IDENTITIES OF THE MULTI-VARIATE INDEPENDENCE POLYNOMIALS FROM HEAPS THEORY

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ABSTRACT. We study and derive identities for the multi-variate independence polynomials from the perspective of heaps theory. Using the inversion formula and the combinatorics of partially commutative algebras we show how the multi-variate version of Godsil type identity as well as the fundamental identity can be obtained from weight preserving bijections. Finally, we obtain a new multi-variate identity involving connected bipartite subgraphs similar to the Christoffel-Darboux type identities obtained by Bencs.

1. INTRODUCTION

Let $G$ be a finite simple connected graph with vertex set $V(G)$. A subset of $V(G)$ is said to be independent if it does not include two adjacent vertices and by convention we allow the empty subset to be independent. The multi-variate independence polynomial of $G$ is defined as

$$I(G, x) := \sum_{S} (-1)^{|S|} \prod_{v \in S} x_v$$

where the sum runs over all independent subsets $S$ of $V(G)$. The aim of this article is to approach certain identities for multi-variate independence polynomials using the inversion formula from heaps theory.

To explain our motivations and results we need some terminologies. One can associate a monoid called the Cartier–Foata monoid to the graph $G$ (see [5]). This monoid is generated by the vertices of $G$ and the defining relations are given by $uv = vu$ if $u, v \in V(G)$ and there is no edge between them. One can prove that the Cartier–Foata monoid of $G$ is equivalent to the monoid of heaps with pieces in $V(G)$ and the concurrency relation is determined by $G$ (see [13]). The fundamental result of Viennot’s general theory of heaps is the inversion lemma (see for example [13] and [4, Theorem 2.1]) which gives a closed formula for the generating function of heaps with all maximal pieces in some fixed subset.

Even though heaps give a geometric interpretation of the elements of the Cartier–Foata monoid, we prefer to work with the Cartier–Foata monoid itself in this paper. Fix a subset $K$ of $V(G)$, and consider the set $\mathcal{P}_K^\emptyset(G)$ that consists of all elements in the monoid that can only end with one of the $v$’s from $K$ (see Section 2.1 for more details). We can assign a weight to each element of $\mathcal{P}_K^\emptyset(G)$ as follows: given $w = u_1 \cdots u_r \in \mathcal{P}_K^\emptyset(G)$, define $\text{wt}(w) = \prod_{i=1}^{r} x_{u_i} \in \mathbb{C}[x_v : v \in V(G)]$.

Then the generating function of $\mathcal{P}_K^\emptyset(G)$ is simply given by

$$\sum_{w \in \mathcal{P}_K^\emptyset(G)} \text{wt}(w) = \frac{I(G - K, x)}{I(G, x)}$$
where $G - K$ is the graph obtained from $G$ by removing the vertices in $K$. The motivation of this work comes from a Godsil’s type identity that has been proved in [3] for one-variable independence polynomials; recall that the one variable independence polynomial is obtained by evaluating $x_v = -x$ for all $v \in V(G)$ in the multi-variate independence polynomial. Given a vertex $u \in G$, Bencs constructed a rooted (stable path) tree $(T, u')$ such that

$$\frac{I(G - u, x)}{I(G, x)} = \frac{I(T - u', x)}{I(T, x)} \quad (1.1)$$

Godsil’s original identity was stated for matching polynomials [7] and was one of the key ingredients in proving that the matching polynomial is real rooted. Furthermore, the importance of this identity is also highlighted in [12] where the authors prove the existence of infinite families of regular bipartite Ramanujan graphs of every degree greater than 2. It is not hard to prove the multi-variate version of Equation (1.1) (the proof goes along the same lines as the proof of [3, Theorem 2.3]). However, both sides of the multi-variate version of Equation (1.1) are the generating functions of certain words from the Cartier–Foata monoid of $G$. More precisely, the left hand side of Equation (1.1) corresponds to the generating function of $P_u^\emptyset (G)$ and the right hand side corresponds to the generating function of $P_u^\emptyset (T)$. So, we have the following natural questions:

- Is there any natural weight preserving bijective map from $P_u^\emptyset (G)$ onto $P_u^\emptyset (T)$ that gives the multi-variate version of Equation (1.1)?
- Using the method of finding weight preserving bijections, is one able to give new proofs of existing identities, generalize them to the multi-variate case and obtain new identities?

We answer the first question affirmatively in this paper. We will also use our approach to get more identities and prove existing identities for multi-variate independence polynomial of $G$. In particular we prove a new multi-variate identity Equation (4.4) involving connected bipartite subgraphs similar to the Christoffel-Darboux type identities obtained by Bencs [2]. This identity seems to be new in the literature.

## 2. Independence Polynomials and Word Decompositions

### 2.1. Let $G$ be a finite simple connected graph, i.e., $G$ contains no loops and multiple edges. The vertex set and edge set of $G$ are denoted as $V(G)$ and $E(G)$ respectively. We denote by $e(u, v)$ the edge between the vertices $u$ and $v$ of $G$. For $u \in V(G)$, we denote by $N_G(u)$ the neighbourhood of $u$ in $G$, $d_G(u) := |N_G(u)|$ the degree of $u$ in $G$ and set $N_G[u] = N_G(u) \cup \{u\}$. For a subset $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of $G$ spanned by the vertices in $S$. Let $P^\emptyset (G)$ denote the partially commutative monoid of $G$ which is generated by the elements of $V(G)$ modulo the relations

$$uv = vu \iff e(u, v) \notin E(G).$$

If $C^\emptyset (G)$ denotes the commutative monoid generated by $V(G)$, we have a canonical monoid morphism $\pi_G : P^\emptyset (G) \to C^\emptyset (G)$. We set $P(G) := P^\emptyset (G)\\{\text{pt}\}$ where we think of the extra point in $P^\emptyset (G)$ as the empty word and introduce further

$$P_{v_1,\ldots,v_r}(G) = \{w \in P(G) : IA(w) \subseteq \{v_1,\ldots,v_r\}\},$$
$$P_{v_1,\ldots,v_r}^c(G) = \{w \in P(G) : IA(w) = \{v_1,\ldots,v_r\}\},$$
\[ P^\emptyset_{v_1, \ldots, v_r}(G) = P_{v_1, \ldots, v_r}(G) \cup \{pt\}, \]
i.e., \( P_{v_1, \ldots, v_r}(G) \) consists of all words that can only end with one of the \( v_i \)'s. For a word \( w = v_1 \cdots v_r \in P(G) \) we write \(|w| = r \) for the length of \( w \) and set \( v(w) = \{|1 \leq j \leq r : v_j = v\}| \) for a vertex \( v \in V(G) \). The initial alphabet of \( w \) is the multiset denoted by \( IA_m(w) \) and defined by \( v \in IA_m(w) \) (counted with multiplicities) if and only if \( w = uv \) for some \( u \in P(G) \). We denote the underlying set by \( IA(w) \).

**Example.** Let us take \( G \) to be the path graph \( P_4 \):

\[ 1 \quad 2 \quad 3 \quad 4 \]

Take \( w = 342111 \in P(G) \), then
\[ |w| = 6, 1(w) = 3, 2(w) = 3(w) = 4(w) = 1, IA_m(w) = \{1, 1, 1, 4\} \text{ and } IA(w) = \{1, 4\}. \]

**2.2.** Given \( w \in P_u(G) \), it has been shown in [1] Proposition 4.3] that there exists unique words \( w_1, \ldots, w_{u(w)} \in P(G) \) such that
\[ w = w_1 \cdots w_{u(w)}, \quad IA_m(w_i) = \{u\} \text{ for all } 1 \leq i \leq u(w). \quad (2.1) \]

If \( u(w) > 1 \), we refer to the decomposition above as the initial alphabet-decomposition or simply ia-decomposition of \( w \). We shall define now the so-called neighborhood decomposition. We write
\[ N_G(u, w) = \{v \in N_G(u) : v(w) > 0\}, \quad d_G(u, w) = |N_G(u, w)|. \]

**Proposition.** Let \( w \in P_u(G) \) with \( u(w) = 1 \) and write \( N_G(u, w) = \{u_1 < \cdots < u_d\} \) where \( d = d_G(u, w) \). Then there exist unique \( w_1, \ldots, w_d \in P^\emptyset(G) \) such that:

(i) \( w = w_1 \cdots w_d u \)
(ii) \( w_i \in P(G) \), then \( IA(w_i) = \{u_i\} \) for all \( 1 \leq i \leq d \)
(iii) \( u_i \notin w_j \) for all \( i < j \).

**Proof.** We proceed by induction on \( d \) where the \( d = 1 \) case is obviously true. So we can assume that \( d > 1 \). We choose \( u_1, u_2 \) such that \( w = u_1 u_2 \) and \( |u_2| \) is maximal with the property that \( u_1 \notin u_2 \). This forces \( IA(u_1) = \{u_1\} \). Since \( d_G(u, u_2) < d_G(u, w) \) we can use induction to get the required decomposition for \( u_2 \). This gives the decomposition for \( w \) with the properties \((i) - (iii)\) once we set \( w_1 = u_1 \). The rest of the proof is concerned with the uniqueness. Assume that \( w = w'_1 \cdots w'_d u \) is another decomposition satisfying the conditions \((i) - (iii)\) of the lemma. Write \( w = w'_1 u' \) then we have \( u_1 \notin u' \). However, the choice of \( w_1 \) implies \(|w'_1| \geq |w'_1| \) and \( u' \) is a subword of \( u_2 \). This forces \(|w'_1| = |u'_1| \), since \( IA(w'_1) = \{u_1\} \). Hence \( u' = u_2 \) and \( w_1 = w'_1 \). Now a simple induction argument completes the proof. \( \square \)

For \( w \in P_u(G) \) with \( u(w) = 1 \), we refer to the decomposition of Proposition [2.2] as the neighbourhood-decomposition or simply nbd-decomposition of \( w \).

**2.3.** A subset \( S \) of \( V(G) \) is said to be independent if there is no edge between the elements of \( S \) in the graph \( G \). We denote by \( I_G \) the set of independent subsets of \( G \) and note that we have \( \emptyset, \{v\} \in I_G \) for each \( v \in V(G) \). The multi-variate independence polynomial of \( G \) is defined as
\[
I(G, x) := \sum_{S \in I_G} (-1)^{|S|} \prod_{v \in S} x_v
\]
and we view it as an element in $\mathbb{C}[x_v : v \in V(G)]$, the polynomial algebra over $\mathbb{C}$ generated by the commuting variables $\{x_v : v \in V(G)\}$. The aim of this article is to approach certain identities for multi-variate independence polynomials using the inversion formula from heap theory. We need the following trivial identifications.

**Lemma.** Let $S \subseteq V(G)$ and $\{K_1, \ldots, K_s\}$ be the set of non-empty independent subsets of the graph $G[S]$.

1. We have a bijection
   $$\mathcal{P}_c^c(K_1(G) \sqcup \cdots \sqcup \mathcal{P}_c^c(K_s(G) \rightarrow \mathcal{P}_S(G)) \quad \text{(2.2)}$$

2. For any independent subset $K \neq \emptyset$ of $S$ we have a bijection
   $$\varphi_K : \mathcal{P}_c^c(G) \rightarrow \mathcal{P}_B^0(\mathcal{N}_G[K]), \ w \mapsto \prod_{y \in K} y$$

**Proof.** We first show that the left hand side of (2.2) is a disjoint union. Let $w \in \mathcal{P}_c^c(K_1(G) \sqcup \mathcal{P}_c^c(K_2(G)$ and $u \in K_1 \setminus K_2$. Then we have $w = w' u$ and thus $u \in IA(w) = K_2$ which is a contradiction. So the left hand side is disjoint. The identity map
   $$\text{Id}_{K_i} : \mathcal{P}_c^c(K_i(G) \rightarrow \mathcal{P}_S(G)$$

for all $i \in \{1, \ldots, s\}$ induces the desired map (2.2) which is clearly bijective. In order to show the second part we first note that the map is well defined. If $z \in IA(\varphi_K(w))$ but $z \notin \mathcal{N}_G[K]$, then we would also have $z \in IA(w) = K$. Hence $z \notin \mathcal{N}_G[K]$. The map is bijective because the inverse map is simply given by multiplication with $\prod_{y \in K} y$. $\square$

**2.4.** The inversion lemma from heap theory [13, Proposition 5.3] states that

$$\frac{I(G - S, x)}{I(G, x)} = \sum_{w = v_1 \cdots v_r \in \mathcal{P}^0_S(G)} x_{v_1} \cdots x_{v_r}, \ S \subseteq V(G)$$

Using the inversion lemma one can derive certain well-known and possibly new identities of independence polynomials and extend them to the multi-variate version. For example, Lemma 2.3 simply implies that (keeping the same notation)

$$I(G - S, x) - I(G, x) = \sum_{i=1}^{s} (\prod_{v \in K_i} x_v) I(G - \mathcal{N}_G[K_i], x) \quad \text{(2.3)}$$

which is known as the fundamental identity if $S$ is singleton. The importance of the identity can be seen for example in [6] where the authors proved that independence polynomials of claw free graphs are real-rooted by using (2.3) when $S$ is a clique. The single variable version of the above identity is the main result of [10].

**3. Weight preserving bijection and Godsil’s identity**

**3.1.** Here we recall the construction of a rooted tree associated with $(G, u)$, where $u \in V(G)$, which is important in Godsil type identity (originally it is stated for the matching polynomial; see [8] and also [3]) which relates the independence polynomial of $G$ to that of the tree. The constructed tree is called a stable-path tree of $G$, for more details we refer the reader to [3] and for an example see Figure 1. Let $V(G) = \{1, \ldots, n\}$ be an enumeration of the vertices of $G$
and let \( N_G(u) = \{ u_1 < \cdots < u_d \} \) where \( u \in V(G) \) and \( d := d_G(u) \). For each vertex \( u \in V(G) \) we will recursively associate a rooted tree \((T_G, u')\) and a surjective graph homomorphism

\[
\ell_G : V(T_G) \to V(G), \ u' \mapsto u
\]
as follows. If \( d = 0 \) then \( G \) is a single vertex and we set \( T_G = \{ u' \} \) as the tree with one vertex \( u' \). If \( d \geq 1 \), we let \( G_i \) be the connected component of \( G[V(G) \setminus \{ u, u_1, u_2, \ldots , u_{i-1} \}] \) containing \( u_i \) and we take the induced total ordering on \( V(G_i) \) that comes from \( V(G) \). Now we have by induction the family of rooted trees \((T_{G_i}, u'_i)\) and the graph homomorphisms

\[
\ell_{G_i} : V(T_{G_i}) \to V(G_i), \ u'_i \mapsto u_i.
\]
Finally we take the disjoint union of rooted trees \((T_{G_i}, u'_i)\) and a new vertex \( u' \), and join the vertex \( u' \) with the vertices \( u'_i, 1 \leq i \leq d \). Clearly the graph \((T_G, u')\) obtained in this way is a rooted tree. Define the map \( \ell_G : V(T_G) \to V(G) \) by \( \ell_G(u') = u \) and \( \ell_G(v) = \ell_{G_i}(v) \) if \( v \in V(T_{G_i}) \). This is clearly a surjective graph homomorphism and the map \( \ell_G \) induces a partial ordering on \( V(T_G) \) as follows: for \( v_1, v_2 \in V(T_G) \), we have

\[
v_1 \geq v_2 \iff \ell_G(v_1) \geq \ell_G(v_2).
\]
We extend this partial order to a total ordering on \( V(T_G) \). The extension of \( \ell_G \) to \( C(T_G) \) is again denoted as \( \ell_G \).

![Figure 1](image_url)

(a) A graph \( G \) with labeled vertices.  
(b) The graph \( T_{G,1} \).

**Figure 1.** A graph with its stable-path tree.

### 3.2.
We freely use the notations that were developed in the earlier sections. We now state and prove the following result.

**Theorem 1.** Let \( G \) be a finite, simple and connected graph. Then there exists a bijection \( \varphi_G : \mathcal{P}_u^\emptyset(G) \to \mathcal{P}_u'(T_G) \) such that \( |\varphi_G(w)| = |w| \) and

\[
\begin{array}{ccc}
\mathcal{P}_u^\emptyset(G) & \xrightarrow{\varphi_G} & \mathcal{P}_u'(T_G) \\
\pi_G & & \pi_{T_G} \\
\mathcal{C}_G^\emptyset & \xleftarrow{\ell_G} & \mathcal{C}_{T_G}^\emptyset
\end{array}
\]

is a commutative diagram.
Proof. We recursively construct the map $\varphi_G$ and its inverse $\psi_G$. If $|V(G)| = 1$, then we set $\varphi_G(u) = u'$ and $\psi_G(u') = u$. So assume that $|V(G)| > 1$ and let $\varphi_H$ be the required map for all finite, connected graphs with $|V(H)| < |V(G)|$. We first consider the case $w \in P_u(G)$ with $u(w) = 1$ and recall that we have the $\text{nbd}$-decomposition $w = w_1 \cdots w_{\text{nd}}$ by Proposition 2.2, where we abbreviate $d = d(u, w)$ in the rest of the proof. From the conditions (ii) and (iii) of Proposition 2.2, it is clear that $w_i \in P_{u_i}(G_i)$ for all $1 \leq i \leq d$. Now since $|V(G_i)| < |V(G)|$, we obtain by induction a family of bijective maps $\varphi_{G_i} : P_{u_i}(G_i) \to P_{u'_i}(T_{G_i})$ satisfying the required properties for all $1 \leq i \leq d$. We define

$$\varphi_G(w) = \varphi_{G_1}(w_1)\varphi_{G_2}(w_2) \cdots \varphi_{G_d}(w_d)u' \quad (3.1)$$

Since the decomposition $w = w_1 \cdots w_{\text{nd}}$ is unique, the above map is well-defined. Clearly the map $\varphi_G$ preserves the $\text{nbd}$-decomposition, i.e., the decomposition in (3.1) is the $\text{nbd}$-decomposition of $\varphi_G(w)$.

Now we extend this map using the $\text{ia}$-decomposition of $w \in P_u(G)$ with $u(w) > 1$. We have $w = w_1 \cdots w_{u(w)}$ satisfying $w_i \in P_u(G)$ and $u(w_i) = 1$ for all $1 \leq i \leq u(w)$. We extend $\varphi_G$ as follows:

$$\varphi_G(w) = \varphi_G(w_1)\varphi_G(w_2) \cdots \varphi_G(w_{u(w)})$$

Again $\varphi_G$ is well-defined by the uniqueness of the decomposition and $\varphi_G$ preserves the $\text{ia}$-decomposition. The fact that $|\varphi_G(w)| = |w|$ holds and that the above diagram commutes follows from the fact that $\ell_G, \pi_G, \pi_{T_G}$ are all homomorphisms and the maps $\varphi_{G_i}$ also satisfy these properties. So it remains to construct the inverse map.

In a similar way, we now define the inverse map $\psi_G$ using the maps $\psi_{G_i} = \varphi_{G_i}^{-1}$. Let $w' \in P_{u'}(T_G)$ be such that $u'(w') = 1$. Again we have the $\text{nbd}$-decomposition $w' = w'_1 \cdots w'_{d(u', w')}$.

We define

$$\psi_G(w') = \psi_{G_1}(w'_1)\psi_{G_2}(w'_2) \cdots \psi_{G_d(u', w')} (w'_{d(u', w')})u$$

As before this is a well-defined map and preserves the $\text{nbd}$-decomposition. Using this, it is easy to see that $\varphi_G \circ \psi_G(w) = w$ and $\psi_G \circ \varphi_G(w') = w'$ for $w \in P_u(G), w' \in P_{u'}(T_G)$ with $u(w) = u'(w') = 1$.

If $w' \in P_{u'}(T_G)$ with $u'(w') > 1$, we extend the map using the $\text{ia}$-decomposition of $w' = w'_1 \cdots w'_{u'(w')}$, namely we set

$$\psi_G(w') = \psi_G(w'_1) \cdots \psi_G(w'_{u'(w')}) \quad (3.2)$$

As before this is a well-defined map and preserves the $\text{ia}$-decomposition. Again we have $\varphi_G \circ \psi_G = \text{Id}_{P_u(T_G)}$ and $\psi_G \circ \varphi_G = \text{Id}_{P_u(G)}$, proving that $\varphi_G$ is a bijection. \hfill \square

3.3. The observation in Section 2.4 together with Theorem 1 immediately imply the multivariate Godsil identity

$$\frac{I(G - u, x)}{I(G, x)} = \frac{\ell_G(I(T_G - u', x))}{\ell_G(I(T_G, x))}$$

We refer also to [11] for different generalizations of this identity.
4. Bipartite graphs and positive sum identities

4.1. Motivated by the Christoffel-Darboux type identities for the independence polynomial obtained in [2] we would like to achieve a similar type identity or a refined version of it without the alternating sign and in a multi-variate version. Our approach will be the same by observing underlying indexing sets.

Let $u, v$ be two distinct vertices of $G$. Given a pair $(w u, w' v) \in \mathcal{P}_u(G) \times \mathcal{P}_v(G)$ and a shortest path $p = v_1 v_2 \cdots v_k$ connecting $u = v_1$ with $v = v_k$ we define a bipartite graph $H = H_1 \sqcup H_2$ by

$$H_1 = IA(w \cdot v_2 \cdot v_4 \cdots), \quad H_2 = IA(w' \cdot v_1 \cdot v_3 \cdots)$$

Note that $v \in H_1$ and $u \in H_2$ if $k$ is even and $u, v \in H_2$ otherwise. We consider the map

$$\mathcal{P}_u(G) \times \mathcal{P}_v(G) \to \bigcup_{H} \mathcal{P}^\emptyset_{Z_1(H)}(G) \times \mathcal{P}^\emptyset_{Z_2(H)}(G) \quad (4.1)$$

where the disjoint union runs over all connected bipartite subgraphs $H$ of $G$ containing the path $p$ and satisfying

$$H_1 \setminus \{v_2, v_4, \ldots\} \subseteq N_G[u], \quad H_2 \setminus \{v_1, v_3, \ldots\} \subseteq N_G[v],$$
$$Z_1(H) = N_G[H_1 \setminus \{v_2, v_4, \ldots\}] \cup (N_G[H_1] \cap N_G[u]),$$
$$Z_2(H) = N_G[H_2 \setminus \{v_1, v_3, \ldots\}] \cup (N_G[H_2] \cap N_G[v]). \quad (4.2)$$

**Proposition.** The map defined in (4.1) is a bijection.

**Proof.** We first show that the map is well-defined. Set $w' = \frac{w \cdot v_2 \cdot v_4 \cdots}{\prod_{z \in H_1} z}$, then we have

$$w \cdot v_2 \cdot v_4 \cdots = w' \prod_{z \in H_1} z \quad \text{and} \quad w = w' \prod_{z \in H_1 \setminus \{v_2, v_4, \ldots\}} z.$$

Assume that a letter $y$ is in the initial alphabet of the word $w'$ which we assume to be non-empty. Suppose $y \in N_G[H_1 \setminus \{v_2, v_4, \ldots\}]$ then we have $y \in Z_1(H)$. Otherwise $y \notin N_G[H_1 \setminus \{v_2, v_4, \ldots\}]$ which implies $y \in IA(w)$, hence $y \in N_G[u]$. Suppose $y \in N_G(H_1)$, then we have $y \in Z_1(H)$. Otherwise $y \notin N_G(H_1)$, then $y \in IA(w \cdot v_2 \cdot v_4 \cdots) = H_1$, again in this case we have $y \in Z_2(H)$. Similar calculation shows that the initial alphabet of the second component lies in $Z_2(H)$. This shows that the map is well-defined. For the bijectivity we construct the inverse map.

Given a bipartite connected graph $H$ containing $p$ (say $v_1, v_3, \cdots \in H_2$) and satisfying (4.2), we define

$$\mathcal{P}^\emptyset_{Z_1(H)}(G) \times \mathcal{P}^\emptyset_{Z_2(H)}(G) \to \mathcal{P}_u(G) \times \mathcal{P}_v(G)$$

$$(\tilde{w}, \tilde{w}') \to \left( \tilde{w} \prod_{y \in H_1 \setminus \{v_2, v_4, \ldots\}} y u, \tilde{w}' \prod_{y \in H_2 \setminus \{v_1, v_3, \ldots\}} y v \right) \quad (4.3)$$
From (4.2) and the definition of $Z_i(H)$, $i = 1, 2$, we know that the above map is well defined. This map induces the inverse of (4.1) since
\[\text{IA}(\tilde{w} \prod_{y \in H_1} y) = H_1, \quad \text{IA}(\tilde{w}' \prod_{y \in H_2} y) = H_2.\]

4.2. As an immediate consequence of Proposition 4.1 we obtain the following identity
\[
\left( \frac{I(G - u, x)}{I(G, x)} - 1 \right) \left( \frac{I(G - v, y)}{I(G, y)} - 1 \right) = \sum_H \prod_{\begin{array}{c}w \in H_1 \setminus \{v_2, v_4, \ldots \} \vspace{-0.5em} \\
w' \in H_2 \setminus \{v_1, v_3 \ldots \} \end{array}} x_w y_{w'} x_u y_v \left( \frac{I(G - Z_1(H), x)}{I(G, x)} \right) \left( \frac{I(G - Z_2(H), y)}{I(G, y)} \right),
\]
where the sum runs over all connected bipartite subgraphs $H$ of $G$ containing the path $p$ and satisfying (4.2) (by convention we denote always by $H_2$ the part which contains $v_1, v_3, \ldots$).

Using
\[I(G - u, x) - I(G, x) = -x_u \frac{\partial I(G, x)}{\partial x_u} \]
we can rewrite (4.4) as follows
\[
\frac{\partial I(G, x)}{\partial x_u} \frac{\partial I(G, y)}{\partial y_v} = \sum_H \prod_{\begin{array}{c}w \in H_1 \setminus \{v_2, v_4, \ldots \} \vspace{-0.5em} \\
w' \in H_2 \setminus \{v_1, v_3 \ldots \} \end{array}} x_w y_{w'} I(G - Z_1(H), x) I(G - Z_2(H), y),
\]
where the sum runs over the same index set as before.

Remark. If there is an edge between $u$ and $v$, then the left hand side of the above identity becomes (after evaluating $x = y$)
\[
\frac{I(G - u, x)}{I(G, x)} I(G - v, x) - \frac{I(G - \{u, v\}, x)}{I(G, x)}.
\]
This part also appeared for example in Gutman’s identity for trees (see [9]) and for general graphs in [2].

4.3. We will now see some examples that explain our results.

Example. Let us consider the path graph $P_4$ (see Figure 2) and take $u = 2$ and $v = 3$. The connected bipartite subgraphs of $P_4$ containing $u, v$ are given in Figure 3.

![Figure 2. Path graph $P_4$](image)

![Figure 3. Connected bipartite subgraphs of $P_4$ containing 2 and 3](image)
In this case we can rewrite the equation \[\text{Equation 4.5}\] as follows:

\[
(I(G - u, x) - I(G, x)) (I(G - v, y) - I(G, y)) = \sum_{H \in \mathcal{H}_1} \prod_{w \in H} x_w y_w I(G - Z_1(H), x) I(G - Z_2(H), y).
\]

(4.5)

It is easy to see that

\[
I(G, x) = 1 - x_1 - x_2 - x_3 - x_4 + x_1x_3 + x_1x_4 + x_2x_4
\]

\[
I(G - u, x) = 1 - x_1 - x_3 - x_4 + x_1x_3 + x_1x_4, \text{ and}
\]

\[
I(G - v, x) = 1 - y_1 - y_2 - y_4 + y_1y_4 + y_2y_4.
\]

This gives

\[
(I(G - u, x) - I(G, x))(I(G - v, y) - I(G, y)) = x_2y_3(1 - x_4)(1 - y_1).
\]

On the other hand, we have the parts arising from the bipartite parts which we list now:

(a) In this case we have

\[
H_1^1 = \{3\}, \quad H_2^1 = \{2\}, \quad Z_1(H^1) = \{2, 3\} = Z_2(H^1)
\]

and

\[
I(G - \{2, 3\}, x) = 1 - x_1 - x_4 + x_1x_4
\]

(b) In this case we have

\[
H_1^2 = \{1, 3\}, \quad H_2^2 = \{2\}, \quad Z_1(H^2) = \{1, 2, 3\}, \quad Z_2(H^2) = \{2, 3\}
\]

and

\[
I(G - \{1, 2, 3\}, x) = 1 - x_4, \quad I(G - \{2, 3\}, y) = 1 - y_1 - y_4 + y_1y_4;
\]

(c) In this case we have

\[
H_1^3 = \{3\}, \quad H_2^3 = \{2, 4\}, \quad Z_1(H^3) = \{2, 3\}, \quad Z_2(H^3) = \{2, 3, 4\}
\]

and

\[
I(G - \{2, 3\}, x) = 1 - x_1 - x_4 + x_1x_4, \quad I(G - \{2, 3, 4\}, y) = 1 - y_1
\]

(d) In this case we have

\[
H_1^4 = \{1, 3\}, \quad H_2^4 = \{2, 4\}, \quad Z_1(H^4) = \{1, 2, 3\}, \quad Z_2(H^4) = \{2, 3, 4\}
\]

and

\[
I(G - \{1, 2, 3\}, x) = 1 - x_1, \quad I(G - \{2, 3, 4\}, y) = 1 - y_1.
\]

If we simplify the RHS of Equation \[\text{Equation 4.5}\] becomes \(x_2y_3(1 - x_4)(1 - y_1)\) which is same as the LHS of Equation \[\text{Equation 4.5}\].

**Example.** Let us consider the path graph \(P_4\) (see Figure 2) and take \(u = 1\) and \(v = 4\). The only connected bipartite subgraphs of \(P_4\) containing \(u, v\) is \(P_4\) itself. In this case, we have

\[
I(G - u, x) = 1 - x_2 - x_3 - x_4 + x_2x_4 \quad \text{and} \quad I(G - v, y) = 1 - y_1 - y_2 - y_3 + y_1y_3.
\]

The LHS of Equation \[\text{Equation 4.5}\] is equal to

\[
x_1y_4(1 - x_3 - x_4)(1 - y_1 - y_2).
\]
On the other hand, we have $H_1 = \{2, 4\}$, $H_2 = \{1, 3\}$, $Z_1(H) = \{1, 2\}$ and $Z_2(H) = \{3, 4\}$. This implies the RHS of Equation 4.5 is equal to

$$x_1 y_4 (1 - x_3 - x_4)(1 - y_1 - y_2),$$

which is same as the LHS of Equation 4.5.

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