Relationship between reversible reasoning and conceptual knowledge in composition of function

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Abstract. The aim of this study was to investigate relationship between reversible reasoning and conceptual knowledge, particularly, mental action deployed by undergraduate students in reasoning reversible in problem situation of composition of function. This case study compares reversible reasoning of three high-achieving undergraduate student. These students were tested before with test measuring their understanding of aspect function. Data collected through think aloud and clinical interviews depict strategies that the participants used to work back to find original function. The result showed that constructing reversible reasoning requires conceptual knowledge with several characteristics (relational-harmonic, relational-visual, relational-pseudo) that explored in detail. Suggestion for further inquiry on improving knowledge about reversible reasoning as mental valuable tools for students in classroom practice.

1. Introduction
Reversibility as a mental process (reversible reasoning) and conceptual knowledge are two keys to developing current mathematical thinking and reasoning [1-3]. Some studies have acknowledged reversible reasoning and conceptual knowledge independently such as addition, subtraction, multiplication, and division [4,5], but few studies that have discussed reversible reasoning related to the relationship between concepts, especially those associated with the way students reverse their thoughts when dealing with fractions, exponents and logarithms, trigonometric function, antiderivative and derivative, function and inverse. Despite its importance, the relationship between reversible reasoning and conceptual knowledge lacks attention from researchers in mathematics education.

Rather than a structural perspective that occurs in negations and compensations [6], this investigation would refer to the reversible reasoning suggested by [7]. This mental process, which is experienced by students in reversing a problem, is known as reversible reasoning. Reversing a problem is not an easy task for students [8-11], because students tend to perceive a direct process and it’s reverse as two different things. Meanwhile, if the students are able to presume the two processes as equivalent and correlated, they can perform reversible reasoning which helps them develop stronger networking between concepts. In short, the students can think about the process from the beginning until the end or vice versa which as a result improve their ability to understand a mathematical problem.
Reversible reasoning occurs based on a problem’s operation, structure, and transformation [1,3,10]. Structurally, in reversing a problem students try to construct the source of the problem from its result. For example, the structure of the problem is identical with a composition of functions as \( f(g(x)) = h(x) \), where \( g(x) \) refers to the input, \( f(x) \) refers to the process, and \( h(x) \) refers to the result. That structure requires conceptual knowledge about the inverse function. Although numerous researchers have investigated inverse function meanings [12,13,14], fewer researchers have focused on students reversible reasoning during solving inverse problems. Researchers who have measured inverse function suggest that most students do not construct inverse function comprehensively. We wondered how students would handle a similar situation with respect to inverse function and how reversible reasoning and conceptual knowledge play a role during they solve the problem. Investigating this became the goal of our study.

Based on the literature above, our research question is as follows: what is the relation of reversible reasoning to conceptual knowledge, after participants draw on inverse function ideas in ways that promote meanings and perceptions about powerful mathematical concepts and connections

2. Method
We managed interviews with 14 students (8 female, 5 male) enlisted in a large university in East Java. Students in the program enrolled in an undergraduate course in advanced calculus during one semester were deliberated for this study and were between 18 and 20 years of age. We select the participants on a volunteer basis from a convenience sample of participants available to the research team in terms of agenda, location, and scheduling. We intentionally determine to struggle with students prior to the onset of their program coursework in order to investigate their inverse function meanings before these courses potentially affected their interpretations. The participants’ comprehensive mathematical experiences allowed us to explore reversible reasoning and conceptual knowledge students around inverse function. Some essential information that might be related to the research questions was assembled about each participant, including the current degree program; undergraduate major; number of, type of, and grade in undergraduate mathematics courses taken; and information about the level. Table 1 outlines most of the background information for each participant.

| Participant (Pseudonym) | Male/Female | Strata | Undergraduate major | Number of undergrad math courses | Undergrad math grade point average |
|-------------------------|-------------|--------|----------------------|----------------------------------|-----------------------------------|
| ADJ                     | Male        | Medium | Science              | 18                               | 3.72                              |
| BUD                     | Male        | Low    | Science              | 16                               | 3.47                              |
| CIN                     | Female      | Medium | Education            | 17                               | 3.55                              |
| DIA                     | Female      | High   | Science              | 21                               | 3.75                              |
| ERI                     | Male        | Medium | Education            | 18                               | 3.81                              |
| FIR                     | Female      | Medium | Education            | 19                               | 3.63                              |
| GIN                     | Female      | Medium | Science              | 18                               | 3.58                              |
| HAS                     | Female      | High   | Education            | 19                               | 3.78                              |
| ITR                     | Female      | High   | Education            | 20                               | 3.65                              |
| RAT                     | Female      | High   | Education            | 21                               | 3.73                              |
| ZUL                     | Male        | Medium | Education            | 18                               | 3.56                              |
| IBN                     | Male        | Low    | Science              | 16                               | 3.71                              |
| EKA                     | Female      | Low    | Science              | 16                               | 3.55                              |
2.1 Data collection
Our data consists of participants’ responses to the task composition of function (presented in Section 2.2). The task encourages participants to write mental activities by utilizing reversible reasoning as a tool to solve problems and conceptual knowledge about inverse functions. Furthermore, the 60 – 90-minute clinical interviews [15] were conducted at the classroom and an auditorium university and were videotaped. All participants were asked the same set of 15 questions with respect to the composition of function. However, we often posed follow-up questions to better perceive and respond to the participants emerging ideas. We emphasized to each participant that we were not concerned with correct or wrong answers and we demanded the students to ‘think aloud’ as they worked through tasks.

2.2 Task design
Our purpose to develop the tasks was to analyze the students’ mental process when they reversed a composition of functions, particularly the way the students associated their understanding with the inverse function. We had developed [16] instrument by providing three types of problems of \( H = f \circ G \): type 1 (known \( f \) and \( G \), find \( H \)), type 2 (known \( f \) and \( H \), find \( G \)), and type 3 (known \( H \) and \( G \), find \( f \)). Type 2 and type 3 were considered able to reveal the students’ reversible reasoning. Some students admitted that the two types of problems (type 2 and type 3) were difficult to solve directly. The students tended to use a procedural strategy to obtain the result. For some students, however, this tendency would lead to their inability in drawing a conclusion. In this article, Type 2 was selected to clarify relationships between reversible reasoning and conceptual knowledge. In addition, that type possible would obtain participants conception, mental activities, deep understanding, and interpretation of the inverse function consist of algebra, geometrical, structural, and reverse order. For instance, we designed the task in Table 2 so determining a rule for \( f(x + 5) \) or determine \( f^{-1}(2) = x \). A participant who understands that the problem requires inverse function ideas in relation to function composition might understand that \((f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x = f(x)\) or a participant whose inverse function meanings are constrained to “switching-and-solving” technique.

| Task | Explanation |
|------|-------------|
| Task #1 | Suppose that \( f(x) = \frac{x^2 + 2x + 6}{x^2 + x - 5} \), find a value of \( x \) so that \( f^{-1}(2) = x \) |
| Task #2 | \( h(x) = \frac{x-5}{x+1} \) and \( g(x) = 2x - 1 \), and \( h \) is the composite of \( h = f \circ g \), how you determine \( f(x + 5) \) |

2.3 Analysis
We marked each interview, digitized written work, transcribed the videos, capturing word and action. To analyze the data, we use open and axial techniques in combination with conceptual analysis techniques [1,3,17] in order to construct explanatory models of students thinking. Each researcher evaluated a part of students explanation, interpreting each student approach to solve each task. We reviewed our observations, seeing for common techniques on specifics task. As patterns advanced, we discovered and revised codes to characterize the techniques we observe. When an researcher was undecided about how to code a particular instance, we watched this instance collectively to reach an agreement regarding the code. This process sometimes generates the refinement of a code or formulation of a new code. Through this iterative process, we established a final set of codes to represent students’ techniques. After we formulated final codes, each researcher re-coded his or her students’ activities to provide the codes satisfactorily described each student’s techniques. Based on the literature, a worked example was coded as an instance of reversible reasoning (mental action/MA) and conceptual knowledge if it concerned any of the following:
Recalling ideas that have been learned about inverses based on specific functions known (MA1); Concluding that whether a function is known to acquire an inverse function (MA2); Recognizing when to interpret superscript “−1” which represents the inverse or exponent function (MA3); Comprehending the sign “o” as a composition of function (MA4); Composing two functions (MA5); Revealing two functions inversely (MA6); Constructing the techniques used to determine the inverse function in various representations (MA7); and Another mental action (MA8)

Trustworthiness is ensured through the determination that data collection is accurate and complete by managing assignments in written form and producing verbal transcripts from each think-aloud and interview after recording. In addition, the validation of coding process and re-coding of several components of reversible reasoning and conceptual knowledge was carried out through discussion with one professor and two doctoral lecturers in the field of mathematics education.

3. Result and Discussion

In total, 8 of 14 the students (see Table 3) showed consistent techniques to determine the inverse function of an algebraic displayed function or in other word students are frequently given an algebraic process to find inverse function: for instance, “switch the χ’s and y’s in the original function and resolve for y. Four students determined the inverse function by geometrical representation with conceiving that inverse function is a reflection over the line, \( y = x \), and two students determined inverse function by recognition of an identity function and existence inverse element. We found students answers to the tasks notably interesting. For instance, ADJ rapidly identified equivalent relationships between \( f^{-1}(x) = 2 \) and \( f(x) = 2 \) without using Venn diagram or graphics representation. Three other students need approximately 10-minutes to find an equivalent relationship constructed through Venn diagram representation (DIA, GIN, HAS).

| Constructing the techniques used to determine the inverse function in various representations (MA7) | # |
|---|---|
| Switched \( x \) and \( y \) | 8 |
| Reflecting the inverse function using line \( y = x \) | 2 |
| Coordinates \((x, y)\) pairs being switched in the inverse function to \((y, x)\) pairs | 2 |
| Coordinate two functions to obtain identity \((f \circ f^{-1})\) | 4 |
| Inverse in reverse order \((g \circ f)^{-1} = f^{-1} \circ g^{-1}\) | 3 |
| Comprehend equivalent relationships between \( f(x) = y \) and \( f^{-1}(y) = x \) without diagram venn | 1 |
| Comprehend equivalent relationships between \( f(x) = y \) and \( f^{-1}(y) = x \) without diagram venn | 3 |

In next section, we discussed our inferences regarding mental activities that characterize reversible reasoning during problem-solving process. First, all participants begin with reading the task repeatedly and loudly. Some researchers [9,17] have pointed out reversible reasoning in term of schema (schema consist of three parts: recognition situation, activity, and result). Reading the task included a recognition situation whose the aim was to establish the initial idea with recognizing the problem situation. Then, they clarify the concepts needed to produce a relation and make initial conjecture with identifying conditions or relationships in the task. Simon et al [3] claimed that level anticipation is a prerequisite for reversible reasoning. We defined The anticipation level as a process of reflecting or recalling a scheme that has been formed before operating a problem situation and used to predict or plan further action. For instance, ITR, ZUL, and RAT explicitly explain this anticipation saying,
Finally, pairs being switched in the inverse function to determine inverse function structural (four was linear $h(x)$, second, and still necessity deep study to that students’ reversible reasoning and so on, I understand that to solve this problem I need to find the inverse function or use other ways... probably I can simplify it without having to find $f(x)$.

3 of the 14 participants (FIR, IBN, EKA) indicates level anticipation analytically with reinterpreting initial concepts of function and inverse function and involving ideas about superscript “$-1$” which represents inverse, comprehending the sign “$o$” as a composition of function, and composing two function. As a result, the significant diversity for all participants lies in the strategy or approach used in representing inverse functions. Suitable with previous findings [12,18,19], several of the students in our study responded to questions about inverse function involves switching x’s and y’s and(re)solving for y, rather than constructing the techniques used to determine the inverse function in various representations, for instance Reflecting the inverse function using line $y = x$, coordinates $(x, y)$ pairs being switched in the inverse function to $(y, x)$ pairs, coordinate two functions to obtain identity, Inverse in reverse order, and comprehend equivalent relationships. Furthermore, according to finding [20] that students needed constantly to remind a definition or to check the property in several specific points when solving inverse function. In our study, several students were able to compare properties of a function or to compare functions, making analytical anticipation to construct an initial idea, consequently, we considered they had constructed reasoning reversible. Eventually, we expand the findings [12] and [19] with an added genetic decomposition for the inverse function namely: algebraic (action of switching $x$ and $y$ and solving it for $y$); geometrical (mapping of function or switched coordinate $(x, y)$ pairs to $(y, x)$ pairs); structural (existence inverse in element group); identity $((f^{-1} \circ f)(x) = i(x))$; and reversing order $((f, g)^{-1} = g^{-1}, f^{-1}$, determine inverse function with working-backward, or diagramatic representation).

The diverse researches have been given much attention in the multiplicative, proportional, and fraction situation. Moreover, recommendation [1] shows that reversible reasoning is sensitive to the numeric feature of problem parameters. However, our findings have been evoking notion ideas to investigate reversible reasoning processes in conceptual rather than operational areas. Consequently, level anticipation is an essential part of reversible reasoning and and still necessity deep study to explain it. We suggest three categorized that e. Our findings show researchers and curriculum designer should consider the role of reversible reasoning and conceptual knowledge of inverse function to supports students developing productive inverse meanings in different representation.

4. Conclusion
In this study, We, suggest the use term of reversible relational-harmonic to describe 2 of the 14 participants demonstrated a strong preference for algebraic and geometric interpretation to determine inverse function. Unlike four another student as reversible relational-visual demonstrated a strong visual interpretation but did not experience for algebraic aspect to conclude inverse. In addition, we found which students inability to explain the inverse ideas but able to solve the problem correctly, we call as reversible relational-pseudo. Finally, we conceived that students’ reversible reasoning and conceptual knowledge are personal and idiosyncratic and can be different from what they study. At last, We have expanded previous research in several ways: first, we administered a more detail analysis of students’ ideas than previous research; second, we investigated relationships reversible reasoning and conceptual knowledge that has accepted little attention; third, we analyzed the theoretical lens that reinforced our making inferences about students’ inverse function meanings based on their techniques addressing a variety of tasks; and fourth, we propose four other components besides the level of anticipation (anticipating) for longitudinal studies consist of activating, constructing, and verifying.
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