A GENERALIZATION OF THE CASSELS-TATE DUAL EXACT SEQUENCE

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Abstract. We extend the well-known Cassels-Tate dual exact sequence for abelian varieties $A$ over global fields $K$ in two directions: we treat the $p$-primary component in the function field case, where $p$ is the characteristic of $K$, and we dispense with the hypothesis that the Tate-Shafarevich group of $A$ is finite.

1. Introduction

Let $K$ be a global field and let $m$ be a positive integer which is prime to the characteristic of $K$ (in the function field case). Let $A$ be an abelian variety over $K$. Then there exists an exact sequence of discrete groups

$$0 \to \Sha(A)(m) \to H^1(K, A)(m) \to \bigoplus_{v} H^1(K_v, A)(m) \to \text{Coker}(H^1(K, A)(m) \to \bigoplus_{v} H^1(K_v, A)(m)) \to 0,$$

where $K_v$ is the henselization of $K$ at $v$, $M(m)$ denotes the $m$-primary component of a torsion abelian group $M$, and $\text{Coker}(H^1(K, A)(m) \to \bigoplus_{v} H^1(K_v, A))$. The Pontrjagyn dual of the preceding exact sequence is an exact sequence of compact groups

$$0 \leftarrow \Sha(A)(m)\hat{*} \leftarrow H^1(K, A)(m)\hat{*} \leftarrow \prod_{v} H^0(K_v, A')(\hat{*}) \leftarrow \text{Coker}(H^1(K, A)(m)\hat{*} \to \bigoplus_{v} H^1(K_v, A)) \leftarrow 0,$$

where $A'$ is the abelian variety dual to $A$ and, for any abelian group $M$, $M\hat{*}$ denotes the $m$-adic completion $\varprojlim_n M/m^n$ of $M$. Now, if $\Sha(A)(m)$ is finite (or, more generally, if $\Sha(A)(m)$ contains no nontrivial elements which are divisible by $m^n$ for every $n \geq 1$), then $\Sha(A)(m)\hat{*}$ and $\text{Coker}(H^1(K, A)(m)\hat{*} \to \bigoplus_{v} H^1(K_v, A))$ are canonically isomorphic to $\Sha(A')(m)$

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and $A_i'(K)^\sim$, respectively, and the preceding exact sequence induces an exact sequence

$$0 \leftarrow \bigoplus (A_i')(m) \leftarrow H^1(K, A)(m)^* \leftarrow \prod_{v} H^0(K_v, A_i') \leftarrow A_i'(K)^\sim \leftarrow 0$$

which is known as the Cassels-Tate dual exact sequence [3, 11]. See [9, Theorem II.5.6(b), p.247]. The aim of this paper is to extend the isomorphism $B(A)(m)^* \simeq A_i'(K)^\sim$ recalled above to the case where $m$ is divisible by the characteristic of $K$ (in the function field case) and no hypotheses are made on $\bigoplus (A_i')$. The following is the main result of the paper. Let $m$ and $n$ be arbitrary positive integers. Set

$$\text{Sel}(A_i')_{m^n} = \text{Ker} \left[ H^1(K, A_i'_{m^n}) \to \bigoplus_{v} H^1(K_v, A_i') \right]$$

and

$$T_m \text{Sel}(A_i') = \lim_{\leftarrow n} \text{Sel}(A_i')_{m^n}.$$ 

Then the following holds\(^1\):

**Main Theorem.** For any positive integer $m$, there exists a natural exact sequence of compact groups

$$0 \leftarrow \bigoplus (A_i')(m)^* \leftarrow H^1(K, A)(m)^* \leftarrow \prod_{v} H^0(K_v, A_i') \leftarrow T_m \text{Sel}(A_i') \leftarrow 0.$$ 

It should be noted that a similar statement holds true if above the henselizations of $K$ are replaced by its completions. See [9, Remark I.3.10, p.58].

This paper grew out of questions posed to the authors by B.Poonen, in connection with the forthcoming paper [10]. We expect that the above theorem will be useful in [op.cit.].

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\(^1\)To see why the exact sequence of the theorem extends the Cassels-Tate dual exact sequence recalled above, see exact sequence (6) below and note that $T_m \bigoplus (A_i'_{m-div})$ vanishes if $\bigoplus (A_i')_{m-div} = 0$. 
with in the relevant proofs of [9] if \( A(K) \) is replaced with \( T_m \text{Sel}(A) \) throughout\(^2\).

2. Settings and notations

Let \( K \) be a global field and let \( A \) be an abelian variety over \( K \). In the function field case, let \( p \) denote the characteristic of \( K \). All cohomology groups below are either Galois cohomology groups or flat cohomology groups. For any non-archimedean prime \( v \) of \( K \), \( K_v \) will denote the field of fractions of the henselization of the ring of \( v \)-integers of \( K \). If \( v \) is an archimedean prime, \( K_v \) will denote the completion of \( K \) at \( v \), and we will write \( H^0(K_v, A) \) for the quotient of \( A(K_v) \) by its identity component. Note that, for any prime \( v \) of \( K \), the group \( H^1(K_v, A) \) is canonically isomorphic to \( H^1(\hat{K}_v, A) \), where \( \hat{K}_v \) denotes the completion of \( K \) at \( v \). See [9, Remark I.3.10(ii), p.58]. Now let \( X \) denote either the spectrum of the ring of integers of \( K \) (in the number field case) or the unique smooth complete curve over the field of constants of \( K \) with function field \( K \) (in the function field case). In what follows, \( U \) denotes a nonempty open subset of \( X \) such that \( A \) has good reduction over \( U \). When \( N \) is a quasi-finite flat group scheme on \( U \), we endow \( H^r(U,N) \) with the discrete topology. Now let \( m \) and \( n \) be arbitrary positive integers, and let \( M \) be an abelian topological group. We will write \( M/m^n \) for \( M/m^nM = M \otimes_{\mathbb{Z}} \mathbb{Z}/m^n \) and \( \hat{M} \) for the \( m \)-adic completion \( \lim \leftarrow n M/m^n \) of \( M \). Further, we set \( \mathbb{Z}_m = \prod_{\ell | m} \mathbb{Z}_\ell \), \( \mathbb{Q}_m = \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Q} \) and define \( M^* = \text{Hom}_{\text{cts}}(M, \mathbb{Q}/\mathbb{Z}) \). Finally, the \( m \)-primary component of a torsion group \( M \) will be denoted by \( M(m) \).

3. Proof of the Main Theorem

Both \( A \) and its dual variety \( A^t \) extend to abelian schemes \( A \) and \( A^t \) over \( U \) (see [2, Ch.1, §1.4.3]). By [5, VIII.7.1(b)], the canonical Poincaré biextension of \( (A^t, A) \) by \( \mathbb{G}_m \) extends to a biextension over \( U \) of \( (A^t, A) \) by \( \mathbb{G}_m \). Further, by [op.cit., VII.3.6.5], (the isomorphism class of) this biextension corresponds to a map \( A^t \otimes^L A \to \mathbb{G}_m[1] \) in the derived category of the category of smooth sheaves on \( U \). This map in turn induces (see [9, p.283]) a canonical pairing \( H^1(U,A^t) \times H^1_c(U,A) \to H^3_c(U,\mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z} \), where the \( H^r_c(U,A) \) are

\(^2\)After this paper was completed, we learned that the existence of a natural duality between \( B(A)(m) \) and \( T_m \text{Sel}(A^t) \) had already been observed by J.W.S.Cassels in the case of elliptic curves over number fields. See [3, p.153]. Therefore, the Main Theorem of this paper may be regarded as a natural generalization of Cassels’ result.
Remark 3.1. The smoothness of $\mathcal{A}$ implies that the groups $H^r(U, \mathcal{A})$ and $H^r_c(U, \mathcal{A})$ agree with the analogous groups defined for the étale topology. See [9, Proposition III.0.4(d), p.272].

For any positive integer $m$ and any $n \geq 1$, the above pairing induces a pairing

$$H^1(U, \mathcal{A}_m^t) \times H^1_c(U, \mathcal{A})/m^n \to \mathbb{Q}/\mathbb{Z}. \quad (1)$$

On the other hand, the map $\mathcal{A}_m^t \otimes L \mathcal{A} \to \mathbb{G}_m[1]$ canonically defines a map $\mathcal{A}_m^t \times \mathcal{A}_m^t \to \mathbb{G}_m$, which induces a pairing

$$H^1(U, \mathcal{A}_m^t) \times H^2_c(U, \mathcal{A}_m^t) \to \mathbb{Q}/\mathbb{Z}. \quad (2)$$

The preceding pairing induces an isomorphism

$$H^2_c(U, \mathcal{A}_m^t) \cong H^1(U, \mathcal{A}_m^t)^*. \quad (3)$$

See [9, Corollary II.3.3, p.217] for the case where $m$ prime to $p$, and [op.cit., Theorem III.8.2, p.361] for the case where $m$ is divisible by $p$. The pairings (1) and (2) are compatible, in the sense that the following diagram commutes:

$$\begin{array}{ccc}
H^1(U, \mathcal{A}_m^t) \times H^1_c(U, \mathcal{A})/m^n & \xrightarrow{\text{id} \times \partial} & \mathbb{Q}/\mathbb{Z} \\
\downarrow \quad \downarrow & & \\
H^1(U, \mathcal{A}_m^t) \times H^2_c(U, \mathcal{A}_m^t) & \to & \mathbb{Q}/\mathbb{Z},
\end{array} \quad (4)$$

where $\partial: H^1_c(U, \mathcal{A})/m^n \to H^2_c(U, \mathcal{A}_m^t)$ is induced by the connecting homomorphism $H^1_c(U, \mathcal{A}) \to H^2_c(U, \mathcal{A}_m^t)$ coming from the exact sequence

$$0 \to \mathcal{A}_m^t \to \mathcal{A} \to \mathcal{A} \to 0.$$

Now define

$$\text{Sel}(A^t)_m = \text{Ker} \left[ H^1(K, \mathcal{A}_m^t) \to \bigoplus_{\text{all } v} H^1(K_v, A^t) \right]$$

and

$$T_m \text{Sel}(A^t) = \lim_{\longrightarrow} \text{Sel}(A^t)_m.$$

By the proof of [9, Proposition I.6.4, p.92], there exists an exact sequence

$$0 \to A^t(K)/m^n \to \text{Sel}(A^t)_m \to \text{III}(A^t)_m \to 0. \quad (5)$$
Taking inverse limits, we obtain an exact sequence
\begin{equation}
0 \to A'(K)^\sim \to T_m \Sel(A') \to T_m \III(A') \to 0. 
\end{equation}

See [1, Proposition 10.2, p.104]. Now define\footnote{In these definitions, the products extend over all primes of \( K \), including the archimedean primes, not in \( U \).}
\[ D^1(U, A^t_{m^n}) = \Ker \left[ H^1(U, A^t_{m^n}) \to \prod_{v \notin U} H^1(K_v, A^t) \right] \]
and
\[ D^1(U, A^t) = \Im \left[ H^1(U, A^t) \to H^1(U, A^t) \right] = \Ker \left[ H^1(U, A^t) \to \prod_{v \notin U} H^1(K_v, A^t) \right]. \]

Note that the pairing \( H^1(U, A^t) \times H^1_c(U, A) \to \Q/\Z \) induces a pairing \( D^1(U, A^t) \times D^1(U, A) \to \Q/\Z \).

By [9, Proposition III.0.4(a), p.271] and the right-exactness of the tensor product functor, there exists a natural exact sequence
\begin{equation}
\bigoplus_{v \notin U} H^0(K_v, A)/m^n \to H^1_c(U, A)/m^n \to D^1(U, A)/m^n \to 0. 
\end{equation}

**Lemma 3.2.** The map \( H^1(U, A^t_{m^n}) \hookrightarrow H^1(K, A^t_{m^n}) \) induces an isomorphism
\[ D^1(U, A^t_{m^n}) \simeq \Sel(A^t_{m^n}). \]

**Proof.** By [9, Lemma II.5.5, p.246] and Remark 3.1 above, the map \( H^1(U, A^t) \hookrightarrow H^1(K, A^t) \) induces an isomorphism
\[ D^1(U, A^t_{m^n}) \simeq \III(A^t_{m^n}). \]

Now \( H^1(U, A^t_{m^n}) \to \prod_{v \notin U} H^1(K_v, A^t) \) factors through \( H^1(U, A^t) \to \prod_{v \notin U} H^1(K_v, A^t) \), which is the zero map (see [9, (5.5.1), p.247] and Remark 3.1 above). Consequently, \( H^1(U, A^t_{m^n}) \hookrightarrow H^1(K, A^t_{m^n}) \) maps \( D^1(U, A^t_{m^n}) \) into \( \Sel(A^t_{m^n}) \). To prove surjectivity, we consider the commutative diagram
\[ \begin{array}{cccccc}
0 & \to & A^t(U)/m^n & A^t(U) & \to & H^1(U, A^t_{m^n}) & \to & H^1(U, A^t)/m^n & \to & 0 \\
\downarrow & & & & \downarrow & & & \downarrow & & \downarrow \\
0 & \to & A^t(K)/m^n & A^t(K) & \to & H^1(K, A^t_{m^n}) & \to & H^1(K, A^t)/m^n & \to & 0.
\end{array} \]

Note that the properness of \( A^t \) over \( U \) implies that the left-hand vertical map in the above diagram is an isomorphism (see [op.cit., p.242]). Now let \( c \in \Sel(A^t)_{m^n} \), write \( c' \) for its image in \( \III(A^t)_{m^n} \) under the map in (5) and let \( \xi' \in D^1(U, A^t)_{m^n} \subset H^1(U, A^t)_{m^n} \) be the pullback of
c' under the isomorphism $D^1(U, \mathcal{A}^t)_{m^n} \simeq \text{III}(A^t)_{m^n}$ recalled above. Then the fact that the left-hand vertical map in the above diagram is an isomorphism implies that $\xi'$ can be pulled back to a class $\xi \in H^1(U, \mathcal{A}_{m^n})$ which maps down to $c$. Clearly $\xi \in D^1(U, \mathcal{A}_{m^n})$, and this completes the proof. \[ \square \]

The following proposition generalizes [9, Theorem II.5.2(c), p.244].

**Proposition 3.3.** There exists a canonical isomorphism $(T_m \text{Sel}(A^t))^* \xrightarrow{\sim} H^2_c(U, \mathcal{A})(m)$.

**Proof.** There exists a commutative diagram

$$
0 \longrightarrow H^1_c(U, \mathcal{A})/m^n \longrightarrow H^2_c(U, \mathcal{A}_{m^n}) \longrightarrow H^2_c(U, \mathcal{A})_{m^n} \longrightarrow 0
$$

where the vertical map is the isomorphism (3). Clearly, the above diagram induces an isomorphism $\text{Coker } c \simeq H^2_c(U, \mathcal{A})_{m^n}$. On the other hand, there exists a natural exact commutative diagram

$$
\bigoplus_{v \notin U} H^0(K_v, \mathcal{A})/m^n \longrightarrow H^1_c(U, \mathcal{A})/m^n \longrightarrow D^1(U, \mathcal{A})/m^n \longrightarrow 0
$$

where the top row is (7), the right-hand vertical map $\psi$ is the composite of the natural map $D^1(U, \mathcal{A})/m^n \rightarrow D^1(U, \mathcal{A}_{m^n})^*$ induced by the pairing $D^1(U, \mathcal{A}^t) \times D^1(U, \mathcal{A}) \rightarrow \mathbb{Q}/\mathbb{Z}$ and the natural map $D^1(U, \mathcal{A}^t)_{m^n} \rightarrow D^1(U, \mathcal{A}_{m^n})^*$, and the left-hand vertical map is induced by the canonical Poincaré biextensions of $(A^t, \mathcal{A})$ by $\mathbb{G}_m$ over $K_v$ for each $v \notin U$. That the latter map is an isomorphism follows from [9, Remarks I.3.5 and I.3.7, pp.53 and 56, and Theorem III.7.8, p.354] and the fact that the pairings defined in [loc.cit.] are compatible with the pairing induced by the canonical Poincaré biextension (see [4, Appendix]). The above diagram and the identification $\text{Coker } c = H^2_c(U, \mathcal{A})_{m^n}$ yield an exact sequence

$$
D^1(U, \mathcal{A})/m^n \rightarrow D^1(U, \mathcal{A}_{m^n})^* \rightarrow H^2_c(U, \mathcal{A})_{m^n} \rightarrow 0
$$

\[\text{4}\text{The commutativity of this diagram follows from that of diagram (4).}\]
Taking direct limits, we obtain an exact sequence

\[ D^1(U, \mathcal{A}) \otimes \mathbb{Q}_m/\mathbb{Z}_m \to (\varprojlim D^1(U, \mathcal{A}^t_m))^* \to H^2_c(U, \mathcal{A})(m) \to 0 \]

But \( D^1(U, \mathcal{A}) \otimes \mathbb{Q}_m/\mathbb{Z}_m = 0 \) since \( D^1(U, \mathcal{A}) \) is torsion and \( \mathbb{Q}_m/\mathbb{Z}_m \) is divisible. Now lemma 3.2 completes the proof. \( \square \)

By Remark 3.1 and [9, proof of Lemma II.5.5, p.247, and Proposition II.2.3, p. 203], there exist exact sequences

\[ H^1(U, \mathcal{A}) \xrightarrow{c_U} \bigoplus_{v \notin U} H^1(K_v, \mathcal{A}) \to H^2_c(U, \mathcal{A}) \]

and

\[ 0 \to H^1(U, \mathcal{A}) \xrightarrow{i_U} H^1(K, \mathcal{A}) \xrightarrow{\lambda_U} \bigoplus_{v \in U} H^1(K_v, \mathcal{A}), \]

where \( c_U \) and \( \lambda_U \) are natural localization maps and \( i_U \) is induced by the inclusion \( \text{Spec} \mathbb{K} \hookrightarrow U \). If \( U \subset V \) is an inclusion of nonempty open subsets of \( X \), then there exists a natural commutative diagram

\[
\begin{array}{ccc}
H^1(V, \mathcal{A}) & \xrightarrow{c_V} & \bigoplus_{v \notin V} H^1(K_v, \mathcal{A}) \\
\downarrow & & \downarrow \\
H^1(U, \mathcal{A}) & \xrightarrow{c_U} & \bigoplus_{v \notin U} H^1(K_v, \mathcal{A}).
\end{array}
\]

Define

\[ B(A)_U = \text{coker} \left[ c_U : H^1(U, \mathcal{A}) \to \bigoplus_{v \notin U} H^1(K_v, \mathcal{A}) \right], \]

which we regard as a subgroup of \( H^2_c(U, \mathcal{A}) \). The preceding diagram shows that an inclusion \( U \subset V \) of nonempty open subsets of \( X \) induces a map \( B(A)_V \to B(A)_U \). Define

\[ B(A) = \varinjlim B(A)_U = \text{coker} \left[ H^1(K, \mathcal{A}) \to \bigoplus_{\text{all } v} H^1(K_v, \mathcal{A}) \right], \]

where the limit is taken over the directed family of all nonempty open subsets \( U \) of \( X \) such that \( A \) has good reduction over \( U \), ordered by

\[ ^5 \text{In the second exact sequence, } "v \in U" \text{ is shorthand for } "v \text{ is a closed point of } U". \]
\( V \leq U \) if and only if \( U \subset V \). For each \( U \) as above and every \( n \geq 1 \), there exists an exact sequence
\[
\bigoplus_{v \notin U} H^1(K_v, A)_m^n \to (B(A)_U)_m^n \to (\text{Im } c_U)/m^n.
\]
Since \( \text{Im } c_U \) is torsion, we conclude that there exists a surjection
\[
\bigoplus_{v \notin U} H^1(K_v, A)(m) \xrightarrow{(8)} B(A)_U(m)
\]
On the other hand, by the proof of [9, Corollary I.6.23(b), p.111], there exists a natural injection \( T_m \text{Sel}(A^i) \hookrightarrow \prod_v H^0(K_v, A^i)^{*} \) and hence a surjection
\[
\bigoplus_{v} (H^0(K_v, A^i)^{*}) \to (T_m \text{Sel}(A^i))^*.
\]
Further, as noted in the proof of Proposition 3.3, the canonical Poincaré biextensions induce an isomorphism
\[
\bigoplus_{v} (H^0(K_v, A^i)^{*}) \simeq \bigoplus_{v} H^1(K_v, A)(m),
\]
whence there exists a surjection
\[
\bigoplus_{v} H^1(K_v, A)(m) \xrightarrow{(9)} (T_m \text{Sel}(A^i))^*.
\]
The maps (8) and (9) fit into a commutative diagram
\[
\begin{array}{ccc}
\bigoplus_{v} H^1(K_v, A)(m) & \xrightarrow{(9)} & (T_m \text{Sel}(A^i))^* \\
& \uparrow & \downarrow \\
\bigoplus_{v \notin U} H^1(K_v, A)(m) & \xrightarrow{(8)} & B(A)_U(m),
\end{array}
\]
where the isomorphism on the top row exists by Proposition 3.3. Taking the direct limit over \( U \) in the above diagram, we conclude that there exists an isomorphism
\[
B(A)(m) \sim (T_m \text{Sel}(A^i))^*,
\]
as desired.

**Remark 3.4.** Recently [7, Theorem 1.2], the Cassels-Tate dual exact sequence has been extended to 1-motives \( M \) over number fields under the assumption that the Tate-Shafarevich group of \( M \) is finite. Now, using [6, Remark 5.10], it should not be difficult to extend this result
to global function fields, provided the p-primary components of the groups involved are ignored, where p denotes the characteristic of K. In this paper we have removed the latter restriction when M is an abelian variety, but the problem remains for general 1-motives M.

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