On the Computational Complexity of Model Checking for Dynamic Epistemic Logic with S5 Models

Ronald de Haan · Iris van de Pol

Abstract  Dynamic epistemic logic (DEL) is a logical framework for representing and reasoning about knowledge change for multiple agents. An important computational task in this framework is the model checking problem, which has been shown to be PSPACE-hard even for S5 models and two agents. We answer open questions in the literature about the complexity of this problem in more restricted settings. We provide a detailed complexity analysis of the model checking problem for DEL, where we consider various combinations of restrictions, such as the number of agents, whether the models are single-pointed or multi-pointed, and whether postconditions are allowed in the updates. In particular, we show that the problem is already PSPACE-hard in (1) the case of one agent, multi-pointed S5 models, and no postconditions, and (2) the case of two agents, only single-pointed S5 models, and no postconditions. In addition, we study the setting where only semi-private announcements are allowed as updates. We show that for this case the problem is already PSPACE-hard when restricted to two agents and three propositional variables.

Ronald de Haan was supported by the Austrian Science Fund (FWF), project J4047. Iris van de Pol was supported by Gravitation Grant 024.001.006 of the Language in Interaction Consortium from the Netherlands Organization for Scientific Research (NWO).

Ronald de Haan
Institute for Logic, Language and Computation
University of Amsterdam
Amsterdam, the Netherlands
E-mail: me@ronalddehaan.eu

Iris van de Pol
Institute for Logic, Language and Computation
University of Amsterdam
Amsterdam, the Netherlands
E-mail: i.p.a.vandepol@gmail.com
1 Introduction

Dynamic epistemic logic (or DEL, for short) is a logical framework for representing and reasoning about knowledge (and belief) change for multiple agents. This framework has applications in philosophy, cognitive science, computer science and artificial intelligence (see, e.g., [7,10,11,15,20,25]). For instance, reasoning about information and knowledge change is an important topic for multi-agent and distributed systems [15].

DEL is a very general and expressive framework, but many settings where the framework is used allow strong restrictions. For instance, in the context of reasoning about knowledge, the semantic models for the logic are often restricted to models that contain only equivalence relations (also called S5 models).

For many of the applications of DEL, computational and algorithmic aspects of the framework are highly relevant. It is important to study the complexity of computational problems associated with the logic to determine to what extent it can be used in practical settings, and what algorithmic approaches are best suited to solve these problems. One important computational task is the problem of model checking, where the question is to decide whether a formula is true in a model.

The complexity of the model checking problem for DEL has been a topic of investigation in the literature. For a restricted fragment of DEL, known as public announcement logic [4,5,21], the model checking problem is polynomial-time solvable [9,17]. The problem of DEL model checking, in its general form, has been shown to be PSPACE-complete [2,14], even in the case of two agents and S5 models. However, these hardness proofs crucially depend on the use of multi-pointed models, and therefore do not apply for the case where the problem is restricted to single-pointed S5 models. This open question was answered with a PSPACE-hardness proof for the restricted case where all models are single-pointed S5 models, but where the number of agents is unbounded [23,24]. It remained open whether these PSPACE-hardness results extend to more restrictive settings (e.g., only two agents and single-pointed S5 models).

Other Related Work Various topics related to DEL model checking have been studied in the literature. For (several restricted variants of) a knowledge update framework based on epistemic logic, the computational complexity of the model checking problem has been investigated [6]. Other related work includes implementations of algorithms for DEL model checking [8,13]. Additionally, research has been done on the complexity of the satisfiability problem for (fragments of) DEL [2,19].

Contributions In this paper we provide a detailed computational complexity analysis of the model checking problem for DEL, restricted to S5 models. We consider various different restricted settings of this problem.

For the case of arbitrary event models, we have the following results.
– We show that the problem is polynomial-time solvable in the case of a single agent and single-pointed S5 models without postconditions (Proposition 1).
– We show that a similar restriction (single agent and single-pointed S5 models) where postconditions are allowed already leads to $\Delta^p_2$-hardness (Theorem 1).
– When multi-pointed event models are allowed, we show that the problem is PSPACE-hard even for the case of a single agent and S5 models without postconditions (Theorem 2).
– For the case where there are two agents, we show that the problem is already PSPACE-hard when restricted to single-pointed S5 models without postconditions and with only three propositional variables (Theorem 3).

An overview of the complexity results for arbitrary event models can be found in Table 1.

Additionally, we consider the setting where instead of arbitrary event models, only semi-private announcements can be used—this is a restricted class of event models. In this setting, the problem is known to be PSPACE-hard, when an arbitrary number of agents is allowed (i.e., when the number of agents is part of the problem input) [23, Theorem 4].

– We show that the problem is already PSPACE-hard in the case where there are only two agents and only three propositional variables (Theorem 4).

| # agents | single- or multi-pointed | postconditions | complexity                      |
|----------|--------------------------|----------------|----------------------------------|
| 1        | single                   | no             | in P (Proposition 1)             |
| 1        | single                   | yes            | $\Delta^p_2$-hard (Theorem 1)    |
| 1        | multi                    | no / yes       | PSPACE-complete (Theorem 2)      |
| $\geq 2$ | single / multi           | no / yes       | PSPACE-complete (Theorem 3)      |

Table 1: Complexity results for the model checking problem for DEL with S5 models and S5 event models.

Roadmap We begin in Section 2 with reviewing basic notions and notation from dynamic epistemic logic and complexity theory. Then, in Section 3 we present the complexity results for the various settings that involve updates with (arbitrary) event models. In Section 4 we present our PSPACE-hardness proof for the setting of semi-private announcements. Finally, we conclude and suggest directions for future research in Section 5.

1 Theorem 1 is a stronger result than Theorem 3—Theorem 1 implies the result of Theorem 3.
2 Preliminaries

We briefly review some basic notions from dynamic epistemic logic and complexity theory that are required for the complexity results that we present in this paper.

2.1 Dynamic Epistemic Logic

We begin by reviewing the syntax and semantics of dynamic epistemic logic. We consider a version of this logic that is often considered in the literature (e.g., by Van Ditmarsch, Van der Hoek and Kooi [1]). After describing the logic that we consider in this paper, we briefly relate it to other variants of dynamic epistemic logic that have been considered in the literature.

We fix a countable set $P$ of propositional variables, and a finite set $A$ of agents. We begin with introducing the basic language of epistemic logic, and its semantics. The semantics of epistemic logic is based on a type of (Kripke) structures called *epistemic models*. Epistemic models are structures that are used to represent the agents’ knowledge about the world and about the other agents’ knowledge.

**Definition 1 (Epistemic models)** An *epistemic model* is a tuple $M = (W, R, V)$, where $W$ is a non-empty set of worlds, $R$ maps each agent $a \in A$ to a relation $R_a \subseteq W \times W$, and $V : P \rightarrow 2^W$ is a function called a valuation. By a slight abuse of notation, we write $w \in M$ for $w \in W$. We also write $v \in R_a(w)$ for $vR_aw$.

A *single-pointed model* is a pair $(M, w)$ consisting of an epistemic model $M$ and a designated (or pointed) world $w \in M$. A *multi-pointed model* is a pair $(M, W_d)$ consisting of an epistemic model $M$ and a subset $W_d$ of designated worlds.

**Definition 2 (Basic epistemic language)** The language $L_{EL}$ of epistemic logic is defined as the set of formulas $\varphi$ defined inductively as follows, where $p$ ranges over $P$ and $a$ ranges over $A$:

$$\varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K_a \varphi.$$  

The formula $\bot$ is an abbreviation for $p \land \neg p$, and the formula $\top$ is an abbreviation for $\neg \bot$. A formula of the form $(\varphi_1 \lor \varphi_2)$ abbreviates $\neg (\neg \varphi_1 \land \neg \varphi_2)$, and a formula of the form $(\varphi_1 \rightarrow \varphi_2)$ abbreviates $(\neg \varphi_1 \lor \varphi_2)$. Moreover, a formula of the form $\neg K_a \varphi$ is an abbreviation for $\neg K_a \neg \varphi$. We call formulas of the form $p$ or $\neg p$ literals. We denote the set of all literals by $\text{Lit}$.

Intuitively, the formula $K_a \varphi$ expresses that ‘agent $a$ knows that $\varphi$ holds in the current situation.’ Next, we define when a formula in the basic epistemic language is true in a world of an epistemic model.

**Definition 3 (Truth conditions for $L_{EL}$)** Given an epistemic model $M = (W, R, V)$, we inductively define the relation $\models \subseteq W \times L_{EL}$ as follows. For
all \( w \in W \):

- \( \mathcal{M}, w \models p \) iff \( w \in V(p) \)
- \( \mathcal{M}, w \models \neg \varphi \) iff \( \mathcal{M}, w \not\models \varphi \)
- \( \mathcal{M}, w \models \varphi_1 \land \varphi_2 \) iff both \( \mathcal{M}, w \models \varphi_1 \) and \( \mathcal{M}, w \models \varphi_2 \)
- \( \mathcal{M}, w \models K_a \varphi \) iff for all \( v \in R_a(w) \), it holds that \( \mathcal{M}, v \models \varphi \)

The statement \( \mathcal{M}, w \models \varphi \) expresses that the formula \( \varphi \) is true in world \( w \) in the model \( \mathcal{M} \).

The framework of dynamic epistemic logic extends the basic epistemic logic with a notion of updates, that are based on another type of structures: event models. These are used to represent the effects of an event on the world and the knowledge of the agents.

**Definition 4 (Event models)** An event model is a tuple \( \mathcal{E} = (E, S, \text{pre}, \text{post}) \), where \( E \) is a non-empty and finite set of possible events, \( S \) maps each agent \( a \in A \) to a relation \( S_a \subseteq E \times E \), \( \text{pre} : E \to \mathcal{L}_{EL} \) is a function that maps each event to a precondition expressed in the epistemic language, and \( \text{post} : E \to 2^{\text{Lit}} \) is a function that maps each event to a set of literals (not containing complementary literals). For convenience, we write \( \top \) to denote an empty postcondition. By a slight abuse of notation, we write \( e \in E \) for \( e \in E \). A single-pointed event model is a pair \( (\mathcal{E}, e) \) consisting of an event model \( \mathcal{E} \) and a designated (or pointed) event \( e \in E \). A multi-pointed event model is a pair \( (\mathcal{E}, E_d) \) consisting of an event model \( \mathcal{E} \) and a subset \( E_d \subseteq E \) of designated events.

The language of dynamic epistemic logic extends the basic epistemic language with update modalities.

**Definition 5 (Dynamic epistemic language)** The language \( \mathcal{L}_{DEL} \) of dynamic epistemic logic is defined as the set of formulas \( \varphi \) defined inductively as follows:

\[
\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid K_a \varphi \mid [\mathcal{E}, e] \varphi \mid [\mathcal{E}, E_d] \varphi,
\]

where \( p \) ranges over \( P \) and \( a \) ranges over \( A \), and where \( (\mathcal{E}, e) \) and \( (\mathcal{E}, E_d) \) are single- and multi-pointed event models, respectively. A formula of the form \( [\mathcal{E}, e] \varphi \) is an abbreviation for \( \neg [\mathcal{E}, e] \neg \varphi \); we use a similar abbreviation \( [\mathcal{E}, E_d] \varphi \) for updates with multi-pointed event models.

The effect of these event models is defined using the following notion of product update.

**Definition 6 (Product update)** Let \( \mathcal{M} = (W, R, V) \) be an epistemic model and let \( \mathcal{E} = (E, S, \text{pre}, \text{post}) \) be an event model. The product update of \( \mathcal{M} \) by \( \mathcal{E} \)

---

2 Alternatively, one can define postconditions using a function \( \text{post} : E \times P \to \mathcal{L}_{EL} \), (see, e.g., [12]). The complexity results in this paper also hold when this alternative definition is used.
is the epistemic model \( M \otimes E = (W', R', V') \) defined as follows, where \( p \) ranges over \( P \) and \( a \) ranges over \( A \):

\[
W' = \{ (w, e) \in W \times E : M, w \models \text{pre}(e) \} \\
R'_a = \{ ((w, e), (w', e')) \in W' \times W' : wR_a w' \text{ and } eS_a e' \} \\
V'(p) = \{ (w, e) \in W' : w \in V(p) \text{ and } \neg p \not\in \text{post}(e) \} \cup \{ (w, e) \in W' : p \in \text{post}(e) \}
\]

Next, we define when a formula in the dynamic epistemic language is true in a world of an epistemic model.

**Definition 7 (Truth conditions for \( L_{\text{DEL}} \))** Given an epistemic model \( M = (W, R, V) \) and a formula \( \varphi \in L_{\text{DEL}} \), we inductively define the relation \( \models \subseteq W \times L_{\text{DEL}} \) as follows. For all \( w \in W \):

\[
M, w \models [E, e] \varphi \iff M, w \models \text{pre}(e) \text{ implies } M \otimes E, (w, e) \models \varphi \\
M, w \models [E, E_d] \varphi \iff M, w \models [E, e] \varphi \text{ for all } e \in E_d
\]

The other cases are identical to Definition 3. Again, the statement \( M, w \models \varphi \) expresses that the formula \( \varphi \) is true in state \( w \) in the model \( M \).

(Having defined the language \( L_{\text{DEL}} \), we could now also change the definition of preconditions in event models to be functions \( \text{pre} : E \to L_{\text{DEL}} \) mapping events to formulas in the dynamic epistemic language \( L_{\text{DEL}} \). The definition of product update would work in an entirely similar way. All results in this paper work for either definition of preconditions \( \text{pre} \).)

We can then define truth of a formula \( \varphi \in L_{\text{DEL}} \) in epistemic models as follows. A formula \( \varphi \) is true in a single-pointed epistemic model \( (M, w) \) if \( M, w \models \varphi \), and a formula \( \varphi \) is true in a multi-pointed epistemic model \( (M, W_d) \) if \( M, w \models \varphi \) for all \( w \in W_d \).

For the purposes of representing knowledge, the relations in epistemic models and event models are often restricted to be equivalence relations, that is, reflexive, transitive and symmetric (see, e.g., [11]). Models that satisfy these requirements are also called \textit{S5 models}, after the axiomatic system that characterizes this type of relations. In the remainder of this paper, we consider only epistemic models and event models that are S5 models. All our hardness results hold for S5 models, as well as for arbitrary models.

For the sake of convenience, we will often depict epistemic models and event models graphically. We will represent worlds with solid dots, events with solid squares, designated worlds and events with a circle or square around them, valuations, preconditions and postconditions with labels next to the dots, and relations with labelled lines between the dots. Since we restrict ourselves to S5 models, and thus to equivalence relations, all relations are symmetric and it suffices to represent relations with undirected lines. Moreover, the reflexive relations are not represented graphically. For a valuation of a world \( w \), we use the literals that the valuation makes true in world \( w \) as a label, and for the preconditions and postconditions of an event \( e \), we use the label \( (\text{pre}(e), \text{post}(e)) \).

Moreover, since all epistemic models and event models that we consider in
this paper have reflexive relations, in order not to clutter the graphical representation of models, we do not explicitly depict the reflexive relations. For an example of an epistemic model with its graphical representation, see Figure 1 and for an example of an event model with its graphical representation, see Figure 2.

![Diagram of epistemic model](image1)

**Fig. 1:** The epistemic model \((M, w_1)\) for the set \(A = \{a, b\}\) of agents and a single proposition \(z\), where \(M = (W, R, V)\), \(W = \{w_1, w_2\}\), \(R_a = \{(w_1, w_1), (w_1, w_2), (w_2, w_1), (w_2, w_2)\}\), \(R_b = \{(w_1, w_1), (w_2, w_2)\}\), and \(V(z) = \{w_1\}\).

![Diagram of event model](image2)

**Fig. 2:** The event model \((E, e_1)\) for the set \(A = \{a, b\}\) of agents and a single proposition \(h\), where \(M = (E, S, \text{pre}, \text{post})\), \(E = \{e_1, e_2\}\), \(R_a = \{(e_1, e_1), (e_1, e_2), (e_2, e_1), (e_2, e_2)\}\), \(R_b = \{(e_1, e_1), (e_2, e_2)\}\), \(\text{pre}(e_1) = \text{pre}(e_2) = \top\), \(\text{post}(e_1) = h\), and \(\text{post}(e_2) = \neg h\).

A particular type of S5 event models that has been considered in the literature are semi-private (or semi-public) announcements [3]. Intuitively, a semi-private announcement publicly announces one of two formulas \(\varphi_1, \varphi_2\) to a subset \(A\) of agents, and to the remaining agents it publicly announces that one of the two formulas is the case, and that the agents in \(A\) learned which one is. A semi-private announcement for formulas \(\varphi_1, \varphi_2\) and a subset \(A \subseteq A\) of agents is represented by the event model in Figure 3.

To illustrate the notion of semi-private announcements, consider the following example scenario. There are two agents, Ayla (\(a\)) and Blair (\(b\)). Ayla flips a coin, which lands either on heads (\(h\)) or on tails (\(\neg h\)), and hides the result of the coin flip from Blair. Blair sees that the coin is flipped and that Ayla knows the result of the coin flip, but Blair herself does not see the result of the coin flip. This semi-private announcement is represented by the event model \(E\) that is depicted in Figure 2 (in the event model depicted in Figure 2 the coin lands on heads).

**Relations to other variants of Dynamic Epistemic Logic** The formalism of dynamic epistemic logic that we consider is based on the one originally introduced by Baltag, Moss, and Solecki [3,5]. Their language only considers...
single-pointed event models. A few years later, Baltag and Moss [3] extended this original language to include regular operators (union, composition and ‘star’) for the update modalities. The language that we consider corresponds to the variant of their language with only the union operator. The language presented by Van Ditmarsch et al. in their textbook [11] resembles the language that we consider, as their framework also allows the union operator for updates, but not the composition or ‘star’ operators. The union operator for update modalities corresponds to allowing multi-pointed event models. Because it simplifies notation, we use multi-pointed models, following the notation of other existing work [10]. Additionally, the language that we consider also allows events to have postconditions, unlike the language presented by Van Ditmarsch et al. [11].

2.2 Computational Complexity

Next, we review some basic notions from computational complexity that are used in the proofs of the results that we present. We assume the reader to be familiar with the complexity classes P and NP, and with basic notions such as polynomial-time reductions. For more details, we refer to textbooks on computational complexity theory (see, e.g., [1]).

The class PSPACE consists of all decision problems that can be solved by an algorithm that uses a polynomial amount of space (memory). Alternatively, one can characterize the class PSPACE as all decision problems for which there exists a polynomial-time reduction to the problem QSAT, that is defined using quantified Boolean formulas as follows. A (fully) quantified Boolean formula (in prenex form) is a formula of the form \( Q_1x_1Q_2x_2\ldots Q_nx_n.\psi \), where all \( x_i \) are propositional variables, each \( Q_i \) is either an existential or a universal quantifier, and \( \psi \) is a (quantifier-free) propositional formula over the variables \( x_1,\ldots,x_n \).

Truth for such formulas is defined in the usual way. The problem QSAT consists of deciding whether a given quantified Boolean formula is true. Moreover, QSAT is PSPACE-hard even when restricted to the case where \( Q_i = \exists \) for odd \( i \) and \( Q_i = \forall \) for even \( i \). (For the proofs of Theorems 2, 3 and 4 we will use reductions from this restricted variant of QSAT.)

Additionally, one can restrict the number of quantifier alternations occurring in quantified Boolean formulas, i.e., the number of times where \( Q_i \neq Q_{i+1} \). For each constant \( k \geq 1 \) number of alternations, this leads to a different complexity class. These classes together constitute the Polynomial Hierarchy. We consider the complexity classes \( \Sigma^p_k \), for each \( k \geq 1 \). The complexity class \( \Sigma^p_k \)
On the Complexity of Model Checking for DEL with S5 Models

consists of all decision problems for which there exists a polynomial-time reduction to the problem $QSAT_k$. Instances of $QSAT_k$ are quantified Boolean formulas of the form $\exists x_1 \ldots \exists x_{\ell_1} \forall x_{\ell_1+1} \ldots \forall x_{\ell_2} \ldots Q_k x_{\ell_{k-1}+1} \ldots Q_k x_{\ell_k} \cdot \psi$, where $Q_k = \exists$ if $k$ is odd and $Q_k = \forall$ if $k$ is even, where $1 \leq \ell_1 \leq \cdots \leq \ell_k$, and where $\psi$ is quantifier-free. The question is to decide whether the quantified Boolean formula is true.

The last complexity class that we consider is $\Delta_p^2$. We give a definition of this class that is based on algorithms with access to an oracle, i.e., a black box that is able to decide certain decision problems in a single operation. Consider the problem $Sat$ of deciding satisfiability of a given propositional formula. The class $\Delta_p^2$ consists of all decision problems that can be solved by a polynomial-time algorithm with access to an oracle for $Sat$. Alternatively, the class $\Delta_p^2$ consists of all decision problems for which there exists a polynomial-time reduction to the problem where one is given a satisfiable propositional formula $\phi$ over the variables $x_1, \ldots, x_n$, and the question is whether the lexicographically maximal assignment that satisfies $\phi$ (given the fixed ordering $x_1 \prec \cdots \prec x_n$) sets variable $x_n$ to true [18]. An assignment $\alpha_1$ is lexicographically larger than an assignment $\alpha_2$ (given the ordering $x_1 \prec \cdots \prec x_n$) if there exists some $1 \leq i \leq n$ such that $\alpha_1(x_i) = 1$, $\alpha_2(x_i) = 0$, and for all $1 \leq j \leq i$ it holds that $\alpha_1(x_j) = \alpha_2(x_j)$.

3 Results for updates with arbitrary S5 models

In this section, we provide complexity results for the model checking problem for DEL when arbitrary event models are allowed for the update modalities in the formulas. For several cases, we prove PSPACE-hardness. Since the problem was recently shown to be in PSPACE for the most general variant of dynamic epistemic logic that we consider in this paper [2,23,24], these hardness results suffice to show PSPACE-completeness.

3.1 Polynomial-time solvability

We begin with showing polynomial-time solvability for the strongest restriction that we consider in this paper: a single agent, single-pointed S5 models, and no postconditions.

**Proposition 1.** The model checking problem for DEL with S5 models is polynomial-time solvable when restricted to instances with a single agent and only single-pointed event models without postconditions.

**Proof** We describe a polynomial-time algorithm that solves the problem. The main idea behind this algorithm is the following. Even though the updates might cause an exponential blow-up in the number of worlds in the model, in this restricted setting, we only need to remember a small number of these worlds.
Concretely, since there is only a single agent $a$, and since there is only a single designated world $w_0$, we only need to remember the set of worlds that are connected with an $a$-relation to the designated world $w_0$. Moreover, among these worlds, we can merge those with an identical valuation. Since the event models contain no postconditions, this (contracted) set of worlds can only decrease after updates, i.e., updates can only remove worlds from this set.

Formally, we can describe this argument as follows. Let $(M, w_0)$ be a single-pointed S5 epistemic model with one agent, and let $(E, e_0)$ be a single-pointed S5 event model with one agent and no postconditions. Then $M \otimes E$ is bisimilar to a submodel $M'$ of $M$, that is, to some $M'$ that can be obtained from $M$ by removing some worlds. Specifically, let $W'$ be the set of worlds in $M$ that are $a$-accessible from $w_0$, and let $E'$ be the set of events in $E$ that are $a$-accessible from $e_0$. Then, let $W'' \subseteq W'$ be the subset of worlds that satisfy the precondition of at least one $e \in E'$. One can straightforwardly verify that $(M \otimes E, (w_0, e_0))$ is bisimilar to the submodel $(M', w_0)$ of $(M, w_0)$ induced by $W''$. Moreover, $M'$ can be computed in polynomial time.

Using this property, we can construct a recursive algorithm to decide whether $M, w \models \varphi$. We consider several cases. In the case where $\varphi = p$ for some $p \in P$, the problem can easily be solved in polynomial time, by simply checking whether $w \in V(p)$. In the case where $\varphi = \neg \varphi_1$, we can recursively call the algorithm to decide whether $M, w \models \varphi_1$, and return the opposite answer. Similarly, for $\varphi = \varphi_1 \land \varphi_2$, we can straightforwardly decide whether $M, w \models \varphi$ by first recursively determining whether $M, w \models \varphi_1$ and whether $M, w \models \varphi_2$. In the case where $\varphi = K_a \varphi_1$, we firstly recursively determine whether $M, w' \models \varphi_1$ for each $w' \in W$ that is $a$-accessible from $w$. This information immediately determines whether $M, w \models K_a \varphi_1$.

Finally, consider the case where $\varphi = [E, e] \varphi_1$. In this case, we firstly recursively decide whether $M, w \models \text{pre}(e)$. If this is not the case, then trivially, $M, w \models \varphi$. Otherwise, we construct the submodel $M'$ of $M$ that is bisimilar to $M \otimes E$. This can be done as described above. In order to do this, we need to decide which states in $W'$ satisfy the precondition of some $e' \in E'$, where $W' \subseteq W$ and $E' \subseteq E$ are defined as explained above. This can be done by recursive calls of the algorithm. Having determined $W'$, and having constructed $M'$, we can now answer the question whether $M, w \models [E, e] \varphi_1$ by using only $M'$, $w$ and $\varphi_1$. We know that $w$ is a world in $M'$, since $M, w \models \text{pre}(e)$. Since $M'$ is bisimilar to $M \otimes E$, it holds that $M \otimes E, (w, e) \models \varphi_1$ if and only if $M', w \models \varphi_1$. Therefore, by recursively calling the algorithm to decide whether $M', w \models \varphi_1$, we can decide whether $M, w \models [E, e] \varphi_1$.

It is straightforward to verify that this recursive algorithm correctly decides whether $M, w \models \varphi$. However, naively executing this recursive algorithm will, in the worst case result in an exponential running time. This is because for the case for $\varphi = [E, e] \varphi_1$, the algorithm makes multiple (say $b \geq 2$) recursive calls for $\text{pre}(e)$, and $\text{pre}(e)$ could contain subformulas of the form $[E', e'] \varphi'$—which in turn triggers multiple recursive calls for $\text{pre}(e')$ for each of the $b$ branches in the recursion tree, and so forth. As the number of these iterations can grow linearly with the input size (say $f(n)$), the recursion tree can be of exponential
size (namely, of size $\geq 2^{f(n)}$). We describe how to modify the algorithm to run in polynomial time, using the technique of memoization. Whenever a recursive call is made to decide whether $\mathcal{N}, u \models \psi$, for some submodel $\mathcal{N}$ of $\mathcal{M}$, some world $w$ in $\mathcal{N}$, and some subformula $\psi$ of $\varphi$, the result of this recursive call is stored in a lookup table. Moreover, before making a recursive call to decide whether $\mathcal{N}, u \models \psi$, the lookup table is consulted, and if an answer is stored, the algorithm uses this answer instead of executing the recursive call.

The number of submodels $\mathcal{N}$ of $\mathcal{M}$ that need to be considered in the execution of the modified algorithm is upper bounded by the number of occurrences of update operators $[E, e]$ in the formula $\varphi$ that is given as input to the problem. Therefore, the size of the lookup table is polynomial in the input size. Moreover, computing the answer for any entry in the lookup table can be done in polynomial time (using the answers for other entries in the lookup table). Therefore, the modified algorithm decides whether $\mathcal{M}, w \models \varphi$ in polynomial time.

3.2 Hardness results for one agent

Next, we consider the restriction where we have a single agent and single-pointed models, but where postconditions are allowed in the event models. In this case, the problem is $\Delta^P_2$-hard.

**Theorem 1** The model checking problem for DEL with S5 models restricted to instances with a single agent and only single-pointed models, but where event models can contain postconditions, is $\Delta^P_2$-hard.

**Proof** To show $\Delta^P_2$-hardness, we give a polynomial-time reduction from the problem of deciding whether the lexicographically maximal assignment that satisfies a given propositional formula $\varphi$ over variables $x_1, \ldots, x_n$ sets the variable $x_n$ to true. Let $\varphi$ be an instance of this problem, with variables $x_1, \ldots, x_n$. We construct a single-pointed epistemic model $(\mathcal{M}, w_0)$ with a single agent $a$ and a DEL-formula $\chi$ whose updates consist of single-pointed event models (that contain postconditions), such that $\mathcal{M}, w_0 \models \chi$ if and only if $x_n$ is true in the lexicographically maximal assignment that satisfies $\varphi$.

In addition to the propositional variables $x_1, \ldots, x_n$, we introduce a variable $z$. Then, we construct the model $(\mathcal{M}, w_0)$ as depicted in Figure 4.

![Fig. 4: The epistemic model $(\mathcal{M}, w_0)$, used in the proof of Theorem 1.](image)

Then, for each $1 \leq i \leq n$, we introduce the single-pointed event model $(E_i, e_i)$ as depicted in Figure 5. Intuitively, these updates will serve to generate, for
each possible truth assignment $\alpha$ to the variables $x_1, \ldots, x_n$, a world that agrees with $\alpha$ (and that sets $z$ to false), in addition to the designated world (where $z$ is true). We will denote the model resulting from updating $(M, w_0)$ subsequently with the updates $(E_1, e_1), \ldots, (E_n, e_n)$ by $(M', w')$.

\[ \langle z, \top \rangle \langle \neg z, x_i \rangle \langle \neg z, \top \rangle \]

Fig. 5: The event model $(E_i, e_i)$, used in the proof of Theorem 1.

Next, for each $1 \leq i \leq n$, we introduce the single-pointed event model $(E'_i, e'_i)$ as depicted in Figure 6. Intuitively, we will use the event models $(E'_i, e'_i)$ to obtain (many copies of) the lexicographically maximal assignment that satisfies $\varphi$. Applying the update $(E'_i, e'_i)$ to $(M', w')$ will set the variable $x_i$ to true in all worlds (that satisfy $\neg z$) if there is an assignment (among the remaining assignments) that satisfies $\varphi$ and that sets $x_i$ to true, and it will set the variable $x_i$ to false in all worlds (that satisfy $\neg z$) otherwise. Then, after applying the updates $(E'_1, e'_1), \ldots, (E'_n, e'_n)$ to $(M', w')$, all worlds in the resulting model will have the same valuation—namely, a valuation that agrees with the lexicographically maximal assignment that satisfies $\varphi$. In particular, the variable $x_n$ is true in this valuation if and only if $x_n$ is true in the lexicographically maximal assignment that satisfies $\varphi$.

\[ \langle z, \top \rangle \langle \neg z \land \hat{K}_a(x_i \land \varphi), x_i \rangle \langle \neg z \land \neg \hat{K}_a(x_i \land \varphi), \neg x_i \rangle \]

Fig. 6: The event model $(E'_i, e'_i)$, used in the proof of Theorem 1.

We then let $\chi = [E_1, e_1][E_2, e_2][E'_1, e'_1] \cdots [E'_n, e'_n] \hat{K}_a x_n$. We now formally show that the lexicographically maximal assignment $\alpha_0$ that satisfies $\varphi$ sets $x_n$ to true if and only if $M, w_0 \models \chi$. In order to do so, we will prove the following claim. The model $(M'', w'') = (M, w_0) \otimes (E_1, e_1) \otimes \cdots \otimes (E_n, e_n) \otimes (E'_1, e'_1) \otimes \cdots \otimes (E'_n, e'_n)$ consists of a world $w''$ that sets $z$ to true and all other variables
to false, and of worlds that set \( z \) to false and that agree with \( \alpha_0 \) on the
variables \( x_1, \ldots, x_n \). Firstly, it is straightforward to verify that \((\mathcal{M}', w') = (\mathcal{M}, w_0) \otimes (\mathcal{E}_1, e_1) \otimes \cdots \otimes (\mathcal{E}_n, e_n)\)
consists of the world \( w' \) and exactly one world corresponding to each truth assignment \( \alpha \) to the
variables \( x_1, \ldots, x_n \).

Then, applying the update \((\mathcal{E}_i', e_i')\) to \((\mathcal{M}', w')\) has two possible outcomes:
either (1) if there exists a model of \( \varphi \) that sets \( x_1 \) to true, then in all worlds
(that set \( z \) to false) the variable \( x_1 \) will be set to true; or (2) if there exists
no model of \( \varphi \) that sets \( x_1 \) to true, then in all worlds (that set \( z \) to false) the
variable \( x_1 \) will be set to false. For each \( 1 < i \leq n \), subsequently applying the
update \((\mathcal{E}_i', e_i')\) has an entirely similar effect. By a straightforward inductive
argument, it then follows that all the worlds in \((\mathcal{M}'', w'')\) that set \( z \) to false
agree with the lexicographically maximal model of \( \varphi \).

Therefore, \( \mathcal{M}, w_0 \models \chi \) if and only if \( x_n \) is true in the lexicographically
maximal model of \( \varphi \), and we can conclude that the problem is \( \Delta^P_2 \)-hard. \( \square \)

When we allow multi-pointed models, the problem turns out to be PSPACE-hard,
even when restricted to a single agent.

**Theorem 2** The model checking problem for DEL with S5 models restricted
to instances with a single agent and no postconditions in the event models,
but where models can be multi-pointed, is PSPACE-hard.

**Proof** In order to show PSPACE-hardness, we give a polynomial-time reduction
from the problem of deciding whether a quantified Boolean formula is true.
Let \( \varphi = \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n \psi \) be a quantified Boolean formula, where \( \psi \)
is quantifier-free (we assume without loss of generality that \( n \) is even). We construct a single-pointed epistemic model \((\mathcal{M}, w_0)\) with one agent \( a \) and a
DEL-formula \( \chi \) (containing updates with multi-pointed event models) such
that \( \mathcal{M}, w_0 \models \chi \) if and only if \( \varphi \) is true.

The first main idea behind this reduction is that we represent truth assignments
to the propositional variables \( x_1, \ldots, x_n \) with connected groups of worlds.
Let \( \alpha \) be a truth assignment to the variables \( x_1, \ldots, x_n \), and let \( x_{i_1}, \ldots, x_{i_\ell} \)
be the variables that \( \alpha \) sets to true. We then represent \( \alpha \) by means of a group
of worlds \( w_0, w_1, \ldots, w_\ell \), where the world \( w_0 \) makes no propositional variable
true, and for each \( 1 \leq j \leq \ell \), world \( w_j \) makes exactly one propositional variable
true (namely, \( x_{i_j} \)). These worlds \( w_0, w_1, \ldots, w_\ell \) are fully connected. This collection of worlds \( w_0, w_1, \ldots, w_\ell \) is what we call the group of worlds corresponding to \( \alpha \).
Moreover, the designated state is \( w_0 \). Consider the truth assignment
\( \alpha = \{ x_1 \mapsto 1, x_2 \mapsto 1, x_3 \mapsto 0, x_4 \mapsto 1 \} \), for example. In Figure 7 we show
the group of worlds that we use to represent this truth assignment \( \alpha \). We let the model \( \mathcal{M} \) be the group of worlds corresponding to the truth assignment \( \alpha_0 \)
that assigns all variables \( x_1, \ldots, x_n \) to true.

The next main idea is that we represent existential and universal quantification
of the propositional variables using the dynamic operators \( \langle \mathcal{E}, E \rangle \)
and \( [\mathcal{E}, E] \), respectively. For each propositional variable \( x_i \) in the quantified
Boolean formula, we introduce the multi-pointed event model \((\mathcal{E}_i, E_i)\) as depicted in Figure 8. We use the event models \( (\mathcal{E}_1, E_1), \ldots, (\mathcal{E}_n, E_n) \), to create
Ronald de Haan, Iris van de Pol

Fig. 7: The group of worlds that we use to represent the truth assignment \( \alpha = \{x_1 \mapsto 1, x_2 \mapsto 1, x_3 \mapsto 0, x_4 \mapsto 1\} \), in the proof of Theorem 2.

(disconnected) groups of worlds (that all have a designated world) that correspond to each possible truth assignment \( \alpha \) to the variables \( x_1, \ldots, x_n \).

Fig. 8: The multi-pointed event model \((E_i, E_i)\) corresponding to variable \( x_i \), used in the proof of Theorem 2.

Using the alternation of diamond dynamic operators and box dynamic operators, we can simulate existential and universal quantification of variables in the formula \( \varphi \). We simulate an existentially quantified variable \( \exists x_i \) by the dynamic operator \( \langle E_i, E_i \rangle \)—a formula of the form \( \langle E_i, E_i \rangle \psi \) is true if and only if \( \langle E_i, e_i \rangle \psi \) is true for some \( e_i \in E_i \). Similarly, we simulate a universally quantified variable \( \forall x_i \) by the dynamic operator \([E_i, E_i]\)—a formula of the form \([E_i, E_i]\psi \) is true if and only if \([E_i, e_i]\psi \) is true for all \( e_i \in E_i \).

Concretely, we let \( \chi = \langle E_1, E_1 \rangle \langle E_2, E_2 \rangle \ldots \langle E_{n-1}, E_{n-1} \rangle \langle E_n, E_n \rangle \chi' \), where \( \chi' \) is the formula obtained from \( \psi \) by replacing each occurrence of a propositional variable \( x_i \) by the formula \( K_a x_i \). We show that \( \varphi \) is a true quantified Boolean formula if and only if \( M, w_0 \models \chi \). In order to do so, we prove the following statement, relating truth assignments \( \alpha \) to the variables \( x_1, \ldots, x_n \) to groups of worlds containing a designated world. The statement that we will prove inductively for all \( 1 \leq i \leq n + 1 \) is the following.

**Statement:** Let \( \alpha \) be any truth assignment to the variables \( x_1, \ldots, x_n \) that sets all variables \( x_1, \ldots, x_n \) to true, and let \( \alpha' \) be the restriction of \( \alpha \) to the variables \( x_1, \ldots, x_{i-1} \). Moreover, let \( M \) be a group of worlds that corresponds to the truth assignment \( \alpha \), containing a designated world \( w \). Then \( Q_i x_i \ldots \exists x_{n-1} \forall x_n, \psi \) is true under \( \alpha' \) if and only if:

- \( w \) makes \( \langle E_i, E_i \rangle \ldots \langle E_n, E_n \rangle \chi' \) true, if \( i \) is odd; and
- \( w \) makes \([E_i, E_i] \ldots [E_n, E_n] \chi' \) true, if \( i \) is even.
The statement for \( i = 1 \) implies that \( M, w_0 \models \chi \) if and only if \( \varphi \) is a true quantified Boolean formula. We show that the statement holds for \( i = 1 \) by showing that the statement holds for all \( 1 \leq i \leq n + 1 \). We begin by showing that the statement holds for \( i = n + 1 \). In this case, we know that \( \alpha = \alpha' \) is a truth assignment to the variables \( x_1, \ldots, x_n \). Moreover, by construction of \( \chi' \) we know that \( w \) makes \( \chi' \) true if and only if \( \alpha \) satisfies \( \psi \). Therefore, the statement holds.

Next, we let \( 1 \leq i \leq n \) be arbitrary, and we assume that the statement holds for \( i + 1 \). We now distinguish two cases: either (1) \( Q_i = \exists \), i.e., the \( i \)-th quantifier of \( \varphi \) is existential, or (2) \( Q_i = \forall \), i.e., the \( i \)-th quantifier of \( \varphi \) is universal.

First, consider case (1). Suppose that \( \exists x_1 \ldots \exists x_{n-1} \forall x_n, \psi \) is true under \( \alpha' \). Then there exists a truth assignment \( \alpha'' \) to the variables \( x_1, \ldots, x_i \) that agrees with \( \alpha' \) on the variables \( x_1, \ldots, x_{i-1} \) and for which \( \forall x_{i+1} \ldots \exists x_{n-1} \forall x_n, \psi \) is true under \( \alpha'' \). Therefore, there exists some event \( e \in E_i \) such that the group \( M' = \{ (v, e) : v \in M \text{ and } M, v \models \text{pre}(e) \} \) of worlds and the world \( w' = (w, e) \), together with the assignment \( \alpha''' \) that agrees with \( \alpha'' \) on the variables \( x_1, \ldots, x_i \), and that sets the variables \( x_{i+1}, \ldots, x_n \) to true, satisfy the requirements for the statement for \( i + 1 \). Then, by the induction hypothesis we know that \( w' \) makes \( [E_{i+1}, E_{i+1}] \ldots [E'_{n-1}, E'_{n-1}][E_n, E_n] \chi' \) true. Therefore, we can conclude that \( w \) makes \( (E_i, E_i) \ldots (E'_{n-1}, E'_{n-1})[E_n, E_n] \chi' \) true.

Conversely, suppose that \( w \) makes \( (E_i, E_i) \ldots (E'_{n-1}, E'_{n-1})[E_n, E_n] \chi' \) true. This can only be the case if there is some event \( e \in E_i \) such that the set \( M' \) and \( w' \) (defined as above) correspond to a truth assignment \( \alpha''' \) (also defined as above). Then, by the induction hypothesis, we know that \( \forall x_{i+1} \ldots \exists x_{n-1} \forall x_n, \psi \) is true under \( \alpha'' \) (obtained from \( \alpha''' \) as above). Therefore, since \( \alpha'' \) extends \( \alpha' \), we can conclude that \( \exists x_1 \ldots \exists x_{n-1} \forall x_n, \psi \) is true under \( \alpha' \).

The argument for case (2) is entirely analogous (yet dual). We omit a detailed treatment of this case. This concludes the inductive proof of the statement for all \( 1 \leq i \leq n + 1 \), and thus concludes our proof that \( M, w_0 \models \chi \) if and only if \( \varphi \) is true.

\[ \square \]

3.3 Hardness results for two agents

Next, we show that when we consider the case of two agents, the model checking problem for DEL is PSPACE-hard, even when we only allow single-pointed models without postconditions.

**Theorem 3** The model checking problem for DEL is PSPACE-hard, even when restricted to the case where the question is whether \( M, w_0 \models [E_1, e_1] \ldots [E_n, e_n] \chi \), where:

- the model \( (M, w_0) \) is a single-pointed S5 model;
- all the \( (E_i, e_i) \) are single-pointed S5 event models without postconditions;
- \( \chi \) is an epistemic formula without update modalities that contains (multiple occurrences of) only three propositional variables; and
there are only two agents.

Proof In order to show PSPACE-hardness, we give a polynomial-time reduction from the problem of deciding whether a quantified Boolean formula is true. Let \( \varphi = \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n. \psi \) be a quantified Boolean formula, where \( \psi \) is quantifier-free. We construct an epistemic model \( (\mathcal{M}, w_0) \) with two agents \( a, b \) and a DEL-formula \( \xi \) such that \( \mathcal{M}, w_0 \models \xi \) if and only if \( \varphi \) is true.

The first main idea behind the reduction is that we use two propositional variables, say \( z_0 \) and \( z_1 \), to represent an arbitrary number of propositions, by creating chains of worlds that represent these propositions. Let \( x_1, \ldots, x_n \) be the propositions that we want to represent. Then we represent a proposition \( x_j \) by a chain of worlds of length \( j + 1 \) that are connected alternatingly by \( b \)-relations and \( a \)-relations. In this chain, the last world is the only world that makes \( z_0 \) true. Moreover, the first world in the chain is the only world that makes \( z_1 \) true. An example of such a chain that we use to represent proposition \( x_3 \) can be found in Figure 9.

![Fig. 9: The chain of worlds that we use to represent proposition \( x_3 \) in the proof of Theorem 3. The first world in the chain is the world \( w_0 \), that is depicted on the left.](image)

The next idea that plays an important role in the reduction is that we will group together (the first worlds) of several such chains to represent a truth assignment to the propositions \( x_1, \ldots, x_n \). Let \( \alpha \) be a truth assignment to the propositions \( x_1, \ldots, x_n \), and let \( x_{i_1}, \ldots, x_{i_\ell} \) be the propositions that \( \alpha \) sets to true (for \( 1 \leq i_1 < \cdots < i_\ell \leq n \)). Then we represent the truth assignment \( \alpha \) in the following way. We take the chains corresponding to the propositions \( x_{i_1}, \ldots, x_{i_\ell} \), and we connect the first world (the world that is labelled with \( w_0 \) in Figure 9) of each two of these chains with an \( a \)-relation. In other words, we connect all these first worlds together in a fully connected clique of \( a \)-relations. Moreover, to this \( a \)-clique of worlds, we add a designated world where both \( z_1 \) and a third propositional variable \( z_2 \) are true. The collection of all worlds in the chains corresponding to the propositions \( x_{i_1}, \ldots, x_{i_\ell} \) and this additional designated world is what we call the group of worlds representing \( \alpha \). For the sake of convenience, we call the world where \( z_1 \) and \( z_2 \) are true the central world of the group of worlds. In Figure 10 we give an example of such a group of worlds that we use to represent the truth assignment \( \alpha = \{ x_1 \rightarrow 1, x_2 \rightarrow 1, x_3 \rightarrow 0, x_4 \rightarrow 1 \} \).

By using the expressivity of epistemic logic, we can construct formulas that extract information from these representations of truth assignments. Intuitively, we can check whether a truth assignment \( \alpha \) sets a proposition \( x_j \)
to true by checking whether the group of worlds representing $\alpha$ contains a chain of worlds of length exactly $j + 1$. Formally, we will define a formula $\chi_j$ for each $1 \leq j \leq n$, which is true in the designated world if and only if the group contains a chain representing proposition $x_j$. We describe how to construct the formulas $\chi_j$. Firstly, we inductively define formulas $\chi^a_j$ and $\chi^b_j$ for all $1 \leq j \leq n$ as follows. Intuitively, the formula $\chi^a_j$ is true in exactly those worlds from which there is an alternating chain that ends in a $z_0$-world, that is of length at least $j$ and that starts with an $a$-relation. Similarly, the formula $\chi^b_j$ is true in exactly those worlds from which there is an alternating chain that ends in a $z_0$-world, that is of length at least $j$, and that starts with a $b$-relation. We let $\chi^a_0 = \chi^b_0 = z_0$, and for each $j > 0$, we let $\chi^a_j = \hat{K}_b(\neg z_1 \land \neg z_2 \land \chi^a_{j-1})$ and $\chi^b_j = \hat{K}_a(\neg z_1 \land \neg z_2 \land \chi^b_{j-1})$. Now, using the formulas $\chi^a_j$, we can define the formulas $\chi_j$. We let $\chi_j = z_1 \land \neg z_2 \land \chi^b_j \land \neg \chi^b_{j-1}$. As a result of this definition, the formula $\chi_j$ is true in exactly those worlds that are the first world of a chain of length exactly $j$.

For example, consider the formula $\chi_2 = z_1 \land \neg z_2 \land \hat{K}_b(\neg z_1 \land \neg z_2 \land \chi^a_{2-1}) \land \neg \hat{K}_b(\neg z_1 \land \neg z_2 \land z_0)$ and consider the group of worlds depicted in Figure 10 [10]. This formula is true only in the first world of the chain of length 2.

The epistemic model $(\mathcal{M}, w_0)$ that we use in the reduction is based on the model $\mathcal{M}_{\alpha_0}$ representing the truth assignment $\alpha_0 : \{x_1, \ldots, x_n\} \rightarrow \{0, 1\}$ that sets all propositions $x_1, \ldots, x_n$ to true. To obtain $\mathcal{M}$, we will add a number of additional worlds to the model $\mathcal{M}_{\alpha_0}$, that we will use to simulate the behavior of the existential and universal quantifiers in the DEL-formula $\chi$ that we will construct below. Specifically, we will add alternating chains of worlds to the model that are similar to the chains that represent the propositions $x_1, \ldots, x_n$. However, the additional chains that we add differ in two aspects from the chains that represent the propositions $x_1, \ldots, x_n$: (1) the additional chains start with an $a$-relation instead of starting with a $b$-relation, and (2) in the first world of the additional chains, the propositional variable $z_2$ is true instead of the variable $z_1$. For each $1 \leq i \leq n$, we add such an additional chain of length $i$, and we connect the first worlds of these additional chains, together with the designated world, in a clique of $b$-relations. To illustrate this, the
model \((\mathcal{M}, w_0)\) that results from this construction is shown in Figure 11 for the case where \(n = 3\). For the sake of convenience, we will denote these additional chains by \(z_2\)-chains, and the chains that represent the propositions \(x_1, \ldots, x_n\) by \(z_1\)-chains (after the propositional variables that are true in the first worlds of these chains).

To check whether an alternating chain of length exactly \(j\), that starts from a \(z_2\)-world with an \(a\)-relation, is present in the model, we define formulas \(\chi'_j\) similarly to the way we defined the formulas \(\chi_j\). Specifically, we let \(\chi'_j = \neg z_1 \land z_2 \land \chi_j^a \land \neg \chi_{j-1}^a\).

We will use the \(z_2\)-chains together with the formulas \(\chi'_j\) to keep track of an additional counter. We will use this counter as a technical trick to implement the simulation of existentially and universally quantified variables in the formula \(\varphi\).

Next, we describe how we can generate all possible truth assignments over the variables \(x_1, \ldots, x_n\) from the initial model \(\mathcal{M}\). We do this in such a way that we can afterwards express the existential and universal quantifications of the formula \(\varphi\) using modal operators in the epistemic language. In order to generate groups of worlds that represent truth assignments \(\alpha\) that differ from the all-ones assignment \(\alpha_0\), we will apply updates that copy the existing worlds but that eliminate (the first worlds of) chains of a certain length. This is the third main idea behind this reduction.

Specifically, we will introduce a single-pointed event model \((\mathcal{E}_i, e_i)\) for each propositional variable \(x_i\), that is depicted in Figure 12. Intuitively, what happens when the update \((\mathcal{E}_i, e_i)\) is applied is the following. All existing groups of worlds will be duplicated, resulting in five copies—this corresponds to the five events \(f^1_i, \ldots, f^5_i\) in the event model. The resulting groups of worlds will be connected corresponding to the relations between the events in the event model. That is, for any existing group of worlds, three of its copies (corresponding to the events \(f^1_i, f^2_i,\) and \(f^3_i\)) will be connected by \(b\)-relations. The second
and third of these copies (the ones corresponding to the events $f^1_i$ and $f^3_i$) will be connected by $a$-relations to the fourth and fifth copy (corresponding to the events $f^4_i$ and $f^5_i$), respectively. Moreover, in the fourth and fifth copy, (the first world of) the $z_2$-chain of length $i$ is removed, and in the fifth copy, (the first world of) the $z_1$-chain of length $i$ is removed as well. These effects of removing (the first worlds of) chains is enforced by the preconditions of the events in the event model.

By applying the updates $(E_1,e_1),\ldots,(E_n,e_n)$, we generate many (in fact, an exponential number of) copies of the model $M$, in each of which certain chains of worlds are removed, and which are connected to each other by means of $a$-relations and $b$-relations in the way described in the previous paragraph. In particular, for each truth assignment $\alpha$ to the propositions $x_1,\ldots,x_n$, there is some group of worlds that corresponds to $\alpha$.

Finally, we construct the DEL-formula $\xi$. We let $\xi = [E_1,e_1] \ldots [E_n,e_n]\xi_1$, where we define $\xi_1$ below. The formula $\xi_1$ exploits the structure of the epistemic model $M'$, that results from updating the model $M$ with the updates $(E_i,e_i)$, to simulate the semantics of the quantified Boolean formula $\varphi$. For each $1 \leq i \leq n+1$, we define $\xi_i$ inductively as follows:

$$
\xi_i = \begin{cases}
\psi' & \text{if } i = n+1, \\
\hat{K}_b\hat{K}_a(z_1 \land z_2 \land \bigwedge_{1 \leq j \leq i} \neg\hat{K}_b\chi_j' \land \bigwedge_{i < j \leq n} \hat{K}_b\chi_j' \land \xi_{i+1}) & \text{for odd } i \leq n, \\
K_bK_a((z_1 \land z_2 \land \bigwedge_{1 \leq j \leq i} \neg\hat{K}_b\chi_j' \land \bigwedge_{i < j \leq n} \hat{K}_b\chi_j') \rightarrow \xi_{i+1}) & \text{for even } i \leq n.
\end{cases}
$$

Here, $\psi'$ is the formula obtained from $\psi$ (the quantifier-free part of the quantified Boolean formula $\varphi$) by replacing each occurrence of a propositional variable $x_i$ by the formula $\hat{K}_a\chi_i$. 

Fig. 12: The event model $(E_i,e_i)$ corresponding to the propositional variable $x_i$, used in the proof of Theorem 3. The events are labelled $f^1_i,\ldots,f^5_i$. 

---

On the Complexity of Model Checking for DEL with S5 Models 19
We use the formulas $\xi_i$ to express the formula $\varphi$ with its existentially and universally quantified variables. Intuitively, the formulas $\xi_i$ navigate through the groups of worlds in the model $\mathcal{M}'$—resulting from updating $\mathcal{M}$ with the event models $(\mathcal{E}_i, e_i)$—as follows. Consider the central world of some group of worlds in the model $\mathcal{M}'$, and consider the formula $\xi_i$ for some odd $i \leq n$. For $\xi_i$ to be true in this world, the formula $\xi_{i+1}$ needs to be true in the central world of some group of worlds that corresponds to one of the events $f^4_i$ or $f^5_i$ from the event model $(\mathcal{E}_i, e_i)$. Similarly, for the formula $\xi_i$ to be true in this world, for even $i \leq n$, the formula $\xi_{i+1}$ needs to be true in the central world of both groups of worlds that correspond to the events $f^4_i$ and $f^5_i$. In this way, for odd $i \leq n$, the formula $\xi_i$ together with the event model $(\mathcal{E}_i, e_i)$ serves to simulate an existential choice of a truth value for the variable $x_i$. Similarly, for even $i \leq n$, the formula $\xi_i$ together with the event model $(\mathcal{E}_i, e_i)$ serves to simulate a universal choice of a truth value for the variable $x_i$.

The model $(\mathcal{M}, w_0)$ and the formula $\xi$ can be constructed in polynomial time in the size of the quantified Boolean formula $\varphi$. Furthermore, in the constructed instance, there are only two agents, the epistemic model $(\mathcal{M}, w_0)$ is a single-pointed S5 model, all event models $(\mathcal{E}_i, e_i)$ are single-pointed S5 models without postconditions, and $\xi_1$ is a formula without update modalities that contains (many occurrences of) only three propositional variables $z_0, z_1, z_2$.

We show that $\varphi$ is a true quantified Boolean formula if and only if $\mathcal{M}, w_0 \models \xi$. In order to do so, we prove the following (technical) statement relating truth assignments $\alpha$ to the propositions $x_1, \ldots, x_n$ and (particular) worlds $w$ in the epistemic model $(\mathcal{M}', w'_0) = (\mathcal{M}, w_0) \otimes (\mathcal{E}_1, e_1) \otimes \cdots \otimes (\mathcal{E}_n, e_n)$. Before we give the statement that we will prove, we observe that every world $w$ that sets both $z_1$ and $z_2$ to true is the central world of some group of worlds that represents a truth assignment $\alpha$ to the propositions $x_1, \ldots, x_n$. For the sake of convenience, we will say that $w$ corresponds to the truth assignment $\alpha$. The statement that we will prove for all $1 \leq i \leq n + 1$ is the following.

**Statement:** Let $\alpha$ be any truth assignment to the propositions $x_1, \ldots, x_{i-1}$. Moreover, let $w$ be any world in the model $(\mathcal{M}', w'_0)$ such that:

1. $w$ makes $z_1$ and $z_2$ true,
2. $w$ makes $K_{x'_1} \chi'_i$ false for all $1 \leq j < i$,
3. $w$ makes $K_{x'_1} \chi'_i$ true for all $1 \leq j \leq n$, and
4. the truth assignment corresponding to $w$ agrees with $\alpha$ on the propositions $x_1, \ldots, x_{i-1}$.

Then the (partially) quantified Boolean formula $Q, x_1 \ldots \exists x_{n-1} \forall x_n. \psi$ is true under $\alpha$ if and only if $w$ makes $\xi_i$ true.

Observe that for $i = 1$, the world $w'_0$ satisfies all four conditions. Therefore, the statement for $i = 1$ implies that $\mathcal{M}, w_0 \models \xi$ if and only if $\varphi$ is a true quantified Boolean formula. Thus, proving this statement for all $1 \leq i \leq n + 1$ suffices to show the correctness of our reduction.

We begin by showing that the statement holds for $i = n + 1$. In this case, we know that $\alpha$ is a truth assignment to the propositions $x_1, \ldots, x_n$.
Moreover, $\xi_{n+1} = \psi'$. By construction of $\psi'$, we know that $w$ makes $\psi'$ true if and only if $\alpha$ satisfies $\psi$. Therefore, the statement holds for $i = n+1$.

Next, we let $1 \leq i \leq n$ be arbitrary, and we assume that the statement holds for $i+1$. That is, the statement holds for every combination of a truth assignment $\alpha$ and a world $w$ that satisfies the conditions. Since $w$ is a world in the model $(\mathcal{M}, w_0) \otimes (\mathcal{E}_1, e_1) \otimes \cdots \otimes (\mathcal{E}_n, e_n)$, and since $w$ makes $z_1$ and $z_2$ true, we know that $w = (w_0, e'_1, \ldots, e'_n)$, for some $e'_1, \ldots, e'_n$, where $e'_j \in \mathcal{E}_j$ for all $1 \leq j \leq n$. We know that $w$ makes $K_b \chi_j'$ true for all $1 \leq j \leq n$. Therefore, we know that for each $i \leq j \leq n$ it holds that $e'_j \in \{f'_1, f'_2, f'_3\}$.

We now distinguish two cases: either (1) $i$ is odd, or (2) $i$ is even. In case (1), the $i$-th quantifier of $\psi$ is existential, and in case (2), the $i$-th quantifier of $\psi$ is universal. First, consider case (1). Suppose that $\exists x_i \ldots \exists x_n \forall x_n. \psi$ is true under $\alpha$. Then there exists some truth assignment $\alpha'$ to the propositions $x_1, \ldots, x_i$ that agrees with $\alpha$ on the propositions $x_1, \ldots, x_i-1$ and that ensures that $\forall x_{i+1} \ldots \exists x_n \forall x_n. \psi$ is true under $\alpha'$. Suppose that $\alpha'(x_i) = 0$; the case for $\alpha'(x_i) = 1$ is entirely similar. Now, consider the worlds $w' = (w_0, e'_1, \ldots, e'_{i-1}, f'_3, e'_{i-1+1}, \ldots, e'_n)$ and $w'' = (w_0, e'_1, \ldots, e'_{i-1}, f'_3, e'_{i+1}, \ldots, e'_n)$. By the construction of $(\mathcal{E}_i, e_i)$, by the semantics of product update, and by the fact that $e'_j \in \{f'_1, f'_2, f'_3\}$, it holds that $(w, w') \in R_b$ and $(w', w'') \in R_b$. Moreover, it is straightforward to verify that $w''$ satisfies conditions (1)–(4), for the truth assignment $\alpha'$. Also, we know that $\forall x_{i+1} \ldots \exists x_n \forall x_n. \psi$ is true under $\alpha'$. Therefore, by the induction hypothesis, we know that $w''$ makes the formula $\xi_{i+1}$ true. It then follows from the definition of $\xi_i$ that $w'$ and $w''$ witness that $w$ makes $\xi_i$ true.

Conversely, suppose that $w$ makes $\xi_i$ true. Moreover, suppose that $w'$ and $w''$ (as defined above) witness this. (The only other possible worlds $u'$ and $w''$ that could witness this are obtained from $w$ by replacing $e'_j$ by $f'_3$ and $f'_j$, respectively. The case where $u'$ and $w''$ witness that $w$ makes $\xi_i$ true is entirely similar.) This means that $w''$ makes $\xi_{i+1}$ true. Then, by the induction hypothesis, it follows that the truth assignment $\alpha'$ to the propositions $x_1, \ldots, x_i$ corresponding to the world $w''$ satisfies $\forall x_{i+1} \ldots \exists x_n \forall x_n. \psi$. Moreover, since $\alpha'$ agrees with $\alpha$ on the propositions $x_1, \ldots, x_i-1$, it follows that $\exists x_i \ldots \exists x_n \forall x_n. \psi$ is true under $\alpha$.

Next, consider case (2). Suppose that $\forall x_1 \ldots \exists x_n \forall x_n. \psi$ is true under $\alpha$. Then for both truth assignments $\alpha'$ to the variables $x_1, \ldots, x_i$ that agree with $\alpha$ it holds that $\exists x_{i+1} \ldots \exists x_n \forall x_n. \psi$ is true under $\alpha'$. The only worlds that satisfy $(z_1 \land z_2 \land \bigwedge_{1 \leq j \leq i} K_b \chi_j' \land \bigwedge_{i < j \leq n} K_b \chi_j')$ and that are accessible from $w$ by a $b$-relation followed by an $a$-relation are the worlds $u_1$ and $u_2$, where $u_1 = (w_0, e'_1, \ldots, e'_{i-1}, f'_3, e'_{i-1+1}, \ldots, e'_n)$ and $u_2 = (w_0, e'_1, \ldots, e'_{i-1}, f'_3, e'_{i+1}, \ldots, e'_n)$. Moreover, the truth assignments $\alpha_1$ and $\alpha_2$ that correspond to $u_1$ and $u_2$, respectively, agree with $\alpha$ on the propositions $x_1, \ldots, x_i$. Because $\forall x_1 \ldots \exists x_n \forall x_n. \psi$ is true under $\alpha$, we know that $\exists x_{i+1} \ldots \exists x_n \forall x_n. \psi$ is true under both $\alpha_1$ and $\alpha_2$. Then, by the induction hypothesis it follows that both $u_1$ and $u_2$ make $\xi_{i+1}$ true. Therefore, $w$ makes $\xi_i$ true.
Conversely, suppose that \( w \) makes \( \xi_i \) true. By the definition of \( \xi_i \), we then know that all worlds that are accessible from \( w \) by a \( b \)-relation followed by an \( a \)-relation and that make \( (z_1 \land z_2 \land \bigwedge_{1 \leq j \leq i} \neg K_b \chi'_j \land \bigwedge_{i < j \leq n} K_b \chi'_j) \) true, also make \( \xi_{i+1} \) true. Consider the worlds \( u_1 \) and \( u_2 \), as defined above. These are both accessible from \( w \) by a \( b \)-relation followed by an \( a \)-relation, and they make \( (z_1 \land z_2 \land \bigwedge_{1 \leq j \leq i} \neg K_b \chi'_j \land \bigwedge_{i < j \leq n} K_b \chi'_j) \) true. Therefore, both \( u_1 \) and \( u_2 \) make \( \xi_{i+1} \) true. Also, the truth assignments \( \alpha_1 \) and \( \alpha_2 \) that correspond to \( u_1 \) and \( u_2 \), respectively, agree with \( \alpha \) on the propositions \( x_1, \ldots, x_{i-1} \). Moreover, the truth assignments \( \alpha_1 \) and \( \alpha_2 \) are both possible truth assignments to the propositions \( x_1, \ldots, x_i \) that agree with \( \alpha \). By the induction hypothesis, the formula \[ \exists x_{i+1} \ldots \exists x_{n-1} \forall x_n. \psi \] is true under both \( \alpha_1 \) and \( \alpha_2 \). Therefore, we can conclude that \[ \forall x_i \ldots \exists x_{n-1} \forall x_n. \psi \] is true under \( \alpha \).

This concludes the inductive proof of the statement for all \( 1 \leq i \leq n + 1 \), and thus concludes our correctness proof. Therefore, we can conclude that the problem is PSPACE-hard.

\[ \square \]

4 Results for semi-private announcements

Next, we consider the model checking problem for DEL when restricted to updates that are semi-private announcements. In fact, PSPACE-hardness for the setting with semi-private announcements (rather than allowing arbitrary event models) already follows from a recent PSPACE-hardness proof for a restricted variant of the model checking problem [23, Theorem 4]. In this PSPACE-hardness result, the number of agents is unbounded, i.e., the number of agents is part of the problem input. We show that the problem is already PSPACE-hard when the number of agents is bounded by a constant \( k \geq 2 \).

Theorem 4 is a stronger result than Theorem 3—Theorem 4 implies the result of Theorem 3. We presented the proof of Theorem 3 in full detail, because it allows us to explain the proof of Theorem 4 in a clear way.

**Theorem 4** The model checking problem for DEL is PSPACE-hard, even when restricted to the case where the question is whether \( M, w_0 \models [\mathcal{E}_1, e_1] \ldots [\mathcal{E}_n, e_n] \chi \), where:

- the model \((\mathcal{M}, w_0)\) is a single-pointed S5 model;
- all the \((\mathcal{E}_i, e_i)\) are (single-pointed) semi-private announcements;
- \( \chi \) is an epistemic formula without update modalities that contains (multiple occurrences of) only three propositional variables; and
- there are only two agents.

**Proof** We modify the proof of Theorem 3 to work also for the case of semi-private announcements. Most prominently, we will replace the event models \((\mathcal{E}_i, e_i)\) that are used in the proof of Theorem 3 (shown in Figure 12) by a number of event models for semi-private announcements. Intuitively, these semi-private announcements will take the role of the event models \((\mathcal{E}_i, e_i)\). In order to make this work, we will also slightly change the initial model \( \mathcal{M} \).
As in the proof of Theorem 3, we give a polynomial-time reduction from the problem of deciding whether a quantified Boolean formula is true. Let \( \varphi = \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n. \psi \) be a quantified Boolean formula, where \( \psi \) is quantifier-free. We construct an epistemic model \((M, w_0)\) with two agents \(a, b\) and a DEL-formula \(\xi\) such that \(M, w_0 \models \xi\) if and only if \(\varphi\) is true.

In the proof of Theorem 3, the initial model \(M\) consisted of a central world (where \(z_1\) and \(z_2\) are true), a number of \(z_1\)-chains (for each \(1 \leq i \leq n\), there is a \(z_1\)-chain of length \(i\)), and a number of \(z_2\)-chains (for each \(1 \leq i \leq n\), there is a \(z_2\)-chain of length \(i\))—and these worlds are connected by \(a\)-relations and \(b\)-relations as shown in Figure 11. To obtain the initial model \(M\) that we use in this proof, we add additional \(z_2\)-chains. Specifically, for each \(1 \leq i \leq 3n\), we will have a \(z_2\)-chain of length \(i\). These additional \(z_2\)-chains are connected to the central world in exactly the same way as the original \(z_2\)-chains (that is, all the first worlds of the \(z_2\)-chains are connected in a \(b\)-clique to the central world).

We will use these additional \(z_2\)-chains to simulate the behavior of the event models \((E_i, e_i)\) from the proof of Theorem 3 with event models corresponding to semi-private announcements. The number of \(z_1\)-chains remains the same. The designated world \(w_0\) is the central world (that is, the only world that makes both \(z_1\) and \(z_2\) true), as in the proof of Theorem 3.

![Fig. 13: The semi-private announcements \((E_1^1, f_1^1)\), \((E_2^2, f_2^2)\) and \((E_3^3, f_3^3)\) used in the proof of Theorem 4.](image)

The event models \((E_i, e_i)\) that are used in the proof of Theorem 3 we replace by the semi-private announcements \((E_1^1, f_1^1)\), \((E_2^2, f_2^2)\), and \((E_3^3, f_3^3)\), as shown in Figure 13. The intuition behind these updates is the following. Firstly, the semi-private announcement \(E_1^1\), shown in Figure 13A, transforms every group
of worlds into two copies, and allows a choice between these two copies when following a $b$-relation. Moreover, in one copy, every (first world of the) $z_2$-chain of length $i + n$ is removed, and in the other copy, every (first world of the) $z_2$-chain of length $i + 2n$ is removed. In other words, the choice between these two copies determines whether the formula $K_b x_{i+n}$ or the formula $K_b x_{i+2n}$ is false in the central world. Then for every group of worlds that does not include a $z_2$-chain of length $i + n$, the semi-private announcement $E_i^2$, shown in Figure 13, creates an $a$-accessible copy where the $z_2$-chain of length $i$ is removed. Similarly, for every group of worlds that does not include a $z_2$-chain of length $i + 2n$, the semi-private announcement $E_i^3$, shown in Figure 14, creates an $a$-accessible copy where both the $z_2$-chain of length $i$ and the $z_1$-chain of length $i$ are removed.

Next, we construct the DEL-formula $\xi$. We let $\xi = [E_1^1, f_1^1][E_1^2, f_1^2][E_1^3, f_1^3] \cdots [E_n^1, f_n^1][E_n^2, f_n^2][E_n^3, f_n^3][\xi_1]$, where $\xi_1$ is defined as follows, similarly to the definition used in the proof of Theorem 3. For each $1 \leq i \leq n + 1$, we define $\xi_i$ inductively as follows:

$$\xi_i = \begin{cases} 
\psi' & \text{if } i = n + 1, \\
K_b(\bigwedge_{1 \leq j \leq i} \neg K_b x_j' \land \bigwedge_{1 \leq j \leq i} \neg K_b x_j' \land \bigwedge_{i < j \leq n} \neg K_b x_j' \land \xi_{i+1}) & \text{for odd } i \leq n, \\
K_b(\bigwedge_{1 \leq j \leq i} \neg K_b x_j' \rightarrow K_a((z_1 \land z_2 \land \bigwedge_{i \leq j \leq n} \neg K_b x_j' \land \bigwedge_{i < j \leq n} \neg K_b x_j' \rightarrow \xi_{i+1})) & \text{for even } i \leq n.
\end{cases}$$

Here, $\psi'$ is the formula obtained from $\psi$ (the quantifier-free part of the quantified Boolean formula $\varphi$) by replacing each occurrence of a propositional variable $x_i$ by the formula $K_a x_i$.

The formulas $\xi_i$ that we defined above are very similar to their counterparts in the proof of Theorem 3 and the idea behind their use in the proof is entirely the same as in the proof of Theorem 3. The only difference is the addition of the subformulas $\bigwedge_{1 \leq j \leq i} \neg K_b x_j'$ after the first modal operator. These additional subformulas are needed to ensure that some additional worlds—that are a by-product of the combination of the semi-private announcements $(E_1^1, f_1^1)$, $(E_1^2, f_1^2)$, and $(E_1^3, f_1^3)$—do not interfere in the reduction.

We show that $\varphi$ is a true quantified Boolean formula if and only if $M, w_0 \models \xi$. In order to do so, as in the proof of Theorem 3 we prove the following (technical) statement relating truth assignments $\alpha$ to the propositions $x_1, \ldots, x_n$ and (particular) worlds $w$ in the epistemic model $(M', w'_0) = (M, w_0) \otimes (E_1^1, f_1^1) \otimes \cdots \otimes (E_n^1, f_n^1)$. Before we give the statement that we will prove, we observe that every world $w$ that sets both $z_1$ and $z_2$ to true is the central world of some group of worlds that represents a truth assignment $\alpha$ to the propositions $x_1, \ldots, x_n$. For the sake of convenience, we will say that $w$ corresponds to the truth assignment $\alpha$. The statement that we will prove for all $1 \leq i \leq n+1$ is the following.

**Statement:** Let $\alpha$ be any truth assignment to the propositions $x_1, \ldots, x_{i-1}$. Moreover, let $w$ be any world in the model $(M', w'_0)$ such that:
1. $w$ makes $z_1$ and $z_2$ true,
2. $w$ makes $K_1 \chi_1$ false for all $1 \leq j < i$,
3. $w$ makes $K_1 \chi_j$ true for all $i \leq j \leq n$, and
4. the truth assignment corresponding to $w$ agrees with $\alpha$ on the propositions $x_1, \ldots, x_{i-1}$.

Then the (partially) quantified Boolean formula $Q, x_i, \ldots, x_{n-1} \forall x_n. \psi$ is true under $\alpha$ if and only if $w$ makes $\xi_i$ true.

Observe that for $i = 1$, the world $w_0$ satisfies all four conditions. Therefore, the statement for $i = 1$ implies that $\mathcal{M}, w_0 \models \xi$ if and only if $\varphi$ is a true quantified Boolean formula. Thus, proving this statement for all $1 \leq i \leq n + 1$ suffices to show the correctness of our reduction.

We begin by showing that the statement holds for $i = n + 1$. In this case, we know that $\alpha$ is a truth assignment to the propositions $x_1, \ldots, x_n$. Moreover, $\xi_{n+1} = \psi'$. By construction of $\psi'$, we know that $w$ makes $\psi'$ true if and only if $\alpha$ satisfies $\psi$. Therefore, the statement holds for $i = n + 1$.

Next, we let $1 \leq i \leq n$ be arbitrary, and we assume that the statement holds for $i+1$. That is, the statement holds for every combination of a truth assignment $\alpha$ and a world $w$ that satisfies the conditions. Since $w$ is a world in the model $(\mathcal{M}, g_0) \otimes (E_1^1, f_1^1) \otimes \cdots \otimes (E_n^3, f_n^3)$, and since $w$ makes $z_1$ and $z_2$ true, we know that $w = (w_0, g_1, f_1^1, \ldots, g_n, f_n^3)$, for some $g_1, f_1^1, \ldots, g_n, f_n^3$, where for each $1 \leq j \leq n$, it holds that $g_j \in \{f_1^1, f_1^2\}$, $f_j \in \{f_2^2, f_2^3\}$, and $g_j^j \in \{f_3^3, f_3^4\}$.

We now distinguish two cases: either (1) $i$ is odd, or (2) $i$ is even. In case (1), the $i$-th quantifier of $\varphi$ is existential, and in case (2), the $i$-th quantifier of $\varphi$ is universal. First, consider case (1). Suppose that $\exists x_i, \ldots, x_{n-1} \forall x_n. \psi$ is true under $\alpha$. Then there exists some truth assignment $\alpha'$ to the propositions $x_1, \ldots, x_i$ that agrees with $\alpha$ on the propositions $x_1, \ldots, x_{i-1}$ and that ensures that $\forall x_{i+1}, \ldots, x_{n-1} \forall x_n. \psi$ is true under $\alpha'$. Suppose that $\alpha'(x_i) = 0$; the case for $\alpha'(x_i) = 1$ is entirely similar. Now, consider the worlds $w' = (w_0, g_1, \ldots, g_{i-1}, f_i^1, g_i^i, \ldots, g_{n-1}, f_n^3)$ and $w'' = (w_0, g_1, \ldots, g_{i-1}, f_i^2, g_i^i, \ldots, g_{n-1}, f_n^3)$.

By the construction of $E_1^1, E_2^2, E_3^3$, and by the semantics of product update, it holds that $(w, w') \in R_0$ and $(w', w'') \in R_3$. Also, we know that $w$ makes $\chi_i$ true. Moreover, it is straightforward to verify that $w''$ satisfies conditions (1)–(4), for the truth assignment $\alpha'$. Also, we know that $\xi_{i+1} = \psi'$. Therefore, by the induction hypothesis, we know that $w''$ makes the formula $\xi_{i+1}$ true. It then follows from the definition of $\xi_i$ that $w'$ and $w''$ witness that $w$ makes $\xi_i$ true.

Conversely, suppose that $w$ makes $\xi_i$ true. Moreover, suppose that $w'$ and $w''$ (as defined above) witness this. It could also be the case that the worlds $w'$ and $w''$ witness this, which are obtained from $w$ by replacing $g_i$ by $f_i^1$, and by replacing $g_i$ by $f_i^2$, and by replacing $g_i$ by $f_i^3$, respectively. The case where $w'$ and $w''$ witness that $w$ makes $\xi_i$ true is entirely similar. (There are also variants of $w'$ and $w''$, and of $u'$ and $u''$, that could witness the fact that $w$ makes $\xi_i$ true. These variants can be obtained by replacing $g_j$, $g_j'$, and $g_j''$—for $i < j \leq n$—ensuring that for all $i < j \leq n$ it holds that neither (1) $g_j = f_j^1$
and \( g'_j = f^j_1 \) nor (2) \( g_j = f^j_2 \) and \( g''_j = f^j_3 \). The following argument is entirely similar for these variants. Therefore, we restrict our attention to the worlds \( w' \) and \( w'' \). The assumption that \( w' \) and \( w'' \) witness that \( w \) makes \( \xi \) true implies that \( w' \) makes \( \bigwedge_{1 \leq j \leq i} \neg \hat{K}_b \chi_j' \) true and that \( w'' \) makes \( \xi_{i+1} \) true. Then, by the induction hypothesis, it follows that the truth assignment \( \alpha' \) to the propositions \( x_1, \ldots, x_i \) corresponding to the world \( w'' \) satisfies \( \forall x_{i+1} \ldots \exists x_n \forall x_n \psi \).

Moreover, since \( \alpha' \) agrees with \( \alpha \) on the propositions \( x_1, \ldots, x_{i-1} \), it follows that \( \exists x_i \ldots \exists x_n-1 \forall x_n \psi \) is true under \( \alpha \).

Next, consider case (2). Suppose that \( \forall x_1 \ldots \exists x_{n-1} \forall x_n \psi \) is true under \( \alpha \). Then for both truth assignments \( \alpha' \) to the variables \( x_1, \ldots, x_i \) that agree with \( \alpha \) it holds that \( \exists x_{i+1} \ldots \exists x_n-1 \forall x_n \psi \) is true under \( \alpha' \). We need to look at those worlds that satisfy \( (z_1 \land z_2 \land \bigwedge_{1 \leq j \leq i} \neg \hat{K}_b \chi_j' \land \bigwedge_{i<j \leq n} \hat{K}_b \chi_j' \) and that are accessible from \( w \) by a \( b \)-relation followed by an \( a \)-relation (where the intermediate world makes \( \bigwedge_{1 \leq j \leq i} \neg \hat{K}_b \chi_j' \) true). For our argument, it suffices to look at the worlds \( u_1 \) and \( u_2 \), where \( u_1 = (w_0, g_1, \ldots, g''_{i-1}, f^j_1, f^j_2, g''_i, g_{i+1}, \ldots, g''_n) \) and \( u'' = (w_0, g_1, \ldots, g''_{i-1}, f^j_1, f^j_2, g''_i, g_{i+1}, \ldots, g''_n) \). (As in the argument for case (1) above, there are variants of these worlds that also satisfy the requirements. The argument for these variants is entirely similar, and therefore we restrict our attention to the worlds \( u_1 \) and \( u_2 \).) The truth assignments \( \alpha_1 \) and \( \alpha_2 \) that correspond to \( u_1 \) and \( u_2 \), respectively, agree with \( \alpha \) on the propositions \( x_1, \ldots, x_{i-1} \). Because \( \forall x_1 \ldots \exists x_{n-1} \forall x_n \psi \) is true under \( \alpha \), we know that \( \exists x_{i+1} \ldots \exists x_n-1 \forall x_n \psi \) is true under both \( \alpha_1 \) and \( \alpha_2 \). Then, by the induction hypothesis it follows that both \( u_1 \) and \( u_2 \) make \( \xi_{i+1} \) true. Therefore, \( w \) makes \( \xi \) true.

Conversely, suppose that \( w \) makes \( \xi \) true. By the definition of \( \xi \), we then know that all worlds that are accessible from \( w \) by a \( b \)-relation followed by an \( a \)-relation (where the intermediate world makes \( \bigwedge_{1 \leq j \leq i} \neg \hat{K}_b \chi_j' \) true) also make \( \xi_{i+1} \) true. Consider the worlds \( u_1 \) and \( u_2 \), as defined above. These are both accessible from \( w \) by a \( b \)-relation followed by an \( a \)-relation (where the intermediate world makes \( \bigwedge_{1 \leq j \leq i} \neg \hat{K}_b \chi_j' \) true), and they agree with \( \xi \) on the propositions \( x_1, \ldots, x_{i-1} \). Moreover, the truth assignments \( \alpha_1 \) and \( \alpha_2 \) are both possible truth assignments to the propositions \( x_1, \ldots, x_i \) that agree with \( \alpha \). By the induction hypothesis, the formula \( \exists x_{i+1} \ldots \exists x_n-1 \forall x_n \psi \) is true under both \( \alpha_1 \) and \( \alpha_2 \). Therefore, we can conclude that \( \forall x_i \ldots \exists x_n-1 \forall x_n \psi \) is true under \( \alpha \).

This concludes the inductive proof of the statement for all \( 1 \leq i \leq n+1 \), and thus concludes our correctness proof. Therefore, we can conclude that the problem is PSPACE-hard. \( \square \)
5 Conclusion

We extended known complexity results for the model checking problem for DEL with a detailed computational complexity analysis. In particular, we studied various restrictions of the problem where all models are S5, including bounds on the number of agents, allowing only single-pointed models, allowing no postconditions, and allowing semi-private announcements rather than updates with arbitrary event models. We showed that the problem is already PSPACE-hard for very restricted settings.

Future research includes extending the computational complexity analysis to additional restricted settings. For instance, it would be interesting to see whether the polynomial-time algorithm for Proposition 11 can be extended to the setting where the models contain only relations that are transitive, Euclidean and serial (KD45 models). In the setting of KD45 models, it would also be interesting to investigate the complexity of the problem for the case where all updates are private announcements (i.e., a public announcement to a subset of agents, where the remaining agents have no awareness that any action has taken place). Moreover, future research includes obtaining upper bounds for the case where we only found lower bounds (i.e., for the case of one agent, a single-pointed models, and single-pointed event models with postconditions, where we showed $\Delta^2_p$-hardness).

Acknowledgements We would like to thank anonymous reviewers for their feedback on previous versions of the paper.

References

1. Arora, S., Barak, B.: Computational Complexity – A Modern Approach. Cambridge University Press (2009)
2. Aucher, G., Schwarzentruber, F.: On the complexity of dynamic epistemic logic. In: Proceedings of the 14th conference on Theoretical Aspects of Rationality and Knowledge (TARK 2013). Chennai, India (2013)
3. Baltag, A., Moss, L.: Logics for epistemic programs. Synthese 139(2), 165–224 (2004)
4. Baltag, A., Moss, L., Solecki, S.: The logic of public announcements, common knowledge, and private suspicions. In: Proceedings of the 7th conference on Theoretical Aspects of Rationality and Knowledge (TARK 1998), pp. 43–56. Morgan Kaufmann Publishers Inc. (1998)
5. Baltag, A., Moss, L., Solecki, S.: The logic of public announcements, common knowledge, and private suspicions. Tech. rep., Indiana University (1999)
6. Baral, C., Zhang, Y.: Knowledge updates: Semantics and complexity issues. Artificial Intelligence 164(1-2), 209–243 (2005)
7. van Benthem, J.: Epistemic logic and epistemology: The state of their affairs. Philosophical Studies 128(1), 49–76 (2006)
8. van Benthem, J., van Eijck, J., Gattinger, M., Su, K.: Symbolic model checking for dynamic epistemic logic – S5 and beyond. Journal of Logic and Computation 28(2), 367–402 (2018)
9. van Benthem, J., van Eijck, J., Kooi, B.: Logics of communication and change. Information and computation 204(11), 1620–1662 (2006)
10. Bolander, T., Andersen, M.B.: Epistemic planning for single and multi-agent systems. Journal of Applied Non-Classical Logics 21(1), 9–34 (2011)
11. van Ditmarsch, H., van der Hoek, W., Kooi, B.P.: Dynamic epistemic logic, *Synthese*, vol. 337. Springer (2008)
12. van Ditmarsch, H., Kooi, B.: Semantic results for ontic and epistemic change. In: G. Bonanno, W. van der Hoek, M. Wooldridge (eds.) Logic and the Foundations of Game and Decision Theory (LOFT 7), pp. 87–118 (2006)
13. van Eijck, J.: DEMO – A demo of epistemic modelling. In: J. van Benthem, D. Gabbay, B. Löwe (eds.) Interactive Logic – Proceedings of the 7th Augustus de Morgan Workshop, vol. 1, pp. 303–362 (2007)
14. van Eijck, J., Schwarzentruber, F.: Epistemic probability logic simplified. In: Advances inModal Logic, pp. 158–177 (2014)
15. Fagin, R., Halpern, J.Y., Moses, Y., Vardi, M.Y.: Reasoning about Knowledge. MIT Press, Cambridge (1995)
16. van der Hoek, W., Wooldridge, M.: Multi-agent systems. In: Handbook of Knowledge Representation, *Foundations of Artificial Intelligence*, vol. 3, pp. 887–928. Elsevier (2008)
17. Kooi, B., van Benthem, J.: Reduction axioms for epistemic actions. In: R. Schmidt, I. Pratt-Hartmann, M. Reynolds, H. Wansing (eds.) AiML-2004: Advances in Modal Logic, Department of Computer Science, University of Manchester, Technical report series, UMCS-04-9-1, pp. 197–211 (2004)
18. Krentel, M.W.: Generalizations of OptP to the Polynomial Hierarchy. Theoretical Computer Science 97(2), 183–198 (1992)
19. Lutz, C.: Complexity and succinctness of public announcement logic. In: H. Nakashima, M.P. Wellman, G. Weiss, P. Stone (eds.) Proceedings of the fifth International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS), pp. 137–143. IFAAMAS (2006)
20. Meyer, J.J., van der Hoek, W.: Epistemic logic for computer science and artificial intelligence (1995)
21. Plaza, J.: Logics of public communications. In: M. Emrich, M. Pfeifer, M. Hadzikadic, Z. Ras (eds.) Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems: poster session program, pp. 201–216. Oak Ridge National Laboratory (1989). (Reprinted as [22])
22. Plaza, J.: Logics of public communications. Synthese 158(2), 165–179 (2007)
23. van de Pol, I.: How Difficult is it to Think that you Think that I Think that ...? A DEL-based Computational-level Model of Theory of Mind and its Complexity. Master’s thesis, University of Amsterdam, the Netherlands (2015)
24. van de Pol, I., van Rooij, I., Szymanik, J.: Parameterized complexity of theory of mind reasoning in dynamic epistemic logic. *Journal of Logic, Language and Information* (2018)
25. Verbrugge, R.: Logic and social cognition: The facts matters, and so do computational models. *Journal of Philosophical Logic* 38(6), 649–680 (2009)