Gaffney-Lazarsfeld Theorem for Homogeneous Spaces

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Abstract. We generalize the Gaffney-Lazarsfeld theorem on higher ramification loci of branched coverings of $\mathbb{P}^n$ to homogeneous spaces with Picard number 1.

1. Introduction

Consider a branched covering $f : X \to Y$ between complex projective smooth varieties. The famous purity of the branch locus says that the branch locus of $f$ must be purely codimension 1 [Zar58]. One can analyze the ramification locus $R$ closely in more general cases. After defining the notion of local degree which means the number of sheets coming together, in [GL80], the notion of higher ramification locus is defined as the set whose points have higher local degree. When $X$ is an irreducible projective variety and $Y$ is a projective space $\mathbb{P}^n$, it is proved by Gaffney and Lazarsfeld in [GL80] that the codimension of higher ramification locus is bounded above by the local degree of its points minus one. It is a natural generalization of the purity of branch locus since every point in the ramification locus has local degree at least 2 and the locus has codimension 1. The method in [GL80] uses induction on the dimension of $\mathbb{P}^n$ and the Fulton-Hansen connectedness theorem [FL81, Laz04].

In order to generalize this result to other projective smooth variety $Y$, first one may need to find suitable subvarieties in $Y$ to run the dimension induction, i.e., a subvariety $Z \subset Y$ with similar property as $Y$ and the preimage $f^{-1}(Z)$ is also irreducible. In [Deb96], it is proved that there are natural subvarieties which are also Grassmannians and their pull-backs are irreducible, e.g., for a branched covering $f : X \to Gr(r,n)$, the preimage $f^{-1}(Gr(r,n-1))$ is irreducible, so one may run the dimension induction for Grassmannians but there would be some gaps. The second thing is the Fulton-Hansen connectedness theorem. There are some generalized forms of Fulton-Hansen connectedness theorem: [Han83, Deb96] for Grassmannians and flagmanifolds, [Fal81] for homogenous spaces, [B96] for weighted projective spaces, and [BR08] for products of weighted projective spaces. On the other hand, [KM99] considers some certain homogeneous spaces of Picard number 1 and then show Barth-Lefschetz type theorems(cf. [Laz04], Theorem 3.2.1) which may be applied to Gaffney-Larzarsfeld theorem for smooth varieties. Another direction is that Debarre [Deb95] proves the Gaffney-Lazarsfeld theorem on abelian varieties by using the similar argument as [GL80](see also [Laz04]).
When it comes to a general homogeneous space with Picard number 1, however, there is no natural subvariety which is suitable for running the dimension induction, and the known forms of the Fulton-Hansen connectedness theorem is too weak to apply in general cases. We attack these problems by using the results in \[\text{Kol13}\] to show that the connectedness theorem holds indeed, and we avoid the dimension induction on target variety by using induction on the higher ramification locus instead. The main theorem of this paper is that if \( f : X \to Y \) is a branched covering from an irreducible projective variety to a homogeneous space \( Y \) with Picard number 1, then the higher ramification locus has anticipated codimension bound. As application, we show that if a projective irreducible variety has low degree branched covering to a projective homogeneous space with Picard number one, it is simply connected.

2. Preliminaries

We assume that all varieties are over \( \mathbb{C} \).

Given a finite surjective mapping \( f : X \to Y \) between irreducible varieties and \( Y \) is smooth. The local degree of a point is
\[
e_f(x) := \deg \left( V(x) \to U(f(x)) \right),
\]
where \( x \in X \), \( U(f(x)) \) is a small neighborhood of \( f(x) \), and
\[
f^{-1}(U(f(x))) = \bigcup_{f(x') = f(x)} V(x')
\]
splits as disjoint union of connected neighborhoods of the inverse images. We have the basic formula
\[
\sum_{f(x') = f(x)} e_f(x') = \deg f.
\]
We can define the higher ramification loci as
\[
R_\ell(f) := \{ x \in X \mid e_f(x) > \ell \}.
\]
(See \[\text{GL80}, \text{Laz04}\] Definition 3.4.7., for more detailed description). Note that \( R_1(f) \) is the usual ramification locus.

We also introduce the notion of non-degeneracy in \[\text{Kol13}\] for subvarieties in a homogeneous space. Let \( Y = G/H \) be a simply connected, quasi-projective, homogeneous space. An irreducible subvariety \( Z \subset Y \) is called degenerate if there is a subgroup \( H \subset K \subset G \) such that \( Z \) is contained in a fiber of the natural projection \( p_k : G/H \to G/K \). \( Z \) is non-degenerate if it is not degenerate. The key point here is that every positive dimensional algebraic subvariety in a projective homogeneous space of Picard number 1 is non-degenerate. For a list of homogeneous spaces with Picard number 1, see \[\text{Mok08}\].

3. Main Theorem

The main theorem is the following:

**Theorem.** Let \( X \) be an irreducible projective variety, \( Y \) is a projective homogeneous space \( G/P \). \( G \) is a connected linear algebraic group. \( Y \) has Picard number 1 and dimension \( n \). \( f : X \to Y \) is a branched covering of degree \( d \). Then \( R_\ell(f) \) is nonempty and
\[
codim X R_\ell(f) \leq \ell, \text{ provided } \ell \leq \min\{d - 1, n\}.
\]
Proof. We use induction on $\ell$. For $\ell = 1$, assume that $f : X \to Y$ is unramified. By [Bor91], (11.16), every parabolic subgroup of $G$ is connected and $P$ is maximal, so $G/P$ is simply connected. Therefore, $Y$ is simply connected and $f$ is actually an isomorphism. So if $d \geq 2$, the morphism $f$ must ramify and $\text{codim } R_1(f) = 1$.

Now assume that the result holds for all $\ell \leq k - 1$. Assume that $k \leq \min\{d - 1, n\}$. We show that $R_k(f) \neq \emptyset$. By the hypothesis of $R_{k-1}(f)$, we can pick an irreducible component $S \subset R_{k-1}(f)$ which has dimension $\geq n - k + 1 \geq 1$. Since $Y$ is a projective homogeneous space of Picard number 1, $f(S)$ is nondegenerate as in the definition of [Kol13]. Take $Y^0 = Y \setminus f(R_1(f))$, $X^0 = f^{-1}(Y^0)$ the unramified part of $f$, and $g \in G$ a general element in the action group $G$ of $Y$ such that $(g \circ f(S)) \cap Y^0 \neq \emptyset$. By [Kol13], Theorem 2.12, the map between fundamental groups $\pi_1((g \circ f|_S)^{-1}(Y^0)) \to \pi_1(Y^0)$ is surjective, and $S' := (g \circ f|_S)^{-1}(Y^0) = S \setminus \text{(closed subset in } S)$ is irreducible. Then the fiber product $X^0 \times_Y S'$ is irreducible and $X \times_Y S_g$ is irreducible where $S_g$ emphasizes the twisted morphism $g \circ f|_S : S \to Y$. Then $X \times_Y S$ is the limit of $X \times_Y S_g$, so it is connected.

\[
\begin{array}{c}
X^0 \times_Y S' \\
\downarrow \\
X^0 \\
\downarrow \\
X \\
\downarrow \\
Y \\
\downarrow \\
Y^0 \\
\downarrow \\
g \circ f \\
\end{array}
\]

If $X \times_Y S = \Delta_S = \{(s, s) \in X \times X \mid s \in S\}$, we have $e_f(x) = d \geq k + 1$ and the result is true. So assume $X \times_Y S \neq \Delta_S$. We already know that $X \times_Y S$ is connected, so there is an irreducible component $T$ of $X \times_Y S$ such that $T \neq \Delta_S$ and $T \cap \Delta_S \neq \emptyset$. We can pick a sequence $(t_h, s_h) \in T$ such that $t_h \neq s_h, f(t_h) = f(s_h)$ for every $h \geq 1$ and $\lim_{h \to \infty} t_h = \lim_{h \to \infty} s_h = s^*$. We have $e_f(t_h) \geq 1, e_f(s_h) \geq k$, hence

$$e_f(s^*) \geq k + 1$$

which means $s^* \in R_k(f) \neq \emptyset$.

To show that $R_k(f)$ indeed has the required codimension count, we use the result obtained by Lazarsfeld:

**Proposition.** ([GL80] [FL81] [Gar88]) Let $f : X \to Y$ be a branched covering of irreducible varieties with $X$ normal and $Y$ nonsingular. If $R_\ell(f)$ is non-empty, then every irreducible component of $R_\ell(f)$ has codimension $\leq \ell$ in $X$.

**Proof.** We show this property by induction on $\ell$. For $\ell = 0$, we have $R_0(f) = X$ and it is clear. Assume it is true for all $\ell < k$ and $R_k(f)$ is non-empty. Take $V \subset X$ an irreducible component of $R_{k-1}(f)$ containing $x \in R_k(f)$. $V$ has dimension $\geq n - k + 1$ by induction hypothesis. If $V$ is also a component of $R_k(f)$, then we are done. So we assume that $V$ is not a component of $R_k(F)$. Since $f$ is a finite surjective and $Y$ smooth, each fiber has the same cardinality counting the multiplicity where the sum is the degree of $f$ ([GL80] [Laz04]). So we have $p \in V, q \in X, p \neq q$ with $f(p) = f(q)$.

Consider the map $F := f \times f |_V : X \times V \to Y \times Y$. Set $\Delta_Y$ as the diagonal set in $Y \times Y$. From the existence of $p, q$, we have a component $T \subset F^{-1}(\Delta_Y)$ with...
$T \neq (V \times V), T \cap (V \times V) \neq \emptyset$. We have $p_1(T \cap (V \times V)) \subset R_k(f)$ where $p_1$ is the projection map. By $F^{-1}(\Delta_Y)$ being connected in dimension $n - k$ (see [Ga88], lemma 2.4), we get $\dim R_k(f) \geq n - k$ which is the desired result. \hfill \Box

Remark. A Noetherian scheme $X$ is connected in dimension $k$ if $\dim X > k$ and $X \setminus T$ is connected for every closed subset $T \subset X$ of dimension $< k$. One can find some properties of connectedness in dimension $k$ in [Laz04], section 3.3.C.

Remark. We are grateful to Professor Lazarsfeld of pointing out that the proposition above is also proved in the thesis [Laz80].

Now we come back to the theorem. Let $\hat{n} : \hat{X} \to X$ be the normalization of $X$. Then $f \circ \hat{n} : \hat{X} \to Y$ is a branched covering satisfying the property above. We have $\hat{n}^{-1}(ss) \subset R_k(f \circ \hat{n})$, so $R_k(f \circ \hat{n})$ has codimension $\leq k$, and $\hat{n}(R_k(f \circ \hat{n}))$ has codimension $\leq k$, which gives that $R_k(f)$ has codimension $\leq k$. So the induction holds and we proved the theorem. \hfill \Box

According to the theorem, we have a corollary about simply connectedness of coverings of low degree as the one mentioned in [GL80].

Corollary. Let $f : X \to Y$ be a branched covering from an irreducible normal projective variety $X$ to a homogeneous space $Y$ with Picard number 1. If $\deg f \leq \dim X$, $X$ is algebraically simply connected.

Proof. The proof is exactly the same as in [Laz04], Corollary 3.4.10. We assume on the contrary that there is an étale covering $\pi : \tilde{X} \to X$ with $\deg \pi \geq 2$. $\tilde{X}$ is irreducible since $X$ is normal. We apply the main theorem to the composition $g := f \circ \pi : \tilde{X} \to Y$,

so there is at least a point $p \in \tilde{X}$ such that $e_g(p) \geq \min \{\deg g, n + 1\} > \deg f$. On the other hand, we have $e_g(p) = e_f(\pi(p))$ since $\pi$ is étale. This is a contradiction. Thus $X$ must be algebraically simply connected. \hfill \Box

In fact, we have the following result:

Corollary. In the same setting of the corollary above, $X$ is actually simply connected.

Proof. The process is to show some surjectivity between fundamental groups and it would imply that $\pi_1(X) = 1$. We have the following lemma:

Lemma. Let $f : X \to Y$ be a finite surjective morphism from an irreducible projective variety to a homogeneous space $Y = G/P$ with dimension $n$ and Picard number one. If $d := \deg f \leq n$, there is a subvariety $T \subset X$ of dimension $\geq n + 1 - d \geq 1$ such that $f$ has only one preimage over $T$.

Proof. By the main theorem, $R_{d-1}(f)$ is nonempty and codimension $R_{d-1}(f) \leq d - 1$. Therefore, there is an irreducible component $T \subset R_{d-1}(f)$ which has dimension $\geq n + 1 - d \geq 1$. Since every point $x \in R_{d-1}(f)$ has local degree $e_f(x) = d$, $f$ has only one preimage over $T$. \hfill \Box

From the lemma, we have a subvariety $T \subset Y$ with positive dimension in the corollary assumption. Take normalization $n : \tilde{T} \to T$ and consider the map...
\( F := (f |_T \circ n, f |_T \circ n) : \tilde{T} \rightarrow Y \times Y \), note that \((X \times \tilde{T}) \times_{Y \times Y} \tilde{T}\) is homeomorphic to \(\tilde{T}\). The map

\[
\pi_1(\tilde{T}) = \pi_1((X \times \tilde{T}) \times_{Y \times Y} \tilde{T}) \rightarrow \pi_1(X \times \tilde{T})
\]

is surjective by using the main result in [Kol13] which appears in the proof of the main theorem above. More specifically, \(Y \times Y\) is homogeneous and \(F(\tilde{T})\) is a nondegenerate irreducible subvariety in \(Y \times Y\). For \(g \in G\) a general element of group action on \(Y \times Y\), \((g \circ F(\tilde{T})) \cap (Y^0 \times Y^0) \neq \emptyset\) where \(Y^0 \times Y^0\) is the unramified locus of \((f, f) : X \times X \rightarrow Y \times Y\). Denote \(\tilde{T}^0 := (g \circ F)^{-1}(Y^0 \times Y^0) \subset \tilde{T}\) and \(X^0 := f^{-1}(Y^0)\), we have the diagram

\[
\begin{array}{ccc}
(X^0 \times \tilde{T}^0) \times_{Y^0 \times Y^0} \tilde{T}^0 & \longrightarrow & \tilde{T}^0 \longrightarrow \tilde{T} \\
\downarrow & & \downarrow \\
X^0 \times \tilde{T}^0 & \longrightarrow & Y^0 \times Y^0 \\
\downarrow & \searrow \swarrow & \downarrow \searrow \swarrow \\
X \times \tilde{T} & \longrightarrow & Y \times Y \\
\end{array}
\]

By [Kol13], Theorem 2.12, \(\pi_1(\tilde{T}^0) \rightarrow \pi_1(Y^0 \times Y^0)\) is surjective. Consider the diagram

\[
\begin{array}{cccc}
\pi_1((X^0 \times \tilde{T}^0) \times_{Y^0 \times Y^0} \tilde{T}^0) & \longrightarrow & \pi_1(X^0 \times \tilde{T}^0) \\
\downarrow & & \downarrow \\
\pi_1(\tilde{T}) & \longrightarrow & \pi_1((X \times \tilde{T}) \times_{Y \times Y} \tilde{T}) & \longrightarrow \pi_1(X \times \tilde{T}) \\
\end{array}
\]

The upper horizontal map is surjective since it is lifted by an étale morphism. The two vertical maps are also surjective because of the open inclusion and irreducibility of \((X \times \tilde{T}) \times_{Y \times Y} \tilde{T}\) and \(X \times \tilde{T}\). So we get the surjection \(\pi_1(\tilde{T}) \rightarrow \pi_1(X \times \tilde{T})\). The surjectivity then implies that \(\pi_1(X) = 1\), \(X\) is simply connected.

**Remark.** The argument for \(Y = \mathbb{P}^n\) can be found in [FL81] (also see [Laz04], Example 3.4.11.) and the proof here is essentially the same way. In [Deb95], Cor. 7.4, Debarre shows the simply connectedness when \(Y\) is Grassmannian and \(f : X \rightarrow Y\) satisfies certain nonempty intersection condition.

Therefore, any irreducible projective variety \(X\) with nontrivial fundamental group can only have finite surjective morphisms to a homogeneous space of Picard number one with degree lower bound \(\dim X + 1\).

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