Abstract

A cograph is a simple graph which contains no path on 4 vertices as an induced subgraph. The vicinal preorder on the vertex set of a graph is defined in terms of inclusions among the neighborhoods of vertices. The minimum number of chains with respect to the vicinal preorder required to cover the vertex set of a graph $G$ is called the Dilworth number of $G$. We prove that for any cograph $G$, the multiplicity of any eigenvalue $\lambda \neq 0, -1$, does not exceed the Dilworth number of $G$ and show that this bound is tight. G. F. Royle [The rank of a cograph, Electron. J. Combin. 10 (2003), Note 11] proved that if a cograph $G$ has no pair of vertices with the same neighborhood, then $G$ has no 0 eigenvalue, and asked if beside cographs, there are any other natural classes of graphs for which this property holds. We give a partial answer to this question by showing that an $H$-free family of graphs has this property if and only if it is a subclass of the family of cographs. A similar result is also shown to hold for the $-1$ eigenvalue.

Keywords: Cograph, Eigenvalue, Dilworth number, Threshold graph

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1 Introduction

A cograph is a simple graph which contains no path on four vertices as an induced subgraph. The family of cographs is the smallest class of graphs that includes the single-vertex graph and is closed under complementation and disjoint union. This property justifies the name ‘cograph’
standing for ‘complement reducible graph’ which was coined in [7]. However, this family of graphs was initially defined under different names [16, 17, 20, 21, 29, 31] and since then has been intensively studied. It is well known that any cograph has a canonical tree representation, called a cotree. This tree decomposition scheme of cographs is a particular case of the modular decomposition [12] that applies to arbitrary graphs. Partly because of this property, cographs are interesting from the algorithmic point of view (see [3, p. 175]). As pointed out in [28], cographs have numerous applications in areas like parallel computing [26] or even biology [11] since they can be used to model series-parallel decompositions. For an account on properties of cographs see [3].

Cographs have also been studied from an algebraic point of view. Based on a computer search, Sillke [30] conjectured that the rank of the adjacency matrix of any cograph is equal to the number of distinct non-zero rows in this matrix. The conjecture was proved by Royle [27]. Since then alternative proofs and extensions of this result have appeared [2, 5, 15, 28]. Furthermore, in [19] an algorithm is introduced for locating eigenvalues of cographs in a given interval. In [13], we present a new characterization of cographs; namely a graph \( G \) is a cograph if and only if no induced subgraph of \( G \) has an eigenvalue in the interval \((-1, 0)\). In [13], it is also shown that the multiplicity of any eigenvalue of a cograph \( G \) does not exceed the total number of duplication and coduplication classes of \( G \) (see Section 2 for definitions) which is not greater than the sum of the multiplicities of 0 and \(-1\) as eigenvalues of \( G \).

In this paper we explore further properties of the eigenvalues (of the adjacency matrix) of a cograph. We consider a relation on the vertex set of a graph \( G \) as follows. We define \( u \prec v \) if the open neighborhood of \( u \) is contained in the closed neighborhood of \( v \). It turns out that ‘\( \prec \)’ is a preorder (that is reflexive and transitive) which is called the vicinal preorder. The minimum number of chains with respect to the vicinal preorder required to cover the vertex set of a graph \( G \) is called the Dilworth number of \( G \). In Section 3, we prove that for any cograph \( G \), the multiplicity of any eigenvalue \( \lambda \neq 0, -1 \), does not exceed the Dilworth number of \( G \) and show that this bound is best possible. This was first conjectured in [13]. In [27], Royle proved that if a cograph \( G \) has no duplications, then \( G \) has no 0 eigenvalue, and asked if beside cographs, there are any other natural classes of graphs for which this property holds. In Section 4, we give a partial answer to this question by showing that an \( H \)-free family of graphs \( \mathcal{F} \) has this property if and only if \( \mathcal{F} \) is a subclass of the family of cographs. A similar result is also shown to hold for the \(-1\) eigenvalue. It is also observed that these results can be stated in terms of the existence of a basis consisting of weight 2 vectors for the eigenspace of either 0 or \(-1\) eigenvalues.

2 Preliminaries

In this section we introduce the notations and recall a basic fact which will be used frequently. The graphs we consider are all simple and undirected. For a graph \( G \), we denote by \( V(G) \) the vertex set of \( G \). For two vertices \( u, v \), by \( u \sim v \) we mean \( u \) and \( v \) are adjacent. If \( V(G) = \)
\{v_1, \ldots, v_n\}, then the adjacency matrix of \(G\) is an \(n \times n\) matrix \(A(G)\) whose \((i, j)\)-entry is 1 if \(v_i \sim v_j\) and 0 otherwise. By eigenvalues and rank of \(G\) we mean those of \(A(G)\). The multiplicity of an eigenvalue \(\lambda\) of \(G\) is denoted by \(\text{mult}(\lambda, G)\). For a vertex \(v\) of \(G\), let \(N_G(v)\) denote the open neighborhood of \(v\), i.e. the set of vertices of \(G\) adjacent to \(v\) and \(N_G[v] = N_G(v) \cup \{v\}\) denote the closed neighborhood of \(v\); we will drop the subscript \(G\) when it is clear from the context. Two vertices \(u\) and \(v\) of \(G\) are called duplicates if \(N(u) = N(v)\) and called coduplicates if \(N[u] = N[v]\). Note that duplicate vertices cannot be adjacent while coduplicate vertices must be adjacent. We write \(u \equiv v\) if \(u\) and \(v\) are either duplicates or coduplicates. A subset \(S\) of \(V(G)\) such that \(N(u) = N(v)\) for any \(u, v \in S\) is called a duplication class of \(G\). Coduplication classes are defined analogously. If \(X \subset V(G)\), we use the notation \(G - X\) to mean the subgraph of \(G\) induced by \(V(G) \setminus X\).

An important subclass of cographs are threshold graphs. These are the graphs which are both a cograph and a split graph (i.e. their vertex sets can be partitioned into a clique and a coclique). For more information see [3, 23].

Remark 1. (Sum rule) Let \(x\) be an eigenvector for eigenvalue \(\lambda\) of a graph \(G\). Then the entries of \(x\) satisfy the following equalities:

\[ \lambda x(v) = \sum_{u \sim v} x(u), \quad \text{for all } v \in V(G). \] (1)

From this it is seen that if \(\lambda \neq 0\), then \(x\) is constant on each duplication class and if \(\lambda \neq -1\), then \(x\) is constant on each coduplication class.

3 Dilworth number and multiplicity of eigenvalues

In this section we recall a preorder on the vertex set of a graph which is defined in terms of open/closed neighborhoods of vertices. In [13], it was conjectured that the multiplicity of eigenvalues of any cograph except for 0, \(-1\) is bounded above by the maximum size of an antichain with respect to this preorder. We prove this conjecture in this section.

Let \(G\) be a graph and consider the following relation on \(V(G)\):

\[ u \prec v \text{ if and only if } \begin{cases} N[u] \subseteq N[v] & \text{if } u \sim v, \\ N(u) \subseteq N(v) & \text{if } u \not\sim v, \end{cases} \]

or equivalently

\[ u \prec v \text{ if and only if } N(u) \subseteq N[v]. \]

It is easily verified that \(\prec\) is a preorder that is reflexive and transitive [10]. This preorder is called vicinal preorder. Note that \(\prec\) is not antisymmetric, since \(u \prec v\) and \(v \prec u\) imply only \(u \equiv v\).

We consider the chains and antichains in \(G\) with respect to the vicinal preorder. The minimum number of chains with respect to the vicinal preorder required to cover \(V(G)\) is called the
Figure 1: A threshold graph: $V_i$’s are cliques, $U_i$’s are cocliques, each thick line indicates the edge set of a complete bipartite subgraph on some $U_i, V_j$

Dilworth number of $G$ and denoted by $\nabla(G)$. This parameter was first introduced in [10] (see also [3]). Also we refer to [23, Chapter 9] for several interesting results on Dilworth number and its connection with other graph theoretical concepts. Note that by Dilworth’s theorem, $\nabla(G)$ is equal to the maximum size of an antichain of $V(G)$ with respect to the vicinal preorder.

Remark 2. (Structure of threshold graphs) As it was observed in [24] (see also [1, 14]), the vertices of any threshold graph $G$ can be partitioned into $t$ non-empty coduplication classes $V_1, \ldots, V_t$ and $t$ non-empty duplication classes $U_1, \ldots, U_t$ such that the vertices in $V_1 \cup \cdots \cup V_t$ form a clique and $N(u) = V_1 \cup \cdots \cup V_i$ for any $u \in U_i$, $1 \leq i \leq t$.

For an illustration of this structure with $t = 5$, see Figure 1.

From the structure of threshold graphs it is clear that if $G$ is a threshold graph, then $\nabla(G) = 1$. In [6, 10], it was observed that the converse is also true. We give its simple argument here.

If $\nabla(G) = 1$, then all the vertices of $G$ form a chain $v_1 \prec v_2 \prec \cdots \prec v_n$. First note that $v_i \sim v_{i+1} \neq v_{i+2}$ is impossible for any $i$; since otherwise $N[v_i] \subseteq N[v_{i+1}]$ and $N(v_{i+1}) \subseteq N(v_{i+2})$. Hence, $v_i \in N(v_{i+1}) \cap N(v_{i+2})$, and thus $v_{i+2} \in N(v_i) \subset N[v_{i+1}]$ which means $v_{i+1} \sim v_{i+2}$, a contradiction. So there must exist some $j$ such that $v_1 \nsim \cdots \nsim v_j \sim \cdots \sim v_n$. It turns out that $G$ is a split graph as the vertices $v_1, \ldots, v_j$ form a coclique and the vertices $v_{j+1}, \ldots, v_n$ form a clique. Note that any induced subgraph of $G$ has Dilworth number 1. As $\nabla(P_4) = 2$, $G$ has no induced subgraph $P_4$, so $G$ is also a cograph which in turn implies that $G$ is a threshold graph.

The next result shows that when vicinal preorder is a total order, i.e. the whole $V(G)$ itself is a chain, a strong constraint is imposed on the multiplicities of the eigenvalues. This result was first proved in [18] (see [13] for a simpler proof).

Lemma 3. ([18]) Let $G$ be a threshold graph. Then any eigenvalue $\lambda \neq 0, -1$ is simple.
Motivated by this result, we investigated the connection between eigenvalue multiplicity and Dilworth number of cographs (as an extension of threshold graphs). As the vertices of any graph $G$ can be partitioned into $\nabla(G)$ chains and so into $\nabla(G)$ threshold subgraphs, we conjectured that the Dilworth number is an upper bound for the multiplicity of any eigenvalues $\lambda \neq 0, -1$ in cographs. We prove this conjecture in the next theorem. Before that we present the following crucial lemma on the structure of cographs.

**Lemma 4.** Let $G$ be a cograph with Dilworth number $k \geq 2$. Then there exists a vertex-partition of $G$ into $k$ threshold graphs such that one of the threshold graphs $H$ of the partition has the property that all the vertices of $H$ have the same neighborhood in $G - V(H)$.

**Proof.** Note that $N_G(u) \subseteq N_G[v]$ if and only if $N_G(v) \subseteq N_G[u]$. It follows if we have $u_1 \prec u_2 \prec \cdots \prec u_\ell$ in $G$, then we have $u_\ell \prec \cdots \prec u_2 \prec u_1$ in $\overline{G}$. This in particular implies that $\nabla(G) = \nabla(\overline{G})$ and that if the assertion holds for $G$, then it also holds for $\overline{G}$. Since connected cographs have disconnected complements (because any cograph is either a union or join of two smaller cographs), we may assume that $G$ is disconnected. Also we may suppose that $G$ has no isolated vertices.

We now proceed by induction on $k \geq 2$. First let $k = 2$. As $G$ is disconnected, it is a union of two threshold graphs for which the assertion trivially holds. Now, let $k \geq 3$. If $G$ has a connected component with Dilworth number at least 2, then we are done by the induction hypothesis. Otherwise, all connected components of $G$ are threshold graphs for which the assertion holds as well. \qed

We are now in a position to prove the main result of the paper.

**Theorem 5.** For any cograph $G$, the multiplicity of any eigenvalue $\lambda \neq 0, -1$, does not exceed the Dilworth number of $G$. Moreover, there are an infinite family of cographs for which this bound is tight.

**Proof.** Let $G$ be a cograph, $\nabla(G) = k$ and $\lambda \neq 0, -1$ be an eigenvalue of $G$. We proceed by induction on $k$. If $k = 1$, then $G$ is a threshold graph and the assertion follows from Lemma 3.

Let $k \geq 2$, and the theorem hold for graphs with Dilworth number at most $k - 1$. If $G$ is disconnected, we are done by the induction hypothesis. Otherwise, all connected components of $G$ are threshold graphs for which the assertion holds as well.

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**Proof.** Let $G$ be a cograph, $\nabla(G) = k$ and $\lambda \neq 0, -1$ be an eigenvalue of $G$. We proceed by induction on $k$. If $k = 1$, then $G$ is a threshold graph and the assertion follows from Lemma 3. Let $k \geq 2$, and the theorem hold for graphs with Dilworth number at most $k - 1$. If $G$ is disconnected, we are done by the induction hypothesis. So we may assume that $G$ is connected.

For a contradiction, assume that $\text{mult}(\lambda, G) \geq k + 1$. By Lemma 4 $G$ contains a threshold subgraph $H'$ such that any two vertices of $H'$ have the same neighborhood in $G - V(H')$ with $G - V(H')$ having Dilworth number $k - 1$. Then $H' = H \cup I$ where $I$ (possibly empty) is the subgraph consisting of isolated vertices of $H'$ and $H$ is a connected threshold graph with at least one edge. Let $V_1, \ldots, V_\ell$ and $U_1, \ldots, U_\ell$ be the partition of $V(H)$ according to Remark 2. Let $u_t \in U_t$. There is a $k$-dimensional subspace $\Omega$ of eigenvectors of $G$ for $\lambda$ which vanishes on $u_t$. Let $x \in \Omega$. Note that $N_H(u_t) = V_1 \cup \cdots \cup V_\ell$. By the sum rule and since $x(u_t) = 0$, $0 = \lambda x(u_t) = \sum_{v \in V_1 \cup \cdots \cup V_\ell} x(v) + \alpha$. 

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where $\alpha$ is the sum of entries of $x$ on the neighbors of $u_t$ outside $H'$; by the way $H'$ is chosen, this is constant for all the vertices of $H'$. Let $v_t \in V_t$. Then

$$N_H(v_t) = U_t \cup V_1 \cup \cdots \cup V_t \setminus \{v_t\}.$$ 

By Remark 1 for all vertices $u \in U_t$ we have $u \equiv u_t$, and thus $x(u) = x(u_t) = 0$. It follows that

$$\lambda x(v_t) = \sum_{v \in N_H(v_t)} x(v) + \alpha$$

$$= -x(v_t) + \sum_{v \in V_1 \cup \cdots \cup V_t} x(v) + \alpha$$

$$= -x(v_t) + \lambda x(u_t)$$

$$= -x(v_t).$$

As $\lambda \neq -1$, it follows that $x(v_t) = 0$. Again by Remark 1 we have $x(v) = 0$ for all $v \in V_t$. Now let $u_{t-1} \in U_{t-1}$. We have $N_H(u_{t-1}) = V_1 \cup \cdots \cup V_{t-1}$, and since $x = 0$ on $V_t$,

$$\lambda x(u_{t-1}) = \sum_{v \in V_1 \cup \cdots \cup V_{t-1}} x(v) + \alpha = \sum_{v \in V_1 \cup \cdots \cup V_t} x(v) + \alpha = \lambda x(u_t) = 0.$$ 

Again, it follows that $x = 0$ on $U_{t-1}$. Let $v_{t-1} \in V_{t-1}$. Then

$$N_H(v_{t-1}) = U_t \cup U_{t-1} \cup V_1 \cup \cdots \cup V_t \setminus \{v_{t-1}\}.$$ 

Therefore,

$$\lambda x(v_{t-1}) = -x(v_{t-1}) + \sum_{v \in U_t \cup U_{t-1} \cup V_1 \cup \cdots \cup V_t} x(v) + \alpha$$

$$= -x(v_{t-1}) + \sum_{v \in V_1 \cup \cdots \cup V_{t-1}} x(v) + \alpha$$

$$= -x(v_{t-1}) + \lambda x(u_{t-1})$$

$$= -x(v_{t-1}).$$

As $\lambda \neq -1$, it follows that $x(v_{t-1}) = 0$. By Remark 1 we have $x(v) = 0$ for all $v \in V_{t-1}$.

Continuing this procedure, we alternately choose vertices $u_{t-2}, v_{t-2}, \ldots, u_1, v_1$ where similar to the above argument we see that $x$ vanishes on $U_{t-2}, V_{t-2}, \ldots, U_1, V_1$. So $x = 0$ on the whole $V(H)$ and we must have $\alpha = 0$ which in turn implies that $x$ is zero on $I$. Hence we conclude that $x$ is zero on $V(H')$. It turns out that if for any $x \in \Omega$ we remove the entries corresponding to the vertices of $H'$ from $x$, the resulting vector is an eigenvector of $G - V(H')$ for eigenvalue $\lambda$. This means that the graph $G - V(H')$ with Dilworth number $k - 1$ has the eigenvalue $\lambda$ with multiplicity $k$, which is a contradiction. Hence the assertion follows.

Finally, we present an infinite family of graphs for which the multiplicity of an eigenvalue $\lambda \neq 0, -1$ is equal to the Dilworth number. It is easy to construct disconnected cographs for which the
equality holds (just take \( k \) copies of a fixed threshold graph). Here we show that the bound can be achieved by connected cographs. Let \( s, k \geq 1 \) and \( G = G(s, k) := K_1 \lor (K_{s \ldots s} \cup (s^2 - s)K_1) \), in which \( K_{s \ldots s} \) is a complete \( k \)-partite graph with parts of size \( s \), and ‘\( \lor \)’ denotes the join of two graphs. Let \( v \) be the vertex of \( G \) adjacent to all the other vertices, \( B_1, \ldots, B_k \) be the parts of the complete \( k \)-partite subgraph, and \( C \) be the set of \( s^2 - s \) pendant vertices. It is easily seen that \( B_1, \ldots, B_k, \{ v \}, C \) make an equitable partition \(^1\) of \( G \), with the quotient matrix

\[
Q = \begin{pmatrix}
1 & 0 \\
: & : \\
1 & 0 \\
s & \cdots & s & 0 & s^2 - s \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix},
\]

where \( J_k \) is the \( k \times k \) all 1’s matrix. Now, we have

\[
Q + sI = \begin{pmatrix}
s & \cdots & s & 1 & 0 \\
: & : & : & : & : \\
s & \cdots & s & 1 & 0 \\
s & \cdots & s & s^2 - s \\
0 & \cdots & 0 & 1 & s
\end{pmatrix}.
\]

It is seen that all the rows of \( Q + sI \) can be obtained by linear combinations of the last two rows. So \( \text{rank}(Q + sI) = 2 \) which means \( \lambda = -s \) is an eigenvalue of \( Q \) with multiplicity \( k \) and thus an eigenvalue of \( G \) with multiplicity at least \( k \). Also the vertices of \( G \) can be partitioned into \( k \) chains with respect to vicinal preorder, namely \( C \cup B_1 \cup \{ v \}, B_2, \ldots, B_k \). Hence \( \nabla(G) \leq k \). \( \Box \)

**Remark 6.** Theorem\(^5\) cannot hold for general graphs. Here we describe a family of counterexamples. For any positive integer \( n \), the **cocktail party graph** \( \text{CP}(n) \) is the graph obtained from the complete graph \( K_{2n} \) by removing a perfect matching. In fact, \( \text{CP}(n) \) is the complete \( n \)-partite graph \( K_{2 \ldots 2} \). Let \( H \) be a graph with vertices \( v_1, \ldots, v_n \), and \( a_1, \ldots, a_n \) be non-negative integers. The **generalized line graph** \( L(H; a_1, \ldots, a_n) \) consists of disjoint copies of \( L(H) \) and \( \text{CP}(a_1), \ldots, \text{CP}(a_n) \) together with all edges joining a vertex \( \{ v_i, v_j \} \) of \( L(H) \) with each vertex in \( \text{CP}(a_i) \) and \( \text{CP}(a_j) \). It is known that all the eigenvalues of generalized line graphs are greater than or equal to \(-2 \). Moreover, by Theorem 2.2.8 of [5], if \( H \) has \( m \) edges and if not all \( a_i \)'s are zero, then

\[
\text{mult}(-2, L(H; a_1, \ldots, a_n)) = m - n + \sum_{i=1}^{n} a_i.
\]  

Consider the generalized line graph \( G = G(k) := L(K_{1,k}; k, 1, \ldots, 1) \). In fact \( G \) is obtained from the graph \( K_k \lor \text{CP}(k) \) by attaching two pendant vertices to each of the vertices of \( K_k \). Let \( u_1, \ldots, u_k \) be the vertices of \( K_k \), \( U_1, \ldots, U_k \) be the parts of \( \text{CP}(k) \), and \( w_{i1}, w_{i2} \) be the pendant vertices of \( K_k \). For the definition and properties of equitable partitions, we refer to [4] p. 24.

\(^1\)For the definition and properties of equitable partitions, we refer to [4] p. 24.
vertices attached to \( u_i \), for \( i = 1, \ldots, k \). Then for each \( i \), \( \{ u_i, w_{i1}, w_{i2} \} \cup U_i \) form a chain, so \( \nabla(G) \leq k \). On the other hand, \( U_1, \ldots, U_k \) is an antichain in the vicinal preorder of \( G \), so \( \nabla(G) \geq k \). Thus we have \( \nabla(G) = k \). However by (2), \( \text{mult}(-2, G) = 2k - 1 \).

**Remark 7.** The bound given in Theorem 5 can be arbitrarily loose. To see this, let \( r_1, \ldots, r_k \) be distinct integers greater than 1 and \( G = K_{r_1, \ldots, r_k} \). Clearly, \( G \) is a cograph with \( \nabla(G) = k \). However, all the non-zero eigenvalues of \( G \) are simple (see [9, Theorem 1]).

### 4 \( H \)-free graphs that only (co)duplications reduce their rank

In [13] a characterization of cographs based on graph eigenvalues was given; namely a graph \( G \) is a cograph if and only if no induced subgraph of \( G \) has an eigenvalue in the interval \((-1, 0)\). In this section we give another characterization for the family of cographs which is based on the presence of eigenvalue 0 or \(-1\) in connection with the existence of duplications or coduplications.

Regarding the presence of eigenvalue 0 in cographs, Royle [27] proved the following result confirming a conjecture by Sillke [30].

**Lemma 8.** If a cograph \( G \) has no duplications, then \( A(G) \) has full rank.

The following result concerning \(-1\) eigenvalues is also implicit in [27] (see also [3, 5, 28]).

**Lemma 9.** If a cograph \( G \) has no coduplications, then \( A(G) + I \) has full rank.

We say that a graph \( G \) satisfies *Duplication-Rank Property* (DRP for short) if the following holds:

\[ G \text{ has a duplication, or } A(G) \text{ has full rank.} \]

Similarly, we say that \( G \) satisfies *Coduplication-Rank Property* (CDRP for short) if:

\[ G \text{ has a coduplication, or } A(G) + I \text{ has full rank.} \]

Lemma 8 says that cographs satisfy DRP. Motivated by this, Royle [27] posed the following question:

*Beside cographs, are there any other natural classes of graphs for which DRP holds?*

For a given graph \( H \), the family of \( H \)-free graphs is the set of all graphs which do not contain \( H \) as an induced subgraph. A ‘natural’ class of graphs is the family of \( H \)-free graphs for a specific graph \( H \). Here we give the answer to the Royle’s question for \( H \)-free families of graphs. The same result is given for graphs with CDRP.

For a graph \( H \), we denote the family of \( H \)-free graphs by \( \mathcal{F}(H) \). We say that \( \mathcal{F}(H) \) satisfies DRP (or CDRP) if any \( G \in \mathcal{F}(H) \) does. Also by \( H \leq G \), we mean that \( H \) is an induced subgraph of \( G \).
The following result shows that if an $H$-free family of graphs satisfies DRP or CDRP it must be contained in the family of cographs.

**Theorem 10.** Let $\mathcal{F}(H)$ be the family of $H$-free graphs.

(i) $\mathcal{F}(H)$ satisfies DRP if and only if $H$ is an induced subgraph of $P_4$.

(ii) $\mathcal{F}(H)$ satisfies CDRP if and only if $H$ is an induced subgraph of $P_4$.

**Proof.** (i) If $H \leq P_4$, then $\mathcal{F}(H) \subseteq \mathcal{F}(P_4)$, and so by Lemma 8, $\mathcal{F}(H)$ satisfies DRP, showing the ‘sufficiency’. For the ‘necessity,’ assume that for a given graph $H$, $\mathcal{F}(H)$ satisfies DRP. We show that $H \leq P_4$.

If $H$ has one or two vertices, then $H \leq P_4$, and we are done. Let $H$ have three vertices. If $H = P_3$ or $H = P_2 \cup P_1$, then $H \leq P_4$. We show that it is impossible that $H = K_3$ or $\overline{K_3}$. Note that $P_5 \in \mathcal{F}(K_3)$ and it does not satisfy DRP as it has no duplications but has a 0 eigenvalue (its 0-eigenvector is illustrated in Figure 2). The graph depicted in Figure 3 belongs to $\mathcal{F}(\overline{K_3})$ but it does not satisfies DRP; it has no duplications and has a 0 eigenvalue (its 0-eigenvector indicated in the picture). Hence the assertion follows for three-vertex graphs.

Suppose that $H$ has four vertices. If $H = P_4$, there is nothing to prove. If $K_3 \leq H$ or $\overline{K_3} \leq H$, then $\mathcal{F}(K_3) \subseteq \mathcal{F}(H)$ or $\mathcal{F}(\overline{K_3}) \subseteq \mathcal{F}(H)$ and so $\mathcal{F}(H)$ does not satisfies DRP. It remains to consider 4-vertex bipartite graphs with no 3-coclique. Besides $P_4$, there are only two such graphs, namely $K_{2,2}$ and $2K_2$. But $P_5$ is a $K_{2,2}$-free graph not satisfying DRP and the graph of Figure 3 is a $2K_2$-free graph not satisfying DRP. Therefore, the assertion holds for four-vertex graphs.

Now let $H$ have five or more vertices. Let $H'$ be a five-vertex graph with $H' \leq H$. As $\mathcal{F}(H') \subseteq \mathcal{F}(H)$, the family $\mathcal{F}(H')$ also satisfies DRP. By the argument for four-vertex graphs, $P_4$ is the only four-vertex graph with $\mathcal{F}(P_4)$ satisfying DRP. It follows that all four-vertex graphs $H$ with $H' \leq H$ must satisfy DRP if $\mathcal{F}(H')$ satisfies DRP.

Figure 2: $P_5$ with its 0-eigenvector shown on the above and $-1$-eigenvector on the below.

Figure 3: A graph in $\mathcal{F}(\overline{K_3}) \cap \mathcal{F}(P_5) \cap \mathcal{F}(2K_2)$ and its 0-eigenvector.
induced subgraphs of $H'$ must be isomorphic to $P_4$. There is a unique graph $H'$ with this property, namely the 5-cycle $C_5$. However, $P_5 \in \mathcal{F}(C_5)$ and it does not satisfy DRP. It turns out that for no graph $H$ with five or more vertices, $\mathcal{F}(H)$ satisfies DRP. This completes the proof.

(ii) Similar to the proof of (i), the ‘sufficiency’ follows from Lemma 9. For the ‘necessity,’ assume that $\mathcal{F}(H)$ satisfies CDRP. There is nothing to prove if $H$ has one or two vertices. Let $H$ have three vertices. If $H = P_3$ or $H = P_2 \cup P_1$, then $H \leq P_4$. Note that $P_5$ does not satisfies CDRP (as shown in Figure 2) but belongs to $\mathcal{F}(K_3)$ and the graph of Figure 4 does not satisfies CDRP but belongs to $\mathcal{F}(K_3)$. Hence the assertion follows for three-vertex graphs.

Suppose that $H$ has four vertices. If $H = P_4$, there is nothing to prove. If $K_3 \leq H$ or $\overline{K_3} \leq H$, then $\mathcal{F}(K_3) \subseteq \mathcal{F}(H)$ or $\mathcal{F}(\overline{K_3}) \subseteq \mathcal{F}(H)$ and so $\mathcal{F}(H)$ does not satisfies CDRP. It remains to consider $H = K_{2,2}$ and $2K_2$. But $P_5$ is a $K_{2,2}$-free graph not satisfying CDRP and the graph of Figure 4 is a $2K_2$-free graph not satisfying CDRP. Therefore, the assertion holds for four-vertex graphs.

Now let $H$ have five or more vertices. Let $H'$ be a five-vertex graph with $H' \leq H$. As $\mathcal{F}(H') \subseteq \mathcal{F}(H)$, the family $\mathcal{F}(H')$ also satisfies CDRP. By the argument for four-vertex graphs, we have that all four-vertex induced subgraphs of $H'$ must be isomorphic to $P_4$ and so $H' = C_5$. However, $P_5 \in \mathcal{F}(C_5)$ and it does not satisfy CDRP. It turns out that for no graph $H$ with five or more vertices, $\mathcal{F}(H)$ satisfies CDRP, completing the proof. □

Any pair of duplicate vertices $u, v$ in a graph $G$ give rise to a null-vector of $A(G)$ of weight two (the vector whose components corresponding to $u, v$ are $1, -1$, and zero elsewhere). Conversely, any null-vector $x$ of $A(G)$ of weight two comes from a pair of duplicate vertices. To see this, suppose that $x_u$ and $x_v$ are the two non-zero components of $x$. As $A(G)$ is a 0, 1-matrix, we must have $x_u = -x_v$. It turns out that the rows of $A(G)$ corresponding to $u$ and $v$ are identical which means that $u$ and $v$ are duplicates. Hence, we observe that any null-vector of $A(G)$ of weight two corresponds with a pair of duplicate vertices in $G$. Similarly, null-vectors of $A(G) + I$ of weight two correspond to pairs of coduplicate vertices. In [23] (see also [13]) it is shown that if $G$ is a cograph, then the null-space of $A(G)$ (resp. $A(G) + I$) has a basis consisting of the weight-two null-vectors corresponding to duplicate (resp. coduplicate) pairs. Hence the following can be deduced from Theorem 10.
Corollary 11. Let $\mathcal{F}(H)$ be the family of $H$-free graphs.

(i) For all graphs $G \in \mathcal{F}(H)$ the null-space of $A(G)$ has a basis consisting of vectors of weight two if and only if $H$ is an induced subgraph of $P_4$.

(ii) For all graphs $G \in \mathcal{F}(H)$ the null-space of $A(G) + I$ has a basis consisting of vectors of weight two if and only if $H$ is an induced subgraph of $P_4$.

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