Asymptotic of Lorentzian Polyhedra Propagator

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Abstract. A certain operator $T = \int_{SL(2,\mathbb{C})} dg Y^\dagger g Y$ can be found in various Lorentzian EPRL calculations. The properties of this operator has been studied here in large $j$ limit. The leading order of $T$ is proportional to the identity operator.

Knowing the operator $T$ one can renormalize spin-foam’s edge self-energy by computing the amplitude of sum of a series of edges with increasing number of vertices and bubbles. This amplitude is calculated and is shown to be convergent.

Moreover some technical tools useful in Lorentzian Spin-Foam calculation has been developed.

Keywords: representation of the Lorentz group, Classical hypergeometric functions, Spin-Foams, quantum gravity

PACS numbers: 02.20.Tw, 02.30.Gp, 04.60.Pp, 04.60.Gw

AMS classification scheme numbers: 22E43, 22E66, 33C05, 83C45

1. Introduction and motivation

Loop quantum gravity and its covariant version - Spin-Foam Models - are promising candidates for a quantum theory of gravity [1][2]. Recent development has shown that within the Spin-Foam Models the EPRL method of calculating the vertex amplitude [4][10] are a way to recover the simplicity constraints of general relativity in the classical limit [3][5]. There are two main classes of the EPRL model: the euclidean EPRL [4][5] - with the $SO(4)$ group as a gauge group of the theory, and the Lorentzian EPRL [4][6] - with the universal covering group of $SO(1,3)$, namely $SL(2,\mathbb{C})$, as a gauge group. In both versions of the model the key role is played by so called EPRL map being the map

$$Y : \bigoplus_j \mathcal{H}_j \to \bigoplus_{\rho} \mathcal{H}_{\rho} \quad (1)$$

where $j$ are the spinlabels of representations of $SU(2)$ and $\mathcal{H}_j$ is the Hilbert space, on which this representation acts, while $\rho$ are the unitary representations of the group $G$ (being either $SO(4)$ or $SL(2,\mathbb{C})$) and $\mathcal{H}_\rho$ is the representation space respectively. In the euclidean case the $Y$ map is given by certain combinations of Clebsh-Gordon coefficients, and the $SO(4)$ representations can be expressed in terms of Wigner matrices, whose properties are well known. These makes the calculations relatively simple and several examples has been studied in detail. Calculations in Lorentzian case however involve matrix elements of unitary $SL(2,\mathbb{C})$ representations, being
combinations of hypergeometric functions [16], what increase a lot difficulty of getting a precise result.

Nevertheless some attempts has been done in the Lorentzian model as well. In the Dipole Cosmology model (DC) [12] the transition amplitude is written explicitly

\[
W(z) = \sum_{\{j_\ell\}} \prod_{\ell=1}^{4} (2j_\ell + 1) e^{-2t\alpha j_{\ell}(j_{\ell}+1) - i\lambda v j_{\ell}} \int_{SL(2,\mathbb{C})} \prod_{\ell=1}^{4} \langle j_\ell | u^\dagger_{n_\ell} Y^\dagger g_Y u_{n_\ell} | j_\ell \rangle \langle j_\ell | T | j_\ell \rangle
\]

with \( \ell' := \int_{SU(2)} du \prod_{\ell=1}^{4} u \cdot u_{n_\ell} | j_\ell \rangle \)

The integral does not have to be known explicitly, the only thing authors need is that it is bound by \( N_0 j_0^3 \) for large \( j_0 \). Such behaviour of this integral is assumed per analogy to the euclidean case [11], however it was not proven rigorously.

The same integral can be found in the calculation of radiative correction to the spin-foam edge (also called the "self energy" of the edge) coming from the "melonic" diagram [15]. In this paper it was shown, that the divergent part coming from a certain bubble is

\[
W_{\Lambda} \sim \Lambda^{6(\mu - 1)} \int_{SL(2,\mathbb{C})^2} \prod_{i=1}^{4} \langle m_i | Y^\dagger g_Y m_i \rangle \langle n_i | Y^\dagger g_Y | \tilde{m}_i \rangle = \Lambda^{6(\mu - 1)T^2}
\]

here however the operator \( T \) appear in more general form, since there are no assumptions on the external spins of the bubble. Thou detailed study of the operator \( T \) gives an insight to behaviour of the divergences appearing in the Spin-Foam models.

Finally the the framework of the graph diagrams [14] allowed to introduce the simple Feynman-like rules to find the expressions on the Spin-foam amplitudes (see also [3]). These rules show clearly, that objects \( \prod_{i=1}^{4} \langle m_i | Y^\dagger g_Y m_i \rangle \langle n_i | Y^\dagger g_Y | \tilde{m}_i \rangle \) are the main building blocks of amplitude of each diagram (spin-foam). Thus understanding of them is needed, and the operator \( T = \int_{SL(2,\mathbb{C})} dg Y^\dagger g_Y \) is the first step in this study.

That’s why we decided to investigate some basic properties of the \( T \) operator.

The paper is divided into two parts: the main text and the technical appendix. In the main text the in section 2 there is a strict definition of the object we study. Section 3 contains some remarks on the Saddle Point Approximation method of integration and points out the issues one has to check to use it in calculating the Lorentzian Polyhedra Propagator. Section 4 is the study of the function we integrate to obtain the Propagator, however to make the text easy to read some parts of the calculations (which are conceptually simple, but technically complicated) are moved to Appendixes. In section 5 we gather the results and point out some possible applications, and in section 6 we briefly conclude.

There are three appendixes: Appendix A is a collection of useful properties of Gauss Hypergeometric Function \( \mathbf{2F1} \), which is the main character of the calculations; in Appendix B we present some properties of the \( SU(2) \)-invariant tensors, and in Appendix C we prove some more technical lemmas.
2. Lorentzian Polyhedra Propagator

To make the paper self-contained, let us fix notation and definitions and recall some mathematical facts.

- The $SU(2)$-elements will be denoted usually by $u$, $v$, the $SL(2, \mathbb{C})$-elements will be denoted by $g$.
- By $\mathcal{H}_j$ we will denote the representation space of $SU(2)$ of spin $j$. For the basis elements of $\mathcal{H}_j$ we will use the bra-ket notation: $|m\rangle_j$. The basis elements of tensor product $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_n}$ are denoted by $|m_1, \ldots, m_n\rangle_{j_1} \otimes \cdots \otimes |m_n\rangle_{j_n}$, or shortly by $|m\rangle_{j_1, \ldots, j_n}$.
- The representation spaces of primary series of $SL(2, \mathbb{C})$ we will denote by $\mathcal{H}^{(p,k)}$, and its basis elements in bra-ket notation are $|j, m\rangle_{p,k}$.
- The Wigner matrices will be denoted by $D^j(u)^m_m := \langle m | u | m' \rangle_j$. The matrix elements of the primary series of $SL(2, \mathbb{C})$ representations are $D^{(p,k)}(g)^j_{m'}^m := \langle j, m | g | j' m' \rangle^{(p,k)}$.
- The EPRL map $Y$ is an injection of $\mathcal{H}_j$ into $\mathcal{H}^{(\gamma,j)}$ given by

$$Y : \mathcal{H}_j \ni |m\rangle_j \mapsto |j, m\rangle^{(\gamma,j)} \in \mathcal{H}^{(\gamma,j)}$$

(see [3][10][3][10]). It is obvious to generalise the EPRL map to tensor products of $\mathcal{H}_j$ spaces: $Y : (|a\rangle \otimes |b\rangle) \mapsto Y |a\rangle \otimes Y |b\rangle$.
- The choice of the basis in $\mathcal{H}^{(p,j)}$ picks a normal subgroup $SU(2) \triangleleft SL(2, \mathbb{C})$ such that given $u \in SU(2)$ the matrix elements are $D^{(p,k)}(u)^j_{m'}^m = \delta^j_j D^j(u)^m_m$ (in other words $u$ commutes with $Y$). This subgroup defines a decomposition of $SL(2, \mathbb{C})$ into $H^3 \rtimes SU(2)$ (and $SU(2) \rtimes H^3$) where $H^3$ is a 3-hyperboloid, which can be parametrised by $\mathbb{R}^3$.

Having defined above, we can proceed to the Lorentzian Polyhedra Propagator.

2.1. Definition

Given a set of spinlabels $j_1, \ldots, j_N$ let us define an operator $T$ acting on $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}$ by the formula

$$T := \int_{SL(2, \mathbb{C})} dg \ Y^\dagger g Y$$

(5)

The matrix elements of $T$ in the spin-$z$ basis can be expressed as

$$\langle m_1, \ldots, m_N | T | m'_1, \ldots, m'_N \rangle = T^{m_1 \cdots m_N}_{m'_1 \cdots m'_N} = \int_{SL(2, \mathbb{C})} \prod_{i=1}^N D^{(\gamma,j)}(g)^{j}_{j_i m'_i} \int_{SU(2)} dk \ du \ Y^\dagger k \cdot u Y$$

(6)

Let us now study some basic properties of $T$.

2.2. Domain and rank

Since each $SL(2, \mathbb{C})$ element $g$ can be decomposed into $g = k \cdot u$ where $u \in SU(2)$ and $k \in H^3$, the $T$ operator can be written as

$$T = \int_{SL(2, \mathbb{C})} \int_{H^3 \rtimes SU(2)} \int_{SU(2)} dk \ du \ Y^\dagger k \cdot u Y$$

(7)
but since the $SU(2)$ elements commute with $Y$:

$$T = \int_{H^3} dk Y^\dagger k Y \int_{SU(2)} du \ u$$

Now note, that $\int_{SU(2)} du =: P_{\text{inv}}$ is the projection onto the $SU(2)$-invariant subspace of the space we are acting on - in this case it is $\mathcal{H}_{\text{inv}} := \text{inv} (\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N})$. Thus

$$T = \hat{A} \cdot P_{\text{inv}}$$

where $\hat{A}$ is some operator.

We can do the same decomposition on the left-hand side:

$$T = \int_{SL(2,\mathbb{C})} dg Y^\dagger g Y = \int_{SU(2) \times H^3} dk du \ Y^\dagger \ k Y =$$

$$= \int_{SU(2)} du \ \int_{H^3} dk Y^\dagger k Y = P_{\text{inv}} \cdot \hat{A}$$

thus

$$T = P_{\text{inv}} \cdot \hat{A} \cdot P_{\text{inv}}$$

So $T$ acts nontrivially only on $\mathcal{H}_{\text{inv}}$. Let us now choose an orthonormal basis $\{\ket{i}\} \in \mathcal{H}_{\text{inv}}$. Let $\ket{i}$ and $\ket{i'}$ be the basis elements. It is enough to investigate the matrix elements of $T$

$$T'_{i',i} := \int_{SL(2,\mathbb{C})} dg \ \bra{i} Y^\dagger g Y \ket{i'} = \int_{SL(2,\mathbb{C})} dg \Phi'_{i,i'}(g)$$

what we shall do in what follows.

2.3. Symmetries

The vector fields on $SL(2,\mathbb{C})$ are spanned by three $SU(2)$ rotation generators $L^i$ and three boost generators $K^i$. The $k$ component of the decomposition $g = k \cdot u$ can be written as $k(\vec{\eta}) = e^{\vec{\eta} \cdot \vec{R}}$. For each vector $\vec{\eta}$ (of length $|\vec{\eta}| =: \eta$) exists an $SU(2)$ element $u_{\vec{\eta}}$ such that $\vec{R}_\eta = u_{\vec{\eta}}^\dagger K^3 u_{\vec{\eta}}$, and thou

$$k(\vec{\eta}) = e^{\vec{\eta} \cdot \vec{R}} = e^{\eta \vec{u}_\eta K^3 u_{\vec{\eta}}} = u_{\vec{\eta}}^\dagger e^{\eta K^3} u_{\vec{\eta}}$$

Thus each $g \in SL(2,\mathbb{C})$ can be expressed as

$$g = u_{\vec{\eta}}^\dagger e^{\eta K^3} u_{\vec{\eta}} \cdot u$$

Let us now investigate, how does $\Phi'_{i,i'}(g)$ depend on $u$ and $\vec{\eta}$.

2.3.1. $SU(2)$ symmetry  Using the fact, that $SU(2)$-elements commute with the EPRL map $Y$ it is straightforward to see that

$$\Phi'_{i,i}(k \cdot u) = \bra{i} Y^\dagger k \cdot u Y \ket{i'} = \bra{i} Y^\dagger k Y u \ket{i'}$$

but since $i'$ is an $SU(2)$-invariant, we have $u \ket{i'} = \ket{i'}$ and thus

$$\Phi'_{i,i}(k \cdot u) = \bra{i} Y^\dagger k Y \ket{i'} = \Phi'_{i,i}(k)$$

so the integrand is $SU(2)$ invariant.
2.3.2. Rotation-of-boost symmetry. Now using \([13]\) lets investigate \(\Phi^\iota_t(k(\vec{\eta}))\):
\[
\Phi^\iota_t(k(\vec{\eta})) = \langle | \ Y^1 u_\eta e^{i\eta K^3} u_\eta K Y | \iota' \rangle = \langle | \ u_\eta K^3 Y u_\eta | \iota' \rangle
\] (17)
again using \(SU(2)\) invariance of \(\iota\) and \(\iota'\) we get
\[
\Phi^\iota_t(k(\vec{\eta})) = \langle | \ Y^1 e^{i\eta K^3} Y | \iota' \rangle = : \Phi^\iota_t(\eta) \tag{18}
\]
Thus the integrand depends only on the length of the boost vector \(\eta\).

2.4. Integral measure

Thanks the \(SU(2)\)-symmetry presented above we can trivially do the \(SU(2)\)-integral, (which gives identity, thanks to normalisation of the Haar measure) and we are left with the integral
\[
T^\nu = \int_{H^3} dk(\vec{\eta}) \Phi^\iota_t(\eta)
\] (19)
According to \([16]\) the measure \(dk\) is
\[
dk(\vec{\eta}) = \frac{1}{(4\pi)^2} d\phi(\vec{\eta}) \sin \theta(\vec{\eta}) d\theta(\vec{\eta}) (\sinh |\vec{\eta}|)^2 d|\vec{\eta}|
\] (20)
where \(\phi(\vec{\eta})\) and \(\theta(\vec{\eta})\) are the spherical angles of the boost direction.

Let us now introduce the measure function
\[
\mu(x) := \left(\frac{\sinh |x|}{4\pi |x|}\right)^2
\] (21)
Now obviously
\[
dk(\vec{\eta}) = \mu(\vec{\eta}) d^3 \vec{\eta}
\] (22)
and this is the measure we will use in further calculations, i.e. we will calculate
\[
T^\nu = \int_{\mathbb{R}^3} d^3 \vec{\eta} \mu(\vec{\eta}) \Phi^\iota_t(\vec{\eta})
\] (23)

3. Strategy of integration

To calculate the \(T\) operator one have to do the integral over \(SL(2,\mathbb{C})\) group. We will not do this integral explicitly, we will find its leading order using the saddle point approximation (SPA) method \([18]\). The SPA allows to express integrals of the form \(\int d^n x \ g(x) e^{-\Lambda f(x)}\) for large values of \(\Lambda\) as a power series in \(\frac{1}{\Lambda}\). The leading order of the integral is determined by the value of integrand in the critical point \(x_0\) of the function \(f(x)\):
\[
\int d^n x \ g(x) e^{-\Lambda f(x)} = \left(\frac{2\pi}{\Lambda}\right) \left(\frac{\partial^2 f}{\partial x^2}\right)_{x_0}^{-\frac{1}{2}} g(x_0) e^{-\Lambda f(x_0)} (1 + O(\Lambda^{-1})) \tag{24}
\]
where \(\frac{\partial^2 f}{\partial x^2}\) is the determinant of the Hessian matrix of the function \(f(x)\). If the function \(f\) has more than one critical point \(\{x_1, \ldots, x_k\}\), than the argument \(x_0\) that appear in the formula \([24]\) is the maximal critical point: such a point \(x_0 \in \{x_1, \ldots, x_k\}\) that \(\Re(-f(x_0))\) is maximal (if there is more then one maximal critical point, than sum over them).

Several assumptions must be satisfied for the formula \([24]\) to be valid. First of all the function \(f(x)\) must be smooth and twice differentiable at the point \(x_0\). Moreover the integrand \(g(x) e^{-\Lambda f(x)}\) must vanish outside a compact region \(\Omega \subset \mathbb{R}^n\) (or decay sufficiently fast with \(|x| \rightarrow \infty\) – see section \([3.3]\)). These properties of our integrand \(\Phi^\iota_t(g)\) will be checked in section \([4]\).
3.1. Non obvious integrand

The euclidean EPRL transition amplitude can be easily expressed in the form of \( \int dx \, g(x) e^{-A_f(x)} \) \([5]\). In the Lorentzian case, which we consider now, the decomposition of the integrand is not obvious. However one can still use the SPA method for the integrand of the form \( \Phi(\Lambda, r) \) defined on \( \mathbb{R} \times \Omega \); if it has a proper large-\( \Lambda \) behaviour and if one can identify the critical points of the exponent part of the integrand \([9]\).

Given a function \( \Phi(\Lambda, r) \) let us define the exponent part of the integrand:

\[
\phi(x) := \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \log (\Phi(\Lambda, x))
\]  

The function \( \phi(x) \) may diverge for some points \( x \). If the points, where it diverges, form a region - it means, that the parameter \( \Lambda \) is not a good largeness parameter, i.e. the exponent grows faster then linear in \( \Lambda \). In such case one should consider another parameter \( \Lambda(\Lambda) \) instead (this is for example the case when integrating \( \Phi(a, x) = e^{-a^2 x^2} \) - the proper largeness parameter is \( a^2 \), not \( a \)). A similar situation is when the limit is identically zero - then the exponent grows slower then linearly in \( \Lambda \) (for example \( \Phi(a, x) = e^{-\sqrt{a}x} \)). If the largeness parameter \( \Lambda \) was chosen properly, we obtain a nontrivial function \( \phi(x) \) (which of course may have some poles or zero points).

The potentially critical points are the critical points of \( \phi \) and the poles of \( \phi \). The poles represent the situation when \( g(x) = 0 \). In this case value of \( \phi(x) \) does not capture behaviour of \( f \), hence one need to consider instead

\[
\phi^n(x) := \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \log (|\nabla|^n \Phi(\Lambda, x))
\]  

where \( n \) is the lowest order of differentiation, for which the limit converges, and \( |\nabla| f := |\sum \partial_i f| \).

Now let us define the family of sets of poles \( B^{(n)} := \{ x \in \Omega : \phi^n(x) = \infty \} \) (with \( \phi^0 = \phi \) ) and a family of sets of critical points \( A^{(n)} := \{ x \in B^{(n-1)} : \nabla \phi^n(x) = 0 \} \) (with \( B^{(1)} = \Omega \)). The set of critical points of the exponent is \( A := \bigcup_{n \geq 0} A^{(n)} \).

For each critical point \( x_i \in A \) let us define the real part of the exponent as

\[
F_i := \Re(\phi^n(x_i)) \quad \text{where } n \text{ such that } x_i \in A^{(n)}
\]  

Now the maximal critical point \( x_{\text{max}} \) is the one, for which \( F_i \) is maximal.

So at the end of the day the integral equals

\[
\int_\Omega d^n x \, \Phi(\Lambda, x) = \left( \frac{2\pi}{\Lambda} \right)^{\frac{n}{2}} \left( \frac{\partial^2 \phi^n}{\partial x^2} \right)_{x_{\text{max}}}^{-\frac{1}{2}} \Phi(\Lambda, x_{\text{max}}) \left( 1 + O(\Lambda^{-1}) \right)
\]  

where \( n \) is such that \( x_{\text{max}} \in A^{(n)} \).

3.2. Spherically symmetric integrand

Consider a spherically symmetric integrand \( \Phi(\Lambda, \vec{x}) = \Phi(\Lambda, r) \), for \( r = |\vec{x}| \), with the maximal critical point at \( r = 0 \) (being in the interior of the region \( \Omega \)). Then the Hessian matrix of the function \( \phi \) is

\[
\frac{\partial^2 \phi}{\partial x^i \partial x^j} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} \frac{df}{dr} + \frac{\partial r}{\partial x^i} \frac{\partial r}{\partial x^j} \frac{d^2 \phi}{dr^2}
\]  

Since \( r = 0 \) is the critical point, the differential \( \frac{df}{dr}|_{r=0} = 0 \), thus the first term vanish.
Further simplification comes from the fact, that
\[ \frac{1}{2} \frac{\partial^2 r^2}{\partial x^i \partial x^j} = \frac{\partial r}{\partial x^i} \frac{\partial r}{\partial x^j} + r \frac{\partial^2 r}{\partial x^i \partial x^j} = \frac{\partial r}{\partial x^i} \frac{\partial r}{\partial x^j} \text{ at } r = 0 \] (30)
and since \( r^2 = \sum (x^i)^2 \), we have
\[ \frac{\partial r}{\partial x^i} \frac{\partial r}{\partial x^j} = \delta_{ij} \text{ at } r = 0 \] (31)
Thus the Hessian matrix is
\[ \left. \frac{\partial^2 \phi}{\partial x^i \partial x^j} \right|_{\bar{x} = 0} = \frac{d^2 \phi}{dr^2} \bigg|_{r=0} \delta_{ij} \] (32)
And the Hessian determinant
\[ \left| \frac{\partial^2 \phi}{\partial x^i \partial x^j} \right|_{\bar{x} = 0} = \left| \frac{d^2 \phi}{dr^2} \right|^N \bigg|_{r=0} \] (33)

3.3. Non compactness of integration range
The SPA method is well defined if the region of integration \( \Omega \) is compact and the integrand vanish at \( \partial \Omega \). However under certain assumptions one can generalise the SPA to the case of noncompact integration range. Assume that exists \( \Lambda_0 \) such that for all \( \Lambda > \Lambda_0 \) the following is true: for each \( \epsilon > 0 \) exists a compact region \( R_\epsilon \subset \Omega \), such that
\[ \int_{\Omega \setminus R_\epsilon} d^N x \, |\Phi(\Lambda, x)| < \epsilon \] (34)
and that \( R_\epsilon \subset R_{\epsilon'} \) for \( \epsilon > \epsilon' \). Then let us introduce for each \( \epsilon \) another compact region \( \bar{R}_\epsilon \), such that \( R_\epsilon \subset \bar{R}_\epsilon \subset \Omega \) and a smooth function \( \chi_\epsilon(x) \) such that
\[ \chi_\epsilon(x) = \begin{cases} 1 & x \in R_\epsilon \\ 0 & x \in \bar{R}_\epsilon \setminus R_\epsilon \end{cases} \] (35)
Obviously \( \int_{\Omega \setminus \bar{R}_\epsilon} d^N x \, |\chi_\epsilon(x)\Phi(\Lambda, x)| < \epsilon \). Now let \( I(\Lambda) := \int_\Omega d^N x \Phi(\Lambda, x) \) and \( I_\epsilon(\Lambda) := \int_{\Omega \setminus \bar{R}_\epsilon} d^N x \chi_\epsilon(x)\Phi(\Lambda, x) \). Obviously for all \( \Lambda > \Lambda_0 \) we have \( |I(\Lambda) - I_\epsilon(\Lambda)| < 2\epsilon \), thus function \( I_\epsilon(\Lambda) \) converges to \( I(\Lambda) \) uniformly with respect to \( \Lambda \). But each integral \( I_\epsilon(\Lambda) \) is in fact integral over a compact region \( \bar{R}_\epsilon \), so it can be calculated using the SPA method:
\[ I_\epsilon(\Lambda) = \left( \frac{2\pi}{\Lambda} \right)^\frac{N}{2} \left( \left| \frac{\partial^2 \phi}{\partial x^2} \right|_{x_\epsilon} \right)^{-\frac{1}{2}} \Phi(\Lambda, x_\epsilon) (1 + O(\Lambda^{-1})) \] (36)
where \( x_\epsilon \) is the maximal critical point of \( \phi \) in the region \( R_\epsilon \). For \( \epsilon \) sufficiently small, for example for \( \epsilon < \frac{1}{\Lambda} (\frac{2\pi}{\Lambda})^{\frac{N}{2}} \left( \left| \frac{\partial^2 \phi}{\partial x^2} \right|_{x_0} \right)^{-\frac{1}{2}} \Phi(\Lambda, x_0) \), the region \( R_\epsilon \) must contain the maximal critical point of \( \phi \), so \( x_\epsilon = x_0 \), thus the leading term does not depend on \( \epsilon \). So the leading term of the function \( I(\Lambda) \) is the limit at \( \epsilon \to 0 \) of the leading terms of \( I_\epsilon(\Lambda) \).
3.4. Multiplication by a $\Lambda$-independent function

Consider now integral of the form

$$\tilde{I}(\Lambda) := \int_\Omega d^N x \mu(\vec{x}) \Phi(\Lambda, \vec{x})$$

(37)

where $\mu(\vec{x})$ is a nonvanishing function. When using the SPA method, the integral $\tilde{I}(\Lambda)$ can be easily related to $I(\Lambda) := \int_\Omega d^N x \Phi(\Lambda, x)$. Indeed, note that presence of $\mu(\vec{x})$ does not affect the exponent part of the integrand:

$$\tilde{\phi}(\vec{x}) = \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \ln \mu(\vec{x}) + \frac{1}{\Lambda} \ln \Phi(\Lambda, \vec{x}) = 0 + \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \ln \Phi(\Lambda, \vec{x}) = \phi(\vec{x})$$

(38)

thus the maximal critical points of the exponent of the integrand does not know about presence of $\mu(\vec{x})$. So using SPA method we get

$$\tilde{I}(\Lambda) = \left( \frac{2\pi}{\Lambda} \right)^\frac{3}{2} \left( \frac{\partial^2 \phi}{\partial \vec{x}^2} \right)^{-\frac{1}{2}} \mu(x_{\text{max}}) \Phi(\Lambda, x_{\text{max}}) \left( 1 + O(\Lambda^{-1}) \right) = \mu(x_{\text{max}}) I(\Lambda)$$

(39)

4. Detailed analysis of the integrand

4.1. Measure and exponent part of the integrand

Let us now investigate the properties of the integrand. Recalling (23) the integrand is

$$\tilde{\Phi}(\eta) = \mu(\eta) \Phi_{\iota,\iota'}(\eta)$$

with

$$\mu(\eta) = \left( \frac{\sinh \eta}{4\pi \eta} \right)^2 \Phi_{\iota,\iota'}(\eta) = \langle \iota | Y^\dagger e^{\eta K} Y | \iota' \rangle$$

(40)

and $\iota, \iota' \in \text{inv}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N})$. Our largeness parameter is $J = \max \{j_i\}_{i=1,\ldots,N}$. Obviously the function $\mu(\eta)$ does not depend on the largeness parameter. Thus, using section 3.4, it is enough to find the critical points of $\Phi_{\iota,\iota'}(\eta)$. From now on we will call the function $\tilde{\Phi}(\eta)$ the full integrand, the function $\Phi_{\iota,\iota'}(\eta)$ the integrand, and the function $\mu(\eta)$ the measure part of the integrand or simply the measure.

In this section we will often go back to the $|m\rangle_j$ basis, so that

$$\Phi_{\iota,\iota'}(\eta) = \sum_{\vec{m},\vec{m}'} \langle \iota | \vec{m} \rangle_j \langle \vec{m}' | \iota' \rangle_j \Phi_{\vec{m},\vec{m}'}(\eta)$$

(41)

where

$$\Phi_{\vec{m},\vec{m}'}(\eta) = \prod_{i=1}^N \langle m_i | Y^\dagger e^{\eta K} Y | m_i' \rangle_j$$

(42)

Since $[K^3, L^3] = 0$, each term is proportional to $\delta_{m_i,m_i'}$, and thus we can define the function

$$f_{m}^{(j)}(\eta) := \langle m | Y^\dagger e^{\eta K} Y | m \rangle_j$$

(43)

such that

$$\Phi_{\vec{m},\vec{m}'}(\eta) = \prod_{i=1}^N \delta_{m_i,m_i'} f_{m_i}^{(j)}(\eta)$$

(44)
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Thanks to (44) it is now obvious, that in the basis $|\vec{m}\rangle$ all the nondiagonal matrix elements are precisely zero (without any approximation).

The exponent part of the integrand is

$$\phi_i^J(\eta) = \lim_{J \to \infty} \phi_i^J(\eta, J) := \lim_{J \to \infty} \frac{1}{J} \ln \left[ \sum_{\vec{m}} \sum_{\vec{m}'} e^{\phi_{\vec{m}}(\eta, J)} \right]$$

Because of the sum under the logarithm it is quite inconvenient object, thus most calculations we will do for the exponent part of the function $\Phi_{\vec{m}}(\eta)$, namely:

$$\phi_{\vec{m}}(\eta) = \lim_{J \to \infty} \phi_{\vec{m}}(\eta, J) := \lim_{J \to \infty} \frac{1}{J} \ln \sum_{i=1}^N f_{\vec{m}_i}(\eta)$$

Note, that

$$\phi_i^J(\eta, J) = \frac{1}{J} \ln \left[ \sum_{\vec{m}} \sum_{\vec{m}'} e^{J \phi_{\vec{m}}(\eta, J)} \right]$$

For later convenience let us introduce parameters $x_i := \frac{j_i}{J}$. Obviously $0 \leq x_i \leq 1$, but one cannot treat them as the small parameters.

4.2. Hypergeometric representation

Let us now focus on the function $f_{\vec{m}}^{(j)}(\eta)$. According to (10) it can be written as

$$f_{\vec{m}}^{(j)}(\eta) = (2j + 1) \left( \frac{2j}{j + m} \right) e^{-(j+m+1)\eta} e^{i\gamma j} \times$$

$$\times \int_0^1 dt \int_0^1 dt' \left( 1 - t \right)^{j-m} \left[ 1 - \left( 1 - e^{-2\eta} \right) \right]^{j-m}$$

Recall now the integral definition of the Gauss’s hypergeometric function:

$$2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \frac{e^{zt}}{(c-zt)^{a}}$$

Comparison of formulas (18) and (49) gives a conclusion, that

$$f_{\vec{m}}^{(j)}(\eta) = e^{-(j+m+1)\eta} e^{i\gamma j} \times 2F_1 \left( j + 1 - i\gamma j, j + 1 + m; 2j + 1; 1 - e^{-2\eta} \right)$$

4.3. Maximum point

To apply the SPA method we need to find the maximal critical point of $\phi_i^J(\eta)$. The natural candidate is $\eta = 0$. Since $f_{\vec{m}}^{(j)}(\eta)|_{\eta=0} = 1$ (see (A.1)), it is obvious to find the value of the integrand

$$\Phi_{\vec{m}}(0) = 1 \quad \text{so} \quad \Phi_i^J = \langle \eta | \eta' \rangle$$

Now let us show, that $\eta = 0$ is the only critical point that counts, i.e. was there any other critical point $\eta_1$. The value of $\Phi_{\vec{m}}(\eta_1)$ would be exponentially suppressed in large $J$ limit. Indeed, one can estimate the modulus of $f_{\vec{m}}^{(j)}(\eta)$ by

$$|f_{\vec{m}}^{(j)}(\eta)| \leq \left( e^{1-2\eta - e^{-2\eta}} \right)^{(j+1)^2 - m^2} \frac{1}{e^{2\eta}}$$
Asymptotic of Lorentzian Polyhedra Propagator

(see Appendix A.3) thus

$$|\Phi^m_{\vec{m}}(\eta_1)| \leq \left(e^{1-2\eta_1-e^{-2\eta_1}}\right)^{\sum_{i=1}^N \frac{(j_i+1)^2-m_i^2}{4(j_i+1)}} = |\mathcal{C}(\eta_1)|^{\sum_{i=1}^N \frac{(j_i+1)^2-m_i^2}{4(j_i+1)}}$$

(53)

Now since

$$|\Phi^m_{i'}(\eta_1)| = \left|\sum_{\vec{m}} \Phi^m_{\vec{m}}(\eta_1) e^{i\vec{m} \cdot \vec{m}'}\right| \leq \left|\sum_{\vec{m}} |\Phi^m_{\vec{m}}(\eta_1)| e^{i\vec{m} \cdot \vec{m}'}\right|$$

(54)

we have

$$|\Phi^m_{i'}(\eta_1)| \leq \left|\sum_{\vec{m}} |\mathcal{C}(\eta_1)|^{\sum_{i=1}^N \frac{(j_i+1)^2-m_i^2}{4(j_i+1)}} e^{i\vec{m} \cdot \vec{m}'}\right|$$

(55)

Now using the fact, that the states $|\iota\rangle$ and $|\iota'\rangle$ are $SU(2)$-gauge invariant we can replace $\sum_{i=1}^N m_i^2$ by $\sum_{i=1}^N \frac{j_i(j_i+1)}{3}$ in the formula above (see Appendix B) and obtain

$$|\Phi^m_{i'}(\eta_1)| \leq |\langle\iota | \iota'\rangle| |\mathcal{C}(\eta_1)|^{\sum_{i=1}^N j_i+1} = |\langle\iota | \iota'\rangle| |\mathcal{C}(\eta_1)|^{\sum_{i=1}^N j_i+1/2}$$

(56)

(see (B.15)). Since $\forall_{\eta_1>0} \mathcal{C}(\eta_1) < 1$ (see Appendix C.1), it is obvious, that

$$|\Phi^m_{i'}(\eta_1)| \ll |\Phi^m_{i'}(0)| \quad \text{for} \quad J \gg 1$$

(57)

4.4. Smoothness check

Having proven that the maximal critical point of $\phi_{i'}$ is at $\eta = 0$ (if any), let us check, whether the exponent part of the integrand is smooth at this point, i.e. if $\left.\frac{d\phi_{i'}}{d\eta}\right|_{\eta=0} = 0$? Using the property [47] we have

$$\left.\frac{d\phi_{i'}}{d\eta}\right|_{\eta=0} = \frac{1}{2} \sum_{\vec{m}} |\vec{m} \cdot \vec{m}'| \cdot e^{i\vec{m} \cdot \vec{m}'} \frac{d\phi_{i'}(\eta)}{d\eta} \bigg|_{\eta=0} = \sum_{\vec{m}} |\vec{m} \cdot \vec{m}'| \left.\frac{d\phi_{i'}}{d\eta}\right|_{\eta=0}$$

(58)

We will show, that $\left.\frac{d\phi_{i'}}{d\eta}\right|_{\eta=0} = 0$ for all $\vec{m}$.

Let us then calculate:

$$\frac{\partial}{\partial \eta} \phi_{\vec{m}}(\eta, J) = \frac{1}{J} \sum_{i=1}^N \frac{d\phi_{i'}(\eta)}{d\eta}$$

(59)

Details of calculations can be found in Appendix A.2.2 the result is

$$\left.\frac{\partial}{\partial \eta} \phi_{\vec{m}}(\eta, J)\right|_{\eta=0} = -\frac{1}{J} \sum_{i=1}^N \delta_i - 2 \frac{a_i b_i}{c_i}$$

(61)
using simple algebra (similarly to (A.14)) we can simplify it to:

\[
\left. \frac{\partial}{\partial \eta} \phi_{\vec{m}}(\eta, J) \right|_{\eta=0} = -i\gamma \sum_{i=1}^{N} m_i \left( J \right) + O \left( J^{-1} \right)
\]  

(62)

Thanks to SU(2) invariance of the states we act on, the sum \( \sum_{i=0}^{N} m_i = 0 \). The second term vanish when the limit \( J \to \infty \) is taken. Thus for all \( \vec{m} \)

\[
\left. \frac{d}{d\eta} \phi_{\vec{m}}(\eta) \right|_{\eta=0} = 0
\]

(63)

so

\[
\left. \frac{d\phi^i}{d\eta} \right|_{\eta=0} = 0
\]

(64)

so the exponent part of the integrand is smooth at the maximal critical point.

4.5. Asymptotics check

As we have already shown in 56, the modulus of the integrand is bounded by

\[
|\Phi^i(\eta)| \leq \left( e^{1-2\eta-e^{-2\eta}} \right)^{1/2} \sum_{i=1}^{N} x_i + \frac{1}{2}
\]

(65)

Thus and for arbitrary small \( \epsilon \) one can find such an \( \eta_\epsilon \) that

\[
\forall J > 1 \quad \left| \int_{\eta_\epsilon}^{\infty} d\eta \mu(\eta) \Phi^i(\eta) \right| < \epsilon
\]

(66)

Indeed, this integral can be estimated by

\[
\left| \int_{\eta_\epsilon}^{\infty} d\eta \mu(\eta) \Phi^i(\eta) \right| \leq \int_{\eta_\epsilon}^{\infty} d\eta |\mu(\eta)\Phi^i(\eta)| \\
\leq \int_{\eta_\epsilon}^{\infty} d\eta \left( \frac{\sinh \eta}{4\pi \eta} \right)^{2} \left( e^{1-2\eta-e^{-2\eta}} \right)^{1/2} \sum_{i=1}^{N} \left( x_i + \frac{1}{2} \right)
\]

(67)

where the last inequality holds for \( \frac{1}{2} \sum_{i=1}^{N} \left( x_i + \frac{1}{2} \right) > 3 \) (see Appendix C.2) - what is for sure true for \( J > 36 \), and since we consider the large \( J \) limit, this condition holds in the limit. As it is shown in Appendix C.3, the inequality \( \frac{1}{2} \left( \frac{e}{4\pi \eta} \right)^{2} \left( e - e^{1-e^{-2\eta}} - e^{1-2\eta-e^{-2\eta}} \right) \) < \( \epsilon \) has always a solution. Thus the assumptions of the lemma of section 3.3 are satisfied, so our integrand \( \Phi^i \) has proper asymptotic behaviour and the SPA method can be applied.

4.6. Second derivative

We need to know the Hessian determinant

\[
\left| \frac{d^2 \phi^i}{d\eta^2} \right|_{\eta=0}
\]

at \( \eta = 0 \), which thanks to the spherical symmetry is equal

\[
\left| \frac{d^2 \phi^i}{d\eta^2} \right|_{\eta=0}^3
\]

(see 3.2).
To calculate the second derivative of $\phi_i'(\eta)$ we will use the equation (58):

$$\frac{d^2\phi_i'}{d\eta^2} = \frac{d}{d\eta} \sum_{m} \overline{m} m' e^{J\phi_m(\eta)} \frac{d\Phi_i'}{d\eta}$$

$$= \sum_{m} \overline{m} m' e^{J\phi_m(\eta)} \frac{d^2\phi_m}{d\eta^2} - \frac{d\Phi_i'}{d\eta} \sum_{m} \overline{m} m' e^{J\phi_m(\eta)} \frac{d\phi_m}{d\eta} \frac{d\phi_m}{d\eta}$$

(68)

The second term in above formula vanish at $\eta = 0$. Indeed, in section 4.4 we have checked, that $\forall m \frac{d\phi_m}{d\eta} \bigg|_{\eta=0} = 0$ up to $O(J^{-1})$ corrections. Thus noting, that $\Phi_i'(0) = \langle \iota | \iota' \rangle$ and $\phi_m(0) = 0$, we have

$$\frac{d^2\phi_i'}{d\eta^2} \bigg|_{\eta=0} = \frac{1}{\langle \iota | \iota' \rangle} \sum_{m} \overline{m} m' \frac{d^2\phi_m}{d\eta^2} \bigg|_{\eta=0}$$

(69)

Let us then analyse $\frac{d^2\phi_m}{d\eta^2} \bigg|_{\eta=0}$. Obviously it decomposes into a sum

$$\frac{d^2\phi_m}{d\eta^2} \bigg|_{\eta=0} = \sum_{i=1}^{N} \frac{1}{J} \frac{d^2 \ln \left| f_{m_i}^{(j_i)} \right|}{d\eta^2} \bigg|_{\eta=0}$$

(70)

Using calculations of Appendix A.2.3 we get (up to $\frac{1}{2}$ corrections)

$$\frac{d^2 \ln \left| f_{m_i}^{(j_i)} \right|}{d\eta^2} \bigg|_{\eta=0} = -(1 + \gamma^2) \frac{(j_i + 1)^2 - m_i^2}{(2j_i + 3)} \left(1 + O(J^{-1})\right)$$

(71)

so (neglecting the $O(J^{-1})$ terms)

$$\frac{d^2\phi_m}{d\eta^2} \bigg|_{\eta=0} = -\frac{1 + \gamma^2}{J} \sum_{i=1}^{N} (j_i + 1)^2 - m_i^2$$

(72)

Now recalling (69) we have

$$\frac{d^2\phi_i'}{d\eta^2} \bigg|_{\eta=0} = \frac{1}{\langle \iota | \iota' \rangle} \sum_{m} \overline{m} m' \left( -\frac{1 + \gamma^2}{J} \right) \sum_{i=1}^{N} (j_i + 1)^2 - m_i^2$$

(73)

Thanks to $SU(2)$ symmetry we may use the lemma of Appendix B and substitute each term $m_i^2$ by $\frac{\hat{\iota} (j_i + 1)}{3}$ (see (B.16)) and then we get

$$\frac{d^2\phi_i'}{d\eta^2} \bigg|_{\eta=0} = -\frac{1 + \gamma^2}{J} \sum_{i=1}^{N} (j_i + 1) \frac{\hat{\iota} (j_i + 1)}{3}$$

(74)

The $\iota$-dependent factor cancels out, because $\sum_{m} \overline{m} m' m = \langle \iota | \iota' \rangle$:

$$\frac{d^2\phi_i'}{d\eta^2} \bigg|_{\eta=0} = -\frac{1 + \gamma^2}{3} \left( \sum_{i=1}^{N} x_i + \frac{1}{J} \right)$$

(75)

Neglecting the order terms simplify the formula a lot and gives

$$\frac{d^2\phi_i'}{d\eta^2} \bigg|_{\eta=0} = -\frac{1 + \gamma^2}{3} \sum_{i=1}^{N} x_i$$

(76)
So the Hessian determinant is
\[ \left| \frac{\partial^2 \phi'}{\partial \eta^2} \right|_{\eta=0} = \left[ \frac{1 + \gamma^2}{3} \sum_{i=1}^{N} x_i \right]^3 \] (77)

5. Summary and applications

5.1. Lorentzian Polyhedra Propagator

Let us summarise all the calculations done in sections 3 and 4. The Lorentzian Polyhedra Propagator \( T \) given by the integral
\[ T_{\ell'} = \int_{\mathbb{R}^3} d^3 \eta \mu(\eta) \Phi_{\ell'}(\eta) \] (78)
for \( J \gg 1 \) can be approximated by the value of integrand at \( \eta = 0 \)
\[ T_{\ell'} = \left( \frac{2\pi}{J} \right)^\frac{5}{2} \left| \frac{\partial^2 \phi}{\partial \eta^2} \right|_{\eta=0} \mu(0) \Phi_{\ell'}(0) \left( 1 + O \left( J^{-1} \right) \right) \] (79)

Noting that \( \mu(0) = \left( \frac{1}{4\pi} \right)^2 \), \( \Phi_{\ell'}(0) = \sum_{m,m'} \bar{m}_m m'_m \delta_{\bar{m},m'} = \langle \ell | \ell' \rangle \), taking the value of \( \left| \frac{\partial^2 \phi}{\partial \eta^2} \right|_{\eta=0} \) from (77) and neglecting the \( \frac{1}{J} \) corrections we get
\[ T_{\ell'} = \left( \frac{1}{4\pi} \right)^2 \left( \frac{2\pi}{J} \right)^\frac{5}{2} \left[ \frac{3}{(1 + \gamma^2) \sum_{i=1}^{N} x_i} \right]^{\frac{3}{2}} \langle \ell | \ell' \rangle \] (80)

This means that the leading order of \( T \) is proportional to identity:
\[ T = \left( \frac{1}{4\pi} \right)^2 \left( \frac{2\pi}{J} \right)^\frac{5}{2} \left[ \frac{3}{(1 + \gamma^2) \sum_{i=1}^{N} x_i} \right]^{\frac{3}{2}} \mathbb{I} \] (81)

The higher order terms of \( T \) are not known yet, but we have already found (see (44)), that in the basis \( |\bar{m} \rangle \) only the diagonal terms are nonzero:
\[ T_{\bar{m} \bar{m}'} = \left( \frac{1}{4\pi} \right)^2 \left( \frac{2\pi}{J} \right)^\frac{5}{2} \left[ \frac{6\pi}{J \left( 1 + \gamma^2 \right) \sum_{i=1}^{N} x_i} \right]^{\frac{3}{2}} + \left( \frac{1}{J} \right)^{\frac{5}{2}} T_{\bar{m} \bar{m}'} \] (82)

5.2. Application in Dipole Cosmology

According to [12] the DC transition amplitude is
\[ W(z) = \sum_{\{j\ell\}} \prod_{\ell=1}^{4} (2j_{\ell} + 1) e^{-2h_{j_{\ell}}(j_{\ell} + 1) - i\lambda \nu_{j_{\ell}} \frac{3}{2} - iz_{j_{\ell}}} \int_{SL(2,\mathbb{C})} d g \prod_{\ell=1}^{4} \langle j_{\ell} | u_{n_{\ell}^0} Y Y Y u_{n_{\ell}^0} | j_{\ell} \rangle \] (83)
Our calculations influence only the term $\langle \iota | T | \iota' \rangle$. Authors assume that it behaves like $\frac{N_0}{j_0^3}$ (with $j_0 = \frac{3z}{4\pi}$ and $N_0$ being a constant for a given graph). The direct formula for the Lorentzian Polyhedra propagator shows, that it is

$$\langle \iota | T | \iota' \rangle = \tilde{N}(x_i) \cdot J^{-\frac{3}{2}} \langle \iota | \iota' \rangle$$

(84)

where the factor $\tilde{N}(x_i)$ depend only on ratios of spins $x_i = \frac{j_i}{J}$, but it does not scale with $J$. The coherent $SU(2)$-intertwiner have norm squared scaling as $j^{-\frac{3}{2}}$ (see [9]), thus the overall scaling of $\langle \iota | T | \iota' \rangle$ is $N \cdot j^{-3}$, as expected. So taking into account the direct calculation of the Lorentzian Polyhedra Propagator, the Dipole Cosmology transition amplitude reproduce the original DC formula of [11, 12]:

$$W(z) = \sum_{\{j_k\}} \frac{N(x_i)}{j_0^3} \prod_{\ell=1}^4 (2j_\ell + 1) e^{-2th_{j_\ell}(j_\ell + 1) - i\lambda_{\ell}j_\ell^2 - iz_j}$$

(85)

with a minor correction: the graph-shape dependent factor $N$ now depends also on ratios of spins.

Using the same summation technique, as in [11], (i.e. summing by integrating the Gaussian integrals) one obtains the same result

$$W(z) = \frac{N_0}{j_0^3} \left(2j_0^4 \sqrt{\frac{\pi}{t}} e^{-\frac{z^2}{8t\hbar}}\right)^4 = N_0 e^{-\frac{z^2}{2t\hbar}}$$

(86)

with $N_0$ being the value of $N(x_i)$ factor calculated for all $x_i = 1$.

5.3. Application in radiative corrections

As we have already recalled, the radiative corrections to the spin-foam edge are proportional to the $T^2$ operator[15]. To be precise, given a cutoff $\Lambda$ on the internal spins of the bubble, the leading order of the transition amplitude are the matrix elements of the following operator

$$W^\Lambda = \Lambda^{6(\mu - 1)\pi^2}$$

(87)

where $\mu$ is the parameter dependent on the model chosen (i.e. it is a degree of the polynomial that is used to define the face amplitude). For $\mu = 1$ the exponent is zero, which means, that the divergence is $O(\ln(\Lambda))$. Our calculation shows, that in large $J$ limit (with $J$ being the maximal external spin of the external faces of the bubble) up to $\frac{1}{J}$ corrections this radiative-corrected edge operator is proportional to the identity with the constant dependent on $\Lambda$ and $J$:

$$W^\Lambda_{\iota,\iota'} = \ln \Lambda \cdot \left(\frac{1}{J}\right)^3 \frac{27}{32\pi} \left[\frac{1}{(1 + \gamma^2)\sum_{i=1}^N x_i}\right]^3 \delta_{\iota,\iota'}$$

(88)

The factor in front of the delta we will call $a(J, \Lambda)$:

$$a(J, \Lambda) := \ln \Lambda \cdot \left(\frac{1}{J}\right)^3 \frac{27}{32\pi} \left[\frac{1}{(1 + \gamma^2)\sum_{i=1}^N x_i}\right]^3$$

(89)

We will use it in the next subsection in the edge-renormalization problem.

Note that since $\Lambda$ is the maximum spin appearing in the bubble causing infinity, whereas $J$ is the maximum spin appearing on the external faces, they cannot be identified. To make the formula [88] correct we have to assume $1 \ll J \ll \Lambda$. 

Asymptotic of Lorentzian Polyhedra Propagator
5.4. Application in edge renormalization

Let us now consider a series of spin-foams with $\kappa_n$ such that they differ only by a number of vertices on one selected edge $e$, i.e. $\kappa_n$ has $n$ vertices (see figure 1). Then the spin-foam operator related to each $\kappa_n$ is

$$W_{\kappa_n} = W_{\kappa_0} \cdot T^n$$  \hspace{1cm} (90)

Obviously $\|T\| < 1$. Indeed, the factor $\alpha := \left(\frac{1}{\pi^2}\right)^2 \left(\frac{6\pi}{(1+\gamma^2)\sum_{i=1}^N x_i}\right)^{3/2} < \frac{0.52}{(1+\gamma^2)\sum_{i=1}^N x_i}^{3/2}$ is less than 1, because $1 + \gamma^2 \geq 1$ and $\sum_{i=1}^N x_i \geq 2$. Moreover by assumption $J \gg 1$. Thus we can sum the leading orders in this series to

$$W_R^{\kappa} = \sum_{n=0}^{\infty} W_{\kappa_n} = W_{\kappa_0} \frac{1}{1 - \alpha J^{-3/2}}$$  \hspace{1cm} (91)

The same procedure can be done for a series of spin-foams $\bar{\kappa}_N$ that differ by a number of bubbles on a 4-valent edge (see figure 2). Then, using (88) we get

$$W_R^{\bar{\kappa}, \text{bubble}} = \sum_{n=0}^{\infty} W_{\bar{\kappa}_n} = W_{\kappa_0} \frac{1}{1 - a(J, \Lambda)}$$  \hspace{1cm} (92)

The above formula is convergent iff $a(J, \Lambda) < 1$.

Assume now for a moment, that the maximum spin $\Lambda$ is the inverse cosmological constant expressed in Plank units, i.e. that $\Lambda = 10^{120}$ (such choice is justified in [15]). Then we get an upper bound:

$$a(J, \Lambda) < \frac{9.28}{J^3}$$  \hspace{1cm} (93)

thus for $J \geq 2^{1/4}$ the sum (92) is convergent. Note, that all our approximations were made in large $J$ limit, so they do not apply for $J \leq 2$, and thus it is possible, that the sum (92) is convergent for all $J$s.
Figure 2. The foams with increasing number of “melonic” bubbles $\tilde{\kappa}_0$, $\tilde{\kappa}_1$, $\tilde{\kappa}_2$, etc. form a geometric series, that can be summed to $\sum_{n=0}^{\infty} \tilde{\kappa}_n = \tilde{\kappa}_0 \frac{1}{1 - \ln \Lambda T}$.

6. Conclusions and further directions

The Lorentzian Polyhedra Propagator is an important object in Spin-foam theory. It clarifies behaviour of the edge amplitude. It adds precision to calculations of Dipole Cosmology and “melonic” bubble divergence. It opens a path to study renormalization.

The tools developed in this study may be useful in direct computation of more complicated transitions amplitudes. The matrix elements of Lorentzian EPRL representation are key objects in Spin-foam calculations, and their properties were investigated here (mainly in the appendices).

A natural following step is to study subleading order of the $T$ operator. We expect $T$ can be approximated by an operator similar to a heat kernel and the subleading order would describe its spread.

Another further direction is to study in more detail the issue of renormalization in Spin-foam models. It contains both the low-$J$ behaviour of bubble divergence, more complicated bubbles (i.e. bubbles more than 4-valent edges) and higher order bubbles.

Acknowledgments

Firstly I would like to thank a lot Carlo Rovelli and Aldo Riello for inspiring discussions and valuable remarks. Many thanks to Wojciech Kamiński, Marcin Kisielowski and Jędrzej Świeżewski for lots of technical advice and to Jerzy Lewandowski for remarks on the draft of the article. I also thank the participants of Emerging Fields Initiative winter conference on canonical and covariant LQG, for many interesting comments.

This work was supported by the Foundation for Polish Science International PhD Projects Programme co-financed by the EU European Regional Development Fund.
Appendix A. Same calculations:

In the appendix let us present some technical details about calculations.

Appendix A.1. Definitions

Let us recall some definitions:

Firstly the Gauss hypergeometric function in its series and integral representation:

\[ \binom{a}{b}{c}{z} := \sum_{k=0}^{\infty} \frac{a^k b^k}{c^k k!} z^k = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \]  (A.1)

where

\[ x^\overline{k} := \prod_{n=0}^{k-1} (x+n) \]  (A.2)

For \(|z| < 1\) the series is convergent. Note that if all arguments \(a, b, c > 0\) and \(0 \leq z < 1\), then the series is sum of real positive numbers, thus in this case \(2F_1(a, b; c; z) \geq 0\).

The function \(f^{(j)}_m(\eta)\):

\[ f^{(j)}_m(\eta) = e^{-(j+1+m-ij\gamma)} 2F_1 \left( j + 1 - ij, j + 1 + m; 2j + 2; 1 - e^{-2\eta} \right) \]  (A.3)

Let us define another function, \(g^{(j)}_m(\eta)\), by setting \(\gamma = 0\) in the above formula:

\[ g^{(j)}_m(\eta) = e^{-(j+1+m)\eta} 2F_1 \left( j + 1, j + 1 + m; 2j + 2; 1 - e^{-2\eta} \right) \]  (A.4)

One can easily see, that \(g^{(j)}_m(\eta) \geq 0\). Indeed, all the parameters of \(2F_1\) function are positive.

In this section it is convenient to consider \(f^{(j)}_m\) and \(g^{(j)}_m\) as functions of \(z(\eta)\):

\[ f^{(j)}_m(\eta) = (1-z)^{(j+1+m-ij)} 2F_1 \left( j + 1 - ij, j + 1 + m; 2j + 2; z \right) \]  (A.5)

We will also use a following shortcut notation:

\[ J := j + 1 \quad G := \gamma j \]  (A.6)

\[ a := J - iG \quad b := J + m \quad c := 2J \quad \delta := J + m - iG \]  (A.7)

thus

\[ f^{(j)}_m(\eta) = (1-z)^{\frac{J}{2}} \binom{a}{b}{c}{z} \]  (A.8)

in case of \(g^{(j)}_m(\eta)\) it is convenient to use also \(\tilde{a} := J\), so

\[ g^{(j)}_m(\eta) = (1-z)^{\frac{b}{2}} \binom{\tilde{a}}{b}{c}{z} \]  (A.9)
Appendix A.2. Some basic properties of \( f_m^{(j)}(\eta) \) function

Appendix A.2.1. Mirror symmetry: \( f_m^{(j)}(z) = f_{-m}^{(j)}(z) \) Let us use the symmetry of \( 2F_1 \) function:

\[
2F_1(a, b; c; z) = (1 - z)^{-a - b} 2F_1(c - a, c - b; c; z)
\]

and the mirror symmetry \( 2F_1(\frac{\alpha}{2}, \frac{b}{2}, \frac{c}{2}, z) = 2F_1(a, b; c; z) \) to relate \( f_m^{(j)}(z) \) with \( f_{-m}^{(j)}(z) \). In our case \( c - a = J + iG = \overline{c} \) and \( c - b = J - m \), and \( c - a - b = -m + iG \), thus for \( 0 \leq z < 1 \) we have:

\[
f_m^{(j)}(z) = (1 - z)^{-J} 2F_1(J - iG, J + m; 2J; z) = (1 - z)^{-J-m+iG} 2F_1(J + iG, J - m; 2J; z) = (1 - z)^{-J-m+iG} f_{-m}^{(j)}(z)
\]

Obviously the same calculation show that \( g_m^{(j)}(z) = g_{-m}^{(j)}(z) \).

Appendix A.2.2. First derivative of \( f_m^{(j)}(z(\eta)) \) Let us now calculate the first derivative of \( f_m^{(j)} \) and \( g_m^{(j)} \) with respect to \( \eta \). Using \( \partial_z 2F_1(a, b; c; z) = \frac{ab}{c} 2F_1(a + 1, b + 1; c + 1, z) \) we get:

\[
\frac{df_m^{(j)}(z(\eta))}{d\eta} = -\delta e^{-\delta \eta} 2F_1(a, b; c; z)
\]

\[
+ e^{-\delta \eta} \frac{dz(\eta)}{d\eta} \frac{ab}{c} 2F_1(a + 1, b + 1; c + 1; z)
\]

using (A.7) and \( z = 1 - e^{-2\eta} \) we get \( \frac{dz}{d\eta} = 2e^{-2\eta} = 2(1 - z) \), \( \frac{ab}{c} = \frac{(J - iG)(J + m)}{2J} = \frac{\delta}{2} - \frac{iGm}{2J} \) so

\[
\frac{df_m^{(j)}(z(\eta))}{d\eta} = -\delta e^{-\delta \eta} 2F_1(a, b; c; z)
\]

\[
+ e^{-\delta} \left( \frac{\delta - \frac{iGm}{J}}{2} \right) 2F_1(a + 1, b + 1; c + 1; z)
\]

At \( \eta = 0 \) we use \( z(0) = 0 \) and \( 2F_1(a; b, c ; 0) = 1 \) to obtain

\[
\frac{df_m^{(j)}(\eta)}{d\eta} \bigg|_{\eta=0} = -\delta + \delta - \frac{iGm}{J} = -\frac{iGm}{J} = -\frac{j m}{j + 1} = -im \left( 1 + O \left( \frac{1}{j} \right) \right)
\]

To find the derivative of \( g_m^{(j)}(\eta) \) we just set \( G = 0 \) in the above formulae and get

\[
\frac{dg_m^{(j)}(z(\eta))}{d\eta} = -\delta e^{-\delta \eta} 2F_1(a, b; c; z) + \delta e^{-\delta + 2\eta} 2F_1(a + 1, b + 1; c + 1; z)
\]

\[
= -\delta e^{-\delta \eta} \left[ 2F_1(a, b; c; z) - (1 - z) 2F_1(a + 1, b + 1; c + 1; z) \right]
\]

We can simplify the hypergeometric term using the series expansion:

\[
2F_1(a, b; c; z) - (1 - z) 2F_1(a + 1, b + 1; c + 1; z) =
\]
\[ = 1 + \sum_{k=1}^{\infty} \frac{\alpha_k b_k}{e^k!} z^k - 1 - \sum_{k=1}^{\infty} \frac{(a + k) k + b + 1}{(c + 1)^k!} z^k + \sum_{k=1}^{\infty} \frac{(a + k - 1) k + b + 1}{(c + 1)^k!} z^k \]

\[ = \sum_{k=1}^{\infty} \frac{\alpha_k b_k}{e^k!} z^k \left[ 1 - \frac{(a + k) k + b + 1}{c(b + k) + \frac{ck}{ab}} \right] \]

\[ = \sum_{k=1}^{\infty} \frac{\alpha_k b_k}{e^k!} z^k \]

\[ = \sum_{k=1}^{\infty} \frac{\alpha_k b_k}{e^k!} \frac{(c - a)(c - b)}{ab(c + k)} \]

\[ = z \frac{(c - a)(c - b)}{c(c + 1)} \sum_{k=1}^{\infty} \frac{(a + 1) k - 1}{(c + 2)^k!} z^{k-1} \]

\[ = z \frac{(c - a)(c - b)}{c(c + 1)} \sum_{k=1}^{\infty} \frac{(a + 1)(b + 1)^k}{(c + 2)^k} z^k = \]

\[ = z \frac{(c - a)(c - b)}{c(c + 1)} {}_2F_1(a + 1, b + 1; c + 2; z) \]

so

\[ \frac{d^2 f_m^{(j)}(z(\eta))}{d\eta^2} = -\delta e^{-\delta \eta} z(\eta) \frac{(c - a)(c - b)}{c(c + 1)} {}_2F_1(a + 1, b + 1; c + 2; z(\eta)) \]  

what we will need in Appendix A.3.

Appendix A.2.3. Second derivative of \( f_m^{(j)}(z(\eta)) \) and \( \ln \left[ f_m^{(j)}(z(\eta)) \right] \) at \( \eta = 0 \)

In section 4.6 we need to know \( \frac{d^2}{d\eta^2} \ln \left[ f_m^{(j)}(z(\eta)) \right] \) at \( \eta = 0 \).

Let us first calculate \( \frac{d^2 f_m^{(j)}(z(\eta))}{d\eta^2} \). By differentiating the equation (A.13) and recalling, that \( (\delta - \frac{iG_m}{\tau}) = 2 \frac{ab}{c} \), we get

\[ \frac{d^2 f_m^{(j)}(z(\eta))}{d\eta^2} = -\delta \left[ -\delta e^{-\delta \eta} {}_2F_1(a, b; c; z(\eta)) + 2 \frac{ab}{c} e^{-(\delta + 2)\eta} {}_2F_1(a + 1, b + 1; c + 1; z(\eta)) \right] \]

\[ - (\delta + 2) 2 \frac{ab}{c} e^{-(\delta + 2)\eta} {}_2F_1(a + 1, b + 1; c + 1; z(\eta)) \]

\[ + 2 \frac{ab}{c} \frac{(a + 1)(b + 1) d(\eta)}{d\eta} e^{-(\delta + 2)\eta} {}_2F_1(a + 2, b + 2; c + 2; z(\eta)) \]  

(A.18)

Fortunately we do not need to know the second derivative for all \( \eta \), it is enough to find its value at \( \eta = 0 \), where \( z(0) = 0 \) and \( {}_2F_1(a, b; c; 0) = 1 \), so the formula simplifies:

\[ \left. \frac{d^2 f_m^{(j)}(z(\eta))}{d\eta^2} \right|_{\eta=0} = -\delta \left[ -\delta e^{-\delta \eta} + 2 \frac{ab}{c} e^{-(\delta + 2)\eta} \right] \]

\[ - (\delta + 2) 2 \frac{ab}{c} e^{-(\delta + 2)\eta} \]

\[ + 2 \frac{ab}{c} \frac{(a + 1)(b + 1) d(\eta)}{d\eta} e^{-(\delta + 2)\eta} \]  

(A.19)

Now using (A.7) one get

\[ \left. \frac{d^2 f_m^{(j)}(z(\eta))}{d\eta^2} \right|_{\eta=0} = 4 \left[ \frac{\delta^2}{4} - \frac{ab \delta}{c} - \frac{ab}{c} \frac{(a + 1)(b + 1)}{(c + 1)} \right] \]  

(A.20)
We are interested in the second differential of \( \ln \left[ f_{m}^{(j)} \left( z(\eta) \right) \right] \). Using the identity

\[
\frac{d^2 \ln f(x)}{dx^2} = \frac{f''(x)f(x) - [f'(x)]^2}{[f(x)]^2} \tag{A.21}
\]

and the fact, that \( f_{m}^{(j)} (z(0)) = 1 \) we have

\[
\left. \frac{d^2 \ln \left[ f_{m}^{(j)} (z(\eta)) \right]}{d\eta^2} \right|_{\eta=0} = \left. \frac{d^2 f_{m}^{(j)} (z(\eta))}{d\eta^2} \right|_{\eta=0} - \left( \frac{df_{m}^{(j)} (z(\eta))}{d\eta} \right|_{\eta=0} \right)^2 \tag{A.22}
\]

putting here the expressions \( \text{ref} \) and \( \text{reff} \) we get

\[
\left. \frac{d^2 \ln \left[ f_{m}^{(j)} (z(\eta)) \right]}{d\eta^2} \right|_{\eta=0} = 4 \left[ \frac{\delta^2}{4} - \frac{ab\delta}{c} - \frac{ab}{c} + \frac{ab \left( a + 1 \right) \left( b + 1 \right)}{c (c + 1)} \right] - \left[ -\delta + 2 \frac{ab}{c} \right]^2
\]

\[
= \delta^2 - 4 \frac{ab\delta}{c} - 4 \frac{ab}{c} + 4 \frac{ab \left( a + 1 \right) \left( b + 1 \right)}{c (c + 1)} - \delta^2 + 4 \frac{ab\delta}{c} - 4 \frac{ab}{c} \tag{A.23}
\]

Now using \( \text{A.7} \) we have

\[
\left. \frac{d^2 \ln \left[ f_{m}^{(j)} (z(\eta)) \right]}{d\eta^2} \right|_{\eta=0} = -4 \left( J^2 + G^2 \right) \left( J^2 - m^2 \right) \left( \frac{J^2}{2J + 1} \right) = - \left( 1 + \frac{G^2}{J^2} \right) \frac{J^2 - m^2}{(2J + 1)} \tag{A.24}
\]

and using \( \text{A.6} \)

\[
\left. \frac{d^2 \ln \left[ f_{m}^{(j)} (z(\eta)) \right]}{d\eta^2} \right|_{\eta=0} = - \left( 1 + \left( \frac{\gamma j}{j + 1} \right)^2 \right) \frac{(j + 1)^2 - m^2}{(2j + 3)} \tag{A.25}
\]

Since \( \frac{\gamma j}{j + 1} = \gamma \left( 1 + O \left( j^{-1} \right) \right) \), the leading order of \( \left. \frac{d^2 \ln \left[ f_{m}^{(j)} (z(\eta)) \right]}{d\eta^2} \right|_{\eta=0} \) is

\[
\left. \frac{d^2 \ln \left[ f_{m}^{(j)} (z(\eta)) \right]}{d\eta^2} \right|_{\eta=0} = - \left( 1 + \gamma^2 \right) \frac{(j + 1)^2 - m^2}{(2j + 3)} \left( 1 + O \left( j^{-1} \right) \right) \tag{A.26}
\]

The term dependent on \( m \) can be further simplified - see \( \text{Appendix B} \).
Appendix A.3. Estimations

It is easy to see, that \( |f_m^{(j)}(\eta)| \leq g_m^{(j)}(\eta) \). Indeed, taking the integral representation of \( _2F_1 \) we see, that

\[
\left| f_m^{(j)}(z(\eta)) \right| = \left| e^{-(j+1+m-i\gamma)\eta} \frac{(2j+1)!}{(j+m)!(j-m)!} \int_0^1 dt \frac{t^j m \left(1-t\right)^{j-m}}{[1-(1-e^{-2\eta}) t]^{(j+1)-i\gamma j}} \right|
\]

\[
= e^{-(j+1+m)\eta} \frac{(2j+1)!}{(j+m)!(j-m)!} \int_0^1 dt \frac{t^j m \left(1-t\right)^{j-m}}{[1-(1-e^{-2\eta}) t]^{(j+1)-i\gamma j}}
\]

\[
\leq e^{-(j+1+m)\eta} \frac{(2j+1)!}{(j+m)!(j-m)!} \int_0^1 dt \frac{t^j m \left(1-t\right)^{j-m}}{[1-(1-e^{-2\eta}) t]^{(j+1)-i\gamma j}}
\]

\[
= e^{-(j+1+m)\eta} \frac{2F_1(j+1, j+1; 2j+2; 1-\eta e^{-2\eta})}{2F_1(j+1, j+1; 2j+2; 1-\eta e^{-2\eta})} = g_m^{(j)}(z(\eta))
\]

Let us now estimate \( g_m^{(j)}(\eta) \) from above and from below.

To estimate \( g_m^{(j)} \) we will estimate its derivative. We will consider bounds of \( \frac{dg_m^{(j)}}{d\eta} \):

\[
\frac{dg_m^{(j)}}{d\eta}(z(\eta)) = -\frac{(J+m)(J-m)}{2(2J+1)} \frac{z_2F_1(J+1, J+1; 2(J+1); z)}{2F_1(J, J+m; 2J; z)}
\]

(A.28)

To simplify the formulae let us introduce a constant \( \omega := \frac{(J+m)(J-m)}{2(2J+1)} = \frac{b(c-a)}{2(c+1)} \).

Appendix A.3.1. Estimation of \( g_m^{(j)}(z) \) from above One can easily see, that for nonnegative \( a, b, c, \)

for \( 0 \leq z < 1 \) and for \( c = 2a \) and \( b \leq c \) the following inequality holds

\[
2F_1(a, b; c; z) \leq 2F_1(a+1, b+1; c+2; z)
\]

(A.29)

Indeed, using the series representation we can compare each term, obtaining:

\[
1 = 1 \quad (k = 0)
\]

\[
\frac{ab}{2a} < \frac{(a+1)(b+1)}{2a+2} \quad (k = 1)
\]

\[
\frac{ab}{(2a)(2a+1)} < \frac{(a+k)(b+k)}{(2a+k)(2a+k+1)} \quad (k \geq 2)
\]

where the last inequality is equivalent to

\[
0 < k^2 a (4a + 2 - b) + k (4a^2 + 2a^2 + ab)
\]

(A.31)

which is satisfied for our assumptions. Thus \( \frac{2F_1(J+1, J+m+1; 2J+2; z)}{2F_1(J, J+m; 2J; z)} \geq z \), so

\[
\frac{dg_m^{(j)}}{d\eta}(z(\eta)) \leq -\frac{(J+m)(J-m)}{2(2J+1)} z(\eta) = -\omega (1-e^{-2\eta})
\]

(A.32)

Integrating the above equations one gets

\[
\ln \left( g_m^{(j)} \right) \big|_0^\eta \leq \frac{\omega}{2} - \omega \eta - \frac{\omega}{2} e^{-2\eta}
\]

(A.33)
so we have the upper bound:

\[ g_m^{(j)}(z(\eta)) \leq \left[ e^{1-e^{-2\eta}-2\eta} \right]^{(j+m)(j-m)} \frac{4^{(j+1)}}{4(j+1)} \] (A.34)

and replacing \( J \) by \( j+1 \) (see (A.6))

\[ g_m^{(j)}(z(\eta)) \leq \left[ e^{1-e^{-2\eta}-2\eta} \right]^{(j+1+m)(j+1-m)} \frac{4^{(j+1)}}{4(j+1)} \] (A.35)

**Appendix A.3.2. Estimation of \( g_m^{(j)}(z) \) from below**  
To find the lower bound of (A.28) let us show, that under even weaker assumptions (i.e. \( a, b, c \geq 0 \) and \( 0 \leq z < 1 \)) the following is true:

\[ \exists \alpha > 0 \ 2F_1(a, b; c; z) \geq z \ 2F_1(a+1, b+1; c+2; z) \] (A.36)

Let us introduce auxiliary quantity \( A_\alpha := 2F_1(a, b; c; z) - z \ 2F_1(a+1, b+1; c+2; z) \) and use the series expansion of the hypergeometric functions:

\[
A_\alpha = \alpha + \sum_{k=1}^{\infty} \left( \frac{\alpha a b c}{c k!} \right) \zeta^k - \sum_{k=1}^{\infty} \left( \frac{(a+1)^{k-1}(b+1)^{k-1}}{(c+2)^{k-1}(k-1)!} \zeta^k \right)
\]

\[
= \alpha + \sum_{k=1}^{\infty} \left( \frac{\alpha a b c}{c k!} \right) \zeta^k \left( \alpha - \frac{c(c+1)k}{ab(c+k)} \right)
\]

\[
= \alpha + \sum_{k=1}^{\infty} \left( \frac{\alpha a b c}{c k!} \right) \zeta^k \left( \frac{a b c + a b c - c(c+1)k}{ab(c+k)} \right)
\]

\[
= \alpha + \sum_{k=1}^{\infty} \left( \frac{\alpha a b c}{c k!} \right) \zeta^k \left( \frac{a a b c c + a b c - c(c+1)k}{ab(c+k)} \right)
\]

\[
= \alpha + \sum_{k=1}^{\infty} \left( \frac{\alpha a b c}{c k!} \right) \zeta^k \left( \frac{a b c + a b c - c(c+1)k}{ab(c+k)} \right)
\]

\[
= \alpha + \sum_{k=1}^{\infty} \left( \frac{\alpha a b c}{c k!} \right) \zeta^k + \left( \frac{a b c + a b c - c(c+1)k}{ab(c+k)} \right)
\]

\[
= \alpha + \frac{a b c + a b c - c(c+1)k}{ab(c+k)}
\]

\[
= \alpha \ 2F_1(a, b; c; z) + \frac{z}{c(c+1)} \left( a b c + a b c - c(c+1) \right) \ 2F_1(a+1, b+1; c+2; z)
\]

For \( \alpha > 0 \) all the elements of the above formula are nonnegative, thus we have \( A_\alpha \geq 0 \) for \( \alpha > 0 \), which proves the lemma. Thus

\[
\frac{z \ 2F_1(a+1, b+1; c+2; z)}{2F_1(a, b; c; z)} \leq \alpha_0
\] (A.38)

and so

\[
\frac{d g_m^{(j)}(z(\eta))}{d \eta} \geq -\omega_0 = -\frac{b(c-b)}{2(c+1)} \frac{c(c+1)}{ab} = -\frac{c}{2a} (c-b)
\] (A.39)

In our case \( c = 2a \), so

\[
\frac{d g_m^{(j)}(z(\eta))}{d \eta} \geq -(c-b) = -(j+1-m)
\] (A.40)

Integrating above formula one gets

\[
g_m^{(j)}(z(\eta)) \geq e^{-\eta(j+1-m)}
\] (A.41)

So finally we have

\[
e^{-\eta(j+1-m)} \leq g_m^{(j)}(z(\eta)) \leq \left[ e^{1-e^{-2\eta}-2\eta} \right]^{(j+1+m)(j+1-m)} \frac{4^{(j+1)}}{4(j+1)}
\] (A.42)
Appendix B. Squared magnetic momentum number

Several times in the calculations above the squared magnetic number $m^2$ appears. It seem to break $SU(2)$ invariance, however when considering the invariant states $|\iota\rangle \in \text{Inv} \left( \mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N} \right)$, one can express such components in terms of gauge invariant quantities, what we will prove below.

At the beginning let us remind, that given an invariant state $|\iota\rangle$ we can decompose it in the magnetic momentum basis

$$ |\iota\rangle = \sum_{m_1, \ldots, m_N} t_{m_1, \ldots, m_N} |m_1\rangle_{j_1} \otimes \cdots \otimes |m_N\rangle_{j_N} =: \sum_{\vec{m}} t_{\vec{m}} |\vec{m}\rangle_{j} $$

(B.1)

note, that

$$ t_{\vec{m}} := \langle \vec{m} | \iota \rangle_{j} $$

(B.2)

We will learn how to compute the formulae of the form

$$ \sum_{\vec{m}} f \left( m_i^2, j_b \right) \sum_{\vec{m}'} t_{\vec{m}'}_{\vec{m}} $$

(B.3)

for a real analytic function $f$.

Appendix B.1. Single squared magnetic momentum number

First let us consider an expression $\sum_{\vec{m}} m_i^2 |m\rangle_{j} |\vec{m}\rangle_{j}$. Now using (B.2) we get

$$ \sum_{\vec{m}} m_i^2 |m\rangle_{j} |\vec{m}\rangle_{j} = \sum_{\vec{m}} \langle \iota | m \rangle_{j} \langle \vec{m} | \iota' \rangle_{j} j_i $$

(B.4)

Now since $\sum_{\vec{m}} |\vec{m}\rangle \langle \vec{m}|$ is the identity operator, we have

$$ \sum_{\vec{m}} m_i^2 |m\rangle_{j} |\vec{m}\rangle_{j} = \langle \iota | \frac{1}{3} L_{z,(i)}^2 |\iota' \rangle $$

(B.5)

Now thanks to $SU(2)$ invariance we have

$$ \langle \iota | \frac{1}{3} L_{z,(i)}^2 |\iota' \rangle = \frac{1}{3} \langle \iota | L_{z,(i)}^2 + L_{y,(i)}^2 + L_{z,(i)}^2 |\iota' \rangle $$

(B.6)

The operator $L_{z,(i)}^2$ is an invariant with eigenvalue $j_i(j_i + 1)$, thus after all we have

$$ \sum_{\vec{m}} m_i^2 |m\rangle_{j} |\vec{m}\rangle_{j} = \frac{j_i(j_i + 1)}{3} \langle \iota | \iota' \rangle $$

(B.7)
Appendix B.2. Real function of $m_i^2$

Consider now a real analytic function $f(m_i^2)$ instead of $m_i^2$. We will start with a polynomial: $(m_i^2)^n$. Since $\hat{L}_{z,i}^{-2}$ is a positive, selfadjoint operator, we can repeat the above procedure and obtain

$$\sum_\vec{m} (m_i^2)^n \tau_\vec{m}^t \vec{m} = |\ell| (\hat{L}_{z,i}^{-2})^n |\ell'| \quad (B.8)$$

Now we can insert an orthonormal intertwiner basis between each two $\hat{L}_{z,i}^{-2}$ operators:

$$\sum_\vec{m} (m_i^2)^n \tau_\vec{m}^t \vec{m} = \sum_{\ell_1, \ldots, \ell_{n-1}} |\ell_1| \hat{L}_{z,(i)}^{-2} |\ell_1 \ell_2 \ldots |\ell_{n-1} \hat{L}_{z,(i)}^{-2} |\ell'| \quad (B.9)$$

Each expression $|\ell_1 \hat{L}_{z,(i)}^{-2} |\ell_{I+1}|$ equals $\frac{1}{3} |\ell_1 \hat{L}_{z,(i)}^{-2} |\ell_{I+1}|$ (see (B.6)) giving the eigenvalue $\frac{\ell_i (\ell_i + 1)}{3}$, thus we have

$$\sum_\vec{m} (m_i^2)^n \tau_\vec{m}^t \vec{m} = [\frac{\ell_i (\ell_i + 1)}{3}]^n |\ell| |\ell'| \quad (B.10)$$

For a real analytic function $f$ more general than the polynomial we expand $f$ in a power series and follow the above steps for each power of $m_i^2$, obtaining

$$\sum_\vec{m} f(m_i^2) \tau_\vec{m}^t \vec{m} = f \left( \frac{\ell_i (\ell_i + 1)}{3} \right) |\ell| |\ell'| \quad (B.11)$$

Appendix B.3. Function of many $m_i^2$s

Consider now a real function $f(m_1^2, \ldots, m_N^2)$. Since the operators $\hat{L}_{z,(i)}^{-2}$ commute for different $i$, for each term $m_i^2$ we can do the procedure of Appendix B.2 separately, obtaining

$$\sum_\vec{m} f(m_1^2, \ldots, m_N^2) \tau_\vec{m}^t \vec{m} = f \left( \frac{\ell_1 (\ell_1 + 1)}{3}, \ldots, \frac{\ell_N (\ell_N + 1)}{3} \right) |\ell| |\ell'| \quad (B.12)$$

Appendix B.4. Function of many $m_i^2$s and many $j_i$s

At least let us consider a real function $f(m_1^2, \ldots, m_N^2, j_1, \ldots, j_N) =: f\left( m^2, \vec{j} \right)$. Note, that given an expression $\sum_\vec{m} f(m^2, \vec{j}) \tau_\vec{m}^t \vec{m}$, the $j_i$-dependent parts do not differ when $\vec{m}$ changes. We can thus follow the procedure of Appendix B.3, treating all $j_i$-dependences as parameters of the function, and obtain the result:

$$\sum_\vec{m} f\left( m^2, \vec{j} \right) \tau_\vec{m}^t \vec{m} = f \left( \frac{\vec{j} (\vec{j} + 1)^3}{3}, \vec{j} \right) |\ell| |\ell'| \quad (B.13)$$
Appendix B.5. Application in the text

In the main text of the article the above lemma is used twice: when the large \( J \) behaviour of the function \( \Phi'(\eta) \) is considered section 4.3 and when the Hessian matrix of \( \phi_i'(\eta) \) is calculated section 4.6.

In the first case we estimate \( |\Phi'(\eta)| = \left| \sum_{\eta} \Phi'(\eta) r_{\eta} t_{\eta}' \right| \leq \left| \sum_{\eta} \Phi(\eta) r_{\eta} t_{\eta}' \right| \), knowing that \( |\Phi(\eta)| \leq |C(\eta)| \sum_{j=1}^{N_i} (j+1) m_j^2 \), where the factor \( C(\eta) \) does not depend on \( m \). We have thus

\[
|\Phi'(\eta)| \leq \left| \sum_{\eta} C(\eta) \sum_{j=1}^{N_i} (j+1) m_j^2 \right| t_{\eta} t_{\eta}' \]

and using our lemma we can substitute each appearance of \( m_j^2 \) by \( \frac{j(j+1)}{3} \), obtaining

\[
|\Phi'(\eta)| \leq |C(\eta)| \sum_{j=1}^{N_i} (j+1) \frac{j(j+1)}{3} \left| \langle \eta | \eta' \rangle \right| \]

In the second case we calculate \( \frac{d^2 \phi_j'}{d\eta^2} \bigg|_{\eta=0} = \frac{1}{\langle \eta | \eta' \rangle} \sum_{\eta} \frac{d^2 \phi_j}{d\eta^2} \bigg|_{\eta=0} t_{\eta} t_{\eta}' m \), knowing that

\[
\frac{d^2 \phi_j}{d\eta^2} \bigg|_{\eta=0} = -\frac{1}{J} \sum_{i=1}^{N_i} \left( \frac{1}{2} \langle \eta | \eta' \rangle \right) = -\frac{1}{3J} \sum_{i=1}^{N_i} (j+1) \left( \frac{1}{3} \langle \eta | \eta' \rangle \right) = \frac{1}{3J} \sum_{i=1}^{N_i} (j+1) \]

Appendix C. Proofs of lemmas used in the text

Appendix C.1. The inequality \( \forall \eta > 0 C(\eta) < 1 \)

Let us show the following inequality

\[
\forall \eta \geq 0 \left( e^{1-2\eta} - e^{-2\eta} \right) \leq 1 \]  

where the equality holds only for \( \eta = 0 \).

Obviously the case \( \eta = 0 \) is satisfied:

\[
e^{1-0} - e^0 = e^0 = 1 \]

Since both sides of the inequality are nonnegative, we can take the logarithm of the inequality

\[
1 - 2\eta - e^{-2\eta} < 0 \]

We can differentiate both sides of the inequality with respect to \( \eta \):

\[
-2 + 2e^{-2\eta} < 0 \]

which is obviously true for \( \eta > 0 \). Thus for \( \eta > 0 \)

\[
1 - 2\eta - e^{-2\eta} = 1 + \int_0^{\eta} (-2 + 2e^{-2\eta}) d\tilde{\eta} < 1 + \int_0^{\eta} 0d\tilde{\eta} = 1 \]

\textit{quad erat demonstrandum.}
Appendix C.2. The estimation in section 56

We want to estimate the integral

\[ I_{\eta} := \int_{\eta}^{\infty} d\eta \left( \frac{\sinh \eta}{4\pi \eta} \right)^2 \left( e^{1-2\eta-e^{-2\eta}} \right)^{\kappa} \]  

for \( \kappa > 3 \) and \( \eta_e > 0 \).

First let us note, that \( \frac{1}{\eta} \leq \frac{1}{\eta_e} \) for all \( \eta \) in the integration range, thus

\[ I_{\eta} \leq \left( \frac{1}{4\pi \eta_e} \right)^2 \int_{\eta_e}^{\infty} d\eta \left( \frac{\sinh \eta}{4\pi \eta} \right)^2 \left( e^{1-2\eta-e^{-2\eta}} \right)^{\kappa} \leq \cdots \]  

Now note, that \( (\sinh \eta)^2 \leq e^{2\eta} \), so

\[ \cdots \leq \left( \frac{1}{4\pi \eta_e} \right)^2 \int_{\eta_e}^{\infty} d\eta e^{2\eta} \left( e^{1-2\eta-e^{-2\eta}} \right)^{\kappa} = \cdots \]  

all the integration rage is the subset of \( x \leq 1 \), so \( e^{-x} \leq e \), thus we can remove the denominator:

\[ \cdots \leq \frac{1}{2} \left( \frac{e}{4\pi \eta_e} \right)^2 \int_0^{1} dx \left( xe^{1-x} \right)^{\kappa-2} \leq \cdots \]  

One can easily prove, that \( \forall x \in [0,1] \), \( x \leq e^{x-1} \). Indeed, for \( x = 1 \) we have \( 1 = 1 \), and the derivative of left-hand side is \( x' = 1 \) is bigger than the derivative of the right-hand side \( (e^{x-1})' = e^{x-1} \). Knowing that, and since \( \kappa > 3 \), we can estimate \( \left( xe^{1-x} \right)^{\kappa-2} \leq xe^{1-x} \), and though

\[ \cdots \leq \frac{1}{2} \left( \frac{e}{4\pi \eta_e} \right)^2 \int_0^{1} dx \left( xe^{1-x} \right) = \cdots \]  

which can be integrated by parts:

\[ \cdots = \frac{1}{2} \left( \frac{e}{4\pi \eta_e} \right)^2 \left[ \left( x+1 \right) e^{1-x} \right]_0^1 = \cdots \]  

Now putting back \( x_e = e^{-2\eta_e} \) we get

\[ I_{\eta} \leq \frac{1}{2} \left( \frac{e}{4\pi \eta_e} \right)^2 \left( e - e^{1-e^{-2\eta_e}} - e^{1-2\eta_e-e^{-2\eta_e}} \right) = \tilde{I}_{\eta_e} \]  

Appendix C.3. Proof of existence of \( \eta_e \)

Now let us proof, that for each \( \epsilon > 0 \) there is \( \eta_e \) such that \( I_{\eta_e} \leq \epsilon \). We will do it by showing (using Darboux theorem), that the equation \( \tilde{I}_{\eta_e} = \epsilon \) has a solution.

Let us check the limits of \( \tilde{I}_{\eta_e} \). For \( \eta_e \to 0 \) we have

\[ \lim_{\eta_e \to 0} \tilde{I}_{\eta_e} = \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \left( e - 2 \right) \left( \frac{1}{\eta_e} \right)^2 = +\infty \]  

(14)
For $\eta_\epsilon \to +\infty$ we have

$$\lim_{\eta_\epsilon \to +\infty} \tilde{I}_{\eta_\epsilon} = \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \left[ \frac{1}{+\infty} (e - e^{1-0} - e^{-\infty}) \right] = 0$$

(C.15)

Thus $\tilde{I}_{\eta_\epsilon}$ runs through all real positive numbers, so for each $\epsilon > 0$ there is $\eta_\epsilon$ being the solution to the equation $\tilde{I}_{\eta_\epsilon}$, and since $I_{\eta_\epsilon} \leq \tilde{I}_{\eta_\epsilon}$ (what we have shown in (C.13)), it is obvious now, that

$$\forall \epsilon > 0 \exists \eta_\epsilon : I_{\eta_\epsilon} \leq \epsilon$$

(C.16)

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