Non-local Josephson effect in Andreev molecules

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We propose the “Andreev molecule”, an artificial quantum system comprised of two closely spaced Josephson junctions. As in a real molecule, the coupling between Josephson junctions in an Andreev molecule occurs through the overlap and hybridization of the junction’s “atomic” orbitals, the Andreev Bound States. One of the striking consequences of molecular hybridization is that the supercurrent flowing through one junction depends on the superconducting phase difference across the other junction. Here we derive the energy spectrum of Andreev molecules and show how gaps open as the inter-junction separation is reduced. We calculate the non-local current-phase relations for arbitrary separation using scattering and tight-binding techniques. We demonstrate the possibility of creating a $\varphi$-junction, i.e. a junction carrying a supercurrent without any superconducting phase difference across it. In order to synthesize and detect Andreev molecules, we propose experiments on devices fabricated only with conventional materials and standard nanofabrication techniques. Andreev molecules are a new class of superconducting quantum devices with potential applications in quantum information, metrology, sensing, and molecular simulation.

I. INTRODUCTION

New physical phenomena often emerge when two interacting objects are juxtaposed. For example the molecule that forms when two hydrogen atoms are brought within a Bohr radius of each other has modified orbitals, a distinct energy spectrum, and completely different chemical properties. Similarly, when the separation between electromagnetic components is smaller than the electromagnetic wavelength or the electronic coherence length, interference effects modify the circuit response. In this work we predict novel, non-local electrical behavior in a system of two closely spaced Josephson junctions and explain how it arises with an analogy to molecular bonding.

When isolated, junctions obey the Josephson relations [1] and their properties are well understood and exploited in various fields such as magnetometry and metrology. Moreover, due to their quantum coherence and potential for integration in large-scale circuits, Josephson junctions serve as qubits for quantum information and computation.

As superconducting circuits are generally composed of several junctions it is natural to ask whether novel behavior emerges when two Josephson junctions are brought close to each other. In circuits with typical operating frequencies and propagation velocities, junctions can interact via the electromagnetic field over distances spanning micrometers to centimeters. This interaction is the best understood and most often utilized in applications [2]. It is not intrinsic in that it depends entirely on the electromagnetic environment which couples the junctions together. Junctions also affect each other over a length scale of microns via the diffusion of quasiparticles but this is only important close to $T_c$, the superconducting transition temperature, or at large bias voltages [3]. This process induces modifications of the superconducting energy gap and critical currents.

As the distance between Josephson junctions becomes comparable to $\xi$, the superconducting coherence length, additional processes mediated by Cooper pairs modify their electrical properties in a dramatic way. Initially, minor effects resulting from this “order-parameter interaction” were calculated for temperatures near $T_c$ using the Ginzburg-Landau equations [3, 4]. More recently, theorists have investigated this problem at zero temperature using Green’s function techniques [5–10]. They calculated non-local out-of-equilibrium supercurrents and demonstrated the existence of $\pi$ shifts in the current-phase relation. They predicted a remarkable phase-locking similar to Shapiro steps which has been measured experimentally in superconducting bi-junctions biased with commensurate voltages [11, 12]. They attribute these phenomena to the formation of entangled Cooper pairs called “quartets.” Although this interpretation is intriguing, many open questions remain: what is the microscopic mechanism involved? How does the interaction depend on the distance between the junctions? Have all non-local effects been revealed? How can one detect these effects and exploit them in devices?

The microscopic basis for the Josephson effect are electronic states called Andreev Bound States (ABS), which form a two-level system or artificial atom [13]. Here, motivated by this similarity between real atoms and Josephson junctions, we provide a clear and intuitive understanding of the Cooper pair mediated interaction. Andreev Bound States are the analog of hydrogen’s atomic orbitals with $\xi$ playing the role of the Bohr radius and determining the spread of the ABS wavefunctions. In the same way that electrons occupy atomic orbitals, quasi-particles (bogolubions) occupy ABS. Every Josephson junction is comprised of one to many ABS depending on its physical characteristics. When two Josephson junctions are brought within a distance $\xi$ of each other, a covalent bond forms just as with the hydrogen molecule.
the wavefunctions hybridize and the energy spectrum is transformed. As a result, in this system of two merged Josephson artificial atoms, which we call the Andreev molecule, novel, non-local Josephson effects emerge.

In Section II, we present the theoretical framework for the Andreev molecule, giving the previous concept of an “order-parameter interaction” a solid microscopic foundation. We calculate the dependence of the Andreev molecule’s wavefunction, energy spectrum, and non-local current-phase relation on the spacing between the two artificial atoms. We show that an Andreev molecule exhibits a ϕ-junction effect and propose a novel device which can be used to demonstrate this behavior. Section III discusses the case of quantum conductors with multiple channels, which more accurately describes a large class of hybrid Josephson junctions. Section IV extends the description of Andreev molecules to two dimensions using a tight-binding approach in which it is possible to incorporate disorder or spatial inhomogeneity. Finally, in section V we propose experiments to detect the formation of Andreev molecules and the emergence of a non-local Josephson effect.

II. ANDREEV MOLECULE IN A ONE-DIMENSIONAL CHANNEL

A. Background

The Josephson effect arises from the coupling of two superconductors via a weak link and the formation of Andreev Bound States (ABS) localized at the interface and whose energies depend on the superconducting phase difference δ across the junction. If the weak link connecting the superconductors is a short electronic channel with transmission τ0, there are two ABS with energies $E_A^{\pm}(\delta)$ smaller than the superconducting gap $\Delta$,

$$E_A^{\pm}(\delta) = \pm \Delta \sqrt{1 - \tau_0 \sin^2 \frac{\delta}{2}},$$

forming a two-level system which can be used to encode quantum information [13]. Occupied ABS carry a non-dissipative phase-dependent current through the junction, the supercurrent $I_A^{\pm}(\delta)$, given by

$$I_A^{\pm}(\delta) = -\frac{1}{\varphi_0} \frac{\partial E_A^{\pm}(\delta)}{\partial \delta},$$

where $\varphi_0 = \hbar/2e$ is the reduced flux quantum. In the low transmission limit $\tau_0 < 1$, or tunnel limit, this expression simplifies to the DC Josephson relation $I(\delta) = I_0 \sin \delta$, with $I_0 = \Delta \tau_0/2\varphi_0$. The wavefunction of a quasiparticle occupying an ABS decays exponentially away from it over a distance $\xi = \xi_0/\sqrt{\tau_0} \sin \frac{\delta}{2}$, where $\xi_0 = \hbar v_F/\Delta$ is the bare superconducting coherence length and $v_F$ is the Fermi velocity [15] (Fig. 1(a) and (b)).

As with real atoms, ABS form a coherent quantum system with spatially localized wavefunctions. As a consequence, if ABS are separated by a distance $l \sim \xi$, the overlap of their wavefunctions leads to their hybridization and the formation of the Andreev artificial molecule. Two microscopic mechanisms underlie this process: (i) double elastic cotunneling of Cooper pairs (dEC) and (ii) double crossed-Andreev reflections (dCAR). (d) The spectrum of the device, plotted for varying $\delta_L$ and fixed $\delta_R = 3\pi/5$ chosen arbitrarily, shows avoided crossings which indicate the formation of Andreev molecules by hybridization of the individual ABS. (e) Andreev molecules can be incorporated into devices exhibiting non-local Josephson effects. In this circuit the supercurrent flowing through the left junction $I_L$ depends on the superconducting phase difference $\delta_R$ across the right junction which is controlled with a magnetic flux $\Phi_R$.

Figure 1. Comparison between local and non-local Josephson effects in an Andreev molecule. (a) In a Josephson junction, the supercurrent is carried by Cooper pairs tunneling between superconductors connected by a weak link. (b) The weak link hosts localized electronic states called Andreev Bound States (ABS) whose energies depend periodically on the superconducting phase difference. (c) When two junctions are placed in close proximity, the ABS wavefunctions overlap, leading to new non-local mechanisms to carry the supercurrent: double elastic cotunneling (dEC) and double crossed Andreev reflection (dCAR). (d) The spectrum of the device, plotted for varying $\delta_L$ and fixed $\delta_R = 3\pi/5$ chosen arbitrarily, shows avoided crossings which indicate the formation of Andreev molecules by hybridization of the individual ABS. (e) Andreev molecules can be incorporated into devices exhibiting non-local Josephson effects. In this circuit the supercurrent flowing through the left junction $I_L$ depends on the superconducting phase difference $\delta_R$ across the right junction which is controlled with a magnetic flux $\Phi_R$. 

As pictured in Fig. 1(a), the supercurrent in a single Josephson junction corresponds to the elastic tunneling of Cooper pairs from the left superconducting electrode to the right one. For two junctions which are located close to each other (1(c)), Cooper pairs can also be transferred directly across the whole device (dEC, 1(c) top, 1). In contrast, dCAR (1(c) bottom, 2) is the joint splitting of two Cooper pairs in the center of the device and their crossed recombination in the left and right electrodes. These processes respectively confer $\delta_L$ - $\delta_R$ and $\delta_L + \delta_R$ dependence to the energy spectra. As a corollary, current

$$I_L(\delta_L, \delta_R)$$

controlled with a magnetic flux $\Phi_R$.
due to dEC is in the same direction, from the left to right electrode, whereas dCAR is in opposite directions, from the central electrode outwards.

To summarize, at large separation $l \gg \xi$, each weak link (Fig. 3(a)) will house independent ABS with spectra as in Fig. 3(b). As the separation is reduced (Fig. 3(c)) hybrid electronic states emerge with energies which now depend on two superconducting phase differences $\delta_L$ and $\delta_R$ rather than a single one (Fig. 3(d)). Incidentally, the current flowing through one of the junctions may depend on the phase across the other one (Fig. 3(e)).

**B. Hamiltonian of the circuit**

The simplest Josephson junction is composed of two superconductors connected by a one-dimensional quantum conductor with a single electronic channel. Experimentally, this has been realized with superconducting atomic contacts [17, 18] or could be envisioned in semiconducting nanowires epitaxially covered with superconducting aluminum [19–21]. An Andreev molecule can be engineered by connecting in series two of these junctions separated by a short distance $l$. The resulting device is then composed of three superconductors connected by two closely-spaced weak links. As illustrated in Fig. 3(c) and Fig. 3(a), by connecting an electrode to the central conductor one can flow current independently through each junction or apply superconducting phase differences $\delta_{L(R)}$ using magnetic fields.

Electrons in such a circuit can be described by a $2 \times 2$ Hamiltonian $H$ in Nambu space

$$H = \begin{pmatrix} H_0 + H_{WL} & \Delta(x) \\ \Delta^*(x) & -H_0 - H_{WL} \end{pmatrix}$$

(3)

where $H_0 = \frac{-\hbar^2}{2m} \partial_x^2 - \mu$ is the single particle energy ($m$ is the electron mass, $\mu$ the chemical potential) and $H_{WL} = U_L \delta(x + l/2) + U_R \delta(x - l/2)$ models scattering at the weak links with amplitudes $U_{L(R)}$. By using $\delta$-Dirac functions for the scatterers we limit the analysis to weak links that are shorter than the superconducting coherence length. The off-diagonal terms of the Hamiltonian describe electron pairing in each superconductor

$$\Delta(x) = \begin{cases} \Delta e^{i\varphi_L} & \text{if } x < -l/2 \\
\Delta & \text{if } |x| < l/2 \\
\Delta e^{i\varphi_R} & \text{if } x > l/2 \end{cases}$$

where the amplitude $\Delta$ is homogeneous along the whole device and by gauge invariance we can choose the phase of the central superconductor to be zero such that $\delta_{L(R)} = \varphi_{L(R)}$. The eigenstates and eigenenergies of $H$ respectively give the orbital states and the spectrum of the Andreev molecule. This minimal model is sufficient to capture the essential features of the non-local Josephson effect.

**C. Resolution of the Hamiltonian**

In order to solve the system we build wave functions $\psi$ that are continuous along the $x$-axis and obey the Bogoliubov-de Gennes (BDG) equation $H\psi = E\psi$, where $E$ is the eigenenergy. The spectrum is composed of discrete Andreev Bound States with energies smaller than the superconducting gap $|E| < \Delta$ and a continuum of states for $|E| > \Delta$.

The sub-gap states of the BDG equation for an infinite superconductor are evanescent waves. These solutions are spinors of electron or hole type $\psi^{\pm}_{e(h)}(x) \propto (u_{e(h)}, v_{e(h)})^T e^{\pm i k_{e(h)} x}$ where $u_{e(h)}$ and $v_{e(h)}$ are coherence factors (see Appendix A) and $k_{e(h)}$ are complex momenta which results in an exponential envelope to the wavefunction. If the superconducting gap is much smaller than the Fermi energy $\Delta \ll E_F$, the momenta can be approximated as $k_{e(h)} \approx k_F \mp i/\xi$ where $k_F$ is the Fermi momentum in the normal state and the coherence length $\xi$ is a function of energy $\xi^{-1} = \xi_0^{-1} \sqrt{1 - \epsilon^2} \ll k_F$, where $\epsilon = E/\Delta$ is the normalized energy. The wave-functions of Andreev molecules are built from the these spinors and defined piece-wise in the following way

$$\psi(x) = \begin{cases} l_x \psi_e^-(x) + l_h \psi_h^+(x) & \text{if } x < -l/2 \\
c^+_e \psi_e^+(x) + c^-_e \psi_e^-(x) + c^+_h \psi_h^+(x) & \text{if } |x| < l/2 \\
r_x \psi_e^+(x) + r_h \psi_h^-(x) & \text{if } x > l/2 \end{cases}$$

We exclude diverging spinors on the left and the right and introduce two sets of four coefficients $\Psi_e = (l_e, r_e, c^+_e, c^-_e)^T$ and $\Psi_h = (l_h, r_h, c^+_h, c^-_h)^T$ giving the respective weight of each component ($l$ for left, $r$ for right and $c^\pm$ for center). Continuity and the BdG Hamiltonian impose conditions on these coefficients which can be expressed as two equations

$$\Psi_e = a(\epsilon) A \Psi_h$$

(4)

and

$$\Psi_h = a(\epsilon) A^* \Psi_e.$$  

(5)

The first coefficient $a(\epsilon) = e^{-i \arccos \epsilon}$ is the probability amplitude for an electron to be Andreev reflected into a hole. The matrix $A = e^{-i \Phi} M^{-1} M^* e^{i \Phi}$ contains the superconducting phase differences via the diagonal matrix $\Phi = \text{diag}(\delta_L/2, \delta_R/2, 0, 0)$ and the scattering amplitudes via the matrix $M$,

$$M = e^{i k_F l/2} \begin{pmatrix} i & 0 & -i e^{-i k_F l} & -i \\
0 & i & -i & -i e^{-i k_F l} \\
-u_L & 0 & -u_L e^{-i k_F l} & u_L^2 \\
u_R & -u_R^2 & u_R e^{-i k_F l} \end{pmatrix},$$

where the reduced scattering potentials are $u_{L(R)} = 1 + iU_{L(R)}/h v_F$. The transmissions $\tau_{L(R)}$ for the junctions are given by $\tau_L = 1/|u_{L(R)}|^2$. In the rest of the
manuscript, we will mostly consider symmetric junctions where $\tau_L = \tau_R = \tau_0$ since the asymmetric case is not qualitatively different. All figures in section II are calculated with scattering parameters $U_L/h\nu_F = U_R/h\nu_F = 0.25$ or transmission $\tau_0 \approx 0.94$.

Equations 4 and 5 have non-trivial solutions only if

\[
\text{Det} \left(1 - a (e^2 A A^*) \right) = 0
\]

which gives the discrete part of the Andreev molecule’s spectrum. For given superconducting phase differences ($\Phi$ matrix), Eq. 6 yields either two or four eigenenergies depending on the distance between the junctions. These states emerge from the hybridization of two pairs of independent ABS belonging to the left and right junctions. In the case of small separation $l < \xi_0$, hybridization may push one of the ABS pairs into the continuum, leaving only two sub-gap states. The ABS pushed into the continuum are referred to as “leaky” Andreev states [22] but unlike ABS are delocalized plane waves [23][25].

**D. Wavefunctions and spectra of Andreev molecules**

Once we have solved for the eigenenergies we obtain the Andreev molecule wavefunctions from equations 4 or 5. These wavefunctions arise from the hybridization of the independent Andreev Bound States at each junction. Fig. 2(a) and (c) show a one-dimensional single channel conductor with weak links separated by a distance $l$, a device which can be fabricated from epitaxially-coated superconducting nanowires [20]. For $l$ comparable to $\xi_0$, the wavefunctions hybridize as shown in Fig. 2.

When the independent ABS energies are degenerate the wavefunctions may be spread out over both junctions. In the general case, degeneracy occurs when a Snell’s law like condition, $\sqrt{\tau_L} \sin(\delta_L/2) = \pm \sqrt{\tau_R} \sin(\delta_R/2)$ is satisfied. For the case of symmetric junctions considered here this equation takes the simple form $\delta_L = \mp \delta_R$ (mod 2$\pi$). For $\delta_L = -\delta_R$, corresponding to dEC, the wavefunction has a significant weight in the central superconductor. On the contrary, for the dCAR process at $\delta_L = \delta_R$ the wavefunctions are zero at the origin and peak at the positions of the weak links. For infinitely far junctions ($l \gg \xi_0$), these two processes are negligible and the wavefunctions are localized at one of the junctions (Fig. 2(c) and (d)) and correspond to distinct, non-overlapping pairs of ABS. As expected the spectrum is that of two isolated junctions (Eq. 1). Only local tunneling of Cooper pairs occurs and there is no non-local Josephson effect.

Fig. 3(a) to (e) shows how the spectrum evolves, for fixed $\delta_R = 3\pi/5$, as the junction separation is reduced from $l = \infty$ to $l = 0$. For moderate distance, $l = 2\xi_0$, ABS of the left and right junction start to hybridize into bonding and anti-bonding states producing avoided crossings around the points of degeneracy where dEC or dCAR is optimal (respectively at $\delta_R = \mp \delta_L$ for symmetric junctions). In the particular case of weak links with perfect transmissions ($\tau_R = \tau_L = 1$), avoided crossings disappear at $\delta_L = \delta_R$ where the dominant non-local process is dCAR. As explained in Section III, this process relies on the interconversion of electrons and holes propagating in the same direction and requires back-scattering, which is suppressed for $\tau_0 = 1$. In contrast avoided crossings at $\delta_L = -\delta_R$ produced by dEC, a momentum conserving process, are preserved for $\tau_0 = 1$. The inequivalence of the two processes of dEC and dCAR lead to the unusual asymmetry about $\delta_L = \pi$ in the spectra for fixed $\delta_R$. Such asymmetry, absent in the spectra of isolated junctions, is permitted by time-reversal invariance which only requires $E(\delta_L, \delta_R) = E(-\delta_L, -\delta_R)$ and not $E(\delta_L, \delta_R) = E(-\delta_L, \delta_R)$.

As the junction separation is reduced, the size of the avoided crossings increases until some of the ABS are partially pushed out of the superconducting gap and into the continuum. For any state with $|E| > \Delta$ the momentum $k_{c(h)}$ must be real and therefore the corresponding ABS are no longer localized for all values of the phase difference. These non-evanescent “leaky” Andreev states, which also exist in asymmetric gap Josephson junctions.
are U ≪ (l) grow as other, leading to the appearance of avoided crossings which Fermi momentum is chosen such that \( \delta_L = \pi \) in the case of separate junctions \( (l/\xi_0 \gg 1, \text{ Fig. } 3(a)) \) and the gap at \( \delta_L = \pi + \delta_R \) in the case of fused junctions \( (l/\xi_0 \ll 1, \text{ Fig. } 3(e)) \). This can be explained by the competition between scattering, which pushes the innermost spectral lines away from zero energy, and molecular hybridization due to dEC (\( \delta_L = -\delta_R \)), which pushes them inward.

When the distance \( l \) becomes negligible compared to \( \xi_0 \), the two junctions fuse into a single junction with twice the scattering amplitude and one pair of ABS pushed entirely into the continuum. Instead of having two junctions in series with transmissions \( \tau_0 \approx 0.94 \) at large separation \( l \gg \xi_0 \), there is now a single junction with a larger scattering amplitude \( U_L + U_R \) (or equivalently a smaller transmission \( \tau'_0 \approx 0.8 \)) and an additional continuum state. The spectrum, shown in Fig. 3(e), is effectively that of a single junction of transmission \( \tau'_0 \) shifted by \( \delta_R = 3\pi/5 \) to account for the fact that the total phase drop is \( \delta = \delta_L - \delta_R \). The mechanism of dCAR is no longer possible since there is no central superconducting electrode and dEC is simply transformed into regular tunneling. The two cases \( l \gg \xi_0 \) and \( l \ll \xi_0 \) are conventional in that the non-local mechanisms are absent and that the energy spectrum can be described by the single-junction expression Eq. 1, albeit with different transmissions and a shift in phase.

The transition from two independent artificial atoms to a single one with an intermediate molecular state can be visualized in Fig. 4(a) where only the positive spectral lines for \( \delta_L = \pm \delta_R \) are plotted as \( l/\xi_0 \) is varied for fixed \( \delta_R = 3\pi/5 \). At large separation \( l/\xi_0 \gg 1 \) all positive ABS converge to the same energy, \( E^+ = (\delta = \pm 3\pi/5, \tau_0) \approx 0.62\Delta \). For the dominant dEC mechanism at \( \delta_L = -\delta_R \) (solid lines), with decreasing \( l/\xi_0 \) the two degenerate ABS of the left and right junction gradually split to form bonding and anti-bonding molecular states until the higher-energy ABS escapes into the continuum. Beyond that point the device hosts only one pair of ABS, symmetric in energy, such that it behaves as a single artificial atom rather than a molecule. Incidentally, around this transition from molecule to atom, the lower ABS reaches a minimum resulting in an overall shape which suggests the interatomic potentials used to describe the formation of molecules. Physically, this minimum corresponds to the point where the single quasiparticle occupying the lower ABS has been maximally localized and isolated from the continuum. At \( l/\xi_0 = 0 \), in contrast to a real interatomic potential, the energy of this lower state does not diverge and is less than the energy at large separation, \( l/\xi_0 \to \infty \).

On the other hand for \( \delta_L = \delta_R \) (dashed lines) dCAR is the dominant mechanism. The corresponding ABS hy-

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Figure 3. Energy spectrum of an Andreev molecule for varying separation. (a) When two single channel superconducting junctions are placed far from each other, their ABS are independent. Energies of ABS in the left junction (blue lines) modulate with \( \delta_L \) (x-axis) while those of the right junction are constant (red lines). (b-c-d) When the junction separation \( l \) is comparable to \( \xi_0 \), the ABS hybridize among each other, leading to the appearance of avoided crossings which grow as \( l \) decreases (magenta regions). The gaps at \( \delta_L = \delta_R \) are due to dCAR and the gaps at \( \delta_L = -\delta_R \) are due to dEC. The external ABS are gradually pushed into the continuum \((|E| > \Delta)\) as a result of hybridization. The dashed line in graph (c) \( (l = \xi_0) \) shows the spectrum for \( U_L = U_R = 0 \) \( (\tau_L = \tau_R = 1) \), where dCAR splitting is absent. (e) For \( l < \xi_0 \), the two junctions merge into a single one, leaving only one pair of ABS. Parameters used for these calculations are \( U_L = U_R = 0.25\hbar v_F \) \( (\tau_L = \tau_R \approx 0.94) \), \( \delta_R = 3\pi/5 \) and the Fermi momentum is chosen such that \( k_F l \gg 1 \) and \( k_F l = \pi/2 \) (mod2\( \pi \)), maximizing the avoided crossings at \( \delta_L = \delta_R \).
bridize as \( l/\xi_0 \) decreases and then escape into the continuum, leaving no discrete states for small \( l/\xi_0 \). As the size \( l \) of the central superconductor approaches zero, the probability for dCAR to happen shrinks to zero and, since the effective phase difference becomes zero, the bound states enter the continuum.

The strength of the two mechanisms dEC and dCAR can be adjusted by tuning \( \delta_R \) as shown in the spectra of Fig. 4(b), where the junction separation is fixed at \( l/\xi_0 = 0.5 \). Different regimes can be identified by the number of ABS. For \( \delta_R \) increasing from 0 to \( \pi \) at first there is only a single dEC line, which is then joined by two dCAR lines whose splitting increases with \( \delta_R \), and then finally by a second dEC line near \( \pi \). The dCAR splitting, or hybridization strength, is maximal at \( \delta_R = \pi \) whereas for dEC this maximum splitting occurs at the point where the second dEC line passes below the gap approaching \( \delta_R = \pi \), and then decreases as \( \delta_R \) goes to \( \pi \). By changing \( \delta_R \) one can thus control the degree of hybridization or the molecular nature of the system. These properties could be exploited for quantum simulations of elaborate molecules, including those with strong Coulomb repulsion since one could use weak links with large charging energy such as carbon nanotubes \[28–31\]. In addition this tunable multi-level qubit may have applications for quantum information. It is also possible that the ability to tune levels into the continuum may be useful for preparing or resetting quantum states.

E. \( \varphi \)-junction behavior and non-local Josephson effect

The formation of an Andreev molecule dramatically modifies the Josephson effect compared to the standard case. The three-terminal devices we described in previous sections are not simply a series connection of two-terminal devices. For two closely spaced junctions the supercurrent through the left junction \( I_L \) will also depend on the phase drop across the right junction \( \delta_R \), making it possible to obtain \( \varphi \)-junctions.

This behavior results from the unusual asymmetry in the spectrum of Fig. 3 with respect to \( \delta_L = 0 \) or \( \pi \). For an isolated Josephson junction (Fig. 3(a)) the spectrum is always symmetric and therefore, according to Eq. 2, the supercurrent cancels at these points. In particular unless the weak link breaks time-reversal symmetry such that \( I_L(\delta_L) \neq -I_L(\delta_L) \), there is no zero-phase supercurrent: \( I_L(\delta_L = 0) = 0 \). However when two junctions are close enough for an Andreev molecule to form their current-phase relations depends on two phases. In this case, time-reversal symmetry imposes \( I_L(-\delta_L,-\delta_R) = -I_L(\delta_L,\delta_R) \), which means that the current \( I_L(\delta_L,\delta_R) \) does not need to cancel at \( \delta_L = 0 \) if \( \delta_R \) is non-zero.

In an Andreev molecule, the total supercurrent can be divided in two contributions: one from the ABS \( I_L^{ABS} \) and one from the continuum \( I_L^{cont} \) such that \( I_L = I_L^{ABS} + I_L^{cont} \). The ABS contribution to the supercurrent flowing through the left junction is obtained similarly to the case of a single junction

\[
I_L^{ABS}(\delta_L,\delta_R) = -\frac{1}{\varphi_0} \sum_{E_{ABS} < 0} \frac{\partial E_{ABS}}{\partial \delta_L}
\]

where we sum over all the ABS of negative energy \( E_{ABS} \). The derivation of the continuum contribution is given in Appendix C.

Figure 4. From artificial atoms to molecules and fusion. (a) Continuous evolution of the positive part of the spectrum at the avoided crossings \( \delta_L = -\delta_R \) (dEC, solid lines) and \( \delta_L = \delta_R \) (dCAR, dashed lines) as a function of \( l/\xi_0 \) for fixed \( \delta_R = 3\pi/5 \). At large separation, ABS are degenerate and each junction behaves as an artificial atom with only one state below the energy gap \( \Delta \). As the separation is reduced, these atoms hybridize to form a molecule, indicated by the splitting due to dEC and dCAR, and eventually fuse into a single new artificial atom leaving only one sub-gap state. (b) The strength of hybridization can be controlled by adjusting \( \delta_R \). Here we see a transition from an Andreev molecule, near \( \delta_R = \pi \), where there are two possible positive-energy states (pair of dashed or solid lines), to a single artificial atom near \( \delta_R \approx 0 \) (mod2\( \pi \)). Parameters used for these calculations are \( U_L = U_R = 0.25 \hbar v_F \) (\( \tau_L = \tau_R \approx 0.94 \)), \( \delta_R = 3\pi/5 \) and \( k_F l = \pi/2 \) (mod2\( \pi \)) with \( k_F l \gg 1 \).
ABS and the continuum contribute significantly when the supercurrent separation is identical to that of a single junction. (b-c-d) As the junction separation becomes small compared to $\xi_0$ and converges to zero for large $l$ with an oscillatory behavior (Fig. 5(a)). These oscillations have a period $\sim \xi_0$ and result from the interference of plane waves from the continuum with momenta $k_L = k_F + l/\xi$ and $k_R = k_F - l/\xi$. The exponential dependence of $I_L^{\phi_R=0}(l/\xi_0)$ is a direct result of the overlap of the ABS wavefunctions and a signature of the non-local nature of the $\varphi$-junction effect. In the case of zero separation ($l = 0$), the device only has two terminals and as expected the supercurrent is zero at $\delta_L = \varphi = \delta_R$ since the total phase drop $\delta = \delta_L - \delta_R$ is also zero. The $\varphi$-supercurrent $I_L^{\phi_R=0}(0)$ for the symmetric case is given by the single junction expression (2) with $\tau_0 = \tau_0'$ and $\delta = -\varphi$. In addition to the rightwards shift of the current-phase relation, since $\tau_0 > \tau_0'$ the overall amplitude decreases as $l/\xi_0 \to 0$ (Fig. 5 top to bottom).

For single junctions in the short limit the continuum carries a negligible supercurrent [32] and one can accurately describe the Josephson effect by considering only the ABS. Here, we note that while in the extreme cases $l \ll \xi_0$ and $l \gg \xi_0$ the supercurrent is entirely carried by ABS (Fig. 5(a) and (e)), this is not the case for intermediate separation $l \approx \xi_0$. The contribution of the continuum (states with $E > \Delta$) becomes comparable in magnitude and tends to flow in the opposite direction. Comparing Fig. 3 and 5 we observe cusps in both contributions to the supercurrent each time a discrete state escapes from the gap. The total supercurrent however evolves smoothly, indicating that the ABS hybridize as much with the continuum as with each other and that the distinction is somewhat artificial.

Andreev molecules can be exploited to engineer non-local Josephson devices. As an example, Fig. 6(b) shows a circuit where two superconducting loops interrupted by Josephson junctions are connected to each other by a single superconducting wire. In the absence of hybridization, assuming the superconducting loops have negligible inductances and that there is no flux in the left loop ($\Phi_L = 0$), we expect that the applied flux $\Phi_R$ results in a phase difference $\delta_R = \Phi_R/\varphi_0$ and a supercurrent in the right loop. However, if the length of this wire is short such that the junction separation is small $l \ll \xi_0$, applying a magnetic flux $\Phi_R$ will not only induce a current in the right loop but also give rise to a non-local current in the left loop, even though $\delta_L$ is zero. This zero-phase supercurrent $I_L^{\phi_R=0}$ depends periodically on $\delta_R$ (Fig. 6(b)).
Figure 6. Zero-phase current and $\phi$-junction behavior. (a) Evolution of the 0-phase current as a function of $l/\xi_0$ for fixed $\delta_R = 3\pi/5$. Contributions of the continuum (dotted lines) and ABS (dashed lines) are indicated. (b) Circuit (top) to demonstrate a non-local Josephson effect. If hybridization between the two junctions is strong enough the current $I_L$ flowing through the left junction can be modulated by the flux $\Phi_R$ threading the right loop (bottom plot, calculated for $l/\xi_0 = 0.5$). Parameters used for these calculations: $U_L = U_R = 0.25\hbar v_F$ ($\tau_L = \tau_R \approx 0.94$), and $k_F l = \pi/2$ (mod $2\pi$) with $k_F l \gg 1$.

with a negative slope near the origin. Experimentally, one could deposit an isolated superconducting layer over the left loop to maintain $\Phi_L = 0$. In order to detect the spontaneous non-local supercurrent $I_L^{\delta_R=0}(\delta_R)$ one could insert a large Josephson tunnel junction to the left loop and make switching measurements as described in Section V.

In light of the calculations presented in this section for symmetric junctions of moderate transmission, non-local Josephson effects are significant as long as the junction separation is comparable to the superconducting coherence length. For devices made with superconducting aluminum with $\xi_0 \sim 200$ nm and nanowires of intermediate transmission ($\tau_0 \gtrsim 0.5$), such effects should be readily observable for a separation of 100 nm, feasible with standard nanofabrication techniques.

The case of a single-channel tunnel junction with $\tau_0 \ll 1$ is treated in Appendix D. Although there are modifications to the ABS spectrum for small separation, the current-phase relation has a negligible non-local contribution. Both effects would be difficult to measure experimentally in realistic devices.

III. MULTI-CHANNEL ANDREEV MOLECULE

In a Josephson junction with multiple conduction channels, each channel hosts independent Andreev Bound States (ABS). The total supercurrent is given by the sum of all of their contributions and the Josephson effect is qualitatively the same as for a single channel. However, when two multi-channel junctions are placed close to each other, each ABS at the left junction can potentially couple to every ABS at the right junction to form complex Andreev molecules. This is analogous to increasing the number of valence electrons leading to Andreev molecules of larger atomic number. Experimentally, the multi-channel case is relevant since many quantum conductors used in weak links have lateral dimensions larger than the Fermi wavelength and thus host a large number of channels.

A convenient approach to extend the single channel case (Section II) to multiple channels is to use the Landauer-Büttiker scattering formalism. Matrices describe the scattering of propagating electrons or holes on three different types of elements: weak links, semi-infinite superconductors, and a superconductor of finite length. Assembling the matrices as pictured in Fig. 7(a), we obtain two Josephson junctions whose ABS may hybridize via scattering in the central superconductor and yield Andreev molecules.

For energies smaller than the superconducting gap ($|E| < \Delta$), electrons (resp. holes) propagating toward a semi-infinite superconductor (blue and red shaded boxes) are Andreev reflected as holes (resp. electrons). The probability amplitude of such an event is $r_A = a(\epsilon) e^{-i\delta_L(R)}$, where $\delta_L(R)$ is the superconducting phase of the the left (right) superconductor, $\epsilon = E/\Delta$, and $a(\epsilon) = e^{-\frac{i}{2} \ln 2 |\epsilon|}$ as before. Since $|r_A|^2 = 1$, an electron or hole incident on a semi-infinite superconductor will always be Andreev reflected.

However, as shown in Fig. 7(b), if the superconductor has a finite length, an electron (resp. hole) can also propagate across the superconducting slab and emerge on the other side as an electron (resp. hole). Since electrons are paired, this mechanism actually describes the cotunneling of two electrons across the superconductor and is the building block of dEC and dCAR. As shown in Appendix
if the slab has perfect interfaces and its phase is set to zero, the probability for an electron of momentum $\pm k_F$ to tunnel through it is given by

$$t_S^\pm = \frac{e^{-l/\xi} \left(1 - a(\epsilon)^2\right)}{1 - e^{-2l/\xi a(\epsilon)^2}} e^{\pm ik_F l}$$

with $\xi = \xi_0/\sqrt{1 - \epsilon^2}$. The Andreev reflection amplitude is given by

$$r_S = a(\epsilon) \frac{(1 - e^{-2l/\xi})}{1 - e^{-2l/\xi a(\epsilon)^2}}$$

and satisfies $|r_S|^2 + |t_S|^2 = 1$ as expected from quasiparticle conservation. For $l \gg \xi_0$, one recovers the Andreev amplitude of a semi-infinite superconductor with $t_S \to 0$ and $r_S \to a(\epsilon)$ while in the opposite case $l \ll \xi_0$ Andreev reflections are suppressed such that all quasiparticles tunnel, $t_S \to 1$.

For imperfect interfaces, electrons (resp. holes) can also be back-scattered (Fig. 7(b), BS process) and crossed-Andreev reflected (Fig. 7(b), CAR process), which consists of tunneling through the superconductor and conversion into a hole (resp. electron) \([35]\). As depicted the CAR process for an electron incident from the left corresponds to first an Andreev reflection and then back-scattering of the retroreflected hole, which then exits toward the right. This mechanism can also be seen as the formation, in the central slab, of a Cooper pair comprised of electrons from both left and right electrodes. The time-reversed process is known as Cooper-pair splitting and has device applications \([35, 36]\). The CAR process does not conserve momentum and is forbidden unless there are imperfect interfaces which permit back-scattering. Indeed, if each interface has a transmission $\tau_0$, the first-order probability for CAR to happen is given by $\tau_0^2 (1 - \tau_0) |r_S t_S|$, which is equal to zero for perfect transmission. When left and right electrodes are superconducting, as in our case, the formation or splitting of two Cooper pairs in the central slab from Cooper pairs in the outer electrodes is called double-CAR.

In the multi-channel case, the effect of scattering on the left and right weak links can simply be described by their normal-state scattering matrices $S_{L(R)}$ (resp. $S_{L(R)}^*$) which relate incoming to outgoing electrons (resp. holes). For weak links containing $N$ conduction channels these matrices are of size $2N \times 2N$. It is convenient to introduce a single global matrix $S_N$, \[
S_N = \begin{pmatrix}
S_L & 0 & 0 & 0 \\
0 & S_R & 0 & 0 \\
0 & 0 & S_L^* & 0 \\
0 & 0 & 0 & S_R^*
\end{pmatrix},
\]
which accounts for scattering at both weak links. Similarly, scattering on superconductors can also be described by a single $8N \times 8N$ matrix

$$S_S = \begin{pmatrix}
S_{ee} & S_{eh} e^{-i\Phi} \\
S_{eh} e^{i\Phi} & S_{hh}
\end{pmatrix} \otimes I_N,$$

with blocks $S_{eh}$ on the anti-diagonal for Andreev reflections,

$$S_{eh} = \begin{pmatrix}
r_A & 0 & 0 & 0 \\
0 & r_S & 0 & 0 \\
0 & 0 & r_S & 0 \\
0 & 0 & 0 & r_A
\end{pmatrix},$$

and blocks $S_{ee}$ and $S_{hh}$ on the diagonal for tunneling through the central superconducting slab,

$$S_{ee} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & t_S & 0 \\
0 & 0 & 0 & t_S \\
0 & 0 & 0 & 0
\end{pmatrix},$$

$S_{hh}$ is obtained from $S_{ee}$ with the transformation $k_F \to -k_F$. The superconducting phases are contained in the diagonal matrix $\Phi = \text{diag}(\delta_L, 0, 0, \delta_R)$ and $I_N$ is the identity matrix of dimension $N \times N$, which accounts for all the conduction channels.

Since the left and right superconducting electrodes act as perfect phase-conjugating mirrors for electrons and holes, the device constitutes a cavity of infinite finesse \([38]\). As a consequence, stationary waves can form due to interference of propagating and counterpropagating states produced by normal and superconducting scattering. As shown in Appendix \(E\) one closed propagation cycle corresponds to the product $S_N S_S$, which means that a constructive interference occurs when \([39]\):

$$\text{det} (I_{8N} - S_N S_S) = 0.$$  

This characteristic equation provides the eigenenergies of the system as a function of the superconducting phases and the scattering properties of the weak links. Solving this equation numerically, we calculated spectra of Andreev molecules comprised of weak links having tens of channels (Appendix \(E\)). For the scattering matrices, we used randomly generated symmetric unitary matrices \([40, 42]\).

In Fig. 8(a) we represent an example of a multi-channel double weak link formed from a highly doped graphene nanoribbon \([43, 44]\). Spectra for various junction separations are shown in Fig. 8(b). If the separation is large, $l \gg \xi_0$, each conduction channel of each junction hosts one pair of ABS and as a consequence the top left spectrum shows additional lines compared to the single channel case. All lines obey the single channel energy Eq. \([1]\) for different values of transmission. For fixed $\delta_R = 3\pi/5$ and variable $\delta_L$, ABS of the right junction (red lines) sit at fixed energies while those of the left junction (blue lines) oscillate periodically within the gap. There is no hybridization so the lines cross without forming gaps. As the junctions are brought closer, multiple
avoided crossings materialize, signaling the formation of Andreev molecules (spectra for $l/\xi_0 = 1, 0.5, 0.1$ in Fig. 8(b)). Similarly to the one-channel case, the amplitude of the avoided crossings increases and discrete states are gradually pushed out into the continuum. At separation $l = 0.1\xi_0$, where the Andreev molecule fuses into a single weak link, only approximately half of the ABS remain in the gap and the states have shifted in phase to the right by $\delta_R = 3\pi/5$.

Overall the spectra of Fig. 8 for the multi-channel case show qualitatively the same behavior as for the Andreev molecule in the single channel case (Fig. 3). The most obvious sign of hybridization remains the breaking of symmetry about the phase $\delta_L = \pi$. Since there are often phase offsets in experiments it may be difficult to be certain that $\delta_L = \pi$. One could instead check for symmetry about the more easily identifiable point, $\delta_L^0 = \pi$, where the ABS are closest to zero in energy. The multi-channel spectra Fig. 8 indicate that the symmetry point $\delta_L^0$ shifts from $\pi$ to $\pi + \delta_R$ as the separation $l/\xi_0$ goes from infinity to zero and that symmetry is broken for $l \lesssim \xi_0$.

IV. EFFECT OF DISORDER ON ANDREEV MOLECULES

Andreev molecules emerge when ABS of different junctions interfere with each other and form a delocalized state. In mesoscopic systems, interference may be destroyed by disorder and one can legitimately wonder whether or not Andreev molecules can form in realistic devices where weak links are connected to thin film diffusive superconductors. Although the superconducting gap $\Delta$ remains largely unaffected by non-magnetic disorder, in the limit $l_0 \ll \xi_0$, where $l_0$ is the mean free path in the diffusive superconductor, the effective superconducting coherence length is reduced to $\sqrt{\xi_0 l_0}$. As a consequence, for an isolated junction, the spread of the wavefunction is reduced but the spectrum, which only depends on $\Delta$, remains unchanged. However in an Andreev molecule, disorder reduces the overlap of wavefunctions of adjacent junctions, hence degrading the probability of dEC and dCAR and changing the spectrum.

The one-dimensional approach we described so far is powerful to obtain a simple understanding of Andreev molecules but it does not describe the effect of disorder. In this section we present a tight-binding model for Josephson devices made with diffusive superconduc-
The first line corresponds to the on-site energy and the system is similar to Eq. 3 except for the kinetic part the superconducting coherence length. Moreover, this approach can easily generalize the concept of Andreev molecule in two-dimensional conditions. Note that $S$ chosen such that $k l / \xi \approx \l_0 / \xi$ each value of $l / \xi \approx \l_0$ and $l = \l_0$ (mod 2). Hybridization of Andreev states is indicated by the appearance of avoided crossings for $N = 20$ channels. Hybridization of Andreev molecules, where $u^j \psi_\up$ and $v^j \psi_\down$ are respectively the electron and hole components, are determined by numerical diagonalization of the tight-binding Hamiltonian (details are given in Appendix F). The total supercurrent through the left junction is then obtained by generalizing Eq. 2

$$I_L (\delta_L, \delta_R) = -\frac{1}{\phi_0} \sum_{\epsilon_k < 0} \frac{\partial \epsilon_k}{\partial \delta_L}$$

where we sum over all negative energy states including the continuum.

To reproduce the results of the preceding sections, we focus on a superconducting double quantum point contact (QPC), as shown in Fig. 8(a). This geometry also corresponds to back-to-back superconducting atomic contacts. We choose the width $W \sim \l_0$ of the weak links such that only one conduction channel is open in each QPC. Within the tight-binding approach, the disorder associated to the diffusive nature of superconductors can be described by giving a random Gaussian distribution to the hopping energies with a mean value $t_0$ and a standard deviation $\delta t$. The timescale over which an electron is scattered is then approximately $\hbar / \delta t$ and an order of magnitude estimate of the mean free path is $l_0 \sim t_0 \alpha / \delta t$ provided that this is smaller than the size of the device.

Fig. 8(b) shows the spectrum obtained for two QPCs separated by twenty sites, which is more than twice the effective superconducting coherence length $\sim 2.5 \sqrt{\xi_0}$. For varying $\delta_L$ and fixed $\delta_R = 3 \pi / 5$, we recover the main features that we observed in the one-dimensional case. We see the appearance of avoided crossings at $\delta_L = 3 \delta R$ between ABS that were originally localized at different junctions. In the absence of disorder (see Appendix F), the hybridization of ABS is revealed by the interference pattern in the wavefunction, which consists of circular waves of period $\l_0$ centered at the QPCs and decaying exponentially into the left and right superconductors. In between the QPCs, the circular waves overlap and interfere leading to parallel fringes demonstrating the hybridization of ABS into an Andreev molecule.

As we add disorder, this regular interference pattern is severely destroyed and the wavefunctions resemble a random speckle pattern. As expected, we also observe that the wavefunctions are more localized than for a ballistic

$$H_0 = \sum_{i,j} (4t - \mu_{ij}) |i, j\rangle \langle i, j| - t \sum_{\sigma = \pm 1} |i + \sigma, j\rangle \langle i, j| + |i, j + \sigma\rangle \langle i, j| \tag{7}$$

The first line corresponds to the on-site energy and the second one to the coupling between adjacent sites by a hopping energy $t = \hbar^2 / 2m^2 \alpha^2$. To describe electrons in continuous media, we require that the Fermi wavelength exceed the lattice spacing ($\l_0 \gg \alpha$), a condition satisfied for $\mu \leq t$. In the following, we set $\mu \approx t$ such that the wavelength and superconducting coherence length are given by $\l_0 \approx 2 \pi \alpha$ and $\xi_0 \approx 2 \pi \alpha / \pi \Delta$, respectively. Since $\alpha$ is the spatial resolution and $\xi_0$ is the spread of Andreev molecules, we choose parameters such that $\xi_0 \gg \l_0$. In the Hamiltonian, we have introduced an on-site chemical potential $\mu_{ij}$ to define the shape of the device. Sites on the boundary of superconductors and weak links with $\mu_{ij} \gg t$ and are therefore kept empty. The eigenenergies $E_k$ and wavefunctions $\psi (i, j) = (\psi^j_\psi, \psi^j_\down)^T$ of Andreev molecules, where $\psi^j_\psi$ and $\psi^j_\down$ are respectively the electron and hole components, are determined by numerical diagonalization of the tight-binding Hamiltonian (details are given in Appendix F). The total supercurrent through the left junction is then obtained by generalizing Eq. 2
Figure 9. Andreev molecule in a double superconducting quantum point contact. (a) Schematic of a device based on two point contacts (QPC) used as weak links between disordered superconducting electrodes (blue, magenta and red). In these electrodes, quasiparticles propagate in a diffusive way. (b) Spectrum of a two-dimensional Andreev molecule as a function of $\delta_L$ at fixed $\delta_R$ for a disordered device ($\delta t/t_0 = 0.2$). (c-d-e) Square modulus of the wave function for three different phases $\delta_L$ at fixed $\delta_R = 3\pi/5$ (see b). The red (resp. blue) dot corresponds to an electronic state localized on the right (resp. left) junction with $\delta_L = 0$ (resp. $\delta_L = \pi$). For the magenta point, hybridization is maximum ($\delta_L = -\delta_R$) and the electronic states are fully delocalized on the two junctions. Parameters used for calculations are either given in the figures or: $\alpha = 1, l = 20, t_0 = 10, \delta t = 0.2, \Delta = 0.5, \mu = 11$ and $W = 6$. This gives $\xi_0 \approx 12.7, l_0 \approx 5, \sqrt{\xi_0 l_0} \approx 8$ and $\lambda_F \approx 6.3$. The number of sites is $200 \times 200$. Separations between the left, right end central superconductors are indicated by dashed lines and colors along one axis. The weak links are small openings in the center of these separations (not shown, see Appendix F).

When the distance between QPCs becomes smaller than $\xi_0$ (Fig. 10), the ABS start to hybridize more strongly since the overlap of their wavefunctions becomes significant. In this regime, the Andreev molecule shows a large non-local Josephson effect. In Fig. 10(a), we show three current-phase relations of the left junction for three different superconducting phase differences across the right junction. As in the one-dimensional case, we observe the appearance of a zero-phase current, which modulates with $\delta_R$ and represents a significant fraction of the maximum supercurrent. Moreover it is clear in this comparison that the critical current of the left junction (i.e. the maximum of $I_L$) modulates with $\delta_R$. The same calculations performed for a ballistic sample (Fig. 10(b)) provides similar results, showing that disorder does not affect qualitatively the formation or behavior of an Andreev molecule.

In conclusion, as for ABS in a single junction, we ob-
serve that the main consequence of disorder on Andreev molecules is to effectively reduce the superconducting coherence length. This imposes the junctions to be close enough such that $l$ is smaller than the diffusive superconducting coherence length, but it does not induce qualitative changes and non-local effects are still substantial. Andreev molecules can thus be detected in devices made of diffusive aluminum since the diffusive superconducting coherence length is 150 nm \cite{45}, as compared to 3 $\mu$m in the ballistic limit. A Josephson device showing non-local effects could therefore be fabricated using conventional techniques of nanofabrication, which have a resolution down to 10 nm.

Figure 10. $\varphi$-junction in disordered two-dimensional devices. (a) Supercurrent $I_L$ flowing through the left junction as a function of $\delta_L$, for three different superconducting phase differences across the right junction $\delta_R$, in a two-dimensional disordered device ($\delta t/t_0 = 0.2$). We choose $l = 4\alpha$ such that $l \lesssim \sqrt{\xi_0 l_0}$. We observe an important 0-phase current $I_{L,0}$, which modulates with $\delta_R$. (b) Current-phase relation calculated for a ballistic device ($\delta t/t_0 = 0$). Here, We choose $l = 8\alpha$ such that $l \lesssim \xi_0$. We observe that non-local effects are qualitatively the same with or without disorder. For these calculations, we use the following parameters with $\alpha = 1$: $l = 4, t_0 = 10, \delta t = 0.2$ (for the diffusive case), $\Delta = 0.5, \mu = 11$ and $W = 6$. This gives $\xi_0 \approx 12.7, l_0 \approx 5, \sqrt{\xi_0 l_0} \approx 8$ (for the diffusive case) and $\lambda_F \approx 6.3$. The number of sites is $200 \times 200$. The supercurrent is obtained from the highest 1000 eigenstates of negative energies found numerically. They include the ABS of negative energies, from zero down to approximately $-3\Delta$.

V. EXPERIMENTAL PROPOSALS TO DETECT ANDREEV MOLECULES

The two main signatures of Andreev molecules at equilibrium are the spectral avoided crossings, which form when ABS hybridize through non-local processes, and the non-local current-phase relations. The non-equilibrium case, in which the junction voltages are non-zero, was considered by \cite{5} and have been investigated experimentally \cite{11,12}. The interpretation of their results is complicated as dynamic phenomena such as mutual phase locking \cite{46} can lead to similar signatures. In contrast the equilibrium experiments we detail below should provide an unambiguous detection of Andreev molecules.

In Fig. 11(a), we propose a setup to perform Josephson spectroscopy of an Andreev molecule in a Josephson device with two junctions (black crosses connected to red, magenta and blue wires forming two loops). A magnetic field $B$ induces a flux $\Phi$ in both of the loops of the device. For independent control of $\varphi_L$ and $\varphi_R$, a gradiometric line with control current $I_d$ (on the right) induces an additional flux coupled only to the left junction loop. (b) Setup for measurement of the current-phase relation of a device hosting Andreev molecules. The fluxes in both loops are controlled independently as in (a). The functioning of these two experiments are described in the text and given references.

Figure 11. Detection of Andreev molecules. (a) Setup to perform Josephson spectroscopy of an Andreev molecule in a Josephson device with two junctions (black crosses connected to red, magenta and blue wires forming two loops). A magnetic field $B$ induces a flux $\Phi$ in both of the loops of the device. For independent control of $\varphi_L$ and $\varphi_R$, a gradiometric line with control current $I_d$ (on the right) induces an additional flux coupled only to the left junction loop. (b) Setup for measurement of the current-phase relation of a device hosting Andreev molecules. The fluxes in both loops are controlled independently as in (a). The functioning of these two experiments are described in the text and given references.
corresponding to the transition energies $E_T$ of the Andreev molecule. The magnetic field $B$ and gradometric control current $I_d$ allow independently tuning the phases $\delta_L$ and $\delta_R$. A full measurement $I_d(V_J, B, I_d)$ would allow reconstructing the spectra of Fig. 3. In addition, assuming that the loops have the same area $A$ and their geometric inductances are negligible, by fixing $I_d = 0$ and varying $B$ so that $\delta_L = -\delta_R = B A/\phi_0$ one can directly probe the dEC transitions. Likewise by fixing $B = 0$ and varying $I_d$ such that $\delta_L = \delta_R \propto I_d$ one can probe dCAR transitions and thereby obtain the hybridization spectrum of Fig. 3(b).

Josephson spectroscopy is well suited for detecting Andreev molecules, as $E_T$ may be comparable to $\Delta/h$ (90 GHz for aluminum), well within the spectrometer bandwidth. Conventional microwave spectroscopy using a resonator coupled to the Andreev molecule is well suited for small hybridization gaps less than 18 GHz, for example when transmissions are low or the separation is large.

Further theoretical work is necessary to determine the selection rules for the transitions. By design, as only the $L$ junction is inside the spectrometer high-frequency loop, emitted photons cannot directly excite the $R$ junction. This implies that only vertical transitions in Fig. 3 are possible. However as the symmetry of the BdG Hamiltonian is different for dEC and dCAR, it is not evident that the selection rules are the same. Since the conventional ABS Hamiltonian has the same symmetry as the dEC Hamiltonian, at least the transitions at $\delta_R = -\delta_L$ should be optically active.

In a second experiment, the existence of a non-local current-phase relation can be directly determined by measurements of the switching current. This type of experiment has been performed on superconducting atomic contacts [47], graphene [48] and carbon nanotubes [49]. For this measurement (Fig. 11[b]), a large Josephson junction of critical current $I_0 \gg \text{max}(I_L, I_R)$ is wired in parallel with one of the junctions ($L$) hosting the Andreev molecule, forming an asymmetric SQUID. The $R$ junction is shorted in a loop enclosing an applied magnetic flux $\varphi_R$ such that its superconducting phase difference is $\delta_R = \varphi_R$. A differential flux line combined with a global applied flux $\Phi = \varphi_0/2$ allows independent tuning of $\delta_L$ and $\delta_R$. Due to the asymmetry, the SQUID critical current $I_{SQ}(\varphi, \varphi_R)$ is essentially given by that of the large junction, effectively at a phase difference of $\pi/2$, modulated by the supercurrent of the $L$ junction, $I_{SQ}(\varphi, \varphi_R) \approx I_0 + I_L(\varphi - \pi/2, \varphi_R)$, where flux quantization constrains $\delta_L = \varphi - \pi/2$. By sending current pulses or ramps and measuring the switching of the SQUID to a non-zero voltage state, one can extract the current-phase relation $I_L(\varphi - \pi/2, \varphi_R)$ and demonstrate that there is a non-local component that depends on $\delta_R$.

VI. CONCLUSION AND OUTLOOK

We have shown that two superconducting weak links separated by a distance on the order of the superconducting coherence length exhibit a non-local Josephson effect, a phenomena attributed to the hybridization of Andreev Bound States between the two weak links and the formation of an Andreev molecule. Our work provides a microscopic understanding of the “order-parameter,” or Cooper-pair mediated, interaction between Josephson junctions. The most prominent signatures of non-local effects are a modification of the Andreev Bound State energy spectrum and the emergence of a $\varphi$-junction, a weak link with non-zero supercurrent at zero phase difference. We propose realistic, technically feasible experiments to detect these signatures.

Modifications to the spectrum can be measured by Josephson junction spectroscopy or conventional microwave techniques, and current switching experiments allow determining the current-phase relation and demonstrating $\varphi$-junction behavior.

Andreev molecules, with three terminals, multiple tunable energy levels, and an unusual $\varphi$-junction effect, may have applications in devices for quantum information, metrology, or magnetometry. The consequences of coupling Andreev bound states by bringing them in close proximity is well demonstrated in the Andreev molecule, but the idea is more general and can be applied to more than two junctions. It is possible to conceive of Andreev “polymers”, a chain of closely spaced junctions such that Andreev bound states hybridize over the whole length. The spectrum of these artificial polymers could be determined by extending the calculations presented here. By controlling the spacing between junctions, one could potentially simulate specific molecules such as polyacetylene (the Su-Schrieffer-Heeger model) with interesting, possibly topological, properties.

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Appendix A: Discrete spectrum of Andreev molecules $|E| < \Delta$

In this Appendix, we give more details on how to obtain the discrete spectrum of Andreev molecules for energies smaller than the superconducting gap $|E| < \Delta$. As explained in the main text, in order to find the discrete spectrum of Andreev molecules, we build wave functions from the eigenstates of an infinite superconductor, which are plane waves of electron or hole type ($\eta = e$ or $h$)

$$\psi_{\eta \pm}^\delta(x) = \frac{1}{\sqrt{L}} \left( u_\eta^\delta, v_\eta^\delta \right)^T e^{\pm ik_\eta x}$$

where $L$ is the total size of the system and we have defined coherence factors

$$u_\eta^\delta = \frac{e^{-i\delta/2}}{\sqrt{2}} \left( 1 \pm \sqrt{1 - \epsilon^{-2}} \right)^{1/2}$$

$$v_\eta^\delta = \text{sign}(\epsilon) \times \frac{e^{i\delta/2}}{\sqrt{2}} \left( 1 \pm \sqrt{1 - \epsilon^{-2}} \right)^{1/2}$$

where $k_\eta(h)$ are complex momenta. If the superconducting gap is much smaller than $E_F$, they can be approximated as $k_\eta(h) \approx k_F \pm i/\xi$ where $k_F$ is the Fermi momentum in the normal state and the coherence length is a function of energy $\xi^{-1} = \xi_0^{-1} \sqrt{1 - \epsilon^{-2}} \ll k_F$. Here $\xi_0 = h v_F/\Delta$ is the bare superconducting coherence length, $v_F$ is the Fermi velocity and $\epsilon = E/\Delta$ is the normalized energy. The wave-functions of Andreev molecules are defined piecewise in the following way

$$\psi(x) = \begin{cases} 
  l_c \psi_c^-(x) + l_h \psi_h^+(x) & \text{if } x < -l/2 \\
  c_e^+ \psi_e^+(x) + c_e^- \psi_e^-(x) & \text{if } -l/2 < x < l/2 \\
  c_h^+ \psi_h^+(x) + c_h^- \psi_h^-(x) + r_c \psi_c^+(x) + r_h \psi_h^-(x) & \text{if } l/2 < x
\end{cases}$$

where we exclude diverging spinors on the left and the right and we introduce two sets of four coefficients $\Psi_e = (l_c, r_c, c_e^+, c_e^-) \ T$ and $\Psi_h = (l_h, r_h, c_h^+, c_h^-) \ T$ giving the respective weight of each component ($l$ for left, $r$ for right and $c$ for the center).

These wave-functions must be continuous at $x = \pm l/2$ in order to avoid unphysical sharp variation of the electronic density. This continuity can be expressed in the following way

$$\psi(\pm l/2^-) = \psi(\pm l/2^+)$$

where the superscripts $\pm$ mean slightly lower or larger than $\pm l/2$. Similarly, the Bogoliubov-de Gennes equation imposes boundary conditions on the derivative of $\psi$

$$\partial_x \psi(\pm l/2^+) - \partial_x \psi(\pm l/2^-) = \frac{2m}{\hbar^2} U_{R(L)} \psi(\pm l/2)$$

due to the $\delta$ Dirac function that we use to model scattering on the weak links. For the term on the right, we choose the definition

$$\psi(\pm l/2) = [\psi(\pm l/2^+) + \psi(\pm l/2^+)]/2$$

As said in the main text, these two conditions give eight equations for eight unknown coefficients

$$\Psi_e = a(\epsilon) A \Psi_h \quad \text{and} \quad \Psi_h = a(\epsilon) A^* \Psi_e$$  \hspace{1cm} (A1)

with $a(\epsilon) = e^{-i \arccos \epsilon}$ the probability amplitude for an electron to be Andreev reflected into a hole at the interface between a normal metal and superconductor. The matrix $A(\Phi) = e^{-i \Phi} M^{-1} M^* e^{i \Phi}$ depends on superconducting phase differences contained in the diagonal matrix $\Phi = \text{diag} (\delta L/2, \delta R/2, 0, 0)$ and scattering amplitude at the weak links through matrix $M$

$$M = e^{i k_F l/2} \begin{pmatrix} 
  i & 0 & -ie^{-ik_c l} & -i \\
  0 & i & -i & -ie^{-ik_c l} \\
  -u_L & 0 & -u_L e^{-ik_c l} & u_L^* \\
  0 & u_R & -u_R^* & u_R e^{-ik_c l}
\end{pmatrix}$$

with reduced scattering potentials

$$u_{L(R)} = 1 + i U_{L(R)}/\hbar v_F,$$

which are directly related to the transmissions of each junction $\tau_{L(R)} = 1/|u_{L(R)}|^2$. Combining the two parts of Eq. (A1) we have non trivial solutions only if

$$\det \left( 1 - a(\epsilon)^2 A(\Phi) A^{-1} (-\Phi) \right) = 0.$$  \hspace{1cm} (A2)

This is equivalent to find energies for which

$$a(\epsilon)^2 \lambda_A = 1,$$

where $\lambda_A$ are eigenvalues of $A(\Phi) A^{-1} (-\Phi)$.

Writing equation $\det (A(\Phi) - \lambda_A A(-\Phi)) = 0$, we can show that $\lambda_A$ are roots of a symmetric polynomial

$$\alpha_0 \lambda_A^4 + \alpha_1 \lambda_A^3 + \alpha_2 \lambda_A^2 + \alpha_3 \lambda_A + \alpha_4 = 0$$

with coefficients given by

$$\alpha_0 = \left| U_{L} U_{R} e^{-i 2k_F l} - e^{-2 \frac{2i}{\hbar} (U_{L} - 1) (U_{R} - 1)} \right|^2 \quad \text{and} \quad \alpha_1 = -4 \alpha_0 + \epsilon_1$$

$$\epsilon_1 = 4 e^{-2 \frac{2i}{\hbar} \sin^2 \left( \frac{\delta_R - \delta_L}{2} \right)} \left[ \left( 1 - e^{-2 \frac{2i}{\hbar}} \right) (U_{L} - 1)^2 + \left( e^{-2 \frac{2i}{\hbar}} - 1 \right) \right] \sin^2 \left( \frac{\delta_R - \delta_L}{2} \right) + 4 \left( 1 - e^{-2 \frac{2i}{\hbar}} \right) (U_{R} - 1)^2 + \left( e^{-2 \frac{2i}{\hbar}} - 1 \right) \sin^2 \left( \frac{\delta_R - \delta_L}{2} \right)$$

and

$$\left\{ \begin{array}{l}
  \alpha_2 = -2 (\alpha_1 + \alpha_0 - \epsilon_2) \\
  \epsilon_2 = 8 \left( e^{-2 \frac{2i}{\hbar}} - 1 \right)^2 \sin^2 \left( \frac{\delta_R - \delta_L}{2} \right) \sin^2 \left( \frac{\delta_R + \delta_L}{2} \right)
\end{array} \right.$$
The four solutions for such polynomial are known and take the following form
\[\lambda_A^{\sigma\sigma'} = 1 - \frac{1}{4a_0} \left( \epsilon_1 + \sigma \sqrt{c_1^2 - 8a_0\epsilon_2} \right)\]
\[-\sigma' \sqrt{2} \left( \epsilon_1 - 4a_0 \right) \left( \epsilon_1 + \sigma \sqrt{c_1^2 - 8a_0\epsilon_2} \right) - 4\epsilon_2a_0\]
with \(\sigma = \pm\) and \(\sigma' = \pm\).

The eigenenergies of the Andreev molecule can be deduced using the relations
\[\lambda_A^{\sigma\sigma'} \times a(\epsilon)^2 = 1\]
or equivalently
\[\epsilon = \frac{\sqrt{\Re \left( \lambda_A^{\sigma\sigma'} \right)} + 1}{2}\]

In the general case, energies satisfying this equation can only be found numerically because coefficients \(\lambda_A^{\sigma\sigma'}\) depend on energy in a non algebraic way. It is however useful to obtain analytical results in particular cases such as infinitely close \((l \approx 0)\) of far junction \((l \rightarrow \infty)\), and we will see in one of the following appendices that we can use it to obtain analytical formula for the Andreev molecule spectrum of two closely spaced tunnel junctions.

From the solutions of Eq. \(A2\) we generally obtain up to four eigenenergies \(E_{ABS}\), from which we can deduce the contribution of ABS to the supercurrent flowing through the left junction
\[I_L^{ABS} = -\frac{1}{\varphi_0} \sum_{E_{ABS} < 0} \frac{\partial E_{ABS}}{\partial \delta_L}\]

In an Andreev molecule, the energies \(E_{ABS}\) depend both on \(\delta_L\) and \(\delta_R\), which leads to non-local phase dependencies of the supercurrent.

Appendix B: Spectral asymmetry in junctions with perfect transmission

In the general case, Eq. \(B1\) does not have simple analytical solutions and it requires a numerical resolution to find Andreev molecule spectra. However, in the limit of large transmission \(\tau_0 \rightarrow 1\) (or equivalently \(u_{L(R)} = 1\)), it simplifies into two compact transcendental equations
\[e^{-\frac{i}{\hbar} \sin \beta / 2} \sin \frac{\delta_L}{2} \sin \frac{\delta_R}{2} = \sin \left( \frac{\delta_L}{2} \pm \beta \right) \sin \left( \frac{\delta_R}{2} \pm \beta \right)\]
where \(\beta = -\arccos \epsilon\) and \(\epsilon = E/\Delta\), from which one can deduce interesting spectral features. For example, we can recover spectra for well known particular cases. If the junctions are infinitely far \((l \rightarrow \infty)\), the left-hand side cancels and we get \(\epsilon = \pm \cos \frac{\delta_{L(R)}}{2}\), which are the usual ABS for two independent perfectly transmitted junctions. Similarly, if the junctions are merged into a single one \((l = 0)\), we recover the spectrum \(\epsilon = \pm \cos (\delta_L - \delta_R)\) of a single junction with a phase drop \(\delta_L - \delta_R\).

Eq. \(B1\) can also yield interesting prediction about Andreev molecules in the general case such as the spectral asymmetry that we mention in the main part of this manuscript. As shown by Eq. \(B1\) For a single Josephson junction, the spectrum is symmetric with respect to \(\delta = \pi\) and \(\epsilon = 0\) and ABS cross at this point for perfect transmission \((\tau_0 = 1)\) such that they become degenerate. For two closely spaced junctions however, this symmetry is broken and the degeneracy point is shifted along the phase axis. If the phase \(\delta_R\) across the right junction is fixed, ABS become degenerate at \(\epsilon = 0\) \((i.e. \beta = -\pi/2)\) and
\[\delta_L = \pi - 2 \arctan \left[ e^{-2l/\xi_0} \tan \frac{\delta_R}{2} \right]\]

Due to ABS hybridization, the degeneracy point shifts from \(\pi\) of \(\varphi \approx 2 \arctan \left[ e^{-2l/\xi_0} \tan \frac{\delta_R}{2} \right]\), which makes the spectrum asymmetric. We notice that this deviation is significant only if two junctions are sufficiently close \((l \approx \xi_0)\) but decreases exponentially when \(l\) becomes large compared to \(\xi_0\).

Appendix C: Eigenstates and current carried by the continuum \(|E| > \Delta\)

The continuum of Andreev molecules is obtained with a similar approach as for the subgap states but with energies larger than the superconducting gap \(|E| > \Delta\). We look for wavefunctions \(\psi_E\) that are continuous along the \(x\)-axis and obey the Bogoliubov-De Gennes (BDG) equation \(H\psi_E = E\psi_E\), where \(E\) is the eigenenergy and \(H\) is given by Eq. \(B3\). They are built from the eigenstates of an infinite superconductor, which are plane waves of electron or hole type \((\eta = e\) or \(h)\)
\[\psi_E^{\delta \eta}(x) = \frac{1}{\sqrt{L}} (u^{\delta \eta}_{1}, u^{\delta \eta}_{2}) e^{\pm ik_{\eta} x}\]

where \(L\) is the total size of the system and we have defined coherence factors
\[\psi^{\delta \eta}(h) = e^{-i\varepsilon/2} \frac{(1 \pm \sqrt{1 - \varepsilon^2})^{1/2}}{\sqrt{2}}\]
\[\psi^{\delta \eta}(e) = \text{sign}(\varepsilon) \times e^{i\varepsilon/2} \frac{(1 \pm \sqrt{1 - \varepsilon^2})^{1/2}}{\sqrt{2}}\]
Assuming \(\Delta \ll E_F\), we can make the approximation \(k_{\eta}(h) \approx k_F \pm 1/\xi\) where \(k_F\) is the Fermi momentum in the normal state and the coherence length is a function of energy \(\xi^{-1} = \xi_0^{-1} \sqrt{1 - \varepsilon^2} \ll k_F\). Here \(\xi_0 = h v_F/\Delta\) is the bare superconducting coherence length, \(v_F\) is the Fermi velocity and \(\varepsilon = E/\Delta\) is the normalized energy.

Since momenta \(k_{\eta}\) are real, there is no bound states but only propagating solutions called scattering states. They are superpositions of an incoming wave and an outgoing wave resulting from scattering at the weak links
\[\psi^{\eta \pm}_E = \psi^{\eta \pm}_{\text{inc}}(x) + \psi^{\eta \pm}_{\text{out}}(x)\]
with four possible types of incoming wave (here Θ is the Heaviside step function)

\[
\begin{align*}
\psi_{\text{inc}}^+ (x) &= \psi_{\text{inc}}^{\delta L} (x) \Theta [-x - l/2] \\
\psi_{\text{inc}}^- (x) &= \psi_{\text{inc}}^{\delta R} (x) \Theta [x - l/2] \\
\psi_{\text{inc}}^h+ (x) &= \psi_{\text{inc}}^{R+} (x) \Theta [x - l/2] \\
\psi_{\text{inc}}^h- (x) &= \psi_{\text{inc}}^{R-} (x) \Theta [-x - l/2]
\end{align*}
\]

and resulting outgoing wave

\[
\psi_{\text{out}}^\eta (x) = \begin{cases} 
\epsilon_1^{\eta \pm} \psi_{\text{inc}}^{\delta L} (x) + \epsilon_2^{\eta \pm} \psi_{\text{inc}}^{R+} (x) & \text{for } x < -\frac{l}{2} \\
\epsilon_3^{\eta \pm} \psi_{\text{inc}}^{\delta L} (x) + \epsilon_4^{\eta \pm} \psi_{\text{inc}}^{R+} (x) + \epsilon_5^{\eta \pm} \psi_{\text{inc}}^{\delta L} (x) + \epsilon_6^{\eta \pm} \psi_{\text{inc}}^{R+} (x) & \text{for } |x| < \frac{l}{2} \\
\epsilon_7^{\eta \pm} \psi_{\text{inc}}^{\delta L} (x) + \epsilon_8^{\eta \pm} \psi_{\text{inc}}^{R+} (x) & \text{for } x > \frac{l}{2}
\end{cases}
\]

For each type of incoming wave, we have a set of eight coefficients giving the respective weight on the left, right and in the center of the device for the outgoing wave. The continuity of the wavefunctions and the conditions imposed on their derivatives by the BDG equation [50] give the relations

\[
\begin{align*}
\Psi_{E}^\eta &= -M_{E}^{-1} \Psi_{\text{inc}}^\eta \\
M_{E} &= \begin{pmatrix} 
m_{L} m_{R}^+ & -m_{L}^+ & 0 \\
0 & -m_{L} & m_{L} m_{R}^+ \\
0 & m_{L} m_{R}^+ & m_{L} m_{R}^+ & 0 \\
0 & 0 & m_{L} m_{R}^+ & m_{L} m_{R}^+
\end{pmatrix}
\end{align*}
\]

and we define three submatrices depending on the parameters of the device

\[
m_{L(R)} = \begin{pmatrix} 
u_{L(R)} & 0 \\
0 & -u_{L(R)}^*
\end{pmatrix}
\]

where \(M_{E}\) is a 8 \times 8 matrix given by (02 are 2 \times 2 matrix of zeros)

\[
M_{E} = \begin{pmatrix} 
m_{L} m_{R}^+ & -m_{L}^+ & 0 \\
0 & -m_{L} & m_{L} m_{R}^+ \\
0 & m_{L} m_{R}^+ & m_{L} m_{R}^+ & 0 \\
0 & 0 & m_{L} m_{R}^+ & m_{L} m_{R}^+
\end{pmatrix}
\]

and we define three submatrices depending on the parameters of the device

\[
m_{L(R)} = \begin{pmatrix} 
u_{L(R)} & 0 \\
0 & -u_{L(R)}^*
\end{pmatrix}
\]

and

\[
m_{L} m_{R}^+ = \begin{pmatrix} 
\epsilon_{L}^{\eta \pm} & a_{\eta \pm} & 0 \\
0 & 0 & -u_{L} a_{\eta \pm}
\end{pmatrix}
\]

with \(a_{\eta} = \text{sign}(\epsilon) |\epsilon - \sqrt{\epsilon^2 - 1}|\) the probability amplitude for an electron of normalized energy \(\epsilon\) to be Andreev reflected at the interface between a normal metal and a superconductor. This gives then four degenerate states for each energy \(E\).

The solutions obtained from Eq. C1 with sets of coefficients C2 can therefore be written in the following way

\[
\psi_{E}^\eta (x) = \frac{1}{V_{E}^\eta (x)} \begin{pmatrix} 
U_{E}^\eta (x) \\
V_{E}^\eta (x)
\end{pmatrix}
\]

where \(U_{E}^\eta\) is the electron part and \(V_{E}^\eta\) is the hole part of the wavefunctions. According to Blonder, Tinkham and Klapwijk [50], such a quasiparticle state carries a charge current given by

\[
I_{E}^{\eta \sigma} (x) = e \frac{\hbar}{m} \text{ Im} \left[ U_{E}^{\eta \sigma} (x)^* \partial_x V_{E}^{\eta \sigma} (x) + V_{E}^{\eta \sigma} (x)^* \partial_x U_{E}^{\eta \sigma} (x) \right]
\]

The total current is then given by summing over all occupied states at \(x = -l/2\)

\[
I_{L}^{\text{cont}} = \sum_{\eta, \sigma} \sum_{|k| < k_{F}} \left| I_{E}^{\eta \sigma} \right| \left( \frac{1}{2} \right)
\]

We substitute the sum by an integral \(\sum_{n} \rightarrow \int_{k} dk \times L/2\pi\) and sum over energy rather than momentum by making use of \(d\xi / dk \approx h^2 k_{F} / m\) and \(d\xi / dE = E / \sqrt{E^2 - \Delta^2}\).
where $\xi_0$ is the kinetic energy. The current carried by the continuum through the left junction is therefore given by

$$I_{L}^{\text{cont}} = \frac{\Delta}{\varphi_0} \sum_{n}\int_{-\Delta}^{\Delta} \text{Im} \left[ U^{\sigma}_{E} \left(\frac{1}{2}\right) \frac{1}{\tau} \partial_{x} U^{\sigma}_{E} \left(\frac{1}{2}\right) + V^{\sigma}_{E} \left(\frac{1}{2}\right) \frac{1}{\tau} \partial_{x} V^{\sigma}_{E} \left(\frac{1}{2}\right) \right] \frac{dE}{2\pi k_F}$$

where $\varphi_0 = 2e/h$ is the flux quantum. The current through the right junction $I_R$ is obtained similarly by taking $x = -1/2$.

**Appendix D: Andreev molecules in tunnel Josephson junctions**

1. **Spectrum for small transmission $\tau_0 \ll 1$**

A wide variety of superconducting devices including most superconducting qubits are based on tunnel Josephson junctions. In that case, ABS are very close to the edge of the superconducting gap ($\epsilon \sim 1$) such that their wave-functions extend over a large distance ($\xi_0/\xi \ll 1$). For devices made, for example, with aluminum, the distance $l$ between the junction is typically comparable to the bare superconducting coherence length $\xi_0$ and the overlap of ABS wave-functions $e^{-2l/\xi} \sim 1$ becomes important (Fig. 12(a)). One could thus naively expect a strong hybridization of their ABS leading to spectacular non-local effects. However, for small transmission $\tau_0$, non-local microscopic mechanisms (dEC and dCAR) are very unlikely as they require double tunneling of Cooper pairs through the barriers with a probability proportional to $\tau_0^2$. This competes with local tunneling of Cooper pairs (EC) whose probability is proportional to $\tau_0 (1 - e^{-2l/\xi})$ where the second factor corresponds to the portion of the ABS wave-functions in the central superconductor. As a consequence, local events will generally dominate unless wave-functions spread over a very large distance $\xi \sim 1/\tau_0$. In standard devices, the distance between tunnel junctions is a few $\mu$m and the transmissions are around $\tau_0 \sim 10^{-6}$ (for $1 \mu$A, $\mu$m$^{-1}$). Wave-functions would thus have thus to extent over more than a meter in order to fulfill this condition, which suggests that non-local effects are weak in these devices and one expects to observe standard Josephson effect.

This reasoning does not take into account the fact that the tunnel barriers form a Fabry-Perot cavity, which leads to interference that drastically affects the transmission. This interference results in the formation of normal discrete electronic levels in the central part of the device that will serve as channels for local and non-local microscopic events to happen. As we will see in this appendix, one can distinguish “off” or “on” resonance conditions, depending on the value of $k_F l$, for which the transmission is respectively reduced or increased and the Josephson effect, including non-local processes, is modified.

One advantage of the tunnel limit is that it provides compact analytical expression of the Andreev spectrum giving insight in the mechanism leading to ABS hybridization. In the tunnel limit $(l/\xi \ll 1)$ in Eq. A3 leads to the following ABS spectrum

$$\epsilon^{\pm} \approx \pm \sqrt{1 - \tau(\epsilon^{\pm}) F^{\pm}(\delta_L, \delta_R)}$$ (D1)

where energy has been normalized by the superconducting gap $\Delta$. This expression is an approximation, which remains correct if $\sin^2(k_F l) \neq 0$ or if the junctions are not too close $l/\xi \gg \tau_0$. Here, $\tau(\epsilon)$ is an energy-dependent effective transmission and $F^{\pm}$ are combinations of trigonometric functions of $\delta_{L(R)}$

$$F^{\pm}(\delta_L, \delta_R) = F_1 \mp \sqrt{F_2^2 - F_1^2}$$

with

$$\begin{align*}
F_1 &= s_L^2 + s_R^2 + \frac{\delta_L}{\pi} s_{LR}^2 \\
F_2 &= 4 \sin^2(k_F l) s_L^2 s_R^2
\end{align*}$$

where we have introduced compact notations

$$\begin{align*}
s_L &= \sin \left(\frac{\delta_L}{2}\right) \\
s_R &= \sin \left(\frac{\delta_R}{2}\right) \\
s_{LR} &= \sin \left(\frac{\delta_L - \delta_R}{2}\right)
\end{align*}$$

Intuitively, one can see that combinations of $s_{L(R)}$ and $s_{LR}$ will result in contributions proportional to $\sin(\delta_L \mp \delta_R)$, emerge respectively from dEC and dCAR and others to $\sin^2 \delta_{L(R)}$ linked to local processes. The secular Eq. D1 reminds of the energy spectrum of ABS (see Eq. 1) in a single junction except that the apparent transmission $\tau(\epsilon)$ depends on $\xi$, and therefore on the energy, and momentum $k_F$. At first order in $l/\xi$, this transmission is given by

$$\tau(\epsilon) \approx \frac{\tau_0}{2} \frac{l/\xi}{(l/\xi)^2 + \sin^2(k_F l)}$$ (D2)

This function of $k_F l$ peaks and becomes larger than the bare transmission of the junctions $\tau_0$ when $k_F l = 0$ (mod$\pi$) but still with $\tau(\epsilon) < 1$ thanks to the condition $l/\xi \gg \tau_0$. This results in periodic Lorentzian shaped peaks of width $l/\xi$. This width is reminiscent of the decay $e^{-l/\xi}$ of quasiparticles propagating between the tunnel barriers in the central part of the device such that interference gradually disappears as $l$ tends to infinity. For other values of $k_F l$, $\tau(\epsilon)$ becomes much smaller. These variations of the transmission cause ABS to respectively go deeper into the superconducting gap or closer to its edge, which makes $\xi$ vary by orders of magnitude. At the same time, the probabilities for local and non-local processes to happen are also affected as they depend on how ABS wave-functions spread over the different parts of the device. This incidentally changes the nature of the Josephson effect as we will subsequently demonstrate.
2. Off resonance ballistic devices

Interference as described above would only be observable if quasiparticles could propagate ballistically in the device and preserve their phases. In hypothetical devices made with ballistic superconductors, effects of ABS hybridization through dEC and dCAR can have spectacular consequences on the properties of the circuits whether it is resonant or not.

Off resonance ($\sin^2(k_Fl) \approx 1$, Fig. 12(b)), the transmission can be approximated as $\tau \approx \tau_{\text{off}}/\sqrt{1 - e^2}$ with $\tau_{\text{off}} = \tau_0/\sqrt{2}\xi_0$ and becomes much smaller than the bare transmissions of the junctions since $e \approx 1$. Under this condition, Eq. D1 at first order in $l/\xi$ leads to the following ABS energies

$$\epsilon_{\text{off}}^{\pm} = \pm \sqrt{1 - \tau_{\text{off}}^2} \left[ s_{L}^2 + s_{R}^2 \mp \sqrt{(s_{L}^2 - s_{R}^2)^2 + \frac{\xi_0^2}{l^2} s_{LR}^2} \right] 2$$

This is very different from the standard ABS spectrum of a single junction (Eq. 1). First, energies depend on two superconducting phase differences $\delta_{L(R)}$ rather than a single one. Moreover the amplitude of ABS oscillation with these phases has an amplitude proportional to $s_{LR}$ rather than $\tau_0$, which drastically suppresses the supercurrent carried by ABS. This is due to destructive interference in the Fabry-Perot resonator formed by the two barriers, and this suppression becomes more efficient as the junctions are placed closer to each other.

On top of these interference effects, the formation of an Andreev molecule, due to hybridization of ABS, materializes in the term proportional to $s_{LR}$, due to dEC, with an amplitude that slowly tends to zero when junctions are far from each other compared to $\xi_0$. As shown in Fig. 12 it manifests in the spectrum as avoided crossings between ABS of the left and right junctions at $\delta_{L} = -\delta_{R}$. Remarkably, the avoided crossing is much less pronounced at $\delta_{L} = \delta_{R}$ and even cancels at first order in $l/\xi$ (Fig. 12(c)). This is because dCAR is strongly suppressed by destructive interference in the central part of the device, while dEC involving a direct tunneling between the left and right part remains important. Similarly to the case of junctions with large transmissions described before, for a fixed $\delta_{R}$, this introduces large asymmetry with respect to $\delta_{L} = \pi$ in the spectra. Note that here we can recover the spectrum of a single junction by taking $l \rightarrow 0$.

The contribution of ABS to the Josephson energy is given by the sum of ABS of negative energies $E_{\text{ABS}} = \Delta \left( \epsilon_{\text{off}}^{\pm} + \epsilon_{\text{off}}^{-} \right)$ and, since $\tau_{\text{off}} \ll 1$, it can be approximated as

$$E_{\text{ABS}} \approx \frac{\Delta}{2} \tau_{\text{off}}^2 \sum_{i=L,R} (1 - \cos \delta_i)^2 + 2 \frac{\xi_0^2}{l^2} s_{LR}$$

where we have discarded a constant term. This expression has the same $2\pi$-periodicity with $\delta_{L(R)}$ than the Josephson energy of a single junction ($E_J \propto \cos \delta$) but also additional harmonics with a $\pi$-periodicity. As a consequence, the current carried by ABS deviates from the usual Josephson relation ($I_J = I_c \sin \delta$)

$$I_{L(R)}^{\text{ABS}} = I_c^{\text{off}} \left[ 4s_{L(R)}^2 \epsilon_{\text{off}}^{\pm} \sin \delta_{L(R)} \right] \frac{\frac{\xi_0^2}{l^2} \sin (\delta_L - \delta_R)}{2} \right] \tag{D3}$$

The off resonance critical current $I_c^{\text{off}} = \Delta \epsilon_{\text{off}}^{\pm} / 2\phi_0$ is much smaller than the bare critical current of individual Josephson junctions $I_c = \Delta \tau_0 / 4\phi_0$ since $\tau_{\text{off}} \ll \tau_0$ and would be therefore only detectable for moderately low transmissions, for example $\tau_0 \approx 0.1$. The second term in Eq. D3 is of non-local nature and is the consequence of ABS hybridization between the junctions. We see that if one junction is polarized with a finite phase, for example with $\delta_R \neq 0$, the ABS will carry a supercurrent through the other junction even when no phase is applied across it (Fig. 12(d)). Maintaining $\delta_{L} = 0$, ABS have thus a perfectly non-local behavior with a 0-phase current $I_{L(R)}^{\text{ABS}}(0, \delta_{R}) = -I_c^{\text{off}} \epsilon_{\text{off}}^{-} / 4\xi_0 \sin (\delta_{R})$. This behavior is however extremely hard to detect experimentally because the continuum is carrying a much larger supercurrent than the ABS. Numerical calculations show us that the total current–phase relation is the same as the one of a single bare junction up to a factor (Fig. 12(e)). In order to see the formation of Andreev molecules, we would need to be able to measure selectively the current carried by ABS and the continuum.

3. On resonance ballistic devices

The behavior of the device is completely different in the resonant case (Fig. 12(f)). If $\sin^2(k_Fl) \approx 0$, the transmission then peaks at $\tau \approx \tau_{\text{on}}/\sqrt{1 - e^2}$ with $\tau_{\text{on}} = \tau_0 \xi_0 / (2l)$. In this configuration, two ABS merge with the continuum at $e = 1$ leaving a spectrum of a single ABS pair

$$\epsilon_{\text{on}}^{\pm} = \pm \sqrt{1 - 2\tau_{\text{on}} \left( s_{L}^2 + s_{R}^2 \right)^{2/3}}$$

The exponent 2/3 of the second term under the square root is due to the joined effects of dEC and dCAR, which are enhanced by the fact that a resonant level lies at the Fermi energy level of the electrodes. This can be understood by looking at the evolution of the ABS spectrum under resonant condition when the left and right junctions are gradually brought together from infinity, which makes non-local processes more and more effective. This causes the formation of larger and larger avoided crossings between ABS of the left and right junction. Once the limit $l/\xi \ll 1$ has been reached, one pair of ABS vanishes into the continuum while the other one lies within the superconducting gap and always remains detached from the edge provided that one of the phases $\delta_{L(R)}$ is finite (Fig. 12(g)). The remaining pair is fully delocalized over
the two junctions. Moreover, due to constructive interference, the amplitude of ABS oscillations is proportional to $\tau_0^{-2/3}$, which can be orders of magnitude larger than for isolated tunnel junctions. Note that, in this expression, we cannot take the limit $l \to 0$ to merge the two junctions and recover the spectrum of a single one because the conditions of validity ($\sin^2 (k_F l) \neq 0$ or $l/\xi \gg \tau_0$) would not be respected and we would obtain transmission $\tau (\epsilon) \sim \tau_0 \xi/l$ larger than 1, which is unphysical.

Similarly to the off resonance case, one can deduce the contribution of ABS to the Josephson energy

$$E_{ABS} \approx -\frac{\Delta}{2} \tau_{on}^{2/3} (2 - \cos \delta_L + \cos \delta_R)^{2/3}$$

as well as the current carried by ABS

$$I^{ABS}_{L(R)} = I^{on}_{c} \sin \delta_{L(R)} (2 - \cos \delta_L - \cos \delta_R)^{-1/3}$$

with the critical current $I^{on}_{c} = \Delta \tau_{on}^{2/3} / (3 \varphi_0)$. Similarly to the off resonance case, the contribution of the continuum is important and the total current–phase relation is the same as the one of a single bare junction (Fig. 12(e) and (h)).

In practice, ballistic superconductors are extremely rare in superconducting circuits, but semiconducting nanowires covered with epitaxial aluminum or Van der Waals materials such as NbSe$_2$ are potential candidates for the observation of an Andreev molecule in ballistic tunnel devices. Non-local effects are however much weaker than in devices of larger transmission and it would be a challenge to detect them. For example, the $\varphi$–junction behavior seems almost suppressed (Fig. 12(e)).

### 4. Diffusive devices

In most experiments involving tunnel Josephson junctions, superconductors are diffusive. Each electronic channel carry a supercurrent through diffusive paths over which electrons take random phases $k_F l$ where $l$ can vary by several Fermi wavelengths. As a consequence, the interference effects that we described previously are suppressed and non-local effects are essentially negligible. The total Josephson energy of the system is obtained by averaging the phase over all channels

$$E_J \approx \frac{\Delta}{\pi} \int_{-\pi/2}^{\pi/2} (\epsilon^+ + \epsilon^-) dk_F l$$

where $\epsilon^{\pm}$ are the energies of ABS of negative energies in Eq. [D1] and the integral is summing over all the possible values of $k_F l$ within one period of $\tau (\epsilon)$. This can either be done numerically or by approximating $\tau (\epsilon)$ as a Dirac delta distribution $\tau_0 \pi \delta (k_F l) / 2$ leading at first order in $\tau_0$ to

$$E_J \approx \frac{\tau_0 \Delta}{2} (\cos \delta_L + \cos \delta_R)$$

![Figure 12](image-url)

Figure 12. Andreev molecules in tunnel junctions. (a) In tunnel junctions, the superconducting coherence length is large and we expect a strong overlap of ABS wavefunctions $\psi$ when two junctions are placed close to each other. (b) When $\sin^2 (k_F l) \approx 1$, the device is set off resonance. In that configuration the density of states in the central part of the device is weak because of destructive interference. This limits the transmission of the two junctions and reduces Josephson effect drastically. (c) The spectrum for a tunnel device off resonance shows a strong hybridization of ABS with large avoided crossings, but the discrete spectrum is very close to the edge of the gap. (d) Off resonance, the supercurrent carried by the left tunnel junction has two contributions: one from the discrete spectrum (dashed line) and the other from the continuum (dotted line). The contribution coming from ABS shows a $\varphi$–junction behavior but it is much smaller than the one from the continuum (here we multiply the ABS contribution by a factor $\tau_0^{-1}$ in order to make it visible). (e) Taking into account both contributions, the current-phase relations of the left tunnel junction is standard $I_L \approx I_c \sin \delta_L$, with no $\varphi$–junction behavior. There is however an increase or decrease of the critical current when setting the device on (continuous line) or off resonance (dashed line). Current-phase relation of a single tunnel junction is given in dotted line. (f) On resonance $\sin^2 (k_F l) = 0$, the density of states in the central part of the device is expected to be large, which favors the flow of supercurrent. (g) The spectrum then shows only one pair of ABS, which are detached from the edge of the gap. ABS go deeper in the gap than in a single junction. (h) The contribution of ABS and continuum to the supercurrent are of the same order of magnitude but in opposite direction. Parameters used for these calculations: $\delta_R = 3 \pi/5$, $l/\xi_0 = 1$, $U_L = U_R = 100 h v_F$ ($\tau_L = \tau_R \approx 10^{-4}$), and $k_F l \gg 1$. 

- $\tau_0 = 2 \pi / \nu_0$
- $\nu_0 = 2 e B / h$
- $F = e B / 2 m$
- $l$, $\xi = \sqrt{\pi F}$
- $c$, $\xi_0 = \sqrt{\pi F} / 2$
- $\varphi_0 = \hbar v_F / 2 e$
- $\tau_0$, $\pi F / e$
- $\Delta$, $2 \Delta / \varphi_0$
- $\varphi_0$, $2 e B / h$
where we have dismissed a constant term. This simply corresponds to the Josephson energy of two independent tunnel Josephson junctions. The next-order correction in \( \tau_0 \) is given by
\[
\delta E_J \approx \frac{5/3}{4\delta_0^2} \approx \frac{s_{LR}^2}{(s_L^2 + s_R^2)^{1/3}}
\]
which is completely negligible since \( \tau_0^{5/3} \ll \tau_0 \).

In fact, even taking into account the contribution of the continuum, the current-phases relations correspond to the standard Josephson formula \( I_{L(R)} \approx \frac{\Phi_0}{2\pi} \sin \delta_{L(R)} \) where \( \Phi_0 \) is the bare critical current of the junctions. The effects of ABS hybridization are therefore hardly measurable in conventional devices with superconducting tunnel junctions. It would be interesting however to explore the implications of a fraction of the supercurrent being carried by the continuum for SQUID measurements or the coherence time of superconducting qubits.

### Appendix E: Scattering approach

In section III we describe the formation of Andreev molecules using scattering formalism. In this approach, electrons and holes \((e\) and \(h)\) are described by an ensemble of waves propagating to the left or the right \((\leftarrow \) or \(\rightarrow)\), which are connected to each other by normal scattering processes at the weak link or Andreev processes on superconductors. These waves can be labeled with two sets of eight coefficients
\[
\begin{align*}
A &= (a_{Le}^+, a_{Le}^-, a_{Re}^+, a_{Re}^-, a_{Lh}^+, a_{Lh}^-, a_{Rh}^+, a_{Rh}^-)^T \\
B &= (b_{Le}^+, b_{Le}^-, b_{Re}^+, b_{Re}^-, b_{Lh}^+, b_{Lh}^-, b_{Rh}^+, b_{Rh}^-)^T
\end{align*}
\]
where \(A\) describes incoming waves propagating towards weak links and \(B\) describes outgoing waves (see Fig. 13) such that
\[
B = S_N A
\]
with
\[
S_N = \begin{pmatrix}
S_L & 0 & 0 & 0 \\
0 & S_R & 0 & 0 \\
0 & 0 & S_L^* & 0 \\
0 & 0 & 0 & S_R^*
\end{pmatrix}
\]
where \(S_{L(R)}\) is the normal state scattering matrix describing scattering of electrons in the left (resp. right) weak link. The equivalent for holes is given by \(S_{L(R)}^*\). The dimensions of \(S_N\) is therefore \(8N \times 8N\) where \(N\) is the number of channels. In this Appendix, we explain first how to describe Andreev processes on the central superconductor of the device and then we demonstrate the characteristic equation giving the Andreev molecule spectrum for \(|e| < 1\).

![Figure 13](image-url)  

#### 1. Scattering on superconducting slab \(|e| < 1\)

In order to understand the effect of having a superconducting slab in the device, we consider that it has perfect interfaces. In practice, scattering at interfaces is inevitable but it can be included in the scattering matrices of the weak links. This implies that there is no backscattering and that Andreev processes on the slab preserves momentum. For example, an incoming electron propagating from the right \(b_{Re}^+\) has a momentum \(+k_F\) and can only be reflected as a left propagating hole \((a_{Rh}^-)\) or transmitted as a right propagating electron \((a_{Re}^+)\), which both have a momentum \(+k_F\). Any other processes on this electron would not conserve momentum.

The probability amplitudes associated to these processes can be found using the continuity of wavefunctions at each interface. If we focus on the subspace of waves with positive momentum, the wavefunction on the left of the superconducting slab is given by
\[
b_{Le}^+ e^{i k_F x + \frac{\pi}{4}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_{Lh}^+ e^{i k_F x + \frac{\pi}{4}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
while in the center it is
\[
c_x^+ \begin{pmatrix} v_0^0 \\ v_0^0 \end{pmatrix} e^{i k_e x} + c_h^+ \begin{pmatrix} v_0^0 \\ v_0^0 \end{pmatrix} e^{i k_h x}
\]
and on the right
\[
a_{Re}^- e^{i k_F x - \frac{\pi}{4}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b_{Rh}^- e^{i k_F x - \frac{\pi}{4}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
The continuity must be preserved at $x = \pm 1/2$, which implies that (we use notations of Appendix A)

\[
\begin{pmatrix}
    b_{Lc}^+ \\
    a_{Lh}^-
\end{pmatrix}
= e^{-\frac{ik_F l}{\xi}}
\begin{pmatrix}
    u_{Le}^0 e^{i/2\xi} & u_{Le}^0 e^{-i/2\xi} \\
    u_{Le}^0 e^{-i/2\xi} & u_{Le}^0 e^{i/2\xi}
\end{pmatrix}
\begin{pmatrix}
    c_e^+ \\
    c_h^-
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
    b_{Lc}^+ \\
    a_{Re}^-
\end{pmatrix}
= e^{\frac{ik_F l}{\xi}}
\begin{pmatrix}
    u_{Le}^0 e^{-i/2\xi} & u_{Le}^0 e^{i/2\xi} \\
    u_{Le}^0 e^{i/2\xi} & u_{Le}^0 e^{-i/2\xi}
\end{pmatrix}
\begin{pmatrix}
    c_e^+ \\
    c_h^-
\end{pmatrix}
\]

We can eliminate coefficients $c_{e(h)}^+$ and express waves moving away from the slab as a function of waves approaching the slab

\[
\begin{pmatrix}
    a_{Lh}^+ \\
    a_{Re}^-
\end{pmatrix}
= \begin{pmatrix}
    r_S & t_S e^{-ik_F l} \\
    t_S e^{ik_F l} & r_S
\end{pmatrix}
\begin{pmatrix}
    b_{Lc}^+ \\
    b_{Rh}^-
\end{pmatrix}
\]

with

\[
t_S = \frac{e^{-l/\xi} \left(1 - a(\epsilon)^2\right)}{1 - e^{-2l/\xi}a(\epsilon)^2}
\]

and

\[
r_S = \frac{a(\epsilon) \left(1 - e^{-2l/\xi}\right)}{1 - e^{-2l/\xi}a(\epsilon)^2}
\]

Here $a(\epsilon) = e^{-i\arccos \epsilon}$ is defined as usual. The same calculation can be done for subspace $-k_F$, which gives the same results except with the transformation $k_F \rightarrow -k_F$.

2. Finding eigenenergies $|\epsilon| < 1$

Matrix $S_S$ given in the main text relates incoming waves $A$ to outgoing waves $B$ (Eq. E1) in a similar way than $S_N$ since one has

\[A = S_SB\]

Combining this equation with Eq. E2 we can write

\[B = S_NS_SB\]

which has non-trivial solution only if

\[\text{det} (I_{8N} - S_NS_S) = 0\]

This last equation is only true for discrete energies of $\epsilon$ corresponding to the eigenstates of Andreev molecules.

Appendix F: Tight-binding approach of the Andreev molecule in two dimensions

In the tight-binding approach that we use to describe Andreev molecule (see Fig. 14(a)), we decompose the space over a $N \times N$ square lattice with lattice constant $a$ in discrete sites labeled $(i,j)$ where $i$ and $j$ are integers. In the corresponding base $|i,j\rangle$, the normal part of the Hamiltonian $H_0 = \frac{\hbar^2}{2m} \left(\partial_x^2 + \partial_y^2\right) - \mu$ becomes

\[H_0 = \sum_{i,j} \left(4t - \mu\right)|i,j\rangle\langle i,j| - t \sum_{\sigma = \pm 1} |i + \sigma,j\rangle\langle i,j| + |i,j + \sigma\rangle\langle i,j|\]

where the first line describes the on-site energy and the second one is the coupling between adjacent sites where the hopping energy is $t = \hbar^2 / 2ma^2$. This approach is a good approximation to describe electrons in continuous media as long as the electronic states considered have a wavelength $\lambda_F \gg a$, which is reached for $\mu \leq t$. In the main text, we choose $\mu = t$ such that $\lambda_F \approx 2\pi a$. Since the length scale of interest is the bare superconducting coherence length $\xi_0 = 2\alpha / \pi \Delta$, we have $\xi_0 > \lambda_F > a$ and our resolution is sufficient to observe effects due to the overlap of ABS wavefunctions.

In order to build a global matrix describing the system, we first define local BDG matrices in Nambu space that describes individual sites of the lattice

\[H_{0D}(i,j) = \begin{pmatrix}
    \epsilon_{i,j} & \Delta e^{i\phi_{i,j}} \\
    \Delta e^{-i\phi_{i,j}} & -\epsilon_{i,j}
\end{pmatrix}\]

where $\epsilon_{i,j} = 4t - \mu + U_{i,j}$, $U_{i,j}$ is a local electrostatic potential, $\Delta = |\Delta| e^{i\phi_{i,j}}$ and $\phi_{i,j}$ is the local superconducting phase for the site labeled $(i,j)$. The $i$-th line of the lattice corresponds then to the assembly of $N$ of these matrices coupled by horizontal hopping terms between adjacent sites $T_{zz} = \text{diag}(-t,t)$

\[H_{1D}(i) = \begin{pmatrix}
    H_{0D}(i,1) & T_{zz} & 0 & 0 \\
    T_{zz} & H_{0D}(i,2) & \ddots & 0 \\
    0 & \ddots & \ddots & T_{zz} \\
    0 & 0 & T_{zz} & H_{0D}(i,N)
\end{pmatrix}\]

The total matrix $H_{2D}$ describing the $N \times N$ lattice in 2 dimensions is then built by coupling $N$ of these lines

\[H_{2D} = \begin{pmatrix}
    H_{1D}(1) & T_{zz} & 0 & 0 \\
    T_{zz} & H_{1D}(2) & \ddots & 0 \\
    0 & \ddots & \ddots & T_{zz} \\
    0 & 0 & T_{zz} & H_{1D}(N)
\end{pmatrix}\]

with vertical hopping diagonal matrices $T_{zz} = \text{diag}(-t,-t,-t,-t)$ of dimensions $2N$. The resulting Hamiltonian $H_{2D}$ is a $2N^2 \times 2N^2$ matrix, where the factor 2 stands for the electron and hole parts of the electronic states.

The energy spectrum of the system corresponds then to the ensemble of eigenvalues of $H$ while the eigenvectors are the quasiparticle wave-functions (see Fig. 14(b)). The spectrum of the Andreev molecule is extracted by selecting subgap states with energies smaller than $\Delta$. In
Figure 14. Tight-binding description. (a) Illustration of the discretization for the tight-binding approach: each site is a circle. For clarity, we draw less site than we actually use for calculations (200 × 200). The weak links are in black and their widths are chosen such that each QPC has a single channel. (b) Real part of $u^{ij}_\psi$, the electron part of the wave function $\psi(i,j)$, when hybridization is maximum ($\delta_L = -\delta_R$) for a device with no disorder. Interference between ABS leads to the formation of fringes between the junctions, which shows the hybridization into an Andreev molecule. The checkerboard patterns in the corners are finite size effects due to reflections on the edges. Parameters used for calculations are either given in the figures or: $\alpha = 1, l = 28, t = 10, \delta t = 0, \Delta = 0, \delta_R = \pi/2, \mu = 11$ and $W = 6$. This gives $\xi_0 \approx 12.7, \bar{l}_0 = \infty$ (or rather the size of the device $N$) and $\lambda_F \approx 6.3$.

In order to model systems with spin-orbit interaction such as InAs semiconducting nanowires, one would need to extend the model by introducing terms that flip the spin while a particle moves over the lattice, which would require to choose a Nambu space not only with electrons of spin up and holes of spin down but also with electrons of spin down and holes of spin up (i.e. $4 \times 4$ matrices for $H_{0D}$ and $T_{\downarrow\uparrow}$ and a $4N$ matrix for $T_{\downarrow\uparrow}$). We do not address this case here, as it is beyond the scope of this work.

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