Multivalued Elliptic Equation with exponential critical growth in $\mathbb{R}^2$ *

Claudianor O. Alves† and Jefferson A. Santos

Abstract

In this work we study the existence of nontrivial solution for the following class of multivalued elliptic problems

$$-\Delta u + V(x)u - \epsilon h(x) \in \partial_t F(x, u) \quad \text{in} \quad \mathbb{R}^2, \quad (P)$$

where $\epsilon > 0$, $V$ is a continuous function verifying some conditions, $h \in (H^1(\mathbb{R}^2))^*$ and $\partial_t F(x, u)$ is a generalized gradient of $F(x, t)$ with respect to $t$ and $F(x, t) = \int_0^t f(x, s) \, ds$. Assuming that $f$ has an exponential critical growth and a discontinuity point, we have applied Variational Methods for locally Lipschitz functional to get two solutions for $(P)$ when $\epsilon$ is small enough.

2000 AMS Subject Classification: 35A15, 35J25, 34A36.

Key words and phrases: exponential critical growth, discontinuous nonlinearity.

1 Introduction

The interest in the study of nonlinear partial differential equations with discontinuous nonlinearities has increased because many free boundary problems arising in mathematical physics may be stated in this form. Among these problems, we have the obstacle problem, the seepage surface problem, and the Elenbaas equation; see for example [15, 16, 17].

Among the typical examples, we have chosen the model for the heat conductivity in electrical media. This model has a discontinuity in its constitutive laws. In fact, considering a domain $\Omega \subset \mathbb{R}^2$ (which in particular could be taken as being the whole space $\mathbb{R}^2$, see [10]) with electrical media, the thermal and electrical conductivity are denoted by $K(x, t)$ and $\sigma(x, t)$, respectively. Here $x$ is in $\Omega$ and $t$ represents the temperature. Since we are considering an electrical media, the function $\sigma$ may have discontinuities in $t$,

---

*Partially supported by Procad and Casadinho CNPQ
†Partially supported by CNPq - Grant 304036/2013-7
and the distribution of the temperature is unknown. The differential equation describing this distribution is

\[- \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( K(x, u(x)) \frac{\partial u(x)}{\partial x_i} \right) = \sigma(x, u(x)).\]

Note that this equation is related to a free boundary problem in which the jump surface of the electrical conductivity is unknown. We describe this surface as being the set

\[\Gamma_{\alpha}(u) = \{ x \in \Omega : u(x) = \alpha, \sigma \text{ is discontinuous at } \alpha \}.\]  

(1.1)

When the thermal conductivity \( K \) is constant, \( \Omega = \mathbb{R}^2 \) and the electrical conductivity \( \sigma \) is given by

\[\sigma(x, t) = H(t - a)f(t) + \epsilon h(x) - V(x)t,\]

with \( f \) having an exponential critical growth, the model becomes

\[- \Delta u + V(x)u = H(u - a)f(u) + \epsilon h(x) \quad \text{in } \mathbb{R}^2.\]  

(1.2)

Here \( H \) is the Heaviside function, that is,

\[H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0, \end{cases}\]

\( \epsilon \) is a positive parameter and \( h \) is a measurable function defined in \( \mathbb{R}^2 \). Note that in this model the jump surface of the solution (1.1) is represented by the set

\[\Gamma_{a}(u) = \{ x \in \mathbb{R}^2 : u(x) = a \}.\]  

(1.3)

Related to Problem (1.2) for the special case of \( a = 0 \), i.e., without jump discontinuities, we cite the works by de Freitas [24] and do ´O, de Medeiros and Severo [28, 29].

A rich literature is available by now on problems with discontinuous nonlinearities, and we refer the reader to Alves, Bertone and Gonçalves [4], Alves and Bertone [5], Alves, Gonçalves and Santos [6], Ambrosetti and Turner [9], Ambrosetti, Calahorrano and Dobarro [10], Badiale and Tarantelo [11], Carl, Le and Motreanu [13], Clarke [14], Chang [15], Carl and Dietrich [18], Carl and S. Heikkila [19, 20], Cerami [21], de Souza, de Medeiros and Severo [25, 26], Hu, Kourogenis and Papageorgiou [33], Montreanu and Vargas [35], Radulescu [37] and their references. Several techniques have been developed or applied in their study, such as variational methods for nondifferentiable functionals, lower and upper solutions, global branching, fixed point theorem, and the theory of multivalued mappings.

After a bibliography review, we did not find any paper involving existence of solution for a class of elliptic problem with discontinuous nonlinearity and exponential critical growth via variational methods for nondifferentiable
functional. Motivated by this fact, in this paper we employ variational techniques to study existence and multiplicity of nonnegative solutions for a large class of multivalued elliptic equations, which includes the equation (1.2). More precisely, we will study the multivalued elliptic equation

$$-\Delta u + V(x)u - \epsilon h(x) \in \partial_t F(x, u) \quad \text{in} \quad \mathbb{R}^2,$$

where $\epsilon > 0$ is a positive parameter, $V$ is a continuous function verifying some technical conditions, $h \in (H^1(\mathbb{R}^2))^*$ and $F(x, t)$ is the primitive of a function $f(x, t)$, which has an exponential critical growth and a discontinuity point, for more details see Section 2.

In $\mathbb{R}^2$, to apply variational methods, the natural growth restriction on the function $f$ is given by the inequality of Trudinger and Moser [34, 38]. More precisely, we say that a function $f$ has an exponential critical growth if there is $\alpha_0 > 0$ such that

$$\lim \frac{|f(s)|}{e^{\alpha s^2}} = 0 \quad \forall \alpha > \alpha_0$$

and

$$\lim \frac{|f(s)|}{e^{\alpha s^2}} = +\infty \quad \forall \alpha < \alpha_0.$$

We would like to mention that problems involving exponential critical growth, with $f$ being a continuous functions, have received a special attention at last years, see for example, [2, 7, 8, 12, 22, 23, 30, 31] and their references. Here, since we intend to get a solution for the differential inclusion (P), we assume that there exists $\alpha_0 > 0$ such that

$$(f_0) \quad \limsup_{t \to +\infty} \frac{\max \{|\xi|; \xi \in \partial_t F(x, t)\}}{e^{\alpha_0 t^2}} < +\infty \text{ uniformly in } x \in \mathbb{R}^2.$$ 

Moreover, assuming a condition at origin like

$$(f_1) \quad \limsup_{t \to 0} \frac{2\max \{|\xi|; \xi \in \partial_t F(x, t)\}}{|t|} < +\infty \text{ uniformly in } x \in \mathbb{R}^2.$$ 

it is easy to check that the functional $\Psi : H^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$\Psi(u) = \int_{\mathbb{R}^2} F(x, u) \, dx$$

is well defined, for more details see Section 4. However, to apply variational methods is better to consider the functional $\Psi$ in a more appropriated domain, that is, $\Psi : L^p(\mathbb{R}^2) \to \mathbb{R}$, for $\Phi(t) = e^{t|t|^2} - 1$. But, once $\Phi$ does not satisfy the $\Delta_2$-condition, we cannot guarantee that given $J \in (L^p(\mathbb{R}^2))^*$, then

$$J(u) = \int_{\mathbb{R}^2} vu \, dx, \quad \forall u \in L^p(\mathbb{R}^2),$$

for some $v : \mathbb{R}^2 \to \mathbb{R}$ mensurable function. For the familiar readers with the study of the differential inclusions, they will observe that the above remark is
bad to apply variational methods, because in general for this type of equations we need to prove that the inclusion below holds

$$\partial \Psi(u) \subset \partial F(x,u) = [\underline{f}(x,u(x)), \overline{f}(x,u(x))]$$

a.e. in \( \mathbb{R}^2 \),

where

$$\underline{f}(x,t) = \lim_{r \downarrow 0} \text{ess inf} \{ f(x,s) : |s-t| < r \}$$

and

$$\overline{f}(x,t) = \lim_{r \downarrow 0} \text{ess sup} \{ f(x,s) : |s-t| < r \}.$$

In Section 4, we analyze this question. In fact, we show that it is enough to consider \( \Psi : E_\Phi(\mathbb{R}^2) \to \mathbb{R} \) where

$$E_\Phi(\Omega) = C^\infty_0(\mathbb{R}^2) \left\| \Phi \right\|.$$

Before to state our main result, we must mention our conditions on \( V, h \) and \( f \), which are the following:

\( (h_0) \) \( h \in (H^1(\mathbb{R}^2))^* \) and \( 0 < \int_{\mathbb{R}^2} h \, dx < +\infty \).

\( (V_1) \) \( V \) is continuous and \( V(x) \geq V_0 > 0, \forall x \in \mathbb{R}^2 \),

\( (V_2) \) \( \frac{1}{V} \in L^1(\mathbb{R}^2) \).

\( (f_2) \) There is \( t_0 \geq 0 \) such that

$$f(x,t) = 0 \quad \text{for} \quad t < t_0 \quad \text{and} \quad \forall x \in \mathbb{R}^2$$

and

$$f(x,t) > 0 \quad \text{for} \quad t > t_0 \quad \text{and} \quad \forall x \in \mathbb{R}^2.$$

\( (f_3) \) \( \limsup_{t \to 0} \frac{2 \max \{ |\xi| : \xi \in \partial F(x,t) \}}{|t|} < \lambda_1 \) uniformly in \( x \in \mathbb{R}^2 \), where

$$\lambda_1 = \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2) \, dx}{\int_{\mathbb{R}^2} |u|^2 \, dx}$$

and

$$E := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)u^2 \, dx < +\infty \right\}.$$

\( (f_4) \) There is a compact set \( K \subset \mathbb{R}^2 \) and constants \( c_3, c_4 > 0 \) and \( \nu > 2 \), such that

$$F(x,t) \geq c_3 t^\nu - c_4, \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \forall x \in K.$$

\( (f_5) \) There is \( \tau > 2 \) verifying

$$0 \leq \tau F(x,t) \leq f(x,t)t, \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad \forall x \in \mathbb{R}^2.$$
(f_6) There are \( p > 2 \) and \( \mu > 0 \) such that

\[
F(x, t) \geq \mu(t - t_0)^p, \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad \forall x \in \mathbb{R}^2.
\]

Here, we would like to point out that the function

\[
f(t) = 2H(t - a)te^{t^2}, \quad \forall t \in \mathbb{R},
\]

verifies \((f_1) - (f_6)\).

Now, we are able to state our main result

**Theorem 1.1** Assume \((V_1) - (V_2)\), \((h_0)\), \((f_0), (f_2) - (f_6)\). Then, there are \( \epsilon_0, \mu^* \) and \( t_1 > 0 \), such that problem \((P)\) possesses a solution \( v_\epsilon \in E \), with \( I_\epsilon(v_\epsilon) = d_\epsilon > 0 \), for all \( \epsilon \in (0, \epsilon_0) \), \( t_0 \in [0, t_1) \) and \( \mu \geq \mu^* \). Moreover, decreasing \( \epsilon_0 \) and \( t_1 \), and increasing \( \mu^* \), if necessary, we have two solutions \( u_\epsilon, v_\epsilon \in E \) with \( I_\epsilon(u_\epsilon) = c_\epsilon < 0 < d_\epsilon = I_\epsilon(v_\epsilon) \).

In the proof of Theorem 1.1, we use variational methods for nondifferentiable functional. A solution is obtained by applying Ekeland’s variational principle, while the other one is obtained by using Mountain Pass Theorem. Here, we would like point out that by applying the above theorem for the function \( f \) given in (1.4), we find two solutions \( u_1, u_2 \in H^1(\mathbb{R}^2) \) for the equation

\[
-\Delta u = 2H(u - a)ue^{u^2} + ch, \quad \text{in} \quad \mathbb{R}^2,
\]

with

\[
||u_i = a|| = 0 \quad \text{for} \quad i = 1, 2.
\]

**Notation:** In this paper, we use the following notations:

- The usual norms in \( L^t(\mathbb{R}^2) \) and \( H^1(\mathbb{R}^2) \) will be denoted by \( |.|_t \) and \( || \cdot || \) respectively.
- \( C \) denotes (possible different) any positive constant.
- \( B_R(z) \) denotes the open ball with center at \( z \) and radius \( R \).
- If \( B \subset \mathbb{R}^2 \) is a measurable set, let us denote by \( |B| \) the Lebesgue’s measure of \( B \).
- \( \Phi \) denotes the N-function \( \Phi(t) = e^{|t|^2} - 1 \).
2 Technical results involving the exponential critical growth

In this section, we will prove some technical lemmas, which are crucial in our approach. Since we will work with exponential critical growth, some versions of the Trudinger-Moser inequality are very important in our arguments. The first version that we would like to recall is due to Trudinger and Moser, see [34] and [38], which claims if $\Omega$ is a bounded domain with smooth boundary, then for any $u \in H^1_0(\Omega)$,

$$
\int_{\Omega} e^{\alpha|u|^2} \, dx < +\infty, \quad \text{for every } \alpha > 0.
$$

(2.1)

Moreover, there exists a positive constant $C = C(\alpha, |\Omega|)$ such that

$$
\sup_{\|u\|_{H^1_0(\Omega)} \leq 1} \int_{\Omega} e^{\alpha|u|^2} \, dx \leq C, \quad \forall \alpha \leq 4\pi.
$$

(2.2)

A version in $H^1(\Omega)$ has been proved by Adimurthi and Yadava [3], and it says that if $\Omega$ is a bounded domain with smooth boundary, then for any $u \in H^1(\Omega)$,

$$
\int_{\Omega} e^{\alpha|u|^2} \, dx < +\infty, \quad \text{for every } \alpha > 0.
$$

(2.3)

Furthermore, there exists a positive constant $C = C(\alpha, |\Omega|)$ such that

$$
\sup_{\|u\|_{H^1(\Omega)} \leq 1} \int_{\Omega} e^{\alpha|u|^2} \, dx \leq C, \quad \forall \alpha \leq 2\pi.
$$

(2.4)

The third version that we will use is due to Cao [12], which is version of the Trudinger-Moser inequality in whole space $\mathbb{R}^2$ and has the following statement:

$$
\int_{\mathbb{R}^2} \left( e^{\alpha|u|^2} - 1 \right) \, dx < +\infty, \quad \text{for all } u \in H^1(\mathbb{R}^2) \text{ and } \alpha > 0.
$$

(2.5)

Besides, given $\alpha < 4\pi$ and $M > 0$, there is a constant $C_1 = C_1(M, \alpha) > 0$ verifying

$$
\sup_{u \in B_M} \int_{\mathbb{R}^2} \left( e^{\alpha|u|^2} - 1 \right) \, dx \leq C_1
$$

(2.6)

where

$$
B_M = \{ u \in H^1(\mathbb{R}^2) : |\nabla u|_2 \leq 1 \text{ and } |u|_2 \leq M \}.
$$

As a consequence from (2.5)-(2.6), we are able to prove some technical lemmas. The first of them is crucial in the study of the $(PS)$ condition for $I_\epsilon$.

Lemma 2.1 Let $\alpha > 0$ and $(u_n)$ be a sequence in $H^1(\mathbb{R}^2)$ with

$$
\limsup_{n \to +\infty} \|u_n\| < \sqrt{\frac{4\pi}{\alpha}}.
$$
Then, there exist \( t > 1, t \) close to 1, and \( C > 0 \) satisfying
\[
\int_{\mathbb{R}^N} \left( e^{\alpha |u_n|^2} - 1 \right)^t \, dx \leq \mathcal{C}, \quad \forall \, n \in \mathbb{N}.
\]

**Proof.** As
\[
\limsup_{n \to \infty} \|u_n\| < \sqrt{\frac{4\pi}{\alpha}},
\]
there are \( m > 0 \) and \( n_0 \in \mathbb{N} \) verifying
\[
\|u_n\|^2 < m < \frac{4\pi}{\alpha}, \quad \forall \, n \geq n_0.
\]
Fix \( t > 1, \) with \( t \) close to 1, and \( \beta > t \) satisfying \( \beta m < \frac{4\pi}{\alpha} \). Then, there exists \( C = C(\beta) > 0 \) such that
\[
\int_{\mathbb{R}^2} \left( e^{\alpha |u_n|^2} - 1 \right)^t \, dx \leq C \int_{\mathbb{R}^2} \left( e^{\alpha \beta m \left( \frac{|u_n|}{\|u_n\|} \right)^2} - 1 \right) \, dx,
\]
for every \( n \geq n_0 \). Hence, by (2.6),
\[
\int_{\mathbb{R}^2} \left( e^{\alpha |u_n|^2} - 1 \right)^t \, dx \leq C_1 \quad \forall \, n \geq n_0,
\]
for some positive constant \( C_1 \). Now, the lemma follows fixing
\[
C = \max \left\{ C_1, \int_{\mathbb{R}^2} \left( e^{\alpha |u_n|^2} - 1 \right)^t \, dx, \ldots, \int_{\mathbb{R}^2} \left( e^{\alpha |u_{n_0}|^2} - 1 \right)^t \, dx \right\}.
\]

**Lemma 2.2** Let \( \beta, M > 0 \) verifying \( \beta M < 4\pi \) and \( q > 2 \). If \( \|u\|^2 \leq M \), then there is \( C = C(\beta, M, q) > 0 \) such that
\[
\int_{\mathbb{R}^2} |u|^q \left( e^{\beta |u|^2} - 1 \right) \, dx \leq C(\beta) \|u\|^q.
\]

**Proof.** In what follows, fix \( t > 1 \) close to 1, such that \( \alpha = t \beta M < 4\pi \). Then, there is a constant \( C > 0 \) such that
\[
\int_{\mathbb{R}^2} \left( e^{\alpha |u|^2} - 1 \right)^t \, dx \leq C \int_{\mathbb{R}^2} \left( e^{\alpha |u|^2} - 1 \right) \, dx.
\]
Note that
\[
\int_{\mathbb{R}^2} \left( e^{\beta |u|^2} - 1 \right) \, dx = \int_{\mathbb{R}^2} \left( e^{\beta |u|} \left( \frac{|u|}{\|u\|} \right)^2 - 1 \right) \, dx \leq \int_{\mathbb{R}^2} \left( e^{\beta M \left( \frac{|u|}{\|u\|} \right)^2} - 1 \right) \, dx.
\]
Thereby, by (2.6),
\[
\int_{\mathbb{R}^2} \left( e^{\beta |u|^2} - 1 \right) \, dx \leq \int_{\mathbb{R}^2} \left( e^{\alpha (\frac{|u|}{\|u\|})^2} - 1 \right) \, dx \leq \sup_{\|v\| \leq 1} \int_{\mathbb{R}^2} \left( e^{\alpha |v|^2} - 1 \right) \, dx = C < +\infty.
\]
From this, the function \( \zeta_u = e^{\beta |u|^2} - 1 \in L^4(\mathbb{R}^2) \) and there is \( C = C(\beta, M) > 0 \) such that
\[
|\zeta_u|_t \leq C, \quad \forall u \in B_M = \{ u \in H^1(\mathbb{R}^2) : \|u\| \leq M \}.
\] (2.7)

Then, by applying the Hölder’s inequality,
\[
\int_{\mathbb{R}^2} |u|^q \left( e^{\beta |u|^2} - 1 \right) dx = \int_{\mathbb{R}^2} |u|^q \zeta_u dx \leq |\zeta_u|_t |u|_t^{q_t}
\]
where \( \frac{1}{t} + \frac{1}{t'} = 1 \). Hence, by Sobolev embedding, there is \( C > 0 \) such that
\[
\int_{\mathbb{R}^2} |u|^q \left( e^{\beta |u|^2} - 1 \right) dx \leq C |\zeta_u|_t \|u\|^q. \] (2.8)

Now, the lemma follows combining (2.7) and (2.8). 

### 3 Preliminaries about Orlicz spaces

In this section, we recall some properties of Orlicz and Orlicz-Sobolev spaces. We refer to [1, 27, 32, 36] for the fundamental properties of these spaces. First of all, we recall that a continuous function \( A : \mathbb{R} \to [0, +\infty) \) is a N-function if:

1. \( A \) is convex.
2. \( A(t) = 0 \iff t = 0. \)
3. \( \lim_{t \to 0} \frac{A(t)}{t} = 0 \) and \( \lim_{t \to +\infty} \frac{A(t)}{t} = +\infty. \)
4. \( A \) is even.

We say that a N-function \( A \) verifies the \( \Delta_2 \)-condition, denote by \( A \in \Delta_2 \), if
\[
A(2t) \leq K_* A(t) \quad \forall t \geq 0,
\]
for some constant \( K_* > 0 \).

The complementary function (or conjugate function) \( \tilde{A} \) associated with \( A \) is given by the Legendre’s transformation, that is,
\[
\tilde{A}(s) = \max_{t \geq 0} \{ st - A(t) \} \quad \text{for} \quad s \geq 0.
\]

The functions \( A \) and \( \tilde{A} \) are complementary each other. Moreover, we also have a Young type inequality given by
\[
st \leq A(t) + \tilde{A}(s) \quad \forall s, t \geq 0. \] (3.1)

In what follows, fixed an open set \( \Omega \subset \mathbb{R}^N \) and a N-function \( A \), we define the Orlicz space associated with \( A \) as
\[
L^A(\Omega) = \left\{ u \in L^1_{loc}(\Omega) : \int_{\Omega} A\left(\frac{|u|}{\lambda}\right) dx < +\infty \quad \text{for some} \quad \lambda > 0 \right\}.
\]
The space $L^A(\Omega)$ is a Banach space endowed with Luxemburg norm given by

$$\|u\|_A = \inf \left\{ \lambda > 0 : \int_\Omega A\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$ 

The convexity of $A$ implies in the inequality below, which will be used later on

$$\|u\|_A \leq 1 \iff \int_\Omega A(|u|) dx \leq 1.$$ \hspace{1cm} (3.2)

Using the inequality (3.1), it is possible to prove a Hölder type inequality, that is,

$$\left| \int_\Omega uv dx \right| \leq 2\|u\|_A \|v\|_{\tilde{A}} \quad \forall u \in L^A(\Omega) \quad \text{and} \quad \forall v \in L^{\tilde{A}}(\Omega).$$

The space $L^A(\Omega)$ is separable and reflexive when $A$ and $\tilde{A}$ satisfy the $\Delta_2$-condition. Moreover, the $\Delta_2$-condition implies that

$$u_n \to u \text{ in } L^A(\Omega) \iff \int_\Omega A(|u_n - u|) dx \to 0.$$  

3.1 The class $K_A(\Omega)$ and the subspace $E_A(\Omega)$

In the study of the Orlicz space $L^A(\Omega)$, we denote by $K_A(\Omega)$ the following set

$$K_A(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega A(|u|) dx \leq 1 \right\}$$

and let us denote by $E_A(\Omega) \subset L^A(\Omega)$ the following subspace

$$E_A(\Omega) = \overline{L^\infty(\Omega)}^{\| \cdot \|_A} \quad \text{if} \quad |\Omega| < +\infty \quad \text{is bounded}$$

or

$$E_A(\Omega) = \overline{C_0^\infty(\Omega)}^{\| \cdot \|_A} \quad \text{if} \quad |\Omega| = +\infty \quad \text{is unbounded.}$$

Using the above notations, it follows that

$$E_A(\Omega) \subset K_A(\Omega) \subset L^A(\Omega),$$

and

$$K_A(\Omega) \subset \{ u \in L^A(\Omega) : \text{dist}(u, E_A(\Omega)) \leq 1 \}. \hspace{1cm} (3.3)$$

It is possible to prove that if $A$ verifies $\Delta_2$-condition, then

$$E_A(\Omega) = K_A(\Omega) = L^A(\Omega).$$

However, if $A$ does not satisfies the $\Delta_2$-condition, we have that $E_A(\Omega)$ is a proper subspace of $L^A(\Omega)$. For example, this situation holds for the N-function
\(\Phi(t) = e^{t^2} - 1\), because it does not verify the \(\Delta_2\)-condition. Moreover, \(L^\Phi(\Omega)^*\) is not reflexive, hence we cannot guarantee that if \(J_0 \in (L^\Phi(\Omega))^*\), then

\[
J_0(u) = \int_{\mathbb{R}^2} vu \, dx, \forall u \in L^\Phi(\mathbb{R}^2),
\]

for some measurable function \(v : \mathbb{R}^2 \to \mathbb{R}\). However, this type of problem does not hold in \((E^\Phi(\Omega))^*\), because if \(J_1 \in (E^\Phi(\Omega))^*\) we know that there exists \(v \in L^\tilde{\Phi}(\mathbb{R}^2)\) such that

\[
J_1(u) = \int_{\mathbb{R}^2} vu \, dx, \forall u \in L^\Phi(\mathbb{R}^2).
\]

**Lemma 3.1** Let \(\xi(t) = \max\{t, t^2\}\) and \(\tilde{\Phi}\) the conjugate function associated with \(\Phi\). Then,

\[
\tilde{\Phi}\left(\frac{\Phi(r)}{r}\right) \leq \Phi(r) \quad \text{and} \quad \tilde{\Phi}(tr) \leq \xi(t)\tilde{\Phi}(r), \; t, r \geq 0.
\]

Hence, \(\tilde{\Phi} \in \Delta_2\), \(E^\tilde{\Phi}(\mathbb{R}^2) = L^\tilde{\Phi}(\mathbb{R}^2)\) and \(L^\tilde{\Phi}(\mathbb{R}^2)\) is separable.

**Proof.** The first inequality follows from [32]. To prove the second one, we recall that

\[
2 \leq \frac{\Phi'(t)t}{\Phi(t)}, \; t \in (0, +\infty).
\]

Fix \(s > 0\), such that \(t = \tilde{\Phi}'(s)\). Since \(\tilde{\Phi}' = (\Phi')^{-1}\) and \(s\tilde{\Phi}'(s) = \tilde{\Phi}(s) + \Phi(\tilde{\Phi}'(s))\), we derive that

\[
2 \leq \frac{\Phi'(\tilde{\Phi}'(s))\tilde{\Phi}'(s)}{\Phi(\tilde{\Phi}'(s))} = \frac{s\tilde{\Phi}'(s)}{s\Phi'(s) - \Phi(s)}
\]

and so,

\[
2s\tilde{\Phi}'(s) - 2\Phi(s) \leq s\tilde{\Phi}'(s),
\]

that is

\[
\frac{s\tilde{\Phi}'(s)}{\Phi(s)} \leq 2.
\]

Now, fixing \(s = \rho r > 0\), we get

\[
\frac{d}{d\sigma} \left(\ln \left(\tilde{\Phi}(\rho r)\right)\right) \leq \frac{2}{\sigma}.
\]

From this,

\[
\tilde{\Phi}(tr) \leq t^2\tilde{\Phi}(r), \; t \geq 1 \text{ and } r \geq 0.
\]

On the other hand, the convexity of \(\tilde{\Phi}\) combines with \(\tilde{\Phi}(0) = 0\) to give

\[
1 \leq \frac{\tilde{\Phi}'(t)t}{\Phi(t)}, \; t \in (0, +\infty).
\]
Using again (3.2), we see that
\[ \tilde{\Phi}(tr) \leq t \tilde{\Phi}(r), \ t \in (0,1] \text{ and } r \geq 0. \] (3.5)

Hence, from (3.4) and (3.5),
\[ \tilde{\Phi}(tr) \leq \xi(t) \tilde{\Phi}(r), \ t, r \geq 0. \]

Now, the conclusion follows from [32].

**Lemma 3.2** Let \( X = H^1_0(\Omega) \) or \( X = H^1(\Omega) \), where \( \Omega \subset \mathbb{R}^2 \) is a smooth bounded domain or \( \Omega = \mathbb{R}^2 \). Then, the embedding \( X \hookrightarrow E\Phi(\Omega) \) is continuous.

**Proof.** From (2.2), (2.4) and (2.6), we have that the embedding \( X \hookrightarrow L\Phi(\Omega) \) is continuous. Then, the lemma follows by demonstrating the inclusion \( X \subset E\Phi(\Omega) \). First of all, by (2.1), (2.3) and (2.5), we know that
\[ \int_{\Omega} (e^{\lambda |u|^2} - 1) dx < +\infty, \ \forall \lambda \geq 0 \text{ and } \forall u \in X, \]
implying that
\[ X \subset K\Phi(\Omega). \] (3.6)

Assume by contradiction that there is \( u_0 \in X \) with \( u_0 \notin E\Phi(\Omega) \). Since \( E\Phi(\Omega) \) is a closed subspace in \( L^\Phi(\Omega) \), we ensure that \( \text{dist}(u_0, E\Phi(\Omega)) > 0 \). Thereby, for
\[ \lambda > \frac{1}{\text{dist}(u_0, E\Phi(\Omega))}, \]
we have that
\[ \text{dist}(\lambda u_0, E\Phi(\Omega)) = \inf \{ ||\lambda u_0 - v||_\Phi; v \in E\Phi(\Omega) \} \]
\[ = \lambda \inf \{ ||u_0 - \frac{v}{\lambda}||_\Phi; v \in E\Phi(\Omega) \} \]
\[ = \lambda \text{dist}(u_0, E\Phi(\Omega)) > 1. \]

Then, by (3.3), \( \lambda u_0 \notin K\Phi(\Omega) \), which contradicts (3.6), because \( \lambda u_0 \in X \). ■

**Lemma 3.3** The embeddings \( E\Phi(\Omega) \hookrightarrow L^{2n}(\Omega) \) are continuous for any \( n \in \mathbb{N} \).

**Proof.** For each \( n \in \mathbb{N}^* \), we know that
\[ \frac{1}{n!} t^{2n} \leq \sum_{k=0}^{+\infty} \frac{1}{k!} t^{2k} = e^{t^2} - 1, \ \forall t \in \mathbb{R}. \]

Then, for each \( u \in E\Phi(\Omega) \)
\[ \frac{1}{n!} \int_{\Omega} \left( \frac{u}{|u|_\Phi} \right)^{2n} dx \leq \int_{\Omega} \left( e^{\left( \frac{|u|_\Phi}{n} \right)^2} - 1 \right) dx \leq 1, \]
leading to
\[ |u|^{2n}_{2n} = \int_{\Omega} |u|^{2n} \leq n! |u|^{2n}_{\Phi}, \]
showing the lemma. ■
4 Some properties of the functional $\Psi$

Let $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be a measurable function for each $t \in \mathbb{R}$ and Locally Lipschitzian for each $x \in \mathbb{R}^2$ verifying:

($f_2$) There is $t_0 \geq 0$ such that

$$f(x, t) = 0 \text{ for } t < t_0 \text{ and } \forall x \in \mathbb{R}^2$$

and

$$f(x, t) > 0 \text{ for } t > t_0 \text{ and } \forall x \in \mathbb{R}^2.$$

($f_*)$ There are $\alpha_0, c_1, c_2 > 0$ and $\alpha_0 > 0$ such that

$$|\xi| \leq c_1|u| + c_2e^{\alpha_0}|u|^2, \ \forall \xi \in \partial F(x, u) \ \text{ and } \forall x \in \mathbb{R}^2,$$

where $F(x, t) = \int_0^t f(x, s)ds$.

**Theorem 4.1** The functional $\Psi : E_\Phi(\Omega) \to \mathbb{R}$ given by

$$\Psi(u) = \int_{\Omega} F(x, u) dx$$

is well defined and $\Psi \in \text{Lip}_{loc}(E_\Phi(\Omega), \mathbb{R})$.

**Proof.** For each $u \in E_\Phi(\Omega)$ and $R > 0$, consider $w, v \in B_R(u) \subset E_\Phi(\Omega)$. By Lebourg’s Theorem, there is $\xi \in \partial F_1(x, \theta)$ with $\theta \in [w, v]$ such that

$$|F(x, w) - F(x, v)| = |\langle \xi, w - v \rangle| \leq |\xi||w - v|.$$

Then by ($f_*$),

$$|F(x, w) - F(x, v)| \leq \left( c_1|\theta| + c_2(e^{\alpha_0}|\theta|^2 - 1) \right) |w - v|,$$

for $\alpha > \alpha_0$ and $\alpha$ close to $\alpha_0$. Setting $\eta(x) = |v(x)| + |w(x)|$, it follows that

$$|F(x, w) - F(x, v)| \leq \left( c_1|\eta| + c_2(e^{\alpha_0}|\eta|^2 - 1) \right) |w - v|,$$

and so,

$$|\Psi(w) - \Psi(v)| \leq \int_{\Omega} \left( c_1|\eta| + c_2(e^{\alpha_0}|\eta|^2 - 1) \right) |w - v| \ dx.$$

By Hölder’s inequality,

$$|\Psi(w) - \Psi(v)| \leq c_1|\eta|_2 |w - v|_2 + c_2 \left( \int_{\Omega} (e^{2\alpha_0}|\eta|^2 - 1) dx \right)^{\frac{1}{2}} |w - v|_2.$$

Once $E_\Phi(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, see Lemma 3.3, we derive that

$$|\Psi(w) - \Psi(v)| \leq c_1(|w - u|_\Phi + |u - v|_\Phi + 2|u|_\Phi) |w - v|_\Phi$$

$$+ c_2 \left( \int_{\Omega} (e^{2\alpha_0}|\eta|^2 - 1) dx \right)^{\frac{1}{2}} |w - v|_\Phi.$$

(4.1)
On the other hand, the convexity of $\Phi$ yields

$$
\int_{\Omega} \left( e^{2\alpha(|w|+|v|)^2} - 1 \right) dx \leq \frac{1}{4} \int_{\Omega} \left( e^{32\alpha|w-u|^2} - 1 \right) dx + \frac{1}{4} \int_{\Omega} \left( e^{32\alpha|v-u|^2} - 1 \right) dx
+ \frac{1}{2} \int_{\Omega} \left( e^{32\alpha|u|^2} - 1 \right) dx.
$$

(4.2)

Now, fixing $R > 0$ verifying $R < \frac{1}{\sqrt{32\alpha_0}}$ and $\alpha$ close to $\alpha_0$ satisfying $R < \frac{1}{\sqrt{32\alpha}}$, we derive that

$$
|\sqrt{32\alpha}(w-u)|_\Phi \leq \sqrt{32\alpha} R \leq 1 \quad \text{and} \quad |\sqrt{32\alpha}(v-u)|_\Phi \leq \sqrt{32\alpha} R \leq 1,
$$

and so,

$$
\int_{\Omega} \left( e^{32\alpha|w-u|^2} - 1 \right) dx \leq 1 \quad \text{and} \quad \int_{\Omega} \left( e^{32\alpha|v-u|^2} - 1 \right) dx \leq 1,
$$

(4.3)

for all $w, v \in B_R(u)$. From (4.2)-(4.3)

$$
\int_{\Omega} \left( e^{2\alpha(|w|+|v|)^2} - 1 \right) dx \leq \frac{1}{4} \left( 2 + 2 \int_{\Omega} \left( e^{32\alpha|u|^2} - 1 \right) dx \right).
$$

(4.4)

Thereby, gathering (4.1) and (4.4),

$$
|\Psi(w) - \Psi(v)| \leq c_1 (R + R + 2|u|_\Phi) |w - v|_\Phi
+ c_2 \frac{1}{4} \left( 2 + 2 \int_{\Omega} \left( e^{32\alpha|u|^2} - 1 \right) dx \right)^{\frac{1}{2}} |w - v|_\Phi
= K(R,u)|w - v|_\Phi, \forall w, v \in B_R(u).
$$

Our next goal is proving the differentiable inclusion

$$
\partial \Psi(u) \subset \int_{\Omega} \partial_t F(x,u) dx, \ u \in E_\Phi(\Omega).
$$

(4.5)

To do this, we need of the following result

**Lemma 4.1** Let $\psi : \mathbb{R} \to \mathbb{R}_+$ be a N-function and

$$
g_n \to g \text{ in } E_\psi(\Omega).
$$

Then, there is $\hat{g} \in E_\psi(\Omega)$ and a subsequence of $\{g_n\}$, denoted by $\{g_{m_k}\}$, such that

(i) $g_{m_k}(x) \to g(x)$ a.e. $x \in \Omega,$

(ii) $|g_{m_k}(x)| \leq \hat{g}(x)$ a.e. $x \in \Omega.$
Proof. As
\[ |g_m - g|_\psi \to 0, \]
we have that
\[ \int_\Omega \psi(g_m - g)dx \leq |g_m - g|_\psi \to 0, \]
implying that there is a subsequence of \( \{g_n\} \), denoted by \( \{g_{m_k}\} \), such that
\[ \psi(g_{m_k} - g)(x) \to 0 \text{ a.e. in } \Omega, \]
and so,
\[ (g_{m_k} - g)(x) = \psi^{-1} \circ \psi(|g_{m_k} - g|)(x) \to 0 \text{ a.e. in } \Omega, \]
that is,
\[ g_{m_k}(x) \to g(x) \text{ a.e. in } \Omega, \]
Now, define
\[ \zeta_m = \sum_{k=1}^{m} |g_{n_{k+1}} - g_{n_k}| \in E_\psi(\Omega), \]
with
\[ |g_{n_{k+1}} - g_{n_k}|_\psi < \frac{1}{2^k}, \quad \forall k \in \mathbb{N}. \]
Hereafter, \( g_k \) denotes \( g_{n_k} \), that is, \( g_k := g_{n_k} \). For \( n \leq m \),
\[ |\zeta_m - \zeta_n|_\psi \leq \sum_{k=n}^{m} |g_{k+1} - g_k|_\psi \leq \sum_{k=n}^{m} \frac{1}{2^k} \to 0, \]
from if follows that \( \{\xi_m\} \subset E_\psi(\Omega) \) is a Cauchy’s sequence in \( E_\psi(\Omega) \). Once
\( E_\psi(\Omega) \) is a Banach space, there exists \( \xi \in E_\psi(\Omega) \) such that
\[ \zeta_m \to \xi \text{ in } E_\psi(\Omega). \]
Then
\[ \int_\Omega \psi(\xi_m - \xi)dx \leq |\zeta_m - \xi|_\psi \to 0, \]
and so,
\[ \zeta_{m_k}(x) \to \xi(x) \text{ a.e. in } \Omega, \]
and
\[ \zeta_{m_k}(x) \leq \xi(x) \text{ a.e. in } \Omega, \quad k \in \mathbb{N}. \]
On the other hand, for \( n \leq m \),
\[ |g_m - g_n|(x) \leq \xi_m(x) \leq \xi(x) \text{ a.e. in } \Omega. \]
Setting \( \hat{g} = \xi + |g| \in E_\psi(\Omega) \) and taking the \( n \to +\infty \), we get
\[ |g_m(x)| \leq \hat{g}(x) \text{ a.e. in } \Omega \quad \forall m \in \mathbb{N}, \]
showing \((ii)\).
Applying Lemma 4.1, there exists \( g \) for some subsequence. Thereby, by (4.8) and (4.9), there exists a subsequence

Moreover,

\[ \partial \Psi(u) \subset \partial_t F(x, u) = [f(x, u(x)), \overline{f}(x, u(x))] \text{ a.e. in } \Omega. \]  

where \( X = H^1_0(\Omega) \) or \( X = H^1(\Omega) \). Here, the above inclusion means that given \( \xi \in \partial \Psi(u) \subset E_\Phi(\Omega)^* \), there is \( \xi \in L^\Phi(\Omega) \) satisfying

- \( (\xi, v) = \int_\Omega \xi v dx, \quad \forall v \in E_\Phi(\Omega), \)
- \( \xi(x) \in \partial F(x, u) = [f(x, u(x)), \overline{f}(x, u(x))] \text{ a.e. in } \Omega. \)

**Proof.** Given \( u, v \in E_\Phi(\Omega) \), let \( \{g_j\} \subset E_\Phi(\Omega) \) with \( g_j \to 0 \) in \( E_\Phi(\Omega) \) and \( \{\lambda_j\} \subset \mathbb{R}^+ \) with \( \lambda_j \to 0 \) verifying

\[
\Psi^0(u; v) = \lim_{j \to +\infty} \int_\Omega \frac{F(u + g_j + \lambda_j v) - F(u + g_j)}{\lambda_j} dx. \tag{4.7}
\]

Setting

\[
F_j(u; v) := \frac{F(u + g_j + \lambda_j v) - F(u + g_j)}{\lambda_j},
\]

the Lebourg’s Theorem guarantees that there is \( \xi_j \in \partial_t F(x, \theta_j) \), with \( \theta_j \in [u + g_j + \lambda_j v, u + g_j] \) such that

\[
|F_j(u; v)| = \frac{1}{\lambda_j} |(\xi_j, \lambda_j v)| \leq |\xi_j||v|.
\]

Hence by \((f_*)\),

\[
|F_j(u; v)| \leq \left(c_1|\theta_j| + c_2(e^{\alpha|\theta_j|^2} - 1)\right)|v|,
\]

for \( \alpha > \alpha_0 \) and \( \alpha \) close to \( \alpha_0 \). Fixing

\[
\beta_j = (|u| + |g_j| + \lambda_j|v|) + (|u| + |g_j|) = 2|u| + 2|g_j| + \lambda_j|v|,
\]

we see that

\[
|F_j(u; v)| \leq \left(c_1|\beta_j| + c_2(e^{\alpha|\beta_j|^2} - 1)\right)|v|. \tag{4.8}
\]

Applying Lemma 4.1, there exists \( g_* \in E_\Phi(\Omega) \) such that

\[
|\beta_j| \leq 2|u| + 2g_* + c|v| \text{ a.e. in } \Omega, \tag{4.9}
\]

for some subsequence. Thereby, by (4.8) and (4.9), there exists a subsequence \( \{F_{j_k}(u; v)\} \) such that

\[
|F_{j_k}(u; v)| \leq \left(c_1(2|u| + 2g_* + c|v|) + c_2(e^{\alpha(2|u| + 2g_* + c|v|)^2} - 1)\right)|v| \in L^1(\Omega).
\]
Applying the Lebesgue’s Theorem,

\[
\Psi^0(u; v) = \lim_{j_k \to +\infty} \int_{\Omega} F_{j_k}(u; v) \, dx = \int_{\Omega} \lim_{j_k \to +\infty} F_{j_k}(u; v) \, dx
\]

\[
\leq \int_{\Omega} F^0(u; v) \, dx = \int_{\Omega} \max\{\langle \xi, v \rangle; \xi \in \partial_t F(x, u)\} \, dx
\]

\[
\leq \int_{\{v < 0\}} f(x, u(x)) \, v \, dx + \int_{\{v > 0\}} \mathcal{F}(x, u) \, v \, dx.
\]

(4.10)

Now, we will show that for each \(\xi \in \partial \Psi(u) \subset (E_{\Phi}(\Omega))^*\), the function \(\hat{\xi} \in L^\Phi(\Omega)\), which satisfies

\[
\langle \xi, w \rangle = \int_{\Omega} \hat{\xi} w \, dx, \quad \forall w \in E_{\Phi}(\Omega),
\]

must verify

\[
\hat{\xi}(x) \in [f(x, u(x)), \mathcal{F}(x, u(x))] \text{ a.e. in } \Omega.
\]

Indeed, assume by contradiction that there is a measurable set \(M \subset \Omega\), with \(0 < |M| < +\infty\), satisfying

\[
\hat{\xi}(x) < f(x, u(x)), \quad x \in M.
\]

(4.11)

Setting \(v = -\chi_M \in E_{\Phi}(\Omega)\), we must have

\[
- \int_M \hat{\xi} \, dx = \int_{\Omega} \hat{\xi} (-\chi_M) \, dx \leq \Psi^0(u, -\chi_M) \leq - \int_M f(x, u(x)) \, dx,
\]

leading to

\[
\int_{\Omega} \hat{\xi} \chi_M \, dx \geq \int_M f(x, u(x)) \, dx,
\]

which contradicts (4.11). Thereby,

\[
\hat{\xi}(x) \geq f(x, u(x)) \text{ a.e. in } \Omega.
\]

The same type of arguments work to show that

\[
\hat{\xi}(x) \leq \mathcal{F}(x, u(x)) \text{ a.e. in } \Omega.
\]

From definition of \(X\), we know that \(X \|_{\Phi} = E_{\Phi}(\Omega)\), then the Lemma combined with chain rule gives

\[
\partial \Psi|_X(u) \subset \partial \Psi(u), \quad \forall u \in X.
\]
5 An application

In this section, we will study the existence of solution for the following class of multivalued elliptic equation

\[-\Delta u + V(x)u - \epsilon h(x) \in \partial_t F(x, u), \quad \text{in} \quad \mathbb{R}^2, \quad (P)\]

where

- \( h \in H^{-1} \), that is, the functional \( \langle h, v \rangle = \int_{\mathbb{R}^2} h v dx \) is continuous in \( H^1(\mathbb{R}^2) \)
  and \( 0 < \int_{\mathbb{R}^2} h \, dx < +\infty \).

- \( F(x, t) = \int_0^t f(x, s) \, ds \), for \((x, t) \in \mathbb{R}^2 \times \mathbb{R}\), where \( f \) verifies \((f_0)\) and \((f_1)\).

Related to the potential \( V : \mathbb{R}^2 \to \mathbb{R} \), we assume that

\((V_1)\) \( V \) is continuous and \( V(x) \geq V_0 > 0 \), \( \forall x \in \mathbb{R}^2 \);

\((V_2)\) \( V \in L^1(\mathbb{R}^2) \).

In order to apply variational methods, we will consider the Hilbert space

\[ E := \left\{ u \in H^1(\mathbb{R}^2) / \int_{\mathbb{R}^2} V(x) |u|^2 dx < +\infty \right\} \]

endowed with the inner product

\[ \langle u, v \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv) \, dx. \]

Associated with the above inner product, we have the norm

\[ \| u \| = \left( \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x) |u|^2) \, dx \right)^{\frac{1}{2}}. \]

using the above information, it is well known that

\((E_1)\) \( E \hookrightarrow L^q(\mathbb{R}^2) \) is a compact embedding for all \( q \geq 1 \), see \[28, 29\]

\((E_2)\) \( E \hookrightarrow H^1(\mathbb{R}^2) \hookrightarrow E_{\Phi}(\Omega) \) is a continuous embedding (see Lemma \[3.2\]).

In the present paper, we say that \( u \in E \) is a solution for \((P)\), if there is \( \rho \in L^1(\mathbb{R}^2) \) such that

\[(i)\] \( \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv) dx - \int_{\mathbb{R}^2} \rho v dx - \epsilon \int_{\mathbb{R}^2} hv dx = 0, \quad v \in E, \)

\[(ii)\] \( \rho(x) \in \partial_t F(x, u(x)) \quad \text{a.e. in} \quad \mathbb{R}^2, \)
The reader is invited to observe that \( u \in E \) is a solution for \((P)\) if, and only if, \( u \) is a critical point of the energy functional associated with \((P)\) given by:

\[
I_\epsilon(u) = \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2)dx - \int_{\mathbb{R}^2} F(x, u)dx - \epsilon \int_{\mathbb{R}^2} hvdx, \ u \in E.
\]

Note that Theorem 4.1 gives that \( I_\epsilon \in \text{Lip}_{\text{loc}}(E; \mathbb{R}) \). From this, using some properties of the generalizity gradient together with Theorem 4.2, given \( u \in E \) and \( w \in \partial I_\epsilon(u) \), there exists \( \rho \in L_{\tilde{\Phi}}(\mathbb{R}^2) \) such that

\[
\langle w, v \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv)dx - \int_{\mathbb{R}^2} \rho vdx - \epsilon \int_{\mathbb{R}^2} hvdx \quad \forall v \in E,
\]

with

\[
\rho(x) \in \partial_t F(x, u(x)) \quad \text{a.e. in } \mathbb{R}^2.
\]

6 Existence of solution via Ekeland’s variational principle

In this section, we will get a solution via Ekeland’s variational principle.

**Lemma 6.1** Assume that \((f_0)\) and \((f_2)-(f_4)\) hold. Then, there are \( \epsilon_0, r, \alpha, \delta > 0 \), such that

\[
c_\epsilon = \inf_{\|u\| \leq r} I_\epsilon(u) < -\delta
\]

and

\[
I_\epsilon(u) \geq \alpha \quad \text{for } \|u\| = r
\]

for all \( \epsilon \in (0, \epsilon_0) \). Here, \( r \) is independent of \( \epsilon \), but \( \alpha \) and \( \delta \) depend on \( \epsilon \). Moreover, the numbers \( \epsilon_0, r, \alpha \) and \( \delta \) do not depend on \( t_0 \) given in \((f_2)\).

**Proof.** Using the conditions on \( F \), given \( \beta \in (0, \lambda_1) \), \( q > 2 \) and \( \alpha > \alpha_0 \) close to \( \alpha_0 \), we have that

\[
F(x, t) \leq \frac{(\lambda_1 - \beta)}{2} |t|^2 + C |t|^q (e^{\alpha |t|^2} - 1), \quad \forall t \in \mathbb{R}.
\]

Then, fixing \( r > 0 \) small enough such that \( \alpha r^2 < 4\pi \) and using Lemma 2.2, we get for \( u \in E \) with \( \|u\| \leq r \),

\[
I_\epsilon(u) \geq \frac{1}{2} \|u\|^2 - \frac{(\lambda_1 - \beta)}{2} |u|^2 - C \|u\|^q - \epsilon \|h\|_\infty \|u\|
= \frac{1}{2} \left( 1 - \frac{(\lambda_1 - \beta)}{\lambda_1} \right) \|u\|^2 - C \|u\|^q - \epsilon \|h\|_\infty \|u\|,
\]
showing that $I_\epsilon$ is bounded from below for $\|u\| \leq r$. Moreover, decreasing $r$ if necessary of a way that
\[
\frac{1}{2}r^2 - Crq \geq \frac{1}{4}r^2,
\]
we derive that
\[
I_\epsilon(u) \geq \frac{1}{4}r^2 - \epsilon \|h\| r, \quad \|u\| = r.
\]
Thereby, choosing $\epsilon_0 > 0$ such that
\[
\alpha \epsilon_0 = \frac{1}{4}r^2 - \epsilon \|h\| r > 0, \quad \forall \epsilon \in (0, \epsilon_0),
\]
we see that
\[
I_\epsilon(u) \geq \alpha_\epsilon \quad \text{for} \quad \|u\| = r, \quad \forall \epsilon \in (0, \epsilon_0).
\]
Now, take $v \in E$ satisfying
\[
\|v\| = 1 \quad \text{and} \quad \int_{\mathbb{R}^2} hv \, dx > 0.
\]
Note that for each $s > 0$,
\[
I_\epsilon(sv) = \frac{s^2}{2} - \int_{\mathbb{R}^2} F(x, sv) \, dx - \epsilon s \int_{\mathbb{R}^2} hv \, dx < \frac{s^2}{2} - \epsilon s \int_{\mathbb{R}^2} hv \, dx.
\]
Fixing $s = s(\epsilon) > 0$ small enough satisfying
\[
\delta = -\frac{s^2}{2} + \epsilon s \int_{\mathbb{R}^2} hv \, dx > 0,
\]
it follows that $\|sv\| < r$ and
\[
I_\epsilon(sv) < -\delta < 0,
\]
implying that
\[
c_\epsilon = \inf_{\|u\| \leq r} I_\epsilon(u) < -\delta < 0.
\]
\begin{flushright}
\[\blacksquare\]
\end{flushright}

**Theorem 6.1** Assume $(V_1), (V_2), (f_0), (f_2)$ and $(f_3)$. Then, problem $(P)$ possesses a solution $u_\epsilon \in E$, with $I_\epsilon(u_\epsilon) = c_\epsilon < -\delta < 0$, for all $\epsilon \in (0, \epsilon_0)$ and $t_0 \in [0, t_\ast)$, with $t_\ast = t_\ast(\epsilon) = \frac{2\delta \epsilon}{\int_{\mathbb{R}^2} hv \, dx} > 0$.

**Proof.** Fix $r > 0$ such that $\alpha_0 r^2 < 4\pi$. Applying the Lemma 6.1 together with Ekeland’s variational principle, there is $\{u_n\} \subset B_r(0)$ verifying

- $I_\epsilon(u_n) \to c_\epsilon$ (as $n \to +\infty$),
- $\lambda_\epsilon(u_n) := \min\{\|\xi\|_{E^*} / \xi \in \partial I_\epsilon(u_n)\} \to 0$ (as $n \to +\infty$).
Next, we fix \( w_n \in \partial I_c(u_n) \) and \( \{\rho_n\} \subset L_\Phi(\mathbb{R}^2) \) verifying
\[
\| w_n \|_{E^*} := \lambda_c(u_n)
\]
\[
\langle w_n, v \rangle = \int_{\mathbb{R}^2} \nabla u_n \nabla v + V(x)u_nvdx - \int_{\mathbb{R}^2} \rho_nvdx - \epsilon \int_{\mathbb{R}^2} hvdx, \quad \forall v \in E, \quad (6.1)
\]
and
\[
\rho_n(x) \in \partial F(x, u_n(x)) \quad \text{a.e. in } \mathbb{R}^2.
\]
We claim that \( \{\rho_n\} \) is bounded in \( L_\Phi(\mathbb{R}^2) \). Indeed, fixing \( p > 4, \alpha > \alpha_0 \) with \( \alpha r^2 < 4\pi \), and using \((f_n)\), we get
\[
\int_{\mathbb{R}^2} \tilde{\Phi}(\rho_n)dx \leq \int_{\mathbb{R}^2} \tilde{\Phi}(c_1|u_n| + c_2|u_n|^p(e^\alpha|u_n|^2 - 1)) dx.
\]
The convexity of \( \tilde{\Phi} \) and the \( \Delta_2 \)-condition combine to give
\[
\int_{\mathbb{R}^2} \tilde{\Phi}(\rho_n)dx \leq \frac{\xi(2c_1)}{2} \int_{\mathbb{R}^2} \tilde{\Phi}(|u_n|)dx + \frac{\xi(2c_2)}{2} \int_{\mathbb{R}^2} \tilde{\Phi}(|u_n|^p(e^\alpha|u_n|^2 - 1)) dx.
\]
By Lemma 3.1 and \((E_1)\), there are positive constants \( C_1, C_2 \) such that
\[
\int_{\mathbb{R}^2} \tilde{\Phi}(\rho_n)dx \leq C_1(|u_n| + |u_n|^2) + C_2 \int_{\mathbb{R}^2} (|u_n|^{p+1} + |u_n|^{2(p+1)}) (e^{\alpha|u_n|^2} - 1) dx.
\]
Recalling that the space \( E \) is continuously embedding in \( L^1(\mathbb{R}^2) \) and \( L^2(\mathbb{R}^2) \), and \( \alpha r^2 < 4\pi \), the Lemma 2.2 yields there is \( C_3 > 0 \) verifying
\[
\int_{\mathbb{R}^2} \tilde{\Phi}(\rho_n)dx \leq C_3, \quad \forall n \in \mathbb{N},
\]
showing that \( \{\rho_n\} \) is a bounded sequence in \( L_\Phi(\mathbb{R}^2) \). From this, the sequence of functionals \( \{\tilde{\rho}_n\} \subset \partial \Psi(u_n) \subset (E_\Phi(\mathbb{R}^2))^* \) associated with \( \{\rho_n\} \) is also bounded in \((E_\Phi(\mathbb{R}^2))^*\), and so, there is \( \tilde{\rho}_0 \in (E_\Phi(\mathbb{R}^2))^* \), such that \( \tilde{\rho}_n \rightharpoonup \tilde{\rho}_0 \) in \((E_\Phi(\mathbb{R}^2))^*\) for some subsequence, that is,
\[
\int_{\mathbb{R}^2} \rho_nvdx = \langle \tilde{\rho}_n, v \rangle \rightarrow \langle \tilde{\rho}_0, v \rangle = \int_{\mathbb{R}^2} \rho_0vdx, \quad \forall v \in E, \quad (6.2)
\]
for some \( \rho_0 \in L_\Phi(\mathbb{R}^2) \).

Now, using the fact that \( \{u_n\} \) is also bounded in \( E \), there is \( u_\epsilon \in E \) such that
\[
u_n \rightharpoonup u_\epsilon \text{ in } E.
\]
From \((6.1)\) and \((6.3)\)
\[
0 = \int_{\mathbb{R}^2} \nabla u_\epsilon \nabla v + V(x)u_\epsilon vdx - \int_{\mathbb{R}^2} \rho_0vdx - \epsilon \int_{\mathbb{R}^2} hvdx, \quad v \in E. \quad (6.4)
\]
To conclude the proof that \( u_\epsilon \) is a solution of \((P)\), we must prove that
\( i \) \( \rho_0(x) \in \partial_t F(x, u_\varepsilon(x)) \) a.e. in \( \mathbb{R}^2 \) and

\( ii \) \( \| u_\varepsilon > t_0 \| > 0. \)

To prove the \( i \), we must show that \( \{ u_n \} \) is strongly convergent to \( u_\varepsilon \) in \( E \), because this fact will imply that \( \rho_0 \in \partial \Psi(u_0) \). This way, by Theorem 4.2

\[ \rho_0(x) \in \partial_t F(x, u_\varepsilon(x)) \ \text{a.e. in} \ \mathbb{R}^2. \]

Related to the second item, the proof is as follows: If \( t_0 = 0 \), then \( \| u_\varepsilon > t_0 \| > 0 \), because \( u_\varepsilon \geq 0 \) and \( u_\varepsilon \neq 0 \). Next, we will consider the case \( t_0 \in (0, t_*) \). Once \( \rho_0, u_\varepsilon \geq 0 \), it follows that

\[ 0 = \| u_\varepsilon \|^2 - \int_{\mathbb{R}^2} \rho_0 u_\varepsilon dx - \epsilon \int_{\mathbb{R}^2} h u_\varepsilon dx \]

\[ \leq \| u_\varepsilon \|^2 - \epsilon \int_{\mathbb{R}^2} h u_\varepsilon dx, \]

that is,

\[ \| u_\varepsilon \|^2 \geq \epsilon \int_{\mathbb{R}^2} h u_\varepsilon dx. \quad (6.5) \]

Arguing by contradiction, we assume that \( \| u_\varepsilon > t_0 \| = 0 \), for some \( t_0 \in (0, t_*) \). Thereby,

\[ f(x, u_\varepsilon(x)) = 0, \ \text{a.e. in} \ \mathbb{R}^2, \]

from where it follows that

\[ \partial_t F(x, u_\varepsilon(x)) = \{ 0 \} \ \text{a.e. in} \ \mathbb{R}^2. \]

Consequently,

\[ \rho_0(x) = 0 \ \text{a.e. in} \ \mathbb{R}^2. \]

On the other hand, by Lemma 6.1 and (6.5),

\[ 0 > -\delta > I_\varepsilon(u_\varepsilon) = \frac{1}{2} \| u_\varepsilon \|^2 - \epsilon \int_{\mathbb{R}^2} h u_\varepsilon dx \geq -\frac{1}{2} \epsilon \int_{\mathbb{R}^2} h u_\varepsilon dx \geq - \frac{t_0}{2} \epsilon \int_{\mathbb{R}^2} h dx, \]

implying that

\[ t_0 \geq \frac{2\delta}{\epsilon \int_{\mathbb{R}^2} h dx} = t_*, \]

which is a contradiction.

**Convergence of \( \{ u_n \} \) to \( u_\varepsilon \) in \( E \):** Hereafter, fix \( \gamma_n := u_n - u_0 \) and recall that \( \gamma_n \rightarrow 0 \) in \( E \). By a direct computation,

\[ \| u_n \|^2 = \| u_\varepsilon \|^2 + \| \gamma_n \|^2 + o_n(1). \]
Moreover, we also have
\[ o_n(1) = \langle w_n, u_n \rangle = \| u_n \|^2 - \int_{\mathbb{R}^2} \rho_n u_n dx - \epsilon \int_{\mathbb{R}^2} h u_n dx - \| u_\epsilon \|^2 + \int_{\mathbb{R}^2} \rho_0 u_\epsilon dx + \epsilon \int_{\mathbb{R}^2} h u_\epsilon dx \]
\[ = \| \gamma_n \|^2 + \left( \int_{\mathbb{R}^2} \rho_n u_n dx - \int_{\mathbb{R}^2} \rho_n u_\epsilon dx \right) + o_n(1) \]
\[ + \left( \int_{\mathbb{R}^2} \rho_n u_\epsilon dx - \int_{\mathbb{R}^2} \rho_n u_n dx \right) + o_n(1) \]
\[ = \| \gamma_n \|^2 - \int_{\mathbb{R}^2} \rho_n \gamma_n dx + o_n(1). \quad (6.6) \]

On the other hand, by \((f_1)\),
\[ \left| \int_{\mathbb{R}^2} \rho_n \gamma_n dx \right| \leq c_1 \int_{\mathbb{R}^2} |u_n| \| \gamma_n \| dx + c_2 \int_{\mathbb{R}^2} |\gamma_n| \left( e^{\alpha |u_n|^2} - 1 \right) dx \]
\[ \leq c_1 |u_n|^{\frac{2}{2}} |\gamma_n|^{\frac{2}{2}} + c_2 \int_{\mathbb{R}^2} |\gamma_n| \left( e^{\alpha |\gamma_n|^2} - 1 \right) dx. \]

Once \(\alpha r^2 < 4\pi\), there is \(q > 1\) close to 1, such that
\[ M = \sup_{n \in \mathbb{N}} \left( \int_{\mathbb{R}^2} \left( e^{\alpha |\gamma_n|^2} - 1 \right)^q dx \right)^{\frac{1}{q}} < +\infty. \]

Thus, by Lemma 2.2 and Hölder inequality
\[ \left| \int_{\mathbb{R}^2} \rho_n \gamma_n dx \right| \leq c_1 |u_n|^{\frac{2}{2}} |\gamma_n|^{\frac{2}{2}} + C |\gamma_n|_{q'}, \]
where \(\frac{1}{q} + \frac{1}{q'} = 1\). Since the embeddings \(E \hookrightarrow L^2(\mathbb{R}^2)\) and \(E \hookrightarrow L^{q'}(\mathbb{R}^2)\) are compact, we can ensure that
\[ \int_{\mathbb{R}^2} \rho_n \gamma_n dx \to 0. \quad (6.7) \]

From (6.6) and (6.7), \(\gamma_n \to 0\) in \(E\), or equivalently \(u_n \to u_\epsilon\) in \(E\), finishing the proof.

7 Existence of solution via Mountain Pass

In this section, we will assume more some conditions on function \(f\), namely \((f_0), (f_2) - (f_0)\). By \((f_3)\), there are \(\bar{\epsilon}, \delta := \tilde{\delta} > 0\), satisfying
\[ 2 \max \{ |\xi|; \xi \in \partial_t F(x, t) \} < (\lambda_1 - \bar{\epsilon}) |t|, \text{ for } |t| \leq \tilde{\delta} \text{ and } x \in \mathbb{R}^2. \]
From Lebourg's Theorem, there are $\theta(t) \in [0, t]$, with $|t| \leq \tilde{\delta}$ and $\xi_0 \in \partial F_t(x, \theta)$ verifying
\[
|F(x, t)| = |F(x, t) - F(x, 0)| = |\xi_0| |t - 0| \leq (\lambda_1 - \tilde{\theta})|t|, \quad x \in \mathbb{R}^2.
\] (7.1)

Now, by $(f_0)$, given $q \geq 2$ and $\alpha > \alpha_0$, there is $C = C(q, \tilde{\delta}) > 0$ such that
\[
|\xi| \leq C|t|^{(q-1)} \left(e^{\alpha|t|^2} - 1\right), \quad \xi \in \partial_t F(x, t), \quad |t| \geq \tilde{\delta} \text{ and } x \in \mathbb{R}^2.
\]

Applying again Lebourg's Theorem
\[
|F(x, t)| \leq C|t|^q \left(e^{\alpha|t|^2} - 1\right), \quad \forall t \in \mathbb{R} \text{ and } \forall x \in \mathbb{R}^2.
\]

From this, for $u \in E$ with $u \neq 0$ and $\|u\| := \eta_0 < \sqrt{\frac{4}{\alpha_0}}$, we see that
\[
\int_{\mathbb{R}^2} |F(x, u)| dx \leq C(q, \tilde{\delta}, \alpha_0) \|u\|^q.
\] (7.2)

Here, we have fixed $\alpha$ close to $\alpha_0$ of the $a$ way that $\alpha \eta_0^2 < 4\pi$.

**Lemma 7.1** Assume that $(f_0)$ and $(f_2) - (f_6)$ hold. Then, there exists $\varphi_0 \in B_r^c(0)$ such that
\[
I_\epsilon(\varphi_0) < \inf_{\|u\|=\pi} I_\epsilon(u), \quad \epsilon \in (0, \epsilon_0],
\]
where $r$ and $\epsilon_0$ are given in Lemma 6.1.

**Proof.** Let $\psi_0 \in C^\infty_0(\mathbb{R}^2) \setminus \{0\}$, $\psi_0 > 0$, with $\text{supp}(\psi_0) \subset K$, where $K \subset \mathbb{R}^2$ is the compact set fixed in $(f_\epsilon)$. In this case, for any $\epsilon > 0$,
\[
I_\epsilon(t\psi_0) \leq \frac{t^2}{2} \|\psi_0\|^2 - c_3 t^\nu \int_{\mathbb{R}^2} \psi_0' \| + c_4 |K| - t \epsilon \int_{\mathbb{R}^2} h \psi_0 dx,
\]
from where it follows that
\[
\lim_{t \to +\infty} I_\epsilon(t\psi_0) = -\infty.
\]

Thus, the lemma follows choosing $\varphi_0 := t\psi_0 \in B_r^c(0)$ with $t$ large enough. 

From Lemmas 6.1 and 7.1 we can use the Mountain Pass Theorem to get a sequence $\{v_n\} \subset E$ verifying
\[
I_\epsilon(v_n) \to d_\epsilon \text{ in } \mathbb{R} \text{ and } \lambda_\epsilon(v_n) := \max\{\|\xi\|_\ast / \xi \in \partial I_\epsilon(v_n)\} \to 0, \quad \text{ (7.3)}
\]
where
\[
d_\epsilon := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} L_\epsilon(\gamma(t)) \quad \text{(mountain pass level)}
\]
and
\[
\Gamma := \{\gamma \in C([0, 1]; E) \mid \gamma(0) = 0 \text{ and } \gamma(1) = \varphi_0\}.
\]

In the sequel, we intend to show that $I_\epsilon$ verifies the $(PS)_{d_\epsilon}$ condition if the parameter $\mu$ given in $(f_6)$ is large enough. To this end, we need of the following lemma.
Lemma 7.2 Let \( \{v_n\} \) be the sequence obtained in (7.3). Then, \( \{v_n\} \) is bounded in \( E \) and
\[
\limsup_{n \to \infty} \| v_n \| \leq \frac{(\tau - 1) \epsilon + \sqrt{\epsilon^2 \left( \frac{\tau - 1}{\tau} \right)^2 + 2d_\epsilon \left( \frac{\tau - 2}{2\tau} \right)}}{2 \left( \frac{\tau - 2}{2\tau} \right)},
\]
where \( \tau \) is given in (f5).

Proof. Let \( w_n \in E^* \) and \( \rho_n \in \partial \Psi(v_n) \) verifying
\[
\| w_n \| = \lambda_\epsilon(v_n) \quad \text{and} \quad \langle w_n, v_n \rangle = \| v_n \|^2 - \int_{\mathbb{R}^2} \rho_n v_n dx - \epsilon \int_{\mathbb{R}^2} hv_n dx.
\]
From (f5) and Theorem 4.2,
\[
d_\epsilon + o_n(1) + o_n(1) \| v_n \| \geq I_\epsilon(v_n) - \frac{1}{\tau} \langle w_n, u_n \rangle
\]
\[
= \left( \frac{1}{2} - \frac{1}{\tau} \right) \| v_n \|^2 + \frac{1}{\tau} \int_{\mathbb{R}^2} \rho_n v_n - F(x, v_n) dx
\]
\[
+ \left( \frac{1}{\tau} - 1 \right) \epsilon \int_{\mathbb{R}^2} hv_n dx \quad (7.4)
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{\tau} \right) \| v_n \|^2 + \left( \frac{1}{\tau} - 1 \right) \epsilon \| h \|_\ast \| v_n \|,
\]
which implies that \( \{v_n\} \) is bounded in \( E \). Moreover, as \( \{v_n\} \) does not converge to \( v = 0 \) in \( E \), we can assume that for some subsequence,
\[
l := \lim_{n \to \infty} \| v_n \| > 0.
\]
Consequently, by (7.4),
\[
d_\epsilon + \left( \frac{\tau - 1}{\tau} \right) \epsilon \| h \|_\ast l \geq \left( \frac{1}{2} - \frac{1}{\tau} \right) l^2,
\]
that is
\[
\left( \frac{1}{2} - \frac{1}{\tau} \right) l^2 - \left( \frac{\tau - 1}{\tau} \right) \epsilon \| h \|_\ast l - d_\epsilon \leq 0.
\]
As \( l > 0 \), we must have
\[
l \leq \frac{(\tau - 1) \epsilon + \sqrt{\epsilon^2 \left( \frac{\tau - 1}{\tau} \right)^2 + 2d_\epsilon \left( \frac{\tau - 2}{2\tau} \right)}}{2 \left( \frac{\tau - 2}{2\tau} \right)},
\]
which completes the proof. 

Lemma 7.3 Assume (f0) – (f6). Then, there are \( \epsilon_1, \mu^* > 0 \) and \( t_1 > 0 \) such that
\[
\frac{(\tau - 1) \epsilon + \sqrt{\epsilon^2 \left( \frac{\tau - 1}{\tau} \right)^2 + 2d_\epsilon \left( \frac{\tau - 2}{2\tau} \right)}}{\left( \frac{\tau - 2}{2\tau} \right)} < \sqrt{\frac{4\pi}{\alpha_0}}.
\]
for all \( \epsilon \in (0, \epsilon_1), \mu \geq \mu^* \) and \( t_0 \in [0, t_1) \).
Proof. Consider the function $\psi_0$ used in the proof of Lemma 7.1. Then,

$$\sup_{t \in [0,t_0]} I_\epsilon(t\psi_0) \leq \frac{t_0^2}{2} \|\psi_0\|^2,$$

and so, there is $t_2 > 0$ such that

$$\sup_{t \in [0,t_0]} I_\epsilon(t\psi_0) \leq \epsilon^2$$

for $t_0 \in [0,t_2)$. On the other hand, by (f6),

$$\sup_{t \geq t_0} I_\epsilon(t\psi_0) \leq \max_{t \geq 0} \left\{ \frac{t^2}{2} \|\psi_0\|^2 - \mu \int_{\Gamma_r^2} \psi_0^p dx \right\} + \mu c_2 t_0^p |supt(\psi_0)|,$$

that is,

$$\sup_{t \geq t_0} I_\epsilon(t\psi_0) \leq \left( \frac{1}{2p^{\frac{p-2}{p}}} - \frac{1}{p^{\frac{p-2}{p}}} \right) \frac{1}{\mu^{\frac{p-2}{p}}} \left( \frac{\|\psi_0\|}{|\psi_0|_p} \right)^{2p} + \mu c_2 t_0^p |supt(\psi_0)|.$$

Now, fix $\mu^* > 0$ such that

$$\left( \frac{1}{2p^{\frac{p-2}{p}}} - \frac{1}{p^{\frac{p-2}{p}}} \right) \frac{1}{\mu^{\frac{p-2}{p}}} \left( \frac{\|\psi_0\|}{|\psi_0|_p} \right)^{2p} \leq \epsilon^2, \quad \forall \mu \geq \mu^*$$

and $t_3 = t_3(\mu, \epsilon) > 0$ such that

$$\mu c_2 t_0^p |supt(\psi_0)| \leq \epsilon^2, \quad \forall t \in [0,t_3].$$

From this, for $t_1 = \min\{t_2, t_3\}$, we must have

$$\sup_{t \geq t_0} I_\epsilon(t\psi_0) \leq 2\epsilon^2,$$

and so,

$$d_\epsilon \leq \max_{t \geq 0} I_\epsilon(t\psi_0) \leq 2\epsilon^2.$$

Hence, there is $c_1 > 0$ independent of $\epsilon$ such that

$$\frac{(\frac{r-1}{r}) \epsilon + \sqrt{\epsilon^2 (\frac{r-1}{r})^2 + 2d_\epsilon (\frac{r-2}{r})}}{(\frac{r-2}{r})} \leq c_1 \epsilon.$$

Then, there is $\epsilon_0 > 0$ such that

$$\frac{(\frac{r-1}{r}) \epsilon + \sqrt{\epsilon^2 (\frac{r-1}{r})^2 + 2d_\epsilon (\frac{r-2}{r})}}{(\frac{r-2}{r})} < \sqrt{\frac{4\pi}{\alpha_0}}, \quad \forall \epsilon \in (0, \epsilon_0).$$

As an immediate consequence of the last lemma, we have the following corollary.
Corollary 7.1 Let \( \{v_n\} \) be the sequence obtained in (7.3). Then, there is \( \epsilon_0 \) such that
\[
\limsup_{n \to +\infty} \|v_n\|^2 < \frac{4\pi}{\alpha_0}, \quad \forall \epsilon \in (0, \epsilon_0).
\]
Moreover, there is a subsequence of \( \{v_n\} \) still denoted by itself, and \( v_\epsilon \in E \) such that \( v_n \to v_\epsilon \) in \( E \).

Proof. The first part of the lemma is an immediate consequence of Lemmas 7.2 and 7.3. The proof of the second part follows the same idea explored in the proof of Theorem 6.1. \( \blacksquare \)

Theorem 7.1 Assume \((V_1) - (V_2)\) and \((f_0) - (f_0)\). Then, there are \( \epsilon_0, \mu^* \) and \( t_1 > 0 \), such that problem \((P)\) possesses a solution \( v_\epsilon \in E \), with \( I_\epsilon(v_\epsilon) = d_\epsilon > 0 \), for all \( \epsilon \in (0, \epsilon_0) \), \( t_0 \in [0, t_1) \) and \( \mu \geq \mu^* \). Moreover, decreasing \( \epsilon_0 \) and \( t_1 \), and increasing \( \mu^* \), if necessary, we have two solutions \( u_\epsilon, v_\epsilon \in E \) with
\[
I_\epsilon(u_\epsilon) = c_\epsilon < 0 < d_\epsilon = I_\epsilon(v_\epsilon).
\]

Proof. The theorem follows applying the Lemmas 6.1 and 7.1 and Corollary 7.1. \( \blacksquare \)

References

[1] A. Adams and J. F. Fournier, Sobolev spaces, 2nd ed., Academic Press, (2003).

[2] A. Adimurthi, Existence of Positive solutions of the semilinear Dirichlet problem with critical growth for the N-Laplacian, Ann. Sc. Norm. Super. Pisa, 17 (1990), 393-413.

[3] A. Adimurthi and S. L. Yadava, Critical exponent problem in \( \mathbb{R}^2 \) with Neumann boundary condition, Comm. Partial Differential Equations, 15 (1990), 461–501.

[4] C. O. Alves, A. M. Bertone and J. V. Goncalves, A variational approach to discontinuous problems with critical Sobolev exponents, J. Math. Anal. App. 265 (2002), 103-127.

[5] C. O. Alves and A.M. Bertone, A discontinuous problem involving the p-Laplacian operator and critical exponent in \( \mathbb{R}^N \), Electron. J. Differential Equations 2003 (2003), 1-10.

[6] C.O. Alves, J.V. Goncalves and J.A. Santos, Strongly Nonlinear Multivalued Elliptic Equations on a Bounded Domain, J. Glob. Optim. 58 (2014), 565-593.

[7] C. O. Alves, João Marcos do Ó and O. H. Miyagaki, On nonlinear perturbations of a periodic elliptic problem in \( \mathbb{R}^2 \) involving critical growth, Nonlinear Anal. 56 (2004), 781–791.
[8] C. O. Alves and D. S. Pereira, *Existence and nonexistence of least energy nodal solution for a class of elliptic problem in \( \mathbb{R}^2 \),* to appear in Topol. Methods Nonlinear Anal.

[9] A. Ambrosetti and R. E. L. Turner, *Some discontinuous variational problems,* Diff. Int. Equations 1 (1988), 341-349.

[10] A. Ambrosetti, M. Calahorrano and F. Dobarro, *Global branching for discontinuous problems,* Comm. Math. Univ. Caroliniae 31 (1990), 213-222.

[11] M. Badiale and G. Tarantello, *Existence and Multiplicity results for elliptic problems with critical growth and discontinuous nonlinearities,* Nonlinear Anal. 29 (1997), 639-677.

[12] D. M. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in \( \mathbb{R}^2 \),* Comm. Partial Differential Equations 17 (1992), 407–435.

[13] S. Carl, V. K. Le and D. Motreanu, *Nonsmooth variational problems and their inequalities. Comparison principles and applications,* Springer Monographs in Mathematics. Springer, New York, (2007).

[14] F.H. Clarke, *Optimization and Nonsmooth Analysis,* John Wiley & Sons, N.Y, 1983.

[15] K.C. Chang, *Variational methods for nondifferentiable functionals and their applications to partial differential equations.* J. math. Analysis Appl. 80 (1981), 102-129.

[16] K. C. Chang, *On the multiple solutions of the elliptic differential equations with discontinuous nonlinear terms,* Sci. Sinica 21 (1978), 139-158.

[17] K. C. Chang, *The obstacle problem and partial differential equations with discontinuous nonlinearities,* Comm. Pure Appl. Math (1978), 139-158.

[18] S. Carl and H. Dietrich, *The weak upper and lower solution method for elliptic equations with generalized subdifferentiable perturbations,* Appl. Anal. 56 (1995), 263-278.

[19] S. Carl and S. Heikkila, *Elliptic equations with discontinuous nonlinearities in \( \mathbb{R}^N \),* Nonlinear Anal. 31 (1998), 217-227.

[20] S. Carl and S. Heikkila, *Elliptic equations with discontinuous nonlinearities in \( \mathbb{R}^N \),* Nonlinear Anal. 30 (1997), 1743-1751.

[21] G. Cerami, *Metodi variazionali nello studio di problemi al contorno con parte nonlineare discontinua,* Rend. Circ. Mat. Palermo 32 (1983), 336-357.

[22] D.G. de Figueiredo, O.H. Miyagaki and B. Ruf, *Elliptic equations in \( \mathbb{R}^2 \) with nonlinearities in the critical growth range,* Calc. Var. 3 (1995), 139-153.
[23] D.G. de Figueiredo, J.M. do Ó and B. Ruf, *On an Inequality by N. Trudinger and J. Moser and related elliptic equations*, Comm. Pure. Appl. Math. 55 (2002), 135-152.

[24] L.R. de Freitas, *Multiplicity of solutions for a class of quasilinear equations with exponential critical growth*, Nonlinear Analysis 95 (2014), 607-624.

[25] M. de Souza, E. de Medeiros and U. Severo, *On a class of quasilinear elliptic problems involving Trudinger-Moser nonlinearities*, J. Math. Anal. Appl. 403 (2013), 357-364.

[26] M. de Souza, E. de Medeiros and U. Severo, *On a class of Nonhomogeneous elliptic problems involving exponential critical growth*, Topol. Methods Nonlinear Anal. 44 (2014), 399-412.

[27] T.K. Donaldson and N.S. Trudinger, *Orlicz-Sobolev spaces and imbedding theorems*, J. Funct. Anal. 8 (1971), 52-75.

[28] J.M. do Ó, E. de Medeiros and U. Severo, *A nonhomogeneous elliptic problem involving critical growth in dimension two*, J. Math. Anal. Appl. 345 (2008), 286-304.

[29] J.M. do Ó, E. de Medeiros and U. Severo, *On a quasilinear nonhomogeneous elliptic equation with critical growth in \( \mathbb{R}^N \)*, J. Diff. Equations 246 (2009), 1363-1386.

[30] J. M. B. do Ó and B. Ruf. *On a Schrödinger equation with periodic potential and critical growth in \( \mathbb{R}^2 \)*. Nonlinear Differential Equations Appl. 13 (2006), 167-192.

[31] J. M. B. do Ó, M. de Souza, E. de Medeiros and U. Severo. *An improvement for the Trudinger-Moser inequality and applications*. J. Differential Equations 256 (2014), 1317-1349.

[32] N. Fukagai, M. Ito and K. Narukawa, *Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on \( \mathbb{R}^N \)*, Funkcial. Ekvac. 49 (2006), 235-267.

[33] S. Hu, N. Kourogenis and N. S. Papageorgiou, *Nonlinear elliptic eigenvalue problems with discontinuities*, J. Math. Anal. Appl. 233 (1999), 406-424.

[34] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. 20 (1971), 1077–1092.

[35] D. Motreanu and C. Varga, *Some critical point results for locally Lipschitz functionals*, Comm. Appl. Nonlinear Anal. 4 (1997), 17-33.

[36] M.N. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York (1985).
[37] V. Radulescu, Mountain pass theorems for non-differentiable functions and applications. Proc. Jpn. Acad. 69 (Ser.A) (1993), 193-198.

[38] N. S. Trudinger, *On imbedding into Orlicz spaces and some application*, J. Math Mech. 17 (1967), 473–484.

Claudianor O. Alves and Jefferson A. Santos
Universidade Federal de Campina Grande,
Unidade Académica de Matemática ,
CEP:58109-970, Campina Grande - PB, Brazil
e-mail: coalves@mat.ufcg.edu.br and jefferson@mat.ufcg.edu.br