On the Stability of Squashed Kaluza-Klein Black Holes

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Abstract

The stability of squashed Kaluza-Klein black holes is studied. The squashed Kaluza-Klein black hole looks like five dimensional black hole in the vicinity of horizon and four dimensional Minkowski spacetime with a circle at infinity. In this sense, squashed Kaluza-Klein black holes can be regarded as black holes in the Kaluza-Klein spacetimes. Using the symmetry of squashed Kaluza-Klein black holes, $SU(2) \times U(1) \simeq U(2)$, we obtain master equations for a part of the metric perturbations relevant to the stability. The analysis based on the master equations gives a strong evidence for the stability of squashed Kaluza-Klein black holes. Hence, the squashed Kaluza-Klein black holes deserve to be taken seriously as realistic black holes in the Kaluza-Klein spacetime.

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I. INTRODUCTION

Recently, higher dimensional black holes have attracted much attention. In particular, many exotic black holes in the asymptotically flat spacetime are found \[1, 2, 3, 4, 5\]. From a realistic point of view, however, the extra dimensions need to be compactified to reconcile the higher dimensional gravity theory with our apparently four-dimensional world. The higher dimensional spacetimes with compact extra dimensions are called Kaluza-Klein spacetimes. The black holes should reside not in the asymptotically flat spacetimes but the asymptotically Kaluza-Klein spacetimes. We call these ‘Kaluza-Klein black holes’. It would be important to study Kaluza-Klein black holes in the general dimensions. In this paper, we will consider five dimensional Kaluza-Klein black holes as a first step.

It is well known that the simplest five-dimensional Kaluza-Klein black hole is the black string which is the direct product of four dimensional Schwarzschild black hole and a circle \[6\]. The topology of the horizon of black strings is \(S^2 \times S^1\). The stability analysis of black strings has been done, and it was shown that black strings are stable when the horizon radius is larger than the scale of compact extra dimension \[7\]. Because of the stability, black strings are natural candidate of Kaluza-Klein black holes.

Interestingly, another possibility has been recognized \[8\]. It is squashed Kaluza-Klein (SqKK) black holes that could also reside in the Kaluza-Klein spacetime. The topology of the horizon of this SqKK black holes is \(S^3\), while it looks like four dimensional black holes with a circle as an internal space in the asymptotic region. SqKK black holes were originally derived as five-dimensional vacuum solutions in the context of Kaluza-Klein theory \[9, 10\]. Recently, much effort has been devoted to reveal the properties of squashed Kaluza-Klein black holes \[11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27\]. Since the horizons of these black holes have the same nature as the five-dimensional black holes, Hawking radiation and quasi-normal modes from SqKK black holes would be different from those seen in four-dimensional black holes even at low energy \[20, 25, 26\]. That means that the extra dimension can be observed through these squashed black holes. These are distinct properties from black strings for which we need to see the excitation of Kaluza-Klein modes to find the extra dimension. However, the stability of SqKK black holes is needed for these arguments to be meaningful.

Related to the stability problem, Bizon et al \[27\] investigated the non-liner purtabation
of Gross-Perry-Sorkin (GPS) monopole which is the zero mass limit of the SqKK black hole. They showed GPS monopole is stable against small perturbations but unstable against large perturbations and collapses to a SqKK black hole. This suggests that the SqKK black hole is a final state of a gravitational collapse in the presence of GPS monopole. Hence, SqKK black holes seem to be stable, although the stability is not yet proved. The purpose of this paper is to study the stability of SqKK black holes directly.

To analyze the stability, it is important to obtain a set of single ordinarily differential equations of motion, the so-called master equations. To achieve this aim, we focus on the symmetry of SqKK black holes, $SU(2) \times U(1) \simeq U(2)$. Since SqKK black holes have the same symmetry as five-dimensional Myers-Perry black holes with equal angular momenta, the analysis of field equations in the degenerate Myers-Perry spacetime [28] can be applicable to SqKK black holes. By doing so, we show that metric perturbations which are supposed to be relevant to the stability can be described by master equations. Using the master equations, we prove the stability of SqKK black holes under these perturbations.

The organization of this paper is as follows. In section II we present the SqKK black holes and discuss the symmetry of these black holes. In section III the formalism to classify metric perturbations is explained. Firstly, we introduce Wigner functions, which are irreducible representation of $U(2)$. The tensor fields are expanded in terms of these Wigner functions in invariant forms. Using the classification based on the symmetry, we find infinite number of master variables. In section IV we derive the master equations for master variables. By analyzing these equations, we give a strong evidence of the stability of SqKK black holes. The final section is devoted to the discussion.

II. SYMMETRY OF SQUASHED KALUZA-KLEIN BLACK HOLES

In this paper, we concentrate on the static SqKK black holes in vacuum whose metric is given by

$$ds^2 = -F(\rho)dt^2 + \frac{K(\rho)^2}{F(\rho)}d\rho^2 + \rho^2 K(\rho)^2[(\sigma^1)^2 + (\sigma^2)^2] + \frac{\rho_0(\rho_0 + \rho_+)}{K(\rho)^2}(\sigma^3)^2. \quad (1)$$

Here, the function $F(\rho)$ and $K(\rho)$ are given by

$$F(\rho) = 1 - \frac{\rho_+}{\rho}, \quad K^2(\rho) = 1 + \frac{\rho_0}{\rho}, \quad (2)$$
where $\rho_+$ and $\rho_0$ are constant parameters. The invariant forms $\sigma^a$ ($a = 1, 2, 3$) of $SU(2)$ are given by

\[
\begin{align*}
\sigma^1 &= -\sin \psi d\theta + \cos \psi \sin \theta d\phi , \\
\sigma^2 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi , \\
\sigma^3 &= d\psi + \cos \theta d\phi ,
\end{align*}
\]

which satisfy $d\sigma^a = 1/2\varepsilon^{abc}\sigma^b \wedge \sigma^c$, where $\varepsilon^{abc}$ is the Levi-Civita symbol. The coordinate ranges are $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 4\pi$.

The angular part of the space, on which the metric (1) is spanned by $\sigma^a$, is topologically $S^3$. The horizon is located at $\rho = \rho_+$, and then its topology is $S^3$. In fact, the radius of $S^2$ is $\sqrt{\rho_+(\rho_++\rho_0)}$ and the radius of the circle is $\sqrt{\rho_0 \rho_+}$. Hence, the geometry is a squashed three-sphere. The asymptotic form of metric at infinity becomes

\[
d s^2 \sim -dt^2 + d\rho^2 + \rho^2 d\Omega^2_2 + \rho_0(\rho_0 + \rho_+)(d\psi + \cos \theta d\phi)^2 ,
\]

where $d\Omega^2_2 = (\sigma^1)^2 + (\sigma^2)^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the metric of $S^2$. From the metric (4), we see the asymptotic geometry has the structure of $S^1$ fibered over $M^4$. Therefore, the extra dimension of spacetime (1) is compactified at infinity, and the scale of compactification $\ell$ is given by

\[
\ell = \sqrt{\rho_0(\rho_0 + \rho_+)} .
\]

In this sense, the spacetime given by the metric (1), which has a squashed horizon, can be regarded as a kind of Kaluza-Klein black holes. Thus, the SqKK black hole has a $S^3$ horizon as a five-dimensional black hole and the asymptotic structure similar to that of a five-dimensional black string. It is well known that there exists Gregory-Laflamme instability [7] in the black string system. On the other hand, five-dimensional Schwarzschild black holes are stable [29, 30]. Therefore, it is interesting to study the stability of squashed black holes.

Apparently, the metric (1) has the $SU(2)$ symmetry generated by Killing vectors $\xi_\alpha$, ($\alpha = x, y, z$):  

\[
\begin{align*}
\xi_x &= \cos \phi \partial_\theta + \frac{\sin \phi}{\sin \theta} \partial_\psi - \cos \theta \sin \phi \partial_\phi , \\
\xi_y &= -\sin \phi \partial_\theta + \frac{\cos \phi}{\sin \theta} \partial_\psi - \cot \theta \cos \phi \partial_\phi , \\
\xi_z &= \partial_\phi .
\end{align*}
\]
The symmetry can be explicitly shown by using the relation $\mathcal{L}_{\xi^\alpha} \sigma^\alpha = 0$, where $\mathcal{L}_{\xi^\alpha}$ is a Lie derivative with respect to $\xi^\alpha$. The dual vectors to $\sigma^\alpha$ are given by

\[
\begin{align*}
  e_1 &= -\sin \psi \partial_\theta + \frac{\cos \psi}{\sin \theta} \partial_\phi - \cot \theta \cos \psi \partial_\psi, \\
  e_2 &= \cos \psi \partial_\theta + \frac{\sin \psi}{\sin \theta} \partial_\phi - \cot \theta \sin \psi \partial_\psi, \\
  e_3 &= \partial_\psi,
\end{align*}
\]  

(7)

and, by definition, they satisfy $\sigma^i_i e^i_j = \delta^i_j$. Let us define the two kind of angular momentum operators

\[
L_\alpha = i \xi^\alpha, \quad W_a = i e^a.
\]  

(8)

where $\alpha, \beta, \cdots = x, y, z$ and $a, b, \cdots = 1, 2, 3$. They satisfy commutation relations

\[
[L_\alpha, L_\beta] = i \epsilon_{\alpha\beta\gamma} L_\gamma, \quad [W_a, W_b] = -i \epsilon_{abc} W_c.
\]  

(9)

They commute each other as $[L_\alpha, W_a] = 0$. From the metric (1), we can also read off the additional $U(1)$ symmetry, which keeps the $S^2$ metric, $\sigma_1^2 + \sigma_2^2$ invariant. Thus, the spatial symmetry of SqKK black holes is $SU(2) \times U(1) \simeq U(2)$, where $e_3$ generates $U(1)$ and $\xi_\alpha (\alpha = x, y, z)$ generate $SU(2)$. As will be seen later, these symmetry yield the separability of equations for the metric perturbations.

It is convenient to define the new invariant forms

\[
\sigma^\pm = \frac{1}{2} (\sigma^1 \mp i \sigma^2).
\]  

(10)

Here, we note that

\[
\mathcal{L}_{W_3} \sigma^\pm = \pm \sigma^\pm, \quad \mathcal{L}_{W_3} \sigma^3 = 0.
\]  

(11)

The dual vectors to $\sigma^\pm$ are

\[
e^\pm = e_1 \pm i e_2.
\]  

(12)

By use of $\sigma^\pm$, the metric (1) can be rewritten as

\[
d\tilde{s}^2 = -F(\rho)dt^2 + \frac{K(\rho)^2}{F(\rho)} d\rho^2 + 4\rho^2 K(\rho)^2 \sigma^+ \sigma^- + \frac{\rho_0(\rho_0 + \rho_+)}{K(\rho)^2} (\sigma^3)^2.
\]  

(13)

\footnote{The metric (1) also has time translation symmetry generated by $\partial/\partial t$.}
III. CLASSIFICATION OF THE METRIC PERTURBATIONS BASED ON THE SYMMETRY

Because the squashed black hole spacetime (1) has the $SU(2) \times U(1)$ symmetry, the metric perturbations can be expanded by the irreducible representation of $SU(2) \times U(1)$. We explain the formalism to obtain master equations for the metric perturbations [28, 31].

Let us construct the representation of $U(2) \cong SU(2) \times U(1)$. The eigenfunctions of $L^2 \equiv L^2_\alpha = W^2_\alpha$ are degenerate, but can be completely specified by eigenvalues of the operators $L_z$ and $W_3$. The eigenfunctions are called Wigner functions, which are defined by

$$L^2 D^J_{KM} = J(J + 1)D^J_{KM} , \quad L_z D^J_{KM} = M D^J_{KM} , \quad W_3 D^J_{KM} = K D^J_{KM} ,$$

where $J, K, M$ are integers satisfying $J \geq 0$, $|K| \leq J$, $|M| \leq J$. From Eqs. (14), we see that $D^J_{KM}$ form the irreducible representation of $SU(2) \times U(1)$. The Wigner functions $D^J_{KM}(x^i)$ are functions defined on $S^3$, i.e., $x^i = \theta, \phi, \psi$, which satisfy the orthonormal relation

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{4\pi} d\psi \sin \theta D^J_{KM}(x^i) D^{J'}_{K'M'}(x^i) = \delta_{JJ'}\delta_{KK'}\delta_{MM'} .$$

Now, we consider metric perturbations $g_{\mu\nu} + h_{\mu\nu}$, where $g_{\mu\nu}$ is the background metric (13). The tensor field $h_{\mu\nu}$ can be classified into three parts, $h_{AB}, h_{Ai}, h_{ij}$, ($A, B = t, \rho$) which behave as scalars, vectors and a tensor under the coordinate transformation of $\theta, \phi, \psi$. The scalars $h_{AB}$ can be expanded by the Wigner functions as

$$h_{AB} = \sum_K h^K_{AB}(x^A) D_K(x^i) ,$$

where we have omitted the index $J, M$, because the metric perturbations with different $J$ and $M$ are decoupled trivially in the perturbed equations.

To decompose the vector part $h_{Ai}$, we construct vector harmonics as

$$D^+_i_{i,K} = \sigma^+_i D^{-}_{K-1} , \quad (|K - 1| \leq J) ,$$

$$D^-_i_{i,K} = \sigma^-_i D^{+}_{K+1} , \quad (|K + 1| \leq J) ,$$

$$D^3_i_{i,K} = \sigma^3_i D_K , \quad (|K| \leq J) .$$

One can check that

$$L^2 D^a_{i,K} = J(J + 1)D^a_{i,K} , \quad L_z D^a_{i,K} = M D^a_{i,K} , \quad W_3 D^a_{i,K} = K D^a_{i,K} ,$$

where
where \( a = \pm, 3 \) and operations are defined by Lie derivatives, that is, \( W_a D^b_{i,K} \equiv \mathcal{L}_{W_a} D^b_{i,K} \) and \( L_\alpha D^a_{i,K} \equiv \mathcal{L}_{L_\alpha} D^a_{i,K} \). In Eq. (17), taking the relation (11) into account, we have shifted the index \( K \) of Wigner functions so that \( D^a_{i,K} \) have the same \( U(1) \) charge \( K \). From Eqs. (18), we see that \( D^a_{i,K} \) form the irreducible representation of \( SU(2) \times U(1) \). Then, \( h_{Ai} \) can be expanded as

\[
h_{Ai}(x^\mu) = \sum_K h^K_{Aa}(x^A) D^a_{i,K}(x^i) . \tag{19}
\]

Similarly, the expansion of tensor part \( h_{ij} \) can be carried out as

\[
h_{ij}(x^\mu) = \sum_K h^K_{ab}(x^A) D^{ab}_{ij,K}(x^i) , \tag{20}
\]

where tensor harmonics \( D^{ab}_{ij,K} \) are defined by

\[
\begin{align*}
D^{++}_{ij,K} &= \sigma^+_i \sigma^+_j D^{K-2}_{ij,K} \quad (|K - 2| \leq J) , \\
D^{+-}_{ij,K} &= \sigma^+_i \sigma^-_j D^{K}_{ij,K} \quad (|K| \leq J) , \\
D^{+3}_{ij,K} &= \sigma^+_i \sigma^3_j D^{K-1}_{ij,K} \quad (|K - 1| \leq J) , \\
D^{-+}_{ij,K} &= \sigma^-_i \sigma^-_j D^{K+2}_{ij,K} \quad (|K + 2| \leq J) , \\
D^{-3}_{ij,K} &= \sigma^-_i \sigma^3_j D^{K+1}_{ij,K} \quad (|K + 1| \leq J) , \\
D^{33}_{ij,K} &= \sigma^3_i \sigma^3_j D^{K}_{ij,K} \quad (|K| \leq J) .
\end{align*}
\]

We have shifted the eigenvalue \( K \) of Wigner functions so that the tensor harmonics \( D^{ab}_{ij,K} \) satisfy

\[
L^2 D^{ab}_{ij,K} = J(J + 1) D^{ab}_{ij,K} , \quad L_z D^{ab}_{ij,K} = M D^{ab}_{ij,K} , \quad W_3 D^{ab}_{ij,K} = K D^{ab}_{ij,K} . \tag{22}
\]

Equations (22) mean that \( D^{ab}_{ij,K} \) form the irreducible representation of \( SU(2) \times U(1) \).

Using the expansions (16), (19) and (20) we can obtain a set of equations for expansion coefficient fields labelled by \( J, M, K \). Because of \( SU(2) \times U(1) \) symmetry no coupling appears between coefficients with different sets of indices \( (J, M, K) \).

Interestingly, without explicit calculation, we can reveal the structure of couplings between coefficients with the same \( (J, M, K) \). First, since the index \( K \) is shifted in the definition of vector and tensor harmonics, then the coefficients \( h^K_{AB}, h^K_{Aa} \) and \( h^K_{ab} \) exist for \( K \) satisfying the inequality listed in the following table:

| \( h_{++} \) | \( h_{A+, h_{+3}} \) | \( h_{AB}, h_{A3}, h_{+-, h_{33}} \) | \( h_{A-}, h_{-3} \) | \( h_{--} \) |
| --- | --- | --- | --- | --- |
| \(|K - 2| \leq J\) | \(|K - 1| \leq J\) | \(|K| \leq J\) | \(|K + 1| \leq J\) | \(|K + 2| \leq J\) |
Therefore, for $J = 0$ modes, we can classify the coefficients by possible $K$ as follows:

$$J = 0;$$

\[
\begin{array}{cccccc}
K = 2 & h_{++} & h_{A+}, h_{+3} & h_{AB}, h_{A3}, h_{+-}, h_{33} & h_{A-}, h_{-3} & h_{--} \\
K = 1 & & & & & \\
K = 0 & & & & & \\
K = -1 & & & & & \\
K = -2 & & & & & \\
\end{array}
\]

Apparently, for $h_{++}$ and $h_{--}$, we can obtain equations for a single variable, respectively. For other sets of coefficients ($h_{A+}, h_{+3}$), ($h_{AB}, h_{A3}, h_{+-}, h_{33}$), ($h_{A-}, h_{-3}$) they are coupled to the coefficients in each set. As we will see later, after fixing the gauge symmetry, we have the master equation for a single variable in each set. In total, there are five master equations, the number of which match to physical degrees of freedom of the gravitational perturbations.

For $J = 1$ modes, we can classify the coefficients as follows:

$$J = 1;$$

\[
\begin{array}{cccccc}
K = 3 & h_{++} & h_{A+}, h_{+3} & h_{AB}, h_{A3}, h_{+-}, h_{33} & h_{A-}, h_{-3} & h_{--} \\
K = 2 & K = 2 & & & & \\
K = 1 & K = 1 & K = 1 & & & \\
K = 0 & K = 0 & K = 0 & & & \\
K = -1 & K = -1 & K = -1 & & & \\
K = -2 & K = -2 & K = -2 & & & \\
K = -3 & & & & & \\
\end{array}
\]

We can see that $h_{++}$ in $(J = 1, M, K = 3)$ modes and $h_{--}$ in $(J = 1, M, K = -3)$ modes are decoupled from other coefficients. It is easy to generalize this fact for arbitrary $J$, and we can also see that $h_{++}$ in $(J, M, K = J + 2)$ modes and $h_{--}$ in $(J, M, K = -(J + 2))$ modes are always decoupled. The perturbation equations for these modes can be reduced to the master equations for the single variables, respectively.
The linearized Einstein equation in vacuum is
\[
\delta G_{\mu \nu} = \frac{1}{2} \left[ \nabla^\rho \nabla_\mu h_{\nu \rho} + \nabla^\rho \nabla_\nu h_{\mu \rho} - \nabla^2 h_{\mu \nu} - \nabla_\mu \nabla_\nu h - g_{\mu \nu} (\nabla^\rho \nabla_\sigma h_{\rho \sigma} - \nabla^2 h) \right] = 0 .
\] (23)
where \( \nabla_\mu \) denotes the covariant derivative with respect to the background metric \( g_{\mu \nu} \) and \( h = g^{\mu \nu} h_{\mu \nu} \). As is mentioned in the previous section, we can obtain master equations for variables in \((J = 0, M = 0, K = 0, \pm 1, \pm 2)\) modes and \((J, M, K = \pm (J + 2))\) modes. We derive these explicitly.

A. zero modes \((J = 0)\) perturbation

In the case \(J = 0\), there are five physical degrees of freedom, namely \(K = \pm 2, \pm 1, 0\) modes. We treat these modes separately.

1. \(K = \pm 2\) modes

In \(K = \pm 2\) modes, there exist two coefficients \(h_{++}\) and \(h_{--}\). We note that these are gauge invariant. We consider only \(h_{++}\) because \(\bar{h}_{++} = h_{--}\), where bar denotes the complex conjugate. We set \(h_{\mu \nu}\)
\[
h_{\mu \nu}(x^\mu) dx^\mu dx^\nu = h_{++}(r)e^{-i\omega t} \sigma^+ \sigma^+ .
\] (24)
Substituting Eq. (24) into Eq. (23), we can get the equation of motion for \(h_{++}\) from \(\delta G_{++} = 0\). In order to rewrite the equation in the Schrödinger form, we introduce the new variable
\[
\Phi_2(\rho) \equiv \frac{1}{\rho^{1/4}(\rho + \rho_0)^{3/4}} h_{++}(\rho) ,
\] (25)
and tortoise coordinate \(\rho_*\) defined by
\[
\frac{d\rho_*}{d\rho} = \frac{K}{F} .
\] (26)
Then, the final form of the equation becomes
\[
- \frac{d^2}{d\rho_*^2} \Phi_2 + V_2(\rho) \Phi_2 = \omega^2 \Phi_2 ,
\] (27)
where the potential \( V_2(\rho) \) is defined by

\[
V_2(\rho) = \frac{\rho - \rho^+}{16\rho^2\rho_0(\rho_+ + \rho_0)(\rho + \rho_0)^3} \left[ 4\rho_+ (\rho_+ + \rho_0)^2 (16\rho_+^2 + 28\rho_+ \rho_0 + 11\rho_0^2) \\
+ (320\rho_+^4 + 960\rho_+^3 \rho_0 + 996\rho_+^2 \rho_0^2 + 391\rho_+ \rho_0^3 + 35\rho_0^4) (\rho - \rho^+) \\
+ 8 (80\rho_+^3 + 182\rho_+^2 \rho_0 + 127\rho_+ \rho_0^2 + 25\rho_0^3) (\rho - \rho^+)^2 \\
+ 32 (20\rho_+^2 + 31\rho_+ \rho_0 + 11\rho_0^2) (\rho - \rho^+)^3 \\
+ 64 (5\rho_+ + 4\rho_0) (\rho - \rho^+)^4 + 64(\rho - \rho^+)^5 \right].
\]

From this expression, we can see \( V_2 > 0 \) in the region \( \rho^+ < \rho < \infty \), explicitly. Typical profiles of the potential \( V_2 \) are plotted in Fig. 1.

We consider that \( \Phi \) is square integrable in the region \( -\infty < \rho^* < \infty \). Then, \( \omega^2 \) is real. Multiplying both sides of Eq. (27) by \( \bar{\Phi}_2 \) we have

\[
-\bar{\Phi}_2 \frac{d^2}{d\rho_*^2} \Phi_2 + V_2(\rho)\bar{\Phi}_2\Phi_2 = \omega^2 \bar{\Phi}_2\Phi_2.
\]

Adding eqs. (29) and its complex conjugate equation, and integrating it, we obtain

\[
\int d\rho_* \left[ \left| \frac{d\Phi_2}{d\rho_*} \right|^2 + V_2(\rho) \left| \Phi_2 \right|^2 \right] - \frac{1}{2} \left[ \frac{d}{d\rho_*} \Phi_2 + \Phi_2 \frac{d}{d\rho_*} \bar{\Phi}_2 \right]_{\rho^*=\infty} = \omega^2 \int d\rho_* \left| \Phi_2 \right|^2.
\]

Because the boundary term vanishes, the positivity of \( V_2 \) means \( \omega^2 > 0 \). Therefore, we have proved that the background metric is stable against the \( K = \pm 2 \) perturbations.

**FIG. 1:** The effective potential \( V_2 \) for \( K = \pm 2 \) mode.
2. $K = \pm 1$ modes

Because of the relation $\tilde{h}_{A^+} = h_{A^-}$ and $\tilde{h}_{+3} = h_{-3}$, we consider only $h_{A^+}$ and $h_{+3}$. We set $h_{\mu\nu}$ as

$$h_{\mu\nu} dx^\mu dx^\nu = 2h_{A^+}(r) e^{-i\omega t} dx^A \sigma^+ + 2h_{+3}(r) e^{-i\omega t} \sigma^+ \sigma^3 .$$

There are three components in Eq. (31). We use the gauge condition \(^2\)

$$h_{+3} = 0 ,$$

which fix completely the gauge freedom. Substituting Eq. (31) and (32) into $\delta G_{A^+} = 0$ and $\delta G_{+3} = 0$, and eliminating $h_{t^+}$ from these equations, we get the master equation for $K = 1$ mode. Defining a new variable

$$\Phi_1(\rho) \equiv \frac{4(\rho - \rho_+)(\rho_+ \rho_0 - \rho(2\rho_0 + \rho))}{\rho^{3/4}(\rho + \rho_0)^{9/4}} h_{\rho^+}(\rho) ,$$

we have the master equation in Shrödinger form:

$$- \frac{d^2}{d\rho^2} \Phi_1(\rho) \Phi_1 = \omega^2 \Phi_1 .$$

The potential $V_1$ reads

$$V_1(\rho) = \frac{\rho - \rho_+}{16\rho_0(\rho_+ + \rho_0)\rho^2(\rho_0 + \rho)^3(\rho + \rho_0 - \rho(2\rho_0 + \rho))^2 \left[4\rho_+^3(\rho_+ + \rho_0)^4(4\rho_+^2 - 8\rho_+ \rho_0 - 11\rho_0^2) + \rho_+^2(\rho_+ + \rho_0)^3(144\rho_+^3 + 48\rho_+^2 \rho_0 - 68\rho_+ \rho_0^2 + 31\rho_0^3)(\rho - \rho_+) + 4\rho_+(\rho_+ + \rho_0)^3(144\rho_+^3 + 152\rho_+^2 \rho_0 + 152\rho_+ \rho_0^2 + 75\rho_0^3)(\rho - \rho_+)^2 + 2(\rho_+ + \rho_0)^2(672\rho_+^4 + 1520\rho_+^3 \rho_0 + 1548\rho_+^2 \rho_0^2 + 781\rho_+ \rho_0^3 + 126\rho_0^4)(\rho - \rho_+)^3 + 4(\rho_+ + \rho_0)^2(504\rho_+^3 + 1032\rho_+^2 \rho_0 + 757\rho_+ \rho_0^2 + 191\rho_0^3)(\rho - \rho_+)^4 + (2016\rho_+^4 + 7072\rho_+^3 \rho_0 + 9164\rho_+^2 \rho_0^2 + 5211\rho_+ \rho_0^3 + 1103\rho_0^4)(\rho - \rho_+)^5 + 8(168\rho_+^3 + 460\rho_+^2 \rho_0 + 411\rho_+ \rho_0^2 + 119\rho_0^3)(\rho - \rho_+)^6 + 96(6\rho_+^2 + 11\rho_+ \rho_0 + 5\rho_0^2)(\rho - \rho_+)^7 + 16(9\rho_+ + 8\rho_0)(\rho - \rho_+)^8 + 16(\rho - \rho_+)^9 \right] .$$

\(^2\) Note that we cannot choose this gauge condition in the case of five dimensional Schwarzschild black hole limit.
Typical profiles of the potential $V_1$ are shown in Fig. 2.

From Fig. 2, we see that this potential $V_1$ contains a negative region. Hence, we hardly show the stability from this form of potential. However, we can overcome this difficulty by using a transformation of the coordinate. We introduce a new radial coordinate $y$ as

$$\frac{d}{dy} = \frac{1}{\beta(\rho)} \frac{d}{d\rho^*},$$  \hspace{1cm} (36)

where $\beta(\rho)$ is some real function and must be non-singular outside of the horizon, $\rho_+ \leq \rho < \infty$. Then, the master equation becomes

$$-\frac{d^2}{dy^2}\Phi_1 - \frac{1}{\beta} \frac{d\beta}{dy} \Phi_1 + \frac{V_1}{\beta^2} \Phi_1 = \frac{\omega^2}{\beta^2} \Phi_1.$$  \hspace{1cm} (37)

Multiply both sides of equation by $\bar{\Phi}_1$ we obtain

$$-\bar{\Phi}_1 \frac{d^2}{dy^2}\Phi_1 - \frac{1}{\beta} \frac{d\beta}{dy} \bar{\Phi}_1 \Phi_1 + \frac{V_1}{\beta^2} \bar{\Phi}_1 \Phi_1 = \frac{\omega^2}{\beta^2} \bar{\Phi}_1 \Phi_1.$$  \hspace{1cm} (38)

Adding eq. (38) and its complex conjugate equation, and integrating it, we obtain the equation

$$\int dy \left[ \frac{\left| \frac{d\Phi_1}{dy} \right|^2 + \bar{V}_1 \left| \Phi_1 \right|^2}{\beta^2} \right] - \frac{1}{2} \left[ \Phi_1 \frac{d}{dy} \Phi_1 + \Phi_1 \frac{d}{dy} \bar{\Phi}_1 + \frac{1}{\beta} \frac{d\beta}{dy} \Phi_1 \bar{\Phi}_1 \right]_{\rho_+ = \infty}^{\rho_+ = -\infty} \rho_+ = \infty$$

$$= \omega^2 \int dy \frac{\left| \Phi_1 \right|^2}{\beta^2},$$  \hspace{1cm} (39)

where

$$\bar{V}_1 = V_1 + \frac{1}{2} \beta^2 \frac{d}{dy} \left( \frac{1}{\beta} \frac{d\beta}{dy} \right).$$  \hspace{1cm} (40)
The boundary terms in (39) vanish because $\Phi_1$ is square-integrable. Therefore, if the deformed effective potential $\tilde{V}$ is positive everywhere, there are no $\omega^2 < 0$ mode. Now, we choose $\beta$ as

$$\beta^2 = \frac{15}{K^2},$$

then the potential becomes

$$\tilde{V}_1 = \frac{\rho - \rho_+}{16\rho_0 (\rho_+ + \rho_0) \rho^3 (\rho_0 + \rho)^3 (\rho_+ \rho_0 - \rho (2\rho_0 + \rho))^3}
\left[ 16\rho_+^3 (\rho_+ - \rho_0)^2 (\rho_+ + \rho_0)^4
+ \rho_+^2 (\rho_+ + \rho_0)^3 (144\rho_+^3 + 48\rho_+^2 \rho_0 + 112\rho_+ \rho_0^2 + 211\rho_0^3) (\rho - \rho_+)
+ 4\rho_+ (\rho_+ + \rho_0)^3 (144\rho_+^3 + 152\rho_+^2 \rho_0 + 152\rho_+ \rho_0^2 + 75\rho_0^3) (\rho - \rho_+)^2
+ 2(\rho_+ + \rho_0)^2 (672\rho_+^4 + 1520\rho_+^3 \rho_0 + 1248\rho_+^2 \rho_0^2 + 361\rho_+ \rho_0^3 + 6\rho_0^4) (\rho - \rho_+)^3
+ 4(\rho_+ + \rho_0)^2 (504\rho_+^3 + 1032\rho_+^2 \rho_0 + 532\rho_+ \rho_0^2 + 11\rho_0^3) (\rho - \rho_+)^4
+ (\rho_+ + \rho_0) (2016\rho_+^3 + 5056\rho_+^2 \rho_0 + 3568\rho_+ \rho_0^2 + 563\rho_0^3) (\rho - \rho_+)^5
+ 32(\rho_+ + \rho_0) (2\rho_+ + \rho_0) (21\rho_+ + 26\rho_0) (\rho - \rho_+)^6
+ 96(\rho_+ + \rho_0) (6\rho_+ + 5\rho_0) (\rho - \rho_+)^7
+ 16 (9\rho_+ + 8\rho_0) (\rho - \rho_+)^8 + 16(\rho - \rho_+)^9 \right].$$

We can see $\tilde{V}_1 > 0$ from above expression. Thus, we have proved the stability for $K = \pm 1$ modes.

3. $K = 0$ mode

For $K = 0$ mode, there are $h_{AB}, h_{A3}, h_{33}, h_{+-}$. We set $h_{\mu\nu}$ as

$$h_{\mu\nu} dx^\mu dx^\nu = h_{AB}(r) e^{-i \omega t} dx^A dx^B + 2h_{A3}(r) e^{-i \omega t} dx^A \sigma^3
+ 2h_{+-}(r) e^{-i \omega t} \sigma^+ \sigma^- + h_{33}(r) e^{-i \omega t} \sigma^3 \sigma^3 .$$

We choose the gauge conditions \(^3\)

$$h_{+-} = 0 , \ h_{tt} = 0 , \ h_{t3} = 0 .$$

\(^3\) Note that for static perturbation, we cannot choose this gauge condition.
Substituting Eq. (43) and (44) into $\delta G_{AB} = 0$, $\delta G_{33} = 0$ and $\delta G_{+-} = 0$, we get the equation of motion for $K = 0$ mode. Because of gauge symmetry and constraint equations, there is only one physical degree of freedom. Introducing the new variable

$$\Phi_0(\rho) \equiv \frac{(\rho + \rho_0)^{5/4}(2\rho + \rho_0)}{\rho^{1/4}(4\rho + 3\rho_0)} h_{33}(\rho).$$  \hspace{1cm} (45)$$

we get the master equation

$$-\frac{d^2}{d{\rho}^2} \Phi_0 + V_0(\rho) \Phi_0 = \omega^2 \Phi_0,$$  \hspace{1cm} (46)$$

where the potential $V_0$ is defined by

$$V_0(\rho) = \frac{\rho - \rho_+}{16\rho^3(\rho + \rho_0)^3(4\rho + 3\rho_0)^2} \left[ 4\rho_+ (64\rho_+^4 + 304\rho_+^3\rho_0 + 516\rho_+^2\rho_0^2 + 375\rho_+\rho_0^3 + 99\rho_0^4) 
+ (1024\rho_+^4 + 3776\rho_+^3\rho_0 + 4656\rho_+^2\rho_0^2 + 2220\rho_+\rho_0^3 + 315\rho_0^4) (\rho - \rho_+) 
+ 48 (32\rho_+^3 + 84\rho_+^2\rho_0 + 65\rho_+\rho_0^2 + 15\rho_0^3) (\rho - \rho_+)^2 
+ 16 (64\rho_+^2 + 100\rho_+\rho_0 + 33\rho_0^2) (\rho - \rho_+)^3 + 128 (2\rho_+ + \rho_0) (\rho - \rho_+)^4 \right].$$  \hspace{1cm} (47)$$

This expression explicitly shows $V_0 > 0$ outside the horizon. Then, we see the stability for $K = 0$ mode. Typical profiles of $V_0$ are shown in Fig. 3.

![Fig. 3: The effective potential $V_0$ for $K = 0$ mode.](image)

**FIG. 3:** The effective potential $V_0$ for $K = 0$ mode.

**B. $K = \pm (J + 2)$ modes perturbation**

As noted in the previous section, the highest and lowest modes, namely $K = \pm (J + 2)$ modes, $h_{++}$ and $h_{--}$ are always decoupled respectively for arbitrary $J$. By the same
procedure in the previous subsection, using a new variable

$$\Phi_J(\rho) \equiv \frac{1}{\rho^{1/4}(\rho + \rho_0)^{3/4}} h_{++}(\rho) .$$  \quad (48)$$

we obtain the master equation

$$- \frac{d^2}{d\rho^2} \Phi_J + V_J(\rho) \Phi_J = \omega^2 \Phi_J ,$$  \quad (49)$$

where the potential $V_J(\rho)$ is defined by

$$V_J(\rho) = \rho - \frac{\rho_+}{16 \rho^3 \rho_0 (\rho_+ + \rho_0) (\rho + \rho_0)^3} \left[ 4 \rho_+ (\rho_+ + \rho_0)^2 (16 \rho_+^2 + 28 \rho_+ \rho_0 + 11 \rho_0^2) 
+ 8 \rho_+ (\rho_+ + \rho_0) (80 \rho_+^2 + 102 \rho_+ \rho_0 + 25 \rho_0^2) (\rho - \rho_+) 
+ 32 (\rho_+ + \rho_0) (20 \rho_+ + 11 \rho_0) (\rho - \rho_+)^3 
+ 64 (5 \rho_+ + 4 \rho_0) (\rho - \rho_+) + 64 (\rho - \rho_+)^5 
+ J \left[ 16 \rho_+(\rho_+ + \rho_0)^3 (4 \rho_+ + 3 \rho_0) 
+ 16 (\rho_+ + \rho_0)^2 (20 \rho_+^2 + 21 \rho_+ \rho_0 + 3 \rho_0^2) (\rho - \rho_+) 
+ 16 (\rho_+ + \rho_0) (40 \rho_+^2 + 53 \rho_+ \rho_0 + 14 \rho_0^2) (\rho - \rho_+) 
+ 16 (\rho_+ + \rho_0) (40 \rho_+ + 23 \rho_0) (\rho - \rho_+)^3 
+ 64 (5 \rho_+ + 4 \rho_0) (\rho - \rho_+) + 64 (\rho - \rho_+)^5 \right] 
+ J^2 \left[ 16 \rho_+(\rho_+ + \rho_0)^4 + 16 (\rho_+ + \rho_0)^3 (5 \rho_+ + \rho_0) (\rho - \rho_+) 
+ 32 (\rho_+ + \rho_0)^2 (5 \rho_+ + 2 \rho_0) (\rho - \rho_+) 
+ 32 (\rho_+ + \rho_0) (5 \rho_+ + 3 \rho_0) (\rho - \rho_+)^3 
+ (80 \rho_+ + 64 \rho_0) (\rho - \rho_+) + 16 (\rho - \rho_+)^5 \right] \right].$$  \quad (50)$$

Clearly, the potential $V_J$ is positive. Then, we confirm the stability against all $K = \pm (J + 2)$ modes.

V. SUMMARY AND DISCUSSION

We have studied the stability of squashed Kaluza-Klein (SqKK) black holes. By utilizing
the symmetry $U(2)$ of the SqKK black holes, we have obtained the master equations for the
metric perturbations labeled by $(J = 0, M = 0, K = 0, \pm 1, \pm 2)$ and $(J, M, K = \pm (J + 2))$.  

15
We have proved the stability of SqKK black holes for these perturbations. Strictly speaking, we have not shown the stability of SqKK black holes completely, because we have analyzed the restricted modes. Empirically, the instability appear in the lower modes. For example, Gregory-Laflamme instability appears in a s-wave. Therefore, our result for \((J = 0, M = 0, K = 0, \pm 1, \pm 2)\) modes gives a strong evidence for stability of the SqKK black holes.

Our stability analysis suggests that the SqKK black holes deserve to be taken seriously as realistic black holes in the Kaluza-Klein spacetime. Because of the stability, the SqKK black holes would be good idealization of black holes created in colliders or in the cosmic history. If so, we can observe the extra dimension through Hawking radiation or quasi-normal modes \([20, 26]\). Namely, the SqKK black holes could be a window to the extra dimension.

There are several directions to be studied. Our method can be applicable to other \(U(2)\) symmetric spacetimes such as five-dimensional Myers-Perry black holes with equal angular momenta \([32]\). The rotating SqKK black holes \([14]\) has also the symmetry \(U(2)\). It is known that the rotation of black holes induces the superradiant instability for massive scalar fields. Since Kaluza-Klein modes of gravitational perturbation are regarded as massive fields from the 4-dimensional point of view, the rotating SqKK black holes may show the superradiant instability. It is interesting to study if it occurs or not by using our formalism. As to other direction, the generalization of squashed black hole to higher dimensions is intriguing.

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