Differential dispersion relations with an arbitrary number of subtractions: a recursive approach

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Making use of a recursive approach, derivative dispersion relations are generalized for an arbitrary number of subtractions. The results for both cross even and odd amplitudes are theoretically consistent at sufficiently high energies and in the region of small momentum transfer.

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Dispersion-relation techniques have been widely used in the investigation of high energy hadron scattering. In this area, dispersion relations are rigorous consequences of local QFT through the bridge represented by the concepts of analyticity and polynomial boundedness \cite{1}. This last condition means that the scattering amplitude, as function of the center-of-mass energy $\sqrt{s}$ and the four momentum transfer squared $t$, satisfies the relation

$$\lim_{|s| \to \infty} f(s,t) \leq c|s|^k,$$

for some finite $k$ and in all directions of the complex cut $s$-plane \cite{1,2}. In the case of integral relations, it was proved that in the unphysical region $0 < t < t_0$, where $t_0$ is a constant, the number of subtractions consistent with minimal conditions for the Froissart bound is two \cite{3}. The same result was obtained in the physical region (forward direction) without the use of the Froissart bound, but with the condition that the forward-scattering amplitude does not become purely real at infinite energies \cite{4}.

The limited usefulness of the integral dispersion relations, associated with its non-local character, led to the introduction of “quasi-local” differential operators \cite{5,6}. These derivative dispersion relations (DDR) have played an important role in the simultaneous investigation of total cross sections and the $\rho$-parameter (ratio of the forward real and imaginary parts of the scattering amplitude), for both $pp$ and $\bar{p}p$ elastic scattering \cite{6}. Also, its mathematical aspects and range of validity have been extensively discussed \cite{6} and, in the case of total hadronic cross sections, the smooth increase of this quantity with the energy assures its validity \cite{7}. Based on the general behaviour of the experimental data on total cross sections and the usual neglecting of the odd (crossing symmetric) amplitude at the highest energies, the form generically used corresponds to the first-order result for an even forward amplitude \cite{6}

$$\Re f_+(s,t=0) = \frac{\pi}{2} \frac{d}{d \ln s} \left[ \Im f_+(s,t=0) \right],$$

with the subtraction constant removed. This result is consistent with the usual convergence bounds inferred from the experimental data presently available.

However, there is recent indication that total cross sections may increase faster than usually expected at the highest energies \cite{9,10}. It has also been shown that a possible failure of polynomial boundedness (characteristic of some non-local approaches and string theory), could be observed at sufficiently high energies \cite{2}. Moreover, the odderon hypothesis (c-odd Regge trajectory) \cite{11} is supported by both perturbative QCD and fits to the experimental data \cite{12} and a recipe to detect its signal at higher energies has been recently proposed \cite{13}.

All these facts originate great expectations concerning the advent of the new generation of colliders, the Relativistic Heavy Ion Collider (RHIC) \cite{14} and the CERN Large Hadron Collider (LHC) \cite{15}. The reason is that energies never reached before in $pp$ collisions shall be investigated. It will also be possible to analyse the correct contribution of the c-odd amplitude and convergence properties of the amplitude with increasing energy, as represented by Eq. (1).

At this stage, even if we do not consider the extreme case of an effective failure of the polynomial boundedness, it may be useful to investigate higher bounds concerning convergence properties. Since the convergence of the integral
relations is controlled by subtractions and quasi-local connections (DDR) play an important role in the analysis of experimental data, it may be suitable to extend a higher number of subtractions to the derivative relations.

To this end, in this work we first treat the singly subtracted derivative relation, introducing a recursive relation. Then we extend the $k$-subtracted relation to the derivative case, for both even and odd amplitudes.

We assume the usual cut structure in the complex energy plane [14] and consider the high energy limit $s \gg m^2$, where $m$ is the proton mass, and the region where $|\xi| \lesssim m^2$. Taking into account the polynomial boundedness, Eq. (1), and neglecting the residues from the poles at the origin, the $k$-subtracted integral dispersion relation for even $(+)$ and odd $(-)$ crossing amplitudes reads [13]

$$
Re f_{\pm}(s, t) = \frac{(s)k}{\pi} P \int_{s_0}^{+\infty} ds' \frac{Im f_{\pm}(s', t)}{(s')^k} \left[ \frac{1}{s' - s} \pm \frac{(-1)^k}{s' + s} \right].
$$

(3)

We first consider an even amplitude and one subtracted case ($k = 1$). Following Ref. [8], after multiplying and dividing by $s^\alpha$ ($\alpha$ being a real parameter), we integrate by parts and introduce the change of variable $s = e^y$, to obtain

$$
Re f_+(s, t) = \frac{s}{\pi} \int_{\ln s_0}^{+\infty} dy \left( \frac{1}{2} \frac{d}{dy} \right) Im f_+(s', t) / s^\alpha.
$$

(4)

In order to perform the integration we shall take into account the following:

(i) If $Im f_+(s', t) / s^\alpha$ is an (real) analytic function of the variable $\xi' = \ln s'$ we can expand it in powers of $\xi' - \xi$. We observe that this is not a trivial assumption.

(ii) It is easy to show that in the asymptotic limits,

$$
\lim_{|\xi' - \xi| \to 0} \ln \coth \frac{1}{2} |\xi' - \xi| = +\infty, \quad \lim_{|\xi' - \xi| \to +\infty} \ln \coth \frac{1}{2} |\xi' - \xi| = 2e^{-|\xi' - \xi|};
$$

(5)

(iii) Since $s \gg s_0$, the result (ii) means that for large $|s' - s|$ the contribution to the integral is small. In this case, we can take $s_0 \to 0$ so that $\ln s_0 \to -\infty$, as the lower limit in Eq. (4).

From (i) and (iii) and assuming that the series may be integrated term by term we obtain

$$
Re f_+(s, t) = s^\alpha \sum_{n=0}^{+\infty} d^{(n)}_{\ln s(n)} (Im f_+(s, t) / s^\alpha) I_n / n!;
$$

(6)

where $I_n$ represents the integral in the variable $\xi'$,

$$
I_n = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\xi' e^{(\alpha - 1)(\xi' - \xi)} \ln \coth \frac{1}{2} |\xi' - \xi| \left( \alpha - 1 + \frac{d}{d\xi'} \right) (\xi' - \xi)^n.
$$

(7)

Denoting $\xi' - \xi \equiv y$ and through integration by parts, this equation may be put in the form

$$
I_n = \frac{1}{\pi} \ln \coth \frac{1}{2} |y| e^{(\alpha - 1)y} |y|^{+\infty} + \frac{1}{\pi} \int_{-\infty}^{+\infty} dy \frac{e^{(\alpha - 1)y} y^n}{\sinh y}.
$$

(8)

From the first term it is easy to see that convergence to a finite value (zero in this case) is obtained only for limited values of the parameter $\alpha$, namely, $\alpha \in (0, 2)$. For $n = 0$ the above integral may be evaluated in the complex plane, $I_0 = \tan \left( \frac{\pi}{2} (\alpha - 1) \right)$. For $n = 1, 2, 3, ...$, the integral $I_n$ may be formally determined if we differentiate the second term in Eq. (8), taking $\alpha$ as variable and changing the order of integration and differentiation. This is allowed since the integrand is continuous and the integral is uniformly convergent. With this we obtain a novel recursive relation

$$
I_n = \frac{d^{(n)} I_0}{d\alpha^{(n)}} = \frac{d^{(n)}}{d\alpha^{(n)}} \left( \tan \left( \frac{\pi}{2} (\alpha - 1) \right) \right).
$$

(9)

Finally, each term of the series in Eq. (8) may be obtained through recursive derivation. Moreover, the result may be put in a closed form which is suitable for theoretical manipulations: Substituting Eq. (8) into Eq. (4) with the tangent expressed by its series we get
The above expression connects the real part of an even amplitude with the derivatives of the imaginary part at the same energy. Due to the derivative character involved, it shall be considered as a quasi-local relation.

In what follows we generalize this result for an arbitrary number of subtractions and also for odd amplitudes. To this aim we begin with the integral relation for an arbitrary number of subtractions. From Eq. (3), for \( k = 2n \) and \( k = 2n - 1 \) we have for the even amplitudes

\[
\text{Ref}_+(s, t) = s^{2n-1} P \int_{s_0}^{\infty} ds' \operatorname{Im} \left( \frac{f_+(s', t)}{s'^{2n-1}} \right) \left[ \frac{2s}{s'^2 - s^2} \right].
\]

Analogously, for \( k = 2n \) and \( k = 2n + 1 \) the odd amplitude reads

\[
\text{Ref}_-(s, t) = s^{2n-1} P \int_{s_0}^{\infty} ds' \operatorname{Im} \left( \frac{f_-(s', t)}{s'^{2n-1}} \right) \left[ \frac{2s^2}{s'^2 - s^2} \right].
\]

Defining

\[
g_+(s', t) = \frac{f_+(s', t)}{s'^{2(n-1)}}
\]

and from Eq. (3) with \( k = 1 \) and (11), it follows that \( g_+ \) satisfies a two-subtracted integral dispersion relation for an even function. Although \( g_+ \) might have a pole at the origin, this does not bring any disagreement with the approach as can easily be verified. In addition, if we assume assertion (i), \( g_+ \) verifies all the necessary conditions for the derivative relation (10), so that

\[
\text{Ref}_+(s, t) = s^{2(n-1)+\alpha} \tan \left[ \frac{\pi}{2} \left( \alpha - 1 + \frac{d}{d \ln s} \right) \right] \operatorname{Im} f_+(s, t) / s^{2(n-1)+\alpha}.
\]

Analogously, defining

\[
g_+(s', t) = \frac{f_-(s', t)}{s'^{2n-1}}
\]

the same arguments lead to

\[
\text{Ref}_-(s, t) = s^{2n-1+\alpha} \tan \left[ \frac{\pi}{2} \left( \alpha - 1 + \frac{d}{d \ln s} \right) \right] \operatorname{Im} f_-(s, t) / s^{2n-1+\alpha}.
\]

Differently from the even parity, this expression shows that the 1st and 2nd subtractions are quite distinct in the odd case.

The essential results of this work are: (a) introduction of a recursive relation in the parameter \( \alpha \), Eq. (9), in order to obtain a compact form for the differential relation, Eq. (10); (b) generalization of the derivative relations for an arbitrary number of subtractions, for both even and odd amplitudes, Eqs (14) and (16) respectively, near the forward scattering.

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