Quantum minimax receiver for ternary coherent state signal in the presence of thermal noise

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Abstract. This paper is concerned with the minimax strategy in quantum signal detection theory. First we show a numerical calculation method for finding a solution to the quantum minimax decision problem in the case that the average probability of decision errors is used as the quality function of a quantum communication system. To verify the numerical calculation method, ternary coherent state signal is considered in the absence of thermal noise. After that, the error probability of the quantum minimax receiver for the ternary coherent state signal in the presence of thermal noise is computed by using this numerical calculation method.

1. Introduction

About three decades ago Hirota and Ikehara discussed the quantum minimax strategy and its application to optical communication systems. Through the discussion, they showed necessary and sufficient conditions of the quantum minimax strategy under the situation that the average probability of decision errors is used as a quality function [1]. Recently their result was extended to the case of the average Bayes cost [2]. The aim of this study is to develop an evaluation method for optical communication systems from the viewpoint of the quantum minimax strategy by showing a concrete example with the help of the necessary and sufficient conditions of the minimax strategy.

In this paper we investigate the error performance of the minimax receiver for ternary coherent state signal, \( \{ |0\rangle, |\alpha\rangle, |-\alpha\rangle \} \). Section 2 gives a short survey of the minimax problem in quantum signal detection. In Section 3, we show a numerical calculation method for finding a solution to the quantum minimax problem. If there is no thermal noise, one can obtain the closed-form expressions of the minimax decision rule, minimax distribution, and minimax value for the ternary coherent state signal, respectively. The comparison between these analytical results and the corresponding numerical calculation results based on the numerical calculation method is done in Section 4. Finally numerical calculation results for the ternary signal in the presence of thermal noise are shown in Section 5, and we give the conclusion of this paper in Section 6.

2. The minimax problem in quantum signal detection

Suppose that there are \( M \) symbols to be transmitted from a sender to a receiver and these symbols are drawn according to a probability distribution \( \mathbf{p} = (p_1, p_2, \ldots, p_M) \). The \( i \)-th symbol is modulated to a quantum state signal, and the signal is transmitted to the receiver through a communication channel. Let \( \hat{\rho}_i \) denote the received signal when the \( i \)-th signal was transmitted,
where $\hat{\rho}_i$ is a density operator on Hilbert space $\mathcal{H}$: $\hat{\rho}_i \geq 0$ and $\text{Tr}\hat{\rho}_i = 1$. Then the receiver performs an appropriate quantum signal detection, which is described by a positive operator-valued measure (POVM) $\Pi = (\hat{\Pi}_1, \hat{\Pi}_2, \ldots, \hat{\Pi}_M)$. At that time, the conditional probability that the receiver chooses symbol $j$ when symbol $i$ was sent is given by $P(j|i) = \text{Tr}\hat{\rho}_i\hat{\Pi}_j$. Similarly, the joint probability $P(i,j)$ is given by $P(i,j) = p_i\text{Tr}\hat{\rho}_i\hat{\Pi}_j$. To evaluate this communication system, we employ the average probability of error,

$$
P_e(\Pi, \mathbf{p}) = \sum_{i \neq j} p_i\text{Tr}\hat{\rho}_i\hat{\Pi}_j = 1 - \sum_{i=1}^{M} p_i\text{Tr}\hat{\rho}_i\hat{\Pi}_i.
$$

In our scenario we assume that the probability distribution $\mathbf{p}$ is unknown, unlike the Bayes decision problem. Under this assumption, we consider the following optimization problem: find $(\Pi^\circ, \mathbf{p}^\circ)$ and $\bar{P}_e$ such that

$$
\min_{\Pi \in \mathcal{D}} \max_{\mathbf{p} \in \mathcal{P}} P_e(\Pi, \mathbf{p}) = \bar{P}_e(\Pi^\circ, \mathbf{p}^\circ) = \bar{P}_e^\circ,
$$

where $\mathcal{D} = \{\Pi = (\hat{\Pi}_1, \ldots, \hat{\Pi}_M)\}$ and $\mathcal{P} = \{\mathbf{p} = (p_1, \ldots, p_M)\}$. For this problem, Hirota and Ikehara [1] showed that the relation $\min_{\Pi \in \mathcal{D}} \max_{\mathbf{p} \in \mathcal{P}} P_e(\Pi, \mathbf{p}) = \bar{P}_e^\circ = \max_{\mathbf{p} \in \mathcal{P}} \min_{\Pi \in \mathcal{D}} P_e(\Pi, \mathbf{p})$ holds and there exists a solution to this problem. Hereafter, we call $\Pi^\circ$ the minimax decision rule, $\mathbf{p}^\circ$ the minimax distribution, and $\bar{P}_e^\circ$ the minimax value. In general, it is not easy to find a closed-form solution to this minimax problem even when the signal states are pure. However, we have obtained necessary and sufficient conditions for the minimax decision rule $\Pi^\circ = (\hat{\Pi}_1^\circ, \hat{\Pi}_2^\circ, \ldots, \hat{\Pi}_M^\circ)$ and minimax distribution $\mathbf{p}^\circ = (p_1^\circ, p_2^\circ, \ldots, p_M^\circ)$ as follows [1, 2]:

$$
\hat{\Pi}_i^\circ(p_i^\circ\hat{\rho}_i - p_j^\circ\hat{\rho}_j)\hat{\Pi}_j^\circ = 0 \quad \forall (i,j),
$$

$$
\text{Tr}\hat{\Pi}_i^\circ\hat{\rho}_i = \text{Tr}\hat{\Gamma}^\circ \quad \forall i \text{ s.t. } p_i^\circ > 0,
$$

$$
\text{Tr}\hat{\Pi}_i^\circ\hat{\rho}_i \geq \text{Tr}\hat{\Gamma}^\circ \quad \forall i \text{ s.t. } p_i^\circ = 0,
$$

where $\hat{\Gamma}^\circ = \sum_{k=1}^{M} p_k^\circ \hat{\rho}_k\hat{\Pi}_k^\circ$, and the minimax value is $\bar{P}_e^\circ = 1 - \text{Tr}\hat{\Gamma}^\circ$.

### 3. Numerical calculation method for finding a minimax solution

As pointed out above it is not easy to find an explicit expression of the minimax solution, $(\Pi^\circ, \mathbf{p}^\circ)$ and $\bar{P}_e^\circ$, in general. This motivates us to consider a numerical calculation method for finding a minimax solution.

To begin with, we choose a probability distribution $\mathbf{p}(0) = (p_1(0), p_2(0), \ldots, p_M(0))$ of the signals arbitrary. Then we can find a Bayes decision rule $\Pi(0) = (\hat{\Pi}_1(0), \hat{\Pi}_2(0), \ldots, \hat{\Pi}_M(0))$ for $\mathbf{p}(0)$ such that

$$
\min_{\Pi \in \mathcal{D}} P_e(\Pi, \mathbf{p}(0)) = P_e(\Pi(0), \mathbf{p}(0)) = \bar{P}_e^{\text{bayes}}(\mathbf{p}(0)) \equiv \bar{P}_e(0).
$$

![Figure 1. Communication model and its quality function](image-url)
Our numerical calculation method starts with these initial values — \((\Pi^{(0)}, p^{(0)})\) and \(\bar{P}_e^{(0)}\) — and iterates the procedures mentioned below until the change of the average probability of error is small enough.

Suppose that the calculation procedure is in the \((n+1)\)-st stage, \((n = 0, 1, \ldots)\). At that time, the \(n\)-th distribution \(p^{(n)} = (p_1^{(n)}, p_2^{(n)}, \ldots, p_M^{(n)})\), Bayes decision rule \(\Pi^{(n)} = (\hat{\Pi}_1^{(n)}, \hat{\Pi}_2^{(n)}, \ldots, \hat{\Pi}_M^{(n)})\), and error probability \(\bar{P}_e^{(n)} \equiv \bar{P}_e(\Pi^{(n)}, p^{(n)})\) have been already recorded in the memory of computer. Now we choose a pair of indeces \((i, j)\) such that \(i \neq j\). For this \((i, j)\), we consider the set of probability distributions of the form
\[
p^{(\text{temp})} = (p_1^{(n)}, p_2^{(n)}, \ldots, s_i, \ldots, t_j, \ldots, p_M^{(n)}),
\] where \(s\) and \(t\) satisfy the conditions \(s \geq 0\), \(t \geq 0\), and \(s + t = 1 - \sum_{k \neq i, j} p_k^{(n)}\). From the concavity of \(\min \bar{P}_e(\Pi^{(n)}, p) = \bar{P}_e^{\text{bayes}}(p)\) with respect to \(p\), we have the inequality
\[
\bar{P}_e(\Pi^{(n)}, p^{(n)}) \leq \max_{s, t} \bar{P}_e^{\text{bayes}}(p^{(\text{temp})}) = \max_{s, t} \min_{\bar{P}_e(\Pi^{(n)}, p^{(\text{temp})})}.
\]
Therefore our task at this stage is to find a pair \((\Pi^*, p^*)\) such that
\[
\max_{s, t} \min_{\bar{P}_e(\Pi^{(n)}, p^{(\text{temp})})} = \bar{P}_e(\Pi^*, p^*).
\]
Since \(s\) and \(t\) are not independent, the maximization part is a single-parameter maximization problem. Therefore, by changing the value \(s\) form \(s = 0\) to \(t = 1 - \sum_{k \neq i, j} p_k^{(n)}\), we can find a solution to this maximization problem. Recall that the Bayes cost reduction algorithm has been well developed [3, 4]. So, one can find such a pair \((p^*, \Pi^*)\) without any technical difficulty. Once we could find \((p^*, \Pi^*)\), we set
\[
\Pi^{(n+1)} \leftarrow \Pi^*, \quad p^{(n+1)} \leftarrow p^*, \quad \bar{P}_e^{(n+1)} \leftarrow \bar{P}_e(\Pi^{(n+1)}, p^{(n+1)}),
\]
and proceed to the next stage by choosing another \((i, j)\).

Finally we check whether the conditions (3)-(5) are satisfied or not, when the increase of \(\bar{P}_e^c\) is small enough. If the conditions are satisfied with sufficiently small numerical error, we halt the calculation; if not, we choose another initial values and repeat the procedures.

4. Ternary pure state signal case

In this section we consider the pure state signal case to verify the numerical calculation method mentioned above. Let \(\mathcal{A} = \{0, +, -\}\) be the alphabet of signaling symbols. For each symbol, we set \(0 \mapsto |0\rangle \equiv |\psi\rangle, + \mapsto |\alpha\rangle \equiv |\psi_+\rangle, - \mapsto -|\alpha\rangle \equiv |\psi_-\rangle\), where \(|0\rangle\) is the vacuum state, and \(|\pm \alpha\rangle\) are the coherent states of light having complex amplitudes \(\alpha\) and \(-\alpha\), respectively. Now we assume that \(\alpha > 0\) for simplicity and define
\[
\begin{align*}
|\phi_1\rangle & \equiv \frac{1}{\sqrt{2(1+\kappa^2)}} |\psi_+\rangle + \frac{1}{\sqrt{2(1-\kappa^2)}} |\psi_-\rangle, \\
|\phi_2\rangle & \equiv \frac{1}{\sqrt{2(1-\kappa^2)}} |\psi_+\rangle - \frac{1}{\sqrt{2(1+\kappa^2)}} |\psi_-\rangle, \\
|\phi_3\rangle & \equiv \frac{\sqrt{1+\kappa}}{\sqrt{1-\kappa^2}} |\psi_0\rangle - \frac{\kappa}{(1-\kappa^2)^{1/4}} |\psi_+\rangle - \frac{\kappa}{(1+\kappa^2)^{1/4}} |\psi_-\rangle,
\end{align*}
\]
where \(\kappa = \exp[-|\alpha|^2/2]\). These vectors form an ordered orthonormal basis, \(\gamma = \{ |\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}\).

By using this orthonormal basis \(\gamma\), the signal states can be represented as
\[
|\psi_0\rangle \doteq [|\psi_0\rangle]_\gamma = \begin{pmatrix} w \\ 0 \\ x \end{pmatrix}, \quad |\psi_+\rangle \doteq [|\psi_+\rangle]_\gamma = \begin{pmatrix} y \\ z \\ 0 \end{pmatrix}, \quad |\psi_-\rangle \doteq [|\psi_-\rangle]_\gamma = \begin{pmatrix} y \\ -z \\ 0 \end{pmatrix},
\]
(12)
Figure 2. (a) The minimax value $P_{e}^{\circ}$ of the ternary coherent state signal in the absence of thermal noise versus the average number $|\alpha|^{2}$ of signal photons. (b)-(c) The corresponding minimax distribution $P^{\circ} = (p_{0}^{\circ}, p_{+}^{\circ}, p_{-}^{\circ})$.

where

$$
\begin{align*}
    w &= \frac{\sqrt{2}\kappa}{\sqrt{1 + \kappa^{2}}} \\
    x &= \frac{1 - \kappa^{2}}{\sqrt{1 + \kappa^{2}}} \\
    y &= \frac{\sqrt{1 + \kappa^{4}}}{\sqrt{2}} \\
    z &= \frac{\sqrt{1 - \kappa^{4}}}{\sqrt{2}}.
\end{align*}
$$

(13)

Let $\Pi^{\circ} = (\hat{\Pi}_{0}^{\circ}, \hat{\Pi}_{+}^{\circ}, \hat{\Pi}_{-}^{\circ})$ denote the minimax decision rule for the ternary coherent state signal. The minimax decision operators are respectively given by $\hat{\Pi}_{0}^{\circ} = |d_{0}\rangle\langle d_{0}|$, $\hat{\Pi}_{+}^{\circ} = |d_{+}\rangle\langle d_{+}|$, and $\hat{\Pi}_{-}^{\circ} = |d_{-}\rangle\langle d_{-}|$ with

$$
\begin{align*}
    |d_{0}\rangle &\doteq \left( \begin{array}{c}
    \sqrt{1 - q^{2}} \\
    0 \\
    q
    \end{array} \right), \\
    |d_{+}\rangle &\doteq \frac{1}{\sqrt{2}} \left( \begin{array}{c}
    q \\
    1 \\
    -\sqrt{1 - q^{2}}
    \end{array} \right), \\
    |d_{-}\rangle &\doteq \frac{1}{\sqrt{2}} \left( \begin{array}{c}
    q \\
    -1 \\
    \sqrt{1 - q^{2}}
    \end{array} \right),
\end{align*}
$$

(14)
where \( q = (\sqrt{4y(y - \sqrt{2x})} + 2w + \sqrt{2xz - yz})/(y^2 - 2\sqrt{2xy} + 2) \). Further, the minimax value \( P_e^o \) and minimax distribution \( p^o = (p^o_0, p^o_+, p^o_-) \) are respectively given by

\[
P_e^o = 1 - \left( \sqrt{1-q^2}w + qx \right)^2,
\]

and

\[
p^o_0 = \frac{\sqrt{1-q^2}y}{\sqrt{2qw + \sqrt{1-q^2}(y - \sqrt{2x})}}, \quad p^o_+ = p^o_- = \frac{qw - \sqrt{1-q^2}x}{2qw + \sqrt{1-q^2}(\sqrt{2y} - 2x)}.
\]

Using Eqs.(15) and (16) one can simulate their behavior explicitly. Figure 2(a) shows the minimax value \( P_e^o \) that is given by Eq.(15), together with the numerical calculation result that is computed by the proposing method. Similarly, Figure 2(b) and 2(c) show the minimax distribution \( p^o \) that is given by Eq.(16) and the corresponding numerical calculation result, respectively. From these figures we see that the numerical calculation results based on the numerical calculation method capture the analytical results. Thus the numerical calculation method works.

5. For the signals in the presence of thermal noise

Let \( \bar{N}_th \) denote the average number of photons of thermal noise. The density operator of the thermal state is given by

\[
\hat{\rho}_{th} = (1 - \nu) \sum_{n=0}^{\infty} \nu^n |n\rangle \langle n|,
\]

where \(|n\rangle\) is the eigenstate of the number operator \( \hat{a}^\dagger \hat{a} \), and the real parameter \( \nu \) (0 \leq \nu \leq 1) is defined as \( \nu = \bar{N}_th/(1 + \bar{N}_th) \), i.e., \( \bar{N}_th = \nu/(1 - \nu) \).

Suppose that the \( i \)-th coherent state signal has amplitude \( \alpha_i \). If each signal is in the pressure of thermal noise, then the corresponding quantum state is given as \( \hat{\rho}_i = \hat{D}(\alpha_i)\hat{\rho}_{th}\hat{D}^\dagger(\alpha_i) \), where \( \hat{D}(\beta) \equiv \exp[\beta \hat{a}^\dagger - \beta^* \hat{a}] \) is the displacement operator. In the case of the ternary coherent state signal, the signal received states are respectively given by

\[
\hat{\rho}_0 = \hat{\rho}_{th}, \quad \hat{\rho}_+ = \hat{D}(\alpha)\hat{\rho}_{th}\hat{D}^\dagger(\alpha), \quad \hat{\rho}_- = \hat{D}(-\alpha)\hat{\rho}_{th}\hat{D}^\dagger(-\alpha),
\]

and the signal-to-noise ratio is defined as \( \text{SNR} = |\alpha|^2/(2\bar{N}_th + 1) \) [3].

To perform numerical calculation, we use the following matrix representation of the signals in the basis \{\(|n\rangle\)\}, according to the literature [5]: \( \hat{\rho}_i \equiv \rho_i = (|m\rangle\langle n|)_{m,n=0,1,2,...} \) with

\[
\langle m|\hat{\rho}_i|n\rangle = \sqrt{m!n!/(1 - \nu)}\nu^n \left( \frac{\alpha_i^*}{\bar{N}_th} \right)^{n-m} \exp[-(1 - \nu)|\alpha_i|^2]\frac{L_n^{(n-m)}[-(1 - \nu)|\alpha_i|^2]}{\nu}
\]

for \( m \leq n \), and \( \langle m|\hat{\rho}_i|m\rangle = (|n\rangle\langle m|)_{m} \) for \( m > n \), where \( L_n^m(x) \) is the the associated Laguerre polynomial. Numerical calculation results are shown in Figure 3 for \( \nu = 0.01, 0.02, 0.05, 0.1, 0.2, \) and 0.5. We observe from Figure 3(b) that every \( p^o_i \) is positive for each \( \nu \). According to Lemma 3 in the literature [2], the relation \( P_e(II^o,q) = P_e^o \) holds for any \( q \in \mathcal{P} \). Thus, in each \( \nu \), the minimax receiver \( II^o \) for the ternary coherent state signal always indicates the same error performance shown in Figure 3(a) even if the true distribution \( q \) of the signals varies.

6. Conclusions

We have shown a numerical calculation method for finding a solution to the quantum minimax decision problem. To verify the numerical calculation method, we have analytically derived the minimax decision operators, minimax distribution, and minimax value for the ternary pure
Figure 3. (a) The minimax value $P^\circ$ of the ternary coherent state signal in the presence of thermal noise versus the signal-to-noise ratio $|\alpha|^2/(2N_{th} + 1)$. (b) The corresponding minimax distribution $p^\circ = (p^\circ_0, p^\circ_+, p^\circ_-)$.

cohherent state signal, respectively. Using these closed-form expressions, we have verified that the numerical calculation method works. Finally the error probability of the quantum minimax receiver for the ternary coherent state signal that is in the pressure of thermal noise was computed by using the numerical calculation method. Since every minimax probability obtained in our simulation is positive, we can expect that the minimax receiver for the ternary coherent state signal always indicates the same error performance even if the true distribution of the signals varies, because of Lemma 3 of the literature [2].

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