Entropy Functions of BPS Black Holes in AdS$_4$ and AdS$_6$

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We find the entropy functions of supersymmetric black holes in AdS$_4$ and AdS$_6$ with electric charges and angular momenta. Extremizing these functions, one obtains the entropies and the chemical potentials of known analytic black hole solutions.

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I. INTRODUCTION

Understanding black holes [1-3] is an important subject in AdS/CFT [4]. In models with supersymmetry, one expects that quantitative analysis at strong coupling would be easier in the BPS sectors of SCFTs. Supersymmetric AdS black holes correspond to thermal ensembles of BPS states, carrying angular momenta and also internal charges (electric charges in AdS). In AdS$_d$ with $d > 3$, supersymmetric black holes have very complicated structures. First of all, it is known that there are no BPS black holes with electric charges only, at zero angular momenta. This is because in the dual field theory, the local BPS operators will reduce to chiral rings which do not have enough numbers of microstates to form black holes: e.g. see Ref. 5 for the case with $d = 5$. With nonzero angular momenta, the solutions appear very involved. See, e.g. Refs. 6 and 7 for $d = 4$, Refs. 8, 9, 10 and 11 for $d = 5$, Ref. 12 for $d = 6$, and Refs. 13 and 14 for $d = 7$.

In Refs. 15 and 16, simple underlying structures of BPS black holes were discovered in AdS$_5$ and AdS$_7$. Firstly, one can obtain the entropies and the chemical potentials of known BPS black holes [8–11] in AdS$_d 	imes S^2$ by extremizing the following entropy function [15],

$$ S(\Delta_I, \omega_i) = \frac{N^2 \Delta_1 \Delta_2 \Delta_3}{2 \omega_1 \omega_2} + \sum_{I=1}^3 \Delta_I Q_I + \sum_{i=1}^2 \omega_i J_i , \quad (1) $$

subject to the constraint $\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 2 \pi i$. Here, $Q_I$ and $J_i$ are $U(1)^3 \subset SO(6)$ electric charges and $U(1)^2 \subset SO(4)$ angular momenta, respectively. The Bekenstein-Hawking entropy of the black hole is the extremal value of $Re(S)$, at one of the extremum solutions for $\Delta_I, \omega_i$ [15]. The black hole chemical potentials are the extremal values of $Re(\Delta_I), Re(\omega_i)$ [17]. Similarly, the properties of known BPS black holes [13,14] in AdS$_7 \times S^4$ can be understood by extremizing the following entropy function [16],

$$ S(\Delta_I, \omega_i) = -\frac{N^3 (\Delta_1 \Delta_2 \Delta_3)^2}{24 \omega_1 \omega_2 \omega_3} + \sum_{I=1}^2 \Delta_I Q_I + \sum_{i=1}^3 \omega_i J_i , \quad (2) $$

subject to the constraint $\Delta_1 + \Delta_2 - \omega_1 - \omega_2 - \omega_3 = 2 \pi i$. $Q_I$ and $J_i$ are $U(1)^3 \subset SO(5)$ electric charges and $U(1)^3 \subset SO(6)$ angular momenta. These extremely simple formulae encode apparently complicated properties of supersymmetric AdS black holes. They triggered interesting follow-up works aiming at microscopic accounts for these black holes [17,18]. In particular Ref. 17, derived Eq. (1) for large AdS$_5$ black holes, from the index of 4d $\mathcal{N} = 4$ Yang-Mills theory (Ref. 17 also found a generalization of Ref. 15 in a different charge sector). References 17 and 19 also provided an anomaly-based microscopic discussion which leads to Eq. (2) for large AdS$_7$.
black holes. Therefore, the simple functions Eqs. (1) and (2) provided very useful inspirations for microscopic studies.

We thus find that it will be very helpful to have entropy function formalisms for supersymmetric AdS black holes in other dimensions. In this note, we provide such functions in AdS$_4$ and AdS$_6$, simplifying the apparently complicated structures of known black hole solutions. These led to better microscopic understandings based on CFT$_3$ [20] and CFT$_5$ duals [21]. In particular, Refs. 20 and 21 statistically accounted for large BPS black holes in AdS$_4$ and AdS$_6$, from the indices of CFT$_3$ and CFT$_5$ duals.

The rest of this note is organized as follows. Section II summarizes the properties of known supersymmetric black holes in AdS$_4 \times S^7$, and show that an entropy function we suggest encodes these properties. Section III makes similar studies with supersymmetric AdS$_6$ black holes. Section IV concludes with remarks.

II. ADS$_4$ BLACK HOLES

1. Black hole solutions

We study the supersymmetric black holes in AdS$_4 \times S^7$ of Ref. 7. These are obtained by taking supersymmetric limits of Ref. 22, also demanding the existence of smooth horizons.

Black holes in AdS$_4 \times S^7$ can carry six kinds of conserved quantities: mass (or energy) $E$, angular momentum $J$ on $S^2$ of global AdS$_4$, and four Cartan charges $Q_I$ ($I = 1, 2, 3, 4$) of SO(8) symmetry on $S^7$. The last four conserved quantities $Q_I$ appear in 4d gravity as $U(1)^4$ electric charges. The convention of Ref. 7 for $Q_I$ is to take four angular momenta acting on the orthogonal 2-planes of $R^8$ related to $S^7$. The most general black holes known to date have pairwise equal electric charges, $Q_1 = Q_3$, $Q_2 = Q_4$. With the last charge restrictions, the four conserved quantities $E$, $J$, $Q_1$, $Q_2$ are labeled by four parameters $m$, $a$, $\delta_1$, $\delta_2$ as Ref. 7.

$$E = \frac{m}{2G\Xi}(\cosh 2\delta_1 + \cosh 2\delta_2) ,$$
$$J = \frac{ma}{2G\Xi}(\cosh 2\delta_1 + \cosh 2\delta_2) ,$$
$$Q_1 = Q_3 = \frac{m}{4G\Xi} \sinh 2\delta_1 ,$$
$$Q_2 = Q_4 = \frac{m}{4G\Xi} \sinh 2\delta_2 ,$$

where $\Xi = 1 - a^2g^2$. The entropy is given by

$$S = \frac{\pi(r_1r_2 + a^2)}{G\Xi} ,$$

where $r_1 = r_+ + 2m \sinh^2 \delta_1$, $r = r_+$ is the location of the event horizon. $G$ is the 4d Newton constant, which will be replaced by microscopic parameters later. (In Ref. 7, all charges and entropy are computed omitting the overall $\frac{1}{G}$ factor, or at $G = 1$. E.g., the entropy is computed by dividing the horizon area by 4, rather than $S = \frac{\pi}{G\Xi}$.) $g$ is a parameter of the 4d gauged supergravity, and is related to the radius $\ell$ of AdS$_4$ as $g = \ell^{-1}$.

The BPS limit of these black holes is given by

$$e^{2\delta_1 + 2\delta_2} = 1 + \frac{2}{ag} ,$$

which corrects a typo of Ref. 7. Only after this correction, the BPS relation

$$E = gJ + \sum_{r=1}^{4} Q_r = gJ + 2Q_1 + 2Q_2$$

is met. A further condition to have a regular horizon is $\Delta_+ = 0$ having a double root at $r = r_+$. (See Ref. 7 for the definition of the function $\Delta_+$.) This happens only after a further tuning of $m$. After the tuning, the horizon location $r = r_+$ is given by

$$r_+ = \frac{2m \sinh \delta_1 \sinh \delta_2}{\cosh(\delta_1 + \delta_2)} ,$$

when $m$ satisfies

$$(mg)^2 = \frac{\cosh^2(\delta_1 + \delta_2)}{e^{\delta_1 + \delta_2} \sinh^2(\delta_1 \sinh(2\delta_1) \sinh(2\delta_2))} .$$

This again corrects the formula $mg = \frac{\cosh(\delta_1 + \delta_2)}{\sinh(\delta_1 + \delta_2) \sinh(2\delta_1) \sinh(2\delta_2)}$ of Ref. 7. The typos found in this paragraph are also reported in Ref. 23.

Taking the BPS limit, the entropy of the supersymmetric black hole is given by

$$S = \frac{2\pi}{g^2G(e^{2\delta_1 + 2\delta_2} - 3)} .$$

The two conditions (Eqs. (5) and (8)) leave two independent parameters among $m$, $a$, $\delta_1$, $\delta_2$. Even after restricting $E$ as Eq. (6) due to the BPS condition, the remaining charges $Q_1$, $Q_2$, $J$ satisfy a relation. Together with $S$, we find the following two relations after taking the BPS limit:

$$S^2 + \frac{\pi}{g^2G} S - 4\pi^2 \frac{2Q_1}{g} \frac{2Q_2}{g} = 0 .$$

Since these equations determine $S$ twice, one will get a charge relation between $Q_1$, $Q_2$, $J$ from the compatibility of two equations. Explicitly, we insert the solution of the first equation to the second equation, demanding two equations have the same solution for $S$. Then, taking the
Defining $\Delta$, one finds
\begin{align}
S &= \frac{\pi}{g^2 G} \frac{J}{\left(\frac{2Q_1}{g} + \frac{2Q_2}{g}\right)} , \\
J &= \frac{1}{2} \left(\frac{2Q_1}{g} + \frac{2Q_2}{g}\right) \left(-1 + \sqrt{1 + 16g^4 G^2 \frac{2Q_1}{g} \frac{2Q_2}{g}}\right).
\end{align}
(11)

Thus, we have explicitly found the charge relation between $Q_1$, $Q_2$, $J$.

The black hole chemical potentials and the free energy $F$ satisfy
\begin{equation}
S = -T^{-1} F(T) + T^{-1} E - T^{-1} \Omega J - T^{-1} \sum_{I=1}^{4} \Phi_I Q_I ,
\end{equation}
(12)
where $T$ is the temperature, $\Omega$ is the angular velocity, and $\Phi_I$'s are the electrostatic potentials. The chemical potentials are evaluated on the horizon. In the BPS limit, we are interested in
\begin{equation}
T = \frac{\Delta_i'}{4\pi(r_1 r_2 + a^2)} \to 0 ,
\end{equation}
(13)
because $\Delta_i$ has a double root at the horizon. On the other hand, as one inserts the value of the variables in the BPS limit, $a = \frac{g \sqrt{e + r_2}}{2 (e + r_2 - 1)}$, $mg$ given by Eq. (8), and then the horizon location $r \to r_+$ (Eq. (7)), one finds
\begin{align}
\Omega &= a \left(1 + \frac{g^2 r_1 r_2}{r_1 r_2 + a^2}\right) \to g , \\
\Phi_1 &= \frac{mr_2 \sinh(2\delta_i)}{r_1 r_2 + a^2} \to 1 , \\
\Phi_2 &= \frac{mr_1 \sinh(2\delta_i)}{r_1 r_2 + a^2} \to 1 .
\end{align}
(14)

Defining $\Delta E$ by $E = \Delta E + 2Q_1 + 2Q_2 + gJ$, one finds that
\begin{equation}
S = -T^{-1} F(T) + T^{-1} \Delta E - T^{-1} (\Omega - g) J - T^{-1} \sum_{I=1}^{4} (\Phi_I - 1) Q_I .
\end{equation}
(15)
The BPS limit satisfies $T \to 0$, $\Delta E \to 0$. One first finds that
\begin{equation}
\omega = -\lim_{T \to 0} (T^{-1} (\Omega - g)) , \quad \Delta_i = -\lim_{T \to 0} (T^{-1} (\Phi_I - 1))
\end{equation}
(16)
are well defined in the BPS limit, by explicitly computing them (although the expressions are very complicated). Since $S$ is also finite in this limit, the ‘BPS free energy’ $F_{\text{BPS}} \equiv \lim_{T \to 0} (T^{-1} (F - \Delta E))$ should also be well defined. So, one finds
\begin{equation}
S = -F_{\text{BPS}} + \omega J + \sum_{I=1}^{4} \Delta_i Q_I
\end{equation}
(17)
in the BPS limit. $-F_{\text{BPS}}$ is to be interpreted as log $Z$, where $Z$ is the BPS partition function of this system. We again stress that the BPS limit is taken by first inserting $aq \to \frac{g^2 r_1}{2 (e + r_2 - 1)}$, $mg \to \sqrt{(\cosh(\delta_i + 2\delta_2) - 1) \cosh^2(\delta_1 + 2\delta_2)}$ and then $r \to 2m \sinh(\delta_1) \sinh(\delta_2)$. This results in quite complicated expressions for $\omega, \Delta_i$. After taking the BPS limit, one can show that they satisfy
\begin{equation}
\Delta_1 + \Delta_2 = \frac{1}{g} \omega \Rightarrow \sum_{I=1}^{4} \frac{g}{2} \Delta_i - \omega = 0 .
\end{equation}
(18)
This is an alternative statement of the charge relation between $Q_1$, $Q_2$, $J$.

2. Entropy function

We now present an entropy function, whose suitable Legendre transformation in $\Delta_i$, $\omega$ yields the entropy $S(Q_i, J)$ and the BPS chemical potentials of the supersymmetric black holes. Our entropy function $S(\Delta_i, \omega; Q_i, J)$ is given by
\begin{equation}
S(\Delta_i, \omega; Q_i, J) = -i \frac{4 \sqrt{2 N} \sqrt{\Delta_i \Delta_2 \Delta_3 \Delta_4}}{3} \omega + \omega J + \sum_{I=1}^{4} \Delta_i Q_I .
\end{equation}
(19)
We extremize $S$ in $\Delta_i, \omega$ with the constraint
\begin{equation}
\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - \omega = 2 \pi i .
\end{equation}
(20)

A microscopic derivation of the entropy function (Eq. (19)) from the CFT3 dual was studied in Ref. 20, in the Cardy limit $\omega \to 0$. Just like AdS$_5$, AdS$_7$ black holes analyzed in Ref. 17, the constraint Eq. (20) is given an interpretation in Ref. 20. Here, the number of M2-branes $N$ is related to the 4d Newton constant $G$ as follows:
\begin{equation}
G_{11} = 16 \pi^7 \ell_P^9 , \quad \ell_{S7} = 2 \ell = \ell_P (2^5 \pi^2 N)^{1/6} \Rightarrow \frac{1}{g^2 G} = \frac{\text{vol}(S^7)}{g^2 G_{11}} = \frac{2 \sqrt{2}}{3} N^{3/2} = \frac{2 \sqrt{2}}{3} N^{3/2} .
\end{equation}
(21)
$\ell_P$ is the 11d Planck scale, $\ell_{S7}$ is the radius of $S^7$, and $\ell$ is the AdS$_4$ radius as defined in the previous subsection. We claim that the external extremal value of $Re(S)$ is the entropy of supersymmetric black holes. We shall check this against the known solutions summarized in the previous subsection, at $Q_1 = Q_3$, $Q_2 = Q_4$ (which is equivalent to $\Delta_1 = \Delta_3$, $\Delta_2 = \Delta_4$). Here, note that the chemical potentials $\Delta_i$, $\omega$ are all complexified. With complex $\Delta_i$, the square root $\sqrt{\Delta_i \Delta_2 \Delta_3 \Delta_4}$ in (19) should be understood as to take the argument of $\Delta_1 \Delta_2 \Delta_3 \Delta_4$ in the principal branch $(-\pi, \pi)$ [20].
We show our claim by extremizing $S$, subject to the constraint (Eq. (20)). We introduce the Lagrange multiplier $\lambda$ and extremize

$$
S = -\frac{4\sqrt{2}N^2}{3} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4} + \omega J + \sum_{i=1}^{4} \Delta_i Q_i + \lambda \left( \sum_{i=1}^{4} \Delta_i - \omega - 2\pi i \right).
$$

The extremum conditions are given by

$$
\lambda + Q_i = \frac{4\sqrt{2}N^2}{3\omega} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \quad (I = 1, \cdots, 4),
$$

$$
\lambda - J = \frac{4\sqrt{2}N^2}{3\omega^2} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}.
$$

Inserting these charges into Eq. (22), to eliminate the appearances of $Q_1$, $J$, one obtains

$$
S = -2\pi \lambda.
$$

Multiplying the four equations on the first line of Eq. (23), one finds

$$
(\lambda + Q_1)(\lambda + Q_2)(\lambda + Q_3)(\lambda + Q_4)
= \frac{64 \pi^6}{81\omega^4} \Delta_1 \Delta_2 \Delta_3 \Delta_4,
$$

$$
= -\frac{2N^3}{9}(\lambda - J)^2.
$$

So, one obtains a very useful expression,

$$
\left( \frac{S}{2\pi i} - Q_1 \right) \left( \frac{S}{2\pi i} - Q_2 \right) \left( \frac{S}{2\pi i} - Q_3 \right) \left( \frac{S}{2\pi i} - Q_4 \right)
= -\frac{2N^3}{9} \left( \frac{S}{2\pi i} + J \right)^2.
$$

One needs care to treat the above expression. While the above is the quartic equation in $S$, only the half of them are the true solutions to Eq. (23) satisfying the constraint Eq. (20). The other halves are the extraneous solutions. Hence, after solving the above equation, one should check whether the resulting solution is a true one.

After extremizing the entropy function, one would generally obtain complex solutions for $S$ by solving Eq. (26). Along the spirit of Ref. 17, we shall generally regard $\text{Re}(S)$ as the entropy at the extremum. See Refs. 17 and 20 for the interpretation of the imaginary part. However, we are primarily interested in comparing our results against the known black hole solutions of Sec. II. 1. Therefore, we impose the charge relation of these black holes and compare the thermodynamic quantities on that surface only. Somewhat remarkably, the charge relation of known black holes will turn out to be $\text{Im}(S) = 0$ at the extremum of our entropy function. So from now on, we demand the existence of a real solution for $S$ in Eq. (26), and compare the results with the known black holes. Demanding real $S$ for real charges $Q_1$, $Q_2$, $Q_3$, $Q_4$, $J$, the complex Eq. (26) is separated into two real equations as follows:

$$
\frac{1}{16\pi^4} S^4 - \frac{\sum_{I<J} Q_I Q_J}{4\pi^2} S^2 + Q_1 Q_2 Q_3 Q_4 = 0,
$$

$$
\frac{N^3}{18\pi^2} S^2 - \frac{2N^3 J^2}{9},
$$

$$
-\frac{\sum_{I<J} Q_I Q_J}{2\pi} S + \frac{\sum_{I<J<K} Q_I Q_J Q_K}{9} S = \frac{2N^3 J}{9\pi} S.(27)
$$

These equations determine $S$ twice as functions of charges. From the compatibility of two equations, one will get a relation of $Q_1$, $J$. Explicitly, one may take the unique positive solution of the second equation and insert it to the first equation, to obtain the charge relation. One can check that this solution is a true solution satisfying Eqs. (23) and (20).

To compare with known black holes summarized in Sec. II. 1, we set $Q_1 = Q_3$, $Q_2 = Q_4$. Then, taking the unique positive solution assuming $Q_1$, $Q_2$, $J > 0$, one obtains

$$
S = \frac{2\pi}{3} \sqrt{9Q_1 Q_2 (Q_1 + Q_2) - 2N^3 J},
$$

$$
0 = 2N^3 J^2 + 2N^3 (Q_1 + Q_2)J - 9Q_1 Q_2 (Q_1 + Q_2)^2.
$$

These can be rearranged as

$$
S = \frac{2\sqrt{2}\pi N^2}{3} \frac{J}{Q_1 + Q_2} = \frac{\pi}{\sqrt{2} g \mathcal{G}} \frac{J}{Q_1 + Q_2},
$$

$$
J = \frac{1}{2} (Q_1 + Q_2) \left(-1 + \sqrt{1 + \frac{18}{N^3} Q_1 Q_2} \right)
$$

$$
= \frac{1}{2} (Q_1 + Q_2) \left(-1 + \sqrt{1 + 16g^2 \mathcal{G}^2 Q_1 Q_2} \right). (29)
$$

One can easily check that this solution indeed satisfies Eqs. (23) and (20), i.e., it is not an extraneous solution. The above expressions are exactly the same as Eq. (11), which we obtained from the supersymmetric black holes. Note that the charges and chemical potentials of the entropy function Eq. (22) are related to those of supersymmetric black holes. Thus, on comparing the results with the known black holes (Eq. (9)), we find that the relation between the chemical potentials of the entropy function (Eq. (20)) is equivalent to that of the supersymmetric black holes (Eq. (18)).

To summarize, our entropy function (Eq. (22)) indeed reproduces the Bekenstein-Hawking entropy of the supersymmetric AdS$_4$ black holes (Eq. (9)) and the corresponding charge/chemical potential relations (Eqs. (11)}
and (18)) at \( Q_1 = Q_3, \ Q_2 = Q_4 \), where solutions are known. Recently, 4 parameter BPS black hole solutions with all different \( Q_i \)'s were discovered in Ref. 24, whose physics is successfully described by our entropy function (Eq. (22)).

### III. ADS\(_6\) BLACK HOLES

#### 1. Black hole solutions

In this section, we study the supersymmetric AdS\(_6\) black holes, and find an entropy function which accounts for their physics. We construct an entropy function for the solution of Ref. 12. The solution may be regarded as describing BPS states of any large \( N \) 5d SCFT dual. For instance, as our favorite example, results in this section may be understood in the context of massive type IIA string theory on warped AdS\(_6\) \( \times S^4/\mathbb{Z}_2 \) product background. This system is dual to 5d \( N = 1 \) SCFT living on \( N \) D4-branes probing the Os\(-\)DS system [25]. The 5d SCFT dual has a gauge theory description, with \( Sp(N) \) gauge group, rank 2 antisymmetric hypermultiplet, and \( N_f \leq 7 \) fundamental hypermultiplets. However, we expect that our general analysis can be embedded to AdS\(_6\) black holes in the backgrounds of Refs. 26, 27 and 28.

The 6d \( N = (1, 0) \) SU(2) gauged supergravity was obtained by a consistent Kaluza-Klein truncation of massive type IIA supergravity on \( S^4/\mathbb{Z}_2 \) [29]. In Ref. 12, the charged rotating AdS\(_6\) black hole solution in this gauged supergravity was obtained. It has four kinds of conserved quantities: mass \( E \), two angular momenta \( J_1, J_2 \), which describe the orthogonal 2-plane rotations on \( S^4 \) in global AdS\(_6\), and one \( U(1) \subset SU(2) \) electric charge \( Q \). They are given in terms of four parameters \( m, a, b, \delta \) of the solution as Ref. 12.

\[
E = \frac{2\pi m}{3G\Xi_a\Xi_b} \left[ \frac{1}{\Xi_a} + \frac{1}{\Xi_b} + \sinh^2 \delta \left(1 + \frac{\Xi_a}{\Xi_b} + \frac{\Xi_b}{\Xi_a}\right) \right], \\
Q = \frac{2\pi m}{G\Xi_a\Xi_b} \sinh 2\delta, \\
J_1 = \frac{2\pi ma}{3G\Xi_a\Xi_b} \left(1 + \Xi_b \sinh^2 \delta\right), \\
J_2 = \frac{2\pi mb}{3G\Xi_a\Xi_b} \left(1 + \Xi_a \sinh^2 \delta\right),
\]

where \( \Xi_a = 1 - a^2 g^2 \) and \( \Xi_b = 1 - b^2 g^2 \). The entropy is given by

\[
S = \frac{2\pi^2 [(r_+^2 + a^2)(r_+^2 + b^2) + 2mr_+ \sinh^2 \delta]}{3G\Xi_a\Xi_b}.
\]

The event horizon is located at \( r = r_+ \). Here, \( G \) is the 6d Newton constant. (In Ref. 12, the unit \( G = 1 \) is used.) \( g \) is a gauge coupling constant in 6d gravity, setting the inverse-radius of AdS\(_6\).

This black hole solution admits the supersymmetric limit without naked closed timelike curves. The BPS condition

\[
E = gJ_1 + gJ_2 + Q
\]

is satisfied if

\[
e^{2\beta} = 1 + \frac{2}{(a + b)g}.
\]

In addition, a smooth horizon exists only if

\[
m = \frac{(a + b)^2(1 + ag)(1 + bg)(2 + ag + bg)}{2(1 + ag + bg)} \sqrt{ab}.
\]

Taking the BPS limit, the entropy of the supersymmetric black hole is given by

\[
S = \frac{2\pi^2 ab(a + b)}{3G(1 - ag)(1 - bg)(1 + ag + bg)}.
\]

The two conditions (Eqs. (34) and (35)) leave two independent parameters among \( m, a, b, \delta \). Even after restricting \( E \) as Eq. (33) from the BPS condition, the remaining charges \( J_1, J_2, Q \) carried by the supersymmetric black holes will satisfy a charge relation. Equivalently, together with \( S \), we find the following two relations:

\[
S^3 - \frac{2\pi^2}{3g^4 G} S^2 - 12\pi^2 \left(\frac{Q}{3g}\right)^2 S + \frac{8\pi^4}{3g^4 G} J_1 J_2 = 0, \\
\frac{Q}{3g} S^2 + \frac{2\pi^2}{9g^2 G} (J_1 + J_2) S - \frac{4\pi^2}{3} \left(\frac{Q}{3g}\right)^3 = 0.
\]

Since these equations determine \( S \) twice, one will get a charge relation between \( J_1, J_2, Q \) from the compatibility of two equations. Explicitly, one may take the unique positive solution of the second equation and insert it to the first equation, to get the charge relation.

The black hole chemical potentials and the free energy \( F \) satisfy

\[
S = -T^{-1} F + T^{-1} E + T^{-1} \Omega_1 J_1 - T^{-1} \Omega_2 J_2 - T^{-1} \Phi Q,
\]

where \( T \) is the temperature, \( \Omega_1, \Omega_2 \) are the angular velocities, and \( \Phi \) is the electrostatic potential. The temperature of the supersymmetric black hole is zero in the BPS smooth horizon limit,
\[ T = \frac{2r^2_c(1+g^2r^2_+)(2a^2+r^2_+ + b^2) - (1-g^2r^2_+)(r^2_+ + a^2)(r^2_+ + b^2) + 8mg^2r_+^3 \sinh^2 \delta - 4m^2g^2 \sinh^4 \delta}{4\pi r_+[(r^2_+ + a^2)(r^2_+ + b^2) + 2mr_+ \sinh^2 \delta]} \rightarrow 0 . \]  

The other chemical potentials in the BPS limit are given by

\[ \Omega_1 = \frac{(1+g^2r^2_+)(r^2_+ + a^2) + 2mg^2r_+ \sinh^2 \delta}{(r^2_+ + a^2)(r^2_+ + b^2) + 2mr_+ \sinh^2 \delta} \rightarrow g, \]
\[ \Omega_2 = \frac{(1+g^2r^2_+)(r^2_+ + b^2) + 2mg^2r_+ \sinh^2 \delta}{(r^2_+ + a^2)(r^2_+ + b^2) + 2mr_+ \sinh^2 \delta} \rightarrow g, \]
\[ \Phi = \frac{(r^2_+ + a^2)(r^2_+ + b^2) + 2mr_+ \sinh^2 \delta}{mr_+ \sinh 2\delta} \rightarrow 1 . \]

Similar to the analysis in Sec. II. 1, the following limits exist,

\[ F_{BPS} = \lim_{T \rightarrow 0} (T^{-1}(F - \Delta E)) , \]
\[ \omega_1 = -\lim_{T \rightarrow 0} (T^{-1}(\Omega_1 - g)) , \]
\[ \Delta = -\lim_{T \rightarrow 0} (T^{-1}(\Phi - 1)) , \]

where \( \Delta E \equiv E - Q - gJ_1 - gJ_2 \). Then, in the zero temperature BPS limit, one obtains

\[ S = -F_{BPS} + \omega_1 J_1 + \omega_2 J_2 + \Delta Q . \]

Using the computed expressions for \( \omega_1, \Delta \), one finds that

\[ \omega_1 + \omega_2 = 3g\Delta . \]

Again, this is the alternative statement of the charge relation of \( J_1, J_2, Q \).

2. Entropy function

We now present an entropy function which encodes the physics of the BPS black holes presented in the previous subsection. The entropy function is given by

\[ S = -i \frac{\pi}{81g^2G} \frac{\Delta^3}{\omega_1 \omega_2} + \Delta Q + \omega_1 J_1 + \omega_2 J_2 + \lambda \left( \Delta - \omega_1 - \omega_2 - 2\pi i \right) , \]

where \( G \) is the 6d Newton constant as before. Having in mind the concrete example of massive IIA supergravity on warped AdS_6 \times S^4/\mathbb{Z}_2 background, one would find \( \frac{1}{g^2G} = \frac{27\sqrt{2}}{8} \frac{x^2}{y^2} \). In that case, a microscopic derivation of the entropy function (Eq. (45)) from the CFT_5 dual was studied in Ref. 21, in the Cardy limit

\[ \omega_{1,2} \rightarrow 0 . \]

Here, we introduced the Lagrange multiplier \( \lambda \) to extremize \( S \) in \( \Delta, \omega_1, \omega_2 \) subject to the constraint

\[ \Delta - \omega_1 - \omega_2 = 2\pi i . \]

Differentiating with respect to the chemical potentials, one obtains

\[ \lambda + Q = i \frac{\pi}{27g^2G} \frac{\Delta^2}{\omega_1 \omega_2} , \]
\[ \lambda - J_1 = i \frac{\pi}{81g^4G} \frac{\Delta^3}{\omega_1^2 \omega_2^2} , \]
\[ \lambda - J_2 = i \frac{\pi}{81g^4G} \frac{\Delta^3}{\omega_1^2 \omega_2^2} . \]

Inserting these back to the original entropy function formula, one obtains

\[ S = -2\pi i \lambda . \]

Multiplying the last two equations of Eq. (47), one obtains

\[ (\lambda - J_1)(\lambda - J_2) = -\left( \frac{\pi}{81g^4G} \right)^2 \frac{\Delta^6}{\omega_1^2 \omega_2^2} = -i \frac{3g^4G}{\pi} (\lambda + Q)^3 . \]

Hence, one obtains

\[ \left( \frac{S}{2\pi i} + J_1 \right) \left( \frac{S}{2\pi i} + J_2 \right) = i \frac{3g^4G}{\pi} \left( \frac{S}{2\pi i} - Q \right)^3 . \]

As in our Sec. II and Ref. 17, we dismiss \( \text{Im}(S) \), focussing on \( \text{Re}(S) \) as our entropy. However, again note that all known supersymmetric AdS_6 black holes have a charge relation. This charge relation will coincide with the condition \( \text{Im}(S) = 0 \) at the saddle point. So we demand real \( S \) for real charges \( Q, J_1, J_2 \). Then, Eq. (50) is separated into two real equations as follows:

\[ S^3 - \frac{2\pi^2}{3g^4G} S^2 - 12\pi^2 Q^2 S + \frac{8\pi^4}{3g^4G} J_1 J_2 = 0 , \]
\[ QS^2 + \frac{2\pi^2}{9g^4G} (J_1 + J_2) S - \frac{4\pi^2}{3} Q^3 = 0 . \]

These equations determine \( S \) twice as functions of charges. Therefore, from the compatibility of two equations, one obtains a charge relation of \( Q, J_1, J_2 \). These two equations of \( S, Q, J_1, J_2 \) (Eq. (51)), derived from the entropy function (Eq. (45)), are exactly the same as those from the supersymmetric black holes (Eq. (38)). Note that the charges and chemical potentials of the entropy function (Eq. (45)) are related to those of the black holes as

\[ S_{BH} = S, \quad J_{i,BH} = J_i, \quad \frac{1}{3g} Q_{BH} = Q , \]
\[ \omega_{i,BH} = \text{Re}(\omega_i), \quad 3g \Delta_{BH} = \text{Re}(\Delta) . \]
The subscripts ‘BH’ denote the black hole quantities, while the others are the quantities used in the entropy function. One can also realize that the relation between the chemical potentials in the entropy function (Eq. (46)) is equivalent to that of the supersymmetric black holes (Eq. (44)).

Thus, our entropy function (Eq. (45)) indeed reproduces the Bekenstein-Hawking entropy of the supersymmetric AdS$_6$ black holes (Eq. (37)), and also their chemical potentials.

IV. CONCLUDING REMARKS

In this note, we presented the entropy functions of supersymmetric black holes in AdS$_4$ and AdS$_6$. Complicated black hole quantities can be very concisely understood from simple extremization principles of these entropy functions. Considering the inspirations given by the similar entropy functions in AdS$_5$ [15] or AdS$_7$ [16] to their microscopic studies, we expect that our entropy functions will also play similar roles.

The entropy function in AdS$_4$, presented in section II, was actually motivated by our microscopic study [20] of BPS states in the M2-brane QFT (see, e.g. Refs. 31, 32 and 33) from its index [30]. In particular, Ref. 20 derived the entropy function (Eq. (19)) for large AdS$_4$ black holes, from the index of the radially quantized SCFT on M2-branes, where the condensation of the magnetic monopole operators gives rise to the novel deconfined $N^2$ degrees of freedom.

One can also derive our results on AdS$_6$ black holes from 5d SCFT duals, for instance, the strong coupling limits of 5d gauge theories on the D4-D8-O8 system [21]. The indices of such 5d SCFTs on $S^4 \times \mathbb{R}$ were explored in Ref. 34, where the problem reduced to studies of the 5d instanton partition functions: see, e.g. Refs. 35, 36 and 37. Later, Ref. 21 derived the entropy function (Eq. (45)) for large AdS$_6$ black holes, from the indices of such 5d SCFTs and their orbifold theories, where the instanton solitons play subtle roles to realize deconfined $N^2$ degrees of freedom. Furthermore, while AdS$_6$ black hole solution known to date has only one electric charge dual to R-charge [21], obtained a more general form of the entropy function, which describes AdS$_6$ black holes carrying various electric charges, dual to R-charge, mesonic charge and baryonic charges, yet to be discovered. For example, when the black hole has one more electric charge dual to the mesonic charge, the numerator $\sim \Delta^3$ of our entropy function (Eq. (45)) is refined to $[(\Delta + m\pi)(\Delta - m\pi)]^3$, where $m \equiv \hat{m} + 2\pi i$ is the chemical potential conjugate to the mesonic charge.

One may think of generalizations of our results on AdS$_4 \times S^7$, to more general 4d $N = 2$ gauged supergravity models arising from string or M-theory. To see a natural possibility of generalization, note that the numerator $\sim \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}$ of our entropy function (Eq. (19)) is the homogeneous degree 2 prepotential of the $U(1)^4$ supergravity [38]. The prepotential is the square root of a degree 4 polynomial. See, e.g. Ref. 39 and 40 for such structures in other backgrounds. We conjecture that, for BPS black holes in 4d $N = 2$ gauged supergravity, an entropy function like (Eq. (19)) can be constructed by replacing the numerator by the prepotential of the theory. Recently, such an entropy function was found in Ref. 24, and also microscopically studied in Ref. 41 from the indices of CFT$_3$ duals in the Cardy limit $\omega \to 0$.

Here, note that similar prepotentials appeared in the entropy functions of magnetic/dyonic AdS$_4$ black holes [42] (see also Ref. 43). The unifying underlying structures for all these entropy functions were microscopically studied in Ref. 41 from CFT$_3$ duals. We finally note that the entropy functions of electric AdS$_6$ black holes in our paper also appear to have some similarities with magnetized black holes in AdS$_6$, with boundaries replaced by more general 4-manifolds [44,45].

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