Double-negation Shift as a constructive principle

Danko Ilik
Laboratory for Complex Systems and Networks
Macedonian Academy of Sciences and Arts
Email: danko.ilik@gmail.com

Abstract—We consider the Double-negation Shift (DNS) as a constructive principle in its own right and its effect on modified realizability (MR) and Dialectica (D) interpretations.

We notice that DNS proves its own MR-interpretation, meaning that a priori one does not have to consider the more complex D-interpretation with Bar Recursion for interpreting Analysis. From the “with truth” variant of MR, we obtain the closure under Weak Church’s Rule. We notice, in contrast, that DNS proves the double negation of the Limited Principle of Omniscience (LPO), hence of the solvability of the Halting Problem, and recall the related fact that DNS refutes a formal version of Church’s Thesis (CT). This shows that intuitionistic Arithmetic plus DNS presents a distinct variant of Constructive Mathematics.

We revisit the standard approach of Kreisel and Spector for showing consistency of Analysis, and show that one can omit the preliminary double-negation-translation phase.

Finally, we formalize the proofs of not-not-LPO and not-CT using a previously introduced constructive logic based on delimited control operators from the theory of programming languages.

I. INTRODUCTION

The logical principle known as Double-negation Shift,

$$\forall n \in \mathbb{N} \neg \neg A(n) \rightarrow \neg \forall n \in \mathbb{N} A(n),$$

(DNS)

that is not provable by intuitionistic logic, has first been isolated by Sigeikatu Kuroda [1]. Its importance for the foundational program has been realized by Kreisel [2], who showed that if one can prove the Double-negation Shift, then one can prove the double-negation-translation of the countable Axiom of Choice (ACω0). Coupled with Gödel’s double-negation translation, that reduces Classical Arithmetic to Intuitionistic Arithmetic [3], and with Gödel’s functional interpretation [4, 5, 6], that interprets Intuitionistic Arithmetic by functionals defined by primitive recursion at higher types (System T), a functional interpretation of DNS allows to extend this reduction (and relative consistency proof) to Classical Analysis. The functional interpretation of DNS has been given by Kreisel and Spector [2] by adding an additional recursion schema to System T, called Bar Recursion.

This method, functional interpretation with bar recursion, has been the main one when it comes to extracting computational content from proofs that use both classical logic and choice, and has had many applications, notably in the Proof Mining approach of Kohlenbach [8].

Recently, the author has shown, based on initial observations of Hugo Herbelin, that one can prove DNS in a system for first-order logic, based on the delimited control operators shift and reset of Danvy and Filinski [9, 10] from Programming Languages Theory. This system of pure logic, in the absence of all axioms, has been shown to be constructive.

The purpose of this paper is to investigate the metamathematical status of DNS in presence of higher-type Intuitionistic Arithmetic (HAω).

In Section II we revise the basic relationships between Double-negation Shift, the Countable Axiom of Choice, and double-negation translations, in the context of HAω.

In Section III we show that DNS proves its own modified realizability interpretation, meaning that modified realizability is a viable (and simpler) alternative to functional interpretation and bar recursion, for extracting programs from proofs in Classical Analysis. This has practical implications for proof assistants based on intuitionistic logic that extract programs by modified realizability (like Coq).

Furthermore, we show from the “with truth” variant of modified realizability interpretation, that HAω+ACω0+DNS has the constructive closure properties of HAωω, among which a version of Church’s Rule. However, DNS is known to refute (formal versions of) Church’s Thesis. We briefly review this argument, that comes back to Gödel, and we argue that HAωω+ACω0+DNS is a variety of Constructive Mathematics that permits to see in a new way the diagonalization arguments that reduce a problem to the non-solvability of the Halting Problem.

In Section IV we also revisit the approach of functional interpretation and bar recursion. We observe, for the first time (to the best of our knowledge), that it is not necessary to perform a double-negation translation phase when interpreting Classical Analysis proofs, that is, that bar recursion already has some kind of double-negation translation built-in.

In Section V we apply the system from [11] to formalize the arguments relating to Church’s Thesis and the Halting Problem, by constructive delimited control operators directly.

In Section VI we mention related works, that have not already been mentioned in the preceding sections, and we give some directions for future work.

We hope that this work will contribute to the program of enriching current proof assistant with constructive principles that are beyond intuitionistic logic, and with programming principles (control operators) that are beyond side-effect-free programming.

II. PRELIMINARIES

In this paper, we will work in the context of higher-type intuitionistic, or Heyting, Arithmetic abbreviated HAωω. Intuitively, one can think of HAωω as a logical system based on
two orthogonal typed lambda calculi. At one level, we have the
typed lambda calculus that is used to denote natural deduction
proofs; this is the logical level, and we call the corresponding
\( \lambda \)-terms: proof terms. At the other level, there is the typed
lambda calculus that constructs individuals of the domain of
quantification (the higher-types level), and the corresponding
\( \lambda \)-terms are called just terms\(^1\). Hence, HA\(^\omega \) stops short of a
dependent type theory, which collapses the two levels into one.
HA\(^\omega \) is also closely connected to Gödel’s System T \(^2\), a
quantifier-free logical theory based on equations between nu-
merical terms constructed by higher-type primitive recursion.

Technically, HA\(^\omega \) has as basis a multi-sorted first-order
logic, with sorts built from \( 0 \) (standing for \( \text{AC} \))
tied to specific types; we usually suppress writing the type of
members of the pair type, and using the projections \( t_1 \) and \( t_2 \),
we destruct them. The constructor of the unit type is denoted \( \top \).

There is a decidable equality relation \( =_0 \) between terms of
type \( 0 \) (numbers). The Peano axioms are stated in terms of
this equality, including the induction axiom schema for \( 0 \) individuals. For the purposes of this paper, we do not need to
be more precise than this, for more information the reader
may take a look at a standard reference book like \([12]\) or \([8]\).

We will consider the following principles, both unprovable
in HA\(^\omega \). The Double-negation Shift at type \( \rho \),
\[
\forall x \neg \neg A(x) \rightarrow \neg \neg \forall x A(x),
\]
(DNS\(^\rho \))
and the Axiom of Choice at types \( \rho \) and \( \sigma \),
\[
\forall x^\rho y^\sigma A(x, y) \rightarrow \exists f^\rho \rightarrow \sigma \forall x A(x, f(x)).
\]
(AC\(^{\rho, \sigma} \))

Particularly important will be the instances DNS\(^0 \) and AC\(^0 \)
(short for AC\(^{0,0} \)). We will write DNS\(^\omega \) and AC\(^\omega \) when we
intend to have DNS\(^0 \)and AC\(^0 \) for all \( \rho \) and \( \sigma \).

Although both DNS\(^0 \) and AC\(^0 \) are unprovable in HA\(^\omega \),
the later is admissible for arithmetical formulas, that is, if
HA\(^\omega \)+AC\(^0 \) proves an arithmetical \( A \), then HA\(^\omega \)-alone proves \( A \). A formula is arithmetical if it contains only type \( 0 \) quantifiers.

HA\(^\omega \) extended with the Law of Excluded Middle is called
PA\(^\omega \), higher-type classical, or Peano, Arithmetic. The relation
between HA\(^\omega \) and PA\(^\omega \) is given by double-negation translation.

---

**Definition 1** (Kuroda’s double-negation translation \(^1\), or the
call-by-value continuation-passing style translation \(^1\). The
double-negation translation of \( A \) is the formula \( A^\perp := \neg \neg A \perp \),
where \( A \perp \) is defined by recursion on the complexity of \( A \) by the
clauses:
\[
P_\perp := P \quad \text{P-prime}
\]
\[
(A \land B)_\perp := A_\perp \land B_\perp \quad \text{HA} \quad \text{HA}
\]
\[
(A \lor B)_\perp := A_\perp \lor B_\perp \quad \text{HA} \quad \text{HA}
\]
\[
(A \rightarrow B)_\perp := A_\perp \rightarrow B_\perp \quad \text{HA} \quad \text{HA}
\]
\[
(\forall x A(x))_\perp := \forall x (A^\perp(x)) \quad \text{HA} \quad \text{HA}
\]
\[
(\exists x A(x))_\perp := \exists x (A^\perp(x)) \quad \text{HA} \quad \text{HA}
\]

**Fact 1** (\([12]\)). In PA\(^\omega \), \( \vdash A \iff A^\perp \). Also, \( \Gamma \vdash A \)
in PA\(^\omega \) if and only if \( \Gamma \perp \vdash A^\perp \) in HA\(^\omega \), where \( \Gamma \perp \) stands for the context
obtained by applying the \( \perp \)-translation to all formulas in \( \Gamma \).

**Fact 2** (Lemma 1.10.9 of \([12]\)). In presence of DNS\(^\omega \), HA\(^\omega \) can prove the equivalence
\[
\neg \neg A \leftrightarrow A^\perp
\]
between the double-negation of \( A \) and the double-negation
translation of \( A \). DNS\(^\omega \) suffices to prove the same statement
for arithmetical formulas \( A \).

**Proof:** The proof term
\[
\lambda a.\lambda k. d(\lambda x.\lambda k’.a x (\lambda a’.\text{dest } a’ \text{ as } (x,e)) \in k(x,e)))
\]
(\( \lambda b. k(\epsilon b) \))
derives
\[
\forall x^\rho \exists y^\sigma A^\perp(x, y) \rightarrow \neg \neg \exists f \forall x \neg \neg A^\perp(x, f(x)),
\]
when the proof terms \( d \) and \( c \) derive the following instances
of DNS\(^\rho \) and AC\(^{\rho, \sigma} \):
\[
d : \forall x \neg \neg \exists y \neg \neg A^\perp(x, y) \rightarrow \neg \neg \forall x \exists y \neg \neg A^\perp(x, y)
\]
\[
c : \forall x \exists y \neg \neg A^\perp(x, y) \rightarrow \exists f \forall x \neg \neg A^\perp(x, f(x))
\]

We call the theory PA\(^\omega \)+AC\(^0 \), Classical Analysis.

---

III. Modified realizability interpretation of DNS

Kreisel’s modified realizability interpretation \([15]\) can be
seen as a precise statement of the Brouwer-Heyting-
Kolmogorov interpretation of intuitionistic logic, although its
original aim was to show the non-derivability of Markov’s
Principle in Heyting Arithmetic. It is this realizability inter-
pretation that is behind the program extraction facilities of proof
assistants based on intuitionistic logic like Coq.

Let us recall the definition and basic properties of modified
realizability relevant to this paper. The proofs of the facts may
be found in standard references as \([12]\), \([16]\), \([8]\).

**Definition 2.** For a term \( t \) and a formula \( A \), the relation \( t \text{ mr } A \)
(\( t \) interprets \( A \) by modified realizability) is defined by recursion
for all \( \rho \) and \( \sigma \).

---

\(^1\) In standard presentations, the terms level is given by a combinator calculus,
but a \( \lambda \)-calculus treatment is also possible, see for example 1.8.4. of \([12]\).

\(^2\) This unusual notation for writing projections is quite efficient when used
in sections \([III]\) and \([IV]\).
on the complexity of $A$, by the following clauses:

$$
t \text{ mrt } P := P \quad \text{if } P \text{-prime, } t = tt$$

$$
t \text{ mrt } A \land B := (t_1 \text{ mrt } A) \land (t_2 \text{ mrt } B)
$$

$$
t \text{ mrt } A \lor B := (t_2 = 0 \rightarrow t_1 \text{ mrt } A) \land (t_2 \neq 0 \rightarrow t_1 \text{ mrt } B)
$$

$$
t \text{ mrt } A \rightarrow B := \forall x (x \text{ mrt } A \rightarrow tx \text{ mrt } B)
$$

$$
t \text{ mrt } \exists y A(y) := t_1 \text{ mrt } A(t_2)
$$

$$
t \text{ mrt } \forall y A(y) := \forall y (ty \text{ mrt } A(y))
$$

**Remark 1.** The type of the term $t$ such that $t \text{ mrt } A$ is determined solely by the logical form of $A$.

**Fact 4** (Soundness of modified realizability). From a derivation of $A_1, A_2, \ldots, A_n \vdash B$ in HA$^\omega$, we can construct a term $t$ in the variables $x_1, x_2, \ldots, x_n$ such that $x_1 \text{ mrt } A_1, x_2 \text{ mrt } A_2, \ldots, x_n \text{ mrt } A_n \vdash t \text{ mrt } B$ in HA$^\omega$.

**Fact 5.** For any $A$ an instance of AC$^\omega$, there is a term $t$ such that $\vdash t \text{ mrt } A$ in HA$^\omega$.

**Theorem 1.** HA$^\omega$+AC$^\omega$+DNS$^\omega$ proves the modified realizability interpretation of DNS$^\omega$.

**Proof:** A term $t$ mrt-interprets DNS$^\omega$ if and only if, by definition of mrt, the following three chains of equivalences hold inside HA$^\omega$:

$$
y \text{ mrt } \forall n^\omega \rightarrow A(n) \quad \forall y ([y \text{ mrt } \forall n^\omega \rightarrow A(n)] \rightarrow [ty \text{ mrt } \forall n^\omega A(n)])
$$

and

$$
y \text{ mrt } \forall n^\omega \rightarrow A(n) \quad \forall n^\omega [y \text{ mrt } \forall A(n) \rightarrow [ynz \text{ mrt } \bot]]
$$

$$
y \text{ mrt } \forall n^\omega \rightarrow A(n) \quad \forall n^\omega [y \text{ mrt } \forall A(n) \rightarrow [yx \text{ mrt } \bot]]
$$

Then, the implication (10) → (17) can be proved using DNS$^\omega$ and AC$^\omega$, where the type $\sigma$ is determined by the logical complexity of the formula $A(n)$.

**Corollary 1.** If the theory PA$^\omega$+AC$^\omega$ proves $A$, then the theory HA$^\omega$+AC$^\omega$+DNS$^\omega$ realizes $\neg A$ with a term $t$ extracted by modified realizability from the original proof. Hence, proofs of arithmetical formulas in Classical Analysis (PA$^\omega$+AC$^\omega$) can be interpreted inside the theory HA$^\omega$+AC$^\omega$+DNS$^\omega$ by modified realizability.

**Proof:** This follows by applying double-negation translation (Fact 1), then eliminating AC$^\omega$ from the context in favor of AC$^\omega$+DNS$^\omega$ (Fact 3), then applying Soundness for mrt (Fact 4), eliminating the mrt-translation of AC$^\omega$ by Fact 5 and eliminating mrt-translated DNS$^\omega$ in favor of non-translated DNS$^\omega$ and AC$^\omega$ (Theorem 1). In the end, one uses Fact 2 to replace $A \rightarrow B$ by the intuitionistically stronger $\neg A$. 

In the rest of this section, we will show that the theory HA$^\omega$+AC$^\omega$+DNS$^\omega$ is constructive, i.e. that HA$^\omega$+AC$^\omega$+DNS$^\omega$ satisfies certain closure rules characteristic of HA$^\omega$ alone.

We will need the following “with truth” variant of modified realizability from [16], [8].

**Definition 3.** The interpretation called modified realizability with truth (mrt) is obtained when the clause for implication from Definition 2 is replaced by

$$
t \text{ mrt } A \rightarrow B := \forall x (x \text{ mrt } A \rightarrow tx \text{ mrt } B) \land (A \rightarrow B).
$$

We need mrt-, because of the following fact that does not hold for simple mrt-realizability.

**Fact 6** ([8]). For any formula $A$, we have that, in HA$^\omega$, $\vdash (t \text{ mrt } A) \rightarrow A$.

**Corollary 2.** From a derivation in HA$^\omega$+AC$^\omega$+DNS$^\omega$ of $A_1, A_2, \ldots, A_n \vdash B$, one can construct a term $t$ in the variables $x_1, x_2, \ldots, x_n$ such that, in HA$^\omega$+AC$^\omega$+DNS$^\omega$, $x_1 \text{ mrt } A_1, x_2 \text{ mrt } A_2, \ldots, x_n \text{ mrt } A_n \vdash t \text{ mrt } B$.

**Proof:** Theorem 1 allows us (by Theorem 3.4.5 of [12] and Theorem 3.5 of [16]) to obtain a soundness theorem for mrt- and mrt-interpretation, for the extension HA$^\omega$+AC$^\omega$+DNS$^\omega$ of HA$^\omega$.

**Corollary 3.** The theory HA$^\omega$+AC$^\omega$+DNS$^\omega$ (and, in particular, also HA$^\omega$+AC$^\omega$+DNS$^\omega$) satisfies:

- the Existence Property (EP): if $A$ is a formula (even an open one) such that $\Gamma \vdash \exists x A(x)$ then there exists a term $t^\omega$, with free variables $\vec{x}$, such that $\vec{x} \text{ mrt } \Gamma \vdash A(t)$;
- the Existence Property (EP$^\omega$) for closed arithmetical formulas $\exists x^\omega A(x)$: if $\vdash \exists x^\omega A(x)$, then there exists $n \in \mathbb{N}$ such that $\vdash A(t)$, where $t$ denotes the representation of $n$ inside HA$^\omega$;
- the Disjunction Property (DP$^\omega$) for closed arithmetical formulas $A \lor B$: if $\vdash A \lor B$, then either $\vdash A$ or $\vdash B$;
- the Weak Church Rule (WCR$^\omega$) for closed arithmetical formulas $\forall x^\omega \exists y^\omega A(x, y)$: if $\vdash \forall x A(x, y)$, then there exists a total recursive function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $n \in \mathbb{N}$, we have that $\vdash A(t, f(n))$.

**Proof:** The proof follows the ones of Corollary 5.24 of [8] and 1.11.7 of [12].

EP$^\omega$ is obtained by Corollary 2 and Fact 6.

EP$^\omega$ follows by EP and the fact that for every closed term $t$ of type 0 we can find $n \in \mathbb{N}$ such that $t = 0$.

DP$^\omega$ is proved like EP$^\omega$, using the mrt-interpretation of $A \lor B$,

$$
\vdash (t = 0 \rightarrow s \text{ mrt } A) \land (t \neq 0 \rightarrow s \text{ mrt } B),
$$
and the fact that, for any $n \in \mathbb{N}$, either $\pi = 0$ or $\pi \neq 0$.

For WCR, note first that the theory HA$^\omega$+AC$^0$+DNS$^0$, like HA$^\omega$, is recursively axiomatizable, that is, there exists a recursive predicate Proof$(k, l)$ formalizing the fact that $k \in \mathbb{N}$ is a code for a derivation of the formula coded by $l \in \mathbb{N}$.

Let $g(n) = \min_m \text{Proof}(j_1 m, \overline{A(\overline{\pi}, j_2 m)})$, where $j_1$ and $j_2$ are the projections of some surjective pairing function. By its definition, $g$ is a partial recursive function.

Now, given $\vdash x^0 \exists y^0 A(x, y)$ and $n \in \mathbb{N}$, we obtain $\vdash \exists y^0 \forall \pi, y \exists m \in \mathbb{N}$ such that $\vdash A(\overline{\pi}, \overline{m})$. We proved that, for every $n$, there exists $m$ s.t. $\vdash A(\overline{\pi}, \overline{m})$. This shows that the function $g$ is total recursive. We may now take $f(n) := j_2(g(n))$ and by definition we have that, for any $n$, $\vdash A(\overline{\pi}, \overline{f(n)})$.

Remark 2. By the work of Joan Rand Moschovakis [17], we know that the theory HA$^\omega$+AC$^0$+DNS$^0$+MP+GC$_1$, where GC$_1$ is a generalization of Brouwer’s Continuity Principle, satisfies the following version of Church’s Rule, where the recursivity of the function $f$ is proven inside the theory.

\[ \forall x^0 \exists y^0 A(x, y) \Rightarrow \exists z^0 \forall x^0 \exists u^0 (T(z, x, u) \land A(x, U(u))). \] (CR$^0$)

That result, obtained by a variant of Kleene’s number realizability could probably also be obtained for the restriction HA$^\omega$+AC$^0$+DNS$^0$ without MP and GC$_1$.

A. HA$^\omega$+AC$^0$+DNS$^0$ as a distinct variety of Constructive Mathematics

In contrast to the constructive closure properties satisfied, the theory HA$^\omega$+AC$^0$+DNS$^0$ has some surprising metamathematical properties that we review in this subsection. Let us first consider the following fact due to Kreisel [13].

Fact 7. The following forms of DNS are intuitionistically equivalent:

\[ \forall x \neg \neg A(x) \rightarrow \neg \neg \forall x A(x) \] (18)
\[ \neg \forall x A(x) \rightarrow \neg \forall x \neg \neg A(x) \] (19)
\[ \neg \neg \forall x A(x) \lor \neg A(x) \] (20)
\[ \neg[\forall x \neg \neg A(x) \land \neg \forall x A(x)] \] (21)

The variant (20) of DNS$^0$ allows to immediately deduce the double-negation of Bishop’s Limited Principle of Omniscience (LPO) [19]: given $f : \mathbb{N} \rightarrow \{0, 1\}$,

\[ \exists n f(n) = 1 \lor \neg \exists n f(n) = 1 \], (LPO)

or, equivalently,

\[ \exists n f(n) = 1 \lor \forall n f(n) = 0 \].

Related to this fact (20), for $A(x) := \exists y T(x, x, y)$ from DNS$^0$ one obtains that it is not the case that the Halting Problem can not be solved.

\[ \neg \forall x (\exists y T(x, x, y) \lor \neg \exists y T(x, x, y)). \] (\neg\neg-HP)

Given the fact that the (single) negation of HP is intuitionistically provable from Church’s Thesis [21], this means that DNS$^0$ refutes Church’s Thesis, like the one formalized by

\[ \forall x^0 \exists y^0 A(x, y) \rightarrow \exists z^0 \forall x^0 \exists u^0 (T(z, x, u) \land A(x, U(u))). \] (CT$^0$)

Fact 8. DNS$^0$ implies $\neg$CT$^0$.

This had already been noticed by Gödel [4], reported by Kreisel [22], and also appears in later print (3.5.20 of [12]).

From Corollary 3 and the above proofs of $\neg$--LPO, $\neg$--HP, and $\neg$--CT$^0$, it follows that the system HA$^\omega$+AC$^0$+DNS$^0$ is a variety of Constructive Mathematics [19] similar to Brouwer’s Intuitionism (that proves $\neg$--CT$^0$), but distinct from Russian Constructivism (that postulates CT$^0$ and proves $\neg$LPO) and from Classical Mathematics (that proves $\neg$CT$^0$, but does not have the Existence Property). Note, however, that, when considered at higher types, DNS$^\omega$, contradicts with certain continuum principles [23].

IV. DIALECTICA INTERPRETATION IN PRESENCE OF DNS

The functional “Dialectica” interpretation was proposed by Gödel [4, 5, 6], before modified realizability was developed, as a way to make precise the somewhat vague notion of construction in the Brouwer-Heijting-Kolmogorov interpretation of intuitionistic implication. The Dialectica interpretation is a realizability technique for intuitionistic Arithmetic (implemented as a program extraction feature in the Minlog proof assistant), that also plays a fundamental role in establishing the consistency of classical Mathematics. Namely, by Spector’s extension of HA$^\omega$ by an equation schema called Bar Recursion (BR), and by the double-negation translation, Classical Analysis (PA$^\omega$+AC$^0$) can be interpreted in HA$^\omega$+BR.

As mentioned in the Introduction, the key in this interpretation is to Dialectica-interpret DNS$^0$ by BR.

In this section, we notice that the standard method for interpreting Classical Analysis can be simplified, by omitting the double-negation translation phase. That is, we show that the Dialectica interpretation in presence of DNS$^0$ already has the double-negation translation built-in.

Definition 4. The term $t$ interprets the formula $A$ by Dialectica, if, for all terms $s$, the relation $|A|^t_s$ holds. This relation is defined by recursion on the complexity of $A$ by the following clauses:

\[ |P|^t_{1^t_s} := P \] if $P$-prime, $t = tt$, $s = tt$
\[ |A \land B|^t_{1^t_s} := |A|^{t_1}_{1^t_s} \land |B|^{t_2}_{1^t_s} \]
\[ |A \lor B|^t_{1^t_s} := (t_2 = 0 \Rightarrow |A|^{t_1}_{t_2}) \lor (t_2 \neq 0 \Rightarrow |B|^{t_2}_{t_2}) \]
\[ |A \rightarrow B|^t_{1^t_s} := |A|^{t_1}_{t_2} \rightarrow |B|^{t_2}_{t_2} \]
\[ |3x A(x)|^t_{1^t_s} := |A(t_2)|^{t_1}_{t_2} \]
\[ |\forall x A(x)|^t_{1^t_s} := |A(s_2)|^{t_1}_{t_2} \]

Fact 9 (Soundness of Dialectica interpretation [12]). From a derivation of $A_1, A_2, \ldots, A_n \vdash B$ in HA$^\omega$, we can construct a term $t$ in the variables $x_1, x_2, \ldots, x_n$ such that $\forall y (A_1)^{|t|}_y, \forall y (A_2)^{|t|}_y, \ldots, \forall y (A_n)^{|t|}_y \vdash \forall y (B)^{|t|}_y$ in HA$^\omega$.

Fact 10 ([12]). For any $A$ an instance of AC$^\omega$, there is a term $t$ such that $\forall y (A)^{|t|}_y$ in HA$^\omega$. 

\[ T(x, z, y) \] is Kleene’s primitive recursive predicate formalizing the fact that the program coded by $x$, returns with result coded by $y$, when run on input coded by $z$. Kleene’s function $U(y)$ is used to extract the actual output from the coded result $y$. 

---

$\neg\neg$-LPO means that for any $n \in \mathbb{N}$, there exists an $m$ such that $\forall n \exists m \text{Proof}(j_1 m, \overline{A(\overline{\pi}, j_2 m)})$, where $j_1$ and $j_2$ are the projections of some surjective pairing function. By its definition, $g$ is a partial recursive function.
and the fact that DNS is our previous work [24], [11] on the
properties of DNS is our previous work [24], [11] on the
non-intuitionistic rules are given in the lower box. There
is the reset rule, that can be used with an empty or with ⊥ being ⊥. In the former case, the role of reset is to begin a
classical proof of ⊥ in the later case, there is no logical
meaning to the reset rule, but there is computational meaning to it, coming from the operational semantics for shift and reset. There is also the shift rule, that allows to use a principle similar to double-negation elimination from classical logic, but only inside a derivation that is ultimately proving ⊥, as guaranteed by a reset rule which must have set the annotation to ⊥ before.

For more explanations and the computational behavior of MQC+, please refer to [11].

We now give proof terms that formalize the proofs of the four alternative versions of DNS from Fact 7
\[ \lambda h. \lambda k. \# k (\lambda x. S k'. h x k') \quad (22) \]
\[ \lambda k. \lambda h. \# k (\lambda x. S k'. h x k') \quad (23) \]
\[ \lambda k. \# k (\lambda x. S k. k' (\text{inr} (\lambda a. k' (\text{inl} a)))) \quad (24) \]
\[ \lambda h. \# (\text{snd} h) (\lambda x. S k. (\text{fst} h) x k) \quad (25) \]

To give the proof term for ¬CT₀, we only need a formalized proof of the implication CT₀ → ¬HP, this proof is
intuitionistic and, in principle, it could be formalized in HA^ω, following, for example, [21].

This technique can be used to refute in MQC⁺+HA^ω any property P that implies ¬HP or ¬LPO. Assume that such a proof term
\[ r : P \rightarrow ¬HP \]
is given. Then,
\[ \lambda p. \# r (\lambda x. S k'. k' (\text{inr} (\lambda a. k' (\text{inl} a)))) : ¬P. \]

Since reducing a problem to the non-solvability of the Halting Problem is a common proof method, this means that all such results can be seen in a new light in the constructive logic HA^ω+AC^0+DNS^0.

VI. Future and related work

In the future, we would like to extend the modified realizability interpretation to the whole of the system MQC⁺+HA^ω. This would allow us to extract a realizer for ¬¬AC^0 by delimited control operators. Currently, it is not known whether DNS^ω captures exactly the additional provability power that MQC⁺ has over intuitionistic predicate logic.

Usually, modified realizability and the Dialectica interpretation are developed using Hilbert-style systems. In this paper, we used natural deduction. For detailed development of mr-
and D-interpretation in the context of natural deduction, the reader may refer to [24].

In the context of Constructive Reverse Mathematics, Wim Veldman has shown that, in presence of Markov’s Principle, the Double-negation Shift is equivalent to Open Induction on Cantor space [26]. Based on his work, in unpublished work [27], Keiko Nakata and the author show how to prove the principle of Open Induction on Cantor Space using delimited control operators.

In [28], Paulo Oliva revisits Spector’s bar recursive interpretation of DNS. In particular, he remarks that the finitary version of DNS, in which “∀” is replaced by a finite conjunction, is provable intuitionistically.

ACKNOWLEDGMENT

I would like to thank Keiko Nakata and Martín Escardó for stimulating discussions around some topics of this paper.

REFERENCES

[1] S. Kuroda, “Intuitionistische untersuchungen der formalistischer logik,” Nagoya Mathematical Journal, no. 3, pp. 35–47, 1951.
[2] G. Kreisel, “Interpretation of analysis by means of constructive functionals of finite types,” ser. Studies in Logic and The Foundations of Mathematics, A. Heyting, Ed. North-Holland Publishing Company Amsterdam, 1959, pp. 101–127.
[3] K. Gödel, “Zur intuitionistischen Arithmetik und Zahlentheorie,” Ergebnisse eines mathematischen Kolloquiums, vol. 4, pp. 34–38, 1933.
[4] ———, In what sense is intuitionistic logic constructive. New York: The Clarendon Press Oxford University Press, 1941, vol. III, pp. 189–200, early lecture on the Dialetica interpretation.
[5] ———, On a hitherto unutilized extension of the finitary standpoint. New York: The Clarendon Press Oxford University Press, 1958, vol. II, pp. 241–251.
[6] ———, An extension of finitary mathematics which has not yet been used. New York: The Clarendon Press Oxford University Press, 1972, vol. II, pp. 271–280.
[7] C. Spector, “Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics,” in Proc. Sympos. Pure Math., Vol. V. Providence, R.I.: American Mathematical Society, 1962, pp. 1–27.
[8] U. Kohlenbach, Applied Proof Theory: Proof Interpretations and Their Use in Mathematics, ser. Springer Monographs in Mathematics. Berlin, Heidelberg: Springer-Verlag, 2008.
[9] O. Danvy and A. Filinski, “A functional abstraction of typed contexts,” Computer Science Department, University of Copenhagen, Tech. Rep., 1989, diKU Rapport 89/12.
[10] ———, “Abstracting control,” in LISP and Functional Programming, 1990, pp. 151–160.
[11] D. Ilik, “Delimited control operators prove double-negation shift,” Annals of Pure and Applied Logic, vol. 163, no. 11, pp. 1549 – 1559, 2012.
[12] A. S. Troelstra, Ed., Metamathematical Investigations of Intuitionistic Arithmetic and Analysis, ser. Lecture Notes in Mathematics. Berlin, Heidelberg, New York: Springer-Verlag, 1973, no. 344.
[13] M. Beeson, “Goodman’s theorem and beyond,” Pacific Journal of Mathematics, vol. 84, no. 1, 1979.
[14] G. D. Plotkin, “Call-by-name, call-by-value and the [lambda]-calculus,” Theoretical Computer Science, vol. 1, no. 2, pp. 125–159, 1975.
[15] G. Kreisel, “On weak completeness of intuitionistic predicate logic,” The Journal of Symbolic Logic, vol. 27, no. 2, pp. 139–158, 1962.
[16] A. S. Troelstra, Realizability, ser. Studies in Logic and the foundations of Mathematics. Elsevier, 1998, vol. 137, ch. 6, pp. 407–473, Handbook of Proof Theory.
[17] J. R. Moschovakis, “Analyzing realizability by troelstra’s methods,” Annals of Pure and Applied Logic, no. 114, pp. 203–225, 2002.
[18] G. Kreisel, “Proof theoretic results on intuitionistic first order arithmetic,” The Journal of Symbolic Logic, vol. 27, no. 3, pp. 379–380, September 1962, abstract for the Meeting of the Association for Symbolic Logic, Leeds 1962.
[19] D. Bridges and F. Richman, Varieties of Constructive Mathematics, ser. London Mathematical Society Lecture Note Series. Cambridge University Press, 1987, no. 97.
[20] F. Richman, “Church’s thesis without tears,” The Journal of Symbolic Logic, vol. 48, no. 3, pp. 797–803, 1983.
[21] S. C. Kleene, Introduction to Metamathematics. North-Holland Publishing Co., 1952.
[22] G. Kreisel, “A survey of proof theory ii,” in Proceedings of the Second Scandinavian Logic Symposium, J. E. Fenstad, Ed. North-Holland Publishing Company, 1971, pp. 109–170.
[23] J. R. Moschovakis, “Classical and constructive hierarchies in extended intuitionistic analysis,” The Journal of Symbolic Logic, vol. 68, no. 3, pp. 1015–1043, September 2003.
[24] D. Ilik, “Constructive completeness proofs and delimited control,” Ph.D. dissertation, École Polytechnique, October 2010.
[25] K. F. Jorgensen, “Finite type arithmetic: Computable existence analysed

and D-interpretation in the context of natural deduction, the reader may refer to [24].

In the context of Constructive Reverse Mathematics, Wim Veldman has shown that, in presence of Markov’s Principle, the Double-negation Shift is equivalent to Open Induction on Cantor space [26]. Based on his work, in unpublished work [27], Keiko Nakata and the author show how to prove the principle of Open Induction on Cantor Space using delimited control operators.

In [28], Paulo Oliva revisits Spector’s bar recursive interpretation of DNS. In particular, he remarks that the finitary version of DNS, in which “∀” is replaced by a finite conjunction, is provable intuitionistically.

ACKNOWLEDGMENT

I would like to thank Keiko Nakata and Martín Escardó for stimulating discussions around some topics of this paper.

REFERENCES

[1] S. Kuroda, “Intuitionistische untersuchungen der formalistischer logik,” Nagoya Mathematical Journal, no. 3, pp. 35–47, 1951.
[2] G. Kreisel, “Interpretation of analysis by means of constructive functionals of finite types,” ser. Studies in Logic and The Foundations of Mathematics, A. Heyting, Ed. North-Holland Publishing Company Amsterdam, 1959, pp. 101–127.
[3] K. Gödel, “Zur intuitionistischen Arithmetik und Zahlentheorie,” Ergebnisse eines mathematischen Kolloquiums, vol. 4, pp. 34–38, 1933.
[4] ———, In what sense is intuitionistic logic constructive. New York: The Clarendon Press Oxford University Press, 1941, vol. III, pp. 189–200, early lecture on the Dialetica interpretation.
[5] ———, On a hitherto unutilized extension of the finitary standpoint. New York: The Clarendon Press Oxford University Press, 1958, vol. II, pp. 241–251.
[6] ———, An extension of finitary mathematics which has not yet been used. New York: The Clarendon Press Oxford University Press, 1972, vol. II, pp. 271–280.
[7] C. Spector, “Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics,” in Proc. Sympos. Pure Math., Vol. V. Providence, R.I.: American Mathematical Society, 1962, pp. 1–27.
by modified realisability and functional interpretation,” Master’s thesis, University of Roskilde, 2001.

[26] W. Veldman, “The principle of open induction on the unit interval [0,1] and some of its equivalents,” Slides, May 2010.

[27] D. Ilik and K. Nakata, “A direct constructive proof of open induction on cantor space,” manuscript.

[28] P. Oliva, “Understanding and using Spector’s bar recursive interpretation of classical analysis,” in CiE, ser. Lecture Notes in Computer Science, A. Beckmann, U. Berger, B. Löwe, and J. V. Tucker, Eds., vol. 3988. Springer, 2006, pp. 423–434.

[29] S. Feferman, Ed., Collected works. Publications 1938–1974. New York: The Clarendon Press Oxford University Press, 1990, vol. II.