THE HODGE CONJECTURE FOR GENERAL PRYM VARIETIES

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INTRODUCTION

We work over \( \mathbb{C} \), the field of complex numbers.

The Prym variety of a double cover \( C \to D \) of a smooth connected projective curve \( D \) by a smooth connected curve \( C \) is defined (see [7]) as the identity component of the kernel of the norm homomorphism \( N : J(C) \to J(D) \) between the Jacobians of the curves. This is an abelian variety polarised by the restriction of the canonical polarisation on \( J(C) \); we denote this variety by \( P(C \to D) \) or simply \( P \) when there is no possibility of ambiguity.

A Hodge class on a variety \( X \) is an integral singular cohomology class on the complex manifold \( X(\mathbb{C}) \) which is represented by a closed differential form of type \((p, p)\). The Hodge conjecture (see [3]) asserts that some multiple of such a class is the cohomology class of an algebraic cycle on \( X \).

Let \( A \) be an abelian variety. The K"unneth decomposition implies that the rational singular cohomology of \( A \times \cdots \times A \) is a direct sum of subquotients of tensor products of \( H^1(A(\mathbb{C}), \mathbb{Q}) \). Hence we have an action of a linear automorphism of this vector space on these cohomology groups. The Mumford-Tate group \( H(A) \) of \( A \) can thus be defined (see [2]) as the group of all linear automorphisms of \( H^1(A(\mathbb{C}), \mathbb{Q}) \) which stabilise all Hodge cycles on the varieties \( A \times \cdots \times A \).

The aim of this note is to show that the Mumford-Tate group \( H(P) \) of a general Prym variety \( P(C \to D) \) is isomorphic to the full symplectic group \( \text{Sp}(2g) \); where the class in \( H^2(P(\mathbb{C}), \mathbb{Q}) = H^2(P(\mathbb{C}), \mathbb{Q}) \) which is stabilised by this group is the first Chern class of the natural polarisation on the Prym variety. Invariant theory (see [4] or [3] and [8]) then implies that the only Hodge cycles on \( P \) are powers (under cup-product) of this polarisation class. In particular, we obtain the Hodge conjecture for \( P \) as a consequence of this result.

As a particular case the Néron-Severi group of a general Prym variety is \( \mathbb{Z} \). This was proved earlier by Pirola (see [8]). We do not give a new proof of that result and use it in an essential way to prove our result.
The outline of the paper is as follows. In section 1 we set out some standard arguments about Mumford-Tate groups in families. In section 2 we use an extension (due to Beauville) of the definition of Prym varieties to the case where $C$ and $D$ are singular curves. The results on Mumford-Tate groups are applied to this larger family of Prym varieties in section three. In addition we use the semi-simplicity of the Mumford-Tate group (see [2]) and the result of Pirola (see [3]) to reduce the problem to an elementary lemma on subgroups of the symplectic group.

1. Mumford-Tate groups in families

Let $f : X \to S$ be a family of smooth projective varieties parametrised by a smooth connected variety $S$. For some positive integer $k$ let $V = R^idf_*\mathbb{Q}^X$ denote the variation of pure Hodge structures of weight $k$ on $S$. More generally we can consider any variation $V$ of Hodge structures of weight $k$ on $S$.

Let $V_{a,b} = V \otimes a \otimes V^* \otimes b$ be the associated tensor variations of pure Hodge structures of weight $(a-b)k$. For every $(a, b)$ such that $(a-b)k = 2p$ is even, we have the nested sequence of analytic subvarieties

$$H_{a,b} := V_{a,b}^Z \cap F^pV_{a,b} \subset V_{a,b}^Z \subset V_{a,b}^C$$

of the complex vector bundle $V_{a,b}^Z$ over $S$ associated with $V^a,b$. The analytic variety $H_{a,b}$ parametrises pairs $(s, c)$, where $s$ is a point of $S$ and $c$ an integral class of type $(p, p)$ in $V_{a,b}^s$, i.e. $c$ is a Hodge cycle.

If $W$ is an irreducible component of $H_{a,b}$ such that the natural map $W \to S$ is open at some point, then $W$ contains an open subset of $V_{a,b}^Z$, hence $W$ is a connected component of $V_{a,b}^Z$. Let $A_{a,b}$ be the union of all such components. The map $A_{a,b} \to S$ makes each component of the former a covering space of $S$.

Now, if $W$ is an irreducible component of $H_{a,b}$ for which the map $W \to S$ is not open at any point then its image in $S$ is a set of measure zero by Sard’s theorem. Let $B$ be the (countable) union of these images as we vary over all the components of $H_{a,b}$ and as we vary $a$ and $b$.

If $s$ is any point of $S$ which is not in $B$, then by the above reasoning, the only points of $H_{a,b}$ that lie over it are in $A_{a,b}$. Let $t$ be any other point of $S$ and $\gamma$ be a path in $S$ connecting $s$ and $t$. We can use $\gamma$ to identify $V_{a,b}^Z,s$ with $V_{a,b}^Z,t$; this then gives an identification of $A_{a,b}^s$ with $A_{a,b}^t$. Hence, under this identification, the collection of Hodge cycles in $V_{a,b}^Z,s$ is contained in the collection of Hodge cycles in $V_{a,b}^Z,t$. Thus the Mumford-Tate group $G_t$ of $V_t$ is identified by $\gamma$ with a subgroup of the Mumford-Tate group $G_s$ of $V_s$. In other words, we have
The Mumford-Tate group at a general point contains (a conjugate of) the Mumford-Tate group at a special point in a variation of Hodge structures over a smooth connected variety.

2. DEGENERATE COVERS

A connected projective curve which has at worst ordinary double points as its singularities is called a semi-stable curve. The dual graph of such a curve has as its vertices the irreducible components; each singular point gives an edge incident on the two vertices corresponding to the components that contain it. We will be interested in semi-stable curves whose dual graph is contractible and hence a tree; such curves are called tree-like. In this case, the first cohomology of the curve is a direct sum of the first cohomology of its components with the induced (pure) Hodge structure. In particular, the Jacobian of a tree-like semi-stable curve is the product of the Jacobians of its components.

A finite morphism \( C \to D \) of semi-stable curves is called a semi-stable cover (or an admissible cover) if

1. This is a topological cover of constant degree of \( D \) outside a finite set of points which includes the singular locus of \( D \).
2. The inverse image of a singular point of \( D \) consists of singular points of \( C \).
3. The order of ramification on the two branches at a singular point of \( C \) must be equal.

This notion was first defined by Beauville [1] for the case of degree two covers (which are the case of interest) and later generalised (see [4]). In these papers, it is shown that the deformations of such a semi-stable cover of tree-like curves are unobstructed. In other words, there is a smooth (open) curve \( S \), a flat morphism \( p : \mathcal{D} \to S \) and a finite flat morphism \( f : \mathcal{C} \to \mathcal{D} \). There is a point \( o \) of \( S \) over which the \( f \) restricts to the given semi-stable cover \( C \to D \). Moreover, the general fibre is a double cover \( C' \to D' \) of a smooth curve \( D' \) by a smooth curve \( C' \).

We are interested in the case of degree two covers \( C \to D \); here the singular points of \( C \) are either unramified on each branch or ramified of order two on each branch. Let us further assume that \( C \) and \( D \) are tree-like. For each component \( D_i \) of \( D \) there are two possibilities:

1. There is exactly one component \( C_i \) of \( C \) that lies over it. The map \( C_i \to D_i \) is a double cover in the usual sense.
2. There are two components \( C'_i \) and \( C''_i \) of \( C \) that lie over \( D_i \) and the given map is an isomorphism between these components and \( D_i \).
The Prym variety can be defined as before as the identity component of the kernel of the natural norm homomorphism between the Jacobians $J(C) \to J(D)$. It follows that the Prym variety is the product of the Prym varieties of the covers $C_i \to D_i$ corresponding to the first case and the Jacobians of the curves $D_i$ corresponding to the second case. In particular, the product of these components gives an abelian variety. Hence we have

The family of Prym varieties can be extended to include the Prym varieties of degenerate tree-like covers. In particular, the Mumford-Tate group of a general Prym variety contains (a conjugate of) the Mumford-Tate group of the Prym variety of any degenerate tree-like cover.

In the special case when $D$ has two exactly components (call them $D_1$ and $D_2$), such a cover can be constructed in one of two ways:

I Let $C_1 \to D_1$ be a double cover that is not branched at the common point $p = D_1 \cap D_2$. Then, $C$ is obtained by attaching to $C_1$ two copies of $D_2$, one at each point lying over $p$.

II Let $C_1 \to D_1$ and $C_2 \to D_2$ be double covers that are both branched at the common point $p$. We obtain $C$ by attaching the curves $C_1$ and $C_2$ along their respective ramification points lying over $p$.

The specific covers that we are interested in are the following.

1. A covering of type (II) which is the degeneration of a double cover $C \to D$ where $D$ is rational and $C$ is of genus $g$. The curves $D_i$ are smooth rational curves. The curve $C_1$ is an elliptic curve and the curve $C_2$ is any hyperelliptic curve of genus $g - 1$.

2. A covering of type (I) which is the degeneration of a double cover $C \to D$ where $D$ has genus at least 2 and the cover is étale. The curve $D_1$ is any elliptic curve, $C_1 \to D_1$ is an étale double cover and $D_2$ is any curve of genus one less than that of $D$.

3. A covering of type (I) which is the degeneration of a double cover $C \to D$ where $D$ has genus at least 1 and the cover is ramified at some point. The curve $D_1$ is any curve of genus one less than that of $D$ and $C_1 \to D_1$ is a double cover ramified at the same number of points as the cover $C \to D$; $D_2$ is any elliptic curve.

As a result we have

**Lemma 1.** We have containments of Mumford-Tate groups as enumerated below.

1. The Mumford-Tate group of a general hyperelliptic curve of genus $g$ contains a conjugate of the product of the Mumford-Tate group
of any elliptic curve with the Mumford-Tate group of any hyper-

elliptic curve of genus $g - 1$.

2. The Mumford-Tate group of the Prym variety of a general étale
cover of a curve of genus $g \geq 2$ contains a conjugate of the
Mumford-Tate group of any curve of genus $g - 1$.

3. The Mumford-Tate group of the Prym variety of a general cover of
a curve of genus $\geq 1$ ramified at $r \geq 1$ points contains a conjugate
of the product of the Mumford-Tate group of any elliptic curve
with the Mumford-Tate group of the Prym variety of any cover of
a curve of genus $g - 1$ which is ramified at $r$ points.

Proof. The first cohomology group of the product of two abelian vari-
eties is the direct sum of the first cohomology groups of the individual
abelian varieties. Moreover, the Hodge cycles on the individual vari-
eties pull-back to give Hodge cycles on the product. Thus it follows
that the Mumford-Tate group of the product contains the product of
the Mumford-Tate groups. The result now follows from the above con-
structions.

3. The Main result

To prove the main result we need the following three lemmas.

Lemma 2 (Pirola). The Néron-Severi group of a general Prym variety
is the free group on 1 generator.

This lemma is proved in [8]. We note that this case includes the case
of a general hyperelliptic curve.

Lemma 3. Let $G$ be a connected semi-simple subgroup of the symplec-
tic group $Sp(2n)$ which contains (a conjugate of) the product $Sp(2a) \times
Sp(2n - 2a)$, then $G$ is either this product or it is the the full symplectic
group

Proof. Let $V$ be the standard representation of $Sp(2n)$. Let $\oplus_{i \in I} W_i$
be its decomposition into isotypical components as a representation of
$G$. Let $V = V_1 \oplus V_2$ be the decomposition of $V$ as a representation of
$Sp(2a) \times Sp(2n - 2a)$. Then each $W_i$ is either $V_1$ or $V_2$ or $V = V_1 \oplus V_2$.
The result follows by dimension counting.

The lemma also follows from the fact that the quotient

$$\frac{sp(2n)}{sp(2a) \times sp(2n - 2a)}$$

of the Lie algebras is an irreducible module over $Sp(2a) \times Sp(2n - 2a)$.
Lemma 4. Let $A$ be an abelian variety of dimension $n$ whose Mumford-Tate group is the product $Sp(2a) \times Sp(2n-2a)$ in $Sp(2n)$. Then the Néron-Severi group of $A$ is of rank at least 2.

Proof. The first cohomology group of $A$ decomposes as a direct sum of two (polarised) sub-Hodge structures. It follows that $A$ is the product of two abelian subvarieties. Hence we have the result. \hfill \Box

Theorem 5. The Mumford-Tate group of a general Prym variety is the full symplectic group.

Proof. We begin with the case where the base curve has genus zero. In this case the Prym varieties are the Jacobians of the corresponding hyperelliptic double cover. The result is classical for elliptic curves which can be considered as the Prym varieties associated with double covers of smooth rational curves branched at 4 points. By induction, let us assume that the result is known for hyperelliptic Jacobians of genus less than $g \geq 2$. The lemma then shows that the Mumford-Tate group of a general hyperelliptic curve of genus $g$ contains $Sp(2) \times Sp(2g-2)$. By the above results we see that thus Mumford-Tate group must be either $Sp(2g)$ or $Sp(2) \times Sp(2g-2)$. In the latter case, the Néron-Severi group of the curve would have rank at least two but by Pirola’s result we know that this is not true for the general hyperelliptic curve. Hence we see that the Mumford-Tate group of a general hyperelliptic curve must be $Sp(2g)$ where $g$ is the genus of the curve.

Now let us consider the case where the cover is unramified. Then we may assume that the base curve of the double cover has genus $g$ at least 2 (else the Prym variety is just a point). In this case the Prym variety has dimension $n = g - 1$. By the lemma we know that the Mumford-Tate group contains the Mumford-Tate group of any curve of genus $g - 1$. In particular, it contains the Mumford-Tate group of an hyperelliptic curve of genus $g - 1$ and hence by the previous paragraph it contains (and is thus equal to) $Sp(2n)$.

Now assume that the base curve of the double cover has genus $g$ at least 1 and the cover is ramified. We argue by induction on the genus of the base curve. We can begin the induction since we already know the result for the hyperelliptic curves. Let us assume that the result is known for base curves of genus less than $g$. By the lemma we know that the Mumford-Tate group contains the product of the Mumford-Tate group of an elliptic curve with the Mumford-Tate group of the Prym variety of a the double of a curve of genus $g - 1$; in other words it contains $Sp(2) \times Sp(2n-2)$ by induction. Now as argued above,
the three lemmas above imply that the Mumford-Tate group must be \(Sp(2n)\). 

**References**

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