DYNAMICS OF COVERING MAPS OF THE ANNULUS.

J.IGLESIAS, A.PORTELA, A.ROVELLA AND J.XAVIER

Abstract. It is often the case that a covering map of the open annulus is semiconjugate to a map of the circle of the same degree. We investigate this possibility and its consequences on the dynamics. In particular, we address the problem of the classification up to conjugacy. However, there are examples which are not semiconjugate to a map of the circle, and this opens new questions.

1. Introduction.

Let \( A \) be the open annulus \((0,1) \times S^1\). If \( f : A \to A \) is a continuous function, then the homomorphism \( f_* \) induced by \( f \) on the first homology group \( H_1(A,\mathbb{Z}) \equiv \mathbb{Z} \), is \( n \to dn \), for some integer \( d \). This number \( d \) is called the degree of \( f \). If \( f \) is a covering map, any point \( x \in A \) has an open neighborhood \( U \) such that \( f^{-1}(U) \) is a disjoint union of \( |d| \) open sets, each of which is mapped homeomorphically onto \( U \) by \( f \).

In this article we consider the dynamics of covering maps \( f : A \to A \) of degree \( d \), with \( |d| > 1 \). Our interest is focused on the existence of a semiconjugacy with \( m_d(z) = z^d \) acting on \( S^1 \). In general, looking for semiconjugacies with maps with known features is useful to classify, to find periodic orbits, to calculate entropy.

Definition 1. A continuous endomorphism \( f : T \to T \), is semiconjugate to \( g : T' \to T' \) (where \( T \) and \( T' \) stand for the annulus or the circle) if there exists a continuous map \( h : T \to T' \) such that \( hf = gh \) and \( h_* \) is an isomorphism.

The case of coverings where \( d = \pm 1 \) (i.e annulus homeomorphisms) has been extensively studied. In particular, much work has been devoted to use the rotations as models for the dynamics of other homeomorphisms. That is, is it possible to decide if a homeomorphism of the annulus that is isotopic to the identity is conjugate or semi-conjugate to a rotation?

The case of the circle is classical: An orientation-preserving homeomorphism of the circle is semi - conjugate to an irrational rotation if and only if its rotation number is irrational, and if and only if it has no periodic points \([\text{Poin}]\). Poincaré’s construction of the rotation number can be generalized to a rotation set for homeomorphisms of the (open or closed) annulus that are isotopic to the identity (see, for example \([\text{BCLP}]\)). If one focuses on pseudo-rotations, the class of annulus homeomorphisms whose rotation set is reduced to a single number \( \alpha \), it is natural to ask the following question: how much does the dynamics of a pseudo-rotation of irrational angle \( \alpha \) look like the dynamics of the rigid rotation of angle \( \alpha \)? (see \([\text{BCLP}], [\text{BCL}]\) for results on this subject). Even in the compact case (i.e irrational pseudo-rotations of the closed annulus) it is known that the dynamics is not conjugate to that of the rigid rotation. Furthermore, examples that are not even semi-conjugate to a rotation in the circle can be constructed using the method of...
Anosov and Katok [AK] (see [FatHe], [FayK], [FayS], [Ha], [He] for examples and further developments about this method).

In contrast, any endomorphism of the closed annulus of degree $d$ is semi-conjugate to $z^d$ on $S^1$. One way to show this (that is developed in this paper) is to generalize the concept of rotation number for endomorphisms and to show that it is a continuous map onto $S^1$ that semi-conjugates with $z^d$ (see Corollary 6 in Section 2.2). However, this is no longer true without compactness. We construct examples of degree $d$ covering maps of the open annulus that are not semiconjugate to $z^d$ on the circle.

The results obtained in [IPRX] give further motivation for our study, as covering maps of the annulus arise naturally when studying surface attractors. A connected set $\Lambda$ is an attracting set of an endomorphism $f$ on a manifold $M$ if there exists a neighborhood $U$ of $\Lambda$ such that the closure of $f(U)$ is a subset of $U$ and $\Lambda = \cap_{n \geq 0} f^n(U)$. The attracting set is called normal if $f$ is a local homeomorphism of $U$ and the restriction of $f$ to $\Lambda$ is a covering of $\Lambda$.

Let $M$ be a compact surface, $f$ an endomorphism of $M$ and $\Lambda$ a normal attractor of $f$. Assume that $f$ is $d:1$ in $\Lambda$. Then, it is shown in [IPRX] that the immediate basin $B_0(\Lambda)$ of $\Lambda$ is an annulus and the restriction of $f$ to it is a $d:1$ covering map. Moreover, if $A$ is an invariant component of $B_0(\Lambda) \setminus \Lambda$, then $A$ is also an annulus and $f$ is a $d:1$ covering of $A$. Finally, if $\Lambda$ is a hyperbolic (normal) attractor, then $\Lambda$ is homeomorphic to a circle and the restriction of $f$ to $\Lambda$ is conjugate to $z \rightarrow z^d$ in $S^1$.

Several questions arise naturally:

1. Is the restriction of a map $f$ to the immediate basin of a normal attracting set semiconjugate to $p_d(z) = z^d$ in $C \setminus \{0\}$?
2. Is the restriction of $f$ to the immediate basin of $\Lambda$ semiconjugate to $z \rightarrow z^d$ in $S^1$?
3. Is the restriction of $f$ to $A$ semiconjugate to $m_d(z) = z^d$ in $S^1$?

We address all of these questions in this paper. The answer to (1) is no, while the answer to (2) and (3) is yes; the proof of these statements is contained in Section 4.

The more general question whether or not any covering map of the open annulus is semiconjugate to $m_d$ has a negative answer as was already pointed out, and a counterexample is given in Section 3.3.

In sections 2 and 3 we find sufficient conditions for the existence of a semiconjugacy with $m_d(z) = z^d$ in $S^1$. In addition, in section 2, we give a method to construct semiconjugacies, based on the rotation number, that applies always to covering maps of the circle. This leads to a classification up to conjugacy for coverings of the circle. At first, we tried to extend, at least in part, this classification to coverings of the annulus. The results obtained throughout this work show however, that a generalization is very difficult, even for maps having empty nonwandering set.

In section 4 we give necessary conditions for an annulus covering to be semiconjugate to $m_d$ in the circle, and obtain a counterexample. This counterexample has empty nonwandering set.

There are few references to coverings of the annulus in the literature. The better known family of examples is given by:

$$(x, z) \in \mathbb{R} \times S^1 \mapsto (\lambda x + \tau(z), z^d),$$
where $\lambda$ is a positive constant less than 1, and $\tau : S^1 \to \mathbb{R}$ is a continuous function. The map with $\tau = 0$ was introduced by Przytycki ([Prz]) to give an example of a map that is Axiom A but not $\Omega$-stable. Then Tsujii ([Tsu]) gave some examples having invariant measures that are absolutely continuous with respect to Lebesgue. In [BKRU] the topological aspects of the attractor are studied and examples of hyperbolic attractors with nonempty interior are given. Note that in these examples the attracting sets cannot be normal, on account of the results of [IPRX] cited above.

In a subsequent work about coverings of the annulus we will investigate some questions about the existence of periodic cycles.

2. Sufficient conditions for the existence of semiconjugacies.

In this section we will show how to construct semiconjugacies of covering maps. The first method, an extension of the rotation number is first defined for circle maps of degree of absolute value greater than one. In the case of circle maps, the rotation number always exist, even if the map is not a covering. This immediately induces a semiconjugacy to $n_d(z) = z^d$. We will give some consequences to the classification of covering maps of the circle in the first subsection. In the second subsection, we will give some sufficient conditions for the existence of the rotation number for maps of the annulus and introduce the shadowing argument. In the final part of this section we introduce the repeller argument, a condition for the existence of semiconjugacies, that in subsequent sections is also seen to be necessary.

2.1. The rotation number. Let $f$ be a continuous self map of $S^1$ of degree $d$, with $|d| > 1$. The circle $S^1$ is considered here as $\{z \in \mathbb{C} : |z| = 1\}$ and its universal covering projection as the map $\pi_0 : \mathbb{R} \to S^1$, $\pi_0(x) = \exp 2\pi i x \in S^1$.

Note that $f$ has a fixed point, and that we may assume that this fixed point is 1. Take $\tilde{f} : \mathbb{R} \to \mathbb{R}$ to be the lift of $f$ that fixes 0.

**Proposition 1.** For every $x \in \mathbb{R}$ the limit

$$\lim_{n \to \infty} \frac{\tilde{f}^n(x)}{d^n}$$

exists and defines a continuous function $\rho(x)$ such that $\rho(x + 1) = \rho(x) + 1$ for every $x$ and $\rho(\tilde{f}(x)) = d\rho(x)$. If $f$ is a covering map, then $\tilde{f}$ is monotonic, so $\rho$ is monotonic.

The proof begins with a simple lemma.

**Lemma 1.** Let $\{y_n\}_{n \geq 0}$ be a sequence of real numbers such that $|y_{n+1} - dy_n| \leq C$ for all $n \geq 0$ for some constant $C$. Then there exists a unique $\bar{y} \in \mathbb{R}$ such that $\{|d^n\bar{y} - y_n|\}_{n \geq 0}$ is bounded. In particular, $y_n/d^n$ converges.

**Proof:** The uniqueness statement is trivial since $|d| > 1$. Note that the sequence $\{\tau_n\}_{n \geq 0}$ defined by $\tau_n = y_n/d^n$ satisfies $|\tau_{n+1} - \tau_n| \leq C/|d|^n$, from which it follows that $\tau_n$ is convergent to some point $\bar{y}$. Notice that

$$|\bar{y} - \tau_n| = \left| \sum_{k=n}^{\infty} \tau_{k+1} - \tau_k \right| \leq \frac{C}{(|d| - 1)|d|^{n-1}}.$$  

This implies $|d^n\bar{y} - y_n| \leq \frac{C|d|}{|d| - 1}$. \hfill $\square$

**Proof of proposition 1.** The map $\tilde{f}$ is a lift of $f$, so $|\tilde{f}(x) - dx| \leq |d|$ for all $x \in \mathbb{R}$. Then for every $x$ and positive $n$ it holds that $|\tilde{f}^{n+1}(x) - d\tilde{f}^n(x)| \leq |d|$, so the
existence of the limit is consequence of the lemma above. To prove the continuity of the limit \( \rho(x) \), take \( n \in \mathbb{N} \), \( x \in \mathbb{R} \) and let \( J \in \mathbb{Z} \) be such that \( f^n(x) \in (J-1,J+1) \). Then, \( f^{n+m}(x) \in (d^m(J-1),d^m(J+1)) \) for every positive \( m \). Then, dividing by \( d^{m+n} \) and taking limit, it comes that \( \rho(x) \) belongs to the closed interval with extreme points \( d^{m}(J-1) \) and \( d^{m}(J+1) \). Given \( \epsilon \), take \( n \) such that \( 2/|d|^n < \epsilon \). Then choose \( \delta \) such that \( f^n(y) \in (J-1,J+1) \) for every \( y \) in the \( \delta \)-neighborhood of \( x \) and apply the above equation also for \( y \) to obtain that \( \rho(y) \) is \( \epsilon \)-close to \( \rho(x) \).

The remaining assertions are immediate.

Given \( z \in S^1 \), take \( x \in \mathbb{R} \) such that \( \exp 2\pi i x = z \) and define \( h_f(z) = \exp 2\pi i \rho(z) \). It follows easily that \( h_f \) is a semiconjugacy.

**Corollary 1.** The function \( \rho \) of the above proposition induces a semiconjugacy \( h_f \) from \( f \) to \( m_d(z) = z^d \).

At the beginning of the next subsection we will explain when this construction can be carried on for annulus maps. But first we will use the previous corollary to obtain some consequences for one dimensional dynamics.

**Definition 2.** Let \( f \) be a covering map of the circle, and \( h_f \) the semiconjugacy to \( m_d \) obtained in Corollary 1. Let \( \Lambda_f \) be the set of points \( x \in S^1 \) such that \( h_f^{-1}(h_f(x)) \) is a nontrivial interval, meaning that it does not reduce to the point \( \{ x \} \).

Some properties are:

1. \( f \) is conjugate to \( m_d \) if and only if \( \Lambda_f \) is empty.
2. If \( I \) is a component of \( \Lambda_f \), then \( f \) is injective in \( I \).
3. \( \Lambda_f \) is completely invariant, meaning \( f^{-1}(\Lambda_f) = \Lambda_f \).
4. If \( x \in h_f(\Lambda_f) \) then \( h_f^{-1}(x) \) is a closed nontrivial interval, because the function \( \rho \) of Proposition 2 and Corollary 1 is monotonic.
5. Any component of the interior of \( \Lambda_f \) is a nontrivial interval; if it is not wandering, then it is (pre) periodic, and every point in it is asymptotic to a periodic point.
6. For \( h_f \) as in Corollary 1, note that \( h_f(\Lambda_f) \) is completely invariant under \( m_d \), so it is dense in \( S^1 \), but is not the whole circle since it is countable.

The first application of the existence of \( h_f \) describes all the transitive covering maps of the circle. A map is transitive if there exists \( x \in S^1 \) such that its forward orbit is dense in \( S^1 \).

**Corollary 2.** Let \( f : S^1 \to S^1 \) be a transitive covering map of degree \( d > 1 \). Then \( f \) is conjugate to \( z^d \).

**Proof:** We know that \( f \) is semi-conjugate to \( z^d \). Recall from property (1) above that \( f \) is conjugate to \( z^d \) if and only if \( \Lambda_f = \emptyset \). Suppose that \( \Lambda_f \) contains a nontrivial interval \( I \). Let \( x \in S^1 \) be such that its forward orbit is dense. Then, there exists \( n \geq 0 \) such that \( f^n(x) \in I \) and \( m > n \) such that \( f^m(x) \in I \). It follows from property (5) above that \( I \) is a periodic interval, contradicting that the forward orbit of \( x \) is dense. \( \square \)

The nonwandering set of every covering of \( f \) is defined as the set of points \( x \) such that for every neighborhood \( U \) of \( x \) there exists an integer \( n > 0 \) such that \( f^n(U) \cap U \neq \emptyset \). The nonwandering set of \( f \) is denoted by \( \Omega(f) \).

**Corollary 3.** If \( f \) is a covering of the circle having degree \( d \) with \( |d| > 1 \), then the set of periodic points of \( f \) is dense in the nonwandering set.
Proof: If $x$ is a nonwandering point of $f$ that is not periodic, then $x$ does not belong to $\Lambda_f$. It follows that the image under $h_f$ of a neighborhood $U$ of $x$ is an interval with nonempty interior in $S^1$. As the $m_d$-periodic points are dense, it follows that the interior of $h_f(U)$ contains a periodic point, say $p$. Then the endpoints of $h_f^{-1}(p)$ must be periodic points of $f$, and one of them is necessarily contained in $U$.

The last application is a classification of coverings of the circle. The classification is roughly speaking, in terms of the following data: given a map $f$, give the degree of $f$, the set $h_f(\Lambda_f)$, that is, which orbits of $m_d$ belong to the image of a Plateau of $h_f$ and an equivalence class of homeomorphisms of the interval for each periodic orbit of $f$ that belongs to $\Lambda_f$. The construction is a little bit complicated as $m_d$ has a lot of self-conjugacies.

First look at the self conjugacies of $m_d$, that is, the set of homeomorphisms of the circle that commute with $m_d$.

**Lemma 2.** The group $G_d$ of self conjugacies of $m_d$ is isomorphic to $D_{d-1}$, the dihedral group.

In other words, if $\{\alpha_j : 0 < j \leq d - 1\}$ denote the $(d - 1)$-roots of unity, $c_j$ is the rotation of angle $\alpha_j$ and $\bar{\alpha}(z) = \bar{z}$ is complex conjugacy, then the set of self conjugacies of $m_d$ is the group generated by the $c_j$ and $\bar{c}$.

**Proof:** First note that the above mentioned homeomorphisms of the circle form a group isomorphic to the dihedral group with $2(d - 1)$ elements. Moreover, all of them commute with $m_d$.

So let $c$ be a homeomorphism of the circle such that $cm_d = m_d c$ and assume first that $c$ preserves orientation and $c(1) = 1$. It follows that $c$ fixes every preimage of 1 under $m_d^n$, for every $n > 0$. As this is a dense set, it follows that $c$ is the identity.

If $c$ does not preserve orientation, then composition with $\bar{c}$ does preserve, so one can assume it preserves. Note that $c(1)$ is a fixed point of $m_d$, so it is a $(d - 1)$-root of unity, say $\alpha_j$. It comes that $c_j^{-1} c = id$ so $c = c_j$. □

Note that the set $h_f(\Lambda_f)$ may contain a periodic orbit; in this case $\Lambda_f$ contains a periodic interval. Assume $I$ is a component of $\Lambda_f$ such that $f^n(I) = I$. Then the restriction of $f^n$ to $I$ is conjugate to a homeomorphism of the interval. Let $\varphi_1 : I_1 \to I_1$ and $\varphi_2 : I_2 \to I_2$ be homeomorphisms, where $I_1$ and $I_2$ are closed intervals. Then $\varphi_1$ and $\varphi_2$ are equivalent if they are topologically conjugate. The quotient space is denoted by $\mathcal{H}$ and the class of a homeomorphism $\varphi$ is denoted by $[\varphi]$. Then one may assign an element $[f^n|_I] \in \mathcal{H}$ to each periodic component $I$ of $\Lambda_f$. As $[f^n|_I] = [f^n|_{f(I)}]$ for every $j > 0$, this correspondence depends on the orbit of the interval and not on the choice of $I$.

Given maps $\tau_1 : \Lambda_i \to \mathcal{H}$, where $\Lambda_i$ is a $m_d$ completely invariant nontrivial subset of $S^1$ for $i = 1, 2$, say that $\tau_1$ is equivalent to $\tau_2$ if there exists $c \in G_d$ such that $c\Lambda_1 = \Lambda_2$ and $\tau_1 = \tau_2 c$. The equivalence class of $\tau$ is denoted by $[\tau]$.

Next we define the data $\mathcal{D}_f$ associated to $f$ as follows:

1. An integer of absolute value greater than one, the degree $d_f$ of $f$.
2. The class $[\tau_f]$, where $\tau_f : h_f(\Lambda_f) \to \mathcal{H}$ is defined as follows: for each $x \in h_f(\Lambda_f) \cap \text{Per}(m_d)$, $\tau_f(x) = [f^n|_{h_f^{-1}(x)}]$, where $n$ is the period of $x$.

When $x \in h_f(\Lambda_f)$ is not periodic, $\tau_f(x)$ is defined as the class of the identity.
Theorem 1. Two covering maps \( f \) and \( g \) are conjugate if and only if \( \mathcal{D}_f = \mathcal{D}_g \).

Proof: Assume first that \( f \) and \( g \) are conjugate, and let \( h \) be a homeomorphism of \( S^1 \) such that \( hf = gh \). It follows that each point \( z \) in \( S^1 \) has the same number of preimages under \( f \) or \( g \). This implies that the degree of \( f \) and \( g \) are equal. Next consider the semiconjugacies \( h_f \) and \( h_g \) such that \( h_ff = m_d h_f \) and \( h_g g = m_d h_g \), given by Corollary 1. Note that \( h \) is a homeomorphism and that \( \Lambda \) is a component of \( \Lambda_g \). Then exists a unique extension of \( f \) such that \( \Lambda \) belongs to \( \Lambda_g \). This implies that given any \( z \in S^1 \), then \( hf \) is constant in \( h^{-1} h_g^{-1} (\{ z \}) \). So it is well defined a function \( \ell(z) = h_f h^{-1} h_g^{-1}(z) \). Note that \( \ell \) is a homeomorphism and that \( \ell(m_d(z)) = h_f h^{-1} h_g^{-1}(m_d(z)) = h_f h^{-1} g h_g^{-1}(z) = h_f h^{-1} g h^{-1}(z) = m_d h_f h^{-1} h_g^{-1}(z) = m_d \ell(z) \), so that \( \ell \) commutes with \( m_d \), which implies that \( \ell \in G_d \). Moreover,

\[
h_f(\Lambda_f) = h_f f(\Lambda_f) = h_f h^{-1}(\Lambda_g) = h_f h^{-1}(g(\Lambda_g)) = h_f h^{-1}(\Lambda_g) = \ell h_g(\Lambda_g).
\]

Finally, note that if \( z \in \Lambda_g \) and \( J = h_g^{-1}(z) \), then \( I := h^{-1}(J) \) is contained in \( \Lambda_f \). If \( z \) is periodic for \( m_d \), with period \( n \), then the restriction of \( g^n \) to \( J \) is conjugated to the restriction of \( f^n \) to \( I \); moreover, as \( \ell(z) = h_f(I) \), it follows that \( \tau_f(\ell(z)) = \tau_g(z) \). The same equation holds trivially when \( z \in \Lambda_g(\Lambda_g) \) is not periodic.

Assume now that \( \mathcal{D}_f = \mathcal{D}_g \), which implies that there exist conjugacies \( h_f \) and \( h_g \) such that \( h_ff = m_d h_f \) and \( h_g g = m_d h_g \). Moreover, there exists a homeomorphism \( c \in G_d \) such that \( c(h_f(\Lambda_f)) = h_g(\Lambda_g) \) and \( \tau_f = \tau_g c \). The conjugacy \( \psi \) between \( f \) and \( g \) will be obtained by solving the equation \( ch_f = h_g \psi \). If \( x \) does not belong to \( \Lambda_f \) then define \( \psi(x) = h_g^{-1}(c(h_f(x))) \); this can be done because \( c(h_f(x)) \) does not belong to \( \Lambda_g \).

Let \( I \) be a component of \( \Lambda_f \), and assume first that it is not (pre-) periodic. Then there is a component \( J \) of \( \Lambda_g \) such that \( h_g(J) = c(h_f(I)) \). As \( I \) and \( J \) are wandering intervals, one can choose any increasing homeomorphism \( \psi \) from \( I \) to \( J \). There exists a unique extension of \( \psi \) to the \( f \)-grand orbit of \( I \) \( (\cup_{n \in Z} f^n(I)) \) such that \( h_g \psi = ch_f \). Note that the image of the \( f \)-grand orbit of \( I \) under \( \psi \) is the \( g \)-grand orbit of \( J \).

Now assume that \( I \) is an \( f \)-periodic interval. Then there exists a component \( J \) of \( \Lambda_g \) such that \( h_g(J) = c(h_f(I)) \), that must be \( g \)-periodic as well, and with the same period, say \( n \). As \( \tau_f(h_f(I)) = \tau_g(ch_f(I)) = \tau_g(h_g(J)) \), it follows that the restrictions of \( f^n \) to \( I \) and \( g^n \) to \( J \) are conjugate, say by a homeomorphism \( \psi \). Then one can proceed as above to extend \( \psi \) to the grand orbit of \( I \). The fact that \( \psi \) is a homeomorphism follows from the equation \( ch_f = h_g \psi \).

The space \( \mathcal{H} \) is uncountable, so the set of conjugacy classes of coverings of the circle is also uncountable. However, restricting to the class of maps having all its periodic points hyperbolic, we state the following question: How many equivalence classes of coverings are there? This question should be rather easy if it is shown that every non pre-periodic orbit of \( m_d \) can be contained in the image of \( h_f(\Lambda_f) \) for some \( f \). However, wandering intervals are forbidden under stronger assumptions.

Corollary 4. The number of equivalence classes of \( C^2 \) covering maps of the circle all of whose periodic points are hyperbolic and critical points are non-flat is countable.

Note that a \( C^2 \) covering \( f \) may have points \( z \) where \( f'(z) = f''(z) = 0 \).
Proof: The corollary follows by two strong theorems. On one hand, Mañé proved that under these hypothesis the set of attracting periodic orbits is finite. See Chapter 4 of [MS], where it is also proved that there cannot be wandering intervals in these cases. \qed

2.2. The shadowing argument. We have seen in the previous subsection some applications of the generalized rotation number for covering maps of the circle. The question now is to what extent this can be carried out when \( f \) is an annulus covering, or just a degree \( d \) \((|d| > 1)\) annulus endomorphism.

Let \( A \) be the open annulus, \( A = (0, 1) \times S^1 \) and \( f \) a degree \( d \) covering map of \( A \). We will use some standard notations: the universal covering projection is the map \( \pi : \tilde{A} = (0, 1) \times \mathbb{R} \rightarrow A \) given by \( \pi(x, y) = (x, \exp(2\pi iy)) \). We will denote by \( \tilde{f} \) any lift of \( f \) to the universal covering, that is, \( \tilde{f} \) is a map satisfying \( f \pi = \pi \tilde{f} \). Note that \( \tilde{f}(x, y + 1) = \tilde{f}(x, y) + (0, d) \). For a point \((x_0, y_0) \in \tilde{A} \), let \((x_n, y_n) = \tilde{f}^n(x_0, y_0)\), where \( n \geq 0 \). As a Corollary of the proof of Proposition\( \ddagger \) we have:

Corollary 5. Assume that there exists a constant \( C \) such that \(|y_1 - dy_0| \leq C\) for every \((x_0, y_0) \in \tilde{A}\). Then, the limit

\[
\lim_{n \to +\infty} \frac{y_n}{d^n}
\]

exists for every \((x_0, y_0)\), and is denoted by \( \rho(x_0, y_0) \). The function \( \rho \) is continuous and satisfies \( \rho \tilde{f}(x_0, y_0) = d \rho(x_0, y_0) \), and \( \rho(x, y + 1) = \rho(x, y) + 1 \).

In particular, the projection of \( \rho \) to the annulus gives a semiconjugacy between \( f \) and \( m_d \).

Moreover, \( \rho(x_0, y_0) \) is the unique real number \( \bar{y} \) for which it holds that \( \sup_n \{|y_n - d^n \bar{y}|\} \) is bounded. In this sense, the orbit of \((x_0, y_0)\) under \( f \) is shadowed by the orbit of \( \rho(x_0, y_0) \) under multiplication by \( d \). We will need to extend these arguments to obtain more consequences.

Assume that \( K \) is a compact subset of \((0, 1)\). Then there exists a constant \( C_K \), depending on \( K \), such that \(|y_1 - dy_0| \leq C_K\) for every \((x_0, y_0) \in K \times [0, 1] \). Now assume that \( y_0 \) is any real number, take \( y_0' \in [0, 1) \) such that \( y_0 - y_0' \) is an integer, and note that \( y_1 = y_1' + d(y_0 - y_0') \), where \( y_1' \) denotes the second coordinate of \( f(x_0, y_0') \). It follows that

\[
|y_1 - dy_0| = |y_1' - dy_0'| \leq C_K
\]

holds for every \( x_0 \in K \) and \( y_0 \in \mathbb{R} \).

Proposition 2. Let \((x_0, y_0) \in \tilde{A} \) be such that \( x_n \in K \) for every \( n \geq 0 \), where \( K \subset (0, 1) \) is compact and \((x_n, y_n) = \tilde{f}^n(x_0, y_0) \). Then there exists a unique \( \bar{y} \in \mathbb{R} \) such that

\[
\sup_{n \geq 0} \{|y_n - d^n \bar{y}|\} < \infty.
\]

Proof: Let \((x_0, y_0) \in \tilde{A} \). By hypothesis \( x_n \in K \) for every \( n \geq 0 \). It follows by equation \( \ddagger \) that \(|y_{n+1} - dy_{n}| \leq C\) for every \( n \geq 0 \), then the assertion of the proposition is consequence of Lemma\( \ddagger \) \( \Box \)

Corollary 6. Assume \( f \) is a covering of the closed annulus \( \bar{A} \). Then there exists a semiconjugacy of \( f \) with the map \( m_d(z) = z^d \) in \( S^1 \).
We say that a subset $X \subseteq H$ is essential, if $i_*(H_1(U, \mathbb{Z})) = \mathbb{Z}$, where $i_* : H_1(U, \mathbb{Z}) \to H_1(A, \mathbb{Z})$ is the induced map in homology by the inclusion $i : U \to A$. We say that a subset $X \subseteq A$ is essential if any neighbourhood of $X$ in $A$ is essential. Equivalently, a subset $X \subseteq A$ is essential if and only if it intersects every connector.

**Remark 1.** If $X \subseteq A$ is essential, and $h : A \to S^1$ is a continuous map satisfying $h_* = \text{Id}$, then $h(X) = S^1$. Indeed, there is a basis of neighborhoods $(U_n)_{n \geq 0}$ of $X$ such that $h(U_n) = S^1$, because $h_* = \text{Id}$ and every neighborhood of $X$ is essential. Then, by continuity $h(X) = S^1$.

**Corollary 7.** If $X \subseteq A$ is compact and $f(X) \subseteq X$, then there exists a continuous function $h : X \to S^1$ such that $h f|_{X} = m_d h$. Moreover, there exists a continuous function $\hat{h} : \pi^{-1}(X) \to \mathbb{R}$ such that $\pi_0 \hat{h} = h \pi|_{\pi^{-1}(X)}$ and $\hat{h}(x, y) + (0, 1) = \hat{h}((x, y)) + (0, 1)$.

Moreover, if $f^{-1}(X) = X$ or $X$ is connected and essential then $h$ is surjective.

**Proof:** Let $\tilde{X} = \pi^{-1}(X)$. Note that as $X$ is compact, there exists a compact set $K \subset (0, 1)$ such that $\tilde{X} \subset K \times \mathbb{R}$. So, we may apply Proposition 2 as in the previous corollary. If $f^{-1}(X) = X$, then $m_d\tilde{h}(X)) = h(X)$ and so $h(X)$ is dense in $S^1$. As $h(X)$ is also compact, $h(X) = S^1$. If $X$ is connected and essential, $h$ must be surjective as was pointed out in the previous remark.

As an application, let $U$ be an open set homeomorphic to an open annulus and $f : U \to f(U)$ a covering map of degree $d$, $|d| > 1$. Assume that $f(U) \subseteq U$ and let $X = \cap_{n \geq 0} f^n(U)$. Then, $f|_X$ is semi-conjugate to $m_d$.

2.3. The repeller argument.

**Definition 3.** A subset $C$ of $A$ is called a connector if it is connected, closed, and has accumulation points in both components of the boundary of $A$. In other words, given $r > 0$ there are points $(x, y)$ and $(x', y')$ in $C$ with $x < r$ and $x' > 1 - r$. A connector is called trivial if its complement contains a connector.

If $f$ is a degree $d$ covering map of the annulus, then each connected component of the preimage of a connector is a connector. The preimage of a trivial connector is equal to the disjoint union of $d$ trivial connectors.

Note also that just one connected component of the complement of a trivial connector contains a connector. This follows from the fact that the union of two disjoint connectors separates the annulus, and so the existence of two connectors in different components of the complement of $C$ implies that $C$ is disconnected. When $C$ is trivial, the component of $A \setminus C$ that contains a connector is called the big component of the complement of $C$.

**Definition 4.** An invariant connector $C$ is repelling for a map $f$ if there exists a neighborhood $U$ of $C$ such that $f(U)$ contains the closure of $U$ and every point in $U$ has a preorbit contained in $U$ that converges to $C$. A connector $C$ is called free for a covering map $f$ if $f(C) \cap C = \emptyset$.

**Proposition 3.** If $f$ is a degree $d$ covering of the open annulus $A$ having a trivial free connector, then there exist $d - 1$ repelling connectors for $f$. 

Proof: Let $C$ be a trivial free connector and $C_1, \ldots, C_d$ the components of $f^{-1}(C)$. Denote by $D_1, \ldots, D_d$ the components of $A \setminus f^{-1}(C)$ that contain connectors (note that $D_1 \cup D_2 \cup \cdots \cup D_d$ is equal to the intersection of the big components of $C_1, \ldots, C_d$).

Moreover, for each $1 \leq i \leq d$, the set $f(D_i)$ is equal to the big component of $A \setminus C$. On the other hand, as $C$ is free, it follows that $f^{-1}(C) \cap C = \emptyset$. Thus $C$ is contained in some $D_i$, say $D_d$, to simplify. It follows that $f(D_i)$ contains the closure of $D_i$ for all $1 \leq i \leq d-1$, but not for $i = d$.

Let $\tilde{D}_i$ be defined as the intersection for $n \leq 0$ of the sets $D_i^{(n)} := \cap_{j=0}^n f^{-j}(D_i)$.

Then $\tilde{D}_i$ is a repelling connector:

1. It is closed in the open annulus because for every $n > 0$ the closure of $D_i^{(n)}$ is contained in $D_i^{(n-1)}$.
2. It is connected because each $D_i^{(n)}$ is connected and the sequence is decreasing.
3. It is an invariant repeller by construction.

□

It also follows from the construction that the complement of each repeller $\tilde{D}_i$ is connected. This will be used in the next result.

Corollary 8. Let $f$ be a degree $d$ covering of the open annulus and assume that there exists a trivial free connector. Then there exists a semiconjugacy between $f$ and $m_d$ in $S^1$.

Proof: By proposition 8 there exist $\tilde{D}_1, \ldots, \tilde{D}_{d-1}$ repellers, that are trivial. The set of $D_j$ as $1 \leq j \leq d-1$ is ordered in such a way that between two consecutive $\tilde{D}_j$ and $\tilde{D}_{j+1}$ there is no other $\tilde{D}_i$ (we consider counterclockwise orientation). Begin defining $h$ in the repellers, in such a way that $h$ is constant in each repeller, and equal to a repeller of $m_d$; in other words, $h(\tilde{D}_j) = \exp(\frac{2\pi i j}{d})$. Consider the component $\Delta_j$ between $\tilde{D}_j$ and $\tilde{D}_{j+1}$. Note that by the construction of the repelling connectors in Proposition 8 any point in $\Delta_i$ has two $f$-preimages in $\Delta_i$, while every point outside $\Delta_i$ has exactly one preimage in $\Delta_i$. It follows that for each $1 \leq j \leq d-1$, the set $f^{-1}(\tilde{D}_j)$ has exactly one component in $\Delta_i$. It follows that there exists a unique way of extending $h$ to the preimage of the connectors in such a way that it still preserves the cyclic order and the semiconjugacy condition. One can argue as above to obtain that in each component of the complement of $f^{-1}(\cup \tilde{D}_i)$ there exists exactly one component of $f^{-2}(\tilde{D}_j)$, and this holds for each $j$.

Recursively define $h$ in $\cup_i f^{-k}(\tilde{D}_j)$. It is claimed now that $h$ can be uniquely extended to a semiconjugacy from $f$ to $m_d$.

Let $x$ arbitrary in the annulus and let $\epsilon = 1/d^n$ be given. Let $U$ be a neighborhood of $x$ not intersecting $\cup_{j=0}^n f^{-j}(\tilde{D}_i)$ for every $i$. Then the image of $U$ under $h$ has length less than $\epsilon$. It follows that $h$ is continuous. The condition $h^* = id$ follows by construction.

□

Corollary 9. Let $f$ be a degree $d$ covering of the open annulus and assume that there exists an invariant trivial connector $C$ ($f(C) = C$). Then there exists a semiconjugacy between $f$ and $m_d$.

Proof: The preimage of a trivial connector $C$ has exactly $d$ components, each one of which is a trivial connector. So, if $C$ is invariant, then $f^{-1}(C)$ is equal to the
union of $C$ plus $d - 1$ connectors $C_1, \ldots, C_{d-1}$. Each of the $C_i$ is a trivial free connector, hence the previous corollary applies directly.

\section*{Lemma 4.}

Let $A$ be contained in a topological disk of $X$.

Proof: Let $z$ be a point of $A$ connected, for every $\gamma$ joining $p$ and $f(p)$ such that $\gamma(t) = (x(t), z(t))$ with $x(t)$ an increasing function. So, $C = \cup_{n \in \mathbb{Z}} \gamma_n$ is an invariant connector, where $\gamma_n = f^n(\gamma)$ for $n \geq 0$, and for $n < 0$, $\gamma_n$ is defined by induction, beginning with $\gamma_{-1}$ being the lift of $\gamma(1-t)$ starting at $x$. Note that $C$ is trivial, due to the choice of $\gamma$ with increasing $x(t)$ and the formula for $f$.

3. Necessary conditions.

Throughout this section we will assume that there exists a semiconjugacy $h$ between a covering map $f$ of degree $d$ ($d > 1$) of the annulus and $m_d(z) = z^d$ in $S^1$. It is as always assumed that the map $h_*$ induced by $h$ on homotopy is an isomorphism.

\section*{Lemma 3.}

Let $p$ and $q$ be fixed points of $f$. The following conditions are equivalent:

1. There exists a curve $\gamma$ from $p$ to $q$ such that $\gamma_n(1) = q$ for every $n > 0$, where $\gamma_n$ is the unique lift of $\gamma$ by $f^n$ that begins at $p$.
2. There exists a curve $\gamma$ from $p$ to $q$ such that $f\gamma$ and $\gamma$ are homotopic with fixed endpoints.
3. $h(p) = h(q)$.

Proof. (1) implies (2): If $f(\gamma)$ is not homotopic to $\gamma$ then $\beta = \gamma^{-1}.f(\gamma)$ is not null-homotopic. If the lift of $\beta$ under $f^n$ is a closed curve for every $n \geq 0$, then the curve $\beta \neq 0$ belongs to $\bigcap_{t \geq 0} f^t(\mathbb{Z})$, where $f_*$ is the induced map in homology. This cannot hold since $f_*$ is multiplication by $d$.

(2) implies (1): If $f(\gamma)$ is homotopic to $\gamma$, then the final point of $\gamma_1$ is $q$, because $\gamma^{-1}.\gamma$ is the lift of the null homotopic curve $\gamma^{-1}.f(\gamma)$. It follows also that $\gamma_1^{-1}.\gamma$ is null, so the same argument shows that $\gamma_2^{-1}.\gamma_1$ is null, and so on.

(3) implies (2): Let $C_{x,y}(A)$ be the quotient space of curves from $x$ to $y$ in the annulus, where two curves are identified whenever there is a homotopy with fixed endpoints between them. It is easy to see that the hypothesis on $h$ imply that $h$ induces a bijection $h_*$ from $C_{x,y}(A)$ to $C_{h(x), h(y)}(S^1)$. Therefore, $h(p) = h(q)$ implies that there exists a curve $\gamma$ from $p$ to $q$ such that $h(\gamma)$ is null-homotopic. Then $hf(\gamma) = (h\gamma)^d$ is also null homotopic. It follows that $f\gamma$ is homotopic to $\gamma$, because $h_{x,y}$ is injective.

(2) implies (3): Let $\gamma$ be as in (2). Then, $h(\gamma) \sim hf(\gamma) \sim m_d h(\gamma)$. But $h(\gamma)$ joins $h(p)$ to $h(q)$, fixed points of $m_d$. So, $h(\gamma) \sim m_d h(\gamma)$ implies $h(p) = h(q)$.

3.1. Connectivity. Denote by $A^*$ the compactification of the annulus with two points, that is $A^* = A \cup \{N, S\}$. Considered with the natural topology, it is homeomorphic to the two-sphere. We will use the spherical metric in $A^*$.

\section*{Lemma 4.}

Let $h$ be a semiconjugacy between $f$ and $m_d$. Then $h^{-1}(z)$ contains a connector, for every $z \in S^1$.

Proof: Let $X = h^{-1}(z)$ and notice that every connected component of $X$ is contained in a topological disk of $A$. Otherwise, $X$ would be an essential subset of $A$ and $h|_X$ would be surjective (see Remark 1), a contradiction.
Suppose that $X$ does not contain a connector. This means that every connected component of $X$ is either a bounded subset of $A$ or accumulates on $N$ but not on $S$ or vice-versa. Let $C_N$ be the set of connected components of $X$ accumulating on $N$, and $C_S$ be the set of connected components of $X$ accumulating on $S$. So, $A' = A \setminus (C_N \cup C_S)$ is homeomorphic to an annulus, as it complement in the sphere $A^*$ has exactly two connected components. As every connected component of $X$ is contained in a topological disk of $A$, the set of bounded connected components is contained in some closed disk $D$ of $A$. So, $U = A' \setminus D$ is an open set contained in $A \setminus X$ that contains an essential loop of $A$. So, $h|_U$ is surjective because $h_* = Id$, contradicting the fact that $h(U)$ does not contain $z$.

The following result completed with Corollary 9 implies that $f$ is semiconjugate to $m_d$ if and only if there exists a nontrivial invariant connector.

**Corollary 10.** With $f$ as above, there exists an invariant trivial connector contained in $h^{-1}(1)$.

**Proof:** Use the previous lemma to obtain a connector $C$ in $h^{-1}(1)$ and assume it is maximal (not contained in a larger connector in $h^{-1}(1)$). As $h$ is a semiconjugacy, $C$ is trivial. If $C$ is not free, then it is invariant, and if it is free, then the repeller argument (section 3) provides an invariant connector. □

We will use the following two propositions to construct an example of $d : 1$ covering of the annulus that is not semi-conjugate to $z^d$.

**Proposition 4.** If there exists a semiconjugacy $h$ from $f$ to $m_d$, then there is a lift $\tilde{h} : (0, 1) \times \mathbb{R} \to \mathbb{R}$ of $h$ and a lift $\tilde{f}$ of $f$ such that $\tilde{hf} = d\tilde{h}$. Moreover $\tilde{h}$ satisfies the following property: Given a compact set $K \subset (0, 1)$ there exists a constant $M$ depending on $K$ such that $|\tilde{h}(x, y) - y| \leq M$ for every $x \in K$ and $y \in \mathbb{R}$.

**Proof:**
Note that for any lift $\hat{h}$ of $h$, $\hat{h}(x, y + 1) = \hat{h}(x, y) + 1$, since $h_* = id$. Take any lift $\tilde{f}'$ of $f$, and note that as $\hat{h}\tilde{f}'$ and $d\hat{h}$ are lifts of the same map, $\hat{h}\tilde{f}' - d\hat{h} = k \in \mathbb{Z}$. Define $f = \tilde{f}' - (0, k)$. So, $\hat{h}\tilde{f} = \hat{h}\tilde{f}' - k = k + d\hat{h} - k = dh$.

There exists a constant $M$ such that $|\hat{h}(x, y) - y| \leq M$ for every $x \in K$ and $y \in [0, 1]$, since this set is compact, but $\hat{h}(x, y + 1) = \hat{h}(x, y) + 1$ implies that the same constant $M$ bounds $|\hat{h}(x, y) - y|$ for $x \in K$ and $y \in \mathbb{R}$.

**3.2. Bounded preimages.** We note $\gamma_1 \land \gamma_2$ the algebraic intersection number between two arcs in $A$ whenever it is defined. In particular, when both arcs are loops, when one of the arcs is proper and the other is a loop or when both arcs are defined on compact intervals but the endpoints of any of the arcs does not belong to the other arc. For convention, we set $c \land \gamma = 1$ if $c : (0, 1) \to A$ and $\gamma : [0, 1] \to A$ verify:

$$c(t) = (t, 1), \quad \gamma(t) = (1/2, e^{2\pi it}).$$

Note that the arc $c$ is a connector whose intersection number with any loop in $A$ gives the homology class of the loop.

If $\alpha$ is a loop in $A$, denote by $j\alpha$ the concatenation of $\alpha$ with itself $j$ times.

**Proposition 5.** Let $f$ be a covering of the open annulus $A$ and assume that $f$ is semiconjugated to $m_d$. Then the following condition holds:
(*) For each compact set $K \subset A$ there exists a number $C_K$ such that: given $\alpha \subset A$ a simple closed curve, $n \geq 1$ and $j \in [1, \ldots, d^{n-1}]$ then any $f^n - \text{lift } \beta$ of $\alpha \omega$ with endpoints in $K$ satisfies $|\beta \wedge c| \leq C_K$.

Proof. Let $h$ be the semiconjugacy between $f$ and $m_d$ and let $\tilde{h}$ and $\tilde{f}$ be the lifts of $h$ and $f$ verifying $\tilde{h}f = dh$. Take $a, b \in (0, 1)$ such that the set $\tilde{K} = [a, b] \times \mathbb{R}$ contains $\pi^{-1}(K)$. By Proposition 4 above, there exists a constant $M$ such that $|\tilde{h}(x, y) - y| \leq M$ whenever $(x, y) \in \tilde{K}$.

Take $\alpha, n, j$ and $\beta$ as in the statement. Let $\tilde{\beta}$ be a lift of $\beta$ to the universal covering. As the endpoints of $\beta$ belong to $K$, then the extreme points $(x_1, y_1)$ and $(x_2, y_2)$ of $\tilde{\beta}$ belong to $\tilde{K}$. Note that it is enough to show that $|y_2 - y_1|$ is bounded by a constant $C_K$. We will prove that this holds with $C_K = 2M + 1$.

Note that $\tilde{f}^n(x_1, y_1)$ and $\tilde{f}^n(x_2, y_2)$ are the endpoints of a lift of $\alpha \omega$ to the universal covering. This means that $|\tilde{f}^n(x_1, y_1) - \tilde{f}^n(x_2, y_2)| = (0, j)$. It follows that $|\tilde{h}(\tilde{f}^n(x_1, y_1)) - \tilde{h}(\tilde{f}^n(x_2, y_2))| = j$.

Then,

$$|d^n \tilde{h}(x_1, y_1) - d^n \tilde{h}(x_2, y_2)| = |\tilde{h}(\tilde{f}^n(x_1, y_1)) - \tilde{h}(\tilde{f}^n(x_2, y_2))| = j \leq d^n,$$

so $|\tilde{h}(x_1, y_1) - \tilde{h}(x_2, y_2)| \leq 1$. Finally, using that the endpoints of $\tilde{\beta}$ belong to $\tilde{K}$, it follows that $|y_1 - y_2| \leq |y_1 - \tilde{h}(x_1, y_1)| + |\tilde{h}(x_1, y_1) - \tilde{h}(x_2, y_2)| + |\tilde{h}(x_2, y_2) - y_2| \leq 2M + 1$.

3.3. Counterexample. Now we construct $f$, a covering of the open annulus for which the condition (*) introduced in Proposition 5 does not hold. This implies that $f$ is not semiconjugate to $m_d$.

Let $\{a_n : n \in \mathbb{Z}\}$ be an increasing sequence of positive real numbers such that $a_n \to 0$ when $n \to -\infty$ and $a_n \to 1$ when $n \to +\infty$. Define the annuli $A_n$ as the product $[a_n, a_{n+1}] \times S^1$, for each $n \in \mathbb{Z}$. Let also $\lambda_n$ be the affine increasing homeomorphism carrying $[0, 1]$ onto $[a_n, a_{n+1}]$. Define $f(x, z) = (\lambda_{n+1}(\lambda_n^{-1}(x)), z^2)$ for $x \in [a_n, a_{n+1}]$, for every $n \leq -1$, that is, $(x, z) \in \cup_{n<0} A_n$.

Assume $f$ constructed until the annulus $A_{n-2}$ for some $n$ and we will show how to construct the restriction of $f$ to $A_{n-1}$. We will suppose that $f(a_k, z) = (a_{k+1}, z^2)$ for every $k \leq n - 1$ and every $z \in S^1$.

Let $\alpha$ be a curve in $A_0$ such that

1. $\alpha$ joins $(a_0, 1)$ with $(a_1, 1)$.
2. The lift $\alpha_0$ of $\alpha$ to the universal covering that begins at $(a_0, 0)$, ends at $(a_1, n)$.
3. $\beta := f^{n-1}(\alpha)$ is simple.

Note that $f^{n-1}$ is already defined in $A_0$. To prove that such an $\alpha$ exists, take first any $\alpha'$ satisfying the first and second conditions. Then $f^{n-1}(\alpha')$ is a curve joining $(a_{n-1}, 1)$ with $(a_n, 1)$. Maybe $f^{n-1}(\alpha')$ is not simple, but there exists a simple curve $\beta$ homotopic to $f^{n-1}(\alpha')$ and with the same extreme points. Then define $\alpha$ as the lift of $\beta$ under $f^{n-1}$ that begins at the point $(a_0, 1)$.

Choose any simple arc $\beta'$ disjoint from $\beta$ and contained in $A_{n-1}$, joining the points $(a_{n-1}, -1)$ and $(a_n, -1)$. Note that $f^{-(n-1)}(\beta')$ is the union of $2^{n-1}$ curves, all of them disjoint from $\alpha$. Choose any one of these curves and denote it by $\alpha'$. Note that it does not intersect $\alpha$. Observe that there is a lift $\alpha_0'$ of $\alpha'$ that begins
in a point \((a_0, t)\) and ends at \((a_1, n + t)\) in the universal covering. Then choose a point \(Y \in \alpha'\) whose lift \(Y'\) in \(\alpha'_0\) has second coordinate greater than \(n\). Also choose a point \(X\) in \(\alpha\) whose lift \(X'\) in \(\alpha_0\) has second coordinate less than \(1/2\).

Observe that \(f^{n-1}(X) \in \beta\) and \(f^{n-1}(Y) \in \beta'\).

The complement of \(\beta \cup \beta'\) in the interior of \(A_{n-1}\) consists of two open discs, each one of which homeomorphic to the complement of \(s\) in the interior of \(A_n\), where \(s\) is the segment \(\{(x, 1) : a_n < x < a_{n+1}\}\). Then it is possible to take a homeomorphism from each of these components and extend it to the boundary in such a way that the image of \(\beta\) is \(s\) and the image of \(\beta'\) is also \(s\), and carrying \(X\) and \(Y\) to the same point \(p \in s\). If the homeomorphisms are taken carefully, they induce a covering \(f\) from \(A_{n-1}\) to \(A_n\). Now take a simple essential closed curve \(\gamma\) contained in \(A_n\) and with base point \(p\). Note that for some \(j \in [1, \ldots, 2^{n-1}]\), the curve \(j\gamma\) lifts under \(f^n\) to a curve joining \(X\) to \(Y\). But the difference between the second coordinates of \(Y'\) and \(X'\) is greater than \(n - 1\). By the remark preceding Proposition 5, it follows that the intersection number of a lift of \(j\gamma\) and the connector \(c\) in \(A_0\) exceeds \(n - 1\). Taking \(K = A_0\) in Proposition 5 note that \(C_K \geq n - 1\), and as this can be done for every positive \(n\), it follows that \(f\) does not satisfy condition (*).

3.4. Inverse limit. Here we will define the inverse limit of a covering \(f\) and prove that if \(f\) has a fundamental domain, then its inverse limit is semiconjugate to the inverse limit of \(m_d\). This shows that for the example given above, even that \(f\) is not semiconjugate to \(m_d\), the semiconjugacy can be defined on inverse limits.

Definition 5. The inverse limit set of a self map \(f\) of a topological space \(X\) (denoted \(X_f\)) is defined as the set of orbits of points in \(X\), endowed with the product topology inherited from the countable product of \(X\). If \(\sigma_f\) denotes the shift map on \(X_f\), then \(\sigma_f\) is a homeomorphism onto \(X_f\) and if \(n \in \mathbb{Z}\), then \(\pi_n \sigma_f = f \pi_n\), where \(\pi_n : X_f \to X\) denotes the projection onto the \(n\)th coordinate. The inverse limit set of the map \(m_d\) in \(S^1\) is denoted by \(S_d\); the map \(\sigma_d := \sigma_{m_d}\) is commonly known as a solenoid.

Definition 6. Let \(f\) be a covering of the annulus \(A\). A fundamental domain for \(f\) is a compact annulus \(A_0\) such that every orbit of \(f\) hits at least once and at most twice in \(A_0\).

It follows that one of the components of the boundary of \(A_0\), denoted \(\partial^- (A_0)\), is the preimage of the other component of the boundary of \(A_0\), denoted \(\partial^+ (A_0)\).

The preimage of a fundamental domain is also a fundamental domain, but not the image. For instance, take \(p_d(z) = z^d\) in the punctured unit disc, \(\mathbb{D} \setminus \{0\}\), let \(\gamma\) be a simple closed curve (close to a circle) centered at the origin but that is not symmetric with respect to the origin, this means there exists a point \(x \in \gamma\) such that \(-x \notin \gamma\). Then \(\gamma^{-1}(\gamma) \cap \gamma = \emptyset\), and \(p_d(\gamma)\) is not a simple curve. It follows that the set of points between \(\gamma\) and \(p_d^{-1}(\gamma)\) is a fundamental domain but its image is not.

If \(f\) has a fundamental domain, then the \(\omega\)-limit set and \(\alpha\)-limit set of every point \(x\) are empty. Furthermore, the nonwandering set of \(f\) is also empty. To prove this, assume that \(x_0\) is a nonwandering point. Then there exists an \(f\)-orbit of \(x_0\), denoted \(\{x_n\}_{n \in \mathbb{Z}}\), contained in \(\Omega(f)\). It follows there exist at least one \(k \in \mathbb{Z}\) such that \(x_k \in A_0\). But a point in \(A_0\) cannot be nonwandering.

Proposition 6. If a covering map \(f\) of the annulus has a fundamental domain, then \(\sigma_f\) is semiconjugate to \(\sigma_d\), with \(d\) the degree of \(f\).
Here, a semiconjugacy is a continuous surjective map $h : X_f \to S_d$ such that $h \sigma_f = \sigma h$.

**Proof:** Let $A_0$ be a fundamental domain for $f$. Then $\sigma_f$ has a fundamental domain $A_0$ defined as the set of points $z$ such that $z_0 \in A_0$. However, the map $\sigma_d$ does not have a fundamental domain; note, for instance, that its nonwandering set is the whole $S_d$. Then the image under a semiconjugacy of the fundamental domain of $\sigma_f$ must be the whole $S_d$. It suffices to construct the semiconjugacy $h$ in $A_0$. One has to define functions $h_n : A_0 \to S^1$ to determine $h = \{h_n\}$. It begins with the choice of a surjective map from $A_0$ to $S^1$, the function $h_0$ will depend just on the 0-coordinate of a point $\bar{z} \in A_0$. It is obvious how to define $h_n$ for $n$ positive, but for negative $n$, one has to determine regions where $f^n$ is injective.

We proceed to do this. Take any simple arc $\gamma_0$ joining a point $x \in \partial^-(A_0)$ with the point $f(x) \in \partial^+(A_0)$. The curve $\gamma_0$ is a connector of $A_0$. Then $f^{-1}(\gamma_0)$ is equal to the union of $d$ simple arcs, each one of which is a connector of $A_{-1} = f^{-1}(A_0)$. We assume that these arcs are enumerated as $\gamma_0^n, \ldots, \gamma_{d-1}$, in such a way that $\gamma_0^n$ has an extreme point in $x$, and the extreme points of $\gamma_j^n$ in $\partial^-(A_0)$ are counterclockwise ordered. By induction, we can define, for each positive $n$, a sequence $\{\gamma_j^n : 0 \leq j \leq d^n - 1\}$ of connectors of $A_{-n}$, such that $\gamma_0^n$ has an extreme point in common with $\gamma_0^{n-1}$. Moreover, if the extreme point of $\gamma_j^n$ in $f^{-n}(\partial^+(A_0))$ is denoted by $x_j^n$, then these points are counterclockwise oriented in the curve $f^{-n}(\partial^+(A_0))$. Note that the restriction of $f^n$ to the open region $D_j^n$ contained in $f^{-n}(A_0)$ and bounded by $\gamma_j^n$ and $\gamma_{j+1}$ is injective, for $0 \leq j \leq d^n - 1$. By convenience, we will denote $\gamma_0^n := \gamma_0^n$.

We will first define a function $\phi : A_0 \to S^1$ satisfying some conditions. The function $\phi$ will be used to define $h_0$; indeed, $h_0(\bar{z})$ will depend only on the value of $z_0$, and will be equal to $h_0(\bar{z})$ if $\bar{z}$ belongs to $A_0$. The conditions imposed on $\phi$ are: 1. $\phi$ is continuous, 2. $\phi$ carries $\gamma_0$ to $\{1\}$, 3. $\phi$ carries the circle $\partial^-(A_0)$ onto $S^1$ in such a way that $f(y) = f(y')$ if and only if $m_d(\phi(y)) = m_d(\phi(y'))$. The last assertion allows to define $\phi(y)$ whenever $y \in \partial^-(A_0)$.

To define $h$, begin with a point $\bar{z} = \{z_n\}_{n \in \mathbb{Z}}$ contained in $A_f$, and assume that $z_0 \in A_0 \setminus \partial^-(A_0)$. Assume also that $z_0 \notin \gamma_0$. Then, for each positive $n$, the point $z_n$ belongs to $D_j^n$ for some (unique) $j$, $0 \leq j \leq d^n - 1$. Then let $h_{-n}(\bar{z})$ be the unique $m_d^n$-preimage of the point $\phi(z_0)$ that belongs to the arc in $S^1$ with extreme points $e^{2\pi i \frac{j}{d^n}}$ and $e^{2\pi i \frac{j+1}{d^n}}$.

Now, if $z_0$ belongs to $\gamma_0$, then $z_n$ belongs to some $\gamma_j^n$. Then define $h_{-n}(\bar{z}) = e^{2\pi i \frac{j}{d^n}}$. For positive $n$, define $h_n(\bar{z}) = m_d^n(\phi(z_0))$.

The equation $\pi_n(h(\bar{z})) = h_n(\bar{z})$ defines a map from $\pi^{-1}(A_0) \cap A_f$ to $S_d$. By construction, each $h_n$ is continuous, from which it follows that $h$ is continuous. Note also that the image of this map is all $S_d$.

This concludes the definition of the restriction of $h$ to the fundamental domain of $\sigma_f$.

To define $h$ in the whole $A_f$, take any $\bar{z} \in A_f$. Then there exists a unique $k \in \mathbb{Z}$ such that $z_k \in A_0 \setminus \partial^-(A_0)$. Define $h(\bar{z}) = \sigma_d^k \sigma_f(\bar{z})^k$. □
4. Basins.

Let $f$ be an endomorphism of a surface $M$ having an attracting set $\Lambda$ which is normal (see definition in the introduction) and has degree $d > 1$; assuming that $f$ has no critical points in the basin $B_0(\Lambda)$ of $\Lambda$, it comes that the restriction of $f$ to the immediate basin of $\Lambda$ is a covering of the same degree. Moreover, if $C$ is a component of $B_0(\Lambda) \setminus \Lambda$, then $C$ is an annulus and if $C$ is invariant, then $f$ is a covering of $C$ (see [IPRX]). We will prove here that under these conditions, $f$ in $C$ is semiconjugate to $m$. We note that this is not an immediate consequence of Corollary 6 because the closure of $C$ is not necessarily a closed annulus.

The following example is illustrating on the situation. The map $f(z) = z^2 - 1$ is a hyperbolic map of the two-sphere having a superattractor at $\infty$. The restriction of $f$ to the basin of $\infty$ is a degree two covering map of the annulus $C \setminus \hat{\mathcal{J}}$ conjugate to $z \rightarrow z^2$ restricted to the exterior of the unit circle, where $J$ is the Julia set of $f$ and $\hat{\mathcal{J}}$ is the filled Julia set. The restriction of $f$ to the Julia set $J$ (that is the boundary of the basin of $\infty$), is also a covering of degree two, but this map is not semiconjugate to $m$. The Julia set is a curve, but is not a simple curve, and this prevents the existence of a semiconjugacy to $m$ in the circle.

The results mentioned above, proved in [IPRX], imply that the Julia set of $f$ cannot be the attracting set of a continuous map of the sphere. Indeed, if $\Lambda$ is a connected attracting set, then there exists a basis $\mathcal{B}$ of neighborhoods of $\Lambda$, each homeomorphic to an annulus. Moreover each $U \in \mathcal{B}$ satisfies that the closure of $f(U)$ is contained in $U$.

**Theorem 2.** If $\Lambda$ is a normal attractor then:
(a) the restriction of $f$ to $\Lambda$ is semiconjugate to $m$,
(b) the restriction of $f$ to an invariant component $C$ of $B_0(\Lambda) \setminus \Lambda$ is also semiconjugate to $m$.

To do the proof we will need first the following corollary of a theorem of F. Riesz and M. Riesz.

**Lemma 5.** There exists a universal covering $\pi_0 : \hat{C} \rightarrow C$, $\{ \pi_0 : \hat{\mathcal{C}} = \{(x, y) \in \mathbb{R}^2 : x \in (0, 1), y \in \mathbb{R}\}$ such that
\begin{enumerate}
  \item $\pi_0^{-1}(\pi_0(x, y)) = \{(x, y + j) : j \in \mathbb{Z}\}$
  \item if $\gamma : (0, 1) \rightarrow C$ is a curve such that $\lim_{t \rightarrow 0} \gamma(t) \in \Lambda$, then for each lift $\hat{\gamma}$ of $\gamma$ it holds that $\lim_{t \rightarrow 0} \hat{\gamma}(t)$ exists.
\end{enumerate}

**Proof.** Note that as $C$ is contained in a surface $M$, then one component $\partial^+ C$ of the boundary of $C$ is contained in $\Lambda$ and the other, $\partial^- C$, is disjoint to $\Lambda$. Consider $C \cup \partial^- C$; identifying $\partial^- C$ to a point $p$ we obtain a simply connected set $\Omega$. Moreover, $\Omega$ is conformally equivalent to the unit disk $\mathbb{D}$ because $\Lambda$ cannot be reduced to a point, as $\pi|_{\Lambda}$ is $d : 1$ covering. Let $q : C \cup \partial^- C \rightarrow \Omega$ denote the quotient map, $q(\partial^- C) = p$. Now let $\varphi : \Omega \rightarrow \mathbb{D}$ be the Riemann map carrying $p$ to 0, and $\pi : (0, 1) \times \mathbb{R} \rightarrow \mathbb{D}$, $\pi(x, y) = (x, e^{2\pi iy}) \in \mathbb{R} \times S^1 = \mathbb{D} \setminus \{0\}$. Finally define $\pi_0 = q^{-1}\varphi^{-1}\pi$, well defined in $\hat{\mathcal{C}}$ since $q$ is injective in $C$. This implies item (1) in the assertion of the lemma.

The second item follows immediately from the following result of F.Riesz and M.Riesz: If $\beta : (0, 1) \rightarrow \Omega$ is a curve such that $\lim_{t \rightarrow 0} \beta(t) \in \partial\Omega$, then the $\lim_{t \rightarrow 0} \varphi(\beta(t))$ also exists. \qed
Let \( \tilde{f} : \tilde{C} \to \tilde{C} \) be a lift of \( f \) under \( \pi_0 \). Note that \( \tilde{f}(x, y + 1) = \tilde{f}(x, y) + (0, d) \), by the first item in the lemma above. We claim that there exists a curve \( \tilde{\gamma} : (0, 1) \to \tilde{C} \) such that \( \lim_{t \to 0} \tilde{\gamma}(t) \) exists and belongs to \( \{1\} \times \mathbb{R} \) and \( \lim_{t \to 0} \tilde{f}(\tilde{\gamma}(t)) \) exists and belongs to \( \{1\} \times \mathbb{R} \). To prove this, take any curve \( \gamma \) in \( C \) with an end in \( \Lambda \), and let \( \tilde{\gamma} \) be any lift under \( \pi_0 \). The second part of Lemma 5 implies that \( \tilde{\gamma} \) has an end in \( \{1\} \times \mathbb{R} \). The same conclusion holds for \( f \tilde{\gamma} \), that is a lift of \( f \gamma \) that also has an end in \( \Lambda \) because \( f \) is defined and continuous in the whole manifold \( M \).

Proof of Theorem 2: The item (a) was proved in the application of page 8. For proof of item (b), let \( K = [1/2, 1] \times \mathbb{R} \). Note that if \((x, y) \in \tilde{C}\), then there exists \( n_0 \) such that \( \tilde{f}^n(x, y) \in \tilde{K} \) for every \( n > n_0 \). Let \((x, y) = \tilde{f}^n(x, y)\) and we claim that \( |y_{n+1} - dy_n| \leq M_R \) for every \( n > n_0 \) and some constant \( M_R \).

Let \( \tilde{\gamma} : [0, 1] \to \tilde{C} \) be a simple curve such that both \( \lim_{t \to 1} \tilde{\gamma}(t) \) and \( \lim_{t \to 0} \tilde{f}(\tilde{\gamma}(t)) \) exist. Let \( t_0 = \min t \in [0, 1] : \tilde{\gamma}(t) \in \tilde{K} \). Again, we may assume that \( t_0 = 0 \) and that \( \tilde{\gamma}(0) = 0 \). Let \( \tilde{\gamma}_n(t) = \tilde{\gamma}(t) + (0, n) \) and take \( n_0 \) such that \( \tilde{\gamma}_{n_0} \cap \tilde{\gamma} = \emptyset \). Given \((x, y) \in \tilde{K}\), take \( t_0 \) such that \( \tilde{\gamma}(t_0) = x \). Let \( \tilde{\gamma}(t) = (\tilde{x}(t), \tilde{y}(t)) \) and \( \tilde{K}_0 = \{(x, y) : \tilde{y}(t_0) \leq y \leq \tilde{y}(t_0) + n_0, \ x \geq \frac{1}{2}\} \).

Note that there exist a constant \( M_{\tilde{K}_0} \) such that \( |(\tilde{f}(x, y))_2 - dy| \leq M_{\tilde{K}_0} \) for all \((x, y) \in \tilde{K}_0\). So, \( |y_{n+1} - dy_n| \leq M_{\tilde{K}_0} \).

This implies that there exists \( \tilde{y} \in \mathbb{R} \) such that \( |y_n - d^n \tilde{y}| \) is bounded (arguing as in Lemma 1). Then \( H(x, y) = \tilde{y} \) defines a semiconjugacy on \( R \) that induce a semiconjugacy on \( C \). □

Two homeomorphisms may have homeomorphic basins without being conjugate. However, when restricted to the trivial dynamics in \( B \setminus \Lambda \), the fact that two fundamental domains are homeomorphic implies that the maps are conjugate. This is not true for coverings in general. We will consider the map \( p_d(z) = z^d \) as acting in \( \mathbb{D}^* = \mathbb{D} \setminus \{0\} \) (we use the notation \( m_d \) for \( z \to z^d \) acting on \( S^1 \)).

**Example 1.** There exists a covering \( f : [0, 1] \times S^1 \to [0, 1] \times S^1 \), admitting a fundamental domain, but whose restriction to \((0, 1) \times S^1 \) is not conjugate to \( p_d \).

Given any degree \( d \) covering \( g \) of \( S^1 \) that is not conjugate to \( m_d \), the map \( \tilde{f} : [0, 1] \times S^1 \to [0, 1] \times S^1 \) given by \( f(x, z) = (x^2, g(z)) \) is not conjugate to \( p_d \). As \( g \) is not conjugate to \( m_d \), there exists a periodic or wandering arc \((a, b) \subset S^1 \). Let \( \Delta = \{(x, z) \in (0, 1) \times S^1 : z \in (a, b)\} \) and \( B \) a closed disc contained in \( \Delta \). Assume there exists a conjugacy \( H \) between \( f \) and \( p_d \). Note that \( f^n(B) \) is a disc for every \( n > 0 \), because \( f^n \) is injective in \( \Delta \). On the other hand, \( H(B) \) is a disc and so \( p_d^n \) is not injective in \( H(B) \) for every large \( n \). This is a contradiction.

**Example 2.** There exists a map \( f : [0, 1] \times S^1 \to [0, 1] \times S^1 \) with fundamental domain, but it is not semiconjugate to \( p_d \).

Note that \( f \) is defined in the closed annulus which implies that is semiconjugate to \( m_d \).

**Proof:** Let \( f(x, z) = (\phi(x, z), z^d) \), where \( \phi \) will be determined. The condition to be imposed on \( f \) is the following: there exists a point \( \Delta \) such that the union for \( n > 0 \) of the sets \( f^{-n}(f^n(P)) \) is dense in an essential annulus \( A_0 \subset A \). Assume that there exists a semiconjugacy \( h \) between \( f \) and \( p_d \). This means that \( hf = p_d h \), that \( h \) is surjective and is the identity on first homotopy group. It follows that \( h(A_0) \) must be a nontrivial circle. If, in addition, the annulus \( A_0 \) is a fundamental
domain for $f$, then the range of $h$ will be a countable union of circles, hence $h$ is not surjective, a contradiction.

Let \( \{a_n : n \in \mathbb{Z}\} \) be an increasing sequence of positive real numbers such that \( a_n \to 0 \) when \( n \to -\infty \) and \( a_n \to 1 \) when \( n \to +\infty \). Define the annuli \( A_n \) as the product \([a_n, a_{n+1}] \times S^1\), for each \( n \in \mathbb{Z}\). Let also \( \lambda_n \) be the linear increasing homeomorphism carrying \([0,1]\) onto \([a_n, a_{n+1}]\).

The construction of \( \phi \) depends on two sequences. First let \( \bar{z} = \{z_n : n < 0\} \) be a preorbit of 1 under \( m_d \), that is, \( m_d(z_n) = z_{n+1} \) for \( n < -1 \), and \( m_d(z_{-1}) = 1 \), and assume also that \( z_{-1} \neq 1 \), which implies that the \( z_n \) are all different. Then take an element \( \bar{\nu} = \{\nu_n\}_{n<0} \in (0,1)^N \).

By appropriately choosing the function \( \phi \), it will come that the map \( f \) will be such that, for some \( P \in A_0 \), the set \( f^{-n}(f^n(P)) \) contains the point \((\lambda_0(\nu_{-n}), z_{-n})\) for every \( n > 0 \).

It is clear that the sequences \( \bar{\nu} \) and \( \bar{z} \) can be chosen in order to make the set \( \bigcup_{n>0} f^{-n}(f^n(P)) \) dense in \( A_0 \), which is a fundamental domain for \( f \) in \( A \).

Let \( P = (\lambda_0(1/2),1) \) be a point determined in the annulus \( A_0 \). To define \( \phi \) we will use the sequences \( \bar{z} \) and \( \bar{\nu} \). The definition of \( \phi \) is by induction beginning in the annulus \( A_0 \). Note that \( \phi \) must carry the annulus \( A_n \) into the segment \([a_{n+1}, a_{n+2}]\). For each \( z \in S^1 \), note that \( \phi(z) := \phi(x,z) \) is a homeomorphism from the segment \([a_n, a_{n+1}]\) onto the segment \([a_{n+1}, a_{n+2}]\).

We will first define \( \phi_1 \) in its whole domain, and then, by induction on \( k \), the restriction of \( \phi \) to \( A_k \).

First define \( \phi_1(x) \): for \( x \in [a_n, a_{n+1}] \) let \( \phi_1(x) = \lambda_{n+1}(\lambda_n^{-1}(x)) \), that is, \( \phi_1 \) is affine, the image of \( P \) under \( f^k \) is equal to \((\lambda_k(1/2),1)\).

Also define \( \phi_{z_{-1}}(\lambda_0(\nu_{-1})) = \lambda_k(1/2) \), this is the only condition asked for this map. It is obvious that \( \phi \) can be extended to the annulus \( A_0 \) so as to satisfy this unique condition (it is used, of course, that \( z_{-1} \neq 1 \)).

Next let \( k > 0 \), and define \( \phi \) in \( A_k \) assuming it is already known in \( A_{k-1} \). Of course, \( f \) is also defined in \( A_0 \cup \cdots \cup A_{k-1} \) and one can iterate \( f^k \) at points in \( A_k \), in particular the image of the point \((\lambda_0(\nu_{-k}), z_{-k-1}) \in A_0 \) under \( f^k \) is a point having second coordinate \( z_{-1} \), denote it by \((x_k, z_{-1}) \) in \( A_k \). Next extend \( \phi \) to \( A_{k+1} \) in order to satisfy only one condition: \( \phi_{z_{-1}}(x_k) = \lambda_{k+1}(1/2) \).

As was pointed out above, \( f^n(P) = (\lambda_n(1/2),1) \) for every \( n \geq 0 \). It follows that

\[
\begin{align*}
    f^n(\lambda_0(\nu_{-n}), z_{-n}) &= f(f^{n-1}(\lambda_0(\nu_{-n}), z_{-n})) = f(x_{n-1}, z_{-1}) \\
    &= (\phi_{z_{-1}}(x_{n-1}), m_d(z_{-1})) = f^n(P),
\end{align*}
\]

as required.

We finish this work with another negative result, negative in the direction of a possible classification of covering maps of the annulus.

Consider the Whitney (or strong) \( C^0 \) topology in the space of covering maps of the annulus, defined as follows: if \( f \in \text{Cov}(A) \) and \( \epsilon : A \to \mathbb{R}^+ \) is a continuous function, then the \( \epsilon \)-neighborhood of \( f \) is

\[
\mathcal{N}_\epsilon(f) = \{ g \in \text{Cov}(A) : d(g(x), f(x)) < \epsilon(x) \forall x \in A \}
\]

where \( d \) is any fixed distance compatible with the topology of \( A \). The notation for the space of \( C^0 \) maps endowed with this topology is \( C^0_W(A) \).
Definition 7. A map $f \in \text{Cov}(A)$ is $C^0_W(A)$ stable if there exists a neighborhood of $f$ such that every map $g$ in this neighborhood is conjugate to $f$.

Theorem 3. The map $p_d(z) = z^d$ is not $C^0_W(A)$ stable if $A$ is the punctured unit disc $A = D^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \}$.

Some remarks before the proof:
1. The distance $d$ is the Euclidean distance in $\mathbb{D}$.
2. If $g \in \text{Cov}(A)$ belongs to a neighborhood of $p_d$ such that $\epsilon(x) \to 0$ as $x \to \partial A$, then $g(z_n) \to p_d(z)$ whenever $\{z_n\}$ is a sequence in $A$ converging to a point $z \in \partial A$. Thus $g$ extends continuously to the boundary, where it coincides with $p_d$.
3. Note that $p_d$ is not $C^0$ stable, when one considers weak topology (define neighborhoods as above but with $\epsilon$ equal to a constant). This is obvious since one can create periodic points in $A$.
4. It is well known that the restriction of $p_d$ to its Julia set is $C^1$ stable. Moreover, the restriction of $p_d$ to $A = \mathbb{C} \setminus \{0\}$ is $C^1_W$ stable (see [PR], Corollary 4).
5. Note that a homeomorphism having a hyperbolic attractor is $C^0_W$ stable when restricted to an invariant component of $B(\Lambda) \setminus \Lambda$. Indeed, the definition of the semiconjugacy can be made in a fundamental domain and then extended to the future and the past. The same construction is not possible for noninvertible maps: note, for example, that the image of a fundamental domain is not necessarily a fundamental domain.
6. Note that the theorem also implies that if $A = \mathbb{C} \setminus \mathbb{D}$, then $p_d$ is not $C^0_W(A)$ stable.

Proof: We will use $f = p_d$ and prove this case, the generalization to arbitrary degree being obvious. Then the function to be perturbed is $f(x \exp(it)) = x^2 \exp(2it)$, where $x$ is positive and $t \in \mathbb{R}$. We will find a perturbation $g$ of $f$ having an invariant set with nonempty interior where $g$ is injective. This does not exist for $f$.

First the perturbation of $t \in \mathbb{R} \to 2t \in \mathbb{R}$. Let $\rho$ and $\rho'$ be positive numbers such that $\rho' \leq \rho$. Then there exists an increasing continuous function $\phi = \phi_{\rho,\rho'} : \mathbb{R} \to \mathbb{R}$ such that $\phi(-\rho,\rho)) = (-\rho',\rho')$, $\phi(0) = 0$ and $|\phi(t) - 2t| \leq 2\rho - \rho'$ for every $t$. Moreover, $\phi(t) = 2t$ whenever $|t| > 2\rho$. Moreover, one can ask the function $(t,\rho,\rho') \to \phi_{\rho,\rho'}(t)$ to be continuous.

Let $\epsilon$ be any positive continuous function defined in $\mathbb{D}^*$. Note that $\epsilon' \leq \epsilon$ implies that the $\epsilon'$-neighborhood of $f$ is contained in the $\epsilon$-neighborhood of $f$. So it can be assumed that for some positive $\delta$, the function $\epsilon$ satisfies $\epsilon(x \exp(it)) = \epsilon(x)$ whenever $x$ is positive and $|t| < \delta$. Then assume also $\epsilon(x) < \delta$.

It is claimed now that there exists a continuous function $\rho : (0,1) \to \mathbb{R}^+$ such that $2\rho(x) < \epsilon(x)$ for every $x \in (0,1)$ and $\rho(x^2) < \rho(x)$ for every $x \leq 1/2$. Indeed, first define $\rho(x)$ in the interval $[1/4, 1/2]$ so that $\rho(x) < 1/2\epsilon(x)$ and $\rho(1/4) < \rho(1/2)$. Then define $\rho$ for $x \in [1/16, 1/4]$, so as to satisfy $\rho(x) < \rho(\sqrt{x})$ and $\rho(x) < 1/2\epsilon(x)$. Then use induction to define it in the remaining fundamental domains of the action of $x \to x^2$ in $(0,1/2)$. It is clear that $\rho$ can be continuously extended to the whole interval $(0,1)$ so as to satisfy $2\rho(x) < \epsilon(x)$.

Next proceed to the definition of $g$, a particular perturbation of $f$. Define

$$g(x \exp(it)) = x^2 \exp(\iota \phi_{\rho(x),\rho(x^2)}(t)).$$

Note first that $g$ is continuous, and defines a covering of the annulus, because the functions $\phi$ used in its definition are all increasing. Moreover, $z = x \exp(it)$
implies \( f(z) = g(z) \) if \( |t| > 2\rho(x) \), in particular, if \( |t| > \delta \). For other values of \( t \) \((|t| \leq \delta)\), and any \( x \in (0,1) \), it comes that:

\[
|g(z) - f(z)| = x^2 \exp(i\phi_{\rho(x)},\rho(x^2)(t)) - \exp(2it)| \leq x^2 \phi_{\rho(x)},\rho(x^2)(t) - 2t
\]

\[
< 2\rho(x) - \rho(x^2) < 2\rho(x) < \epsilon(x) = \epsilon(z)
\]

Therefore \( g \) belongs to the \( \epsilon \)-neighborhood of \( f \). It remains to show that \( g \) is not conjugate to \( f \). Note that the set \( R := \{ x \exp(it) : x < 1/2, |t| < \rho(x) \} \) is forward invariant under \( g \), because \( g(x \exp(it)) = x^2 \exp(i\phi(t)) \) and \( |\phi(t)| < \rho(x^2) \) if \( |t| < \rho(x) \). But \( g \) is injective on \( R \), and \( R \) has nonempty interior. Thus \( f \) and \( g \) cannot be conjugate. \( \square \)

5. Some final comments and questions.

The problem of classifying coverings is very complicated, we cannot even imagine a classification of a neighborhood of the "simplest" map \( p_d \). One simple question, whose answer we still don’t know is if every Whitney \( C^0 \) perturbation of \( p_d \) has a covering by fundamental domains. In the perturbation made above, the invariant foliation by circles centered at the origin is preserved.

It may be easy to see that no covering of \( A \) of degree \( |d| > 1 \) is \( C^0 \) \( W \) stable, but we won’t give a proof of this here.

The question of the existence of periodic points for covering maps of the annulus will be considered in a following article. For example it will be proved that a covering of degree greater than one having an invariant continuum must have fixed points.

It is natural to consider the rotation number for coverings of the annulus as defined in section 2.1. That is, given a lift \( F : (0,1) \times \mathbb{R} \to (0,1) \times \mathbb{R} \) of \( f \), take a point \( (x_0, y_0) \in (0,1) \times \mathbb{R} \) and define

\[
\rho(x_0, y_0) = \lim_{n \to +\infty} \frac{y_n}{d^n},
\]

whenever this limit exists, and where \( (x_n, y_n) = F^n(x_0, y_0) \). See Corollary \( \ref{cor:rotation-number} \) where conditions are given to assure the existence of the limit. Of course, the converse of that Corollary is not true. What conclusions can be drawn if, for example, this limit exists for every point? Does it necessarily define a continuous function?

References

[AK] D. V. Anosov, A. B. Katok. New examples in smooth ergodic theory. Ergodic diffeo- morphisms, Trans. Moscow Math. Soc., 23 (1970), 1-35.

[BCLP] F. Béguin, S. Crovisier, F. LeRoux, A. Patou. Pseudo-rotations of the closed annulus: variation on a theorem of J. Kwapisz. Nonlinearity, 17 (2004), 1427-1453.

[BCL] F. Béguin, S. Crovisier, F. LeRoux. Pseudo-rotations of the open annulus. Bull. Braz. Math. Soc. (N.S.), 37 (2006), no. 2, 275-306.

[BKRU] R.Bamón, J.Kiwi, J.Rivera-Letelier, R.Urzúa. On the topology of solenoidal attractors of the cylinder. Ann. I. H. Poincaré 23 (2006), 209-236.

[Bue] J.Buescu. Exotic Attractors. Progress in Math. Vol 153 Birkhauser (1997)

[FatHe] A. Fathi, M. Herman. Existence de difféomorphismes minimaux. Dynamical Systems, Vol.I-Warsaw, Astérisque, Soc. Math. France, 49 (1984), 37-59.

[FayK] B. Fayad, A. Katok. Constructions in elliptic dynamics. Ergodic Theory Dynam. Systems, 24 (2004), no 5, 1477-1520.

[FayS] B. Fayad, M. Saprykina. Weak mixing disc and annulus diffeomorphisms with ar-bitary Liouville rotation number on the boundary. Ann. Sci. École Norm. Sup., (4) 38 (2005), 339-364.
[Ha] M. Handel. A pathological area preserving $C^\infty$ diffeomorphism of the plane. Proc. Amer. Math. Soc., 86 (1982), no 1, 163-168.

[He] Herman, Michael. Construction of some curious diffeomorphisms of the Riemann sphere. J. London Math. Soc. (2) 34 (1986), no. 2, 375-384.

[IPR] J. Iglesias, A. Portela and A. Rovella. Structurally stable perturbations of polynomials in the Riemann sphere. Ann. de l’Inst. H. Poincaré, (C) (2009) vol 25, no. 6, 1209-1220

[IPRX] J. Iglesias, A. Portela, A. Rovella. and J. Xavier. Attracting sets on surfaces. to appear in Proc. Amer. Math. Soc.

[MS] W. de Melo and S. van Strien One -Dimensional Dynamics. Springer-Verlag (1993)

[Poin] H. Poincaré. Mémoire sur les courbes définies par une équation différentielle. J. Math. Pure.

[Tho] C. Thomassen. The Jordan-Schonflies theorem and the classification of surfaces. Amer. Math. Monthly. 99(2), (1992), 116-130.

[Tsu] M. Tsujii. Fat solenoidal attractors. Nonlinearity 14 (2001), 1011-1027.

J. Iglesias, UNIVERSIDAD DE LA REPÚBLICA. FACULTAD DE INGENIERÍA. IMERL. JUlio Herrera y Reissig 565. C.P. 11300. MONTEVIDEO, URUGUAY

E-mail address: jorgei@fing.edu.uy

A. Portela, UNIVERSIDAD DE LA REPÚBLICA. FACULTAD DE INGENIERÍA. IMERL. JUlio Herrera y Reissig 565. C.P. 11300. MONTEVIDEO, URUGUAY

E-mail address: aldo@fing.edu.uy

A. Rovella, UNIVERSIDAD DE LA REPÚBLICA. FACULTAD DE CIENCIAS. CENTRO DE MATEMÁTICA. IGUÁ 4225. C.P. 11400. MONTEVIDEO, URUGUAY

E-mail address: leva@cmat.edu.uy

J. Xavier, UNIVERSIDAD DE LA REPÚBLICA. FACULTAD DE INGENIERÍA. IMERL. JUlio Herrera y Reissig 565. C.P. 11300. MONTEVIDEO, URUGUAY

E-mail address: jxavier@fing.edu.uy