Can chemotactic effects lead to blow-up or not in two-species chemotaxis-competition models?

Masaaki Mizukami, Yuya Tanaka and Tomomi Yokota

Abstract. This paper deals with the two-species chemotaxis-competition models
\[
\begin{align*}
    u_t &= d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), \\
    v_t &= d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v), \\
    \tau w_t &= d_3 \Delta w + \alpha u + \beta v - h(u, v, w),
\end{align*}
\]
where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary, and $h = \gamma w$ or $h = \frac{1}{|\Omega|} \int_{\Omega} (\alpha u + \beta v) \, dx$. In the case that $\kappa_1 = \lambda_1 = \kappa_2 = \lambda_2 = 2$ and $h = \gamma w$, it is known that smallness conditions for the chemotactic effects lead to boundedness of solutions in, for example, [21]. However, the case that the chemotactic effects are large seems not to have been studied yet; therefore, it remains to consider the question whether the solution is bounded also in the case that the chemotactic effects are large. The purpose of this paper is to give a negative answer to this question.

Mathematics Subject Classification. Primary 35K51; Secondary 35B44, 92C17.

Keywords. Chemotaxis, Lotka–Volterra, Boundedness, Finite-time blow-up.

1. Introduction

Background. A Lotka–Volterra competition model represents a simplified scenario of the coexistence of two competing species. Moreover, if the species have random spatial motion, the model reads
\[
\begin{align*}
    u_t &= d_1 \Delta u + \mu_1 u(1 - u - a_1 v), \\
    v_t &= d_2 \Delta v + \mu_2 v(1 - a_2 u - v)
\end{align*}
\]
(see, e.g., [22]), where $d_1, d_2, \mu_1, \mu_2, a_1, a_2 > 0$ are constants and $u, v$ idealize the population densities of the two species. In a mathematical view, the diffusion and competition terms lead to boundedness and stabilization of solutions. Indeed, there is a rich literature on large time behavior of solutions to the above model (see, e.g., [5, 7, 8, 13, 16, 18, 19]). In addition to the random spatial motion, if the model includes concentration phenomena of the two species, how do solutions behave? In order to consider this question, we next focus on a two-species chemotaxis-competition model proposed by Tello and Winkler [24], that is,
\[
\begin{align*}
    u_t &= d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), \\
    v_t &= d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v), \\
    \tau w_t &= d_3 \Delta w + \alpha u + \beta v - \gamma w,
\end{align*}
\]

Y. Tanaka: Partially supported by JSPS KAKENHI Grant Number JP22J11193.
T. Yokota: Partially supported by JSPS KAKENHI Grant Number JP21K03278.

Published online: 25 October 2022
where \( \chi_1, \chi_2, d_3, \alpha, \beta, \gamma > 0 \) and \( \tau \in \{0, 1\} \) are constants and \( w \) shows the concentration of the signal substance. This model describes a situation in which multi species move toward higher concentrations of the signal substance and, moreover, compete with each other. Here the structural feature of this model is that it involves not only the diffusion and competition terms but also the chemotaxis terms. In particular, the diffusion and competition terms indicate stabilization of the species, whereas the chemotaxis terms represent concentration of the species. Therefore, it is one of the fundamental themes whether the solutions remain bounded or not.

Now let us collect known results in the above models. We shall recall the parabolic–parabolic–elliptic case (\( \tau = 0 \)). In this case, boundedness and stabilization were obtained under smallness conditions for \( \chi_1 \) and \( \chi_2 \) in the cases that \( a_1, a_2 \in (0, 1) \) ([4,24]) and that \( a_1 > a_2 \) ([23]); moreover, these conditions in [4,23,24] were improved in some cases in [21]; particularly, the conditions for boundedness in [21] were given by \( 0 < \chi_1 \leq \frac{d_{\mu_1}}{a}, \chi_2 \leq \frac{d_{\mu_2}a}{\beta} \cdot \frac{n}{n-2} \) and \( 0 < \chi_2 \leq \frac{d_{\mu_2}a}{\beta} \cdot \frac{n}{n-2} \). Similarly, in the fully parabolic case (\( \tau = 1 \)), the results on global existence and asymptotic stability were obtained under smallness conditions for \( \chi_1 \) and \( \chi_2 \) in [1,17,20].

The challenge of proving blow-up. The above studies indicate that smallness of chemotactic effects entails boundedness and stabilization of solutions to chemotaxis-competition models. However, to the best of our knowledge, there is no result in the case that chemotactic effects are large. Therefore, the following question arises:

Does the solution remain bounded in the case that the chemotactic effects are large?

The purpose of this work is to give a negative answer to this question.

Review of chemotaxis systems. In order to consider the above question, it is useful to give an overview of the known results for the parabolic–elliptic chemotaxis system with logistic source,

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla w) + \lambda u - \mu u^\kappa, \\
    0 &= \Delta w - w + u,
\end{align*}
\]

where \( \chi, \lambda, \mu > 0 \) and \( \kappa > 1 \). As to this model, the results on boundedness and blow-up were obtained in [3,11,25,27,29]. Indeed, Tello and Winkler [25] proved boundedness of solutions in the cases that \( \kappa = 2 \) and \( \chi < \frac{n}{n-2} \mu \) and that \( \kappa > 2 \) and \( \chi > 0 \), whereas Winkler [29] derived finite-time blow-up under some smallness condition for \( \kappa > 1 \); in a simplified parabolic–elliptic chemotaxis system with logistic source, smallness conditions for \( \kappa \) leading to finite-time blow-up were established in higher-dimensional cases by Winkler [27] and in three- and four-dimensional cases by Black et al. [3]; moreover, Fuest [11] succeeded in improving the conditions for \( \kappa \) in [3,27] and showing finite-time blow-up under largeness conditions for \( \chi \) in the case \( \kappa = 2 \) in higher-dimensional cases. In summary, in the one-species model, boundedness and blow-up are determined by the sizes of \( \chi \) and \( \kappa \).

Main results. From the review of the one-species chemotaxis systems with logistic source, it is expected that a behavior of solutions depends on chemotactic effects and competitive effects also in the two-species model. In this context, we consider two-species chemotaxis models with generalized competition terms, which read

\[
\begin{align*}
    u_t &= d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u (1 - u^{\kappa_1 - 1} - a_1 v^{\lambda_1 - 1}), \\
    v_t &= d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v (1 - a_2 u^{\lambda_2 - 1} - v^{\kappa_2 - 1}), \\
    0 &= d_3 \Delta w + \alpha u + \beta v - h(u, v, w), \\
    \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla u \cdot \nu = 0, \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega, \quad t > 0.
\end{align*}
\]
where $\Omega \subset \mathbb{R}^n \ (n \geq 2)$ is a bounded domain with smooth boundary $\partial \Omega$; $\nu$ is the outward normal vector to $\partial \Omega$; $d_1, d_2, d_3, \chi_1, \chi_2, \mu_1, \mu_2, a_1, a_2, \alpha, \beta > 0$ and $\kappa_1, \kappa_2, \lambda_1, \lambda_2 > 1$;

$$h(u, v, w) := \begin{cases} \gamma w \quad (\gamma > 0) & \text{(Keller–Segel type (cf. [14, 15]))}, \\ \frac{1}{|\Omega|} \int_{\Omega} (\alpha u + \beta v) \, dx & \text{(Jäger–Luckhaus type (cf. [12]))} \end{cases}$$ (1.2)

the initial data $u_0, v_0$ are assumed to be nonnegative functions; in particular, in the case that $h$ is of Jäger–Luckhaus type, we consider the model (1.1) with

$$\int_{\Omega} w \, dx = 0, \quad t > 0.$$ (1.3)

From the preceding consideration, we aim to show boundedness and blow-up of solutions to (1.1). Here we briefly state our main results as follows:

- **Prevention of blow-up, i.e., boundedness (Theorem 3.1):** If $n \geq 2$ and

$$0 < \chi_1 < \begin{cases} \frac{\alpha \mu_1}{\beta} \cdot \frac{n}{n-2}, & \text{if } \kappa_1 > 2, \lambda_1 > 2, \\ \frac{a_1 d_3 \mu_1}{\beta} \cdot \frac{n}{n-2}, & \text{if } \kappa_1 = 2, \lambda_1 > 2, \\ \min \left\{ \frac{d_3 \mu_1}{\alpha}, \frac{a_1 d_3 \mu_1}{\beta} \right\} \cdot \frac{n}{n-2}, & \text{if } \kappa_1 = 2, \lambda_1 = 2 \end{cases}$$

and

$$0 < \chi_2 < \begin{cases} \frac{\alpha d_3 \mu_2}{\beta} \cdot \frac{n}{n-2}, & \text{if } \kappa_2 > 2, \lambda_2 > 2, \\ \frac{a_2 d_3 \mu_2}{\alpha} \cdot \frac{n}{n-2}, & \text{if } \kappa_2 = 2, \lambda_2 > 2, \\ \min \left\{ \frac{d_3 \mu_2}{\beta}, \frac{a_2 d_3 \mu_2}{\alpha} \right\} \cdot \frac{n}{n-2}, & \text{if } \kappa_2 = 2, \lambda_2 = 2, \end{cases}$$

then for all nonnegative initial data, (1.1) has a bounded solution.

- **Finite-time blow-up in the case $h = \gamma w$ (Theorem 4.1):** If

$$\max \{ \kappa_1, \lambda_1, \kappa_2, \lambda_2 \} < \begin{cases} \frac{7}{6} & \text{if } n \in \{3, 4\}, \\ \frac{1}{2(n-1)} & \text{if } n \geq 5, \end{cases}$$

then for some nonnegative initial data, (1.1) has a blow-up solution.

- **Finite-time blow-up in the case $h = \frac{1}{|\Omega|} \int_{\Omega} (\alpha u + \beta v) \, dx$ (Theorem 5.1):** If $n \geq 5$, $\lambda_1 = \lambda_2 = 2$ and

$$\chi_1 > \begin{cases} \frac{a_1 d_3 \mu_1}{\beta} \cdot \frac{n}{n-4}, \quad & \text{if } \kappa_1 < 2, \\ \max \left\{ \frac{d_3 \mu_1}{\alpha}, \frac{a_1 d_3 \mu_1}{\beta} \right\} \cdot \frac{n}{n-4}, \quad & \text{if } \kappa_1 = 2 \end{cases}$$

or

$$n \geq 5, \quad \lambda_1 = \lambda_2 = 2$$

and

$$\chi_1 > \frac{a_1 d_3 \mu_1}{\beta}$$

and

$$\chi_2 > \begin{cases} \frac{a_2 d_3 \mu_2}{\alpha} \cdot \frac{n}{n-4}, \quad & \text{if } \kappa_2 < 2, \\ \max \left\{ \frac{d_3 \mu_2}{\beta}, \frac{a_2 d_3 \mu_2}{\alpha} \right\} \cdot \frac{n}{n-4}, \quad & \text{if } \kappa_2 = 2, \end{cases}$$

then for some nonnegative initial data, (1.1) has a blow-up solution.

**Main ideas and plan of the paper.** We first collect some preliminary facts on local existence of solutions to (1.1) and estimates for masses of $u$ and $v$ in Sect. 2. In Sect. 3 we show boundedness of solutions to (1.1),
the proof of which is based on [21]. The key point is to establish an $L^p$-estimate for $u$ and an $L^q$-estimate for $v$ by constructing the differential inequalities
\[
\frac{1}{p} \cdot \frac{d}{dt} \int_{\Omega} u^p \, dx \leq C_1 \int_{\Omega} u^p \, dx - C_2 \int_{\Omega} u^{p+\kappa_1 - 1} \, dx + C_3
\]
and
\[
\frac{1}{q} \cdot \frac{d}{dt} \int_{\Omega} v^q \, dx \leq C_4 \int_{\Omega} v^q \, dx - C_5 \int_{\Omega} v^{q+\kappa_2 - 1} \, dx + C_6.
\]
In Sect. 4 we prove finite-time blow-up in the case that $h$ is of Keller–Segel type. Referring to [29], we will introduce the functions
\[
\phi_U(t) := \int_0^{s_0} s^{-b}(s_0 - s) U(s, t) \, ds \quad \text{and} \quad \phi_V(t) := \int_0^{s_0} s^{-b}(s_0 - s) V(s, t) \, ds
\]
with $s_0 \in (0, R^n)$ and $b \in (0, 1)$, where
\[
U(s, t) := \int_0^{s^n} \rho^{n-1} u(\rho, t) \, d\rho \quad \text{and} \quad V(s, t) := \int_0^{s^n} \rho^{n-1} v(\rho, t) \, d\rho.
\]
In this case the goal is to obtain a superlinear differential inequality for $\phi_U + \phi_V$, and toward this goal, we need to control the integrals
\[
\int_0^{s_0} s^{-b}(s_0 - s) \left( \int_0^s U_s(\sigma, t) V_s^{\lambda_1 - 1}(\sigma, t) \, d\sigma \right) \, ds
\]
and
\[
\int_0^{s_0} s^{-b}(s_0 - s) \left( \int_0^s V_s(\sigma, t) U_s^{\lambda_2 - 1}(\sigma, t) \, d\sigma \right) \, ds
\]
which come from the competition terms. To this end, we derive pointwise upper estimates for $u$ and $v$ (see Lemma 4.2) by an argument similar to that in [29, Lemma 3.3]. Finally, in Sect. 5 we show finite-time blow-up in the case that $h$ is of Jäger–Luckhaus type. In this case, by the structure of the third equation in (1.1), it is sufficient to establish a superlinear differential inequality for $\phi_U$ or $\phi_V$ with the parameter $b > 1$. The key to obtaining the inequality for $\phi_U$ or $\phi_V$ is to derive concavity of $U$ and $V$ (see Lemma 5.2). Here, this property will be shown by using a comparison principle, which is a different method in [28, Lemma 2.2].

2. Preliminaries

We first recall the known result about local existence of solutions to (1.1), which is proved by a standard fixed point argument as in [6,23].

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^n \ (n \geq 2)$ be a bounded domain with smooth boundary and $d_1, d_2, d_3, \chi_1, \chi_2, \mu_1, \mu_2, a_1, a_2, \alpha, \beta > 0, \kappa_1, \kappa_2, \lambda_1, \lambda_2 > 1$ and let $h$ satisfy (1.2). Assume that
\[
u_0, \nu_0 \in C^0(\Omega)
\]
are nonnegative.

Then there exist $T_{\text{max}} \in (0, \infty]$ and a unique triplet $(u, v, w)$ of functions
\[
u, v, w \in C^0(\Omega \times [0, T_{\text{max}}]) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})),
\]
which solves (1.1) classically, and fulfills (1.3) in the case that $h$ is of Jäger–Luckhaus type. Moreover, $u, v \geq 0$ in $\Omega \times (0,T_{\text{max}})$ and
\[
\text{if } T_{\text{max}} < \infty, \quad \text{then } \lim_{t \to T_{\text{max}}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty.
\] (2.2)

Furthermore, if $u_0, v_0$ are radially symmetric, then so are $u, v, w$ for any $t \in (0,T_{\text{max}})$.

We next show that the masses of $u$ and $v$ can be controlled by $\mu_1, u_0$ and $\mu_2, v_0$, respectively.

**Lemma 2.2.** Assume that $u_0, v_0 \in C^0(\bar{\Omega})$ are nonnegative. Then
\[
\int_\Omega u(x,t) \, dx \leq e^{\mu_1 t} \int_\Omega u_0(x) \, dx \quad \text{and} \quad \int_\Omega v(x,t) \, dx \leq e^{\mu_2 t} \int_\Omega v_0(x) \, dx
\]
for all $t \in (0,T_{\text{max}})$.

**Proof.** By integrating the first and second equations in (1.1), and using that $\mu_1 > 0$ and $\mu_2 > 0$, it follows that
\[
\frac{d}{dt} \int_\Omega u \, dx \leq \mu_1 \int_\Omega u \, dx \quad \text{and} \quad \frac{d}{dt} \int_\Omega v \, dx \leq \mu_2 \int_\Omega v \, dx
\]
for all $t \in (0,T_{\text{max}})$. These inequalities imply the conclusion of this lemma. \qed

### 3. Boundedness

In this section let us denote by $(u, v, w)$ the local classical solution of (1.1) on $[0,T_{\text{max}})$ given in Lemma 2.1. We shall prove the following theorem which asserts global existence and boundedness in (1.1).

**Theorem 3.1.** Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary, and let $d_1, d_2, d_3, \chi_1, \chi_2, a_1, a_2, \alpha, \beta > 0$. Assume that $\mu_1$ and $\mu_2$ fulfill
\[
0 < \chi_1 < \begin{cases} 
\infty & \text{if } \kappa_1 > 2, \lambda_1 > 2, \\
\frac{d_1 a_2 d_3 \mu_1}{\beta} \cdot \frac{n}{n-2} & \text{if } \kappa_1 = 2, \lambda_1 > 2, \\
\min \{ \frac{d_1 a_2 d_3 \mu_1}{\alpha}, \frac{a_1 d_3 \mu_1}{\beta} \} \cdot \frac{n}{n-2} & \text{if } \kappa_1 > 2, \lambda_1 = 2,
\end{cases}
\]
and
\[
0 < \chi_2 < \begin{cases} 
\infty & \text{if } \kappa_2 > 2, \lambda_2 > 2, \\
\frac{d_2 a_2 d_3 \mu_2}{\alpha} \cdot \frac{n}{n-2} & \text{if } \kappa_2 = 2, \lambda_2 > 2, \\
\min \{ \frac{d_2 a_2 d_3 \mu_2}{\beta}, \frac{a_1 d_3 \mu_2}{\alpha} \} \cdot \frac{n}{n-2} & \text{if } \kappa_2 > 2, \lambda_2 = 2,
\end{cases}
\]
and that (2.1) is satisfied. Then the corresponding solution $(u, v, w)$ of (1.1) is global and bounded in the sense that $T_{\text{max}} = \infty$ and
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C
\]
for all $t \geq 0$ with some $C > 0$.

**Remark 3.1.** In the case that $\kappa_1 = \lambda_1 = \kappa_2 = \lambda_2 = 2$, the conditions (3.1) and (3.2) are the same as in [21, (6)]. Thus this theorem is a generalization of [21, Theorem 1.1].

For the proof of Theorem 3.1 we derive an $L^p$-estimate for $u$ with some $p > \frac{n}{2}$ and an $L^q$-estimate for $v$ with some $q > \frac{n}{2}$. 
Lemma 3.2. Assume that (2.1) holds.

(i) If $\mu_1$ satisfies (3.1), then for some $p > \frac{n}{2}$ there exists $C_1 > 0$ such that
$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C_1$$
for all $t \in (0, T_{\text{max}})$.

(ii) If $\mu_2$ satisfies (3.2), then for some $q > \frac{n}{2}$ there exists $C_2 > 0$ such that
$$\|v(\cdot, t)\|_{L^q(\Omega)} \leq C_2$$
for all $t \in (0, T_{\text{max}})$.

Proof. We let $p > \frac{n}{2}$ be fixed later. Multiplying the first equation in (1.1) by $u^{p-1}$ and integrating it over $\Omega$, we have
$$\frac{1}{p} \cdot \frac{d}{dt} \int_{\Omega} u^p \, dx = \int_{\Omega} u^{p-1} \Delta u \, dx - \chi_1 \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla w) \, dx$$
$$+ \mu_1 \int_{\Omega} u^p (1 - u^{\kappa_1-1} - a_1 v^{\lambda_1-1}) \, dx$$
$$= -\frac{4(p-1)}{p^2} d_1 \int_{\Omega} |\nabla u^p|^2 \, dx - \frac{p-1}{p} \chi_1 \int_{\Omega} u^p \Delta w \, dx$$
$$+ \mu_1 \int_{\Omega} u^p \, dx - \mu_1 \int_{\Omega} u^{p+\kappa_1-1} \, dx - a_1 \mu_1 \int_{\Omega} u^{p+\lambda_1-1} \, dx$$
for all $t \in (0, T_{\text{max}})$. From the third equation in (1.1) and the nonnegativity of $h$, it follows that
$$-\frac{p-1}{p} \chi_1 \int_{\Omega} u^p \Delta w \, dx \leq \frac{p-1}{p} \cdot \frac{\alpha \chi_1}{d_3} \int_{\Omega} u^{p+1} \, dx + \frac{p-1}{p} \cdot \frac{\beta \chi_1}{d_3} \int_{\Omega} u^p v \, dx$$
and so, we see that
$$\frac{1}{p} \cdot \frac{d}{dt} \int_{\Omega} u^p \, dx \leq -\frac{4(p-1)}{p^2} d_1 \int_{\Omega} |\nabla u^p|^2 \, dx$$
$$+ \frac{p-1}{p} \cdot \frac{\alpha \chi_1}{d_3} \int_{\Omega} u^{p+1} \, dx + \frac{p-1}{p} \cdot \frac{\beta \chi_1}{d_3} \int_{\Omega} u^p v \, dx$$
$$+ \mu_1 \int_{\Omega} u^p \, dx - \mu_1 \int_{\Omega} u^{p+\kappa_1-1} \, dx - a_1 \mu_1 \int_{\Omega} u^{p+\lambda_1-1} \, dx$$
(3.5)
for all $t \in (0, T_{\text{max}})$. We first prove (3.3) in the case that $\kappa_1 = 2$ and $\lambda_1 = 2$. In this case, the above inequality gives
$$\frac{1}{p} \cdot \frac{d}{dt} \int_{\Omega} u^p \, dx \leq \mu_1 \int_{\Omega} u^p \, dx - \left( \mu_1 - \frac{p-1}{p} \cdot \frac{\alpha \chi_1}{d_3} \right) \int_{\Omega} u^{p+1} \, dx$$
$$- \left( a_1 \mu_1 - \frac{p-1}{p} \cdot \frac{\beta \chi_1}{d_3} \right) \int_{\Omega} u^p v \, dx.$$
and
\[ a_1 \mu_1 - \frac{p-1}{p} \cdot \frac{\beta \chi_1}{d_3} > 0. \] (3.7)

Then, noting that \(-\left(a_1 \mu_1 - \frac{p-1}{p} \cdot \frac{\beta \chi_1}{d_3}\right) \int_\Omega u^p v \, dx \leq 0\), by using Hölder’s inequality we derive that

\[ \frac{1}{p} \cdot \frac{d}{dt} \int_\Omega u^p \, dx \leq \mu_1 \int_\Omega u^p \, dx - c_1 \left( \int_\Omega u^p \, dx \right)^{\frac{p+1}{p}} \]

for all \( t \in (0, T_{\text{max}}) \), where \( c_1 := \left( \mu_1 - \frac{p-1}{p} \cdot \frac{\alpha \chi_1}{d_3} \right) |\Omega|^{-\frac{1}{p}} > 0 \). Putting \( Y(t) := \int_\Omega u^p(x, t) \, dx \), we thereby have

\[ Y'(t) \leq \rho_1 Y(t) - \rho_2 Y^{\frac{p+1}{p}}(t) \quad \text{for all } t \in (0, T_{\text{max}}). \]

Then the comparison principle asserts that \( Y(t) \leq \max \{ \int_\Omega u_0^p \, dx, \left( \frac{c}{c_1} \right)^p \} \). Hence (3.3) results from this. Next, we consider the case that \( \kappa_1 > 2 \) and \( \lambda_1 > 2 \). For all \( p > \frac{n}{2} \) it follows from Young’s inequality that

\[ \frac{p-1}{p} \cdot \frac{\alpha \chi_1}{d_3} \int_\Omega u^p v \, dx \leq \mu_1 \int_\Omega u^{p+\lambda_1-1} \, dx + c_2 \]

(3.8)

with some \( c_2 > 0 \), and

\[ \frac{p-1}{p} \cdot \frac{\beta \chi_1}{d_3} \int_\Omega u^p v \, dx \leq a_1 \mu_1 \int_\Omega u^{p+\lambda_1-1} \, dx + c_3 \int_\Omega u^p \, dx \]

(3.9)

with some \( c_3 > 0 \). Combining these inequalities with (3.5), we obtain

\[ \frac{1}{p} \cdot \frac{d}{dt} \int_\Omega u^p \, dx \leq c_4 \int_\Omega u^p \, dx - \frac{\mu_1}{2} \int_\Omega u^{p+\lambda_1-1} \, dx + c_2 \]

for all \( t \in (0, T_{\text{max}}) \), where \( c_4 := \mu_1 + c_3 \). This entails that \( Y'(t) \leq c_4 Y(t) - c_5 Y^{\frac{p+\lambda_1-1}{p}}(t) \) for all \( t \in (0, T_{\text{max}}) \) with some \( c_5 > 0 \). Thus we see that \( Y(t) \leq \max \{ \int_\Omega u_0^p \, dx, \left( \frac{c}{c_5} \right)^{\frac{p}{p-1}} \} \), which leads to (3.3). Similarly, taking \( p \) satisfying (3.6) and using (3.9) in the case that \( \kappa_1 = 2 \) and \( \lambda_1 > 2 \), and picking \( p \) fulfilling (3.7) and applying (3.8) in the case that \( \kappa_1 > 2 \) and \( \lambda_1 = 2 \), we can confirm that (3.3) holds. Moreover, we can show that (3.4) holds for all \( t \in (0, T_{\text{max}}) \) with some \( q > \frac{n}{2} \) by an argument similar to that in the proof of (3.3) under the condition (3.2).

With these estimates for \( u \) and \( v \) at hand, we can show Theorem 3.1.

**Proof of Theorem 3.1.** Since the proof is similar to that in [21, Lemma 2.4], let us briefly state the proof. By making use of Lemma 3.2, we can establish an \( L^p \)-estimate for \( u \) for some \( p > \frac{n}{2} \) and an \( L^q \)-estimate for \( v \) for some \( q > \frac{n}{2} \). Thus, for all \( r \in \left( \frac{n}{2}, \min\{p, q\} \right) \cap (0, n) \) there exists \( c_1 > 0 \) such that \( \|u(\cdot, t)\|_{L^r(\Omega)} + \|v(\cdot, t)\|_{L^r(\Omega)} \leq c_1 \) for all \( t \in (0, T_{\text{max}}) \). Hence, applying elliptic regularity theory (see, e.g., [9, Theorem 19.1]) to the third equation in (1.1) and the Sobolev embedding theorem, we can take \( c_2 > 0 \) such that

\[ \|\nabla w(\cdot, t)\|_{L^{\frac{np}{p-2n}}(\Omega)} \leq c_2 \]

for all \( t \in (0, T_{\text{max}}) \), and moreover, we invoke the standard semigroup technique (see, e.g., [2, Proof of Lemma 3.2]) to find \( c_3 > 0 \) such that

\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq c_3 \]

for all \( t \in (0, T_{\text{max}}) \), which in conjunction with (2.2) implies the end of the proof. \( \square \)
4. Finite-time blow-up in a model of Keller–Segel type

In this section we consider the system (1.1) in the case that \( h \) is of Keller–Segel type as follows:

\[
\begin{align*}
\dot{u} &= d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla v) + \mu_1 u (1 - u^{\kappa_1 - 1} - a_1 v^{\lambda_1 - 1}), \quad x \in \Omega, \; t > 0, \\
\dot{v} &= d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v (1 - a_2 u^{\kappa_2 - 1} - v^{\lambda_2 - 1}), \quad x \in \Omega, \; t > 0, \\
0 &= d_3 \Delta w + \alpha u + \beta v - \gamma w, \quad x \in \Omega, \; t > 0, \\
\nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, \quad x \in \partial \Omega, \; t > 0, \\
u(x, 0) &= u_0(x), \; v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

where \( \Omega = B_R(0) \subset \mathbb{R}^n \) \((n \geq 3)\) is a ball with some \( R > 0 \), and \( u_0, v_0 \) satisfy (2.1) and \( u_0, v_0 \) are radially symmetric.

Moreover, let \((u, v, w)\) be the local classical solution of (4.1) on \([0, T_{\max})\) given in Lemma 2.1. We now state the following theorem which guarantees finite-time blow-up.

**Theorem 4.1.** Let \( \Omega = B_R(0) \subset \mathbb{R}^n \) \((n \geq 3)\) and let \( d_1, d_2, d_3, \chi_1, \chi_2, \mu_1, \mu_2, a_1, a_2 > 0 \) and \( \alpha, \beta, \gamma > 0 \) as well as \( \kappa_1, \kappa_2, \lambda_1, \lambda_2 > 1 \). Assume that \( \kappa_1, \kappa_2, \lambda_1 \) and \( \lambda_2 \) satisfy that

\[
\max \{ \kappa_1, \lambda_1, \kappa_2, \lambda_2 \} < \begin{cases} 7 \over 8 & \text{if } n \in \{3, 4\}, \\
1 + \frac{1}{2(n-1)} & \text{if } n \geq 5. \end{cases}
\]

Then, for all \( L > 0 \), \( M_0 > 0 \) and \( \tilde{M}_0 \in (0, M_0) \) there exists \( r_* \in (0, R) \) with the following property: If \( u_0, v_0 \) satisfy (2.1), (4.2) and

\[
\int_{\Omega} (u_0(x) + v_0(x)) \, dx = M_0 \quad \text{and} \quad \int_{B_{r_*}(0)} (u_0(x) + v_0(x)) \, dx \geq \tilde{M}_0
\]

as well as

\[
u_0(x) + v_0(x) \leq L |x|^{-n(n-1)} \quad \text{for all } x \in \Omega,
\]

then the corresponding solution \((u, v, w)\) of (4.1) blows up in finite time in the sense that \( T_{\max} < \infty \) and

\[
\lim_{t \uparrow T_{\max}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty.
\]

**Remark 4.1.** If \( v = 0 \) in (4.1), then \( \lambda_1, \kappa_2 \) and \( \lambda_2 \) can be taken smaller than \( \kappa_1 \), and thus (4.3) is rewritten as

\[
\kappa_1 < \begin{cases} 7 \over 8 & \text{if } n \in \{3, 4\}, \\
1 + \frac{1}{2(n-1)} & \text{if } n \geq 5. \end{cases}
\]

This is the same condition as in [29, (1.4)], which means that the result in [29, Theorem 1.1] is generalized to that in the two-species model.

The proof of Theorem 4.1 is based on [29]. In what follows, we regard \( u(x, t), v(x, t) \) and \( w(x, t) \) as functions of \( r := |x| \) and \( t \). Also, we introduce the mass accumulation functions \( U, V \) and \( W \) as

\[
U(s, t) := \int_0^s \rho^{n-1} u(\rho, t) \, d\rho,
\]

\[
V(s, t) := \int_0^s \rho^{n-1} v(\rho, t) \, d\rho
\]
and

\[ W(s, t) := \int_0^{s_t} \rho^{n-1} w(\rho, t) \, d\rho \]

for \( s \in [0, R^n] \) and \( t \in [0, T_{\text{max}}] \). Moreover, we define the functions \( \phi_U, \phi_V, \psi_U \) and \( \psi_V \) as

\[ \phi_U(t) := \int_0^{s_0} s^{-b}(s_0 - s)U(s, t) \, ds, \]

\[ \phi_V(t) := \int_0^{s_0} s^{-b}(s_0 - s)V(s, t) \, ds \]

and

\[ \psi_U(t) := \int_0^{s_0} s^{-b}(s_0 - s)U(s, t)U_s(s, t) \, ds, \]

\[ \psi_V(t) := \int_0^{s_0} s^{-b}(s_0 - s)V(s, t)V_s(s, t) \, ds \]

for \( t \in [0, T_{\text{max}}] \) with suitably chosen \( s_0 \in (0, R^n) \) and \( b \in (0, 1) \). Here, we note that \( \phi_U, \phi_V \in C^0([0, T_{\text{max}}]) \cap C^1((0, T_{\text{max}})) \).

To prove Theorem 4.1, we will establish an ordinary differential inequality for \( \phi_U + \phi_V \). As a preparation of the proof, we show pointwise upper estimates for \( u \) and \( v \).

**Lemma 4.2.** For all \( L > 0 \), \( M_0 > 0 \) and \( \varepsilon > 0 \) there exist \( C_1 > 0 \) and \( C_2 > 0 \) such that if \( u_0, v_0 \) satisfy (2.1), (4.2) and \( \int_\Omega(u_0(x) + v_0(x)) \, dx = M_0 \) as well as (4.5), then

\[ u(r, t) \leq C_1 r^{-n(n-1) - \varepsilon} \quad \text{and} \quad v(r, t) \leq C_2 r^{-n(n-1) - \varepsilon} \]

for all \( r \in (0, R) \) and \( t \in (0, \min\{1, T_{\text{max}}\}) \).

**Proof.** We put \( \tilde{u}(x, t) := e^{-\nu t} u(x, t) \). Then we see from (4.1) that

\[
\begin{cases}
\tilde{u}_t \leq d_1 \Delta \tilde{u} - \chi_1 \nabla \cdot (\tilde{u} \nabla w), & x \in \Omega, \, t > 0, \\
\nabla \tilde{u} \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial \Omega, \, t > 0, \\
\tilde{u}(x, 0) = u_0(x), & x \in \Omega.
\end{cases}
\]

(4.8)

Also, we have \( \int_\Omega \tilde{u}(\cdot, 0) \, dx \leq M_0 \). Next we verify that

\[ \int_\Omega |x|^{(n-1)\theta} |\nabla w(x, t)|^{\theta} \, dx \leq c_1 \]

for all \( t \in (0, \min\{1, T_{\text{max}}\}) \) with some \( \theta > n \) and \( c_1 > 0 \). Since the third equation in (4.1) yields \( \int_\Omega (\alpha u + \beta v) \, dx = \gamma \int_\Omega w \, dx \), we can observe from the third equation in (4.1) that

\[ |r^{n-1} w_r| = \left| -\frac{1}{d_3} \int_0^r \rho^{n-1} (\alpha u + \beta v - \gamma w) \, d\rho \right| \leq \frac{2}{d_3} \int_0^R \rho^{n-1} (\alpha u + \beta v) \, d\rho. \]
Estimating the right-hand side by Lemma 2.2 and the condition \( \int_{\Omega}(u_0 + v_0) \, dx = M_0 \), we see that for all \( r \in (0, R) \) and \( t \in (0, \min\{1, T_{\max}\}) \),
\[
|r^{n-1}w_r| \leq \frac{2(\alpha \epsilon^{\mu_1} + \beta \epsilon^{\mu_2})M_0}{d_3 \omega_n} =: c_2,
\]
where \( \omega_n := n|B_1(0)| \). Given \( \epsilon > 0 \), let us choose \( \theta > n \) so large such that
\[
n(n - 1) + \epsilon > \frac{n(n - 1)\theta}{\theta - n}.
\]
Then, due to (4.10) we can derive (4.9) with \( c_1 = c_2 [\Omega] \). Thanks to (4.5), (4.8), (4.9) and (4.11), we can apply [10, Theorem 1.1] to find \( c_3 > 0 \) such that \( \hat{u}(x,t) \leq c_3|x|^{-n(n-1)\theta} \) for all \( x \in \Omega \) and \( t \in (0, \min\{1, T_{\max}\}) \), which means the first claim in (4.7). Similarly, we have the second claim in (4.7).

We next derive differential inequalities for \( \phi_U \) and \( \phi_V \) as in [29, Lemmas 4.1, 4.3 and 4.5].

**Lemma 4.3.** Let \( s_0 \in (0, R^n) \). Assume that \( \kappa_1, \kappa_2 > 1 \) and \( b \in \left( 1 - \frac{2}{n}, \min\{1, 2 - \frac{4}{n}\} \right) \) satisfy
\[
(n - 1)(\max\{\kappa_1, \kappa_2\} - 1) < \frac{b}{2}.
\]
Then, for all \( L > 0 \), \( M_0 > 0 \) and \( \epsilon > 0 \) there exist \( C_2 > 0, C_3 > 0 \) and \( C_4 > 0 \) such that if \( u_0, v_0 \) satisfy (2.1), (4.2) and \( \int_{\Omega}(u_0(x) + v_0(x)) \, dx = M_0 \) as well as (4.5), then
\[
\phi_U'(t) \geq -C_1 s_0 \frac{\sqrt{\psi_U'(t)}}{\psi_U(t)}
+ \frac{\alpha \chi_1 n}{d_3} \psi_U(t) - (b + 1) \frac{\gamma \chi_2 n}{d_3} s_0 \int_0^s s^{-b-1} U(s,t) W(s,t) \, ds
- C_2 s_0^{-n(n-1)\kappa_1} \frac{\sqrt{\psi_U(t)}}{\psi_U(t)}
- a_1 \mu_1 n \kappa_1^{-1} \int_0^s s^{-b} (s_0 - s) \left( \int_0^s U_s(\sigma,t) \nu_s^\kappa_1^{-1}(\sigma,t) \, d\sigma \right) \, ds
\]
and
\[
\phi_V'(t) \geq -C_4 s_0 \frac{\sqrt{\psi_V'(t)}}{\psi_V(t)}
+ \frac{\beta \chi_2 n}{d_3} \psi_V(t) - (b + 1) \frac{\gamma \chi_2 n}{d_3} s_0 \int_0^s s^{-b-1} V(s,t) W(s,t) \, ds
- C_3 s_0^{-n(n-1)\kappa_2} \frac{\sqrt{\psi_V(t)}}{\psi_V(t)}
- a_2 \mu_2 n \kappa_2^{-1} \int_0^s s^{-b} (s_0 - s) \left( \int_0^s V_s(\sigma,t) \nu_s^\kappa_2^{-1}(\sigma,t) \, d\sigma \right) \, ds
\]
for all \( t \in (0, \min\{1, T_{\max}\}) \).

**Proof.** We first derive a differential inequality for \( \phi_U \). The first equation in (4.1) is rewritten as
\[
u_{11} r^{1-n}(r^{n-1}u_r)_r - \chi_1 r^{1-n}(r^{n-1}u_w)_r + \mu_1 u(1 - \nu^{\kappa_1-1} - a_1 v^{\lambda_1-1})
\]
for all \( r \in (0, R) \) and \( t \in (0, T_{\max}) \). Also, we multiply the third equation in (4.1) by \( r^{n-1} \) and integrate it over \( [0, s^\pi] \) to obtain
\[
s^{1-n}w_r(s^\pi,t) = -\frac{\alpha}{d_3} U(s,t) - \frac{\beta}{d_3} V(s,t) + \frac{\gamma}{d_3} W(s,t).
\]
Thus, noticing that \( U_s(s, t) = \frac{1}{n} u(s^{\frac{1}{n}}, t), \) \( V_s(s, t) = \frac{1}{n} v(s^{\frac{1}{n}}, t), \) \( W_s(s, t) = \frac{1}{n} w(s^{\frac{1}{n}}, t) \) and \( U_{ss}(s, t) = \frac{1}{n^2} s^{\frac{1}{n}-1} u_r(s^{\frac{1}{n}}, t) \), we can verify from (4.15) and (4.16) that

\[
U_t = d_1 n^2 s^{2-\frac{2}{n}} U_{ss} + \frac{\alpha \chi_1 n}{d_3} U_s + \frac{\beta \chi_1 n}{d_3} U_s V - \frac{\gamma \chi_1 n}{d_3} U_s W
\]

\[
+ \mu_1 U - \mu_1 n^{\kappa_1-1} \int_0^s U_s^{\kappa_1}(\sigma, t) d\sigma - a_1 \mu_1 n^{\lambda_1-1} \int_0^s U_s(\sigma, t)V_s^{\lambda_1-1}(\sigma, t) d\sigma
\]

for all \( s \in (0, R^n) \) and \( t \in (0, T_{\text{max}}) \). In the light of this identity we infer that

\[
\phi'_U(t) \geq d_1 n^2 \int_0^{s_0} s^{2-\frac{2}{n}-b}(s_0 - s) U_{ss} ds
\]

\[
+ \frac{\alpha \chi_1 n}{d_3} \psi_U(t) - \frac{\gamma \chi_1 n}{d_3} \int_0^{s_0} s^{-b}(s_0 - s) U_s W ds
\]

\[
- \mu_1 n^{\kappa_1-1} \int_0^{s_0} s^{-b}(s_0 - s) \left( \int_0^s U_s^{\kappa_1}(\sigma, t) d\sigma \right) ds
\]

\[
- a_1 \mu_1 n^{\lambda_1-1} \int_0^{s_0} s^{-b}(s_0 - s) \left( \int_0^s U_s(\sigma, t)V_s^{\lambda_1-1}(\sigma, t) d\sigma \right) ds
\]

\[=: I_1 + I_2 + I_3 + I_4 + I_5 \] (4.17)

for all \( t \in (0, T_{\text{max}}) \). We next show estimates for \( I_1, I_3 \) and \( I_4 \). By arguments similar to those in \[29, \) estimate (4.4) and Lemma 4.3, it follows that

\[
I_1 \geq -c_1 s_0^{\frac{3n+2}{n} - \frac{2}{n}} \sqrt{\psi_U(t)} \] (4.18)

for all \( t \in (0, T_{\text{max}}) \) with some \( c_1 > 0 \). As to \( I_4 \), by Fubini’s theorem we deduce that

\[
I_4 = -\mu_1 n^{\kappa_1-1} \int_0^{s_0} \left( \int_0^s s^{-b}(s_0 - s) ds \right) U_s^{\kappa_1}(\sigma, t) d\sigma
\]

\[
\geq -\mu_1 n^{\kappa_1-1} \int_0^{s_0} \left( \int_0^s s^{-b} ds \right) (s_0 - \sigma) U_s^{\kappa_1}(\sigma, t) d\sigma
\]

\[
geq -\frac{\mu_1 n^{\kappa_1-1}}{1-b} \int_0^{s_0} (s_0 - s) U_s^{\kappa_1}(s, t) ds \] (4.19)

so that by virtue of (4.12) we apply \[29, Lemma 4.5\] to find \( c_2 > 0 \) such that

\[
I_4 \geq -c_2 s_0^{-(n-1)(\kappa_1-1) + \frac{3n+2}{n} - \varepsilon} \sqrt{\psi_U(t)} \] (4.20)

for all \( t \in (0, \min\{1, T_{\text{max}}\}) \). With regard to \( I_3 \), as in \[29, \) estimate (4.5)], by integration by parts we can make sure that

\[
I_3 \geq -(b+1) \frac{\gamma \chi_1 n}{d_3} \int_0^{s_0} s^{-b-1} UW ds \] (4.21)
for all \( t \in (0, T_{\text{max}}) \). By (4.17), (4.18), (4.20) and (4.21) we arrive at (4.13). The inequality (4.14) is derived by a similar argument. \( \square \)

We next present an estimate for the integrals involving \( W \) in (4.13) and (4.14). Since the following lemma has been obtained in [26, Lemma 4.4], we provide only the statement of the lemma.

**Lemma 4.4.** Let \( s_0 \in (0, R^n) \) and \( b \in (0, \min\{1, 2 - \frac{4}{n}\}) \). For all \( L > 0 \) and \( M_0 > 0 \) there exists \( C > 0 \) such that if \( u_0, v_0 \) satisfy (2.1), (4.2) and \( \int_{\Omega}(u_0(x) + v_0(x)) \, dx = M_0 \) as well as (4.5), then

\[
- (b + 1) \frac{\gamma \lambda_1}{d_3} s_0 \int_0^{s_0} s^{-b-1} U(s, t) W(s, t) \, ds - (b + 1) \frac{\gamma \lambda_2}{d_3} s_0 \int_0^{s_0} s^{-b-1} V(s, t) W(s, t) \, ds
\]

\[
\geq - C s_0^{1-b+\frac{2}{n}} - C s_0^\frac{2}{n} (\psi_U(t) + \psi_V(t))
\]

for all \( t \in (0, \min\{1, T_{\text{max}}\}) \).

We estimate the last terms in (4.13) and (4.14).

**Lemma 4.5.** Let \( s_0 \in (0, R^n) \). Assume that \( \lambda_1, \lambda_2 > 1 \) and \( b \in (0, 1) \) satisfy

\[
(n - 1)(\max\{\lambda_1, \lambda_2\} - 1) < \frac{b}{2}.
\]

Then, for all \( L > 0 \), \( M_0 > 0 \) and \( \varepsilon > 0 \) there exist \( C_1 > 0 \) and \( C_2 > 0 \) such that if \( u_0, v_0 \) satisfy (2.1), (4.2) and \( \int_{\Omega}(u_0(x) + v_0(x)) \, dx = M_0 \) as well as (4.5), then

\[
- a_1 \mu_1 n^{\lambda_1 - 1} \int_0^{s_0} s^{-b}(s_0 - s) \left( \int_0^s U_s(\sigma, t)V_{s_0}^{\lambda_1 - 1}(\sigma, t) \, d\sigma \right) \, ds
\]

\[
\geq - C_1 s_0^{-(n-1)(\lambda_1 - 1) + \frac{3 \lambda_1}{2} - \varepsilon} \sqrt{\psi_U(t)}
\]

and

\[
- a_2 \mu_2 n^{\lambda_2 - 1} \int_0^{s_0} s^{-b}(s_0 - s) \left( \int_0^s V_s(\sigma, t)U_{s_0}^{\lambda_2 - 1}(\sigma, t) \, d\sigma \right) \, ds
\]

\[
\geq - C_2 s_0^{-(n-1)(\lambda_2 - 1) + \frac{3 \lambda_2}{2} - \varepsilon} \sqrt{\psi_V(t)}
\]

for all \( t \in (0, \min\{1, T_{\text{max}}\}) \).

**Proof.** It is sufficient to show (4.22), because we can confirm (4.23) by a similar argument. Given \( \varepsilon > 0 \), let us fix \( \eta > 0 \) fulfilling

\[
\frac{\eta}{n} (\lambda_1 - 1) \leq \min\{\varepsilon, 1\} \quad \text{and} \quad (n - 1)(\lambda_1 - 1) + \frac{\eta}{n} (\lambda_1 - 1) < \frac{b}{2}.
\]

Then Lemma 4.2 (with \( \varepsilon = \eta \)) ensures that

\[
v(r, t) \leq c_1 r^{-(n-1) - \eta}
\]

for all \( r \in (0, R) \) and \( t \in (0, \min\{1, T_{\text{max}}\}) \) with some \( c_1 > 0 \). By an argument similar to that in (4.19), we infer from Fubini’s theorem that

\[
- a_1 \mu_1 n^{\lambda_1 - 1} \int_0^{s_0} s^{-b}(s_0 - s) \left( \int_0^s U_s(\sigma, t)V_{s_0}^{\lambda_1 - 1}(\sigma, t) \, d\sigma \right) \, ds
\]

\[
\geq - \frac{a_1 \mu_1 n^{\lambda_1 - 1}}{1 - b} s_0^{1-b} \int_0^{s_0} (s_0 - s) U_s V_{s_0}^{\lambda_1 - 1} \, ds.
\]
Moreover, from (4.24) there exists $c_2 > 0$ such that
\[
V_s^{\lambda_1 - 1}(s, t) = \left( \frac{1}{n} v(s^{\frac{1}{n}}, t) \right)^{\lambda_1 - 1} \leq c_2 s^{-(n-1)(\lambda_1 - 1) - \frac{2}{n}(\lambda_1 - 1)}
\]
for all $s \in (0, R^n)$ and $t \in (0, \min\{1, T_{\text{max}}\})$, and thus we derive that
\[
s_0^{1-b} \int_0^{s_0} (s_0 - s) U_s V_s^{\lambda_1 - 1} ds \leq c_2 s_0^{1-b} \int_0^{s_0} s^{-(n-1)(\lambda_1 - 1) - \frac{2}{n}(\lambda_1 - 1)} (s_0 - s) U_s ds
\]
for all $t \in (0, \min\{1, T_{\text{max}}\})$. Combining this inequality with (4.25) and using an argument similar to that in the proof of [29, Lemma 4.5], we can verify that (4.22) holds.

We can now pass to the proof of Theorem 4.1.

**Proof of Theorem 4.1.** Let us show that there are $b \in \left( 1 - \frac{2}{n}, \min\{1, 2 - \frac{4}{n}\} \right)$, $s_1 \in (0, R^n)$, $c_1 > 0$ and $c_2 > 0$ such that for any $s_0 \in (0, s_1)$,
\[
\phi'_U(t) + \phi'_V(t) \geq c_1 s_0^{-(3-b)} (\phi_U(t) + \phi_V(t))^2 - c_2 s_0^{1-b + \frac{2}{n}}
\]
for all $t \in (0, \min\{1, T_{\text{max}}\})$. By virtue of (4.3), in the case $n = 3$ we can make sure that $(n - 1)(\max\{\kappa_1, \lambda_1, \kappa_2, \lambda_2\} - 1) < \frac{1}{3} = 2 - \frac{2}{n}$, whereas in the case $n \geq 4$ we have $(n - 1)(\max\{\kappa_1, \lambda_1, \kappa_2, \lambda_2\} - 1) < \frac{1}{2}$. Thus we take $b \in \left( 1 - \frac{2}{n}, \min\{1, 2 - \frac{4}{n}\} \right)$ such that
\[
(n - 1)(\max\{\kappa_1, \lambda_1, \kappa_2, \lambda_2\} - 1) < \frac{b}{2}.
\]
Also, let us pick $\varepsilon > 0$ fulfilling
\[
2\varepsilon \leq 1 - \frac{2}{n}.
\]
Then we can apply Lemmas 4.3–4.5 to find $c_3 > 0$ and $c_4 > 0$ such that
\[
\phi'_U(t) + \phi'_V(t) \geq -c_3 s_0^{\frac{3-b}{2} - \frac{\varepsilon}{2}} (\sqrt{\psi_U(t)} + \sqrt{\psi_V(t)})
+ c_4 (\psi_U(t) + \psi_V(t))
- c_3 s_0^{1-b + \frac{2}{n} - \frac{\varepsilon}{2}} \sqrt{\psi_U(t)} + c_3 s_0^{-(n-1)(\kappa_1-1)+\frac{2}{n}-\varepsilon} \sqrt{\psi_U(t)} - c_3 s_0^{-(n-1)(\kappa_2-1)+\frac{2}{n}-\varepsilon} \sqrt{\psi_V(t)}
- c_3 s_0^{-(n-1)(\lambda_1-1)+\frac{2}{n}-\varepsilon} \sqrt{\psi_U(t)} - c_3 s_0^{-(n-1)(\lambda_2-1)+\frac{2}{n}-\varepsilon} \sqrt{\psi_V(t)}
\]
for all $t \in (0, \min\{1, T_{\text{max}}\})$. Moreover, Young’s inequality yields that there exists $c_5 > 0$ such that
\[
\phi'_U(t) + \phi'_V(t) \geq \frac{3}{4} c_4 (\psi_U(t) + \psi_V(t)) - c_3 s_0^{\frac{2}{n}} (\psi_U(t) + \psi_V(t))
- c_5 s_0^{1-b + \frac{2}{n}} \left( \frac{2}{s_0^{\frac{2}{n}}} + 1 + s_0^{2-\frac{2}{n}-2(n-1)(\kappa_1-1)-2\varepsilon} + s_0^{2-\frac{2}{n}-2(n-1)(\kappa_2-1)-2\varepsilon} + s_0^{2-\frac{2}{n}-2(n-1)(\lambda_1-1)-2\varepsilon} + s_0^{2-\frac{2}{n}-2(n-1)(\lambda_2-1)-2\varepsilon} \right)
\]
for all $t \in (0, \min\{1, T_{\text{max}}\})$. Here, we choose $s_1 \in (0, R^n)$ such that $c_3 s_0^{\frac{2}{n}} \leq \frac{c_4}{4}$, and then we see that $c_3 s_0^{\frac{2}{n}} \leq \frac{c_4}{4}$ for all $s_0 \in (0, s_1)$. Since it follows from (4.27), (4.28) and the condition $b < \min\{1, 2 - \frac{4}{n}\}$ that
we infer from the relation \( s_0 < R^n \) that

\[
\phi_U(t) + \phi_V(t) \geq \frac{c_4}{2} (\psi_U(t) + \psi_V(t)) - c_6 s_0^{1-b+\frac{n}{2}}
\]

(4.29)

for all \( s_0 \in (0, s_1) \) and \( t \in (0, \min\{1, T_{\text{max}}\}) \), where

\[
c_6 := c_5 (R^{2n-6} + 1 + R^{2n-2n(n-1)(\kappa_1-1)-2n\varepsilon} + R^{2n-2n(n-1)(\kappa_2-1)-2n\varepsilon})
\]

+ \( R^{2n-2n(n-1)(\lambda_1-1)-2n\varepsilon} + R^{2n-2n(n-1)(\lambda_2-1)-2n\varepsilon} \).

Since [29, Lemma 4.4] provides \( c_7 > 0 \) such that

\[
\psi_U(t) \geq c_7 s_0^{-(3-b)} \phi_U(t) \quad \text{and} \quad \psi_V(t) \geq c_7 s_0^{-(3-b)} \phi_V(t)
\]

for all \( t \in (0, T_{\text{max}}) \), these inequalities together with (4.29) lead to (4.26).

Now let us prove (4.6). We take \( s_0 \in (0, s_1) \) so small such that

\[
\sqrt{\frac{c_2}{c_1}} \frac{1}{s_0} + \frac{2}{c_1} s_0 \leq \frac{\tilde{M}_0}{2^{3-b} \omega_n},
\]

and then the relation

\[
\frac{\tilde{M}_0}{2^{3-b} \omega_n} s_0^{2-b} \geq \sqrt{\frac{c_2}{c_1}} s_0^{2-b+\frac{n}{2}} + \frac{2}{c_1} s_0^{3-b}
\]

holds. Let us put \( r_* := \left( \frac{s_0}{4} \right)^{\frac{1}{2}} \in (0, R) \). Moreover, we suppose that the initial data \( u_0, v_0 \) satisfy (2.1), (4.2), (4.4) and (4.5). Then we can make sure that \( \phi_U(0) + \phi_V(0) \geq \frac{\tilde{M}_0}{2^{3-b} \omega_n} s_0^{2-b} \) by [29, estimate (5.5)]. Thus, as in the proof of [11, Lemma 4.6] (with \( d_1(s_0) = c_1 s_0^{-(3-b)}, d_2(s_0) = c_2 s_0^{1-b+\frac{n}{2}} \) and \( \phi(s_0) = \frac{\tilde{M}_0}{2^{3-b} \omega_n} s_0^{2-b} \)), we can establish that \( T_{\text{max}} \leq \frac{1}{2} \), and hence (4.6) results from (2.2).

**5. Finite-time blow-up in a model of Jäger–Luckhaus type**

In this section we deal with the system (1.1) with \( \lambda_1 = \lambda_2 = 2 \) in the case that \( h \) is of Jäger–Luckhaus type. Namely, we consider the system

\[
\begin{align*}
\frac{u_t}{h} & = d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - w^{\kappa_1-1} - a_1 v), \quad x \in \Omega, \ t > 0, \\
v_t & = d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - w^{\kappa_2-1}), \quad x \in \Omega, \ t > 0, \\
0 & = d_3 \Delta w + \alpha u + \beta v - \frac{1}{|\mathcal{W}|} \int_{\Omega} (\alpha u + \beta v) \, dx, \quad x \in \Omega, \ t > 0, \\
\nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

(5.1)

with

\[
\int_{\Omega} w \, dx = 0, \quad t > 0,
\]

where \( \Omega = B_R(0) \subset \mathbb{R}^n \) \( (n \geq 5) \) is a ball with some \( R > 0 \), and the initial data \( u_0, v_0 \) satisfy the condition (2.1) and

\[
u_0, v_0 \text{ are radially symmetric and nonincreasing with respect to } |x|.
\]

(5.2)
Let \((u, v, w)\) be the unique local classical solution of (5.1) on \([0, T_{\text{max}}]\) given in Lemma 2.1. Moreover, we put
\[
\mathcal{M}(t) := \frac{1}{|\Omega|} \int_\Omega (\alpha u + \beta v) \, dx.
\]
The main theorem in this section reads as follows.

**Theorem 5.1.** Let \(\Omega = B_R(0) \subset \mathbb{R}^n \) (\(n \geq 5\)) and let \(d_1, d_2, d_3, \chi_1, \chi_2, a_1, a_2, \alpha, \beta > 0\) and \(\kappa_1, \kappa_2 \in (1, 2]\). Assume that \(\mu_1\) and \(\mu_2\) satisfy that
\[
\chi_1 > \begin{cases} \frac{a_1 d_3 \mu_1}{\beta} \cdot \frac{n}{n-4} \max \left\{ \frac{d_3 \mu_1}{\alpha}, \frac{a_1 d_3 \mu_1}{\beta} \right\} \cdot \frac{n}{n-4} & \text{if } \kappa_1 < 2, \\ \frac{a_2 d_3 \mu_2}{\alpha} & \text{if } \kappa_1 = 2 \end{cases} \quad \text{and} \quad \chi_2 > \frac{a_2 d_3 \mu_2}{\alpha}. \tag{5.3}
\]
Then, for all \(M_0 > 0\) and \(\bar{M}_0 \in (0, M_0)\) there exists \(r_* \in (0, R)\) with the following property: If \(u_0, v_0\) satisfy (2.1) and (5.2) as well as
\[
\int_\Omega u_0(x) \, dx = M_0 \quad \text{and} \quad \int_\Omega u_0(x) \, dx \geq \bar{M}_0, \tag{5.4}
\]
then the corresponding solution \((u, v, w)\) of (5.1) blows up in finite time in the sense that \(T_{\text{max}} < \infty\) and
\[
\lim_{t \nearrow T_{\text{max}}} (||u(\cdot, t)||_{L^\infty(\Omega)} + ||v(\cdot, t)||_{L^\infty(\Omega)}) = \infty.
\]

**Remark 5.1.** If \(v = 0, d_1 = d_2 = 1, \chi_1 = 1, \kappa_1 = 2\) and \(\alpha = 1\) in (5.1), then the condition (5.3) is the same as in [11, (1.4b)]. Thus this theorem is a generalization of [11, Theorem 1.1] in the higher-dimensional case.

**Remark 5.2.** If \(\mu_1\) and \(\mu_2\) fulfill that
\[
\chi_1 > \frac{a_1 d_3 \mu_1}{\beta} \quad \text{and} \quad \chi_2 > \begin{cases} \frac{a_2 d_3 \mu_2}{\alpha} \cdot \frac{n}{n-4} \max \left\{ \frac{d_3 \mu_2}{\beta}, \frac{a_2 d_3 \mu_2}{\alpha} \right\} \cdot \frac{n}{n-4} & \text{if } \kappa_2 < 2, \\ \frac{a_2 d_3 \mu_2}{\alpha} & \text{if } \kappa_2 = 2, \end{cases}
\]
then, by replacing \(u_0\) with \(v_0\) in (5.4), a similar result can be derived.

Now, let us define the functions \(U, V\) and \(W\) as in Sect. 4. Furthermore, we introduce the functions \(\phi_U, \phi_V, \psi_U\) and \(\psi_V\) defined in Sect. 4 with \(b \in (1, 2]\).

**Remark 5.3.** As to the functions \(\phi_U\) and \(\phi_V\), even though we have taken \(b \in (1, 2]\) unlike, e.g., [3, 28, 29] and Sect. 4, we see that these functions are well-defined and that \(\phi_U \) and \(\phi_V\) belong to \(C^0([0, T_{\text{max}}]) \cap C^1((0, T_{\text{max}}))\), since we can confirm from the following lemma that \(s^{-b}(s_0 - s)U(s, t) \leq C s^{1-b}(s_0 - s)\) for all \(t \in [0, T_{\text{max}}]\) with some \(C > 0\) and that for any \(t_0 \in (0, T_{\text{max}})\) and \(t_1 \in (t_0, T_{\text{max}})\), \(|\frac{d}{dt} s^{-b}(s_0 - s)U(s, t)| \leq \tilde{C} s^{1-b}(s_0 - s)\) holds for all \(t \in (t_0, t_1)\) with some constant \(\tilde{C} > 0\) (for details see the proof of [11, Lemma 4.1]).

We first show concavity of the mass accumulation functions \(U\) and \(V\), which is obtained by the comparison principle.

**Lemma 5.2.** Assume that (2.1) and (5.2) hold. If \(\mu_1\) and \(\mu_2\) satisfy that
\[
0 < \mu_1 < \frac{\beta \chi_1}{a_1 d_3} \quad \text{and} \quad 0 < \mu_2 < \frac{\alpha \chi_2}{a_2 d_3}, \tag{5.5}
\]
then
\[
U_{ss}(s, t) \leq 0 \quad \text{and} \quad V_{ss}(s, t) \leq 0 \tag{5.6}
\]
and moreover,
\[
U_s(s, t) \leq \frac{U(s, t)}{s} \leq U_s(0, t) \quad \text{and} \quad V_s(s, t) \leq \frac{V(s, t)}{s} \leq V_s(0, t)
\] (5.7)
for all \( s \in (0, R^n) \) and \( t \in [0, T_{\max}) \).

**Proof.** Since (5.7) is proved by using the mean value theorem and (5.6), we need only to show (5.6).

To see this, we establish that \( u_r \leq 0 \) and \( v_r \leq 0 \) for all \( r \in (0, R) \) and \( t \in (0, T_{\max}) \) because we have \( U_{ss}(s, t) = \frac{1}{n} s^{1-n} u_r(s^{\frac{1}{n}}, t) \) and \( V_{ss}(s, t) = \frac{1}{n} s^{1-n} v_r(s^{\frac{1}{n}}, t) \) for all \( s \in (0, R^n) \) and \( t \in (0, T_{\max}) \). By an approximation argument as in Step 2 in the proof of [28, Lemma 2.2], we can assume without loss of generality that \( u_0, v_0 \in C^2(\Omega) \) with \( \nabla u_0 \cdot \nu = \nabla v_0 \cdot \nu = 0 \). Then this assumption guarantees that
\[
u_r, v_r \in C^0([0, R] \times [0, T_{\max})) \cap C^{2,1}((0, R) \times (0, T_{\max})).
\]

From the third equation in (5.1) it follows that
\[
\chi_1 r^{1-n}(r^{n-1} u w_r)_r = \chi_1 u_r w_r + \chi_1 u \cdot r^{1-n}(r^{n-1} w_r)_r
\]
\[
= \chi_1 u_r w_r + \chi_1 u \cdot \left( -\frac{\alpha}{d_3} u - \frac{\beta}{d_3} v + \frac{1}{d_3} \bar{M}(t) \right)
\]
\[
= \chi_1 u_r w_r - 2 \frac{\alpha \chi_1}{d_3} u^2 - \frac{\beta \chi_1}{d_3} u v + \chi_1 \frac{d_3}{d_3} \bar{M}(t) u,
\]
and so we obtain
\[
u_r = (d_1 r^{n-1}(r^{n-1} u w_r)_r - \chi_1 r^{1-n}(r^{n-1} u w_r)_r + \mu_1 u(1 - u_r - a_1 v))_r
\]
\[
= d_1 u_{rr} + d_1 \frac{n-1}{r} u_{rr} - d_1 \frac{n-1}{r^2} u_r
\]
\[
- \chi_1 u_{rr} w_r - \chi_1 u_{rr} w_r + 2 \frac{\alpha \chi_1}{d_3} u_{rr} + \frac{\beta \chi_1}{d_3} u_r v + \frac{\beta \chi_1}{d_3} u w_r - \chi_1 \frac{d_3}{d_3} \bar{M}(t) u_r
\]
\[
+ \mu_1 u_r - \mu_1 \kappa_1 u_{r-1} u_r - a_1 \mu_1 u_r - a_1 \mu_1 u w_r
\]
for all \( r \in (0, R) \) and \( t \in (0, T_{\max}) \). Moreover, this identity is rewritten as
\[
u_r = d_1 u_{rr} + A_1(r, t) u_{rr} + B_1(r, t) u_r + C_1(r, t) v_r,
\] (5.8)
where
\[
A_1(r, t) := d_1 \frac{n-1}{r} - \chi_1 w_r(r, t),
\]
\[
B_1(r, t) := -d_1 \frac{n-1}{r^2} - \chi_1 w_{rr}(r, t) + 2 \frac{\alpha \chi_1}{d_3} u(r, t) + \frac{\beta \chi_1}{d_3} v(r, t) - \chi_1 \frac{d_3}{d_3} \bar{M}(t)
\]
\[
+ \mu_1 - \mu_1 \kappa_1 u_{r-1} u_r - a_1 \mu_1 v(r, t)
\]
and
\[
C_1(r, t) := \frac{\beta \chi_1}{d_3} u(r, t) - a_1 \mu_1 u(r, t)
\]
for \( r \in (0, R) \) and \( t \in (0, T_{\max}) \). Now let us fix \( T \in (0, T_{\max}) \) and denote by \( (u_r)_+ \) and \( (v_r)_+ \) the positive part functions of \( u_r \) and \( v_r \), respectively. Then we have
\[
(u_r)_+(0, t) = (u_r)_+(R, t) = 0,
\] (5.9)
because the regularity and radial symmetry of \( u \) imply that \( (u_r)_+(0, t) = 0 \) and the Neumann boundary condition in (5.1) gives \( (u_r)_+(R, t) = 0 \). In view of (5.8) we can verify from integration by parts and (5.9)
that

\[
\frac{1}{2} \cdot \frac{d}{dt} \int_0^R r (u_r)_+^2 \, dr
\]

\[
= \int_0^R r (u_r)_+ [d_1 u_{rrr} + A_1(r,t) u_{rr} + B_1(r,t) u_r + C_1(r,t) v_r] \, dr
\]

\[
= -d_1 \int_0^R \left( \frac{1}{2} [(u_r)_+]_r + r([u_r]_+)_r \right) \, dr + \frac{1}{2} \int_0^R r A_1(r,t) [(u_r)_+]_r \, dr
\]

\[
+ \int_0^R r B_1(r,t) (u_r)_+^2 \, dr + \int_0^R r C_1(r,t) (u_r)_+ v_r \, dr
\]

\[
\leq \frac{1}{2} \int_0^R r A_1(r,t) [(u_r)_+]_r \, dr + \int_0^R r B_1(r,t) (u_r)_+^2 \, dr + \int_0^R r C_1(r,t) (u_r)_+ v_r \, dr
\]

\[=: J_1 + J_2 + J_3 \tag{5.10} \]

for all \( t \in (0,T) \). Recalling the definition of \( A_1 \), we obtain from (5.9) and integration by parts that

\[
J_1 = \frac{d_1(n-1)}{2} \int_0^R [(u_r)_+]_r \, dr - \chi_1 \int_0^R r w_r [(u_r)_+]_r \, dr
\]

\[
= \frac{d_1(n-1)}{2} \left[ [(u_r)_+(R,t) - (u_r)_+(0,t)] \right]
\]

\[
- \chi_1 R w_r(R,t) \cdot (u_r)_+(R,t) + \chi_1 \int_0^R w_r(u_r)_+^2 \, dr + \chi_1 \int_0^R r w_{rr}(u_r)_+^2 \, dr
\]

\[
\leq \chi_1 \int_0^R w_r(u_r)_+^2 \, dr + \chi_1 \|w_{rr}\|_{L^\infty((0,R) \times (0,T))} \int_0^R r(u_r)_+^2 \, dr.
\]

Here the third equation in (5.1) yields

\[
w_r = -\frac{1}{d_3} r^{1-n} \int_0^r \rho^{n-1} (\alpha u + \beta v) \, d\rho + \frac{1}{d_3 n} M(t) r \leq \frac{1}{d_3 n} M(t) r, \tag{5.11}
\]

which together with Lemma 2.2 means that

\[
J_1 \leq \frac{\chi_1}{d_3 n} M(t) \int_0^R r(u_r)_+^2 \, dr + \chi_1 \|w_{rr}\|_{L^\infty((0,R) \times (0,T))} \int_0^R r(u_r)_+^2 \, dr
\]

\[
\leq c_1 \int_0^R r(u_r)_+^2 \, dr \tag{5.12}
\]
for all $t \in (0, T)$, where
\[
c_1 := \frac{\chi_1}{d_3 \eta n |\Omega|} \left( \alpha e^{\mu_1 T} \int_{\Omega} u_0 \, dx + \beta e^{\mu_2 T} \int_{\Omega} v_0 \, dx \right) + \chi_1 \|w_{rr}\|_{L^\infty((0, R) \times (0, T))}.
\]

We next estimate $J_2$. Noticing from the third equation in (5.1) and (5.11) that
\[
-w_{rr} = \frac{n-1}{r} w_r + \frac{\alpha}{d_3} u + \frac{\beta}{d_3} v - \frac{1}{d_3} \bar{M}(t)
\]
\[
\leq -\frac{1}{d_3 n} \bar{M}(t) + \frac{\alpha}{d_3} u + \frac{\beta}{d_3} v
\]
and using the nonpositivity of $-d_1 \frac{n-1}{r^2}$, $-\frac{\alpha}{d_3} \bar{M}(t)$, $-\mu_1 \kappa_1 u^{\kappa_1-1}$ and $-a_1 \mu_1 v$, we infer that
\[
B_1(r, t) \leq \frac{3\alpha \chi_1}{d_3} u + \frac{2\beta \chi_1}{d_3} v + \mu_1
\]
\[
\leq \frac{3\alpha \chi_1}{d_3} \|u\|_{L^\infty((0, R) \times (0, T))} + \frac{2\beta \chi_1}{d_3} \|v\|_{L^\infty((0, R) \times (0, T))} + \mu_1 =: c_2,
\]
and hence
\[
J_2 \leq c_2 \int_0^R r(u_r)_+^2 \, dr \quad (5.13)
\]
for all $t \in (0, T)$. With regard to $J_3$, the first condition in (5.5) ensures that $\frac{\beta \chi_1}{d_3} - a_1 \mu_1 > 0$, and thereby we have
\[
C_1(r, t) \leq \left( \frac{\beta \chi_1}{d_3} - a_1 \mu_1 \right) \|u\|_{L^\infty((0, R) \times (0, T))}.
\]

Thanks to this inequality, $J_3$ is estimated as
\[
J_3 \leq \left( \frac{\beta \chi_1}{d_3} - a_1 \mu_1 \right) \|u\|_{L^\infty((0, R) \times (0, T))} \int_0^R r(u_r)_+ (v_r)_+ \, dr
\]
\[
\leq c_3 \int_0^R r(u_r)_+^2 \, dr + c_3 \int_0^R r(v_r)_+^2 \, dr \quad (5.14)
\]
for all $t \in (0, T)$, where $c_3 := \frac{1}{2} \left( \frac{\beta \chi_1}{d_3} - a_1 \mu_1 \right) \|u\|_{L^\infty((0, R) \times (0, T))}$. Therefore, a combination of (5.12)–(5.14) with (5.10) implies that
\[
\frac{1}{2} \frac{d}{dt} \int_0^R r(u_r)_+^2 \, dr \leq c_4 \int_0^R r(u_r)_+^2 \, dr + c_3 \int_0^R r(v_r)_+^2 \, dr
\]
for all $t \in (0, T)$, where $c_4 := c_1 + c_2 + c_3$. Similarly, under the second condition in (5.5), we can observe from the second equation in (5.1) that
\[
\frac{1}{2} \frac{d}{dt} \int_0^R r(v_r)_+^2 \, dr \leq c_5 \int_0^R r(u_r)_+^2 \, dr + c_6 \int_0^R r(v_r)_+^2 \, dr
\]
with some $c_3 = c_5(T) > 0$ and $c_6 = c_6(T) > 0$, and thus find $c_7 = c_7(T) > 0$ such that
\[
\frac{1}{2} \cdot \frac{d}{dt} \int_0^R r((u_r)_+^2 + (v_r)_+^2) \, dr \leq c_7 \int_0^R r((u_r)_+^2 + (v_r)_+^2) \, dr
\]
for all $t \in (0, T)$. Since $u_{0r} \leq 0$ and $v_{0r} \leq 0$ for all $r \in [0, R]$ by (5.2), the above inequality provides that $u_r \leq 0$ and $v_r \leq 0$ for all $r \in (0, R)$ and $t \in (0, T)$. Letting $T \nearrow T_{\text{max}}$, we arrive at the conclusion. 

We next establish a differential inequality for $\varphi_U$ by using estimates in [11, Lemmas 4.3 and 4.4].

**Lemma 5.3.** Let $s_0 \in (0, R^n)$ and $b \in (1, 2 - \frac{4}{n})$. Assume that $\mu_1$ and $\mu_2$ satisfy (5.5). Then there exist $C_1 > 0$ and $C_2 > 0$ such that
\[
\varphi_U'(t) \geq -C_1 s_0^{\frac{2-b}{2}} \sqrt{\varphi_U(t)} + \frac{\alpha \chi_1 n}{d_3} \varphi_U(t) + \frac{\beta \chi_1 n}{d_3} \int_0^{s_0} s^{b}(s_0 - s) U_s(s, t)V(s, t) \, ds
\]
\[
- \frac{C_2 s_0^{2-b}}{4} \sqrt{\varphi_U(t)} - \mu_1 n \kappa_1^{-1} \int_0^{s_0} s^{b}(s_0 - s) \left( \int_0^{s} U_s^{\kappa_1}(\sigma, t) \, d\sigma \right) \, ds
\]
\[
- a_1 \mu_1 n \int_0^{s_0} s^{b}(s_0 - s) \left( \int_0^{s} U_s(\sigma, t)V_s(\sigma, t) \, d\sigma \right) \, ds
\]
(5.15)

for all $t \in (0, \min\{1, T_{\text{max}}\})$.

**Proof.** Since we can verify from the third equation in (5.1) that
\[
s^{1-b} w_r(s^{\frac{b}{2}}, t) = -\frac{\alpha}{d_3} U(s, t) - \frac{\beta}{d_3} V(s, t) + \frac{1}{d_3 n} \mathcal{M}(t)s
\]
for all $s \in (0, R^n)$ and $t \in (0, T_{\text{max}})$, we obtain from (4.15) and this identity that
\[
U_t = d_1 n^2 s^{2-b} U_{ss} + \frac{\alpha \chi_1 n}{d_3} UU_s + \frac{\beta \chi_1 n}{d_3} U_s V - \frac{\chi_1}{d_3} \mathcal{M}(t)s U_s
\]
\[
+ \mu_1 U - \mu_1 n \kappa_1^{-1} \int_0^{s} U_s^{\kappa_1}(\sigma, t) \, d\sigma - a_1 \mu_1 n \int_0^{s} U_s(\sigma, t)V_s(\sigma, t) \, d\sigma
\]
for all $s \in (0, R^n)$ and $t \in (0, T_{\text{max}})$. In view of this identity we deduce that
\[
\varphi_U'(t) \geq d_1 n^2 \int_0^{s_0} s^{2-b}(s_0 - s) U_{ss} \, ds + \frac{\alpha \chi_1 n}{d_3} \varphi_U(t)
\]
\[
+ \frac{\beta \chi_1 n}{d_3} \int_0^{s_0} s^{b}(s_0 - s) U_s V \, ds - \frac{\chi_1}{d_3} \mathcal{M}(t) \int_0^{s_0} s^{1-b}(s_0 - s) U_s \, ds
\]
\[
- \mu_1 n \kappa_1^{-1} \int_0^{s_0} s^{b}(s_0 - s) \left( \int_0^{s} U_s^{\kappa_1}(\sigma, t) \, d\sigma \right) \, ds
\]
\[
- a_1 \mu_1 n \int_0^{s_0} s^{b}(s_0 - s) \left( \int_0^{s} U_s(\sigma, t)V_s(\sigma, t) \, d\sigma \right) \, ds
\]
=: \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5 + \tilde{I}_6
\]
(5.16)
for all \( t \in (0, T_{\text{max}}) \). Let us next show estimates for \( \tilde{I}_1 \) and \( \tilde{I}_4 \). With regard to \( \tilde{I}_1 \), owing to (5.5) we can employ (5.7), and so by an argument similar to that in the proof of [11, Lemma 4.3] we provide \( c_1 > 0 \) such that

\[
\tilde{I}_1 \geq -c_1 s_0^{\frac{3-b}{2}} \frac{1}{2} \sqrt{\psi_U(t)}
\]  

(5.17)

for all \( t \in (0, T_{\text{max}}) \). As to \( \tilde{I}_4 \), arguing as in [29, Lemma 4.3], we can take \( c_2 > 0 \) such that

\[
\psi_U(t) \geq c_2 s_0^{-(3-b)} \phi_U^2(t)
\]  

(5.18)

for all \( t \in (0, T_{\text{max}}) \), so that we rely on (5.7) and this inequality to see that

\[
\tilde{I}_4 \geq -\frac{\chi_1}{d_3} M(t) \phi_U(t) \geq -\frac{\chi_1}{c_2 d_3} \frac{2}{2} M(t) \sqrt{\psi_U(t)}
\]  

(5.19)

for all \( t \in (0, T_{\text{max}}) \). Moreover, it follows from Lemma 2.2 that for all \( t \in (0, \min\{1, T_{\text{max}}\}) \),

\[
M(t) \leq \alpha e^{\mu_1} \int_\Omega u_0 \, dx + \beta e^{\mu_2} \int_\Omega v_0 \, dx,
\]

which in conjunction with (5.19) yields

\[
\tilde{I}_4 \geq -c_3 s_0^{\frac{b-2}{2}} \sqrt{\psi_U(t)}
\]  

(5.20)

for all \( t \in (0, \min\{1, T_{\text{max}}\}) \), where \( c_3 := \frac{\chi_1}{\sqrt{c_2 d_3}} (\alpha e^{\mu_1} \int_\Omega u_0 \, dx + \beta e^{\mu_2} \int_\Omega v_0 \, dx) \). Accordingly, a combination of (5.16), (5.17) and (5.20) entails (5.15). \( \square \)

Next, we estimate the fifth term on the right-hand side of (5.15). The following lemma generalizes [11, (i) in Lemma 4.2].

**Lemma 5.4.** Let \( s_0 \in (0, R^\alpha) \) and \( b \in (1, 2) \). Assume that \( \mu_1 \) and \( \mu_2 \) satisfy (5.5). Then

\[
-\mu_1 n^{\kappa_1-1} \int_0^{s_0} s^{-b}(s_0 - s) \left( \int_0^s U_s^{\kappa_1}(\sigma, t) \, d\sigma \right) \, ds \geq \begin{cases} 
-\frac{C s_0^{-(3-b)\kappa_1}}{2 \kappa_1} \psi_U^2(t) & \text{if } \kappa_1 \in (1, 2), \\
-\frac{\mu_1 n}{b-1} \psi_U(t) & \text{if } \kappa_1 = 2
\end{cases}
\]

for all \( t \in (0, T_{\text{max}}) \) with some \( C > 0 \).

**Proof.** In the case \( \kappa_1 = 2 \) this lemma can be obtained by the same argument as in the proof of [11, (i) in Lemma 4.2]. Therefore we consider the case \( \kappa_1 \in (1, 2) \). Noting that \( b > 1 \), by Fubini’s theorem we have

\[
-\mu_1 n^{\kappa_1-1} \int_0^{s_0} s^{-b}(s_0 - s) \left( \int_0^s U_s^{\kappa_1}(\sigma, t) \, d\sigma \right) \, ds
\]

\[
= -\mu_1 n^{\kappa_1-1} \int_0^{s_0} \left( \int_{\sigma}^s s^{-b}(s_0 - s) \, ds \right) U_s^{\kappa_1}(\sigma, t) \, d\sigma
\]

\[
\geq -\mu_1 n^{\kappa_1-1} \int_0^{s_0} \left( \int_{\sigma}^s s^{-b} \, ds \right) (s_0 - \sigma) U_s^{\kappa_1}(\sigma, t) \, d\sigma
\]

\[
\geq -\frac{\mu_1 n^{\kappa_1-1}}{b-1} \int_0^{s_0} s^{1-b}(s_0 - s) U_s^{\kappa_1}(s, t) \, ds
\]  

(5.21)
for all \( t \in (0, T_{\text{max}}) \). Since we can apply Lemma 5.2 by the condition (5.5), we know that \( U_s(s, t) \leq s^{-1}U(s, t) \) for all \( s \in (0, R^n) \) and \( t \in (0, T_{\text{max}}) \), and hence it follows that

\[
- \frac{\mu_1 n^{\kappa_1 - 1}}{b - 1} \int_0^{s_0} s^{1-b}(s_0 - s)U_s^{\kappa_1} \, ds \geq - \frac{\mu_1 n^{\kappa_1 - 1}}{b - 1} \int_0^{s_0} s^{1-\frac{\kappa_1}{2} - b}(s_0 - s)(U_U)^{\frac{\kappa_1}{2}} \, ds.
\]

(5.22)

Moreover, by using H"older’s inequality we can deduce that

\[
\int_0^{s_0} s^{1-\frac{\kappa_1}{2} - b}(s_0 - s)(U_U)^{\frac{\kappa_1}{2}} \, ds
\]

\[
\leq \left( \int_0^{s_0} s^{1-b}(s_0 - s) \, ds \right)^{1 - \frac{\kappa_1}{2}} \left( \int_0^{s_0} s^{-b}(s_0 - s)U_U \, ds \right)^{\frac{\kappa_1}{2}}
\]

\[
= c_1 s_0^{(3-b) \frac{2-\kappa_1}{2} - b} \psi_{U_U}^{\frac{\kappa_1}{2}}(t)
\]

(5.23)

for all \( t \in (0, T_{\text{max}}) \), where \( c_1 := \left( \frac{1}{2-b(3-b)} \right)^{(3-b) \frac{2-\kappa_1}{2}} \). A combination of (5.21)–(5.23) leads to the conclusion of this lemma. \( \square \)

As a final preparation for the proof of Theorem 5.1, we show that the sixth term on the right-hand side of (5.15) is estimated by the third term.

**Lemma 5.5.** Let \( s_0 \in (0, R^n) \) and \( b \in (1, 2) \). Assume that \( \mu_1 \) and \( \mu_2 \) satisfy (5.5). Then

\[
- a_1 \mu_1 n \int_0^{s_0} s^{-b}(s_0 - s) \left( \int_0^s U_s(\sigma, t)V_s(\sigma, t) \, d\sigma \right) \, ds
\]

\[
\geq - \frac{a_1 \mu_1 n}{b - 1} \int_0^{s_0} s^{-b}(s_0 - s)U_s(s, t)V(s, t) \, ds
\]

(5.24)

for all \( t \in (0, T_{\text{max}}) \).

**Proof.** As in the proof of Lemma 5.4, Fubini’s theorem yields

\[
- a_1 \mu_1 n \int_0^{s_0} s^{-b}(s_0 - s) \left( \int_0^s U_s(\sigma, t)V_s(\sigma, t) \, d\sigma \right) \, ds
\]

\[
\geq - \frac{a_1 \mu_1 n}{b - 1} \int_0^{s_0} s^{-b}(s_0 - s)U_s(s, t)V_s(s, t) \, ds
\]

for all \( t \in (0, T_{\text{max}}) \). Also, from Lemma 5.2 we obtain that \( V_s(s, t) \leq s^{-1}V(s, t) \) for all \( s \in (0, R^n) \) and \( t \in (0, T_{\text{max}}) \), which together with the above inequality implies (5.24). \( \square \)

We are now in the position to prove Theorem 5.1.

**Proof of Theorem 5.1.** We first find \( b \in \left( 1, 2 - \frac{4}{n} \right) \) that there exist \( c_1 > 0 \) and \( c_2 > 0 \) such that for any \( s_0 \in (0, R^n) \),

\[
\phi'_U(t) \geq c_1 s_0^{(3-b) - 1} \phi_U^2(t) - c_2 s_0^{3-b-\frac{4}{n}}
\]

(5.25)
for all \( t \in (0, \min\{1, T_{\text{max}}\}) \). To see this, we first consider the case \( \kappa_1 \in (1, 2) \). In this case, the condition for \( \chi_1 \) in (5.3) implies

\[
2 - \frac{4}{n} > 1 + \frac{a_1 d_3 \mu_1}{\beta \chi_1}.
\]  

(5.26)

Thus we can choose \( b_1 \in \left(1 + \frac{a_1 d_3 \mu_1}{\beta \chi_1}, 2 - \frac{4}{n}\right) \). Then, applying Lemmas 5.3–5.5, we have

\[
\phi_U'(t) \geq -c_3 s_0^{\frac{3-b_1}{2}} \sqrt{\psi_U(t)} + \frac{\alpha x_1 n}{d_3} \psi_U(t) + \left(\frac{\beta x_1 n}{d_3} - \frac{a_1 \mu_1 n}{b_1 - 1}\right) \int_0^{s_0} s^{-b_1} (s_0 - s) U_s V ds
\]

\[
- c_3 s_0^{\frac{3-b_1}{2}} \sqrt{\psi_U(t)} - c_3 s_0 (3-b_1)^{\frac{2-s_1}{2}} \psi_U'(t)
\]

(5.27)

for all \( t \in (0, \min\{1, T_{\text{max}}\}) \) with some \( c_3 > 0 \). Noting from the condition \( b_1 > 1 + \frac{a_1 d_3 \mu_1}{\beta \chi_1} \) that

\[
\left(\frac{\beta x_1 n}{d_3} - \frac{a_1 \mu_1 n}{b_1 - 1}\right) \int_0^{s_0} s^{-b_1} (s_0 - s) U_s V ds \geq 0
\]

(5.28)

and using Young’s inequality to the first, fourth and fifth terms on the right-hand side of (5.27), we obtain \( c_4 > 0 \) such that

\[
\phi_U'(t) \geq \frac{\alpha x_1 n}{2 d_3} \psi_U(t) - c_4 s_0^{3-b_1-\frac{4}{n}} (1 + 2 s_0^\frac{4}{n})
\]

for all \( t \in (0, \min\{1, T_{\text{max}}\}) \). Hence, from (5.18) and the fact that \( s_0 < R^n \) it follows that

\[
\phi_U'(t) \geq c_5 s_0 (3-b_1) \phi_U(t) - c_4 \left(1 + 2 R^4\right) s_0^{3-b_1-\frac{4}{n}}
\]

for all \( t \in (0, \min\{1, T_{\text{max}}\}) \) with some \( c_5 > 0 \), which means that (5.25) holds with \( c_1 = c_5 \) and \( c_2 = c_4 \left(1 + 2 R^4\right) \). Next, we consider the case \( \kappa_1 = 2 \). In the light of the condition for \( \chi_1 \) in (5.3) we can make sure that (5.26) and the relation \( 2 - \frac{4}{n} > 1 + \frac{d_3 \mu_1}{\alpha x_1}, \) hold, and thereby take \( b_2 \in \left(1 + \max\left\{\frac{a_1}{\beta}, \frac{1}{\alpha}\right\}, \frac{d_3 \mu_1}{\chi_1}, 2 - \frac{4}{n}\right) \). By means of Lemmas 5.3–5.5, there exists \( c_6 > 0 \) such that
\[ \phi'_U(t) \geq -c_6 s_0^{-\frac{3-b_2}{2}} \frac{2}{\pi} \sqrt{\psi_U(t)} + \left( \frac{\alpha \chi_1 n}{d_3} - \frac{\mu_1 n}{b_2 - 1} \right) \psi_U(t) \]

\[ + \left( \frac{\beta \chi_1 n}{d_3} - \frac{a_1 \mu_1 n}{b_2 - 1} \right) \int_0^{s_0} s^{-b_2} (s_0 - s) U_s V ds \]

\[ - c_6 s_0^{-\frac{3-b_2}{2}} \sqrt{\psi_U(t)} \]

for all \( t \in (0, \min\{1, T_{\text{max}}\}) \). By the condition \( b_2 > 1 + \max \left\{ \frac{a_1}{\beta}, \frac{1}{\alpha} \right\} \cdot \frac{d_3 \mu_1}{\chi_1} \), the identity (5.28) with \( b_1 \) replaced with \( b_2 \) holds. Furthermore, we deduce that

\[ \frac{\alpha \chi_1 n}{d_3} - \frac{\mu_1 n}{b_2 - 1} > 0. \]

Therefore, as in the case \( \kappa_1 \in (1, 2) \), Young’s inequality and (5.18) as well as the fact that \( s_0 < R^n \) provide that

\[ \phi'_U(t) \geq c_7 s_0^{\frac{3-b_2}{2}} \phi_U^2(t) - c_8 s_0^{\frac{3-b_2}{2}} \]

for all \( t \in (0, \min\{1, T_{\text{max}}\}) \) with some \( c_7 > 0 \) and \( c_8 > 0 \).

We next show that \( T_{\text{max}} \) is finite. Now we pick \( b \in (1, 2 - \frac{4}{n}) \) such that (5.25) holds. Let us fix \( s_0 \in (0, R^n) \) so small satisfying

\[ \sqrt{\frac{c_2}{c_1}} s_0^{1-\frac{2}{n}} + \frac{2}{c_1} s_0 \leq \frac{\tilde{M}_0}{2^{3-b} \omega_n}, \]

and hence

\[ \frac{\tilde{M}_0}{2^{3-b} \omega_n} s_0^{2-b} \geq \sqrt{\frac{c_2}{c_1}} s_0^{3-b-\frac{2}{n}} + \frac{2}{c_1} s_0^{3-b}. \]

Then, putting \( r_* := \left( \frac{s_0}{4} \right)^{\frac{1}{n}} \in (0, R) \) and taking the initial data \( u_0 \) with (2.1), (5.2) and (5.4), we obtain that \( T_{\text{max}} < \infty \) as in the proof of Theorem 4.1, which means the end of the proof.

\[ \square \]

Acknowledgements

The authors would like to express thanks to the referees for their careful reading and helpful comments.

Author contributions Y. Tanaka wrote the main manuscript. M. Mizukami and T. Yokota directed this project. All authors reviewed the manuscript.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

[1] Bai, X., Winkler, M.: Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics. Indiana Univ. Math. J. 65, 553–583 (2016)

[2] Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M.: Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues. Math. Model Method Appl. Sci. 25, 1663–1763 (2015)
[3] Black, T., Fuest, M., Lankeit, J.: Relaxed parameter conditions for chemotactic collapse in logistic-type parabolic-elliptic Keller-Segel systems. Z. Angew. Math. Phys. 72, 96 (2021)
[4] Black, T., Lankeit, J., Mizukami, M.: On the weakly competitive case in a two-species chemotaxis model. IMA J. Appl. Math. 81, 860–876 (2016)
[5] Brown, P.N.: Decay to uniform states in ecological interactions. SIAM J. Appl. Math. 38, 22–37 (1980)
[6] Cieślak, T., Winkler, M.: Finite-time blow-up in a quasilinear system of chemotaxis. Nonlinearity 21, 1057–1076 (2008)
[7] Conway, E.D., Smoller, J.A.: A comparison technique for systems of reaction-diffusion equations. Commun. Partial Differ. Equ. 2, 679–697 (1977)
[8] de Mottoni, P., Rothe, F.: Convergence to homogeneous equilibrium state for generalized Volterra-Lotka systems with diffusion. SIAM J. Appl. Math. 37, 648–663 (1979)
[9] Friedman, A.: Partial Differential Equations. Holt, Rinehart and Winston Inc, New York-Montreal, Que-London (1969)
[10] Fuest, M.: Blow-up profiles in quasilinear fully parabolic Keller-Segel systems. Nonlinearity 33, 2306–2334 (2020)
[11] Fuest, M.: Approaching optimality in blow-up results for Keller-Segel systems with logistic-type dampening. NoDEA Nonlinear Differ. Equ. Appl. 28, 16 (2021)
[12] Jäger, W., Luckhaus, S.: On explosions of solutions to a system of partial differential equations modelling chemotaxis. Trans. Am. Math. Soc. 329, 819–824 (1992)
[13] Kan-on, Y., Yanagida, E.: Existence of non-constant stable equilibria in competition diffusion equations. Hiroshima Math. J. 23, 193–221 (1993)
[14] Keller, E.F., Segel, L.A.: Traveling bands of chemotactic bacteria: a theoretical analysis. J. Theor. Biol. 30, 235–248 (1971)
[15] Keller, E.F., Segel, L.A.: Initiation of slime mold aggregation viewed as an instability. J. Theor. Biol. 26, 399–415 (1970)
[16] Kishimoto, K., Weinberger, H.F.: The spatial homogeneity of stable equilibria of some reaction-diffusion systems on convex domains. J. Differ. Equ. 58, 15–21 (1985)
[17] Lin, K., Mu, C., Wang, L.: Boundedness in a two-species chemotaxis system. Math. Method Appl. Sci. 38, 5085–5096 (2015)
[18] Lou, Y., Ni, W.-M.: Diffusion, self-diffusion and cross-diffusion. J. Differ. Equ. 131, 79–131 (1996)
[19] Matano, H., Mimura, M.: Pattern formation in competition-diffusion systems in nonconvex domains. Publ. Res. Inst. Math. Sci. 19, 1049–1079 (1983)
[20] Mizukami, M.: Boundedness and asymptotic stability in a two-species chemotaxis-competition model with signal-dependent sensitivity. Discret. Contin. Dyn. Syst. Ser. B 22, 2301–2319 (2017)
[21] Mizukami, M.: Boundedness and stabilization in a two-species chemotaxis-competition system of parabolic-parabolic-elliptic type. Math. Meth. Appl. Sci. 41, 234–249 (2018)
[22] Murray, J.D.: Mathematical Biology. II Spatial Models and Biomedical Applications, Interdisciplinary Applied Mathematics, 3rd edn. Springer, New York (2003)
[23] Stinner, C., Tello, J.I., Winkler, M.: Competitive exclusion in a two-species chemotaxis model. J. Math. Biol. 68, 1607–1626 (2014)
[24] Tello, J.I., Winkler, M.: Stabilization in a two-species chemotaxis system with a logistic source. Nonlinearity 25, 1413–1425 (2012)
[25] Tello, J.I., Winkler, M.: A chemotaxis system with logistic source. Commun. Partial Differ. Equ. 32, 849–877 (2007)
[26] Tu, X., Qiu, S.: Finite-time blow-up and global boundedness for chemotaxis system with strong logistic dampening. J. Math. Anal. Appl. 486, 123876 (2020)
[27] Winkler, M.: Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction. J. Math. Anal. Appl. 384, 261–272 (2011)
[28] Winkler, M.: A critical blow-up exponent in a chemotaxis system with nonlinear signal production. Nonlinearity 31, 2031–2056 (2018)
[29] Winkler, M.: Finite-time blow-up in low-dimensional Keller-Segel systems with logistic-type superlinear degradation. Z. Angew. Math. Phys. 69, 40 (2018)
Yuya Tanaka and Tomomi Yokota  
Department of Mathematics  
Tokyo University of Science  
1-3, Kagurazaka, Shinjuku-ku  
Tokyo 162-8601  
Japan  
e-mail: yuya.tns.6308@gmail.com  

Tomomi Yokota  
e-mail: yokota@rs.tus.ac.jp  

(Received: July 29, 2022; revised: October 5, 2022; accepted: October 7, 2022)