Bosonic interference as a complementary resource for implementation of quantum walks

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Quantum walks are interesting simple models for describing various fundamental processes in nature ranging from chaos or photosynthesis to universal quantum computation. Their implementation usually comprises a single particle and $O(n^2)$ optical elements, where $n$ is the size of the quantum walk space. The question arises if this can be achieved with fewer or complementary resources. Here we show that this can be done using a multi-particle Hong–Ou–Mandel (HOM) interference. It employs $O(n)$ indistinguishable bosons and a single beam splitter to achieve a similar quantum walk. In addition we show that a quantum-to-classical transition takes place if the particles become distinguishable. This approach establishes a link between the fundamental indistinguishability of quantum particles and the wavelike coherent nature of the walk.

**Introduction.** A quantum walk (QW) is the quantum analogy of the classical random walk in which an initially localized particle evolves into a superposition over many different positions in space [1–3]. Usually, the space in which the walk takes place is discrete and can be represented as a graph. Standard implementations of QWs on a graph of size $O(n)$ require $O(n^2)$ elements in an experimental setup (beam splitters in the optical case) that build up an underlying graph structure in which a single particle takes $t = n$ steps. The natural question arises: can a QW be achieved with fewer or complementary resources?

Here we introduce a complementary platform for observation of quantum walks, based on the multiphoton Hong–Ou–Mandel (HOM) interference [4] (HOM QW). It requires a single beam splitter of reflectivity $r$ and $n$ particles to implement a walk on a chain of size $O(n)$ for $t$ that depends on $r$. Our model provides a new insight into the quantum-to-classical transition triggered by a partial distinguishability of interfering particles. Although we focus on photonic implementation, our results apply to a broad class of light and matter bosonic systems.

The distinctive property of a QW is statistics with variance scaling quadratically in time (ballistic spread). QWs are interesting simple models for describing various fundamental processes in nature ranging from chaos [5], topological phases [6, 7] or photosynthesis [8–11] to universal quantum computation [12] and quantum search algorithms [13–15]. In the presence of decoherence, QW statistics turn to the classical one—the binomial distribution [16]. Recently, numerous elementary QWs have been performed, most notably using quantum optics [17–26], but they have been limited to relatively few steps. Increasing the number of vertices in the graphs underlying QWs is important in order to implement larger instances of QW algorithms, or observe QW evolution over longer times. This is a central challenge, difficult to defeat using conventional approaches.

We show that Hong–Ou–Mandel (HOM) interference of multiphoton Fock states in Fig. 1b, reproduces the same or similar quantum probability distributions describing a QW with ballistic spread, as those resulting from the multi-mode interference of a single photon in an array of beam splitters in Fig. 1a, a paradigmatic platform for realization of a QW. The HOM QW belongs to the continuous-time domain [2, 27]. By adjusting the

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number of interfering photons and the single-particle reflectivity of a beam splitter, it immediately generates the probability distribution which, in usual walks, is obtained after many steps. It simulates perfect state transfer through spin chains \cite{28, 29} and reproduces the quantum-to-classical transition \cite{19, 21}. We discuss feasibility of its experimental demonstration and conclude the paper with open problems. We stress that we are not aiming towards practical solutions for applications based on QWs, but rather at revealing intriguing phenomena related to multiphoton interference. It constitutes a realistic resource for quantum technologies and allows exciting fundamental research in quantum mechanics.

Quantum walk simulated by HOM interference. A QW comprises at least one ‘walker’, e.g. a particle like a photon, traversing a graph. A continuous-time quantum walk (CTQW) \cite{2} takes place in the position Hilbert space $H_P$ where the basis states $|\Delta_i\rangle$ correspond to vertices $\Delta_i$ in the graph. In the case of a QW on a line, the position can take any integer value. The evolution of the walker is generated by a Hamiltonian $H$ which sets amplitudes of hopping from vertex $\Delta_i$ to $\Delta_j$ per unit time, $a_{ij} = (\Delta_j | H | \Delta_i)$. They are zero for disjoint vertices. The walker, initially in a state $|\Delta\rangle$, after time $t$ is in a state $|\Delta_{\text{out}}\rangle = e^{-iHt} |\Delta\rangle$, where the variable $\Delta_{\text{out}}$ is governed by a doubly-peaked probability distribution $p(\Delta_{\text{out}})$ with variance scaling quadratically with time. Here we take $\hbar = 1$. Surprisingly, unlike in the case of discrete-time quantum walks (DTQW), even in an infinitesimal time difference the walker can be transferred arbitrarily far from the starting position, albeit with low amplitude.

The Hong–Ou–Mandel interference \cite{4} is observed for two photons impinging on a beam splitter, a quantum device described by the Hamiltonian

$$H_{BS} = a^\dagger b e^{i\varphi} + a b^\dagger e^{-i\varphi}.$$ \hspace{1cm} (1)

The annihilation operators $a$ and $b$ correspond to the interfering modes and $\varphi$ is the phase difference between the reflected and transmitted fields behind the beam splitter. We assume $\varphi = \pi$. If the probabilities of reflection and transmission are equal, two identical photons will always leave together through the same exit port—photon bunching. Similar effects hold for multiphoton Fock-number states: photons will go out only in certain configurations, e.g. such that the difference between the occupations of the exit ports will be even, whereas an odd difference will never occur \cite{30–32}. These interference effects, consisting of enhancing some events and canceling others, lie at the heart of the QW simulated by multiphoton HOM interference.

We now show that for two interfering Fock states $|S - N\rangle = \frac{\langle S | e^{\frac{-S}{2}} \rangle}{\sqrt{|S - N|}} |0\rangle$ and $|N\rangle = \frac{\langle 1 | e^{\frac{-1}{2}} \rangle}{\sqrt{N}} |0\rangle$, $H_{BS}$ generates a CTQW on a line of length $S + 1$ with specific, inhomogeneous jump amplitudes. The total number of photons $S$ entering the beam splitter is a conserved quantity for $H_{BS}$. However, the distribution of particles is not. We parametrize the input state $|S - N\rangle |N\rangle$ using the initial mode occupation difference $\Delta = S - 2N$ and denote it $|\Delta\rangle$. Behind the beam splitter, the infinitesimal evolution turn initially well-defined $|\Delta\rangle$ into the superposition

$$H_{BS}|\Delta\rangle = q_{\Delta-2,\Delta}|\Delta-2\rangle + q_{\Delta+2,\Delta}|\Delta+2\rangle.$$ \hspace{1cm} (2)

This suggests the physical interpretation of $\Delta$ as the initial position of the walker in the graph, from where it jumps to the right $|\Delta + 2\rangle$ or left $|\Delta - 2\rangle$. Interestingly, because the length of the jump is two in the position of the particle, the HOM QW is confined to a finite region $\Delta \in \{-S, -S + 2, \ldots, S - 2, S\}$ with the total number of positions $S + 1$. The jump amplitudes

$$q_{\Delta,\Delta-2} = q_{\Delta,\Delta+2} = \frac{1}{\sqrt{2}} \sqrt{(S+\Delta)(S-\Delta+2)}$$ \hspace{1cm} (3)

describe an important class of QWs. The Hamiltonian \eqref{1} corresponds to the Schwinger representation of the $S_z$ spin–$\frac{S}{2}$ matrix, and a two-mode Fock state $|S - N\rangle |N\rangle$ can be associated with a fictitious particle of total spin $\frac{S}{2}$ and eigenvalue of the $S_z$ component $\frac{\Delta}{2}$. Under these mappings, $q_{\Delta,\Delta-2}$ reproduce the couplings enabling perfect state transfer in linear spin chains \cite{29} and the HOM QW describes rotation of the $S_z$ component.

The evolution in the HOM QW is governed by the beam-splitter unitary $U_{BS} = \exp[-iH_{BS}t]$. A single photon state $|1,0\rangle$ impinging on a beam splitter evolves into $\cos t |1,0\rangle - i \sin t |0,1\rangle$, thus for the single particle case we can define reflectivity $r = \sin^2 t$. The spectrum of $H_{BS}$ is harmonic and the evolution is periodic. After time $t = \frac{\pi}{2}$ any input state $|\Delta\rangle$ is mapped to the opposite vertex of the graph $|\Delta\rangle$. For $t = 2\pi$ the evolution becomes the unit operator and a recurrence is observed, i.e. the QW returns to the initial state.

It is now straightforward to unveil the probability distribution of the final positions of a walker, created by $U_{BS}$. Behind the beam splitter, the interfering Fock states are turned into a superposition with population difference in the exit modes $\Delta_{\text{out}}$ given by

$$p(\Delta_{\text{out}}) = A \left[ \sum_k \left( \frac{f_{\Delta+k}}{f_{\Delta-k}} \right)^k \left( \frac{f_{\Delta-k}}{f_{\Delta+k}} \right)^{\Delta_{\text{out}}-k} \right]^2,$$ \hspace{1cm} (4)

where $A = \frac{f_{\Delta_{\text{out}}} f_{\Delta_{\text{out}}}}{f_{\Delta} f_{\Delta-k}} r (1 - r)^S \left( \frac{f_{\Delta-k}}{f_{\Delta}} \right)^{\Delta_{\text{out}}-k}$, $k$ runs from $\max \{0, \Delta_{\text{out}} - \Delta\}$ to $\min \{\Delta_{\text{out}}, S + 1\}$, $f_{\Delta} = \frac{S + \Delta}{2}$. (For the explicit form of the output superposition and an easy-to-read form of \eqref{4} see the Supplementary Material). In Fig. 2 we plot \eqref{4} for various starting positions in the HOM QW performed with $S = 50$ photons. In Fig. 2a, the walker starts in the middle of the line, i.e. $\Delta = 0$, and its walk spreads uniformly in both directions, thus the distribution develops symmetrically with respect to this point. This scenario takes place if the two interfering Fock states are equal. If the initial position is shifted from the middle, $\Delta = -30$ in Fig. 2b, the distribution is asymmetric for all times except for $t = \pi/4$. 

we show how the doubly-peaked probability distribution shown blue in Fig. 2a modifies if particles become more and more distinguishable. With increasing $y$, full cancellation of certain events is impossible and the two peaks gradually shift to the center of the line and merge to create the binomial distribution for $y = \pi$, typical of the classical random walk. Distinguishability mimics decoherence because it leads to interference of the multiphoton states with the vacuum state. In this way it causes quantum particles to behave as classical ones, whose statistics cannot mimic quantum coherence and therefore we observe a transition from the quantum to the classical domain.

Two-dimensional quantum walk. Since a beam splitter sees orthogonal polarizations independently, the above model generalizes to a two-dimensional case. This requires that the particle evolves independently in each direction, i.e. there are no operations entangling the two polarizations. One simply considers two simultaneously interfering pairs of states: $|S - N\rangle$ and $|N\rangle$ in horizontal and $|S - M\rangle$ and $|M\rangle$ in vertical polarization. Thus now, each input beam is a two-mode beam, with fully distinguishable modes. This scenario differs from the previous decoherence model in that this time detectors are maximally distinguishable if $y = \pi/2$ (fully indistinguishable), b) $y = \pi/4$, c) $y = \pi/12$, d) $y = \pi/6$, e) $y = \pi/3$ and f) $y = \pi/2$ (fully distinguishable).

FIG. 3. (Color online) Modification of the probability distribution depicted in Fig. 2a in blue ($\Delta = 0$, $t = \pi/4$), for various degrees of distinguishability of photons: a) $y = 0$ (fully indistinguishable), b) $y = \pi/4$, c) $y = \pi/12$, d) $y = \pi/6$, e) $y = \pi/3$ and f) $y = \pi/2$ (fully distinguishable).
modes instead of polarization, it is possible to further extend the model to arbitrary dimension.

**DT vs. HOM QW.** A click of a detector in Fig. 1a, announces position of a photon in space. This is a first-order coherence measurement, in its spirit similar to the double-slit Young experiment. This setup provides the paradigmatic platform for observation of DTQWs [18–20, 25]. The evolution of a walker is decomposed into two stages, given by coin and step operators [3]. The coin (wave plates) coherently manipulates the direction of the walker in the graph, while the step updates its position according to the coin value.

The setup in Fig. 1b involves two detectors and the information extracted from the physical system comes from correlations between their measurement outcomes, witnessing second-order correlation for the two interfering Fock input states.

Intuitively, these two setups elicit very different information. Our work however, demonstrates that they are complementary resources in realization of QWs. By employing HOM interference, we transfer the random variable from position to particle number. Now, the walk is not a function of physical time, but a fixed, adjustable parameter of the setup, the reflectivity of the beam splitter. Although the HOM QW is energetically pricey, the size of the space of the walk scales linearly with the energy of the input Fock states, it is modest in space and time. The final distribution is reached instantaneously, which in terms of DTQWs corresponds to a quantum leap rather than a walk in steps. Provided that pure, single-mode Fock states are used, this avoids blocking of the walk during its evolution due to e.g. Anderson localization [33]. Moreover, the distributions presented in Fig. 2 can be reproduced not only approximately but exactly by a DTQW shown in Fig. 1a, provided that specific position-dependent coins are selected [34].

**Feasibility.** The key challenge in the optical demonstration of the HOM QW is the creation of pure, single-mode, multiphoton Fock states. They could be conditionally prepared from two-mode squeezed vacuum (SV) states, which possess perfect photon-number correlations between the two modes, called the signal and idler, respectively. SV states are produced by parametric down conversion sources, pumped with a high intensity laser. The idler would be used as a trigger and measured by a photon counting detector. Then the signal would end up in a known Fock state. For the HOM QW, one would need two such sources and detectors. SV states with mean photon number of the order of ten are accessible in laboratories [35, 36]. For measurements, one could use superconducting transition-edge sensors (TESs) [37], which reach photon-counting efficiencies near 100% and have extremely well-resolved photon-number peaks, up to around ten photons [38].

**Conclusions.** We have described two complementary implementations regarding the origin of QWs: a single photon scattering on an array of beam splitters and many photons scattering on a single beam splitter. We have shown that a one-dimensional QW, in the form of a ‘quantum leap’, can be implemented with multiphoton HOM interference. By exploiting the polarization or spectral degrees of freedom, the scheme may be generalized to higher-dimensional QWs. In conventional QW protocols, $O(n^2)$ optical elements are required to implement a walk with $n$ position states, which becomes challenging for large walks. Our approach represents a substantial saving in optical elements, by trading away elements for photon number - we reduce the complexity of the system to having just a single beam splitter, but at the expense of $O(n)$ photons as opposed to just one.

Our model replicates the crucial features of QWs: the quantum probability distribution with ballistic spread and quantum-to-classical transition triggered by partial distinguishability of photons. Interestingly, this approach establishes a link between the fundamental indistinguishability of quantum particles and the wavelike coherent nature of the walk. Quantum-optical implementation of the HOM QW in the range of ca. ten steps is within the reach of current technology (Fig. 2 for $S = 10$ is shown in the Supplementary Material).

Although our investigations were carried out within the specific framework of photonic implementation, the insights are of general relevance, because the HOM-like effect can be observed for arbitrary bosons: in neutron diffraction [39], diffraction of X rays [40], for surface plasmon polaritons [41] and atomic Bose-Einstein condensates [42, 43]. Thus, bosonic interference provides an implementation platform for simulation of a QW.

Our result leads to some open problems. We have demonstrated a QW scenario for specific transition amplitudes between the neighbouring positions. Is it possible to generalize this result to arbitrary couplings? Here one can consider the QW with Bose-Einstein condensates, or in other matter systems in which interactions between particles cannot be neglected. It is therefore natural to ask: how does this interaction influence the dynamics of the walk and the observed probability distribution?

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Supplementary Material:
Bosonic interference as a complementary resource for implementation of quantum walks

Supplementary Material gives some technical details of computations which lead to the results presented in the main text. It does not contain any new discussion. We mainly present here the proof and graphs illustrating explicitly the quadratic-in-time scaling of the variances for the probability distribution describing the HOM QW, as well as the results obtained for lower photon numbers, easily accessible in laboratories.

I. QW PROBABILITY DISTRIBUTION \( p(\Delta_{\text{out}}) \)

We will now consider an unbalanced beam splitter with reflectivity \( r \), which transforms the input annihilation operators \( a \) and \( b \) in the following way: \( a_r = \sqrt{1-r} a + \sqrt{r} b \), \( a_l = -\sqrt{r} a + \sqrt{1-r} b \). Note that \( r \) is the single-photon reflectivity because it gives the probability of reflection of a photon in the case if a single photon impinges on the beam splitter. The incoming Fock state \( \ket{K}_a \ket{L}_b \), where \( \ket{K} = \frac{(\sqrt{a})^K}{\sqrt{K!}} \ket{0} \) and \( \ket{L} = \frac{(\sqrt{b})^L}{\sqrt{L!}} \ket{0} \), is turned into the following superposition

\[
\ket{\psi_{\text{out}}} = \frac{1}{\sqrt{K!L!}} (\sqrt{1-r} a_r^L - \sqrt{r} a_l^L) (\sqrt{r} a_l^K + \sqrt{1-r} a_r^K) \ket{0}
\]

\[
= \frac{1}{\sqrt{K!L!}} \sum_{k=0}^{K} \sum_{l=0}^{L} \binom{K}{k} \binom{L}{l} (-1)^{K-k} \sqrt{r}^{K-k+l} \sqrt{1-r}^{L-l+k} a_r^l a_l^k K-k+L-l \ket{0}.
\]

We are interested in the probability distribution of measurement outcomes of photon counting detectors located behind the beam splitter \( \langle p, q | \psi_{\text{out}} \rangle^2 \). Two Fock states \( \ket{K} \) and \( \ket{L} \) can always be parametrized by the total photon number \( S = K + L \) and the population difference \( \Delta = K - L \). In this case \( \ket{K} = \ket{\frac{S+\Delta}{2}} \) and \( \ket{L} = \ket{\frac{S-\Delta}{2}} \). Since \( p + q = S \) as well, and \( p - q = \Delta_{\text{out}} \) out, the probability distribution reads

\[ p(\Delta_{\text{out}}) = \frac{(S-\Delta)^S}{(S+\Delta)^S} \binom{S}{\Delta}(1-r)^S \left(1 - \frac{\Delta}{2S} \right) \]

\[ \times \begin{bmatrix}
\min\left(\frac{S-\Delta}{2}, \frac{S+\Delta}{2}\right)
\sum_{k=\max(0, \frac{\Delta-\Delta}{2})}^{\frac{S+\Delta}{2}} \frac{S-\Delta}{k} \left(1 - \frac{\Delta}{2S} \right)^k \left(1 - \frac{\Delta}{r} \right)^k
\end{bmatrix}^2.
\]

The above distribution corresponds to Eq. (4) in the main text. It is shown for \( S = 10 \) in Fig. 4 below.

II. BALLISTIC SPREAD OF \( p(\Delta_{\text{out}}) \)

At first let us note that the beam-splitter Hamiltonian

\[ H_{BS} = a^\dagger b + ab^\dagger e^{-i\varphi} \]

(Eq. (1) in the main text), for \( \varphi = \pi \), corresponds to the Schwinger representation of the \( S_z \) spin-\( \frac{1}{2} \) matrix, defined as \( S_z = \frac{a^\dagger b + ab^\dagger}{2} \). Secondly, a two-mode Fock state \( \ket{S - N} \ket{N} \) can be associated with a fictitious particle of total spin \( \frac{1}{2} \) and eigenvalue of the \( S_z \) component equal to \( \frac{1}{2} \), where \( S_z = a^\dagger a - b^\dagger b \). Now \( \ket{S - N} \ket{N} \equiv \ket{\Delta} \). Here \( \Delta = S - 2N \) denotes the initial mode occupation difference.

Under these mappings, the HOM QW is in one-to-one correspondence with a spin-\( \frac{1}{2} \) particle whose evolution is given by the Hamiltonian proportional to the \( S_z \) operator. In this picture the position operator (of the walker in the graph) is proportional to the \( S_z \) operator and the states with well defined position \( \ket{x} \) correspond to the eigenstates of \( S_z \): \( \ket{\frac{S}{2}, \frac{\Delta}{2}} \), where \( x \equiv \Delta \) (in the main text we have also shown that the physical interpretation of \( \Delta \) corresponds to the position of a walker in the graph.). It is therefore straightforward to show that in the Heisenberg picture the \( S_z \) evolves as

\[ S_z(t) = S_z \cos t + S_y \sin t. \]

Note that \( S_y \) is a linear combination of rising \( S_+ = a^\dagger b \) and lowering \( S_- = b^\dagger a \) operators, \( S_y = i(S_+ - S_-) \), therefore for the states \( \ket{\frac{S}{2}, \frac{\Delta}{2}} \) the average value of \( S_y \) is zero and we obtain

\[ \langle S_z(t) \rangle = \langle S_z \rangle \cos t. \]

Moreover,

\[ S_z^2(t) = S_z^2 \cos^2 t + S_y^2 \sin^2 t + \langle S_y, S_z \rangle \cos t \sin t. \]

Due to the same reason as above, for the states \( \ket{\frac{S}{2}, \frac{\Delta}{2}} \) the average value of \( S_z^2(t) \) equals

\[ \langle S_z^2(t) \rangle = \langle S_z^2 \rangle \cos^2 t + \langle S_y^2 \rangle \sin^2 t. \]
In addition, since $|\frac{S_z}{2}, \frac{\Delta}{2}\rangle$ are eigenstates of $S_z$, we have $\langle S_z^2 \rangle = \langle S_z \rangle^2$. The value $\langle S_z^2 \rangle$ can be evaluated in the following way. At first, we note that

$$S_y^2 = \frac{-(S_- - S_+)^2}{4} = -S_-^2 - S_+^2 + S_+ S_- + S_- S_+.$$  \hspace{1cm} (SM.8)

For the states $|\frac{S_z}{2}, \frac{\Delta}{2}\rangle$ we have

$$\langle S_y^2 \rangle = \frac{\langle S_+ S_- \rangle + \langle S_- S_+ \rangle}{4} = \frac{S_+ (S_+ + 1) - (\frac{\Delta}{2})^2}{2}.$$  \hspace{1cm} (SM.9)

Therefore

$$\text{Var}[S_z(t)] = \langle S_z^2(t) \rangle - \langle S_z(t) \rangle^2 = \frac{S_+ (S_+ + 1) - (\frac{\Delta}{2})^2}{2} \sin^2 t.$$  \hspace{1cm} (SM.10)

The variance oscillates because the HOM QW takes place on a finite line. However, for short times the boundary effects cannot play any significant role. Indeed, by approximating $\sin t \approx t$ we get

$$\text{Var}[S_z(t)] \approx \frac{1}{4} \left( S_+^2 + \Delta^2 + S \right) t^2,$$  \hspace{1cm} (SM.11)

which confirms ballistic spreading. In the main text we denote $\text{Var}[S_z(t)]$ by $\text{Var}(\Delta_{\text{out}})$.

FIG. 4. (Color online) Probability distributions of final positions of a walker in the HOM CTQW performed with $S = 10$ photons, obtained for initial position of the walker a) $\Delta = 0$, b) 4, c) 10, as a function of the time of the walk: $t \approx \pi/10$ – green, $t \approx \pi/7$ – red, $t = \pi/4$ – blue, and $t \approx 4\pi/10$ – gray.

FIG. 5. Variance of the probability distributions of final positions of a walker in the HOM CTQW as a function of the time of the walk, for various values of $S$ and $\Delta$.