Continuous phase transition and critical behaviors of 3D black hole with torsion

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Abstract
We study the phase transition and the critical behavior of the BTZ black hole with torsion obtained in (1+2)-dimensional Poincaré gauge theory. According to Ehrenfest’s classification, when the parameters in the theory are arranged properly, the BTZ black hole with torsion may possess the second-order phase transition which is also a smaller mass/larger mass black hole phase transition. Nevertheless, the critical behavior is different from the one in the van der Waals liquid/gas system. We also calculated the critical exponents of the relevant thermodynamic quantities, which are the same as the ones obtained in the Hořava-Lifshitz black hole and the Born–Infeld black hole.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The laws of black hole thermodynamics and the usual laws of thermodynamics have very similar forms. The good agreement indicates that black hole is also a thermodynamic system. Phase transitions and critical phenomena are important characteristics of usual thermodynamics. Thus, the natural question to ask is whether there also exists phase transition in the black hole thermodynamics. In fact, since the pioneering work of Davies \cite{1} and the well-known Hawking–Page phase transition \cite{2} were proposed, the question has been answered partly. The phase transitions and critical phenomena in \textit{four-dimensional and higher} dimensional AdS black holes have been studied extensively \cite{3–6}. Recently, some
interesting works on asymptotically anti-de Sitter black holes have been done, which show that there exists phase transition similar to the van der Waals liquid/gas phase transition [7–15].

The properties of black hole are relevant to the dimension of spacetime. Thus, we want to know whether the similar phase transition exists for lower dimensional black holes. The key advantage of lower dimensional black holes lies in the simplicity of the construction. Although just a mathematical abstraction, lower dimensional black holes can be applied to physical reality in some special cases. Hence, it can be interesting to investigate the possibility that a lower dimensional black hole does exhibit a phase transition. BTZ black hole is an important solution of general relativity with negative cosmological constant in three-dimensional (3D) spacetime [16, 17]. The BTZ black hole is free of singularity and closely related with the recent developments in gravity, gauge theory and string theory [18].

In general relativity and many other theories of gravity, curvature plays an essential role, while torsion has received less attention. However, torsion also has its geometrical meaning and plays some roles in gravitation theory. Since the 1970s, many theories of gravity with torsion have been proposed, such as Poincaré gauge gravity, de Sitter gauge gravity, teleparallel gravity, \( f(T) \) gravity, etc. In particular, Mielke and Baekler proposed a model of 3D gravity with torsion (the MB model), which also has BTZ black hole as solution [19–23]. This model aroused the following research on the thermodynamics of the BTZ black hole with torsion \( \text{(BTZT black hole for short)} \) and AdS/CFT with torsion [24–26].

In [27], we have verified that for the BTZT black hole, phase transition may exist. In this paper, we will investigate the type of the phase transition and calculate the critical exponents. Although the BTZ solution is the same as the one obtained in GR, the different actions will make their thermodynamics very different. The modified action in the MB model will modify the conserved charges such as mass and angular momentum. Correspondingly, the entropy of BTZ black hole and the first law of black hole thermodynamics will also be changed. It is shown that the heat capacity of the BTZ black hole in the MB model is not always positive any more, but changes signs at some points and may diverge at the critical point. Thus, for the BTZT black hole in the MB model, phase transition exists. According to Ehrenfest’s classification, we also consider the Gibbs free energy, the isothermal compressibility and the expansion coefficient as functions of temperature. It is shown that the kind of phase transition for the BTZT black hole belongs to the second-order one or continuous one.

The paper is arranged as follows: in the next section, we simply introduce the MB model and its BTZ-like solution and the corresponding thermodynamic quantities. In section 3, according to Ehrenfest’s classification, we will analyze the type of the phase transition of the BTZT black hole in the extended phase space. In section 4, the case with non-extended phase space will be discussed and the critical behaviors are investigated. We make some concluding remarks in section 5.

2. Mielke–Baekler model and the 3D black hole with torsion

First, we should review the topological 3D gravity model with torsion proposed by Mielke and Baekler [19, 20], which is a natural generalization of Riemannian GR with a cosmological constant. Defining curvature and torsion 2-forms out of \( \omega^a_b \) and coframe \( e^a \) by

\[
T^a = de^a + \omega^a_b \wedge e^b, \tag{2.1}
\]

\[
R^a_{\ b} = d\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b}, \tag{2.2}
\]

the gravitational action is written as
\[ I = \int 2 \chi e^a \wedge R_a = \frac{\Lambda}{3} \epsilon_{abc} e^a \wedge e^b \wedge e^c + \alpha_3 \left( \omega^a \wedge d\omega_a + \frac{1}{3} \epsilon_{abc} \omega^a \wedge \omega^b \wedge \omega^c \right) + \alpha_4 e^a \wedge T_a, \]

(2.3) where the dual expression, \( R_a \) and \( \omega_a \) are defined by \( R_{ab} = \epsilon^{abc} R_c \) and \( \omega_{ab} = \epsilon^{abc} \omega_c \). In equation (2.3) the first term corresponds to the Einstein–Cartan action, with \( \chi = \frac{1}{16\pi G} \). The second one is the cosmological term. The last two terms are the Chern–Simons term and the Nieh–Yan (N–Y) term, which should be given particular attention.

The N–Y form is a special 2-form only for the Riemann–Cartan geometry [28, 29]. On the four-dimensional manifold \( M \), it can be written as

\[ N = T^a \wedge T_a + R_{ab} \wedge e^a \wedge e^b = dQ_{NY}, \]

(2.4) \[ Q_{NY} = e^a \wedge T_a. \]

The N–Y form is a kind of Chern–Simons form and will have its application to manifolds with boundaries and reflect the role of torsion in geometry.

After variation to \( \omega_{ab} \) and \( e^a \), two vacuum equations can be obtained from the MB action (2.3)

\[ T^a = \frac{p}{2} \epsilon^{a_k} e^b \wedge e^c, \]

(2.5) \[ R^a = \frac{q}{2} \epsilon^{a_k} e^b \wedge e^c, \]

(2.6) with the two constant coefficients \( p, q \) defined by \( p = \frac{\alpha_3 + \alpha_4 \chi}{\alpha_4 - \chi} \) and \( q = -\frac{\alpha_3^2 + \chi}{\alpha_4 - \chi} \).

The curvatures in Einstein–Cartan geometry can be connected to their counterparts in Riemannian geometry. In particular, in 3D spacetime, the equations above can be simplified to equations without torsion

\[ \tilde{R}^a = \frac{\Lambda_{\text{eff}}}{2} \epsilon^{a_k} e^b \wedge e^c, \]

(2.7) where \( \tilde{R}^a \) is the curvature without torsion and \( \Lambda_{\text{eff}} = q - \frac{1}{p} \) is the effective cosmological constant. One can let \( \Lambda_{\text{eff}} = -\frac{1}{p} < 0 \) to construct an asymptotically anti-de Sitter space.

As in the 3D Einstein equation, equation (2.7) has the well-known BTZ solution. But in this case, torsion is contained in the gravitational action. The metric is

\[ ds^2 = -N(r)^2 \, dt^2 + \frac{1}{N(r)^2} \, dr^2 + r^2 (d\phi + N_\phi \, dt)^2 \]

(2.8) where

\[ N(r)^2 = \frac{r^2}{l^2} - M_0 + \frac{J_0^2}{4r^2}, \quad N_\phi(r) = \frac{J_0}{2r}. \]

(2.9) Here, we have considered \( 8G = 1 \). This metric is the same as the one in GR, except that \( l = 1/\sqrt{-\Lambda_{\text{eff}}} \) here and a constant torsion [21]. For this metric, there are two horizons: the outer one \( r_+ \) and the inner one \( r_- \). From \( N^2(r) = 0 \), one can obtain the expressions of both horizons:

\[ r_{\pm}^2 = \frac{M_0 l^2}{2} (1 \pm \Delta), \quad \Delta = \left| 1 - (J_0/M_0 l)^2 \right|^{1/2}. \]

(2.10) Conversely, \( M_0 \) and \( J_0 \) can be expressed as follows:

\[ M_0 = \frac{r_+^2 + r_-^2}{l^2}, \quad J_0 = \frac{2r_+ r_-}{l}. \]

(2.11)
Hawking radiation is just a kinematic effect, which only depends on the event horizon and is irrelevant to the dynamical equations and the gravitational theories. Therefore, the temperature of BTZ black hole in the MB model has the similar form as in GR, which is
\[ T = \frac{r_+^2 - r_-^2}{2\pi F r_+}. \] (2.12)
Certainly because of the existence of \( l \), the temperature is relevant to the coefficients \( \alpha_3, \alpha_4, \Lambda \) of MB Lagrangian. Define
\[ \Omega_H = \frac{g_{ij}}{g_{\phi\phi}} \bigg|_{r_+} = \frac{J_0}{2r_+^2}, \] (2.13)
which can be regarded as the angular velocity of BTZ black hole.

Because of the existence of the topological terms, the asymptotically behavior is different from the one for Einstein–Cartan theory. Blagojevic \textit{et al} have proved that the gravitational conserved charges in the MB model should be \[ M = M_0 + 2\pi \alpha_3 \left( \frac{p M_0}{2} - \frac{J_0}{l} \right) = aM_0 - \frac{b}{l} J_0, \quad J = J_0 + 2\pi \alpha_3 \left( \frac{p J_0}{2} - M_0 \right) = aJ_0 - bM_0, \] (2.14)
where we have defined \( a = 1 + \pi \alpha_3 p, b = 2\pi \alpha_3 \). Obviously, when \( \alpha_3 = 0 \), they will return to their conventional interpretation as energy and angular momentum, as with the BTZ metric in general relativity.

Correspondingly, the entropy can be derived:
\[ S = 4\pi r_+ + 4\pi^2 \alpha_3 \left( pr_+ - \frac{2r_+}{l} \right) = 4\pi \left( ar_+ - \frac{b}{l} r_- \right). \] (2.15)
It differs from the Bekenstein–Hawking result by an additional term and will coincide with Solodukhin’s result if \( \alpha_3^p = 0 \) \[ 31\]. Black hole entropy is not always equated with one quarter of the event horizon area. In fact, it is related to the gravitational theory under consideration. It can be easily verified that in the MB model, the entropy, temperature and the conserved charges not only satisfy the first law of thermodynamics
\[ dM = T \, dS + \Omega_H \, dJ, \] (2.16)
but also fulfil the Smarr-like formula
\[ M = \frac{1}{4} TS + \Omega_H J. \] (2.17)
The further implicates that with torsion, the BTZ black hole can still be treated as a thermodynamic system and the thermodynamic laws still hold. It should be noted that in the expression of the entropy of the BTZ black hole, no torsion exists explicitly, only \( \alpha_4 \) in \( p \) implicitly. In particular, when \( \alpha_3 = 0 \), the entropy in equation (2.15) returns to the usual BTZ black hole entropy. It means that the N–Y term \( \alpha_4 e^\theta \wedge T_\theta \) influences the conserved charges and the exact form of entropy only when the CS term exists.

3. Phase transition in extended phase space

Ehrenfest had attempted to classify the phase transitions. Phase transitions connected with an entropy discontinuity are called discontinuous or first-order phase transitions, and phase transitions where the entropy is continuous are called continuous or second-order/higher order phase transitions. More precisely, for the first-order phase transition, the Gibbs free energy \( G(T, P, \ldots) \) should be continuous and its first derivative with respect to the external fields
\[ S = -\frac{\partial G}{\partial T} \bigg|_{(P, \ldots)}, \quad V = \frac{\partial G}{\partial P} \bigg|_{(T, \ldots)} \] (3.1)
is discontinuous at the phase transition points.
For the second-order phase transition, the Gibbs free energy \( G(T, P, \ldots) \) and its first derivative are both continuous, but the second derivative of \( G \) will diverge at the phase transition points like the specific heat \( C_p \), the compressibility \( \kappa \), the expansion coefficient \( \alpha \):

\[
C_p = T \frac{\partial S}{\partial T} \bigg|_P = -T \frac{\partial^2 G}{\partial T^2} \bigg|_P, \quad \kappa = -\frac{1}{V} \frac{\partial V}{\partial P} \bigg|_T = -\frac{1}{V} \frac{\partial^2 G}{\partial P \partial T} \bigg|_T, \quad \alpha = -\frac{1}{V} \frac{\partial V}{\partial T} \bigg|_P = -\frac{1}{V} \frac{\partial^2 G}{\partial P^2} \bigg|_P.
\]

(3.2)

In this sense, because the heat capacity is always positive, there is no second-order phase transition for BTZ black hole obtained in GR. This property of BTZ black hole can also be verified by the method of thermodynamic curvature [32, 33].

To utilize Ehrenfest’s classification, we consider variable cosmological constant and relate it to the pressure [10, 11, 14, 15, 34, 35]. The first law of thermodynamics for the BTZT black hole should be

\[
dM = T \, dS + \Omega \, dJ + V \, dP,
\]

(3.3)

where \( P = \frac{\Omega}{\kappa \pi} \) and \( V = \frac{\partial M}{\partial \Omega} \bigg|_{S, J} \) is the corresponding thermodynamic volume. Therefore, the mass of black hole is no more internal energy, but should be interpreted as the thermodynamic enthalpy, namely \( H = M(S, P, J) \) [10, 11, 34, 35]. The first law of black hole thermodynamics represented by the internal energy \( U(S, V, J) \) reads

\[
dU = T \, dS + \Omega \, dJ - P \, dV.
\]

(3.4)

where \( U = H - PV \).

For BTZT black hole, one can express mass \( M \) as functions of \( S, J, P \), which are

\[
H_{\pm} = M_{\pm} = \frac{1}{8\pi^2 b^2} \left[ aS^2 + 8\pi^2 abJ \pm S\sqrt{(a^2 - 8\pi b^2 P)(S^2 + 16\pi^2 bJ)} \right].
\]

(3.5)

One can substitute the expressions of \( S, M, J \) into equation (3.5) to test and verify it directly. This result can also be verified easily by differentiating the mass \( M_{\pm} \) with the entropy \( S \) to obtain the Hawking temperature (2.12).

It should be noted that when \( M \) is expressed with \( r_+, r_-, l \), its form is unique. When we express it with the thermodynamic quantities, two different forms \( M_{\pm} \) appear. In fact, \( M_{\pm} \) depend on the relation between \( al \) and \( b \). They are established under different conditions

- \( b^2 \leq a^2 l^2 \) or \( b^2 < \frac{a^2 l^2}{8\pi^2 P} \), the expression \( M_- \) is right. At this time, to keep the expression in square root has physical meaning, \( S^2 + 16\pi^2 bJ \geq 0 \) should also be satisfied.
- \( b^2 \geq a^2 l^2 \) or \( b^2 \geq \frac{a^2 l^2}{8\pi^2 P} \), the expression \( M_+ \) should be used. Similarly, to keep the expression in square root has physical meaning, \( S^2 + 16\pi^2 bJ \leq 0 \) should be satisfied.

According to equations (2.14) and (2.15), when \( |b| \geq |al| \), \( M, J, S \) may be negative. In fact, for gravities with higher derivative terms, there is the possibility for negative entropy and energy which depend on the parameters of higher derivative terms [36]. Although black holes behave as thermodynamic systems, they also show some exotic behaviors, the most known one is that the entropy of black holes is proportional to area and not the volume. Therefore, it is understandable if black holes exhibit some strange thermodynamic properties. Below, we will show that the condition \( |b| \geq |al| \) is the key for the BTZT black hole to have phase transition.

The temperature of BTZT black hole can be evaluated according to \( S, P, J \):

\[
T_{-}(S, J, P) = \left. \frac{\partial M_-}{\partial S} \right|_{P, J} \quad \text{or} \quad T_{+}(S, J, P) = \left. \frac{\partial M_+}{\partial S} \right|_{P, J}.
\]

(3.6)

They look different when expressed with the thermodynamic quantities \( S, P, J \) because they correspond to different conditions for the parameters \( a, b, l \). When replacing the
thermodynamic variables \((S, J, P)\) with the geometric ones \((r_+, r_-, l)\), the two expressions can be unified and equation (2.12) can turn up.

According to equations (3.5) and (3.6), the specific heats at constant pressure and constant angular momentum can be calculated easily:

\[
C_- = \frac{\partial M_-}{\partial T} \bigg|_{P, J} = T_- \frac{1}{\frac{T_-}{S^-}} \left|_{P, J} \right., \quad C_+ = \frac{\partial M_+}{\partial T} \bigg|_{P, J} = T_+ \frac{1}{\frac{T_+}{S^+}} \left|_{P, J} \right. \tag{3.7}
\]

With the geometric quantities, the heat capacity can be written as

\[
C_P = C_\pm = \frac{4\pi r_+^2 (r_+^2 - r_-^2) (b^2 - a^2 l^2)}{l(b r_+ (3r_+^2 + r_-^2) - a l r_+ (r_+^2 + 3r_-^2))}. \tag{3.8}
\]

It is more appropriate to study the phase transition of the BTZT black holes according to geometric quantities \(r_+, r_-, l\). Because the conditions \(r_+ \geq r_-, M_0 > 0, J_0 > 0\) must be fulfilled. If employing the thermodynamic quantities completely, one may omit these conditions. Below, we will set \(l = 1\) and analyze the phase transition of the BTZT black hole numerically. According to Ehrenfest’s classification, we should first derive the Gibbs free energy \(G\):

\[
G = H - TS = M - TS. \tag{3.9}
\]

3.1. \(a^2 l^2 \geq b^2\)

One can easily plot the \(G_- - T, S - T, C_- - T\) curves as shown in figure 1. Obviously \(G_-, S, C_-\) are all continuous function of temperature \(T\). No turning point and divergence turn up, which means no second-order phase transition happens in this case. This conclusion can also be supported by the method of thermodynamic curvature [33, 37].

3.2. \(a^2 l^2 \leq b^2\)

In this case, we first plot the \(C_+ - r_+\) curves for different values of \(a, b\). In figure 2(a) and (b), we show that there is no divergent point when \(a, b\) take opposite signs. The phase transition may happen only when \(a, b\) are both positive and negative. Under the given conditions, one can easily derive the position of the divergent point, \(r_c \approx 5.522\). Obviously, the phase transition is
a smaller mass/larger mass black hole phase transition. When $a > 0$, $b > 0$, the smaller black hole is stable because the heat capacity is positive, when $a < 0$, $b < 0$, the other way around, the larger black hole is stable.

One should further analyze the value of $M, J, S$. We only consider the two cases which have the phase transitions. In figure 3, it is shown that under the condition $a^2l^2 \leq b^2$ when $a, b$ are both positive, the corrected mass $M$ and the entropy $S$ are positive at the divergent point $r_c$, while $J$ is negative; for the case with negative $a, b$, the situation is just the opposite. Thus, we conclude that for the BTZT black hole if the phase transition can happen, the $M, J, S$ cannot be all positive.

Now, we investigate the type of the phase transition for the BTZT black hole according to Ehrenfest’s classification. To calculate the isothermal compressibility $\kappa$ and the expansion coefficient $\alpha$, we should first obtain the thermodynamic volume:
Although $M_+$ and $M_-$ are different, when considering the conditions for $M_\pm$ one can find that
\begin{equation}
\frac{\partial V}{\partial P} \bigg|_{S,J} = \frac{\partial M_+}{\partial P} \bigg|_{S,J} = \frac{\partial M_-}{\partial P} \bigg|_{S,J}.
\end{equation}
(3.10)

Inversely, one can derive the pressure
\begin{equation}
P = \frac{-16\pi^2bJS^2 - S^4 + 4\pi^2a^2V^2}{32\pi^3b^2V^2}.
\end{equation}
(3.12)

According to the above equations, one can derive the isothermal compressibility $\kappa$ and the expansion coefficient $\alpha$. We can plot the curves of $G - T$, $S - T$, $C - T$, $\kappa - T$ and $\alpha - T$ in figure 4. Similarly, only when the parameters $a$, $b$ have the same sign, the phase transition for the BTZT black hole can turn up. The critical temperature lies at $T_c = 0.85$. As shown in the figure, the Gibbs free energy and the entropy are continuous functions of temperature, while the heat capacity, the isothermal compressibility and the expansion coefficient all diverge at the critical point. Therefore, the phase transition at this critical point is the second-order phase transition or continuous one.

According to equations (3.11) and (3.12) and $U = H - PV$, we can obtain the internal energy
\begin{equation}
U = U(S, V, J) = -\frac{(S^2 - 2\pi aV)(-2\pi aV + 16\pi^2bJ + S^2)}{32\pi^3b^2V}.
\end{equation}
(3.15)
Figure 4. The Gibbs free energy, entropy, the heat capacity at constant pressure and the isothermal compressibility and the expansion coefficient as functions of temperature for BTZT black hole for the choices of $l = 1$, $r_+ = 1$, $a = \pm 1$, $b = \pm 2$ and $r_+ \geq r_-$. For the two cases, $a = 1$, $b = 2$ and $a = -1$, $b = -2$, the $\kappa - T$ and the $\alpha - T$ curves are both concurrent, respectively.

from which one can easily derive the temperature as functions of $S, V, J$:

$$T(S, V, J) = \frac{S(2a\pi V - 8\pi^2 bJ - S^2)}{8b^2\pi^3 V}. \quad (3.16)$$
From equation (3.12) and (3.16), one can derive the equation of state between the pressure $P$, the temperature $T$ and the volume $V$ by eliminating $S$. Thus, one can obtain the pressure $P$ as function of $V, T, J$ in principle. But the expression is too lengthy and obviously it will depend on the value of $J$. Below we will analyze the $P-V$ relation by means of the static scaling law [38]. The dimensional analysis implies that the $P$ and $T$ are both homogeneous functions of the variables $S, V, J$, since $P \rightarrow \lambda P$, $T \rightarrow \lambda T$ when $V \rightarrow \lambda^2 V$, $S \rightarrow \lambda S, J \rightarrow \lambda^2 J$. Thus, $P$ and $T$ are in fact the functions of two independent variables. The same logic also applies to the internal energy $U$. So, we can take advantage of the scaling character to redefine the functions and the variables. One can take

$$t = \frac{T}{S}, \quad p = \frac{V}{S^2}, \quad j = \frac{J}{S^2}. \quad (3.17)$$

In this way, the entropy $S$ can be eliminated in equations (3.12) and (3.16) and they are simplified to be

$$t = \frac{2\pi v - 8\pi^2 \beta j - 1}{8\pi^2 \beta^2 v}, \quad p = -\frac{16\pi^2 \beta j + 1 - 4\pi^2 v^2}{32\pi^2 \beta^2 v^2}. \quad (3.18)$$

Furthermore, removing the $j$ and combing the two equations together, one can obtain

$$p = \frac{t}{2v} + \frac{1}{8b^2 \pi^2} - \frac{1}{8b^2 \pi^2 v} + \frac{1}{32b^2 \pi^2 v^2}. \quad (3.19)$$

The great advantage of the above relation lies at it is irrelevant to $J$ and is more universal. Although the $p-v$ structure is not the same as the $P-V$ structure, it can reflect some properties of the system. The critical point occurs at the point where

$$\frac{\partial p}{\partial v} = 0, \quad \frac{\partial^2 p}{\partial v^2} = 0. \quad (3.20)$$

But unfortunately, the above equations do not have solutions. One can also easily plot the $p-v$ curves of equation (3.19) at different temperatures. We can draw conclusions that for the BTZT black hole, there is no the similar phase structure and critical behavior to the van der Waals liquid–gas system.

4. Critical behaviors in non-extended phase space

In the non-extended phase space, the $l$ or $\Lambda$ should be considered as constant. Thus, equation (3.5) is modified to

$$M_{\pm} = \frac{1}{8\pi^2 \beta} \left[ aS^2 + 8\pi^2 abJ \pm \frac{S}{l} \sqrt{(a^2 l^2 - b^2)(S^2 + 16\pi^2 b J)} \right]. \quad (4.1)$$

When considering the $J-\Omega$ relations, some rotating black holes such as Kerr-AdS black hole will exhibit similar critical behavior to the van der Waals liquid–gas system [39, 40]. Thus, we analyze for the BTZT black hole whether there is similar conclusion. The $J-\Omega$ relation can be easily obtained:

$$J = -\frac{4\pi^2 T^2 (-2a\Omega^2 + b l^2 \Omega^2 + b)}{(l^2 \Omega^2 - 1)^2}. \quad (4.2)$$

One may try to derive the critical point according to

$$\left. \frac{\partial J}{\partial \Omega} \right|_T = 0, \quad \left. \frac{\partial^2 J}{\partial \Omega^2} \right|_T = 0. \quad (4.3)$$
But it can be easily verified that also no solution exists. In this case, the heat capacity $C$ is the same as equation (3.8). However, the isothermal compressibility should be defined as [40]

$$\kappa_T = \frac{\partial \Omega}{\partial J} \bigg|_T = -\frac{(l^2\Omega^2 - 1)^3}{8\pi^2 l^2 T^2 (3a^2\Omega^2 + a - b^2\Omega^3 - 3b\Omega)}.$$  (4.4)

We plot the $\kappa_T - T$ curves as in figure 5. In this case, the critical temperature still lies at $T_c = 0.85$. In order to see the thermodynamic behavior near the critical point, the critical exponents can be introduced as

$$J - J_c \sim |\Omega - \Omega_c|^{\delta}, \quad \Omega - \Omega_c \sim |T - T_c|^{\beta}$$
$$C_J \sim |T - T_c|^{-\alpha}, \quad \kappa_T \sim |T - T_c|^{-\gamma}.$$  (4.5)

From equation (4.2), at the critical point $T = T_c$, the first derivative of $J$ over $\Omega$ satisfies

$$\left. \frac{\partial J}{\partial \Omega} \right|_{T_c} = 0.$$  (4.6)

But the second derivative can be calculated as

$$\left. \frac{\partial^2 J}{\partial \Omega^2} \right|_{T_c} = \pm 322.187 \neq 0$$  (4.7)

for the two cases $a = 1, b = 2, l = 1$ and $a = -1, b = -2, l = 1$. Thus,

$$J - J_c = \left. \frac{\partial^2 J}{\partial \Omega^2} \right|_{T_c} (\Omega - \Omega_c)^2 + O((\Omega - \Omega_c)^3)$$  (4.8)

which means $\delta = 2$.

According to equation (3.8)

$$C_J = \frac{4\pi r_c^2 (b^2 - a^2l^2)(-2a l r_c \sqrt{a^2l^2r_c^3 - b^2r_c^3 - b Jl^2 + 2 a l r_c} + 2b^2 r_c^2 + b J l^2)}{l (2a^2 l r_c^3 + 2a^2 l r_c^3 + \sqrt{a^2l^2r_c^3 - b^2r_c^3 - b Jl^2 + b J l^2 - 2b J l^2 - b l^2 r_c^3})}.$$  (4.9)

One can set

$$T = T_c (1 + \epsilon), \quad r_\pm = r_c (1 + \Delta),$$  (4.10)
where $|\epsilon|, |\Delta| \ll 1$. Because the entropy $S$ and the temperature $T$ can both be expressed as $S = S(r_+, J), T = T(r_+, J)$, and

$$C_J = T \frac{\partial S}{\partial T} |_{r_+, J} = \frac{\partial S}{\partial r_+} |_{r_+, J}. \quad (4.11)$$

According to figure 2, at the critical point $r = r_c$

$$\left( \frac{\partial T}{\partial r_+} \right) |_{r=r_c} = 0. \quad (4.12)$$

Moreover one can easily verify $\left( \frac{\partial^2 T}{\partial r_+^2} \right) |_{r=r_c} \neq 0$. Therefore, in a sufficiently small neighborhood of $r_c$, one can expand $T$ in terms of $r_+$ as

$$T(r_+) = T(r_c) + \frac{1}{2} \left( \frac{\partial^2 T}{\partial r_+^2} \right) |_{r=r_c} r_+^2 \Delta^2 + \mathcal{O}(\Delta^3), \quad (4.13)$$

from which we obtain

$$\Delta = \frac{\epsilon}{D^{1/2}}, \quad (4.14)$$

where

$$D = \frac{r_c^2}{2T_c} \left( \frac{\partial^2 T}{\partial r_+^2} \right) |_{r=r_c}. \quad (4.15)$$

Substitute equation (4.10) into equation (4.9), we can derive that the critical behavior of $C_J$ is described by

$$C_J \approx \frac{A}{\epsilon^{1/2}}, \quad (4.16)$$

where $A$ is a function of $a, b, l, J_c, D$ and very complicated. Here, we do not give the detailed expression. Comparing equation (4.16) with equation (4.5), one can find that $\alpha = 1/2$.

To calculate $\beta$, we first derive the $\Omega$ as function of $r_+, J$

$$\Omega(r_+, J) = \frac{\sqrt{a^2r_+^2 - b^2r_+^2 +BJ^2 + alr_+}}{br_+}. \quad (4.17)$$

For fixed $a, b, l$ and the critical $J_c$,

$$\Omega(r_+, J) = \Omega(r_c, J_c) + \left( \frac{\partial \Omega}{\partial r_+} \right) |_{r=r_c} (r_+ - r_c) + \text{higher order terms}. \quad (4.18)$$

Ignoring the higher order terms, we finally obtain

$$\Omega(r_+, J) - \Omega(r_c, J_c) = \frac{J_c}{r_c^2 \sqrt{a^2r_+^2 - b^2r_+^2 - BJ\Delta^2 - 1/2D^{1/2}}|T - T_c|^{1/2}}. \quad (4.19)$$

Therefore $\beta = 1/2$.

Following the previous approach, one can express the $\kappa_T$ as function of $r_+, J$. Utilizing equation (4.10) we can obtain

$$\kappa_T \approx \frac{B}{\Delta} = \frac{BD^{1/2}}{\epsilon^{1/2}} = \frac{BD^{1/2}/\epsilon^{1/2}}{|T - T_c|^{1/2}}. \quad (4.20)$$

which means $\gamma = 1/2$. Therefore, the critical exponents $\alpha, \beta, \gamma, \delta$ have the same values as the ones obtained in the Hořava-Lifshitz black hole and the Born–Infeld black hole [41, 42]. Obviously, they obey the scaling symmetry like the ordinary thermodynamic systems

$$\alpha + 2\beta + \gamma = 2, \quad \alpha + \beta(\delta + 1) = 2, \quad \gamma(\delta + 1) = (2 - \alpha)(\delta - 1), \quad \gamma = \beta(\delta - 1). \quad (4.21)$$
5. Discussion and conclusion

In this paper, we adopted Ehrenfest’s classification to study the phase transition of the BTZ black hole with torsion obtained in the MB topological gravitational model. Although the gravitational action contains torsion, the metric part of the BTZ black hole with torsion looks like the usual BTZ solution. Because of the existence of the Chern–Simons term and the Nieh–Yan term, the conserved charges for the BTZ black hole should be modified. Inclusion of these topological terms makes the thermodynamic properties and critical behaviors of BTZ black hole with torsion very different from the ones of the usual BTZ black hole obtained in GR.

By treating the effective cosmological constant as a thermodynamic pressure, in the extended phase space, we completely followed the standard of Ehrenfest to explore the type of the phase transition of the BTZ black hole with torsion. It is shown that when \(|a| \leq |b|\), the Gibbs free energy and entropy are continuous functions of temperature; however, the heat capacity \(C_P\), the isothermal compressibility \(\kappa\) and the expansion coefficient \(\alpha\) are all divergent at the critical point. This means this kind of phase transition for the BTZ black hole with torsion is continuous or second order. Nevertheless, the phase transition and critical behavior are different from the ones in the van der Waals liquid/gas system. Because \(a, b\) here are related to the parameters \(\alpha_3, \alpha_4\) in the action of MB model. Thus, whether phase transition can happen depends not only upon the black hole solutions, but also upon the gravitational actions.

Moreover, we also considered the non-extended phase space. In this case, no direct thermodynamic analogy for the isothermal compressibility exists. Thus, we employed another form and named it \(\kappa_T\). It is shown that the \(\kappa_T\) also diverge at the same critical point. Therefore, in the non-extended phase space, the phase transition is also the second order. The critical exponents for the BTZ black hole are also calculated, which are the same as the ones obtained in the Hořava-Lifshitz black hole and the Born–Infeld black hole. Is this just a coincidence, or is there some inherent reason, still need consideration further.

Although we discussed the 3D topological model with torsion, the results have included the torsion-free case which corresponds to the topologically massive gravity (TMG) [43]. For the TMG, the field equations of which are also solved by the BTZ metric (CS-BTZ solution). The conserved charges and the entropy are modified to be [31, 44, 45]

\[
M = M_0 - \frac{\beta}{L^2} J_0, \quad J = J_0 - \beta M_0, \quad S = 4\pi \left( r_+ - \frac{\beta}{L} r_+ \right). \tag{5.1}
\]

Here, \(\beta\) is the Chern–Simons coupling constant and \(L\) is the usual cosmological radius. Obviously, the phase transition and the critical behaviors for the BTZ black hole in the TMG correspond to the \(a = 1\) case of the BTZT black hole in the MB model. Therefore, when \(|\beta| > L\), phase transition also exists in the CS-BTZ black hole. Similarly, in this time, the mass, angular momentum and the entropy cannot all be positive.

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