Outliers and Random Noises in System Identification: a Compressed Sensing Approach

Weiyu Xu\textsuperscript{a} Erwei Bai\textsuperscript{b} Myung Cho\textsuperscript{c}

\textsuperscript{a}Dept. of Electrical and Computer Engineering
University of Iowa, Iowa City, Iowa 52242

\textsuperscript{b}Dept. of Electrical and Computer Engineering
University of Iowa, Iowa City, Iowa 52242
School of Electronics, Electrical Engineering and Computer Science
Queen’s University, Belfast, UK

\textsuperscript{c}Dept. of Electrical and Computer Engineering
University of Iowa, Iowa City, Iowa 52242

Abstract

In this paper, we consider robust system identification under sparse outliers and random noises. In this problem, system parameters are observed through a Toeplitz matrix. All observations are subject to random noises and a few are corrupted with outliers. We reduce this problem of system identification to a sparse error correcting problem using a Toeplitz structured real-numbered coding matrix. We prove the performance guarantee of Toeplitz structured matrix in sparse error correction. Thresholds on the percentage of correctable errors for Toeplitz structured matrices are established. When both outliers and observation noise are present, we have shown that the estimation error goes to 0 asymptotically as long as the probability density function for observation noise is not “vanishing” around 0. No probabilistic assumptions are imposed on the outliers.

Key words: system identification, least absolute deviation, $\ell_1$ minimization, Toeplitz matrix, compressed sensing

1 Introduction

In a linear system identification setting, an unknown system parameter vector $x \in \mathbb{R}^m$ is often observed through a Toeplitz matrix $H \in \mathbb{R}^{n \times m}$ ($n \geq m$), namely

$$y = Hx,$$

where $y = (y_1, y_2, \ldots, y_n)^T$ is the system output and the Toeplitz matrix $H$ is equal to

$$
\begin{bmatrix}
    h_{-m+2} & h_{-m+3} & \ldots & h_1 \\
    h_{-m+3} & h_{-m+4} & \ldots & h_2 \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{-m+n+1} & \ldots & \ldots & h_n
\end{bmatrix},
$$

with $h_i, -m + 2 \leq i \leq n$, being the system input sequence.

If there is no interference or noise in the observation $y$, one can then simply recover $x$ from matrix inversion. However, in applications, the observations $y$ are corrupted by noises and a few elements can be exposed to large-magnitude gross errors or outliers. Such outliers
can happen with the failure of measurement devices, measurement communication errors and the interference of adversary parties. Mathematically, when both additive observation noise and outliers are present, the observation $\mathbf{y}$ can be written as

$$\mathbf{y} = H\mathbf{x} + \mathbf{e} + \mathbf{w}, \quad (1.2)$$

where $\mathbf{e}$ is a sparse outlier vector with $k \ll n$ non-zero elements, and $\mathbf{w}$ is a measurement noise vector with each element usually being assumed to be i.i.d. random variables. We further assume $m$ is fixed and $n$ can increase, which is often the case in system identifications [6].

If only random measurement errors are present, the least-square solutions generally provide an asymptotically good estimate. However, the least-square estimate breaks down in the presence of outliers. Thus, it is necessary to protect the estimates from both random noise and outliers. Research along this direction has attracted a significant amount of attention, for example, [1,3,4,5,6,8]. An effective way is to visually inspect the residual plot and change the obviously erroneous measurements “by hand” to an appropriately interpolated values [6]. The approach does not however always work. Another approach was the idea of few violated constraints [1] in the setting of the bounded error parameter estimation. The other two popular methods in the statistical literature to deal with the outliers are the least median squares and the least trimmed squares [5]. Instead of minimizing the sum of the residual squares, the least median squares yields the smallest value for the median of squared residuals computed from the entire data set and the least trimmed squares try to minimize the sum of squared residues over a subset of the given data. Both have shown robustness against the outliers [5]. The problem is their computational complexity. Both algorithms are nonlinear and in fact combinatorial in nature. This limits their practical applications if $n$ and/or $m$ are not small or even modest. The most popular way to deal with the outliers in the statistical literature is the the least absolute deviation estimate ($\ell_1$ minimization) which has been extensively studied [2,9,10,11,25]. Instead of searching for all the $\binom{n}{k}$ possibilities for the locations of outliers, [2,9,10] proposed to minimize the least absolute deviation:

$$\min_{\mathbf{x}} \| \mathbf{y} - H\mathbf{x} \|_1. \quad (1.3)$$

Under the assumption that the error $\mathbf{e} + \mathbf{w}$ is an i.i.d. random sequence with a common density which has median zero and is continuous and positive in the neighborhood of zero, the difference between the unknown $\mathbf{x}$ and its estimate is asymptotically Gaussian of zero mean [2]. The problem is that the assumption of i.i.d. of median zero on the unknown outliers is very restrictive and seldomly satisfied in reality. We study the least absolute deviation estimator or $\ell_1$ minimization from the compressed sensing point of view and show that i.i.d. of median zero on the outliers are unnecessary. In fact only the number of outliers relative to the total number of data length plays a role.

Recovering signals from outliers or errors have been studied [9,10,11,15,16,23,25]. In their setting, each element of the nonsingular $(n - m) \times n$ matrix $A$ such that $AH = 0$, is assumed to be i.i.d. random variables following a certain distribution, for example, Gaussian distribution or Bernoulli distribution. These types of matrices have been shown to obey certain conditions such as restricted isometry conditions [9] so that (1.3) can correctly recover $\mathbf{x}$ when there are only outliers present; and can recover $\mathbf{x}$ approximately when both outliers and measurement noise exist. However, in the system identification problem, $H$ has a natural Toeplitz structure and the elements of $H$ are not independent but correlated. The natural question is whether (1.3) also provides performance guarantee for recovering $\mathbf{x}$ with a Toeplitz matrix. We provide a positive answer in this paper.

Despite the fact that the elements of Toeplitz matrices are correlated, we are able to show in this paper that Toeplitz structured matrices also enable the successful recovery of $\mathbf{x}$ by using (1.3). These results are first established for Toeplitz Gaussian matrices, where the system input sequence $h_i, -m + 2 \leq i \leq n$, are i.i.d. Gaussian random variables. We then extend our results to Toeplitz matrix with non-Gaussian input sequences. The main contribution of this paper is the establishment of the performance guarantee of Toeplitz structured matrices in parameter estimation in the presence of both outliers and random noises. In particular, we calculated the thresholds on the sparsity $k$ such that the parameter vector can be perfectly calculated in the presence of a unknown outlier vector with no more than $k$ non-zero elements using (1.3). When both outliers and observation noise are present, we have shown that the estimation error goes to 0 asymptotically as long as the probability density function for observation noise is not “vanishing” around 0.

We also like to point out that there is a well known duality between compressed sensing [12,14] and sparse error detection [9,10]: the null space of sensing matrices in compressed sensing corresponds to the tall matrix $H$ in sparse error corrections. Toeplitz and circulant matrices have been studied in compressed sensing in several papers [18,19,20]. In these papers, it has been shown that Toeplitz matrices are good for recovering sparse vectors from undersampled measurements. In contrast, in our model of parameter estimation in the presence of outliers, the signal itself is not sparse and the linear system involved is overdetermined rather underdetermined. Also, the null space of a Toeplitz matrix does not necessarily correspond to another Toeplitz matrix; so the problem studied in this paper is essentially different from
those studied in [18,19,20].

The rest of this paper is organized as follows. In Section 2, we derive performance bounds on the number of outliers we can correct when only outliers are present. In Section 3, we derive the estimation of system parameters when both outliers and random noises are present. In Section 4, we extend our results to non-Gaussian inputs in system identification. In Section 5, we provide the numerical results and in Section 6 conclude our paper by discussing extensions and future directions.

2 With Only Outliers

We establish one main result regarding the threshold of successful recovery of the system parameter vector x by ℓ₁-minimization.

**Theorem 2.1** Let H be an n × m Toeplitz Gaussian matrix as in (1.1), where m is a fixed positive integer and h_i, −m + 2 ≤ i ≤ n are i.i.d. N(0,1) Gaussian random variables. Suppose that y = Hx + e, where e is a sparse outlier vector with no more than k non-zero elements. Then there exists a constant c₁ > 0 and a constant β > 0 such that, with probability 1 − e^−c₁n as n → ∞, the n × m Toeplitz matrix H has the following property.

For every x ∈ Rᵐ and every error e with its support K satisfying |K| = k ≤ βn, x is the unique solution to (1.3). Here the constant 0 < β < 1 can be taken as any number such that for some constant μ > 0 and 0 < δ < 1,

\[ \beta \log(1/β) + (1 - β)\log\left(\frac{1}{1 - β}\right) + mβ\log(2) + \frac{mμ^2}{2} \]

\[ + \log(Φ(μ√m)) + \left(\frac{1}{2m - 1} - β\right)\log(2) \]

\[ + \frac{1}{2}μ^2(1 - δ)^2 + \log(1 - Φ(μ(1 - δ))) < 0 \]

where Φ(t) = \frac{1}{\sqrt{2π}} \int_{−∞}^{t} e^{-\frac{x^2}{2}} dx is the cumulative distribution function for the standard Gaussian random variable.

**Remark:** The derived correctable fraction of errors β depends on the system dimension m.

We first outline the overall derivation strategy, and then go on to prove Theorem 2.1.

Our derivation is based on checking the following now-well-known theorem for ℓ₁ minimization (see [26], for example).

**Theorem 2.2** (1.3) can recover the parameter vector x exactly whenever ∥e∥₀ ≤ k, if and only if for every vector z ∈ Rᵐ ≠ 0, ∥(Hz)_K∥₁ < ∥(Hz)_{K^c}∥₁ for every subset K ⊆ \{1, 2, ..., n\} with cardinality |K| = k, where K^c = \{1, 2, ..., n\} \ K.

The difficulty of checking this condition is that the elements of H are not independent random variables and that the condition must hold for every vector in the subspace generated by H. We adopt the strategy of discretizing the subspace generated by H, see [11,17,21].

It is obvious that we only need to consider Hz for z ∈ Rᵐ with ∥z∥₂ = 1. We then pick a finite set V = {v_1, ..., v_N} called γ-net on \{z∥z∥₂ = 1\} for a constant γ > 0: in a γ-net, for every point z from \{z∥z∥₂ = 1\}, there is a v_i in V such that ∥z − v_i∥₂ ≤ γ. We subsequently establish the property in Theorem 2.2 for all the points in γ-net V, before extending the property in Theorem 2.2 to Hz with ∥z∥₂ = 1 not necessarily belonging to the γ-net.

Following this method, we divide the derivation into Lemmas 2.3, 2.5 and 2.6, which lead to Theorem 2.1.

We now start to derive the results in Theorem 2.1. We first show the concentration of measure phenomenon for Hz, where z ∈ Rᵐ is a single vector with ∥z∥₂ = 1, in Lemmas 2.3, 2.5.

**Lemma 2.3** Let ∥z∥₂ = 1. For any ε > 0, when n is large enough, with probability at least 1 − 2e^−εn^2 as n → ∞, it holds that

\[ (1 - ε)S ≤ \|Hz\|₁ ≤ (1 + ε)S \]

where S = nE(|X|) and X is a random variable following the Gaussian distribution N(0,1). Namely there exists a constant c₂ > 0, such that

\[ (1 - ε)S ≤ \|Hz\|₁ ≤ (1 + ε)S \]

holds with probability 1 − 2e^−εn as n → ∞.

**Proof:** Our derivations rely on the concentration of measure inequalities and the Chernoff bounds for Gaussian random variables [21,22].

**Proposition 2.4** (Gaussian concentration inequality for Lipschitz functions) Let f : Rᵈ → R be a function which is Lipschitz with constant L (i.e. for all a ∈ Rᵈ and b ∈ Rᵈ, |f(a) − f(b)| ≤ L∥a − b∥₂). Then for any t, we have

\[ P(|f(X) − E(f(X))| ≥ t) ≤ 2e^{−\frac{t^2}{2L^2}} \]

where X is a vector of d i.i.d. standard Gaussian random variables N(0,1).

We show that, for any ∥z∥₂ = 1, the function f(h) = ∥Hz∥₁ is a function of Lipschitz constant \(\sqrt{m(n+m-1)}\), where h = (h_{m+2}, h_{m+1}, ..., h_n).
For two vectors $h^1$ and $h^2$, by the triangular inequality for $\ell_1$ norm and the Cauchy-Schwarz inequality,

$$|f(h^1) - f(h^2)| \leq \sum_{i=-m+2}^{n} |(h^1)_i - (h^2)_i| \times |z_1|$$

$$\leq \sqrt{n + m - 1} \left\| h^1 - h^2 \right\|_2 \sqrt{m} ||z||_2$$

$$= \sqrt{(n + m - 1)m} ||h^1 - h^2||_2,$$

since $||z||_2 = 1$. Then a direct application of the Gaussian concentration inequality above leads us to Lemma 2.3.

\[\Box\]

**Lemma 2.5** Let $||z||_2 = 1$ and $0 < \delta < 1$ be a constant. Then there exists a threshold $\beta \in (0, 1)$ and a constant $c_3 > 0$ (depending on $m$ and $\beta$), such that, with a probability $1 - e^{-c_3m}$, for all subsets $K \subseteq \{1, 2, ..., n\}$ with cardinality $\frac{|K|}{n} \leq \beta$,

$$||(Hz)_K||_1 \leq \frac{1 - \delta}{2 - \delta} ||Hz||_1.$$

**Proof**: Note that for a vector $z$ from the unit Euclidean norm in $\mathbb{R}^n$, we have $\sum_{i \in K} |(Hz)_i| \leq \sum_{i \in K} \sum_{j=1}^m |H_{i,j}| |z_j|$, and by the Cauchy-Schwarz inequality,

$$\sum_{i \in K} \sum_{j=1}^m |H_{i,j}| |z_j| \leq \sum_{j \in J} \sqrt{m} |h_j|,$$

where $h_j$ is an element of $h$ and $J \subseteq \{-m + 2, ..., n\}$ is the set of elements $j$ such that $h_j$ is involved in $H_K$. So the cardinality $|J| \leq mk$. By the definition of Toeplitz matrices, the number of rows in $H$ that involve only $h_j$’s with $j$ coming from $\{-m + 2, ..., n\} \setminus J$, is at least $n - k \times (2m - 1)$. Among these rows involving only $h_j$’s with $j$ coming from $\{-m + 2, ..., n\} \setminus J$, one can pick at least $\frac{n-k \times (2m-1)}{2m-1}$ rows, such that these rows involve distinct $h_j$’s from each other.

Thus for a fixed vector $z$, there exists a set $I \subseteq \{1, 2, ..., n\}$ with cardinality at least $\frac{n-k \times (2m-1)}{2m-1}$ such that $(Hz)_i, i \in I$, are independent $N(0, 1)$ Gaussian variables; moreover, these $(Hz)_i, i \in I$, are independent from those $h_j$’s with $j \in J$. Thus for a fixed set $K$, the probability that $||(Hz)_K||_1 > \frac{1-\delta}{2-\delta} ||Hz||_1$ is smaller than the probability that

$$\sqrt{m} \sum_{i=1}^{mk} |h_i| \geq (1-\delta) \sum_{j=1}^{n-k \times (2m-1)} |h_j|,$$

where $h_i$’s and $h_j$’s are all i.i.d. $N(0, 1)$ Gaussian random variables.

Now we use the Chernoff bound,

$$P(\sqrt{m} \sum_{i=1}^{mk} |h_i| \geq (1-\delta) \sum_{j=1}^{n-k \times (2m-1)} |h_j|) \leq \min_{\mu \geq 0} E \left\{ e^{\mu(\sqrt{m} \sum_{i=1}^{mk} |h_i|) - (1-\delta) \sum_{j=1}^{n-k \times (2m-1)} |h_j|} \right\}$$

$$= \min_{\mu \geq 0} E \left\{ e^{\mu(\sqrt{m} \sum_{i=1}^{mk} |h_i|) - (1-\delta) \sum_{j=1}^{n-k \times (2m-1)} |h_j|} \right\}$$

After simple algebra, we have

$$E \left\{ e^{\mu(\sqrt{m} |h_i|)} \right\} = 2e^{\frac{\mu^2 m}{2}} F(\mu \sqrt{m})$$

and

$$E \left\{ e^{-\mu(1-\delta) |h_i|} \right\} = 2e^{\frac{\mu^2 (1-\delta)^2}{2}} (1 - F(\mu (1-\delta)))$$

where $F(t)$ is the cumulative distribution function for a standard Gaussian random variable $N(0, 1)$.

Putting this back into the Chernoff bound and notice that there are at most $\binom{n}{k}$ possible support sets $K$ with cardinality $k$, the probability $P$ that

$$\sqrt{m} \sum_{i=1}^{mk} |h_i| \geq \beta \sum_{i=1}^{n-k \times (2m-1)} |h_i|$$

is violated for at least one support set $K$ is upper bounded by the union bound

$$\log \left( \binom{n}{k} \right) + mk \log(2) + \frac{m \mu^2}{2} + \log(F(\mu \sqrt{m}))$$

$$+ \frac{n}{2m-1} k \log(2) + \frac{\mu^2 (1-\delta)^2}{2} + \log(Q((1-\delta)u)),$$

where $Q((1-\delta)u) = 1 - F((1-\delta)u)$ is the Gaussian tail function.

Let $k = \beta n$, and thus $\log(\binom{n}{k}) / n \rightarrow H(\beta)$ as $n \rightarrow \infty$, where $H(\beta) = \beta \log(1/\beta) + (1-\beta) \log(1/(1-\beta))$ is the en-
entropy function. As long as, for a certain \( \mu > 0 \),
\[
H(\beta) + m\beta \times [\log(2) + \frac{m\mu^2}{2} \log(\mu/\sqrt{m})] + \\
(1 - \beta)\log(2) + \frac{\mu^2(1 - \delta)^2}{2} \log(Q(\mu(1 - \delta)))]
\]
\[< 0,
\]
then \( \beta \) is within the correctable error region for all the support sets \( K \) with high probability. In fact, this quantity can always be made smaller than 0 if we take \( \beta \) small enough. To see this, we can first take \( \mu \) large enough, such that \( \log(2) + \frac{1}{2}\mu^2(1 - \delta)^2 + \log(1 - F(\mu(1 - \delta))) \] < 0.

This is always doable, because the tail function \( 1 - F(\mu(1 - \delta)) \) satisfies
\[
1 - F(\mu(1 - \delta)) \leq \frac{1}{\sqrt{2\pi}\mu} e^{-\frac{(\mu(1 - \delta))^2}{2}}
\]
for \( \mu(1 - \delta) > 0 \). Fixing that \( \mu \), we can then take \( \beta \) sufficiently small such that \( \log(P) < 0 \), as \( n \to \infty. \)

We have so far only considered the condition in Theorem 2.2 for a single point \( z \) on the \( \gamma \)-net. By a union bound on the size of \( \gamma \)-net, Lemma 2.3 and 2.5 indicate that, with overwhelming probability, the recovery condition in Theorem 2.2 holds for the discrete points on \( \gamma \)-net. The following lemma formally proves this fact, and then extends the proof of the recovery condition for every point in the set \( \{z \| z \parallel_2 = 1 \} \).

**Lemma 2.6** There exist a constant \( c_4 > 0 \) such that when \( n \) is large enough, with probability \( 1 - e^{-c_4n} \), the Gaussian Toeplitz matrix \( H \) has the following property:
for every \( z \in \mathbb{R}^m \) and every subset \( K \subseteq \{1, \ldots, n\} \) with \( |K| \leq \beta n \), \( \sum_{i \in K} |(Hz)_i| - \sum_{i \in K} |(Hz)_i| \geq \delta S \), where \( \delta > 0 \) is a constant.

**Proof** For any given \( \gamma > 0 \), there exists a \( \gamma \)-net \( V = \{v_1, \ldots, v_N\} \) of cardinality less than \( (1 + \frac{1}{\gamma})^m \). Since each row of \( H \) has \( m \) i.i.d. \( N(0, 1) \) entries, elements of \( H_{v_j} \), \( 1 \leq j \leq N \), are (not independent) \( N(0, 1) \) entries. Applying a union bound on the size of \( \gamma \)-net, Lemmas 2.5 and 2.3 imply that for every \( v_j \in V \), for some \( \delta > 0 \) and for any constant \( \epsilon > 0 \), with probability \( 1 - 2e^{-c\epsilon} \), for some \( c > 0 \),
\[
\|(Hv_j)_{K}\parallel_1 \leq \frac{(1 - \delta)(1 + \epsilon)}{2 - \delta} S
\]
\[\geq (1 - \epsilon) S \leq \|Hv_j\parallel_1 \leq (1 + \epsilon) S
\]
hold simultaneously for every vector \( v_j \) in \( V \).

For any \( z \) such that \( \|z\parallel_2 = 1 \), there exists a point \( v_0 \) (we change the subscript numbering for \( V \) to index the order in \( V \) such that \( \|z - v_0\parallel_2 \leq \gamma_1 \leq \gamma \). Let \( z_1 \) denote \( z - v_0 \), then \( \|z_1 - v_j\parallel_2 \leq \gamma_2 \leq \gamma \) for some \( v_j \) in \( V \). Repeating this process, we have \( z = \sum_{j \geq 0} v_j \), where \( \gamma_0 = 1, \gamma_j \leq \gamma_j \) and \( v_j \in V \).

Thus for any \( z \in \mathbb{R}^m, \) \( z = \|z\parallel_2 \sum_{j \geq 0} \gamma_j v_j \). For any index set \( K \) with \( |K| \leq \beta n \),
\[
\sum_{i \in K} |(Hz)_i| = \|z\parallel_2 \sum_{i \in K} |(\sum_{j \geq 0} \gamma_j H_{v_j})_i|
\]
\[\geq \|z\parallel_2 \sum_{i \in K} \sum_{j \geq 1} \gamma_j |(Hv_j)_i| - \sum_{j \geq 1} \gamma_j |(Hv_j)_i|
\]
\[\geq \|z\parallel_2 ((1 - \epsilon) S - \sum_{j \geq 1} \gamma_j |(Hv_j)_i|
\]
\[\geq S \|z\parallel_2 (1 - \epsilon) - \frac{\gamma(1 + \epsilon)}{1 - \gamma}.
\]
So \( \sum_{i \in K} |(Hz)_i| - \sum_{i \in K} |(Hz)_i| \geq S \|z\parallel_2 (1 - \epsilon) - \frac{\gamma(1 + \epsilon)}{1 - \gamma} - 2(\frac{(1 - \delta)(1 + \epsilon)}{2 - \delta}) \). For a given \( \delta \), we can pick \( \gamma \) and \( \epsilon \) small enough such that \( \sum_{i \in K} |(Hz)_i| - \sum_{i \in K} |(Hz)_i| \geq \delta S \|z\parallel_2 \), satisfying the condition in Theorem 2.2. □

**Proof** (of Theorem 2.1). A direct consequence of Theorem 2.2 and Lemmas 2.5, 2.3, and 2.6. □

So far, we have considered the uniform performance guarantee of \( \ell_1 \) minimization for all the sets with cardinality up to \( k \). Instead, given a support set \( K \) for the outliers (though we do not know what the support set is before performing \( \ell_1 \) minimization), the correctable sparsity \( \frac{|K|}{n} \) of outliers can go to 1. This result is formally stated in Theorem 2.7.

**Theorem 2.7** Take an arbitrary constant \( 0 < \beta < 1 \) and let \( y = Hx + e \), where \( H \) is a Gaussian Toeplitz
matrix, and \( e \) is an outlier vector with no more than \( k = \beta n \) non-zero elements. When \( n \to \infty \), \( x \) can be recovered perfectly using \( \ell_1 \) minimization from \( e \) with high probability.

**Proof** The statement follows from Theorem 3.1, specialized to the case \( w = 0 \). \( \square \)

### 3 With Both Outliers and Observation Noises

We further consider Toeplitz matrix based system identification when both outliers and random observation errors are present, namely, the observation \( y = Hx + e + w \), where \( e \) is a sparse outlier vector with no more than \( k \) non-zero elements, and \( w \) is the vector of additive observation noises. We show that, under mild conditions, the identification error \( \| \hat{x} - x \|_2 \) goes to 0 even when there are both outliers and random observation errors, where \( \hat{x} \) is the solution to (1.3).

**Theorem 3.1** Let \( m \) be a fixed positive integer and \( H \) be an \( n \times m \) Toeplitz matrix \((m < n)\) in (1.1) with each element \( h_i \), \(-m+2 \leq i \leq n\), being i.i.d. \( N(0,1) \) Gaussian random variables. Suppose

\[
y = Hx + e + w,
\]

where \( e \) is a sparse vector with \( k \leq \beta n \) non-zero elements \((\beta < 1 \text{ is a constant})\) and \( w \) is the observation noise vector. For any constant \( t \), we assume that, with high probability as \( n \to \infty \), at least \( \alpha(t)n \) \((\text{where } \alpha(t) > 0 \text{ is a constant depending on } t)\) elements in \( w + e \) are no bigger than \( t \) in amplitude.

Then \( \| \hat{x} - x \|_2 \to 0 \) with high probability as \( n \to \infty \), where \( \hat{x} \) is the solution to (1.3).

We remark that, in Theorem 3.1, the condition on the unknown outlier vector is merely \( \beta < 1 \), and the condition on the random noise \( w \) is weaker than the usual condition of having i.i.d. elements with median 0 [2]. In fact, if \( w \) is independent from \( e \), and the elements of \( w \) are i.i.d. random variables following a distribution which is not “vanishing” in an arbitrarily small region around 0 (namely the cumulative distribution function \( F(t) \) satisfies that \( F(t) - F(-t) > 0 \) for any \( t > 0 \)). Note that the probability density function \( f(t) \) (if allowed to be 0, however), the conditions in Theorem 3.1 will be satisfied.

To see that, first observe that \((1 - \beta)n \) elements of the outlier vector are zero. If elements of \( w \) are i.i.d. following a probability density function \( f(s) \) that is not “vanishing” around \( s = 0 \), with probability converging to one as \( n \to \infty \), at least \( \{F(t) - F(-t)\}(1 - \beta)(1 - \epsilon)n = \alpha(t)n \) elements of the vector \( e + w \) are no bigger than \( t \), where \( \epsilon > 0 \) is an arbitrarily small number. Gaussian distributions, exponential distributions, Gamma distributions and many other distributions for \( w \) all satisfy such conditions in Theorem 3.1. This greatly broadens the existing results, e.g., in [2], which requires \( f(0) > 0 \) and does not accommodate outliers. Compared with analysis in compressed sensing [9,13], this result is for Toeplitz matrix in system identification and applies to observation noises with non-Gaussian distributions. The results in this paper also improve on the performance bounds in [25], by showing that the identification error actually goes to 0.

**Proof** (of Theorem 3.1) \( \| y - H \hat{x} \|_1 \) can be written as \( \| H(x - \hat{x}) + e + w \|_1 \). We argue that for any constant \( t > 0 \), with high probability as \( n \to \infty \), for all \( \hat{x} \) such that \( \| x - \hat{x} \|_1 = t \), \( \| H(x - \hat{x}) + e + w \|_1 > \| e + w \|_1 \), contradicting the assumption that \( \hat{x} \) is the solution to (1.3).

To see this, we cover the sphere \( Z = \{ z : \| z \|_2 = 1 \} \) with a \( \gamma \)-net \( V \). We first argue that for every discrete point \( t v_j \) with \( v_j \) from the \( \gamma \)-net, \( \| H t v_j + e + w \|_1 > \| e + w \|_1 \); and then extend the result to the set \( tZ \).

Let us denote

\[
g(h,t) = \| H t v_j + e + w \|_1 - \| e + w \|_1
\]

where \( l_i = (e + w)_i \) for \( 1 \leq i \leq n \). We note that \( (H t v_j) \) is a Gaussian random variable \( N(0, \sigma^2) \). Let \( X \) be a Gaussian random variable \( N(0, \sigma^2) \), then for an arbitrary number \( l \),

\[
E \{ ||l + tX| - |l|| \} = \frac{2}{\sqrt{2\pi \sigma}} \int_0^\infty xe^{-\frac{(x+|l|)^2}{2\sigma^2}} dx
\]

\[
= \frac{2}{\pi \sigma} e^{-\frac{|l|^2}{2\sigma^2}} - 2l|1 - \Phi(\frac{|l|}{\sigma})|
\]

which is a decreasing nonnegative function in \( |l| \). From this, \( E\{g(h,t)\} = \sum_{i=1}^n \{ \sqrt{2}e^{-\frac{|l|^2}{2\sigma^2}} - 2|l|(1 - \Phi(\frac{|l|}{\sigma})) \} \).

When \( |l| \leq t \) and \( \sigma = 1 \),

\[
E \{ ||l + tX| - |l|| \} = \frac{\sqrt{2}e^{-\frac{|l|^2}{2}}}{\pi} - 2|l|(1 - \Phi(1))
\]

\[
\geq 0.1666t.
\]

It is not hard to verify that \( |g(a,t) - g(b,t)| \leq \sum_{i=1}^n t\sqrt{m}|a_i - b_i| \leq t\sqrt{mn}|a_i - b_i|_2 \) and \( g(h,t) \) has a Lipschitz constant (for \( h \)) no bigger than \( t\sqrt{mn} \). Then by the concentration of measure phenomenon for
Gaussian random variables (see [21,22]),

\[ P(g(h, t) \leq 0) = P\left(\frac{g(h, t) - E(g(h, t))}{t/\sqrt{nm}} \leq -\frac{E(g(h, t))}{t/\sqrt{nm}}\right) \]

\[ \leq 2e^{-\frac{c_5}{2}t/\sqrt{nm}} \]

If there exists a constant \( c(t) \) such that, as \( n \to \infty \), at least \( c(t)n \) elements have magnitudes smaller than \( t_0 \), then the numerator in \( tV \) and the corresponding probability \( P(g(h, t) \leq 0) \) behaves as \( 2e^{-\Theta(n)} \). This is because when \( |\mathbf{y}| \) is a sufficiently small constant

\[ x \in \mathbb{R}^n \]

By the same reasoning, \( g(h, t) \leq c t \) holds with probability no more than \( e^{-c_5n} \) for each discrete point from the \( \gamma \)-net \( V \), where \( c > 0 \) is a sufficiently small constant and \( c_5 > 0 \) is a constant which may only depend on \( c \). Since there are at most \((1 + 2\gamma)^n \) points from the \( \gamma \)-net, \( tV \), by a simple union bound, with probability \( 1 - e^{-c_5n} \) as \( n \to \infty \), \( g(h, t) > c t \) holds for all points from the \( \gamma \)-net \( tV \), where \( c_6 > 0 \) is a constant and \( c_5 \) can be taken as an arbitrarily small constant. Following similar \( \gamma \)-net proof techniques for Lemmas 2.3, 2.5 and 2.6, if we choose a sufficiently small constant \( \epsilon > 0 \) and accordingly a sufficiently small constant \( \gamma > 0 \), \( g(h, t) > 0.5c t \) holds simultaneously for every point in the set \( tV \) with high probability \( 1 - e^{-c_5n} \), where \( c_7 > 0 \) is a constant.

Notice if \( g(h, t) > 0 \) for \( t = t_1 \), then necessarily \( g(h, t) > 0 \) for \( t = t_2 > t_1 \). This is because \( g(h, t) \) is a convex function in \( t \geq 0 \) and \( g(h, 0) = 0 \). So if \( g(h, t) > 0.5c t \) holds with high probability for every point \( tV \), necessarily \( \|\mathbf{x} - \mathbf{x}'\|_2 < t \), because \( \mathbf{x} \) minimizes the objective in (1.3). Because we can pick arbitrarily small \( t \), \( \|\mathbf{x} - \mathbf{x}'\|_2 \to 0 \) with high probability as \( n \to \infty \). \( \square \)

4 Extensions to Non-Gaussian Inputs

In previous sections, we have considered Gaussian inputs for system identifications. In fact, our results also extend to non-Gaussian inputs. We illustrate this by considering a widely applied input for system identification, the pseudo-random binary sequence (PRBS) which is a random binary sequence taking values \( \pm 1 \) with the equal probability. To simply our analysis, we only consider strong performance bounds of uniformly correcting every possible set of outliers with cardinality smaller than \( k \). By noting that the PRBS input is actually an i.i.d. Bernoulli sequence, we first state the main result.

**Theorem 4.1** Let \( H \) be an \( n \times m \) Toeplitz matrix as in (1.1), where \( m \) is a fixed positive integer and \( h_i, -m < i < 0 \) are i.i.d. Bernoulli random variables taking values \(+1\) and \(-1\) with equal probability. Suppose that \( y = Hx + e \), where \( e \) is a sparse outlier vector with no more than \( k \) non-zero elements. Then there exists a constant \( c_1 > 0 \) and a constant \( \beta > 0 \) such that, with probability \( 1 - e^{-c_1n} \) as \( n \to \infty \), the \( n \times m \) Toeplitz matrix \( H \) has the following property: for every \( x \in \mathbb{R}^m \) and every error \( e \) with its support \( K \) satisfying \( |K| = k \leq \beta n \), \( x \) is the unique solution to (1.3).

In order to prove the results, again we only need to consider \( Hz \) for \( z \in \mathbb{R}^m \) with \( \|z\|_2 = 1 \). As in Section 2, we adopt the strategy of discretizing the subspace generated by \( H \). We pick a finite set \( V = \{v_1, \ldots, v_N\} \) called \( \gamma \)-net on \( \{z\|z\|_2 = 1\} \) for a constant \( \gamma > 0 \): in a \( \gamma \)-net, for every point \( z \) from \( \{z\|z\|_2 = 1\} \), there is a \( v_i \) such that \( \|z - v_i\|_2 \leq \gamma \). We subsequently establish the property in Theorem 2.2 for the points in the \( \gamma \)-net \( V \), before extending the results to every point \( Hz \) with \( \|z\|_2 = 1 \).

We first need a few lemmas and the concentration of measure phenomenon for \( Hz \), where \( z \in \mathbb{R}^m \) is a single vector with \( \|z\|_2 = 1 \). To this end, we need to use the McDiarmid’s inequality.

**Theorem 4.2** (McDiarmid’s inequality [7]) Let \( X_1, \ldots, X_n \) be independent random variables taking values in a set \( \chi \), and let \( F: \chi \times \cdots \times \chi \to \mathbb{R} \) be a function with the property that if one freezes all but the \( i \)-th coordinate of \( F(x_1, \ldots, x_n) \) for some \( 1 \leq i \leq n \), then \( F \) only fluctuates by most \( c_i > 0 \), thus,

\[ |F(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - F(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n)| \leq c_i \]

for all \( x_j \in \chi \) and \( x'_j \in \chi \), for \( 1 \leq i, j \leq n \). Then for any \( \lambda > 0 \), one has

\[ P(|F(X) - E[F(X)]| \geq \lambda) \leq 2e^{-\frac{\lambda^2}{2\sigma^2}} \]

where \( \sigma^2 := \sum_{i=1}^n c_i^2 \).

**Lemma 4.3** Let \( \|z\|_2 = 1 \). For any \( \epsilon > 0 \), with probability \( 1 - 2e^{-\frac{\epsilon^2}{2\sigma^2}} \), it holds that

\[ S - c \epsilon \leq \|Hz\|_1 \leq S + c \epsilon \]

where \( S = nE[|X|] \), and \( X = \sum_{i=1}^m z_i h_i \) with \( h_i \)'s being independent Bernoulli random variables.

**Proof** Define \( f(h_{-m+2}, \ldots, h_n) = \|Hz\|_1 \). We note that, by changing the value of only one Bernoulli variable \( h_i \), \( f(h_{-m+2}, \ldots, h_n) \) changes at most by \( 2\|z\|_1 \leq 2\sqrt{m} \).
The conclusion of this lemma then follows from applying the McDiarmid’s inequality. □

**Lemma 4.4** Let $\|z\|_2 = 1$, $\beta \in (0, 1)$ and $\epsilon$ be an arbitrarily small positive number. Let $K \subseteq \{1, 2, \ldots, n\}$ be an arbitrary subset with cardinality $|K| \leq \beta$. With a probability of at least $1 - 2e^{-\frac{\epsilon^2}{2m+2\epsilon^2}m}$,

$$\frac{|K|}{n} S - \epsilon n \leq \|(Hz)_K\|_1 \leq \frac{|K|}{n} S + \epsilon n,$$

where $S = nE(|X|)$, and $X = \sum_{i=1}^{m} z_i h_i$ with $h_i$’s being independent Bernoulli random variables.

**Proof** Define $f(h_{m+2}, \ldots, h_n) = \|(Hz)_K\|_1$. Then again, the conclusion follows from applying the McDiarmid’s inequality. □

We also have simple upper and lower bounds on $S$ in Lemma 4.3 and Lemma 4.4.

**Lemma 4.5** For any $z$ with $\|z\|_2 = 1$,

$$\frac{n}{2\sqrt{m}} \leq S \leq n\sqrt{m}$$

**Proof** Since $\|z\|_2 = 1$, there must exist one $1 \leq j \leq m$ such that $|z_j| \geq \frac{1}{\sqrt{m}}$. So no matter what $\sum_{i=1, i\neq j}^{m} z_i h_i$ is, there is always a probability of one half such that $\sum_{i=1, i\neq j}^{m} z_i h_i + z_j h_j \geq \frac{1}{\sqrt{m}}$. This leads to the lower bound.

The upper bound is from $E\{|X|\} \leq \sum_{i=1}^{m} |z_i| |h_i| \leq \sqrt{m}$. □

Combining Lemma 4.3, Lemma 4.4 and Lemma 4.5, by a simple union bound over the $\gamma$-net, we have:

**Lemma 4.6** Let $\beta \in (0, 1)$, and let $K \subseteq \{1, 2, \ldots, n\}$ be an arbitrary subset with cardinality $\frac{|K|}{n} \leq \beta$. With a probability of at least $1 - 4(1 + \frac{2}{\beta})^m e^{-\frac{\epsilon^2}{2m+2\epsilon^2}m}$, the following two statements hold true simultaneously for all the points $z$ of the $\gamma$-net $V$:

- $\frac{n}{2\sqrt{m}} - \epsilon n \leq \|Hz\|_1$;

\[\|Hz\|_1 \leq \beta n\sqrt{m} + \epsilon n.\]

By using the same “zooming” technique as in Section 2, and a simple union bound over $\binom{n}{|K|}$ possible subsets with cardinality $K$, we can get the following lemma which extends the result from the $\epsilon$-net to $\{z\|z\|_2 = 1\}$.

**Lemma 4.7** There exist a constant $c_3 > 0$ and a sufficiently small constant $\beta > 0$, such that when $n$ is large enough, with probability $1 - e^{-c_3n}$, the Bernoulli Toeplitz matrix $H$ has the following property: for every $z \in \mathbb{R}^m$ and every subset $K \subseteq \{1, \ldots, n\}$ with $|K| \leq \beta n$, $\sum_{i \in K} |(Hz)_i| - \sum_{i \in K} |(Hz)_i| \geq \delta' n$, where $\delta' > 0$ is a sufficiently constant.

**Proof** of Theorem 4.1. A direct consequence of Theorem 4.2 and lemmas 4.3-4.7. □

### 5 Numerical Evaluations

Based on Theorem 2.1, we calculate the strong thresholds in Figure 1 for different values of $m$ by optimizing over $\mu > 0$ and $\delta$. As $m$ increases, the correlation length in the matrix $H$ also increases and the corresponding correctable number of errors decreases (but the bound $\beta = k/n > 0$ always exists).

We then evaluate in Figure 2 the $\ell_2$-norm error $\|\hat{x} - x\|_2$ of $\ell_1$ minimization for Gaussian Toeplitz matrices under both outliers and i.i.d. observation noises of different probability distributions: Gamma distribution with shape parameter $k = 2$ and scale $\frac{1}{\sqrt{a}}$; standard Gaussian distribution $N(0, 1)$; and exponential distribution with...
mean $\frac{\sqrt{2}}{2}$. These distributions are chosen such that the observation noises have the same expected energy. The system parameter $m$ is set to 5 and the system state $x$ are generated as i.i.d. standard Gaussian random variables. We randomly pick $\frac{n}{2}$ i.i.d. $N(0, 100)$ Gaussian outliers with random support for the error vector $e$. For all these distributions, the average error goes to 0 (we also verified points beyond $n > 1000$). What is interesting is that the error goes to 0 at different rates. Actually, as implied by the proof of Theorem 3.1, for random noises with bigger probability density functions around 0, the system identification error may be smaller. In fact, the Gamma distribution has the worst performance because its probability density function is smaller around the origin (actually 0 at the origin), while the exponential distribution has the best performance, because it has the largest probability density function around 0.

To illustrate the asymptotic convergence and limited data length performance, we give two simulation examples. The first example is a five dimensional FIR system. The input $h_i$’s are an i.i.d. Gaussian sequence of zero mean and variance $2^2$ and the random noises are i.i.d. Gaussian of zero mean and variance $0.2^2$. The system parameter vector $x$ is randomly generated according to Gaussian of zero mean and unit variance. The number of non-zero elements in the outlier vector is $\text{rand} \times 0.2 \times n$ where $\text{rand}$ is the random variable uniformly in $[0,1]$ which loosely translates into that 10% of the data are outliers on average. The non-zero elements of the outliers are randomly generated according to Gaussian of mean=100 and variance $50^2$. The locations of the non-zeros elements are randomly assigned in each simulation. Figure 3 shows the simulation results of the estimation error for different $n$ which are the average of 10 Monte Carlo runs respectively. An example of outliers is shown.
in Figure 4. Since outliers are heavily biased with very large magnitudes if non-zero, the signal to noise ratio is very small $SNR = -30 dB$. Clearly, even at this very low level of SNR, the identification results are still very good as predicted by the analysis. The second example is a line $y = \left( z, 1 \right) \begin{bmatrix} k \\ b \end{bmatrix} + noise$ where the true but unknown $x = \left( k, b \right)^T = (0.2, 0.2)^T$. Because of noise that is an i.i.d. Gaussian of zero mean and variance $0.2^2$, the actual data set, shown in Figure 5 as circles, is given by Table 1. First the least squares estimate $(0.1958, 0.2286)^T$ is calculated which is close to the true but unknown $x = (0.2, 0.2)^T$. Now, a single outlier at $z = 10$ is added as a square in Figure 5. For this single outlier, the least squares estimate becomes $(0.6503, -1.1351)^T$ in dash-dot line, way off the true but unknown $x$. Finally, the $\ell_1$ optimization as discussed in the paper is applied to have an estimate $(0.2109, 0.1955)^T$ in bold-dash line in Figure 5. Virtually, the $\ell_1$ optimization eliminates the effect of the outlier even for a very limited data case.

6 Concluding Remarks

We have re-visited the least absolute deviation estimator or $\ell_1$ minimization from a compressed sensing point of view and shown that the exact system parameter vector can be recovered as long as the outlier is sparse in the sense that the number of non-zero elements has to be bound by $\beta n$ for some $\beta < 1$. No any probabilistic assumption is imposed on the unknown outlier. Further, in the presence of both the outliers and random noises, the system parameter vector can still be recovered if some mild conditions on the random noises are met.

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