The geometric sense of R. Sasaki connection

Alexey V. Shchepetilov
Department of Physics, Moscow State University,
119992 Moscow, Russia*

Abstract

For the Riemannian manifold $M^n$ two special connections on the sum of the tangent bundle $TM^n$ and the trivial one-dimensional bundle are constructed. These connections are flat if and only if the space $M^n$ has a constant sectional curvature $\pm 1$. The geometric explanation of this property is given. This construction gives a coordinate free many-dimensional generalization of the connection from the paper: R. Sasaki 1979 Soliton equations and pseudospherical surfaces, Nuclear Phys., 154 B, pp. 343-357. It is shown that these connections are in close relation with the imbedding of $M^n$ into Euclidean or pseudoeuclidean $(n+1)$-dimension spaces.

Keywords: Riemannian space, bundle, connection, curvature, zero-curvature representation.

PACS number: 02.40.Ky

Mathematical Subject Classification: primary 53C07; secondary 53C42, 35Q53.

*email: alexey@quant.phys.msu.su
1 Introduction

In the paper [1] there was proposed the formula for some local connection on a 2-dimensional real Riemannian manifold \( M^2 \), which has played a big role in the theory of nonlinear integrable partial differential equations. The construction of this connection is as follows\(^1\).

Let \( g \) be a Riemannian metric on \( M^2 \), \( \nabla \) the corresponding Levi-Civita connection on \( T M^2 \), \( \{e_1, e_2\} \) be a moving orthonormal frame on some open domain \( U \subset M^2 \) and \( \{\omega^1, \omega^2\} \) a corresponding moving coframe. The relations \( \nabla(e_i) = \omega^j_i \otimes e_j \) define the connection 1-form matrix \( \omega^j_i \) with respect to the frame \( \{e_1, e_2\} \). The orthonormality of this frame implies that \( \omega^1_1 = \omega^2_2 = 0 \), \( \omega^2_1 = -\omega^1_2 \). The Levi-Civita connection is torsion-free that gives the following structural equations:

\[
d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = \omega^1 \wedge \omega^3. \tag{1}
\]

The Gaussian curvature \( K \) of the space \( M^2 \) is defined by

\[
d\omega^3 = K \omega^1 \wedge \omega^2. \tag{2}
\]

R. Sasaki proposed to consider a matrix 1-form on \( M^2 \):

\[
A = \frac{1}{2} \begin{pmatrix}
\omega^2_1 & -\omega^1_1 + \omega^1_2 \\
-\omega^1_1 - \omega^2_2 & -\omega^2_1
\end{pmatrix}, \tag{3}
\]

as a new connection form for some (non-specified) bundle over \( M^2 \). The key property of the matrix 1-form \( A \) is that it satisfies the null curvature condition \( \Omega \equiv dA + A \wedge A \equiv 0 \) iff \( K \equiv -1 \) on \( U \).

In some preceding (for example [2]) and many consequent papers (some of the most recent are [3] – [7]) there were discussed different matrix 1-forms, depending on a function \( u \) (or some functions) of some independent variables, such that the null curvature condition for this form is equivalent to one of the well known nonlinear partial differential equations (KdV, mKdV, sine-Gordon, sinh-Gordon, non-linear Schrödinger, Burgers) possessing many conservation laws and reach symmetry groups. R. Sasaki was the first who connect the matrix 1-form \( A \) with the surfaces of constant negative curvature. In the paper [8] there was defined a class of differential equations \( F[u] = 0 \), which could be obtained as a null curvature condition for the form \( A \), depending on the function \( u \).

However, it seems that the geometric meaning of the connection \( \tilde{\nabla}^h \), corresponding to the matrix 1-form \( A \), remained unclear. Firstly, the definition \( A \) is valid only for a local trivialization of potential unknown bundle, because it might happen that there is no global moving frame on \( M^2 \), for example for \( M^2 = S^2 \). Secondly, according to [3] the matrix 1-form \( A \) is a \( sl(2, \mathbb{R}) \)-valued 1-form. It seems a bit strange, because it is defined for an arbitrary metric \( g \), and the corresponding group \( SL(2, \mathbb{R}) \) is the isometry group only for \( M^2 \) equal to the hyperbolic plane \( \mathbb{H}^2 \). Thirdly, the forms \( \omega^1, \omega^2 \) and \( \omega^3 \) play the similar role in \( \tilde{\nabla}^h \), but their geometric sense is quite different. The Levi-Civita connection 1-form for the tangent bundle \( TM^2 \) with respect to the moving frame \( \{e_1, e_2\} \) is

\[
\begin{pmatrix}
0 & \omega^3_1 \\
-\omega^3_2 & 0
\end{pmatrix}. \tag{4}
\]

It is contained in \( A \) with the strange factor \( \frac{1}{2} \), violating the geometric sense of this summand due to nonlinear relation between a connection 1-form \( A \) and a curvature 2-form \( \Omega \).

\(^1\)The following description of the Sasaki construction is slightly different from the original one for the better agreement with the sequel. Particularly we choose the sign in the null curvature condition in more geometric way.
At last it seems to be unclear how to generalize this construction for higher dimensions. Below we will answer on all these questions.

Note the difference of R. Sasaki connection from Sasakian geometry introduced in [9].

2 Reformulation of Sasaki’s construction

Denote
\[
\sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
the base in the Lie algebra \( \mathfrak{sl}(2, \mathbb{R}) \). The commutative relation for it are:
\[
[\sigma_1, \sigma_2] = \sigma_3, \quad [\sigma_2, \sigma_3] = -\sigma_1, \quad [\sigma_3, \sigma_1] = -\sigma_2. \tag{5}
\]
Then the connection matrix 1-form (3) can be expressed as
\[
A = \sigma_1 \omega^1 + \sigma_2 \omega^2 + \sigma_3 \omega^3 \tag{6}
\]
and the corresponding curvature form will be
\[
\Omega = A + \frac{1}{2} [A, A] = \sigma_1 d\omega^1 + \sigma_2 d\omega^2 + \sigma_3 d\omega^3 + [\sigma_1, \sigma_2] \omega^1 \wedge \omega^2 + [\sigma_1, \sigma_3] \omega^1 \wedge \omega^1 + [\sigma_2, \sigma_3] \omega^2 \wedge \omega^2.
\]
Here we used a standard notation \([B, C] = \sum_i B_i C^i \omega^i \wedge \omega^j\), where \(B = \sum_i B_i \omega^i\), \(C = \sum_i C_i \omega^i\); \(B_i, C_i\) are coefficients in Lie algebra \(\mathfrak{g}\) and \(\omega^i\) are scalar differential 1-forms.

When \(\mathfrak{g}\) is a matrix algebra it is obvious that \(B \wedge C = \frac{1}{2} [B, C]\). Hence the condition \(\Omega = 0\) depends only on relations (1), (2) and commutative relations in the algebra \(\mathfrak{sl}(2, \mathbb{R})\).

It is well known that Lie algebras \(\mathfrak{sl}(2, \mathbb{R}), \mathfrak{so}(2, 1)\) and \(\mathfrak{su}(1, 1)\) are isomorphic, so we can change elements \(\sigma_1, \sigma_2, \sigma_3\) in (6) by the equivalent elements from \(\mathfrak{so}(2, 1)\):
\[
\bar{\sigma}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \bar{\sigma}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \bar{\sigma}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
with the same commutative relations (5). After this substitution the 1-form \(A\) becomes:
\[
\bar{A} = \bar{\sigma}_1 \omega^1 + \bar{\sigma}_2 \omega^2 + \bar{\sigma}_3 \omega_2 \tag{7}
\]
where expression
\[
\bar{\sigma}_3 \omega_2 = \begin{pmatrix} 0 & \omega_2^1 & 0 \\ -\omega_2^1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
contains the Levi-Civita connection form
\[
\begin{pmatrix} 0 & \omega_2^1 \\ -\omega_2^1 & 0 \end{pmatrix} \tag{8}
\]
as a direct summand. Due to the last fact it is now possible to give a geometric interpretation of the Sasaki’s connection.
Let $E = \mathbb{R} \times M^2$ be a trivial one dimensional vector bundle over $M^2$ and $F = TM^2 \oplus E$ be a direct sum of two bundles over the same base. Define an indefinite metric $\tilde{g}_h$ on each fiber $T_x M^2 \oplus \mathbb{R}$ of $F$ as a direct sum of the metric $g$ and the metric $(x, y) = -xy, x, y \in \mathbb{R}$. Let $e$ be a unit vector in $\mathbb{R}$. Thus $\{e_1, e_2, e\}$ is a moving frame in $F$ and the connection 1-form (7) defines a covariant derivation $\tilde{\nabla}^h$:

$$
\tilde{\nabla}^h e_1 = -\omega^1_2 \otimes e_2 + \omega^1 \otimes e, \quad \tilde{\nabla}^h e_2 = \omega^2_1 \otimes e_1 + \omega^2 \otimes e, \quad \tilde{\nabla}^h e = \omega^1 \otimes e_1 + \omega^2 \otimes e_2, 
$$

which conserves the metric $\tilde{g}_h$. It is easily seen that when $M^2$ is the hyperbolic plane $\mathbb{H}^2$, imbedded in the standard way as a one sheet of two-sheeted hyperboloid into the pseudoeuclidean space $\mathbb{E}^{2,1}$, then $F$ is simply the trivial bundle $E^{2,1} \times \mathbb{H}^2$. In this case $E$ is the normal bundle over hyperboloid $\mathbb{H}^2 \subset \mathbb{E}^{2,1}$. This explains the null-curvature for $\tilde{\nabla}^h$ when $M^2 = \mathbb{H}^2$, because $\tilde{\nabla}^h$ on $\mathbb{E}^{2,1} \times \mathbb{H}^2$ is the restriction of the flat Levi-Civita connection on $TE^{2,1} = \mathbb{E}^{2,1} \times \mathbb{E}^{2,1}$.

To be sure that the connection $\tilde{\nabla}^h$ is well-defined on the whole bundle $F$ for the general metric $g$ on $M^2$ we can rewrite $\tilde{\nabla}^h$ as follows. Let $\xi + fe = \xi^1 e_1 + \xi^2 e_2 + fe$ is a direct expansion of an arbitrary section of $F$ over $U$, where $f$ is a smooth function on $M^2$ and $\xi$ is a sections of $TM^2$. Then due to (9) we obtain:

$$
\tilde{\nabla}^h_X (\xi + fe) = X(\xi^1) e_1 + X(\xi^2) e_2 - \xi^1 \omega^1_2(X) e_2 + \xi^2 \omega^1_2(X) e_1 + f(\omega^1(X) e_1 + \omega^2(X) e_2) + (X(f) + \xi^1 \omega^1(X) + \xi^2 \omega^2(X)) e = \nabla^h_X \xi + fX + (X(f) + g(X, \xi)) e, 
$$

where $X$ is a vector field on $M^2$. It is obvious that this formula gives the definition for $\tilde{\nabla}^h$ on the whole bundle $F$.

It is possible to change the connection on the bundle $F$ in such a way that it will be flat iff $g$ is the metric of constant positive curvature $K = 1$. To do this we should write the connection 1-form $A$ as:

$$
A = \begin{pmatrix}
0 & \omega^1_2 & \omega^1 \\
-\omega^1_2 & 0 & \omega^2 \\
-\omega^1 & -\omega^2 & 0
\end{pmatrix}.
$$

The corresponding derivation will be

$$
\tilde{\nabla}^h_X (\xi + fe) = \nabla_X \xi + fX + (X(f) - g(X, \xi)) e. 
$$

We see that now $A$ is a $\mathfrak{so}(3)$ valued differential form and the derivation $\tilde{\nabla}^s$ conserves the positively defined metric $\tilde{g}_s$ on fibers which is the direct sum of the metric $g$ on $TM^2$ and the metric $(x, y) = xy, x, y \in \mathbb{R}$ on $\mathbb{R}$.

3 Generalization on higher dimensions

The formulas (10) and (11) make possible the immediate generalization of this construction on higher dimensions. Now $M^n$ becomes a $n$-dimensional Riemannian manifolds and $F$ is the bundle $TM^n \oplus E$, where $E = M^n \times \mathbb{R}$. Define the connections $\tilde{\nabla}^s$ by (11) and connection $\tilde{\nabla}^h$ by the right hand side of (10), where $e$ again is the unit element of the fiber $\mathbb{R}$ of $F$. The definitions for metrics $\tilde{g}_h$ and $\tilde{g}_s$ are the same as in the previous section.

It is well-known that the Riemannian tensor $R$ on a manifold with constant sectional curvature $K$ satisfies the following relation:

$$
R(X, Y)Z = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]} = K(g(Y, Z)X - g(X, Z)Y).
$$
We denote such tensor as $R_K$. Let us calculate the curvature tensor $R^{h,s}$ corresponding to the connections $\tilde{\nabla}^h$ and $\tilde{\nabla}^s$ on $F$:

$$R^{h,s}(X, Y)\tilde{\xi} = \tilde{\nabla}^h_X \tilde{\nabla}^h_Y \tilde{\xi} - \tilde{\nabla}^h_Y \tilde{\nabla}^h_X \tilde{\xi} - \tilde{\nabla}^{h,s}_{[X,Y]} \tilde{\xi},$$

where $\tilde{\xi} = \xi + fe$ is a section of $F$. From \[(10)\] we obtain:

$$\tilde{\nabla}^h_X \tilde{\nabla}^h_Y (\xi + fe) = \nabla_X \nabla_Y \xi + \nabla_X (fY) + (Y(f) - g(Y, \xi)) X + (X(Y(f) - g(Y, \xi)) - g(X, \nabla_Y \xi + fY)) e,$n

so

$$R^s(X, Y)\tilde{\xi} = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi + \nabla_X (fY) - \nabla_Y (fX) + Y(f)X - X(f)Y - g(Y, \xi) X + g(X, \xi) Y + \{(X \circ Y(f) - Y \circ X(f)) - X(g(\xi, Y)) + Y(g(X, \xi)) - g(X, \nabla_Y \xi) + g(Y, \nabla_X \xi) - f([X, Y][f] - g([X, Y], Y) Y - ([X, Y]f) - g([X, Y], \xi) e = R(X, Y)\xi + f([X, Y], \xi) e = R(X, Y)\xi - R_1(X, Y)\xi,$n

due to the equality $K_\varphi(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$ for the torsion $K_\varphi$ of the Levi-Civita connection and the condition $\nabla_X g = 0$. Reasoning in the similar way, we obtain:

$$R^h(X, Y)\tilde{\xi} = R(X, Y)\xi - R_{-1}(X, Y)\xi.$$

Thus the connection $\tilde{\nabla}^h$ is flat iff $M^n$ is a space of the constant sectional curvature $-1$ and the connection $\tilde{\nabla}^s$ is flat iff $M^n$ is a space of the constant sectional curvature $1$.

We can verify that the connection $\tilde{\nabla}^h$ conserves the metric $\tilde{g}_h$ and the connection $\tilde{\nabla}^s$ conserves the metric $\tilde{g}_s$. Indeed, let $\tilde{\xi}_i = \xi_i + f_i e$, $i = 1, 2$ be sections of $F$. Then

$$\tilde{\nabla}_X^s (\tilde{g}_s(\tilde{\xi}_1, \tilde{\xi}_2)) = X(g(\xi_1, \xi_2) + f_1 f_2) = g(\nabla_X \xi_1, \xi_2) + g(\xi_2, \nabla_X \xi_1) + X(f_1 f_2).$$

On the other hand

$$\tilde{g}_s \left( \tilde{\nabla}_X^h \tilde{\xi}_1, \tilde{\xi}_2 \right) + \tilde{g}_s \left( \tilde{\nabla}_X^h \tilde{\xi}_2, \tilde{\xi}_1 \right) = \tilde{g}_s (\nabla_X \xi_1 + f_1 X + (X(f_1) - g(X, \xi_1)) e, \xi_2 + f_2 e) + \tilde{g}_s (\xi_1 + f_1 e, \nabla_X \xi_2 + f_2 X + (X(f_2) - g(X, \xi_2)) e) = g(\nabla_X \xi_1, f_1 X, \xi_2) + (X(f_1) - g(X, \xi_1)) f_2 + g(\xi_1, \nabla_X \xi_2 + f_2 X) + (X(f_2) - g(X, \xi_2)) f_1 = g(\nabla_X \xi_1, \xi_2) + g(\xi_2, \nabla_X \xi_1) + X(f_1) f_2 + X(f_2) f_1.$$

The last two equalities give

$$\left( \tilde{\nabla}_X^s \tilde{g}_s \right) (\tilde{\xi}_1, \tilde{\xi}_2) \equiv \tilde{\nabla}_X^s \left( \tilde{g}_s(\tilde{\xi}_1, \tilde{\xi}_2) \right) - \tilde{g}_s \left( \tilde{\nabla}_X^s \tilde{\xi}_1, \tilde{\xi}_2 \right) - \tilde{g}_s \left( \tilde{\xi}_1, \tilde{\nabla}_X^s \tilde{\xi}_2 \right) = 0.$$

A similar reasoning gives $\tilde{\nabla}_X^h \tilde{g}_h = 0$. The conservation of these metrics means that the corresponding connection 1-form $A$ is so($n + 1$) valued for $\tilde{\nabla}^s$ and so($n, 1$) valued for $\tilde{\nabla}^h$ with respect to orthonormal moving frames.

Let now $M^n$ be a simply connected space with constant sectional curvature $\pm 1$. Considering the standard models for this space as a submanifold of Euclidean (for $K = 1$) or pseudoeuclidean (for $K = -1$) spaces, we see that the bundle $F$ is isomorphic to the trivial bundle $\mathbb{E}^{n+1} \times \mathbb{H}^n$ for $K = -1$ and to $\mathbb{E}^{n+1} \times \mathbb{S}^n$ for $K = 1$, where $\mathbb{E}^{n+1}$ is the $(n+1)$-dimensional Euclidean space and $\mathbb{E}^{n,1}$ is the $(n+1)$-dimensional pseudoeuclidean space of signature $(n, 1)$. In these cases the connection $\tilde{\nabla}^{h,s}$ is the restriction of the flat Levi-Civita connection for $TE^{n,1} = \mathbb{E}^{n,1} \times \mathbb{E}^{n,1}$ onto $\mathbb{E}^{n,1} \times \mathbb{H}^n$ or of the flat Levi-Civita connection for $TE^{n+1} = \mathbb{E}^{n+1} \times \mathbb{E}^{n+1}$ onto $\mathbb{E}^{n+1} \times \mathbb{S}^n$. 
4 Discussion

The common point of view [11], [12] is that the R. Sasaki connection is based only on internal geometry of surfaces. However the connection 1-form \( \mathfrak{g} \) possesses the additional (with respect to internal geometry) matrix structure. Here we interpreted this additional structure as the trivial one-dimensional summand in the corresponding vector bundle. In the case of the constant sectional curvature this summand becomes a normal bundle of the hypersurfaces. Thus our interpretation means a "virtual" imbedding of the initial space \( M^n \) into the space \( \mathbb{E}^{n+1} \) or \( \mathbb{E}^{n,1} \). This "virtual" imbedding becomes actual when \( M^n \) is a space with constant sectional curvature \( \pm 1 \).

In the papers [13], [14] a multi-dimensional generalization of Sine-Gordon equation \( u_{xy} = \sin u \) was given as an imbedding condition of \( M^n \) into \( \mathbb{E}^{2n-1} \). On the other hand it is well-known [1] that the condition \( dA + A \wedge A \) for matrix 1-form \( A \) given by \( \mathfrak{g} \) is equivalent to the Sine-Gordon equation whenever differential forms \( \omega^1, \omega^2 \) are parameterized by the function \( u \) in a definite way. The generalization of R. Sasaki connection given in this paper seems to be quite natural, so it can lead to another multi-dimensional generalization of the Sine-Gordon equation.

After finishing the present paper the author has received the letter of Jack Lee from University of Washington (to whom the author expresses his deep gratitude), who has pointed to the paper [15] of M. Min-Oo. In that paper the connections \( \tilde{\nabla}^h \) on \( TM \oplus E \) was constructed under the name hyperbolic Cartan connection in order to prove the hyperbolic version of the positive mass theorem. There are no any links with the theory of integrable partial differential equations and particularly with the Sasaki’s construction. Thus the present paper establishes the connection between pure geometrical construction of M. Min-Oo and the well known construction from the theory of integrable partial differential equations.

References

[1] R. Sasaki 1979 Soliton equations and pseudospherical surfaces, Nuclear Phys., V. 154 B, pp. 343-357.
[2] M. Crampin 1978 Solitons and \( SL(2,\mathbb{R}) \), Phys. Lett. A, V.66, pp.170-172.
[3] E.G. Reyes 1998 Pseudo-spherical surfaces and integrability of evolution equations, J. Diff. Equations, V. 147, pp. 195-230.
[4] Q. Ding 2000 The \( NLS^- \) equation and it’s \( SL(2,\mathbb{R}) \) structure, J. Phys. A, V.33, L325-L329.
[5] M.V. Foursov, P.J. Olver, E.G. Reyes 2001 On formal integrability of evolution equations and local geometry of surfaces, Diff. Geom. Appl., V. 15, pp. 183-199.
[6] J. Inoguchi 1999 Darboux transformations on timelike constant mean curvature surfaces, J. Geom. Phys. V. 32, pp. 57-78.
[7] M. Marvan 2002 Scalar second-order evolution equations possesing an irreducible \( sl_2 \)-valued zero-curvature representation, J. Phys. A, V. 35, pp. 9431-9439.
[8] S.S. Chern, K. Tenenblat 1986 Pseudospherical surfaces and evolution equations, Stud. Appl. Math., V. 74, pp. 55-83.
Discussion

[9] S. Sasaki 1960 On differentiable manifolds with certain structures which are closely related to almost contact structure, Tôhoku Math. J., V. 2, PP. 459-476.

[10] S. Kobayashi, K. Nomizu 1963 Foundations of differential geometry, V.1. Interscience publishers, N. Y.

[11] A. Sym 1985 Soliton surfaces and their applications, in Geometric Aspects of the Einstein Equations and Integrable Systems: Proc. Conf. (Scheveningen, The Nederlands, 26-31 Aug. 1984) (Lecture notes in physics, V.239), ed. R. Martini, Berlin: Springer, pp. 154-231.

[12] M. Antonowicz, A. Sym 1985 New integrable nonlinearities from affine geometry, Phys. Lett. A, V.112, pp.1-2.

[13] C.-L. Terng 1980 A higher dimensional generalization of the Sine-Gordon equation and it’s solution theory, Annals of Math., V.111, pp.491-510.

[14] R. Beals, K. Tenenblat 1988 Inverse scattering and the Bäcklund transformation for the generalized wave and generalized sine-Gordon equation, Stud. Appl. Math., V.78, pp. 227-256.

[15] M. Min-Oo 1989 Scalar curvature rigidity of asymptotically hyperbolic spin manifolds, Math. Ann. V.285 , pp. 527–539