SUBALGEBRAS THAT COVER OR AVOID CHIEF FACTORS
OF LIE ALGEBRAS

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Abstract. We call a subalgebra \( U \) of a Lie algebra \( L \) a \( CAP \)-subalgebra of \( L \) if for any chief factor \( H/K \) of \( L \), we have \( H \cap U = K \cap U \) or \( H + U = K + U \). In this paper we investigate some properties of such subalgebras and obtain some characterizations for a finite-dimensional Lie algebra \( L \) to be solvable under the assumption that some of its maximal subalgebras or 2-maximal subalgebras be \( CAP \)-subalgebras.

1. The covering and avoidance property

Throughout, \( L \) will denote a finite-dimensional Lie algebra over a field \( F \). Let
\[
0 = A_0 \subset A_1 \subset \ldots \subset A_n = L
\]
be a chief series for \( L \). The subalgebra \( U \) avoids the factor algebra \( A_i/A_{i-1} \) if \( U \cap A_i = U \cap A_{i-1} \); likewise, \( U \) covers \( A_i/A_{i-1} \) if \( U + A_i = U + A_{i-1} \). We say that \( U \) has the covering and avoidance property of \( L \) if \( U \) either covers or avoids every chief factor of \( L \). We also say that \( U \) is a \( CAP \)-subalgebra of \( L \). The corresponding concepts in group theory have been studied extensively and have proved useful in characterising finite solvable groups and some of their subgroups (see, for example, [12], [15] and [7]). In Lie algebras, some parallel results have been obtained by a number of authors, and this paper is intended to be a further contribution to that work.

There are a number of ways in which \( CAP \)-subalgebras arise. We say that \( A_i/A_{i-1} \) is a Frattini chief factor if \( A_i/A_{i-1} \subseteq \phi(L/A_{i-1}) \); it is complemented if there is a maximal subalgebra \( M \) of \( L \) such that \( L = A_i + M \) and \( A_i \cap M = A_{i-1} \). When \( L \) is solvable it is easy to see that a chief factor is Frattini if and only if it is not complemented. For a subalgebra \( B \) of \( L \) we denote by \([B:L]\) the set of all subalgebras \( S \) of \( L \) with \( B \subseteq S \subseteq L \), and by \([B:L]_{\max}\) the set of maximal subalgebras in \([B:L]\); that is, the set of maximal subalgebras of \( L \) containing \( B \). We define the set \( I \) by \( i \in I \) if and only if \( A_i/A_{i-1} \) is not a Frattini chief factor of \( L \). For each \( i \in I \) put
\[
\mathcal{M}_i = \{ M \in [A_{i-1},L]_{\max} : A_i \nsubseteq M \}.
\]
Then \( U \) is a prefattini subalgebra of \( L \) if
\[
U = \bigcap_{i \in I} M_i \text{ for some } M_i \in \mathcal{M}_i.
\]
It was shown in [13] that, when $L$ is solvable, this definition does not depend on the choice of chief series, and that the prefrattini subalgebras of $L$ cover the Frattini chief factors and avoid the rest; that is, they are $CAP$-subalgebras of $L$.

Further examples were given by Stitzinger in [10], where he proved the following result (see [10] for definitions of the terminology used).

**Theorem 1.1** ([10, Theorem 2]). Let $F$ be a saturated formation of solvable Lie algebras, and let $U$ be an $F$-normaliser of $L$. Then $U$ covers every $F$-central chief factor of $L$ and avoids every $F$-eccentric chief factor of $L$.

The chief factor $A_i/A_{i-1}$ is called central if $[L, A_i] \subseteq A_{i-1}$ and eccentric otherwise. A particular case of the above result is the following theorem, due to Hallahan and Overbeck.

**Theorem 1.2** ([6, Theorem 1]). Let $L$ be a metanilpotent Lie algebra. Then $C$ is a Cartan subalgebra of $L$ if and only if it covers the central chief factors and avoids the eccentric ones.

In group theory an important class of $CAP$-subgroups is given by the normally embedded (also called strongly pronormal) subgroups (see [5] page 251). In a sense, the natural analogue of this concept in Lie algebras is to call a subalgebra $U$ of $L$ strongly pronormal if every Cartan subalgebra of $U$ is also a Cartan subalgebra of $U^L$, the ideal closure of $U$ in $L$. Such subalgebras satisfy a number of the same properties as those of their group-theoretic counterparts. However, they are not necessarily $CAP$-subalgebras, even when $L$ is metabelian, as the following example shows.

**Example 1.1.** Let $L$ be the four-dimensional real Lie algebra with basis $e_1$, $e_2$, $e_3$, $e_4$ and multiplication $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$, $[e_1, e_4] = -e_2$ and $[e_2, e_4] = e_1$, other products being zero. Then $A = \mathbb{R}e_1 + \mathbb{R}e_2$ is a minimal abelian ideal of $L$ and $U = \mathbb{R}e_1 + \mathbb{R}e_3$ is strongly pronormal in $L$ (since the Cartan subalgebras of $U$ are of the form $\mathbb{R}(\alpha e_1 + e_3)$ ($\alpha \in \mathbb{R}$) and these are also Cartan subalgebras of $U^L = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$). However, $U \cap A = \mathbb{R}e_1 \neq U \cap 0$ and $U + A = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3 \neq U + 0$, so $U$ is not a $CAP$-subalgebra of $L$.

It is worth noting at this point that it is known that if $\dim L < |F|$, then $L$ possesses a Cartan subalgebra (see [1, Corollary 1.2]). However, the existence of Cartan subalgebras has not been shown in general. The best result known to the author is the following, proved by Siciliano.

**Proposition 1.3** ([8 Proposition 2.2]). Let $L$ be a minimal example of a finite-dimensional Lie algebra over a field $F$ without Cartan subalgebras. Then

1. $F$ is a finite field and $|F| \leq \dim L$;
2. $L$ is not a restricted Lie algebra; and
3. $L$ is semisimple.

An alternative approach which does yield examples of $CAP$-subalgebras will be given in the next section.

2. Elementary results

In this section we collect together some properties of $CAP$-subalgebras and give characterisations of simple and of supersolvable Lie algebras in terms of them. If $U$
is a subalgebra of $L$, the core of $U$, denoted $U_L$, is the largest ideal of $L$ contained in $U$.

**Lemma 2.1.** Let $B$ be a subalgebra of $L$ and $H/K$ a chief factor of $L$. Then

(i) $B$ covers $H/K$ if and only if $B \cap H + K = H$; and

(ii) $B$ avoids $H/K$ if and only if $(K + B) \cap H = K$.

(iii) If $B \cap H + K$ is an ideal of $L$, then $B$ covers or avoids $H/K$. In particular, ideals are CAP-subalgebras.

(iv) The non-trivial Lie algebra $L$ is simple if and only if it has no non-trivial proper CAP-subalgebras.

(v) $B$ covers or avoids $H/K$ if and only if there exists an ideal $N$ with $N \subseteq B \cap K$ and $B/N$ covers or avoids $(H/N)/(K/N)$ respectively. Furthermore, $B$ is a CAP-subalgebra of $L$ if and only if there exists an ideal $N$ of $L$ such that $N \subseteq B$ and $B/N$ is a CAP-subalgebra of $L/N$.

(vi) Let $C$ be a subalgebra containing $B$. If $H/K$ is covered (respectively, avoided) by $B$, then so is $(H \cap C)/(K \cap C)$.

**Proof.** (i), (ii) These are straightforward.

(iii) Since $B \cap H + K$ is an ideal of $L$, we have that $B \cap H + K = H$ or $B \cap H + K = K$. The former implies that $B$ covers $H/K$, by (i); the latter yields that $(K + B) \cap H = (B \cap H) + K = K$, whence $B$ avoids $H/K$, by (ii).

(iv) This is straightforward.

(v) Let $N = (B \cap K)_L$. Then

$$B + H = B + K \iff \frac{B}{N} + \frac{H}{N} = \frac{B}{N} + \frac{K}{N},$$

and

$$B \cap H = B \cap K \iff \frac{B}{N} \cap \frac{H}{N} = \frac{B}{N} \cap \frac{K}{N}.$$  

(vi) This is straightforward. \qed

A subalgebra $U$ of $L$ will be called *ideally embedded* in $L$ if $I_L(U)$ contains a Cartan subalgebra of $L$, where $I_L(U) = \{x \in L : [x, U] \subseteq U\}$ is the *idealiser* of $U$ in $L$. Clearly, any subalgebra containing a Cartan subalgebra of $L$ and any ideal of $L$ is ideally embedded in $L$. Then we have the following extension of Theorem 1.2.

**Theorem 2.2.** Let $L$ be a metanilpotent Lie algebra and let $U$ be ideally embedded in $L$. Then $U$ is a CAP-subalgebra of $L$.

**Proof.** Let $C \subseteq I_L(U)$ be a Cartan subalgebra of $L$ and let $N$ be the nilradical of $L$. Then $(C + N)/N$ is a Cartan subalgebra of $L/N$ and $L/N$ is nilpotent, so $L = C + N$. Let $H/K$ be a chief factor of $L$. Then $[N, H] \subseteq K$ so $U \cap H + K$ is an ideal of $L$. The result now follows from Lemma 2.1 (iii) \qed

We define the *nilpotent residual*, $\gamma_\infty(L)$, of $L$ to be the smallest ideal of $L$ such that $L/\gamma_\infty(L)$ is nilpotent. Clearly this is the intersection of the terms of the lower central series for $L$. Then the *lower nilpotent series* for $L$ is the sequence of ideals $N_i(L)$ of $L$ defined by $N_0(L) = L$, $N_{i+1}(L) = \gamma_\infty(N_i(L))$ for $i \geq 0$. Then we have the following extension of Theorem 2.2.

**Corollary 2.3.** Let $L$ be any solvable Lie algebra and let $U$ be an ideally embedded subalgebra of $L$ with $K = N_2(L) \subseteq U$. Then $U$ is a CAP-subalgebra of $L$. 
Proof. Let \( C \subseteq I_L(U) \) be a Cartan subalgebra of \( L \). Then \((C + K)/K\) is a Cartan subalgebra of \( L/K \), and \( I_L(U/K) \supseteq (I_L(U) + K)/K \supseteq (C + K)/K \), so \( U/K \) is ideally embedded in \( L/K \). Moreover, \( L/K \) is metanilpotent. It follows from Theorem 2.2 that \( U/K \) is a \( \text{CAP}\)-subalgebra of \( L/K \). But now Lemma 2.4 (v) yields that \( U \) is a \( \text{CAP}\)-subalgebra of \( L \). \( \square \)

Let \( U \) be a subalgebra of \( L \) and \( B \) an ideal of \( L \). Then \( U \) is said to be a \textit{supplement} to \( B \) in \( L \) if \( L = U + B \). Another set of examples of \( \text{CAP}\)-subalgebras, which don’t require \( L \) to be solvable, is given by the next result.

**Theorem 2.4.** Let \( L \) be any Lie algebra, let \( U \) be a supplement to an ideal \( B \) in \( L \), and suppose that \( B^k \subseteq U \) for some \( k \in \mathbb{N} \). Then \( U \) is a \( \text{CAP}\)-subalgebra of \( L \).

**Proof.** Let \( L = B + U \) and let \( H/K \) be a chief factor of \( L \). Then \( K + [B, H] = H \) or \( K \). Suppose first that \( K + [B, H] = H \). Then \([B, H] \subseteq K + [B, [B, H]]\) and a simple induction argument shows that \( H \subseteq K + B^k \) for all \( k \geq 1 \). Hence \( H \subseteq K + U \), which yields that \( H + U = K + U \).

So suppose now that \( K + [B, H] = K \), whence \([B, H] \subseteq K \). Then \( K + U \cap H \) is an ideal of \( L \), and the result now follows from Lemma 2.4 (iii). \( \square \)

**Lemma 2.5.** Let \( U \) be a \( \text{CAP}\)-subalgebra of \( L \) and let \( B \) be an ideal of \( L \). Then \( B + U \) is a \( \text{CAP}\)-subalgebra of \( L \).

**Proof.** Let \( H/K \) be a chief factor of \( L \). If \( U + H = U + K \), then \( B + U + H = B + U + K \), so suppose that \( U \cap H = U \cap K \). Similarly, since \( B \) is a \( \text{CAP}\)-subalgebra, by Lemma 2.1 (iii), we can suppose that \( B \cap H = B \cap K \).

Then \( \frac{B + H}{B + K} \cong \frac{(B + H)/B}{(B + K)/B} \cong \frac{H/B \cap H}{K/B \cap K} \cong \frac{H}{K} \) is a chief factor of \( L \). If \( U + B + H = U + B + K \), the result is clear, so suppose that \( U \cap (B + H) = U \cap (B + K) \).

Let \( x \in (B + U) \cap H \). Then \( x = b + u \) for some \( b \in B \), \( u \in U \), and \( x \in H \). It follows that \( u \in (B + H) \cap U = (B + K) \cap U \), so that \( x \in (B + K) \cap H = K + B \cap H = K \). Thus \((B + U) \cap H \subseteq (B + U) \cap K \). But the reverse inclusion is clear and the result follows. \( \square \)

The next result gives the dimension of \( \text{CAP}\)-subalgebras in terms of the chief factors that they cover.

**Lemma 2.6.** Let \( U \) be a \( \text{CAP}\)-subalgebra of \( L \), let \( 0 = A_0 < A_1 < \ldots < A_n = L \) be a chief series for \( L \) and let \( \mathcal{I} = \{ i : 1 \leq i \leq n, U \text{ covers } A_i/A_{i-1} \} \). Then
\[
\dim U = \sum_{i \in \mathcal{I}} (\dim A_i - \dim A_{i-1}).
\]

**Proof.** We use induction on \( n \). The result is clear if \( n = 1 \). So suppose it holds for all Lie algebras with chief series of length \( < n \), and let \( L \) have a chief series of length \( n \). Then \( U + A_1/A_1 \) is a \( \text{CAP}\)-subalgebra of \( L/A_1 \), by Lemmas 2.5 and 2.4 (v). Moreover,
\[
\dim(U + A_1/A_1) = \sum_{i \in \mathcal{I}, i \neq 1} (\dim A_i - \dim A_{i-1}),
\]
by the inductive hypothesis. If \( U \) covers \( A_1/A_0 \), then
\[
\dim U = \dim(U + A_1) = \dim(U + A_1/A_1) + \dim A_1 = \sum_{i \in \mathcal{I}} (\dim A_i - \dim A_{i-1}).
\]
If $U$ avoids $A_1/A_0$, then
\[
\dim U = \dim(U/U \cap A_1) = \dim(U + A_1/A_1) = \sum_{i \in I} \dim A_i - \dim A_{i-1}.
\]

Finally in this section we consider supersolvable Lie algebras, that is, Lie algebras all of whose chief factors are one-dimensional.

**Proposition 2.7.** Let $H/K$ be a chief factor of $L$. Then every one-dimensional subalgebra of $L$ covers or avoids $H/K$ if and only if $\dim(H/K) = 1$.

**Proof.** If $x \in K$, then $Fx = Fx \cap H = Fx \cap K$, so $Fx$ avoids $H/K$. If $x \notin H$, then $0 = Fx \cap H = Fx \cap K$, so again $Fx$ avoids $H/K$. If $x \in H \setminus K$, then $Fx$ does not avoid $H/K$, and $Fx$ covers $H/K$ if and only if $H = K + Fx$, whence the result. \hfill \Box

**Corollary 2.8.** Every one-dimensional subalgebra of $L$ is a CAP-subalgebra of $L$ if and only if $L$ is supersolvable.

**Proposition 2.9.** If $L$ is supersolvable, then every subalgebra of $L$ is a CAP-subalgebra.

**Proof.** Let $U$ be a subalgebra of $L$ and let $H/K$ be a chief factor of $L$. Suppose first that $U \cap H \subseteq K$. Then $U \cap H \subseteq U \cap K \subseteq U \cap H$, whence $U \cap H = U \cap K$. So suppose now that $U \cap H \nsubseteq K$. Then, since $\dim(H/K) = 1$, we have that $H = K + U \cap H$, whence $H + U = K + U$. \hfill \Box

3. **Some characterisations of solvable algebras**

In this section we are seeking characterisations of solvable Lie algebras in terms of CAP-subalgebras. The results are analogues of those for groups as obtained in [15, Section 3], but the proofs are different. A subalgebra $U$ of a Lie algebra $L$ is called a c-ideal of $L$ if there is an ideal $C$ of $L$ such that $L = U + C$ and $U \cap C \leq U$; c-ideals were introduced in [14]. First we need the following result.

**Proposition 3.1.** Let $L$ be a Lie algebra over a field $F$ which has characteristic zero, or is algebraically closed and of characteristic greater than 5, with minimal ideal $A$ and maximal subalgebra $M$. If $M$ is solvable and $M \cap A = 0$, then $L$ is solvable.

**Proof.** Clearly $L = M \oplus A$. But now $M$ is a c-ideal of $L$ and it follows from [14] Theorems 3.2 and 3.3 that $L$ is solvable, a contradiction again. \hfill \Box

**Corollary 3.2.** Let $L$ be a Lie algebra over a field $F$ which has characteristic zero, or is an algebraically closed field and of characteristic greater than 5. Then $L$ is solvable if and only if there is a maximal subalgebra $M$ of $L$ such that $M$ is a solvable CAP-subalgebra of $L$.

**Proof.** If $L$ is solvable it is easy to see that every maximal subalgebra of $L$ is a CAP-subalgebra of $L$. So suppose now that $L$ is the smallest non-solvable Lie algebra which has a solvable maximal subalgebra $M$ that is a CAP-subalgebra of $L$. If $M_L \neq 0$, then $L/M_L$ must be solvable, whence $L$ is solvable, a contradiction. Hence $M_L = 0$. Now $L$ cannot be simple, by Lemma 2.1 (iv), so let $A$ be a minimal ideal of $L$ with $A \nsubseteq M$. Since $M$ is a CAP-subalgebra we have $M \cap A = 0$. But then $L$ is solvable, by Proposition 3.1 a contradiction. \hfill \Box
The Lie algebra $L$ is called monolithic with monolith $A$ if $A$ is the unique minimal ideal of $L$. We denote by $\phi(L)$ the Frattini ideal of $L$. If all of the maximal subalgebras of $L$ are CAP-subalgebras of $L$ we can deduce solvability without any restrictions on the field $F$.

**Theorem 3.3.** Let $L$ be a Lie algebra over any field $F$. Then $L$ is solvable if and only if all of its maximal subalgebras are CAP-subalgebras.

**Proof.** If $L$ is solvable it is easy to see that every maximal subalgebra of $L$ is a CAP-subalgebra of $L$. So suppose that $L$ is the smallest non-solvable Lie algebra all of whose maximal subalgebras are CAP-subalgebras. Then $L$ is not simple, by Lemma 2.1 (iv), so let $A$ be a minimal ideal of $L$. By the minimality of $L$, $L/A$ is solvable. If $L$ has two different minimal ideals $A_1$ and $A_2$, then $L/A_1, L/A_2$ and hence $L \cong L/(A_1 \cap A_2)$ is solvable. It follows that $L$ is monolithic with monolith $A$.

Let $M$ be any maximal subalgebra of $L$. Since $M$ is a CAP-subalgebra of $L$ we have that either $M + A = M$, whence $A \subseteq M$, or $M \cap A = 0$. If the former holds for every maximal subalgebra $M$, then $A \subseteq \phi(L)$, whence $A$ is abelian and $L$ is solvable. Thus, the latter must hold for some maximal subalgebra $K$. But, for any such maximal subalgebra $K$, $L = K \oplus A$ and $K \cong L/A$ is a solvable $c$-ideal of $L$. Moreover, if $M$ is a maximal subalgebra of $L$ with $A \subseteq M$, then $M/A$ is a maximal subalgebra of $L/A$ and so is a $c$-ideal of $L/A$, by [14, Theorem 3.1]. It follows that $M$ is a $c$-ideal of $L$, by [14, Lemma 2.1]. Hence $L$ is solvable, by [14, Theorem 3.1]. This contradiction establishes the result. \qed

Let $M$ be a maximal subalgebra of $L$ and let $K$ be a maximal subalgebra of $M$. Then we call $K$ a 2-maximal subalgebra of $L$. Next we consider Lie algebras in which every 2-maximal subalgebra is a CAP-subalgebra of $L$. If $x \in L$ we put $C_L(x) = \{y \in L : [y, x] = 0\}$, the centraliser of $x$ in $L$. We say that $L$ has the one-and-a-half generation property if, given any $x \in L$, there exists $y \in L$ such that the subalgebra generated by $x$ and $y$, $\langle x, y \rangle$, is $L$. First we need the following result concerning simple Lie algebras with a one-dimensional maximal subalgebra.

**Theorem 3.4.** Let $L$ be a simple Lie algebra over a perfect field $F$ of characteristic zero or $p > 3$. Then $L$ has a one-dimensional maximal subalgebra if and only if $L$ is three-dimensional simple and $\sqrt{F} \subsetneq F$.

**Proof.** Suppose that $L$ has a one-dimensional maximal subalgebra $Fx$. Clearly $L$ has rank one and $Fx$ is a Cartan subalgebra of $L$. Let $\Gamma$ denote the centroid of $L$. Since $\Gamma x$ is an abelian subalgebra of $L$, we have that $\Gamma x < C_L(x) = Fx$. So $\Gamma = F$, and $L$ is central-simple. Suppose that $\dim L > 3$. It follows from [2] that $L$ is a form of an Albert-Zassenhaus algebra. Moreover, $L$ has the one-and-a-half generation property. For, given any $y \in L$, either $y = \alpha x$ for some $\alpha \in F$, in which case $\langle y, z \rangle = L$ for any $z \notin Fx$, or else $y \notin Fx$, and then $\langle y, x \rangle = L$. Thus, $L$ is a form of a Zassenhaus algebra, by [3].

Let $K$ be a splitting field for the minimal polynomial of $ad x$ over $F$, and let $G$ be the Galois group of $K$ over $F$. Let $\sigma \in G$. Then $\sigma' = 1 \otimes \sigma$ is a Lie automorphism of $L \otimes_F K = L_K$. As $K$ is a Galois extension of $F$, an element of $L_K$ lies in $L$ if and only if it is fixed by $\sigma'$ for every $\sigma \in G$. Now $L_K$ has a unique maximal subalgebra $M$ containing $Kx$ of codimension one in $L_K$ and $\sigma'$ must fix $M$. It follows that $(M \cap L)_K = M$ (see [4, p. 54]) and so $M \cap L$ is a subalgebra of $L$ of
codimension one in $L$. We must have $M \cap L = Fx$, which is impossible. Hence $L$ is three-dimensional simple and, as is well known, has a one-dimensional maximal subalgebra if and only if $\sqrt{F} \not\subseteq F$.

The converse is easy. □

**Theorem 3.5.** Let $L$ be a Lie algebra over any field $F$, in which every 2-maximal subalgebra of $L$ is a CAP-subalgebra. Then either

(i) $L$ is solvable, or

(ii) $L$ is simple and every maximal subalgebra of $L$ is one-dimensional; in particular, if $F$ is perfect and of characteristic zero or $p > 3$, $L$ is three-dimensional simple and $\sqrt{F} \not\subseteq F$.

**Proof.** Suppose first that $L$ is simple. Then every 2-maximal is 0 and so every maximal subalgebra of $L$ is one-dimensional, which is case (ii). So let $A$ be a minimal ideal of $L$. Suppose first that $A$ is a maximal subalgebra of $L$. Then every maximal subalgebra of $A$ is a 2-maximal subalgebra of $L$ and so is a CAP-subalgebra of $L$. It follows that every maximal subalgebra of $A$ is 0 and hence that $\dim A = 1$. Also, by the maximality of $A$, $\dim(L/A) = 1$ and $L$ is solvable.

So now assume that $A$ is not a maximal subalgebra of $L$ and that $L$ is a minimal counter-example. Suppose first that $L/A$ is as in (ii). Let $Fx + A$ be a maximal subalgebra of $L$ and let $K$ be a 2-maximal subalgebra of $L$ with $Fx \subseteq K \subset Fx + A$. Clearly $A \not\subseteq K$, so $K \cap A = 0$, since $K$ is a CAP-subalgebra of $L$. Now $L/A$ is a chief factor of $L$ and $K \not= 0$, so $L = K \oplus A = Fx + A$, a contradiction.

Thus $L/A$ is solvable and $L$ is monolithic, as in Theorem [3.3] If $A \subseteq \phi(L)$, then $A$ is solvable and hence so is $L$. Thus, $\phi(L) = 0$ and $L = M + A$ for some maximal subalgebra $M$ of $L$. Suppose that $M \cap A \neq 0$. Let $K$ be a maximal subalgebra of $M$ with $M \cap A \subseteq K$. Then $K$ is a 2-maximal subalgebra of $L$ and so either $K + A = A$, yielding $A \subseteq K \subseteq M$, or $M \cap A \subseteq K \cap A = 0$, both of which are contradictions. It follows that $M \cong L/A$ is a solvable c-ideal, as is any maximal subalgebra of $L$ not containing $A$. But every maximal subalgebra containing $A$ is a c-ideal, as in Theorem [3.3] and the result follows similarly. □

**Example 3.1.** Note that there are solvable Lie algebras with 2-maximal subalgebras which are not CAP-subalgebras. For example, let $L = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$ with $[e_1, e_3] = -[e_3, e_1] = e_2$, $[e_2, e_3] = -[e_3, e_2] = -e_1$ and all other products zero. Then $A = \mathbb{R}e_1 + \mathbb{R}e_2$ is a minimal ideal of $L$ and $U = \mathbb{R}e_1$ is a 2-maximal subalgebra of $L$. However, $A + U = A \neq U = 0 + U$ and $A \cap U = U \neq 0 = 0 \cap U$, so $U$ is not a CAP-subalgebra of $L$.

**Lemma 3.6.** Let $L$ be a solvable Lie algebra. Then there is a 2-maximal subalgebra $K$ of $L$ which is an ideal of $L$, and hence a CAP-subalgebra of $L$.

**Proof.** If $\dim(L/L^2) > 1$ there is clearly a 2-maximal subalgebra of $L$ containing $L^2$, so suppose that $\dim(L/L^2) = 1$. Let $L = L^2 + Fx$, and let $L^2/K$ be a chief factor of $L$. Suppose that $K + Fx \subset U$, where $U$ is a subalgebra of $L$. Then $[U \cap L^2, L] = [U \cap L^2, L^2] + [U \cap L^2, Fx] \subseteq U \cap L^2$, since $L^{(2)} = [L^2, L^2] \subseteq K$. It follows that $L^2 \subseteq U \cap L^2$, whence $K + Fx$ is a maximal subalgebra and $K$ a 2-maximal subalgebra of $L$. □

Finally we seek to characterise Lie algebras having a solvable 2-maximal subalgebra which is a CAP-subalgebra of $L$. 

Theorem 3.7. Let L be a Lie algebra over a field F which has characteristic zero. Then L has a solvable 2-maximal subalgebra K of L that is a CAP-subalgebra of L if and only if either

(i) L is solvable, or
(ii) $L = R \oplus S$, where R is the (solvable) radical of L (possibly 0), S is three-dimensional simple and $\sqrt{F} \not\subseteq F$.

Proof. Suppose that K is a solvable 2-maximal subalgebra of L that is a CAP-subalgebra of L. Consider $R + K = L$ and that R is the radical of L. Then $R + K$ is a solvable subalgebra of L. If $R + K = L$, we case (i). So suppose that $R + K \neq L$. Let $L = R \oplus S$ where $S = S_1 \oplus \ldots \oplus S_n$, $S_i$ is a simple ideal of S and put $J_i = R + S_1 \oplus \ldots \oplus S_i$ for $i = 0, \ldots, n$ (where $J_0 = R$). Suppose that $K \subseteq J_i$. Since $J_i / J_{i-1}$ is a chief factor of L we have that $J_i = K + J_i = K + J_{i-1}$ or $K = K \cap J_i = K \cap J_{i-1}$. The former implies that $J_i / J_{i-1} \cong K/K \cap J_{i-1}$, which is impossible as $J_i / J_{i-1}$ is simple and $K/K \cap J_{i-1}$ is solvable. It follows that $K \subseteq J_{i-1}$, from which $K \subseteq R$, since $K \subseteq J_n$.

Let M be a maximal subalgebra of L containing K as a maximal subalgebra. Suppose that $R \not\subseteq M$, so that $L = R + M$. Then $K \subseteq M \cap R \subseteq M$, so either $M \cap R = M$ or $M \cap R = K$. The former implies that $M \subseteq R$, which is impossible; the latter is also impossible, since $S \cong L/R \cong M/M \cap R$ and $M \cap R$ is not maximal in M. Hence $K \subseteq R \subseteq M$. It follows that $K = R$, from which (ii) easily follows.

It is easy to see that algebras as in (i) and (ii) have a solvable 2-maximal subalgebra which is a CAP-subalgebra.

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