Role of the cosmological constant in the holographic description of the early universe

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Abstract

We investigate the role of the cosmological constant in the holographic description of a radiation-dominated universe $C_2/R^4$ with a positive cosmological constant $\Lambda$. In order to understand the nature of cosmological term, we first study the newtonian cosmology. Here we find two aspects of the cosmological term: entropy ($\Lambda \to S_{\Lambda}$) and energy ($\Lambda \to E_{\Lambda}$). Also we solve the Friedmann equation parametrically to obtain another role. In the presence of the cosmological constant, the solutions are described by the Weierstrass elliptic functions on torus and have modular properties. In this case one may expect to have a two-dimensional Cardy entropy formula but the cosmological constant plays a role of the modular parameter $\tau(C_2, \Lambda)$ of torus. Consequently the entropy concept of the cosmological constant is very suitable for establishing the holographic entropy bounds in the early universe. This contrasts to the role of the cosmological constant as a dark energy in the present universe.

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1 Introduction

Nowadays the cosmological constant plays an important role in several fields: cosmology, astronomy, particle physics and string theory. The reason is twofold. One is that the inflation turned out to be a successful tool to resolve the problems of the hot big bang model [1]. Thanks to the recent observations of the cosmic microwave background anisotropies and large scale structure galaxy surveys, it has become widely accepted by the cosmology community [2]. The idea of primordial inflation is based on the very early universe dominance of vacuum energy density of a hypothetical scalar field, the inflaton. This produces the quasi-de Sitter spacetime [3] and during the slow-roll period, the equation of state can be approximated by the vacuum state as \( p \approx -\rho \) like \( p_\Lambda = \omega \rho_\Lambda, \omega = -1 \) for the cosmological constant \( \Lambda \) [4]. The other is that an accelerating universe (with positive cosmological constant) has recently proposed to interpret the astronomical data of supernova. In this case, the cosmological constant has been identified with a dark exotic form of energy that is smoothly distributed and which contributes 2/3 to the critical density of the present universe\(^1\).

On the other hand we have to build cosmology from the quantum gravity for completeness, but now we are far from it. Although we are lacking for a complete understanding of the quantum gravity, there exists the holographic principle. This principle is mainly based on the idea that for a given volume \( V \), the state of maximal entropy is given by the largest black hole that fits inside \( V \). ’t Hooft and Susskind [6] argued that the microscopic entropy \( S \) associated with the volume \( V \) should be less than the Bekenstein-Hawking entropy: \( S \leq A/4G \) in the units of \( c = \hbar = 1 \) [7]. Here the horizon area \( A \) of a black hole equals the surface area of the boundary of \( V \). That is, if one reconciles quantum mechanics and gravity, the observable degrees of freedom of our three-dimensional universe comes from a two-dimensional surface. Actually holographic area bounds limit the number of physical degrees of freedom in the bulk spacetime.

The implications of the holographic principle for the early universe have been investigated in the literature. Following an earlier work by Fischler and Susskind [8] and works in [9] [10], it was argued that the maximal entropy inside the universe is given by the Hubble entropy. This geometric entropy plays an important role in establishing the cosmological holographic principle in the early universe. Roughly speaking, the total matter entropy should be less than or equal the Bekenstein-Hawking entropy of the Hubble-size black hole (\( \approx H V_H / 4G_{n+1} \)) times the number (\( N_H \approx V / V_H \)) of Hubble regions

\(^1\)Recently, the dark form of energy is classified according to the equation of state : quintessence with \(-1 < \omega < -1/3\), cosmological constant with \( \omega = -1 \), and phantom energy with \( \omega < -1 \) [5].
in the early universe. That is, the Hubble entropy as an upper bound on the total matter entropy is proportional to \(H V/4G_{n+1}\). Furthermore, Verlinde fixed the prefactor as \((n-1)\) and proposed the new holographic bounds Eq.1.3 in a radiation-dominated phase by introducing three entropies \([11]\): Bekenstein-Verlinde entropy \(S_{BV}\), Bekenstein-Hawking entropy \(S_{BH}\), and Hubble entropy \(S_H\). As an example, such a radiation-dominated phase is provided by a conformal field theory (CFT) with a large central charge which is dual to the AdS-black hole \([12]\). In this case it appeared an interesting relationship between the Friedmann equation governing the cosmological evolution and the square root form of entropy-energy relation, called Cardy-Verlinde formula \([13]\). Although the Friedmann equation has a geometric origin and the Cardy-Verlinde formula is designed only for the matter content, it suggested that both may arise from a single fundamental theory. However, this approach remains obscure for a radiation-dominated universe with a positive cosmological constant \([14]\). This is mainly due to the unclear role of the cosmological constant in the holographic description of the early universe.

In this work we will clarify the role of the cosmological term in the early universe. For this purpose we introduce the newtonian cosmology and the parametric solution to the Friedmann equation. We will show that the geometric entropy interpretation of the cosmological term plays an important role in establishing the holographic entropy bound for a radiation-dominated universe with a positive cosmological constant. Finally we wish to point out the different roles of the cosmological constant in the early universe and in the present universe.

The relevant equation is an \((n+1)\)-dimensional Friedmann-Robertson-Walker (FRW) metric with \(k=1\)

\[
ds^2 = -dt^2 + R(t)^2 d\Omega_n^2,
\]

where \(R\) is the scale factor of the universe and \(d\Omega_n^2\) denotes the line element of an \(n\)-dimensional unit sphere. A cosmological evolution is determined by the two Friedmann equations

\[
H^2 = \frac{16\pi G_{n+1} E}{n(n-1)V} - \frac{1}{R^2} + \frac{1}{l_{n+1}^2}, \tag{1.2}
\]

\[
\dot{H} = -\frac{8\pi G_{n+1}}{n-1} \left( \frac{E}{V} + p \right) + \frac{1}{R^2}, \tag{1.3}
\]

where \(H\) represents the Hubble parameter with the definition \(H = \dot{R}/R\) and the overdot stands for derivative with respect to the cosmic time \(t\), \(E\) is the total energy of matter filling the universe, and \(p\) is its pressure. \(V\) is the volume of the universe, \(V = R^n \Omega_n\) with \(\Omega_n\) being the volume of an \(n\)-dimensional unit sphere, and \(G_{n+1}\) is the newtonian
constant in \((n + 1)\) dimensions. Here we assume the equation of state for any matter: \(p = \omega \rho, \rho = E/V\). For our purpose, we include the curvature radius of de Sitter space \(l_{n+1}\) which relates to the cosmological constant via \(1/l_{n+1}^2 = 2\Lambda_{n+1}/n(n-1)\). For \(n = 3\) case, we use the notation of \(G, \Lambda\) instead of \(G_4, \Lambda_4\).

The organization of this paper is as follows. In section 2, we study the newtonian cosmology. Section 3 is devoted to solving the Friedmann equation in a parametrical way to find out the role of the cosmological term. The cosmological holographic bounds for a radiation-dominated universe without/with a positive cosmological constant are discussed in section 4. Finally we discuss our results in section 5.

## 2 Newtonian cosmology

In order to understand the cosmological term \(\Lambda\) in the Friedmann equation, let us study the newtonian cosmology in (3+1) dimensions. Even though the newtonian cosmology is valid for the matter-dominated universe (that is, it is non-relativistic), this approach is useful for understanding the origin of the cosmological term. We propose that the universe consists of a number of galaxies with their mass \(m_i\) and position \(\mathbf{r}_i(t) = \mathbf{r}_i(t)\hat{r}\) as measured from a fixed origin \(O\). Then the kinetic energy of the system \(T\) is given by

\[
T = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{r}_i^2. \tag{2.1}
\]

The total gravitational potential energy \(V\) is

\[
V_g = -G \sum_{i<j}^{n} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \tag{2.2}
\]

Assuming that there exists a cosmological force acting on the \(i\)-th galaxy of the form \(\mathbf{F}_i = \frac{\Lambda}{3} m_i \mathbf{r}_i\) with a constant \(\Lambda\) leads to the cosmological potential energy

\[
V_c = -\frac{\Lambda}{6} \sum_{i=1}^{n} m_i r_i^2. \tag{2.3}
\]

Then the total energy \(E\) of this system is given by

\[
E = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{r}_i^2 - G \sum_{i<j}^{n} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} - \frac{\Lambda}{6} \sum_{i=1}^{n} m_i r_i^2. \tag{2.4}
\]

Suppose that the distribution and motion of the system is known at some fixed epoch \(t = t_0\). By the cosmological principle of homogeneity and isotropy, the radial motion at any time \(t\) is then given by \(r_i(t) = S(t) r_i(t_0)\) where \(S(t)\) is a universal function of time.
which is the same for all galaxies and is called the scale factor. Substituting this into Eq. (2.4) leads to

\[ E = A \dot{S}(t)^2 - \frac{B}{S(t)} - D S(t)^2, \]  

(2.5)

where the coefficients are positive constants given by

\[ A = \frac{1}{2} \sum_{i=1}^{n} m_i [r_i(t_0)]^2, \quad B = G \sum_{i<j} \frac{m_i m_j}{|r_i(t_0) - r_j(t_0)|}, \quad D = \Lambda \frac{6}{6} \sum_{i=1}^{n} m_i [r_i(t_0)]^2 = \frac{\Lambda}{3} A. \]  

(2.6)

This is one form of the cosmological differential equation for the scale factor \( S(t) \). If the universe with \( \Lambda = 0 \) is expanding, \( A \)-term decreases since the total energy remains constant as \( B \)-term decreases. Therefore the expansion must slow down. If \( \Lambda \) is positive, all galaxies experience a cosmic repulsion, pushing them away from the origin out to infinity. If \( \Lambda \) is negative, all galaxies experience a cosmic attraction towards the origin. Introducing a new scale factor with \( R(t) = \mu S(t) \), Eq. (2.5) takes the form

\[ \dot{R}^2 = \frac{C_1}{R} + \frac{\Lambda}{3} R^2 - k, \]  

(2.7)

where the constants \( C_1 \) and \( k \) are defined by \( C_1 = B \mu^3 / A \) and \( k = -\mu^2 E / A \). When \( E = 0 \), \( \mu \) is arbitrary. However, if \( E \neq 0 \), one may choose \( \mu^2 = A / |E| \) so that \( k = 1, 0, -1 \). This equation is exactly the same form of the Friedmann equation of relativistic cosmology. Although there exist ambiguities in determining the cosmological parameters \( C_1 \) and \( k \), one finds that the cosmological term has a slightly different origin from others. The term in the left-hand side of Eq. (2.7) originates from the kinetic energy, the first term (last term) in the right-hand side come from the potential energy (total energy) whereas the second term from the constant cosmological repulsion or attraction. We are interested in the role of the cosmological term in the holographic description of cosmology. As are shown in Eqs. (2.5) and (2.6), a shape of the cosmological term is similar to the kinetic term which can be expressed as the Hubble entropy. On the other hand its nature belongs to the \( B \)-potential term that can be transformed into the energy term. These two pictures will be used for confirming the cosmological holographic bounds for a radiation-dominated universe with a cosmological constant.

\footnote{Similarly, assuming the five-dimensional newton potential \( V_{5g} = -G_5 \sum_{i<j} \frac{m_i m_j}{|r_i - r_j|^2} \), one can find the equation for a radiation-dominated universe in four-dimensional spacetime as \( \dot{R}^2 = \frac{C_2}{R} + \frac{\Lambda}{3} R^2 - k \). Even though it is a non-relativistic approach to obtain a relativistic matter of radiation, this may provide us a hint for interpreting the cosmological term in the Friedmann equation.}
3 Parametric cosmological solutions

There exists another approach to establishing the cosmological holographic principle. In this case, it seems that the cosmological constant plays a role of a parameter in deriving a Cardy formula on torus. In this section, we study this approach to investigate a role of the cosmological term explicitly.

3.1 Case without a cosmological constant

In general we have three cosmological parameters $C, \Lambda, k$. Let us first consider the matter-dominated Friedmann equation with $\Lambda = 0, k = 1$

$$\dot{R}^2 = \frac{C_1}{R} - 1$$

with $C_1 = 8\pi G \rho_{m0}/3$. Here the energy density for a matter-dominated universe is given by $\rho_m = E_m/V = \rho_{m0}/R^3$. Introducing an arc parameter $\eta$ (radians of arc distance on $S^3$), one finds the solution \[15\]

$$R(\eta) = \frac{C_1}{2} (1 - \cos \eta), \quad t(\eta) = \frac{C_1}{2} (\eta - \sin \eta).$$

(3.2)

The range of $\eta$ from start of expansion to end of recontraction is $2\pi$ and the curve of $R(t)$ is cycloid. The limiting form of law of expansion at the early times is given by

$$R \approx \frac{C_1}{4} \eta^2, \quad t \approx \frac{C_1}{12} \eta^3 \rightarrow R \approx (9C_1/4)^{1/3} t^{2/3}$$

(3.3)

which is consistent with the solution to the matter-dominated universe.

Now we consider the radiation-dominated Friedmann equation

$$\dot{R}^2 = \frac{C_2}{R^2} - 1$$

(3.4)

with $C_2 = 8\pi G \rho_{r0}/3$. The energy density for a radiation-dominated universe is given by $\rho_r = E_r/V = \rho_{r0}/R^4$. Introducing the same arc parameter $\eta$, one finds the solution\[3\]

$$R(\eta) = \sqrt{C_2} \sin \eta, \quad t(\eta) = \sqrt{C_2} (1 - \cos \eta).$$

(3.5)

\[3\]This is the same form of the entropy solution $S_H(\eta) = S_{BV} \sin \eta, \quad S_{BH}(\eta) = S_{BV} (1 - \cos \eta)$ to the circular relation of the holographic entropies with $\Lambda = 0$: $S_H^2 + (S_{BV} - S_{BH})^2 = S_{BV}^2$ \[11\]. Here $\eta$ corresponds to the conformal time coordinate via $R d\eta = (n - 1) dt$. $S_{BV}$ is constant, $S_H$ and $S_{BH}$ change with time.
The range of $\eta$ from start of expansion to end of recontraction is $\pi$ and the curve of $R(t)$ is semicircle. The limiting form of law of expansion at the early times are given by

$$R \approx \sqrt{C_2 \eta}, \quad t \approx \frac{\sqrt{C_2}}{2} \eta^2 \quad \rightarrow \quad R \approx 2^{1/2} C_2^{1/4} t^{1/2}$$

(3.6)

which leads to the well-known solution for the radiation-dominated universe. The parametric solutions to the Friedmann equation with $\Lambda = 0$ are determined by the elementary trigonometric functions. But their nature is different: one is cycloid and the other is semicircle.

### 3.2 Case with a cosmological constant

We start with the matter-dominated Friedmann equation with $\Lambda \neq 0, k = 1$

$$\ddot{R} = \frac{C_1}{R} + \frac{\Lambda}{3} R^2 - 1.$$  

(3.7)

Introducing an idea of elliptic curves on torus $T^2$, one finds the solution expressed in terms of the Weierstrass function as

$$R(u, \tau) = \frac{3C_1}{12\wp(u + \epsilon, \tau) + 1}, \quad t(u, \tau) = \sqrt{3 \Lambda \left[ \log \left( \frac{\sigma(u + \epsilon - v_0)}{\sigma(u + \epsilon + v_0)} \right) + 2u \zeta(v_0) \right]},$$

(3.8)

where $\wp(z|\tau), \sigma(z|\tau), \zeta(z|\tau)$ are the Weierstrass’ family of functions: Weierstrass, Weierstrass sigma, Weierstrass zeta functions, respectively. $u(C_1, \Lambda)$ is the complex coordinate and $\tau(C_1, \Lambda)$ is a modular parameter. These two describing a torus are actually functions of both $C_1$ and $\Lambda$. $\epsilon$ is a constant of integration. The Weierstrass function $\wp$ satisfies the equation of an elliptic curve, a Riemann surface of genus 1 (torus)

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

(3.9)

where the cubic invariants are given by

$$g_2 = \frac{1}{12}, \quad g_3 = \frac{1}{216} - \frac{\Lambda C_1^2}{48}.$$  

(3.10)

Also it is a meromorphic modular form of weight 2 under $SL(2, \mathbb{Z})$ transformation,

$$\wp \left( \frac{az + b}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 \wp(z, \tau).$$

(3.11)

Differentiating Eq.(3.9) twice leads to the KdV nonlinear differential equation of soliton physics in a time-independent way

$$\wp(z)^{''''} = 12\wp(z)\wp(z)^{''}.$$  

(3.12)
Now we consider the radiation-dominated Friedmann equation
\[ \dot{R}^2 = \frac{C_2}{R^2} + \frac{\Lambda}{3} R^2 - 1. \] (3.13)

In this case the solution is given by [17]
\[ R(v, \tilde{\tau}) = \sqrt{\frac{3C_2}{12\phi(v + \epsilon, \tilde{\tau}) + 1}}, \quad t(v, \tilde{\tau}) = \frac{1}{2} \int R(v, \tilde{\tau})dv \] (3.14)

where \( v(C_2, \Lambda) \) is the complex coordinate and \( \tilde{\tau}(C_2, \Lambda) \) is a modular parameter. These two describing a new torus are functions of both \( C_2 \) and \( \Lambda \). Here the cubic invariants are given by
\[ g_2 = \frac{1}{12} - \frac{\Lambda C_2}{12}, \quad g_3 = \frac{1}{216} - \frac{\Lambda C_2}{144}. \] (3.15)

From Eqs. (3.10) and (3.15), if \( \Lambda = 0 \), one finds that discriminant is zero (\( \Delta = 0 \)). The solutions to this case are no longer given by the elliptic functions and do not have modular properties. These were previously discussed in Sec.3.1. Assuming a CFT with \((L_0, c)\) on a torus, a partition function with modular parameter \( \tau \) can be introduced as
\[ Z(\tau(C_1, \Lambda)) = \text{Tr} q^{L_0-c/24}, \quad q = e^{2\pi i \tau}, \] (3.16)

where we suppress the \( \tilde{\tau} \)-part for simplicity. Making use of the modular properties of this partition function, we may find the density of states and a two-dimensional Cardy formula for a matter-dominated universe
\[ S_{\text{matter}} = 2\pi \sqrt{\frac{c}{6} \left(L_0 - \frac{c}{24}\right)}. \] (3.17)

Similarly, by assuming a CFT with \((\tilde{L}_0, \tilde{c})\), we expect to have \( S_{\text{radiation}} = \) for a radiation-dominated universe, which is the same form as in Eq. (3.17). These may lead to the chain connections: Friedmann equation \( \rightarrow \) Weierstrass equation \((\phi) \rightarrow \) torus with \( \tau \rightarrow \) CFT partition function \((Z(\tau(C_1, \Lambda))) \rightarrow \) Cardy formula. However, we don’t know exactly what kind of a CFT is suitable for our purpose. Further, the cosmological parameters of \( \Lambda, C_1, C_2 \) are used only for determining the geometry of a torus itself. This presumed mapping from the Friedmann equation on \( R^1 \times S^3 \) into the Cardy formula on torus \((T^2)\) is not clearly justified. Actually we do not obtain a direct definition of the quantities of \( L_0(\tilde{L}_0), c(\tilde{c}) \) appearing in the Cardy formula as a function of \( \Lambda, C_1, C_2 \).

Consequently the existence of a Cardy formula from the solution to the Friedmann equation is not clear and even if it is found, the role of the cosmological constant \( \Lambda \) always remains as a modular parameter of torus. Also we note that the Verlinde’s map from the Friedmann equation to the Cardy-Verlinde formula is based on a CFT with a large central charge on 3-sphere of radius \( R \) \((S^3)\) not torus \((T^2)\).
4 Cosmological holographic bounds

In this section we study two aspects of the cosmological term in the holographic description of the universe: a look of entropy ($\Lambda \rightarrow S_{\Lambda}$) and a look of energy ($\Lambda \rightarrow E_{\Lambda}$). In order to study the first aspect, we introduce four holographic entropies which are necessary for making the holographic description of a radiation-dominated universe with a positive cosmological constant [11, 14]:

- Bekenstein–Verlinde entropy: $S_{BV} = \frac{2\pi}{n} ER$,
- Bekenstein–Hawking entropy: $S_{BH} = (n - 1) \frac{V}{4G_{n+1} R^3}$,
- Hubble entropy: $S_{H} = (n - 1) \frac{HV}{4G_{n+1}}$,
- Cosmological entropy: $S_{\Lambda} = (n - 1) \frac{V}{4G_{n+1} l_{n+1}}$. (4.1)

$S_{BV} \leq S_{BH}$ is supposed to hold for a weakly self-gravitating universe ($HR \leq 1$), while $S_{BV} \geq S_{BH}$ works when the universe is in the strongly self-gravitating phase ($HR \geq 1$). It is interesting to note that for $HR = Hl_{n+1} = 1$, one finds that four entropies are identical: $S_{BV} = S_{BH} = S_{H} = S_{\Lambda}$. In the holographic approach, it is useful to consider $S_{BV}$ not really as an entropy but rather as the energy. And the remaining three belong to the geometric entropy. Then the first Friedmann equation (1.2) can be expressed in terms of the above four entropies as

$$S_{H}^2 + (S_{BV} - S_{BH})^2 = S_{BV}^2 + S_{\Lambda}^2. \quad (4.2)$$

In this section we no longer consider the matter-dominated case because one cannot transform the first Friedmann equation into the cosmological Cardy-Verlinde formula to find the cosmological holographic bounds. This is mainly because its energy-density is given by $\rho_m = \rho_{m0}/R^3$ and the solution $R(t)$ is expressed as a cycloid. In this case, the above entropies are not suitable for representing the cosmological holographic bounds. On the other hand, for a radiation-dominated case, there does not exist any difficulty in representing the Friedmann equation in terms of the above four entropies. In this case we have $\rho_r = \rho_{r0}/R^4$ and $R(t)$ is expressed as the semicircle. As is shown in the

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4Although Bousso argued that a cosmological constant did not carry a genuine matter entropy [18], there is no contradiction to introducing the geometric entropy. $S_{\Lambda}$ was constructed by analogy of the Hubble entropy $S_{H}$. But $S_{\Lambda}$ is closely related to the maximal de Sitter entropy of $S_{dS}$. Explicitly this is given by the Bekenstein-Hawking entropy of the de Sitter cosmological horizon ($(n - 1)V_{dS}/4G_{n+1} l_{n+1} \approx S_{dS} = A/4G_{n+1}$) times the number ($N_{dS} = V/V_{dS}$) of de Sitter regions in the early universe.
footnote 3, the same nature of entropy solution can be obtained by substitution: \( R(\eta) \leftrightarrow S_H(\eta), \ t(\eta) \leftrightarrow S_{BH}(\eta), \ \sqrt{c_2} \leftrightarrow S_{BV}. \) This is why we will study a radiation-dominated universe without/with the cosmological constant.

4.1 Radiation-dominated universe without a cosmological constant

We start with \( \Lambda_{n+1} = 0 \) case because this case gives us a concrete relation. We define a quantity \( E_{BH} \) which corresponds to energy needed to form a universe-size black hole: 
\[
S_{BH} = \frac{(n-1)V}{4G_{n+1}R} = 2\pi E_{BH}R/n.
\]
With this quantity, the Friedmann equations (1.2) and (1.3) can be further cast to the cosmological entropy-energy relation (cosmological Cardy-Verlinde formula) and the cosmological Smarr formula respectively
\[
S_H = \frac{2\pi R}{n} \sqrt{E_{BH}(2E - E_{BH})},
\]
\[
E_{BH} = n(E + pV - T_H S_H),
\]
(4.3)
where the Hubble temperature \( (T_H) \) as the minimum temperature during the strongly gravitating phase is given by \( T_H = -\frac{\dot{H}}{2\pi H}. \) These are another representation of the two Friedmann equations expressed in terms of holographic quantities. On the other hand, we propose that the entropy of a radiation-matter and its Casimir energy can be described by the Cardy-Verlinde formula and the Smarr formula respectively
\[
S = \frac{2\pi R}{n} \sqrt{E_c(2E - E_c)},
\]
\[
E_c = n(E + pV - TS).
\]
(4.4)
The first denotes the entropy-energy relation, where \( S \) is the entropy of a CFT-like radiation living on an \( n \)-dimensional sphere with radius \( R \) \( (S^n) \) and \( E \) is the total energy of the CFT. Further the second represents the relation between a non-extensive part of the total energy (Casimir energy) and thermodynamic quantities. Here \( E_c \) and \( T \) stand for the Casimir energy of the system and the temperature of radiation with \( \omega = 1/3. \) Actually the above equations correspond to thermodynamic relations for the CFT-radiation which are originally independent of the geometric Friedmann equations. Suppose that the entropy of radiation in the FRW universe can be described by the Cardy-Verlinde formula. Then comparing (4.3) with (4.4), one finds that if \( E_{BH} = E_c, \) then \( S_H = S \) and \( T_H = T. \) At this stage we introduce the Hubble bound for entropy, temperature and Casimir energy [11]
\[
S \leq S_H, \ T \geq T_H, \ E_c \leq E_{BH}, \text{ for } HR \geq 1
\]
(4.5)
which shows inequalities between geometric quantities and matter contents. The Hubble entropy bound can be saturated by the entropy of a radiation-matter filling the universe when its Casimir energy \( E_c \) is enough to form a universe-size black hole. If this happens, equations (4.3) and (4.4) coincide exactly. This implies that the first Friedmann equation somehow knows the entropy formula of a square-root form for a radiation-matter filling the universe. As an example, one considers a moving brane universe in the background of the five-dimensional Schwarzschild-AdS black hole. Savonije and Verlinde [12] found that when this brane crosses the black hole horizon, the Hubble entropy bound is saturated by the entropy of black hole (=the entropy of the CFT-radiation). At this moment the Hubble temperature and energy \((T, E_{BH})\) equal to the temperature and Casimir energy \((T, E_c)\) of the CFT-radiation dual to the AdS black hole respectively.

### 4.2 Radiation-dominated universe with a positive cosmological constant: a look of entropy

For a radiation-dominated universe with \( \Lambda_{n+1} \neq 0 \), we have to introduce the cosmological D-entropy \( S_D \) and D-temperature \( T_D \) as [14]

\[
S_D = \sqrt{|S_H^2 - S_{\Lambda}^2|}, \quad T_D = -\frac{\dot{H}}{2\pi \sqrt{H^2 - 1/l^2_{n+1}}}. \tag{4.6}
\]

We note that the cosmological D-entropy \( S_D \) is constructed by analogy of the static D-bound\(^5\). \( T_D \) is the lower bound of the temperature during the strongly gravitating phase with a positive cosmological constant. Here we insist that the first three entropies appeared in Eq. (4.1) are still applicable for describing the radiation-dominated universe with \( \Lambda_{n+1} \neq 0 \) without any modification. As a check point, one can recover the radiation-dominated universe without a cosmological constant, as \( \Lambda_{n+1} \to 0 \):

\[
S_{\Lambda} \to 0, \quad S_D \to S_H, \quad T_D \to T_H. \tag{4.7}
\]

Using \( S_D \), one finds from Eq. (4.2) the entropy relation

\[
S_D^2 + (S_{BV} - S_{BH})^2 = S_{BV}^2, \tag{4.8}
\]

which is the same relation for \( \Lambda_{n+1} = 0 \) case in the footnote 3. Hence \( S_{BV} \) is constant, \( S_D = S_{BV} \sin \eta \) and \( S_{BH} = S_{BV} (1 - \cos \eta) \) change with time. Then Eqs. (1.3) and (4.3) can

\(^5\)Suppose \( M \) is asymptotically de Sitter space. Then the entropy of matter in \( M \) is bounded by the difference (D) between the entropy of exact de Sitter space and the Bekenstein-Hawking entropy of the apparent cosmological horizon in \( M \) of asymptotically de Sitter space.
be rewritten as the cosmological Cardy-Verlinde and cosmological Smarr formulas

\[
S_D = \frac{2\pi R}{n} \sqrt{E_{BH}(2E - E_{BH})}, \\
E_{BH} = n(E + pV - T_D S_D),
\]

while the entropy and Casimir energy of the CFT-radiation can be expressed as

\[
S = \frac{2\pi R}{n} \sqrt{E_c(2E - E_c)}, \\
E_c = n(E + pV - TS).
\]

As is shown in Eq.(4.7), the cosmological D-entropy plays the same role as the Hubble entropy does in the case without a positive cosmological constant. That is, it is also a geometric entropy during the strongly gravitating phase with a positive cosmological constant.

Now we are in a position to see how the entropy bounds are changed here. The first Friedmann equation in Eq.(1.2) can be rewritten as

\[
(HR)^2 - \frac{R^2}{l_{n+1}^2} = 2 \frac{S_{BV}}{S_{BH}} - 1.
\]

Using this relation, in case of \(\Lambda_{n+1} = 0\), one finds that \(HR \geq 1 \rightarrow S_{BV} \geq S_{BH}\), while \(HR \leq 1 \rightarrow S_{BV} \leq S_{BH}\). Hence this leads to the Hubble entropy bound of \(S \leq S_H\) for \(HR \geq 1\), whereas the Bekenstein-Verlinde entropy bound of \(S \leq S_{BV}\) for \(HR \leq 1\). For \(\Lambda_{n+1} \neq 0\), it is shown that \((HR)^2 - \frac{R^2}{l_{n+1}^2} \geq 1 \rightarrow S_{BV} \geq S_{BH}\), while \((HR)^2 - \frac{R^2}{l_{n+1}^2} \leq 1 \rightarrow S_{BV} \leq S_{BH}\). Thus this leads to the cosmological D-bound for entropy, temperature, and Casimir energy for the strongly gravitating phase:

\[
S \leq S_D, \quad T \geq T_D, \quad E_c \leq E_{BH}, \quad \text{for} \quad HR \geq \sqrt{1 + \frac{R^2}{l_{n+1}^2}},
\]

whereas the Bekenstein-Verlinde entropy bound is found for the weakly gravitating phase:

\[
S \leq S_{BV}, \quad \text{for} \quad HR \leq \sqrt{1 + \frac{R^2}{l_{n+1}^2}}.
\]

When the cosmological D-entropy bound is saturated by the entropy \(S\) of a CFT-radiation, equations (4.9) and (4.10) coincide, just like the case without the cosmological constant. We note that one cannot find the relation of \(S_D = S_{BV} = S_{BH}\) for \(HR = 1\), unless \(\Lambda_{n+1} = 0\).

12
4.3 Radiation-dominated universe with a positive cosmological constant: a look of energy

If the cosmological term in Eq. (1.2) takes a closer look of the potential energy term, then we can incorporate this into the Bekenstein-Verlinde entropy. Noting that the Bekenstein-Verlinde entropy $S_{BV}$ is really considered as an energy, the cosmological term appears in an additive form of energy in the cosmological Cardy-Verlinde formula Eq. (4.3) without introducing $S_D$. Introducing the corresponding energy $E_\Lambda = \frac{\Lambda n}{8\pi G_n+1} \frac{E_{\Lambda n}}{V}$ (equivalently, the last term in Eq. (1.2) is given by $\frac{1}{t_{n+1}} = \frac{16\pi G_{n+1}}{n(n-1)} E_{\Lambda n}$), the Friedmann equations take the form instead of Eq. (4.8) [19]

$$S_H = \frac{2\pi R}{n} \sqrt{E_{BH} \left[2(E + E_\Lambda) - E_{BH}\right]},$$
$$E_{BH} = n(E + pV - T_H S_H).$$

(4.14)

On the other hand, the entropy and Casimir energy of the CFT-radiation remains unchanged as

$$S = \frac{2\pi R}{n} \sqrt{E_c \left(2E - E_c\right)},$$
$$E_c = n(E + pV - TS).$$

(4.15)

The above two forms do not resemble each other, because on the CFT side, it is hard to incorporate the bulk cosmological term into the Cardy-Verlinde formula. In terms of naive power counting, the vacuum energy (cosmological term) corresponds to a relevant operator in CFT. And this leads to power divergences. In this case it is not easy to obtain the cosmological holographic bounds like Eqs. (4.5) and (4.12). Furthermore, introducing a related entropy $\bar{S}_\Lambda = \frac{2\pi R}{n} E_\Lambda$ like $S_{BH} = \frac{2\pi R}{n} E_{BH}$, the entropy relation in Eq. (4.2) is changed into an ugly form as

$$S_H^2 + (S_{BV} + \bar{S}_\Lambda - S_{BH})^2 = (S_{BV} + \bar{S}_\Lambda)^2.$$  

(4.16)

Here $S_{BV} + \bar{S}_\Lambda$ does not remain constant during the cosmological evolution unlike $\Lambda_{n+1} = 0$ case shown in the footnote 3 and Eq. (4.8) for $\Lambda_{n+1} \neq 0$ case.

5 Discussion

In this work we discuss the role of the cosmological constant in the early universe. Especially for the a radiation-dominated universe $\rho_r = \rho_{r0}/R^4$ with a positive cosmological constant $\Lambda$, we confirm the cosmological holographic bounds Eq. (1.2) if the cosmological
constant is considered as an entropy ($\Lambda \rightarrow S_\Lambda$). Here the entropy concept originates from the Hubble entropy $S_H$ which plays a crucial role in establishing the cosmological holographic principle in the radiation-dominated universe. We note here that the two entropies $S_H$ and $S_\Lambda$ are regarded as the geometric entropy but not the genuine matter entropy like $S$ for a CFT-radiation matter.

Taking a genuine view of energy ($\Lambda \rightarrow E_\Lambda$), one cannot establish the cosmological holographic bounds in the early universe. For the matter-dominated universe without/with a positive cosmological constant, we cannot achieve the cosmological holographic bounds because of its energy density nature with $\rho_m = \rho_{m0}/R^3$. Further, for the pure de Sitter case without any radiation, one cannot derive the cosmological holographic bounds \[20\].

Finally, considering the cosmological constant term as a candidate of dark energy in the present universe, its role of the geometric entropy in the holographic description of the early universe emerges as an opposite one. If this view is correct, our work implies a duality of the cosmological constant: (geometric) entropy in the early universe and (dark) energy in the present universe.

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