MULTIDIMENSIONAL COSMOLOGY
WITH A GENERALIZED MAXWELL FIELD:
INTEGRABLE CASES

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We consider multidimensional cosmologies in even-dimensional space-times \((D = 2n)\) containing perfect fluid and a multidimensional generalization of the Maxwell field \(F_{A_1...A_n}\) preserving its conformal invariance. Among models with an isotropic physical 3-space some integrable cases are found: vacuum models (which are integrable in the general case) and some perfect fluid models with barotropic equations of state. All of them contain a component of the \(F\) field appearing as an additional scalar in 4 dimensions. A two-parameter family of spatially flat models and four one-parameter families, including non-spatially flat models, have been obtained (where the parameters are constants from the fluid equation of state). All these integrable models admit the inclusion of a massless scalar field or an additional fluid with the maximally stiff equation of state. Basic properties of vacuum models in the physical conformal frame are outlined.

1. Introduction

This paper continues a study of exact solutions to the gravitational field equations including a generalization of the Maxwell field to \(D\)-dimensional space-times \((n = 2, 3, ...),\) began in Ref. [1]. Unlike the conventional straightforward use of the Maxwell field in \(D\) dimensions, this generalization preserves the invariance of the field under multidimensional conformal transformations.

This generalization (Generalized Maxwell Field, GMF) is represented by a \((D/2)\)-form, where \(D = 2n\) is the space-time dimension:

\[
F = dU, \quad \text{or} \quad F_{A_1...A_n} = n! \partial_{[A_1} U_{A_2...A_n]} \tag{1}
\]

where \(U\) is a potential \((n - 1)\)-form and square brackets denote alternation. The field \(F\) is invariant with respect to the gauge transformation

\[
U \rightarrow U + dZ \tag{2}
\]

where \(Z\) is an arbitrary \((n - 2)\)-form; in 4 dimensions this is the conventional gradient transformation of the Maxwell field.

The GMF was introduced in Refs. [1, 2, 3], where some particular cosmological models with the GMF were studied; in [1] we discussed in some details several families of spherically symmetric solutions, with combinations of electric, magnetic and quasiscalar type components of the GMF, with possible inclusion of a massless, minimally coupled scalar field \(\varphi = \varphi^{\min}\).

Some cases of interest were found, namely, black holes with a somewhat unusual mass and charge dependence of the Hawking temperature.

This paper presents some classes of exact cosmological solutions, including those of [2, 3, 4] as special cases. Just as in those papers, only Friedmann-like models, isotropic in the three physical dimensions, are considered, therefore the GMF can have only “quasiscalar” components, behaving as scalars in the physical space-time. The solutions found here contain only one such component. Further generalizations with a number of such components and even a number of \(p\)-forms, as discussed in field models like M-theory (see e.g. [6] and references therein), are possible.

The aim of this paper is basically to indicate the integrable cases. A more detailed description of the solutions will be given elsewhere.

Only the behaviour of vacuum solutions is briefly outlined, some of them exhibiting attractive features. In particular, solutions with a hyperbolic 3-space, have a linearly expanding physical space at their late-time asymptotic, with internal-space scale factors tending to finite constant values which (bearing in mind the usual assumption that the internal spaces are compact) may be arbitrarily small.

Throughout the paper capital Latin indices range over all \(D\) space-time dimensions, Greek ones take the values 0,1,2,3, indices like \(a_1, \ldots, a_n\) refer to internal
spaces and \( i, j \) enumerate the factor spaces.

2. Generalized Maxwell field and 3-isotropic cosmology.

Field equations

Consider general relativity in a Riemannian space-time \( V^D \) \((D = 2n, n \geq 2)\), in the presence of the GMF and perfect fluid (matter), with the total action

\[
S = \int d^Dx \sqrt{g} \left[ R/(2\kappa) + (-1)^{n-1}F^2 + L_m \right]
\]

\[
F^2 \equiv F^{A_1...A_n}F_{A_1...A_n},
\]

where \( R = D^2 \) is the scalar curvature corresponding to the \( D \)-metric \( g_{AB} \), \( g = |\text{det} g_{AB}| \), \( \kappa \) is the \( D \)-dimensional gravitational constant, \( F \) is the GMF and \( L_m \) is the matter Lagrangian.

The \( F \)-sector of the action is invariant under conformal mappings of \( V^D \) \((g_{AB} \rightarrow f(x^A)g_{AB})\) provided the components \( F_{A_1...A_n} \) are unchanged in such transformations (the potential \( U \) from \( \Box \) may experience a gauge transformation \( \Box \) where \( Z \) is some \((n-2)\)-form depending on the conformal factor \( f \)).

The field equations are

\[
G^B_A \equiv R^B_A - \frac{1}{2}\delta^B_A R = -\kappa T^B_A,
\]

\[
\nabla_A F^{A_1...A_n} = 0,
\]

where the energy-momentum tensor (EMT) \( T^B_A \) is a sum of contributions from \( F \) and matter. The EMT of the \( F \) field is

\[
T^B_{m,A} = (-1)^{n-1}(nF_{AC_2...C_n}F^{B_2...C_n} - \frac{1}{2}\delta^B_A F^2).
\]

An alternative form of \( \Box \) is

\[
R^B_A = -\kappa T^B_{m,A} - \kappa \left(T^B_{m,A} - \frac{1}{D-2}\delta^B_A T^m_m\right)
\]

where \( T^B_{m,A} \) is the EMT of all matter except the \( F \) field.

In what follows we will assume that the physical 3-space is isotropic and the GMF has only one (up to index permutations) nontrivial component compatible with such isotropy, namely,

\[
F_{0a_2...a_n},
\]

where the index 0 corresponds to time and all the others to extra dimensions; moreover, we restrict ourselves to the simplest space-time structure compatible with such an \( F \) field. Evidently, among the \( 2n-4 \) internal dimensions Eq. \( \Box \) distinguishes two subsets, with and without the \( F \) field, or, in other words, \( n-1 \) coordinates from the remaining \( n-3 \). Thus for \( n > 3 \) there must be at least two internal factor spaces with different dynamics.

Accordingly, we consider \( V^{2n} \) with the structure

\[
V^D = \mathbb{R} \times M_1 \times M_2 \times M_3,
\]

where \( \mathbb{R} \) corresponds to time, \( M_1 \) is the external isotropic 3-space (of curvature \( K = 0, \pm 1 \)), \( M_2 \) and \( M_3 \) are Ricci-flat internal spaces, \( \dim M_2 = n - 1 \), \( \dim M_3 = n - 3 \), so that \( M_4 \) is parametrized by the coordinates whose numbers belong to \( a_k \) in \( \Box \) and \( M_3 \) to the remaining ones.

The metric is assumed in the form

\[
ds^2 = e^{2\gamma}d\tau^2 - e^{2\beta_1}ds_1^2 - e^{2\beta_2}ds_2^2 - e^{2\beta_3}ds_3^2
\]

where \( \gamma, \beta \) are functions of the time variable \( \tau \) and \( ds_i \) are the \( \tau \)-independent metrics in \( M_i \).

For the perfect fluid we assume a pressure isotropic in each factor space, so that its EMT is

\[
T^B_{m,A} = \text{diag}(\rho, [-p_1]_3, [-p_2]_{n-1} [-p_3]_{n-3})
\]

where \( \rho \) is the density, \( p_i \) are the pressures in the corresponding directions and the symbol \( [x]_k \) means that a quantity \( x \) occupies \( k \) positions along the diagonal. We admit that \( \rho \) and \( p_i \) are connected by the equations of state

\[
p_i = (a_i - 1)\rho, \quad a_i = \text{const},
\]

so that a physical range of \( a_i \) is between 0 (vacuum-like state) and 2 (stiff matter).

It should be noted that the 4-dimensional part of the \( D \)-metric \( \Box \) does not coincide with the metric to be used to interpret actual cosmological observations. Indeed, real space-time measurements (such as Solar-system experiments or cosmological redshift measurements) rest on the constancy of atomic quantities (the atomic system of measurements). Thus, the modern definition of reference length is connected with a certain spectral line, determined essentially by the Rydberg constant and ultimately by the electron and nucleon masses. Therefore it is reasonable to apply as a physical one such a metric (to be denoted \( g_{\mu\nu}^{a} \)) conformally related to \( g_{\mu\nu} \), with which masses of bodies of nongravitational matter, like atomic particles, do not change from point to point. This happens if the matter Lagrangian appears in the total Lagrangian in 4 dimensions without any \( x^k \)-dependent factor.

The choice of \( g_{\mu\nu}^{a} \) depends on how \( L_m \) appears in the original action. In our case \( \Box \), just as in Ref. \( \Box \), it is easy to check that the metric in the “atomic gauge” should have the form

\[
g_{\mu\nu}^{a} = e^{c/2}g_{\mu\nu}
\]

where \( c \) is the volume factor of extra dimensions, in our case \( c = (n-1)\beta_2 + (n-3)\beta_3 \). Thus, in particular, the physical 3-dimensional scale factor \( a(\tau) \) is

\[
a(\tau) \equiv \exp[\beta_1 + \frac{1}{3}(n-1)\beta_2 + \frac{1}{3}(n-3)\beta_3].
\]
We are now ready to write down the field equations explicitly. Let us use the harmonic time coordinate $\tau$, so that
\[ \gamma(\tau) \equiv \sum_{i=1}^{3} N_i \beta_i(\tau), \quad N_i = \dim M_i. \tag{15} \]

The Maxwell-like equation \[ F_{045} = q/\sqrt{g}, \quad q = \text{const} \tag{16} \]
where due to \[ F = e^{2\gamma}. \] So the EMT of the $F$ field may be written as
\[ T^B_{FA} = \frac{1}{n!} q^2 e^{-6\beta_1-2(n-3)\beta_3} \times \text{diag}(1, [-1]_3, [1]_{n-1}, [-1]_{n-3}). \tag{17} \]

For perfect fluid the conservation law $\nabla_B T^B_{FA} = 0$ after integration leads to
\[ \rho = \rho_0 \exp\left(-\sum_{i=1}^{3} N_i a_i \beta_i \right), \quad \rho_0 = \text{const} \tag{18} \]
and its EMT is presented in the form
\[ T^B_{m} = \rho_0 e^{-2\gamma+2y} \times \text{diag}(1, [1-a]_3, [1-a_2]_{n-1}, [1-a_3]_{n-3}) \tag{19} \]
where we have introduced the function
\[ 2y = \sum_{i=1}^{3} (2-a_i) N_i \beta_i. \tag{20} \]

With \[ (14) \] and \[ (17) \], the nonzero components of the Ricci tensor are
\[ R_{\gamma} = e^{-2\gamma} \left( \gamma - \gamma^2 + \sum_{i=1}^{3} N_i \beta_i^2 \right), \tag{21} \]
\[ R_{\alpha}^{\alpha} = \delta_{\alpha}^{\alpha} e^{2\beta_1} K_1 (N_1 - 1) + e^{-2\gamma} \beta_i, \tag{22} \]
where $K_1 = 0, \pm 1$ are the factor space curvatures; we will assume $K_2 = K_3 = 0$ and only for the physical space treat all the three variants, corresponding to different Friedmann-Robertson-Walker models.

The spatial components of the Einstein equations now read:
\[ 2K_1 e^{2\gamma+2\beta_1} + \beta_1 = Q^2 e^{2x} + A_1 K_1 \rho_0 e^{2y}, \tag{23} \]
\[ \beta_2 = -Q^2 e^{2x} + A_2 K_1 \rho_0 e^{2y}, \tag{24} \]
\[ \beta_3 = Q^2 e^{2x} + A_3 K_1 \rho_0 e^{2y}, \tag{25} \]
where
\[ x \equiv (n-1)\beta_2, \]
\[ A_1 = a_i - A \beta_2, \quad A \equiv \sum_{i=1}^{3} N_i a_i, \]
\[ Q^2 = \frac{1}{2} n! Krq^2. \tag{26} \]

Eqs. \[ (23) \]–\[ (25) \] have a first integral (the energy integral), coinciding with the $t$ component of the Einstein equations:
\[ 3K_1 e^{2\gamma+2\beta_1} + \frac{1}{2} \beta_1 \gamma^2 - \frac{1}{2} \sum_{i=1}^{3} N_i \beta_i^2 = Q^2 e^{2x} + \kappa \rho_0 e^{2y}. \tag{27} \]

In what follows we try to solve Eqs. \[ (23) \]–\[ (25) \]; \[ (27) \] is used to obtain an additional relation between the integration constants and simultaneously to verify the correctness of the solutions.

3. Vacuum solutions with the $F$ field

In the case $\rho_0 = 0 \Rightarrow \rho \equiv p_i = 0$ one easily solves Eq. \[ (24) \]:
\[ e^{-(n-1)\beta_2} = \frac{1}{n} \sqrt{(n-1)Q^2} \cosh[h(\tau - \tau_2)] \tag{28} \]
with the integration constants $h > 0$ and $\tau_2$. As now $\beta_2 + \beta_3 = 0$, $\beta_3$ is also easily found:
\[ \beta_3 = -\beta_2 + br + b_3, \quad b, b_3 = \text{const}. \tag{29} \]
Lastly, \[ (23) \] can be combined with \[ (24) \] and \[ (25) \] to give
\[ (\gamma + \beta_1)^2 + 4K_1 e^{2\gamma+2\beta_1} = 0, \tag{30} \]
whence
\[ e^{-z} \equiv e^{\beta_1 - \gamma} = \begin{cases} 2s(k, \tau), & K_1 = -1 \ (k \in \mathbb{R}); \\ s(k, \tau), & K_1 = 0, \ (k \in \mathbb{R}); \\ 2k^{-1} \cos k \tau, & K_1 = 1, \ (k > 0), \end{cases} \tag{31} \]
where
\[ s(k, \tau) \equiv \begin{cases} k^{-1} \sinh k \tau, & k > 0; \\ k \tau, & k = 0; \\ k^{-1} \sin k \tau, & k < 0; \end{cases} \tag{32} \]
k = const and one more constant is eliminated by shifting the origin of $\tau$. The integration is completed.

A substitution to the energy integral \[ (27) \] gives:
\[ 3K_1 = \frac{4}{n-1} h^2 + (n-1)(n-3)b^2, \tag{33} \]
so that, in particular, for $K_1 = -1$ we have $k > 0$ and $e^{-z} = (2/k) \sinh k \tau$.

The physical components of the metric are (see \[ (3) \], \[ (4) \]):
\[ g^{*}_{00} = e^{2\gamma}, \quad g^{*}_{0\tau} = \exp\left[3z - \beta_2 - \frac{1}{2}(n-3)br\right]; \]
\[ a^*(\tau) = \exp\left[z - \beta_2 - \frac{1}{2}(n-3)br\right]. \tag{34} \]

The model properties strongly depend on the dimension $2n$ and specific values of the integration constants. Thus, the cosmic proper time $t = \int e^{\gamma^*} dt$
is finite or infinite depending on the convergence or divergence of the integral, etc. One can conclude the following.

1. The harmonic time \( \tau \in \mathbb{R} \) for \( K_1 = 0, +1 \) and (with no generality loss) \( \tau \in (0, \infty) \) for \( K_1 = -1 \).

2. The internal-space scale factor \( e^{\beta_2} \) tends to zero as \( \tau \to \pm \infty \), i.e. at both ends of the evolution process for \( K_1 = 0, +1 \) and at one end for \( K_1 = -1 \). In the latter case, \( \beta_2 \) tends to a finite limit as \( \tau \to 0 \).

3. Another scale factor \( e^{\beta_3} \) (existing only for \( n > 3 \)) tends to infinity either as \( \tau \to +\infty \), or as \( \tau \to -\infty \), or in both limits (if \( |b| < h/2 \)). Again, it is finite as \( \tau \to 0 \) (\( K_1 = -1 \)).

4. For \( n = 3 \) (\( D = 6 \)), Eq. (33) gives just \( 3k^2 = 2h^2 \), therefore the solution behaviour is uniquely characterized in terms of the proper time \( t \). Thus, for \( K_1 = 0 \) (spatially flat world) the evolving model has the asymptotics (where arbitrary constants are chosen with no loss of generality)

\[
\begin{align*}
t &\sim e^{(\sqrt{3}/2 + 1/4)h|\tau|} \to 0, \\
a(t) &\sim e^{-(\sqrt{1}/6 + 1/4)h|\tau|} \sim t^{0.162}, \\
e^{\beta_2} &\sim e^{-|\tau|/2} \sim t^{0.514} \\
e^{\beta_3} &\sim e^{-|\tau|/2} \sim t^{-0.34}
\end{align*}
\]

— initial singularity, and

\[
\begin{align*}
t &\sim e^{(\sqrt{3}/2 + 1/4)h|\tau|} \to 0, \\
a(t) &\sim e^{(\sqrt{1}/6 + 1/4)h|\tau|} \sim t^{0.447}, \\
e^{\beta_2} &\sim e^{-|\tau|/2} \sim t^{-0.34} \\
e^{\beta_3} &\sim e^{-|\tau|/2} \sim t^{-0.34}
\end{align*}
\]

— expansion in physical space and contraction in extra dimensions.

For \( K_1 = +1 \) (spherical world), the time symmetry of the evolution is broken only by the shift \( t_2 \) in \( e^x \) which does not affect the asymptotics. The evolution proceeds between two singularities like (35).

For \( K_1 = -1 \) (hyperbolic world) the model has a singularity like (35) at \( \tau = \infty \); on the other hand,

\[
\begin{align*}
t &\sim 1/\sqrt{\tau} \to \infty; \\
a(t) &\sim 1/\sqrt{t} \sim t \quad \text{as} \quad \tau \to 0,
\end{align*}
\]

i.e. there is a linear expansion (contraction) asymptotic with the internal scale factor \( e^{\beta_2} \) tending to a finite limit.

5. For \( n > 3 \), the advent of one more constant \( b \) makes the evolution of \( a(t) \) more diverse. One can observe, in particular, that, for any set of constants, there is always an asymptotic like (33) for \( K_1 = -1 \) and at least one singularity like (35) (with, in general, other numerical characteristics) for \( K = +1 \).

It should be stressed that the behaviour of \( a(t) \) and even the finiteness or infiniteness of time intervals crucially depend on the choice of the conformal frame. Thus, the present vacuum solutions for \( n = 3 \) coincide with those of Ref. [3], but the conclusions are different since there the authors considered the metric \( g_{\mu\nu} \) as the physical one.

In what follows we only indicate the integrable cases, postponing their detailed analysis to further publications.

4. Solutions with matter: \( K_1 = 0 \)

The unknown function \( y \) is excluded from Eq. (24) in two cases:

1. if \( y \) is proportional to \( x \) (i.e., to \( \beta_2 \)),
2. if \( A_2 = 0 \).

1. This possibility is realized if \( a_1 = a_3 = 2 \) (stiff matter equation of state in \( M_1 \) and \( M_3 \)). Then we have

\[
\hat{\beta}_1 = \beta_3 = -\beta_2,
\]

so that it is sufficient to integrate (24) for \( \beta_2 \), then \( \beta_1 \) and \( \beta_3 \) will be easily found. Eq. (24) reduces to the form

\[
\ddot{\beta}_2 + Q^2 e^{2(n-1)\beta_2} + (1 - \frac{1}{2}a_1) e^{(n-1)(2-a_2)}\beta_2 = 0.
\]

Thus it belongs to the type

\[
\ddot{x} + B_1 e^{2x} + B_2 e^{2cx} = 0,
\]

\[
B_1, B_2, c = \text{const.}
\]

Its first integral is

\[
\ddot{x}^2 + B_1 e^{2x} + (B_2/c) e^{2cx} = \text{const}
\]

which is evidently integrable by quadratures. The solution is expressed in elementary functions in the cases \( c = 1/2, 1, 2 \). By (39) this corresponds to

\[
a_2 = 1, 0, -2,
\]

the first two values belonging to the physical range.

So we have obtained a class of solutions which can be labelled

\[
(2n, 0, F \mid 2, a_2, 2),
\]

where the first three symbols designate the dimension, spatial curvature (in \( M_1 \)) and a nonzero \( F \) field, while the remaining three indicate the coefficients \( a_i \).

2. The relation \( A_2 = 0 \) is explicitly written as

\[
2 + (n-1)a_2 = 3a_1 + (n-3)a_3.
\]

So the equation for \( \beta_2 \) is integrated in the form (28). Then, if we combine Eqs. (23)–(25) to form \( \ddot{y} \) in the
left-hand side, it turns out that, in the right-hand side, the coefficient before \( e^{2x} \) vanishes and we obtain
\[
2y = 3(2-a_1)(a_1-a_2) + (n-3)(2-a_3)(a_3-a_2) e^{2y}
\]
which is easily integrated like \( \text{(30)} \). So we know \( \beta \) and \( y \) \( \left(\text{25}\right) \) whose right-hand side is zero, namely,
\[
(a_3-a_2)\beta_1 + (a_3-a_1)\beta_2 + (a_2-a_1)\beta_3 = 0. \quad \left(\text{46}\right)
\]
In the general case Eqs. \( \left(\text{45}\right) \) and \( \left(\text{46}\right) \) are independent, so after trivially solving \( \left(\text{46}\right) \) the integration is completed. In the exceptional case when they are not, one can find \( \beta_1 \) from \( \left(\text{23}\right) \), since both \( x \) and \( y \) are already known.

Thus for \( K_1 = 0 \) we have obtained two families of solutions: \( \left(\text{43}\right) \) (one-parameter) and
\[
(2n,0,F | a_1,a_2,a_3), \quad \left(\text{47}\right)
\]
constrained by \( \left(\text{44}\right) \) (two-parameter). Their only common element is the trivial case of all \( a_i = 2 \), when all \( A_i = 0 \) and the solution is reduced to vacuum up to the relation between the integration constants that follows from the energy integral. This case of stiff matter in the whole space coincides with that of a minimally coupled scalar field \( \varphi(\tau) \), to be discussed later.

5. Solutions with matter: \( K_1 = \pm 1 \)

The same combination of \( \left(\text{23} - 2\right) \) that has led to Eq. \( \left(\text{8}\right) \) in the vacuum case, for \( \rho_0 \neq 0 \) reads:
\[
(\gamma - \beta_1)\gamma + 4K_1 e^{2\gamma - 2\beta_1} = \kappa \rho_0 (2 - a_1) e^{2y}. \quad \left(\text{48}\right)
\]
One can again select two cases of its integrability:
1. \( a_1 = 2 \) [then \( \left(\text{48}\right) \) coincides with \( \left(\text{30}\right) \)] and
2. \( y \) is proportional to \( (\gamma - \beta_1) \) and \( \left(\text{48}\right) \) reduces to \( \left(\text{10}\right) \).

1. \( a_1 = 2 \), stiff matter in physical space. We are left with Eqs. \( \left(\text{24}\right) \) and \( \left(\text{25}\right) \) with two unknowns \( x \) and \( y \). Eq. \( \left(\text{24}\right) \) is easily integrated (reduces to \( \left(\text{28}\right) \)) when \( A_2 = 0 \), that is
\[
(n-1)a_2 - (n-3)a_3 = 4. \quad \left(\text{49}\right)
\]
This condition is of interest only for \( n > 3 \), since for \( n = 3 \) it reduces to \( a_2 = 2 \), i.e., the trivial case \( a_1 = a_2 = 2 \).

For \( n > 3 \), the function \( y \) can be found from an equation similar to \( \left(\text{44}\right) \) but with another constant before \( e^{2y} \). This completes the integration.

We thus obtain the 1-parameter family of models
\[
(2n, \pm 1,F | 2,a_2,a_3), \quad n > 3 \quad \left(\text{50}\right)
\]
constrained by \( \left(\text{49}\right) \).

For \( a_1 = 2, n = 3 \), there is no \( \beta_3 \), and Eq. \( \left(\text{24}\right) \) takes the form
\[
2\beta_2 = -2Q^2 e^{4\beta_2} + \kappa \rho_0 (a_2 - 2) e^{2\beta_2(2-a_2)} \quad \left(\text{51}\right)
\]
belonging to the type \( \left(\text{40}\right) \). It is integrable in elementary functions for \( a_2 = 1 \) (no pressure) and \( a_2 = 0 \) (vacuum-like state). Thus we obtain the models
\[
(6, \pm 1,F | 2,a_2). \quad \left(\text{52}\right)
\]

Another case with an equation like \( \left(\text{51}\right) \) exists for \( n > 3 \): if \( a_1 = a_3 = 2 \), then
\[
2y = (n-1)(2-a_2)\beta_2. \quad \left(\text{53}\right)
\]
Thus Eq. \( \left(\text{24}\right) \) is solved; then, knowing the combination \( \gamma - \beta_1 \) and \( \beta_2 \) (hence also \( y \)), it is straightforward to find \( \beta_3 \) from \( \left(\text{23}\right) \). Thus we obtain a family of models similar to \( \left(\text{43}\right) \)
\[
(2n, \pm 1,F | 2,a_2,2), \quad n > 3. \quad \left(\text{54}\right)
\]

2. \( a_1 \neq 2 \); to solve \( \left(\text{48}\right) \), let us require that the two 3-vectors of coefficients by \( \beta_i \) in the expressions for \( y \) and \( \gamma - \beta_1 \) be parallel:
\[
\left( \frac{3(2-a_1)}{(n-1)(2-a_2)} \right) \sim \left( \frac{2}{n-3} \right), \quad \left(\text{55}\right)
\]
whence
\[
a_2 = a_3 = \frac{3}{2}a_1 - 1. \quad \left(\text{56}\right)
\]

In this case also automatically \( A_2 = A_3 = 0 \), so that \( \beta_2 \) and \( \beta_3 \) are found quite easily, thus completing the integration. We get the models
\[
(2n, \pm 1,F | a_1,\frac{3}{2}a_1 - 1,\frac{3}{2}a_1 - 1). \quad \left(\text{57}\right)
\]

Thus we have obtained 4 one-parameter families of models with nonzero spatial curvature. In each case, to obtain a final form of the solution, it is necessary to substitute the results of integration to the first integral \( \left(\text{27}\right) \) to find a relation between the integration constants.

6. Inclusion of a massless scalar field

So far we have been working with the action \( \left(\text{3}\right) \). However, all the above solutions can easily include a massless, minimally coupled scalar field \( \varphi = \phi^{\text{min}} \) with the action
\[
S[\phi^{\text{min}}] = \int \sqrt{g}(\nabla \varphi)^2. \quad \left(\text{58}\right)
\]
Then, in the space-time considered, the scalar field equation for \( \varphi = \varphi(\tau) \) is immediately integrated to give
\[
\varphi = \varphi_0 + \varphi_1 \tau; \quad \varphi_0, \varphi_1 = \text{const}. \quad \left(\text{59}\right)
\]
The EMT of $\phi^{\text{min}}$ coincides up to a constant factor with that of stiff matter (when all $a_i = 2$).

It is easily verified that $\phi^{\text{min}}$ does not contribute to the right-hand side of Eqs. (1) and therefore in no way affects the solution process described above. It only affects the system through its contribution to the first integral (27) where one should add the term $\kappa \varphi^2_1$ to the right-hand side. The same applies if one adds one more component of matter in the form of a perfect fluid with stiff equation of state ($a_1 = 2$): then the addition is just $\kappa \rho_0^2 = \text{const}$. This addition certainly changes some properties of the solutions.

Due to the conformal invariance of the $F$ field, $D$-dimensional conformal transformations lead to models of generalized scalar-tensor theories of gravity. Vacuum solutions are then generalized in a straightforward way, whereas perfect fluids will in general acquire a nontrivial interaction with scalar fields.

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