Research Article

Novel Construction of Copulas Based on \((\alpha, \beta)\) Transformation for Fuzzy Random Variables

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The paper introduces a method for the construction of bivariate copulas with the usage of specific values of the parameters \(\alpha\) and \(\beta\) \((\alpha, \beta)\) transformation) and the parameters \(\kappa\) and \(\lambda\) in their domain. The produced bivariate copulas are defined in four subrectangles of the unit square. The bounds of the produced copulas are investigated, while a novel construction method for fuzzy copulas is introduced, with the usage of the produced copulas via \((\alpha, \beta)\) transformation in four subrectangles of the unit square. Following this construction procedure, the production of an infinite number of copulas and fuzzy copulas could be possibly achieved. Some applications of the proposed methods are presented.

1. Introduction

Copulas are a significant member of aggregation functions on the unit interval \([0, 1]\). The ability to construct aggregation functions with numerous processes and methods is of great importance, since it is essential for the researchers to not deviate from the real-life data. Sklar [1] presented the concept of copulas, by means of a mathematical tool that describes the stochastic dependence structure within random variables. There are several procedures regarding the construction of copulas, based on given ones, in the literature, such as the construction of asymmetric multivariate copulas [2], which is connected with the product of copulas and the generalization of Archimedean copulas. Another construction of copulas, produced by the gluing of two or more copulas, is presented in [3]. In [4], three types of ordinal sums based on product copula are introduced as construction methods. A representation via the g-ordinal sums of copulas is introduced in [5]. In [6], a method for the construction of bivariate copulas by the modifications of given copulas on some subrectangles of the unit square is contained. Two different representations of 2-increasing aggregation functions, via the lower and the upper margins and a copula, are provided in [7]. The construction of copulas as a patchwork-like assembly of arbitrary copulas, with nonoverlapping rectangles as patches, is included in [8]. In [9], the set of copulas with the given horizontal section was studied and extended. The family of \((\alpha, \beta)\)-homogenous copulas was introduced in [10]. A general construction of copulas with given a horizontal and a vertical section is introduced in [11]. One of the most important methods is the flipped and survival copulas [12]. Those are special cases of the \((\alpha, \beta)\)-transformation, which is a more general construction method [12].

On the other hand, regarding the real-life problems, researchers may handle data possibly imprecise. In order to deal with imprecise or vague information, fuzzy sets [13] are the most adequate tools for someone to establish. In [14], the fuzzy random variables are provided in order to represent the relationship between random experiments results and nonstatistical imprecise data. Thus, the notion of fuzzy copulas is introduced in [15] to describe the stochastic dependence structure between two fuzzy random variables.

This paper shares two main purposes. The first is to provide a novel construction method for copulas, more general than the existing ones, based on the
function, with margins $F_X$ and $F_Y$. If $F_X^{-1}$ and $F_Y^{-1}$ are the inverses of $F_X$ and $F_Y$, respectively, then

$$C(u, v) = F_{XY}(F_X^{-1}(u), F_Y^{-1}(v)), \quad \forall (u, v) \in [0, 1] \times [0, 1].$$  (4)

As it was mentioned in Section 1, a construction method of copulas is $(\alpha, \beta)$-transformation that is defined [12] for parameters $\alpha$ and $\beta$, that is, $(\alpha, \beta) \in [0, 1]^2$. The transformation of a copula $C$ into the copula $C_{\alpha, \beta}$ is defined on the unit square by

$$C_{\alpha, \beta}(x, y) = V_C([\alpha(1-x), x + \alpha(1-x)] \times [\beta(1-y), y + \beta(1-y)]).$$  (5)

As a result, for specific values of the parameters $\alpha$ and $\beta$ (in their domain), different copulas can be constructed. In case that $\alpha = 0$ and $\beta = 0$, we obtain the copula:

$$C_{0,0}(x, y) = C(x, y).$$  (6)

In case that $\alpha = 0$ and $\beta = 1$, we obtain the copula:

$$C_{0,1}(x, y) = y - C(x, 1 - y).$$  (7)

In case that $\alpha = 1$ and $\beta = 0$, we obtain the copula:

$$C_{1,0}(x, y) = x - C(1 - x, y).$$  (8)

In case that $\alpha = 1$ and $\beta = 1$, we obtain the copula:

$$C_{1,1}(x, y) = x + y - 1 + C(1 - x, 1 - y) = \tilde{C}(x, y).$$  (9)

In the first case, we get the original copula; in the second and third cases, we get the flipped copulas; and in the last case, we get the survival copula.

2.2 Fuzzy Sets, Notions, and Definitions. Let $X$ be a universal set. Each function $\tilde{A}: X \rightarrow [0, 1]$ is called a fuzzy set of $X$, where $\mu_A(x)$’s interpretation is the membership degree of $x$ in the fuzzy set $\tilde{A}$. Crisp (classical) sets are special cases of fuzzy sets, with $\tilde{A}(x) = 0$, or $\tilde{A}(x) = 1$. The $\alpha$-cuts of a fuzzy set $\tilde{A}$ are defined by $\tilde{A}_\alpha = \{x: \mu_A(x) \geq \alpha\}$, $\forall \alpha \in (0, 1]$, where $\tilde{A}_{[0]}$, which is called support, is the closure in the topology of $X$ of the union of all the $\alpha$-cuts [19], i.e., $\tilde{A}_{[0]} = \bigcup_{\alpha \in (0, 1]} \tilde{A}_\alpha = \{x: \mu_A(x) > 0\}$. Now, a fuzzy set $\tilde{A}$ of $\mathbb{R}$ is called a fuzzy number if

1. $\exists x \in \mathbb{R}: \mu_A(x) = 1$, which means that $\tilde{A}$ is normal
2. $\forall k \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}$; we have

$$\mu_A((1-k)x_1 + kx_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\},$$  (10)

which means that $\tilde{A}$ is convex fuzzy set
3. $\forall \alpha \in [0, 1], \tilde{A}_\alpha$ is a nonempty compact interval in $\mathbb{R}$, which means that $\tilde{A}$ has compact support

The interval of the $\alpha$-cuts is denoted by $\tilde{A}_\alpha = [\tilde{A}_\alpha^L, \tilde{A}_\alpha^R]$, where $\tilde{A}_\alpha^L = \inf\{x: x \in \tilde{A}_\alpha\}$ and $\tilde{A}_\alpha^R = \sup\{x: x \in \tilde{A}_\alpha\}$. We denote the set of all fuzzy
numbers by $\mathcal{F}(\mathbb{R})$. In [20], based on [21], some of the operations of $\alpha$-cuts were presented as follows.

Let $\bar{A}, \bar{B} \in \mathcal{F}(\mathbb{R})$ and $\bar{A}_a = [\bar{A}_a^L, \bar{A}_a^R]$, $\bar{B}_a = [\bar{B}_a^L, \bar{B}_a^R]$ be their $\alpha$-cuts, respectively. Then, $\forall \alpha \in [0, 1]$, and the fuzzy addition and the scalar multiplication were defined as follows:

$$[\bar{A} \oplus \bar{B}]_a = \left[\bar{A}_a^L + \bar{B}_a^L, \bar{A}_a^R + \bar{B}_a^R\right],$$

$$[\lambda \otimes \bar{A}]_a = [\lambda, 1] \left[\bar{A}_a^L, \bar{A}_a^R\right] = \left[\lambda \bar{A}_a^L, \lambda \bar{A}_a^R\right],$$

respectively, where the scalar $\lambda \in \mathbb{R}^+$ was identified as the interval $[\lambda, \lambda]$.

**Definition 3** (see [22]). Let $\bar{A}$ be a fuzzy number ($\bar{A} \in \mathcal{F}(\mathbb{R})$) and $x \in \mathbb{R}$. Then, the index $Cr: \mathcal{F}(\mathbb{R}) \times \mathbb{R} \rightarrow [0, 1]$ is defined by

$$Cr[\bar{A} \leq x] = \frac{1}{2} \left(\sup_{y \leq x} \mu_\bar{A}'(y) + 1 - \sup_{y \geq x} \mu_\bar{A}'(y)\right).$$

(12)

This gives the credibility degree, that is, $\bar{A}$ is less than or equal to $x$.

**Remark 1** (see [22]). Let $\bar{A} \in \mathcal{F}(\mathbb{R})$ and $x \in \mathbb{R}$, then

$$Cr[\bar{A} \leq x] = 1,$$

if and only if

$$\bar{A}_0^R \leq x.$$  

(14)

2. For any fixed $\bar{A}$, $Cr[\bar{A} \leq x]$ is a nondecreasing function with respect to $x$, i.e.,

$$\forall x_1 < x_2, \quad Cr[\bar{A} \leq x_1] \leq Cr[\bar{A} \leq x_2].$$

(15)

3 $Cr$ is self-dual, i.e.,

$$Cr[\bar{A} > x] = 1 - Cr[\bar{A} \leq x].$$

(16)

**Definition 4** (see [23]). Let $\bar{A} \in \mathcal{F}(\mathbb{R})$ and $\alpha \in (0, 1]$, then the $\alpha$-pessimistic value of $\bar{A}$ is given by $\bar{A}_\alpha = \inf \{x \in \bar{A}_0: Cr[\bar{A} \leq x] \geq \alpha\}$. Furthermore, $\bar{A}_\alpha$ is a nondecreasing function of $\alpha \in (0, 1]$.

**Remark 2** (see [24]). For a given $\bar{A} \in \mathcal{F}(\mathbb{R})$, let $\bar{A}_\alpha$ be defined $\forall \alpha \in (0, 1]$, by

$$\bar{A}_\alpha = \begin{cases} \bar{A}_a^L, & \alpha \in \left(0, \frac{1}{2}\right], \\ \bar{A}_a^R, & \alpha \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

(17)

Then, the $\alpha$-cuts of $\bar{A}$ are given by

$$\bar{A}_\alpha = \left[\bar{A}_{\alpha/2}, \bar{A}_{1-(\alpha/2)}\right].$$

(18)

**Lemma 1** (see [15]). Let $\bar{A}, \bar{B} \in \mathcal{F}(\mathbb{R})$ and let $\lambda \in \mathbb{R}$. Then,

$$\bar{A} \oplus \lambda \bar{B} = \lambda \bar{A} + \bar{B},$$

$$\lambda \otimes \bar{A}_\alpha = \begin{cases} \lambda \bar{A}_\alpha, & \lambda \geq 0, \\ \lambda \bar{A}_{1-\alpha}, & \lambda \leq 0. \end{cases}$$

(19)

**Proof.** The proof of Lemma 1 can be found in [15].

**Definition 5** (see [24]). Let $\bar{A}, \bar{B} \in \mathcal{F}(\mathbb{R})$. If $\bar{A} \leq \bar{B}$, then $\bar{A} < \bar{B}$, and if $\bar{A} = \bar{B}$, then $\bar{A} = \bar{B}$.

**Example 1.** Let $x \in \mathbb{R}$ and $\bar{A}$ be a nonsymmetric triangular fuzzy number with membership function given by

$$\mu_\bar{A}'(x) = \begin{cases} 0, & -\infty < x \leq 5, \\ \frac{x - 5}{3}, & 5 \leq x \leq 8, \\ 1, & x = 8, \\ 9 - x, & 8 \leq x < 9, \\ 0, & 9 \leq x < +\infty. \end{cases}$$

(20)

Then, the credibility degree that $\bar{A}$ is less than or equal to $x$ is given by

$$Cr[\bar{A} \leq x] = \begin{cases} 0, & -\infty < x \leq 5, \\ \frac{x - 5}{6}, & 5 \leq x \leq 8, \\ \frac{x - 7}{2}, & 8 \leq x < 9, \\ 1, & 9 \leq x < +\infty. \end{cases}$$

(21)

As a result, we obtain the $\alpha$-pessimistic values of $\bar{A}$ by

$$\bar{A}_\alpha = \begin{cases} 6\alpha + 5, & 0 < \alpha \leq \frac{1}{2}, \\ 2\alpha + 7, & \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

(22)

2.3. Fuzzy Random Variables. The concept of fuzzy random variables is one of the most adequate tools to handling the results of random experiments, expressed in nonexact terms. In order to integrate randomness and vagueness, random fuzzy sets and random fuzzy numbers [25], are introduced. In most real-life problems, the nature of the data of the experiments is affected by fuzziness, and the procedure of
the extraction of the data of the experiments is affected by randomness. Thus, the definition of fuzzy random variables to be considered in this paper was given in [24], as in the following definition.

**Definition 6.** (see [24]). Let $\Omega$ be the set of all possible outcomes of a random experiment, let $\mathcal{F}$ be the $\sigma$-algebra of the subsets of $\Omega$, and let $\mathbb{P}$ be the probability measure on the measurable space $(\Omega, \mathcal{F})$. Now, suppose that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ describes the random experiment. If $\forall \alpha \in [0, 1]$, $\bar{X}_\alpha: \Omega \longrightarrow \mathbb{R}$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, then $\bar{X}: \Omega \longrightarrow \mathcal{F}(\mathbb{R})$ is called a fuzzy random variable.

The notion of fuzzy random variables was introduced in [14, 26]. In [27], the notion of fuzzy random variables was formalized with the following approach: let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If $\forall \alpha \in [0, 1]$, the two mappings $\bar{X}^L_\alpha: \Omega \longrightarrow \mathbb{R}$ and $\bar{X}^R_\alpha: \Omega \longrightarrow \mathbb{R}$ are random variables, then $\bar{X}: \Omega \longrightarrow \mathcal{F}(\mathbb{R})$ is a fuzzy random variable.

**Remark 3.** (see [28]). In the next relationships, summarize the data of the two dimensional variable $(\bar{X}^L_\alpha, \bar{X}^R_\alpha)$, in the one-dimensional variable $\bar{X}_\alpha$.

$$\bar{X}_\alpha = \begin{cases} \bar{X}^L_{2\alpha}, & 0 < \alpha \leq \frac{1}{2} \\ \bar{X}^R_{2(1-\alpha)}, & \frac{1}{2} \leq \alpha \leq 1 \end{cases}$$

$$\bar{X}_{[\alpha]} = [\bar{X}_{\alpha/2}, \bar{X}_{1-(\alpha/2)}], \quad \alpha \in (0, 1].$$

**Definition 7** (see [15]). If $\bar{X}_\alpha$ and $\bar{Y}_\alpha$ are independent $\forall \alpha \in [0, 1]$, then $\bar{X}$ and $\bar{Y}$ are independent fuzzy random variables. If $\bar{X}_\alpha$ and $\bar{Y}_\alpha$ are identically distributed $\forall \alpha \in [0, 1]$, then $\bar{X}$ and $\bar{Y}$ are identically distributed fuzzy random variables.

### 3. The Novel Construction Method of Copulas

In this section, a novel construction method of copulas is provided, via the $(\alpha, \beta)$-transformation, in four different subrectangles of the unit square. This becomes feasible by the jointing process of the four cases produced in Section 2.1 for specific values of the parameters $\alpha$ and $\beta$, with the adequate adjustments. The construction is achieved through the following theorem.

**Theorem 1.** Let $\kappa$ and $\lambda$ be fixed in $[0, 1]$ and $C: [0, 1] \times [0, 1] \longrightarrow [0, 1]$ be any copula. Then, the function $C$, defined by

$$C(x, y) = \begin{cases} C\left(\frac{x}{\kappa}, \frac{y}{\lambda}\right) \kappa \lambda, & (x, y) \in [0, \kappa] \times [0, \lambda], \\ x - C\left(\frac{x}{\kappa}, \frac{1-y}{1-\lambda}\right) \kappa (1-\lambda), & (x, y) \in [0, \kappa] \times [\lambda, 1], \\ y - C\left(\frac{1-x}{1-\kappa}, \frac{y}{\lambda}\right) (1-\kappa)\lambda, & (x, y) \in [\kappa, 1] \times [0, \lambda], \\ x + y - 1 + C\left(\frac{1-x}{1-\kappa}, \frac{1-y}{1-\lambda}\right) (1-\kappa)(1-\lambda), & (x, y) \in [\kappa, 1] \times [\lambda, 1], \end{cases}$$

**Proof.** The proof that the function $C$ is well defined and the boundary conditions of Definition 1 are satisfied is straightforward. In order to prove that $C$ is double-

increasing, the proof that $C$ is double-increasing in each one of the four rectangles of the domain is needed. For the first rectangle, we have $0 \leq x_1 \leq x_2 \leq \kappa$ and $0 \leq y_1 \leq y_2 \leq \lambda$, and with the usage of the first branch of $C$, we obtain

$$V_\alpha\left([x_1, x_2] \times [y_1, y_2]\right) = \kappa \lambda V_C\left(\left[\frac{x_1}{\kappa}, \frac{x_2}{\kappa}\right] \times \left[\frac{y_1}{\lambda}, \frac{y_2}{\lambda}\right]\right) \geq 0$$

(25)
since $\kappa, \lambda \in [0,1]$ and $C$ is double-increasing as a copula. Hence, $C$ is double-increasing in the rectangle $[0, \kappa] \times [0, \lambda]$. For the rectangle $[0, \kappa] \times [\kappa, 1]$, we have $0 \leq x_1 \leq x_2 \leq \kappa$ and

$$V_\kappa ([x_1, x_2] \times [y_1, y_2]) = \kappa (1 - \lambda) \left( V_C \left( \left[ \frac{x_1}{\kappa}, \frac{x_2}{\kappa} \right] \times \left[ \frac{1 - y_2}{1 - \lambda}, \frac{1 - y_1}{1 - \lambda} \right] \right) \right) \geq 0$$

(26)

since $\kappa, 1 - \lambda \in [0,1]$ and $C$ is double-increasing as a copula. Hence, $C$ is double-increasing in the rectangle $[\kappa, 1] \times [0, \lambda]$. For the rectangle $[\kappa, 1] \times [\lambda, 1]$, we have $\kappa \leq x_1 \leq x_2 \leq 1$ and

$$V_\kappa ([x_1, x_2] \times [y_1, y_2]) = (1 - \kappa) \lambda \left( V_C \left( \left[ \frac{1 - x_2}{1 - \kappa}, \frac{1 - x_1}{1 - \kappa} \right] \times \left[ \frac{1 - y_2}{\lambda}, \frac{1 - y_1}{\lambda} \right] \right) \right) \geq 0$$

(27)

since $1 - \kappa, \lambda \in [0,1]$ and $C$ is double-increasing as a copula. Hence, $C$ is double-increasing in the rectangle $[\kappa, 1] \times [0, \lambda]$. Finally, $C$ is double-increasing in the rectangle $[0, 1]^2$. As a result, $C$ is a copula, and the proof is completed.

\[ \mathcal{C}_1 (x, y) = \begin{cases} 
6xy \\
9x + 8y - 12xy \\
12x^2y + 6x - 6xy - 9x^2 \\
12xy^2 - 8y^2 + 6y - 6xy \\
9x^2 + 8y^2 - 12x^2y - 12xy^2 - 30xy - 15x - 14y + 6
\end{cases} \]

\[ \mathcal{C}_1 (x, y) = \begin{cases} 
\frac{9x + 8y - 12xy}{2} \\
\frac{12x^2y + 6x - 6xy - 9x^2}{2} \\
\frac{12xy^2 - 8y^2 + 6y - 6xy}{-8y - 9x + 9 + 12xy} \\
\frac{9x^2 + 8y^2 - 12x^2y - 12xy^2 - 30xy - 15x - 14y + 6}{9x + 8y - 5 - 12xy} 
\end{cases} \]

\[ \mathcal{C}_1 (x, y) = \begin{cases} 
\frac{12x^2y + 6x - 6xy - 9x^2}{12xy^2 - 8y^2 + 6y - 6xy} \\
\frac{9x^2 + 8y^2 - 12x^2y - 12xy^2 - 30xy - 15x - 14y + 6}{9x + 8y - 5 - 12xy} 
\end{cases} \]

\[ \mathcal{C}_1 (x, y) = \begin{cases} 
\frac{9x + 8y - 12xy}{2} \\
\frac{12x^2y + 6x - 6xy - 9x^2}{12xy^2 - 8y^2 + 6y - 6xy} \\
\frac{9x^2 + 8y^2 - 12x^2y - 12xy^2 - 30xy - 15x - 14y + 6}{9x + 8y - 5 - 12xy} 
\end{cases} \]

The produced copula of Example 1, is presented in Figure 2.

Fréchet [29] and Hoeffding [30] introduced the Fréchet–Hoeffding [12] bounds of copulas for any $x, y \in [0,1]$, $\max(x + y - 1, 0) \leq C (x, y) \leq \min(x, y)$. In order to present the bounds of the produced copulas $C'$, we provide the following theorem.

**Theorem 2.** Let $\kappa$ and $\lambda$ be fixed in $[0, 1]$ and $C$ and $C'$ be two copulas of Theorem 1. Then, the functions $C_-$ and $C^+$ with

Note: The text contains mathematical expressions and inequalities that are not fully rendered in this text format. For a complete understanding, please refer to the original document or PDF.
Choose the copula C.

Transform the copula C into the copula \( C_{\alpha, \beta} \) for the cases:

\((\alpha = 0, \beta = 0), (\alpha = 0, \beta = 1), (\alpha = 1, \beta = 0), (\alpha = 1, \beta = 1)\)

and use Theorem 1 in order to construct the copula C for the adequate values of \( \kappa \) and \( \lambda \).

Obtain the copula \( \zeta \) via the copula C.

Stop

**Figure 1:** Flowchart of copula construction process.

**Figure 2:** Produced copula of \( C_1 \) with \( \kappa = 2/3 \) and \( \lambda = 3/4 \) (left axis represents \( y \), right axis represents \( x \), and vertical axis represents \( C_1(x, y) \)).
\( \text{Dom} \mathcal{C}_- = \text{Dom} \mathcal{C}^- = [0, 1]^2 \) and \( \text{Ran} \mathcal{C}_- = \text{Ran} \mathcal{C}^- = [0, 1] \),
given by

\[
\mathcal{C}_-(x, y) = \begin{cases} 
\max[0, \lambda x + \kappa y - \kappa \lambda], \\
x - \min[(1 - \lambda)x, \kappa(1 - y)], \\
y - \min[(1 - \kappa)y, \lambda(1 - x)], \\
x + y - 1 + \max[0, (1 - \lambda)(1 - x) + (1 - \kappa)(1 - y) - (1 - \kappa)(1 - \lambda)], \\
\min[\lambda x, \kappa y], \\
x - \max[0, (1 - \lambda)x + \kappa(1 - y) - (1 - \kappa)], \\
y - \max[0, \lambda(1 - x) + (1 - \kappa)y - (1 - \kappa)\lambda], \\
x + y - 1 + \min[(1 - \lambda)(1 - x)(1 - \kappa)(1 - \lambda)], \\
\end{cases}
\]

\( \mathcal{C}^-(x, y) = \begin{cases} 
\max[0, \lambda x - \kappa \lambda], \\
x - \min[(1 - \kappa)x], \\
y - \min[(1 - \lambda)y], \\
x + y - 1 + \max[0, (1 - \kappa)(1 - x) + (1 - \lambda)(1 - y) - (1 - \kappa)(1 - \lambda)], \\
\min[\lambda x, \kappa y], \\
x - \max[0, (1 - \kappa)x + \kappa(1 - y) - (1 - \lambda)], \\
y - \max[0, \lambda(1 - x) + (1 - \kappa)y - (1 - \lambda)\lambda], \\
x + y - 1 + \min[(1 - \kappa)(1 - x)(1 - \lambda)(1 - \lambda)], \\
\end{cases}
\]

respectively, are copulas, and \( \mathcal{C}_-(x, y) \leq \mathcal{C}(x, y) \leq \mathcal{C}^-(x, y) \),
for every copula, \( \forall x, y \in [0, 1] \).

**Proof.** The proof that \( \mathcal{C}_- \) and \( \mathcal{C}^- \) have \( \text{Dom} \mathcal{C}_- = \text{Dom} \mathcal{C}^- = [0, 1]^2 \) and \( \text{Ran} \mathcal{C}_- = \text{Ran} \mathcal{C}^- = [0, 1] \), and that satisfy the boundary conditions of copulas is straightforward. In addition, for the proof that \( \mathcal{C}_- \) and \( \mathcal{C}^- \) are double-increasing, we have to examine this in each one of the four rectangles of their domain. For the function \( \mathcal{C}^- \), we have that \( \forall x_1, x_2 \in [0, \kappa] \) and \( \forall y_1, y_2 \in [0, \lambda] \), with \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \):

\[
\kappa(1 - \lambda) \left( \min \left[ \frac{1}{1 - \lambda}, \frac{1}{1 - \kappa} \right] \times \frac{1}{1 - \lambda} \right) \geq 0
\]

since \( \kappa, \lambda \in [0, 1] \), and \( M \) is double-increasing as a copula. Hence, \( \mathcal{C}^- \) is double-increasing in \( [0, \kappa] \times [0, \lambda] \). Next, \( \forall x_1, x_2 \in [0, \kappa] \) and \( \forall y_1, y_2 \in [\lambda, 1] \), with \( 0 \leq (x_1 / \kappa) \leq (x_2 / \kappa) \leq 1 \) and \( \lambda \leq y_1 \leq y_2 \leq \lambda \Rightarrow 0 \leq ((1 - y_1) / (1 - \lambda)) \leq ((1 - y_2) / (1 - \lambda)) \leq 1 \):

\[
(1 - \kappa) \left( \min \left[ \frac{1}{1 - \lambda}, \frac{1}{1 - \kappa} \right] \times \frac{1}{1 - \lambda} \right) \geq 0
\]

since \( x, y \in [0, 1] \) and \( M \) is double-increasing as a copula. Hence, \( \mathcal{C}^- \) is double-increasing in \( [0, \kappa] \times [\lambda, 1] \). Next, \( \forall x_1, x_2 \in [\kappa, 1] \) and \( \forall y_1, y_2 \in [0, \lambda] \), with \( \kappa \leq x_1 \leq x_2 \leq 1 \Rightarrow 0 \leq ((1 - x_1) / (1 - \kappa)) \leq ((1 - x_2) / (1 - \kappa)) \leq 1 \) and \( \kappa \leq y_1 \leq y_2 \leq \lambda \Rightarrow 0 \leq (y_1 / \lambda) \leq (y_2 / \lambda) \leq 1 \):

\[
(1 - \kappa) \left( \min \left[ \frac{1}{1 - \lambda}, \frac{1}{1 - \kappa} \right] \times \frac{1}{1 - \lambda} \right) \geq 0
\]

since \( 1 - \kappa, 1 - \lambda \in [0, 1] \), and \( M \) is double-increasing as a copula. Hence, \( \mathcal{C}^- \) is double-increasing in \( [\kappa, 1] \times [0, \lambda] \). Next, \( \forall x_1, x_2 \in [\kappa, 1] \) and \( \forall y_1, y_2 \in [\lambda, 1] \), with \( \kappa \leq x_1 \leq x_2 \leq 1 \Rightarrow 0 \leq (y_1 / \lambda) \leq (y_2 / \lambda) \leq 1 \) and \( \lambda \leq y_1 \leq y_2 \leq 1 \Rightarrow 0 \leq (y_1 / \lambda) \leq (y_2 / \lambda) \leq 1 \):

\[
(1 - \kappa) \left( \min \left[ \frac{1}{1 - \lambda}, \frac{1}{1 - \kappa} \right] \times \frac{1}{1 - \lambda} \right) \geq 0
\]

since \( 1 - \kappa, 1 - \lambda \in [0, 1] \), and \( M \) is double-increasing as a copula. Hence, \( \mathcal{C}^- \) is double-increasing in \( [\kappa, 1] \times [\lambda, 1] \).

For the cases of the other three rectangles, with the same approach as the case of the first rectangle, we obtain the desirable equations. Hence, \( \mathcal{C}_- \) and \( \mathcal{C}^- \) are copulas, and the bounds of every copula \( \mathcal{C} \), and as a result, the proof is completed.

The present construction method of copulas that can be achieved through Theorem 1, and based on the fact that \( \mathcal{C}_- \neq \mathcal{C}^- \), of Theorem 2, can lead us in the result that there exists an infinite number of copulas that can be constructed via this transformation. The disadvantage of this method is that, for the construction of copula \( \mathcal{C} \), we are using for each of the four branches only the copula \( \mathcal{C} \).

**Example 3.** Let \( \kappa = 7/8, \lambda = \sqrt{2}/2 \), and copulas \( \mathcal{C}_- \) and \( \mathcal{C}^- \) of Theorem 2. Then, the produced copulas for those specific values of \( \kappa, \lambda \) are given by
The plots of $G_-$ and $G^-$ are illustrated in Figures 3 and 4, respectively.

4. The Novel Construction Method of Copulas for Fuzzy Random Variables

The extension of copulas was achieved in [15], through the notion of fuzzy copula functions of two fuzzy random variables $\tilde{X}$ and $\tilde{Y}$ at $(x, y) \in [0,1] \times [0,1]$, with the following $\alpha$-cuts:

$$[\tilde{C}(x, y)]_{[a]} = \left[ F\left( F_X^{-1}(x)_{[a/2]}, F_Y^{-1}(y)_{[a/2]} \right), F\left( F_X^{-1}(x)_{1-(a/2)}, F_Y^{-1}(y)_{1-(a/2)} \right) \right].$$

As a result, $\tilde{C}$ is a joint fuzzy distribution function [15]. The following proposition examines the properties of the fuzzy copula.

**Proposition 1** (see [15]). For a fuzzy copula $\tilde{C}(x, y)$, the following conditions hold:

$$\tilde{C}(x, 0) = \bar{0}, \quad \tilde{C}(x, 1) = \bar{x}, \quad \tilde{C}(1, y) = \bar{y},$$

$$\tilde{C}(x_2, y_2) \oplus \tilde{C}(x_1, y_1) > \tilde{C}(x_1, y_2) \oplus \tilde{C}(x_2, y_1),$$

where $x, y, x_1, x_2, y_1, y_2 \in [0,1]$, with $x_1 < x_2$ and $y_1 < y_2$. 

$$\begin{align*}
G_-(x, y) &= \left\{ \begin{array}{ll}
\max \left\{ 0, \frac{\sqrt{2} x - 7 y}{2} - \frac{7 \sqrt{2}}{16} \right\}, & (x, y) \in \left[ 0, \frac{7}{8} \right] \times \left[ 0, \frac{\sqrt{2}}{2} \right], \\
x - \min \left\{ \frac{(2 - \sqrt{2})x - 7(1 - y)}{2}, \frac{7 \sqrt{2}}{8} \right\}, & (x, y) \in \left[ 0, \frac{7}{8} \right] \times \left[ \frac{\sqrt{2}}{2}, 1 \right], \\
y - \min \left\{ \frac{y \sqrt{2} (1 - x)}{2}, \frac{7 \sqrt{2}}{8} \right\}, & (x, y) \in \left[ \frac{7}{8}, 1 \right] \times \left[ 0, \frac{\sqrt{2}}{2} \right], \\
x + y - 1 + \max \left\{ 0, \frac{16 - 7 \sqrt{2} - 8 (2 - \sqrt{2})x - 2y}{16} \right\}, & (x, y) \in \left[ \frac{7}{8}, 1 \right] \times \left[ \frac{\sqrt{2}}{2}, 1 \right], \\
\end{array} \right. \\
G^-(x, y) &= \left\{ \begin{array}{ll}
\min \left\{ \frac{\sqrt{2} x - 7 y}{2}, \frac{7 \sqrt{2}}{8} \right\}, & (x, y) \in \left[ 0, \frac{7}{8} \right] \times \left[ 0, \frac{\sqrt{2}}{2} \right], \\
x - \max \left\{ 0, \frac{(2 - \sqrt{2})x - 7 y}{2} + \frac{7 \sqrt{2}}{16} \right\}, & (x, y) \in \left[ 0, \frac{7}{8} \right] \times \left[ \frac{\sqrt{2}}{2}, 1 \right], \\
y - \max \left\{ 0, \frac{3 \sqrt{2} - \sqrt{2} x}{2} + \frac{y}{8} \right\}, & (x, y) \in \left[ \frac{7}{8}, 1 \right] \times \left[ 0, \frac{\sqrt{2}}{2} \right], \\
x + y - 1 + \frac{\min \left\{ (2 - \sqrt{2})(1 - x) - (1 - y) \right\}}{2}, & (x, y) \in \left[ \frac{7}{8}, 1 \right] \times \left[ \frac{\sqrt{2}}{2}, 1 \right]. \\
\end{array} \right. 
\end{align*}$$

(36)
The proof of this proposition can be found in [15]. Next, inspired by the fuzzy copula for fuzzy random variables, we propose a novel method for the construction of fuzzy copulas. This is achieved through the following theorem.

**Theorem 3.** Let $\kappa$ and $\lambda$ be fixed in $[0, 1]$, $\mathcal{C} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be any fuzzy copula and $\mathcal{C} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be produced by any copula $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$ with the usage of Theorem 1. Then, $\mathcal{C}^+(x, y)$ at $(x, y) \in [0, 1] \times [0, 1]$, defined by

$$
\mathcal{C}^+(x, y) = \begin{cases} 
\overline{C}(\frac{x}{\kappa}, \frac{y}{\lambda}) \otimes (\kappa \lambda), & (x, y) \in [0, \kappa] \times [0, \lambda], \\
\overline{x} \oplus \overline{\mathcal{C}}(\frac{x}{\kappa}, \frac{1 - y}{1 - \lambda}) \otimes (\kappa (\lambda - 1)), & (x, y) \in [0, \kappa] \times [\lambda, 1], \\
\overline{y} \oplus \overline{\mathcal{C}}(\frac{1 - x}{1 - \kappa}, \frac{y}{\lambda}) \otimes ((\kappa - 1) \lambda), & (x, y) \in [\kappa, 1] \times [0, \lambda], \\
\overline{x} \oplus \overline{y} \oplus (-1) \overline{\mathcal{C}}(\frac{1 - x}{1 - \kappa}, \frac{1 - y}{1 - \lambda}) \otimes ((1 - \kappa)(1 - \lambda)), & (x, y) \in [\kappa, 1] \times [\lambda, 1], \\
\end{cases}
$$

\[ (39) \]
is the fuzzy copula of the fuzzy random variables $\tilde{X}$ and $\tilde{Y}$.

Proof. For the first branch of $\tilde{C}$, we have that, for $x = 0$ and $\forall y \in [0, \lambda]$,

$$[\tilde{C}(0, y)]_{[0]} = \left[C\left(\frac{\lambda}{\lambda - y}\right)\right]_{[0]}$$

$$= \left[F\left(F^{-1}_X(0)\right)_{1/2}, (F^{-1}_Y(\frac{\lambda}{\lambda - y}))_{1/2}\right]_{[0]} \kappa \lambda,$$

$$= \left[F\left(F^{-1}_X(0)\right)_{1-(\alpha/2)}, \left(F^{-1}_Y(\frac{\lambda}{\lambda - y})\right)_{1-(\alpha/2)}\right]_{[0]} \kappa \lambda,$$

$$= \left[F^{-1}_X(0)^{-1}, (F^{-1}_Y(\frac{\lambda}{\lambda - y}))_{1-(\alpha/2)}\right]_{[0]} \kappa \lambda,$$

$$= \left[F^{-1}_Y(\frac{\lambda}{\lambda - y})_{\alpha/2}, F^{-1}_Y(\frac{\lambda}{\lambda - y})_{1-(\alpha/2)}\right]_{[0]} \kappa \lambda.$$  \hspace{1cm} (40)

$$= [0, 0] = \bar{0}_{[0]}, \ \forall \alpha \in (0, 1].$$

As a result, $\tilde{C}(0, y) = \bar{0}$. The proof that $\tilde{C}(x, 0) = \bar{0}$ can be considered in the same manner. The next step is to examine the second condition that holds in the special case of $\kappa = 1$ and $\lambda = 1$. In this case, $\tilde{C}$ becomes $\tilde{C}$, and the proof is straightforward. In the last step, we have that $\forall x_1, x_2 \in [0, \kappa]$ and $\forall y_1, y_2 \in [0, \lambda]$, where $x_1 < x_2$ and $y_1 < y_2$ and

$$\tilde{C}(x_2, y_2) \preceq \tilde{C}(x_1, y_1) \preceq \tilde{C}(x_1, y_2) \preceq \tilde{C}(x_2, y_1).$$  \hspace{1cm} (41)

Since $\kappa, \lambda \geq 0$, $\forall \alpha \in (0, 1]$, we have that

$$\kappa \lambda \preceq \left(\tilde{C}(x_2, y_2) \oplus \tilde{C}(x_1, y_1)\right)_{[0]}$$

$$> \kappa \lambda \preceq \left(\tilde{C}(x_1, y_1) \oplus \tilde{C}(x_2, y_1)\right)_{[0]} \Rightarrow \tilde{C}(x_2, y_2) \oplus \tilde{C}(x_1, y_1) \preceq \tilde{C}(x_1, y_2) \oplus \tilde{C}(x_2, y_1).$$  \hspace{1cm} (42)

As a result, the proof for the first branch is completed. Next, for the case of the second branch, of $\tilde{C}$, we have that $\forall x \in [0, \kappa]$ and for $y = 1$,

$$\left[\tilde{x} \oplus \tilde{C}\left(\frac{\lambda}{\lambda - y}\right) \oplus (\kappa(\lambda - 1))\right]_{[\alpha]} =$$

$$= \left[\tilde{x}_{1-(\alpha/2)} - F\left(F^{-1}_X(\frac{\lambda}{\lambda - y})\right)_{1-(\alpha/2)}, (F^{-1}_Y(0))_{1-(\alpha/2)}\right] \kappa(1 - \lambda),$$

$$\tilde{x}_{1-(\alpha/2)} - F\left(F^{-1}_X(\frac{\lambda}{\lambda - y})\right)_{1-(\alpha/2)} (F^{-1}_Y(0))_{1-(\alpha/2)} \kappa(1 - \lambda)$$

$$= \left[\tilde{x}_{1-(\alpha/2)} - F\left(F^{-1}_X(\frac{\lambda}{\lambda - y})\right)_{1-(\alpha/2)} (F^{-1}_Y(0))_{1-(\alpha/2)}\right] \kappa(1 - \lambda),$$

$$\tilde{x}_{1-(\alpha/2)} - F\left(F^{-1}_X(\frac{\lambda}{\lambda - y})\right)_{1-(\alpha/2)} (F^{-1}_Y(0))_{1-(\alpha/2)} \kappa(1 - \lambda)$$

$$= \left[\tilde{x}_{1-(\alpha/2)} - F\left(F^{-1}_X(\frac{\lambda}{\lambda - y})\right)_{1-(\alpha/2)} (-\infty)\right] \kappa(1 - \lambda),$$

$$\tilde{x}_{1-(\alpha/2)} - F\left(F^{-1}_X(\frac{\lambda}{\lambda - y})\right)_{1-(\alpha/2)} (-\infty) \kappa(1 - \lambda).$$  \hspace{1cm} (43)

$$\tilde{x}_{1-(\alpha/2)} - F\left(F^{-1}_X(\frac{\lambda}{\lambda - y})\right)_{1-(\alpha/2)} (-\infty) \kappa(1 - \lambda).$$

$$\tilde{x}_{1-(\alpha/2)} - F\left(F^{-1}_X(\frac{\lambda}{\lambda - y})\right)_{1-(\alpha/2)} (-\infty) \kappa(1 - \lambda).$$
Hence, we obtain
\[
\bar{\mathbf{\gamma}}(x, 0) = \bar{x}.
\]  

For the examination of the third condition, we have that \( \forall x_1, x_2 \in [0, \kappa] \) and \( \forall y_1, y_2 \in [\lambda, 1] \), where \( x_1 < x_2 \) and \( y_1 < y_2 \) and

\[
\begin{align*}
(x_2)_a - \left( C \left( \frac{x_1 - y_1}{\kappa} \frac{1}{1 - \lambda} \right)^{\kappa(1 - \lambda)} \right)_{1-a} + (x_1)_a - \left( C \left( \frac{x_1 - y_2}{\kappa} \frac{1}{1 - \lambda} \right)^{\kappa(1 - \lambda)} \right)_{1-a} > \\
\left( C \left( \frac{x_1 - y_1}{\kappa} \frac{1}{1 - \lambda} \right)^{\kappa(1 - \lambda)} \right)_{1-a} + (x_1)_a - \left( C \left( \frac{x_1 - y_2}{\kappa} \frac{1}{1 - \lambda} \right)^{\kappa(1 - \lambda)} \right)_{1-a} \equiv & \left( C \left( \frac{x_1 - y_1}{\kappa} \frac{1}{1 - \lambda} \right)^{\kappa(1 - \lambda)} \right)_{1-a} + \left( C \left( \frac{x_1 - y_2}{\kappa} \frac{1}{1 - \lambda} \right)^{\kappa(1 - \lambda)} \right)_{1-a}.
\end{align*}
\]

This holds if we consider that since \( \lambda \leq y_1 < y_2 \leq 1 \), then \( 0 \leq (1 - y_2)/(1 - \lambda) \leq (1 - y_1)/(1 - \lambda) \leq 1 \). Hence, the proof of the second branch is completed. The cases of the third and fourth branches of \( \mathbf{\gamma} \) can be examined in a similar way as the first and second branches of \( \mathbf{\gamma} \). Hence, the proof that \( \mathbf{\gamma}(x, y) \) is a fuzzy copula is completed. \( \square \)

The flowchart of Figure 5 illustrates the novel fuzzy copula construction process.

Now, in order to illustrate the novel construction of fuzzy copula, the following example is provided, based on an example that may be found in [15].

**Example 4.** Let \( X \) and \( Y \) be random variables that have joint distribution function, given by

\[
F_{XY}(x, y) = \frac{(x + 1)(1 - e^{-x})}{(x - 1)e^{-y} + 2}, \quad (x, y) \in [-1, 1] \times [0, \infty].
\]

(46)

As a result, \( X \sim U[-1, 1] \) and \( Y \sim Weibull(1, 1) \). Let \( \bar{\Theta} = (2, 1, 4)_T \) and \( \bar{\Gamma} = (8, 5, 9)_T \) be two nonsymmetric triangular fuzzy numbers and let \( \bar{X} = \bar{\Theta} + X \) and \( \bar{Y} = \bar{\Gamma} + Y \). Hence, we have that \( \bar{X} = (X + 2, 1, 4)_T \) and \( \bar{Y} = (Y + 8, 5, 9)_T \). Therefore, we obtain the \( \alpha \)-pessimistic values of \( \bar{X} \) and \( \bar{Y} \), respectively, by

\[
\begin{align*}
\bar{X}_a & = \begin{cases} 
X + 2\alpha + 1, & 0 < \alpha \leq \frac{1}{2} \\
X + 4\alpha, & \frac{1}{2} \leq \alpha \leq 1,
\end{cases} \\
\bar{Y}_a & = \begin{cases} 
Y + 6\alpha + 5, & 0 < \alpha \leq \frac{1}{2} \\
Y + 2\alpha + 7, & \frac{1}{2} \leq \alpha \leq 1.
\end{cases}
\end{align*}
\]

(47)

Based on the fact that \( \forall \alpha \in (0, 1) \), \( \bar{X}_a \) has uniform distribution and \( \bar{Y}_a \) has Weibull distribution. Hence,

\[
\begin{align*}
\bar{X}_a & \sim \begin{cases} 
U(2\alpha, 4\alpha - 1), & 0 < \alpha \leq \frac{1}{2} \\
U(4\alpha - 1, \alpha + 1), & \frac{1}{2} \leq \alpha \leq 1,
\end{cases} \\
\bar{Y}_a & \sim \begin{cases} 
\text{Weibull}(6\alpha + 5, 1, 1), & 0 < \alpha \leq \frac{1}{2} \\
\text{Weibull}(2\alpha + 7, 1, 1), & \frac{1}{2} \leq \alpha \leq 1.
\end{cases}
\end{align*}
\]

(48)

Also, we have the \( \alpha \)-pessimistic values of \( \bar{\Theta} \) and \( \bar{\Gamma} \) given as follows:
Transform the fuzzy copula $C$ into the fuzzy copula $C_{\alpha,\beta}$ for the cases:

- $(\alpha = 0, \beta = 0)$, $(\alpha = 0, \beta = 1)$,
- $(\alpha = 1, \beta = 0)$, $(\alpha = 1, \beta = 1)$

and use Theorem 4 in order to construct the fuzzy copula $\tilde{C}$ for the adequate values of $\kappa$ and $\lambda$.

\[
\tilde{\Theta}_a = \begin{cases} 
2\alpha + 1, & 0 < \alpha \leq \frac{1}{2} \\
4\alpha, & \frac{1}{2} \leq \alpha \leq 1 \\
6\alpha + 5, & 0 < \alpha \leq \frac{1}{2} \\
2\alpha + 7, & \frac{1}{2} \leq \alpha \leq 1.
\end{cases}
\]

\[
\tilde{V}_a = \begin{cases} 
2\alpha + 1, & 0 < \alpha \leq \frac{1}{2} \\
4\alpha, & \frac{1}{2} \leq \alpha \leq 1 \\
6\alpha + 5, & 0 < \alpha \leq \frac{1}{2} \\
2\alpha + 7, & \frac{1}{2} \leq \alpha \leq 1.
\end{cases}
\]

Hence, we obtain the next $\alpha$-pessimistic values:

\[
(F_X (x))_a = \begin{cases} 
\frac{x - \tilde{\Theta}_{1-a} + 1}{2}, & x \in \tilde{\Theta}_{1-a} - 1, \tilde{\Theta}_{1-a} + 1 \\
1 - e^{\tilde{V}_{1-a}-1}, & y \in [\tilde{V}_{1-a}, \infty].
\end{cases}
\]

\[
((F_X (u))_a)^{-1} = 2u + \tilde{\Theta}_{1-a} - 1,
\]

\[
((F_Y (v))_a)^{-1} = \tilde{V}_{1-a} - \ln(1 - v).
\]

Also, we have that

\[
(F_X^{-1} (u))_a = 2u + \tilde{\Theta}_{1-a} - 1,
\]

\[
(F_Y^{-1} (v))_a = \tilde{V}_{1-a} - \ln(1 - v).
\]

Thus, we obtain the left and right parts of the $\alpha$-cuts of the fuzzy copula by

\[
\tilde{C}_a(u, v) = F_{XY}((F_X^{-1} (u))_{\alpha/2}, (F_Y^{-1} (v))_{\alpha/2})
\]

\[
= \frac{2u + \tilde{\Theta}_{\alpha/2}}{2u + \tilde{\Theta}_{\alpha/2} - 2} e^{\tilde{V}_{\alpha/2} (1 - v)} + 2,
\]

\[
\tilde{C}_a^{-1}(u, v) = F_{XY}^{-1}((F_X^{-1} (u))_{1-(\alpha/2)}, (F_Y^{-1} (v))_{1-(\alpha/2)})
\]

\[
= \frac{2u + \tilde{\Theta}_{1-(\alpha/2)}}{2u + \tilde{\Theta}_{1-(\alpha/2)} - 2} e^{\tilde{V}_{1-(\alpha/2)} (1 - v)} + 2.
\]

where \([\tilde{C}_a(u, v)]_{\alpha} = [\tilde{C}^L_a(u, v), \tilde{C}^R_a(u, v)]\). Now, based on Theorem 3, we have that, for the first branch of $\tilde{C}$,
\[
[\bar{\mathcal{G}}(u, v)]_{[a]} = \left[ \bar{G}_a^L(u, v), \bar{G}_a^R(u, v) \right], \quad \text{(54)}
\]
where

\[
\bar{G}_a^L(u, v) = C_a^L\left(\frac{u}{\lambda}, \frac{v}{\lambda}\right) = \left(\frac{(2u/k) + \bar{\Theta}_{a/2}}{(2u/k) + \bar{\Theta}_{a/2} - 2}\right) e^{-T_{a/2}} (1 - ((1 - v)/\lambda)) \kappa(\lambda - 1),
\]
\[
\bar{G}_a^R(u, v) = C_a^R\left(\frac{u}{\lambda}, \frac{v}{\lambda}\right) = \left(\frac{(2u/k) + \bar{\Theta}_{1-(a/2)}}{(2u/k) + \bar{\Theta}_{1-(a/2)} - 2}\right) e^{-T_{1-(a/2)}} (1 - ((1 - v)/\lambda)) \kappa(\lambda - 1).
\]

\[\forall u \in [0, k], \text{ and } v \in [0, \lambda].\]

Next, for the second branch of \( \bar{\mathcal{G}} \), we have that \( \forall u \in [0, k] \) and \( v \in [\lambda, 1] \),

\[
[\bar{\mathcal{G}}(u, v)]_{[a]} = \left[ \bar{G}_a^L(u, v), \bar{G}_a^R(u, v) \right], \quad \text{(56)}
\]

where

\[
\bar{G}_a^L(u, v) = u_{a/2} + \left(\frac{2(1-u)/(1-k)}{2(1-u)/(1-k) + \bar{\Theta}_{a/2}}\right) e^{-T_{a/2}} (1 - ((1 - v)/\lambda)) (\kappa(\lambda - 1)),
\]
\[
\bar{G}_a^R(u, v) = \bar{v}_{1-(a/2)} + \left(\frac{2((1-u)/(1-k)) + \bar{\Theta}_{1-(a/2)}}{2((1-u)/(1-k)) + \bar{\Theta}_{1-(a/2)} - 2}\right) e^{-T_{1-(a/2)}} (1 - (v/\lambda)) (\kappa - 1)\lambda.
\]

Finally, for the case of the fourth branch of \( \bar{\mathcal{G}} \), we have that \( \forall u \in [\lambda, 1] \) and \( v \in [0, \lambda] \),

\[
[\bar{\mathcal{G}}(u, v)]_{[a]} = \left[ \bar{G}_a^L(u, v), \bar{G}_a^R(u, v) \right], \quad \text{(60)}
\]

where

\[
\bar{G}_a^L(u, v) = u_{a/2} + \nu_{a/2} - 1 + \left(\frac{2(1-u)/(1-k)}{2(1-u)/(1-k) + \bar{\Theta}_{a/2}}\right) e^{-T_{a/2}} (1 - ((1 - v)/\lambda)) (1 - \kappa)(1 - \lambda),
\]
\[
\bar{G}_a^R(u, v) = \bar{u}_{1-(a/2)} + \bar{v}_{1-(a/2)} - 1 + \left(\frac{2((1-u)/(1-k)) + \bar{\Theta}_{1-(a/2)}}{2((1-u)/(1-k)) + \bar{\Theta}_{1-(a/2)} - 2}\right) e^{-T_{1-(a/2)}} (1 - (v/\lambda)) (1 - \kappa)(1 - \lambda).
\]

(61)
In Example 3 of [15], the production of a fuzzy copula \( \tilde{C} \) was established. In the illustrative Example 4 of the present paper, the generalization of this result was presented with the usage of \((\alpha, \beta)\) transformation, in four different subrectangles of the unit square.

5. Conclusions

We provided a method for the construction of copulas in four subrectangles of the unit square, via the \((\alpha, \beta)\) transformation, and we introduced the upper and lower bounds of the produced copulas of this method. Also, we developed a method to construct fuzzy copulas for fuzzy random variables. With the usage of those methods, we can conclude that the construction of an infinite number of copulas and fuzzy copulas can be achieved. Each one of the produced copulas and fuzzy copulas could be applied based on their adequacy in real-life problems. It may also be of interest considering these construction methods in the case of \(n\)-dimensional copulas \(n > 2\). On the other hand, the case in which these construction methods could possibly be defined in more than four subrectangles of the unit square may be examined. In addition, the extension of those methods for Intuitionistic fuzzy sets [16, 18] and Pythagorean fuzzy sets [31] could be possibly achieved, in order to develop aggregation operators for multiple attribute decision making algorithms. These topics are the basis for our future investigations.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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