Strong gravitational lensing by a strongly naked null singularity

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Abstract

We study strong gravitational lensing in a static, spherically symmetric, naked singularity space-time, without a photon sphere. The nature of the singularity is found to be light-like. We discuss the characteristic lensing features of this naked singularity in the strong deflection limit. In spite of the absence of a photon sphere in this space-time, the bending angle of light diverges, as it approaches the singularity. However, unlike black holes, it is found that the nature of this divergence is non-logarithmic, and we derive an analytic formula for the same. Moreover, the relativistic rings produced due to strong lensing by the singularity are found to be well separated from each other, making them easy to resolve and possibly detect. These features are expected to be important in the study of strong lensing by ultra compact objects, especially ones without event horizons.

1 Introduction

General relativity (GR) is one of the most successful theories of gravity till date. It has so far passed many experimental tests, of which the deflection of light in a gravitational field is one of the most significant ones. In fact, this was the very first experimental test that was used to validate GR more than a century ago, in 1919 [1, 2]. Most of these tests deal with gravity in the weak field limit, for example gravity around the sun, or planets in our solar system. With the advent of modern technology, it now looks possible to probe deeper, i.e into the strong gravity regime. Indeed, the Event Horizon Telescope, and the recent imaging of the center of the M87 galaxy ([3]-[8]), along with the recent detection of gravitational waves by the LIGO-VIRGO collaboration (see, for example [9, 10]), has started a new era in observational astronomy to probe GR in regions of strong gravity.

Ubiquitous in such studies of strong gravity are black holes – singularities covered by an event horizon. It is commonly believed that the centers of galaxies are inhabited by supermassive black holes. This has led to a flurry of activities in the recent past which concern gravitational lensing in black hole backgrounds. Close to a black hole horizon, gravity might become strong, and it is imperative that here one goes beyond the weak field limit to understand lensing from these objects. In recent years such studies have been extended to objects that do not have an event horizon. One such space-time that has received a lot of attention of late is that of a naked singularity. Although the cosmic censorship conjecture prohibits the formation of naked singularities [11], a plethora of recent studies in the literature have shown that

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it is possible to form such singularities as the end product of gravitational collapse of massive objects under suitable initial conditions ([12]-[18]). Therefore, the study of naked singularities, or more generally, horizonless compact objects, are of interest. In this regard, a natural question that arises is how to distinguish such objects from black holes. Once again, gravitational lensing in the strong deflection limit acts as a significant tool to observationally distinguish between black holes and naked singularities, or horizonless objects. The pioneering work on gravitational lensing from naked singularities by Virbhadra and Ellis [19] appeared almost two decades ago. Ever since, different aspects of gravitational lensing by various horizonless objects, such as naked singularities ([20]-[26]), wormholes ([27]-[49]), Bosonic stars [50], gravastars [51] etc. have been studied and analyzed in detail. Recently, there have also been many other studies on the observational aspects of ultra compact objects ([52]-[55]) in the context of strong lensing. It has been observed from these studies that, while such objects may sometimes produce strong lensing features mimicking those of black holes, they might also differ dramatically from black holes, thus opening a possibility for their observational detection.

In this paper, we extend such analysis, and consider a naked singularity recently proposed by Joshi et al. in [56], and study its lensing features in the strong deflection limit. A curvature singularity (with or without an event horizon) can, in general, be classified into three categories, namely space-like, time-like, and light-like or null. For example, the Schwarzschild singularity is space-like, the Reissner-Nordstrom (RN) singularity (for both the black hole and naked singularity cases) is time-like, and for references on null singularity, one may refer to [57, 58]. Null singularities are generally formed at late times inside the RN or Kerr black holes. The inner Cauchy horizons (which are inside the outer event horizons) in these black holes generally break up into two kinds of null singularities as the black holes become old [57, 58]. The interesting feature of the singularity that we consider in this paper is that it is found to be both null and naked. Moreover, as is well known, we can further classify a naked singularity based on whether it is cloaked by a photon sphere or not [19]. If a naked singularity is within a photon sphere, it is called a weakly naked singularity (WNS), otherwise it is termed as a strongly naked singularity (SNS). They have their own characteristic strong lensing features, different from each other. A WNS generally produces strong lensing features similar to that of black holes, i.e., the bending angle diverges at some critical limit in both cases. A WNS can thus in principle act as a black hole mimicker. Whereas, a SNS that does not having any photon sphere, is expected to show crucial differences from a WNS or a black hole. The naked singularity space-time that we consider in this paper also does not contain a photon sphere [56], so it falls in the SNS category. Remarkably, what we will show in sequel is that the bending angle still diverges at a specific critical limit, just like a WNS or a black hole. Further, it is well known that for strong lensing due to photon spheres (i.e., in black holes or WNSs), the characteristic divergence of the bending angle is logarithmic. Here, on the other hand, we show that this divergence is non-logarithmic, and derive an analytic formula for the same. These are novel results on strong lensing from a naked null singularity, which will be established in the rest of the paper.

This paper is organized as follows. In section 2, we recall some features of the space-time that we study. Then we briefly discuss the properties of null geodesics in this space-time in section 3. The next section 4 deals with the bending of light in the strong deflection limit by the naked singularity, which is followed by a study of different observables in this scenario in section 5. We then conclude with a summary and future directions in section 6. Throughout this work, we have used \( c = G = 1 \) units, unless
2 Properties of the Naked Singularity Spacetime

We start with a static, spherically symmetric space-time, recently proposed by Joshi et al. in [56], with the following line element expressed in \((t, r, \theta, \phi)\) coordinates as

\[
ds^2 = -\frac{dt^2}{(1 + \frac{M}{r})^2} + \left(1 + \frac{M}{r}\right)^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\]

where \(M\) is the ADM mass of the space-time. The metric components \(g_{tt}\) and \(g_{rr}\) can be expanded in powers of \(\frac{M}{r}\) as

\[
g_{tt} = -\left(1 + \frac{M}{r}\right)^{-2} = -\left[1 - \frac{2M}{r} + 3 \left(\frac{M}{r}\right)^2 - \cdots\right],
\]

\[
g_{rr} = \left(1 + \frac{M}{r}\right)^2 = \frac{1}{(1 + \frac{M}{r})^{-2}} = \left[1 - \frac{2M}{r} + 3 \left(\frac{M}{r}\right)^2 - \cdots\right].
\]

Therefore, in the large \(r\) limit, it resembles the Schwarzschild space-time and in the asymptotically infinite \((r \to \infty)\) limit, it reduces to the flat Minkowski space-time. Expressions of the Ricci and Kretschmann scalars of this space-time appear in [56], and it can be shown that both diverge at \(r = 0\), which corresponds to a curvature singularity of this space-time at \(r = 0\). Moreover, the \(g_{rr}\) term in Eq. (1) is always finite and positive for the whole range of \(r > 0\). Therefore, the space-time does not contain any event or absolute horizons and hence the space-time represented by the metric in Eq. (1) contains a globally naked singularity at \(r = 0\). It is shown in [56] that the space-time is seeded by an anisotropic fluid, which satisfies all the required energy conditions. It is also shown that the equation of state parameter of the anisotropic fluid tends to the value \(-\frac{1}{3}\) as \(r \to 0\), and becomes \(\frac{1}{3}\) in the limit \(r\) tends to infinity.

It is important to study the causal structure of the space-time. These are best represented by the corresponding maximally extended space-time diagrams, or the Carter-Penrose (CP) diagrams. Figure (1) shows the CP diagram of the space-time of Eq.(1). The diagram consists of two distinct regions, denoted by I and II. They are identical copies of each other, but causally disconnected. The different kinds of infinities are denoted by the following standard notations: \(\mathcal{I}^\pm\) denote the future and past null infinities respectively, \(i^\pm\) denote the future and past time-like infinities respectively, and \(i^0\) denotes the space-like infinity. The hyper-surface \(r = 0\) representing the singularity is denoted by the zig-zag line, corresponding to light-like geodesics. As a result, the singularity at \(r = 0\) will be light-like or null in nature. Therefore, in addition to being globally naked, the singularity is also light-like, which is an interesting and novel feature of this space-time.

3 Characteristics of null geodesics

Let us now consider the nature of null geodesics in this space-time. Since the space-time is spherically symmetric, we can choose \(\theta = \pi/2\) without any loss of generality. Moreover, it being a static, spherically
Figure 1: Carter-Penrose Diagram of the space-time represented by the metric in Eq. (1). There are two different regions in the diagram, denoted by I and II. They are completely identical, but causally disconnected. The corresponding past null infinity (I⁻), future null infinity (I⁺), past time-like infinity (i⁻), future time-like infinity (i⁺), and space-like infinity (i⁰) are shown. The surface r = 0 which represents the singularity is shown by the zig-zag line.

symmetric space-time, it admits two constants of motion, namely the energy E and the z-component of angular momentum L. The corresponding t and φ geodesic equations for photon motion are given by

\[ \dot{t} = \left(1 + \frac{M}{r}\right)^2 E, \quad \text{and} \quad \dot{\phi} = \frac{L}{r^2}, \quad (3) \]

where ‘overdot’ represents a derivative with respect to the affine parameter along a null geodesic. From the normalization of the four velocity of photons \( u^\mu u_\mu = 0 \), we have

\[ \dot{r}^2 + \frac{L^2}{r^2 \left(1 + \frac{M}{r}\right)^2} = E^2 \quad \text{or} \quad r^2 + V_{\text{eff}}(r) = E^2, \quad (4) \]

where \( V_{\text{eff}}(r) = \frac{L^2}{(r + M)^2} \) is the effective potential for photon motion. At the turning point \( (r_{tp}) \), we have \( \dot{r} = 0 \) which yields from Eq. (4),

\[ V_{\text{eff}}(r_{tp}) = \frac{L^2}{(r_{tp} + M)^2} = E^2 \quad \text{or} \quad \frac{L}{E} = b(r_{tp}) = \sqrt{(r_{tp} + M)^2} = (r_{tp} + M), \quad (5) \]

where \( b = L/E \) represents the impact parameter of a light ray, which is a constant of motion for that ray. In Fig. (2), we have plotted the effective potential, and the corresponding light trajectories of the space-time of Eq.(1). It is interesting to note from Fig. (2(a)) that \( V_{\text{eff}}(r) \) is finite everywhere, and does not contain any extremum within the whole range of \( r \in [0, \infty) \). The value of \( V_{\text{eff}} \) at the location of the singularity, i.e., at \( r = 0 \) is given by \( V_{\text{eff}}|_{r=0} = \frac{L^2}{M^2} \). Remember that the maximum of \( V_{\text{eff}}(r) \) corresponds to the position of what is known as a photon sphere (in spherically symmetric space-times), which represents a spherical surface consisting of unstable light rings. The most significant feature of such a photon sphere is that light coming from a distant source either crosses the surface of photon sphere and spirals into it without ever coming out of the surface, or goes very close it, takes a number
of turns around it and finally moves out towards infinity. As a result, the sky of a distant observer gets divided into two distinct regions, a completely dark region devoid of light (known as the shadow region), and a bright region surrounding the shadow. Therefore, the photon sphere, in this case, plays the most significant role in producing the shadow region.

Since $V_{\text{eff}}(r)$ of the naked singularity space-time does not have any extremum, as can be seen from Fig. (2(a)), it does not possess any photon sphere, and naively we do not expect it to produce a shadow. Interestingly however, as shown in [56], this space-time indeed produces a shadow. From Eq. (5), the value of the impact parameter ($b$), for which a light ray has a turning point just at the singularity, can be obtained as $b_{r_{\text{tp}}=0} = M$. This corresponds to the critical ray forming the boundary of the shadow. In Fig. (2(b)), we have shown two sets of light rays – one representing a ray having impact parameter $b = 1.6M > b_{\text{cr}}$ (blue ray), and the other one having impact parameter $b < b_{\text{cr}} = M$ (magenta ray). We can see that any ray of light having an impact parameter $b > M$ will take a turn at some finite $r_{\text{tp}} > 0$. On the contrary, for light rays with impact parameters $b < M$, their geodesics will be incomplete and will be terminated at the singularity. Therefore, all such rays coming from a distant source are captured by the singularity, producing the shadow. It is worth noting here that, in case of the Schwarzschild space-time having the same ADM mass $M$, the corresponding critical impact parameter producing a shadow is given by $b_{\text{cr}}^{\text{Sch}} = 3\sqrt{3}M$. Hence, the size of the shadow of the naked singularity space-time of Eq.(1) is smaller than that of the Schwarzschild shadow. This feature has also been reported in [56] in case of a spherical accretion by the Schwarzschild black hole, and the naked singularity under consideration. We also have shown the boundaries of the shadow regions for both naked singularity (black circle) and Schwarzschild (red circle) space-times in Fig. (2(b)) for illustrative purposes.
4 Bending of light in the strong field limit

An important property of a space-time having a photon sphere is that as light rays come very close to the photon sphere, they take a number of turns before reaching the observer. As a result, the bending angle of light becomes large, and diverges at the location of the photon sphere. Since, in the present naked singularity space-time, the shadow is produced without a photon sphere, it will be interesting to study how the light bending angle behaves in such a scenario. For this purpose, let us re-arrange Eq. (4) to the form

\[ \dot{r} = \pm \sqrt{E^2 - \frac{L^2}{r^2 (1 + \frac{M}{r})^2}} = \pm \sqrt{E^2 - \frac{L^2}{(r + M)^2}}. \]  

(6)

Combining the above equation and the second expression of Eq. (3), we obtain

\[ \frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} = \pm \frac{L}{r^2 \sqrt{E^2 - \frac{L^2}{(r + M)^2}}} = \pm \frac{b}{r^2 \sqrt{1 - \frac{b^2}{(r + M)^2}}}. \]  

(7)

Again, from Eq. (5), we know \( b(r_{tp}) = (r_{tp} + M) \). Therefore, Eq. (7) can be re-written as

\[ \frac{d\phi}{dr} = \pm \frac{(r_{tp} + M)}{r^2 \sqrt{1 - \frac{(r_{tp} + M)^2}{(r + M)^2}}} = \pm \frac{(r_{tp} + M) (r + M)}{r^2 \sqrt{r^2 + 2 (r - r_{tp}) M - r_{tp}^2}}. \]  

(8)

whose integration gives,

\[ \phi(r) = \phi_{\infty} \pm \int_{r_{tp}}^{\infty} \frac{(r_{tp} + M) (r + M)}{r^2 \sqrt{r^2 + 2 (r - r_{tp}) M - r_{tp}^2}} dr \]  

(9)

where \( \phi_{\infty} \) is the initial azimuthal angle made by the light ray when it originates from the source asymptotically. Therefore, the true angle of deflection of light due to the gravitational field produced by the naked singularity is given by

\[ \alpha(r_{tp}) = 2 \left| \phi(r_{tp}) - \phi_{\infty} \right| - \pi, \text{ i.e., } \alpha(r_{tp}) = 2 \int_{r_{tp}}^{\infty} \frac{(r_{tp} + M) (r + M)}{r^2 \sqrt{r^2 + 2 (r - r_{tp}) M - r_{tp}^2}} dr - \pi. \]  

(10)

The integration in eq.(10) is elementary, and we obtain the analytic expression of \( \alpha(r_{tp}) \) as

\[ \alpha(r_{tp}) = \frac{2M (r_{tp} + M)}{r_{tp} (r_{tp} + 2M)} + \frac{4 (r_{tp} + M)^3}{r_{tp}^3 (r_{tp} + 2M)^{3/2}} \arctan \left( \sqrt{1 + \frac{2M}{r_{tp}}} \right) - \pi. \]  

(11)

Equivalently, the bending angle can also be obtained in terms of the impact parameter \( b \), and is given by

\[ \alpha(b) = \frac{2Mb}{(b^2 - M^2)} + \frac{4b^3}{(b^2 - M^2)^{3/2}} \arctan \left( \sqrt{\frac{b + M}{b - M}} \right) - \pi. \]  

(12)

As we can see from Eqs. (11) and (12), the bending angle diverges in the limit of the critical impact parameter, i.e., \( b \to b_{cr} = M \) or equivalently in the limit \( r_{tp} \to 0 \). To find out the nature of divergence
of \( \alpha \) in these limiting cases, we first need to use the expansion of \( \arctan(x) \) for the \( x \geq 1 \) case, given as
\[
\arctan(x) = \frac{x}{2} - \frac{1}{3x^2} - \frac{1}{5x^4} + \cdots \text{, for } x \geq 1.
\]
Therefore, the expressions of \( \alpha(r_{tp}) \) and \( \alpha(b) \) can be expanded respectively as
\[
\alpha(r_{tp}) = \left[ \frac{2\pi(r_{tp} + M)^3}{r_{tp}^{3/2}(r_{tp} + 2M)^{3/2}} - \frac{2(r_{tp} + M)(2r_{tp} + 3M)}{(r_{tp} + 2M)^2} + \frac{4(r_{tp} + M)^3}{3(r_{tp} + 2M)^3} - \frac{4r_{tp}(r_{tp} + M)^3}{5(r_{tp} + 2M)^4} + \cdots \right] - \pi ,
\]
\[
\alpha(b) = \left[ \frac{2\pi(b/M)^3}{(b^2 - M^2)} - \frac{2(b/M)(2b/M + 1)}{(b^2 - M^2)^{3/2}} + \frac{4(b/M)^3}{3(b^2 - M^2)^3} - \frac{4(b/M)^3}{5(b^2 - M^2)^3} + \cdots \right] - \pi .
\]
Hence, the divergences of \( \alpha(r_{tp}) \) in the limit \( r_{tp} \to 0 \), and \( \alpha(b) \) in the limit \( b \to b_{cr} = M \) respectively take the form
\[
\lim_{r_{tp} \to 0} \alpha(r_{tp}) = \frac{\pi}{\sqrt{2}} \left( \frac{M}{r_{tp}} \right)^{3/2} - \frac{4}{3} - \pi + \mathcal{O}[r_{tp}] , \quad (14)
\]
\[
\lim_{b \to b_{cr}} \alpha(b) = \frac{(\pi/\sqrt{2})}{(b_{cr} - 1)^{3/2}} - \frac{4}{3} - \pi + \mathcal{O}\left( \left( \frac{b}{b_{cr}} - 1 \right) \right) . \quad (15)
\]
From Eqs. (14) and (15), we observe that the nature of the divergence of \( \alpha \) in the limits \( b \to b_{cr} = M \) or \( r_{tp} \to 0 \) is non-logarithmic and is of polynomial type, which is a characteristic feature of this naked singularity. For completeness, let us recapitulate the different kinds of divergences of the bending angle in different situations.

- **Case-1:** If a space-time contains only a photon sphere \( (r_{ph}) \), which corresponds to the critical impact parameter \( b = b_{ph} \), the bending angle diverges in the limit \( r \to r_{ph} \) or \( b \to b_{ph} \) as the light ray approaches the photon sphere from the \( b > b_{ph} \) side. The corresponding divergence is logarithmic which was first shown by Bozza in [59], and later refined by Tsukamoto in [60]. If we write the metric of a general static, spherically symmetric space-time as
\[
ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2 \theta d\phi^2),
\]
then the corresponding formula for the bending angle in the limit \( b \to b_{ph} \) will be given by [60]
\[
\alpha(b) = -\bar{a} \log \left( \frac{b}{b_{ph}} - 1 \right) + \bar{b} + \mathcal{O}[(b - b_{ph}) \log (b - b_{ph})],
\]
where the expressions of \( \bar{a} \) and \( \bar{b} \) are
\[
\bar{a} = \sqrt{\frac{2AB}{C''A - CA''}} \bigg|_{r_{ph}}, \quad \bar{b} = \bar{a} \log \left[ r_{ph}^2 \left( \frac{C''}{C} - \frac{A''}{A} \right) \right]_{r_{ph}} + I_R(r_{ph}) - \pi,
\]
with a prime denoting a derivative with respect to \( r \), and \( I_R(r_{ph}) \) is a constant that can be found in [60].
• **Case-2:** If a space-time contains both a photon \((r_{\text{ph}})\) and an anti-photon \((r_{\text{aph}})\) sphere,\(^1\) the bending angle diverges in the limit \(r \to r_{\text{ph}}\) or \(b \to b_{\text{ph}}\) from both the \(b > b_{\text{ph}}\) and the \(b < b_{\text{ph}}\) sides. The nature of divergence due to photon sphere, i.e., from the \(b > b_{\text{ph}}\) side, is the same as described in Case-1. On the contrary, though the divergence due to the anti-photon sphere, i.e., from the \(b < b_{\text{ph}}\) side, is also logarithmic in nature, its functional dependence is different from Case-1. The scenario has been discussed in detail in [52], and the corresponding formula of divergence from \(b < b_{\text{ph}}\) side is obtained as \([52]\),

\[
\alpha(b) = -\bar{a} \log \left( \frac{b_{\text{ph}}^2}{b^2} - 1 \right) + \bar{b} + \mathcal{O} \left[ (b_{\text{ph}}^2 - b^2) \log(b_{\text{ph}}^2 - b^2) \right],
\]

(19)

where the expressions of \(\bar{a}\) and \(\bar{b}\), in this case, are

\[
\bar{a} = 2 \sqrt{\frac{2AB}{C''A - CA''}} \left|_{r_{\text{ph}}} \right., \quad \bar{b} = \bar{a} \log \left[ 2r^2 \left( \frac{C''}{C} - \frac{A''}{A} \right) \left( \frac{r}{r_c} - 1 \right) \right] \left|_{r_{\text{ph}}} \right. + I_R(r_c) - \pi,
\]

(20)

and \(r_c\) \((< r_{\text{ph}})\) is the radius for which \(V_{\text{eff}}(r_c) = V_{\text{eff}}(r_{\text{ph}})\) [52]. Therefore, it evident that the logarithmic divergences due to a photon sphere (Case-1) and an anti-photon sphere (Case-2) are quite different, which result in some important observational consequences as described in detail in [52].

• **Case-3:** There exists a critical situation in between the previous two cases, namely when the photon and anti-photon spheres merge together to form a point of inflection in the effective potential. Let us denote this point of inflection as \(r_0\) \(= r_{\text{ph}} = r_{\text{aph}}\), and the corresponding impact parameter as \(b_0\) \(= b_{\text{ph}}\). This scenario has been recently addressed by Tsukamoto in [61] for the Damour-Solodukhin wormhole. The importance of this analysis is that the corresponding divergence of the bending angle turns out to be non-logarithmic, given by the following expression [61],

\[
\alpha(b) = \frac{\bar{a}}{(\frac{b}{b_0} - 1)^{1/4}} + \bar{b} + \mathcal{O} \left[ (\frac{b}{b_0} - 1)^{3/4} \right],
\]

(21)

where \(\bar{a}\) and \(\bar{b}\) are given by

\[
\bar{a} = 2^{7/4} \times 3^{-1/4} = 2.5558, \quad \bar{b} = -\frac{4}{3} + I_R(r_0) - \pi = -2.1078.
\]

(22)

In the present analysis, we have seen from Eq. (14), or Eq. (15) that the space-time of Eq.(1) also admits non-logarithmic divergence, as in Case-3. Importantly, the nature of divergence is different from that of Case-3. In this case, the corresponding bending angle is [from Eq. (15)] turns out to be

\[
\alpha(b) = \frac{\bar{a}}{(\frac{b}{b_{\text{cr}}} - 1)^{3/2}} + \bar{b} + \mathcal{O} \left[ (\frac{b}{b_{\text{cr}}} - 1)^{3/2} \right],
\]

(23)

\(^1\)By an anti-photon sphere, we refer to the minimum of the effective potential, just like photon sphere corresponds to the maximum of the effective potential.
where \( \bar{a} = \frac{\pi}{\sqrt{2}} = 2.2214 \) and \( \bar{b} = -\frac{4}{3} - \pi = -4.4749 \), with \( b_{\text{cr}} = M \).

For a space-time having a photon sphere, a light ray with an impact parameter exactly equal to its critical value will arrive at the surface of the sphere tangentially, and in principle, remain there. Naturally, such a ray will have an infinite bending angle. For rays having impact parameters just above the critical value, their turning points will be close to the photon sphere, and they will make a large number of rotations before escaping to the observer. In other words, as the value of the impact parameter approaches its critical value from a larger one, the corresponding turning points of the rays will be closer to the photon sphere, and the closer the turning points are towards the photon sphere, the more will be the bending angle. Therefore, the divergence of bending angle at the photon sphere occurs naturally. On the other hand, the notion of such a divergence of bending angle without a photon sphere is interesting, and deserves a careful analysis.

![Figure 3](image_url)

**Figure 3:** Plots of bending angle \( \alpha \) as a function of the impact parameter \( b \) (in units of \( M \)) [Fig. 3(a)], and a representative trajectory of light having impact parameter just greater than its critical value [Fig. 3(b)].

In Fig. 3, we have plotted the bending angle \( \alpha \) as functions of the impact parameter \( b \), and the trajectory of a typical ray of light having impact parameter just greater than the critical value for the present space-time. We can see from Fig. 3(a) that \( \alpha(b) \) diverges as \( b \to b_{\text{cr}} = M \). Figure 3(b) is particularly important for us to understand the reason of this divergence. Since the singularity is light-like, light rays originating from a distant source can come very close to it, or in fact, in the critical case, they can even skim along the singularity without being truly captured by it. So, even the critical ray, after reaching at the singularity and taking a turn there, can in principle reach the observer. In doing so, such rays take many turns (infinite in the critical case) around the singularity before reaching the observer to produce a large deflection. Only the rays having impact parameters less than the critical value are captured by the singularity, and the corresponding geodesics terminate without reaching the observer. Therefore, the nature of this naked singularity itself causes the bending angle to diverge.
5 Observables in Strong lensing

In this section, we consider various observables of gravitational lensing, in the strong deflection limit. The lens diagram is shown in Fig. (4). A ray of light is emitted by the source $S$ and received by the observer $O$, encountering the gravitational field produced by the lensing object $L$ and then producing the image $I$. Both the source and the observer are situated at an asymptotically flat region. OLP is the optic axis, $D_{OS}$, $D_{SL}$, and $D_{LS}$ represent the observer-source, source-lens, and the lens-observer distances respectively. The angular positions of the source ($S$) and the image ($I$) with respect to the optic axis are denoted by the angles $\delta$ and $\theta$ respectively. The impact parameter, and the bending angle are represented by $b$ and $\alpha$ respectively.

\[
\tan \delta = \tan \theta - \frac{D_{LS}}{D_{OS}} \left[ \tan \theta + \tan(\alpha - \theta) \right], \tag{24}
\]

where $\delta$ and $\theta$ are respectively the angular positions of the source and the image with respect to the optic axis, as shown in Fig. (4). $D_{OS}, D_{LS}$ represent the observer-source distance and the lens-source distance respectively, and $\alpha$ is the total bending angle. The impact parameter $b$ can be written in terms of $\theta$ as

\[
b = D_{OL} \sin \theta = \kappa \sin \theta, \quad \text{[where, } \kappa = D_{OL}], \tag{25}\]

with $D_{OL}$ being the observer-lens distance. The radial ($\mu_r$), tangential ($\mu_t$), and total ($\mu$) magnifications of the image are expressed as

\[
\mu_r = \left( \frac{\partial \delta}{\partial \theta} \right)^{-1}, \quad \mu_t = \left( \frac{\sin \delta}{\sin \theta} \right)^{-1}, \quad \text{and} \quad \mu = \mu_t \cdot \mu_r = \left( \frac{\sin \delta}{\sin \theta} \frac{\partial \delta}{\partial \theta} \right)^{-1}. \tag{26}\]

We shall use the three relations of Eq. (26) for the rest of our calculations. Since we have an analytic formula for the bending angle, we can find out the expressions of the quantities in analytic forms. Using Eqs. (12) and (25), we get

\[
\alpha(\theta) = \frac{2M\kappa \sin \theta}{(\kappa^2 \sin^2 \theta^2 - M^2)} + \frac{4\kappa^3 \sin^3 \theta^3}{(\kappa^2 \sin^2 \theta^2 - M^2)^{3/2}} \arctan \left( \frac{\kappa \sin \theta + M}{\kappa \sin \theta - M} \right) - \pi. \tag{27}\]
Using this $\alpha(\theta)$ in Eq. (24), we obtain the source position ($\delta$) as a function of the image position ($\theta$), and from $\delta(\theta)$, it is straightforward to evaluate the expressions of the magnifications as functions of $\theta$. Note from Eq. (26) that, when $\delta = 0$, i.e., when the source, lens and observer are perfectly aligned along the optic axis, $\mu_t$ diverges. In this scenario, the images form perfect concentric rings in the observer’s sky, which are known as Einstein rings (when the bending of light is less than 2$\pi$), or relativistic Einstein rings (when light bends by more than 2$\pi$). Moreover, the singularities in magnification in the image plane are known as critical curves and the corresponding locations in the source plane are called caustics, that produce the critical curves. The singularities in $\mu_r$ correspond to radial critical curves (RCCs) in the image plane and radial caustics (RCs) in the source plane, and similarly, the singularities in $\mu_t$ represent tangential critical curves (TCCs) in the image plane and tangential caustics (TCs) in the source plane. Clearly therefore, the rings represent the TCCs, and correspondingly, $\delta = 0$ gives the TC.

Now, to obtain the angular positions of the rings, we put $\delta = 0$ in Eq. (24) and find

$$\tan[\alpha(\theta) - \theta] = \frac{D_{OS}}{D_{LS}} \left(1 - \frac{D_{LS}}{D_{OS}}\right) \tan \theta.$$  \hspace{1cm} (28)

In our calculation, we have considered, $D_{OL} = D_{LS} = \frac{1}{2} D_{OS}$. Therefore, the above equation yields

$$\tan[\alpha(\theta) - \theta] = \tan \theta, \quad \Rightarrow \quad \alpha(\theta) = m\pi + 2\theta, \quad [m = 1, 2, 3, \cdots].$$  \hspace{1cm} (29)

Since only the rays which take one or more complete rotations around the lens during its travel from the source to the observer can reach the observer, only even values of $m$ will contribute to the ring formation in the observer’s sky. Thus, the position of the $n$-th ring ($\theta_n$) can be conveniently determined from the points of intersections of the curves, $y_1(\theta_n) = \alpha(\theta_n)$ and $y_2(\theta_n) = 2n\pi + 2\theta_n$, with $n = 1, 2, \cdots$. Even if we do not assume the condition, $D_{OL} = D_{LS} = \frac{1}{2} D_{OS}$, we can still find out $\theta_n$ by similarly evaluating the points of intersections of the two curves, $y_1(\theta_n) = \tan[\alpha(\theta_n) - \theta_n]$ and $y_2(\theta_n) = \frac{D_{OS}}{D_{LS}} \left(1 - \frac{D_{LS}}{D_{OS}}\right) \tan \theta_n$, with $n = 1, 2, \cdots$. Note that $n = 1$ represents the outermost ring, and as we increase $n$, the angular positions of the relativistic rings decrease, and the corresponding bending angle increases. The $n = \infty$ case represents the innermost critical ring which forms the edge of the shadow region, and its angular position is denoted by $\theta_\infty$. This ring is produced by the critical ray having impact parameter $b = b_{cr} = M$.

In Fig. (5), we have plotted $\mu_r$ and $\mu_t$ of the images as functions of the image positions ($\theta$). In the plots, the following parameter values are used:

$$M = 4.31 \times 10^6 \ M_\odot, \quad D_{OL} = D_{LS} = 7.86 \ \text{kpc}, \quad \text{and} \quad D_{OL} = \frac{1}{2} D_{OS}.$$  

The above mass and distance corresponds to the supermassive central object (considered as a black hole, Sgr A*) of our galaxy. Moreover, the unit of $\theta$ is expressed in arcseconds in the radial magnification plot, and in microarcseconds in the tangential magnification plot. We can see from Fig. (5(a)) that the radial magnification never diverges, which signifies the absence of RCC or correspondingly RC in this space-time. It can be shown that $\mu_r$ diverges only when $\frac{\partial \alpha(\theta)}{\partial \theta} > 0$. Whereas, in the present scenario, $\frac{\partial \alpha(\theta)}{\partial \theta}$ is always negative, as $\alpha(\theta)$ continuously increases with decreasing $\theta$. Therefore, the radial magnifications of the images always remain finite ranging from zero to one. On the other hand, Fig. (5(b)) shows that the tangential magnification $\mu_t$ corresponding to ($n = 1$) diverges at the angular position ($\theta_1$), which
Figure 5: Radial ($\mu_r$) and tangential ($\mu_t$) magnifications as functions of the image angular position ($\theta$). Here, 5(a) represents the variation of $\mu_r$, and 5(b) corresponds to the plot of $\mu_t$. In this case, the point of divergence of $\mu_t$ denotes the angular position of the first (outermost) relativistic Einstein ring. Here, $\theta$ is expressed in arc second in the left figure (radial magnification), and in microarc second in the right figure (tangential magnification).

marks the position of the outermost relativistic Einstein ring.

To perform a comparative analysis between the Schwarzschild black hole and the naked singularity, the angular positions of a few relativistic Einstein rings (bending angle $> 2\pi$), and the corresponding separations between two consecutive rings, defined as $s_n = \theta_n - \theta_{(n+1)}$, are tabulated in Table 1. To this end, it should be noted that, apart from these relativistic rings, there exists one more ring well outside this relativistic ring system, for which the bending angle is very small. This additional ring is known as the Einstein ring denoted as $\theta_E$ in Table 1. The angular positions (in arcseconds) of this ring for both the Schwarzschild black hole and the naked singularity are found out to be almost equal. It signifies that, in the weak field limit, the naked singularity space-time behaves like the Schwarzschild one, as stated earlier.

Things change dramatically in the strong deflection limit. As can be seen from Table 1, the rings in Schwarzschild space-time are very close to each other and converge quickly. In fact, the angular separation between the first and the last ring ($\theta_1 - \theta_\infty$) is only 0.035497 microarcseconds. From the values of angular positions of the rings in Schwarzschild space-time, we see that only the first ring can be resolved from the rest. On the other hand, in the naked singularity space-time, the rings are well separated from each other, and many rings can be individually resolved from the ring system before they finally become convergent. To visualize this feature, we have plotted, in Fig. (6), the first five relativistic rings, as well as the last one which forms the edge of the shadow, as seen by the observer.
Table 1: Angular positions of the Einstein ring ($\theta_E$), and a few relativistic Einstein rings ($\theta_n$) (bending angle $> 2\pi$) are shown for both the Schwarzschild black hole, and the naked singularity space-time. We have also shown the separation between two consecutive rings, $s_n = \theta_n - \theta_{n+1}$. Here, we have assumed the parameter values, $M = 4.31 \times 10^6 M_\odot$, $D_{\text{OL}} = D_{\text{LS}} = 7.86 \text{kpc}$, and $D_{\text{OL}} = \frac{1}{2} D_{\text{OS}}$, which correspond the supermassive central object (considered as a black hole, Sgr A$^*$) of our Milky Way galaxy. The angles are expressed in microarcsecond. Note that $\theta_E$ has value in the order of arcsecond.

|        | Schwarzschild | Naked Singularity of metric (1) |
|--------|---------------|----------------------------------|
| $\theta_E$ | $1.49745 \times 10^6$ | $1.49747 \times 10^6$ |
| $\theta_1$ | 28.280347     | 9.60106                          |
| $\theta_2$ | 28.24494      | 7.75320                          |
| $\theta_3$ | 28.24484      | 7.10409                          |
| $\theta_4$ | —             | 6.76498                          |
| $\theta_\infty$ | 28.24485     | 5.43573                          |
| $s_1$      | 0.035407      | 1.84786                          |
| $s_2$      | 0.00009       | 0.64911                          |
| $s_3$      | —             | 0.33911                          |
| $s_{50}$   | —             | 0.00290                          |
| $\theta_1 - \theta_\infty$ | 0.035497    | 4.16533                          |

Figure 6: Relativistic Einstein rings as seen in the observer’s sky. The outermost ring is shown in blue, and the innermost one forming the edge of the shadow is shown in black. The second, third, fourth and fifth rings are denoted in magenta, brown, green, and red color respectively.
From the figure also, we observe that the rings are well separated, and therefore, it acts as the most crucial and prominent distinguishing feature of this naked singularity. These features may prove to be critical for the detection of such kinds of naked singularities in futuristic experiments.

6 Summary

Currently, observational signatures of black holes and horizonless objects attract much attention. The Event Horizon Telescope has opened up the possibility of experimentally probing gravity in its strongest regime, near the event horizon of a black hole or close to a singularity. These might provide important clues towards understanding the nature of quantum gravity. A number of studies have appeared in the recent literature about how to make observational distinctions between objects with and without event horizons. A broad consensus in this regard is that sometimes the horizonless objects do possess distinguishing signatures from that of black holes, while in many cases, they are indistinguishable, and mimic black holes. In this scenario, it is important and interesting to glean further insight into the differences between gravitational lensing between black holes or horizonless objects.

In this paper, we have performed an extensive study of gravitational lensing in the strong deflection limit by a strongly naked null singularity proposed recently in [56]. We have shown that the naked singularity is light-like, and then we have discussed its lensing properties. We have uncovered novel lensing phenomena that this space-time might give rise to. First, we have shown that, in spite of being a strongly naked singularity, the bending angle of light diverges in a critical limit, which normally happens in a space-time having a photon sphere. Moreover, we have found that the nature of this divergence is non-logarithmic, contrary to black holes, and we have derived an analytic formula for the same. We have also analyzed various observables in this space-time, and have found that the relativistic Einstein rings are well separated, and so, many of them can be easily resolved as compared to the Schwarzschild case. We believe that all these results are novel, and add significantly to the existing literature.

An immediate extension of this work would be to find a rotating version of such kind of a singularity, and to study its characteristic features as far as strong gravitational lensing is concerned. We leave it for a future work.

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