We consider the energy representation for the gauge group. The gauge group is the set of $C^\infty$-mappings from a compact Riemannian manifold to a semi-simple compact Lie group. In this paper, we obtain irreducibility of the energy representation of the gauge group for any dimension of $M$. To prove irreducibility for the energy representation, we use the fact that each operator from a space of test functionals to a space of generalized functionals is realized as a series of integral kernel operators, called the Fock expansion.

**KEY WORDS:** gauge group, energy representation, irreducibility, white noise calculus

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1 Introduction

In this paper, we discuss irreducibility of the energy representation of the gauge group. The gauge group is the set of all $C^\infty$-mappings with compact support from a Riemannian manifold $M$ to a semi-simple compact Lie group $G$. Then the energy representation of the gauge group is studied in $[2]$, $[3]$, $[5]$, $[6]$, and $[8]$. The first definition of the energy representation appears in $[15]$ and I. Gelfand et al. proved irreducibility in case of simple compact Lie group and $I. Gelfand$ et al. proved irreducibility in case of simple compact Lie group studied in $[2]$, $[3]$, $[5]$, $[6]$, and $[8]$. In section 4 of this paper, we shall obtain irreducibility of the energy representation of $G$ for any dimension of $M$ when $M$ is compact.

We present a brief sketch of our idea for proof of irreducibility. In order to prove irreducibility for the case $\dim M \geq 2$, all authors of previous works used results of a Gaussian measure $\mu$ on the real vector space $E^\ast$ of distributions. On the other hand, in order to analyze operators on $L^2(E^\ast,\mu)$, we can use the theory of the spaces $(E)$ and $(E)^\ast$, and of operators from $(E)$ to $(E)^\ast$, called the white noise calculus. Here $(E)$ is a space of test functionals and $(E)^\ast$ is a space of generalized functionals. The white noise calculus is introduced by T. Hida in 1975, and led us to important consequences concerning to an analysis of the Boson system. (See $[12]$.) In the white noise calculus, a continuous linear operators from $(E)$ to $(E)^\ast$ is realized as a series of operators $\Xi_{l,m}(\kappa_{l,m}), l, m \in \mathbb{Z}_{\geq 0}$ such that

$$\Xi_{l,m}(\kappa_{l,m}) = \int \kappa_{l,m}(s_1, \ldots, s_l, t_1, \ldots, t_m) \partial_{s_1}^s \ldots \partial_{s_l}^s \partial_{t_1}^t \ldots \partial_{t_m}^t ds_1 \ldots ds_l dt_1 \ldots dt_m$$

where $\partial_{s}^s$ is a creation operator, $\partial_{t}^t$ is an annihilation operator, and $\kappa_{l,m}$ is a kernel distribution. $\Xi_{l,m}(\kappa_{l,m})$ is called an integral kernel operator with a kernel distribution $\kappa_{l,m}$ and this realization is called Fock expansion for a continuous linear operator from $(E)$ to $(E)^\ast$.

The merit of using the Fock expansion is that we can determine the commutant by direct algebraic computation, that is, the problem of irreducibility becomes easy comparatively. It is not exaggeration to say like this since we have the following reason. We know that one possible way to determine the commutant is to apply Tomita-Takesaki modular theory of the von Neumann algebras. However, the existence of the cyclic and separating vector for von Neumann algebra generated by the energy representation is not obvious. In fact, in $[2]$ and $[3]$, S. Albeverio et al. succeeded in proving reducibility for the energy representation of a subgroup of $H(S^1, G)$ only in the cyclic component with respect to the vacuum vector. However, they failed to show that vacuum vector is cyclic.

We should remark the relation between our result and the result of previous works $[2]$ and $[3]$. Albeverio et al. $[2]$ considered the energy representation of the Sobolev-Lie group $H(S^1, G)$. They showed that the cyclic component of the vacuum of the energy representation of $H(S^1, G)$ is unitary equivalent to

$$(UR(\phi)f)(\eta) := \left(\frac{d\mu(\eta\phi)}{d\mu(\eta)}\right)^{1/2} f(\eta\phi),$$

$\phi \in H(S^1, G), \quad f \in L^2(\{\eta \in C(S^1, G) \mid \eta(0) = \eta(2\pi) = e\}; \mu),$
and showed reducibility of $U^R$. Here $H(S^1, G)$ is the completion of all smooth loops $\phi$ satisfying $\phi(0) = \phi(2\pi) = e$. On the other hand, our result is related with the group $C^\infty(S^1, G)$, where $\phi \in C^\infty(S^1, G)$ needs not satisfy the condition $\phi(0) = \phi(2\pi) = e$. Therefore, our result on irreducibility is not in conflict with reducibility obtained in [2] and [3]. The difference of two above-mentioned results on irreducibility is also seen in the work of V. Jones and A. Wassermann. It is well-known fact that the level $\ell$ projective unitary representation $\pi \otimes \ell$ of $C^\infty(S^1, G)$ is irreducible. However A. Wassermann proved that the von Neumann algebra $\pi \otimes \ell(\{ \phi \in C^\infty(S^1, G) | \phi(\theta) = e \text{ for all } \theta \in [\pi, 2\pi] \})''$ is type $III_1$ factor (Theorem A of [16]).

Now we present the organization of this paper. In section 2, we briefly sketch the white noise calculus and prepare for our analysis of the energy representation. In section 3, we define the gauge group and its energy representation. In section 4, the main theorem 4.1 is mentioned and proved.

2 Survey of the white noise calculus

In this section, we introduce the white noise calculus. The details of this section are in [12].

**Definition 2.1.** Let $H$ be a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle_0$. Let $A$ be a self-adjoint operator defined on a dense domain $D(A)$. Let $\{ \lambda_j \}_{j \in \mathbb{N}}$ be eigenvalues of $A$ and $\{ e_j \}_{j \in \mathbb{N}}$ be normalized eigenvectors for $\{ \lambda_j \}_{j \in \mathbb{N}}$, i.e., $A e_j = \lambda_j e_j$ and $|e_j|_0 = 1$ for all $j \in \mathbb{N}$. Moreover, we also assume the following two conditions:

(i) $\{ e_j \}_{j \in \mathbb{N}}$ is a C.O.N.S. of $H$,

(ii) Multiplicity of $\{ \lambda_j \}_{j \in \mathbb{N}}$ is finite and $1 < \lambda_1 \leq \lambda_2 \leq \ldots \rightarrow \infty$.

Then we have the following properties.

(1) For $p \in \mathbb{Z}_{\geq 0}$ and $x, y \in D(A^p)$, let $\langle x, y \rangle_p := \langle A^p x, A^p y \rangle_0$. Then $\langle \cdot, \cdot \rangle_p$ is an inner product on $D(A^p)$. Moreover, $D(A^p)$ is complete with respect to the norm $|\cdot|_p$, that is, the pair $E_p := (D(A^p), \langle \cdot, \cdot \rangle_p)$ is a Hilbert space.

(2) For $p \geq 0$, let $j_{p,p+1} : E_{p+1} \rightarrow E_p$ be the inclusion map. Then every inclusion map is continuous and has a dense image. For $q \geq p \geq 0$, let $j_{p,q} := j_{p,p+1} \circ \ldots \circ j_{q-1,q} : E_q \rightarrow E_p$.

Then $\{ E_p, j_{p,q} \}$ is a reduced projective system.

(3) A countable Hilbert space $E := \lim_{\leftarrow} E_p = \bigcap_{p \geq 0} E_p$ constructed from the pair $(H, A)$ is a reflexive Fréchet space. We call $E$ a CH-space simply.

(4) From (3), we have $E^* = \lim_{\rightarrow} E_p^*$ as a topological vector space, i.e. the strong topology on $E^*$ and the inductive topology on $\lim_{\rightarrow} E_p^*$ coincide.
(5) Let \( p \in \mathbb{Z}_{\geq 0} \) and \( \langle x, y \rangle_{-p} := \langle A^{-p}x, A^{-p}y \rangle_0 \). Then \( \langle \cdot, \cdot \rangle_{-p} \) is an inner product on \( H \).

(6) For \( p \geq 0 \), let \( E_{-p} \) be the completion of \( H \) with respect to the norm \( | \cdot |_{-p} \). Then we can consider the inclusion map \( i_{-(p+1),-p} : E_{-p} \to E_{-(p+1)} \), and for \( q \geq p \geq 0 \) let

\[
    i_{-q,-p} := i_{-q,-q+1} \circ \ldots \circ i_{-(p+1),-p} : E_{-p} \to E_{-q}.
\]

Then \( \{ E_{-p}, i_{-q,-p} \} \) is an inductive system. Moreover, \( E_{-p} \) and \( E_{-p}^\ast \) are anti-linear isomorphic and isometric. Thus, from (4), we have

\[
    E^\ast = \lim_{\to} E_{-p} = \bigcup_{p \geq 0} E_{-p}.
\]

Furthermore, we require for the operator \( A \) that there exists \( \alpha > 0 \) such that \( A^{-\alpha} \) is a Hilbert-Schmidt class operator, namely

\[
    \delta^2 := \sum_{j=1}^{\infty} \lambda_j^{-2\alpha} < \infty. \tag{2.1}
\]

From this condition, \( E \) (resp. \( E^\ast \)) is a nuclear space. Thus we can define the \( \pi \)-tensor topology \( E \otimes_\pi E \) (resp. \( E^\ast \otimes_\pi E^\ast \)) of \( E \) (resp. \( E^\ast \)). If there is no danger of confusion, we will use the notation \( E \otimes E \) (resp. \( E^\ast \otimes E^\ast \)) simply.

We denote the canonical bilinear form on \( E^\ast \times E \) by \( \langle \cdot, \cdot \rangle \). We have the following natural relation between the canonical bilinear form on \( E^\ast \times E \) and the inner product on \( H \):

\[
    \langle f, g \rangle = \langle \overline{f}, g \rangle_0
\]

for all \( f \in H \) and \( g \in E \).

**Definition 2.2.** Let \( X \) be a Hilbert space, or a CH-space.

1. Let \( g_1, \ldots, g_n \in X \). We define the symmetrization \( s_n(g_1 \otimes \ldots \otimes g_n) \) of \( g_1 \otimes \ldots \otimes g_n \in X^\otimes n \) as follows.

\[
    s_n(g_1 \otimes \ldots \otimes g_n) := g_1 \hat{\otimes} \ldots \hat{\otimes} g_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} g_{\sigma(1)} \otimes \ldots \otimes g_{\sigma(n)},
\]

where \( \mathfrak{S}_n \) is the set of all permutations of \( \{1, 2, \ldots, n\} \).

2. If \( f \in X^\otimes n \) satisfies \( s_n(f) = f \), then we call \( f \) symmetric. We denote the set of all symmetric elements of \( X^\otimes n \) by \( X^\otimes n \) and we call \( X^\otimes n \) the \( n \)-th symmetric tensor of \( X \). Then \( s_n \) is a projection from \( X^\otimes n \) to \( X^\otimes n \).

3. For \( F \in (X^\otimes n)^\ast \) and \( \sigma \in \mathfrak{S}_n \), let \( F^\sigma \) be an element of \( (X^\otimes n)^\ast \) satisfying

\[
    \langle F^\sigma, g_1 \otimes \ldots \otimes g_n \rangle := \langle F, g_{\sigma^{-1}(1)} \otimes \ldots \otimes g_{\sigma^{-1}(n)} \rangle, \quad g_i \in X.
\]

Then we define the symmetrization \( s_n(F) \) as follows.

\[
    s_n(F) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} F^\sigma.
\]
(4) If \( F \in (X^{\otimes n})^* \) satisfies \( s_n(F) = F \), we call \( F \) symmetric. We denote the set of all symmetric elements of \( (X^{\otimes n})^* \) by \( (X^{\otimes n})_{\text{sym}}^* \).

From the above discussion, we obtain a Gelfand triple:

\[
E \subset H \subset E^*.
\]

Next, we define the Fock space and the second quantization of a linear operator.

**Definition 2.3.** Let \( H \) be a Hilbert space and \( A \) be a linear operator on \( H \).

1. Let

\[
\Gamma_b(H) := \left\{ \sum_{n=0}^{\infty} f_n \mid f_n \in H^{\otimes n}, \sum_{n=0}^{\infty} n! |f_n|^2 < +\infty \right\},
\]

\[
\langle \sum_{n=0}^{\infty} f_n, \sum_{n=0}^{\infty} g_n \rangle_0 := \sum_{n \in \mathbb{Z}_0} n! \langle f_n, g_n \rangle_0.
\]

Then we call \( \Gamma_b(H) \) the *Boson Fock space*. The Boson Fock space \( \Gamma_b(H) \) is a Hilbert space with respect to the inner product \( \langle \cdot, \cdot \rangle_0 \).

2. Let “id” be the identity operator on \( H \) and \( \text{id}_m := \text{id} \otimes \ldots \otimes \text{id} \). Let

\[
\Gamma_b(A) := \sum_{n=0}^{\infty} A^{\otimes n},
\]

\[
d\Gamma_b(A)^{(n)} := \sum_{j=1}^{n} \text{id}_{j-1} \otimes A \otimes \text{id}_{n-j}, \quad \text{and}
\]

\[
d\Gamma_b(A) := \sum_{n=0}^{\infty} d\Gamma_b(A)^{(n)}.
\]

Then we call \( \Gamma_b(A) \) the *second quantization* of \( A \) and \( d\Gamma_b(A) \) the *differential second quantization* of \( A \).

**Definition 2.4.** Let \( H \) be a complex Hilbert space and \( A \) be a self-adjoint operator on \( H \) satisfying the conditions (i) and (ii) in definition 2.1 and (2.1). Then we can define a CH-space \( (E) \) constructed from \( (\Gamma_b(H), \Gamma_b(A)) \) and we obtain a Gelfand triple:

\[
(E) \subset \Gamma_b(H) \subset (E)^*.
\]

**Corollary 2.5.** Let \( \phi := \sum_{n=0}^{\infty} f_n \in \Gamma_b(H) \), \( f_n \in H^{\otimes n} \). Then \( \phi \in (E) \) if and only if \( f_n \in E^{\otimes n} \) for all \( n \geq 0 \). Moreover, it follows that

\[
\|\phi\|_p := \|\Gamma_b(A)^p \phi\|_0 < +\infty
\]

for all \( p \geq 0 \).
Therefore we obtain $F$, such that

Definition 2.6. Let $X$, $Y$ be locally convex spaces. $\mathcal{L}(X,Y)$ is the set of all continuous linear operators from $X$ to $Y$.

Lemma 2.7. Let $X$ (resp. $Y$) be a locally convex space with seminorms $\{ | \cdot |_q \}_{q \in Q}$ (resp. $\{ | \cdot |_p \}_{p \in P}$). Then a linear operator $V : X \to Y$ is continuous, namely, $V \in \mathcal{L}(X,Y)$ if and only if for any $p \in P$, there exist $q \in Q$ and $C > 0$ such that

$$|Vf|_p \leq C|f|_q, \quad f \in X.$$

In order to discuss an integral kernel operator, we define a contraction of tensor products.

Definition 2.8. Let $H$ be a complex Hilbert space and $A$ be a self-adjoint operator on $H$ satisfying the conditions (i) and (ii) in definition 2.1 and (2.1). Let $e(i) := e_{i_1} \otimes \ldots \otimes e_{i_l}$, \( i := (i_1, \ldots, i_l) \in \mathbb{N}^l \).

1. For $F \in (E^{\otimes(l+m)})^*$, let

$$|F|_{l,m;p,q}^2 := \sum_{i,j} |\langle F, e(i) \otimes e(j) \rangle|^2 |e(i)|^2_p |e(j)|^2_q$$

where $i$ and $j$ run over the whole of $\mathbb{N}^l$ and $\mathbb{N}^m$ respectively.

2. For $F \in (E^{\otimes(l+m)})^*$ and $g \in E^{\otimes(l+n)}$, we define a contraction $F \otimes I g \in (E^{m+n})^*$ of $F$ and $g$ as follows.

$$F \otimes I g := \sum_{j,k} \left( \sum_{i} \langle F, e(j) \otimes e(i) \rangle \langle g, e(k) \otimes e(i) \rangle \right) e(j) \otimes e(k)$$

where $i$, $j$, and $k$ run over the whole of $\mathbb{N}^l$, $\mathbb{N}^m$, and $\mathbb{N}^n$ respectively.

We check well-definedness of the contraction. For any $F \in (E^{\otimes(l+m)})^*$ there exists $p \geq 0$ such that $|F|_{l,m;p,-p} = |F|_{-p} < +\infty$. We note that $|e(i)|_{-p} = 1$ for all $p \geq 0$ and $|e(i)|_p \leq |e(i)|_{p+q}$ for $p \in \mathbb{Z}$ and $q \geq 0$. Then we have

$$|F \otimes I g|_{-p}^2 = |F \otimes I g|_{l,m;-p,-p}^2 = \sum_{j,k} \left( \sum_{i} |\langle F, e(j) \otimes e(i) \rangle \langle g, e(k) \otimes e(i) \rangle| \right) |e(j)|_{-p}^2 |e(k)|_{-p}^2$$

$$\leq \sum_{j,k} \left( \sum_{i} |\langle F, e(j) \otimes e(i) \rangle|^2 |e(i)|_{-p}^2 \right) \left( \sum_{j} |\langle g, e(k) \otimes e(i') \rangle|^2 |e(i')|_{-p}^2 \right) |e(j)|_{-p}^2 |e(k)|_{-p}^2$$

Therefore we obtain $F \otimes I g \in (E^{\otimes(l+m)})^*$.

Now we define an integral kernel operator.
Definition 2.9 (Integral kernel operator). Let $\kappa \in (E^{\otimes (l+m)})^*$. For $\phi := \sum_{n=0}^{\infty} f_n \in (E), f_n \in E\hat{\otimes}^n$, let
\[
\Xi_{l,m}(\kappa)\phi := \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} s_{l+m} \left( \kappa \otimes_m f_{m+n} \right).
\]
Then $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E)^*)$. We call $\Xi_{l,m}(\kappa)$ an integral kernel operator with a kernel distribution $\kappa$.

As for integral kernel operators, see section 4.3 of [12]. Note that the following map
\[
(E^{\otimes (l+m)})^* \ni \kappa \mapsto \Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E)^*)
\]
is not injective. We define
\[
s_{l,m}(\kappa) := \frac{1}{l!m!} \sum_{\sigma \in \mathfrak{S}_l \times \mathfrak{S}_m} \kappa^\sigma,
\]
where $\kappa^\sigma$ is defined in definition 2.2 (3). Put
\[
(E^{\otimes (l+m)})_{\text{sym}(l,m)}^* := \{ \kappa \in (E^{\otimes (l+m)})^* | s_{l,m}(\kappa) = \kappa \}.
\]

Lemma 2.10. The map
\[
(E^{\otimes (l+m)})_{\text{sym}(l,m)}^* \ni \kappa \mapsto \Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E)^*)
\]
is injective. Moreover, for $\kappa \in (E^{\otimes (l+m)})_{\text{sym}(l,m)}^*$ and $\kappa \in (E^{\otimes (l'+m')})_{\text{sym}(l',m')}^*$, if $\Xi_{l,m}(\kappa) = \Xi_{l',m'}(\kappa')$, then $l = l'$, $m = m'$ and $s_{l,m}(\kappa) = s_{l,m}(\kappa')$.

Due to this lemma, the map
\[
\bigoplus_{l,m=0}^{\infty} (E^{\otimes (l+m)})_{\text{sym}(l,m)}^* \ni \{ \kappa_{l,m} \}_{l,m=0}^{\infty} \mapsto \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E),(E)^*) \quad (2.2)
\]
is injective.

Proposition 2.11 (Fock expansion). The map (2.2) is surjective, i.e., for any $\Xi \in \mathcal{L}((E),(E)^*)$, there exists an unique $\{ \kappa_{l,m} \}_{l,m=0}^{\infty}$, $\kappa_{l,m} \in (E^{\otimes (l+m)})_{\text{sym}(l,m)}^*$ such that
\[
\Xi \phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \quad \phi \in (E) \quad (2.3)
\]
where the sum of the right hand side of (2.3) converges in $(E)^*$. If $\Xi \in \mathcal{L}((E),(E))$, then
\[
\kappa_{l,m} \in E^{\hat{\otimes}l} \otimes (E^{\otimes m})_{\text{sym}}^*, \quad l, m \geq 0
\]
and the sum of the right hand side of (2.3) converges in $(E)$.

Proof. See section 4.4 and 4.5 of [12].
Proposition 2.12. For $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E), (E)^*)$ it follows that
\[
\Xi_{l,m}(\kappa_{l,m})^* = \Xi_{m,l}(t_{m,l}(\kappa_{l,m}))
\]
where the map $t_{m,l}$ is defined by
\[
\langle t_{m,l}(\kappa_{l,m}), \eta \otimes \zeta \rangle := \langle \kappa_{l,m}, \zeta \otimes \eta \rangle, \quad \eta \in E^\otimes_m, \zeta \in E^\otimes_l.
\]

By the way, for $\Xi_{l,m}(\kappa) \in \mathcal{L}((E), (E)^*)$ and $\Xi_{l,m'}(\lambda) \in \mathcal{L}((E), (E))$, we have $\Xi_{l,m}(\kappa)\Xi_{l,m'}(\lambda) \in \mathcal{L}((E), (E)^*)$. From proposition 2.11 we can infer that $\Xi_{l,m}(\kappa)\Xi_{l,m'}(\lambda)$ is expressed as a sum of integral kernel operators.

Definition 2.13. Let $\kappa \in (E^\otimes(l+m))^*$, $\lambda \in E^\otimes l' \otimes (E^\otimes m')^*$ and $m \wedge l' := \min\{m, l'\}$. For $0 \leq k \leq m \wedge l'$, we define $S_{l-k}^{l'}\lambda^{l'-k}(\kappa \circ_k \lambda) \in (E^\otimes(l+l'+m+m-2k))^*$ as follows.
\[
S_{l-k}^{l'}\lambda^{l'-k}(\kappa \circ_k \lambda) := \sum_{i,j,i',j'} \sum_{h} \langle \kappa(e(i) \otimes e(j) \otimes e(h)) \times \langle \lambda(e(h) \otimes e(i') \otimes e(j')) \rangle e(i) \otimes e(i') \otimes e(j) \otimes e(j'),
\]
where $i, j, i', j'$, and $h$ runs over the whole of $N^l, N^{m-k}, N^{l'-k}, N^{m'}$, and $N^k$ respectively.

We check well-definedness of $S_{l-k}^{l'}\lambda^{l'-k}(\kappa \circ_k \lambda)$. Now there exists $p \geq 0$ such that $|\kappa|_{-p} = |\kappa|_{l-m-k-p} < +\infty$. Note that for any $p' \geq 0$ there exists $q' = q'(l', m'; p') \geq 0$ such that $|\lambda|_{l-m'p'-p'(p'+q')} < \infty$. Let $p' := p$ and $q := q'(l', m'; p)$. Then
\[
|S_{l-k}^{l'}\lambda^{l'-k}(\kappa \circ_k \lambda)|_{l+l'-k, m+m'-k; -p, -(p+q)} \leq \sum_{i,j,i',j'} \sum_{h} |\langle \kappa(e(i) \otimes e(j) \otimes e(h)) \rangle|_p |e(h)|_{-p}^2 \times \sum_{h'} |\langle \lambda(e(h') \otimes e(i') \otimes e(j')) \rangle|_p |e(h')|_p |e(i')|_{-q} \times |e(j)|_{-p} \times |e(j')|_{-(p+q)}.
\]

Since $\inf \text{Spec}(A) > 1$, we have
\[
|e(j)|_{-(p+q)} = |e(j)|_{-p} |e(j)|_{-q} \leq |e(j)|_{-p}
\]
and $|e(i')|_{-p} \leq |e(i')|_{p}$. This implies that
\[
|S_{l-k}^{l'}\lambda^{l'-k}(\kappa \circ_k \lambda)|_{l+l'-k, m+m'-k; -p, -(p+q)} \leq |\kappa|_{-p} |\lambda|_{l'-k, m'-k; -(p+q)} < +\infty.
\]

Proposition 2.14. For $\Xi_{l,m}(\kappa) \in \mathcal{L}((E), (E)^*)$ and $\Xi_{l,m'}(\lambda) \in \mathcal{L}((E), (E))$, it follows that
\[
\Xi_{l,m}(\kappa)\Xi_{l,m'}(\lambda) = \sum_{k=0}^{m \wedge l'} \sum_{k=0}^{l'} k! \binom{m}{k} \binom{l'}{k} \Xi_{l+l'-k, m+m'-k}(S_{l-k}^{l'}\lambda^{l'-k}(\kappa \circ_k \lambda)).
\]

The formal part of the proof of this proposition is the same as proposition 7.3 of [10]. From definition 2.11 the analytic part of the proof of this proposition is obvious.

Finally, we define an useful tool for an analysis of a Fock space.
Definition 2.15. For \( f \in H \),

\[
\exp(f) := \sum_{n=0}^{\infty} \frac{1}{n!} f^\otimes n \in \Gamma_b(H)
\]
is called an exponential vector or a coherent vector.

The following lemmas are well-known facts.

Lemma 2.16. \( \{\exp(f) \mid f \in H\} \) generates \( \Gamma_b(H) \). Moreover, \( \{\exp(f) \mid f \in E\} \) spans a dense subspace of \( (E) \).

3 The gauge group and its representation

Let \( G \) be a Lie group and \( \mathfrak{g} \) be the Lie algebra of \( G \) and \( \mathfrak{g}^c \) be the complexification of \( \mathfrak{g} \). Let “Ad” be the adjoint representation of \( G \). For the complexification of “Ad”, we use the same notation.

Now let \( G \) be a semi-simple compact Lie group. Then the Killing form \( K \) on \( \mathfrak{g} \) is negative-definite. Thus \( (-K) \) is an inner product on \( \mathfrak{g} \) and hence we can define an inner product \( (\cdot,\cdot)_{\mathfrak{g}} \) of \( \mathfrak{g}^c \) via the polarization identity. The representation \( (\mathfrak{g}^c,\text{Ad}) \) of \( G \) is a unitary representation with respect to the inner product \( (\cdot,\cdot)_{\mathfrak{g}} \).

For \( X, Y \in \mathfrak{g} \), let

\[
\text{ad}(X)Y := [X,Y]
\]
where \([X,Y]\) be the Lie bracket of \( \mathfrak{g} \). Then \( \text{ad}(X) \) is a representation of the Lie algebra \( \mathfrak{g} \) on the vector space \( \mathfrak{g} \). The following lemma is a well-known fact about the Lie algebra \( \mathfrak{g} \) and its Killing form.

Lemma 3.1. Let \( \mathfrak{g} \) be a Lie algebra, and \( K \) be the Killing form on \( \mathfrak{g} \). Then

\[
K(X,\text{ad}(Z)Y) = -K(\text{ad}(Z)X,Y)
\]
for all \( X, Y, Z \in \mathfrak{g} \). Moreover, for \( X, Y, Z \in \mathfrak{g}^c \), we have

\[
(X,\text{ad}(Z)Y)_{\mathfrak{g}} = (-\text{ad}(Z)X,Y)_{\mathfrak{g}}
\]
where

\[
Z_1 + \sqrt{-1}Z_2 := Z_1 - \sqrt{-1}Z_2.
\]
for \( Z_1, Z_2 \in \mathfrak{g} \).

We shall use this lemma in the following section.

Next, we define the gauge group and its representation. Let \( M \) be a Riemannian manifold without boundary and \( (\cdot,\cdot)_x \) be a inner product on \( T^*_xM \) determined by the Riemannian structure of \( M \).

Let \( C^\infty_c(M,G) \) be the set of all \( C^\infty \)-mappings \( \psi : M \to G \) with compact support. We call \( C^\infty_c(M,G) \) the gauge group. Let \( C^\infty_c(M,\mathfrak{g}) \) be the set of all \( C^\infty \)-mappings \( \Psi : M \to \mathfrak{g} \) with compact support. This is the “Lie algebra” of \( C^\infty_c(M,G) \).

Let \( \Omega^1(M) \) be the space of real-valued 1-forms on \( M \) with compact support and \( \Omega^1(M,\mathfrak{g}) := \Omega^1(M) \otimes \mathfrak{g}^c \). We can define a natural inner product on \( \Omega^1(M,\mathfrak{g}) \) as follows. First, let

\[
\langle \omega_x \otimes X, \omega'_x \otimes X' \rangle_x := (\omega_x,\omega'_x)_{\mathfrak{g}}(X,X')_{\mathfrak{g}}
\]
for all $\omega_x, \omega'_x \in T^*_x M$ and $X, X' \in \mathfrak{g}^c$. For each $x \in M$, $\langle \cdot, \cdot \rangle_x$ is an inner product on $T^*_x M \otimes \mathfrak{g}^c$. Then

$$\langle f, g \rangle_0 := \int_M \langle f(x), g(x) \rangle_x \, dv,$$

where $dv$ is the volume measure on $M$, and where $f, g \in \Omega^1(M, \mathfrak{g})$. This is an inner product on $\Omega^1(M, \mathfrak{g})$. We denote the completion of $\Omega^1(M, \mathfrak{g})$ with respect to the inner product $\langle \cdot, \cdot \rangle_0$ by $H(M, \mathfrak{g})$.

Let

$$(V(\psi)f)(x) := [\text{id}_{T^*_x M} \otimes \text{Ad}(\psi(x))]f(x), \quad x \in M$$

for all $\psi \in C^\infty_c(M, G)$ and $f \in H(M, \mathfrak{g})$. Then $V(\psi)$ is a unitary operator on the Hilbert space $H(M, \mathfrak{g})$. We call $V$ the adjoint representation of gauge group $C^\infty_c(M, G)$.

For $\psi \in C^\infty_c(M, G)$, we define the right logarithmic derivative $\beta(\psi) \in \Omega^1(M, \mathfrak{g})$ as follows.

$$(\beta(\psi))(x) := (d\psi)_x \psi(x)^{-1}.$$ 

$\beta(\psi)$ satisfies

$$\beta(\psi \cdot \varphi) = V(\psi)\beta(\varphi) + \beta(\psi),$$

where $\psi \cdot \varphi$ is defined by the pointwise multiplication. The relation (3.2) is called the Maurer-Cartan cocycle.

**Definition 3.2.** Let $U(\psi)$ be an unitary representation on the on the Boson Fock space $\Gamma_h(H(M, \mathfrak{g}))$ determined by

$$U(\psi) \exp(f) := \exp \left( -\frac{1}{2} |\beta(\psi)|^2 \right) \exp (-\langle \beta(\psi), V(\psi)f \rangle_0) \exp (V(\psi)f + \beta(\psi))$$

for $f \in H(M, \mathfrak{g})$ and $\psi \in C^\infty_c(M, G)$. We call $U$ the energy representation of the gauge group $C^\infty_c(M, G)$.

Now we construct a CH-space $E$ by using the Hilbert space $H(M, \mathfrak{g})$ and a self-adjoint operator on $H(M, \mathfrak{g})$. Let $M$ be a compact Riemann manifold without boundary. Let $\Delta$ be the Bochner Laplacian on $\Omega^1(M)$ determined by the Levi-Civita connection on $M$ and $H(M)$ be the completion of $\Omega^1(M)$. Then $\Delta + 2$ is an essentially self-adjoint operator on $\Omega^1(M)$. (This is shown by considering the complexification of $\Delta + 2$.) Let $A$ be a closed extension of $(\Delta + 2) \otimes \text{id}_\mathfrak{g}$. Then there exists a C.O.N.S. $\{e_i\}_{i \in \mathbb{N}}$ of $H(M)$ consisting of eigenvectors of the essentially self-adjoint operator $\Delta + 2$ on $\Omega^1(M)$. For any C.O.N.S. $\{u_j\}_{j=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}^c$,

$$\{e(i,j) := e_i \otimes u_j \mid i \in \mathbb{N}, 1 \leq j \leq \dim \mathfrak{g} \}$$

is a C.O.N.S. of $H(M, \mathfrak{g})$. This C.O.N.S. of $H(M, \mathfrak{g})$ and the essentially self-adjoint operator $A$ satisfy the condition (i), and (ii) of definition 2.1 and (2.1). (As for the general theory of Laplacian on a vector bundle, see chapter 1 of [7].) For the sake of the calculation in section 4 we take a C.O.N.S. of $\mathfrak{g}^c$ as follows.

Let $\Delta$ be a root system of $\mathfrak{g}^c$ and $\Delta'$ be a set of all positive roots of $\Delta$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\{H_1, \ldots, H_{\dim \mathfrak{h}}\}$ be a C.O.N.S. of $\mathfrak{h}$. Let $X_\alpha, \alpha \in \Delta$ be a normalized element of $\mathfrak{g}^c$ such that $[H, X_\alpha] = \alpha(H) X_\alpha$ for all $H \in \mathfrak{h}$. Then we have a C.O.N.S.

$$\{H_1, \ldots, H_{\dim \mathfrak{h}}, X_\alpha, X_{-\alpha} \mid \alpha \in \Delta' \}$$

(3.4)
of a complex vector space \( g^c \) with respect to the inner product on \( g^c \).

Thus we obtain the CH-space \( E \) constructed from \((H(M, g), A)\).

Next, we construct a CH-space \( (E) \) by using the Hilbert space \( \Gamma_b(H(M, g)) \) and a self-adjoint operator on \( \Gamma_b(H(M, g)) \). We write a C.O.N.S. of \((H(M, g))^\otimes n \) \( (n \geq 1) \) in terms of \( H(M, g) \). Put \( \dim g = N_0 \), \( \dim h = N_1 \), and

\[
\Delta' = \{\alpha_1, \ldots, \alpha_{N_2}\}, \quad (N_2 := \#\Delta' = \frac{1}{2}(N_0 - N_1)).
\]

Let

\[
u_j := \begin{cases} H_j, & \text{if } 1 \leq j \leq N_1, \\ X_{\alpha_j - N_1}, & \text{if } N_1 + 1 \leq j \leq N_1 + N_2, \\ X_{-\alpha_j - (N_1 + N_2)}, & \text{if } N_1 + N_2 + 1 \leq j \leq N_1 + 2N_2 = N_0 \end{cases}
\]

Fix \( d \in \{1, 2, \ldots, n\} \). Let

\[
i := (i_1, \ldots, i_1, i_2, \ldots, i_2, \ldots, i_d, \ldots, i_d) \in \mathbb{N}^n, \\
N(1) + N(2) + \ldots + N(d) = n, \quad i_1 < i_2 < \ldots < i_d.
\]

For this \( i \in \mathbb{N}^n \), we define \( j \in \{1, 2, \ldots, N_0 = \dim g\} \) as follows.

\[
j = (j(i_1, 1), \ldots, j(i_1, N(1)), j(i_2, 1), \ldots, j(i_2, N(2)), j(i_d, 1), \ldots, j(i_d, N(d)))
\]

where \( j(i, k) \in \{1, 2, \ldots, N_0\} \) satisfies the following conditions: for each \( i \in \mathbb{N} \) if \( k_1 < k_2 \), then \( j(i, k_1) \leq j(i, k_2) \).

Let \( \Lambda(n) \) be the subset of \( \mathbb{N}^n \times \{1, 2, \ldots, N_0\}^n \) which consists of all \((i, j)\). For \((i, j) \in \Lambda(n)\), let

\[
\hat{e}(i, j) := e(i_1, j(i_1, 1)) \otimes \ldots \otimes e(i_1, j(i_1, N(1))) \\
\otimes e(i_2, j(i_2, 1)) \otimes \ldots \otimes e(i_2, j(i_2, N(2))) \\
\ldots \otimes e(i_d, j(i_d, 1)) \otimes \ldots \otimes e(i_d, j(i_d, N(d))),
\]

then \( \{\hat{e}(i, j) \mid (i, j) \in \Lambda(n)\} \) is a C.O.N.S. of \( H(M, g)^\otimes n, \ n \geq 1 \). This C.O.N.S. of \( \Gamma_b(H(M, g)) \) and the essentially self-adjoint operator \( \Gamma_b(A) \) satisfy the condition (i), and (ii) of definition 2.1 and \( (2.1) \).

Therefore we obtain a CH-space \( (E) \) constructed from \((\Gamma_b(H(M, g)), \Gamma_b(A))\).

In this paper, we discuss irreducibility of the energy representation \( U \) of \( C^\infty(M, G) \) with the help of the white noise calculus, however, it is difficult to deal with the energy representation \( U \) directly. Thus we treat not the representation of “Lie group” \( C^\infty(M, G) \) but the representation of “Lie algebra” \( C^\infty(M, g) \).

We introduce a proposition for the differentiability of a operator \( V(\psi) \) on the CH-space \( E \).

Proposition 3.3. Let \( \psi_t(x) := \exp(t \Psi(x)) \) for \( \Psi \in C^\infty(M, g) \) and \( t \in \mathbb{R} \). Then \( \{V(\psi_t)\}_{t \in \mathbb{R}} \) is a regular one-parameter subgroup of \( GL(E) \), namely, for any \( p \geq 0 \) there exists \( q \geq 0 \) such that

\[
\lim_{t \to 0} \sup_{f \in E: \|f\|_q \leq 1} \left\| \frac{V(\psi_t)f - f - V(\Psi)f}{t} \right\|_p = 0,
\]

where

\[
(V(\Psi)f)(x) := [id_{T_x^* M} \otimes \text{ad}(\Psi(x))](f)(x), \quad x \in M
\]

for all \( f \in E \).
Proof. Note that there exists $C(\Psi, p) > 0$ such that
\[
\|V(\Psi)f\|_p \leq C(\Psi, p)\|f\|_p, \quad f \in \Omega^1(M, g)
\]
for each $\Psi \in C^\infty(M, g)$ and $p \in \mathbb{N}$, i.e. $V(\Psi) \in \mathcal{L}(E, E)$. (See proposition 2.5 of [14].) Since
\[
(V(\psi_t)f)(x) = [\text{id}_{T^*_xM} \otimes \text{Ad}(\exp(t \Psi(x)))]f(x) = [\text{id}_{T^*_xM} \otimes \exp(t \text{ad}(\Psi(x)))]f(x) = \left(\sum_{k=0}^\infty \frac{1}{k!}(t \text{V}(\Psi))^k\right) f(x)
\]
for each $x \in M$, we have
\[
\left\|\frac{V(\psi_t)f - f}{t} - V(\Psi)f\right\|_p \leq \frac{1}{t} \sum_{k=2}^\infty \frac{1}{k!} \left\|(t \text{V}(\Psi))^k\right\| f \|_{p} \leq \frac{1}{t} \sum_{k=2}^\infty \frac{1}{k!} (t \Omega(\Psi, p))^k \|f\|_p
\]
\[
\leq t \left(\sum_{k=0}^\infty \frac{1}{k!} \Omega(\Psi, p)^k\right) \|f\|_p = t \exp(C(\Psi, p)) \|f\|_p
\]
for $0 < t < 1$. This implies that
\[
\lim_{t \to 0} \sup_{\|f\|_p \leq 1} \left\|\frac{V(\psi_t)f - f}{t} - V(\Psi)f\right\|_p = 0.
\]

This proposition plays a crucial role of our proof of the differentiability of the energy representation $U(\psi_t)$, that is,

**Lemma 3.4.** Let $\psi_t(x) := \exp(t \Psi(x))$ for $\Psi \in C^\infty(M, g)$ and $t \in \mathbb{R}$. Then $\{U(\psi_t)\}_{t \in \mathbb{R}}$ is a regular one-parameter subgroup of $GL((E))$ with infinitesimal generator $d\Gamma_b(V(\Psi)) + D_{\Psi}^\ast - D_{\Psi}$.

Proof. Let $\mu$ be the Gaussian measure on the real vector space $E_{Re}^\ast$ of distributions. Then the Boson Fock space is identified with the space of complex valued $L^2$-functions on $E_{Re}^\ast$ with respect to $\mu$. For $\eta \in E_{Re}^\ast$, let
\[
(T_\eta \phi)(x) := \exp\left(-\frac{1}{2}\|\eta\|^2\right) \phi(x)\phi(x - 2\eta), \quad \phi \in L^2(E_{Re}^\ast, C; \mu).
\]
Here
\[
\phi_\eta(x) := \sum_{n=0}^\infty \frac{1}{n!} \langle x^\otimes n, \eta^\otimes n \rangle, \quad x \in E_{Re}^\ast.
\]
Then $T_\eta(\psi_t)$ satisfies $U(\psi_t) = T_\eta(\psi_t)\Gamma_b(V(\psi))$ and $T_{\psi_t}(\psi_t) = T_{\psi_t}$. (See section 5.7 of [12].) Moreover, from proposition 5.4.5 of [12] and proposition 3.3 $\Gamma_b(V(\psi_t))$ is a regular one-parameter subgroup of $GL((E))$ with infinitesimal generator $d\Gamma_b(V(\Psi))$. Thus $U(\psi_t)$ is a regular one-parameter subgroup of $GL((E))$ with infinitesimal generator $d\Gamma_b(V(\Psi)) + D_{\Psi}^\ast - D_{\Psi}$.
Let \( \pi(\Psi) := d\Gamma_b(V(\Psi)) + D^*d\Psi - Dd\Psi \). Then \( \pi(\Psi) \in \mathcal{L}((E), (E)) \) and \( \pi(\Psi) \) has the following expression:

\[
\pi(\Psi) = \Xi_{1,1}(\text{id} \otimes V(\Psi))^\tau + \Xi_{1,0}(d\Psi) - \Xi_{0,1}(d\Psi),
\]

where \( \tau \in (E \otimes E)^* \) is defined by

\[
\langle \tau, f \otimes g \rangle := \langle f, g \rangle, \quad f, g \in E.
\]

(See proposition 4.5.3 of [12].) To avoid notational complexity, we put \( \lambda_{1,1} = (\text{id} \otimes V(\Psi))^\tau \) and \( \lambda_{1,0} = d\Psi \) for \( \Psi \in C^\infty(M, g) \), namely

\[
\pi(\Psi) = \Xi_{1,1}(\lambda_{1,1}) + \Xi_{1,0}(\lambda_{1,0}) - \Xi_{0,1}(\lambda_{0,1}).
\]

Let

\[
\tilde{\pi}(\Psi) := -\pi(\Psi)^*, \quad \Psi \in C^\infty(M, g).
\]

Then \( \tilde{\pi}(\Psi) \in \mathcal{L}((E)^*, (E)^*) \). Moreover, we can check that \( \pi(\Psi)^|(E) = -\pi(\Psi) \) by using proposition 2.12 and lemma 3.1. In fact, since we have \( t_{1,1}(\lambda_{1,1}) = -\lambda_{1,1} \), it satisfies that

\[
\Xi_{1,1}(\lambda_{1,1}) = -\Xi_{1,1}(\lambda_{1,1}).
\]

4 Irreducibility of the energy representation

We now give the main theorem of this paper.

**Theorem 4.1.** Let \( M \) be a compact Riemannian manifold without boundary. Then the energy representation \( \{U(\psi) | \psi \in C^\infty(M, G)\} \) is irreducible.

Let \( \Xi \in \mathcal{L}(\Gamma_b(H(M, g)), \Gamma_b(H(M, g))) \) satisfy

\[
U(\exp(t\Psi))\Xi = \Xi U(\exp(t\Psi)) \tag{4.1}
\]

for all \( \Psi \in C^\infty(M, g), t \in \mathbb{R} \). Note that the restriction \( \Xi|(E) \) of \( \Xi \) to \( (E) \) is a continuous linear operator from \( (E) \) to \( (E)^* \) and \( \pi(\Psi)^* \) is a continuous linear operator on \( (E)^* \). Then (4.1) implies

\[
\tilde{\pi}(\Psi)\Xi = \Xi \pi(\Psi) \tag{4.2}
\]

for all \( \Psi \in C^\infty(M, g) \) as a continuous linear operator from \( (E) \) to \( (E)^* \). Our main problem is to find \( \Xi \) satisfying (4.2).

**Lemma 4.2.** Let

\[
\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}).
\]

be the Fock expansion for a continuous linear operator \( \Xi \) from \( (E) \) to \( (E)^* \). If \( \Xi \) satisfies (4.2), then \( \kappa_{l,m}, l, m \geq 0 \) satisfy the following relations:

\[
s_{l,0} \left( S_{0,0}^0(\kappa_{l,1} \circ_1 \lambda_{1,0}) \right) = s_{l,0} \left( l S_{0,0}^{l-1}(\lambda_{1,1} \circ_1 \kappa_{l,0}) - (l + 1)S_{0,0}^0(\lambda_{0,1} \circ_1 \kappa_{l+1,0}) \right), \tag{4.3}
\]

12
\[ s_{0,m} \left( -S_{0,m}^0(\lambda_{0,1} \circ_1 \kappa_{1,m}) \right) = s_{0,m} \left( m S_{m-1}^0(\kappa_{0,m} \circ_1 \lambda_{1,1}) + (m + 1) S_{m}^0(\kappa_{0,m+1} \circ_1 \lambda_{1,0}) \right), \]  
(4.4)

\[ s_{l,m} \left( m S_{m-1}^l(\kappa_{l,m} \circ_1 \lambda_{1,1}) + (m + 1) S_{m}^l(\kappa_{l,m+1} \circ_1 \lambda_{1,0}) \right) = s_{l,m} \left( l S_{0,m}^{l-1}(\lambda_{1,1} \circ_1 \kappa_{l,m}) - (l + 1) S_{0}^{l}(\lambda_{0,1} \circ_1 \kappa_{l+1,m}) \right) \]  
(4.5)

for all \( l, m \geq 1 \) and
\[ -\lambda_{0,1} \circ_1 \kappa_{1,0} = \kappa_{0,1} \circ_1 \lambda_{1,0}. \]  
(4.6)

Proof.

\[
\left( \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \right) \Xi_{1,0}(\lambda_{1,0})
= \sum_{l,m=0}^{\infty} \sum_{k=0}^{m \wedge l} k! \binom{m}{k} \left( \binom{1}{k} \Xi_{l+1-k,m-k} \left( S_{m-k}^{l-k}(\kappa_{l,m} \circ_k \lambda_{1,0}) \right) \right)
= \Xi_{0,0}(\kappa_{0,1} \circ_1 \lambda_{1,0})
+ \sum_{l=1}^{\infty} \Xi_{l,0} \left( S_{0}^{l-1}(\kappa_{l-1,0} \circ \lambda_{1,0}) + S_{0}^{l-1}(\kappa_{l,1} \circ_1 \lambda_{1,0}) \right)
+ \sum_{m=1}^{\infty} \Xi_{0,m} \left( (m+1) S_{m}^0(\kappa_{0,m+1} \circ_1 \lambda_{1,0}) \right)
+ \sum_{l,m=1}^{\infty} \Xi_{l,m} \left( S_{m}^{l-1}(\kappa_{l-1,m} \circ \lambda_{1,0}) + (m+1) S_{0}^{l}(\kappa_{l,m+1} \circ_1 \lambda_{1,0}) \right)
\]

and

\[
\left( \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \right) \Xi_{0,1}(\lambda_{0,1})
= \sum_{l,m=0}^{\infty} \Xi_{l,m+1} \left( S_{l}^m(\kappa_{l,m} \circ \lambda_{0,1}) \right)
= \sum_{m=1}^{\infty} \Xi_{0,m} \left( S_{m-1}^0(\kappa_{0,m-1} \circ \lambda_{0,1}) \right) + \sum_{l,m=1}^{\infty} \Xi_{l,m} \left( S_{m-1}^{l}(\kappa_{l,m-1} \circ \lambda_{0,1}) \right)
\]
and
\[
\left( \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \right) \Xi_{1,1}(\lambda_{1,1})
= \sum_{l,m=0}^{\infty} \sum_{k=0}^{m} k! \binom{m}{k} \Xi_{l+1-k,m+1-k} \left( S_{m-k}^{l-1-k}(\kappa_{l,m} \circ_k \lambda_{1,1}) \right)
= \sum_{m=1}^{\infty} \Xi_{0,m} \left( m S_{m-1}^{0} \circ_{0} \Xi_{0,m} \right. 
+ \sum_{l,m=1}^{\infty} \Xi_{l,m} \left( S_{m-1}^{l-1}(\kappa_{l-1,m} \circ \lambda_{1,1}) + m S_{m-1}^{l-1}(\kappa_{l,m} \circ \lambda_{1,1}) \right)
\]

Thus,
\[
\left( \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \right) \pi(\Psi)
= \Xi_{0,0}(\kappa_{0,0} \circ_{0} \lambda_{1,0})
\]
\[
+ \sum_{l=1}^{\infty} \Xi_{l,0} \left( S_{0}^{l-1}(\kappa_{l-1,0} \circ \lambda_{1,0}) + S_{0}^{l}(\kappa_{l,1} \circ \lambda_{1,0}) \right)
\]
\[
+ \sum_{m=1}^{\infty} \Xi_{0,m} \left( m S_{m-1}^{0} \circ_{0} \Xi_{0,m} \right.
+ \sum_{l,m=1}^{\infty} \Xi_{l,m} \left( S_{m-1}^{l-1}(\kappa_{l-1,m} \circ \lambda_{1,1}) + m S_{m-1}^{l-1}(\kappa_{l,m} \circ \lambda_{1,1}) \right)
\]

On the other hand, it follows that
\[
\tilde{\pi}(\Psi) \left( \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \right) = - \left\{ \left( \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \right) \pi(\Psi) \right\}^{*}
\]
as a continuous linear operator from \((E)\) to \((E)^{*}\). When we consider
\[
s_{m+m',k,l+l',k,l+m'+l'} \left( S_{m-k}^{l} \circ_{k} \lambda \right)
= s_{m+m'-k,l+l'} \left( S_{m-k}^{l'} \circ_{k} (t_{m,l}(\lambda) \circ_{k} t_{m,l}(\kappa)) \right)
\]
for all \(\kappa \in (E^{\otimes(l+m)})^{*}\) and \(\lambda \in E^{\otimes l'} \otimes \left( E^{\otimes m'} \right)^{*}\), the relations (4.7), (4.8), and proposition 2.12.
imply

\[ \tilde{\pi}(\Psi) \left( \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \right) = -\Xi_{0,0}(\lambda_{0,1} \circ \kappa_{1,0}) + \sum_{l=1}^{\infty} \Xi_{l,0} \left( l \, S_{0}^{l-1}(\lambda_{1,1} \circ \kappa_{l,0}) + S_{0}^{l-1}(\lambda_{1,0} \circ \kappa_{l,1,0}) - (l + 1) S_{0}^{l-1}(\lambda_{0,1} \circ \kappa_{l,1,0}) \right) \]

\[ + \sum_{m=1}^{\infty} \Xi_{0,m} \left( -S_{0, m-1}(\lambda_{0,1} \circ \kappa_{0,m-1}) - S_{0, m-1}(\lambda_{0,1} \circ \kappa_{1,m}) \right) \]

\[ + \sum_{l,m=1}^{\infty} \Xi_{l,m} \left( S_{1}^{l-1}(\lambda_{1,1} \circ \kappa_{l,1,m-1}) + l \, S_{0}^{l-1}(\lambda_{1,1} \circ \kappa_{l,m}) \right) \]

\[ + S_{1}^{l-1}(\lambda_{1,0} \circ \kappa_{l,1,m}) - S_{1}^{l-1}(\lambda_{0,1} \circ \kappa_{l,m-1}) - (l + 1) S_{0, m-1}(\lambda_{0,1} \circ \kappa_{l,1,m}) \right) \].

Therefore we have

\[ s_{l,0} \left( S_{0}^{l-1}(\kappa_{l,1,0} \circ \lambda_{l,0}) + S_{0}^{l}(\kappa_{l,1} \circ \lambda_{l,0}) \right) \]

\[ = s_{l,0} \left( l \, S_{0}^{l-1}(\lambda_{1,1} \circ \kappa_{l,0}) + S_{0}^{l-1}(\lambda_{1,0} \circ \kappa_{l,1,0}) - (l + 1) S_{0}^{l-1}(\lambda_{0,1} \circ \kappa_{l,1,0}) \right) , \]

\[ s_{0,m} \left( m \, S_{m-1}^{0}(\kappa_{0,m} \circ \lambda_{1,0}) + (m + 1) \, S_{m-1}^{0}(\kappa_{0,m+1} \circ \lambda_{1,0}) - S_{m-1}^{0}(\kappa_{0,m+1} \circ \lambda_{0,1}) \right) \]

\[ = s_{0,m} \left( -S_{0, m-1}(\lambda_{0,1} \circ \kappa_{0,m-1}) - S_{0, m-1}(\lambda_{0,1} \circ \kappa_{1,m}) \right) , \]

\[ s_{l,m} \left( S_{m-1}^{l-1}(\kappa_{l,m} \circ \lambda_{l,1}) + m S_{m-1}^{l}(\kappa_{l,m} \circ \lambda_{l,1}) \right) \]

\[ + S_{m-1}^{l-1}(\kappa_{l,m} \circ \lambda_{l,0}) + (m + 1) S_{m-1}^{l}(\kappa_{l,m+1} \circ \lambda_{l,1}) - S_{m-1}^{l}(\kappa_{l,m+1} \circ \lambda_{0,1}) \right) \]

\[ = s_{l,m} \left( S_{1}^{l-1}(\lambda_{1,0} \circ \kappa_{l,1,m-1}) + l \, S_{0}^{l-1}(\lambda_{1,1} \circ \kappa_{l,m}) \right) \]

\[ + S_{0, m-1}^{l}(\lambda_{1,0} \circ \kappa_{l,1,m}) - S_{1, m-1}^{l}(\lambda_{0,1} \circ \kappa_{l,m-1}) - (l + 1) S_{0, m-1}^{l}(\lambda_{0,1} \circ \kappa_{l,1,m}) \right) , \]

and

\[-\lambda_{0,1} \circ \kappa_{1,0} = \kappa_{0,1} \circ \lambda_{1,0} \]

for all \( l, m \geq 1 \). On the other hand, from \( \lambda_{1,0}, \lambda_{0,1} \in E \) and the definition of \( S_{\beta}^{\alpha} \), we have

\[ s_{l,0} \left( S_{0}^{l}(\kappa_{1,1,0} \circ \lambda_{1,0}) \right) = s_{l,0} \left( S_{0}^{l-1}(\lambda_{1,0} \circ \kappa_{1,0}) \right) , \]

\[ s_{0,m} \left( S_{1}^{0}(\lambda_{0,1} \circ \kappa_{0,m-1}) \right) = s_{0,m} \left( S_{m-1}^{0}(\kappa_{0,m-1} \circ \lambda_{0,1}) \right) , \]

\[ s_{l,m} \left( S_{m}^{l}(\lambda_{1,0} \circ \kappa_{l,1,m-1}) \right) = s_{l,m} \left( S_{m}^{l-1}(\kappa_{l,1,m} \circ \lambda_{1,0}) \right) , \]

\[ s_{l,m} \left( S_{1}^{l-1}(\lambda_{0,1} \circ \kappa_{l,m-1}) \right) = s_{l,m} \left( S_{m-1}^{l}(\kappa_{l,m-1} \circ \lambda_{0,1}) \right) , \]

and

\[ s_{l,m} \left( S_{1}^{l-1}(\lambda_{1,1} \circ \kappa_{l,1,m-1}) \right) = s_{l,m} \left( S_{m-1}^{l}(\kappa_{l,1,m-1} \circ \lambda_{1,1}) \right) \]

for all \( l, m \geq 1 \). Hence we obtain 14.3 - 14.6.
Corollary 4.3. $\kappa_{l,m}$, $l, m \geq 1$ satisfy the following conditions:

$$s_{l,0} \left( S_{0,0}^{m-1}(\lambda_{1,1} \circ \kappa_{l,0}) \right) = 0, \quad (4.9)$$

$$s_{0,m} \left( S_{m-1}^{0}(\kappa_{l,m}) \right) = 0, \quad \text{and} \quad (4.10)$$

$$l_{s_{l,m}} \left( S_{0,0}^{m-1}(\lambda_{1,1} \circ \kappa_{l,m}) \right) = m_{s_{l,m}} \left( S_{m-1}^{0}(\kappa_{l,m} \circ \lambda_{1,1}) \right). \quad (4.11)$$

Proof. $C^\infty(M, g)$ contains constant functions. Thus if $\Psi(x) = \text{Const.} \in g$ i.e., $\lambda_{1,0} = \lambda_{0,1} = 0$, then we obtain (4.9) – (4.11).

Now, for $l \geq 0, m \geq 1$, we have

$$S_{m-1}^{0}(\kappa_{l,m} \circ \lambda_{1,1})$$

$$= \sum \sum \langle \kappa_{l,m}, e(i) \otimes e(j) \otimes e(h) \rangle \langle e(h), V(\Psi)e(j') \rangle e(i) \otimes e(j) \otimes e(j')$$

$$= \sum \langle \kappa_{l,m}, e(i) \otimes e(j) \otimes V(\Psi)e(j') \rangle e(i) \otimes e(j) \otimes e(j')$$

$$= (\text{id}_{l+m-1} \otimes V(\Psi))^* \kappa_{l,m} = (\text{id}_{l+m-1} \otimes V(\Psi))^* \kappa_{l,m}$$

In the same manner, it follows that

$$S_{0,0}^{m-1}(\lambda_{1,1} \circ \kappa_{l,m}) = -(V(\Psi) \otimes \text{id}_{l+m-1})^* \kappa_{l,m} = -(V(\Psi)^* \otimes \text{id}_{l+m-1}) \kappa_{l,m}$$

for $l \geq 1, m \geq 0$.

On the other hand, we have

$$s_n(A \otimes \text{id}_{n-1}) = s_n(\text{id} \otimes A \otimes \text{id}_{n-2}) = \ldots = s_n(\text{id}_{n-1} \otimes A) s_n$$

$$= \frac{1}{n} \text{d} \Gamma_b(A)(m). \quad (4.12)$$

for $A \in \mathcal{L}(E^*, E^*)$ by direct computation. From (4.12), (4.3) – (4.5) are equivalent to

$$\text{d} \Gamma_b(V(\Psi)^*(l)) \kappa_{l,0} = s_{l,0} \left( (l + 1)S_{0,0}^{0}(\lambda_{0,1} \circ \kappa_{l+1,0}) + S_{0,0}^{0}(\kappa_{l,1} \circ \lambda_{1,0}) \right), \quad (4.13)$$

$$-\text{d} \Gamma_b(V(\Psi)^*(m)) \kappa_{0,m} = s_{0,m} \left( (m + 1)S_{m,0}^{0}(\kappa_{0,m+1} \circ \lambda_{1,0}) + S_{0,m}^{0}(\lambda_{0,1} \circ \kappa_{1,m}) \right), \quad \text{and} \quad (4.14)$$

$$\left( \text{d} \Gamma_b(V(\Psi)^*(l)) \otimes \text{id}_m + \text{id}_l \otimes \text{d} \Gamma_b(V(\Psi)^*(m)) \right) \kappa_{l,m}$$

$$= s_{l,m} \left( (l + 1)S_{0,m}^{0}(\lambda_{0,1} \circ \kappa_{l+1,m}) + (m + 1)S_{m,0}^{0}(\kappa_{l,m+1} \circ \lambda_{1,0}) \right) \quad (4.15)$$

respectively for all $\Psi \in C^\infty(M, g)$. In particular, (4.9) – (4.11) become

$$\text{d} \Gamma_b(V(\Psi)^*(l)) \kappa_{l,0} = 0, \quad (4.16)$$

$$\text{d} \Gamma_b(V(\Psi)^*(m)) \kappa_{0,m} = 0, \quad (4.17)$$

$$\left\{ \text{d} \Gamma_b(V(\Psi)^*(l)) \otimes \text{id}_m + \text{id}_l \otimes \text{d} \Gamma_b(V(\Psi)^*(m)) \right\} \kappa_{l,m} = 0. \quad (4.18)$$
Lemma 4.4. For \( l, m \geq 0 \), \( l + m \neq 0 \), we have
\[
\kappa_{l,m} = \sum \langle \kappa_{l,m}, \widehat{e}(i,j) \otimes \widehat{e}(i',j') \rangle \widehat{e}(i,j) \otimes \widehat{e}(i',j'), \tag{4.19}
\]
where the sum is over all \((i,j) \in \Lambda(l)\) and \((i',j') \in \Lambda(m)\) satisfying the condition:
\[
\sum_{1 \leq p \leq N_2} \alpha_p(H)(n_{p,+}(j) - n_{p,-}(j) + n_{p,+}(j') - n_{p,-}(j')) = 0
\]
for all \( H \in \mathfrak{h} \). Here
\[
n_{p,+}(j) := \#\{q \in \{1, 2, \ldots, n\} | j_q = N_1 + p\},
n_{p,-}(j) := \#\{q \in \{1, 2, \ldots, n\} | j_q = N_1 + N_2 + p\},
\]
for \( j = (j_1, \ldots, j_n) \in \{1, 2, \ldots, N\}^n \) and \( 1 \leq p \leq N_2 \).

Proof. Since
\[
d\Gamma_b(V(H)^{(n)})\widehat{e}(i,j) = \sum_{1 \leq p \leq N_2} \alpha_p(H)(n_{p,+}(j) - n_{p,-}(j))\widehat{e}(i,j)
\]
for \( H \in \mathfrak{h} \), we have
\[
\{d\Gamma_b(V(H)^{(l)} \otimes \text{id}_m)\} \kappa_{l,m}
= \sum_{(i,j) \in \Lambda(l)} \sum_{(i',j') \in \Lambda(m)} \sum_{1 \leq p \leq N_2} \langle \kappa_{l,m}, \{d\Gamma_b(V(H)^{(l)} \otimes \text{id}_m)\}\widehat{e}(i,j) \otimes \widehat{e}(i',j') \rangle \widehat{e}(i,j) \otimes \widehat{e}(i',j')
= \sum_{(i,j) \in \Lambda(l)} \sum_{(i',j') \in \Lambda(m)} \sum_{1 \leq p \leq N_2} \alpha_p(H)(n_{p,+}(j) - n_{p,-}(j)) \langle \kappa_{l,m}, \widehat{e}(i,j) \otimes \widehat{e}(i',j') \rangle \widehat{e}(i,j) \otimes \widehat{e}(i',j')
\]
\[
\tag{4.20}
\]
where \((i,j)\) runs over the whole of \( \Lambda(l) \) satisfying the conditions: there exists \( p \) such that \( n_{p,+}(j) \neq n_{p,-}(j) \). In the same manner,
\[
\left(\text{id}_l \otimes d\Gamma_b(V(H)^{(m)})\right) \kappa_{l,m}
= \sum_{(i,j) \in \Lambda(l)} \sum_{(i',j') \in \Lambda(m)} \sum_{1 \leq p \leq N_2} \alpha_p(H)(n_{p,+}(j') - n_{p,-}(j')) \langle \kappa_{l,m}, \widehat{e}(i,j) \otimes \widehat{e}(i',j') \rangle \widehat{e}(i,j) \otimes \widehat{e}(i',j'). \tag{4.21}
\]
where \((i',j')\) runs over the whole of \( \Lambda(m) \) satisfying the conditions: there exists \( p \) such that \( n_{p,+}(j') \neq n_{p,-}(j') \). Thus we obtain \eqref{4.19} from \eqref{4.18}.

When the operator \( d\Gamma_b(V(u_{k+N_1})^*) \), \( 1 \leq k \leq N_2 \) act \( \kappa_{l,m} \), we need the following lemma.

Lemma 4.5. \( V(X_\alpha)^*|E = -V(X_{-\alpha}) \),
that is, if \( 1 \leq k \leq N_2 \), then
\[
V(u_{k+N_1})^*|E = -V(u_{k+N_1+N_2}).
\]
Proof. Let $U_\alpha$ and $V_\alpha$, $\alpha \in \Delta'$ be elements defined by
\[
X_\alpha = \frac{U_\alpha + \sqrt{-1}V_\alpha}{|U_\alpha + \sqrt{-1}V_\alpha|_g}, \quad X_{-\alpha} = \frac{U_\alpha - \sqrt{-1}V_\alpha}{|U_\alpha - \sqrt{-1}V_\alpha|_g}.
\] (4.22)
And let $\mathfrak{g}_R$ be a compact real form of $\mathfrak{g}$ and $\mathfrak{h}_R$ be a Cartan subalgebra of $\mathfrak{g}_R$. If $\{\sqrt{-1}H'_i \mid i = 1, 2, \ldots, N_1\}$ is a basis of $\mathfrak{h}_R$, then
\[
\{\sqrt{-1}H'_1, \ldots, \sqrt{-1}H'_{N_1}, U_\alpha, V_\alpha \mid \sqrt{-1}H'_i \in \mathfrak{h}_R, \alpha \in \Delta'\}
\] is a basis of $\mathfrak{g}_R$, called Cartan-Weyl basis. (More details of this basis are in \[13\].)

Since $g$ is compact, there exists an automorphism $\sigma$ of $\mathfrak{g}$ such that $\sigma(\mathfrak{g}_R) = \mathfrak{g}$. Hence we can regard the Cartan-Weyl basis of $\mathfrak{g}_R$ as a basis of $\mathfrak{g}$.

From lemma 3.1 for $f, g \in E$, we have
\[
\langle V(X_\alpha)^* f, g \rangle = \langle \overline{f}, V(X_\alpha)g \rangle_0 = \langle -V(X_\alpha) \overline{f}, g \rangle_0 = \langle -\overline{V(X_\alpha) f}, g \rangle.
\]
If we note that the identification of $E_{-p}$ and $E^*_{-p}$ under the anti-linear isomorphism (see definition \[21\]), then this implies that $V(X_\alpha)^* f = -V(X_\alpha) f$. Since $\overline{X_\alpha} = X_{-\alpha}$ from (4.22), we obtain this lemma.

**Lemma 4.6.** $\kappa_{1,0} = 0$ and $\kappa_{0,1} = 0$.

**Proof.** (4.19) implies that
\[
\kappa_{1,0} = \sum_{(i,j) \in \mathcal{A}(1); 1 \leq j \leq N_1} \langle \kappa_{1,0}, e(i, j) \rangle e(i, j).
\]
For $1 \leq k \leq N_2$, if the operator $d\Gamma_b(V(u_{k+N_1})^*)$ act $\kappa_{1,0}$, then we obtain
\[
0 = d\Gamma_b(V(u_{k+N_1})^*)\kappa_{1,0} = -\sum_{i=1}^\infty \sum_{1 \leq j \leq N_1} \langle \kappa_{1,0}, e(i, j) \rangle \alpha_k(u_j)e(i, k + N_1 + N_2)
\]
with the help of
\[
V(u_{k+N_1})^* e(i, j) = -V(u_{k+N_1} + N_2)e(i, j) = -e_i \otimes \text{ad}(u_{k+N_1} + N_2)u_j
\]
\[
= -\alpha_k(u_j)e(i, k + N_1 + N_2).
\]
This implies
\[
\sum_{1 \leq j \leq N_1} \langle \kappa_{1,0}, e(i, j) \rangle \alpha_k(u_j) = 0
\]
for all $i \in \mathbb{N}$, $k \in \{1, 2, \ldots, N_2\}$. Since $\mathfrak{h}^*$ is generated by linear combinations of $\{\alpha_k\}_{k=1}^{N_2}$, we can choose basis $\{\alpha_{k_1}, \ldots, \alpha_{k_{N_1}}\}$ of $\mathfrak{h}^*$. Then the matrix $(\alpha_k(u_j))_{1 \leq i, j \leq N_1} \in \text{Mat}(N_1, \mathbb{C})$ is invertible. Therefore
\[
\langle \kappa_{1,0}, e(i, j) \rangle = 0
\]
for all $i \in \mathbb{N}$ and $j \in \{1, 2, \ldots, N_1\}$, i.e., $\kappa_{1,0} = 0$. In the same manner, $\kappa_{0,1} = 0$.

**Lemma 4.7.** $\kappa_{i,1} = 0$ and $\kappa_{1,m} = 0$.  

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Proof. In (4.18), let \( \Psi = u_{k+N_1} \) for \( 1 \leq k \leq N_2 \) and \( m = 1 \). When we consider (4.19), we have

\[
(d \Gamma_b(V(u_{k+N_1})^*)^l \otimes \text{id}) \kappa_{l,1}
\]

\[
= \sum_{(i,j) \in \mathbb{N}} \sum_{1 \leq j \leq N_1} \langle \kappa_{l,1}, \tilde{e}(i,j) \otimes e(i,j) \rangle \ [d \Gamma_b(V(u_{k+N_1})^*)^l \tilde{e}(i,j)] \otimes e(i,j),
\]

and

\[
(\text{id} \otimes d \Gamma_b(V(u_{k+N_1})^*)^l) \kappa_{l,1}
\]

\[
= \sum_{(i,j) \in \mathbb{N}} \sum_{1 \leq j \leq N_1} \langle \kappa_{l,1}, \tilde{e}(i,j) \otimes e(i,j) \rangle \alpha_k(u_j) \tilde{e}(i,j) \otimes e(i, k + N_1 + N_2),
\]

where \((i,j)\) runs over the whole of \( \Lambda(l) \) satisfying

\[
\sum_{1 \leq p \leq N_2} \alpha_p(H)(n_{p,+}(j) - n_{p,-}(j)) = 0. \tag{4.23}
\]

If \( 1 \leq j \leq N_1 \) and \( 1 \leq k \leq N_2 \), then \([d \Gamma_b(V(u_{k+N_1})^*)^l \tilde{e}(i,j)] \otimes e(i,j) \) and \( \tilde{e}(i,j) \otimes e(i, k + N_1 + N_2) \) are orthogonal each other with respect to the inner product on \( H(M, \mathfrak{g})^\otimes l \otimes H(M, \mathfrak{g}) \). Thus, \((\text{id} \otimes d \Gamma_b(V(u_{k+N_1})^*)^l)) \kappa_{l,1} = 0 \) and hence

\[
\sum_{1 \leq j \leq N_1} \langle \kappa_{l,1}, \tilde{e}(i,j) \otimes e(i,j) \rangle \alpha_k(u_j) = 0 \tag{4.24}
\]

for all \( i \in \mathbb{N} \), \( 1 \leq k \leq N_2 \), and all \((i,j)\) \( \in \Lambda(l) \) satisfying (4.23).

Since we can select \( k_1, k_2, \ldots, k_{N_1} \in \{1, 2, \ldots, N_2\} \) such that a matrix \((\alpha_{k_i}(u_{j}))_{1 \leq i,j \leq N_1} \in \text{Mat}(N_1, \mathbb{C})\) is invertible, we obtain

\[
\langle \kappa_{l,1}, \tilde{e}(i,j) \otimes e(i,j) \rangle = 0
\]

for all \( i \in \mathbb{N} \), \( 1 \leq j \leq N_1 \) and all \((i,j)\) \( \in \Lambda(l) \) satisfying (4.23), i.e., \( \kappa_{l,1} = 0 \).

In the same manner, we also have \( \kappa_{l,m} = 0 \).

**Lemma 4.8.** \( \kappa_{l,0} = 0 \) and \( \kappa_{0,m} = 0 \) for all \( l, m \geq 1 \).

**Proof.** We prove them by the induction. We have only to show \( \kappa_{l,0} = 0 \). The case of \( \kappa_{1,0} = 0 \) has been proved. Let \( \kappa_{l,0} = 0 \). Since \( \kappa_{l,1} = 0 \) and (4.15), we have

\[
0 = s_{l,0} \bigg( (l + 1) \sum_{0}^{0} \langle \lambda_{0,1}, 0, \kappa_{l+1,0} \rangle \bigg)
\]

\[
= \sum_{(i,j) \in \Lambda(l)} \sum_{(i,j) \in \Lambda(1)} \langle \lambda_{0,1}, e(i,j) \rangle \langle \kappa_{l+1,0}, e(i,j) \otimes \tilde{e}(i,j) \rangle \tilde{e}(i,j)
\]

\[
= \sum_{(i,j) \in \Lambda(l)} \langle \kappa_{l+1,0}, \lambda_{0,1} \otimes \tilde{e}(i,j) \rangle \tilde{e}(i,j)
\]

and hence

\[
\langle \kappa_{l+1,0}, \lambda_{0,1} \otimes \tilde{e}(i,j) \rangle = 0 \tag{4.25}
\]

for all \((i,j)\) \( \in \Lambda(l) \). This implies that

\[
\langle \kappa_{l+1,0}, d\Psi \otimes \tilde{e}(i,j) \rangle = 0, \tag{4.26}
\]

\[
\langle \kappa_{l+1,0}, V(\Psi) d\Psi' \otimes \tilde{e}(i,j) \rangle = 0 \tag{4.27}
\]
for all \((i,j) \in \Lambda(l)\), and \(\Psi, \Psi' \in C^\infty(M, g)\). (4.26) is obvious. We show (4.27).

For each \(\Psi, \Psi' \in C^\infty(M, g)\) and \(|s|, |t| \ll 1\), there exists a unique \(\Phi_{s,t} \in C^\infty(M, g)\) such that

\[
\exp(t\Psi) \exp(s\Psi') = \exp(\Phi_{s,t}).
\]

Since

\[
d\Phi_{s,t} = \beta(\exp(\Phi_{s,t})) = \beta(\exp(t\Psi) \exp(s\Psi'))
\]

\[
= V(\exp(t\Psi))\beta(\exp(s\Psi')) + \beta(\exp(t\Psi))
\]

\[
= sV(\exp(t\Psi))d\Psi' + td\Psi
\]

and (4.26), we have

\[
0 = \langle \kappa_{l+1,0}, d\Phi_{s,t} \otimes \hat{e}(i,j) \rangle
\]

\[
= s \langle \kappa_{l+1,0}, V(\exp(t\Psi))d\Psi' \otimes \hat{e}(i,j) \rangle + t \langle \kappa_{l+1,0}, d\Psi \otimes \hat{e}(i,j) \rangle
\]

\[
= s \langle \kappa_{l+1,0}, V(\exp(t\Psi))d\Psi' \otimes \hat{e}(i,j) \rangle.
\]

Hence \(\kappa_{l+1,0}\) satisfies (4.27) by considering the differential of the above equation at \(t \in \mathbb{R}\).

By the way, \(H(M, g)\) is generated by

\[
\{d\Psi, V(\Psi)d\Psi' \mid \Psi, \Psi' \in C^\infty(M, g)\}.
\]

(See lemma 3.5 of [2].) Thus the following relation is a direct consequence of (4.26) and (4.27):

\[
\langle \kappa_{l+1,0}, e(i, j) \otimes \hat{e}(i, j) \rangle = 0
\]

for all \((i, j) \in \Lambda(1)\) and \((i, j) \in \Lambda(l)\). Therefore we obtain \(\kappa_{l+1,0} = 0\).

In the same manner, we can show \(\kappa_{0,m} = 0\) for all \(m \geq 1\).

\[\blacksquare\]

**Lemma 4.9.** \(\kappa_{l,m} = 0\) for all \((l, m) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0, 0)\}\), that is, we obtain theorem 4.1

**Proof.** We prove this statement by the induction. We have already shown the case of \(l = 1\), i.e., \(\kappa_{1,m} = 0\) for all \(m \geq 0\). Let \(\kappa_{l,m} = 0\) for all \(m \geq 0\). Then we show \(\kappa_{l+1,m} = 0\) for all \(m \geq 0\). Fix \(m \geq 0\). Since \(\kappa_{l,m} = 0\) and \(\kappa_{l,m+1} = 0\) and (4.5), we have

\[
0 = z_{l,m} \left( (l + 1)S_{0,m}^l \lambda_{0,1} \right) \kappa_{l+1,m}
\]

\[
= (l + 1) \sum_{(i,j) \in \Lambda(l)} \sum_{(i',j') \in \Lambda(m)} \langle \kappa_{l+1,m}, \lambda_{0,1} \hat{e}(i,j) \otimes \hat{e}(i',j') \rangle \hat{e}(i,j) \otimes \hat{e}(i',j').
\]

This shows \(\langle \kappa_{l+1,m}, \lambda_{0,1} \hat{e}(i,j) \otimes \hat{e}(i',j') \rangle = 0\) for all \((i, j) \in \Lambda(l)\), \((i', j') \in \Lambda(m)\). Thus we can show

\[
\langle \kappa_{l+1,m}, e(i, j) \otimes \hat{e}(i, j) \otimes \hat{e}(i', j') \rangle = 0
\]

for all \((i, j) \in \Lambda(1)\), \((i, j) \in \Lambda(l)\), and \((i', j') \in \Lambda(m)\). This implies \(\kappa_{l+1,m} = 0\). Since \(m \geq 0\) is arbitrary, the proof has been completed.

\[\blacksquare\]

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