \) commute if \( r^b, r, r^{b^{-1}} \) commute. Let \( a \) be a generator of \( B(2,3) \). Then \( r^{ca}r^{ca^{-1}} = 1 \) and all these elements commute, but \( r^{ca}, r^{ca^{-1}} \) commute with \( r \). This implies that all the elements \( r^b, b \in B(2,3) \) commute with \( r \). After translation by \( B(2,3) \) we obtain that all the elements \( r^g \) commute.

\[ \square \]

Lemma 6.13 Let \( T \) be the group generated by \( r^b, b \in B(2,3) \). Then \( T \) is an abelian group with 4 generators.
Proof Indeed the set of cyclic subgroups which don’t lie in the center generate a family of relations. Since we have established that $T$ is an abelian group we shall write them in the additive form $r^x + r^{ax} + r^{a^2x} = 0$, $x \in B(2,3)a$ but not in the center $C$. Denote by $T_S$ a subgroup of $T$ generated by a subset $S \subset B(2,3)$. Let $A$ be an abelian subgroup generated by $a$ and $c$ which generates the center of $B(2,3)$. The summation over orbits of cyclic noncentral subgroups gives zero. Thus if we consider $T_A$ modulo a subgroup $T_C$ corresponding to the center we obtain $r^g + r^{gh} = 0, g \notin C$. Hence the elements $r^g = -r^{g^{-1}}$ and $r^{gc} = r^g$ modulo subgroup $r^c, r$. We obtain that $r^a$ generates $T_A$ modulo the subgroup $(r_c, r)$ and a sum over any orbit of $C$ is also zero.

Thus we have $r(x^{-1}) = -r - r^x$ and $r^x + r^y + r^{-1}y^{-1}x^{-1} = 0$ for the elements $x, y$ which lie in one abelian subgroup of $B(3,2)$. The same is true modulo $r^c$ since we can apply the same argument to the quotient of $B(3,2)$ by the center. Hence $r^x + r^y = r^{xy}$ (modulo($r^c$)) for any $x, y \in B(2,3)$. In particular $r^a, r^b, r^c, r$ generate the group $T$.

Lemma 6.14 $T$ is an elementary abelian group.

Proof We have $r^2 = 2r^x - r$ and $r = 3r^x - 2r$. Hence $3r = 3r^x$ for any $x$. Hence $3(r - r^x) = 0$ for any $x$. Thus the elements of zero degree in $T$ constitute an elementary 3-group $T_0$ which is a normal subgroup of $P^9(3,3)$. The quotient $T/T_0$ is a cyclic group. The group $P^9(3,3)/T_0$ is a central extension of $B(2,3)$. Since $P^9(3,3)$ is generated by elements of order 3 we obtain that $T/T_0 = \mathbb{Z}_3$.

Corollary 6.3 The number of elements in $T$ is $3^4$.

Hence the number of elements in $P^9(3,3)$ is equal to $3^7$ and coincides with the number of elements in $B(3,3)$. Since there exists a surjective map $p : P^9(3,3) \to B(3,3)$ the groups coincide. Thus we have proved that group $P^9(n,3)$ coincide with $B(n,3)$ for all $n$.

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