SOME REMARKS ON POHOZAEV-TYPE IDENTITIES

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Abstract. The aim of this note is to discuss in more detail the Pohozaev-type identities that have been recently established by the author, P. Laurain and T. Rivière in \cite{3} in the framework of half-harmonic maps defined either on $\mathbb{R}$ or on the sphere $S^1$ with values into a closed manifold $N^n \subset \mathbb{R}^m$. Weak half-harmonic maps are critical points of the following nonlocal energy

\begin{equation}
L_{\mathbb{R}}^{1/2}(u) := \int_{\mathbb{R}} |(-\Delta)^{1/4} u|^2 \, dx \quad \text{or} \quad L_{S^1}^{1/2}(u) := \int_{S^1} |(-\Delta)^{1/4} u|^2 \, d\theta.
\end{equation}

If $u$ is a sufficiently smooth critical point of (1) then it satisfies the following equation of stationarity

\begin{equation}
\frac{du}{dx} \cdot (-\Delta)^{1/2} u = 0 \quad \text{a.e in } \mathbb{R} \quad \text{or} \quad \frac{\partial u}{\partial \theta} \cdot (-\Delta)^{1/2} u = 0 \quad \text{a.e in } S^1.
\end{equation}

As it was announced in \cite{3}, by using the invariance of (2) in $S^1$ with respect to the trace of the Möbius transformations of the 2 dimensional disk we derive a countable family of relations involving the Fourier coefficients of weak half-harmonic maps $u : S^1 \rightarrow N^n$. In the same spirit we also provide as many Pohozaev-type identities in 2-D for stationary harmonic maps as conformal vector fields in $\mathbb{R}^2$ generated by holomorphic functions.

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1. Introduction

The notion of weak 1/2-harmonic maps into a $n$-dimensional closed (compact without boundary) manifold $N^n \subset \mathbb{R}^m$ has been introduced by Tristan Rivière and the author in \cite{7,8}. Since then the theory of fractional harmonic maps have received a lot of attention in view of their application to important geometrical problems (see e.g \cite{2,6} for an overview of the theory).
These maps are critical points of the fractional energy on $\mathbb{R}^k$

\[
\mathcal{L}^{1/2}(u) := \int_{\mathbb{R}^k} |(-\Delta)^{1/4}u|^2 \, dx^k
\]

within

\[
\dot{H}^{1/2}(\mathbb{R}^k, \mathcal{N}^n) := \left\{ u \in \dot{H}^{1/2}(\mathbb{R}^k, \mathbb{R}^m) ; u(x) \in \mathcal{N}^n \text{ for a.e. } x \in \mathbb{R}^k \right\}.
\]

Precisely they satisfy

\[
\frac{d}{dt} \mathcal{L}^{1/2}(\Pi(u + t\varphi)) \bigg|_{t=0} = 0,
\]

where $\varphi \in C^1_c(\mathbb{R}^k, \mathbb{R}^m)$, $\Pi : \mathcal{U} \to \mathcal{N}^n$ is any fixed $C^2$ projection, defined on some tubular neighborhood $\mathcal{U}$ of $\mathcal{N}^n$.

The homogeneous fractional Sobolev space $\dot{H}^{1/2}(\mathbb{R}^k, \mathbb{R}^m)$ can be defined as follows

\[
\dot{H}^{1/2}(\mathbb{R}^k) := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^k) ; \|u\|_{\dot{H}^{1/2}(\mathbb{R}^k)}^2 := \int_{\mathbb{R}^{2k}} \frac{|u(x) - u(y)|^2}{|x - y|^2} \, dx^k \, dy^k < \infty \right\}.
\]

The fractional Laplacian $(-\Delta)^{1/4}u$ can be defined by

\[
(-\Delta)^{1/4}u := \lim_{\epsilon \to 0} \mathcal{F}^{-1}
\left[
(\epsilon^2 + 4\pi^2|\xi|^2)^{1/4} \mathcal{F}u\right],
\]

provided that the limit exists in $\mathcal{S}'(\mathbb{R})$\(^{(1)}\).

We observe that if $u \in \dot{H}^{1/2}(\mathbb{R}^k, \mathbb{R}^m)$, then $(-\Delta)^{1/4}u$ exists, lies in $L^2(\mathbb{R}^k)$ and is given by

\[
(-\Delta)^{1/4}u = \mathcal{F}^{-1}
\left[
(2\pi|\xi|)^{1/2} \hat{u}\right],
\]

(see for instance Lemma B.5 in [5] and the references therein).

Weak 1/2-harmonic maps satisfy the Euler-Lagrange equation

\[
\nu(u) \wedge (-\Delta)^{1/2}u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k),
\]

\(^{(1)}\) We denote respectively by $\mathcal{S}(\mathbb{R})$ the spaces of (real or complex) Schwartz functions. Given a function $\varphi \in \mathcal{S}(\mathbb{R})$, we denote either by $\hat{\varphi}$ or by $\mathcal{F}\varphi$ the Fourier transform of $\varphi$, i.e.

\[
\hat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = \int_{\mathbb{R}} \varphi(x)e^{-2\pi i \xi x} \, dx.
\]
where $\nu(z)$ is the Gauss Map at $z \in \mathcal{N}^n$ taking values into the grassmannian $\tilde{\text{Gr}}_{m-n}(\mathbb{R}^m)$ of oriented $m-n$ planes in $\mathbb{R}^m$ which is given by the oriented normal $m-n$-plane to $T_z\mathcal{N}^n$.

One of the main results in [8] is the following Theorem

**Theorem 1.1.** Let $\mathcal{N}^n$ be a $C^2$ closed submanifold of $\mathbb{R}^m$ and let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N}^n)$ be a weak $1/2-$harmonic map into $\mathcal{N}^n$. Then $u \in \bigcap_{0<\delta<1} C^{0,\delta}_{loc}(\mathbb{R}, \mathcal{N}^n)$.

Finally a bootstrap argument leads to the following result (see [5] for the details of this argument).

**Theorem 1.2.** Let $\mathcal{N}^n \subset \mathbb{R}^m$ be a $C^k$ closed submanifold of $\mathbb{R}^m$, with $k \geq 2$, and let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N}^n)$ be a weak $1/2$-harmonic. Then

$$u \in \bigcap_{0<\delta<1} C^{k-1,\delta}_{loc}(\mathbb{R}, \mathcal{N}^n).$$

In particular, if $\mathcal{N}^n$ is $C^\infty$ then $u \in C^\infty(\mathbb{R}, \mathcal{N}^n)$.

We remark that if $P_{-i} : \mathbb{S}^1 \setminus \{-i\} \to \mathbb{R}$, $P_{-i}(\cos(\theta) + i \sin(\theta)) = \frac{\cos(\theta)}{1 + \sin(\theta)}$ is the classical stereographic projection whose inverse is given by

$$(6) \quad P_{-i}^{-1}(x) = \frac{2x}{1 + x^2} + i \left(-1 + \frac{2}{1 + x^2}\right),$$

then the following relation between the $1/2$-Laplacian in $\mathbb{R}$ and in $\mathbb{S}^1$ holds:

**Proposition 1.1.** Given $u : \mathbb{R} \to \mathbb{R}^m$, we set $v := u \circ P_{-i} : \mathbb{S}^1 \to \mathbb{R}^m$. Then $u \in L^\frac{1}{2}(\mathbb{R})$ (2) if and only if $v \in L^1(\mathbb{S}^1)$. In this case

$$(7) \quad (-\Delta)^{\frac{1}{2}} v(e^{i\theta}) = \left((-\Delta)^{\frac{1}{2}} u(P_{-i}(e^{i\theta}))\right) \frac{1 + \sin \theta}{\cos(\theta)} \text{ in } \mathcal{D}'(\mathbb{S}^1 \setminus \{-i\}).$$

Observe that $(1 + \sin(\theta))^{-1} = |P'_{-i}(\theta)|$, hence we have

$$\int_0^{2\pi} (-\Delta)^{\frac{1}{2}} v(e^{i\theta}) \varphi(e^{i\theta}) \, d\theta = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u(x) \varphi(P_{-i}^{-1}(x)) \, dx \quad \text{for every } \varphi \in C^\infty_0(\mathbb{S}^1 \setminus \{-i\}).$$

(2)We recall that $L^\frac{1}{2}(\mathbb{R}) := \left\{ u \in L^1_{loc}(\mathbb{R}) : \int_{\mathbb{R}} \frac{|u(x)|^2}{1 + x^2} \, dx < \infty \right\}$
For the proof of Proposition 1.1 we refer for instance to [2] or [4].

From Proposition 1.1 it follows that \( u \in \dot{H}^{1/2}(\mathbb{R}) \) is a 1/2-harmonic map in \( \mathbb{R} \) if and only if \( v := u \circ \mathcal{P} \in \dot{H}^{1/2}(S^1) \) is a 1/2 harmonic map in \( S^1 \).

In the study of quantization properties of half-harmonic maps we established in [3] new Pohozaev-type identities for the half Laplacian in one dimension, that we are going to present below.

We first consider the fundamental solution \( G \) of the fractional heat equation:

\[
\begin{aligned}
\partial_t G + (-\Delta)^{1/2} G &= 0 \quad x \in \mathbb{R}, \ t > 0 \\
G(0, x) &= \delta_0 \quad t = 0.
\end{aligned}
\]

It is given by

\[
G(t, x) = \frac{1}{\pi} \frac{t}{x^2 + t^2}.
\]

The following equalities hold

\[
\partial_t G = \frac{1}{\pi} \frac{x^2 - t^2}{(t^2 + x^2)^2}, \quad \partial_x G = -\frac{1}{\pi} \frac{2xt}{(t^2 + x^2)^2}.
\]

**Theorem 1.3. [Pohozaev Identity in \( \mathbb{R} \)]** Let \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \) be such that

\[
\frac{du}{dx} \cdot (-\Delta)^{1/2} u = 0 \quad \text{a.e in } \mathbb{R}.
\]

Assume that for some \( u_0 \in \mathbb{R} \)

\[
\int_{\mathbb{R}} |u - u_0| \ dx < +\infty, \quad \int_{\mathbb{R}} \left| \frac{du}{dx}(x) \right| \ dx < +\infty.
\]

Then the following identity holds

\[
\left| \int_{\mathbb{R}} \partial_t G(t, x)(u(x) - u_0) \ dx \right|^2 = \left| \int_{\mathbb{R}} \partial_x G(t, x)(u(x) - u_0) \ dx \right|^2 \quad \text{for all } t \in \mathbb{R}. \quad \square
\]

We get an analogous formula in \( S^1 \). By identifying \( S^1 \) with \([\pi, \pi] \) we consider the following problem

\[
\begin{aligned}
\partial_t F + (-\Delta)^{1/2} F &= 0 \quad \theta \in [-\pi, \pi), \ t > 0 \\
F(0, \theta) &= \delta_0(x) \quad \theta \in [-\pi, \pi].
\end{aligned}
\]
The solution of (12) is given by

$$F(\theta, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{-|n|} e^{in\theta} = \frac{e^{2t} - 1}{e^{2t} - 2e^t \cos(\theta) + 1}.$$

In this case we have

$$\partial_t F(t, \theta) = -2e^t \frac{e^{2t} \cos(\theta) - 2e^t + \cos(\theta)}{(e^{2t} - 2e^t \cos(\theta) + 1)^2}$$

and

$$\partial_\theta F(t, \theta) = -2e^t \frac{\sin(\theta)(e^{2t} - 1)}{(e^{2t} - 2e^t \cos(\theta) + 1)^2}.$$

Then the following holds

**Theorem 1.4. [Pohozaev Identity on $S^1$]** Let $u \in \mathcal{W}^{1,2}(S^1, \mathbb{R}^m)$ be such that

$$\frac{\partial}{\partial \theta} \cdot ((-\Delta)^{1/2} u = 0 \quad \text{a.e in } S^1. \quad (13)$$

Then the following identity holds

$$\left| \int_{S^1} u(z) \partial_t F(z) \, d\theta \right|^2 = \left| \int_{S^1} u(z) \partial_\theta F(z) \, d\theta \right|^2. \quad (14)$$

From (14) one deduces in particular that

$$\left| \int_0^{2\pi} u(e^{i\theta}) \cos(\theta) \, d\theta \right|^2 = \left| \int_0^{2\pi} u(e^{i\theta}) \sin(\theta) \, d\theta \right|^2. \quad \square \quad (15)$$

For the proof of Theorem 1.3 and Theorem 1.4 we refer the reader to [3].

We could have solved (8) by requiring $G(0, x) = \delta_{x_0}$, with $x_0 \in \mathbb{R}$ and we would have obtained as many corresponding Pohozaev-type formulas.

We observe that if $u$ is a smooth critical point of (3) in $\mathbb{R}$ then it is stationary as well, namely it is critical with respect to the variation of the domain:

$$\left( \frac{d}{da} \int_\mathbb{R} |(-\Delta)^{1/4} u(x + aX(x))|^2 \, dx \right)_{a=0} = 0$$

where $X: \mathbb{R}^2 \to \mathbb{R}^2$ is a $C^1_c(R^2)$ vector field.

Actually any variation the form $u(x + aX(x)) = u(x) + a \frac{du(x)}{dx} X(x) + o(a^2)$ can be interpreted as being a variation in the target with $\varphi(x) = \frac{du(x)}{dx} X(x)$. 


From (16) we get the so-called equation of stationarity:

\[ 0 = \int_{\mathbb{R}} [(-\Delta)^{1/2}(u(x+aX(x))) \cdot \frac{d}{da} (u(x+aX(x)))] \bigg|_{a=0} \, dx = \int_{\mathbb{R}} (-\Delta)^{1/2} (u(x)) \cdot \frac{du(x)}{dx} X(x) \, dx. \]

By the arbitrariness of \( X \) and the smoothness of \( u \) from (17) we deduce that

\[ (-\Delta)^{1/2} u(x) \cdot \frac{du}{dx}(x) = 0 \quad x \in \mathbb{R}. \]

In an analogous way if \( u \) is a smooth critical point of the fractional energy (3) in \( S^1 \), it also satisfies

\[ \left( \frac{d}{da} \int_{S^1} |(-\Delta)^{1/4}(u(z+aX(z)))|^2 \, d\sigma(z) \right) \bigg|_{a=0} = 0 \]

where \( X: S^1 \to \mathbb{R}^2 \) is \( C^1(S^1) \) vector field.

\[ (-\Delta)^{1/2} (u(z)) \cdot \partial_z u(z) = 0 \quad z \in S^1. \]

Therefore the assumptions of Theorem 1.3 and Theorem 1.4 are satisfied by sufficiently smooth 1/2-harmonic maps.

We have now to give some explanations why these identities belong to the Pohozaev identities family. These identities are produced by the conformal invariance of the highest order derivative term in the Lagrangian from which the Euler Lagrange is issued. For instance the Dirichlet energy

\[ \mathcal{L}(u) = \int_{\mathbb{R}^2} |\nabla u|^2 \, dx^2 \]

is conformal invariant in 2-D, whereas the following fractional energy

\[ \mathcal{L}^{1/2}_{\mathbb{R}}(u) = \int_{\mathbb{R}} |(-\Delta)^{1/4} u|^2 \, dx \]

is conformal invariant in 1-D. The infinitesimal perturbations issued from the dilations produce in (20) and (21) respectively the following infinitesimal variations of these highest order terms

\[ \sum_{i=1}^2 x_i \frac{\partial u}{\partial x_i} \cdot \Delta u \quad \text{in 2-D and} \quad x \frac{du}{dx} \cdot (-\Delta)^{1/2} u \quad \text{in 1-D} \]

Such kind of perturbations play an important role in establishing Pohozaev-type identities.
In two dimensions, integrating the identity (17) on a ball $B(x_0, r)$ ($x_0 \in \mathbb{R}^2, r > 0$) gives the following balancing law between the radial part and the angular part of the energy classically known as Pohozaev identity.

**Theorem 1.5.** Let $u \in W^{2,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^m)$ such that

(22) \[ \frac{\partial u}{\partial x_i} \cdot \Delta u = 0 \quad \text{a.e in } B(0, 1) \]

for $i = 1, 2$. Then it holds

(23) \[ \int_{\partial B(x_0, r)} \left| \frac{1}{r} \frac{\partial u}{\partial \theta} \right|^2 \, d\theta = \int_{\partial B(x_0, r)} \left| \frac{\partial u}{\partial r} \right|^2 \, d\theta \]

for all $r \in [0, 1]$.

In 1 dimension one might wonder what corresponds to the 2 dimensional dichotomy between radial and angular parts. Figure 1 is intended to illustrate the following correspondence of dichotomies respectively in 1 and 2 dimensions.
In this note we make the observation that by exploiting the invariance of the equation (13) with respect to the trace of Möbius transformations of the disk in \( \mathbb{R}^2 \) of the form 
\[ M_{\alpha,a}(z) := e^{i\alpha} \frac{z-a}{1-az}, \quad \alpha \in \mathbb{R}, a \in (-1,1) \]
we can derive from (15) a countable family of relations involving the Fourier coefficients of solutions of (13). This fact has been already announced in the paper [3]. We heard that the proof of this property has been recently obtained also in the work of preparation [1] by using a different approach.

Given \( u: S^1 \rightarrow \mathbb{R}^m \) we define its Fourier coefficients for every \( k \geq 0 \):
\[
\begin{align*}
    a_k &= \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \cos k\theta \, d\theta \\
    b_k &= \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \sin k\theta \, d\theta.
\end{align*}
\]

The following result holds.

**Proposition 1.2. [Relations of the Fourier coefficients on \( S^1 \)]** Let \( u \in W^{1,2}(S^1, \mathbb{R}^m) \) satisfy (13). Then for every \( n \geq 2 \) it holds
\[
\sum_{k=1}^{n-1} (n-k)k(a_ka_{n-k} - b_kb_{n-k}) = 0
\]
and
\[
\sum_{k=1}^{n-1} (n-k)k(a_kb_{n-k} + b_ka_{n-k}) = 0. \quad \square
\]

\((3)\) We recall that since \( M_{\alpha,a}(z) \) is conformal with \( M'_{\alpha,a}(z) \neq 0 \) we have
\[
(-\Delta)^{1/2}(u \circ M_{\alpha,a}(z)) = (-\Delta)^{1/2}u_a = e^{\lambda_{\alpha,a}}((-\Delta)^{1/2}u) \circ M_{\alpha,a}(z),
\]
where \( \lambda_{\alpha,a}(z) = \log(|\frac{\partial M_{\alpha,a}}{\partial \theta}(z)|), \quad z \in S^1 \)
We conclude this introduction by mentioning that in the paper [9] the authors obtain a different Pohozaev identity for bounded weak solutions to the following problem

\[
\begin{cases}
(-\Delta)^s u = f(u) & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega
\end{cases}
\]

where \( s \in (0, 1) \) and \( \Omega \subset \mathbb{R}^n \) is a bounded domain. As a consequence of their Pohozaev identity they get nonexistence results for problem (27) with supercritical nonlinearity in star-shaped domains.

This paper is organized as follows. In section 2 we prove Proposition 1.2 and in section 3 we provide infinite many Pohozaev formulas for stationary harmonic maps in 2-D in correspondence to conformal vector fields in \( \mathbb{C} \) generated by the holomorphic functions. In particular we will prove a generalization of Theorem 1.5. This can be useful for different purposes.

2. Proof of Proposition 1.2

From Theorem 1.4 it follows that \( u \) satisfies in particular

\[
\left| \int_0^{2\pi} u(e^{i\theta}) \cos(\theta) \, d\theta \right|^2 = \left| \int_0^{2\pi} u(e^{i\theta}) \sin(\theta) \, d\theta \right|^2
\]

We can rewrite (28) as follows

\[
\left| \int_0^{2\pi} u(e^{i\theta}) \Re(d e^{i\theta}) \right|^2 = \left| \int_0^{2\pi} u(e^{i\theta}) \Im(d e^{i\theta}) \right|^2.
\]

Given \( a \in \mathbb{R} \) with \( |a| < 1 \) and \( \alpha \in \mathbb{R} \) we consider the Möbius map \( M_{\alpha,a}(z) := e^{i\alpha} \frac{z-a}{1-az} \) and we define

\[
u_{a,\alpha}(e^{i\theta}) := u \circ M_{\alpha,a}(z).
\]

Since the condition (13) is invariant with respect to Möbius transformations for every \( \alpha \in \mathbb{R} \) and for every \( a \in (-1, 1) \) we get

\[
\left| \int_0^{2\pi} u \left( e^{i\alpha} \frac{z-a}{1-az} \right) \Re(d e^{i\theta}) \right|^2 = \left| \int_0^{2\pi} u \left( e^{i\alpha} \frac{z-a}{1-az} \right) \Im(d e^{i\theta}) \right|^2.
\]

or equivalently

\[
\left| \Re \left( \int_0^{2\pi} u \left( e^{i\alpha} \frac{e^{i\theta} - a}{1 - ae^{i\theta}} \right) \, de^{i\theta} \right) \right|^2 = \left| \Im \left( \int_0^{2\pi} u \left( e^{i\alpha} \frac{e^{i\theta} - a}{1 - ae^{i\theta}} \right) \, de^{i\theta} \right) \right|^2.
\]
We set
\[ e^{i\varphi} := e^{i\alpha} \frac{e^{i\theta} - a}{1 - ae^{i\theta}}, \]
which implies that
\[ e^{i\theta} = \frac{e^{i(\varphi - \alpha)} + a}{1 + ae^{i(\varphi - \alpha)}} \]
\[ d(e^{i\theta}) = \frac{1 - a^2}{(1 + ae^{i(\varphi - \alpha)})^2} d(e^{i(\varphi - \alpha)}) \]

By plugging (32) and (33) into (31) and dividing by \((1 - a^2)\) we get
\[ |\Re\left( \int_0^{2\pi} u(e^{i\varphi}) \frac{e^{-i\alpha}}{(1 + ae^{i(\varphi - \alpha)})^2} d(e^{i\varphi}) \right)|^2 = |\Im\left( \int_0^{2\pi} u(e^{i\varphi}) \frac{e^{-i\alpha}}{(1 + ae^{i(\varphi - \alpha)})^2} d(e^{i\varphi}) \right)|^2. \]

Observe that for all \(|z| < 1\) we have
\[ \frac{z}{(1 + z)^2} = \sum_{n=1}^{\infty} n(-1)^{n-1}z^n \]

In particular
\[ \frac{e^{i(\varphi - \alpha)}}{(1 + ae^{i(\varphi - \alpha)})^2} = \sum_{n=1}^{\infty} n(-1)^{n-1}a^{n-1}e^{in(\varphi - \alpha)} \]
and
\[ \Re\left( \frac{e^{i(\varphi - \alpha)}}{(1 + ae^{i(\varphi - \alpha)})^2} \right) = \sum_{n=1}^{\infty} n(-1)^{n-1}a^{n-1}\cos(n(\varphi - \alpha)) \]
\[ \Im\left( \frac{e^{i(\varphi - \alpha)}}{(1 + ae^{i(\varphi - \alpha)})^2} \right) = \sum_{n=1}^{\infty} n(-1)^{n-1}a^{n-1}\sin(n(\varphi - \alpha)) \]

We can write
\[ |\Re\left( \int_0^{2\pi} u(e^{i\varphi}) \frac{e^{i(\varphi - \alpha)}}{(1 + ae^{i(\varphi - \alpha)})^2} d(\varphi) \right)|^2 \]
\[ = \sum_{n=1}^{\infty} (-1)^{n-1}a^{n-1} \sum_{k=1}^{n-1} (n-k)k \left( \int_0^{2\pi} u(e^{i\varphi}) \cos(k(\varphi - \alpha))d\varphi \right) \left( \int_0^{2\pi} u(e^{i\varphi}) \cos((n-k)(\varphi - \alpha))d\varphi \right) \]
and

\[ \left| 3 \left( \int_0^{2\pi} u(e^{i\varphi}) \frac{e^{i(\varphi - \alpha)}}{(1 + ae^{i(\varphi - \alpha)})^2} d\varphi \right) \right|^2 = \sum_{n=1}^{\infty} (-1)^n a^{n-1} \sum_{k=1}^{n-1} (n-k)k \left( \int_0^{2\pi} u(e^{i\varphi}) \sin(k(\varphi - \alpha)) d\varphi \right) \left( \int_0^{2\pi} u(e^{i\varphi}) \sin((n-k)(\varphi - \alpha)) d\varphi \right) \]

The identity (34) and the relations (38), (39) imply that for every \( n \geq 2 \) we obtain the following identities

\[ \sum_{k=1}^{n-1} (n-k)k \left( \int_0^{2\pi} u(e^{i\varphi}) \cos(k(\varphi - \alpha)) d\varphi \right) \left( \int_0^{2\pi} u(e^{i\varphi}) \cos((n-k)(\varphi - \alpha)) d\varphi \right) = \sum_{k=1}^{n-1} (n-k)k \left( \int_0^{2\pi} u(e^{i\varphi}) \sin(k(\varphi - \alpha)) d\varphi \right) \left( \int_0^{2\pi} u(e^{i\varphi}) \sin((n-k)(\varphi - \alpha)) d\varphi \right) . \]

From (40) we can deduce a countable family of relations between the Fourier coefficients of the map \( u \). Precisely if we set for every \( n \geq 1 \)

\[ \begin{cases} a_n := \frac{1}{2\pi} \int_0^{2\pi} u(e^\theta) \cos n\theta \, d\theta \\ b_n = \frac{1}{2\pi} \int_0^{2\pi} u(e^\theta) \sin n\theta \, d\theta, \end{cases} \]

we get

\[ \sum_{k=1}^{n-1} (n-k)k \left[ (\cos(k\alpha)a_k + \sin(k\alpha)b_k) (\cos((n-k)\alpha)a_{n-k} + \sin((n-k)\alpha)b_{n-k}) \right. \]

\[ - (\cos(k\alpha)b_k - \sin(k\alpha)a_k) (\cos((n-k)\alpha)b_{n-k} - \sin((n-k)\alpha)a_{n-k}) \right] = 0 \]

The identity (41) can be rewritten as follows

\[ \cos(n\alpha) \left( \sum_{k=1}^{n-1} (n-k)k(a_k a_{n-k} - b_k b_{n-k})) + \sin(n\alpha) \left( \sum_{k=1}^{n-1} (n-k)k(a_k b_{n-k} + b_k a_{n-k}) \right) \right) = 0. \]

The relation (42) yields (25) and (26) because of the linear dependence of \( \cos(n\alpha) \) and \( \sin(n\alpha) \).
We observe that for $n = 2$ we obtain:

\begin{equation}
(|a_1|^2 - |b_1|^2) \cos(2\alpha) - 2a_1 \cdot b_1 \sin(2\alpha) = 0. \tag{43}
\end{equation}

Since $\alpha \in \mathbb{R}$ is arbitrary we get

\[
\begin{cases}
|a_1| = |b_1| \\
a_1 \cdot b_1 = 0
\end{cases}
\]

If $n = 3$ we get

\begin{equation}
4(a_1 \cdot a_2 - b_1 \cdot b_2) \cos(3\alpha) - 4(a_1 \cdot b_2 + b_1 \cdot a_2) \sin(3\alpha) = 0. \tag{44}
\end{equation}

The relation (44) gives

\[
\begin{cases}
a_1 \cdot a_2 = b_1 \cdot b_2 \\
a_1 \cdot b_2 = -a_2 \cdot b_1.
\end{cases}
\]

If $n = 4$ we get

\[
\begin{cases}
|a_2|^2 - |b_2|^2 = \frac{3}{2}(b_1 \cdot b_3 - a_1 \cdot a_3) \\
a_2 \cdot b_2 = -\frac{3}{4}(a_1 \cdot b_3 + b_1 \cdot a_3).
\end{cases}
\]

We can conclude the proof. □

3. POHOZAEV IDENTITIES FOR THE LAPLACIAN IN $\mathbb{R}^2$

In this section we derive Pohozaev identities in 2-D in the same spirit of the previous section. In the following Theorem 3.1 we combine ideas from [10] and [11]. Precisely we multiply the equation satisfied for instance by sufficiently smooth harmonic maps by the fundamental solution of the heat equation and a holomorphic vector field $X : \mathbb{C} \to \mathbb{C}$.

We mention that the use of the fundamental solution to get Pohozaev-type identities and monotonicity formulas has been performed in [11] to study the heat flow. In Chapter 9 of [10] the authors derived in the context of Ginzburg-Landau equation generalized Pohozaev identities for the so-called $\rho$- conformal vector fields $X = (X^1, \ldots, X^n)$, where $\rho$ is a given function defined in a 2 dimensional domain. In the case $\rho \equiv 1$ then the $\rho$- conformal vector fields are exactly conformal vector fields generated by the holomorphic functions.
We recall that the fundamental solution of the heat equation

\[
\begin{aligned}
\partial_t G + (-\Delta) G &= 0 \quad t > 0 \\
G(0, x) &= \delta_{x_0} \quad t = 0.
\end{aligned}
\]  

is given by \(G(x, t) = (4\pi t)^{-1/2} e^{-\frac{|x-x_0|^2}{4t}}\).

**Theorem 3.1.** [Pohozev in \(\mathbb{R}^2\)] Let \(u \in W^{2,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^m)\) such that

\[
\frac{\partial u}{\partial x_i} \cdot \Delta u = 0 \quad \text{a.e in } \mathbb{R}^2
\]

for \(i = 1, 2\). Assume that

\[
\int_{\mathbb{R}^2} |\nabla u(x)|^2 dx < +\infty.
\]

Then for all \(x_0 \in \mathbb{R}^2\), \(t > 0\) and every \(X = X_1 + iX_2 : \mathbb{C} \to \mathbb{C}\) holomorphic function the following identity holds

\[
2 \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} |x - x_0| \left( \frac{\partial u}{\partial \nu} \cdot \frac{\partial u}{\partial X} \right) dx = \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} ((x - x_0) \cdot X) |\nabla u|^2 dx.
\]

In the particular case where \(X = x - x_0\) with \(x_0 \in \mathbb{R}^2\) then for all \(t > 0\) the following identity holds

\[
\iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} |x - x_0|^2 \left| \frac{\partial u}{\partial \nu} \right|^2 dx^2 = \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \left| \frac{\partial u}{\partial \theta} \right|^2 dx^2.
\]
Proof. We multiply the equation (46) by $X_i e^{-\frac{|x-x_0|^2}{4t}}$ and we integrate

\begin{align*}
0 &= \sum_{k,i=1}^{2} \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} X_i \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_k^2} dx \\
&= \sum_{k,i=1}^{2} \frac{1}{2t} \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} (x_k - x_{k,0}) X_i \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} dx \\
&- \sum_{k,i=1}^{2} \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \frac{\partial X_i}{\partial x_k} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} dx \\
&- \sum_{k,i=1}^{2} \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} X_i \frac{\partial^2 u}{\partial x_k \partial x_i} \frac{\partial u}{\partial x_k} dx \\
&= \frac{1}{2t} \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} (X \cdot \nabla u) \frac{\partial u}{\partial v} |x - x_0| dx \\
&- \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \left[ \frac{\partial X_1}{\partial x_1} \frac{\partial X_1}{\partial x_2} \frac{\partial \nabla u}{\partial x_1} \left( \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial X_2}{\partial x_1} \right) (\frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2}) \right] dx \\
&- \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} X_1 \frac{\partial}{\partial x_i} |\nabla u|^2 dx \\
&= \frac{1}{2t} \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} (X \cdot \nabla u) \frac{\partial u}{\partial v} |x - x_0| dx \\
&- \frac{1}{4t} \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} (x - x_0) \cdot X |\nabla u|^2 dx \\
&- \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \frac{\partial X_1}{\partial x_1} |\nabla u|^2 \\
&+ \frac{1}{2} \sum_{i=1}^{2} \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \frac{\partial X_i}{\partial x_i} |\nabla u|^2 dx \\
&= \frac{1}{2} \sum_{i=1}^{2} \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \frac{\partial X_i}{\partial x_i} |\nabla u|^2 dx
\end{align*}

By using the fact that

\begin{align*}
\frac{1}{2} \sum_{i=1}^{2} \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \frac{\partial X_i}{\partial x_i} |\nabla u|^2 dx &= \int \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \frac{\partial X_1}{\partial x_1} |\nabla u|^2 \\
\end{align*}

(4) We use the Einstein summation convention
we obtain from (50) that
\[
\begin{align*}
2 \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} (X \cdot \nabla u) \frac{\partial u}{\partial \nu} |x-x_0| dx &= \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} (x-x_0) \cdot X \nabla |u|^2 dx.
\end{align*}
\]

In particular if \( X = (x-x_0) \) by using that \( \nabla u = (\frac{\partial u}{\partial \nu}, |x-x_0|^{-1} \frac{\partial u}{\partial \theta}) \), from (51) we get the identity
\[
\begin{align*}
\int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} |x-x_0|^2 \left| \frac{\partial u}{\partial \nu} \right|^2 dx &= \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \left| \frac{\partial u}{\partial \theta} \right|^2 dx
\end{align*}
\]
and we conclude.

In Theorem 3.2 we get infinite many Pohozaev identities over balls in correspondence to holomorphic vector fields \( X = X_1 + iX_2 : \mathbb{C} \to \mathbb{C} \) for maps \( u \in W^{2,2}_{loc}(\mathbb{R}^2, \mathbb{R}^m) \) satisfying (46)

**Theorem 3.2. [Pohozev in \( \mathbb{R}^2 \)- Ball Case]** Let \( u \in W^{2,2}_{loc}(\mathbb{R}^2, \mathbb{R}^m) \) such that

\[
\frac{\partial u}{\partial x_i} \cdot \Delta u = 0 \quad a.e \text{ in } \mathbb{R}^2
\]

for \( i = 1, 2 \). Then for all \( x_0 \in \mathbb{R}^2, r > 0 \) and every \( X = X_1 + iX_2 : \mathbb{C} \to \mathbb{C} \) holomorphic function the following identity holds
\[
\begin{align*}
2 \int_{\partial B(x_0, r)} \frac{\partial u}{\partial \nu} \nabla u \cdot X dx &= \int_{\partial B(x_0, r)} X \cdot \nu |\nabla u|^2 dx
\end{align*}
\]

In the particular case where \( X = x-x_0 \) with \( x_0 \in \mathbb{R}^2 \), then for all \( r > 0 \) the following identity holds
\[
\begin{align*}
2 \int_{\partial B(x_0, r)} \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma &= \int_{\partial B(x_0, r)} |\nabla u|^2 d\sigma
\end{align*}
\]

or
\[
\begin{align*}
\int_{\partial B(x_0, r)} \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma &= \frac{1}{r^2} \int_{\partial B(x_0, r)} \left| \frac{\partial u}{\partial \theta} \right|^2 d\sigma
\end{align*}
\]

or
\[
\begin{align*}
\int_{\partial B(x_0, r)} \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma &= \frac{1}{r^2} \int_{\partial B(x_0, r)} \left| \frac{\partial u}{\partial \theta} \right|^2 d\sigma
\end{align*}
\]
**Proof.** We multiply the equation (53) by $X_i$ and we integrate over $B(x_0, r)$:

$$0 = \int_{B(x_0, r)} X_i \frac{\partial^2 u}{\partial x_i \partial x_k^2} dx$$

$$= \int_{B(x_0, r)} \frac{\partial}{\partial x_k} \left( X_i \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} \right) dx$$

$$- \int_{B(x_0, r)} \frac{\partial X_i}{\partial x_k} \frac{\partial u}{\partial x_i} dx$$

$$- \frac{1}{2} \int_{B(x_0, r)} X_i \frac{\partial |\nabla u|^2}{\partial x_i} dx$$

$$= \int_{\partial B(x_0, r)} (X \cdot \nabla u) \left( \frac{\partial u}{\partial \nu} \right) d\sigma$$

$$- \frac{1}{2r} \int_{\partial B(x_0, r)} X \cdot (x - x_0) |\nabla u|^2 dx$$

$$+ \frac{1}{2} \int_{B(x_0, r)} \frac{\partial X_i}{\partial x_i} |\nabla u|^2 dx - \int_{B(x_0, r)} \frac{\partial X_i}{\partial x_k} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} dx.$$  

By using the Cauchy Riemann equations one deduces that the last term in (57) is zero and therefore

$$\int_{\partial B(x_0, r)} (X \cdot \nabla u) (\nabla u \cdot (x - x_0)) d\sigma = \frac{1}{2} \int_{\partial B(x_0, r)} X \cdot (x - x_0) |\nabla u|^2 d\sigma.$$

and we conclude.  

\[\square\]

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