RECOVERY OF INHOMOGENEITIES AND BURIED OBSTACLES

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Abstract. In this paper we consider the unique determination of inhomogeneities together with possible buried obstacles by scattering measurements. Under the assumption that the buried obstacles have only planar contacts with the inhomogeneities, we prove that one can recover both of them by knowing the associated scattering amplitude at a fixed energy.

1. Introduction

We shall be concerned with the unique determination of a medium together the possible buried obstacles by making scattering measurement far away from the (unknown/inaccessible) object. There are no global identifiability results available in literature on recovering both of them by knowing the scattering amplitude at a fixed energy. The existing results are either based on knowing the outside inhomogeneity in advance to recover only the included obstacle ([8]); or by making use of measurement data with frequency from an open interval ([3]), that is, much more data are utilized than needed. Moreover, it is noted that the uniqueness result in [3] cannot be generalized to three dimensions since its argument involves conformally mapping the domain containing the support of the inhomogeneity onto an annulus. We would like to mention that the global uniqueness in the determination of scatterers consisting of sole mediums or obstacles by scattering amplitude at a fixed energy has been widely known as sophisticatedly established. The result for recovering a medium was obtained by Nachman ([10]) which is based on the use of complex geometric optics (CGO) solutions due to Sylvester-Uhlmann ([12]); and for recovering an obstacle was obtained by Kirsch-Kress ([7]) which is based on the use of singular sources due to Isakov ([5]). We also refer to the monographs [1] and [5] for a comprehensive discussion and related literature.

In this paper, in combination of the two methodologies of using CGO solutions and singular sources we are able to prove the global identifiability of both the scattering medium and the possible buried obstacles. Our restrictive assumption is that the buried obstacles have only planar contacts with the inhomogeneities. In general, we cannot recover an obstacle which is completely buried inside the inhomogeneity. But the exposure part of the obstacle to the exterior of the medium can be arbitrarily small. On the other hand, our proofs indicate that if the obstacle is enclosed entirely in the medium but known in advance, then one can recover the
surrounding medium by the corresponding scattering amplitude. In the rest of this section, we give a brief formulation of the direct and inverse scattering problems.

Let \( D \subset \mathbb{R}^3 \) represent the obstacle which is a bounded domain with connected Lipschitz complement \( \Gamma := \mathbb{R}^3 \setminus \overline{D} \); that is, we include in our discussion the case of multiple obstacle components. Further let \( B \) be a sufficiently large ball such that \( \overline{D} \subset B \). Let \( q \in L^\infty(\Gamma) \) with \( \Im q \geq 0 \) and \( \text{supp}(1 - q) \subset B \setminus D \) represent the scattering medium, namely, the refractive index. We consider the following scattering problem for the time-harmonic plane wave \( u^i(x) := \exp\{i\kappa x \cdot \theta\} \)

\[
\begin{cases}
(D + q\kappa^2)u = 0 & \text{in } \Gamma, \\
B u = 0 & \text{on } \partial D, \\
M u = 0 & \text{in } \mathbb{R}^3,
\end{cases}
\]

where \( u := u^i + u^s \) with \( u^s \) the so-called scattered field and \( B \) is the boundary operator which gives a Dirichlet boundary condition \( u|_{\partial D} = 0 \) corresponding to a sound-soft obstacle \( D \). Moreover, the last equation is the well-known Sommerfeld radiation condition given by

\[
\lim_{r \to \infty} r^i \left( \frac{\partial u^s}{\partial r} - i\kappa u^s \right) = 0 \quad r = |x|,
\]

which holds uniformly for all directions \( \hat{x} := x/|x| \in S^2 \). It is known that \( u \) has the following asymptotic representation (see [1])

\[
u(x; \theta, \kappa) = \exp\{i\kappa \theta \cdot x\} + \frac{\exp\{i\kappa |x|\}}{|x|} A(\hat{x}, \theta, \kappa) + O(|x|^{-2}).
\]

The function \( A \) is called the scattering amplitude (or the far-field pattern) with \( \hat{x} \), \( \theta \) and \( \kappa \) denoting, respectively, the observation direction, the incident direction and the wave number. The inverse scattering problem consists in the determination of the obstacle \( D \) and the scattering medium \( q \) by knowing \( A(\hat{x}, \theta, \kappa) \) for a fixed \( \kappa > 0 \) and all \( \hat{x} \in S^2, \theta \in S^2 \).

The paper is organized as follows. In section 2 we present the class of admissible scatterers and give a brief study of the forward scattering problem. Section 3 is devoted to the unique determination of a scatterer with the buried obstacle. In Section 4 we indicate how to determine the surrounding medium when the obstacle is buried inside completely but known a priori.

2. Class of admissible scatterers and the direct scattering problem

In order to state our uniqueness results we need first to introduce a suitable class \( \mathcal{C} \) of admissible scatterers. We begin by fixing some notations which shall be used throughout the rest of the paper. For any \( x \in \mathbb{R}^3 \) and \( r > 0 \), with \( B_r(x) \) we denote the open ball of center \( x \) and radius \( r \). Let \( \Pi_l \) with an index \( l \in \mathbb{N} \) represent a simply connected subset of some plane \( \Pi_l \) in \( \mathbb{R}^3 \), and moreover, \( \mathcal{R}_l \) represent the reflection in \( \mathbb{R}^3 \) with respect to \( \Pi_l \). Let \( C \) denote a generic constant which may be changed in different inequalities but must be fixed and finite in a given relation. Finally, “\( a \lesssim b \)” shall refer to “\( a \leq C b \)”.

**Definition 2.1.** We say that \( \Sigma(D, q, \Omega) \) is a scatterer of class \( \mathcal{C} \) with obstacle \( D \) and scattering medium \( q \) if it satisfies the following assumptions

1. \( \Sigma \) is a compact set in \( \mathbb{R}^3 \) with connected complement \( \Sigma_\infty := \mathbb{R}^3 \setminus \Sigma \).
ii) \( \Sigma = \overline{\Omega} \cup D \), where \( D \) is a \( C^{2,1} \) domain with connected complement \( \Gamma := \mathbb{R}^3\setminus D \) and \( \Omega \subset \Gamma \) is an open set.

iii) \( q(x) \in L^\infty(\Omega) \) with \( 3q \geq 0 \) and \( \text{supp}(1-q) = \overline{\Omega} \). Moreover, \( 1-q \in C^{0,1}(\overline{\Omega}) \) and there exists a constant \( \epsilon_0 > 0 \) such that \( |1-q(x)| \geq \epsilon_0 \) for all \( x \in \overline{\Omega} \), i.e., \( 1-q \) has jump across \( \partial \overline{\Omega} \).

iv) The medium and the obstacle have only planar contact in the sense that \( \partial \Omega \cap \partial D = \bigcup_{l=1}^{N_0} \Gamma_l \), where \( N_0 \) is a finite integer and \( \Gamma_l \subset \partial \Omega_l \) with each \( \Omega_l \) a connected component of \( \Omega \), \( l = 1, 2, \ldots, N_0 \). Moreover, if we set \( \Omega_0 := \Omega - \bigcup_{l=1}^{N_0} \Omega_l \), then \( \Omega_l \cap \Omega_{l'} = \emptyset \) if \( l \neq l' \) for \( 0 \leq l, l' \leq N_0 \).

v) Set \( \Gamma_{int} := \bigcup_{l=1}^{N_0} \Gamma_l \) and \( \partial D_{ext} := \partial D \setminus \Gamma_{int} \), \( \partial \Omega_{ext} := \partial \Omega \setminus \Gamma_{int} \), \( \partial \Omega_{int} \) are \( C^2 \) continuous.

vi) \( (\Omega_1 \cup \mathcal{R}_1) \cap (\Omega_{l'} \cup \mathcal{R}_{l'}) = \emptyset \) for \( 1 \leq l, l' \leq N_0 \) and \( l \neq l' \); and \( (\Omega_l \cup \mathcal{R}_{\rho}) \cap \overline{\Omega}_0 = \emptyset \) for \( 1 \leq l \leq N_0 \).

Clearly, according to our definition, a scatterer \( \Sigma(D, q, \Omega) \in \mathcal{C} \) is composed of an impenetrable obstacle \( D \) and the surrounding medium \( q \) with support in \( \overline{\Omega} \); and the \( \Gamma_{int} \) part of the obstacle \( D \) is buried in the inhomogeneity. Moreover, it is noted that we admit multiple scattering components. In fact, if we let \( \sigma \) denote a connected component of \( \Sigma \), then it may be an obstacle, or the support of a scattering medium, or the two combined together with the obstacle buried inside the inhomogeneity. Hence, an admissible scatterer is much general which may consist of multiple components being obstacles, or scattering mediums, or the combination of the two with the obstacles as inclusions.

The following is a remark on the geometric and topological assumptions of the admissible scatterers concerning our subsequent uniqueness study.

**Remark 2.2.** The \( C^{0,1} \) and \( C^2 \) regularity assumptions, respectively, on the refractive index \( q \) in iii) and on \( \partial \Omega_{ext} \) of the scatterer in v) are only needed for the subsequent uniqueness theorem in determining the location and shape of a scatterer. Though at certain point, such regularity requirement can be weakened, we choose to work with a consistent assumption to ease the exposition. The topological assumption in vi) is only needed for the subsequent uniqueness theorem in determining the scattering medium \( q \) provided the buried obstacle have been recovered.

Next, we consider the direct scattering problem with a scatterer \( \Sigma(D, q, \Omega) \in \mathcal{C} \). Starting from now on, we fix the wave number to be \( \kappa_0 > 0 \). Let \( \mathcal{L}_q := \Delta + \kappa_0^2 q \) denote the Schrödinger operator. Using the fact that \( (\Delta + \kappa_0^2) u^i = 0 \), the forward scattering problem is reformulated as

\[
(2.1) \quad \mathcal{L}_q u^i = f_q(u^i) \quad \text{in} \quad \mathcal{G}, \quad B u^i = g(u^i) \quad \text{on} \quad \partial \mathcal{G} \quad \text{and} \quad \mathcal{M} u = 0,
\]

where \( f_q(u^i) = \kappa_0^2 (1-q) u^i \) and \( g(u^i) = -u^i|_{\partial \mathcal{G}}. \) In the sequel, for each \( \rho > 0 \), we set \( \mathcal{G}_\rho := \mathcal{G} \cap B_\rho(0) \). To study the direct scattering problem, it is convenient to introduce the notation

\[
H_{loc}^1(\mathcal{G}) = \{ u \in \mathcal{D}'(\mathcal{G}); u|_{\mathcal{G}_\rho} \in H^1(\mathcal{G}_\rho) \} \quad \text{for each} \quad \rho > 0 \quad \text{such that} \quad \Sigma \subset B_\rho(0) \}.
\]

Then, one can show the well-posedness of the direct scattering problem in the space \( H_{loc}^1(\mathcal{G}) \). In fact, the uniqueness is easily derived by using the Rellich uniqueness theorem (see Lemma 6.1 in \[5\]). For the existence, by using the Lax-Phillips
method, one can reduce the problem to a corresponding one in the bounded domain \( \Omega \) (see Chapter 6 in [5]), which has been well understood (see [2] and [9]). Moreover, we have the well-known elliptic stability estimate, that is, \( \|u\|_{H^1(G)} \lesssim \|f\|_{L^2(\Omega)} + \|g(u)\|_{H^1/2(\partial G)} \). However, for our subsequent uniqueness study in the inverse problem, we need an integral representation of the solution, which has been given in [7]. To this end, we let \( \Phi(x,y) := e^{i\kappa_0|y-x|}/|x-y| \) be the fundamental solution to the differential operator \((\Delta + \kappa_0^2)\) and introduce the following potential operators:

\[
(2.2) \quad \text{SL}(x) = \int_{\partial G} \Phi(x,y)\psi(y)dS_y, \quad \text{DL}(x) = \int_{\partial G} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\psi(y)dS_y
\]

where \( \nu \) is the interior normal to \( G \), and

\[
(2.3) \quad V_q\psi(x) = \kappa_0^2 \int_{\Omega} \Phi(x,y)[1 - q(y)]\psi(y)dS_y.
\]

SL and DL are well-known as the single- and double-layer potential operators, while \( V_q \) is known as the volume potential operator; we refer [1] and [9] for a detailed study and relevant mapping properties. For the forward scattering problem, we have the following theorem which is readily modified from Theorem 2.2 in [7].

**Theorem 2.3.** \( u^* \in H^1_{loc}(G) \cap C(G) \) is a solution of (2.1) if \( u^*|_{\Omega} \in C(\Omega) \) has the form

\[
(2.4) \quad u^*(x) = -V_q u^*(x) + (DL + i\kappa_0 SL)\psi(x) + r(x) \quad x \in \Omega,
\]

where \( r(x) := -V_q u^*(x) \) and \( \psi(x) \in C(\partial \Omega) \) satisfies

\[
(2.5) \quad \psi(x) = 2T V_q(x) - 2(T DL + i\kappa_0 T SL)\psi(x) + t(x) \quad x \in \partial G
\]

where \( t(x) := 2T V_q u^*(x) \) and \( T \) is the one-sided trace operator for \( G \). Moreover, we have

i) The system (2.4)-(2.5) of integral equations is uniquely solvable in \( C(\Omega) \times C(\partial \Omega) \) for \( (r,t) \in C(\Omega) \times C(\partial \Omega) \) and depends continuously on \( r \) and \( t \).

ii) The system (2.4)-(2.5) of integral equations is uniquely solvable in \( L^2(\Omega) \times C(\partial \Omega) \) for \( (r,t) \in L^2(\Omega) \times C(\partial \Omega) \) and depends continuously on \( r \) and \( t \).

It is remarked that in our uniqueness study of determining the obstacle, we would essentially make use the continuity of scattered field in the exterior domain. Next we introduce a more singular point source than \( \Phi(x,y) \) which is given for every fixed \( x_0 \in \mathbb{R}^3 \) by

\[
(2.6) \quad \Psi(y,x_0) = h^{(1)}_1(\kappa_0 \rho)P_1(\cos(\psi)),
\]

where \( h^{(1)}_1 \) is the spherical Hankel function of the first kind of order one and \( P_1 \) is the Legendre polynomial of order one; and \( (\rho, \phi, \psi) \) is the spherical coordinate of \( y-x_0 \). \( \Psi(y,x) \) is known as the spherical wave function and we refer to [1] for related study. It is noted that \( \Psi(y,x_0) \) has quadratic singularity only at the point \( y = x_0 \) which comes from that of the spherical Hankel function; that is, \( (y-x_0)^2 \Psi(y,x_0) \) is smooth over \( \mathbb{R}^3 \).

We conclude this section with an approximation property of point sources by linear combination of plane waves.
Lemma 2.4. Let $E \subset \mathbb{R}^3$ be a compact set and $x_0 \in \mathbb{R}^3 \setminus E$ be fixed. Then there exist sequences $v_n(y)$ and $\omega_n(y)$ in the span of plane waves

$$\mathcal{E} := \text{span}\{e^{i\kappa_0 y \cdot \theta} : \theta \in \mathbb{S}^2\}$$

such that

(2.7) \hspace{1cm} \|v_n - \Phi(\cdot, x_0)\|_{C^1(E)} \to 0 \quad \text{as} \quad n \to \infty.

and

(2.8) \hspace{1cm} \|\omega_n - \Psi(\cdot, x_0)\|_{C^1(E)} \to 0 \quad \text{as} \quad n \to \infty.

Proof. This follows from Lemma 3.2 in [7] by noting that $\Phi(\cdot, x_0)$ and $\Psi(\cdot, x_0)$ are smooth solutions for the Helmholtz equation in any domain that does not contain $x_0$. See also Lemma 5 in [11]. \hfill \Box

3. Recovery of inhomogeneities together with buried obstacles

In this section, we show the uniqueness in determining a scatterer $\Sigma(D, q, \Omega) \in C$ by its corresponding scattering amplitude $A(\hat{x}, \theta, \kappa_0)$. The main result is stated as follows.

Theorem 3.1. $\Sigma(D, q, \Omega) \in C$ is uniquely determined by knowledge of the far-field pattern $A(\hat{x}, \theta, \kappa_0)$ for arbitrarily fixed $\kappa_0 > 0$ and all $\hat{x}, \theta \in \mathbb{S}^2$.

The proof of Theorem 3.1 is proceeded in three steps, which we shall outline briefly in the following. In the first step, we recover the exterior boundary of the scatterer, namely $\partial \Sigma$, disregarding the interior medium and obstacle. This is based on the use of the singular sources $\Psi(\cdot, x_n)$ with $x_n$ approaching a point $x_0$ which may either lies on the exterior boundary part of the medium or the exterior boundary part of the obstacle. It is shown the corresponding scattered waves will blow up in the limiting case. We would like to remark that the use of point source with quadratic singularity is also considered in [11] to determine the support of a scattering medium. In the second step, we show that one can distinguish the exterior medium boundary from the exterior obstacle boundary, and thus can determine the inside obstacle. This is based on the use of singular source $\Phi(\cdot, x_n)$ with $x_n$ approaching an exterior boundary point $x_0$ of the scatterer. It is shown that the scattered wave will blow up in the limiting case when $x_0$ lies on the boundary of the obstacle, whereas scattered wave remains bounded when $x_0$ lies on the boundary of the medium. In the final step, we recover the medium along the line of the Sylvester-Uhlmann methodology (see [12]). In doing this, we first derive a novel approximation result of Runge type (see Lemma 3.4). The result is remarkable since it enables us to use CGO solutions in different medium components with different complex phases. Next, it is natural to construct the almost complex exponential solutions which vanish on the interior boundary of the obstacle. Since the medium and the buried obstacle have only planar contact, this can be carried out by making reflections of the CGO solutions with respect to the contact planes. In [6], similar idea of implementing reflection of solutions have been used to prove a uniqueness in inverse conductivity problem with local Cauchy data on the boundary. We would like to note that in [6] the inaccessible part of the boundary is assumed to be on a single plane.
3.1. Unique determination of Σ. By contradiction, let \( \hat{\Sigma} := \Sigma(\hat{D}, \hat{q}, \hat{\Omega}) \in C \) be a scatterer such that \( \hat{\Sigma} \neq \Sigma \) and \( A(\hat{x}, \theta) = A(x, \theta) \) for \( \hat{x}, \theta \in S^2 \), where \( A \) and \( \hat{A} \) are respectively the scattering amplitudes of \( \Sigma \) and \( \hat{\Sigma} \) corresponding to the incident plane waves \( \exp(i\kappa_n x \cdot \theta) \). Let \( \Lambda \) be the (unique) unbounded connected component of \( \mathbb{R}^3 \setminus (\Sigma \cup \hat{\Sigma}) \). We denote by \( u(x, \theta) \) and \( \hat{u}(x, \theta) \), respectively, the total fields corresponding to \( \Sigma \) and \( \hat{\Sigma} \). Then, by the Rellich uniqueness theorem, we know \( u(x, \theta) = \hat{u}(x, \theta) \) in \( \Lambda \) for all \( \theta \in S^2 \). Since \( \Sigma \neq \hat{\Sigma} \) and both are connected, we easily see that either \((\mathbb{R}^3 \setminus \hat{\Lambda}) \setminus \emptyset \) or \((\mathbb{R}^3 \setminus \Lambda) \setminus \emptyset \). Without loss of generality, we assume the former case and set \( \Sigma^* = (\mathbb{R}^3 \setminus \hat{\Lambda}) \setminus \emptyset \). It is obvious that \( \partial \Sigma^* \subset \partial \Lambda \cup \partial \Sigma \subset \partial \Sigma \cup \partial \Sigma^* \). According to Definition 2.1, \( \partial \Sigma = \partial D_{ext} \cup \partial \Omega_{ext} \). Let \( x_0 \in \partial \Sigma^* \setminus \partial \Sigma \subset (\partial D_{ext} \cup \partial \Omega_{ext}) \setminus \Sigma \). We next distinguish two cases that \( x_0 \in \partial D_{ext} \setminus \Sigma \) and \( x_0 \in \partial \Omega_{ext} \setminus \Sigma \). In the following, we fix \( \rho_0 > 0 \) be sufficiently large such that \( \bar{\Sigma} \cup \hat{\Sigma} \subset B_{\rho_0}(0) \) and let \( G_{\rho_0} \) and \( \hat{G}_{\rho_0} \), respectively, denote \((\mathbb{R}^3 \setminus \hat{D}) \cap B_{\rho_0}(0) \) and \((\mathbb{R}^3 \setminus \hat{D}) \cap B_{\rho_0}(0) \).

**Case 1.** \( x_0 \in \partial D_{ext} \setminus \Sigma \). Let \( \tau_0 > 0 \) be sufficiently small such that \( B_{\tau_0}(x_0) \subset \Sigma_{\infty} \) and \( B_{\tau_0}(x_0) \cap \hat{\Omega} = \emptyset \). Set \( S := \partial D_{ext} \cap B_{\tau_0}(x_0) \). Without loss of generality, we assume that \( S \subset \partial D_{ext} \cap \partial \Lambda \). Obviously, \( B_{\tau_0}(x_0) \) is divided by \( S \) into two parts and we denote by \( B_{\tau_0}^+ \) the one contained in \( \Lambda \). We now consider the two scattering problems corresponding to \( \Sigma(D,q,\Omega) \) and \( \hat{\Sigma}(\hat{D},\hat{q},\hat{\Omega}) \) with the incident fields being the point sources \( \Psi(\cdot, x) \) for \( x \in B_{\tau_0}^+ \). Let \( \omega^s(\cdot, x) \) and \( \hat{\omega}^s(\cdot, x) \) denote, respectively, the scattered fields. Since the scattered waves coincide in \( \Lambda \) for all plane waves, by using Lemma 2.4 it is straightforward to show that \( \omega^s(\cdot, x) = \hat{\omega}^s(\cdot, x) \) in \( \Lambda \) for \( x \in B_{\tau_0}^+ \). Next, it is observed that \( B_{\tau_0}^+ \subset \Sigma_{\infty} \), and hence \( \Psi(\cdot, x) \) with \( x \in B_{\tau_0}^+ \) is smooth in \( \hat{\Sigma} \). By using the expansions of the spherical Hankel functions, one can verify directly that \( \| \Psi(\cdot, x) \|_{C^1(\Sigma)} \leq C \) for \( x \in B_{\tau_0}^+ \). From the well-posedness of the forward scattering problem, we see \( \| \omega^s(\cdot, x) \|_{C^1(\hat{G}_{\rho_0})} \leq C \); see related discussion in Section 2.

Then, we choose \( h > 0 \) such that the sequence
\[
(3.1) \quad x_n := x_0 + \frac{h}{n} \nu(x_0) \quad n = 1, 2, \ldots
\]
is contained in \( B_{\tau_0}^+ \), where \( \nu(x_0) \) is the outward normal to \( \partial \hat{D} \) at \( x_0 \). By our discussion made earlier, \( \| \omega^s(x_n, x_n) \| \leq C \) uniformly for \( n \geq 1 \). On the other hand, referring to Lemma 3 in [11], we know \( \| \Psi(x_0, x_n) \| \to \infty \) as \( n \to \infty \), and hence by using the Dirichlet boundary condition of \( \hat{\omega}^s \) on \( \partial \hat{D} \)
\[
(3.2) \quad |\omega^s(x_0, x_n)| = |\hat{\omega}^s(x_0, x_n)| = | - \Psi(x_0, x_n)| \to \infty \quad \text{as} \quad n \to \infty.
\]
This obviously gives a contradiction.

**Case 2.** \( x_0 \in \partial \Omega_{ext} \setminus \Sigma \). Similar to Case 1, we let \( B_{\tau_0}(x_0) \) be a sufficiently small ball such that \( B_{\tau_0}(x_0) \subset \Sigma_{\infty} \) and \( B_{\tau_0}(x_0) \cap \hat{D} = \emptyset \). Moreover, let \( S := \partial \Omega_{ext} \cap B_{\tau_0}(x_0) \) which is assumed to lie entirely on \( \partial \Lambda \), and let \( B_{\tau_0}^+ \) denote the part of \( B_{\tau_0}(x_0) \) contained in \( \Lambda \). By a same argument as that for Case 1, we know \( \| \omega^s(\cdot, x) \|_{C^1(\hat{G}_{\rho_0})} \leq C \) for \( x \in B_{\tau_0}^+ \). Clearly, in order to get a contradiction, we only need to show that \( \hat{\omega}^s(\cdot, x) \) reveals singular behavior near \( x_0 \). To this end, let \( x_n, n = 1, 2, \ldots \) be as defined in (3.1). It is first observed that \( |V_q \Psi(x_n, x_0)| \leq 1/|x_n - x_0| \) (see Lemma 4 in [11]). Hence \( \| V_q \Psi(x_n, x_0) \|_{L^2(\hat{G}_{\rho_0})} \leq C \) uniformly for \( n \in N \). Moreover, noting \( x_n \)'s are contained in \( B_{\tau_0}^+ \) which is...
away from \( \tilde{D} \). \( \| \mathcal{F} V_q \psi(\cdot, x_n) \|_{C(\partial D)} \leq C \) uniformly for \( n \in \mathbb{N} \). By Theorem 2.2 ii), \( \| \hat{\omega}^s(\cdot, x_n) \|_{L^2(\Omega)} \leq C \) and \( \| \psi(\cdot, x_n) \|_{C(\partial \Omega)} \leq C \), where \( \psi(\cdot, x_n) \) is the density in (2.3) corresponding to the incident waves \( \Psi(\cdot, x_n) \). Next, using the mapping properties that \( V_q \) maps \( L^2(\Omega) \) continuously into \( C(\tilde{G}_p) \), and \( SL \psi \) and \( DL \psi \) map \( C(\partial \Omega) \) continuously into \( C(\tilde{G}_p) \) (see [1]), we know \( |V_q \hat{\omega}^s(x_0, x_n)| \leq C \) and \( |(DL + i\kappa_0 SL)\psi(x_0, x_n)| \leq C \) uniformly for \( n \in \mathbb{N} \). On the other hand, referring to Lemma 3 in [11], we know

\[
|V_q \psi(x_0, x_n)| \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.
\]

Hence, by using the relation given in (2.4)

\[
|\hat{\omega}^s(x_0, x_n)| \geq |V_q \psi(x_0, x_n)| - |V_q \hat{\omega}^s(x_0, x_n)| - |(DL + i\kappa_0 SL)\psi(x_0, x_n)| \rightarrow \infty
\]

as \( n \rightarrow \infty \), which then yields a similar contradiction to that in (3.2). \( \square \)

3.2. Recovery of the obstacle \( D \). Let \( \Sigma(D, q, \Omega) \) and \( \tilde{\Sigma}(\tilde{D}, \tilde{q}, \tilde{\Omega}) \) be the two scatterers considered in subsection 3.1, we next show \( D = \tilde{D} \). Since \( \Sigma = \tilde{\Sigma} \) and both \( \Sigma(D, q, \Omega) \) and \( \tilde{\Sigma}(\tilde{D}, \tilde{q}, \tilde{\Omega}) \) belongs to class \( C \), one only need to show that

\[
\partial D_{ext} = \partial \tilde{D}_{ext}
\]

which then implies \( \Gamma_{int} = \tilde{\Gamma}_{int} \) and \( \partial \Omega_{ext} = \partial \tilde{\Omega}_{ext} \). In fact, due to assumptions iv) and vi) in Definition 2.1 it is easily seen that each planar uniformly for \( C \). 

\[
\text{hence, letting } B := \tilde{B}, \text{ and } \tilde{\kappa} \text{ is not a Dirichlet eigenvalue for } -\Delta \text{ neither in } B. \text{ Moreover, we require that the homogeneous Dirichlet problem for } L_{\tilde{q}} \text{ has only trivial solution in } H^1_0(B \setminus \tilde{D}). \text{ Setting } w = u - \tilde{u}, \text{ we see}
\]

\[
w = 0 \quad \text{in } B \setminus \Sigma,
\]

and hence

\[
w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial \Omega_{ext} := \partial \tilde{\Omega} \setminus \Gamma_{int}.
\]

3.3. Unique determination of the scattering medium \( q \). In view of the results in subsection 3.1 and 3.2, we only need to show that if \( \Sigma(D, q, \Omega) \) and \( \Sigma(D, \tilde{q}, \Omega) \) produce the same scattering amplitude, then \( q = \tilde{q} \).

Let \( B := B_{\rho}(0) \) with suitably selected \( \rho > 0 \) such that \( \Sigma \subset B \), and \( \kappa_0^2 \) is not a Dirichlet eigenvalue for \( -\Delta \) neither in \( B \). Moreover, we require that the homogeneous Dirichlet problem for \( L_{\tilde{q}} \) has only trivial solution in \( H^1_0(B \setminus \tilde{D}) \). Setting

\[
w = u - \tilde{u}, \text{ we see}
\]

\[
w = 0 \quad \text{in } B \setminus \Sigma,
\]

and hence

\[
w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial \Omega_{ext} := \partial \Omega \setminus \Gamma_{int}.
\]
where $\nu$ is unit outward normal to $\partial \Omega$. Moreover, by noting that $D$ is a sound-soft obstacle,

$$w = 0 \quad \text{on } \partial D := \partial D_{\text{ext}} \cup \Gamma_{\text{int}}.$$  

It is also straightforward to verify that $u \in H^1(\Omega)$ satisfies the following differential equation

$$\Delta w + \kappa_0^2 q w = \kappa_0^2 \delta_q \tilde{u},$$

where $\delta_q = q - \tilde{q}$. Next, we define

$$\mathcal{H}_{q,\Gamma_{\text{int}}} := \{ v \in H^1(\Omega); L_q v = 0 \text{ in } \Omega \text{ and } v = 0 \text{ on } \Gamma_{\text{int}} \}.$$  

Multiplying both sides of (3.7) by an arbitrary $v \in \mathcal{H}_{q,\Gamma_{\text{int}}}$ and using Green’s formula, we have

$$\int_{\Omega} \kappa_0^2 \delta_q \tilde{u} v \, dx = \int_{\Omega} (L_q w)v - (L_q v)w \, dx = \int_{\partial \Omega} \frac{\partial w}{\partial \nu} v - w \frac{\partial v}{\partial \nu} \, dS_x.$$  

In terms of the relations in (3.5)-(3.6), this further yields

$$\int_{\Omega} \kappa_0^2 \delta_q \tilde{u} v \, dx = 0.$$  

Equivalently, (3.9) is read as

$$\sum_{l=0}^{N_0} \int_{\Omega_l} \delta_q \tilde{u} v \, dx = \int_{\Omega} \delta_q \tilde{u} v \, dx = 0,$$

where it is recalled that $\Omega_0$ is separated from the inhomogeneity while $\Omega_l$ has planar contact with the inhomogeneity at $\Gamma_l$ for $l = 1, 2, \ldots, N_0$. We next divide our argument into three steps.

**Step I. Denseness argument and two approximation results**

Define

$$\mathcal{H}_{q,B\setminus\tilde{D}} := \{ \phi \in H^1(B\setminus\tilde{D}); L_{\tilde{q}} \phi = 0 \text{ in } B\setminus\tilde{D} \text{ and } \phi = 0 \text{ on } \partial D \}.$$  

and

$$\mathcal{H}_{q,\Gamma_{\text{int}}} := \{ \phi \in H^1(\Omega); L_{\tilde{q}} \phi = 0 \text{ in } \Omega \phi = 0 \text{ on } \Gamma_{\text{int}} \}.$$  

We shall show the following two lemmata at the end of the present subsection.

**Lemma 3.2.** The set of total fields $\{u(x; \theta, \tilde{q}); \theta \in S^2\}$ to (1.1) is complete in $\mathcal{H}_{q,B\setminus\tilde{D}}$ with respect to the $L^2(B\setminus\tilde{D})$-norm.

**Lemma 3.3.** Any $\phi \in \mathcal{H}_{q,\Gamma_{\text{int}}}$ can be $L^2(\Omega)$-approximated by distributions in $\mathcal{H}_{q,B\setminus\tilde{D}}$.

Combining Lemmata 3.2 and 3.3, we easily see

**Lemma 3.4.** The set of total fields $\{u(x; \theta, \tilde{q}); \theta \in S^2\}$ to (1.1) is complete in $\mathcal{H}_{q,\Gamma_{\text{int}}}$ with respect to the $L^2(\Omega)$-norm.

By Lemma 3.4, we have from (3.10) that

$$\int_{\Omega} \delta_q \phi v \, dx = 0, \quad \forall \phi \in \mathcal{H}_{q,\Gamma_{\text{int}}}, \forall v \in \mathcal{H}_{q,\Gamma_{\text{int}}}.$$
Step II. Construction of the CGO solutions vanishing on the buried boundary of the obstacle

In this step, we construct special complex geometric optics solutions to the Schrödinger operator \( \mathcal{L}_p \) with compactly supported \( p \in L^\infty(\mathbb{R}^3) \). Let \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \). We introduce
\[
e(1) = (\xi_1^2 + \xi_2^2)^{-1/2}(\xi_1, \xi_2, 0), \quad e(3) = (0, 0, 1)
\]
and the unit vector \( e(2) \) to form an orthonormal basis \( e(1), e(2), e(3) \) in \( \mathbb{R}^3 \). The coordinate of \( x \in \mathbb{R}^3 \) in this basis is denoted by \( (x_1, x_2, x_3) \). It is observed that
\[
\xi = (\xi_1, 0, \xi_3)_e, \quad \xi_1 = (\xi_1^2 + \xi_3^2)^{1/2}
\]
and
\[
x \cdot y = \sum_{l=1}^{3} x_ly_l = \sum_{l=1}^{3} x_le_l.
\]
Define
\[
\zeta(1) = (\frac{\xi_1}{2} - \tau \xi_3, i|\xi|)\left(\frac{1}{4} + \tau^2\right)^{1/2} \frac{\xi_1}{2} + \tau \xi_1, e,
\]
\[
\zeta(1)^* = (\frac{\xi_1}{2} - \tau \xi_3, i|\xi|)\left(\frac{1}{4} + \tau^2\right)^{1/2} \left(\frac{\xi_1}{2} - \tau \xi_1\right), e,
\]
\[
\zeta(2) = (\frac{\xi_1}{2} + \tau \xi_3, -i|\xi|)\left(\frac{1}{4} + \tau^2\right)^{1/2} \frac{\xi_1}{2} + \tau \xi_1, e,
\]
\[
\zeta(2)^* = (\frac{\xi_1}{2} + \tau \xi_3, -i|\xi|)\left(\frac{1}{4} + \tau^2\right)^{1/2} \left(\frac{\xi_1}{2} + \tau \xi_1\right), e,
\]
where \( \tau \) is a positive real number. By straightforward calculations, one can verify that
\[
\zeta(l) \cdot \zeta(l) = \zeta^*(l) \cdot \zeta^*(l) = 0 \quad l = 1, 2.
\]
From the geometric interpretation of the inner product for vectors in \( \mathbb{R}^3 \), we further see that for any unitary matrix \( U \in \mathbb{R}^{3 \times 3} \)
\[
U \zeta(l) \equiv U \zeta^*(l) \equiv U \zeta^*(l) = 0 \quad l = 1, 2.
\]
Next, we construct special CGO solutions in each sub-domain \( \Omega_l \), \( 1 \leq l \leq N_0 \), of \( \Omega \) for \( \mathcal{L}_p \). To ease our exposition, we fix an arbitrary \( \Omega_l \) for the following construction. We denote \( p_l \) the restriction of \( p \) on \( \Omega_l \). Then, we extend \( p_l \in L^\infty(\Omega_l) \) to \( \mathbb{R}^3 \) as follows,
\[
p_l(x) = \begin{cases} 
p & \text{for } x \in \Omega_l, \\
\mathcal{R}p & \text{for } x \in \mathcal{R}\Omega_l, \\
0 & \text{for } x \in \mathbb{R}^3 \setminus (\Omega_l \cup \mathcal{R}\Omega_l),
\end{cases}
\]
where and in the following, for a function \( f(x), x \in \mathbb{R}^3 \), we denote by \( \mathcal{R}f(x) = f(\mathcal{R}x) \). That is, \( \hat{p}_l \in L^\infty(\mathbb{R}^3) \) is an odd symmetric function with respect to \( \Pi_l \).

Let \( U_l \in \mathbb{R}^{3 \times 3} \) be a unitary matrix such that \( U_l^T \Pi_l = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_3 = c_l\} \), where \( c_l \) is a constant. Because of the relations in (3.16), it is known that there are CGO solutions of the form (see (12))
\[
e^{iU_l \zeta(l) \cdot x}(1 + \omega_{1,l}), \quad e^{iU_l \zeta(2) \cdot x}(1 + \omega_{2,l})
\]
to the equation \( \mathcal{L}_p u = 0 \) in \( \mathbb{R}^3 \), where
\[
\|\omega_{1,l}\|_{L^2(\Omega_l)} + \|\omega_{2,l}\|_{L^2(\Omega_l)} = 0 \quad \text{as } \tau \to \infty,
\]
with $B_0 \subset \mathbb{R}^3$ a ball containing $\Omega_l \cup \hat{\mathcal{R}} \Omega_l$.

Set
\begin{align}
\psi_1(x) &= e^{U_l \zeta(1) \cdot x}(1 + \omega_{1,l}) - e^{U_l \zeta(1) \cdot \hat{\mathcal{R}} x}(1 + \hat{\mathcal{R}} \omega_{1,l}), \\
\psi_2(x) &= e^{U_l \zeta(2) \cdot x}(1 + \omega_{2,l}) - e^{U_l \zeta(2) \cdot \hat{\mathcal{R}} x}(1 + \hat{\mathcal{R}} \omega_{2,l}).
\end{align}

We know that $\psi_1, \psi_2 \in H^2(\Omega_l \cup \hat{\mathcal{R}} \Omega_l)$ solve the differential equation $L_{\hat{\mathcal{R}}l} \phi = 0$, and
\[ \psi_1 = \psi_2 = 0 \quad \text{on} \quad \Gamma_l. \]

Next, we investigate the product of $\psi_1$ and $\psi_2$ for the subsequent use. It is first observed that
\begin{equation}
\tag{3.21}
e^{U_l \zeta(1) \cdot x}(1 + \omega_{1,l}) = e^{U_l \zeta(1) \cdot U_l^T x}(1 + \hat{\omega}_{1,l}) = e^{\zeta(1) \cdot U_l^T x}(1 + \hat{\omega}_{1,l}),
\end{equation}
where $\hat{\omega}_{1,l}(U_l^T x) = \omega_{1,l}(x)$. Setting $y = U_l^T x$ for $x \in \mathbb{R}^3$, we further have
\begin{equation}
\tag{3.22}
e^{U_l \zeta(1) \cdot x}(1 + \omega_{1,l}) = e^{\zeta(1) \cdot y}(1 + \hat{\omega}_{1,l}(y)).
\end{equation}
In similar manner, we can treat $e^{U_l \zeta(1) \cdot \hat{\mathcal{R}} x}(1 + \hat{\mathcal{R}} \omega_{1,l}), e^{U_l \zeta(2) \cdot x}(1 + \omega_{2,l})$ and $e^{U_l \zeta(2) \cdot \hat{\mathcal{R}} x}(1 + \hat{\mathcal{R}} \omega_{2,l})$ to get
\begin{equation}
\tag{3.23}
e^{U_l \zeta(1) \cdot \hat{\mathcal{R}} x}(1 + \hat{\mathcal{R}} \omega_{1,l}) = e^{\zeta(1) \cdot \hat{\mathcal{R}} y}(1 + \hat{\mathcal{R}} \hat{\omega}_{1,l}(y)), \\
e^{U_l \zeta(2) \cdot \hat{\mathcal{R}} x}(1 + \hat{\mathcal{R}} \omega_{2,l}) = e^{\zeta(2) \cdot y}(1 + \hat{\omega}_{2,l}(y)),
\end{equation}
where $\hat{\mathcal{R}}$ is the reflection with respect to $U_l^T \Pi_l = \{(y_1, y_2, y_3) \in \mathbb{R}^3; y_3 = c_l\}$. We remind that obviously $\hat{\mathcal{R}}(y_1, y_2, y_3) = (y_1, y_2, 2c_l - y_3)$. In the following, we denote $\hat{\mathcal{R}}y$ by $y^*$ and $\hat{\mathcal{R}} \hat{\omega}_{1,l}(y)$ by $\hat{\omega}_{1,l}^*(y), \alpha = 1, 2$. Now,
\begin{equation}
\tag{3.24}
\psi_1(x) \psi_2(x) = \big[e^{\zeta(1) \cdot y}(1 + \hat{\omega}_{1,l}) - e^{\zeta(1) \cdot y^*}(1 + \hat{\omega}_{1,l}^*) \big] \\
\times \big[e^{\zeta(2) \cdot y}(1 + \hat{\omega}_{1,l}^*) - e^{\zeta(2) \cdot y^*}(1 + \hat{\omega}_{1,l}^*) \big] \\
= e^{\zeta(1) + \zeta(2) \cdot y}(1 + \hat{\omega}_{1,l})(1 + \hat{\omega}_{1,l}^*) \\
- e^{\zeta(1) + \zeta(2) \cdot y^*}(1 + \hat{\omega}_{1,l}^*)(1 + \hat{\omega}_{1,l}^*) \\
- e^{\zeta(1) + \zeta(2) \cdot y}(1 + \hat{\omega}_{1,l})(1 + \hat{\omega}_{1,l}^*) \\
+ e^{\zeta(1) + \zeta(2) \cdot y^*}(1 + \hat{\omega}_{1,l}^*)(1 + \hat{\omega}_{1,l}^*)
\end{equation}
where $y = U_l^T x$. Here we note that in (3.21)
\begin{equation}
\tag{3.25}
\zeta(1) \cdot (0, 0, 2c_l) = (\xi_3 + 2\tau \xi_{1e})c_l, \\
\zeta(2) \cdot (0, 0, 2c_l) = (-\xi_3 + 2\tau \xi_{1e})c_l,
\end{equation}
both are real numbers.

**Step III. Concluding the proof**

With the above preparations, we can conclude the proof as follows. First, we fix an $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$. Next, as in Step II, we construct CGO solutions for the
operators $L_\eta$ and $L_{\bar{\eta}}$, respectively as $\psi_1$ and $\psi_2$ in (3.20), in each subdomain $\Omega_l$ with $\xi^l := U_l^T \eta$ replacing the $\xi$ in (3.14) in each $\Omega_l$, where $U_l^T \in \mathbb{R}^{3 \times 3}$ are unitary matrices such that there are constants $c_l$ and $U_l^T \Pi_l = \{(y_1, y_2, y_3) \in \mathbb{R}^3; y_3 = c_l\}$ for $l = 1, 2, \ldots, N_0$. Whereas for CGO solutions in $\Omega_0$, we take $\xi^0 := \eta$ in (3.14) for defining the complex phases and let them be given as those in (3.13) without the rotation matrix $U_l$. That is, we need not the rotation of the subdomain $\Omega_l$, nor the reflection of CGO solutions in $\Omega_0$. Noting that the sub-domains $\Omega_l$'s of $\Omega$ are disjoint from each other, these CGO solutions constructed in each subdomain are patched together to yield, respectively solutions $\phi \in \mathcal{H}_{q,T_{\text{in}}}$ and $v \in \mathcal{H}_{q,T_{\text{in}}}$. Then, in view of (3.13) and (3.24), we have

$$0 = \int_{\Omega} \delta_q \phi v \, dx = \int_{\Omega_0} \delta_0^q \phi v \, dx + \sum_{l=1}^{N_0} \int_{\Omega_l} \delta_0^l \phi v \, dx$$

(3.26)

$$= \int_{\Omega_0} \delta_0^0 e^{i\eta \cdot x} (1 + \omega_{1,0})(1 + \omega_{2,0}) \, dx + \sum_{l=1}^{N_0} \int_{\Omega_l} \delta_0^l \left[ e^{i\xi^l \cdot y} (1 + \omega_{1,l})(1 + \omega_{2,l}) - e^{i(\xi^l_0 - 0.2\tau \xi^l_1 \cdot y_1 + \xi^l_1)} e^{i\xi^l \cdot (2 - 0.0, 0.2c_l)} (1 + \omega_{1,l})(1 + \omega_{2,l}) - e^{i(\xi^l_0 + 0.2\tau \xi^l_1 \cdot y_1 - \xi^l_1)} e^{i\xi^l \cdot (0, 0, 0.2c_l)} (1 + \omega_{1,l})(1 + \omega_{2,l}) + e^{i\xi^l \cdot y^*} (1 + \omega_{1,l}^*)(1 + \omega_{2,l}^*) \right] \, dx,$$

where $\delta_0^l$ is the restriction of $\delta_0$ on $\Omega_l$. Clearly, the moduli of all exponents are bounded by 1 by noting (3.25). Now, we let $\tau \to \infty$ in (3.20). Due to (3.19), the limits of all terms containing $\omega_{\sigma,0}^l$, $\omega_{\sigma,1}^l$ and $\omega_{\sigma,1}^*$, $\sigma = 1, 2$, are zero. By the Riemann-Lebesgue Lemma,

$$\lim_{\tau \to \infty} \int_{\Omega_l} \delta_0^l \Longrightarrow e^{i(\xi^l_0 \cdot 0.2\tau \xi^l_1 \cdot y_1 + \xi^l_1)} e^{i\xi^l \cdot (2 - 0.0, 0.2c_l)} \, dx = 0,$$

$$\lim_{\tau \to \infty} \int_{\Omega_l} \delta_0^l \Longrightarrow e^{i(\xi^l_0 \cdot 0.2\tau \xi^l_1 \cdot y_1 - \xi^l_1)} e^{i\xi^l \cdot (0, 0, 0.2c_l)} \, dx = 0,$$

provided $\xi^l_{1,c} \neq 0$. Define

$$\mathcal{A} := \{ \eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3; \xi^l_{1,c} \neq 0 \text{ with } \xi^l = U_l^T \eta \text{ for } l = 1, 2, \ldots, N_0 \}.$$

Obviously, $\mathcal{A}$ is an open set in $\mathbb{R}^3$. Now, summarizing the above discussion by letting $\eta \in \mathcal{A}$, we have obtained from (3.26) that

$$0 = \sum_{l=1}^{N_0} \int_{\Omega_l} \tilde{q}_l (e^{i\xi^l \cdot y} + e^{i\xi^l \cdot y^*}) \, dx + \int_{\Omega_0} \delta_0^0 \phi v \, dx$$

(3.27)

$$= \sum_{l=1}^{N_0} \int_{\Omega_l} \tilde{q}_l (e^{iU_l^T \eta \cdot y} + e^{iU_l^T \eta \cdot y^*}) \, dx + \int_{\Omega_0} \delta_0^0 \phi v \, dx$$

$$= \sum_{l=1}^{N_0} \int_{\Omega_l} \tilde{q}_l (e^{i\eta \cdot y} + e^{i\eta \cdot y^*}) \, dx + \int_{\Omega_0} \delta_0^0 \phi v \, dx.$$
As in Step II, we extend \( \delta_q \) to \( \mathbb{R}^3 \) by patching together those \( \delta_{q,l}^i \)'s in \( \Omega_l \cup \mathcal{R}_l \Omega_l \) which is obtained by even extension of \( \delta_{q,l}^i \) in \( \Omega_l \) with respect to \( \Pi_l \) (cf. \( \text{(3.17)} \)), and letting it be zero in \( (\mathbb{R}^3 \setminus \Omega_0) \setminus \bigcup_{l=1}^{N_0} (\Omega_l \cup \mathcal{R}_l \Omega_l) \). This is possible by our assumption vi) in Definition 2.1 that \( (\Omega_l \cup \mathcal{R}_l \Omega_l) \cap (\Omega_{l'} \cup \mathcal{R}_{l'} \Omega_{l'}) = \emptyset \) and \( (\Omega_l \cup \mathcal{R}_l \Omega_l) \cap \Omega_0 = \emptyset \) for \( 1 \leq l, l' \leq N_0 \) and \( l \neq l' \). Hence, we further have from \( \text{(3.27)} \) that
\begin{equation}
(3.28) \quad \int_{\mathbb{R}^3} \delta_q(x) e^{i\eta \cdot x} \, dx = 0,
\end{equation}
for all \( \eta \in \mathcal{A} \). Since \( \delta_q(x) \) is compactly supported, the LHS of \( \text{(3.28)} \) is analytic with respect to \( \eta \). So, we see that \( \text{(3.28)} \) holds for all \( \eta \in \mathbb{R}^3 \). Now, \( \delta_q = 0 \) by the uniqueness of inverse Fourier transform. The proof is completed. \( \Box \)

**Proof of Lemma 3.2.** By contradiction, we assume there exists \( \bar{f} \in L^2(\mathbb{R}^3 \setminus \mathcal{D}) \) such that
\begin{equation}
(3.29) \quad \int_{B^3 \setminus \mathcal{D}} f(x) u(x; \theta, \bar{q}) \, dx = 0
\end{equation}
for all total fields \( \bar{u}(x) := u(x; \theta, \bar{q}) \) to \( \text{(1.1)} \) with \( \theta \in S^2 \); whereas
\begin{equation}
(3.30) \quad \int_{B^3 \setminus \mathcal{D}} f \phi \, dx \neq 0
\end{equation}
for some \( \phi \in \mathcal{H}_{\Pi_l} B_1 D \). We extend \( f \) to be zero in \( \mathbb{R}^3 \setminus \mathcal{B} \). Let \( u^* \in H^1_{\text{loc}}(G) \) be the unique solution to
\begin{equation}
(3.31) \quad \left\{ \begin{array}{ll}
\mathcal{L}_q u^*(x) = f(x) & x \in G, \\
u u^* = 0 & \text{on } \partial \mathcal{D},
\end{array} \right.
\end{equation}
and \( \mathcal{M} u^* = 0 \), namely \( u^* \) satisfies the radiation condition. In view of \( \text{(3.29)} \) and \( \text{(3.31)} \), we see
\begin{equation}
(3.32) \quad \int_{B^3 \setminus \mathcal{D}} \bar{u} \mathcal{L}_q u^* = 0.
\end{equation}
By further noting \( \mathcal{L}_q u = 0 \), we have from \( \text{(3.32)} \) with the help of Green's formula that
\begin{equation}
(3.33) \quad 0 = \int_{B^3 \setminus \mathcal{D}} (\mathcal{L}_q u^*) \bar{u} - u^* (\mathcal{L}_q \bar{u}) \, dx
= \int_{\partial \mathcal{B}} \frac{\partial u^*}{\partial \nu} \bar{u} - \frac{\partial \bar{u}}{\partial \nu} u^* \, dS_x + \int_{\partial \mathcal{D}} \frac{\partial u^*}{\partial \nu} \bar{u} - \frac{\partial \bar{u}}{\partial \nu} u^* \, dx
= \int_{\partial \mathcal{B}} \frac{\partial u^*}{\partial \nu} \bar{u} - \frac{\partial \bar{u}}{\partial \nu} u^* \, dS_x,
\end{equation}
where \( \nu \) is the exterior normal to corresponding domains and in the last equality we have made use of boundary conditions \( \bar{u} = u^* = 0 \) on \( \partial \mathcal{D} \). Next, using the fact \( \bar{u}(x, \theta) = \bar{u}^*(x, \theta) + u^i(x, \theta) = \bar{u}^* + e^{i\kappa \cdot x} \), we have from \( \text{(3.33)} \)
\begin{equation}
(3.34) \quad \int_{\partial \mathcal{B}} \frac{\partial u^*}{\partial \nu} u^i - u^i \frac{\partial u^*}{\partial \nu} \, dS_x = - \int_{\partial \mathcal{B}} \frac{\partial u^*}{\partial \nu} \bar{u}^* - \bar{u}^* \frac{\partial \bar{u}^*}{\partial \nu} \, dS_x.
\end{equation}
Since \( (\Delta + \kappa^2_0) u^* = (\Delta + \kappa^2_0) \bar{u}^* = 0 \) in \( \mathbb{R}^3 \setminus \mathcal{B} \) and both \( u^* \) and \( \bar{u}^* \) satisfies the radiation condition, we see that the RHS of \( \text{(3.34)} \) vanishes identically, and hence
\begin{equation}
(3.35) \quad \int_{\partial \mathcal{B}} \frac{\partial u^*}{\partial \nu} u^i - u^i \frac{\partial u^*}{\partial \nu} \, dS_x = 0.
\end{equation}
Then we define $\omega^*$ to be the unique solution to $(\Delta + \kappa_0^2)\omega^* = 0$ in $B$ with Dirichlet boundary data $\omega^* = u^*$ on $\partial B$. It is remarked that the unique existence is guaranteed by our earlier assumption that $\kappa_0^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $B$. Noting that $(\Delta + \kappa_0^2)u^i = 0$ in $B$, we have from Green’s formula

$$0 = \int_{\partial B} \frac{\partial \omega^*}{\partial \nu} u^i - \omega^* \frac{\partial u^i}{\partial \nu} \, dS_x = \int_{\partial B} \frac{\partial \omega^*}{\partial \nu} u^i - u^* \frac{\partial u^i}{\partial \nu} \, dS_x,$$

which together with (3.35) further yields

$$\int_{\partial B} \left( \frac{\partial \omega^*}{\partial \nu} - \frac{\partial u^*}{\partial \nu} \right) e^{i\kappa_0 x \cdot \theta} \, dx = 0 \quad \text{for all} \quad \theta \in S^2. \quad \text{(3.37)}$$

Since $\kappa_0^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $B$, $\{e^{i\kappa_0 x \cdot \theta}|_{\partial B}; \theta \in S^2\}$ is dense in $L^2(\partial B)$ (cf. [4]). Hence, one can conclude from (3.37) that $\frac{\partial \omega^*}{\partial \nu} = \frac{\partial u^*}{\partial \nu}$ on $\partial B$. Now, if we set $\Psi$ to be $u^*$ in $\mathbb{R}^3 \setminus \overline{B}$ and $\omega^*$ in $B$, then it is an entire solution to $\Delta + \kappa_0^2$ and satisfies the radiation condition as $u^*$ does. Clearly, $\Psi$ must be identically zero. In doing this, we have shown that $u^* = 0$ in $\mathbb{R}^3 \setminus \overline{B}$. Finally, again by using Green’s formula, we have

$$\int_{B \setminus \overline{D}} f \phi \, dx = \int_{B \setminus \overline{D}} (\mathcal{L}_q u^*) \phi - u^* (\mathcal{L}_q \phi) \, dx = \int_{\partial B \cup \partial D} \frac{\partial u^*}{\partial \nu} \phi - u^* \frac{\partial \phi}{\partial \nu} \, dS_x = 0,$$

where in the last equality we have made use of homogeneous boundary conditions $u^* = 0$ on $\partial B$ and $u^* = \phi = 0$ on $\partial \overline{D}$. This obviously contradicts to (3.30), thus completing the proof.

\textbf{Proof of Lemma 3.3} We assume contrarily that there exits $\tilde{f} \in L^2(B \setminus \overline{D})$ supported in $\Omega$ such that

$$\int_{\Omega} f u \, dx = 0 \quad \forall u \in \mathcal{H}_{\tilde{q}, B \setminus \overline{D}}, \quad \text{(3.38)}$$

but

$$\int_{\Omega} f v \, dx \neq 0 \quad \text{for some} \quad v \in \mathcal{H}_{\tilde{q}, \Gamma_{int}}. \quad \text{(3.39)}$$

Let $u^* \in H^1_0(B \setminus \overline{D})$ be the unique solution to $\mathcal{L}_q u^* = f$. Here it is noted that the unique existence is guaranteed by our earlier requirement that $B$ is chosen such that the homogeneous Dirichlet problem for the partial differential operator $\mathcal{L}_q$ has only trivial solution in $B \setminus \overline{D}$. Then, in view of (3.38) and with the help of Green’s formula, we have by straightforward calculations that

$$0 = \int_{\Omega} f u \, dx = \int_{\Omega} (\mathcal{L}_q u^*) u - u^* (\mathcal{L}_q u) \, dx = \int_{\Omega} \frac{\partial u^*}{\partial \nu} u - u^* \frac{\partial u^*}{\partial \nu} \, dS_x = \int_{(\partial \Omega \setminus \Gamma_{int}) \cup \partial D_{ext}} \frac{\partial u^*}{\partial \nu} u - u^* \frac{\partial u^*}{\partial \nu} \, dS_x = \int_{\partial \Sigma} \frac{\partial u^*}{\partial \nu} u \, dx = \int_{\partial B} \frac{\partial u^*}{\partial \nu} u \, dx,$$

$$\text{(3.40)}$$
where \( \nu \) is the exterior normal to corresponding domains. In the above deduction, we have made use of the boundary conditions \( u = 0 \) on \( \partial D = \Gamma_{int} \cup \partial D_{ext} \) and \( u^* = 0 \) on \( \partial D \cup \partial B \). It is clear that \( u \) can be arbitrary smooth function on \( \partial B \). So, we have from (3.40) that \( \partial u^*/\partial \nu = 0 \) on \( \partial B \). Hence, by the unique continuation principle, we know \( u^* = 0 \) in \( (B \setminus \bar{\Sigma} \). Now, again by using Green’s formula, we have

\[
\int_{\Omega} f v \, dx = \int_{\Omega} (L_qu^*)v - u^*(L_qv) \, dx = \int_{\partial\Omega} \frac{\partial u^*}{\partial \nu} v - u^* \frac{\partial v}{\partial \nu} \, dS_x = \int_{\partial\Omega, \Gamma_{int}} \frac{\partial u^*}{\partial \nu} v - u^* \frac{\partial v}{\partial \nu} \, dS_x + \int_{\Gamma_{int}} \frac{\partial u^*}{\partial \nu} v - u^* \frac{\partial v}{\partial \nu} \, dS_x = 0,
\]

where we have made use of the homogeneous boundary conditions \( u^* = \frac{\partial u^*}{\partial \nu} = 0 \) on \( \partial\Omega \setminus \Gamma_{int} \) and \( u^* = v = 0 \) on \( \Gamma_{int} \). This obviously contradicts to (3.39), which completes the proof.

\[ \square \]

4. Recovery of scattering medium with known included obstacle

As can be seen from the argument in subsection 3.2, in order to recover the buried obstacle, one has to assume that the obstacle is partly exposed to the exterior of the medium. In this final section, we would like to remark an interesting case that one can recover the surrounding medium even if the obstacle is buried completely but known \textit{a priori}. We would only give a simple example though one can appeal for a more general study.

Let \( D \) be a bounded polyhedron in \( \mathbb{R}^3 \) and \( G = \mathbb{R}^3 \setminus \bar{D} \). We denote by \( F_l, l = 1, 2, \ldots, m \) the faces of \( D \). For each \( F_l \), we let \( \Omega_l \subset G \) be a bounded Lipschitz domain such that \( \partial\Omega_l \cap \partial D = F_l, l = 1, 2, \ldots, m \). We further assume that all \( \Omega_l \)'s are simply connected and satisfy a topological requirement as that given in vi), Definition 2.1. Let \( q \in L^\infty(G) \) such that \( supp(1-q) = \cup_{l=1}^m \partial\Omega_l \). Clearly, the obstacle \( D \) is now completely included in the scattering medium. For such a scatterer, we would like to remark that by using Lax-Phillips method, one can still show the unique existence of a solution \( u^* \in H^1_{loc}(G) \) to the forward scattering problem 2.1. It is also readily seen that the approximation result in Lemma 3.3 still holds. Hence, all our arguments in subsection 3.3 remain valid to show the unique determination of the scattering medium \( q \) provided the obstacle \( D \) is known in advance.

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