Sharp estimates for the Neumann functions and applications to quantitative photo-acoustic imaging in inhomogeneous media

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Abstract

We obtain sharp $L^p$ and Hölder estimates for the Neumann function of the operator $\nabla \cdot \gamma \nabla - ik$ on a bounded domain. We also obtain quantitative description of its singularity. We then apply these estimates to quantitative photo-acoustic imaging in inhomogeneous media. The problem is to reconstruct the optical absorption coefficient of a diametrically small anomaly from the absorbed energy density.

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1 Introduction and motivation

The purpose of this paper is to derive sharp estimates of the Neumann function of the operator $\nabla \cdot \gamma \nabla - ik$ and its derivatives, where $\gamma$ is an (scalar) elliptic coefficient defined on a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \geq 3$) and $k$ is a positive constant. The Neumann function of $\nabla \cdot \gamma \nabla - ik$ in $\Omega$ is the function $N : \Omega \times \Omega \rightarrow \mathbb{C} \cup \{\infty\}$ satisfying

\[
\begin{cases}
-(\nabla \cdot \gamma \nabla - ik)N(\cdot, y) = \delta_y & \text{in } \Omega, \\
\gamma \nabla N(\cdot, y) \cdot n = 0 & \text{on } \partial \Omega,
\end{cases}
\]

for all $y \in \Omega$, where $\delta_y$ is the Dirac mass at $y$ and $n$ is the outward unit normal vector field on $\partial \Omega$ (see subsection 22 for a precise definition of the Neumann function). The function $N(x, y)$ has a singularity at $x = y$. We are particularly interested in describing in a quantitative manner the singularity of $N(x, y)$ and its dependence on the parameter $k$.

The investigation of this paper is motivated by quantitative photo-acoustic imaging, particularly by the recent work 2.

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The purpose of quantitative photo-acoustic imaging is to image the optical absorption coefficient from the absorbed energy. The absorbed energy is obtained from boundary measurements of the pressure wave induced by the photoacoustic effect. We refer to [1] and references therein for recent development on this inverse problem. Reconstruction of the optical absorption coefficient, $\mu_a$, from the absorbed energy, $A$, is more delicate than the reconstruction of the absorbed energy from the pressure wave since $\mu_a$ is related to $A$ in an implicit and non-linear way (see Section 3). One direction of research in quantitative photo-acoustic imaging is to reconstruct the absorption coefficient of diametrically small unknown anomalies. In [2, 3], efficient methods to reconstruct $\mu_a$ from $A$ are proposed and implemented numerically when there is a small absorbing anomaly in the background medium. The methods use in an essential way an asymptotic expansion of $A$ in terms of $\mu_a$ when the diameter of the anomaly tends to 0. The asymptotic expansion is derived using estimates of the Neumann function under the assumption that the scattering coefficient of the medium is constant. In order to extend the results of [2, 3] to inhomogeneous media, we shall derive sharp estimates of the Neumann function of problem (1.1), which is exactly what this paper aims at.

To describe the kinds of results obtained in this paper, let us fix a point $z \in \Omega$ ($z$ indicates the location of the anomaly), and let $\gamma^* := \gamma(z)$. Let $\Gamma(x) := -1/(4\pi|x|)$ be a fundamental solution of the Laplacian in three dimensions. Then, we will show by precise estimates depending on $k$ that the singularity of $N(x, z)$ for $x$ near $z$ is of the form $\frac{1}{\gamma^*}\Gamma(x - z)$. We also show that the singularity of the derivatives of $N(x, z)$ is given by the derivatives of $\frac{1}{\gamma^*}\Gamma(x - z)$. We also derive $L^p$, pointwise, and Hölder estimates of the Neumann function $N$. We then use these estimates to derive an asymptotic expansion in inhomogeneous media where the scattering coefficient $\mu_s$ is not constant.

This paper is organized as follows. In Section 2, we derive $L^p$ and pointwise estimates of the Neumann function $N$. In Section 3, we show how these estimates can be used for reconstructing the absorption coefficient of a small absorbing anomaly.

## 2 Estimates for Neumann functions

This section is devoted to the study of the Neumann function for the operator $L$ given by

$$Lu = \nabla \cdot (\gamma \nabla u) -iku$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ with $d \geq 3$. Here, we assume that $k$ is a positive constant satisfying $k \geq k_0$ for some $k_0 > 0$ and $\gamma : \Omega \to \mathbb{R}$ satisfies the uniform ellipticity condition

$$\nu \leq \gamma(x) \leq \nu^{-1}, \quad \forall x \in \Omega,$$

for some constant $\nu \in (0, 1]$.

We first introduce some (standard) notation and definitions that will be used throughout the paper. Let $\Omega \in \mathbb{R}^d \ (d \geq 3)$ be a bounded Lipschitz domain. We call diam($\Omega$) the least upper bound of the distances between pairs of points in $\Omega$. We say that a function $f$ on $\Omega$ admits a modulus of continuity $\theta$ if $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that

$$|f(x) - f(y)| \leq \theta(|x - y|), \quad \forall x, y \in \Omega.$$

For $0 < \lambda < 1$ and $f \in C^{0,\lambda}(\Omega)$, we let $[f]_{0,\lambda,\Omega}$ denote the $\lambda$-Hölder seminorm of $f$ in $\Omega$; i.e.,

$$[f]_{0,\lambda,\Omega} = \sup_{x, y \in \Omega; x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\lambda}.$$
For $p \geq 1$ and $m$ a non-negative integer, we define the space $W^{m,p}(\Omega)$ as the family of all $m$ times weakly differentiable functions in $L^p(\Omega)$, whose weak derivatives of orders up to $m$ are functions in $L^p(\Omega)$. We let $W_0^{m,p}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$, where $C_0^\infty(\Omega)$ is the set of all infinitely differentiable functions with compact supports in $\Omega$. We use $C_{loc}^{\alpha,\beta}(\Omega)$ and $W^{m,p}_{loc}(\Omega)$ to denote the local spaces of functions belonging respectively to $C^{0,\alpha}(\Omega')$ and $W^{m,p}(\Omega')$ for all $\Omega' \subset \subset \Omega$. We write $u \in L^p(\Omega; \mathbb{C})$ (or $u \in W^{m,p}(\Omega; \mathbb{C})$, etc.) to emphasize that $u$ is a complex valued function. We recall that for $m = 1$ and $p = 2$, the spaces $W^{1,2}(\Omega; \mathbb{C})$ and $W_0^{1,2}(\Omega; \mathbb{C})$, equipped with the inner product

$$\langle u, v \rangle := \int_\Omega \nabla u \cdot \nabla v + uv \, dx,$$

are Hilbert spaces. Finally, for $p > 1$ and $q$ being its conjugate exponent, i.e., $1/p + 1/q = 1$, we use $W^{-1,q}(\Omega; \mathbb{C})$ and $W_0^{-1,q}(\Omega; \mathbb{C})$ to respectively denote the dual spaces to $W^{1,p}(\Omega; \mathbb{C})$ and $W^{1,2}(\Omega; \mathbb{C})$.

Our main result in this section is the following.

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^1$ domain. Let $\gamma \in C^{0,\lambda}(\overline{\Omega})$ for some $\lambda \in (0, 1)$. Let $N$ be the Neumann function of $L$ in $\Omega$. For $y \in \Omega$, denote $\gamma_0 = \gamma(y)$ and let $N_0$ be the Neumann function for $L_0 = \nabla \cdot \gamma_0 \nabla - ik$ in $\Omega$. Then we have

$$|N(x, y) - N_0(x, y)| \leq C|x - y|^{2-d+\lambda}, \quad \forall x \in \Omega, \quad x \neq y,$$

(2.2)

where $C$ is a constant depending only on $d, \nu, k_0, \lambda, \Omega$, and $[\gamma]_{0,\lambda,\Omega}$. Also, if $0 < |x - y| < d_y/2$, where $d_y = \text{dist}(y, \partial \Omega)$, then we have

$$|\nabla_x (N(x, y) - N_0(x, y))| \leq C \left(|x - y|^{1-d+\lambda} + k|x - y|^{3-d+\lambda}\right),$$

(2.3)

where the constant $C$ depends on $\text{diam} \Omega$ as well. Moreover, if we assume further that $\gamma \in C^{1,\lambda}(\overline{\Omega})$, then for all $x \in \Omega$ satisfying $0 < |x - y| < d_y/2$, we have

$$|\nabla_x(N(x, y) - N_0(x, y))| \leq C|x - y|^{1-d+\lambda},$$

(2.4)

$$|\nabla^2_x(N(x, y) - N_0(x, y))| \leq C \left(|x - y|^{-d+\lambda} + k|x - y|^{2-d+\lambda}\right),$$

(2.5)

where $C$ depends only on $||\gamma||_{C^{1,\lambda}(\overline{\Omega})}, \ d, \nu, k_0, \lambda, \Omega$, and $\text{diam} \Omega$.

In this section, we first consider the Neumann boundary value problems for the operators $L$ and its adjoint $L^*$ given by

$$L^* := \nabla \cdot \gamma \nabla + ik.$$

(2.6)

Then we give a definition of a Neumann function. Next, we construct Neumann functions, $N$ and $N^*$, of respectively $L$ and $L^*$ in $\Omega$. Our construction of $N$ and $N^*$ holds for a Lipschitz bounded domain $\Omega$ and a coefficient $\gamma$ uniformly continuous on $\Omega$. If we further assume that $\Omega$ is of class $C^1$, then we are able to derive $L^p$ estimates for the operators $L$ and $L^*$ with Neumann boundary conditions on $\partial \Omega$. Finally, based on the following global pointwise bound for the Neumann function $N$:

$$|N(x, y)| \leq C|x - y|^{2-d} \quad \text{for all } x, y \in \Omega \text{ with } x \neq y,$$

(2.7)

where $C$ depends only on $d, \nu, \Omega, k_0$, and $\theta$ (a modulus of continuity of $\gamma$), we describe the local behavior of $N$ such as (2.2). Assuming that $\gamma \in C^{0,\lambda}(\overline{\Omega})$, for $0 < \lambda < 1$, we prove that estimates (2.2)–(2.5) hold.
Estimates of \([2.8]\)-type were derived for the Dirichlet Green's function of \(L\) with \(k = 0\) and \(\gamma \in L^\infty(\Omega)\) in \([18, 12]\). Under the further assumption that the principal coefficients are uniformly continuous of belong to the class VMO, they were generalized to the vectorial case in \([9, 7, 13, 15]\) and to the periodic case in \([5, 16]\).

### 2.1 Neumann boundary value problem

We begin with the weak formulation of the Neumann boundary value problem

\[
\begin{aligned}
-Lu &= f + \nabla \cdot F \quad \text{in } \Omega, \\
(\gamma \nabla u + F) \cdot n &= g \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(2.8)

where \(f \in L^1_{\text{loc}}(\Omega; \mathbb{C})\), \(F \in L^1_{\text{loc}}(\Omega; \mathbb{C}^d)\), and \(g \in L^1_{\text{loc}}(\partial \Omega; \mathbb{C})\). We say that \(u \in W^{1,1}_{\text{loc}}(\Omega)\) is a weak solution of problem \((2.8)\) if the following identity holds:

\[
\int_{\Omega} (\gamma \nabla u \cdot \nabla \phi + i ku \overline{\phi}) \, dx = \int_{\Omega} (\nabla \phi \cdot F - \overline{\nabla \phi}) \, dx + \int_{\partial \Omega} g \overline{\phi} \, d\sigma, \quad \forall \phi \in C^\infty(\Omega; \mathbb{C}).
\]

Let \(H = W^{1,2}(\Omega; \mathbb{C})\). We define the sesquilinear form \(B(\cdot, \cdot) : H \times H \to \mathbb{C}\), associated to the operator \(L\), as

\[
B(u, v) := \int_{\Omega} (\gamma \nabla u \cdot \nabla v + i ku \overline{v}) \, dx.
\]

It is easy to check that \(B\) is bounded and coercive.

Let \(f \in L^{2d/(d+2)}(\Omega; \mathbb{C})\), \(F \in L^2(\Omega; \mathbb{C}^d)\), and \(g \in L^2(\partial \Omega; \mathbb{C})\). Then by the Sobolev embedding and the trace theorem, we find that

\[
F(v) := \int_{\Omega} (\nabla \phi \cdot F - \overline{\nabla \phi}) \, dx + \int_{\partial \Omega} g \overline{\phi} \, d\sigma
\]

is a bounded skew-linear functional on \(H\). Therefore, by the Lax-Milgram lemma, we find that there exists a unique \(u \in H\) such that

\[
B(u, v) = F(v), \quad \forall v \in H.
\]

We have thus shown that if \(f \in L^{2d/(d+2)}(\Omega; \mathbb{C})\), \(F \in L^2(\Omega; \mathbb{C}^d)\), and \(g \in L^2(\partial \Omega; \mathbb{C})\), then problem \((2.8)\) has a unique weak solution \(u \in W^{1,2}(\Omega; \mathbb{C})\). Since \(C^\infty(\overline{\Omega}; \mathbb{C})\) is dense in \(W^{1,2}(\Omega; \mathbb{C})\), we find that \(u\) satisfies following identity:

\[
\int_{\Omega} (\gamma \nabla u \cdot \nabla v + i ku \overline{v}) \, dx = \int_{\Omega} (\nabla \phi \cdot F - \overline{\nabla \phi}) \, dx + \int_{\partial \Omega} g \overline{\phi} \, d\sigma, \quad \forall v \in W^{1,2}(\Omega; \mathbb{C}).
\]  

(2.9)

Let \(L^*\) be given by \((2.6)\). By the same reasoning, we find that there exists a unique weak solution \(u\) in \(W^{1,2}(\Omega; \mathbb{C})\) of problem

\[
\begin{aligned}
-L^*u &= f + \nabla \cdot F \quad \text{in } \Omega, \\
(\gamma \nabla u + F) \cdot n &= g \quad \text{on } \partial \Omega,
\end{aligned}
\]

provided \(f \in L^{2d/(d+2)}(\Omega; \mathbb{C})\), \(F \in L^2(\Omega; \mathbb{C}^d)\), and \(g \in L^2(\partial \Omega; \mathbb{C})\); i.e.,

\[
\int_{\Omega} (\gamma \nabla u \cdot \nabla v - i ku \overline{v}) \, dx = \int_{\Omega} (\nabla \phi \cdot F - \overline{\nabla \phi}) \, dx + \int_{\partial \Omega} g \overline{\phi} \, d\sigma, \quad \forall v \in W^{1,2}(\Omega; \mathbb{C}).
\]  

(2.10)
2.2 Definition of the Neumann function

We say that a function $N : \Omega \times \Omega \to \mathbb{C} \cup \{\infty\}$ is a Neumann function of $L$ in $\Omega$ if it satisfies the following properties:

i) $N(\cdot, y) \in W^{1,1}_{\text{loc}}(\Omega)$ and $N(\cdot, y) \in W^{1,2}(\Omega \setminus B_r(y))$ for all $y \in \Omega$ and $r > 0$.

ii) $N(\cdot, y)$ is a weak solution of
\[
\begin{cases}
-LN(\cdot, y) = \delta_y & \text{in } \Omega, \\
\gamma \nabla N(\cdot, y) \cdot n = 0 & \text{on } \partial \Omega,
\end{cases}
\]
for all $y \in \Omega$ in the sense
\[
\int_{\Omega} (\gamma(x) \nabla_x N(x, y) \cdot \nabla \phi(x) + ikN(x, y)\overline{\phi(x)}) \, dx = \overline{\phi(y)}, \quad \forall \phi \in C^\infty(\overline{\Omega}; \mathbb{C}).
\]

iii) For any $f \in C^\infty_{\text{c}}(\Omega; \mathbb{C})$, the function $u$ given by
\[
u(x) := \int_{\Omega} N(y, x) f(y) \, dy \tag{2.11}
\]
is the unique solution in $W^{1,2}(\Omega)$ of problem
\[
\begin{cases}
-L^* u = f & \text{in } \Omega, \\
\gamma \nabla u \cdot n = 0 & \text{on } \partial \Omega.
\end{cases} \tag{2.12}
\]

We remark that part iii) of the above definition gives the uniqueness of a Neumann function. Indeed, let $\tilde{N}(x, y)$ be another function satisfying the above properties. Then by the uniqueness of a solution in $W^{1,2}(\Omega; \mathbb{C})$ of problem (2.12), we have
\[
\int_{\Omega} (\tilde{N} - N)(y, x) f(y) \, dy = 0, \quad \forall f \in C^\infty_{\text{c}}(\Omega; \mathbb{C}),
\]
and thus we conclude that $N = \tilde{N}$ a.e. in $\Omega \times \Omega$.

2.3 Local boundedness estimates

Let $B_R = B_R(x_0)$ be the ball of radius $R$ centered at $x_0$, and let $u \in W^{1,2}(B_R)$ be a weak solution of $-Lu = 0$ in $B_R$. For $0 < \rho < R$, let $\eta$ be a smooth cut-off function satisfying
\[0 \leq \eta \leq 1, \quad \text{supp} \ \eta \subset B_R, \quad \eta \equiv 1 \text{ on } B_\rho, \quad \text{and} \quad |\nabla \eta| \leq 2/(R - \rho).
\]

By taking $\eta^2 \overline{\eta}$ as a test function, we get
\[
\int_{B_R} \gamma \eta^2 |\nabla u|^2 \, dx = - \int_{B_R} 2\gamma \overline{\eta} \nabla u \cdot \nabla \eta \, dx + ik \int_{B_R} \eta^2 |u|^2 \, dx.
\]

By taking real parts in the above and using Cauchy’s inequality, we get
\[
\int_{B_R} \gamma \eta^2 |\nabla u|^2 = - \Re \int_{B_R} 2\gamma \overline{\eta} \nabla u \cdot \nabla \eta \, dx \leq \frac{1}{2} \int_{B_R} \gamma \eta^2 |\nabla u|^2 \, dx + 2 \int_{B_R} \gamma |\nabla \eta|^2 |u|^2 \, dx. \tag{2.13}
\]
Therefore, we obtain Caccioppoli’s inequality
\[
\int_{B_r} |\nabla u|^2 \, dx \leq \frac{C}{(R-\rho)^2} \int_{B_R} |u|^2 \, dx,
\]
where \( C = C(\nu) \).

Next, we consider the operator \( L_0 \) defined by
\[
L_0 u = \nabla \cdot (\gamma_0 \nabla u) - iku = \gamma_0 \Delta u - iku,
\]
where \( \gamma_0 \) is a constant satisfying the condition \( \ref{eq:2.13} \). Let \( u \in W^{1,2}(B_1) \) be a weak solution of \(-L_0 u = 0\). Since \( L_0 \) has constant coefficients, we may apply \( \ref{eq:2.14} \) iteratively to get
\[
\|u\|_{W^{m,2}(B_{1/2})} \leq C(m,\nu)\|u\|_{L^2(B_1)}, \quad m = 1, 2, \ldots.
\]

By the Sobolev embedding theorem, we then have
\[
\sup_{B_{1/2}} |u| \leq C(d)\|u\|_{W^{m,2}(B_{1/2})} \leq C(d,\nu)\|u\|_{L^2(B_1)},
\]
where \( m = \lfloor d/2 \rfloor + 1 \). Here and throughout this paper \([s]\) denotes the smallest integer not less than \( s \). Since the above estimate does not depend on \( k \), by a scaling argument we conclude that if \( u \in W^{1,2}(B_R) \) is a weak solution of \(-L_0 u = 0\) in \( B_R \), then we have
\[
\sup_{B_{R/2}} |u| \leq C(d,\nu)R^{-d/2}\|u\|_{L^2(B_R)}.
\]

Similarly, if \( u \in W^{1,2}(B_R) \) is a weak solution of \(-L_0 u = 0\) in \( B_R \), then we have
\[
\sup_{B_{R/2}} |\nabla u| \leq CR^{-d/2}\|\nabla u\|_{L^2(B_R)}.
\]

It follows from the above estimate that for all \( 0 < \rho < r \leq R \), we have
\[
\int_{B_{\rho}} |\nabla u|^2 \, dx \leq C(\rho/r)^d \int_{B_{\rho}} |\nabla u|^2 \, dx,
\]
where \( C = C(d,\nu) \). Indeed, in the case when \( \rho < r/2 \), we utilize \( \ref{eq:2.16} \) to get the above estimate; otherwise, we may simply take \( C = 2^d \) in \( \ref{eq:2.17} \).

Observe that the same estimates are valid for \( u \in W^{1,2}(B_R) \) satisfying \(-L_0^* u = 0\) weakly in \( B_R \), where \( L_0^* \) is defined as \( L_0^* = \nabla \cdot \gamma_0 \nabla + ik \).

**Lemma 2.2** Assume that \( \gamma \in C^0(\overline{\Omega}) \) and let \( \theta \) be a modulus of continuity of \( \gamma \). Let \( B_R = B_R(x_0) \subset \Omega \) and let \( u \in W^{1,2}(B_R;\mathbb{C}) \) be a weak solution of either \(-Lu = f + \nabla \cdot F \) or \(-L^* u = f + \nabla \cdot F \) in \( B_R \), where \( f \in L^q(B_R;\mathbb{C}) \) with \( q > d/2 \) and \( F \in L^p(B_R;\mathbb{C}^d) \) with \( p > d \). Then \( u \) is locally Hölder continuous in \( B_R \) and the following estimate holds:
\[
R^{\lambda_0}[u]_{0,\lambda_0;B_{R/2}} \leq C_0 \left( R^{-d/2}\|u\|_{L^2(B_R)} + R^{2-d/q}\|f\|_{L^q(B_R)} + R^{1-d/p}\|F\|_{L^p(B_R)} \right),
\]
where \( \lambda_0 \in (0,1) \) and \( C_0 \) are constants depending on \( d,\nu,p,q,\theta \), and \( \lambda_0 \)-Hölder seminorm of \( u \) in \( D \). Moreover, for any \( p_0 > 0 \) and \( 0 < \rho < R \), we have
\[
\sup_{B_{\rho}} |u| \leq C \left( (R-\rho)^{-d/p_0} \left( \int_{B_R} |u|^{p_0} \, dx \right)^{1/p_0} + R^{2-d/q}\|f\|_{L^q(B_R)} + R^{1-d/p}\|F\|_{L^p(B_R)} \right)^{1/p_0},
\]
where \( C \) depends on \( d,\nu,p,q,p_0 \), and \( \theta \).
Proof. We consider the case when $u$ is a weak solution of

$$-Lu = f + \nabla \cdot F \quad \text{in } B_R. \quad (2.20)$$

The proof for the other case is identical. Let $R_0 > 0$ be a number to be fixed later. Let $y \in B_R$ and $0 < r \leq R_0$ be arbitrary but fixed. Denote $\gamma_0 = \gamma(y)$ and let $L_0$ be defined as in (2.15). Observe that $u$ is a weak solution of

$$-L_0u = f + \nabla \cdot F + \nabla \cdot ((\gamma - \gamma_0)\nabla u) \quad \text{in } B_R.$$

Let $w \in W^{1,2}_0(B_r(y))$ be the unique weak solution of

$$\begin{cases}
-L_0w = f + \nabla \cdot F + \nabla \cdot ((\gamma - \gamma_0)\nabla u) \quad \text{in } B_r(y), \\
w = 0 \quad \text{on } \partial B_r(y).
\end{cases}$$

Then $w$ satisfies the following identity:

$$\int_{B_r(y)} (\gamma_0|\nabla w|^2 + ik|w|^2) \, dx = \int_{B_r(y)} (f \nabla \cdot F - \nabla \cdot (\gamma - \gamma_0)\nabla u \cdot \nabla w) \, dx.$$  

Taking the real parts in the above and using Sobolev embedding, Poincaré inequality, and Hölder’s inequalities, we may deduce that

$$\int_{B_r(y)} |\nabla u|^2 \, dx \leq C \int_{B_r(y)} |\nabla u|^2 \, dx + C \int_{B_r(y)} |f|^2 |\nabla u|| |\nabla w|| L^2(B_r(y)) \, dx.$$  

Denote $\lambda_1 = 2 - d/q$ and $\lambda_2 = 1 - d/p$. From the above inequality, we obtain

$$\|\nabla u\|_{L^2(B_r(y))} \leq C \|f\|_{L^p(B_r(y))} \|\nabla w\|_{L^2(B_r(y))} + C \theta(r^2) \|\nabla u\|_{L^2(B_r(y))}.$$  

On the other hand, observe that $v := u - w$ satisfies $-L_0v = 0$ weakly in $B_r(y)$. Therefore, by (2.17), for $0 < \rho < r$, we get

$$\int_{B_r(y)} |\nabla u|^2 \, dx \leq 2 \int_{B_r(y)} |\nabla v|^2 \, dx + 2 \int_{B_r(y)} |\nabla w|^2 \, dx.$$

By Campanato’s iteration argument (see, for instance, [10] Lemma 2.1, p. 86), we find that if $\theta(R_0)$ is small enough, then for all $0 < \rho < r \leq R_0$ we have

$$\int_{B_r(y)} |\nabla u|^2 \, dx \leq C \left( \frac{\rho}{r} \right)^d |\nabla w|^2 \, dx + C \rho^{d-2+2\lambda_1} \|f\|^2_{L^p(B_r(y))} + C \rho^{d-2+2\lambda_2} \|F\|^2_{L^p(B_r(y))},$$  

where $\lambda_1 = 2 - d/q$ and $\lambda_2 = 1 - d/p$. From the above inequality, we obtain

$$\|\nabla u\|_{L^2(B_r(y))} \leq C \|f\|_{L^p(B_r(y))} \|\nabla w\|_{L^2(B_r(y))} + C \theta(r^2) \|\nabla u\|_{L^2(B_r(y))}.$$  

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where \(0 < \lambda_0 < \min(\lambda_1, \lambda_2) = \min(2 - d/q, 1 - d/p)\). The above estimate (via Morrey’s characterization of Hölder continuous functions in terms of Dirichlet integrals; see, for instance, [20, Theorem 3.5.2]) implies that \(u\) is locally Hölder continuous in \(B_R\) and, in particular, we have the estimate

\[
R^{2\lambda_0}[u]_0^{2, \lambda_0; B_{R/4}} \leq C \left( R^{2-d} \int_{B_{R/2}} |\nabla u|^2 \, dx + R^{2(2-d/q)} \|f\|^2_{L^q(B_R)} + R^{2(1-d/p)} \|F\|^2_{L^p(B_R)} \right).
\]

(2.21)

Let \(\eta\) be a smooth cut-off function satisfying

\[
0 \leq \eta \leq 1, \quad \text{supp } \eta \subset B_R, \quad \eta \equiv 1 \text{ on } B_{R/2}, \quad \text{and } |\nabla \eta| \leq 4/R.
\]

By taking \(\eta^2\pi\) as a test function in (2.20), we get

\[
\int_{B_{R/2}} \gamma \eta^2 |\nabla u|^2 \, dx + ik \int_{B_{R/2}} \eta^2 |u|^2 \, dx
\]

\[
= - \int_{B_{R/2}} 2\gamma \eta \nabla u \cdot \nabla \eta \, dx + \int_{B_{R/2}} \eta^2 f \, dx + \int_{B_{R/2}} \eta^2 F \cdot \nabla \eta \, dx + \int_{B_{R/2}} 2\eta \pi F \cdot \nabla \eta \, dx.
\]

By taking the real parts in the above and using Cauchy’s inequality, we get

\[
\int_{B_{R/2}} |\nabla u|^2 \, dx \leq CR^{-2} \int_{B_R} |u|^2 \, dx + CR^2 \int_{B_R} |f|^2 \, dx + C \int_{B_R} |F|^2 \, dx.
\]

By Hölder’s inequality, we then obtain

\[
\int_{B_{R/2}} |\nabla u|^2 \, dx \leq CR^{-2} \int_{B_R} |u|^2 \, dx + CR^{2+d-2d/q} \|f\|^2_{L^q(B_R)} + CR^{d-2d/p} \|F\|^2_{L^p(B_R)}.
\]

By combining (2.21) and the above inequality, we get (2.18) via a standard covering argument.

Observe that for any \(x \in B_{R/2}\), we have

\[
|u(x)| \leq |u(x')| + |u(x) - u(x')| \leq |u(x')| + R^{\lambda_0}[u]_0^{0, \lambda_0; B_{R/2}}, \quad \forall x' \in B_{R/2}.
\]

By taking average with respect to \(x'\) in \(B_{R/2}\) and then using (2.18) and Hölder’s inequality we get

\[
\sup_{B_{R/2}} |u| \leq C \left( R^{-d/2} \|u\|_{L^2(B_R)} + R^{2-d/q} \|f\|_{L^q(B_R)} + R^{1-d/p} \|F\|_{L^p(B_R)} \right).
\]

By using a standard iteration argument (see [11, pp. 80–82]), we obtain (2.19) from the above inequality. This completes the proof.

\[-\]

2.4 Construction of Neumann functions

The aim of this subsection is to construct Neumann functions of \(L\) and \(L^*\) in \(\Omega\) and derive their basic properties. The following theorem holds.
Theorem 2.3 Assume \( \gamma \in C^0(\Omega) \). Then there exist Neumann functions \( N(x, y) \) and \( N^\ast(x, y) \) of respectively \( L \) and \( L^\ast \) in \( \Omega \). Moreover, there exists \( \lambda_0 \in (0, 1) \) such that \( N(\cdot, y), N^\ast(\cdot, y) \in C^0,\lambda_0(\Omega \setminus \{y\}) \) for all \( y \in \Omega \) and the identity,

\[
N^\ast(x, y) := \overline{N}(y, x), \quad \forall x, y \in \Omega, \quad x \neq y,
\]

(2.22)

holds. Furthermore, the following estimates hold uniformly in \( y \in \Omega \), where we denote \( d_y = \text{dist}(y, \partial \Omega) \):

i) \( \|N(\cdot, y)\|_{L^{2d/(d-2)}(\Omega \setminus B_r(y))} + \|\nabla N(\cdot, y)\|_{L^2(\Omega \setminus B_r(y))} \leq C r^{1-d/2} \) for all \( r \in (0, d_y) \).

ii) \( \|N(\cdot, y)\|_{L^p(B_r(y))} \leq C r^{2-d+p} \) for all \( r \in (0, d_y) \), where \( p \in [1, \frac{d}{d-2}] \).

iii) \( |\{x \in \Omega : |N(x, y)| > t\}| \leq Ct^{-d/(d-2)} \) for all \( t > d_y^2 \).

iv) \( \|\nabla N(\cdot, y)\|_{L^p(B_r(y))} \leq C r^{2-d+p} \) for all \( r \in (0, d_y) \), where \( p \in [1, \frac{d}{d-2}] \).

v) \( |\{x \in \Omega : |\nabla_x N(x, y)| > t\}| \leq Ct^{-d/(d-1)} \) for all \( t > d_y^2 \).

vi) \( |N(x, y)| \leq C|x - y|^{2-d} \) whenever \( 0 < |x - y| < d_y/2 \).

vii) \( |N(x, y) - N(x', y)| \leq C|x - x'|^{\lambda_0} |x - y|^{2-d-\lambda_0} \) if \( 2|x - x'| < |x - y| < d_y/2 \).

In the above, \( C \) is a constant depending on \( d, \nu, k_0, \Omega, \) and \( \theta \); it depends on \( p \) as well in ii) and iv). The estimates i) – vii) are also valid for \( N^\ast(\cdot, y) \). Finally, if \( q > d/2 \) and \( p > d \), then for any \( f \in L^q(\Omega, \mathbb{C}) \), \( F \in L^p(\Omega; \mathbb{C}^d) \) and \( g \in L^2(\partial \Omega; \mathbb{C}) \), the function \( u \) given by

\[
u(x) := \int_{\Omega} (N(x, y)f(y) - \nabla_y N(x, y) \cdot F(y)) \, dy + \int_{\partial \Omega} N(x, y)g(y) \, d\sigma(y)
\]

(2.23)

is the unique solution in \( W^{1,2}(\Omega) \) of problem (2.2).

Proof. We follow the strategy used in [6], which in turn is based on [13]. Let us fix a function \( \Phi \in C^\infty_c(\mathbb{R}^d) \) such that \( \Phi \) is supported in \( B_1(0) \), \( 0 \leq \Phi \leq 2 \), and \( \int_{\mathbb{R}^d} \Phi \, dx = 1 \). Let \( y \in \Omega \) be fixed but arbitrary. For any \( \epsilon > 0 \), we define

\[
\Phi_\epsilon(x) = \epsilon^{-d} \Phi((x - y)/\epsilon).
\]

Let \( v_{\epsilon, y} \) be the unique weak solution in \( W^{1,2}(\Omega; \mathbb{C}) \) of problem

\[
\left\{
\begin{array}{ll}
-Lv = \Phi_\epsilon & \text{in } \Omega, \\
\gamma \nabla v \cdot n = 0 & \text{on } \partial \Omega.
\end{array}
\right.
\]

(2.24)

We define the “averaged Neumann function” \( N^\epsilon(\cdot, \cdot) \) by

\[
N^\epsilon(\cdot, y) = v = v_{\epsilon, y}.
\]

Then \( N^\epsilon(\cdot, y) \) satisfies the following identity (c.f. [20]):

\[
\int_{\Omega} (\gamma \nabla N^\epsilon(\cdot, y) \cdot \nabla \phi + i k N^\epsilon(\cdot, y) \overline{\phi}) \, dx = \int_{\Omega \cap B_r(y)} \Phi_\epsilon \overline{\phi} \, dx, \quad \forall \phi \in W^{1,2}(\Omega; \mathbb{C}).
\]

(2.25)
By taking $\phi = N^\epsilon(\cdot, y) = v$ in (2.25), we get
\[
\int_\Omega |\nabla v|^2 \, dx = \Re \int_\Omega (|\nabla v|^2 + ik|v|^2) \, dx = \Re \int_{\Omega \cap B_r(y)} \Phi_r \overline{v} \, dx \leq C \epsilon^{(2-d)/2} \|v\|_{W^{1,2}(\Omega)},
\]
where the last inequality follows from the Sobolev embedding, namely,
\[
\left| \int_{\Omega \cap B_r(y)} \Phi_r \overline{v} \, dx \right| \leq C \|\Phi_r\|_{L^{2d/(d+2)}(B_r(y))} \|v\|_{W^{1,2}(\Omega)} \leq C \epsilon^{(2-d)/2} \|v\|_{W^{1,2}(\Omega)}.
\]
Similarly, we get
\[
\int_\Omega k|v|^2 \, dx = \Im \int_\Omega (|\nabla v|^2 + ik|v|^2) \, dx = \Im \int_{\Omega \cap B_r(y)} \Phi_r \overline{v} \, dx \leq C \epsilon^{(2-d)/2} \|v\|_{W^{1,2}(\Omega)}.
\]
Therefore, we have
\[
\|N^\epsilon(\cdot, y)\|_{W^{1,2}(\Omega)} \leq C \epsilon^{(2-d)/2}, \tag{2.26}
\]
where $C = C(d, \nu, k_0)$.

Let $R \in (0, d_y)$ be arbitrary, but fixed. Assume that $f \in C^\infty(\Omega; \mathbb{C})$ is supported in $B_R = B_R(y) \subset \Omega$. Let $u$ be a unique weak solution in $W^{1,2}(\Omega; \mathbb{C})$ of problem (2.12). We then have the following identity (c.f. (2.10)):
\[
\int_\Omega (|\nabla w \cdot \nabla u + ikw\overline{u}|) \, dx = \int_\Omega w \overline{f} \, dx, \quad \forall w \in W^{1,2}(\Omega; \mathbb{C}). \tag{2.27}
\]
Then by setting $\phi = u$ in (2.25) and setting $w = N^\epsilon(\cdot, y) = v$ in (2.27), we get
\[
\int_\Omega N^\epsilon(x, y) \overline{f(x)} \, dx = \int_{\Omega \cap B_r(y)} \Phi_r \overline{v} \, dx. \tag{2.28}
\]
Also, by taking $w = u$ in (2.27), we see that
\[
\int_\Omega |\nabla u|^2 \, dx + ik \int_\Omega |u|^2 \, dx = \int_\Omega u \overline{f} \, dx.
\]
Taking the real and imaginary parts in the above and using the Sobolev embedding and Hölder’s inequality
\[
\int_\Omega |\nabla u|^2 \, dx = \Re \int_\Omega u \overline{f} \, dx \leq C \|f\|_{L^{2d/(d+2)}(\Omega)} \|u\|_{W^{1,2}(\Omega)},
\]
\[
k \int_\Omega |u|^2 \, dx = \Im \int_\Omega u \overline{f} \, dx \leq C \|f\|_{L^{2d/(d+2)}(\Omega)} \|u\|_{W^{1,2}(\Omega)}.
\]
Therefore, we obtain
\[
\|u\|_{W^{1,2}(\Omega)} \leq C \|f\|_{L^{2d/(d+2)}(\Omega)}, \tag{2.29}
\]
where $C = C(d, \nu, k_0)$. From (2.19) in Lemma (2.2) with $p_0 = 2d/(d - 2)$, it follows that
\[
\|u\|_{L^\infty(B_{R/2})} \leq C \left( R^{1-d/2} \|u\|_{L^{2d/(d-2)}(\Omega)} + R^2 \|f\|_{L^\infty(B_R)} \right).
\]
Furthermore, (2.29) yields
\[ \|u\|_{L^{2d/(d-2)}(\Omega)} \leq CR^{1+d/2}\|f\|_{L^\infty(B_R)}, \]
provided that \( f \) is supported in \( B_R \). Therefore, by combining the above two inequalities, we have
\[ \|a\|_{L^\infty(B_{R/2})} \leq CR^2\|f\|_{L^\infty(B_R)}, \tag{2.30} \]
where \( C \) depends on \( d, \nu, \Omega, \) and \( \theta \). By (2.28) and (2.30), we find that for all \( \epsilon \in (0, R/2) \) and \( R \in (0, d_y) \),
\[ \left| \int_{B_R} N^{\epsilon}(\cdot, y) f \, dx \right| \leq CR^2\|f\|_{L^\infty(B_R)}, \forall f \in C_c^\infty(B_R; \mathbb{C}). \]

Therefore, by duality, we conclude that
\[ \|N^{\epsilon}(\cdot, y)\|_{L^1(B_R(\eta))} \leq CR^2, \quad \forall \epsilon \in (0, R/2), \quad \forall R \in (0, d_y). \]

Now, for any \( x \in \Omega \) such that \( 0 < |x - y| < d_y/2 \), let us take \( R := 2|x - y|/3 \). Notice that if \( \epsilon < R/2 \), then \( N^{\epsilon}(\cdot, y) \in W^{1,2}(B_R(x)) \) and satisfies \(-L N^{\epsilon}(\cdot, y) = 0\) in \( B_R(x) \). Then by (2.19) in Lemma 2.2, we have
\[ |N^{\epsilon}(x, y)| \leq CR^{-d}\|N^{\epsilon}(\cdot, y)\|_{L^1(B_{r}(x))} \leq CR^{-d}\|N^{\epsilon}(\cdot, y)\|_{L^1(B_{r/2}(y))} \leq Cr^{2-d}. \]

We have thus shown that for any \( x, y \in \Omega \) satisfying \( 0 < |x - y| < d_y/2 \), we have
\[ |N^{\epsilon}(x, y)| \leq C|x - y|^{2-d}, \quad \forall \epsilon < |x - y|/3. \tag{2.31} \]

Next, fix \( r \in (0, d_y/2) \) and \( \epsilon \in (0, r/6) \). Let \( \eta \) be a smooth function on \( \mathbb{R}^d \) satisfying
\[ 0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } \mathbb{R}^d \setminus B_r, \quad \eta \equiv 0 \text{ on } B_{r/2}, \quad \text{and } |\nabla \eta| \leq 4/r. \tag{2.32} \]
We set \( \phi = \eta^2v = \eta^2N^{\epsilon}(\cdot, y) \) in (2.25) to get
\[ \int_{\Omega} \gamma \eta^2 |\nabla v|^2 \, dx + ik \int_{\Omega} \eta^2 |v|^2 \, dx = - \int_{\Omega} 2\gamma \eta \bar{\eta} \nabla v \cdot \nabla \eta \, dx. \]

By taking the real part in the above and using Cauchy’s inequality, we get (c.f. (2.23))
\[ \int_{\Omega} \gamma \eta^2 |\nabla N^{\epsilon}(x, y)|^2 \, dx \leq 4 \int_{\Omega} \gamma |\nabla \eta|^2 |N^{\epsilon}(x, y)|^2 \, dx. \]

We then use (2.31) to obtain
\[ \int_{\Omega} \gamma |\nabla N^{\epsilon}(x, y)|^2 \, dx \leq Cr^{-2} \int_{B_r(y) \setminus B_{r/2}(y)} |x - y|^{2(2-d)} \, dx \leq Cr^{2-d}. \]

Therefore, for all \( 0 < \epsilon < r/6 \), we have
\[ \|\nabla N^{\epsilon}(\cdot, y)\|_{L^2(\Omega, B_r(y))} \leq Cr^{(2-d)/2}. \]
In the case when $\epsilon \geq r/6$, we obtain from (2.20) that
\[
\| \nabla N^\epsilon(\cdot, y) \|_{L^2(\Omega \setminus B_r(y))} \leq \| \nabla N^\epsilon(\cdot, y) \|_{L^2(\Omega)} \leq C\epsilon^{(2-d)/2}.
\]
By combining the above two inequalities, we obtain
\[
\| \nabla N^\epsilon(\cdot, y) \|_{L^2(\Omega \setminus B_r(y))} \leq C\epsilon^{(2-d)/2}, \quad \forall r \in (0, d/2), \quad \forall \epsilon > 0. \quad (2.33)
\]
Observe that (2.31) also implies
\[
\| N^\epsilon(\cdot, y) \|_{L^{4d/(d-2)}(\Omega \setminus B_r(y))} \leq C\epsilon^{(2-d)/2}, \quad \forall \epsilon \in (0, r/6).
\]
On the other hand, if $\epsilon \geq r/6$, then (2.26) implies
\[
\| N^\epsilon(\cdot, y) \|_{L^{4d/(d-2)}(\Omega \setminus B_r(y))} \leq C\| N^\epsilon(\cdot, y) \|_{W^{1,2}(\Omega)} \leq C\epsilon^{(2-d)/2}.
\]
By combining the above two estimates, we obtain
\[
\| N^\epsilon(\cdot, y) \|_{L^{4d/(d-2)}(\Omega \setminus B_r(y))} \leq C\epsilon^{(2-d)/2}, \quad \forall r \in (0, d/2), \quad \forall \epsilon > 0. \quad (2.34)
\]
From the obvious fact that $d_y/2$ and $d_y$ are comparable to each other, we find by (2.33) and (2.34) that for all $0 < r < d_y$ and $\epsilon > 0$, we have
\[
\| N^\epsilon(\cdot, y) \|_{L^{4d/(d-2)}(\Omega \setminus B_r(y))} + \| \nabla N^\epsilon(\cdot, y) \|_{L^2(\Omega \setminus B_r(y))} \leq C\epsilon^{(2-d)/2}. \quad (2.35)
\]
From (2.35) it follows that (see [13, pp. 147–148])
\[
\begin{align*}
| \{ x \in \Omega : |N^\epsilon(x, y)| > t \} | \leq Ct^{-(d/(d-2))}, & \quad \forall t > d_y^{2-d}, \quad \forall \epsilon > 0, \quad (2.36) \\
| \{ x \in \Omega : |\nabla N^\epsilon(x, y)| > t \} | \leq Ct^{-(d/(d-1))}, & \quad \forall t > d_y^{1-d}, \quad \forall \epsilon > 0. \quad (2.37)
\end{align*}
\]
It is routine to derive the following strong type estimates from the above weak type estimates (2.36) and (2.37) (see, for instance, [13, p. 148]):
\[
\begin{align*}
\| N^\epsilon(\cdot, y) \|_{L^p(B_r(y))} & \leq C\epsilon^{2-d/p}, \quad \forall r \in (0, d_y), \quad \forall \epsilon > 0, \quad \forall p \in [1, \frac{d}{d-1}], \quad (2.38) \\
\| \nabla N^\epsilon(\cdot, y) \|_{L^p(B_r(y))} & \leq C\epsilon^{1-d/p}, \quad \forall r \in (0, d_y), \quad \forall \epsilon > 0, \quad \forall p \in [1, \frac{d}{d-1}]. \quad (2.39)
\end{align*}
\]
From (2.33), (2.38), and (2.39), it follows that there exists a sequence $\{\epsilon_n\}_{n=1}^{\infty}$, tending to zero and a function $N(\cdot, y)$ such that $N^\epsilon_n(\cdot, y)$ converges to $N(\cdot, y)$ weakly in $W^{1,p}(B_y)$ for $1 < p < d/(d-1)$ and all $r \in (0, d_y)$ and also that $N^\epsilon_n(\cdot, y)$ converges to $N(\cdot, y)$ weakly in $W^{1,2}(\Omega \setminus B_r(y))$ for all $r \in (0, d_y)$; see [13, p. 159] for the details. Then it is routine to check that $N(\cdot, y)$ satisfies the properties i) and ii) at the beginning of Section 2.2, and also the estimates i) – v) in the theorem; see [13, Section 4.1].

We now turn to the pointwise bound for $N(x, y)$. For any $x \in \Omega$ such that $0 < |x - y| < d_y/2$, set $R := 2|x - y|/3$. Notice that (2.33) implies that $N(\cdot, y) \in W^{1,2}(B_{R_R}(x))$ and satisfies $-LN(\cdot, y) = 0$ weakly in $B_{R}(x)$. Then, by (2.19) in Lemma 2.2 and the estimate ii) in the theorem, we have
\[
|N(x, y)| \leq CR^{-d}\| N(\cdot, y) \|_{L^1(B_{R_R}(x))} \leq CR^{-d}\| N(\cdot, y) \|_{L^1(B_{3R}(y))} \leq C|x - y|^{2-d}.
\]
We have thus shown that the estimate vi) in the theorem holds. Then, it is routine to see that the estimate vii) in the theorem follows from (2.18) in Lemma 2.2 and the above estimate.
Next, let \( x \in \Omega \setminus \{ y \} \) be fixed but arbitrary, and let \( \bar{N}^\epsilon(\cdot, x) \in W^{1,2}(\Omega; \mathbb{C}) \) be the averaged Neumann function of the adjoint operator \( L^* \) in \( \Omega \), where \( 0 < \epsilon' < d_x \). Then we have

\[
\int_{\Omega} (\gamma \nabla \bar{N}^\epsilon(z, x) \cdot \nabla \bar{\psi}(z) - ik\bar{N}^\epsilon(z, x)\bar{\psi}(z)) \, dz = \int_{\Omega \cap B_{\epsilon'}(x)} \Phi_{\epsilon'}(z)\bar{\psi}(z) \, dz, \tag{2.40}
\]

for all \( \psi \in W^{1,2}(\Omega; \mathbb{C}) \). By setting \( \phi = \bar{N}^\epsilon(\cdot, x) \) in (2.25) and \( \psi = N^\epsilon(\cdot, y) \) in (2.40) and then taking complex conjugate, we obtain

\[
\int_{\Omega \cap B_{\epsilon'}(x)} \Phi_{\epsilon'}(\cdot, y) \, dz = \int_{\Omega \cap B_{\epsilon'}(y)} \Phi_{\epsilon'}(\cdot, x) \, dz.
\]

Let \( N^* (\cdot, x) \) be a Neumann function of \( L^* \) in \( \Omega \) obtained from \( \bar{N}^\epsilon_m(\cdot, x) \), where \( \{ \epsilon_m \}_{m=1}^\infty \) is a sequence tending to 0. Then, by following the same steps as in \cite{13} p. 151, we conclude

\[
N(x, y) = \bar{N}^\epsilon(y, x),
\]

which obviously implies the identity (2.22). We remark that by following similar lines of reasoning as in \cite{13} p. 151, we find

\[
N^\epsilon(x, y) = e^{-d} \int_\Omega \Phi \left( \frac{z - y}{\epsilon} \right) N(x, z) \, dz,
\]

and thus we have in fact the following pointwise convergence:

\[
\lim_{\epsilon \to 0} N^\epsilon(x, y) = N(x, y), \quad \forall x, y \in \Omega, \quad x \neq y. \tag{2.41}
\]

Now, let \( u \) be the unique solution in \( W^{1,2}(\Omega; \mathbb{C}) \) of problem (2.12) with \( f \in C_\infty(\Omega; \mathbb{C}) \). By Lemma 2.2, we find that \( u \) is continuous in \( \Omega \). By setting \( w = N^\epsilon(\cdot, y) \) in (2.27) and setting \( \phi = u \) in (2.26), we get

\[
\int_{\Omega} N^\epsilon(x, y)f(x) \, dx = \int_{\Omega \cap B_{\epsilon'}(y)} \Phi_{\epsilon'} u \, dx.
\]

We take the limit \( \epsilon \to 0 \) above and then take complex conjugate to get

\[
u(y) = \nu(x, y)f(x) \, dx,
\]

which is equivalent to (2.11). We have shown that \( N(x,y) \) satisfies the property iii) in Section 2.2. and thus that \( N(x,y) \) is the unique Neumann function of the operator \( L \) in \( \Omega \).

Finally, let \( f \in L^2(\Omega; \mathbb{C}) \) with \( q > d/2 \) and \( g \in L^2(\partial \Omega; \mathbb{C}) \), and let \( u \) be the unique weak solution in \( W^{1,2}(\Omega; \mathbb{C}) \) of problem (2.28); see Section 2.4. Then \( u \) satisfies the identity (2.40). By setting \( v = \bar{N}^\epsilon(\cdot, x) \) in (2.3) and setting \( \psi = u \) in (2.40), we get

\[
\int_{\Omega} (\bar{N}^\epsilon(z, x) - \nabla \bar{N}^\epsilon(z, x)) f(z) \, dz + \int_{\partial \Omega} \bar{N}^\epsilon(z, x)g(z) \, d\sigma(z) = \int_{\Omega \cap B_{\epsilon'}(x)} \Phi_{\epsilon'} u \, dz.
\]

By Lemma 2.2 we again find that \( u \) is H"older continuous in \( \Omega \). Then by proceeding similarly as above and using (2.22), we obtain

\[
u(x) = \nu(x, y)f(y) - \nabla_y N(x, y) \cdot F(y) \, dy + \nu(x, y)g(y) \, d\sigma(y),
\]

which is the formula (2.23). The proof is complete. \( \square \)
2.5 \( L^p \) estimates

We now assume that \( \Omega \) is a bounded \( C^1 \) domain. In the following lemma we obtain \( L^p \) estimates for the operator \( L \) with uniformly continuous coefficient \( \gamma \).

**Lemma 2.4** Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^1 \) domain and assume that \( \gamma \in C^0(\overline{\Omega}) \). Let \( q \in (1,d) \), \( p \in (1,\infty) \), and \( s = \min(q^*,p) \), where \( q^* = dq/(d-q) \). For each \( f \in L^q(\Omega;\mathbb{C}) \) and \( F \in L^p(\Omega;\mathbb{C}^d) \), there is a unique weak solution \( u \in W^{1,s}(\Omega) \) to

\[
- Lu = f + \nabla \cdot F \quad \text{in} \quad \Omega,
\]

\[
(\gamma \nabla u + F) \cdot n = 0 \quad \text{on} \quad \partial \Omega.
\]

Moreover, the following estimate holds:

\[
\|u\|_{W^{1,s}(\Omega)} \leq C \left( \|f\|_{L^q(\Omega)} + \|F\|_{L^p(\Omega)} \right),
\]

where \( C \) depends on \( d,\nu,k_0,p,q,\Omega, \) and \( \theta \).

**Proof.** Note that in the case when \( f \equiv 0 \), the proof for estimate (2.43) reduces to

\[
\|u\|_{W^{1,p}(\Omega)} \leq C \|F\|_{L^p(\Omega)}.
\]

In this case the proof for the existence and uniqueness of weak solution \( u \in W^{1,p}(\Omega) \) as well as the estimate (2.44) follow essentially from the same argument as in [17].

We consider the case when \( f \) is not identically zero. Observe that \( L^q(\Omega) \subset W^{-1,q^*}(\Omega) \) with the estimate

\[
\|f\|_{W^{-1,q^*}(\Omega)} \leq C \|f\|_{L^q(\Omega)}, \quad \text{where} \quad C = C(d,\Omega).
\]

Then by [3] Corollary 9.3, there exists a unique weak solution \( v \) in \( W^{1,q^*}(\Omega) \) of the Neumann problem

\[
\begin{align*}
\Delta v &= f - \frac{1}{|\Omega|} \int_{\Omega} f \, dy \quad \text{in} \quad \Omega, \\
\partial v / \partial n &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

where \( |\Omega| \) is the volume of \( \Omega \). Moreover, \( v \) satisfies the estimate

\[
\|\nabla v\|_{L^{q^*}(\Omega)} \leq C \|f - \frac{1}{|\Omega|} \int_{\Omega} f \, dy\|_{W^{-1,q^*}(\Omega)} \leq C \|f\|_{L^q(\Omega)}.
\]

Then, we apply estimate (2.44) with \( F + \nabla v + (\frac{1}{|\Omega|} \int_{\Omega} f \, dy)x \) and \( s \) in place of \( F \) and \( p \), respectively, and use Hölder’s inequality to get estimate (2.43).

We denote by \( L^{p,\infty}(\Omega) \) the usual weak \( L^p \) space. The following lemma is a variant of Lemma 2.4 in the weak Lebesgue spaces.

**Lemma 2.5** Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^1 \) domain and assume that \( \gamma \in C^0(\overline{\Omega}) \). Let \( q \in (1,d) \), \( p \in (1,\infty) \), and \( s = \min(q^*,p) \), where \( q^* = dq/(d-q) \). If \( f \in L^q(\Omega;\mathbb{C}) \) and \( F \in L^{p,\infty}(\Omega;\mathbb{C}^d) \), there is a weak solution \( u \) of problem (2.42) that satisfies the estimate

\[
\|\nabla u\|_{L^{p,\infty}(\Omega)} \leq C \left( \|f\|_{L^q(\Omega)} + \|F\|_{L^{p,\infty}(\Omega)} \right),
\]
and, for \( s < d \), the following estimate as well:

\[
\|u\|_{L^{s,\infty}(\Omega)} \leq C \left( \|f\|_{L^{s,\infty}(\Omega)} + \|F\|_{L^{p,\infty}(\Omega)} \right).
\]

Moreover, there is uniqueness of weak solutions to (2.42) in the sense that if \( \tilde{u} \) is a solution in \( W^{1,d}(\Omega) \) for some \( t > 1 \), then \( u = \tilde{u} \).

**Proof.** The lemma follows immediately from Lemma 2.4 by applying [7, Lemma 1] to the solution operator \( T : F \mapsto u \) as well as to the map \( f \mapsto v \) in (2.45).

\[\blacksquare\]

**Lemma 2.6** Let \( \Omega \) and \( \gamma \) satisfy the same assumptions as in Lemma 2.4. There exists a constant \( C_1 > 0 \) such that the following holds: For any \( f \in C_\infty^\infty(\Omega; \mathbb{C}) \), let \( u \in W^{1,2}(\Omega; \mathbb{C}) \) be the unique weak solution of

\[
\begin{aligned}
- Lu &= f \quad \text{in } \Omega, \\
\gamma \nabla u \cdot n &= 0 \quad \text{on } \partial \Omega \quad \text{or} \quad - L^* u &= f \quad \text{in } \Omega, \\
\gamma \nabla u \cdot n &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Then for all \( x \in \Omega \) and \( 0 < R < \text{diam}(\Omega) \), we have

\[
\|u\|_{L^\infty(\Omega \cap B_{R/2}(x))} \leq C_1 \left( R^{-d/2}\|u\|_{L^2(\Omega \cap B_R(x))} + R^2\|f\|_{L^\infty(\Omega \cap B_R(x))} \right).
\]

The constant \( C_1 \) depends on \( d, \nu, \Omega, \) and \( \theta \).

**Proof.** We will only consider the case when \( u \) is a weak solution of \( -Lu = f \) with zero conormal data. By Lemma 2.4, we find that \( u \in W^{1,p}(\Omega) \) for all \( p \in (1, \infty) \) and

\[
\|\nabla u\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)} \leq C \|f\|_{L^\infty(\Omega)}.
\]

Let \( v = \zeta u \), where \( \zeta : \mathbb{R}^d \to \mathbb{R} \) is a smooth function to be chosen later. Observe that \( v \) is a weak solution of the problem

\[
\begin{aligned}
- Lv &= \tilde{f} + \nabla \cdot \tilde{F} \quad \text{in } \Omega, \\
(\gamma \nabla v + \tilde{F}) \cdot n &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where

\[
\tilde{f} := \zeta f - \gamma \nabla \zeta \cdot \nabla u, \quad \tilde{F} := -\gamma u \nabla \zeta.
\]

Let \( x \in \Omega \) and \( 0 < R < \text{diam}(\Omega) \) be arbitrary but fixed. For any \( y \in \Omega \cap B_R(x) \) and \( 0 < \rho < r \leq R \), we choose the function \( \zeta \) to be such that

\[
0 \leq \zeta \leq 1, \quad \text{supp } \zeta \subset B_r(y), \quad \zeta \equiv 1 \text{ on } B_\rho(y), \quad \text{and } |\nabla \zeta| \leq 2/(r-\rho).
\]

For any \( p \in (1, \infty) \), we set \( q = pd/(p+d) \) and apply Lemma 2.4 together with Hölder’s inequality to get

\[
\|\nabla u\|_{L^p(\Omega_\rho)} \leq C \left( r^{1+d/p}\|f\|_{L^\infty(\Omega_\rho)} + (r-\rho)^{-1}\|\nabla u\|_{L^{pd/(p+d)}(\Omega_\rho)} + (r-\rho)^{-1}\|u\|_{L^p(\Omega_\rho)} \right),
\]

(2.46)
where we use the notation \( \Omega_r = \Omega_r(y) = \Omega \cap B_r(y) \). Now, fix \( p > d \), and let \( m = \lfloor d(1/2 - 1/p) \rfloor \),
\[
p_j = \frac{pd}{d + p_j} \quad \text{and} \quad r_j = \rho + \frac{(r - \rho) j}{m}, \quad j = 0, \ldots, m.
\]

Then we apply (2.46) iteratively to get
\[
\| \nabla u \|_{L^p(\Omega_r)} \leq \sum_{j=1}^{m} C^j \left( \frac{m}{r - \rho} \right)^{j-1} r_j^{1+d/p_j-1} \| f \|_{L^\infty(\Omega_{r_j})} + \sum_{j=1}^{m} C^j \left( \frac{m}{r - \rho} \right)^j \| u \|_{L^{p_j}(\Omega_{r_j})} + C^m \left( \frac{m}{r - \rho} \right)^m \| \nabla u \|_{L^{p_m}(\Omega_{r_m})}.
\]

Notice that \( 1 < p_m \leq 2 \). By using Hölder’s inequality we then obtain
\[
\rho^{-d(1/2 - 1/p)} \| \nabla u \|_{L^2(\Omega_r)} \leq C \left( \frac{r}{r - \rho} \right)^{m-1} r^{1+d/p} \| f \|_{L^\infty(\Omega_r)} + C \left( \frac{r}{r - \rho} \right)^m r^{d(1/p - 1/2)} \| \nabla u \|_{L^2(\Omega_r)}.
\]

If we take \( r = R/4 \) and \( \rho < r/2 = R/4 \) in the above, then for all \( y \in \Omega_{R/4} \), we get
\[
\left( \rho^{-(d-2+2(1-d/p))} \int_{\Omega_{R/4}} |\nabla u|^2 dz \right)^{1/2} \leq CR^{1-d/p} \| f \|_{L^\infty(\Omega_{R/2})} + CR^{-1} \| u \|_{L^p(\Omega_{R/2})} + CR^{d(1/p - 1/2)} \| \nabla u \|_{L^2(\Omega_{R/2})} = A(R) \quad (2.47)
\]

Hereafter in the proof, we shall denote \( \Omega_R = \Omega_R(x) \). Then by Morrey-Campanato’s theorem (see [11 Section 3.1]), for all \( z, z' \in \Omega_{R/4} \), we have
\[
|u(z) - u(z')| \leq CR^{1-d/p} A(R),
\]
where \( A(R) \) is as defined in (2.37). Therefore, for any \( z \in \Omega_{R/4} \) we have
\[
|u(z)| \leq |u(z')| + |u(z) - u(z')| \leq |u(z')| + CR^{1-d/p} A(R), \quad \forall z' \in \Omega_{R/4}.
\]

By taking average over \( z' \in \Omega_{R/4} \) in the above and using the definition of \( A(R) \), we obtain
\[
\sup_{\Omega_{R/4}} |u| \leq \frac{1}{\Omega_{R/4}} \int_{\Omega_{R/4}} |u(z')| dz' + CR^2 \| f \|_{L^\infty(\Omega_R)} + CR^{-d/p} \| u \|_{L^p(\Omega_R)} + CR^{1-d/2} \| \nabla u \|_{L^2(\Omega_{R/2})}.
\]

Then by using Hölder’s inequality and Caccioppoli’s inequality, we get
\[
\sup_{\Omega_{R/4}} |u| \leq CR^2 \| f \|_{L^\infty(\Omega_R)} + CR^{-d/p} \| u \|_{L^p(\Omega_R)} + CR^{1-d/2} \| \nabla u \|_{L^2(\Omega_R)}.
\]

By using a standard argument (see [11 pp. 80–82]), we derive from the above inequality
\[
\sup_{\Omega_{R/2}} |u| \leq CR^2 \| f \|_{L^\infty(\Omega_R)} + CR^{-d/2} \| u \|_{L^2(\Omega_R)}.
\]

The proof is complete.
2.6 Global estimates for Neumann function

The next theorem provides global pointwise bound for the Neumann function $N$.

**Theorem 2.7** Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^1$ domain and assume that $\gamma \in C^0(\Omega)$. Let $N(x,y)$ be the Neumann function of $L$ in $\Omega$ as constructed in Theorem 2.3. Then we have the following global pointwise bound for the Neumann function:

\[
|N(x,y)| \leq C|x-y|^{2-d} \quad \text{for all } x, y \in \Omega \text{ with } x \neq y,
\]

where $C$ depends on $d, \nu, \Omega$, and $\theta$. Moreover, for all $y \in \Omega$ and $0 < r < \text{diam}(\Omega)$, we have

i) $\|N(\cdot, y)\|_{L^{2d/(d-2)}(\Omega; B_r(y))} + \|\nabla N(\cdot, y)\|_{L^2(\Omega \setminus B_r(y))} \leq Cr^{1-d/2}$.

ii) $\|N(\cdot, y)\|_{L^p(\Omega \setminus B_r(y))} \leq Cr^{2-d+d/p}$ for $p \in [1, \frac{d}{d-2})$.

iii) $|\{ x \in \Omega : |N(x, y)| > t \}| \leq C t^{-d/(d-2)}$ for all $t > 0$.

iv) $\|\nabla N(\cdot, y)\|_{L^p(\Omega \setminus B_r(y))} \leq C r^{1-d+d/p}$ for $p \in [1, \frac{d}{d-1})$.

v) $|\{ x \in \Omega : |\nabla_x N(x, y)| > t \}| \leq C t^{-d/(d-1)}$ for all $t > 0$.

vi) $|N(x, y) - N(x', y)| \leq C|x-x'|^{\lambda_0} |x-y|^{2-d-\lambda_0}$ if $|x-x'| < |x-y|/2$ for some $\lambda_0 \in (0, 1)$.

In the above, $C$ is a constant depending on $d, \nu, k_0, \Omega$, and $\theta$; it depends on $p$ as well in ii) and iv). Estimates i) – vi) are also valid for the Neumann function $N^*(x, y)$ of the adjoint $L^*$.

**Proof.** Let $y \in \Omega$ be arbitrary, but fixed. Assume that $f \in C_c^\infty(\Omega; \mathbb{C})$ is supported in $\Omega_R(y) = \Omega \cap B_R(y)$ and let $u$ be the unique weak solution in $W^{1,2}(\Omega; \mathbb{C})$ of problem \[ (2.12) \]

Then we have the identities \[ (2.27) \] and \[ (2.28) \] as in the proof of Theorem 2.3. Also, we have estimate \[ (2.29) \], and thus by Sobolev embedding theorem, we get

\[
\|u\|_{L^{2d/(d-2)}(\Omega)} \leq C \|f\|_{L^{2d/(d-2)}(\Omega)} \leq CR^{(2+d)/2} \|f\|_{L^\infty(\Omega_R(y))}, \quad \text{(2.49)}
\]

where $C = C(d, \nu, \Omega)$. Then by Lemma \[ 2.3 \] and \[ (2.49) \], we obtain

\[
\|u\|_{L^\infty(\Omega_{R/2}(y))} \leq CR^2 \|f\|_{L^\infty(\Omega_R(y))}. \quad \text{(2.50)}
\]

Hence, by \[ (2.28) \] and \[ (2.50) \], we conclude that

\[
\left| \int_{\Omega_{R/2}(y)} N^*(x, y)f(x) \, dx \right| \leq CR^2 \|f\|_{L^\infty(\Omega_R(y))}, \quad \forall f \in C_c^\infty(\Omega_R(y); \mathbb{C}), \quad \forall \epsilon \in (0, R/2).
\]

Therefore, by duality, we conclude from \[ (2.51) \] that

\[
\|N^*(\cdot, y)\|_{L^1(\Omega_{R/2}(y))} \leq CR^2, \quad \forall \epsilon \in (0, R/2). \quad \text{(2.52)}
\]

Next, recall that the $v = N^*(\cdot, y)$ is the unique weak solution in $W^{1,2}(\Omega; \mathbb{C})$ of problem \[ (2.24) \]. Let $x \in \Omega$, $r > 0$, and $\epsilon > 0$ be such that $B_{\epsilon}(y) \cap B_r(x) = \emptyset$. Then Lemma \[ 2.4 \] implies that

\[
\|N^*(\cdot, y)\|_{L^\infty(\Omega_{\epsilon/2}(x))} \leq CR^{-d/2} \|N^*(\cdot, y)\|_{L^2(\Omega_{R}(x))}. \quad \text{(2.53)}
\]
By a standard iteration argument (see [11, pp. 80–82]), we then obtain from (2.54) that
\[ \|N^\epsilon(\cdot, y)\|_{L^\infty(\Omega, r)} \leq Cr^{-\frac{d}{3}}\|N^\epsilon(\cdot, y)\|_{L^1(\Omega, \epsilon)} \] (2.54)
Now, for any \( x \in \Omega \setminus \{y\} \), take \( R = 3r = 3|x - y|/2 \). Then by (2.54) and (2.52), we obtain for all \( \epsilon \in (0, r) \) that
\[ |N^\epsilon(x, y)| \leq Cr^{-d}\|N^\epsilon(\cdot, y)\|_{L^1(\Omega, \epsilon)} \leq Cr^{-\frac{d}{3}}\|N^\epsilon(\cdot, y)\|_{L^1(\Omega, \epsilon)} \leq C|x - y|^2. \]
Therefore, by using (2.41), we may take the limit \( \epsilon \to 0 \) in the above and obtain (2.48).

To derive estimates i) – vi) in the theorem, we need to repeat some steps in the proof of Theorem 2.3 with a little modification. Let \( v = N^\epsilon(\cdot, y) \), where \( 0 < \epsilon < \min(d_{y, r})/6 \) and \( 0 < r < \text{diam}(\Omega) \). Let \( \eta \) be a smooth function on \( \mathbb{R}^d \) satisfying the conditions (2.32). We set \( \phi = \eta^2 v \) in (2.23) and obtain
\[ \int_\Omega (\gamma\eta^2 \nabla v \cdot \nabla \eta + ikv \eta) dx + \int_\Omega 2\eta^4 \nabla \eta \cdot \nabla \eta dx = 0, \]
where we used the fact that \( \eta^2 \Phi \equiv 0 \). By using Cauchy’s inequality we get
\[ \int_\Omega |\nabla N^\epsilon(\cdot, y)|^2 dx \leq C \int_\Omega |\nabla \eta|^2 |N^\epsilon(\cdot, y)|^2 dx. \]
By using the pointwise bound for \( N^\epsilon(x, y) \) obtained above, we get
\[ \int_{\Omega \setminus B_r(y)} |\nabla N^\epsilon(\cdot, y)|^2 dx \leq Cr^{-2} \int_{B_r(y) \setminus B_{r/2}(y)} |x - y|^{4-2d} dx \leq Cr^{2-d}. \]
By taking the limit \( \epsilon \to 0 \) in the above, we get
\[ \|\nabla N(\cdot, y)\|_{L^2(\Omega, r)} \leq Cr^{(2-d)/2}, \quad 0 < r < \text{diam}(\Omega). \]
Observe that the pointwise bound (2.48) together with the above estimate yields
\[ \|N(\cdot, y)\|_{L^2(\Omega, r)} + \|\nabla N(\cdot, y)\|_{L^2(\Omega, r)} \leq Cr^{(2-d)/2}, \quad 0 < r < \text{diam}(\Omega), \]
where \( C \) depends on \( d, \nu, \Omega, \) and \( \theta \).

By following literally the same steps used in deriving (2.36) – (2.39) from (2.33), and using the fact that \( |\Omega| < \infty \), we obtain estimates i) – v) from (2.48) and (2.55).

Finally, we remark that the proof of Lemma 2.6 in fact implies that there exist constants \( \lambda_0 \in (0, 1) \) and \( C_1 > 0 \), which depend on \( d, \nu, \Omega, \) and \( \theta \), such that for all \( x \in \Omega \) and \( 0 < R < \text{diam}(\Omega) \), the following holds: Let \( u \) be a weak solution in \( W^{1,2}(\Omega_R(x)) \) of either
\[ -Lu = 0 \quad \text{in} \quad \Omega \cap B_R(x), \quad \gamma \nabla u \cdot n = 0 \quad \text{on} \quad \partial \Omega \cap B_R(x), \]
or \[ -Lu = 0 \quad \text{in} \quad \Omega \cap B_R(x), \quad \gamma \nabla u \cdot n = 0 \quad \text{on} \quad \partial \Omega \cap B_R(x), \]
then we have
\[ R^{\lambda_0} \|u\|_{L^2(\Omega_{R/2})} \leq C_1 R^{-d/2} \|u\|_{L^2(\Omega_R)}. \]
By utilizing the above estimate and modifying the proof for estimate vii) in Theorem 2.3, we have vii), and the proof is complete. \( \Box \)
2.7 Proof of Theorem 2.1

We are now ready to prove Theorem 2.1. Let \( u = N(\cdot, y) - N_0(\cdot, y) \). Observe that Theorem 2.7 implies that \( u \in W^{1,q}(\Omega) \) for \( 1 \leq q < d/(d-1) \), and also that we have

\[
\int_{\Omega} (\gamma \nabla u \nabla \phi + ik u \phi) \, dx = \int_{\Omega} (\gamma - \gamma_0) \nabla N_0(\cdot, y) \nabla \phi \, dx, \quad \forall \phi \in C^\infty(\Omega; \mathbb{C}).
\]

In other words, \( u \) is a weak solution in \( W^{1,q}(\Omega) \) of the problem

\[
\begin{cases}
-Lu = -\nabla \cdot F & \text{in } \Omega, \\
(\gamma \nabla u + F) \cdot n = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( F = (\gamma - \gamma_0) \nabla N_0(\cdot, y) \).

Note that

\[
|\nabla_x N_0(x, y)| \leq C|x - y|^{-d/2}, \quad \forall x, y \in \Omega, \quad x \neq y.
\]

Indeed, for any \( x \in \Omega \) with \( x \neq y \), we set \( R = |x - y|/2 \) and apply (2.16) and estimate i) in Theorem 2.7 to obtain

\[
|\nabla_x N_0(x, y)| \leq CR^{-d/2}\|\nabla N_0(\cdot, y)\|_{L^2(\Omega \setminus B_R(y))} \leq CR^{1-d},
\]

which obviously implies (2.56). Moreover, by repeating the same argument, we have

\[
|\nabla_x N_0(x, y)| \leq C|x - y|^{2-d-k}, \quad \forall x, y \in \Omega, \quad x \neq y, \quad k = 1, 2, \ldots
\]

We then obtain

\[
|F(x)| \leq C[\gamma]_{0,\lambda,\Omega}|x - y|^{-d/\alpha}, \quad \forall x \in \Omega, \quad x \neq y,
\]

where \( \alpha = d/(d-1-\lambda) \), and hence \( F \in L^2(\Omega) \) for all \( q < \alpha \). It then follows from Lemma 2.4 that \( u \in W^{1,q}(\Omega) \) for all \( q \in (1, \alpha) \). In fact, by Lemma 2.5 we have

\[
\|u\|_{L^{\alpha,\alpha}(\Omega)} + \|\nabla u\|_{L^{\alpha,\infty}(\Omega)} \leq C.
\]

Let \( v = \zeta u \), where \( \zeta : \mathbb{R}^d \to \mathbb{R} \) is a smooth function to be fixed later. Observe that \( v \) is a weak solution of the problem

\[
\begin{cases}
-Lv = \tilde{f} + \nabla \cdot \tilde{F} & \text{in } \Omega, \\
(\gamma \nabla v + F) \cdot n = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where

\[
\tilde{f} := -\nabla \zeta \cdot F - \gamma \nabla \zeta \cdot \nabla u, \quad \tilde{F} := \zeta F - \gamma u \nabla \zeta.
\]

Notice that if \( \zeta \equiv 0 \) on a neighborhood of \( y \), then we have \( \tilde{f} \in L^q(\Omega) \) and \( \tilde{F} \in L^{q'}(\Omega) \) for all \( q \in (1, \alpha) \). By Lemma 2.4 we have \( v \in W^{1,q}(\Omega) \) and thus, we find that \( u \in W^{1,q}_{\text{loc}}(\Omega \setminus \{y\}) \).

By repeating the above argument, if necessary, we conclude that \( u \in W^{1,q}_{\text{loc}}(\Omega \setminus \{y\}) \) for some \( s > d \), and thus we have \( u \in L^q_{\text{loc}}(\Omega \setminus \{y\}) \).

Next, for \( x \in \Omega \) with \( x \neq y \), let \( R = |x - y|/2 \). For any \( x' \in \Omega \cap B_R(x) \) and \( 0 < \rho < r \leq R \), we choose the function \( \zeta \) to be such that

\[
0 \leq \zeta \leq 1, \quad \text{supp } \zeta \subset B_r(x'), \quad \zeta \equiv 1 \text{ on } B_\rho(x'), \quad \text{and } |\nabla \zeta| \leq 2/(r - \rho).
\]
Notice that for all $q \in (1, d)$, we have the following estimates, where we write $\Omega_\rho = \Omega_\rho(x') = \Omega \cap B_\rho(x')$ for the simplicity of notation,
\[
\|\nabla v\|_{L^{t, \infty}(\Omega_\rho)} \leq \|\nabla v\|_{L^{t, \infty}(\Omega)},
\]
\[
\|\nabla \zeta \cdot F\|_{L^{t, \infty}(\Omega)} \leq \|\nabla \zeta\|_{L^\infty} \|F\|_{L^{t, \infty}(\Omega)},
\]
\[
\|\nabla \zeta \cdot \nabla u\|_{L^{t, \infty}(\Omega)} \leq \|\nabla \zeta\|_{L^\infty} \|\nabla u\|_{L^{t, \infty}(\Omega)},
\]
\[
\|\zeta\|_{L^{t, \infty}(\Omega)} \leq \|\zeta\|_{L^\infty} \|F\|_{L^{t, \infty}(\Omega)},
\]
\[
\|u \nabla \zeta\|_{L^{t, \infty}(\Omega)} \leq \|\nabla \zeta\|_{L^\infty} \|u\|_{L^{t, \infty}(\Omega)},
\]
Therefore, by Lemma 2.5 applied to $v$, we have for all $t \in (d/(d - 1), \infty)$
\[
\|\nabla u\|_{L^{t, \infty}(\Omega_\rho)} \leq C \left( (r - \rho)^{-1} r^{1/d} \|F\|_{L^\infty(\Omega)} + (r - \rho)^{-1} \|\nabla u\|_{L^{d/(d + 1), \infty}(\Omega)} \right).
\]
Now, fix $s > d$ and let $m = [d(1/\alpha - 1/s)]$,
\[
s_j = \frac{sd}{d + s_j} \quad \text{and} \quad r_j = \rho + \frac{(r - \rho)j}{m}, \quad j = 0, \ldots, m.
\]
Recall that if $E$ is a bounded set and $0 < q < \infty$, then
\[
\|f\|_{L^{q, \infty}(E)} \leq \|f\|_{L^q(E)} \leq \sqrt{\frac{p}{p - q}} |E|^{1/q - 1/p} \|f\|_{L^{p, \infty}(E)}.
\]
With the aid of (2.61), we apply (2.60) repeatedly and argue as in the proof of Lemma 2.6 to obtain
\[
\rho^{-d(1/2 - 1/s)} \|\nabla u\|_{L^2(\Omega_\rho)} \leq C \left( \frac{r}{r - \rho} \right)^m r^{d/s} \|F\|_{L^\infty(\Omega)} + C \left( \frac{r}{r - \rho} \right)^{m - 1} r^{d/s} \|F\|_{L^\infty(\Omega)}
\]
\[
+ Cr^{-1} \left( \frac{r}{r - \rho} \right)^m \|u\|_{L^t(\Omega)} + C \left( \frac{r}{r - \rho} \right)^m r^{d(1/s - 1/\alpha)} \|u\|_{L^{t, \infty}(\Omega)}.
\]
If we take $r = R/4$ and $\rho < r/2 = R/4$ in the above, then for all $x' \in \Omega_{R/4}(x)$, we get
\[
\left( \rho^{-(d - 2 + (1 - d)/s)} \int_{\Omega_{R/4}(x')} |\nabla u|^2 \, dz \right)^{1/2} \leq CR^{d/s} \|F\|_{L^\infty(\Omega_R(x))}
\]
\[
+ CR^{-1} \|u\|_{L^t(\Omega_R(x))} + CR^{d(1/s - 1/\alpha)} \|u\|_{L^{t, \infty}(\Omega)},
\]
which is analogous to (2.37) in the proof of Lemma 2.6. Then by utilizing (2.59) and proceeding as in the proof of Lemma 2.6, we obtain
\[
\sup_{\Omega_{R/4}} |u| \leq CR^{-d} \|u\|_{L^1(\Omega_R)} + CR \|F\|_{L^\infty(\Omega)} + CR^{1 - d/\alpha}, \quad \Omega_R = \Omega_R(x).
\]
By Lemma 2.5 and (2.61) again, we get
\[
\|u\|_{L^1(\Omega_R)} \leq CR^{2+\lambda} \|u\|_{L^{t, \infty}(\Omega_R)} \leq CR^{2+\lambda}.
\]
Combining the above two inequalities and using (2.63), we get
\[
|N(x, y) - N_0(x, y)| = |u(x)| \leq CR^{2-d+\lambda} + CR^{1-d/\alpha} \leq C|x - y|^{2-d+\lambda}.
\]
This completes the proof of (2.2).

Next, we turn to the proof of (2.3). Let \( u = N(\cdot, y) - N_0(\cdot, y) \) as before. Observe that \( u \) satisfies
\[
-L_0u = \nabla \cdot (F + (\gamma - \gamma_0)\nabla u) \quad \text{in} \quad \Omega.
\]
Let \( x \in \Omega \) satisfy \( 0 < |x - y| < d_y/2 \) and let \( R = |x - y|/2 \) as before. For any \( x' \in B_{R/2}(x) \) and \( 0 < r \leq R/2 \), let \( w \) be the unique weak solution in \( W^{1,2}_0(B_r(x')) \) of the problem
\[
\begin{cases}
-\gamma_0 \Delta w = -iku + \nabla \cdot (F + (\gamma - \gamma_0)\nabla u) & \text{in} \ B_r(x'),

w = 0 & \text{on} \ \partial B_r(x').
\end{cases}
\]
Then \( w \) satisfies the following identity:
\[
\int_{B_r(x')} \gamma_0 \nabla w \cdot \nabla w \, dz = - \int_{B_r(x')} (iku \nabla w - (F - F_r) \cdot \nabla w - (\gamma - \gamma_0)\nabla u \cdot \nabla w) \, dz, \quad (2.63)
\]
where we use the notation
\[
F_r = F_{x',r} = \frac{1}{|B_r(x')|} \int_{B_r(x')} F \, dz.
\]
Notice that by Hölder’s inequality and the Sobolev inequality, we have
\[
\left| \int_{B_r(x')} iku \nabla w \right| \leq Ck \left( \int_{B_r(x')} |u|^{2d/(d+2)} \right)^{(d+2)/2d} \left( \int_{B_r(x')} |\nabla w|^2 \right)^{1/2} \leq Ck \gamma_0^{(d+2)/2} \|u\|_{L^\infty(B_R)} \left( \int_{B_r(x')} |\nabla w|^2 \right)^{1/2}.
\]
Also, by Hölder’s inequality, we have
\[
\left| \int_{B_r(x')} (F - F_r) \cdot \nabla w \right| \leq \left( \int_{B_r(x')} |F - F_r| \right)^{1/2} \left( \int_{B_r(x')} |\nabla w|^2 \right)^{1/2} \leq |F|_{0,\lambda;B_R} r^\lambda |B_r|^{1/2} \left( \int_{B_r(x')} |\nabla w|^2 \right)^{1/2} \leq C |F|_{0,\lambda;B_R} r^{\lambda+d/2} \left( \int_{B_r(x')} |\nabla w|^2 \right)^{1/2}.
\]
Similarly, we estimate
\[
\left| \int_{B_r(x')} (\gamma - \gamma_0)\nabla u \cdot \nabla w \right| \leq \|\gamma - \gamma_0\|_{L^\infty(B_r(x'))} \left( \int_{B_r(x')} |\nabla w|^2 \right)^{1/2} \left( \int_{B_r(x')} |\nabla u|^2 \right)^{1/2} \leq |\gamma|_{0,\lambda;B_R} r^\lambda \left( \int_{B_r(x')} |\nabla u|^2 \right)^{1/2} \left( \int_{B_r(x')} |\nabla w|^2 \right)^{1/2}.
\]
Therefore, by using Cauchy’s inequalities, we derive from (2.64) and the above estimates that
\[
\int_{B_r(x')} |\nabla w|^2 \, dz \leq Ck^{d+2}\|u\|_{L^\infty(B_{2r})}^2 + Cr^{d+2}\|F\|_{\mathbb{L}^2(B_r)}^2 + Cr\|\gamma\|_{\mathbb{L}^2(B_r)}^2 \int_{B_r(x')} |\nabla u|^2 \, dz,
\]
where we use abbreviation \( B_R = B_R(x) \).

Notice that \( v = u - w \) satisfies
\[
\Delta v = 0 \quad \text{in} \quad B_r(x').
\]

By well-known estimates for harmonic functions (see, for instance, \([10], p. 78\)), we get
\[
\int_{B_{\rho}(x')} |\nabla v - (\nabla v)_\rho|^2 \, dz \leq C(\rho/r)^{d+2} \int_{B_{\rho}(x')} |\nabla v - (\nabla v)_\rho|^2 \, dz, \quad \forall \rho \in (0, r).
\]

Then by using the triangle inequality, we get for all \( 0 < \rho < r \) that
\[
\int_{B_{\rho}(x')} |\nabla u - (\nabla u)_\rho|^2 \, dz \leq 2 \int_{B_{\rho}(x')} |\nabla v - (\nabla v)_\rho|^2 \, dz + 2 \int_{B_{\rho}(x')} |\nabla w - (\nabla w)_\rho|^2 \, dz
\]
\[
\leq C(\rho/r)^{d+2} \int_{B_{\rho}(x')} |\nabla v - (\nabla v)_\rho|^2 \, dz + 2 \int_{B_{\rho}(x')} |\nabla w|^2 \, dz
\]
\[
\leq C(\rho/r)^{d+2} \int_{B_{\rho}(x')} |\nabla u - (\nabla u)_\rho|^2 \, dz + C \int_{B_{\rho}(x')} |\nabla w|^2 \, dz,
\]

where we have used the well known fact that
\[
\inf_{c \in \mathbb{R}} \int_{B_{\rho}(x')} |f - c|^2 \, dz = \int_{B_{\rho}(x')} |f - f_{\rho}|^2 \, dz.
\]

By combining the above inequality and (2.64), we get for all \( 0 < \rho < r \) that
\[
\int_{B_{\rho}(x')} |\nabla u - (\nabla u)_\rho|^2 \, dz \leq C \left( \frac{\rho}{r} \right)^{d+2} \int_{B_{\rho}(x')} |\nabla u - (\nabla u)_\rho|^2 \, dz
\]
\[
+ Ck^{2+d+2}\|u\|_{L^\infty(B_{2r})}^2 + Cr^{d+2}\|F\|_{\mathbb{L}^2(B_r)}^2 + Cr\|\gamma\|_{\mathbb{L}^2(B_r)}^2 \int_{B_r(x')} |\nabla u|^2 \, dz. \tag{2.65}
\]

On the other hand, by setting \( \epsilon = d/s \) and \( \rho = r \) in (2.64), we get
\[
\left( r^{-(d-2\epsilon)} \int_{B_{\rho}(x')} |\nabla u|^2 \, dz \right)^{1/2} \leq CR\|F\|_{L^\infty(B_r)} + CR^{-1}\|u\|_{L^\infty(B_r)} + CR^{-d/\alpha}.
\]

Combining the above inequalities, for all \( x' \in B_{R/2}(x) \) and \( 0 < \rho < r \), we get
\[
\int_{B_{\rho}(x')} |\nabla u - (\nabla u)_\rho|^2 \leq C \left( \frac{\rho}{r} \right)^{d+2} \int_{B_{\rho}(x')} |\nabla u - (\nabla u)_\rho|^2 + Ck^{2+d+2}\|u\|_{L^\infty(B_{2r})}^2 + Cr\|F\|_{\mathbb{L}^2(B_r)}^2 \int_{B_r(x')} |\nabla u|^2 \, dz
\]
\[
+ C\|\gamma\|_{\mathbb{L}^2(B_r)}^2 r^{d+2\lambda-2\epsilon} \left( R^\epsilon\|F\|_{L^\infty(B_r)} + R^{-1}\|u\|_{L^\infty(B_r)} + R^{-d/\alpha} \right)^2.
\]
By Campanato’s iteration lemma, for all \( x' \in B_{R/2}(x) \) and \( 0 < r \leq R/2 \), we have

\[
\int_{B_r(x')} |\nabla u - (\nabla u)_2|^2 \lesssim C \left( \frac{r}{R} \right)^{d+2\beta} \int_{B_r(x')} |\nabla u|^{2} + C k \int_{B_r(x')} \left( |\nabla u|^{2} + R^{2\lambda - 2\beta} \|u\|^{2}_{L^{\infty}(B_r(x'))} + C |\nabla [F]_{0,\lambda} R^{-2\lambda - 2\beta} \right) + C |\nabla u|^{2} + R^{2\lambda - 2\beta} \|u\|^{2}_{L^{\infty}(B_r(x'))} + R^{-d/\alpha} \right)^{2},
\]

where we set \( \beta := \lambda - \epsilon \in (0, 1) \). Therefore, by Campanato’s theorem, we obtain

\[
R^\alpha [\nabla u]_{0,\beta;B_{r/2}} \leq CR^{-d/2} \|\nabla u\|_{L^2(B_{r/2})} + C k R \|u\|_{L^\infty(B_{r/2})} + CR^\lambda [F]_{0,\lambda;B_{r/2}} + C |\nabla u|_{B_{r;2}} \leq CR^{-d/2} \|\nabla u\|_{L^2(B_{r/2})} + C R^{-d/2} R^{1-d+\lambda}.
\]

Also, observe that

\[
[F]_{0,\lambda;B_{r/2}} \leq R^\lambda \left[ \gamma \right]_{0,\lambda;B_{r/2}} + \left[ \gamma \right]_{0,\lambda;B_{r/2}} \|\nabla N_0(\cdot, y)\|_{L^\infty(B_{r/2})} \leq CR^{-d},
\]

where \( C \) depends on \( d, \nu, \lambda, \Omega, \) and \( \left[ \gamma \right]_{0,\lambda;\Omega} \). Therefore,

\[
R^\beta [\nabla u]_{0,\beta;B_{r/2}} \leq C \left( R^{1-d+\lambda} + k R^{3-d+\lambda} + R^{1-d+2\lambda} \right) \leq CR^{1-d+\lambda} (1 + k R^2), \tag{2.66}
\]

where we used the assumption that \( \Omega \) is bounded in the last step. By proceeding as in the proof of Lemma 2.9, we derive from (2.66) that

\[
\left\| \nabla u \right\|_{L^2(B_{r/4})} \leq CR^{1-d+\lambda} (1 + k R^2).
\]

This completes the proof of (2.63).

Now, let us assume that \( \gamma \in C^{1,\lambda}(\Omega) \). Let \( x \in \Omega \) satisfy \( 0 < |x - y| < d_y/2 \). We again set \( R = |x - y|/2 \) and write \( B_R = B_R(x) \). Observe that \( u \) satisfies

\[
-\gamma \Delta u = f \quad \text{in} \quad B_R,
\]

where

\[
f := \nabla \gamma \cdot \nabla u - iku + ik \gamma_0^{-1} (\gamma - \gamma_0) N_0(\cdot, y) + \nabla \gamma \cdot \nabla N_0(\cdot, y).
\]

We claim that \( f \in C^{0,\lambda}(\overline{B_R}) \). Indeed, observe that by feeding estimate (2.3) back to (2.63) and repeating the above steps, we obtain an improved version of estimate (2.66), namely,

\[
\left\| \nabla u \right\|_{0,\lambda;B_{r/2}} \leq CR^{1-d} (1 + k R^2).
\]

Therefore, we obtain

\[
\left\| \nabla \gamma \cdot \nabla u \right\|_{0,\lambda;B_R} \leq \left\| \nabla \gamma \right\|_{0,\lambda;\Omega} \left\| \nabla u \right\|_{L^\infty(B_R)} + \left\| \nabla \gamma \right\|_{L^\infty(\Omega)} \left\| \nabla u \right\|_{0,\lambda;B_R} \leq CR^{1-d+\lambda} (1 + k R^2) + CR^{1-d} (1 + k R^2) \leq CR^{1-d} (1 + k R^2),
\]

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where we have used the assumption that $\Omega$ is bounded. Also, by taking $s = d/(1 - \lambda)$ in (2.62), we find that for all $x' \in B_{R/2}$ and $\rho \leq R/4$, we have
\[
\left( \rho^{-(d-2+2\lambda)} \int_{B_\rho(x')} |\nabla u|^2 \, dz \right)^{1/2} \leq CR^{1-\lambda}||F||_{L^\infty(B_R)} + CR^{-\lambda}||u||_{L^\infty(B_R)} + CR^{2-d} \leq CR^{2-d}.
\]
From the above inequality and a standard covering argument, we find that
\[
[iku]_{0,\lambda;B_R} = k[u]_{0,\lambda;B_R} \leq CR^{2-d}.
\]
In a similar fashion, with the aid of (2.57), we also estimate
\[
[\nabla u]_{0,\lambda;B_{R/2}} \leq CR^{1-d},
\]
where we again used that $\text{diam} \Omega < \infty$. Then the interior Schauder estimate yields
\[
[\nabla u]_{0,\lambda;B_{R/4}} \leq C \left( [f]_{0,\lambda;B_R} + R^{-2-\lambda}||u||_{L^\infty(B_R)} \right) \leq CR^{2}(1 + kR^2).
\]
On the other hand, by the standard $L^2$ estimates, we have
\[
||\nabla^2 u||_{L^2(B_{R/2})} \leq C \left( R^{-1}||\nabla u||_{L^2(B_R)} + ||\nabla u||_{L^2(B_{3R/2})} + ||\nabla F||_{L^2(B_R)} \right)
\leq C \left( (R^{-2} + R^{-1})||u||_{L^2(B_{3R/2})} + (R^{-1} + 1)||F||_{L^2(B_{3R/2})} + ||\nabla F||_{L^2(B_R)} \right)
\leq CR^{d/2}R^{-d+\lambda}(1 + R^{-1-\lambda} + R) \leq CR^{d/2}R^{-d+\lambda}.
\]
Therefore, we have
\[
\sup_{B_{R/4}} ||\nabla^2 u|| \leq CR^{d/2}||\nabla^2 u||_{L^2(B_{R/2})} + CR^{\lambda}||\nabla^2 u||_{0,\lambda;B_{R/2}}
\leq CR^{-d+\lambda} + CR^{-d+\lambda}(1 + kR^2) \leq CR^{-d+\lambda}(1 + kR^2).
\]
We have thus proved (2.5). Finally, we prove (2.3) as follows. Notice that $v := \partial u/\partial x_i$, for $i = 1, \ldots, d$, satisfies
\[-Lv = \nabla \cdot \tilde{F}, \quad \text{where} \quad \tilde{F} = (\partial \gamma/\partial x_i)\nabla u + \partial F/\partial x_i.
\]
Let $R = |x - y|/2$ as before and applying (2.19) in Lemma 2.2 to $v$, we obtain
\[
\sup_{B_{R/2}} |v| \leq C \left( R^{-d/2}||\nabla u||_{L^2(B_R)} + R^{1-d/2}||\tilde{F}||_{L^2(B_R)} \right).
\]
Notice that
\[
||\tilde{F}||_{L^2(B_R)} \leq C||\nabla \gamma||_{L^\infty(\Omega)} \left( ||\nabla u||_{L^2(B_R)} + ||\nabla N_0(\cdot,y)||_{L^2(B_R)} + R||\nabla^2 N_0(\cdot,y)||_{L^2(B_R)} \right)
\leq C||\nabla u||_{L^2(B_R)} + CR^{d/2}R^{1-d}.
\]
On the other hand, by (2.62), we find
\[ R^{-d/2} \| \nabla u \|_{L^2(B_R)} \leq C \left( \| F \|_{L^\infty(B_{2R})} + R^{-1} \| u \|_{L^\infty(B_{2R}/2)} + R^{-d/\alpha} \right) \leq CR^{1-d+\lambda}. \]

Combining together, we obtain
\[ \sup_{B_{R/2}} |\nabla u| \leq C \left( R^{1-d+\lambda} + R^{2-d+\lambda} + R^{2-d} \right) \leq CR^{1-d+\lambda}, \]

where \( C \) depends on \( \| \nabla \gamma \|_{L^\infty(\Omega)} \) and \( \text{diam } \Omega \) as well as on \( d, \nu, k_0, \lambda, \Omega \). This proves estimate (2.4). The proof of Theorem 2.1 is now complete.

Remark 2.8 We remark that for \( z \in \Omega \) fixed, we may choose \( \epsilon = \epsilon(z) > 0 \) so small that for all \( y \in B_{\epsilon}(z) \), we have \( d_y > 4 \epsilon \), and hence (2.2)-(2.5) hold for all \( x, y \in B_{\epsilon}(z) \). We also note that in the proof of (2.2), it is enough to assume that \( \gamma \in C^1(\Omega) \) not \( \gamma \in C^{1,\lambda}(\Omega) \). Also, if we assume \( \gamma \in C^2(\Omega) \), then instead of (2.5), we have
\[ |\nabla^2 x N(x,y) - \nabla^2 x N_0(x,y)| \leq C|x-y|^{-d+\lambda}, \quad \forall x \in \Omega \text{ satisfying } 0 < |x-y| < d_y/2, \]

where \( C \) depends on \( |\gamma|_{C^2(\Omega)} \) and \( \text{diam } \Omega \) as well as on \( d, \nu, k_0, \lambda, \Omega \). The proof for (2.67) is analogous to that for (2.4). Moreover, if \( d \geq 4 \), then we may take \( \gamma = 1 \) in (2.2), (2.4), and (2.67), since in that case, we may take \( \alpha = d/(d-2) < d \) in (2.6).

3 Applications to quantitative photo-acoustic imaging

In this section we deal with the problem of quantitative photo-acoustic imaging to reconstruct the optical absorption coefficient from the absorbed energy density. The absorbed energy density can be reconstructed using the measurements of the acoustic wave on the boundary of the medium. See, for instance, [1, 21].

Reconstruction of the optical absorption coefficient, \( \mu_a \), from the absorbed energy density, \( A(x) \), is subtle since \( \mu_a \) is related to \( A(x) \) in a nonlinear and implicit way. In fact, \( \mu_a \) is related to \( A(x) \) by
\[ A = \mu_a \Phi \]  
(3.1)

Here \( \Phi \) is the light fluence which depends on the distribution of scattering and absorption within \( \Omega \), as well as the light sources. Let \( \mu_s \) be the scattering coefficient. The function \( \Phi \) is related to \( \mu_a \) through the diffusion equation
\[ \left( \frac{i \omega}{c} + \frac{\mu_a(x)}{3} \nabla \cdot \frac{1}{\mu_a(x) + \mu_s(x)} \nabla \right) \Phi(x) = 0 \quad \text{in } \Omega, \]

(3.2)

with the boundary condition
\[ \frac{1}{3(\mu_a(x) + \mu_s(x))} \frac{\partial \Phi}{\partial \nu} = g \quad \text{on } \partial \Omega, \]

(3.3)

where \( g \) denotes the light source and \( \omega \) is a given frequency. Equation (3.2) is derived based on the diffusion approximation to the transport equation which holds when \( \mu_s \gg \mu_a \). See, for instance, [1] [14]. Note that in [2], the boundary condition is a Robin boundary
condition. However, it is easy to check that all the estimates derived in \[2\] Section 2 hold for the Neumann boundary condition (3.3).

We restrict ourselves to the three-dimensional case and suppose that the medium contains a small absorbing anomaly whose absorption coefficient is to be reconstructed. The small unknown anomaly $D$ is modeled as

$$D = z + \epsilon B,$$

where $z$ represents the location of $D$, $B$ is a reference domain which contains the origin, and $\epsilon$ is a small parameter representing the diameter of the anomaly. We assume that the anomaly is away from the boundary $\partial \Omega$, namely

$$\text{dist}(z, \partial \Omega) \geq C_0$$

for some constant $C_0$. Since $D$ is small and absorbing, and the background absorption is quite small compared to the scattering, we may assume that

$$\mu_a(x) = \mu_a \chi_D(x)$$

where $\mu_a$ is a constant and $\chi_D$ is the characteristic function of $D$. Then, (3.2) and (3.3) may be approximated by

$$\begin{cases}
\left( \frac{i \omega}{c} + \mu_a \chi_D(x) - \frac{1}{3} \nabla \cdot \frac{1}{\mu_s(x)} \nabla \right) \Phi(x) = 0 & \text{in } \Omega, \\
\frac{1}{3 \mu_s(x)} \frac{\partial \Phi}{\partial \nu} = g & \text{on } \partial \Omega.
\end{cases}$$

Since $D$ is small, we may regard $\Phi$ as a perturbation of $\Phi^{(0)}$ which is the solution of

$$\begin{cases}
\left( \frac{i \omega}{c} - \frac{1}{3} \nabla \cdot \frac{1}{\mu_s(x)} \nabla \right) \Phi^{(0)}(x) = 0 & \text{in } \Omega, \\
\frac{1}{3 \mu_s(x)} \frac{\partial \Phi^{(0)}}{\partial \nu} = g & \text{on } \partial \Omega.
\end{cases}$$

The reconstruction methods in \[2\] deeply rely on the following asymptotic formula $\Phi - \Phi^{(0)}$, which was obtained under the assumption that $\mu_s$ is constant:

$$(\Phi - \Phi^{(0)})(x) \approx 3 \epsilon^2 \mu_a \mu_s \Phi^{(0)}(z) \tilde{N}_B(0) - \epsilon \frac{\mu_a}{\mu_s} S_B[\nu](0) \cdot \nabla \Phi^{(0)}(z),$$

where $\tilde{N}_B$ be the Newtonian potential of $B$, which is given by

$$\tilde{N}_B(x) := \int_B \Gamma(x - y) \, dy, \quad x \in \mathbb{R}^3,$$

and $S_B$ is the single layer potential associated to $B$, which is given for a density $\psi \in L^2(\partial B)$ by

$$S_B[\psi](x) := \int_{\partial B} \Gamma(x - y) \psi(y) \, d\sigma(y), \quad x \in \mathbb{R}^3.$$
Here $\Gamma$ is the fundamental solution to the Laplacian in three dimensions, i.e.,
\[
\Gamma(x) := -\frac{1}{4\pi|x|}.
\]

The purpose of this section is to show that the asymptotic expansion (3.8) holds even when $\mu_s$ is variable. The following theorem holds.

**Theorem 3.1** Let $\Omega$ be a bounded $C^2$-domain in $\mathbb{R}^3$. Let $D = z + \epsilon B$, where $B$ is a bounded Lipschitz domain in $\mathbb{R}^3$ containing the origin. Suppose that $\mu_a$ is given by (3.7) and $\mu_s \in C^{1,\lambda}([\Omega])$, $\lambda \in (0,1)$, and set $\bar{\mu}_s := \mu_s(z)$. Then, as $\epsilon \to 0$, we have
\[
(\Phi - \Phi^{(0)})(z) \approx 3\epsilon^2 \mu_a \bar{\mu}_s \Phi^{(0)}(z) + \epsilon \frac{\mu_a}{\bar{\mu}_s} S_B[v](0) \cdot \nabla \Phi^{(0)}(z), \quad (3.11)
\]
where the error term is less than
\[
C\left(c^{1+\mu} \mu_a \bar{\mu}_s^{3/2}(1 + \epsilon \sqrt{\mu_a}) \left(\epsilon^2 \mu_a \bar{\mu}_s + \frac{\mu_a}{\bar{\mu}_s}\right) + \epsilon \sqrt{\mu_a} \left(\epsilon^2 \mu_a \bar{\mu}_s + \frac{\mu_a}{\bar{\mu}_s} \right)^2 + \epsilon^2 \mu_a\right),
\]
for $p > 3$ and some constant $C$ depending on $||\mu_s||_{C^{1,\lambda}}, \lambda, \Omega, \omega/c$, and $g$.

Since the proof is essentially the same as that in [2], we only outline the proof without much details.

Let $N(x, y)$ be the Neumann function of the operator $\frac{i\omega}{c} - \frac{1}{y} \nabla \cdot \frac{1}{\mu_a(x)} \nabla$ on $\Omega$. Then one can show by following the same lines of the proof of [2] Lemma 2.1 that for any $x \in \Omega$,
\[
(\Phi - \Phi^{(0)})(x) = \mu_a \int_D \Phi(y) N(x, y) \, dy
+ \frac{1}{3} \int_D \left(\frac{1}{\mu_a + \mu_s(y)} - \frac{1}{\mu_s(y)}\right) \nabla \Phi(y) \cdot \nabla_y N(x, y) \, dy. \quad (3.12)
\]
Let $N_0(x, y)$ be the Neumann function of $\frac{i\omega}{c} - \frac{1}{y} \nabla \Delta$ on $\Omega$. We suppose that $\frac{\partial}{\partial y_c}$ satisfy the ellipticity condition (2.1). It follows from Theorem 2.1 that there is a constant $C$ such that
\[
|N(x, y) - N_0(x, y)| \leq C\bar{\mu}_s |x - y|^{-1+\lambda},
|\nabla_x (N(x, y) - N_0(x, y))| \leq C\bar{\mu}_s |x - y|^{-2+\lambda},
|\nabla_x^2 (N(x, y) - N_0(x, y))| \leq C \left(\bar{\mu}_s |x - y|^{-3+\lambda} + \bar{\mu}_s^2 |x - y|^{-1+\lambda}\right),
\]
for all $x, y \in D$ provided that $\omega$ is bounded. On the other hand, it is proved in [2] Lemma 2.2 that there is a constant $C$ such that
\[
|N_0(x, y) - 3\bar{\mu}_s \Gamma(x - y)| \leq C\bar{\mu}_s^{3/2},
|\nabla_x (N_0(x, y) - 3\bar{\mu}_s \Gamma(x - y))| \leq C \left(\bar{\mu}_s^2 + \bar{\mu}_s^{3/2} |x - y|^{-1}\right),
|\nabla_x^2 (N_0(x, y) - 3\bar{\mu}_s \Gamma(x - y))| \leq C \left(\bar{\mu}_s^{5/2} + \bar{\mu}_s^{3/2} |x - y|^{-2}\right),
\]
for all $x, y \in D$ provided that $\epsilon \sqrt{\mu_a}$ is sufficiently small. Therefore, if we put
\[
R(x, y) = N(x, y) - 3\bar{\mu}_s \Gamma(x - y), \quad (3.13)
\]
we obtain the following lemma.
Lemma 3.2 Let $R$ be defined by (3.13). There exists a constant $C$ such that

$$|R(x, y)| \leq C \left( \bar{\mu} s^{3/2} + \bar{\mu} s |x - y|^{-1+\lambda} \right),$$  \hspace{1cm} (3.14)

$$|\nabla_x R(x, y)| \leq C \left( \bar{\mu} s^2 + \bar{\mu} s^{3/2} |x - y|^{-1} + \bar{\mu} s |x - y|^{-2+\lambda} \right),$$  \hspace{1cm} (3.15)

$$|\nabla_x^2 R(x, y)| \leq C \left( \bar{\mu} s^{5/2} + \bar{\mu} s^{3/2} |x - y|^{-2} + \bar{\mu} s |x - y|^{-3+\lambda} + \bar{\mu} s^2 |x - y|^{-1+\lambda} \right).$$  \hspace{1cm} (3.16)

We introduce some notation following [2]. Let

$$n(x) := \int_D N(x, y) \, dy, \quad x \in D,$$  \hspace{1cm} (3.17)

and define a multiplier $M$ by

$$M[f](x) := \mu a n(x) f(x).$$  \hspace{1cm} (3.18)

We then define two operators $N$ and $R$ by

$$N[f](x) := 3 \mu a \bar{\mu} s \int_D (f(y) - f(x)) \Gamma(x - y) \, dy$$
$$+ \bar{\mu} s \int_D \left( \frac{1}{\mu a + \mu s} - \frac{1}{\mu s} \right) \nabla f(y) \cdot \nabla \Gamma(x - y) \, dy,$$  \hspace{1cm} (3.19)

$$R[f](x) := \mu a \int_D (f(y) - f(x)) R(x, y) \, dy$$
$$+ \frac{1}{3} \int_D \left( \frac{1}{\mu a + \mu s} - \frac{1}{\mu s} \right) \nabla f(y) \cdot \nabla y R(x, y) \, dy.$$  \hspace{1cm} (3.20)

Then, (3.12) can be rewritten as

$$(I - M)[\Phi] - (N + R)[\Phi] = \Phi(0) \quad \text{on} \quad D,$$  \hspace{1cm} (3.21)

where $I$ is the identity operator.

For $\eta > 0$, define

$$T_\eta[f](x) = \int_D \frac{f(y)}{|x - y|^{1-\eta}} \, dy, \quad x \in D.$$  \hspace{1cm} (3.22)

Then one can show using Hölder’s inequality that

$$\|T_\eta[f]\|_{L^p(D)} \leq C \epsilon^\eta \|f\|_{L^p(D)}$$  \hspace{1cm} (3.23)

for all $p > \frac{2}{\eta}$.

We fix $\lambda$ so that $\lambda > \frac{1}{2}$. Suppose that $\epsilon \sqrt{\bar{\mu} s}$ and $\frac{\mu a}{\mu s}$. Using (3.23) one can show that

$$\|N[f]\|_{W^{1,p}(D)} \leq C \left( \epsilon^2 \mu a \bar{\mu} s + \frac{\mu a}{\mu s} \right) \|\nabla f\|_{L^p(D)}.$$  \hspace{1cm} (3.24)

One can also show using (3.14), (3.16) and (3.22) that

$$\|R[f]\|_{W^{1,p}(D)} \leq C \epsilon \sqrt{\bar{\mu} s} \left( \mu a \bar{\mu} s \epsilon^2 + \frac{\mu a}{\mu s} \right) \|\nabla f\|_{L^p(D)}.$$  \hspace{1cm} (3.25)

Therefore, the following lemma holds.
Lemma 3.3  Let $p > 3$. Then there exists a constant $C$ such that (3.23) and (3.24) hold.

The rest of derivation of (3.9) is exactly the same as in [2]. But we briefly describe it for the readers’ sake. From (3.21), we get

$$\Phi = \sum_{j=0}^{\infty} \left( (I - \mathcal{M})^{-1}(N + \mathcal{R}) \right)^{j}(I - \mathcal{M})^{-1} [\Phi^{(0)}],$$

which converges in $W^{1,p}(D)$ for all $p > 3$. We then obtain

$$\Phi(x) = (I - \mathcal{M})^{-1} [\Phi^{(0)}](x) + (N + \mathcal{R})(I - \mathcal{M})^{-1} [\Phi^{(0)}](x) + E(x), \quad x \in D,$$

where the error term $E$ satisfies

$$\|E\|_{W^{1,p}(D)} \leq C\varepsilon \mu s \mu a (1 + \varepsilon \sqrt{\mu s}) \left( \varepsilon^2 \mu a \bar{\mu s} + \frac{\mu a}{\bar{\mu s}} \right) \|\Phi^{(0)}\|_{W^{1,p}(D)}.$$

Then (3.9) follows from (3.26) and the error of the approximation is bounded by a constant times

$$\varepsilon^{1 + 1/p} \mu a \bar{\mu s}^{3/2} \left( 1 + \varepsilon \sqrt{\mu s} \right) \left( \varepsilon^2 \mu a \bar{\mu s} + \frac{\mu a}{\bar{\mu s}} \right) + \varepsilon \sqrt{\mu s} \left( \varepsilon^2 \mu a \bar{\mu s} + \left( \frac{\mu a}{\bar{\mu s}} \right)^2 \right) + \varepsilon^2 \mu a.$$

We emphasize that approximation (3.9) is valid under the assumption that $\varepsilon \sqrt{\mu s}$ and $\frac{\mu a}{\bar{\mu s}}$ are small, which indicates that the size and absorption coefficient of the anomaly are much smaller than the scattering coefficient.

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