Third order extensions of 3d Chern-Simons interacting to gravity:

Hamiltonian formalism and stability

D. S. Kaparulin,* I. Yu. Karataeva† and S. L. Lyakhovich‡

Physics Faculty, Tomsk State University, Tomsk 634050, Russia

Abstract

We consider inclusion of interactions between 3d Einstein gravity and the third order extensions of Chern-Simons. Once the gravity is minimally included into the third order vector field equations, the theory is shown to admit a two-parameter series of symmetric tensors with on-shell vanishing covariant divergence. The canonical energy-momentum is included into the series. For a certain range of the model parameters, the series include the tensors that meet the weak energy condition, while the canonical energy is unbounded in all the instances. Because of the on-shell vanishing covariant divergence, any of these tensors can be considered as an appropriate candidate for the right hand side of Einstein’s equations. If the source differs from the canonical energy momentum, the coupling is non-Lagrangian while the interaction remains consistent with any of the tensors. We reformulate these not necessarily Lagrangian third order equations in the first order formalism which is covariant in the sense of 1 + 2 decomposition. After that, we find the Poisson bracket such that the first order equations are Hamiltonian in all the instances, be the original third order equations Lagrangian or not. The brackets differ from canonical ones in the matter sector, while the gravity admits the usual PB’s in terms of ADM variables. The Hamiltonian constraints generate lapse, shift and gauge transformations of the vector field with respect to these Poisson brackets. The Hamiltonian constraint, being the lapse generator, is interpreted as strongly conserved energy. The matter contribution to the Hamiltonian constraint corresponds to 00-component of the tensor included as a source in the right hand side of Einstein equations. Once the 00-component of the tensor is bounded, the theory meets the usual sufficient condition of classical stability, while the original field equations are of the third order.

1 Introduction

Various higher derivative field theories are studied once and again over many decades for several reasons. Among the most frequently mentioned advantages of the higher derivative systems are the better convergence properties at classical and quantum level comparing to the analogues without higher derivatives. For discussion of various types of higher derivative models we refer to the paper [1] and references therein. The higher derivative theories are also notorious for the instability problem. The simplest stability test – boundedness of energy – is usually failed by the models with higher order equations of

* dsc@phys.tsu.ru
† karin@phys.tsu.ru
‡ sll@phys.tsu.ru
motion. The best known exception – $f(R)$-gravity \cite{2} – is stable due to very strong second class constraints. Because of that, on the constrained surface, the Hamiltonian is bounded, so the theory meets the sufficient condition for stability.

Even if the energy is unbounded for general higher-derivative dynamics, the theory is not necessarily unstable. If another bounded conserved quantity exists, it stabilizes dynamics at least at classical level. For example, the free fourth-order Pais-Uhlenbeck (PU) oscillator \cite{3} performs bi-harmonic oscillations, being obviously stable, while the canonical Ostrogradsky Hamiltonian is unbounded in this model. The stability is due to the mere fact that the sum of energies of two harmonics conserves, being the positive quantity, while the canonical energy is a difference of energies of the two oscillations. The true problem reveals itself once one attempts to go beyond the free classical theory, either quantizing the model or/and switching on the interaction at classical level. As the canonical Hamiltonian formalism involves unbounded Ostrogradsky Hamiltonian, being the difference of energies of harmonics, the spectrum is unbounded. That results in instability at quantum level. Various methods are discussed to cure the problem, see for example \cite{4,5,6}. These attempts are not systematically extended beyond the free PU oscillator. It is also noticed that some special interactions of PU model can have the isles of stability in classical dynamics, see \cite{7,8,9}, while it is unstable globally.

In the paper \cite{1}, a systematic way is suggested to switch on the interactions in certain class of higher derivative systems without breaking stability of free theory. This class of higher derivative systems encompasses both PU oscillator with various extensions, and a range of field theories. The class of higher derivative dynamics with stable interactions has been further extended in the paper \cite{10}. The starting point for this scheme of switching on stable interactions is that the free higher derivative theory should admit a series of conserved quantities such that includes the canonical energy. Then the method implies to switch on the interaction in such a way that is compatible with conservation of certain representative of this series. Another aspect of this scheme is that any of the conserved quantities of the same series, not just canonical energy, is connected to same the symmetry of the equations. The connection is established by the extension of Noether theorem suggested in the paper \cite{11}. This implies that the field equations admit Lagrange anchor, which has been first introduced in ref. \cite{12} to BRST embed and quantize not necessarily Lagrangian systems. The Lagrange anchor, being admitted by the equations of motion, connects the conserved quantities with symmetries \cite{11} irrespectively to existence of the action functional for the equations. One more crucial feature of the Lagrange anchor is that it makes the theory Hamiltonian in the first order formalism \cite{13}. This means, if the higher derivative equations admit multiple Lagrange anchors, the first order formulation is a multi-Hamiltonian theory. Given the Lagrange anchor in free theory, the interaction can be switched in such a way that provides conservation of the quantity connected by the anchor with the symmetry if the latter is unbroken at interacting level \cite{14}. In particular, if the conserved quantity is bounded in free theory, the dynamics will remain stable upon inclusion of interaction, at least at perturbative level. One more detail of this scheme of introducing the interactions is that the interacting theory will admit constrained Hamiltonian formulation even if the vertices are non-Lagrangian in the original higher derivative equations.

The free PU oscillator provides a simplest example of a multi-Hamiltonian higher derivative system, as it has been noticed more than a decade ago, \cite{15,16}. The multi-Hamiltonian formulations are also known for various extensions of the free PU oscillator \cite{17,18}. It has been also found that the interactions can be included into the PU equations leaving the dynamics stable \cite{11}. Furthermore, the stable higher derivative PU equation with interaction still admits Hamiltonian formulation in the corresponding first order formalism, though the stable vertices are non-Lagrangian in the higher derivative equations of motion \cite{19}.

One of the frequently discussed higher derivative field theories is the third order extension of Chern–Simons \cite{20}. It has
been recently found that the model is multi-Hamiltonian at free level \([21]\). To the best of our knowledge, it is the first known explicit example of higher derivative field theory that admits multi-Hamiltonian structure. It is also found that the stable interaction can be included with spin \(1/2\) in this model \([21]\). Even though the stable interaction is non-Lagrangian, the theory still admits Hamiltonian formulation at interacting level \([21]\), so it can be quantized.

In this paper, we study the coupling of the third order extension of Chern-Simons to Einstein’s gravity. At free level, the theory admits a continuous series of conserved tensors found in \([10]\) that includes canonical energy-momentum. The series involves bounded quantities if the free third order field equations describe reducible unitary representations, while the canonical energy-momentum is unbounded in every instance.

We begin with inclusion of minimal coupling to gravity into the third order field equations

\[
(m^{-1} * d * d * d + \alpha_2 * d + \alpha_1 m * d)A = 0,
\]

where \(A = A_\mu(x)dx^\mu\), \(\mu = 0, 1, 2\) is the vector field, \(m\) is the constant with the dimension of mass, \(\alpha_1, \alpha_2\) are the dimensionless constant parameters, \(d\) denotes the exterior derivative, and \(*\) stands for the Hodge star operator,

\[
(*dA)^\mu = \frac{1}{\sqrt{g}} \varepsilon^{\mu\nu\rho} \partial_\nu A_\rho.
\]

All the tensor indices are raised and lowered by the spacetime metric \(g_{\mu\nu}\). The signature of the metric is mostly negative. The 3d Levi-Civita symbol \(\varepsilon^{\mu\nu\rho}\) is the tensor density, \(\varepsilon^{012} = 1\).

If field equations (1) are considered in Minkowski space, they admit a series conserved tensors found in \([10]\). As we shall demonstrate, the minimal covariant extension of this series leads us to the covariantly transverse tensors

\[
\nabla_\nu T^{\mu\nu}(A, g; \alpha, \beta, \gamma) \approx 0, \quad \forall \alpha, \beta, \gamma.
\]

Here \(\alpha\) are the parameters involved in the field equations (1), while \(\beta, \gamma\) are the independent real parameters labeling the representatives of the series of tensors. The sign \(\approx\) means the on-shell equality with respect to the equations (1). The explicit expressions for the series of tensors \(T\) are provided in the next section. Once the tensors (3) are on shell covariantly transverse, any representative of the series can be considered as an admissible right hand side for the Einstein equations:

\[
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} (R - \Lambda) = - \frac{1}{2} T^{\mu\nu}(A, g; \alpha, \beta, \gamma).
\]

If the rhs of these equations is the canonical energy-momentum for the field \(A\) subject to the equations (1), then the equations (1), (4) form a Lagrangian system. Otherwise, it is not Lagrangian. Be the equations Lagrangian or not, the system (1), (4) is fully consistent once the tensor \(T\) on the rhs of (4) is transverse on shell (1). There are two obvious facts indicating the consistency of field equations (1) and (4): (i) the system has explicit gradient gauge symmetry for the field \(A\), and the equations are diffeomorphism invariant; (ii) there are gauge identities between the

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We adopt the following definitions for covariant derivative \(\nabla_\mu\), curvature tensor \(R^{\rho}_{\mu\nu\rho}\) and Ricci tensor \(R_{\mu\nu}\):

\[
\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\sigma_{\nu\rho} A^\sigma, \quad R^{\rho}_{\mu\nu\rho} = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\rho\mu} + \Gamma^\sigma_{\mu\nu} \Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\rho\nu} \Gamma^\rho_{\sigma\mu}, \quad R_{\mu\nu} = R^{\sigma}_{\mu\nu\sigma}.
\]

where \(\Gamma\) is the Christoffel symbol; \(\Lambda\) is the cosmological constant.
equations which are the same in Lagrangian and non-Lagrangian case – the divergence vanishes identically of the equations (1), while the covariant divergence of equations (4) vanishes because of (3). The orders of equations, symmetries, and identities are the same in all the cases, Lagrangian and non-Lagrangian. These data are sufficient to define the degree of freedom number for the theory being formulated in covariant form, without explicit recourse to Hamiltonian constrained analysis. The formula (8) of the paper [22] allows one to easily compute the local degree of freedom number in a covariant way. The computation gives that the number is four, i.e. it is the same as for the equations (1) in Minkowski space. This means consistency, because in three dimensions, Einstein’s gravity does not have local DoF by itself.

In the next section, we elaborate on the on-shell transverse tensors admitted by the equations (1). In particular, we use the ADM space decomposition to clarify the structure of the tensors. As we shall see, the series of tensors $T(\alpha, \beta, \gamma)$ (3) includes the representatives that meet the so-called weak energy condition (abbreviated as WEC), once the parameters $\alpha, \beta$ meet certain conditions. The dynamics is stable once the tensor in the rhs of Einstein equations (4) meets the WEC. The canonical energy-momentum, being included in the series, does not meet the condition, so the stable interactions are inevitably non-Lagrangian. In the section three, we reformulate the field equations (1), (4) in the first order formalism with respect to the time derivatives. This is done making use of $1 + 2$ decomposition in the ADM variables. Then, we find the Poisson bracket such that the first order equations read as a constrained Hamiltonian system in all the instances, be the original system (1), (4) Lagrangian or not. The Poisson bracket is not canonical in general, and it involves the parameters $\alpha, \beta, \gamma$ from the rhs of the Einstein’s equations (4). All the Hamiltonian constraints are of the first class with respect to this bracket. The constraints include the Hamiltonian generators of lapse and shift transformations, and also the generator of gauge transformations for the vector field. The matter contribution to the lapse constraint can be positive for certain range of the model parameters $\alpha, \beta$ involved in the field equations (1), (4). If the rhs of Einstein’s equations (4) is the canonical energy-momentum (that corresponds to $\beta_1 = 1, \beta_2 = 0$), the matter contribution to the lapse constraint will be unbounded for any $\alpha$. In this case, the constrained Hamiltonian formalism is canonically equivalent to Ostrogradsky formulation of the Lagrangian theory. The formulations with the bounded Hamiltonian are inequivalent to this case, because no canonical transformation can turn any on-shell bounded quantity into an unbounded one and vice versa.

2 Tensors with on-shell vanishing covariant divergence and stability

In this section, we find the series of on-shell covariantly transverse second rank symmetric tensors, and study the weak energy condition for these tensors.

Let us introduce abbreviations

$$F = *dA, \quad G = *d*dA.$$ (5)

Obviously, $F$ and $G$ are gauge invariant quantities. Also notice that from the definition (5) immediately follows that the one-forms $F, G$ are co-closed, so the covariant divergence identically vanishes of the vectors $F^\mu, G^\mu$,

$$d*dF \equiv 0, \quad d*dG \equiv 0 \quad \Leftrightarrow \quad \nabla_\mu F^\mu \equiv 0, \quad \nabla_\mu G^\mu \equiv 0.$$ (6)

2 We mean the usual definition for the local DoF number – it is the number of independent Cauchy data per point of space.

3 The accessory parameter $\gamma$ has a different meaning. On shell, the tensors $T(\alpha, \beta, \gamma)$ become $\gamma$-independent. So this parameter labels representatives in the same equivalence class of the on-shell transverse tensors. Given the parameters $\alpha$ defining the the field equations (1), we have the two parameters $\beta$ labeling inequivalent on-shell transverse tensors.
In terms of $F, G$, the third order equations \[ (1) \] read

\[ E \equiv m^{-1} \ast dG + \alpha_2 G + \alpha_1 m F \approx 0 \ \Rightarrow \ E^\mu \equiv m^{-1} \frac{1}{\sqrt{g}} \epsilon^{\mu\rho\nu} \partial_\rho G_\nu + \alpha_2 G^\mu + \alpha_1 m F^\mu \approx 0. \quad (7) \]

In the paper \[ (10) \], two independent on-shell conserved symmetric tensors are found for these equations in Minkowski space. The minimal covariant extension of these tensors \[ (1) \] read

\[ T_{(1)}^{\mu\nu}(A, g; \alpha) = \frac{1}{m} \left[ (G^\mu F^\nu + G^\nu F^\mu - g^{\mu\nu} G_\rho F_\rho) + \alpha_2 m (F^\mu F^\nu - \frac{1}{2} g^{\mu\nu} F_\rho F^\rho) \right], \quad (8) \]

\[ T_{(2)}^{\mu\nu}(A, g; \alpha) = \frac{1}{m^2} \left[ (G^\mu G^\nu - \frac{1}{2} g^{\mu\nu} G_\rho G^\rho) - \alpha_1 m^2 (F^\mu F^\nu - \frac{1}{2} g^{\mu\nu} F_\rho F^\rho) \right], \quad (9) \]

where $\alpha_1, \alpha_2$ are the parameters of the third-order extension of Chern-Simons \[ (1) \]. Upon account for the identities \[ (6) \], the covariant divergence of the tensors \[ (8), (9) \] is seen to vanish on shell:

\[ \nabla_\nu T_{(1)}^{\mu\nu}(A, g; \alpha) = -\frac{1}{\sqrt{g}} \epsilon^{\mu\rho\nu} F_\rho E_\nu \approx 0, \quad \nabla_\nu T_{(2)}^{\mu\nu}(A, g; \alpha) = -\frac{1}{m} \frac{1}{\sqrt{g}} \epsilon^{\mu\rho\nu} G_\rho E_\nu \approx 0. \quad (10) \]

Notice that any on-shell vanishing tensor is on-shell transverse. In the third-order Chern-Simons theory \[ (1) \], one of these trivial tensors is relevant for constructing Hamiltonian formulation. We chose it in the form

\[ T_{(3)}^{\mu\nu}(A, g; \alpha, \beta, \gamma) = \frac{1}{m^2} \left[ \frac{\beta_1^2 + \gamma_1}{\beta_1 - \beta_2 \alpha_2 - \gamma_1} \left( E^\mu G^\nu + E^\nu G^\mu - g^{\mu\nu} E_\rho G^\rho \right) + \right. \]

\[ \left. \frac{1}{m} \frac{\beta_1 \beta_2 + \gamma (\beta_1 \alpha_2 - \beta_2 \alpha_1)}{\beta_1 - \beta_2 \alpha_2 - \gamma_1} \left( E^\mu F^\nu + E^\nu F^\mu - g^{\mu\nu} F_\rho F^\rho \right) \right], \quad (11) \]

where $E^\mu$ denote the lhs of equations \[ (7) \], and $\beta_1, \beta_2, \gamma$ are constant dimensionless parameters such that $\beta_1 - \beta_2 \alpha_2 - \gamma_1 \neq 0$.

As the tensors \[ (8), (9), (11) \] have on-shell vanishing covariant divergence, any linear combination of these tensors is covariantly transverse on shell. Given the field equations \[ (1) \], we have the three-parameter series of on-shell transverse tensors

\[ T^{\mu\nu}(A, g; \alpha, \beta, \gamma) = \beta_1 T_{(1)}^{\mu\nu}(A, g; \alpha) + \beta_2 T_{(2)}^{\mu\nu}(A, g; \alpha) + T_{(3)}^{\mu\nu}(A, g; \alpha, \beta, \gamma) = \]

\[ \frac{\beta_1}{m} \left[ (G^\mu F^\nu + G^\nu F^\mu - g^{\mu\nu} G_\rho F^\rho) + \alpha_2 m (F^\mu F^\nu - \frac{1}{2} g^{\mu\nu} F_\rho F^\rho) \right] + \]

\[ \frac{\beta_2}{m^2} \left[ (G^\mu G^\nu - \frac{1}{2} g^{\mu\nu} G_\rho G^\rho) - \alpha_1 m^2 (F^\mu F^\nu - \frac{1}{2} g^{\mu\nu} F_\rho F^\rho) \right] + \]

\[ \frac{1}{m^2} \frac{\beta_1^2 + \gamma_1}{\beta_1 - \beta_2 \alpha_2 - \gamma_1} \left( E^\mu G^\nu + E^\nu G^\mu - g^{\mu\nu} E_\rho G^\rho \right) + \frac{1}{m} \frac{\beta_1 \beta_2 + \gamma (\beta_1 \alpha_2 - \beta_2 \alpha_1)}{\beta_1 - \beta_2 \alpha_2 - \gamma_1} \left( E^\mu F^\nu + E^\nu F^\mu - g^{\mu\nu} F_\rho F^\rho \right) \right], \quad (12) \]

where $\beta_1, \beta_2, \gamma$ are the independent parameters that label representatives of the series, while $\alpha_1, \alpha_2$ specify the field equations. Once the tensor $T^{\mu\nu}(A, g; \alpha, \beta, \gamma)$ is included into rhs of Einstein’s equations \[ (4) \], $\beta_1, \beta_2$ being the factors at the on-shell non-
vanishing contributions \( (8), (9) \), can be understood as coupling constants responsible for the interaction between gravity and matter. The accessory parameter \( \gamma \) is involved into on-shell vanishing term. So, it accounts for possible contributions to the rhs of equations \( (4) \) such that vanish on account of the field equations \( (1) \).

The equations \( (1) \) for the vector field, being considered alone, apart from Einstein’s equations \( (4) \), follow from the least action principle for the functional

\[
S_A = \frac{1}{2} \int \! \star A \wedge (m^{-1} \star d \star d + \alpha_2 \star d \star d + \alpha_1 m \star d) A, \quad \frac{\delta S_A}{\delta A_\mu} \equiv \sqrt{g} E^\mu.
\]  

The quantity \( T^{\mu \nu}_{(1)}(A, g; \alpha) \) \( (8) \), being the first constituent of the series of on-shell transverse tensors \( (12) \), is the canonical stress-energy tensor for the action,

\[
T^{\mu \nu}_{(1)}(A, g; \alpha) \equiv \frac{2}{\sqrt{g}} \delta S_A \delta g_{\mu \nu}.
\]  

while \( T^{\mu \nu}_{(2)}(A, g; \alpha) \) \( (9) \) is a different independent tensor. If the scalar curvature of the metric and cosmological constant are added to the matter action \( (13) \), the corresponding Lagrange equations will be Einstein’s ones \( (4) \) with canonical stress-energy tensor \( T^{\mu \nu}_{(1)}(A, g; \alpha) \) \( (8) \) in the right hand side. Contribution of \( T^{\mu \nu}_{(2)}(A, g; \alpha) \) \( (9) \) into the right hand side of equations \( (4) \) is on-shell non-trivial. This contribution is non-Lagrangian, though it is fully consistent as it has been already explained in the introduction.

Below we elaborate on the issue of stability of the system \( (1), (4) \). As we shall see, the stability can be achieved with certain representatives of the series \( (12) \) in the right hand side of Einstein’s equations, while inclusion of a pure canonical stress-energy results in instability.

Various assumptions are known about energy-momentum tensor which can provide stability of dynamics of gravity coupled to the matter. These assumptions are usually referred to as energy conditions. For general discussion of energy conditions we refer to the books [23, 24]. In this paper, we examine the weak energy condition (WEC), which implies that the scalar

\[
T^{\mu \nu}(A, g; \alpha, \beta, \gamma) \xi_\mu \xi_\nu \geq 0,
\]  

is bounded from below on the mass shell \( (7) \) for arbitrary timelike vector \( \xi_\mu(x), \xi^2 > 0 \). This condition means that the observer will always measure a positive energy density of the matter when traveling by any timelike path. Since the Hamiltonian of the theory is the phase space equivalent of energy, the theory, whose matter satisfies the WEC \( (15) \), has a good chance to admit a Hamiltonian formulation with bounded Hamiltonian of the matter. The latter can be viewed as the stability condition in the usual sense.

Once the WEC \( (15) \) is imposed onto the tensor \( T(\alpha, \beta, \gamma) \) \( (12) \), it restricts the admissible range of parameters \( \alpha, \beta \). Now, we are going to find these restrictions explicitly. Let us choose a special coordinate system such that the timelike vector \( \xi \) has the canonical form \( \xi_\mu = (1, 0, 0) \). In this coordinate system, the inequality \( (15) \) reads

\[
T^{00}(A, g; \alpha, \beta, \gamma) \geq 0.
\]  

The lhs of this expression is a bilinear form in the variables \( F, G \). The coefficients of the form depend on the metric. To simplify the dependence on metric, we use the ADM variables that suites well to the problems where the explicit decomposition
in space and time has to be done. The ADM variables read

\[ N \equiv \frac{1}{\sqrt{g^{00}}}, \quad N_i \equiv g_{0i}, \quad g_{ij} \equiv g_{ij}, \quad i, j = 1, 2. \]  

(17)

The 3d metric \( g_{\alpha \beta} \) and its inverse can be expressed in terms of the ADM variables,

\[ g_{\alpha \beta} = \left( \begin{array}{cc} N^2 + N_s N^s & N_j \\ N_i & g_{ij} \end{array} \right), \quad g^{\alpha \beta} = \left( \begin{array}{cc} N^{-2} & -N^j N^{-2} \\ -N^i N^{-2} & g^{ij} + N^i N^j N^{-2} \end{array} \right), \]  

(18)

\[ \hat{g}^{i \alpha} \hat{g}_{ij} = \delta^i_j, \quad N^i \equiv \hat{g}^{ij} N_j. \]  

(19)

For the vector fields \( E \) (7), \( F, G \) (5), we introduce the following 1 + 2 decomposition:

\[ Z_\alpha = (N \hat{E}^0 + N^r Z_r, Z_i), \quad \hat{E}^0 = NZ^0, \quad Z = \{E, G, F\}, \]  

(20)

where \( E_i, G_i, F_i \) are space components of the forms (7), (5), and \( E^0, G^0, F^0 \) are the time components of the 3d vectors \( F^{sr}, G^{\alpha} \). The quantities \( \hat{E}^0, \hat{G}^0, \hat{F}^0 \) read

\[ \theta \equiv \hat{E}^0 = \frac{1}{\sqrt{g}} \epsilon^{sr} (m^{-1} \partial_r G_r + \alpha_2 \partial_r F_r + \alpha_1 m \partial_s A_r), \quad \hat{E}^0 = \frac{1}{\sqrt{g}} \epsilon^{sr} \partial_s F_r, \quad \hat{F}^0 = \frac{1}{\sqrt{g}} \epsilon^{sr} \partial_s A_r, \]  

(21)

where \( \epsilon^{12} = 1 \). Here, the mass shell condition \( \theta = 0 \) corresponds to the Gauss law constraint in the model (1).

In the notation (20), (21), we rewrite the WEC (15) in the ADM variables

\[ T^{00} = \frac{1}{N^2} \beta_1 \left[ (\hat{G}^0 \hat{E}^0 - \hat{g}^{sr} G_s G_r) + \alpha_2 m \frac{1}{2} (\hat{F}^0 \hat{F}^0 - \hat{g}^{sr} F_s F_r) \right] + \]  

\[ + \frac{1}{N^2} \beta_2 \frac{1}{m} \left[ \frac{1}{2} (\hat{G}^0 \hat{G}^0 - \hat{g}^{sr} G_s G_r) - \alpha_1 m \frac{1}{2} (\hat{F}^0 \hat{F}^0 - \hat{g}^{sr} F_s F_r) \right] + \]  

\[ + \frac{1}{N^2} \frac{1}{m^2} \beta_3 \left( \hat{G}^0 \hat{G}^0 - \hat{g}^{sr} E_s G_r \right) + \frac{1}{N^2} \frac{1}{m^2} \beta_4 \left( \hat{F}^0 \hat{F}^0 - \hat{g}^{sr} E_s F_r \right) \geq 0. \]  

(22)

We evaluate the lhs of this inequality on the mass shell (21), being equivalent to the original equations (1). On shell, \( \theta = 0, E_i = 0 \), while the rest of the expression is a bilinear form in the variables \( G_i, F_i, A_i \). Only 4 of these variables give independent Cauchy data for the model (1): one of components \( A_i \) can be set to zero by gradient gauge transformation, and one of components \( G_i \) is fixed by the Gauss law constraint \( \theta = 0 \). Choosing initial data in the form

\[ G_i = m \partial_i \xi, \quad F_i = \partial_i \zeta, \quad A_i = 0, \]  

(23)

with \( \xi(x), \zeta(x) \) being the arbitrary functions of spacetime coordinates, we automatically satisfy the Gauss law constraint, while the condition (22) reads

\[ -\frac{1}{2} \beta_1 \hat{g}^{ij} \partial_i \xi \partial_j \xi + \beta_2 \hat{g}^{ij} \partial_i \xi \partial_j \zeta - \frac{1}{2} (\beta_1 \alpha_2 - \beta_2 \alpha_1) \hat{g}^{ij} \partial_i \zeta \partial_j \zeta \geq 0, \quad \forall \xi(x), \zeta(x). \]  

(24)
Considering this expression as the quadratic form in the gradient vectors \( \partial_i \xi, \partial_i \zeta \), and using the Sylvester criterion to ensure that the form is positive semidefinite, we get two restrictions on the parameters \( \alpha, \beta \),

\[
\beta_2 \geq 0, \quad -\beta_1^2 + \beta_1 \beta_2 \alpha_2 - \beta_2^2 \alpha_1 \geq 0.
\] (25)

These two conditions are also sufficient to meet WEC (15), because, in this case, \( T^{00}(\alpha, \beta, \gamma) \) is a positive (semi-)definite quadratic form of the variables \( G^0, \tilde{F}^0, G_i, F_i \). There is a special case when the quadratic form \( T^{00}(\alpha, \beta, \gamma) \) is degenerate. We skip the degenerate case in this paper. The non-degeneracy requirement restricts the parameters by the condition

\[
\beta_1^2 - \beta_1 \beta_2 \alpha_2 + \beta_2^2 \alpha_1 \neq 0.
\] (26)

This restriction is assumed in all the considerations below.

Relations (25), (26) determine the range of the parameters \( \alpha, \beta \) that results in the stable theory (1), (4). The consistency for these relations implies that the parameters \( \alpha \) have to meet the condition

\[
\alpha_2^2 - 4 \alpha_1 > 0.
\] (27)

Under this condition, the Minkowski space limit of the third order Chern-Simons equations (1) transforms by reducible unitary representation of the Poincaré group, see ref. [10]. If \( \alpha_1 \neq 0 \), it describes the pair of self-dual massive spins 1 with different masses. The self-dual massive spin models are well known in 3d [25, 26]. If \( \alpha_1 = 0 \), the set of sub-representations includes a massless spin 1 and a massive spin 1 subject to a self-duality condition. Once relation (27) is not satisfied, the Minkowski space limit of the equations (1) transforms by a non-unitary representation of the Poincaré group. The non-unitary representation does not decompose into irreducible ones. In this case, any tensor of the series (12) does not meet the WEC (15), so the dynamics is unstable with any \( T \) of the series.

The general conclusion is that the model (1) with any parameters \( \alpha_1, \alpha_2 \) admits a series of consistent couplings with gravity described by tensors (10) included in the rhs of Einstein’s equations (4). If the WEC (15) is imposed, it ensures the stability of dynamics. From the WEC, the restriction follow on the parameters \( \alpha_1, \alpha_2 \) of the extended CS equations (1). Given the parameters \( \alpha_1, \alpha_2 \), the parameters \( \beta_1, \beta_2 \) define the admissible tensor (12) in the rhs of Einstein’s equations (4). If the WEC is imposed, the parameters \( \beta \) have to meet the conditions (25). Under these conditions the third order theory (1) remains stable being coupled to Einstein’s gravity.

3 The first order formulation

The existence of Hamiltonian formulation of the model turns out indifferent to stability. So, we develop the first order formalism for equations (1), (4) and seek for the Hamiltonian structure in a uniform way for any tensor of the series (12) included in the rhs of equations (4), be the model stable or unstable. Let us notice once again, that the field equations (1), (4) are non-Lagrangian in all the instances besides \( \beta_2 = 0 \) which results in the unstable theory, be the flat space limit stable or not.

As we use mostly negative signature of metric in 1+2 dimensions, the quadratic form like \((k^0)^2 - \tilde{g}^{ij} k_i k_j\) is positive for any vector \( k \).

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When the Hamiltonian formulation is constructed for the diffeomorphism-invariant theories, it is convenient to represent the spacetime as a foliation whose leaves are the spacelike hypersurfaces. Locally, the spacetime is understood as a normal bundle to the spacelike hypersurface which is referred to as space. The coordinate on the fiber of normal bundle is considered as time. We suppose that we have the coordinates $x^\mu$, $\mu = 0, 1, 2$ such that $x^i$, $i = 1, 2$ are the space coordinates, while $x^0$ is the time. The ADM variables (17), (18), (19) are very convenient to describe the metric once the spacetime is decomposed in space and time.

Let us specify the notation related to the ADM parametrization of metric and curvature. In this section, $\dot{*}g_{ij}$ denotes the covariant derivative with respect to the space metric $\dot{g}_{ij}$, all the space indices are lowered and raised by $\dot{g}_{ij}$ and $\dot{g}^{ij}$. The scalar curvature of the space metric $\dot{g}_{ij}$ is $\ddot{R}$.

Once the derivatives of metric are to be considered in terms of decomposition in space and time, the $2d$ tensor of extrinsic curvature of the spacelike hypersurface $x^0 = \text{const}$ is a relevant structure. It reads

$$ K_{ij} = \frac{1}{2N} \left[ \dot{*}N_j + \dot{*}N_i - \dot{g}_{ij} \right]. \quad (28) $$

Hereinafter, the dot denotes derivative by $x^0$. To absorb the time derivatives of metric, we use the variable $\pi^{ij}$, which is canonically conjugate momentum to $g_{ij}$ in Einstein’s gravity without matter. As the matter contribution to the Einstein’s equations (4), being expressed in terms of $F, G$ (5), does not involve derivatives of metric, we expect that the inclusion of matter does not change the canonical momentum of gravity. In terms of extrinsic curvature, the momentum reads

$$ \pi^{ij} = \sqrt{\dot{g}} \left( \dot{g}^{is} \dot{g}^{jr} - \dot{g}^{ij} \dot{g}^{sr} \right) K_{sr} \quad \Leftrightarrow \quad K_{ij} = \frac{1}{\sqrt{\dot{g}}} \left( \dot{g}_{is} \dot{g}^{jr} - \dot{g}_{ij} \dot{g}^{sr} \right) \pi^{sr}. \quad (29) $$

The trace of $\pi^{ij}$ is denoted by $\pi \equiv \dot{g}_{sr} \pi^{sr}$. As is seen from the definition, $\pi^{ij}$ and its trace $\pi$ are correspondingly $2d$ tensor and scalar densities.

To depress the order of the field equations (1), we introduce the variables $F_i, G_i, i = 1, 2$ absorbing the first and second time derivatives of the vector field $A$:

$$ F_i = \frac{1}{N \sqrt{\dot{g}}} \left( \dot{g}_{is} \epsilon^{sr} \left( \partial_r A_0 - \dot{\dot{A}}_r \right) + N_i \epsilon^{sr} \partial_s A_r \right), \quad (30) $$

$$ G_i = \frac{1}{N \sqrt{\dot{g}}} \left\{ \dot{g}_{is} \epsilon^{sr} \left( \partial_r \left( N \frac{1}{\sqrt{\dot{g}}} e^{kl} \partial_k F_l + N^k F_k \right) \right) + N_i \epsilon^{sr} \partial_s G_r \right\}. \quad (31) $$

The variables $F_i, G_i$ are the space components of the differential forms $F, G$ (5) in three dimensions, i.e. these can be viewed as the reduction of the spacetime forms to the hypersurface $x^0 = \text{const}$.

In terms of these variables, the field equations (1), (4) read as the first order system:

$$ \dot{A}_i = N \sqrt{\dot{g}} \dot{g}_{is} \epsilon^{sr} F_r + N^k \epsilon_{ki} \epsilon^{sr} \partial_s A_r + \partial_i A_0; \quad (32) $$

$$ \dot{F}_i = N \sqrt{\dot{g}} \dot{g}^{is} \epsilon^{sr} G_r + N^k \epsilon_{ki} \epsilon^{sr} \partial_s F_r + \partial_i \left( N \frac{1}{\sqrt{\dot{g}}} e^{sr} \partial_s A_r + N^s F_s \right); \quad (33) $$
The equations (34) represent the i-th component of the original field equations (1). The equation (35) is 0-component of the equations (1), and it is understood as Gauss law constraint of this theory (21). Relation (32) is equivalent to definition (30) of the variable $F_i$, while (33) is a definition of $G_i$ (31) being resolved with respect to $\hat{F}$. The equations (37), (38) and (39) represent the space-space, time-time, and space-time components of Einstein’s equations (4), with all the time derivatives of the matter fields being expressed by means of equations (32), (35), (36). The equations (38), (39) are the constraints as they do not involve time derivatives of the variables of the first order formulation. These equations can be thought of as energy and momentum constraints, given their dependence on gravitational variables. Also notice that the matter contribution to the energy constraint (38) corresponds to the expression

$$\dot{G}_i = -\alpha_2 m^2 \sqrt{g} \varepsilon_{i8} \hat{g}^{sr} G_r - \alpha_1 m^2 N \sqrt{g} \varepsilon_{i8} \hat{g}^{sr} F_r - \alpha_2 m N \varepsilon_{ki} \varepsilon^{sr} \partial_s F_r - \alpha_1 m^2 N \varepsilon_{ki} \varepsilon^{sr} \partial_s A_r +$$

$$+ \partial_i \left( N \sqrt{g} \varepsilon^{sr} \partial_s F_r + N^s G_s \right);$$

$$\theta \equiv \frac{1}{\sqrt{g}} \varepsilon^{sr} (m^{-1} \partial_r G_r + \alpha_2 \partial_r F_r + \alpha_1 m \partial_r A_r) = 0;$$

$$\dot{g}_{ij} = \hat{\nabla}_i N_j + \hat{\nabla}_j N_i - 2N \sqrt{g} \left( \pi_{ij} - \hat{g}_{ij} \pi \right);$$

$$\pi^{ij} = -\pi^{i8} \hat{\nabla}_s N^j - \pi^{j8} \hat{\nabla}_s N^i + \sqrt{g} \hat{\nabla}_s \left( \frac{1}{\sqrt{g}} \pi^{ij} N^s \right) +$$

$$+ \frac{1}{2} N \sqrt{g} \left( \pi_{sr} \varepsilon^{sr} - \pi^{2} \right) + \sqrt{g} \left[ \hat{\nabla}^i \hat{\nabla}^j N - \hat{g}^{ij} \hat{\nabla}^s N \right] - \frac{1}{2} N \sqrt{g} \hat{g}^{ij} \Lambda -$$

$$- \frac{1}{2} \beta_1 m \sqrt{g} \varepsilon^{ik} \hat{g}^{jl} \left[ \left( G_k F_l + G_l F_k - \hat{g}_{kl} \left( \hat{g}^{00} \hat{F}^0 + \hat{g}^{sr} G_s F_r \right) \right) + \alpha_2 m \left( F_k F_l - \frac{1}{2} \hat{g}_{kl} \left( \hat{g}^{00} \hat{F}^0 + \hat{g}^{sr} F_s F_r \right) \right) \right] -$$

$$- \frac{1}{2} \beta_2 m \sqrt{g} \varepsilon^{ik} \hat{g}^{jl} \left[ \left( G_k G_l - \frac{1}{2} \hat{g}_{kl} \left( \hat{g}^{00} \hat{G}^0 + \hat{g}^{sr} G_s G_r \right) \right) - \alpha_1 m^2 \left( F_k F_l - \frac{1}{2} \hat{g}_{kl} \left( \hat{g}^{00} \hat{F}^0 + \hat{g}^{sr} F_s F_r \right) \right) \right];$$

$$\tau \equiv \frac{1}{\sqrt{g}} \left( \pi^2 - \pi_{sr} \pi^{sr} \right) - \hat{R} + \Lambda +$$

$$+ \frac{\beta_1}{m} \left[ \left( \hat{g}^{00} \hat{F}^0 - \hat{g}^{sr} G_s F_r \right) + \alpha_2 m \frac{1}{2} \left( \hat{g}^{00} \hat{F}^0 - \hat{g}^{sr} F_s F_r \right) \right] +$$

$$+ \frac{\beta_2}{m^2} \left[ \frac{1}{2} \left( \hat{g}^{00} \hat{G}^0 - \hat{g}^{sr} G_s G_r \right) - \alpha_1 m^2 \frac{1}{2} \left( \hat{g}^{00} \hat{F}^0 - \hat{g}^{sr} F_s F_r \right) \right] +$$

$$+ \frac{1}{m^2} \left( \frac{\beta_1 \beta_2 + \gamma \beta_1}{\beta_1 - \beta_2 \alpha_2 - \gamma \alpha_1} \hat{g}^{00} \theta + \frac{1}{m} \frac{\beta_1 \beta_2 + \gamma (\beta_1 \alpha_2 - \beta_2 \alpha_1)}{\beta_1 - \beta_2 \alpha_2 - \gamma \alpha_1} \hat{F}^0 \theta \right) = 0;$$

$$\tau_i \equiv -2 \hat{\nabla}_s \left( \frac{1}{\sqrt{g}} \pi^s \right) + \frac{\beta_1}{m} \left[ \left( \hat{g}^{00} F_i + G_i \hat{F}^0 \right) + \alpha_2 m \hat{F}^0 F_i \right] + \frac{\beta_2}{m^2} \left[ \hat{g}^{00} G_i - \alpha_1 m^2 \hat{F}^0 F_i \right] +$$

$$+ \frac{1}{m^2} \left( \frac{\beta_1 \beta_2 + \gamma \beta_1}{\beta_1 - \beta_2 \alpha_2 - \gamma \alpha_1} \hat{g}^{00} \theta + \frac{1}{m} \frac{\beta_1 \beta_2 + \gamma (\beta_1 \alpha_2 - \beta_2 \alpha_1)}{\beta_1 - \beta_2 \alpha_2 - \gamma \alpha_1} \hat{F}_i \theta \right) = 0,$
The relations (36) are equivalent to the definition of momentum (29) and extrinsic curvature (28) in terms of metrics.

The equivalence of these first order equations to the original ones can be easily verified. The relations (32), (33), (36) allow one to express the variables $G, F, \pi$ in terms of $A, g$ in the form (28), (29), (30), (31). Upon substitution $G, F, \pi$ as functions of $A, g$ and their derivatives into the rest of equations, one arrives at the original system (1), (4).

Notice that the first order equations (32)-(39) cannot be deduced from the original equations by any Legendre transformation, because the system (1), (4) is non-Lagrangian with the general tensor (12) inserted in the rhs of Einstein’s equations (4).

In the next section, we find the Poisson bracket among the variables $A_i, F_i, G_i, \tilde{g}_{ij}, \pi_{ij}$ such that the equations (32)-(39) represent the first class constrained Hamiltonian system, with $A_0, N, N^i$ being the Lagrange multipliers at constraints $\theta, \tau, \tau_i$.

### 4 Poisson brackets and Hamiltonian equations

In previous section, we have described the first order formulation for the system of the third order extension of Chern-Simons (1) coupled to Einstein’s gravity (4) through transverse not necessarily canonical energy-momentum tensor (12). The first order formulation includes the evolutionary equations resolved with respect to the time derivatives of $A_i, F_i, G_i, \tilde{g}_{ij}, \pi_{ij}$ (32), (33), (34), (36), (37). There are also constraints (35), (38), (39) imposed on the same variables. The time derivatives of $A_0, N, N^i$ are not involved in the equations, while the variables themselves are linearly included in the rhs of evolutionary equations (32), (33), (34), (36), (37). This structure resembles the equations of Hamiltonian constrained system. By itself, this structure does not mean that the Poisson bracket exists such that the rhs of evolutionary equations are the Poisson brackets with Hamiltonian being the linear combination of constraints. Moreover, any system of reparameterization invariant differential equations can be cast into the first order normal form like that [(27)]:

\[
\dot{y}^I = Z^I_\alpha(y)\lambda^\alpha, \quad (40)
\]
\[
T_a(y) = 0. \quad (41)
\]

These equations constitute the constrained Hamiltonian system if the Poisson bracket \{y^I, y^J\} exists of the variables $y^I$ such that

\[
Z^I_\alpha(y)\lambda^\alpha = \{y^I, T_a\lambda^\alpha\} \equiv \{y^I, y^J\}\partial_J T_a\lambda^\alpha, \quad \text{(mod $T_a(y)$)}, \quad (42)
\]

where (mod $T_a(y)$) means that the equality holds true up to the terms vanishing on constraints $T$. Once the Poisson brackets exist obeying (42), the Hamiltonian is defined as the linear combination of constraints,

\[
H(y, \lambda) = \lambda^\alpha T_a(y), \quad (43)
\]

and the equations (40), (41) read as the constrained Hamiltonian system,

\[
\dot{y} = \{y, H(y, \lambda)\}, \quad T(y) = 0 \quad (44)
\]
Not any system of the normal form \([40], [41]\) admits the Poisson brackets obeying the conditions \([42]\). Given the equations of motion \([40]\), and constraints \([41]\), the relations \([42]\) can be considered as equations that define the bracket \(\{y^i, y^j\}\) which endows the system \([40], [41]\) with the structure of constrained Hamiltonian dynamics \([44]\). For general discussion of the Poisson structures compatible with normal forms of dynamical equations see \([27]\). Below, we are seeking for the Poisson brackets such that meet the conditions \([42]\) for the first order formulation \([32]-[39]\) of the original third order system \([1], [4]\).

The first order formulation \([32]-[39]\) of the original third order equations \([1], [4]\) have the form \([40], [41]\), with \([32]-[34], [36], [37]\) corresponding to the evolutionary equations \([40]\). Relations \([35], [38], [39]\) are constraints. We chose the ansatz which endows the system \([40], [41]\) with the structure of constrained Hamiltonian dynamics \([44]\). For general discussion of the Hamiltonian form \([44]\). This leads us to the system of linear algebraic equations \([42]\) for unknown Poisson brackets of the form \([44]\). This ansatz includes five independent parameters \(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma\) involved in the constraints. The parameters \(\alpha\) specify the equations for the vector field \([1]\). The parameters \(\beta, \gamma\) define the matter contribution to the rhs of Einstein’s equations \([4]\). These parameters are involved into the lapse and shift constraints \(\tau, \tau_i, \theta\) \([38], [39]\). The coefficient \(k_0\) is a constant factor at the Lagrange multiplier \(A_0\). We introduce \(k_0\) to conveniently control an overall multiplier at the gradient term \(\partial_i A_0\) in equations \([32]\). As the constraints are defined modulo overall non-vanishing factors, the redefinition is inessential.

We seek for the Poisson bracket assuming that the evolutionary equations \([32], [33], [34], [35], [36], [37]\) should have the Hamiltonian form \([44]\). This leads us to the system of linear algebraic equations \([42]\) for unknown Poisson brackets of the fields \(A_i, F_i, G_i, g_{ij}, \pi^i\):

\[
\{ A_i, H \}_{\alpha, \beta, \gamma} = N \sqrt{g} \varepsilon_{is} \hat{g}^{sr} F_r + \chi^k \varepsilon_{ki} \hat{e}^{sr} \partial_s A_r + \partial_i A_0; \tag{46}
\]

\[
\{ F_i, H \}_{\alpha, \beta, \gamma} = N \sqrt{g} \varepsilon_{is} \hat{g}^{sr} G_r + \chi^k \varepsilon_{ki} \hat{e}^{sr} \partial_s F_r + \partial_i \left( N \frac{1}{\sqrt{g}} \hat{e}^{sr} \partial_s A_r + N^s F_s \right); \tag{47}
\]

\[
\{ G_i, H \}_{\alpha, \beta, \gamma} = -\alpha_2 N \sqrt{g} \varepsilon_{is} \hat{g}^{sr} G_r - \chi^k \varepsilon_{ki} \hat{e}^{sr} \partial_s F_r - \chi^l \varepsilon_{kl} \hat{e}^{sr} \partial_s A_r + \chi^l \varepsilon_{kl} \hat{e}^{sr} \partial_s A_r + \partial_i \left( N \frac{1}{\sqrt{g}} \hat{e}^{sr} \partial_s A_r + N^s G_s \right); \tag{48}
\]

\[
\{ g_{ij}, H \}_{\alpha, \beta, \gamma} = \nabla_i N_j + \nabla_j N_i - 2N \frac{1}{\sqrt{g}} (\pi_{ij} - \hat{g}_{ij} \pi); \tag{49}
\]

\[
\{ \pi^i, H \}_{\alpha, \beta, \gamma} = -N \hat{\nabla} i N^j + \hat{\nabla} j N^i - 2N \frac{1}{\sqrt{g}} (\pi^i - \hat{g}^{ij} \pi^j) + \frac{1}{2} N \frac{1}{\sqrt{g}} \hat{g}^{ij} (\pi_{sr} \hat{g}^{sr} - \pi^2) + \sqrt{g} \left( \hat{\nabla} ^i \hat{\nabla} j N - \hat{g}^{ij} \hat{\nabla} s N \right) - \frac{1}{2} N \sqrt{g} \hat{g}^{ij} \pi^r = -\frac{1}{2} \frac{\beta_1}{m} N \sqrt{g} \hat{g}^{ik} \hat{g}^{jl} \left[ G_k F_l + G_l F_k - \hat{g}_{kl} (\hat{G}^0 \hat{F}^0 + \hat{g}^{sr} G_s F_r) \right] + \alpha_2 m \left( F_k F_l - \frac{1}{2} \hat{g}_{kl} \left( \hat{F}^0 \hat{F}^0 + \hat{g}^{sr} F_s F_r \right) \right) - \frac{1}{2} \frac{\beta_2}{m^2} N \sqrt{g} \hat{g}^{ik} \hat{g}^{jl} \left[ G_k G_l - \frac{1}{2} \hat{g}_{kl} (\hat{G}^0 \hat{G}^0 + \hat{g}^{sr} G_s G_r) \right] - \alpha_1 m \left( F_k F_l - \frac{1}{2} \hat{g}_{kl} \left( \hat{F}^0 \hat{F}^0 + \hat{g}^{sr} F_s F_r \right) \right). \tag{50}
\]
Given the Hamiltonian \( (45) \), the solution to this system reads

\[
\{ \hat{g}_{ij}(\vec{x}), \pi^k(\vec{y}) \}_{\alpha,\beta,\gamma} = \frac{1}{2} \left( \delta_i^\alpha \delta_j^\beta + \delta_j^\alpha \delta_i^\beta \right) \delta(\vec{x} - \vec{y}); \tag{51}
\]

\[
\{ G_i(\vec{x}), G_j(\vec{y}) \}_{\alpha,\beta,\gamma} = m^3 \frac{\beta_1 (\alpha_1 - \alpha_2^2) + \beta_2 \alpha_1 \alpha_2}{\beta_1^2 - \beta_1 \beta_2 \alpha_2 + \beta_2^2 \alpha_1} \varepsilon_{ij} \delta(\vec{x} - \vec{y}); \tag{52}
\]

\[
\{ F_i(\vec{x}), G_j(\vec{y}) \}_{\alpha,\beta,\gamma} = m^2 \frac{\beta_1 \alpha_2 - \beta_2 \alpha_1}{\beta_1^2 - \beta_1 \beta_2 \alpha_2 + \beta_2^2 \alpha_1} \varepsilon_{ij} \delta(\vec{x} - \vec{y}); \tag{53}
\]

\[
\{ A_i(\vec{x}), G_j(\vec{y}) \}_{\alpha,\beta,\gamma} = \frac{m}{\beta_1^2 - \beta_1 \beta_2 \alpha_2 + \beta_2^2 \alpha_1} \varepsilon_{ij} \delta(\vec{x} - \vec{y}); \tag{54}
\]

\[
\{ A_i(\vec{x}), F_j(\vec{y}) \}_{\alpha,\beta,\gamma} = \frac{\gamma}{\beta_1^2 - \beta_1 \beta_2 \alpha_2 + \beta_2^2 \alpha_1} \varepsilon_{ij} \delta(\vec{x} - \vec{y}), \tag{55}
\]

\[
\{ \hat{g}_{ij}, \pi^k \}_{\alpha,\beta,\gamma} = \{ \pi^{ij}, \pi^{kl} \}_{\alpha,\beta,\gamma} = \{ \hat{g}_{ij}, A_k \}_{\alpha,\beta,\gamma} = \{ \hat{g}_{ij}, F_k \}_{\alpha,\beta,\gamma} = \{ \pi^{ij}, G_k \}_{\alpha,\beta,\gamma} = \{ \pi^{ij}, A_k \}_{\alpha,\beta,\gamma} = \{ \pi^{ij}, F_k \}_{\alpha,\beta,\gamma} = \{ \pi^{ij}, G_k \}_{\alpha,\beta,\gamma} = 0, \tag{56}
\]

where the vectors \( \vec{x}, \vec{y} \) label the space points, and \( \delta(\vec{x} - \vec{y}) \) is the \( 2d \) \( \delta \)-function. As is seen from the equations (51)-(57), the Poisson bracket of the gravity variables \( \hat{g}_{ij}, \pi^k \), (51) is canonical. This happens because the Einstein’s equations without matter are Lagrangian in themselves, while the matter contribution to the Einstein’s equations (4), being expressed in terms of the variables \( F, G \), does not involve derivatives of the metric. The Poisson brackets (52)-(56) of the vector variables \( A, F, G \) involve the parameters \( \alpha, \beta, \gamma \) of the model (1), (4), and they are non-canonical. The parameter \( \gamma \) contributes to the bracket (56) between space components of the original vector field \( A_i \), while the Poisson bracket of the physical observables \( G_i, F_i, F_0 \) does not depend on \( \gamma \). This is no surprise because coupling constants in the theory (1), (4) are \( \beta_1, \beta_2 \), while \( \gamma \) labels equivalent representatives of one and the same theory, so it can only contribute to the brackets of non-gauge invariant quantities.

All the Poisson brackets are well-defined if the \( 00 \)-component \( T^{00}(\alpha, \beta) \) of the matter contribution to Einstein’s equations (4) is a non-degenerate quadratic form of the fields \( F, G \), see condition (26). In this way we see that the theory (1), (4) is Hamiltonian with the Hamilton function (45) and Poisson brackets (51)-(57).

Given the Poisson brackets (51)-(57), consider the PB algebra of the constraints \( \tau \) (38), \( \tau_i \) (39), \( \theta \) (35). Introducing the test functions \( \mathcal{M}(x), \mathcal{M}^i(x), A(x) \), we define the functional of constraints \( T[\mathcal{M}], D[\mathcal{M}], \Theta[A] \):

\[
T[\mathcal{M}] = \int d^d x \sqrt{\hat{g}} \mathcal{M}(x) \tau(x), \quad D[\mathcal{M}] = \int d^d x \sqrt{\hat{g}} \mathcal{M}^i(x) \tau_i(x), \quad \Theta[A] = \int d^d x \sqrt{\hat{g}} A(x) \theta(x). \tag{58}
\]

Then, these functional have the following Poisson brackets:

\[
\{ T[\mathcal{M}], T[N] \} = D[N \hat{\nabla}^i \mathcal{M} - \mathcal{M} \hat{\nabla}^i N],
\]

13
As is seen, the PBs of constraints $\tau, \tau_i$ correspond to the algebra of lapse and shift transformations modulo the Gauss law constraint $\theta$. The Gauss law constraint $\theta$ has identically vanishing Poisson brackets with all the constraints. These PB relations indicate that the constraints $\tau, \tau_i$ generate the lapse and shift transformations for all the fields in theory, while the Gauss law constraint $\theta$ generates the gradient gauge transformations for components of original vector field $A_i$ as it does in free theory without matter.

Let us provide explicit expressions for Poisson brackets of the fields $\hat{g}_{ij}, \pi^{ij}, G_i, F_i, A_i$ with the constraints $T\{M, D[M], \Theta, A\}$:  

\[
\{ T\{M, D[M], \Theta, A\} \}_a = 0, \quad \{ D[M], \Theta, A\}_a = 0, \quad \{ \Theta, \Theta, A\}_a = 0, \quad \{ \Theta, \Theta, \Theta\}_a = 0, \quad \{ T[M], \Theta, A\}_a = 0, \quad \{ D[M], \Theta, A\}_a = 0, \quad \{ \Theta, \Theta, A\}_a = 0, \quad \{ \Theta, \Theta, \Theta\}_a = 0.
\]

\[
k_1 = \frac{\beta_1^2 - \beta_1 \beta_2 \alpha_2 + \beta_2 \alpha_1}{\beta_1 - \beta_2 \alpha_2 - \gamma \alpha_1}, \quad k_2 = \frac{\gamma^2 (\beta_1 \alpha_2 - \beta_2 \alpha_1) + 2 \gamma \beta_1 \beta_2 + \beta_2^2}{(\beta_1 - \beta_2 \alpha_2 - \gamma \alpha_1)^2}, \quad k_3 = \frac{1}{m} \frac{\gamma (\beta_1^2 - \beta_1 \beta_2 \alpha_2 + \beta_2^2 \alpha_1)}{(\beta_1 - \beta_2 \alpha_2 - \gamma \alpha_1)^2}.
\]  

(59)
\[
\{ G_i, D[M] \}_{\alpha, \beta, \gamma} = \mathcal{M}^* \partial_\nu G_i + G_\nu \partial_\lambda \mathcal{M}^* - \frac{\beta_1}{\beta_1 - \beta_2 \alpha_2 - \gamma \alpha_1} m \mathcal{M}^* \sqrt{g} \varepsilon_{\alpha_1 \beta_1 \gamma};
\]
(67)

\[
\{ \hat{g}_{ij}, D[M] \}_{\alpha, \beta, \gamma} = \mathcal{M}^* \partial_\nu g_{ij} + g_{\nu i} \partial_\lambda \mathcal{M}^* + g_{\nu j} \partial_\lambda \mathcal{M}^*;
\]
(68)

\[
\{ \pi^{ij}, D[M] \}_{\alpha, \beta, \gamma} = -\pi^{is} \partial_\nu M^j - \pi^{js} \partial_\nu M^i + \partial_\nu (\mathcal{M}^* \pi^{ij});
\]
(69)

\[
\{ A_i, \Theta[A] \}_{\alpha, \beta, \gamma} = \frac{\beta_1 - \beta_2 \alpha_2 - \gamma \alpha_1}{\beta_1^2 - \beta_1 \beta_2 \alpha_2 + \beta_2^2 \alpha_1} \partial_\lambda A,
\]
(70)

\[
\{ F_i, \Theta[A] \}_{\alpha, \beta, \gamma} = \{ G_i, \Theta[A] \}_{\alpha, \beta, \gamma} = \{ \hat{g}_{ij}, \Theta[A] \}_{\alpha, \beta, \gamma} = \{ \pi^{ij}, \Theta[A] \}_{\alpha, \beta, \gamma} = 0,
\]
(71)

where \(A^0\) denotes the time component of 3d vector \(A^\alpha\), and \(\hat{F}^0, \hat{G}^0\) are introduced in (24). On the shell of Gauss law constraint (55) and evolutionary equations of motion (32), (33), (34), (36), (37), the Poisson brackets (61)–(64), (66)–(69) correspond to the transformation law for the quantities \(F, G, \hat{g}, \pi\) under the action of infinitesimal diffeomorphisms. The constraint functional \(T[M]\) generates shifts along the timelike vector \(\xi = (M/N, -MN^i/N)\), while \(D[M]\) acts by space shifts, with \(\zeta = (0, M^i)\) being the shift vector. These geometric interpretations suggest that the constraints \(\tau\) and \(\tau_i\) generate the lapse and shift transformations for each tensor quantity \(O(F, G, \hat{g}, \pi)\),

\[
\{ O(F, G, \hat{g}, \pi), T[M] \}_{\alpha, \beta, \gamma} = L_\xi O(F, G, \hat{g}, \pi), \quad \xi = (M/N, -MN^i/N),
\]
(72)

\[
\{ O(F, G, \hat{g}, \pi), D[M] \}_{\alpha, \beta, \gamma} = L_\zeta O(F, G, \hat{g}, \pi), \quad \zeta = (0, M^i).
\]
(73)

The time derivatives are defined in the rhs of equations (72) by the equations of motion (32), (37). Relation (73) holds true modulo Gauss law constraint (55). The brackets of the field \(A\) with \(T[M], D[M]\) represent the lapse and shift transformations with certain admixture of gradient gauge transformations for the vector potential

\[
\{ A_i, T[M] \}_{\alpha, \beta, \gamma} = (L_\xi A)_i - \partial_\lambda (M N A^\lambda), \quad \{ A_i, D[M] \}_{\alpha, \beta, \gamma} = (L_\zeta A)_i - \partial_\lambda (M^* A^\lambda), \quad i = 1, 2.
\]
(74)

The admixture of gauge transformation seems to be admissible because the vector potential is not a gauge invariant quantity. It is not even 1-form, being rather 1-gerbe. Because of that, it seems natural that the lapse and shift transformations for \(A\) may involve an admixture of the gauge transformation adding the exact 1-form to the gerbe. The gauge invariant physical observables \(F\) and \(G\) transform under the lapse and shift in the usual way, as one-forms. The latter fact identifies the constraints \(\tau, \tau_i\) with the lapse and shift Hamiltonian generators.

In constrained Hamiltonian formalism, the strongly conserved energy of the model is usually identified with the constraint that generates lapse transformations. In the model (11), (12), the matter contribution to the lapse constraint \(\tau\) (38) is given by the 00-component \(T^{00}(\alpha, \beta, \gamma)\) (22) of the on-shell covariantly transverse tensor \(T^{\mu\nu}(\alpha, \beta, \gamma)\) (12). Under the relations (25), the matter contribution to the lapse constraint is bounded. This fact provides another evidence of stability of the model.

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*The relation (74) is understood in the same sense as (72), (73), i.e. with the time derivatives defined by equations of motion, and modulo Gauss constraint.*
5 Concluding remarks

In this paper we focus at three aspects of the third order extension of Chern-Simons. At first, we introduce gravity into the field equations \([1]\) in a minimal way. Then we notice that the theory admits a two-parameter series of on-shell covariantly transverse tensors \([2]\). This leaves some freedom in consistent inclusion of the matter into the rhs of Einstein’s equations \([4]\) because any of the transverse tensors fits well to this role. Second, we see that some of the admissible couplings with gravity meet the weak energy condition, so they are stable, while the interaction through the canonical energy-momentum breaks stability. The third point is that the inclusion of interaction with gravity through the non-canonical energy-momentum, being a non-Lagrangian coupling, still admits constrained Hamiltonian formulation of the corresponding first order equations. So, the theory admits quantization while the interaction is not necessarily Lagrangian.

Let us mention in the end, that even though Einstein’s gravity does not have its own local degrees of freedom in 3d, inclusion of interaction with matter subject to higher derivative equations is a nontrivial issue, especially from the viewpoint of maintaining stability. Even if the matter dynamics is stable in the flat space due to conservation of certain tensor which differs from the canonical energy-momentum, it is unclear a priori why and how the stability can persist in the non-flat Einstein’s space that corresponds to the non-trivial energy-momentum in the rhs. This work suggests the pattern of construction that can answer to this question. The construction, in fact, is not too sensitive to the dynamical content of gravity. One can expect that the similar pattern should work in higher dimensions, where Einstein’s gravity has its own local degrees of freedom. In \(d > 3\), however, the explicit construction of consistent and stable coupling of gravity with higher derivative matter can become more complicated in certain respect. The construction of transverse tensors meeting the WEC for higher derivative field equations seems basically following the same pattern in \(d > 3\) as in \(d = 3\). It is the non-canonical construction of constrained Hamiltonian formalism which may seem a more complex issue in \(d > 3\).

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