Multiple resource demands and viability in multiplex networks

Byungjoon Min\textsuperscript{1} and Kwang-Il Goh\textsuperscript{1,}\textsuperscript{2}

\textsuperscript{1}Department of Physics, Korea University, Seoul 136-713, Korea

(Dated: January 9, 2014)

Many complex systems demand manifold resources to be supplied from distinct channels to function properly, i.e., water, gas, and electricity for a city. Here, we study a model for viability of such systems demanding more than one type of vital resources produced and distributed by source nodes in multiplex networks. We found a rich variety of behaviors such as discontinuity, bistability, and hysteresis in the fraction of viable nodes with respect to the density of networks and the fraction of source nodes. Our result suggests that viability in multiplex networks is not only exposed to the risk of abrupt collapse but also suffers excessive complication in recovery.

Our life in modern society relies on interrelated infrastructure networks including water supply networks, gas supply networks, and power grid systems \cite{1}. For this reason, the network of connections between public utilities and consumers has been studied for a long time \cite{2}. Over the last decade, there have been a series of attempts to understand robustness of networks including infrastructure networks using their connectivity properties in physics and network science communities \cite{3,4,5}. Such studies on individual networks, however, could not fully assess the vulnerability of real infrastructure systems because many real systems are multiplex and interdependent \cite{6,7,8,9}. Moreover these systems often demand multiple classes of resources to be supplied through different layers of networks \cite{10}. As a result, the vulnerability of interdependent infrastructure systems can be far beyond that expected in an individual network because the damage in one layer is not localized therein but able to provoke avalanche collapse leading to an abrupt breakdown of the whole system \cite{11,12,13}.

Most previous studies on interdependent and multiplex networks have assumed that mutual connectivity, that is the simultaneous connectivities through each and every network layer, is the requirement for an active node \cite{14,15}. Such condition is plausible for the networks in which connections by itself can provide function, such as road networks and Internet. This condition, however, is not sufficient for the systems in which the source nodes generate products and distribute them along the links to the neighbors, such as the power grid and the water supply networks. For such kind of systems, simultaneous connectivities with source nodes such as power plants in power grid and water sources in water supply networks through a series of functioning nodes are essential for the proper functioning, or to be viable as we will call it. In this paper, we introduce and study a simple model of viability of multiplex systems requiring supports of more than one type of vital resources produced from a fixed set of source nodes. By presenting algorithms to identify the set of viable nodes and the analytic solutions to the problem, we illustrate novel features of system behaviors characterized by the multiple resource demands.

We found that the final fraction of viable nodes exhibits discontinuous jumps, bistability, and hysteresis with the density of links and source nodes. The discontinuous jumps indicate a potential danger of abrupt collapse of the system similar to the previous study of cascading failures in interdependent networks \cite{16}. Furthermore, the strong hysteresis with respect to the link density suggests that after collapse, more addition of links compared with the link density before collapse is required to restore viability of networks as the previous level. We also examine the effect of the number of the resource layers and find the expression for the critical point at which the discontinuity disappears for \( n \)-layer Erdős-Rényi (ER) multiplex networks. In addition, our model can also be interpreted as a unifying model of mutual percolation \cite{17,18,19,20} and cooperative epidemics on multiplex networks \cite{21,22}.

Consider a network with \( n \)-multiple layers, where each layer of the network corresponds to a certain infrastructural network. We assume that a given fraction of source nodes, \( \rho \), generates and distributes resources essential to be viable. Another key assumption of this model is that only viable nodes can function properly and transmit resources further to their connected neighbors. Then, a node is viable only if it can reach, via the viable nodes, to a source node in each and every layer. We present two algorithms to identify the set of viable nodes, called the cascade of activations (CA) and deactivations (CD).

An example of the CA and the CD algorithms applied to a multiplex network with two layers is illustrated in Fig. 1. For the CA algorithm, initially all nodes except source nodes are unviable. Each step, unviable nodes that are linked to viable nodes through each and every layer (colored green in Fig. 1) are activated to become viable. This cascade of activations continues until no nodes newly become viable. For the CD algorithm, initially all nodes including source nodes are viable. Each step, the nodes that do not reach the source nodes in any layers (colored green in Fig. 1) are deactivated to become unviable. This cascade of deactivations continues until there are no nodes to deactivate. In the limit \( N \to \infty \), the final fraction of viable nodes \( V \) is called the viability of the system. Note that the viability \( V \) obtained from the
CA and the CD algorithms can be in general different from each other: the viable nodes for the CA are always viable for the CD as well, but not \textit{vice versa}.

One can also compute the viability analytically for a locally-tree-like structure, as follows. To calculate viability, we first consider the probability \( w_i \) that a randomly chosen node reached by following an \( i \)-type link is not viable. Each node has \( k = (k_1, \ldots, k_n) \) degrees over different layers drawn from the joint degree distribution \( p(k) \). Given the initial fraction of randomly distributed source nodes \( \rho \), on locally-tree like structures \( u_i \) can be expressed as the self-consistency equations,

\[
1 - u_i = \rho + (1 - \rho) \sum_k \frac{k_i p(k)}{z_i} (1 - u_i^{k_i-1}) \prod_{j=1}^{n'} (1 - u_j^{k_j}),
\]

where \( z_i \) is the mean degree of the \( i \)-layer network, and the prime denotes the product with the index \( i \) excluded. The first term is the probability that a randomly chosen node is a source node. And the second term is the probability that a node is connected with viable nodes through each type of links. The mean final fraction of viable nodes \( V \), the viability, is the same with the probability that a randomly chosen node is viable. Therefore, \( V \) can be similarly expressed as

\[
V = \rho + (1 - \rho) \sum_k p(k) \prod_i (1 - u_i^{k_i}).
\]

By solving Eqs. (1–2) with given \( p(k) \) and \( \rho \), one obtains the viability \( V \).

We illustrate the basic features of the model with a specific example of a randomly-coupled multiplex network with two Erdős-Rényi (ER) layers. For simplicity we take the mean degrees of two layers to be the same, denoted as \( z \), and then Eqs. (1) and (2) reduce as

\[
V = \rho + (1 - \rho)(1 - e^{-zV})^2.
\]

By solving Eq. (3) graphically, \( V \) can be obtained for given \( \rho \) and \( z \). The typical behaviors of the graphical solution of Eq. (3) with \( \rho = 0.04 \) and \( z = 2, 2.5, \) and 4 are shown in Fig. 2. There is one stable solution for \( z = 2 \) and 4, corresponding to low (\( V \approx 0.05 \)) and high viability (\( V \approx 0.96 \)), respectively. In these cases, the outcomes of the CA and the CD algorithms coincide, being identical to \( V \). On the other hand, when \( z = 2.5 \), there are two stable solutions, \( V \approx 0.06 \) and 0.68, corresponding to the CA and the CD cases, respectively. Such a bistability implies that the viability of networks is dependent on the initial fraction of viable nodes. The bistability region is bounded by the saddle node bifurcations at \( z_{CA} = 2.35 \) and \( z_{CD} = 2.88 \) for \( \rho = 0.04 \).

In the \((\rho, z)\) phase diagram (Fig. 3a), the low and high viability phases are separated by two lines, indicating the loci of saddle node bifurcations. Defining \( h(V) = V - \rho - (1 - \rho)(1 - e^{-zV})^2 \), the locations of saddle node bifurcations can be determined by imposing the conditions, \( h(R) = h'(R) = 0 \). The solid line corresponds to the points at which the viability for the CA algorithm undergoes a discontinuous change, and the dashed line indicates that for the CD algorithm. Between two lines (hysteresis region), there are two possible stable solutions, and \( V \) is determined by its initial value. For example, for the CA algorithm, \( V \) keeps to be in low viable state until the abrupt jump at the solid line. For the CD case, high viability sustains until abrupt collapse at the dashed line. Two lines merge at the critical point located at \( (\rho_c, z_c) = \left( \frac{3}{2} \log 2 + \frac{3}{2} \right) \) derived by the conditions, \( h(V) = h'(V) = h''(V) = 0 \). Above \( \rho_c \), as \( z \) increases, the viability changes gradually without dis-
V \rho \Phi (V)

FIG. 3. (a) Phase diagram of viability with respect to \rho and z. Solid and dashed lines indicate the locations of a discontinuous jump for the CA and the CD algorithm, respectively. Filled circle at (\rho_c, z_c) = (0.04, 2), (c) (0.04, 2.5), (d) (0.04, 4), and (e) (\rho_c, z_c). Local minima which give the viability V are denoted by filled circles.

continuity.

For more intuitive understanding of the behavior of V, we define a potential function \Phi (V) = -\int_0^V h(x)dx. For the randomly coupled two ER networks, the potential function is obtained as

\Phi (V) = -V + \frac{V^2}{2} - \frac{1 - \rho}{2z} (4e^{-zV} - e^{-2zV} - 3).

(4)

V descends along \Phi (V) from its initial value, and so finally remains at local minima which are stable solutions of Eq. (3) (Fig. 3b-e). Especially, in the hysteresis region (Fig. 3c), two local minima in double wells correspond to the CA and the CD algorithm, respectively, and the local maximum corresponds to an unstable fixed point.

The bistability implies a hysteresis in viability of multiplex networks (Fig. 4). In order to demonstrate the hysteresis explicitly, let us suppose the following scenario of a sequence of systemic collapse and subsequent recovery of viability. Initially the system is in high viability state with well-established networks. As z decreases by random failures of links, V abruptly collapses at z_{CD}. After collapse, if we try to restore viability to the level before the collapse, a far more link addition is needed up to z_{CA} which is much larger than the point of collapse z_{CD}. Thus, the bistability induces the hysteresis with z which hinders and complicates recovery from the low viability state. Our result shows that multiple resource demands produce not only a potential danger of abrupt collapse but also severe complication in recovery. Also, note that the hysteresis does not occur in single networks, n = 1.

To manage the potential risk of collapse and complication in recovery, there can be two possible strategies: More suppliers and denser networking. First, as the number of source nodes increases, meaning that \rho increases, the gap between z_{CA} and z_{CD} decreases and eventually disappears for above \rho_c as shown in Fig. 5a. Therefore, by placing enough source nodes, i.e., \rho > \rho_c, one can avoid both discontinuity and hysteresis. Another way to maintain high viability is dense networking. When z_c < z < z_{MP} where z_c = \log 2 + 3/2 \approx 2.193 and z_{MP} = 2.455 \ldots, the threshold of mutual percolation for 2-layer ER networks [9, 13], V as a function of \rho shows discontinuous jumps and the hysteresis (Fig. 5b). In case z > z_{MP}, however, the high viability corresponding to the CD algorithm is always guaranteed as long as \rho \neq 0. Thus, the network can maintain a highly viable state for z > z_{MP}, even with extremely small density of sources.

It is straightforward to extend to a n-resource demand problem on n-layer multiplex networks. For example, in n-layer ER networks, V can be obtained by

\begin{equation}
V = \rho + (1 - \rho)(1 - e^{-zV})^n.
\end{equation}

The results are qualitatively the same with 2-layer cases.
displaying discontinuity and hysteresis. The critical point is obtained as $(\rho_c, z_c) = (\frac{C_2 \log n}{C_1 + C_2 \log n}, \log n + \frac{C_1}{C_2})$, where $C_1 = n^n - (n-1)^n$ and $C_2 = n(n-1)^{n-1}$. $\rho_c$ and $z_c$ increase with $n$ and $(\rho_c, z_c) \to (1, \infty)$ in the limit $n \to \infty$. Therefore, as $n$ increases, the higher $\rho$ is needed to avoid discontinuity and hysteresis, increasing the system’s susceptibility to the abrupt collapse and excessive recovery. We can also make the problem more realistic that each source can supply only one kind of resource, rather than all resources. Defining $\rho_i$ as the fraction of $i$-type source nodes randomly distributed over the network, for the 2-layer ER networks, $u_i$ and $V$ can be obtained by solving the following equations similarly Eq. (1) and (2),

$$1 - u_i = \sum_k \frac{k_i p(\vec{k})}{z_i} \left[(1 - \rho_i)(1 - \rho_j)(1 - u_i^{k_i-1})(1 - u_j^{k_j})\right]$$

$$+ \rho_i(1 - \rho_j)(1 - u_j^{k_j}) + \rho_j(1 - \rho_i)(1 - u_i^{k_i-1}) + \rho_i \rho_j,$$

(6)

$$V = \sum_k p(\vec{k}) \left[(1 - \rho_i)(1 - \rho_j)(1 - u_i^{k_i})(1 - u_j^{k_j})\right]$$

$$+ \rho_i(1 - \rho_j)(1 - u_j^{k_j}) + \rho_j(1 - \rho_i)(1 - u_i^{k_i}) + \rho_i \rho_j.$$

(7)

For the case of equal $\rho_i$’s, $V$ can be simply expressed as

$$V = [\rho + (1 - \rho)(1 - e^{-zV})]^2.$$  

(8)

The results are again qualitatively the same but with different critical point $(\rho_c, z_c) = (0.347\ldots, 2.702\ldots)$, suggesting the increased potential risk of abrupt collapse on a broader range of system parameters.

Finally, worthwhile to note is the relation between our model and other dynamics including mutual percolation, bootstrap percolation, and cooperative epidemics. The mutual percolation (equivalently cascading failures in interdependent networks) [9, 13] corresponds to the CD algorithm of our model in the limit $\rho \to 0$. In this aspect, our model can be regarded as a generalization of mutual percolation on multiplex networks. On the other hand, the CA algorithm can be regarded as a variant of bootstrap percolation applicable to multiplex networks [15, 19]. From the perspective of epidemic spreading and social contagion, the condition of viability can be interpreted as cooperative infection [17, 18] and contagion [20] on multiplex networks. Therefore, our simple model unifies various percolation and epidemic spreading on multiplex networks.

To summarize, in this paper we have introduced and studied a simple model to assess viability of multiplex systems demanding connections to resources through multiple different channels. The model exhibits rich phenomenology such as a hysteresis and discontinuous jumps in viability with the mean degree and the fraction of source nodes, and the critical point for general $n$-layer of ER networks was obtained. Our result warns that recovery processes after systemic collapse can be exceedingly costly due to the hysteresis in viability in multiplex systems.

This work was supported by Basic Science Research Program through the NRF grant funded by MSIP (No. 2011-0014191).

* kgoh@korea.ac.kr

[1] S. M. Rinaldi, J. P. Peerenboom, and T. K. Kelly, IEEE Contr. Syst. Mag. 21, 11 (2001).
[2] D. E. Kullmann, Mathematics Magazine, 52, 299 (1979).
[3] R. Albert, H. Jeong, and A.-L. Barabási, Nature 406, 378 (2000).
[4] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. 85, 5468 (2000).
[5] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. 85, 4626 (2000).
[6] P. Holme, B. J. Kim, C. N. Yoon, and S. K. Han, Phys. Rev. E 65, 056109 (2002).
[7] V. Latora and M. Marchiori, Phys. Rev. E 71, 015103 (2005).
[8] V. Rosato, L. Issacharoff, F. Tiriticco, S. Meloni, S. Porcellinis, and R. Setola, Int. J. Crit. Infrastruct. 4, 63 (2008).
[9] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, Nature 464, 7291 (2010).
[10] C. D. Brummitt, R. M. D’Souza, and E. A. Leicht, Proc. Natl. Acad. Sci. USA 109, E608 (2012).
[11] W. Li, A. Bashan, S. V. Buldyrev, H. E. Stanley, and S. Havlin, Phys. Rev. Lett. 108, 228702 (2012).
[12] A. Bashan, Y. Berezin, S. V. Buldyrev, and S. Havlin, Nat. Phys. 9, 667 (2013).
[13] S.-W. Son, G. Bizhani, C. Christensen, P. Grassberger, and M. Paczuski, EPL 97, 16006 (2012).
[14] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett. 109, 248701 (2012).
[15] G. J. Baxter, S. N. Dorogovtsev, J. F. F. Mendes, and D. Cellai, [arXiv:1312.3814] (2013).
[16] H. K. Janssen, M. Müller, and O. Stenull, Phys. Rev. E 70 026114 (2004).
[17] G. Bizhani, M. Paczuski, and P. Grassberger, Phys. Rev. E 86, 011128 (2012).
[18] L. Chen, F. Ghanbarnejad, W. Cai, and P. Grassberger, EPL 104, 50001 (2013).
[19] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. E 82, 011103 (2010).
[20] C. D. Brummitt, K.-M. Lee, and K.-I. Goh, Phys. Rev. E 85, 045102(R) (2012).