Non-integrability of some higher-order Painlevé equations in Liouville sense

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Abstract

In this paper we study the equation
\[ w^{(4)} = 5w''(w^2 - w') + 5w(w')^2 - w^5 + (\lambda z + \alpha)w + \gamma, \]
which is one of the higher-order Painlevé equations (i.e. equations in the polynomial class having the Painlevé property). Like the classical Painlevé equations, this equation admits a Hamiltonian formulation, Bäcklund transformations and families of rational and special functions.

We prove that this equation considered as a Hamiltonian system with parameters $\gamma/\lambda = 3k, \gamma/\lambda = 3k - 1, k \in \mathbb{Z}$, is not integrable in Liouville sense by means of rational first integrals. To do that we use Ziglin - Morales-Ruiz - Ramis approach.

Then we study the integrability of the second and the third members of the PII-hierarchy. Again as in the previous case it turns out that the normal variational equations are particular cases of the generalized confluent hypergeometric equations whose differential Galois groups are non-commutative and hence, they are obstructions to integrability.

Keywords: Painlevé type equations, Hamiltonian systems, Differential Galois groups, generalized confluent hypergeometric equations

2010 MSC: 70H05, 70H07, 34M55, 37J30

1 Introduction

The Painlevé property for a system of differential equations is the property that its general solution is without movable critical points. Equations with this property are called equations of Painlevé type.
In the beginning of 20th century Painlevé and Gambier investigated this property for second-order ordinary differential equations. They proved that there are fifty equations possessing the Painlevé property. Among them six equations turned out to be new at that time, now called classical Painlevé equations. Although derived in a pure mathematical way, the six Painlevé equations have appeared in many physical applications: in the description of nonlinear waves, in statistical mechanics, in the theory of quantum gravity, in topological field theory, in plasma physics, in the theory of random matrix models and so on.

The classical Painlevé equations have many remarkable properties, in particular they admit a Hamiltonian formulation. In [19] Morales-Ruiz asked the question about the integrability as Hamiltonian systems of classical Painlevé equations which have particular rational solutions. This question was answered affirmatively for $P_{II}$ with values of the parameter $\alpha = n \in \mathbb{Z}$. For the recent development in the study of the non-integrability of the other Painlevé equations we refer to Stoyanova [27]. In a very recent paper Žoladek and Filipuk [30] have proved that any of the classical Painlevé equations does not admit a first integral which is an elementary function of the variables with exception of some known cases.

It is natural to extend the question for non-integrability to the higher-order Painlevé equations. Consider the following fourth-order nonlinear ordinary differential equation

$$w^{(4)} = 5w''(w^2 - w') + 5w(w')^2 - w^5 + (\lambda z + \alpha)w + \gamma,$$

where $\lambda, \alpha, \gamma$ are complex parameters.

This equation appears as a group-invariant reduction of the modified Kaup-Kupershmidt (or Sawada-Kotera) equation, see for instance Hone [7], Kudryashov [10]. Then it appears as equation F-XVIII in the classification made by Cosgrove [3] of all fourth- and fifth-order equations with Painlevé property. It is also studied by V. Gromak [5] from different points of view. It is proven in [10] that (1.1) with $\lambda = 1, \alpha = 0$ has no polynomial first integral.

Like the classical Painlevé equations, this equation admits a Hamiltonian formulation, Bäcklund transformations and families of rational and special functions. For instance:

- when $\lambda = 0, \gamma \neq 0$ it is solved in terms of hyperelliptic functions;
- when $\lambda = 0, \gamma = 0$ it is solved via elliptic functions;
- when $\gamma = -\lambda/2$, $w(z)$ can be expressed in terms of two Painlevé I solutions.

Further, we assume that $\lambda \neq 0$.

The equation (1.1) possesses two families of rational solutions:

I) $\gamma/\lambda = 3k, k \in \mathbb{Z}$

$$k = 0, \ w = 0; \quad k = 1, \ w = -\frac{3\lambda}{\alpha + \lambda z}; \text{ etc.}$$

II) $\gamma/\lambda = 3k - 1, k \in \mathbb{Z}$
\[ k = 0, w = \frac{\lambda}{\alpha + \lambda z}; \quad k = 1, w = -\frac{2\lambda}{\alpha + \lambda z}; \quad \text{etc.} \quad (1.3) \]

In fact, these two families are the only rational solutions of (1.1) (Gromak [5] Theorem 3).

Denote \( q_1(z) := w(z), \epsilon^2 = 1 \). Then the equation (1.1) can be presented as two equivalent 2 + 1/2 degrees of freedom Hamiltonian systems with

\[
H_\epsilon = \frac{1}{2} p_2^2 + \frac{7 - 9\epsilon}{12} q_2^2 + p_1 q_2 - \frac{1 + 3\epsilon}{4} p_1 q_1^2 + \frac{3\epsilon - 1}{4} q_2 (\lambda z + \alpha) + \left( \gamma + \frac{3\epsilon - 1}{4} \lambda \right) q_1. \quad (1.4)
\]

The corresponding system of equations is (\( ' = d/dz \)):

\[
\begin{align*}
q_1' &= q_2 - \frac{3\epsilon + 1}{4} q_1^2, & p_1' &= \frac{1 + 3\epsilon}{2} p_1 q_1 - \gamma - \frac{3\epsilon - 1}{4} \lambda \\
q_2' &= p_2, & p_2' &= -p_1 - \frac{7 - 9\epsilon}{4} q_2^2 - \frac{3\epsilon - 1}{4} (\lambda z + \alpha).
\end{align*}
\quad (1.5)
\]

There exist Bäcklund transformations (see Gromak [5]) \( T_1, T_2 \) and \( T := T_2 T_1 \) for the equation (1.1) acting on the parameters in the following way:

\[
\begin{align*}
T_1(\lambda) &= \lambda, & T_1(\alpha) &= \alpha, & T_1(\gamma) &= -\gamma - \lambda, \\
T_2(\lambda) &= \lambda, & T_2(\alpha) &= \alpha, & T_2(\gamma) &= -\gamma + 2\lambda, \\
T(\gamma) &= \gamma + 3\lambda, & T^{-1}(\gamma) &= \gamma - 3\lambda.
\end{align*}
\quad (1.6-1.8)
\]

Gromak has shown that these Bäcklund transformations are birational. It is easy to be seen that they are canonical also.

We can extend in a natural way the Hamiltonian system (1.5) to a three degrees of freedom autonomous system by denoting \( \hat{H}(q_1, q_2, z, p_1, p_2, F) := H_\epsilon + F \). Then we have

\[
\begin{align*}
\frac{dq_1}{ds} &= q_2 - \frac{3\epsilon + 1}{4} q_1^2, & \frac{dp_1}{ds} &= \frac{1 + 3\epsilon}{2} p_1 q_1 - \gamma - \frac{3\epsilon - 1}{4} \lambda, \\
\frac{dq_2}{ds} &= p_2, & \frac{dp_2}{ds} &= -p_1 - \frac{7 - 9\epsilon}{4} q_2^2 - \frac{3\epsilon - 1}{4} (\lambda z + \alpha), \\
\frac{dz}{ds} &= 1, & \frac{dF}{ds} &= -\lambda - \frac{3\epsilon - 1}{4} q_2.
\end{align*}
\quad (1.9)
\]

Our first result is the following

**Theorem 1.** The Hamiltonian system (1.9) with parameters \( \gamma/\lambda = 3k, \gamma/\lambda = 3k - 1, k \in \mathbb{Z} \) is not integrable in Liouville sense by means of rational first integrals.

We use as a tool Ziglin-Morales-Ruiz-Ramis Theorem. We obtain a particular solution to (1.9), write the normal variational equation and study its Galois group which appears to be large.
The paper is organized as follows. In section 2 we summarize the necessary facts about the integrability of Hamiltonian systems and the theory of linear equations with singular points and its relation to the differential Galois theory. In [9] N. Katz and O. Gabber have calculated the Galois groups of some classes of linear equations using purely algebraic arguments—global characterization of semisimple algebras. In section 3 we apply their result to prove Theorem 1. In section 4 we recover the Katz’s result about the Galois group for our particular linear equation by giving its topological generators. It turns out that the linear equations which appear here are the confluent generalized hypergeometric equations. We use the approach of Duval and Mitschi [4] for the calculation of the formal monodromy, exponential torus and Stokes matrices in the corresponding case.

It is found that the confluent generalized hypergeometric equations have appeared also along the study of other higher-order Painlevé equations. In section 5 we prove that the second and the third members of the \( P_{II} \)-hierarchy are non-integrable in the Hamiltonian context for the particular values of the parameters. Again the differential Galois groups of confluent generalized hypergeometric equations are obstructions to integrability. We conjecture that the higher members in the \( P_{II} \)-hierarchy satisfy the same property.

\section{Theory}

In this section we recall some notions and facts about integrability of Hamiltonian systems in complex domain, the Ziglin–Morales-Ruiz–Ramis–Simó theory and their relations with differential Galois groups of linear equations.

We may take as a definition for integrability of a Hamiltonian systems with \( n \) degrees of freedom the existence of \( n \) (almost everywhere) independent first integrals in involution.

Consider a Hamiltonian system
\[
\dot{x} = X_H(x), \quad t \in \mathbb{C}, \quad x \in M
\]  
(2.1)
corresponding to an analytic Hamiltonian \( H \), defined on the complex \( 2n \)-dimensional manifold \( M \). Suppose the system (2.1) has a non-equilibrium solution \( \Psi(t) \). Denote by \( \Gamma \) its phase curve. We can write the equation in variation (VE) along this solution
\[
\dot{\xi} = DX_H(\Psi(t))\xi, \quad \xi \in T_\Gamma M.
\]  
(2.2)
Further, using the integral \( dH \) we can reduce the variational equation. Consider the normal bundle of \( \Gamma \), \( F := T_\Gamma M/TM \) and let \( \pi : T_\Gamma M \to F \) be the natural projection. The equation (2.2) induces an equation on \( F 
\[
\dot{\eta} = \pi_*(DX_H(\Psi(t)))(\pi^{-1}\eta), \quad \eta \in F.
\]  
(2.3)
which is called the normal variational equation (NVE).

It is natural to assume that if the system (2.1) is integrable, then the linear equations (VE) and (NVE) are also integrable.

Then the main result of Ziglin–Morales-Ruiz–Ramis–Simó [18] theory is:
Theorem 2. Suppose that the Hamiltonian system (2.1) has $n$ meromorphic first integrals in involution. Then the identity component $G^0$ of the Galois group of the variational equation is Abelian.

Next we consider a linear non-autonomous system (or equivalently a linear homogeneous differential equation)

$$y' = A(x)y, \quad y \in \mathbb{C}^n,$$

with $x \in \mathbb{CP}^1$ (which is enough for our purposes) and $A \in \text{gl}(n, \mathbb{C}(x))$. Let $S := \{x_1, \ldots, x_s\}$ be the set of singular points of (2.4) and let $Y(x)$ be a fundamental solution of (2.4) at $x_0 \in \mathbb{C} \setminus S$. The continuation of $Y(x)$ along a nontrivial loop on $\mathbb{CP}^1$ defines a linear automorphism of the space of solutions, called the monodromy transformation. Analytically this transformation can be presented in the following way. The linear automorphism $\Delta_\gamma$, associated with a loop $\gamma \in \pi_1(\mathbb{CP}^1 \setminus S, x_0)$ corresponds to multiplication of $Y(x)$ from the right by a constant matrix $M_\gamma$, called monodromy matrix

$$\Delta_\gamma Y(x) = Y(x)M_\gamma.$$

The set of these matrices form the monodromy group [28, 31].

We may attach another object to the system (2.4) - a differential Galois group. A differential field $K$ is a field with a derivation $\partial = \frac{\partial}{\partial x}$, i.e. an additive mapping satisfying Leibnitz rule. A differential automorphism of $K$ is an automorphism commuting with the derivation.

The coefficient field in (2.4) is $K = \mathbb{C}(x)$. Let $y_{ij}$ be the elements of the fundamental matrix $Y(x)$. Let $L(y_{ij})$ be the extension of $K$ generated by $K$ and $y_{ij}$ - a differential field. This extension is called Picard-Vessiot extension. Similarly to classical Galois Theory we define the Galois group $G := \text{Gal}_K(L) = \text{Gal}(L/K)$ to be the group of all differential automorphisms of $L$ leaving the elements of $K$ fixed. The Galois group is, in fact, an algebraic group. It has a unique connected component $G^0$ which contains the identity and which is a normal subgroup of finite index. The Galois group $G$ can be represented as an algebraic linear subgroup of $\text{GL}(n, \mathbb{C})$ by

$$\sigma(Y(x)) = Y(x)R_\sigma,$$

where $\sigma \in G$ and $R_\sigma \in \text{GL}(n, \mathbb{C})$.

We can do the same locally at $a \in \mathbb{CP}^1$, replacing $\mathbb{C}(x)$ by the field of germs of meromorphic functions at $a$. In this way we can speak of a local differential Galois group $G_a$ of (2.4) at $a \in \mathbb{CP}^1$, defined in the same way for Picard-Vessiot extensions of the field $\mathbb{C}\{x-a\}/(x-a)^{-1}$.

One should note that by its definition the monodromy group is contained in the differential Galois group of the corresponding system.

We say that two linear systems $y' = A(x)y$ and $z' = B(x)z$ are $K$-equivalent if the latter is obtained from the first by a $K$-linear change $y = Pz, P \in \text{GL}(n, \mathbb{C})$ and $B = P^{-1}AP - P^{-1}P'$. Next, we review some facts from the theory of linear systems with singularities. Assuming $x_1 = 0$, locally the system (2.4) can be written as (apparently the behavior at $\infty$ can be obtained as the behavior at $\tau = 0$ after $\tau = 1/x$)

$$x^{r+1}y' = B(x)y,$$

(2.5)
where $B(x)$ is holomorphic at 0 and $r \in \mathbb{Z}$. If $r < 0$ the point $x = 0$ is a regular point, if $r = 0$ the point $x = 0$ is regular singular, and if $r > 1$ the point $x = 0$ is irregular singular. The number $r$ is called Poincaré rank. The system with only regular singularities is called Fuchsian system. For such systems we have:

**Theorem 3.** *(Schlesinger [25])*  
The Galois group coincides with the Zariski closure in $\text{GL}(n, \mathbb{C})$ of the monodromy group.

Now we briefly recall the Ramis description of local Galois group of (2.4) at 0 which we assume to be an irregular singularity of rank $r$.  

Let $K = \mathbb{C}\{x\}[x^{-1}]$ be the field of convergent Laurent series near 0 (field of formal Laurent series), $K_t = \mathbb{C}\{t\}[t^{-1}]$ be the same objects with respect to the variable $t$ and $A \in \text{gl}(n, K)$. It is known from the classical theory that there exists a formal fundamental solution to (2.5):

$$\hat{Y}(t) = \hat{H}(t)x^L e^{Q(t)},$$  \hspace{1cm} (2.6)

where $t^\sigma = x (\sigma \in \mathbb{N}^*)$, $\zeta = e^{2\pi i/\sigma}$, $L = \text{Mat}(n, \mathbb{C})$, $\hat{H} \in \text{GL}(n, \hat{K})$ and $Q = \text{diag}(q_1, \ldots, q_n)$, $q_i \in t^{-i}\mathbb{C}[\frac{1}{t}]$, $i = 1, \ldots, n$. The integer $\sigma$ is called ramification degree at 0.

First we recall the formal invariants of (2.5). The change of variable $x \to xe^{2\pi i}$ commutes with the derivation, so it defines an element $\hat{m} \in G$, the formal monodromy ($t \to t\zeta$ commutes with the corresponding derivation). Relative to $\hat{Y}$, the automorphism $\hat{m}$ can be represented by a matrix $\hat{M}$:

$$\hat{Y}(t\zeta) = \hat{Y}(t)\hat{M}.$$ \hspace{1cm} (2.7)

By definition the exponential torus $T$ of (2.4) relative to $\hat{Y}$ is the group of the differential $\hat{K}_t$-automorphisms of the differential extension

$$\hat{K}_t(e^Q) = \hat{K}_t(e^{q_1}, e^{q_2}, \ldots, e^{q_n}) \text{ of } \hat{K}_t.$$

$T$ is isomorphic to $(\mathbb{C}^*)^l$, where $l$ is the rank of $\mathbb{Z}$-module generated by the $q_i$.

The matrix $\hat{M}$, clearly invariant by $\hat{K}$-equivalence is a formal invariant of (2.5). The same thing applies to the exponential torus $T$.

Let $V_d(\alpha)$ be an open sector in $\mathbb{C}^* \setminus \{0\}$ with its vertex at 0:

$$V_d(\alpha) = \{x \in \mathbb{C}^*|0 < |x| < R, d - \frac{\alpha}{2} < \arg(x) < d - \frac{\alpha}{2}\}$$

and let $f$ be a holomorphic function on $V_d(\alpha)$. We say that $f$ is asymptotic to $\hat{f} = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[x]$ on $V_d(\alpha)$ (in Poincaré sense) if, for every closed subsector $W \subset V_d(\alpha)$ there exists a positive constant $M_{W,n}$, such that for every $x \in W$

$$|x|^{-n} |f(x) - \sum_{m=0}^{n-1} a_m x^m| \leq M_{W,n}.$$
for every \( n \). We write \( f \sim \hat{f} \) on \( V_d(\alpha) \).

By the classical theory (Sibuya [26], Wasow [29], Martinet, Ramis [12]) we have the following result:

**Theorem 4.** For the system \([2.5]\) with a formal solution \([2.6]\), there exists an actual solution \( Y = H^x e^{\Omega} \), where \( H \in \text{GL}(n, \mathbb{C}\{x\}) \) has asymptotic expansion \( \hat{H} \) (\( H \sim \hat{H} \) and \( Y \sim \hat{Y} \)) in any open angular sector with opening \( \pi/r \).

We want to extend the solution \( Y \) to sectors with opening greater than \( \pi/r \). For this purpose we define:

- a Stokes ray as a direction where, for some \( i, j = 1, \ldots, n \), one has \( \text{Re}[q_i(t) - q_j(t)] = 0 \).
- a singular ray is a direction of maximal decay for some \( \exp(q_i - q_j) \), i.e., a bisecting ray of a maximal sector where \( \text{Re}[q_i(t) - q_j(t)] < 0 \).

Let \( d \) be a singular direction for \([2.5]\) at \( x = 0 \), let \( d^+ \) and \( d^- \) be nearby directions with arguments \( d^+ = d + \varepsilon, d^- = d - \varepsilon \). Then \( V^\pm = V_{d^\pm}(\pi/r) \) are two overlapping sectors containing \( d \). Let \( Y^- \) and \( Y^+ \) be actual solutions of \([2.5]\), such that \( Y^- \sim \hat{Y} \) in \( V^- \) and \( Y^+ \sim \hat{Y} \) in \( V^+ \). Hence, we have two actual solutions \( Y^-, Y^+ \) over \( V_d(\pi/r) \) (by analytic continuation to this sector, \( \varepsilon \to 0 \)). Then there exists \( S_d \in \text{GL}(n, \mathbb{C}) \), such that

\[
Y^- = Y^+ S_d.
\]

The matrix \( S_d \) is called Stokes matrix (or multiplier) with respect to \( d \) and \( \hat{Y} \). The Stokes matrices are unipotent. Moreover, they are invariant under \( K \)-equivalence, that is, they are analytic invariants for \([2.5]\) (see also Balser, Jurkat, Lutz [1]).

The actual solutions \( Y \) are usually obtained by summation procedure of \( \hat{H} \) along non-singular directions in maximal sectors. We will not recall here the summation theory developed by Ramis (see, for instance [12, 28] for more details) because in our particular case there exist fundamental systems of solutions near the irregular point in suitable sectors expressed by Meijer G-function [14].

Finally, we have a theorem that generalize the Schlesinger’s result for the Fuchsian case.

**Theorem 5.** (Ramis) With respect to the formal solution \([2.6]\) the analytic Galois group of \([2.5]\) at \( 0 \) is the Zariski closure in \( \text{GL}(n, \mathbb{C}) \) of the subgroup generated by the formal monodromy \( \hat{M} \), the exponential torus \( \mathcal{T} \) and the Stokes matrices \( S_d \) for all singular rays.

This theorem can be naturally extended to the global setting [15].
3 Proof of Theorem 1.

Consider first the family of rational solutions (I). Take $\gamma/\lambda = 0$ or $k = 0$ and $w = 0$. Then it is straightforward to be seen that

$$w = q_1 = 0, \quad q_2 = 0, \quad p_1 = \frac{1 - 3\varepsilon}{4}(\lambda s + \alpha), \quad z = s, \quad F = F_0 = \text{const.} \quad (3.1)$$

is a particular solution.

Let $\xi_j = dq_j, \eta_j = dp_j, j = 1, 2, \xi_3 = ds, \eta_3 = dF$ are the variations. The variational equations (VE) along te solution (3.1) takes the form

$$\begin{align*}
\xi_1' &= \xi_2, \quad \eta_1' = \frac{1 + 3\varepsilon}{2}p_1\xi_1, \\
\xi_2' &= \eta_2, \quad \eta_2' = -\eta_1 + \frac{1 - 3\varepsilon}{4}\lambda\xi_3, \\
\xi_3' &= 0, \quad \eta_3' = -\frac{3\varepsilon - 1}{4}\xi_2. 
\end{align*} \quad (3.2)$$

Then the normal variational equation (NVE) is

$$\begin{align*}
\xi_1' &= \xi_2, \quad \eta_1' = \frac{1 + 3\varepsilon}{2}p_1\xi_1, \\
\xi_2' &= \eta_2, \quad \eta_2' = -\eta_1. 
\end{align*} \quad (3.3)$$

Reducing (3.3) to a single equation yields

$$\xi_1^{(4)} = (\lambda z + \alpha)\xi_1. \quad (3.4)$$

For this equation $\infty$ is an irregular singular point. After changing the independent variable $z \to \lambda z + \alpha$ we get ($\partial = d/dz$)

$$L_1\xi_1 = 0, \quad L_1 = \partial^4 + az, \quad a := -1/\lambda^4. \quad (3.5)$$

The operator $L_1$ is usually called Airy type operator. It is irreducible and it is obviously invariant under $\partial \to -\partial$, hence, $L_1$ is selfdual in the terminology of Katz [9]. In the same paper Katz [9] has found that the identity component of the Galois group of $L_1\xi_1 = 0$ is $G^0 = \text{Sp}(4, \mathbb{C})$ which is clearly non-commutative. Therefore, by the Theorem 2 the Hamiltonian system (1.9) is not integrable in a neighborhood of the particular solution (3.1).

Further we consider the second family of rational solutions (II). Take $\gamma/\lambda = -1$ or $k = 0$ and $w = \frac{1}{s + \alpha/\lambda}$. Then we have the following particular solution to (1.9)

$$\begin{align*}
w &= q_1 = \frac{1}{s + \frac{\alpha}{\lambda}}, \quad q_2 = \frac{3}{4}(s + \frac{\alpha}{\lambda})^2, \quad p_2 = -\frac{3}{2}(s + \frac{\alpha}{\lambda})^3, \\
p_1 &= \frac{1 - 3\varepsilon}{4}(\lambda s + \alpha), \quad z = s, \quad F = \frac{3\lambda s - \varepsilon}{4} + F_0. \quad (3.6)
\end{align*}$$
The (VE) along this solution is
\[\xi'_1 = \xi_2 - \frac{1 + 3\varepsilon}{2}q_1\xi_1, \quad \eta'_1 = \frac{1 + 3\varepsilon}{2}(p_1\xi_1 + q_1\eta_1),\]
\[\xi'_2 = \eta_2, \quad \eta'_2 = -\eta_1 + \frac{1 - 3\varepsilon}{4}\lambda\xi_3 + \frac{9\varepsilon - 7}{2}q_2\xi_2, \quad (3.7)\]
\[\xi'_3 = 0, \quad \eta'_3 = -\frac{3\varepsilon - 1}{4}\xi_2.\]

Then the (NVE) becomes
\[\xi'_1 = \xi_2 - \frac{1 + 3\varepsilon}{2}q_1\xi_1, \quad \eta'_1 = \frac{1 + 3\varepsilon}{2}(p_1\xi_1 + q_1\eta_1),\]
\[\xi'_2 = \eta_2, \quad \eta'_2 = -\eta_1 + \frac{9\varepsilon - 7}{2}q_2\xi_2. \quad (3.8)\]

Again reducing (3.8) to a single equation gives
\[\xi^{(4)}_1 - \frac{10}{(z + \frac{\alpha}{\lambda})^2}\xi''_1 + \frac{20}{(z + \frac{\alpha}{\lambda})^3}\xi'_1 - \left[ \frac{20}{(z + \frac{\alpha}{\lambda})^4} + \lambda \left( z + \frac{\alpha}{\lambda} \right) \right] \xi_1 = 0. \quad (3.9)\]

We make some transformations in order to put this equation in appropriate form. First we shift the independent variable \(z + \frac{\alpha}{\lambda} \rightarrow z\). Then we make a change \(x = \frac{\lambda^3}{5}\), which is a finite branched covering that preserves the identity component of the Galois group. As a result we get
\[\xi^{(4)}_1 + \frac{24}{5x}\xi^{(3)}_1 + \frac{86}{25x^2}\xi''_1 + \frac{4}{125x^3}\xi'_1 - \left( \frac{4}{125x^4} + \frac{1}{x^3} \right) \xi_1 = 0.\]

Finally, we put \(u := x^{2/5}\xi_1\) from where denoting \(\delta = xd/dx\) we obtain
\[L_2u := \delta \left( \delta - \frac{2}{5} - 1 \right) \left( \delta + \frac{1}{5} - 1 \right) \left( \delta + \frac{2}{5} - 1 \right) u - xu = 0. \quad (3.10)\]

The operator \(L_2\) in (3.10) is a particular case of so called Kloosterman operators
\[Kl = \prod_{i=1}^{n}(\delta - a_i) + \lambda x \quad (3.11)\]

with \(n = 4, \lambda = -1, a_1 = 0, a_2 = 7/5, a_3 = 4/5, a_4 = 3/5\). Katz has found that (see [9] Theorem 4.5.3, pp. 59-60) the Galois group of \(L_2u = 0\) is actually \(\langle e^{2\pi i/5}\rangle\text{Sp}(4,\mathbb{C})\), which is noncommutative (more generally, \(\langle e^{2\pi i/N}\rangle\text{Sp}(n,\mathbb{C})\) denotes a subgroup of \(\text{GL}(n,\mathbb{C})\), where \(\text{Sp}(n,\mathbb{C})\) is clear and \(\langle e^{2\pi i/N}\rangle\) is a cyclic group of order \(N\), \(N\) is prime to \(n\), consisting of matrices of the kind \(e^{2\pi ik/N}\mathbb{I}\), \(0 \leq k \leq N\) and \(\mathbb{I}\) is the unit matrix).

Hence, by Theorem 2 the Hamiltonian system (1.9) is not integrable in a neighborhood of the particular solution (3.6).
To finish the proof we recall that one can obtain the family (I) \( \gamma/\lambda = 3k, k \in \mathbb{Z} \) by applying the canonical transformations \( T^k, T^{-k} \) on \( \gamma/\lambda = 0 \) and \( w = 0 \). Since for this member we have just proved that the Hamiltonian system is non-integrable by rational first integrals, any member of the family (I) satisfies the same property. Applying the same arguments, we conclude the non-integrability of the Hamiltonian systems corresponding to the every member of the second family (II). This ends the proof of Theorem 1.

4 Stokes matrices

In this section we explicitly compute the differential Galois group of (3.10) using the approach taken by Duval and Mitschi [4, 15, 16] based on obtaining the topological generators of the Galois group, namely the formal and analytical invariants of the equation. We focus only on the equation (3.10) because the other equation (3.5) after the change of the independent variable \( x = z^5/(5^4\lambda^4) \) becomes

\[
\delta \left( \frac{2}{5} \right) \left( \frac{3}{5} - 1 \right) \left( \frac{4}{5} - 1 \right) \xi_1 - x\xi_1 = 0,
\]

which is of similar kind as (3.10).

The following equation

\[
D_{qp}(y) = \left[ (-1)^{q-p} x^p \prod_{j=1}^{p} (\delta + \mu_j) - \prod_{j=1}^{q} (\delta + \nu_j - 1) \right] y = 0, \tag{4.1}
\]

is called generalized confluent hypergeometric equation since it generalizes the classical confluent Kummer equation. Here \( \delta = xd/dx, 0 \leq p \leq q, \mu_j, \nu_j \in \mathbb{C}, \mu_i - \mu_j \notin \mathbb{Z} \). For this equation the point 0 is a regular singularity and \( \infty \) is an irregular singularity. For such kind of equations the local Galois group \( G_0 \) is a subgroup of \( G_\infty \), so the global Galois group is \( G = G_\infty \). In what follows we need some notations:

1) \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n; \)
2) \( \langle \alpha \rangle_m = \prod_{j=1}^{m} (\alpha_j + 1) \ldots (\alpha_j + m - 1); \)
3) For \( a \in \mathbb{C}^p, b \in (\mathbb{C} \setminus \mathbb{Z})^q \), let :

\[
_{p}F_{q}(a; b | x) = \sum_{n \geq 0} \langle a \rangle_n \langle b \rangle_n \frac{x^n}{n!} \tag{4.2}
\]

be the generalized hypergeometric series and

\[
G_{p q}^{m n} \left( x \mid \frac{a}{b} \right) = \frac{1}{2\pi i} \int_{C} \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} x^s ds, \tag{4.3}
\]
be the Meijer G-function \[4\] and \( C \) is a suitable path in the complex plane. We can always assume that \( p < q \). It is clear that \( _pF_q \) can be expressed as a \( G \)-function.

4) Let \( \tilde{C} \) be the Riemann surface of the logarithm, \( l_1, l_2 \in \mathbb{R} : l_1 < l_2 \). By a sector we understand the following set:

\[ \theta(l_1, l_2) := \{ x \in \tilde{C} \mid \arg x \in (l_1, l_2) \}; \]

5) \( \sigma = q - p, \quad \zeta = e^{2\pi i / \sigma}, \quad \lambda = \frac{1}{2}(\sigma + 1) + \sum_{j=1}^{p} \mu_j - \sum_{j=1}^{q} \nu_j. \)

If \((\nu_j - \mu_k) \notin \mathbb{Z}\) for \( j = 1, \ldots, q \) and \( k = 1, \ldots, p \), there exists a basis of solutions to \( (4.1) \) near \( x = 0 \) given in terms of \( G_{p,q}^1 \) or \( p_{F_{q-1}} \). Similarly, in a neighborhood of \( x = \infty \) there exists a fundamental system of solutions expressed in terms of \( G_{p,q}^{1,p} \) and \( G_{p,q}^{0,0} \) (see Meijer, Luke \[11\]). We specialize them for our particular case.

Duval and Mitschi have calculated the formal invariants (formal monodromy and exponential torus) and the analytic invariants (Stokes matrices) for some families of the equations \( (4.1) \) assuming \( p \geq 1 \). Their calculations can be adapted to the case \( p = 0 \) in which we are. It is obvious that the Kloosterman equations are nothing but \( D_{q0} \) type equation. Note that these equations are generically irreducible (see Beukers, Brownawell, Heckmann \[2\] and Duval, Mitschi \[11\] for the proof).

In the rest of the section we carry out detailed calculations in our particular case for the reader’s convenience.

Let us rewrite \( L_2 \) from \( (3.10) \) in the form

\[ L_2 = (\delta + \nu_1 - 1)(\delta + \nu_2 - 1)(\delta + \nu_3 - 1)(\delta + \nu_4 - 1) - x \quad (4.4) \]

which is of type \( D_{40} \) with \( \nu_1 = 1, \nu_2 = -2/5, \nu_3 = 1/5, \nu_4 = 2/5 \).

The fundamental system of solutions near \( x = 0 \) of \( L_2 \xi_1 = 0 \) is

\[ \left\{ _0F_3 \left( \frac{-2}{5}, \frac{1}{5}, \frac{2}{5} \mid x \right); \quad _0F_3 \left( \frac{4}{5}, \frac{8}{5}, \frac{3}{5} \mid x \right); \quad _0F_3 \left( \frac{2}{5}, \frac{6}{5}, \frac{9}{5} \mid x \right); \quad _0F_3 \left( \frac{8}{5}, \frac{9}{5}, \frac{12}{5} \mid x \right) \right\}. \quad (4.5) \]

Then the monodromy \( M_0 \) around \( x = 0 \) becomes

\[ M_0 = \text{diag}(1, e^{2\pi i/5}, e^{2\pi i 3/5}, e^{2\pi i 4/5}). \quad (4.6) \]

Since 0 is regular singular, the monodromy generates the local Galois group \( G_0 \).

Recall that in our case we have \( \sigma = 4, \zeta = i, \lambda = 5/2 - 6/5 \). We prefer using letters \( \zeta \) and \( \lambda \) instead their particular values.

Let us turn to the description of the local Galois group \( G_\infty \). To define a fundamental system near \( x = \infty \) we need one more function. Let \( C \) be a path in the complex plane, connecting \(-i\infty \) and \( i\infty \) and enclosing the points \( n - \nu_j, j = 1, \ldots, 4; n \in \mathbb{N} \). The following function

\[ G_0(x) := G^{40}_{0,4} \left( x \mid \frac{\nu}{\nu} \right) = \frac{1}{2\pi i} \int_C \Gamma(1 - \nu - s)x^s ds \quad (4.7) \]

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is a solution of $L_2\xi_1 = 0$, holomorphic in $\theta(-2\pi, 2\pi)$. The analytic continuation of $G_0$ in the sector $\theta(-5\pi, 5\pi)$ admits the following asymptotic expansion at $x \to \infty$ (Propositions 1.2, 1.3 Duval, Mitschi [4] or Luke [11])

$$e^{-4x^{1/4}}x^{\lambda/4}\Theta(x),$$

where $\Theta$ is a formal series in $x^{-1/4}$. It is straightforward that

$$G_0(xe^{-2\pi ih}), \quad h \in \mathbb{Z}$$

are also solutions to the same equation. In order to get a fundamental solution near $x = \infty$, one needs to pick four of them in our case. Next, we need a particular version of Meijer’s Expansion Theorem which expresses every G-function as a finite sum of the functions $G_0(xe^{-2\pi ih})$.

**Proposition 1.** (see [4] Proposition 1.5). Let $x \in \theta(3\pi, 5\pi)$. Then the following identity holds

$$G_0(x) = A_1 G_0(xe^{-2\pi i}) + A_2 G_0(xe^{-4\pi i}) + B_0 G_0(xe^{-8\pi i}) + B_1 G_0(xe^{-6\pi i}),$$

where

$$A_h = -\frac{d^h}{dx^h}((1-xe^{-2\pi ih}))_{x=0}, \quad B_h = e^{2\pi i\lambda} \frac{d^h}{dx^h}((1-xe^{-2\pi ih}))_{x=0}.$$  

The formal solutions of $D_{q\nu}$ at $\infty$ are known [2, 11] and can be verified by computer in our particular case. It is more convenient from now on to use a new variable $t = x^{1/4}$. Denote by $L_2$ the equation obtained after the change of variable. Then we have

$$\Theta_1(t) = e^{-4\zeta^{-1}t}\Theta(\zeta^{-1}t), \quad \Theta_0(t) = e^{-4\zeta^0t}\Theta(\zeta^0t),$$

$$\Theta_2(t) = e^{-4\zeta^1t}\Theta(\zeta^1t), \quad \Theta_3(t) = e^{-4\zeta^2t}\Theta(\zeta^2t),$$

where $\Theta(t) \in \mathbb{C}[t^{-1}]$. We denote the basis of formal solutions at $\infty$ of $L_2$ by

$$\Sigma(t) = \{\Theta_{-1}(t), \Theta_{0}(t), \Theta_{1}(t), \Theta_{2}(t)\}.$$  

In this basis the formal monodromy is $\Sigma(\zeta t) = \Sigma(t)\tilde{M}_\infty$ :

$$\tilde{M}_\infty = e^{2\pi i\lambda/4} \begin{pmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

Since in our case

$$q_{-1}(t) = -4\zeta^{-1}t, \quad q_0(t) = -4\zeta^0t, \quad q_1(t) = -4\zeta^1t, \quad q_2(t) = -4\zeta^2t,$$
the exponential torus is $\mathcal{T} \cong (\mathbb{C}^*)^2$ and can be presented by

$$\mathcal{T} = \{ \text{diag}(t_1, t_2, t_1^{-1}, t_2^{-1}) \}, \quad t_1, t_2 \in \mathbb{C}^*. \tag{4.15}$$

The Stokes rays can easily calculated from (4.14) to be

$$\arg t = n\frac{\pi}{4}, \quad n = 0, \ldots, 7. \tag{4.16}$$

Similarly the singular rays $d_s$, i.e., the rays bisecting the sectors $\text{Re}(q_i(t) - q_j(t)) < 0$ turn out to be the same as (4.16).

Let us define the sectors

$$\theta_n = \theta \left( -\frac{\pi}{2} + \frac{n-1}{4}\pi, \frac{n}{4}\pi \right), \quad n = 0, \ldots, 7. \tag{4.17}$$

The following Proposition is proven by Ramis [22] for the general confluent hypergeometric equation $D_{q_p}$. We reformulate it for our particular case.

**Proposition 2.** For every sector $\theta_n, n \in [0, 1, 2, \ldots, 7]$, there exists unique basis of solutions $\Sigma_n(t)$ of $L_2$ in $\theta_n$ with asymptotic expansion at $\infty \Sigma(t)$.

This solution corresponds to "summation" of $\Sigma(t)$ along a direction in the sector $\theta \left( \frac{n-1}{4}\pi, \frac{n}{4}\pi \right)$. As we will see in the sequel we won’t need summation because there exist fundamental systems of actual solutions in $\theta_n$.

In these notations the Stokes matrix corresponding to the singular ray $\frac{n}{4}\pi, n \in [0, 1, \ldots, 7]$ is defined via

$$\Sigma_n(t) = \Sigma_{n+1}(t)S_n, \quad t \in \theta_n \bigcap \theta_{n+1}. \tag{4.18}$$

**Proposition 3.** Suppose $n$ and $n'$ belong to $[0, 1, \ldots, 7]$ and $n' - n = 2$. Then

$$\Sigma_{n'}(\zeta t) = \Sigma_n(t)\hat{M}_\infty.$$  

**Proof:** If $t \in \theta_n$, then $\zeta t \in \theta_{n'}$ and $\Sigma_{n'}(\zeta t)\hat{M}_\infty^{-1}$ is a basis of solutions to $L_2$ in $\theta_n$ which admits the asymptotic expansion $\Sigma_n(t)$. The uniqueness from Proposition [22] gives the desired result.

**Proposition 4.** Let $n \in [0, 1, \ldots, 7]$ and $n = 2m + r$. Then

$$S_n = \hat{M}_{\infty}^{-m} S_r \hat{M}_{\infty}^{m}. \tag{4.19}$$

**Proof:** From the relation (4.18) after changing variables we obtain $\Sigma_n(\zeta t) = \Sigma_{n+1}(\zeta t)S_n$. Proposition [3] gives that

$$\Sigma_{n-2}(t)\hat{M}_\infty = \Sigma_{n+1-2}(t)\hat{M}_\infty S_n.$$  

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This procedure repeated $m$ times yields

$$\Sigma_r(t)\tilde{M}_\infty^m = \Sigma_{r+1}(t)\tilde{M}_\infty^m S_n.$$ 

But by definition we have $\Sigma_r(t) = \Sigma_{r+1}(t)S_r$ from where the result is immediate. 

\[ \square \]

This Proposition reduces the calculation of the Stokes matrices to $S_0$ and $S_1 := S_{\pi/4}$ only.

**Proposition 5.** The function $V(t) = \Theta_0(t^4)$ is asymptotic to $\Theta_0$ in $\theta\left(-\frac{3\pi}{4}, \frac{5\pi}{4}\right)$ when $t \to \infty$.

This is a reformulation of (4.7) and (4.8) in terms of the variable $t$.

\[ \square \]

**Proposition 6.** If $t \in \theta\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$, the following identity holds

$$V(t) - e^{2\pi i \lambda} V(te^{-2\pi i}) = A_1 V(t\zeta^{-1}) + A_2 V(t\zeta^{-2}) + B_1 V(te^{-2\pi i \zeta}). \quad (4.19)$$

This a version of the formula (4.10) in terms of the variable $t$ ($B_0 = e^{2\pi i \lambda}, \zeta = e^{2\pi i/4}$).

\[ \square \]

Next, we find fundamental systems of actual solutions near $\infty$ only in $\theta_0, \theta_1, \theta_2$ since we need only $S_0$ and $S_1$.

**Proposition 7.** (see Proposition 4.8 [4]) Let $n = 0, 1, 2$. The following sets of solutions form fundamental systems of actual solutions in $\theta_n$:

$$\Sigma_n(t) := \{Y_{n,j}(t), j \in \{-1, 0, 1, 2\}\},$$

where

$$Y_{n,-1}(t) = \zeta^\lambda V(t\zeta^{-1}), \quad n = 0, 1, 2,$$

$$Y_{n,0}(t) = V(t), \quad n = 0, 1, 2,$$

$$Y_{n,1}(t) = \zeta^{-\lambda} V(t\zeta), \quad n = 0, 1. \quad (4.20)$$

$$Y_{0,2}(t) = \begin{cases} 
\zeta^{2\lambda} [V(t\zeta^{-2}) - A_1 V(t\zeta)], & t \in \theta(-\frac{3\pi}{4}, \frac{\pi}{4}), \\
\zeta^{2\lambda} [e^{2\pi i \lambda} V(te^{-2\pi i \zeta^2}) + A_2 V(t) + B_1 V(t\zeta^{-1})], & t \in \theta(-\frac{\pi}{4}, \frac{3\pi}{4}). 
\end{cases} \quad (4.23)$$

$$Y_{1,2}(t) = \begin{cases} 
\zeta^{2\lambda} [V(t\zeta^{-2}) - A_1 V(t\zeta) - A_2 V(t)], & t \in \theta(-\frac{\pi}{4}, \frac{3\pi}{4}), \\
\zeta^{2\lambda} [e^{2\pi i \lambda} V(te^{-2\pi i \zeta^2}) + B_1 V(t\zeta^{-1})], & t \in \theta(-\frac{\pi}{4}, \frac{3\pi}{4}). 
\end{cases} \quad (4.24)$$

$$Y_{2,1}(t) = \begin{cases} 
\zeta^{2\lambda} [V(t\zeta) - A_1 V(t)], & t \in \theta(-\frac{\pi}{4}, \frac{3\pi}{4}), \\
\zeta^{2\lambda} [e^{2\pi i \lambda} V(te^{-2\pi i \zeta^2}) + A_2 V(t\zeta^{-1}) + B_1 V(t\zeta^{-2})], & t \in \theta(-\frac{\pi}{4}, \frac{\pi}{4}). 
\end{cases} \quad (4.25)$$

$$Y_{2,2}(t) = \begin{cases} 
\zeta^{2\lambda} [V(t\zeta^2) - A_1 V(t\zeta) - A_2 V(t) - B_1 V(t\zeta^{-1})], & t \in \theta(-\frac{\pi}{4}, \frac{\pi}{4}), \\
\zeta^{2\lambda} e^{2\pi i \lambda} V(te^{-2\pi i \zeta^2}), & t \in \theta(-\frac{\pi}{4}, \pi). 
\end{cases} \quad (4.26)$$

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Proposition 8. \((4.19)\) The Stokes matrices \(S_0\) and \(S_1\) are given by the following formulas:

\[
\begin{align*}
S_0 &= \mathbb{I} + \zeta^{-2\lambda}A_2E_{24}, \\
S_1 &= \mathbb{I} + \zeta^{-\lambda}A_1E_{23} + e^{-i\pi\lambda}\zeta^{-\lambda}B_1E_{14}.
\end{align*}
\]

**Proof:** By definition \(\Sigma(t) = \Sigma(t+1)\Sigma_n\), \(t \in \theta_n \cap \theta_{n+1}\). For \(t \in \theta_0 \cap \theta_1 = \theta\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\) from Proposition 7 we get

\[
\begin{align*}
Y_{0,-1}(t) &= Y_{1,-1}(t), & Y_{0,0}(t) &= Y_{1,0}(t), \\
Y_{0,1}(t) &= Y_{1,1}(t), & Y_{0,2}(t) &= Y_{1,2}(t) + \zeta^{-2\lambda}A_2Y_{1,0}(t).
\end{align*}
\]

Then \(\{Y_{0,-1}(t), Y_{0,0}(t), Y_{0,1}(t), Y_{0,2}(t)\} = \{Y_{1,-1}(t), Y_{1,0}(t), Y_{1,1}(t), Y_{1,2}(t)\}\) gives \(S_0\). Similarly, for \(t \in \theta_1 \cap \theta_2 = \theta\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)\) we have

\[
\begin{align*}
Y_{1,-1}(t) &= Y_{2,-1}(t), & Y_{1,0}(t) &= Y_{2,0}(t), \\
Y_{1,1}(t) &= Y_{2,1}(t) + \zeta^{-\lambda}A_1Y_{2,0}, & Y_{1,2}(t) &= Y_{2,2}(t) + \zeta^{-2\lambda}B_1\zeta^{-\lambda}Y_{2,-1}(t).
\end{align*}
\]

Then \(\{Y_{1,-1}(t), Y_{1,0}(t), Y_{1,1}(t), Y_{1,2}(t)\} = \{Y_{2,-1}(t), Y_{2,0}(t), Y_{2,1}(t), Y_{2,2}(t)\}\) gives \(S_1\) since \(\zeta^{-2\lambda} = e^{-i\pi\lambda}\).

From the above Proposition we know the Stokes matrices \(S_0\) and \(S_1\). In our case easy calculations give that

\[
\begin{align*}
S_0 &= \mathbb{I} + aE_{24}, & S_1 &= \mathbb{I} + bE_{23} + cE_{14},
\end{align*}
\]

Using Proposition 6 we combine some of them in order to obtain proper asymptotic in \(\theta_n\). It remains to verify the validity of the formulas \((4.23), (4.24), (4.25), (4.26)\) in the intersection of their definition intervals. We check only \((4.23)\) since the rest are treated in the same way. So, we need to verify that

\[
V(t\zeta^2) - A_1V(t\zeta) = e^{2\pi\lambda}V(te^{-2\pi\zeta^2}) + A_2V(t) + B_1V(t\zeta^{-1})
\]

is valid in \(\theta\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)\). Using \((4.19)\) from Proposition 6 \((\zeta^4 = e^{2\pi i})\) and making the change \(t \to t\zeta^2\) gives the needed result.

\[\square\]
where \( a = 2i \), \( c = i e^{-\lambda} e^{2\pi i/5} \), \( b = ic \).

Then Proposition \( \square \) gives the other Stokes matrices obtained from \( S_0, S_1 \):

\[
S_2 = I + aE_{13}, \quad S_4 = I + aE_{42}, \quad S_6 = I + aE_{31},
\]

\[
S_3 = I + cE_{12} + icE_{43}, \quad S_5 = I + cE_{41} + icE_{32}, \quad S_7 = I + cE_{34} + icE_{21}.
\]

(4.29)

Now consider the (connected) subgroup topologically generated by the Stokes matrices and the exponential torus \( G_s = \{ S_j, \mathcal{T} \} \) which is normal in the Galois group \( G \) [15]. Hence, by Theorem 5 \( G \) is topologically generated by \( G_s \) and \( \hat{M}_\infty \). Let \( \mathcal{G}_s \) be the Lie algebra of \( G_s \) (\( \mathcal{G}_s \subset \text{sl}(4, \mathbb{C}) \)). Denote \( s_j \in \mathcal{G}_s \) such that \( S_j = \exp s_j \). Recall that \( a \in \mathbb{C}, c \in e^{2\pi i/5} \mathbb{C} \). Additionally we have that \( [s_2, s_6] = a^2 d_1, d_1 = E_{11} - E_{33} \) and \( [s_0, s_4] = a^2 d_2, d_2 = E_{22} - E_{44} \).

The Lie algebra \( \mathcal{G}_s \) admits the following basis:

\[
\mathcal{B} := \{ s_0, s_2, s_4, s_6, d_1, d_2, s_1, s_3, s_5, s_7 \}.
\]

(4.30)

Hence, \( \mathcal{G}_s \) consists of all matrices \( V \) such that \( V^T J_1 + J_1 V = 0 \), where \( J_1 \) is the following skew-symmetric matrix

\[
J_1 = \begin{pmatrix}
0 & 0 & i\beta & 0 \\
0 & 0 & 0 & \beta \\
-i\beta & 0 & 0 & 0 \\
0 & -\beta & 0 & 0
\end{pmatrix}, \quad \beta^4 = -1.
\]

Therefore, we get that \( G_s \cong \langle e^{2\pi i/5} \rangle \text{Sp}(4, \mathbb{C}) \). Note that from (4.6) \( G_0 \) is indeed a subgroup of \( G_s \). Furthermore, \( \hat{M}_\infty^T J_1 \hat{M}_\infty \neq J_1 \), so we obtain that \( G^0 = \langle e^{2\pi i/5} \rangle \text{Sp}(4, \mathbb{C}) \) as claimed.

5 Non-integrability of the second and the third members of the Painlevé II - hierarchy

We have proved that a fourth-order Painlevé equation is non-integrable in Liouville sense for discrete family of parameters. It turns out that the variational equations along certain rational solutions are well known generalized hypergeometric equations whose Galois groups can be found. Since generically these groups are large then the non-integrability comes from the Morales-Ruiz–Ramis–Simó theorem.

We briefly mention an interesting relation concerning the linear equations that have appeared in this paper. Let \( X \) be a smooth complex projective Fano variety. One can define quantum differential equations on \( X \) (see e.g. [6, 17] and the references there for details). In the case when the quantum equation is a linear ordinary differential equation J. Morales and M. van der Put [17] confirm Dubrovin’s conjecture that the Gram matrix of \( X \) coincides with the Stokes matrix of the quantum differential equation (up to certain equivalence). It appears that for \( X = \mathbb{P}^{n-1} \) the quantum differential operator is Airy type operator \( \partial^n - z \),

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and for the weighted projective spaces \( P(w_0, \ldots, w_n) \) the quantum differential operator is of Kloosterman type or \( D_{q0} \) for certain \( q \). The classical Stokes matrices are computed then for these operators using "multisummation" and the "monodromy identity" (see [17] for details).

It is interesting to note that generalized hypergeometric functions and generalized confluent hypergeometric equations are related also with other Painlevé equations. For instance, the classical dilogarithm
\[
\text{Li}_2(z) = -\int_0^z \ln(1-s) \frac{ds}{s},
\]
whose nontrivial monodromy plays essential role in proving the non-integrability of some Painlevé VI equations studied by Horozov and Stoyanova [8], is related to the generalized hypergeometric function as
\[
\text{Li}_2(z) = z_3 F_2(1, 1; 2, 2|z).
\]
The polylogarithms \( \text{Li}_k \) have similar representations.

Let us turn our attention to other higher-order Painlevé equations which admit Hamiltonian formulation. Consider the \( P_{\Pi} \)-hierarchy which is given by
\[
P^{(n)}_{\Pi} : \left( \frac{d}{dz} + 2w \right) L_n[w' - w^2] + \sum_{l=1}^{n-1} t_l \left( \frac{d}{dz} + 2w \right) L_l[w' - w^2] = z w + \alpha_n, \quad n \geq 1, \quad (5.1)
\]
where \( L_n \) is the operator defined by the recursion relation
\[
\frac{d}{dz} L_{n+1} = \left[ \frac{d^3}{dz^3} + 4(w' - w^2) \frac{d}{dz} + 2(w' - w^2) \right] L_n; \quad L_0[w' - w^2] = \frac{1}{2} \quad (5.2)
\]
and \( t_l \) and \( \alpha_n \) are arbitrary complex parameters. A particular member of (5.1) is a nonlinear ODE of order \( 2n, n \geq 1 \). Some authors consider all \( t_l \) trivial. The first three members of the \( P_{\Pi} \)-hierarchy are:
\[
P^{(1)}_{\Pi} : w'' - 2w^3 = z w + \alpha_1, \quad (5.3)
\]
\[
P^{(2)}_{\Pi} : w^{(4)} - 10w(ww'' + w'^2) + 6w^5 + t_1(w'' - 2w^3) = z w + \alpha_2, \quad (5.4)
\]
\[
P^{(3)}_{\Pi} : w^{(6)} - 14w^{(4)}w^2 - 56w^{(3)}w'w + 70w^{(2)}(w^4 - w'^2) + 140w^3w'^2 - 42w(w'')^2 - 20w^7 + t_1[w'' - 2w^3] + t_2[w^{(4)} - 10w(ww'' + w'^2) + 6w^5] = z w + \alpha_3. \quad (5.5)
\]
The equation \( P^{(2)}_{\Pi} \) appears in Cosgrove [3] as F-XVII.

We are interested in integrability of the Hamiltonian systems corresponding to these equations. The Hamiltonian for the \( P^{(1)}_{\Pi} \) is known long ago from Okamoto [21] (and also for the other classical Painlevé equations). The Hamiltonian structure for the \( P_{\Pi} \)-hierarchy is found by Mazzocco and Mo [13]. We study the Liouville integrability of the Hamiltonian systems corresponding to the first three members of the \( P_{\Pi} \)-hierarchy which are manageable.

Consider first the Hamiltonian for the \( P^{(1)}_{\Pi} \), namely
\[
H^{(1)} = 4p^2 + \frac{1}{4}q + \frac{1}{4}pq^2 + 2pz - \frac{1}{2}q\alpha_1, \quad (5.6)
\]
where \( q = 4w, \ p = \frac{1}{2}(w' - w^2 - \frac{s}{w}) \). Extending (5.6) in a natural way to a two degrees of freedom autonomous Hamiltonian system \( \hat{H}_1 = H^{(1)} + F \), one can find \(' = d/ds\) the corresponding equations

\[
\begin{align*}
q' &= 8p + \frac{1}{4}q^2 + 2z, \quad z' = 1, \\
p' &= -\frac{1}{4} - \frac{1}{2}pq + \frac{1}{2} \alpha_1, \quad F' = -2p.
\end{align*}
\]

The system (5.7) admits the following particular solution when \( \alpha_1 = 0 \)

\[
q = 0, \quad p = -\frac{1}{4}s, \quad z = s, \quad F = \frac{s^2}{4} + F_0.
\]

The (NVE) along (5.8) is

\[
\xi''_1 = z\xi_1.
\]

This is the Airy equation whose Galois group is \( G = \text{SL}(2, \mathbb{C}) \cong \text{Sp}(2, \mathbb{C}) \). Then the Hamiltonian system (5.7) is not integrable with rational first integrals, but we know that from Morales-Ruiz [20].

Next we consider the Hamiltonian for the PII:

\[
H^{(2)} = \frac{q_2}{16} + 2zp_2 - 16p_1^2p_2 + 16p_2^2 + \frac{q_1q_2p_2}{8} + \frac{p_1p_2q_2^2}{16} + \frac{\alpha_2(p_1q_2 - q_1)}{8} + t_1(8p_1 - t_1)p_2,
\]

where \( q_j, p_j, j = 1, 2 \) are expressible via \( w \) and its derivatives. Extending as usual to a three degrees of freedom autonomous Hamiltonian system \( \hat{H}_2 = H^{(2)} + F \) we get

\[
\begin{align*}
q'_1 &= -32p_1p_2 + \frac{1}{16}p_2q_2 + \frac{1}{8}q_2\alpha_2 + 8t_1p_2, \\
q'_2 &= 2z - 16p_1^2 + 32p_2 + \frac{1}{8}q_1q_2 + \frac{1}{16}p_1q_2^2 + t_1(8p_1 - t_1), \\
p'_1 &= -\frac{1}{8}p_2q_2 + \frac{1}{8}\alpha_2, \\
p'_2 &= -\frac{1}{16} - \frac{1}{8}p_2q_1 - \frac{1}{8}p_1p_2q_2 - \frac{1}{8}\alpha_2p_1, \\
z' &= 1, \quad F' = -2p_2.
\end{align*}
\]

The system (5.11) admits the following particular solution when \( \alpha_2 = 0 \)

\[
q_1 = q_2 = 0, \quad p_1 = \frac{t_1}{4}; \quad p_2 = -\frac{s}{16}; \quad z = s, \quad F = \frac{s^2}{16} + F_0.
\]

The (NVE) along the solution (5.12) reduced to a single linear equation is

\[
\xi^{(4)}_1 - \frac{5}{s}\xi^{(3)}_1 + \left(\frac{12}{s^2} - \frac{t_1s}{16}\right)\xi''_1 + \left(\frac{t_1}{16} - \frac{12}{s^3}\right)\xi'_1 - \frac{s^3}{256}\xi_1 = 0.
\]
Here we take the case \( t_1 = 0 \) which is simpler. After introducing the new independent variable \( z = s^7/(2^87^4) \) the above equation becomes

\[
\delta \left( \delta + \frac{2}{7} - 1 \right) \left( \delta + \frac{3}{7} - 1 \right) \left( \delta + \frac{5}{7} - 1 \right) \xi_1 - z\xi_1 = 0,
\]

(5.14)

which is an equation of type \( D_{40} \xi_1 = 0 \) with \( \nu_1 = 1, \nu_2 = 2/7, \nu_3 = 3/7, \nu_4 = 5/7 \). In a similar way as in Section 4 (or referring to Katz [9]) one can obtain that the identity component of the Galois group of (5.14) is \( G^0 = \langle e^{2\pi i/7} \rangle \text{Sp}(4, \mathbb{C}) \), which is not commutative. Hence, the Hamiltonian system corresponding to the higher-order Painlevé equation \( P^{(2)}_{\text{II}} \) is not integrable in Liouville sense.

Finally, let us write the Hamiltonian for \( P^{(3)}_{\text{II}} \).

\[
H^{(3)} = 64p_1^4 - 192p_1^2p_2 + 128p_1p_3 + \frac{1}{64}p_3q_3^2 - \frac{1}{64}p_1q_2^2 + 64p_2^2 - \frac{1}{32}q_1q_2 + 2zp_1 + \frac{q_3}{64} - \frac{1}{32}\alpha_3q_3 + 8t_1(p_1^2 - p_2) + t_2(4p_1^2t_2 - 4p_2t_2 - 32p_3^4 + 64p_1p_2 - 2p_1t_1).
\]

(5.15)

Extending as usual to a four degrees of freedom autonomous Hamiltonian system \( \hat{H}_3 = H^{(3)} + F \), we obtain

\[
q_1' = 256p_1^3 - 384p_1p_2 + 128p_3 - \frac{q_2^2}{64} + 2z + 16p_1t_1 + 8p_1t_2^2 - 96t_2p_1^2 + 64p_2t_2 - 2t_1t_2, \\
q_2' = -192p_1^2 + 128p_2 - 8t_1 - 4t_2^2 + 64p_1t_2, \\
q_3' = 128p_1 + \frac{1}{64}q_3^2, \\
p_1' = \frac{1}{32}q_2, \\
p_2' = \frac{1}{32}p_1q_2 + \frac{1}{32}q_1, \\
p_3' = -\frac{1}{32}p_3q_3 - \frac{1}{64} + \frac{1}{32}\alpha_3, \\
z' = 1, \quad F' = -2p_1.
\]

(5.16)

Here we consider only the case \( t_1 = t_2 = 0 \). When \( \alpha_3 = 0 \) the system (5.16) admits the following particular solution

\[
q_1 = q_2 = q_3 = 0, \quad p_1 = p_2 = 0, \quad p_3 = \frac{s}{64}, \quad z = s, \quad F = F_0 = \text{const}.
\]

(5.17)

The (NVE) along the solution (5.17), reduced to a single linear equation becomes

\[
\xi_1^{(6)} - \frac{4}{s}\xi_1^{(5)} + \frac{12}{s^2}\xi_1^{(4)} - \frac{24}{s^3}\xi_1^{(3)} + \frac{24}{s^4}\xi_1^{(2)} - s\xi_1 = 0.
\]

(5.18)
After changing the independent variable by $x = s^7/7^6$ we obtain
\[
\delta \left( \delta + \frac{1}{7} - 1 \right) \left( \delta + \frac{2}{7} - 1 \right) \left( \delta + \frac{3}{7} - 1 \right) \left( \delta + \frac{4}{7} - 1 \right) \left( \delta + \frac{6}{7} - 1 \right) \xi_1 - x \xi_1 = 0, \quad (5.19)
\]
which is an equation of type $D_{60} \xi_1 = 0$ with $\nu_1 = 1, \nu_2 = 1/7, \nu_3 = 2/7, \nu_4 = 3/7, \nu_5 = 4/7, \nu_6 = 6/7$. We proceed in a similar way as in Section 4. Here
\[
\sigma = q - p = 6, \quad \zeta = e^{2\pi i/6}, \quad \lambda = \frac{1}{2} - \frac{2}{7}.
\quad (5.20)
\]
The local monodromy at 0 is clear. It is more convenient to use a new variable $t = x^{1/6}$.

The basis of the formal solutions is straightforward (see for instance [2, 4]) and the formal monodromy is
\[
\hat{M}_\infty = e^{2\pi i \lambda/6} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.
\quad (5.21)
\]
The exponential torus is again $\mathcal{T} \cong (\mathbb{C}^*)^2$
\[
\mathcal{T} = \{ \text{diag}(t_1, t_2, t_1^{-1} t_2, t_1^{-1}, t_2^{-1}, t_1) \}, \quad t_1, t_2 \in \mathbb{C}^*.
\quad (5.22)
\]
The Stokes rays and the singular rays are
\[
\arg t = n \frac{\pi}{6}, \quad n = 0, \ldots, 11.
\]
We define the sectors $\theta_n = \theta \left( -\frac{\pi}{2} + \frac{n-1}{6} \pi, \frac{\pi}{2} + \frac{n}{6} \pi \right), \quad n = 0, \ldots, 11$. Again we need only $S_0$ and $S_1 := S_{\pi/6}$ in order to obtain all Stokes matrices. The Stokes matrices $S_0$ and $S_1$ are given by (see Theorem 5.2 [4])
\[
S_0 = I + aE_{45} + bE_{36} + cE_{21}, \quad S_1 = I + dE_{35} + f E_{26},
\quad (5.23)
\]
where $a = \zeta^{-\lambda}A_1, b = \zeta^{-3\lambda}A_3, c = \zeta^\lambda e^{-2\pi i \lambda}B_1, d = \zeta^{-2\lambda}A_2, f = \zeta^{-\lambda} e^{-2\pi i \lambda}B_2$. Using (5.20) and (4.11) we get
\[
f = i \zeta^{-\lambda} e^{2\pi i/7}, \quad d = f^2, \quad c = \frac{i}{j}, \quad b = i, \quad a = i f (f^3 = -1).
\]
The matrices $S_0$ and $S_1$ together with the Proposition 4 for $n = 0, \ldots, 11$ give the rest of the
Stokes matrices

\[ S_2 = \mathbb{I} + ifE_{34} + iE_{25} + \frac{i}{f}E_{16}, \quad S_3 = \mathbb{I} + f^2E_{24} + fE_{15}, \quad S_4 = \mathbb{I} + ifE_{23} + iE_{14} + \frac{i}{f}E_{65}, \]
\[ S_5 = \mathbb{I} + f^2E_{13} + fE_{64}, \quad S_6 = \mathbb{I} + ifE_{12} + iE_{63} + \frac{i}{f}E_{54}, \quad S_7 = \mathbb{I} + f^2E_{62} + fE_{53}, \]
\[ S_8 = \mathbb{I} + ifE_{61} + iE_{52} + \frac{i}{f}E_{43}, \quad S_9 = \mathbb{I} + f^2E_{51} + fE_{42}, \quad (5.24) \]
\[ S_{10} = \mathbb{I} + ifE_{56} + iE_{41} + \frac{i}{f}E_{32}, \quad S_{11} = \mathbb{I} + f^2E_{46} + fE_{31}. \]

Now consider again the (connected) subgroup topologically generated by the Stokes matrices and the exponential torus \( G_s = \{S_j, T\} \). Let \( G_s \) be the Lie algebra of \( G_s \) (\( G_s \subset \text{sl}(6, \mathbb{C}) \)). Denote again \( s_j \in G_s \) such that \( S_j = \exp s_j, \ j = 0, \ldots, 11 \) and \( \tau_1 = E_{11} - E_{33} - E_{44} + E_{66} \) and \( \tau_2 = E_{22} + E_{33} - E_{55} - E_{66} \) which belong to Lie \( T \). Direct calculations yield

\[ [s_0, s_3] = 2iE_{25}, \quad [s_0, s_9] = -2iE_{41}, \quad [s_6, s_9] = 2iE_{52}, \]
\[ [s_2, s_{11}] = -2iE_{36}, \quad [s_2, s_5] = 2iE_{14}, \quad [s_4, s_7] = 2iE_{63}, \]

and hence, \( E_{14}, E_{41}, E_{25}, E_{52}, E_{36}, E_{63} \in G_s \).

Additionally we have that the elements

\[ B_1 := E_{11} - E_{44} = [s_8, s_2] - \tau_2, \quad B_2 := E_{22} - E_{55} = [s_9, s_3] - B_1, \quad B_3 := E_{33} - E_{66} = B_1 - \tau_1, \]

and

\[ B_4 := \frac{1}{f}E_{12} - E_{54} = \frac{i}{f^2}([s_6, \tau_2] - 2iE_{63}), \quad B_5 := \frac{1}{f}E_{23} - E_{65} = -if^2([\tau_1, s_4] - 2iE_{14}), \]
\[ B_6 := fE_{21} - E_{45} = -if^2([\tau_2, s_0] - 2iE_{36}), \quad B_7 := fE_{32} - E_{56} = -if^2([s_{10}, \tau_1] - 2iE_{41}), \]
\[ B_8 := \frac{1}{f^2}E_{16} + E_{34} = \frac{i}{f}([\tau_2, s_2] - 2iE_{25}), \quad B_9 := \frac{1}{f^2}E_{61} + E_{43} = -if([s_8, \tau_2] - 2iE_{52}) \]

also belong to \( G_s \).

Then the Lie algebra \( G_s \) admits the following basis:

\[ \mathcal{B} := \{E_{14}, E_{41}, E_{25}, E_{52}, E_{36}, E_{63}, s_1, s_3, s_5, s_7, s_9, s_{11}, B_j, j = 1, \ldots, 9\}. \quad (5.25) \]

Hence, \( G_s \) consists of all matrices \( V \) such that \( V^T J_1 + J_1 V = 0 \), where \( J_1 \) is the following skew-symmetric matrix

\[ J_1 = \begin{pmatrix} 0 & 0 & 0 & f^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -f^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -f & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \]
and we get that $G \cong \langle e^{2\pi i/7} \rangle \operatorname{Sp}(6, \mathbb{C})$. Note that $\hat{M}_\infty^T J_1 \hat{M}_\infty \neq J_1$, so we obtain only that $G^0 = \langle e^{2\pi i/7} \rangle \operatorname{Sp}(6, \mathbb{C})$ which recovers the Katz’s result. Hence, the Hamiltonian system corresponding to the higher-order Painlevé equation $P^{(3)}_\Pi$ is not integrable in Liouville sense. Summarizing we have the following

Theorem 6. Suppose that

(i) $t_1 = \alpha_2 = 0$. Then the Hamiltonian system corresponding to $P^{(2)}_\Pi$ is not integrable by means of rational first integrals;

(ii) $t_1 = t_2 = \alpha_3 = 0$. Then the Hamiltonian system corresponding to $P^{(3)}_\Pi$ is not integrable by means of rational first integrals.

The study of the other members of the $P_\Pi$-hierarchy is technically involved. However, we think that the (NVE) along certain nontrivial solutions reduced to single equations are of type $D_q \xi = 0$ with $q$ even. Since the identity components of their Galois groups are roughly speaking $\operatorname{Sp}(q, \mathbb{C})$, the autonomous Hamiltonian systems corresponding to these equations are non-integrable.

The result of Theorem 6 can be extended to the entire orbits of parameters using Bäcklund transformations and other special solutions found recently by Sakka [23, 24]. This issue will be addressed elsewhere.

Acknowledgements. The authors thank Ivan Dimitrov for useful discussion. O.C. acknowledges partial support by Grant 059/2014 with Sofia University.

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