Uniqueness and Non-Uniqueness of Static Vacuum Black Holes in Higher Dimensions

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We prove the uniqueness theorem for asymptotically flat static vacuum black hole solutions in higher-dimensional space-times. We also construct infinitely many non-asymptotically flat regular static black holes on the same space-time manifold with the same spherical topology.

§1. Introduction

With the development of string theory, black holes in higher-dimensional space-times have come to play a fundamental role in physics.1 Furthermore, the possibility of black hole production in high energy experiments has recently been suggested in the context of the so-called TeV gravity.2 A TeV-size black hole in TeV gravity is small enough to be well approximated by an asymptotically flat black hole in higher dimensions. To predict phenomenological results, we need reliable knowledge about higher-dimensional black holes. However, some essential features of black hole theory have not so far been fully explored. Among these, the equilibrium problem for black holes is one of most important issues. The final equilibrium state of the black hole is known to drastically simplify in the case of four space-time dimensions, because of the uniqueness properties of static or stationary black hole solutions. The uniqueness theorem for the vacuum black hole is well established in four-dimensional space-times.3–5 (See also Ref. 6 for comprehensive review.) Although this no hair property is fundamental to the nature of black holes, it is at the same time a quite non-trivial result derived from the Einstein equations. So far, there is no evidence that in higher dimensions the final state of the black hole is unique. Remarkably, five-dimensional stationary vacuum black holes are not unique; there is a Myers-Perry solution,1 which is a generalization of the Kerr solution to arbitrary dimensions, while Emparan and Reall7 have recently found five-dimensional rotating black ring solutions with the same angular momenta and mass but with an event horizon homeomorphic to $S^2 \times S^1$. In the static case, such a counter-example has not yet been presented. The only known asymptotically flat static vacuum black hole is the $n$-dimensional hyperspherically symmetric Schwarzschild-Tangherlini solution.8 We
shall show in what follows that there are no others.\footnote{After submitting this manuscript to gr-qc, we were informed by M. Anderson that a similar proof has been given by Hwang.\footnote{9}} Our proof can be extended to charged dilatonic cases.\footnote{10} However, it is interesting to note that, as we shall expand upon below, if one drops the condition of asymptotic flatness but still insists that the space-time have the same topology as the Schwarzschild-Tangheelerlini solution, then the uniqueness property fails badly: There exist discrete infinities of solutions.

In four dimensions, there are essentially two ways of proving the uniqueness of the Schwarzschild solution. The first is Israel’s original proof,\footnote{3} which cannot be generalized to higher dimensions in a simple manner. For example, it uses the Gauss-Bonnet theorem to evaluate the surface integral of the Ricci curvature of each level set of the length of the time-like Killing vector field. This can be done only in the case that the level set is two dimensional. Another proof was given by Bunting and Masood-ul-Alam.\footnote{4} This method uses a corollary of the positive mass theorem to establish that the domain of outer communication is spatially conformally flat. Then an identity involving the Bach-Weyl tensor of the spatial geometry is used to prove that the conformal flatness of the spatial geometry implies spherical symmetry. In higher dimensions, this argument does not hold, since the Bach-Weyl tensor is no longer a conformal tensor.

\section{Proof of uniqueness}

The first half of our reasoning is parallel to Bunting-Masood-ul-Alam’s method, and therefore we shall only briefly describe this part here. In general, the metric of an \(n\)-dimensional static space-time has the form

\[
ds^2 = -V^2 dt^2 + g_{ij} dx^i dx^j, \tag{2.1}\]

where \(V\) and \(g_{ij}\) are independent of \(t\) and are regarded as quantities on the \(t = \text{[constant]}\) hypersurface \(\Sigma\). The event horizon \(H\) is a Killing horizon located at the level set \(V = 0\), which is assumed to be non-degenerate. In fact non-degeneracy follows from Smarr’s formula relating the mass, surface gravity and area of the horizon. We impose the vacuum Einstein equations,

\[
D_i D^i V = 0 \tag{2.2}
\]

and

\[
^{(n-1)}R_{ij} - \frac{1}{V} D_i D_j V = 0, \tag{2.3}
\]

where \(D_i\) and \(^{(n-1)}R_{ij}\) denote a covariant derivative and the Ricci tensor defined on \((\Sigma, g_{ij})\), respectively.

In asymptotically flat space-times, one can find an appropriate coordinate system in which the metric has an asymptotic expansion of the form

\[
V = 1 - \frac{C}{r^{n-3}} + O(1/r^{n-2}) \tag{2.4}
\]
and

\[ g_{ij} = \left( 1 + \frac{2}{n - 3} \frac{C}{r^{n-3}} \right) \delta_{ij} + O(1/r^{n-2}), \quad (2.5) \]

where \( C = \text{[constant]} \) represents the ADM mass (up to a constant factor) and \( r := \sqrt{\sum_i (x^i)^2} \).

Consider the two conformal transformations

\[ \tilde{g}_{ij} = \Omega_{\pm}^2 g_{ij}, \quad (2.6) \]

where

\[ \Omega_{\pm} = \left( \frac{1 \pm V}{2} \right)^{2/(n-3)}. \quad (2.7) \]

Then, we have two manifolds \((\Sigma^\pm, g_{ij})\). On \( \Sigma^+ \), the asymptotic behavior of the metric becomes

\[ \tilde{g}_{ij}^+ = \delta_{ij} + O \left( \frac{1}{r^{n-2}} \right). \quad (2.8) \]

On \( \Sigma^- \), we have

\[ \tilde{g}_{ij}^- dx^i dx^j = \frac{(C/2)^{4/(n-3)}}{r^4} \left( dr^2 + r^2 d\Omega_{n-2}^2 \right) + O(1/r^5) \]

\[ = (C/2)^{4/(n-3)} \left( dR^2 + R^2 d\Omega_{n-2}^2 \right) + O(R^5), \quad (2.9) \]

where \( d\Omega_{n-2}^2 \) denotes the round sphere metric and \( R := 1/r \) has been defined.

Pasting \((\Sigma^\pm, g_{ij}^\pm)\) across the level set \( V = 0 \) and adding a point \( \{p\} \) at \( R = 0 \), we can construct a complete regular surface \( \tilde{\Sigma} = \Sigma^+ \cup \Sigma^- \cup \{p\} \). It can be shown that the Ricci curvature on \( \tilde{\Sigma} \) vanishes. Furthermore, Eq. (2.8) implies that the total mass also vanishes on \( \tilde{\Sigma} \). As a consequence of the positive mass theorem, \(^{13},^{14}\) such a surface \( \tilde{\Sigma} \) must be flat. In other words, \( g_{ij} \) can be written in the conformally flat form

\[ g_{ij} = v^{4/(n-3)} \delta_{ij}, \quad (2.10) \]

\[ v := \frac{2}{1 + V}. \quad (2.11) \]

Then, the Einstein equation (2.2) reduces to the Laplace equation on the Euclidean \((n-1)\)-space

\[ \nabla_0^2 v = 0, \quad (2.12) \]

where \( \nabla_0 \) denotes the flat connection.

On the other hand, since the function \( V \) is a harmonic function, it can be used as a local coordinate in a neighbourhood \( U \subset \Sigma \) of each connected component of the horizon. Let \( \{x^A\} \) be coordinates on level sets of \( V \) such that their trajectories are orthogonal to each level set. Then, the metric on \( \Sigma \) can be written in the form

\[ g_{ij} dx^i dx^j = \rho^2 dV^2 + h_{AB} dx^A dx^B. \quad (2.13) \]
From Eq. (2.3) and the conformal flatness of $\Sigma$, the Riemann tensor on $\Sigma$ becomes
\[
^{(n-1)}R_{ijkl} = \frac{2}{n-3} \frac{1}{V} \left( g_{i[k}D_{l]}D_jV - g_{j[k}D_{l]}D_iV \right). \tag{2.14}
\]
The $n$-dimensional Riemann invariant can be calculated using the formulae
\[
^{(n)}R_{ijkl} = \frac{1}{n-1} R_{ijkl}, \tag{2.15}
\]
\[
^{(n)}R_{0i0j} = V D_i D_jV, \tag{2.16}
\]
\[
D_i D_jV = \frac{1}{\rho} k_{ij} - \frac{2}{\rho^2} n_{(i} D_{j)} \rho + \frac{\partial V \rho}{\rho^2} n_i n_j, \tag{2.17}
\]
where $n = \rho dV$ and $k_{AB}$ denote the unit normal and the extrinsic curvature of the $V = \text{[constant]}$ surface, respectively. Thus, we obtain
\[
^{(n)}R_{\mu\nu\lambda\rho}^{(n)}R_{\mu\nu\lambda\rho} = \frac{4(n-2)}{(n-3)V^2} (D_i D_jV)(D^i D^j V) = \frac{4(n-2)}{(n-3)V^2 \rho^2} \left[ k_{AB} k^{AB} + k^2 + 2D_A \rho D^A \rho \right], \tag{2.18}
\]
where $D_A$ denotes the covariant derivative on each level set of $V$.

The requirement that the event horizon $H$ be a regular surface leads to the condition
\[
k_{AB}|_H = 0, \tag{2.19}
\]
\[
D_A \rho |_H = 0. \tag{2.20}
\]
In particular, $H$ is a totally geodesic surface in $\Sigma$.

Let us consider how the event horizon is embedded into the base space $(\tilde{\Sigma}, \delta_{ij})$. We can adopt the following expression for the base space metric:
\[
\delta_{ij} dx^i dx^j = \rho^2 dv^2 + \tilde{h}_{AB} dx^A dx^B. \tag{2.21}
\]
The event horizon is located at the $v = 2$ surface, $\tilde{H}$. Then, the extrinsic curvature $\tilde{k}_{AB}$ of the level set $v = \text{[constant]}$ can be expressed as
\[
\tilde{k}_{AB} = v^{-2/(n-3)} k_{AB} + \frac{2}{n-3} \frac{v^{(n-1)/(n-3)}}{\rho} \tilde{h}_{AB}. \tag{2.22}
\]
Thus we obtain
\[
\tilde{k}_{AB} = \frac{2^{2(n-4)/(n-3)}}{(n-3)\rho_H^{n-3}} \tilde{h}_{AB}, \tag{2.23}
\]
on $\tilde{H}$. In other words, the embedding of $\tilde{H}$ into the Euclidean $(n-1)$-space is totally umbilical. It is known that such an embedding must be hyperspherical; that is, each connected component of $\tilde{H}$ is a geometric sphere with a certain radius $r_0$. 

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determined by the value of $\rho|_{\bar{H}}$. The embedding of a hypersphere into the Euclidean space is known to be rigid,\(^\text{12}\) which means that we can always locate one connected component of $\bar{H}$ at the $r = r_0$ surface of $\bar{\Sigma}$ without loss of generality.

Now we have a boundary value problem for the Laplace equation on the base space $\Omega := E^{n-1} \setminus B^{n-1}$ with the Dirichlet boundary conditions. The system is characterized by a parameter $\rho|_{\bar{H}}$, which fixes the radius of the inner boundary $\bar{H} = \partial B^{n-1}$. The field equation is the Laplace equation (2.12) with the Dirichlet condition

$$v|_{\bar{H}} = 2$$

and the asymptotic decay condition

$$v = 1 + O(1/r^{n-3}),$$

with $r \to +\infty$. Let $v_1$ and $v_2$ be solutions of this boundary value problem. Integration of the Green identity

$$\nabla_0^2 \left[ \frac{1}{2} (v_1 - v_2)^2 \right] = \nabla_0 (v_1 - v_2) \cdot \nabla_0 (v_1 - v_2) + (v_1 - v_2) \nabla_0^2 (v_1 - v_2)$$

over $\Omega$ gives

$$\int_{r=+\infty}^{r} (v_1 - v_2) \frac{\partial}{\partial r} (v_1 - v_2) dS = \int_{\Omega} |\nabla_0 (v_1 - v_2)|^2 dV. \quad (2.27)$$

Since the surface integral vanishes due to the boundary condition imposed, the integrand in the volume integral must be identically zero: $v_1 - v_2 = \text{[constant]}$. Then, from the boundary condition, two solutions must be identical ($v_1 = v_2$), which shows the uniqueness of the solution of this boundary value problem if the horizon is connected: The only $n$-dimensional asymptotically flat static vacuum black hole with non-degenerate regular event horizon is the Schwarzschild-Tangherlini family.

In fact, we assumed that the horizon is connected to obtain the above theorem. However, we can remove this assumption as follows. Let us consider the evolution of the level surface in Euclidean space. From the Gauss equation in Euclidean space, we obtain the evolution equation for the shear $\tilde{\sigma}_{AB} := \tilde{k}_{AB} - \tilde{k} h_{AB}/(n - 2)$:

$$\mathcal{L}_{\tilde{n}} \tilde{\sigma}_{AB} = \tilde{\sigma}_A^C \tilde{\sigma}_{CB} + \frac{1}{n-2} \tilde{h}_{AB} \tilde{\sigma}_{CD} \tilde{\sigma}^{CD} - \frac{1}{\tilde{\rho}} \left( \tilde{\nabla}_A \tilde{\nabla}_B - \frac{1}{n-2} \tilde{h}_{AB} \tilde{D}^2 \right) \tilde{\rho}. \quad (2.28)$$

Using $\Delta_0 v = 0$, we can derive the equation for $\tilde{\nabla}_A \ln \tilde{\rho}$ as

$$\mathcal{L}_{\tilde{n}} \tilde{\nabla}_A \ln \tilde{\rho} = \tilde{k} \tilde{\nabla}_A \ln \tilde{\rho} + \tilde{D}_A \tilde{k}. \quad (2.29)$$

For the trace part of $\tilde{k}_{AB}$, we have

$$\mathcal{L}_{\tilde{n}} \tilde{k} = -\tilde{\sigma}_{AB}^2 - \frac{1}{n-2} \tilde{k}^2 - \frac{1}{\tilde{\rho}} \tilde{D}^2 \tilde{\rho} \quad (2.30)$$
and
\[ \mathcal{L}_n D_A \tilde{k} = \tilde{D}_A \mathcal{L}_n \tilde{k} + (\tilde{D}_A \ln \tilde{\rho})(\mathcal{L}_n \tilde{k}). \] (2.31)

From the above equations, we can see that
\[ \tilde{\sigma}_{AB} = 0, \quad \tilde{D}_A \tilde{\rho} = 0 \quad \text{and} \quad \tilde{D}_A \tilde{k} = 0; \] (2.32)
that is, each level surface of \( v \) is totally umbilic, and hence spherically symmetric, which implies that the metric is isometric to the Schwarzschild-Tangherlini solution. This is, of course, a local result, since we consider only the region containing no saddle points of the harmonic function \( v \). To obtain the global result, we need a further assumption, such as analyticity. However, the assumption that there is no saddle point may be justified as follows. At a saddle point \( \rho = 0 \), the level surface of \( v \) is multi-sheeted; that is, the embedding of the level surfaces is singular there. One can find at least one level surface such that \( k_{AB} \neq 0 \) near the saddle point. Then, Eq. (2.18) implies that the saddle point is singular. If the horizon is not connected, this naked singularity must exist to compensate for the gravitational attraction between black holes.

### §3. Non-uniqueness

We have shown that if space-time is assumed to be asymptotically flat then the only regular static black hole is given by the Schwarzschild-Tangherlini metric. However, other non-asymptotically flat solutions may be obtained by replacing the metric of the round \((n - 2)\)-sphere by any other Einstein manifold whose Ricci-curvature has the same magnitude as that of a unit round \((n - 2)\)-sphere. In particular, we can replace the round metric on \( S^5, S^6, \cdots, S^9 \) by the infinite sequences of Einstein metrics found recently by Bohm. \(^{15}\) These have the form
\[ d\theta^2 + a^2(\theta)g_p + b^2(\theta)g_q, \] (3.1)
where \( g_m \) is the unit round metric on the \( m \)-sphere, \( p + q = n - 3 \) and neither \( p \) nor \( q \) is 1. In general, these metrics are inhomogeneous with isometry groups \( SO(p + 1) \times SO(q + 1) \). The round metric on \( S^{n-2} \) is given by \( a = \sin \theta \) and \( b = \cos \theta \). In addition, Bohm demonstrated the existence of infinite sequences of smooth Einstein metrics that converge to singular metrics of finite volume. By a theorem of Bishop, \(^{16}\) the volume of these metrics is always less than that of the round metric. It follows that for fixed temperature, the associated static black holes always have smaller Bekenstein-Hawking entropy than the Schwarzschild-Tangherlini black hole. For this reason, we believe that these metrics are all unstable. Presumably, they decay to the Schwarzschild-Tangherlini solution.

Clearly, Bohm metrics may also be used in D3-branes solutions and applied to the AdS/CFT correspondence. The volume may then be related to the central charge of a conformal field theory. \(^{17}\) In this way we obtain a further connection between the entropy of horizons, geometry and the central charges of quantum field theories, where now Bishop’s theorem provides a universal bound for the central charge.
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