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**Precursor phenomena of nucleations of quantized vortices in the presence of a uniformly moving obstacle in Bose-Einstein condensates**

the date of receipt and acceptance should be inserted later

**Keywords** superfluidity, Bose-Einstein condensation, critical velocity, quantized vortex

**Abstract** We investigate the excitation and the fluctuation of Bose-Einstein condensates in a two-dimensional torus with a uniformly moving Gaussian potential by solving Gross-Pitaevskii equation and the Bogoliubov equation. A scaling law of the energy gap in the finite system is found. This scaling law suggests that dynamical critical phenomena occur in this system. Near the critical velocity, we show that low-energy local density fluctuations are enhanced. These can be regarded as precursor phenomena of the vortex nucleation.

PACS numbers: 03.75.Kk, 67.85.-d, 67.25.dg, 67.85.De

1 Introduction

Persistent current, which is a dissipationless infinite lifetime flow, is one of the most intriguing phenomenon in condensed matter physics[1]. Owing to the development of experimental techniques of cold atomic gases[2], this remarkable state has been observed in experiments of Bose-Einstein condensates(BECs) in annular traps[3,4,5,6,7]. An annular geometry provides us a suitable stage for studying fundamental properties of superfluidity because they emerge more clearly than in simply-connected systems[1].

One of the most important issues of superfluidity is the existence of a critical velocity[8,9], above which superfluidity breaks down. In cold atomic gases experiments with blue-detuned laser beams, the critical velocity has been observed in simply-connected[10,11,12,13] and multiply-connected[7] systems. In both cases, nucleations of the quantized vortices have been observed above the

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critical velocity. The observed critical velocity is lower than that of Landau’s prediction[8]. This arises from inhomogeneity of the systems. In order to study the vortex nucleation above the critical velocity, we must treat the non-uniform systems.

Our goal is to reveal a mechanism of the breakdown of superfluidity due to the vortex nucleation from fundamental properties of the non-uniform system such as excitation spectra and fluctuations. Theoretical studies of breakdown of superfluidity have been done by the Gross-Pitaevskii (GP) equation[14] in various setups[15-16-17-18-19-20-21-22-23-24-25-26-27-28-29-30-31-32-33-34]. These studies have reported many nontrivial phenomena such as nonlinear dynamics of solitons or vortices etc. However, the properties of excitation spectrum related to the vortex nucleation in systems without translational or rotational symmetry have not been studied so far. Exception is soliton nucleation in infinite 1D system with a point-like defect or higher dimensional system with a sheet-like potential[30]. This work showed that the dynamical local density fluctuations are enhanced near the critical velocity. This phenomenon is a precursor effect of breakdown of superfluidity. Since both solitons and quantized vortices are topological defects that kill superfluidity, we can expect that the dynamical density fluctuations are enhanced near the critical velocity as a precursor effect of nucleation of vortices.

In this paper, we study the stability of a BEC confined in a two-dimensional torus with a moving Gaussian potential by the GP and the Bogoliubov equation[35]. Our setup corresponds to a simplified model of the recent experiment of a BEC in a ring trap with a rotating weak link[7]. Since we focus on the effect of the Gaussian potential, we neglect the surface and the centrifugal effects.

2 Model

We consider an N-Boson system confined in a two-dimensional torus \([-L/2, L/2] \times [-L/2, L/2]\) with a moving Gaussian potential \(U(r + vt)\), where a velocity \(v \equiv ve_x(v > 0)\) of the potential is parallel to the unit vector \(e_x\) of \(x\)-direction. In the mean-field approximation, the GP equation[14] in the moving frame with the potential is given by

\[
-\frac{\hbar^2}{2m} \nabla^2 \Psi(r) + U(r)\Psi(r) + g|\Psi(r)|^2\Psi(r) = \mu\Psi(r),
\]

where \(\Psi(r)\) is the condensate wave function(order parameter of BEC), \(U(r) \equiv U_0 \exp\left[-(r/d)^2\right]\) is the Gaussian potential, \(g(>0)\) is the strength of the repulsive interaction, and \(\mu\) denotes the chemical potential that is determined by the condition \(N = \int d\mathbf{r} |\Psi(r)|^2\). We impose the twisted periodic boundary condition

\[
\Psi(r + Le_x) = e^{i\frac{mvL}{\hbar}}\Psi(r), \quad \Psi(r + Le_y) = \Psi(r)
\]

where \(e_x\) denotes the unit vector of \(y\)-direction. We note that the boundary condition[2] does not change under the transformation \(v \rightarrow v + 2\pi\hbar n/(mL)\), where \(n\) corresponds to the winding number.
The Bogoliubov equation is given by

\[
\begin{bmatrix}
\mathcal{L} & -g|\Psi(r)|^2 \\
g|\Psi^*(r)|^2 & -\mathcal{L}^* \\
\end{bmatrix}
\begin{bmatrix}
u_i(r) \\
v_i(r) \\
\end{bmatrix} = E_i \begin{bmatrix}
u_i(r) \\
v_i(r) \\
\end{bmatrix},
\]

(3)

\[
\mathcal{L} \equiv -\frac{\hbar^2}{2m} \nabla^2 + U(r) - \mu + 2g|\Psi(r)|^2,
\]

(4)

where \(\nu_i(r)\) and \(v_i(r)\) are the wave functions of the excited state \(i\) and \(E_i\) is the excitation energy of the state \(i\). The wave functions of the excited states satisfy the following orthonormal conditions \(\int dr [\nu_i(r)\nu_j^*(r) - \nu_i(r)v_j^*(r)] = \delta_{ij}\) and \(\int dr [\nu_i(r)v_j(r) - \nu_i(r)v_j(r)] = 0\) and the boundary conditions \(u_i(r + Le_x) = e^{-im\xi}h\nu_i(r), u_i(r + Le_y) = u_i(r), v_i(r + Le_x) = e^{-im\xi}h\nu_i(r),\) and \(v_i(r + Le_y) = v_i(r)\).

We take the unit of the length, velocity, and energy, respectively, as the healing length \(\xi \equiv \hbar/\sqrt{mg0}\), the sound velocity \(v_s \equiv \sqrt{g0/m}\), and \(\varepsilon_0 \equiv \hbar^2/m\xi^2 = g0\), where \(n_0 \equiv N/L^2\) represents a mean-particle density. Here, we set \(1/\sqrt{n_0\xi^2} = 0.1\).

Our numerical calculations are performed in the \(k\)-space, that is, we expand \(\Psi(r), \nu_i(r),\) and \(v_i(r)\) with the plane wave \(e^{ikr}\), where \(k \equiv (2\pi/L)(n_xe_x + n_ye_y)\) and \(n_x\) and \(n_y\) are integers. Rewriting the GP and the Bogoliubov equations in terms of the expansion coefficients, we solve them numerically. The solutions of the GP equation are obtained by the imaginary time evolution method. The Bogoliubov equation is diagonalized by the same technique as that used in Ref. [36]. Note that we have checked that the present results do not depend on the number of basis.

3 Results

First, we define the critical velocity in our system. In many cases, the critical velocity is defined by the point where the excitation spectra exhibit the anomalous behavior such as the emergence of excitations with negative energy (Landau instability) or complex energy (dynamical instability). However, our system does not exhibit these anomalies as shown later. In our system, a phase slip, which changes the winding number, occurs at a certain velocity. Therefore, we define the critical velocity as the point where the phase slip occurs. In the present work, we focus on the winding number \(n = 0\) state.

Here, we present the stable stationary solution of the GP equation near the critical velocity in Fig. [1]. In the presence of the strong potential, we find that the vortex pair appears in the low density region. This vortex pair is called ghost vortex pair (GVP[39,31]. Although GVP exists, the stationary solution is stable because the excitation spectra, which will be shown later, do not have anomaly. Other physical quantities except local phase spectral function, which will be discussed later, do not also have singularities. The reason why the stationary solution is stable can be understood that the GVP is pinned to the potential. Free vortex motions that cause the energy dissipation do not occur by the pinning. In other parameter regions, especially in weak potential strength cases, the GVP does not appear in...
our numerical calculations. We cannot conclude whether or not GVP always exists in the presence of an obstacle potential because it is difficult to calculate near the critical velocity.

Now, we focus on the lowest excitation energy because this mode can induce the instability. The lowest excitation energy corresponds to an energy gap in the finite size system. We show the velocity dependence of the energy gap in Fig. 2(a). We find no anomalous behavior of the excitation spectra, that is, $\epsilon_i \geq 0$ and $\text{Im}(\epsilon_i) = 0$ for all excited states. This point is different from the properties observed in the translational\[8] or the rotational symmetric systems\[33]. These systems exhibit the Landau instability.

Although there is no anomalous behavior of the excitation spectra, we can find two characteristic behaviors of the energy gap: one is the linearly decreasing of the energy gap as a function of the velocity $v$ in the small $v$ region. This result is reflected in that of the uniform system, in which the energy gap is given by $\Delta_{\text{uni}} = \frac{2\pi g\hbar_0}{(L/\xi)}[-v/v_c + \sqrt{\pi^2/(L/\xi)^2 + 1}]$. Another one is the steeply decreasing of the energy gap in the vicinity of the critical velocity. In order to characterize this behavior, we assume the fitting function $\Delta = \Delta_0 \left( (v_c - v)/v_c \right)^c$, where $\Delta_0$, $v_c$, and $c$ are fitting parameters and perform the fitting to four sets of data points close to the critical velocity. The results are shown in Fig. 2(b). Resultant fitting parameters are shown in Table 1. Data points near the critical velocity collapse to a single power law as a function of $(v_c - v)/v_c$. We obtain the value of the exponent $c \sim 0.25$. Note that the exponent is nearly independent of the potential height, width, and the system size within our calculations (see Table 1).

Regardless of our calculations of the excitation spectra in the finite size system, our results indicate that the energy gap vanishes at $v = v_c$. From the viewpoint of non-linear physics, the origin of this behavior can be understood by the bifurcation of the solution of the differential equation. According to the bifurcation theory\[40], eigenvalues of a linearized equation of an original differential equation become zero at the bifurcation point. Here, the original equation, the linearized equation, and the bifurcation point correspond to the GP equation, the Bogoliubov equation, and the critical velocity, respectively. Additionally, the eigenvalues obey a scaling law near the bifurcation point, which is reflected in the types of the bifurcation. Our power law behavior ($c = 0.25$) is consistent with a Hamiltonian saddle
node bifurcation reported in Ref. [22]. Physically, the existence of the scaling law of the energy gap implies that the characteristic time scale diverges toward the critical velocity. In that sense, we can regard this scaling law as an evidence for dynamical critical phenomena [31].

Up to here, we revealed the characteristic properties of the excitation spectra in a non-uniform finite size system. However, we cannot obtain the detailed properties of the instability at the critical velocity only from the behavior of the energy gap. In order to investigate the details of the instability, we calculate fluctuations from the wave functions of the excited states. In the case of soliton nucleation, Kato and Watabe [30] introduced a local density spectral function $I_n(r, \varepsilon)$. In the Bogoliubov approximation, $I_n(r, \varepsilon)$ is given by

$$I_n(r, \varepsilon) \equiv \sum_i |\delta n_i(r)|^2 \delta(\varepsilon - \varepsilon_i), \quad (5)$$

$$\delta n_i(r) \equiv \Psi^*(r) u_i(r) - \Psi(r) v_i(r). \quad (6)$$

According to Ref. [30], the low-energy density fluctuations increase toward the critical velocity around the obstacle.

In our system, it is difficult to calculate the spectral function because of the discretized eigenvalues in finite size system. Instead of calculating the spectral function, we plot the matrix element (6) in Fig. 3. In the small velocity regime, the low-energy density fluctuation is small because of the existence of the energy gap (see Fig. 3(a)). The low-energy local density fluctuations grow with increasing the velocity shown in Figs. 3(b) and (c). Combining the scaling law of the energy gap and the behavior of the dynamical density fluctuations, we can expect that the zero-energy eigenstate that contributes the density fluctuations exists at $v = v_c$. In the previous work for the soliton nucleation in infinite 1D system [41], the zero-mode related to the density fluctuations was obtained by analytical calculations and plays a crucial role of the breakdown of superfluidity. Therefore, in the case of the vortex nucleation, the density fluctuation is also expected to be a key to understand the breakdown of superfluidity.

Besides the local density spectral function (eq. (5)), Watabe and Kato [42] also defined the local phase spectral function. In the Bogoliubov approximation, this is given by

$$I_{\phi}(r, \varepsilon) = \sum_i |\delta \phi_i(r)|^2 \delta(\varepsilon - \varepsilon_i), \quad \delta \phi_i(r) \equiv \frac{\Psi^*(r) u_i(r) + \Psi(r) v_i(r)}{2|\Psi(r)|^2}. \quad (7)$$
They showed that the local phase spectral function is not useful to detect the critical velocity. The local phase fluctuation is nearly independent of the velocity because the zero mode related to the phase fluctuations always exist as a result of the $U(1)$ symmetry breaking. Furthermore, there is no phase singularity in 1D system. However, in our system, the matrix element of the phase fluctuation diverges locally because of the presence of the GVP. Although we can detect the existence of the GVP from the divergence of the matrix element of the local phase fluctuation, the critical velocity cannot be detected from it because the GVP appears for $v < v_c$.

Table 1 Fitting results.

| $L/\xi$ | $U_0/\varepsilon_0$ | $d/\xi$ | exponent         | $L/\xi$ | $U_0/\varepsilon_0$ | $d/\xi$ | exponent         |
|--------|---------------------|--------|------------------|--------|---------------------|--------|------------------|
| 32     | 1                   | 2.0    | 0.24879(6)       | 32     | 10                  | 2.5    | 0.25139(7)       |
| 32     | 1                   | 2.5    | 0.24880(5)       | 32     | 20                  | 2.0    | 0.24994(5)       |
| 32     | 5                   | 2.0    | 0.24849(6)       | 32     | 20                  | 2.5    | 0.25130(6)       |
| 32     | 5                   | 2.5    | 0.25125(3)       | 48     | 10                  | 2.5    | 0.24875(9)       |
| 32     | 10                  | 2.0    | 0.24905(9)       | 64     | 10                  | 2.5    | 0.24862(6)       |

4 Summary and discussion

In summary, we studied the excitation spectra and the fluctuations of a BEC in a 2D torus with a uniformly moving Gaussian potential by solving the GP and the Bogoliubov equation. Unlike translational or rotational symmetric systems, the Landau instability and the dynamical instability were not observed. We found that the first excited energy (or the energy gap) obeys the scaling law. This implies that the dynamical critical phenomena occur. In order to characterize the instability, we investigated the fluctuations and found the enhancement of dynamical low-energy local density fluctuation toward the critical velocity. These behaviors can be regarded as precursor effects of the vortex nucleation.

Our future work is to elucidate the relation between enhancement of the dynamical density fluctuations and the breakdown of superfluidity (or energy dissipation). It is difficult to measure the local density spectral function experimentally. We must find the relation between the local density spectral function and other
physical quantities such as transport coefficients. In the case of the Landau instability, the authors of Ref. [26] showed that the energy dissipation and the density fluctuation are related through the drag force by using the perturbative approach. Although their approach is not applicable for the vortex and soliton nucleation, their results give us a clue to the relation between the vortex nucleation and the energy dissipation.

Acknowledgements We thank S. Watabe for fruitful discussions. M. K. acknowledges the support of a Grant-in-Aid for JSPS Fellows (23054032) and (24543061) from JSPS and (20029007) from MEXT in Japan.

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