CM Periods, CM Regulators, and Hypergeometric Functions, I

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Abstract. We prove the Gross–Deligne conjecture on CM periods for motives associated with $H^2$ of certain surfaces fibered over the projective line. Then we prove for the same motives a formula which expresses the $K_r$-regulators in terms of hypergeometric functions $I_F$, and obtain a new example of non-trivial regulators.

1 Introduction

Periods and regulators of a motive over a number field are very important invariants, whose arithmetic significance can be seen from their conjectural relations with values of the $L$-function at integers. Such conjectures include those of Birch–Swinnerton-Dyer, Deligne, Bloch, Beilinson and Bloch–Kato. If the motive has complex multiplication (CM) by a number field, especially by an abelian field, those invariants take a special form.

If $A$ is an abelian variety with CM by a subfield of the $N$-th cyclotomic field, its periods are written in terms of values of the gamma function at $\frac{1}{N} \mathbb{Z}$. When $A$ is an elliptic curve, the formula is due to Lerch [15] and was rediscovered by Chowla–Selberg [8]. Gross [13] gave a geometric proof of a generalization of the formula and proposed a conjecture for any motivic Hodge–de Rham structure with CM by an abelian field, whose precise form was given by Deligne. Using Shimura’s monomial relation [23], Anderson [1] proved the formula for CM abelian varieties by reducing to the case of Fermat curves.

In this paper, we study a surface $X$ fibered over $\mathbb{P}^1$ (t-line) with the general fiber defined by $y^p = x^a(1-x)^b(t^l-x)^{p-b}$, where $l$ and $p$ are distinct prime numbers. It admits an action of $\mu_{12}$ and its second cohomology modulo the image of classes supported at singular fibers gives a Hodge–de Rham structure $H = (H_{\text{DR}}, H_B)$ with multiplication by $K := \mathbb{Q}(\mu_{12})$ (see §2.2). We shall prove that $H_B$ is one-dimensional over $K$ (Theorem 4.12). For each embedding $\chi : K \to \mathbb{C}$, let $H^\chi$ be the eigencompomenent. We shall determine its period and the Hodge type independently, and prove the Gross–Deligne conjecture.
**Theorem 1.1** (Period formula, see Theorem 5.4)  
For each $\chi: K \to \mathbb{C}$, let $\chi(\zeta_p) = \zeta_p^m$, $\chi(\zeta_1) = \zeta_1^m$, and put $\alpha = \{ \frac{n}{p} \}$, $\beta = \{ \frac{n}{p} \}$, $\mu = \{ \frac{n}{p} \}$. Then we have

$$\text{Per}(H^1) = B(\beta, \mu)B(1 - \beta, \beta - \alpha + \mu),$$

where $K' := \mathbb{Q}(\mu_{21p})$, and the Gross–Deligne conjecture holds.

On the other hand, regulators of the Fermat curve of degree $N$ are written in terms of values at 1 of hypergeometric functions $_2F_1$ with parameters in $\frac{1}{2}\mathbb{Z}$ [18]. The conjectural relation with $L$-values is verified for some cases in [19,20]. Recall that the beta function is related to the value of Gauss’ hypergeometric function $_2F_1$ at 1. It is also suggestive that the classical polylogarithm can be written as

$$\text{Li}_k(x) = x \cdot _{k+1}F_k\left(\begin{array}{c} 1,1,\ldots,1 \\ 2,\ldots,2 \end{array} ; x\right),$$

and hence special values of Dirichlet $L$-functions are written in terms of $_kF_k$-values.

For the surface $X$, we consider the Beilinson regulator [7] from the motivic cohomology to the Deligne cohomology

$$r_{\otimes}: H^3_{\text{eff}}(X, \mathbb{Q}(2)) \to H^3_{\text{eff}}(X_{\mathbb{C}}, \mathbb{Q}(2)).$$

In terms of algebraic $K$-theory, we have $H^3_{\text{eff}}(X, \mathbb{Q}(2)) = (K_1(X) \otimes \mathbb{Q})^{(2)}$ (the second eigenspace for the Adams operations). Let $Z_1$ be the union of fibers over $\mu_1$ and consider the image of $H^3_{\text{eff},Z_1}(X, \mathbb{Q}(2)) \to H^3_{\text{eff}}(X, \mathbb{Q}(2))$. The Deligne cohomology can be regarded as functionals on $F^1H^2_{\text{dR}}(X)$ up to periods, and we restrict them to $F^3H^3_{\text{dR}}$.

**Theorem 1.2** (Regulator formula, see Theorem 6.5)  
Let $\chi$ be an embedding such that $H^3_{\text{dR}} \subset F^1H^3_{\text{dR}}$. Then, for any $z \in H^3_{\text{eff},Z_1}(X, \mathbb{Q}(2))$ and $\omega \in H^3_{\text{dR}}$, we have

$$r_{\otimes}(z)(\omega) = K. B(1 - \alpha, \beta) \cdot _3F_2\left(\begin{array}{c} 1 - \alpha, \beta, \beta - \alpha + \mu \\ 1 - \alpha + \beta, \beta - \alpha + \mu + 1 \end{array} ; 1\right),$$

where $\alpha, \beta, \mu$ are as before.

Moreover, we shall show the non-vanishing of the regulator image under a mild assumption (Theorem 6.6).

Regarding these examples, it is tempting to ask if the regulators and hence the $L$-values of a motive with CM by an abelian field can be written in terms of values of $_kF_k$, with $k$ depending on the weight. In a forthcoming paper [4], we shall study more general fibrations of varieties over $\mathbb{P}^1$ with multiplication by a number field whose relative $H^1$ has a special type of monodromy.

Concerning the period conjecture, there is a result of Maillot–Roessler [16] using Arakelov theory on the absolute value of the period. Recently, Fresán [12] proved the formula for the alternating product of the determinants for any smooth projective variety with a finite order automorphism by reducing to a result of Saito–Terasoma [22]. Since we prove dim$_{\mathbb{Q}} H_B = 1$ and $H^1(X) = H^2(X) = 0$, the Gross–Deligne conjecture for our $H$ follows from Fresán’s result. However, we need our precise computations.
for the study of regulators. Our method is quite different from previous works mentioned above. A crucial step is to compute explicitly Deligne’s canonical extension \( \mathcal{H}_c \) of the Gauss–Manin connection on the relative first de Rham cohomology. Our fibration is smooth outside \( D := \{ 0, \infty \} \cup \mu_1 \), and there is a connection
\[
\nabla : \mathcal{H}_c \rightarrow \Omega^1_{\mathbb{C}}(\log D) \otimes \mathcal{H}_c.
\]

We will describe it explicitly and determine the Hodge structure of \( H \). The 1-periods of the fiber are Gauss hypergeometric functions \( _2F_1 \). By the integral representation of Euler type, the 2-periods of \( X \) are first written in terms of \( _3F_2 \)-values, which then turn out to be \( _2F_1 \)-values. The conjecture follows by comparing these computations.

It is more delicate in general to compute the regulators of given motivic elements, even for a fibration of curves. Here we use a new technique [3], originally unpublished, but now included in the appendix of the present paper. Via the canonical extension, we shall represent elements of \( F^1H_{dR} \) by certain rational 2-forms. Then the regulators are expressed as integrals of those rational forms over Lefschetz thimbles, which are again written in terms of \( _3F_2 \)-values.

This paper proceeds as follows. In Section 2, we fix the setting and compute the 1-periods of the fiber and 2-periods of \( X \). In Section 3, we determine the Gauss–Manin connection and the canonical extension. In Section 4, we determine the Hodge structure and show that \( H_B \) is one-dimensional over \( K \). In Section 5, we give a basis of \( F^1H_{dR} \) and verify the Gross–Deligne conjecture. In Section 6, we prove the regulator formula and discuss the non-vanishing. The appendix, due to the first author, provides the technique to compute the regulators.

### 1.1 Notations

Throughout this paper, \( \overline{\mathbb{Q}} \) denotes the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \). For each positive integer \( N \), \( \mu_N \) denotes the group of \( N \)-th roots of unity and we put \( \zeta_N = e^{2\pi i/N} \). For a real number \( x \), we write \( x = \lfloor x \rfloor + \{ x \} \) with \( \lfloor x \rfloor \in \mathbb{Z} \), \( 0 \leq \{ x \} < 1 \), and put \( \lfloor -x \rfloor = -\lfloor x \rfloor \). For \( \alpha \in \mathbb{C} \) and an integer \( n \geq 0 \), \( (\alpha)_n = \prod_{i=0}^{n-1} (\alpha + i) \) is the Pochhammer symbol and the generalized hypergeometric function is defined by
\[
 {}_pF_q \left( \begin{array}{c} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{x^n}{n!}
\]

We often drop the subscripts from \( {}_pF_q \). It converges at \( x = 1 \) when \( \text{Re}(\sum \beta_j - \sum \alpha_i) > 0 \). We use the standard notation for the product of \( \Gamma \)-values
\[
 \Gamma \left( \begin{array}{c} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} \right) = \frac{\prod_{i=1}^{p} \Gamma (\alpha_i)}{\prod_{j=1}^{q} \Gamma (\beta_j)}
\]

For a variety \( X \) over \( \overline{\mathbb{Q}} \), \( H^n_{dR}(X) = H^n_{dR}(X/\overline{\mathbb{Q}}) \) denotes the algebraic de Rham cohomology and \( H^n(X, \mathbb{Q}) \) denotes the Betti cohomology of the analytic manifold \( X(\mathbb{C}) \), or the associated mixed Hodge structure.
2 Preliminaries

2.1 The Setting

Let $p$, $l$ be distinct prime numbers and $a$, $b$, $c$ be integers with $0 < a, b, c < p$ (we shall soon assume that $b + c = p$). We define a fibration of curves $f : X \to \mathbb{P}^1$ as follows. Let $g : Y \to \mathbb{P}^1$ be a proper flat morphism over $\mathbb{Q}$ whose fiber $Y_t$ at $t \in \mathbb{P}^1$ is the normalization of the curve defined by $y^p = x^a (1-x)^b (t-x)^c$. Then $g$ is smooth outside $\{0,1,\infty\}$ and, by the Riemann–Hurwitz formula, the genus of the generic fiber is $p - 1$. The fiber $Y_1$ is a union of $\mathbb{P}^1$ intersecting transversally with each other. We have an automorphism $\sigma$ of order $p$ of $Y$ over $\mathbb{P}^1$ defined by $\sigma(x, y) = (x, \xi_p y)$.

Let $g^{(1)} : Y^{(1)} \to \mathbb{P}^1$ be the base change of $g$ by the morphism $\mathbb{P}^1 \to \mathbb{P}^1, t \mapsto t^l$. The action of $\sigma$ extends naturally to $Y^{(1)}$. On the other hand, the automorphism $\tau(t) = \xi_t$ of $\mathbb{P}^1$ induces an automorphism $\tau$ of $Y^{(1)}$ over $Y$. There is a desingularization $X$ of $Y^{(1)}$ such that $\sigma$ and $\tau$ extend to automorphisms of $X$ respectively over $\mathbb{P}^1$ and $Y$ (for example, if one takes a sequence of blow-ups only at the singular points, then $\sigma$ and $\tau$ extend automatically). As a result, we obtain a fibration $f : X \to \mathbb{P}^1$ of curves in the commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y^{(1)} \\
\downarrow f & & \downarrow g \\
\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1
\end{array}
$$

and for $t \notin \{0, \infty\} \cup \mu_1$, the fiber $X_t$ is isomorphic to $Y_t$.

2.2 CM Hodge–de Rham structures

A Hodge–de Rham structure is a quadruple $H = (H_{\text{DR}}, H_B, \iota, F^*)$ consisting of

- a finite-dimensional $\overline{\mathbb{Q}}$-vector space $H_{\text{DR}}$,
- a finite-dimensional $\mathbb{Q}$-vector space $H_B$,
- an isomorphism
  $$
  \iota : H_{\text{DR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to H_B \otimes_{\mathbb{Q}} \mathbb{C},
  $$
  and
- a descending filtration $F^* H_{\text{DR}}$ that induces a Hodge structure on $H_B$ via $\iota$.

For a proper smooth variety $X$ over $\overline{\mathbb{Q}}$, its $n$-th de Rham and Betti cohomology groups, the comparison isomorphism, and the Hodge filtration define a Hodge–de Rham structure $H^n(X)$.

Let $K$ be a finite extension of $\mathbb{Q}$. We say that $H$ admits a $K$-multiplication if we are given $K$-actions on $H_{\text{DR}}$ and $H_B$ that are compatible with $\iota$ and $F^*$. Moreover, we say that $H$ has CM by $K$ if $\dim_K H_B = 1$. For each embedding $\chi : K \to \mathbb{C}$, let $H_{\text{DR}}^\chi$, $H_B^\chi := (H_B \otimes_{\mathbb{Q}} \mathbb{C})^\chi$ denote the subspace on which $K$ acts as the multiplication via $\chi$. If $\dim_K H_B = 1$, then these subspaces are 1-dimensional over $\overline{\mathbb{Q}}$. Choosing any bases $\omega_{\text{DR}} \in H_{\text{DR}}^\chi$ and $\omega_B \in H_B^\chi$, we define the period $\text{Per}(H^x) \in \mathbb{C}^*$ by $\iota(\omega_{\text{DR}}) = \text{Per}(H^x) \omega_B$.

By the ambiguity of the choices, $\text{Per}(H^x)$ is only well defined up to $\mathbb{Q}^*$. If $(H_{\text{DR}}, F^*)$ is already defined over $K$, the period is well defined up to $K^*$. 

Let $X$ be as in Section 2.1 and let $Z = X \times \mathbb{P}^1 \left( \{ 0, \infty \} \cup \mu_1 \right)$ be the union of the singular fibers. Note that $Z$ is stable under the actions of $\sigma$ and $\tau$. Put $R = \mathbb{Q}[\sigma, \tau], K = \mathbb{Q}((\mu_p))$ and regard $K$ as an $R$-algebra by $\sigma \mapsto \zeta_p, \tau \mapsto \zeta$. The Hodge–de Rham structure we consider in this paper is $H := \text{Coker}(H^2_Z(X) \to H^2(X)) \otimes_R K$. It admits a $K$-multiplication, and we shall show that $\text{rank}_K H_B = 1$ (Theorem 4.12). An embedding $\chi: K \to \mathbb{C}$ is identified with an element $h \in (\mathbb{Z}/p\mathbb{Z})^\times$ such that $\chi(\zeta^p) = \zeta^h$. If

$$\text{Coker}(H^2_Z(X) \to H^2(X)) = \bigoplus_{m \in \mathbb{Z}/1\mathbb{Z}, n \in \mathbb{Z}/p\mathbb{Z}} H^{(m,n)}$$

denotes the decomposition into the eigenspaces on which $\tau$ (resp. $\sigma$) acts by $\zeta^m$ (resp. $\zeta^n$), we have $H = \bigoplus_{m \neq 0, n \neq 0} H^{(m,n)}$.

### 2.3 Periods of the Fiber

For $n = 1, \ldots, p - 1$ and integers $i, j, k$, put a rational 1-form on $Y_t$ by

$$\omega_{ij}^k = \frac{x^i(1-x)^j(t-x)^k}{y^n} dx.$$

Then we have

$$\sigma^* \omega_{ij}^k = \zeta_p^n \omega_{ij}^k. \tag{2.1}$$

Let $0 < t < 1$ and $\delta_0$ be a path on $Y_t$ from $(0, 0)$ to $(t, 0)$ defined by

$$x = ts, \quad y = \sqrt[p\cdot\text{deg}(1-x^p)]{(t-x)^p}.$$

Let $\delta_1$ be a path on $Y_t$ from $(t, 0)$ to $(1, 0)$ defined by

$$x = t + (1-t)s, \quad y = e^{\varepsilon} \sqrt[p\cdot\text{deg}(1-x^p)]{(x-t)^p},$$

where we put

$$\varepsilon = \begin{cases} 1 & \text{if } p = 2, \\ -1 & \text{if } p \text{ is odd.} \end{cases}$$

If we put $\kappa_m = (1 - \sigma), \delta_m, (m = 0, 1)$, these define 1-cycles on $Y_t$, and we have

$$\int_{\kappa_m} \omega_{ij}^k = \int_{\delta_m} (1 - \sigma) \omega_{ij}^k = (1 - \zeta_p^n) \int_{\delta_m} \omega_{ij}^k. \tag{2.2}$$

**Lemma 2.1** Fix integers $i, j, k \geq 0$. For $n = 1, \ldots, p - 1$, put

$$\alpha = \frac{na}{p} - i, \quad \beta = \frac{nb}{p} - j, \quad \gamma = \frac{nc}{p} - k.$$

Then we have

$$\int_{\delta_b} \omega_{ij}^k = B(1 - \alpha, 1 - \gamma) \cdot t^{1 - \alpha - \gamma} F \left( \begin{array}{c} 1 - \alpha, \beta \\ 2 - \alpha - \gamma \end{array}; t \right),$$

$$\int_{\delta_1} \omega_{ij}^k = e^{\beta y} B(1 - \beta, 1 - \gamma) \cdot (1 - t)^{1 - \beta - \gamma} F \left( \begin{array}{c} \alpha, 1 - \beta \\ 2 - \beta - \gamma \end{array}; 1 - t \right).$$
Proof The first equality follows directly from Euler’s integral representation of the Gauss hypergeometric function $_2F_1$:

\[ B(b, c - b) \cdot F\left( \frac{a, b}{c} ; t \right) = \int_0^1 (1 - tx)^{-a} x^{b-1} (1 - x)^{c-b-1} dx \]

(let \( a = \beta, b = 1 - \alpha, c = 2 - \alpha - \gamma \)). The second one follows from the same formula and the transformation formula

\[ F\left( \frac{a, c - b}{c} ; 1 - \frac{1}{t} \right) = t^a \cdot F\left( \frac{a, b}{c} ; 1 - t \right). \]

2.4 Cohomology of the Fiber

We have decompositions

\[ H^i(Y_1, \mathbb{C}) = \bigoplus_{n=1}^{p-1} H^i(Y_1, \mathbb{C})^{(n)}, \quad H_1(Y_1, \mathbb{Q}(\mu_p)) = \bigoplus_{n=1}^{p-1} H_1(Y_1, \mathbb{Q}(\mu_p))^{(n)}, \]

where \( (n) \) denotes the subspace on which \( \sigma^* \) (resp. \( \sigma_\ast \)) acts as the multiplication by \( \zeta_p^n \). Note that \( H^1(Y_1, \mathbb{C})^{(n)} = 0 \) since \( Y_1/\mu_p \) is a rational curve. The natural paring induces a non-degenerate pairing \( H^1(Y_1, \mathbb{C})^{(n)} \otimes H_1(Y_1, \mathbb{Q}(\zeta_p))^{(n)} \rightarrow \mathbb{C} \). We shall give bases of these spaces under a certain assumption.

Lemma 2.2 Let \( n = 1, \ldots, p - 1 \) and \( i, j, k \geq 0 \) be integers.

(i) If \( p \nmid a + b + c \), then \( \omega_n^{ijk} \) is a differential form of the second kind.

(ii) Moreover, \( \omega_n^{ijk} \) is holomorphic if and only if

\[ i \geq \left\lfloor \frac{na + 1}{p} \right\rfloor - 1, \quad j \geq \left\lfloor \frac{nb + 1}{p} \right\rfloor - 1, \quad k \geq \left\lfloor \frac{nc + 1}{p} \right\rfloor - 1, \]

\[ i + j + k \leq \left\lfloor \frac{n(a + b + c) - 1}{p} \right\rfloor - 1. \]

Proof See [2, (18)] (but see [2, (13)] for the correct sign in the fourth inequality).

Henceforth, we assume \( b + c = p \). Then the condition \( p \nmid a + b + c \) is automatically satisfied. By Lemma 2.2, \( \omega_n^{ijk} \) is holomorphic if and only if

\[ i = \left\lfloor \frac{na + 1}{p} \right\rfloor - 1, \quad j = \left\lfloor \frac{nb + 1}{p} \right\rfloor - 1, \quad k = \left\lfloor \frac{nc + 1}{p} \right\rfloor - 1, \]

and we write this \( \omega_n^{ijk} \) simply as \( \omega_n \). The \( \alpha, \beta, \gamma \) in Lemma 2.1 become

\[ \alpha = \left\lfloor \frac{na}{p} \right\rfloor, \quad \beta = \left\lfloor \frac{nb}{p} \right\rfloor, \quad \gamma = \left\lfloor \frac{nc}{p} \right\rfloor = 1 - \beta. \]

In particular, \( 0 < \alpha, \beta, \gamma < 1 \). Although these depend on \( n \), we shall suppress \( n \) from the notation. By Lemma 2.1, we have

\[ \int_{\delta_n} \omega_n = B(1 - \alpha, \beta) \cdot t^{\alpha - \beta} F\left( \frac{1 - \alpha, \beta}{1 - \alpha + \beta} ; t \right), \]

\[ \int_{\delta_i} \omega_n = -\epsilon^{p\beta} B(1 - \beta, \beta) \cdot F\left( \frac{\alpha, 1 - \beta}{1} ; 1 - t \right). \]
For each \( n \), let \( i, j, k \) be as above and put \( \eta_n = \omega_n^{i,j+1,k} \). Then \( \beta \) is replaced by \( \beta - 1 \) in Lemma 2.1 and we obtain

\[
\int_{s_0} \eta_n = B(1 - \alpha, \beta) \cdot t^{\beta - \alpha} F\left(\frac{1 - \alpha, \beta - 1}{1 - \alpha + \beta} ; t\right),
\]

(2.4)

\[
\int_{s_1} \eta_n = -c^{pB}B(1 - \beta, \beta) \cdot (1 - \beta)(1 - t)F\left(\frac{\alpha, 2 - \beta}{2} ; 1 - t\right).
\]

Here we used \( B(2 - \beta, \beta) = (1 - \beta)B(1 - \beta, \beta) \).

**Proposition 2.3** Let \( n = 1, \ldots, p - 1 \) and \( 0 < t < 1 \). Then \( \{ \omega_n, \eta_n \} \) is a basis of \( H^1(Y_t, \mathbb{C})^{(n)} \).

**Proof** By (2.1), (2.2), (2.3), and (2.4), \( \omega_n, \eta_n \) are non-trivial elements of \( H^1(Y_t, \mathbb{C})^{(n)} \). Since \( \omega_n \) is holomorphic and \( \eta_n \) is not, they are linearly independent. Since

\[
\dim H^1(Y_t, \mathbb{C}) = 2(p - 1),
\]

the proposition follows.

**Proposition 2.4** Let \( n = 1, \ldots, p - 1 \) and \( 0 < t < 1 \).

(i) The projections of \( \kappa_0, \kappa_1 \) form a basis of \( H_2(Y_t, \mathbb{Q}(p))^{(n)} \).

(ii) As a \( \mathbb{Q}[\sigma] \)-module, \( H_1(Y_t, \mathbb{Q}) \) is generated by \( \kappa_0 \) and \( \kappa_1 \).

**Proof** The period matrix is

\[
M_n(t) = \left(\begin{array}{cc}
\int_{s_0} \omega_n & \int_{s_0} \eta_n \\
\int_{s_1} \omega_n & \int_{s_1} \eta_n
\end{array}\right).
\]

It suffices to show that \( \det M_n(t) \neq 0 \). Since \( \prod_{p=1}^{p-1} \det M_n(t) \) is constant, it coincides with its limit as \( t \to 1 \). Hence the proposition follows from the lemma below.

**Lemma 2.5** We have

\[
\lim_{t \to 1} \det M_n(t) = c^{pB} \left(1 - \xi_p^{n}\right)^2 \cdot \frac{B(\beta, 1 - \beta)}{1 - \alpha}.
\]

**Proof** By (2.2), (2.3), (2.4), we have

\[
\det M_n(t) = -c^{pB} \left(1 - \xi_p^{n}\right)^2 B(1 - \alpha, \beta)B(1 - \beta, \beta)t^{\beta - \alpha}
\]

\[
\times \det\left(\begin{array}{cc}
F\left(\frac{1 - \alpha, \beta}{1 - \alpha + \beta} ; t\right) & F\left(\frac{1 - \alpha, \beta - 1}{1 - \alpha + \beta} ; t\right) \\
F\left(\frac{\alpha, 1 - \beta}{1 - \alpha + \beta} ; 1 - t\right) & F\left(\frac{\alpha, 2 - \beta}{1 - \alpha + \beta} ; 1 - t\right)
\end{array}\right).
\]

First, we have

\[
\lim_{t \to 1} (1 - t)^{\beta} \left(\begin{array}{c}
1 - \alpha, \beta \\
1 - \alpha + \beta
\end{array}\right) = 0.
\]

This follows from the transformation formula (cf. [11, p. 74 (2)])

\[
F\left(\frac{1 - \alpha, \beta}{1 - \alpha + \beta} ; t\right) = \frac{1}{B(1 - \alpha, \beta)} \sum_{n=0}^{\infty} \frac{(1 - \alpha)\beta_n}{(n!)^2} (k_n - \log(1 - t))(1 - t)^n,
\]

\[
k_n := 2\psi(n + 1) - \psi(1 - \alpha + n) - \psi(\beta + n)
\]
where $\psi(t) = \Gamma'(t)/\Gamma(t)$ is the digamma function. On the other hand, by Euler's formula, we have

$$F \left( \frac{1 - \alpha, \beta - 1}{1 - \alpha + \beta} ; 1 \right) = \Gamma \left( \frac{1 - \alpha + \beta}{2 - \alpha, \beta} \right) = \frac{1}{(1 - \alpha)B(1 - \alpha, \beta)}.$$ 

Hence the lemma follows.

### 2.5 Periods of $X$

Now we consider the fibration $f : X \to \mathbb{P}^1$. Recall that $X_t \simeq Y_t$. By abuse of notation, for each $t = 0, 1$, let $\delta_t$ (resp. $\kappa_t$) be the path (resp. loop) on $X_t$ which corresponds to the one on $Y_t$ defined in §2.3. For each $s$, let $\Delta_s$ be the 2-simplex obtained by sweeping $\delta_t$ along $0 \leq t \leq 1$. Since $\delta_t$ is vanishing as $t \to s$, the Lefschetz thimble $(1 - \sigma)_* \Delta_s$ has boundary on the fiber $X_{1-t}$. We shall use $(1 - \sigma)_* \Delta_1$ (resp. $(1 - \sigma)_* \Delta_0$) to compute the periods (resp. regulators). Again by abuse of notation, let $\omega_n$ denote the pullback to $X$ of the rational 1-form $\omega_n$ on $Y$ defined in §2.4. For $n = 1, \ldots, p - 1$ and an integer $m$, define rational 2-forms on $X$ by

$$\omega_{m,n} = t^m \frac{dt}{t} \wedge \omega_n, \quad \eta_{m,n} = t^m \frac{dt}{t} \wedge \eta_n.$$ 

We have evidently, $(t^i \sigma^j)^* \omega_{m,n} = \gamma^m \lambda_i^j \omega_{m,n}$ and $(t^i \sigma^j)^* \eta_{m,n} = \gamma^m \lambda_i^j \eta_{m,n}$.

**Proposition 2.6** Let $n = 1, \ldots, p - 1$ and $\alpha = \left\{ \frac{na}{p} \right\}$, $\beta = \left\{ \frac{nb}{p} \right\}$ as before. For an integer $m$, put $\mu = m/1$.

(i) If $\mu > \alpha - \beta$, then we have

$$\int_{\Delta_1} \omega_{m,n} = -\frac{\epsilon^{p\beta}}{1} B(\beta, \mu) B(1 - \beta, \beta - \alpha + \mu),$$

$$\int_{\Delta_1} \eta_{m,n} = -\frac{\epsilon^{p\beta}(1 - \beta)}{1(1 - \alpha + \mu)} B(\beta, \mu) B(1 - \beta, \beta - \alpha + \mu).$$

(ii) We have

$$\int_{\Delta_0} \omega_{m,n} = \frac{B(1 - \alpha, \beta)}{l(\beta - \alpha + \mu)} F \left( \frac{1 - \alpha, \beta - \alpha + \mu}{1 - \alpha + \beta, \beta - \alpha + \mu + 1} ; 1 \right),$$

$$\int_{\Delta_0} \eta_{m,n} = \frac{B(1 - \alpha, \beta)}{l(\beta - \alpha + \mu)} F \left( \frac{1 - \alpha - 1, \beta - \alpha + \mu}{1 - \alpha + \beta, \beta - \alpha + \mu + 1} ; 1 \right).$$

**Proof** Recall the integral representation of $_3F_2$ (cf. [24, (4.1.2)]):

$$\Gamma \left( \begin{array}{c} c, c - c \\ e \end{array} \right) F \left( \begin{array}{c} a, b, c \\ d, e \end{array} ; t \right) = \int_0^1 F \left( \begin{array}{c} a, b \\ d \end{array} ; tx \right) x^{c-1}(1-x)^{e-c-1} \, dx.$$
By (2.3), we have
\[
\int_{\Delta t} \omega_{m,n} = -\varepsilon^{\beta} B(\beta, 1 - \beta) \int_0^1 F \left( \frac{\alpha, 1 - \beta}{1} ; 1 - t \right) t^{m - 1} dt \\
= -\varepsilon^{\beta} \frac{B(\beta, 1 - \beta)}{l} \int_0^1 F \left( \frac{\alpha, 1 - \beta}{1} ; 1 - t \right) t^{l - 1} dt \\
= -\varepsilon^{\beta} \frac{B(\beta, 1 - \beta)}{l} \int_0^1 F \left( \frac{\alpha, 1 - \beta}{1} ; t \right) (1 - t)^{l - 1} dt \\
= -\varepsilon^{\beta} \frac{B(\beta, 1 - \beta)}{l\mu} F \left( \frac{\alpha, 1 - \beta, 1}{1, \mu + 1} ; 1 \right) \\
= -\varepsilon^{\beta} \frac{B(\beta, 1 - \beta)}{l\mu} F \left( \frac{\alpha, 1 - \beta}{\mu + 1} ; 1 \right),
\]
which converges by the assumption. Using Euler’s formula
\[
F \left( \frac{a, b}{c} ; 1 \right) = \Gamma \left( \frac{c, c - a - b}{c - a, c - b} \right) \left( \text{Re}(c - a - b) > 0 \right)
\]
and the functional equations
\[
\Gamma(x + 1) = x\Gamma(x), \quad B(x, y) = \Gamma \left( \frac{x, y}{x + y} \right),
\]
we obtain the first equality of (i). The others follow similarly, using (2.4) for $\eta_{m,n}$. \qed

3 Canonical Extension

In this section, we compute the Gauss–Manin connection of the fibration and determine its canonical extension to $\mathbb{P}^1$.

3.1 Gauss–Manin Connection

Let us start with the fibration $g: Y \to \mathbb{P}^1$; for a while, $t$ denotes the coordinate of the base scheme of $g$. Put $T = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $Y_T = Y \times_{\mathbb{P}^1} T$. Then the restriction $g: Y_T \to T$ is smooth. Put
\[
\mathcal{H} = R^1 g_* \Omega^1_{Y_T/T}, \quad \Omega^1_T = \Omega^1_{T/\mathbb{P}^1},
\]
and let $\nabla: \mathcal{H} \to \Omega^1_T \otimes \mathcal{H}$ be the Gauss–Manin connection. For each $n = 1, \ldots, p - 1$, let $\mathcal{H}^{(n)} \subset \mathcal{H}$ be the subbundle on which $\sigma^* \sigma$ acts as the multiplication by $\sigma^n$. Then $\mathcal{H}^{(n)}$ is locally generated by $\omega_n, \eta_n$ as defined in §2.4, and the Hodge filtration $F^i \mathcal{H}^{(n)}$ is generated by $\omega_n$.

**Proposition 3.1** For $n = 1, \ldots, p - 1$, the Gauss–Manin connection
\[
\nabla: \mathcal{H}^{(n)} \to \Omega^1_T \otimes \mathcal{H}^{(n)}
\]
is given by
\[
(\nabla \omega_n, \nabla \eta_n) = \frac{dt}{l} \otimes \left( \omega_n, \eta_n \right) \begin{pmatrix} 1 - \beta & 0 \\ 0 & 1 - \alpha \end{pmatrix} \begin{pmatrix} -1 & -1 \\ (1 - t)^{-1} & 1 \end{pmatrix},
\]
where we put $\alpha = \{ \frac{n}{p} \}, \beta = \{ \frac{n}{p} \}$ as before.
Proof We use the following standard derivation relations among Gauss hypergeometric functions [24, (1.4.1), (1.4.6)]:

(3.1) \[
\frac{d}{dt} F\left( \frac{a,b}{c} \mid t \right) = \frac{ab}{c} F\left( \frac{a+1,b+1}{c+1} \mid t \right),
\]

(3.2) \[
\frac{d}{dt} \left( t^{c-1} F\left( \frac{a,b}{c} \mid t \right) \right) = (c-1)t^{c-2} F\left( \frac{a,b}{c-1} \mid t \right).
\]

We also use the following contiguous relations (see [24, (1.4.1), (1.4.3), (1.4.5), (1.4.9), (1.4.13)]):

(3.3) \[
(c - 2a + (a - b)t) F + a(1 - t) F[a + 1] = (c - a) F[a - 1],
\]

(3.4) \[
(c - a - b) F + a(1 - t) F[a + 1] = (c - b) F[b - 1],
\]

(3.5) \[
(c - a - 1) F + aF[a + 1] = (c - 1) F[c - 1],
\]

(3.6) \[
(a - 1 + (1 + b - c) t) F + (c - a) F[a - 1] = (c - 1)(1 - t) F[c - 1],
\]

(3.7) \[
c(1 - t) F + (c - a) t F[c + 1] = c F[b - 1].
\]

Here, \( F = F\left( \frac{a,b}{c} \mid t \right) \) and the notation \( F[a + 1] \), for example, means \( F\left( \frac{a+1,b}{c} \mid t \right) \).

We are reduced to show

(3.8) \[
\frac{d}{dt} M_n(t) = M_n(t) \begin{pmatrix} 1 - \beta & 0 \\ 0 & 1 - \beta \end{pmatrix} \begin{pmatrix} -1 & -1 \\ (1 - t)^{-1} & 1 \end{pmatrix}.
\]

We prove this for each row vector. For the first row vector, put

\[
(f(t), g(t)) = \left( t^\beta F\left( \frac{1 - \alpha, \beta}{1 - \alpha + \beta} \mid t \right), t^\beta F\left( \frac{1 - \alpha, \beta - 1}{1 - \alpha + \beta} \mid t \right) \right).
\]

First, consider the case \( \alpha \neq \beta \). By (3.2), we have

\[
\frac{d}{dt} (f(t), g(t)) = \left( (\beta - \alpha) t^\beta F\left( \frac{1 - \alpha, \beta}{1 - \alpha + \beta} \mid t \right), (\beta - \alpha) t^\beta F\left( \frac{1 - \alpha, \beta - 1}{1 - \alpha + \beta} \mid t \right) \right).
\]

Applying (3.6) to \( F\left( \frac{\beta, 1 - \alpha}{1 - \alpha + \beta} \mid t \right) \), we obtain

\[
\frac{d}{dt} f(t) = -(1 - \beta) f(t) + (1 - \alpha)(1 - t)^{-1} g(t).
\]

Applying (3.5) to \( F\left( \frac{\beta, 1 - \alpha}{1 - \alpha + \beta} \mid t \right) \), we obtain \( \frac{d}{dt} g(t) = -(1 - \beta) f(t) + (1 - \alpha) g(t) \).

Hence we are done. Now consider the case \( \alpha = \beta \). Then

\[
(f(t), g(t)) = \left( F\left( \frac{1 - \alpha, \alpha}{1} \mid t \right), F\left( \frac{1 - \alpha, \alpha - 1}{1} \mid t \right) \right).
\]

By (3.1), we have

\[
\frac{d}{dt} (f(t), g(t)) = \left( (1 - \alpha) a F\left( \frac{2 - \alpha, 1 + \alpha}{2} \mid t \right), -(1 - \alpha)^2 F\left( \frac{2 - \alpha, \alpha}{2} \mid t \right) \right).
\]

Applying (3.7) to \( F\left( \frac{2 - a, 1 + \alpha}{1} \mid t \right) \), we have

(3.9) \[
\frac{d}{dt} f(t) = a(1 - t) F\left( \frac{2 - \alpha, 1 + \alpha}{1} \mid t \right) - a F\left( \frac{2 - \alpha, \alpha}{1} \mid t \right).
\]
Applying (3.4) to \( F \left( \frac{1 - \alpha, 1 + \alpha}{1} ; t \right) \), we have
\[
(1 - \alpha)(1 - t)F \left( \frac{2 - \alpha, 1 + \alpha}{1} ; t \right) = F \left( \frac{1 - \alpha, 1 + \alpha}{1} ; t \right) - \alpha f(t).
\]
Applying (3.3) to \( F \left( \frac{\alpha, 1 - \alpha}{1} ; t \right) \), we have
\[
\alpha(1 - t)F \left( \frac{1 - \alpha, 1 + \alpha}{1} ; t \right) = (2\alpha - 1)(1 - t) f(t) + (1 - \alpha) g(t).
\]
Applying (3.4) to \( F \left( \frac{1 - \alpha, \alpha}{1} ; t \right) \), we have
\[
(1 - t)F \left( \frac{2 - \alpha, \alpha}{1} ; t \right) = g(t).
\]
Combining (3.9)–(3.12), we obtain \( t \frac{d}{dt} g(t) = (1 - \alpha) \left( -f(t) + (1 - t)^{-1} g(t) \right) \). Applying (3.7) to \( F \left( \frac{\alpha, 2 - \alpha}{1} ; t \right) \), we have
\[
t \frac{d}{dt} g(t) = (1 - \alpha) \left( -F \left( \frac{1 - \alpha, \alpha}{1} ; t \right) + (1 - t) F \left( \frac{2 - \alpha, \alpha}{1} ; t \right) \right)
\]
\[
= (1 - \alpha) \left( -f(t) + g(t) \right).
\]
In both cases \( \alpha \neq \beta \) and \( \alpha = \beta \), we have proved (3.8) for the first row vector. For the second row vector, put
\[
(u(t), v(t)) = \left( F \left( \frac{\alpha, 1 - \beta}{1} ; 1 - t \right), (1 - \beta)(1 - t) F \left( \frac{\alpha, 2 - \beta}{2} ; 1 - t \right) \right).
\]
Then by (3.1) and (3.2) we have
\[
\frac{d}{dt} (u(t), v(t)) = -(1 - \beta) \left( \alpha F \left( \frac{\alpha + 1, 2 - \beta}{2} ; 1 - t \right), F \left( \frac{\alpha, 2 - \beta}{1} ; 1 - t \right) \right).
\]
Applying (3.7) to \( F \left( \frac{\alpha, 2 - \beta}{1} ; 1 - t \right) \), we obtain
\[
(1 - \beta) \frac{d}{dt} v(t) = -\alpha u(t) + (1 - \alpha) v(t).
\]
Applying (3.4) to \( F \left( \frac{\alpha, 2 - \beta}{2} ; 1 - t \right) \), we have
\[
(1 - \beta) \frac{d}{dt} u(t) = (\beta - \alpha)(1 - t)^{-1} v(t) - (1 - \beta) \beta : F \left( \frac{\alpha, 1 - \beta}{2} ; 1 - t \right).
\]
Applying (3.6) to \( F \left( \frac{2 - \beta, \alpha}{2} ; 1 - t \right) \), we have
\[
(1 - \beta) \beta : F \left( \frac{\alpha, 1 - \beta}{2} ; 1 - t \right) = -(1 - \beta)(1 - t)^{-1} (1 - \alpha) v(t) - \beta \frac{d}{dt} v(t)
\]
\[
= (1 - \beta) \left( u(t) - (1 - t)^{-1} v(t) \right).
\]
Combining (3.14) and (3.15), we obtain
\[
\frac{d}{dt} u(t) = -(1 - \beta) u(t) + (1 - \alpha)(1 - t)^{-1} v(t).
\]
Hence we have proved (3.8) for the second row vector.
3.2 Canonical Extension

Now we return to the fibration $f : X \to \mathbb{P}^1$, and from now on $t$ denotes the coordinate of the base scheme of $f$. Put $D = \{0, \infty\} \cup \mu_1$, $T = \mathbb{P}^1 \setminus D$, $U = X \times_{\mathbb{P}^1} T$, $\mathcal{H} = R^1 f_* \Omega^*_U/T$, and let $\nabla : \mathcal{H} \to \Omega^1_T \otimes \mathcal{H}$ be the Gauss–Manin connection. The following is immediate from Proposition 3.1.

**Proposition 3.2** For $n = 1, \ldots, p - 1$, the Gauss–Manin connection $\nabla : \mathcal{H}^{(n)} \to \Omega^1_T \otimes \mathcal{H}^{(n)}$ is given by

$$
(\nabla \omega_n, \nabla \eta_n) = \begin{cases} 
\frac{dt}{t} \otimes (\omega_n, \eta_n) \begin{pmatrix} 1 - \beta & 0 & -1 \\ 0 & 1 - \alpha & 1 \end{pmatrix} & \text{if } \alpha \neq \beta, \\
\frac{ds}{s} \otimes (\omega_n, \eta_n) \begin{pmatrix} 1 - \beta & 0 & 1 \\ 0 & 1 - \alpha & -1 \end{pmatrix}, & s = 1/t.
\end{cases}
$$

Let $j : T \to \mathbb{P}^1$ denote the embedding. Let $\Omega^1_{\mathbb{P}^1}(\log D)$ be the sheaf of differentials on $\mathbb{P}^1$ with logarithmic poles along $D$. Then Deligne’s canonical extension ([9, 5.1]) $\nabla : \mathcal{H}_c \to \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathcal{H}$ is defined to be the unique sub-bundle of $j_! \mathcal{H}$ satisfying the following properties:

- $\nabla(\mathcal{H}_c) \subset \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathcal{H}$,
- for each $t \in D$, all the eigenvalues of $\text{Res}_t(\nabla)$ lie in the interval $[0, 1)$, where $\text{Res}_t(\nabla)$ denotes the residue at $t$ of the connection matrix.

In fact, we have $\mathcal{H}_c = R^1 f_* \Omega^*_X(\log Z)$ (recall $Z = X \times_{\mathbb{P}^1} \{0, \infty\}$) by Steenbrink [25, (2.18), (2.20)]. This is determined as follows.

**Proposition 3.3** For $n = 1, \ldots, p - 1$, local bases of $\mathcal{H}^{(n)}_c$ at $t \in D$ are given as follows.

$$
\mathcal{H}^{(n)}_c |_0 = \begin{cases} 
(\omega_n - \eta_n, t^{(1-\beta)}(1 - \beta) \omega_n - (1 - \alpha) \eta_n) & \text{if } \alpha \neq \beta, \\
(\omega_n, \eta_n) & \text{if } \alpha = \beta,
\end{cases}
$$

$$
\mathcal{H}^{(n)}_c |_\infty = \begin{cases} 
(t^{1-\beta})((1 - \alpha - \beta) \omega_n + (1 - \alpha) t^{-1} \eta_n), t^{(1-\beta)} \eta_n & \text{if } \alpha + \beta \\ t^{1-\beta} \omega_n, t^{1-\beta} \eta_n & \text{if } \alpha + \beta = 1,
\end{cases}
$$

$$
\mathcal{H}^{(n)}_c |_\zeta = \langle \omega_n, \eta_n \rangle (\zeta \in \mu_1).
$$

The residue matrices with respect to these bases are

$$
\text{Res}_0(\nabla) = \begin{pmatrix} 0 & 0 & (\beta - \alpha) \\ 0 & 0 & (1 - \alpha) \beta \\ 0 & 0 & 1 \end{pmatrix}, & \text{if } \alpha \neq \beta, \\
\begin{pmatrix} 0 & (\beta - \alpha) \\ 0 & (1 - \alpha) \beta \\ 0 & 1 \end{pmatrix}, & \text{if } \alpha = \beta,
$$

$$
\text{Res}_\infty(\nabla) = \begin{pmatrix} (1-\beta) & 0 & 0 \\ 0 & t^{(1-\beta)}(1 - \beta) & 0 \\ 0 & 0 & (1 - \alpha) \beta \end{pmatrix}, & \text{if } \alpha + \beta \\ \begin{pmatrix} (1-\beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1 - \alpha) \beta \end{pmatrix}, & \text{if } \alpha + \beta = 1,
$$

$$
\text{Res}_\zeta(\nabla) = -(1 - \alpha) \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}.
$$

**Proof** Let $A$ be the matrix of the connection from Proposition 3.2. For each $t \in D$, we shall find a matrix $P$ with coefficients in local sections of $j_! \mathcal{O}_U$ such that $(\omega_n, \eta_n)P$...
is a local basis of $\mathcal{H}_{e}$ at $t$. The connection matrix with respect to this basis is given by the gauge transformation $A_{P} := P^{-1} AP + P^{-1} P'$, where $P' = \frac{d}{dt} P$. For $t = 0$, we let

\[ P = \begin{pmatrix} 1 & 1 - \beta \\ -1 & 1 - \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tau^{(\alpha - \beta) t} \end{pmatrix} \]

if $\alpha \neq \beta$, and $P = I$ (the unit matrix) if $\alpha = \beta$. For $t = \zeta \in \mu_{t}$, we let $P = I$. Finally for $t = \infty$, we let

\[ P = \begin{pmatrix} 1 & 0 \\ 0 & \tau^{-t} \end{pmatrix} \begin{pmatrix} 1 - \alpha - \beta & 0 \\ 1 - \alpha & 1 \end{pmatrix} \begin{pmatrix} \tau^{(1 - \beta) t} & 0 \\ 0 & \tau^{(\alpha t)} \end{pmatrix} \]

if $\alpha + \beta \neq 1$, and

\[ P = \begin{pmatrix} \tau^{(\alpha t)} & 0 \\ 0 & \tau^{(\alpha t) - t} \end{pmatrix} \]

if $\alpha + \beta = 1$. Then one verifies that $A_{P}$ satisfies the desired properties and its residue is given as stated.

To see the Hodge filtration, we rewrite the above bases as follows.

**Corollary 3.4** Let $n = 1, \ldots, p - 1$.

\[ \mathcal{H}_{e}^{(n)} \mid_{z = 0} = \begin{cases} \{ \omega_{n}, \tau^{(\beta - n) t}((1 - \beta) \omega_{n} - (1 - \alpha) \eta_{n}) \} & \text{if } \alpha \leq \beta, \\
\{ \tau^{(\alpha - \beta) t} \omega_{n}, \omega_{n} - \eta_{n} \} & \text{if } \alpha > \beta. \end{cases} \]

\[ \mathcal{H}_{e}^{(n)} \mid_{z = \infty} = \begin{cases} \{ \tau^{(1 - \beta) t} \omega_{n}, \tau^{(\alpha t) - t} \eta_{n} \} & \text{if } |\alpha| \geq ((1 - \beta) t), \\
\{ \tau^{(\alpha t)} \omega_{n}, \tau^{(1 - \beta) t}((1 - \alpha - \beta) \omega_{n} + (1 - \alpha) \tau^{-t} \eta_{n}) \} & \text{if } |\alpha| < ((1 - \beta) t). \end{cases} \]

\[ \mathcal{H}_{e}^{(n)} \mid_{z = \zeta} = \langle \omega_{n}, \eta_{n} \rangle \quad (\zeta \in \mu_{t}). \]

Write $\mathcal{O} = \mathcal{O}_{\mu_{t}}$ and define $F^{1} \mathcal{H}_{e} = \mathcal{H}_{e} \cap j_{*}(F^{1} \mathcal{H})$. Then we immediately have the following corollary.

**Corollary 3.5** Let $n = 1, \ldots, p - 1$.

(i) We have $F^{1} \mathcal{H}_{e}^{(n)} = \mathcal{O}(i) t^{i} \omega_{n}$ with

\[ (i, j) = \begin{cases} \{ ((1 - \beta) t, 0) & \text{if } |\alpha| > ((1 - \beta) t), \alpha \leq \beta, \\
((1 - \beta) t - [(\alpha - \beta) t], [(\alpha - \beta) t]) & \text{if } |\alpha| > ((1 - \beta) t), \alpha > \beta, \\
(\alpha t, 0) & \text{if } |\alpha| < ((1 - \beta) t), \alpha \leq \beta, \end{cases} \]

(ii) According to the four cases as above, we have

\[ \text{Gr}^{1}_{\mu_{t}} \mathcal{H}_{e}^{(n)} = \begin{cases} \mathcal{O}(\left(-(1 - \alpha) t\right)] + [(\beta - \alpha) t]) t^{-(\beta - \alpha) t}((1 - \beta) \omega_{n} - (1 - \alpha) \eta_{n}), \\
\mathcal{O}(\left((1 - \alpha) t\right)] \omega_{n} - \eta_{n}, \\
\mathcal{O}(\left((\beta - \alpha) t\right)] - [(\beta t)] t^{-(\beta - \alpha) t} \\
\mathcal{O}((\beta - t]) \prod t \omega_{n} - (1 - \beta) \omega_{n} + (1 - \alpha) \eta_{n}, \\
\mathcal{O}((1 - \alpha) t] \left((1 - \alpha - \beta) t^{i} \omega_{n} - (1 - \alpha) \omega_{n} - \eta_{n}) \right). \end{cases} \]
Here, by abuse of notation, the images of $\omega_n$, $\eta_n$ in $Gr^1_\mathcal{F}(\mathcal{H}_\varepsilon^{(n)})$ are denoted by the same letters.

**Corollary 3.6** For each $\xi \in \mu_1$, $X_\xi$ is a normal crossing divisor in $X$ with rational irreducible components.

**Proof** By Proposition 3.3, the local monodromy of $H^i(X_\xi, \mathbb{Q})$ at $t = \xi$ is unipotent, hence $X_\xi$ is normal crossing [21, Theorem 1]. By the Clemens–Schmid exact sequence [17, §4 (a)], $H^i(X_\xi, \mathbb{Q})$ is the kernel of the log local monodromy $N: H^i(X_t, \mathbb{Q}) \to H^i(X_t, \mathbb{Q})$. The cohomology group $H^i(X_t, \mathbb{Q})$ carries a limiting mixed Hodge structure and $N$ is a morphism of mixed Hodge structures of type $(−1, −1)$. Since rank $N = \frac{1}{2} \dim H^i(X_t, \mathbb{Q})$ by Proposition 3.3, we have Gr$^{(0)} H^i(X_t, \mathbb{Q}) = 0$ and $W_0 H^i(X_t, \mathbb{Q}) = \text{Ker}(N)$. Hence $H^i(X_\xi)$ is of pure weight 0, and all the irreducible components of $X_\xi$ are rational.

## 4 Hodge Numbers

In this section, we determine the Hodge numbers of the eigencomponents of our $H$ and prove that it has CM by $K$, i.e., $\dim_K H_B = 1$.

### 4.1 Localization Sequence

Let the notations be as in Section 3.2 and put $Z = X \setminus U$. We have the localization sequence $H^2_\mathcal{F}(X) \to H^2(X) \to H^2(U) \to H^2_\mathcal{F}(X) \to H^2(X)$ both for the de Rham and Betti cohomologies. Let $(Z)$ denote the image of the first map. Recall that we defined (2.2) the Hodge–de Rham structure $H = H^2(X)/(Z) \otimes_K K$.

**Proposition 4.1** $H^1(X) = H^3(X) = 0$.

**Proof** By Poincaré duality, it suffices to show $H^1(X, \mathbb{Q}) = 0$. Since $H^1(X, \mathbb{Q}) \to W_1 H^1(U, \mathbb{Q})$, where $W_1$ denotes the weight filtration, it suffices to show the vanishing of the latter. Using the Leray spectral sequence, we have an exact sequence

$$0 \longrightarrow H^1(T, \mathbb{Q}) \longrightarrow H^1(U, \mathbb{Q}) \longrightarrow H^0(T, R^1 f_* \mathbb{Q}) \longrightarrow 0.$$

By the computation of Res$_{\omega_k} (\nabla)$ in Proposition 3.3, for $n = 1, \ldots, p − 1$, the local monodromy around $t = \infty$ of $H^i(X_\xi, \mathbb{C})^{(n)}$ does not have 1 as an eigenvalue. Hence we have $H^0(T, R^1 f_* \mathbb{Q}) = 0$ (recall that $H^1(X_t, \mathbb{C})^{(0)} = 0$). Since $H^1(T, \mathbb{Q})$ is of weight 2, we have $W_1 H^1(U, \mathbb{Q}) = 0$.

As a result, we have an exact sequence on the de Rham side [14, Chapter II, Theorem (3.3), Proposition (3.4)]

$$0 \longrightarrow H^2_{\text{dR}}(X)/(Z) \longrightarrow H^2_{\text{dR}}(U) \longrightarrow H^2_{\text{dR}}(Z) \longrightarrow 0.$$

The middle term is described by the canonical extension as follows. The Leray spectral sequence yields an exact sequence

$$0 \longrightarrow H^1(T, \mathcal{M}) \longrightarrow H^2_{\text{dR}}(U) \longrightarrow H^0(T, R^2 f_* \Omega^1_{X/\mathbb{C}}) \longrightarrow 0.$$
Since $\sigma^*$ acts on $R^2 f_* \Omega^1_{U/T}$ trivially, we have $H^1(T, \mathcal{H}^{(n)}) \simeq H^2_{\text{DR}}(U)^{(n)}$ for $n = 1, \ldots, p - 1$. Put a complex of sheaves on $\mathbb{P}^1$ as $\mathcal{E} = [\mathcal{H} \xrightarrow{\nabla} \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathcal{H}]$. Then the map of complexes

$$
\mathcal{H} \xrightarrow{j_*} \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathcal{H}
$$

induces an isomorphism $H^0(\mathbb{P}^1, \mathcal{E}) \cong H^1(T, \mathcal{H})$, and the first group carries a mixed Hodge structure [26, Theorem (4.1)] and its Hodge filtration is given as follows [26, (4.10)]:

(4.1) \quad F^0 H^1(\mathbb{P}^1, \mathcal{E}) = H^1(\mathbb{P}^1, \mathcal{E}),

(4.2) \quad F^1 H^1(\mathbb{P}^1, \mathcal{E}) = H^1(\mathbb{P}^1, \mathcal{F} \mathcal{H} \rightarrow \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathcal{H}),

(4.2) \quad F^2 H^1(\mathbb{P}^1, \mathcal{E}) = H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes F^1 \mathcal{H}).

It follows that

$$
\text{Gr}_F^0 H^1(\mathbb{P}^1, \mathcal{E}) = H^1(\mathbb{P}^1, \text{Gr}_F^0 \mathcal{H}),
$$

$$
\text{Gr}_F^1 H^1(\mathbb{P}^1, \mathcal{E}) = \text{Coker}(H^0(\mathbb{P}^1, F^1 \mathcal{H}) \rightarrow H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes \text{Gr}_F^0 \mathcal{H})),
$$

where $\nabla$ is the map induced from the composition of $\nabla$ and the projection $\mathcal{H} \rightarrow \text{Gr}_F^0 \mathcal{H}$.

### 4.2 Residues

For each $t \in D$, let $\partial_t : H^2_{\text{DR}}(U) \rightarrow H^1_{1}(X_t)$ be the $t$-component of the coboundary map $\partial$. Let $N_t \subset \mathcal{H}_{e,t}$ be the image of the composite

$$
\Gamma(U_t, \mathcal{H}) \xrightarrow{\nabla} \Gamma(U_t, \Omega^1_{\mathbb{P}^1}(\log t) \otimes \mathcal{H}) \xrightarrow{\text{Res}_t} \mathcal{H}_{e,t},
$$

where $U_t$ is a small open neighborhood of $t$. Then it is not difficult to show that the diagram

$$
\begin{array}{ccc}
H^1(\mathbb{P}^1, \mathcal{E}) & \xrightarrow{c} & H^2_{\text{DR}}(U) \\
\downarrow \text{Res}_t & & \downarrow \partial_t \\
\mathcal{H}_{e,t}/N_t & \xrightarrow{n} & H^1_{1}(X_t)
\end{array}
$$

commutes, where the lower map is an isomorphism. The following is immediate from Proposition 3.3.

**Proposition 4.2** \quad For $n = 1, \ldots, p - 1$, we have

$$
N^{(n)}_0 = \left\{ t^{(\alpha - \beta)} \iota(1 - \beta) \omega_n - (1 - \alpha) \eta_n \right\},
$$

$$
N^{(n)}_\infty = \mathcal{H}_{e,\infty},
$$

$$
N^{(n)}_\zeta = \left\{ \eta_n \right\} \quad \text{for } \zeta \in \mu_1.
$$
Therefore, we have

$$\dim H^1_{dR}(X_t)^{(n)} = \begin{cases} 1 & \text{if } t = 0 \text{ or } t \in \mu_1, \\ 0 & \text{if } t = \infty. \end{cases}$$

Later, we shall use the following.

**Lemma 4.3** Let $n = 1, \ldots, p - 1$.

(i) If $\alpha \leq \beta$, then $t^m \omega_n|_{t=0} \in N_0^{(n)}(\alpha)$ if $m > 0$, and $\not\in N_0^{(n)}(\alpha)$ if $m = 0$.

(ii) If $\alpha > \beta$, then $t^m \omega_n|_{t=0} \in N_0^{(n)}(\beta)$ if $m \geq [(\alpha - \beta)]$.

**Proof** By Corollary 3.4 and Proposition 4.2, this is trivial except when $\alpha > \beta$ and $m = [(\alpha - \beta)]$. In this case, we have

$$t^m \omega_n|_{t=0} = t^m \omega_n|_{0} + \frac{1 - \alpha}{\alpha - \beta} t^m (\omega_n - \eta_n)|_{t=0} = \frac{t^m ((1 - \beta) \omega_n - (1 - \alpha) \eta_n)|_{t=0}}{\alpha - \beta} \in N_0^{(n)}(\beta).$$

\[\square\]

### 4.3 Hodge Numbers

For each $n = 1, \ldots, p - 1$, we obtained an exact sequence

$$0 \longrightarrow \frac{(H^2_{dR}(X)/(\Lambda))}{(n)} \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}^{(n)}) \longrightarrow \frac{\operatorname{Res} \mathcal{H}_{\epsilon,0}(n) / \mathcal{N}^{(n)}_{\epsilon,0} \oplus \bigoplus_{\epsilon \in \mu_1} \mathcal{H}_{\epsilon,0}^{(n)} / \mathcal{N}^{(n)}_{\epsilon,0}}{0} \longrightarrow 0. \tag{4.3}$$

First, we give a basis of $F^2$. By (4.1), we have an embedding

$$\iota: F^2(\frac{(H^2_{dR}(X)/(\Lambda))}{(n)}) \rightarrow \Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes F^1 \mathcal{H}^{(n)}).$$

By this, we identify $F^2(\frac{(H^2_{dR}(X)/(\Lambda))}{(n)})$ with the elements of the right-hand side having trivial residues. Recall the rational 2-forms $\omega_{m,n} = t^m \frac{dt}{t} \otimes \omega_n$.

**Proposition 4.4** For each $n = 1, \ldots, p - 1$, a basis of $F^2(\frac{(H^2_{dR}(X)/(\Lambda))}{(n)})$ is given by $\{\omega_{m,n} \mid m \in I_n^2\}$, where

$$I_n^2 = \{ m \mid \max \{1, [(\alpha - \beta)] \} \leq m \leq \min \{[\alpha 1], [(1 - \beta)] \} \}.$$ 

In particular, $\dim F^2(\frac{(H^2_{dR}(X)/(\Lambda))}{(n)}) = \min \{[\alpha 1], [(1 - \beta)] \} - \max \{0, [(\alpha - \beta)] \}$.

**Proof** Let $F^1 \mathcal{H}^{(n)} = \mathcal{O}(i) t^j \omega_n$ be as in Corollary 3.5 (i). One easily sees that a basis of $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes F^1 \mathcal{H}^{(n)})$ is given by

$$\omega_{m,n}(j \leq m \leq i + j), \quad t^i \frac{dt}{t - \zeta} \otimes \omega_n (\zeta \in \mu_1).$$

For the first type, the residues at $\zeta \in \mu_1$ are trivial. By Lemma 4.3, $\operatorname{Res}_0(\omega_{m,n}) = t^m \omega_n$ is trivial for $m \geq j$ unless $\alpha \leq \beta$ and $m = 0$. For the second type, it has trivial residues.
except at \( \zeta \) and
\[
\text{Res}_t \left( t^i \frac{dt}{t-\zeta} \otimes \omega_n \right) = t^i \omega_n,
\]
which is non-trivial by Proposition 4.2. These show that a basis of
\[
F^2(H^{2,n}_{\text{dR}}(X)/(\mathbb{Z}))^{(n)}
\]
is given by \( \omega_{m,n} \) with \( j \leq m \leq i + j \) and \( m \neq j = 0 \) if \( \alpha \leq \beta \). Hence the proposition follows from Corollary 3.5 (i).

Since \( (H^{e,0}/N_0)^{(n)} \) and \( (H^{e,\zeta}/N_\zeta)^{(n)} \) are all 1-dimensional, the above proof implies the following.

**Corollary 4.5** For \( n = 1, \ldots, p - 1 \), we have
\[
\text{Res}(F^2H^1(\mathbb{P}^1, \mathcal{E}^{(n)})) = \begin{cases} (H^{e,0}/N_0)^{(n)} \oplus (H^{e,\zeta}/N_\zeta)^{(n)} & \text{if } \alpha < \beta, \\ (H^{e,\zeta}/N_\zeta)^{(n)} & \text{if } \alpha > \beta. \end{cases}
\]

**Corollary 4.6** Suppose that \( p < 1 \). Then we have \( F^2(H^{2,n}_{\text{dR}}(X)/(\mathbb{Z}))^{(n)} \neq 0 \) for any \( n = 1, \ldots, p - 1 \).

**Proof** Since \( \alpha, 1 - \beta \geq 1/p \), we have \( l \alpha, l(1 - \beta) > 1 \). Since \( \beta \geq 1/p \) and \( \alpha \leq 1 - 1/p \), we have \( (\alpha - \beta)l < a_l - 1, (1 - \beta)l - 1 \). Hence we have \( I_z \neq \emptyset \).

Now we determine the other Hodge numbers.

**Lemma 4.7** Let \( n = 1, \ldots, p - 1 \).
(i) If \( \alpha \leq \beta \), then we have \( \Gr^1_F(H^{2,n}_{\text{dR}}(X)/(\mathbb{Z}))^{(n)} = \Gr^1_F H^1(\mathbb{P}^1, \mathcal{E}^{(n)}) \).
(ii) If \( \alpha > \beta \), then we have an exact sequence
\[
0 \to \Gr^1_F(H^{2,n}_{\text{dR}}(X)/(\mathbb{Z}))^{(n)} \to \Gr^1_F H^1(\mathbb{P}^1, \mathcal{E}^{(n)}) \to \text{Res}_0 (H^{e,0}/N_0)^{(n)} \to 0.
\]

**Proof** By (4.3) and Corollary 4.5, we are left to show the non-triviality of \( \text{Res}_0 \) in the case (ii). If \( |\alpha l| \geq |(1 - \beta) l| \), consider
\[
\frac{dt}{t(1 - t^l)} \otimes (\omega_n - \eta_n).
\]
By Corollary 3.5 (ii), this is an element of \( H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes \Gr^0_F H^{e,0}) \). Its residue at 0 is \( \omega_n - \eta_n \neq 0 \mod N_0 \) by Proposition 4.2. If \( |\alpha l| < |(1 - \beta) l| \), consider similarly
\[
\frac{dt}{t(1 - t^l)} \otimes ((1 - \alpha - \beta) t^l \omega_n - (1 - \alpha)(\omega - \eta_n)),
\]
whose residue at 0 is \(- (1 - \alpha)(\omega_n - \eta_n) \neq 0 \mod N_0 \).

**Proposition 4.8** For each \( n = 1, \ldots, p - 1 \), we have
\[
\dim \Gr^1_F(H^{2,n}_{\text{dR}}(X)/(\mathbb{Z}))^{(n)} = |\alpha l| - |(1 - \beta) l| + |\alpha - \beta| l.
\]
Proof First we show that the map
\[
\nabla: H^0(\mathbb{P}^1, F^1 \mathcal{H}_e(n)) \to H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1(\log D)} \otimes \text{Gr}^0 F \mathcal{H}_e(n))
\]
is injective. Let \( F^1 \mathcal{H}_e(n) = \mathcal{O}(i) t^j \omega_n \) as in Corollary 3.5 (i). Then \( H^0(\mathbb{P}^1, F^1 \mathcal{H}_e(n)) \) has a basis \( \{ \omega_{m,n} \mid j \leq m \leq i + j \} \), and
\[
\nabla \omega_{m,n} = \frac{dt}{t} t^m \left( (m - l(1 - \beta)) \omega_n + \frac{l(1 - \alpha)}{1 - t^l} \eta_n \right) = l(1 - \alpha) \frac{dt}{t} t^m \eta_n \neq 0
\]
modulo \( H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1(\log D)} \otimes \mathcal{O}(n)) \). Since \( 0 \leq i < l \) in every case, \( \omega_{m,n} \) belong to different eigenspaces with respect to the \( \tau \)-action. Hence the non-vanishing implies the injectivity.

By Corollary 3.5 (ii), we have \( \text{Gr}^0 F \mathcal{H}_e(n) = \mathcal{O}(k) \), where
\[
k := \begin{cases} 
-[(1 - \alpha)l] + [(\beta - \alpha)l] & \text{if } |a| \geq |(1 - \beta)l|, \alpha \leq \beta, \\
-(1 - \alpha)l & \text{if } |a| \geq |(1 - \beta)l|, \alpha > \beta, \\
[(\beta - \alpha)l] - [\beta l] & \text{if } |a| < |(1 - \beta)l|, \alpha \leq \beta, \\
-\beta l & \text{if } |a| < |(1 - \beta)l|, \alpha > \beta.
\end{cases}
\]
Note that \( k < 0 \) in any case. One sees that \( H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1(\log D)} \otimes \mathcal{O}(k)) \) has a basis
\[
\frac{t^m}{1 - t^l} \frac{dt}{t} \otimes \omega_n \quad (0 \leq m \leq i + k).
\]
By (4.2) and the above injectivity, we have
\[
\dim \text{Gr}^0 F^1 \mathcal{H}(\mathbb{P}^1, \mathcal{O}(n)) = \dim H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1(\log D)} \otimes \mathcal{O}(k)) - \dim H^0(\mathbb{P}^1, \mathcal{O}(i)) = (l + k + 1) - (i + 1) = l + k - i.
\]
By Corollary 3.5 (i) and Lemma 4.7, we obtain the desired formula.

Corollary 4.9 Assume that \( p < l \) and \( p > 2 \) when \( a = b \). Then we have
\[
\text{Gr}^0 F^1 \mathcal{H}(\mathbb{P}^1, \mathcal{O}(n)) = 0
\]
for any \( n = 1, \ldots, p - 1 \).

Proof If \( a \neq b \), then \(|a - b| \geq \frac{1}{p} \geq 1\). If \( a = b \), then \( \alpha \neq 1 - \alpha \) since \( p > 2 \), and hence \(|a| - |(1 - a)l| \geq 1\).

Proposition 4.10 For each \( n = 1, \ldots, p - 1 \), we have
\[
\dim \text{Gr}^0 F^1 \mathcal{H}(\mathbb{P}^1, \mathcal{O}(n)) = \min\{|(1 - a)l|, |\beta l|\} - \max\{0, |(\beta - a)l|\}.
\]

Proof By (4.2), Corollary 4.5, and Lemma 4.7, we have
\[
\text{Gr}^0 F^1 \mathcal{H}(\mathbb{P}^1, \mathcal{O}(n)) = H^1(\mathbb{P}^1, \mathcal{O}(n)) = H^1(\mathbb{P}^1, \mathcal{O}(k)),
\]
where \( k \) is as in the proof of Proposition 4.8. Since \( k < 0 \), we have
\[
\dim H^1(\mathbb{P}^1, \mathcal{O}(k)) = \dim H^0(\mathbb{P}^1, \mathcal{O}(k - 2)) = -k - 1.
\]
Hence the proposition follows.
Remark 4.11 In fact, Proposition 4.10 is equivalent to the dimension formula in Proposition 4.4. Note that the complex conjugation switches \( n \) (resp. \( \alpha, \beta \)) and \( p - n \) (resp. \( 1 - \alpha, 1 - \beta \)).

Theorem 4.12 The Hodge–de Rham structure \( H = (H^2(X)/\langle Z \rangle) \otimes_K K \) has CM by \( K \), i.e., \( \dim_K H_B = 1 \).

Proof Combining Propositions 4.4, 4.8, and 4.10, one verifies that
\[
\dim(H^2_{\text{dR}}(X)/\langle Z \rangle)^{(n)} = l - 1
\]
for each \( n = 1, \ldots, p - 1 \). It follows that \( \dim_K H_B \leq (l - 1)(p - 1) = [K: \mathbb{Q}] \). It remains to show that \( H \neq 0 \), for which it suffices to show that \( \tau \) is not the identity on \( H^2_{\text{dR}}(X)/\langle Z \rangle \). If \( p < l \), this follows from Proposition 4.4 and Corollary 4.6. The general case follows from Proposition 5.2 below.

5 Periods

We compute the periods of our \( H \) and verify the Gross–Deligne conjecture, for which it will suffice to consider \( F^1H_{\text{dR}} \).

5.1 Basis of \( F^1H_{\text{dR}} \)

Recall that, by (4.3), we can identify \( F^1(H^2_{\text{dR}}(X)/\langle Z \rangle)^{(n)} \) with the elements of
\[
F^1H^1(\mathbb{P}^1, \mathcal{E}^{(n)})
\]
having trivial residues. Furthermore, they are identified with rational 2-forms by the following lemma. Put \( T_1 = \mathbb{P}^1 \setminus \{0, \infty\} \).

Lemma 5.1 For each \( n = 1, \ldots, p - 1 \), there is a natural injection
\[
i: F^1(H^2_{\text{dR}}(X)/\langle Z \rangle)^{(n)} \to \Gamma(T_1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes F^1 \mathbb{H}^{(n)})
\]

Proof By (4.1) and (4.3), it suffices to show the existence of an injection
\[
H^1(\mathbb{P}^1, F^1 \mathcal{E}^{(n)}) \to \Gamma(T_1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes F^1 \mathbb{H}^{(n)})
\]
where we put \( F^1 \mathcal{E} = [F^1 \mathbb{H} \to \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathbb{H}^{(n)}] \). Consider the commutative diagram in Figure 1, where the right vertical sequence is exact. By Proposition 3.3, \( \nabla \) is an isomorphism on \( T_1 \). Therefore, we have an isomorphism
\[
\Gamma(T_1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes F^1 \mathbb{H}^{(n)}) \xrightarrow{\sim} H^1(T_1, F^1 \mathcal{E}^{(n)})
\]
It remains to show the injectivity of \( H^1(\mathbb{P}^1, F^1 \mathcal{E}^{(n)}) \to H^1(T_1, F^1 \mathcal{E}^{(n)}) \). This follows from the fact that \( H^1(\mathbb{P}^1, F^1 \mathcal{E}) \to H^1(T_1, F^1 \mathcal{E}) \) is injective and \( H^1(\mathbb{P}^1, \mathcal{E}) \to H^1(T_1, \mathcal{E}) \) is an isomorphism.


Under the identification via $\iota$, we have the following.

**Proposition 5.2** For each $n = 1, \ldots, p - 1$, a basis of $F^1(H^2_{\text{DR}}(X)/(Z))^{(n)}$ is given by

\[ \{ \omega_{m,n} \mid m \in I_n^1 \}, \text{ where} \]

\[ I_n^1 := \begin{cases} \{-\lfloor (\beta - \alpha)l \rfloor, \ldots, -1\} \cup \{1, \ldots, \max\{\lfloor a l \rfloor, \lfloor (1 - \beta) l \rfloor\}\} & \text{if } \alpha < \beta, \\ \{1, \ldots, \max\{\lfloor a l \rfloor, \lfloor (1 - \beta) l \rfloor\}\} & \text{if } \alpha \geq \beta. \end{cases} \]

Recall that $\alpha = \{ \frac{na}{p} \}$, $\beta = \{ \frac{nb}{p} \}$.

**Proof** It is routine to verify that $|I_n^1| = \dim F^1(H^2_{\text{DR}}(X)/(Z))^{(n)}$ using Propositions 4.4 and 4.8. Therefore, it suffices to show that

\[ \omega_{m,n} \in F^1(H^2_{\text{DR}}(X)/(Z))^{(n)} \]

if $m \in I_n^1$. We construct Čech cocycles representing elements of $H^1(\mathbb{P}^1, F^1\mathcal{E}^{(n)})$ with trivial residues which correspond to $\omega_{m,n}$. Take a covering $\mathbb{P}^1 = U_0 \cup U_\infty$, where $U_0 := \mathbb{P}^1 \setminus \{ \infty \}$, $U_\infty := \mathbb{P}^1 \setminus \{ 0 \}$; note that $T_1 = U_0 \cap U_\infty$. A Čech cocycle in this case is a triple

\[ (\psi, \varphi_0, \varphi_\infty) \in \Gamma(T_1, F^1\mathcal{E}^{(n)}) \oplus \bigoplus_{i=0,\infty} \Gamma(U_i, \Omega^1_{\text{DR}}(\log D) \otimes \mathcal{H}_e^{(n)}) \]
satisfying \( \nabla \psi = \varphi_0 |_{\tau_1} - \varphi_\infty |_{\tau_1} \). We construct such cocycles in four ways. By Proposition 3.2, we have

\[
I^{-1} \nabla (t^m \omega_n)
\]

(5.1) \( = (\mu - 1 + \beta) \omega_{m,n} + \frac{1 - \alpha}{1 - t^l} \eta_{m,n} \)

(5.2) \( = \left( \mu - \alpha - \frac{1 - \alpha - \beta}{1 - t^l} \right) \omega_{m,n} + \frac{t^l}{1 - t^l} \left( (1 - \alpha - \beta) \omega_{m,n} + \frac{1 - \alpha}{1 - t^l} \eta_{m,n} \right) \)

(5.3) \( = (\mu + (1 - \beta) \frac{t^l}{1 - t^l}) \omega_{m,n} - \frac{1 - \alpha}{1 - t^l} (\omega_{m,n} - (1 - \alpha) \eta_{m,n}) \)

(5.4) \( = (\mu - \alpha + \beta + (1 - \alpha) \frac{1 - t^l}{1 - t^l}) \omega_{m,n} - \frac{1 - \alpha}{1 - t^l} (\omega_{m,n} - \eta_{m,n}) \).

Put \( j = \max \{0, \lfloor (\alpha - \beta) l \rfloor \}, k = \min \{\lfloor \alpha l \rfloor, \lfloor (1 - \beta) l \rfloor \} \).

(i) Suppose that \( \lfloor \alpha l \rfloor \geq \lfloor (1 - \beta) l \rfloor \). Let \( \psi = I^{-1} t^m \omega_n \),

\[
\varphi_0 = (\mu - 1 + \beta) \omega_{m,n}, \quad \varphi_\infty = -\frac{1 - \alpha}{1 - t^l} \eta_{m,n}.
\]

By (5.1) and Corollary 3.4, these define a cocycle if \( j \leq m \leq \lfloor \alpha l \rfloor \), \( m \neq 0 \).

(ii) Suppose that \( \lfloor \alpha l \rfloor < \lfloor (1 - \beta) l \rfloor \). Then by (5.2) and Corollary 3.4, \( \psi = I^{-1} t^m \omega_n \),

\[
\varphi_0 = \left( \mu - \alpha - \frac{1 - \alpha - \beta}{1 - t^l} \right) \omega_{m,n},
\]

\[
\varphi_\infty = -\frac{t^l}{1 - t^l} \left( (1 - \alpha - \beta) \omega_{m,n} + (1 - \alpha) t^{-1} \eta_{m,n} \right)
\]

define a cocycle if \( j \leq m \leq \lfloor (1 - \beta) l \rfloor \). To kill the residues, we use Lemma 5.3 below. Then by letting

\[
\varphi_0 = (\mu - \alpha) \omega_{m,n}, \quad \varphi_\infty = (1 - \alpha - \beta) \omega_{m,n} - \frac{1 - \alpha}{1 - t^l} \eta_{m,n},
\]

we obtain an element of \( F^1 (H^2_{\text{dR}} (X)/\mathbb{Z}) \) for \( j \leq m \leq \lfloor (1 - \beta) l \rfloor \), \( m \neq 0 \).

(iii) Suppose that \( \alpha \leq \beta \). Then by (5.3) and Corollary 3.4, \( \psi = -I^{-1} t^m \omega_n \),

\[
\varphi_0 = \frac{1}{1 - t^l} \left( (1 - \beta) \omega_{m,n} - (1 - \alpha) \eta_{m,n} \right), \quad \varphi_\infty = \left( \mu + (1 - \beta) \frac{t^l}{1 - t^l} \right) \omega_{m,n}
\]

define a cocycle if \( -\lfloor (\beta - \alpha) l \rfloor \leq m \leq k \). If \( m < 0 \), we can kill the residues using Lemma 5.3, and \( \varphi_0 = (1 - \beta) \omega_{m,n} - \frac{1 - \alpha}{1 - t^l} \eta_{m,n} \), and \( \varphi_\infty = \mu \omega_{m,n} \) define an element of \( F^1 (H^2_{\text{dR}} (X)/\mathbb{Z}) \) for \( -\lfloor (\beta - \alpha) l \rfloor \leq m < 0 \).

(iv) Finally suppose that \( \alpha > \beta \). Then, by (5.4) and Corollary 3.4, \( -I^{-1} t^m \omega_n \),

\[
\varphi_0 = \frac{1 - \alpha}{1 - t^l} (\omega_{m,n} - \eta_{m,n}), \quad \varphi_\infty = \left( \mu - \alpha + \beta + (1 - \alpha) \frac{t^l}{1 - t^l} \right) \omega_{m,n}
\]

define a cocycle if \( 0 \leq m \leq k \). If \( m \neq 0 \), we can use Lemma 5.3 to kill the residues and

\[
\varphi_0 = (1 - \alpha) \omega_{m,n} - \frac{1 - \alpha}{1 - t^l} \eta_{m,n}, \quad \varphi_\infty = (\mu - 1 + \beta) \omega_{m,n}
\]
define an element of $F^1(H^2 \mathbb{R}(X)(Z))^{(n)}$ for $0 < m \leq k$. Combining (iii) and (i) or (ii), we obtain the first case of the proposition. For the second case, combine (iv) and (i) or (ii), just noting that $k > j - 1 = \lfloor (\alpha - \beta)l \rfloor$.

**Lemma 5.3** If $j \leq m < l$, $m \neq 0$, then

$$
\frac{1}{(1-t)^j} \otimes \omega_{m,n} \in \Gamma(\mathcal{P}^1, \Omega^1_{\mathcal{P}^1}(\log D) \otimes \mathcal{H}^{(n)}),
$$

and it has trivial residues at $t = 0, \infty$.

**Proof** This is immediate from Corollary 3.4 and Lemma 4.3.

**5.2 Period Formula**

We prove the period formula which verifies the conjecture of Gross–Deligne [13, §4] (but see Remark 5.6 below). We identify an embedding $\chi: K \to \mathbb{C}$ with the element $h \in (\mathbb{Z}/l\mathbb{Z})^\times$ such that $\chi(\zeta_p) = \zeta_p^h$, and write $H^{(h)}$ instead of $H^\chi$. For each $h \in (\mathbb{Z}/l\mathbb{Z})^\times$, let $(p(h), 2 - p(h))$ be the Hodge type of $H^{(h)}$. Put $K' = \mathbb{Q}(\mu_{2l}) (K = K'$ if $l$ is odd).

**Theorem 5.4** Define a function $\varepsilon: \mathbb{Z}/l\mathbb{Z} \to \mathbb{Z}$ by

$$
\varepsilon(i) = \begin{cases} 
1 & \text{if } i \equiv 1b, p, l(p - b), l(b - a) + p \pmod{l}, \\
-1 & \text{if } i \equiv 1b + p, l(p - a) + p \pmod{l}, \\
0 & \text{otherwise}.
\end{cases}
$$

Then, for any $h \in (\mathbb{Z}/l\mathbb{Z})^\times$, we have

$$
p(h) = \sum_{i \in \mathbb{Z}/l\mathbb{Z}} \varepsilon(i) \left\{ -\frac{hi}{lp} \right\} \text{ and } \text{Per}(H^{(h)}) \sim_{K'} \prod_{i \in \mathbb{Z}/l\mathbb{Z}} \Gamma\left(\frac{hi}{lp}\right) \varepsilon^{(i)}.\]

**Proof** For real numbers $x, y$ with $0 < x, y < 1, x + y \neq 1$, put

$$
\delta(x, y) := \{ -x \} + \{ -y \} - \{ -(x + y) \} = \begin{cases} 
1 & \text{if } x + y < 1, \\
0 & \text{if } x + y > 1.
\end{cases}
$$

Then we have $\varphi(h) := \sum_i \varepsilon(i) \left\{ -\frac{hi}{lp} \right\} = \delta(\beta, \mu) + \delta(1 - \beta, \{\beta - \alpha + \mu\})$, where we put $\alpha = \{ ha/p \}, \beta = \{ hb/p \}, \mu = \{ h/l \}$. First, we have $\varphi(h) = 2$ if and only if

$$
\beta + \mu < 1, \quad 1 - \beta + \{\beta - \alpha + \mu\} < 1.
$$

Letting $m = l\mu$, the first condition becomes $m < (1 - \beta)l$, i.e., $m \leq \lfloor (1 - \beta)l \rfloor$. Similarly, the second condition is equivalent to

$$
(\alpha \leq \beta, m < a l) \quad \text{or} \quad (\alpha > \beta, (\alpha - \beta)l < m < a l).
$$

Comparing with Proposition 4.4, we have $p(h) = 2$ if and only if $\varphi(h) = 2$. Secondly, since $p(h) + p(-h) = \varphi(h) + \varphi(-h) = 2$, we have $p(h) = 0$ if and only $\varphi(h) = 0$. Since $p(h), \varphi(h) \in \{0, 1, 2\}$, we have $p(h) = \varphi(h)$ for any $h$.

For the second statement, we compute the periods over the 2-cycle

$$
(1 - \tau), (1 - \sigma), \Delta_1.
$$
Since \((1 - \zeta_i)(1 - \zeta_p)\) is invertible in \(K\), it reduces to the periods over \(\Delta_1\) (Proposition 2.6 (i)). First consider the two cases:

(i) \(\alpha \leq \beta\) and \(p(h) \geq 1\),

(ii) \(\alpha > \beta\) and \(p(h) = 2\).

By Propositions 4.4 and 5.2, \(H^k\) is generated by \(\omega_{m,n}\) satisfying \([\alpha - \beta]l \leq m\) in both cases, which is equivalent to \(\alpha - \beta < \mu := \frac{m}{l}\). This is the assumption of Proposition 2.6 (i) and we obtain the desired formula.

The other cases are reduced to the ones above. If we replace \(\chi\) with \(\chi^{-1}\), then \(h\) (resp. \(\alpha, \beta, p(h)\)) is replaced with \(-h\) (resp. \(1 - \alpha, 1 - \beta, 2 - p(h)\)). By Lemma 5.5, the cup-product \(H^2(X) \otimes H^2(X) \to \mathbb{Q}(-2)\) induces an auto-duality on \(H\), under which \(H^k\) is dual to \(H^{k^*}\). Hence we have \(\text{Per}(H^{k}) \cdot \text{Per}(H^{k^*}) \sim_{K^*} (2\pi i)^2\). On the other hand, recall the reflection formula

\[
\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x} \sim_{K^*} 2\pi i,
\]

for any \(x \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}\). Therefore, the case where \(\alpha \leq \beta\) and \(p(h) = 0\) (resp. \(\alpha > \beta\) and \(p(h) \geq 1\)) is equivalent to case (ii) (resp. (i)).

\[\text{Lemma 5.5} \quad \text{Put } H^2(X)_{\mathbb{Z}} = \text{Ker}(H^2(X) \to H^2(\mathbb{Z})). \text{ Then the composition}
\]

\[H^2(X)_{\mathbb{Z}} \to H^2(X) \to H^2(X)/\mathbb{Z}
\]

\[\text{induces an isomorphism of Hodge–de Rham structures } H^2(X)_{\mathbb{Z}} \otimes_{\mathbb{R}} K \cong H.
\]

\[\text{Proof} \quad \text{This follows from the fact that the kernel of the composite}
\]

\[H^2_*(X, \mathbb{C}) \to H^2(X, \mathbb{C}) \to H^2(Z, \mathbb{C})
\]

is one-dimensional by Zariski's lemma [6, III, (8.2)].

\[\text{Remark 5.6} \quad \text{Our definition of } \varepsilon \text{ is slightly different from [13]; } \varepsilon(i) \text{ here is } \varepsilon(-i),
\]

where Gross looks at the values \(\Gamma(1 - \frac{1}{2}(h/p))\varepsilon(i)\). The former conforms to the definition of the Stickelberger element as

\[
\sum_{h \in \mathbb{Q}(\mu_N)} \left\{\frac{-h}{N}\right\} \sigma_h^{-1},
\]

where \(\sigma_h \in \text{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})\) sends an \(N\)-th root of unity to its \(h\)-th power.

6 Regulators

After explaining the regulator map we are considering, we prove Theorem 1.2 from the introduction and its consequences on the non-vanishing.

6.1 Formulation

The Deligne cohomology of \(X_{\mathbb{C}} := X \times_{\text{Spec } \mathbb{Q}} \text{Spec } \mathbb{C}\) with coefficients in \(\mathbb{Q}(2)\) is defined to be the hypercohomology of the complex \(\mathbb{Q}(2) \to \Omega^1_{X_{\mathbb{C}}} \to \Omega^1_{X_{\mathbb{C}}/\mathbb{C}}\), where \(\mathbb{Q}(2) := (2\pi i)^2\mathbb{Q}\) is placed in degree 0. Consider the Beilinson regulator map [7]
from the motivic cohomology \( r_\#: H^3_{\text{dR}}(X, \mathbb{Q}(2)) \to H^3_{\text{dR}}(X_C, \mathbb{Q}(2)) \). We have a natural isomorphism \( H^2_{\text{dR}}(X_C, \mathbb{Q}(2)) \cong H^2(X, \mathbb{C})/(F^2 + H^2(X, \mathbb{Q}(2))) \), and the Carlson isomorphism

\[
H^2(X, \mathbb{C})/(F^2 + H^2(X, \mathbb{Q}(2))) \cong \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2))).
\]

Here MHS denotes the abelian category of \( \mathbb{Q} \)-mixed Hodge structures. By Poincaré duality \( H^2(X, \mathbb{Q}(2)) \cong H_2(X, \mathbb{Q}) \), we obtain an identification

\[
H^3_{\text{dR}}(X_C, \mathbb{Q}(2)) \cong \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H_2(X, \mathbb{Q})).
\]

Let \( Z \subset X \) be as before and consider the regulator map

\[
r_{\#Z}: H^3_{\text{dR}, Z}(X, \mathbb{Q}(2)) \to H^3_{\text{dR}, Z}(X, \mathbb{Q}(2)) \cong H_1(Z, \mathbb{Q})
\]

from the motivic cohomology supported on \( Z \). Since \( H_1(X, \mathbb{Q}) = 0 \) by Proposition 4.1, we have an exact sequence of mixed Hodge structures

\[
H_2(Z, \mathbb{Q}) \to H_2(X, \mathbb{Q}) \to H_2(X, Z; \mathbb{Q}) \to H_1(Z, \mathbb{Q}) \to 0.
\]

If we denote the image of the first map by \( \langle Z \rangle \), we have the connecting homomorphism \( \rho: H_1(Z, \mathbb{Q}) \cap H^{0,0} \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H_2(X, \mathbb{Q})/\langle Z \rangle) \), where \( H^{0,0} \) denotes the Hodge \((0,0)\)-component of \( H_1(Z, \mathbb{C}) \). By the lemma and Remark 6.2, \( \rho \) describes the restriction of \( r_{\#Z} \) to the image of \( H^3_{\text{dR}, Z}(X, \mathbb{Q}(2)) \).

**Lemma 6.1** The diagram below is commutative up to sign.

\[
\begin{array}{ccc}
H^3_{\text{dR}, Z}(X, \mathbb{Q}(2)) & \xrightarrow{r_{\#Z}} & H_1(Z, \mathbb{Q}) \cap H^{0,0} \\
\downarrow & & \downarrow \rho \\
H^3_{\text{dR}}(X, \mathbb{Q}(2)) & \xrightarrow{r_{\#}} & H^3_{\text{dR}}(X_C, \mathbb{Q}(2)) \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H_2(X, \mathbb{Q})/\langle Z \rangle) & \cong & \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H_2(X, \mathbb{Q}))
\end{array}
\]

where the vertical maps are the natural ones.

**Proof** See [5, Theorem 11.2].

**Remark 6.2** The right vertical arrow is surjective since \( \text{Ext}^1_{\text{MHS}} = 0 \). Its kernel is topologically generated by decomposable elements, i.e., the image of

\[
(\text{CH}_1(Z) \otimes \mathbb{Q}\text{[e]}) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^3_{\text{dR}, Z}(X, \mathbb{Q}(2)).
\]

Also, it is not difficult to show that \( r_{\#Z} \) is surjective.

### 6.2 Regulator Formula

Now we regard the extension classes as functionals (up to period functionals). Let \( H^2(X)_Z = \text{Ker}(H^2(X) \to H^2(Z)) \) as before. Since \( H^2(X, \mathbb{Q})_Z \cong (H_2(X, \mathbb{Q})/\langle Z \rangle)^* \), we have

\[
\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H_2(X, \mathbb{Q})/\langle Z \rangle) \cong (F^1H^2(X, \mathbb{C})_Z)^*/\text{Image} H_2(X, \mathbb{Q}),
\]

where \( * \) denotes the \( \mathbb{C} \)-linear dual. By Lemma 5.5, \( \rho \) induces a map

\[
\rho: (H_1(Z, \mathbb{Q}) \cap H^{0,0}) \otimes_{\mathbb{R}} K \to (F^1H_{\text{C}})^*/H_3^d.
\]
where $H_C := H_B \otimes_{\mathbb{Q}} \mathbb{C}$ and $H^\vee$ denotes the dual Hodge–de Rham structure of $H$.

Put $Z_1 = \bigcup_{\zeta \in \mu_1} X_\zeta$. We shall describe the restriction of $\rho$ to $H_1(Z_1, \mathbb{Q}) \otimes_K \mathbb{C}$, recall that $H_1(Z_1, \mathbb{Q}) \subset H^{0,0}$ (Corollary 3.6). We have, in fact, the following.

**Lemma 6.3** We have an isomorphism $H_1(Z_1, \mathbb{Q}) \otimes_K \mathbb{C} \overset{\sim}{\to} H_1(Z, \mathbb{Q}) \otimes_K \mathbb{C}$.

**Proof** By Proposition 4.2, $\tau$ acts trivially on $H_1(X_0, \mathbb{Q})$ and $H_1(X_\infty, \mathbb{Q}) = 0$.

Let $(1 - \sigma)_0 \in H_2(X, Z_1 ; \mathbb{Q})$ be the Lefschetz thimble defined in Section 2.5, and let $H_2(X, Z_1 ; \mathbb{Q})_{\text{Let}} \subset H_2(X, Z_1 ; \mathbb{Q})$ denote the $R$-submodule generated by this element.

**Lemma 6.4** The restriction of the boundary map

$$\partial : H_2(X, Z_1 ; \mathbb{Q})_{\text{Let}} \otimes_K \mathbb{C} \to H_1(Z_1, \mathbb{Q}) \otimes_K \mathbb{C}$$

is surjective and $H_1(Z_1, \mathbb{Q}) \otimes_K \mathbb{C}$ is one-dimensional over $K$.

**Proof** By Proposition 4.2, $\dim_K H_1(X_\zeta, \mathbb{Q}) = p - 1$ for $\zeta \in \mu_1$. Since $\tau$ permutes the components of $Z_1$, $H_1(Z_1, \mathbb{Q}) \otimes_K \mathbb{C}$ is one-dimensional over $K$. Whereas $K_0$ and $K_1$ generate $H_1(X_\zeta, \mathbb{Q})$ (Proposition 2.4 (ii)), $K_1$ vanishes as $t \to 1$ by definition. Therefore $K_0$ does not vanish, i.e., $\partial((1 - \sigma)_0, \Delta_0)$ is non-trivial in $H_1(X_\zeta, \mathbb{Q})$, hence is in $H_1(Z_1, \mathbb{Q}) \otimes_K \mathbb{C}$.

Now we state our main theorem. For $x \in K$, let $x_*$ (resp. $x^*$) denote its action on homology (resp. cohomology). Since $1 - \zeta_p$ is invertible in $K$, we write

$$((1 - \zeta_p)^{-1})(1 - \sigma)_0 \in H_1(X, Z_1 ; \mathbb{Q}) \otimes_K \mathbb{C}$$

simply as $\Delta_0$. For each $m$ and $n$, define an embedding $\chi_{m,n} : K \to \mathbb{C}$ by

$$\chi_{m,n}(\zeta_1) = \zeta_1^m, \quad \chi_{m,n}(\zeta_p) = \zeta_p^n.$$

**Theorem 6.5** Let $y \in H_1(Z_1, \mathbb{Q}) \otimes_K \mathbb{C}$ and take $x \in K$ such that $y = x_* \partial \Delta_0$. Let $\{\omega_{m,n} \mid n = 1, \ldots, p - 1, m \in \mathbb{Z}_n\}$ be the basis of $\mathcal{F}_Y H_{\text{dR}}$ given in Proposition 5.2. Then we have

$$\rho(y)(\omega_{m,n}) = \chi_{m,n}(x) \frac{B(1 - \alpha, \beta)}{l(\beta - \alpha + \mu)} \cdot F \left( \frac{1 - \alpha, \beta - \alpha + \mu}{1 - \alpha + \beta - \alpha + \mu + 1} ; 1 \right),$$

where $\alpha = \{ na \}, \beta = \{ nb \}, \mu = \frac{m}{r}$.

**Proof** We apply Theorem A.3 of the appendix to our situation, where $D = Z_1$ and $X^o = X \setminus (X_0 \cup X_\infty)$ (see the proof of Lemma 5.1). Note that $H_C = H_{\text{dR}}^2(X_\zeta) \otimes_K \mathbb{C}$ by Lemma 5.5 since $\tau$ acts trivially on $H_{\text{dR}}^2(\ell(\mathbb{P}_1))$ (see §A.2 for the notations).

Put $\Gamma = (1 - \tau)(1 - \sigma)_0$. Since $\Gamma \in H_2(X, Z_1 ; \mathbb{Q})$ does not necessarily come from $H_2(X^o, Z_1 ; \mathbb{Q})$, we take a detour. Let $\Gamma'$ be the Lefschetz thimble given by sweeping $(1 - \sigma)_0, \Delta_0$ along the path $k_1 + k_2 + k_3$ in $T \setminus \{ 0, \infty \}$, where $k_1$ is the line segment from $\zeta$ to $\zeta' (\epsilon > 0)$, $k_2$ is the arc from $\epsilon \xi$ to $\xi$, and $k_3$ is the line segment from $\xi$ to 1. Then $\Gamma' \in H_2(X^o, Z_1 ; \mathbb{Q})$ and $y := \partial(\Gamma) = \partial(\Gamma')$. Theorem A.3 yields $\rho(y)(\omega_{m,n}) = \int_{\Gamma'} \omega_{m,n}$. The right integral is computed similarly as Proposition 2.6.
(ii), and letting $\epsilon \to 0$, we obtain the theorem for $x = (1-\zeta_i)(1-\zeta_p)$. The general case follows by the cyclicity of $H_1(Z_1, \mathbb{Q}) \otimes_R K$.

### 6.3 Non-vanishing

We prove the non-vanishing of $\rho$ under a mild assumption. The situation is different depending on whether $a + b = p$ or not.

If $a + b \neq p$, the regulator does not vanish even in the Deligne cohomology with $\mathbb{R}$-coefficients, or equivalently, the extension group of $\mathbb{R}$-mixed Hodge structures

$$\text{Ext}^1_{\text{RMHS}}(\mathbb{R}, H_{\mathbb{R}}) \simeq (F^1H_C)^*/H'^*_{\mathbb{R}},$$

where $H_{\mathbb{R}} = H_{\mathbb{R}} \otimes \mathbb{Q}\mathbb{R}$, $H_C = H \otimes \mathbb{Q}\mathbb{C}$. Note that $\dim_{\mathbb{R}}(F^1H_C)^*/H'^*_{\mathbb{R}} = \text{dim}_{\mathbb{C}} \text{Gr}^P_1 H_{\text{dR}}$.

Let $\rho_{\mathbb{R}}: H_1(Z_1, \mathbb{Q}) \otimes_R K \to (F^1H_C)^*/H'^*_{\mathbb{R}}$ be the composition of $\rho$ and the natural surjection.

**Theorem 6.6** Suppose that $p < l$ and $a + b \neq p$ (so $p > 2$). Then $\rho_{\mathbb{R}}$ is non-trivial. In particular, $\dim_{\mathbb{Q}} \rho_{\mathbb{R}}(H_1(Z_1, \mathbb{Q}) \otimes_R K) = (l-1)(p-1)$.

**Proof** By restricting the functionals to $F^1H_{\mathbb{R}} := F^1H_{\mathbb{C}} \cap H_{\mathbb{R}}$ and taking the imaginary part, we obtain a $K \cap \mathbb{R}$-linear map $\rho_{\mathbb{R}}: H_1(Z_1, \mathbb{Q}) \otimes_R K \to \text{Hom}(F^1H_{\mathbb{R}}, i\mathbb{R})$. For each $n = 1, \ldots, p-1$, we have $\alpha \neq 1 - \beta$ by the assumption. Hence $|\alpha - (1 - \beta)| > 1/p > 1/l$ and there exists an $m$ satisfying

$$\min\{|a l|, |(1 - \beta) l|\} < m \leq \max\{|a l|, |(1 - \beta) l|\}.$$

Then we have $\omega_{m,n} \in \text{Gr}^P_1 H_{\text{dR}}$ by Propositions 4.4 and 5.2. Since $m > |(\alpha - \beta) l|$, we have $\mu := m/l > \alpha - \beta$, hence we can apply Proposition 2.6 (i) to compute the period

$$\Omega_{m,n} := \int_{\Delta_1} \omega_{m,n} = \frac{(-1)^{p\beta}}{l} B(\beta, \mu) B(1 - \beta, \beta - \alpha + \mu).$$

Put a normalization as $\bar{\omega}_{m,n} = \Omega_{m,n}^{-1} \omega_{m,n}$. Then we have

$$\int_{x/\Delta_1} \bar{\omega}_{m,n} = \int_{\Delta_1} x^* \bar{\omega}_{m,n} = \chi_{m,n}(x),$$

for any $x \in K$. If we let $n' = p - n$, $\alpha' = \{n' a/p\} = 1 - \alpha$, $\beta' = \{n' b/p\} = 1 - \beta$, $m' = l - m$, and $\mu' = \{m'/l\} = 1 - \mu$, then these satisfy the assumption (6.1). Hence, $\bar{\omega}_{m',n'}$ is defined and we have $\int_{x/\Delta_1} \bar{\omega}_{m',n'} = \chi_{m,n}(x)$, for any $x \in K$. Since $H_{\mathbb{R}}'$ is generated as a $K$-module by $(1 - \zeta_i)^{-1}(1 - \zeta_p)^{-1}(1 - \tau_1)\ldots(1 - \tau_\alpha)(1 - \sigma)\Delta_1$, that we simply denote $\Delta_1$ as before, we have $\bar{\omega}_{m,n} = \bar{\omega}_{m',n'}$ and hence

$$\bar{\omega}_{m,n} + \bar{\omega}_{m',n'} \in F^1H_{\mathbb{R}}.$$

Define the regulator as

$$R_{m,n} := \int_{\Delta_0} \omega_{m,n} = \frac{B(1 - \alpha, \beta)}{l(\beta - \alpha + \mu)} \cdot F\left(\frac{1 - \alpha, \beta, \beta - \alpha + \mu}{1 - \alpha + \beta, \beta - \alpha + \mu + 1}, 1\right).$$
By Theorem 6.5, for any \( y \in H_1(Z_1, \mathbb{Q}) \) corresponding to \( x \in K \) as in Theorem 6.5 we have

\[
\rho_\mathbb{R}'(y)(\bar{\omega}_{m,n} + \bar{\omega}_{m',n'}) = \text{Im} \left( \chi_{m,n}(x) \Omega_{m,n}^{-1} R_{m,n} + \chi_{m,n}(x) \Omega_{m',n'}^{-1} R_{m',n'} \right)
\]

\[
= \text{Im} \left( \chi_{m,n}(x) \right) \left( \Omega_{m,n}^{-1} R_{m,n} - \Omega_{m',n'}^{-1} R_{m',n'} \right).
\]

Since \( \Omega_{m,n} \Omega_{m',n'} < 0 \) and \( R_{m,n}, R_{m',n'} > 0 \), the above does not vanish for \( x \in K \setminus \mathbb{R} \). Hence \( \rho_\mathbb{R} \) is non-trivial. Since \( \rho_\mathbb{R} \) is \( K \)-linear, the second assertion follows.

The non-vanishing of \( \rho \) is a more subtle problem. For the case \( a + b = p \), we have the following criterion.

**Proposition 6.7** Let \( p, l \) be distinct prime numbers and suppose that \( a + b = p \). If \( \rho \) is trivial, then there exists an \( x \in K \) such that \( R_{m,n} = \chi_{m,n}(x) \Omega_{m,n} \), for any \( n = 1, \ldots, p-1 \), and \( m \in \mathbb{Z}_l \) such that \( \frac{m}{l} > \left\{ \frac{na}{p} \right\} - \left\{ \frac{nb}{p} \right\} \).

**Proof** Let \( y = \partial \Delta_0 \) and suppose that \( \rho(y) = 0 \). Since \( H^{'\dagger}_\mathbb{R} \) is generated by \( \Delta_1 \) over \( K \), there exists an \( x \in K \) such that \( \rho(y) \) is represented by the functional \( \int_{x \Delta_1} \). If \( m, n \) are as in the statement, then \( \int_{r \Delta_1} \omega_{m,n} = \int_{\Delta_1} \chi^\ast \omega_{m,n} = \chi_{m,n}(x) \Omega_{m,n} \) by the definition. Hence the proposition follows.

**Example 6.8** If \( p = 2 \), then \( \alpha = \beta = 1/2 \) and \( Y \) is nothing but the Legendre family of elliptic curves. By Proposition 4.8, we have \( \text{Gr}_\mathbb{R}^p H_{\text{DR}} = 0 \) and the Deligne cohomology with \( \mathbb{R} \)-coefficients is trivial. Since the condition \( \frac{m}{l} > \left\{ \frac{na}{p} \right\} - \left\{ \frac{nb}{p} \right\} = 0 \) is automatically satisfied, Proposition 6.7 is, in fact, an equivalence. If, for example, \( l = 3 \), then \( \rho \) is trivial if and only if

\[
\sqrt{3} \left( \frac{\Gamma(\frac{5}{8})}{\Gamma(\frac{1}{8})} \right)^2 \cdot \mathbf{F} \left( \frac{1}{2}, \frac{1}{3} ; \frac{1}{2}, \frac{1}{3} ; 1 \right) \in \mathbb{Q}.
\]

Here we used \( \mathbb{Q}(\zeta_5) \cap i\mathbb{R} = \sqrt{3}i\mathbb{Q} \).

**A Appendix: (M. Asakura) Fibration of Curves and Extension of Motives**

In this appendix, we develop a technique that was used in the proof of the regulator formula (Theorem 6.5) to compute regulators for a fibration of curves and motivic elements constructed from degenerating fibers [3].

**A.1 Relative Cohomology**

Let \( V \) be a quasi-projective smooth surface over \( \mathbb{C} \). Let \( D \subset V \) be a chain of curves. Let \( \pi : \tilde{D} \to D \) be the normalization and \( \Sigma \subset D \) be the set of singular points. Let \( s : \Sigma := \pi^{-1}(\Sigma) \to \tilde{D} \) be the inclusion. There is an exact sequence

\[
0 \to \mathcal{O}_D \xrightarrow{\pi^*} \mathcal{O}_{\tilde{D}} \xrightarrow{s^*} \mathcal{C}_\Sigma / \mathcal{C}_D \to 0,
\]
where $C_{\Sigma} = \text{Maps}(\widetilde{\Sigma}, \mathbb{C}) = \text{Hom}(\widetilde{\Sigma}, \mathbb{C})$ and $\pi^*, s^*$ are the pull-backs. For a smooth manifold $M$, let $\mathcal{A}^q(M)$ denote the space of smooth differential $q$-forms on $M$ with coefficients in $\mathbb{C}$. We define $\mathcal{A}^*(D)$ to be the mapping fiber of $s^*: \mathcal{A}^*(\widetilde{D}) \to C_{\Sigma}/C_{\Sigma}$:

$$
\mathcal{A}^0(\widetilde{D}) \xrightarrow{s^* \circ d} C_{\Sigma}/C_{\Sigma} \oplus \mathcal{A}^1(\widetilde{D}) \xrightarrow{0 \circ d} \mathcal{A}^2(\widetilde{D}),
$$

where the first term is placed in degree 0. Then $H^q_{\text{dR}}(D) = H^q(\mathcal{A}^*(D))$ is the de Rham cohomology of $D$, which fits into the exact sequence

$$
\cdots \to H^0_{\text{dR}}(\widetilde{D}) \to C_{\Sigma}/C_{\Sigma} \to H^1_{\text{dR}}(D) \to H^1_{\text{dR}}(\widetilde{D}) \to \cdots.
$$

We have the natural pairing

$$
\langle \cdot, \cdot \rangle_D: H_1(D, \mathbb{Z}) \otimes H^1_{\text{dR}}(D) \to \mathbb{C}, \quad \eta \otimes z \mapsto \int_{\Gamma} c(\pi^{-1}(\eta)),
$$

where $z$ is represented by $(c, \eta) \in C_{\Sigma}/C_{\Sigma} \oplus \mathcal{A}^1(\widetilde{D})$ with $d\eta = 0$ and $d$ denotes the boundary of homology cycles.

We define $\mathcal{A}^*(V, D)$ to be the mapping fiber of $\widetilde{i}^*: \mathcal{A}^*(V) \to \mathcal{A}^*(\widetilde{D})$, the pull-back by $\widetilde{i}: \widetilde{D} \to V$:

$$
\mathcal{A}^0(V) \xrightarrow{\partial} \mathcal{A}^0(\widetilde{D}) \oplus \mathcal{A}^1(V) \xrightarrow{\partial} C_{\Sigma}/C_{\Sigma} \oplus \mathcal{A}^1(\widetilde{D}) \oplus \mathcal{A}^2(V) \xrightarrow{\partial} \cdots.
$$

Then the relative de Rham cohomology is defined by $H^q_{\text{dR}}(V, D) = H^q(\mathcal{A}^*(V, D))$ and fits into the exact sequence

(A.1) $\cdots \to H^q_{\text{dR}}(V, D) \to H^q_{\text{dR}}(V, D) \to H^q_{\text{dR}}(V, D) \to H^q_{\text{dR}}(D) \to \cdots.$

An element of $H^q_{\text{dR}}(V, D)$ is represented by

(A.2) $(c, \eta, \omega) \in C_{\Sigma}/C_{\Sigma} \oplus \mathcal{A}^1(\widetilde{D}) \oplus \mathcal{A}^2(V)$

that satisfies $\widetilde{i}^* \omega = d\eta$ and $d\omega = 0$. The natural pairing

$$
\langle \cdot, \cdot \rangle_{V, D}: H_2(V, D; \mathbb{Z}) \otimes H^2_{\text{dR}}(V, D) \to \mathbb{C}
$$

is given by

$$
\langle \Gamma, z \rangle_{V, D} = \int_{\Gamma} \omega - (\partial \Gamma, (c, \eta))_D = \int_{\Gamma} \omega - \int_{\Gamma} \eta + c(\pi^{-1}(\partial \Gamma)).
$$

The complexes $\mathcal{A}^*(V)$ and $\mathcal{A}^*(D)$ are canonically equipped with Hodge and weight filtrations; then $(\mathbb{Q}_V, \mathcal{A}^*(V), F^*, W_*)$ and $(\mathbb{Q}_D, \mathcal{A}^*(D), F^*, W_*)$ become cohomological mixed Hodge complexes in the sense of [10, (8.1.2)]. The Hodge and weight filtrations on $\mathcal{A}^*(V, D)$ are induced from them and the data

$$(\mathbb{Q}_{V, D}, \mathcal{A}^*(V, D), F^*, W_*)$$

becomes a cohomological mixed Hodge complex as well. Hence we have an exact sequence

$$
\cdots \to H^q_{\text{dR}}(D, \mathbb{Q}) \to H^q_{\text{dR}}(V, D; \mathbb{Q}) \to H^q_{\text{dR}}(V, \mathbb{Q}) \to H^q_{\text{dR}}(D, \mathbb{Q}) \to \cdots
$$

of mixed Hodge structures which is compatible with (A.1). Taking its dual, we obtain an exact sequence

$$
0 \to H_2(V, \mathbb{Q})/H_2(D) \to H_2(V, D; \mathbb{Q}) \xrightarrow{\partial} H_1(D, \mathbb{Q}) \to H_1(V, \mathbb{Q}).
$$
Since $H_1(V, \mathbb{Q}) \cap H^{0,0} = 0$, we obtain the coboundary map

(A.3) \[ \rho_{V,D} : H_1(D, \mathbb{Q}) \cap H^{0,0} \to \text{Ext}_{\text{MHS}}(\mathbb{Q}, H_2(V, \mathbb{Q})/H_2(D)) \]

to the extension group of mixed Hodge structures. If we put

\[ H^2_{\text{dR}}(V)_D := \text{Ker}[H^2_{\text{dR}}(V) \to H^2_{\text{dR}}(D)], \]

then we have the Carlson isomorphism

(A.4) \[ \text{Ext}_{\text{MHS}}(\mathbb{Q}, H_2(V, \mathbb{Q})/H_2(D)) \cong \text{Coker}[H_2(V, \mathbb{Q}) \to (F^1H^2_{\text{dR}}(V)_D)\ast], \]

where $\ast$ denotes the $\mathbb{C}$-linear dual and the map is the natural pairing. Under this identification, the map $\rho_{V,D}$ is described as follows. For $y \in H_1(D, \mathbb{Q}) \cap H^{0,0}$, take a $\Gamma \in H_2(V, D ; \mathbb{Q})$ such that $\vartheta(\Gamma) = y$. Then we have

(A.5) \[ \rho_{V,D}(y) = [\omega \mapsto \langle \Gamma, \omega_{V,D} \rangle_{V,D}], \]

where $\omega_{V,D} \in F^1H^2_{\text{dR}}(V, D)$ is a lifting of $\omega$, on which the pairing does not depend.

### A.2 Rational Forms

For a given $\omega$, it is usually complicated to compute an analytic lifting $\omega_{V,D}$ explicitly. In the following situation, we shall be able to associate a rational 2-form via Deligne’s canonical extension, which gives a simple expression of $\rho_{V,D}$.

Let $C$ be a projective smooth curve over $\mathbb{C}$ and $f : X \to C$ be a fibration of curves with connected general fiber that admits a section $e : C \to X$. Henceforth, we use the algebraic de Rham cohomology groups [14] and identify them with the analytic ones in the previous paragraph. For a Zariski open set $S \subset C$, let $V = f^{-1}(S)$ and put

\[ H^2_{\text{dR}}(V)_0 = \text{Ker}[H^2_{\text{dR}}(V) \to \prod_{s \in S} H^2_{\text{dR}}(f^{-1}(s)) \times H^2_{\text{dR}}(e(S))], \]

\[ H^2_{\text{dR}}(V, D)_0 = \text{Ker}[H^2_{\text{dR}}(V, D) \to H^2_{\text{dR}}(V)/H^2_{\text{dR}}(V)_0]. \]

Then we have an exact sequence of mixed Hodge structures

(A.6) \[ H^1_{\text{dR}}(V) \to H^1_{\text{dR}}(D) \to H^2_{\text{dR}}(V, D)_0 \to H^2_{\text{dR}}(V)_0 \to 0. \]

The arrows are strictly compatible with the Hodge and weight filtrations. In particular, $F^1H^2_{\text{dR}}(V, D)_0 \to F^1H^2_{\text{dR}}(V)_0$ is surjective. Later, we shall use the following.

**Lemma A.1** Let $g : V' \to V$ be a birational transformation that is an isomorphism outside $D$ and put $D' = g^{-1}(D)$. Then the pull-back $g^*$ induces isomorphisms

\[ H^2_{\text{dR}}(V)_0 \simeq H^2_{\text{dR}}(V')_0 \quad \text{and} \quad H^2_{\text{dR}}(V, D)_0 \simeq H^2_{\text{dR}}(V', D')_0. \]

**Proof** By (A.6) it is enough to show isomorphisms

\[ H^1_{\text{dR}}(V) \simeq H^1_{\text{dR}}(V'), \quad H^1_{\text{dR}}(D) \simeq H^1_{\text{dR}}(D'), \quad H^2_{\text{dR}}(V)_0 \simeq H^2_{\text{dR}}(V')_0. \]
The first one is an easy exercise. Let $X'$ be a smooth compactification of $V'$ such that $X' \setminus D' = X \setminus D$ and consider the commutative diagram with exact rows

\[
\begin{array}{ccccccc}
H^2_{\text{dr}}(X') & \xrightarrow{a^*} & H^2_{\text{dr}}(X' \setminus D') & \longrightarrow & H^1_{\text{dr}}(D') & \longrightarrow & H^3_{\text{dr}}(X') & \xrightarrow{a^*} & H^3_{\text{dr}}(X' \setminus D') \\
\downarrow{g_*} & & \downarrow{g_*} & & \downarrow{g_*} & & \downarrow{g_*} & & \\
H^2_{\text{dr}}(X) & \xrightarrow{b^*} & H^2_{\text{dr}}(X \setminus D) & \longrightarrow & H^1_{\text{dr}}(D) & \longrightarrow & H^3_{\text{dr}}(X) & \xrightarrow{b^*} & H^3_{\text{dr}}(X \setminus D). \\
\end{array}
\]

The second isomorphism follows from the fact that

\[
\text{Image}(a^n) = \text{Image}(b^n) = W_n H^2_{\text{dr}}(X \setminus D).
\]

The last isomorphism follows from the commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H^2_{\text{dr}}(V)_0 & \longrightarrow & H^2_{\text{dr}}(V \setminus D)_0 & \longrightarrow & H^1_{\text{dr}}(D) \\
\downarrow{g^*} & & \downarrow{g^*} & & \downarrow{g^*} & & \\
0 & \longrightarrow & H^2_{\text{dr}}(V')_0 & \longrightarrow & H^2_{\text{dr}}(V' \setminus D')_0 & \longrightarrow & H^1_{\text{dr}}(D')
\end{array}
\]

with exact rows.

Now fix a Zariski open set $S \subset C$ such that $U := f^{-1}(S) \to S$ is smooth. Put $T = C \setminus S$ and $Z = X \setminus U$. Let $\nabla: \mathcal{H} \to \Omega^1_C(\log T) \otimes \mathcal{H}$ be the Deligne canonical extension of the Gauss–Manin connection $(\mathcal{H} = R^1f_*\Omega^*_{U/S}, \nabla)$. Put $F^1\mathcal{H} = j_*F^1\mathcal{H} \cap \mathcal{H}$, where $j: S \to C$ and $\text{Gr}_F^0\mathcal{H} = \mathcal{H}/F^1\mathcal{H}$. Let $\nabla: F^1\mathcal{H} \to \Omega^1_C(\log T) \otimes \text{Gr}_F^0\mathcal{H}$ be the $\mathcal{O}_C$-linear map induced from $\nabla$. In what follows, we assume the following.

\((*)\) The map $\nabla$ is generically bijective.

Let $C^\circ \subset C$ be a Zariski open set on which $\nabla$ is bijective and put $X^\circ := f^{-1}(C^\circ)$. Note that $S \notin C^\circ$ in general and $X^\circ \to C^\circ$ is not necessarily smooth. Then the commutative diagram

\[
\begin{array}{ccccccc}
\Omega^1_C(\log T) \otimes F^1\mathcal{H} & \xrightarrow{\nabla} & \Omega^1_C(\log T) \otimes F^1\mathcal{H} \\
\uparrow{\nabla} & & \uparrow{\nabla} \\
F^1\mathcal{H} & \xrightarrow{\nabla} & \Omega^1_C(\log T) \otimes F^1\mathcal{H} \\
\downarrow{=} & & \downarrow{=} \\
F^1\mathcal{H} & \xrightarrow{\nabla} & \Omega^1_C(\log T) \otimes \text{Gr}_F^0\mathcal{H} \\
\downarrow{0} & & \downarrow{0}
\end{array}
\]
induces an isomorphism
\[ \Lambda^\circ \colonequals \Gamma(C^\circ, \Omega^1_C(\log T) \otimes F^1 CH) \xrightarrow{\sim} H^1(C^\circ, F^1 CH) \rightarrow \Omega^1_C(\log T) \otimes CH. \]

Note that \( \Lambda^\circ \subseteq \Gamma(X^\circ, \Omega^2_X(\log Z)) \).

**Lemma A.2** There are natural injections \( F^1 H^2_{\text{dr}}(X)_0 \hookrightarrow F^1 H^2_{\text{dr}}(U)_0 \rightarrow \Lambda^\circ \).

**Proof** The first injectivity follows from Zariski's lemma [6, III, (8.2)]. Since
\[ H^2_{\text{dr}}(U)_0 \cong H^1(S, CH \rightarrow \Omega^1_S \otimes CH) \cong H^1(C, CH \rightarrow \Omega^1_C(\log T) \otimes CH) \]
and
\[ F^1 H^1(S, CH \rightarrow \Omega^1_S \otimes CH) = H^1(C, F^1 CH \rightarrow \Omega^1_C(\log T) \otimes CH) \]
[26, §5], the second injectivity follows from that of \( F^1 H^2_{\text{dr}}(U)_0 \rightarrow F^1 H^2_{\text{dr}}(U \cap X^\circ)_0 \).

Define \( \Lambda(X) \subset \Lambda(U) \subset \Lambda^\circ \) to be the images of \( F^1 H^2_{\text{dr}}(X)_0, F^1 H^2_{\text{dr}}(U)_0 \), respectively. By the commutative diagram

\[
\begin{array}{ccc}
F^1 H^2_{\text{dr}}(X)_0 & \rightarrow & F^1 H^2_{\text{dr}}(U)_0 \\
\downarrow \cong & & \downarrow \cong \\
\Lambda(X) & \rightarrow & \Lambda(U) \\
\downarrow & & \downarrow \\
& & H^0(X^\circ, \Omega^2_X(\log Z)/\Omega^2_X) \\
\end{array}
\]

we have \( \Lambda(X) \subseteq \Gamma(X^\circ, \Omega^2_X) \). For any cohomology class \( \omega \in F^1 H^2_{\text{dr}}(X)_0 \), let \( \omega^\circ \in \Lambda(X) \) denote the corresponding rational 2-form.

**A.3 Main Result**

Now let \( D \subset X^\circ \) be a finite union of fibers. We give a description of the map
\[ \rho_{X, D} : H^0(D, \mathbb{Q}) \cap H^0(\cdot) \rightarrow \text{Coker}[H^2(X, \mathbb{Q}) \rightarrow (F^1 H^2_{\text{dr}}(X)_0]^* \]
induced from (A.3), (A.4), and the inclusion \( F^1 H^2_{\text{dr}}(X)_0 \subset F^1 H^2_{\text{dr}}(X)_D \). Note that this factors through \( \rho_{X^\circ, D} \). We regard an element \( \eta \in \Lambda^\circ \) as an element of \( \mathcal{O}^2(X^\circ) \).

For the dimension reasons, we have \( \Gamma^0 \eta = 0 \) and \( d\eta = 0 \). Hence \( (0, 0, \eta) \) as in (A.2) defines a cohomology class \( \eta \in H^2_{\text{dr}}(X^\circ, D) \). Note that \( \eta \) does not necessarily belong to \( F^1 \). For any \( \omega \in F^1 H^2_{\text{dr}}(X)_0 \), write \( \tilde{\omega} \) instead of \( \omega^\circ \).

**Theorem A.3**

(i) For any \( \omega \in F^1 H^2_{\text{dr}}(X)_0 \), we have \( \tilde{\omega} \in F^1 H^2_{\text{dr}}(X^\circ, D)_0 \) and it lifts \( \omega|_{X^\circ} \).

(ii) For any \( \gamma \in H^1(D, \mathbb{Q}) \cap H^0(\cdot) \), choose \( \Gamma \in H^2(X^\circ, D) \) such that \( \partial(\Gamma) = \gamma \). Then we have \( \rho_{X, D}(\gamma) = [\omega \rightarrow f^* \omega^\circ] \).

**Proof** By (A.5), assertion (ii) follows immediately from (i). By Lemma A.1, we may assume that \( D_{\text{red}} \) and \( Z_{\text{red}} \) are divisors with normal crossings. It suffices to prove the
case where \( D = f^{-1}(P), P \in C^* \). For a Zariski sheaf \( \mathcal{F} \), let \((\check{C}^*(\mathcal{F}), \delta)\) denote its Čech complex. First, \( H^2_{\text{dr}}(X) \) is given by the cohomology in the middle of the complex
\[
\check{C}^1(\mathcal{O}_X) \times \check{C}^0(\Omega^1_X) \xrightarrow{\delta_1} \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\Omega^1_X) \times \check{C}^0(\Omega^2_X) \\
\xrightarrow{\delta_2} \check{C}^2(\mathcal{O}_X) \times \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\Omega^2_X).
\]
A description of \( H^2_{\text{dr}}(U) = H^2(X, \Omega^*_X(\log Z)) \) is given similarly. Finally, \( H^2_{\text{dr}}(X, D) \) is given by the complex
\[
\check{C}^1(\mathcal{O}_X) \times \check{C}^0(\mathcal{O}_D \oplus \Omega^1_X) \xrightarrow{\delta_1} \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\mathcal{O}_D \oplus \Omega^1_X) \times \check{C}^0(\mathcal{O}_D \oplus \Omega^1_X \oplus \Omega^2_X) \\
\xrightarrow{\delta_2} \check{C}^2(\mathcal{O}_X) \times \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\mathcal{O}_D \oplus \Omega^1_X \oplus \Omega^2_X).
\]
Let \( \omega \in F^1 H^2_{\text{dr}}(X)_0 \) and take its representative \( z = (0) \times (\alpha_i) \times (\beta_i) \in \text{Ker}(\mathcal{F}_2) \). Since \( \omega \in F^1 H^2_{\text{dr}}(X)_D \), there exists \((\epsilon_i) \in \check{C}^0(\Omega^1_D)\) such that \( \alpha_i|_{\mathcal{O}_D} = \epsilon_j - \epsilon_i \). If we put \( z_{X,D} = (0) \times (0, \alpha_i) \times (0, \epsilon_i, \beta_i) \), then \( z_{X,D} \in \text{Ker}(\mathcal{F}_4) \). By the definition of the Hodge filtration, it represents a class \( \omega_{X,D} \in F^1 H^2_{\text{dr}}(X, D) \) that lifts \( \omega \). Let \( \omega_{X,D}|_{X^o} \) be its image in \( H^2_{\text{dr}}(X^o, D) \).

Let \( \tilde{\omega} \in H^2_{\text{dr}}(X^o, D) \) be the class of the Čech cocycle \( \bar{z}:=(0) \times (0, 0) \times (0, 0, \omega^o) \). The group \( H^2(C^*, F^1 \mathcal{H}_e \to \Omega^1_e(\log T) \otimes \mathcal{H}_e) \) is given by the complex
\[
\check{C}^0(F^1 \mathcal{H}_e|_{C^*}) \xrightarrow{\delta_1} \check{C}^1(F^1 \mathcal{H}_e|_{C^*}) \times \check{C}^0(\Omega^1_e(\log T) \otimes \mathcal{H}_e|_{C^*}) \\
\xrightarrow{\delta_2} \check{C}^2(F^1 \mathcal{H}_e|_{C^*}) \times \check{C}^1(\Omega^1_e(\log T) \otimes \mathcal{H}_e|_{C^*}).
\]
By the definition of \( \omega^o \), there exists \( y = (v_i) \in \check{C}^0(F^1 \mathcal{H}_e|_{C^*}) \) such that \( \mathcal{F}_5(y) = (\alpha_i) \times (\beta_i) \times (0) \times (0, \omega^o) \), i.e., \( v_j - v_i = \alpha_{ij}, dv_i = \beta_i - \omega^o \). Hence we have
\[
z_{X,D}|_{X^o} - \bar{z} = (0) \times (0, v_j - v_i) \times (0, \epsilon_i, dv_i).
\]
It is clear that this vanishes in \( H^2_{\text{dr}}(X^o, D) \), hence \( \tilde{\omega} \) lifts \( \omega|_{X^o} \).

We are left to show that the class of \( \tilde{\omega} \) lies in \( F^1 \). Let \( V \) be a sufficiently small neighborhood of \( D \) so that we have an exact sequence
\[
0 \to \Omega^1_V \to \Omega^1_V(\log D) \xrightarrow{\text{Res}} \tilde{\iota}_* \mathcal{O}_D \to 0.
\]
Since \( H^2_{\text{dr}}(X^o, D)/F^1 \to H^2_{\text{dr}}(V, D)/F^1 \) is injective, it suffices to show the claim after restricting to \( V \). Since \( \text{Res}(v_i) - \text{Res}(v_i) = \text{Res}(\alpha_{ij}) = 0, (\text{Res}(v_i)) \) defines a class \( e \in H^2(\tilde{D}, \mathcal{O}_D) \). Consider the composite
\[
H^0(\tilde{D}, \mathcal{O}_D) \xrightarrow{\delta} H^1(V, \Omega^1_V) \xrightarrow{\gamma} H^1(\tilde{D}, \Omega^1_D) \approx H^2_{\text{dr}}(\tilde{D}),
\]
where \( \delta \) is the connecting map. Then \((\tilde{i}^* \circ \delta)(e)\) is represented by \((\alpha_{ij}|_{\tilde{D}}) \in \check{C}^1(\Omega^1_D)\).

Therefore, under the above isomorphism, \((\tilde{i}^* \circ \delta)(e)\) corresponds to \( \tilde{i}^*(\omega) = 0 \). Let \( t \in \mathcal{O}_{C,p} \) be a uniformizer at \( P \). By Zariski’s lemma [6, III, (8.2)], \( \text{Ker}(\tilde{i}^* \circ \delta) \) is one-dimensional and generated by \( \text{Res}(d_t/\tau) \). Hence there exists a constant \( c \) such that \( \theta_t := v_t - c d_t/\tau \) has no pole along \( D \). By replacing \( v_t \) with \( \theta_t \) and taking \( \epsilon_t = \theta_t|_{\mathcal{O}_D} \), we see that \( \omega_{X,D}|_V - \tilde{\omega}|_V \) is the image of \( F^1 H^1_{\text{dr}}(V) \to H^2_{\text{dr}}(V, D) \). Hence we obtain \( \tilde{\omega} \in F^1 \) and the proof is complete. \[\blacksquare\]
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