SOME OBSERVATIONS ON
COMPACT INDESTRUCTIBLE SPACES

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Abstract. Inspired by a recent work of Dias and Tall, we show that a compact indestructible space is sequentially compact. We also prove that a Lindelöf $T_2$ indestructible space has the finite derived set property and a compact $T_2$ indestructible space is pseudoradial.

A compact space is indestructible if it remains compact in any countably closed forcing extension. This is a particular case of the notion of Lindelöf indestructibility, whose study was initiated by Tall in [9]. A space is compact indestructible if and only if it is compact and Lindelöf indestructible. A nice connection of Lindelöf indestructibility with certain infinite topological game was later discovered by Scheepers and Tall [8].

$G^{\omega_1}_1(\mathcal{O}, \mathcal{O})$ denotes the game of length $\omega_1$ played on a topological space $X$ by two players I and II in the following way: at the $\alpha$-th inning player I choose an open cover $U_\alpha$ of $X$ and player II responds by taking an element $U_\alpha \in U_\alpha$. Player II wins if and only if $\{U_\alpha : \alpha < \omega_1\}$ covers $X$.

Proposition 1. ([8], Theorem 1) A Lindelöf space $X$ is indestructibly Lindelöf if and only if player I does not have a winning strategy in $G^{\omega_1}_1(\mathcal{O}, \mathcal{O})$.

Recently, Dias and Tall [4] started to investigate the topological structure of compact indestructible spaces. In particular, they proved that a compact $T_2$ indestructible space contains a non-trivial convergent sequence ([4], Corollary 3.4).

The aim of this short note is to strengthen the above result, by showing that indestructibility actually gives even more than sequential compactness (Theorem 3). However, indestructibility forces a compact space to be sequentially compact in the absolute general case, that is by assuming no separation axiom (Theorem 1). The same proof, with minor changes, will show that a Lindelöf $T_2$ indestructible space has the finite derived set property (Theorem 2).

As usual, $A \subseteq^* B$ means $|A \setminus B| < \aleph_0$ (mod finite inclusion).
Theorem 1. Every compact indestructible space is sequentially compact.

Proof. Let $X$ be a compact indestructible space and assume that $X$ is not sequentially compact. Our task is to show that in this case player I would have a winning strategy in the game $G_1^{|\omega|}(\mathcal{O}, \mathcal{O})$. Fix a countable infinite set $A \subseteq X$ with no infinite convergent subsequence. For each $x \in X$ there is an open set $U_x$ such that $x \in U_x$ and $|A \setminus U_x| = \aleph_0$. The first move of player I is the open cover $U_0 = \{ U_x : x \in X \}$. If player II responds by choosing $U_{\{x_0\}} \in U_0$, then let $A_{\{x_0\}} = A \setminus U_{\{x_0\}}$. For each $x \in X$ there is an open set $U_{(x_0,x)}$ such that $x \in U_{(x_0,x)}$ and $|A_{\{x_0\}} \setminus U_{(x_0,x)}| = \aleph_0$. The second move of player I is the open cover $U_1 = \{ U_{(x_0,x)} : x \in X \}$. If player II responds by choosing $U_{(x_0,x_1)} \in U_1$, then let $A_{(x_0,x_1)} = A_{\{x_0\}} \setminus U_{(x_0,x_1)}$. Again, for each $x \in X$ player I chooses an open set $U_{(x_0,x_1,x)}$ such that $x \in U_{(x_0,x_1,x)}$ and $|A_{(x_0,x_1)} \setminus U_{(x_0,x_1,x)}| = \aleph_0$. At the $\omega$-th inning of the game, the moves of the two players have defined a function $f : \omega \to X$ and a decreasing chain of sets $\{ A_f|n : n < \omega \}$. Player I chooses an infinite set $B \subseteq A$ satisfying $B \subseteq^* A_f|n$ for each $n < \omega$ and for each $x \in X$ an open set $U_{f|x}$ such that $x \in U_{f|x}$ and $|B \setminus U_{f|x}| = \aleph_0$. Then, at the $\omega$-th inning player I plays the open cover $U_0 = \{ U_{f|x} : x \in X \}$. If player II responds by choosing $U_{f|x}$, then let $x_0 = x$ and $A_f = B \setminus U_{f|x}$. In general, at the $\alpha$-th inning the moves of the two players have already defined a function $f : \alpha \to X$ and a mod finite decreasing family $\{ A_f|\beta : \beta < \alpha \}$ of infinite subsets of $A$. Then, player I fixes an infinite set $B \subseteq A$ such that $B \subseteq^* A_f|\beta$ for each $\beta < \alpha$ and plays the open cover $U_\alpha = \{ U_{f|x} : x \in X \}$, where $x \in U_{f|x}$ and $|B \setminus U_{f|x}| = \aleph_0$. If the responses of player II is $U_{f|x}$, then let $x_\alpha = x$, $A_f = B \setminus U_{f|x}$ and so on.

At the end of the game, we have a function $g : \omega_1 \to X$ and a mod finite decreasing chain $\{ A_g|\alpha : \alpha < \omega_1 \}$ of infinite subsets of $A$. The set resulting from the moves of player II is the collection $\mathcal{V} = \{ U_g|\alpha+1 : \alpha < \omega_1 \}$. For any finite set of ordinals $\alpha_0, \ldots, \alpha_m < \omega_1$, taking some $\beta < \omega_1$ such that $\alpha_i < \beta$ for $i \leq m$, we see that the infinite set $A_g|\beta$ has a finite intersection with each $U_g|\alpha_{i+1}$ and therefore the subcollection $\{ U_g|\alpha_{i+1} : i \leq m \}$ cannot cover $X$. Since $\mathcal{V}$ does not have finite subcovers, the compactness of $X$ implies that the whole $\mathcal{V}$ cannot cover $X$. Thus, player I wins the game, in contrast with Proposition 1. □

Recall that a topological space $X$ has the finite derived set (briefly FDS) property provided that every infinite set of $X$ contains an infinite subset with at most finitely many accumulation points (see for instance [2]). Since in a $T_2$ space a convergent sequence has only one accumulation point, we see that if a $T_2$ space has a countable infinite set $A$ violating the finite derived set property, then for each infinite set $B \subseteq A$ and each point $x \in X$ there must be an open set $U_x$ such that $x \in U_x$ and $|B \setminus U_x| = \aleph_0$. Notice, however, that for this much less than $T_2$ is needed. For instance, it suffices for the space to be SC, namely that every convergent sequence together with the limit point is a closed subset (see [2]).

With this observation in mind, we can modify the above proof to get the following

Theorem 2. A Lindelöf $T_2$ indestructible space has the finite derived set property.

Proof. Let $X$ be a Lindelöf $T_2$ indestructible space and assume that $X$ does not have the FDS property. As in the proof of Theorem 1, our task is to show that in this case player I would have a winning strategy in the game $G_1^{|\omega|}(\mathcal{O}, \mathcal{O})$. Fix a countable infinite set $A \subseteq X$ witnessing the failure of the FDS property. Taking
into account the paragraph before the theorem, for each infinite set $B \subseteq A$ and each $x \in X$ there is an open set $U_x$ such that $x \in U_x$ and $|B \setminus U_x| = \aleph_0$. Now, the strategy of player I is exactly the same as that in the proof of Theorem 1. At the end of the game, the set resulting from the moves of player II is again the collection $V = \{U_{g|\alpha+1} : \alpha < \omega_1\}$. We claim that $V$ cannot cover $X$. Otherwise, by the Lindelöfness of $X$, there should exist a countable set of ordinals $S \subseteq \omega_1$ such that the subcollection $\{U_{g|\alpha+1} : \alpha \in S\}$ would cover $X$. Taking some $\beta < \omega_1$ such that $\alpha < \beta$ for each $\alpha \in S$, we see that the infinite set $A_{g|\beta}$ does not have accumulation points in $X$, in contrast with the supposed failure of the FDS property in $A$. Thus, $V$ cannot cover $X$ and again player I wins the game. \[\square\]

The above theorem provides new informations on the topological structure of a Lindelöf indestructible space.

We will finish by showing that for $T_2$ spaces Theorem 1 can be improved.

Proposition 2. ([4], Corollary 3.3) A compact $T_2$ space which is not first countable at any point is destructible.

Recall that a topological space $X$ is pseudoradial provided that for any non-closed set $A \subseteq X$ there exists a well-ordered net $S \subseteq A$ which converges to a point outside $A$. For more on these spaces see [3].

Clearly every compact pseudoradial space is sequentially compact, but the converse may consistently fail [5].

Theorem 3. Any compact $T_2$ indestructible space is pseudoradial.

Proof. Let $X$ be a compact $T_2$ indestructible space and let $A$ be a non-closed subset. Let $\lambda$ be the smallest cardinal such that there exists a non-empty $G_\lambda$-set $H \subseteq \overline{A} \setminus A$. As $X$ is indestructible, so is the subspace $H$. Hence, by Proposition 2, $H$ is first countable at some point $p$. Clearly, $\{p\}$ is a $G_\lambda$-set in $X$ and so there are open sets $\{U_\alpha : \alpha < \lambda\}$ satisfying $\{p\} = \bigcap\{U_\alpha : \alpha < \lambda\} = \bigcap\{\overline{U_\alpha} : \alpha < \lambda\}$. The minimality of $\lambda$ ensures that for each $\alpha < \lambda$ we may pick a point $x_\alpha \in A \cap \bigcap\{U_\beta : \beta < \alpha\}$. The compactness of $X$ implies that the well-ordered net $\{x_\alpha : \alpha < \lambda\}$ converges to $p$ and we are done. \[\square\]

Notice that the indestructibility of a compact space is stronger than pseudoradiality: the Example in section 3 of [4] is a compact $T_2$ pseudoradial space which is destructible.

The fact that pseudoradiality is a weakening of sequentiality and the well-known fact that compact spaces of countable tightness are sequential under PFA [1], might suggest that a compact $T_2$ indestructible space of countable tightness is always sequential. But this is not the case: the one-point compactification of the Ostaszewski’s space [7] is a non-sequential compact $T_2$ space of countable tightness which is indestructible having cardinality $\aleph_1$ (see [4]).

Theorem 3 is no longer true for Lindelöf spaces. Koszmider and Tall constructed [6] a model of ZFC+CH where there exists a regular Lindelöf $P$-space $X$ of cardinality $\aleph_2$ without Lindelöf subspaces of size $\aleph_1$. Such a space does not have convergent well-ordered nets of length $\aleph_1$. Therefore, $X$ is not pseudoradial because it obviously contains non-closed subsets of cardinality $\aleph_1$. On the other hand, it is easy to check that a Lindelöf $P$-space is indestructible (see e.g. [8]).
Recall that a space $X$ satisfies the selection principle $S_{1}^{\omega_1}(\mathcal{O}, \mathcal{O})$ provided that for any family $\{U_\alpha : \alpha < \omega_1\}$ of open covers of $X$ one may pick an element $U_\alpha \in U_\alpha$ in such a way that the collection $\{U_\alpha : \alpha < \omega_1\}$ covers $X$.

It is clear that any compact indestructible space satisfies $S_{1}^{\omega_1}(\mathcal{O}, \mathcal{O})$ and the example described in section 3 of [4] shows that the previous implication is consistently not reversible. Such example is a compact LOTS and so it is sequentially compact. An obvious question then arises:

**Question 1.** Let $X$ be a compact (or compact $T_2$) space satisfying $S_{1}^{\omega_1}(\mathcal{O}, \mathcal{O})$. Is $X$ sequentially compact?

A space answering the above question in the negative would provide a compact space satisfying $S_{1}^{\omega_1}(\mathcal{O}, \mathcal{O})$ which is “more destructible” than the mentioned example in [4].

We can also formulate a weaker version of the problem.

**Question 2.** Is it true that any compact $T_2$ space satisfying $S_{1}^{\omega_1}(\mathcal{O}, \mathcal{O})$ contains a non-trivial convergent sequence?

An interesting feature of the above question is that any counterexample to it turns out to be an Efimov’s space, that is a compact $T_2$ space containing no copy of $\beta \omega$ and no non-trivial convergent sequence.

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