The Hamiltonian of Einstein affine-metric formulation of General Relativity

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Abstract

It is shown that the Hamiltonian of the Einstein affine-metric (first order) formulation of General Relativity (GR) leads to a constraint structure that allows the restoration of its unique gauge invariance, four-diffeomorphism, without the need of any field dependent redefinition of gauge parameters as is the case for the second order formulation. In the second order formulation of ADM gravity the need for such a redefinition is the result of the non-canonical change of variables [arXiv: 0809.0097]. For the first order formulation, the necessity of such a redefinition “to correspond to diffeomorphism invariance” (reported by Ghalati [arXiv: 0901.3344]) is just an artifact of using the Henneaux-Teitelboim-Zanelli ansatz [Nucl. Phys. B 332 (1990) 169], which is sensitive to the choice of linear combination of tertiary constraints. This ansatz cannot be used as an algorithm for finding a gauge invariance, which is a unique property of a physical system, and it should not be affected by different choices of linear combinations of non-primary first class constraints. The algorithm of Castellani [Ann. Phys. 143 (1982) 357] is free from such a deficiency and it leads directly to four-diffeomorphism invariance for first, as well as for second order Hamiltonian formulations of GR. The distinct role of primary first class constraints, the effect of considering different linear combinations of constraints, the canonical transformations of phase-space variables, and their interplay are discussed in some detail for Hamiltonians of the second and first order formulations of metric GR. The first order formulation of Einstein-Cartan theory, which is the classical background of Loop Quantum Gravity, is also discussed.

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I. INTRODUCTION

We reconsider the Hamiltonian of General Relativity (GR) by using its equivalent first order form, the affine-metric formulation of Einstein \[1\]. The reason for returning to this old and apparently solved problem is twofold.

Firstly, in the literature this problem is claimed to have been solved 50 years ago by Arnowitt, Deser and Misner (ADM) \[2\]. The comparison of the ADM Hamiltonian with the Dirac Hamiltonian \[3\] shows some similarities (not equivalence) \[4\]; but Dirac based his derivation on second order, metric, GR. Preliminary results on the GR Hamiltonian for an equivalent first order formulation, based on slightly different but equivalent set of variables (a linear combination of affine connections), leads to a different conclusion; in particular, the necessity to have tertiary constraints \[5, 6\], contrary to the ADM treatment of the same problem. Recently, and for the first time, the Dirac analysis of the first order formulation was completed by Ghalati and McKeon \[7\] with an explicit demonstration of the closure of the Dirac procedure \[8\] and with the explicit form of the tertiary constraints given. This differs from the ADM Hamiltonian formulation and the reason for this discrepancy lies in the solving of the first class constraints, as indicated and discussed in \[5, 6, 7\]. According to the Dirac procedure, only second class constraints can be solved and the Poisson brackets of the remaining phase-space variables might be modified (Dirac brackets) \[9, 10\].

Secondly, the Dirac analysis for systems with first class constraints cannot be considered complete without the restoration of gauge transformations that, in accordance with the Dirac conjecture \[8\], needs all first class constraints. Some steps of such a restoration and partial transformation for one phase-space variable, \(h^{00} = \sqrt{-g}g^{00}\), was recently reported by Ghalati \[11\] based on earlier obtained in \[7\] first class constraints. In \[11\] the transformation of \(h^{00}\) was found using the approach proposed by Henneaux, Teitelboim and Zanelli (HTZ) \[12\]. This transformation is different from four-diffeomorphism and the author of \[11\] concluded that a field dependent redefinition of gauge parameters is needed for the derived transformation “to correspond to diffeomorphism invariance” \[11\]. This result is puzzling because in the Hamiltonian formulation of the second order GR, the necessity for such a field dependent redefinition is the result of a non-canonical change of variables \[4\]. Without such changes, the four-diffeomorphism follows without any redefinition of gauge parameters as was demonstrated for the Dirac formulation \[3\], as well as for the oldest Hamiltonian
formulation of GR due to Pirani, Schild and Skinner (PSS) \cite{13} in \cite{4,14}. The equivalence of two formulations was demonstrated in \cite{15}. In formulation of \cite{7}, the redefinitions of phase-space variables that might affect the result obtained in \cite{11} are canonical, so one expects the complete and direct restoration of four-diffeomorphism without the need for any field dependent redefinition of gauge parameters, as is found in the second order formulation. The gauge invariance is a unique characteristic of a theory and equivalent second and first order formulations should give the same gauge invariance. The only difference in the second and first order formulations (considered in \cite{4,14} and \cite{11}, respectively) lies in the methods of the restoration of gauge invariance that were used. For the Hamiltonian of the second order GR, the Castellani algorithm \cite{16} was used for both Dirac \cite{4} and PSS \cite{14} formulations; whereas in \cite{11} the HTZ ansatz was employed.

We want to clarify these two discrepancies simultaneously; and this dictates our choice of first order formulation. We perform our analysis for the standard and more familiar affine-metric formulation due to Einstein \cite{1} because this formulation was the starting point in the ADM analysis \cite{2}. The affine-metric formulation, and one used in \cite{5,6,7,11}, are both equivalent to the second order formulation. The equivalence of affine-metric and metric GR was demonstrated by Einstein \cite{1} and for a different variable which is a linear combination of affine connections used in \cite{7}, the equivalence was explicitly demonstrated in Appendix A of \cite{5}. Both first order formulations lead to similar Hamiltonians, as will become clear in the course of the calculations for the affine-metric formulation presented here and from the comparison of our results with ones obtained in \cite{7}. Such a choice of variables (affine connections or a linear combination of affine connections) cannot be responsible for appearance of a different gauge invariance. There are some purely technical advantages in the parts of the calculations for one formulation over another that we will comment on in the course of our calculations; but they are not crucial, and neither formulation gives any overall advantage in calculation efficiency. The only “advantages” of affine-metric formulation that we want to mention, is the manifest covariant structure of primary first class constraints and possibility of performing a direct comparison with known transformations for affine connections.

We perform this analysis for all dimensions higher than two as a specialization to the four dimensional case does not have any advantages or peculiarities. The standard expression “four-diffeomorphism” that we will use is equally well applied to all dimensions. The main
goal of our article is a thorough analysis of all the steps of calculation and, in particular, of the canonicity of all changes of phase-space variables that are used. All of these “technicalities” are our main concern. Our article is not an essay on the Hamiltonian formulation, but it is the Hamiltonian formulation itself, with all the steps of calculations and with sufficient details that anyone can repeat or check our derivations. So this article is the detailed proof that the Hamiltonian of the affine-metric GR and its constraints lead directly to four-diffeomorphism invariance without any redefinition of gauge parameters. This is exactly as it was shown in [4, 14] for the second order formulation of GR, if we use exactly the same algorithm of restoration of gauge invariance [16].

The plan of the paper is as follows. In the next Section, by using Dirac procedure, the Hamiltonian of the affine-metric formulation is obtained by performing the Hamiltonian reduction, i.e. the elimination of the phase-space variables associated with second class constraints. In Section 3 closure of the Dirac procedure is demonstrated for a particular choice of tertiary constraints and the algebra of Poisson brackets (PBs) among tertiary constraints is compared with a similar algebra of secondary constraints of the second order formulation. In Section 4 the effect of different choices of linear combinations of non-primary\(^1\) first class constraints are considered and their interplay with canonical transformations of phase-space variables is discussed. Section 5 provides arguments for a special role of primary first class constraints in the Hamiltonian formulation of gauge invariant theories. The examples from the Hamiltonian formulation of the second order metric GR as well as the Hamiltonian formulation of the first order Einstein-Cartan theory are discussed and the effect of unjustified manipulations with primary constraints is illustrated by an example of the Hamiltonian of Loop Quantum Gravity (LQG). In Section 6 using the Castellani algorithm we restore as in [11] the partial transformations, but for all the phase-space variables of the reduced Hamiltonian and we obtain all terms with temporal derivatives of fields and gauge parameters in the gauge transformations that coincide with four-diffeomorphism. We show that different choices of combinations of non-primary constraints do not affect the gauge transformations if we use the correct method to restore gauge invariance. In Section 7 we demonstrate the sensitivity of HTZ ansatz to a choice of tertiary constraints, contrary to the Castellani al-

\(^1\) We call non-primary constraints all secondary, tertiary constraints, etc. In the literature they are sometimes all called secondary; but because we consider two particular formulations with secondary and tertiary (not some general case) this terminology seems to be preferable to avoid confusion.
gorithm. Such an ambiguity and dependence of gauge invariance on a choice of a linear combination of non-primary first class constraints is in contradiction to the uniqueness of gauge invariance, which is an important property of a theory. This explains the origin of the puzzling result reported in [11]. The reason for the failure of the HTZ ansatz is discussed, which is also related to a special role of the primary constraints - the true Masters of Hamiltonians for gauge invariant theories. In the Appendix A the details of the solution for the secondary second class constraints that were used in Section 2 to find the reduced Hamiltonian is given.

II. THE HAMILTONIAN

We start from the first order, affine-metric, Einstein action [1]

\[ S_E (g^{\alpha\beta}, \Gamma_\alpha^\lambda) = \int L (g^{\alpha\beta}, \Gamma_\alpha^\lambda) \, dx^D \]  

with the Lagrange density function

\[ L (g^{\alpha\beta}, \Gamma_\alpha^\lambda) = -\sqrt{-g} g^{\alpha\beta} (\Gamma_\alpha^\lambda, \lambda - \Gamma_\alpha^{\lambda, \beta} + \Gamma_\sigma^\lambda \Gamma_\sigma^\beta - \Gamma_\alpha^\lambda \Gamma_\sigma^\beta \lambda) \]  

where the metric, \( g^{\mu\nu} \), and affine connection, \( \Gamma_\alpha^\lambda \), are treated as independent variables, \( g = \det (g_{\mu\nu}) \), and \( D \) is the dimension of spacetime. Greek letters are used for “spacetime” indices \( (\mu = 0, 1, ..., D - 1) \) and Latin letters for “space” indices \( (k = 1, ..., D - 1) \).

The first step in passing to the Hamiltonian formulation is the explicit separation of terms with temporal derivatives (“kinetic” part of \( L \)). For (2) we obtain

\[ L_{\text{kin}} = -\sqrt{-g} g^{km} \Gamma^0_{k0,m} - \sqrt{-g} g^{0k} (\Gamma^0_{k0,0} - \Gamma^m_{k0,m}) + \sqrt{-g} g^{00} \Gamma^k_{0k,0}. \]

This suggests the following field redefinition:

\[ \Gamma^0_{km} = \Sigma_{km}, \quad \Gamma^0_{k0} = 2\Sigma_{k0}, \quad \Gamma^m_{k0} = \Sigma^m_{km} - \frac{1}{D - 1} \delta^k_m \Sigma_{00}, \]  

where \( \Sigma^m_{0m} \) is a traceless field, \( \Sigma^k_{0k} = 0 \), i.e. \( \Sigma_{00} = -\Gamma^k_{0k} \).

This redefinition does not affect the following components:

\[ \Gamma^\mu_{00} = \Gamma^\mu_{00}, \quad \Gamma^m_{kp} = \Gamma^m_{kp}. \]
After integration by parts and a change of variables, (4), the “kinetic” part of the Lagrangian becomes diagonal

\[ L_{\text{kin}} = (\sqrt{-g} g^{km})_\mu \Sigma_{km} + 2 (\sqrt{-g} g^k_0)_\mu \Sigma_{k0} + (\sqrt{-g} g^{00})_\mu \Sigma_0 = (\sqrt{-g} g^{\alpha\beta})_\mu \Sigma_{\alpha\beta}. \]  

and “potential” part of the Lagrangian (terms without “velocities”) is

\[ L_{\text{pot}} = (\sqrt{-g} g^{00})_k \Gamma^k_{00} + 2 (\sqrt{-g} g^{p0})_k \Sigma^k_{0p} - (\sqrt{-g} g^{k0})_k \left( \Gamma^0_{00} - \frac{D - 3}{D - 1} \Sigma_{00} \right) \]

\[ + (\sqrt{-g} g^{pq})_k \Gamma^k_{pq} - 2 (\sqrt{-g} g^{pk})_k \left( \Sigma_{0p} + \Gamma^m_{pm} \right) \]

\[ - \sqrt{-g} g^{00} \left( - \Sigma_{00} \Gamma^0_{00} - 2 \Sigma_{0k} \Gamma^k_{00} - \Sigma^m_{0k} \Sigma^k_{0m} - \frac{1}{D - 1} \Sigma_{00} \Sigma_{00} \right) \]

\[ -2 \sqrt{-g} g^{k0} \left( - 2 \Sigma_{00} \Sigma_{0k} - \Sigma_{00} \Gamma^m_{km} + \Gamma^p_{mp} \Sigma^m_{0k} + \Sigma_{km} \Gamma^m_{00} - \Gamma^p_{km} \Sigma^m_{00} \right) \]

\[ - \sqrt{-g} g^{km} \left[ \Gamma^0_{00} \Sigma_{km} + 2 \Sigma_{0p} \Gamma^p_{km} + 2 \Gamma^q_{pq} \Gamma^k_{km} - 4 \Sigma_{0k} \Sigma_{0m} \right] \]

\[ -4 \Sigma_{0k} \Gamma^q_{mp} - \Gamma^p_{kp} \Gamma^q_{mp} - 2 \Sigma_{kp} \Sigma_{0p} - \frac{D - 3}{D - 1} \Sigma_{km} \Sigma_{00} - \Gamma^p_{kq} \Gamma^q_{mp} \right]. \]

Using the above variables and by performing Legendre transformation, we obtain the total Hamiltonian

\[ H_T = \hat{g}^{\alpha\beta} P_{\alpha\beta} + \dot{\Sigma}_{\alpha\beta} \Pi_{\alpha\beta} + \dot{\Gamma}^0_{\mu} \Pi^0_{\mu} + \dot{\Sigma}^k_{0m} \Pi^m_{k0} + \dot{\Gamma}^m_{kp} \Pi^k_{np} - (\sqrt{-g} g^{\alpha\beta})_\mu \Sigma_{\alpha\beta} - L_{\text{pot}}. \]  

The new set of independent variables, (4) and (5), and their conjugate momenta obey the fundamental Poisson brackets (PBs)

\[ \{ g^{\alpha\beta}(\vec{x}), P_{\mu\nu}(\vec{y}) \} = \Delta^{\alpha\beta}_{\mu\nu} \delta(\vec{x} - \vec{y}), \quad \{ \Sigma_{\alpha\beta}, \Pi^{\mu\nu} \} = \Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2} \left( \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} + \delta^{\mu}_{\beta} \delta^{\nu}_{\alpha} \right), \]  

\[ \{ \Gamma^\mu_{00}, \Pi^0_{\nu} \} = \delta^\mu_{\nu}, \quad \{ \Sigma^k_{0m}, \Pi^0_{q} \} = \delta^k_q \Delta^0_{0m} - \frac{1}{D - 1} \delta^k_m \Delta^0_{0q}, \quad \{ \Gamma^m_{kp}, \Pi^{pq} \} = \delta^m_a \Delta^{pq}_{kp}. \]
Here, only the first PB is written in a complete form. Further we will omit the delta functions and the dependence on “space” vectors $\vec{x}$, $\vec{y}$ to shorten our notation, except in the cases where the derivatives of the delta functions appear; and where it is important to indicate with respect to what argument the differentiation is performed.

We could equally well start from (2) to obtain the Hamiltonian and then perform canonical transformations in phase space from the original pairs $(g^{\alpha\beta}, P_{\alpha\beta}^{\alpha\beta})$ and $(\Gamma_{\alpha\sigma}^{\lambda}, \Pi_{\lambda}^{\alpha\sigma})$ to a new set of variables: $(g^{\alpha\beta}, P_{\alpha\beta}^{\alpha\beta})$, $(\Sigma_{\alpha\beta}, \Pi^{\alpha\beta})$, $(\Sigma_{0m}^{k}, \Pi_{0m}^{k})$, $(\Gamma_{00}^{\mu}, \Pi_{00}^{\mu})$ and $(\Gamma_{m}^{\mu k}, \Pi_{m}^{\mu k})$. This is a canonical transformation that is automatically guaranteed for a linear and invertible redefinition of fields.

As in any first order formulation, the Hamiltonian analysis leads to primary constraints equal in number to the number of independent fields

$$P_{\alpha\beta} - \sqrt{-g}E_{\alpha\beta}^{\mu\nu}\Sigma_{\mu\nu} \approx 0, \quad \Pi^{\alpha\beta} \approx 0, \quad \Pi_{00}^{\mu} \approx 0, \quad \Pi_{0m}^{0m} \approx 0, \quad \Pi_{kp}^{k} \approx 0$$

(11)

where we used

$$(\sqrt{-g}g^{\alpha\beta})_{0} = \sqrt{-g}E_{\mu\nu}^{\alpha\beta}f_{0}^{\mu\nu}$$

(12)

with

$$E_{\mu\nu}^{\alpha\beta} \equiv \Delta_{\mu\nu}^{\alpha\beta} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}. \quad \text{(13)}$$

Among the primary constraints (11) we have one pair of second class constraints with the following PB

$$\{P_{\alpha\beta} - \sqrt{-g}E_{\alpha\beta}^{\mu\nu}\Sigma_{\mu\nu}, \Pi^{\rho\sigma}) = -\sqrt{-g}E_{\alpha\beta}^{\rho\sigma}, \quad \text{(14)}$$

which can be easily eliminated (they are of a special form; and therefore, the Dirac brackets among the remaining variables are the same as the corresponding PBs [10]). The solution for this pair is

$$\Pi^{\rho\sigma} = 0, \quad \Sigma_{\mu\nu} = \frac{1}{\sqrt{-g}}I^{\alpha\beta}_{\mu\nu}P_{\alpha\beta}^{\alpha\beta}$$

(15)

where

$$I_{\mu\nu}^{\alpha\beta} \equiv \Delta_{\mu\nu}^{\alpha\beta} - \frac{1}{D-2}g_{\mu\nu}g^{\alpha\beta}, \quad I_{\mu\nu}^{\alpha\beta}E_{\alpha\beta}^{\gamma\sigma} = \Delta^{\gamma\sigma}_{\mu\nu}. \quad \text{(16)}$$
Substitution of solution (15) into the Hamiltonian (8) leads to (the first Hamiltonian reduction)

\[ H_T = \dot{\Gamma}^{\mu}_{00} \Pi_{\mu}^{00} + \dot{\Sigma}^{k}_{0m} \Pi^{0m}_{k} + \dot{\Gamma}^{m}_{kp} \Pi^{kp}_{m} - L_{pot} \left( \Sigma_{\mu\nu} = \frac{1}{\sqrt{-\tilde{g}}} I^{\alpha\beta}_{\mu\nu} P_{\alpha\beta} \right). \] (17)

The appearance of the combinations \( \sqrt{-g} g^{\alpha\beta} \) and \( \frac{1}{\sqrt{-g}} I^{\alpha\beta}_{\mu\nu} P_{\alpha\beta} \) in the Hamiltonian and the invertability of \( I^{\alpha\beta}_{\mu\nu} \) suggest the following canonical change of variables:

\[ h^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta}, \quad \pi_{\mu\nu} = \frac{1}{\sqrt{-g}} I^{\alpha\beta}_{\mu\nu} P_{\alpha\beta}. \] (18)

Such a change obviously preserves the relation (using (12) and (16))

\[ \dot{h}^{\alpha\beta} \pi_{\alpha\beta} = \dot{g}^{\alpha\beta} P_{\alpha\beta}. \] (19)

So, this transformation is canonical [17] and it can be also checked explicitly that

\[ \{ h^{\alpha\beta}, \pi_{\mu\nu} \} = \left\{ \sqrt{-g} g^{\alpha\beta}, \frac{1}{\sqrt{-g}} I^{\rho\sigma}_{\mu\nu} P_{\rho\sigma} \right\} = \Delta^{\alpha\beta}_{\mu\nu}. \] (20)

Equation (19) is a necessary and sufficient condition for canonicity [17]. With such a canonical change, the Hamiltonian becomes much simpler; moreover, because the Lagrangian (2) is linear in \( \sqrt{-g} g^{\alpha\beta} \), the Hamiltonian written in terms of new fields is polynomial. (So, the often stated polynomiality of the tetrad Hamiltonian constraints in Ashtekar variables [18] as one of the advantages of the Hamiltonian formulation of the first order, tetrad-spin connection, Einstein-Cartan theory, is actually not something special since the first order affine-metric formulation is also polynomial in fields after the canonical transformation (18) is performed).

Substitution of (18) into (8) constitutes the first reduction and gives the total Hamiltonian with fewer variables

\[ H_T = \dot{\Gamma}^{\mu}_{00} \Pi_{\mu}^{00} + \dot{\Sigma}^{k}_{0m} \Pi^{0m}_{k} + \dot{\Gamma}^{m}_{kp} \Pi^{kp}_{m} + H_c, \] (21)

where the canonical Hamiltonian, \( H_c \), is

\[ H_c = -\Gamma^{k}_{00} \left( h^{00}_{,k} + 2h^{00}_{,k0} + 2h^{m0}_{,km} \right) - \Gamma^{00}_{00} \left( -h^{0k}_{,k} + h^{00}_{,00} - h^{km}_{,km} \right) \] (22)
\[-\hbar^00 \frac{1}{D-1} \pi^{00}_0 \pi^{00}_0 - \frac{D-3}{D-1} (h^{k0}_k + h^{km}_k \pi^{km}_k) \pi^{00}_0 - (4h^{k0}_0 \pi^{00}_0 + 4h^{km}_m \pi^{00}_m) - 2h^{pk}_p \pi^{00}_k \]

\[-\hbar^00 \sum_k^m \sum_{0m}^k + 2h^{k0}_0 \left( \Gamma^p_{mp} \sum_{0k}^m - \Gamma^p_{km} \sum_{0m}^m \right) - 2h^{km}_m \pi^{kp}_k \sum_{0m}^m - 2h^{p0}_k \sum_{0p}^k \]

\[+ h^{km} \left( 2 \Gamma^q_{pq} \Gamma^p_{km} - \Gamma^p_{kp} \Gamma^q_{mq} - \Gamma^p_{kq} \Gamma^q_{mp} \right) - 2h^{k0}_0 \pi^{km}_0 + 2h^{km}_m \left( \pi^{00}_0 \Gamma^m_{km} - 2\pi^{00}_0 \Gamma^q_{mq} \right) - h^{pq}_{kq} \Gamma^k_{pq} + h^{pk}_k \Gamma^m_{pm} \]

After the elimination of the primary second class constraints (15) (the first Hamiltonian reduction) and after performing the canonical transformation (18), we must continue the Dirac procedure and consider the time development of the remaining primary constraints. Two of them give the following secondary constraints:

\[\dot{\Pi}^{00}_0 = \{\Pi^{00}_0, H_c\} = -h^{0k}_{k} + h^{00}_0 \pi^{00}_0 - h^{km}_m \pi^{km}_m \equiv \chi^{00}_0, \quad (23)\]

\[\dot{\Pi}^{00}_k = \{\Pi^{00}_k, H_c\} = h^{00}_0 + 2h^{00}_0 \pi^{00}_k + 2h^{m0}_m \pi^{km}_m \equiv \chi^{00}_k, \quad (24)\]

and the only non-zero PB among them is

\[\{\chi^{00}_0, \chi^{00}_k\} = -\chi^{00}_k. \quad (25)\]

The secondary constraints \(\chi^{00}_\mu\) obviously (just from their field content) have zero PBs with all of the primary constraints:

\[\{\chi^{00}_\mu, \Pi^{00}_\nu\} = 0, \quad \{\chi^{00}_\mu, \Pi^{0m}_k\} = 0, \quad \{\chi^{00}_\mu, \Pi^{kp}_m\} = 0, \quad (26)\]

so they are first class, at least, at this stage of the Dirac procedure.

Two remaining primary constraints, \(\Pi^{0m}_k \approx 0 \) and \(\Pi^{kp}_m \approx 0\), lead to the secondary constraints:

\[\dot{\Pi}^{0m}_k = \{\Pi^{0m}_k, H_c\} = -\delta H_c \frac{\delta}{\delta \sum_{0m}^k} = \chi^{0m}_k, \quad (27)\]

\[\dot{\Pi}^{kp}_m = \{\Pi^{kp}_m, H_c\} = -\delta H_c \frac{\delta}{\delta \Gamma^{km}_{kp}} = \chi^{kp}_m. \quad (28)\]
And, because $H_c$ has quadratic contributions in the corresponding coordinates $(\Sigma^k_{0m}, \Gamma^m_{kp})$, we have secondary constraints, which are second class, and two additional pairs of constraints, $(\Pi^m_{0m}, \chi^m_{0m})$ and $(\Pi^m_{kp}, \chi^m_{kp})$, which can be solved, and the corresponding pairs of canonical variables, $(\Pi^m_k, \Sigma^k_{0m})$ and $(\Pi^m_{kp}, \Gamma^m_{kp})$, can be eliminated without affecting the PBs of remaining fields, as pairs $(\Pi^m_{0m}, \chi^m_{0m})$ and $(\Pi^m_{kp}, \chi^m_{kp})$ are also of a special form. Solution of these constraints is given in Appendix A. Substitution of the solutions into the Hamiltonian (the second Hamiltonian reduction) gives us the following total Hamiltonian

$$H_T \left( \Gamma^\mu_{00}, \Pi^\mu_{00}, h^{\alpha\beta}, \pi_{\alpha\beta} \right) = \dot{\Gamma}^\mu_{00} \Pi^\mu_{00} + H_c$$

with the canonical part $H_c$

$$H_c = -\Gamma^\mu_{00} \chi^\mu_{00} + H'_c,$$

where $H'_c$ is the part of the canonical Hamiltonian after extracting the term $-\Gamma^\mu_{00} \chi^\mu_{00}$ (secondary constraints):

$$H'_c = -h^{00} \frac{1}{D-1} \dot{\pi}_{00} \pi_{00} - \frac{D-3}{D-1} D^0_k \dot{\pi}_{0k} - \left( 4 h^{k0} \pi_{00} + 4 h^{km} \pi_{0m} - 2 h^{pk} \right) \pi_{0k}$$

$$+ \frac{1}{h^{00}} D^0_k D^0_n - \frac{1}{D-1} \frac{1}{h^{00}} D^m_n D^0_k$$

$$+ \frac{1}{2} D^x_i h_{bz} D^y_z - \frac{1}{4} h_{ab} e^{nx} D^0_n D^y_z + \frac{1}{4} D - 2 e^{nx} h_{ab} D^0_n h_{yz} D^y_x$$

Here we use the notation introduced by Dirac in the second order formulation of GR as it naturally arises in the course of calculation (see Appendix A):

$$e^{nq} \equiv h^{nq} - \frac{h^{0n} h^{0q}}{h^{00}}, \quad e^{nq} h_{qp} = \delta^n_p.$$ (32)

Note that $e^{nq}$ is a short-hand notation, not a new variable. We introduce the combinations $D^{0b}_a$ and $D^{ab}_n$, and their explicit forms are calculated in Appendix and given by:

$$D^{0k}_m = h^{k0}_{,m} + h^{bk} \pi_{bm},$$

$$D^{kq}_m = -2 h^{kq} \pi_{0m} + h^{kq}_{,m} - \frac{h^{0q}}{h^{00}} D^0_k D^0_m - \frac{h^{k0}}{h^{00}} D^q_k D^0_m + \frac{1}{D-1} \left( \frac{1}{h^{00}} D^0_c - \pi_{00} \right) \left( h^{0q} \delta^k_m + h^{k0} \delta^q_m \right).$$ (34)
The next step of the Dirac procedure is the time development of the secondary first class constraints \( \dot{\chi}^{00}_\mu = \{\chi^{00}_\mu, H_T\} \), to check whether they produce tertiary constraints. At this step of the analysis, we can proceed in two different ways. The direct way is to obtain \( \{\chi^{00}_\mu, H_T\} \), and single out already known secondary constraints and form tertiary constraints from what is left. Another way is to isolate terms with secondary constraints, in addition to \( \Gamma^{\mu}_{00}\chi^{00}_\mu \), in the total Hamiltonian before calculating \( \{\chi^{00}_\mu, H_T\} \). The terms in \( H_T \) proportional to the secondary constraints \( A^{\nu}_{00}\chi^{00}_\nu \) will give PB \( \{\chi^{00}_\mu, A^{\nu}_{00}\chi^{00}_\nu\} \) with the result which is proportional to the secondary constraints because of (25). In this case, in order to find the tertiary constraints (if any), we have to consider the PBs of secondary constraints with the rest of the Hamiltonian, what is left after all terms with secondary constraints were isolated. Note that considering linear combinations of constraints is perfectly consistent with the Dirac procedure (as a linear combination of constraints is also a constraint).

Performing the direct calculation, we use the following simple PBs of the secondary constraint \( \chi^{00}_0 \) with the combinations presented in (30)-(34):

\[
\left\{ \chi^{00}_0, \frac{1}{h^{00}} \right\} = \frac{1}{h^{00}}, \quad \left\{ \chi^{00}_0, h^{00} \right\} = -h^{00}, \quad \left\{ \chi^{00}_0, \pi^{00} \right\} = \pi^{00}, \quad (35)
\]

\[
\left\{ \chi^{00}_0, h^{0k} \right\} = 0, \quad \left\{ \chi^{00}_0, D^{0k}_m \right\} = 0, \quad (36)
\]

\[
\left\{ \chi^{00}_0, 4h^{k0}\pi^{00} + 4h^{km}\pi^{0m} - 2h^{pk}_p \right\} = 4h^{k0}\pi^{00} + 4h^{km}\pi^{0m} - 2h^{pk}_p, \quad (37)
\]

as well as

\[
\left\{ \chi^{00}_0, h^{kq}_m - 2h^{kq}\pi^{0m} \right\} = h^{kq}_m - 2h^{kq}\pi^{0m} \quad (38)
\]

that, in combination with (35) and (36), gives

\[
\left\{ \chi^{00}_0, D^{pk}_m \right\} = D^{pk}_m. \quad (39)
\]

Using definition (32) we also find:

\[
\left\{ \chi^{00}_0, e^{km} \right\} = e^{km}, \quad \left\{ \chi^{00}_0, h^{km} \right\} = -h^{km}. \quad (40)
\]

The first PB in (40) requires calculations; but the second is just the result of \( e^{km}h^{km} = \delta^m_n \). Similar PBs for \( \chi^{00}_k \) are:
\[ \{ \chi^0, e^{km} \} = 0, \quad \{ \chi^0, h_{km} \} = 0, \tag{41} \]

\[ \{ \chi^0, h^{00} \} = 0. \tag{42} \]

We would like to mention here that we originally analyzed the first order of the EH action in different variables \([5, 6]\) where we used another combination

\[ \xi^\alpha_{\alpha\beta} = \Gamma^\alpha_{\alpha\beta} - \frac{1}{2} \left( \delta^\alpha_{\alpha} \Gamma^\sigma_{\beta\sigma} + \delta^\alpha_{\beta} \Gamma^\sigma_{\alpha\sigma} \right) \tag{43} \]

and discussed the unavoidable appearance of tertiary constraints. These variables, (43), simplify the calculations because the “kinetic” part of the Lagrangian becomes diagonal automatically: \( L_{kin} = -\sqrt{-g} g^{\alpha\beta} \xi^0_{\alpha\beta} \). But this simplification appears only in the first steps of the Hamiltonian analysis. These variables were used by Ghalati and McKeon to find the Hamiltonian for the first order formulation of GR and, for the first time, they explicitly demonstrated the closure of the Dirac procedure \([7]\). Firstly, our choice to start from the affine connections that lead to combinations (4) and (5) is dictated by our goal to compare our results with \([2]\), where the authors claimed that they used Palatini formulation,\(^3\) which is the affine-metric formulation due to Einstein. Secondly, the variables (4)-(5) also diagonalize the “kinetic” part of the Lagrangian (6); and, what is more important, the primary first class constraints \( \Pi^0_{\mu} \) appear in the covariant form. In addition, if we used \( \xi^\alpha_{\alpha\beta} \) from (43) instead of \( \Gamma^\alpha_{\alpha\beta} \) then some of the brackets (35)-(39) would be more complicated. For example, (39) became non-local (proportional to derivatives of delta functions). In this case, it would be impossible to use simple associative properties of PBs for the terms of \( H'_c \) in the second and third lines of (31) when calculating \( \{ \chi^0, H'_c \} \). Whereas, using the Hamiltonian in terms of \( \Gamma^\lambda_{\alpha\beta} \) and properties of the above PBs the calculation of \( \{ \chi^0, H'_c \} \) is greatly simplified, for example, from (39) and (40) it immediately follows that \( \{ \chi^0, D^z b_z D^y x D^x z \} = D^z b_z D^y x D^x z \), etc.

One more advantage in using the original variables \( \Gamma^\lambda_{\alpha\beta} \), is the simplification of restoration of gauge invariance; we will not need to restore the gauge transformations of \( \Gamma^\lambda_{\alpha\beta} \) from that of

\(^2\) This is generalization of change of variables that we found considering the Hamiltonian of 2D in \([19]\). Later we learned that such a combination was known before (see Horava \([20]\) for any dimension and Kijowski for four-dimesional case \([21]\)).

\(^3\) This formulation was originally introduced by Einstein \([1]\), but continues to be mistakenly attributed to Palatini \([22]\) (see also Palatini’s original paper \([23]\) and its English translation \([24]\)).
The above arguments are mainly technical and the real advantage is just a manifestly covariant form of the primary first class constraints. As we worked with both $\xi^\alpha_{\beta\gamma}$ and $\Gamma^\alpha_{\beta\gamma}$, we have to admit that there is no overall advantage in a particular choice of variables and the total amount of calculation (difficulties) is conserved.

For the time development of $\chi_0^{00}$ we have

$$\dot{\chi}_0^{00} = \{\chi_0^{00}, H_T\} = \{\chi_0^{00}, -\Gamma^\mu_{00} \chi^{00}_\mu\} + \{\chi_0^{00}, H'_c\} = \Gamma^k_{00} \chi_0^{00} + \{\chi_0^{00}, H'_c\}$$

(44)

where using properties (35)-(40), the last PB can be just read off

$$\{\chi_0^{00}, H'_c\} = H'_c + \left(2h^{k0} \pi_{00} + 2h^{km} \pi_{0m} - h^{pk}_p\right)_k.$$

(45)

Note that the right-hand side of (45) is not the Hamiltonian density and we cannot neglect this spatial derivative.

Because all PBs (35)-(40) are local (no derivatives of delta functions), using them and associative properties of PB makes the result (45) almost obvious. The first term in (44), $\Gamma^k_{00} \chi_0^{00}$, is proportional to the secondary constraints. The rest is given by (45) and to find out whether we have closure of Dirac procedure at this stage or if the next generation of constraints appears, we first have to find in (45) the combinations proportional to the secondary constraints. If there are contributions, which are not proportional to secondary constraints in (45), then we have tertiary constraints.

We proceed as follows. Using secondary constraints (23) and (24) we express some fields in terms of constraints and remaining variables, substitute them into the Hamiltonian, isolate the contributions, which are proportional to secondary first class constraints, and then work with what is left. Note that according to the Dirac procedure we cannot solve first class constraints, as it was done in [2] and which is shown especially clearly by Faddeev in [25]; and the consequence of solving first class constraints was discussed in [3, 6, 7]. These reexpressions, to single out contributions proportional to secondary first class constraints, after calculation of PB (15) or before, as it was done in [7], is consistent with the possibility of using linear combinations of non-primary first class constraints (the role primary first class constraints will be discussed in Section 5).

The only two phase-space variables that can be unambiguously re-expressed (by algebraic operations) using secondary constraints (23), (24) are two momenta:
\[ \pi_{00} = \frac{1}{h_{00}} D_{k}^{0k} + \frac{1}{h_{00}} \chi_{0}^{00}, \quad (46) \]

\[ \pi_{0k} = -\frac{1}{2} \frac{1}{h_{00}} h_{00}^{0k} - \frac{1}{h_{00}} h_{00}^{m0} \pi_{km} + \frac{1}{2} \frac{1}{h_{00}} \chi_{00}^{0k}. \quad (47) \]

After substitution of (46) and (47) into the Hamiltonian (31), we separate terms proportional to \( \chi_{\mu}^{00} \) and write the rest of the Hamiltonian as a sum of three contributions of different order in \( \pi_{km} \) (e.g. \( H_{c}''(2) \) is of the second order in \( \pi_{km} \), etc.). Performing the same operations with (45) we obtain the following form of the Hamiltonian \( H_{c}' \)

\[ H_{c}' = H_{c}''(2) + H_{c}''(1) + H_{c}''(0) + A^{\mu} \chi_{\mu}^{00} + S_{m}^{m} \quad (48) \]

and equation (45) can be rewritten as

\[ \{ \chi_{00}^{00}, H_{c}' \} = H_{c}''(2) + H_{c}''(1) + H_{c}''(0) + \{ \chi_{00}^{00}, A^{\mu} \chi_{\mu}^{00} \}, \quad (49) \]

where

\[ H_{c}''(2) = \frac{1}{h_{00}} \left[ e^{kp} e^{mq} (\pi_{mp} \pi_{kq} - \pi_{kp} \pi_{mq}) \right], \quad (50) \]

\[ H_{c}''(1) = \frac{h_{0k}}{h_{00}} \left[ 2 (e^{mq} \pi_{mq})_{,k} - 2 (e^{mq} \pi_{kq})_{,m} - \frac{1}{h_{00}} (e^{np} h_{00}^{0n})_{,k} \pi_{np} \right], \quad (51) \]

\[ H_{c}''(0) = \frac{1}{h_{00}} \left[ - (e^{km} h_{00}^{0k})_{,mk} + \frac{1}{2} (e^{xh} h_{00}^{00})_{,y} h_{00}^{0y} \right] \]

\[ -\frac{1}{4} h_{ab} h_{b} e^{nx} (e^{ab} h_{00}^{00})_{,n} \frac{1}{h_{00}} (e^{y} h_{00}^{00})_{,x} + \frac{1}{4} D - 2 e^{nx} h_{ab} (e^{ab} h_{00}^{00})_{,x} h_{yz} \frac{1}{h_{00}} (e^{yz} h_{00}^{00})_{,x} \right], \quad (52) \]

\[ A^{0} = -\frac{1}{D-1} \pi_{00} \quad (53) \]

\[ \frac{1}{D-1} \left[ \frac{1}{h_{00}} (h_{0y} \delta_{x}^{0} + h_{x}^{0} \delta_{y}^{0}) + \frac{1}{2} D_{y} h_{b} e^{nx} D_{n} h_{00}^{00} \right] h_{00} h_{00}^{00} + \frac{1}{4} D - 2 e^{nx} h_{ab} D_{n}^{ab} h_{00}^{00} h_{yz} \]
\[- \frac{1}{D} - \frac{1}{h^{00}} \left( h^{00} \delta_n^b + h^{00} \delta_n^a \right) \left[ \frac{1}{2} h_{bz} \delta^a_n D_a - \frac{1}{4} h_{ay} h_{bz} e^{nx} D_x + \frac{1}{4} \frac{1}{D - 2} e^{nx} h_{ab} h_{yz} D_x \right], \]

\[ A^k = -2 \frac{h^{00} h^{km}}{h^{00}} \langle \pi_00 \rangle - 2 \frac{h^{k0}}{h^{00}} \langle \pi_00 \rangle + \frac{h^{k0}}{h^{00}} \langle \pi_00 \rangle + 2 \frac{h^{km}}{h^{00}} \langle \pi_00 \rangle \]

\[ - \frac{h^{yz}}{h^{00}} \left[ \frac{1}{2} D_y h_{bz} - \frac{1}{4} h_{ay} h_{bz} e^{nk} D_n + \frac{1}{4} \frac{1}{D - 2} e^{nk} h_{ab} D_n h_{yz} \right] \]

\[ - \frac{h^{ab}}{h^{00}} \left[ \frac{1}{2} h_{bz} D_a - \frac{1}{4} h_{ay} h_{bz} e^{kz} D_x + \frac{1}{4} \frac{1}{D - 2} e^{kz} h_{ab} h_{yz} D_x \right] \]

and

\[ \frac{e^{kq}}{D_m} = h^{kq} \frac{1}{h^{00}} h_{,m}^{00} + 2 h^{kq} \frac{1}{h^{00}} h^{00}_{,m} + h^{kq} \frac{1}{h^{00}} D_0^{00} - \frac{h^{kq}}{h^{00}} D_m^{00}. \]

\[ A^0 \] and \( A^k \) are the functions which depend on the fields and derivatives and their explicit form is not needed for further calculations in this article; they are given only for completeness. Our main interest is in the contributions, which are not proportional to secondary constraints, \( H''_c(i) \). The explicit form of \( S^m \) is found by comparison of the parts not proportional to the secondary constraints in (45) with \( H'_c \)

\[ S^m = -2 h^{00} \frac{e^{kq}}{h^{00}} \pi_{kq} + 2 h^{00} e^{mq} \pi_{kq} - h^{00} \frac{h^{00}}{h^{00}} + h^{00} \frac{h^{00}}{h^{00}} + \frac{1}{h^{00}} \left( e^{km} h^{00} \right) \]

And, as in the previous steps of the Hamiltonian reduction, writing \( h^{km} \) in terms of \( e^{km} \) makes the expressions more transparent. Note that (48), contrary to (45), is the Hamiltonian density and a surface term can be neglected in subsequent calculations.

Using (48) and (50)-(52) the calculation of the time development of \( \chi^{00}_k \) is straightforward

\[ \dot{\chi}^{00}_k = \left\{ \chi^{00}_k, H_c \right\} = -\Gamma^{00}_0 \chi^{00}_k + \left\{ \chi^{00}_k, H'_c \right\} \]

where

\[ \left\{ \chi^{00}_k, H'_c \right\} = 2 (e^{mq} \pi_{mq})_k - 2 (e^{mq} \pi_{kq})_m - \frac{1}{h^{00}} \left( e^{np} h^{00} \right)_k \pi_{np} + \left\{ \chi^{00}_p, A^n \chi^{00}_p \right\}. \]

Note that \( \left\{ \chi^{00}_k, H'_c(2) \right\} = \left\{ \chi^{00}_k, H'_c(0) \right\} = 0 \) which is based on simple PBs of \( \chi^{00}_k \) with combinations of fields presented in (41)-(42).
The last term in (57) gives contributions proportional to $\chi_{00}^{\mu}$, whereas the first three terms in (57) cannot be expressed as a linear combination of $\chi_{00}^{\mu}$; moreover they coincide with the expression in square brackets of $H''_{c} (1)$ in (51). This part of (57), which is not proportional to the secondary constraints, can be chosen to be called a tertiary constraint

$$
\tau_{k}^{00} \equiv 2 (e^{mq} \pi_{mq})_{,k} - 2 (e^{mq} \pi_{kq})_{,m} - \frac{1}{h_{00}} (e^{np} h_{00}^{00})_{,k} \pi_{np}.
$$

(58)

Taking into account (58), the bracket $\{\chi_{00}^{00}, H''_{c}\}$ can be rewritten as

$$
\{\chi_{00}^{00}, H''_{c}\} = H''_{c} (2) + \frac{h_{0k}^{00}}{h_{00}} \tau_{k}^{00} + H''_{c} (0) + \{\chi_{00}^{00}, A_{\mu} \chi_{00}^{\mu}\},
$$

so the terms which are not proportional to already known constraints should be called a new, tertiary constraint. Using the exact expressions of $H''_{c} (2)$ and $H''_{c} (0)$ from (50) and (52) we can name the following combination as a tertiary constraint

$$
\tau_{00}^{00} \equiv e^{kp} e^{mq} (\pi_{mp} \pi_{kq} - \pi_{kp} \pi_{mq}) - (e^{km} h_{00}^{00})_{,mk} + \frac{1}{2} (e^{xb} h_{00}^{00})_{,y} h_{0k}^{00} \frac{1}{h_{00}} (e^{yz} h_{00}^{00})_{,x}
$$

(59)

$$
- \frac{1}{4} h_{xy} h_{bz} e^{nx} (e^{ab} h_{00}^{00})_{,n} \frac{1}{h_{00}} (e^{yz} h_{00}^{00})_{,x} + \frac{1}{4} \frac{1}{D - 2} e^{nx} h_{ab} (e^{ab} h_{00}^{00})_{,n} h_{yz} \frac{1}{h_{00}} (e^{yz} h_{00}^{00})_{,x}.
$$

The canonical Hamiltonian written in terms of the constraints $\chi_{00}^{\mu}$ and $\tau_{00}^{00}$ is

$$
H_{c} = -\Gamma_{00}^{\mu} \chi_{00}^{\mu} + \frac{1}{h_{00}} \tau_{00}^{00} + \frac{h_{0k}^{00}}{h_{00}} \tau_{k}^{00} + A_{\mu} \chi_{00}^{\mu}.
$$

(60)

Note that the choice of tertiary constraints is not unique. For example, if we start from $\chi_{00}^{00}$ we would name the whole combination $H''_{c} (2) + H''_{c} (1) + H''_{c} (0)$ in (49) a tertiary constraint $\tau_{00}^{00}$, because in this case $\tau_{00}^{00}$ has not yet been found. Such arbitrariness in the choice of constraints looks ambiguous. Firstly, it does not contradict the Dirac procedure as any linear combination of constraints is also a constraint. Secondly, and we will show this below, if the correct method of the restoration of gauge invariance is used then the final result does not depend on a choice of tertiary constraints.

We have chosen such a form of tertiary constraints, (58) and (59), because calculations of PBs among secondary and these tertiary constraints is almost manifest due to the simple
properties of the PBs of the secondary constraints with their combinations presented here (e.g. (35)-(40) and (41)-(42)). It is easy to show that

\[ \{ \chi^{00}_\mu, \tau^{00}_\nu \} = 0. \]  

(61)

The brackets \( \{ \tau^{00}_\mu, A^\mu \chi^{00}_\mu \} \) are proportional to the secondary constraints because of (61). In addition, because of the relatively simple form of \( \tau^{00}_\mu \) ((58) and (59)), the calculation of PBs among them is not inordinately tedious. We must find these PBs to prove the closure of the Dirac procedure; and if it closes for one choice of tertiary constraints, then it closes for any combination of them. For these constraints, \( \tau^{00}_\mu \), the only possibility to have a closure is to demonstrate that

\[ \{ \tau^{00}_\mu, \tau^{00}_\nu \} = 0 \text{ or } \sim \tau^{00}_\sigma, \]  

(62)

as PBs of combinations (58) and (59) cannot form secondary constraints just because of their field content. In the next Section, we consider the calculations of PBs (62) and the closure of the Dirac procedure.

**III. CALCULATION OF \( \{ \tau^{00}_\mu, \tau^{00}_\nu \} \), CLOSURE OF THE DIRAC PROCEDURE AND ALGEBRA OF TERTIARY CONSTRAINTS**

Even considering relatively simple combinations \( \tau^{00}_\mu \) ((58) and (59)) as a choice of tertiary constraints, the calculation of the PBs among them is a laborious procedure. This fact and a variety of other choices of tertiary constraints were the reason why the first attempts to prove a closure of the Dirac procedure for first order formulation of GR were not finished [5, 6], where the variables (43) were used; only later the proof was completed in [7, 26] and the expectations outlined in [5, 6] are thus realized. The presence of derivatives of fields imposes some additional complications and the best way of dealing with such calculations is to use test functions [9] that were demonstrated in some detail for constraints of Yang-Mills theory [27]. More details of calculation using test functions were given and applied to the first order formulation of GR in [7, 11]. The PBs among constraints can be written in the form with the explicit presence of test functions (e.g. see work of Faddeev [25], Section 3), which might be useful for further calculations (e.g. restoration of gauge invariance, analysis
of different choices of tertiary constraints, etc.) compared with the standard form which contains derivatives of delta functions.

Introducing test functions, \( f(x) \) and \( g(y) \), that have a zero PB with phase space variables, we calculate

\[
\int \int d\vec{x} \, d\vec{y} \left\{ f(\vec{x}) \tau_{\mu}^{00}(\vec{x}), g(\vec{y}) \tau_{\nu}^{00}(\vec{y}) \right\} = \\
\int \int d\vec{x} \, d\vec{y} \, (\ldots) \, \vec{\tau} \, \delta(\vec{x} - \vec{y}) = \int d\vec{x} \, (\ldots) \, \vec{x} \, \tau_{\nu}^{00}(\vec{y}) \, \tau_{\mu}^{00}(\vec{x}) = (\ldots) \, \vec{x} \, .
\]

where, to shorten the notation, we omit the integrals, i.e.

\[
\{ f(\vec{x}) \tau_{\mu}^{00}(\vec{x}), g(\vec{y}) \tau_{\nu}^{00}(\vec{y}) \} = (\ldots) \, \vec{x} \, . \tag{64}
\]

After long but straightforward calculation, (63) can be presented in the following form for \( \mu, \nu = 0, i \):

\[
\{ f(\vec{x}) \tau_{i}^{00}(\vec{x}), g(\vec{y}) \tau_{j}^{00}(\vec{y}) \} = f_{j}g_{i} - f_{i}g_{j} \, \tau_{0}^{00} \, \tau_{0}^{00} \, , \tag{65}
\]

\[
\{ f(\vec{x}) \tau_{0}^{00}(\vec{x}), g(\vec{y}) \tau_{0}^{00}(\vec{y}) \} = f_{i}g_{0} - f_{0}g_{i} \, \tau_{i}^{00} \, , \tag{66}
\]

\[
\{ f(\vec{x}) \tau_{0}^{00}(\vec{x}), g(\vec{y}) \tau_{0}^{00}(\vec{y}) \} = f_{i}g_{0} - f_{0}g_{i} \, \tau_{0}^{00} \, . \tag{67}
\]

\[
\{ f(\vec{x}) \tau_{i}^{00}(\vec{x}), g(\vec{y}) \tau_{0}^{00}(\vec{y}) \} = f_{i}g_{0} - f_{0}g_{i} \, \tau_{0}^{00} \, . \tag{68}
\]

From these expressions we can also obtain the standard form of PBs with delta functions if we rearrange (63) in the form without derivatives of the test functions (note that to do this for (65)-(68) derivatives of delta functions are unavoidable)

\[
\int \int d\vec{x} \, d\vec{y} \left\{ f(\vec{x}) \tau_{\mu}^{00}(\vec{x}), g(\vec{y}) \tau_{\nu}^{00}(\vec{y}) \right\} = \int \int d\vec{x} \, d\vec{y} \, f(\vec{x}) \, (\ldots) \, \vec{\tau} \, \delta(\vec{x} - \vec{y}) = \int d\vec{x} \, (\ldots) \, \vec{x} \, g(\vec{y}) \, . \tag{69}
\]

For (65)-(67) we obtain:

\[
\{ \tau_{i}^{00}(\vec{x}), \tau_{j}^{00}(\vec{y}) \} = \tau_{i}^{00}(x) \, \partial_{y}^{j} \delta(x - y) - \tau_{i}^{00}(y) \, \partial_{x}^{j} \delta(x - y) \, , \tag{70}
\]
\[
\{ \tau^{00}_0 (\vec{x}) , \tau^{00}_0 (\vec{y}) \} = -\hbar^{00} (x) e^{k} (x) \tau^{00}_p (x) \partial_k \delta (x-y) + \hbar^{00} (y) e^{k} (y) \tau^{00}_p (y) \partial_k \delta (x-y),
\]
(71)

\[
\{ \tau^{00}_0 (\vec{x}) , \tau^{i0}_0 (\vec{y}) \} = \tau^{00}_0 (x) \partial^i \delta (x-y) - \tau^{00}_0 (y) \partial_i \delta (x-y),
\]
(72)

\[
\{ \tau^{i0}_0 (\vec{x}) , \tau^{00}_i (\vec{y}) \} = \tau^{i0}_0 (x) \partial^0 \delta (x-y) - \tau^{i0}_0 (y) \partial^0 \delta (x-y),
\]
(73)

(here we use the notation: \( \partial^i_k = \frac{\partial}{\partial x^i} \) or \( \partial^i_l = \frac{\partial}{\partial y^i} \)).

As a consistency check we can integrate the above expressions with \( \int d\vec{y} f (\vec{x}) g (\vec{y}) \ldots (\vec{x}, \vec{y}) \), which leads us back to the previous form (65)-(67). So, these, (65)-(68) and (70)-(73) are two different but equivalent forms of the constraint algebra.

This algebra of constraints (in one form or another) is equivalent with the algebra found in [7], which becomes clear after canonical transformations are performed (see next Section) and it is also the same as the algebra of constraints (but secondary) given by Faddeev [25] and Teitelboim [28] (despite different expressions for constraints themselves).

The conventional form of the algebra of secondary constraints for the conventional Hamiltonian formulation of the second order EH action, known also as “Dirac’s algebra” or “hypersurface deformation algebra”, is:

\[
\{ \mathcal{H}_L (x) , \mathcal{H}_L (x') \} = e^{r s} (x) \mathcal{H}_s (x) \delta_{r(x)} (x-x') - e^{r s} (x') \mathcal{H}_s (x') \delta_{r(x')} (x-x'),
\]
(74)

\[
\{ \mathcal{H}_s (x) , \mathcal{H}_L (x') \} = \mathcal{H}_L (x) \delta_{s(x)} (x-x'),
\]
(75)

\[
\{ \mathcal{H}_r (x) , \mathcal{H}_s (x') \} = \mathcal{H}_s (x) \delta_{r(x)} (x-x') - \mathcal{H}_r (x') \delta_{s(x')} (x-x');
\]
(76)

where \( \mathcal{H}_L \) and \( \mathcal{H}_s \) are the “Hamiltonian” and “diffeomorphism” constraints, respectively.

The algebra (74)-(76) can be found in slightly different forms, for example, in [8, 29, 30, 31, 32, 33].

Comparing (74)-(76) with (70)-(73) one can notice a difference. It is not related to the variety of notations used in the literature, but to the obviously less symmetric form of (75).
in the conventional algebra, contrary to (73) and (68). This discrepancy must be clarified. Let us trace out the origin of the algebra (74)-(76) starting from its first name “Dirac’s algebra”. It appeared for the first time in the Dirac book [8], where he referred to his paper [34]. But in [34] he derived this algebra for the motion of space-like surfaces, as PBs among tangential and normal to a surface variables, not as PBs among the secondary constrains of GR. Kuchar in [30], by geometrical reasoning, showed how \( \mathcal{H}_L \) and \( \mathcal{H}_s \) (“super-Hamiltonian” and “super-momenta”, in his terminology) “represent the set of deformations of space-like hypersurfaces”. Probably, after that the name “hypersurface deformation algebra” appeared. In [32] Teitelboim reconstructed this algebra by “a simple geometrical argument based exclusively on the path independence of the dynamical evolution”, i.e. on the ‘motion’ of a three-dimensional cut in a four-dimensional manifold of hyperbolic signature”. In his later work [28] this algebra was altered by another one where the PB (75) was replaced by (71). This transition was not explained and left unnoticed, which is strange, especially because when presenting the new algebra he referred to his old paper [32] where the algebra is different. This algebra of PBs among secondary constraints was derived by Faddeev in [25] where he considered the first order formulation of GR; in his paper, the algebra among constraints is written in the form of (65)-(68). The same algebra was presented by Ghalati and McKeon [7]. Our calculation also results in (65)-(68), or equivalently in (70)-(73).

In [35] analyzing the Dirac derivation of [34], the author made a conclusion that “Dirac’s derivation of the constraint algebra cannot be considered satisfactory”. This statement must be clarified. Dirac did not derive the algebra of constraints as they were not even known and appeared only a few years after [3]; and in this article there is no statement that constraints satisfy “Dirac algebra” of [34]. This conclusion was made by other authors and without calculation.

To answer the question why the discrepancy, the difference between (73) and (75), appears, we are planning to revisit our analysis of the Dirac formulation of the second order of EH action given in [4], where a different choice of secondary constraints was used. The results will be reported elsewhere.

If we are interested only in a demonstration of closure of the Dirac procedure and in restoration of four-diffeomorphism, we need to consider the time development of all first class constraints started from primary, i.e. to calculate PB with Hamiltonian. In this case, the algebra of particularly chosen tertiary constraints is not important. In addition, one
can completely avoid the non-locality of these PBs (i.e. derivatives of test functions in (65)-(68) or derivatives of delta functions in (70)-(73)) as the PB of a constraint with the Hamiltonian is always local (one integration has to be performed as Hamiltonian in field theories is integral of Hamiltonian density). For example, PBs between tertiary constraints and the Hamiltonian is defined as

\[ \{ \tau_{\mu}^{00}, H_{\mathcal{T}} \} = \left\{ f(\vec{x}) \tau_{\mu}^{00}(\vec{x}), \int d\vec{y} g(\vec{y}) H_{\mathcal{T}}(\vec{y}) \right\}. \]

As the Hamiltonian for the EH action is a linear combination of the constraints, what we actually need to calculate are the following PBs

\[ \left\{ f(\vec{x}) \tau_{\mu}^{00}(\vec{x}), \int d\vec{y} g^\nu(\vec{y}) \tau_{\nu}^{00}(\vec{y}) \right\} \]  

where, according to our choice of constraints (60), we have to put

\[ g_0(\vec{y}) = \frac{1}{h_{00}(\vec{y})} \]

and

\[ g^k(\vec{y}) = \frac{h_{0i}(\vec{y})}{h_{00}(\vec{y})}. \]

Using (70)-(73) we can easily find:

\[ \left\{ \tau_{i}^{00}(\vec{x}), \int d\vec{y} \tau_{j}^{00}(\vec{y}) \frac{h_{0j}(\vec{y})}{h_{00}(\vec{y})} \right\} = -\left( \tau_{i}^{00} \frac{h_{0j}}{h_{00}} \right)_{,j} - \tau_{j}^{00} \left( \frac{h_{0j}}{h_{00}} \right)_{,i}, \]  

\[ \left\{ \tau_{0}^{00}(\vec{x}), \int d\vec{y} \tau_{0}^{00}(\vec{y}) \frac{1}{h_{00}(\vec{y})} \right\} = \left( e^{kp}_{\rho} \tau_{0}^{00} \right)_{,k} + h^{00} e^{kp}_{\rho} \tau_{0}^{00} \left( \frac{1}{h_{00}} \right)_{,k}, \]  

\[ \left\{ \tau_{0}^{00}(\vec{x}), \int d\vec{y} \tau_{i}^{00}(\vec{y}) \frac{h_{0i}(\vec{y})}{h_{00}(\vec{y})} \right\} = -\tau_{0}^{00} \left( \frac{h_{0i}}{h_{00}} \right)_{,i} - \tau_{0}^{00} \left( \frac{1}{h_{00}} \frac{h_{0i}}{h_{00}} \right)_{,i}, \]  

\[ \left\{ \tau_{i}^{00}(\vec{x}), \int d\vec{y} \tau_{0}^{00}(\vec{y}) \frac{1}{h_{00}(\vec{y})} \right\} = -\tau_{0}^{00} \left( \frac{1}{h_{00}} \right)_{,i} - \tau_{0}^{00} \left( \frac{1}{h_{00}} \right)_{,i}. \]

Equally well we can use the PBs, (65)-(68), where by integration by parts we have to move the derivative from a test function \( f \), and put \( f = 1 \) at the end of the calculations (see, e.g. [27]). This will result in the same expressions (78)-(81). We also want to emphasize that these calculations can be performed directly, without any reference to a particular form of the algebra of constraints, which depends on our choice of constraints (see next Section).

The above PBs, (78)-(81), complete the proof of closure of the Dirac procedure that the PBs of the tertiary constraints with the Hamiltonian are proportional to already known constraints.
In next two Sections we will discuss the role of different choices of constraints and canonical transformations, which is important in general, but also will be needed for the discussion of methods of restoration of gauge symmetry (Sections 6 and 7) that, in accordance with the Dirac conjecture [8], is generated by the full set of first class constraints.

Both equivalent forms of the PB algebra among tertiary constraints, which we chose out of many possible combinations, (65)-(68) and (70)-(73), might be useful in calculations that we need to perform, especially for the general analysis of a role of different linear combinations of non-primary first class constraints. A particular choice of tertiary constraints can lead to considerable simplification in some parts of the analysis; but the algebra with derivatives of delta functions or test functions can be completely avoided [4, 14], and are not needed for proof of closure nor for the restoration of gauge invariance using the Castellani algorithm [16].

IV. LINEAR COMBINATIONS OF TERTIARY CONSTRAINTS, CANONICAL TRANSFORMATIONS AND THEIR INTERPLAY

Considering the time development of secondary constraints in Section 2 we demonstrated that tertiary constraints can be defined in different ways and all such combinations do not contradict the Dirac procedure. All the different choices are linear combinations of each other, i.e. the Hamiltonian formulation of affine-metric GR provides an example of the theory with the non-artificial appearance of different linear combinations of constraints. We would like to discuss this apparent ambiguity of the Hamiltonian procedure. We restrict our discussion to different combinations of tertiary constraints that we initially defined in (58), (59) and the corresponding part of the canonical Hamiltonian

\[
H''_c = \frac{1}{\hbar^{00}} \tau^{00} + \frac{\hbar^{0k}}{\hbar^{00}} \tau_{k}^{00} .
\]

Of course, more choices exist if the secondary constraints are also used in such redefinitions; but it will just make the calculations more involved. Some conclusions can be made based on simple examples. Note that the role of first class primary constraints is quite special and will be discussed in next Section.
One possible choice of tertiary constraints is

\[ \bar{\tau}_0^{00} = \frac{1}{\hbar_{00}} \tau_0^{00}, \quad \bar{\tau}_k^{00} = \tau_k^{00}. \]  

(84)

In terms of these constraints, the part, \( H''_c \), of the canonical Hamiltonian \( \textbf{[83]} \) is

\[ \bar{H}''_c = \bar{\tau}_0^{00} + \frac{\hbar_{00}^{0k}}{\hbar_{00}^{00}} \bar{\tau}_k^{00}. \]  

(85)

The PBs among constraints \( \bar{\tau}_0^{00} \) and \( \bar{\tau}_k^{00} \) can be easily found by using their relations with the original choice \( \textbf{[84]} \) and the corresponding algebra of constraints \( \textbf{[65]} - \textbf{[68]} \), e.g.

\[ \{ f(\overrightarrow{x}), g(\overrightarrow{y}) \bar{\tau}_0^{00}(\overrightarrow{x}), \bar{\tau}_i^{00}(\overrightarrow{y}) \} = -f \frac{1}{\hbar_{00}^{00}} h_{i,i}^0 g_{i,0}^{00} + f_{i} g_{i,0}^{00} - f_{i,0}^{00} g_{i}. \]  

(86)

The whole algebra of \( \bar{\tau}_0^{00} \) and \( \bar{\tau}_k^{00} \) can be calculated and the closure of the Dirac procedure can be demonstrated using these combinations.

Another choice is:

\[ \tilde{\tau}_0^{00} = \frac{1}{\hbar_{00}^{00}} \tau_0^{00} + \frac{\hbar_{00}^{0k}}{\hbar_{00}^{00}} \tau_k^{00}, \quad \tilde{\tau}_k^{00} = \tau_k^{00}. \]  

(87)

that leads to a very simple expression for the corresponding part of the canonical Hamiltonian

\[ \tilde{H}''_c = \tilde{\tau}_0^{00}, \]  

(88)

and again the PB algebra of constraints \( \tilde{\tau}_0^{00} \) and \( \tilde{\tau}_k^{00} \) can be found using \( \textbf{[87]} \) and \( \textbf{[65]} - \textbf{[68]} \), e.g.

\[ \{ f(\overrightarrow{x}), \tilde{\tau}_0^{00}(\overrightarrow{x}), g(\overrightarrow{y}) \tilde{\tau}_i^{00}(\overrightarrow{y}) \} = f_{i} g_{i,0}^{00} - f_{i,0}^{00} g_{i} + f_{i,0}^{00} \frac{\hbar_{00}^{0k}}{\hbar_{00}^{00}} \tau_k^{00} - f_{i,0}^{00} \frac{\hbar_{00}^{0k}}{\hbar_{00}^{00}} \tau_k^{00}. \]  

(89)

Is there any physical significance in a particular choice of constraints? Of course, there are some possible technical (computational) advantages; but considering different linear combinations of tertiary constraints should not affect the physical results. The simplest argument is to convert our reduced total Hamiltonian into the corresponding Lagrangian by inverse Legendre transformation

23
\[ L = \dot{\Gamma}^{\mu}_{00} \Pi_{\mu}^{00} + \dot{h}^{\alpha \beta} \pi_{\alpha \beta} - H_T = \dot{h}^{\alpha \beta} \pi_{\alpha \beta} - \Gamma^{\alpha}_{00} \chi^{00}_{\alpha} - H'' - A^{\mu}_{00} \chi^{00}_{\mu}, \tag{90} \]

where \( H'' \) is a functional, \( H''(h^{00}, h^{0k}, h^{km}, \pi_{km}) \). Whatever combination we consider, \( (83), (85) \) or \( (88) \), we have the same Lagrangian \( (H'' = \tilde{H}'' = \hat{H}'') \) and by calling some parts of these Lagrangians by ‘tertiary constraints’, which is nothing more than a short notation at the Lagrangian level, we cannot affect the physics; and in particular, the gauge invariance should not change. So, if a choice of tertiary constraints cannot influence gauge invariance, then an algebra of PBs for this particular choice should not bear any physical significance.

For the Hamiltonian formulation of the second order metric GR, we can make the same conclusion, but, of course, for possible choices of secondary first class constraints and their algebra, contrary to a broadly accepted view that a particular choice of constraints/algebra has some physical significance that is even reflected in special names given to one particular choice, “Hamiltonian” and “spatial diffeomorphism” constraints. These particular combinations become special only after a non-canonical change of variables is made (see discussion in next Section). Related to the combinations of constraints idea of the Master Constraint Programme is of limited interest as any physical results cannot depend on a particular choice of non-primary first class constraints; in other words, in the “society” of non-primary constraints there is no place for a Master. One additional conclusion that is connected to the freedom to choose combinations of tertiary constraints is more technical and related to the methods of the restoration of gauge symmetry, based on a full set of first class constraints (the Dirac conjecture). Gauge invariance should be independent of a choice of non-primary first class constraints. So any method, which is sensitive to redefinition of constraints (gives different transformations) is not correct (see, Sections 6, 7).

For any Hamiltonian formulation, if change of phase-space variables is performed then it must be canonical. Also such a change should not affect the physical properties of a system, in particular, it should not change its gauge invariance. In the second order formulation of GR we considered the connection of two Hamiltonians, Dirac’s and PSS, and demonstrated that they are related by a canonical transformation and both lead to the same gauge transformation which is four diffeomorphism invariance \[ 4, 14, 15 \]. In the passage from PSS to the Dirac formulation, we followed Dirac’s idea: to simplify primary constraints. He achieved this by modifying the original Lagrangian. We worked in phase space and performed the canonical transformation \[ 15 \]. In first order formulation of GR that we consider here, the
primary first class constraints already have the simplest possible form: pure momenta con-
jugate to $\Gamma_{00}^\mu$, and any further simplification is impossible. We should have different reasons
to look for canonical transformations. One such a reason is to simplify the expressions for
the secondary constraints, for example:

$$
\chi_{00}^0 = -h_{,k}^0 + h^{00}_k\pi_{00} - h^{km}_k\pi_{km} = -h_{,k}^0 + h^{00}_k\tilde{\pi}_{00} = \chi_{00}^0,
$$

(91)

$$
\chi_k^{00} = h_{,k}^{00} + 2h^{00}_k\pi_{0k} + 2h^{m0}_k\pi_{km} = h_{,k}^{00} + 2h^{00}_k\tilde{\pi}_{0k} = \chi_k^{00},
$$

(92)
i.e. to introduce new momenta

$$
\tilde{\pi}_{00} = \pi_{00} - \frac{h^{km}}{h_{00}^{00}}\pi_{km},
$$

(93)

$$
\tilde{\pi}_{0k} = \pi_{0k} + \frac{h^{m0}}{h_{00}^{00}}\pi_{km}.
$$

(94)

These two redefinitions, (93) and (94), are algebraic and invertible; but this is not enough
to preserve the canonicity of phase-space variables, and it must be accompanied by a change
of the remaining phase-space variables, which are involved in redefinitions (93)-(94). Such
a necessary and sufficient condition for the transformation to be canonical is

$$
h_{0\alpha\beta}^{\alpha\beta} = \tilde{h}_{0 \alpha\beta}^{\alpha\beta}.
$$

(95)

Let us restrict our search for canonical transformations by assuming that

$$
h_{00} = \tilde{h}_{00}, \quad h^{0k} = \tilde{h}^{0k}.
$$

(96)

(This restriction has to be relaxed if it is not possible to satisfy (95).) After substitution of
new variables in terms of old into (95), and some simple rearrangements we obtain

$$
\tilde{h}_{00}^{km}\bar{\pi}_{km} = \left(h_{00}^{km} - h^{0k}\tilde{h}^{00}\right)_{,0}\frac{1}{h_{00}^{00}}\pi_{km}.
$$

(97)

To fulfill this condition we have to define:

$$
\bar{\pi}_{km} = \frac{1}{h_{00}^{00}}\pi_{km}, \quad \tilde{h}_{00}^{km} = h_{00}^{km} - h^{0k}\tilde{h}^{00}\tilde{h}^{0m} = h_{00}^{00}e^{km}.
$$

(98)
Of course, that the transformation which involves (93), (94), (96) and (98) is canonical can be checked by direct calculation of the PBs among all new variables; and this should lead to:

\[
\{\bar{h}^\alpha{}^\beta, \bar{\pi}^\nu{}^\mu\} = \{h^\alpha{}^\beta, \pi^\nu{}^\mu\}, \quad (99)
\]

\[
\{\bar{h}^\alpha{}^\beta, \bar{h}^\nu{}^\mu\} = \{h^\alpha{}^\beta, h^\nu{}^\mu\}\}_{h^\alpha{}^\beta, \pi^\nu{}^\mu} = 0,
\]

\[
\{\bar{\pi}^\alpha{}^\beta, \bar{\pi}^\nu{}^\mu\} = \{\pi^\alpha{}^\beta, \pi^\nu{}^\mu\}\}_{h^\alpha{}^\beta, \pi^\nu{}^\mu} = 0.
\]

Note, that the second equation in (98) was used by Faddeev [25], but without any discussion of canonicity, i.e. necessary changes for the rest of the variables. In Faddeev’s approach all of the problems related to such a change of variables are hidden because, in addition, some first class constraints were solved. This is against the Dirac procedure and it eliminates the possibility to restore gauge invariance as all first class constraints are needed (a simple example can be found in [5]).

Equally well, instead of a simplification of secondary constraints, we can try to use a combination \(e^{km}\) which naturally appeared when the secondary second class constraints were solved (the same combination, \(e^{km}\), was used in the second order GR by Dirac [3]). We used \(e^{km}\) as a short-hand notation; but it is possible to find such a canonical transformation that converts \(e^{km}\) into a new variable. Let us introduce a variable

\[
\bar{h}^{km} \equiv e^{km} = h^{km} - \frac{h^0{}^k h^0{}^m}{h^0{}^0}.
\]

As in the previous case, we will restrict our search by imposing

\[
h^0{}^0 = \bar{h}^0{}^0, \quad h^0{}^k = \bar{h}^0{}^k
\]

and use the same condition as (95)

\[
h^\alpha{}^\beta_0 \pi^\alpha{}^\beta_0 = \bar{h}^\alpha{}^\beta_0 \bar{\pi}^\alpha{}^\beta_0.
\]

Substitution of \(h^{km}\) from (100) into (102) gives
\[ \tilde{h}_{ik}^{km} \pi_{km} + 2 h_{0k}^{0m} \left( \pi_{0k} + h_{00}^{0m} \pi_{km} \right) + h_{00}^{00} \left( \pi_{00} - h_{0k}^{0m} h_{00}^{0m} \pi_{km} \right) = \tilde{h}_{0k}^{km} \pi_{\alpha\beta} + 2 \tilde{h}_{0k}^{0k} \pi_{0k} + \tilde{h}_{00}^{00} \pi_{00} \]

(103)

and the redefinition of momenta follows:

\[ \bar{\pi}_{km} = \pi_{km} , \]

(104)

\[ \bar{\pi}_{0k} = \pi_{0k} + h_{00}^{0m} \pi_{km} , \]

(105)

\[ \bar{\pi}_{00} = \pi_{00} - h_{0k}^{0m} h_{00}^{0m} \pi_{km} . \]

(106)

It is easy to check that (104)-(106), together with (100) and (101), give the same PBs as (99) and so these transformations are canonical.

Another possible argument to find canonical transformations is to look at expressions for tertiary constraints (or rather our first choice of tertiary constraints) (58), (59) and try to simplify them. There is one obvious combination, \( h_{00} e^{km} \). And it can be used to build canonical transformations; but it will lead to the same transformation that we have already considered in the first example (98). If we use it together with (93)-(94), we will get the canonical transformations, which simplify secondary and tertiary constraints simultaneously.

Substitution of (98) into constraints (58) and (59) gives:

\[ \bar{\tau}_{ik}^{00} = 2 \left( \tilde{h}^{mq} \bar{\pi}_{mk} \right)_{,k} - 2 \left( \tilde{h}^{mq} \bar{\pi}_{kq} \right)_{,m} - \tilde{h}_{ik}^{np} \bar{\pi}_{np} , \]

(107)

\[ \bar{\tau}_{00} = \tilde{h}_{kp}^{mq} \left( \pi_{mp} \bar{\pi}_{kq} - \pi_{kp} \bar{\pi}_{mq} \right) \]

where \( \tilde{h}_{km} \) is defined as \( \tilde{h}_{pq}^{qm} h_{mp} = \delta_{q}^{p} \) and related to the original variables by \( \tilde{h}_{km} = h_{km} \).

These constraints, (107) and (108), are very similar to the constraints obtained in [11] for the first order formulation based on variables (43). This is not a surprise, as both first order formulations are equivalent to the EH action. Up to a simple rearrangement and with a different notation the expressions corresponding to (107)-(108) in [11] are the following:
is equivalent with Eq. (56) of \[11\] and in (108), only the first term differs in sign from the corresponding term in Eq. (58) of \[11\].

The effect of canonical transformations on the algebra of constraints is simple and it preserves its form (form-invariance). In general, a canonical change of variables leads to changes in constraints

\[(q, p) \rightarrow (Q, P) : \psi_\mu (q, p) \rightarrow \Psi_\mu (Q, P)\]

and if the algebra of constraints in old variables is

\[\{\psi_\mu, \psi_\nu\} = c_{\mu\nu}^\gamma \psi_\gamma,\]

where \(c_{\mu\nu}^\gamma\) are structure functions, then in new variables its form should be preserved and given by

\[\{\Psi_\mu, \Psi_\nu\} = C_{\mu\nu}^\gamma \Psi_\gamma\]

with the simple condition on the structure functions

\[C_{\mu\nu}^\gamma (Q, P) = c_{\mu\nu}^\gamma (q, p)_{q=q(Q,P), p=p(Q,P)} .\]

For the first time such properties were demonstrated for the canonical transformation in linearized gravity \[36\] and later for a complete formulation \[15\]. In the Hamiltonian formulation of the first order EH action, the form-invariance of the algebra of constraints is also preserved after canonical transformations. For example, our secondary \(\chi_{0}^{\mu} \) (23)-(24) and tertiary \(\tau_{00}^{\mu} \) (58)-(59) constraints have the algebra of PBs given in (25), (61), (70)-(73). Performing the canonical transformations (93), (94), (96), (98), we obtain new secondary \(\tilde{\chi}_{0}^{\mu}\) (91)-(92) and tertiary \(\tilde{\tau}_{00}^{\mu}\) (107)-(108) constraints and calculate new PBs among them. The algebra of new PBs is related to the old one exactly as is described by (109)-(112), i.e. it is form-invariant.

In the first order formulation of the EH action, we consider the canonical change of variables that does not affect the primary variables - fields for which their corresponding momenta are primary first class constraints. So the corresponding algebra of all first class primary constraints is form-invariant (in new and old variables primary first class constraints have zero PBs with the rest of constraints). In the second order, PSS/Dirac, formulation,
canonical transformations also affect the primary constraints; but the form-invariance of the whole algebra is preserved (see more details in \[4, 14, 15\] and discussion in next Section).

In this Section we considered two operations: using different linear combinations of tertiary constraints and canonical transformations that involve only non-primary variables. What is the relationship between these two operations? We showed that they are independent in the following sense. Considering different choices of tertiary constraints for the same canonical transformation produces different algebra among constraints, but preserves its form-invariance. For example, compare $\tau_{\mu}^{00}$ (84) with $\tau_{\mu}^{00}$ for the canonical transformation given by (93), (94), (96), (98). The PBs for them are different (compare, for example, (86) and (89)); but the form-invariance is preserved in accordance with (109)-(112). Applying different canonical transformations to the same choice of tertiary constraints modifies the constraints and structure functions, but it also preserves the form-invariance of the algebra of constraints. We briefly discussed examples of such operations with tertiary first class constraints and canonical transformations that did not involve primary variables (and primary first class constraints). We will consider operations with them in the next Section.

V. THE ROLE OF PRIMARY FIRST CLASS CONSTRAINTS

Let us discuss the special properties and distinct role of primary first class constraints in the Hamiltonian formulation of gauge invariant theories. In the previous Section we discussed and demonstrated, by examples, the independence of two operations: the choice of combinations of non-primary first class constraints and canonical transformations of phase-space variables. We will show that these two operations are not independent any more for primary first class constraints, which are either pure canonical momenta as, for example, in the Hamiltonians of the affine-metric formulation and the metric formulation due to Dirac \[3\], or pure momenta plus some extra contributions, as in the metric formulation due to Pirani, Schild and Skinner (PSS) \[13\] (the oldest one). In the PSS formulation, primary constraints are originated from terms in the GR Lagrangian linear in the temporal derivatives (“velocities”) of the $g_{0\mu}$ components of the metric tensor, and in the Dirac formulation or any first order formulation of gauge invariant theories (e.g. affine-metric or tetrad-spin connection), from variables without temporal derivatives in the corresponding Lagrangians. Momenta conjugate to such variables are primary constraints and at the same time they
are part of a phase space of a considered system. This part of a phase space is often and mistakenly neglected in Hamiltonian formulations of GR (e.g. see discussion on p. 47 of [4] and references therein).

In the Dirac approach to the Hamiltonian formulation of constrained systems, all variables are treated on an equal footing and each variable has the corresponding momentum. Moreover, variables that are often neglected in the Hamiltonian formulation of GR even have a special name given by Bergmann: “primary”, that reflects their importance. In monographs on constrained dynamics, e.g. [9, 10], and in non-GR Hamiltonians (Maxwell, Yang-Mills), primary constraints are always present and are part of the total Hamiltonian (name given by Dirac [8]).

For the Hamiltonian of first order metric-affine GR any change of the primary first class constraints would be artificial as they are already in the simplest possible form. So, to discuss canonical transformations that involve primary first class constraints we refer to two Hamiltonian formulations of the same theory, PSS and Dirac. Constraints and structure functions of their algebra are quite different [4, 14]; but in both cases the complete sets of first class constraints lead to the same gauge invariance, as it should be. And this invariance, derivable from the constraints, is the four-dimensional diffeomorphism that follows directly with no need for a field dependent redefinition of the gauge parameters. For the ADM formulation the gauge transformations differ from four-diffeomorphism and the only so-called “correspondence” with diffeomorphism [37], or “diffeomorphism-induced” [38], or “specific metric-dependent diffeomorphism” [39], etc. can be accomplished.

The two total Hamiltonians of PSS and Dirac, $H_{PSS}^T$ and $H_{Dirac}^T$ respectively, are given by [14] and [4]:

$$H_{PSS}^T = \dot{g}_{0\rho} \left( \pi_{PSS}^{0\rho} - \varphi_{0\rho} (\pi, g) \right) + g_{0\rho} \mathcal{H}_{PSS}^{0\rho} (\pi^{km}, g_{\mu\nu}) ,$$

(113)

$$H_{Dirac}^T = \dot{g}_{0\rho} \pi^{0\rho}_{Dirac} + g_{0\rho} \mathcal{H}_{Dirac}^{0\rho} (\pi^{km}, g_{\mu\nu}) .$$

(114)

In [15] it was explicitly shown that the phase-space variables of the two formulations are related by a canonical transformation that can be performed if one wants to simplify the primary first class constraints of the PSS formulation. Note that Dirac [3] found a suitable change in the EH Lagrangian by adding to it two total derivatives, which does not affect the
equations of motion, but the primary constraints can be brought into simple form. In [15] it was shown that the same simplification can be accomplished at the Hamiltonian level using canonical transformations in its phase space and, of course, with the same result. What is important is that the comparison of two formulations [15] with different constraints and structure functions in the algebras of constraints gives exactly relations (109)-(112). Note that primary first class constraints of the Dirac and PSS formulations have non-zero PBs with a particular choice of secondary constraints that were used in [14] and [4]; but this part of the algebra and structure functions also preserves form-invariance under a canonical transformation. So, conditions (109)-(112) are satisfied by all first class constraints; and this is in complete correspondence with the Dirac conjecture [8] that all first class constraints generate gauge symmetry. It is to be expected that all relations amongst them must be preserved under canonical transformations to keep invariance in tact.

As in a first order formulation, different combinations of secondary constraints for the Dirac Hamiltonian (114) can also be considered; and one particular combination was discussed in [4]

\[ H_T = \dot{g}_{0\rho} \pi^{0\rho} + (-g^{00})^{-1/2} \tilde{H} - \frac{g^{0i}}{g^{00}} \tilde{H}_i. \]  

(115)

Note, there are some similarities with our first choice of tertiary constraints for part \( H_c \) (83). One particular choice, out of many possible linear combinations of constraints, as in the case of first order formulation (where a few choices of tertiary constraints were considered in previous Section), is not special and cannot affect the gauge invariance of this Hamiltonian; so it has to lead to a complete restoration of four-dimensional diffeomorphism with field independent gauge parameters as was stated in [4]. Of course, the algebra of these combinations is different compared with algebra of constraints for (114); but all the different choices should not affect the physical results. Similar combinations can also be constructed for (113); and under canonical transformations, the form-invariance is also preserved, which is the same interplay of linear combinations of non-primary first class constraints and canonical transformations that we illustrated for tertiary constraints in previous Section.

Now we briefly discuss a connection of (115) with the conventional formulation due to Arnowitt, Deser and Misner (ADM) [2] in which the total Hamiltonian is (see e.g. [16])

\[ H_T = \dot{N}P + \dot{N}^i P_i + N\dot{H} + N^i \dot{H}_i \]  

(116)

31
where the secondary constraints are exactly the same as Dirac’s and are known as the “Hamiltonian” \( \hat{\mathcal{H}} \) and the “spatial diffeomorphism constraint” \( \hat{\mathcal{H}}_i \). According to Pullin [40] “It [ADM paper [41]] bases the formulation on the Palatini action principle”, which is actually the affine-metric formulation of Einstein [1] (see footnote 3). But it is clear that it differs from our results for the affine-metric formulation (29), (60) and rather obviously have, at least, some similarities with Dirac’s second order formulation (115) where the coefficients in front of constraints for one out of many possible combinations were redefined and called new variables:

\[
N = (-g^{00})^{-1/2}, \tag{117}
\]

\[
N^i = -\frac{g^{0i}}{g^{00}} \tag{118}
\]

which are known as “lapse” and “shift” functions\(^4\).

As in the examples from the previous Section, because we are working in a phase space we cannot just use some invertible transformations like (117)-(118) and simply write the new total Hamiltonian as (115). To preserve canonicity, the change of variables (117)-(118), which is obviously invertible, has to be accompanied by a change of the rest of phase-space variables or, at least, some of them (as in the examples considered in previous Section). But, as in ADM approach, the space-space components of the metric tensor, \( g_{km} \), are not changed and are exactly the same as in the Dirac formulation

\[
\left( g_{km}, \pi^{km} \right)_{Dirac} = \left( g_{km}, \pi^{km} \right)_{ADM} ; \tag{119}
\]

and the search for new momenta has to be restricted by

\[
\dot{g}_{0\rho} \pi^{0\rho} = \dot{N} P + \dot{N}^i P_i . \tag{120}
\]

\(^4\) Note that even the names used in this formulation manifest their non-covariant nature and shows the distinction of these variables, or different roles that they play in the ADM formulation. Their names appeared soon after the original works of ADM were published. To the best of our knowledge, it was Wheeler who coined the names of these variables, “lapse” and “shift” in [33]. DeWitt in [29] reserved the name “Hamiltonian constraint” only for \( \hat{\mathcal{H}} \) as it is a “particularly important constraint”, probably, to reflect its distinction from \( \hat{\mathcal{H}}_i \), although \( \hat{\mathcal{H}} \) and \( \hat{\mathcal{H}}_i \) are both the part of the Hamiltonian.
But it is impossible to find such a transformation that preserves (120) with the additional condition (119). This unavoidably leads to a conclusion that passing from (115) to (116) is not a canonical transformation (an interested reader can find more detail in Section 4 of [4]). The calculation of one simple PB is enough to prove non-canonicity (see Eq. (152) of [4])

\[
\{ N, \pi^{km} \} = \left\{ (-g^{00})^{-1/2}, \pi^{km} \right\} \neq 0.
\]  

(121)

So, because the ADM Hamiltonian and Dirac Hamiltonian of GR are not related canonically, any connection between them is lost; but the disappearance of four-dimensional diffeomorphism cannot be explained just by this fact. Even if the ADM formulation is considered as a model, not related to the Hamiltonian formulation of metric GR, one can argue that it still might have diffeomorphism invariance and base his arguments on the fact that the number of the primary first class constraints are still the same, and this number defines the number of gauge parameters. Note, if the ADM Hamiltonian (116) is treated as a model, then the lapse and shift functions are canonical variables of this formulation. But if one claims that (116) is a canonical formulation of GR, then it is not the case because ADM variables are not related canonically to the metric tensor and its momentum. To accommodate these two possible understandings we will use quotation marks for “canonical” ADM Hamiltonian\(^5\).

It is not possible to find a canonical transformation, part of which constitutes (117)-(118) with the corresponding momenta (120) and which simultaneously preserves the condition that the space-space components of \(g_{\mu\nu}\) remains the same as in the Dirac formulation (119). But by relaxing this too restrictive condition (119), the canonical transformation can be found and its form was given in [4] (see Eqs. (156)-(158)) and the problems that arise with such transformations were discussed [4]. Note that such a transformation, of course,

\(^5\) The common statements as in [42] “Unfortunately, the canonical treatment breaks the symmetry between space and time in general relativity and the resulting algebra of constraints is not the algebra of four diffeomorphism” has a double meaning. If “canonical” is understood as the formulation of ADM this is a true statement; but for GR, for which ADM formulation is not canonical, Pullin’s statement is wrong. We would like to note that in the Hamiltonian formulation of GR the covariance is not manifest, however, it is not broken as the gauge symmetry of GR, four-diffeomorphism, is recovered in manifestly covariant form (see [4, 14]). So, the Hamiltonian formulation of GR does not break the main property of the Einstein GR: general covariance.
converts the Dirac Hamiltonian into a form which is different from ADM anyway. So, for systems with constraints, even canonical transformations can lead to some problems; and for such systems the canonicity of the transformations is the only necessary condition to have equivalent formulations. Note that our conclusion based on a particular model, is in contradiction with general discussion of canonical transformations for constraints systems of where the authors stated that the condition of canonicity is “too strong” for constraint systems; but in our opinion, based on a particular theory, it is too weak, at least for covariant theories with first class constraints.

One obvious, at least for covariant theories, problem with the ADM variables is in the original transformation (117)-(118) and it is related to simple dimensional analysis and to a special role of the primary constraints. If we can find canonical transformations, they have to preserve (120) so, in particular, we will have the following part in \( H_T \)

\[
H_T = \dot{N}P + \dot{N}^i P_i + ... \tag{122}
\]

with simple primary first class constraints \( P \) and \( P_i \). However, if fields have a physical dimension, then the components of the metric tensor should have the same dimension. It is obvious that the lapse and shift functions defined by (117)-(118) have different dimensions in terms of the dimension of the metric tensor (\( \dim N^i = 0 \) and \( \dim N = 1/\sqrt{\dim g^{00}} \)); so the corresponding momenta in (122) (which are primary constraints in such formulation) should have different dimensions as well. And such a dimensional mismatch of primary constraints for a covariant theory guarantees the failure of this formulation to preserve covariance and so such a transformation, even being canonical, has to be rejected in any Hamiltonian formulation of a covariant system. This conclusion is based on the following arguments related to the special role of primary constraints in the derivation of gauge invariance. In the Castellani procedure [16], which we used to restore four-diffeomorphism invariance for the PSS and Dirac formulations [14] and [4], the gauge generator is started from primary first class constraints

\[
G = \partial_0^{(n)} \varepsilon^\mu P_\mu + ... \tag{123}
\]

where \( \partial_0^{(n)} \) is the temporal derivative of \( n \)-th order of the gauge parameters \( \varepsilon^\mu \), \( n \) depends on the number of generation of the constraints (if secondary constraints are present then
$n = 1$, if tertiary: $n = 2$, etc.) For a covariant theory $\varepsilon^\mu$ should be a true four-vector. Again based on dimensional analysis, if the primary constraints have a different dimension, then the components of the gauge parameters also have different dimensions. It is the well-known fact that four quantities combined together do not necessarily form a true four-vector as their components must transform in the same way under general coordinate transformations. As we can see from (117) and (118), $N$ and $N^i$ transform differently, because they are defined in terms of particular components of the metric tensor in non-covariant way (see again (117) and (118)). This property is also transferred to the corresponding momenta and gauge parameters. If, at least one gauge parameter has a different dimension or transforming property from the remaining parameters (which is exactly the case here), it is just impossible to combine them into the four-vector gauge parameter, which is needed for four-diffeomorphism. To conclude: even in the case of canonical change of variables, but with a mismatch of their dimensions (at least for variables which correspond to primary constraints) the covariance is lost. So, introduction of lapse and shifts functions by itself, whether they are a part of a canonical transformation or not, unavoidably destroys covariance and in turn the equivalence with the original covariant theory. Please note that there are different approaches to the restoration of gauge invariance, where a generator is built on other principles and is actually started from the end of the constraint chains (e.g. see [12]), i.e. from secondary constraints (in second order formulation) multiplied by the gauge parameters

$$G = \ldots + \varepsilon \hat{H} + \varepsilon^i \hat{H}_i.$$ (124)

In this case the conclusion is the same, because if lapse and shift functions have different dimensions, so do the “Hamiltonian” and “spatial diffeomorphism” constraints and the corresponding gauge parameters in (124). We will discuss in detail the methods of restoration of gauge invariance in Sections 6 and 7 with application to a first order affine-metric formulation of GR. Here we would like just to add, that in both methods, the initial assumption of either the Castellani algorithm or HTZ ansatz is the independence of gauge parameters of fields; and this assumption is used in the iterative procedure to a find generator. So any field dependent redefinition of gauge parameters, as advocated in many articles (i.e. [37, 38, 39]), that is performed after completion of the procedure, is in complete contradiction with the initial assumptions for both approaches.
Now we will consider one additional example, the Hamiltonian formulation of the Einstein-Cartan (EC) theory, which is a little bit aside of the main topic of this article. But it provides an illustration of even further (possibly general) restriction on the manipulations with primary constraints (canonical variables), which cannot be illustrated using metric or affine-metric formulations.

The Hamiltonian formulation of the Einstein-Cartan theory in its first order form, the so-called tetrad-spin connection formulation was discussed in many works (i.e. [46], [47]). As in the Hamiltonian formulation of affine-metric GR, it leads to second class constraints that should be eliminated, and after the Hamiltonian reduction leads to the following total Hamiltonian (up to a total spatial derivative) [45, 48]

\[ H_T = \dot{e}_0^\rho \pi^0_\rho + \dot{\omega}_0^{\alpha\beta} \pi^0_{\alpha\beta} + e_0^\rho \chi^0_\rho + \omega_0^{\alpha\beta} \chi^0_{\alpha\beta}. \]  

(125)

In the 3D case, to obtain (125) is a simple task because there are no secondary second class constraints [48]. In the 4D case, the different methods (specific to this dimension) of solving secondary constraints were used. One particularly transparent method is the introduction of Darboux coordinates (specifically constructed only for 4D) due to Bañados and Contreras [49]. Of course, the Dirac procedure can be used in any dimension higher than two,\(^6\) which gives the same Hamiltonian (125), and that was shown in [45]. Direct calculations are involved and a considerable simplification occurs when the Darboux coordinates (common to all dimensions) are used [50].

The notation can vary from paper to paper, but it is usually explained in detail. The equation (125) is written in the notation used in [45]; and it is very close to what can be found in the first “gauge-free” formulation\(^7\) of the Einstein-Cartan Hamiltonian, due to Castellani, van Nieuwenhuizen and Pilati [46], where the canonical variables (after elimination of secondary second class constraints) are

\[ e_\mu(\rho), \pi^\mu(\rho), \omega_0^{\alpha\beta}, \pi^0_{\alpha\beta}. \]  

(126)

\(^6\) As for metric EH action, the second and the first order, affine-metric, formulations are equivalent only in dimensions higher than two, the same thing happens for EC action: tetrad and tetrad-spin connection formulations are equivalent also when \( D > 2 \) (see [43], Section II).

\(^7\) “Gauge-free” means without fixing a gauge at the beginning of analysis. Such a fixing is in contradiction with the Dirac procedure as a gauge cannot be fixed before a gauge symmetry is found.
The form of (125), which is a nice covariant expression for the total Hamiltonian of the first order EC action, was known for a long time. But it is difficult to find it in this form, with a few rare exceptions (e.g. [51]) where the first two terms of (125) are given in Eq. (2.4) and the last two in Eq. (3.3)). Because the “canonical” formulation, in accordance with the conventional wisdom, is equivalent with the presence of lapse and shift functions, the change of variables is always performed to introduce them\(^8\) instead of completion of the Dirac analysis (proof of closure) and restoration of gauge invariance for (125). (Some steps in the analysis of (125) can be found in [45, 48].)

The Hamiltonian in terms of lapse and shift functions for tetrads becomes (this form is much easier to find in literature, contrary to (125), e.g. [52])

\[
H_T = \dot{NP} + \dot{N}^i P_i + \omega_{0(\alpha\beta)} \pi^0(\alpha\beta) + N\tilde{\mathcal{H}} + N^i \tilde{\mathcal{H}}_i + \omega_{0(\alpha\beta)} \chi^{0(\alpha\beta)}.
\] (127)

Note that introduction of lapse and shift functions guarantees the disappearance of covariance in the formulation. We have already discussed the effect of the ADM variables, which have different dimensions; and any hope to have a covariant formulation is lost. They also, as in metric formulations, are not canonical for a transformation from (125) to (127) if they are introduced with restriction on the rest of tetrad components and corresponding momenta (which is always the case).

The canonicity of transformation in the phase space, i.e. from the complete set of canonical variables of (125) \((e_{\mu(\rho)}, \pi^{\mu(\rho)}, \omega_{0(\alpha\beta)}, \pi^{0(\alpha\beta)})\) \([45, 46, 51]\) to variables of (127) \((N, N^i, P_i, e_{k(\rho)}, \pi^{k(\rho)}, \omega_{0(\alpha\beta)}, \pi^{0(\alpha\beta)})\), has never been discussed and it has the same deficiency as for passing from the Dirac to ADM formulations in the second order metric EH action. Moreover, exactly as in the metric case, only the part of variables is involved in such a change

\[
e_{0(\rho)}, \pi^{0(\rho)} \rightarrow N, N^i, P, P_i,
\] (128)

\[
(e_{k(\rho)}, \pi^{k(\rho)}, \omega_{0(\alpha\beta)}, \pi^{0(\alpha\beta)})_{EC} = (e_{k(\rho)}, \pi^{k(\rho)}, \omega_{0(\alpha\beta)}, \pi^{0(\alpha\beta)})_{ADM}
\] (129)

where the lapse (117) and shift (118) functions have to be expressed in terms of the original

\(^8\) Such a transition is justified by either “it is more convenient” [51] or “it is useful” [46].
phase-space variables (126) of the reduced Hamiltonian (125) using \( g^{\mu\nu} = e^\mu_{(\rho)} e^{\nu(\rho)} \)

\[
N = \left(-e^0_{(\rho)} e^{0(\rho)}\right)^{-1/2},
\]

(130)

\[
N^i = -\frac{e^0_{(\gamma)} e^{i(\gamma)}}{e^0_{(\rho)} e^{0(\rho)}},
\]

(131)

As in the metric case, there are no canonical transformations for the subset of phase-space variables (128) if the rest of variables of (126) is not involved. As in the transition from the Dirac to ADM variables it was enough to find one PB (121) which proves non-canonicity of the ADM variables, for the tetrad formulation at least one PB is also non-zero

\[
\{ N^i, \pi^{k(\lambda)} \} = \frac{\delta N^i}{\delta e^{k(\lambda)}} \neq 0.
\]

(132)

So, introduction of lapse and shift functions in the tetrad formulation is also a non-canonical transformation. Actually, based on previous analysis, it is obvious that such a formulation (even with adjustments for canonicity that will change constraints) will create a dimensional mismatch of variables and gauge parameters, and so destroy the covariance in exactly the same way as lapse and shift functions destroy it for the metric GR. The combination “Dirac-ADM” is not correct, as well as “ADM-Einstein-Hilbert”, so the ADM and Einstein-Cartan formulations are not compatible. All problems related to the “spatial diffeomorphism” constraint safely propagate into the tetrad formulation of GR and such the Hamiltonian (127) is not the Hamiltonian of the original theory. In addition, this new theory is not covariant by construction. However, the use of ADM variables for first order tetrad-spin connection formulation is much more interesting example compared to its metric counterpart; and this is the main reason to include the Einstein-Cartan Hamiltonian in our discussion about the role of primary first class constraints.

Here one can make an additional and simple observation related to the role of primary first class constraints. Constructing the generator for formulation (125), one obtains

\[
G = \dot{\varepsilon}_{(\rho)} \pi^{0(\rho)} + \dot{\varepsilon}_{(\alpha\beta)} \pi^{0(\alpha\beta)} + ...
\]

(133)

with two gauge parameters, “rotational” \((\varepsilon_{(\alpha\beta)})\) and “translational” \((\varepsilon_{(\rho)})\), both with “internal” indices (which correspond to motion in the tangent space), and which lead to translational and rotational invariance in the internal space in 3D \([48, 53]\), as well as in all higher
dimensions \[45, 54\]. It must be emphasized that among the many papers on the Hamiltonian formulation of EC action, the work of \[46\] is an exception because this is the only one where lapse and shift functions are just a short-hand notation, so non-canonical transformations \([130], [131]\) were not performed. Should one apply the Castellani procedure (see next Section) starting from the primary constraints of \[46\] the gauge invariance, translation and rotation in the internal space, would follow (not a diffeomorphism, which is a symmetry, but not the \textit{gauge} symmetry of the EC action).

In contrast, if the generator is built for \([127] 52\) (let us forget for a moment about non-canonicity) then it becomes

\[
G = \dot{\varepsilon} P + \dot{\varepsilon}^i P_i + \dot{\varepsilon}_{(\alpha\beta)}\pi^{0(\alpha\beta)} + ... \tag{134}
\]

which also has two gauge parameters. One, as before, corresponds to rotation in the \textit{internal space} \((\varepsilon_{(\alpha\beta)})\); but the second (actually, there are two of them, \(\varepsilon\) and \(\varepsilon^i\)) corresponds to the translation in the \textit{external space}: \(\varepsilon\) for time lapse and \(\varepsilon^i\) for shift in a space-like surface. Because \(P\) and \(P_i\) are momenta conjugate to lapse and shift functions, which have different dimensions and rules of transformation, then \(P\) and \(P_i\) also do not form a true four-vector, and so are the parameters \(\varepsilon\) and \(\varepsilon^i\) (it follows from \([134]\)). Even if by relaxing the conditions of \([129]\) and finding some canonical transformations, and even with a match of dimensions, such a change of gauge symmetry, from translation in an internal space to lapse and shift in an external space would be strange.

Is it possible to have equivalence in such a case? This would mean that the Hamiltonian formulation does not give a unique \textit{gauge} invariance; but this conclusion is hard to accept.

Why does one, having the simplest possible constraints presented in \([125]\), have to perform some manipulations, besides the desire to have a “canonical” formulation, i.e. to have lapse and shift functions that destroy covariance? We think that for field theories, changes of variables (even canonical, if we can find such) should be restricted by the requirement to also preserve a tensorial character of primary variables. Maybe, such a condition is equivalent to the preservation of form-invariance of the algebra of constraints \([109]-[112]\). There are still many questions that need to be clarified. Dirac introduced his procedure as an outline of a general approach, together with his famous conjecture about the connection of first class constraints and gauge invariance. Only after the methods of restoration of gauge symmetry were developed, it became clear that the Dirac conjecture is correct and it can be used as a
procedure. But there are still many other questions, especially related to field theories, that remain to be answered.

Based on the examples considered in this Section, we can make a conclusion that for constrained field theories the canonicity of change of phase-space variables is a necessary, but not a sufficient condition. To keep an equivalence of two Hamiltonian formulations, the canonical transformations that lead to the mismatch of the dimensions for primary variables (which is permissible for mechanical systems or field theories without first class constraints); and also for canonical transformations that change tensorial character of primary variables, must be disregarded. Primary constraints are the true Masters of the Hamiltonian formulations of gauge theories. They do not need a master constraint programme as they are primary constraints; and manipulations with them are restrictive, especially for covariant theories. In the formulations considered here where primary constraints are in the simplest possible form (e.g. \((\Gamma_{\mu 00}, \Pi_{\mu 00})\)), it is better not to change them at all, to keep in tact the primary properties of the system (or there should be a very good reason to do the changes).

The reasons behind converting covariant expressions for the total Hamiltonian of the second order metric GR \((\Gamma_{13})\)-\((\Pi_{13})\) and the first order tetrad-spin connection formulation \((125)\) into non-covariant expressions \((116)\), and \((127)\), which destroy any hope of having covariant results (e.g. four-diffeomorphism for metric GR), remains a mystery to us. (We prefer to avoid speculation and leave it for the History of Science to figure out what happened 50 years ago, and why Einstein’s general covariance became unimportant.) But what is even more mysterious is the unquestionable acceptance of the ADM formulation by the majority of practitioners despite its inconsistencies and despite (not often) the strong voices that express concern about its deficiency, contradiction with GR, and even give “hints” about the source of the problems. We provide just a few such statements. Hawking 30 years ago concluded \[55\]: “The split into three spatial dimensions and one time dimension seems to be contrary to the whole spirit of relativity.” There appear more recently the statements of Pons \[39\]: “Being non-intrinsic, the 3+1 decomposition is somewhat at odds with a generally covariant formalism, and difficulties arise for this reason” and Rovelli \[56\]: “The very foundation of general covariant physics is the idea that the notion of a simultaneity surface over the universe is devoid of physical meaning”. The warning about the sources of problems in further constructions based on the ADM formulation was given by Landsman \[57\]: “the lack of covariance of the ADM approach, which is especially dangerous in connection with
quantum field theory”.

Finally, we repeat the “hint” given by Isham and Kuchar [58] “Thus the full group of spacetime diffeomorphism has somehow got lost in making the transition from the Hilbert action to the Dirac-ADM action”. This statement (without long calculation or a long chain of logical constructions) immediately leads to the conclusion that if a transition from one action (Einstein-Hilbert) to another (ADM) was performed by a change of variables and something got lost then “somehow” is exactly the change of variables that were performed in such a transition. We strongly object to the use of combination “Dirac-ADM”, as nothing “got lost” in the Dirac Hamiltonian formulation based on the Einstein-Hilbert action [3].

Last year, 50 years after publication of the Dirac Hamiltonian of GR [3], it was explicitly demonstrated [4] that his Hamiltonian of the second order metric GR leads directly to four-diffeomorphism without any, even numerical, redefinition of gauge parameters. This clearly demonstrates that four-diffeomorphism has not “got lost” if one abandons the idea of making a transition from the Einstein-Hilbert action to ADM action (i.e. if the ADM variables “got lost” instead). In [4] we argued (see Eq. (163) of [4]) that, based on equivalence of Hamiltonian and Lagrangian methods, it should be possible to demonstrate non-equivalence of the ADM and EH actions also at the pure Lagrangian level (without going to a phase space); but we did not formulate the criteria and did not show a proof. Recently we demonstrated that the Einstein-Cartan action is invariant under translation in a tangent space [54] using the pure Lagrangian method (to verify the strong indication of the presence of this gauge symmetry in the Hamiltonian formulation [45]). This also allows us to formulate the condition for equivalence of two actions at the pure Lagrangian level for singular systems: if transition (field redefinition) from one singular action (e.g. Einstein-Hilbert) to another singular action (e.g. ADM-inspired tetrad) is performed by an invertible change of variables (the necessary condition for equivalence) we can find differential identities for both of them and obtain the corresponding transformations for their sets of variables. These transformations, for equivalent formulations, must be derivable one from another by using the same (or inverse) redefinition of fields as in transition from one action to another10.

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9 We are thankful to Shestakova [59] for this observation and in this article we are also trying do keep track of important anniversaries.

10 The “loss” of four-diffeomorphism in the ADM formulation was demonstrated by Banerjee at al [60] (see Section 6 of [60]) where symmetries of the ADM Lagrangian and the Einstein-Hilbert Lagrangian
This year, one year after 50 anniversary of Dirac’s paper \[3\], it was a celebration dedicated to another paper on the same subject \[2\] that took place in Texas A&M University on November 7 and 8, 2009 “ADM-50: A celebration of current GR Innovation”. This event marked 50 years of loss of four-diffeomorphism and covariance in the “canonical” formulation of covariant GR. In the above mentioned article \[2\] the authors claimed to consider the Hamiltonian formulation of GR in Palatini form, i.e. the formulation of Einstein \[1\]. This is also the main subject of our paper, but with different results and without loss of four-diffeomorphism (see next Section).

The transition from the EH action to the ADM action, which is responsible for a loss of four-diffeomorphism, continues to propagate into new fields. A particular example is Loop Quantum Gravity (LQG) - one of the major players in the quest for quantization of gravity. According to Thiemann \[62\] “One of the reasons why LQG is gaining in its degree of popularity as compared to string theory is that LQG has ‘put its cards on the table’”. Let us look at the LQG “cards” that were put together in the recent article \[63\] (see first paragraph of Introduction) that describes the meaning of LQG: “…LQG is a mathematically rigorous\(^\text{11}\) quantization of general relativity (GR)... It is inspired by the formulation of GR as a canonical dynamical theory... The total Hamiltonian of GR is a linear combination of the Gauss constraint, the spatial diffeomorphism constraints and the Hamiltonian constraint. Thus the dynamics of GR are essentially the gauge transformations generated by constraints”. Is it canonical formulation of the covariant GR if the “spatial diffeomorphism” constraint is present? It is \[127\] that cannot have covariance and it is not equivalent to the Einstein-Cartan GR. The Hamiltonian of LQG can be written in different variables; but they are originated from lapse and shift functions, that is why, the “spatial diffeomor-

\(^\text{11}\) In our taste, “rigorous” in combination with mathematics is a tautology, but because the rival of LQG employs elegant mathematics, the use of just mathematics, probably, sounds a little bit weak.
"Phism" constraint is always there, as in the ADM gravity (116). Moreover, and this was already demonstrated (e.g. [64, 65, 66]), all new variables which are used in LQG can be converted into ADM variables by canonical transformations, i.e. new formulations are equivalent to the ADM formulation. The gauge transformation generated by constraints of the ADM-inspired Hamiltonian of tetrad gravity is the “spatial diffeomorphism”; but the transformations generated by the Hamiltonian of EC action are different [45, 54]. So, in reality LQG, using rigorous mathematics, studies quantization of the ADM-inspired model and its gauge invariance and, because it is not covariant, all well-known results follow: quantization of three-dimensional surfaces [67], Lorentz violation, etc. The suspicion of people not working in this field about how a covariant theory of GR can lead to such results is correct. The explanation of these effects originated from the classical Hamiltonian mechanics and the theory of canonical transformations. These are no quantum effects, but they just manifest the dependence of quantum effect on a classical background, the ADM Hamiltonian. To show this, neither rigorous nor elegant, but just a mathematical condition, the Poisson brackets (121) and (132), that must be zero to keep two Hamiltonians equivalent and the notion of a phase space are needed. This is what any outsider to this field can and should read from the LQG “cards”. This situation can be perfectly described using one of the famous paradoxical forms of Wheeler (e.g. see [68]): LQG is quantum “gravitation without gravitation” in Einstein’s sense.

We discuss recent, current and coming soon (see next paragraph) anniversaries/celebrations. But let us also mention a lesser known anniversary: 200 years ago Siméon-Denis Poisson published his work “Sur la Variation des Constantes arvitraires dans les questions de Mécanique” in the Journal de l’École Polytechnique, Tome VIII, p. 266, Décembre 1809, Paris. This is the first appearance of what has become known as the Poisson brackets; and we have used them to show that transition from EH or EC variables to ADM variables is not canonical (see (121) and (132)).

Let us return from the past to present days, and continue with one more quotation from Thiemann [62]: “The ‘rules of the game’ have been written and are not tinkered with”. And that is an important standard to which current research in GR must be held. The rules of the game are those published by Poisson, in 1809, and by Einstein at the beginning of the twentieth century. These are important rules which must not be tinkered with, and must not be ignored.
However, seems to us that the main rule of the LQG “game” is to work with the ADM-inspired action which is not covariant and, because of this, allows to obtain non-covariant results but, at the same time, to present this as properties of the Einstein-Cartan action, as scientific community at large still associates gravity with the name of Einstein. Neither outside view on LQG (Nicolai, at al [69]) nor inside view on the same subject (Thiemann [62]) disobey this rule, and, as a result, LQG is also approaching its 25 anniversary.

Of course, there is a freedom of research (or variety of “games” in nowadays language) and one can choose to play with the ADM-inspired metric or tetrad models in a framework of non-covariant theories; but one should not be surprised by the results which are not covariant. Or one can study the covariant theories of GR (Einstein or Einstein-Cartan) and in this case one ought be surprised if non-covariant results follow.

We choose to work with the covariant Einstein formulation of GR and do not convert his brilliant theory into “common currency”\(^\text{12}\), i.e. we do not make any non-canonical change of variables and we return to the total Hamiltonian \((29)\), \((60)\) of the first order affine-metric Einstein GR, the main subject of this article. The Hamiltonian formulation of any system leading to first class constraints cannot be considered as complete without deriving the corresponding gauge invariance. In addition, a restoration of gauge invariance is an important consistency check, especially if we investigate such a complicated theory as GR.

In the course of our calculations it becomes clear that, despite different starting points in our current consideration (affine-metric action \((2)\)) and those based on different set of variables (see \((41)\)) that were discussed in \([5, 6]\), and especially due to work of Ghalati and McKeon \([7]\) where the closure of Dirac procedure was demonstrated for the first time, the results are very similar. One can say it is remarkable or surprising, but nothing here is either “remarkable” or “surprising” because these results come from two equivalent first order formulations of the same theory (see Appendix of \([5]\)). Of course, there are some differences; but secondary and tertiary constraints (for the same particular choice) are exactly the same after the canonical transformations \((93)-(98)\) were performed. Neither our formulation, nor \([7]\) were converted into a “canonical” form (despite that some attempts to make a formulation “canonical” were made in the novel approach of \([11]\)); and one should expect the complete

\(^{12}\) The name given by Pullin \([40]\) to the ADM formulation in his Editorial note for republication of \([41]\) in
restoration of four-diffeomorphism as it was obtained for the second order formulation of metric GR [4, 14, 15].

The first steps of the restoration of gauge symmetry were made in [11], with a truly surprising result for a covariant theory: the need for a field dependent redefinition of gauge parameters to make the transformations of $h^{00}$ (the only one that was calculated\textsuperscript{13}) “correspond to diffeomorphism invariance” [11], exactly as in the conventional “canonical” formulations of metric and tetrad gravities [37, 38, 39]. Why does the Hamiltonian formulation of the first order EH action, obtained without non-canonical changes of variables, give so different result compared to the Hamiltonian formulation of the second order of GR? In the restoration of gauge invariance, we used the Castellani algorithm [16] but in [11] the different method, the HTZ ansatz [12], was employed. This is the only difference, and a comparison of the two different methods is the best point to start the search for understanding of the apparently different gauge transformations for first and second order formulations of GR. The field dependent redefinition of gauge parameters of these two formulations could be just an artifact of a particular method (if in one it is needed, but in another it is not). In the next Section, we first apply the Castellani algorithm to the first order formulation to obtain, as in [11], partial transformations (which is enough to make a conclusion about the necessity of a field dependent redefinition of gauge parameters). And then, we compare this with the results obtained using the HTZ ansatz.

VI. CASTELLANI ALGORITHM

In his paper [16] Castellani illustrated the application of his method by considering Yang-Mills theory (the system with only primary and secondary first class constraints). We also used this method to restore gauge invariance in the Hamiltonian formulation of the second order metric GR [4, 14] and of the first order, tetrad-spin connection, formulation of Einstein-Cartan action in the three dimensional case [48]. In the Hamiltonian of the first-order affine-metric GR, we have tertiary constraints; but the procedure of [16] is general and can be applied to systems with any number of generations of constraints. We are not

\textsuperscript{13} To avoid any confusion, we would like to emphasize that converting the results of direct calculations into the ADM form, which is extensively discussed in [11], was not used in the calculation of these transformations.
aware of an application of the Castellani procedure to a realistic Hamiltonian with tertiary constraints, and such an application is interesting by itself. Our main goal in this, and in the following Section, is to analyze the appearance of a field dependent redefinition of gauge parameters, not a complete restoration of gauge invariance. So as in [11], we will calculate transformations only partially but for all fields, to see whether this method produces correct terms in the transformations of all fields, contrary to [11], where the transformation of only one field was found.

In the Castellani algorithm [16] the generator of the gauge transformations for the Hamiltonian with first class constraints for the system with tertiary constraints is given by

$$ G = \varepsilon^\mu G_{(0)}^\mu + \dot{\varepsilon}^\mu G_{(1)}^\mu + \ddot{\varepsilon}^\mu G_{(2)}^\mu $$

(135)

where $\varepsilon^\mu$ are the gauge parameters and $\dot{\varepsilon}^\mu, \ddot{\varepsilon}^\mu$ are their temporal derivatives. The number of gauge parameters and their tensorial dimension are uniquely defined by primary first class constraints, so for the formulation considered, the number of parameters is equal to the dimension of spacetime, $D$. The functions $G_{(i)}^\mu$ are defined by the following iterative procedure (see Eq. (16b) and for more details see also Section 5 of [16])

$$ G_{(2)}^\mu = \Pi_{\mu}^{00}, $$

(136)

$$ G_{(1)}^\mu + \{G_{(2)}^\mu, H_T\} = \int d\vec{y}^\nu \alpha^\nu_{\mu}(\vec{x}, \vec{y}) \Pi_{\nu}^{00}(\vec{y}), $$

(137)

$$ G_{(0)}^\mu + \{G_{(1)}^\mu, H_T\} = \int d\vec{y}^\nu \beta^\nu_{\mu}(\vec{x}, \vec{y}) \Pi_{\nu}^{00}(\vec{y}), $$

(138)

$$ \{G_{(0)}^\mu, H_T\} = \text{primary}. $$

(139)

Note that only primary constraints enter equations (136)-(139) explicitly. The functions $G_{(2)\mu}$ are uniquely defined as primary constraints. The functions, $G_{(1)\mu}$ and $G_{(0)\mu}$, in general, are not just secondary or tertiary constraints because, for example, different linear combinations of tertiary constraints can be considered (see (83), (85), and (87)). This makes this method insensitive to our choice of combinations of non-primary constraints and gives the same gauge invariance regardless of what combinations we will call ‘tertiary constraints’. We
will illustrate this important point in more detail: for complicated theories the possibility of working with different combinations of constraints, without destroying its unique gauge symmetry, could give significant computational advantages.

Using PBs among the first class constraints and the total Hamiltonian, which is given by (29)-(30) with (48) and (50)-(52), we can solve (136) for \( G^{(1)}_{\mu} \)

\[
G^{(1)}_{\mu} = - \{ G^{(2)}_{\mu}, H_T \} + \alpha^\nu_{\mu} \Pi^0_\nu = - \chi^0_\mu + \alpha^\nu_{\mu} \Pi^0_\nu .
\]  

(140)

In this equation the secondary constraints \( \chi^0_\mu \) unambiguously appear through \( \{ \Pi^0_\mu, H_T \} \); and \( G^{(1)}_{\mu} \) becomes a secondary plus linear combination of primary constraints with coefficient-functions \( \alpha^\nu_{\mu} \) that have to be found. To shorten the notation, we will not write integrals in equations that involve coefficient functions, which in general might also depend on fields and their derivatives (see, i.e. [4, 14]). Only in the case of contributions with derivatives, which is specific to field theories, a more careful treatment is needed for finding the expressions of the corresponding coefficient-functions. In this Section we restrict ourselves to some simple steps of procedure mainly to discuss its important properties and to show that it provides a strong indication for the correct restoration of the gauge transformations for all fields. We will consider only a part of the transformations that are produced by simple contributions from coefficient-functions, which do not involve spatial derivatives, so our short notation without integrals will not lead to any ambiguity. The complete restoration of diffeomorphism invariance from the constraint structure of the Hamiltonian of first order affine-metric GR using the Castellani method is in progress and, of course, the complete calculations involve the spatial derivatives of the fields and integral form of expressions with coefficient-functions becomes important. Details of such calculations will be reported elsewhere.

At the next step of the procedure, using (138) and the above result (140) the functions \( G^{(0)}_{\mu} \) can be found. The total Hamiltonian (in our condensed notation) is

\[
H_T = \dot{\Gamma}_0^\mu \Pi^0_\mu - \Gamma_0^\mu \chi^0_\mu + H'_c .
\]  

(141)

In such a form, (50)-(52), \( H'_c \) is independent of "primary fields" \( \Gamma_0^\mu \) (fields for which conjugate momenta are primary constraints) and this allows us to perform a few steps of calculation, which are independent of a particular choice of tertiary constraints. Using this
form of the Hamiltonian, (141) and after some rearrangement, (138) becomes

\[ G(0)_\mu = -\{\chi^{00}_\mu, \Gamma^{00}_c\} + \{\chi^{00}_\mu, H'_c\} - \Pi^{00}_\nu \{\alpha^{00}_\mu, \Pi^{00}_\nu\} - \alpha^{00}_\mu \chi^{00}_\nu + \Pi^{00}_\nu \{\alpha^{00}_\mu, \chi^{00}_\gamma\} \Gamma^{00}_\gamma \]

\[ -\Pi^{00}_\nu \{\alpha^{00}_\mu, H'_c\} + \beta^{00}_\mu \Pi^{00}_\gamma. \]  

(142)

At this stage of the calculation, both coefficient-functions \( \alpha^{00}_\mu \) and \( \beta^{00}_\mu \) enter (142) while at the previous step only \( \alpha^{00}_\mu \) was present in (140).

The last equation of Castellani algorithm, (139), serves to find unspecified coefficient-functions \( \alpha^{00}_\mu \) and \( \beta^{00}_\mu \) as all terms proportional to the secondary and tertiary constraints must be identically zero

\[ \{G(0)_\mu, \hat{\Gamma}^{00}_\nu \Pi^{00}_\nu - \Gamma^{00}_c \chi^{00}_\nu + H'_c\} = \text{primary}. \]  

(143)

After \( \alpha^{00}_\mu \) and \( \beta^{00}_\mu \) are found, we have the following generator

\[ G = \varepsilon^{\mu} G(0)_\mu + \dot{\varepsilon}^{\mu} G(1)_\mu + \ddot{\varepsilon}^{\mu} G(2)_\mu = \]  

(144)

\[ \varepsilon^{\mu} \left( -\{\chi^{00}_\mu, \chi^{00}_\nu\} \Gamma^{00}_c + \{\chi^{00}_\mu, H'_c\} \right) - \{\varepsilon^{\mu} \Pi^{00}_\nu, \hat{\Gamma}^{00}_c \Pi^{00}_\alpha\} + \alpha^{00}_\mu \chi^{00}_\nu + \beta^{00}_\mu \Pi^{00}_\gamma \right) + \dot{\varepsilon}^{\mu} \left( -\chi^{00}_\mu + \alpha^{00}_\mu \Pi^{00}_0 \right) + \ddot{\varepsilon}^{\mu} \Pi^{00}_\mu \]

that allows us to find the transformations of fields or combinations of fields, \( F \), using

\[ \delta F = \{G, F\}. \]  

(145)

Here, we again do not specify an \( H'_c \), which can be written in different ways, as its form depends on a choice of tertiary constraints. Obviously a particular choice of tertiary constraints can only affect the transformations through the coefficient-functions. The transformations generated by the part of the generator (144), without coefficient-functions is

\[ \delta F = \{\varepsilon^{\mu} \left( -\{\chi^{00}_\mu, \chi^{00}_\nu\} \Gamma^{00}_c + \{\chi^{00}_\mu, H'_c\} \right) - \dot{\varepsilon}^{\mu} \chi^{00}_\mu + \ddot{\varepsilon}^{\mu} \Pi^{00}_\mu, F\} \]

(146)

and it produces the same result, whatever combination of tertiary constraints is considered. The complete restoration of the gauge generator, and especially the gauge transformations
of all fields, is a technically involved problem. Solutions of second class constraints must be used or we have to go to the reduced Lagrangian (90), which corresponds to the reduced Hamiltonian, and find the momenta in terms of the coordinates from the equations of motion, as it is described in the HTZ paper [12]. We restrict our discussion to the first and relatively simple steps of calculations: to single out a method of restoration that does not lead to contradictions.

Let us start from the derivation of the partial transformations of $\Gamma^\mu_{00}$ to illustrate that the Castellani algorithm is independent of a choice of tertiary constraints. We will show that it leads to the correct contributions to the gauge transformations for all fields; and that these transformations are equivalent to diffeomorphism without any field dependent or even numerical redefinitions of gauge parameters.

First of all, we write $G_{(0)\mu}$ in components that allow us to explicitly calculate some PBs

$$G_{(0)0} = \Gamma^k_{00} \chi^0_k + \{\chi^0_0, H'_c\} - \Pi^0_\nu \frac{\delta G^\nu}{\delta \Gamma^\gamma_{00}} \dot{\Gamma}^\gamma_{00}$$

$$- \alpha^\nu_0 \chi^0_\nu + \Pi^0_\nu \{\alpha^\nu_0, \chi^0_\gamma\} \Gamma^\gamma_{00} - \Pi^0_\nu \{\alpha^\nu_0, H'_c\} + \beta^0_0 \Pi^0_\gamma, \quad (147)$$

$$G_{(0)p} = -\chi^0_p \Gamma^0_{00} + \{\chi^0_p, H'_c\} - \Pi^0_\nu \frac{\delta G^\nu}{\delta \Gamma^\gamma_{00}} \dot{\Gamma}^\gamma_{00}$$

$$- \alpha^\nu_p \chi^0_\nu + \Pi^0_\nu \{\alpha^\nu_p, \chi^0_\gamma\} \Gamma^\gamma_{00} - \Pi^0_\nu \{\alpha^\nu_p, H'_c\} + \beta^0_0 \Pi^0_\gamma. \quad (148)$$

The condition (143) gives for (147)-(148) the following terms, which are not proportional to primary constraints,

$$\{G_{(0)0}, H_T\} = \chi^0_k \dot{\Gamma}^k_{00} - \Gamma^k_{00} \chi^0_k \dot{\Gamma}^0_{00} + \Gamma^k_{00} \{\chi^0_k, H'_c\}$$

$$- \{\{\chi^0_0, H'_c\}, \chi^0_\alpha\} \Gamma^\alpha_{00} + \{\{\chi^0_0, H'_c\}, H'_c\} - \delta \frac{\alpha^\nu_0}{\delta \Gamma^\gamma_{00}} \dot{\Gamma}^\gamma_{00} \chi^0_\nu$$

$$- \chi^0_\nu \frac{\delta \alpha^\nu_0}{\delta \Gamma^\gamma_{00}} \dot{\Gamma}^\gamma_{00} + \{\alpha^\nu_0 \chi^0_\nu, \chi^0_\alpha\} \Gamma^\alpha_{00} - \alpha^\nu_0 \{\chi^0_0, H'_c\} - \chi^0_0 \{\alpha^\nu_0, H'_c\}$$

$$+ \{\alpha^\nu_0, \chi^0_\alpha\} \Gamma^\alpha_{00} \chi^0_\nu - \{\alpha^\nu_0, H'_c\} \chi^0_\nu + \beta^0_0 \chi^0_\gamma = 0.$$
\[
\left\{ G_{(0)p}, \dot{\Gamma}_{\alpha}^{00} \Pi_{\alpha}^{00} - \Gamma_{\alpha}^{00} \chi_{\alpha}^{00} + H'_{c} \right\} = (150)
\]

\[
-\chi_{p}^{00} \dot{\Gamma}_{00}^{0} + \left\{ -\chi_{p}^{00} \Gamma_{00}^{0}, -\Gamma_{00}^{0} \chi_{00}^{00} \right\} - \Gamma_{00}^{0} \left\{ \chi_{p}^{00}, H'_{c} \right\} - \left\{ \left\{ \chi_{p}^{00}, H'_{c} \right\}, \chi_{00}^{00} \right\} \Gamma_{00}^{0} + \left\{ \chi_{p}^{00}, H'_{c} \right\} \right.
\]

\[
-\dot{\Gamma}_{00}^{\alpha} \delta \alpha_{p}^{\nu} - \chi_{00}^{00} \left\{ \alpha_{p}^{\nu}, \Pi_{\alpha}^{00} \right\} \dot{\Gamma}_{00}^{\alpha} + \left\{ \alpha_{p}^{\nu}, \chi_{00}^{00} \right\} \Gamma_{00}^{0} - \alpha_{p}^{\nu} \left\{ \chi_{00}^{00}, H'_{c} \right\} - \chi_{00}^{00} \left\{ \alpha_{p}^{\nu}, H'_{c} \right\}
\]

\[
+ \left\{ \alpha_{p}^{\nu}, \chi_{00}^{00} \right\} \Gamma_{00}^{0} \chi_{00}^{00} - \left\{ \alpha_{p}^{\nu}, H'_{c} \right\} \chi_{00}^{00} + \beta_{p}^{\nu} \chi_{00}^{00} = 0.
\]

The only terms in (149)-(150) that can give contributions proportional to tertiary constraints (hidden in \( H'_{c} \)) are:

\[
+ \Gamma_{00}^{k} \left\{ \chi_{k}^{00}, H'_{c} \right\} - \left\{ \left\{ \chi_{00}^{00}, H'_{c} \right\}, \chi_{00}^{00} \right\} \Gamma_{00}^{k} + \left\{ \left\{ \chi_{00}^{00}, H'_{c} \right\}, H'_{c} \right\} \right. - \alpha_{0}^{0} \left\{ \chi_{00}^{00}, H'_{c} \right\} = 0, (151)
\]

\[
- \Gamma_{00}^{0} \left\{ \chi_{p}^{00}, H'_{c} \right\} - \left\{ \left\{ \chi_{p}^{00}, H'_{c} \right\}, \chi_{00}^{00} \right\} \Gamma_{00}^{0} + \left\{ \left\{ \chi_{p}^{00}, H'_{c} \right\}, H'_{c} \right\} \right. - \alpha_{p}^{0} \left\{ \chi_{p}^{00}, H'_{c} \right\} = 0. (152)
\]

Equations (151)-(152) allow us to find the coefficient-functions \( \alpha_{\nu}^{\mu} \). Note that (151)-(152) are algebraic equations which are written in a general form and are independent of a particular choice of tertiary constraints. But to find the coefficient-functions we have to expand the expressions (151)-(152) and collect terms proportional to the secondary and tertiary constraints. To do this we have to specify our choice. A few choices were discussed in Section 4; but let us start from one particular combination, (60) with the simplest PBs among constraints, namely one that leads to zero PBs among secondary with tertiary constraints: \( \{ \chi_{00}^{00}, \tau_{\mu}^{00} \} = 0 \). The simplicity of the PBs was also the reason in [7] to use these combinations to prove closure of the Dirac procedure. It was precisely this choice that led to the conclusion in [11] that diffeomorphism invariance does not follow directly and a field dependent redefinition of parameters is needed to find the “correspondence” with diffeomorphism using HTZ ansatz [12]. So, we specify \( H'_{c} \) (note that until this moment all equations were independent of a choice of tertiary constraints) and use the following form
\[ H'_c = \frac{1}{h^{00}} \tau^{00} + h^{0m} \tau^{00} + A^0 \chi^{00} + A^m \chi^{00}. \]  

(153)

With this \( H'_c \), and keeping only terms that lead to tertiary constraints (needed to find \( \alpha^\nu_{\mu} \)) from (151) and (153), we have

\[ 2\Gamma^k_{00} \tau^{00} + \left( \frac{1}{h^{00}} \tau^{00} + \frac{h^{0m}}{h^{00}} \tau^{00} \right) \Gamma^0_{00} - \alpha^0_0 \left( \frac{1}{h^{00}} \tau^{00} + \frac{h^{0m}}{h^{00}} \tau^{00} \right) - \alpha^k_0 \tau^{00} + \left\{ \frac{1}{h^{00}} \tau^{00} + \frac{h^{0m}}{h^{00}} \tau^{00} + \{ \chi^0_{00}, A^0 \chi^{00} + A^m \chi^{00} \} , H'_c \right\} = 0. \]  

(154)

Let us, for simplicity, restrict our calculations even further and consider only the dependence of the coefficient-functions \( \alpha^\nu_{\mu} \) on \( \Gamma^\mu_{00} \). From the first line of (154) we can uniquely find how \( \alpha_0^0 \) and \( \alpha_k^0 \) depend on \( \Gamma^\mu_{00} \) (there are no contributions proportional to \( \Gamma^\mu_{00} \) in the second line of (154)). Combining together the terms in the first line of (154) that are proportional to the tertiary constraints we obtain

\[ \left( 2\Gamma^k_{00} - \alpha^k_0 + \frac{h^{0k}}{h^{00}} \Gamma^0_{00} - \frac{h^{0k}}{h^{00}} \alpha^0_0 \right) \tau^{00} + \left( \frac{1}{h^{00}} \Gamma^0_{00} - \frac{1}{h^{00}} \alpha^0_0 \right) \tau^{00} = 0. \]  

(155)

The second bracket of (155) gives

\[ \alpha^0_0 (\Gamma) = \Gamma^0_{00}. \]  

(156)

Using this result and the first bracket of (155), we obtain

\[ \alpha^k_0 (\Gamma) = 2\Gamma^k_{00}. \]  

(157)

Similarly, from (152) we find

\[ \alpha^0_p (\Gamma) = 0, \quad \alpha^k_p (\Gamma) = -\delta^k_p \Gamma^0_{00}. \]  

(158)

With these results we can also calculate the contributions to the coefficient-function \( \beta^\nu_{\mu} \) from the corresponding expressions that are proportional to the secondary constraints. In (149)–(150), keeping only terms with \( \beta^\nu_{\mu} \) and those with temporal derivatives of \( \dot{\Gamma}^\mu_{00} \), we have

\[ \chi^0_k \dot{\Gamma}^k_{00} - \dot{\Gamma}^0_{00} \frac{\delta \alpha^\nu_{\mu}}{\delta \Gamma^\mu_{00}} \chi^\nu_0 - \chi^0_0 \frac{\delta \alpha^\nu_{\mu}}{\delta \Gamma^\mu_{00}} \dot{\Gamma}^\alpha_{00} + \beta^0_0 \chi^0_0 + \beta^0_p \chi^0_p = 0, \]  

(159)
\[-\chi_p^{00}\dot{\Gamma}_0^0 - \frac{\delta\alpha^\nu}{\delta\Gamma_0^0}\Gamma_0^0\chi_\nu^{00} - \chi_\nu^{00}\frac{\delta\alpha^\nu}{\delta\Gamma_0^0}\dot{\Gamma}_0^0 + \beta_p^0\chi_0^{00} + \beta_p^k\chi_k^{00} = 0, \tag{160}\]

which after substitution of (156)-(158) gives

\[\beta_0^0\left(\dot{\Gamma}\right) = 2\dot{\Gamma}_0^{00}, \quad \beta_0^k\left(\dot{\Gamma}\right) = 3\dot{\Gamma}_0^k, \tag{161}\]

\[\beta_p^0\left(\dot{\Gamma}\right) = 0, \quad \beta_p^k\left(\dot{\Gamma}\right) = -\delta_p^k\dot{\Gamma}_0^0. \tag{162}\]

Let us check the consistency of these simple partial results and consider variation of \(\delta\Gamma_0^\mu\)

\[\delta\Gamma_0^\mu = \{G, \Gamma_0^\mu\}. \tag{163}\]

For the relevant part of the generator (terms proportional to primary constraints \(\Pi_0^{00}\) and found in (156)-(158) and (161)-(162) \(\Gamma\)-dependent parts of \(\alpha_\mu^\nu\) and \(\beta_\mu^\nu\)), we have

\[G = \varepsilon^0\left(\dot{\Gamma}_0^0\Pi_0^{00} + \dot{\Gamma}_0^k\Pi_k^{00}\right) + \varepsilon^0\Gamma_0^0\Pi_0^{00} + \varepsilon^0\varepsilon^0\Gamma_0^0\Pi_0^{00} + \varepsilon^0\Pi_0^{00} + \varepsilon^0\Pi_0^{00}. \tag{164}\]

This part of the generator gives

\[
\begin{align*}
\delta\Gamma_0^0 &= -\varepsilon^0\dot{\Gamma}_0^0 - \varepsilon^0\Gamma_0^0 - \varepsilon^0, \tag{165} \\
\delta\Gamma_0^p &= -\varepsilon^0\dot{\Gamma}_0^p - 2\varepsilon^0\Gamma_0^p + \varepsilon^0\Gamma_0^0 - \varepsilon^p. \tag{166}
\end{align*}
\]

We have to compare these partial results with the well-known transformations of the components, \(\Gamma_0^\mu\), under diffeomorphism invariance. Using the general expression for the transformations of affine-connections under diffeomorphism,

\[\delta_{\text{diff}}\Gamma_\mu^\lambda = -\varepsilon_\mu^\lambda + \Gamma_\mu^\rho\varepsilon^\rho - \varepsilon^\rho\Gamma_\mu^\rho - \Gamma_\mu^\rho\varepsilon_\rho - \Gamma_\rho^\lambda\varepsilon_\mu, \tag{167}\]

for two particular components of (165) and (166), we obtain

\[
\delta_{\text{diff}}\Gamma_0^0 = -\varepsilon_0^0\dot{\Gamma}_0^0 - \varepsilon_0^0\partial_0\varepsilon^0 - \varepsilon_0^0\partial_0\dot{\Gamma}_0^0 + \Gamma_0^k\partial_k\varepsilon^0 - \varepsilon^k\partial_k\Gamma_0^0 - 2\left(\Sigma_{0k} + \Gamma_{km}^0\right)\partial_0\varepsilon^k, \tag{168}
\]

52
\[ \delta_{\text{diff}} \Gamma^b_{00} = -\varepsilon^b_{00} + \Gamma^0_{00} \partial_0 \varepsilon^b + \Gamma^b_k \partial_k \varepsilon^b \]

\[ -\varepsilon^0 \partial_0 \Gamma^b_{00} - \varepsilon^k \partial_k \Gamma^b_{00} - 2\Gamma^b_{00} \partial_0 \varepsilon^0 - 2\Sigma^b_{0k} \partial_0 \varepsilon^k + \frac{2}{D - 1} \Sigma_{00} \partial_0 \varepsilon^b. \] (169)

Here we used the redefinitions (4), (15) and (18) (from last two it follows that \( \pi_{\mu\nu} = \Sigma_{\mu\nu} \)).

We do not perform such substitutions for \( \Sigma^m_{0k} \) and \( \Gamma^m_{km} \) as they do not affect the parts of the transformations that we shall compare with our partial result (165)-(166) (they do not have terms that depend on primary variables \( \Gamma^\mu_{00} \)).

Comparing (165)-(166) and (168)-(169), we can see that all terms with temporal derivatives of the parameter \( \varepsilon^\mu \) (up to the second order which requires tertiary constraints, as we argued in [5, 6]) and temporal derivatives of primary fields, in the part of the generator (165)-(166) we considered, coincide with diffeomorphism transformation (168)-(169) without any need for field dependent redefinition of the gauge parameters.

If the transformations for some fields that were calculated using the same generator require a field dependent redefinition of gauge parameters to “correspond” to diffeomorphism transformations, then such a redefinition will obviously destroy transformations (165)-(166). If that happens then either the Hamiltonian formulation is not correct (ordinary mistakes are possible in such long calculations or non-canonical change of variables were performed) or the method of restoration of the gauge invariance is incorrect or, perhaps, sensitive to a choice of constraints; in that case such a method cannot be called an algorithm.

There are a few more contributions to the transformations of different fields that we can obtain using the part of the generator (144), even without knowledge of the coefficient-functions \( \alpha^\nu_{\mu} \) and \( \beta^\nu_{\mu} \). This part of (144), after substitution of expressions for secondary constraints (23) and (24), is

\[ G' = -\dot{\varepsilon}^\mu \chi^0_{\mu} = -\dot{\varepsilon}^0 \left( -h^0_{,k} + h^0_{00} \pi_{00} - h^{km}_{,km} \pi_{km} \right) - \dot{\varepsilon}^k \left( h^0_{,k} + 2h^{00}_{00} \pi_{0k} + 2h^{m0}_{m0} \pi_{km} \right). \] (170)

It allows us to find some contributions to the transformations of \( h^{\alpha\beta} \)

\[ \delta h^{\alpha\beta} = \{ G', h^{\alpha\beta} \} \] (171)

that for different components leads to:
\[ \delta h^{pq} = \{ -\dot{\varepsilon}^0 (-h^{km}\pi_{km}) - \dot{\varepsilon}^k (2h^{m0}\pi_{km}) , h^{pq} \} = -\dot{\varepsilon}^0 h^{pq} + \dot{\varepsilon}^p h^{q0} + \dot{\varepsilon}^q h^{p0}, \] (172)

\[ \delta h^{0p} = \{ -\dot{\varepsilon}^k (2h^{00}\pi_{0k}) , h^{0p} \} = \dot{\varepsilon}^p h^{00}, \] (173)

\[ \delta h^{00} = \{ -\dot{\varepsilon}^0 (h^{00}\pi_{00}) , h^{00} \} = \dot{\varepsilon}^0 h^{00}. \] (174)

Using the known transformations for \( g^{\alpha\beta} \) under diffeomorphism invariance and the definition of \( h^{\alpha\beta} \) (18), one obtains

\[ \delta_{\text{diff}} h^{\mu\nu} = h^{\mu\lambda} \varepsilon^\nu_{,\lambda} + h^{\nu\lambda} \varepsilon^\mu_{,\lambda} - (h^{\mu\nu} \varepsilon^\lambda_{,\lambda}), \] (175)

which for different components of \( h^{\alpha\beta} \) gives:

\[ \delta_{\text{diff}} h^{00} = h^{00} \varepsilon^0_{,0} + 2h^{0p} \varepsilon^0_{,p} - \varepsilon^0 h^{00} - \varepsilon^p h^{00} - h^{00} \varepsilon^p_{,p}, \] (176)

\[ \delta_{\text{diff}} h^{0p} = h^{00} \varepsilon^p_{,0} + h^{pm} \varepsilon^0_{,m} + h^{pm} \varepsilon^0_{,m} - \varepsilon^0 h^{00} - h^{0p} \varepsilon^m_{,m} - \varepsilon^m h^{0p}_{,m}, \] (177)

\[ \delta_{\text{diff}} h^{pq} = h^{00} \varepsilon^q_{,0} + h^{0p} \varepsilon^q_{,p} + h^{pm} \varepsilon^q_{,m} + h^{pm} \varepsilon^q_{,m} - h^{pq} \varepsilon^0_{,0} - h^{pq} \varepsilon^m_{,m} - \varepsilon^0 h^{pq}_{,0} - \varepsilon^m h^{pq}_{,m}. \] (178)

Similar to transformations of \( \Gamma^{\mu}_{00} \) (163)-(166), all found in the (172)-(174) contributions, are present in (176)-(178) and cover all terms with temporal derivatives of the gauge parameter \( \varepsilon^\mu \). So, field dependent redefinition of gauge parameters is also not needed.

The same part of the generator (170) can be used to find partial transformations of \( \pi_{\alpha\beta} \)

\[ \delta \pi_{\alpha\beta} = \{ G' , \pi_{\alpha\beta} \}, \] (179)

which gives for components

\[ \delta \pi_{pq} = \{ \dot{\varepsilon}^0 h^{km}\pi_{km}, \pi_{pq} \} = \dot{\varepsilon}^0 \pi_{pq}, \] (180)

\[ \delta \pi_{0p} = \{ \varepsilon^0 h^{0k} - \dot{\varepsilon}^k 2h^{m0}\pi_{km}, \pi_{0p} \} = -\frac{1}{2} \dot{\varepsilon}^0 \pi_{0p} - \dot{\varepsilon}^k \pi_{kp}, \] (181)
\[ \delta \pi_{00} = \{- \dot{\epsilon}^0 \left( h_{00} \pi_{00} \right) - \dot{\epsilon}^k \left( h_{0k} \pi_{00} + 2 h_{k0} \pi_{0k} \right), \pi_{00} \} = -\dot{\epsilon}^0 \pi_{00} + \dot{\epsilon}^k - 2 \dot{\epsilon}^k \pi_{0k} \].

(182)

Returning to our original redefinitions, \(15\) and \(18\), we find that \(\pi_{\alpha\beta} = \Sigma_{\alpha\beta}\). Now we can compare \(180\)-\(182\) with the transformations of \(\pi_{\alpha\beta}\) under diffeomorphism by using

\[ \delta_{\text{diff}} \pi_{\alpha\beta} = \delta_{\text{diff}} \Sigma_{\alpha\beta} \]  

(183)

and redefinition \(44\), which allows us to express \(\Sigma_{\alpha\beta}\) in terms of \(\Gamma^\mu_{\alpha\beta}\) (transformation of \(\Gamma^\mu_{\alpha\beta}\) under diffeomorphism is known \(167\)). From \(183\), for the components of \(\pi_{\alpha\beta}\), we obtain:

\[ \delta_{\text{diff}} \pi_{pq} = -\dot{\epsilon}_{pq}^0 + \pi_{pq}^0 \dot{\epsilon}^0 + \Gamma^m_{pq} \dot{\epsilon}^m_m - \epsilon^0 \pi_{pq,0} \]

\[- \epsilon^m \pi_{pq,m} - \left( 2 \pi_{0p} + \Gamma^m_{pm} \right) \epsilon^0_q - \pi_{pm} \epsilon^m_q - \left( 2 \pi_{0q} + \Gamma^m_{qm} \right) \epsilon^0_p - \pi_{qm} \epsilon^m_p , \]  

(184)

\[ \delta_{\text{diff}} \pi_{0p} = -\frac{1}{2} \dot{\epsilon}_{0p}^0 - \pi_{0p}^0 \dot{\epsilon}^0 + \Sigma^m_{0p} \dot{\epsilon}^m_m \]

\[- \frac{1}{2} \frac{D + 1}{D - 1} \pi_{00} \dot{\epsilon}_{0p}^0 - \dot{\epsilon}^0 \pi_{00,0} - \epsilon^m \pi_{00,m} - \frac{1}{2} \Gamma^0_{00} \dot{\epsilon}_{0p}^0 - \pi_{m0} \dot{\epsilon}^m_p + \frac{1}{2} \dot{\epsilon}_{pm}^m , \]  

(185)

\[ \delta_{\text{diff}} \pi_{00} = \epsilon^m_{0m} - 2 \pi_{0m0} \dot{\epsilon}^m_m - \epsilon^0 \pi_{00,0} - \epsilon^k \pi_{00,k} + \Gamma^m_{00} \dot{\epsilon}^m_m - \pi_{00} \dot{\epsilon}^0_0 . \]  

(186)

Here, as in \(168\)-\(169\), the solutions for \(\Gamma^\mu_{pq}\) and \(\Sigma^m_{0p}\) must also be substituted (but it is not needed at this stage). Compare \(180\)-\(182\) with the diffeomorphism transformation \(184\)-\(186\); we see that again all terms with temporal derivatives of the gauge parameter \(\epsilon^\mu\) are exactly the same, including two contributions with mixed spatio-temporal derivatives.

Note also that the last two transformations, \(176\)-\(178\) and \(180\)-\(182\), follow just from an explicit form of the secondary constraints and obviously cannot be affected (in the Castellani procedure) by a choice of tertiary constraints. The transformations \(165\)-\(166\) were obtained using the explicit form of coefficient-functions for which a particular choice of tertiary constraints \(153\) was used; and there is possibility that different choices can affect the result. Of course, from physical point of view, this is impossible and if such happens, it would be an indication of problems with a method of the restoration of gauge invariance.
So, let us consider the effect of a different choice of constraints; e.g. one which we discussed in Section 4. To be explicit, let us consider (87) for which the Hamiltonian takes the form

\[ H'^{c'} = \tilde{\tau}^{00} + A^0 \chi^{00}_0 + A^m \chi^m_{m0}. \]  

(187)

For this choice, contribution (155) is modified

\[ + 2\Gamma^k_{00} \tilde{\tau}^{00}_k + \tilde{\tau}^{00}_0 \Gamma^0_{00} - \alpha^0_0 \chi^{00}_0 - \alpha^k_0 \chi^{00}_k; \]  

(188)

but solutions for coefficient-functions are the same as those given by (156)-(157). The remaining contributions calculated earlier, for the original choice of constraints, are also the same. One can easily check another choice that we considered in Section 4, (85), and verify that it also preserves the parts of coefficient-functions proportional to \( \Gamma^\mu_{00} \). These particular examples illustrate the general statement made in Section 4 that a choice of linear combinations of constraints (or their PBs algebra) should not affect physical results (should not, in particular, affect gauge invariance).

The above examples of partial restoration of gauge invariance using the Castellani procedure, which were considered, lead to partial contributions to the transformations of all fields presented in the reduced Hamiltonian of affine-metric formulation of GR. All contributions found are exactly the same as the corresponding terms of diffeomorphism transformations; and there is no need for a field dependent redefinition of gauge parameters. Moreover, all these contributions are independent of a choice of tertiary constraints, as it should be, if we respect the concept that the gauge invariance is a unique characteristic of a gauge invariant theory. Of course, to make the final conclusion, full calculations must be performed and all contributions have to be found. These calculations are straightforward, but quite laborious; the results will be reported elsewhere. Our main goal here is to find the reason for the contradictory result of [11] about necessity to have “correspondence” with diffeomorphism invariance. Now, after we show that diffeomorphism invariance follows directly from the Castellani procedure; but when used in [11] the HTZ ansatz does not give correct transformations. One can draw the conclusion that the HTZ ansatz cannot be considered to be an algorithm for the restoration of the gauge invariance and the need of field dependent redefinitions found in [11] is just an artifact of the method used. In next Section we explicitly demonstrate the failure of HTZ and discuss the possible reasons for this failure.
In this Section we turn our attention to the HTZ approach \cite{12}. This approach was used in \cite{11} where the gauge invariance obtained for the first order Einstein-Hilbert action differs from the results of the previous Section where we follow the Castellani procedure. According to \cite{11}, the gauge invariance of first order formulation of GR is not a four-diffeomorphism and a field dependent redefinition of gauge parameters is needed to “correspond to diffeomorphism invariance”.

The appearance of such a “correspondence” in \cite{11} cannot be explained by a non-canonical change of variables, as it was in the case of ADM model \cite{4}. We have the Hamiltonian of the same theory and one method, Castellani (see previous Section), leads to four-diffeomorphism and another, HTZ (see \cite{11}), does not. This is clearly a deficiency of HTZ approach and this is exactly what we want to investigate and discuss in this Section.

There are two major differences between the HTZ approach and the Castellani algorithm. The first one is the use of a so-called extended formalism: instead of the total Hamiltonian (as in Castellani’s) the extended formalism is the starting point of the HTZ approach, where all primary constraints are included in the Hamiltonian with Lagrange multipliers. The second one is so-called HTZ ansatz - a generator of the gauge transformation is assumed to be a linear combination of all first class constraints and all of them enter explicitly \cite{12}, contrary to Castellani’s generator (136)-(139) where only primary first class constraints explicitly enter a generator. Moreover, the HTZ iterative procedure \cite{12} to find field dependent coefficients in front of the constraints (see (189)), is started from the end of the constraint chains (e.g. start from tertiary constraints for the first order affine-metric GR).

The question of a “total versus extended” Hamiltonian will not be discussed here. We only would like to mention that in the particular application of the HTZ approach the “gauge fixing” of the Lagrange multipliers of non-primary first class constraints is used. This effectively converts the extended Hamiltonian into a total Hamiltonian (see Eq. (4.1) of \cite{12}). Moreover, different variations of the HTZ method were developed; and some of them are based entirely on the total Hamiltonian, as in Castellani procedure (e.g. \cite{72}) without even mentioning the extended Hamiltonian; but they lead to the same results as HTZ approach. So, our interest is the HTZ ansatz, which is also the essential part of all modifications of this approach (e.g. see Eq. (5) of \cite{72}). In \cite{11} the gauge transformations
were obtained “using a method very similar to the method of HTZ”. We will consider the HTZ ansatz following the original paper \[12\] (the reprint of this paper can also be found in Sections 3.2-3.3 of the book \[71\]).

According to the HTZ ansatz \[12\] for a system with first class constraints, the generator is simply a linear combination of all first class constraints. As in the Castellani algorithm, all second class constraints should be eliminated in the preliminary step and the Dirac brackets should be calculated \[12\]; this is exactly what we did in Section 2. In the case of the Hamiltonian with tertiary constraints, the HTZ generator is given by (see Eq. (4.2a) of \[12\])

\[
G = a^{\mu_1} \phi_{\mu_1} + a^{\mu_2} \phi_{\mu_2} + a^{\mu_3} \phi_{\mu_3} .
\]  

(189)

We slightly adjust the HTZ notation to make it more transparent for a covariant theory. \(\phi_{\mu_i}\) are the first class constraints of different generations \(i = 1, 2, 3\) (primary, secondary, and tertiary) and \(a^{\mu_i}\) are functions of the canonical variables and inexpressible velocities (\(\dot{\Gamma}_0\) in the considered formulation) that are iteratively defined from equation (see Eq. (4.2b) of \[12\] for \(i \geq 2\))

\[
\frac{D a^{\mu_i}}{D t} + \{a^{\mu_i}, H_c\} + \dot{\Gamma}_0^{\nu_i} \{a^{\mu_i}, \phi_{\nu_1}\} - \sum_{j \geq i-1} a^{\nu_j} V_{\nu_j}^{\mu_i} - \dot{\Gamma}_0^{\nu_1} \sum_{j \geq i} a^{\beta_j} C_{\nu_1 \beta_j}^{\mu_i} = 0
\]  

(190)

where \(C_{\nu_1 \beta_j}^{\mu_i}\) and \(V_{\nu_j}^{\mu_i}\) are structure functions in the PBs of the primary constraints with the rest of constraints (\(C_{\nu_1 \beta_j}^{\mu_i}\)) and in PB of any constraints with the canonical Hamiltonian (\(V_{\nu_j}^{\mu_i}\))

\[
\{\phi_{\mu_1}, \phi_{\nu_3}\} = \sum_{i \leq s} C_{\mu_1 \nu_3}^{\mu_i} \phi_{\mu_i} ,
\]  

(191)

\[
\{H_c, \phi_{\mu_1}\} = \sum_{i \leq s+1} V_{\nu_1}^{\mu_i} \phi_{\nu_i} ,
\]  

(192)

\(\frac{D}{D t}\) is a short notation for (see Eq. (3.4c) of \[12\])

\[
\frac{D}{D t} = \frac{\partial}{\partial t} + \dot{\Gamma}_0^{\nu} \frac{\partial}{\partial \Gamma_0^{\nu}} + \ddot{\Gamma}_0^{\nu} \frac{\partial}{\partial \dot{\Gamma}_0^{\nu}} + ...
\]  

(193)
According to \[12\], one has to start from the equation of the highest order and “without loss of generality” take \(a^{\mu_3} = \varepsilon^\mu\), i.e. functions (gauge parameters), which are independent of the canonical variables that leads for \(i = 3\) to

\[
\frac{D\varepsilon^\mu}{Dt} + \{\varepsilon^\mu, H_c\} + \hat{\Gamma}^\nu_1 \{\varepsilon^\mu, \phi_\nu_1\} - \sum_{j \geq 2} a^{\nu_3} V^{\mu_3}_{\nu_3} - \hat{\Gamma}^\nu_1 \sum_{j \geq i} a^{\beta_j} C_{\nu_1,\beta_j}^{\mu_i} = 0. \tag{194}
\]

We would like to emphasize that the independence of the gauge parameters of the canonical variables is the starting point of this iterative procedure; and without this assumption one will face the problem of solving variational equations instead of algebraic ones. Despite the computational problems with field dependent parameters, there are no a priori criteria of what possible dependence should be assumed. Let us, as the authors of \[12\] suggested and as it was done in \[11\] (see Eq. (145) of \[11\]), take \(a^{\mu_3} \equiv \varepsilon^\mu\); i.e. as independent of the phase-space variables function of spacetime coordinates.

This equation, (194), can be simplified as the result of independence of \(\varepsilon^\mu\) of the canonical variables: \(\frac{D}{Dt} = \frac{\partial}{Dt}\), and both P Bs in (194) are zero. Because the HTZ ansatz (189) and structure functions (191)-(192) explicitly depend on a choice of constraints, we have to specify our choice from outset (note that in previous Section, using the Castellani algorithm, we were able to perform some calculations without referring to the explicit form of the tertiary constraints). Partial restoration of gauge invariance using the HTZ approach for first order GR was discussed in \[11\] for a particular choice of tertiary constraints. So, we also consider the same combinations, (58) and (59) (the same choice was used in previous Section which led directly to four-diffeomorphism when the Castellani procedure was applied). For these constraints the corresponding canonical Hamiltonian is

\[
H_c = -\Gamma^\nu_0 \chi^0_\nu + \frac{1}{h^0_0} \tau^0_0 + \frac{h^0_0}{h^0_0} \tau^0_0 m + A^0_0 \chi^0_0 + A^m_0 \chi^0_m. \tag{195}
\]

The P Bs among the chosen constraints are the simplest, \(\{\chi^0_\nu, \tau^0_0\} = 0\), and primary constraints have zero PBs with all secondary and tertiary constraints making all \(C_{\nu_1,\mu_i} = 0\) (see (191)), which leads to a simple form of equation (194)

\[
\frac{\partial \varepsilon^\mu}{\partial t} - a^{\nu_3} V^{\mu_3}_{\nu_3} - \varepsilon^\nu V^{\mu_3}_{\nu_3} = 0. \tag{196}
\]

The explicit form of the structure functions \(V^{\mu_3}_{\nu_3}\) (192) for this choice of tertiary constraints is
\[ V_{0_2}^{m_3} = -\frac{1}{h_{00}} \], \quad \[ V_{0_2}^{m_3} = -\frac{h_{0m}}{h_{00}} \], \quad \[ V_{p_2}^{m_3} = \delta_{p}^{m} \], \quad \[ V_{p_2}^{0_3} = 0 \],  

(197)

which allows to solve equation (196), which is algebraic with respect to \( a^{\nu_2} \)

\[ a^{0_2} = -h_{00} \frac{\partial \varepsilon_{0}}{\partial t} + h_{00} \varepsilon_{\nu_3}^{0} \],  

(198)

\[ a^{p_2} = \frac{\partial \varepsilon_{p}}{\partial t} - h_{0p} \frac{\partial \varepsilon_{0}}{\partial t} + h_{0p} \varepsilon_{\nu_3}^{0} - \varepsilon_{\nu_3}^{0} \].  

(199)

Using (78)-(81), the remaining structure functions \( V_{\nu_3}^{\mu_3} \) can be easily found (note that they have only dependence on canonical variables \( h_{0\alpha} \)), which makes \( a^{\mu_2} \) to be functions of the gauge parameters and variables \( h_{0\alpha} \). This immediately allows us to find the partial transformations of \( h^{\alpha\beta} \) using part of the generator (189) with \( a^{\mu_1} \) and \( a^{\mu_2} \) (already found)

\[ \delta h^{\alpha\beta} = \{ h^{\alpha\beta}, G \} = \{ h^{\alpha\beta}, \ldots + a^{\mu_2} \chi_{0}^{00} + \varepsilon_{\nu}^{00} \} = \frac{\delta}{\delta \pi^{\alpha\beta}} \left( \ldots + a^{\mu_2} \chi_{0}^{00} + \varepsilon_{\nu}^{00} \right). \]  

(200)

This is especially simple for the component \( h_{00} \), as the corresponding momenta are present only in the secondary constraints, which lead to

\[ \delta h^{00} = a^{\mu_2} \frac{\delta \chi_{\mu}^{00}}{\delta \pi^{00}} = a^{0_2} h^{00}, \]  

(201)

\[ \delta h^{0p} = a^{\mu_2} \frac{\delta \chi_{\mu}^{00}}{\delta \pi^{0p}} = a^{p_2} h^{00}. \]  

(202)

After substitution of explicit form of \( V_{\nu_3}^{\mu_3} \) into (198)-(199), we obtain:

\[ a^{0_2} = -h_{00} \left[ \frac{\partial \varepsilon_{0}}{\partial t} + \varepsilon_{m} \frac{h_{0m}}{h_{00}} - \varepsilon_{0} \left( \frac{h_{0m}}{h_{00}} \right)_{,m} - \varepsilon_{m} \left( \frac{1}{h_{00}} \right)_{,m} + \varepsilon_{m} \frac{1}{h_{00}} \right] \],  

(203)

\[ a^{p_2} = \varepsilon_{p} - h_{0p} \varepsilon_{0} + h_{00} \varepsilon_{\mu\nu} \left( \frac{1}{h_{00}} \right)_{,m} \varepsilon_{0} - \varepsilon_{\mu\nu} \varepsilon_{0} - \left( \frac{h_{0p}}{h_{00}} \right)_{,m} \varepsilon_{m} - \left( \frac{h_{0m}}{h_{00}} \right)_{,m} \varepsilon_{0} - \left( \frac{h_{0m} h_{0p}}{h_{00}} \right) \varepsilon_{0}. \]  

(204)

Equation (203) is exactly the same as Eq. (146) of [11]; but (204) has different signs in front of a few terms, as compared to the similar Eq. (147) of [11]. These transformations,
are not transformations under four-diffeomorphism (see (176)-(177) in previous Section) and at most the so-called “correspondence” can be found by a field dependent redefinition of the gauge parameters. Note that such a “redefinition” is not needed when the Castellani algorithm is employed. The “appropriate” redefinition of gauge parameters was found in [11]:

\[ \varepsilon^0 = -\frac{1}{h_{00}}\xi^0, \]  
\[ \varepsilon^i = \xi^i - \frac{h_{0i}}{h_{00}}\xi^0. \]  

Substitution of these expressions into (203) gives

\[ \delta h^{00} = h^{00}\xi^0_0 + 2h^{0p}\xi^0_0 - \xi^0_0h_{,0}^{00} - \xi^p h_{,p}^{00} - h^{00}\xi^p_0. \]  

As we mentioned above, our (204) has a few different signs compared to the similar Eq. (147) of [11]; this can probably explain why only a transformation \( \delta h^{00} \) was provided in [11]. Substitution of (205) and (206) into (204) gives

\[ \delta h^{0p} = h^{00}\xi^p_0 + h^{0m}\xi^p_0 + h^{pm}\xi^0_0 - \xi^0_0h_{,0}^{0p} - h^{0p}\xi^m_0 - \xi^m h_{,0}^{0p}. \]  

According to the author of [11] (203) “is IDENTICAL [Capital letters are ours] with diffeomorphism invariance ... IF we substitute” (205) and (206), i.e. perform field dependent redefinition of gauge parameters, which by the initial assumption of the HTZ ansatz should be independent of the phase-space variables.

What is the significance of such a “correspondence”, especially if there is a different method that leads directly to four-diffeomorphism? Firstly, by the transformations produced by the HTZ ansatz are (203)-(204) and their derivation was based on the assumption (that was used in the course of derivation) that the gauge parameters are field independent. So, a field dependent redefinition of the gauge parameters is just a manipulation, not a derivation; and this manipulation is in contradiction with what was used to derive the gauge transformations. Secondly, why is this particular redefinition of parameters chosen? If the gauge invariance of some theory is not known a priori, then it is meaningless to seek a correspondence to this unknown gauge invariance precisely because it is unknown. In such a case what should be called the gauge invariance if such manipulations are allowed?
Gauge invariance is a unique and very important characteristic of a theory, and neither a Hamiltonian formulation nor methods of restoration that lead to such ambiguities (or the need for such manipulations) can be accepted. Returning to the discussion in Section 5, it is quite obvious that the HTZ method cannot give a covariant result for this choice of tertiary constraints because these constraints, \((58)\) and \((59)\), have different dimensions, as do the corresponding gauge parameters. In the Castellani procedure the choice of linear combinations of tertiary constraints is irrelevant and any combination leads to the same unique gauge transformation, whereas in the HTZ method, at least some of the combinations of tertiary constraints definitely cannot lead to a covariant result. This gives a limitation on the application of this method and imposes severe restrictions on the possible operations with non-primary first class constraints.

Let us take a different combination of tertiary constraints which gives \(H''_c\) of \((187)\), one that we used in previous Section where we showed that in this case the Castellani procedure produced the same gauge transformations as using any other combination. Let us return to \((196)\), which is the first step of the iterative procedure, and its form is the same for any choice of tertiary constraints (only structure functions will be different). Now we consider the following linear combinations of tertiary constraints

\[
\tilde{\tau}^{00}_0 = \frac{1}{h^{00}} \tau^{00}_0 + \frac{\tilde{h}^{0m}}{h^{00}} \tau^{00}_m, \quad \tilde{\tau}^{00}_m = \tau^{00}_m.
\]  

(209)

In terms of these constraints the canonical Hamiltonian is

\[
H_c = -\Gamma^{\nu \mu \nu} \chi^{00}_\nu + \tilde{\tau}^{00}_0 + A^0 \chi^{00}_0 + A^m \chi^{00}_m.
\]  

(210)

In this case the structure functions \(V_{\nu_2}^{\mu_3}\) of the HTZ ansatz become (as \(\{H_c, \chi^{00}_\mu\} = -\tilde{\tau}^{00}_\mu + \text{terms proportional to } \chi^{00}_\mu\))

\[
V_{\nu_2}^{\mu_3} = -\delta^{\mu}_\nu
\]  

(211)

and the solution to equation \((196)\) is

\[
a^{\mu_2} = -\frac{\partial \varepsilon^{\mu}}{\partial t} + \varepsilon^\nu V_{\nu_3}^{\mu_3}.
\]  

(212)

Even without specifying the structure functions \(V_{\nu_3}^{\mu_3}\), which are more complicated compared to the previous choice of tertiary constraints, some contributions to the transformations can
be calculated using the general expression (8)

$$\delta h^{\alpha\beta} = \{h^{\alpha\beta}, G\} = \left\{h^{\alpha\beta}, \ldots + \frac{\partial \varepsilon^\mu}{\partial t} \chi^{00}_\mu + \ldots\right\} = \frac{\partial \varepsilon^\mu}{\partial t} \delta \chi^{00}_\mu.$$  \hspace{1cm} (213)

This partial contribution to the transformations leads exactly to the same result as in (180)-(182) (for this choice of constraints, (209), equation (213) equals to (179) in previous Section), which are part of four-diffeomorphism without any need for redefinition of gauge parameters, in full correspondence with the original assumption of their independence of phase-space variables.

This is a clear demonstration of the sensitivity of the HTZ ansatz to the choice of tertiary constraints. It cannot be considered an algorithm if such ambiguities are possible. We have already discussed this in Section 4 (see (90)) using the arguments that invariance of the corresponding Lagrangian cannot depend on a choice of tertiary constraints.

The simple examples, (201)-(204) and (213), are related to calculation of a small part of the full generator and transformations that it produces, more discrepancies might appear if full calculations are performed. If the HTZ ansatz for each choice of linear combinations of tertiary first class constraints gives different transformations, it cannot be considered a reliable method of finding gauge invariance, which must be unique characteristics of a system. The only possibility to reconcile the ambiguities of the HTZ ansatz with a unique gauge invariance of a system is an existence of one particular choice of tertiary constraints and this choice must be specified in the HTZ approach. How can such a combination of tertiary constraints be found? It makes no sense to even try. Especially, since it is not clear at all whether it is necessary to do this if there exists the Castellani algorithm that allows us to work with any combination of tertiary constraints without affecting the unique gauge invariance of a system; and, at the same time, we can pick a combination of constraints with which it is easier to perform calculations.

The possible existence of one special combination of constraints gives too strong restriction on the HTZ approach and is opposite to the advantages of the method that were stated in [12]. In particular, according to the authors [12], if the restriction that structure functions become structure constants is imposed (see Eq. (4.5b) [12]) then the generators can be written in the form given previously by Castellani. In reality, the opposite is true: only with exactly such a restriction, (211), the HTZ ansatz gives some meaningful results, contrary to the Castellani algorithm, which is independent on a choice of linear combinations of
tertiary constraints, as it should be in a reliable algorithm. For completeness of discussion of the HTZ ansatz we provide our arguments on what restrictions might be imposed on a choice of constraints in the HTZ ansatz to recover the same invariance with the Castellani algorithm (which is free from such restrictions). These arguments are based on calculations that were performed in [4] where to restore the four-diffeomorphism in Dirac’s Hamiltonian formulation of second order GR we used both methods: Castellani’s and one based on the HTZ ansatz developed in [72]. We obtained the same result, four-diffeomorphism with no need for any field dependent redefinition of parameters, by using both methods; and this, as we understood now, was an “accidental” result. The reason for this is a very special choice of secondary constraints, as in (114), that we used in [4]; and they were defined as the result of the PBs of primary constraints with the Hamiltonian. This is the only case where the HTZ ansatz might work. One realistic field-theoretical example considered by the authors of [72] is Yang-Mills theory, which also leads to known gauge transformations; it was also considered by Castellani [16] to illustrate the use of his algorithm. But in case of Yang-Mills theory, the secondary constraints were also defined as PBs of the corresponding primary constraints with the Hamiltonian. It seems to us that only with such a restriction on what we should call a secondary constraint the HTZ ansatz reproduces the same results as the Castellani algorithm. In the first order affine-metric formulation of GR, tertiary constraints also appear. It is not clear how to find the “right” combination of constraints. The HTZ method might work if we define a “tertiary constraint” as everything that is produced as a time development of the corresponding secondary first class constraint. In contrast, according to the Dirac procedure at every step, when we consider a time development of known constraints, we have, first of all, to single out constraints which are already found, then the remaining part can be called a new constraint. In the formulation considered here, first order metric-affine GR, PBs of secondary first class constraints with the Hamiltonian give the expressions in which we can isolate terms proportional to a linear combination of secondary constraints. But this linear combination is not unique, and we do not have any prescription to find which linear combination is preferable. In addition, if all linear combinations of secondary constraints are equally good, why do we have to use a particular and very special linear combination of tertiary constraints? From our point of view, it is preferable to use the method which is free from these problems.

Another important observation, based on our calculation in the second order EH action
using the Castellani or [72] methods is that the amount of calculation is the same in both cases; so there is no advantage in using the HTZ ansatz even if we can figure out what restrictions should be imposed for the general case with long chains of constraints. In contrast, our partial calculations in the previous Section show that the independence of the final result on which linear combinations of tertiary constraints are used, gives us an advantage as some of them can drastically simplify the calculations.

The sensitivity of the HTZ ansatz and the problems related to it provide an additional illustration of importance of primary first class constraints in the Hamiltonian formulation that we discussed in Section 5. They are the true Masters of the Hamiltonian formulation and because the Castellani approach uses explicitly only primary constraints, it gives correct results and takes care of the possible redefinition of all non-primary constraints. But methods based on the HTZ ansatz, which starts from non-primary constraints, is an attempt to interchange the roles of constraints and this is the reason for its failure. The HTZ ansatz has some similarities with the non-canonical change of variables in the ADM formulation: both treat primary constraints as unimportant and both emphasize the role of non-primary constraints, either by starting the iterative procedure from them or by completely fixing one particular combination of secondary constraints that are reflected in giving the special names: “Hamiltonian” and “spatial diffeomorphism” constraints. Such names are in contradiction with the covariance of General Relativity and with the Dirac conjecture that all first class constraints are responsible for gauge invariance. For example, among eight constraints of the second order formulation of GR in four-dimensional spacetime, if we separate three of them and call them a “spatial diffeomorphism” constraint, for what invariance are the five remaining constraints responsible?

The origin of the failure of the HTZ ansatz is made especially clear from another article [73], which was published almost simultaneously with [12]. In [73], the Castellani procedure for construction of a generator of gauge transformations is considered, and the authors remark that “the problem is complicated by the fact that the chain algorithm [the name used for the Castellani algorithm [136,139]] strongly depends on the representation of the primary first-class constraints surface that is adopted. More precisely, if the primary first-class constraints are

$$\phi_{a_1} = 0, \quad a_1 = 1, ..., m_1$$ (214)
and can each be taken as the head of a chain in [our \(136\)], it is in general not true that the equivalent constraints

\[
\phi'_{a_1} = M_{a_1}^{b_1} (q, p) \phi_{b_1} = 0
\]

(215)
also lead to consistent solution of [our \(136-139\)]. So, algorithm is not invariant under [our \(215\)]. An algorithm ... that is invariant under redefinition [our \(215\)] has been recently proposed in [our \([12]\)]."

So, according to \([73]\) the HTZ ansatz is built on the assumption of invariance under \((215)\) while the Castellani algorithm is not.

Of course, the Hamiltonian formulation “strongly depends” on the primary first class constraints; but this is not a “problem”. It is just the nature of the Hamiltonian formulation that is reflected in the Castellani algorithm\(^{14}\). As we discussed in Section 5, primary first class constrains are special due to their dual nature: they are constraints and, at the same time, they are phase-space variables. As constraints, they allow redefinition \((215)\) without any restriction. If primary constraints are pure momenta (as it happens in most cases), then \((215)\) would be also the redefinition of phase-space variables and it must be canonical!
So this is not a surprise that an “algorithm” which allows an arbitrary redefinition \((215)\), leads to loss of a unique gauge invariance because in general such a redefinition does not correspond to a canonical transformation; and consequently, the new total Hamiltonian will not be equivalent to the original one, as well as its original gauge invariance will not be restored without some field dependent redefinition of gauge parameters (at best). So any “algorithm” that is invariant under arbitrary \((215)\) will unavoidably lead to the loss of gauge invariance for majority of choices of \(M_{a_1}^{b_1} (q, p)\).

Calculations with the Hamiltonian of GR are involved and to illustrate the devastating effect of \((215)\) the simple Hamiltonian formulation can be considered, e.g. Maxwell Electrodynamics for which the total Hamiltonian \([5, 9]\) is

\[
H_T = \dot{A}^0 \pi_0 - A^0 \partial_k \pi^k - \frac{1}{2} \pi_k \pi^k + \frac{1}{4} (\partial_k A_m - \partial_m A_k) (\partial^k A^m - \partial^m A^k).
\]

\(^{14}\) This kind of problem is similar to what is stated in the first line of “Golden Oldie” \([70]\): “The general coordinate invariance underlying the theory of relativity creates basic problems in the analysis of the dynamics of the gravitational field”. ADM variables is the solution of the problem with covariance.
One can take, for example, the following redefinition

\[ \pi'_0 = A^k \pi_k \pi_0 \]

and try to consider the time development of this new primary constraint, closure of the Dirac procedure, algebra of constraints and the restoration of gauge invariance in this case.

VIII. CONCLUSION

In this paper we presented a detailed derivation of the Hamiltonian for the first order affine-metric formulation of GR, including restoration of its unique gauge invariance, four-dimensional diffeomorphism, using the Castellani algorithm, and we demonstrated that four-diffeomorphism can be lost as the result of a non-canonical change of variables or by using methods of restoration which are sensitive to a choice of linear combinations of non-primary first class constraints. These results are based on mathematical derivations and as such do not need any interpretation or discussion. They can be disproved by indication of mistake(s) or must be accepted. One additional, “conventional”, option is just to ignore them saying that it is the well-known fact that “the canonical treatment breaks the symmetry between space and time in general relativity and the resulting algebra of constraints is not the algebra of four diffeomorphism” \[42\] and, because of this, only by some unjustified manipulations the “correspondence to diffeomorphism invariance” can be accomplished.

We address this article to the readers who make their judgement based on the results, not on the correspondence of results to “conventional wisdom”. Such readers, as well as “conventional” ones, do not need a long conclusion to make their minds and, because of this, we stop our discussion here.

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APPENDIX A: SOLVING THE SECOND CLASS CONSTRAINTS

We outline the last step of the Hamiltonian reduction, the elimination of two additional pairs of the phase-space variables \((\Pi^m_0, \Sigma^k_{0m})\) and \((\Pi^p_{kp}, \Lambda^m_{kp})\). The part of the canonical Hamiltonian, \(H_c\), with terms proportional to \(\Sigma^k_{0m}\) and \(\Gamma^p_{km}\) (two last lines of (22)), the only source of contributions into secondary second class constraints \((27)\) and \((28)\), is

\[
H_c \left( \Sigma^k_{0m}, \Gamma^p_{km} \right) = -\hbar^{00} \Sigma^k_{0m} \Sigma^k_{0m} + 2\hbar^{k0} \left( \Gamma^p_{mp} \Sigma^m_{0k} - \Gamma^p_{km} \Sigma^m_{0k} \right) - \hbar^{km} 2\pi_{kp} \Sigma^p_{0m} - 2\hbar^{k0} \Sigma^k_{0m}, \tag{A1}
\]

\[
+ \hbar^{km} \left( 2\Gamma^q_{pq} \Gamma^p_{km} - \Gamma^p_{kp} \Gamma^q_{qm} - \Gamma^p_{kp} \Gamma^q_{mp} \right) - 2\hbar^{k0} \pi_{00} \Gamma^p_{km} + 2\hbar^{km} \left( \pi_{0p} \Gamma^p_{km} - 2\pi_{0k} \Gamma^q_{mp} \right) - \hbar^{pq} \pi_{0k} + 2\hbar^{pk} \Gamma^m_{pm}.\]

The first line of (A1) can be presented in the following form

\[
H_c \left( \Sigma^k_{0m}, \right) = -\hbar^{00} \Sigma^k_{0m} \Sigma^k_{0m} - 2\Sigma^m_{0k} \tilde{D}^0_{km} \tag{A2}
\]

where

\[
\tilde{D}^0_{km} = \hbar^{k0} \pi_{qm} + \hbar^{q0} \Gamma^k_{qm} - \hbar^{k0} \Gamma^k_{mp}. \tag{A3}
\]

Note that \(\tilde{D}^0_{km}\) is not a traceless combination.

The secondary constraint \(\chi^m_{0m}, (27)\), can be obtained from the variation of \(H_c\) (using the fundamental PB \(\{\Sigma^k_{0m}, \Pi^p_{0q}\}\) from (10))

\[
\hat{\Pi}^m_k = \left\{ \Pi^m_k, H_c \right\} = -\frac{\delta H_c}{\delta \Sigma^k_{0m}} = \chi^m_{0m} = \hbar^{00} \Sigma^k_{0m} + \tilde{D}^0_{km} - \frac{\delta^m_{k}}{D - 1} \tilde{D}^0_{m}. \tag{A4}
\]

This constraint obviously has a non-zero PB with the primary constraint \(\Pi^p_{0q}\), and this pair of constraints \((\Pi^m_{0m}, \chi^m_{0m})\) is of second class and the corresponding pair of variables, \((\Pi^m_{0m}, \Sigma^k_{0m})\), can be eliminated. Solving \(\chi^m_{0m} = 0\) from (A4) for \(\Sigma^m_{0k}\) and substituting the pair

\[
\]
\[ \Pi_{q}^{0p} = 0, \quad \Sigma_{0k}^{m} = -\frac{1}{\hbar^{00}} \left( \tilde{D}_{k}^{0m} - \frac{1}{D-1} \tilde{D}_{n}^{0n} \delta_{k}^{m} \right) \]  \hspace{1cm} (A5)

into (22) gives the next reduction with the following change in the canonical Hamiltonian \( H_{c} \),

\[ H_{c} (\Sigma_{0m}^{k}) = H_{c} (\Sigma_{0m}^{k} \text{ from Eq. (A5)}) = -\frac{1}{\hbar^{00}} \tilde{D}_{b}^{0a} \tilde{D}_{a}^{0b} + \frac{1}{D-1} \frac{1}{\hbar^{00}} \tilde{D}_{n}^{0n} \tilde{D}_{a}^{0a}. \]  \hspace{1cm} (A6)

Separating in \( \tilde{D}_{b}^{0a} \) contributions proportional to \( \Gamma_{qm}^{k} \) we write

\[ \tilde{D}_{m}^{0k} = D_{m}^{0k} + D_{m}^{0k} (\Gamma_{qm}^{k}) \]  \hspace{1cm} (A7)

where

\[ D_{m}^{0k} = h_{m}^{k0} + h_{q}^{k} \pi_{qm}, \]  \hspace{1cm} (A8)

and

\[ D_{m}^{0k} (\Gamma_{qm}^{k}) = h_{0}^{0} \Gamma_{qm}^{k} - h_{m}^{k0} \Gamma_{mp}^{q}. \]  \hspace{1cm} (A9)

Note that the trace of \( D_{m}^{0k} (\Gamma_{qm}^{k}) \) is zero and for (A6); we obtain

\[ H_{c} (\Sigma_{0m}^{k}) = -\frac{1}{\hbar^{00}} D_{b}^{0a} D_{a}^{0b} + \frac{1}{D-1} \frac{1}{\hbar^{00}} D_{n}^{0n} D_{a}^{0a} - \frac{1}{\hbar^{00}} \frac{1}{D-1} D_{b}^{0a} (\Gamma_{qm}^{k}) D_{a}^{0b} (\Gamma_{qm}^{k}) \]  \hspace{1cm} (A10)

The first two terms in (A10) correspond to the second line of (31) and the last two terms of (A10) must be combined with the second line of (A1). Now the part of the canonical Hamiltonian that depends on \( \Gamma_{qm}^{k} \) can be written as

\[ H_{c} (\Gamma_{qm}^{k}) = e^{km} (2 \Gamma_{pq}^{q} \Gamma_{km}^{p} - \Gamma_{kp}^{p} \Gamma_{mq}^{q} - \Gamma_{kq}^{q} \Gamma_{mp}^{p}) - \tilde{D}_{m}^{qk} \Gamma_{qm}^{k}. \]  \hspace{1cm} (A11)

In the quadratic part of (A11), the combination \( e^{km} \) naturally arises and \( \tilde{D}_{m}^{qk} \), in the part linear in \( \Gamma_{qm}^{k} \), can be written in manifestly symmetric form

\[ \tilde{D}_{m}^{qk} = -2 \hbar^{kq} \pi_{0m} + h_{m}^{kq} - \frac{h_{0}^{q0}}{\hbar^{00}} D_{m}^{0k} - \frac{h_{0}^{k0}}{\hbar^{00}} D_{m}^{0q}. \]  \hspace{1cm} (A12)
\[
+ \left( \frac{\hbar^{0} \pi_{00} + 2h^{0k} \pi_{0p} - h^{kp} + \frac{h^{c0}}{h^{00}} D^{0k}}{\delta_{m}^{k}} \right) \delta_{m}^{q} + \left( \frac{h^{0q} \pi_{00} + 2h^{pq} \pi_{0p} - h^{qp} + \frac{h^{c0}}{h^{00}} D^{0q}}{\delta_{m}^{k}} \right) \delta_{m}^{k}
\]

where \( D^{0k}_{m} \) is defined as (A8) (or (33) in the main text). This part of \( H_{c} \), (A11), leads to the secondary second class constraints (28)

\[
\dot{\Pi}_{x}^{yz} = \{ \Pi_{x}^{yz}, H_{c} \} = -\frac{\delta H_{c}}{\delta \Gamma_{y}^{x}} = \chi_{x}^{yz} = 0. \quad (A13)
\]

and the last pair of phase-space variables can be eliminated: \( \Pi_{m}^{kp} \). Performing the variation \( \frac{\delta H_{c}}{\delta \Gamma_{y}^{x}} \) in (A13) we obtain

\[
2e^{\nu z} \Gamma_{x}^{\nu} - e^{k y} \Gamma_{k x}^{z} - e^{k z} \Gamma_{k x}^{y} + \delta_{y}^{z} \left( e^{k m} \Gamma_{k m}^{y} - e^{k y} \Gamma_{k p}^{p} \right) + \delta_{x}^{y} \left( e^{k m} \Gamma_{k m}^{z} - e^{k z} \Gamma_{k p}^{p} \right) = \tilde{D}_{x}^{yz}. \quad (A14)
\]

The way to solve (A14) for \( \Gamma_{x}^{m} \) is similar to the Einstein proof of equivalence of the first and second order formulations of metric GR [1] (see also [5], Appendix A). The solution of (A14) is based on the subsequent elimination of “traces”. Contracting (A14) with \( \delta_{y}^{x} \) we obtain

\[
e^{k m} \Gamma_{k m}^{z} - e^{k z} \Gamma_{k p}^{p} = \frac{1}{D - 1} \tilde{D}_{x}^{xz}. \quad (A15)
\]

The left-hand side of (A15) is exactly the combination which appears in brackets with a Kronecker delta in equation (A14). This allows us to write (A14) as

\[
2e^{\nu z} \Gamma_{x}^{\nu} - e^{k y} \Gamma_{k x}^{z} - e^{k z} \Gamma_{k x}^{y} + \delta_{y}^{z} \left( e^{k m} \Gamma_{k m}^{y} - e^{k y} \Gamma_{k p}^{p} \right) + \delta_{x}^{y} \left( e^{k m} \Gamma_{k m}^{z} - e^{k z} \Gamma_{k p}^{p} \right) = \tilde{D}_{x}^{yz}. \quad (A16)
\]

This combination, \( D_{x}^{yz} \), is also used in the reduced Hamiltonian (31); and its explicit form is written in (34).

We still have to eliminate the trace \( \Gamma_{x}^{q} \) in (A16). Contracting (A16) with \( h_{y z} \) we find

\[
\Gamma_{x}^{q} = \frac{1}{2 (D - 2)} h_{y z} D_{x}^{yz} \quad (A17)
\]

and equation (A16) takes the final form

\[
- e^{k y} \Gamma_{k x}^{z} - e^{k z} \Gamma_{k x}^{y} = D_{x}^{yz} - e^{yz} \frac{1}{(D - 2)} h_{p q} D_{x}^{pq} \equiv \hat{D}_{x}^{yz}. \quad (A18)
\]
The solution of (A18) can be found using the Einstein permutation, as it was done in [1] and [5] (Appendix A). We need to have three free indices in the same position. To obtain this, we contract (A18) with $e^{nx}$ to obtain

$$-e^{nx} e^{kx} \Gamma^z_{kx} - e^{nx} e^{kz} \Gamma^y_{yx} = e^{nx} \hat{D}^{yz}_x. \tag{A19}$$

Now we perform a permutation in indices $n, y$ and $z$ and add the resulting equations in the following order $(nyz) + (zny) - (yzn)$. That gives us

$$e^{nx} e^{kz} \Gamma^y_{kx} = -\frac{1}{2} \left( e^{nx} \hat{D}^{yz}_x + e^{zx} \hat{D}^{ny}_x - e^{yz} \hat{D}^{zn}_x \right). \tag{A20}$$

Contracting (A20) with $h_{an} h_{bz}$ we obtain the solution for $\Gamma^y_{ba}$,

$$\Gamma^y_{ba} = -\frac{1}{2} \left( h_{bz} \hat{D}^{yz}_a + h_{an} \hat{D}^{ny}_b - h_{an} h_{bz} e^{yz} \hat{D}^{zn}_x \right). \tag{A21}$$

The last pair of phase-space variables $(\Pi^{yz}_x, \Gamma^{x}_{yz})$ can be eliminated (the last Hamiltonian reduction).

Substitution of (A21) into the part of the canonical Hamiltonian (A11) gives the third line of (31) that is expressed in terms of the combination $D^{yz}_x$. Note that in (A11) we have $\hat{D}^{yz}_x$ and by using our redefinition (A16) can be also written in terms of $D^{yz}_x$

$$\hat{D}^{yz}_x = D^{yz}_x - D^{mu}_x \delta^z_x - D^{pz}_x \delta^y_x. \tag{A22}$$

This completes the Hamiltonian reduction of the affine-metric formulation of GR.

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