**PT-Symmetric Waveguides**

Denis Borisov and David Krejčířík

**Abstract.** We introduce a planar waveguide of constant width with non-Hermitian \( PT \)-symmetric Robin boundary conditions. We study the spectrum of this system in the regime when the boundary coupling function is a compactly supported perturbation of a homogeneous coupling. We prove that the essential spectrum is positive and independent of such perturbation, and that the residual spectrum is empty. Assuming that the perturbation is small in the supremum norm, we show that it gives rise to real weakly-coupled eigenvalues converging to the threshold of the essential spectrum. We derive sufficient conditions for these eigenvalues to exist or to be absent. Moreover, we construct the leading terms of the asymptotic expansions of these eigenvalues and the associated eigenfunctions.

**Mathematics Subject Classification (2000).** 35P15, 35J05, 47B44, 47B99.

**Keywords.** Non-self-adjointness, \( J \)-self-adjointness, waveguides, \( PT \)-symmetry, Robin boundary conditions, Robin Laplacian, eigenvalue and eigenfunction asymptotics, essential spectrum, reality of the spectrum.

1. **Introduction**

There are two kinds of motivations for the present work. The first one is due to the growing interest in spectral theory of non-self-adjoint operators. It is traditionally relevant to the study of dissipative processes, resonances if one uses the mathematical tool of complex scaling, and many others. The most recent and conceptually new application is based on the potential quantum-mechanical interpretation of non-Hermitian Hamiltonians which have real spectra and are invariant under a simultaneous \( P \)-parity and \( T \)-time reversal. For more information on the subject, we refer to the pioneering work [3] and especially to the recent review [2] with many references.

The other motivation is due to the interesting phenomena of the existence of bound states in quantum-waveguide systems intensively studied for almost two decades. Here we refer to the pioneering work [12] and to the reviews [10, 21]. In
these models the Hamiltonian is self-adjoint and the bound states – often without classical interpretations – correspond to an electron trapped inside the waveguide.

In this paper we unify these two fields of mathematical physics by considering a quantum waveguide modelled by a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian. Our main interest is to develop a spectral theory for the Hamiltonian and demonstrate the existence of eigenvalues outside the essential spectrum. For non-self-adjoint operators the location of the various essential spectra is often as much as one can realistically hope for in the absence of the powerful tools available when the operators are self-adjoint, notably the spectral theorem and minimax principle.

In the present paper we overcome this difficulty by using perturbation methods to study the point spectrum in the weak-coupling regime. In certain situations we are also able to prove that the total spectrum is real.

Let us now briefly recall the notion of $\mathcal{PT}$-symmetry. If the underlying Hilbert space of a Hamiltonian $H$ is the usual realization of square integrable functions $L^2(\mathbb{R}^n)$, the $\mathcal{PT}$-symmetry invariance can be stated in terms of the commutator relation

\[
(\mathcal{PT})H = H(\mathcal{PT}),
\]

where the parity and time reversal operators are defined by $(\mathcal{P}\psi)(x) := \psi(-x)$ and $T\psi := \overline{\psi}$, respectively. In most of the $\mathcal{PT}$-symmetric examples $H$ is the Schrödinger operator $-\Delta + V$ with a potential $V$ satisfying (1.1), so that $H^* = THT$ where $H^*$ denotes the adjoint of $H$. This property is known as the $T$-self-adjointness of $H$ in the mathematical literature [11], and it is not limited to $\mathcal{PT}$-symmetric Schrödinger operators. More generally, given any linear operator $H$ in an abstract Hilbert space $\mathcal{H}$, we understand the $\mathcal{PT}$-symmetry property as a special case of the $J$-self-adjointness of $H$:

\[
H^* = JHJ,
\]

where $J$ is a conjugation operator, i.e.,

\[
\forall \phi, \psi \in \mathcal{H}, \quad (J\phi, J\psi)_{\mathcal{H}} = (\psi, \phi)_{\mathcal{H}}, \quad J^2 \psi = \psi.
\]

This setting seems to be adequate for a rigorous formulation of $\mathcal{PT}$-symmetric problems, and alternative to that based on Krein spaces [22, 24].

The nice feature of the property (1.2) is that $H$ “is not too far” from the class of self-adjoint operators. In particular, the eigenvalues are found to be real for many $\mathcal{PT}$-symmetric Hamiltonians [28, 9, 22, 8, 26, 7, 20]. However, the situation is much less studied in the case when the resolvent of $H$ is not compact.

The spectral analysis of non-self-adjoint operators is more difficult than in the self-adjoint case, partly because the residual spectrum is in general not empty for the former. One of the goals of the present paper is to point out that the existence of this part of spectrum is always ruled out by (1.2):

**Fact.** Let $H$ be a densely defined closed linear operator in a Hilbert space satisfying (1.2). Then the residual spectrum of $H$ is empty.