THE ZAKHAROV-KUZNETSOV EQUATION IN HIGH DIMENSIONS: SMALL INITIAL DATA OF CRITICAL REGULARITY

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Abstract. The Zakharov-Kuznetsov equation in spatial dimension $d \geq 5$ is considered. The Cauchy problem is shown to be globally well-posed for small initial data in critical spaces and it is proved that solutions scatter to free solutions as $t \to \pm \infty$. The proof is based on i) novel endpoint non-isotropic Strichartz estimates which are derived from the $(d-1)$-dimensional Schrödinger equation, ii) transversal bilinear restriction estimates, and iii) an interpolation argument in critical function spaces. Under an additional radiality assumption, a similar result is obtained in dimension $d = 4$.

1. Introduction

This paper is concerned with the Zakharov-Kuznetsov equation

$$\partial_t u + \partial_{x_1} \Delta u = \partial_{x_1} u^2 \quad \text{in } \mathbb{R} \times \mathbb{R}^d \quad \text{on } \mathbb{R}^d$$

where $d \geq 2$, $u = u(t,x), \ (t,x) = (t,x_1, \ldots, x_d) \in \mathbb{R} \times \mathbb{R}^d$, $u$ is real-valued, and $\Delta$ denotes the Laplacian with respect to $x$.

The Zakharov–Kuznetsov equation was introduced in [13] as a model for propagation of ion-sound waves in magnetic fields. The Zakharov–Kuznetsov equation can be seen as a multidimensional extension of the well-known Korteweg–de Vries (KdV) equation. In contrast to the KdV equation, the Zakharov–Kuznetsov equation is not completely integrable, but possesses two invariants,

$$M(u) := \int_{\mathbb{R}^d} u^2 dx, \quad E(u) := \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u|^2 + \frac{1}{3} u^3 dx.$$

In the following, $H^s(\mathbb{R}^d)$ denotes the standard $L^2$-based inhomogeneous Sobolev space and $B^s_{2,1}(\mathbb{R}^d)$ is the Besov refinement, and the dotted versions their homogeneous counterparts, see below for definitions. The scale-invariant regularity threshold for (1.1) is $s_c = \frac{d-4}{2}$.

Before we state our main results, let us briefly summarize the progress which has been made regarding the well-posedness problem associated to (1.1). In the two-dimensional case, Faminski˘ı [3] established global well-posedness in the energy space $H^1(\mathbb{R}^2)$. Later, Linares and Pastor [9] proved local well-posedness in $H^s(\mathbb{R}^2)$ for $s > 3/4$, before Grünrock and Herr [4] and Molinet and Pilod [11] showed...
local well-posedness for \( s > 1/2 \). Recently, the second author \cite{7} proved local well-posedness for \( s > -1/4 \). In dimension \( d = 3 \), Linares and Saut \cite{10} obtained local well-posedness in \( H^s(\mathbb{R}^3) \) for \( s > 9/8 \). Ribaud and Vento \cite{12} proved local well-posedness for \( s > 1 \) and in \( B_{2,1}^{1,1}(\mathbb{R}^3) \). The global well-posedness in \( H^s(\mathbb{R}^3) \) for \( s > 1 \) was obtained by Molinet and Pilod in \cite{11}. Recently, in dimensions \( d \geq 3 \), local well-posedness in \( H^s(\mathbb{R}^d) \) in the full subcritical range \( s > s_c \) was proved in \cite{5}, which implies global well-posedness in \( H^1(\mathbb{R}^d) \) if \( 3 \leq d \leq 5 \) and in \( L^2(\mathbb{R}^3) \). We refer the reader to these papers for a more thorough account on the Zakharov–Kuznetsov equation, and more references.

In the present paper, we address the problem of global well-posedness and scattering for small initial data in critical spaces. By well-posedness we mean existence of a (mild) solution, uniqueness of solutions (in some subspace) and (locally Lipschitz) continuous dependence of solutions on the initial data. We say that a global solution \( u \in C(\mathbb{R}, H^s(\mathbb{R}^d)) \) of (1.1) scatters as \( t \to \pm \infty \), if there exist \( u_\pm \in H^s(\mathbb{R}^d) \) such that

\[
\|u(t) - e^{itS}u_\pm\|_{H^s(\mathbb{R}^d)} \to 0 \quad (t \to \pm \infty).
\]

Here, \( e^{itS} \) denotes the unitary group generated by the skew-adjoint linear operator \( S = -\partial_t \Delta \), so that \( e^{itS}u_\pm \) solves the linear homogeneous equation.

Our first main result covers small data in dimension \( d = 5 \).

**Theorem 1.1.** For \( d = 5 \), the Cauchy problem (1.1) is globally well-posed for small initial data in \( B_{2,1}^{\infty}(\mathbb{R}^5) \), and solutions scatter as \( t \to \pm \infty \). The same result holds in \( B_{2,1}^{3,1}(\mathbb{R}^d) \).

In dimensions \( d \geq 6 \), we can extend this to Sobolev regularity.

**Theorem 1.2.** For \( d \geq 6 \), the Cauchy problem (1.1) is globally well-posed for small initial data in \( H^{\infty}(\mathbb{R}^d) \), and solutions scatter as \( t \to \pm \infty \). The same result holds in \( \dot{H}^{\infty}(\mathbb{R}^d) \).

Note that in \( d = 6 \) this result includes the energy space \( \dot{H}^1(\mathbb{R}^6) \).

If we restrict to initial data which is radial in the last \( (d-1) \) variables (see below for definitions), we obtain small data global well-posedness and scattering in the critical Sobolev spaces for any dimension \( d \geq 4 \).

**Theorem 1.3.** For \( d \geq 4 \), the Cauchy problem (1.1) is globally well-posed for small data in \( H_{\text{rad}}^{s_0}(\mathbb{R}^d) \), and solutions scatter as \( t \to \pm \infty \). The same result holds for radial data in \( H^{s_0}_{\text{rad}}(\mathbb{R}^d) \).

As the proof shows, the radiality assumption can be weakened to an angular regularity assumption, but we do not pursue this. One of the most interesting special cases here is \( d = 4 \), when \( s_c = 0 \), hence the result covers the radial \( L^2 \) space.

The main idea of this paper is to combine a new set of non-isotropic Strichartz estimates with the bilinear transversal estimate and an interpolation argument in critical function spaces.

The paper is structured as follows: In Subsection 1.1 we introduce notation. In Section 2 we derive Strichartz type estimates which are based on the well-known Strichartz estimates for the \((d-1)\)-dimensional Schrödinger equation and allow us to treat the case \( d = 5 \). In Section 3 we combine this with the bilinear transversal estimate and an interpolation argument, which leads to a proof of Theorem 1.2. Finally, in Section 4 we discuss an variation of these ideas under the additional radiality assumption and a proof of Theorem 1.3.
1.1. Notation. We write $x' = (x_2, \ldots, x_d)$, $D_{x_j} = -i \partial_j$, $D = (-\Delta)^{\frac{1}{2}}$, $|\nabla x'|^s = \mathcal{F}^{-1}_x (\xi')^s \mathcal{F}_x$, and $\langle \nabla x' \rangle^s = \mathcal{F}^{-1}_x (\xi')^s \mathcal{F}_x$. Here and in the sequel we denote the Fourier transform of $u$ in time, space, and the first spatial variable, by $\mathcal{F}_t u$, $\mathcal{F}_x u$, $\mathcal{F}_{x_1} u$, and $\mathcal{F}_{x'}$, respectively. $\mathcal{F}_{t,x} u = \hat{u}$ denotes the Fourier transform of $u$ in time and space. Choose a non-negative bump function $\psi \in C_0^\infty (\mathbb{R})$ supported in the interval $(1/2, 2)$ with the property that $\sum_{N \in 2\mathbb{Z}} \psi (r/N) = 1$ for $r > 0$, and set $\psi_N = \psi (\cdot / N)$. For $N$, $\lambda \in 2\mathbb{Z}$, we define (spatial) frequency projections $P_N, Q_\lambda$ as the Fourier multipliers with symbols $\psi_N (|\xi|)$, $\psi_\lambda (|\xi_1|)$, and $\psi_M (|\xi'|)$, respectively, where $(\tau, \xi) = (\tau, \xi_1, \xi') = (\tau, \xi_1, \ldots, \xi_d) \in \mathbb{R} \times \mathbb{R}^d$ are temporal and spatial frequencies. In addition, we define

$$P_{\leq 1} = \sum_{1 \geq N \in 2^\mathbb{Z}} P_N, \quad Q_{\leq 1} = \sum_{1 \geq \lambda \in 2^\mathbb{Z}} Q_\lambda, \quad R_{\leq 1} = \sum_{1 \geq M \in 2^\mathbb{Z}} R_M.$$

As usual, the Sobolev space $H^s (\mathbb{R}^d)$ is defined as the completion of the Schwartz functions with respect to the norm

$$\|f\|_{H^s (\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}},$$

and the (smaller) Besov space $B^s_{2,1} (\mathbb{R}^d)$ as the completion of the Schwartz functions $\mathcal{S} (\mathbb{R}^d)$ with respect to the norm

$$\|f\|_{B^s_{2,1} (\mathbb{R}^d)} = \|P_{\leq 1} f\|_{L^2} + \sum_{N \in 2^\mathbb{Z}} N^s \|P_N f\|_{L^2}.$$ 

Similarly, for $s \geq 0$, the homogeneous Sobolev space $\dot{H}^s (\mathbb{R}^d)$ is defined as the completion of the Schwartz functions with respect to the norm

$$\|f\|_{\dot{H}^s (\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}},$$

and the homogeneous Besov space $\dot{B}^s_{2,1} (\mathbb{R}^d)$ as the completion of the Schwartz functions with respect to the norm

$$\|f\|_{\dot{B}^s_{2,1} (\mathbb{R}^d)} = \sum_{N \in 2^\mathbb{Z}} N^s \|P_N f\|_{L^2}.$$ 

The radial subspaces $H^s_{\text{rad}} (\mathbb{R}^d)$ and $\dot{H}^s_{\text{rad}} (\mathbb{R}^d)$ are defined by the requirement that $f(x_1, x') = f(x_1, y')$ if $|x'| = |y'|$, i.e. for fixed $x_1$, the functions are radial in $x'$. Finally, the Duhamel operator is denoted by

$$\mathcal{I}(F)(t) := \int_0^t e^{(t-t')s} F(t') \, dt'.$$

2. Strichartz estimates and the proof of Theorem \ref{thm:main}.

For $d \geq 2$, we say $(q, r)$ is $(d-1)$-admissible if

$$2 \leq q \leq \infty, \quad 2/q = d \leq (d-1)/(2 - 1/r), \quad (d, q, r) \neq (3, 2, \infty).$$

**Theorem 2.1.** Let $d \geq 2$ and $(q_1, r_1), (q_2, r_2)$ be $(d-1)$-admissible. Then, we have

$$\|D_{x_1}^s e^{tS} f\|_{L^q_{t} L^r_{x_1} L^2_{x_2}} \lesssim \|f\|_{L^2}, \quad (2.1)$$

$$\|D_{x_1}^s + \frac{1}{4} IF\|_{L^q_{t} L^r_{x_1} L^2_{x_2}} \lesssim \|F\|_{L^q_{t} L^r_{x_2} L^2_{x_1}}, \quad (2.2)$$

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where \(1/q'_2 = 1 - 1/q_2\) and \(1/r'_2 = 1 - 1/r_2\).

Proof. Let \(\Delta_{\nu'} = \sum_{j=2}^d \partial_{x_j}^2\). For fixed \(\xi_1 \in \mathbb{R}\), define \(V_{\xi_1}(t)f(x') := (e^{-it\xi_1 \Delta_{\nu'}}f)(x')\). Since \(V_{\xi_1}(t/\xi_1) = e^{it\Delta_{\nu'}/\xi_1}\), for \(f \in \mathcal{S}(\mathbb{R}^{d-1})\) and \(F \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^{d-1})\), the Strichartz estimates of Schrödinger equations in \(\mathbb{R}^{d-1}\) imply

\[
\|\xi_1 \frac{\partial}{\partial t} V_{\xi_1}(t)f\|_{L^q_t L^{r'}_x} \lesssim \|f\|_{L^{s'}_x}, \tag{2.3}
\]
\[
\left\| \int_0^t |\xi_1 \frac{\partial}{\partial t} V_{\xi_1}(t-t')F(t')dt' \right\|_{L^q_t L^{r'}_x} \lesssim \|F\|_{L^{s'}_t L^{r'}_x}, \tag{2.4}
\]
see [6] Theorem 1.2 for details. We deduce from the Plancherel’s Theorem, Minkowski’s inequality and (2.3) that

\[
\|D^{\frac{1}{2}}_x e^{it\xi_1 \Delta_{\nu'}}f\|_{L^q_t L^{r'}_x L^{s'}_1} = \left( \int_{\mathbb{R}} \left\| \int_0^t |\xi_1 \frac{\partial}{\partial t} V_{\xi_1}(t-t') \mathcal{F} x_1 f(t')dt' \right\|_{L^q_t L^{r'}_x}^2 d\xi_1 \right)^{\frac{1}{2}} \lesssim \|f\|_{L^q_x},
\]
which is (2.1). Similarly, by (2.4),

\[
\|D^{\frac{1}{2}}_x \frac{1}{\xi_1} \mathcal{F}(F)\|_{L^q_t L^{r'}_x L^{s'}_1} = \left( \int_{\mathbb{R}} \left\| \int_0^t |\xi_1 \frac{\partial}{\partial t} V_{\xi_1}(t-t') \mathcal{F} x_1 (F)(t')dt' \right\|_{L^q_t L^{r'}_x}^2 d\xi_1 \right)^{\frac{1}{2}} \lesssim \|F\|_{L^{s'}_t L^{r'}_x L^{q'}_1},
\]
which is (2.2).

Now, we can complete the proof of Theorem 1.1. Recall that \(d = 5\) implies \(s_c = 1/2\).

**Definition 2.2.** We define

\[
\|u\|_{\tilde{L}^{\frac{1}{2}}_x} := \|P_{\leq 1} u\|_{L^\infty L^2_x} + \|P_{\leq 1} D^{\frac{1}{2}}_x u\|_{L^q_t L^{r'}_x L^{s'}_1} + \sum_{N \in 2^\mathbb{Z}} \|P_N D^{\frac{1}{2}}_x u\|_{L^q_t L^{r'}_x L^{s'}_1},
\]

\[
\|u\|_{\tilde{L}^{\frac{3}{2}}_x} := \sum_{N \in 2^\mathbb{Z}} \|P_N u\|_{L^\infty L^2_x} + \|P_N D^{\frac{3}{2}}_x u\|_{L^q_t L^{r'}_x L^{s'}_1},
\]

and the corresponding Banach spaces.

By the standard argument involving the contraction mapping principle, it suffices to prove the following:

**Proposition 2.3.** Let \(d = 5\). Then, we have

\[
\|\mathcal{I}(\partial_x (u_1 u_2))\|_{\tilde{L}^{\frac{1}{2}}_x} \lesssim \|u_1\|_{\tilde{L}^{\frac{1}{2}}_x} \|u_2\|_{\tilde{L}^{\frac{1}{2}}_x}, \quad \|\mathcal{I}(\partial_x (u_1 u_2))\|_{\tilde{L}^{\frac{3}{2}}_x} \lesssim \|u_1\|_{\tilde{L}^{\frac{3}{2}}_x} \|u_2\|_{\tilde{L}^{\frac{3}{2}}_x}.
\]

**Proof.** Let \(N_{\max} = \max(N_1, N_2, N_3)\) and \(N_{\min} = \min(N_1, N_2, N_3)\). For any \(N \in 2^\mathbb{Z}\), Theorem 2.1 gives

\[
\|P_N \mathcal{I}(\partial_x (u_1 u_2))\|_{L^\infty L^2_x} + \|P_N D^{\frac{1}{2}}_x \mathcal{I}(\partial_x (u_1 u_2))\|_{L^q_t L^{r'}_x L^{s'}_1} \lesssim \|P_N D^{\frac{1}{2}}_x (u_1 u_2)\|_{L^q_t L^{r'}_x L^{s'}_1}.
\]
Further, we obtain
\[
\| P_N \partial_t^\frac{d}{2} (u_{N_1} u_{N_2}) \|_{L_t^2 L_x^\frac{4}{d+4}, L_x^2} \leq N^\frac{d}{2} \min \| P_N \partial_t^\frac{d}{2} u_{N_1} \|_{L_t^\infty L_x^2} + N^\frac{d}{2} \min \| u_{N_1} \|_{L_t^\infty L_x^2} \| D_x^\frac{d}{2} u_{N_2} \|_{L_t^2 L_x^\lambda_1, L_x^2, L_x^2}
\]
from the Kato-Ponce inequality and the Bernstein inequality. This can be summed up both in the homogeneous and in the inhomogeneous version. □

This argument also implies the scattering claim, since it implies that the Duhamel integral converges to a free solutions as \( t \to \pm \infty \). We omit the details of this standard argument.

3. Transversal estimates and the proof of Theorem 1.2

**Lemma 3.1.** Let \( d \geq 2 \) and \( f_{N_1, \lambda_1} = Q_{\lambda_1} P_N f, g_{N_2, \lambda_2} = Q_{\lambda_2} P_N g \). For all \( \lambda_j, N_j \in 2^\mathbb{N} \) such that
\[
\| \nabla \varphi (\xi) - \nabla \varphi (\eta) \| \geq \max \{ \lambda_1, \lambda_2 \} N_{\max},
\]
for all \( \xi \in \text{supp} \widehat{f}_{N_1, \lambda_1}, \eta \in \text{supp} \widehat{g}_{N_2, \lambda_2} \), it holds that
\[
\| P_N (e^{iBT} f_{N_1, \lambda_1} e^{iBT} g_{N_2, \lambda_2}) \|_{L_t^2 L_x^2} \leq \left( \frac{N_{\min}^{d-1}}{\max \{ \lambda_1, \lambda_2 \} N_{\max}} \right)^\frac{1}{2} \| f_{N_1, \lambda_1} \|_{L^2} \| g_{N_2, \lambda_2} \|_{L^2}.
\]
(3.1)

This is an instance of the well-known bilinear transversal estimate, e.g. a special case of [1] Lemma 2.6, where a proof can be found.

Next, we recall the definitions of \( U^p \) and \( V^p \) spaces, which have been introduced in [8] the dispersive PDE context. We refer the reader to [1] and the references therein for further details. For \( 1 \leq p < \infty \), we call a function \( a : \mathbb{R} \to L^2 (\mathbb{R}^d) \) a \( p \)-atom, if there exists a finite partition \( J = \{ (- \infty, t_1), [t_2, t_3], \ldots, [t_K, \infty) \} \) of the real line such that
\[
a(t) = \sum_{J \in J} 1(t) f_J, \quad \sum_{J \in J} \| f_J \|_{L^2}^p \leq 1.
\]

Now, \( U^p \) is defined as the space of all \( u : \mathbb{R} \to L^2 (\mathbb{R}^d) \), such that there exists an atomic decomposition \( u = \sum_{j=1}^\infty c_j a_j \), where \( (c_j) \in \ell^1 (\mathbb{N}) \) and the \( a_j \)'s are \( p \)-atoms. Then, \( \| u \|_{U^p} = \inf \sum_{j=1}^\infty |c_j| \) is a norm (the infimum is taken with respect to all possible atomic decompositions), so that \( U^p \) is a Banach space. Further, let \( V^p \) denote the space of all right-continuous functions \( v : \mathbb{R} \to L^2 (\mathbb{R}^d) \), such that
\[
\| v \|_{V^p} = \| v \|_{L_t^\infty L_x^2} + \sup \left( \sum_{j \in \mathbb{Z}} \| v(t_j) - v(t_{j-1}) \|_{L_x^2}^p \right)^\frac{1}{p} < \infty,
\]
where the supremum is taken over all increasing sequences \( (t_j) \). Now, we define the atomic space \( U^q_{S} = e^{-S} U^p \) with norm \( \| u \|_{U^q_{S}} = \| e^{-S} u \|_{U^p} \), and \( V^q_{S} = e^{-S} V^p \) with norm \( \| u \|_{V^q_{S}} = \| e^{-S} u \|_{V^p} \).

There is the embedding \( V^p_{S} \subset U^q_{S} \) if \( p < q \), see [8] Lemma 6.4. Due to the atomic structure of \( U^q_{S} \) and the Strichartz estimate (2.1), we have
\[
\| D_x^\frac{d}{2} u \|_{L_t^2 L_x^2} \lesssim \| u \|_{U^q_{S}} \quad \text{(3.2)}
\]
for \( (d-1) \)-admissible pairs, and \( \| u \|_{U^q_{S}} \) may be replaced by \( \| u \|_{V^q_{S}} \) for non-endpoint pairs, i.e. when \( q > 2 \).
Let $\lambda_{\max} := \max(\lambda_1, \lambda_2, \lambda_3)$ and $\lambda_{\min} := \min(\lambda_1, \lambda_2, \lambda_3)$. We use the shorthand notation $u_N := P_N u$, $u_{N, \lambda} := Q_\lambda P_N u$, etc.  

**Proposition 3.2.** Let $d \geq 6$ and the pair $(q, r)$ be $(d - 1)$-admissible with $2 < q < \frac{2(d-3)}{d-6}$, and let $\varepsilon > 0$. Suppose

$$|\nabla \varphi(\xi) - \nabla \varphi(\eta)| \gtrsim \max\{\lambda_1, \lambda_2\} N_{\max},$$

for all $\xi \in \supp \hat{u}_N, \lambda_1, \eta \in \supp \hat{u}_N, \lambda_2$. Then, for all $\lambda, N \in \mathbb{Z}^+$,

$$\left\| P_{N_3} Q_{\lambda} (u_{N_1, \lambda} u_{N_2, \lambda}) \right\|_{L^q_t L^r_x L^2_z} \lesssim \frac{\lambda_{\max}}{\lambda_{\min}} \frac{1}{N_{\min}} \frac{d-3-q}{d-1} \left\| u_{N_1, \lambda_1} \right\|_{V^2_q} \left\| u_{N_2, \lambda_2} \right\|_{V^2_r}.$$

**Proof.** By symmetry, we may assume that $\lambda_1 \sim \lambda_{\max}$. For a sufficiently small $\varepsilon > 0$, we define the $(d - 1)$-admissible pairs $(q_1, r_1)$ and $(q_2, r_2)$ by

$$\left(\frac{1}{q_1}, \frac{1}{r_1}\right) = \left(\frac{1}{2} - \varepsilon, \frac{1}{2} - \frac{1 - 2\varepsilon}{d - 1}\right), \quad \left(\frac{1}{q_2}, \frac{1}{r_2}\right) = \left(\frac{d - 3}{4} - \frac{d - 3}{2q} + \varepsilon, \frac{d - 3}{q(d - 1)} + \frac{1 - 2\varepsilon}{d - 1}\right).$$

In addition, letting

$$\frac{1}{\alpha} = \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{1}{\beta} = \frac{1}{r_1} + \frac{1}{r_2},$$

by using (3.2), we have

$$\left\| P_{N_3} Q_{\lambda} (u_{N_1, \lambda} u_{N_2, \lambda}) \right\|_{L^q_t L^r_x L^2_z} \lesssim \frac{\lambda_{\max}}{\lambda_{\min}} \frac{1}{N_{\min}} \frac{d-3-q}{d-1} \left\| u_{N_1, \lambda_1} \right\|_{L^{q_1}_t L^{r_1}_x L^2_z} \left\| u_{N_2, \lambda_2} \right\|_{L^{q_2}_t L^{r_2}_x L^2_z} \lesssim \frac{\lambda_{\max}}{\lambda_{\min}} \frac{1}{N_{\min}} \frac{d-3-q}{d-1} \left\| u_{N_1, \lambda_1} \right\|_{V^2_q} \left\| u_{N_2, \lambda_2} \right\|_{V^2_r}. \tag{3.4}$$

Lemma 3.1 immediately extends from free solutions to $2$-atomic. Therefore, the atomic structure of $U^2$ implies

$$\left\| P_{N_3} Q_{\lambda} (u_{N_1, \lambda} u_{N_2, \lambda}) \right\|_{L^q_t L^r_x L^2_z} \lesssim \left(\frac{N^{d-1}}{\lambda_{\max} N_{\max}}\right)^{\theta} \left\| u_{N_1, \lambda_1} \right\|_{V^2_q} \left\| u_{N_2, \lambda_2} \right\|_{V^2_r}. \tag{3.5}$$

For $\theta = \frac{2}{d-3}$, it is observed that

$$\frac{\theta + 1}{\alpha - \frac{2}{d-3}} = 1 - \frac{1}{q}, \quad \frac{\theta + 1}{\beta - \frac{2}{d-3}} = 1 - \frac{1}{r}.$$

Now, we interpolate (3.4) and (3.5) to obtain (3.3). More precisely, we follow the argument in [1] p. 1203: For brevity, we set $u := u_{N_1, \lambda}, v := u_{N_2, \lambda}$. Then, [8] Lemma 6.4 implies that there exist decompositions $u = \sum_{k=1}^\infty u_k$, such that $\hat{u}_k \subset \supp \hat{u}$, and for any $q \geq 2$ we have $\|u_k\|_{L^q_u} \lesssim 2^{k(\frac{2}{q} - 1)} \|u\|_{V^2_q}$, and the analogous decomposition for $v$. Then, by convexity, we obtain

$$\left\| P_{N_3} Q_{\lambda} (u v) \right\|_{L^q_t L^r_x L^2_z} \lesssim \sum_{k, k' \in \mathbb{N}} \left\| P_{N_3} Q_{\lambda} (u_k v_{k'}) \right\|_{L^q_t L^r_x L^2_z} \lesssim \sum_{k, k' \in \mathbb{N}} \left\| P_{N_3} Q_{\lambda} (u_k v_{k'}) \right\|_{L^q_t L^r_x L^2_z} \left\| P_{N_3} Q_{\lambda} (u_k v_{k'}) \right\|_{L^q_t L^r_x L^2_z}^{1-\theta}.$$

Estimates (3.4) and (3.5) further imply

$$\left\| P_{N_3} Q_{\lambda} (u v) \right\|_{L^q_t L^r_x L^2_z} \lesssim \left(\sum_{k, k' \in \mathbb{N}} 2^{k(\frac{2}{q} - 1)} 2^{k'(\frac{2}{r} - 1)}\right)^{\frac{\theta}{\theta - \frac{2}{q} - \frac{2}{r}}} \frac{\lambda_{\max}}{\lambda_{\min}} \frac{d-3-q}{d-1} \frac{\lambda_{\max}}{\lambda_{\min}} \left\| u \right\|_{V^2_q} \left\| v \right\|_{V^2_r}.$$

Since $q_1, q_2 > 2$, the sums converge, and the proof of (3.3) is complete. \qed
We may assume $|\nabla|$ which implies the claim since $\partial$. 

**Proof.** Firstly, we consider the case \( \max(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^d \) such that $\gamma_1 + \gamma_2 - \gamma_3 \in R_{\lambda_{\max}}$. Then we have either

\[
\text{supp} \, \hat{u}_{N_1, \lambda_1} \subset R_{\lambda_{\max}}, \quad \text{or} \quad \text{max} \left\{ \xi \in \mathbb{R}^d \mid \xi - \gamma_i \in R_{\lambda_{\max}} \right\}. 
\]

Then we have either

\[
|\nabla \varphi(\xi) - \nabla \varphi(\eta)| \gtrsim \lambda_{\max} \gamma_i N_{\max}, 
\]

for all $\xi \in \text{supp} \, \hat{u}_{N_1, \lambda_1}$, $\eta \in \text{supp} \, \hat{u}_{N_2, \lambda_2}$, or

\[
|\nabla \varphi(\eta) - \nabla \varphi(\xi)| \gtrsim \lambda_{\max} \gamma_i N_{\max}, 
\]

for all $\eta \in \text{supp} \, \hat{u}_{N_2, \lambda_2}$, $\xi \in \text{supp} \, \hat{u}_{N_3, \lambda_3}$.

**Proof.** Firstly, we consider the case max(|\xi'|, |\eta'|, |\zeta'|) $\ll N_{\max}$. We deduce from $\partial_1 \varphi(\xi) = 3\xi_1^2 + |\xi'|^2$ that

\[
|\partial_1 \varphi(\xi) - \partial_1 \varphi(\eta)| + |\partial_1 \varphi(\eta) - \partial_1 \varphi(\xi + \eta)| \gtrsim 3|\xi_1^2 - \eta_1^2| + 3|\xi_1 + \eta_1| - |\zeta'|^2 - |\eta'|^2 - |\xi'|^2 \gtrsim N_{\max}^2, 
\]

which implies the claim since $|\nabla^2 \varphi(\xi) \ll |\xi|$.

Next we assume max(|\xi'|, |\eta'|, |\zeta'|) $\sim N_{\max}$. For all $\xi \in \text{supp} \, \hat{u}_{N_1, \lambda_1}$, $\eta \in \text{supp} \, \hat{u}_{N_2, \lambda_2}$, $\xi + \eta \in \text{supp} \, \hat{u}_{N_3, \lambda_3}$, we will show

\[
\sum_{j=2}^{d} \left| \partial_j \varphi(\xi) - \partial_j \varphi(\eta) \right| \gtrsim \lambda_{\max} N_{\max}. 
\]

We may assume |\xi'| $\ll N_{\max}, \lambda_1 \sim \lambda_{\max}$. For $2 \leq j \leq d$, it is observed that $\partial_j \varphi(\xi) = 2\xi_j$. Then, for \(3.9\), it suffices to show

\[
|\xi_1 \zeta' - \eta_1 \eta'| + |\eta_1 \eta' - (\xi_1 + \eta_1)(\zeta' + \eta')| \gtrsim \lambda_1 N_1. 
\]
Since $|\xi'| \sim N_{\text{max}}, \lambda_1 \sim \lambda_{\text{max}}$, if either $\lambda_{\text{min}} \ll \lambda_{\text{max}}$ or $\min(|\eta'|, |\xi' + \eta'|) \ll N_{\text{max}}$ holds, we easily verify (3.10). Then we assume $\lambda_1 \sim \lambda_2 \sim \lambda_3$ and $|\xi'| \sim |\eta'| \sim |\xi' + \eta'|$. We observe

$$|\eta_1\eta' - (\xi_1 + \eta_1)(\xi' + \eta')| = |\eta_1\eta' - (\xi_1 + \eta_1)(\xi' - \frac{\eta_1}{\xi_1}\eta' + \frac{\eta'}{\xi_1})|$$

$$\geq |\eta_1\eta' - (\xi_1 + \eta_1)(1 + \frac{\eta_1}{\xi_1})\eta'| - |(1 + \frac{\eta_1}{\xi_1})(\xi_1\xi' - \eta_1\eta')|$$

$$= |1 + \frac{\eta_1}{\xi_1} + \frac{\xi_1}{\eta_1}| |\eta_1\eta'| - |(1 + \frac{\eta_1}{\xi_1})(\xi_1\xi' - \eta_1\eta')|.$$

Since $|\alpha + \alpha^{-1} | \geq 2$ for any $\alpha \in \mathbb{R}$, this completes the proof of (3.10).

From (3.9), without loss of generality, we can assume that there exist $\xi_0 \in \text{supp}_\xi \hat{u}_{N_1,\lambda_1}$, $\eta_0 \in \text{supp}_\xi \hat{u}_{N_2,\lambda_2}$ such that

$$\sum_{j=2}^{d} |\partial_j \varphi(\xi_0) - \partial_j \varphi(\eta_0)| \geq \lambda_{\text{max}}N_{\text{max}}. \quad (3.11)$$

For $2 \leq j, k \leq d$ and all $\xi \in \text{supp}_\xi \hat{u}_{N_1,\lambda_1}$, $\eta \in \text{supp}_\xi \hat{u}_{N,\lambda_2}$, since $|\partial_\lambda \partial_j \varphi(\xi) + |\partial_\lambda \partial_j \varphi(\eta)| \lesssim N_{\text{max}}$ and $|\partial_\lambda \partial_j \varphi(\xi)| + |\partial_\lambda \partial_j \varphi(\eta)| \lesssim \lambda_{\text{max}}$, we get

$$|\partial_j \varphi(\xi) - \partial_j \varphi(\xi_0)| + |\partial_j \varphi(\eta) - \partial_j \varphi(\eta_0)| \ll \lambda_{\text{max}}N_{\text{max}},$$

for all $\xi \in \text{supp}_\xi \hat{u}_{N_1,\lambda_1}$, $\eta \in \text{supp}_\xi \hat{u}_{N_2,\lambda_2}$. This estimate and (3.11) yield the claim.

Now we define the solution spaces as $Y^s := C(\mathbb{R}; H^s(\mathbb{R}^d) \cap \langle \nabla_x \rangle^{-s} V_2^2)$ and $\hat{Y}^s := C(\mathbb{R}; \hat{H}^s(\mathbb{R}^d) \cap \langle \nabla_x \rangle^{-s} V_2^2)$, with norms

$$\|u\|_{Y^s} := \left( \sum_{N \in 2^\mathbb{Z}} \langle N \rangle^{2s} \|P_N u\|_{V_2^2} \right)^{1/2},$$

$$\|u\|_{\hat{Y}^s} := \left( \sum_{N \in 2^\mathbb{Z}} N^{2s} \|P_N u\|_{V_2^2} \right)^{1/2},$$

respectively.

**Proposition 3.4.** Let $d \geq 6$. Then we have

$$\|I(\partial_{x_1}(u_1u_2))\|_{Y^{s_+}} \lesssim \|u_1\|_{Y^{s_+}}\|u_2\|_{Y^{s_+}}, \quad \|I(\partial_{x_1}(u_1u_2))\|_{\hat{Y}^{s_+}} \lesssim \|u_1\|_{\hat{Y}^{s_+}}\|u_2\|_{\hat{Y}^{s_+}}.$$  

**Proof.** We show first that there exists $\varepsilon > 0$ such that for any $N_1, N_2, N_3 \in 2^\mathbb{Z}$ we have

$$\left| \int \int P_{N_1} u_1 P_{N_2} u_2 \partial_{x_1} P_{N_3} u_3 dxdt \right| \lesssim N_{\text{max}}^{s_+} \|P_{N_1} u_1\|_{V_2^2}^3 \|P_{N_2} u_2\|_{V_2^2}^3 \|P_{N_3} u_3\|_{V_2^2}. \quad (3.12)$$

As before, we use the shorthand notation $u_{N_j} := P_{N_j} u_j, u_{N,\lambda_j} := Q_{\lambda_j} P_{N_j} u_j, \text{etc.}$. Obviously, (3.12) is implied by

$$\sum_{\lambda_1, \lambda_2, \lambda_3 \in 2^\mathbb{Z}} \lambda_1 \left| \int \int u_{N_1,\lambda_1} u_{N_2,\lambda_2} u_{N_3,\lambda_3} dxdt \right| \lesssim N_{\text{max}}^{s_+} \|P_{N_1} u_1\|_{V_2^2}. \quad (3.13)$$

Now we show (3.13). After harmless decompositions, we may assume that there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}^d$ such that $\gamma_1 + \gamma_2 - \gamma_3 \in \mathcal{R}_{4\lambda_{\text{max}},4N_{\text{max}}}$. Lemma (3.3) provides either $|\nabla \varphi(\xi) - \nabla \varphi(\eta)| \ll \lambda_{\text{max}} N_{\text{max}}$ for all $\xi \in \text{supp}_\xi \hat{u}_{N_1,\lambda_1}, \eta \in \text{supp}_\xi \hat{u}_{N_2,\lambda_2}$, and $u_{N_1,\lambda_1} u_{N_2,\lambda_2} u_{N_3,\lambda_3}$ is a solution to

$$\frac{\partial u_{N_1,\lambda_1} u_{N_2,\lambda_2} u_{N_3,\lambda_3}}{\partial \xi} = \mathcal{L}_{\gamma_1} \mathcal{L}_{\gamma_2} \mathcal{L}_{\gamma_3} u_{N_1,\lambda_1} u_{N_2,\lambda_2} u_{N_3,\lambda_3},$$
supp₂πμ₁ or |∇φ(η) − ∇φ(ξ)| ≥ λmaxNmax for all η ∈ supp₂πμ₁ and ξ ∈ supp₂πμ₁. For the former case, it follows from the Hörmander’s inequality, the Strichartz estimate (3.2), and the bilinear estimate (3.3) that

\[ \sum_{λ_1, λ_2, λ_3 ∈ 2\mathbb{Z}} λ_3 \left| \int u_{N_1, λ_1} u_{N_2, λ_2} u_{N_3, λ_3} dxdt \right| \]

\[ ≤ \sum_{λ_1 ≤ N_1 (i=1,2,3)} λ_1 \| P_{N_1} Q_{λ_1} (u_{N_1, λ_1} u_{N_2, λ_2}) \|_{L^3_t L^9_x \cap L^3_t L^9_x} \| u_{N_1, λ_1} \|_{L^3_t L^9_x \cap L^3_t L^9_x} \]

\[ ≤ N_{\min}^{α_1 + \frac{1}{2} - \frac{d-1}{r} + \frac{d-1}{q} + \frac{d-1}{q} + \frac{d-1}{q}} \| u_{N_1} \|_{V^3_{\beta}} \| u_{N_2} \|_{V^3_{\beta}} \| u_{N_3} \|_{V^3_{\beta}}. \]

Here, the pair (q, r) should satisfy the hypothesis of Proposition 3.2 and we have used λmax ≤ Nmax and λmin ≤ Nmin. In the similar way, the latter case is treated as follows:

\[ \sum_{λ_1, λ_2, λ_3 ∈ 2\mathbb{Z}} λ_3 \left| \int u_{N_1, λ_1} u_{N_2, λ_2} u_{N_3, λ_3} dxdt \right| \]

\[ ≤ \sum_{λ_1 ≤ N_1 (i=1,2,3)} λ_3 \| P_{N_1} Q_{λ_1} (u_{N_2, λ_2} u_{N_3, λ_3}) \|_{L^3_t L^9_x \cap L^3_t L^9_x} \| u_{N_1, λ_1} \|_{L^3_t L^9_x \cap L^3_t L^9_x} \]

\[ ≤ N_{\min}^{α_1 + \frac{1}{2} - \frac{d-1}{r} + \frac{d-1}{q} + \frac{d-1}{q} + \frac{d-1}{q}} \| u_{N_1} \|_{V^3_{\beta}} \| u_{N_2} \|_{V^3_{\beta}} \| u_{N_3} \|_{V^3_{\beta}}. \]

Finally, we explain why (3.12) implies Proposition 3.4. By duality, see e.g. [1, Lemma 7.3], we obtain

\[ \| P_{N_1} T_δ (P_{N_1} u_1 P_{N_2} u_2) \|_{V^3_{\beta}} ≤ N_{\min}^{α_1} \left( \frac{N_{\min}}{N_{\max}} \right)^{\varepsilon} \| P_{N_1} u_1 \|_{V^3_{\beta}} \| P_{N_2} u_2 \|_{V^3_{\beta}}. \]

This can be easily summed up. \( \square \)

Again, the proof of Theorem 1.2 is a straight-forward application of the contraction mapping principle. The scattering claim follows from the well-known fact that functions in \( V^2 \) have limits at \( ±\infty \).

4. Radial Strichartz estimates and the proof of Theorem 1.3

We first prove a variant of the Strichartz estimates in (2.1) for functions which, for fixed \( x_1 \), are radial in \( x' \).

**Theorem 4.1.** Let \( d ≥ 3 \) and \( 2 ≤ q, r ≤ \infty \) satisfy

\[ \frac{2}{q} ≤ \frac{(2d - 3) \left( \frac{1}{2} - \frac{1}{r} \right)}{①}, \quad (d, q, r) ≠ (3, 2, \infty), \quad (q, r) ≠ \left( 2, \frac{2(2d - 3)}{2d - 5} \right). \]

and let \( σ = -\frac{d-1}{q} + \frac{d-1}{r} + \frac{1}{2} \). Then, for all functions \( f ∈ L^2_{rad}(\mathbb{R}^d) \), we have

\[ \| D^\frac{1}{2}_{x_1} |\nabla x'|^σ e^{iS} f \|_{L^q_t L^r_x \cap L^q_t L^r_x} \lesssim \| f \|_{L^2_x}. \] (4.1)

The proof follows the exact same lines as the proof of Theorem 2.1 but with the Strichartz estimates for the \( (d-1) \)-dimensional Schrödinger equation from [6] replaced by the radial version obtained in [2, Theorem 1.1].
Lemma 4.2. Let $d \geq 2$ and $f_{N_1, \lambda_1, M_1} = R_M Q_{\lambda_1} P_{N_1} f$, $g_{N_2, \lambda_2, M_2} = R_M Q_{\lambda_2} P_{N_2} g$.

(i) Suppose that there exists $\ell \in \{2, \ldots, d\}$ such that
$$|\partial_\ell \varphi(\xi) - \partial_\ell \varphi(\eta)| \gtrsim N_{\max}^2,$$
for all $\xi \in \text{supp} \, \hat{f}_{N_1, \lambda_1, M_1}$, $\eta \in \text{supp} \, \hat{g}_{N_2, \lambda_2, M_2}$. Then it holds that
$$\|P_{N_1}(e^{iS} f_{N_1, \lambda_1, M_1} e^{iS} g_{N_2, \lambda_2, M_2})\|_{L^2_x L^{\infty}_t} \lesssim \left( \frac{\min\{\lambda_1, \lambda_2\} \min\{M_1, M_2\}^{-d-2}}{N_{\max}^2} \right)^\frac{1}{2} \|f_{N_1, \lambda_1, M_1}\|_{L^2_x} \|g_{N_2, \lambda_2, M_2}\|_{L^2_x}. \quad (4.2)$$

(ii) Suppose that
$$|\partial_1 \varphi(\xi) - \partial_1 \varphi(\eta)| \gtrsim N_{\max}^2,$$
for all $\xi \in \text{supp} \, \hat{f}_{N_1, \lambda_1, M_1}$, $\eta \in \text{supp} \, \hat{g}_{N_2, \lambda_2, M_2}$. Then it holds that
$$\|P_{N_1}(e^{iS} f_{N_1, \lambda_1, M_1} e^{iS} g_{N_2, \lambda_2, M_2})\|_{L^2_x L^{\infty}_t} \lesssim \left( \frac{\min\{M_1, M_2\}^{-d}}{N_{\max}^2} \right)^\frac{1}{2} \|f_{N_1, \lambda_1, M_1}\|_{L^2_x} \|g_{N_2, \lambda_2, M_2}\|_{L^2_x}. \quad (4.3)$$

As above, the proof of this lemma follows from [1, Lemma 2.6]. As above, it immediately extends to $U_{3/2}$-functions.

Let $Y^s_{rad}$ and $\tilde{Y}^s_{rad}$ be the subspaces of $Y^s$ and $\tilde{Y}^s$ of functions which, for fixed $x_1$, are radial in $x'$. Then, the key for the proof of Theorem 1.3 is the following

Proposition 4.3. Let $d \geq 4$. Then we have
$$\|\mathcal{I}(\partial_{x_1}(u_1 u_2))\|_{Y^s_{rad} \tilde{Y}^s_{rad}} \lesssim \|u_1\|_{Y^s_{rad}} \|u_2\|_{\tilde{Y}^s_{rad}}, \quad \|\mathcal{I}(\partial_{x_1}(u_1 u_2))\|_{\tilde{Y}^s_{rad} \tilde{Y}^s_{rad}} \lesssim \|u_1\|_{\tilde{Y}^s_{rad}} \|u_2\|_{\tilde{Y}^s_{rad}}. \quad (4.4)$$

Proof. For $i = 1, 2, 3$, we use $u_i := R_M Q_{\lambda_i} P_{N_i} u$. As in the proof of Proposition 3.4 it suffices to show
$$\sum_{\lambda_i, M_i} \lambda_{\max} \left| \int u_1 u_2 u_3 dx dt \right| \lesssim N_{\max}^3 \prod_{i=1}^3 \|u_{N_i}\|_{V^s_3}. \quad (4.5)$$

Here and in the sequel, all functions are implicitly assumed to satisfy the radially hypothesis. Let
$$\left( \frac{1}{q_1}, \frac{1}{r_1} \right) = \left( \frac{1}{2} - \varepsilon, \frac{2d - 5}{2(2d - 3)} \right),$$
$$\left( \frac{1}{q_2}, \frac{1}{r_2} \right) = \left( \frac{d - 1}{2d - 4} \left( \frac{2d - 3}{2} \varepsilon + \frac{d - 1}{2d - 3} \varepsilon \right), \frac{d - 1}{2d - 4} \left( \frac{2d - 3}{2} \varepsilon + \frac{d - 1}{2d - 3} \varepsilon \right) \right),$$
$$\left( \frac{1}{q_3}, \frac{1}{r_3} \right) = \left( 2, \frac{2}{d - 3} \right).$$

Then we have
$$\|R_M Q_{\lambda} P_{N} u\|_{L^q_{t} L^r_{x}} \lesssim \lambda^{-\frac{1}{r_1}} M^{-\frac{d - 1}{2d - 3} + 2\varepsilon} \|R_M Q_{\lambda} P_{N} u\|_{U_{3/2}^s}, \quad (4.6)$$
$$\|R_M Q_{\lambda} P_{N} u\|_{L^q_{t} L^r_{x}} \lesssim \lambda^{-\frac{1}{r_2}} \|R_M Q_{\lambda} P_{N} u\|_{U_{3/2}^s}, \quad (4.7)$$
$$\|R_M Q_{\lambda} P_{N} u\|_{L^q_{t} L^r_{x}} \lesssim \lambda^{-\frac{1}{r_3}} M^{-\frac{1}{2d - 4} \left( \frac{2d - 3}{2} \varepsilon + \frac{d - 1}{2d - 3} \varepsilon \right)} \|R_M Q_{\lambda} P_{N} u\|_{U_{3/2}^s}. \quad (4.8)$$

By symmetry of (4.4) we may assume $N_3 \lesssim N_1 \sim N_2$, $\lambda_2 \lesssim \lambda_1$, and then it is enough to consider the following three cases:

(1) $M_1 \sim N_1$, $M_2 \sim N_2$, (2) $M_1 \sim N_1$, $M_2 \ll N_1$, (3) $M_1 \ll N_1$, $M_2 \ll N_1$. 

(1) First, we assume \( M_1 \sim N_1, M_2 \sim N_2 \). By using (3.5) and (1.7) we obtain
\[
\left| \int \int u_1 u_2 u_3 dx dt \right| \lesssim \lambda_{\min}^\frac{1}{2} \| u_1 \|_{L^\infty_t L^1_x} \| u_2 \|_{L^{d-1}_t L^d_x} \| u_3 \|_{L^2_t L^{2d-4}_x} \| u_3 \|_{L^2_t L^{2d-4}_x} \| u_3 \| \prod_{i=1,2,3} \| u_i \|_{V^\frac{d}{2}_3},
\]
which completes (4.4).

(2) In the case \( M_1 \sim N_1, M_2 \ll N_1 \), it is observed that \( \lambda_2 \sim M_3 \sim N_1 \). Then, without loss of generality, we may assume \( \lambda_3 \lesssim \lambda_1 \sim N_1 \). In the case \( \lambda_3 \ll \lambda_1 \), for all \( \xi \in \text{supp}_\xi \tilde{u}_1, \eta \in \text{supp}_\xi \tilde{u}_2 \) such that \( \xi + \eta \in \text{supp}_\xi \tilde{u}_3 \), we observe
\[
|\tau_1 - \varphi(\xi)| + |\tau_2 - \varphi(\eta)| + |\tau_1 + \tau_2 - \varphi(\xi + \eta)| \\
\gtrsim |\varphi(\xi + \eta) - \varphi(\xi) - \varphi(\eta)| \\
\gtrsim |\xi| |\eta|^2 - |\xi_1 + \eta_1| |\xi + \eta|^2 + \xi_1^2 - \xi_1 \eta_1 + \eta_1^2 - |\eta_1|^2 \gtrsim N_1^3.
\]
Thus we can assume that at least one of \( u_1, u_2, u_3 \) satisfies \( \text{supp}_\xi \tilde{u}_i \subset \{ (\tau, \xi) | |\tau - \varphi(\xi)| \gtrsim N_1^3 \} \). We easily see that this condition verifies the claim by utilizing Theorem 2.1 and (3.2). For example, if \( \text{supp}_\xi \tilde{u}_1 \subset \{ (\tau, \xi) | |\tau - \varphi(\xi)| \gtrsim N_1^3 \} \), using Bernstein’s inequality and Theorem 4.1 we obtain
\[
\left| \int \int u_1 u_2 u_3 dx dt \right| \lesssim \| u_1 \|_{L^\infty_t L^2_x} \| u_2 \|_{L^{d-1}_t L^d_x} \| u_3 \| \prod_{i=1,2,3} \| u_i \|_{V^\frac{d}{2}_3}.
\]
Next we consider the case \( \lambda_1 \sim \lambda_2 \sim \lambda_3 \sim N_1 \). Since \( M_2 \ll M_1 \sim \lambda_1 \), we may assume that there exists \( \ell \in \{ 2, \ldots, d \} \) such that \( |\partial_\ell \varphi(\xi) - \partial_\ell \varphi(\eta)| \gtrsim N_1^2 \) for \( \xi \in \text{supp}_\xi \tilde{u}_1, \eta \in \text{supp}_\xi \tilde{u}_2 \). Then, from (4.12) we get
\[
\| u_1 u_2 \|_{L^\infty_t L^2_x} \lesssim M_2^{\frac{d}{2d-4}} N_1^{\frac{1}{2}} \| u_1 \|_{V^{3}_{d/2}} \| u_2 \|_{V^{3}_{d/2}}. \tag{4.8}
\]
On the other hand, for \((\frac{1}{\alpha}, \frac{1}{\beta}) = (\frac{1}{q_1} + \frac{1}{q_2}, \frac{1}{r_1} + \frac{1}{r_2})\), we have
\[
\| u_1 u_2 \|_{L^{\alpha}_t L^{\beta}_x} \lesssim \lambda_{\min}^{\frac{1}{2}} \| u_1 \|_{L^{\infty}_t L^{2d-4}_x} \| u_2 \|_{L^2_t L^{2d-4}_x} \| u_3 \|_{L^2_t L^{2d-4}_x} \| u_3 \| \prod_{i=1,2,3} \| u_i \|_{V^{\frac{d}{2}_3}}. \tag{4.9}
\]
We notice that \( \alpha, \beta \geq 1 \) and \( q_1, q_2 > 2 \) if \( \varepsilon > 0 \) is chosen sufficiently small. Let \( \theta = \frac{2(1-\frac{d}{2d-4}) - 3\varepsilon}{(d-1)(2d-4) - 4(2d-3)\varepsilon} \). Then, since
\[
\frac{\theta}{\alpha} + \frac{1 - \theta}{2} = \frac{1}{q_1}, \quad \frac{\theta}{\beta} + \frac{1 - \theta}{2} = \frac{1}{r_1},
\]
by interpolating the above two estimates (with a similar argument as in the proof of Proposition 3.4), we have

$$\|u_1 u_2\|_{L_t^q L_x^r L_y^1 L_z^1} \lesssim M_2^{\frac{1}{2} - (1 - \theta)} N_1^{\frac{1}{2} + \frac{d}{2q} + \frac{3}{2} + \frac{3\theta}{2q} + \frac{\theta}{2q} + 3\varepsilon + (1 + \theta)} \|u_1\|_{V_0^1} \|u_2\|_{V_0^1}. \quad (4.12)$$

This and (4.3) yield

$$\left| \int \int u_1 u_2 u_3 dx dt \right| \lesssim \|u_1 u_2\|_{L_t^q L_x^r L_y^1 L_z^1} \|u_3\|_{L_t^{q_1} L_x^{r_1} L_y^{i_1} L_z^{i_1}} \lesssim M_2^{\frac{1}{2} - (1 - \theta)} N_1^{\frac{1}{2} + \frac{d}{2q} + \frac{3}{2} + \frac{3\theta}{2q} + \frac{\theta}{2q} + 3\varepsilon + (1 + \theta)} \prod_{i=1,2,3} \|u_i\|_{V_0^1} \lesssim M_2^{\varepsilon + \varepsilon} N_1^{-1 - \varepsilon} \prod_{i=1,2,3} \|u_i\|_{V_0^1}.$$  

(3) We deal with the last case $M_1 \ll N_1, M_2 \ll N_1$. By symmetry, we assume $M_2 \leq M_1$. Assume first that $M_1 \gtrsim (\lambda_3 N_1)^{\frac{1}{2}}$ which implies $\lambda_3 \ll \lambda_1 \sim \lambda_2 \sim N_1$. Thus, we observe that $|\partial_1 \phi(\xi) - \partial_1 \phi(\eta)| \gtrsim N_1^2$ for $\xi \in \text{supp} \tilde{u}_2, \eta \in \text{supp} \tilde{u}_3$. (4.3) implies

$$\|u_2 u_3\|_{L_t^q L_x^r L_y^1 L_z^1} \lesssim M_2^{\frac{1}{2} - (1 - \theta)} N_1^{-1} \|u_2\|_{U_0^2} \|u_3\|_{U_0^2}.$$ 

While, similarly to the above observation, we get

$$\|u_2 u_3\|_{L_t^q L_x^r L_y^1 L_z^1} \lesssim \lambda_2^{-\frac{d}{2q}} \lambda_3^{-\frac{d}{2q}} M_2^{\frac{1}{2} - (1 - \theta) - \frac{3\theta}{2q} + 3\varepsilon + (1 + \theta)} \prod_{i=1,2,3} \|u_i\|_{V_0^1} \|u_2\|_{U_0^2} \|u_3\|_{U_0^2}.$$ 

Interpolating the above two, we get

$$\|u_2 u_3\|_{L_t^q L_x^r L_y^1 L_z^1} \lesssim \lambda_2^{-\frac{d}{2q}} \lambda_3^{-\frac{d}{2q}} M_2^{\frac{1}{2} - (1 - \theta) - \frac{3\theta}{2q} + 3\varepsilon + (1 + \theta)} N_1^{-1 + \theta} \prod_{i=1,2,3} \|u_i\|_{V_0^1} \|u_2\|_{U_0^2} \|u_3\|_{U_0^2}.$$ 

Consequently, it follows from $M_1 \gtrsim \max\{M_{\min}, (\lambda_3 N_1)^{\frac{1}{2}}\}$ that

$$\left| \int \int u_1 u_2 u_3 dx dt \right| \lesssim \|u_1\|_{L_t^{q_1} L_x^{r_1} L_y^{i_1} L_z^{i_1}} \|u_2 u_3\|_{L_t^{q_1} L_x^{r_1} L_y^{i_1} L_z^{i_1}} \lesssim \lambda_1^{-\frac{d}{2q}} \lambda_3^{-\frac{d}{2q}} M_1^{\frac{1}{2} - (1 - \theta) - \frac{3\theta}{2q} + 3\varepsilon + (1 + \theta)} N_1^{-1 + \theta} \prod_{i=1,2,3} \|u_i\|_{V_0^1} \|u_2\|_{U_0^2} \|u_3\|_{U_0^2} \lesssim \lambda_1^{-1} \lambda_2^{-1} M_2^{\varepsilon + \varepsilon} N_1^{-2\varepsilon} \prod_{i=1,2,3} \|u_i\|_{V_0^1}.$$ 

In the case $M_1 \ll (\lambda_3 N_1)^{\frac{1}{2}}$, we easily observe that at least one of $u_1, u_2, u_3$ satisfies $\text{supp} \tilde{u}_i \subset \{(\tau, \xi) \mid |\tau - \varphi(\xi)| \gtrsim \lambda_3 N_1^2\}$. Indeed, $M_2 \lesssim M_1 \ll (\lambda_3 N_1)^{\frac{1}{2}}$ yields

$$|\tau_1 - \varphi(\xi)| + |\tau_2 - \varphi(\eta)| + |\tau_1 + \tau_2 - \varphi(\xi + \eta)| \gtrsim |\varphi(\xi + \eta) - \varphi(\xi) - \varphi(\eta)| \gtrsim 3|\xi_1 \eta_1 (\xi_1 + \eta_1)| - 10(|\xi_1| + |\eta_1|)(|\xi'|^2 + |\eta'|^2) \gtrsim \lambda_3 N_1^2,$$

for all $\xi \in \text{supp} \tilde{u}_1, \eta \in \text{supp} \tilde{u}_2$ which satisfy $\xi + \eta \in \text{supp} \tilde{u}_3$. In the case $\text{supp} \tilde{u}_1 \subset \{(\tau, \xi) \mid |\tau - \varphi(\xi)| \gtrsim \lambda_3 N_1^2\}$ and $M_2 \lesssim M_3$, since $M_2 \lesssim M_1 \ll (\lambda_3 N_1)^{\frac{1}{2}}$,
it follows from the Strichartz estimates (3.2) and Bernstein’s inequality that
\[
\left| \int_0^t u_1 u_2 u_3 dx dt \right| \lesssim \|u_1\|_{L_t^6 L_x^2} \|u_2\|_{L_t^4 L_x^4}^2 \|u_3\|_{L_t^\infty L_x^2} \lesssim \lambda_1^{\frac{1}{2}} \lambda_2^{\frac{1}{2} + 2\varepsilon} \lambda_3^{-\frac{1}{2}} \prod_{i=1,2,3} \|u_i\|_{L_t^6 L_x^2}^{\frac{1}{2} - 2\varepsilon} N_1^{-\frac{1}{2}} \prod_{i=1,2,3} \|u_i\|_{L_t^6 L_x^2}^\infty.
\]

The other cases are treated similarly.
\[\square\]

As above, Theorem 1.3 follows by the standard argument.

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