ELECTROWEAK STRINGS, SPHALERONS AND MAGNETIC FIELDS

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1. INTRODUCTION

In this talk I would like to discuss three main topics. First, I will connect electroweak strings and the sphaleron in a more elaborate manner than that described in my paper with George Field\(^1\) and discuss the important issue of the stability of electroweak strings. Secondly, I would like to initiate a discussion of the formation of electroweak strings and their evolution by painting a heuristic picture of the phase transition if the phase transition is weakly first order or second order. Inextricably tied to this picture is a possible scenario for baryogenesis and it is hoped that the present discussion will inspire more rigorous work to either confirm or reject the scenario. Finally, I will make a few comments on the generation of magnetic fields at the electroweak phase transition. Kari Enqvist has already described the basic idea of how this scenario works\(^2\) and the purpose of these comments is to clarify some of the issues that have not been addressed in the literature. I will also describe the scenario in an alternate way that makes an interesting connection between electroweak strings and primordial magnetic fields.

2. THE Z-STRING AND THE SPHALERON

The bosonic sector of the standard model of the electroweak interactions is described by the Lagrangian:

\[
L = L_W + L_B + L_\Phi - V(\Phi) \tag{2.1}
\]

where,

\[
L_W = -\frac{1}{4} W_{\mu\nu a} W^{\mu\nu a} \tag{2.2}
\]
where $W_{\mu \nu}^a$ and $Y_{\mu \nu}$ are the field strengths for the $SU(2)$ and $U(1)$ gauge fields $W_\mu^a$ and $Y_\mu$ respectively. Also,

$$L_Y = -\frac{1}{4} Y_{\mu \nu} Y^{\mu \nu}$$  \hfill (2.3)$$

$L_Y$ is the Lagrangian density for the $SU(2)$ and $U(1)$ gauge fields.

Also,

$$L_\Phi = |D_\lambda \Phi|^2 \equiv \left( \partial_\lambda - \frac{1}{2} ig \tau^a W^a_\lambda - \frac{1}{2} ig' Y_\lambda \right) \Phi \right|^2$$  \hfill (2.4)$$

where $\Phi$ is a complex doublet. In addition, we define

$$V(\Phi) = \lambda (\Phi^\dagger \Phi - \eta^2 / 2)^2 ,$$  \hfill (2.5)$$

The electroweak model has two different string solutions$^{3,4}$, namely, the $W-$string and the $Z-$string$^{5,6,7}$. The latter is the lighter of the two and is better studied and hence we will not discuss the $W-$string but only comment that some of the ensuing discussion directly applies to the $W-$string as well.

The $Z-$string is a solution to the classical equations of motion following from the Lagrangian in (2.1). In cylindrical coordinates ($r, \theta, z$) it is:

$$\Phi = \frac{\eta}{\sqrt{2}} f(r)e^{im \theta} \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad Z_\mu = -\frac{\alpha r}{\alpha r} \delta_{\mu \theta}$$  \hfill (2.9)$$

with all the other gauge fields equal to zero. The functions $f(r)$ and $v(r)$ can be evaluated numerically (for example, see Ref. 8). The string solution in (2.9) is precisely a Nielsen-Olesen vortex$^9$ that has been embedded in the electroweak model$^{3,4}$.

The field configuration in (2.9) describes a tube of energy density localized around the $z$-axis. At the center of this tube the Higgs field vanishes. The tube contains a flux of Z-magnetic field whose value is:

$$F_Z = \frac{4 \pi}{\alpha} = \frac{4 \pi}{e} \sin \theta_W \cos \theta_W$$  \hfill (2.10)$$

where, $e = g \sin \theta_W$ is the electric charge on an electron. The energy per unit length of the string is:

$$\mu = \pi \eta^2 M(\beta)$$  \hfill (2.11)$$
where $M(1) = 1$ and the tendency of $M$ is to increase with increasing $\beta$. But the
dependence of $M$ on $\beta$ is not too strong in the neighbourhood of $\beta = 1$ and so we
shall set $M = 1$ whenever we need numerical estimates. (For a plot of $\mu$ as a function
of $\beta$, see Ref. 8.)

The solution in (2.9) describes an infinite $Z$–string along the z-axis but one can
think of finite string configurations in the form of closed loops\(^{10}\) or open segments.
Finite string configurations in vacuum would not be static but would oscillate. And if
the configuration is large compared to the string thickness, the dynamics is described
quite accurately by the Nambu-Goto action. With time such oscillating loops and
segments will radiate away their energy and convert into particles. If the loops and
segments are not in vacuum but in a plasma that strongly interacts with the strings,
the dynamics will be very different and one expects that the oscillations will be
damped and the radiation suppressed.

A finite segment of $Z$–string terminates on magnetic monopoles\(^{5}\). The electro-

cmagnetic flux emanating from a monopole is:

$$F_A = \frac{4\pi}{\alpha} \tan \theta_W = \frac{4\pi}{e} \sin^2 \theta_W.$$  \hspace{1cm} (2.12)

One way to understand the presence of monopoles at the end of $Z$–strings is to note
that the $Z$ gauge field is a linear superposition of the $W^3$ and $Y$ fields as given in
eqn. (2.6). Then, when the string terminates, the $Y$ flux cannot terminate because
it is a $U(1)$ gauge field and the $Y$ magnetic field is divergenceless. Therefore some
field must continue even beyond the end of the string. This has to be the massless
field of the theory, that is, the electromagnetic field.

Following Nambu, the asymptotic Higgs field configurations of a monopole and
an antimonopole are\(^{5}\):

$$\Phi_m = \begin{pmatrix} \cos(\theta_m/2) \\ \sin(\theta_m/2) e^{i\phi} \end{pmatrix}, \quad \Phi_{\bar{m}} = \begin{pmatrix} \sin(\theta_{\bar{m}}/2) \\ \cos(\theta_{\bar{m}}/2) e^{i\phi} \end{pmatrix}$$  \hspace{1cm} (2.13)

where $\theta_m$ and $\theta_{\bar{m}}$ are spherical angles defined with the monopole and antimonopole
at the origin respectively and we have rescaled the Higgs field so that the vacuum
manifold is given by $\Phi^\dagger \Phi = 1$. The gauge fields are taken to be so that the covariant
derivative of the Higgs field vanishes.

$$gW^a_\mu = -\epsilon^{abc} n^b \partial_\mu n^c + i \cos^2 \theta_W n^a(\Phi^\dagger \partial_\mu \Phi - \partial_\mu \Phi^\dagger \Phi)$$  \hspace{1cm} (2.14a)

$$gY_\mu = -i \sin^2 \theta_W (\Phi^\dagger \partial_\mu \Phi - \partial_\mu \Phi^\dagger \Phi).$$  \hspace{1cm} (2.14b)

where, $n^a$ is defined in (2.7). For the special case $\theta_W = 0$, this is equivalent to the
usual

$$W_\mu = -i g \frac{2}{2}(\partial_\mu U)U^{-1}.$$  \hspace{1cm} (2.15)
where, \( U \) is a \( 2 \times 2 \) unitary matrix defined by

\[
\Phi = U (1, 0)^T
\]  

(2.16)

Note that the monopole configuration has a singularity along the negative z-axis since the Higgs field becomes multi-valued when we set \( \theta_m = \pi \). Similarly the antimonopole has a singularity along the positive z-axis (\( \theta_{\bar{m}} = 0 \)). These singularities tell us the location of the \( Z^- \)string that is attached to the monopole and the antimonopole.

Once we have the monopole and antimonopole configurations, we can patch them together to get the field configuration for a finite segment of \( Z^- \)string:

\[
\Phi_{m\bar{m}}(\gamma) = \left( \begin{array}{c} \cos(\Theta/2) \\ \sin(\Theta/2) e^{i\phi} \end{array} \right)
\]

(2.17)

where,

\[
\cos\Theta \equiv \cos\theta_m - \cos\theta_{\bar{m}} + 1.
\]

(2.18)

It is straightforward to check that (2.17) yields the monopole field configuration close to the monopole (\( \theta_{\bar{m}} \to 0 \)) and the antimonopole configuration close to the antimonopole (\( \theta_m \to \pi \)). It also yields a string singularity along the straight line joining the monopole and antimonopole (\( \theta_m = \pi, \theta_{\bar{m}} = 0 \)).

What is important for us is that there are other Higgs field configurations that also describe monopoles and antimonopoles. These are given by global \( U(1) \) transformations of (2.13). Therefore we will write

\[
\Phi_m = e^{i\gamma} \left( \begin{array}{c} \cos(\theta_m/2) \\ \sin(\theta_m/2) e^{i\phi} \end{array} \right), \quad \Phi_{\bar{m}} = e^{i\gamma} \left( \begin{array}{c} \sin(\theta_{\bar{m}}/2) \\ \cos(\theta_{\bar{m}}/2) e^{i\phi} \end{array} \right)
\]

(2.19)

This seemingly trivial observation is very useful because it allows us to construct \( Z^- \)string segments which are twisted. Consider the Higgs field configuration:

\[
\Phi_{m\bar{m}}(\gamma) = \left( \begin{array}{c} \sin(\theta_m/2) \sin(\theta_{\bar{m}}/2) e^{i\gamma} + \cos(\theta_m/2) \cos(\theta_{\bar{m}}/2) \\ \sin(\theta_m/2) \cos(\theta_{\bar{m}}/2) e^{i\phi} - \cos(\theta_m/2) \sin(\theta_{\bar{m}}/2) e^{i(\phi-\gamma)} \end{array} \right)
\]

(2.20)

together with the gauge fields given by eqn. (2.14). When we take the limit \( \theta_{\bar{m}} \to 0 \) we find the monopole configuration of (2.13) and when we take \( \theta_m \to \pi \) the configuration is that of the antimonopole of eqn. (2.19) provided we perform the rotation \( \phi \to \phi + \gamma \). The monopole and antimonopole in (2.20) also have the usual string singularity joining them. This means that the configuration in (2.20) describes a monopole and antimonopole pair that are joined by a \( Z^- \)string segment that is twisted by an angle \( \gamma \).

Now we will calculate the Chern-Simons number (which is, loosely speaking, the baryon number) of the twisted segment of string described in (2.20) and with gauge fields in (2.14).
Let me assume that \( \gamma \) is a rational fraction of \( 2\pi \) so that we can write \( \gamma = 2\pi p/q \) where \( p \) and \( q \) are integers. Then we can take \( q \) twisted segments, each of which is described by eqns. (2.20) and (2.14), and join them up - the antimonopole of one segment can be brought to annihilate the monopole of another segment - to form a closed loop. In this way we will get a loop of \( Z \)–string that is twisted by an angle \( 2\pi p \).

Now we need to calculate the Chern-Simons number of this loop. The Chern-Simons number is defined as

\[
CS = \frac{N_F}{32\pi^2} \int d^3x \epsilon_{ijk} \left[ g^2 \left( W^{aij} W^{ak} - \frac{g}{3} \epsilon_{abc} W^{ai} W^{bj} W^{ck} \right) - g' Y^{ij} Y^{ik} \right].
\] (2.21)

where, we have included the number of families \( N_F \) in the definition. For a closed loop of \( Z \)–string, this expression simplifies considerably since the only non-vanishing gauge field is the \( Z \) gauge field. In this circumstance, (2.21) reduces to

\[
CS = N_F \frac{\alpha^2}{32\pi^2} \cos(2\theta_W) \int d^3x \vec{Z} \cdot (\vec{\nabla} \times \vec{Z})
\] (2.22)

We can now easily calculate the Chern-Simons number of the twisted loop by using the result that if we have a twisted flux loop of a gauge field \( \vec{A} \) with flux \( F \) and twist \( 2\pi p \), then,

\[
\int d^3x \vec{A} \cdot (\vec{\nabla} \times \vec{A}) = 2 F^2 p.
\] (2.23)

This result is well-known to people working in hydrodynamics and astrophysics\(^{11}\) but less known in the particle physics community. So we give a quick sketch of the derivation.

We first note that the internal twisting of a flux tube is equivalent to a linking of two different flux tubes\(^{12}\). Hence we can restrict ourselves to evaluating the left-hand side of (2.23) for two untwisted flux tubes that are linked \( p \) times.

We work in the gauge \( \vec{\nabla} \cdot \vec{A} = 0 \) and the magnetic field is given by \( \vec{B} = \vec{\nabla} \times \vec{A} \). Therefore, if we are given a magnetic field configuration, the gauge field can be found from

\[
\vec{A}(\vec{x}) = -\frac{1}{4\pi} \int d^3x' \vec{B}(\vec{x}') \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}.
\] (2.24)

Denoting the integral on the left-hand side of (2.23) by \( I \), we then have:

\[
I = \frac{1}{4\pi} \int d^3x \vec{B}(\vec{x}) \cdot \int d^3x' \vec{B}(\vec{x}') \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}
\] (2.25)

and then perform integrations over the cross-section of the flux tubes. This has the effect

\[
I = (2F^2) \left[ -\frac{1}{4\pi} \oint df \cdot \oint dg \times \frac{\vec{f} - \vec{g}}{|\vec{f} - \vec{g}|} \right]
\] (2.26)
where, we have assumed that both flux tubes carry the same flux $F$ and that their locations are given by $\vec{f}$ and $\vec{g}$. The expression in square brackets is the Gauss linkage formula\textsuperscript{13} for the curves $\vec{f}$ and $\vec{g}$ and this proves (2.23).

Now, using (2.23) (with $\vec{A}$ replaced by $\vec{Z}$) in (2.22) and inserting the value of the $Z$ flux in the string (eqn. (2.10)), we get the Chern-Simons number of the $Z$–string loop that is twisted by $2\pi p$:

$$CS = N_F \cos(2\theta_W) p$$  \hspace{1cm} (2.27)

Since the loop was built out of $q$ segments and $\gamma = 2\pi p/q$, the Chern-Simons number of one segment is

$$CS = N_F \cos 2\theta_W \frac{\gamma}{2\pi}.$$  \hspace{1cm} (2.28)

So far we have been working with string segments having arbitrary twist. But now consider the case, $\gamma = \pi/\cos(2\theta_W)$. With this twist, the Chern-Simons number is $N_F/2$ - precisely that of the sphaleron\textsuperscript{14}!

Given that the segment with twist $\pi/\cos(2\theta_W)$ has Chern-Simons number equal to that of the sphaleron, it is natural to ask if some deformation of it will yield the sphaleron. This deformation is not hard to guess for the $\theta_W = 0$ case. In this case, if we let the segment size shrink to zero, we have $\theta_m = \theta_{\bar{m}} = \theta$ and the Higgs field configuration of (2.20) gives:

$$\Phi_{m\bar{m}}(\gamma = \pi) = \begin{pmatrix} \cos \theta \\ \sin \theta \ e^{i\phi} \end{pmatrix}.$$  \hspace{1cm} (2.29)

And this is exactly the field configuration of Nick Manton’s $SU(2)$ sphaleron\textsuperscript{6}. (Note that the gauge fields continue to be given by (2.14) or, equivalently in this case, by (2.15).)

Encouraged by this successful connection between the twisted string segment and the sphaleron in the $\theta_W = 0$ case, we can conjecture that the twisted segment of $Z$–string with Chern-Simons number $N_F/2$ will collapse into the sphaleron for any $\theta_W$. Therefore the asymptotic Higgs field configuration for the sphaleron can be obtained by letting $\theta_m = \theta_{\bar{m}} = \theta$ in (2.20) and taking $\gamma = \pi/\cos(2\theta_W) = \gamma_S$. Denoting the sphaleron asymptotic Higgs field configuration by $\Phi_S$, we conjecture

$$\Phi_S = \begin{pmatrix} \sin^2(\theta/2) - \cos^2(\theta/2) \ e^{i\gamma_S} \\ \sin(\theta/2) \cos(\theta/2) \ e^{i\phi}(1 - e^{-i\gamma_S}) \end{pmatrix}$$  \hspace{1cm} (2.30)

and the asymptotic gauge fields are given by (2.14). If this conjecture is true, we should be able to find a solution to the classical equations of motion with this ansatz for the asymptotic fields. Of course, we would first need to insert a suitable radial dependence in the ansatz and then solve the field equations.
On physical grounds it seems reasonable that there should be a critical value of twist at which one can get a static solution for a $Z$–string segment. This is because the segment likes to shrink under its own tension but the twist prevents the shrinkage and is equivalent to a repulsive force between the monopole and antimonopole. Then, if the string is sufficiently twisted, the attractive force due to the tension and the repulsive force due to the twist will balance and a static solution can exist. So far we have been assuming that the only dynamics of the segment is towards collapsing or expanding. However, since we are dealing with twisted segments, we should also include the rotational dynamics associated with twisting and untwisting. So, while any twist greater than a certain critical twist will successfully prevent the segment from collapsing, only a special value of the twist will give a static solution to the rotational dynamics. Furthermore, we expect that this solution will be unstable towards rotations that twist and untwist the string segment. This would be the unstable mode of the sphaleron.

A question that I have frequently been asked is that how can an object have fractional Chern-Simons number? This question is easily answered: only the vacua have integer Chern-Simons number; outside the vacua, the Chern-Simons number can be anything. But then, what would happen when this object having fractional Chern-Simons number decays into particles? We know that the change in the Chern-Simons number of the object equals the change in the baryon number, then how can fractional baryon number be produced? The answer to this question is currently under investigation by Eddie Farhi and collaborators$^{15}$ and requires a quantum treatment of the decay of the fractionally charged object. The belief is that the equality of baryon number change to the change in Chern-Simons number should be viewed as an equality of expectation values. So the change in baryon number will always be integral and only upon averaging will one get a fractional change in baryon number.

This completes what I wanted to say about the connection of the $Z$–string and the sphaleron. (Other discussions of this connection may be found in Refs. 4, 16.) Just to summarize, twisted segments of $Z$–string carry Chern-Simons number in proportion to their twist and when the twist is $\pi/cos(2\theta_W)$, the Chern-Simons number is $N_F/2$. We conjecture that the string segment with this special value of twist is precisely an extended sphaleron.

3. $Z$–STRING STABILITY

A non-trivial and very important aspect of studying electroweak strings is to understand their (meta-)stability. In an exhaustive analysis done in collaboration with Margaret James and Leandros Perivolaropoulos, we plotted the region of parameter space in which the $Z$–string is stable towards small perturbations$^{17}$. The conclusion is that one can only have stable strings for $sin^2\theta_W \gtrsim 0.9$, that is, for $g' \gtrsim 3g$. The
experimentally determined value of $\sin^2 \theta_W$ is 0.23 and is deep inside the instability region.

Recently there have been a few papers that have shed more light on the issue of stability for $\theta_W$ not too large. First there was a paper by Warren Perkins that showed that there is an instability towards developing $W$ fields for $\sin^2 \theta_W \lesssim 0.8$. Then Manuel Barriola, Martin Bucher and I used a simple argument to show that the string is unstable in the case $\theta_W = \pi/4$. In this instability, the upper component of the Higgs field grows, the lower component diminishes in magnitude and the gauge field spreads. To see the explicit form of the instability, denote the unperturbed electroweak string solution by $\Phi_0$ and $Z_j^{(0)}$ and consider the sequence of field configurations labeled by a parameter $\xi$:

$$
\Phi(\vec{x}; \xi) = \cos \xi \Phi_0(\cos \xi \vec{x}) + \sin \xi \Phi_\perp
$$

(3.1)

$$
Z_j(\vec{x}; \xi) = \cos \xi Z_j^{(0)}(\cos \xi \vec{x})
$$

where, $Z_j$ is given in terms of the $SU(2)_L \times U(1)_Y$ gauge fields in eqn. (2.6) together with $n^a = (0, 0, 1)^T$ (note!), and,

$$
\Phi_\perp = \frac{\eta}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

(3.2)

For $\xi = 0$, the configuration is the embedded defect solution and for $\xi = \pi/2$, the configuration describes the vacuum. We then have,

$$
\Phi^\dagger(\vec{x}; \xi)\Phi(\vec{x}; \xi) = \cos^2 \xi \Phi_0^\dagger \Phi_0 + \sin^2 \xi \eta^2
$$

$$
W_{ij}^a(\vec{x}; \xi)W_{ij}^a(\vec{x}; \xi) = \cos^4 \xi W_{ij}^a(\vec{x})W_{ij}^a(\vec{x}),
$$

$$
Y_{ij}^a(\vec{x}; \xi)Y_{ij}^a(\vec{x}; \xi) = \cos^4 \xi Y_{ij}^a(\vec{x})Y_{ij}^a(\vec{x}),
$$

$$
D_\bar{i}\Phi(\vec{x}; \xi) = \cos^2 \xi \left[ \partial_\bar{i} + i\alpha Z_i(\vec{x})\mathcal{T}^Z \right] \Phi_0(\vec{x})
$$

(3.3)

$$
V[\Phi(\vec{x}, \xi)] = \cos^4 \xi V[\Phi_0(\vec{x})]
$$

where,

$$
\mathcal{T}^Z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
$$

(3.4)

From (3.3) we get

$$
[D_\bar{i}\Phi(\vec{x}; \xi)]^\dagger [D_\bar{i}\Phi(\vec{x}; \xi)] = \cos^4 \xi (D_\bar{i}\Phi_0(\vec{x}))^\dagger (D_\bar{i}\Phi_0(\vec{x}))
$$

(3.5)

and hence, the total energy of the configuration is

$$
E(\xi) = \cos^2 \xi E(\xi = 0).
$$

(3.6)

This explicitly shows a sequence of configurations that start at the string ($\xi = 0$) and end up in the vacuum ($\xi = \pi/2$) and with monotonically decreasing energy.
Although in the sequence of configurations considered above, only the $Z$ field is written as being non-zero, this is really misleading because $\Phi$ is no longer proportional to $(0, 1)^T$. So the gauge fields in the above configurations actually include $W^\pm$ fields and the instability demonstrated above is towards developing $W$ fields. If the parameter $\xi$ is replaced by a parameter along the length of the string ($z$), the above sequence of configurations are slices of a string along the $z$ axis that has terminated in a monopole. So the instability described above is the instability of a string to break-up by forming monopoles and antimonopoles.

Frans Klinkhamer and Poul Olesen$^{19}$, and, independently, Margaret James$^{20}$, have constructed a two parameter family of configurations - a two sphere in configuration space - with the string on top of the sphere for $\sin^2 \theta_W \lesssim 0.7$. This picture implies that the string has at least two unstable modes for small $\theta_w$. However, it seems likely that there is only one unstable mode of the string solution under small perturbations since the two modes that Klinkhamer and Olesen, and, James, have found are gauge equivalent when the perturbations are taken to be small at the solution$^{21}$. If this is true, we might reasonably guess that the only unstable perturbative mode is towards breaking up of strings by forming monopole-antimonopole pairs$^*$. 

There is a simple way to understand why the $W$ fields develop and destabilize the $Z$ string$^{23,18}$. For this consider the energy of a spin $s$ particle in a uniform magnetic field $\vec{B}$ along the $z$-direction:

$$E^2 = k_z^2 + m^2 + (2n + 1)eB - 2e\vec{B} \cdot \vec{s}$$

(3.7)

where, the charge on the particle is $e$ and the g-factor of the particle (in the last term) is taken to be 2. The right-hand side contains the $z$-momentum and mass contribution to the energy. Then there is the contribution of the various Landau levels labeled by the integer $n$ and finally there is the coupling of the magnetic moment (spin) to the magnetic field. Clearly if $s = 1$, it is possible for the right-hand side to be negative if

$$B > \frac{m^2}{e}$$

(3.8)

and hence there is an instability of the magnetic field to condensing spin one particles or, to “$W$ condensation”.

In the case of the $Z$--string, the magnetic field $\vec{B}$ in (3.7) is the $Z$--magnetic field within the string and the spin-1 particles are the $W$ bosons. It can be checked that the g-factor is indeed 2 as has been assumed in writing (3.7) and the $Z$ charge on the

* The physics of the two unstable modes is not completely clear to me. Perhaps the constructions of Ref. 19,20 are counting the instabilities towards forming monopoles that are twisted in the way described in Sec. 2.
$W$ boson is $e_Z = g\cos\theta_W$. Therefore the conditions are right for the $W$ condensation argument to apply to the $Z$ string. The only condition that is not immediately satisfied is the assumption in (3.7) that the magnetic field is uniform. But if the string is not too thin relative to the $W$ boson Compton wavelength, one is justified in thinking of the $Z$ magnetic field of the string as being uniform. This is precisely the case for low values of $\sin^2 \theta_W$. For small values of $\theta_W$, we also see that $e_Z = g\cos\theta_W$ is not small and so the critical $Z$ magnetic field required for $W$ condensation (eqn. (3.8)) can be relatively small. These facts show that $Z$ strings are unstable towards $W$ condensation for small $\theta_W$.

A point to remember is that we have only been considering infinite strings so far. The instability towards break-up will not apply when the string length is less than a certain critical length since the formation of monopoles requires certain gradients along the length of the string. The critical length of string below which break-up is not possible is likely to be a few times the size of the monopole ($\sim m_W^{-1}$). Such short segments are not string-like in appearance but can still be important as they can carry Chern-Simons number just as in the case of longer segments.

Are there any new instabilities of a finite segment that are absent in the case of the infinite string? As the string segment is not topological, there is a possibility that the string segment could destabilize as a whole and disappear into the vacuum. But this does not appear to be possible since the evolution of the string segment has to conserve magnetic charge and so the only way a monopole can disappear is to annihilate with an antimonopole. Therefore the only new instability of a finite segment of string is towards dynamical collapse. If the segment has angular momentum or is twisted, this dynamical instability may not be too severe. (Nambu has given rough estimates for the life-time of a long string segment with angular momentum towards radiative decay\(^5\).)

There are some circumstances under which the stability of the $Z$–string improves. Thermal effects can be shown to improve stability\(^{24}\) but the improvement is more so that strings with higher values of $\beta$ become stable and strings with small values of $\theta_W$ remain unstable. Rick Watkins and I showed that if there are particles bound to the string, the string can be stable down to lower values of $\sin^2 \theta_W$. In our analysis\(^{25}\), we found stable strings down to about $\sin^2 \theta_W = 0.5$ but did not find stable $Z$–strings with $\sin^2 \theta_W = 0.23$.

The picture then is the following: if we start with a long (infinite) $Z$–string in the case when $\theta_w$ is small, the string will rapidly break up into small segments of a critical length - of the order of a few times the string thickness - which will then survive until they radiate away their energy and angular momentum.

If we are willing to consider extensions of the standard model, stable strings are
possible. The popular two Higgs model does not yield stable strings\textsuperscript{26} but more complicated models, such as, left-right models can yield stable strings\textsuperscript{24}. Other extensions can also give topologically stable strings with $Z$–flux in them\textsuperscript{27}.

4. ELECTROWEAK STRINGS AT THE PHASE TRANSITION

I will assume that the electroweak phase transition is second order or weakly first order - after hearing the various talks at this conference, this seems likely. This means that the correlation length at the phase transition is of the order $T^{-1}$ where $T_c$ is the critical temperature at which the phase transition occurs. Under this assumption the temperature ($T$) dependent vacuum expectation value of $\Phi$ is given by:

$$<|\Phi|> = \frac{\eta}{\sqrt{2}} \left(1 - \frac{T^2}{T_c^2}\right)^{1/2}. \quad (4.1)$$

Therefore the mass per unit length of the $Z$–string follows from (2.11):

$$\mu(T) \approx \pi M(\beta) <|\Phi|^2 = \frac{\pi \eta^2}{2} \left(1 - \frac{T^2}{T_c^2}\right) M(\beta). \quad (4.2)$$

Let us now define the “Hagedorn temperature” $T_H$ as the temperature at which

$$T_H = \sqrt{\frac{3\mu(T_H)}{2\pi}}. \quad (4.3)$$

This gives,

$$T_H = T_c \sqrt{\frac{x}{1+x}}, \quad \text{with} \quad x \equiv \frac{3M(\beta)\eta^2}{4T_c^2}. \quad (4.4)$$

For $\beta = 1$ (that is, $m_H = m_Z$), $\eta = 250$GeV and $T_c = 100$GeV, this gives $T_H \approx 0.9T_c = 90$GeV. Note that, while $T_H$ is quite close to $T_c$, the time required for the universe to cool from $T_c$ to $T_H$ is about $10^{16}T_H^{-1}$.

The Hagedorn temperature is significant in the study of string statistical mechanics because it is the temperature that marks a phase transition in a string network. Below the Hagedorn temperature, a string network prefers to break up into the smallest possible loops or segments. Above the Hagedorn temperature, the network is dominated by long strings. These results have been obtained by studying the statistical mechanics of a box of strings and it is believed that the results apply to fundamental strings as well as strings that arise as defects in a phase transition.

How can we understand these results? The basic point is that the number of states of a string grows exponentially with length while the Boltzmann distribution decays exponentially with length. It is found that the partition function depends on an integrand proportional to

$$\exp \left[ \left( \sqrt{\frac{2\pi}{3\mu(T)}} - \frac{1}{T} \right) E \right] \quad (4.5)$$
where $E$ is the energy of the string. Above the Hagedorn temperature, the number of states associated with the long strings wins over the Boltzmann suppression and the network settles into long (infinite) strings.

There are various subtleties in studying string statistical mechanics and the above argument is only meant to suggest that something strange should happen above the Hagedorn temperature which is still below the critical temperature. For those wishing to pursue this line of argument further, a paper summarizing earlier work on the statistical analysis of strings as well as applying string statistical mechanics to cosmic strings, is the paper by Dave Mitchell and Neil Turok. In this paper, the authors also treat the statistical mechanics of open strings where the monopoles at the end of strings and the strings themselves form during the same cosmological phase transition - exactly the case relevant to electroweak strings! However, Mitchell and Turok ignore string-string interactions in their analysis and this may be a bad approximation to make. In particular, important processes such as string break-up and reconnections have not been taken into account in the analysis. For this reason, we cannot import the results of Ref. wholesale but can only take them as an indication that the density of strings may be significant at temperatures near the Hagedorn temperature but still below the critical temperature.

There is another way of seeing that something bizarre may happen above the Hagedorn temperature. For $T_c > T > T_H$, the tension in the string is smaller than the thermal excitations due to interaction with the plasma. Therefore the strings are “hot” and the tension in the string can be ignored compared to thermal excitations. This would mean that points on a string segment would diffuse and the distance between any two points on a string with grow with time. So the length of the segment would grow and a given segment would not be able to shrink and disappear. All that a hot segment of electroweak string can do is to grow in length and break up. The smaller pieces that are formed due to the fragmentation would also not be able to shrink and would keep growing bigger. Then the density of strings would become high and the likelihood of strings merging to form bigger strings would increase. In this way one is led to conclude that, at temperatures lower than $T_c$ but higher than $T_H$, the strings will be relatively long. This is in contrast to the picture where long strings are Boltzmann suppressed.

These arguments lead us to the picture that, for $T_c > T > T_H$, the plasma contains a significant density of long strings. As the temperature falls, the separation between strings increases. If the phase transition is first order, the strings get separated by the process of bubble formation whereas if the phase transition is second order, the inter-string separation grows larger continuously. At $T = T_H$ there is a phase transition in the string network itself and long strings begin to break-up at this temperature. If the strings are metastable, the strings break-up by nucleating
monopoles and antimonopoles and this would be a first order phase transition in the string network itself. If the strings are unstable, the break-up is more like a second order phase transition in the string network. In this picture, the long electroweak strings at $T_H$ are genuine relics of the unbroken phase and are not due to thermal fluctuations (since these ceased at $T_c$).

Now we follow the evolution to temperatures below the Hagedorn temperature. As the universe cools below the Hagedorn temperature, the infinite strings will break up and yield an exponential distribution of string segments and the number of string segments of length $l$ will be proportional to $\exp[-al]$ where $a > 0$ is a constant. With further cooling, the tension in the strings starts becoming important and the segments start shrinking and decaying into radiation. All this will happen relatively quickly and so we can say that the string network disappears at a temperature $T \sim T_H$. As the network disappears, it will produce baryons and anti-baryons since, from our results in Sec. 2, twisted and linked strings carry baryon number. If there is sufficient CP violation in the dynamics of the string network, the decay of strings would lead to a production of net baryon number.

The above scenario is a scenario of baryogenesis at the Hagedorn temperature $T_H$ and it is a concern that subsequent sphaleron transitions might erase any baryon number produced at $T_H$. To check if this happens, we find the temperature $T_S$ at which the sphaleron transition rate first falls below the Hubble expansion rate:

\[
T_S \exp \left[ -\frac{M_S(T_S)}{T_S} \right] = \frac{T_S^2}{m_{Pl}}. \tag{4.6}
\]

where, $M_S(T)$ is the temperature dependent mass of the sphaleron and $m_{Pl}$ is the Planck mass. Now we know that, for $\theta_W = 0$,

\[
M_S(T) = 2B(\lambda/\alpha_W) \frac{M_W(T)}{\alpha_W} \tag{4.7}
\]

where, $B$ is a weakly dependent function of the coupling constant ratio $\lambda/\alpha_W$, $\alpha_W = g^2/4\pi$ and $M_W(T)$ is the temperature dependent mass of the $W$-boson. Writing $m_W = M_W(0)$ and approximating $\ln(T_S)$ by $\ln(T_c)$ we get,

\[
T_S = T_c \left[ 1 + \left\{ \frac{\alpha_W T_c}{2B m_W \ln \left( \frac{m_{Pl}}{T_c} \right)} \right\}^2 \right]^{-1/2}. \tag{4.8}
\]

Now comparing $T_S$ to $T_H$ (eqn. (4.4)), we find that $T_S$ is larger than $T_H$ provided

\[
\left[ \frac{\alpha_W \eta}{4m_W} \ln \left( \frac{m_{Pl}}{T_c} \right) \right]^2 3M(\beta) < [B(\lambda/\alpha_W)]^2 \tag{4.9}
\]
With $\alpha_W = 1/30$, $\eta = 250GeV$, $m_W = 80GeV$ and $\ln(m_{Pl}/T_c) = 17\ln(10)$, this condition gives

$$3M(\beta) < [B(\lambda/\alpha_W)]^2.$$  \hspace{1cm} (4.10)

The value of $B$ ranges from 1.5 to 2.7 - at least for the $SU(2)$ sphaleron (that is, for the $\theta_W = 0$ case) - and is roughly 1.9 when $\beta = 1$ ($m_H = m_Z$) at which point $M(\beta) = 1$. Therefore the condition (4.9) is satisfied in the parameter range of interest and we have $T_H < T_S$. For $\theta_W \neq 0$ too, we expect that this condition will be satisfied for an interesting range of parameters.

The above argument assures us that sphaleron transitions cannot completely erase the baryon number that would be generated by the string network at the Hagedorn temperature. The exact fraction of baryon number that survives can be found by studying the equations of detailed balance in an expanding universe.

While the above scenario seems plausible to me, it has several weaknesses. The first weakness is that we do not understand the formation of electroweak strings at the electroweak phase transition. The arguments given above suggest that electroweak strings should be present in significant numbers above the Hagedorn temperature but we still do not have a method to get a rigorous estimate of quantities such as the number density of strings, or, the length distribution. The second weakness is that the production of baryons over antibaryons in the process of string decay requires CP violation which is thought to be very weak in the standard model. This, however, is a problem with any scenario of electroweak baryogenesis and it is quite common to consider more strongly CP violating extensions of the standard model. One could also take this approach with electroweak strings. On the other hand, it would be more satisfying if there were some unusual source of CP violation in the monopole-antimonopole system within the standard model that may not be present in the usual particle interactions. These issues are presently being investigated.

5. MAGNETIC FIELDS

In this last section, I would like to discuss the generation of cosmological magnetic fields arising at the electroweak phase transition\textsuperscript{32}. Kari Enqvist has discussed the scenario in some detail and my purpose is to make a few clarifying remarks about some of the assumptions that go into the scenario. Then I would like to describe a connection between electroweak strings and primordial magnetic fields.

The basic idea is that, as the Higgs field acquires a vacuum expectation value, currents are produced that lead to a magnetic field. Specifically, one defines the electromagnetic gauge field as in eqn. (2.6) but the electromagnetic field strength as:

$$F_{\mu\nu}^{em} = \sin\theta_W n^a W^{a}_{\mu\nu} + \cos\theta_W Y_{\mu\nu} - i4g^{-1}\eta^{-2}\sin\theta_W[(D_\mu\Phi)\dagger D_\nu\Phi - (D_\nu\Phi)\dagger D_\mu\Phi].$$  \hspace{1cm} (5.1)
So far, all that we have done is to define the electromagnetic field strength and have not included any dynamics. The dynamics comes in when we estimate the various terms in (5.1) in a plasma at high temperature. Suppose we are interested in a macroscopic volume of size \( L \) which is much larger than the thermal wavelength \( T^{-1} \). We expect that the field strengths \( W_{\mu \nu}^a \) and \( Y_{\mu \nu} \), when averaged over the volume of size \( L \), will be exponentially decreasing with increasing \( L \) since the plasma is neutral on such scales and has no net currents either. However, the covariant derivative is expected to fall off as a power law in \( L \) as it is simply the covariant gradient of a scalar field. For example, the covariant gradient may be estimated as the change in the value of \( \Phi \) occurring over the scale \( L \) - therefore, \( |D_\mu \Phi| \sim \eta/L \). This is the hinge on which the production of magnetic fields at the electroweak phase transition rests and one can give several arguments why this seems reasonable. (Though none of the arguments rigorously proves the assumption.)

We now give the arguments that indicate that the covariant derivative falls off as a power law and not exponentially with \( L \). The motivation for expecting this behaviour comes from the Kibble argument. If the covariant derivative vanishes, there is complete correlation between distant parts of the universe which have never been in causal contact and this is inadmissible. This argument is strictly valid for global symmetries where the phases in \( \Phi \) have physical meaning and so the argument can legitimately be questioned for gauge symmetries. On the other hand, note that we are dealing with a system at high temperatures when the energy is distributed among the various degrees of freedom. These include the covariant derivative of \( \Phi \) and it is natural to assume that the covariant derivative takes on a value given by energy equipartition. Whereas \( W_{\mu \nu}^a \) and \( Y_{\mu \nu} \) fall off exponentially with \( L \) due to vanishing currents in the plasma, there is no current density in the particles (fermions etc.) in the plasma that is responsible for the last term in (5.1) and so a power law fall off is possible.

Here we will give another argument for why magnetic fields should be produced at the electroweak phase transition. This argument is consistent with the theme of the talk as it suggests that there is a source for the magnetic field and the source is none other than the electroweak monopoles present at the ends of electroweak strings. As in Sec. 4, we expect electroweak monopoles to be produced at the electroweak phase transition and then to go away at a somewhat lower temperature. Therefore, imagine a distribution of monopoles and antimonopoles in a plasma with the magnetic lines of force running from monopoles to antimonopoles. With time, the strings shrink and the monopoles and antimonopoles at the ends of the string annihilate. However, the magnetic lines of force are glued to the plasma because the plasma is a very good electrical conductor. So the magnetic lines of force survive even
when the monopoles themselves annihilate and disappear. If the lines were present only on very small scales, they too would eventually disappear (as the conductivity of the plasma decreases with time). But this is not likely; instead we expect that the magnetic lines of force will percolate much like the percolation of a cosmic string network. With time, the small scale curvature on the magnetic lines of force will disappear and the lines will straighten out but there will always be lines of force present on large scales where they are frozen in the plasma. Hence, at any epoch a relic magnetic field will be present.

The magnetic field strength is roughly given by \( B \sim T^2 \) where \( T \) is the temperature of the universe provided we assume that the field is frozen-in on the smallest scales (of order \( T^{-1} \)). However, this is an incorrect assumption as the frozen-in scale is much larger than the thermal wavelength. If the frozen-in scale is denoted by \( l_f \), we have, \( l_f \sim 10^{12}T^{-1} \). Below this scale, the plasma is unimportant for the evolution of the magnetic field and so the field will smooth itself out. Using a flux average to estimate the field strength, we find \( B \sim 10^{-12}T^2 \) and this field is coherent on a scale \( l_f \). At the electroweak scale, this gives \( B \sim 10^{12}G \) with a coherence scale of \( 10^{-5}cms \).

The presence of strong magnetic fields in the very early universe leads to very interesting physics and is currently under investigation by several groups. Here, I would only like to remark that, if the above arguments connecting electroweak strings with magnetic fields is correct, the direct or indirect observation of such a magnetic field would immediately yield information about the cosmological electroweak phase transition and about electroweak strings!

Acknowledgements:

I thank the organizers of this meeting and especially Filipe Freire for hosting this conference and making it so enjoyable. This work was supported by the National Science Foundation.

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