Propagation of analyticity for a class of nonlinear hyperbolic equations

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Abstract

We consider the hyperbolic semilinear equations of the form

\[ \partial_t^m u + a_1(t) \partial_t^{m-1} \partial_x u + \cdots + a_m(t) \partial_x^m u = f(u), \]

where \( f(u) \) is entire analytic, with characteristic roots satisfying the condition

\[ \lambda_i^2(t) + \lambda_j^2(t) \leq M(\lambda_i(t) - \lambda_j(t))^2, \quad \text{for} \quad i \neq j, \]

and we prove that, if the \( a_h(t) \) are analytic, each solution bounded in \( C^\infty \) enjoys the propagation of analyticity; while if \( a_h(t) \in C^\infty \), this property holds for those solutions which are bounded in some Gevrey class.

1 Introduction

The linear operator

\[ \mathcal{L} U = U_t + \sum_{h=1}^n A_h(t, x) U_{x_h} \quad \text{on} \quad [0, T] \times \mathbb{R}^n, \]

(1)

where the \( A_h \)'s are \( N \times N \) matrices, \( U \in \mathbb{R}^N \), is hyperbolic when, for all \( \xi \in \mathbb{R}^n \), the matrix \( \sum A_h(t, x) \xi_h \) has real eigenvalues \( \lambda_j(t, x, \xi), 1 \leq j \leq N \).

Denoting by \( \mu(\lambda) \) the multiplicity of the eigenvalue \( \lambda \), we call multiplicity of \( \Pi \) the integer \( m = \max_{t,x,\xi} \max_j \{ \mu(\lambda_j(t, x, \xi)) \} \). The case \( m = 1 \) corresponds to the strictly hyperbolic systems.

We study the regularity of solutions to nonlinear weakly hyperbolic system, in particular, semilinear systems

\[ \mathcal{L} U = f(t, x, U), \]

(2)
where \( U : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^N \), and \( f(t, x, U) \) is a \( \mathbb{R}^N \)-valued, analytic function, typically a polynomial in the scalar components of \( U \).

More precisely, assuming the coefficients of \( \mathcal{L} \) analytic in \( x \), we investigate under which additional assumptions a solution \( U(t, x) \) of (2), analytic at the initial time, keeps its analyticity, i.e., satisfies

\[
U(0, \cdot) \in \mathcal{A}(\mathbb{R}^n) \quad \Rightarrow \quad U(t, \cdot) \in \mathcal{A}(\mathbb{R}^n) \quad \forall \ t \in [0, T] \quad (3)
\]

Actually, we consider two versions of (3), the first weaker and the second one stronger than (3):

\[
U(0, \cdot) \in \mathcal{A}_{L^2}(\mathbb{R}^n) \quad \Rightarrow \quad U(t, \cdot) \in \mathcal{A}_{L^2}(\mathbb{R}^n) \quad \forall \ t \in [0, T], \quad (4)
\]

\[
U(0, \cdot) \in \mathcal{A}(\Gamma_0) \quad \Rightarrow \quad U(t, \cdot) \in \mathcal{A}(\Gamma_t) \quad \forall \ t \in [0, T], \quad (5)
\]

where \( \mathcal{A}_{L^2}(\mathbb{R}^n) \) is the class of (analytic) functions \( \varphi(x) \in H^\infty \) such that \( \|\partial^j \varphi\|_{L^2} \leq C \Lambda^j j! \), while \( \Gamma \) is a cone of determinacy for the operator \( \mathcal{L} \) with base \( \Gamma_0 \) (at \( t = 0 \)) and sections \( \{\Gamma_t\} \).

The propagation of analyticity is a natural property for nonlinear hyperbolic equations. Indeed, on one side, the theorem of Cauchy-Kovalewsky ensures the validity of (3) in some time interval \([0, \tau]\) (the problem is to prove that \( \tau = T \)), on the other side, by the Bony-Schapira’s theorem, the Cauchy problem for any linear (weakly) hyperbolic system is globally well posed in the class of analytic functions.

The first results of analytic propagation go back to Lax (\([L], 1953\)) who considered (2) with \( n = 1 \) in the strictly hyperbolic case, and proved (5) for those solutions which are a priori bounded in \( C^1 \). Later on Alinhac and Métivier (\([AM], 1984\)) extended this results to several space dimensions, but assuming that \( U(t, \cdot) \) is bounded in \( H^s(\mathbb{R}^n) \) for \( s \) greater than some \( \bar{s}(n) \).

In the weakly hyperbolic (nonlinear) case, the first results were concerning a second order equation of the form

\[
\mathcal{L}_0 u \equiv \sum_{i,j}^{1,n} \partial_{x_i}(a_{ij}(t, x) \partial_{x_j} u) = f(u), \quad \sum a_{ij} \xi_i \xi_j \geq 0, \quad (6)
\]

with \( f(u), a_{ij}(t, x) \) analytic:

**Theorem A** (\([S], 1989\))

i) In the special case when \( a_{ij} = \beta_0(t) \alpha_{ij}(x) \), a solution of (6) enjoys (5) as long as remains bounded in \( C^\infty \).

ii) In the general case, a solution \( u(t, \cdot) \) enjoys (5) provided it is bounded in some Gevrey class \( \gamma^s \) with \( s < 2 \).

We recall that the Cauchy problem for any strictly hyperbolic linear system is globally wellposed in \( C^\infty \). On the other hand, the Cauchy problem for the
linear equation \( \mathcal{L}_0 u = 0 \), i is globally wellposed in \( \mathcal{C}^\infty \) n the special case (i), whereas it is only globally wellposed in \( \gamma^s \) for \( s < 2 \) in the general case (ii). Thus, it is natural to formulate the following

**Conjecture** In order to get the analytic propagation for a given solution to a weakly hyperbolic system \( \mathcal{L} U = f(t, x, U) \), it is sufficient to assume a priori that \( U(t, \cdot) \) is bounded in some functional class \( \mathcal{X} \) in which the Cauchy problem for the linear systems \( \mathcal{L} U + B(t, x)U = f(t, x) \) is globally well posed.

[Typically the space \( \mathcal{X} \) is equal to \( \mathcal{C}^\infty \) or to some Gevrey class \( \gamma^s \)]

In the case when \( \mathcal{L} \) is a weakly hyperbolic operator of the **general type** (1), this Conjecture says that a solution \( U(t, \cdot) \) enjoys the analytic propagation a long as remains bounded in some Gevrey class \( \gamma^s \) of order \( s < m/(m-1) \), where \( m \) is the multiplicity of \( \mathcal{L} \). Indeed, Bronshtein’s Theorem ([B], 1979) states that, for any linear system \( \mathcal{L} U + B(t, x)U = f(t, x) \) with analytic coefficients in \( x \), the Cauchy problem is well-posed in these Gevrey classes.

Actually, this fact was proved in two special cases: time depending coefficients, and one space variable. More precisely:

**Theorem B** ([DS], 1999) A solution of

\[
U_t + \sum_{j=1}^{n} A_j(t) U_{x_j} = f(t, x, U), \quad x \in \mathbb{R}^n,
\]

satisfies (4) as long as \( U(t, \cdot) \) remains bounded in some \( \gamma^s \) with \( s < m/(m-1) \).

**Theorem C** ([ST], 2010) A solution of

\[
U_t + A(t, x) U_x = f(t, x, U), \quad x \in \mathbb{R},
\]

satisfies (4) as long as \( U(t, \cdot) \) remains bounded in some \( \gamma^s \) with \( s < m/(m-1) \).

The study of the general case (coefficients depending on \( (t, x) \), and \( n \geq 2 \)) is in progress.

**Open Problem.** To prove the sharpness of the bound \( s < m/(m-1) \) in Theorems B or C. In particular: to construct a hyperbolic nonlinear system admitting a solution \( U \in \mathcal{C}^\infty(\mathbb{R}^2) \) which is analytic on the halfplane \( \{t < 0\} \) but non analytic at some point of the line \( t = 0 \). This kind of questions is related to the so called **Nonlinear Holmgren Theorem** (see [M]).

**Acknowledgments.** We are indebted to Giovanni Taglialatela for his help to the drawing of this paper.
2 Main results

Hence, we consider the scalar equations of the form
\[ \mathcal{L}u \equiv \partial_t^m u + a_1(t) \partial_t^{m-1} \partial_x u + \cdots + a_m(t) \partial_x^m u = f(u), \]
for a second order equation, which satisfying the condition
\[ \Delta = \lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_m(t), \]

where
\[ \Delta(t) = a_2^2(t) - 4 a_2(t) \geq c a_1^2(t); \]
along with
\[ \Delta(t) \geq c (a_1(t)a_2(t) - 9 a_3(t))^2, \]

while for a third order equation, it becomes
\[ \Delta(t) \geq -c a_2, \text{ or equivalently } \Delta \geq c a_3. \]

Condition (8) for the linear equation \( \mathcal{L}u = 0 \) was introduced in [CO] as a sufficient (and almost necessary) condition for the wellposedness in \( C^\infty \). A different proof of such a result, based on the quasi-symmetrizer, was given in [KS], where, also the case of non-analytic coefficients was considered: it was proved that, if \( a_h(t) \in C^\infty([0,T]) \) and (8) is fulfilled, then the Cauchy problem for \( \mathcal{L}u = 0 \) is well posed in each Gevrey class \( \gamma^s \), \( s \geq 1 \).

By these existence results, it is natural to expect some kind of analytic propagation for the solutions which are bounded in \( C^\infty \) in case of analytic coefficients, or for those which are bounded in some Gevrey class \( \gamma^s \) in case of \( C^s \) coefficients.

Actually, introducing the analytic, and Gevrey classes
\[ A_{L^2} = \{ \varphi(x) \in C^\infty(\mathbb{R}) : \| \partial^j \varphi \|_{L^p(\mathbb{R})} \leq C \Lambda^j j! \}, \]
\[ \gamma^s_{L^2} = \{ \varphi(x) \in C^\infty(\mathbb{R}) : \| \partial^j \varphi \|_{L^p(\mathbb{R})} \leq C \Lambda^j j!^s \}, \]
where \( s \geq 1 \), we prove:
Theorem 1  Assume that the $a_j(t)$’s are analytic functions on $[0,T]$. Then, for any solution of (7) satisfying
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}} |\partial_t^h \partial_x^j u(t,x)| \, dx < \infty, \quad \forall j \in \mathbb{N},
\] (10)
\[
\partial_t^h u(0, \cdot) \in \mathcal{A}_{L^2},
\] (11)
for $h = 0, 1, \ldots, m - 1$, it holds
\[
u
u
\] (12)
Under the same assumptions, we have also
\[
u
u
\] (13)
Theorem 2  If the $a_j(t)$’s are $C^\infty$ functions on $[0,T]$, the implication (11) $\implies$ (12) holds true for those solutions which belong to $C^{m}([0,T], \gamma^s_{L^2})$ for some $s \geq 1$.

Proof of Theorem 1.  For the sake of simplicity, we shall perform the proof only in the case when the nonlinear term $f(u)$ is a monomial function, the general case requiring only minor additional computations. Thus, for a given integer $\nu \geq 1$, we consider the equation
\[
\partial_t^m u + a_1(t) \partial_t^{m-1} \partial_x u + \cdots + a_m(t) \partial_x^m u = u^\nu.
\] (14)
Putting
\[
\hat{u}(t, \xi) = \int_{-\infty}^{+\infty} e^{-i\xi x} u(t,x) \, dx,
\]
\[
V(t, \xi) = \begin{pmatrix} (i\xi)^{m-1} \hat{u} \\ (i\xi)^{m-2} \hat{u}' \\ \vdots \\ \hat{u}^{(m-1)} \end{pmatrix}, \quad F(t, \xi) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t, \xi) \end{pmatrix},
\] (15)
and
\[
A(t) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ a_m(t) & \cdots & a_2(t) & a_1(t) \end{pmatrix},
\] (16)
we transform equation (14) into the ODE’s system
\[
V' + i\xi A(t)V = F(t, \xi),
\] (17)
where
\[
f(t, \xi) = \hat{u} \ast \cdots \ast \hat{u}.
\] (18)
Our target is to prove that, if
\[
\int_{\mathbb{R}} |\xi|^j |V(t, \xi)| \, d\xi \leq K_j < \infty \quad \forall \, j, \quad \forall \, t \in [0, T],
\]
(19)
\[
\int_{\mathbb{R}} |\xi|^j |V(0, \xi)| \, d\xi \leq C \Lambda_j \lambda_j
\]
(20)
then, for some new constants \(\tilde{C}, \tilde{\Lambda}\), it holds
\[
\int_{\mathbb{R}} |\xi|^j |V(t, \xi)| \, d\xi \leq \tilde{C} \tilde{\Lambda} \lambda_j \lambda_j
\]
(21)
Indeed, (20) is an easy consequence of (11); while (10) implies that
\[
\{ \partial_t^j \partial_x^k u(t, \cdot) \}
\]
is bounded in \(L^\infty(\mathbb{R})\) for all \(j\), whence (19). Finally, taking into account that
\[
|V(t, \xi)| \leq K < \infty \quad \text{(by (10))},
\]
we see that (21) implies (12).
To get this target, we firstly prove an apriori estimate for the linear system
(17), without taking (18) into account. We follow [KS], but some modifications are needed in order to get an estimate suitable to the nonlinear case. The main tool is the theory of quasi-symmetrizer developed in [J] and [DS].

Recalls on quasi-symmetrizer.

[DS]: For any matrix of the form (16) with real eigenvalues, we can find a family of Hermitian matrices
\[
Q_\varepsilon(t) = Q_0(t) + \varepsilon^2 Q_1(t) + \cdots + \varepsilon^{2(m-1)} Q_{m-1}(t)
\]
(22)
such that the entries of the \(Q_\varepsilon(t)\)'s are polynomial functions of the coefficients \(a_1(t), \ldots, a_m(t)\) (in particular inherit their regularity in \(t\)), and
\[
C^{-1} \varepsilon^{2(m-1)} |V|^2 \leq (Q_\varepsilon(t)V, V) \leq C |V|^2
\]
(23)
\[
\left| (Q_\varepsilon(t)A(t) - A(t)Q_\varepsilon(t))V, V \right| \leq C \varepsilon^{1-m} (Q_\varepsilon(t)V, V).
\]
(24)
for all \(V \in \mathbb{R}^m\), \(0 < \varepsilon \leq 1\).

[KS]: If the eigenvalues of \(A(t)\) satisfy the condition (8), then \(Q_\varepsilon(t)\) is a nearly diagonal matrix, i.e., it satisfies, for some constant \(c > 0\), independent on \(\varepsilon\),
\[
(Q_\varepsilon(t)V, V) \geq c \sum_{j=1}^{m} q_{\varepsilon, ij}(t)v_j^2, \quad \forall \, V \in \mathbb{R}^m,
\]
(25)
where \(q_{\varepsilon, ij}\) are the entries of \(Q_\varepsilon\), \(v_j\) the scalar components of \(V\). □

In our assumptions, the \(a_h(t)\)'s are analytic functions on \([0, T]\), consequently also the entries \(q_{\varepsilon, ij}(t), 1 \leq i, j \leq m\) of the matrix \(Q_\varepsilon(t)\) will be analytic.
Therefore, putting together all the isolated zeroes of these functions, we form a partition of \([0, T]\), independent on \(\varepsilon\),

\[
0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T,
\]

such that, for each \(r, i, j\), it holds:

either \(q_{r,ij} \equiv 0\), or \(q_{r,ij}(t) \neq 0 \quad \forall t \in I_h = [t_{h-1}, t_h[.
\]

Now, let us notice that, by Cauchy-Kovalewsky, if at some point \(t\) a solution to (14) satisfies \(\partial_t^h u(t, \cdot) \in A_{L^2}(R)\) for all \(h \leq m - 1\), then the same holds in a right neighborhood of \(t\). Thus, it will be sufficient to put ourselves inside one of the intervals \(I_1, \ldots, I_N\). In other words it is not restrictive to assume that, for each \(r, i, j\),

\[
\text{either } q_{r,ij} \equiv 0, \quad \text{or } q_{r,ij}(t) \not\equiv 0 \quad \forall 0 \leq t < T.
\]

Therefore, by the analyticity of \(q_{r,ij}(t)\) we easily derive that

\[
|q'_{r,ij}(t)| \leq \frac{C}{T-t} |q_{r,ij}(t)| \quad \text{on } [0, T[.
\]

Next, following [KS], for any fixed \(\xi \in \mathbb{R}\) we prove two different apriori estimates for a solution \(V(t, \xi)\) of (17): a Kovalewskian estimate in a (small) left neighborhood of \(T\), \([T - \tau, T[\), and a hyperbolic estimate on \([0, \tau]\).

[In the following \(C, C_j\) will be constants depending on the coefficients of \((14)\)]

**Lemma 1** Let \(V(t, \xi)\) be a solution of (17) on \([0, T[\), and put

\[
E_{\varepsilon}(t, \xi) = (Q_{\varepsilon}(t) V(t, \xi), V(t, \xi)).
\]

Then, for any fixed \(\xi \in \mathbb{R}\), the following estimates hold:

\[
\partial_t |V(t, \xi)| \leq \frac{C_0}{T} |\langle \xi \rangle | |V(t, \xi)| + |F(t, \xi)|,
\]

\[
\partial_t \sqrt{E_{\varepsilon}(t, \xi)} \leq C_0 \left( \frac{1}{T-t} + \varepsilon |\langle \xi \rangle| \right) \sqrt{E_{\varepsilon}(t, \xi)} + C_0 |F(t, \xi)|,
\]

\(C_0\) a constant depending only on the coefficients of the equation, and on \(T\).

In particular, putting

\[
E_* = E_{\varepsilon_*}, \quad \text{where } \varepsilon_* = \langle \xi \rangle^{-1}, \quad \langle \xi \rangle = 1 + |\xi|,
\]

\((17)\) gives

\[
(\sqrt{E_*})' \leq C_0 \left( \frac{1}{T-t} + 1 \right) \sqrt{E_*} + C_0 |F(t, \xi)|.
\]
Proof: As an easy consequence of (17), we get (30) with
\[ C_0 \geq \max_{t \in [0, T]} \| A(t) \|, \quad C_0 \geq 1. \]

To prove (31) we differentiate (29) in time. Recalling (23) we find
\[
E'(t, \xi) = (Q'_\xi V, V) + (Q\xi V', V) + (Q\xi V, V')
\leq K_\epsilon(t, \xi) E_\epsilon(t, \xi) + C_1 |F(t, \xi)| \sqrt{E_\epsilon(t, \xi)}
\]
where \( V = V(t, \xi) \) and
\[
K_\epsilon(t, \xi) = \frac{|(Q'_\xi V, V)|}{(Q\xi V, V)} + |\xi| \frac{|((Q\xi A - A^*Q\xi)V, V)|}{(Q\xi V, V)}.
\]

We have to prove that
\[
K_\epsilon(t, \xi) \leq C \left( \frac{1}{T - t} + \epsilon |\xi| \right) \quad \forall t \in [0, T[.
\]
Let us firstly note that the second quotient in (34) is estimated by \( C\epsilon \) by the property (24) of our quasi-symmetrizer. To estimate the first quotient, apply to the nearly diagonality of the matrix \( Q\xi(t) \), i.e., (25): recalling (22), and noting that \( |q_{r,ij}| \leq \sqrt{q_{r,ii}q_{r,jj}} \) (since \( Q\xi(t) \) is a symmetric matrix \( \geq 0 \)), it follows
\[
|(Q'_\xi V, V)| \leq \sum_{r=0}^{m-1} \varepsilon^{2r} \sum_{ij} |q'_{r,ij}| |v_i v_j| \leq C (T - t)^{-1} \sum_{r} \varepsilon^{2r} \sum_{ij} |q_{r,ij}| |v_i v_j|
\leq C (T - t)^{-1} \sum_{r} \varepsilon^{2r} \sum_{j} q_{r,jj} v_j^2 = C (T - t)^{-1} q_{\epsilon,jj} v_j^2
\leq C_1 (T - t)^{-1} (Q\xi V, V).
\]
This completes the proof of (35), hence of (31). \( \square \)

Next, we define
\[
\tau(\xi) = T - |\xi|^{-1},
\]
\[
\Phi(t, \xi) = C_0 \min \left\{ (T - t)^{-1} + 1, |\xi| \right\} = \begin{cases} C_0 \left( (T - t)^{-1} + 1 \right) & \text{on } [0, \tau(\xi)] \\ C_0 |\xi| & \text{on } [\tau(\xi), T[ \end{cases}
\]
\[
\rho(t, \xi) = \int_t^T \Phi(s, \xi) \, ds.
\]
Therefore, by (30) and (33) it follows
\[
\begin{align*}
\{V(t, \xi)|' \leq & \Phi(t, \xi) |V(t, \xi)| + C_0 |F(t, \xi)| \quad \text{on } [\tau(\xi), T[ \\
\sqrt{E_+(t, \xi)}' \leq & \Phi(t, \xi) \sqrt{E_+(t, \xi)} + C_0 |F(t, \xi)| \quad \text{on } [0, \tau(\xi)]
\end{align*}
\] (39)
and thus, since \(\rho' = -\Phi\),
\[
\partial_t \left\{ e^{\rho(t, \xi)} |V(t, \xi)| \right\} \leq C_0 e^{\rho(t, \xi)} |F(t, \xi)| \quad \text{for } \tau(\xi) \leq t \leq T
\]
\[
\partial_t \left\{ e^{\rho(t, \xi)} \sqrt{E_+(t, \xi)} \right\} \leq C_0 e^{\rho(t, \xi)} |F(t, \xi)| \quad \text{for } 0 \leq t \leq \tau(\xi).
\]
By integrating in time, we find (omitting \(\xi\) everywhere)
\[
e^{\rho(t)} |V(t)| \leq e^{\rho(\tau)} |V(\tau)| + C_0 \int_\tau^t e^{\rho(s)} |F(s)| \, ds \quad (40)
\]
\[
e^{\rho(\tau)} \sqrt{E_+(\tau)} \leq e^{\rho(0)} \sqrt{E_+(0)} + C_0 \int_0^\tau e^{\rho(s)} |F(s)| \, ds \quad (41)
\]
Now, by (23) with \(\varepsilon = \langle \xi \rangle^{-1}\), we know that
\[
C^{-1} \langle \xi \rangle^{-2(1-m)} |V(t, \xi)|^2 \leq E_+(t, \xi) \leq C |V(t, \xi)|^2;
\]
hence we derive, form (40) and (41),
\[
e^{\rho(t)} |V(t)| = C_1 \langle \xi \rangle^{m-1} e^{\rho(\tau)} \sqrt{E_+(\tau)} + C_0 \int_\tau^t e^{\rho(s)} |F(s)| \, ds
\]
\[
\leq C_1 \langle \xi \rangle^{m-1} \left\{ e^{\rho(0)} \sqrt{E_+(0)} + \int_0^\tau e^{\rho(s)} |F(s)| \, ds \right\} + C_0 \int_\tau^t e^{\rho(s)} |F(s)| \, ds
\]
\[
\leq C_2 \langle \xi \rangle^{m-1} \left\{ e^{\rho(0)} \sqrt{E_+(0)} + C_0 \int_0^t e^{\rho(s)} |F(s)| \, ds \right\}.
\]
Recalling the definitions (37) and (38) of \(\Phi\) and \(\rho\), we get
\[
\rho(0, \xi) = \int_0^T \Phi(s, \xi) \, ds \leq C_0 \int_0^{\tau(\xi)} \left\{ \frac{1}{T-t} + 1 \right\} \, dt + (T - \tau(\xi)) \langle \xi \rangle
\]
and hence we derive, since \(\partial_t \rho < 0\) and \(\tau(\xi) = T - |\xi|^{-1}\),
\[
\rho(t, \xi) \leq C (\log \langle \xi \rangle + 1) \quad \text{for all } t \in [0, T]. \quad (42)
\]
Therefore we obtain, for some integer \(N\),
\[
e^{\rho(t, \xi)} |V(t, \xi)| \leq C \langle \xi \rangle^N |V(0, \xi)| + C \langle \xi \rangle^{m-1} \int_0^t e^{\rho(s, \xi)} |F(s, \xi)| \, ds. \quad (43)
\]
By the way, we note that the last inequality ensures the wellposedness in $C^\infty$ of the Cauchy problem for the linear system (17).

Let us go back to the nonlinear equation $L u = u^\nu$. For our purpose we must consider a more general equation, namely

$$L u = u_1 \cdots u_\nu,$$

where the $u_j = u_j(t, x)$ are given functions (actually, some $x$-derivatives of $u$). In such a case, the function $F$ in (17) is

$$F(t, \xi) = \hat{u}_1 \ast \cdots \ast \hat{u}_\nu,$$  \hspace{1cm} (44)

where the convolutions are effected w.r. to $\xi$, and thus

$$|F(t, \xi)| \leq \int_{\xi_1 + \cdots + \xi_\nu = \xi} |\hat{u}_1(t, \xi_1) \cdots \hat{u}_\nu(t, \xi_\nu)| \, d\sigma(\xi_1, \ldots, \xi_\nu).$$

We notice that the function $\xi \mapsto \min\{C, |\xi|\}$ is a sub-additive; consequently for each fixed $t$ (see (37), (38)) the function $\Phi(t, \xi)$, hence also $\rho(t, \xi)$, is sub-additive in $\xi$. On the other hand, $\xi \mapsto \langle \xi \rangle$ is sub-multiplicative. Thus one has, for $\xi = \xi_1 + \cdots + \xi_\nu$,

$$\rho(t, \xi) \leq \rho(t, \xi_1) + \cdots + \rho(t, \xi_\nu), \quad \langle \xi \rangle^{m-1} \leq \langle \xi_1 \rangle^{m-1} \cdots \langle \xi_\nu \rangle^{m-1},$$

whence, by (44), it follows the pointwise estimate

$$e^{\rho(t, \xi)} \langle \xi \rangle^{m-1} |F| \leq \left( e^{\rho(t, \xi)} \langle \xi \rangle^{m-1} |\hat{u}_1| \right) \ast \cdots \ast \left( e^{\rho(t, \xi)} \langle \xi \rangle^{m-1} |\hat{u}_\nu| \right).$$

Now, if $V_j(t, \xi)$ are the vectors formed as $V(t, \xi)$ (see (15)), with $u_j$ in place of $u$, we have

$$\langle \xi \rangle^{m-1} |\hat{u}_j(t, \xi)| \leq |V_j(t, \xi)|, \quad j = 1, \ldots, \nu,$$

and thus, going back to (43), we obtain

$$e^{\rho(t, \xi)} |V(t, \xi)| \leq C \langle \xi \rangle^N |V(0, \xi)| + C \int_0^t \left( e^{\rho(s, \xi)} |V_1| \ast \cdots \ast e^{\rho(s, \xi)} |V_\nu| \right)(s, \xi) \, ds.$$

Finally, we integrate in $\xi \in \mathbb{R}$ to get

$$\mathcal{E}(t, u) \leq C \int_{\mathbb{R}} |V(0, \xi)| \langle \xi \rangle^N d\xi + C \int_0^t \mathcal{E}(s, u_1) \cdots \mathcal{E}(s, u_\nu) \, ds,$$  \hspace{1cm} (45)

where we define the $C^\infty$-energy

$$\mathcal{E}(t, u) = \int_{\mathbb{R}} e^{\rho(t, \xi)} |V(t, \xi)| \, d\xi.$$  \hspace{1cm} (46)
We emphasize that, by virtue of our assumption (19), and (42), we have
\[ E(t, u) \leq M_0 < \infty \quad (0 \leq t \leq T). \tag{47} \]

Differentiating \( j \) times in \( x \) the equation \( \mathcal{L}u = u^\nu \), we get
\[ \mathcal{L}(\partial^j u) = j! \sum_{h_1 + \cdots + h_\nu = j} \frac{\partial^{h_1} u \cdots \partial^{h_\nu} u}{h_1! \cdots h_\nu!} \quad \text{(where } \partial = \partial_x), \]
and to this equation we apply the estimate (45) with \( u_j = \partial^j u \). We obtain:
\[ \frac{E_j(t)}{j!} \leq C \int_{\mathbb{R}} |V_j(0, \xi)| \langle \xi \rangle^N d\xi + C \sum_{|h| = j} \int_0^t \frac{E_{h_1}(s)}{h_1!} \cdots \frac{E_{h_\nu}(s)}{h_\nu!} ds, \tag{48} \]

where \( V_j(t, \xi) \) is the vector associated to \( u_j \equiv \partial^j u \), and
\[ E_j(t) = E(t, \partial^j u). \]

Putting
\[ \alpha_j(t) = \int_{\mathbb{R}} |V_j(0, \xi)| \langle \xi \rangle^N d\xi + j! \sum_{|h| = j} \int_0^t \frac{E_{h_1}(s)}{h_1!} \cdots \frac{E_{h_\nu}(s)}{h_\nu!} ds, \]
we rewrite (48) as
\[ E_j(t) \leq C \alpha_j(t). \tag{49} \]

Next, we introduce the super-energies
\[ \mathcal{F}(t, u) = \sum_{0}^{\infty} E_j(t) \frac{r(t)^j}{j!}, \tag{50} \]
\[ \mathcal{G}(t, u) = \sum_{0}^{\infty} \alpha_j(t) \frac{r(t)^j}{j!}, \quad \mathcal{G}^1(t, u) = \sum_{1}^{\infty} \alpha_j(t) \frac{r(t)^{j-1}}{(j-1)!}, \tag{51} \]

where \( r(t) \) is a decreasing, positive function on \([0, T]\) to be defined later. By differentiating in time, we find
\[ \mathcal{G}' = \sum_{0}^{\infty} \alpha_j(t) \frac{r^j}{j!} + \sum_{1}^{\infty} \alpha_j(t) \frac{r^{j-1}}{(j-1)!} r' = \sum_{j=0}^{\infty} \sum_{|h| = j} \frac{E_{h_1}}{h_1!} \cdots \frac{E_{h_\nu}}{h_\nu!} + r' \mathcal{G}^1 = \mathcal{F}' + r' \mathcal{G}^1, \]
and hence, noting that $\mathcal{F}(t) \leq C \mathcal{G}(t)$ by (49),
\[
\mathcal{G}' \leq C'' \mathcal{G}'' + r' \mathcal{G}^1. \tag{52}
\]
Now, noting that (by (19) and (47))
\[
\alpha_0(t) = \int_R |V(0, \xi)| \langle \xi \rangle^N d\xi + \int_0^t \mathcal{E}(s) ds \leq K_N + M_0 \equiv M,
\]
by the definition (51) of $\mathcal{G}(t)$ it follows
\[
\mathcal{G}(t) \leq \alpha_0(t) + r(t) \mathcal{G}^1(t) \leq M + r(t) \mathcal{G}^1(t).
\]
From this inequality it follows, arguing by induction w.r. to $\nu$,
\[
\mathcal{G}'' \leq M'' + r \mathcal{G}^1 (\mathcal{G} + M)^{\nu-1},
\]
consequently (52) gives (putting $\phi(\mathcal{G}) = C'' (M + \mathcal{G})^{\nu-1}$)
\[
\mathcal{G}' \leq \mathcal{G}^1 \{ r' + r \phi(\mathcal{G}) \} + (CM)^\nu. \tag{53}
\]
On the other hand, by virtue of our assumption (20), we see that
\[
\mathcal{G}(0, u) = \sum_{j=0}^{\infty} \left\{ \int_R |V_j(0, \xi)| \langle \xi \rangle^N d\xi \right\} \frac{r(0)^j}{j!} < \infty.
\]
provided $r(0) \equiv r_0$ is small enough. Therefore, taking
\[
L = \mathcal{G}(0, u) + (CM)^\nu T, \quad r(t) = r_0 e^{-\phi(L)t}, \tag{54}
\]
we can derive from (53) the estimate
\[
\mathcal{G}(t, u) < L \quad \text{for all} \quad t \in [0, T]. \tag{55}
\]
**Proof of (55).** Since $L > \mathcal{G}(0)$, this estimate holds true in a right neighborhood of $t = 0$ by Cauchy-Kovalewsky. Then assuming that, for some $\tau_* < T$, (55) holds for all $t < \tau_*$ but not at $t = \tau_*$, we have $\mathcal{G}(\tau_*) = L$, and hence, with $r(t)$ as in (54),
\[
r'(t) + r(t) \phi(\mathcal{G}(t)) \leq r'(t) + r(t) \phi(L) \leq 0 \quad \text{on} \quad [0, \tau_*[.
\]
This yields a contradiction; indeed, by (53),
\[
\mathcal{G}(t) \leq \mathcal{G}(0) + (CM)^\nu \tau_* < L \quad \text{on} \quad [0, \tau_*].
\]
Conclusion of the Proof of Theorem 1. Recalling that \( F(t, u) \leq C G(t, u) \), (55) says that \( F(t, u) < CL \) on \([0, T]\). Therefore, by (50), we get our goal (21):

\[
\int_{\mathbb{R}} |V(t, \xi)||\xi|^j \, d\xi \leq \int_{\mathbb{R}} e^{\rho(t, \xi)} |V(t, \xi)||\xi|^j \, d\xi = \mathcal{E}(t, \partial^j u) \leq F(t) r(t)^{-j} j! \leq CL \left \{ r_0 e^{\phi(L)T} \right \}^j j! = \widetilde{C} \widetilde{\Lambda} \eta^{j+1} j!.
\]

To prove (13), i.e., the global analyticity of the solution \( u \) in \((t, x)\), it is sufficient to resort to Cauchy-Kovalewski.

Remark 2 The previous proof of (55) is somewhat formal, since it assumes not only that \( G(t) < \infty \), but also that \( G^1(t) < \infty \) on \([0, \tau^*] \). To make the proof more precise we must replace the radius function \( r(t) \) by \( r_\eta(t) = \eta \exp(-\phi(L)t) \), \( \eta < 1 \), and apply the previous computation to the corresponding functions \( G_\eta(t) \) and \( G^1_\eta(t) \). Finally let \( \eta \to 1 \) (see [ST] for the details).

Proof of Theorem 2. The proof is not very different from that of Thm.1, thus we give only a sketch of it.

The main difference is that the entries \( q_{r,ij}(t) \) are no more analytic, but only \( C^\infty \), hence (28) fails. However, for any function \( f \in C^k([0, T]) \) it holds

\[
|f'(t)| \leq \Lambda(t) |f(t)|^{1-1/k} \|f\|_{C^k([0, T])},
\]

for some \( \Lambda \in L^1(0, T) \) [this was proved in [CJS] in the case \( f(t) \geq 0 \), and in [T] in the general case]. Therefore, recalling that \( Q_\varepsilon(t) \) is a nearly diagonal matrix, and proceeding as in [KS], we get, for all integer \( k \geq 1 \),

\[
|(Q_\varepsilon(t)V(t, \xi), V(t, \xi))| \leq \Lambda_k(t) (Q_\varepsilon(t)V(t, \xi), V(t, \xi))^{1-1/k} |V(t, \xi)|^{2/k} \tag{56}
\]

for some \( \Lambda_k \in L^1(0, T) \), independent of \( \varepsilon \). Differently from Thm. 1, we need now to consider only the hyperbolic energy

\[
E_\varepsilon(t, \xi) = (Q_\varepsilon(t)V, V) \quad \text{with} \quad \varepsilon = |\xi|^{-1}.
\]

Thanks to (56), we prove (for every integer \( k \geq 1 \)) the estimate

\[
\left \{ \sqrt{E_\varepsilon(t, \xi)} \right \}' \leq C_0 \Phi(t, \xi) \sqrt{E_\varepsilon(t, \xi)} + C_0 |F(t, \xi)|
\]

on all the interval \([0, T] \), where

\[
\Phi(t, \xi) = \Lambda_k(t) |\xi|^{2(m-1)/k} + 1
\]

Note that \( \Phi \) is sub-additive w.r. to \( \xi \) as soon as \( k \geq 2(m-1) \).
Next, putting
\[ \rho(t, \xi) = \int_t^T \Phi(t, \xi) d\xi \equiv |\xi|^{2(m-1)/k} \int_t^T \Lambda_k(s) \, ds + (T-t), \]
we define the Gevrey-energy
\[ E(t, u) = \int_{\mathbb{R}} e^{\rho(t, \xi)} \sqrt{E_+(t, \xi)} \, d\xi. \]
We conclude as in the proof of Thm.1.

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