A general criterion for the entanglement of two indistinguishable particles

GianCarlo Ghirardi†
Department of Theoretical Physics of the University of Trieste, and
International Centre for Theoretical Physics “Abdus Salam”, and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy

and

Luca Marinatto‡
Department of Theoretical Physics of the University of Trieste, and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy

Abstract

We relate the notion of entanglement for quantum systems composed of two identical constituents to the impossibility of attributing a complete set of properties to both particles. This implies definite constraints on the mathematical form of the state vector associated to the whole system. We then analyze separately the cases of fermion and boson systems, and we show how the consideration of both the Schmidt number of the fermionic and bosonic analog of the Schmidt decomposition of the global state vector and the von Neumann entropy of the one-particle reduced density operators can supply us with a consistent criterion for detecting entanglement. In particular, the consideration of the von Neumann entropy is particularly useful in deciding whether the correlations of the considered states are simply due to the indistinguishability of the involved particles or they are a genuine manifestation of the entanglement. The treatment leads to a full clarification of the subtle aspects of entanglement of two identical constituents which have been a source of embarrassment and of serious misunderstandings in the recent literature.

Key words: Entanglement; Identical particles; von Neumann entropy.

PACS: 03.65.Ta, 03.67.-a, 89.70.+c

1 Introduction

Quantum entanglement, considered by Schrödinger “the characteristic trait of Quantum Mechanics, the one that enforces its entire departure from classical line of thoughts” [1], has played a central role in the historical development of Quantum Mechanics and nowadays it constitutes an essential resource for many aspects of quantum information and quantum computation theory. In fact, the possibility of performing reliable teleportation processes, of generating unconditionally secure private keys in cryptography, or of devising quantum algorithms allowing

---

*Work supported in part by Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy
†e-mail: ghirardi@ts.infn.it
‡e-mail: marinatto@ts.infn.it
to solve certain computational problems in a more efficient way than the best known classical methods, is essentially based on the peculiar properties of entangled states. However, in spite of the fact that entangled states involving identical constituents are widely used in the experimental implementations of the above mentioned (and many others) processes, the very notion of entanglement for such ubiquitous physical systems seems to be too often misunderstood, or not understood at all in the current scientific literature on the subject. The most frequent misinterpretations arise in connection with the Symmetrization Postulate of Quantum Mechanics which requires definite symmetry properties for the state vectors associated to systems of identical particles. Their non-factorized form\(^1\) seems to suggest, when compared with the well-known case of systems composed of distinguishable particles, the occurrence of an unavoidable form of entanglement when identical particles enter into play.

In a recent paper \cite{2} (see also \cite{3}) we have analyzed in great detail the problem of entanglement dealing both with systems of two (or more) distinguishable or identical particles. In accordance with the position of the founding fathers of Quantum Mechanics \cite{1,4}, we have strictly related the non-occurrence of entanglement to the possibility of attributing complete sets of properties to both constituents of the composite system. In this way we have been able to formulate an unambiguous criterion for deciding whether a given state vector is entangled or not which works both for the case of distinguishable and identical constituents. It has to be stressed that, contrary to what has sometimes been stated, non-entangled states involving identical constituents can actually occur.

Obviously, in the case of distinguishable particles, our criterion is equivalent (as we have shown in Ref. \cite{2}) to the commonly used criteria to identify whether a system is entangled or not which involve the consideration of the Schmidt number of the biorthonormal decomposition of the state or, equivalently, the evaluation of the von Neumann entropy of the reduced statistical operators.

The situation is radically different in the case of composite systems involving identical constituents. In the literature, various authors \cite{5,6,7} have suggested to identify the entangled or non-entangled nature of a system of two identical constituents by resorting to natural generalizations of the above mentioned criteria. In so doing they have met various difficulties which emerge when one compares the obtained results with the analogous ones for the case of distinguishable particles. Moreover the procedure yields apparently contradictory results for the fermion and the boson cases \cite{6,7,8}.

In this paper we show that the source of the problems stays in not having appropriately taken into account the fact that, even in the case in which it is physically legitimate and correct to consider a state as non-entangled - so that one knows the properties of the constituents - there is an unavoidable lack of information about the actual situation of the constituents arising from their identity. Furthermore, we prove that resorting to the general analysis of Refs. \cite{2,3} one can get a complete clarification of the matter and we present a unified criterion for detecting entanglement in systems of identical particles such that: i) it involves both the Schmidt number of the fermionic and bosonic analog of the Schmidt decomposition of the state vectors and the von Neumann entropy of the reduced single particle statistical operators; ii) it applies equally well for fermions and bosons; iii) it is in complete accordance with our original criterion.

\(^1\)The only important exception being represented by the peculiar case of identical bosons in the same state.
2 Entanglement and properties

In this section we briefly review the arguments of Ref. [2] which show that the correct way to identify the entanglement is to relate it to the impossibility of attributing precise properties to the (identical) constituents of a two-particle system \( S = S_1 + S_2 \). As is well known, in the case of non-entangled distinguishable particles the factorized nature of the state vector \( |\psi(1,2)\rangle = |\phi_1\rangle \otimes |\chi_2\rangle \) is a necessary and sufficient condition for being allowed to claim that subsystem \( S_1 \) objectively possesses the (complete set of) properties associated to the state vector \( |\phi_1\rangle \) and subsystem \( S_2 \) those associated to \( |\chi_2\rangle \). We stress that in the considered case we know not only the properties which are possessed but also to which system they refer. Obviously, in the case of identical constituents one cannot resort to the factorizability criterion to claim that the two systems are non-entangled, otherwise one would be led to conclude (mistakenly) that non-entangled states cannot exist (exception made for two bosons in the same state) since the necessary symmetry requirements forbid the occurrence of factorized states.

However, this naive and inappropriate conclusion derives from taking a purely formal attitude about the problem, without paying the due attention to the physically meaningful conditions which, when satisfied, allow one to legitimately state that two systems are non-entangled. Such conditions are, primarily, those of being allowed to claim that one particle possesses a precise and complete set of properties and the other one exhibits analogous features. Obviously, one must always keep clearly in mind that it is absurd to pretend to individuate the particles, i.e. to identify which one possesses one set of properties and which one the other set (in the case in which such sets are different).

Considerations of this kind have led us to identify the following physically appropriate criterion characterizing non-entangled states of two identical particles:

**Definition 2.1** The identical constituents \( S_1 \) and \( S_2 \) of a composite quantum system \( S = S_1 + S_2 \) are non-entangled when both constituents possess a complete set of properties.

Obviously, we still have to make fully precise the meaning of the expression “both constituents possess a complete set of properties”. To this purpose we resort, first of all, to the following Definition:

**Definition 2.2** Given a composite quantum system \( S = S_1 + S_2 \) of two identical particles described by the normalized state vector \( |\psi(1,2)\rangle \), we will say that one of the constituents possesses a complete set of properties iff there exists a one-dimensional projection operator \( P \), defined on the single particle Hilbert space \( \mathcal{H} \), such that:

\[
\langle \psi(1,2) | \mathcal{E}_P(1,2) | \psi(1,2) \rangle = 1
\]

(2.1)

where

\[
\mathcal{E}_P(1,2) = P^{(1)} \otimes [I^{(2)} - P^{(2)}] + [I^{(1)} - P^{(1)}] \otimes P^{(2)} + P^{(1)} \otimes P^{(2)}.
\]

(2.2)

Condition (2.1) gives the probability of finding at least one of the two identical particles (of course we cannot say which one) in the state associated to the one-dimensional projection operator\(^2\) \( P \).

\(^2\)We remark that one could drop the last term in the expression of Eq. (2.2), getting an operator whose expectation value gives the probability of finding precisely one particle in the state onto which \( P \) projects. In the case of identical fermions this makes no difference, but for bosons it would not cover the case of both particles being in the same state.
Since any state vector is a simultaneous eigenvector of a complete set of commuting observables, condition (2.1) allows to attribute to at least one of the particles the complete set of properties (eigenvalues) associated to the considered set of observables.

At this point we must distinguish two cases: if the two identical particles are fermions, then one can immediately prove (see Ref. [2]) that Eq. (2.1) implies that there exists another one-dimensional projection operator \( Q \), which is orthogonal to \( P \), such that the operator \( E_Q \), which has the expression (2.2) with \( Q \) replacing \( P \), also satisfies

\[
\langle \psi(1,2) | E_Q(1,2) | \psi(1,2) \rangle = 1.
\]

(2.3)

Obviously, in such a case, since it is simultaneously true that “there is at least one particle having the properties associated to \( P \)” and “there is at least one particle having the properties associated to \( Q \)” and, moreover, such properties are mutually exclusive due to the orthogonality of the projection operators, we can legitimately claim that, in the state \( |\psi(1,2)\rangle \), one particle (it is meaningless to ask which one) has the complete set of properties associated to \( P \) and one the complete set associated to \( Q \). As we have shown in Ref. [2] our request for non-entanglement leads, in the fermion case, to the following Theorem:

**Theorem 2.1** The identical fermions \( S_1 \) and \( S_2 \) of a composite quantum system \( S = S_1 + S_2 \) described by the pure normalized state \( |\psi(1,2)\rangle \) are non-entangled iff \( |\psi(1,2)\rangle \) is obtained by antisymmetrizing a factorized state.

In the boson case the situation is slightly different since the two particles can be in the same state. As before, the requirement that at least one of the constituents possesses a complete set of properties implies the existence of another one-dimensional projection operator \( Q \) such that \( E_Q(1,2) \) satisfies Eq. (2.3). The basic difference with the previous case derives from the fact that now \( Q \) is not necessarily orthogonal to \( P \). Two remarks are appropriate:

- The only two instances allowing to claim that both particles have a complete set of properties correspond either to \( Q = P \) or to \( Q \) orthogonal to \( P \). In fact, when \( Q \) is not orthogonal to \( P \) but it is different from it, while the two statements “at least one particle has the properties associated to \( P \)” and “at least one particle has the properties associated to \( Q \)” are true, they do not imply that “one particle has the properties associated to \( P \) and the other those associated to \( Q \)”, since it may happen (the probability of such an occurrence being different from 0 and 1) that both particles are found to have the properties associated to \( P \) (or to \( Q \)) in a measurement process.

- The case in which \( P = Q \), i.e. the one in which both particles have the same properties, is the only one in which the unavoidable ambiguities ensuing from the identity of the constituents disappear and one can claim that he knows completely the “state” or “the complete set of properties” characterizing each individual constituent.

The above considerations, as the reader can easily grasp and as it has been proven in Ref. [2], lead to the following Theorem:

**Theorem 2.2** The identical bosons of a composite quantum system \( S = S_1 + S_2 \) described by the pure normalized state \( |\psi(1,2)\rangle \) are non-entangled iff either the state is obtained by
symmetrizing a factorized product of two orthogonal states or if it is the product of the same state for the two particles.

Concluding, we have shown that, when one deals with the problem of the entanglement using the necessary logical rigour and making appeal to the physical meaning of entanglement itself, then the process of (anti)symmetrizing a state vector does not necessarily lead to an entangled state.

Besides the physical motivations we have presented, there are other compelling reasons which show that our criterion for non-entanglement is the correct one. It is in fact easy to show that, when the conditions of Theorems 2.1 or 2.2 are satisfied, it is not possible to take advantage of the form of the state vector to perform teleportation processes or to violate Bell’s inequality. These facts further strengthen our conclusion that the state is non-entangled.

To clarify our argument we resort to an extremely simple example. Let us consider the following state of two identical spin 1/2 fermions, in which we have denoted as\(|\psi(1,2)\rangle = \frac{1}{\sqrt{2}} \left[ |z\uparrow\rangle_1 |R\rangle_1 \otimes |z\downarrow\rangle_2 |L\rangle_2 - |z\downarrow\rangle_1 |L\rangle_1 \otimes |z\uparrow\rangle_2 |R\rangle_2 \right]. \tag{2.4}\)

Such a state, which can be obtained by antisymmetrizing a factorized state (that is, \(|\psi(1,2)\rangle\)), makes perfectly legitimate a statement of the type\(^3\): “there is one particle at Right having spin up along the z-direction and the spatial properties associated to |R\rangle and a particle at Left having spin down along the z-direction and the spatial properties associated to |L\rangle”. As a consequence this state does not imply any (non-local) correlation between spin-measurements performed in the two space-like separated regions Right and Left and thus it cannot violate Bell’s inequality. This follows immediately from the fact that, when the state is the one of Eq. (2.4), we have for the mean value \(E(\vec{a},\vec{b})\) of the product of the outcomes of two spin measurements along the directions \(\vec{a}\) and \(\vec{b}\) in the two regions Right and Left, the following expression:

\[
E(\vec{a},\vec{b}) = \langle \psi | \left[ \vec{\sigma}^{(1)} \cdot \vec{a} P^{(1)}_R \otimes \vec{\sigma}^{(2)} \cdot \vec{b} P^{(2)}_L + \vec{\sigma}^{(1)} \cdot \vec{b} P^{(1)}_L \otimes \vec{\sigma}^{(2)} \cdot \vec{a} P^{(2)}_R \right] |\psi \rangle = \\
\langle z\uparrow | \vec{\sigma} \cdot \vec{a} |z\uparrow\rangle \langle z\downarrow | \vec{\sigma} \cdot \vec{b} |z\downarrow\rangle \tag{2.5}\]

where we have denoted as \(P_R\) and \(P_L\) the projection operators on the closed linear manifolds of the spatial wave functions with compact support in the Right and Left regions respectively. The occurrence of a factorized product of two mean values implies that no choice of the unit vectors \(\vec{a}, \vec{b}, \vec{c}\) and \(\vec{d}\) can lead to a violation of Bell’s inequality [9]:

\[
|E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| + |E(\vec{b},\vec{d}) + E(\vec{c},\vec{d})| \leq 2 \tag{2.6}\]

On the other hand, let us consider a state of the form:

\[
\langle \phi(1,2) \rangle = \frac{1}{2} \left[ |z\uparrow\rangle_1 |z\downarrow\rangle_2 - |z\downarrow\rangle_1 |z\uparrow\rangle_2 \right] \otimes \left[ |R\rangle_1 |L\rangle_2 + |L\rangle_1 |R\rangle_2 \right], \tag{2.7}\]

\(^3\)Note that in making our statement we somehow individuate our constituents by making reference to the fact that they lie in different spatial regions. This remark is useful for the analysis which will follow.
which is the one used in the Bohm version of the EPR argument of incompleteness. Since it cannot be obtained by antisymmetrizing a factorized product of two orthogonal states it is a genuine entangled state according to our criterion. Correspondingly, one can easily prove that such a state exhibits the non-local features leading to a violation of Bell’s inequality for a proper choice of the orientations \( \vec{a}, \vec{b}, \vec{c} \) and \( \vec{d} \).

In spite of their simplicity, the just considered examples show clearly why vectors displaying the apparent form of an entangled state, such as the one of Eq. (2.4), must be considered as non-entangled, in perfect agreement with our criterion.

### 3 The entanglement criteria

As we have already stressed, when one deals with quantum systems composed of two distinguishable particles, the appearance of entanglement is equivalent to the impossibility of writing the state vector of the compound system \( |\psi(1, 2)\rangle \) as a tensor product of two single-particle states. This in turn implies two well-known formal facts: i) by resorting to the Schmidt decomposition, the global state turns out to be non-entangled if and only if its associated Schmidt number (that is, the number of non-zero coefficients in such a decomposition) equals 1; ii) the state is non-entangled if and only if the von Neumann entropy of the reduced statistical operator associated to both particles is equal to zero.

Both facts have clear physical implications and a precise meaning: the first refers to the possibility of attributing a complete and precise set of properties to each constituent, the second ensures us that we have the most complete and exhaustive information allowed by quantum theory about the situation of each constituent. In fact in a factorized state each component subsystem is associated to a precise state vector and the reduced statistical operator for one of the two particles, for example the one labeled by 1, i.e. \( \rho^{(1)} = Tr^{(2)}[|\psi(1, 2)\rangle\langle\psi(1, 2)|] \), turns out to be a projection operator onto a one-dimensional manifold. Correspondingly its von Neumann entropy \( S(\rho^{(1)}) \equiv -Tr[\rho^{(1)} \log \rho^{(1)}] \) equals zero. This result is correct since such a quantity measures the lack of information about the single-particle subsystem and there is, in fact, no uncertainty at all concerning the state which must be attributed to it.

When passing to the more subtle case of interest, that is to systems composed of two identical constituents, the relations between entanglement and both the Schmidt number and the von Neumann entropy of the reduced statistical operators become less clear and require a careful analysis. The purpose of this Section is to clarify the matter and to present a criterion for determining whether a state is entangled or not which is: i) based on the consideration of both the Schmidt number and the von Neumann entropy; ii) consistent with our original criterion summarized in the Definition 2.1; iii) equally applicable to fermion and boson systems; and finally iv) able to unify in a consistent way various criteria which appeared recently in the literature [5, 6, 7].

We limit our considerations to the case of a finite dimensional single particle Hilbert space and, in accordance with the above remarks, we deal separately with the fermion and the boson cases, since they exhibit quite different features.

\(^4\)As is well known, given a statistical operator \( \rho \), its Von Neumann entropy is defined as \( S(\rho) \equiv -Tr[\rho \log \rho] \) where the base of the log function is the number \( e \). However, in the present paper, we will rescale such a quantity and follow the information theory convention of using all logarithms in base 2.
3.1 The fermion case

The notion of entanglement for systems composed of two identical fermions has been discussed in Ref. [5] where a “fermionic analog of the Schmidt decomposition” has been exhibited. Such a decomposition is based on a nice extension to the set of the antisymmetric complex matrices of a well known theorem holding for antisymmetric real matrices (see for example [10]). The Theorem of [5] states that:

**Theorem 3.1.1** For any antisymmetric \((N \times N)\) complex matrix \(A\) (that is, \(A \in \mathcal{M}(N, \mathbb{C})\) and \(A^T = -A\)) there exists a unitary transformation \(U\) such that \(A = UZU^T\), with \(Z\) a block-diagonal matrix of the sort

\[
Z = \text{diag}(Z_0, Z_1, \ldots, Z_M), \quad Z_0 = 0, \quad Z_i = \begin{bmatrix} 0 & z_i \\ -z_i & 0 \end{bmatrix},
\]

where \(Z_0\) is the \((N - 2M) \times (N - 2M)\) null matrix and \(z_i\) are complex numbers. Equivalently, \(Z\) is the direct sum of the the \((N - 2M) \times (N - 2M)\) null matrix and the \(M (2 \times 2)\) complex antisymmetric matrices \(Z_i\).

The fermionic Schmidt decomposition follows from an application of Theorem 3.1.1 to systems composed of identical fermions:

**Theorem 3.1.2** [fermionic Schmidt decomposition]: Any state vector \(|\psi(1, 2)\rangle\) describing two identical fermions of spin \(s\) and, consequently, belonging to the antisymmetric manifold \(A(\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1})\), can be written as:

\[
|\psi(1, 2)\rangle = \frac{1}{\sqrt{2}} \sum_{i=1}^{(2s+1)/2} a_i \cdot |2i-1\rangle_1 \otimes |2i\rangle_2 - |2i\rangle_1 \otimes |2i-1\rangle_2,
\]

where the states \(\{|2i-1\rangle, |2i\rangle\}\) with \(i = 1 \ldots (2s+1)/2\) constitute an orthonormal basis of \(\mathbb{C}^{2s+1}\), and the complex coefficients \(a_i\) (some of which may vanish) satisfy the normalization condition \(\sum_i |a_i|^2 = 1\).

Following the authors of Ref. [5], the number of non-zero coefficients \(a_i\) appearing in the decomposition (3.2) is called the *Slater number* of \(|\psi(1, 2)\rangle\), and it is the fermionic analog of the Schmidt number. The relation of such a number with the notion of entanglement has been made explicit in the papers [6, 7], where a state displaying the form of Eq. (3.2) is called entangled if and only if its Slater number is strictly greater than one. It is worth noticing that such a condition turns out to be totally equivalent to our Definition 2.1. In fact, given an arbitrary two particle state \(|\psi(1, 2)\rangle\), suppose that its fermionic Schmidt decomposition has Slater number equal to one. According to Theorem 3.1.2, this means that there exist two orthonormal vectors \(|1\rangle\) and \(|2\rangle\) belonging to \(\mathbb{C}^{2s+1}\) such that:

\[
|\psi(1, 2)\rangle = \frac{1}{\sqrt{2}} [ |1\rangle \otimes |2\rangle_2 - |2\rangle_1 \otimes |1\rangle_2 ], \quad \langle 1\rangle_2 = 0.
\]

Since the state can be obtained by antisymmetrizing the product state \(|1\rangle \otimes |2\rangle_2\), the state must be considered as non-entangled in accordance with our criterion. Vice versa, any state obtained
by antisymmetrizing a factorized state has Slater number equal to 1. On the contrary, if the Slater number is greater (or equal) to two, the form Eq. (3.2) of the state shows immediately that it cannot be obtained by antisymmetrizing a product of two orthonormal vectors, so that the state must be considered as a genuinely entangled one.

So far the criterion of Ref. [5, 6, 7] and ours agree completely; however, some problems arise when one calculates the von Neumann entropy of the reduced density operator associated to one of the two particles. In fact from Eq. (3.2) one gets:

$$\rho^{(1)} \equiv Tr^{(2)}[|\psi(1,2)\rangle\langle\psi(1,2)|] = \sum_{i} \frac{|a_i|^2}{2} \cdot [|2i-1\rangle_i\langle 2i-1| + |2i\rangle_i\langle 2i|].$$

(3.4)

The von Neumann entropy for such an operator, which is already in its diagonal form, can be easily calculated:

$$S(\rho^{(1)}) \equiv -Tr^{(1)}[\rho^{(1)} \log \rho^{(1)}] = -\sum_{i} |a_i|^2 \log \frac{|a_i|^2}{2} = 1 - \sum_{i} |a_i|^2 \log |a_i|^2.$$

(3.5)

It follows trivially that:

$$S(\rho^{(1)}) \geq 1 \quad \forall |a_i|^2 \in [0,1] \quad \sum_{i} |a_i|^2 = 1.$$

(3.6)

In analogy with the case of distinguishable particles, one could be tempted to regard such a quantity as a measure of entanglement. But, according to the authors of Ref. [6] and [8], this naturally raises the following two puzzling issues: i) $S(\rho^{(1)})$ of Eq. (3.5) attains its minimum value $S_{min} = 1$ in correspondence of a state with Slater number equal to one (which, in accordance with their position is assumed to identify a non-entangled state), contrary to what happens for a (non-entangled) state of distinguishable particles with the same Schmidt number, for which the value of the entropy still takes its minimum value which, however, equals 0; ii) moreover, in the case of two identical bosons, the minimum of the analogous quantity, as we will show later, is null. This seems to imply that the von Neumann entropy is inappropriate to deal with our problem since it gives different measures of entanglement for bosons and fermions states.

The problematic aspects of this situation derive entirely from not having taken correctly into account the real meaning of the von Neumann entropy as a measure of the uncertainty about the state of a quantum system. To be more precise, let us consider a state $|\psi(1,2)\rangle$ with Slater number equal to 1, i.e. a non-entangled state according to the Slater number criterion, like the one of Eq. (3.3). As we have stressed in Section 1, in this situation we can attribute definite quantum states $|1\rangle$ and $|2\rangle$ to the particles but, since they are totally indistinguishable, we cannot know whether, for example, particle 1 is associated to the state $|1\rangle$ or to the state $|2\rangle$. As a natural consequence, the reduced density operator of each particle and its associated von Neumann entropy reflect such an unavoidable ignorance. Actually, they are equal to:

$$\rho^{(1 \text{ or } 2)} = \frac{1}{2} [|1\rangle\langle 1| + |2\rangle\langle 2|] \quad \Rightarrow \quad S(\rho^{(1 \text{ or } 2)}) = 1.$$

(3.7)

\footnote{Since the state vector of the compound system $|\psi(1,2)\rangle$ is antisymmetric under the exchange of the two particles, its associated density operator $\rho^{(1+2)} \equiv |\psi(1,2)\rangle\langle\psi(1,2)|$ is a symmetric operator and the reduced density operators of the two particles are equal, that is $\rho^{(1)} \equiv Tr^{(2)}[\rho^{(1+2)}] = \rho^{(2)} \equiv Tr^{(1)}[\rho^{(1+2)}].$}
It should be obvious that we cannot pretend that the operator $\rho^{(1 \text{ or } 2)}$ of Eq. (3.7) describes the properties of precisely the first or of the second particle of the system: once again, due to the subtle implications of the identity in quantum mechanics, such an operator describes correctly the properties of a randomly chosen particle (a particle which cannot be better identified). Accordingly, in this case, the quantity $S(\rho^{(1 \text{ or } 2)}) = 1$ correctly measures the uncertainty concerning the quantum state to attribute to each of the two identical physical subsystems, and, in this situation, it cannot be regarded as a measure of the entanglement of the whole state.

A counterpart of this is the fact that the only quantum correlations exhibited by the state $|\psi(1,2)\rangle$ of Eq. (3.3) are those related to the exchange properties of the indistinguishable fermions. As we have stressed before, such correlations cannot be used to violate Bell’s inequality or to perform any teleportation process. Accordingly, they cannot be considered as a manifestation of entanglement.

Continuing our analysis, if we consider a state vector with Slater number strictly greater than 1, i.e. an entangled state according to the Slater number criterion, the von Neumann entropy of the associated reduced density operator turns out to be strictly greater than 1 (see Eq. (3.6)). Besides the minimum amount of uncertainty deriving from the indistinguishability of the particles, there is now the additional ignorance naturally connected with the genuine entanglement of the state. In this case, in fact, we cannot identify two quantum states which can be attributed$^6$ to the pair of particles (contrary to the previously described situation).

To conclude this Subsection, we exhibit the criteria for detecting entanglement involving the Slater number (from now on, we will call it Schmidt number) or the von Neumann entropy, in the case of two identical fermions. Such criteria are totally consistent with our original requirement that definite properties can be attributed to both component subsystems of a non-entangled state:

**Theorem 3.1.3** A state vector $|\psi(1,2)\rangle$ describing two identical fermions is non-entangled iff its Schmidt number is equal to 1 or equivalently iff the von Neumann entropy of the one-particle reduced density operator $S(\rho^{(1 \text{ or } 2)})$ is equal to 1.

In full agreement with the previous considerations we can say that the von Neumann entropy is a measure both of the amount of uncertainty deriving from the indistinguishability of the involved particles and of the possible uncertainty related to the entangled nature of the states (that is, the impossibility to attribute two definite state vectors to the pair of particles). This quantity, in the case of fermions, is strictly greater than one and the greater it is, the larger is the amount of entanglement of the state.

Before coming to the boson case a last remark is appropriate. Let us consider a non-entangled state of two fermions which has the form of Eq. (3.3). Suppose we choose two arbitrary orthonormal vectors $|\bar{1}\rangle$ and $|\bar{2}\rangle$ belonging to the two dimensional manifold spanned by the states $|1\rangle$ and $|2\rangle$, such that:

$$|1\rangle = \alpha|\bar{1}\rangle + \beta|\bar{2}\rangle \quad |2\rangle = -\beta^*|\bar{1}\rangle + \alpha^*|\bar{2}\rangle.$$  

$^6$With this expression we mean that a measurement aimed to test whether there is one particle in state $|1\rangle$ and one in state $|2\rangle$ gives with certainty a positive outcome.
with $|\alpha|^2 + |\beta|^2 = 1$.

Substituting these expressions in Eq. (3.3) we have:

$$|\psi(1, 2)\rangle = \frac{1}{\sqrt{2}} [ |\bar{1}\rangle_1 \otimes |\bar{2}\rangle_2 - |\bar{2}\rangle_1 \otimes |\bar{1}\rangle_2 ].$$  (3.9)

Therefore, when a state is obtained by antisymmetrizing the product of two orthogonal single particle states, the same state can also be obtained by antisymmetrizing the product of any pair of orthogonal states belonging to the two-dimensional manifold spanned by the original states. This is not surprising but it raises some questions concerning the problem of the assignment of definite properties to the constituents. In fact, Eq. (3.9) shows that, just as we can claim that in state of Eq. (3.3) there is one particle with the properties associated to $|1\rangle$ and one with the properties associated to $|2\rangle$, we can make an analogous statement with reference to any two arbitrary orthogonal states $|\bar{1}\rangle$ and $|\bar{2}\rangle$ of the same manifold. However, this peculiar feature is only apparently problematic. In fact it seems so because here we are confining our considerations to the spin degrees of freedom. When one takes into account also the space degrees of freedom one realizes that the only physically interesting situations are those associated to states like the one of Eq. (2.4), that is those in which one wants, for example, to investigate the spin properties of a particle which has a precise location.  

As we have seen, in such a state one can make precise claims about the spin state of the particle at Right or at Left, respectively, contrary to what happens for a state like of Eq. (2.7) which is genuinely entangled. This point has been exhaustively discussed in Ref. [2], to which we refer the reader for a more detailed analysis.

### 3.2 The boson case

Let us now pass to the more delicate case of two identical bosons. In such a case, dealing with the problem of their entanglement requires great care in order to avoid possible misunderstandings. Let us start, as before, by considering the bosonic analogue of the Schmidt decomposition for an arbitrary state vector $|\psi(1, 2)\rangle$ belonging to the symmetric manifold $S(\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1})$ describing two identical bosons, and let us recall a well-known theorem of matrix analysis (the so-called Takagi’s factorization theorem [11], a particular instance of a more general theorem named Singular Value Decomposition) which is particularly useful for such a case:

**Theorem 3.2.1** For any symmetric $(N \times N)$ complex matrix $B$ (that is, $B \in \mathcal{M}(N, \mathbb{C})$ and $B^T = B$) there exists a unitary transformation $U$ such that $B = U \Sigma U^T$, where $\Sigma$ is a real nonnegative diagonal matrix $\Sigma = \text{diag}[b_1, \ldots, b_N]$. The columns of $U$ are an orthonormal set of eigenvectors of $BB^\dagger$ and the diagonal entries of $\Sigma$ are the nonnegative square roots of the corresponding eigenvalues.

It is worth noticing that the columns of the unitary operator $U$ must be built by choosing a precise set of eigenvectors of $BB^\dagger$ (to be determined in an appropriate manner, see [11]). Now, the bosonic analog of the Schmidt Decomposition turns out to be a trivial consequence of the just mentioned Theorem 3.2.1.
Theorem 3.2.2 [bosonic Schmidt decomposition]: Any state vector describing two identical s-spin boson particles \(|\psi(1,2)\rangle\) and, consequently, belonging to the symmetric manifold \(S(\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1})\) can be written as:

\[
|\psi(1,2)\rangle = \sum_{i=1}^{2s+1} b_i |i\rangle_1 \otimes |i\rangle_2 ,
\]

where the states \(|i\rangle\), with \(i = 1, \ldots, 2s + 1\), constitute an orthonormal basis for \(\mathbb{C}^{2s+1}\), and the real nonnegative coefficients \(b_i\) are the diagonal elements of the matrix \(\Sigma\) and satisfy the normalization condition \(\sum_i b_i^2 = 1\).

In analogy with the case of two distinguishable particles and of two identical fermions, one could naively be inclined to term entangled any bosonic state whose Schmidt number (that is, the number of non-zero coefficients in the decomposition (3.10)) is strictly greater than one (as clearly stated, for example, in Ref. \[6\]). We note that this would amount to consider non-entangled only factorized states in which the two bosons are in the same state. While it is obvious that such states must be considered as non-entangled also according to our criterion, and, moreover, for them one knows precisely the properties of both constituents so that no uncertainty remains concerning which particle has which property, they do not exhaust the cases of non-entangled pairs of identical bosons. In fact, if one considers a state \(|\psi(1,2)\rangle\) which has Schmidt number equal to two and is obtained by symmetrizing the tensor product of two orthonormal states \(|\phi\rangle\) and \(|\chi\rangle\)

\[
|\psi(1,2)\rangle = \frac{1}{\sqrt{2}} \left[ |\phi\rangle_1 \otimes |\chi\rangle_2 + |\chi\rangle_1 \otimes |\phi\rangle_2 \right],
\]

in full accordance with our Theorem 3.2.2, one must consider this state as a non-entangled one since it is possible to attribute well definite state vectors to both particles (of course, once again, we cannot say which particle is associated to the state \(|\phi\rangle\) or \(|\chi\rangle\) because of their indistinguishability). Actually, as the reader has certainly grasped, such a state has precisely the same conceptual status and exhibits the same physical features as the non-entangled states of a pair of fermions. If we resort to the bosonic Schmidt decomposition of Theorem 3.2.2 (taking into account that, in this peculiar case, the symmetric matrix \(B\) of the coefficients of the state (3.11) yields an operator \(BB^\dagger\) with a degenerate eigenvalue) we find that the state (3.11) has the following form:

\[
|\psi(1,2)\rangle = \frac{1}{\sqrt{2}} \left[ |1\rangle_1 \otimes |1\rangle_2 + |2\rangle_1 \otimes |2\rangle_2 \right],
\]

where \(|1\rangle = 1/\sqrt{2} (|\phi\rangle + |\chi\rangle)\) and \(|2\rangle = i/\sqrt{2} (|\phi\rangle - |\chi\rangle)\) are orthogonal states. Thus, an uncritical application of the same criterion used for fermion particles does not work for this specific case, since it would lead to consider as entangled a state which is not such on physical grounds.

Before proceeding some comments are appropriate. First, due to the degeneracy of the eigenvalues of \(BB^\dagger\), if one chooses any pair of orthonormal vectors which are a linear combination, with real coefficients, of the above states \(|1\rangle\) and \(|2\rangle\):

\[
|\bar{1}\rangle = \alpha|1\rangle + \beta|2\rangle , \quad |\bar{2}\rangle = -\beta|1\rangle + \alpha|2\rangle ,
\]
where \( \alpha, \beta \in \mathbb{R} \) and \( \alpha^2 + \beta^2 = 1 \), one has:

\[
|\psi(1, 2)\rangle = \frac{1}{\sqrt{2}} \left[ |\bar{1}\rangle_1 |\bar{1}\rangle_2 + |\bar{2}\rangle_1 |\bar{2}\rangle_2 \right],
\]

\[(3.14)\]

and the form of Eq. \((3.12)\) is preserved with \(|\bar{1}\rangle\) and \(|\bar{2}\rangle\) replacing \(|1\rangle\) and \(|2\rangle\) respectively. Thus, as in the case of distinguishable particles, when degeneracy occurs, the Schmidt decomposition is not unique.

This situation recalls the one we have met in the case of two non-entangled fermions. However, the possibility of writing \(|\psi(1, 2)\rangle\) in different Schmidt forms, contrary to what happens with fermions, is absolutely unproblematic from the physically interesting point of view of the properties possessed by the two particles. Actually, the pair of orthonormal states whose symmetrization leads to the state of Eq. \((3.11)\) is uniquely determined in the present case. Accordingly, for identical bosons there is no ambiguity concerning the properties which allow us to claim that in the state of Eq. \((3.11)\) “there is one particle with the properties identified by \(|\phi\rangle\) and one with the properties identified by \(|\chi\rangle\)”.

It is illuminating to stress the difference between two ways of looking at the properties of the constituents according to what emerges naturally when one expresses the state in the form of Eq. \((3.11)\) or of Eq. \((3.12)\). Actually, if one looks at the form of Eq. \((3.12)\), without taking into account that it has been obtained by symmetrizing a factorized pair of orthogonal states, one is naturally led to make different statements, statements which have a probabilistic character, i.e., to assert that “with probability 1/2 both particle are (in the sense they will be found to be) in state \(|1\rangle\) (or \(|\bar{1}\rangle\)) and, with the same probability they are in state \(|2\rangle\) (or \(|\bar{2}\rangle\))”.

The situation becomes more intricate when we consider the remaining set of symmetrized states with Schmidt number equal to 2, that is those obtained by symmetrizing a factorized product of two non-orthogonal states \(|\phi\rangle\) and \(|\chi\rangle\):

\[
|\psi(1, 2)\rangle = \frac{1}{\sqrt{2}(1 + |\langle \chi | \phi \rangle|^2)} \left[ |\phi\rangle_1 \otimes |\chi\rangle_2 + |\chi\rangle_1 \otimes |\phi\rangle_2 \right], \quad \langle \chi | \phi \rangle \neq 0.
\]

\[(3.15)\]

It is easy to show that, by defining the vector \(|\phi_\perp\rangle\) as the unique normalized vector orthogonal to \(|\phi\rangle\) and lying in the two-dimensional manifold spanned by \(|\phi\rangle\) and \(|\chi\rangle\), the state of Eq. \((3.15)\) can be written as \(^9\):

\[
|\psi(1, 2)\rangle = a|\phi\rangle_1 \otimes |\phi\rangle_2 + \frac{b}{\sqrt{2}} \left[ |\phi\rangle_1 \otimes |\phi_\perp\rangle_2 + |\phi_\perp\rangle_1 \otimes |\phi\rangle_2 \right],
\]

\[(3.16)\]

with certain complex coefficients \(a, b \neq 0\) satisfying the normalization condition \(|a|^2 + |b|^2 = 1\) and depending on the modulus of the scalar product \(|\langle \chi | \phi \rangle|\) in the following way:

\[
|b| = \left(1 - |\langle \chi | \phi \rangle|^2\right)^{1/2} \left(1 + |\langle \chi | \phi \rangle|^2\right)^{-1/2}.
\]

In this case, the criteria adopted in Refs. \([6, 7, 8]\) correctly lead to declare the state of Eq. \((3.16)\) as a genuine entangled one. Since such a state is obtained by symmetrizing a factorized product of two non-orthogonal vectors, it is impossible to attribute to both particles well definite properties, so that it has to be considered as entangled also within our approach. We stress that

\(^9\)We notice that one could have equivalently defined the vector \(|\chi_\perp\rangle\) as the unique normalized vector orthogonal to \(|\chi\rangle\) and lying in the two-dimensional manifold spanned by \(|\phi\rangle\) and \(|\chi\rangle\), without modifying the forthcoming conclusions.
its Schmidt number is equal to 2 precisely as the one of the state of Eq. (3.11). In fact, by resorting to Theorem 3.2.1 we may explicitly calculate, starting from Eq. (3.16), the unitary its Schmidt number is equal to 2 precisely as the one of the state of Eq. (3.11). In fact, by
orthogonal states $|\psi\rangle$ by the non-entangled state of Eq. (3.11), which has been obtained by symmetrizing the two
An example of a state whose Schmidt decomposition is precisely of this sort is represented on a factorized single particle basis including $|\phi\rangle$ and $|\phi_\perp\rangle$. The resulting bosonic Schmidt decomposition of state (3.16) turns out to be:

$$|\psi(1, 2)\rangle = \sqrt{\frac{1 + \sqrt{1 - |b|^4}}{2}} |1\rangle_1 \otimes |1\rangle_2 + \sqrt{\frac{1 - \sqrt{1 - |b|^4}}{2}} |2\rangle_1 \otimes |2\rangle_2 \quad |b| \in (0, 1),$$

where the orthonormal states $|1\rangle$ and $|2\rangle$ are defined as:

$$|1\rangle = \frac{|b|e^{i\theta_a/2}}{\sqrt{2 - |a|^2 + \sqrt{1 - |b|^4}}} \left[ |\phi\rangle + \frac{\sqrt{1 - |b|^4} - |a|^2}{\sqrt{2ab^*}} |\phi_\perp\rangle \right],$$

$$|2\rangle = \frac{i|b|e^{i\theta_a/2}}{\sqrt{2 - |a|^2 - \sqrt{1 - |b|^4}}} \left[ |\phi\rangle - \frac{\sqrt{1 - |b|^4} + |a|^2}{\sqrt{2ab^*}} |\phi_\perp\rangle \right],$$

with $a = |a|e^{i\theta_a}$, $a$ and $b \neq 0$ and $|a|^2 + |b|^2 = 1$. It is worth noticing that, contrary to what happens for the non-entangled state of Eq. (3.11), the Schmidt decomposition of Eq. (3.15) is now uniquely determined since the two eigenvalues of the operator $BB^\dagger$ are distinct. More generally, the Schmidt decomposition of Eq. (3.10) is always unique when the eigenvalues of the operator $BB^\dagger$ are non-degenerate, as it happens for the biorthonormal decomposition of states describing distinguishable particles.

Concluding, in the boson case, the consideration of the Schmidt number alone to detect the entanglement of a state fails since, as the previous examples have clearly shown, there exist bosonic states with Schmidt number equal to 2 which can be entangled as well as non-entangled. A complete and satisfactory criterion for distinguishing entangled from non-entangled states should involve, as we will see, both the consideration of the Schmidt number criterion and the von Neumann entropy of the reduced density operator of each single particle.

For this purpose let us consider the most general state with Schmidt number equal precisely to 2:

$$|\psi(1, 2)\rangle = b_1|1\rangle_1 \otimes |1\rangle_2 + b_2|2\rangle_1 \otimes |2\rangle_2,$$

where $b_1^2 + b_2^2 = 1$. The single particle reduced density operator and its associated von Neumann entropy are:

$$\rho^{(1 \text{ or } 2)} = b_1^2 |1\rangle_1 \langle 1| + b_2^2 |2\rangle_2 \langle 2|,$$

$$S(\rho^{(1 \text{ or } 2)}) = -b_1^2 \log b_1^2 - b_2^2 \log b_2^2.$$
those descending from the exchange of the two identical particles, the von Neumann entropy $S(\rho^{(1 \text{ or } 2)})$ is correctly equal to 1 and our ignorance concerns only which particle has to be associated to which state.

In the second case the two coefficients are different, that is $b_1 \neq b_2$. Every state which has been obtained by symmetrizing a factorized product of two non-orthogonal states $|\phi\rangle$ and $|\chi\rangle$ like that of Eq. (3.16), has precisely this property, as it is apparent by looking at the state of Eq. (3.17). In this case the reduced density operator describing one randomly chosen particle and its associated von Neumann entropy are equal to:

$$\rho^{(1 \text{ or } 2)} = \frac{1 + \sqrt{1 - |b|^4}}{2} |1\rangle\langle 1| + \frac{1 - \sqrt{1 - |b|^4}}{2} |2\rangle\langle 2|,$$

$$S(\rho^{(1 \text{ or } 2)}) = -\frac{1 + \sqrt{1 - |b|^4}}{2} \log \frac{1 + \sqrt{1 - |b|^4}}{2} - \frac{1 - \sqrt{1 - |b|^4}}{2} \log \frac{1 - \sqrt{1 - |b|^4}}{2}.$$

(3.21)

It is easy to prove that in this case the von Neumann entropy, belongs to the open interval $(0, 1)$.

Then, in a measurement process, there is a probability $(1 + \sqrt{1 - |b|^4})/2$ (which is greater than $1/2$) that both bosons be found in the same physical state $|1\rangle$ and a probability $(1 - \sqrt{1 - |b|^4})/2$ (which is obviously less than $1/2$) to find them both in the orthogonal state $|2\rangle$. Accordingly, in this situation we have more information about the single particle state than in the case of the non-entangled state of Eq. (3.11) and, as a consequence, the von Neumann entropy is strictly less than 1.

Finally, there is only one case left unanalyzed, that is the one in which the Schmidt number of the considered state of two bosons is greater or equal to 3. In this situation, the state is a genuine entangled one since it cannot be obtained by symmetrizing a factorized product of two orthogonal states and the von Neumann entropy of the reduced density operators is such that $S(\rho^{(1 \text{ or } 2)}) \in (0, \log(2s + 1)]$. Like before, the entropy is close to zero when, in the Schmidt decomposition of the state, one of its coefficients is very close to one, implying that there is a very high probability of being right in claiming that both bosons are (in the usual quantum sense “will be found to be if a measurement is performed”) in the state associated to the largest coefficient. On the contrary, the entropy equals $\log(2s + 1)$ when the decomposition involves all the basis states of $\mathbb{C}^{2s+1}$ with equal weights (a maximally entangled state). Moreover, when the entropy lies within the interval $(0, 1)$, there is a precise state such that the probability of finding a (randomly chosen) particle in it, in an appropriate measurement, is greater than $1/2$ while, when the entropy lies in the interval $[1, \log(2s + 1))$, the most probable state in which we can find a particle has a probability which is smaller (or equal) to $1/2$. Obviously, the identification of this mentioned privileged states requires the knowledge of the Schmidt decomposition since they are those associated to the largest $b_i$ appearing decomposition of Eq. (3.10).

It is important to stress that, since the entropy can actually take the value 1 also for an entangled state with Schmidt number greater than 2, the von Neumann entropy criterion does not allow, by itself, an unambiguous identification of non-entangled states.

Thus, according to our analysis, there is a consistent way of combining the von Neumann entropy with the Schmidt number criteria in order to obtain a physically meaningful way of

---

10 In fact, if this was true, the rank of the reduced density operator would be equal to two, in contradiction with the fact that a Schmidt number greater or equal to three implies a rank equal or greater to three.
checking whether a state of two identical bosons can be considered entangled or not. It is expressed by the following:

**Theorem 3.2.3** A state vector $|\psi(1,2)\rangle$ describing two identical bosons is non-entangled iff either its Schmidt number is equal to 1, or the Schmidt number is equal to 2 and the von Neumann entropy of the one-particle reduced density operator $S(\rho^{(1 or 2)})$ is equal to 1. Alternatively, one might say that the state is non-entangled iff its von Neumann entropy is equal to 0, or it is equal to 1 and the Schmidt number is equal to 2.

4 Summary

To conclude our paper we summarize the previous results and give a general overview of the problem of characterizing the entanglement of quantum systems composed of two identical particles. First of all we recall that this problem can be dealt with in a completely rigorous way by following our original criterion [2] expressed in the Definition 2.1. However, here we are interested in analyzing the criteria based on the determination of the Schmidt number of the (fermionic or bosonic) Schmidt decomposition of the state vector of the composite system and on the evaluation of the von Neumann entropy of the reduced density operators associated to each single constituent. As usual, we deal separately with the cases of two identical fermions and of two identical bosons, respectively.

We start with a system of two identical fermions:

**Two identical fermions**

1. Schmidt Number of $|\psi(1,2)\rangle = 1 \iff S(\rho^{(1 or 2)}) = 1 \iff$ Non-entangled state

2. Schmidt Number of $|\psi(1,2)\rangle > 1 \iff S(\rho^{(1 or 2)}) > 1 \iff$ Entangled state

In the first case the state can be obtained by antisymmetrizing a tensor product of two orthogonal single particle states, while in the second case it cannot.

In the considered case the two criteria are, as indicated, fully equivalent and they identify precisely the same states as our criterion. It is interesting to notice that, since identical fermions cannot be in the same state, at least the uncertainty corresponding to the impossibility of identifying which particle has which property, when there is no entanglement, is always present. This is why the minimum value of the entropy turns out to be equal to 1. In such a case, when one makes a precise claim concerning the state of a randomly chosen particle, there is a probability equal to 1/2 that the claim is correct. On the other hand, when the Schmidt number is greater than 1, or, equivalently, the entropy is larger than 1, there is no state such that the statement “this particle, if subjected to the appropriate measurement, will be found in such a state” has a probability equal or greater than 1/2. Thus, the non-entangled state is the one in which one has the maximum information about the state of the system.

The case of two identical bosons is slightly more articulated. We recall the conclusions we have reached:
Two identical bosons

1. Schmidt Number of $|\psi(1, 2)\rangle = 1 \overset{\text{def}}{=} S(\rho^{1\text{or}2}) = 0 \Rightarrow \text{Non-entangled state}$

2. Schmidt Number of $|\psi(1, 2)\rangle = 2$ and $S(\rho^{1\text{or}2}) \in (0, 1) \Rightarrow \text{Entangled state}$

3. Schmidt Number of $|\psi(1, 2)\rangle = 2$ and $S(\rho^{1\text{or}2}) = 1 \Rightarrow \text{Non-entangled state}$

4. Schmidt Number of $|\psi(1, 2)\rangle > 2 \Rightarrow S(\rho^{1\text{or}2}) \in (0, \ln(2s + 1)) \Rightarrow \text{Entangled state}$

In the first case the state is the tensor product of the same single particle vector; in the second case the state can be obtained by symmetrizing two non-orthogonal vectors and thus no definite properties can be attributed to both subsystems. In the third case the state can be obtained by symmetrizing a tensor product of two orthogonal vectors while in the last one it involves more than two linearly independent single particle states.

The above list exhibits some interesting features. First of all it shows clearly that the Schmidt number cannot be used, unless it has the value 0, to identify non-entangled states, since there are both entangled and non-entangled states with Schmidt number equal to 2. It also shows that the von Neumann entropy criterion does not allows, by itself, a clear-cut identification of the non-entangled states since it can take the value 1 both for a non-entangled state of Schmidt number 2 and for an entangled state of Schmidt number greater than 2.

As discussed in this paper, the von Neumann entropy supplies us, by itself, with an important information concerning the state of a randomly chosen constituent: when it lies in the interval $(0, 1)$ there is a precise state such that the probability of finding it in an appropriate measurement is greater than $1/2$, while, when it takes a value greater than 1, the most probable state in which we can find a particle has a probability smaller than $1/2$.

5 Conclusions

In this paper we have discussed in general the problem of deciding whether a state describing a system of two identical particles is entangled or not. We have recalled the general criteria and results derived in Refs. [2, 3], which make explicit reference to the possibility of attributing a complete set of objective properties to both constituents. We have shown how this attitude allows to clarify the somehow puzzling situation which one meets when resorting to the consideration either of the Schmidt number of the Schmidt decomposition or of the von Neumann entropy of the reduced statistical operator for detecting entanglement. Our analysis has made clear that some alleged difficulties of the now mentioned approaches derive simply from not having appropriately taken into account the peculiar role of the identity within the quantum formalism. In particular, we have shown how the consideration of the von Neumann entropy allows to determine whether the uncertainty concerning the states arises simply from the identity of the particles or it is also a genuine consequence of the entanglement. With reference to this problem we stress once more that, when all our lack of information derives simply from the identity, the existing correlations cannot be used as a quantum mechanical resource to implement any teleportation procedure or to violate Bell’s inequality, contrary to what happens for an entangled state.
References

[1] E.Schrödinger, Naturwissenschaften, 23, 807 (1935); English translation in: Proc. Am. Philos. Soc., 124, 323 (1980).

[2] G.C.Ghirardi, L.Marinatto and T.Weber, Journal of Statistical Physics, 108, 49 (2002).

[3] G.C.Ghirardi and L.Marinatto, Fortschritte der Physik, 51, No.4-5, 379 (2003).

[4] A.Einstein, B.Podolsky and N.Rosen, Phys. Rev., 47, 777 (1935).

[5] J.Schliemann, J.I.Cirac, M.Kus, M.Lewenstein, and D.Loss, Phys. Rev. A, 64, 022303 (2001).

[6] R.Paskauskas and L.You, Phys. Rev. A, 64, 042310 (2001).

[7] Y.S.Li, B.Zeng, X.S.Liu and G.L.Long, Phys. Rev. A, 64, 054302 (2001).

[8] H.M.Wiseman and J.A.Vaccaro, Phys. Rev. Lett., 91, 097902 (2003).

[9] J.F.Clauser, M.A.Horne, A.Shimony and R.A.Holt, Phys. Rev. Lett, 26, 880 (1969).

[10] M.L.Mehta Matrix theory: selected topics and useful results, Les Editions de Physique, (1989).

[11] R.A.Horn and C.R.Johnson, Matrix analysis, Cambridge University Press, Cambridge (1986).