Creating Edge Element from Four Node Quadrilateral Element

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Abstract. When nodal based FEM is applied to vector field problems, some spurious eigen values appear. This can be avoided by adopting edge based FEM in the formulation. A separate conversion algorithm is used which takes the nodal information of quadrilateral element as input. Edge element is created from four node quadrilateral element to solve the 2D eigen value problem.

1. Introduction

Finite element method (FEM) has been extensively used in Electromagnetic application [1-3]. Nodal based FEM have following drawbacks [4-13):

1. In Eigen value problems presence of spurious modes can not be avoided due to lack of proper continuity condition at material interfaces.
2. Nodal based elements can not capture singular eigen values due to sharp corners and edges.
3. Due to nodal continuity requirement, these elements cannot give direct solutions in terms of electric and magnetic field variables. We have to follow potential formulations.
4. Vector fields require special elements due to their tangential continuity and normal discontinuity across material interfaces.

These drawbacks are alleviated by adopting Edge element in FEM implementation. Whitney introduced these elements in the field of FEM in electromagnetics. These are constructed by curl and divergence conforming spaces called Whitney spaces. In this element [4-17] electric fields are along the edge of the element satisfying required tangential continuity.

2. Conversion of node to edge

In edge based FEM, element information are required to be supplied in edge data structure. In order to attain this, a separate conversion algorithm is required which uses the nodal data as input and gives the edge information as output. Each edge of the element is formed by connecting two nodes of the element. Here we have used a conversion algorithm which converts the nodal element data of a quadrilateral element into edge element data. Such algorithm is essential as most of the mesh generator create nodal element data structure.
In this conversion algorithm, a simple finite element quadrilateral mesh as shown in Fig. 1 is used. Global nodal connectivity of discretized domain is shown in Fig. 2. Fig. 3 shows the local nodal connectivity of quadrilateral element and Fig. 4 is the desired local edge connectivity of such element. Four edges $e_1, e_2, e_3$ and $e_4$ are formed by connecting local node sets $(1, 2), (4, 3), (1, 4)$ and $(2, 3)$ respectively. In this algorithm, the outer loop runs over the total number of discretized quadrilateral elements. The inner most loop runs over the total number of edges ($nlocedge$) of each element. Inside the local edge loop with the help of the local nodal connectivity as shown in Fig. 3 and local edge connectivity as shown in Fig. 4 two end nodes of the edge of quadrilateral element are returned.
Table 1. Nodeedge array of nodes of the discretized domain.

| Global Node | Total no. of connected edges (nodeedgenum) | Connecting edge and corresponding other node(nodeedge(1:8)) |
|-------------|-------------------------------------------|----------------------------------------------------------|
|             |                                           | 1st edge  | Other node | 2nd edge  | Other node | 3rd edge  | Other node | 4th edge  | Other node |
| 1           |                                           | 2         | -          | 4         | -          | 3         | -          | -         | -          |
| 2           |                                           | 3         | -3         | 1         | 2          | 5         | -6         | 2         | -          |
| 3           |                                           | 2         | 5          | 6         | -6         | 2         | -          | -         | -          |
| 4           |                                           | 3         | -1         | 1         | 4          | 5         | 8          | 7         | -          |
| 5           |                                           | 4         | -2         | 2         | -4         | 4         | 7          | 6         | 9          | 8          |
| 6           |                                           | 3         | -5         | 3         | -7         | 5         | 11         | 9         | -          |
| 7           |                                           | 2         | -8         | 4         | 10         | 8         | -          | -         | -          |
| 8           |                                           | 3         | -9         | 5         | -10        | 7         | 12         | 9         | -          |
| 9           |                                           | 2         | -11        | 6         | -12        | 8         | -          | -         | -          |

2.1 Formation of edge connectivity array

The information of the two end nodes (starting and end nodes) of each local edge, last assigned global edge and existing edge connectivity array are supplied to another subroutine. Here in this subroutine checking is performed whether any edge is already existing between the current starting and end nodes with the help of nodeedge array. It stores the information of other edges connected with the current starting node in the odd columns and the corresponding end nodes in the even columns of each row. This array also have the direction information of the edge. Table 1 shows the information about the nodeedge array for all the global nodes of the discretized domain with the quadrilateral element. If there is no current edge is found in between the two end nodes then existing global edge number is incremented to digit 1. This is assigned to the current edge and this information is updated in the existing edge connectivity array (edgearr). Table 2 is the edge connectivity of quadrilateral elements of the discretized domain as shown in Fig. 1. Fig. 5 shows the edge connectivity of each element, arrow shows the direction of the edge pointed. For each global node total number of edges shared are counted by one global counter and this information is stored in nodeedgenum array. After successful updation of various arrays local edge loop runs for next local edge. This process repeats for all the edges of the element.

Table 2. Edge connectivity array of elements of the discretized domain.

| Element number | Edge connectivity (Global edge no.) |
|----------------|-------------------------------------|
| 1              | 1, 2, 3, 4                          |
| 2              | 2, 5, 6, 7                          |
| 3              | 8, 9, 4, 10                         |
| 4              | 9, 11, 7, 12                        |

Figure 5. Edge connectivity of elements
3. Numerical Analysis

The governing differential equation under harmonic excitation can be given as [3]

\[ \nabla \times \left( \frac{1}{\mu_r} \nabla \times E \right) - k_0^2 \varepsilon_r E = -i \omega \mu_0 j, \]  

(1)

where \( \xi = \sqrt{-1} \), \( k_0 = \omega^2 / c^2 \) is the wave number, \( \omega \) is the excitation frequency, \( c \) is the speed of light in vacuum, \( E \) is the electric field, \( j \) is current density, \( \mu_r = \mu / \mu_0 \), \( \varepsilon_r = \varepsilon / \varepsilon_0 \) are the relative permeability and relative permittivity, and \( \varepsilon_0 \) and \( \mu_0 \) are the permittivity and permeability for vacuum. Assuming no external load condition, \( j \) is to be zero, the above equation reduces to

\[ \nabla \times \left( \frac{1}{\mu_r} \nabla \times E \right) - k_0^2 \varepsilon_r E = 0. \]  

(2)

This equation is used to solve the eigen value problems.

To validate the edge element implemented, standard 2D eigen value problem in computational field as reported in [18] is considered. For the problem considered we have assumed \( \varepsilon_r = \mu_r = 1.0 \). In order to find the square of the eigen values equation (2) is considered.

3.1 Square domain with perfectly conducting sides

Square domain having side length of \( \pi \) is considered. All the sides/boundaries of square domain are perfectly conducting. \( E \times n = 0 \) is considered as the boundary condition where \( n \) is the normal of the sides of the square. Here the domain is discretized with the 256 quadrilateral elements (Q4) to perform numerical analysis. In Table. 3 square of eigen values are presented. These values are compared with analytical results reported in [18] and numerical results(nodal based) from [19]. Here, B9 is the conventional 9-node quadrilateral element. The percentage of error with analytical results against the numerical results of edge element are calculated as shown in Table. 3. It is observed that error lies below 0.8%. Proposed element(mesh) results are showing good match with the analytical and numerical results of nodal elements. It also gave correct multiplicity of eigen values. For all the elements first non zero eigen value occurred after stating the number of zeros generated at machine precision level. These zeros signifies the approximation of null space.

| Analytical (Benchmark) | Nodal element (B9) | Edge element (Q4) | % Error between analytical results and edge element results |
|------------------------|--------------------|-------------------|----------------------------------------------------------|
| 1                      | 1.000033           | 1.000803          | 0.0803                                                   |
| 1                      | 1.000033           | 1.000803          | 0.0803                                                   |
| 2                      | 2.000066           | 2.001607          | 0.0803                                                   |
| 4                      | 4.002049           | 4.012868          | 0.3217                                                   |
| 4                      | 4.002049           | 4.012868          | 0.3217                                                   |
| 5                      | 5.002081           | 5.013671          | 0.2734                                                   |
| 5                      | 5.002081           | 5.013671          | 0.2374                                                   |
| 8                      | 8.004097           | 8.025735          | 0.3217                                                   |
| 9                      | 9.022487           | 9.065245          | 0.7249                                                   |
| 9                      | 9.022487           | 9.065245          | 0.7249                                                   |
| 10                     | 10.022520          | 10.066048         | 0.6605                                                   |
| 10                     | 10.022520          | 10.066048         | 0.6605                                                   |
| 13                     | 13.024535          | 13.078112         | 0.6009                                                   |
| 13                     | 13.024535          | 13.078112         | 0.6009                                                   |
| Number of computed zeros | -                  | 961               | 220                                                       |

Table 3: \( k_0^2 \) on the square domain for different elements.

4
4. Conclusion

Edge element based FEM will nullify the presence of spurious modes and singularities in modal analysis and harmonic analysis. In implementation of FEM information of the edges of the element are to be supplied. With the conversion algorithm 4-node quadrilateral element data is converted to 4-edge quadrilateral element data. Square domain of dimension π is discretized with the quadrilateral edge element to perform eigen analysis and is compared with the analytical values.

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