Mean values and moments of arithmetic functions over number fields

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Abstract
For an odd integer \( d > 1 \) and a finite Galois extension \( K/\mathbb{Q} \) of degree \( d \), Lü and Yang (J Number Theory 131:1924–1938, 2011) obtained an asymptotic formula for the mean values of the divisor function for \( K \) over square integers. In this article, we obtain the same for finitely many number fields of odd degree and pairwise coprime discriminants, together with the moment of the error term arising here, following the method adapted by Shi (An Stiint Univ Al I Cuza Mat (N.S.) 62:615–621, 2016). We also define the sum of divisor function over number fields and find the asymptotic behaviour of the summatory function of two number fields taken together.

1 Introduction
By an arithmetic function \( f \), we mean a function \( f \) from \( \mathbb{N} \) to \( \mathbb{C} \). These functions appear in many branches of mathematics, particularly in analytic number theory and thus it is extremely useful to know the behaviour of them. One of the many arithmetic functions of special interest to number theorists is the divisor function, that counts the number of divisors of a natural number \( n \), and is defined by the following equation.

\[
\tau(n) = \sum_{d \mid n} 1.
\]

By looking at a few small values of \( \tau(n) \), we observe that the behaviour is quite irregular. But instead of \( \tau(n) \), if we consider the quantity \( \sum_{n \leq x} \tau(n) \), for a large positive real number \( x \), we get a much well-behaved function of \( x \). In fact, we have the following asymptotic formula

\[
\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(x^\theta),
\]

where \( \gamma \) is the Euler’s constant and \( \theta \) is a real number with \( 0 < \theta < 1 \). The determination of the exact order of the error term is famously known as the Dirichlet divisor problem.

Likewise, we may consider analogous problems in algebraic number fields. For a number field \( K \) and positive integers \( k \) and \( n \), it is a natural question to ask for the number of ways we can write \( n \) as a product of the norms of \( k \) ideals in the ring of integers \( \mathcal{O}_K \) of \( K \). More precisely, we define

\[
\tau^K_k(n) = \sum_{N(a_1...a_k)=n} 1,
\]

where

\[
\omega(n) = \sum_{d\mid n} \mu(d)
\]

is the Möbius function, and \( \nu(n) \) is the number of distinct prime factors of \( n \).
and study its asymptotic behaviour.

It is easy to see from (1) that $\tau^K_k(n)$ is a multiplicative function of $n$. The problem of finding an asymptotic formula for the function $\sum_{n \leq x} \tau^K_k(n)$ is known as the $k$-dimensional divisor problem in that number field. Panteleeva [14] considered this problem for quadratic and cyclotomic fields, providing an asymptotic formula for both the cases as follows.

**Theorem 1** (Quadratic field case) [14] Let $K = \mathbb{Q}(\sqrt{D})$ for some square-free integer $D$. Then for every integer $k \geq 1$,

$$\sum_{n \leq x} \tau^K_k(n) = xP_k(\log x) + \theta x^{1-\frac{10}{\pi}k^{-\frac{3}{2}}} (C \log x)^{2k}$$

is valid for all real number $x$ satisfying $(\log x)^2 \geq D$, where $P_k$ is a polynomial of degree $k - 1$, $\theta$ is a complex number satisfying $|\theta| \leq 1$ and $C$ is an absolute constant.

**Theorem 2** (Cyclotomic field case) [14] Let $k$ and $t \geq 1$ be integers and let $\zeta_t$ be a primitive $t$-th root of unity. Then for $K = \mathbb{Q}(\zeta_t)$, we have

$$\sum_{n \leq x} \tau^K_k(n) = xP_k(\log x) + \theta x^{1 - \frac{1}{\phi(t)}k^{-\frac{3}{2}}} (C \log x)^{\phi(t)k},$$

where $P_k$ is a polynomial of degree $k - 1$, $\theta$ is a complex number satisfying $|\theta| \leq 1$, $C$ is an absolute constant and $\phi$ stands for the Euler’s phi-function.

It is also interesting to deal with different positive integers $k_1, \ldots, k_l - 1$ and $k_l \geq 2$ rather than just a single $k$. In [15], Panteleeva addressed this problem for $K = \mathbb{Q}$ and proved the following.

**Theorem 3** [15] Let $l \geq 1$ and $k_1, \ldots, k_l \geq 2$ be integers. Then

$$\sum_{n \leq x} \tau_{k_1}(n) \cdots \tau_{k_l}(n) = xP_m(\log x) + \theta x^{1 - \frac{2}{\phi(m)}m^{-\frac{3}{2}}} (C \log x)^m$$

is valid for all $x$ such that $\log x \geq m$, where $m = \prod_{i=1}^{l} k_i$, $P_m$ is a polynomial of degree $m - 1$, $\theta$ is a complex number satisfying $|\theta| \leq 1$ and $C$ is an absolute constant.

In [4], Deza and Varukhina extended Theorem 3 to quadratic and cyclotomic fields. Later, Lü [10] generalized their results to a finite Galois extension of $\mathbb{Q}$ as follows.

**Theorem 4** [10] Let $K/\mathbb{Q}$ be a Galois extension of degree $d$. Then for integers $l \geq 1$, $k_1, \ldots, k_l \geq 2$ and any $\epsilon > 0$, we have

$$\sum_{n \leq x} \tau_{k_1}^K(n) \cdots \tau_{k_l}^K(n) = xP_m(\log x) + O\left(x^{1 - \frac{3}{md(\log x)^{1+\epsilon}}} \right),$$

where $m = k_1 \cdots k_l d^{l-1}$ and $P_m$ is a polynomial of degree $m - 1$.

Later, Lü and Ma [11] extended Theorem 4 for several number fields and proved the following.

**Theorem 5** [11] Let $l \geq 1$ and $k_1, \ldots, k_l \geq 2$ be given integers. For each $i \in \{1, 2, \ldots, l\}$, let $K_i$ be a number field having discriminant $D_i$ and degree $d_i$. Suppose that $\gcd(D_i, D_j) = 1$
for \( i \neq j \). Then for any \( \epsilon > 0 \), we have
\[
\sum_{n \leq x} \tau_{K_1}^{k_1}(n) \ldots \tau_{K_l}^{k_l}(n) = xP_m(\log x) + O\left(x^{1 - \frac{3}{md_1 + \ldots + d_l} + \epsilon}\right),
\]
where \( m = k_1 \ldots k_l \) and \( P_m \) is a polynomial of degree \( m - 1 \).

From the formula
\[
\sum_{n \leq x} \tau(n) = \sum_{n \leq x} \sum_{d|n} 1 = \sum_{ab \leq x} 1,
\]
one sees that the above expression counts the number of lattice points lying in the first quadrant as well as below the hyperbola \( XY = x \). Similarly, the quantity
\[
\sum_{n \leq x} \tau(n^2) = \sum_{n \leq x} \sum_{d|n^2} 1 = \sum_{n \leq x} \sum_{ab = n^2} 1
\]
counts the number of lattice points lying in the first quadrant as well as below the hyperbola \( XY = x \), where \( x \) is a perfect square. This geometric interpretation motivates to study the asymptotic behaviour of divisor functions over square integers. Along this direction, Lü and Yang [13] proved the following.

**Theorem 6** [13] Let \( d \geq 3 \) be an odd integer and let \( K \) be a Galois extension of degree \( d \) over \( \mathbb{Q} \). Then for any integer \( k \geq 2 \) and any \( \epsilon > 0 \), we have
\[
\sum_{n \leq x} \tau_{K}^{k}(n^2) = xP_m(\log x) + O\left(x^{1 - \frac{3}{md_1 + \ldots + d_l} + \epsilon}\right),
\]
where \( m = \frac{k^2d + k}{2} \) and \( P_m \) is a polynomial of degree \( m - 1 \).

We now consider the problem related to the order of magnitude of a function. In general, to determine the precise order of magnitude of an arithmetic function is very hard. But it is very useful to find the following limiting behaviour.

\[
\limsup_{x \to \infty} \frac{f(x)}{g(x)} = + \infty \quad \text{and} \quad \liminf_{x \to \infty} \frac{f(x)}{g(x)} = - \infty
\]
for some function \( f(x) \) and a positive function \( g(x) \).

For example, if \( r_2(n) \) stands for the number of representations of \( n \) as a sum of two squares, then it is known that
\[
R(x) = \sum_{n \leq x} r_2(n) = \pi x + E(x).
\]

Hardy [5] and Ingham [7] proved that
\[
\limsup_{x \to \infty} \frac{E(x)}{x^{3/2}} = + \infty \quad \text{and} \quad \liminf_{x \to \infty} \frac{E(x)}{x^{3/2}} = - \infty.
\]

In [2], Chandrasekharan and Narasimhan extended the above formulae for a large class of arithmetical functions. We now briefly describe their result proved in [2].

Suppose we have the following functional equation
\[
\Delta(s)\phi(s) = \Delta(\delta - s)\psi(\delta - s),
\]
where \( \delta \) is a real number, \( \Delta(s) = \prod_{\nu=1}^{N} \Gamma(\alpha_{\nu}s + \beta_{\nu}), \alpha_{\nu} > 0 \) satisfying \( M = \sum_{\nu=1}^{N} \alpha_{\nu} \geq 1 \) and \( \beta_{\nu} \)'s are complex numbers. We consider solutions of (2) in the form of Dirichlet series.
Let \( \phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \) for \( \text{Re}(s) > \alpha \) and \( \psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \) for \( \text{Re}(s) > \beta \), where \( \alpha \) and \( \beta \) are complex numbers. Also, let \( A(x) = \sum_{n \leq x} a_n \), \( B(x) = \sum_{n \leq x} b_n \) and \( Q(x) = \frac{1}{2\pi i} \int_C \phi(s) \frac{x^s}{s} ds \), where \( C \) is a suitably chosen contour containing all the singularities of the integrand. The following is a special case of a theorem proved in [2].

**Theorem 7** [2] Let \( \Delta, \phi, \psi, M \) be defined as above such that (2) holds. Let \( \{\mu_n\} \) contain a subsequence \( \{\mu_{n_k}\} \) such that \( \mu_{\frac{1}{2}M n} \) is representable as a linear combination of the numbers \( \{\mu_{\frac{1}{2}M n_k}\} \) with coefficients \( \pm 1 \), unless \( \mu_{\frac{1}{2}M n} = \mu_{\frac{1}{2}M n-r} \) for some \( r \), in which case \( \mu_{\frac{1}{2}M n} \) has no other representations. Suppose

\[
\sum_{n=1}^{\infty} \frac{|\text{Re}(b_{n_k})|}{\mu_{\frac{1}{2}M n_k}} = +\infty \quad \text{holds.} \tag{3}
\]

Then we have

\[
\limsup_{x \to \infty} \frac{\text{Re}(A(x) - Q(x))}{x^\theta} = +\infty \tag{4}
\]

and

\[
\liminf_{x \to \infty} \frac{\text{Re}(A(x) - Q(x))}{x^\theta} = -\infty, \tag{5}
\]

where \( \theta = \frac{\delta}{2} - \frac{1}{4M} \).

If we replace \( \text{Re}(b_{n_k}) \) by \( \text{Im}(b_{n_k}) \), then we obtain analogous statements for \( \text{Im}(A(x) - Q(x)) \) in place of \( \text{Re}(A(x) - Q(x)) \).

### 2 Statements of theorems

In this section we state our results. First, we generalize Theorem 6 to finitely many number fields and our precise statement is as follows.

**Theorem 8** Let \( l \geq 1 \) and \( k_1, \ldots, k_l \geq 2 \) be integers. Let \( K_1, \ldots, K_{l-1} \) and \( K_l \) be finite Galois extensions of \( \mathbb{Q} \) with discriminants \( D_1, \ldots, D_{l-1} \) and \( D_l \), respectively. Suppose that \( |K_i : \mathbb{Q}| = d_i \geq 3 \) is odd for all \( i = 1, \ldots, l \) and \( \gcd(D_i, D_j) = 1 \) for \( i \neq j \). Then for any \( \epsilon > 0 \), we have

\[
\sum_{n \leq x} \tau_{k_1}(n^2) \cdots \tau_{k_l}(n^2) = xP_m(\log x) + O \left( x^{1-\frac{3}{m+1}+\epsilon} \right), \tag{6}
\]

where \( m = \prod_{i=1}^{l} k_i^2 d_i + k_i 2 \) and \( P_m \) stands for a polynomial of degree \( m - 1 \).

We note that in Theorem 8, we are required to have \( d_i \) odd for each \( i \). But if we restrict ourselves to two number fields, one among them being quadratic, then we can have a similar result for the mean values of the ideal counting function \( a_K(n) \). In [17], Yang found an asymptotic formula for the product of the ideal counting functions of two quadratic fields taken together (see also [12]). In the next theorem of ours, we consider the same problem for a quadratic field and an odd degree number field. More precisely, the statement is as follows.
Theorem 9 Let $K_1$ and $K_2$ be two number fields with discriminants $D_{K_1}$ and $D_{K_2}$, respectively such that $[K_1 : \mathbb{Q}] = 2$ and $[K_2 : \mathbb{Q}] = d$ with $d \geq 3$ odd. Suppose that $\gcd(D_{K_1}, D_{K_2}) = 1$. Then for any integer $l \geq 1$ and any $\epsilon > 0$, we have

$$\sum_{n \leq x} a_{K_1}(n)^l a_{K_2}(n)^l = xP_m(\log x) + O\left(x^{1 - \frac{3}{2d^2} + \epsilon}\right).$$

We now turn our attention towards the error term arising in Theorem 8. Keeping the same notations, if we define $\Delta(x) := \sum_{n \leq x} \tau_{K_1}^K(n^2) \ldots \tau_{K_l}^K(n^2) - xP_m(\log x)$, then we get

$$\Delta(x) = O\left(x^{1 - \frac{3}{md} + \epsilon}\right).$$

One of the useful ways to understand $\Delta(x)$ is via the higher order moments. Recently, in [16], Shi obtained the second order moment of the error term arising in the estimation of $\sum_{n \leq x} \tau_{K_1}^K(n) \ldots \tau_{K_l}^K(n)$. Here, adapting the methods and techniques used in [16], we calculate the second order moment for the error term $\Delta(x)$ appearing in Theorem 8. More precisely, we prove the following.

Theorem 10 Under the same notations and hypotheses as in Theorem 8, let

$$\Delta(x) := \sum_{n \leq x} \tau_{K_1}^K(n^2) \ldots \tau_{K_l}^K(n^2) - xP_m(\log x).$$

Then for a given $\epsilon > 0$, we have

$$\int_1^X \Delta^2(x)dx = O\left(x^{3 - \frac{6}{md} + \epsilon}\right).$$

(7)

Here, the implied constant depends only on $\epsilon$.

Now, for a positive integer $a$, we define the generalized sum of divisor function in a number field $K$ by the formula

$$\sigma_a^K(n) = \sum_{N(a)|n} (N(a))^a.$$

The precise statements of our results regarding the estimation of the summatory function are given below.

Theorem 11 Let $K$ be a number field of degree $d$. Then for any positive integer $a \geq 1$ and any $\epsilon > 0$, we have

$$\sum_{n \leq x} \sigma_a^K(n) = \frac{c_K \zeta(1 + a)}{1 + a} x^{1 + a} + O\left(x^{(1+a) - \frac{3}{2d^2} + \epsilon}\right),$$

where $c_K$ is the residue of $\zeta_K(s)$ at $s = 1$.

Theorem 12 Let $K_1$ and $K_2$ be two number fields of degree $d_1$ and $d_2$ and discriminant $D_{K_1}$ and $D_{K_2}$, respectively. Suppose $K_1$ and $K_2$ satisfy the following conditions.

(i) Both $K_1$ and $K_2$ are Galois over $\mathbb{Q}$;
(ii) $\gcd(D_{K_1}, D_{K_2}) = 1$. 

Then for positive integers \(a, b\) and any \(\epsilon > 0\), we have,

\[
\sum_{n \leq x} \sigma_{a}^{K_1}(n)\sigma_{b}^{K_2}(n) = \frac{\zeta(1 + a + b)\zeta_{K_1}(1 + b)\zeta_{K_2}(1 + a)}{\zeta_{K_1,K_2}(1 + a + b)} x^{1+a+b} + O\left(x^{(1+a+b)-\frac{3}{2(a+b)+\epsilon}}\right),
\]

where \(c_{K_1,K_2}\) is the residue of \(\zeta_{K_1,K_2}(s)\) at \(s = 1\).

**Remark 1** Theorem 12 can be generalized for any finite number of number fields with pairwise coprime discriminants.

Finally, as an application of Theorem 7, we get the following corollary.

**Corollary 2.1** Let \(K\) be a quadratic field with discriminant \(D_K\). Then we have

\[
\limsup_{x \to \infty} \frac{\sum_{n \leq x} \tau_{K}^{K}(n) - xP_{k}(\log x)}{x^{1 - \frac{1}{4k}}} = +\infty,
\]

and

\[
\liminf_{x \to \infty} \frac{\sum_{n \leq x} \tau_{K}^{K}(n) - xP_{k}(\log x)}{x^{1 - \frac{1}{4k}}} = -\infty,
\]

where \(P_{k}\) is a polynomial of degree \(k - 1\) appearing in the main term in the asymptotic formula of \(\sum_{n \leq x} \tau_{K}^{K}(n)\).

**Remark 2** For the function \(\sigma_{a}^{K}(n)\), the functional equation exists for the corresponding \(L\)-function \(\zeta(s) \cdot \zeta_{K}(s-a)\) but it does not satisfy equation (2). Therefore, corresponding results similar to (4) and (5) cannot be obtained using Theorem 7.

**Remark 3** In this paper, for most of the cases we do not have the functional equation for the \(L\)-series corresponding to the sumatory function of a given arithmetic function. Instead of that, we get an expression of the form \(L(s) = \tau_{k}^{K}(s) \cdot U(s)\) for some Dirichlet series \(U(s)\) which is absolutely convergent for \(\text{Re}(s) > \frac{1}{2}\). Therefore, it is quite difficult to obtain expressions similar to (4) and (5).

### 3 Preliminaries

In this section, we list all the necessary results required for the proofs of Theorems 8–12. For that, let us start with a few fundamental properties of the Dedekind zeta-function associated to a number field. For an algebraic number field \(K\), the Dedekind zeta-function of \(K\) is defined by

\[
\zeta_{K}(s) = \sum_{a \subseteq \mathcal{O}_{K}} \frac{1}{N(a)^{s}} \text{ for } \text{Re}(s) > 1,
\]

where the sum runs over all the non-zero ideals in the ring \(\mathcal{O}_{K}\) and the series is absolutely convergent in the half-plane \(\text{Re}(s) > 1\). When we expand the above expression in the form of a Dirichlet series, it has the following form.

\[
\zeta_{K}(s) = \sum_{a \subseteq \mathcal{O}_{K}} \frac{1}{N(a)^{s}} = \sum_{n=1}^{\infty} \frac{a_{K}(n)}{n^{s}} \text{ for } \text{Re}(s) > 1.
\]
Here, $a_K(n)$ denotes the number of ideals in $O_K$ having norm $n$, often called the \textit{ideal counting function} of the number field $K$. Our first lemma asserts that $a_K(n)$ is a multiplicative function of $n$ and the proof can be found in [3].

\textbf{Lemma 1} [3] For a number field $K$, let $a_K(n)$ be the number of ideals in $O_K$ with norm $n$. Then $a_K(n)$ is a multiplicative function of $n$ and for any $\epsilon > 0$,

$$a_K(n) \ll \epsilon n^\epsilon$$

Since the coefficients of the Dirichlet series (8) are multiplicative, $\zeta_K(s)$ admits the following Euler product expansion.

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s} = \prod_p \left( 1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \ldots \right) \text{ for Re}(s) > 1.$$ 

As we are dealing with several number fields at a time, it will be quite useful to know the behaviour of the ideal counting function at prime arguments in the compositum of two number fields.

\textbf{Lemma 2} [11] Let $K$ and $L$ be two number fields with discriminants $D_K$ and $D_L$, respectively. Suppose that $\gcd(D_K, D_L) = 1$. Then for any prime number $p$, we have

$$a_{KL}(p) = a_K(p) a_L(p).$$

The next couple of lemmas, which relate the divisor function of a finite Galois extension to the ideal counting function, will play a crucial role in the course of the proofs of our theorems.

\textbf{Lemma 3} [13] Let $k \geq 2$ be an integer and let $K/\mathbb{Q}$ be a finite Galois extension of odd degree $d$. Then

$$\tau^K_k(p^2) = \frac{k^2 d + k}{2} a_K(p)$$

holds for all but finitely many prime numbers $p$.

\textbf{Lemma 4} [10] Let $K/\mathbb{Q}$ be finite Galois extension of degree $d$. Then for any positive integer $k$, the relation

$$a_K(p)^k = d^{k-1} a_K(p)$$

holds for all but finitely many prime numbers $p$.

Now, we state the \textit{Phragman–Lindelöf} hypothesis which deals with the estimation of an analytic function in a given strip.

\textbf{Lemma 5} [9] Let $a, b$ be two real numbers with $a < b$ and let $f$ be an analytic function defined on an open neighbourhood of the strip $a \leq \sigma \leq b$, for some real numbers $a < b$, such that $|f(s)| = O(|s|^A)$ for some $A \geq 0$ and for all $s$ satisfying $a \leq \text{Re}(s) \leq b$.

(i) Assume that $|f(s)| \leq M$ for all $s$ on the boundary of the critical strip, i.e, for $\sigma = a$ or $\sigma = b$. Then we have $|f(s)| \leq M$ for all $s$ in the strip.

(ii) If there exist real numbers $M_a, M_b, \alpha, \beta$ such that for all $t \in \mathbb{R}$,

$$|f(a + it)| \leq M_a(1 + |t|^\alpha)$$
Lemma 6 Let \( f(s) = \sum_{n=1}^{\infty} a_n n^s \) and \( g(s) = \sum_{n=1}^{\infty} b_n n^s \) be two Dirichlet series both of which are absolutely convergent in \( \Re(s) > 1 \) and satisfy the following conditions:

(i) Both \( a_n \) and \( b_n \) are positive and multiplicative functions of \( n \),

(ii) For any \( \epsilon > 0 \), we have \( a_n \ll \epsilon n^\sigma \) and \( b_n \ll \epsilon n^\sigma \),

(iii) \( a_p = b_p \) for all but finitely many prime numbers \( p \).

Then \( f(s) = g(s) \cdot U(s) \), where \( U(s) \) is a Dirichlet series which is absolutely convergent in \( \Re(s) > \frac{1}{2} \) and uniformly convergent in \( \Re(s) \geq \frac{1}{2} + \epsilon \) for any \( \epsilon > 0 \).

Proof Since both \( a_n \) and \( b_n \) are multiplicative functions of \( n \), \( f(s) \) and \( g(s) \) admit Euler product expansions. For \( \Re(s) = \sigma > 1 \), consider \( f(s) \cdot g(s)^{-1} = \prod_p U_p(s) \), where

\[
U_p(s) = \left( 1 + \frac{a_p}{p^s} + \sum_{m=2}^{\infty} \frac{a_{pm}}{p^{ms}} \right) \cdot \left( 1 + \frac{b_p}{p^s} + \sum_{m=2}^{\infty} \frac{b_{pm}}{p^{ms}} \right)^{-1}.
\]

Let \( U'_p(s) = \sum_{m=2}^{\infty} \frac{a_{pm}}{p^{ms}} \) and \( U''_p(s) = \sum_{m=2}^{\infty} \frac{b_{pm}}{p^{ms}} \). Then using condition (ii), we get

\[
U'_p(s) \ll \epsilon \sum_{m=2}^{\infty} \frac{p^{me}}{p^{ms}} = \frac{1}{p^{\sigma - \epsilon}} \cdot \frac{1}{p^{\sigma - \epsilon}} - 1 \ll \epsilon \frac{1}{p^{2\sigma - 2\epsilon}}.
\]

Similarly, \( U''_p(s) \ll \frac{1}{p^{2\sigma - 2\epsilon}} \). Hence we obtain

\[
U_p(s) = \left( 1 + \frac{a_p}{p^s} + O_\epsilon \left( \frac{1}{p^{2\sigma - 2\epsilon}} \right) \right) \cdot \left( 1 + \frac{b_p}{p^s} + O_\epsilon \left( \frac{1}{p^{2\sigma - 2\epsilon}} \right) \right)^{-1}.
\]

Now from (9) and the hypotheses (i) and (iii), we conclude that \( U_p(s) = 1 + O(p^{\sigma - 2\sigma}) \) for all but finitely many primes \( p \).

Hence \( \prod_p U_p(s) \) converges if and only if \( \sum_p \frac{1}{p^{2\sigma - \epsilon}} \) converges. In other words, for \( \sigma > \frac{1}{2} \), \( U(s) = \prod_p U_p(s) \) defines an absolutely convergent Dirichlet series and therefore we obtain \( f(s) = g(s) \cdot U(s) \). \( \square \)

Now, we record the following estimate, which can be seen in [6], regarding the growth of the Dedekind zeta-function of a number field on the half-line \( \Re(s) = \frac{1}{2} \).
**Lemma 7** [6] Let \( K \) be a number field of degree \( d \). Then for any real number \( t \geq 1 \) and \( \epsilon > 0 \), we have
\[
\zeta_K \left( \frac{1}{2} + it \right) = O \left( t^{d + \epsilon} \right),
\]
where the implied constant depends only on the number field \( K \).

Lastly, the following lemma provides the Perron’s formula in general set-up.

**Lemma 8** ([8], Equation (A.10)) Let \( a(n) \) be an arithmetic function satisfying \( |a(n)| \ll \Phi(n) \), where \( \Phi \) is an increasing function of \( n \) and
\[
\sum_{n=1}^{\infty} |a(n)|n^{-\alpha} \ll (\sigma - 1)^{-\alpha}
\]
as \( \sigma \to 1^+ \) and for some real number \( \alpha > 0 \).

Let \( 1 < b \ll 1, T > 0 \) and \( x \geq 1 \). Let us define \( F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \). Then we have
\[
\sum_{n \leq x} a(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s) \frac{x^s}{s} ds + O \left( \frac{x^{b-1}(b-1)^{-\alpha}}{\Phi(2x) \log(2x)} \right) + O \left( \Phi(2x) \right).
\]

### 4 Proof of Theorem 8

Our strategy, to prove Theorem 8, is via studying the Dedekind zeta-function for a suitable number field and its connection to a particular Dirichlet series. For integers \( l \geq 1 \) and \( k_1, \ldots, k_l \geq 2 \), we define
\[
\tau_{K_{k_1}, \ldots, K_{k_l}}(n) = \sum_{n=1}^{\infty} \frac{\tau_{K_{k_1}}(n^2) \ldots \tau_{K_{k_l}}(n^2)}{n^s} \text{ for } Re(s) > 1. \tag{10}
\]

Using the well-known result \( \tau(n) = O(n^\epsilon) \) for any \( \epsilon > 0 \), we get that for any integer \( k \geq 1 \) and any number field \( K \),
\[
\tau^K_k(n) = O \left( \sum_{n=n_1 \ldots n_k} \tau_k(n_1) \ldots \tau_k(n_k) \right) = O(n^\epsilon),
\]
which also implies that \( \tau^K_k(n^2) = O(n^\epsilon) \). Thus the Dirichlet series defined in (10) is absolutely convergent in the half-plane \( Re(s) > 1 \).

Since each \( \tau^K_{k_i}(n) \) is multiplicative, so are the coefficients \( \tau^K_{k_1}(n^2) \ldots \tau^K_{k_l}(n^2) \) of \( n^{-s} \) in (10). Thus in \( Re(s) > 1 \), by using Lemma 2 and Lemma 3, we have
\[
\tau^{K_{k_1}, \ldots, K_{k_l}}(n) = \prod_p \left( 1 + \frac{\tau^{K_{k_1}}_{k_1}(p^2) \ldots \tau^{K_{k_l}}_{k_l}(p^2)}{p^{s}} + \frac{\tau^{K_{k_1}}_{k_1}(p^4) \ldots \tau^{K_{k_l}}_{k_l}(p^4)}{p^{2s}} + \ldots \right)
\]
\[
= \prod_p \left( 1 + \sum_{i=1}^{l} \frac{k_i^2 d_i + k_i}{2} a_{K_{k_1} \ldots K_{k_l}}(p) \right) \frac{1}{p^{s}} + \ldots \right).
\]
for all but finitely many primes.

On the other hand, if we consider the Dedekind zeta-function for the compositum of the number fields $K_1, \ldots, K_l$, then we get for $Re(s) > 1$, 

$$\prod_{i=1}^{l} \left( \frac{k_i^2 d_i + k_i}{2} \right)^{\frac{1}{2}} = \prod_{p} \left( 1 + \frac{\prod_{i=1}^{l} \left( \frac{k_i^2 d_i + k_i}{2} \right) a_{K_1 \ldots K_l}(p)}{p^s} + \ldots \right)$$

Hence we see that the coefficients of $p^{-s}$ in the Euler product expansions of $L_{K_1, \ldots, K_l}(s)$ and $(\zeta_{K_1 \ldots K_l}(s))_{i=1}^{l}$ are same for all but finitely many prime numbers $p$. Thus by Lemma 6, we have 

$$L_{K_1, \ldots, K_l}(s) = (\zeta_{K_1 \ldots K_l}(s))_{i=1}^{l} \cdot \zeta(s),$$

where $\zeta(s)$ is a Dirichlet series, absolutely convergent in $Re(s) > \frac{1}{2}$ and uniformly convergent in $Re(s) \geq \frac{1}{2} + \epsilon$, for any $\epsilon > 0$. From (11), we can say that $L_{K_1, \ldots, K_l}(s)$ has an analytic continuation to the half-plane $Re(s) > \frac{1}{2}$, except for a pole at $s = 1$ of order

$$\prod_{i=1}^{l} \left( \frac{k_i^2 d_i + k_i}{2} \right).$$

Now using Lemma 8, we get 

$$\sum_{n \leq x} \tau_{K_1}(n^2) \tau_{K_2}(n^2) = \frac{1}{2\pi i} \left[ \int_{b-iT}^{b+iT} L_{K_1 \ldots K_l}(s) \frac{x^2}{s} ds + O \left( \frac{x^{1+\epsilon}}{T} \right) \right],$$

where $b = 1 + \epsilon$ and $T$ is a parameter satisfying $1 \leq T \leq x$, to be appropriately chosen later.

Now, we move the line of integration to $Re(s) = \frac{1}{2} + \epsilon$, and consider the rectangular contour formed by joining the points $1 \frac{1}{2} + \epsilon - iT, \frac{1}{2} + \epsilon + iT, b + iT$ and $b - iT$, successively. From (12), by using Cauchy residue theorem, we get 

$$\sum_{n \leq x} \tau_{K_1}(n^2) \tau_{K_2}(n^2) = \frac{1}{2\pi i} \left[ \int_{1/2 + \epsilon - iT}^{1/2 + \epsilon + iT} L_{K_1 \ldots K_l}(s) \frac{x^2}{s} ds + \text{Res}_{s=1} L_{K_1 \ldots K_l}(s) \frac{x^2}{s} + O \left( \frac{x^{1+\epsilon}}{T} \right) \right],$$

where $I_1 = \frac{1}{2\pi i} \int_{1/2 + \epsilon - iT}^{1/2 + \epsilon + iT} L_{K_1 \ldots K_l}(s) \frac{x^2}{s} ds$, $I_2 = \frac{1}{2\pi i} \int_{1/2 + \epsilon - iT}^{1/2 + \epsilon + iT} L_{K_1 \ldots K_l}(s) \frac{x^2}{s} ds$, and $m = \prod_{i=1}^{l} \left( \frac{k_i^2 d_i + k_i}{2} \right)$ is a polynomial of degree $m-1$.

Now, by using Lemma 5 and Lemma 7, we get 

$$\zeta_{K_1 \ldots K_l}(\sigma + it) \ll (1 + |t|) \frac{e^{1/2}}{T(1 - \sigma + \epsilon)}.$$  

Raising both sides of (13) to $m$-th power, we get 

$$\zeta_{K_1 \ldots K_l}^m(\sigma + it) \ll (1 + |t|) \frac{e^{m/2}}{T(1 - \sigma + \epsilon)}.$$  

(14)
Since $U(s)$ is absolutely convergent in the region $\text{Re}(s) > \frac{1}{2}$, it is bounded in that region. Thus by (11) and (14), we get

$$I_1 = \frac{1}{2\pi i} \int_{1+\epsilon+IT}^{1+\epsilon-IT} L_{K_1,\ldots,K_l}(s) x^s ds$$

$\ll \int_{-T}^T \left| L_{k_{1,\ldots,k_l}} \left( \frac{1}{2} + \epsilon + it \right) \right| \left| x^{\frac{1}{2} + \epsilon + it} \right| dt$

$\ll x^{1+\epsilon} + x^{\frac{1}{2}+\epsilon} \left( \int_{-T}^{-1} \left| L_{k_{1,\ldots,k_l}} \left( \frac{1}{2} + \epsilon + it \right) \right| t dt + \int_{1}^{T} \left| L_{k_{1,\ldots,k_l}} \left( \frac{1}{2} + \epsilon + it \right) \right| t dt \right)$

$\ll x^{1+\epsilon} + x^{\frac{1}{2}+\epsilon} \left( \int_{1}^{T} \left| L_{k_{1,\ldots,k_l}} \left( \frac{1}{2} + \epsilon - it \right) \right| t dt + \int_{1}^{T} \left| L_{k_{1,\ldots,k_l}} \left( \frac{1}{2} + \epsilon + it \right) \right| t dt \right)$

Combing the above estimations, we obtain

$$\sum_{n \leq x} \prod_{k_1}^{K_1}(n^2) \ldots \prod_{k_l}^{K_l}(n^2) = xP_m(\log x) + O(x^{1+\epsilon} T^{-1/\gamma - \frac{3}{2} \epsilon}) + O(x^{\frac{1}{2} + \epsilon} T^{-5/6 - \frac{3}{2} \epsilon})$$

By taking $T = x^{\frac{3}{md_1 - d_l + 6}}$, we get

$$\sum_{n \leq x} \prod_{k_1}^{K_1}(n^2) \ldots \prod_{k_l}^{K_l}(n^2) = xP_m(\log x) + O(x^{1-\frac{3}{md_1 - d_l + 6} + \epsilon})$$

This completes the proof of Theorem 8.

**5 Proof of Theorem 9**

We consider the Dirichlet series, defined by

$$L_{K_1,\ldots,K_l}(s) = \sum_{n=1}^{\infty} \frac{a_{K_1}(n)^l a_{K_2}(n)^l}{n^s} \text{ for Re}(s) > 1.$$
primes $p$. Comparing this with $\zeta_{K_1K_2}(s)(2d)^{l-1}$, we see that the coefficients of $p^{-s}$ in the Euler product of these two Dirichlet series are same for all but finitely many prime numbers $p$. Thus by Lemma 6, we have

$$L_{K_1K_2}(s) = \zeta_{K_1K_2}(s)(2d)^{l-1} \cdot U(s)$$

where $U(s)$ is a Dirichlet series, absolutely convergent for $Re(s) > 1/2$. The rest of the proof follows exactly the same line of argument as that of Theorem 8 and hence we omit it. \(\square\)

### 6 Proof of Theorem 10

From equation (13), we have

$$\sum_{n \leq x} \tau_{K_1}^{K_1}(n^2) \cdots \tau_{K_j}^{K_j}(n^2) = xP_m(\log x) + I_1 + I_2 + I_3 + O\left(\frac{x^{1+\epsilon}}{T}\right).$$

In order to prove Theorem 10, we need to find upper bounds of the following integrals.

$$\int_1^X I_j^2(x)dx, \quad j = 1, 2, 3 \quad \text{and} \quad \int_1^X \left(\frac{x^{1+\epsilon}}{T}\right)^2 dx.$$

It is easy to see that,

$$\int_1^X \left(\frac{x^{1+\epsilon}}{T}\right)^2 dx \ll \frac{x^{3+\epsilon}}{T^2}.$$

We also need to use the following inequality to estimate $\int_1^X I_j^2(x)dx$ .

$$\int_{-T}^T \frac{dt_2}{1 + |t_1 - t_2|} \ll \log(2T). \quad (15)$$

Using inequalities (11), (14) and (15), we get

$$\int_1^X I_j^2(x)dx = \frac{1}{4\pi^2} \int_1^X \left[ \int_{-T}^T L_{K_1 \cdots K_j}^{K_1 \cdots K_j} \left(\frac{1}{2} + \epsilon + i\tau\right) \frac{x^{1/2+\epsilon+i\tau}}{1/2 + \epsilon + i\tau_1} dt_1 \right] \frac{x^{1/2+\epsilon+i\tau}}{1/2 + \epsilon + i\tau_2} dt_2 dx$$

$$= \frac{1}{4\pi^2} \int_{-T}^T \int_{-T}^T L_{K_1 \cdots K_j}^{K_1 \cdots K_j} \left(\frac{1}{2} + \epsilon + i\tau_1\right) L_{K_1 \cdots K_j}^{K_1 \cdots K_j} \left(\frac{1}{2} + \epsilon + i\tau_2\right) \left(\int_{-T}^T x^{1/2+\epsilon+i(t_1-t_2)} dt_1\right) dt_2 dx$$

$$\ll X^{2+2\epsilon} \int_{-T}^T \int_{-T}^T \left[ L_{K_1 \cdots K_j}^{K_1 \cdots K_j} \left(\frac{1}{2} + \epsilon + i\tau_1\right) \left| L_{K_1 \cdots K_j}^{K_1 \cdots K_j} \left(\frac{1}{2} + \epsilon + i\tau_2\right) \right| \right] \frac{dt_1 dt_2}{(1 + |t_1|)(1 + |t_2|)(1 + |t_1 - t_2|)}$$

$$\ll X^{2+2\epsilon} \int_{-T}^T \int_{-T}^T \left[ \frac{L_{K_1 \cdots K_j}^{K_1 \cdots K_j} \left(\frac{1}{2} + \epsilon + i\tau_1\right)}{(1 + |t_1|)^2} \right] \left[ \frac{L_{K_1 \cdots K_j}^{K_1 \cdots K_j} \left(\frac{1}{2} + \epsilon + i\tau_2\right)}{(1 + |t_2|)^2} \right] \frac{dt_1 dt_2}{1 + |t_1 - t_2|}$$

$$\ll X^{2+2\epsilon} \log(2T) \int_{-T}^T \left[ \frac{L_{K_1 \cdots K_j}^{K_1 \cdots K_j} \left(\frac{1}{2} + \epsilon + i\tau_1\right)}{(1 + |t_1|)^2} \right] \frac{dt_1}{1 + |t_1 - t_2|}$$

$$\ll X^{2+2\epsilon} \log(2T) + X^{2+2\epsilon} \log(2T) \int_1^T \left[ \frac{\zeta_{K_1 \cdots K_j}^{m} \left(1/2 + \epsilon + i\tau\right)}{U \left(1/2 + \epsilon + it\right)} \right]^2 dt$$
Again using inequalities (11) and (14), for \( j = 2, 3 \) we get
\[
I_j(x) \ll \int_{1/2 + \epsilon}^{1+\epsilon} x^\sigma |\epsilon_{K_1 \ldots K_j}(\sigma + iT)| T^{-1} d\sigma \ll \max_{1/2+\epsilon \leq \sigma \leq 1+\epsilon} x^\sigma T^{-\frac{md_j - dt_j}{2} - 1+\epsilon}.
\]

Therefore, for \( j = 2, 3 \) we have
\[
\int_1^X I_j^2(x) dx \ll \frac{X^{2+2\epsilon}}{T^2} + X^{2+2\epsilon} T^{-\frac{md_j - dt_j}{2} - 1+\epsilon}.
\]

Let us choose, \( T = X^{\frac{3}{md_j - dt_j}} \). For this choice of \( T \), from above calculations, we get
\[
\int_1^X \Delta^2(x) dx = O \left( X^{3 - \frac{6}{md_j - dt_j} + 1+\epsilon} \right).
\]

\[\Box\]

7 Proof of Theorem 11

For \( a \geq 1 \), we write \( L_K(s) = \sum_{n=1}^{\infty} \frac{\sigma^K_a(n)}{n^s} \). From the definition of \( \sigma^K_a(n) \), it is easy to see that \( L_K(s) \) is convergent for \( \text{Re}(s) > 1 + a \).

Observe that, \( L_K(s) = \zeta(s) \zeta_K(s - a) \). Now by using Lemma 8, we get
\[
\sum_{n \leq x} \sigma^K_a(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L_K(s) \frac{x^s}{s} ds + O \left( \frac{x^{1+a+\epsilon}}{T} \right),
\]
where \( b = 1 + a + \epsilon \) and \( T \) is a parameter satisfying \( 1 \leq T < x^{1+a} \) and to be appropriately chosen later.

Now, we move the line of integration to \( \text{Re}(s) = \frac{1}{2} + a \), and consider the rectangular contour formed by joining the points \( \frac{1}{2} + a - iT, \frac{1}{2} + a + iT, b + iT \) and \( b - iT \) successively. Since \( \zeta(s) \) is bounded along the above vertical and horizontal line, so it is enough to consider the behaviour of Dedekind zeta function \( \zeta_K(s - a) \). Also observe that, at \( s = \frac{1}{2} + a + it \) the value of \( \zeta_K(s - a) \) is \( \zeta_K \left( \frac{1}{2} + it \right) \).

From this point onwards, by following the same argument as given in the proof Theorem 8, we get the desired result. \[\Box\]

8 Proof of Theorem 12

We consider \( L(s) = \sum_{n=1}^{\infty} \frac{\sigma^K_{a_1}(n)\sigma^K_{a_2}(n)}{n^s} \). From the definition of \( \sigma^K_{a_j}(n) \), \( j = 1, 2 \) it is clear that \( L(s) \) converges absolutely for \( \text{Re}(s) > 1 + a + b \).

Now from Lemma 2, it easily follows that \( \sigma^K_{a_1}(n)\sigma^K_{a_2}(n) \) is a multiplicative function of \( n \).

Also, we notice that for a number field \( K \),
\[
\sigma^K_a(p) = \sum_{N(a)|p} N(a) \sigma_a = 1 + \sum_{N(a)=p} p^a = 1 + p^a \sum_{N(a)=p} 1 = 1 + a_K(p) \cdot p^a.
\]
Now, by Lemma 4 and the following equality
\[ a_K(p) = \begin{cases} p; & \text{if } p \text{ splits completely} \\
0; & \text{otherwise} \end{cases} \]
we get for \( Re(s) > 1 + a + b \)
\[
L(s) = \sum_{n=1}^{\infty} \frac{\sigma_a^{K_1}(n)\sigma_b^{K_2}(n)}{n^s} = \prod_p \left( 1 + \frac{\sigma_a^{K_1}(p)\sigma_b^{K_2}(p)}{p^s} + \ldots \right) = \prod_p \left( 1 + \frac{1 + p^a a_{K_1}(p) + p^b a_{K_2}(p) + p^{a+b} a_{K_3}(p)}{p^s} + \ldots \right)
\]
for all but finitely many prime numbers \( p \).

Now, comparing this with the Euler product expansion of \( \zeta(s) \cdot \zeta_K(s-a) \cdot \zeta_K(s-b) \cdot \zeta_{K_3}(s-a-b) \), we see that the coefficients of \( p^{-s} \) are same for all but finitely many primes \( p \). Hence, by Lemma 6, we have \( L(s) = \zeta(s) \cdot \zeta_K(s-a) \cdot \zeta_K(s-b) \cdot \zeta_{K_3}(s-a-b) \cdot L'(s) \), where \( L'(s) \) is a Dirichlet series, which is absolutely convergent for \( Re(s) > \frac{1}{2} + a + b \).

Again by following the same argument as given in the proof Theorem 8, we get our desired result.

\[ \square \]

9 Proof of Corollary 2.1

We know that the \( L \)-series corresponding to \( \sum_{n \leq x} \zeta^K_k(n) \) is \( \zeta^k_K(s) \). Moreover, \( \zeta^k_K(s) \) satisfies the functional equation
\[
\left( \frac{\sqrt{|D_K|}}{2\pi} \right)^s \Gamma(s) \zeta^k_K(s) = \left( \frac{\sqrt{|D_K|}}{2\pi} \right)^{1-s} \Gamma(1-s) \zeta^k_K(1-s) \text{ if } D_K < 0
\]
and
\[
\left( \frac{\sqrt{|D_K|}}{\pi} \right)^s \Gamma^2 \left( \frac{s}{2} \right) \zeta^k_K(s) = \left( \frac{\sqrt{|D_K|}}{\pi} \right)^{1-s} \Gamma^2 \left( \frac{1-s}{2} \right) \zeta^k_K(1-s) \text{ if } D_K > 0.
\]

Now, suppose first that \( K \) is an imaginary quadratic field. Then \( \zeta^k_K(s) \) satisfies the functional equation
\[
\left( \frac{\sqrt{|D_K|}}{2\pi} \right)^{ks} \Gamma^k(s) \zeta^k_K(s) = \left( \frac{\sqrt{|D_K|}}{2\pi} \right)^{(k-1)s} \Gamma^k(1-s) \zeta^k_K(1-s)
\]
Therefore, keeping the same notations as in Theorem 7, we have \( \Delta(s) \phi(s) = \Delta(1-s) \phi(1-s) \), where \( M = k, \delta = 1, \phi(s) = \left( \frac{\sqrt{|D_K|}}{2\pi} \right)^{ks} \zeta^k_K(s), \Delta(s) = \Gamma^k(s) \) and \( \lambda_n = \left( \frac{\sqrt{|D_K|}}{2\pi} \right)^k \cdot n \).

Thus we see that \( \phi(s) = \zeta^k_K(s) \) is a solution of \( \Delta(s) \phi(s) = \Delta(1-s) \psi(1-s) \). Hence from Theorem 7, our result follows. Similarly, working with the functional equation for real quadratic fields and using Theorem 7, we get the desired result for real quadratic field case.

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