A note on subgroup commutativity degrees of finite groups

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Abstract

In this note we give some new results concerning the subgroup commutativity degree of a finite group $G$. These are obtained by considering the minimum of subgroup commutativity degrees of all sections of $G$.

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1 Introduction

In the last years there has been a growing interest in the use of probability in finite group theory. One of the most important aspects which have been studied is the probability that two elements of a finite group $G$ commute. It is called the commutativity degree of $G$ and is denoted by $d(G)$. Inspired by this concept, in [8] (see also [9]) we introduced a similar notion for the subgroups of $G$, called the subgroup commutativity degree (or the subgroup permutability degree) of $G$. This quantity is defined by

$$
sd(G) = \frac{1}{|L(G)|^2} |\{(H, K) \in L(G)^2 \mid HK = KH\}| = \frac{1}{|L(G)|^2} |\{(H, K) \in L(G)^2 \mid HK \in L(G)\}| \tag{1}
$$
(where \( L(G) \) denotes the subgroup lattice of \( G \)) and it measures the probability that two subgroups of \( G \) commute, or equivalently the probability that the product of two subgroups of \( G \) be a subgroup of \( G \).

We recall that for a finite group \( G \) we have \( sd(G) = 1 \) if and only if \( G \) is an Iwasawa group, i.e. a nilpotent modular group (see [6, Exercise 3, p. 87]). A complete description of these groups is given by a well-known Iwasawa’s result (see Theorem 2.4.13 of [6]). In particular, we infer that \( sd(G) = 1 \) for all Dedekind groups \( G \).

A well-known result by Gustafson [3] concerning the commutativity degree states that if \( d(G) > 5/8 \) then \( G \) is abelian, and we have \( d(G) = 5/8 \) if and only if \( G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Note that the similar problem for the subgroup commutativity degree does not have a solution, i.e. there is no constant \( c \in (0, 1) \) such that if \( sd(G) > c \) then \( G \) is Iwasawa, as shows Theorem 2.15 of [1].

In the following we will study this problem by replacing the condition "\( sd(G) > c \)" with the stronger condition "\( sd^*(G) > c \)". where
\[
.sd^*(G) = \min\{sd(S) \mid S \text{ section of } G\}.
\]
It was suggested by the fact that a \( p \)-group is modular if and only if each of its sections of order \( p^3 \) is. Moreover, if a \( p \)-group is not modular then it contains a section isomorphic to \( D_8 \), the dihedral group of order 8, or to \( E(p^3) \), the non-abelian group of order \( p^3 \) and exponent \( p \) for \( p > 2 \) (see Lemma 2.3.3 of [6]). Note that a similar condition for the cyclic subgroup commutativity degree led in [11] to a criterion for a finite group to be an Iwasawa group. We will prove that the condition \( sd^*(G) > 23/25 \) implies the modularity for finite nilpotent groups \( G \), and also that it implies the solvability for arbitrary finite groups \( G \). Then we will show the non-existence of a constant \( c \in (0, 1) \) such that if \( sd^*(G) > c \) then \( G \) is Iwasawa, extending the above mentioned result of Aivazidis.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [7]. For subgroup lattice concepts we refer the reader to [6].

2 Main results

**Theorem 1.** Let \( G \) be a finite nilpotent group such that \( sd^*(G) > 23/25 \). Then \( G \) is modular, and consequently an Iwasawa group.
Proof. Being nilpotent, $G$ can be written as a direct product of its Sylow subgroups $G_i$, $i = 1, 2, ..., k$. Clearly, for each $i$ we have

$$sd^*(G_i) \geq sd^*(G) \geq \frac{23}{25}.$$ 

Assume that $G_i$ is not modular. Then there is a section $S$ of $G_i$ such that $S \cong D_8$ or $S \cong E(p^3)$ for $p > 2$. We can easily check that

$$sd(E(p^3)) = \frac{3p^3 + 12p^2 + 16p + 16}{(p^2 + 2p + 4)^2} < \frac{23}{25} = sd(D_8)$$

and therefore $sd(S) \leq 23/25$, a contradiction. Thus $G_i$ is modular, for all $i = 1, 2, ..., k$, which implies that $G$ is itself a modular group. 

Remark. The constant $23/25$ in Theorem 1 can be decreased for $p$-groups with $p > 2$ by observing that such a group cannot have sections isomorphic to $D_8$. Thus, a finite $p$-group $G$ of odd order which satisfies

(1) $$sd^*(G) > \frac{3p^3 + 12p^2 + 16p + 16}{(p^2 + 2p + 4)^2}$$

is always an Iwasawa group. We also observe that

$$\frac{3p^3 + 12p^2 + 16p + 16}{(p^2 + 2p + 4)^2} < \frac{3}{p}, \text{ for all } p$$

and therefore the above statement remains true by replacing the condition (1) with the more elegant condition

$$sd^*(G) \geq \frac{3}{p}.$$ 

In what follows we will study what can be said about an arbitrary finite group $G$ satisfying $sd^*(G) > 23/25$. A first answer is given by the following theorem.

Theorem 2. Let $G$ be a finite group such that $sd^*(G) > 23/25$. Then $G$ is solvable.

Proof. Assume that $G$ is not solvable. Then it contains a section isomorphic to one of the following groups:
- PSL(2, p), where \( p > 3 \) is a prime such that \( 5 \mid p^2 + 1 \);

- PSL(2, 3^p), where \( p \geq 3 \) is a prime;

- PSL(2, 2^p), where \( p \) is a prime;

- Sz(2^p), where \( p \geq 3 \) is a prime;

- PSL(3, 3).

It is well-known that PSL(2, q) has a subgroup isomorphic to \( D_{q+1} \) for \( q \) odd and a subgroup isomorphic to \( D_{2(q+1)} \) for \( q = 2^p \) (see e.g. [2]). Then the first three groups above have a section isomorphic to \( D_8 \) or to \( D_{2r} \) with \( r \geq 3 \) a prime. But

\[
\text{sd}(D_{2r}) = \frac{7r + 9}{(r + 3)^2} < \frac{23}{25},
\]

a contradiction. A similar contradiction is obtained for Sz(2^p) since it contains a subgroup of type \( D_{2(2^p-1)} \). Finally, PSL(3, 3) has a subgroup isomorphic to \( A_4 \) and

\[
\text{sd}(A_4) = \frac{16}{25} < \frac{23}{25},
\]

contradicting again our hypothesis. This completes the proof.

Next we try to see whether the condition \( \text{sd}^*(G) > 23/25 \) (or the more general condition \( \text{sd}^*(G) > c \)) implies that \( G \) is nilpotent. We start by providing an example of a Schmidt group \( S_1 \) of order \( p^r q \) for which we are able to compute explicitly \( \text{sd}^*(S_1) \). It also has the property that if \( p \) and \( q \) are suitably chosen, then \( \text{sd}^*(S_1) \) tends to 1 when \( p \) tends to infinity. This will be the main ingredient of the proof of Theorem 3.

**Example.** Let \( S \) be a Schmidt group, i.e. a finite non-nilpotent group all of whose proper subgroups are nilpotent. By [5] (see also [1]) it follows that \( S \) is a solvable group of order \( p^m q^n \) (where \( p \) and \( q \) are different primes) with a unique Sylow \( p \)-subgroup \( P \) and a cyclic Sylow \( q \)-subgroup \( Q \), and hence \( S \) is a semidirect product of \( P \) by \( Q \). Moreover, we have:

- if \( Q = \langle y \rangle \) then \( y^a \in Z(S) \);

- \( Z(S) = \Phi(S) = \Phi(P) \times \langle y^a \rangle, S' = P, P' = (S')' = \Phi(P) \);

- \( |P/P'| = p^r \), where \( r \) is the order of \( p \) modulo \( q \);
- if $P$ is abelian, then $P$ is an elementary abelian $p$-group of order $p^r$ and $P$ is a minimal normal subgroup of $S$;

- if $P$ is non-abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^r$.

We infer that $S_1 = S/Z(S)$ is also a Schmidt group of order $p^r q$ which can be written as semidirect product of an elementary abelian $p$-group $P_1$ of order $p^r$ by a cyclic group $Q_1$ of order $q$ (note that $S_3$ and $A_4$ are examples of such groups). It is easy to see that $S_1$ does not contain subgroups of order $p^i q$ for $i = 1, 2, ..., r - 1$. Then

$$L(S_1) = L(P_1) \cup \{Q_1^x \mid x \in S_1\} \cup \{S_1\}$$

and so

$$|L(S_1)| = a_{r,p} + p^r + 1,$$

where $a_{r,p}$ denotes the total number of subgroups of $P_1$. By [10] the numbers $a_{r,p}$ can be written as

$$a_{r,p} = f_r(p), \text{ where } f_r \in \mathbb{Z}[X] \text{ and } \deg(f_r) = \lfloor r^2/4 \rfloor.$$

Since $S_1/1$ is the unique non-abelian section of $S_1$, one obtains

$$sd^*(S_1) = sd(S_1).$$

Let $Q_1^1, Q_1^2, ..., Q_1^{p^r}$ be the conjugates of $Q_1$. Then the pairs of commuting subgroups of $S_1$ are:

- $(X, Y)$ with $X, Y \leq P_1$,
- $(X, S_1)$ and $(S_1, X)$ with $X \leq P_1$,
- $(Q_i^1, 1)$ and $(1, Q_i^1)$, $i = 1, 2, ..., p^r$,
- $(Q_i^1, P_1)$ and $(P_1, Q_i^1)$, $i = 1, 2, ..., p^r$,
- $(Q_i^1, S_1)$ and $(S_1, Q_i^1)$, $i = 1, 2, ..., p^r$,
- $(Q_i^1, Q_j^1)$, $i = 1, 2, ..., p^r$,
- $(S_1, S_1)$.

It follows that
\[
\text{sd}^*(S_1) = \frac{a_{r,p}^2 + 2a_{r,p} + 7p^r + 1}{(a_{r,p} + p^r + 1)^2} = \frac{1 + \frac{2}{a_{r,p}} + \frac{7p^r}{a_{r,p}^2} + \frac{1}{a_{r,p}^2}}{1 + \frac{p^{2r}}{a_{r,p}^2} + \frac{1}{a_{r,p}^2} + \frac{2p^r}{a_{r,p}^2} + \frac{2}{a_{r,p}^2}}.
\]

**Theorem 3.** There is no constant \( c \in (0, 1) \) such that if \( \text{sd}^*(G) > c \) then \( G \) is Iwasawa.

**Proof.** Let \((p_n)_{n \geq 1}\) and \((q_n)_{n \geq 1}\) be two strictly increasing sequences of primes such that the order \( r_n \) of \( p_n \) modulo \( q_n \) is greater than 4. It follows that

\[
[r_n^2/4] > r_n, \quad \forall \ n \geq 1.
\]

For every \( n \geq 1 \), let \( G_n \) be a semidirect product of an elementary abelian \( p_n \)-group \( P_n \) of order \( p_n^{r_n} \) by a cyclic group of order \( q_n \) generated by an element \( x_n \) which permutes the elements of a basis of \( P_n \) cyclically. Then \((G_n)_{n \geq 1}\) are Schmidt groups of order \( p_n^{r_n}q_n \) such as \( S_1 \) in our example, and so (2) and (3) lead to

\[
\lim_{n \to \infty} \text{sd}^*(G_n) = 1,
\]

completing the proof.

Inspired by the above results, we came up with the following conjecture.

**Conjecture 4.** Let \( G \) be a finite group such that \( \text{sd}^*(G) > 23/25 \). Then \( G \) is either an Iwasawa group or a Schmidt group.

**Remark.** The above example also leads to two new classes of finite groups whose subgroup commutativity degree vanishes asymptotically. For \( i = 1, 2 \), let \((p_n^i)_{n \geq 1}\) and \((q_n^i)_{n \geq 1}\) be two strictly increasing sequences of primes such that the order \( r_n^i \) of \( p_n^i \) modulo \( q_n^i \) is 2 and 3, respectively. Let \((G_n^i)_{n \geq 1}\) be the Schmidt groups of order \((p_n^i)^{r_n^i}q_n^i\) constructed as in the proof of Theorem 3. Then (2) implies that

\[
\lim_{n \to \infty} \text{sd}^*(G_n^i) = \lim_{n \to \infty} sd(G_n^i) = 0, \quad i = 1, 2,
\]
as desired.
Finally, we formulate another natural problem concerning our study.

**Open problem.** Describe the structure of finite groups $G$ satisfying $sd^*(G) = 23/25$.

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