On the origin of inflation by using exotic smoothness

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In this paper we discuss a spacetime having the topology of \( S^3 \times \mathbb{R} \) but with a different smoothness structure leading to a geometric model for inflation, called geometric inflation. In particular this spacetime is not globally hyperbolic and we obtain a time line with a spatial topology change from the 3-sphere to a homology 3-sphere and back. The topology of the spacetime remains invariant. Among the infinite possible smoothness structures of this spacetime, we choose a homology 3-sphere constructed from the knot \( 8_{10} \) with hyperbolic geometry, i.e. admitting a homogenous metric of negative scalar curvature. We discuss the accelerated expansion for FLRW cosmology caused by the topology change. In contrast to other inflation models, this process stops after a finite time. Alternatively, the topology change can be also described by a \( SU(2) \) -valued scalar field. Then we calculate the expansion rate (having more than 60 e-folds) and the energy / time scale. The coupling to matter is also interpreted geometrically and the reheating process (as well the supercooled expansion during inflation) is naturally obtained. The model depends only on a single parameter, a topological invariant of the homology 3-sphere, and assumes a Planck size universe of \( S^3 \) -topology. The dependence of the model on the initial state and the a geometric interpretation of quantum fluctuations are also discussed.

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I. INTRODUCTION

Because of the influx of observational data, recent years have witnessed enormous advances in our understanding of the early universe. To interpret the present data, it is sufficient to work in a regime in which spacetime can be taken to be a smooth continuum as in general relativity, setting aside fundamental questions involving the deep Planck regime. However, for a complete conceptual understanding as well as interpretation of the future, more refined data, these long-standing issues will have to be faced squarely. As an example, can one show from first principles that the smooth spacetime of general relativity is valid at the onset of inflation? In this paper we will focus mainly on this question about the origin of inflation. Inflation is today the main theoretical framework that describes the early Universe and that can account for the present observational data [39]. In thirty years of existence [33, 13], inflation has survived, in contrast with earlier competitors, the tremendous improvement of cosmological data. In particular, the fluctuations of the Cosmic Microwave Background (CMB) had not yet been measured when inflation was invented, whereas they give us today a remarkable picture of the cosmological perturbations in the early Universe. In nearly all known models, the inflation period is caused by one or more scalar field(s) [12].

A. The model

In this paper we try to derive an inflationary phase from first principles. The spacetime has the topology of $S^3 \times \mathbb{R}$ and is smoothable (i.e. a smooth 4-manifold) [50] but (we assume) it is not diffeomorphic to $S^3 \times \mathbb{R}$. What does it mean? Every manifold is defined by a collection of charts, the atlas, including also the transition functions between the charts. From the physical point of view, charts are the reference frames. The transition functions define the structure of the manifold, i.e. transition functions are homeomorphisms (topological manifold) or diffeomorphisms (smooth manifold). Two (smooth) atlases are compatible (or equivalent) if their union is a (smooth) atlas again. The equivalence class (the maximal atlas) is called a differential structure [70]. In dimension smaller than 4, there is only one differential structure, i.e. the topology of these manifolds define uniquely its smoothness properties. In contrast, beginning with dimension 4 there is the possibility of more than one differential structure. But 4-manifolds are really special: here there are many examples of 4-manifolds with infinite many differential structures (countable for compact and uncountable for non-compact 4-manifolds including $\mathbb{R}^4$). Among these differential structures there is one exceptional, the standard differential structure. We will illustrate these standard structure for our spacetime $S^3 \times \mathbb{R}$. The 3-sphere $S^3$ has an unique differential structure (the standard differential structure) which extends to $S^3 \times \mathbb{R}$. All other differential structures (also called misleadingly ”exotic smoothness structures”) can never split smoothly into $S^3 \times \mathbb{R}$. We denote it by $S^3 \times_\theta \mathbb{R}$. Our main hypothesis is now:

Main hypothesis: The spacetime has the topology $S^3 \times \mathbb{R}$ but having the differential structure $S^3 \times_\theta \mathbb{R}$.

In [23] the first $S^3 \times_\theta \mathbb{R}$ was constructed and we will use this construction here. The details of the construction including the foliation can be found in section [11]. One starts with a homology 3-sphere $\Sigma$, i.e. a compact 3-manifold $\Sigma$ with the same homology as the 3-sphere but non-trivial fundamental group, see for instance [11]. The Poincare sphere is one example of a homology 3-sphere. Now we consider the 4-manifold $\Sigma \times [0,1]$ with fundamental group $\pi_1(\Sigma \times [0,1]) = \pi_1(\Sigma)$. By a special procedure (the plus construction, see [46, 58]), one can ”kill” the fundamental group $\pi_1(\Sigma)$ in the interior of $\Sigma \times [0,1]$. This procedure transforms a non-contractable closed curve (as element of $\pi_1(\Sigma)$) to a contractable curve (denoted as ”killing the fundamental group” above). It will result in a 4-manifold $W$ with boundary $\partial W = -\Sigma \cup S^3$ ($-\Sigma$ is $\Sigma$ with opposite orientation), a so-called cobordism between $\Sigma$ and $S^3$. The gluing $-W \cup_\Sigma W$ along $\Sigma$ with the boundary $\partial(-W \cup_\Sigma W) = -S^3 \cup S^3$ defines one piece of the exotic $S^3 \times_\theta \mathbb{R}$. The whole construction can be extended to both directions to get the desired exotic $S^3 \times_\theta \mathbb{R}$ (see [23, 68] and the subsection [HIC] for the details). There is one critical point in the construction: the 4-manifold $W$ is not a smooth manifold. As Freedman [24] showed, the 4-manifold $W$ always exists topologically but, by a result of Gompf [30] (using Donaldson...
The energy scale using Mostow rigidity \cite{50}, subsection III D. Then we obtain the expansion rate $C_S$ Chern-Simons invariant which only depend on 3-manifold $\Sigma$. The parameter $\vartheta$ this process is part of every exotic smoothness structure $S^3 \times_\vartheta \mathbb{R}$, i.e. we obtain the mathematical fact.

**Fact**: In the spacetime $S^3 \times_\vartheta \mathbb{R}$ we have a change of the spatial topology from the 3-sphere to some homology 3-sphere $\Sigma$ and back but without changing the topology of the spacetime.

Now we have to discuss the choice of the homology 3-sphere $\Sigma$. At first, usually every homology 3-sphere is the boundary of a topological, contractable 4-manifold \cite{24} but this homology 3-sphere $\Sigma$ never bounds a smooth, contractable 4-manifold. Secondly, every homology 3-sphere can be constructed by using a knot \cite{57}. One starts with the complement $S^3 - (D^2 \times K)$ of a knot $K$ (a smooth embedding $S^3 \to S^3$) and glue in a solid torus $D^2 \times S^1$ using a special map (a $\pm 1$ Dehn twist). The resulting 3-manifold $\Sigma(K)$ is a homology 3-sphere. For instance the trefoil knot $3_1$ (in Rolfsen notation \cite{57}) generates the Poincare sphere by this method (with $-1$ Dehn twist).

Our model $S^3 \times_\vartheta \mathbb{R}$ starts with a 3-sphere as spatial topology. Now we will be using a powerful tool for the following argumentation, Thurston's geometrization conjecture \cite{63} proved by Perelman \cite{53–55}. According to this theory, only the 3-sphere and the Poincare sphere carry a homogenous metric of constant positive scalar curvature (spherical geometry or Bianchi IX model) among all homology 3-spheres. Also other homology 3-spheres \cite{72} are able to admit a homogenous metrics. There is a close relation between Thurston's geometrization theory and Bianchi models in cosmology \cite{11,2}. Most of the (irreducible) homology 3-spheres have a hyperbolic geometry (Bianchi V model), i.e. a homogenous metric of negative curvature. Here we will choose such a hyperbolic homology 3-sphere. As an example we choose the knot $8_{10}$ leading to the hyperbolic homology 3-sphere $\Sigma(8_{10})$. This choice is not arbitrary. According to the construction above, we need a homology 3-sphere which does not bound a smooth, contractable 4-manifold. The homology 3-sphere $\Sigma(8_{10})$ fulfills this condition. For more details we refer to subsection III C.

### B. Results of the Paper

At first we will give an overview of our assumptions:

1. The spacetime is $S^3 \times_\vartheta \mathbb{R}$ (with topology $S^3 \times \mathbb{R}$) containing a homology 3-sphere $\Sigma$ (as cross section).

2. This homology 3-sphere $\Sigma$ is a hyperbolic 3-manifold (with negative scalar curvature).

According to the mathematical facts above, the 3-sphere is changed to $\Sigma$ and back in $S^3 \times_\vartheta \mathbb{R}$, i.e. we obtain a topology change of the spatial cosmos. In the next section we will discuss the implication for the cosmology (using the Robertson-Walker metric). Then we obtain the relation

$$\dot{a}(t) \cdot \left(\frac{\rho_0}{a^2} + 2\dot{\vartheta}\right) > 0$$

for the scaling parameter $a(t)$ leading to an accelerated expansion $\dot{a} > 0$ for $\frac{\rho_0}{a^2} \ll 1$. After a short excursion in the differential topology of exotic 4-manifold, we will construct explicitly the foliation of $S^3 \times_\vartheta \mathbb{R}$ and discuss the physical properties like causality and global hyperbolicity in section III. The main results can be found in section IV. At first we will show that the change of the 3-sphere $S^3$ to $\Sigma$ can be described by a $SU(2)$ valued scalar field with double well potential as interaction. Then we will calculate the expansion rate by analyzing the topology-changing process. This process is determined by one parameter

$$\vartheta = \frac{3 \cdot \text{vol}(\Sigma)}{2 \cdot C_S(\Sigma)}$$

which only depend on 3-manifold $\Sigma$. The parameter $\vartheta$ is a relation between two topological invariants of $\Sigma$, the Chern-Simons invariant $C_S(\Sigma)$ (see the appendix D) and the volume $\text{vol}(\Sigma)$ (an invariant for a hyperbolic 3-manifold using Mostow rigidity \cite{50}, subsection III D). Then we obtain the expansion rate

$$l_{\text{scale}} = \exp(\vartheta)$$

the energy scale

$$\varepsilon_{\text{scale}} = 1 + \frac{\vartheta^2}{2} + \frac{\vartheta^3}{4} + \frac{\vartheta^4}{3!}$$
the time scale
\[ t_{\text{scale}} = \sum_{n=0}^{5} \frac{\vartheta^n}{n!} \]
and the decrease of the temperature \( T_0 \) start temperature and \( T_1 \) end temperature during the inflation (supercooled expansion). For the concrete model of a hyperbolic 3-manifold \( \Sigma = \Sigma(8_{10}) \), generated by the knot \( 8_{10} \), we obtain the following values:

| \( \vartheta \) | \( l_{\text{scale}} \) | \( e_{\text{scale}} \) | \( t_{\text{scale}} \) | \( \frac{T_0}{T_1} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 83.131... | 1.3 \cdot 10^{36} | 115172.2606... | 3.5 \cdot 10^{14} | 115172.2606... |

If we assume a Planck state (a state of Planck size having Planck energy along Planck time) then we will obtain the measurable values

| size after inflation | Energy scale | time scale | supercooled temperature \( (T_0 =) \) |
|----------------------|--------------|------------|-----------------------------------------|
| \( \approx 10 m \)   | \( \approx 10^{14} \) GeV | \( \approx 10^{-37} \) s | \( T_1 \approx 10^{27} \) K |

Finally we will also discuss the coupling to the matter and the reheating after inflation. Furthermore, we will obtain the model of parametric resonance naturally.

II. FLRW COSMOLOGY FOR A TOPOLOGY CHANGE

The analysis of the WMAP data do not exclude the case that our universe is a compact 3-manifold with a slightly positive curvature [59]. So, we are able to choose the topology of the spacetime to be \( S^3 \times \mathbb{R} \). But we remark that \( \mathbb{R}^4 \) topology of the spacetime can be also chosen. Clearly this spacetime admits a Lorentz metric (given by the topological condition to admit a non-vanishing vector field normal to \( S^3 \)). But we weaken the condition of global hyperbolicity otherwise it induces a diffeomorphism [12] [13] to \( S^3 \times \mathbb{R} \). We will discuss the implications in subsection III B. The spacetime has the topology of \( S^3 \times \mathbb{R} \) and is smoothable (i.e. a smooth 4-manifold) [56] but (we assume) it is not diffeomorphic to \( S^3 \times \mathbb{R} \). This exotic \( S^3 \times \mathbb{R} \) will be denoted by \( S^3 \times \varrho \mathbb{R} \). As stated above and explained in section III this \( S^3 \times \varrho \mathbb{R} \) contains a homology 3-sphere and we have a topology change from \( S^3 \) to \( \Sigma \) and back. Now we will study the geometry and topology changing process more carefully. Let us consider the Robertson-Walker metric (with \( c = 1 \))

\[ ds^2 = dt^2 - a(t)^2 h_{ik} dx^i dx^k \]

in Friedmann–Lematre–Robertson–Walker (FLRW) cosmology with the scaling function \( a(t) \). This metric do not depend on the topology of the spacetime. One assumes only the local splitting \( N \times [0,1] \) with \( N \) a 3-manifold with homogenous metric and coordinates \( x^i \) and a (finite) time variable \( t \).

At first we assume a spacetime \( S^3 \times \mathbb{R} \) with increasing function \( a(t) \) fulfilling the Friedman equations

\[ \left( \frac{\dot{a}(t)}{a(t)} \right)^2 + \frac{k}{a(t)^2} = \kappa \frac{\rho}{3} \]  
\[ 2 \left( \frac{\ddot{a}(t)}{a(t)} + \left( \frac{\dot{a}(t)}{a(t)} \right)^2 \right) + \frac{k}{a(t)^2} = -\kappa p \]

derived from Einsteins equation

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \]

with the gravitational constant \( \kappa \) and the energy-momentum tensor of a perfect fluid

\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu} \]
with the (time-dependent) energy density $\rho$ and the (time-dependent) pressure $p$. The spatial cosmos has the scalar curvature $3R$

$$3R = \frac{3k}{a^2}$$

(5)
rom the 3-metric $h_{i\kappa}$ and we obtain the 4-dimensional scalar curvature $R$

$$R = \frac{6}{a^2} \left( \ddot{a} \cdot a + \dot{a}^2 + k \right).$$

(6)

Let us consider the model $S^3 \times \mathbb{R}$ with positive spatial curvature $k = +1$. In case of dust matter ($p = 0$) only, one obtains a closed universe. Now we consider our model of an exotic $S^3 \times_\theta \mathbb{R}$. As explained above, the foliation of $S^3 \times_\theta \mathbb{R}$ must contain a hyperbolic homology 3-sphere $\Sigma(8_{10})$ (with negative scalar curvature). But then we have a transition from a space with positive curvature to a space with negative curvature and back. To model this behavior, we consider a time-dependent parameter $k(t)$ in the curvature

$$3R(t) = \frac{3k(t)}{a^2}$$

with the following conditions:

1. The change of the geometry from spherical $k > 0$ to hyperbolic $k < 0$ happens at $t_0$,

2. $k(t) > 0$ for $t \ll t_0$ and $t \gg t_0$.

The change of the topology is an abrupt process. So at first we will take the path $\Sigma(8_{10}) \times (t_0, t_0 + \Delta t)$ for a suitable time interval and discuss the equation (1) for $t > t_0$:

$$-1 = k(t_0 + \Delta t) = \frac{\rho_0}{a(t_0 + \Delta t)} - (\dot{a}(t_0 + \Delta t))^2$$

and the path $S^3 \times (t_0 - \Delta t, t_0)$ for $t < t_0$:

$$+1 = k(t_0 - \Delta t) = \frac{\rho_0}{a(t_0 - \Delta t)} - (\dot{a}(t_0 - \Delta t))^2$$

The difference of these equations (representing the equation at $t_0$) is given by

$$0 > \lim_{\Delta t \to 0} \frac{k(t_0 + \Delta t) - k(t_0 - \Delta t)}{2\Delta t} = \frac{d}{dt} \left( \frac{\rho_0}{a(t)} - \dot{a}(t)^2 \right)_{t=t_0}$$

$$= -\ddot{a}(t) \cdot \left( \frac{\rho_0}{a^2} + 2\dot{a} \right).$$

Now lets take $0 \approx \frac{\rho_0}{3a^2} \ll 1$ with good accuracy and $\dot{a} > 0$ (expansion) then we obtain an accelerated expansion $\ddot{a} > 0$. The calculation above is true for every kind of function $k(t)$, jumping from $+1$ for $t < t_0$ to $-1$ for $t > t_0$. This model is only a motivation for a model with a spatial topology change. For instance, the energy-momentum tensor has to change also and we will analyze it in subsection 5A.

Now we will reverse the argumentation. Let us assume a solution with accelerated expansion. If we interpret the equation (1) as a relation to calculate $k(t)$ then the corresponding function $k(t)$ has to reflect the properties discussed above. In particular, we will consider a model with a balance between dust matter ($p = 0$) (having the scaling behavior $\rho \sim a^{-3}$) and the expansion. For $t \ll t_0$ and $t \gg t_0$ we have a cosmos of constant radius. As we argue above, in the period around $t_0$ we obtained an accelerated expansion. Now we choose a solution of the Friedmann equation (1) with an accelerated expansion but with a (nearly) constant radius for $t \ll t_0$ and $t \gg t_0$. To reflect all these properties in one function, we choose the solution

$$a(t) = \exp(\tanh(\zeta(t - t_0)))$$

(7)

also visualized in the right figure of Fig. 1. Now we reverse the argumentation and interpret the Friedmann equation to calculate the curvature function $k(t)$

$$k(t) = \frac{\rho_0}{a(t)} - (\dot{a}(t))^2$$
for dust matter. Of course, this approach is only a approximation to visualize our model. Finally we obtain

\[ k(t) = \rho_0 \exp\left(-\tanh(\zeta(t-t_0))\right) - \zeta^2 \left(1 - \tanh^2(\zeta(t-t_0))\right) \exp\left(2 \cdot \tanh(\zeta(t-t_0))\right). \]

This function is plotted in the left figure of Fig. 1 for special parameters confirming the conditions above and the topology change (encoded into the change of the curvature). In particular we obtain again a positive acceleration \( \ddot{a}(t) \) in some time interval. Furthermore the acceleration \( \ddot{a}(t) \) is also negative or the inflation process stops. Finally the topology change of the space

\[ \text{spherical 3-sphere} \rightarrow \text{hyperbolic homology 3-sphere} \quad (8) \]

produces an accelerated expansion of \( a(t) \). We will later prove that the expansion rate has to be exponential which motivates to denote this model as inflation. But in contrast to the usual inflation models, we derive this behavior from first principles using the spacetime \( S^3 \times_\theta \mathbb{R} \) (with a non-standard differential structure). Furthermore in our inflation model, the growing stops, i.e. our inflation is not eternal. In subsection IV B we will discuss the reason for this behavior.

### III. PRELIMINARIES: THE EXOTIC \( S^3 \times \mathbb{R} \)

In this section we will describe the construction and some properties of the exotic \( S^3 \times_\theta \mathbb{R} \). Some background can be found in the book [6] (suitable for physicists) or in the math books [31, 38].

#### A. Smoothness on manifolds

If two manifolds are homeomorphic but non-diffeomorphic, they are exotic to each other. The smoothness structure is called an exotic smoothness structure. The implications for physics are tremendous because we rely on the smooth calculus to formulate field theories. Thus different smoothness structures have to represent different physical situations leading to different measurable results. But it should be stressed that exotic smoothness is not exotic physics. Exotic smoothness is a mathematical structure which should be further explored to understand its physical relevance.

Usually one starts with a topological manifold \( M \) and introduces structures on them. Then one has the following ladder of possible structures:

\[ \text{Topology} \rightarrow \text{piecewise-linear(PL)} \rightarrow \text{Smoothness} \rightarrow \text{bundles, Lorentz, Spin etc.} \rightarrow \text{metric, geometry,...} \]
We do not want to discuss the first transition, i.e. the existence of a triangulation on a topological manifold. The following basic facts should the reader keep in mind for any $n$-dimensional manifold $M^n$:

1. The maximal differentiable atlas $\mathcal{A}$ of $M^n$ is the smoothness structure.

2. To determine a smoothness structure it suffices to give a single maximal differentiable atlas. Thus $\mathbb{R}^n$ has an unique smoothness structure containing the identity map of $\mathbb{R}^n$ (standard smoothness structure, $(\mathbb{R}^n, id_{\mathbb{R}^n})$ is the atlas).

3. It is difficult to define the standard smoothness structure on a general 4-manifold $M$. One way to get around this difficulty is the usage of the instability of all exotic smoothness structures in dimension 4. Stable smoothness structures are able to extend from a smoothness structure on $M$ to $M \times \mathbb{R}^k$ (see [37] for the notation of a stable CAT structure). The classification theory of smoothness structure [37, 51] for all manifolds of dimension greater than 5 implies (together with a result of Quinn [50] about the vanishing of $\pi_4(TOP/O) = 0$) that the smoothness structure of $M \times \mathbb{R}^k$ is unique for all $k > 0$ (up to diffeomorphisms). Here we have to assume that the Kirby-Siebenmann invariant vanishes. We call this smoothness structure the standard smoothness structure of $M \times \mathbb{R}^k$. Then one can extend this smoothness structure to $M$ by restriction. All other possible smoothness structures non-diffeomorphic to the standard smoothness structure are called exotic smoothness structures.

4. The existence of a smoothness structure is necessary to introduce Riemannian or Lorentzian structures on $M$, but the smoothness structure do not further restrict the Lorentz structure.

We want to close this subsection with a general remark: the number of non-diffeomorphic smoothness structures is finite for all dimensions $n \neq 4$ [37]. In dimension four there are many examples of compact 4-manifolds with infinite finite and many examples of non-compact 4-manifolds with uncountable infinite many non-diffeomorphic smoothness structures.

B. Global hyperbolicity and decomposition by handles

Before we start the investigation of the proposed model, we will discuss some more general physical implications. Firstly we consider the existence of a Lorentz metric, i.e. a 4-manifold $M$ (the spacetime) admits a Lorentz metric if (and only if) there is a non-vanishing vector field. In case of a compact 4-manifold $M$ we can use the Poincare-Hopf theorem to state: a compact 4-manifold admits a Lorentz metric if the Euler characteristic vanishes $\chi(M) = 0$. But in a compact 4-manifold there are closed time-like curves (CTC) contradicting the causality or more exactly: the chronology violating set of a compact 4-manifold is non-empty (Proposition 6.4.2 in [34]). Non-compact 4-manifolds $M$ admit always a Lorentz metric and a special class of these 4-manifolds have an empty chronology violating set. If $S$ is an acausal hypersurface in $M$ (i.e., a topological hypersurface of $M$ such that no pair of points of $M$ can be connected by means of a causal curve), then $D^+(S)$ is the future Cauchy development (or domain of dependence) of $S$, i.e. the set of all points $p$ of $M$ such that any past-inextensible causal curve through $p$ intersects $S$. Similarly $D^-(S)$ is the past Cauchy development of $S$. If there are no closed causal curves, then $S$ is a Cauchy surface if $D^+(S) \cup S \cup D^-(S) = M$. As shown in [12], the existence of a Cauchy surface implies that $M$ is diffeomorphic to $S \times \mathbb{R}$.

This strong result is also connected with the concept of global hyperbolicity. A spacetime manifold $M$ without boundary is said to be globally hyperbolic if the following two conditions hold:

1. Absence of naked singularities: For every pair of points $p$ and $q$ in $M$, the space of all points that can be both reached from $p$ along a past-oriented causal curve and reached from $q$ along a future-oriented causal curve is compact.

2. Chronology: No closed causal curves exist (or Causality holds on $M$).

Usually condition 2 above is replaced by the more technical condition Strong causality holds on $M$ but as shown in [11] instead of strong causality one can write simply the condition causality (and strong causality will hold under causality plus condition 1 above).

Then together with the diffeomorphism between $M$ and $S \times \mathbb{R}$ we can conclude that all (non-compact) 4-manifolds $S \times \mathbb{R}$ are the only 4-manifolds which admit a globally hyperbolic Lorentz metric [12]. The existence of a Cauchy surface $S$ implies global hyperbolicity of the spacetime and its unique representation by $S \times \mathbb{R}$ (up to diffeomorphism). But as shown in [10], also the metric is determined (up to isometry) by global hyperbolicity.
Theorem 1 If a spacetime \((M,g)\) is globally hyperbolic, then it is isometric to \((\mathbb{R} \times S, -f \cdot d\tau^2 + g_\tau)\) with a smooth positive function \(f : \mathbb{R} \to \mathbb{R}\) and a smooth family of Riemannian metrics \(g_\tau\) on \(S\) varying with \(\tau\). Moreover, each \(\{t\} \times S\) is a Cauchy slice.

Furthermore in [13] it was shown:

- If a compact spacelike submanifold with boundary of a globally hyperbolic spacetime is acausal then it can be extended to a full Cauchy spacelike hypersurface \(S\) of \(M\), and
- for any Cauchy spacelike hypersurface \(S\) there exists a function as in Th. [1] such that \(S\) is one of the levels \(\tau = \text{constant}\).

But what are the implications of global hyperbolicity in the exotic case? At first, the existence of a Lorentz metric is a purely topological condition which will be fulfilled by all non-compact 4-manifolds independent of the smoothness structure. But by considering global hyperbolicity the picture changes. An exotic spacetime \(M = S \times_\theta \mathbb{R}\) homeomorphic to \(S \times \mathbb{R}\) is not diffeomorphic to \(S \times \mathbb{R}\). The Cauchy surface \(S\) is a 3-manifold with an unique smoothness structure (up to diffeomorphisms) – the standard structure. So, the smooth product \(S \times \mathbb{R}\) must be admit the standard smoothness structure. But the diffeomorphism [12] between \(M\) and \(S \times \mathbb{R}\) is necessary for global hyperbolicity. Therefore an exotic \(S \times_\theta \mathbb{R}\) is never globally hyperbolic but admits a Lorentz metric. Generally we have an exotic \(S \times_\theta \mathbb{R}\) with a Lorentz metric such that the projection \(S \times_\theta \mathbb{R} \to \mathbb{R}\) is a time-function (that is, a continuous function which is strictly increasing on future directed causal curves). But then the exotic \(S \times_\theta \mathbb{R}\) has no closed causal curves and must contain naked singularities [73].

With this result in mind, one should ask for the physical interpretation of naked singularities. To visualize the problem, we consider the following toy model: a non-trivial surface (see Fig. 2) connecting two circles which can be deformed to the usual cylinder. This example can be described by the concept of a cobordism. A cobordism \((W,M_1,M_2)\) between two \(n\)–manifolds \(M_1,M_2\) is a \((n+1)\)–manifold \(W\) with \(\partial W = M_1 \sqcup M_2\) (ignoring the orientation). Then there exists a smooth function \(f : W \to [0,1]\) with isolated critical points (vanishing first derivative) such that \(f^{-1}(0) = M_1, f^{-1}(1) = M_2\). By general position arguments, one can assume that all critical points of \(f\) occur in the interior of \(W\). In this setting \(f\) is called a Morse function on a cobordism. For every critical point of \(f\) (vanishing first derivative) one adds a so-called \(k\)–handle \(D^k \times D^{n-k}\). In our example in Fig. 2 we add a 2-handle \(D^2 \times D^0\) (the maximum) and a 1-handle \(D^1 \times D^1\) (the saddle). But obviously this cobordism is diffeomorphic to the trivial cobordism \(S^1 \times [0,1]\) (the two boundary components are diffeomorphic to each other). Therefore the 2-/1-handle pair is "killed" in this case. The 2-handle and the 1-handle differ in one direction whereas the Morse function has a maximum for the 2-handle and a minimum for the 1-handle. The left graph of Fig. 3 visualizes this fact. Furthermore the sequence of graphs from the left to right represents the process to "kill" the handle pair.

In our example of an exotic \(S \times_\theta \mathbb{R}\), we consider a (non-compact) cobordism between homeomorphic boundary components (the two Cauchy surfaces at infinity, i.e. \("S \times \{-\infty\} \text{ and } S \times \{+\infty\}\)\)). By a result of [48], a cobordism of this kind (a so-called h-cobordism) contains only handles of complement dimension in the interior. But these handles can be killed: the details of the construction can be found in [48]. Here we will give only some general remarks. Any \(0-/1-\)handle pair as well any \(n-/(n+1)\)-handle pair (remember the h-cobordism is \(n+1\)-dimensional) can be killed by a general procedure. The killing of a \(k-/(k+1)\)-handle pair depends on a special procedure, the Whitney trick. For 4- and 5-dimensional h-cobordisms (between 3- and 4-manifolds, respectively) we cannot use the Whitney trick. This failure lies at the heart of the problem to classify 3- and 4-manifolds (see [31]).
In case of the spacetime we are interested in a 4-dimensional h-cobordism between 3-manifolds. Here we can kill the $0−/1$− and the $3−/4$−handle pair of the h-cobordism. As a result we get extra $1−/2$− and $2−/3$−handle pairs. If the Whitney trick works in this case, we can also kill these pairs of handles. But it is known that the Whitney trick only works topologically [24]. The exotic manifolds $S \times_0 \mathbb{R}$ (as non-compact examples) are counterexamples that the (infinite) pairs of handles never cancel each other. Lets start with $1−/2$−handle pair. The critical point of the Morse function with index 1 (the Morse function has a minimum in one directions (saddle point)) corresponds to the 1-handle. Then the Morse function of index 2 corresponds to the 2-handle. Each pair of handles is connected to each other, i.e. the direction representing the minimum of a 1-handle is connected with one direction representing the maximum of the 2-handle. This connection between the 1- and the 2-handle is non-trivial, i.e. there are extra intersection points between the (attaching) sphere of the 1-handle and the (belt) sphere of the 2-handle (see [31] Prop. 4.2.9). The $2−/3$−handle pair is dual to the $1−/2$−handle pair, i.e. the number of $2−/3$−handle pairs is equal to the number of $1−/2$−handle pairs. Each handle of the pair represents the neighborhood of a naked singularity. Then these singularities appear only pairwise. The Morse vector field (gradient of the Morse function) vanishes but we get a saddle having negative curvature (no curvature singularity or quasi-regular singularity, also called locally extendible singularity [21]). There is growing evidence for the appearance of this singularity without violating causality (causal continuity, see [19]). Thus it is very probable that the property of exotic smoothness requiring the appearance of naked singularities in form of non-canceling 2-handle pairs can be interpreted in a consistent physically way: In [9] we merged exotic smoothness with fermions and bosons obtaining the consequence that exotic smoothness implies naked singularities as well as particles. We do not think this is an accident – we conjecture that the naked singularities can be seen as the geometrically consequence of the particles.

C. Constructing the exotic $S^3 \times \mathbb{R}$

Let $\Sigma$ be a homology 3-sphere which do not bound a contractable, smooth 4-manifold. According to Freedman [24], every homology 3-sphere bounds a contractable, topological 4-manifold but not every of these 4-manifolds is smoothable. Now we consider the following pieces: $W_1$ is a cobordism between $\Sigma$ and its one-point complement $\Sigma \setminus pt.$ and $W_2$ is a cobordism between $\Sigma \setminus pt.$ and $\Sigma \setminus pt.$ The manifold $W = \ldots \cup W_2 \cup W_1 \cup W_1 \cup W_2 \cup W_2 \cup W_2 \cup \ldots$ (see [23]) is homeomorphic to $(S^3 \times \mathbb{R})$ (using the proper h-cobordism theorem in [24]) but not diffeomorphic to it, i.e. $W = S^3 \times_0 \mathbb{R}$. The construction of the pieces $W_1, W_2$ rely heavily on the concept of a Casson handle. We do not plan to overload this subsection and shift the definition of a Casson handle to the [3].

Now we consider a Casson handle $CH$ and its 3-stage tower $T^3_{\partial}$. By the embedding theorem of Freedman (see [24], Theorem 1), one can construct another 3-stage tower $T^3_{1}$ inside of $T^3_{\partial}$ (increase the number of self-intersections of the core). This process can be done infinitely. Lets take the example of an homology 3-sphere $\Sigma$ constructed below. The fundamental group is generated by one generator. The attachment of one 2-handle transforms this generator (a non-contractable curve) into a contractable curve but produces a non-trivial 2-handle. This 2-handle can be eliminated by attaching a 3-handle. At the end we obtain a trivial fundamental group and a 2−/3−handle pair is dual to the 1−/2−handle pair. The critical point of the Morse vector field (gradient of the Morse function) vanishes but we get a saddle having negative curvature (no curvature singularity or quasi-regular singularity, also called locally extendible singularity [21]).

For completeness, we will construct one example of a hyperbolic homology 3-sphere which does not bound a contractable 4-manifold. Then we can use the procedure above to get an exotic $S^3 \times_0 \mathbb{R}$. For that purpose we consider a knot $K$, i.e. a smooth embedding $S^1 \rightarrow S^3$. This knot $K$ can be thicken to $\partial N(K) = K \times D^2$ and one obtains the knot complement $C(K) = S^3 \setminus N(K)$ with boundary $\partial C(K) = T^2$. By the attachment of a solid torus $D^2 \times S^1$ using a $-1$ Dehn twist, one obtains the manifold $\Sigma = C(K) \cup (D^2 \times S^1)$, a homology 3-sphere [57]. The geometry of $\Sigma$ is determined by the knot complement $C(K)$. A fundamental result of Thurston [63] states that most knot...
complements are hyperbolic 3-manifolds. Examples are the figure-8 knot 4₁, the 3-twist knot 5₂ or the knot 8₁₀ (in Rolfsen notation [57]). Therefore we have to look for a knot inducing a hyperbolic knot complement and leading to a homology 3-sphere which does not bound a smooth contractable 4-manifold. As Freedman [24] showed, every homology 3-sphere bound a contractable, topological 4-manifold. But Donaldson [15] found the first example, the Poincare sphere, of a homology 3-sphere which fails to do it. The Poincare sphere is generated by the trefoil knot [52]. From the knot-theoretical point of view, we have to look for a knot concordant to the trefoil knot [44].

One example is given by the knot 8₁₀ (see [45] and Fig. 4). Then the homology 3-sphere Σ constructed from this knot has all desired properties.

D. Intermezzo: Geometric structures on 3-manifolds and Mostow rigidity

A connected 3-manifold N is prime if it cannot be obtained as a connected sum of two manifolds N₁ # N₂ (see the appendix A for the definition) neither of which is the 3-sphere S³ (or, equivalently, neither of which is the homeomorphic to N). Examples are the 3-torus T³ and S¹ × S² but also the Poincare sphere. According to [47], any compact, oriented 3-manifold is the connected sum of an unique (up to homeomorphism) collection of prime 3-manifolds (prime decomposition). A subset of prime manifolds are the irreducible 3-manifolds. A connected 3-manifold is irreducible if every differentiable submanifold S homeomorphic to a sphere S² bounds a subset D (i.e. ∂D = S) which is homeomorphic to the closed ball D³. The only prime but reducible 3-manifold is S¹ × S². For the geometric properties (to meet Thurston's geometrization theorem) we need a finer decomposition induced by incompressible tori. A properly embedded connected surface S ⊂ N is called 2-sided [74] if its normal bundle is trivial, and 1-sided if its normal bundle is nontrivial. A 2-sided connected surface S other than S² or D² is called incompressible if for each disk D ⊂ N with D ∩ S = ∂D there is a disk D' ⊂ S with ∂D' = ∂D. The boundary of a 3-manifold is an incompressible surface. Most importantly, the 3-sphere S³, S² × S¹ and the 3-manifolds S³/Γ with Γ ⊂ SO(4) a finite subgroup do not contain incompressible surfaces. The class of 3-manifolds S³/Γ (the spherical 3-manifolds) include cases like the Poincare sphere (Γ = I* the binary icosahedron group) or lens spaces (Γ = Zₚ the cyclic group). Let Kᵢ be irreducible 3-manifolds containing incompressible surfaces then we can N split into pieces (along embedded S²)

\[ N = K₁ # \cdots # Kₙ₁ # S¹ × S² # S³/Γ, \]

where #ₙ denotes the n-fold connected sum and Γ ⊂ SO(4) is a finite subgroup. The decomposition of N is unique up to the order of the factors. The irreducible 3-manifolds K₁, . . . , Kₙ are able to contain incompressible tori and one can split Kᵢ along the tori into simpler pieces K = H ∪ T₂ G [55] (called the JSJ decomposition). The two classes G and H are the graph manifold G and hyperbolic 3-manifold H (see Fig. 5). The hyperbolic 3-manifold H has a torus boundary T² = ∂H, i.e. H admits a hyperbolic structure in the interior only. One property of hyperbolic 3- and 4-manifolds is central: MOSTOW RIGIDITY. As shown by Mostow [50], every hyperbolic n−manifold n > 2 has this property: Every diffeomorphism (especially every conformal transformation) of a hyperbolic n−manifold is induced by an isometry. Therefore one cannot scale a hyperbolic 3-manifold. Then the volume vol( ) and the curvature are topological invariants but for later usages we combine the curvature and the volume into the Chern-Simons invariant CS( ) (see appendix D). Together with the prime and JSJ decomposition

\[ N = (H₁ ∪ T₂ G₁) # \cdots # (Hₙ₁ ∪ T₂ Gₙ₁) # S¹ × S² # S³/Γ, \]
we can discuss the geometric properties central to Thurston's geometrization theorem: Every oriented closed prime 3-manifold can be cut along tori (JSJ decomposition), so that the interior of each of the resulting manifolds has a geometric structure with finite volume. Now, we have to clarify the term “geometric structure”. A model geometry is a simply connected smooth manifold $X$ together with a transitive action of a Lie group $G$ on $X$ with compact stabilizers. A geometric structure on a manifold $N$ is a diffeomorphism from $N$ to $X/G$ for some model geometry $X$, where $\Gamma$ is a discrete subgroup of $G$ acting freely on $X$. It is a surprising fact that there are also a finite number of three-dimensional model geometries, i.e. 8 geometries with the following models: spherical $(S^3, O_4(\mathbb{R}))$, Euclidean $(\mathbb{E}^3, O_3(\mathbb{R}) \times \mathbb{R}^3)$, hyperbolic $(\mathbb{H}^3, O_{1,3}(\mathbb{R})^+)$, mixed spherical-Euclidean $(S^2 \times \mathbb{R}, O_3(\mathbb{R}) \times \mathbb{R} \times \mathbb{Z}_2)$, mixed hyperbolic-Euclidean $(\mathbb{H}^2 \times \mathbb{R}, O_{1,3}(\mathbb{R})^+ \times \mathbb{R} \times \mathbb{Z}_2)$ and 3 exceptional cases called $SL_2$ (twisted version of $\mathbb{H}^2 \times \mathbb{R}$), NIL (geometry of the Heisenberg group as twisted version of $\mathbb{E}^3$), SOL (split extension of $\mathbb{R}^2$ by $\mathbb{R}$, i.e. the Lie algebra of the group of isometries of the 2-dimensional Minkowski space). We refer to [60] for the details.

E. The foliation of the exotic $S^3 \times \theta \mathbb{R}$

A delicate structure for an exotic 4-manifolds is a codimension-1 foliation. In appendix [C], we give a short account in foliations and foliated cobordism including a construction of a codimension-1 foliation of $S^3$ using a polygon in the hyperbolic plane $\mathbb{H}^2$.

Let $W = S^3 \times \theta \mathbb{R}$ be an exotic non-compact 4-manifold admitting a codimension-1 foliation. But the exoticness of $W$ restricts the possible foliations. At first we do not have the obvious foliation with leaves $S^3 \times \{t\}$ and $t \in \mathbb{R}$, otherwise we can use the unique smoothness structure of the 3-sphere to induce the standard smoothness structure on $W$ contradicting the exoticness. Furthermore, by the same reasons, there is no smoothly embedded 3-sphere in $W$. For the construction of the foliation we consider the following decomposition of the 3-sphere:

$$S^3 = \left( S^3 \setminus \bigcup_n (D^2 \times S^1) \right) \cup_{T^2} \bigcup_n (D^2 \times S^1)$$

used in the Dehn surgery (see subsection [III C]). In the notation of the previous subsection, the manifold $W_1$ is a cobordism between $\Sigma$ and $\Sigma \setminus pt$. which is homeomorphic to a cobordism between $\Sigma$ and $S^3 \setminus pt$. In the construction of $\Sigma$ above we used Dehn surgery along the knot $K = S_{10}$. Dually we can decompose $\Sigma$ like

$$\Sigma = (S^3 \setminus (D^2 \times S^1)) \cup_{T^2, -1} (K \times D^2)$$

where $\cup_{T^2, -1}$ denotes the $-1$ Dehn twist. Now we choose the $n$–component link

$$L_{\Sigma} = K \sqcup_{n-1} S^1$$

and write for $\Sigma$

$$S^3 = \left( S^3 \setminus \bigcup_n (D^2 \times S^1) \right) \cup_{T^2, -1} (D^2 \times L_{\Sigma}) .$$

Therefore $\Sigma$ and $S^3$ agreed on the subset $(S^3 \setminus \bigcup_n (D^2 \times S^1))$ and we are able to construct a cobordism $W_3$ with

$$\partial W_3 = \left( S^3 \setminus \bigcup_n (D^2 \times S^1) \right) \sqcup - \left( S^3 \setminus \bigcup_n (D^2 \times S^1) \right) .$$

Figure 5: Torus (JSJ-) decomposition, $H_i$ hyperbolic manifold, $S_i$ Graph-manifold, $T_i$ Tori
A foliation of $W_3$ cannot be the trivial one, otherwise it will contradict the exotic smoothness structure. But we can choose a foliation as a foliated cobordism inducing a foliation of $(S^3 \setminus \bigcup_n (D^2 \times S^1))$. From the topological point of view, $W_3$ is given by

$$ W_3 = \left( S^3 \setminus \bigcup_n (D^2 \times S^1) \right) \times [0, 1]. $$

The foliation of $W_3$ is induced from the foliation of the boundary, i.e. a foliation of $(S^3 \setminus \bigcup_n (D^2 \times S^1))$. But this foliation is constructed from a foliation of a polygon $P$ with $n$ vertices in the hyperbolic plane $\mathbb{H}^2$ (see appendix C). We will describe this foliation in the appendix. These foliations are related to (geodesic) laminations of the disk (see Fig. 6 for an example). The leaves of the foliation of the polygon $P$ are curves $\gamma$ starting and ending at the boundary. Then the leaves of the foliation of $(S^3 \setminus \bigcup_n (D^2 \times S^1))$ are of the form $\gamma \times S^1$. Finally the leaves of $W_3$ are $\gamma \times S^1 \times [0, 1]$. The remaining components of the exotic $W = S^3 \times_\theta \mathbb{R}$ are a cobordism $W(L_\Sigma)$ between $D^2 \times L_\Sigma$ and $\bigcup_n D^2 \times S^1$ equipped with Reeb foliations (see appendix C and Fig. 9). We will later see that this cobordism is the topological expression for the creation of matter (see subsection V A).

IV. THE PROPERTIES OF THE INFLATION MODEL AND ITS DESCRIPTION

In the model above, we assume implicitly that the transition from $k = +1$ to $k = -1$ is an abrupt process. But topology never claimed how fast is a concrete transition. Therefore we have to analyze the structure of the spacetime (i.e. $S^3 \times_\theta \mathbb{R}$) from the geometrical and topological point of view to determine this transition.

In the following we will use the strategy:

1. The exotic $S^3 \times_\theta \mathbb{R}$ is build from a hyperbolic homology 3-sphere $\Sigma$ which does not bound a smooth contractable 4-manifold. Part of the construction is a suitable embedded Casson handle (or at least a 3-stage tower). We refer to appendix B for a description of the Casson handle.

2. The change from $S^3$ to $\Sigma$ is described by infinite pairs of $1 - /2$–handles (or dually by $2 - /3$–handle pairs) which is canceled by the Casson handle allowing a Morse-theoretic description by using a scalar field.

3. Every stage of the Casson handle is a collection of immersed 2-disks. The immersed disk is also described by a canceling $1 - /2$–handle pair. From the Morse-theoretical point of view, every immersed disk is a pair of saddle points with a contribution to the negative curvature.

4. For every stage $N$ of the Casson handle we obtain a contribution

$$ \frac{1}{N!} \left( \frac{3 \cdot Vol(\Sigma)}{CS(\Sigma)} \right)^N $$

to the curvature, which is expressed by the Chern-Simons invariant (see D). The combinatorial factor is related to the embedding of the Casson handle (using a hyperbolic metric).
5. The embedding of the Casson handle uses a hyperbolic metric and we obtain an exponential increase of the curvature. The whole process (starting with the 3-sphere) stops after the formation of Σ (by using Mostow rigidity, see subsection III.D). If the 3-sphere has radius \( L_\rho \) (the Planck length) the we will obtain for the radius \( a_0 \) of Σ

\[
a_0 = L_\rho \cdot \exp \left( \frac{3 \cdot \text{Vol}(\Sigma)}{2 \cdot CS(\Sigma)} \right)
\]

A. Morse-theoretic description of the topological transition \( S^3 \to \Sigma \)

As explained above the transition \( S^3 \to \Sigma \) (written as cobordism \( W_1 \), see above) is an expression of the exotic smoothness structure. Every topological transition is reflected in the critical values of the Morse function. The difference between \( S^3 \) and \( \Sigma \) is represented by the fundamental group \( \pi_1(S^3) = 0 \) and \( \pi_1(\Sigma) \neq 0 \). Every generator of the fundamental group \( \pi_1(\Sigma) \) can be “killed” by a pair of 2-/3-handles producing two critical values in the Morse function (of index 2 and 3, respectively). These critical points are naked singularities (see the discussion in subsection III.B). The Morse vector field (gradient of the Morse function) vanishes but we get a saddle having negative curvature (no curvature singularity or quasi-regular singularity, also called locally extendible singularity [21]). There is growing evidence for the appearance of this singularity without violating causality (causal continuity, see [19]). A pair of 2-/3-handles can be canceled making the interior of the cobordism \( W_1 \) to a simple-connected 4-manifold. But to start the cancellation process we need a special embedded disk or better a Casson handle. The existence of the exotic smoothness structure forbids the cancellation of the handle pairs in the Casson handle. The generic case is given by the following model. Clue a 1-/2-handle pair (dual to a 2-/3-handle pair) to a 0-handle. The 0-handle is modeled by a minimum in Morse theory. But the 1-/2-handle pair generates a maximum and a minimum which can be canceled (see Fig. 5). Unfortunately exotic smoothness forbids this cancellation and we obtain an extra pair of maximum/minimum. Let us consider the commutative diagram,

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\psi} & \mathbb{R} \\
\phi \downarrow & \circlearrowleft & \uparrow id \\
S^3 & \xrightarrow{\psi} & \mathbb{R}
\end{array}
\]

The map \( \psi \) is a Morse function of the 3-sphere \( S^3 \) and \( \Psi \) is the Morse function of \( \Sigma \). At the 3-sphere there are only two critical points: one maximum and one minimum (the two poles). A short calculation of the homology groups of the cobordism \( W_1 \) between \( S^3 \) and \( \Sigma \) shows that the homology-groups \( H_0(M) = H_3(M) = \mathbb{Z} \) are the only non-trivial groups of the cobordism. Furthermore the evaluation of the exact sequence of the pair \((W_1, \partial W_1)\) gives the result that \( H_0(W_1) \) is generated by one of the boundary components, e.g. by \( H_0(S^3) \) and \( H_3(W_1) \) by the other one, i.e. \( H_3(\Sigma) \). The transition \( y = \phi(x) \) represented by \( W_1 \) maps the Morse function \( \psi(y) = ||y||^2 \) on \( S^3 \) to the Morse function \( \Psi(x) = ||\phi(x)||^2 \).

A field-theoretic discussion of the Morse-theoretic result starts with the following simple idea: The 3-sphere is isomorphic to \( SU(2) \) and so one can define a formal group-operation on \( S^3 \). Then the map \( \phi : \Sigma \to S^3 \) is a map \( \phi : \Sigma \to SU(2) = \mathbb{R} \otimes SU(2) \) which can be interpreted as a \( SU(2) \)-valued scalar-field over \( \Sigma \). Witten has derived the Morse-theory from the field-theoretic properties [63]. His ansatz gives a field-theoretic construction (a \( \sigma \) model) of the dynamics leading to a “tunneling path” between two critical points. In our case, the field of the Witten-Construction is the \( SU(2) \)-scalar \( \phi \) over \( \Sigma \). The 1-/2-handle pair above and the 0-handle is represented by a Morse function with two minima and one maxima:

\[
\psi(y) = ||y||^4 - ||y||^2
\]

on \( S^3 \). But we describe the transition \( S^3 \to \Sigma \) by a \( SU(2) \)-valued scalar field \( \phi : \Sigma \to SU(2) \) and obtain

\[
\Psi(x) = \phi(x)^4 - \phi(x)^2
\]

the Morse function at \( \Sigma \) (where we substitute \( ||\phi||^n \) by \( \phi^n \)). We remark that there is a certain freedom in the choice of the potential. Here we assume the equality of the two minima but a potential like

\[
\Psi(x) = \phi(x)^4 - \phi(x)^3 - \phi(x)^2
\]

is also possible. Currently we have no idea to assign the potential. But we interpret this Morse function as a potential for the scalar field and obtain the expression

\[
\mathcal{L}_\phi = D_\mu \phi \cdot D^\mu \phi + V(\phi) = D_\mu \phi \cdot D^\mu \phi + \frac{\partial}{2} (\phi^4 - \phi^2),
\]
as Lagrangian of the scalar field where $\rho_\phi$ is the energy density of $\phi$ which is directly related to the curvature of the Morse function. The coupling to gravity can be realized by adding the Einstein-Hilbert action. Obviously this model has a symmetry-breaking phase whose meaning will be investigated in a forthcoming paper. If we are able to cancel the handle pair then the Morse function is reduced to the Morse function for the 0-handle, i.e.

$$\psi(y) = ||y||^2$$

and we obtain the potential $V(\phi) = \phi^2$ in the Lagrangian (11). The cancellation process can be used to obtain the energy of symmetry breaking (see subsection IV.C).

B. Curvature-contribution of the Casson handle

The idea of a Casson handle can be simply expressed: to kill the $1-\frac{1}{2}$--handle pair one needs an embedded disk. But in dimension 4, it fails and one obtains an immersed disk (a disk with self-intersections) only. This immersed disk (also represented by a canceling $1-\frac{1}{2}$--handle pair) can be changed to an embedded disk by embedding a disk again etc. The Casson handle is the expression of this procedure ad infinitum. For more details about Casson handles we refer to appendix B.

The following points are central in the following argumentation: firstly the appearance of the Chern-Simons invariant defined by the scalar curvature (therefore related to the geometry) and secondly the usage of an infinite construction, the so-called Casson handle, to obtain the topological structure of the transition $S^3$ to $\Sigma$. The first point, the Chern-Simons invariant, will determine the scale factor and the second point, the Casson handle, will give the exponential behavior.

Step 1: As shown by Witten [66–68], the action

$$\int_{\Sigma}^{3} R \sqrt{h} \, d^3x = L \cdot CS(\Sigma)$$

(12)

for every 3-manifold $\Sigma$ is related to the Chern-Simons invariant $CS(\Sigma)$ (see D). We argued above that, because of Mostow rigidity, $\Sigma$ has an invariant volume. Then the scaling factor $L$ is independent of the volume and we obtain

$$L \cdot CS(\Sigma_0, A) = L^3 \cdot \frac{CS(\Sigma)}{L^2} = \int_{\Sigma} \frac{CS(\Sigma)}{L^2 \cdot vol(\Sigma)} \sqrt{h} \, d^3x$$

(13)

by using

$$L^3 \cdot vol(\Sigma) = \int_{\Sigma} \sqrt{h} \, d^3x.$$  

Together with (5), we can compare the kernels of the integrals (12) and (13) to get for every time

$$\frac{3k}{a^2} = \frac{CS(\Sigma)}{L^2 \cdot vol(\Sigma)}.$$  

Finally we obtain the scaling factor

$$\vartheta = \frac{a^2}{L^2} = \frac{3 \cdot vol(\Sigma)}{|CS(\Sigma)|}$$

(14)

where we set $k = -1$ which cancels the negative sign of $CS(\Sigma)$. We set $CS(\Sigma)$ instead of $|CS(\Sigma)|$ in the following.

Step 2: The hyperbolic structure (induced by the hyperbolic structure of $\Sigma$) is an (geometric) expression for an accelerated expansion. It is an amazing fact that 4-dimensional hyperbolic structures show also Mostow rigidity (see subsection III D). Then we obtain a constant term in the action and in the Friedmann equation

$$\left(\frac{a}{\dot{a}}\right)^2 = \frac{1}{L^2}$$

with respect to the length scale $L$ of the hyperbolic structure. In the following we will work with quadratic expressions because we will mainly argue with the curvatures. Then we obtain

$$da^2 = \frac{a^2}{L^2} \, dt^2 = \vartheta \, dt^2$$

(15)
by using the scale $\vartheta$.

Step 3: Therefore we have to understand which submanifold is the source of the negative curvature. In the construction of the exotic $S^3 \times \mathbb{R}$ (see subsection IIIC) we used a non-trivial Casson handle. A Casson handle is an infinite construction of failures to embed a disk best described by a tree: the vertex is a kinky handle (a thickened disk with self-intersections) with branches (the parts away from the self-intersection) representing the branches. A schematic picture of the first three stages can be found in Fig. 7. Alternatively, a Casson handle can be described by a tree of 1-/2-handle pairs (see [31]). A 1- and a 2-handle is represented locally by the Morse functions, respectively,

$$x^2_0 + x^2_1 + x^2_2 - x^2_3, \quad x^2_0 + x^2_1 - x^2_2 - x^2_3,$$

i.e. by saddles having negative curvature (see Fig. 8). Now, the idea of the calculation is the sum over these negative curvature using the local relation (15) to arrive at a formula for $a(t)$ during the inflation process.

Step 4: By using the tree of the Casson handle, we obtain a countable infinite sum of contributions for (15). Before we start we will clarify the geometry of the Casson handle. The discussion of the Morse functions above uncovers the hyperbolic geometry of the Casson handle. Therefore the tree corresponding to the Casson handle must be interpreted as a metric tree with hyperbolic structure in $\mathbb{H}^2$ and metric $ds^2 = (dx^2 + dy^2)/y^2$. The direction of the increasing levels $n \to n+1$ is identified with $dy^2$ and $dx^2$ is the number of edges for a fixed level with scaling parameter $\vartheta$. The contribution of every level in the tree is determined by the previous level best expressed in the scaling parameter $\vartheta$.

An immersed disk at level $n$ needs at least one disk to resolve the self-intersection point. This disk forms the level $n+1$ but this disk is contained in the previous disk. So we obtain for $da^2|_{n+1}$ at level $n+1$

$$da^2|_{n+1} \sim \vartheta \cdot da^2|_n$$

up to a constant. By using the metric $ds^2 = (dx^2 + dy^2)/y^2$ with the interpretation $(y^2 \to n+1, dx^2 \to \vartheta)$ we obtain for the change $dx^2/y^2$ along the $x-$direction (i.e. for a fixed $y$) $\vartheta^{n+1}/(n+1)!$. This change determines the scaling from the level $n$ to $n+1$, i.e.

$$da^2|_{n+1} = \frac{\vartheta}{n+1} \cdot da^2|_n = \frac{\vartheta^{n+1}}{(n+1)!} \cdot da^2|_0$$

and after the whole summation (as substitute for an integral for the discrete values) we obtain for the relative scaling

$$a^2 = \sum_{n=0}^{\infty} (da^2|_n) = a^2_0 \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \vartheta^n = a_0 \cdot \exp (\vartheta) = a_0 \cdot l_{\text{scale}}$$ (16)
with $da^2_0 = a_0^2$. So, if we start with the radius $a_0$ of $S^3$ and arrive at $a$ for $\Sigma$ driven by inflation then we will get

$$a = a_0 \cdot \exp \left( \frac{\vartheta}{2} \right) = a_0 \cdot \exp \left( \frac{3 \cdot \text{vol}(\Sigma)}{2 \cdot CS(\Sigma)} \right).$$

The formula reflects only the changes made to obtain $\Sigma$. That is the reason for the appearance of the Chern-Simons invariant $CS(\Sigma)$ only. If we set the starting 3-sphere of Planck size $a_0 = L_P$ then one obtains

$$a = L_P \cdot \exp \left( \frac{3 \cdot \text{vol}(\Sigma)}{2 \cdot CS(\Sigma)} \right).$$

(17)

C. Energy, time and expansion scale

Now we will calculate the scaling factor for our inflation model above. Therefore we choose $\Sigma = \Sigma(8_{10})$ and using the package SnapPea by J. Weeks[75] to obtain the values for $\text{vol}(\Sigma(8_{10}))$ and $CS(\Sigma(8_{10}))$:

$$\text{vol}(\Sigma(8_{10})) = 8.65115...$$
$$CS(\Sigma(8_{10})) = 0.15616...$$

leading to the expansion factor

$$l_{\text{scale}} = \exp \left( \frac{3 \cdot \text{vol}(\Sigma)}{2 \cdot CS(\Sigma)} \right) \approx \exp(83.131...) \approx 1.3 \cdot 10^{36}.$$

This factor (more than 60 $e$-folds) is enough to explain the homogeneity and the flatness of the universe. For the energy scale we need the Morse-theoretic model from subsection [IV A]. In our model we have a simple interpretation of the potential $\varphi^4 - \varphi^2$ for the $SU(2)$--valued scalar field $\varphi$. If the $1 - /2$--handle pair can be canceled then the potential will be reduced to $\varphi^4$ or (by a simple transformation) to $\varphi^2$. But then there is no symmetry-breaking phase. So, if we cancel the handle pair then we stop the inflation. The cancellation of the handle pair represents the energy which is needed to implement the inflation. Therefore we have to determine the number of levels which are needed to cancel the handle pair. In [25] Freedman showed that three levels are needed to construct a disk which cancels the handle pair. Therefore the first three levels determine the energy of the symmetry breaking needed for the inflation process. This argument gives the energy scaling factor

$$e_{\text{scale}} = \sum_{n=0}^{3} \frac{\vartheta^n}{n!} = 1 + \frac{\vartheta}{1} + \frac{\vartheta^2}{4} + \frac{\vartheta^3}{6}$$

reducing the energy $E_0$ (before the symmetry breaking) to

$$E_{\text{scale}} = \frac{E_0}{e_{\text{scale}}}.$$

For the scaling

$$\vartheta = \frac{3 \cdot \text{vol}(\Sigma)}{2 \cdot CS(\Sigma)} \approx 83.131...$$

we obtain the energy scaling factor

$$e_{\text{scale}} = 1 + \frac{\vartheta}{1} + \frac{\vartheta^2}{2} + \frac{\vartheta^3}{6} \approx 115172.2606$$

(18)

as a sum of the first three contributions of the Casson handle. Assuming the Planck energy at the beginning $E_0 = E_{\text{Planck}}$, then the energy

$$E_{\text{scale}} = \frac{E_{\text{Planck}}}{e_{\text{scale}}} \approx 1.0591 \cdot 10^{14} \text{GeV}$$

(19)

is obtained. For the determination of the time scale, we have to look for the periodic structure in the Casson handle. By using Freedmans reembedding theorems [24], we know that a Casson handle consists of a 5-level substructures...
which build a Casson handle\[76\]. Therefore the periodic structure is given by these 5-levels (the ”atoms” of the Casson handle). Then we obtain for the time scale

\[
t_{\text{scale}} = \sum_{n=0}^{5} \frac{\vartheta^n}{n!} \approx 3.5 \cdot 10^7
\]

and if the inflation starts at Planck time

\[
T_{\text{scale}} = T_{\text{Planck}} \cdot t_{\text{scale}} \approx 3 \cdot 10^{-36} \text{s}
\]

which also agrees with the current measurements.

V. REHEATING

As explained in the previous sections, if there is an transition from a spherical geometry \( (k = +1) \) to a hyperbolic geometry \( (k = -1) \) then there is a period of exponential growth. After this period there is the usual expansion. But now we have a problem: hyperbolic 3-manifolds do not scale! This property, also called Mostow rigidity (see subsection III D), is central for hyperbolic manifolds. In particular, the volume of hyperbolic 3-manifolds is a topological invariant. Above we used this property to explain why inflation ends after a finite time. So, after the formation of the hyperbolic 3-manifold there is no further expansion. But there is a simple way to overcome this difficulty. Let \( \Sigma \) be the hyperbolic homology 3-sphere. The 3-manifold

\[
\Sigma \# P = (\Sigma \setminus D^3) \cup S^2 \left( P \setminus D^3 \right)
\]

as connected sum of \( \Sigma \) and a non-hyperbolic homology sphere \( P \) is a solution to this problem. The hyperbolic 3-manifold \( \Sigma \) cannot scale (by Mostow rigidity) but the other homology 3-sphere \( P \) can change its size. The usage of a 3-sphere instead of \( P \) is also excluded because \( \Sigma \# S^3 = \Sigma \). If the curvature of \( \Sigma \# P \) is dominated by the negative curvature of the hyperbolic 3-manifold \( \Sigma \) then we obtain also the inflation as described above. Therefore in a realistic theory of geometric inflation we need a more complex 3-manifold. But here we will stop making these remarks and go over to the reheating.

A. Matter coupling

The inflation, as a process to form \( \Sigma \), will stop after the formation of \( \Sigma \) (as transition from the 3-sphere \( S^3 \)). Therefore we have to understand the meaning of \( \Sigma \). The appearance of \( \Sigma \) in \( S^3 \times \theta \mathbb{R} \) is a sign for the exotic smoothness structure. In [9] we studied the effect of exotic smoothness to the Einstein-Hilbert action. The main claim was the derivation of the Dirac and the Yang-Mills action using the properties of the exotic smoothness structure. In particular we identified special 3-manifolds with fermions and bosons:

1. Fermions: complements \( S^3 \setminus (K \times D^2) \) of a knot \( K \), the complement is a hyperbolic 3-manifold with boundary a torus

2. Bosons: vector (gauge) bosons as torus bundles \( (T^2 \times [0,1]) \) as local structure, no hyperbolic geometry.

We will shortly introduce the model now applied to our \( S^3 \times \theta \mathbb{R} \). In subsection IIIE we described the foliation of \( S^3 \times \theta \mathbb{R} \). In particular we obtained a cobordism \( W(L_\Sigma) \) between \( D^2 \times L_\Sigma \) and \( \bigcup_n D^2 \times S^1 \). \( n - 1 \) components of the link \( L_\Sigma \) are trivial and not linked with each other. Therefore the cobordism \( W(L_\Sigma) \) reduces to a cobordism \( W(K) \) between \( N(K) = K \times D^2 \) and \( S^1 \times D^2 \) (solid torus). In section III we used the Einstein equation [3] to derive some properties of the model. But this equation is obtained from the Einstein-Hilbert action and we considered the restriction

\[
S_{\text{EH}}(W(K)) = \int_{W(K)} R \sqrt{g} d^4x
\]

to the cobordism \( W(K) \). Furthermore we restrict the action to a small neighborhood \( N(K) \times [0,1] \) of the boundary \( N(K) \) using a product metric.
\[ ds^2 = d\theta^2 + h_{ik}dx^i dx^k \]  
with coordinate \( \theta \) on \([0, 1]\) and metric \( h_{ik} \) on \( N(K) \). By the ADM formalism with the lapse \( N \) and shift function \( N^i \) one gets a relation between the 4-dimensional \( R \) and the 3-dimensional scalar curvature \( R_{(3)} \) (see [49] (21.86) p. 520)

\[ \sqrt{g} R \, d^4x = N\sqrt{h} \, (R_{(3)} + |n|^2((trK)^2 - trK^2)) \, d\theta \, d^3x \]

with the normal vector \( n \) and the extrinsic curvature \( K \). The embedding \( N(K) \hookrightarrow N(K) \times [0, 1] \) can be chosen so that \( K = \text{const.} \). Then we obtain the action

\[ S_{EH}(N(K) \times [0, 1]) = \int_{N(K) \times [0, 1]} R \sqrt{g} d^4x = \int_{N(K)} R_{(3)} \sqrt{h} N \, d^3x = S_{EH}(N(K)). \]

The integral over \( N(K) = K \times D^2 \) is completely determined by the boundary (the disk is flatly embedded). It can be calculated as a term over the boundary \( \partial N(K) = K \times S^1 \), a knotted torus, i.e. we obtain

\[ S_{EH}(N(K)) = \int_{N(K)} R_{(3)} \sqrt{h} N \, d^3x = \int_{\partial(N(K))} X \sqrt{\gamma} d^2x \]

\[ = S_{EH}(\partial(N(K))) \]

where \( X \) is a 2-dimensional expression for the boundary term of the Einstein-Hilbert action. We will use the same symbol for the 2-dimensional metric \( h \) and its restriction to the boundary submanifold. Now we are looking for the action at the boundary. As shown by York [69], the fixing of the conformal class of the spatial metric in the ADM formalism leads to a boundary term which can be also found in the work of Hawking and Gibbons [27]. Also Ashtekar et al. [3] [4] discussed the boundary term in the Palatini formalism. The main reason for the introduction of the boundary term came from the Hamiltonian formulation of Einstein's theory. It has been known since the 1960's (see [49] section 21.4-21.8) that in the Hamiltonian quantization of gravity it is essential to include boundary terms in the action, as this allows to define consistently the momentum conjugate to the metric. This makes it necessary to modify the Einstein-Hilbert action by adding to it a surface integral term so that the variation of the action becomes well defined and yields the Einstein field equations. All these discussions enforce us to choose the following action term at the boundary \( \partial(N(K)) \)

\[ S_{EH}(\partial(N(K))) = \int_{\partial(N(K))} H_\partial \sqrt{\gamma} d^2x \]

(22)

with \( H_\partial \) as mean curvature of \( \partial(N(K)) \), i.e. the trace of the second fundamental form. If we consider \( \partial(N(K)) \times [0, 1] \) then \( H_\partial \) is constant along \([0, 1]\) and we obtain the integral relation

\[ S_{EH}(N(K) \times [0, 1]) = S_{EH}(\partial N(K) \times [0, 1]) = \int_{\partial(N(K)) \times [0, 1]} H_\partial \sqrt{\gamma} d^2x \, d\theta \]

(23)

using the product metric (20) and the splitting (21). The action (23) above is completely determined by the knotted torus \( \partial N(K) = K \times S^1 \) and its mean curvature \( H_{\partial N(K)} \). This knotted torus is an immersion of a torus \( S^1 \times S^1 \) into \( \mathbb{R}^3 \). The well-known Weierstrass representation can be used to describe this immersion. As proved in [26] [10] there is an equivalent representation via spinors. This so-called Spin representation of a surface gives back an expression for \( H_{\partial N(K)} \) and the Dirac equation as geometric condition on the immersion of the surface. As we will show below, the term (23) can be interpreted as Dirac action of a spinor field. Here we have an immersion of a torus \( I : T^2 = S^1 \times S^1 \to \mathbb{R}^3 \) with image the knotted torus \( im(I) = T(K) = K \times S^1 \) that is the boundary \( \partial N(K) \) of \( N(K) \). This immersion \( I \) can be defined by a spinor \( \varphi \) on \( T^2 \) fulfilling the Dirac equation

\[ D\varphi = H\varphi \]

(24)

with \( |\varphi|^2 = 1 \) (or an arbitrary constant) (see Theorem 1 of [26]). As discussed above a spinor bundle over a surface splits into two sub-bundles \( S = S^+ \oplus S^- \) with the corresponding splitting of the spinor \( \varphi \) in components

\[ \varphi = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} \]
and we have the Dirac equation

\[ D \varphi = \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix} \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} = H \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} \]

with respect to the coordinates \((z, \bar{z})\) on \(T^2\).

In dimension 3, the spinor bundle has the same fiber dimension as the spinor bundle \(S\) (but without a splitting \(S = S^+ \oplus S^-\) into two sub-bundles). Now we define the extended spinor \(\phi\) over \(T^2 \times [0, 1] = S^1 \times S^3 \times [0, 1]\) via the restriction \(\phi|_{T^2} = \varphi\). The spinor \(\phi\) is constant along the normal vector \(\partial_N \phi = 0\) fulfilling the 3-dimensional Dirac equation

\[ D^{3D} \phi = \begin{pmatrix} \partial_N & \partial_z \\ \partial_{\bar{z}} & -\partial_N \end{pmatrix} \phi = H \phi \]

(25)

induced from the Dirac equation (24) via restriction and where \(|\phi|^2 = \text{const.}\). Especially one obtains for the mean curvature of \(\partial N(K) \times [0, 1] = K \times S^3 \times [0, 1]\) (up to a constant from \(|\phi|^2\))

\[ H = \bar{\phi} D^{3D} \phi. \]

(26)

This mean curvature can be put into the action (23) to obtain the 3-dimensional Dirac action

\[ S_{EH}(\partial N(K) \times [0, 1]) = \int_{[0,1] \times \partial N(K)} \bar{\phi} D^{3D} \phi \sqrt{g} d\theta d^2x. \]

(27)

For the extension of this action to 4 dimensions, we consider a slightly more general case. Let \(\iota : \Sigma \hookrightarrow M\) be an immersion of the 3-manifold \(\Sigma\) into the 4-manifold \(M\) with the normal vector \(\vec{N}\). The spin bundle \(S_M\) of the 4-manifold splits into two sub-bundles \(S^\pm_M\) where one subbundle, say \(S^+_M\), can be related to the spin bundle \(S^\Sigma\) of the 3-manifold. Then the spin bundles are related by \(S^\Sigma = \iota^* S^+_M\) with the same relation \(\phi = \iota_* \Phi\) for the spinors \((\phi \in \Gamma(S^\Sigma)\) and \(\Phi \in \Gamma(S^+_M))\). Let \(\nabla^\Sigma_X, \nabla^S_X\) be the covariant derivatives in the spin bundles along a vector field \(X\) as section of the bundle \(T\Sigma\). Then we have the formula

\[ \nabla^\Sigma_X(\Phi) = \nabla^S_X \phi - \frac{1}{2}(\nabla_X \vec{N}) \cdot \vec{N} \cdot \phi \]

(28)

with the obvious embedding \(\phi \mapsto \begin{pmatrix} \phi \\ 0 \end{pmatrix} = \Phi\) of the spinor spaces. The expression \(\nabla_X \vec{N}\) is the second fundamental form of the immersion where the trace \(\text{tr}(\nabla_X \vec{N}) = 2H\) is related to the mean curvature \(H\). Then from (28) one obtains a similar relation between the corresponding Dirac operators

\[ D^{4D} \Phi = D^{3D} \phi - H \phi \]

(29)

with the Dirac operator \(D^{3D}\) defined via (25). Together with equation (25) we obtain

\[ D^{4D} \Phi = 0 \]

(30)

i.e. \(\Phi\) is a parallel spinor. This Dirac equation is obtained by varying the action

\[ \delta \int_M \bar{\Phi} D^{4D} \Phi \sqrt{g} d^4x = 0 \]

(31)

Importantly, this variation has a different interpretation in contrast to varying the 3-dimensional action. Both variations look very similar. But in (31) we vary over smooth maps \(\Sigma \to M\) which are not conformal immersions (i.e. represented by spinors \(\Phi\) with \(D^{4D} \Phi \neq 0\)). Only the choice of the extremal action selects the conformal immersion among other smooth maps. Especially the spinor \(\Phi\) (as solution of the 4-dimensional Dirac equation) is localized at the immersed 3-manifold \(\Sigma\). The 3-manifold \(\Sigma\) moves along the normal vector (see the relation (28) between the covariant derivatives representing a parallel transport). Therefore the 3-dimensional action (27) can be extended to the whole 4-manifold (but for a spinor \(\Phi\) of fixed chirality). Especially we have a unique fermionic action on the manifold \(M\).
Applied to our example, we obtain for the Einstein-Hilbert action

$$S_{EH}(N(K) \times [0,1]) = \int_{N(K) \times [0,1]} \bar{\Phi} D^{4D} \Phi \sqrt{g} \, d^4x$$

the 4-dimensional Dirac action. But then we have the close relation between the knot

Knot $K = 8_{10} \leftrightarrow$ Spinor $\Phi$

forming our result

The formation of the hyperbolic homology 3-sphere $\Sigma$ produces spinorial matter.

But we may ask: Is this kind of matter the correct baryonic matter in the cosmological sense? By using the hyperbolic structure on $\Sigma$ we can simple claim: Yes. Hyperbolic 3-manifolds do not scale by Mostow rigidity (see subsection [III D]). So, if we consider the model $\Sigma#P$ above, then the size of $P$ can be increase whereas $\Sigma$ remains constant.

Therefore an energy density associated to $\Sigma$ scales like $a^{-3}$ if $a$ is the scale of $P$. We obtain the correct scaling behavior of matter in cosmology. The appearance of interactions demand a more complex knot and we refer to our paper [9] for the details.

B. Supercooled expansion

For a discussion of the temperature, we have to consider the degrees of freedom. In a topological model, one usually considers the topological degrees, i.e. the topology-determining submanifolds. In case of 3-manifolds, the fundamental group is the main structure. For $S^3 \to \Sigma = \Sigma(8_{10})$, we have the transition of the trivial fundamental group $\pi_1(S^3) = 0$ (containing only the unit element, a contractable closed curve) to the infinite fundamental group

$$\pi_1(\Sigma(8_{10})) = \langle a, b, c, d, f \mid d = adfc; b^{-1} = adb^{-1}af = df^{-1}d; ac^{-1}a^{-1}c = f^{-1}; a^2d = ca^{-1}bafbdca^{-1}c \rangle$$

with five generators (and four relations). The new degrees of freedom are cause the lowering of the temperature during the inflation phase. We obtain a supercooled expansion. The correct temperature value is directly related to the energy scale [18] by the Boltzmann constant $k_B$. Explicitly we have the energy fraction

$$\frac{E_0}{E_1} = e^{\text{scale}}$$

between the energy $E_0$ at the beginning and the energy $E_1$ at the end of the inflation. But this formula is identical to [19] and we obtain the temperature fraction

$$\frac{E_0}{E_1} = \frac{T_0}{T_1} = e^{\text{scale}} = 1 + \vartheta + \vartheta^2 + \frac{\vartheta^3}{4} \approx 115172.2606$$

using [18], $E = k_B \cdot T$ and the value for $\vartheta \approx 83.131...$. Then we obtain the decrease of the temperature of about 115000 K (from the Planck temperature $10^{32}$ K to $10^{27}$ K) in agreement with the usual inflation models.

C. Reheating

At the end of the inflation, the hyperbolic homology 3-sphere $\Sigma$ is formed. But by the results of subsection [IV A] matter is created now. The inflation phase ends and the energy is now transferred to the matter. For an explicit construction of the dynamics, we have to use our Morse-theoretic model of subsection [IV A]. In the process to cancel the handle pair we have to use a Casson handle. We learned above that 3-levels of the Casson handle are enough to cancel the handle pair but at the same time new 1-/2-handle pairs appear. For the following argumentation we consider the simplest Casson handle (represented by the unbranched tree $T_0$), i.e. an unbranched chain of 1-/2-handle pairs. As explained in subsection [IV A], the cancellation of the handle pair is equivalent to the potential $V(\phi) = \phi^2$ but the Casson handle introduces new (infinite many) 1-/2-handle pairs. From the Morse-theoretic point of view, these handles are attached to the critical point of $\phi^2$, i.e. to $\phi = 0$. The Morse function $\Psi$ in the commutative diagram [10] must be modified for the Casson handle $CH$. Now we have to consider a Morse function $\Psi' : \Sigma \cup CH \to \mathbb{R}$. The Casson handle is an periodic structure [30] (or more mathematically: an end-periodic manifold) and we need an extra
parameter for the Morse function reflecting the period. We model the handle pairs by a periodic function like sine or cosine. So, the Morse function for \( CH \) is \( \sin(k) \) with the new parameter \( k \). The attachment to the critical value \( \phi = 0 \) can be realized by the product \( \phi^2 \cdot \sin(k) \). Then it modifies the whole potential to

\[
V(\phi) = \phi^2 + \phi^2 \cdot \sin(k)
\]

and we obtain the new Lagrangian \( \mathcal{L} \) modifying (11)

\[
\mathcal{L}_\phi = D_\mu \phi \cdot D^\mu \phi + A \cdot \phi^2 + B \cdot \phi^2 \cdot \sin(k)
\]

But we should keep one thing in mind: in the derivation of the expansion rate in subsection IV B we choose time as the direction of the growing tree. Therefore the parameter \( k = C \cdot t \) is proportional to time \( t \). In the following we consider the time-dependent changes of the scalar field, the fluctuations, denoted by \( \chi(t) \). For these fluctuations we obtain the Lagrangian

\[
\mathcal{L}_\chi = \dot{\chi}^2 + \chi^2 (A + B \cdot \sin(C \cdot t))
\]

and the equation (the dot is the time-derivative)

\[
\ddot{\chi} + \chi (A + B \cdot \sin(C \cdot t)) = 0
\]

also called Mathieu equation (for parametric resonance). At least for the fluctuation \( \chi \), we obtain also the model of parametric resonance as the basic process for reheating and matter creation.

Currently, the model is not fully realistic. It shows the main features of an inflation model but it do not contain the right particle types (like leptons or quarks). But in a forthcoming work we will address this question further.

D. Inhomogeneities

At the end of the previous subsection, we emphasized that our model is only a simplification. For a more realistic model, we need a more complex 3-manifold consisting on a connected sum

\[
\Sigma = \Sigma_1 \# \Sigma_2 \# \cdots
\]

of homology 3-spheres. In this paper we showed that hyperbolic 3-manifolds induces inflation. But a critical look into the arguments of the derivation uncovers a different possibility. A hyperbolic 3-manifold is (uniquely) characterized by the fact that every plane in a hyperbolic 3-manifolds has a negative curvature [77]. In subsection III D we also described the other 3-geometries. In the list, one can find geometries like \( \mathbb{H}^2 \times \mathbb{R} \), \( \tilde{SL}_2 \) and SOL with negative curvature along some plane. These geometries are also able to generate an accelerated expansion but in a weaker sense. Therefore a sum of (geometrically) different homology 3-spheres induces an inhomogeneous inflation process. But we will study it in a forthcoming paper more carefully.

E. Dependence on initial conditions and Quantum fluctuations

One problem of the current inflation theory is the choice of the initial conditions: only a tiny class of conditions show inflation (see Penrose [52]). In contrast to this discussion of usual inflation, our model is more robust. The cause of the inflation is a topological transition explaining the robustness of the process. Indeed, the model is independent of the concrete smoothness structure. We have to demand only that the smoothness structure is not the standard one. But the richness of exotic smoothness structures imply that the choice of the standard smoothness structure is of measure zero. Secondly we need a homology 3-sphere with negative curvature. As one learns from Thurston's work, most of the geometric structures on 3-manifolds are hyperbolic structures. In particular, there is only homology 3-sphere not diffeomorphic to the 3-sphere which does carry a geometric structure of positive curvature: the Poincare sphere. Again, only a small finite number among infinite configurations do not lead to inflation. Therefore our model has a converse behavior: nearly all initial conditions are producing an inflationary scenario.

In the inflation model, quantum fluctuations are amplified to be the cause of structure formation in the cosmos. Therefore we need a counterpart of the quantum fluctuations in our geometric model. In [78], we constructed a geometric quantum state by using a wild embedding. Let \( i : K \to M \) be an embedding, i.e. \( i(K) \) is homeomorphic to \( K \). An embedding is called wild, if \( i(K) \) is an infinite polyhedron (a triangulation with infinite many triangles) which can be never reduced to a finite polyhedron. Otherwise, an embedding is called tame. Famous examples of
VI. CONCLUSION

In this paper we developed the theory of geometric inflation. According to this model, an inflationary phase in the cosmic evolution is caused by the exotic smoothness structure of our spacetime. The exotic smoothness structure is constructed by a hyperbolic homology 3-sphere Σ. The exponential expansion has its origin in the hyperbolic structure of the spacetime. This expansion is determined by a single parameter θ, the fraction of two topological invariants for the hyperbolic homology 3-sphere: the volume and the Chern-Simons invariant. Furthermore we obtain expressions for all relevant quantities like energy, time and temperature depending only on the parameter invariants for the hyperbolic homology 3-sphere: the volume and the Chern-Simons invariant. Furthermore we obtain expressions for all relevant quantities like energy, time and temperature depending only on the parameter θ. The coupling to matter can be also expressed geometrically (using the spinor representation of embedded surfaces). Then the reheating process has also a geometrical counterpart and we obtain naturally the model of parametric resonance. Finally we discuss a geometric interpretation of quantum fluctuations. One question remains: But what is about the inflation without quantum effects? Fortunately, there is growing evidence that the differential structures constructed above (i.e. exotic smoothness in dimension 4) is directly related to quantum gravitational effects [5, 20]. Maybe we touch only the tip of the iceberg.

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Appendix A: Connected and boundary-connected sum of manifolds

Now we will define the connected sum # and the boundary connected sum ♯ of manifolds. Let M, N be two n-manifolds with boundaries ∂M, ∂N. The connected sum M♯N is the procedure of cutting out a disk Dn from the interior int(M) \ Dn and int(N) \ Dn with the boundaries Sn−1 ∪ ∂M and Sn−1 ∪ ∂N, respectively, and gluing them together along the common boundary component Sn−1. The boundary ∂(M♯N) = ∂M ∪ ∂N is the disjoint sum of the boundaries ∂M, ∂N. The boundary connected sum M♯N is the procedure of cutting out a disk Dn−1 from the boundary ∂M \ Dn−1 and ∂N \ Dn−1 and gluing them together along Sn−2 of the boundary. Then the boundary of this sum M♯N is the connected sum ∂(M♯N) = ∂M ∪ ∂N of the boundaries ∂M, ∂N.

Appendix B: Casson handle

Above we have seen the determination of the smoothness structure for the compact 4-manifold M by the Akbulut cork A, a contractable 4-manifold with boundary a homology 3-sphere, and by the involution τ : ∂A → ∂(M \ A). So, why we need the Casson handle? Let’s start again with a h-cobordism W between M0 and M. As shown by Freedman [24] W is topologically trivial, i.e. W is homeomorphic to M0 × [0, 1]. Thus, as explained above, one can cancel the 2 –/3–handle pairs by using the Whitney trick, i.e. one is able to embed a special 2-disk D2 or equivalently a 2-handle D2 × D2. Usually one fails in dimension four to do it but Casson [15] found an infinite construction, the Casson handle, homotopic to D2 × S2. Via a “tour de force” Freedman [24] proved that every Casson handle is homeomorphic to D2 × R2, the open 2-handle, leading to the topological triviality of the h-cobordism W. All these constructions are relative to the boundary ∂D2 × R2, i.e. the attachment of the Casson handle is extremely important. But by [17] W is in general not diffeomorphic to M0 × [0, 1]. The reason is very simple: the Casson handle is not diffeomorphic to D2 × R2 (relative to the boundary ∂D2 × R2) [23, 24]. As explained above, this Casson handle determines the involution τ of the boundary ∂A and therefore the embedding of the Akbulut cork A.
Let consider now the basic construction of the Casson handle $CH$. Let $M$ be a smooth, compact, simple-connected 4-manifold and $f : D^2 \to M$ a (codimension-2) mapping. By using diffeomorphisms of $D^2$ and $M$, one can deform the mapping $f$ to get an immersion (i.e. injective differential) generically with only double points (i.e. $\# [f^{-1}(f(x))] = 2$) as singularities \cite{25}. But to incorporate the generic location of the disk, one is rather interesting in the mapping of a 2-handle $D^2 \times D^2$ induced by $f \times id : D^2 \times D^2 \to M$ from $f$. Then every double point (or self-intersection) of $f(D^2)$ leads to self-plumbings of the 2-handle $D^2 \times D^2$. A self-plumbing is an identification of $D^2_0 \times D^2$ with $D^2_1 \times D^2$ where $D^2_0, D^2_1 \subset D^2$ are disjoint sub-disks of the first factor disk\cite{78}. Consider the pair $(D^2 \times D^2, \partial D^2 \times D^2)$ and produce finitely many self-plumbings away from the attaching region $\partial D^2 \times D^2$ to get a kinky handle $(k, \partial^k)$ where $\partial^k$ denotes the attaching region of the kinky handle. A kinky handle $(k, \partial^k)$ is a one-stage tower $(T_1, \partial^*-T_1)$ and an $(n+1)$-stage tower $(T_{n+1}, \partial^*-T_{n+1})$ is an $n$-stage tower union kinky handles $\bigcup_{\ell=1}^n (T_\ell, \partial^\ell-T_\ell)$ where two towers are attached along $\partial^\ell-T_\ell$. Let $T_n$ be (interior$T_n) \cup \partial^\ell-T_n$ and the Casson handle

$$CH = \bigcup_{\ell=0}^n T^-_\ell$$

is the union of towers (with direct limit topology induced from the inclusions $T_n \hookrightarrow T_{n+1}$).

The main idea of the construction above is very simple: an immersed disk (disk with self-intersections) can be deformed into an embedded disk (disk without self-intersections) by sliding one part of the disk along another (embedded) disk to kill the self-intersections. Unfortunately the other disk can be immersed only. But the immersion can be deformed to an embedding by a disk again etc. In the limit of this process one “shifts the self-intersections into infinity” and obtains\cite{79} the standard open 2-handle $(D^2 \times \mathbb{R}^2, \partial D^2 \times \mathbb{R}^2)$.

A Casson handle is specified up to (orientation preserving) diffeomorphism (of pairs) by a labeled finitely-branching tree with base-point $*$, having all edge paths infinitely extendable away from *. Each edge should be given a label + or −. Here is the construction: $t \to CH$. Each vertex corresponds to a kinky handle; the self-plumbing number of that kinky handle equals the number of branches leaving the vertex. The sign on each branch corresponds to the sign of the associated self plumbing. The whole process generates a tree with infinite many levels. In principle, every tree with a finite number of branches per level realizes a corresponding Casson handle. For every labeled based tree $Q$, let us describe a subset $U_Q$ of $D^2 \times D^2$. Now we will construct a $(U_Q, \partial D^2 \times D^2)$ which is diffeomorphic to the Casson handle associated to $Q$. In $D^2 \times D^2$ embed a ramified Whitehead link with one Whitehead link component for every edge labeled by + (plus) leaving * and one mirror image Whitehead link component for every edge labeled by − (minus) leaving *. Corresponding to each first level node of $Q$ we have already found a (normally framed) solid torus embedded in $D^2 \times \partial D^2$. In each of these solid tori embed a ramified Whitehead link, ramified according to the number of + and − labeled branches leaving that node. We can do that process for every level of $Q$. Let the disjoint union of the (closed) solid tori in the $n$th family (one solid torus for each branch at level $n$ in $Q$) be denoted by $X_n$. $Q$ tells us how to construct an infinite chain of inclusions:

$$\ldots \subset X_{n+1} \subset X_n \subset X_{n-1} \subset \ldots \subset X_1 \subset D^2 \times \partial D^2$$

and we define the Whitehead decomposition $Wh_Q = \bigcap_{n=1}^{\infty} X_n$ of $Q$. $Wh_Q$ is the Whitehead continuum \cite{64} for the simplest unbranched tree. We define $U_Q$ to be

$$U_Q = D^2 \times D^2 \setminus (D^2 \times \partial D^2 \cup \text{closure}(Wh_Q))$$

alternatively one can also write

$$U_Q = D^2 \times D^2 \setminus \text{cone}(Wh_Q) \quad \text{(B1)}$$

where cone() is the cone of a space. As Freedman (see \cite{24} Theorem 2.2) showed $U_Q$ is diffeomorphic to the Casson handle $CH_Q$ given by the tree $Q$. We will later use this construction in the determination of the boundary $\partial N(A)$.

**Appendix C: Foliation, foliated cobordism and foliations of $S^4$**

A codimension $k$ foliation\cite{83} of an $n$-manifold $M^n$ (see the nice overview article \cite{111}) is a geometric structure which is formally defined by an atlas $\{\phi_i : U_i \to M^n\}$, with $U_i \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$, such that the transition functions have the form

$$\phi_{ij}(x,y) = (f(x,y), g(y)), \quad [x \in \mathbb{R}^{n-k}, y \in \mathbb{R}^k]$$

To be more precise one has
Definition 1 Let $(M, \mathcal{A})$ be an $n-$dimensional smooth manifold (of differential structure $\mathcal{A}$) with boundary: we take $\phi_\lambda : U_\lambda \to V_\lambda \subset \mathbb{R}^n_+ \subset \mathbb{R}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ for a chart $(U_\lambda, \phi_\lambda) \in \mathcal{A}$. Let $k$ with $0 \leq k \leq n$ be an integer. Let $\mathcal{F} = \{L_\alpha \mid \alpha \in A\}$ be a family of arcwise connected subsets $L_\alpha$ of the manifold $M$. We say that $\mathcal{F}$ is a $k-$dimensional smooth foliation of $M$ if it satisfies the following rules.

1. $L_\alpha \cap L_\beta = \emptyset$ for $\alpha, \beta \in A$ with $\alpha \neq \beta$.
2. $\bigcup_{\alpha \in A} (L_\alpha) = M$
3. Given a point $p$ in $M$, there exists a chart $(U_\lambda, \phi_\lambda) \in A$ about $p$ such that for $L_\alpha$ with $U_\lambda \cap L_\alpha \neq \emptyset$, $\alpha \in A$, each (arcwise) connected component of $\phi_\lambda(U_\lambda \cap L_\alpha)$ is of the form

$$\{(x_1, \ldots, x_n) \in \phi_\lambda(U_\lambda) \mid x_{k+1} = c_{k+1}, x_{k+2} = c_{k+2}, \ldots, x_n = c_n\}$$

where $c_{k+1}, c_{k+2}, \ldots, c_n$ are constants determined by the (arcwise) connected component.
4. For $p \in \partial M \cap L_\alpha$ we denote the boundary by $\partial \mathcal{F} = \{L_\alpha \mid \alpha \in A, L_\alpha \subset \partial M\}$

We call $L_\alpha$ a leaf of the foliation $\mathcal{F}$. The foliation $\mathcal{F}$ is also referred to as a smooth codimension $n-k$ foliation or a smooth foliation of codimension $n-k$.

Intuitively, a foliation is a pattern of $(n-k)$-dimensional stripes - i.e., submanifolds - on $M^n$, called the leaves of the foliation, which are locally well-behaved. The tangent space to the leaves of a foliation $\mathcal{F}$ forms a vector bundle over $M^n$, denoted $TF$.

Now we will discuss an important equivalence relation between foliations, cobordant foliations.

Definition 2 Let $M_0$ and $M_1$ be two closed, oriented $m$-manifolds with codimension-$q$ foliations. Then these foliated manifolds are said to be foliated cobordant if there is a compact, oriented $(m+1)$-manifold with boundary $\partial W = M_0 \sqcup M_1$ and with a codimension-$q$ foliation $\mathcal{F}$ transverse to the boundary. Every leaf $L_\alpha$ of the foliation $\mathcal{F}$ induces leaves $L_\alpha \cap \partial W$ of foliations $\mathcal{F}_{M_0}, \mathcal{F}_{M_1}$ on the two components of the boundary $\partial W$.

The resulted foliated cobordism classes $[\mathcal{F}_M]$ of the manifold $M$ form an abelian group $C\mathcal{F}_{m,q}(M)$ under disjoint union $\sqcup$ (inverse $\bar{M}$, unit $S^q \times S^{m-q}$, see [61] §29).

One of the first examples of a nontrivial foliation is known as Reeb foliation. Let $f : (-1, 1) \to \mathbb{R}$ be a smooth function $f(t) = \exp(t^2/(1-t^2)) - 1$. But every function $f$ with

$$f(0) = 0 \quad f(t) \geq 0 \quad f(t) = f(-t) \quad -1 < t < 1$$

$$\lim_{t \to \pm 1} \frac{df}{dt} = \infty \quad \lim_{t \to \pm 1} \frac{d^k}{dt^k} \left(\frac{1}{f(t)}\right) = 0 \quad k = 0, 1, 2, \ldots$$

will also work. Define subsets $L_\alpha'$, $0 \leq \alpha < 1$, and $L_\pm'$ of $D^1 \times S^1$ by

$$L_\alpha' = \{t, \exp(2\pi(\alpha + f(t))i) \mid -1 < t < 1\}, \quad L_\pm' = \{(\pm 1, e^{2\pi i\theta}) \mid 0 \leq \theta < 1\}$$

defining a smooth foliation of $D^1 \times S^1$(see the left Fig. 9). The join of two copies results in a smooth foliation of the
torus $T^2 = (D^1 \times S^1) \cup (D^1 \times S^1)$. This example can be generalized to the solid torus $D^2 \times S^1$ by defining the subsets $L_\alpha$ by

$$L_\alpha = \{ x, \exp(2\pi (\alpha + f(|x|))i) | x \in \text{int}(D^2) \}, \ 0 \leq \alpha < 1$$

where $|x|$ is the distance between the origin of the disc and the point $x$ in the interior $\text{int}(D^2)$. The family of sets $L_\alpha$ and $L_{\alpha'}$, $L_k$ for the boundary $\partial(D^2 \times S^1) = T^2$ forms a smooth foliation of $D^2 \times S^1$ which is known as Reeb foliation (see the right Fig. 9).

In [62] Thurston constructed a foliation of the 3-sphere $S^3$ which depends on a polygon $P$ in the hyperbolic plane $\mathbb{H}^2$ so that two foliations are non-cobordant if the corresponding polygons have different areas. We will present this construction now (see also the book [61] chapter VIII for the details).

Consider the hyperbolic plane $\mathbb{H}^2$ and its unit tangent bundle $T_1 \mathbb{H}^2$, i.e the tangent bundle $T \mathbb{H}^2$ where every vector in the fiber has norm 1. Thus the bundle $T_1 \mathbb{H}^2$ is a $S^1$-bundle over $\mathbb{H}^2$. There is a foliation $\mathcal{F}$ of $T_1 \mathbb{H}^2$ invariant under the isometries of $\mathbb{H}^2$ which is induced by bundle structure and by a family of parallel geodesics on $\mathbb{H}^2$. The foliation $\mathcal{F}$ is transverse to the fibers of $T_1 \mathbb{H}^2$. Let $P$ be any convex polygon in $\mathbb{H}^2$. We will construct a foliation $\mathcal{F}_P$ of the three-sphere $S^3$ depending on $P$. Let the sides of $P$ be labeled $s_1, \ldots, s_k$ and let the angles have magnitudes $\alpha_1, \ldots, \alpha_k$. Let $Q_\epsilon$ be the closed region bounded by $P \cup P'$, where $P'$ is the reflection of $P$ through $s_1$. Let $Q_\epsilon$, be $Q$ minus an open $\epsilon$-disk about each vertex. If $\pi : T_1 \mathbb{H}^2 \to \mathbb{H}^2$ is the projection of the bundle $T_1 \mathbb{H}^2$, then $\pi^{-1}(Q)$ is a solid torus $Q \times S^1$ (with edges) with foliation $\mathcal{F}_1$ induced from $\mathcal{F}$. For each $i$, there is an unique orientation-preserving isometry of $\mathbb{H}^2$, denoted $I_i$, which matches $s_i$ point-for-point with its reflected image $s_i'$. We glue the cylinder $\pi^{-1}(s_i \cap Q_\epsilon)$ to the cylinder $\pi^{-1}(s_i' \cap Q_\epsilon)$ by the differential $dI_i$ for each $i > 1$, to obtain a manifold $M = (S^2 \setminus \{k \text{ punctures}\}) \times S^1$, and a (glued) foliation $\mathcal{F}_2$, induced from $\mathcal{F}_1$. To get a complete $S^3$, we have to glue-in $k$ solid tori for the $k$ $S^1 \times$ punctures. Now we choose a linear foliation of the solid torus with slope $\alpha_k/\pi$ (Reeb foliation). Finally we obtain a smooth codimension-1 foliation $\mathcal{F}_P$ of the 3-sphere $S^3$ depending on the polygon $P$.

Appendix D: Chern-Simons invariant

Let $P$ be a principal $G$ bundle over the 4-manifold $M$ with $\partial M \neq 0$. Furthermore let $A$ be a connection in $P$ with the curvature

$$F_A = dA + A \wedge A$$

and Chern class

$$C_2 = \frac{1}{8\pi^2} \int_M \text{tr}(F_A \wedge F_A)$$

for the classification of the bundle $P$. By using the Stokes theorem we obtain

$$\int_M \text{tr}(F_A \wedge F_A) = \int_{\partial M} \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \quad (D1)$$

with the Chern-Simons invariant

$$CS(\partial M, A) = \frac{1}{8\pi^2} \int_{\partial M} \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \quad (D2)$$

Now we consider the gauge transformation $A \to g^{-1}Ag + g^{-1}dg$ and obtain

$$CS(\partial M, g^{-1}Ag + g^{-1}dg) = CS(\partial M, A) + k$$

with the winding number

$$k = \frac{1}{24\pi^2} \int_{\partial M} (g^{-1}dg)^3 \in \mathbb{Z}$$

of the map $g : M \to G$. Thus the expression

$$CS(\partial M, A) \mod 1$$
is an invariant, the Chern-Simons invariant. Now we will calculate this invariant. For that purpose we consider the functional \( D_2 \) and its first variation vanishes

\[ \delta CS(\partial M, A) = 0 \]

because of the topological invariance. Then one obtains the equation

\[ dA + A \wedge A = 0, \]

i.e. the extrema of the functional are the connections of vanishing curvature. The set of these connections up to gauge transformations is equal to the set of homomorphisms \( \pi_1(\partial M) \to SU(2) \) up to conjugation. Thus the calculation of the Chern-Simons invariant reduces to the representation theory of the fundamental group into \( SU(2) \). In [22] the authors define a further invariant

\[ \tau(\Sigma) = \min \{ CS(\alpha) | \alpha : \pi_1(\Sigma) \to SU(2) \} \]

for the 3-manifold \( \Sigma \). This invariant fulfills the relation

\[ \tau(\Sigma) = \frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{R}} tr(F_A \wedge F_A) \]

which is the minimum of the Yang-Mills action

\[ \left| \frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{R}} tr(F_A \wedge F_A) \right| \leq \frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{R}} tr(F_A \wedge *F_A) \]

i.e. the solutions of the equation \( F_A = \pm *F_A \). Thus the invariant \( \tau(\Sigma) \) of \( \Sigma \) corresponds to the self-dual and anti-self-dual solutions on \( \Sigma \times \mathbb{R} \), respectively. Or the invariant \( \tau(\Sigma) \) is the Chern-Simons invariant for the Levi-Civita connection.
This 3-sphere is very complicated embedded (a so-called wild embedding). We will discuss the corresponding implications in subsection V E.

Any non-compact manifold $M$ admits stably causal metrics (that is, those with a time function). So, if $M$ is not diffeomorphic to some product $S \times \mathbb{R}$, all these (causally well behaved) metrics must contain naked singularities. We thank M. Sánchez for the explanation of this result.

The ‘sides’ of $S$ then correspond to the components of the complement of $S$ in a tubular neighborhood $S \times [0, 1] \subset N$.

In complex coordinates the plumbing may be written as $(z, w) \mapsto (w, z)$ or $(z, w) \mapsto (\bar{w}, \bar{z})$ creating either a positive or negative (respectively) double point on the disk $D^2 \times 0$ (the core).

In the proof of Freedman [24], the main complications come from the lack of control about this process.

In general, the differentiability of a foliation is very important. Here we consider the smooth case only.