Global solvability of a predator-prey model with predator-taxis and prey-taxis

Jianping Wang

School of Applied Mathematics, Xiamen University of Technology, Xiamen, 361024, China

Mingxin Wang

School of Mathematics, Harbin Institute of Technology, Harbin 150001, China

Abstract. This paper is concerned with a diffusive predator-prey model with predator-taxis and prey-taxis. Based on the Schauder fixed point theorem, we prove the global existence, uniqueness and boundedness of the classical solutions under the conditions that the predator-taxis and prey-taxis effects are weak enough.

Keywords: Predator-prey model; Predator-taxis; Prey-taxis; Existence and boundedness.

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1 Introduction and main results

Attractive prey-taxis describes the biological phenomenon that the predator move towards higher concentrations of prey. It was first observed in [9] that the ladybugs (predators) in area-restricted search tend to move toward areas with high aphids (prey) density to increase the efficiency of predation. Since the pioneer work of [9], the prey-taxis systems have been widely investigated by many authors. The general form of the prey-taxis system with constant taxis coefficient is

\[
\begin{align*}
    u_t &= d_1 \Delta u - \chi \nabla \cdot (u \nabla v) - uh(u) + c_1 u F(v), \\
    v_t &= d_2 \Delta v + g(v) - u F(v),
\end{align*}
\]

where the unknown functions \( u(x, t), v(x, t) \) represent the density of the predator and prey, respectively. The term \( h(u) \) describes the mortality rate of predators. The function \( g(v) \) is the growth function of prey. \( F(v) \) is the functional response function accounting for the intake rate of predators as a function of prey density and the parameter \( c_1 \) is the conversion rate. The term \(-\chi \nabla \cdot (u \nabla v)\) represents the prey-taxis effect, where \( \chi \) is a positive constant.

The global existence, uniqueness and boundedness of solutions of (1.1) have been studied by many authors, see, for example, [6, 8, 11, 15, 18, 20] and the references therein. Especially, it was discovered in [8, 18] that the system is global solvable in two space dimension, while smallness

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\[\text{2E-mail: jianping0215@163.com}
\[\text{3Corresponding author. E-mail: nxwang@hit.edu.cn} \]
assumption for \( \chi \) can prevent blow up in high dimensions. For a parabolic-elliptic version of (1.1), the global existence of solutions and global stability of a spatial homogeneous equilibrium were established in [17].

Repulsive predator-taxis explains the phenomenon that prey move away from the gradient of predator. The example is the presence of bass (predator) restricts crayfish (prey) foraging and increases anti-predator behaviour such as shelter seeking [7]. A typical form of predator-taxis system is

\[
\begin{align*}
    u_t &= d_1 \Delta u - uh(u) + c_1 u F(v), \\
    v_t &= d_2 \Delta v + \xi \nabla \cdot (v \nabla u) + g(v) - u F(v),
\end{align*}
\]

Here \( \xi \nabla \cdot (v \nabla u) \) represents the repulsive predator-taxis mechanism, where the constant \( \xi > 0 \). For \( F(v) \leq kv \) with \( k > 0 \) and sufficiently small \( \xi \), the global existence and boundedness of solutions, existence and stability of coexistence steady state solutions as well as Turing instability are given in [19].

Assume from now on that \( h(u) = a_1 + b_1 u, \quad g(v) = a_2 v - b_2 v^2 \) and \( F(v) = v \). By combing the above two taxis mechanisms, it arrives at the following system

\[
\begin{align*}
    u_t &= d_1 \Delta u - \chi \nabla \cdot (u \nabla v) + u(-a_1 - b_1 u + c_1 v), \quad x \in \Omega, \quad t > 0, \\
    v_t &= d_2 \Delta v + \xi \nabla \cdot (v \nabla u) + v(a_2 - b_2 v - u), \quad x \in \Omega, \quad t > 0, \\
    \partial u \big|_{\partial \Omega} &= \partial v \big|_{\partial \Omega} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

(1.2)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( \nu \) denotes the outward normal vector on \( \partial \Omega \), and constants \( \chi, \xi, a_i, b_i, c_i > 0 \), \( i = 1, 2 \). System (1.2) is also referred to as a pursuit-evasion model (1.14).

Despite the well development of the prey-taxis system, the surveys of the predator-prey system with predator-taxis and prey-taxis are at an early stage. In [12, 13], the global existence and large time behavior of weak solutions are constructed in a bounded interval in one space dimension. It is shown in [3] that, under some conditions on the initial data, system (1.2) admits global classical solutions near homogeneous steady states and these solutions converge to the homogeneous steady states. In [4], global weak solutions to a variant of (1.2) with nonlinear diffusion and saturated taxis sensitivity are constructed. The spatial pattern formation induced by the prey-taxis and predator-taxis is clarified in [16]. As it has been stated in [12, 13, 3], it is more challenging to analysis (1.2) compared with the single-taxis system, even for the local existence of solutions. This article proves the global existence of the classical solutions provided the taxis mechanisms are weak enough.

Notations of Hölder spaces from [11] and [2, Chapter 3] are very important for our conclusion. We denote \( Q_T = \Omega \times (0, T] \) with \( T \in (0, \infty) \) and

\[
\| \cdot \|_p = \| \cdot \|_{L^p(\Omega)}, \quad \| \cdot \|_{2+\alpha, \Omega} = \| \cdot \|_{C^{2+\alpha}(\Omega)}, \quad \| \cdot \|_{i+\alpha, \overline{Q}_T} = \| \cdot \|_{C^{i+\alpha}, \overline{Q}_T}, \quad i = 0, 1, 2
\]

for the simplicity. Especially, \( \| \cdot \|_{0, \overline{Q}_T} = \| \cdot \|_{C(\overline{Q}_T)} \). Throughout this paper the initial data \( u_0, v_0 \)
are supposed to satisfy

\[
  u_0, v_0 \in C^{2+\alpha}(\bar{\Omega}) \text{ with } \alpha \in (0,1), \quad u_0, v_0 \geq 0 \text{ in } \bar{\Omega}, \quad \frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0 \text{ on } \partial\Omega.
\]

We state the main result as follows:

**Theorem 1.1.** Let \( n \geq 1 \). Then there exist \( k > 0 \) and \( C > 0 \) such that for \( 0 < \chi, \xi < k \), the problem

\[
  \begin{align*}
    u_t - a(x,t)\Delta u + \sum_{i=1}^{n} b_i D_i u - cu &= f, & x \in \Omega, & t \in (0,T], \\
    \frac{\partial u}{\partial \nu} &= 0, & x \in \partial\Omega, & t \in [0,T], \\
    u(x,0) &= \phi(x), & x \in \bar{\Omega},
  \end{align*}
\]

has a unique global solution \((u,v) \in [C^{2,1}(\bar{\Omega} \times [0,\infty))]^2, \) and

\[
  \|u(\cdot,t)\|_{C^{2,1}(\bar{\Omega})} + \|v(\cdot,t)\|_{C^{2,1}(\bar{\Omega})} \leq C \text{ for all } t \in (0,\infty).
\]

The idea of proving Theorem 1.1 is inspired by [1]. Based on the Schauder-type estimates, we first derive a priori estimates. And then by use of the Schauder fixed point theorem, we prove the existence and boundedness of solutions of the problem (1.2). Finally, we show the uniqueness. We remark that under the assumption that the mortality rates are density dependent (i.e., \( b_1, b_2 > 0 \)), one can obtain similar conclusions for other form of (1.2) via following the arguments leading to Theorem 1.1.

## 2 Global existence and boundedness

### 2.1 A basic lemma and some notations

**Lemma 2.1.** ([12] Theorem 3.1) (i) There is a function \( K : (0,1) \times (0,\infty)^2 \to (0,\infty) \) with the following property:

Let \( \alpha \in (0,1), \ 0 < \varepsilon \leq k, \ T \in [1,\infty), \ a,b_1,c,f \in C^{\alpha,\alpha/2}(\overline{Q}_T), \ \phi \in C^{2,\alpha}(\Omega), \) with \( \max\{|a|_{\alpha,\overline{Q}_T},|b_1|_{0,\overline{Q}_T},|c|_{0,\overline{Q}_T}\} \leq k \) and \( a \geq \varepsilon \) on \( \overline{Q}_T \). If \( u \in C^{2,1}(\overline{Q}_T) \) solves

\[
  \begin{align*}
    u_t - a(x,t)\Delta u + \sum_{i=1}^{n} b_i D_i u - cu &= f, & x \in \Omega, & t \in (0,T], \\
    \frac{\partial u}{\partial \nu} &= 0, & x \in \partial\Omega, & t \in [0,T], \\
    u(x,0) &= \phi(x), & x \in \bar{\Omega},
  \end{align*}
\]

then

\[
  |u|_{\alpha,\overline{Q}_T} \leq K(\alpha,k,\varepsilon) \left( |f|_{0,\overline{Q}_T} + \|\phi\|_{2,\Omega} + |u|_{0,\overline{Q}_T} \right).
\]

(ii) There is a function \( L : (0,1) \times (0,\infty)^2 \to (0,\infty) \) with the following property:

Let \( \alpha, k, \varepsilon, T, a, b_1, c, f, \phi, u \) be given as in (i), but with the assumption: \( \max\{|a|_{\alpha,\overline{Q}_T},|b_1|_{0,\overline{Q}_T},|c|_{0,\overline{Q}_T}\} \leq k \) replaced by the stronger condition: \( \max\{|a|_{\alpha,\overline{Q}_T},|b_1|_{\alpha,\overline{Q}_T},|c|_{\alpha,\overline{Q}_T}\} \leq k \).

Then

\[
  |u|_{2+\alpha,\overline{Q}_T} \leq L(\alpha,k,\varepsilon) \left( |f|_{\alpha,\overline{Q}_T} + \|\phi\|_{2+\alpha,\Omega} + |u|_{0,\overline{Q}_T} \right).
\]
For the later use, we next introduce some notations. For $0 < \alpha < 1$ and
\[
\begin{align*}
\rho &= \min \{d_1, d_2, b_1, b_2\}, \\
\sigma &= \max \{d_1, d_2, b_1, b_2, a_1, \frac{3a_2}{\rho}, \frac{3c_1}{\rho}, \|u_0\|_{2+\alpha, \Omega}^{\frac{1}{2}}, \|v_0\|_{2+\alpha, \Omega}\},
\end{align*}
\]
we define
\[
\begin{align*}
\rho &= \min \{d_1, d_2, b_1, b_2, a_1, \frac{3a_2}{\rho}, \frac{3c_1}{\rho}, \|u_0\|_{2+\alpha, \Omega}^{\frac{1}{2}}, \|v_0\|_{2+\alpha, \Omega}\},
\end{align*}
\]
\begin{equation}
\tag{2.1}
\end{equation}
we define
\[
\begin{align*}
h_1 &= h_1(\rho, \sigma) = \sigma(1 + \rho + 2\sigma), \\
h_2 &= h_2(\rho, \sigma) = \sigma(1 + \rho), \\
h_3 &= h_3(\rho, \sigma) = \sigma(1 + \rho\sigma + \sigma^2), \\
h_4 &= h_4(\rho, \sigma) = \sigma(1 + \rho\sigma),
\end{align*}
\]
and
\[
\begin{align*}
P &= 2 \max \{K(\alpha, h_1, \rho), K(\alpha, h_3, \rho)\}, \\
R &= \min \left\{ \frac{3\rho}{\sigma + \sigma^3}, \frac{\rho}{(2\sigma^2(1+2P)+2)|L_2|}, \frac{\rho}{\sigma^3(\rho + \sigma)(1+2P)+2|L_4|} \right\},
\end{align*}
\]
\begin{equation}
\tag{2.2}
\end{equation}
where $L_2 = L(\alpha, h_2, \rho), L_4 = L(\alpha, h_4, \rho), L, K$ are determined by Lemma 2.1.

\subsection*{2.2 Existence of solutions}

The following lemma provides a priori estimates for $v$ with given solution component $u$ and small $\xi$.

\textbf{Lemma 2.2.} Let $T \in [1, \infty), \alpha \in (0, 1), \sigma, \rho, P, R$ be given by (2.1) and (2.2). Assume that
\[
0 < \xi \leq R/3
\]
and $u \in C^{2+\alpha, 1+\alpha/2}(\overline{Q_T})$ satisfying
\[
0 \leq u \leq \sigma R, \ |u|_{\alpha, \overline{Q_T}} \leq \sigma PR, \ |u|_{2+\alpha, \overline{Q_T}} \leq \rho.
\]

Then the problem
\[
\begin{align*}
\tilde{v}_t - d_2 \Delta \tilde{v} - \frac{\sigma\xi}{R} \nabla u \cdot \nabla \tilde{v} - \left( \frac{\sigma\xi}{R} \Delta u + a_2 - \frac{\sigma b_2}{R} \right) \tilde{v} = 0, & \quad x \in \Omega, \ t \in (0, T], \\
\frac{\partial \tilde{v}}{\partial \nu} = 0, & \quad x \in \partial \Omega, \ t \in [0, T], \\
\tilde{v}(x, 0) = \tilde{v}_0(x) := \frac{R}{\sigma} v_0(x), & \quad x \in \Omega.
\end{align*}
\]

admits a unique solution $\tilde{v} \in C^{2+\alpha, 1+\alpha/2}(\overline{Q_T})$. Moreover,
\[
0 \leq \tilde{v} \leq R, \ |\tilde{v}|_{\alpha, \overline{Q_T}} \leq PR, \ |\tilde{v}|_{2+\alpha, \overline{Q_T}} \leq \rho.
\]

\textbf{Proof.} From (2.1), it is easy to get
\[
d_2, b_2 \in [\rho, \sigma], \ a_2 \in [0, \rho \sigma/3].
\]
Since $\tilde{v}_0(x) := \frac{R}{\sigma} v_0(x)$, by (1.3) and (2.1), we have
\[
\tilde{v}_0 \in C^{2+\alpha}(\Omega), \ \frac{\partial \tilde{v}_0}{\partial \nu} |_{\partial \Omega} = 0, \ \tilde{v}_0 \geq 0, \ \|\tilde{v}_0\|_{2+\alpha, \Omega} \leq R.
\]
**Step 1: Existence, uniqueness and boundedness.** Thanks to \( \tilde{v}(x,0) = \tilde{v}_0(x) \geq 0 \) for \( x \in \Omega \), it is easy to see that \( \bar{v}(x,t) \equiv 0 \) is a lower solution of (2.5).

Let \( \bar{v}(x,t) \equiv R \) for \( (x,t) \in \overline{Q}_T \). It follows from (2.3), (2.4) and (2.7) that

\[
\left( \frac{\sigma \xi}{R} \Delta u + a_2 - \frac{\sigma u}{R} - \frac{\sigma b_2}{R} \tilde{v} \right) \bar{v} \leq \left( \frac{\sigma}{3} |\Delta u| + \frac{\sigma \rho}{3} - \sigma \rho \right) R \leq 0 \quad \text{in} \quad \overline{Q}_T.
\]

Moreover, \( \tilde{v}_0 \leq R \) due to (2.8). Hence, \( \bar{v} \) is an upper solution of (2.5). Making use of the upper and lower solutions method, one can easily obtain the existence and uniqueness of classical solution to (2.5). Hence, the problem (2.5) has a unique solution \( \tilde{v} \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T) \) and

\[
0 \leq \tilde{v} \leq R \quad \text{on} \quad \overline{Q}_T.
\]  

(2.9)

**Step 2: The regularity.** For the convenience, we let

\[
r(x,t) := \frac{\sigma \xi}{R} \Delta u + a_2 - \frac{\sigma u}{R} - \frac{\sigma b_2}{R} \tilde{v}.
\]

By (2.3), (2.4), (2.7) and (2.9), we find

\[
\max \left\{ d_2, \frac{\sigma \xi}{R} \left| \nabla u \right|_{0,\overline{Q}_T}, \left| r \right|_{0,\overline{Q}_T} \right\} \leq \sigma + \rho \sigma + 2 \sigma^2 = h_1(\rho, \sigma) \quad \text{and} \quad d_2 \geq \rho.
\]

This combined with (2.9) and (2.8) enables us to apply Lemma 2.1 (i) to (2.5) to get

\[
|\tilde{v}|_{0,\overline{Q}_T} \leq K(\alpha, h_1(\rho, \sigma), \rho) \left( ||\tilde{v}_0||_{C^2(\tilde{\Omega})} + |\tilde{v}|_{0,\overline{Q}_T} \right) \\
\leq 2K(\alpha, h_1(\rho, \sigma), \rho) R \\
\leq PR,
\]

(2.10)

where we used (2.2) in the last step. This shows the second estimate of (2.6).

Next, we shall prove the last inequality of (2.6). Rewriting (2.5) as

\[
\begin{cases}
\tilde{v}_t - d_2 \Delta \tilde{v} - \frac{\sigma \xi}{R} \nabla u \cdot \nabla \tilde{v} - \left( \frac{\sigma \xi}{R} \Delta u + a_2 \right) \tilde{v} = - \left( \frac{\sigma u}{R} - \frac{\sigma b_2}{R} \tilde{v} \right) \tilde{v}, & x \in \Omega, \ t \in (0,T], \\
\frac{\partial \tilde{v}}{\partial \nu} = 0, & x \in \partial \Omega, \ t \in [0,T], \\
\tilde{v}(x,0) = \tilde{v}_0(x), & x \in \tilde{\Omega}.
\end{cases}
\]  

(2.11)

Making use of (2.3), (2.4) and (2.7), it follows that

\[
d_2 \geq \rho \quad \text{and} \quad \max \left\{ d_2, \frac{\sigma \xi}{R} \left| \nabla u \right|_{0,\overline{Q}_T}, \left| \frac{\sigma \xi}{R} \Delta u + a_2 \right|_{0,\overline{Q}_T} \right\} \leq \sigma + \rho \sigma =: h_2(\rho, \sigma).
\]  

(2.12)

Noting that

\[
|fg|_{\alpha,\overline{Q}_T} \leq |f|_{0,\overline{Q}_T} |g|_{0,\overline{Q}_T} + |f|_{0,\overline{Q}_T} |g|_{\alpha,\overline{Q}_T} + |f|_{\alpha,\overline{Q}_T} |g|_{0,\overline{Q}_T}
\]

(2.13)

holds for all \( f, g \in C^{\alpha,\alpha/2}(\overline{Q}_T) \). In view of (2.13), (2.10), and (2.4), we have

\[
\left| \left( - \frac{\sigma u}{R} - \frac{\sigma b_2}{R} \tilde{v} \right) \tilde{v} \right|_{\alpha,\overline{Q}_T} \leq 2 \sigma^2 R(1 + 2P).
\]

(2.14)
Recalling that \( \|\tilde{v}_0\|_{2+\alpha,\Omega} \leq R \) due to (2.8). Based on (2.12) and (2.14), using Lemma 2.1(ii) to (2.11), we find

\[
|\tilde{v}|_{2+\alpha,\overline{Q}_T} \leq L(\alpha, h_2(\rho, \sigma), \rho)
\left( \left\| -\frac{\sigma u}{R} - \frac{\sigma b_2\tilde{v}}{R} \right\|_{\alpha,\overline{Q}_T} + \|\tilde{v}_0\|_{2+\alpha,\Omega} + |\tilde{v}|_{0,\overline{Q}_T} \right)
\leq L(\alpha, h_2(\rho, \sigma), \rho)(2\sigma^2(1+2P)+2)R
\leq (2\sigma^2(1+2P)+2)L_2R \leq \rho.
\]

Here we used (2.2) in the last deduction. The proof is end. \( \square \)

The following lemma provides the a priori estimates for \( u \) when \( \chi \) is small enough and \( v \) is given and satisfies (2.6).

**Lemma 2.3.** Let \( T \in [1, \infty) \), \( 0 < \alpha \leq 1 \), and \( \rho, \sigma, P, R \) be given by (2.1) and (2.2). Assume that

\[
0 < \chi \leq \sigma R/3 \tag{2.15}
\]

and \( v \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T) \) with

\[
0 \leq v \leq R, \ |v|_{\alpha,\overline{Q}_T} \leq PR, \ |v|_{2+\alpha,\overline{Q}_T} \leq \rho. \tag{2.16}
\]

Then the problem

\[
\begin{cases}
\tilde{u}_t - d_1\Delta \tilde{u} + \frac{\sigma\chi}{R} \nabla v \cdot \nabla \tilde{u} - \left( -\frac{\sigma}{R} \Delta v - a_1 + \frac{\sigma c_1}{R} v - \frac{\sigma b_1}{R} \tilde{u} \right) \tilde{u} = 0, & x \in \Omega, \ 0 < t \leq T, \\
\frac{\partial \tilde{u}}{\partial \nu} = 0, & x \in \partial \Omega, \ 0 \leq t \leq T, \\
\tilde{u}(x,0) = \tilde{u}_0(x) := \frac{R}{\sigma} u_0, & x \in \bar{\Omega}
\end{cases} \tag{2.17}
\]

has a unique solution \( \tilde{u} \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T) \) which satisfies

\[
0 \leq \tilde{u} \leq \sigma R, \ |\tilde{u}|_{\alpha,\overline{Q}_T} \leq \sigma PR, \ |\tilde{u}|_{2+\alpha,\overline{Q}_T} \leq \rho. \tag{2.18}
\]

**Proof.** The proof is similar to that of Lemma 2.2. It follows from (2.11) that

\[
d_1, b_1 \in [\rho, \sigma], \ a_1 \in [0, \sigma], \ c_1 \in [0, \rho \sigma/3]. \tag{2.19}
\]

By (1.3) and (2.1), we have

\[
\tilde{u}_0 \in C^{2+\alpha}(\bar{\Omega}), \ \frac{\partial \tilde{u}_0}{\partial \nu} \bigg|_{\partial \Omega} = 0, \ \tilde{u}_0 \geq 0, \ \|\tilde{u}_0\|_{2+\alpha,\bar{\Omega}} \leq \sigma R. \tag{2.20}
\]

**Step 1: Existence, uniqueness and boundedness.** Due to \( \tilde{u}_0 \geq 0 \) on \( \bar{\Omega} \), it is easily seen that \( \underline{u} \equiv 0 \) is the lower solution of (2.17).

Let \( \bar{u}(x,t) \equiv \sigma R \) in \( \overline{Q}_T \). From (2.15), (2.16) and (2.19), we find

\[
\left( -\frac{\sigma\chi}{R} \Delta v - a_1 + \frac{\sigma c_1}{R} v - \frac{\sigma b_1}{R} \bar{u} \right) \bar{u} \leq \left( \frac{\sigma^2}{3} |\Delta v| + \frac{\rho \sigma^2}{3} - \rho \sigma^2 \right) \sigma R \leq 0 \quad \text{in} \quad Q_T.
\]
Hence, it is easy to verify that \( \bar{u} \) is an upper solution of (2.17). The upper and lower solutions method shows that problem (2.17) admits a unique solution \( \bar{u} \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T) \) which satisfies
\[
0 \leq \bar{u} \leq \sigma R. \tag{2.21}
\]
This establishes the first inequality in (2.18).

**Step 2: The regularity (2.18).** For the convenience, let
\[
g(x,t) = -\frac{\sigma \chi}{R} \Delta v - a_1 + \frac{\sigma c_1}{R} v - \frac{\sigma b_1}{R} \bar{u}.
\]
Thanks to (2.15), (2.16), (2.19) and (2.21), we have
\[
d_1 \geq \rho \quad \text{and} \quad \max \left\{ d_1, \frac{\sigma \chi}{R} |\nabla v|_{0,\overline{Q}_T}, |g|_{0,\overline{Q}_T} \right\} \leq \sigma + \sigma^2 \rho + \sigma^3 = h_3(\rho, \sigma). \tag{2.22}
\]
We then use Lemma 2.1 (i) to infer that
\[
|\bar{u}|_{\alpha,\overline{Q}_T} \leq K(\alpha, h_3(\rho, \sigma), \rho) (\|\bar{u}_0\|_{2,\bar{\Omega}} + |\bar{u}|_{0,\overline{Q}_T})
\leq 2K(\alpha, h_3(\rho, \sigma), \rho) \sigma R
\leq \sigma PR, \tag{2.23}
\]
where we have used (2.22), (2.20), (2.21) and (2.22). This proves the second estimate of (2.18).

It remains to show the last inequality of (2.18). To achieve this, we need to rewrite (2.17) as
\[
\begin{aligned}
\begin{cases}
\bar{u}_t - d_1 \Delta \bar{u} + \frac{\sigma \chi}{R} \nabla v \cdot \nabla \bar{u} - \left( -\frac{\sigma \chi}{R} \Delta v - a_1 \right) \bar{u} = \left( \frac{\sigma c_1}{R} v - \frac{\sigma b_1}{R} \bar{u} \right) \bar{u}, & x \in \Omega, \ 0 < t \leq T, \\
\frac{\partial \bar{u}}{\partial \nu} = 0, & x \in \partial \Omega, \ 0 \leq t \leq T, \\
\bar{u}(x,0) = \bar{u}_0(x), & x \in \bar{\Omega}
\end{cases}
\end{aligned}
\tag{2.24}
\]
Making use of (2.13), (2.16) and (2.19), we have
\[
d_1 \geq \rho \quad \text{and} \quad \max \left\{ d_1, \frac{\sigma \chi}{R} |\nabla v|_{\alpha,\overline{Q}_T}, \left| -\frac{\sigma \chi}{R} \Delta v - a_1 \right|_{\alpha,\overline{Q}_T} \right\} \leq \sigma + \rho \sigma^2 = h_4(\rho, \sigma).
\]
This enables us to apply Lemma 2.1 (ii) to (2.24) to derive
\[
|\bar{u}|_{2+\alpha,\overline{Q}_T} \leq \mathcal{L}(\alpha, h_4(\rho, \sigma), \rho) \left( \left| \left( \frac{\sigma c_1}{R} v - \frac{\sigma b_1}{R} \bar{u} \right) \bar{u} \right|_{\alpha,\overline{Q}_T} + \|\bar{u}_0\|_{2+\alpha,\bar{\Omega}} + |\bar{u}|_{0,\overline{Q}_T} \right). \tag{2.25}
\]
Similar to the derivation of (2.14), by using (2.13), (2.19), (2.16), (2.21) and (2.23), one can obtain
\[
\left| \left( \frac{\sigma c_1}{R} v - \frac{\sigma b_1}{R} \bar{u} \right) \bar{u} \right|_{\alpha,\overline{Q}_T} \leq \sigma^3 R(\rho + \sigma)(1 + 2P).
\]
This combined with (2.25), (2.20), (2.21) and (2.2) implies
\[
|\bar{u}|_{2+\alpha,\overline{Q}_T} \leq \mathcal{L}(\alpha, h_4(\rho, \sigma), \rho) \left( \left| \left( \frac{\sigma c_1}{R} v - \frac{\sigma b_1}{R} \bar{u} \right) \bar{u} \right|_{\alpha,\overline{Q}_T} + \|\bar{u}_0\|_{2+\alpha,\bar{\Omega}} + |\bar{u}|_{0,\overline{Q}_T} \right)
\leq \mathcal{L}(\alpha, h_4(\rho, \sigma), \rho) [\sigma^3(\rho + \sigma)(1 + 2P) + 2\sigma] R
= \left[ \sigma^3(\rho + \sigma)(1 + 2P) + 2\sigma \right] \mathcal{L}_4 R
\leq \rho.
\]
This gives the last estimation of (2.18) and hence completes the proof. \(\Box\)
In what follows, we shall consider
\[
\begin{aligned}
& u_t = d_1 \Delta u - \frac{\sigma \chi}{R} \nabla \cdot (u \nabla v) + \left( -a_1 \frac{\sigma c_1}{R} v - \frac{\sigma b_1}{R} u \right) u, \quad x \in \Omega, \quad t \in (0, T], \\
& v_t = d_2 \Delta v + \frac{\sigma \xi}{R} \nabla \cdot (v \nabla u) + \left( a_2 - \frac{\sigma}{R} u - \frac{\sigma b_2}{R} v \right) v, \quad x \in \Omega, \quad t \in (0, T], \\
& \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t \in [0, T], \\
& u(x, 0) = \frac{R}{\sigma} u_0(x), \quad v(x, 0) = \frac{R}{\sigma} v_0(x), \quad x \in \bar{\Omega}.
\end{aligned}
\] (2.26)

Lemma 2.4. Let \( T \in [1, \infty), \alpha \in (0, 1), \) and \( \rho, \sigma, P, R \) be given by (2.1) and (2.2). Assume that \( \chi \in (0, \sigma R/3] \) and \( \xi \in (0, R/3] \).

Then there exists \((\tilde{u}, \tilde{v}) \in (C^{2+\alpha,1+\alpha/2}(\bar{Q}_T))^2\) which solves (2.26). And \((\tilde{u}, \tilde{v})\) satisfies
\[
0 \leq \tilde{u} \leq \sigma R, \quad |\tilde{u}|_{\alpha, \bar{Q}_T} \leq \sigma PR, \quad |\tilde{u}|_{2+\alpha, \bar{Q}_T} \leq \rho,
\]
and
\[
0 \leq \tilde{v} \leq R, \quad |\tilde{v}|_{\alpha, \bar{Q}_T} \leq PR, \quad |\tilde{v}|_{2+\alpha, \bar{Q}_T} \leq \rho.
\]

Proof. We first define
\[
\Sigma = \left\{ u, v \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_T) : \begin{array}{l}
0 \leq u \leq \sigma R, \quad 0 \leq v \leq R, \quad |u|_{\alpha, \bar{Q}_T} \leq \sigma PR, \\
|v|_{\alpha, \bar{Q}_T} \leq PR, \quad \max\{|u|_{2+\alpha, \bar{Q}_T}, |v|_{2+\alpha, \bar{Q}_T}\} \leq \rho
\end{array} \right\}
\]

Define
\[
W = [C^{2+\alpha/2,1+\alpha/4}(\bar{Q}_T)]^2.
\]

It is easy to see that \( W \) is a Banach space endowed with norm
\[
\| (u, v) \|_W = |u|_{2+\alpha/2, \bar{Q}_T} + |v|_{2+\alpha/2, \bar{Q}_T} \quad \text{for} \quad (u, v) \in W.
\]

Moreover, \( \Sigma \) is a compact convex subset of \( W \).

For the given \((u, v) \in \Sigma, \) let \( \tilde{v} = M(u) \) be the unique solution of (2.5) obtained by Lemma 2.2 and \( \tilde{u} = N(v) \) be the solution of (2.17) given by Lemma 2.3. Set \( \mathcal{F}(u, v) = (\tilde{u}, \tilde{v}) \). It follows from Lemma 2.2 and Lemma 2.3 that \( \tilde{u}, \tilde{v} \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_T) \) and
\[
0 \leq \tilde{u} \leq \sigma R, \quad |\tilde{u}|_{\alpha, \bar{Q}_T} \leq \sigma PR, \quad |\tilde{u}|_{2+\alpha, \bar{Q}_T} \leq \rho,
\]
and
\[
0 \leq \tilde{v} \leq R, \quad |\tilde{v}|_{\alpha, \bar{Q}_T} \leq PR, \quad |\tilde{v}|_{2+\alpha, \bar{Q}_T} \leq \rho. \quad (2.27)
\]

Hence, \( \mathcal{F} \) maps from \( \Sigma \) into itself.

In order to apply the Schauder fixed point theorem, we will prove that \( \mathcal{F} \) is continuous in the norm \( \| \cdot \|_W \) of the Banach space \( W \). For \((u_i, v_i) \in \Sigma, i = 1, 2, \) let
\[
\tilde{v}_i = M(u_i), \quad \tilde{u}_i = N(v_i), \quad u = u_1 - u_2, \quad v = v_1 - v_2, \quad \tilde{u} = \tilde{u}_1 - \tilde{u}_2, \quad \tilde{v} = \tilde{v}_1 - \tilde{v}_2.
\]
Clearly, $\tilde{v} = \tilde{v}_1 - \tilde{v}_2$ satisfies
\[
\begin{cases}
\tilde{v}_1 - d_2 \Delta \tilde{v} - \frac{\sigma \xi}{R} \nabla u_1 \cdot \nabla \tilde{v} - \left( \frac{\sigma \xi}{R} \Delta u_1 + a_2 - \frac{\sigma}{R} u_1 - \frac{\sigma b_2}{R} \left( \tilde{v}_1 + \tilde{v}_2 \right) \right) \tilde{v} \\
\quad = \frac{\sigma \xi}{R} \nabla \tilde{v}_2 \cdot \nabla u + \frac{\sigma \xi}{R} \tilde{v}_2 \Delta u - \frac{\sigma}{R} \tilde{v}_2 u, \quad x \in \Omega, \ t \in (0, T], \\
\partial_t \tilde{v} = 0, \quad x \in \partial \Omega, \ t \in [0, T], \\
\tilde{v}(x, 0) = 0, \quad x \in \bar{\Omega}.
\end{cases}
\]

It is clear that $|\varphi|_{\alpha/2, \bar{Q}_T} \leq 3|\varphi|_{\alpha, \bar{Q}_T}$ for any $\varphi \in C^{\alpha, \alpha/2}(\bar{Q}_T)$. Since $\tilde{v}_i$ satisfy Equation (2.27) for $i = 1, 2$ and $(u_1, v_1) \in \Sigma$, we have
\[
|u_1|_{2+\alpha, \bar{Q}_T} \leq \rho, \quad |\tilde{v}_i|_{\alpha, \bar{Q}_T} \leq PR \text{ for } i = 1, 2.
\]

And hence there is $C_1 > 0$ such that
\[
|\nabla \tilde{v}_1|_{\alpha/2, \bar{Q}_T} \leq C_1.
\]

Similarly, one can find $C_2 > 0$ such that
\[
|\nabla \tilde{v}_2|_{2+\alpha/2, \bar{Q}_T} \leq C_2 |u|_{2+\alpha/2, \bar{Q}_T}.
\]

In view of the parabolic Schauder theory (cf. [10, Theorem IV.5.3]), there is $C_3 > 0$ which depends on $\alpha, T, \Omega, C_1$ such that
\[
|\tilde{v}|_{2+\alpha/2, \bar{Q}_T} \leq C_3 \left( |\nabla \tilde{v}_2| \nabla u + \frac{\sigma \xi}{R} \tilde{v}_2 \Delta u - \frac{\sigma}{R} \tilde{v}_2 u \right)_{\alpha/2, \bar{Q}_T} \leq C_2 C_3 |u|_{2+\alpha/2, \bar{Q}_T}.
\]

We next estimate $\tilde{u}$. It is easy to see that $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$ satisfies
\[
\begin{cases}
\tilde{u}_1 - d_1 \Delta \tilde{u} + \frac{\sigma \chi}{R} \nabla v_1 \cdot \nabla \tilde{u} + \left( \frac{\sigma \chi}{R} \Delta v_1 + a_1 - \frac{\sigma c_1}{R} v_1 + \frac{\sigma b_1}{R} (\tilde{u}_1 + \tilde{u}_2) \right) \tilde{u} \\
\quad = -\frac{\sigma \chi}{R} \nabla \tilde{u}_2 \cdot \nabla v - \frac{\sigma \chi}{R} \tilde{u}_2 \Delta v + \frac{\sigma c_1}{R} \tilde{u}_2 v, \quad x \in \Omega, \ 0 < t \leq T, \\
\partial_t \tilde{u} = 0, \quad x \in \partial \Omega, \ 0 \leq t \leq T, \\
\tilde{u}(x, 0) = 0, \quad x \in \bar{\Omega}.
\end{cases}
\]

Similar to the above, there exist $C_4, C_5 > 0$ such that
\[
|\nabla \tilde{u}_1|_{\alpha/2, \bar{Q}_T}, \quad \left| \frac{\sigma \chi}{R} \nabla v_1 + a_1 - \frac{\sigma c_1}{R} v_1 + \frac{\sigma b_1}{R} (\tilde{u}_1 + \tilde{u}_2) \right|_{\alpha/2, \bar{Q}_T} \leq C_4,
\]

and
\[
\left| -\frac{\sigma \chi}{R} \nabla \tilde{u}_2 \cdot \nabla v - \frac{\sigma \chi}{R} \tilde{u}_2 \Delta v + \frac{\sigma c_1}{R} \tilde{u}_2 v \right|_{2+\alpha/2, \bar{Q}_T} \leq C_5 |v|_{2+\alpha/2, \bar{Q}_T}.
\]
Again by the parabolic Schauder theory, there is $C_0 > 0$ such that

$$|\hat{u}|_{2+\alpha/2,\overline{Q}_T} \leq C_0|v|_{2+\alpha/2,\overline{Q}_T}.$$  

This combined with (2.28) yields that

$$\|F(u_1, v_1) - F(u_2, v_2)\|_W = \|(\hat{u}, \hat{v})\|_W$$

$$= |\hat{u}|_{2+\alpha/2,\overline{Q}_T} + |\hat{v}|_{2+\alpha/2,\overline{Q}_T}$$

$$\leq \max\{C_2C_3, C_0\} \left( |u|_{2+\alpha/2,\overline{Q}_T} + |v|_{2+\alpha/2,\overline{Q}_T} \right)$$

$$\leq \max\{C_2C_3, C_0\} \|(u_1, v_1) - (u_2, v_2)\|_W.$$  

This shows that $F$ is continuous in the norm $\| \cdot \|_W$ of the Banach space $W$.

Making use of the Schauder fixed point theorem (cf. [5, Theorem 11.1]), there exists $(\hat{u}, \hat{v}) \in \Sigma$ such that $F(\hat{u}, \hat{v}) = (\hat{u}, \hat{v})$. Hence, problem (2.26) admits a solution $(\hat{u}, \hat{v})$. And the desired estimates follow from the definition of $\Sigma$.

In view of Lemma 2.4, we can prove the existence of solutions of (1.2).

**Lemma 2.5.** Let $\alpha \in (0, 1)$, and $\rho, \sigma, P, R$ be given by (2.1) and (2.2). Assume that

$$\chi \in (0, \sigma R/3) \quad \text{and} \quad \xi \in (0, R/3].$$

Then there exists a solution $(u, v) \in [C^2,1(\overline{Q}_T)]^2$ solving (1.2) on $[0, T]$ for any $T \in [1, \infty)$, and $(u, v)$ satisfies

$$0 \leq u \leq \sigma^2, \quad |u|_{\alpha,\overline{Q}_T} \leq P\sigma^2, \quad |u|_{2+\alpha,\overline{Q}_T} \leq \rho\sigma/R.$$  

(2.29)

and

$$0 \leq v \leq \sigma, \quad |v|_{\alpha,\overline{Q}_T} \leq P\sigma, \quad |v|_{2+\alpha,\overline{Q}_T} \leq \rho\sigma/R.$$  

(2.30)

**Proof.** Let $T \in [1, \infty)$ and $(\hat{u}, \hat{v})$ be the solution of (2.26) obtained in Lemma 2.4 i.e.,

$$\begin{cases}
\hat{u}_t = d_1\Delta \hat{u} - \frac{\sigma\chi}{R} \nabla \cdot (\hat{u} \nabla \hat{u}) + \left(-a_1 + \frac{\sigma c_1}{R} \hat{v} - \frac{\sigma b_1}{R} \hat{u}\right) \hat{u}, & x \in \Omega, \quad t \in (0, T],

\hat{v}_t = d_2\Delta \hat{v} + \frac{\sigma\xi}{R} \nabla \cdot (\hat{v} \nabla \hat{u}) + \left(a_2 - \frac{\sigma}{R} \hat{u} - \frac{\sigma b_2}{R} \hat{v}\right) \hat{v}, & x \in \Omega, \quad t \in (0, T],

\frac{\partial \hat{u}}{\partial \nu} = \frac{\partial \hat{v}}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0, T],

\hat{u}(x, 0) = \frac{R}{\sigma} u_0(x), \quad \hat{v}(x, 0) = \frac{R}{\sigma} v_0(x), & x \in \Omega.
\end{cases}$$

By letting $u = \frac{R}{\sigma} \hat{u}$ and $v = \frac{R}{\sigma} \hat{v}$, then it is easy to verify that $(u, v)$ solves (1.2) on $[0, T]$ and fulfills (2.29) and (2.30).
2.3 Uniqueness of solution

The coming lemma asserts that the solution obtained in Lemma 2.5 is the unique solution of (1.2).

**Lemma 2.6.** Let $\alpha \in (0,1)$, and $\rho, \sigma, P, R$ be given by (2.1) and (2.2). Assume that
\[
\chi \in (0, \sigma R/3] \quad \text{and} \quad \xi \in (0, R/3].
\] (2.31)
Then there is a unique solution $(u, v) \in [C^{2,1}(\Omega_T)]^2$ of the problem (1.2) on $[0, T]$ for $T \in [1, \infty)$.

**Proof.** On the one hand, let $u, v \in [C^{2,1}(\Omega_T)]^2$, be a solution of (1.2). By using the maximum principle, it is easy to see that $u, v \geq 0$. Clearly, there is $C_0 > 0$ such that
\[
|u|_{2,\Omega_T}^2, |v|_{2,\Omega_T}^2 \leq C_0.
\] (2.32)
On the other hand, suppose that $(u_2, v_2)$ is the solution of (1.2) obtained in Lemma 2.5. Then $(u_2, v_2)$ satisfies (2.29) and (2.30), and hence
\[
0 \leq u_2 \leq \sigma^2 \quad \text{and} \quad 0 \leq v_2 \leq \sigma.
\] (2.33)
Let $w = u_1 - u_2$ and $z = v_1 - v_2$. Then $w$ and $z$ satisfy
\[
\begin{aligned}
&w_t - d_1 \Delta w + \chi \nabla \cdot (w \nabla v_1) + [a_1 - c_1 v_1 + b_1(u_1 + u_2)] w = -\chi \nabla \cdot (u_2 \nabla z) + c_1 u_2 z, \quad x \in \Omega, \ 0 < t \leq T, \\
&\frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, \ 0 \leq t \leq T, \\
&w(x, 0) = 0, \quad x \in \bar{\Omega},
\end{aligned}
\]
and
\[
\begin{aligned}
&z_t - d_2 \Delta z - \xi \nabla \cdot (z \nabla u_1) + (-a_2 + u_1 + b_2(v_1 + v_2)) z = \xi \nabla \cdot (v_2 \nabla w) - v_2 w, \quad x \in \Omega, \ 0 < t \leq T, \\
&\frac{\partial z}{\partial \nu} = 0, \quad x \in \partial \Omega, \ 0 \leq t \leq T, \\
&z(x, 0) = 0, \quad x \in \bar{\Omega}.
\end{aligned}
\]
Making use of the testing procedure, by (2.1), (2.31), (2.33) and (2.32), we have that, for some $C_1, C_2 > 0$,
\[
\begin{aligned}
\frac{1}{2} & \frac{d}{dt} \int_\Omega w^2 \, dx = -d_1 \int_\Omega |\nabla w|^2 \, dx + \chi \int_\Omega w \nabla w \cdot \nabla v_1 \, dx + \chi \int_\Omega u_2 \nabla z \cdot \nabla w \, dx \\
&\quad - \int_\Omega w^2 [a_1 - c_1 v_1 + b_1(u_1 + u_2)] \, dx + c_1 \int_\Omega u_2 w z \, dx \\
&\leq - \frac{d_1}{2} \int_\Omega |\nabla w|^2 \, dx + \frac{\chi^2}{2d_1} \int_\Omega w^2 |\nabla v_1|^2 \, dx + \frac{\chi}{2} \int_\Omega u_2 |\nabla w|^2 \, dx \\
&\quad + \frac{\chi}{2} \int_\Omega u_2 |\nabla z|^2 \, dx + c_1 \int_\Omega v_1 w^2 \, dx + c_1 \int_\Omega u_2 w z \, dx
\end{aligned}
\]
\[ \left( -\frac{\rho}{2} + \frac{\sigma^3 R}{6} \right) \int_\Omega |\nabla w|^2 dx + \frac{\sigma R}{6} \int_\Omega |\nabla z|^2 dx + C_1 \int_\Omega (w^2 + z^2) dx, \quad (2.34) \]

and

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega z^2 dx = -d_2 \int_\Omega |\nabla z|^2 dx - \xi \int_\Omega z \nabla z \cdot \nabla u_1 dx - \xi \int_\Omega v_2 \nabla w \cdot \nabla z dx \]
\[ - \int_\Omega z^2 (a_2 + u_1 + b_2(v_1 + v_2)) dx - \int_\Omega v_2 wz dx \]
\[ \leq -\frac{d_2}{2} \int_\Omega |\nabla z|^2 dx + \frac{\xi^2}{2d_2} \int_\Omega z^2 |\nabla u_1|^2 dx + \frac{\xi}{2} \int_\Omega v_2 |\nabla w|^2 dx \]
\[ + \frac{\xi}{2} \int_\Omega v_2 |\nabla z|^2 dx + a_2 \int_\Omega z^2 dx - \int_\Omega v_2 wz dx \]
\[ \leq \left( -\frac{\rho}{2} + \frac{\sigma R}{6} \right) \int_\Omega |\nabla z|^2 dx + \frac{\sigma R}{6} \int_\Omega |\nabla w|^2 dx + C_2 \int_\Omega (w^2 + z^2) dx. \quad (2.35) \]

It follows from (2.34) and (2.35) that

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega (w^2 + z^2) dx \leq \left( -\frac{\rho}{2} + \frac{\sigma R}{6} + \frac{\sigma^3 R}{6} \right) \left( \int_\Omega |\nabla w|^2 dx + \int_\Omega |\nabla z|^2 dx \right) \]
\[ + (C_1 + C_2) \int_\Omega (w^2 + z^2) dx. \quad (2.36) \]

From the definition of \( R \), we have \( R \leq 3\rho/(\sigma + \sigma^3) \) and hence

\[ -\frac{\rho}{2} + \frac{\sigma R}{6} + \frac{\sigma^3 R}{6} \leq 0. \]

This combined with (2.36) yields

\[ \frac{d}{dt} \int_\Omega (w^2 + z^2) dx \leq 2(C_1 + C_2) \int_\Omega (w^2 + z^2) dx. \]

Noting that \( w(x, 0) = z(x, 0) = 0 \), Gronwall’s lemma asserts that \( w = z = 0 \). This completes the proof. \( \square \)

**Proof of Theorem 1.1.** Combining the conclusions of Lemmas 2.5, 2.6, and using the arbitrariness of \( T \geq 1 \) we know that the solution \((u, v)\) of (1.2) exists uniquely and globally, and the estimate (1.4) holds. \( \square \)

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