LREE of a Dynamical Unstable Dp-brane

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Abstract

We derive the left-right entanglement entropy (LREE) for a bosonic Dp-brane. This brane has tangential dynamics and has been dressed by a $U(1)$ gauge potential, the Kalb-Ramond field and a tachyon field. For this purpose, the Rényi entropy will be computed and then, by taking a special limit of it, the LREE will be obtained. Besides, the behavior of the LREE under the tachyon condensation process will be evaluated. In addition, after the transition of the system, i.e. the collapse of the unstable brane, the second law of the thermodynamics on the LREE will be checked. We find that preserving the second law imposes some conditions on the parameters of the setup.

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1 Introduction

The entanglement entropy, as an appropriate measure for entanglement, has been widely studied in the different contexts. For instance, this quantity is employed in many-body quantum systems to study the quantum phases and phase transitions [1], [2]. Besides, connections between the entanglement entropy and black hole entropy were found [3], [4]. More appealing works were done in the AdS/CFT area, which reveal some connections between the entanglement entropy and gravity [5], [6].

Since a boundary state encodes the whole properties of the associated D-brane it is a suitable tool for studying the D-branes, their interactions and so on [7]-[15]. In this paper, by applying the boundary state formalism, our goal is to investigate a special property of the D-branes, which is called left-right entanglement entropy (LREE) [16]-[19]. Thus, we figure out the LREE of a bosonic unstable Dp-brane with a tangential linear motion and rotation which has been dressed by the Kalb-Ramond field, a $U(1)$ gauge potential and a tachyon field of the open string spectrum. Traditionally, in order to measure the entanglement in a bipartite system the system is geometrically divided to subsystems [20]. In our approach, similar to the Ref. [16], the division occurs in the Hilbert space. Precisely, the two subsystems are the left- and right-moving modes of closed strings which appear in the expansion of the boundary state as a favorable bipartite system.

On the other hand, according to the literature, e.g. Ref. [21], presence of the open string tachyon on a D-brane obviously makes it unstable. Consequently, it decays to an unstable lower dimensional D-brane through the tachyon condensation process. The resultant intermediate brane eventually collapses to the closed string vacuum or decays to a lower dimensional stable brane [22]-[27]. Since the open string tachyon field lives on our brane we were motivated to investigate effect of the tachyon condensation on the LREE corresponding to our unstable brane. Besides, since the similarity of the thermal and entanglement entropies has been demonstrated [28]-[31], we were stimulated to examine the second law of thermodynamics for changing the LREE via the condensation of the tachyon.

In fact, the corresponding LREE of a D-brane potentially possesses a connection with
the entropies of the black holes \[3, 4\]. Therefore, the LREE of our setup may find a relation with the entropies of the rotating-moving charged black holes.

Note that the LREE was originally studied by P. Zayas and N. Quiroz for a one-dimensional boundary state in a 2D CFT with the Dirichlet or the Neumann boundary condition \[16\]. Then, they extended their work to the case of a bare-stationary Dp-brane \[17\]. We shall apply their approach to compute the LREE of our setup.

The paper is organized as follows. In Sec. 2, we shall introduce the boundary state, associated with a dynamical-dressed unstable Dp-brane. Then the interaction amplitude between two parallel Dp-branes, which is necessary for the calculation of the LREE, will be introduced. In Sec. 3, we compute the LREE for the foregoing Dp-brane. In Sec. 4, the effect of the tachyon condensation on the LREE will be calculated, and some thermodynamical interpretations will be presented. Section 5 is devoted to the conclusions.

2 The boundary state and interaction amplitude

2.1 The boundary state

Consider a Dp-brane with tangential dynamics in the presence of the antisymmetric field \( B_{\mu\nu}(X) \), a \( U(1) \) gauge potential \( A_\alpha(X) \) and an open string tachyon field \( T(X) \). In order to introduce the corresponding boundary state, we start with the action

\[
S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} \! d^2\sigma \left( \sqrt{-g} g^{\mu\nu} \partial_\mu X^\mu \partial_\nu X^\nu + \varepsilon^{\mu\nu} B_{\mu\nu} \partial_\mu X^\mu \partial_\nu X^\nu \right) + \frac{1}{2\pi\alpha'} \int_{\partial\Sigma} \! d\sigma \left( A_\alpha \partial_\sigma X^\alpha + \omega_{\alpha\beta} J^{\alpha\beta} + T(X^\alpha) \right),
\]

where we shall use the sets \( \{x^\alpha|\alpha = 0,1,\cdots,p\} \) and \( \{x^i|i = p+1,\cdots,d-1\} \) to show the parallel and perpendicular directions to the brane worldvolume, respectively. We apply the reliable gauge \( A_\alpha = -\frac{1}{2} F_{\alpha\beta} X^\beta \) with the constant field strength \( F_{\alpha\beta} \), and adopt the tachyon profile as \( T = \frac{1}{2} U_{\alpha\beta} X^\alpha X^\beta \) with the constant symmetric matrix \( U_{\alpha\beta} \). The spacetime and worldsheet metrics and the Kalb-Ramond field are taken to be constant with \( G_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1,1,\cdots,1) \). The antisymmetric matrix \( \omega_{\alpha\beta} \) denotes the spacetime
angular velocity of the brane inside its worldvolume and \( J_{\tau}^{\alpha\beta} = X^\alpha \partial_\tau X^\beta - X^\beta \partial_\tau X^\alpha \) represents the angular momentum density.

Comparing the setup of this paper with that of our previous work [32], we can say the branes of both papers have been dressed by the same Kalb-Ramond field and the same internal \( U(1) \) gauge potential. Besides, both branes have the same tangential dynamics. The only difference between them is the instability of the present brane, which is induced by the tachyon field. We shall observe that the associated LREE of the unstable brane is very different from the LREE of the stable brane.

Variation of the action with respect to \( X^\mu \) gives the equation of motion and following boundary state equations

\[
(\Delta_{\alpha\beta} \partial_\tau X^\beta + \mathcal{F}_{\alpha\beta} \partial_\sigma X^\beta + B_{\alpha\beta} \partial_\sigma X^i + U_{\alpha\beta} X^\beta)_{\tau=0} |B_x\rangle = 0,
\]

\[
(X^i - y^i)_{\tau=0} |B_x\rangle = 0,
\]

(2.2)

where \( \Delta_{\alpha\beta} = \eta_{\alpha\beta} + 4\omega_{\alpha\beta} \) and \( \mathcal{F}_{\alpha\beta} = B_{\alpha\beta} - F_{\alpha\beta} \). The transverse vector \( y^i \) determines the brane position. Using the mode expansion of \( X^\mu \) for the propagating closed string and the decomposition \( |B_x\rangle = |B_{(osc)}\rangle \otimes |B_{(0)}\rangle \) one can rewrite these equations in terms of the string oscillators

\[
\left[ \left( \Delta_{\alpha\beta} - \frac{4\omega_{\alpha\beta}}{2m} U_{\alpha\beta} \right) \alpha^\beta_m + \left( \Delta_{\alpha\beta} + \frac{4\omega_{\alpha\beta}}{2m} U_{\alpha\beta} \right) \tilde{\alpha}^-_m \right] |B_{(osc)}\rangle = 0,
\]

\[
(2\alpha' \Delta_{\alpha\beta} p^\beta + U_{\alpha\beta} x^\beta) |B_{(0)}\rangle = 0,
\]

(2.3)

for the tangential directions and

\[
(\alpha^i_m - \tilde{\alpha}^i_m) |B_{(osc)}\rangle = 0,
\]

\[
(x^i - y^i) |B_{(0)}\rangle = 0,
\]

(2.4)

for the perpendicular directions.

Applying the quantum mechanical methods, particularly the coherent state formalism,
the zero-mode and oscillating parts of the boundary state find the features

\[
|B(0)\rangle = \frac{T_p}{2\sqrt{\det(U/4\pi\alpha')}} \int_{-\infty}^{\infty} \prod_{\alpha=0}^{p} \exp \left[ i\alpha' \sum_{\beta \neq \alpha} (U^{-1}\Delta + \Delta^{T} U^{-1})_{\alpha\beta} p^\alpha p^\beta \right] \\
+ \frac{i\alpha'}{2} (U^{-1}\Delta + \Delta^{T} U^{-1})_{\alpha\alpha} (p^\alpha)^2 \right] |p^\alpha\rangle dp^\alpha \\
\times \prod_{i=p+1}^{d-1} \left[ \delta(x^i - y^i)|p^i = 0 \right],
\]

(2.5)

\[
|B_{osc}\rangle = \prod_{n=1}^{\infty} \left[ \det M(m) \right]^{-1} \exp \left[ -\sum_{m=1}^{\infty} \left( \frac{1}{m} \alpha_m^\mu S(m)_{\mu\nu} \alpha_m^\nu \right) \right] |0\rangle_{\alpha} |0\rangle_{\tilde{\alpha}},
\]

(2.6)

where \(T_p\) is the tension of the Dp-brane. The matrix \(S(m)_{\mu\nu}\) is defined by \(S(m)_{\mu\nu} = (Q(m)_{\alpha\beta} \equiv (M(m)^{-1} N(m))_{\alpha\beta}, -\delta_{ij}), \) in which

\[
M(m)_{\alpha\beta} = \Delta_{\alpha\beta} - F_{\alpha\beta} + \frac{i}{2m} U_{\alpha\beta}, \\
N(m)_{\alpha\beta} = \Delta_{\alpha\beta} + F_{\alpha\beta} - \frac{i}{2m} U_{\alpha\beta}.
\]

(2.7)

In fact, the parameters of the setup, which were appeared in the boundary state \(|B_x\rangle\), are not independent. That is, the first equation of Eqs. (2.3) enables us to express the left- and right-moving oscillators in terms of each other. If we choose the set \(\{\alpha_m^\alpha, \tilde{\alpha}_m^\alpha | m \in \mathbb{N}\}\) the boundary state manifestly possesses the matrix \(Q(m)_{\alpha\beta}\), and if we select the set \(\{\tilde{\alpha}_m^\alpha, \alpha_{-m}^\alpha | m \in \mathbb{N}\}\) it will contain the new matrix \(\left[ Q_{(-m)}^{-1} \right]_{\alpha\beta} \). Equality of these matrices leads to the condition \(Q(m) Q_{(-m)}^{\dagger} = 1\), which gives

\[
\Delta U = U \Delta^{T}, \\
\Delta F = F \Delta^{T}.
\]

(2.8)

The normalization prefactors of Eqs. (2.5) and (2.6) come from the disk partition function. For example, look at the Eq. (2.6). For the constant Kalb-Ramond field the second term of the action (2.1) reduces to a boundary term. Therefore, we receive a total boundary action which includes the matrices \(F_{\alpha\beta}, \omega_{\alpha\beta}\) and \(U_{\alpha\beta}\). By computing the partition function via the boundary action the quantities \(\{\det M(m) | m = 1, 2, 3, \cdots \}\)
appear in the normalization factor. Similar normalization factors can be found, e.g., in Refs. [14, 15].

Beside the matter part of the boundary state, there is a contribution by the conformal ghosts too,

\[ |B_{gh}⟩ = \exp \left[ \sum_{n=1}^{∞} \left( c_{-n} \bar{b}_{-n} - b_{-n} \tilde{c}_{-n} \right) \right] c_0 + \tilde{c}_0 \frac{1}{2} |q = 1⟩ |\bar{q} = 1⟩. \] (2.9)

Thus, the total bosonic boundary state takes the form

\[ |B⟩ = |B_{(osc)}⟩ ⊗ |B_{(0)}⟩ ⊗ |B_{gh}⟩. \] (2.10)

### 2.2 The interaction amplitude

The calculation of the left-right entanglement entropy needs to extract the partition function. Thus, we introduce the interaction amplitude of two dynamical Dp-branes which are dressed by the foregoing fields. For this purpose, we consider the tree-level diagram of a closed string which propagates between two such Dp-branes. The amplitude can be computed by the overlap of the total boundary states \(|B_{1,2}⟩\) via the propagator \(D\) of the exchanged closed string

\[ \mathcal{A} = \langle B_{1}|D|B_{2}⟩, \]

\[ D = 2\alpha' \int_{0}^{∞} dt \, e^{-tH}, \] (2.11)

where \(H\) is the total Hamiltonian of the propagating closed string, including the ghost part. Hence, the amplitude is given by

\[ \mathcal{A} = \frac{T_p^2 \alpha' V_{p+1}}{16(2\pi)^{d-p-1} \sqrt{\det(U_1/4\pi\alpha') \det(U_2/4\pi\alpha')}} \prod_{m=1}^{∞} \left[ \det \left( M^\dagger_{(m)} M_{(m)} \right) \right]^{-1} \]

\[ \times \left[ 1 - Q_{(m)}^\dagger Q_{(m)} e^{-4m\pi t} \right]^{-1} \left( 1 - e^{-4m\pi t} \right)^{p-d+3} \right], \] (2.12)

where \(V_{p+1}\) represents the worldvolume of each brane. The first exponential comes from the zero-point energy, the two factors next to it originate from zero modes. In addition,
the determinant part in the last line is the contribution of the Neumann oscillators while
the factor $\prod_{m=1}^{\infty} (1 - e^{-4\pi m t})^{p-d+3}$ is due to the Dirichlet oscillators and the conformal
ghosts.

3 The corresponding LREE to the dynamical-dressed unstable D$p$-brane

In a composite quantum system, which includes some subsystems, entanglement clearly
relates the various parts of the system. The quantum state of each subsystem is not in-
dependent of the states of the other subsystems. Entanglement entropy is an appropriate
quantity for measuring the entanglement among the subsystems.

Consider a bipartite system which comprises the subsystems A and B. If the pure
state of the composite system is denoted by $|\psi\rangle$, then the density operator of this state
is given by $\rho = |\psi\rangle\langle\psi|$, which satisfies the probability conservation $\text{Tr}\rho = 1$. Therefore,
the reduced density operator for the subsystem A is given by the partial trace over the
subsystem B, i.e., $\rho_A = \text{Tr}_B\rho$.

For measuring the entanglement, we choose the entanglement and Rényi entropies.
The former quantity is given by the von Neumann formula $S = -\text{Tr}(\rho_A \ln \rho_A)$ [33], and
the latter is defined by $S_n = \frac{1}{1-n} \ln \text{Tr} \rho_A^n$, where $n \geq 0$ and $n \neq 1$. By taking the limit
$n \to 1$, the entanglement entropy can be derived from the Rényi entropy [34].

3.1 The density operator of the configuration

As we know the Hilbert space of string theory can be represented by the direct product
of two subspaces L and R with the left- and right-moving modes as its bases. That is,
the Hilbert space possesses the factorized form $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$. By imposing the Virasoro
constraints, which leads to the same excitation number on the left and right sectors, we
receive the physical Hilbert space. Therefore, these constraints very weakly relate the left
and right sectors, and in principle they are still independent. Explicitly, the most general
state for the closed strings has the feature $|\psi\rangle = |\psi\rangle_L \otimes |\psi\rangle_R$, where

$$
|\psi\rangle_L = \prod_{k=1}^{\infty} \frac{1}{\sqrt{n_k!}} \left( \frac{\alpha_{-k}^{\mu_k}}{\sqrt{k}} \right)^{n_k} |0\rangle,
$$

$$
|\psi\rangle_R = \prod_{k=1}^{\infty} \frac{1}{\sqrt{m_k!}} \left( \frac{\tilde{\alpha}_{-k}^{\mu_k}}{\sqrt{k}} \right)^{m_k} |0\rangle,
$$

such that $\sum_{k=1}^{\infty} kn_k = \sum_{k=1}^{\infty} km_k$. We see that the sets $\{n_k|k = 1, 2, 3, \cdots\}$ and $\{m_k|k = 1, 2, 3, \cdots\}$, up to the mentioned constraint, are essentially independent. This implies that the physical Hilbert space is still a product of the left and right sectors.

Besides, a boundary state satisfies the physical constraint $(L_n - \tilde{L}_{-n})|B\rangle = 0$ for any $n \in \mathbb{Z}$, where $L_n$ and $\tilde{L}_{-n}$ are the Virasoro operators. However, the boundary state can be decomposed to the left- and right-moving modes via Schmidt procedure [35, 36]. Thus, we can take the total boundary state (2.10) as our composite system. Since the matrix $S_{(m)\mu\nu}$ in Eq. (2.6) has nontrivial elements, our system is an entangled composite system with the left-right entanglement.

For a given boundary state $|B\rangle$ if we define the density matrix as $\rho = |B\rangle \langle B|$, because of the divergence of the inner product $\langle B|B\rangle$, we don’t receive the probability conservation $\text{Tr}\rho = 1$. Hence, we have to introduce a finite correlation length $\epsilon$ to acquire the condition $\text{Tr}\rho = 1$, [29, 37]. Thus, the density matrix is redefined by

$$
\rho = \frac{e^{-\epsilon H}|B\rangle \langle B| e^{-\epsilon H}}{Z(2\epsilon)},
$$

where the denominator is fixed by the probability condition as

$$
Z(2\epsilon) = \langle B| e^{-2\epsilon H} |B\rangle
$$

$$
= \frac{T_p^2 V_{p+1}}{8(2\pi)^{d-p-1}} \frac{\prod_{m=1}^{\infty} |\det M_{(m)}|^{-2} e^{(d-2)\pi\epsilon/3}}{\det(U/8\pi)} \left( \frac{1}{2\sqrt{\epsilon}} \right)^{d-p-1}
$$

$$
\times \prod_{m=1}^{\infty} \left( \det \left[ 1 - Q_{(m)}^\dagger Q_{(m)} e^{-8m\pi\epsilon} \right]^{-1} (1 - e^{-8m\pi\epsilon})^{p-d+3} \right). \tag{3.2}
$$

Eq. (2.12) implies that $Z(2\epsilon)$ is the tree-level amplitude of a closed string which propagates for the time $t = 2\epsilon$ between the two identical Dp-branes. Since the Dp-branes are identical and have been located in the same position the indices 1 and 2 were omitted and the $y$-dependence was also removed. Note that we have taken $\alpha' = 2$. 

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3.2 The LREE of the setup

The reduced density matrix for the subsystem L, i.e. $\rho_L$, is obtained by taking the trace over the right-moving oscillators. In order to find the Rényi entropy, we apply the replica trick which simplifies the computation of $\text{Tr}\rho_L^n$ with real $n$,

$$\text{Tr}\rho_L^n \sim \frac{Z(2n\epsilon)}{Z^n(2\epsilon)} \equiv \frac{Z_n(L)}{Z^n},$$

(3.3)

where $Z_n(L)$ is the replicated form of the partition function. Note that we shall use the partition function (3.2) to calculate the above relation, and then we express it in terms of the Dedekind eta-function

$$\eta(q) = q^{1/12} \prod_{m=1}^{\infty} (1 - q^{2m}),$$

(3.4)

where $q = e^{-4\pi\epsilon}$. In the limit $\epsilon \to 0$ the variable $q$ approaches to one. To avoid this value of $q$ we apply the open/closed worldsheet duality via the transformation $4\epsilon \to 1/4\epsilon$ to go to the open string channel. Consequently, as $\epsilon$ goes to zero the new variable $\tilde{q} = e^{-\pi/4\epsilon}$ vanishes and we can utilize the expansion of the Dedekind $\eta$-function for small argument.

Therefore, we receive the following relation

$$\frac{Z_n}{Z^n} \approx \frac{1 - n \text{Tr} \left(Q^\dagger(m)Q(m)\right) e^{-m\pi/2\epsilon} - n \left[\text{Tr} \left(Q^\dagger(m)Q(m)\right)\right]^2 e^{-m(1/\epsilon+1/n)} + \frac{n}{2} \left[\text{Tr} \left(Q^\dagger(m)Q(m)\right)\right] e^{-m\pi/\epsilon}}{1 - n \left[\text{Tr} \left(Q^\dagger(m)Q(m)\right)\right]^2 e^{-m\pi/\epsilon} + \frac{n}{2} \left[\text{Tr} \left(Q^\dagger(m)Q(m)\right)\right] e^{-m\pi/\epsilon}}},$$

(3.5)

where $q = e^{-4\pi\epsilon}$. In the limit $\epsilon \to 0$ the variable $q$ approaches to one. To avoid this value of $q$ we apply the open/closed worldsheet duality via the transformation $4\epsilon \to 1/4\epsilon$ to go to the open string channel. Consequently, as $\epsilon$ goes to zero the new variable $\tilde{q} = e^{-\pi/4\epsilon}$ vanishes and we can utilize the expansion of the Dedekind $\eta$-function for small argument.
up to the order $\mathcal{O}(\exp(-3\pi/2\epsilon))$. The factor $K_p$ was entered into Eq. (3.5) via Eq. (3.2), and possesses the following definition

$$K_p = \frac{T_p^2 V_{p+1}}{8 (2\pi)^{d-p-1}} \prod_{m=1}^\infty |\det M(m)|^{-2} \frac{\det(U/8\pi)}{\det(U)'^2}.$$  (3.6)

Now, by taking the limit $n \to 1$ of the Rényi entropy we acquire the entanglement entropy

$$S_p \approx \ln K_p + \frac{d-p-1}{2} (2 \ln 2 + \ln \epsilon - 1) + \frac{(d-2)\pi}{24\epsilon} e^{-(d-2)\pi/2\epsilon}$$

$$+ \sum_{m=1}^\infty \left\{ \text{Tr} \left( Q_{(m)}^\dagger Q_{(m)} \right) + d-p-3 \right\} \left( 1 - \frac{m\pi}{2\epsilon} \right) e^{-m\pi/2\epsilon}$$

$$+ \left[ \text{Tr} \left( Q_{(m)}^\dagger Q_{(m)} \right)^2 + d-p-3 \right] \left( \frac{1}{2} - \frac{m\pi}{2\epsilon} \right) e^{-m\pi/\epsilon}.$$  (3.7)

The first term is concerned to the boundary entropy of the brane. The second term comes from the zero-modes, and the next terms are contributions of the oscillators and conformal ghosts. The brane dynamics and the background fields were accumulated in the first term and series. One of the prominent effects of the tachyonic field is the appearance of all mode numbers $\{m \in \mathbb{N}\}$ in the LREE.

By quenching the tachyon field the mode dependence of the matrices $Q_{(m)}$ and $M_{(m)}$ disappears. In this case by employing the formulas $\sum_{m=1}^\infty x^m = \frac{x}{1-x}$ and $\sum_{m=1}^\infty mx^m = \frac{x}{(1-x)^2}$ for small $x$ where $x = e^{-\pi/2\epsilon}$ and $x = e^{-\pi/\epsilon}$ for the second and third lines of Eq. (3.7), respectively, the LREE, up to the term $\ln K_p$, exactly reduces to Eq. (3.7) of Ref. [32]. Now return to the term $\ln K_p$. In fact, by assuming $\det U \neq 0$ the inverse of the matrix $U$ was appeared in Eq. (2.5). Thus, in the subsequent formulas, e.g. Eq. (3.7), we are not allowed to put the tachyon matrix away. However, by applying the regularization scheme $\prod_{m=1}^\infty a \to 1/\sqrt{a}$ the infinite product in Eq. (3.6) for $U \to 0$ reduces to $|\det M|$ of Eq. (3.7) of the Ref. [32]. Hence, for the case $U \to 0$ the term $\ln K_p$ also is consistent with its counterpart in Ref. [32].

By turning off all background fields and the brane dynamics we acquire the LREE of a bare-static D$p$-brane, see Sec. 3.3 of Ref. [32]. In this special setup, for the configuration $p = 1$ and $d = 3$, the leading terms are exactly compatible with Ref. [16]. In fact, the case $p = 1$ and $d = 3$ is the simplest setup which was originally considered for computing
the LREE in Ref. [16]. It represents a warm-up calculation, and is corresponding to a one-dimensional boundary state in a 3-dimensional target spacetime.

3.3 Connection between the LREE and thermodynamic entropy

By defining a temperature, proportional to the inverse of the correlation length $\epsilon$, we can specify the thermal interpretation of our system. The partition function of the dressed-dynamical brane was defined by Eq. (3.2). In the limit of $\beta = 2\epsilon \to 0$, which exhibits the high temperature of the thermal system, the thermodynamic entropy of the system is given by

$$S_{\text{th}} = \beta^2 \frac{\partial}{\partial \beta} \left( -\frac{1}{\beta} \ln Z \right)$$

$$\approx \ln K_p + \frac{d - p - 1}{2} \left( 2 \ln 2 + \ln \frac{\beta}{2} - 1 \right) + \frac{(d - 2)\pi}{12\beta}$$

$$+ \sum_{m=1}^{\infty} \left\{ \text{Tr} \left( Q_{(m)}^\dagger Q_{(m)} \right) + d - p - 3 \right\} \left( 1 - \frac{m\pi}{\beta} \right) e^{-m\pi/\beta}$$

$$+ \left[ \text{Tr} \left( Q_{(m)}^\dagger Q_{(m)} \right) ^2 + d - p - 3 \right] \left( \frac{1}{2} - \frac{m\pi}{\beta} \right) e^{-2m\pi/\beta} \right\}, \quad (3.8)$$

up to the order $O(\exp(-3\pi/\beta))$. This demonstrates that the thermal entropy of our system exactly matches with its LREE counterpart, i.e. (3.7). There are some other papers which also reveal such connections, e.g. see the Refs. [28]-[31].

Due to the presence of the tachyon field this thermal entropy has a more general form than that of the Ref. [32]. We observed that the LREE of our setup, for the special case $U \to 0$, reduced to the LREE of the stable dressed-dynamical brane [32]. With the same logic and mathematical tools, the thermal entropy (3.8) for a vanishing tachyon field also is consistent with that of the stable brane [32].

4 Effect of the tachyon condensation on the LREE

According to the literature, e.g. Ref. [21], adding an open string tachyonic mode to a single D-brane or to a group of D-branes makes them unstable. That is, in the
tachyon condensation process the brane dimension is decreased \[22\], so that eventually there will remain the closed string vacuum or an intermediate stable brane. In the tachyon condensation phenomenon at least one of the elements of the tachyon matrix \[U_{\alpha \beta}\] becomes infinite. For simplicity we impose the condensation of the tachyon field only in the \[x^p\]-direction, i.e., we apply the limit \[U_{pp} \to \infty\].

For simplification, at first we obtain the LREE in the limit of large tachyon field, i.e., we apply \[U \gg 2m(\Delta - F)\] with the consideration of the conditions \[22,28\]. Thus, we receive

\[
\tilde{S}_p \approx \ln K_p + \frac{d - p - 1}{2} (2 \ln 2 + \ln \epsilon - 1) + \frac{(d - 2)\pi}{24\epsilon} \\
+ \sum_{m=1}^{\infty} \left\{ [d - 2 - 512 m^2 \text{Tr}(\omega^2 U^{-2})] \left(1 - \frac{m\pi}{2\epsilon}\right) e^{-m\pi/2\epsilon} \\
+ [d - 2 - 1024 m^2 \text{Tr}(\omega^2 U^{-2})] \left(\frac{1}{2} - \frac{m\pi}{2\epsilon}\right) e^{-m\pi/\epsilon}\right\},
\]

up to the order \(O(U^{-3})\). We observe that the large tachyon field conveniently quenches the total field strength, unless in the first term.

Now we condensate the tachyon in the \(x^p\)-direction. The first term of Eq. \[(4.1)\] takes the limit

\[
\lim_{U_{pp} \to \infty} \ln K_p = \ln(2\pi L_p) + \ln K_{p-1} \\
\equiv \gamma + \ln K_{p-1},
\]

where \(L_p\) is the length of the \(x^p\)-direction of the brane. For obtaining this we utilized the reliable relation \(T_p = T_{p-1}(2\pi\sqrt{\alpha'})\) and the regularization scheme \(\prod_{n=1}^{\infty} n \to \sqrt{2\pi}\). The second term of Eq. \[(4.1)\] can be rewritten as

\[
\frac{d - p - 1}{2} (2 \ln 2 + \ln \epsilon - 1) = \frac{d - (p - 1) - 1}{2} (2 \ln 2 + \ln \epsilon - 1) + \Gamma.
\]

Moreover, the limit of the factor \(\text{Tr}(\omega^2 U^{-2})\) is given by \(\text{Tr}(\omega^2 U^{-2})'\) where the prime represents a \(p \times p\) matrix. Adding all these together we acquire

\[
\lim_{U_{pp} \to \infty} \tilde{S}_p = \tilde{S}_{p-1} + \lambda, \\
\lambda \equiv \ln(\pi L_p) - \frac{1}{2}(\ln \epsilon - 1).
\]
After the tachyon condensation the Dp-brane reduces to a D\((p - 1)\)-brane. The corresponding entanglement entropy of the resultant brane is specified by \(\tilde{S}_{p-1}\). The extra entropy \(\lambda\) may be associated to the created closed strings during the collapse of the initial brane. The structure of the entropy \(\lambda\) elaborates that for the larger (smaller) value of the brane length \(L_p\), there is a greater (smaller) number of the released closed strings and consequently we obtain the larger (smaller) entropy \(\lambda\).

### 4.1 The LREE and second law of thermodynamics

Due to the similar behavior of the thermal entropy and the LREE [28]-[31], we were motivated to examine the second law of thermodynamics for our LREE in the tachyon condensation process. During this evolution the system goes from an initial state, which is the initial Dp-brane, to a final state, i.e., the subsequent D\((p - 1)\)-brane and the released closed strings. Hence, the LREE changes from the initial entropy \(S_i\) to the final entropy \(S_f\),

\[
S_i = \tilde{S}_p, \quad S_f = \lim_{U_{pp} \to \infty} \tilde{S}_p = \tilde{S}_{p-1} + \lambda. \tag{4.5}
\]

In order to check the second law, i.e. \(S_f > S_i\), we should verify the validity of the inequality \(\tilde{S}_{p-1} + \lambda - \tilde{S}_p > 0\), in which

\[
\tilde{S}_{p-1} + \lambda - \tilde{S}_p = \ln \pi - \ln \left( \frac{\det U'}{\det U} \right) - \sum_{m=1}^{\infty} \left\{ \ln \left( \frac{\det M'_m}{\det M_m} \right)^2 \right. \\
- 512 \ m^2 \left[ \text{Tr} \left( \omega^2 U^{-2} \right) - \text{Tr} \left( \omega^2 U'^{-2} \right) \right] \\
\times \left[ \left(1 - \frac{m \pi}{2 \epsilon} \right) e^{-m \pi/2 \epsilon} + \left(1 - \frac{m \pi}{\epsilon} \right) e^{-m \pi/\epsilon} \right], \tag{4.6}
\]

where the primes show the \(p \times p\) matrices. Positivity of this quantity for \(\epsilon \to 0\) imposes the following condition on the parameters of the setup

\[
\det \left( U \prod_{m=1}^{\infty} M_m^2 \right) > \frac{1}{\pi} \det \left( U' \prod_{m=1}^{\infty} M'_m^2 \right). \tag{4.7}
\]
In fact, this is a minimal condition for preserving the second law of thermodynamics. In other words, the whole quantity in the left-hand-side of Eq. (4.6) for any arbitrary and small value of “ε” should be positive.

As a simple example, look at a stretched D1-brane along the $x^1$-direction. Because of the Lorentz symmetry this object can not possess tangential dynamics, thus, we have $\omega = 0$. There exists a total electric field $\mathcal{E}$ along this D-string. After the tachyon condensation along the direction $x^1$, there will remain a D0-brane and a collection of the released closed strings. Accordingly, the Eq. (4.7) eventuates to the following restriction

$$U_{11} > \frac{2}{\pi} \ln \left( \frac{1 + \sqrt{1 + 4/\pi}}{2} \right).$$

(4.8)

Note that in the previous section we have explicitly chosen very large matrix elements for the tachyon matrix. Therefore, the above condition has been already satisfied.

5 Conclusion

We computed the left-right entanglement entropy of a non-stationary D$p$-brane in the presence of an internal $U(1)$ gauge potential, the Kalb-Ramond field and a tachyon field. For obtaining this, we employed the boundary state formalism and the amplitude of interaction between two parallel and identical dynamical-dressed D$p$-branes. Presence of the various parameters in the configuration dedicated a generalized form to the LREE. Thus, the LREE can be accurately adjusted to any desirable value via these parameters.

By applying the partition function we conveniently computed the thermodynamical entropy, associated with the dressed-dynamical unstable D$p$-brane. We compared this entropy with the LREE of the system. By a redefinition of the temperature we observed that both entropies exactly are the same.

In comparison with the LREE of a dynamical-dressed stable D$p$-brane [32], presence of the tachyon field imposed the contributions of all mode numbers of the closed string to the LREE via an infinite product and an infinite series. Similar additional contributions also occurred for the thermodynamical entropy of the system. As expected, when we turn off the tachyon field the results of this paper completely reduce to the results of the
dynamical-dressed stable Dp-brane [32].

We examined the behavior of the LREE under the condensation of the tachyon. We observed that the associated LREE to the Dp-brane was decomposed to the LREE of a dynamical-dressed D(p − 1)-brane and an entropy which may be corresponded to the released closed strings after the Dp-brane collapse. The latter entropy logarithmically depends on the length of the lost direction of the initial brane.

Finally, because of the similar behavior of the thermal entropy and the LREE, we were persuaded to check the second law of thermodynamics for our LREE via the process of the tachyon condensation. We saw that preserving the second law obligates the parameters of the initial setup to obey a minimal condition. We explicitly figured out the condition for the case of a D-string, which imposed a lower bound on the matrix element $U_{11}$.

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