$C^\infty$ SOLUTIONS OF THE NAVIER-STOKES EQUATIONS ON $\mathcal{R}^3 \times [0, \infty)$

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Abstract. When $\nu$ is $> 0$, there are functions $u^{\nu}$ and $p^{\nu} \in C^\infty$ on $\mathcal{R}^3 \times [0, \infty)$ which thereon satisfy the Navier-Stokes equations

$$\partial_t u^{\nu} + \sum_j u_j^{\nu} \partial_j u^{\nu} = \nu \Delta u^{\nu} - \text{grad } p^{\nu}$$

$$\text{div } u^{\nu} = 0$$

$$u^{\nu}(x, 0) = u_o(x)$$

and

$$\sup_{t \in [0, \infty)} |u^{\nu}(t)|_{L^\infty} = |u_o|_{L^\infty} \text{ and }$$

$$\sup_{t \in [0, \infty)} |u^{\nu}(t)|_{L^2} = |u_o|_{L^2}$$

when the initial condition $u_o$ is a divergence free member of $L^2$ which has for each multi-index $k'$ a weak derivative $D^{k'} u_o$ which is $\in L^2$.

Such equations model the flow of an incompressible, viscous fluid that fills all of $\mathcal{R}^3$ and which is not subject to gravity or another external force.

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1
1. Introduction

This paper shows that there are functions \( u^\nu(x, t; u_o) \) and \( p^\nu(x, t; u_o) \in C^\infty \) on \( \mathcal{R}^3 \times [0, \infty) \) which thereon satisfy the (1.1) \( \nu, u_o \) Navier-Stokes equations

\[
\begin{align*}
\partial_t u^\nu + \sum_j u_j^\nu \partial_j u^\nu &= \nu \Delta u^\nu - \text{grad} \ p^\nu \\
\text{div} \ u^\nu &= 0 \\
\|u^\nu(x, 0) &= u_o(x) \]
\end{align*}
\]

(1.1)

In such equations, \( \nu > 0 \) and the initial condition \( u_o \) is a divergence free member of \( L^2 \) which has for each multi-index \( k' \) a weak derivative \( D^{k'} u_o \) which is \( \in L^2 \).

The (1.1) \( \nu, u_o \) Navier-Stokes equations model the flow of an incompressible, viscous fluid that fills all of \( \mathcal{R}^3 \) and which is not subject to gravity or another external force. We also show that

\[
\sup_{t \in [0, \infty)} \|u^\nu(t)\|_{L^\infty} = |u_o|_{L^\infty} \quad \text{and} \quad \sup_{t \in [0, \infty)} \|u^\nu(t)\|_{L^2} = |u_o|_{L^2}
\]

Leray [7] (1934) showed that there is a \( C^\infty \), local-in-time solution \( u^\nu(x, t; u_o) \) of the (1.1) \( \nu, u_o \) Navier-Stokes equations on \( \mathcal{R}^3 \times [0, T^\nu_{\text{bup}}(u_o)) \) for certain initial conditions \( u_o \) and that if a blow up time \( T^\nu_{\text{bup}}(u_o) \) is finite then

\[
\sup_{t \in [0, T]} \|u^\nu(t; u_o)\|_{L^\infty} \rightarrow \infty \quad \text{as} \quad T \rightarrow \text{the finite} \ T^\nu_{\text{bup}}(u_o)
\]

(1.2)

Leray’s paper also showed that a \( u^\nu(x, t; u_o) \) with a finite \( T^\nu_{\text{bup}}(u_o) \) can after \( T^\nu_{\text{bup}}(u_o) \) be continued as a weak solution on \( \mathcal{R}^3 \times [0, \infty) \), that \( u^\nu(x, t; u_o) \) is a global-in-time solution when the initial condition \( u_o \) is sufficiently small and that any global-in-time \( u^\nu(x, t; u_o) \) satisfies

\[
\sup_{t \in [T, \infty)} \|u^\nu(t; u_o)\|_{L^\infty} \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty
\]

Caffarelli-Kohn-Nirenberg [3] (1982) extended Scheffer [10] (1976) to limit the way in which a solution \( u^\nu(x, t; u_o) \) can blow up at a finite time and reduced the upper bound for the size of the set on which a blow
up can occur. Lin [8] (1998) simplified the Caffarelli-Kohn-Nirenberg paper. See Fefferman [5] (2006).

Our contribution to the mathematics of the \((1.1)_{\nu,u_o}\) Navier-Stokes equations is the inequality

$$\sup_{t \in [0,T_{\text{bup}}(u_o))] \left| u^{\nu=1}(t; u_o) \right|_{L^\infty} = \left| u_o \right|_{L^\infty}$$

which with the \((1.2)\) blow up condition shows that \(u^{\nu=1}(x, t; u_o)\) and a related \(p^{\nu=1}(x, t; u_o)\) are a global-in-time solution of the \((1.1)_{\nu=1,u_o}\) Navier-Stokes equations.

Prior to establishing such inequality, we establish de novo certain results, such as the existence of local-in-time solutions \(u^{\nu=1}(x, t; u_o)\) which satisfy the blow up condition

$$|u^{\nu=1}|_{L^\infty \times [0,T]} \to \infty \text{ as } T \to \text{ a finite } T_{\text{bup}}(u_o)$$

(1.3)

without using Leray’s paper as authority, which may not be readily accessible to some readers. We do not discuss the history of any of our results as the literature of these equations is vast. See, for example, the references in [2].

After we establish the existence of the global-in-time \(u^{\nu=1}(x, t; u_o)\) and \(p^{\nu=1}(x, t; u_o)\), we show that such functions are \(\in C^\infty\) and that scaling rules identify global-in-time solutions of the \((1.1)_{\nu,u_o}\) Navier-Stokes equations for \(\nu > 0\).

Section 2 of this paper contains definitions and discusses certain related matters. In particular, we use the definitions

$$\mathcal{R}^3 \times [0,T] := \{(x, t) \mid x \in \mathcal{R}^3 \text{ and } t \in [0,T]\}$$

$$\left| u^{\gamma=0}_{\eta=0} \right|_{L^\infty \times [0,T]} := \sup_{\eta \in [0,T]} \left| u^{\gamma=0}(\eta) \right|_{L^\infty}$$

and variations thereof. We also define various function spaces. \(C^m\) is the space of functions \(v : \mathcal{R}^3 \to \mathcal{R}^3\) which are continuous on \(\mathcal{R}^3\) and each derivative \(D^{k'} v\) in which the multi-index \(k'\) satisfies \(1 \leq |k'| \leq m\) is also so continuous.
\( H^m \) is the space of functions \( v : \mathcal{R}^3 \rightarrow \mathcal{R}^3 \) for which
\[
|v|_{H^m} := \sum_{0 \leq |k'| \leq m} |D^{k'}v|_{L^2} < \infty
\]

\( H^{m,df} \) is the divergence free subset of \( H^m \). When \( m \geq 2 \), one member \( v \) of each equivalence class which is a member of \( H^m \) is \( \in C^{m-2} \) and satisfies
\[
\sum_{0 \leq |k'| \leq m-2} |D^{k'}v|_{L^\infty} \leq [ \ldots ]_{2.14,m} |v|_{H^m}
\]
Thus, any weak derivative \( \partial_k v \) of \( v \in H^3 \) is a bounded and continuous function of \( x \in \mathcal{R}^3 \).

\( H^m \times [a,b] \) is the space of functions \( v : \mathcal{R}^3 \times [a,b] \rightarrow \mathcal{R}^3 \) for which \( v(t) \) is \( \in H^m \) when \( t \in [a,b] \) and \( v(t) \) is a function of \( t \in [a,b] \) which is continuous in the \( H^m \) norm. When \( v \in H^m \times [a,b] \), the functional
\[
|v|_{H^m \times [a,b]} := \sup_{t \in [a,b]} |v(t)|_{H^m}
\]
is finite because a continuous function on a compact \( [a,b] \) takes its supremum thereon. \( H^{m,df} \times [a,b] \) is the divergence free subset of \( H^m \times [a,b] \). \( H^m \times [a,b] \) and \( H^{m,df} \times [a,b] \) are defined in the way that \( H^m \times [a,b] \) and \( H^{m,df} \times [a,b] \) are defined, but with the half-open interval \( [a,b) \) in place of \( [a,b] \).

\( W^{m,1} \) is the space of functions \( v : \mathcal{R}^3 \rightarrow \mathcal{R}^3 \) for which
\[
|v|_{W^{m,1}} := \sum_{0 \leq |k'| \leq m} |D^{k'}v|_{L^1} < \infty
\]

Section 3 establishes inequalities for certain convolutions and derivatives thereof. Section 4 establishes the existence of the Helmholtz-Hodge decomposition
\[
v = Gv + P_v
\]
of each \( v \in L^2 \) and shows when \( v \in \cap_m [H^m \cap W^{m,1}] \) that \( Gv \) is the gradient of a scalar which is \( \in \cap_m H^{m,df} \) and \( P_v \) is divergence free. Section 5 establishes inequalities for certain integrals and the derivatives of such integrals.
Section 6 shows that there is for \( \gamma \geq 0, m \geq 7 \) and \( u_o \in \bigcap_m H^{m,df} \) a \( T_{bup}^{\gamma, m}(u_o) > 0 \) and a function

\[
u^{\gamma, m}(x, t; u_o) \in H^{m,df} \times [0, T_{bup}^{\gamma, m}(u_o))
\]

which satisfies the \((6.1)_{k'=0, \gamma, u_o}\) integral equation

\[
u^{\gamma, m}(x, t) = \int_0^t \int_{\mathbb{R}^3} \sum_j \hat{P} \left[ \partial_j K^\dagger(t - \eta) \right](y) \ J_\gamma(u_j^{\gamma, m})u^{\gamma, m}(x - y, \eta) \ dyd\eta
+ \int_{\mathbb{R}^3} K(y, t) \ u_o(x - y) \ dy
\]
on \( \mathbb{R}^3 \times [0, T_{bup}^{\gamma, m}(u_o)) \).

In such integral equation, \( K^\dagger(y, t) \) is the \( 3 \times 3 \) diagonal matrix the diagonal of which is the scalar valued function

\[
K(y, t) = [4\pi t]^{-3/2} \exp\left(-\frac{|y|^2}{4t}\right)
\]
(1.4)

\( K^\dagger(y, t) \) is the \( i \)th row of \( K^\dagger(y, t) \). \( \hat{P} \left[ \partial_j K^\dagger(t) \right](y) \) is the \( 3 \times 3 \) matrix the \( i \)th row of which is \( \hat{P} \left[ \partial_j K^\dagger(t) \right](y), \hat{P} := (-1)P, \)

\[
J_\gamma > 0(u_j^{\gamma, m}) \text{ is the convolution } m_{\gamma > 0}^\dagger * u_j^{\gamma, m}
\]
in which the mollifier

\[
m_{\gamma > 0}(y) := \gamma^{-3} \ m^\dagger(y\gamma^{-1})
\]

and \( m^\dagger \) is the standard, \( C_0^\infty \) mollifier of [4] Appendix C.5, \( J_{\gamma = 0}(u_j^{\gamma, m}) \) is \( = u_j^{\gamma, m} \), and \( J_\gamma(u_j^{\gamma, m})u^{\gamma, m} \) is a column vector.

Section 7 shows that each derivative \( \partial_t u^{\gamma, m}(x, t; u_o) \) exists on \( \mathbb{R}^3 \times [0, T_{bup}^{\gamma, m}(u_o)) \) and is \( \in H^{m-2,df} \times [0, T_{bup}^{\gamma, m}(u_o)), \) that \( u^{\gamma, m} \) satisfies the

\[(7.1)_{k'=0, \gamma, u_o} \ \text{pde} \]

\[
\partial_t u^{\gamma, m}(x, t) + \mathcal{P} \left[ \sum_j \ J_\gamma(u_j^{\gamma, m})\partial_j u^{\gamma, m}(t) \right](x) = \Delta u^{\gamma, m}(x, t)
\]

\[
u^{\gamma, m}(x, 0) = u_o(x)
\]
on the strip \( \mathbb{R}^3 \times [0, T_{bup}^{\gamma, m}(u_o)) \) where \( u^{\gamma, m} \) is defined, that

\[
|u^{\gamma, m}|_{L^2 \times [0, T_{bup}^{\gamma, m}(u_o))} = |u_o|_{L^2}
\]
(1.5)
and that \( u^{\gamma,m} \) satisfies the blow up condition
\[
|u^{\gamma,m}|_{H^m \times [0,T]} \to \infty \text{ as } T \to \text{ a finite } T^{\gamma,m}_{\text{bup}}(u_o)
\]
(1.6)

Section 8 first shows that each \( u^{\gamma,m} \) also satisfies a second blow up condition
\[
|u^{\gamma,m}|_{L^\infty \times [0,T]} \to \infty \text{ as } T \to \text{ a finite } T^{\gamma,m}_{\text{bup}}(u_o)
\]
(1.7)
and then shows that the functions \( u^{\gamma,m}(x,t;u_o) \) with the same \( \gamma \geq 0 \) and initial condition \( u_o \in \cap m H^{m,df} \) are equal to a function
\[
u(x,t;u_o) \in \cap m [H^{m,df} \times [0,T^{\gamma,m}_{\text{bup}}(u_o)]]
\]
that the derivative \( \partial_t u^{\gamma}(x,t;u_o) \) exists on the strip where \( u^{\gamma} \) is defined, that
\[
T^{\gamma=0}_{\text{bup}}(u_o) \geq [\ldots]_{6.19} |u_o|_{H^2}^{-2} \text{ and } T^{\gamma>0}_{\text{bup}}(u_o) = \infty
\]
and that \( u^{\gamma}(x,t;u_o) \) satisfies every equation, inequality or relation which any \( u^{\gamma,m}(x,t;u_o) \) to which \( u^{\gamma} \) is equal satisfies.

Section 9 first establishes the scaling rule
\[
u^{\gamma>0}(x,t;u_o^\alpha) = \alpha u^{\gamma>0}(x\alpha, t\alpha^2; u_o)
\]
in which \( u_o^{\alpha>0}(x) := \alpha u_o(x\alpha) \) and then for \( x \in \mathcal{R}^3, t \geq 0 \) and \( \alpha > 0 \) that
\[
|u^{\gamma>0}(x,t\alpha^{-2};u_o^\alpha)| \\
\leq [\ldots]_{[t\alpha^{-2}]1/2} \sup_{\eta \in [0,t\alpha^{-2}]^1/2} \left| J_{\gamma>0}[u_{j}(x,\eta;u_o^\alpha)] \right|_{L^\infty(x \in \mathcal{R}^3)} \\
\sup_{\eta \in [0,t\alpha^{-2}]} \left| u^{\gamma>0}(x,\eta;u_o^\alpha) \right|_{L^\infty(x \in \mathcal{R}^3)} + |u_o^\alpha|_{L^\infty}
\]
which with the scaling rule
\[
|J_{\gamma>0}[u_{j}(x,\eta;u_o^\alpha)]|_{L^\infty} \leq |m^j|_{L^2} \gamma^{-3/2} |u_o|_{L^2} \alpha^{-1/2} \text{ and } |u_o^\alpha|_{L^\infty} = \alpha |u_o|_{L^\infty}
\]
shows that
\[
\alpha |u^{\gamma>0}(t;u_o)|_{L^\infty(x \in \mathcal{R}^3)} \\
\leq [\ldots](\gamma, |m^j|_{L^2}, |u_o|_{L^2}) t^{1/2} \alpha^{-1/2} \sup_{\eta \in [0,t]} \left| \alpha u^{\gamma>0}(x,\eta;u_o) \right|_{L^\infty(x \in \mathcal{R}^3)} + \alpha |u_o|_{L^\infty}
\]
and then that
\[
\sup_{t \in [0, \infty]} |u^{\gamma>0}(x, t; u_o)|_{L^\infty(x \in \mathbb{R}^3)} \leq |u_o|_{L^\infty}
\]

With the Arzela-Ascoli theorem and the Cantor diagonalization argument, such section shows that \(u^{\gamma>0} \to u^{\gamma=0}\) in the \(L^\infty\) norm on each compact subset of \(\mathbb{R}^3 \times [0, \infty)\) and with the dominated convergence theorem and uniqueness result that
\[
\sup_{t \in [0, T^{\gamma=0}(u_o))] } |u^{\gamma=0}(x, t; u_o)|_{L^\infty(x \in \mathbb{R}^3)} \leq |u_o|_{L^\infty}
\]

Thus, \(u^{\gamma=0}(x, t; u_o)\) is a global-in-time solution of the \((6.1)_{\nu=0, \gamma, u_o}\) integral equation and of its related pde on \(\mathbb{R}^3 \times [0, \infty)\).

Section 10 shows that \(u^{\gamma=0}\) and \(p^{\gamma=0}\) are \(C^\infty\) on \(\mathbb{R}^3 \times [0, \infty)\), \(u^{\gamma=0}\) is \(\in \cap_m [H^{m,df} \times [0, \infty)]\) and \(p^{\gamma=0}\) is \(\in \cap_m [H^m \times [0, \infty)]\). Thereafter, section 10 identifies \(u^{\gamma=0}\) and \(p^{\gamma=0}\) as \(u^{\nu=1}\) and \(p^{\nu=1}\), which it shows satisfy the \((1.1)_{\nu=1, u_o}\) pde on \(\mathbb{R}^3 \times [0, \infty)\).

Section 11 shows for \(\nu > 0\) that the functions
\[
\begin{align*}
{u'}(x, t; u_o) &:= u^{\nu=1}(x^{\nu^{-1}}, t^{\nu^{-1}}; u_o(x^{\nu})) \\
{p'}(x, t; u_o) &:= p^{\nu=1}(x^{\nu^{-1}}, t^{\nu^{-1}}; u_o(x^{\nu}))
\end{align*}
\]
are a solution of the \((1.1)_{\nu, u_o}\) Navier-Stokes equations on \(\mathbb{R}^3 \times [0, \infty)\) and are \(C^\infty\) on \(\mathbb{R}^3 \times [0, \infty)\). In addition, it shows that \(u'\) is \(\in \cap_m [H^{m,df} \times [0, \infty)]\), that \(p'\) is \(\in \cap_m [H^m \times [0, \infty)]\) and that
\[
\begin{align*}
|{u'}|_{L^\infty \times [0, \infty)} &= |u_o|_{L^\infty} \quad \text{and} \\
|{u'}|_{L^2 \times [0, \infty)} &= |u_o|_{L^2}
\end{align*}
\]

Our results are a solution of Option A of the Millennium Problem for the Navier-Stokes equations of the Clay Mathematics Institute as the rapidly decaying initial conditions for which such problem seeks a \(C^\infty\) solution of the Navier-Stokes equations on \(\mathbb{R}^3 \times [0, \infty)\) are members of \(\cap_m H^{m,df}\).
2. FUNCTION SPACES AND OTHER INFRASTRUCTURE

2.1. General. We write the space variable as

\[ x = (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3) \text{ or } z = (z_1, z_2, z_3) \]

\( B(r) \) is the closed ball in \( \mathbb{R}^3 \) of radius \( r > 0 \) that is centered at the origin. \( \partial B(r) \) is the boundary of \( B(r) \), and \( \mathbb{R}^3 \setminus B(r) \) is the complement of \( B(r) \). The strip \( \mathbb{R}^3 \times [a, b] \) (alternatively, \( \mathbb{R}^3 \times [a, b) \)) contains those tuples \((x, t)\) for which \( x \in \mathbb{R}^3 \) and \( t \in \) the closed interval \([a, b] \) (alternatively, in the half-open interval \([a, b)\)).

\[ \ldots \] is a finite number, \([ \ldots ]_{x.xx} \] is a finite number that is defined in or near \((x.xx)\) hereof, \([ \leq 0 \] is a non-negative number and \([ \leq 0 ](u_o) \) is a non-negative function of \( u_o \).

\[ a := \ldots \] is the definition of a

2.2. Vectors \( \in \mathbb{R}^3 \). The scalar \( v_i \) is the \( i^{th} \) component of the vector \( v \in \mathbb{R}^3 \), \( |v_i| \) is the absolute value of \( v_i \) and

\[ [a]^i \] is the sparse vector \( \in \mathbb{R}^3 \)

the \( i^{th} \) component of which is \( = a \) and the other two components of which are \( = 0 \). In addition

\[ K^i(y, t) := [K(y, t)]^i \]  

(2.1)

The function \( K(y, t) \) is defined in (1.4).

When \( u \) and \( v \) are vectors \( \in \mathbb{R}^3 \), the inner product

\[ (u, v)_2 := \sum_i u_i v_i \]  

(2.2)

Norms on a vector \( v \in \mathbb{R}^3 \) include

\[ |v|_2 := (v, v)^{1/2} \]

\[ |v|_1 := \sum_i |v_i| \quad \text{and} \]

\[ |v|_{\infty} := \sup_i |v_i| \]  

(2.3)
2.3. **Functions on \( \mathbb{R}^3 \).** A function, such as \( u \) or \( v \), is defined on \( \mathbb{R}^3 \) except as the discussion otherwise provides. When \( u \) and \( v \) are such vector valued functions on \( \mathbb{R}^3 \), the inner product

\[
\langle u, v \rangle_{\mathbb{R}^3} := \int_{\mathbb{R}^3} (u(x), v(x))_2 \, dx
\]

Integrals and measurable sets are determined in the Lebesgue theory of integration.

Norms on a vector valued function \( v \) include

\[
|v|_{L^2} := \langle v, v \rangle_{\mathbb{R}^3}^{1/2} = \left[ \sum_i |v_i|_{L^2}^2 \right]^{1/2} = \left[ \sum_i \int_{\mathbb{R}^3} v_i^2(x) \, dx \right]^{1/2}
\]

\[
|v|_{H^m} := \sum_{0 \leq |k'| \leq m} |D^{k'} v|_{L^2}
\]

\[
|v|_{L^1} := \sum_i \int_{\mathbb{R}^3} |v_i(x)|_1 \, dx \text{ and}
\]

\[
|v|_{L^\infty} := \sup_k \left[ \sup_{x \in \mathbb{R}^3} |v_k(x)| \right]
\]

\( (2.4) \)

As we use the \( L^\infty \) norm only for continuous functions \( v \), the definition of \( |v|_{L^\infty} \) does not exclude values of \( v \) in a set of measure zero. The above definitions show that

\[
|\langle u, v \rangle_{\mathbb{R}^3}| \leq |u|_{L^1} \, |v|_{L^\infty}
\]

\( (2.5) \)

We abbreviate \( H^m(\mathbb{R}^3) \) and \( C^\infty_o(\mathbb{R}^3) \) as \( H^m \) and \( C^\infty_o \) and abbreviate other function spaces on \( \mathbb{R}^3 \) in the same way.

A norm of a scalar valued function on \( \mathbb{R}^3 \) is determined as though that function is the first component of a vector valued function that carries \( \mathbb{R}^3 \to \mathbb{R}^3 \) the second and third components of which vanish.

\( \partial_j v \) is the partial derivative with respect to \( x_j \) of the function \( v \). When \( u \) and \( v \) have enough derivatives, the Laplacian

\[
\Delta v := \sum_j \partial^2_j v
\]
and the divergence
\[ \text{div } v := \sum_i \partial_i v_i \] (2.6)

When \( u \) is divergence free
\[ \sum_j \partial_j [u_j v] = \sum_j u_j \partial_j v \] (2.7)

and when \( v \) is divergence free
\[
\text{div} \left[ \sum_j u_j \partial_j v \right] = \sum_i \partial_i [u_j \partial_j v_i] = \sum_{i,j} \partial_i u_j \partial_j v_i
\] (2.8)

The components of a multi-index \( k' = (k'_1, k'_2, k'_3) \) are non-negative integers
\[ |k'| := k'_1 + k'_2 + k'_3 \text{ and } \]
\[ D^{k'} := \partial_1^{k'_1} \partial_2^{k'_2} \partial_3^{k'_3} \]
\( \alpha' \text{ and } \beta' \text{ are also multi-indices and } \)
\[ \alpha' + \beta' := (\alpha'_1 + \beta'_1, \alpha'_2 + \beta'_2, \alpha'_3 + \beta'_3) \]

\( D^{k'} v_i \) is a derivative (alternatively, a "strong derivative") or a weak derivative as the discussion provides. The weak derivative \( D^{k'} v_i \) of a scalar valued function \( v_i \), if it exists, satisfies
\[
\int_{\mathbb{R}^3} (-1)^{|k'|} D^{k'} \phi(x) \ v_i(x) \ dx = \int_{\mathbb{R}^3} \phi(x) \ D^{k'} v_i(x) \ dx
\] (2.9)

for each scalar valued \( \phi \in C_0^\infty \).

As \( D^{\alpha'} \) and \( D^{\beta'} \) commute on \( \phi \in C_0^\infty \), (2.9) shows that the weak derivative
\[ D^{\alpha'} D^{\beta'} v_i = D^{\beta'} D^{\alpha'} v_i \] (2.10)
SMOOTH SOLUTIONS OF NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3 \times [0, \infty)$

when $|\alpha'| + |\beta'|$ is $\leq m$ and $v$ is $\in H^m$. The components of the weak
derivative of a vector valued function are the weak derivatives of the
components of that vector valued function.

When $v$ is $\in H^{|k'|}$, allowing $n \to \infty$ in

$$
\int_{\mathbb{R}^3} (-1)^{|k'|} D^{k'} \phi_n(x) \ v_i(x) \ dx = \int_{\mathbb{R}^3} \phi_n(x) \ D^{k'} v_i(x) \ dx
$$

in which $\phi_n$ is a sequence of scalar valued members of $C^\infty_o$ that $\to q$ in
the $H^{|k'|}$ norm establishes the integration by parts equation

$$
\int_{\mathbb{R}^3} (-1)^{|k'|} D^{k'} q(x) \ v_i(x) \ dx = \int_{\mathbb{R}^3} q(x) \ D^{k'} v_i(x) \ dx
$$

(2.11)

As the chain rule shows that $D^k_y [v_i(x - y)] = D^k_x [v_i(x - y)] (-1)^{|k'|}$,
(2.11) shows that

$$
\int_{\mathbb{R}^3} D^k_y q(y) \ v_i(x - y) \ dy = \int_{\mathbb{R}^3} q(y) \ D^k_x [v_i(x - y)] \ dy
$$

(2.12)

$C^m$ is the space of functions $v$ which carry $\mathcal{R}^3 \to \mathcal{R}^3$ each member
$v$ of which is continuous on $\mathcal{R}^3$ and each derivative $D^{k'} v$ in which
$1 \leq |k'| \leq m$ exists and is continuous on $\mathcal{R}^3$.

$H^m$ is the space of functions $v$ which carry $\mathcal{R}^3 \to \mathcal{R}^3$ for which the
norm

$$
|v|_{H^m} := \sum_{0 \leq |k'| \leq m} |D^{k'} v|_{L^2} \text{ is } < \infty
$$

(2.13)

Therein, $D^{k'} v$ is a weak derivative. $H^m$ is a Banach space, [1] §3.3, in
which $C^\infty_o$ is dense. [1] §3.25. $H^{m,df}$ is the divergence free subspace of
$H^m$.

When $m$ is $\geq 2$, one member of each equivalence class which is a
member of $H^m$ is $\in C^{m-2}$ and satisfies

$$
\sum_{0 \leq |k'| \leq m-2} |D^{k'} v|_{L^\infty} \leq \ldots \ [2.14, \ m] |v|_{H^m}
$$

(2.14)
Equation (6.5). Therein, $D^{k'}v$ is a weak derivative. Thus, a derivative $\partial_{k'} v$ of a function $v \in H^3$ is a bounded and continuous function of $x \in \mathcal{R}^3$ and

$$\cap_{m \geq 0} H^m$$

is a subset of $C^\infty$

$$\text{(2.15)}$$

$W^{m,1}$ is the space of functions $v : \mathcal{R}^3 \to \mathcal{R}^3$ for which the norm

$$|v|_{W^{m,1}} := \sum_{0 \leq |k'| \leq m} |D^{k'}v|_{L^1} \text{ is } < \infty$$

$$\text{(2.16)}$$

Each derivative in (2.16) is a weak derivative.

$C^\infty$ is the space of functions $v : \mathcal{R}^3 \to \mathcal{R}^3$ each derivative $D^{k'}v$ of which exists on $\mathcal{R}^3$. $C^\infty_o$ is the subspace of $C^\infty$ each member of which vanishes outside of a compact subset of $\mathcal{R}^3$. $\langle C^\infty_o, L^2 \rangle$ is $C^\infty_o$ with the topology of the $L^2$ norm.

**Lemma 2.1.** Let $u, v \in H^m$ and the multi-index $k'$ satisfy $1 \leq |k'| \leq m$. Then there are non-negative integers $c(\alpha', \beta', k')$ so that the weak derivative

$$D^{k'}[u_j v] = \sum_{\alpha' + \beta' = k'} c(\alpha', \beta', k') D^{\alpha'} u_j D^{\beta'} v$$

$$\text{(2.17)}$$

$$\sup_{\alpha' + \beta' = k'} c(\alpha', \beta', k') := [\ldots ]_{2.18} < \infty$$

$$\text{(2.18)}$$

the weak derivative

$$D^{k'}[u_j v] \in H^{m-|k'|}$$

$$\text{(2.19)}$$

and

$$|u_j v|_{W^{m,1}} \leq [\ldots ]_{2.20, m} |u_j|_{H^m} |v|_{H^m}$$

$$\text{(2.20)}$$
When \( m \) is \( \geq 6 \) and \( \alpha' + \beta' = k' \)

at least one of \( |\alpha'| \) and \( |\beta'| \) is \( \leq m - 2 \)

(2.21)

\[
|u_j v|_{H^m} \leq \left[ \ldots \right]_{2.22, m} |u|_{H^m} |v|_{H^m}
\]

(2.22)

\[
|D^{k'}[u_j v]|_{L^\infty} \leq \left[ \ldots \right]_{2.23, m} |u|_{H^m} |v|_{H^m}
\]

when \( 0 \leq |k'| \leq m - 2 \)

(2.23)

and

\[
|D^{k'}[u_j v]|_{L^1} \leq \left[ \ldots \right]_{2.24, m} |u|_{H^m} |v|_{H^m}
\]

(2.24)

Proof. We first establish (2.17). When \( u_j \) and \( v \) are \( \in C^\infty_0 \), the calculus expands \( D^{k'}[u_j v] \) into the right side of (2.17) and identifies the integers \( c(\alpha', \beta', k') \) and \( \left[ \ldots \right]_{2.18}(k') \).

As \( C^\infty_0 \) is dense in \( H^m \), we let \( u_j^n \) and \( v^n \) \( \in C^\infty_0 \to u_j \) and \( v \) in the \( H^m \) norm. As the derivative \( D^{k'}[u_j^n v^n] \) is also \( \in C^\infty_0 \), (2.9) shows that

\[
\int_{\mathbb{R}^3} (-1)^{|k'|} D^{k'} \phi(x) [u_j^n v^n](x) \, dx = \int_{\mathbb{R}^3} \phi(x) D^{k'}[u_j^n v^n](x) \, dx
\]

(2.25)

which with (2.17) is

\[
= \int_{\mathbb{R}^3} \phi(x) \left[ \sum_{\alpha'+\beta'=k'} c(\alpha', \beta', k') [D^{\alpha'} u_j^n D^{\beta'} v^n](x) \right] \, dx
\]

(2.26)

Customary algebra and a Hölder inequality show that

\[
|D^{\alpha'} u_j^n D^{\beta'} v^n - D^{\alpha'} u_j D^{\beta'} v|_{L^1} \\
\leq |D^{\alpha'} u_j^n - D^{\alpha'} u_j|_{L^2} |D^{\beta'} v^n|_{L^2} + |D^{\alpha'} u_j^n|_{L^2} |D^{\beta'} v - D^{\beta'} v^n|_{L^2}
\]

which as \( |D^{\alpha'} u_j^n - D^{\alpha'} u_j|_{L^2} \) and \( |D^{\beta'} v - D^{\beta'} v^n|_{L^2} \to 0 \) shows that

\[
|D^{\alpha'} u_j^n D^{\beta'} v^n - D^{\alpha'} u_j D^{\beta'} v|_{L^1} \to 0
\]
and that (2.26)

$$\rightarrow \int_{\mathbb{R}^3} \phi(x) \left[ \sum_{\alpha' + \beta'' = k'} c(\alpha', \beta', k') \ D^{\alpha'} u_j \ D^{\beta''} v(x) \right] \ dx$$

In the same way, a Hölder inequality shows that

$$|u^n_j v^n - u_j v|_{L^1} \leq |u^n_j - u_j|_{L^2} \ |v^n|_{L^2} + |u_j|_{L^2} \ |v^n - v|_{L^2}$$

which shows that

$$|u^n_j v^n - u_j v|_{L^1} \to 0$$

and the left side of (2.25)

$$\to \int_{\mathbb{R}^3} (-1)^{|k'|} D^{k'} \phi(x) \ u_j v(x) \ dx$$

As a result

$$\int_{\mathbb{R}^3} (-1)^{|k'|} D^{k'} \phi(x) \ [u_j v](x) \ dx$$

$$= \int_{\mathbb{R}^3} \phi(x) \left[ \sum_{\alpha' + \beta'' = k'} c(\alpha', \beta', k') \ [D^{\alpha'} u_j D^{\beta''} v](x) \right] \ dx$$

The proof of (2.17) is complete.

That there are a finite number of \(c(\alpha', \beta', k')\) establishes (2.18).

As (2.17) shows that \(u_j v\) is \(\in H^m\), \(D^{k'} u_j v\) is \(\in H^{m-|k'|}\). The proof of (2.19) is complete.

We now establish (2.20). With (2.17) and the definition of the \(W^{m,1}\) norm

$$|u_j v|_{W^{m,1}} = \sum_{0 \leq |k'| \leq m} \ |D^{k'} [u_j v]|_{L^1}$$

$$\leq \sum_{0 \leq |k'| \leq m} \left[ \sum_{\alpha' + \beta'' = k'} c(\alpha', \beta', k') \ |D^{\alpha'} u_j D^{\beta''} v|_{L^1} \right]$$

which with a Hölder inequality is

$$\leq \sum_{0 \leq |k'| \leq m} \left[ \sum_{\alpha' + \beta'' = k'} \ ... \ (m) \ |D^{\alpha'} u_j|_{L^2} \ |D^{\beta''} v|_{L^2} \right]$$

$$\leq \left[ \ ... \right]_{2.20, m} |u_j|_{H^m} \ |v|_{H^m}$$

The proof of (2.20) is complete.
We now establish (2.21) by contradiction. If neither of $|\alpha'|$ and $|\beta'|$ were $\leq m - 2$, then each would be $\geq m - 1$ and

$$m \text{ would be } \geq |\alpha'| + |\beta'|$$

which would be $\geq (m - 1) + (m - 1) = 2m - 2$

Thus, $m$ would be $\leq 2$ and would not, as we have assumed, be $\geq 6$.

We now establish (2.22). With (2.17)

$$|u_jv|_{H^m} = \sum_{0 \leq |k'| \leq m} |D^k[u_jv]|_{L^2}$$

$$\leq \sum_{0 \leq |k'| \leq m} \left[ \sum_{\alpha'+\beta'=k'} c(\alpha', \beta', k') |D^{\alpha'}u_jD^{\beta'}v|_{L^2} \right]$$

As at least one of $|\alpha'|$ and $|\beta'|$ in each summand is $\leq m - 2$, (2.21), the $L^\infty$ norm of one of $D^{\alpha'}u_j$ and $D^{\beta'}v$ in each summand is $\leq [\ldots]_{2,14,m}$ multiplied by the $H^m$ norm thereof, (2.14), and the $L^2$ norm of the other weak derivative is $\leq$ the $H^m$ norm thereof. As a result

$$|u_jv|_{H^m} \leq [\ldots]_{2,22,m} |u_j|_{H^m} |v|_{H^m}$$

We now establish (2.23). As $D^{k'}u_jv$ is $\in H^{m-|k'|}$, (2.19), $D^{k'}u_jv$ is $\in H^2$ when $|k'| \leq m - 2$. Then with (2.14)

$$|D^{k'}u_jv|_{L^\infty} \leq [\ldots]_{2,14,2} |D^{k'}u_jv|_{L^2}$$

$$\leq |u_jv|_{H^m} \leq [\ldots]_{2,20,m} |u_j|_{H^m} |v|_{H^m}$$

The proof of (2.23) is complete.

We now establish (2.24). With (2.17)

$$D^{k'}[u_jv] = \sum_{\alpha'+\beta'=k'} c(\alpha', \beta', k') D^{\alpha'}u_j D^{\beta'}v$$

and with a Hölder inequality

$$|D^{k'}[u_jv]|_{L^1} \leq \sum_{\alpha'+\beta'=k'} c(\alpha', \beta', k') |D^{\alpha'}u_j|_{L^2} |D^{\beta'}v|_{L^2}$$

$$\leq [\ldots]_{2,24,m} |u_j|_{H^m} |v|_{H^m}$$

□
2.4. **Functions on** \( R^3 \times [a, b] \). \( H^m \times [a, b] \) is the space of functions \( v : R^3 \times [a, b] \to R^3 \) for which \( v(t) \) is in \( H^m \) when \( t \) is in \( [a, b] \) and the function \( v(t) \) is a function of \( t \in [a, b] \) which is continuous in the \( H^m \) norm. The continuity of \( v(t) \) in the \( H^m \) norm shows that

\[
|v|_{H^m \times [a,b]} := \sup_{t \in [a,b]} \left[ \sum_{0 \leq |k'| \leq m} |D^{k'} v(t)|_{L^2} \right] < \infty
\]  

(2.27)

as \([a, b]\) is compact and that \( v(t) \) is uniformly continuous in the compact \([a, b]\) in the \( H^m \) norm. The right side of (2.27) is the norm on \( H^m \times [a, b] \).

\( H^{m,df} \times [a, b] \) is the divergence free subset of \( H^m \times [a, b] \). \( H^m \times [a, b] \) and \( H^{m,df} \times [a, b] \) are defined in the same way but with the half-open interval \([a, b)\) in place of \([a, b]\).

For \( v : R^3 \times [a, b] \to R^3 \), the norm

\[
|v|_{L^\infty \times [a,b]} := \sup_{t \in [a,b]} |v(t)|_{L^\infty}
\]

As we use the \( L^\infty \) norm only for continuous functions \( v \), the definition of \( |v|_{L^\infty \times [a,b]} \) does not exclude values of \( v \) in a set of measure zero.

A norm of a scalar valued function on \( R^3 \times [a, b] \) is determined as though it is the first component of a vector valued function that carries \( R^3 \times [a, b] \to R^3 \) the second and third components of which vanish.

\( C^\infty \) on \( R^3 \times [a, b] \) is the space of functions \( v : R^3 \times [a, b] \to R^3 \) each derivative of which exists on \( R^3 \times [a, b] \). \( C^\infty_o \) on \( R^3 \times [a, b] \) is the subspace of \( C^\infty \) on \( R^3 \times [a, b] \) each member of which vanishes outside of a compact subset of \( R^3 \times [a, b] \).

**Lemma 2.2.** \( H^m \times [a, b] \) is a Banach space.

**Proof.** As \( H^m \times [a, b] \) is a normed vector space, we need only show that \( H^m \times [a, b] \) is complete in its norm. When the sequence \( v^n \in H^m \times [a, b] \) is a Cauchy sequence in the \( H^m \times [a, b] \) norm and \( t \) is in \([a, b]\)

\[
|v^{n'}(t) - v^n(t)|_{H^m} \leq |v^{n'} - v^n|_{H^m \times [a,b]} := o_{2.28} (\min \ [n', n])
\]  

(2.28)

which shows when \( t \) is in \([a, b]\) that the sequence \( v^{n'}(t) \) is a Cauchy sequence in \( H^m \). As a result, \( v^{n'}(t) \) converges in the \( H^m \) norm to
a function $v(t) \in H^m$. The functions $v(t)$ define the function $v$ on $\mathcal{R}^3 \times [a, b]$ which we now show is $\in H^m \times [a, b]$.

Allowing $m \to \infty$ in (2.28) shows that

$$|v(t) - v^n(t)|_{H^m} \leq o_{2.28}(n)$$

and that

$$|v - v^n|_{H^m \times [a, b]} = \sup_{t \in [a, b]} |v(t) - v^n(t)|_{H^m} \leq o_{2.28}(n)$$

As a result, $v^n$ converges in the $H^m \times [a, b]$ norm to a function $v$ which is defined on $\mathcal{R}^3 \times [a, b]$ and the $H^m \times [a, b]$ norm of which is finite.

We now show that $v(t)$ is a function of $t \in [a, b]$ which is continuous in the $H^m$ norm. With $t$ and $t + \Delta t$ are $\in [a, b]$

$$|v(t + \Delta t) - v(t)|_{H^m} \leq |v(t + \Delta t) - v^n(t + \Delta t)|_{H^m} + |v^n(t + \Delta t) - v^n(t)|_{H^m} + |v^n(t) - v(t)|_{H^m}$$

That $v^n \to v$ in the $H^m \times [a, b]$ norm shows that the first and third summands are each $< \epsilon/3$ when $n$ is $\geq$ a sufficiently large $n_0(\epsilon)$. Therewith

$$|v(t + \Delta t) - v(t)|_{H^m} \leq \epsilon/3 + |v^{n_0(\epsilon)}(t + \Delta t) - v^{n_0(\epsilon)}(t)|_{H^m} + \epsilon/3$$

The continuity of $v^{n_0(\epsilon)}$ on $[a, b]$ in the $H^m$ norm shows that the right side is $< \epsilon$ when $\Delta t$ is sufficiently small. $\square$

**Lemma 2.3.** When $v$ is $\in H^4 \times [a, b]$

$$v$$

is uniformly continuous on $\mathcal{R}^3 \times [a, b]$ (2.29)

and when $v$ is $\in H^6 \times [a, b]$

$$v, v_j, v$$

and $\Delta v$ are uniformly continuous on $\mathcal{R}^3 \times [a, b]$ (2.30)

and

$$|v|_{L^\infty \times [a, b]} \leq |v|_{H^2 \times [a, b]} < \infty$$

(2.31)
Proof. We first establish (2.29). With the triangle inequality
\[
|v(x + \Delta x, t + \Delta t) - v(x, t)|_{L^\infty}
\leq |v(x + \Delta x, t + \Delta t) - v(x + \Delta x, t)|_{L^\infty}
+ |v(x + \Delta x, t) - v(x, t)|_{L^\infty}
\]
(2.14) shows that the first summand in the right side is
\[
\leq [\ldots] |v(x + \Delta x, t + \Delta t) - v(x + \Delta x, t)|_{H^2}
\]
which is \(o(\Delta t)\) as \(v \in H^4 \times [a, b]\) and is continuous on \([a, b]\) in the \(H^2\) norm. The second summand in the right side is \(\leq |\text{grad } v|_{L^\infty \times [a, b]} \Delta x \leq |v|_{H^3 \times [a, b]} \Delta x\). (2.14). As a result
\[
|v(x + \Delta x, t + \Delta t) - v(x, t)|_{L^\infty}
\leq [\ldots] o(\Delta t) + [\ldots] |v|_{H^3 \times [a, b]} \Delta x
\]
The proof of (2.29) is complete.

We now establish (2.30). When \(v \in H^6 \times [a, b]\), \(v_j\) and \(v\) are uniformly bounded, (2.14), and uniformly continuous on \(R^3 \times [a, b]\), (2.29), with the result that \(v_j v\) is uniformly continuous on \(R^3 \times [a, b]\). As \(\Delta v\) is \(H^4 \times [a, b]\) in this case, \(\Delta v\) is also uniformly continuous on \(R^3 \times [a, b]\). The proof of (2.30) is complete.

With (2.14)
\[
|v|_{L^\infty \times [a, b]} \leq [\ldots] |v|_{H^2 \times [a, b]} < \infty
\]
The proof of (2.31) is complete. \(\square\)

2.5. \(|z|^{-1}\) and certain derivatives thereof.

Lemma 2.4. Let \(z \neq 0\) be \(R^3\) and \(r > 0\). Then
\[
\partial_i |z|^{-1} = -z_i |z|^{-3}
\]
\[
|\partial_i |z|^{-1}| \leq |z|^{-2}
\]
\[
|\partial_i |z|^{-1}|_{L^2(R^3 \setminus B(r))} \leq [\ldots]_{2.32c} r^{-1/2} \text{ and}
\]
\[
D^{k'}|z|^{-1} \text{ is } \in L^2(R^3 \setminus B(r))
\]
(2.32)
\[ \partial_i |z|^{-3} = -3z_i |z|^{-5} \]
\[ |\partial_k \partial_i |z|^{-1}| \leq [\ldots]_{2.33b} |z|^{-3} \text{ and} \]
\[ \partial_k \partial_i |z|^{-1} = |z|^{-3} + 3z_k z_i |z|^{-5} \]

(2.33)

\[ |z|^{-1}|L^1(B(r))| \leq [\ldots]_{2.34a} r^2 \]
\[ |\partial_i |z|^{-1}|L^1(B(\Omega))| \leq [\ldots]_{2.34b} r \]
\[ |\partial_i |z|^{-1}|L^1(\partial B(r))| \leq [\ldots]_{2.34c} \]
\[ |\partial_j \partial_k |z|^{-1}|L^1(\partial B(r))| \leq [\ldots]_{2.34d} r^{-1} \text{ and} \]
\[ |\partial_i \partial_j \partial_k |z|^{-1}|L^1(\mathbb{R}^3 \backslash B(r))| \leq [\ldots]_{2.34e} r^{-1} \]

(2.34)

**Proof.** As \( |z| = [\sum_i z_i^2]^{1/2} \), the chain rule shows that
\[ \partial_i |z|^{-1} = -1/2 [\sum_i z_i^2]^{-3/2} 2z_i = -z_i |z|^{-3} \]

The proof of the first line of (2.32) is complete. That the absolute value of \( z_i/|z| \) is \( \leq 1 \) establishes the second line of (2.32).

We now establish the third line of (2.32). With the first line of (2.32)
\[ |\partial_i |z|^{-1}|_{L^2(\mathbb{R}^3 \backslash B(r))} \]
\[ = [\int_{r \leq |z|} \left| -z_i/|z|^3 \right|^2 d\mathbb{S} \] \( \left[ \int_{r \leq |z|} z_i^2/|z|^6 d\mathbb{S} \right]^{1/2} \]
\[ \text{ which after changing to spherical coordinates in such integral is} \]
\[ \leq [\int_{r \leq |z|} |z|^{-4} \left[ \ldots \right]|z|^2 d[|z|] ]^{1/2} \leq [\ldots]_{2.32e} r^{-1/2} \]

(2.35)

The proof of the third line of (2.32) is complete.

We establish the fourth line of (2.32) below.

We now establish the first line of (2.33). With the chain rule
\[ \partial_i |z|^{-3} = -3 |z|^{-4} \partial_i |z| \]
\[ = -3 |z|^{-4} z_i |z| = -3 z_i |z|^{-5} \]

The proof of the first line of (2.33) is complete.
We now establish the second line of (2.33). With \( k \neq i \) and (2.32) (first line)
\[
\partial_k \partial_i |z|^{-1} = \partial_k [-z_i/|z|^3]
\]
which with (2.33) (first line) is
\[
= -z_i(-3)z_k |z|^{-5} = [\ldots] z_i z_k |z|^{-5}
\]
the absolute value of which is \( \leq [\ldots] |z|^{-3} \). The proof of the second line of (2.33) is complete.

When \( k = i \) and (2.33) (first line)
\[
\partial_k \partial_i |z|^{-1} = \partial_k [-z_k |z|^{-3}] = -|z|^{-3} + 3z_k z_k |z|^{-5}
\]
(2.36)
the absolute value of which is also \( \leq [\ldots] |z|^{-3} \). (2.36) establishes the third line of (2.33).

The first line of (2.34) follows after a change to spherical coordinates in an integral. The second line of (2.34) follows from the second line of (2.32) and a change to spherical coordinates. The third line of (2.34) follows from the second line of (2.32) as the surface area of \( \partial B(r) \) is
\[
= [\ldots] r^2.1
\]

We now establish the fourth line of (2.34). As the absolute value of \( \partial_i \partial_j \partial_k |z|^{-1} \) is \( \leq [\ldots] r^{-3} \), (2.33) (second line), and the surface area of \( \partial B(r) \) is \( = [\ldots] r^2 \). The proof of the fourth line of (2.34) is complete.

We now establish the fifth line of (2.34). When \( i, j \) and \( k \) are different
\[
\partial_i \partial_j \partial_k |z|^{-1} = [\ldots] z_i z_j z_k |z|^{-7}
\]
the absolute value of which is \( \leq [\ldots] |z|^{-4} \). With (2.36)
\[
\partial_i \partial_j \partial_k |z|^{-1}
= \partial_i [-|z|^{-3} + 3z_k z_k |z|^{-5}]
= [\ldots] z_i |z|^{-5} + [\ldots] z_k z_k z_i |z|^{-7}
\]
the absolute value of which is also \( \leq [\ldots] |z|^{-4} \).

\(^1\)http://math2.org/math/geometry/areasvols.htm. Accessed 27 December 2024.
In the remaining case
\[
\partial_k \partial_k \partial_k |z|^{-1} = \partial_k \left[ - |z|^{-3} + 3z_k \partial_k |z|^{-5} \right] = \left[ \ldots \right] z_k |z|^{-5} + \left[ \ldots \right] z_k |z|^{-5} + \left[ \ldots \right] z_k z_k z_k |z|^{-7}
\]
the absolute value of which is also \( \leq \left[ \ldots \right] |z|^{-4} \). Therewith
\[
\int_{R^3 \setminus B(r)} | \partial_i \partial_j \partial_k |z|^{-1} | \; dz 
\leq \int_{r \leq |z|} \left[ \ldots \right] |z|^{-4} |z|^2 \, d|z| \leq \int_{r \leq |z|} \left[ \ldots \right] |z|^{-2} \, d|z| = \left[ \ldots \right]_{2.34e} r^{-1}
\]
for all i, j and k. The proof of the fifth line of (2.34) is complete.

We now establish the fourth line of (2.32). Our earlier results show that
\[
|D^{k'}| |z|^{-1} | \text{ is } \leq \left[ \ldots \right] (k') |z|^{-|k'| - 1}
\]
when \( 1 \leq |k'| \leq 4 \). Such inequality extends to larger values of \( |k'| \). We leave the details to the reader. The extended inequality with the calculation at (2.35) completes the proof of the fourth line of (2.32).

3. Certain Convolutions

This section first studies the convolutions
\[
\hat{m} \ast v(x) := \int_{R^3} \hat{m}(y) \, v(x - y) \, dy
\]
in which the mollifier \( \hat{m} \) is \( \in L^1 : R^3 \to R^1 \) and the function \( v \) carries \( R^3 \to R^3 \). Such mollifiers \( \hat{m} \) include
\[
K(y, t) := (4\pi t)^{-3/2} \exp(-|y|^2/4t)
\]
in which t is \( > 0 \) and
\[
|K(t)|_{L^1} = 1
\]
(3.1)
and
\[
m^\dagger_\gamma(y) := \gamma^{-3} \, m^\dagger(y \gamma^{-1})
\]
in which \( \gamma \) is \( > 0 \) and
\[
|m^\dagger_\gamma|_{L^1} = 1
\]
(3.2)
$m^\dagger$ is the standard, $C^\infty_o$ mollifier of [4] Appendix C.5; it vanishes outside of $B(1)$. We use
\begin{align*}
J_{\gamma>0}(v_j) &:= m^\dagger_\gamma * v_j \quad \text{and} \\
J_{\gamma=0}(v_j) &:= v_j
\end{align*}
(3.3)
to abbreviate certain convolutions and the limit thereof. As $K(t)$ and $m^\dagger_\gamma$ are $L^1 \cap L^\infty$, each is also $L^2$.

This section also studies the convolutions
\[
\int_{\mathbb{R}^3} \partial_i |z|^{-1} v(x - z) \, dz
\]
in which $v$ is $\in \cap_m \left[ H^m \cap W^{m,1} \right]$ and the mollifier $\partial_i |z|^{-1}$ is $L^1(B(r))$ and is also $L^2(\mathbb{R}^3 \setminus B(r))$.

**Lemma 3.1.** Let $\hat{m} \in L^1 \cap L^2 : \mathcal{R}^3 \to \mathcal{R}^1$, $k'$ be a multi-index for which $0 \leq |k'| \leq m$ and $v \in H^m$. Then the convolution
\[
\hat{m} * D^{k'} v (x) = \int_{\mathbb{R}^3} \hat{m}(y) \, D^{k'} v_k(x - y) \, dy
\]
(3.4)
is a bounded, uniformly continuous function of $x \in \mathcal{R}^3$ which is measurable on $\mathcal{R}^3$
the weak derivative
\[
D^{k'} [\hat{m} * v] = \hat{m} * D^{k'} v
\]
(3.5)
\[
|\hat{m} * D^{k'} v|_{L^\infty} \leq |\hat{m}|_{L^1} \, |D^{k'} v|_{L^\infty}
\]
when $0 \leq |k'| \leq m - 2$
(3.6)
\[
|D^{k'} [m^\dagger_{\gamma>0} * v]|_{L^\infty} \leq |D^{k'} m^\dagger_{\gamma>0}|_{L^2} \, \gamma^{-3/2-|k'|} \, |v|_{L^2}
\]
(3.7)
and
\[
|\hat{m} * v|_{H^m} \leq |\hat{m}|_{L^1} \, |v|_{H^m}
\]
(3.8)
Proof. We first establish (3.4). As $\hat{m}$ and $D^{k'}v$ are in $L^2$, a Hölder inequality shows that the convolution

$$\int_{\mathbb{R}^3} \hat{m}(y) \ D^{k'}v_k(x - y) \ dy$$

(3.9)

is a bounded function of $x \in \mathbb{R}^3$. The convolution is also uniformly continuous on $\mathbb{R}^3$, [1] §2.23, and is, as a result, measurable thereon. [1] §1.42. The proof of (3.4) is complete.

We now establish (3.5) for each component $v_k$ of $v \in H^m$. As the convolution

$$\int_{\mathbb{R}^3} \hat{m}(y) \ D^{k'}v_k(x - y) \ dy$$

is a bounded function of $x \in \mathbb{R}^3$, (3.4)

$$\phi(x) \left[ \int_{\mathbb{R}^3} \hat{m}(y) \ D^{k'}v_k(x - y) \ dy \right]$$

$$= \int_{\mathbb{R}^3} \phi(x) \ \hat{m}(y) \ D^{k'}v_k(x - y) \ dy$$

for each scalar valued $\phi \in C^\infty_0$, and as each such $\phi$ is in $L^1$

$$\int_{\mathbb{R}^3} \phi(x) \left[ \int_{\mathbb{R}^3} \hat{m}(y) \ D^{k'}v_k(x - y) \ dy \right] \ dx$$

$$= \int_{\mathbb{R}^3} \left[ \int_{\mathbb{R}^3} \phi(x) \ \hat{m}(y) \ D^{k'}v_k(x - y) \ dy \right] \ dx$$

(3.10)

for each such $\phi$. 
Fubini’s theorem\(^2\) and the inequality in the margin\(^3\) show that the right side of (3.10) is equal to its related iterated integral

\[
\int_{R^3} \left[ \int_{R^3} \phi(x) \, \widehat{m}(y) \, D^k v_k(x - y) \, dx \right] \, dy
\]

which after integrating by parts in the interior integral thereof, (2.12), is

\[
= \int_{R^3} \left[ \int_{R^3} (-1)^{|k'|} D^k \phi(x) \, \widehat{m}(y) \, v_k(x - y) \, dx \right] \, dy
\]

(3.11)

---

\(^2\) When \(f(x, y)\) is a scalar valued, measurable function on \(R^{m+n}\) and one of the integrals

\[
\int_{R^{m+n}} |f(x, y)| \, dxdy
\]

\[
\int_{R^m} \left[ \int_{R^n} |f(x, y)| \, dx \right] \, dy \quad \text{and} \quad \int_{R^m} \left[ \int_{R^n} |f(x, y)| \, dy \right] \, dx
\]

(0.1)

is finite, Fubini’s theorem provides that such integrals are well defined and are equal to each other. \([1]\) §1.54. As a corollary, the integrals

\[
\int_{R^{m+n}} f(x, y) \, dxdy
\]

\[
\int_{R^n} \left[ \int_{R^m} f(x, y) \, dx \right] \, dy \quad \text{and} \quad \int_{R^m} \left[ \int_{R^n} f(x, y) \, dy \right] \, dx
\]

are also equal to each other. On occasion, we use Fubini’s theorem by first reversing the order of integration in an iterated integral and then showing that the related, iterated integral is \(<\infty\).

---

\(^3\) A Hölder inequality in the interior integral below shows that

\[
\int_{R^3} \left[ \int_{R^3} |\phi(x)| \, |\widehat{m}(y)| \, |D^k v_k(x - y)| \, dx \right] \, dy
\]

is \(\leq |\widehat{m}|_{L^1} \, |\phi|_{L^2} \, |D^k v_k|_{L^2}\)

when \(\phi\) is \(\in C_o^\infty\).
Fubini’s theorem and the inequality in the margin\(^4\) then show that (3.11) is equal to its related iterated integral
\[
\int_{\mathbb{R}^3} \left[ \int_{\mathbb{R}^3} (-1)^{|k'|} D^k \phi(x) \, \hat{m}(y) \, v_k(x - y) \, dy \right] \, dx
\]
which (3.10) (with \(k' = 0\) and \(D^k \phi\) in place of \(\phi\)) shows is
\[
= \int_{\mathbb{R}^3} (-1)^{|k'|} D^k \phi(x) \left[ \int_{\mathbb{R}^3} \hat{m}(y) \, v_k(x - y) \, dy \right] \, dx
\]
Thus, the left side of (3.10), which is
\[
\int_{\mathbb{R}^3} \phi(x) \left[ \int_{\mathbb{R}^3} \hat{m}(y) \, D^k v_k(x - y) \, dy \right] \, dx
\]
is
\[
= \int_{\mathbb{R}^3} (-1)^{-|k'|} D^k \phi(x) \left[ \int_{\mathbb{R}^3} \hat{m}(y) \, v_k(x - y) \, dy \right] \, dx
\]
for every scalar valued \(\phi \in C_0^\infty\). The proof of (3.5) for \(v_k\) is complete. The proof of (3.5) is also complete.

We now establish (3.6). When \(0 \leq |k'| \leq m - 2\), each \(v_k\) is \(\in L^\infty\). (2.14). The definition of the \(L^\infty\) norm shows that
\[
| \hat{m} \ast v |_{L^\infty} = \sup_k \left| [\hat{m} \ast v]_k \right|_{L^\infty} = \sup_k | \hat{m} \ast v_k |_{L^\infty}
\]
which with (2.5) is
\[
\leq \sup_k \left[ | \hat{m} |_{L^1} \, | v_k |_{L^\infty} \right] = | \hat{m} |_{L^1} \, | v |_{L^\infty}
\]
The proof of (3.6) is complete.

We now establish (3.7). As \(v \in H^m\), (3.5) shows that
\[
D^k [ m^\dagger_{\gamma > 0} \ast v ] = \int_{\mathbb{R}^3} m^\dagger_{\gamma > 0}(y) \, D^k v(x - y) \, dy
\]
(3.12)
which integration by parts, (2.12), shows is

$$= \int_{\mathbb{R}^3} D^k m_{\gamma > 0}^\dagger(y) \ v(x - y) \ dy$$

A Hölder inequality then shows that

$$|D^k [m_{\gamma > 0}^\dagger * v]|_{L^\infty} \leq \sup_k \left[ |D^k m_{\gamma > 0}^\dagger|_{L^2} \ |v_k|_{L^2} \right]$$

which with the chain rule is

$$= \gamma^{-3} |D^k m_{\gamma > 0}^\dagger(y \gamma^{-1})|_{L^2(y \in \mathbb{R}^3)} \ \gamma^{-|k'|} \ |v|_{L^2}$$

and which after changing variables in an integral is

$$= \gamma^{-3} |D^k m_{\gamma > 0}^\dagger|_{L^2} \ \gamma^{-3/2} \ \gamma^{-|k'|} \ |v|_{L^2}$$

$$= |D^k m_{\gamma > 0}^\dagger|_{L^2} \ \gamma^{-3/2-|k'|} \ |v|_{L^2}$$

The proof of (3.7) is complete.

We now establish (3.8). We first establish the intermediate result

$$|\hat{m} * v_k|_{L^2} \leq |\hat{m}|_{L^1} \ |v_k|_{L^2}$$

(3.13)

As (3.4) shows that the convolution $\int_{\mathbb{R}^3} \hat{m}(y) \ v_k(x - y) \ dy$ is a bounded and measurable function of $x \in \mathbb{R}^3$, the absolute value thereof is also bounded and measurable, [1] §1.42, and as in the proof of (3.5)

$$\left| \int_{\mathbb{R}^3} \phi(x) \ \int_{\mathbb{R}^3} \hat{m}(y) \ v_k(x - y) \ dy \ dx \right|$$

is well defined for $\phi \in C^\infty_o$ and is

$$\leq \int_{\mathbb{R}^3} \left[ \int_{\mathbb{R}^3} |\phi(x)| \ |\hat{m}(y)| \ |v_k(x - y)| \ dy \right] dx$$

(3.14)

Fubini’s theorem and the note 5 inequality with $k' = 0$ show that the right side of (3.14) is equal to its related, iterated integral

$$\int_{\mathbb{R}^3} \left[ \int_{\mathbb{R}^3} |\phi(x)| \ |\hat{m}(y)| \ |v_k(x - y)| \ dx \right] dy$$

which with an Hölder inequality in its interior integral is

$$\leq |\phi|_{L^2} \ |\hat{m}|_{L^1} \ |v_k|_{L^2}$$
The Hölder Inequality Converse then shows that
\[ |\widehat{m} \ast v_k|_{L^2} = \left| \int_{\mathbb{R}^3} \widehat{m}(y) v_k(x - y) \, dy \right|_{L^2} \]
is \leq |\widehat{m}|_{L^1} |v_k|_{L^2}
The proof of (3.13) is complete.

With the definition of the \( L^2 \) norm
\[ |\widehat{m} \ast v|_{L^2} = \left[ \sum_k |\widehat{m} \ast v_k|^2_{L^2} \right]^{1/2} \]
which with (3.13) is
\[ \leq \left[ \sum_k \left( |\widehat{m}|_{L^1} |v_k|_{L^2} \right)^2 \right]^{1/2} = |\widehat{m}|_{L^1} \left[ \sum_k |v_k|^2_{L^2} \right]^{1/2} \]
\[ = |\widehat{m}|_{L^1} |v|_{L^2} \]
(3.15)
The definition of the \( H^m \) norm then shows that
\[ |\widehat{m} \ast v|_{H^m} = \sum_{0 \leq |k'| \leq m} |D^{k'}[\widehat{m} \ast v]|_{L^2} \]
\[ = \sum_{0 \leq |k'| \leq m} |\widehat{m} \ast D^{k'}v|_{L^2} \]
(3.5). Therewith, (3.15) shows that
\[ |\widehat{m} \ast v|_{H^m} \leq \sum_{0 \leq |k'| \leq m} |\widehat{m}|_{L^1} |D^{k'}v|_{L^2} = |\widehat{m}|_{L^1} |v|_{H^m} \]
The proof of (3.8) is complete. □

As \( |m^1|_{L^1} = 1 \), (3.6) and (3.8) show that \( J_\gamma \) does not increase an \( L^\infty \) norm or an \( H^m \) norm. Hereafter, we use such results, on occasion, without reference to (3.6) or (3.8).

---

5 The “Hölder Inequality Converse” is the proposition that a measurable function \( v \), such as \( \widehat{m} \ast v_k \), is \( \in L^2 \) if
\[ \sup_{\phi \in C^\infty_0, \ |\phi|_{L^2}=1} \left| \langle |v(x)|, \phi(x) \rangle_{\mathbb{R}^3} \right| \ is \ < \infty \]
in which case, \( |v|_{L^2} \) is equal to that finite supremum. See [1] §2.7 (\( C^\infty_0 \) is dense in \( L^2 \)).
We next study the convolutions
\[
\int_{\mathbb{R}^3} \partial_i |z|^{-1} v(x - z) \, dz
\]
in which \( v \) is \( \in \cap_m[H^m \cap W^{m,1}] \) and the mollifier \( \partial_i |z|^{-1} \) is \( \in L^1(B(r)) \) and is also \( \in L^2(\mathbb{R}^3 \setminus B(r)) \).

**Lemma 3.2.** When \( v \) is \( \in \cap_m[H^m \cap W^{m,1}] \) and \( k' \) is a multi-index, the weak derivative
\[
D^{k'} \left[ \int_{\mathbb{R}^3} \partial_i |z|^{-1} v_k(x - z) \, dz \right] = \int_{\mathbb{R}^3} \partial_i |z|^{-1} D^{k'} v_k(x - z) \, dz
\]
(3.16)
\[
\left| \int_{\mathbb{R}^3} \partial_i |z|^{-1} v_k(x - z) \, dz \right|_{H^m} \leq [\ldots]_{2.34b} |v_k|_{H^m} + [\ldots]_{2.32c} |v_k|_{W^{m,1}}
\]
(3.17)
\[
\int_{\mathbb{R}^3} \partial_i |z|^{-1} v_k(x - z) \, dz \text{ is } \in \cap_m H^m
\]
(3.18)
and
\[
\int_{\mathbb{R}^3} \partial_k |z|^{-1} \text{div} v(x - z) \, dz = \partial_k \left[ \int_{\mathbb{R}^3} \sum_i \partial_i |z|^{-1} v_i(x - z) \, dz \right]
\]
(3.19)

**Proof.** We first study the integral
\[
\int_{\mathbb{R}^3} \partial_i |z|^{-1} \psi_{left}^\epsilon(z) v_k(x - z) \, dz
\]
in which \( \epsilon > 0 \), \( \psi_{left}^\epsilon(z) \) is a scalar valued \( C^\infty \) function which is \( = 1 \) on \( B(1) \), \( \epsilon \in [0, 1] \) on \( B(1 + \epsilon) \setminus B(1) \) and vanishes outside of \( B(1 + \epsilon) \).

As \( \psi_{left}^\epsilon(z) \) is \( \in [0, 1] \) when \( z \) is \( \in \mathbb{R}^3 \)
\[
| \partial_i |z|^{-1} \psi_{left}^\epsilon(z) |_{L^1} \leq | \partial_i |z|^{-1} |_{L^1(B(1+\epsilon))}
\]
is \( \leq [\ldots]_{2.34b} (1 + \epsilon) 
\]
(3.20)
As $v_k$ is $\in \cap_m [H^m \cap W^{m,1}]$, which is a subset of $\cap_m H^m$, (3.5) shows that the weak derivative
\[ D^{k'} \left[ \int_{\mathbb{R}^3} \left[ \partial_i |z|^{-1} \psi^\varepsilon_{\text{left}}(z) \right] v_k(x - z) \, dz \right] \]
is equal to
\[ \int_{\mathbb{R}^3} \left[ \partial_i |z|^{-1} \psi^\varepsilon_{\text{left}}(z) \right] D^{k'} v_k(x - z) \, dz \]
(3.21)
and (3.8) shows that
\[ |D^{k'} \left[ \int_{\mathbb{R}^3} \left[ \partial_i |z|^{-1} \psi^\varepsilon_{\text{left}}(z) \right] v_k(x - z) \, dz \right] |_{L^2} \]
\[ \leq \left| \partial_i |z|^{-1} \psi^\varepsilon_{\text{left}}(z) \right|_{L^1} |D^{k'} v_k|_{L^2} \]
\[ \leq \left[ \cdots \right]_{2.34b} (1 + \varepsilon) |D^{k'} v_k|_{L^2} \]
(3.22)

We now study the integral
\[ \int_{\mathbb{R}^3} \partial_i |z|^{-1} \psi^\varepsilon_{\text{right}}(z) \, v_k(x - z) \, dz \]
in which
\[ \psi^\varepsilon_{\text{right}}(z) := 1 - \psi^\varepsilon_{\text{left}}(z) \]
(3.23)
which is a scalar valued $C^\infty$ function on $\mathbb{R}^3$ which vanishes in $B(1)$, is $\in [0, 1]$ on $B(1 + \varepsilon) \setminus B(1)$ and is $= 1$ on $\mathbb{R}^3 \setminus B(1 + \varepsilon)$.

Changing variables in a convolution shows that
\[ \int_{\mathbb{R}^3} [\partial_i |z|^{-1} \psi^\varepsilon_{\text{right}}(z)] \, v_k(x - z) \, dz \]
\[ = \int_{\mathbb{R}^3} [\partial_i |x - z|^{-1} \psi^\varepsilon_{\text{right}}(x - z)] \, v_k(z) \, dz \]
As $v_k$ is $\in \cap_m [H^m \cap W^{m,1}]$, which is a subset of $L^1$ and $D^{k'} [\partial_i |x - z|^{-1} \psi^\varepsilon_{\text{right}}(x - z)]$ is a function of $z$ which with $x$ fixed is $\in L^2$, (2.32) (fourth line), the weak derivative
\[ D^{k'} \left[ \int_{\mathbb{R}^3} [\partial_i |x - z|^{-1} \psi^\varepsilon_{\text{right}}(x - z)] \, v_k(z) \, dz \right] \]
\[ = \int_{\mathbb{R}^3} D^{k'} \left[ \partial_i |x - z|^{-1} \psi^\varepsilon_{\text{right}}(x - z) \right] \, v_k(z) \, dz \]
(3.5), which after integrating by parts ($D^k v_k \in L^2$) is
\[
= \int_{\mathbb{R}^3} \partial_i |x - z|^{-1} \psi_{\text{right}}^\epsilon(x - z) \, D^k v_k(z) \, dz
\]
which after again changing variables in the convolution shows that the weak derivative
\[
D^k x \left[ \int_{\mathbb{R}^3} \partial_i |z|^{-1} \psi_{\text{right}}^\epsilon(z) \, v_k(x - z) \, dz \right]
\]
is
\[
= \int_{\mathbb{R}^3} \partial_i |z|^{-1} \psi_{\text{right}}^\epsilon(z) \, D^k v_k(x - z) \, dz
\]
(3.24)

Therewith, (3.8) shows that
\[
| D^k \left[ \int_{\mathbb{R}^3} \partial_i |z|^{-1} \psi_{\text{right}}^\epsilon(z) \, v_k(x - z) \, dz \right] |_{L^2} \leq | \partial_i |z|^{-1} \psi_{\text{right}}^\epsilon(z) |_{L^2} | D^k v_k |_{L^1}
\]
which as $\psi_{\text{right}}^\epsilon$ vanishes on $B(1)$ is
\[
\leq | \partial_i |z|^{-1} |_{L^2(\mathbb{R}^3 \setminus B(1))} | D^k v_k |_{L^1} \leq [ \ldots ]_{2.32c} | D^k v_k |_{L^1}
\]
(3.25)

(3.21) and (3.24) then show that
\[
D^k \left[ \int_{\mathbb{R}^3} \partial_i |z|^{-1} \, v_k(x - z) \, dz \right]
\]
\[
= \int_{\mathbb{R}^3} \partial_i |z|^{-1} \psi_{\text{left}}^\epsilon(z) \, D^k v_k(x - z) \, dz
\]
\[
+ \int_{\mathbb{R}^3} \partial_i |z|^{-1} \psi_{\text{right}}^\epsilon(z) \, D^k v_k(x - z) \, dz
\]
with which (3.23) shows that
\[
D^k \left[ \int_{\mathbb{R}^3} \partial_i |z|^{-1} \, v_k(x - z) \, dz \right] = \int_{\mathbb{R}^3} \partial_i |z|^{-1} \, D^k v_k(x - z) \, dz
\]
The proof of (3.16) is complete.

(3.22) and (3.25) show that
\[
| D^k \left[ \int_{\mathbb{R}^3} \partial_i |z|^{-1} \, v_k(x - z) \, dz \right] |_{L^2} \leq [ \ldots ]_{2.34b} (1 + \epsilon) | D^k v_k |_{L^2} + [ \ldots ]_{2.32c} | D^k v_k |_{L^1}
\]
Allowing \( \epsilon \to 0 \) and then summing over those multi-indices \( k' \) for which \( 0 \leq |k'| \leq m \) completes the proof of (3.17) and of (3.18).

We now establish (3.19). When \( r > 0 \)
\[
\int_{\mathbb{R}^3} \partial_i |z|^{-1} v_i (x - z) \, dz
= \int_{B(r)} \partial_i |z|^{-1} v_i (x - z) \, dz + \int_{\mathbb{R}^3 \setminus B(r)} \partial_i |z|^{-1} v_i (x - z) \, dz
\]
(3.26)

which after using the Gauss-Green theorem, [4] C.2, to integrate by parts in the second summand in the right side of (3.26) is
\[
= \int_{B(r)} \partial_i |z|^{-1} v_i (x - z) \, dz
+ \int_{\partial[\mathbb{R}^3 \setminus B(r)]} |z|^{-1} v_i (x - z) \, d\Sigma(r) + \int_{\mathbb{R}^3 \setminus B(r)} |z|^{-1} \partial_i v_i (x - z) \, dz
\]
(3.27)

Therein, \( d\Sigma(r) \) is the measure on the surface of \( \mathbb{R}^3 \setminus B(r) \), and \( \vec{n}_{r,i}(z) \) is the \( i^{th} \) component of the outward pointing unit normal to \( \partial[\mathbb{R}^3 \setminus B(r)] \) at \( z \).

The absolute value of the first (3.27) summand is
\[
\leq |\partial_i |z|^{-1}|_{L^1(B(r))} |v_i|_{L^\infty} \leq [ \ldots ]_{2.346} |v_i|_{L^\infty} r
\]
v is \( \in L^\infty \) because \( v_i \) is \( \in \cap_m [H^m \cap W^{m,1}] \). (2.14). The absolute value of the second (3.27) summand is
\[
\leq [ \ldots ] r^2 r^{-1} |v_i|_{L^\infty} \leq [ \ldots ] |v_i|_{L^\infty} r
\]
as the measure of \( \partial[\mathbb{R}^3 \setminus B(r)] \) is \( = [ \ldots ] r^2 \). The monotone convergence theorem, [1] §1.48, shows that the third (3.27) summand
\[
\to \int_{\mathbb{R}^3} |z|^{-1} \partial_i v_i (x - z) \, dz \text{ as } r \to 0
\]

Allowing \( r \to 0 \) in (3.27) shows that
\[
\int_{\mathbb{R}^3} \partial_i |z|^{-1} v_i (x - z) \, dz = \int_{\mathbb{R}^3} |z|^{-1} \partial_i v_i (x - z) \, dz
\]
(3.28)
when \( v_i \) is \( \in \cap_m [H^m \cap W^{m,1}] \).
As $\partial_i v_i$ is also $\in \cap_m [H^m \cap W^{m,1}]$, (3.28) also shows that

$$\int_{\mathbb{R}^3} \partial_k |z|^{-1} [\partial_i v_i] (x-z) \, dz = \int_{\mathbb{R}^3} |z|^{-1} \partial_k \partial_i v_i (x-z) \, dz$$

(3.29)

and as (3.16) shows that

$$\partial_k \left[ \int_{\mathbb{R}^3} \partial_i |z|^{-1} v_i (x-z) \, dz \right] = \int_{\mathbb{R}^3} |z|^{-1} \partial_k \partial_i v_i (x-z) \, dz$$

(3.29) shows that

$$= \int_{\mathbb{R}^3} \partial_k |z|^{-1} \partial_i v_i (x-z) \, dz$$

The proof of (3.19) is complete. \hfill \Box

4. $\mathcal{G}, \mathcal{P} : L^2 \to L^2$

This section establishes the existence of the Helmholtz-Hodge decomposition

$$v = \mathcal{G} v + \mathcal{P} v$$

(4.1)

of $v \in L^2$ and shows when $v$ is $\in \cap_m [H^m \cap W^{m,1}]$ that $\mathcal{G} v$ is the gradient of a scalar valued member of $\cap_m H^{m,df}$ and $\mathcal{P} v$ is divergence free.

We first define

$$\mathcal{G}^\dagger v(x) := (4\pi)^{-1} \int_{\mathbb{R}^3} grad |z|^{-1} \operatorname{div} v (x-z) \, dz$$

(4.2)

for $v \in \cap_m [H^m \cap W^{m,1}]$ and then define

$$\mathcal{P}^\dagger v(x) := v(x) - \mathcal{G}^\dagger v(x)$$

(4.3)

for such $v$. After we study $\mathcal{G}^\dagger$ and $\mathcal{P}^\dagger$, we extend those operators to the operators $\mathcal{G}$ and $\mathcal{P}$ which are defined on $L^2$. 
Lemma 4.1. Let \( v \) and \( \hat{v} \) be \( \in \cap_m [H^m \cap W^{m,1}] \) and \( \alpha \) and \( \beta \) be scalars. Then

\[
v = G^\dagger v + P^\dagger v
\]

(4.4)

\[
G^\dagger [\alpha v + \beta \hat{v}] = \alpha G^\dagger v + \beta G^\dagger \hat{v}
\quad \text{and}
\]

\[
P^\dagger [\alpha v + \beta \hat{v}] = \alpha P^\dagger v + \beta P^\dagger \hat{v}
\]

(4.5)

\( G^\dagger v \) and \( P^\dagger v \) are \( \in \cap_m H^m \)

(4.6)

and

\[
G^\dagger v(x) = (-1) \, \text{grad} \left[ (-4\pi)^{-1} \int_{\mathbb{R}^3} \sum_i \partial_i |z|^{-1} \, v_i(x-z) \, dz \right] = (-1) \, \text{grad} \, p_v(x)
\]

(4.7)

where \( p_v(x) \) is a scalar valued member of \( \cap_m H^m \).

Proof. (4.3) establishes (4.4). (4.2) and (4.3) show that \( G^\dagger \) and \( P^\dagger \) satisfy (4.5). As \( \text{div} \, v \) is \( \in \cap_m [H^m \cap W^{m,1}] \), (3.18) shows that \( G^\dagger v \) is \( \in \cap_m H^m \). (4.3) then shows that \( P^\dagger v \) is \( \in \cap_m H^m \), which completes the proof of (4.6). (4.2) and (3.19) establish (4.7), and (3.18) shows that \( p_v \) is a scalar valued member of \( \cap_m H^m \). \(\square\)

Lemma 4.2. Let \( v \) and \( \hat{v} \) be \( \in \cap_m [H^m \cap W^{m,1}] \). Then

\( G^\dagger v = 0 \) and \( v = P^\dagger v \)

when \( \text{div} \, v = 0 \)

(4.8)

\[
\text{div} \, G^\dagger v = \text{div} \, v \quad \text{and}
\]

\[
\text{div} \, P^\dagger v = 0
\]

(4.9)

\[
(\langle G^\dagger v, P^\dagger \hat{v} \rangle_{\mathbb{R}^3} = 0
\]

(4.10)
〈G^tv, ̇v〉_{R^3} = 〈v, G^t ̇v〉_{R^3} \quad and
〈P^tv, ̇v〉_{R^3} = 〈v, P^t ̇v〉_{R^3}
(4.11)

D^{k'} G^tv = G^t D^{k'} v \quad and
D^{k'} P^tv = P^t D^{k'} v
(4.12)

|v|_{H^m}^2 = |G^tv|_{H^m}^2 + |P^tv|_{H^m}^2
|G^tv|_{H^m} \leq |v|_{H^m} \quad and
|P^tv|_{H^m} \leq |v|_{H^m}
(4.13)

and with β ≠ 0 and v_β(x) := v(xβ)

[G^tv_β] (x) = [G^tv] (xβ) \quad and
[P^tv_β] (x) = [P^tv] (xβ)
(4.14)

Proof. We first establish (4.8). (4.2) shows that G^tv = 0 when div v = 0. (4.3) then shows that P^tv = v for such v. The proof of (4.8) is complete.

We now establish (4.9). With the (4.2) definition of G^t

div G^tv(x) = \sum_k \partial_k \left[ (4\pi)^{-1} \int_{R^3} \partial_k |z|^{-1} \ div v(x - z) \ dz \right]
Then with (3.16) and r > 0

div G^tv(x)
= (4\pi)^{-1} \sum_k \left[ \int_{B(r)} \partial_k |z|^{-1} \partial_k \ div v(x - z) \ dz \right] 
+ (4\pi)^{-1} \sum_k \int_{R^3 \setminus B(r)} \partial_k |z|^{-1} \partial_k \ div v(x - z) \ dz
(4.15)
which (3.16) and the Gauss-Green theorem in the second summand in
the right side of (4.15) show is

\[
\begin{align*}
&= (4\pi)^{-1} \sum_k \left[ \int_{B(r)} \partial_k \partial v(x-z) \, dz \right] \\
&\quad + (4\pi)^{-1} \int_{\mathbb{R}^3 \setminus B(r)} \sum_k \partial^2_k \partial v(x-z) \, dz \\
&\quad + (4\pi)^{-1} \sum_k \int_{\partial B(r)} \partial_k \partial v(x-z) \, d\Sigma(r)
\end{align*}
\]

(4.16)

The absolute value of the first (4.16) summand is

\[
\leq (4\pi)^{-1} \sum_k \left| \partial_k \partial v(x-z) \right| L^1(B(r)) \left| \partial_k \partial v \right| L^\infty
\]

(4.17)

(3.6), which with (2.34) (second line) and (2.14) \(|\partial_k \partial v| L^\infty \leq [...]|v| H^4\) is

\[
\leq [...][...,2,346, r [...]|v| H^4 = o(r)
\]

(4.18)

The second (4.16) summand vanishes because (2.33) (third line)
shows that

\[
\sum_k \partial^2_k \partial v(x-z) = \sum_k \left[ -|z|^{-3} + 3 \frac{z_k z_k}{|z|^3} \right]
\]

\[
= -3 |z|^{-3} + 3 \sum_k z_k^2 |z|^{-5} = 0
\]

(4.19)

In the third (4.16) summand, \(\partial_k \partial v = \frac{-z_k}{|z|^3}\), (2.32) (first line),
which as the outward pointing unit normal \(\vec{n}_{r,k}(z) = -\frac{z_k}{|z|}\) and
\[ |z| = r \text{ is} \]
\[
= (4\pi)^{-1} \int_{\partial B(r)} \left[ \sum_k \left( -z_k / r^3 \right) \text{div} \ v (x - z) \left( -z_k / r \right) \right] d\Sigma(r)
\]
\[
= (4\pi)^{-1} \int_{\partial B(r)} \sum_k z_k^2 / r^4 \text{div} \ v (x - z) d\Sigma(r)
\]
\[
= (4\pi)^{-1} \int_{\partial B(r)} r^{-2} \text{div} \ v (x - z) d\Sigma(r)
\]
(4.20)

As \(4\pi r^2\) is the measure of \(\partial B(r)^6\) and \(\text{div} \ v (x - z)\) is uniformly continuous in \(z\), Lemma 2.3

\[(4.16) \rightarrow \text{div} \ v(x) \text{ as } r \rightarrow 0\]

which shows that

\[ \text{div} \ \mathcal{G}^t v(x) = \text{div} \ v(x) \]
(4.21)

The proof of the first line of (4.9) is complete.

We now establish the second line of (4.9). As \(v \in \cap_m [H^1 \cap W^{m,1}]\), (4.6), differentiation of (4.4) shows that

\[ \text{div} \ v = \text{div} \ \mathcal{G}^t v + \text{div} \ \mathcal{P}^t v \]

The first line of (4.9) then shows that \(0 = \text{div} \ \mathcal{P}^t v\). The proof of (4.9) is complete.

We now establish (4.10). As (4.7) shows that \(\mathcal{G}^t \hat{v} \) is \(\text{grad} \ p_\hat{v}\) for a scalar valued \(p_\hat{v} \in \cap_m H^m\)

\[ \langle \mathcal{G}^t v, \mathcal{P}^t \hat{v} \rangle_{\mathbb{R}^3} = \langle \text{grad} \ p_\hat{v}, \mathcal{P}^t v \rangle_{\mathbb{R}^3} \]

which after integrating by parts is

\[ = (-1) \int_{\mathbb{R}^3} p_\hat{v}(x) \text{div} \ \mathcal{P}^t v(x) \ dx = 0 \]
(4.9) (second line). The proof of (4.10) is complete.

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6 http://math2.org/math/geometry/areavsols.htm. Accessed 27 December 2024.
We now establish (4.11). As \( \hat{v} = G^\dagger \hat{v} + P^\dagger \hat{v} \), (4.4)
\[
\langle \hat{G}^\dagger v, \hat{v} \rangle_{\mathbb{R}^3} = \langle G^\dagger v, G^\dagger \hat{v} + P^\dagger \hat{v} \rangle_{\mathbb{R}^3}
\]
As \( G^\dagger v \) is orthogonal to \( P^\dagger \hat{v} \), (4.10)
\[
\langle G^\dagger v, \hat{v} \rangle_{\mathbb{R}^3} = \langle G^\dagger v, G^\dagger \hat{v} \rangle_{\mathbb{R}^3}
\]
In the same way
\[
\langle v, G^\dagger \hat{v} \rangle_{\mathbb{R}^3} = \langle G^\dagger v, G^\dagger \hat{v} \rangle_{\mathbb{R}^3}
\]
which establishes the first line in (4.11). The same analysis, but with \( P^\dagger \) in place of \( G^\dagger \), establishes the second line of (4.11).

We now establish (4.12). With (4.2) and (3.16)
\[
D^k [G^\dagger v](x) = (4\pi)^{-1} \int_{\mathbb{R}^3} \text{grad} \ |z|^{-1} D^k \text{div} \ v(x - z) \, dz
\]
which as \( D^k \) and \( \text{div} \) commute on \( v \in \bigcap_m [H^m \cap W^{m,1}] \subset C^\infty \) is
\[
= (4\pi)^{-1} \int_{\mathbb{R}^3} \text{grad} \ |z|^{-1} \text{div} \ D^k v(x - z) \, dz
\]
which with (4.2) shows that
\[
D^k [G^\dagger v] = G^\dagger [D^k v]
\]
(4.22)
The proof of the first line of (4.12) is complete.

We now establish the second line of (4.12). As \( G^\dagger v \) and \( P^\dagger v \) are \( \in \bigcap_m H^m \), (4.6), differentiation of (4.4) shows that
\[
D^k v = D^k [G^\dagger v] + D^k [P^\dagger v]
\]
(4.23)
(4.4). The (4.4) decomposition of \( D^k v \) is
\[
D^k v = G^\dagger [D^k v] + P^\dagger [D^k v]
\]
which with (4.22) is
\[
D^k v = D^k [G^\dagger v] + P^\dagger [D^k v]
\]
(4.24)
The difference between (4.23) and (4.24) shows that \( D^k [P^\dagger v] = P^\dagger [D^k v] \). The proof of (4.12) is complete.
We now establish (4.13). As \( v \in \cap_m [H^m \cap W^{m,1}] \), the (4.4) decomposition of \( D^{k'} v \) shows that
\[
|D^{k'} v|_{L^2}^2 = \langle G^\dagger D^{k'} v + P^\dagger D^{k'} v, G^\dagger D^{k'} v + P^\dagger D^{k'} v \rangle_{\mathbb{R}^3}
\]
which with customary mathematics is
\[
= |G^\dagger D^{k'} v|_{L^2}^2 + \langle G^\dagger D^{k'} v, P^\dagger D^{k'} v \rangle_{\mathbb{R}^3} + \langle P^\dagger D^{k'} v, G^\dagger D^{k'} v \rangle_{\mathbb{R}^3} + |P^\dagger D^{k'} v|_{L^2}^2
\]
As \( D^{k'} v \in \cap_m [H^m \cap W^{m,1}] \), the second and third summands vanish, (4.10), which shows that
\[
|D^{k'} v|_{L^2}^2 = |G^\dagger v|_{L^2}^2 + |P^\dagger v|_{L^2}^2
\]
Summing over those \( k' \) for which \( 0 \leq |k'| \leq m \) establishes the first line of (4.13). The other two lines of (4.13) are a corollary of the first line.

We now establish (4.14). As \( \beta \neq 0 \), the chain rule shows that \( v_\beta(x) := v(x,\beta) \) is \( \in \cap_m [H^m \cap W^{m,1}] \). Then with (4.2)
\[
[G^\dagger v_\beta]_k(x) = (4\pi)^{-1} \int_{\mathbb{R}^3} \partial_k |z|^{-1} \text{div} \left[ v_\beta(x - z) \right] \, dz
\]
\[
= (4\pi)^{-1} \int_{\mathbb{R}^3} \partial_k |z|^{-1} \text{div} \left[ v((x - z)\beta) \right] \, dz
\]
With (2.32) (first line) and customary mathematics
\[
[G^\dagger v_\beta]_k(x) = (4\pi)^{-1} \int_{\mathbb{R}^3} (-1) \, z_k |z|^{-3} \text{div} \, v(x\beta - z\beta) \, \beta \, d[z\beta] \, \beta^{-3}
\]
\[
= (4\pi)^{-1} \int_{\mathbb{R}^3} (-1) \, [z_k \beta] |z\beta|^{-3} \text{div} \, v(x\beta - z\beta) \, d[z\beta]
\]
which after changing variables in the integral is
\[
= (4\pi)^{-1} \int_{\mathbb{R}^3} \partial_k |z|^{-1} \text{div} \, v(x\beta - z) \, dz
\]
(4.25)
which shows that
\[
[G^\dagger v_\beta](x) = [G^\dagger v](x\beta)
\]
(4.26)
The proof of the first line of (4.14) is complete.
The (4.4) decomposition of $v_{\beta}$ shows that

$$v_{\beta}(x) = [P^\dagger v_{\beta}](x) + [G^\dagger v_{\beta}](x)$$

which as $v_{\beta}(x) := v(x\beta)$ shows that

$$v(x\beta) = [P^\dagger v_{\beta}](x) + [G^\dagger v](x\beta)$$

(4.27)

The (4.4) decomposition of $v$ at $x\beta$ is

$$v(x\beta) = [P^\dagger v](x\beta) + [G^\dagger v](x\beta)$$

(4.28)

With (4.25), the difference between (4.27) and (4.28) then show that

$$[P^\dagger v_{\beta}](x) = [P^\dagger v](x\beta)$$

The proof of (4.14) is complete. □

Lemma 4.3. Let $v$ and $\hat{v}$ be $\in L^2$ and $\alpha$ and $\beta$ be scalars. Then $G^\dagger$ and $P^\dagger$ extend to operators $G$ and $P$ which carry $H^m \rightarrow H^m$ and satisfy

$$v = Gv + Pv$$

(4.29)

$$G[\alpha v + \beta \hat{v}] = \alpha Gv + \beta G\hat{v} \text{ and}$$

$$P[\alpha v + \beta \hat{v}] = \alpha Pv + \beta P\hat{v}$$

(4.30)

and when $v$ is $\in H^m$

$$|Gv|_{H^m} \leq |v|_{H^m} \text{ and}$$

$$|Pv|_{H^m} \leq |v|_{H^m}$$

(4.31)

Proof. We define $Gv$ when $v$ is $\in L^2$ to be the limit in the $L^2$ norm of $G^\dagger v^n$ for any sequence $v^n \in \cap_m [H^m \cap W^{m,1}]$ that $\rightarrow v$ in the $L^2$ norm. There is at least one sequence that $\rightarrow v \in L^2$ because

$$\langle C_\infty^\infty, L^2 \rangle \subset \langle \cap_m [H^m \cap W^{m,1}], L^2 \rangle$$

which is dense $\in \langle L^2, L^2 \rangle$

As the limit of any such sequence $G^\dagger v^n$ in which $v^n \rightarrow v$ is independent of the sequence, [9] Chap. 7 Sec. 5 Prop. 13, $G$ is well defined on $L^2$. 
Such definition shows that the operator norm of $G$ on $\langle L^2, L^2 \rangle$ is equal to the operator norm of $G^\dagger$ on $\langle \cap_m [H^m \cap W^m.1], L^2 \rangle$. As the sequence $v^n := v \in \cap_m [H^m \cap W^m.1] \to v$, $G = G^\dagger$ on $\cap_m [H^m \cap W^m.1]$.

In the same way, $G^\dagger$ extends to an operator $G^m$ that is defined on $H^m$. The definition of $G^m v$ is the limit in the $H^m$ norm of $G^\dagger v^n$ for any sequence $v^n \in \cap_m [H^m \cap W^m.1]$ that $\to v$ in the $H^m$ norm. As before, there is for each $v \in H^m$ at least one such sequence and $G^m$ is well defined on $H^m$.

As $H^m$ is complete, $G^m$ carries $H^m \to H^m$, and the norm of $G^m$ on $\langle H^m, H^m \rangle$ is equal to the norm of $G^\dagger$ on $\langle \cap_m [H^m \cap W^m.1], H^m \rangle$. As any sequence $v^n \in C^\infty_o$ that $\to v$ in the $H^m$ norm also converges to $v$ in the $L^2$ norm, $G^m v$ is equal to $G v$ when $v \in H^m$. Thus

$$|G v|_{H^m} \leq |v|_{H^m}$$

for such $v$, which establishes the first line of (4.31) and shows that $G$ carries $H^m \to H^m$.

We extend $P^\dagger$ to $P$ in the same way and in doing so, we establish the second line of (4.31) and show that $P$ carries $H^m \to H^m$.

When the sequences $v^n$ and $\hat{v}^n$ are $\in \cap_m [H^m \cap W^m.1]$, (4.4) and (4.5) show that

$$v^n = G^\dagger v^n + P^\dagger v^n$$

$$G^\dagger [\alpha v^n + \beta \hat{v}^n] = \alpha G^\dagger v^n + \beta G^\dagger \hat{v}^n$$

$$P^\dagger [\alpha v^n + \beta \hat{v}^n] = \alpha P^\dagger v^n + \beta P^\dagger \hat{v}^n$$

When $v^n$ and $\hat{v}^n$ converge to $v$ and $\hat{v}$ in the $L^2$ norm, passing to the limit in the $L^2$ norm establishes (4.29) and (4.30). \hfill \square

We now extend Lemma 4.2 to the operators $G$ and $P$.

**Lemma 4.4.** Let $v$ and $\hat{v}$ be $\in L^2$. Then

$$G v = 0 \text{ and } v = P v$$

when $v$ is $\in H^1$ and the weak derivative $\text{div } v = 0$

(4.32)
\[
\text{div } \mathcal{G} v = \text{div } v \text{ and } \\
\text{div } \mathcal{P} v = 0
\]
when \( v \) is \( H^1 \)

\[
\langle \mathcal{G} v, \mathcal{P} \hat{v} \rangle_{\mathbb{R}^3} = 0
\]

\[
\langle \mathcal{G} v, \hat{v} \rangle_{\mathbb{R}^3} = \langle v, \mathcal{G} \hat{v} \rangle_{\mathbb{R}^3} \text{ and } \\
\langle \mathcal{P} v, \hat{v} \rangle_{\mathbb{R}^3} = \langle v, \mathcal{P} \hat{v} \rangle_{\mathbb{R}^3}
\]

\[
D^k \mathcal{G} v = \mathcal{G} D^k v \text{ and } \\
D^k \mathcal{P} v = \mathcal{P} D^k v
\]
when \( v \) is \( H^{k'} \)

and

\[
|v|^2_{H^m} = |\mathcal{G} v|^2_{H^m} + |\mathcal{P} v|^2_{H^m}
\]

\[
|\mathcal{G} v|_{H^m} \leq |v|_{H^m} \text{ and } \\
|\mathcal{P} v|_{H^m} \leq |v|_{H^m}
\]
when \( v \) is \( H^m \)

\[
\mathcal{P}[v] \text{ is } \in \cap_m [H^m \times [0, \infty)]
\]
when \( v \) is \( \in \cap_m [H^m \times [0, \infty)] \)

\[
(4.38)
\]

**Proof.** We now establish (4.32). As \( C^\infty_0 \) is a subset of \( \cap_m [H^m \cap W^{m,1}] \) and is dense in \( H^1 \), we let \( v^n \in C^\infty_0 \) converge in the \( H^1 \) norm to a weakly divergence free \( v \in H^1 \). Then with (4.4)

\[
\langle \mathcal{G}^\dagger v^n, v^n \rangle_{\mathbb{R}^3} = \langle \mathcal{G}^\dagger v^n, \mathcal{G}^\dagger v^n \rangle_{\mathbb{R}^3} + \langle \mathcal{G}^\dagger v^n, \mathcal{P}^\dagger v^n \rangle_{\mathbb{R}^3}
\]

As \( \langle \mathcal{G}^\dagger v^n, \mathcal{P}^\dagger v^n \rangle_{\mathbb{R}^3} = 0, \) (4.10)

\[
\langle \mathcal{G}^\dagger v^n, \mathcal{G}^\dagger v^n \rangle_{\mathbb{R}^3} = \langle \mathcal{G}^\dagger v^n, v^n \rangle_{\mathbb{R}^3}
\]

which is

\[
= \langle \mathcal{G}^\dagger v^n, v^n - v \rangle_{\mathbb{R}^3} + \langle \mathcal{G}^\dagger v^n, v \rangle_{\mathbb{R}^3}
\]
As $G^\dagger v^n$ is the gradient of a scalar which is $\in \cap_m H^m$, (4.8), and the weak derivative $\text{div} \, v$ vanishes, integration by parts shows that
\[
\langle G^\dagger v^n, v \rangle_{\mathbb{R}^3} = \langle v^n, \text{div} \, v \rangle = 0
\]
(4.9). Thus
\[
\langle G^\dagger v^n, G^\dagger v^n \rangle_{\mathbb{R}^3} = \langle G^\dagger v^n, v^n - v \rangle_{\mathbb{R}^3}
\]
and
\[
\limsup_{n \to \infty} \| G^\dagger v^n \|_{L^2}^2 \leq \| G^\dagger v \|_{L^2} \limsup_{n \to \infty} \| v^n - v \|_{L^2}
\]
the right side of which $\to 0$ as $v^n \to v$ in the $H^1$ norm and also in the $L^2$ norm. Thus $\| Gv \|_{L^2}^2 = 0$ for such $v$. The proof of the first line of (4.32) is complete.

As (4.29) shows that $v = Gv + \mathcal{P}v$. The first line of (4.32) shows that $v = \mathcal{P}v$. The proof of (4.32) is complete.

We now establish (4.33). As before, we let $v^n \in C^\infty_0 \to v \in H^1$ in the $H^1$ norm. With (4.9)
\[
\text{div} \, G^\dagger v^n = \text{div} \, v^n
\]
Passing to the limit in the $H^1$ norm establishes the first line of (4.33).

As Lemma 4.3 shows that $Gv$ and $\mathcal{P}v$ are $\in H^1$, differentiating the (4.29) decomposition of $v \in H^1$ establishes
\[
\text{div} \, v = \text{div} \, Gv + \text{div} \, \mathcal{P}v
\]
As $\text{div} \, v = \text{div} \, Gv$, $0 = \text{div} \, \mathcal{P}v$. The proof of (4.33) is complete.

We now establish (4.34), (4.35), (4.36) and (4.37). With $v^n$ and $\hat{v}^n \in C^\infty_0$, Lemma 4.2 shows that
\[
\langle G^\dagger v^n, \mathcal{P}^\dagger \hat{v}^n \rangle_{\mathbb{R}^3} = 0
\]
(4.39)
\[
\langle G^\dagger v^n, \hat{v}^n \rangle_{\mathbb{R}^3} = \langle v^n, G^\dagger \hat{v}^n \rangle_{\mathbb{R}^3} \quad \text{and}
\]
\[
\langle \mathcal{P}^\dagger v^n, \hat{v}^n \rangle_{\mathbb{R}^3} = \langle v^n, \mathcal{P}^\dagger \hat{v}^n \rangle_{\mathbb{R}^3}
\]
(4.40)
\[
D^k G^\dagger v^n = G^\dagger D^k v^n \quad \text{and}
\]
\[
D^k \mathcal{P}^\dagger v^n = \mathcal{P}^\dagger D^k v^n
\]
(4.41)
\[
|v^n|_{H^m}^2 = |G^\dagger v^n|_{H^m}^2 + |\mathcal{P}^\dagger v^n|_{H^m}^2 \\
|G^\dagger v^n|_{H^m} \leq |v^n|_{H^m} \quad \text{and} \\
|\mathcal{P}^\dagger v^n|_{H^m} \leq |v^n|_{H^m}
\]
(4.42)

With sequences \(v^n\) and \(\hat{v}^n\) ∈ \(C^\infty_o\) that converge to \(v\) and \(\hat{v}\) in the \(L^2\) norm, passage to the limit in (4.39) and (4.40) establishes (4.34) and (4.35). So passing to the limit in (4.41) and (4.42) in respect of sequences \(v^n\) and \(\hat{v}^n\) ∈ \(C^\infty_o\) that converge to \(v\) and \(\hat{v}\) in the \(H^k\) norm establishes (4.36) and (4.37).

We now establish (4.38). With (4.37)
\[
|\mathcal{P}[v(t)]|_{H^m} \leq |v(t)|_{H^m}
\]
and
\[
|\mathcal{P}[v(t + \Delta t) - v(t)]|_{H^m} \leq |v(t + \Delta t) - v(t)|_{H^m}
\]
which completes the proof of (4.38).

5. \[\int_{T-\Delta T}^T \int_{\mathbb{R}^3} (\widehat{\mathcal{P}}[\partial_j K^i(T - \eta)](y), J_\gamma(v_j) v (x - y, \eta))_2 \, dyd\eta \]

In this section, we establish upper bounds for the integral
\[
\int_{T-\Delta T}^T \int_{\mathbb{R}^3} \widehat{\mathcal{P}}[\partial_j K^i(T - \eta)](y) J_\gamma(v_j) v (x - y, \eta) \, dyd\eta
\]
(5.1)
and for certain derivatives thereof.

In (5.1), \(K^i(y, t)\) is the 3 × 3 diagonal matrix the diagonal of which is the scalar valued
\[
K(y, t) = [4\pi t]^{-3/2} \exp \left(-|y|^2/4t\right)
\]
which, \([4]\) §2.3.1 shows, satisfies
\[
|K(t)|_{L^1} = 1 \quad \text{and} \\
\partial_t K(y, t) = \Delta K(y, t)
\]
(5.2)

\(K^i(y, t)\) is the \(i^{th}\) row of \(K^i(y, t)\), \(\widehat{\mathcal{P}}[\partial_j K^i(t)](y)\) is the 3 × 3 matrix the \(i^{th}\) row of which is \(\widehat{\mathcal{P}}[\partial_j K^i(t)](y)\), \(\widehat{\mathcal{P}} := (-1)\mathcal{P}\), \(J_\gamma(v_j)\) is a scalar and \(J_\gamma(v_j)v\) is a column vector.
Therewith, the $i^{th}$ component of the (5.1) integral is
\[
\int_{T-\Delta T}^{T} \int_{\mathbb{R}^3} (\hat{P} [\partial_j K^i(T - \eta)](y), J_\gamma(v_j) v (x - y, \eta)) \, dy \, d\eta
\]
\[
= \int_{T-\Delta T}^{T} \int_{\mathbb{R}^3} \sum_k \hat{P} [\partial_j K^i(T - \eta)](y) J_\gamma(v_j) v_k (x - y, \eta) \, dy \, d\eta
\]
(5.3)

which with the definition
\[
D^{k'} p_{j,i,k}(y, T - \eta) := \hat{P} [D^{k'} \partial_j K^i(T - \eta)](y)
\]
(5.4)

for each multi-index $k'$ is
\[
= \int_{T-\Delta T}^{T} \int_{\mathbb{R}^3} \sum_k p_{j,i,k}(y, T - \eta) J_\gamma(v_j) v_k (x - y, \eta) \, dy \, d\eta
\]
(5.5)

**Lemma 5.1.** With $t > 0$ and $k'$ a multi-index
\[
|D^{k'} p_{j,i,k}(t)|_{L^2} \leq |P [D^{k'} \partial_j K^i(t)]|_{L^2} \leq |D^{k'} \partial_j K^i(t)|_{L^2} \leq [\ldots]_{5.6}(k') \, t^{-|k'|/2-5/4}
\]
(5.6)

\[
|D^{k'} p_{j,i,k}(t)|_{L^\infty} \leq |P [D^{k'} \partial_j K^i(t)]|_{L^\infty} \leq [\ldots]_{5.7}(k') \, t^{-|k'|/2-2}
\]
(5.7)

\[
\sup_{t \geq \epsilon > 0} |D^{k'} p_{j,i,k}(t)|_{L^\infty} \leq [\ldots]_{5.7}(k') \, \epsilon^{-|k'|/2-2}
\]
(5.8)

and
\[
|D^{k'} p_{j,i,k}(t)|_{L^1} \leq |D^{k'} \partial_j K^i(t)|_{L^1} = [\ldots]_{5.9}(k') \, t^{-|k'|/2-1/2}
\]
(5.9)

**Proof.** We first establish (5.6). With $t > 0$ and $\xi := y t^{-1/2}$
\[
K(y, t) = [\ldots] \, t^{-3/2} \exp (-|\xi|^2/4)
\]
the chain rule shows that
\[
\partial_y K(y, t) = [\ldots] \, \partial_\xi [\exp(-|\xi|^2/4)] \, \xi_j (-1/2) \partial_y \xi_j \, t^{-3/2}
\]
and as $\partial_{y_j} \xi_j = t^{-1/2}$

$$\partial_{y_j} K(y, t) = [ \ldots ] \exp(-|\xi|^2/4) \xi_j t^{-1/2 - 3/2}$$

The chain rule also shows that

$$D^k \partial_j K(y, t) = [ \ldots ] q_{k'}(\xi) \exp(-|\xi|^2/4) t^{-|k'|/2 - 2}$$

(5.10)

where $q_{k'}(\xi)$ is a polynomial of order $|k'|$ in the components of $\xi$. Therewith

$$|D^k \partial_j K^i(t)|_{L^2}$$

$$= \left[ \int_{\mathbb{R}^3} \left[ D^k K(y, t) \right]^2 \, dy \right]^{1/2}$$

$$\leq \left[ \int_{\mathbb{R}^3} \left[ q_{k'}(yt^{-1/2}) \exp(-yt^{-1/2})^2 |yt^{-1/2}|^{3/2} \right]^2 \, d[yt^{-1/2}] t^{3/2} \right]^{1/2} t^{-|k'|/2 - 2}$$

$$\leq \left[ \int_{\mathbb{R}^3} \left[ q_{k'}(\xi) \exp(-|\xi|^2/4) \right]^2 \, d\xi \right]^{1/2} t^{-|k'|/2 - 5/4}$$

$$\leq [ \ldots ] 5.6(k') t^{-|k'|/2 - 5/4}$$

As $\mathcal{P}$ does not increase an $L^2$ norm, (4.37)

$$|\mathcal{P}[D^k \partial_j K^i(t)]|_{L^2}$$

$$\leq |D^k \partial_j K^i(t)|_{L^2} \leq [ \ldots ] 5.6(k') t^{-|k'|/2 - 5/4}$$

and as the $L^2$ norm of each component of $\mathcal{P}[D^k \partial_j K^i(t)]$ is $\leq$ the $L^2$ norm thereof

$$|D^k \partial_{j,i,k}(t)|_{L^2} \leq |\mathcal{P}[D^k \partial_j K^i(t)]|_{L^2}$$

The proof of (5.6) is complete.

We now establish (5.7). With (5.10)

$$|\mathcal{P}[D^k \partial_j K^i(t)](y)|_{L^\infty(y \in \mathbb{R}^3)}$$

$$= [ \ldots ] |\mathcal{P}[q_{k'}(yt^{-1/2}) \exp(-yt^{-1/2})^2 |yt^{-1/2}|]^{i} |_{L^\infty(y \in \mathbb{R}^3)} t^{-|k'|/2 - 2}$$

which (4.14) shows is

$$= [ \ldots ] |\mathcal{P}[q_{k'}(y) \exp(-|y|^2/4)]^{i} (yt^{-1/2}) |_{L^\infty(y \in \mathbb{R}^3)} t^{-|k'|/2 - 2}$$

$$= [ \ldots ] |\mathcal{P}[q_{k'}(y) \exp(-|y|^2/4)]^{i} |_{L^\infty(y \in \mathbb{R}^3)} t^{-|k'|/2 - 2}$$

which with (2.14) is

$$\leq [ \ldots ] |\mathcal{P}[q_{k'}(y) \exp(-|y|^2/4)] |_{H^2} t^{-|k'|/2 - 2}$$
which with (4.37) is
\[
\leq [ \ldots ] q_k'(y) \exp (-|y|^2/4) \bigg|_{H^2} t^{-|k'|/2-2} \\
\leq [ \ldots ] 5.7(k') t^{-|k'|/2-2}
\]
As the \( L^\infty \) norm of each component of \( \mathcal{P}[D^{k'} \partial_j K^i(t)] \) is \( \leq \) the \( L^\infty \) norm thereof
\[
|D^{k'} p_{j,i,k}(t)|_{L^\infty} \leq |\mathcal{P}[D^{k'} \partial_j K^i(t)]|_{L^\infty}
\]
The proof of (5.7) is complete.

We now establish (5.8). As \( t^{-|k'|/2-2} \) increases as \( t \) decreases, (5.8) is a corollary of (5.7).

We now establish (5.9). With (5.10)
\[
|D^{k'} \partial_j K^i(y, t)|_{L^1}
= \int_{\mathbb{R}^3} |q_{k'}(\xi) \exp (-|\xi|^2/4t)| dy \ t^{-|k'|/2-2}
\]
which after changing variables in the integral is
\[
= \int_{\mathbb{R}^3} |q_{k'}(\xi) \exp (-|\xi|^2/4t)| \ d[\xi] t^{-1/2} t^{-|k'|/2} t^{-1/2}
\]
As the \( L^1 \) norm of each component of \( \mathcal{P}[D^{k'} \partial_j K^i(t)] \) is \( \leq \) the \( L^1 \) norm thereof
\[
|D^{k'} p_{j,i,k}(t)|_{L^1} \leq |\mathcal{P}[D^{k'} \partial_j K^i(t)]|_{L^1} \leq [ \ldots ] 5.9(k') t^{-|k'|/2-1/2}
\]
The proof of (5.9) is complete. \( \square \)

**Lemma 5.2.** With \( t > 0 \)
\[
|p_{j,i,k}(t)|_{L^1} \leq [ \ldots ] 5.11 \ t^{-1/2}
\]
(5.11)
and with \( 0 < T - \Delta T < T \)
\[
\int_{T-\Delta T}^T |p_{j,i,k}(T - \eta)|_{L^1} \ d\eta \leq [ \ldots ] 5.12 \Delta T^{1/2}
\]
(5.12)
SMOOTH SOLUTIONS OF NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3 \times [0, \infty)$

and

$$\int_{T - \Delta T}^T |\tilde{P}[\partial_j K^i(T - \eta)](y)|_{L^1} \, d\eta \leq [\ldots]_{5.13} \Delta T^{1/2}$$

(5.13)

Proof. We first establish (5.11). With the definition of $K^i(t)$

$$\mathcal{G}[\partial_j K^i(t)](y)$$

\[ = [\ldots] \mathcal{G} \left[ \partial_j \left[ t^{-3/2} \exp(-|yt^{-1/2}|^2/4) \right] \right]^i(y) \]

\[ = [\ldots] \mathcal{G} \left[ t^{-3/2} \exp(-|yt^{-1/2}|^2/4) \left[ (-1) 2 y_j t^{-1/2} t^{-1/2} \right] \right]^i(y) \]

\[ = [\ldots] \mathcal{G} \left[ \exp(-|yt^{-1/2}|^2/4) y_j t^{-1/2} \right]^i(y) t^{-3/2} t^{-1/2} \]

which as (5.6) and (5.9) show that $\mathcal{G}[\partial_j K^i(t)](y)$

\[ = [\ldots] \mathcal{G} \left[ \exp(-|yt^2/4|) y_j \right]^i(y) t^{-3/2} t^{-1/2} \]

which as

$$\exp(-|y|^2/4) y_j$$

is

\[ = [\ldots] \partial_j \exp(-|y|^2/4) \]

is

\[ = [\ldots] \mathcal{G} \left[ \partial_j \exp(-|y|^2/4) \right]^i(y) t^{-3/2} t^{-1/2} \]

As a result

\[ \left| \mathcal{G}[\partial_j K^i(t)] \right|_{L^1} = [\ldots] \left| \mathcal{G}[\partial_j K^i(1)] \right|_{L^1} t^{-1/2} \]

(5.14)

We now show that $\left| \mathcal{G}[\partial_j K^i(1)] \right|_{L^1}$ is finite. As $\left[ \partial_j \exp(-|y|^2/4) \right]^i$

is $\in \cap_m \left[ H^m \cap W^{m,1} \right]$

$$\left[ \mathcal{G}[\partial_j \exp(-|y|^2/4)] \right]^i_k$$

\[ = \int_{\mathbb{R}^3} \partial_k |z|^{-1} \text{div}_y \left[ \partial_j \exp(-|y - z|^2/4) \right]^i \, dz \]

(4.2), which as only the $i^{th}$ component of $[\ldots]^i$ is non-zero is

\[ = \int_{\mathbb{R}^3} \partial_k |z|^{-1} \partial_i \partial_j \exp(-|y - z|^2/4) \, dz \]
Then with the scalar valued functions \( \psi_{\text{left}}^\epsilon(z) \) and \( \psi_{\text{right}}^\epsilon(z) \) which we defined in the proof of Lemma 3.2

\[
[\mathcal{G} \left[ \partial_j \exp \left( -|y|^2/4 \right) \right]^i_k] \\
= \int_{\mathbb{R}^3} \partial_k |z|^{-1} \partial_i \partial_j \exp \left( -|y - z|^2/4 \right) dz \\
= \int_{\mathbb{R}^3} \partial_k |z|^{-1} \psi_{\text{left}}^{\epsilon=0.1}(z) \partial_i \partial_j \exp \left( -|y - z|^2/4 \right) dz \\
+ \int_{\mathbb{R}^3} \partial_k |z|^{-1} \psi_{\text{right}}^{\epsilon=0.1}(z) \partial_i \partial_j \exp \left( -|y - z|^2/4 \right) dz
\]

(5.15)

where \( \psi_{\text{left}}^{\epsilon=0.1}(z) \) is \( C^\infty \), is \( \in [0, 1] \) on \( \mathbb{R}^3 \), is \( \equiv 1 \) on \( B(1) \), and vanishes outside of \( B(1.1) \), and \( \psi_{\text{right}}^{\epsilon=0.1}(z) \) is \( C^\infty \), is \( \in [0, 1] \) on \( \mathbb{R}^3 \), vanishes in \( B(0.9) \) and is \( \equiv 1 \) on \( \mathbb{R}^3 \setminus B(1.1) \).

After twice integrating by parts in the second integral in the right side of (5.15), the definition of \( \psi_{\text{left}}^{\epsilon=0.1}(z) \) and that of \( \psi_{\text{right}}^{\epsilon=0.1}(z) \) show that

\[
[\mathcal{G} \left[ \partial_j \exp \left( -|y|^2/4 \right) \right]^i_k] \\
= \int_{\mathbb{R}^3} \partial_k |z|^{-1} \partial_i \partial_j \exp \left( -|y - z|^2/4 \right) dz \\
= \int_{B(1.1)} \partial_k |z|^{-1} \psi_{\text{left}}^{\epsilon=0.1}(z) \partial_i \partial_j \exp \left( -|y - z|^2/4 \right) dz \\
+ \int_{B(1.1) \setminus B(0.9)} \partial_j \partial_i \left[ \partial_k |z|^{-1} \psi_{\text{right}}^{\epsilon=0.1}(z) \right] \exp \left( -|y - z|^2/4 \right) dz \\
+ \int_{\mathbb{R}^3 \setminus B(1.1)} \partial_j \partial_i \partial_k |z|^{-1} \exp \left( -|y - z|^2/4 \right) dz
\]

Therewith

\[
| \mathcal{G} \left[ \partial_j \exp \left( -|y|^2/4 \right) \right]^i_k |_{L^1} \\
\leq \int_{\mathbb{R}^3} \left[ \int_{B(1.1)} |\partial_k |z|^{-1}| \left| \partial_i \partial_j \exp \left( -|y - z|^2/4 \right) \right| dz \right] dy \\
+ \int_{\mathbb{R}^3} \left[ \int_{B(1.1) \setminus B(0.9)} |\partial_j \partial_i [\partial_k |z|^{-1} \psi_{\text{right}}^{\epsilon=0.1}(z)]| \exp \left( -|y - z|^2/4 \right) dy \right] dz \\
+ \int_{\mathbb{R}^3} \left[ \int_{\mathbb{R}^3 \setminus B(1.1)} |\partial_j \partial_i \partial_k |z|^{-1}| \exp \left( -|y - z|^2/4 \right) dy \right] dz
\]
which with Fubini’s theorem to reverse the order of integration in each iterated integral is

\[
\leq \int_{B(1,1)} |\partial_\ell z|^{-1} \left[ \int_{R^3} \left[ |\partial_\ell \partial_\jmath \exp (-|y - z|^2/4) dy \right] dz \\
+ \int_{B(1,1) \setminus B(0,9)} |\partial_\ell \partial_\jmath [\partial_\ell z|^{-1}\psi_{\text{right}}^\ell(z)]| \left[ \int_{R^3} \exp (-|y - z|^2/4) dy \right] dz \\
+ \int_{R^3 \setminus B(1,1)} |\partial_\ell \partial_\jmath [\partial_\ell z|^{-1}| \left[ \int_{R^3} \exp (-|y - z|^2/4) dy \right] dz
\]
\]

which as the integrals with respect to \( y \) above are independent of \( z \) is

\[
\leq \left[ \ldots \right] \left[ \int_{B(1,1)} |\partial_\ell z|^{-1} \right] dz \\
+ \left[ \ldots \right]_{2.34b \ 1.1} + \left[ \ldots \right]_{2.34e \ 0.9^{-1}} < \infty
\]

(5.16)

As \( \partial_\ell \partial_\jmath [\partial_\ell z|^{-1}\psi_{\text{right}}^\ell(z)] \in C^\infty \) and therefore bounded on the compact annulus \( B(1.1) \setminus B(0.9) \) the measure of which is finite, portions of Lemma 2.4 show that (5.16) is

\[
\leq \left[ \ldots \right] \left[ \ldots \right]_{2.34b \ 1.1} + \left[ \ldots \right]_{2.34e \ 0.9^{-1}} < \infty
\]

(5.17)

(5.14) then shows that

\[
\left[ \mathcal{G} K^\ell(t) \right]_{L^1} \quad \text{which is } = \left[ \ldots \right] \left[ \mathcal{G} K^\ell(1) \right]_{L^1} t^{-1/2} \text{ is } < \infty
\]

(5.18)

With the (4.4) decomposition of \( \partial_\jmath K^\ell(t) \) and the triangle inequality

\[
| \mathcal{P} [\partial_\ell K^\ell(t)] |_{L^1} \leq | \partial_\jmath K^\ell(t) |_{L^1} + | \mathcal{G} [\partial_\ell K^\ell(t)] |_{L^1}
\]

(5.19)

which with (5.9) and (5.18) shows that

\[
| p_{j,j,k}(t) |_{L^1} = | \mathcal{P} [\partial_\ell K^\ell(t)] |_{L^1} \\
\leq \left[ \ldots \right] t^{-1/2} + \left[ \ldots \right] t^{-1/2} \leq \left[ \ldots \right] t^{-1/2}
\]
The proof of (5.11) is complete. (5.12) and (5.13) are a corollary of (5.11).

\[\square\]

**Lemma 5.3.** Let \(0 \leq T - \Delta T < T\), \(m \geq 7\), \(v\) and \(\hat{v}\) be \(H^m \times [T - \Delta T, T]\), and \(k'\) be a multi-index for which \(0 \leq |k'| \leq m\). Then the weak derivative

\[
D^{k'} \left[ \int_{T - \Delta T}^{T} \hat{P} \left[ \partial_j K^\dagger (T - \eta) \right](y) J^\gamma (v_j) \hat{v} (x - y, \eta) \ dyd\eta \right]
= \int_{T - \Delta T}^{T} \hat{P} \left[ \partial_j K^\dagger (T - \eta) \right](y) D^{k'} \left[ J^\gamma (v_j) \hat{v} \right] (x - y, \eta) \ dyd\eta
\]

(5.20)

\[
\left| \int_{T - \Delta T}^{T} \hat{P} \left[ \partial_j K^\dagger (T - \eta) \right](y) D^{k'} \left[ J^\gamma (v_j) \hat{v} \right] (x - y, \eta) \ dyd\eta \right|_{L^\infty}
\leq [ \ldots ]_{5.21} \Delta T^{1/2} \ |D^{k'}[J^\gamma (v_j) \hat{v}]|_{L^\infty \times [T - \Delta T, T]}
\]

when \(|k'| \leq m - 2\)

(5.21)

\[
\left| \int_{T - \Delta T}^{T} \hat{P} \left[ \partial_j K^\dagger (T - \eta) \right](y) \left[ J^\gamma (v_j) v - J^\gamma (\hat{v}_j) \hat{v} \right] (x - y, \eta) \ dyd\eta \right|_{L^\infty}
\leq 2 [ \ldots ]_{5.21} \Delta T^{1/2} \ |v - \hat{v}|_{L^\infty \times [T - \Delta T, T]}
\]

\[
\max \left[ \ |v|_{L^\infty \times [T - \Delta T, T]}, \ |\hat{v}|_{L^\infty \times [T - \Delta T, T]} \right]
\]

when \(|k'| \leq m - 2\)

(5.22)

\[
\left| \int_{T - \Delta T}^{T} \hat{P} \left[ \partial_j K^\dagger (T - \eta) \right](y) J^\gamma (v_j) \hat{v} (x - y, \eta) \ dyd\eta \right|_{H^m}
\leq [ \ldots ]_{5.23} \Delta T^{1/2} \ |v|_{H^m \times [T - \Delta T, T]} \ |\hat{v}|_{H^m \times [T - \Delta T, T]}
\]

(5.23)
\[ \left| \int_{T-\Delta T}^{T} \int_{\mathbb{R}^3} \hat{P} [\partial_j k^i (T-\eta)] (y) \left[ J_\gamma (v_j) v - J_\gamma (\hat{v}_j) \hat{v} \right] (x-y, \eta) \, dy \, d\eta \right|_{H^m} \]
\[ \leq 2 \left[ \ldots \right]_{5.23} \Delta T^{1/2} \left| \dot{v} \right|_{H^{m} \times [T-\Delta T, T]} \max \left[ \left| v \right|_{H^{m} \times [T-\Delta T, T]}, \left| \dot{v} \right|_{H^{m} \times [T-\Delta T, T]} \right] \]
(5.24)

and

\[ \left| \int_{T-\Delta T}^{T} \int_{\mathbb{R}^3} K(y, \eta) \, v(x-y, \eta) \, dy \, d\eta \right|_{H^m} \]
\[ \leq \left[ \ldots \right]_{5.25} \Delta T \left| v \right|_{H^{m} \times [T-\Delta T, T]} \]
(5.25)

\left[ \ldots \right]_{5.21}, \left[ \ldots \right]_{5.23} \text{ and } \left[ \ldots \right]_{5.25} \text{ are independent of } \gamma, \, T, \, v \text{ and } \hat{v}. \]

Proof. We first establish (5.20). With \(0 < \epsilon < \Delta T \leq T\) and \(k'\) for which \(0 \leq |k'| \leq m\), a change in variables and Fubini's theorem shows that

\[ \int_{T-\Delta T}^{T-\epsilon} \int_{\mathbb{R}^3} p_{j,i,k} (y, T-\eta) \, D^{k'} [J_\gamma (v_j) \hat{v}_k] (x-y, \eta) \, dy \, d\eta \]
\[ = \int_{T-\Delta T}^{T-\epsilon} \left[ \int_{\mathbb{R}^3} p_{j,i,k} (x-y, T-\eta) \, D^{k'} [J_\gamma (v_j) \hat{v}_k] (y, \eta) \, dy \right] \, d\eta \]
(5.26)

the absolute value of which on \(\mathbb{R}^3\) is

\[ \leq \int_{T-\Delta T}^{T-\epsilon} \left| p_{j,i,k} (T-\eta) \right|_{L^\infty} \left| D^{k'} [J_\gamma (v_j) \hat{v}_k] (\eta) \right|_{L^1} \, d\eta \]

which with (5.8) and (2.24) is

\[ \leq \Delta T \left[ \ldots \right]_{5.7} \epsilon^{-2} \left[ \ldots \right]_{2.24, m} \left| v \right|_{H^{m} \times [T-\Delta T, T]} \left| \dot{v} \right|_{H^{m} \times [T-\Delta T, T]} < \infty \]
(5.27)

which shows that the left side of (5.26) is a bounded function of \(x \in \mathbb{R}^3\).
With $x$ and $\Delta x \in \mathcal{R}^3$
\[
\left| \int_{T-\Delta T}^{T-\epsilon} \int_{\mathcal{R}^3} p_{j,i,k}(x+\Delta x - y, T - \eta) \, D^k [J_\gamma(v_j) \hat{v}_k] (y, \eta) \, dyd\eta \right|
- \left| \int_{T-\Delta T}^{T-\epsilon} \int_{\mathcal{R}^3} p_{j,i,k}(x-y, T - \eta) \, D^k [J_\gamma(v_j) \hat{v}_k] (y, \eta) \, dyd\eta \right|
\leq \Delta T \left[ \cdots \right] \epsilon^{-5/2} \left| \Delta x \right|_\infty
\]
\[
\left[ \cdots \right] 2.22, m \left| v \right|_{H^m \times [T-\Delta T,T]} \left| \hat{v} \right|_{H^m \times [T-\Delta T,T]} < \infty
\]
(5.28)
\]
which shows that the left side of (5.26) is a continuous function of $x \in \mathcal{R}^3$ which, as a result, is measurable on $\mathcal{R}^3$.

(5.27) and (5.28) then show that
\[
\int_{\mathcal{R}^3} \phi(x) \left[ \int_{T-\Delta T}^{T-\epsilon} \int_{\mathcal{R}^3} p_{j,i,k}(y, T - \eta) \, D^k [J_\gamma(v_j) \hat{v}_k] (x - y, \eta) \, dyd\eta \right] \, dx
= \int_{\mathcal{R}^3} \left[ \int_{T-\Delta T}^{T-\epsilon} \int_{\mathcal{R}^3} \phi(x) \, p_{j,i,k}(y, T - \eta) \, D^k [J_\gamma(v_j) \hat{v}_k] (x - y, \eta) \, dyd\eta \right] \, dx
\]
(5.29)
\]
for each scalar valued $\phi \in C^\infty_0$.

Fubini’s theorem and the inequality in the margin show that the right side of (5.29) is equal to its related, iterated integral
\[
\int_{T-\Delta T}^{T-\epsilon} \int_{\mathcal{R}^3} \left[ \int_{\mathcal{R}^3} \phi(x) \, p_{j,i,k}(y, T - \eta) \, D^k [J_\gamma(v_j) \hat{v}_k] (x - y, \eta) \, dx \right] \, dyd\eta
\]
\[
\int_{T-\Delta T}^{T-\epsilon} \left[ \int_{\mathcal{R}^3} \phi(x) \left| p_{j,i,k}(y, T - \eta) \right| \left| D^k [J_\gamma(v_j) \hat{v}_k](x - y, \eta) \right| \, dx \right] \, dyd\eta
\leq \left[ \cdots \right] 5.12 \Delta T^{1/2} \left| \phi \right|_{L^2} \left| D^k [J_\gamma(v_j) \hat{v}_k] \right|_{L^2 \times [T-\Delta T,T]} < \infty
\]

\footnote{A Hölder inequality shows that}
SMOOTH SOLUTIONS OF NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3 \times [0, \infty)$

which after integrating by parts in the interior integral is

$$
= \int_{\mathbb{R}^3} \left[ \int_{T-\epsilon}^{T} \int_{\mathbb{R}^3} \left[ (-1)^{|k'|} D^{k'} \phi(x) \, p_{j,i,k} (y, T - \eta) \, J_{\gamma}(v_j) \hat{v}_k (x - y, \eta) \right] \, dy \, d\eta \right] \, dx
$$

As the (5.26) integral in which $k' = 0$ is a bounded and measurable function of $x \in \mathbb{R}^3$, $(-1)^{|k'|} D^{k'} \phi(x)$ can move outside of the interior integral. Thus

$$
\int_{\mathbb{R}^3} \phi(x) \left[ \int_{T-\epsilon}^{T} \int_{\mathbb{R}^3} p_{j,i,k}(y, T - \eta) \, D^{k'}[J_{\gamma}(v_j)\hat{v}_k] \, (x - y, \eta) \, dy \, d\eta \right] \, dx
$$

and as a result that

$$
\int_{\mathbb{R}^3} \phi(x) \left[ \int_{T-\epsilon}^{T} \int_{\mathbb{R}^3} p_{j,i,k}(y, T - \eta) \, D^{k'}[J_{\gamma}(v_j)\hat{v}_k] \, (x - y, \eta) \, dy \, d\eta \right] \, dx
$$

$$
= \int_{\mathbb{R}^3} \left( -1 \right)^{|k'|} D^{k'} \phi(x)
$$

$$
\left[ \int_{T-\epsilon}^{T} \int_{\mathbb{R}^3} \left[ p_{j,i,k}(y, T - \eta) \, J_{\gamma}(v_j)\hat{v}_k \, (x - y, \eta) \right] \, dy \, d\eta \right] \, dx
$$

$$
+ \int_{\mathbb{R}^3} \phi(x) \left[ \int_{T-\epsilon}^{T} \int_{\mathbb{R}^3} p_{j,i,k}(y, T - \eta) \, D^{k'}[J_{\gamma}(v_j)\hat{v}_k] \, (x - y, \eta) \, dy \, d\eta \right] \, dx
$$

$$
- \int_{\mathbb{R}^3} \left( -1 \right)^{|k'|} D^{k'} \phi(x)
$$

$$
\left[ \int_{T-\epsilon}^{T} \int_{\mathbb{R}^3} \left[ p_{j,i,k}(y, T - \eta) \, J_{\gamma}(v_j)\hat{v}_k \, (x - y, \eta) \right] \, dy \, d\eta \right] \, dx
$$

(5.30)
for $\epsilon > 0$ and every scalar valued $\phi \in C^\infty_0$. We note for our later use that the absolute value of the last two summands in (5.30) is

$$\leq |\phi|_{L^2} \left| \int_{T-\epsilon}^T \int_{\mathbb{R}^3} p_{j,i,k}(y, T-\eta) \, D^{k'} [J_\gamma(v_j) \hat{v}_k] \, (x - y, \eta) \, dyd\eta \right|_{L^2}
$$

$$+ |D^{k'} \phi|_{L^2} \left| \int_{T-\epsilon}^T \int_{\mathbb{R}^3} \left[ p_{j,i,k}(y, T-\eta) \, J_\gamma(v_j) \hat{v}_k \, (x - y, \eta) \right] \, dyd\eta \right|_{L^2}
$$

(5.31)

We now show the integral in which the integrand is the the absolute value of the integrand in the (5.26) integral. The analysis which begins at (5.26) shows that the integral

$$\int_{T-\Delta T}^{T-\epsilon} \int_{\mathbb{R}^3} \left| p_{j,i,k}(y, T-\eta) \right| \left| D^{k'} [J_\gamma(v_j) \hat{v}_k] \, (x - y, \eta) \right| \, dyd\eta
$$

(5.32)

is

$$\leq \Delta T \left[ \ldots \right]_{5.7} \epsilon^{-2} \left[ \ldots \right]_{2.24, m} |v|_{H^m \times [T-\Delta T, T]} \, |\hat{v}|_{H^m \times [T-\Delta T, T]}
$$

which shows that the (5.32) integral is a bounded function of $x \in \mathbb{R}^3$. A similar calculation shows that the scalar $|p_{j,i,k}(x - y, T - \eta)|$ is continuous$^8$ in $x \in \mathbb{R}^3$. Thus, the (5.32) integral is a bounded and measurable function of $x \in \mathbb{R}^3$.

Therewith

$$\int_{\mathbb{R}^3} \int_{T-\Delta T}^{T-\epsilon} \int_{\mathbb{R}^3} \phi(x) \, \left| p_{j,i,k}(y, T-\eta) \right| \left| D^{k'} [J_\gamma(v_j) \hat{v}_k] \, (x - y, \eta) \right| \, dyd\eta \, dx
$$

$$= \int_{\mathbb{R}^3} \int_{T-\Delta T}^{T-\epsilon} \int_{\mathbb{R}^3} \phi(x) \, |p_{j,i,k}(y, T-\eta)| \left| D^{k'} [J_\gamma(v_j) \hat{v}_k] \, (x - y, \eta) \right| \, dyd\eta \, dx
$$

(5.33)

$^8$ The inequality

$$\left| |p_{j,i,k}(x + \Delta x - y, T - \eta)| - |p_{j,i,k}(x - y, T - \eta)| \right|
$$

$$\leq |p_{j,i,k}(x + \Delta x - y, T - \eta)| - p_{j,i,k}(x - y, T - \eta)|
$$

is an application of the inequality

$$||b| - |a|| \leq |b - a|$$
for each scalar valued $\phi \in C^\infty_0$. The right side of (5.33), as Fubini’s theorem and the inequality in note 8 show, is equal to its related, iterated integral

$$\int_{T-\Delta T}^{T-\epsilon} \int_{R^3} \left[ \int_{R^3} \phi(x) \left| p_{j,i,k}(y,T-\eta) \right| \left| D^{k'} [J_\gamma(v_j)\hat{v}_k] (x-y,\eta) \right| dx \right] dy d\eta$$

the absolute value of which on $R^3$, as a Hölder inequality shows, is

$$\leq \left[ \ldots \right]_{5.12} \Delta T^{1/2} |\phi|_{L^2} \left| D^{k'} [J_\gamma(v_j)\hat{v}_k] \right|_{L^2 \times [T-\Delta T,T]}$$

which with (5.33) shows that

$$\left| \int_{R^3} \phi(x) \left[ \int_{T-\Delta T}^{T-\epsilon} \int_{R^3} p_{j,i,k}(y,T-\eta) \left| D^{k'} [J_\gamma(v_j)\hat{v}_k] (x-y,\eta) \right| dy d\eta \right] dx \right|$$

is

$$\leq \left[ \ldots \right]_{5.12} \Delta T^{1/2} |\phi|_{L^2} \left[ \ldots \right]_{2.24,m} |v|_{H^m \times [T-\Delta T,T]} |\hat{v}|_{H^m \times [T-\Delta T,T]}$$

(5.34)

for every scalar valued $\phi \in C^\infty_0$.

As the (5.34) interior integral is measurable, the Hölder Inequality Converse shows that

$$\left| \int_{T-\Delta T}^{T-\epsilon} \int_{R^3} p_{j,i,k}(y,T-\eta) \left| D^{k'} [J_\gamma(v_j)\hat{v}_k] (x-y,\eta) \right| dy d\eta \right|_{L^2}$$

$$\leq \left[ \ldots \right]_{5.12} \Delta T^{1/2} \left[ \ldots \right]_{2.24,m} |v|_{H^m \times [T-\Delta T,T]} |\hat{v}|_{H^m \times [T-\Delta T,T]}$$

The monotone convergence theorem then shows that

$$\left| \int_{T-\Delta T}^{T} \int_{R^3} p_{j,i,k}(y,T-\eta) \left| D^{k'} [J_\gamma(v_j)\hat{v}_k] (x-y,\eta) \right| dy d\eta \right|_{L^2}$$

$$\leq \left[ \ldots \right]_{5.12} \Delta T^{1/2} \left[ \ldots \right]_{2.24,m} |v|_{H^m \times [T-\Delta T,T]} |\hat{v}|_{H^m \times [T-\Delta T,T]}$$

(5.35)
With \( \epsilon > 0 \) in place of \( \Delta T \) and \( k' \) a multi-index for which \( 0 \leq |k'| \leq m \), (5.35) shows that

\[
\left| \int_{T-\epsilon}^{T} \int_{\mathbb{R}^3} p_{j,i,k}(y, T - \eta) \, D^{k'}[J_{\gamma}(v_j)\hat{v}]_k(x - y, \eta) \, dyd\eta \right|_{L^2} \\
\leq \left| \int_{T-\epsilon}^{T} \int_{\mathbb{R}^3} |p_{j,i,k}(y, T - \eta)| \, |D^{k'}[J_{\gamma}(v_j)\hat{v}]_k(x - y, \eta)| \, dyd\eta \right|_{L^2} \\
\leq \left[ \ldots \right] 5.12 \epsilon^{1/2} \left[ \ldots \right] 2.24,m \, |v|_{H^m \times [T-\Delta T, T]} \, |\hat{v}|_{H^m \times [T-\Delta T, T]} \text{ for every scalar valued } \phi \in C^\infty_0 \text{ and each multi-index } k' \text{ for which } 0 \leq |k'| \leq m.
\]

(5.36)

which shows that (5.31) \( \to 0 \) as \( \epsilon \to 0 \). As a result

\[
\int_{\mathbb{R}^3} \phi(x) \left[ \int_{T-\Delta T}^{T} \int_{\mathbb{R}^3} p_{j,i,k}(y, T - \eta) \, D^{k'}[J_{\gamma}(v_j)\hat{v}]_k(x - y, \eta) \, dyd\eta \right] dx \\
= \int_{\mathbb{R}^3} (-1)^{|k'|} D^{k'}\phi(x) \\
\left[ \int_{T-\Delta T}^{T} \int_{\mathbb{R}^3} \left[ p_{j,i,k}(y, T - \eta) \, J_{\gamma}(v_j)\hat{v}_k \, (x - y, \eta) \right] dyd\eta \right] dx
\]

(5.37)

for every scalar valued \( \phi \in C^\infty_0 \) and each multi-index \( k' \) for which \( 0 \leq |k'| \leq m \).

The summation of (5.37) over \( k \) shows that

\[
D^{k'} \left[ \int_{T-\Delta T}^{T} \int_{\mathbb{R}^3} \left( \hat{\mathcal{P}}[\partial_j K^i(T - \eta)](y), J_{\gamma}(v_j)v \, (x - y, \eta) \right)_2 \, dyd\eta \right] \\
= \int_{T-\Delta T}^{T} \int_{\mathbb{R}^3} \left( \hat{\mathcal{P}}[\partial_j K^i(T - \eta)](y), D^{k'}[J_{\gamma}(v_j)v] \, (x - y, \eta) \right)_2 \, dyd\eta
\]

which establishes the \( i^{th} \) component of (5.20). The proof of (5.20) is complete.
We now establish (5.21). As \( D^{k'}[J_i(v_j)\hat{v}] \) is in \( H^2 \times [T - \Delta T, T] \) when \( 0 \leq |k'| \leq m - 2 \), (2.23), and is, as a result, in \( L^\infty \times [T - \Delta T, T] \), (2.14)

\[
\left| \int_{T - \Delta T}^{T} \left\langle \hat{P}[\partial_j K^i(T - \eta)](y), D^{k'}[J_i(v_j)\hat{v}](x - y, \eta) \right\rangle d\eta \right| \leq \int_{T - \Delta T}^{T} \left[ \hat{P}[\partial_j K^i(T - \eta)] \right] L^1 \left| D^{k'}[J_i(v_j)\hat{v}](\eta) \right| L^\infty d\eta
\]

which (5.21) shows is

\[
\leq \int_{T - \Delta T}^{T} \left[ \hat{P}[\partial_j K^i(T - \eta)] \right] L^1 \left| D^{k'}[J_i(v_j)\hat{v}](\eta) \right| L^\infty d\eta
\]

The proof of (5.21) is complete.

We now establish (5.22). Customary algebra shows that the left side of (5.22) is

\[
\left| \int_{T - \Delta T}^{T} \int_{\mathbb{R}^3} \hat{P}[\partial_j K^i(T - \eta)](y) \left[ [J_i(v_j) - J_i(\hat{v}_j)]v + J_i(\hat{v}_j)[v - \hat{v}] \right] (x - y, \eta) dyd\eta \right| \leq 2 \int_{T - \Delta T}^{T} \left| v - \hat{v} \right| L^\infty \times [T - \Delta T, T] \]

which (5.21) shows is

\[
\leq 2 \int_{T - \Delta T}^{T} \left| v - \hat{v} \right| L^\infty \times [T - \Delta T, T] \]

The proof of (5.22) is complete.

We now establish (5.23). The definition of the \( H^m \) norm and (5.30) show that the left side of (5.23) is

\[
\left| \sum_{0 \leq |k'| \leq m} D^{k'} \left[ \int_{T - \Delta T}^{T} \int_{\mathbb{R}^3} \hat{P}[\partial_j K^i(T - \eta)](y) D^{k'}[J_i(v_j)\hat{v}](x - y, \eta) dyd\eta \right] \right| L^2
\]

which with (5.20) and the triangle inequality is

\[
\leq \sum_{i, 0 \leq |k'| \leq m} \left| \int_{T - \Delta T}^{T} \int_{\mathbb{R}^3} \hat{P}[\partial_j K^i(T - \eta)](y) [J_i(v_j)\hat{v}](x - y, \eta) dyd\eta \right| L^2
\]

which with (5.5) is

\[
\leq \sum_{i,k, 0 \leq |k'| \leq m} \left| \int_{T - \Delta T}^{T} \int_{\mathbb{R}^3} p_{i,j,k}(y, T - \eta) D^{k'}[J_i(v_j)\hat{v}_k](x - y, \eta) dyd\eta \right| L^2
\]

which with (5.35) is

\[
\leq \sum_{i,k, 0 \leq |k'| \leq m} \left[ \cdots \right] \Delta T^{1/2} \left| v_j \right| H^m \times [T - \Delta T, T] \left| \hat{v} \right| H^m \times [T - \Delta T, T]
\]
The proof of (5.23) is complete.

We now establish (5.24). With customary mathematics, the left side of (5.24) is
\[
\left| \int_{T-\Delta T}^{T} \int_{\mathbb{R}^3} \hat{P} [\partial_j K^\dagger(T-\eta)](y) \left[ \left( J_\gamma(v_j) - J_\gamma(\hat{v}_j) \right) v + J_\gamma(\hat{v}_j) [v - \hat{v}] \right] (x - y, \eta) \, dyd\eta \right|_{H^m \times [T-\Delta T, T]}
\]
which with (5.23) is
\[
\leq [ \ldots ]_{5.23} \Delta T^{1/2} \left[ |v - \hat{v}|_{H^m \times [T-\Delta T, T]} |v|_{H^m \times [T-\Delta T, T]} \right. \\
+ \left. |\hat{v}|_{H^m \times [T-\Delta T, T]} |v - \hat{v}|_{H^m \times [T-\Delta T, T]} \right]
\leq 2 [ \ldots ]_{5.23} \Delta T^{1/2} |v - \hat{v}|_{H^m \times [T-\Delta T, T]} \\
\max \left[ |v|_{H^m \times [T-\Delta T, T]}, |\hat{v}|_{H^m \times [T-\Delta T, T]} \right]
\]
The proof of (5.24) is complete.

A variation of the proof of (5.23) in which $K(y, \eta)$ replaces $\hat{P}[\partial_j K^\dagger]$ establishes (5.25).

We leave the proof of the last sentence of this lemma to the reader. \hfill \Box

6. $u_{\gamma,m}$ satisfies the $(6.1)_{k',\gamma,u_o}$ integral equation when $0 \leq |k'| \leq m$

This section shows for $\gamma \geq 0$, $m \geq 7$ and $u_o \in \cap_m H^{m,df}$ that there is a function $u_{\gamma,m}(x, t; u_o) \in H^{m,df} \times [T_{\text{bup}}(u_o), T]$ which on the strip $\mathcal{R}^3 \times [0, T_{\text{bup}}(u_o))$ satisfies each $(6.1)_{k',\gamma,u_o}$ integral equation
\[
D^{k'} u_{\gamma,m}(x, t) \\
= \int_0^t \int_{\mathbb{R}^3} \sum_j \hat{P} [\partial_j K^\dagger(t-\eta)](y) D^{k'} [J_\gamma(u^{\gamma,m}_j) u_{\gamma,m}] (x - y, \eta) \, dyd\eta \\
+ \int_{\mathbb{R}^3} K(y, t) D^{k'} u_o(x - y) \, dy
\]
(6.1)
in which $0 \leq |k'| \leq m$. 
We begin with a uniqueness result for a solution of the \( (6.1)_{k'=0,\gamma,u_0} \) integral equation.

**Lemma 6.1.** When \( T > 0 \), there is at most one function \( u \in H^7 \times [0,T] \) which is a solution of the \( (6.1)_{k'=0,\gamma,u_0} \) integral equation on \( \mathbb{R}^3 \times [0,T] \).

**Proof.** If \( u \) and \( v \) satisfy the \( (6.1)_{k'=0,\gamma,u_0} \) integral equation on \( \mathbb{R}^3 \times [0,T] \), then

\[
[u - v](x,t) = \int_0^t \int_{\mathbb{R}^3} \sum_j \widehat{P}[\partial_j K^+(t-\eta)](y) \left[ [J_\gamma(u_j) - J_\gamma(v_j)] u + J_\gamma(v_j) [u - v] \right] (x-y,\eta) \, dy \, d\eta
\]

(6.2)

for \( (x,t) \in \mathbb{R}^3 \times [0,T] \) as \( u(0) = v(0) = u_0 \). (5.22) then shows that

\[
|u - v|_{L^\infty \times [0,t]} \leq \ldots t^{1/2} |u - v|_{L^\infty \times [0,t]} \max \left[ |u|_{L^\infty \times [0,T]}, |v|_{L^\infty \times [0,T]} \right]
\]

(6.3)

when \( t \in [0,T] \). With \( t_* \) as a positive number for which

\[
\ldots t_*^{1/2} \max \left[ |u|_{L^\infty \times [0,T]}, |v|_{L^\infty \times [0,T]} \right] \leq 2^{-1} \text{ and } Nt_* = T \text{ for a positive integer } N
\]

(6.3) shows that

\[
|u - v|_{L^\infty \times [0,t_*]} \leq 2^{-1} |u - v|_{L^\infty \times [0,t_*]}
\]

As \( u \) and \( v \) are continuous on \( \mathbb{R}^3 \times [0,T] \), Lemma 2.3, \( u = v \) on \( \mathbb{R}^3 \times [0,t_*] \), and (6.2) then shows that

\[
[u - v](x,t) = \int_{t_*}^T \int_{\mathbb{R}^3} \sum_j \widehat{P}[\partial_j K^+(t-\eta)](y) \left[ [J_\gamma(u_j) - J_\gamma(v_j)] u + J_\gamma(v_j) [u - v] \right] (x-y,\eta) \, dy \, d\eta
\]

for \( (x,t) \in [t_*,T] \). Thus, \( N - 1 \) repetitions of the foregoing shows that \( u = v \) on \( \mathbb{R}^3 \times [0,Nt_*] = \mathbb{R}^3 \times [0,T] \). \( \square \)
Theorem 6.2. Let $\gamma \geq 0$, $m \geq 7$, $u_o \in \cap_{m'} H^{m',df}$. Then there is a function
\[ u^{\gamma,m}(x, t; u_o) \in H^{m,df} \times [0, T^{\gamma,m}_{bup}(u_o)) \]
which on $R^3 \times [0, T^{\gamma,m}_{bup}(u_o))$ satisfies each (6.1)$_{k', \gamma, u_o}$ integral equation in which $0 \leq |k'| \leq m$.

The strip $R^3 \times [0, T^{\gamma,m}_{bup}(u_o))$ contains every strip $R^3 \times [0, T]$ on which there is a solution of the (6.1)$_{k'=0, \gamma, u_o}$ integral equation,
\[ 0 < \ldots T^{\gamma,m+1}_{bup}(u_o) \leq T^{\gamma,m}_{bup}(u_o) \leq \ldots \leq T^{\gamma,7}_{bup}(u_o) \]
and
\[ u^{\gamma,m+1}(x, t; u_o) = u^{\gamma,m}(x, t; u_o) \]
on $R^3 \times [0, T^{\gamma,m+1}_{bup}(u_o))$
\[ (6.6) \]

Proof. For $\gamma \geq 0$, we define the operator $S^\gamma$ on the function $v \in H^m \times [0, T]$ as
\[ S^\gamma(v)(x, 0) := u_o(x) \]
for $x \in R^3$ and $t = 0$ and
\[ S^\gamma(v)(x, t) := \int_0^t \int_{R^3} \sum_j \hat{P}[\partial_j K^j(t - \eta)](y) J_v(y) J(v_j) v(x - y, \eta) \, dy \, d\eta \]
\[ + \int_{R^3} K(y, t) u_o(x - y) \, dy \]
for $(x, t) \in R^3 \times (0, T]$.

For $T > 0$, we let $U^m \times [0, T]$ be the subset of $H^m \times [0, T]$ each member $u$ of which satisfies
\[ u(x, 0) = u_o(x) \text{ for } x \in R^3 \text{ and} \]
\[ |u|_{H^m \times [0, T]} \leq 2 |u_o|_{H^m} \]
\[ (6.7) \]
$U^m \times [0, T]$ is non-empty as the function $u(x, t) = u_o(x)$ is a member thereof. Such set is also a complete, closed subset of $H^m \times [0, T]$. Lemma 2.2.
The next paragraphs show that $S^{\gamma}(u)$ when $u \in U^m \times [0, T]$ is a function of $t \in [0, T]$ which is continuous in the $H^m$ norm. With $0 < t < t + \Delta t \leq T$

$$S^{\gamma}(u)(x, t + \Delta t) - S^{\gamma}(u)(x, t)$$

$$= \int_{0}^{t + \Delta t} \int_{\mathbb{R}^3} \sum_{j} \hat{P} \left[ \partial_{j} K^\dagger(t + \Delta t - \eta) \right](y) J_{\gamma}(u_j)u(x - y, \eta) \, d\eta$$

$$+ \int_{\mathbb{R}^3} K(y, t + \Delta t) \, u_o(x - y) \, dy$$

$$- \left[ \int_{0}^{t} \int_{\mathbb{R}^3} \sum_{j} \hat{P} \left[ \partial_{j} K^\dagger(t - \eta) \right](y) J_{\gamma}(u_j)u(x - y, \eta) \, dyd\eta$$

$$+ \int_{\mathbb{R}^3} K(y, t) \, u_o(x - y) \, dy \right]$$

which after dividing the first double integral in the right side into two integrals and repositioning the other double integral is

$$= \int_{\Delta t}^{t + \Delta t} \int_{\mathbb{R}^3} \sum_{j} \hat{P} \left[ \partial_{j} K^\dagger(t + \Delta t - \eta) \right](y)$$

$$[J_{\gamma}(u_j)u(x - y, \eta) - J_{\gamma}(u_j)u(x - y, \eta - \Delta t)] \, dyd\eta$$

$$+ \int_{0}^{\Delta t} \int_{\mathbb{R}^3} \sum_{j} \hat{P} \left[ \partial_{j} K^\dagger(t + \Delta t - \eta) \right](y) J_{\gamma}(u_j)u(x - y, \eta) \, dyd\eta$$

$$+ \int_{\mathbb{R}^3} [K(y, t + \Delta t) - K(y, t)] \, u_o(x - y) \, dy$$

(6.8)

(5.24) shows that the $H^m$ norm of the first (6.8) double integral is

$$\leq \left[ \ldots \right] t^{1/2} \sup_{\eta \in [\Delta t, t + \Delta t]} |u(\eta) - u(\eta - \Delta t)|_{H^m} |u|_{H^m \times [0, t + \Delta t]}$$

which as $t \leq t + \Delta t$ is $\leq T$ and $u$ is $\in U^m \times [0, T]$ is

$$\leq \left[ \ldots \right] T^{1/2} \sup_{\eta \in [\Delta t, T]} |u(\eta) - u(\eta - \Delta t)|_{H^m} 2 \, |u_o|_{H^m}$$

which is $= o_{6,8}(\Delta t; T)$

because each $u \in U^m \times [0, T]$ is a function of $t \in [0, T]$ which is continuous in the $H^m$ norm.
(5.23) and (6.7) then show that the $H^m$ norm of the second (6.8) double integral is
\[
\leq \left[ \ldots \right] \Delta t^{1/2} \left[ \frac{1}{2} |u_o|_{H^m} \right]^2
\]
As $t$ is $> 0$, the calculus shows that the (6.8) single integral is
\[
= \int_{R^3} \left[ \int_{t}^{t+\Delta t} \partial_\eta K(y, \eta) \, d\eta \right] u_o(x - y) \, dy
\]
which with (5.2) and Fubini’s theorem is
\[
= \int_{t}^{t+\Delta t} \left[ \int_{R^3} \Delta_y K(y, \eta) \, u_o(x - y) \, dy \right] \, d\eta
\]
which after integration by parts is
\[
= \int_{t}^{t+\Delta t} \left[ \int_{R^3} K(y, \eta) \, \Delta u_o(x - y) \, dy \right] \, d\eta
\]
the $H^m$ norm of which, (5.25) shows, is
\[
\leq \Delta t \, |\Delta u_o|_{H^m} \leq \left[ \ldots \right] \Delta t \, |u_o|_{H^{m+2}}
\]
The foregoing shows when $u$ is $\in U^m \times [0, T]$ and $0 < t < t + \Delta t \leq T$ that
\[
|S^\gamma(u)(t + \Delta t) - S^\gamma(u)(t)|_{H^m}
\leq o_{6.8} (\Delta t; T) + \left[ \ldots \right] \Delta t^{1/2} \left[ u_o \right]_{H^m}^2 + \left[ \ldots \right] \Delta t \, |u_o|_{H^{m+2}}
\]
which we restate for $0 < t_1 < t_2 \leq T$ as
\[
|S^\gamma(u)(t_2) - S^\gamma(u)(t_1)|_{H^m}
\leq o_{6.8} (t_2 - t_1; T) + \left[ \ldots \right] [t_2 - t_1]^{1/2} \left[ u_o \right]_{H^m}^2
+ \left[ \ldots \right] [t_2 - t_1] \, |u_o|_{H^{m+2}}
\]
\[
= o_{6.8} (t_2 - t_1)
\]
(6.9)
Thus, $S^\gamma(u)(t)$ when $u$ is $\in U^m \times [0, T]$ is a function of $t$ in the half-open interval $(0, T]$ which is continuous in the $H^m$ norm.

We now show when $u$ is $\in U^m \times [0, T]$ that $S^\gamma(u)$ is continuous at $t = 0$ in the $H^m$ norm. When $\Delta t$ is $> 0$, (5.2) shows that
\[
\left[ \int_{R^3} K(y, \Delta t) \, dy \right] u_o(x) = u_o(x)
\]
As a result thereof

\[ S^\gamma(u)(x, \Delta t) - u_o(x) = \int_0^{\Delta t} \int_{\mathbb{R}^3} \sum_j \tilde{\mathcal{P}}[\partial_j K^\gamma(\Delta t - \eta)](y) J_y(u_j)u(x-y, \eta) \ dy \ d\eta \]

\[ + \int_{\mathbb{R}^3} K(y, \Delta t) [u_o(x-y) - u_o(x)] \ dy \]

(6.10)

The \( H^m \) norm of the (6.10) double integral is, as (5.23) and (6.7) show,

\[ \leq [ \ldots ] \Delta t^{1/2} \left[ 2 |u_o|_{H^m} \right]^2 \]

The \( H^m \) norm of the (6.10) single integral is

\[ = \sum_{0 \leq |k'| \leq m} \left| \int_{\mathbb{R}^3} K(y, \Delta t) \left[ D^{k'} u_o(x-y) - D^{k'} u_o(x) \right] \ dy \right|_{L^2} \]

(6.11)

which with \( r > 0 \), \( \psi^r_{\text{left}} \) as the characteristic function of \( B(r) \) and \( \psi^r_{\text{right}} \) as the characteristic function of \( \mathbb{R}^3 \setminus B(r) \) is

\[ \leq \sum_{0 \leq |k'| \leq m} \left| \int_{\mathbb{R}^3} K(y, \Delta t) \psi^r_{\text{left}}(y) \left[ D^{k'} u_o(x-y) - D^{k'} u_o(x) \right] \ dy \right|_{L^2} \]

\[ + \sum_{0 \leq |k'| \leq m} \left| \int_{\mathbb{R}^3} K(y, \Delta t) \psi^r_{\text{right}}(y) \left[ D^{k'} u_o(x-y) - D^{k'} u_o(x) \right] \ dy \right|_{L^2} \]

(6.12)

We now develop an upper bound for each (6.12) (first line) summand. As \( u_o \) is in \( \cap_m H^{m, df} \), (3.6) shows that

\[ \left| \int_{\mathbb{R}^3} K(y, \Delta t) \psi^r_{\text{left}}(y) \left[ D^{k'} u_o(x-y) - D^{k'} u_o(x) \right] \ dy \right|_{L^\infty} \]

is \[ \leq \left| K(y, \Delta t) \psi^r_{\text{left}} \right|_{L^1} 2 \left| D^{k'} u_o \right|_{L^\infty} \]

\[ \leq [ \ldots ] \left| D^{k'} u_o \right|_{H^{k'+2}} \]
and is a bounded function of $x \in \mathcal{R}^3$. Moreover
\[
\left| \int_{\mathcal{R}^3} K(y, \Delta t) \psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x + \Delta x - y) - D^{k'} u_o(x + \Delta x) \right] dy \right|
\]
\[
- \left| \int_{\mathcal{R}^3} K(y, \Delta t) \psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] dy \right|_{L^\infty}
\]
which is
\[
= \left| \int_{\mathcal{R}^3} K(y, \Delta t) \psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x + \Delta x - y) - D^{k'} u_o(x - y) \right] dy \right|
\]
\[
- \left| \int_{\mathcal{R}^3} K(y, \Delta t) \psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x + \Delta x) - D^{k'} u_o(x) \right] dy \right|_{L^\infty}
\]
is
\[
\leq \left| K(y, \Delta t) \psi_{\text{left}}^r \right|_{L^1} \left| \text{grad} D^{k'} u_o \right|_{L^\infty} \Delta x
\]
\[
\leq \left[ \ldots \right] |u_o|_{H^{|k'|+3}} \Delta x
\]
which shows that
\[
\int_{\mathcal{R}^3} K(y, \Delta t) \psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x + \Delta x - y) - D^{k'} u_o(x + \Delta x) \right] dy
\]
(6.13)
is a continuous function of $x \in \mathcal{R}^3$ and is therefore a measurable function of $x \in \mathcal{R}^3$.

Customary mathematics then shows that
\[
\int_{\mathcal{R}^3} \phi(x) \left[ \int_{\mathcal{R}^3} K(y, \Delta t) \psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] dy \right] dx
\]
\[
= \int_{\mathcal{R}^3} \left[ \int_{\mathcal{R}^3} K(y, \Delta t) \psi_{\text{left}}^r(y) \phi(x) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] dy \right] dx
\]
(6.14)
for each multi-index $k'$ and every scalar valued $\phi \in C^\infty_0$. With Fubini’s theorem, the right side of (6.14) is equal to its related iterated integral
\[
\int_{\mathcal{R}^3} \left[ \int_{\mathcal{R}^3} K(y, \Delta t) \phi(x) \psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] dx \right] dy
\]
A Hölder inequality in the interior integral thereof then shows that the $L^\infty$ norm of the left side of (6.14) is

\[
\leq \int_{\mathbb{R}^3} K(y, \Delta t) \, dy \\
|\phi|_{L^2} \sup_{y \in \mathbb{R}^3} |\psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] |_{L^2} \\
\leq |\phi|_{L^2} \sup_{y \in \mathbb{R}^3} |\psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] |_{L^2}
\]

Thus, the absolute value of the left side of (6.14), which is

\[
\left| \int_{\mathbb{R}^3} \phi(x) \left[ \int_{\mathbb{R}^3} K(y, \Delta t) \psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] \, dy \right] \, dx \right| \\
= \leq |\phi|_{L^2} \sup_{|y| \leq r} |\psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] |_{L^2}
\]

(6.15)

for every multi-index $k'$ and every scalar valued $\phi \in C^\infty_0$.

As the integral in the left side of (6.15) is a measurable function of $x \in \mathbb{R}^3$, (6.13), the Hölder Inequality Converse shows that

\[
\left| \int_{\mathbb{R}^3} K(y, \Delta t) \psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] \, dy \right|_{L^2} \\
\leq \sup_{|y| \leq r} |\psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] |_{L^2}
\]

for each multi-index $k'$ the right side of which is

\[
= \sup_{|y| \leq r} \left| \int_0^1 \partial_\theta \left[ D^{k'} u_o(x - \theta y) \right] \, d\theta \right|_{L^2} \\
= \sup_{|y| \leq r} \left| \int_0^1 \left( \text{grad} \, D^{k'} u_o(x - \theta y), y \right)_2 \, d\theta \right|_{L^2}
\]

which with a Hölder inequality in the $(\text{grad} \, D^{k'} u_o(x - \theta y), y)_2$ inner product is

\[
\leq \sup_{|y| \leq r} \left| \int_0^1 |\text{grad} \, D^{k'} u_o(x - \theta y)|_2 \cdot 1 \cdot |y|_2 \, d\theta \right|_{L^2}
\]

which with another Hölder inequality is

\[
\leq \sup_{|y| \leq r} \left| \left[ \int_0^1 |\text{grad} \, D^{k'} u_o(x - \theta y)|_2^2 \, d\theta \right]^{1/2} \left[ \int_0^1 1^2 \, d\theta \right]^{1/2} \right|_{L^2}
\]
which with the definition of the $L^2$ norm is
\[
\leq \sup_{|y| \leq r} \left[ \int_{R^3} \left[ \int_0^1 \left| \nabla D^{k'} u_o(x - \theta y) \right| \frac{2}{d} \, d\theta \right]^2 \, dx \right]^{1/2} r
\]
\[
\leq \sup_{|y| \leq r} \left[ \int_0^1 \left[ \int_{R^3} \left| \nabla D^{k'} u_o(x - \theta y) \right| \frac{2}{d} \, dx \right] \, d\theta \right]^{1/2} r
\]
which after reversing the order of integration is
\[
\leq \sup_{|y| \leq r} \left[ \int_0^1 \left[ \int_{R^3} \left| \nabla D^{k'} u_o(x - \theta y) \right| \frac{2}{d} \, dx \right] \, d\theta \right]^{1/2} r
\]
\[
\leq \sup_{|y| \leq r} \left[ \int_0^1 \left[ \int_{R^3} \left| \nabla D^{k'} u_o(x - \theta y) \right| \frac{2}{d} \, dx \right] \, d\theta \right]^{1/2} r
\]
\[
\leq |\nabla D^{k'} u_o|_{L^2} r \leq \ldots \right) \left| u_o \right|_{H^{m+1}} r
\]
which with the analysis which begins in this paragraph shows that
\[
\left| \int_{R^3} K(y, \Delta t) \psi_{\text{left}}^r(y) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] \, dy \right|_{L^2}
\]
\[
is \leq |\nabla D^{k'} u_o|_{L^2} r \leq \ldots \right) \left| u_o \right|_{H^{m+1}} r
\]

We now develop an upper bound for each (6.11) (second line) summand. With (3.8)
\[
\left| \int_{R^3} K(y, \Delta t) \psi_{\text{right}}^r(y) \left[ D^{k'} u_o(x - y) - D^{k'} u_o(x) \right] \, dy \right|_{L^2}
\]
\[
\leq \left| K(y, \Delta t) \psi_{\text{right}}^r(y) \right|_{L^1} \left| D^{k'} u_o(x - y) - D^{k'} u_o(x) \right|_{L^2}
\]
\[
\leq \left| K(y, \Delta t) \psi_{\text{right}}^r(y) \right|_{L^1} 2 \left| D^{k'} u_o \right|_{L^2}
\]
(6.16)

The definition of $K(y, \Delta t)$ and the definition of $\psi_{\text{right}}^r$ show that
\[
| K(y, \Delta t) \psi_{\text{right}}^r(y) |_{L^1}
\]
\[
= \ldots \right) \Delta t^{-3/2} \int_{R^3} \exp \left[ - |y|_2^2 / 4 \Delta t \right] \, dy
\]
\[
= \ldots \right) \int_{R^3} \exp \left[ - |y|_2^2 / 4 \Delta t \right] \, dy
\]
which after changing variables in the integral is
\[
= \ldots \right) \int_{R^3} \exp \left[ - |y|_2^2 / 4 \right] \, dy := o(\Delta t^{1/2} / r)
which \( \to 0 \) as \( \Delta t^{1/2}/r \to 0 \). Thus, the left side of (6.16) is
\[
\leq o(\Delta t^{1/2}/r) \ |u_o|_{H^m}
\]

The discussion which begins at (6.10) shows for any \( r > 0 \) that
\[
|S^\gamma(u)(\Delta t) - u_o|_{H^m}
\leq [ \ldots ] \Delta t^{1/2} |u_o|^2_{H^m}
+ [ \ldots ] |u_o|_{H^{m+1}} r + [ \ldots ] o(\Delta t^{1/2}/r) \ |u_o|_{H^m}.
\]

We fix \( r > 0 \) and so small that the second summand is \(< \epsilon/3 \) and then fix \( \Delta t > 0 \) and so small that each of the two other summands is \(< \epsilon/3 \).

The proof that \( S^\gamma(u)(t) \) when \( u \in U^m \times [0, T] \) is a function of \( t \in [0, T] \) which is continuous thereon in the \( H^m \) norm is complete.

We now show when \( T > 0 \) and sufficiently small that \( S^\gamma \) carries \( U^m \times [0, T] \) into itself and is a contraction thereon. The definition of \( S^\gamma \) shows that
\[
|S^\gamma(u)|_{H^m \times [0, T]} \leq \sum_j \int_0^T \int_{\mathbb{R}^3} \tilde{P} \left[ \partial_j K^\gamma(t - \eta) \right] (y) \ J_\gamma(u_j) u \ (x - y, \eta) \ dyd\eta \ |u|_{H^m}
+ \int_{\mathbb{R}^3} K(y, t) \ u_o(x - y) \ dy \ |u|_{H^m}
\]
when \( t \in [0, T] \), which with (5.23) shows that
\[
|S^\gamma(u)|_{H^m \times [0, T]}
\leq [ \ldots ] \ T^{1/2} \ |J_\gamma(u_j)|_{H^m \times [0, T]} \ |u|_{H^m \times [0, T]} + [u_o]_{H^m}
\]
and when \( u \in U^m \times [0, T] \), (6.7) shows that
\[
|S^\gamma(u)|_{H^m \times [0, T]}
\leq [ \ldots ] \ T^{1/2} \ [2 \ |u_o|_{H^m}^2 + [u_o]_{H^m}
\leq [ \ldots ]_{6.17} \ T^{1/2} \ |u_o|_{H^m}^2 + [u_o]_{H^m}
\]
\[
(6.17)
\]
in which \( [ \ldots ]_{6.17} \) is independent of \( T \).
When $u$ and $v$ are $\in U^m \times [0, T]$, (5.24) and (6.7) show that

$$|S^\gamma(u) - S^\gamma(v)|_{H^m \times [0, T]} \leq \cdots T^{1/2} |u - v|_{H^m \times [0, T]} \max \left[ |u|_{H^m \times [0, T]}, |v|_{H^m \times [0, T]} \right]$$

$$\leq \cdots 6.18 T^{1/2} |u - v|_{H^m \times [0, T]} |u_o|_{H^m}$$

(6.18)

in which $\cdots 6.18$ is independent of $T$.

Thus, there is $T_{6.19}^m(u_o) = \cdots 6.19 |u_o|_{H^m} > 0$ which when $\cdots 6.19$ is $> 0$ and is sufficiently small shows that

$$|S^\gamma(u)|_{H^m \times [0, T_{6.19}^m(u_o)]} \leq 2 |u_o|_{H^m} \text{ and}$$

$$|S^\gamma(u) - S^\gamma(v)|_{H^m \times [0, T_{6.19}^m(u_o)]} \leq 2^{-1} |u - v|_{H^m \times [0, T_{6.19}^m(u_o)]}$$

(6.19)

when $u$ and $v$ are $\in U^m \times [0, T_{6.19}^m(u_o)]$.

As the definition of $S^\gamma$ shows that $S^\gamma(u) \big|_{t=0} = u_o$, the first line of (6.19) completes the proof that $S^\gamma$ carries $U^m \times [0, T_{6.19}^m(u_o)]$ into itself, and the second line thereof shows that $S^\gamma$ is a contraction on $U^m \times [0, T_{6.19}^m(u_o)]$. Thus, the contraction mapping theorem shows that there is a function

$$u_{\gamma,m} \in U^m \times [0, T_{6.19}^m(u_o)]$$

which is a fixed point of $S^\gamma$ and is, as a result, a solution of the $(6.1)_{k'=0,\gamma,u_o}$ integral equation on $\mathcal{R}^3 \times [0, T_{6.19}^m(u_o)]$.

We now show that $u_{\gamma,m}$ is divergence free on $\mathcal{R}^3 \times [0, T_{6.19}^m(u_o)]$. As $u_{\gamma,m}$ satisfies the $(6.1)_{k'=0,\gamma,u_o}$ integral equation $\mathcal{R}^3 \times [0, T_{6.19}^m(u_o)]$, (5.3), Fubini’s theorem and (5.20) show that

$$\partial_t u_{\gamma,m}^i(x,t)$$

$$= \int_0^t \int_{\mathcal{R}^3} \sum_j \left( \hat{P} \left[ \partial_j K^j(t-\eta) \right] (y), \partial_i [J_{\gamma}(u_{\gamma,m}^j)u_{\gamma,m}^j (x - y, \eta)] \right) dy \right] d\eta$$

$$+ \int_{\mathcal{R}^3} K(y,t) [u_o]_i (x - y) dy$$

(6.20)

on $\mathcal{R}^3 \times [0, T_{6.19}^m(u_o)]$. 
We then move \( \hat{P} \) to the right side of the (\( \cdot, \cdot \))\(_2\) inner product, (4.35), and to the right of \( \partial_i \), (4.36), integrate by parts to move each \( \partial_j \) to the right side of that inner product and to the right of \( \partial_i \), (2.10), and to the right of \( \hat{P} \). (4.36). Thus

\[
\partial_i u_i^{\gamma,m}(x,t) = \int_0^t \left[ \int_{\mathbb{R}^3} \left( K^i(y, t - \eta), \partial_i \left[ \hat{P} \left[ \sum_j \partial_j [J_\gamma(u_j^{\gamma,m}(\eta)] u^{\gamma,m}(\eta)](x - y) \right] \right)_2 \right] d\eta \right. \\
+ \int_{\mathbb{R}^3} K(y, t) [u_o]_i (x - y) dy
\]

which as \( K^i(y, T - \eta) \) is a vector the \( i^{th} \) column of which is \( K(y, T - \eta) \) and the other components of which vanish shows that

\[
\partial_i u_i^{\gamma,m}(x,t) = \int_0^t \int_{\mathbb{R}^3} K(y, t - \eta) \partial_i \left[ \hat{P} \left[ \sum_j \partial_j [J_\gamma(u_j^{\gamma,m}(\eta)] u^{\gamma,m}(\eta)] \right]_i (x - y, \eta) d\eta \right. \\
+ \int_{\mathbb{R}^3} K(y, t) [u_o]_i (x - y) dy
\]

(6.21)

which when summed over \( i \) shows that

\[
\text{div} \ u^{\gamma,m}(x,t) = \int_0^t \int_{\mathbb{R}^3} K(y, t - \eta) \text{div}_x \left[ \hat{P} \left[ \sum_j \partial_j [J_\gamma(u_j^{\gamma,m})u^{\gamma,m}(\eta)] \right] (x - y, \eta) d\eta \right. \\
+ \int_{\mathbb{R}^3} K(y, t) \text{div} \ u_o (x - y) dy
\]

(6.22)

on \( \mathbb{R}^3 \times [0, T_{6.19}(u_o)] \).

As \( u^{\gamma,m} \) is \( \in H^7 \times [0, T_{6.19}(u_o)] \), \( J_\gamma(u_j^{\gamma,m}) u^{\gamma,m} \) is \( \in H^7 \times [0, T_{6.19}(u_o)] \), Lemma 2.1,

\[
\sum_j \partial_j [J_\gamma(u_j^{\gamma,m})u^{\gamma,m}] \in H^6 \times [0, T_{6.19}(u_o)]
\]
and (4.33) then shows that
\[ \text{div } \hat{P} \left[ \sum_j \partial_j [J_\gamma(u_j^\gamma)] u^\gamma (\eta) \right] = 0 \]
which shows that the double integral in (6.22) vanishes. As \( u_o \) is divergence free, the single integral in (6.22) also vanishes, which shows that \( u^\gamma \) is divergence free on \( \mathcal{R}^3 \times [0, T_{6.19}(u_o)] \).

We now extend the strip on which \( u^\gamma \) is defined. We consider the numbers \( T \geq T_{6.19}(u_o) \) for which there is a function \( u_{1, m}^\gamma \) which satisfies the (6.1) \( k'=0, \gamma, u_o \) integral equation on \( \mathcal{R}^3 \times [0, T] \). As \( m \geq 7 \), any \( u_{1, m}^\gamma \) and \( u_{2, m}^\gamma \) are equal to each other on any strip where each is defined. Lemma 6.1.

As a result, the functions \( u_{1, m}^\gamma \) define a single function \( \mathcal{T}_{1, m}^\gamma (u_o) \) which satisfies the (6.1) \( k'=0, \gamma, u_o \) integral equation on \( \mathcal{R}^3 \times [0, T] \). The proof of (6.5) and of (6.6) is complete.

The application of the operator \( D^{k'} \) in which \( 1 \leq |k'| \leq m \) to the (6.1) \( k'=0, \gamma, u_o \) integral equation shows, with (5.20) and (3.5), that
\[
D^{k'} u^\gamma(x, t) = \int_0^t \int_{\mathcal{R}^3} \hat{P} \left[ \partial_j K(t - \eta) \right] (y) D^{k'} [J_\gamma(u_j^\gamma)] u^\gamma (x - y, \eta) \, dy d\eta \\
+ \int_{\mathcal{R}^3} K(y, t) D^{k'} u^\gamma_{bup}(x - y) \, dy
\]
(6.23)
for those \( k' \) for which \( 1 \leq |k'| \leq m \) on \( \mathcal{R}^3 \times [0, T^{\gamma, m}_{bup}(u_o)] \). Thus, \( u^\gamma \) satisfies each (6.1) \( k', \gamma, u_o \) integral equation on such strip.

On occasion, we hereafter use the functions \( u^\gamma \) without reminding the reader that \( \gamma \) is \( \geq 0 \), \( m \) is \( \geq 7 \) and \( u_o \) is \( \in \cap_m H^{m,d} \).
Lemma 6.3. For a multi-index $k'$ for which $0 \leq |k'| \leq m - 1$, $u_{\gamma,m}(x, t; u_o)$ satisfies the (6.24)\textsubscript{$k'^{\circ}, \gamma, u_o$} integral equation

$$D^{k'} u_{\gamma,m}(x, t) = \int_0^t \int_{\mathbb{R}^3} K(y, t - \eta) \hat{P} \left[ \sum_j D^{k'} [J_{\gamma}(u_{\gamma,m}(\eta))] \partial_j u_{\gamma,m}(\eta) \right] (x - y) \, dy \, d\eta + \int_{\mathbb{R}^3} K(y, t) D^{k'} u_o(x - y) \, dy$$

(6.24)

on $\mathbb{R}^3 \times [0, T_{\text{bup}}(u_o))$.

Proof. The (6.23) equation can be evolved by steps, as we showed above, to show that $D^{k'} u_{\gamma,m}(x, t)$

$$= \int_0^t \int_{\mathbb{R}^3} K(y, t - \eta) \hat{P} \left[ D^{k'} \sum_j \partial_j [J_{\gamma}(u_{\gamma,m}(\eta))] u_{\gamma,m}(\eta) \right] (x - y) \, dy \, d\eta + \int_{\mathbb{R}^3} K(y, t) D^{k'} u_o(x - y) \, dy$$

for those multi-indices $k'$ for which $0 \leq |k'| \leq m - 1$. As $u_{\gamma}$ is divergence free

$$\sum_j \partial_j [J_{\gamma}(u_{\gamma,m}(\eta))] u_{\gamma,m} = \sum_j J_{\gamma}(u_{\gamma,m}(\eta)) \partial_j u_{\gamma,m}$$

the use of which in (6.25) completes the proof of (6.24). \qed

7. $u_{\gamma,m}$ satisfies the (7.1)\textsubscript{$k'^{\circ}, \gamma, u_o$} PDEs when $0 \leq |k'| \leq m - 7$

The section shows that $u_{\gamma,m}$ has for each multi-index $k'$ for which $0 \leq |k'| \leq m - 7$ a derivative $\partial_t D^{k'} u_{\gamma,m}$ on $\mathbb{R}^3 \times [0, T_{\text{bup}}(u_o))$ which is $H^{m-|k'|-2, df} \times [0, T_{\text{bup}}(u_o))$ and satisfies the (7.1)\textsubscript{$k'^{\circ}, \gamma, u_o$} pde

$$\partial_t D^{k'} u_{\gamma,m}(x, t) + \mathcal{P} \left[ \sum_j D^{k'} [J_{\gamma}(u_{\gamma,m}(t))] \partial_j u_{\gamma,m}(t) \right](x) = \Delta D^{k'} u_{\gamma,m}(x, t)$$

$$D^{k'} u_{\gamma,m}(x, 0) = D^{k'} u_o(x)$$

(7.1)
on $\mathcal{R}^3 \times [0, T_{bup}^{\gamma,m}(u_o))$. This section also shows that $u^{\gamma,m}$ satisfies the inequality

$$\left| u^{\gamma,m}\right|_{L^2 \times [0,T_{bup}^{\gamma,m}(u_o))} \leq |u_o|_{L^2}$$

and satisfies the blow up condition

$$\left| u^{\gamma,m}\right|_{H^m \times [0,T]} \to \infty \text{ as } T \to \text{ a finite } T_{bup}^{\gamma,m}(u_o)$$

**Theorem 7.1.** Let $\gamma$ be $\geq 0$, $m$ be $\geq 7$, $u_o$ be $\in \cap_m H^{m,df}$ and $k'$ be a multi-index for which $0 \leq |k'| \leq m - 7$. Then $u^{\gamma,m} = u^{\gamma,m}(x, t; u_o)$ has on the strip $\mathcal{R}^3 \times [0, T_{bup}^{\gamma,m}(u_o))$ a derivative

$$\partial_t D^{k'} u^{\gamma,m} \text{ which is } \in H^{m-|k'|-2,df} \times [0, T_{bup}^{\gamma,m}(u_o))$$

and satisfies the (7.1)$_{k',\gamma,u_o}$ pde on such strip.

**Proof.** For a multi-index $k'$ for which $0 \leq |k'| \leq m - 7$, we define the scalar valued functions

$$v_k(x, t) := D^{k'} u_k^{\gamma,m}(x, t)$$

$$g_k(x, t) := \left[ \hat{P} \left[ D^{k'} \sum_j J_j(u_j^{\gamma,m}) \partial_t u^{\gamma,m}(t) \right] \right]_k(x) \text{ and}$$

$$[v_o]_k := D^{k'} [u_o]_k$$

(7.2)

for $k = 1, 2$ and $3$ on $\mathcal{R}^3 \times [0, T_{bup}^{\gamma,m}(u_o))$.

With such definitions

$$v_k \text{ is } \in H^{m-|k'|} \times [0, T_{bup}^{\gamma,m}(u_o)) \subseteq H^7 \times [0, T_{bup}^{\gamma,m}(u_o))$$

$$\Delta v_k \text{ is } \in H^{m-|k'|-2} \times [0, T_{bup}^{\gamma,m}(u_o)) \subseteq H^5 \times [0, T_{bup}^{\gamma,m}(u_o))$$

and with (5.20) and (3.5)

$$\Delta v_k(x, t) = \int_0^t \int_{\mathcal{R}^3} K(y, t - \eta) \Delta g_k(x - y, \eta) \, dy \, d\eta$$

$$+ \int_{\mathcal{R}^3} K(y, t) \Delta [v_o]_k(x - y) \, dy$$

(7.4)
That
\[ u_{\gamma,m} \in H^m \times [0, T_{bup}^{\gamma,m}(u_0)) \subseteq H^7 \times [0, T_{bup}^{\gamma,m}(u_0)) \]
shows, with Lemma 2.1, that
\[ J_{\gamma}(u_{\gamma,m}^j) \partial_j u_{\gamma,m} \in H^{m-1} \times [0, T_{bup}^{\gamma,m}(u_0)), \]
\[ \subseteq H^6 \times [0, T_{bup}^{\gamma,m}(u_0)) \]
as \( \tilde{P} \) does not increase an \( H^m \) norm, (4.37), that
\[ g_k \in H^{m-1-|k'|} \times [0, T_{bup}^{\gamma,m}(u_0)) \]
\[ \subseteq H^6 \times [0, T_{bup}^{\gamma,m}(u_0)) \]
(7.5)
and with (5.2) that
\[ g_k(x,t) = \Delta t^{-1} \int_0^{t+\Delta t} \int_{\mathbb{R}^3} K(y,t+\Delta t - \eta) dyd\eta \] \[ g_k(x,t) \]
(7.6)
As \( m - |k'| \geq 7 \), (6.24) then shows that
\[ v_k(x,t) = \int_0^t \int_{\mathbb{R}^3} K(y,t-\eta) g_k(x-y, \eta) dyd\eta \]
\[ \quad + \int_{\mathbb{R}^3} K(y,t) [v_0]_k(x-y) dy \]
(7.7)
and that
\[ \Delta t^{-1} \left[ v_k(x,t+\Delta t) - v_k(x,t) \right] \]
\[ = \Delta t^{-1} \left[ \int_0^{t+\Delta t} \int_{\mathbb{R}^3} K(y,t+\Delta t - \eta) g_k(x-y, \eta) dyd\eta \right. \]
\[ + \int_{\mathbb{R}^3} K(y,t+\Delta t) [v_0]_k(x-y) dy \]
\[ \left. - \int_0^t \int_{\mathbb{R}^3} K(y,t-\eta) g_k(x-y, \eta) dyd\eta \right] \]
\[ + \int_{\mathbb{R}^3} K(y,t) [v_0]_k(x-y) dy \]
which with customary mathematics shows that

\[
\Delta t^{-1} \left[ v_k(x, t + \Delta t) - v_k(x, t) \right] = \Delta t^{-1} \int_t^{t+\Delta t} \int_{\mathbb{R}^3} K(y, t + \Delta t - \eta) g_k(x - y, \eta) \, dy \, d\eta \\
+ \Delta t^{-1} \int_0^t \int_{\mathbb{R}^3} [K(y, t + \Delta t - \eta) - K(y, t - \eta)] g_k(x - y, \eta) \, dy \, d\eta \\
+ \Delta t^{-1} \int_{\mathbb{R}^3} [K(y, t + \Delta t) - K(y, t)] [v_0]_k(x - y) \, dy \\
\tag{7.8}
\]

After subtracting the (7.6) equation and the (7.4) equation from the (7.8) equation

\[
\Delta t^{-1} \left[ v_k(x, t + \Delta t) - v_k(x, t) \right] - g_k(x, t) - \Delta v_k(x, t) = \Delta t^{-1} \int_t^{t+\Delta t} \int_{\mathbb{R}^3} K(y, t + \Delta t - \eta) [g_k(x - y, \eta) - g_k(x, t)] \, dy \, d\eta \\
+ \Delta t^{-1} \int_0^t \int_{\mathbb{R}^3} [K(y, t + \Delta t - \eta) - K(y, t - \eta)] g_k(x - y, \eta) \, dy \, d\eta \\
+ \Delta t^{-1} \int_{\mathbb{R}^3} [K(y, t + \Delta t) - K(y, t)] [v_0]_k(x - y) \, dy \\
- \left[ \int_0^t \int_{\mathbb{R}^3} K(y, t - \eta) \Delta g_k(x - y, \eta) \, dy \, d\eta + \int_{\mathbb{R}^3} K(y, t) \Delta [v_0]_k(x - y) \, dy \right] \\
\tag{7.9}
\]
which with customary algebra shows that

\[ \text{[left side of (7.9)]} \quad (x, t, \Delta t; T) \]

\[ = \left[ \Delta t^{-1} \int_t^{t+\Delta t} \int_{\mathbb{R}^3} K(y, t + \Delta t - \eta) \left[ [g_k(x - y, \eta) - g_k(x, \eta)] + [g_k(x, \eta) - g_k(x, t)] \right] dy d\eta \right] \]

\[ + \left[ \Delta t^{-1} \int_0^t \int_{\mathbb{R}^3} [K(y, t + \Delta t - \eta) - K(y, t - \eta)] g_k(x - y, t) dy d\eta \right. \]

\[ - \left. \int_0^t \int_{\mathbb{R}^3} K(y, t - \eta) \Delta g_k(x - y, \eta) dy d\eta \right] \]

\[ + \left[ \Delta t^{-1} \int_{\mathbb{R}^3} [K(y, t + \Delta t) - K(y, t)] [v_o]k(x - y) dy \right. \]

\[ - \left. \int_{\mathbb{R}^3} K(y, t) \Delta [v_o]k(x - y) \right] \]

\[ := A_{7.10}(x, t, \Delta t; T) + B_{7.10}(x, t, \Delta t; T) + C_{7.10}(x, t, \Delta t; T) \]

(7.10)

We now assume that

\[ 0 < t - \Delta t < T < T_{\text{bup}}^\gamma (u_o) \quad \text{and} \]

\[ \Delta x = \Delta t^{1/4} \]

(7.11)

Therewith

\[ A_{7.10}(x, t, \Delta t; T) \]

which is

\[ = \Delta t^{-1} \int_t^{t+\Delta t} \int_{\mathbb{R}^3} K(y, t + \Delta t - \eta) [g_k(x - y, \eta) - g_k(x, \eta)] dy d\eta \]

\[ + \Delta t^{-1} \int_t^{t+\Delta t} \int_{\mathbb{R}^3} K(y, t + \Delta t - \eta) [g_k(x, \eta) - g_k(x, t)] dy d\eta \]

and is with Fubini’s theorem

\[ = \Delta t^{-1} \int_t^{t+\Delta t} \left[ \int_{|y| \leq \Delta x} K(y, t + \Delta t - \eta) [g_k(x - y, \eta) - g_k(x, \eta)] dy \right] d\eta \]

\[ + \Delta t^{-1} \int_t^{t+\Delta t} \left[ \int_{\Delta x < |y|} K(y, t + \Delta t - \eta) [g_k(x - y, \eta) - g_k(x, \eta)] dy \right] d\eta \]

\[ + \Delta t^{-1} \int_t^{t+\Delta t} \left[ \int_{\mathbb{R}^3} K(y, t + \Delta t - \eta) [g_k(x, \eta) - g_k(x, t)] dy \right] d\eta \]
As \( g_k \) is \( \in H^6 \times [0, T] \), it is uniformly continuous on \( \mathcal{R}^3 \times [0, T] \). Lemma 2.3. As a result

\[
|A_{7.10}(x, t, \Delta t; T)| \\
\leq \Delta t^{-1} \int_{t}^{t+\Delta t} |K(t + \Delta t - \eta)|_{L_1} o_{7.12a}(\Delta x; T) \ d\eta \\
+ \Delta t^{-1} \left| \int_{t}^{t+\Delta t} \left[ \int_{\Delta x < |y|/2} K(y, t + \Delta t - \eta) \left( g_k(x - y, \eta) - g_k(x, \eta) \right) \ dy \right] \ d\eta \right| \\
+ \Delta t^{-1} \int_{t}^{t+\Delta t} |K(t + \Delta t - \eta)|_{L_1} o_{7.12b}(\Delta t; T) \ d\eta \\
(7.12)
\]

which with \( |K(t + \Delta t - \eta)|_{L_1} = 1 \), (5.2), and \( \Delta x = \Delta t^{1/4} \) is

\[
\leq o_{7.12a}(\Delta t^{1/4}; T) + o_{7.12c}(\Delta t^{1/4}; T) \\
+ \Delta t^{-1} \left| \int_{t}^{t+\Delta t} \left[ \int_{\Delta x < |y|/2} K(y, t + \Delta t - \eta) \left( g_k(x - y, \eta) - g_k(x, \eta) \right) \ dy \right] \ d\eta \right| \\
(7.13)
\]

in which the sum \( o_{7.12a}(\Delta t^{1/4}; T) + o_{7.12c}(\Delta t^{1/4}; T) \to 0 \) as \( \Delta t \to 0 \).

With the (5.2) definition of \( K(y, t) \), the third (7.13) summand is

\[
\leq \Delta t^{-1} 2 |g_k|_{L^\infty \times [0, T]} \int_{t}^{t+\Delta t} \left[ \int_{\Delta x < |y|/2} \left( \cdots \right) \left( t + \Delta t - \eta \right)^{-3/2} \right. \\
\left. \exp \left[ - |y|^2/4(t + \Delta t - \eta) \right] \ dy \right] \ d\eta \\
\leq \left[ \cdots \right] \Delta t^{-1} |g_k|_{L^\infty \times [0, T]} \\
\int_{t}^{t+\Delta t} \left[ \int_{\Delta x/(t+\Delta t-\eta) < |y|/2/(t+\Delta t-\eta)} \exp \left[ - |y|^2/4(t + \Delta t - \eta) \right] \ d[y[t + \Delta t - \eta]^{-1/2}] \right] \ d\eta
\]

which after changing variables in the interior integral is

\[
\leq \left[ \cdots \right] \Delta t^{-1} |g_k|_{L^\infty \times [0, T]} \int_{t}^{t+\Delta t} \left[ \int_{\Delta x/(t+\Delta t-\eta)^{1/2} < |y|/2} \exp \left[ - |y|^2/4 \right] \ dy \right] \ d\eta \\
(7.14)
\]
SMOOTH SOLUTIONS OF NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3 \times [0, \infty)$

and that $\eta$ is $\in [t, t + \Delta t]$ shows that

$$0 \leq t + \Delta t - \eta \leq \Delta t$$

and that

$$\Delta x/\Delta t^{1/2} \leq \Delta x/(t + \Delta t - \eta)^{1/2}$$

As the (7.14) interior integral increases as $\Delta x/(t + \Delta t - \eta)^{1/2}$ decreases, (7.14) is

$$\leq \left[ \ldots \right] \Delta t^{-1} |g_k|_{L^\infty \times [0, T]} \int_{t}^{t+\Delta t} d\eta \left[ \int_{\Delta x/\Delta t^{1/2} < |y|^2}^{t^{1/2}} \exp \left[ -|y|^2/4 \right] dy \right]$$

which as $\Delta x = \Delta t^{1/4}$ is

$$\leq \left[ \ldots \right] \Delta t^{-1} |g_k|_{L^\infty \times [0, T]} \int_{\Delta t^{-1/4} < |y|^2}^{t^{-1/4}} \exp \left[ -|y|^2/4 \right] dy$$

$$\leq \left[ \ldots \right] |g_k|_{L^\infty \times [0, T]} o_{7.15}(\Delta t; T)$$

(7.15)

which with (2.14) shows that

$$|A_{7,10}(x, t, \Delta t; T)|$$

$$\leq o_{7.12a}(\Delta t^{1/4}; T) + o_{7.12c}(\Delta t^{1/4}; T) + \left[ \ldots \right] |g_k|_{H^2 \times [0, T]} o_{7.15}(\Delta t; T)$$

which $\to 0$ as $\Delta t \to 0$.

We now develop an upper bound for the absolute value of

$$B_{7,10}(x, t, \Delta t; T),$$

which is

$$= \Delta t^{-1} \int_{0}^{t} \int_{\mathbb{R}^3} \left[ K(y, t + \Delta t - \eta) - K(y, t - \eta) \right] g_k(x - y, \eta) \, dy \, d\eta$$

$$- \int_{0}^{t} \int_{\mathbb{R}^3} K(y, t - \eta) \, \Delta g_k(x - y, \eta) \, dy \, d\eta$$

which the calculus shows is

$$= \Delta t^{-1} \int_{0}^{t} \int_{\mathbb{R}^3} \left[ \int_{0}^{\Delta t} \partial_\theta K(y, t - \eta + \theta) \, d\theta \right] g_k(x - y, \eta) \, dy \, d\eta$$

$$- \int_{0}^{t} \int_{\mathbb{R}^3} K(y, t - \eta) \, \Delta g_k(x - y, \eta) \, dy \, d\eta$$

(7.16)
which with Fubini’s theorem, \( \partial_t K = \Delta K \), (5.2), and integration by parts is

\[
\Delta t^{-1} \int_0^{\Delta t} \left[ \int_0^t \left[ \int_{\mathbb{R}^3} \Delta K(y, t - \eta + \theta) \, g_k(x - y, \eta) \, dy \right] \, d\eta \right] \, d\theta \\
- \Delta t^{-1} \int_0^{\Delta t} \left[ \int_0^t \int_{\mathbb{R}^3} K(y, t - \eta) \, \Delta g_k(x - y, \eta) \, dy \, d\eta \right] \, d\theta
\]

in which \( 0 \leq \theta \leq t \), and which after integrating by parts is

\[
\Delta t^{-1} \int_0^{\Delta t} \left[ \int_0^t \left[ \int_{\mathbb{R}^3} \left( K(y, t - \eta + \theta) - K(y, t - \eta) \right) \right] \, \Delta g_k(x - y, \eta) \, dy \right] \, d\eta \right] \, d\theta
\]

which with a repetition of the discussion which begins at (7.16) is

\[
\Delta t^{-1} \int_0^{\Delta t} \left[ \int_0^t \left[ \int_{\mathbb{R}^3} K(y, t - \eta + \theta) \, \Delta^2 g_k(x - y, \eta) \, dy \right] \, d\theta' \right] \, d\eta \right] \, d\theta
\]

the absolute value of which is

\[
\leq \Delta t^{-1} \int_0^{\Delta t} \left[ \int_0^t \left[ \int_{\mathbb{R}^3} |\Delta^2 g_k|_{L^\infty} \, d\theta' \right] \, d\eta \right] \, d\theta
\leq \Delta t^{-1} \Delta t \, t \, \theta \, |\Delta^2 g_k|_{L^\infty}
\]

which as \( \theta \) is \( \leq \Delta t \) and \( t \) is \( \leq T \), (2.14) shows is

\[
\leq \Delta t \, T \, [ \ldots ] \, |g_k|_{H^6 \times [0, T]}
\]

(7.17)

which shows that

\[
|B_{7,10}(x, t, \Delta t; T)| \leq \Delta t \, T \, [ \ldots ] \, |g_k|_{H^6 \times [0, T]}
\]

We now develop an upper bound for the absolute value of \( C_{7,10}(x, t, \Delta t; T) \). With the definition thereof and the calculus

\[
C_{7,10}(x, t, \Delta t; T)
= \Delta t^{-1} \int_{\mathbb{R}^3} \left[ \int_0^{\Delta t} \partial_t K(y, t + \theta) \, d\theta \, [v_0]_k(x - y) \right] \, dy
\]

\[
- \int_{\mathbb{R}^3} K(y, t) \, \Delta [v_0]_k(x - y) \, dy
\]
which with Fubini’s theorem is

\[
\Delta t^{-1} \int_0^{\Delta t} \left[ \int_{\mathbb{R}^3} \left[ \partial_t K(y, t + \theta) \left[ v_o \right]_k(x - y) \right] \, dy \right] \, d\theta
- \Delta t^{-1} \int_0^{\Delta t} \left[ \int_{\mathbb{R}^3} K(y, t) \left[ \Delta [v_o]_k(x - y) \right] \, dy \right] \, d\theta
\]

(7.18)

which with \( \partial_t K = \Delta K \) and after integrating by parts is

\[
= \Delta t^{-1} \int_0^{\Delta t} \left[ \int_{\mathbb{R}^3} \left[ K(y, t + \theta) - K(y, t) \right] \left[ v_o \right]_k(x - y) \, dy \right] \, d\theta
\]

which with the calculus is

\[
= \Delta t^{-1} \int_0^{\Delta t} \left[ \int_{\mathbb{R}^3} \left[ \int_0^\theta \partial_t K(y, t + \theta') \, d\theta' \right] \left[ v_o \right]_k(x - y) \, dy \right] \, d\theta
\]

which with Fubini’s theorem, \( \partial_t K = \Delta K \) and integration by parts is

\[
= \Delta t^{-1} \int_0^{\Delta t} \left[ \int_0^\theta \left[ \int_{\mathbb{R}^3} K(y, t + \theta') \left[ \Delta [v_o]_k(x - y) \right] \, dy \right] \, d\theta' \right] \, d\theta
\]

the absolute value of which is

\[
\leq \Delta t^{-1} \int_0^{\Delta t} \left[ \int_0^\theta |\Delta^2 [v_o]_k|_{L^\infty \times [0, T]} \, d\theta' \right] \, d\theta
\leq \Delta t^{-1} \Delta t \theta |\Delta^2 [v_o]_k|_{L^\infty \times [0, T]}
\]

and as (7.18) shows that \( 0 \leq \theta \leq \Delta t \), (2.14) shows that

\[
|C_{7.10}(x, t, \Delta t; T)| \leq \Delta t \left[ \ldots \right] |v_o|_{H^6}
\]

The discussion which begins at (7.11) shows that

\[
\left| \Delta t^{-1} \left[ v_k(x, t + \Delta t) - v_k(x, t) \right] - g_k(x, t) - \Delta v_k(x, t) \right|
\leq o_{7.12a}(\Delta t^{1/4}; T) + o_{7.12c}(\Delta t^{1/4}; T) + \left[ \ldots \right] |g_k|_{H^2 \times [0, T]} \, o_{7.15}(\Delta t; T)
+ \Delta t \, T \left[ \ldots \right] |g_k|_{H^6 \times [0, T]} + \Delta t \left[ \ldots \right] |v_o|_{H^6}
= o_{7.19}(\Delta t; T)
\]

(7.19)

when \((x, t) \in \mathbb{R}^3 \times (0, T)\). As a result

\[
\lim_{\Delta t \to 0^+} \Delta t^{-1} \left[ v_k(x, t + \Delta t) - v_k(x, t) \right] = g_k(x, t) + \Delta v_k(x, t)
\]

(7.20)
When $0 < t - \Delta t < t$, we restate (7.19) as
\[
\left| \frac{\Delta t^{-1}}{} \left[ v_k(x, t) - v_k(x, t - \Delta t) \right] - g_k(x, t - \Delta t) - \Delta v_k(x, t - \Delta t) \right| \leq o_{7.19}(\Delta t; T) 
\]
(7.21)

As $g_k$ and $\Delta v_k$ are $\in H^5 \times [0, T]$, (7.3), (7.4) and are, as a result, uniformly continuous on $\mathcal{R}^3 \times [0, T]$, Lemma 2.3
\[
\lim_{\Delta t \to 0^-} \frac{\Delta t^{-1}}{} \left[ v_k(x, t + \Delta t) - v_k(x, t) \right] = g_k(x, t) + \Delta v_k(x, t) 
\]
(7.22)

which with (7.20) shows that
\[
\partial_t v_k(x, t) = g_k(x, t) + \Delta v_k(x, t) 
\]
when $(x, t)$ is $\in \mathcal{R}^3 \times (0, T)$. Then with $0 < t_o < t < T$.
\[
v_k(x, t) - v_k(x, t_o) = \int_{t_o}^t \partial_t v_k(x, \eta) \, d\eta = \int_{t_o}^t g_k(x, \eta) + \Delta v_k(x, \eta) \, d\eta 
\]
As $v_k$, $\Delta v_k$ and $g_k$ are uniformly continuous on $\mathcal{R}^3 \times [0, T]$, Lemma 2.3, allowing $t_o \to 0^+$ shows that
\[
v_k(x, t) - v_k(x, 0) = \int_0^t g_k(x, \eta) + \Delta v_k(x, \eta) \, d\eta 
\]
for $(x, t) \in \mathcal{R}^3 \times [0, T)$.

Dividing each side by $t > 0$ and allowing $t \to 0^+$ shows that
\[
\partial_t v_k(x, t) = g_k(x, t) + \Delta v_k(x, t) 
\]
(7.23)
on $\mathcal{R}^3 \times [0, T)$. As $T$ is any number which is $\in (0, T_{bup}^\gamma(u_o))$ (7.23) extends to $(x, t) \in \mathcal{R}^3 \times [0, T_{bup}^\gamma(u_o))$. In particular, $\partial_t v_k$ has a one-sided derivative at $t = 0$.

As a result, $\partial_t D^{k'} u_{\gamma,m}$ exists and $u_{\gamma,m}$ satisfies
\[
\partial_t D^{k'} u_{\gamma,m}(x, t) = \hat{P} \left[ D^{k'} \sum_j [J_j[u\gamma,m(t)] \partial_j u_{\gamma,m}(t)] \right](x) + \Delta D^{k'} u_{\gamma,m}(x, t) \]
\[
D^{k'} u_{\gamma,m}(x, 0) = D^{k'} u_o(x) 
\]
on $\mathbb{R}^3 \times [0, T_{\text{bup}}^\gamma(u_o))$.

As $\hat{\mathcal{P}} = (-1)^k \mathcal{P}$, $u^\gamma,m$ satisfies each $(7.1)_{k',\gamma,u_o}$ pde on $\mathbb{R}^3 \times [0, T_{\text{bup}}^\gamma(u_o))$, and (7.3) and (7.5) show that

$$\partial_t[D^k u^\gamma,m] \in H^{m-|k'|-2} \times [0, T_{\text{bup}}^\gamma(u_o))].$$

□

**Lemma 7.2.** Each $u^\gamma,m$ satisfies

$$|u^\gamma,m|_{L^2 \times [0,T_{\text{bup}}^\gamma(u_o))] \leq |u_o|_{L^2}$$

(7.24)

**Proof.** We earlier showed that $u^\gamma,m$ is $\in H^{^7,df} \times [0, T_{\text{bup}}^\gamma(u_o))$, Theorem 6.2, that $\Delta u^\gamma,m$ is $\in H^{5,df} \times [0, T_{\text{bup}}^\gamma(u_o))$ and that $\partial_t u^\gamma,m$ is $\in H^{5,df} \times [0, T_{\text{bup}}^\gamma(u_o))$. Theorem 7.1.

As $u^\gamma,m$ satisfies the $(7.1)_{k'=0,\gamma,u_o}$ pde, customary mathematics and Fubini’s theorem show that

$$\int_{\mathbb{R}^3} \left[ \int_0^T \left( u^\gamma,m(x,t), \partial_t u^\gamma,m(x,t) \right)_2 dt \right] dx$$

$$+ \int_0^T \left\langle u^\gamma,m(t), \mathcal{P} \left[ \sum_j J_j(u^\gamma,m_j) \partial_j u^\gamma,m(t) \right] \right\rangle_{\mathbb{R}^3} dt$$

$$= \int_0^T \left\langle u^\gamma,m(t), \Delta u^\gamma,m(t) \right\rangle_{\mathbb{R}^3} dt$$

(7.25)

when $0 \leq T < T_{\text{bup}}^\gamma(u_o))$.

As $\partial_t u^\gamma,m$ is continuous on $\mathbb{R}^3 \times [0, T]$, Lemma 2.3, the first summand on the left side of (7.25) is

$$= \int_{\mathbb{R}^3} \left[ \int_0^T 2^{-1} \partial_t(u^\gamma,m(x,t), u^\gamma,m(x,t))_2 dt \right] dx$$

$$= 2^{-1} \left[ \left\langle u^\gamma,m(T), u^\gamma,m(T) \right\rangle_{\mathbb{R}^3} - \left\langle u_o, u_o \right\rangle_{\mathbb{R}^3} \right]$$

(7.26)
As $P$ is symmetric in the $\langle \cdot, \cdot \rangle_{R^3}$ inner product, (4.35), and $P$ fixes the divergence free $u^{\gamma,m}$, (4.32), the second summand in the left side of (7.25) is

$$\int_0^T \left\langle u^{\gamma,m}(t), \sum_j J_\gamma(u^{\gamma,m}_j) \partial_j u^{\gamma,m}(t) \right\rangle_{R^3} dt$$

(7.27)

which as (3.5) shows that $J_\gamma(u^{\gamma,m})$ is divergence free, (2.7) shows that (7.27) is

$$\int_0^T \left\langle u^{\gamma,m}(t) , 2^{-1} \sum_j J_\gamma(u^{\gamma,m}_j) \partial_j u^{\gamma,m}(t) \right\rangle_{R^3} dt$$

$$+ \int_0^T \left\langle u^{\gamma,m}(t), 2^{-1} \sum_j \partial_j [J_\gamma(u^{\gamma,m}_j) u^{\gamma,m}(t)] \right\rangle_{R^3} dt$$

which after integrating by parts in the second summand and then moving the scalar $J_\gamma(u^{\gamma,m}_j)$ to the left side of the inner product is

$$= 2^{-1} \left[ \int_0^T \sum_j \left\langle u^{\gamma,m}(t), J_\gamma(u^{\gamma,m}_j) \partial_j u^{\gamma,m}(t) \right\rangle_{R^3} dt $$

$$- \int_0^T \sum_j \left\langle J_\gamma(u^{\gamma,m}_j) \partial_j u^{\gamma,m}(t), u^{\gamma,m}(t) \right\rangle_{R^3} dt \right] = 0$$

Integration by parts in the right side of (7.25) shows that such right side is

$$= \int_0^T \sum_j (-1) \left\langle \partial_j u^{\gamma,m}(t), \partial_j u^{\gamma,m}(t) \right\rangle_{R^3}$$

Therewith, (7.25) and customary algebra show that

$$2^{-1} \left\langle u^{\gamma,m}(T), u^{\gamma,m}(T) \right\rangle_{R^3}$$

$$+ \int_0^T \sum_j \left\langle \partial_j u^{\gamma,m}(t), \partial_j u^{\gamma,m}(t) \right\rangle_{R^3} = 2^{-1} \left\langle u_0, u_0 \right\rangle_{R^3}$$

when $T$ is $[0, T_{bup}^{\gamma,m}(u_0))$. □
Lemma 7.3. When \( T > 0 \), there is at most one function \( u \in H^7,d f \times [0,T] \) the derivative \( \partial_t u \) of which exists on \( \mathcal{R}^3 \times [0,T] \) and is \( \in H^5,d f \times [0,T] \) which function is a solution of the (7.1)_{k'=0,\gamma,u_o} pde

\[
\partial_t u(x,t) + \sum_j P \left[ J_\gamma (u_j) \partial_j u \right](\eta)(x) = \Delta u(x,t)
\]

\[ u(x,0) = u_o(x) \]

on \( \mathcal{R}^3 \times [0,T] \).

Proof. Any function \( u \) which is a solution of the (7.1)_{k'=0,\gamma,u_o} pde and its derivative \( \partial_t u \) are continuous on \( \mathcal{R}^3 \times [0,T] \) as each is \( \in H^5 \times [0,T] \). Lemma 2.3. As \( \partial_j u \) is \( \in H^6 \times [0,T] \), \( |\partial_j u|_{L^\infty} \times [0,T] \) is finite. (2.14).

If \( v \) were also a solution of the (7.1)_{k'=0,\gamma,u_o} pde, then \( u - v \) would satisfy

\[
\partial_t [u - v](x,t) + \sum_j P \left[ J_\gamma (u_j - v_j) \partial_j u \right](\eta) = \Delta [u - v](x,t)
\]

\[ [u - v](x,0) = 0 \]

on \( \mathcal{R}^3 \times [0,T] \). Customary mathematics would then show that

\[
\int_0^t \langle [u - v](\eta), \partial_t [u - v](\eta) \rangle_{\mathcal{R}^3} d\eta
+ \int_0^t \langle [u - v](\eta), \sum_j P \left[ J_\gamma (u_j - v_j) \partial_j u \right](\eta) \rangle_{\mathcal{R}^3} d\eta
+ \int_0^t \langle [u - v](\eta), \sum_j P \left[ J_\gamma (v_j) \partial_j [u - v](\eta) \right] \rangle_{\mathcal{R}^3} d\eta
= \int_0^t \langle [u - v](\eta), \Delta [u - v](\eta) \rangle_{\mathcal{R}^3} d\eta
\]

(7.28)

for \( t \in [0,T] \).

The proof of Lemma 7.2 shows that the first summand on the left side of (7.28) is

\[
= 2^{-1} \left[ \langle u - v, u - v \rangle_{\mathcal{R}^3}(t) - \langle u - v, u - v \rangle_{\mathcal{R}^3}(0) \right]
\]
which as \( u(x, 0) = v(x, 0) \) for \( x \in \mathbb{R}^3 \) is

\[
= 2^{-1} \langle u - v, u - v \rangle_{\mathbb{R}^3}(t)
\]

for \( t \in [0, T] \). The proof of Lemma 7.2 shows that the third summand on the left side of (7.28) vanishes, and that the right side thereof is \( \leq 0 \).

Therewith, (7.28) shows that

\[
2^{-1} \langle u - v, u - v \rangle_{\mathbb{R}^3}(t) + [ \geq 0 ]
\]

\[
= (-1) \sum_j \int_0^t \langle [u - v](\eta), J_j(u_j - v_j)(\eta) \partial_j u \rangle_{\mathbb{R}^3} d\eta
\]

(7.29)

Customary mathematics and a Hölder inequality in the right side of (7.29) show that

\[
\left| u(t) - v(t) \right|_{L^2}^2
\]

\[
\leq 2 \int_0^t \left| [u - v](\eta) \right|_{L^2} \sum_j |J_j[u_j - v_j](\eta)|_{L^2} d\eta \sup_j |\partial_j u|_{L^\infty \times [0,T]} \]

\[
\leq \left[ \ldots \right] (T) \int_0^t \left| u(\eta) - v(\eta) \right|_{L^2}^2 d\eta
\]

when \( t \) is \( \in [0, T] \). Gronwall’s inequality, [4] Appendix B.2.k, then shows that

\[
u(x, t) = v(x, t) \text{ a.e. } \in \mathbb{R}^3
\]

for \( t \in [0, T] \). As \( u(x, t) \) and \( v(x, t) \) are continuous on \( \mathbb{R}^3 \times [0, T] \), Lemma 2.3, \( u = v \) on \( \mathbb{R}^3 \times [0, T] \).

\[\square\]

**Lemma 7.4.** Let \( T_{bup} > 0, v \in H^7 \times [0, T_{bup}) \), \( h \) and \( \partial_t v \) be \( \in H^{5,df} \times [0, T_{bup}) \) and

\[
\partial_t v(x, t) = \Delta v(x, t) + h(x, t)
\]

\[
v(x, 0) = v_o(x)
\]

(7.30)
when \((x, t) \in \mathcal{R}^3 \times [0, T_{\text{bup}}]\). Then

\[
v(x, t) = \int_0^t \int_{\mathcal{R}^3} K(y, t - \eta) \ h(x - y, \eta) \ dyd\eta + \int_{\mathcal{R}^3} K(y, t) \ v_0(x - y) \ dy
\]

(7.31)

when \((x, t) \in \mathcal{R}^3 \times [0, T_{\text{bup}}]\).

Proof. When \(x\) and \(y\) are \(\in \mathcal{R}^3\) and \(\eta\) is \(\in [0, T_{\text{bup}}]\), the (7.30) pde shows that

\[
K(y, t - \eta) \ \partial_{\eta}v(x - y, \eta) = K(y, t - \eta) \left[ \Delta v(x - y, \eta) + h(x - y, \eta) \right] \\
v(x, 0) = v_0(x)
\]

With \(0 < t - \Delta t < t < T_{\text{bup}}\) and Fubini’s theorem

\[
\int_{\mathcal{R}^3} \left[ \int_0^{t - \Delta t} K(y, t - \eta) \ \partial_{\eta}v(x - y, \eta) \ d\eta \right] \ dy \\
= \int_0^{t - \Delta t} \int_{\mathcal{R}^3} K(y, t - \eta) \ \Delta v(x - y, \eta) \ dyd\eta \\
+ \int_0^{t - \Delta t} \int_{\mathcal{R}^3} K(y, t - \eta) \ h(x - y, \eta) \ dyd\eta
\]

(7.32)

The left side of (7.32), after integrating by parts in the interior integral therein, is

\[
= \int_{\mathcal{R}^3} \left[ \int_0^{t - \Delta t} \partial_{\eta} \left[ K(y, t - \eta) \ v(x - y, \eta) \right] \\
- \partial_{\eta} \left[ K(y, t - \eta) \right] \ v(x - y, \eta) \ d\eta \right] \ dy
\]

which is

\[
= \int_{\mathcal{R}^3} K(y, \Delta t) \ v(x - y, t - \Delta t) \ dy - \int_{\mathcal{R}^3} K(y, t) \ v_0(x - y) \ dy \\
+ \int_0^{t - \Delta t} \left[ \int_{\mathcal{R}^3} \partial_{\eta} K(y, t - \eta) \ v(x - y, \eta) \ dy \right] \ d\eta
\]
With (5.2), integration by parts in the third (7.32) summand shows that the left side of (7.32) is
\[
\int_{R^3} K(y, \Delta t) v(x - y, t - \Delta t) \, dy - \int_{R^3} K(y, t) v_o(x - y) \, dy \\
+ \int_0^{t-\Delta t} \left[ \int_{R^3} K(y, t - \eta) \Delta v(x - y, \eta) \, dy \right] \, d\eta
\]
and (7.32) and Fubini’s theorem show that
\[
\int_{R^3} K(y, \Delta t) v(x - y, t - \Delta t) \, dy - \int_{R^3} K(y, t) v_o(x - y) \, dy \\
+ \int_0^{t-\Delta t} \int_{R^3} K(y, t - \eta) \Delta v(x - y, \eta) \, dy \, d\eta
\]
\[
= \int_0^{t-\Delta t} \int_{R^3} K(y, t - \eta) \Delta v(x - y, \eta) \, dy \, d\eta \\
+ \int_0^{t-\Delta t} \int_{R^3} K(y, t - \eta) h(x - y, \eta) \, dy \, d\eta
\]
(7.33)

Eliminating the double integral that appears on each side of (7.33) and allowing \( \Delta t \to 0^+ \) shows, as \( v \) is continuous on \( R^3 \times [0, T_{\text{bup}}(u_o)] \), that
\[
v(x, t) = \int_0^t \int_{R^3} K(y, t - \eta) h(x - y, \eta) \, dy \, d\eta \\
+ \int_{R^3} K(y, t) v_o(x - y) \, dy
\]
\hspace{10cm} \square

**Lemma 7.5.** Each \( u^{\gamma,m} \) satisfies the blow up condition
\[
|u^{\gamma,m}|_{H^m \times [0,T]} \to \infty \text{ as } T \to \text{ a finite } T_{\text{bup}}^{\gamma,m}(u_o)
\]
(7.34)

**Proof.** The proof is by contradiction. If \( |u^{\gamma,m}(T)|_{H^m} \to \infty \) as \( T \to \) a finite \( T_{\text{bup}}^{\gamma,m}(u_o) \), then there is an increasing sequence
\[
T_n \in (0, T_{\text{bup}}^{\gamma,m}(u_o)) \text{ which } \to \text{ the finite } T_{\text{bup}}^{\gamma,m}(u_o)
\]
and which satisfies
\[
|u^{\gamma,m}(T_n)|_{H^m} \text{ is } \leq [\ldots]_{7.35} < \infty
\]
(7.35)
As $u^{\gamma,m}$ is $\in H^m \times [0,T_{\text{up}}(u_o))$, it is a solution of the $(6.1)_{k'=0,\gamma,u_o}$ integral equation, Theorem 6.2, and of the related $(7.1)_{k'=0,\gamma,u_o}$ pde on $\mathcal{R}^3 \times [0,T_n]$. Theorem 7.1. It also has a one-sided derivative $\partial_t u^{\gamma,m}(x,0)$ and a one-sided derivative $\partial_t u^{\gamma,m}(x,T_n)$ the latter of which is

$$
= (-1) \mathcal{P} \left[ \sum_j \left[ J_\gamma(u_j^{\gamma,m}(T_n)) \partial_j u^{\gamma,m}(T_n) \right] \right] (x) + \Delta u^{\gamma,m}(x,T_n)
(7.36)
$$

We let $u^{\gamma,m,n} \in H^m \times [0, \ldots \ ]6.19 |u(T_n)|_{H^m}^{-2}$ be the solution of the integral equation

$$
u^{\gamma,m,n}(x,t) = \int_0^t \int_{\mathcal{R}^3} \sum_j \hat{\mathcal{P}}(\partial_j K^j(t-\eta))(y) \left[ J_\gamma(u_j^{\gamma,m,n}) u^{\gamma,m,n} \right] (x-y,\eta) \, dy \, d\eta
+ \int_{\mathcal{R}^3} K(y,t) \ u^{\gamma,m}(x-y,T_n) \, dy
(7.37)
$$
on $\mathcal{R}^3 \times [0, \ldots \ ]6.19 |u(T_n)|_{H^m}^{-2}$, Theorem 6.2, and which Theorem 7.1 shows is a solution of the $(7.1)_{k'=0,\gamma,u_o}$ pde

$$
\partial_t u^{\gamma,m,n}(x,t) + \mathcal{P} \left[ \sum_j \left[ J_\gamma(u_j^{\gamma,m,n}(t)) \partial_j u^{\gamma,m,n}(t) \right] \right] (x)
= \Delta u^{\gamma,m}(x,t)
$$

$$
u^{\gamma,m,n}(x,0) = u^{\gamma,m}(x,T_n)
(7.38)
$$

It has a one-sided derivative $\partial_t u^{\gamma,m,n}$ at $t = 0$ the value of which at $(x,0)$ is equal to the $(7.36)$ one-sided derivative of $u^{\gamma,m}$ at $T_n$.

Thus, the function $u^*$ which is

$$
u^{\gamma,m} \text{ on } \mathcal{R}^3 \times [0,T_n]
$$

and is the translate of $u^{\gamma,m,n}$ to $\mathcal{R}^3 \times [T_n,T_n+[ \ldots \ ]6.19 |u(T_n)|_{H^m}^{-2}]$ is $\in H^m \times [0,T_n+[ \ldots \ ]6.19 |u(T_n)|_{H^m}^{-2}]$ and is a solution of the $(7.1)_{k'=0,\gamma,u_o}$ pde on the strip where it is defined.
Lemma 7.4 shows that such \( u^* \) satisfies the \((6.1) k'=0,\gamma,u_o \) integral equation

\[
u^*(x,t) = \int_0^t \int_{R^3} \sum_j \hat{P} [\partial_j \mathcal{K}^\dagger(t-\eta)](y) \left[ J_\gamma(u^*_j) u^* \right] (x-y,\eta) \, dy \, d\eta + \int_{R^3} K(y,t) \, u_o(x-y) \, dy
\]

on the strip where \( u^* \) is defined.

The definition of \( T_{bup}^{\gamma,m}(u_o) \) in Theorem 6.2 as the supremum of those \( T > 0 \) for which there is a function \( \in H^{m,\text{df}} \times [0,T] \) which satisfies the \((6.1) k'=0,\gamma,u_o \) integral equation on \( R^3 \times [0,T] \) shows that

\[
T_n + [ \ldots ]_{6.19} \, |u_o|^{-2}_{H^m} \, is \, < \, T_{bup}^{\gamma,m}(u_o)
\]

(7.39)

As \( T_n \to T_{bup}^{\gamma,m}(u_o) \) as \( n \to \infty \)

\[
T_n + [ \ldots ]_{6.19} \, |u_o|^{-2}_{H^m} \, is \, > \, the \, finite \, T_{bup}^{\gamma,m}(u_o)
\]

when \( n \) is sufficiently large. The contradiction shows that \( u^{\gamma,m} \) satisfies the (7.34) blow up condition. \( \square \)

**Lemma 7.6.** Let \( u^{\gamma,m} = u^{\gamma,m}(x,t;u_o) \), \( k' \) be a multi-index for which \( 0 \leq |k'| \leq m \) and

\[
0 \leq T - \Delta T < T < T_{bup}^{\gamma,m}(u_o)
\]

Then the weak derivative

\[
D^{k'}_x \left[ \int_0^{T-\Delta T} \int_{R^3} \hat{P} [\partial_j \mathcal{K}^\dagger(T-\eta)](y) \, J_\gamma(u^{\gamma,m}_j) \, u^{\gamma,m}(x-y,\eta) \, dy \, d\eta \right]
\]

\[
= \int_0^{T-\Delta T} \int_{R^3} \hat{P} [\partial_j \mathcal{K}^\dagger(T-\eta)](y) \, D^{k'}_x [J_\gamma(u^{\gamma,m}_j) \, u^{\gamma,m}](x-y,\eta) \, dy \, d\eta
\]

and is also

\[
= \int_0^{T-\Delta T} \int_{R^3} \hat{P} [D^{k'} \partial_j \mathcal{K}^\dagger(T-\eta)](y) \, J_\gamma(u^{\gamma,m}_j) \, u^{\gamma,m}(x-y,\eta) \, dy \, d\eta
\]

(7.40)
\[
\left| \int_0^{T-\Delta T} \int_{\mathbb{R}^3} \hat{P} [ D^{k'} \partial_j \mathcal{K}^i (T - \eta)] (y) J_\gamma (u_j^{\gamma,m}) u^{\gamma,m} (x - y, \eta) \ dy \ d\eta \right|_{L^\infty} \\
\leq \left[ \ldots \right]_{7.41} (k') \left[ \Delta T^{-|k'|/2-1} - T^{-|k'|/2-1} \right] |u_\alpha|_{L^2}^2
\]
(7.41)

and
\[
\left| \int_0^{T-\Delta T} \int_{\mathbb{R}^3} \hat{P} [ D^{k'} \partial_j \mathcal{K}^i (T - \eta)] (y) J_\gamma (u_j^{\gamma,m}) u^{\gamma,m} (x - y, \eta) \ dy \ d\eta \right|_{L^2} \\
\leq \left[ \ldots \right]_{7.42} (k') \left[ \Delta T^{-|k'|/2-1/4} - T^{-|k'|/2-1/4} \right] |u_\alpha|_{L^2}^2
\]
(7.42)

\[ \left[ \ldots \right]_{7.41} (k') \text{ and } \left[ \ldots \right]_{7.42} (k') \text{ are independent of } \gamma, m, T \text{ and } u^{\gamma,m}. \]

**Proof.** We now establish the two results in the paragraph which contains (7.40). With customary mathematics, the left side of (7.40) is
\[
= D^{k'} \left[ \int_0^T \int_{\mathbb{R}^3} \hat{P} [ \partial_j \mathcal{K}^i (T - \eta)] (y) J_\gamma (u_j^{\gamma,m}) u^{\gamma,m} (x - y, \eta) \ dy \ d\eta \right. \\
- \left. \int_{T-\Delta T}^T \int_{\mathbb{R}^3} \hat{P} [ \partial_j \mathcal{K}^i (T - \eta)] (y) J_\gamma (u_j^{\gamma,m}) u^{\gamma,m} (x - y, \eta) \ dy \ d\eta \right]
\]
which as \( u^{\gamma,m} \in H^m \times [0, T] \), (5.20) shows, is
\[
= \left[ \int_0^T \int_{\mathbb{R}^3} \hat{P} [ \partial_j \mathcal{K}^i (T - \eta)] (y) D^{k'} [ J_\gamma (u_j^{\gamma,m}) u^{\gamma,m} ] (x - y, \eta) \ dy \ d\eta \right. \\
- \left. \int_{T-\Delta T}^T \int_{\mathbb{R}^3} \hat{P} [ \partial_j \mathcal{K}^i (T - \eta)] (y) D^{k'} [ J_\gamma (u_j^{\gamma,m}) u^{\gamma,m} ] (x - y, \eta) \ dy \ d\eta \right]
\]
\[
= \int_0^{T-\Delta T} \int_{\mathbb{R}^3} \hat{P} [ \partial_j \mathcal{K}^i (T - \eta)] (y) D^{k'} [ J_\gamma (u_j^{\gamma,m}) u^{\gamma,m} ] (x - y, \eta) \ dy \ d\eta \]
(7.43)

which establishes the first result in the paragraph which contains (7.40).

Integration by parts in each component of the last (7.43) integral moves \( D^{k'} \) to the left of \( \hat{P} [ \partial_j \mathcal{K}^i (T - \eta)] (y) \), (2.12), which after moving
$D^{k'}$ to the right of $\hat{P}$, (4.36), is

$$= \int_0^{T-\Delta T} \int_{\mathbb{R}^3} \hat{P} [D^{k'} \partial_j K^i (T - \eta)] (y) J_\gamma (u^{\gamma;m}_j) u^{\gamma;m} (x - y, \eta) \, dy \, d\eta$$

The proof of (7.40) is complete.

We now establish (7.41). The left side of (7.41) is

$$= \sup_i \left| \int_0^{T-\Delta T} \left[ \int_{\mathbb{R}^3} (\hat{P} [D^{k'} \partial_j K^i (T - \eta)]) (y), J_\gamma (u^{\gamma;m}_j) u^{\gamma;m} (x - y, \eta)) dy \right] \, d\eta \right|_{L^\infty}$$

which with (2.5) in the interior integral is

$$\leq \sup_i \left| \int_0^{T-\Delta T} \left[ \left| \hat{P} [D^{k'} \partial_j K^i (T - \eta)] \right|_{L^\infty} \right|_{[0, T-\Delta T]} \left| J_\gamma (u^{\gamma;m}_j) u^{\gamma;m} (\eta) \right|_{L^1} \, d\eta \right|$$

which with (5.7) and a Hölder inequality is

$$\leq \left[ \ldots \right]_{7.44(k')} \int_0^{T-\Delta T} (T - \eta)^{-|k'|/2 - 2} \left| J_\gamma (u^{\gamma;m}_j) (\eta) \right|_{L^2} \left| u^{\gamma;m}_k (\eta) \right|_{L^2} \, d\eta$$

which with (7.24) is

$$\leq \left[ \ldots \right]_{7.41(k')} \left[ \Delta T^{-|k'|/2 - 1} - T^{-|k'|/2 - 1} \right] \left| u^{\gamma;m}_k \right|_{L^2} (7.45)$$

The proof of (7.41) is complete.

We now establish (7.42). With (2.5)

$$\left| \int_0^{T-\Delta T} \left[ \int_{\mathbb{R}^3} D^{k'} p_{j,i,k} (y, T - \eta) J_\gamma (u^{\gamma;m}_j) u^{\gamma;m}_k (x - y, \eta) \, dy \right] \, d\eta \right|_{L^\infty} \leq \int_0^{T-\Delta T} \left( D^{k'} p_{j,i,k} (T - \eta) \right|_{L^\infty} \left| J_\gamma (u^{\gamma;m}_j) u^{\gamma;m}_k (\eta) \right|_{L^1} \, d\eta$$

(7.45)
which with (5.7), a Hölder inequality and (7.24) is

\[
\leq \int_0^{T-\Delta T} \left[ \ldots \right] (k') (T - \eta)^{-|k'|/2-2} \ d\eta \ |u_o|_{L^2}^2 \\
\leq \left[ \ldots \right] (k') \Delta T^{-|k'|/2-1} |u_o|_{L^2}^2 \tag{7.46}
\]

Thus, the integral

\[
\int_0^{T-\Delta T} \int_{\mathbb{R}^3} D^{k'} p_{j,i,k}(y, T - \eta) J_\gamma(u_j^{\gamma,m}) u_k^{\gamma,m} (x - y, \eta) \ dyd\eta \tag{7.47}
\]

is a bounded function of \( x \in \mathbb{R}^3 \).

As \( D^{k'} p_{j,i,k} \) is continuous in \( \mathbb{R}^3 \times [0, T] \), (5.7), the (7.47) convolution is a continuous function of \( x \in \mathbb{R}^3 \), which is, as a result, measurable on \( \mathbb{R}^3 \). As a result

\[
\int_{\mathbb{R}^3} \phi(x) \left[ \int_0^{T-\Delta T} \int_{\mathbb{R}^3} D^{k'} p_{j,i,k}(y, T - \eta) J_\gamma(u_j^{\gamma,m}) u_k^{\gamma,m} (x - y, \eta) \ dyd\eta \right] \ dx \\
= \int_{\mathbb{R}^3} \left[ \int_0^{T-\Delta T} \phi(x) \int_{\mathbb{R}^3} D^{k'} p_{j,i,k}(y, T - \eta) J_\gamma(u_j^{\gamma,m}) u_k^{\gamma,m} (x - y, \eta) \ dyd\eta \right] \ dx \tag{7.48}
\]

for every scalar valued \( \phi \in C^\infty_o \), which after changing variables in the right side thereof and then reversing the order of integration is

\[
= \int_0^{T-\Delta T} \int_{\mathbb{R}^3} \left[ \int_{\mathbb{R}^3} \phi(x) \ D^{k'} p_{j,i,k}(x - y, T - \eta) \ dx \right] \ J_\gamma(u_j^{\gamma,m}) u_k^{\gamma,m} (y, \eta) \ dyd\eta \tag{7.49}
\]

A Hölder inequality in the (7.49) interior integral shows that the \( L^\infty \) norm of each side of (7.48) is

\[
\leq \int_0^{T-\Delta T} \int_{\mathbb{R}^3} |\phi|_{L^2} \ |D^{k'} p_{j,i,k}(T - \eta)|_{L^2} \ |J_\gamma(u_j^{\gamma,m}) u_k^{\gamma,m} (y, \eta)| \ dyd\eta
\]
which with (5.6) and other mathematics shows that

$$\left| \int_{\mathbb{R}^3} \phi(x) \left[ \int_0^{T-\Delta T} \int_{\mathbb{R}^3} D^{k'} p_{j,i,k} (y, T - \eta) \, J_\gamma (u_j^{\gamma,m}) u_k^{\gamma,m} \, (x - y, \eta) \, dyd\eta \right] \, dx \right|_{L^\infty}$$

is

$$\leq \left| \phi \right|_{L^2} \int_0^{T-\Delta T} \left[ \ldots \right](k') \left| T - \eta \right|^{-|k'|/2-5/4} \, d\eta \, |u_o|_{L^2}^2$$

$$\leq \left| \phi \right|_{L^2} \left[ \ldots \right](k') \left[ \Delta T^{-|k'|/2-1/4} - T^{-|k'|/2-1/4} \right] |u_o|_{L^2}^2$$

(7.50)

for every scalar valued $\phi \in C^\infty_c$ on $\mathbb{R}^3$.

As the (7.50) interior integral is measurable, the Hölder Inequality Converse shows that

$$\left| \int_0^{T-\Delta T} \int_{\mathbb{R}^3} D^{k'} p_{j,i,k} (y, T - \eta) \, J_\gamma (u_j^{\gamma,m}) u_k^{\gamma,m} \, (x - y, \eta) \, dyd\eta \right|_{L^2}$$

is

$$\leq \left[ \ldots \right](k') \left[ \Delta T^{-|k'|/2-1/4} - T^{-|k'|/2-1/4} \right] |u_o|_{L^2}^2$$

(7.51)

The triangle inequality and (5.5) show that the left side of (7.42) is

$$\leq \sum_{i,k} \left| \int_0^{T-\Delta T} \int_{\mathbb{R}^3} D^{k'} p_{j,i,k} (y, T - \eta) \, J_\gamma (u_j^{\gamma,m}) u_k^{\gamma,m} \, (x - y, \eta) \, dyd\eta \right|_{L^2}$$

which with (7.51) is

$$\leq \left[ \ldots \right]_{7.42}(k') \left[ \Delta T^{-|k'|/2-1/4} - T^{-|k'|/2-1/4} \right] |u_o|_{L^2}^2$$

The proof of (7.42) is complete.

We leave the proof of the last sentence of this lemma to the reader. □

**Theorem 7.7.** Let $u^{\gamma,m} = u^{\gamma,m}(x,t;u_o)$, $\Delta T > 0$ and

$$0 \leq T - \Delta T < T < T_{bup}^{\gamma,m}(u_o)$$
Then for those multi-indices $k'$ for which $0 \leq |k'| \leq m - 2$

$$|D^{k'} u^γ,m(T)|_{L^∞}$$

$$\leq \left[ \ldots \right]_{7.52a} ΔT^{1/2} |u^γ,m|_{L^∞×[0,T]} |D^{k'} u^γ,m|_{L^∞×[0,T]}$$

$$+ \left[ \ldots \right]_{7.52b}(k') ΔT^{1/2} \sum_{α'+β'=k'} |D^{α'} u^γ,m|_{L^∞×[0,T]} |D^{β'} u^γ,m|_{L^∞×[0,T]}$$

$$+ \left[ \ldots \right]_{7.52c}(k') \left[ ΔT^{-|k'|/2-1/4} - T^{-|k'|/2-1} \right] |u_o|^2_{L^2} + |D^{k'} u_o|_{L^∞}$$

(7.52)

and for those multi-indices $k'$ for which $0 \leq |k'| \leq m$

$$|D^{k'} u^γ,m(T)|_{L^2}$$

$$\leq \left[ \ldots \right]_{7.53a} ΔT^{1/2} |u^γ,m|_{L^∞×[0,T]} |D^{k'} u^γ,m|_{L^2×[0,T]}$$

$$+ \left[ \ldots \right]_{7.53b}(k') ΔT^{1/2} \sum_{j, \; α'+β'=k'} |J_γ(D^{α'} u^γ,m_j) D^{β'} u^γ,m|_{L^2×[0,T]}$$

$$+ \left[ \ldots \right]_{7.53c}(k') \left[ ΔT^{-|k'|/2-1/4} - T^{-|k'|/2-1} \right] |u_o|^2_{L^2} + |D^{k'} u_o|_{L^2}$$

(7.53)

The numbers $\left[ \ldots \right]_{7.52a}$, and $\left[ \ldots \right]_{7.53a}$ are independent of $u^γ,m$, $γ$, and $T$.

Proof. We first establish (7.52). We begin by dividing each (6.1)$_{k',γ,u_o}$ integral equation into two pieces

$$D^{k'} u^γ,m(x, T; u_o)$$

$$= \int_{T-ΔT}^{T} \int_{R^3} \sum_j \hat{P}[∂_j K^γ(T-η)](y) D^{k'} [J_γ(u^γ,m_j) u^γ,m] (x - y, η) dydη$$

$$+ \int_{0}^{T-ΔT} \int_{R^3} \sum_j \hat{P}[D^{k'} ∂_j K^γ(T-η)](y) J_γ(u^γ,m_j) u^γ,m (x - y, η) dydη$$

$$+ \int_{R^3} K(y, T) D^{k'} u_o(x - y) dy$$

(7.54)
for those multi-indices $|k'|$ for which $0 \leq |k'| \leq m - 2$.

As $u_{\gamma,m}^\gamma$ is $\in H^T \times [0, T]$, $u_{\gamma,m}^\gamma$ satisfies the inequalities of Lemma 5.3 with $u_{\gamma,m}^\gamma$ in place of $v$. (5.21) then shows that the $L^\infty$ norm of the first summand in the right side of (7.54) is

$$\leq \sum_j \Delta T^{1/2} \left| D^{k'} [J_\gamma (u_j^\gamma)^{\gamma,m}] \right|_{L^\infty \times [T - \Delta T, T]}$$

which with (2.17) to expand $D^{k'} [J_\gamma (u_j^\gamma)^{\gamma,m}]$, the (2.18) upper bound for the coefficients $c(\alpha', \beta', k')$ and (3.6) ($J_\gamma$ does not increase an $L^\infty$ norm) is $\leq$ the first two summands in the right side of (7.52).

(7.41) shows that the $L^\infty$ norm of the second summand in the right side of (7.54) is $\leq$ the third summand in the right side of (7.52), and (3.6) shows that the $L^\infty$ norm of the single integral in the right side of (7.54) is $\leq$ the fourth summand in the right side of (7.52).

We now establish (7.53). (5.22), (2.17) and (2.18) show that the $L^2$ norm of the first summand in the right side of (7.54) is

$$\leq \sum_j \Delta T^{1/2} \left[ \left| J_\gamma (u_j^\gamma)^{\gamma,m} \right|_{L^2 \times [0, T]} D^{k'} u_{\gamma,m}^{\gamma,m} \left| \right|_{L^2 \times [0, T]} \right]$$

$$+ \sum_{j, \alpha', \beta'} \Delta T^{1/2} \left| D^{\alpha'} [J_\gamma (u_j^\gamma)^{\gamma,m}] \right|_{L^2 \times [0, T]} u_{\gamma,m}^{\gamma,m} \left| \right|_{L^2 \times [0, T]}$$

which with (3.6) and (3.8) is

$$\leq \sum_j \Delta T^{1/2} \left[ \left| u_j^\gamma \left( \right|_{L^\infty \times [0, T]} \right|_{L^2 \times [0, T]} \right] D^{k'} u_{\gamma,m}^{\gamma,m} \left| \right|_{L^2 \times [0, T]}$$

$$+ \sum_{j, \alpha', \beta'} \Delta T^{1/2} \left| J_\gamma (D^{\alpha'} u_j^\gamma)^{\gamma,m} \right|_{L^2 \times [0, T]} u_{\gamma,m}^{\gamma,m} \left| \right|_{L^2 \times [0, T]}$$

which is equal to the sum of the first two summands in the right side of (7.53).
(7.42) shows that the $L^2$ norm of the second (7.54) summand is

$$\left[ \cdots (k') \right] \left( \Delta T^{-|k'|/2-1/4} - T^{-|k'|/2-1/4} \right) |u_o|^2_{L^2}$$

which is equal to the third summand in the right side of (7.53), and (3.8) shows that the $L^2$ norm of the third (7.54) summand is $\leq$ the fourth summand in the right side of (7.53).

We leave the proof of the last sentence of this theorem to the reader. □

8. $u^\gamma(x, t; u_o) = u^{\gamma,m}(x, t; u_o)$, when $m \geq 7$

This section shows that the functions $u^{\gamma,m}(x, t; u_o)$ with the same $\gamma \geq 0$ and initial condition $u_o \in \cap_m H^m_{df}$ are equal to a function

$$u^\gamma(x, t; u_o) \in \cap_m \left[ H^m_{df} \times [0, T_{bup}^\gamma(u_o)) \right]$$

(8.1)

that each derivative $\partial_k D^k u^\gamma(x, t; u_o)$ exists on the strip where $u^\gamma$ is defined and is $\in \cap_m [H^m_{df} \times [0, T_{bup}^\gamma(u_o))$, that

$$T_{bup}^{\gamma=0}(u_o) \geq \left[ \cdots \right]_{6.19} |u_o|^2_{H^7} \text{ and } T_{bup}^{\gamma>0}(u_o) = \infty$$

and that each $u^\gamma(x, t; u_o)$ satisfies every equation, inequality or other relation which any $u^{\gamma,m}(x, t; u_o)$ to which $u^\gamma$ is equal satisfies.

We first establish another blow up condition which $u^{\gamma,m}$ satisfies.

**Theorem 8.1.** Each $u^{\gamma,m}(x, t; u_o)$ satisfies the blow up condition

$$|u^{\gamma,m}|_{L^\infty \times [0, T]} \to \infty \text{ as } T \to \text{ a finite } T_{bup}^{\gamma,m}(u_o)$$

(8.2)

**Proof.** As each $u^{\gamma,m}(x, t; u_o)|_{u_o=0}$ is $= 0$ on $\mathcal{R}^3 \times [0, \infty)$, Lemma 6.1, no $u^{\gamma,m}(x, t; u_o)|_{u_o=0}$ has a finite blow up time, and we assume in this proof that $u_o$ is $\neq 0$. 
We now show for each multi-index \( k' \) for which \( 1 \leq |k'| \leq m - 2 \) that each \( u_{\gamma,m}(x, t; u_o) \) satisfies the inequality

\[
|D^{k'} u_{\gamma,m}|_{L^\infty \times [0,T]} \leq \ldots \tag{8.3}
\]

when \( T \) is in \([0, T_{bup}^{\gamma,m}(u_o)]\). Each function \([ \ldots ]_{8.3}^{k',m} \) is a polynomial (with a finite number of summands) in \( |u_{\gamma,m}|_{L^\infty \times [0,T]} \) (an "(8.3) polynomial") the coefficients in which are non-negative and are independent of \( u_{\gamma,m} \) (that is, independent of \( u_{\gamma,m}(t) \) for \( t > 0 \)) and of \( \gamma \) and \( T \).

For the convenience of the reader, we write out (7.52)

\[
|D^{k'} u_{\gamma,m}(T)|_{L^\infty} \leq \ldots \tag{7.52'}
\]

which \( u_{\gamma,m} \) satisfies when the multi-index \( k' \) satisfies \( 0 \leq |k'| \leq m - 2 \) and

\[
0 \leq T - \Delta T < T < T_{bup}^{\gamma,m}(u_o) \tag{8.4}
\]

For each \( u_{\gamma,m} \), those \( T \in [0, T_{bup}^{\gamma,m}(u_o)] \) for which

\[
[ \ldots ]_{7.52a}^{T^{1/2}} |u_{\gamma,m}|_{L^\infty \times [0,T]} \leq 2^{-1} \tag{8.5}
\]

satisfy

\[
[ \ldots ]_{7.52b}(k') T^{1/2} \leq [ \ldots ]_{7.52b}(k') [ \ldots ]_{7.52a}^{1} 2^{-1} |u_{\gamma,m}|_{L^\infty \times [0,T]}^{-1} \tag{8.6}
\]

which as \( 0 < |u_o|_{L^\infty} \leq |u_{\gamma,m}|_{L^\infty \times [0,T]} \) is

\[
[ \ldots ]_{7.52b}(k') [ \ldots ]_{7.52a}^{1} 2^{-1} |u_o|_{L^\infty}^{-1} := [ \ldots ]_{8.6}(k', u_o)
\]
which is independent of \( u_{\gamma,m} \setminus u_o, \gamma \) and \( T \).

Then with \( \Delta T \) equal to \( T \)

\[
[ \ldots ]_{7.52a} \Delta T^{1/2} |u_{\gamma,m}|_{L^\infty \times [0,T]} \leq 2^{-1}
\]

\[
[ \ldots ]_{7.52b}(k') \Delta T^{1/2}
\]

\[
\leq [ \ldots ]_{7.52a}^{-1} 2^{-1} |u_o|_{L^\infty}^{-1} := [ \ldots ]_{8.6}(k', u_o)
\]

and

\[ T - \Delta T = 0 < T < T_{bup}^{\gamma,m}(u_o) \]

which shows that such \( \Delta T \) and \( T \) satisfy (8.4).

(7.52') then shows for such \( T \) that \( u_{\gamma,m} \) satisfies

\[
|D^{k'} u_{\gamma,m}(T)|_{L^\infty}
\]

\[
\leq 2^{-1} |D^{k'} u_{\gamma,m}|_{L^\infty \times [0,T]}
\]

\[
+ [ \ldots ]_{8.6}(k', u_o) \sum_{\alpha' + \beta' = k'} |D^{\alpha'} u_{\gamma,m}|_{L^\infty \times [0,T]} |D^{\beta'} u_{\gamma,m}|_{L^\infty \times [0,T]}
\]

\[
+ |D^{k'} u_o|_{L^\infty}
\]

(8.7)

We now consider those \( T \in [0, T_{bup}^{\gamma,m}(u_o)) \) for which

\[
[ \ldots ]_{7.52a} T^{1/2} |u_{\gamma,m}|_{L^\infty \times [0,T]} > 2^{-1}
\]

As the left side thereof is a continuous, increasing function of \( T \), there is for each such \( T \) a unique \( T_* \in (0, T_{bup}^{\gamma,m}(u_o)) \) which is \( < T \) which satisfies

\[
[ \ldots ]_{7.52a} T_*^{1/2} |u_{\gamma,m}|_{L^\infty \times [0,T_*]} = 2^{-1}
\]

(8.8)

and a unique \( \Delta T < T_* < T < T_{bup}^{\gamma,m}(u_o) \) which satisfies

\[
[ \ldots ]_{7.52a} \Delta T^{1/2} |u_{\gamma,m}|_{L^\infty \times [0,T]} = 2^{-1}
\]

(8.9)
Thus $\Delta T$ and $T$ satisfy (8.4), and (8.9) shows that
\[ \left[ \ldots \right]_{7.52b}(k') \Delta T^{1/2} \leq \left[ \ldots \right]_{7.52a}(k') 2^{-1} |u_o|^{-1} := \left[ \ldots \right]_{8.6}(k', u_o) \]
(8.10)
and that
\[ \Delta T^{-|k'|/2-1} = \left[ \ldots \right]_{7.52a}(k') 2^{|k'|+2} |u^{\gamma,m}|^{k'+2}_{L^\infty \times [0,T]} := \left[ \ldots \right]_{8.11}(k') |u^{\gamma,m}|^{k'+2}_{L^\infty \times [0,T]} \]
(8.11)
Therewith, (7.52'), (8.9), (8.10) and (8.11) show that
\[ |D^{k'} u^{\gamma,m}(T)|_{L^\infty} \leq 2^{-1} |D^{k'} u^{\gamma,m}_{L^\infty \times [0,T]}| + \sum_{\alpha' + \beta' = k'} |D^{\alpha'} u^{\gamma,m}|_{L^\infty \times [0,T]} |D^{\beta'} u^{\gamma,m}|_{L^\infty \times [0,T]} + \left[ \ldots \right]_{8.11}(k') |u^{\gamma,m}|^{k'+2}_{L^\infty \times [0,T]} |u_o|_{L^2}^2 + |D^{k'} u_o|_{L^\infty} \]
(8.12)
when $T \in [T^*, T^{\gamma,m}_{bup}(u_o)]$, and (8.7) shows that $u^{\gamma,m}$ satisfies (8.12) for all $T \in [0, T^{\gamma,m}_{bup}(u_o)]$. $\left[ \ldots \right]_{8.6}(k', u_o)$ and $\left[ \ldots \right]_{8.11}(k')$ are independent
of $u^{\gamma,m} \setminus u_o, \gamma$ and $T$.

As the right side of (8.12) is a non-decreasing function of $T \in [0, T^{\gamma,m}_{bup}(u_o)]$, it is also an upper bound for $|D^{k'} u^{\gamma,m}|_{L^\infty \times [0,T]}$. Then with customary mathematics
\[ |D^{k'} u^{\gamma,m}|_{L^\infty \times [0,T]} \leq \sum_{\alpha' + \beta' = k'} |D^{\alpha'} u^{\gamma,m}|_{L^\infty \times [0,T]} |D^{\beta'} u^{\gamma,m}|_{L^\infty \times [0,T]} + \left[ \ldots \right]_{8.13b}(k') |u^{\gamma,m}|^{k'+2}_{L^\infty \times [0,T]} |u_o|_{L^2}^2 + 2 |D^{k'} u_o|_{L^\infty} \]
(8.13)
when $T \in [0, T_{\text{bup}}(u_0))$ and $k'$ is a multi-index for which $|k'|$ is $\leq m - 2$. \[ \ldots \] $8.13a(k', u_0)$ and $[ \ldots ]8.13b(k')$ are independent of $u_{\gamma,m} \setminus u_o, \gamma$ and $T$.

When $|k'| = 1$, the first line in the right side of (8.13) contains no summand, and

$$|\partial_j u^\gamma \cdot m|_{L^\infty \times [0, T]} \leq \ldots 8.13b(k') \ |u^\gamma \cdot m|^3_{L^\infty \times [0, T]} |u_o|^2_{L^2} + 2 |\partial_j u_o|_{L^\infty}$$

when $T \in [0, T_{\text{bup}}(u_o))$, which establishes (8.3) when $|k'| = 1$.

We now establish (8.3) for those $k'$ for which $2 \leq |k'| \leq m - 2$ by induction on $m^\dagger$. The induction hypothesis for $m^\dagger$ for which $1 \leq m^\dagger < m - 2$ is that (8.3) has been established for those $k'$ for which $1 \leq |k'| \leq m^\dagger$.

For those $k'$ for which $|k'| = m^\dagger + 1$, (8.13) shows that

$$|D^{k'} u^\gamma \cdot m|_{L^\infty \times [0, T]} \leq \ldots 8.13a(k', u_0) \sum_{\alpha' + \beta' = k'} \left| D^{\alpha'} u^\gamma \cdot m \right|_{L^\infty \times [0, T]} \left| D^{\beta'} u^\gamma \cdot m \right|_{L^\infty \times [0, T]}$$

$$+ \ldots 8.13b(k') \left| u^\gamma \cdot m \right|^2_{L^\infty \times [0, T]} |u_o|^2_{L^2} + 2 |D^{k'} u_o|_{L^\infty}$$

The induction hypothesis shows that the functionals $|D^{\alpha'} u^\gamma \cdot m|_{L^\infty \times [0, T]}$ and $|D^{\beta'} u^\gamma \cdot m|_{L^\infty \times [0, T]}$ which appear in the right side are each $\leq a (8.3)$ polynomial the product of which is an (8.3) polynomial. Then for those $k'$ for which $|k'| = m^\dagger + 1$

$$|D^{k'} u^\gamma \cdot m|_{L^\infty \times [0, T]} \leq \ldots 8.13a(k', u_0) \sum_{\alpha' + \beta' = |k'|} \left(8.3\right) \text{ polynomial}$$

$$+ \ldots 8.13b(k') \left| u^\gamma \cdot m \right|^{|k'| + 3}_{L^\infty \times [0, T]} |u_o|^2_{L^2} + 2 |D^{k'} u_o|_{L^\infty}$$

the right side of which is an (8.3) polynomial as $[ \ldots ]8.13a(k', u_o)$ and $[ \ldots ]8.13b(k')$ are independent of $u_{\gamma,m} \setminus u_o, \gamma$ and $T$.

Thus, the induction hypothesis for $m^\dagger$ for which $1 \leq m^\dagger < m - 2$ implies the induction hypothesis for $m^\dagger + 1$. The proof of (8.3) for $1 \leq |k'| \leq m - 2$ is complete.
We now show when \( k' \) satisfies \( 0 \leq |k'| \leq m \) that each \( u_{\gamma,m}^r(x, t; u_o) \) satisfies the inequality

\[
|D^{k'} u_{\gamma,m}^r|_{L^2[0, T]} \leq \left[ \ldots \right]_{8.14} \left( |u_{\gamma,m}^r|_{L^\infty[0, T]}^{1/2}; u_o \right)
\]

(8.14)

when \( T \in [0, T_{\text{bup}}(u_o)) \). Each function \( \left[ \ldots \right]_{8.14}^{k',m} \) is a polynomial (with a finite number of summands) in \( |u_{\gamma,m}^r|_{L^\infty[0, T]} \) (an "(8.14) polynomial") the coefficients in which are non-negative and are independent of \( u^r_{\gamma,m \setminus u_o}, \gamma \) and \( T \).

A variation of the analysis which begins with (8.5) and which ends with (8.13) in which we use (7.53) in place of (7.52) shows that

\[
|D^{k'} u_{\gamma,m}^r|_{L^2[0, T]}
\leq \left[ \ldots \right]_{8.15a}^{k',m} \sum_{j, \alpha' + \beta' = k'} |J_j(D^\alpha' u_{\gamma,m}^r) D^\beta' u_{\gamma,m}^r|_{L^2[0, T]}
\]

\[
+ \left[ \ldots \right]_{8.15b}^{k'} |u_{\gamma,m}^r|_{L^\infty[0, T]}^{1/2} |u_o|_2^2 + 2 |D^{k'} u_o|_L^2
\]

(8.15)

for \( T \in [0, T_{\text{bup}}(u_o)) \) and each multi-index \( k' \) for which \( 0 \leq |k'| \leq m \) and in which \( \left[ \ldots \right]_{8.15a}^{k',m} \) and \( \left[ \ldots \right]_{8.15b}^{k'} \) are independent of \( u_{\gamma,m \setminus u_o}, \gamma \) and \( T \).

(7.24) shows that

\[
|u_{\gamma,m}^r|_{L^2[0, T_{\text{bup}}(u_o)]} \leq |u_o|_L^2
\]

which establishes (8.14) for \( k' = 0 \). With \( |k'| = 1 \), the first sum in the right side of (8.15) contains no summand and

\[
|\partial_j u|_{L^2[0, T]} \leq \left[ \ldots \right]_{8.15b}^{k'} |u_{\gamma,m}^r|_{L^\infty[0, T]}^{3/2} |u_o|_L^2 + 2 |\partial_j u_o|_{L^\infty}
\]

which establishes (8.14) for \( |k'| = 1 \).

We now establish (8.14) for \( k' \) for which \( 2 \leq |k'| \leq m \) by induction on \( m^\dagger \). The induction hypothesis for \( m^\dagger \) for which \( 1 \leq m^\dagger < m \) is that (8.14) has been established for those \( k' \) for which \( 1 \leq |k'| \leq m^\dagger < m \).
For those $k'$ for which $|k'| = m^\dagger + 1$, (8.15) shows that

$$|D^{k'} u^{\gamma,m}|_{L^2 \times [0,T]} \leq \left( \ldots \right)_{18a} (k', u_0) \sum_{\alpha' + \beta' = k'}_{1 \leq |\alpha'|, |\beta'| \leq m^\dagger} |J_\gamma (D^{\alpha'} u^{\gamma,m}) D^{\beta'} u^{\gamma,m}|_{L^2 \times [0,T]} + \left( \ldots \right)_{18a} (k') |u^{\gamma,m}|_{L^\infty \times [0,T]} |u_0|_{L^2}^2 + 2 |D^{k'} u_0|_{L^2}

(8.16)

In each $|J_\gamma (D^{\alpha'} u^{\gamma,m}) D^{\beta'} u^{\gamma,m}|_{L^2 \times [0,T]}$ in the right side of (8.16), one of $D^{\alpha'} u^{\gamma,m}$ and $D^{\beta'} u^{\gamma,m}$ is $L^\infty \times [0,T]$, (2.14), (2.21). As $J_\gamma$ does not increase an $L^\infty$ norm or an $L^2$ norm, each $|J_\gamma (D^{\alpha'} u^{\gamma,m}) D^{\beta'} u^{\gamma,m}|_{L^2 \times [0,T]}$ is $\leq$ the product of an (8.3) polynomial in $|u^{\gamma,m}|_{L^\infty \times [0,T]}$ and an (8.14) polynomial in $|u^{\gamma,m}|_{L^\infty \times [0,T]}$, the product of which is an (8.14) polynomial in $|u^{\gamma,m}|_{L^\infty \times [0,T]}$. As a result

$$|D^{k'} u^{\gamma,m}|_{L^2 \times [0,T]} \leq \left( \ldots \right)_{18a} (u_0) \sum_{\alpha' + \beta' = k'}_{1 \leq |\alpha'|, |\beta'| \leq m^\dagger} (8.14) \text{ polynomial}

+ \left( \ldots \right)_{18a} (k') |u^{\gamma,m}|_{L^\infty \times [0,T]} |u_0|_{L^2}^2 + 2 |D^{k'} u_0|_{L^2}

(8.17)

for $T \in [0, T_{\text{bup}}(u_0))$ the right side of which is an (8.14) polynomial as $\left( \ldots \right)_{18a} (k', u_0)$ and $\left( \ldots \right)_{18a} (k')$ are independent of $u^{\gamma,m}, u_0, \gamma$ and $T$.

Thus, the induction hypothesis for $m^\dagger$ for which $1 \leq m^\dagger < m$ implies the induction hypothesis for $m^\dagger + 1 \leq m$. The proof of (8.16) for $0 \leq |k'| \leq m$ is complete.

Summing (8.16) over $k'$ for which $0 \leq |k'| \leq m$ shows that

$$|u^{\gamma,m}|_{H^m \times [0,T]} \leq \sum_{0 \leq |k'| \leq m} \left( \ldots \right)_{18,14} |u^{\gamma,m}|_{L^\infty \times [0,T]} |u_0|_{L^2}^2 + 2 |D^{k'} u_0|_{L^2}

= \left( \ldots \right)_{18,18} m \sum_{0 \leq |k'| \leq m} |u^{\gamma,m}|_{L^\infty \times [0,T]} |u_0|_{L^2}^2

(8.18)

for $T \in [0, T_{\text{bup}}(u_0))$ where the function $\left( \ldots \right)_{18,18} m \sum_{0 \leq |k'| \leq m} |u^{\gamma,m}|_{L^\infty \times [0,T]} |u_0|_{L^2}^2$ is a polynomial (with a finite number of summands) in $|u^{\gamma,m}|_{L^\infty \times [0,T]}$. 

As (7.34) shows that \(|u^{\gamma,m}|_{H^m \times [0,T]} \to \infty\) as \(T \to \) a finite \(T_{bup}^{\gamma,m}(u_o)\), (8.18) shows that
\[
\left[ ... \right]_{8.18} \left( |u^{\gamma,m}|_{L^\infty \times [0,T]}^{1/2}; u_o \right) \to \infty
\]
as \(T\) so converges. As \(\left[ ... \right]_{8.18} \left( |u^{\gamma,m}|_{L^\infty \times [0,T]}^{1/2}; u_o \right)\) is a polynomial (with a finite number of summands) in \(|u^{\gamma,m}|_{L^\infty \times [0,T]}\) the coefficients in which are non-negative, \(|u^{\gamma,m}|_{L^\infty \times [0,T]}\) must \(\to \infty\) as \(T\) so converges. The proof of (8.2) is complete. \(\square\)

**Theorem 8.2.** The functions \(u^{\gamma,m}(x,t; u_o)\) with the same \(\gamma \geq 0\) and initial condition \(u_o \in \bigcap_m H^{m,df}\) are equal to a function
\[
u^\gamma(x,t; u_o) \in \bigcap_m [H^{m,df} \times [0, T_{bup}^\gamma(u_o))]\]
(8.19)
each derivative \(\partial_t D_k^\gamma u^\gamma(x,t; u_o)\) of which exists on the strip where \(u^\gamma\) is defined and is \(\in \bigcap_m [H^{m,df} \times [0, T_{bup}^\gamma(u_o))]\), that
\[
\begin{align*}
T_{bup}^\gamma(u_o) & \geq \left[ ... \right]_{6.19} |u_o|_{L^\infty}^{-2} \\
T_{bup}^{\gamma > 0}(u_o) & = \infty
\end{align*}
\]
(8.20)
and that \(u^\gamma(x,t; u_o)\) satisfies every equation, inequality or other relation which any \(u^{\gamma,m}(x,t; u_o)\) to which it is equal satisfies. In particular, \(u^\gamma\) satisfies the blow up condition
\[
|u^\gamma|_{L^\infty \times [0,T]} \to \infty \text{ as } T \to \text{ a finite } T_{bup}^\gamma(u_o)
\]
(8.21)
each \((6.1)_{k',\gamma,u_o}\) integral equation on \(\mathcal{R}^3 \times [0, T_{bup}^\gamma(u_o))\), each \((7.1)_{k',\gamma,u_o}\) pde thereon
\[
|\partial_j u^{\gamma>0}|_{L^\infty \times [0,T]} \leq \sup_{|k'|=1} \left[ ... \right]_{8.3} \left( |u^{\gamma>0}|_{L^\infty \times [0,T]}; u_o \right) \text{ and }
\]
\[
|\partial_t u^{\gamma>0}|_{L^\infty \times [0,T]} \leq \left[ ... \right] \left[ ... \right]_{8.18} \left( |u^{\gamma>0}|_{L^\infty \times [0,T]}^{1/2}; u_o \right)^2
\]
\[
+ \left[ ... \right]_{8.18} \left( |u^{\gamma>0}|_{L^\infty \times [0,T]}^{1/2}; u_o \right)
\]
(8.22)
where \(\left[ ... \right]_{8.3}\) and \(\left[ ... \right]_{8.18}\) are independent of \(\gamma\).
Proof. As each $u_{\gamma,m}>7(x, t; u_o)$ satisfies the (6.1)$k'=0, \gamma, u_o$ integral equation, it is $u_{\gamma}^{7}(x, t; u_o)$ on $\mathbb{R}^3 \times [0, T_{bup}^{\gamma,m}>7(u_o))$. Lemma 6.1. Thus

$$|u_{\gamma,m}>7|_{L^\infty \times [0, T]} = |u_{\gamma,7}|_{L^\infty \times [0, T]}$$

(8.23)

when $T$ is $\in [0, T_{bup}^{\gamma,m}>7(u_o))$.

If a blow up time $T_{bup}^{\gamma,m}>7(u_o)$ is finite, then each side of (8.23) $\to \infty$ as $T \to T_{bup}^{\gamma,m}=7(u_o)$. Thus, $T_{bup}^{\gamma,m}=7(u_o)$ which is $= T_{bup}^{\gamma,m}>7(u_o)$ is finite, and the finite $T_{bup}^{\gamma,m}=7(u_o)$ is equal to each $T_{bup}^{\gamma,m}>7(u_o)$. In the other case every $T_{bup}^{\gamma,m}(u_o)$ is equal to $T_{bup}^{\gamma,m}=7(u_o) = \infty$.

With $T_{bup}^{\gamma}(u_o)$ as the common value of $T_{bup}^{\gamma,m}(u_o)$, Theorem 6.2 shows that

$$T_{bup}^{\gamma}=0(u_o) \geq \begin{bmatrix} \ldots \end{bmatrix}_{6.19} |u_o|_{H^7}^{-2}$$

which establishes the first line of (8.20).

We then let $u_{\gamma}(x, t; u_o)$ be the common value of $u_{\gamma,m}$ on $\mathbb{R}^3 \times [0, T_{bup}^{\gamma}(u_o))$. As $u_{\gamma}$ is $= u_{\gamma,m}^{7}$ and each $u_{\gamma,m}$ is $\in H^{m, df} \times [0, T_{bup}^{\gamma}(u_o))$ $u_{\gamma}$ is $\in \cap_{m} [H^{m, df} \times [0, T_{bup}^{\gamma}(u_o))]$.

In the same way, $\partial_t D^{k'} u_{\gamma}$ exists on $\mathbb{R}^3 \times [0, T_{bup}^{\gamma}(u_o))$ and is equal to each $\partial_t D^{k'} u_{\gamma,m}$ for which $m$ is $\geq |k'| + 7$. Thus

$$\partial_t D^{k'} u_{\gamma} \in \cap_{m \geq |k'| + 7} [H^{m-|k'|-2, df} \times [0, T_{bup}^{\gamma}(u_o))]$$

$$= \cap_{m \geq 5} [H^{m, df} \times [0, T_{bup}^{\gamma}(u_o))]$$

We now establish the second line of (8.20). With $\Delta T = T$ and $k' = 0$, (7.52) with $u_{\gamma}$ in place of $u_{\gamma,m}$ shows that

$$u_{\gamma}(x, T; u_o)$$

$$= \int_0^T \int_{\mathbb{R}^3} \sum_j \hat{P} \left[ \partial_j K^i(T - \eta) \right] (y) J_\gamma(u_{\gamma}^j) u_{\gamma} (x - y, \eta) dy d\eta$$

$$+ \int_{\mathbb{R}^3} K(y, T) D^{k'} u_o(x - y) dy$$
on $\mathcal{R}^3 \times [0, T]$. (5.21) with $\gamma > 0$ and $t \geq 0$ shows that
\[
|u^{\gamma>0}(t; u_o)|_{L^\infty} 
\leq \left[ \ldots \right] t^{1/2} \sup_{\eta \in [0,t]} |m^{\gamma>0}_j(x-y, \eta; u_o)|_{L^\infty(x \in \mathcal{R}^3)} 
\sup_{\eta \in [0,t]} |u^{\gamma>0}(x-y, \eta; u_o)|_{L^\infty(x \in \mathcal{R}^3)} + |u_o|_{L^\infty}
\]
(8.24)

which with (3.7) shows that
\[
|u^{\gamma>0}(t; u_o)|_{L^\infty} 
\leq \left[ \ldots \right] 8.25 t^{1/2} \left[ m^{\gamma>0}_j \gamma^{-3/2} |u_o|_{L^2} \sup_{\eta \in [0,t]} |u^{\gamma>0}(\cdot, \eta; u_o)|_{L^\infty} + |u_o|_{L^\infty}\right]
\]
(8.25)

and as the right side of (8.25) is a non-decreasing function of $t$
\[
|u^{\gamma>0}(\cdot; u_o)|_{L^\infty \times [0,t]} 
\leq \left[ \ldots \right] 8.26 \left[ m^{\gamma>0}_j \gamma^{-3/2} |u_o|_{L^2} \sup_{\eta \in [0,t]} |u^{\gamma>0}(\cdot, \eta; u_o)|_{L^\infty \times [0,t]} + |u_o|_{L^\infty}\right]
\]
(8.26)

when $t$ is $\in [0, T^\gamma_{\text{bup}}(u_o))$.

With $\Delta T(\gamma)$ as the positive solution of
\[
\left[ \ldots \right] 8.26 \left[ m^{\gamma>0}_j \gamma^{-3/2} |u_o|_{L^2} \sup_{\eta \in [0,t]} |u^{\gamma>0}(\cdot, \eta; u_o)|_{L^\infty \times [0,t]} + |u_o|_{L^\infty}\right] = 2^{-1}
\]
(8.27)

which depends upon $|u_o|_{L^2}$ and not on $|u_o|_{L^\infty}$, (8.26) shows that
\[
|u^{\gamma>0}(\cdot; u_o)|_{L^\infty \times [0,\Delta T(\gamma)]} \leq 2^{-1} |u^{\gamma>0}(\cdot; u_o)|_{L^\infty \times [0,\Delta T(\gamma)]} + |u_o|_{L^\infty}
\]
that
\[
|u^{\gamma>0}(\cdot; u_o)|_{L^\infty \times [0,\Delta T(\gamma)]} \leq 2 |u_o|_{L^\infty}
\]
which shows that
\[
0 < \Delta T(\gamma) < T^\gamma_{\text{bup}}(u_o)
\]
As $|u(t)|_{L^2}$ is a non-increasing function of $t \geq 0$, iteration of such analysis shows that

$$|u^{>0}|_{L^\infty \times [0, N\Delta T(\gamma)]} \leq 2^N |u_o|_{L^\infty}$$

and

$$N \Delta T(\gamma) < T_{bup}^{>0}(u_o)$$

which shows that $T_{bup}^{>0}(u_o) = \infty$. The proof of the last line of (8.20) is complete.

The last sentence of the first paragraph of this theorem and the second paragraph of this theorem are obvious. □

We now write out (8.26) for our later use.

**Lemma 8.3.** When $u_o$ is $\in \cap_m H^{m,f}$, $u^{>0}(x, t; u_o)$ satisfies

$$|u^{>0}(t; u_o)|_{L^\infty} \leq [\ldots]_{8.26} (|m^+|_{L^2}, \gamma) t^{1/2} |u_o|_{L^2} |u^{>0}(\cdot; u_o)|_{L^\infty \times [0, t]} + |u_o|_{L^\infty}$$

(8.28)

for $t \in [0, \infty)$.

9. $u^{=0}(x, t; u_o)$ IS A GLOBAL-IN-TIME SOLUTION

This section shows that $u^{=0}(x, t; u_o)$ is a global-in-time solution of the (6.1)$_{k'=0, \gamma=0, u_o}$ integral equation and of its related (7.1)$_{k'=0, \gamma=0, u_o}$ pde.

We first show that

$$|u^{>0}|_{L^\infty \times [0, \infty)} = |u_o|_{L^\infty}$$

(9.1)

and then show that the functions $u^{>0}(x, t; u_o)$ converge in the $L^\infty$ norm on each compact set

$$\Omega_N := \{ (x, t) \mid 0 \leq |x| \leq N \text{ and } 0 \leq t \leq N \}$$

in which $N$ is a positive integer to a function $u^*$ which is continuous on $\mathcal{R}^3 \times [0, \infty)$, satisfies the (6.1)$_{k'=0, \gamma=0, u_o}$ integral equation on $\mathcal{R}^3 \times [0, \infty)$ and also satisfies

$$|u^*|_{L^\infty \times [0, \infty)} = |u_o|_{L^\infty}$$

(9.2)
Lemma 6.1 then shows that $u_{\gamma}^\gamma = 0$ is $u^*$ on $\mathcal{R}^3 \times [0, T_{\text{bup}}^{\gamma=0}(u_o))]$ which shows as $u_{\gamma}^\gamma = 0$ does not have a finite blow up time that $u_{\gamma}^\gamma = 0$ is a global-in-time solution of the $(6.1)_{k'=0, \gamma=0, u_o}$ integral equation and of its related pde. (8.21).

We begin with a scaling rule for the global-in-time solution $u_{\gamma}(x, t; u_o^\alpha)$ of the $(6.1)_{k'=0, \gamma, u_o^\alpha}$ integral equation and its related pde on $\mathcal{R}^3 \times \infty$ in which $\gamma > 0$ and the initial condition is

$$u_o^\alpha(x) := \alpha u_o(x\alpha)$$

(9.3)

and $\alpha > 0$.

**Lemma 9.1.** Let $u_o \in \cap_m H^{m, df}$ and $\alpha > 0$. Then each

$$u_{\gamma}^{>0}(x, t; u_o^\alpha) = \alpha u_{\gamma}^{>0}(x\alpha, t\alpha^2; u_o)$$

(9.4)

on $\mathcal{R}^3 \times [0, \infty)$.

*Proof.* We first show that $\alpha u_{\gamma}^{>0}(x\alpha, t\alpha^2; u_o)$ satisfies the $(7.1)_{k'=0, \gamma, u_o^\alpha}$ pde on $\mathcal{R}^3 \times [0, \infty)$. The calculation

$$\alpha u_{\gamma}^{>0}(x\alpha, t\alpha^2; u_o)|_{t=0} = \alpha u_{\gamma}^{>0}(x\alpha, 0; u_o) = \alpha u_o(x\alpha) = u_o^\alpha(x)$$

shows that $\alpha u_{\gamma}^{>0}(x\alpha, t\alpha^2; u_o)$ satisfies the initial condition of the $(7.1)_{k'=0, \gamma, u_o^\alpha}$ pde.

The chain rule then shows that

$$\partial_t[\alpha u_{\gamma}^{>0}(x\alpha, t\alpha^2; u_o)]$$

$$+ \mathcal{P} \left[ \sum_j J_{\gamma>0}[\alpha u_j^{>0}(x\alpha, t\alpha^2; u_o)] \partial_j[\alpha u_{\gamma}^{>0}(x\alpha, t\alpha^2; u_o)] \right](x)$$

$$- \Delta[\alpha u_{\gamma}^{>0}(x\alpha, t\alpha^2; u_o)]$$

(9.5)
is
\[ \alpha \partial_t u^{>0}(x, t\alpha^2; u_o) + \mathcal{P} \left[ \sum_j \alpha J_{>0}[u_j^{>0}(x, t\alpha^2; u_o)] \alpha \partial_j u^{>0}(x, t\alpha^2; u_o) \right](x) \]
\[ - \alpha \Delta u^{>0}(x, t\alpha^2; u_o) \alpha^2 \]
(9.6)
on \mathbb{R}^3 \times [0, \infty).

(2.20), (2.22) and the chain rule show that the argument of \( \mathcal{P} \) in (9.6) at \( t\alpha^{-2} \) is \( \in \cap_m [H^m \cap W^{m,1}] \). (4.14) then shows that (9.6) is
\[ \alpha u^{>0}(x, t\alpha^2; u_o) \]
\[ + \mathcal{P} \left[ \sum_j [J_{>0}[u_j^{>0}(\cdot; u_o)] \partial_j u^{>0}(\cdot; u_o) (t\alpha^2) \right](x) \]
\[ - \Delta u^{>0}(x, t\alpha^2; u_o) \alpha^3 \]
(9.7)
which vanishes for all \( (x, t\alpha) \in \mathbb{R}^3 \times [0, \infty) \) as \( u^{>0}(x, t; u_o) \) satisfies the (7.1)\( k'=0, \gamma>0, u_o \) pde on \( \mathbb{R}^3 \times [0, \infty) \). Theorem 8.2.

Thus, (9.5) also vanishes on \( \mathbb{R}^3 \times [0, \infty) \), which shows that \( \alpha u^{>0}(x, t\alpha^2; u_o) \) satisfies the (7.1)\( k'=0, \gamma>0, u_o \) pde on \( \mathbb{R}^3 \times [0, \infty) \).

As \( u^{>0}(x, t; u_o) \) and its derivative \( \partial_t u^{>0}(x, t; u_o) \) are \( \in \cap_m [H^{m,d}] \times [0, \infty) \), Theorem 8.2, the chain rule shows that \( \alpha u^{>0}(x, t\alpha^2; u_o) \) and its derivative \( \partial_t [\alpha u^{>0}(x, t\alpha^2; u_o)] \) are also \( \in \cap_m [H^{m,d}] \times [0, \infty) \). As \( u^{>0}(x, t; u_o) \) and \( \alpha u^{>0}(x, t\alpha^2; u_o) \) are in such function space, Lemma 7.3 shows that such functions are equal to each other on \( \mathbb{R}^3 \times [0, \infty) \) where each function is defined.

We now establish infrastructure for the proof of the next theorem. The definition of \( u_o^\alpha \) establishes the first line below.
\[ |u_o^\alpha|_{L^\infty} = |u_o|_{L^\infty} \alpha \]
\[ |u_o(x\alpha)|_{L^2} = |u_o|_{L^2} \alpha^{-3/2} \text{ and} \]
\[ |u_o^\alpha|_{L^2} = |u_o|_{L^2} \alpha^{-1/2} \]
(9.8)
The calculation

$$|u_o(x\alpha)|_{L^2} = \left[ \int_{\mathbb{R}^3} (u_o(x\alpha), u_o(x\alpha))_2 d[x\alpha] \, \alpha^{-3} \right]^{1/2} = |u_o|_{L^2} \, \alpha^{-3/2}$$

establishes the second line of (9.8). Multiplying the second line of (9.8) by $\alpha$ establishes the third line thereof.

**Theorem 9.2.** Each $u^\gamma = u^\gamma(x, t; u_o)$ is defined on $\mathcal{R}^3 \times [0, \infty)$, satisfies

$$|u^\gamma|_{L^\infty \times [0, \infty)} = |u_o|_{L^\infty}$$

(9.9)

and is a solution of the $(6.1)_{k'=0, \gamma, u_o}$ integral equation and of its related $(7.1)_{k'=0, \gamma, u_o}$ pde on $\mathcal{R}^3 \times [0, \infty)$.

The functions $u^{\gamma>0}(x, t; u_o)$ converge as $\gamma \to 0^+$ to $u^{\gamma=0}(x, t; u_o)$ in the $L^\infty$ norm on each $\Omega_N$.

**Proof.** We first establish (9.9) for $u^{\gamma>0}(x, t; u_o)$, which is defined on $\mathcal{R}^3 \times [0, \infty)$ and is a solution of the $(6.1)_{k'=0, \gamma, u_o}$ integral equation and of its related $(7.1)_{k'=0, \gamma, u_o}$ pde on $\mathcal{R}^3 \times [0, \infty)$.

With $\alpha > 0$, $t \geq 0$ and (8.28) ($u_o^\alpha \in \cap_m H^{m,df}$)

$$|u^{\gamma>0}(x, t\alpha^{-2}; u_o^\alpha)| \leq \left[ \ldots \right]_{8.26} (|m^\dagger|_{L^2}, \gamma) \, [t\alpha^{-2}]^{1/2} \, |u_o^\alpha|_{L^2} \, |u^{\gamma>0}(\eta; u_o^\alpha)|_{L^\infty \times [0, t\alpha^{-2}]} + |u_o^\alpha|_{L^\infty}$$

Then with the (9.4) scaling rule, (9.8) (third line) and (9.8) (first line)

$$\alpha \, |u^{\gamma>0}(x\alpha, t; u_o)| \leq \left[ \ldots \right] (|m^\dagger|_{L^2}, \gamma) \, t^{1/2} \, \alpha^{-1} \, |u_o|_{L^2} \, \alpha^{-1/2} \sup_{\eta \in [0, t\alpha^{-2}]} \alpha \, |u^{\gamma>0}(\eta\alpha^{2}; u_o)|_{L^\infty} + \alpha \, |u_o|_{L^\infty}$$

Customary mathematics then show that

$$|u^{\gamma>0}(t; u_o)|_{L^\infty} \leq \left[ \ldots \right] (|m^\dagger|_{L^2}, \gamma) \, t^{1/2} \, \alpha^{-3/2} \, |u^{\gamma>0}(\eta; u_o)|_{L^\infty \times [0, t]} + |u_o|_{L^\infty}$$
SMOOTH SOLUTIONS OF NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3 \times [0, \infty)$

As $u^{\gamma>0}$ is $\in \cap_m [H^{m,df} \times [0, \infty)]$

$$|u^{\gamma>0}(\cdot; u_o)|_{L^\infty \times [0,t]}$$

is finite for $t \geq 0$

and passage to the limit as $\alpha \to \infty$ with $t$ and $[\quad]$ fixed shows that

$$|u^{\gamma>0}(\cdot; u_o)|_{L^\infty \times [0,\infty)} \leq |u_o|_{L^\infty}$$

and as $|u^{\gamma>0}(0; u_o)|_{L^\infty} = |u_o|_{L^\infty}$

$$|u^{\gamma>0}(\cdot; u_o)|_{L^\infty \times [0,\infty)} = |u_o|_{L^\infty}$$

(9.10)

We now extend (9.10) to $u^{\gamma=0}(x; t; u_o)$. We first show that the functions $u^{\gamma>0} = u^{\gamma>0}(x, t; u_o)$ are equicontinuous on $\mathbb{R}^3 \times [0, \infty)$. With $T \geq 0$ and (8.22)

$$|\partial_j u^{\gamma>0}|_{L^\infty \times [0,T]} \leq \sup_{|k'|=1} [\quad]^{k',7} \left(|u^{\gamma>0}|_{L^\infty \times [0,T]}; u_o\right)$$

which with (9.10) is

$$\leq \sup_{|k'|=1} [\quad]^{k',7} \left(|u_o|_{L^\infty}; u_o\right)$$

(9.11)

which is independent of $\gamma$.

As $u^{\gamma>0}(x, t; u_o)$ satisfies the $(7.1)_{k'=0,\gamma>0, u_o}$ pde on $\mathbb{R}^3 \times [0, \infty)$

$$|\partial_t u^{\gamma>0}|_{L^\infty \times [0,T]} \leq \mathcal{P} \left[ \sum_j J_{\gamma>0}(u^{\gamma>0}_j) \partial_j u^{\gamma>0} \right]_{L^\infty \times [0,T]} + |\Delta u^{\gamma>0}|_{L^\infty \times [0,T]}$$

which with (2.14) is

$$\leq [\quad] \mathcal{P} \left[ \sum_j J_{\gamma>0}(u^{\gamma>0}_j) \partial_j u^{\gamma>0} \right]_{H^6 \times [0,T]} + [\quad] |u^{\gamma>0}|_{H^2 \times [0,T]}$$

As $u^{\gamma>0}$ is $\in \cap_m [H^{m,df} \times [0, \infty))$, Theorem 8.2, the argument of $\mathcal{P}$ above is $\in \cap_m H^m$, and as $\mathcal{P}$ does not increase an $H^6$ norm, (4.37)

$$|\partial_t u^{\gamma>0}|_{L^\infty \times [0,T]} \leq [\quad] \mathcal{P} \left[ \sum_j J_{\gamma>0}(u^{\gamma>0}_j) \partial_j u^{\gamma>0} \right]_{H^6 \times [0,T]} + [\quad] |u^{\gamma>0}|_{H^7 \times [0,T]}$$
which with (2.22) is
\[
\leq [ \ldots ] |u_j^{\gamma > 0}|_{H^6 \times [0, T]} |\partial_j u^{\gamma > 0}|_{H^6 \times [0, T]} + [ \ldots ] |u^{\gamma > 0}|_{H^7 \times [0, T]}
\]
\[
\leq [ \ldots ] |u_j^{\gamma > 0}|_{H^7 \times [0, T]}^2 + [ \ldots ] |u^{\gamma > 0}|_{H^7 \times [0, T]}
\]
(9.10) and (8.22) then show that
\[
|\partial_t u^{\gamma > 0}|_{L^\infty \times [0, \infty)}
\]
\[
\leq [ \ldots ] \left[ [ \ldots ]^m_{8, 18} \left( |u_o|_{L^\infty}^{1/2}; u_o \right)^2 + [ \ldots ]^m_{8, 18} \left( |u_o|_{L^\infty}^{1/2}; u_o \right) \right]
\]
x which is independent of \( \gamma \). Our proof that the functions \( u^{\gamma > 0} \) are equicontinuous on \( \mathbb{R}^3 \times [0, \infty) \) is complete.

As each \( |u^{\gamma > 0}|_{L^\infty \times [0, \infty)} = |u_o|_{L^\infty} \), (9.10), the Arzela-Ascoli theorem, [4] C.8, shows that the functions \( u^{\gamma > 0} \) are pre-compact in the \( L^\infty \) norm on each compact set \( \Omega_N \).

Cantor’s diagonalization argument then shows that there is a sequence of functions \( u^\gamma_n \in \cap_m H^{m,d,f} \), each of which is continuous on \( \mathbb{R}^3 \times [0, \infty) \), Lemma 2.3, which converges as \( n \to \infty \) and \( \gamma_n \to 0^+ \) in the \( L^\infty \) norm on each \( \Omega_N \) to a function \( u^* \), which is continuous on \( \mathbb{R}^3 \times [0, \infty) \), and satisfies
\[
|u^*|_{L^\infty \times [0, \infty)} \leq |u_o|_{L^\infty}
\]

As each \( u^\gamma_n \) is a solution of the (6.1) \( k' = 0, \gamma_n, u_o \) integral equation on \( \mathbb{R}^3 \times [0, \infty) \)
\[
u^\gamma_n(x, T)
\]
\[
= \int_0^T \int_{\mathbb{R}^3} \left( \sum_j \hat{P} \left[ \partial_j K^i(T - \eta) \right] (y), J_{\gamma_n}(u^\gamma_n) u^\gamma_n(x - y, \eta) \right)_2 \, dy \, d\eta
\]
\[
+ \int_{\mathbb{R}^3} K(y, T) [u_o]_i(x - y) \, dy
\]
(9.12)
on \( \mathbb{R}^3 \times [0, \infty) \). The preceding paragraph shows that the left side of (9.12) \( \to u^*_i \) as \( \gamma_n \to 0^+ \).

We now fix \( T \in [0, \infty) \). As (3.6) and (9.10) (\( \gamma_n > 0 \) show that
\[
|J_{\gamma_n}(u^\gamma_n)|_{L^\infty \times [0, T]} \leq |u^\gamma_n|_{L^\infty \times [0, T]} \leq |u_o|_{L^\infty}
\]
the absolute value of the integrand in the (9.12) double integral is
\[
\leq \sum_{j} |\hat{P}[\partial_j K^i(T-\eta)](y)|_1 |u_o|^2_{L^\infty}
\]

The function \(\hat{P}[\partial_j K^i(T-\eta)](y)\) is when \(T\) is fixed \(\in L^1\) on \(\mathbb{R}^3 \times [0, T]\). Lemma 5.2.

As \(m_\gamma(y)\) vanishes outside of \(B(\gamma)\), [4] Appendix C.5, and \(u_j^\gamma\) is equicontinuous on \(\mathbb{R}^3 \times [0, \infty)\)
\[
|J_\gamma(u_j^\gamma)(x, \eta) - u_j^\gamma(x, \eta)|_{L^\infty \times [0, T]} \leq [\ldots] \gamma_n
\]
which shows that
\[
J_\gamma(u_j^\gamma)(x - y, \eta) \to u^\gamma(x - y, \eta)
\]
as \(\gamma_n \to 0^+\) on \(\mathbb{R}^3 \times [0, \infty)\). Thus, the integrand in the (9.12) double integral converges as \(\gamma_n \to 0^+\) a.e. on \(\mathbb{R}^3 \times [0, T]\) to
\[
\left( \sum_{j} \hat{P}[\partial_j K^i(T-\eta)](y), u_j^\gamma u^*(x - y, \eta) \right)_2
\]
The dominated convergence theorem then shows that (9.12) converges as \(\gamma_n \to 0^+\) to
\[
u_i^\gamma(x, T) = \int_0^T \int_{\mathbb{R}^3} \left( \sum_{j} \hat{P}[\partial_j K^i(T-\eta)](y), u_j^\gamma u^*(x - y, \eta) \right)_2 dyd\eta
\]
\[
+ \int_{\mathbb{R}^3} K(y, T) [u_o]_i(x - y) dy
\]
(9.13)
when \(x \in \mathbb{R}^3\).

(9.13) is the \(i^{th}\) component of the \((6.1)_{k'=0, \gamma, u_o}\) integral equation which \(u^\gamma=0\) satisfies on \(\mathbb{R}^3 \times [0, T^\gamma=0_{\text{bup}}(u_o))\). Theorem 8.2. As \(u^\gamma=0\) and \(u^*\) are in \(L^\infty\) on each \(\mathbb{R}^3 \times [0, T]\) and are continuous thereon, Lemma 6.1 shows that
\[
u^\gamma=0 = u^* \text{ on } \mathbb{R}^3 \times [0, T]
\]
when \(T < T^\gamma=0_{\text{bup}}(u_o)\) which shows that
\[
|u^\gamma=0|_{L^\infty \times [0, T^\gamma=0_{\text{bup}}(u_o))} = |u_o|_{L^\infty}
\]
Thus, $T_{\text{bup}}^{\gamma=0}(u_o) = \infty$, (8.21), and
\[ |u^{\gamma=0}|_{L^\infty \times [0, \infty)} = |u_o|_{L^\infty} \]
The proof of (9.9) for $\gamma \geq 0$ is complete.

We now show by contradiction that $u^{\gamma>0}$ converges as $\gamma \to 0^+$ to $u^{\gamma=0}$ on each $\Omega_N$. If $u^{\gamma>0}$ does not so converge, there is a subsequence of $u^{\gamma}$ that is bounded away from $u^{\gamma=0}$ in the $L^\infty$ norm on an $\Omega_N$. The proof above shows that a subsequence of that non-converging sequence converges in the $L^\infty$ norm on that $\Omega_N$ to $u^{\gamma=0}$. The contradiction completes the proof.

\[ \square \]

10. \hspace{1cm} \textit{u^{\gamma=0} and } p^{\gamma=0} \textit{ are a } C^\infty \textit{ solution of the } (1.1)_{\nu=1, u_o} \textit{ pde}

This section first shows that $u^{\gamma=0}(x, t; u_o)$ and a related scalar valued $p^{\gamma=0}(x, t; u_o)$ are $\in C^\infty$ on $\mathcal{R}^3 \times [0, \infty)$ and then shows that $u^{\gamma=0}$ and $p^{\gamma=0}$ satisfy the $(1.1)_{\nu=1, u_o}$ pde on $\mathcal{R}^3 \times [0, \infty)$.

The next three lemmas are infrastructure for the proof thereafter that $u^{\gamma=0}$ and $p^{\gamma=0}$ are $\in C^\infty$ on $\mathcal{R}^3 \times [0, \infty)$. In this section, a statement that a derivative is $\in \cap_m[H^{m,df} \times [0, \infty)]$ is to be read to include the statement that the derivative exists on $\mathcal{R}^3 \times [0, \infty)$.

**Lemma 10.1.** Let the functions $v_{\text{left}}$ and $v_{\text{right}}$ be $\in \cap_m[H^m \times [0, \infty)]$. Then
\[ v_{\text{left}} v_{\text{right}} \text{ is } \in \cap_m[H^m \times [0, \infty)] \]
(10.1)
and
\[ \int_{\mathcal{R}^3} \partial_k |z|^{-1} v_{\text{left}} v_{\text{right}} (x - z, t) \, dz \]
\[ \text{is } \in \cap_m[H^m \times [0, \infty)] \]
(10.2)

**Proof.** We first establish (10.1). As $v_{\text{left}}$ and $v_{\text{right}}$ are $\in \cap_m[H^m \times [0, \infty)]$, each function $v_{\text{left}}(t)$ $v_{\text{right}}(t)$ is $\in \cap_m H^m$ when $t$ is $\geq 0$. (2.22).
We complete the proof of (10.1) by showing that \( v_{\text{left}}(t) \) \( v_{\text{right}}(t) \) is a function of \( t \in [0, \infty) \) which is continuous in each \( H^m \) norm. With the definition of the \( H^m \) norm

\[
\left| v_{\text{left}} v_{\text{right}}(t + \Delta t) - v_{\text{left}} v_{\text{right}}(t) \right|_{H^m}
= \sum_{0 \leq |k'| \leq m} \left| D^{k'} v_{\text{left}} v_{\text{right}}(t + \Delta t) - D^{k'} v_{\text{left}} v_{\text{right}}(t) \right|_{L^2}
\]

(10.3)

The (2.17) expansion of \( D^{k'} [v_{\text{left}}(t) v_{\text{right}}(t)] \) shows that the \( L^2 \) norm of the (10.3) summand which is indexed by \( k' \) is

\[
\leq \sum_{\alpha' + \beta' = k'} c(\alpha', \beta', k') \left| D^{\alpha'} v_{\text{left}}(t + \Delta t) \left[ D^{\beta'} v_{\text{right}}(t + \Delta t) - D^{\beta'} v_{\text{right}}(t) \right] D^{\alpha'} v_{\text{left}}(t) D^{\beta'} v_{\text{right}}(t) \right|_{L^2}
\]

As \( v_{\text{left}} \) and \( v_{\text{right}} \) are \( \in \cap_m [H^m \times [0, \infty)] \), we use

\[
v_{\text{left}}' := D^{\alpha'} v_{\text{left}} \text{ and } v_{\text{right}}' := D^{\beta'} v_{\text{right}}
\]

which are also \( \in \cap_m [H^m \times [0, \infty)] \). Customary mathematics then show that

\[
v_{\text{left}}' v_{\text{right}}' (x, t + \Delta t) - v_{\text{left}}' v_{\text{right}}' (x, t) = v_{\text{left}}' (x, t + \Delta t) v_{\text{right}}' (x, t + \Delta t) - v_{\text{left}}' (x, t) v_{\text{right}}' (x, t)
+ \left[ v_{\text{left}}' (x, t + \Delta t) - v_{\text{left}}' (x, t) \right] v_{\text{right}}' (x, t)
\]

(10.4)

when \( 0 \leq t < t + \Delta t < T < \infty \).

Therewith, the \( L^2 \) norm of the right side of (10.4) is

\[
\leq \left| v_{\text{left}}'(t + \Delta t) \right|_{L^\infty} \left| v_{\text{right}}'(t + \Delta t) - v_{\text{right}}'(t) \right|_{L^2}
+ \left| v_{\text{left}}'(t + \Delta t) - v_{\text{left}}'(t) \right|_{L^2} \left| v_{\text{right}}' \right|_{L^\infty}
\]

which as each of \( v_{\text{left}}' \) and \( v_{\text{right}}' \) is a function of \( t \in [0, \infty) \) which is continuous in each \( H^m \) norm is with (2.14)

\[
\leq \left[ \ldots \right] \left| v_{\text{left}}' \right|_{H^2 \times [0, T]} \left| v_{\text{right}}'(t + \Delta t) - v_{\text{right}}'(t) \right|_{L^2}
+ \left| v_{\text{left}}'(t + \Delta t) - v_{\text{left}}'(t) \right|_{L^2} \left| v_{\text{right}}' \right|_{H^2 \times [0, T]} \right]
\]
which shows when $0 \leq t_1 < t_2 < T < \infty$ that

$$\leq \left[ \ldots \right] \left[ |v_\text{left}'|_{H^2 \times [0,T]} |v_\text{right}'(t_2) - v_\text{right}'(t_1)|_{L^2} + |v_\text{left}'(t_2) - v_\text{left}'(t_1)|_{L^2} |v_\text{right}'|_{H^2 \times [0,T]} \right]$$

which $\to 0$ as $t_2 - t_1 \to 0$ and $T$ is fixed. The proof of (10.1) is complete.

We now establish (10.2). As $v_\text{left}$ and $v_\text{right}$ are $\in \cap_m[H^m \times [0, \infty)]$, (2.20) and (2.22) show when $t \geq 0$ that

$$v_\text{left}(t) v_\text{right}(t) \in \cap_m[H^m \cap W^{m,1}]$$

that

$$D^{k'} \left[ \int_{\mathbb{R}^3} \partial_k|z|^{-1} v_\text{left} v_\text{right} (x - z, t) \, dz \right]$$

is $= \int_{\mathbb{R}^3} \partial_k|z|^{-1} D^{k'} [v_\text{left} v_\text{right}] (x - z, t) \, dz$

for each multi-index $k'$, which the (2.17) expansion of $D^{k'} [v_\text{left} v_\text{right}]$ shows is

$$= \sum_{\alpha'+\beta'=k'} c(\alpha', \beta', k') \int_{\mathbb{R}^3} \partial_k|z|^{-1} D^{\alpha'} v_\text{left} D^{\beta'} v_\text{right} (x - z, t) \, dz$$

which is $\in \cap_m H^m$, (3.18)

(10.5)

We complete the proof of (10.2) by showing that each (10.5) integral is a function of $t \in [0, \infty)$ which is continuous in each $H^m$ norm.

With $v_\text{left}'$ and $v_\text{right}'$ as before, each integral in the right side of (10.5) is equal to

$$\int_{\mathbb{R}^3} \partial_k|z|^{-1} v_\text{left}' v_\text{right}'(x - z, t) \, dz$$

Then with $0 \leq t < t + \Delta t < T < \infty$

$$\int_{\mathbb{R}^3} \partial_k|z|^{-1} v_\text{left}' v_\text{right}' (x - z, t + \Delta t) \, dz$$

$$- \int_{\mathbb{R}^3} \partial_k|z|^{-1} v_\text{left}' v_\text{right}' (x - z, t) \, dz$$

(10.6)
is
\[
\int_{R^3} \partial_k |z|^{-1} \left[ v'_\text{left}(x - z, t + \Delta t) \left[v'_\text{right}(x - z, t + \Delta t) - v'_\text{right}(x - z, t)\right]
+ \left[v'_\text{left}(x - z, t + \Delta t) - v'_\text{left}(x - z, t)\right] v'_\text{right}(x - z, t) \right] dz
\]
the \(L^2\) norm of which, (3.17) shows, is
\[
\leq \left[ \ldots \right]_{3.34b} \left| v'_\text{left}(x - z, t + \Delta t) \left[v'_\text{right}(x - z, t + \Delta t) - v'_\text{right}(x - z, t)\right] \right|_{L^2}
+ \left| v'_\text{left}(x - z, t + \Delta t) - v'_\text{left}(x - z, t) \right|_{L^2} \left| v'_\text{right}(x - z, t) \right|_{L^2}
\]
which with customary mathematics and a Hölder inequality is
\[
\leq \left[ \ldots \right] \left[ \left| v'_\text{left}(t + \Delta t) \right|_{L^\infty} \left| v'_\text{right}(t + \Delta t) - v'_\text{right}(t) \right|_{L^2}
+ \left| v'_\text{left}(t + \Delta t) - v'_\text{left}(t) \right|_{L^2} \left| v'_\text{right}(t) \right|_{L^\infty}
+ \left| v'_\text{left}(t + \Delta t) \right|_{L^2} \left| v'_\text{right}(t + \Delta t) - v'_\text{right}(t) \right|_{L^2}
+ \left| v'_\text{left}(t + \Delta t) - v'_\text{left}(t) \right|_{L^2} \left| v'_\text{right}(t) \right|_{L^2} \right]
\]
which as \(v'_\text{left}\) and \(v'_\text{right}\) are each a function of \(t \in [0, \infty)\) which is continuous in each \(H^m\) norm is
\[
\leq \left[ \ldots \right] \left[ \left| v'_\text{left} \right|_{H^2 \times [0,T]} \left| v'_\text{right}(t + \Delta t) - v'_\text{right}(t) \right|_{L^2}
+ \left| v'_\text{left}(t + \Delta t) - v'_\text{left}(t) \right|_{L^2} \left| v'_\text{right} \right|_{H^2[0,T]}
+ \left| v'_\text{left} \right|_{L^2 \times [0,T]} \left| v'_\text{right}(t + \Delta t) - v'_\text{right}(t) \right|_{L^2}
+ \left| v'_\text{left}(t + \Delta t) - v'_\text{left}(t) \right|_{L^2} \left| v'_\text{right} \right|_{L^2 \times [0,T]} \right]
\]
which shows when \(0 \leq t_1 < t_2 < T < \infty\) that
\[
\leq \left[ \ldots \right] \left[ \left| v'_\text{left} \right|_{H^2 \times [0,T]} \left| v'_\text{right}(t_2) - v'_\text{right}(t_1) \right|_{L^2}
+ \left| v'_\text{left}(t_2) - v'_\text{left}(t_1) \right|_{L^2} \left| v'_\text{right} \right|_{H^2[0,T]}
+ \left| v'_\text{left} \right|_{L^2 \times [0,T]} \left| v'_\text{right}(t_2) - v'_\text{right}(t_1) \right|_{L^2}
+ \left| v'_\text{left}(t_2) - v'_\text{left}(t_1) \right|_{L^2} \left| v'_\text{right} \right|_{L^2 \times [0,T]} \right]
\]
which shows that the $L^2$ norm of (10.6) $\to 0$ as $t_2 - t_1 \to 0$ and $t_1$ is fixed.

The proof of (10.2) is complete. $\square$

**Lemma 10.2.** Let the function $v$ and its derivative $\partial_t v$ be $\in \cap_m [H^m \times [0, \infty)]$. Then

$$\partial_t \mathcal{P}[v(t)](x) \text{ exists on } \mathcal{R}^3 \times [0, \infty) \text{ and}$$

thereon is $= \mathcal{P}[\partial_t v(t)](x)$

(10.7)

and

$$\mathcal{P}[\partial_t v] \text{ is } \in \cap_m [H^{m,df} \times [0, \infty)]$$

(10.8)

**Proof.** As $\hat{\mathcal{P}}$ is linear on the function space $L^2$, (4.30)

$$\Delta t^{-1} \left[ \mathcal{P}[v(t + \Delta t)] - \mathcal{P}[v(t)](x) \right] = \mathcal{P} \left[ \Delta t^{-1} \left[ v(t + \Delta t) - v(t) \right] \right](x)$$

As $\partial_t v$ is $\in \cap_m [H^m \times [0, \infty)]$, $\partial_t v$ is continuous on $\mathcal{R}^3 \times [0, \infty)$, Lemma 2.3, and the calculus shows that

$$\Delta t^{-1} \left[ \mathcal{P}[v(t + \Delta t)] - \mathcal{P}[v(t)] \right](x)$$

is $= \mathcal{P} \left[ \Delta t^{-1} \int_0^{\Delta t} \left[ \partial_t v(x, t + \theta) - \partial_t v(x, t) \right] d\theta \right] + \mathcal{P}[\partial_t v(t)]$

(10.9)

when $0 \leq t < t + \Delta t < T < \infty$.

Then with (2.14)

$$[\ldots]_{10.21}^{-1} \left\| \mathcal{P} \left[ \Delta t^{-1} \int_0^{\Delta t} \left[ \partial_t v(x, t + \theta) - \partial_t v(x, t) \right] d\theta \right] \right\|_{L^\infty}$$

is $\leq \left\| \mathcal{P} \left[ \Delta t^{-1} \int_0^{\Delta t} \left[ \partial_t v(x, t + \theta) - \partial_t v(x, t) \right] d\theta \right] \right\|_{H^2}$
which with the definition of the $H^2$ norm is
\[
\sum_{0 \leq |k'| \leq 2} \left| D^{k'} \left[ \mathcal{P} \left[ \Delta t^{-1} \int_t^{t+\Delta t} [\partial_t v(x, t + \theta) - \partial_t v(x, t)] d\theta \right] \right] \right|_{L^2}
\]
which as $D^{k'}$ commutes with $\mathcal{P}$ and $\mathcal{P}$ does not increase an $L^2$ norm, (4.37), is
\[
\leq \sum_{0 \leq |k'| \leq 2} \left| \Delta t^{-1} \left[ \int_0^{\Delta t} 1 \cdot [D^{k'}\partial_t v(x, t + \theta) - D^{k'}\partial_t v(x, t)] d\theta \right] \right|_{L^2}
\]
which as each of the derivatives $D^{k'}$ of $\partial_t v$ is a function of $t$ which is uniformly continuous in the $L^2$ norm on compact subsets of $[0, \infty)$, Lemma 2.3, is
\[
\leq \sum_{0 \leq |k'| \leq 2} \left| \Delta t^{-1/2} \left[ \int_0^{\Delta t} [D^{k'}\partial_t v(x, t + \theta) - D^{k'}\partial_t v(x, t)]^2 d\theta \right]^{1/2} \right|_{L^2}
\]
which with a Hölder inequality in the interior integral is
\[
\leq \sum_{0 \leq |k'| \leq 2} \left| \Delta t^{-1/2} \left[ \int_0^{\Delta t} [D^{k'}\partial_t v(x, t + \theta) - D^{k'}\partial_t v(x, t)]^2 d\theta \right]^{1/2} \right|_{L^2}
\]
which with the definition of the $L^2$ norm is
\[
= \sum_{0 \leq |k'| \leq 2} \Delta t^{-1/2} \left[ \int_{R^3} \left[ \int_0^{\Delta t} [D^{k'}\partial_t v(x, t + \theta) - D^{k'}\partial_t v(x, t)]^2 d\theta \right]^{1/2} \right]^2 \left| dx \right|^{1/2}
\]
which with Fubini’s theorem is
\[
= \sum_{0 \leq |k'| \leq 2} \Delta t^{-1/2} \left[ \int_0^{\Delta t} \left[ \int_{R^3} [D^{k'}\partial_t v(x, t + \theta) - D^{k'}\partial_t v(x, t)]^2 dx \right] d\theta \right]^{1/2}
\]
\[
\leq \sup_{0 \leq \theta \leq \Delta t} \left| \partial_t v(t + \theta) - \partial_t v(t) \right|_{L^2}^{1/2}
\]
\[
\leq \sup_{0 \leq \theta \leq \Delta t} \left| \partial_t v(t + \theta) - \partial_t v(t) \right|_{H^2}^{1/2}
\]
Thus, the analysis which begins at (10.9) shows that

\[ \Delta t^{-1} \left[ \mathcal{P}[v(t + \Delta t)] - \mathcal{P}[v(t)] \right] \right\|_{L^\infty} \leq \left( \ldots \right) \sup_{0 \leq \theta \leq \Delta t} \left| \partial_t v(t + \theta) - \partial_t v(t) \right|_{H^2} \]

which shows when \( 0 \leq t_1 < t_2 < T < \infty \) that

\[ \left[ t_2 - t_1 \right]^{-1} \left[ \mathcal{P}[v(t_2)] - \mathcal{P}[v(t_1)] \right] \right\|_{L^\infty} \leq \left( \ldots \right) \sup_{0 \leq \theta \leq [t_2 - t_1]} \left| \partial_t v(t_1 + \theta) - \partial_t v(t_1) \right|_{H^2} \]

which as \( \partial_t v \) is uniformly continuous in \([0, T] \) in the \( H^2 \) norm shows that

\[ \lim_{\Delta t \to 0} \Delta t^{-1} \left[ \mathcal{P}[v(t + \Delta t)] - \mathcal{P}[v(t)] \right](x) = \mathcal{P}[\partial_t v](x) \]

when \((x, t) \in \mathcal{R}^3 \times [0, \infty)\). The proof of (10.7) is complete.

As \( \partial_t v(t) \) is \( \in \cap_m[H^{m,d} \times [0, \infty)] \), (4.38) establishes (10.8). \( \square \)

**Lemma 10.3.** Let the functions \( v_{\text{left}} \) and \( v_{\text{right}} \) be \( \in \cap_m[H^{m} \times [0, \infty)] \) and the derivatives \( \partial_t v_{\text{left}} \) and \( \partial_t v_{\text{right}} \) be \( \in \cap_m[H^{m} \times [0, \infty)] \). Then

\[
\partial_t \left[ \int_{\mathcal{R}^3} \partial_k |z|^{-1} v_{\text{left}} v_{\text{right}} (x - z, t) \, dz \right] = \int_{\mathcal{R}^3} \partial_k |z|^{-1} \partial_t [v_{\text{left}} v_{\text{right}}] (x - z, t) \, dz \\
\text{and is } \in \cap_m[H^{m} \times [0, \infty)]
\]

(10.10)

**Proof.** As \( v_{\text{left}}, v_{\text{right}}, \partial_t v_{\text{left}} \) and \( \partial_t v_{\text{right}} \) are \( \in \cap_m[H^{m} \times [0, \infty)] \), each is continuous on \( \mathcal{R}^3 \times [0, \infty) \). Lemma 2.3. With customary mathematics, the derivative

\[
\partial_t [v_{\text{left}} v_{\text{right}}] \text{ is } = v_{\text{left}} \partial_t v_{\text{right}} + [\partial_t v_{\text{left}}] v_{\text{right}}
\]

(10.1) shows that such derivative is \( \in \cap_m[H^{m} \times [0, \infty)] \), and as a result it is continuous on \( \mathcal{R}^3 \times [0, \infty) \). Lemma 2.3.
We now study the function
\[ \ldots ]_{10.11}(t; \Delta t)
\[ := \Delta t^{-1} \left[ \int_{R^3} \partial_k |z|^{-1} \left[ v_{\text{left}} v_{\text{right}} (x - z, t + \Delta t) - v_{\text{left}} v_{\text{right}} (x - z, t) \right] \, dz \right] \tag{10.11} \]

when \( 0 \leq t < t + \Delta t < T < \infty \). With the calculus
\[ \ldots ]_{10.11}(t; \Delta t)
\[ = \Delta t^{-1} \int_{R^3} \partial_k |z|^{-1} \left[ \int_0^{\Delta t} \left[ \partial_t [v_{\text{left}} v_{\text{right}}] (x - z, t + \theta) \, d\theta \right] \right] \, dz \]

which with Fubini’s theorem is
\[ = \Delta t^{-1} \int_0^{\Delta t} \left[ \int_{R^3} \partial_k |z|^{-1} \partial_t [v_{\text{left}} v_{\text{right}}] (x - z, t + \theta) \, dz \right] \, d\theta \]

which after adding and subtracting the same number thereto is
\[ = \Delta t^{-1} \int_0^{\Delta t} \left[ \int_{R^3} \partial_k |z|^{-1} \left[ \partial_t [v_{\text{left}} v_{\text{right}}] (x - z, t + \theta) \right. \right.
\[ \left. \left. - \partial_t [v_{\text{left}} v_{\text{right}}] (x - z, t) \right] \, dz \right] \, d\theta \]
\[ + \int_{R^3} \partial_k |z|^{-1} \partial_t [v_{\text{left}} v_{\text{right}}] (x - z, t) \, dz \] \tag{10.12}

The absolute value of the first (10.12) summand is
\[ \leq \sup_{0 \leq \theta \leq \Delta t} \left| \int_{B(1)} \partial_k |z|^{-1} \left[ \partial_t [v_{\text{left}} v_{\text{right}}] (x - z, t + \theta) \right. \right.
\[ \left. \left. - \partial_t [v_{\text{left}} v_{\text{right}}] (x - z, t) \right] \, dz \right| \]
\[ + \sup_{0 \leq \theta \leq \Delta t} \left| \int_{R^3 \setminus B(1)} \partial_k |z|^{-1} \left[ \partial_t [v_{\text{left}} v_{\text{right}}] (x - z, t + \theta) \right. \right.
\[ \left. \left. - \partial_t [v_{\text{left}} v_{\text{right}}] (x - z, t) \right] \, dz \right| \]
which with (2.5) in the first summand and a Hölder inequality in the second summand is

\[
\leq \left[ \cdots \right]_{2.34b} \cdot 1 \cdot \sup_{0 \leq \theta \leq \Delta t} \left| \partial_t \left[ v_{\text{left}} v_{\text{right}} \right](x - z, t + \theta) \right| - \partial_t \left[ v_{\text{left}} v_{\text{right}} \right](x - z, t) \left| L^\infty(z \in B(r)) \right.
\]

\[
+ \left[ \cdots \right]_{2.32c} \cdot 1^{1/2} \cdot \sup_{0 \leq \theta \leq \Delta t} \left| \partial_t \left[ v_{\text{left}} v_{\text{right}} \right](x - z, t + \theta) \right| - \partial_t \left[ v_{\text{left}} v_{\text{right}} \right](x - z, t) \left| L^2(z \in \mathbb{R}^3) \right.
\]

When \(0 \leq t_1 < t_2 < T < \infty\), (10.11) and the foregoing show that

\[
\left| t_2 - t_1 \right|^{-1} \left[ \int_{\mathbb{R}^3} \partial_k |z|^{-1} \left[ v_{\text{left}} v_{\text{right}}(x - z, t_2) - v_{\text{left}} v_{\text{right}}(x - z, t_1) \right] dz \right] \left| L^\infty \right.
\]

\[
\leq \left[ \cdots \right] \sup_{0 < \theta \leq t_2 - t_1} \left| \partial_t \left[ v_{\text{left}} v_{\text{right}} \right](x - z, t + \theta) - \partial_t \left[ v_{\text{left}} v_{\text{right}} \right](x - z, t) \right| \left| H^2 \right.
\]

Allowing \(t_2 \rightarrow t_1\) completes the proof of (10.10).

The expansion

\[
\partial_t \left[ v_{\text{left}} v_{\text{right}} \right] = v_{\text{left}} \partial_t v_{\text{right}} + [\partial_t v_{\text{left}}] v_{\text{right}}
\]

and (10.1) show that the right side of (10.10) is \(\in \bigcap_m [H^m \times [0, \infty)]\). \(\square\)

We now show that \(u^{\gamma=0}\) is \(\in C^\infty\) on \(\mathcal{R}^3 \times [0, \infty)\). Herein, \(\partial^n D^{k'} u^{\gamma=0}\) is any operator which is equal to a rearrangement of the factors of \(\partial^n D^{k'}\).

**Lemma 10.4.** Let the integer \(n\) be \(\geq 0\) and \(k'\) be a multi-index. Then each derivative \(\partial^n D^{k'} u^{\gamma=0}\) exists on \(\mathcal{R}^3 \times [0, \infty)\) and is \(\in \bigcap_m [H^{m,df} \times [0, \infty)]\).

**Proof.** Theorems 8.2 and 9.2 show that

\[
D^{k'} u^{\gamma=0} \in \bigcap_m [H^{m,df} \times [0, \infty)]
\]

(10.13)
We let the induction hypothesis for an integer \( n \geq 0 \) be that for each multi-index \( k' \) the derivative
\[
\partial_{i}^{n} D^{k'} \left[ \sum_{j} u_{j}^{\gamma=0} \partial_{j} u^{\gamma=0} \right](x, t)
\]
is \( \in \cap_{m} [H^{m,df} \times [0, \infty)] \) and
\[
is = \left[ \sum_{i} [ ... ] \partial_{i}^{\leq n}_{i,j} D^{k',t} u_{j(i)}^{\gamma=0} \partial_{j(i)} u^{\gamma=0} \right](x, t)
on R^{3} \times [0, \infty)
\]
(10.14)

and that for each integer \( n' \) for which \( 0 \leq n' \leq n \) the derivative
\[
\partial_{i}^{n'+1} D^{k'} u^{\gamma=0}
\]
is \( \in \cap_{m} [H^{m,df} \times [0, \infty)] \)
and satisfies the pde
\[
\partial_{i}^{n'+1} D^{k'} u^{\gamma=0} (x, t) = \hat{P} \left[ \partial_{i}^{n'} D^{k'} \left[ \sum_{j} u_{j}^{\gamma=0}(t) \partial_{j} u^{\gamma=0}(t) \right] \right] (x) + \partial_{i}^{n'} D^{k'} \Delta u^{\gamma=0} (x, t)
on R^{3} \times [0, \infty)
\]
(10.15)

We now establish (10.14) for \( n = 0 \). As Theorems 8.2 and 9.2 show that \( u^{\gamma=0} \) is \( \in \cap_{m} [H^{m,df} \times [0, \infty)] \), (10.1) shows that
\[
\sum_{j} u_{j}^{\gamma=0} \partial_{j} u^{\gamma=0} \text{ is } \in \cap_{m} [H^{m} \times [0, \infty)]
\]
(10.16)
and (2.17) shows that
\[
D^{k'} \left[ \sum_{j} u_{j}^{\gamma=0} \partial_{j} u^{\gamma=0} \right](x, t)
\]
\[
= \left[ \sum_{i} [ ... ] \partial_{i}^{\leq n}_{i,j} D^{k',t} u_{j(i)}^{\gamma=0} \partial_{j(i)} u^{\gamma=0} \right](x, t)
on R^{3} \times [0, \infty)
\]
(10.17)
The proof of (10.14) for \( n = 0 \) is complete.
We now establish (10.15) for \( n = 0 \). For each operator \( \partial_t D^{k'} \) there are multi-indices \( k'_1 \) and \( k'_2 \) so that

\[
\partial_t D^{k'} = D^{k'_1} \partial_t D^{k'_2}
\]

Theorems 8.2 and 9.2 then show that

\[
\partial_t D^{k'_2} u^{\gamma=0} \in \cap_m [H^{m,df} \times [0, \infty)]
\]

and satisfies the pde

\[
\partial_t D^{k'_2} u^{\gamma=0} (x, t) = \hat{P} \left[ D^{k'_2} \left( \sum_j u^{\gamma=0}_j(t) \partial_j u^{\gamma=0}(t) \right) \right] (x) + D^{k'_2} \Delta u^{\gamma=0} (x, t)
\]

on \( \mathcal{R}^3 \times [0, \infty) \)

(10.17)

As (10.16) shows that the argument of \( \hat{P} \) in (10.17) is \( \in \cap_m [H^m \times [0, \infty)] \), (4.38) shows that the first summand in the right side of (10.17) is \( \in \cap_m [H^{m,df} \times [0, \infty)] \). (10.13) shows that the second summand in the right side of (10.17) is \( \in \cap_m [H^{m,df} \times [0, \infty)] \).

As each summand in the right side of (10.17) is \( \in \cap_m [H^{m,df} \times [0, \infty)] \), the application of \( D^{k'_1} \) to each side of (10.17) shows that

\[
D^{k'_1} \partial_t D^{k'_2} u^{\gamma=0} \in \cap_m [H^{m,df} \times [0, \infty)]
\]

and with (4.36) that

\[
\partial_t D^{k'} u^{\gamma=0} (x, t) = \hat{P} \left[ D^{k'} \left( \sum_j u^{\gamma=0}_j(t) \partial_j u^{\gamma=0}(t) \right) \right] (x) + D^{k'} \Delta u^{\gamma=0} (x, t)
\]

on \( \mathcal{R}^3 \times [0, \infty) \)

The proof of (10.15) for \( n = 0 \) is complete, and the proof of the induction hypothesis for \( n = 0 \) is complete.

We now show that the induction hypothesis for \( n \geq 0 \) implies (10.14) of the induction hypothesis for \( n + 1 \). With \( k'_2 \) a multi-index, (10.14)
of the induction hypothesis shows that
\[
\frac{\partial_t^n D^k}{\partial x} \left[ \sum_j u_j^{\gamma=0} \partial_j u^{\gamma=0} \right](x, t)
\]
is \(\in \cap_m[H^m \times [0, \infty)]\) and
\[
is = \left[ \sum_i [ ... ] \right] \quad_i \frac{\partial_t^{|\leq n|, l}}{\partial x} D^{k_1, l} u_j^{\gamma=0} \quad_i \frac{\partial_t^{|\leq n|, r}}{\partial x} D^{k_1, r} \partial_j u^{\gamma=0}(x, t)
\]
(10.18)

As (10.13), (10.14) of the induction hypothesis and (10.15) of the induction hypothesis show that each multiplicand and the \(\partial_t\) derivative of each multiplicand in the right side of (10.18) is \(\in \cap_m[H^{m, df} \times [0, \infty)]\), customary mathematics show that the \(\partial_t\) derivative of the right side of (10.18) is
\[
= \sum_{i'} [ ... ] i' \left[ \frac{\partial_t^{|\leq n+1|, l}}{\partial x} D^{k_1, l} u_j^{\gamma=0}(t) \quad_i \frac{\partial_t^{|\leq n+1|, r}}{\partial x} D^{k_1, r} \partial_j u^{\gamma=0}(t) \right]
\]
(10.19)

and as each multiplicand in (10.19) is \(\in \cap_m[H^{m, df} \times [0, \infty)]\), (10.1) shows that
\[
\partial_t \frac{\partial_t^n D^k}{\partial x} \left[ \sum_j u_j^{\gamma=0}(t) \partial_j u^{\gamma=0}(t) \right] \quad_i \in \cap_m[H^m \times [0, \infty)]
\]
\[
= \sum_{i'} [ ... ] i' \left[ \frac{\partial_t^{|\leq n+1|, l}}{\partial x} D^{k_1, l} u_j^{\gamma=0}(t) \quad_i \frac{\partial_t^{|\leq n+1|, r}}{\partial x} D^{k_1, r} \partial_j u^{\gamma=0}(t) \right]
\]
(10.20)

The proof of (10.14) of the induction hypothesis for \(n + 1\) is complete.

We now show that the induction hypothesis for \(n \geq 0\) implies (10.15) of the induction hypothesis for \(n + 1\). For each operator \(\partial_t^{n+2} D^k\) there are multi-indices \(k_1'\) and \(k_2'\) so that
\[
\partial_t^{n+2} D^k = D^{k_1} \partial_t^{n+1} D^{k_2}
\]
(10.21)

(10.15) of the induction hypothesis for \(n\) then shows that
\[
\partial_t^{n+1} D^{k_2} u^{\gamma=0} \quad_i \in \cap_m[H^{m, df} \times [0, \infty)]
\]
and satisfies the pde

\[
\begin{align*}
\frac{\partial_{t}^{n+1} D^{k_2} u^{\gamma=0}}{n+1} (x, t) &= \widehat{P} \left[ \partial_{t}^{n} D^{k_2} \left[ \sum_{j} u_{j}^{\gamma=0}(t) \partial_{j} u^{\gamma=0}(t) \right] \right](x) + \partial_{t}^{n} D^{k_2} \Delta u^{\gamma=0} (x, t) \\
& \text{on } \mathcal{R}^{3} \times [0, \infty) \\
\end{align*}
\]

(10.22)

(10.20) shows that

\[
\partial_{t} \left[ \widehat{P} \left[ \partial_{t}^{n} D^{k_2} \left[ \sum_{j} u_{j}^{\gamma=0}(t) \partial_{j} u^{\gamma=0}(t) \right] \right](x) \right] \\
\text{is } = \widehat{P} \left[ \partial_{t} \partial_{t}^{n} D^{k_2} \left[ \sum_{j} u_{j}^{\gamma=0}(t) \partial_{j} u^{\gamma=0}(t) \right] \right](x) \\
\text{which with (4.38) is } \in \cap_{m}[H^{m,df} \times [0, \infty)]
\]

(10.15) of the induction hypothesis for n also shows that the \( \partial_{t} \) derivative of the second summand in the right side of (10.22) is \( \in \cap_{m}[H^{m,df} \times [0, \infty)] \).

As each summand in the right side of (10.22) is \( \in \cap_{m}[H^{m,df} \times [0, \infty)] \), the \( \partial_{t} \) derivative of the left side of (10.22) which is

\[
\begin{align*}
\frac{\partial_{t}^{n+2} D^{k_2} u^{\gamma=0}}{n+2} (x, t) \text{ is } \in \cap_{m}[H^{m,df} \times [0, \infty)] \\
\text{(10.23)}
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial_{t}^{n+2} D^{k_2} u^{\gamma=0}}{n+2} (x, t) &= \widehat{P} \left[ \partial_{t}^{n+1} D^{k_2} \left[ \sum_{j} u_{j}^{\gamma=0}(t) \partial_{j} u^{\gamma=0}(t) \right] \right](x) + \partial_{t}^{n+1} D^{k_2} \Delta u^{\gamma=0} (x, t) \\
\text{(10.24)}
\end{align*}
\]

The proof of (10.15) for \( n+1 \) is complete. The proof that the induction hypothesis for \( n \geq 0 \) implies the induction hypothesis for \( n+1 \) is complete. \( \square \)
We now define the scalar valued function

\[ p_{\gamma=0} (x, t) = (-4\pi)^{-1} \int_{\mathbb{R}^3} \sum_{i,j} \partial_i |z|^{-1} u_j^{\gamma=0} \partial_j u_i^{\gamma=0} (x - z, t) \, dz \]

(10.25)

**Lemma 10.5.** Let the integer \( n \geq 0 \) and \( k' \) be a multi-index. Then each derivative \( \partial^n D^{k'} p_{\gamma=0} \) exists on \( \mathbb{R}^3 \times [0, \infty) \) and is \( \in \cap_m [H^m \times [0, \infty)] \).

**Proof.** The induction hypothesis for \( n \geq 0 \) is that

\[ \partial^n D^{k'} p_{\gamma=0} (x, t) \]

is \( (-4\pi)^{-1} \int_{\mathbb{R}^3} \sum_{i,j} \partial_i |z|^{-1} \partial^n D^{k'} \left[ \sum_j u_j^{\gamma=0} \partial_j u_i^{\gamma=0} \right] (x - z, t) \, dz \]

and is \( \in \cap_m [H^m \times [0, \infty)] \)

(10.26)

for such \( n \) and each multi-index \( k' \).

As \( u_j^{\gamma=0} (t) \) and \( \partial_j u_i^{\gamma=0} (t) \) are \( \in \cap_m [H^m \times [0, \infty)] \), \( \sum_j u_j^{\gamma=0} (t) \partial_j u_i^{\gamma=0} (t) \) is \( \in \cap_m [H^m \cap W^{m,1}] \), and (3.16) shows for each multi-index \( k' \) that

\[ D^{k'} p_{\gamma=0} (x, t) \]

\[ = (-4\pi)^{-1} \int_{\mathbb{R}^3} \sum_i \partial_i |z|^{-1} D^{k'} \left[ \sum_j u_j^{\gamma=0} \partial_j u_i^{\gamma=0} (x - z, t) \right] \, dz \]

(10.27)

The (10.14) expansion of \( D^{k'} \left[ \sum_j u_j^{\gamma=0} \partial_j u_i^{\gamma=0} \right] \) and (10.2) show that the right side of (10.29) is \( \in \cap_m [H^m \times [0, \infty)] \). The proof of the induction hypothesis for \( n = 0 \) is complete.

We now show that the induction hypothesis for \( n \geq 0 \) implies the induction hypothesis for \( n + 1 \). For each differential operator \( \partial^{n+1} D^{k'} \), there are multi-indices \( k'_1 \) and \( k'_2 \) so that

\[ \partial^{n+1} D^{k'} = D^{k'_1} \partial_t \partial^n D^{k'_2} \]

(10.28)
The induction hypothesis for \( n \) then shows that
\[
\frac{\partial^n D^{k'}}{\partial t^n} p^{\gamma=0}(x, t)
= (-4\pi)^{-1} \int_{\mathbb{R}^3} \sum_i \partial_i |z|^{-1} \frac{\partial^n D^{k'}}{\partial t^n} \left[ \sum_j u_j^{\gamma=0} \partial_j u_i^{\gamma=0}(x - z, t) \right] \, dz
\]
which is \( \in \cap_m[H^m \times [0, \infty)) \)

(10.29)

As (10.14) shows that
\[
\frac{\partial^n D^{k'}}{\partial t^n} \left[ \sum_j u_j^{\gamma=0} \partial_j u_i^{\gamma=0}(x - z, t) \right] \, dz
= \sum_{i'} \left[ \ldots \right] \nu \left[ \partial^{[\leq n |\nu, r]} D^{k_{i'}} u_j^{\gamma=0} \nu(t) \partial^{[\leq n |\nu, r]} D^{k_{i'}} \partial_j u_i^{\gamma=0}(t) \right]
\]

(10.30)

and that each multiplicand in the right side of (10.30) is \( \in \cap_m[H^m \times [0, \infty)] \) and has a \( \partial_t \) derivative which is \( \in \cap_m[H^m \times [0, \infty)] \), Lemma 10.3 shows that
\[
\partial_t \left[ \int_{\mathbb{R}^3} \sum_i \partial_i |z|^{-1} \frac{\partial^n D^{k'}}{\partial t^n} \left[ \sum_j u_j^{\gamma=0} \partial_j u_i^{\gamma=0}(x - z, t) \right] \, dz \right]
\]
is \( = \int_{\mathbb{R}^3} \sum_i \partial_i |z|^{-1} \partial_t \frac{\partial^n D^{k'}}{\partial t^n} \left[ \sum_j u_j^{\gamma=0} \partial_j u_i^{\gamma=0}(x - z, t) \right] \, dz \)
and is \( \in \cap_m[H^m \times [0, \infty)] \)

which with (10.30) shows that
\[
\partial_t \partial_t^n D^{k' \nu} \left[ \sum_j u_j^{\gamma=0} \partial_j u_i^{\gamma=0}(x - z, t) \right] \, dz
= \int_{\mathbb{R}^3} \sum_i \partial_i |z|^{-1} \partial_t \frac{\partial^n D^{k'}}{\partial t^n} \left[ \sum_j u_j^{\gamma=0} \partial_j u_i^{\gamma=0}(x - z, t) \right] \, dz
\]

(10.31)

and that each side of (10.31) is \( \in \cap_m[H^m \times [0, \infty)] \).
Applying \( D_k' \) to each side of (10.31) and using (3.16) to move \( D_k' \) to the right of the integral sign on the right side shows that

\[
\frac{\partial}{\partial t} u_n^{+1} + D_k' p^{+1} = 0 \quad (x, t)
\]

which is \( \in \cap_m [H^m \times [0, \infty)] \)

which establishes the induction hypothesis for \( n + 1 \).

□

We now show that

\[
u^{+1}(x, t; u_o) := u^{+0}(x, t; u_o) \quad \text{and} \quad p^{+1}(x, t; u_o) := p^{+0}(x, t; u_o)
\]

(10.32)

are \( \in C^\infty \) on \( \mathbb{R}^3 \times [0, \infty) \) and satisfy the \((1.1)_{\nu^{+1}, u_o} \) pde on \( \mathbb{R}^3 \times [0, \infty) \). See (10.25) for the definition of \( p^{+0} \).

**Theorem 10.6.** The functions \( u^{+1}(x, t; u_o) \) and \( p^{+1}(x, t; u_o) \) are \( \in C^\infty \) on \( \mathbb{R}^3 \times [0, \infty) \), each derivative \( \partial^n t D_k' u^{+1}(x, t; u_o) \) is \( \in \cap_m [H^m,df \times [0, \infty)] \) and each derivative \( \partial^n t D_k' p^{+1}(x, t; u_o) \) is \( \in \cap_m [H^m \times [0, \infty)] \).

\( u^{+1} \) and \( p^{+1} \) satisfy the pde

\[
\partial_t u^{+1}(x, t; u_o) + \sum_j u_j^{+1} \partial_j u^{+1}(x, t; u_o) = \Delta u^{+1}(x, t; u_o) - \text{grad } p^{+1}(x, t; u_o)
\]

(10.33)

on \( \mathbb{R}^3 \times [0, \infty) \) and

\[
|u^{+1}|_{L^\infty \times [0, \infty)} = |u_o|_{L^\infty} \quad \text{and} \quad |u^{+1}|_{L^2 \times [0, \infty)} = |u_o|_{L^2}
\]

(10.34)

**Proof.** Lemma 10.4 and Lemma 10.5 establish the first paragraph of this theorem. Theorems 8.2 and 9.2 show that \( u^{+1} \) satisfies the \((7.1)_{\nu^{+1}, u_o} \) pde

\[
\partial_t u^{+1}(x, t) + \mathcal{P} \left[ \sum_j u_j^{+1}(\cdot, t) \partial_j u^{+1}(\cdot, t) \right](x) = \Delta u^{+1}(x, t)
\]

(10.35)
on $\mathcal{R}^3 \times [0, \infty)$. As the argument of $P$ in (10.35) is $\in \cap_m [H^m \cap W^{m,1}]$ when $t$ is fixed
\[ \sum_j u_j = 1 \partial_j u = 1(x, t) \]
\[ = P \left[ \sum_j u_j = 1 \partial_j u = 1(t) \right] (x) + \mathcal{G} \left[ \sum_j u_j = 1 \partial_j u = 1(t) \right] (x) \]
(4.4), and as a result
\[ \partial_t u = 1(x, t) + \sum_j u_j = 1 \partial_j u = 1(x, t) \]
\[ = u = 1(x, t) + \mathcal{G} \left[ \sum_j u_j = 1(t) \partial_j u = 1(t) \right] (x) \]
on $\mathcal{R}^3 \times [0, \infty)$. (4.7) then shows that
\[ \mathcal{G} \left[ \sum_j u_j = 1(t) \partial_j u = 1(t) \right] (x) \]
\[ = - \text{grad} \left[ (-4\pi)^{-1} \int_{\mathcal{R}^3} \sum_{i,j} \partial_i |z|^{-1} [u_i = 1 \partial_j u = 1] (x - z, t) \, dz \right] \]
\[ = - \text{grad} p = 1(x, t) \]
the proof of (10.33) is complete.

Theorem 9.2 establishes the first line of (10.34), and (7.24) establishes the second line thereof.

11. $u^\nu$ and $p^\nu$ are $C^\infty$ solutions of the (1.1)$_{\nu, u_o}$ pde on $\mathcal{R}^3 \times [0, \infty)$

This section shows for $\nu > 0$ and $u_o \in \cap_m H^{m,df}$ that
\[ u^\nu(x, t; u_o) := u^\nu=1(x^\nu^{-1}, t^\nu^{-1}; u_o(x^\nu)) \text{ and } \]
\[ p^\nu(x, t; u_o) := p^\nu=1(x^\nu^{-1}, t^\nu^{-1}; u_o(x^\nu)) \]
(11.1)
are $\in C^\infty$ on $\mathcal{R}^3 \times [0, \infty)$ and satisfy the (1.1)$_{\nu, u_o}$ pde on $\mathcal{R}^3 \times [0, \infty)$. 
Theorem 11.1. Let $u_o$ be $\in \cap_m H^{m,df}$ and $\nu$ be $>0$. Then $u^\nu = u^\nu(x,t;u_o)$ and $p^\nu = p^\nu(x,t;u_o)$ are $\in C^\infty$ on $R^3 \times [0,\infty)$, $u^\nu$ and its derivatives are $\in \cap_m [H^{m,df} \times [0,\infty)]$ and $p^\nu$ and its derivatives are $\in \cap_m [H^{m} \times [0,\infty)]$.

$u^\nu$ and $p^\nu$ satisfy the $(1.1)_{\nu,u_o}$ pde

$$\partial_t u^\nu(x,t;u_o) + \sum_j u^\nu_j \partial_j u^\nu(x,t;u_o) = \nu \Delta u^\nu(x,t;u_o)$$

$$- \text{grad } p^\nu(x,t;u_o)$$

(11.2)

on $R^3 \times [0,\infty)$ and $u^\nu$ satisfies

$$|u^\nu|_{L^2 \times [0,\infty)} = |u_o|_{L^2} \text{ and }$$

$$|u^\nu|_{L^\infty \times [0,\infty)} = |u_o|_{L^\infty}$$

(11.3)

Proof. We first establish the first paragraph of the theorem. As the chain rule shows that $u_o(x\nu) \in \cap_m H^{m,df}$, $u^{\nu=1}(x,t;u_o(x\nu)) \in \cap_m [H^{m,df} \times [0,\infty)]$. Theorem 8.2, Theorem 9.2.

The chain rule shows that

$$\partial_i^{n \geq 0} D^{k'} u^\nu(x,t;u_o) = \partial_i^{n \geq 0} D^{k'} u^{\gamma=0}(x^{\nu^{-1}}, t^{\nu^{-1}}; u_o(x\nu)) \nu^{-n-|k'|}$$

and as $u^{\gamma=0}$ is $\in C^\infty$ on $R^3 \times [0,\infty)$, $u^\nu$ is $\in C^\infty$ on $R^3 \times [0,\infty)$, and customary mathematics also shows that $u^\nu$ is $\in \cap_m [H^{m,df} \times [0,\infty)]$.

In the same way, $p^\nu$ is $\in C^\infty$ on $R^3 \times [0,\infty)$ and is $\in \cap_m [H^{m} \times [0,\infty)]$.

We now establish (11.2) and (11.3). The calculation

$$u^\nu(x,t;u_o)|_{t=0} = u^{\gamma=0}(x^{\nu^{-1}},0;u_o(x\nu)) = u_o([x^{\nu^{-1}}]_{\nu}) = u_o(x)$$

shows that $u^\nu$ satisfies the initial condition of the (11.2) pde. With the definition of $u^\nu$ and that of $p^\nu$, the first line of the (11.2) pde is

$$\partial_t \left[u^{\gamma=0}(x^{\nu^{-1}}, t^{\nu^{-1}}; u_o(x\nu))\right]$$

$$+ \sum_j u^{\gamma=0}_j(x^{\nu^{-1}}, t^{\nu^{-1}}; u_o(x\nu)) \partial_j u^{\gamma=0}(x^{\nu^{-1}}, t^{\nu^{-1}}; u_o(x\nu))$$

$$- \nu \left[\Delta u^{\gamma=0}(x^{\nu^{-1}}, t^{\nu^{-1}}; u_o(x\nu))\right] - \text{grad } [p^{\gamma=0}(x^{\nu^{-1}}, t^{\nu^{-1}}; u_o(x\nu))]$$
which the chain rule shows is
\[
\begin{align*}
&= \left[ \partial_t u^\gamma=0 (x\nu^{-1}, t\nu^{-1}; u_o(x\nu)) \\
&\quad + \sum_j [u_j^\gamma=0 \partial_j u^\gamma=0 ] (x\nu^{-1}, t\nu^{-1}; u_o(x\nu)) \\
&\quad - \Delta u^\gamma=0 (x\nu^{-1}, t\nu^{-1}; u_o(x\nu)) - [\text{grad } p^\gamma=0] (x\nu^{-1}, t\nu^{-1}; u_o(x\nu)) \right] \nu^{-1}
\end{align*}
\]
which is the 7.1 \( k'=0, \gamma=0, u_o(x\nu) \) pde \( u^\gamma=0 \) satisfies on \( \mathcal{R}^3 \times [0, \infty) \). The proof that \( u^\nu \) and \( p^\nu \) satisfy the (11.2) pde is complete.

(9.9) and the definition of \( u^\nu \) establish the first line of (11.3).

We now establish the second line of (11.3). With \( t \in [0, \infty) \) and customary mathematics
\[
\begin{align*}
|u^\nu(t)|^2_{L^2} &= \int_{\mathbb{R}^3} \left( u^\gamma=0(x\nu^{-1}, t\nu^{-1}; u_o(x\nu)), u^\gamma=0(x\nu^{-1}, t\nu^{-1}; u_o(x\nu)) \right)_2 \, d[x\nu^{-1}] \nu^3 \\
&= \int_{\mathbb{R}^3} \left( u^\gamma=0(x, t\nu^{-1}; u_o(x\nu)), u^\gamma=0(x, t\nu^{-1}; u_o(x\nu)) \right)_2 \, dx \, \nu^3
\end{align*}
\]
which after changing variables in the integral is
\[
\begin{align*}
&= \int_{\mathbb{R}^3} \left( u^\gamma=0(x, t\nu^{-1}; u_o(x\nu)), u^\gamma=0(x, t\nu^{-1}; u_o(x\nu)) \right)_2 \, dx \, \nu^3
\end{align*}
\]
which with (7.24) is \( \leq |u_o(x\nu)|^2_{L^2} \nu^3 \). (9.16) (first line) then shows that
\[
|u^\nu|^2_{L^2 \times [0, \infty)} \leq |u_o|^2_{L^2} \nu^{-3} \nu^3 = |u_o|^2_{L^2}
\]
The proof of the second line of (11.3) is complete.

The functions \( u^\nu \) and \( p^\nu \) satisfy the conditions of (1), (2), (3), (6) and (7) of Option A of the Millennium Prize Problem of the Clay Mathematics Institute for the Navier-Stokes equations. The class of initial conditions, for which such prize problem requires such a solution are those which are divergence free and satisfy
\[
|D^{k'} u_o(x)| \leq C_{k', K} \text{ on } \mathcal{R}^3 \text{ for any } k' \text{ and } K
\]
A calculation which we leave to the reader shows that the \( L^2 \) norm of each derivative \( D^{k'} u_o(x) \) is \( \in L^2 \). Thus, any such function is \( \in \cap_m H^{m, df} \).
SMOOTH SOLUTIONS OF NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3 \times [0, \infty)$ 131

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