Beating standard quantum limit via two-axis magnetic susceptibility measurement

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We report a metrology scheme which measures magnetic susceptibility of an atomic spin ensemble along the x and z direction and produces parameter estimation with precision beating the standard quantum limit. The atomic ensemble is initialized via one-axis spin squeezing with optimized squeezing time and parameter \( \phi \) to be estimated is assumed as uniformly distributed between 0 and 2\( \pi \). One estimation of \( \phi \) can be produced with every two magnetic susceptibility data measured along the two axis respectively, which has imprecision scaling \( (1.43 \pm 0.02)/N^{0.687 \pm 0.003} \) with respect to the number \( N \) of atomic spins. The measurement scheme is easy to implement and thus one step towards practical application of quantum metrology.

I. INTRODUCTION.

Metrology is the cornerstone of scientific research and technology development. Quantum mechanics provides new opportunities to metrology and attracts attention from different fields [1–5]. Given \( N \) probes, the lowest parameter estimation imprecision allowed by classical theory is of the order of \( 1/\sqrt{N} \), which is the so-called standard quantum limit (SQL). It comes from the well-known central limit theorem. Quantum metrology harnesses quantum features such quantum entanglement and spin squeezing to beat the standard quantum limit. The lowest parameter estimation imprecision can be of the order \( 1/N \) when quantum effects are fully utilized and thus result in a quadrature enhancement of estimation precision. It is the so-called Heisenberg limit (HL) [6, 7].

Every metrology scheme involves three stages: 1) probe state initialization; 2) sensing parameter field; 3) measurement of probes and information extraction. Among them, the second stage depends on the parameter field to be estimated. What we can do is to choose suitable prob systems and design initialization and measurement procedure carefully, which correspond to the first stage and the final stage design respectively. Quantum entanglement and spin squeezing are generated in atomic ensembles which can host \( 10^6 \sim 10^8 \) atoms. It is a platform of great potential in quantum metrology, because of its sheer size as well as the strong quantum entanglement that can be generated between so many probes with technology available to many labs [8–11]. Atoms in ensembles are indistinguishable. Thus neither manipulation nor measurement on individual probe is possible as required in many proposed metrology schemes [12–16]. Magnetic susceptibility \( \hat{J}_n \) along a specific axis is the most common and convenient physical quantity to be measured of an atomic ensemble. When the average magnetic susceptibility \( \langle \hat{J}_n \rangle \) is used to estimate the parameter field \( \phi \), the corresponding estimation imprecision would be \( \Delta \phi = \Delta J_n \sqrt{\frac{\partial \langle \hat{J}_n \rangle}{\partial \phi}} \). Here \( \Delta J_n = \sqrt{\langle (\hat{J}_n - \langle \hat{J}_n \rangle)^2 \rangle} \) is the standard deviation of the magnetic susceptibility along direction \( n \). Thus great fluctuation \( \Delta J_n \) would result in poor estimation. It is widely believed that “low-noise detection” is necessary to improve parameter estimation precision [8–11]. It has attracted plenty attention in the endeavor to suppress the fluctuation of \( J_n \) and improve the single-atom resolution [17, 18]. Others have worked to circumvent the single-atom resolution requirement. The main idea is to use nonlinear atomic interaction to suppress the fluctuation of \( J_n \), which is the so-called echo procedure. The echo procedure can utilize the same nonlinear interaction which is employed to generation multipartite entanglement on the first stage [19–22]. We find that measuring magnetic susceptibilities along two mutually orthogonal axes perpendicular to parameter field \( \phi \) can produce parameter estimation precision beyond SQL. The importance of our work is two-fold. Firstly, it shows that neither single-atom resolution nor nonlinear interaction are necessary on final measurement stage of metrology to achieve precision improvement beyond SQL. Secondly, it employs the most convenient measurement and thus is one step further to achieve metrology gain beyond SQL promised by quantum mechanics in practice.

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II. ONE-AXIS SPIN SQUEEZING

We consider an atomic ensemble consist of \( N \) atoms. They firstly are initialized to the coherent spin state along \( x \) direction. We then use the one-axis spin squeezing Hamiltonian \( \hat{J}_z^2 \) to generate multipartite entanglement among the atoms. Following the standard procedure of one-axis spin squeezing, a little adjustment along the \( x \) direction would also be implemented [23, 24],

\[
|\psi_0\rangle = e^{i\delta_{\text{adj}} \hat{J}_x} e^{-i t_s \hat{J}_z^2} |+\rangle ^{\otimes N}.
\] (1)

The adjustment is \( \delta_{\text{adj}} = \frac{1}{2} \arctan \frac{R}{A} \) with \( A = 1 - (\cos 2 t_s)^{N-2} \) and \( B = 4 \sin t_s (\cos t_s)^{N-2} \). Here, \(|+\rangle\) represents a single atomic spin directing along the positive \( x \) direction. We use the convention \( \hbar = 1 \). The squeezing time \( t_s \) (squeezing strength included) can be adjusted in experiment [8–11, 19, 20]. That is the whole initialization stage. As shown in Fig. 1 (a) and (b), the effect of the spin squeezing is suppressing fluctuation on the \( z \) axis direction at the expense of increasing the fluctuation of spin along the two other directions.

III. INTERFERENCE PROCEDURE

The atomic spins would be sent through a magnetic field pointing to the positive \( y \) direction. Let us assume the overall quantum channel is \( \hat{U}_\phi = e^{-i \phi \hat{J}_y} \). The output state of the atomic ensemble is

\[
|\psi_\phi\rangle = e^{-i \phi \hat{J}_y} |\psi_0\rangle.
\] (2)

By measuring atomic ensemble coming out of the magnetic field, we expect to estimate the value of \( \phi \) and thus the value of the magnetic field.

IV. MAGNETIC SUSCEPTIBILITY MEASUREMENTS ALONG TWO DIRECTIONS.

It is widely believed that there is always optimal measurement for a particular metrology scheme. In many cases, it can be proven that such kind of optimal measurement exists [6]. Others believe that adaptive feedback measurement can give the optimal information extraction [12–16]. Most of these measurement procedures are too complex to be realized on atomic ensembles. Magnetic susceptibility measurement is easy to carry out on atomic ensembles. In most cases, it is not optimal measurement scheme. Our goal here is to use the most simple measurement to achieve as high as possible metrology precision.

The reason that we choose two measurement directions of the magnetic susceptibility is of three fold. Firstly, the price to pay is small. We usually need to collect a sufficient amount data to infer \( \phi \). Hence the initialization-interference-measurement procedure has to be repeated \( R \) times with \( R \) sufficiently big. We divide the \( R \) experiments into two equal groups, of which each group measuring magnetic susceptibility along a different direction from the other group. Let us assume the worst case where magnetic susceptibility measurement on one of the direction provide no information of \( \phi \). Then only \( R/2 \) data sample are of use. The imprecision \( \Delta \phi \) would increase by \( \sqrt{2} \) which is relatively small when \( N \) of the order of \( 10^3 \sim 10^5 \). Secondly, there is actually a simple correspondence between the parameter field \( \phi \) and the magnetic susceptibility on the \( x - z \) plane of the Bloch sphere given the atom ensemble is in an one-axis spin squeezed state (1), c.f. Fig. 1(d), Fig. 1(d)

\[
\langle \hat{J}_\varphi \rangle = \langle \hat{J}_x \rangle \sin \varphi + \langle \hat{J}_z \rangle \cos \varphi.
\] (3)

There is only one single peak of the \( \langle \hat{J}_\varphi \rangle \) curve which is located at \( \varphi = \pi/2 \). When the atomic ensemble initialized in an one-axis spin squeezed state comes out of the
In our scheme, we employ a slightly different evaluation formula from (5)

\[
\sin \phi_{\text{est}} = -\frac{j_z}{\sqrt{j_z^2 + j_x^2}} \quad \text{and} \quad \cos \phi_{\text{est}} = j_x / \sqrt{j_z^2 + j_x^2}.
\]

(6)

\(j_z\) and \(j_x\) are a pair of measurement results of magnetic susceptibility along the \(x\) and \(z\) direction respectively. The two experiment measurement data produce one estimation \(\phi_{\text{est}}\) of the real parameter \(\phi\).

**A. Performance analysis**

We have simulated our metrology scheme for up to 2480 atomic spins. \(R = 2\) magnetic susceptibility measurements are carried out for estimating each \(\phi\), of which one is along the \(x\) axis and the other along the \(z\) axis. The two measurement results are submitted in (6) to produce an estimation \(\phi_{\text{est}}\) of \(\phi\). We choose the \(M = 1000\) parameter \(\phi\) randomly between 0 and \(2\pi\). The performance of our scheme is estimated as

\[
(\Delta \phi)^2 = \frac{R}{M} \sum_{i=1}^{M} \left[ \min \left\{ |\phi^{(i)} - \phi_{\text{est}}^{(i)}|, 2\pi - |\phi^{(i)} - \phi_{\text{est}}^{(i)}| \right\} \right]^2.
\]

(7)

Note that \(R\) in the above equation is to eliminate the metrology contribution of \(R\) the size of data collected to estimate \(\phi\). \(\phi\) is cyclic with period \(2\pi\) and \(\phi = 0\) and \(\phi = 2\pi\) are considered to be of the same value. We have optimized the squeezing time \(t_s\) to achieve the lowest \(\Delta \phi\).

Ref. [24] tells us that the entanglement behavior of one-axis spin squeezed state have a time scale of \(1/N^{2/3}\) and its quantum Fisher information reach its maximum in the region of \(t_s \gtrsim 1/N^{1/2}\). Hence we choose the range of

FIG. 2. (Color online) Performance of the two-axis magnetic susceptibility measurement scheme for estimating \(\phi\). (a) Scaling of the imprecision \(\Delta \phi\) versus atomic ensemble size \(N\). (b) The optimal squeezing time \(t_s\) for different atomic ensemble size \(N\). (c) The imprecision \(\Delta \phi\) given by 1000 atomic spins with different squeezing time. There is a single valley of the \(\Delta \phi\) versus atomic ensemble size \(N\) curve and the corresponding optimal squeezing time (marked red star) is employed in our scheme. (d) Fisher information of one-axis spin squeezed state given optimized squeezing time as shown in (b).
We measure $t_s$ to be $[0, 2/N^{1/2}]$ and optimize $t_s N^{2/3}$ instead of $t_s$.

As shown in Fig. 2(a), our two-axis magnetic susceptibility measurement scheme can produce parameter estimation with imprecision below standard quantum limit for $N \gtrsim 16$. The scaling $\Delta \phi$ with respect to atomic ensemble size is

$$\Delta \phi = \frac{1.43 \pm 0.02}{N^{0.687 \pm 0.003}}. \tag{8}$$

The optimal $t_s$ is around the value of $0.4/N^{2/3}$, as shown in Fig. 2(b). Given the optimized squeezing time $t_s$, Fisher information $F$ of the corresponding one-axis spin squeezed state scales $F = 0.833 N^{1.42}$ with respect to $N$. It matches the scaling of $(\Delta \phi)^2$ with respect to $N$.

V. SUMMARY AND OUTLOOKS.

The magnetic susceptibility fluctuations along two orthogonal axis cannot be both very small. This is prohibited by the Heisenberg uncertainty relation. And sometimes $\Delta J_x$ as well as $\Delta J_z$ can be of the order of $\sqrt{N}$, c.f. Fig. 1(d). Our two-axis magnetic susceptibility measurement scheme can circumvent the worst effect of these fluctuation. By employing our scheme, neither sophisticated control schemes [12–16] nor echo procedure induced by nonlinear interaction [19–22] is needed to extract information from a quantum spin ensemble and SQL can be beaten. Besides, we need only two measurements to provide one estimation and thus small number of data is enough to ensure good estimation. It is interesting to ask whether HL can be reached or approached with such a simple measurement scheme.

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