Hartman-Wintner-type inequality for fractional differential equations with $k$-Prabhakar derivative

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Abstract

In present paper, Hartman-Wintner-type inequality is established for a nonlocal fractional boundary value problem involving $k$-Prabhakar fractional derivative.

Mathematics Subject Classification: 26A33; 33E12; 34A08; 34A40. 

Keywords: Hartman-Wintner-type inequality; Fractional boundary value problem, nonlocal boundary conditions; $k$-Prabhakar derivative.

1 Introduction

In 1951, Hartman and Wintner [12] consider the boundary value problem

\[
\begin{cases}
    x''(t) + q(t)x(t) = 0, & a < t < b, \\
    x(a) = x(b) = 0,
\end{cases}
\]

and if (1.1) has a nontrivial solution then they proved the following inequality

\[
\int_a^b (b - s)(s - a)q^+(s)ds > b - a, \tag{1.2}
\]

where $q^+(s) = \max\{q(s), 0\}$.

In 1907, A. Lyapunov [15] obtained the following remarkable inequality if (1.1) has nontrivial solution

\[
\int_a^b |q(s)|ds > \frac{4}{b - a}. \tag{1.3}
\]

This Lyapunov inequality (1.3) can be deduced from (1.2) using the following fact

\[
\max_{s \in [a, b]} (b - s)(s - a) = \frac{(b - a)^2}{4}. \tag{1.4}
\]
Many generalizations and extensions of inequality (1.3) are exist in the literature [3, 2, 5, 4, 18, 16, 17, 25]. Recently, some Lyapunov type inequalities were obtained for different fractional boundary value problem using various differential operators [10, 11, 13, 14, 22, 24, 19, 1].

In [6] Cabrera and et al. considered the nonlocal fractional boundary value problem

\[
\begin{align*}
& D_\alpha^a x(t) + q(t)x(t) = 0, \quad a < t < b, \\
& x(a) = x'(a) = 0, x'(b) = \beta x(\xi),
\end{align*}
\]

where \( D_\alpha^a \) denotes the standard Riemann-Liouville fractional derivative of order \( \alpha \), \( a < \xi < b, \ 0 \leq \beta (\xi - a)^{\alpha - 1} < (\alpha - 1)(b - a)^{\alpha - 2} \), \( q(t) \) is continuous real valued function on \([a, b]\), and obtained the following Hartman-Wintner-type inequality

\[
\int_a^b (b-s)^{\alpha - 2}(s-a)|q(s)|ds \geq \left( 1 + \frac{\beta (b-a)^{\alpha - 1}}{(\alpha - 1)(b-a)^{\alpha - 2} - \beta (\xi - a)^{\alpha - 1}} \right)^{-1} \Gamma(\alpha).
\]

More recently authors in [21] obtained the Hartman-Wintner-type inequality for following nonlocal fractional boundary problem with Prabhakar derivative

\[
\begin{align*}
& D_{\rho,\mu,\omega,a}^\gamma x(t) + q(t)x(t) = 0, \quad a < t < b, \quad 2 < \mu \leq 3, \\
& x(a) = x'(a) = 0, x'(b) = \beta x(\xi),
\end{align*}
\]

In this paper, we consider the following nonlocal fractional boundary value problem

\[
\begin{align*}
& \mu D_{\rho,\beta,\omega,a+}^\gamma y(t) + q(t)y(t) = 0, \quad a < t < b, \quad 2 < \beta \leq 3, \\
& y(a) = y'(a) = 0, y'(b) = \alpha y(\xi),
\end{align*}
\]

where \( \mu D_{\rho,\beta,\omega,a+}^\gamma \) denotes the \( k \)-Prabhakar derivative of order \( \beta \), \( a < \xi < b, 0 \leq \alpha (\xi - a)^{\beta - 1} < (\beta - 1)(b - a)^{\beta - 2} \), \( q : [a, b] \rightarrow \mathbb{R} \) is real valued continuous function and obtained the Hartman-Wintner-type inequality for problem (1.8).

2 Preliminaries

In this section, we give some basic definitions and lemmas that will be necessary to us in the sequel.
Definition 2.1 [7] The k-Mittag-Leffler function is denoted by $E_{k,\alpha,\beta}^\gamma(z)$ and is defined as

\[ E_{k,\alpha,\beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma_{n,k})}{\Gamma(\gamma_n + \beta)n!} z^n, \]  

(2.1)

where $k \in \mathbb{R}^+$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$; $\Gamma_k(x)$ is the k-Gamma function and $(\gamma)_{n,k} = \frac{\Gamma_{x,(\gamma+mk)}}{\Gamma_{x,(\gamma)}}$ is the pochhammer k-symbol.

Definition 2.2 [8] Let $\alpha, \beta, \gamma \in \mathbb{C}$, $k \in \mathbb{R}^+$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\phi \in L^1([0,b]), (0 < x < b \leq \infty)$. The k-Prabhakar integral operator involving k-Mittag-Leffler function is defined as

\[ (kP_{\alpha,\beta,\omega,\phi}(x)) = \int_0^x \frac{(x-t)^{\frac{\beta-1}{\gamma}}}{k} E_{k,\alpha,\beta}^\gamma[\omega(x-t)^{\frac{\beta}{\gamma}}] \phi(t) \, dt, \quad (x > 0) \]  

(2.2)

where

\[ kE_{\alpha,\beta,\omega}^\gamma(t) = \begin{cases} \frac{t^{\frac{\beta-1}{\gamma}}}{k} E_{k,\alpha,\beta}^\gamma(\omega t^{\frac{\beta}{\gamma}}), & t > 0; \\ 0, & t \leq 0. \end{cases} \]  

(2.3)

Definition 2.3 [8] Let $k \in \mathbb{R}^+$, $\rho, \beta, \gamma, \omega \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $m = [\frac{\alpha}{k}] + 1$, $f \in L^1([0,b])$. The k-Prabhakar derivative is defined as

\[ kD_{\rho,\beta,\omega,\gamma}^\gamma f(x) = \left( \frac{d}{dx} \right)^m k^m P_{\rho,\rho,\gamma - \beta,\omega}^\gamma f(x). \]  

(2.5)

Lemma 2.1 [8] Let $\alpha, \beta, \gamma \in \mathbb{C}$, $k \in \mathbb{R}^+$, $\Re(\alpha) > 0$; $\Re(\beta) > 0$, $\phi \in L^1(\mathbb{R}_0^+)$ and $|\omega k(ks)^{\frac{\beta}{\gamma}}| < 1$ then

\[ \mathcal{L}\{kP_{\rho,\beta,\omega,\phi}(x)\} = \mathcal{L}\{kE_{\alpha,\beta,\omega}^\gamma(t)\} \mathcal{L}\{\phi\} = (ks)^{\frac{\beta}{\gamma}} (1 - \omega k(ks)^{\frac{\beta}{\gamma}}) \mathcal{L}\{\phi\}. \]  

(2.6)

Lemma 2.2 [8] The Laplace transform of k-Prabhakar derivative (2.5) is

\[ \mathcal{L}\{kD_{\rho,\beta,\omega,\gamma}^\gamma f(x)\} = (ks)^{\frac{\beta}{\gamma}} (1 - \omega k(ks)^{\frac{\beta}{\gamma}}) \mathcal{L}\{F(s)\} - \sum_{n=0}^{m-1} k^{n+1} s^n \left( kD_{\rho,\beta,\omega}^\gamma f(0^+) \right). \]  

(2.7)

For the case $[\frac{\alpha}{k}] + 1 = m = 1$,

\[ \mathcal{L}\{kD_{\rho,\beta,\omega,\gamma}^\gamma y(x)\} = (ks)^{\frac{\beta}{\gamma}} (1 - \omega k(ks)^{\frac{\beta}{\gamma}}) \mathcal{L}\{y(x)\} - k(kP_{\rho,\beta,\omega,\gamma}^\gamma y(0^+). \]

with $|\omega k(ks)^{\frac{\beta}{\gamma}}| < 1$. 

3
Lemma 2.3 \cite{21} If \( f(x) \in C(a, b) \cap L(a, b) \); then \( kD_{\rho, \beta, \omega, a}^{\gamma} P_{\rho, \beta, \omega, a}^{\gamma} f(x) = f(x) \) and if \( f(x), kD_{\rho, \beta, \omega, a}^{\gamma} P_{\rho, \beta, \omega, a}^{\gamma} f(x) \in C(a, b) \cap L(a, b) \), then for \( c_j \in \mathbb{R} \) and \( m - 1 < \beta \leq m \), we have

\[
kP_{\rho, \beta, \omega, a}^{\gamma} D_{\rho, \beta, \omega, a}^{\gamma} f(x) = f(x) + c_0 (x - a)^{\frac{\beta}{\alpha} - 1} E_{k, \rho, \beta}^\gamma (\omega (x - a)^{\frac{\beta}{\alpha}})
+ c_1 (x - a)^{\frac{\beta}{\alpha} - 2} E_{k, \rho, \beta - 1}^\gamma (\omega (x - a)^{\frac{\beta}{\alpha}}) \\
+ c_2 (x - a)^{\frac{\beta}{\alpha} - 3} E_{k, \rho, \beta - 2}^\gamma (\omega (x - a)^{\frac{\beta}{\alpha}}) + \ldots \\
+ c_{m - 1} (x - a)^{\frac{\beta}{\alpha} - m} E_{k, \rho, \beta - (m - 1)}^\gamma (\omega (x - a)^{\frac{\beta}{\alpha}}) \tag{2.8}
\]

Lemma 2.4 \cite{21} Let \( k \in \mathbb{R}^+, \rho, \beta, \gamma, \omega \in \mathbb{C}, \Re(\alpha) > 0; \Re(\beta) > 0 \) then for any \( j \in \mathbb{N} \) we have

\[
d^j \frac{dx^j}{dx^j} \left[ x^{\frac{\beta}{\alpha} - 1} E_{k, \rho, \beta}^\gamma (\omega x^{\frac{\beta}{\alpha}}) \right] = \frac{x^{\frac{\beta}{\alpha} - (j + 1)}}{k^j} E_{k, \rho, \beta - j}^\gamma (\omega x^{\frac{\beta}{\alpha}}) \tag{2.9}
\]

3 Main Results

Theorem 3.1 Assume that \( 2 < \beta \leq 3 \) and \( y \in C[a, b] \). If the nonlocal fractional boundary value problem \( (1.1) \) has unique nontrivial solution, then it satisfies

\[
y(t) = \int_a^b G(t, s) q(s) y(s) ds \\
+ \frac{\alpha (t - a)^{\frac{\beta}{\alpha} - 1} E_{k, \rho, \beta}^\gamma (\omega (t - a)^{\frac{\beta}{\alpha}})}{(b - a)^{\frac{\beta}{\alpha} - 2} E_{k, \rho, \beta - 1}^\gamma (\omega (b - a)^{\frac{\beta}{\alpha}}) - \alpha (\xi - a)^{\frac{\beta}{\alpha} - 1} E_{k, \rho, \beta}^\gamma (\omega (\xi - a)^{\frac{\beta}{\alpha}})} \int_a^b G(\xi, s) q(s) y(s) ds,
\]

where the Green’s function \( G(t, s) \) is defined as follows

\[
G(t, s) = \begin{cases} 
\frac{(t-a)^{\frac{\beta}{\alpha} - 1} E_{k, \rho, \beta}^\gamma (\omega (t-a)^{\frac{\beta}{\alpha}})}{E_{k, \rho, \beta - 1}^\gamma (\omega (b-a)^{\frac{\beta}{\alpha}})}, & a \leq s \leq t \leq b, \\
\frac{(b-a)^{\frac{\beta}{\alpha} - 2} E_{k, \rho, \beta - 1}^\gamma (\omega (b-a)^{\frac{\beta}{\alpha}})}{(b-a)^{\frac{\beta}{\alpha} - 2} E_{k, \rho, \beta - (m-1)}^\gamma (\omega (b-a)^{\frac{\beta}{\alpha}})}, & a \leq t \leq s \leq b.
\end{cases}
\tag{3.1}
\]
Proof. From lemma 2.3, the general solution to (1.8) in \( C[a, b] \) can be written as follows

\[
y(t) = c_0 (t-a)^{\frac{\beta}{\pi}} E_{k,p,\beta}(\omega(t-a)^\xi) + c_1 (t-a)^{\frac{\beta+1}{\pi}} E_{k,p,\beta-k}(\omega(t-a)^\xi)
+ c_2 (t-a)^{\frac{\beta-3}{\pi}} E_{k,p,\beta-2k}(\omega(t-a)^\xi) - \int_a^t \frac{(t-s)^{\frac{\beta-1}{\pi}}}{k} E_{k,p,\beta}(\omega(t-s)^\xi) q(s)y(s) ds.
\]

(3.2)

Employing the first boundary condition \( y(a) = y'(a) = 0 \) we obtain \( c_1 = c_2 = 0 \). Therefore the solution (3.2) becomes

\[
y(t) = c_0 (t-a)^{\frac{\beta}{\pi}} E_{k,p,\beta}(\omega(t-a)^\xi) - \int_a^t \frac{(t-s)^{\frac{\beta-1}{\pi}}}{k} E_{k,p,\beta-k}(\omega(t-s)^\xi) q(s)y(s) ds.
\]

(3.3)

For second boundary condition we find

\[
y'(t) = c_0 \frac{(t-a)^{\frac{\beta-2}{\pi}}}{k} E_{k,p,\beta-k}(\omega(t-a)^\xi) - \int_a^t \frac{(t-s)^{\frac{\beta-2}{\pi}}}{k^2} E_{k,p,\beta-k}(\omega(t-s)^\xi) q(s)y(s) ds.
\]

Employing the second boundary condition \( y'(b) = \alpha y(\xi) \) we get

\[
\Rightarrow c_0 \frac{(b-a)^{\frac{\beta-2}{\pi}}}{k} E_{k,p,\beta-k}(\omega(b-a)^\xi) - \int_a^b \frac{(b-s)^{\frac{\beta-2}{\pi}}}{k^2} E_{k,p,\beta-k}(\omega(b-s)^\xi) q(s)y(s) ds = \alpha c_0 (\xi - a)^{\frac{\beta-1}{\pi}} E_{k,p,\beta}(\omega(\xi - a)^\xi) - \alpha \int_a^\xi \frac{(\xi - s)^{\frac{\beta-1}{\pi}}}{k} E_{k,p,\beta}(\omega(\xi - s)^\xi) q(s)y(s) ds,
\]

\[
\Rightarrow c_0 \left[ \frac{(b-a)^{\frac{\beta-2}{\pi}}}{k} E_{k,p,\beta-k}(\omega(b-a)^\xi) - \alpha (\xi - a)^{\frac{\beta-1}{\pi}} E_{k,p,\beta}(\omega(\xi - a)^\xi) \right] = \\
\int_a^b \frac{(b-s)^{\frac{\beta-2}{\pi}}}{k^2} E_{k,p,\beta-k}(\omega(b-s)^\xi) q(s)y(s) ds
- \alpha \int_a^\xi \frac{(\xi - s)^{\frac{\beta-1}{\pi}}}{k} E_{k,p,\beta}(\omega(\xi - s)^\xi) q(s)y(s) ds,
\]

\[
\Rightarrow c_0 = \frac{1}{\frac{(b-a)^{\frac{\beta-2}{\pi}}}{k} E_{k,p,\beta-k}(\omega(b-a)^\xi) - \alpha (\xi - a)^{\frac{\beta-1}{\pi}} E_{k,p,\beta}(\omega(\xi - a)^\xi)
- \times \int_a^b \frac{(b-s)^{\frac{\beta-2}{\pi}}}{k^2} E_{k,p,\beta-k}(\omega(b-s)^\xi) q(s)y(s) ds
- \alpha \int_a^\xi \frac{(\xi - s)^{\frac{\beta-1}{\pi}}}{k} E_{k,p,\beta}(\omega(\xi - s)^\xi) q(s)y(s) ds}.
\]
Thus the solution \( y(t) \) becomes
\[
y(t) = \frac{(t-a)^{\frac{\alpha}{k}} - 1}{(b-a)^{\frac{\alpha}{k}}} \frac{E_{k,\rho,\beta}(\omega(t-a)^{\frac{\alpha}{k}})}{E_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\alpha}{k}}) - \alpha(\xi - a)^{\frac{\alpha}{k}} - 1 E_{k,\rho,\beta}(\omega(\xi - a)^{\frac{\alpha}{k}})}
\times \int_a^b \frac{(b-s)^{\frac{\alpha}{k}} - 2}{k^2} E_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\alpha}{k}}) q(s) y(s) ds
\times \int_a^b \frac{(b-s)^{\frac{\alpha}{k}} - 2}{k^2} E_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\alpha}{k}}) q(s) y(s) ds
\times \int_a^b \frac{(b-s)^{\frac{\alpha}{k}} - 2}{k^2} E_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\alpha}{k}}) q(s) y(s) ds
\times \int_a^b \frac{(b-s)^{\frac{\alpha}{k}} - 2}{k^2} E_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\alpha}{k}}) q(s) y(s) ds.
\]

Taking into account that
\[
E_{k,\rho,\beta}(\omega(t-a)^{\frac{\alpha}{k}})
\]

we have
\[
g(t) = 1 + \frac{\alpha(\xi - a)^{\frac{\alpha}{k}} - 1}{(b-a)^{\frac{\alpha}{k}}} E_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\alpha}{k}}) - \alpha(\xi - a)^{\frac{\alpha}{k}} - 1 E_{k,\rho,\beta}(\omega(\xi - a)^{\frac{\alpha}{k}})
\]

\[
1 + \frac{\alpha(\xi - a)^{\frac{\alpha}{k}} - 1}{(b-a)^{\frac{\alpha}{k}}} E_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\alpha}{k}}) - \alpha(\xi - a)^{\frac{\alpha}{k}} - 1 E_{k,\rho,\beta}(\omega(\xi - a)^{\frac{\alpha}{k}})
\]

\[
1 + \frac{\alpha(\xi - a)^{\frac{\alpha}{k}} - 1}{(b-a)^{\frac{\alpha}{k}}} E_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\alpha}{k}}) - \alpha(\xi - a)^{\frac{\alpha}{k}} - 1 E_{k,\rho,\beta}(\omega(\xi - a)^{\frac{\alpha}{k}})
\]
\[
\begin{align*}
&\times \frac{(t-a)^{\alpha-1}E_{\kappa,\rho,\beta}^{\gamma}(\omega(t-a)^{\frac{\alpha}{\rho}})}{(b-a)^{\frac{\alpha}{\rho}}-E_{\kappa,\rho,\beta-k}(\omega(b-a)^{\frac{\alpha}{\rho}})} \int_a^b \frac{(b-s)^{\frac{\alpha}{\rho}}-2}{k^2} E_{\kappa,\rho,\beta-k}(\omega(b-s)^{\frac{\alpha}{\rho}})q(s)y(s)ds \\
&\quad - \frac{(b-a)^{\frac{\alpha}{\rho}}-2}{k^2} E_{\kappa,\rho,\beta-k}(\omega(b-a)^{\frac{\alpha}{\rho}}) - \alpha(\xi-a)^{\frac{\alpha}{\rho}}-1 E_{\kappa,\rho,\beta}(\omega(\xi-a)^{\frac{\alpha}{\rho}}) \\
&\quad \times \int_a^\xi \frac{(\xi-s)^{\frac{\alpha}{\rho}}-1}{k^2} E_{\kappa,\rho,\beta}(\omega(\xi-s)^{\frac{\alpha}{\rho}})q(s)y(s)ds \\
&\quad - \int_a^t \frac{(t-s)^{\frac{\alpha}{\rho}}-1}{k^2} E_{\kappa,\rho,\beta}(\omega(t-s)^{\frac{\alpha}{\rho}})q(s)y(s)ds.
\end{align*}
\]

On simplifying,

\[
\begin{align*}
y(t) &= \frac{(t-a)^{\alpha-1}E_{\kappa,\rho,\beta}^{\gamma}(\omega(t-a)^{\frac{\alpha}{\rho}})}{(b-a)^{\frac{\alpha}{\rho}}-E_{\kappa,\rho,\beta-k}(\omega(b-a)^{\frac{\alpha}{\rho}})} \int_a^b \frac{(b-s)^{\frac{\alpha}{\rho}}-2}{k^2} E_{\kappa,\rho,\beta-k}(\omega(b-s)^{\frac{\alpha}{\rho}})q(s)y(s)ds \\
&\quad + \frac{(t-a)^{\alpha-1}E_{\kappa,\rho,\beta}(\omega(t-a)^{\frac{\alpha}{\rho}})}{(b-a)^{\frac{\alpha}{\rho}}-E_{\kappa,\rho,\beta-k}(\omega(b-a)^{\frac{\alpha}{\rho}})} \int_a^b \frac{(b-s)^{\frac{\alpha}{\rho}}-2}{k^2} E_{\kappa,\rho,\beta-k}(\omega(b-s)^{\frac{\alpha}{\rho}})q(s)y(s)ds \\
&\quad + \left[ \frac{(b-a)^{\alpha-1}E_{\kappa,\rho,\beta}(\omega(\xi-a)^{\frac{\alpha}{\rho}})}{E_{\kappa,\rho,\beta-k}(\omega(b-a)^{\frac{\alpha}{\rho}}) - \alpha(\xi-a)^{\frac{\alpha}{\rho}}-1 E_{\kappa,\rho,\beta}(\omega(\xi-a)^{\frac{\alpha}{\rho}})} \right] \\
&\quad \times \int_a^\xi \frac{(\xi-s)^{\frac{\alpha}{\rho}}-1}{k^2} E_{\kappa,\rho,\beta}(\omega(\xi-s)^{\frac{\alpha}{\rho}})q(s)y(s)ds \\
&\quad + \left[ \frac{(b-a)^{\alpha-1}E_{\kappa,\rho,\beta}(\omega(\xi-a)^{\frac{\alpha}{\rho}})}{E_{\kappa,\rho,\beta-k}(\omega(b-a)^{\frac{\alpha}{\rho}}) - \alpha(\xi-a)^{\frac{\alpha}{\rho}}-1 E_{\kappa,\rho,\beta}(\omega(\xi-a)^{\frac{\alpha}{\rho}})} \right] \\
&\quad \times \int_a^\xi \frac{(\xi-s)^{\frac{\alpha}{\rho}}-1}{k^2} E_{\kappa,\rho,\beta}(\omega(\xi-s)^{\frac{\alpha}{\rho}})q(s)y(s)ds \\
&\quad - \frac{(b-a)^{\frac{\alpha}{\rho}}-2}{k^2} E_{\kappa,\rho,\beta-k}(\omega(b-a)^{\frac{\alpha}{\rho}}) - \alpha(\xi-a)^{\frac{\alpha}{\rho}}-1 E_{\kappa,\rho,\beta}(\omega(\xi-a)^{\frac{\alpha}{\rho}}) \\
&\quad \times \int_a^\xi \frac{(\xi-s)^{\frac{\alpha}{\rho}}-1}{k^2} E_{\kappa,\rho,\beta}(\omega(\xi-s)^{\frac{\alpha}{\rho}})q(s)y(s)ds \\
&\quad - \int_a^t \frac{(t-s)^{\frac{\alpha}{\rho}}-1}{k^2} E_{\kappa,\rho,\beta}(\omega(t-s)^{\frac{\alpha}{\rho}})q(s)y(s)ds.
\end{align*}
\]
Further, on rearranging the terms, we have

\[ y(t) = \int_a^t \left[ (t - a)^{\frac{\alpha}{2}} \frac{\partial}{\partial t} E_{k,\rho,\beta}^\gamma (\omega(t - a)^\frac{\alpha}{2}) \frac{(b - s)^{\frac{\alpha}{2}} - 2}{k^2} E_{k,\rho,\beta - k}^\gamma (\omega(b - s)^\frac{\alpha}{2}) - \frac{(t - s)^{\frac{\alpha}{2}} - 1}{k} E_{k,\rho,\beta}^\gamma (\omega(t - s)^\frac{\alpha}{2}) \right] q(s)y(s)ds + \int_a^b \left[ (t - a)^{\frac{\alpha}{2}} \frac{\partial}{\partial t} E_{k,\rho,\beta}^\gamma (\omega(t - a)^\frac{\alpha}{2}) \frac{(b - s)^{\frac{\alpha}{2}} - 2}{k^2} E_{k,\rho,\beta - k}^\gamma (\omega(b - s)^\frac{\alpha}{2}) - \frac{(t - s)^{\frac{\alpha}{2}} - 1}{k} E_{k,\rho,\beta}^\gamma (\omega(t - s)^\frac{\alpha}{2}) \right] q(s)y(s)ds + \int_a^b \frac{(b - s)^{\frac{\alpha}{2}} - 2}{k^2} E_{k,\rho,\beta - k}^\gamma (\omega(b - s)^\frac{\alpha}{2}) q(s)y(s)ds \]

therefore the solution \( y(t) \) becomes

\[ y(t) = \int_a^b G(t, s)q(s)y(s)ds + \frac{\alpha(t - a)^{\frac{\alpha}{2}} - 1}{E_{k,\rho,\beta - k}^\gamma (\omega(b - a)^\frac{\alpha}{2}) - \alpha(\xi - a)^{\frac{\alpha}{2}} - 1} E_{k,\rho,\beta}^\gamma (\omega(\xi - a)^\frac{\alpha}{2}) \int_a^b G(\xi, s)q(s)y(s)ds, \]

where the Green’s function \( G(t, s) \) is given by \( 3.1 \).

**Theorem 3.2** The Green’s function \( 3.1 \) satisfies the following properties:
(a) \( G(t, s) \geq 0 \), for all \( (t, s) \in [a, b] \times [a, b] \);
(b) \( G(t, s) \) is nondecreasing function with respect to the first variable;
(c) \( 0 \leq G(a, s) \leq G(t, s) \leq G(b, s) \), \( (t, s) \in [a, b] \times [a, b] \).
proof (a). For proof see, (Theorem 3.2, in [21])
proof (b). Proof of this follows from (Theorem 2, in [9])
Proof (c). Proof of this follows from (b).

Theorem 3.3 Suppose that problem (1.8) has a nontrivial continuous solution, then

\[
\int_a^b \left( (b-a)^{\frac{\alpha}{\beta}} E_{k,\rho,\beta}^\gamma (\omega(b-a)^{\frac{\alpha}{\beta}} (b-s)^{\frac{\alpha}{\beta}-\frac{2}{k}} E_{k,\rho,\beta-k}^\gamma (\omega(b-s)^{\frac{\alpha}{\beta}}) \right. \\
\left. - \frac{b-s}{k} E_{k,\rho,\beta-k}^\gamma (\omega(b-s)^{\frac{\alpha}{\beta}}) \right) |q(s)| ds \\
\geq \left( 1 + \frac{\alpha(b-a)^{\frac{\alpha}{\beta}-1} E_{k,\rho,\beta}^\gamma (\omega(b-a)^{\frac{\alpha}{\beta}}) - \alpha(\xi-a)^{\frac{\alpha}{\beta}-1} E_{k,\rho,\beta}^\gamma (\omega(\xi-a)^{\frac{\alpha}{\beta}}) )}{(b-a)^{\frac{\alpha}{\beta}-}\frac{1}{k} E_{k,\rho,\beta}^\gamma (\omega(b-a)^{\frac{\alpha}{\beta}})} \right)^{-1}
\]

Proof. Consider the Banach space

\[ C[a,b] = \{ u : [a,b] \rightarrow \mathbb{R} \mid u \text{ is continuous} \}
\]
equipped with norm \( \|u\| = \max \{ |u(t)| : a \leq t \leq b \}, u \in C[a,b] \).

By theorem 3.1 a solution \( y \in C[a,b] \) of (1.8) has the expression for \( a \leq t \leq b \),

\[
y(t) = \int_a^b G(t,s)q(s)y(s) ds \\
+ \frac{\alpha(t-a)^{\frac{\alpha}{\beta}-1} E_{k,\rho,\beta}^\gamma (\omega(t-a)^{\frac{\alpha}{\beta}}) - \alpha(\xi-a)^{\frac{\alpha}{\beta}-1} E_{k,\rho,\beta}^\gamma (\omega(\xi-a)^{\frac{\alpha}{\beta}}) )}{(b-a)^{\frac{\alpha}{\beta}-}\frac{1}{k} E_{k,\rho,\beta}^\gamma (\omega(b-a)^{\frac{\alpha}{\beta}})} \int_a^b G(\xi,s)q(s)y(s) ds.
\]

From this, for any \( t \in [a,b] \), we have

\[
|y(t)| \leq \|y\| \int_a^b |G(t,s)||q(s)| ds \\
+ \frac{\alpha(t-a)^{\frac{\alpha}{\beta}-1} E_{k,\rho,\beta}^\gamma (\omega(t-a)^{\frac{\alpha}{\beta}}) - \alpha(\xi-a)^{\frac{\alpha}{\beta}-1} E_{k,\rho,\beta}^\gamma (\omega(\xi-a)^{\frac{\alpha}{\beta}}) )}{(b-a)^{\frac{\alpha}{\beta}-}\frac{1}{k} E_{k,\rho,\beta}^\gamma (\omega(b-a)^{\frac{\alpha}{\beta}})} \int_a^b |G(\xi,s)||q(s)| ds,
\]

therefore,

\[
|y(t)| \leq \|y\| \int_a^b |G(b,s)||q(s)| ds \\
+ \frac{\alpha(b-a)^{\frac{\alpha}{\beta}-1} E_{k,\rho,\beta}^\gamma (\omega(b-a)^{\frac{\alpha}{\beta}}) \|y\| \infty}{(b-a)^{\frac{\alpha}{\beta}-}\frac{1}{k} E_{k,\rho,\beta}^\gamma (\omega(b-a)^{\frac{\alpha}{\beta}})} \int_a^b |G(b,s)||q(s)| ds,
\]
which yields
\[
\|y\|_\infty \leq \|y\|_\infty \left(1 + \frac{\alpha(b-a)^{\frac{\beta}{k}-1}E_{k,\rho,\beta}^v(\omega(b-a)^{\frac{1}{k}})}{(b-a)^{\frac{\beta}{k}-2}E_{k,\rho,\beta-k}^v(\omega(b-a)^{\frac{1}{k}}) - \alpha(\xi - a)^{\frac{\beta}{k}-1}E_{k,\rho,\beta}^v(\omega(\xi - a)^{\frac{1}{k}})}\right) \\
\times \int_a^b G(b, s)|q(s)|ds,
\]

As \(y\) is a nontrivial solution, we have
\[
1 \leq \left(1 + \frac{\alpha(b-a)^{\frac{\beta}{k}-1}E_{k,\rho,\beta}^v(\omega(b-a)^{\frac{1}{k}})}{(b-a)^{\frac{\beta}{k}-2}E_{k,\rho,\beta-k}^v(\omega(b-a)^{\frac{1}{k}}) - \alpha(\xi - a)^{\frac{\beta}{k}-1}E_{k,\rho,\beta}^v(\omega(\xi - a)^{\frac{1}{k}})}\right) \\
\times \int_a^b |G(b, s)||q(s)|ds.
\]

\[
\int_a^b |G(b, s)||q(s)|ds \geq \left(1 + \frac{\alpha(b-a)^{\frac{\beta}{k}-1}E_{k,\rho,\beta}^v(\omega(b-a)^{\frac{1}{k}})}{(b-a)^{\frac{\beta}{k}-2}E_{k,\rho,\beta-k}^v(\omega(b-a)^{\frac{1}{k}}) - \alpha(\xi - a)^{\frac{\beta}{k}-1}E_{k,\rho,\beta}^v(\omega(\xi - a)^{\frac{1}{k}})}\right)^{-1},
\]
therefore
\[
\int_a^b \left[\frac{(b-a)^{\frac{\beta}{k}-1}E_{k,\rho,\beta}^v(\omega(b-a)^{\frac{1}{k}})(b-s)^{\frac{\beta}{k}-2}E_{k,\rho,\beta-k}^v(\omega(b-s)^{\frac{1}{k}})}{(b-a)^{\frac{\beta}{k}-2}E_{k,\rho,\beta-k}^v(\omega(b-a)^{\frac{1}{k}}) - \alpha(\xi - a)^{\frac{\beta}{k}-1}E_{k,\rho,\beta}^v(\omega(\xi - a)^{\frac{1}{k}})}\right]q(s)|ds \\
\geq \left(1 + \frac{\alpha(b-a)^{\frac{\beta}{k}-1}E_{k,\rho,\beta}^v(\omega(b-a)^{\frac{1}{k}})}{(b-a)^{\frac{\beta}{k}-2}E_{k,\rho,\beta-k}^v(\omega(b-a)^{\frac{1}{k}}) - \alpha(\xi - a)^{\frac{\beta}{k}-1}E_{k,\rho,\beta}^v(\omega(\xi - a)^{\frac{1}{k}})}\right)^{-1}.
\]
Hence the result.

4 Conclusion

In this chapter, we obtained more general results than in [21]. The results in [21] can be obtained for particular values of \(k\) and \(\beta\) as \(k = 1\) and \(\beta = \mu\) in Green’s function in Theorem 3.1. and Hartman-Wintner-type inequality in Theorem 3.3.
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