The Extropy of Concomitants of Generalized Order Statistics from Huang–Kotz–Morgenstern Bivariate Distribution

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In this paper, we study the extropy for concomitants of \( m \)-generalized order statistics (\( m \)-GOSs) from Huang–Kotz–Farlie–Gumbel–Morgenstern (HK-FGM) bivariate distribution. Moreover, the cumulative residual extropy (CREX) and negative cumulative extropy (NCEX) are depicted. Furthermore, the empirical technique in conjunction with the concomitant of \( m \)-GOS is used to investigate the problem of estimating the CREX and NCEX. The concomitants of order statistics and record values are offered as some applications of these findings.

1. Introduction

The classic Farlie–Gumbel–Morgenstern (FGM) family of the marginals \( F_X \) and \( F_Y \) was treated by Huang and Kotz [1] as a polynomial-type single parameter extension. The distribution function (DF) which they suggested is

\[
F_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + \lambda (1 - F_X(x))(1 - F_Y(y))], \quad p > 0.
\]

(1)

The corresponding probability density function (PDF) is given by

\[
f_{X,Y}(x,y) = f_X(x)f_Y(y)[1 + \lambda (1 - (1 + p)F_X(x))(1 - (1 + p)F_Y(y))],
\]

denoted by HK-FGM.

The allowable range of the association parameter \( \lambda \) is \(-1/p^2 \leq \lambda \leq 1/p\), and the range for correlation coefficient is \(- (p + 2)^{-2}\) \( \min(1, p^2) \leq \rho \leq 3p(p + 2)^{-3}\). Huang and Kotz [1] revealed that using model (1), the positive correlation between marginal distributions can be enhanced to \( \approx 0.39 \), while the maximum negative correlation remains \(-1/3\). Furthermore, when considering model (1) with uniform marginals, the range \( 0 < p < 1 \) results in a fast decreasing positive correlation, while the allowable range progressively widens. In addition, at \( p = 1 \), the highest negative correlation is found. As a result, we will just look at the situation \( p \geq 1 \). It is worth noting that the HK-FGM family’s easy analytical form piqued the curiosity of many researchers who wanted to use (and generalize) this model in a variety of fields, e.g., Abd Elgawad et al. [2], Bairamov and Kotz [3], Barakat et al. [4], and Fisher and Klein [5].

Kamps [6] proposed a unified model for ordered random variables (RVs) called generalized order statistics (GOSs), which encompass order statistics (OSs), sequential OSs, record values, \( k \)-record values, Pfeifer’s records, and progressive type II censored OSs as special instances. Many key types of ordered RVs, such as OSs, sequential OSs, lower record values, \( k \)-records, and type II censored OSs, are found in the \( m \)-GOS subclass of GOSs. Let \( n \in \mathbb{N}, m \geq 1, k \geq 0 \) be parameters such that \( y_i = k + (n - i)(m + 1), \quad i = 1, 2, \ldots, n \). The RVs \( X_{r,n,m,k} \), \( r = 1, 2, \ldots, n \), are said to be \( m \)-GOSs based on the DF \( F_X(x) \), if their joint PDF is of the form
\[ f_{1,2,...,n}(x_1, x_2, \ldots, x_n) = k \left( \prod_{j=1}^{n-1} y_j \right) \left( \prod_{i=1}^{n-1} F_x(x_i) f(x_i) \right) F_x^{-1}(x_n) f(x_n), \] (3)

The marginal PDF of \( r \)-th \( m \)-GOS, \( X_{(r,m,k)} \), \( 1 \leq r \leq n \), is given by (cf. [6])

\[ f_{X_{(r,m,k)}}(x) = \frac{C_{r-1}}{(r-1)!} \int_{F_x^{-1}(x)}^1 \int_{F_y^{-1}(y)}^1 \int_{F_z^{-1}(z)}^1 \cdots \int_{F_{m+i}^{-1}(m+i)}^1 f_{X,Y_1,Y_2}^{r-1}(x_1, x_2, \ldots, x_r) f_{X,Y_1,Y_2}^{r-2}(x_2, x_3, \ldots, x_r) \cdots f_{X,Y_1,Y_2}^{r-(m-1)}(x_{m-1}, x_m) f_{X,Y_1,Y_2}^{r-m}(x_m) dx_1 dx_2 \cdots dx_r, \]

where \( F_x = 1 - F_{X_1} \), \( F_y = 1 - F_{Y_1} \), \( F_z = 1 - F_{Z_1} \), etc., for the \( X \)-sample, then \( Y \)-variates paired with these \( m \)-GOSs are called the concomitants of \( m \)-GOSs and denoted by \( Y_{(r,m,k)} \), \( r = 1, 2, \ldots, n \). The PDF of the concomitant of \( r \)-th \( m \)-GOS is given by (see, e.g., Abd Elgawad et al. [2] and Alawady et al. [14])

\[ f_{Y_{(r,m,k)}}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_{X_{(r,m,k)}}(x) dx, \] (5)

where \( f_{Y|X}(y|x) \) is the conditional PDF of \( Y \) given \( X \). As a complement dual of Shannon entropy, Lad et al. [18] proposed entropy as a new measure of uncertainty for an absolutely continuous nonnegative RV \( X \), defined by

\[ J(X) = \frac{1}{2} \int_{-\infty}^{\infty} \left( f_X(x) \right)^2 dx \]

\[ = \frac{1}{2} \int_{0}^{1} \phi(u) du, \] (6)

It is obvious that \( J(X) \leq 0 \). One of the statistical applications of entropy is to score the forecasting distributions using the total log scoring rule (for more details about this measure, see Husseiny et al. [19]). Recently, Almaspoor et al. [20] studied the measures of entropy for concomitants of GOs in the FGM family. A number of characterizations, as well as entropy lower bounds for GOs and record values, were spotlighted by Qiu [21]. Qiu and Jia [22] looked into residual entropy using GOs, whereas Qiu and Jia [23] looked into entropy estimators used in uniformity testing. The entropy properties of mixed systems were studied by Qiu et al. [24].

A test of uniformity-based entropy was used by Zamanzade and Mahdizadeh [25] to compare ranked set sampling to basic random sampling. Mohamed et al. [26] used fractional and weighted cumulative residual entropy measurements as well as discussed some of its features to test uniformity. Jahanshahi et al. [27] proposed cumulative residual entropy (CREX) as a measure of uncertainty for RV. The CREX is defined as

\[ \xi^r(X) = -\frac{1}{2} \int_{0}^{\infty} F_X^r(x) dx. \] (7)

It is always negative. As a result, the negative CREX (NCREX) can be expressed as

\[ \xi(X) = \frac{1}{2} \int_{0}^{\infty} F_X^2(x) dx. \] (8)

Newly, Tahmasebi and Toomaj [28] suggested a negative cumulative entropy (NCEX) analogous to (8), defined as

\[ \varphi \xi(X) = \frac{1}{2} \int_{0}^{\infty} \left[ 1 - F_X^2(x) \right] dx \]

\[ = \int_{0}^{1} \phi(u) du, \] (9)

where \( \phi(u) = (1 - u^2)/2, 0 < u < 1 \).

In this paper, we aim to investigate the entropy for concomitants of \( m \)-GOSs in the HK-FGM family. Therefore, the rest of this paper is organized as follows. In Section 2, we begin by calculating the entropy measure of \( Y_{(r,m,k)} \) in the HK-FGM family. Some results of CREX and NCEX for \( Y_{(r,m,k)} \) are obtained. In Section 3, the problem of estimating the NCREX and NCEX using the empirical NCREX and NCEX for concomitants of \( m \)-GOs is discussed.

2. Extropy and Some of Its Related Measures in Concomitants of \( m \)-GOSs Based on HK-FGM

In the present section, we derive the measures of extropy, CREX, and NCEX for the concomitant \( Y_{(r,m,k)} \) of \( m \)-GOSs based on HK-FGM family. Some of these measures properties are revealed.

2.1. Extropy in Concomitants of \( m \)-GOSs from HK-FGM

Let \( X \sim F_X \) and \( Y \sim F_Y \). Since the conditional PDF of \( Y_{(r,m,k)} \) given \( X_{(r,m,k)} = x \) is \( f_{Y_{(r,m,k)}|X_{(r,m,k)}}(y|x) = f_{Y|X}(y|x) \) (cf. Abd Elgawad et al. [2]), we can easily find the PDF of \( Y_{(r,m,k)} \) in \( m \)-GOS as follows:

\[ f_{Y_{(r,m,k)}}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_{X_{(r,m,k)}}(x) dx, \] (5)

where \( f_{Y|X}(y|x) \) is the conditional PDF of \( Y \) given \( X \). As a complement dual of Shannon entropy, Lad et al. [18] proposed entropy as a new measure of uncertainty for an absolutely continuous nonnegative RV \( X \), defined by

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where \( \phi(x) = \infty \), if \( x \) is noninteger, and \( \phi(x) = x \), if \( x \) is integer. Now, by making the transformation

\[
U \sim \text{uniformly RV on} \quad (0, 1) \quad \text{with respect to} \quad u, \text{i.e.,} \quad q(u) = Q'(u). \text{ Therefore, the corresponding quantile-based } J(Y_{[r,n,m,k]}) \text{ can be written as}
\]

\[
J(Y_{[r,n,m,k]}) = J(Y) - \frac{\lambda C_{r,n,m,k}^*}{2} E\left[ \left( \frac{1 - (1 + p)U^p}{q(u)} \right)^2 \right] - \lambda C_{r,n,m,k}^* E\left[ \frac{(1 - (1 + p)U^p)}{q(u)} \right].
\]
Remark 1. If \( m = 0 \) and \( k = 1 \), the \( m \)-GOSs turn into OSs. The PDF and CDF of the concomitant of the \( r \)th OS \( Y_{[r:n]} \) are given by

\[
f_{[r:n]}(y) = f_Y(y)\left[1 + \lambda \Delta_{r,n,p}(1 - (1 + p)F_Y(y))\right],
\]

and

\[
F_{[r:n]}(y) = F_Y(y)\left[1 + \lambda \Delta_{r,n,p}(1 - F_Y(y))\right],
\]

respectively, where \( \Delta_{r,n,p} = 1 - ((1 + p)\beta(r + p, n - r + 1))\beta(r, n - r + 1) \). According to (15), the extropy measure of \( Y_{[r:n]} \) is

\[
j(Y_{[r:n]}) = J(Y) - \frac{\lambda(p - np/n + p)^2}{2} E\left[\frac{(1 - (1 + p)U)^2}{q(u)}\right] - \lambda p - np E\left[\frac{(1 - (1 + p)U)^2}{q(u)}\right],
\]

and

\[
j(Y_{[1:n]}) = J(Y) - \frac{\lambda(1 - (1 + p)p!/(n + p)!)^2}{2} E\left[\frac{(1 - (1 + p)U)^2}{q(u)}\right] - \lambda \left[1 - \frac{(1 + p)p!}{(n + p)!}\right] E\left[\frac{(1 - (1 + p)U)^2}{q(u)}\right].
\]

**Theorem 1.** Let \( Y_{[r:n]} \) be the concomitant of the \( r \)th OS from HK-FGM; then, we have

\[
j(Y_{[r+1:n]}) - j(Y_{[r:n]}) = \lambda \Omega_{r,n,p} E\left[\frac{1 - (1 + p)U}{q(u)}\right] - \frac{(\lambda \Omega_{r,n,p})^2 - 2\lambda^2 \Delta_{r,n,p} \Omega_{r,n,p}}{2} E\left[\frac{(1 - (1 + p)U)^2}{q(u)}\right],
\]

and

\[
j(Y_{[r+2:n]}) - j(Y_{[r:n]}) = \lambda \frac{2r + p + 1}{r + 1} \Omega_{r,n,p} E\left[\frac{1 - (1 + p)U}{q(u)}\right] - \frac{2r + p + 1}{r + 1} \Omega_{r,n,p} E\left[\frac{1 - (1 + p)U}{q(u)}\right],
\]

where \( \Omega_{r,n,p} = p(p + 1)\beta(r + p, n - r + 1)/r\beta(r, n - r + 1) \).

\[
\Delta_{r+1,n,p} = \Delta_{r,n,p} - \frac{p(p + 1)\beta(r + p, n - r + 1)}{r\beta(r, n - r + 1)}.
\]

**Proof.** It is easy to check that
\[ \Delta_{r+1,n} = \frac{p(p+1)(2r+b+1)\beta(r+p,n-r+1)}{r(r+1)\beta(r,n-r+1)} \]  
\( (24) \)

**Theorem 2.** Let \( Y_{[r,n]} \) be the concomitant of the \( r \)th OS from HK-FGM; then, we have

\[ J(Y_{[r-1,n]}) - J(Y_{[r,n]}) = \lambda \Omega^*_0E \left[ \frac{1 - (1 + p)Up}{q(u)} \right] \]

\[ - \left( \frac{(\lambda \Omega^*_0)^2 - 2\lambda^2 \Delta_{r,n} \Omega^*_0 E}{2} \right) \left[ \left( \frac{1 - (1 + p)Up}{q(u)} \right)^2 - \left( \frac{1 - (1 + p)Up}{q(u)} \right)^2 \right], \quad (25) \]

and

\[ J(Y_{[r-2,n]}) - J(Y_{[r,n]}) = \lambda \frac{2n + p - 1}{n-1} \Omega^*_0 E \left[ \frac{1 - (1 + p)Up}{q(u)} \right] \]

\[ - \left( \frac{(\lambda (2n + p - 1/n - 1)\Omega^*_0 E}{2} \right) \left( \frac{1 - (1 + p)Up}{q(u)} \right)^2 \left( \frac{1 - (1 + p)Up}{q(u)} \right)^2 \right], \quad (26) \]

where \( \Omega^*_0E = \frac{p(p+1)\beta(r+p,n-r+1)/n\beta(r,n-r+1)}{n(n-1)\beta(r,n-r+1)} \).

**Proof.** It is easy to check that

\[ \Delta_{r,n} = \frac{p(p+1)\beta(r+p,n-r+1)}{n\beta(r,n-r+1)} \]

and

\[ F_{X,Y}(x,y) = \left( 1 - \exp\left( \frac{-x}{\theta_1} \right) \right) \left( 1 - \exp\left( \frac{-y}{\theta_2} \right) \right) \left[ 1 + \lambda \left( 1 - \exp\left( \frac{-x}{\theta_1} \right) \right) \left( 1 - \left( 1 - \exp\left( \frac{-y}{\theta_2} \right) \right)^p \right) \right], \quad x, y > 0. \]  
\( (29) \)

By using (15), we get the entropy in \( Y_{[r,n]} \) as

\[ J(Y_{[r,n]}) = -\frac{1}{4\theta_2} \left[ \frac{1}{\theta_2} \left( \frac{3p^2}{\theta_2(4p^2 + 10p + 4)} \right) \right] \]

\[ - \lambda \Delta_{r,n} \left[ \frac{p}{\theta_2(2p+4)} \right], \quad \theta_2 > 0. \]  
\( (30) \)

**Example 1.** Assume that \( X \) and \( Y \) have a joint exponential distribution as HK-FGM (denoted by HK-FGM-ED)

**Example 2.** Assume that \( X \) and \( Y \) have a joint logistic distribution as HK-FGM

\[ F_{X,Y}(x,y) = \left( 1 + \exp\left( -x \right) \right)^{-1} \left( 1 + \exp\left( -y \right) \right)^{-1} \]

\[ \cdot \left[ 1 + \lambda \left( 1 - \left( 1 + \exp\left( -x \right) \right)^{-p} \right) \left( 1 - \left( 1 + \exp\left( -y \right) \right)^{-p} \right) \right], \]

\[ -\infty < x < \infty, -\infty < y < \infty. \]  
\( (31) \)
By using (15), we have

\begin{equation}
J(Y_{[r:n]}) = - \frac{1}{12} \left[ \frac{\lambda \Delta_{r:n:p}}{2} \right]^2 \left[ \frac{1}{6} - \frac{2(1 + p)}{p^2 + 5p + 6} + \frac{(p + 1)^2}{4p^2 + 10p + 6} \right]
\end{equation}

(32)

\begin{equation}
- \lambda \Delta_{r:n:p} \left[ \frac{1}{6} - \frac{(1 + p)}{p^2 + 5p + 6} \right].
\end{equation}

**Example 3.** Assume that \(X\) and \(Y\) have a joint exponentiated exponential distribution as HK-FGM

\begin{equation}
F_{X,Y}(x, y) = (1 - \exp(-\theta_1 x))^{\alpha_1} (1 - \exp(-\theta_2 y))^{\alpha_2} \left[ 1 + \lambda \left( 1 - \exp(-\theta_1 x) \right)^{\alpha_1} \right] \left( 1 - \exp(-\theta_2 y) \right)^{\alpha_2}. \tag{33}
\end{equation}

Then, we have

\begin{equation}
J(Y_{[r:n]}) = \frac{\theta_2 \alpha_2}{4(1 - 2\alpha_2)} \left[ \frac{\lambda \Delta_{r:n:p}}{2} \right]^2 \left[ \frac{\theta_2 \alpha_2}{2(1 - 2\alpha_2)} - \frac{2(1 + p)\theta_2 \alpha_2}{(p + 2)(\alpha_2(p + 1) - 1)} + \frac{\alpha^2}{(p + 1)(2\alpha_2(p + 1) - 1)} \right]
\end{equation}

(34)

**Remark 2.** Assume that \((X_i, Y_i), i = 1, 2, \ldots\), is a sequence of bivariate RVs drawn from a continuous DF of a random vector \((X, Y)\). The value of \(Y\) that corresponds to the \(r\)th upper record \(R_r\), \(r > 1\), pertaining to the variate \(X\) will be referred to as the \(r\)th concomitant of the record value \(R_r\), and it will be denoted by \(Y_{[r]}\). The record values and their corresponding concomitants are applicable in real-world experiments like lifetime studies, sporting events, and other experimental fields. The record values, record times, and inter-record times were the subject of a statistical discussed by Chandler [30]. Applications of record values and their concomitants have been discussed in Houchens [31], Ahsanullah [32], and Husseiny et al. [19]. The record value is a special case of the \(m\)-GOSs with \(m = -1\) and \(k = 1\). Therefore, the PDF and DF for \(R_{[r]}\) are obtained as

\begin{equation}
g_{[r]}(y) = f_Y(y) \left[ 1 + \lambda C_{r:p}(1 - (1 + p)F_Y(y)) \right],
\end{equation}

\begin{equation}
G_{[r]}(y) = F_Y(y) \left[ 1 + \lambda C_{r:p}(1 - F_Y(y)) \right], \tag{35}
\end{equation}

where \(C_{r:p} = 1 - (1 + p) \sum_{i=0}^{p} (-1)^i \binom{p}{i} i^{(1-1)^i} \). Therefore, the extropy measure for \(R_{[r]}\) is obtained as follows:

\begin{equation}
J(R_{[r]}) = J(Y) - \frac{\left[ \lambda C_{r:p} \right]^2}{2} E \left[ \frac{(1 - (1 + p)U^2)}{q(u)} \right]
\end{equation}

\begin{equation}
- \lambda C_{r:p} E \left[ \frac{(1 - (1 + p)U^2)}{q(u)} \right]. \tag{36}
\end{equation}

2.2. CREX in Concomitants of \(m\)-GOSs Based on HK-FGM. The CREX measures are provided for the concomitant \(Y_{[r,n,m,k]}\) of the \(r\)th \(m\)-GOS as follows:

\begin{equation}
\xi(Y_{[r,n,m,k]}) = \frac{-1}{2} \int_0^{\infty} F_{Y_{[r,n,m,k]}}(y) \text{d}y
\end{equation}

\begin{equation}
= \xi(Y) + \lambda C_{r,n,m,k:p} \int_0^{\infty} F_Y(y) F_{Y_Y}(y) (1 - F^p_Y(y)) \text{d}y
\end{equation}
Example 4. For the HK-FGM bivariate uniform distribution (HK-FGM-UD) with DF

\[ F_{X,Y}(x, y) = \frac{xy}{\theta_1 \theta_2} \left[ 1 + \lambda \left( 1 - \left( \frac{x}{\theta_1} \right)^p \right) \left( 1 - \left( \frac{y}{\theta_2} \right)^p \right) \right], \tag{38} \]

\(0 < x < \theta_1, 0 < y < \theta_2.\)

By using (37), we get the CREX in \(Y_{[r,n]}\) as

\[ \xi^*(Y_{[r,n]}) = -\frac{\theta_2}{6} + \lambda \Delta_{r,n,k} \frac{\theta_2 p (p + 5)}{6 (p + 2) (p + 3)} - \left[ \lambda C_{r,n,k} \right]^2 \frac{\theta_2^2 (2 p + 9 p + 9)}{3 (2 p^2 + 9 p + 9)}. \tag{39} \]

2.3. NCEX in Concomitants of m–GOSs from HK-FGM.

For the concomitant of \(m\)–GOS, the NCEX is given by

\[ \mathcal{G} \xi(Y_{(r,n,m,k)}) = \frac{1}{2} \int_0^\infty [1 - \tilde{F}_{Y_{(r,n,m,k)}}(y)] dy \]

\[ = \frac{1}{2} \int_0^\infty \left[ 1 - \left[ F_Y(y)^2 + \lambda C_{r,n,m,k}^p \left( 1 - F_Y^p(y) \right) \right] ^2 \right] dy \]

\[ = \mathcal{G} \xi(Y) - \lambda C_{r,n,m,k}^p E[U^2 (1 - U^p)] q(u) \]

\[ - \left[ \lambda C_{r,n,m,k}^p \right]^2 \frac{E[U^2 (1 - U^p)^2]}{2} q(u), \tag{40} \]

where \(\mathcal{G} \xi(Y)\) is the NCEX of the RV \(Y\).

3. Estimating of NCREX and NCEX

In this section, we use empirical estimators to calculate the NCREX and NCEX for concomitant \(Y_{(r,n,m,k)}\). We will now look at the problem of using the empirical NCREX to estimate the NCREX for concomitants. Let \((X_i, Y_i)\), where \(i = 1, 2, \ldots\), be a HK-FGM sequence. Using the relation \(-\xi^* = \xi\), the empirical NCREX of \(Y_{(r,n,m,k)}\) may be calculated as follows:

\[ \tilde{\xi}(Y_{(r,n,m,k)}) = \frac{1}{2} \int_0^\infty [1 - \tilde{F}_{Y_{(r,n,m,k)}}(y)]^2 dy \]

\[ = \frac{1}{2} \int_0^\infty \left[ 1 - F_Y(y) - \lambda C_{r,n,m,k}^p F_Y(y) \left( 1 - F_Y^p(y) \right) \right]^2 dy \]

\[ = \frac{1}{2} \sum_{j=1}^{n-1} \int_{z_j}^{z_{j+1}} \left[ 1 - F_Y(y) \right]^2 - 2 \lambda C_{r,n,m,k}^p F_Y(y) \left( 1 - F_Y^p(y) \right) + \lambda C_{r,n,m,k}^p \left[ F_Y(y) \left( 1 - F_Y^p(y) \right) \right]^2 dy \]
\[
\begin{align*}
E[\tilde{Y}(\mathbf{r}, \mathbf{n})] & = \frac{1}{2} n \sum_{j=1}^{n-1} U_j \left[ \left(1 - \frac{j}{n}\right)^2 - 2\lambda \xi_1 \mu \left(1 - \frac{j}{n}\right) \left(1 - \left(\frac{j}{n}\right)^2\right) \right], \\
\text{Var}[\tilde{Y}(\mathbf{r}, \mathbf{n})] & = \frac{1}{2n} \sum_{j=1}^{n-1} \left[ \left(1 - \frac{j}{n}\right)^2 - 2\lambda \xi_1 \mu \left(1 - \frac{j}{n}\right) \left(1 - \left(\frac{j}{n}\right)^2\right) \right] \left[ \left(1 - \frac{j}{n}\right)^2 - 2\lambda \xi_1 \mu \left(1 - \frac{j}{n}\right) \left(1 - \left(\frac{j}{n}\right)^2\right) \right].
\end{align*}
\]

(41)

where \( U_j = Z_{(j+1)} - Z_{(j)}, \ j = 1, 2, \ldots, n - 1 \), are the sample spacings based on ordered random samples \( Y_j \).

Example 5. If \((X_i, Y_i), i = 1, 2, \ldots, n\) is a random sample from HK-FGM-ED, the sample spacings \( U_j \) are independent and exponentially distributed, with a mean \( 1/\theta_2 (n - j) \) (for more details, see [30]). According to Pyke [33], the empirical NCREX expectation and variance based on \( Y_{[r, n]} \) are as follows:

\[
\begin{align*}
E[\tilde{Y}(\mathbf{r}, \mathbf{n})] & = \frac{1}{2\theta_2} \sum_{j=1}^{n-1} \frac{1}{(n-j)} \left[ \left(1 - \frac{j}{n}\right)^2 - 2\lambda \xi_1 \mu \left(1 - \frac{j}{n}\right) \left(1 - \left(\frac{j}{n}\right)^2\right) \right] \left[ \left(1 - \frac{j}{n}\right)^2 - 2\lambda \xi_1 \mu \left(1 - \frac{j}{n}\right) \left(1 - \left(\frac{j}{n}\right)^2\right) \right]. \\
\text{Var}[\tilde{Y}(\mathbf{r}, \mathbf{n})] & = \frac{1}{4\theta_2^2} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n-j}\right) \left[ \left(1 - \frac{j}{n}\right)^2 - 2\lambda \xi_1 \mu \left(1 - \frac{j}{n}\right) \left(1 - \left(\frac{j}{n}\right)^2\right) \right] \left[ \left(1 - \frac{j}{n}\right)^2 - 2\lambda \xi_1 \mu \left(1 - \frac{j}{n}\right) \left(1 - \left(\frac{j}{n}\right)^2\right) \right].
\end{align*}
\]

(42)

(43)

Example 6. The sample spacings \( U_j \) for HK-FGM-UD with \( \theta_1 = \theta_2 = 1 \) are independent of the beta distribution with parameters 1 and \( n \). According to Pyke [33], the expectation and variance of empirical NCREX based on \( R_{[r]} \) are as follows:

\[
\begin{align*}
E[\tilde{Y}(\mathbf{r}, \mathbf{n})] & = \frac{1}{n+1} \sum_{j=1}^{n} \left[ \left(1 - \frac{j}{n}\right)^2 - 2\lambda \xi_1 \mu \left(1 - \frac{j}{n}\right) \left(1 - \left(\frac{j}{n}\right)^2\right) \right] \left[ \left(1 - \frac{j}{n}\right)^2 - 2\lambda \xi_1 \mu \left(1 - \frac{j}{n}\right) \left(1 - \left(\frac{j}{n}\right)^2\right) \right], \\
\text{Var}[\tilde{Y}(\mathbf{r}, \mathbf{n})] & = \frac{n}{(n+1)^2 (n+2)} \sum_{j=1}^{n} \left[ \left(1 - \frac{j}{n}\right)^2 - 2\lambda \xi_1 \mu \left(1 - \frac{j}{n}\right) \left(1 - \left(\frac{j}{n}\right)^2\right) \right] \left[ \left(1 - \frac{j}{n}\right)^2 - 2\lambda \xi_1 \mu \left(1 - \frac{j}{n}\right) \left(1 - \left(\frac{j}{n}\right)^2\right) \right].
\end{align*}
\]

(44)

(45)

Also, the empirical NCEX of \( Y_{[r, \mu, \lambda]} \) can be obtained as follows:

\[
\begin{align*}
\tilde{Y}(\mathbf{r}, \mathbf{n}) & = \frac{1}{2} \int_0^1 \left[ 1 - \tilde{F}_y^2(\mathbf{r}, \mathbf{n}) \right] dy \\
& = \frac{1}{2} \int_0^1 \left[ 1 - \left[ \tilde{F}_y(\mathbf{r}, \mathbf{n}) \left(1 + \lambda \xi_1 \mu \left(1 - \tilde{F}_y(\mathbf{r}, \mathbf{n})\right)\right) \right]^2 \right] dy \\
& = \frac{1}{2} \sum_{j=1}^{n} \left[ 1 - \tilde{F}_y^2(\mathbf{r}, \mathbf{n}) - 2\lambda \xi_1 \mu \left(1 - \tilde{F}_y(\mathbf{r}, \mathbf{n})\right) \left(1 - \tilde{F}_y(\mathbf{r}, \mathbf{n})\right) \right] \left[ \left(1 - \tilde{F}_y(\mathbf{r}, \mathbf{n})\right)^2 \left(1 - \tilde{F}_y(\mathbf{r}, \mathbf{n})\right)\right] dy \\
& = \frac{1}{2} \sum_{j=1}^{n} \left[ 1 - \tilde{F}_y^2(\mathbf{r}, \mathbf{n}) - 2\lambda \xi_1 \mu \left(1 - \tilde{F}_y(\mathbf{r}, \mathbf{n})\right) \left(1 - \tilde{F}_y(\mathbf{r}, \mathbf{n})\right) \right] \left[ \left(1 - \tilde{F}_y(\mathbf{r}, \mathbf{n})\right)^2 \left(1 - \tilde{F}_y(\mathbf{r}, \mathbf{n})\right)\right] dy.
\end{align*}
\]
Table 1: $E[\xi(Y_{[r,n]})]$ and $\text{Var}[\xi(Y_{[r,n]})]$ from HK-FGM-ED at $r = 2$.

| $n$ | $\theta_2$ | $\lambda = -0.25$ | $\lambda = -0.15$ | $\lambda = 0.15$ | $\lambda = 0.5$ |
|-----|-------------|-------------------|-------------------|------------------|----------------|
| 5   | 0.5         | 0.4689            | 0.4413            | 0.3590           | 0.2642         |
| 5   | 1           | 0.2345            | 0.2206            | 0.1795           | 0.1321         |
| 5   | 2           | 0.1172            | 0.1103            | 0.0897           | 0.0661         |
| 10  | 0.5         | 0.5577            | 0.5444            | 0.3862           | 0.2394         |
| 10  | 1           | 0.2788            | 0.2572            | 0.1931           | 0.1197         |
| 10  | 2           | 0.1394            | 0.1286            | 0.0965           | 0.0598         |
| 15  | 0.5         | 0.5838            | 0.5367            | 0.3973           | 0.2379         |
| 15  | 1           | 0.2919            | 0.2684            | 0.1986           | 1189           |
| 15  | 2           | 0.1459            | 0.1342            | 0.0993           | 0.0595         |
| 20  | 0.5         | 0.5958            | 0.5472            | 0.4034           | 0.2392         |
| 20  | 1           | 0.2979            | 0.2736            | 0.2017           | 0.1196         |
| 20  | 2           | 0.1489            | 0.1368            | 0.1009           | 0.0598         |

Figure 1: Continued.
Table 1 displays the values of $E[\tilde{\xi}(Y_{[r,n]}])$ and $\text{Var}[\tilde{\xi}(Y_{[r,n]}))]$ based on HK-FGM-ED by computing (42) and (43). The following properties can be extracted from Table 1.

1. Generally, with fixed $n$ and $\theta_2$, the values of $E[\tilde{\xi}(Y_{[r,n]}))$ and $\text{Var}[\tilde{\xi}(Y_{[r,n]}))]$ decrease as the value of $\lambda$ increases.

2. Generally, with fixed $n$ and $\theta_2$, the values of $E[\tilde{\xi}(Y_{[r,n]}))$ and $\text{Var}[\tilde{\xi}(Y_{[r,n]}))]$ decrease as the value of $\theta_2$ increases.

Figure 1 shows the relation between NCREX and empirical NCREX of $R_{[r]}$ from HK-FGM-UD at $n = 100$. The following properties can be extracted from Figure 1.

1. NCREX and empirical NCREX have similar values when $p = 1$ and $-1 \leq \lambda \leq 1$ and especially when $\lambda = 0$ for all values of $r$.

2. The value of NCREX is greater than the value of empirical NCREX for $p > 1$ and $0 < \lambda \leq 1$.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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