Distributed Consensus Algorithms in Sensor Networks: Quantized Data and Random Link Failures

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Abstract

The paper studies the problem of distributed average consensus in sensor networks with quantized data and random link failures simultaneously. We consider two versions of the algorithm: unbounded quantizers (algorithm QC) and bounded quantizers (algorithm QCF). To achieve consensus, dither (small noise) is added to the sensor states before quantization. We show by stochastic approximation techniques that consensus is asymptotically achieved with probability one to a finite random variable. For the QC algorithm we show that the mean-squared error (m.s.e.) can be made arbitrarily small by tuning the link weight sequence, at a cost of the convergence rate. For the QCF algorithm we study the tradeoffs between how far away is this limiting random variable from the desired average, the consensus convergence rate, the quantizer parameters, and the network topology. We cast these tradeoff issues as an optimal quantizer design that we solve. A numerical study illustrates the design tradeoffs.

Keywords: Consensus, quantized, random link failures, stochastic approximation, convergence

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I. INTRODUCTION

This paper is concerned with consensus in networks, e.g., a sensor network, when the data exchanges among nodes in the network (sensors, agents) are quantized. Consensus is broadly understood as individuals in a community achieving a consistent view of the World by interchanging information regarding their current state with their neighbors. Considered in the early work of Tsitsiklis et. al. ([1], [2]), it has received considerable attention in recent years and arises in numerous applications including: load balancing, [3], alignment, flocking, and multi-agent collaboration, e.g., [4], [5], vehicle formation, [6], gossip algorithms, [7], tracking, data fusion, [8], and distributed inference, [9]. We refer the reader to the recent overviews on consensus, which include [10], [11].

Consensus is a distributed iterative algorithm where the sensor states evolve on the basis of local interactions. Reference [5] used spectral graph concepts like graph Laplacian and algebraic connectivity to prove convergence for consensus under several network operating conditions (e.g., delays and switching networks, i.e., time varying). Our own work has been concerned with designing topologies that optimize consensus with respect to the convergence rate, [12], [9]. Topology design is concerned with two issues: 1) the definition of the graph that specifies the neighbors of each sensor—i.e., with whom should each sensor exchange data; and 2) the weights used by the sensors when combining the information received from their neighbors to update their state. Reference [13] considers the problem of weight design, when the topology is specified, in the framework of semi-definite programming. References [14], [15] considered the impact of different topologies on the convergence rate of consensus, in particular, regular, random, and small-world graphs, [16].

In wireless sensor network applications, bandwidth and power allocated to the inter-sensor communication channels are limited, which precludes exchange of high precision (analog) data among sensors. Also, the underlying random environment in applications, result in random data packet dropouts. This motivates us to consider the consensus problem with quantized communication and random inter-sensor link failures (see also [17], where parts of the results of this paper were presented.) In this paper, we consider two variations on consensus with quantized data: 1) Quantized Consensus (QC) where the alphabet of the quantizers at the sensors is countable to account for possibly unbounded initial sensor states; and 2) Quantized Consensus with Finite (QCF) number of bits, i.e., the alphabet is finite. To avoid divergence due to error accumulation, we add a small amount of noise, dither, to the data before quantization and use a sequence of weights that satisfy a persistence condition—their sum diverges, while their square sum is finite. In both the algorithms we capture the randomness of the network topology by assuming that the time-varying Laplacian sequence, \( \{L(i)\}_{i \geq 0} \) is independent with mean \( \mathbb{L} \), such that \( \lambda_2(\mathbb{L}) > 0 \) (we just require the network to be connected on the average.) We do not make any distributional assumptions on the link failure model. During the same iteration, the link failures can be spatially dependent, i.e., correlated across different edges of the network. This model subsumes the erasure network model, where the link failures are independent both over space and time. Wireless sensor networks motivate this model since interference among the sensors communication correlates the link failures over space, while over time, it is still reasonable to assume that the channels are memoryless or independent. We also note that the above assumption \( \lambda_2(\mathbb{L}) > 0 \) does not require the individual random instantiations of \( L(i) \) to be connected; in fact, it is possible to have all the
instantiations to be disconnected. This enables us to capture a broad class of asynchronous communication models, for example, the random asynchronous gossip protocol analyzed in [18] satisfies $\lambda_2 (L) > 0$ and hence falls under this framework. We cast the QC and QCF algorithms as distributed stochastic iterative algorithms and prove their convergence using stochastic approximation theory. We prove that these quantized consensus algorithms lead to a.s. consensus of the sensor states to a finite random variable. For the QC algorithm, we show that there exists an interesting trade-off between the m.s.e. (between the limiting random variable and the desired initial average) and the convergence: by tuning the link weight sequence appropriately, it is possible to make the m.s.e. arbitrarily small (irrespective of the quantization step-size), though penalizing the convergence rate. For the QCF algorithm, we show that asymptotically the quantized states get within a ball of radius $\epsilon$, $\epsilon$-consensus, of the desired average with high probability. We establish analytically several tradeoffs between consensus closeness, consensus convergence rate, quantizer parameters, and network topology, which we cast as an optimal quantizer design problem. We illustrate these tradeoffs numerically. We consider generalizations of the above algorithms, which include relaxing the temporal independence assumption on the link failure to temporally Markovian failures and a general class of time-varying quantization step-size models.

Distributed consensus with quantized transmission has been studied recently in [19], [20], [21], [22] with respect to time-invariant topologies, while [23] considers the quantized consensus problem for a certain class of time-varying topologies. The algorithm in [19] is restricted to integer-valued initial sensor states, where at each iteration the sensors exchange integer-valued data. It is shown there that the sensor states are asymptotically close (in the appropriate sense defined there) to the desired average, but may not reach absolute consensus. In [20], the authors interpret the noise in the consensus algorithm studied in [24] as quantization noise and show by simulation with a small network that the variance of the quantization noise is reduced as the algorithm iterates and the sensors converge to a consensus. References [21], [25] study probabilistic quantized consensus. Each sensor updates its state at each iteration by probabilistically quantizing its current state (which is equivalent to dithering, as established in [26]) and linearly combining it with the quantized versions of the states of the neighbors. They show that the sensor states reach consensus a.s. to a quantized level. In [22] a worst case analysis is presented on the error propagation of consensus algorithms with quantized communication for various classes of time-invariant network topologies, while [27] addresses the impact of more involved encoding/decoding strategies, beyond the uniform quantizer. Consensus algorithms with general imperfect communication (including quantization) in a certain class of time-varying topologies has been addressed in [23], which assumes that there exists a window of fixed length, such that the union of the network graphs formed within that window is strongly connected.

The resulting m.s.e. of the algorithms in [19], [20], [21], [22], [23] is proportional to $\Delta^2$, the quantization step-size. This means that if the step-size is large, these algorithms will lead to a large m.s.e. On the other hand, our algorithm is equipped with a scalar control parameter $s$ (associated with the time-varying link weight sequence), which can be tuned to make the m.s.e. as small as we want, irrespective of how large the step-size $\Delta$ is. This is significant, because certain applications which rely on accuracy, may call for very small m.s.e. for being useful. Our algorithm achieves this by simply tuning the parameter $s$ appropriately. The parameter $s$ can then be viewed
as a scalar control parameter, which can be used to trade-off between precision (m.s.e.) and convergence rate. More specifically, if a cost structure is imposed on the consensus problem, where the objective is a function of the m.s.e. and the convergence rate, one may obtain the optimal scaling $s$ minimizing the cost from the pareto-optimal curve generated by varying $s$. Another important feature of our algorithm is the simultaneous treatment of quantized transmission with a very generic model of random link failures, which capture a broad class of random synchronous and asynchronous communication models. Finally, our approach accommodates generalizations which permit temporally Markovian link failures and a large class of time-varying quantization step-size models, as explained in the paper.

We comment briefly on the organization of the main sections of the paper. Section II summarizes relevant background, including spectral graph theory and average consensus, and presents the dithered quantized consensus problem with the dither satisfying the Schuchman conditions. Sections III and IV consider the convergence of the QC and QCF algorithms. They show a.s. convergence to a random variable, whose m.s.e. is fully characterized. Section IV also studies tradeoffs among different quantizer parameters, e.g., number of bits and quantization step, and the network topology to achieve optimal performance under a constraint on the number of levels of the quantizer. Finally, Section V concludes the paper.

II. CONSENSUS WITH QUANTIZED DATA: PROBLEM STATEMENT

This section presents very briefly preliminaries needed for the analysis of the consensus algorithm with quantized data. The set-up is standard, see the introductory sections of relevant recent papers on consensus, in particular, the companion manuscript [28].

A. Preliminaries: Average Consensus

We consider consensus in the context of spectral graph theory where the sensor network at time index $i$ is represented by an undirected, simple, connected graph $G(i) = (V, E(i))$. The vertex and edge sets $V$ and $E(i)$, with cardinalities $|V| = N$ and $|E(i)| = M(i)$, collect the sensors and communication channels or links among sensors in the network at time $i$. The network topology at time $i$, i.e., with which sensors does each sensor communicate with, is described by the $N \times N$ discrete Laplacian $L(i) = L^T(i) = D(i) - A(i) \geq 0$. The matrix $A(i)$ is the adjacency matrix of the connectivity graph at time $i$, a $(0, 1)$ matrix where $A_{nk}(i) = 1$ signifies that there is a link between sensors $n$ and $k$ at time $i$. The diagonal entries of $A(i)$ are zero. The diagonal matrix $D(i)$ is the degree matrix, whose diagonal $D_{nn}(i) = d_n(i)$ where $d_n(i)$ is the degree of sensor $n$, i.e., the number of links of sensor $n$ at time $i$. The neighbors of a sensor or node $n$, collected in the neighborhood set $\Omega_n(i)$, are those sensors $k$ for which entries $A_{nk}(i) \neq 0$. It can be shown that the Laplacian is positive semidefinite; in case the network is connected at time $i$, the corresponding algebraic connectivity or Fiedler value is positive, i.e., the second eigenvalue of the Laplacian $\lambda_2(L(i)) > 0$, where the eigenvalues of $L(i)$ are ordered in increasing order. For detailed treatment of graphs and their spectral theory see, for example, [29], [30], [31].
Distributed Average Consensus. Let the sensors in a sensor network measure the data \( x_n(0), n = 1, \ldots, N \). We collect these data in the vector \( x(0) = [x_1(0) \cdots x_N(0)]^T \in \mathbb{R}^{N \times 1} \). The goal of distributed average consensus is to compute the average \( r \) of these initial data
\[
r = x_{\text{avg}}(0) = \frac{1}{N} \sum_{n=1}^{N} x_n(0)
\]
by local data exchanges among neighboring sensors. In (2), the column vector \( 1 \) has all entries equal to 1. Consensus is an iterative algorithm where at iteration \( i \) each sensor updates its current state \( x_n(i) \) by a weighted average of its current state and the states of its neighbors. Standard consensus assumes a fixed connected network topology, i.e., the links stay online permanently, the communication is noiseless, and the data exchanges are analog. Under mild conditions, the states of all sensors reach consensus, converging to the desired average \( r \), see [5], [13].
\[
\lim_{i \to \infty} x(i) = r 1
\]
where \( x(i) = [x_1(i) \cdots x_N(i)]^T \) is the state vector that stacks the state of the \( N \) sensors at iteration \( i \). We assume here the framework of standard consensus, but consider the data exchanges to be quantized. In [28], we considered consensus when the topologies are random (links fail or become alive at random times) and the links are noisy (analog noise) simultaneously.

B. Dithered Quantization: Schuchman Conditions

We write the sensor updating equations for standard consensus with quantized data and random link failures (packet dropouts) as
\[
x_n(i+1) = [1 - \alpha(i)d_n(i)] x_n(i) + \alpha(i) \sum_{l \in \Omega_n(i)} f_{nl,i} [x_l(i)], \ 1 \leq n \leq N
\]
where: \( \alpha(i) \) is the weight at iteration \( i \); and \( \{f_{nl,i}\}_{1 \leq n, l \leq N, i \geq 0} \) is a sequence of functions (possibly random) modeling the quantization effects. Note that in (4), the weights \( \alpha(i) \) are the same across all links—the equal weights consensus, see [13]—but the weights may change with time. Also, the degree \( d_n(i) \) and the neighborhood \( \Omega_n(i) \) of each sensor \( n, n = 1, \cdots, N \) are dependent on \( i \) emphasizing the topology may be random time-varying.

**Quantizer.** Each inter-sensor communication channel uses a uniform quantizer with quantization step \( \Delta \). We model the communication channel by introducing the quantizing function, \( q(\cdot) : \mathbb{R} \to \mathcal{Q} \),
\[
q(y) = k \Delta, \ (k - \frac{1}{2}) \Delta \leq y < (k + \frac{1}{2}) \Delta
\]
where \( y \in \mathbb{R} \) is the channel input. Writing
\[
q(y) = y + e(y)
\]
where $e(y)$ is the quantization error; we have

$$-\frac{\Delta}{2} \leq e(y) < \frac{\Delta}{2}, \quad \forall y$$

Conditioned on the input, the quantization error $e(y)$ is deterministic.

We consider two cases. In the first, quantized consensus (QC), the quantization alphabet

$$Q = \{k\Delta \mid k \in \mathbb{Z}\}$$

is countably infinite. In the second, quantized consensus with finite (QCF) alphabet, the alphabet is finite. The QCF quantizer model is in Section IV. The QC quantizer alphabet (8) may model the problem where there is no prior knowledge about the range of the initial data measured by the sensors, while QCF assumes that these data are bounded by a fixed constant. The QC results are used to study the convergence of QCF.

We discuss briefly why a naive approach to consensus will fail (see [26] for a similar discussion.) If we use directly the quantized state information, the functions $f_{n,t,i}(\cdot)$ in eqn. (4) are

$$f_{n,t,i}(x_l(i)) = q(x_l(i))$$

$$= x_l(i) + e(x_l(i))$$

Equations (4) take then the form

$$x_{n}(i + 1) = \left[ (1 - \alpha(i))d_{n}(i))x_{n}(i) + \alpha(i) \sum_{l \in \Omega_n(i)} x_l(i) \right] + \alpha(i) \sum_{l \in \Omega_n(i)} e(x_l(i))$$

The non-stochastic errors (the most right terms in (11)) lead to error accumulation. If the network topology remains fixed (deterministic topology,) the update in eqn. (11) represents a sequence of iterations that, as observed above, conditioned on the initial state, which then determines the input, are deterministic. If we choose the weights $\alpha(i)$’s to decrease to zero very quickly, then (11) may terminate before reaching the consensus set. On the other hand, if the $\alpha(i)$’s decay slowly, the quantization errors may accumulate, thus making the states unbounded.

In either case, the naive approach to consensus with quantized data fails to lead to a reasonable solution. This failure is due to the fact that the error terms are not stochastic. To overcome these problems, we introduce in a controlled way noise (dither) to randomize the sensor states prior to quantizing the perturbed stochastic state. We will show that, under appropriate conditions, the resulting quantization errors possess nice statistical properties, leading to the quantized states reaching consensus (in an appropriate sense to be defined below.) Dither places consensus with quantized data in the framework of distributed consensus with noisy communication links; we will apply stochastic approximation arguments to study the limiting behavior; we also used stochastic approximation to show convergence of two versions of consensus with noise that we introduced in [28], the $A - ND$ and $A - NC$ algorithms.

**Schuchman conditions.** We assume that the dither the dither that we add to randomize the quantization effects
satisfies a special condition that we consider now. Let \( \{y(i)\}_{i \geq 0} \) and \( \{\nu(i)\}_{i \geq 0} \) be arbitrary sequences of random variables, and \( q(\cdot) \) be the quantization function (5). When dither is added before quantization, the quantization error sequence, \( \{\varepsilon(i)\}_{i \geq 0} \), is

\[
\varepsilon(i) = q(y(i) + \nu(i)) - (y(i) + \nu(i))
\]  

This corresponds to subtractively dithered systems, see [32], [33].

It can be shown that, if the dither sequence, \( \{\nu(i)\}_{i \geq 0} \), satisfies the Schuchman conditions, [34], then the quantization error sequence, \( \{\varepsilon(i)\}_{i \geq 0} \), in (12) is i.i.d. uniformly distributed on \([-\Delta/2, \Delta/2]\) and independent of the input sequence \( \{y(i)\}_{i \geq 0} \) (see [35], [36], [32]). A sufficient condition for \( \{\nu(i)\} \) to satisfy the Schuchman conditions is for it to be a sequence of i.i.d. random variables uniformly distributed on \([-\Delta/2, \Delta/2]\) and independent of the input sequence \( \{y(i)\}_{i \geq 0} \). In the sequel, the dither \( \{\nu(i)\}_{i \geq 0} \) satisfies the Schuchman conditions. Hence, the quantization error sequence, \( \{\varepsilon(i)\} \), is i.i.d. uniformly distributed on \([-\Delta/2, \Delta/2]\) and independent of the input sequence \( \{y(i)\}_{i \geq 0} \).

C. Dithered Quantized Consensus With Random Link Failures: Problem Statement

We now return to the problem formulation of consensus with quantized data with dither added. Introducing the sequence, \( \{\nu_{nl}(i)\}_{i \geq 0, 1 \leq n, l \leq N} \), of i.i.d. random variables, uniformly distributed on \([-\Delta/2, \Delta/2]\), the state update equation for quantized consensus is:

\[
x_n(i + 1) = (1 - \alpha(i)d_n(i)) x_n(i) + \alpha(i) \sum_{l \in \Omega_n(i)} q[x_l(i) + \nu_{nl}(i)], \quad 1 \leq n \leq N
\]  

This equation shows that, before transmitting its state \( x_l(i) \) to the \( n \)-th sensor, the sensor \( l \) adds the dither \( \nu_{nl}(i) \), then the channel between the sensors \( n \) and \( l \) quantizes this corrupted state, and, finally, sensor \( n \) receives this quantized output. Using eqn. (12), the state update is

\[
x_n(i + 1) = (1 - \alpha(i)d_n(i)) x_n(i) + \alpha(i) \sum_{l \in \Omega_n(i)} [x_l(i) + \nu_{nl}(i) + \varepsilon_{nl}(i)]
\]  

The random variables \( \nu_{nl}(i) \) are independent of the state \( x(j) \), i.e., the states of all sensors at iteration \( j \), for \( j \leq i \). Hence, the collection \( \{\varepsilon_{nl}(i)\} \) consists of i.i.d. random variables uniformly distributed on \([-\Delta/2, \Delta/2]\), and the random variable \( \varepsilon_{nl}(i) \) is also independent of the state \( x(j) \), \( j \leq i \).

We rewrite (14) in vector form. Define the random vectors, \( \Upsilon(i) \) and \( \Psi(i) \in \mathbb{R}^{N \times 1} \) with components

\[
\Upsilon_n(i) = -\sum_{l \in \Omega_n(i)} \nu_{nl}(i)
\]  

\[
\Psi_n(i) = -\sum_{l \in \Omega_n(i)} \varepsilon_{nl}(i)
\]  

The the \( N \) state update equations in (14) become in vector form

\[
x(i + 1) = x(i) - \alpha(i) [L(i)x(i) + \Upsilon(i) + \Psi(i)]
\]  

where $\Upsilon(i)$ and $\Psi(i)$ are zero mean vectors, independent of the state $x(i)$, and have i.i.d. components. Also, if $|\mathcal{M}|$ is the number of realizable network links, eqns. (15) and (16) lead to

$$E\left[\|\Upsilon(i)\|^2\right] = E\left[\|\Psi(i)\|^2\right] \leq \frac{|\mathcal{M}|\Delta^2}{6}, i \geq 0$$

(18)

**Random Link Failures:** We now state the assumption about the link failure model to be adopted throughout the paper. The graph Laplacians are

$$L(i) = \overline{L} + \tilde{L}(i), \ \forall i \geq 0$$

(19)

where $\{L(i)\}_{i \geq 0}$ is a sequence of i.i.d. Laplacian matrices with mean $\overline{L} = E[L(i)]$, such that $\lambda_2(\overline{L}) > 0$ (we just require the network to be connected on the average.) We do not make any distributional assumptions on the link failure model. During the same iteration, the link failures can be spatially dependent, i.e., correlated across different edges of the network. This model subsumes the erasure network model, where the link failures are independent both over space and time. Wireless sensor networks motivate this model since interference among the sensors communication correlates the link failures over space, while over time, it is still reasonable to assume that the channels are memoryless or independent. We also note that the above assumption $\lambda_2(\overline{L}) > 0$ does not require the individual random instantiations of $L(i)$ to be connected; in fact, it is possible to have all the instantiations to be disconnected. This enables us to capture a broad class of asynchronous communication models, for example, the random asynchronous gossip protocol analyzed in [18] satisfies $\lambda_2(\overline{L}) > 0$ and hence falls under this framework.

More generally, in the asynchronous set up, if the sensors nodes are equipped with independent clocks whose ticks follow a regular random point process (the ticking instants do not have an accumulation point, which is true for all renewal processes, in particular, the Poisson clock in [18]), and at each tick a random network is realized with $\lambda_2(\overline{L}) > 0$ independent of the the networks realized in previous ticks (this is the case with the link formation process assumed in [18]) our algorithm applies.

We denote the number of network edges at time $i$ as $M(i)$, where $M(i)$ is a random subset of the set of all possible edges $\mathcal{E}$ with $|\mathcal{E}| = N(N-1)/2$. Let $\mathcal{M}$ denote the set of realizable edges. We then have the inclusion

$$M(i) \subset \mathcal{M} \subset \mathcal{E}, \ \forall i$$

(20)

It is important to note that the value of $M(i)$ depends on the link usage protocol. For example, in the asynchronous gossip protocol considered in [18], at each iteration only one link is active, and hence $M(i) = 1$.

**Independence Assumptions:** We assume that the Laplacian sequence $\{L(i)\}_{i \geq 0}$ is independent of the dither sequence $\{\varepsilon_{nl}(i)\}$.

**Persistence condition:** To obtain convergence, we assume that the gains $\alpha(i)$ satisfy the following.

$$\alpha(i) > 0, \sum_{i \geq 0} \alpha(i) = \infty, \sum_{i \geq 0} \alpha^2(i) < \infty$$

(21)

Condition (21) assures that the gains decay to zero, but not too fast. It is standard in stochastic adaptive signal processing and control; it is also used in consensus with noisy communications in [37], [28].
Markov property. Denote the natural filtration of the process \( X = \{x(i)\}_{i \geq 0} \) by \( \{\mathcal{F}_i^X\}_{i \geq 0} \). Because the dither random variables \( \nu_{nl}(i), 1 \leq n, l \leq N \), are independent of \( \mathcal{F}_i^X \) at any time \( i \geq 0 \), and, correspondingly, the noises \( \Upsilon(i) \) and \( \Psi(i) \) are independent of \( x(i) \), the process \( X \) is Markov.

III. Consensus With Quantized Data: Unbounded Quantized States

We consider that the dynamic range of the initial sensor data, whose average we wish to compute, is not known. To avoid quantizer saturation, the quantizer output takes values in the countable alphabet (8), and so the channel quantizer has unrestricted dynamic range. This is the quantizer consensus (QC) algorithm. Section IV studies the quantized consensus finite-bit (QCF) algorithm, a modification of QC, where the initial sensor data is bounded (the dynamic range is known \textit{a priori},) enabling the use of channel quantizers that take only a finite number of output values (finite-bit quantizers).

We comment briefly on the organization of the remaining of this section. Subsection III-A proves the a.s. convergence of the QC algorithm. We characterize the performance of the QC algorithm and derive expressions for the mean-squared error in Subsection III-B. The tradeoff between m.s.e. and convergence rate is studied in Subsection III-C. Finally, we present generalizations to the approach in Subsection III-D.

A. QC Algorithm: Convergence

We start with the definition of the consensus subspace \( \mathcal{C} \) given as

\[
\mathcal{C} = \{x \in \mathbb{R}^{N \times 1} \mid x = a1, a \in \mathbb{R}\}
\]  

(22)

We show that (17), under the model in Subsection II-C, converges a.s. to a finite point in \( \mathcal{C} \).

Define the component-wise average as

\[
x_{\text{avg}}(i) = \frac{1}{N}1^T x(i)
\]

(23)

We prove the a.s. convergence of the QC algorithm in two stages. Theorem 2 proves that the state vector sequence \( \{x(i)\}_{i \geq 0} \) converges a.s. to the consensus subspace \( \mathcal{C} \). Theorem 3 then completes the proof by showing that the sequence of component-wise averages, \( \{x_{\text{avg}}(i)\}_{i \geq 0} \) converges a.s. to a finite random variable \( \theta \). The proof of Theorem 3 needs a basic result on convergence of Markov processes and follows the same theme of analysis as employed in [28].

Stochastic approximation: Convergence of Markov processes. We state a slightly modified form, suitable to our needs, of a result from [38]. We start by introducing notation, following [38], see also [28].

Let \( X = \{x(i)\}_{i \geq 0} \) be Markov in \( \mathbb{R}^{N \times 1} \). The generating operator \( \mathcal{L} \) is

\[
\mathcal{L}V (i, x) = \mathbb{E} [V (i + 1, x(i + 1)) \mid x(i) = x] - V (i, x) \text{ a.s.}
\]

(24)

for functions \( V(i, x), i \geq 0, x \in \mathbb{R}^{N \times 1} \), provided the conditional expectation exists. We say that \( V(i, x) \in D_\mathcal{L} \) in a domain \( A \), if \( \mathcal{L}V(i, x) \) is finite for all \( (i, x) \in A \).
Let the Euclidean metric be \( \rho(\cdot) \). Define the \( \epsilon \)-neighborhood of \( B \subset \mathbb{R}^{N \times 1} \) and its complementary set

\[
U_\epsilon(B) = \left\{ x \mid \inf_{y \in B} \rho(x, y) < \epsilon \right\}
\]

\[
V_\epsilon(B) = \mathbb{R}^{N \times 1} \setminus U_\epsilon(B)
\]

(25)

(26)

Theorem 1 (Convergence of Markov Processes) Let: \( X \) be a Markov process with generating operator \( \mathcal{L} \); \( V(i, x) \in D_\mathcal{L} \) a non-negative function in the domain \( i \geq 0, x \in \mathbb{R}^{N \times 1} \), and \( B \subset \mathbb{R}^{N \times 1} \). Assume:

1) Potential function:

\[
\inf_{i \geq 0, x \in V_\epsilon(B)} V(i, x) > 0, \quad \forall \epsilon > 0
\]

\[
V(i, x) \equiv 0, \quad x \in B
\]

\[
\lim_{x \to B} \sup_{i \geq 0} V(i, x) = 0
\]

(27)

(28)

(29)

2) Generating operator:

\[
\mathcal{L}V(i, x) \leq g(i)(1 + V(i, x)) - \alpha(i)\varphi(i, x)
\]

(30)

where \( \varphi(i, x), i \geq 0, x \in \mathbb{R}^{N \times 1} \) is a non-negative function such that

\[
\inf_{i, x \in V_\epsilon(B)} \varphi(i, x) > 0, \quad \forall \epsilon > 0
\]

\[
\alpha(i) > 0, \quad \sum_{i \geq 0} \alpha(i) = \infty
\]

\[
g(i) > 0, \quad \sum_{i \geq 0} g(i) < \infty
\]

(31)

(32)

(33)

Then, the Markov process \( X = \{x(i)\}_{i \geq 0} \) with arbitrary initial distribution converges a.s. to \( B \) as \( i \to \infty \)

\[
\mathbb{P} \left( \lim_{i \to \infty} \rho(x(i), B) = 0 \right) = 1
\]

(34)

Proof: For proof, see [38], [28].

Theorem 2 (a.s. convergence to consensus subspace) Consider the quantized distributed averaging algorithm given in eqns. (17). Then, for arbitrary initial condition, \( x(0) \), we have

\[
\mathbb{P} \left[ \lim_{i \to \infty} \rho(x(i), \mathcal{C}) = 0 \right] = 1
\]

(35)

Proof: The key idea of the proof is to show that the quantized iterations satisfy the assumptions of Theorem 1.

Define the potential function, \( V(i, x) \), for the Markov process \( X \) as

\[
V(i, x) = x^T \mathcal{T} x
\]

(36)

Then, using the properties of \( \mathcal{T} \) and the continuity of \( V(i, x) \),

\[
V(i, x) \equiv 0, \quad x \in \mathcal{C} \quad \text{and} \quad \lim_{x \to \mathcal{C}} \sup_{i \geq 0} V(i, x) = 0
\]

(37)
For \( x \in \mathbb{R}^{N \times 1} \), we clearly have \( \rho(x, C) = \| x_{C^\perp} \| \). Using the fact that \( x^T \mathcal{L} x \geq \lambda_2(\mathcal{L}) \| x_{C^\perp} \|^2 \) it then follows

\[
\inf_{i \geq 0, x \in V_i(c)} V(i, x) \geq \inf_{i \geq 0, x \in V_i(c)} \lambda_2(\mathcal{L}) \| x_{C^\perp} \|^2 \geq \lambda_2(\mathcal{L}) \epsilon^2 > 0 \tag{38}
\]

since \( \lambda_2(\mathcal{L}) > 0 \). This shows, together with (37), that \( V(i, x) \) satisfies (27)–(29).

Now consider \( \mathcal{L} V(i, x) \). We have using the fact that \( \tilde{L}(i)x = \tilde{L}(i)x_{C^\perp} \) and the independence assumptions

\[
\mathcal{L} V(i, x) = \mathbb{E} \left[ (x(i) - \alpha(i) \tilde{L} x(i) - \alpha(i) \tilde{L} x(i) - \alpha(i) \Psi(i))^T \mathcal{L} (x(i) - \alpha(i) \tilde{L} x(i)) \right.
\]

\[ - \alpha(i) \tilde{L}(i)x(i) - \alpha(i) \Psi(i)) \big| x(i) = x \big] - x^T \mathcal{L} x \]

\[
\leq -2\alpha(i)x^T \mathcal{L} x + \alpha^2(i) \lambda_2^2(\mathcal{L}) \| x_{C^\perp} \|^2 + 2^{\alpha(i)} \lambda_N(\mathcal{L}) \mathbb{E} \left[ \lambda^2 \left( \tilde{L}(i) \right) \right] \| x_{C^\perp} \|^2
\]

\[ + 2\alpha(i) \lambda_N(\mathcal{L}) \mathbb{E} \left[ \| \Psi(i) \|^2 \right] \right]^{1/2} + \alpha^2(\lambda_N(\mathcal{L})) \mathbb{E} \left[ \| \Psi(i) \|^2 \right] \]

\[ + \alpha^2(i) \lambda_N(\mathcal{L}) \mathbb{E} \left[ \| \Psi(i) \|^2 \right] \tag{39} \]

We now use the fact that \( x^T \mathcal{L} x \geq \lambda_2(\mathcal{L}) \| x_{C^\perp} \|^2 \), the eigenvalues of \( \tilde{L}(i) \) are not greater than \( 2N \) in magnitude and eqn. (18) to get

\[
\mathcal{L} V(i, x) \leq -2\alpha(i)x^T \mathcal{L} x + \left( \frac{\alpha^2(i) \lambda_2^2(\mathcal{L})}{\lambda_2(\mathcal{L})} + \frac{4\alpha^2(i) \lambda_N^2(\mathcal{L})}{\lambda_2(\mathcal{L})^2} \right) x^T \mathcal{L} x + \frac{2\alpha^2(i) \lambda_N \lambda_2^2}{3} \tag{40} \]

where

\[
\varphi(i, x) = 2x^T \mathcal{L} x, \quad g(i) = \alpha^2(i) \left( \frac{\lambda_2^2(\mathcal{L})}{\lambda_2(\mathcal{L})} + \frac{4\lambda_N^2(\mathcal{L})}{\lambda_2(\mathcal{L})^2} \right) \]

\[ + \frac{2\lambda_N \lambda_2^2}{3} \tag{41} \]

Clearly, \( \mathcal{L} V(i, x) \) and \( \varphi(i, x), g(i) \) satisfy the remaining assumptions (30)–(33) of Theorem 1; hence,

\[
\mathbb{P} \left[ \lim_{i \to \infty} \rho(x(i), C) = 0 \right] = 1 \tag{42} \]

The convergence proof for QC will now be completed in the next Theorem.

**Theorem 3 (Consensus to finite random variable)** Consider (17), with arbitrary initial condition \( x(0) \in \mathbb{R}^{N \times 1} \) and the state sequence \( \{ x(i) \}_{i \geq 0} \). Then, there exists a finite random variable \( \theta \) such that

\[
\mathbb{P} \left[ \lim_{i \to \infty} x(i) = \theta \mathbb{1} \right] = 1 \tag{43} \]

**Proof:** Define the filtration \( \{ F_i \}_{i \geq 0} \) as

\[
F_i = \sigma \left\{ x(0), \{ L(j) \}_{0 \leq j < i}, \{ Y(j) \}_{0 \leq j < i}, \{ \Psi(j) \}_{0 \leq j < i} \right\} \tag{44} \]

We will now show that the sequence \( \{ x_{\text{avg}}(i) \}_{i \geq 0} \) is an \( L_2 \)-bounded martingale w.r.t. \( \{ F_i \}_{i \geq 0} \). In fact,

\[
x_{\text{avg}}(i + 1) = x_{\text{avg}}(i) - \alpha(i) Y(i) - \alpha(i) \Psi(i) \tag{45} \]
where $\Upsilon(i)$ and $\Psi(i)$ are the component-wise averages given by
\[
\Upsilon(i) = \frac{1}{N} \mathbf{1}^T \Upsilon(i), \quad \Psi(i) = \frac{1}{N} \mathbf{1}^T \Psi(i)
\] (46)

Then,
\[
\mathbb{E} \left[ x_{\text{avg}}(i + 1) | F_i \right] = x_{\text{avg}}(i) - \alpha(i) \mathbb{E} \left[ \Upsilon(i) | F_i \right] - \alpha(i) \mathbb{E} \left[ \Psi(i) | F_i \right]
\]
(47)
\[
= x_{\text{avg}}(i) - \alpha(i) \mathbb{E} \left[ \Upsilon(i) \right] - \alpha(i) \mathbb{E} \left[ \Psi(i) \right]
\]
\[
= x_{\text{avg}}(i)
\]

where the last step follows from the fact that $\Upsilon(i)$ is independent of $F_i$, and
\[
\mathbb{E} \left[ \Psi(i) | F_i \right] = \mathbb{E} \left[ \Psi(i) | x(i) \right]
\]
(48)
\[
= 0
\]

because $\Psi(i)$ is independent of $x(i)$ as argued in Section II-B.

Thus, the sequence $\{x_{\text{avg}}(i)\}_{i \geq 0}$ is a martingale. For proving $L_2$ boundedness, note
\[
\mathbb{E} \left[ x_{\text{avg}}^2(i + 1) \right] = \mathbb{E} \left[ x_{\text{avg}}(i) - \alpha(i) \Upsilon(i) - \alpha(i) \Psi(i) \right]^2
\]
(49)
\[
= \mathbb{E} \left[ x_{\text{avg}}^2(i) \right] + \alpha^2(i) \mathbb{E} \left[ \Upsilon^2(i) \right] + \alpha^2(i) \mathbb{E} \left[ \Psi^2(i) \right] + 2 \alpha^2(i) \mathbb{E} \left[ \Upsilon(i) \Psi(i) \right]
\]
\[
\leq \mathbb{E} \left[ x_{\text{avg}}^2(i) \right] + \alpha^2(i) \mathbb{E} \left[ \Upsilon^2(i) \right]
\]
\[
+ \alpha^2(i) \mathbb{E} \left[ \Psi^2(i) \right] + 2 \alpha^2(i) \left( \mathbb{E} \left[ \Upsilon^2(i) \right] \right)^{1/2} \left( \mathbb{E} \left[ \Psi^2(i) \right] \right)^{1/2}
\]

Again, it can be shown by using the independence properties and (18) that
\[
\mathbb{E} \left[ \Upsilon^2(i) \right] = \mathbb{E} \left[ \Psi^2(i) \right] \leq \frac{|M| \Delta^2}{6 N^2}
\] (50)

where $M$ is the number of realizable edges in the network (eqn. (20)). It then follows from eqn. (49) that
\[
\mathbb{E} \left[ x_{\text{avg}}^2(i) \right] \leq \mathbb{E} \left[ x_{\text{avg}}^2(i) \right] + \frac{2 \alpha^2(i) |M| \Delta^2}{3 N^2}
\]
(51)

Finally, the recursion leads to
\[
\mathbb{E} \left[ x_{\text{avg}}^2(i) \right] \leq x_{\text{avg}}^2(0) + \frac{2 |M| \Delta^2}{3 N^2} \sum_{j \geq 0} \alpha^2(j)
\]
(52)

Thus $\{x_{\text{avg}}(i)\}_{i \geq 0}$ is an $L_2$-bounded martingale; hence, it converges a.s. and in $L_2$ to a finite random variable $\theta$ ([39]). In other words,
\[
\mathbb{P} \left[ \lim_{i \to \infty} x_{\text{avg}}(i) = \theta \right] = 1
\] (53)

Again, Theorem 2 implies that as $i \to \infty$ we have $x(i) \to x_{\text{avg}}(i)$ a.s. This and (53) prove the Theorem. ■
B. QC Algorithm: Mean-Squared Error

Theorem 3 shows that the sensors reach consensus asymptotically and in fact converge a.s. to a finite random variable \( \theta \). Viewing \( \theta \) as an estimate of the initial average \( r \) (see eqn. (1)), we characterize its desirable statistical properties in the following Lemma.

Lemma 4 Let \( \theta \) be as given in Theorem 3 and \( r \), the initial average, as given in eqn. (1). Define

\[
\zeta = E[\theta - r]^2
\]

(54)

to be the m.s.e. Then, we have:

1) Unbiasedness:

\[
E[\theta] = r
\]

2) M.S.E. Bound:

\[
\zeta \leq \frac{2|\mathcal{M}|\Delta^2}{3N^2} \sum_{j \geq 0} \alpha^2(j)
\]

Proof: The proof follows from the arguments presented in the proof of Theorem 3 and is omitted. We note that the m.s.e. bound in Lemma 4 is conservative. Recalling the definition of \( M(i) \), as the number of active links at time \( i \) (see eqn. (20)), we have (by revisiting the arguments in the proof of Theorem 3)

\[
\zeta \leq \frac{2\Delta^2}{3N^2} \sum_{j \geq 0} \alpha^2(j) E[|M(i)|^2]
\]

(55)

(Note that the term \( \sum_{j \geq 0} \alpha^2(j) E[|M(i)|^2] \) is well-defined as \( E[|M(i)|^2] \leq |\mathcal{M}|^2, \forall i \). In case, we have a fixed (non-random) topology, \( M(i) = \mathcal{M}, \forall i \) and the bound in eqn. (55) reduces to the one in Lemma 4. For the asynchronous gossip protocol in [18], \( |M(i)| = 1, \forall i \), and hence

\[
\zeta_{\text{gossip}} \leq \frac{2\Delta^2}{3N^2} \sum_{j \geq 0} \alpha^2(j)
\]

(56)

Lemma 4 shows that, for a given \( \Delta \), \( \zeta \) can be made arbitrarily small by properly scaling the weight sequence, \( \{\alpha(i)\}_{i \geq 0} \). We formalize this by introducing some notation here, which will be used in the sequel. Given an arbitrary weight sequence, \( \{\alpha(i)\}_{i \geq 0} \), which satisfies the persistence condition (21), define the scaled weight sequence, \( \{\alpha_s(i)\}_{i \geq 0} \), as

\[
\alpha_s(i) = s\alpha(i), \forall i \geq 0
\]

(57)

where, \( s > 0 \), is a constant scaling factor. Clearly, such a scaled weight sequence satisfies the persistence condition (21), and the m.s.e. \( \zeta_s \) obtained by using this scaled weight sequence is given by

\[
\zeta_s \leq \frac{2|\mathcal{M}|\Delta^2s^2}{3N^2} \sum_{j \geq 0} \alpha^2(j)
\]

(58)

showing that, by proper scaling of the weight sequence, the m.s.e. can be made arbitrarily small.

However, reducing the m.s.e. by scaling the weights in this way will reduce the convergence rate of the algorithm and, this tradeoff is considered in the next subsection.
C. QC Algorithm: Convergence Rate

The QC algorithm falls under the framework of stochastic approximation algorithms and, hence, a detailed convergence rate analysis can be done through the ODE method (see, for example, [40]). We do not pursue it in this paper; rather, we present a simpler convergence rate analysis, involving the mean state vector sequence only. From the asymptotic unbiasedness of $\theta$,

$$\lim_{i \to \infty} \mathbb{E}[x(i)] = r1$$  \hspace{1cm} (59)

Our objective is to determine the rate at which the sequence $\{\mathbb{E}[x(i)]\}_{i \geq 0}$ converges to $r1$.

**Lemma 5** Without loss of generality, make the assumption

$$\alpha(i) \leq \frac{2}{\lambda_2(\bar{L}) + \lambda_N(\bar{L})}, \forall i$$  \hspace{1cm} (60)

(We note that this holds eventually, as the $\alpha(i)$ decrease to zero.) Then,

$$\|\mathbb{E}[x(i)] - r1\| \leq \left(e^{-\lambda_2(\bar{L}) (\sum_{0 \leq j \leq i-1} \alpha(j))}\right) \|\mathbb{E}[x(0)] - r1\|$$  \hspace{1cm} (61)

**Proof:** We note that the mean state propagates as

$$\mathbb{E}[x(i+1)] = (I - \alpha(i)\bar{L}) \mathbb{E}[x(i)], \forall i$$  \hspace{1cm} (62)

The proof then follows from [28] and is omitted. ■

It follows from Lemma 5 that the rate at which the sequence $\{\mathbb{E}[x(i)]\}_{i \geq 0}$ converges to $r1$ is closely related to the rate at which the weight sequence, $\alpha(i)$, sums to infinity. On the other hand, to achieve a small bound $\zeta$ on the m.s.e, see lemma 54 in Subsection III-B, we need to make the weights small, which reduces the convergence rate of the algorithm. The parameter $s$ introduced in eqn. (57) can then be viewed as a scalar control parameter, which can be used to trade-off between precision (m.s.e.) and convergence rate. More specifically, if a cost structure is imposed on the consensus problem, where the objective is a function of the m.s.e. and the convergence rate, one may obtain the optimal scaling $s$ minimizing the cost from the pareto-optimal curve generated by varying $s$. This is significant, because the algorithm allows one to trade off m.s.e. vs. convergence rate, and in particular, if the application requires precision (low m.s.e.), one can make the m.s.e. arbitrarily small irrespective of the quantization step-size $\Delta$. It is important to note in this context, that though the algorithms in [21], [19] lead to finite m.s.e., the resulting m.s.e. is proportional to $\Delta^2$, which may become large if the step-size $\Delta$ is chosen to be large.

Note that this tradeoff is established between the convergence rate of the mean state vectors and the m.s.e. of the limiting consensus variable $\theta$. But, in general, even for more appropriate measures of the convergence rate, we expect that, intuitively, the same tradeoff will be exhibited, in the sense that the rate of convergence will be closely related to the rate at which the weight sequence, $\alpha(i)$, sums to infinity.
D. QC Algorithm: Generalizations

The QC algorithm can be extended to handle more complex situations of imperfect communication. For instance, we may incorporate Markovian link failures (as in [28]) and time-varying quantization step-size with the same theme of analysis. The case of time-varying quantization may be relevant in many practical communication networks, where because of a bit-budget, as time progresses the quantization may become coarser (the step-size increases). It may also arise if one considers a rate allocation protocol with vanishing rates as time progresses (see [41]). In that case, the quantization step-size sequence, \( \{ \Delta(i) \}_{i \geq 0} \) is time-varying with possibly

\[
\limsup_{i \to \infty} \Delta(i) = \infty
\]

(63)

Also, as suggested in [26], one may consider a rate allocation scheme, in which the quantizer becomes finer as time progresses. In that way, the quantization step-size sequence, \( \{ \Delta(i) \}_{i \geq 0} \) may be a decreasing sequence.

Generally, in a situation like this to attain consensus the link weight sequence \( \{ \alpha(i) \}_{i \geq 0} \) needs to satisfy a generalized persistence condition of the form

\[
\sum_{i \geq 0} \alpha(i) = \infty, \quad \sum_{i \geq 0} \alpha^2(i)\Delta^2(i) < \infty
\]

(64)

Note, when the quantization step-size is bounded, this reduces to the persistence condition assumed earlier. We state without proof the following result for time-varying quantization case.

**Theorem 6** Consider the QC algorithm with time-varying quantization step size sequence \( \{ \Delta(i) \}_{i \geq 0} \) and let the link weight sequence \( \{ \alpha(i) \}_{i \geq 0} \) satisfy the generalized persistence condition in eqn. (64). Then the sensors reach consensus to an a.s. finite random variable. In other words, there exists an a.s. finite random variable \( \theta \), such that,

\[
P \left[ \lim_{i \to \infty} x_n(i) = \theta, \ \forall n \right] = 1
\]

(65)

Also, if \( r \) is the initial average, then

\[
E \left[ (\theta - r)^2 \right] \leq \frac{2|M|}{3N^2} \sum_{i \geq 0} \alpha^2(i)\Delta^2(i)
\]

(66)

It is clear that in this case also, we can trade-off m.s.e. with convergence rate by tuning a scalar gain parameter \( s \) associated with the link weight sequence.

IV. CONSENSUS WITH QUANTIZED DATA: BOUNDED INITIAL SENSOR STATE

In this section, we consider consensus with quantized data when the initial sensor states are bounded, and this bound is known a priori. In this case, we show that finite bit quantizers (quantizers, whose outputs take only a finite number of values) will suffice. The algorithm QCF that we now consider is a modification of the QC algorithm of Section III. The good performance of the QCF algorithm relies mainly on the fact that, if the initial sensor states are bounded, then the state sequence, \( \{ x(i) \}_{i \geq 0} \) generated by the QC algorithm remains uniformly bounded with
high probability. In that case, channel quantizers with only a finite dynamic range will perform very well with high probability.

We next briefly state the QCF problem in Subsection IV-A. Then, Subsection IV-B shows that with high probability the sample paths generated by the QC algorithm are uniformly bounded, when the initial sensor states are bounded. Subsection IV-C proves that QCF achieves asymptotic consensus. Finally, Subsections IV-D and IV-E analyze its statistical properties, performance, and tradeoffs.

A. QCF Algorithm: Statement

The QCF algorithm modifies the QC algorithm by restricting the alphabet to be finite. It assumes that the initial sensor state $x(0)$, whose average we wish to compute, is known to be bound. Of course, even if the initial state is bounded, the states of QC can become unbounded. The good performance of QCF is a consequence of the fact that, as our analysis will show, the states $\{x(i)\}_{i \geq 0}$ generated by the QC algorithm when started with a bounded initial state $x(0)$ remain uniformly bounded with high probability.

The following are the assumptions underlying QCF:

1) Bounded initial state. The QCF initial state $\bar{x}(0)$ is bounded to the set $B$ known à priori

$$B = \{ y \in \mathbb{R}^{N \times 1} \mid |y_n| \leq b < +\infty \}$$  \hspace{1cm} (67)

for some $b > 0$.

2) Uniform quantizers and finite alphabet. Each inter-sensor communication channel in the network uses a uniform $\lceil \log_2(2p + 1) \rceil$ bit quantizer with step-size $\Delta$, where $p > 0$ is an integer. In other words, the quantizer output takes only $2p + 1$ values, and the quantization alphabet is given by

$$\bar{Q} = \{ l\Delta \mid l = 0, \pm 1, \cdots, \pm p \}$$  \hspace{1cm} (68)

Clearly, such a quantizer will not saturate if the input falls in the range $[-(p - 1/2)\Delta, (p + 1/2)\Delta]$; if the input goes out of that range, the quantizer saturates.

3) Uniform i.i.d. noise. Like with QC, the $\{\nu_{nl}(i)\}_{i \geq 0, 1 \leq n, l \leq N}$ are a sequence of i.i.d. random variables uniformly distributed on $[-\Delta/2(a/2), \Delta/2]$.

4) The link failure model is the same as used in QC.

Given this setup, we present the distributed QCF algorithm, assuming that the sensor network is connected. The state sequence, $\{\bar{x}(i)\}_{i \geq 0}$ is given by the following Algorithm.

The last step of the algorithm can be distributed, since the network is connected.

B. Probability Bounds on Uniform Boundedness of Sample Paths of QC

The analysis of the QCF algorithm requires uniformity properties of the sample paths generated by the QC algorithm. This is necessary, because the QCF algorithm follows the QC algorithm till one of the quantizers get
Algorithm 1: QCF

Initialize
\( \tilde{x}_n(0) = x_n(0), \forall n; \)
\( i = 0; \)
begin
while \( \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n(i)} |(\tilde{x}_l(i) + \nu_{nl}(i))| < (p + 1/2)\Delta \) do
\( \tilde{x}_n(i + 1) = (1 - \alpha(i)d_n(i))\tilde{x}_n(i) + \alpha(i)\sum_{l \in \Omega_n(i)} q(\tilde{x}_l(i) + \nu_{nl}(i)), \forall n; \)
\( i = i + 1; \)
end
Stop the algorithm and reset all the sensor states to zero

overloaded. The uniformity properties require establishing statistical properties of the supremum over the sample paths, which is carried out in this subsection. We show that the state vector sequence, \( \{x(i)\}_{i \geq 0} \), generated by the QC algorithm is uniformly bounded with high probability. The proof follows by splitting the sequence \( \{x(i)\}_{i \geq 0} \) as the sum of the sequences \( \{x_{avg}(i)\}_{i \geq 0} \) and \( \{x_{C\perp}(i)\}_{i \geq 0} \) for which we establish uniformity results. The proof is lengthy and uses mainly maximal inequalities for submartingale and supermartingale sequences.

Recall that the state vector at any time \( i \) can be decomposed orthogonally as
\[
x(i) = x_{avg}(i) + x_{C\perp}(i)
\]
where the consensus subspace, \( C \), is given in eqn. (22). We provide probability bounds on the sequences \( \{x_{avg}(i)\}_{i \geq 0} \) and \( \{x_{C\perp}(i)\}_{i \geq 0} \) and then use an union bound to get the final result.

We need the following result.

Lemma 7 Consider the QC algorithm stated in Section II and let \( \{x(i)\}_{i \geq 0} \) be the state sequence generated. Define the function \( W(i, x) \), \( i \geq 0, \ x \in \mathbb{R}^{N \times 1} \), as
\[
W(i, x) = (1 + V(i, x)) \prod_{j \geq i} [1 + g(j)]
\]
where \( V(i, x) = x^T L x \) and \( \{g(j)\}_{j \geq 0} \) is defined in eqn. (41). Then, the process \( \{W(i, x(i))\}_{i \geq 0} \) is a non-negative supermartingale with respect to the filtration \( \mathcal{F}_i \) defined in eqn. (44).

Proof: From eqn. (40) we have
\[
\mathbb{E} [V(i + 1, x(i + 1)) | x(i)] \leq -\alpha(i)\varphi(i, x(i)) + g(i) [1 + V(i, x(i))] + V(i, x(i))
\]
\( \text{The above function is well-defined because the term } \prod_{j \geq i} [1 + g(j)] \text{ is finite for any } j, \text{ by the persistence condition on the weight sequence.} \)
We then have
\[
\mathbb{E}[W(i + 1, x(i + 1)) \mid \mathcal{F}_i] = \mathbb{E}
\left[
(1 + V(i + 1, x(i + 1))) \prod_{j \geq i + 1} [1 + g(j)] \mid x(i)
\right]
\]
\[
= \prod_{j \geq i + 1} [1 + g(j)] (1 + \mathbb{E}[V(i + 1, x(i + 1)) \mid x(i)])
\]
\[
\leq \prod_{j \geq i + 1} [1 + g(j)] (1 - \alpha(i) \varphi(i, x(i)) + g(i) [1 + V(i, x(i))] + V(i, x(i)))
\]
\[
= -\alpha(i) \varphi(i, x(i)) \prod_{j \geq i + 1} [1 + g(j)] + [1 + V(i, x(i))] \prod_{j \geq i} [1 + g(j)]
\]
\[
= -\alpha(i) \varphi(i, x(i)) \prod_{j \geq i + 1} [1 + g(j)] + W(i, x(i))
\] (72)

Hence \(\mathbb{E}[W(i + 1, x(i + 1)) \mid \mathcal{F}_i] \leq W(i, x(i))\) and the result follows.

The next Lemma bounds the sequence \(\{x_{c^\perp}(i)\}_{i \geq 0}\).

**Lemma 8** Let \(\{x(i)\}_{i \geq 0}\) be the state vector sequence generated by the QC algorithm, with the initial state \(x(0) \in \mathbb{R}^{N \times 1}\). Consider the orthogonal decomposition:
\[
x(i) = x_{avg}(i) 1 + x_{c^\perp}(i), \ \forall i
\] (73)

Then, for any \(a > 0\), we have
\[
P\left[\sup_{j \geq 0} \|x_{c^\perp}(j)\|^2 > a\right] \leq \frac{(1 + x(0)^T \mathcal{L} x(0)) \prod_{j \geq 0} (1 + g(j))}{1 + a \lambda_2(\mathcal{L})}
\] (74)

where \(\{g(j)\}_{j \geq 0}\) is defined in eqn. (41).

**Proof:** For any \(a > 0\) and \(i \geq 0\), we have
\[
\|x_{c^\perp}(i)\|^2 > a \implies x(i)^T \mathcal{L} x(i) \geq a \lambda_2(\mathcal{L})
\] (75)

Define the potential function \(V(i, x)\) as in Theorem 2 and eqn. (36) and the \(W(i, x)\) in (70) in Lemma 7. It then follows from eqn. (75) that
\[
\|x_{c^\perp}(i)\|^2 > a \implies W(i, x(i)) > 1 + a \lambda_2(\mathcal{L})
\] (76)

By Lemma 7, the process \((W(i, x(i)), \mathcal{F}_i)\) is a non-negative supermartingale. Then by a maximal inequality for non-negative supermartingales (see [42]) we have for \(a > 0\) and \(i \geq 0\),
\[
P\left[\max_{0 \leq j \leq i} W(j, x(j)) \geq a\right] \leq \frac{\mathbb{E}[W(0, x(0))]}{a}
\] (77)

Also, we note that
\[
\left\{\sup_{j \geq 0} W(j, x(j)) > a\right\} \iff \bigcup_{i \geq 0} \left\{\max_{0 \leq j \leq i} W(j, x(j)) > a\right\}
\] (78)

Since \(\max_{0 \leq j \leq i} W(j, x(j)) > a\) is a non-decreasing sequence of sets in \(i\), it follows from the continuity of
probability measures and eqn. (76)
\[ P \left[ \sup_{j \geq 0} \| x_{C}(j) \|^2 > a \right] = \lim_{i \to \infty} P \left[ \max_{0 \leq j \leq i} \| x_{C}(j) \|^2 > a \right] \]
\[ \leq \lim_{i \to \infty} P \left[ \max_{0 \leq j \leq i} W(j, x(j)) > 1 + a \lambda_2(L) \right] \]
\[ \leq \lim_{i \to \infty} E \left[ W(0, x(0)) \right] \]
\[ \leq \frac{a}{(1 + x(0)^2 L x(0)) \prod_{j \geq 0} (1 + g(j))} \quad (80) \]

Next, we provide probability bounds on the uniform boundedness of \( \{ x_{avg}(i) \}_{i \geq 0} \).

**Lemma 9** Let \( \{ x_{avg}(i) \}_{i \geq 0} \) be the average sequence generated by the QC algorithm, with an initial state \( x(0) \in \mathbb{R}^{N \times 1} \). Then, for any \( a > 0 \),
\[ P \left[ \sup_{j \geq 0} | x_{avg}(j) | > a \right] \leq \frac{\left[ x_{avg}^2(0) + \frac{2|M|\Delta^2}{3N^2} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{a} \quad (81) \]

**Proof:** It was shown in Theorem 3 that the sequence \( \{ x_{avg}(i) \}_{i \geq 0} \) is a martingale. It then follows that the sequence, \( \{ | x_{avg}(i) | \}_{i \geq 0} \), is a non-negative submartingale (see [39]).

The submartingale inequality then states that for \( a > 0 \)
\[ P \left[ \max_{0 \leq j \leq i} | x_{avg}(j) | \geq a \right] \leq \frac{E \left[ | x_{avg}(i) | \right]}{a} \quad (82) \]
Clearly, from the continuity of probability measures,
\[ P \left[ \sup_{j \geq 0} | x_{avg}(j) | > a \right] = \lim_{i \to \infty} P \left[ \max_{0 \leq j \leq i} | x_{avg}(j) | > a \right] \quad (83) \]
Thus, we have
\[ P \left[ \sup_{j \geq 0} | x_{avg}(j) | > a \right] \leq \lim_{i \to \infty} \frac{E \left[ | x_{avg}(i) | \right]}{a} \quad (84) \]
(the limit on the right exists because \( x_{avg}(i) \) converges in \( L_1 \).)

Also, we have from eqn. (52), for all \( i \),
\[ E \left[ | x_{avg}(i) | \right] \leq \left[ E \left[ | x_{avg}(i) |^2 \right] \right]^{1/2} \]
\[ \leq \left[ x_{avg}^2(0) + \frac{2|M|\Delta^2}{3N^2} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2} \quad (85) \]
Combining eqns. (84,85), we have
\[ P \left[ \sup_{j \geq 0} | x_{avg}(j) | > a \right] \leq \frac{\left[ x_{avg}^2(0) + \frac{2|M|\Delta^2}{3N^2} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{a} \quad (86) \]
Theorem 10  Let \( \{x(i)\}_{i \geq 0} \) be the state vector sequence generated by the QC algorithm, with an initial state \( x(0) \in \mathbb{R}^{N \times 1} \). Then, for any \( a > 0 \),

\[
\mathbb{P} \left[ \sup_{j \geq 0} \|x(j)\| > a \right] \leq \frac{\left[ 2N \beta \mathbb{E}(0) + \frac{4M \lambda^2}{3N} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{a} + \frac{(1 + x(0)^T L x(0)) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{a^2}{2} \lambda(L)}
\]

(87)

where \( \{g(j)\}_{j \geq 0} \) is defined in eqn. (41).

**Proof:**  Since, \( \|x(j)\|^2 = N \beta \mathbb{E}(i) + \|x_{c_i}^\perp\|^2(j) \), we have

\[
\mathbb{P} \left[ \sup_{j \geq 0} \|x(j)\|^2 > a \right] \leq \mathbb{P} \left[ \sup_{j \geq 0} N \beta \mathbb{E}(j) > \frac{a}{2} \right] + \mathbb{P} \left[ \sup_{j \geq 0} \|x_{c_i}^\perp(j)\|^2 > \frac{a}{2} \right]
\]

(88)

We thus have from Lemmas 8 and 9,

\[
\mathbb{P} \left[ \sup_{j \geq 0} \|x(j)\|^2 > a \right] \leq \frac{\left[ 2N \beta \mathbb{E}(0) + \frac{4M \lambda^2}{3N} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{a} + \frac{(1 + x(0)^T L x(0)) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{a^2}{2} \lambda(L)}
\]

(89)

We now state as a Corollary the result on the boundedness of the sensor states, which will be used in analyzing the performance of the QCF algorithm.

**Corollary 11**  Assume that the initial sensor state, \( x(0) \in \mathcal{B} \), where \( \mathcal{B} \) is given in eqn. (67). Then, if \( \{x(i)\}_{i \geq 0} \) is the state sequence generated by the QC algorithm starting from the initial state, \( x(0) \), we have, for any \( a > 0 \),

\[
\mathbb{P} \left[ \sup_{1 \leq n \leq N, j \geq 0} |x_n(j)| > a \right] \leq \frac{\left[ 2N \beta \mathbb{E}(0) + \frac{4M \lambda^2}{3N} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{a} + \frac{(1 + N \lambda \mathbb{E}(0)) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{a^2}{2} \lambda(L)}
\]

(90)

where \( \{g(j)\}_{j \geq 0} \) is defined in eqn. (41).

**Proof:**  We note that, for \( x(0) \in \mathcal{B} \),

\[
x^2_{\mathbb{E}}(0) \leq b^2, \quad x(0)^T L x(0) \leq N \lambda \mathbb{E}(L) b^2
\]

(91)
From Theorem 10, we then get,

\[
P \left[ \sup_{1 \leq n \leq N, j \geq 0} |x_n(j)| > a \right] \leq P \left[ \sup_{j \geq 0} \|x(j)\| > a \right]
\]

\[
\leq \frac{2N x^2(0) + \frac{4|M| \Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j)^{1/2}}{a}
\]

\[
+ \frac{(1 + x(0)^T \bar{L} x(0)) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{a^2}{2} \lambda_2(\bar{L})}
\]

\[
\leq \frac{2N b^2 + \frac{4|M| \Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j)^{1/2}}{a}
\]

\[
+ \frac{(1 + N \lambda_N(\bar{L}) b^2) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{a^2}{2} \lambda_2(\bar{L})}
\]

\[
(92)
\]

\[
(93)
\]

\[
(94)
\]

\[
(95)
\]

\[
(96)
\]

\[
(97)
\]

C. Algorithm QCF: Asymptotic Consensus

We show that the QCF algorithm, given in Subsection IV-A, converges a.s. to a finite random variable and the sensors reach consensus asymptotically.

**Theorem 12** (QCF: a.s. asymptotic consensus) Let \( \{\tilde{x}(i)\}_{i \geq 0} \) be the state vector sequence generated by the QCF algorithm, starting from an initial state \( \tilde{x}(0) = x(0) \in B \). Then, the sensors reach consensus asymptotically a.s. In other words, there exists an a.s. finite random variable \( \tilde{\theta} \) such that

\[
P \left[ \lim_{i \to \infty} \tilde{x}(i) = \tilde{\theta} 1 \right] = 1
\]

**Proof:** For the proof, consider the sequence \( \{x(i)\}_{i \geq 0} \) generated by the QC algorithm, with the same initial state \( x(0) \). Let \( \theta \) be the a.s. finite random variable (see eqn. 42) such that

\[
P \left[ \lim_{i \to \infty} x(i) = \theta 1 \right] = 1
\]

It is clear that

\[
\tilde{\theta} = \begin{cases} 
\theta & \text{on } \left\{ \sup_{1 \leq n \leq N} \sup_{i \in \Omega_n(i)} |x_i(i) + \nu_{nl}(i)| < (p + \frac{1}{2}) \Delta \right\} \\
0 & \text{otherwise}
\end{cases}
\]

In other words, we have

\[
\tilde{\theta} = \theta \mathbb{I} \left( \sup_{1 \leq n \leq N} \sup_{i \in \Omega_n(i)} |x_i(i) + \nu_{nl}(i)| < (p + \frac{1}{2}) \Delta \right)
\]

where \( \mathbb{I}(\cdot) \) is the indicator function. Since \( \left\{ \sup_{1 \leq n \leq N} \sup_{i \in \Omega_n(i)} |x_i(i) + \nu_{nl}(i)| < (p + 1/2) \Delta \right\} \) is a measurable set, it follows that \( \tilde{\theta} \) is a random variable.
D. QCF: \(\epsilon\)-Consensus

Recall the QCF algorithm in Subsection IV-A and the assumptions 1)-4). A key step is that, if we run the QC algorithm, using finite bit quantizers with finite alphabet \(\tilde{\mathcal{Q}}\) as given in eqn. (68), the only way for an error to occur is for one of the quantizers to saturate. This was, in fact, the main intuition behind the design of the QCF algorithm.

Theorem 12 shows that the QCF sensor states asymptotically reach consensus, converging a.s. to a finite random variable \(\tilde{\theta}\). The next series of results address the question of how close is this consensus to the desired average \(r\) in (1). Clearly, this depends on the QCF design: 1) the quantizer parameters (like the number of levels \(2p + 1\) or the quantization step \(\Delta\)); 2) the random network topology ; and 3) the gains \(\alpha\).

We define the following performance metrics which characterize the performance of the QCF algorithm.

**Definition 13 (Probability of \(\epsilon\)-consensus and consensus-consistent)** The probability of \(\epsilon\)-consensus is defined as

\[
T(G, b, \alpha, \epsilon, p, \Delta) = \mathbb{P}\left[ \lim_{i \to \infty} \sup_{1 \leq n \leq N} |\tilde{x}_n(i) - r| < \epsilon \right] \tag{98}
\]

Note that the argument \(G\) in the definition of \(T(\cdot)\) emphasizes the influence of the network configuration, whereas \(b\) is given in eqn. (67).

The QCF algorithm is consensus-consistent\(^2\) iff for every \(G, b, \epsilon > 0\) and \(0 < \delta < 1\), there exists quantizer parameters \(p, \Delta\) and weights \(\{\alpha(i)\}_{i \geq 0}\), such that

\[
T(G, b, \alpha, \epsilon, p, \Delta) > 1 - \delta \tag{99}
\]

Theorem 15 characterizes the probability of \(\epsilon\)-consensus, while Proposition 16 considers several tradeoffs between the probability of achieving consensus and the quantizer parameters and network topology, and, in particular, shows that the QCF algorithm is consensus-consistent.

**Lemma 14** Let \(\tilde{\theta}\) be defined as in Lemma 12, with the initial state \(\tilde{x}(0) = x(0) \in \mathcal{B}\). The desired average, \(r\), is given in (1). Then, for any \(\epsilon > 0\), we have

\[
\mathbb{P}\left[ |\bar{\theta} - r| \geq \epsilon \right] \leq \frac{2|\mathcal{M}| \Delta^2}{3N^2 \epsilon^2} \sum_{j \geq 0} \alpha^2(j) + \frac{2Nb^2 + \frac{4|\mathcal{M}| \Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j)}{p \Delta} \tag{100}
\]

\[
+ \frac{(1 + N \lambda \mathcal{L} b^2) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{p \Delta^2}{\lambda^2} \mathcal{L}} \tag{101}
\]

where \(\{g(j)\}_{j \geq 0}\) is defined in eqn. (41).

**Proof:** For the proof, consider the sequence \(\{x(i)\}_{i \geq 0}\) generated by the QC algorithm, with the same initial

\(^2\)Consensus-consistent means for arbitrary \(\epsilon > 0\), the QCF quantizers can be designed so that the QCF states get within an \(\epsilon\)-ball of \(r\) with arbitrary high probability. Thus, a consensus-consistent algorithm trades off accuracy with bit-rate.
state $x(0)$. Let $\theta$ be the a.s. finite random variable (see eqn. 42) such that
\[
P \left[ \lim_{i \to \infty} x(i) = \theta 1 \right] = 1
\]  
(102)

We note that
\[
P \left[ |\tilde{\theta} - r| \geq \epsilon \right] = P \left[ (|\tilde{\theta} - r| \geq \epsilon) \cap (\tilde{\theta} = \theta) \right] + P \left[ (|\tilde{\theta} - r| \geq \epsilon) \cap (\tilde{\theta} \neq \theta) \right] \\
= P \left[ (|\theta - r| \geq \epsilon) \cap (\theta = \theta) \right] + P \left[ (|\tilde{\theta} - r| \geq \epsilon) \cap (\tilde{\theta} \neq \theta) \right] \\
\leq P \left[ |\theta - r| \geq \epsilon \right] + P \left[ \tilde{\theta} \neq \theta \right]
\]  
(103)

From Chebyshev’s inequality, we have
\[
P \left[ |\theta - r| \geq \epsilon \right] \leq \frac{\mathbb{E} \left[ |\theta - r|^2 \right]}{\epsilon^2} \leq \frac{2|M|\Delta^2}{3N^2\epsilon^2} \sum_{j \geq 0} \alpha^2(j)
\]  
(104)

Next, we bound $P \left[ \tilde{\theta} \neq \theta \right]$. To this end, we note that
\[
\sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n(i)} |x_l(i) + \nu_n(l(i))| \leq \sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n(i)} |x_l(i)| + \sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n(i)} |\nu_n(l(i))| \\
\leq \sup_{i \geq 0} \sup_{1 \leq n \leq N} |x_n(i)| + \sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n(i)} |\nu_n(l(i))| \\
\leq \sup_{i \geq 0} \sup_{1 \leq n \leq N} |x_n(i)| + \frac{\Delta}{2}
\]  
(105)

Then, for any $\delta > 0$,
\[
P \left[ \tilde{\theta} \neq \theta \right] = P \left[ \sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n(i)} |x_l(i) + \nu_n(l(i))| \geq \left( p + \frac{1}{2} \right) \Delta \right] \\
\leq P \left[ \sup_{i \geq 0} \sup_{1 \leq n \leq N} |x_n(i)| + \frac{\Delta}{2} \geq \left( p + \frac{1}{2} \right) \Delta \right] \\
= P \left[ \sup_{i \geq 0} \sup_{1 \leq n \leq N} |x_n(i)| \geq p\Delta \right] \\
\leq P \left[ \sup_{i \geq 0} \sup_{1 \leq n \leq N} |x_n(i)| > p\Delta - \delta \right] \\
\leq \frac{2Nb^2 + \frac{4|M|\Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j)}{p\Delta - \delta} + \frac{\left( 1 + N\lambda_N(L)b^2 \right) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{(p\Delta - \delta)^j}{2} \lambda_2(L)}
\]  
(106)
where, in the last step, we use eqn. (92.) Since the above holds for arbitrary \( \delta > 0 \), we have

\[
\mathbb{P} \left[ \bar{\theta} \neq \theta \right] \leq \lim_{\delta \to 0} \left[ \frac{2 N b^2 + \frac{4 M \Delta^2}{3 N} \sum_{j \geq 0} \alpha_2(j)}{p \Delta - \delta} \right]^{1/2} + \frac{\left(1 + N \lambda_N (\mathcal{L}) b^2 \prod_{j \geq 0}(1 + g(j))\right)}{1 + \frac{(p \Delta - \delta)^2}{2} \lambda_2(\mathcal{L})}
\]

Combining eqns. (103,104,107), we get the result.

We now state the main result of this Section, which provides a performance guarantee for QCF.

**Theorem 15 (QCF: Probability of \( \epsilon \)-consensus)** For any \( \epsilon > 0 \), the probability of \( \epsilon \)-consensus \( T(G, b, \alpha, \epsilon, p, \Delta) \) is bounded below

\[
\mathbb{P} \left[ \lim_{i \to \infty} \sup_{1 \leq n \leq N} |\tilde{x}_n(i) - r| < \epsilon \right] > 1 - \frac{2 |M| \Delta^2}{3 N \epsilon^2} \sum_{j \geq 0} \alpha_2(j) - \frac{\left(2 N b^2 + \frac{4 M \Delta^2}{3 N} \sum_{j \geq 0} \alpha_2(j)\right)^{1/2}}{p \Delta} - \frac{\left(1 + N \lambda_N (\mathcal{L}) b^2 \prod_{j \geq 0}(1 + g(j))\right)}{1 + \frac{(p \Delta - \delta)^2}{2} \lambda_2(\mathcal{L})}
\]

where \( \{g(j)\}_{j \geq 0} \) is defined in eqn. (41).

**Proof:** It follows from Theorem 12 that

\[
\lim_{i \to \infty} \tilde{x}_n(i) = \bar{\theta} \text{ a.s., } \forall 1 \leq n \leq N
\]

The proof then follows from Lemma 14.

The lower bound on \( T(\cdot) \), given by (108), is uniform, in the sense that it is applicable for all initial states \( x(0) \in \mathcal{B} \).

Recall the scaled weight sequence \( \alpha_s \), given by eqn. (57). We introduce the zero-rate probability of \( \epsilon \)-consensus, \( T^z(G, b, \epsilon, p, \Delta) \) by

\[
T^z(G, b, \epsilon, p, \Delta) = \lim_{s \to 0} T(G, b, \alpha_s, \epsilon, p, \Delta)
\]

The next proposition studies the dependence of the \( \epsilon \)-consensus probability \( T(\cdot) \) and of the zero-rate probability \( T^z(\cdot) \) on the network and algorithm parameters.

**Proposition 16 (QCF: Tradeoffs)**  

1) **Limiting quantizer.** For fixed \( G, b, \alpha, \epsilon, \), we have

\[
\lim_{\Delta \to 0, \ p \Delta \to \infty} T(G, b, \alpha, \epsilon, p, \Delta) = 1
\]
Since, this holds for arbitrary $\epsilon > 0$, we note that, as $\Delta \to 0$, $p\Delta \to \infty$,

$$
\mathbb{P} \left[ \lim_{i \to \infty} \tilde{x}(i) = r \right] = \lim_{\epsilon \to 0} \mathbb{P} \left[ \lim_{i \to \infty} \sup_{1 \leq n \leq N} |\tilde{x}_n(i) - r| < \epsilon \right] = \lim_{\epsilon \to 0} \lim_{\Delta \to 0, p\Delta \to \infty} T(G, b, \alpha, \epsilon, p, \Delta)
$$

(112)

In other words, the QCF algorithm leads to a.s. consensus to the desired average $r$, as $\Delta \to 0$, $p\Delta \to \infty$. In particular, it shows that the QCF algorithm is consensus-consistent.

2) **zero-rate $\epsilon$-consensus probability.** Then, for fixed $G, b, \epsilon, p, \Delta$, we have

$$
T^\varepsilon(G, b, \epsilon, p, \Delta) \geq 1 - \frac{(2Nb^2)^{1/2}}{p\Delta} - \frac{1 + N\lambda_N(L)b^2}{1 + \frac{p\Delta^2}{2} \lambda_2(L)}
$$

(113)

3) **Optimum quantization step-size $\Delta$.** For fixed $G, b, \epsilon, p$, the optimum quantization step-size $\Delta$, which maximizes the probability of $\epsilon$-consensus, $T(G, b, \alpha, \epsilon, p, \Delta)$, is given by

$$
\Delta^\varepsilon(G, b, \alpha, \epsilon, p) = \arg \inf_{\Delta \geq 0} \left[ \frac{2|M|\Delta^2}{3N^2\epsilon^2} \sum_{j \geq 0} \alpha^2(j) + \left[ \frac{2Nb^2 + \frac{4M\Delta^2}{N}\sum_{j \geq 0} \alpha^2(j)}{p\Delta} \right]^{1/2} \right]
$$

\begin{equation}
+ \left[ \frac{(1 + N\lambda_N(L)b^2) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{p\Delta^2}{2} \lambda_2(L)} \right]
\end{equation}

(114)

where $\{g(j)\}_{j \geq 0}$ is defined in eqn. (41).

**Proof:** For item 2), we note that, as $s \to 0$,

$$
\sum_{j \geq 0} \alpha^2(j) \to 0, \quad \prod_{j \geq 0} (1 + g_s(j)) \to 1
$$

The rest follows by simple inspection of eqn. (108).

We comment on Proposition 16. Item 1) shows that the algorithm QCF is consensus-consistent, in the sense that we can achieve arbitrarily good performance by decreasing the step-size $\Delta$ and the number of quantization levels, $2p + 1$, appropriately. Indeed, decreasing the step-size increases the precision of the quantized output and increasing $p$ increases the dynamic range of the quantizer. However, the fact that $\Delta \to 0$ but $p\Delta \to \infty$ implies that the rate of growth of the number of levels $2p + 1$ should be higher than the rate of decay of $\Delta$, guaranteeing that in the limit we have asymptotic consensus with probability one.

For interpreting item 2), we recall the m.s.e. versus convergence rate tradeoff for the QC algorithm, studied in Subsection III-B. There, we considered a quantizer with a countably infinite number of output levels (as opposed to the finite number of output levels in the QCF) and observed that the m.s.e. can be made arbitrarily small by rescaling the weight sequence. By Chebyshev’s inequality, this would imply, that, for arbitrary $\epsilon > 0$, the probability of $\epsilon$-consensus, i.e., that we get within an $\epsilon$-ball of the desired average, can be made as close to 1 as we want.
However, this occurs at a cost of the convergence rate, which decreases as the scaling factor $s$ decreases. Thus, for the QC algorithm, in the limiting case, as $s \rightarrow 0$, the probability of $\epsilon$-consensus (for arbitrary $\epsilon > 0$) goes to 1; we call “limiting probability” the zero-rate probability of $\epsilon$-consensus, justifying the m.s.e. vs convergence rate tradeoff.\textsuperscript{3} Item 2) shows, that, similar to the QC algorithm, the QCF algorithm exhibits a tradeoff between probability of $\epsilon$-consensus vs. the convergence rate, in the sense that, by scaling (decreasing $s$), the probability of $\epsilon$-consensus can be increased. However, contrary to the QC case, scaling will not lead to probability of $\epsilon$-consensus arbitrarily close to 1, and, in fact, the zero-rate probability of $\epsilon$-consensus is strictly less than one, as given by eqn. (113). In other words, by scaling, we can make $T(G, b, \alpha_s, \epsilon, p, \Delta)$ as high as $T^z(G, b, \epsilon, p, \Delta)$, but no higher.

We now interpret the lower bound on the zero-rate probability of $\epsilon$-consensus, $T^z(G, b, \epsilon, p, \Delta)$, and show that the network topology plays an important role in this context. We note, that, for a fixed number, $N$, of sensor nodes, the only way the topology enters into the expression of the lower bound is through the third term on the R.H.S. Then, assuming that, $N \lambda_{N(L)} b^2 \gg 1, \frac{p^2 \Delta^2}{2} \lambda_2(L) \gg 1$

we may use the approximation

$$
1 + N \lambda_{N(L)} b^2 \approx \frac{2 N b^2}{p^2 \Delta^2} \frac{\lambda_N(L)}{\lambda_2(L)}
$$

(115)

Let us interpret eqn. (115) in the case, where the topology is fixed (non-random). Then for all $i$, $L(i) = \overline{L} = L$. Thus, for a fixed number, $N$, of sensor nodes, topologies with smaller $\lambda_{N(L)}/\lambda_2(L)$, will lead to higher zero-rate probability of $\epsilon$-consensus and, hence, are preferable. We note that, in this context, for fixed $N$, the class of non-bipartite Ramanujan graphs give the smallest $\lambda_{N(L)}/\lambda_2(L)$ ratio, given a constraint on the number, $M$, of network edges (see [9]).

Item 3) shows that, for given graph topology $G$, initial sensor data, $b$, the link weight sequence $\alpha$, tolerance $\epsilon$, and the number of levels in the quantizer $p$, the step-size $\Delta$ plays a significant role in determining the performance. This gives insight into the design of quantizers to achieve optimal performance, given a constraint on the number of quantization levels, or, equivalently, given a bit budget on the communication.

In the next Subsection, we present some numerical studies on the QCF algorithm, which demonstrate practical implications of the results just discussed.

E. QCF: Numerical Studies

We present a set of numerical studies on the quantizer step-size optimization problem, considered in Item 3) of Proposition 16. We consider a fixed (non-random) sensor network of $N = 230$ nodes, with communication topology given by an LPS-II Ramanujan graph (see [9]), of degree 6.\textsuperscript{4} We fix $\epsilon$ at .05, and take the initial sensor data bound, $b$, to be 30. We numerically solve the step-size optimization problem given in (114) for varying number of levels,

\textsuperscript{3}Note that, for both the algorithms, QC and QCF, we can take the scaling factor, $s$, arbitrarily close to 0, but not zero, so that, these limiting performance values are not achievable, but we may get arbitrarily close to them.

\textsuperscript{4}This is a 6-regular graph, i.e., all the nodes have degree 6.
2p + 1. Specifically, we consider two instances of the optimization problem: In the first instance, we consider the weight sequence, $\alpha(i) = .01/(i+1), (s = .01)$, and numerically solve the optimization problem for varying number of levels. In the second instance, we repeat the same experiment, with the weight sequence, $\alpha(i) = .001/(i + 1), (s = .001)$. As in eqn. (114), $\Delta^*(G, b, \alpha_s, \epsilon, p)$ denotes the optimal step-size. Also, let $T^*(G, b, \alpha_s, \epsilon, p)$ be the corresponding optimum probability of $\epsilon$-consensus. Fig. 1 on the left plots $T^*(G, b, \alpha_s, \epsilon, p)$ for varying $2p + 1$ on the vertical axis, while on the horizontal axis, we plot the corresponding quantizer bit-rate $BR = \log_2(2p + 1)$. The two plots correspond to two different scalings, namely, $s = .01$ and $s = .001$ respectively. The result is in strict agreement with Item 2) of Proposition 16, and shows that, as the scaling factor decreases, the probability of $\epsilon$-consensus increases, till it reaches the zero-rate probability of $\epsilon$-consensus.

Fig. 1 on the right plots $\Delta^*(G, b, \alpha_s, \epsilon, p)$ for varying $2p + 1$ on the vertical axis, while on the horizontal axis, we plot the corresponding quantizer bit-rate $BR = \log_2(2p + 1)$. The two plots correspond to two different scalings, namely, $s = .01$ and $s = .001$ respectively. The results are again in strict agreement to Proposition 16 and further show that optimizing the step-size is an important quantizer design problem, because the optimal step-size value is sensitive to the number of quantization levels, $2p + 1$.

---

**V. Conclusion**

The paper considers distributed average consensus with quantized information exchange and random inter-sensor link failures. We address two versions of the problem: when the quantizers’ alphabet is unbounded, the QC algorithm; and when the quantizers alphabet is bounded, the QCF algorithm. To achieve consensus, we add dither to the sensor states before quantization. We demonstrate that the sensor states achieve consensus to a random variable whose mean is the desired average. We show that the variance of this random variable can be made small by tuning parameters of the algorithm (rate of decay of the gains), the network topology, and quantizers parameters. We
consider several metrics of performance and show that with high probability the sensors states achieve \( \epsilon \)-consensus, i.e., they stay within a ball of radius \( \epsilon \) of the true desired average. Given a finite bit-budget, we cast the quantizer design as an optimization problem. Analytical expressions and a numerical study illustrate this design problem and several interesting tradeoffs among design parameters.

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