Second Order Expansion for the Nonlocal Perimeter Functional

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Abstract: The seminal results of Bourgain et al. (Optimal Control and Partial Differential Equations, IOS, Amsterdam, 2001) and Dávila (Calc Var Partial Differ Equ 15(4):519–527, 2002) show that the classical perimeter can be approximated by a family of nonlocal perimeter functionals. We consider a corresponding second order expansion for the nonlocal perimeter functional. In a special case, the considered family of energies is also relevant for a variational model for thin ferromagnetic films. We derive the $\Gamma$–limit of these functionals as $\epsilon \to 0$. We also show existence for minimizers with prescribed volume fraction. For small volume fraction, the unique, up to translation, minimizer of the limit energy is given by the ball. The analysis is based on a systematic exploitation of the associated symmetrized autocorrelation function.

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1. Introduction and Statement of Main Results

The seminal results of Bourgain et al. [8] and Dávila [19] show that the classical perimeter can be approximated by a family of nonlocal perimeter functionals. In a
For a family of kernels $K_\epsilon : \mathbb{R}^d \to [0, \infty)$, $K_\epsilon \in L^1(B_1)$, $K_\epsilon(z) = K_\epsilon(|z|)$ and which asymptotically have a singularity at the origin as $\epsilon \to 0$ (see (3)–(4) below). Here, $\mathbb{T}_\ell$ is the flat torus with sidelength $\ell > 0$, Furthermore, $M_\epsilon$ is defined by

$$M_\epsilon := \omega_d^{-1} \int_0^1 K_\epsilon(t) t^{d-1} dt,$$

for $\epsilon > 0$ and where $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$. In this paper we consider the corresponding second order expansion for the nonlocal perimeter functional, i.e. we consider the family of energies

$$\mathcal{E}_\epsilon(u) := M_\epsilon\int_{\mathbb{T}_\ell} |\nabla u| \, dx - \int_{\mathbb{R}^d} K_\epsilon(z) \int_{\mathbb{T}_\ell} \frac{|u(x+z) - u(x)|}{2|z|} \, dx \, dz. \quad (2)$$

We assume that the kernels satisfy $K_\epsilon \to K_0$ for $\epsilon \to 0$ almost everywhere in $\mathbb{R}^d$, where the convergence is monotonically increasing in $B_1$ with

$$\lim_{\delta \to 0} \int_{\delta}^\infty K_0(r) r^{d-1} \, dr = \infty \quad (3)$$

and

$$\int_{\delta}^\infty K_0(r) r^{d-2} \, dr < \infty \quad \forall \delta > 0. \quad (4)$$

We note that $K_0$ is nonnegative, because the kernels $K_\epsilon$ are assumed to be nonnegative. In particular, this includes the following kernels

$$K_\epsilon^{(1)}(r) = r^{-d} \chi_{(0, \infty)}(r) \quad (5)$$

$$K_\epsilon^{(2)}(r) = r^{-d+\epsilon} \quad (6)$$

and

$$K_\epsilon^{(3)}(r) = r^{-q} \chi_{(0, \infty)}(r) \quad \text{for } q > d. \quad (7)$$

We denote the corresponding energies by $\mathcal{E}_\epsilon^{(i)}$ for $i = 1, 2, 3$. When $d = 2$, $\mathcal{E}_\epsilon^{(1)}$ is a sharp interface version of a problem from micromagnetism for thin plates with perpendicular anisotropy $[31, 34, 39, 42]$. In this setting, $\{ u = 1 \}$ and $\{ u = 0 \}$ are the magnetic domains where the magnetization points either upward or downward. The interfacial energy penalizes interfaces between the magnetic domains, while the nonlocal term describes the dipolar magnetostatic interaction energy between the domains. For $\mathcal{E}_\epsilon^{(2)}$, the nonlocal term in the energy (2) is the so-called fractional perimeter $P_\epsilon(\Omega)$ with $s = 1 - \epsilon$ of the set $\Omega := \text{spt } u$ (see e.g. $[9–11, 35]$), and the functional is the second order asymptotic expansion of the fractional perimeter as $s \nearrow 1$. The third energy $\mathcal{E}_\epsilon^{(3)}$ shows that our model also includes some cases where the nonlocal term in the energy is more singular in terms of its scaling than the perimeter.

In the first two cases (5)–(6), the nonlocal term and the perimeter term in (2) both formally have the same scaling for $\epsilon = 0$. However, due to the non-integrability of the kernel $r^{-d}$ we have $M_\epsilon \to \infty$, and both terms in (2) are infinite for $\epsilon = 0$. In our
analysis we show that the singularity of these two terms cancels in the leading order, and we calculate the remainder which describes the limiting behaviour of (2) for $\epsilon \to 0$. Our main result establishes compactness and $\Gamma$-convergence for the family of energies (2):

**Theorem 1.1.** Let $K_\epsilon \not\to K_0$ where $K_0$ satisfies (3)–(4).

(i) (Compactness) For any family $u_\epsilon \in BV(T_\ell; \{0, 1\})$ with $\sup_\epsilon \mathcal{E}_\epsilon(u_\epsilon) < \infty$, there exists $u \in BV(T_\ell; \{0, 1\})$ and a subsequence (not relabelled) such that

$$u_\epsilon \to u \text{ in } L^1(T_\ell) \quad \text{and} \quad \int_{T_\ell} |\nabla u_\epsilon| \to \int_{T_\ell} |\nabla u| \quad \text{for } \epsilon \to 0.$$ 

(ii) ($\Gamma$-convergence) $\mathcal{E}_\epsilon \rightharpoonup \mathcal{E}_0$ in the $L^1$–topology, where

$$\mathcal{E}_0(u) := \frac{1}{2} \int_{\mathbb{R}^d} \int_{T_\ell} K_0(|z|) \left( \frac{|z|}{|z|} \cdot \nabla u(x) \right) \chi_{B_1}(z) - \frac{|u(x + z) - u(x)|}{|z|} \right) dxdz \quad \text{(8)}$$

for $u \in BV(T_\ell; \{0, 1\})$ and $\mathcal{E}_0(u) = +\infty$ otherwise.

In fact, statement (i) of Theorem 1.1 is a consequence of the fact that we have a uniform $BV$ estimate for our family of energies (2) as stated in Proposition 3.1. The $\Gamma$-convergence result demonstrates that the functional $\mathcal{E}_0$, given in (8), is the second order of the asymptotic expansion for the nonlocal perimeter functional and quantifies asymptotically the error in the approximation (1). For the notion of $\Gamma$–convergence we refer to e.g. [18,20].

A key observation for our analysis is that both interfacial energy and nonlocal energy can be expressed solely in terms of the autocorrelation function

$$C_u(z) := \int_{T_\ell} u(x + z)u(x) \, dx.$$ 

In particular, the energies $\mathcal{E}_\epsilon$ for $\epsilon \geq 0$ can be written in terms of the autocorrelation function in the form

$$\mathcal{E}_\epsilon(u) = \int_{\mathbb{R}^d} \frac{C_u(z) - C_u(0) - z \cdot \nabla C_u(0) \chi_{B_1}(z)}{|z|} K_\epsilon(z) \, dz$$

for $\epsilon \geq 0$, cf. Proposition 2.6 for the radially symmetrized expression. We note that the new formulated energy has a simpler structure: it is linear in the space of autocorrelation functions. This leads to easier proofs for the $\Gamma$-convergence for more general kernels compared with the existing literature, e.g. [12,39]. We note that the method of autocorrelation function can also be used for a simpler proof of Dávila’s result [19], see “Appendix A”. We have chosen a periodic setting, but our methods can also be used to derive corresponding results for the corresponding full space problem.

We next comment on our assumptions on the kernels $K_\epsilon$: Since in this paper we are interested in the second order asymptotic expansion for the perimeter, we assume that (3) holds, i.e. $K_\epsilon \to K_0$ as $\epsilon \to 0$ for some measurable function $K_0 : \mathbb{R}^d \to \mathbb{R}_+$ with $\|K_0\|_{L^1(\mathbb{R}^d)} = \infty$ where the convergence is monotone in $B_1$. Furthermore, the assumption (4) ensures that the energy is well-defined. We also note that the monotone convergence of $K_\epsilon$ is used for the proof of the $\Gamma$-convergence of $\mathcal{E}_\epsilon$ (cf. Theorem 3.2) to avoid the concentration effect. It might be possible to weaken this assumption, but this was not our aim in this paper.
We note that the use of the autocorrelation function is tailored to sharp interface problems, since the perimeter functional can be encoded as the derivative of its symmetrized version (cf. Prop. 2.4(v)). In particular, the extension of the above result to the corresponding diffuse interface model with a $W^{1,p}$ norm instead of the perimeter term is not straightforward, though it would certainly be interesting to explore this question. Another interesting aim would be to extend the result to non–radial kernels. This might be feasible using the method of the autocorrelation function.

While autocorrelation functions are natural and often used tools in physics and stochastic geometry, they seem not have been used in the context of nonlocal isoperimetric problems before. We note that the idea of linearization of the problem by a change of configuration space also appears in the formulation of quantum field theory as well as in the theory of optimal transport, for introductions see e.g. [44,46]. In both cases the problem gets linear but the configuration space becomes very complex.

We next consider the minimization problem for energies $E_\epsilon$ with prescribed volume fraction, i.e. for some fixed $\lambda \in [0, 1]$ we assume

$$\int_{\mathbb{T}_\ell} u \, dx = \lambda. \quad (9)$$

We first note that for any $\epsilon \geq 0$, the minimization problem has a solution:

**Proposition 1.2 (Existence of minimizers).** For each $\lambda \in [0, 1]$ and each $\epsilon \geq 0$ there exists a minimizer $u$ of $E_\epsilon$ in the class of functions $u \in BV (\mathbb{T}_\ell; \{0, 1\})$ which satisfy (9).

The proof for Proposition 1.2 is based on the uniform $BV$–bound in Proposition 3.1.

Next we note that the energy $E_\epsilon (\chi/\Omega)$ only depends on the boundary $\partial \Omega$ for any set $\Omega \subset \mathbb{R}^d$. In fact, this can be seen from

$$E_\epsilon (u) = \mathcal{E}_\epsilon (1 - u) \quad \forall \epsilon \geq 0,$$

which follows from the symmetry property for the autocorrelation function (Proposition 2.3(v)). In particular, the minimal energy for prescribed volume fraction is symmetric with respect to $\lambda = \frac{1}{2} \forall \epsilon \geq 0$.

In the following, we discuss properties of minimizers of the limit energy for the special choice (5) or (6) of the kernel, i.e. $K_0(r) = r^{-d}$, and the corresponding energy is denoted by $\mathcal{E}_0^{(1)}$. We first prove that when the volume is sufficiently small, the unique minimizer (up to translation) is given by a single ball:

**Theorem 1.3 (Ball as minimizer for small mass).** Let $K_0(r) = r^{-d}$. Then there exists $m_0 = m_0(d) > 0$, such that if

$$\lambda \leq \min \left\{ \frac{m_0}{|\mathbb{T}_\ell|}, \frac{\omega_d}{4^d} \right\},$$

the unique, up to translation, minimizer $u$ of $\mathcal{E}_0^{(1)}$ in $BV (\mathbb{T}_\ell; \{0, 1\})$ with constraint (9) is a single ball.

Theorem 1.3 actually holds for a slightly larger class of kernels, cf. Proposition 3.3. The proof relies on the sharp quantitative isoperimetric inequality in [22] and a careful study of the limit energy using the autocorrelation function. The optimality of the ball as minimizers has been shown for related models with interfacial term and competing nonlocal energy such as the Ohta-Kawasaki energy [7,21,29,32,33,40]. The difference in our model in comparison to these previous results is that the nonlocal energy has
the same critical scaling as the perimeter, which requires a particularly careful analysis. We note that Theorem 1.3 is concerned with the case of small volume instead of small volume fraction, as the smallness assumption for $\lambda$ is not uniform in $\ell$.

To study the minimizers for small volume fraction, we calculate the energy $\delta^{(1)}_0$ for periodic stripes and lattice balls. For this, we allow that the centers of the balls are arranged in an arbitrary Bravais lattice which requires a slightly more general formulation of our energy (cf. Definition 4.1). In particular, we show that in $2d$ when the volume fraction is almost equal, equally distributed stripes have strictly smaller energy than all lattice ball configurations:

**Proposition 1.4 (Ball vs. stripe patterns).** Let $d = 2$ and $K_0(r) = r^{-2}$. Then there is $c_0 > 0$ such that if $|\lambda - \frac{1}{2}| < c_0$, then the energy of equally distributed stripes is strictly smaller than the energy of periodic ball configurations.

The precise statement and the proofs are given in Sect. 4.

**Remark 1.5 (Relation to domain formation in thin ferromagnetic films).** As mentioned before our energy for $d = 2$ is a sharp interface version of a model which describes the domain formation in thin–films where the preferred magnetization direction is perpendicular to the film. These films exhibit the formation of magnetic domains, where the magnetization is almost constant and either points upward ($u = 1$) or downward ($u = 0$). These domains either have approximately the shape of stripe domains or bubble domains, depending on the volume fraction of the two phases. The volume fraction in turn can be finetuned by the application of an external magnetic field. With the solution of the limit model we can explain the bubble domain shapes in the small volume fraction case. It would be interesting to show if periodic stripe domain patterns are indeed the minimizer for near equal volume fraction as seems to be suggested by the experimental pictures. As a standard reference for the theory of magnetic domains, we refer to [28].

**Previous literature and related models.** We note that isoperimetric problems have been extensively researched in the mathematical community. A prototype model which has been investigated is the sharp–interface Ohta-Kawasaki energy, where the system is described by a sharp interface term together with a nonlocal Coulomb interaction term (or more general Riesz interaction term), see e.g. [1,2,13–15,17,25,26,30,38]. However, our model is different from these models, as the nonlocal term asymptotically has the same (or higher) scaling as the interfacial energy.

When $u \in BV(\mathbb{R}^2; \{0, 1\})$ and $K_\epsilon$ is defined in (5), the energy has been studied by Muratov and Simon in [39], where the authors show among other results the $\Gamma$-convergence of the energy. The $\Gamma$-convergence for the energies (6) is studied by Cesaroni and Novaga [12]. Our model includes the models considered in both [39] and [12] (in the periodic setting). We generalize some results of these papers, however, with a different strategy of proofs and in a more general setting. Moreover, we characterize the minimizers of the limit energy when the volume is small, and show that in $2d$ stripes have strictly smaller energy than lattice of balls if the volume fraction is close to $1/2$. These have not been considered in [12,39].

The model (2) has also been investigated in a series of papers by Pegon [43], Merlet and Pegon [37], Goldman, Merlet and Pegon [27]. The authors consider in particular, the subcritical case when there is a smaller than one constant in front of the nonlinear term. In particular, in [37,43] they show optimality of the ball for large mass (which corresponds to small $\epsilon$ in our case), and in [27] they consider regularity of minimizers.
Structure of paper. In Sect. 2 we introduce the autocorrelation function, collect and prove some of their basic properties. We write the energy in terms of the autocorrelation function and use it to explore some properties of the energy. We also derive different formulations of the energy. In Sect. 3, we give the proof of Theorem 1.1 and Theorem 1.3. The stripes and balls configurations are studied in Sect. 4.

Notation. Throughout the paper unless specified we denote by $C$ a positive constant depending only on $d$. By $\mathbb{T}_\ell := \mathbb{R}^d / (\ell \mathbb{Z})^d$ we denote the flat torus in $\mathbb{R}^d$ with side length $\ell > 0$. We repeatedly identify functions on $\mathbb{T}_\ell$ with $\mathbb{R}^d$–periodic functions on $\mathbb{R}^d$ using the canonical projection $\Pi : \mathbb{R}^d \to \mathbb{T}_\ell$. Similarly, any set $\Omega \subset \mathbb{T}_\ell$ can be identified with its periodic extension onto $\mathbb{R}^d$.

Let $M$ be the space of signed Radon measures of bounded variation on $\mathbb{T}_\ell$. For $\mu \in M$ we write $\| \mu \| := |\mu|$ for its total variation. For any function of bounded variation $u \in BV(\mathbb{T}_\ell)$, we analogously write $\| \nabla u \| = |\nabla u|$ for some $\sigma : \mathbb{T}_\ell \to S^{d-1}$. By the structure theorem for BV functions we have $\nabla u = \sigma |\nabla u|$ for some $\sigma : \mathbb{T}_\ell \to S^{d-1}$. By a slight abuse of notation we sometimes write

$$\int_{\mathbb{R}^d} |\omega \cdot \nabla u(x)| \, dx := \int_{\mathbb{R}^d} |\omega \cdot \sigma(x)| \, d|\nabla u|.$$

Given a measurable set $\Omega \subset \mathbb{T}_\ell$ (or $\Omega \subset \mathbb{R}^d$), the Lebesgue measure of $\Omega$ is denoted by $|\Omega|$. If $\Omega$ has finite perimeter, the perimeter of $\Omega$ is denoted by $P(\Omega) := \| \nabla \chi_{\Omega} \|$. Furthermore, $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$ and $\sigma_{d-1} = d \omega_d$ is the area of the $(d - 1)$–dimensional unit sphere $S^{d-1} \subset \mathbb{R}^d$.

2. Autocorrelation Functions and Energy

In this section we give various different formulations for our energy, in particular in terms of the autocorrelation function.

2.1. Autocorrelation functions. In this section we introduce the autocorrelation function in our setting and show some properties of the autocorrelation function. At the end of the section, we compute explicitly the autocorrelation function for a single ball and stripes. The autocorrelation function of a characteristic function is defined as follows:

**Definition 2.1 (Autocorrelation functions).** Let $\ell > 0$. For $u \in L^1(\mathbb{T}_\ell)$, we define the autocorrelation function $C_u : \mathbb{R}^d \to \mathbb{R}$ by

$$C_u(z) := \int_{\mathbb{T}_\ell} u(x + z)u(x) \, dx \quad \text{for a.e. } z \in \mathbb{R}^d. \tag{10}$$

Its radially symmetrized version $c_u : [0, \infty) \to \mathbb{R}$ is given by

$$c_u(r) := \int_{\partial B_r} C_u(y) \, dy = \int_{S^{d-1}} C_u(r \omega) \, d\omega \quad \text{for a.e. } r \geq 0.$$

We can also write the autocorrelation function (10) as $C_u = \frac{1}{|T_\ell|} u \ast g u$, where $g u(x) := u(-x)$. We note that both $C_u$ and $c_u$ are dimensionless. Moreover, they are invariant under translation and reflection with respect to the origin. We note that in this article,
we only use the autocorrelation function for characteristic functions, in which case we have \( C_u : \mathbb{R}^d \to [0, 1] \) and \( c_u : [0, \infty) \to [0, 1] \), and both functions are everywhere defined and continuous.

Autocorrelation functions are widely used in other fields such as physics and stochastic geometry: One common use of these functions is to study spectral properties of observed patterns in theoretical physics (e.g. astrophysics). In the field of stochastic geometry, the autocorrelation function is also called covariogram or set covariance (cf. e.g. [36] and [4]). A common question in this field is to reconstruct a set from its autocorrelation function. One conjecture is e.g. whether the autocorrelation function for a convex set can characterize the set uniquely up to translation and reflection [36].

We note that the definition of autocorrelation function can be extended to more general configurations:

Remark 2.2 (General periodic configurations). Let \( \Lambda \subset \mathbb{R}^d \) be a periodicity cell and assume that \( u : \mathbb{R}^d \to \mathbb{R} \) is \( \Lambda \)-periodic, i.e. \( u(x + \Lambda) = u(x) \). Then the autocorrelation function \( C_u \) can be defined by

\[
C_u(z) = \lim_{\ell \to \infty} \int_{[-\ell, \ell]^d} u(x + z)u(x) \, dx. \tag{11}
\]

Alternatively, one can define it by replacing \( \mathbb{T}_\ell \) by the periodicity cell in (10), and it is easy to see that the two definitions coincide. In fact, the definition (11) is also well-defined for more general functions such as almost periodic functions, cf. eg. [16]. The symmetrized autocorrelation function is defined accordingly. Equation (11) shows in particular that \( C_u \) (and \( c_u \)) do not depend on the choice of periodicity cell.

Some properties of the autocorrelation function are derived in e.g. [23, Prop. 11]. However, otherwise we have not found many results about analytical properties of the autocorrelation function in the mathematical literature. We hence collect some basic properties about \( C_u \) in the following proposition:

**Proposition 2.3** (Properties of \( C_u \)). Let \( u, \tilde{u} \in L^1(\mathbb{T}_\ell; \{0, 1\}) \). Then \( C_u \in C^0(\mathbb{R}^d) \), \( C_u \) is \( \mathbb{T}_\ell \)-periodic and \( C_u(-z) = C_u(z) \). Furthermore,

\[
\begin{align*}
(i) \quad C_u(0) &= \|C_u\|_{L^\infty(\mathbb{R}^d)} = \frac{1}{|\mathbb{T}_\ell|} \|u\|_{L^1(\mathbb{T}_\ell)} \\
(ii) \quad \|C_u - C_{\tilde{u}}\|_{L^\infty(\mathbb{R}^d)} &\leq \frac{2}{|\mathbb{T}_\ell|} \|\tilde{u} - u\|_{L^1(\mathbb{T}_\ell)} \\
(iii) \quad \|C_u\|_{L^1(\mathbb{T}_\ell)} &\leq \frac{1}{|\mathbb{T}_\ell|} \|u\|_{L^1(\mathbb{T}_\ell)}^2 \\
(iv) \quad |C_u(z_2) - C_u(z_1)| &\leq \frac{2}{|\mathbb{T}_\ell|} \|u\|_{L^1(\mathbb{T}_\ell)} \\
v) \quad C_{1 - u} &= C_u + 1 - \frac{2}{|\mathbb{T}_\ell|} \|u\|_{L^1(\mathbb{T}_\ell)}.
\end{align*}
\]

If \( u \in BV(\mathbb{T}_\ell; \{0, 1\}) \) then \( C_u \in C^{0,1}(\mathbb{R}^d) \). Furthermore,

\[
\begin{align*}
(vi) \quad \|\partial_i C_u\|_{L^\infty(\mathbb{R}^d)} &\leq \frac{1}{2|\mathbb{T}_\ell|} \|\partial_i u\|, \quad \forall i \in \{1, \ldots, d\} \\
vii) \quad \partial_{ij} C_u &= -\frac{1}{|\mathbb{T}_\ell|} \partial_i u \ast \partial_j u \in \mathcal{M} \quad \text{where } \hat{u}(x) := u(-x) \text{ and} \\
&\quad \|\partial_{ij} C_u\| \leq \frac{1}{|\mathbb{T}_\ell|} \|\partial_i u\| \|\partial_j u\|, \quad \forall i, j \in \{1, \ldots, d\}.
\end{align*}
\]
Proof. The periodicity and symmetry of $C_u$ follows directly from the periodicity of $u$ and the change of variable $y = x - z$.

(i)-(iv): For (i) we note that $|\mathbb{T}_\ell| C_u(z) = |\Omega^\text{per} \cap (\Omega - z)|$ for $\Omega := \text{spt} u \subset \mathbb{R}^d$ and $\Omega^\text{per} = \bigcup_{\ell \in (\mathbb{Z}^d)'} (\Omega + e)$. Hence, clearly $C_u \in C^0(\mathbb{R}^d, [0, \infty))$ and takes the maximum at $z = 0$. Together with the periodicity this also implies $C_u \in L^\infty(\mathbb{R}^d)$. By the triangle inequality and since $\parallel \tilde{u} \parallel_{L^\infty}, \parallel u \parallel_{L^\infty} \leq 1$, for any $z \in \mathbb{R}^d$ we have

$$|C_u(z) - C_{\tilde{u}}(z)| \leq \frac{1}{|\mathbb{T}_\ell|} \left( \parallel (u - \tilde{u})(\cdot + z) \parallel_{L^1(\mathbb{T}_\ell)} + \parallel u - \tilde{u} \parallel_{L^1(\mathbb{T}_\ell)} \right) = \frac{2}{|\mathbb{T}_\ell|} \parallel u - \tilde{u} \parallel_{L^1(\mathbb{T}_\ell)}.$$

The identity (iii) follows directly from Fubini and since $u$ is periodic. Estimate (iv) is given in [23, Prop.5].

(vi): By periodicity and since $|u(x + z) - u(x)| = |u(x + z) - u(x)|^2$ we get

$$C_u(0) - C_u(z) = \int_{\mathbb{T}_\ell} u(x)^2 - u(x)u(x + z)dx = \frac{1}{2} \int_{\mathbb{T}_\ell} |u(x + z) - u(x)|dx \leq \frac{\parallel \nabla u \parallel_{L^\infty}}{2|\mathbb{T}_\ell|} \forall z \in \mathbb{R}^d. \quad (12)$$

Together with (iv), this yields (vi).

(vii): We choose a sequence $u_\epsilon \in C_c^\infty(\mathbb{T}_\ell)$ with $\nabla u_\epsilon \overset{\ast}{\rightharpoonup} \nabla u$ in $M$. For $u_\epsilon \in C_c^\infty(\mathbb{T}_\ell)$ and with the notation $(\mathcal{G} f)(x) := f(-x)$ we have

$$\partial_{ij} C_{u_\epsilon}(z) = \partial_i \int_{\mathbb{T}_\ell} \partial_j u_\epsilon(x + z) u_\epsilon(x) \, dx = \partial_i \int_{\mathbb{T}_\ell} \partial_j u_\epsilon(y) u_\epsilon(y - z) \, dy$$

$$= -\int_{\mathbb{T}_\ell} \partial_j u_\epsilon(y) \partial_i u_\epsilon(y - z) \, dy = -\frac{1}{|\mathbb{T}_\ell|} (\partial_j u \ast \mathcal{G} \partial_i u)(z). \quad (13)$$

We note that (13) with $g := \partial_i u_\epsilon$ and $f := \partial_j u_\epsilon$ defines a bilinear operator $M \times M \to M$ with $(g, f) \mapsto g \ast \mathcal{G} f$. Since $\parallel \mu_1 \ast \mu_2 \parallel_M \leq \parallel \mu_1 \parallel_M \parallel \mu_2 \parallel_M$, we get $\phi_\epsilon \ast \psi_\epsilon \rightharpoonup \phi \ast \psi$ for all $\phi_\epsilon, \psi_\epsilon \in M$ with $\phi_\epsilon \rightharpoonup \phi, \psi_\epsilon \rightharpoonup \psi$ in $M$. Passing to the limit in the weak formulation of (13) one gets $\partial_{ij} C_u \in M$ and moreover $|\mathbb{T}_\ell| \partial_{ij} C_u = -\partial_j u \ast \mathcal{G} \partial_i u$.

(v): Follows directly from the definition. \hfill \Box

We note that some of the assertions in Proposition 2.3 can be easily generalized to the case of $u \in BV(\mathbb{T}_\ell)$ in which case an additional norm $\parallel u \parallel_{L^\infty}$ has to be added on the right hand side.

In the following we derive properties of the symmetrized autocorrelation. Similarly, these properties extend to general functions $u \in BV(\mathbb{T}_\ell)$. However, in the general case we have to replace $\parallel \nabla u \parallel$ by $\parallel D_3 u \parallel$ in (v) where $D_3 u$ is the singular part of the distributional derivative of $u$, cf. [3].

**Proposition 2.4** (Properties of $c_u$). Let $u, \tilde{u} \in L^1(\mathbb{T}_\ell; [0, 1])$. Then $c_u \in C^0([0, \infty))$ and $c_u$ is uniformly continuous. Furthermore,

(i) $c_u(0) = \parallel c_u \parallel_{L^\infty([0, \infty))} = \frac{1}{|\mathbb{T}_\ell|} \parallel u \parallel_{L^1(\mathbb{T}_\ell)}$. 


(ii) \(|c_u - c_u^\prime|_{L^\infty([0,\infty))} \leq \frac{2}{|T_\ell|} \|u - \bar{u}\|_{L^1(T_\ell)}
.

(iii) \(|c_u(r_2) - c_u(r_1)| \leq c_u(0) - c_u(r_2 - r_1) \quad \forall 0 \leq r_1 \leq r_2 < \infty.

(iv) \(- \frac{1}{r} (c_u(0) - c_u(r)) = \frac{1}{2} \int_{S^{d-1}} \frac{1}{r} |u(x) - u(x + r \omega)| \, dx \omega \quad \forall r > 0.

If \(u \in BV(T_\ell; \mathbb{R})\) then \(c_u' \in BV_{loc}((0, \infty))\) and \(c_u'(0)\) exists. Furthermore,

(v) The one-sided derivative \(c_u'(0)\) exists and we have

\[
c_u'(0) = -\|c_u'\|_{L^\infty([0,\infty))} = -\frac{\omega_{d-1}}{\sigma_{d-1}} \|\nabla u\|_{T_\ell}.
\]

In particular,

\[
c_u'(r) - c_u'(0) \geq 0 \quad \text{for a.e. } r > 0,
\]

\[
c_u(r) - c_u(0) - r c_u'(0) \geq 0 \quad \text{for all } r > 0.
\]

(vi) (Asymptotics as \(r \to \infty\)) When \(d \geq 2\) and \(r \ell \geq 1\) we have

\[
\frac{1}{\ell} |c_u(r) - c_u(0)| + |c_u'(r)| \leq \frac{C}{r^\frac{d-1}{2} \ell^{\frac{d-2}{2}}} \|\nabla u\|^2.
\]

(vii) Let \(\Omega = \text{spt } u\) with outer unit normal \(v\). Then for each \(\phi \in C_c^\infty((0, \infty))\) and with \(z := y - x\) we have

\[
\int_0^\infty c_u'(r) \phi'(r) \, dr = \frac{1}{\sigma_{d-1}|T_\ell|} \int_0^\infty \int_{\partial \Omega} (v(y) \cdot \frac{z}{|z|})(v(x) \cdot \frac{x}{|x|}) |\frac{\partial \phi}{\partial |z|}| \, dx dy,
\]

where \(\Omega^{per} \subset \mathbb{R}^d\) are the periodic copies of \(\Omega\).

Proof. (i)--(iii): Follows from Proposition 2.3(i) by taking the spherical mean.

(iv): Taking the spherical average of (12) and by (i), we get (iv).

(v): By (iv) and [23, Prop. 11] (noting that the identity also holds for periodic functions) the derivative \(c_u'(0)\) exists and we have

\[
-c_u'(0) \overset{(iv)}{=} \lim_{r \to 0^+} \frac{1}{2} \int_{S^{d-1}} \int_{T_\ell} \frac{1}{r} |u(x) - u(x + r \omega)| \, dx \omega \omega
\]

\[
= \frac{1}{2} \int_{T_\ell} \int_{S^{d-1}} |\omega \cdot \nabla u(x)| \, d\omega dx = \frac{\omega_{d-1}}{\sigma_{d-1}} \|\nabla u\|_{T_\ell}.
\]

For the last identity, we have used that

\[
\int_{S^{d-1}} |\omega \cdot e_1| \, d\omega = \frac{2\sigma_{d-2}}{d-1} = 2\omega_{d-1}. \quad (14)
\]

To conclude we note that by (iii) \(|c_u'\|_{L^\infty([0,\infty))} \leq -c_u'(0)\) and hence

\[
c_u(r) - c_u(0) - r c_u'(0) \geq -r \|c_u'\|_{L^\infty} - r c_u'(0) \geq 0.
\]

(vi): For \(k \in \left(\frac{2\pi}{\ell}\right)^d\), let \(\alpha_k\) be the Fourier coefficient of \(C_u\). Since \(C_u(x) = C_u(-x) \in \mathbb{R} \forall x \in T_\ell\), we have \(\alpha_{-k} = \alpha_k \in \mathbb{R}\) and

\[
\alpha_k := \int_{T_\ell} C_u(x)e^{-ik \cdot x} \, dx = \int_{T_\ell} C_u(x) \cos(k \cdot x) \, dx.
\]
Then
\[ C_u(x) = \sum_{k \in \{\frac{2\pi}{T}\mathbb{Z}\}^d} \alpha_k e^{ik \cdot x} = \sum_{k \in \{\frac{2\pi}{T}\mathbb{Z}\}^d} \alpha_k \cos(k \cdot x). \] (15)

Integrating by parts and since \( \|D^2C_u\| \leq \frac{1}{|T|^2} \|\nabla u\|^2 \) (Prop. 2.3(vii)), we get
\[ \|\nabla u\|^2 \leq \|\nabla u\|^2 \leq \|\nabla u\|^2 \] (16)

Thus the series (15) is absolutely convergent. Then

\[ c_u(r) = \sum_k \alpha_k \int_{S^{d-1}} e^{ik \cdot \omega} \, d\omega = C_d \sum_k \alpha_k \int_0^\pi \cos(|k| r \cos \theta) (\sin \theta)^{d-2} \, d\theta, \]

since \( \int_0^\pi \sin(|k| r \cos \theta) (\sin \theta)^{d-2} \, d\theta = 0 \). Note that \( \alpha_0 = c(0)^2 \) and thus
\[ c_u(r) - c_u(0)^2 = C_d \sum_k \alpha_k J_{d-2}(r|k|) (r|k|)^{2-d}, \] (17)

where \( J_\alpha \) are the Bessel functions of the first kind. The decay estimate for \( |c_u(r) - c_u(0)| \)
then follows from the estimate \( |J_\alpha(t)| \leq \sqrt{\frac{2}{\pi t}} + \Theta(\frac{1}{t^2}) \) for \( \alpha \geq 0 \) and \( t \geq 1 \) (cf. [41, 10.17]). Indeed, with \( q = k \ell \) we have
\[ \frac{c(r) - c(0)^2}{\|\nabla u\|^2} \leq C_d r^{1-d} \ell^{-2d} \sum_{k \in \{\frac{1}{\ell}\mathbb{Z}\}^d} k^{-3-d} \leq C_d r^{1-d} \ell^{3-\frac{3d}{2}} \sum_{q \in \mathbb{Z}^d} q^{-3-d}. \]

and the geometric sum on the right hand side is bounded. Using the recurrence relation \( \frac{d}{dt}(t^{-\alpha} J_\alpha(t)) = -t^{-\alpha} J_{\alpha+1}(t) \) we obtain (the series in (17) is absolutely and uniformly convergent for all \( r > 0 \) by (16))
\[ c_u'(r) \stackrel{(17)}{=} -(2\pi)^{\frac{d}{2}} \sum_{k \neq 0} \frac{|k| |a_k|}{(r|k|)^{d-2}} J_{d-2}(r|k|), \] (18)

The decay estimate for (18) then follows analogously as for (17). This completes the proof for (vi).

(vii): For any test function \( \phi \in C^\infty_c((0, \infty)) \) we calculate
\[ \int_0^\infty c_u'(r) \phi'(r) \, dr = -\int_0^\infty \int_{S^{d-1}} \frac{d}{d\omega} (\omega \cdot \nabla C_u(r\omega)) \phi(r) \, d\omega dr \]
\[ = -\frac{1}{\sigma_{d-1}} \int_{\mathbb{R}^d} \frac{1}{|z|^{d+1}} (z, D^2 C_u(z)) \phi(|z|) \, dz, \]

where we have integrated by parts in the last line. By Proposition 2.3
\[ (z, D^2 C_u z) = -\frac{1}{|\nabla u|} (z \cdot \nabla u) \ast B(z \cdot \nabla u) \in M. \]

Since \( \nabla u = -v \eta^{d-1} \partial \Omega \) and with \( z := y - x \) we then get
\[ \int_0^\infty c_u'(r) \phi'(r) \, dr = \frac{1}{\sigma_{d-1} |\nabla u|} \int_{\partial \Omega} \int_{\partial \Omega^{per}} \frac{(v(y) \cdot z)(v(x) \cdot z)}{|z|^{d+1}} \phi(|z|) \, dy dx. \]

For \( \text{spt} \phi \in [\delta, \infty) \) the expression above is estimated by \( C(\delta) \|\phi\|_{L^\infty} \). This shows that \( c_u' \in BV_{loc}((0, \infty)) \). □
Lemma 2.5 (Autocorrelation functions for balls and stripes).

(i) Let \( u = \chi_{B_{\rho}} \) for some \( \rho \in (0, \ell^4) \). Then for \( r \in [0, 2\rho] \) we have

\[
c_u(r) = \frac{2\omega_{d-1}}{|T_{\ell}|} \left( \int_{\frac{r}{2\rho}}^{1} (1 - t^2)^{\frac{d-1}{2}} \, dt \right) \rho^d.
\]

(ii) Let \( u = \chi_S \), where \( S = \bigcup_{i=1}^{k} [a_i, b_i] \times [0, \ell]^{d-1} \) is a finite union of stripes. Let \( d_0 = \min_i |b_i - a_i| > 0 \). Then \( c_u \) is affine linear near 0, i.e.

\[
c_u(r) = \frac{\|u\|_{L^1(T_{\ell})}}{|T_{\ell}|} - \left( \frac{\omega_{d-1}}{\sigma_{d-1}} \frac{\|\nabla u\|}{|T_{\ell}|} \right) r \quad \text{for } r \in (0, d_0).
\]

Proof. (i): By invariance and using cylindrical coordinates we have

\[
c_u(r) = \frac{1}{|T_{\ell}|} |B_{\rho} \cap B_{\rho}(re_d)| = \frac{2\omega_{d-1}}{|T_{\ell}|} \int_{\frac{\rho}{2}}^{\rho} (\rho^2 - s^2)^{\frac{d-1}{2}} \, ds.
\]

for \( r \in [0, 2\rho] \) (see Fig. 1) and

\[
c_u'(r) = -\frac{\omega_{d-1}}{|T_{\ell}|} \left( 1 - \frac{r^2}{4\rho^2} \right)^{\frac{d-1}{2}} \rho^{d-1}.
\]

(ii): For given \( S \) and \( r \in (0, d_0) \) we have

\[
c_u(0) - c_u(r) = \frac{k\ell^{d-1}}{|T_{\ell}|} \int_{S^{d-1}} |r\omega \cdot e_1| \, d\omega = \frac{2k\omega_{d-1}\ell^{d-1}}{|T_{\ell}|} r = -c_u'(0)r.
\]

Then the claim follows from Proposition 2.4(i) and (v). \( \square \)
2.2. Properties of the energies. In this section, we express our energies in terms of the autocorrelation function and use this expression to explore their properties. We first note that by Proposition 2.4 our energies can be expressed in terms of the symmetrized autocorrelation function $c_u$:

**Proposition 2.6** (Energy in terms of $c_u$). Let $d \geq 1$. Let $\epsilon \geq 0$ and suppose that $K_\epsilon, K_0$ satisfy (3)–(4). Then for any $u \in BV(\mathbb{T}_\ell; \{0, 1\})$ we have

$$\mathcal{E}_\epsilon(u) = \sigma_{d-1} \int_0^\infty \left( c_u(r) - c_u(0) - r c'_u(0) \right) \chi_{(0,1)}(r) K_\epsilon(r) r^{d-2} dr.$$  \hspace{1cm} (20)

Furthermore, there is $C = C(K_0, d) > 0$ such that $\mathcal{E}_\epsilon(u) \geq -C$ for $\epsilon \geq 0$.

**Proof.** We first note that the integral in (20) is well-defined in $\mathbb{R} \cup \{\pm \infty\}$: Indeed, in view of Proposition 2.4(i)(v) we have $c_u(r) - c_u(0) - r c'_u(0) \geq 0$ and $c_u(0) - c_u(r) \geq 0 \forall r > 0$. Thus the integrand is nonnegative in $(0, 1)$ and negative in $(1, \infty)$. This combined with our assumptions (3)–(4) on the kernels $K_\epsilon$ and that $c_u \in C^{0,1}((0, \infty))$ gives that the integral (20) is well-defined in $\mathbb{R}$ for $\epsilon > 0$, and is well-defined in $\mathbb{R} \cup \{+\infty\}$ for $\epsilon = 0$. To show the uniform lower bound we observe that

$$\mathcal{E}_\epsilon(u) \geq -\sigma_{d-1} \int_1^\infty \left( c_u(0) - c_u(r) \right) K_\epsilon(r) r^{d-2} dr \geq -C,$$

where the second inequality follows from $c_u(r) \geq 0$ and the expression of $c_u(0)$ in Proposition 2.4(i), and in the last inequality we have used the assumption (4). In view of Proposition 2.4(i)(ii) and using polar coordinates we have

$$\mathcal{E}_\epsilon(u) = -\sigma_{d-1} c'_u(0) \int_0^1 K_\epsilon(r) r^{d-1} dr$$

$$- \int_0^\infty \left[ \int_{S^{d-1}} \frac{1}{2} \int_{\mathbb{T}_\ell} |u(x + r\omega) - u(x)| dx \right] K_\epsilon(r) r^{d-2} d\omega dr.$$

Then (20) directly follows from Proposition 2.4(iv).  \hspace{1cm} \Box

The truncation at $r = 1$ is arbitrary. More generally, we have:

**Remark 2.7** (Different truncation). Let $\epsilon \geq 0$ and let $u \in BV(\mathbb{T}_\ell, \{0, 1\})$. Then for any $r_0 > 0$ we have

$$\mathcal{E}_\epsilon(u) = \sigma_{d-1} \int_0^\infty \left[ c_u(r) - c_u(0) - r c'_u(0) \chi_{(0,r_0)}(r) \right] K_\epsilon(r) r^{d-2} dr$$

$$+ \sigma_{d-1} c'_u(0) \int_1^{r_0} K_\epsilon(r) r^{d-1} dr.$$

Although the energies are linear in terms of the autocorrelation functions, they are not linear in $u$. Indeed, they are superlinear in the following sense:
Lemma 2.8 (Interaction energy). Let $\epsilon \geq 0$. Suppose that $u_1, u_2 \in BV(\mathbb{T}_\ell; \{0, 1\})$ have essentially disjoint support, i.e. $u_1 + u_2 \in BV(\mathbb{T}_\ell; \{0, 1\})$ and $\|D(u_1 + u_2)\| = \|Du_1\| + \|Du_2\|$. Then the interaction energy satisfies

$$
\mathcal{J}_\epsilon(u_1, u_2) := \mathcal{E}_\epsilon(u_1 + u_2) - \mathcal{E}_\epsilon(u_1) - \mathcal{E}_\epsilon(u_2) \geq 0
$$

and for $\Omega := \text{spt } u_i$, $i = 1, 2$ we have

$$
\mathcal{J}_\epsilon(u_1, u_2) = \frac{2}{|\mathbb{T}_\ell|} \sum_{e \in (\ell \mathbb{Z})^d} \int_{\Omega_1 + e} \int_{\Omega_2} K_\epsilon(|x - y|) \, dx \, dy.
$$

Proof. Let $u := u_1 + u_2$. By assumption, we have $c_\epsilon(0) = c_{u_1}(0) + c_{u_2}(0)$ and $c_\epsilon'(0) = c_{u_1}'(0) + c_{u_2}'(0)$. Together with $u = u_1 + u_2$ the assertion follows from the energy expression (20) and

$$
\mathcal{J}_\epsilon(u_1, u_2) = \sigma_{d-1} \int_0^\infty (c_{u_1+u_2}(r) - c_{u_1}(r) - c_{u_2}(r)) K_\epsilon(r) r^{d-2} \, dr
$$

$$
= \int_{\mathbb{R}^d} \int_{\mathbb{T}_\ell} (u(x)u(x + z) - \sum_{i=1,2} u_i(x)u_i(x + z)) \frac{K_\epsilon(|z|)}{|z|} \, dx \, dz.
$$

Using the symmetry of the sum and substituting the supports of $u_i$, we obtain the desired expression for $\mathcal{J}_\epsilon$. □

The space of functions with finite limit energy is quite complicated: the condition $u \in BV(\mathbb{T}_\ell; \{0, 1\})$ is in general only a necessary but not sufficient condition for a configuration to have finite energy. Below we construct a BV function with infinite energy $\mathcal{E}_0^{(1)}$ using the nonnegativity of the interaction energy in Lemma 2.8:

Remark 2.9 (BV-function with infinite energy $\mathcal{E}_0^{(1)}$). Let $(B_{r_k}(x_k))_{k=1}^\infty$ be a sequence of disjoint balls in $\mathbb{T}_\ell \subset \mathbb{R}^2$ with radius $r_k = \frac{1}{k(\ln k)^2}$. Let $u := \sum_{k=2}^\infty \chi_{B_{r_k}(x_k)}$. On one hand, it is immediate from $\sum_{k=2}^\infty r_k < \infty$ that $u \in BV(\mathbb{T}_\ell; \{0, 1\})$. On the other hand, from Lemma 2.8 and the estimate of the energy for a single disk (cf. Lemma 4.4(i) below) we have

$$
\mathcal{E}_0^{(1)}(u) \geq \sum_{k=2}^\infty \mathcal{E}_0^{(1)}(u_k) \geq C \sum_{k=2}^\infty (-\ln r_k) r_k = +\infty.
$$

We end this section by a remark on the finite energy configurations for $\mathcal{E}_0^{(3)}$:

Remark 2.10 (Finite $\mathcal{E}_0^{(3)}$ energy configurations). The family of sets which have finite energy $\mathcal{E}_0^{(3)}$ becomes increasingly smaller as $q$ increases. Indeed, as $q$ gets larger the (symmetrized) autocorrelation function $c_u$ needs to converge faster to its affine approximation at 0 in order to get a finite energy (cf. (20) in Proposition 2.6). Indeed, if $u = \chi_{B_{\rho}}$ with $\rho > 0$, then by (19) in Lemma 2.5 one has $c_u^{(r)}(0) = 0$ and $c_0 < c_u^{(3)}(r) < C_0$ for $r \in (0, \rho)$, where $c_0$ and $C_0$ are positive constants depending on $d$ and $\rho$. In this case using Remark 2.7 and the Taylor expansion of $c_u$ at 0 one can express the energy $\mathcal{E}_0^{(3)}(u)$ as
\[
\hat{E}_0^{(3)}(u) = \sigma_{d-1} \int_0^\rho c_u^{(3)}(\xi_r)r^{-q+d+1} \, dr + \sigma_{d-1} \int_0^\infty (c_u(r) - c_u(0)) r^{-q+d-2} \, dr \\
+ \sigma_{d-1} c'_u(0) \int_1^\rho r^{-q+d-1} \, dr.
\]

Since the second and third integrals above are finite, then from the upper and lower bound for \(c_u^{(3)}\) one has that \(\hat{E}_0^{(3)}(u)\) is finite for \(q < d + 2\) and infinite for \(q \geq d + 2\). If \(u = \chi_{\Omega}\) where \(\Omega\) consists of multiple stripes, then it follows from Lemma 2.5(ii) that \(c_u^{(k)}(r) = 0\) in \((0, d_0)\) for some \(d_0 > 0\) and \(k = 2, 3, \ldots\). Thus in this case the energy \(\hat{E}_0^{(3)}(u)\) is finite for all \(q > d\).

### 2.3. Equivalent formulations of the energy.

In this section, we give some further geometric representations of the limit energy \(\hat{E}_0\). These representations are in particular helpful for the calculation of the energy for specific configurations. We note that the assumption \(\|K_0\|_{L^\infty(\mathbb{R}^d)} = \infty\) is used in the proof of Lemma 2.11 below. Hence, the results do not hold for the approximating energies \(\hat{E}_\epsilon\). Throughout the section \(F : (0, \infty) \to [0, \infty)\) and \(G : (0, \infty) \to \mathbb{R}\) are defined by

\[
F(r) := \int_r^\infty K_0(t) t^{d-2} \, dt, \quad G(r) := \int_1^r F(t) \, dt,
\]

(21)

It follows from (4) that \(F\) is continuous and thus \(G\) is \(C^1\). We note that \(F \geq 0\) and \(F' \leq 0\) by the assumptions on the kernel. For \(K_0(r) = r^{-d}\) we have \(F(r) = \sigma_{d-1} r^{-1}\) and \(G(r) = \sigma_{d-1} \ln r\).

We start with an auxiliary lemma:

**Lemma 2.11** (Asymptotics for \(r \to 0\).) Let \(u \in BV(T_\ell; \{0, 1\})\). Suppose that (3)–(4) hold. If \(\hat{E}_0(u) < \infty\) then there are sequences \(\delta_j, \bar{\delta}_j > 0\) with \(\delta_j, \bar{\delta}_j \to 0\) such that

(i) \(\lim_{j \to \infty} F(\delta_j) (c_u(\delta_j) - c_u(0) - c'_u(0)\delta_j) = 0\),

(ii) The points \(\bar{\delta}_j > 0\) are Lebesgue points of \(c'_u\) and \(c''_u(\bar{\delta}_j) = 0\) for all \(j\). Furthermore, we have \(\lim_{j \to \infty} G(\bar{\delta}_j) (c'_u(\bar{\delta}_j) - c'_u(0)) = 0\).

**Proof.** With the representation (20) of the energy and recalling that \(F'(r) = -K_0(r) r^{d-2}\), we have

\[
-\sigma_{d-1} \int_0^1 (c_u(r) - c_u(0) - c'_u(0)r) F'(r) \, dr \leq \hat{E}_0(u) + C_0 \|c_u\|_{L^\infty} < \infty, \tag{22}
\]

for some constant \(C_0 = C_0(K_0)\).

(i): Assuming that (i) does not hold, we hence may assume that there are constants \(\delta_0 > 0\) and \(c > 0\) such that

\[
F(r) (c_u(r) - c_u(0) - c'_u(0)r) \geq c > 0 \quad \text{for all } r \in (0, \delta_0) \tag{23}
\]

noting that the second factor and hence the left hand side above is nonnegative by Proposition 2.4(v). By the definition (21) of \(F\) and in view of (4) we have \(F(r) \to \infty\) and hence \(\ln F(r) \to \infty\) as \(r \to 0\). On the other hand,
\[
\lim_{r \to 0} \left( \ln F(r) \right) = \ln F(\delta) - \int_0^{\delta_0} \frac{F'(t)}{F(t)} \, dt
\]

using that \( F' \leq 0 \). We recall that \( \ln F(\delta_0) < \infty \) by (3), which together with (22) yields that the right hand side above is finite, which contradicts the observation that \( \lim_{r \to 0} \ln F(r) \to \infty \) and concludes the proof.

(ii): Let \( n \in (0, \infty) \) be the set of points such that \( \epsilon'_u([s]) \neq 0 \) or such that \( s \) is not a Lebesgue point of \( \epsilon'_u \). Since the (signed) Radon measure \( \epsilon''_u \) has locally finite total variation on \((0, \infty)\) for \( u \in BV(\mathbb{R}^d; \{0, 1\}) \) (cf. Proposition 2.4(vii)) and by the Lebesgue differentiation theorem, the set \( n \subset (0, \infty) \) is a Lebesgue null set. We note that \( (\epsilon'_u(r) - \epsilon'_u(0))G(r) \geq 0 \) for all \( r \in (0, \infty) \) \( \setminus n \) by Proposition 2.4(v) and that \( G(r) \leq 0 \) for \( r \in (0, 1) \). Assuming by contradiction as before we hence may assume that there is \( \delta_0 \in (0, 1) \) and \( c > 0 \) such that

\[
-(\epsilon'_u(r) - \epsilon'_u(0))G(r) > c > 0 \quad \text{for all } r \in (0, \delta_0) \setminus n.
\]

We first note that integration by parts yields

\[
\lim_{r \to 0} G(r) \overset{(21)}{=} \lim_{r \to 0} \left( F(1) - r F(r) - \int_r^1 t F(t) \, dt \right)
\]

\[
\overset{(21)}{=} \lim_{r \to 0} \left( F(1) - r \int_1^\infty K_0(t) t^{d-2} \, dt - \int_r^1 K_0(t) t^{d-2} (t - r) \, dt \right)
\]

\[
\leq C - \frac{1}{2} \lim_{r \to 0} \int_{2r}^1 K_0(t) t^{d-1} \, dt \overset{(3)}{=} -\infty,
\]

where we have used that \( K_0 \geq 0 \).

Now, let \( \delta_j \) be the sequence from (i). Applying an integration by parts to (22) from \( \delta_j \) to 1 and using (i) we have

\[
\lim_{j \to \infty} \int_{\delta_j}^{\delta_0} (\epsilon'_u(r) - \epsilon'_u(0)) F(r) \, dr < \infty.
\]

Analogously as for (i) one then gets from the fact that \( G' = F \) and the contradiction assumption (24) that

\[
\ln |G(\delta_j)| = \ln |G(\delta_0)| - \int_{\delta_j}^{\delta_0} \frac{G'(r)}{G(r)} \, dr
\]

\[
\overset{(24)}{\leq} \ln |G(\delta_0)| + \frac{1}{c} \int_{\delta_j}^{\delta_0} (\epsilon'_u(r) - \epsilon'_u(0)) G'(r) \, dr,
\]

where we also used that \( G \leq 0 \) in \((0, 1)\). Taking \( j \to \infty \) in the above inequality, using (26) as well as \( \ln |G(\delta_0)| < \infty \), one gets \( \lim_{j \to \infty} \ln |G(\delta_j)| < \infty \), which is a contradiction to (25).

With Lemma 2.11 at hand we can express the energy using geometric quantities of \( \Omega := \text{spt } u \):
Let \( \bar{\delta}_j \to 0 \) be any sequence as in Lemma 2.11(ii). Assume that there is a sequence \( R_j \to \infty \) such that \( G(R_j) \frac{d-1}{2} \to 0 \) as \( j \to \infty \), \( R_j \) is a Lebesgue point for \( \mathbf{c}_u' \) and \( \mathbf{c}_u''(\{R_j\}) = 0 \) for all \( j \in \mathbb{N} \). Then with \( z := x - y \) we have

\[
|\mathbb{T}_\ell|\mathcal{E}_0(u) = -\omega_{d-1} P(\Omega) F(1) \\
+ \int_{\partial \Omega} \int_{\partial \Omega} G(|z|) \frac{F(|z|)}{|z|} \frac{z}{|z|} \cdot \nu(x) \, d\omega \,dydx \\
+ \int_{\partial \Omega} \int_{\partial \Omega} \frac{F(|z|)}{|z|} \frac{z}{|z|} \cdot \nu(x) \, d\omega \,dydx.
\]

where \( P_j := \{(x, y) \in \partial \Omega \times \partial \Omega_{\text{per}} : \bar{\delta}_j \leq |x - y| \leq R_j\} \) with \( \partial \Omega_{\text{per}} \subset \mathbb{R}^d \) being the periodic copies of \( \Omega \).

Proof. (i): We start with the representation (20) of \( \mathcal{E}_0 \) in Proposition 2.6. We note that \( F(R) \to 0 \) as \( R \to \infty \) which follows from (4). Integrating by parts both integrals in (20) (for the first integral of (20) we integrate by parts from \( \delta_j \) to 1, where the sequence \( \delta_j \) is from Lemma 2.11(i), and then let \( \delta_j \to 0 \) yields

\[
\mathcal{E}_0(u) = \sigma_{d-1} \mathbf{c}_u'(0) F(1) + \sigma_{d-1} \int_0^1 \left( \mathbf{c}_u'(r) - \mathbf{c}_u'(0) \right) F(r) dr \\
+ \sigma_{d-1} \int_1^\infty \mathbf{c}_u'(r) F(r) dr \\
=: I_1 + I_2 + I_3.
\]

Here we have used that the integrand for \( I_2 \) is nonnegative and thus the limit exists. We note that \( I_1 = -\frac{\omega_{d-1} P(\Omega)}{\mathbb{T}_\ell} F(1) \), which follows from Proposition 2.4(v). To calculate \( I_2 \) and \( I_3 \) we first express \( \mathbf{c}_u'(r) \) using the geometric quantities. Using \( \nabla u(x) = -v(x) \mathcal{H}^{d-1}[\partial \Omega] \) and with \( \Omega_r(x) := \frac{\Omega - x}{r} \), for a.e. \( r > 0 \) we have

\[
\mathbf{c}_u'(r) = \int_{\mathbb{T}_\ell} \int_{\mathbb{T}_\ell} \omega \cdot \nabla u(y) u(y - r\omega) \, dyd\omega \\
= \frac{1}{\sigma_{d-1}|\mathbb{T}_\ell|} \int_{\partial \Omega} \int_{\mathbb{T}_\ell} \omega \cdot \nu(x) \, d\omega dx.
\]

In the limit as \( r \to 0 \) we get

\[
\mathbf{c}_u'(0) = \frac{1}{\sigma_{d-1}|\mathbb{T}_\ell|} \int_{\partial \Omega} \int_{\mathbb{T}_\ell} \omega \cdot \nu(x) \, d\omega dx.
\]
Inserting (29) and (30) into the integrand of $I_2$ and by Fubini we get

$$I_2 = \frac{1}{|T_\ell|} \int_{\partial \Omega} \int_0^1 F(r) \left( \int_{S^{d-1} \cap \Omega_r(x)} \omega \cdot v(x) d\omega - \int_{S^{d-1} \cap H^-(x)} \omega \cdot v(x) d\omega \right) dx dr.$$ 

Observing that for each $v \in S^{d-1}$ fixed, $f(K) := -\int_{S^{d-1} \cap K} \omega \cdot v d\omega$ achieves its maximum when $K = H^-(x)$, we have

$$I_2 = \frac{1}{|T_\ell|} \int_{\partial \Omega} \int_0^1 F(r) \int_{S^{d-1} \cap \Omega_r(x) \Delta H^-(x)} |\omega \cdot v(x)| d\omega dr dx. \quad (31)$$

Similarly, inserting (29) into the integrand of $I_3$ we have

$$I_3 = \frac{1}{|T_\ell|} \int_{\partial \Omega} \int_1^\infty F(r) \int_{S^{d-1} \cap \Omega_r(x)} \omega \cdot v(x) d\omega dr dx.$$ 

The desired identity then follows from (28), (31) and (31) with $\omega = \frac{y-x}{|y-x|} \in S^{d-1}$ with $y \in B_1(x)$ and $r = |y-x|$ and by a change from polar coordinates to Cartesian coordinates.

(ii): We choose a sequence $\tilde{\delta}_j \to 0$ such that Lemma 2.11(ii) holds. Let $R_j \to \infty$ be the sequence which satisfies the assumptions in (ii). Let $\eta_t(r) \in \text{Lip}(\mathbb{R})$, $t \in (0, 1)$, be the cut-off function such that $\eta_t = 1$ in $(\tilde{\delta}_j + t, R_j - t)$, $\eta_t = 0$ outside $(\tilde{\delta}_j - t, R_j + t)$ and it is affine in $(\tilde{\delta}_j - t, \tilde{\delta}_j + t)$ and $(R_j - t, R_j + t)$. Starting point is the representation (28) of the energy. Integrating by parts (recalling that $c_u' \in \text{BV}_{\text{loc}}(0, \infty)$ from Proposition 2.4(vii)) and using that $G(1) = 0$ we have

$$\int_{\tilde{\delta}_j - t}^{R_j} \left( c_u'(r) - c_u'(0) \right) F(r) \eta_t(r) dr + \int_1^\infty c_u'(r) F(r) \eta_t(r) dr$$

$$= - \int_{\tilde{\delta}_j - t}^{R_j} G(r) \eta_t(r) dc_u'' - \left( \int_{\tilde{\delta}_j - t}^{\tilde{\delta}_j + t} c_u'(r) G(r) dr - c_u'(0) G(\tilde{\delta}_j - t) \right)$$

$$- \int_{R_j - t}^{R_j + t} c_u'(r) G(r) dr.$$ 

Letting $t \to 0$, since $c_u' \in L^\infty$, $F$ and $G$ are locally bounded in $(0, \infty)$, we have from the dominated convergence theorem that the LHS converges to $\int_{\tilde{\delta}_j}^1 (c_u'(r) - c_u'(0)) F(r) dr + \int_1^\infty c_u'(r) F(r) dr$. Since $\tilde{\delta}_j$ and $R_j$ are not atomic points for the measure $c_u''$, the first term on the RHS converge to $-\int_{\tilde{\delta}_j}^{R_j} G(r) dc_u''$ as $t \to 0$. The rest terms on the RHS converge to $-(c_u'(\tilde{\delta}_j) - c_u'(0)) G(\tilde{\delta}_j) - c_u'(R_j) G(R_j)$ as $t \to 0$, as $G \in C^1((0, \infty))$ and $\tilde{\delta}_j$ and $R_j$ are Lebesgue points of $c_u'$. Thus we have

$$\int_{\tilde{\delta}_j}^1 (c_u'(r) - c_u'(0)) F(r) dr + \int_1^\infty c_u'(r) F(r) dr$$

$$= -\int_{\tilde{\delta}_j}^{R_j} G(r) dc_u'' - \left( c_u'(\tilde{\delta}_j) - c_u'(0) \right) G(\tilde{\delta}_j) - c_u'(R_j) G(R_j).$$
By our choice of the sequences \( \tilde{\delta}_j \), the second term on the right hand side vanishes as \( j \to \infty \). By our assumption as well as Proposition 2.4(vi)(v) the term related to the evaluation at \( R_j \) vanishes as \( j \to \infty \). Adding the expressions for short-range and long-range interaction together and recalling Proposition 2.4(vii), we obtain the claimed equivalent form of the energy. \( \square \)

**Remark 2.13 (Principal value interpretation of energy formula).**

(i) When \( K_0(r) = r^{-d} \), Proposition 2.12(i) holds without the assumption \( \dot{\epsilon}_0(u) < \infty \). This is because \( \lim_{r \to 0^+} \frac{c_u(r) - c_u(0)}{r} = c_u'(0) \) for any \( u \in BV(\mathbb{T}_\ell; [0, 1]) \) by Proposition 2.4(v) and \( F(r) = \sigma_{d-1} r^{-1} \), then Lemma 2.11(i) holds for any \( \delta \to 0 \) without the assumption \( \dot{\epsilon}_0(u) < \infty \). Thus in this special case the energy formula (i) holds for all \( u \in BV(\mathbb{T}_\ell; [0, 1]) \).

(ii) The integrand in the energy formula (27) is not absolutely integrable in general if either \( \tilde{\delta} = 0 \) or \( R = \infty \). An example for this is a configuration which consists of stripe domains, cf. Lemma 4.3.

### 3. Proof of the Theorems

#### 3.1. Proof of Theorem 1.1.

We establish the compactness and \( \Gamma \)-convergence of the nonlocal energy \( \dot{\epsilon}_\epsilon(u) \) as \( \epsilon \to 0 \). We first establish the compactness result:

**Proposition 3.1 (BV-bound and compactness).** Let \( K_{\epsilon}, K_0 \) satisfy (3)–(4).

(i) For \( u \in BV(\mathbb{T}_\ell; [0, 1]) \) there are constants \( C = C(K_0, d) > 0 \) such that

\[
\|\nabla u\| \leq C (\|\mathbb{T}_\ell\|_\epsilon(u) + C\|u\|_{L^1}).
\]

(ii) For any family \( u_\epsilon \in BV(\mathbb{T}_\ell; [0, 1]) \) with \( \sup_\epsilon \dot{\epsilon}_\epsilon(u_\epsilon) < \infty \), there exists \( u \in BV(\mathbb{T}_\ell; [0, 1]) \) and a subsequence (not relabelled) such that

\[ u_\epsilon \to u \text{ in } L^1 \quad \text{and} \quad \|\nabla u_\epsilon\| \to \|\nabla u\| \quad \text{for} \ \epsilon \to 0. \]

**Proof.** (i): We use the expression of the energy \( \dot{\epsilon}_\epsilon(u) \) in terms of the autocorrelation function in Proposition 2.6. By the assumptions (3)–(4) as well as \( K_\epsilon \not\to K_0 \), there are \( C_0, C_1 > 0 \) and a measurable set \( A \subset (0, 1) \), which depends on \( K_0 \), such that for sufficiently small \( \epsilon > 0 \) we have

\[
C_0 \leq \int_A K_\epsilon(r) r^{d-1} \, dr \quad \text{and} \quad \int_{A \cup (1, \infty)} K_\epsilon(r) r^{d-2} \, dr \leq C_1. \tag{32}
\]

Using (32) as well as \( c_u(r) - c_u(0) - c_u'(0) r \geq 0 \) we then get

\[
-c_u'(0) \int_A K_\epsilon(r) r^{d-1} \, dr \leq \sigma_{d-1}^{-1} \dot{\epsilon}_\epsilon(u) + \int_{A \cup (1, \infty)} (c_u(0) - c_u(r)) K_\epsilon(r) r^{d-2} \, dr \\
\leq \sigma_{d-1}^{-1} \dot{\epsilon}_\epsilon(u) + C_1 c_u(0),
\]

where we have used \( c_u(r) \geq 0 \) and (32) for the last estimate. Recalling the expression for \( c_u(0) \) and \( c_u'(0) \) in Proposition 2.4(i) and (v) , we thus obtain the desired estimate from the above inequality.

(ii): By (i) we have \( \|\nabla u_\epsilon\| \leq C \). By the compactness of the BV functions, there is a subsequence of \( u_\epsilon \) (which we still denote by \( u_\epsilon \)) and \( u \in BV(\mathbb{T}_\ell, [0, 1]) \), such
that \( u_\varepsilon \to u \) in \( L^1(\mathbb{T}_\ell) \) and \( \|\nabla u\| \leq \liminf_{\varepsilon \to 0} \|\nabla u_\varepsilon\| \). To show the convergence of \( \|\nabla u_\varepsilon\| \), by Proposition 2.4 it suffices to show \( c'_{u_\varepsilon}(0) \to c'_u(0) \). Assume \( c'_{u_\varepsilon}(0) \to \alpha \) for some \( \alpha \in \mathbb{R} \) along a subsequence. By the lower semi-continuity of the BV-norm we have \( \alpha \leq c'_u(0) \). It remains to show that \( \alpha \geq c'_u(0) \): By Fatou’s lemma for any \( \delta \in (0, 1) \),

\[
\int_{\delta}^{1} (c_u(r) - c_u(0) - \alpha r) K_0(r) r^{d-2} \, dr \leq \liminf_{j \to \infty} \sigma_{d-1}^{-1} \varepsilon_{\delta_j}^{-1}(u_{\delta_j}) + C_1 \leq C.
\]

On the other hand, by Proposition 2.4

\[
\int_{\delta}^{1} (c_u(r) - c_u(0) - c'_u(0) r) K_0(r) r^{d-2} \, dr \geq 0.
\]

Taking the difference of the above two inequalities yields

\[
\int_{\delta}^{1} (c'_u(0) - \alpha) K_0(r) r^{d-1} \, dr \leq C.
\]

Since \( \int_{0}^{\infty} K_0(r) r^{d-1} \, dr = \infty \) and \( K_0 \geq 0 \), then \( \int_{\delta}^{\infty} K_0(r) r^{d-1} \, dr \to \infty \) as \( \delta \to 0 \). This implies that \( c'_u(0) \leq \alpha \), particularly one has \( -c'_u(0) \geq \limsup_{\varepsilon \to 0} -c'_{u_\varepsilon}(0) \).

As a consequence of Proposition 3.1, we get the existence of minimizers for \( \varepsilon_\varepsilon \) with prescribed volume fraction:

**Proof for Proposition 1.2.** Let \( \varepsilon \geq 0 \) be fixed. It follows from Lemma 2.5 (ii) that \( \varepsilon_\varepsilon(u) < +\infty \) if \( u \) is a stripe configuration, and furthermore \( \varepsilon_\varepsilon(u) \geq -C \) for all \( u \in BV(\mathbb{T}_\ell; \{0, 1\}) \) (cf. Proposition 2.6). Hence the infimum of \( \varepsilon_\varepsilon(u) \) exists and is finite. Let \( \{u_\varepsilon\} \) be a minimizing sequence for \( \varepsilon_\varepsilon \). Then by Proposition 3.1(ii) up to a subsequence \( u_\varepsilon \to u \) in \( L^1(\mathbb{T}_\ell) \) and \( \|\nabla u_\varepsilon\| \to \|\nabla u\| \) for some \( u \in BV(\mathbb{T}_\ell; \{0, 1\}) \). In particular \( u \) satisfies (9). Hence \( c_{u_\varepsilon} \to c_u \) pointwisely by Proposition 2.4(ii) and \( c'_{u_\varepsilon}(0) \to c'_u(0) \) by Proposition 2.4(v). Applying Fatou’s lemma (in \( (0, 1) \)) and dominated convergence theorem (in \( (1, \infty) \)) to the autocorrelation function formulation of \( \varepsilon_\varepsilon \) in Proposition 2.6, we obtain that \( \varepsilon_\varepsilon(u) \leq \liminf_{\varepsilon \to \infty} \varepsilon_\varepsilon(u_k) \). This implies that \( u \) is a minimizer.

At the end of the section we establish the \( \Gamma \)-convergence of \( \varepsilon_\varepsilon \):

**Theorem 3.2 (\( \Gamma \)-limit).** Suppose that (3)–(4) hold. Then \( \varepsilon_\varepsilon \uparrow \varepsilon_0 \) in the \( L^1 \)-topology, where \( \varepsilon_0 \) is given in (8).

**Proof.** We use the representation of \( \varepsilon_\varepsilon \) from Proposition 2.6, i.e.

\[
\varepsilon_\varepsilon(u) = \sigma_{d-1} \int_{0}^{\infty} \left( c_u(r) - c_u(0) - c'_u(0) r \chi_{(0,1)}(r) \right) K_\varepsilon(r) r^{d-2} \, dr.
\]

(i): \( \text{Liminf inequality.} \) We need to show that for any sequence \( u_\varepsilon \to u \) in \( L^1 \) and \( \sup \varepsilon \varepsilon_\varepsilon(u_\varepsilon) \leq C < \infty \) we have

\[
\varepsilon_0(u) \leq \liminf_{\varepsilon \to 0} \varepsilon_\varepsilon(u_\varepsilon).
\]

By Proposition 3.1(i) the sequence \( u_\varepsilon \) is uniformly bounded in \( BV(\mathbb{T}_\ell; \{0, 1\}) \). This together with the \( L^1 \) convergence of \( u_\varepsilon \) and Proposition 3.1(ii) yields that \( u \in BV(\mathbb{T}_\ell; \{0, 1\}) \) and \( \|\nabla u_\varepsilon\| \to \|\nabla u\| \). Then by Proposition 2.4 we have \( c_{u_\varepsilon} \to c_u \) uniformly and
Thus by Fatou’s lemma (applied to the integral over \((0, 1)\)) and the dominated convergence theorem (applied to the integral over \((1, \infty)\)) one has

\[
\mathcal{E}_0(u) \leq \sigma_d \liminf_{\epsilon \to 0} \int_0^\infty \left( c_{u_\epsilon}(r) - c_{u_\epsilon}(0) - c_{u_\epsilon}'(0)r \chi_{(0,1)}(r) \right) K_\epsilon(r) r^{d-2} dr,
\]

which yields (33).

(ii): Limsup inequality. Let \(u \in BV(\mathbb{T}_\ell, \{0, 1\})\) with \(E_0(u) < \infty\). Since \(K_\epsilon \rightharpoonup K\) by our assumption, by the monotone convergence theorem for the constant sequence \(u_\epsilon:=u\) we then obtain \(\limsup_{\epsilon \to 0} E_\epsilon(u) \leq E_0(u)\). \(\square\)

3.2. Proof of Theorem 1.3. In this section, we give the proof of Theorem 1.3, i.e., we show that if the volume \(\|u\|_{L^1(\mathbb{T}_\ell)} = \lambda |\mathbb{T}_\ell|\) is sufficiently small depending on the dimension, then the minimal energy is attained by a single ball. We give the proof for a class of energies which in particular includes the energy \(E_0^{(1)}\) in Theorem 1.3. More precisely, we assume that there is some \(M = M(d) > 0\) such that for all \(\eta \in (0, \frac{1}{2})\) we have

\[
\int_{\eta}^1 K_0(r)r^{d-1}dr \geq M \max \left\{ \frac{1}{\eta^2} \int_0^\eta K_0(r)r^{d+1}dr, \eta \int_{\eta}^1 K_0(r)r^{d-2}dr \right\} \tag{34}
\]

We note that the assumption (34) holds for \(K_0^{(3)}(r) = r^{-q}\) for \(d \leq q \leq d + \alpha_0\) for some \(\alpha_0 = \alpha_0(d) > 0\). Then Theorem 1.3 is a consequence of the next proposition:

Proposition 3.3 (Balls as minimizers for small volume). Suppose that (34) holds. Then the energy of of a single ball is finite. Furthermore, there exists \(m_0 = m_0(d, K_0) > 0\) such that if

\[
\lambda \leq \min \left\{ \frac{m_0}{|\mathbb{T}_\ell|}, \frac{\alpha_0}{4d} \right\},
\]

then the unique, up to translation, minimizer \(u\) of \(E_0\) in \(BV(\mathbb{T}_\ell; \{0, 1\})\) with constraint (9) is a single ball.

Proof. The proof is based on a contradiction argument. Similar proofs have been made using the sharp isoperimetric deficit (cf. [22]) e.g. in [7,21,29,32,33,40]. We adapt these arguments to the autocorrelation function formulation of our model. The novelty in our proof is that the nonlocal term has the same scaling as the perimeter term.

The isoperimetric deficit of the bounded Borel set \(\Omega \subset \mathbb{R}^d\) is defined by

\[
D(\Omega) := \frac{1}{|\partial B|} |\partial \Omega| - 1,
\]

where \(B \subset \mathbb{R}^d\) is the ball with \(|B| = |\Omega|\). The sharp estimate on the isoperimetric deficit in [22] entails that

\[
\min \left\{ \frac{|\Omega \Delta B|}{|B|} : B \subset \mathbb{R}^d \text{ is a ball with } |B| = |\Omega| \right\} \leq C_d \delta^2, \tag{35}
\]

where \(\delta := \min\{1, D(\Omega)\}\), noting that the left hand side of (35) is called the Fraenkel asymmetry of \(\Omega\).
Let \( u = \chi_{\Omega} \in BV(\mathbb{T}_\ell; [0, 1]) \) be a minimizer in the class of functions \( u \in BV(\mathbb{T}_\ell; [0, 1]) \) which satisfy (9). Let \( u_0 = \chi_B \), where \( B \subset \mathbb{R}^n \) is the ball with \( |B| = |\Omega| \) with radius \( \rho:= (\omega_d^{-1} \lambda |\mathbb{T}_\ell|)^{\frac{1}{d}} \in (0, \ell) \), which realizes the minimality in (35). By the definition of \( \delta \) we have \( \|\nabla u\| \geq (1 + \delta)\|\nabla u_0\| \). If \( \delta = 0 \) there is nothing to prove, so that we assume \( \delta > 0 \) in the following noting that \( \delta = \delta(\rho) \) in general.

Let \( \eta > 0 \). By assumption (4) we have

\[
X_\eta := \int_{\eta}^{1} K_0(r)r^{d-1} dr \to \infty \quad \text{as} \quad \eta \to 0.
\]

Since \( c_u(r) - c_u(0) - rc'_u(0) \geq 0 \) and \( K_0 \geq 0 \), for \( \eta \in (0, \rho) \) we have

\[
\frac{\dot{\varepsilon}_0(u)}{\sigma_{d-1}} \geq - \int_{\eta}^{1} c'_u(0)K_0(r)r^{d-1} dr - \int_{\eta}^{\infty} (c_u(0) - c_u(r))K_0(r)r^{d-2} dr
\]

\[
= - X_\eta c'_u(0) - \int_{\eta}^{\infty} (c_u(0) - c_u(r))K_0(r)r^{d-2} dr
\]

\[
\geq -(1 + \delta)X_\eta c'_u(0) - \int_{\eta}^{\infty} (c_u(0) - c_u(r))K_0(r)r^{d-2} dr.
\]

Using that \( \dot{\varepsilon}_0(u) \leq \dot{\varepsilon}_0(u_0) \) this implies

\[
-\delta c'_u(0)X_\eta \leq \int_{0}^{\eta} (c_{u_0}(r) - c_{u_0}(0) - rc'_u(0))K_0(r)r^{d-2} dr
\]

\[
+ \int_{\eta}^{\infty} (c_{u_0}(r) - c_u(r))K_0(r)r^{d-2} dr =: I_1 + I_2.
\]

By Lemma 2.5 and since \( \ell \geq 4\rho \), we have \(|\mathbb{T}_\ell|\|c''_u(r)\| \leq C\rho^{d-3} \) for \( r \leq \rho \) and \(|\mathbb{T}_\ell|\|c_{u_0}\|_{L^\infty} + \rho \|c'_u\|_{L^\infty} \) \( \leq C \rho^{d} \) for \( r \geq \frac{\rho}{2} \). This implies

\[
|c_{u_0}(r) - c_{u_0}(0) - rc'_u(0)| \leq \frac{C}{|\mathbb{T}_\ell|} \rho^{d-3} \rho^{d-3} \forall r \geq 0.
\]

We note that (37) together with the assumption (34) implies that the energy of a single ball is finite. Using (37) we then get

\[
I_1 \leq \frac{C}{|\mathbb{T}_\ell|} \rho^{d-3} \int_{0}^{\eta} K_0(r)r^{d+1} dr.
\]

By an application of Proposition 2.4(ii) we get

\[
I_2 \leq \frac{\|u - u_0\|_{L^1(\mathbb{T}_\ell)}}{|\mathbb{T}_\ell|} \int_{\eta}^{\infty} K_0(r)r^{d-2} dr \leq \frac{C\delta^2 \rho^d}{|\mathbb{T}_\ell|} \int_{\eta}^{\infty} K_0(r)r^{d-2} dr,
\]

where we have used the estimate of the isoperimetric deficit (35) and the definition of \( \delta \) in the last inequality. Combining (36), (38) and (39) and recalling \(-c'_u(0) = \frac{\alpha_d-1}{|\mathbb{T}_\ell|} \rho^{d-1} \) we get

\[
\delta \rho^{d-1} X_\eta \leq C \rho^{d-3} \int_{0}^{\eta} K_0(r)r^{d+1} dr + C \delta^2 \rho^d \int_{\eta}^{\infty} K_0(r)r^{d-2} dr.
\]
Inserting the assumption (34) to the above inequality and by (4), we obtain
\[ \delta X_\eta \leq \frac{C}{M} \left( \frac{\eta^2}{\rho^2} + \frac{\delta^2 \rho}{\eta} \right) X_\eta + C_{d,K} \delta^2 \rho. \]

With the choice \( \eta := \delta \rho \), the above inequality gives \( X_\eta \leq C_{1+\delta} M X_\eta + C_{d,K} \delta \rho \). Taking \( M = 4C \) and recalling that \( \delta \leq 1 \) we thus get \( X_\eta \leq C_{1+\delta} M X_\eta + C_{d,K} \delta \rho \). Since \( X_\eta \rightarrow \infty \) as \( \eta \rightarrow 0 \), then necessarily we have \( \rho \geq \eta \geq C \) for some \( C = C(d, K) > 0 \). We thus get a contradiction if \( \lambda |T_\ell| \leq m_0 \) for some \( m_0 = m_0(d, K) \) sufficiently small.

**Remark 3.4.** Proposition 3.3 is for small volume configurations instead of the volume fraction, as the smallness condition on \( \lambda \) is not independent of \( \ell \). Actually it follows from Lemma 4.4 below that when the volume fraction \( \lambda \) is sufficiently small, periodic balls have smaller energy than a single ball in \( T_\ell \) for sufficiently large \( \ell \).

### 4. Stripes and Balls Configurations

In this section we explicitly compute the limit energy for stripes and lattice balls for given volume fraction. Throughout this section we assume the kernel is the prototypical \( K_0^{(1)}(r) = r^{-d} \). In the previous sections we have used the technical assumption that the configurations are \( T_\ell \)-periodic for some arbitrary periodicity \( \ell \). To compare the energy for configurations with fixed volume fraction, it is natural to consider more general configurations. We hence write

**Definition 4.1 (Energy for generalized configurations).** Let \( u : \mathbb{R}^d \rightarrow \{0, 1\} \) be a measurable function such that
\[
C_u(z) = \lim_{R \to \infty} \int_{[-R, R]^d} u(x + z) u(x) \, dx
\]
exists. Then we set
\[
\mathcal{E}_0(u) = \int_{\mathbb{R}^d} \frac{C_u(z) - C_u(0) - z \cdot \nabla C_u(0) \chi_{B_1}(z)}{|z|} K_0(z) \, dz. \tag{40}
\]

As noted in Remark 2.2, this energy is independent of the lattice \( \Lambda \) where it arises from. In particular, the definition (40) is consistent with and generalizes the definition of the energy \( \mathcal{E}_0 \) in Theorem 1.1(ii). We also note that we analogously get a more general definition of the energies \( \mathcal{E}_\epsilon \). However, these will not be used in this article.

**Remark 4.2 (Minimization in generalized configurations).** It is natural to study the minimization problem among generalized configurations with fixed volume fraction \( \lambda = \lim_{\ell \to \infty} \int_{[-\ell, \ell]^d} u(x) \, dx \). One could minimize \( \mathcal{E}_0 \) in Definition 4.1 in the space of autocorrelation functions
\[
\mathcal{A}_\lambda := \left\{ C_u : \mathbb{R}^d \to \mathbb{R} : C_u(0) = \lambda \right\}.
\]

Here \( u \) is any function in \( BV_{loc}(\mathbb{R}^d ; \{0, 1\}) \) such that \( C_u \) is well-defined. One can show that \( \mathcal{A}_\lambda \) is convex. Even though the limit energy is a linear functional in terms of autocorrelation functions, the problem still seems to be very hard since it is difficult to characterize the space \( \mathcal{A}_\lambda \).
In the rest of the section we calculate and compare the energy for stripe and ball configurations with the prototypical kernel \( K_0^{(1)}(r) = r^{-d} \) (and the corresponding energy is denoted by \( \mathcal{E}_0^{(1)} \)). We recall the generalized harmonic number \( H_q \) with index \( q \geq 0 \) is given by

\[
H_q := \int_0^1 \frac{1 - t^q}{1 - t} \, dt. \tag{41}
\]

**Lemma 4.3 (Energy of stripes).** Let \( d \geq 1 \). Let \( \Omega \) be a periodic set of stripes of width \( d_0 > 0 \), distance \( d_1 > 0 \) and volume fraction \( \lambda := \frac{d_0}{d_0 + d_1} \). More precisely, for \( a := d_0 + d_1 \) let

\[
\Omega := \bigcup_{j \in \mathbb{Z}} (ja, ja + d_0) \times \mathbb{R}^{d-1}.
\]

Then the following holds:

(i) The function \( u := \chi_{\Omega} \) is \( \mathbb{T}_a \)-periodic and

\[
\mathcal{E}_0^{(1)}(u) = -\frac{2\lambda \omega_{d-1}}{d_0} \left( 1 + \frac{1}{2} H_{d-1} + \ln \left( \frac{d_0 \sin(\pi \lambda)}{\pi \lambda} \right) \right),
\]

where \( H_q \) is the harmonic number (cf. (41)).

(ii) Among configurations with prescribed \( \lambda \in (0, 1) \) the minimal energy is

\[
e_S(\lambda) := \min_{\lambda \text{ fixed, stripes}} \mathcal{E}_0^{(1)}(u) = -2\omega_{d-1} e^{-\frac{1}{2} H_{d-1}} \sin(\pi \lambda). \tag{42}
\]

with optimal width \( d_{opt} = \frac{\pi \lambda}{\sin(\pi \lambda)} e^{-\frac{1}{2} H_{d-1}} \)

**Proof.** For the calculation we reduce the problem to the calculation of the interaction energy between two parallel slices. Note that the configuration is \( \mathbb{T}_a \)-periodic.

(i): By Proposition 2.12(ii) the interaction energy between two parallel slices \( \{0\} \times [0, a]^{d-1} \) and \( \{\rho\} \times \mathbb{R}^{d-1} \), \( \rho > 0 \), with opposite outer normal reads

\[
I(\rho) := \frac{1}{|\mathbb{T}_a|} \int_{\{0\} \times [0, a]^{d-1}} \int_{\{|\rho\} \times \mathbb{R}^{d-1}} \ln \left( \frac{1}{|y - x|^{d-1} |y' - x'|} \right) \, dy \, dx
\]

\[
= -\frac{\rho^2}{2a} \int_{\mathbb{R}^{d-1}} \ln(\rho^2 + |y'|^2) \frac{dx'}{(\rho^2 + |y'|^2)^{d-1}} d'y',
\]

where we used that the inner integral in the first line above is independent of \( x \) and \( y_1 - x_1 = \rho \). An explicit calculation yields

\[
I(\rho) = -\frac{\omega_{d-2}}{2a} \int_0^\infty \ln(\rho^2 + (1 + t^2)^{d-1}) \, dt
\]

\[
= -\frac{\omega_{d-1}}{a} \left( \ln \rho + \frac{1}{2} H_{d-1} \right).
\]
By Proposition 2.12(ii) with $I(0) := 0$ and $I(-\rho) := I(\rho)$ for $\rho < 0$ we get

$$\mathcal{E}^{(1)}_{0}(u) = -\frac{\omega d-1}{a} P(\Omega) + \frac{2}{a^d} \sum_{k \in \mathbb{Z}} \left( I(ka + d_0) - I(ka) \right)$$

$$= -\frac{\omega d-1}{a} P(\Omega) + \frac{2}{a^d} \left( I(d_0) + \sum_{k=1}^{\infty} \left( I(ka + d_0) - 2I(ka) + I(ka - d_0) \right) \right).$$

The infinite sum in the first line above is taken in the p.v. sense. Inserting the formula for $I(\rho)$ and since $P(\Omega) = 2a^{d-1}$, we arrive at

$$\mathcal{E}^{(1)}_{0}(u) = -\frac{2\omega d-1}{a} \left( \ln d_0 + 1 + \frac{1}{2} H_{d-1} + \sum_{k=1}^{\infty} \ln \frac{(ka + d_0)(ka - d_0)}{(ka)^2} \right).$$

Using $d_0 = \lambda a$ and the Euler’s formula $\prod_{n=1}^{\infty} \frac{k^2 - \lambda^2}{k^2} = \frac{\sin(\pi \lambda)}{\pi \lambda}$ we arrive at

$$\mathcal{E}^{(1)}_{0}(u) = -\frac{2\lambda \omega d-1}{d_0} \left( \ln d_0 + 1 + \frac{1}{2} H_{d-1} + \ln \left( \frac{\sin(\pi \lambda)}{\pi \lambda} \right) \right).$$

(ii):

The result follows by a standard calculation.

As a competitor we next consider a lattice of balls, arranged on a Bravais lattice. We need to fix some notations: For a set of linearly independent vectors $v_i \in \mathbb{R}^d$, $1 \leq i \leq d$ we consider the Bravais lattice

$$\Lambda := \mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_d$$

By $|\Lambda|$ we denote the volume of the periodicity cell, i.e.

$$|\Lambda| := |\{[0, 1)v_1 \oplus \ldots \oplus [0, 1)v_d\}|.$$

The energy for lattice balls can then be formulated in terms of the Appell series $F_4$, which in turn can be expressed using the Pochhammer symbols $(a)_n$ given by $(a)_0 := 1$ for $n = 0$ and $(a)_n := a(a + 1) \cdots (a + n - 1)$ for $n \geq 1$. We set

$$\mathcal{H}_d(t) := F_4\left( \frac{3}{2}, \frac{d+1}{2}; \frac{d+2}{2}, \frac{d+2}{2}; t, t \right) = \sum_{m,n=0}^{\infty} \frac{(\frac{3}{2}m+n)(\frac{d+1}{2})^{m+n}}{m!)^{n!}} t^{m+n}.$$

With these notations, we have

**Lemma 4.4** (Energy of ball configurations). Let $d \geq 1$. Let $\Lambda \subset \mathbb{R}^d$ be a Bravais lattice and let $\rho > 0$. Let $\Omega = \bigcup_{q \in \Lambda} B_\rho(q)$ be a periodic set of balls with centers on $\Lambda$ with radius

$$\rho \leq \frac{1}{2} \min \{ \text{dist}(p, q) : p, q \in \Lambda \}.$$
Then

(i) The function $u := \chi_\Omega$ is $\Lambda$–periodic and we have

$$
\mathcal{E}^{(1)}_0(u) = -\omega_d \left( \ln(2\rho) + 1 - \frac{1}{2} \frac{H_{d-1}}{2^{d-1}} \left| \partial B_\rho \right| + I_\Lambda \right) + I_\Lambda^2,
$$

where $H_q$ is defined in (41) and where

$$
I_\Lambda := \frac{1}{|\Lambda|} \sum_{q \in \Lambda \setminus \{0\}} \mathcal{H}_d \left( \frac{r^2}{|q|^2} \right) |q|^d,
$$

where the Appell series $\mathcal{H}_d$ is defined in (43).

(ii) Let $\Lambda_0$ be a fixed lattice with $|\Lambda_0| = 1$ and let $e_{B, \Lambda_0}(\lambda)$ be the minimal energy among lattices of the form $\Lambda = a \Lambda_0$ for $a > 0$ and for prescribed volume fraction $\lambda = |B_\rho|/|\Lambda|$. Then

$$
e_{B, \Lambda_0}(\lambda) = -(2\omega_d - 1) e^{-\frac{1}{2} H_{d-1} \lambda} + \sum_{e \in \Lambda_0 \setminus \{0\}} \frac{1}{|e|^{d+1}} (\lambda e + \Theta(\lambda^{2+d}))
$$

for all $\lambda \leq \lambda_0$, where $\lambda_0 = \lambda_0(\Lambda_0)$ is the largest volume fraction which can be realized by balls in the lattice. The radius of the optimal ball configuration is given by $\rho_{opt} = \frac{1}{2} e^{\frac{1}{2} H_{d-1} \lambda} + \Theta(1)$ as $\lambda \to 0$.

Proof. (i): We introduce the full-space (radially-symmetrized) autocorrelation function for the single ball $\tilde{u} := \chi_{B_\rho}$ by

$$
\tilde{c}_u(r) := \int\int_{\mathbb{R}^d} \tilde{u}(x + r\omega)\tilde{u}(x) \, dx \, d\omega
$$

Using Lemma 2.8, we decompose the energy as

$$
\mathcal{E}^{(1)}_0(u) = \frac{1}{|\Lambda|} \left( E_{\text{self}}(\rho) + I_{\text{int}}(\Lambda, \rho) \right).
$$

Here, $E_{\text{self}}$ is the self–interaction energy of a single ball, i.e.

$$
E_{\text{self}}(\rho) := \sigma_d \int_0^1 \tilde{c}_u(r) - \tilde{c}_u(0) - \tilde{c}_u'(0)r \, dr + \sigma_d \int_1^\infty \tilde{c}_u(r) - \tilde{c}_u(0) \, dr.
$$

Furthermore, the interaction energy of a single ball $B_\rho$ with other copies is

$$
I_{\text{int}}(\Lambda, \rho) := \sum_{q \in \Lambda \setminus \{0\}} \int_{B_\rho(q)} \int_{B_\rho(0)} \frac{1}{|x - y|^{d+1}} \, dx \, dy.
$$

Computation of $E_{\text{self}}(\rho)$: By definition we have $\tilde{c}_u(r) = 0$ for $r \geq 2\rho$. Using Remark 2.7 and integration by parts we have
Using the formula
\[ \tilde{\varrho}_{u}(0) \ln(2\rho) + \int_{0}^{\rho} \frac{\tilde{\varrho}_{u}(r) - \tilde{\varrho}_{u}(0) - r \tilde{\varrho}'_{u}(0)}{r^{2}} dr - \tilde{\varrho}_{u}(0) \]
and
\[ I \]
Since \( \tilde{\varrho}_{u}(r) = |\Lambda|c_{u}(r) \) for \( 0 \leq r \leq 2\rho \), in view of (19) and with the change of variables \( r = \rho \sqrt{1 - t} \) we get
\[ \int_{0}^{\rho} \frac{\tilde{\varrho}_{u}^{'}(r)}{\tilde{\varrho}_{u}(0)r} dr = -\frac{1}{2} \int_{0}^{1} \frac{1 - r^{2}}{1 - t} dt = \frac{1}{2} H_{d-1} \]
Using the formula \( \tilde{\varrho}_{u}^{'}(0) = -\frac{\omega_{d-1}}{\omega_{d-1}} |\partial B_{\rho}| \) we arrive at
\[ E_{\text{self}}(\rho) = \omega_{d-1} \left( \ln \frac{1}{2\rho} - 1 + \frac{1}{2} H_{d-1} \right) |\partial B_{\rho}|. \] (47)

**Estimate of \( I_{\text{int}}(\Lambda, \rho) \):**
For the estimate of \( I_{\text{int}}(\Lambda, \rho) \) we note that by (53) in “Appendix B” we have
\[ \frac{I_{\text{int}}(\Lambda, \rho)}{|\Lambda|} = |B_{\rho}|^{2} I_{\Lambda}. \]
Together with (47) and (46), this yields (i).
(ii): We note that \( I_{\Lambda} = \frac{1}{a^{d+1}} I_{\Lambda_{0}} \), where
\[ I_{\Lambda_{0}} = \sum_{e \in \Lambda_{0} \setminus \{0\}} \mathcal{H}_{d} \left( \frac{\rho_{e}^{2}}{|e|^{2d+4}} \right), \quad \bar{e} = \frac{\rho}{a}, \]
Using this and \( \lambda = \frac{|B_{\rho}|}{a^{d+1}} \), the averaged interaction energy can be rewritten in terms of \( \rho \) and \( \lambda \) as
\[ \frac{I_{\text{int}}(\Lambda, \rho)}{|\Lambda|} = \frac{\lambda^{2}}{a} I_{\Lambda_{0}} = \frac{\lambda^{2} + \frac{1}{2}}{\omega_{d} \rho} I_{\Lambda_{0}}. \] (48)
From Lemma B.1 and that \( \frac{\rho}{|e|} \leq \frac{1}{2} \), which follows from (44), we have that \( \sum_{e \in \Lambda_{0} \setminus \{0\}} \frac{1}{|e|^{d+1}} \leq I_{\Lambda_{0}} \leq C \sum_{e \in \Lambda_{0} \setminus \{0\}} \frac{1}{|e|^{d+1}}. \)
Using \( \lambda = \frac{|B_{\rho}|}{|\Lambda|} \), we can further express \( E_{\text{self}} \) in terms of \( \lambda \) and \( \rho \) as
\[ \frac{E_{\text{self}}(\rho)}{|\Lambda|} = \frac{d \omega_{d-1} \lambda}{\rho} \left( \ln \frac{1}{2\rho} - 1 + \frac{1}{2} H_{d-1} \right), \]
(49)
Thus the self–interaction energy in (49) is of leading order in \( \lambda \) for \( \lambda \ll 1 \) for fixed lattice \( \Lambda_{0} \). By minimization (49) in \( \rho \) we obtain
\[ \min_{\rho} \frac{E_{\text{self}}(\rho)}{|\Lambda|} = \frac{E_{\text{self}}(\rho_{\text{opt}})}{|\Lambda|} = -\left( 2 \omega_{d-1} d e^{-\frac{1}{2} H_{d-1}} \lambda \right) \text{ with } \rho_{\text{opt}} = \left( 2 e^{\frac{1}{2} H_{d-1}} \lambda \right). \]
Since expression (48) is of lower order in \( \lambda \), and since \( \frac{d^{2} E_{\text{self}}(\rho_{\text{opt}})}{d \rho^{2}} > c_{d} > 0 \), (ii) follows by a standard application of the inverse function theorem. \( \square \)
We compare the minimal energy for stripes and balls in (ii) of Lemma 4.3 and Lemma 4.4 with fixed volume fraction $\lambda$.

**Proposition 4.5** (Stripes vs balls). Let $K_0^{(1)}(r) = r^{-d}$ and let $\Lambda_0$ be a Bravais lattice with $|\Lambda_0| = 1$. Let $e_S, e_B, \Lambda_0$ be defined in (45) and (42). Then there are $\lambda_0 = \lambda_0(d, \Lambda_0) > 0$ and $\delta_0 > 0$ independent of $\Lambda_0$ such that

(i) if $0 < \lambda < \lambda_0$, then $e_B, \Lambda_0(\lambda) < e_S(\lambda)$.

(ii) if $d = 2$ and $|\lambda - \frac{1}{2}| < \delta_0$, then $e_S(\lambda) < e_B, \Lambda_0(\lambda)$.

**Proof.** (i): Comparing the leading order (in $\lambda$) terms of $e_S(\lambda)$ and $e_B, \Lambda_0(\lambda)$ (which is independent of $\Lambda_0$) and using that

$$\lim_{\lambda \to 0} \frac{e_S(\lambda)}{2\omega_{d-1}\lambda} = -e^{-\frac{1}{2}H_{d-1}} > -de^{-\frac{1}{2}H_{d-1}} = \lim_{\lambda \to 0} \frac{e_B, \Lambda_0(\lambda)}{2\omega_{d-1}\lambda}$$

for all $d \geq 2$ we can conclude the assertion.

(ii): For large volume fraction the interaction energy is no more of lower order. Moreover, noting that balls have smaller self-interaction energy than stripes, we have to estimate the lower bound of interaction energy for balls. When $d = 2$, by Lemma 4.3(ii) and since $H^1 = 2 = 2 \ln 2$ we have

$$e_S\left(\frac{1}{2}\right) = -\frac{2e}{\pi}.$$

Now we estimate the lower bound for the interaction energy among Bravais lattices in $\mathbb{R}^2$. Given any Bravais lattice $\Lambda \subset \mathbb{R}^2$ with $a^2 = |\Lambda|$ and $\Lambda_0 = \frac{a}{d} \Lambda$, from (53) in Lemma B.1 and using that $\lambda a^2 = \pi \rho^2$ we have

$$I_{\text{int}}(\Lambda, \rho) \geq \frac{|B\rho|^2}{a^3} \sum_{e \in \Lambda_0 \setminus \{0\}} \frac{1}{|e|^3} = \frac{\lambda}{\sqrt{\pi} \rho} \sum_{e \in \Lambda_0 \setminus \{0\}} \frac{1}{|e|^3}.$$  

Thus combining with the self-interaction energy in Lemma 4.4 we have that for $u = \bigcup_{q \in \Lambda} B\rho(q)$ with $0 < \rho \leq \frac{1}{2} \min_{p,q \in \Lambda} |p-q|$ and $\lambda = \frac{|B\rho|}{|\Lambda|} = \frac{1}{2}$,

$$\varepsilon_0(u) \geq -\frac{2}{\rho} \ln(4\rho) + \frac{\zeta(\Lambda_0)}{4\sqrt{2} \pi \rho}, \quad \zeta(\Lambda_0) := \sum_{e \in \Lambda_0 \setminus \{0\}} \frac{1}{|e|^3}. \quad (50)$$

By Rankin [45], $\zeta(\Lambda_0)$ attains the minimum at the triangle lattice $H_{\text{nor}} := \sqrt{\frac{2}{\sqrt{3}}} (\mathbb{Z}(1,0) \oplus \mathbb{Z}(\frac{1}{2}, \frac{\sqrt{3}}{2}))$ among all $2d$ Bravais lattices with volume 1, and furthermore from its explicit expression we have that $\zeta(H_{\text{nor}}) > 8$. Thus (50) together with the standard minimization gives

$$\min_{\Lambda_0} e_B, \Lambda_0\left(\frac{1}{2}\right) > \min_{\rho} \left( -\frac{2}{\rho} \ln(4\rho) + \frac{8}{4\sqrt{2} \pi \rho} \right) > e_S\left(\frac{1}{2}\right).$$

Since the energy functionals are continuous in $\lambda$, then there is $\delta_0 > 0$ independent of $\Lambda_0$ such that if $\lambda \in (\frac{1}{2} - \delta_0, \frac{1}{2} + \delta_0)$ stripes have strict smaller energy than any $2d$ lattice balls with the volume fraction $\lambda$. □
Remark 4.6 (Triangular lattice in 2d). We conjecture that in 2d for sufficiently small volume fraction $\lambda$, the triangle lattice $H_{nor}$ has the smallest energy among all lattice of balls. Indeed, we recall from Lemma 4.4(i) that the energy consists of the self-interaction energy and the interaction energy. In view of (49), the self-interaction energy is independent of the lattice $\Lambda_0$. For the interaction energy, cf. (48), we expect that $I_{\Lambda_0}$ achieves its minimum for the triangle lattice $H_{nor}$ if $\lambda$ is sufficiently small. To prove such a result one would (at least) need to extend the methods in [6] to the case of non–integrable potentials and we leave it for the future work.

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Declarations

Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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Appendix A. Connection to the Results by Dávila

We start by recalling the classical result by Dávila [19]. Let $K_\epsilon : \mathbb{R}^d \to \mathbb{R}_+$ be a family of nonnegative radially symmetric kernels, which satisfy

$$\lim_{\epsilon \to 0} \frac{\|K_\epsilon\|_{L^1(\mathbb{R}^d \setminus B_\delta)}}{\|K_\epsilon\|_{L^1(\mathbb{R}^d)}} = 0 \quad \text{for all } \delta > 0. \quad (51)$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded set with finite perimeter. Then Dávila showed that for $u := \chi_\Omega$ one has

$$\lim_{\epsilon \to 0} \frac{1}{\|K_\epsilon\|_{L^1(\mathbb{R}^d)}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x + z) - u(x)|}{|z|} K_\epsilon(z) \, dx \, dz = \frac{2\omega_{d-1}}{\sigma_{d-1} \|\nabla u\|}. \quad (52)$$

Using the autocorrelation function, one can give a simpler proof for (52) as we explain below: Let

$$\tilde{c}_u(r) := \int_{S^{d-1}} \int_{\mathbb{R}^d} u(x + z)u(x) \, dz$$
be the (full space) radially symmetrized autocorrelation function. One can easily get the analogous statements as in Proposition 2.4 for the full space symmetrized autocorrelation function. Then it follows from assumptions (51) on \( K_\epsilon \) as well as the existence of \( \tilde{c}'_u(0) \), cf. Proposition 2.4, that

\[
\lim_{\epsilon \to 0} \frac{\sigma_{d-1}}{\| K_\epsilon \|_{L^1(\mathbb{R}^d)}} \int_0^\infty \left( \tilde{c}_u(0) - \tilde{c}_u(r) \right) K_\epsilon(r) r^{d-2} \, dr = -\tilde{c}'_u(0).
\]

With this at hand, (52) directly follows from writing the above equation in terms of \( u \) (cf. Proposition 2.4(iv) and (v) in the non-periodic setting). We note that the decay of \( K_\epsilon \) at infinity is not needed when \( \Omega \) is bounded. Furthermore, instead of \( \| K_\epsilon \|_{L^1(\mathbb{R}^d)} \) it suffices to normalize the LHS of (52) by \( \| K_\epsilon \|_{L^1(B_0)} \) for arbitrary \( r_0 > 0 \).

**Appendix B. Interaction Energy of Two Balls**

In this section we provide the explicit expression for the full-space interaction energy between two balls.

**Lemma B.1** (Interaction energy between two balls in full space). Let \( q \in \mathbb{R}^d \). Then for \( 0 < \rho \leq \frac{|q|}{2} \) we have

\[
\int_{B_\rho(q)} \int_{B_\rho(0)} \frac{1}{|x - y|^{d+1}} \, dx \, dy = \frac{1}{|q|^{d+1}} \mathcal{H}_d \left( \frac{\rho^2}{|q|^2} \right),
\]

where \( \mathcal{H}_d(t) \) is given in (43). In particular, for \( C = C_d > 0 \) we have

\[
\frac{3(d+1)}{d+2} \frac{\rho^2}{|q|^{d+3}} \leq \frac{1}{|q|^{d+1}} \mathcal{H}_d \left( \frac{\rho^2}{|q|^2} \right) - \frac{1}{|q|^{d+1}} \leq \frac{C \rho^2}{|q|^{d+3}}.
\]  

**Proof.** By scaling and rotation invariance, we may assume that \( \rho = 1 \), \( \lambda := |q| \geq 2 \) and \( q = |q| e_1 \). We note that the Fourier transform for \( \chi_{B_1(0)} \) is

\[
\hat{\chi}_{B_1(0)}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{B_1(0)} e^{-i\xi \cdot x} \, dx = |\xi|^{-\frac{d}{2}} J_{\frac{d}{2}}(|\xi|),
\]

where \( J_\nu \) is the Bessel function of first kind. Using that \( B_1(0) =: B_1 \) and \( B_1(q) = B_1 + q \) are disjoint, we have

\[
X := \int_{B_1(q)} \int_{B_1(0)} \frac{1}{|x - y|^{d+1}} \, dx \, dy = -\frac{1}{2} \int_{\mathbb{R}^d} \frac{1}{|h|^{d+1}} \int_{\mathbb{R}^d} (\chi_{B_1}(x + h) - \chi_{B_1}(x)) (\chi_{B_1+q}(x + h) - \chi_{B_1+q}(x)) \, dx \, dh.
\]

By Plancherel and with the change of variable \( h \mapsto \frac{h}{|h|} \) we have

\[
X = -\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\xi \cdot q} |\hat{\chi}_{B}(\xi)|^2 \left| \frac{e^{i\xi \cdot h} - 1}{|h|^{d+1}} \right|^2 \, d\xi \, dh
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\xi \cdot q} |\hat{\chi}_{B}(\xi)|^2 \left| \frac{e^{ih_1} - 1}{|h|^{d+1}} \right|^2 \, d\xi \, dh
\]
\[ -c_d \int_{\mathbb{R}^d} e^{i\xi \cdot q} |\mathcal{X}_B(\xi)|^2 |\xi| \,d\xi, \quad (55) \]

where

\[ c_d := \frac{1}{2} \int_{\mathbb{R}^d} \frac{|e^{ih} - 1|^2}{|h|^{d+1}} \,dh = \frac{\pi \frac{d+1}{2}}{\Gamma\left(\frac{d+1}{2}\right)}. \]

Plugging (54) into (55) and with \( s := |\xi|/\lambda \), \( \xi_1 = |\xi| \cos \theta \) and \( t := \cos \theta \) we get

\[ X = -c_d \int_{\mathbb{R}^d} e^{i\xi \cdot \lambda} |J_d\left(\frac{s}{\lambda}\right)|^2 |\xi|^{-d-1} \,d\xi \]
\[ = -c_d \frac{\sigma_d - 2}{\lambda} \int_0^\infty |J_d\left(\frac{s}{\lambda}\right)|^2 \int_{-1}^1 e^{its} (1 - t^2)^{-\frac{d-3}{2}} \,dt \,ds \]
\[ = -c_d \frac{(2\pi)^d}{\lambda} \int_0^\infty J_d^{2-2}(s)|J_d\left(\frac{s}{\lambda}\right)|^2s^{-\frac{d-2}{2}} \,ds = \frac{\pi^d}{\Gamma\left(\frac{d+1}{2}\right)} \mathcal{H}_d\left(\frac{1}{\lambda^2}\right), \]

where the last equality is due to [5, eq. (7.1)]. Taking the average of \( X \) by dividing \( \omega_d^2 \) and since \( \omega_d = \frac{\pi^d}{\Gamma\left(\frac{d+1}{2}\right)} \), we obtain (43). From the definition we see that \( \mathcal{H}_d(t) \) is monotonically increasing in \( t \) with \( \mathcal{H}_d(t) \geq 1 + \frac{3(d+1)}{d+2} t \). This yields the lower bound in (53). The upper bound follows since \( \mathcal{H}_d(t) = 1 + \mathcal{O}(t) \) as \( t \to 0 \).

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