On Hopf algebroid structure of $\kappa$-deformed Heisenberg algebra

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The $(4+4)$-dimensional $\kappa$-deformed quantum phase space as well as its $(10+10)$-dimensional covariant extension by the Lorentz sector can be described as Heisenberg doubles: the $(10+10)$-dimensional quantum phase space is the double of $D = 4$ $\kappa$-deformed Poincaré Hopf algebra $\mathbb{H}$ and the standard $(4+4)$-dimensional space is its subalgebra generated by $\kappa$-Minkowski coordinates $\vec{x}_\mu$ and corresponding commuting momenta $\vec{p}_\mu$. Every Heisenberg double appears as the total algebra of a Hopf algebroid over a base algebra which is in our case the coordinate sector. We exhibit the details of this structure, namely the corresponding right bialgebroid and the antipode map. We rely on algebraic methods of calculation in Majid-Ruegg bicrosproduct basis. The target map is derived from a formula by J-H. Lu. The coproduct takes values in the bimodule tensor product over a base, what is expressed as the presence of coproduct gauge freedom.

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I. INTRODUCTION

It is often convenient to consider covariant phase spaces, that is to include symmetries along with the space (or spacetime) generators. Noncommutative $\kappa$-deformed Minkowski space has been complemented with appropriate momenta into a noncommutative phase space in a number of works. It has been recently argued that this noncommutative phase space has a structure of a (topological) Hopf algebroid \cite{1,2} and the same for general $\kappa$-phase spaces have a structure of a Heisenberg double \cite{3,4}; a Hopf algebraic generalization of a Heisenberg algebra prominently used in quantum group theory \cite{5,6}. Furthermore, J-H. Lu \cite{8} proved that a finite dimensional Heisenberg double has a structure of a Hopf algebroid. Her description can be generalized to some infinite dimensional situations, including to the $\kappa$-phase spaces; the advantage of covariant phase space description as well as of Heisenberg double packaging are among the main motivations for this work.

While Hopf algebras are quantum analogues of groups \cite{9–11}, Hopf algebroids are quantum analogues of groupoids. Indeed, algebra of functions on a group is a commutative Hopf algebroid, while functions on a groupoid form a commutative Hopf algebroid. To drive our expectations in physics let us recall some example of a groupoid. Groupoid is a many object version of a group: it consists of objects (units) and arrows between them. Each arrow has a source and target object, we can compose two arrows, $g$ and $f$ into $g \circ f$ if the target of $f$ is the same as the source of $g$; each arrow has its composition inverse and the composition is associative. The main example is a transformation (action) groupoid which for a group $G$ acting on a manifold $M$ encodes both space $M$ and the action in a single groupoid. The objects are points of $M$ and arrows are pairs $(g, m) \in G \times M$. The source of $(m, g)$ is $m$ and the target of $(m, g)$ is $g \cdot m$. This way we know what happens to point $m$ when we act by $g$: the information is in the target. Thus both the space and its symmetries are encoded in the action groupoid. Notice again that the arrows form the product $G \times M$ of a symmetry object and the space.

If we replace a group $G$ by Hopf algebra $\mathbb{H}$ of functions on a group and manifold $M$ by an algebra $B$ of coordinate functions on $M$, we expect that $G \times M$ will be replaced by some sort of a tensor product $\mathbb{H} \otimes B$ and the action of $G$ on $M$ dualizes to coaction of $\mathbb{H}$ on $B$. In a typical construction of that type, like in this article, the tensor product is in fact a semidirect product (in Hopf algebra literature called the smash product algebra). Such examples are noncommutative analogues of transformation groupoids. In general a Hopf algebroid over base $B$ is a $B$-bialgebroid with an antipode map (see Section IV) where $B$-bialgebroid entails the following data:

- The role of the quantum arrow space of a bialgebroid or Hopf algebroid $\mathcal{H}$ is taken by the total algebra $H$ and the base $B$ is the base algebra.
- One supplies the (dual versions of) source map $\alpha$ and target map $\beta$ which are now a homomorphism $\alpha : B \to H$ and an antihomomorphism $\beta : B \to H$ of algebras such that their images commute in $H$:

$$[\alpha(b), \beta(c)] = 0, \quad b, c \in B.$$

This way the formula $b.h.c := h\beta(b)\alpha(c)$ equips $H$ with $B$-bimodule structure (this choice leads to right $B$-bialgebroid as opposed to left bialgebroids where $b.h.c = \alpha(b)\beta(c)h$).
There is a coproduct $\Delta : H \to H \otimes H$ which is however coassociative only when projected to the equivalence classes in the tensor product of $B$-bimodules $H \otimes_B H = (H \otimes H)/I_B$ \cite{8,12,14,15}, where $I_B$ is the left ideal in $H \otimes H$ generated by all elements of the form
\[\alpha(b) \otimes 1 - 1 \otimes \alpha(b), \quad b \in B.\]

Regarding that $I_B$ is not a 2-sided ideal \cite{8,12,14,15} (it is 2-sided in some special cases, for instance for the undeformed Heisenberg algebra), the quotient is not an algebra. Thus the Hopf algebraic requirement that the effective coproduct $\Delta : H \to H \otimes_B H$ is multiplicative (that is $\Delta(bc) = \Delta(b)\Delta(c)$) does not make sense in general. A subtle solution to make sense of the multiplicativity of $\Delta$ is the following: one has to require that the image of $\Delta$ is within some subbimodule of $H \otimes_B H$ where the factorwise product is still well defined; such a subspace exists, it is the Takeuchi product $H \times_B H \subset H \otimes_B H$, see e.g. \cite{4,12,13,15,16,10}.

Hopf algebras come not only from groups (as function algebras) but also from Lie algebras (as universal enveloping algebras). This means that they can also encode infinitesimal symmetries. Similarly, Hopf algebroids can also encode the infinitesimal actions. This explains that the Heisenberg algebra which entails coordinates but also their $\kappa$-deformation scheme the general covariant $\kappa$-deformed phase space is provided by the Heisenberg double $\mathcal{H} = \mathbb{H} \times \mathbb{H}$ (see e.g. \cite{8,5}), where $\mathbb{H} = U_\kappa(g)$ describe $\kappa$-deformed Poincaré–Hopf algebra \cite{17,18,19} and $\overline{\mathbb{H}}$ is the Hopf algebra describing dual $\kappa$-deformed quantum Poincaré group \cite{20}. Heisenberg double is a special case of smash (or crossed) product algebra $H \times_B H$, where $V$ is an $H$-module algebra \cite{6}. In this paper we employ the general property (see e.g. \cite{8}, Sect. 6) that Heisenberg double algebra is equipped with the Hopf algebraic structure. In recent literature (see e.g. \cite{2}) the bialgebroid structures of deformed standard quantum phase spaces $(\hat{x}_\mu, p_\mu)$ with $\kappa$-Minkowski space–time sector
\[\hat{x}_0, \hat{x}_i \equiv -\frac{i}{\kappa} \hat{x}_i, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad (1)\]
and commuting fourmomenta $\hat{p}_\mu$ were studied by embedding into canonical quantum phase space algebra (we put $\hbar = 1$)
\[x_\mu, x_\nu = [p_\mu, p_\nu] = 0, \quad [x_\mu, p_\nu] = i\eta_{\mu\nu}. \quad (2)\]
Relation (1) permits the following general class of realizations of quantum phase spaces
\[\hat{x}_\mu = f^\nu_\mu(p)x_\nu, \quad \hat{p}_\mu = p_\mu, \quad (3)\]
where $f^\nu_\mu(p)$ are chosen in consistency with relations (1), (2) (Jacobi identities) and provide large variety of quantum phase spaces with space–time algebra described by relations (1). This approach lacks structural indication how to obtain the covariant action of $\kappa$-deformed Poincaré–Hopf algebra, which is a part of full definition of quantum $\kappa$-deformed Minkowski space.

Here the $\kappa$-deformed quantum phase space is constructed as the Heisenberg double of $D = 4$ $\kappa$-deformed Poincaré–Hopf algebra; this method is first presented in \cite{3}. Such construction contains built-in $\kappa$-covariance of $\kappa$-deformed quantum phase space first observed for $\kappa$-Minkowski space–time sector in \cite{17}. In Majid–Ruegg basis \cite{17} we obtain that both $\kappa$-Poincaré–Hopf algebra $\mathbb{H}$ and $\kappa$-Poincaré group $\overline{\mathbb{H}}$ are described by two dual bicrossproduct structures \cite{2,21,22}, namely
\[\mathbb{H} = U(so(1,3)) \triangleleft \triangleright \mathcal{T}^4 \quad \overset{\text{duality}}{\longleftrightarrow} \quad \overline{\mathbb{H}} = \overline{T}_\kappa \triangleleft \triangleright \mathcal{L}^6, \quad (4)\]
where $\mathcal{L}^6$ describe the functions of Abelian Lorentz parameters $\lambda_{\mu\nu}$ which are dual to $U(so(3,1))$ and $\mathcal{T}^4$ is the fourmomentum sector dual to the algebra $\overline{T}_\kappa$ describing noncommutative functions of $\kappa$-deformed Minkowski coordinates (see (1)). The $\kappa$-Poincaré covariance of fourmomentum sector $\mathcal{T}^4$ can be derived from the bicrossproduct structure of $\mathbb{H}$ (see (1)). The $\kappa$-deformed Poincaré algebra $\mathbb{H}$ acts on standard $\kappa$-deformed quantum phase space $\mathcal{T}^4 \times \overline{T}_\kappa^4$ in a covariant way. Further, the covariant action of $\mathbb{H}$ on $\mathcal{L}^6$ follows from the duality of $\mathcal{L}^6$ and $U(so(3,1))$ algebras as well as the semidirect product of the coalgebra sectors in $\mathbb{H}$ and $\overline{\mathbb{H}}$.

In Section 6 of \cite{3}, J-H. Lu has given explicit formulas for a Hopf algebroid structure on a Heisenberg double of any finite dimensional Hopf algebra, which is here replaced by $\kappa$-Poincaré Hopf algebra $\mathbb{H}$ (which is $\infty$-dimensional as a Hopf algebra, but the recipes still work). To have the formulas explicit, in this article we calculate the details of the
Hopf algebroid structure for the Heisenberg double $H^{(4,4)} \equiv H_{(p,x)} = T^4 \times \tilde{T}^4$, where the dual Hopf algebras $T^4, \tilde{T}^4$ describe respectively the momentum and coordinate sectors

$$
T^4 \begin{cases}
[\hat{p}_\mu, \hat{p}_\nu] = 0, \\
\Delta(\hat{p}_i) = \hat{p}_i \otimes e^{-\hat{p}_i} + 1 \otimes \hat{p}_i,
\end{cases} \quad (5)
$$

$$
\tilde{T}^4 \begin{cases}
[\hat{x}_0, \hat{x}_i] = -\frac{i}{\kappa} \hat{x}_i, \\
[\hat{x}_i, \hat{x}_j] = 0, \\
\Delta(\hat{x}_\mu) = \hat{x}_\mu \otimes 1 + 1 \otimes \hat{x}_\mu.
\end{cases} \quad (6)
$$

Most earlier important examples of Hopf algebroids over noncommutative base are also related to physics. It has been argued out in [24] that low dimensional QFT-s allow for weak Hopf algebra symmetries where weak roughly means that the coproduct and unit are not compatible, $\Delta(1) \neq 1 \otimes 1$; usually one dispenses with weak units passing to appropriate quotient Hopf algebras, but at the cost of having nonphysical zero-norm states. The data of a weak Hopf algebra has an equivalent description as a (special kind of a) Hopf algebroid. Similarly, the dynamical quantum Yang-Baxter equation (dQYB) which describes dynamical quantum group related with Lie algebra $\mathfrak{g}$ are described [25, 27] by a Hopf algebroid with a commutative base algebra $B$ dual to the Cartan subalgebra of $\mathfrak{g}$ [26]. It has been shown in [14] how to reduce dQYB to the ordinary Yang-Baxter equation in a Hopf algebroid framework. As a vector space, this algebroid is built from the algebraic sector of the quantum group by tensoring with the space encoding the dynamical parameter. This conceptually attractive approach is useful for further generalizations and application of twists [28].

All algebras in the paper are over the field $\mathbb{C}$ of complex numbers and the opposite algebra to $B$ is dual to the Cartan subalgebra of $\mathfrak{g}$ [25, 27]. We freely use the Sweedler notation for the comultiplication $\Delta(b) = \sum b(1) \otimes b(2)$ with or without summation sign. The antipode for a Hopf algebra is denoted by $S$, but the antipode for a Hopf algebroid by $\tau$. We mention that in this paper we follow research partly explained in our last publication [23]; however the derivation of the target map in Section III is new.

II. \(\kappa\)-POINCARÉ–HOPF ALGEBRA AND ITS DUAL GROUP

We present here the $\kappa$-Poincaré–Hopf algebra $\mathbb{H}$ as the basic object and the $\kappa$-Poincaré group as its dual Hopf algebra. Their duality will play the major role in the next section on the Heisenberg double.

A. $\kappa$-Poincaré–Hopf algebra $\mathbb{H}$

In the following we use the conventions for the indices $\mu, \nu, \lambda, \sigma = 0, 1, 2, 3$; $i, j = 1, 2, 3$ and metric $g_{\mu\nu} = (-1, 1, 1, 1)$. We denote the $\kappa$-Poincaré algebra generators by $(\hat{p}_\mu, \hat{m}_{\mu\nu})$ and set $\hbar = 1$. Then the $\kappa$-Poincaré–Hopf algebra $\mathbb{H}$ in bicrossproduct basis [17] has the following form

- algebra sector:

$$
[\hat{m}_{\mu\nu}, \hat{m}_{\lambda\sigma}] = i (g_{\mu\sigma} \hat{m}_{\nu\lambda} + g_{\nu\lambda} \hat{m}_{\mu\sigma} - g_{\mu\lambda} \hat{m}_{\nu\sigma} - g_{\nu\sigma} \hat{m}_{\mu\lambda}),
$$

$$
[\hat{m}_{ij}, \hat{p}_\mu] = -i (g_{ij\mu} \hat{p}_\mu - g_{j\mu i} \hat{p}_i),
$$

$$
[\hat{m}_{i0}, \hat{p}_i] = i \hat{p}_i,
$$

$$
[\hat{m}_{i0}, \hat{p}_j] = i \partial_{ij} \left( \kappa \sinh \left( \frac{\hat{p}_0}{\kappa} \right) e^{-\frac{\hat{p}_0}{\kappa}} + \frac{1}{\kappa^2} \hat{p}_0^2 \right) - \frac{i}{\kappa} \hat{p}_i \hat{p}_j.
$$

- coalgebra sector:

$$
\Delta(\hat{m}_{ij}) = \hat{m}_{ij} \otimes I + I \otimes \hat{m}_{ij},
$$

$$
\Delta(\hat{m}_{k0}) = \hat{m}_{k0} \otimes e^{-\frac{\hat{p}_0}{\kappa}} + I \otimes \hat{m}_{k0} + \frac{1}{\kappa} \hat{m}_{kl} \otimes \hat{p}_l
$$

$$
\Delta(\hat{p}_0) = \hat{p}_0 \otimes I + I \otimes \hat{p}_0,
$$

$$
\Delta(\hat{p}_k) = \hat{p}_k \otimes e^{-\frac{\hat{p}_0}{\kappa}} + I \otimes \hat{p}_k
$$

- counit:

$$
\epsilon(\hat{p}_\mu) = 0, \quad \epsilon(\hat{m}_{\mu\nu}) = 0.
$$

(9)
– Hopf algebra antipode:

\[
S(\hat{m}_{ij}) = -\hat{m}_{ij}, \quad S(\hat{m}_{i0}) = -\hat{m}_{i0} + \frac{\hat{p}_i}{2}, \\
S(\hat{p}_i) = -\hat{e}_i, \quad S(\hat{p}_0) = -\hat{p}_0.
\]

(10)

B. The concept of a Hopf pairing

We say that Hopf algebra \( H = (A, m, \Delta, S, \epsilon) \) is in duality with Hopf algebra \( \hat{H} = (A^*, m^*, \Delta^*, S^*, \epsilon^*) \) if there is a vector space pairing \( \langle, \rangle : A^* \otimes A \to \mathbb{C} \) such that

\[
\langle e^*, ab \rangle = \langle \Delta(e^*), a \otimes b \rangle,
\]

\[
\langle e^* \otimes d^*, \Delta(a) \rangle = \langle e^* d^*, a \rangle,
\]

\[
\langle a^*, 1_A \rangle = \epsilon^*(a^*), \quad \langle 1_{A^*}, a \rangle = \epsilon(a) \quad \text{and} \quad \langle S(b^*), a \rangle = \langle b^*, S(a) \rangle.
\]

Any two Hopf algebras in duality act one on another. Namely,

\[
a^* \triangleright a := a_{(1)} \langle a^*, a_{(2)} \rangle, \quad a^* \triangleleft a := \langle a^*_{(1)}, a \rangle a^*_{(2)},
\]

Using (11) one can directly verify that these actions are Hopf (in other words \[6\], \( A \) is a left \( \hat{H} \)-module algebra and \( A^* \) is a right \( H \)-module algebra), that is

\[
a^* \triangleright (ab) = (a^* \triangleright a)(a^* \triangleright b) = a_{(1)} \langle a^*_{(1)}, a_{(2)} \rangle b_{(1)} \langle a_{(2)}^*, b_{(2)} \rangle.
\]

C. \( \kappa \)-Poincaré quantum group \( \hat{H} \)

We introduce the \( \kappa \)-Poincaré quantum group as the dual vector space \( \hat{H} \) to \( H \) with generators \( \hat{p}_\mu, \hat{m}_{\lambda \mu} \) via the following canonical duality relations

\[
\langle \hat{p}_\mu, \hat{p}_\nu \rangle = i\delta^\mu_\nu, \quad \langle \hat{m}_{\lambda \mu}, \hat{m}_{\lambda \rho} \rangle = i(\delta^\mu_\rho g_{\lambda \sigma} - \delta^\rho_\mu g_{\lambda \sigma}) = i(\delta^\mu_\rho g_{\lambda \sigma} + \delta^\rho_\mu g_{\lambda \sigma})
\]

\[
\langle \hat{p}_\mu, \hat{m}_{\lambda \mu} \rangle = \langle \hat{m}_{\lambda \mu}, \hat{p}_\sigma \rangle = 0, \quad \langle \hat{m}_{\lambda \mu}, I \rangle = \delta^\mu_\nu,
\]

(12)

The pairing for vector spaces has to be given for vector space basis but in the case of Hopf algebra duality it is enough to specify the pairing just on the algebra generators. Indeed, for products of generators we can use (11).

From (12) we obtain the commutation relations defining \( \kappa \)-Poincaré group \[21, 22\] in the following form

– algebra sector:

\[
[\hat{p}_\mu, \hat{p}_\nu] = \frac{i}{\kappa}(\delta^\mu_\nu \hat{p}_\nu - \delta^\nu_\mu \hat{p}_\mu), \quad [\hat{m}_{\lambda \mu}, \hat{p}_\nu] = 0
\]

\[
[\hat{m}_{\lambda \mu}, \hat{m}_{\lambda \nu}] = \frac{i}{\kappa} \left( (\hat{m}_{\lambda \mu}^\nu - \delta^\nu_\mu) \hat{m}_{\lambda \nu} + (\hat{m}_{\lambda \nu}^\mu - \delta^\mu_\nu) \hat{m}_{\lambda \mu} \right)
\]

(13)

– coalgebra sector:

\[
\Delta(\hat{p}_\mu) = \hat{p}_\nu \otimes \hat{p}_\mu + \hat{p}_\mu \otimes I
\]

\[
\Delta(\hat{m}_{\lambda \mu}) = \hat{m}_{\lambda \nu} \otimes \hat{m}_{\nu \mu}
\]

(14)

– counit \( \epsilon \) and antipode \( S \):

\[
\epsilon(\hat{p}_\mu) = \delta^\mu_\nu, \quad S(\hat{m}_{\lambda \mu}) = g^\mu_{\rho \nu} \hat{m}_{\nu \sigma} = \hat{m}_{\lambda \sigma}^\mu
\]

\[
\epsilon(\hat{m}_{\lambda \mu}) = 0, \quad S(\hat{p}_\mu) = -\lambda_\nu^\mu \hat{p}_\nu
\]

(15)

In the Heisenberg double algebra \( H^{(10,10)} = H \times \hat{H} \) the commutation relations (7) and (13) are supplemented by the following relations obtained from (12), (8) and (13)

The generalized covariant \( \kappa \)-deformed phase space is described by sets of commutators (7), (12), (14) and (18). The coproducts (8) and (14) realize the coalgebraic homomorphism of relations (7) and (13), but the relations (14), (15) will be mapped into the coalgebra only in the bialgebroid framework.

The Hopf subalgebra \( T_\kappa = U_\kappa(g) \) (\( \kappa \)-Minkowski space-time sector) is the subalgebra generated by the 4 generators \( \hat{x}_\mu \) only.
III. HEISENBERG DOUBLE

Heisenberg double is a construction of an associative algebra $A \times A^*$ (the notation suggests that it is a special case of a smash product algebra) containing a Hopf algebra $H$ and its dual Hopf algebra $\hat{H}$ as the analogues of the coordinate and momentum sectors within the standard Heisenberg algebra. It is the tensor product vector space $A \otimes A^*$ with the nontrivial algebra structure given by the cross relations.

A. Cross relations

The two tensor factors do not commute, but satisfy the cross relations

\[
(1 \otimes a^*)(a \otimes 1) = (a_1^* > a) \otimes a_2^* = a_1 < a_2^* > \otimes a_2^*,
\]

while $(a \otimes 1)(1 \otimes a^*) = a \otimes a^*$. Regarding that we know that $A$ is the first and $A^*$ the second tensor factor in practice we may concatenate and skip the $\otimes$ sign. Thus $a \otimes a^* = aa^*$ and (10) reads $a^*a = a_1 < a_2^* > \otimes a_2^*$.

Using coproducts (5), (14) we calculate (10) the cross relations (16)

\[
\begin{align*}
[\hat{p}_k, \hat{x}_l] &= -i \delta_{kl} \hat{m}, \\
[\hat{x}_l, \hat{m}] &= i \delta_{kl}, \\
[\hat{x}_0, \hat{m}] &= 0,
\end{align*}
\]

where $\hat{m}_{\lambda\sigma} = g^{\mu\rho} \hat{m}_{\lambda\rho}$ and $\hat{m}^\mu_{\lambda} = g^{\mu\rho} \hat{m}_{\mu\lambda}$.

The general covariant $\kappa$-deformed quantum phase space $\mathcal{H}^{(10,10)}$ is generated by $H, \hat{H}$ with the above cross relations. The standard $\kappa$-deformed phase space $\mathcal{H}^{(4,4)} \subset \mathcal{H}^{(10,10)}$ is its subalgebra generated by $\hat{x}^\mu$ and $\hat{p}_\nu$ only. The quotient of $\mathcal{H}^{(10,10)}$ by the relations $\hat{x}^\mu = \delta^\mu_\nu$ is the covariant $\kappa$-deformed DSR algebra. The subalgebra with generators $\hat{x}^\mu$, $\hat{p}_\nu$ and $\hat{m}_{\lambda\sigma}$ is dual to the $\kappa$-deformed DSR algebra.

B. Heisenberg double as a right bialgebroid

The Heisenberg double $\mathcal{H}$ of any finite-dimensional Hopf algebra $\mathbb{H}$ is shown in the Section 6 of [8] to have the structure of a Hopf $A^*$-algebroid where the base $B := A^*$ is the underlying algebra of $\mathbb{H}$. Our Hopf algebroid is an instance of an infinite-dimensional version of that algebra. We shall neglect the mathematical questions of completions (see e.g. [4]).

The recipes of J-H. Lu [8] are the following. The source $A^* \cong \mathbb{C} \otimes A^* \rightarrow A \times A^*$). For the target Lu suggests a formula involving the canonical element. In Section 6 of Lu [8] a formula for the target map $\beta$ of the left bialgebroid structure on a Heisenberg double of finite dimensional Hopf algebra is written out; this formula involves the canonical element in $A \otimes A^*$. Given a vector space basis $\{f^I\}$ of $A^*$ and the dual basis $\{h_I\}$ of $A^*$ characterized by $\langle f^I, h_J \rangle = \delta^I_J$, the canonical element is $\sum_I f^I \otimes h_I$. If $A$ is infinite dimensional then the dimension of $A^*$ is of even bigger cardinality than that of $A$, hence it requires some care to make sense of the canonical element. Still, the dual vectors $f^I$ are well defined and, in cases like ours, the infinite sum converges in a (formal) completion $A \hat{\otimes} A^*$. M. Stojić has pointed to us a right bialgebroid version of Lu’s formula for $\beta$, namely

\[
\beta(t) = \sum_{I,J} f^I S^{-1} f^J \otimes h_I h_J \in A \times A^*
\]

where $\sum_I f^I \otimes h_I$ is the canonical element.

For the effective comultiplication $\Delta : H \rightarrow H \otimes_B H$ (where $B$ is the base) J-H. Lu [8] defines

\[
\Delta(a \otimes b^*) = \sum (a_{(1)} \otimes 1) \otimes_B (a_{(2)} \otimes b^*), \quad a, b \in A^*,
\]

where $\Delta_B(a) = \sum a_{(1)} \otimes a_{(2)}$ is the coproduct in the Hopf algebra $\mathbb{H}$. Of course, in (19) we can simply write $a_{(1)} \otimes a_{(2)} b^*$. 
In the case of $\mathcal{H}^{(4,4)}$, this coproduct $\Delta$ is given on the generators by the formulae

$$
\Delta(\hat{x}^\mu) = 1 \otimes \hat{x}^\mu, \\
\Delta(\hat{p}_k) = \hat{p}_k \otimes \hat{e}^{\mu}, \\
\Delta(\hat{p}_0) = \hat{p}_0 \otimes I + I \otimes \hat{p}_0.
$$

(21)

**C. Counit and Fock actions**

The property of the counit $\epsilon : H \to B$ which one wants to preserve from the Hopf algebra case is its relation to the coproduct, that is

$$
(\epsilon \otimes_B \text{id}) \Delta = (\text{id} \otimes_B \epsilon) \Delta = \text{id}
$$

(22)

This requirement makes sense in view of the canonical identification of the $B$-bimodules $B \otimes_B H \cong H \otimes_B B \cong H$. Taking into account the $B$-bimodule structure on $H$ (used in the bimodule tensor product over $B = A^*$), this amounts to $h_{(1)} \alpha (\epsilon (h_{(2)})) = h = h_{(2)} \beta (\epsilon (h_{(1)}))$. However, it is generally not possible to force the counit to be an algebra homomorphism. Instead it satisfies the weaker properties 12, 15

$$
\epsilon (\alpha (\epsilon (h))) h' = \epsilon (\epsilon (h)) h'
$$

(23)

In the case of the Heisenberg double $A \times A^*$ define

$$
\epsilon (a \otimes b^*) = \epsilon_A(a) b^*, \\
a \otimes b^* \in A \times A^*
$$

(24)

Then $(f, h) \to f \triangleleft h = \epsilon (\alpha (f)) h$ is a right action, where $f \in \widehat{T}^4_\kappa$ and $h, h' \in \mathcal{H}^{(4,4)} = T^4 \times \widehat{T}^4_\kappa$. This action should be viewed as a deformed Fock space where the unit $1 =: \langle 0 |$ is the right Fock vacuum. Indeed, $\langle 0 | \hat{p}_\mu = 0$ and $\langle 0 | \hat{x}^\mu = \hat{x}^\mu$ and the usual normal ordering and commuting procedures for the evaluation of long expressions in coordinates and momenta on the vacuum apply.

The counit $\epsilon$ satisfies defining equations (22) and on the generators of $\mathcal{H}^{(4,4)}$ is given by

$$
\epsilon (\hat{x}_\mu) = \hat{x}_\mu, \\
\epsilon (\hat{p}_\mu) = 0, \\
\epsilon (1) = 1.
$$

(25)

In our case, the identity $\epsilon (\alpha (\hat{x}_\mu)) = \epsilon (\beta (\hat{x}_\mu)) = \hat{x}_\mu$ holds.

**D. Calculating the target map**

To apply the formula (19) for $\beta$ to the $(4 + 4)$-dimensional $\kappa$-phase space we first compute the canonical element. This is easier in the undeformed case; the results is then used to compute the deformed case. Then the monomials $\hat{p}_I$ in momentum operators $\hat{p}_\mu$ are the dual elements, up to proportionality constants, namely by (22) and (11) we obtain

$$
\langle x^I, \hat{p}_I \rangle_0 = i |I| !
$$

where $I = (i_0, i_1, i_2, i_3)$ is a multiindex, $|I| = -i_0 + i_1 + i_2 + i_3$ and $I! = i_0! i_1! i_2! i_3!$. The canonical element is therefore

$$
\sum_{i_0, i_1, i_2, i_3 = 0} ^\infty \frac{i |I| !}{I!} \hat{p}_I \otimes x^I = \exp (i \sum_{\mu=0} ^3 \hat{p}_\mu \otimes x^\mu).
$$

(26)

There is now an isomorphism $\xi$ of coalgebras \(\widehat{\mathbb{R}}\) from the undeformed coordinate algebra (polynomials in $x^\mu$) to the deformed coordinate algebra (polynomials in $\hat{x}^\mu$). The inverse $\xi^{-1}$ of this isomorphism can be computed from the commutation relations as follows. In the right bialgebroid case, $\hat{x}^\mu$ is realized as the operator $-i [\hat{x}^\mu, \hat{p}_\sigma] x^\sigma$ where $\hat{p}_\sigma = -i \partial_\sigma$ and the operators act to the right. Then, an arbitrary expression $h$ in $\hat{x}^\mu$ acts from the right as a differential operator to (the right-hand version of) the Fock vacuum, the result is $\xi^{-1}(h) = \langle 0 | h_{\text{oper}}$. In our case,

$$
\hat{x}_k = x_k, \\
\hat{x}_0 = x_0 - \frac{i}{\kappa} \sum_{j=1} ^3 \partial_j x_j.
$$
Using this, one can calculate that
\[
\langle 0| \exp(-ip_0\hat{x}_0) \exp(ip_j\hat{x}_j) = \exp(-ip_0x_0 + ip_jx_j)
\]
(27)
where \( p_0, p_j \) are just numbers – not operators and the summation over \( i, \sum_{j=1}^{3} p_i x_i \) is understood.

In fact we should find some functions \( F^\mu(k_0, k_1, k_2, k_3) \) so that
\[
\langle 0| \exp(iF^\mu(k)\hat{x}_\mu) = \exp(-ik_0x_0 + ik_jx_j)
\]
but this involves more involved calculations of functions \( F^\mu \). We instead take a simpler looking product of exponentials, on the expense of need for further care of the ordering in the remaining computation below. The crucial observation [7] is now that the deformed pairing \( \langle , \rangle \) can be expressed in terms of the undeformed pairing \( \langle , \rangle_0 \) and the map \( \xi \),
\[
\langle \xi(x^j), \hat{p}^i \rangle = \langle x^j, \hat{p}^i \rangle_0.
\]
Regarding that \( \xi \) is an isomorphism of vector spaces, this means that it is wise to take as basis \( \xi(x^j) \) and preserve the dual basis and we get the canonical element \( \sum_{i} \left( \frac{i\hbar}{\kappa} \right) \hat{p}^i \otimes \xi(x^j) = (1 \otimes \xi) \exp(-i\hat{p}_0 \otimes x^0) \exp(\pm i\hat{p}_0 \otimes x^0) \exp(-i\hat{p}_i \hat{x}_i) \).

Here we used that the first tensor product commutes with the second, hence behaving as formal commuting variable in the calculation above.

Note that the equality of the sums \( \sum_{i} S^{-1}f^i \otimes h_1 = \sum_{i} f^i \otimes S^{-1}h_1 \). Indeed, the antipode \( S \) of is an isomorphism as well and \( (Sf, S^{-1}h) = \langle f, SS^{-1}h \rangle = \langle f, h \rangle \) because the pairing is Hopf (11). Therefore \( f^i = Sf^i \) also form a basis and \( h_1 = S^{-1}h_1 \) are the dual vectors; regarding that the canonical element does not depend on the choice of basis,
\[
\sum_{i} S^{-1}f^i \otimes h_1 = \sum_{i} S^{-1}Sf^i \otimes S^{-1}h_1 = \sum_{i} f^i \otimes S^{-1}h_1.
\]
In our case, \( h_1 \) are the elements in the enveloping algebra and the antipode contributes to the sign, thus for \( \sum_{i} f^i \otimes S^{-1}h_1 \) we get the same group-like exponential (26) with the minus sign, that is its multiplicative inverse. Noting that \( A \) is in our case commutative, the order of \( f_1 \) and \( f_J \) can interchange in the Lu formula (11) and we obtain
\[
\beta(h) = \zeta \left( e^{i\hat{p}_0 \otimes \hat{x}_0} e^{-i\hat{p}_0 \otimes \hat{x}_0} (1 \otimes h) e^{i\hat{p}_0 \otimes \hat{x}_0} e^{-i\hat{p}_0 \otimes \hat{x}_0} \right)
\]
(28)
where \( \zeta \) is the isomorphism of vector spaces sending the tensor product algebra \( A \otimes A^* = T \otimes T^4 \) to the smash product algebra \( T \times T^4 \). This isomorphism is the identity on the vector space level but not a homomorphism of algebras, hence the expressions in terms of products of algebra generators may look different.

We want to apply this formula (29) to the generators \( \hat{x}_0 \) and \( \hat{x}_i \); to this end first we apply the conjugation with the inner exponentials and then with the outer exponentials. Using ad-expansion \( e^{X}Ye^{-X} = e^{adX}(Y) \) and inductive calculations with the exponential series, in the case of \( h = \hat{x}_i \) the inner conjugation in (29) gives \( \exp(-i\hat{p}_0 \otimes \hat{x}_0)(1 \otimes \hat{x}_i) \exp(\hat{p}_0 \otimes \hat{x}_0) = \exp(-\frac{\hat{p}_0}{\kappa}) \otimes \hat{x}_i \) and the outer conjugation leaves the result intact, hence after applying \( \zeta \) we obtain
\[
\beta(\hat{x}_i) = \exp(-\frac{\hat{p}_0}{\kappa}) \hat{x}_i.
\]
(29)

For \( h = \hat{x}_0 \) the inner conjugation is not affecting \((1 \otimes h)\) and the outer conjugation gives
\[
\beta(\hat{x}_0) = \zeta \left( e^{i\hat{p}_0 \otimes \hat{x}_0}(1 \otimes \hat{x}_0)e^{-i\hat{p}_0 \otimes \hat{x}_0} \right) = \zeta \left( 1 \otimes \hat{x}_0 - \frac{1}{\kappa} \hat{p}_k \otimes \hat{x}_k \right) = \hat{x}_0 - \frac{1}{\kappa} \hat{p}_k \hat{x}_k.
\]
(30)

E. The coproduct gauge freedom

If we view the coproduct as an algebra map \( \Delta : H \to H \otimes_c H \) then it is coassociative only up to elements in certain subspace of \( H \otimes_c H \otimes_c H \) and \( \Delta \) itself can also be redefined up to elements in \( \overline{I}_B \subset H \otimes_c H \). This freedom of coproducts is parametrized by the coproduct gauges. If we consider coproduct as means to build the realizations of algebra \( H \) on tensor spaces, which corresponds in the physical context to the Fock space realizations and multiparticle states, various possibilities of such tensoring depending on the physical choice of particular dynamical model are possible (see also Section V). In such a framework the gauge-invariant object is the algebra \( H \), and its tensorial realizations are gauge-dependent, different for various dynamical models with Fock-like extensions. The Hopf algebroid structure in mathematical sense deals with strictly coassociative coproduct, which we may call effective, where the freedom in the left ideal \( \overline{I}_B \) is quotiented out \( \mathcal{I} \), hence \( \Delta : H \to H \otimes_B H = (H \otimes_c H)/\mathcal{I}_B \).
Let us compute the ideal $\mathcal{I}_B$ using $\beta$ from (29) and (30):
\[
\alpha(\hat{x}_i) \otimes 1 - 1 \otimes \beta(\hat{x}_i) = \hat{x}_i \otimes 1 - 1 \otimes e^{-\frac{\rho}{\kappa}} \hat{x}_i,
\]
\[
\alpha(\hat{x}_i) \otimes 1 - 1 \otimes \beta(\hat{x}_i) = \hat{x}_i \otimes 1 - 1 \otimes \hat{x}_0 + 1 \otimes \frac{1}{\kappa} \hat{p}_k \hat{x}_k.
\]
(31)

The expressions on the right hand side of equations (31) generate $\mathcal{I}_B$ as a left ideal.

We can arrive to the same ideal or its parts by considering coproduct freedom using elementary arguments with tensors. Thus, if $\Delta(\hat{x}^\mu) = 1 \otimes \hat{x}^\mu$ then any other homomorphism $\Delta$ which is of the form
\[
\tilde{\Delta}(\hat{x}_\mu) = \Delta(\hat{x}_\mu) + \Lambda_\mu(\hat{x}, \hat{p}) = \hat{x}_\rho \otimes \theta^\rho_\mu(\hat{p}),
\]
where $\Delta(\hat{x}^\mu) = 1 \otimes \hat{x}^\mu$ and $\theta^\rho_\mu(\hat{p})$ is the tensor to be determined satisfies
\[
[\Delta(\hat{x}_\mu), \Lambda_\nu] + [\Lambda_\mu, \Lambda_\nu] = C^{(\kappa)_\rho}_\mu(\Lambda_\rho),
\]
(33)
\[
[\Delta(\hat{p}_\mu), \Lambda_\nu] = 0.
\]
(34)

where $C^{(\kappa)_\rho}_\mu = \frac{1}{\kappa}(\delta^\rho_\mu \delta^\nu_\eta - \delta^\nu_\eta \delta^\rho_\mu)$ are the structure constants, $[\hat{x}_\mu, \hat{x}_\nu] = C^{(\kappa)_\rho}_\mu \hat{x}_\rho$.

The relations (33)–(34) are required if the transformation $\Delta(\hat{x}_\mu) \rightarrow \tilde{\Delta}(\hat{x}_\mu)$ is to describe the coproduct gauge. Postulating that $(\Delta(\hat{x}^\mu), \Delta(\hat{p}_\nu))$ satisfies the quantum phase space algebra relations which $\hat{x}^\mu, \hat{p}_\nu$ do one algebraically derives the conditions fixing the tensor $\theta^\rho_\mu(\hat{p})$, namely
\[
\tilde{\Delta}(\hat{x}_i) = \hat{x}_i \otimes e^{-\frac{\rho}{\kappa}}, \quad \tilde{\Delta}(\hat{x}_0) = \hat{x}_0 \otimes 1 + \frac{1}{\kappa} \hat{x}_i \otimes e^{\frac{\rho}{\kappa}} \hat{p}_i.
\]
(35)

As follows from (29) one gets
\[
\Lambda_\mu = \hat{x}_i \otimes e^{\frac{\rho}{\kappa}} - 1 \otimes \hat{x}_i,
\]
\[
\Lambda_0 = \hat{x}_0 \otimes 1 - 1 \otimes \hat{x}_0 + \frac{1}{\kappa} \hat{x}_i \otimes e^{\frac{\rho}{\kappa}} \hat{p}_i.
\]
(36)

Thus we obtained that $\Lambda$s are in the ideal $\mathcal{I}_B$, namely they are of the form
\[
\Lambda_\mu(\hat{x}, \hat{p}) = (1 \otimes \theta^\rho_\mu)(\alpha(\hat{x}_\rho) \otimes 1 - 1 \otimes \beta(\hat{p}_\rho)) \in \mathcal{I}_B
\]
(37)

Similar tensors $R_\mu = \hat{x}_\mu \otimes 1 - \hat{p}_\rho \otimes \hat{x}_\rho$, where $\hat{p}_\rho$ is the matrix inverse to $\theta^\rho_\mu$ (see (32)), have been introduced in [1] for the canonical twisted Heisenberg algebra and considered in [2] for the $\kappa$-deformed quantum phase space generated by the $\kappa$-deformed Poincaré–Hopf algebra with classical Poincaré algebra sector. Note that our $\hat{x}^\mu$ corresponds to $\hat{y}^\mu$ from [1] rather than their $\hat{x}^\mu$ which is a more convenient generator for the description of the left bialgebroid structure.

Analogous analysis as for $\Lambda_\mu$ can be made for higher order tensors [23].

IV. ANTIPODE

Hopf algebra is a bialgebroid with an antipode which is a linear map $\tau : H \rightarrow H$. In bialgebroids, the source $\alpha$ and target $\beta$ together play the role of the unit and it is natural to require $m(\tau \otimes_B id)\Delta = \alpha \epsilon$ and $m(id \otimes_B id)\Delta = \beta \epsilon$ as an extension of the axioms for the antipode in Hopf algebra theory. The first of the two equations is however problematic: $m(\tau \otimes_B id)\Delta = \alpha \epsilon$ in general, hence the right hand side with effective $\Delta$ being defined up to gauge in $\mathcal{I}_B$ is not well defined; only a subclass of gauges could satisfy this equation. J-H. Lu [8] avoids this by restricting to a subclass of gauges; in fact she introduces a linear map $\gamma : H \otimes_B H \rightarrow H \otimes H$ which is a section of the projection $H \otimes H \rightarrow H \otimes_B H$, so that $\Delta, = \gamma \circ \Delta$ is a choice of gauge. Her axioms (here in the right bialgebroid version) are thus nonsymmetric:
\[
\tau \beta = \alpha,
\]
(38)
\[
m(\tau \otimes_B id)(\gamma \circ \Delta) = \alpha \epsilon,
\]
(39)
\[
m(id \otimes_B \tau)\Delta = \beta \epsilon \tau.
\]
(40)

Now the left hand side of (40) does not depend on coproduct gauge for $\Delta$. An approach by G. Böhm [15] is symmetric: she supplies both a right $B$-bialgebroid $\mathcal{H}_R$ and a left $B^{op}$-bialgebroid $\mathcal{H}_L$ structure on the same total
algebra $\mathcal{H} = \mathcal{H}_R = \mathcal{H}_L$ and the antipode $\tau : \mathcal{H} \to \mathcal{H}$ is a linear antiautomorphism satisfying a number of axioms involving both left and right bialgebroid structures; the coproducts of $\mathcal{H}_L$ and $\mathcal{H}_R$ are also compatible as well as the counit, source and target data for the two bialgebroids. In $\mathcal{B}$ a subalgebra $\mathcal{B}$ in $H \otimes \mathcal{H}$ is singled out within which all gauge choices satisfy (39), i.e. $\gamma \circ \Delta$ can be replaced by any $\Delta$ landing within $\mathcal{B}$. The latter approach has been axiomatized in (30).

To compute the antipode, we can either again follow the Heisenberg double formulas for $\tau$ in Section 6 of (8) which, like for $\beta$, also involve the canonical elements. But this recipe boils down to the following direct reasoning. For the momentum sector $T^4$, $\tau$ agrees with the Hopf-algebraic antipode: $\tau(\hat{p}_0) = S(\hat{p}_0) = -\hat{p}_0$ and $\tau(\hat{p}_1) = S(\hat{p}_1) = -e^{\hat{p}_0}\hat{p}_1$ (see (10)). For the coordinate sector, formulas for $\tau$ are forced by the equations $\tau(\beta(x_i)) = x_i$, $\tau(\beta(x_i)) = \hat{x}_i$, $\tau(\beta(x_0)) = \hat{x}_0 = \frac{3i}{\kappa}$.

\[ \tau^2(\hat{p}_0) = \hat{p}_0, \quad \tau^2(\hat{x}_i) = \hat{x}_i, \quad \tau^2(\hat{x}_0) = \hat{x}_0 - \frac{3i}{\kappa}. \]

V. DISCUSSION AND OUTLOOK

The general aim of this paper is to promote the application of Hopf algebroid structure to the description of quantum-deformed phase spaces. We consider here the algebraic model of phase spaces constructed as Heisenberg doubles or equivalently smash products of Drinfeld quantum enveloping algebra $\mathbb{H}$ acting upon dual quantum group $\mathbb{H}$. In bialgebroid framework an important difference with bialgebra is the generalization of unity to the noncommutative base algebra. If we consider the quantum-deformed phase spaces as Heisenberg doubles the base algebra is provided by the whole algebraic sector of quantum group $\mathbb{H}$. We add that our basic calculational result consists in detailed presentation of $\kappa$-deformed quantum phase space as physically interesting example of Hopf bialgebroid (see also [23]).

In the paper we introduced the notion of coproduct gauges describing a freedom of coproducts in the Hopf algebroid framework. For quantum phase spaces the coproduct gauges describe model-dependent ways of composing the global two–particle phase space from the phase space coordinates of two of its constituents.

In a simple case of D=3 nonrelativistic phase space the momenta coproducts describe global 2–particle momentum as follows

\[ \Delta(p_i) = p_i \otimes 1 + 1 \otimes p_i \quad \longleftrightarrow \quad p_i^{(1+2)} = p_i^{(1)} + p_i^{(2)} \]

and the formula for 2–particle center–of–mass coordinate is given by well-known formula

\[ x_i^{(1+2)} = \frac{m_1}{m_1 + m_2} x_i^{(1)} + \frac{m_2}{m_1 + m_2} x_i^{(2)} \]

which can be represented as well by the following one–parameter class of coproduct gauges $\Lambda_i$ (see also [23], formula (34) in the limit $\kappa \to \infty$)

\[ \Delta(x_i) = x_i \otimes 1 + \tilde{\Lambda}_i \]

\[ \tilde{\Lambda}_i = (\alpha - 1)(x_i \otimes 1 - 1 \otimes x_i) \quad \alpha = \frac{m_i}{m_1 + m_2}. \]

The coproduct gauges describe different ways of composing center-of-mass coordinate $x_i^{(1+2)}$ for different composite systems, characterized by the masses $(m_1, m_2)$ of its constituents. From this point of view therefore the coproduct gauge freedom is not unphysical as in the standard gauge theories, but characterizes different choices of dynamical systems leading to the same phase space algebra (Poisson algebra in classical mechanics, (deformed) Heisenberg algebra in quantum theory).

One can also consider the formulae describing $D = 4$ relativistic center of mass coordinates $\hat{x}_i^{(1+2)}$ (see e.g. [31, 32]) which can be described as the bialgebroid coproduct, with coproduct gauges depending as well on negative powers of energy and three–momenta components. In such a way by considering the (10 + 10) dimensional phase space $\mathcal{H}^{(10,10)}$. 

containing Lorentz algebra and the dual Lorentz group parameters, one can describe in principle various relativistic center-of-mass coordinates for two particles with arbitrary spin.

Finally we present an outlook. We shall point out some ways of extending the presented result, employing the notion of bialgebroid structure:

i) The bialgebroid formulae for various phase space algebra bases.

In this paper (see also [23]) we consider the $\kappa$-deformed phase space generated by Majid–Ruegg basis [17] of corresponding $\kappa$-deformed Poincaré–Hopf algebra. In such a basis the standard $\kappa$-deformed phase space is described by centrally extended 8-dimensional Lie algebra. Such Lie algebraic structure is preserved only under the linear change ($\hat{\rho}_\mu \rightarrow \hat{\rho}'_\mu = \alpha_\mu^\rho \hat{\rho}_\rho$) of momenta basis. The bicrossproduct structure of Poincaré–Hopf algebra which implies quantum covariance properties of phase space remains however valid if we change the fourmomenta basis in arbitrary nonlinear way

$$\hat{\rho}_\mu \rightarrow \hat{\rho}'_\mu = F_\mu(\hat{\rho}) \quad F_\mu(0) = 0. \quad (47)$$

In particular one can choose $F_\mu(\hat{\rho})$ in a way leading to classical Poincaré algebra in algebraic sector of $\kappa$-deformed Poincaré algebra [33–35]. In such a way we get more complicated nonabelian formulae for the composition law of the phase space algebra the Lorentz group parameters these ideas were considered earlier [36–40]). The noncanonical pair of dual generators ($\hat{\lambda}_\mu^\nu, \hat{\pi}_\lambda^\rho$) satisfying the relation (17) can be also replaced by a canonical pair ($\tilde{\omega}_\mu^\nu, \tilde{\pi}_\lambda^\rho$) where $\tilde{\lambda}_\mu^\nu = \delta_\mu^\nu + \tilde{\omega}_\mu^\nu$ and $(\eta_{\mu\nu} = diag(-1, 1, 1, 1))$

$$\left[\tilde{\lambda}_\mu^\nu, \tilde{\omega}_\lambda^\rho\right] = i \left(\delta_\mu^\beta \delta^\nu_\lambda - \delta^\lambda_\rho \delta_\nu^\beta\right) \quad (48)$$

consistent with $\tilde{\omega}_\mu^\nu = -\tilde{\omega}_\nu^\rho$ and the formula $\tilde{m}_\rho^\nu = \tilde{\omega}_\rho^\nu, \tilde{\pi}_\lambda^\rho = \tilde{\omega}_\lambda^\rho$. The calculations of the target map and coproduct gauge freedom for $\mathcal{H}^{(10,10)}$ are now under consideration.

ii) Hopf algebroid structure of generalized phase space $\mathcal{H}^{(10,10)}$.

In order to describe the global phase coordinates for two particles with relativistic spin one should include into the phase space algebra the Lorentz group parameters $\hat{\lambda}_\mu^\nu$ and dual Lorentz algebra generators $\hat{m}_{\mu\nu}$ (for undeformed case these ideas were considered earlier [36–40]). The noncanonical pair of dual generators ($\hat{\lambda}_\mu^\nu, \hat{m}_\rho^\nu$) satisfying the relation (17) can be also replaced by a canonical pair ($\tilde{\omega}_\mu^\nu, \tilde{\pi}_\lambda^\rho$) where $\tilde{\lambda}_\mu^\nu = \delta_\mu^\nu + \tilde{\omega}_\mu^\nu$ and $(\eta_{\mu\nu} = diag(-1, 1, 1, 1))$

$$\left[\tilde{\lambda}_\mu^\nu, \tilde{\omega}_\lambda^\rho\right] = i \left(\delta_\mu^\beta \delta^\nu_\lambda - \delta^\lambda_\rho \delta_\nu^\beta\right) \quad (48)$$

consistent with $\tilde{\omega}_\mu^\nu = -\tilde{\omega}_\nu^\rho$ and the formula $\tilde{m}_\rho^\nu = \tilde{\omega}_\rho^\nu, \tilde{\pi}_\lambda^\rho = \tilde{\omega}_\lambda^\rho$. The calculations of the target map and coproduct gauge freedom for $\mathcal{H}^{(10,10)}$ are now under consideration.

iii) Hopf superalgebroid structure of deformed supersymmetric quantum – deformed phase space.

Following the analogy respecting the “sign rules” between Hopf algebras and Hopf superalgebras one can define the $Z_2$-graded algebraic structure of Hopf superbialgebroids (see e.g. [41]). In order to extend supersymmetrically the $\kappa$-deformed phase space presented in this paper one can use already known results about the supersymmetric extension of $\kappa$-deformed Poincaré and its dual quantum Poincaré supergroup [22, 42]. For such a case one should firstly employ the supersymmetric extension of Heisenberg double, in particular for $\kappa$-deformed Poincaré–Hopf algebra.

iv) Twisted bialgebroids and the deformation of quantum phase spaces.

One can extend the twist deformation technique of Hopf algebras to Hopf algebroids [14]. In such a way one can obtain from Hopf algebroids with commutative base algebra (e.g. standard Heisenberg algebra) the Hopf algebroid with noncommutative base, which can be effectively calculated in terms of twist factor by so-called $*$-product formalism. In particular one can apply this technique to dynamical quantum deformations, described by parameter – dependent classical r-matrices satisfying dynamical YB equation (for the link of dynamical quantum deformations with Hopf algebroids see e.g. [14, 28]).

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