SOLUTIONS OF WEINSTEIN EQUATIONS REPRESENTABLE BY BESSEL POISSON INTEGRALS OF BMO FUNCTIONS

JORGE J. BETANCOR, ALEJANDRO J. CASTRO, JUAN C. FARÍÑA, AND L. RODRÍGUEZ-MESA

Abstract. We consider the Weinstein type equation $\Delta u = 0$ on $(0, \infty) \times (0, \infty)$, where $\Delta = \partial_t^2 + \partial_x^2 - \frac{\lambda^2}{x^2}$, with $\lambda > 1$. In this paper we characterize the solutions of $\Delta u = 0$ on $(0, \infty) \times (0, \infty)$ representable by Bessel-Poisson integrals of BMO-functions as those ones satisfying certain Carleson properties.

1. Introduction

The space $BMO(\mathbb{R}^n)$ of bounded mean oscillation functions in $\mathbb{R}^n$ was introduced by John and Nirenberg ([31]) in the context of partial differential equations. A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is in $BMO(\mathbb{R}^n)$ provided that

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\,dx < \infty,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$. Here, $|B|$ denotes the Lebesgue measure of $B$ and $f_B$ represents the average of $f$ on $B$, that is, $f_B = \frac{1}{|B|} \int_B f(x)\,dx$. By identifying those functions that differ by a constant, $(BMO(\mathbb{R}^n), \| \cdot \|_{BMO(\mathbb{R}^n)})$ is a Banach space.

A celebrated result of Fefferman and Stein ([31]) establishes that $BMO(\mathbb{R}^n)$ is the dual space of the Hardy space $H^1(\mathbb{R}^n)$. The spaces $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ turned out to be the correct substitutes for $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$, respectively, as the domain and the target spaces of operators appearing in harmonic analysis.

Since Fefferman and Stein’s paper ([31]) appeared, the space of bounded mean oscillation functions has motivated the investigations of many mathematicians (see, for instance, [13], [15], [18], [19], [21], [22], [24], [35], [36], [38], [41], [43], [47], [48], [52] and [53]).

The space $BMO(\mathbb{R}^n)$ is closely connected to certain positive measures in $\mathbb{R}^{n+1}_+$ known as Carleson measures. These measures were introduced by Carleson to solve the corona problem ([13]). A positive measure $\mu$ on $\mathbb{R}^{n+1}_+$ is called a Carleson measure when

$$\|\mu\|' := \sup_Q \frac{\mu(Q \times (0, \ell(Q)))}{|Q|} < \infty,$$

where the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$. Here $\ell(Q)$ denotes the length of the edge of $Q$.

If $f$ is a measurable function on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |f(x)|(1 + |x|)^{-n-1}\,dx < \infty$, then, for every $t > 0$, the Poisson integral $P_t(f)$ of $f$ is defined by

$$P_t(f)(x) = \int_{\mathbb{R}^n} P_t(x - y)f(y)\,dy, \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

where

$$P_t(z) = \frac{\Gamma(n+1/2)}{\pi^{n+1/2}} \frac{t}{(|z|^2 + t^2)\Gamma(n+1/2)}, \quad z \in \mathbb{R}^n \text{ and } t > 0.$$

The characterization of the bounded mean oscillation functions via Carleson measures was given by Fefferman and Stein.

Theorem A. ([31] p. 145) Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then, $f \in BMO(\mathbb{R}^n)$ if, and only if, $\int_{\mathbb{R}^n} |f(x)|(1 + |x|)^{-n-1}\,dx < \infty$ and the measure $t|\nabla P_t(f)(x)|^2\,dx\,dt$ is Carleson in $\mathbb{R}^{n+1}_+$, where $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n}, \partial_t)$.

Date: Monday 28th February, 2022.
2010 Mathematics Subject Classification. 42B37, 42B35, 35J15.
Key words and phrases. Weinstein equation, Bessel Poisson integral, BMO, Carleson measure.
This paper was partially supported by MTM2010/17974. The second author was also supported by an FPU grant from the Government of Spain.
Some versions of this result for $BMO$-type spaces associated with operators have been established in the last decade (see [6], [23], [25], [35] and [40], amongst others).

Theorem [A] was completed by Fabes, Johnson and Neri ([28] and [29]). An harmonic function $u$ defined on $\mathbb{R}^{n+1}_+$ is said to be in $HMO(\mathbb{R}^{n+1}_+)$ provided that

$$
\sup_{Q} \frac{1}{|Q|} \int_{Q} |\nabla u(x,t)| \frac{dx dt}{t} < \infty,
$$

where the supremum is taken over all cubes in $\mathbb{R}^n$. Theorem [A] implies that $P_t(BMO(\mathbb{R}^n)) \subseteq HMO(\mathbb{R}^{n+1}_+)$ suitably understood. The equality is established in the following.

**Theorem B.** ([29] Theorem 1.0). A function $u \in HMO(\mathbb{R}^{n+1}_+)$ if, and only if, there exits $f \in BMO(\mathbb{R}^n)$ such that $u(x,t) = P_t(f)(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$.

Our objective in this paper is to establish a version of Theorem B in the Bessel operator context.

The study of harmonic analysis associated with Bessel operators was began in a systematic way by Muckenhoupt and Stein ([42]). In the last decade Bessel harmonic analysis has been developed by Muckenhoupt and Stein ([42]). In the last decade Bessel harmonic analysis has been developed by Muckenhoupt and Stein ([42]). In the last decade Bessel harmonic analysis has been developed by Muckenhoupt and Stein ([42]). In the last decade Bessel harmonic analysis has been developed by Muckenhoupt and Stein ([42]). In the last decade Bessel harmonic analysis has been developed by Muckenhoupt and Stein ([42]).
Theorem C. ([8] Theorem 1.1). Let \( \lambda > 0 \). Assume that \( f \in L^1_{\text{loc}}(0, \infty) \). Then, the following assertions are equivalent.

(i) \( f \in BMO_0(\mathbb{R}) \).

(ii) \( (1 + x^2)^{-1} f \in L^1(0, \infty) \) and

\[
d\gamma_f(x,t) = \frac{|t \partial_t P^\lambda_t(f)(x)|^2}{t} \, dx dt
\]

is a Carleson measure on \((0, \infty) \times (0, \infty)\).

Moreover, the quantities \( \|f \|_{BMO_0(\mathbb{R})}^2 \) and \( \|\gamma_f\|_\mathcal{C} \) are equivalent.

Remark. Another characterization of \( BMO_0(\mathbb{R}) \), slightly different to (ii) in Theorem C and that will be used in Section 3, is given in Lemma 3.1.

If \( \Omega \subseteq (0, \infty) \times \mathbb{R} \) we say that a function \( u \in C^2(\Omega) \) is \( \lambda \)-harmonic provided that

\[
\partial^2_x u(x,t) + B_{\lambda,x} u(x,t) = 0, \quad (x,t) \in \Omega.
\]

The operator \( \partial^2_x + B_{\lambda,x} \) is related to the Weinstein operator associated with the generalized axially symmetric potential theory (see [4] and the references there). We can write \( B_{\lambda,x} = -D^*_x D_{x,\lambda} \), where \( D_{x,\lambda} = x_1^\lambda \partial_x x_1^{-\lambda} \) and \( D^*_x \) is the formal adjoint operator of \( D_{x,\lambda} \) in \( L^2(0, \infty) \). We define the \( \lambda \)-gradient \( \nabla_{\lambda} \) by

\[
\nabla_\lambda = (D_{\lambda,x}, \partial_t).
\]

The main result of this paper is the following.

Theorem 1. Let \( \lambda > 1 \). Assume that \( u \) is a \( \lambda \)-harmonic function on \((0, \infty) \times (0, \infty) \) such that \( x^{-\lambda} u(x,t) \in C^\infty(\mathbb{R} \times (0, \infty)) \) and is even in the \( x \)-variable. Then, the following assertions are equivalent.

(i) There exists \( f \in BMO(\mathbb{R}) \) such that \( u(x,t) = P^\lambda_t(f)(x), (x,t) \in (0, \infty) \times (0, \infty) \).

(ii) The measure

\[
d\mu_\lambda(x,t) = |(\nabla_\lambda u(x,t))|^2 \, dx dt
\]

is Carleson on \((0, \infty) \times (0, \infty) \). Moreover, the quantities \( \|f\|_{BMO(\mathbb{R})}^2 \) and \( \|\mu_\lambda\|_\mathcal{C} \) are equivalent.

Note that the property (ii) in Theorem 1 is stronger than the condition (ii) in Theorem C.

In the next sections we prove Theorem 1. In the sequel by \( C \) we always denote a positive constant not necessarily the same in each occurrence.

2. Proof of (i) \( \Rightarrow \) (ii) in Theorem 1

As it can be observed along the proof, this part of Theorem 1 is valid for \( \lambda > 0 \).

Assume that \( u(x,t) = P^\lambda_t(f)(x), (x,t) \in (0, \infty) \), for a certain \( f \in BMO_0(\mathbb{R}) \). According to Theorem C the measure

\[
d\gamma_f(x,t) = \frac{|t \partial_t P^\lambda_t(f)(x)|^2}{t} \, dx dt
\]

is Carleson on \((0, \infty) \times (0, \infty) \). Moreover, we have that

\[
\|\gamma_f\|_\mathcal{C} \leq C \|f\|_{BMO_0(\mathbb{R})}^2,
\]

where \( C > 0 \) does not depend on \( f \).

We are going to see that the measure

\[
d\rho_f(x,t) = |t D_{\lambda,x} P^\lambda_t(f)(x)|^2 \, dx dt
\]

is Carleson on \((0, \infty) \times (0, \infty) \) and that

\[
\|\rho_f\|_\mathcal{C} \leq C \|f\|_{BMO_0(\mathbb{R})}^2,
\]

for certain \( C > 0 \) which does not depend on \( f \).

Let \( I = (a,b) \) where \( 0 \leq a < b < \infty \). We decompose \( f \) as follows

\[
f = (f - f_{2I}) \chi_{2I} + (f - f_{2I}) \chi_{(0,\infty) \setminus 2I} + f_{2I} =: f_1 + f_2 + f_3.
\]

Here \( 2I = (x_1 - |I|, x_1 + |I|) \cap (0, \infty) \) and \( x_1 = (a + b)/2 \).
We will prove that
\[ \frac{1}{|I|} \int_I \left| \int_I |tD_{\lambda,x}P_t^\lambda(f_j)(x)|^2 \frac{dx dt}{t} \right|^2 \leq C \|f\|_{BMO_o(R)}^2, \quad j = 1, 2, 3, \]
for certain $C > 0$ independent of $I$ and $f$.

2.1. Proof of (4) for $j = 1$. We introduce the Littlewood-Paley function $g_\lambda$ defined by
\[ g_\lambda(F)(x) = \left( \int_0^\infty |tD_{\lambda,x}P_t^\lambda(F)(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad x \in (0, \infty), \]
for every $F \in L^2(0, \infty)$.

**Lemma 2.1.** Let $\lambda > 0$. The Littlewood-Paley function $g_\lambda$ is a bounded (sublinear) operator from $L^2(0, \infty)$ into itself.

**Proof.** We consider the Hankel transformation $h_\lambda$ defined by
\[ h_\lambda(F)(x) = \int_0^\infty \sqrt{xy} J_{\lambda-1/2}(xy) F(y) dy, \quad x \in (0, \infty), \]
for every $F \in L^1(0, \infty)$. Here $J_\nu$ denotes the Bessel function of the first kind and order $\nu$. The transformation $h_\lambda$ can be extended from $L^1(0, \infty) \cap L^2(0, \infty)$ to $L^2(0, \infty)$ as an isometry of $L^2(0, \infty)$, where $h_\lambda^{-1} = h_{\lambda^{-1}}$ (\cite{31} Ch. VIII).

Let $F \in L^2(0, \infty)$. According to \cite{42} (16.1')) we have that
\[ P_t^\lambda(F)(x) = h_{\lambda}(e^{-yt}h_{\lambda}(F))(x), \quad x, t \in (0, \infty). \]
Since $\frac{d}{dz}(z^{-\nu}J_{\nu}(z)) = -z^{-\nu}J_{\nu+1}(z)$, $z \in (0, \infty)$, and $e^{-yt}h_{\lambda}(F) \in L^1(0, \infty)$, $t > 0$, we get
\[ D_{\lambda,x}P_t^\lambda(F)(x) = -h_{\lambda+1}(ye^{-yt}h_{\lambda}(F))(x), \quad x, t \in (0, \infty). \]

Then,
\[
\|g_\lambda(F)\|_{L^2(0, \infty)}^2 = \int_0^\infty \int_0^\infty |tD_{\lambda,x}P_t^\lambda(F)(x)|^2 \frac{dx dt}{t}
\]
\[
= \int_0^\infty \int_0^\infty |h_{\lambda+1}(ye^{-yt}h_{\lambda}(F)(y))(x)|^2 dx dt
\]
\[
= \int_0^\infty \int_0^\infty y^2 e^{-2yt}|h_{\lambda}(F)(y)|^2 dy dt
\]
\[
= \int_0^\infty y^2 |h_{\lambda}(F)(y)|^2 \int_0^\infty te^{-2yt} dy dt = \frac{1}{4} \|h_{\lambda}(F)\|_{L^2(0, \infty)}^2 = \frac{1}{4} \|F\|_{L^2(0, \infty)}^2.
\]

Lemma 2.1 leads to
\[
\frac{1}{|I|} \int_I \left| \int_I |tD_{\lambda,x}P_t^\lambda(f_1)(x)|^2 \frac{dx dt}{t} \right|^2 \leq \frac{1}{|I|} \int_I \int_0^\infty |g_\lambda(f_1)(x)|^2 dx \leq C \frac{1}{|I|} \int_0^\infty \int_0^\infty |f_1(y)|^2 dy
\]
\[
= \frac{C}{|I|} \int_{2I} |f(y) - f_{2I}|^2 dy \leq C \|f\|_{BMO_o(R)}^2,
\]
being $C$ independent of $I$ and $f$.

2.2. Proof of (4) for $j = 2$. First of all we establish the following estimation for the kernel $D_{\lambda,x}P_t^\lambda(x,y), \quad x, y, t \in (0, \infty)$.

**Lemma 2.2.** Let $\lambda > 0$. Then,
\[ |D_{\lambda,x}P_t^\lambda(x,y)| \leq \frac{C}{(x-y)^2 + t^2}, \quad x, y, t \in (0, \infty). \]

**Proof.** We write the following decomposition
\[
D_{\lambda,x}P_t^\lambda(x,y) = -\frac{4\lambda(\lambda + 1)}{\pi} (xy)\lambda t \int_0^\pi (\sin \theta)^{2\lambda-1} (x-y \cos \theta) ((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+2} d\theta
\]
\[
=: P_t^{\lambda,1}(x,y) + P_t^{\lambda,2}(x,y), \quad x, y, t \in (0, \infty),
\]
where
\[ P_t^\lambda(x,y) = -\frac{4\lambda(\lambda+1)}{\pi}(xy)^{\lambda t} \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda-1}(x-y \cos \theta)}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+2}} d\theta, \quad x, y, t \in (0, \infty). \]

We have that
\[ |P_t^\lambda(x,y)| \leq C(xy)^{\lambda t} \int_0^{\pi/2} \frac{x+y}{((x-y)^2 + t^2 + 2xy)^{\lambda+2}} d\theta, \quad x, y, t \in (0, \infty). \]

On the other hand, since
\[ |x - y \cos \theta| \leq |x - y| + \min\{x, y\}(1 - \cos \theta), \quad x, y \in (0, \infty), \quad \theta \in \mathbb{R}, \]
and \( \sin \theta \sim \theta \) and \( 2(1 - \cos \theta) \sim \theta^2, \theta \in [0, \pi/2] \), it follows that
\[ |P_t^\lambda(x,y)| \leq C(P_t^{\lambda,1}(x,y) + P_t^{\lambda,2}(x,y)), \quad x, y, t \in (0, \infty), \]
where
\[ P_t^{\lambda,1}(x,y) = (xy)^{\lambda t}|x-y| \int_0^{\pi/2} \frac{\theta^{2\lambda-1}}{((x-y)^2 + t^2 + x\theta^2)^{\lambda+2}} d\theta, \quad x, y, t \in (0, \infty), \]
and
\[ P_t^{\lambda,2}(x,y) = (xy)^{\lambda t} \min\{x, y\} \int_0^{\pi/2} \frac{\theta^{2\lambda+1}}{((x-y)^2 + t^2 + x\theta^2)^{\lambda+2}} d\theta, \quad x, y, t \in (0, \infty). \]

We get
\[ P_t^{\lambda,1}(x,y) \leq C(xy)^{\lambda t}|x-y| \int_0^{\pi/2} \frac{\theta^{2\lambda-1}}{((x-y)^2 + t^2 + x\theta^2)^{\lambda+2}} d\theta \leq C \frac{|x-y|^2}{(x-y)^2 + t^2}, \quad x, y, t \in (0, \infty), \]
and
\[ P_t^{\lambda,2}(x,y) \leq C(xy)^{\lambda t} \min\{x, y\} \int_0^{\pi/2} \frac{\theta^{2\lambda+1}}{((x-y)^2 + t^2 + x\theta^2)^{\lambda+2}} d\theta \leq C \frac{\min\{x, y\}}{(x-y)^2 + t^2}, \quad x, y, t \in (0, \infty). \]

Hence,
\[ |P_t^\lambda(x,y)| \leq \frac{C}{(x-y)^2 + t^2}, \quad x, y, t \in (0, \infty). \]
Thus, the result follows from \([5], [6] \) and \([7]\). \( \square \)

We now proceed as in \([6]\), pp. 468-469. By Lemma 2.2 we can write
\[ |D_{\lambda,x} P_t^\lambda(f_2)(x)| \leq C \int_{(0,\infty)\setminus I} \frac{|f(y) - f_{2I}|}{(x-y)^2 + t^2} dy \leq C \int_{(0,\infty)\setminus I} \frac{|f(y) - f_{2I}|}{(x-y)^2 + t^2} dy \leq C \int_{I} \frac{1}{|I|} \frac{1}{2k} \left( \frac{1}{2k} \right) \int_{2k+1} dy \leq C \frac{1}{|I|} \|f\|_{BMO}\. \]

In the last inequality we have taken into account \([33]\) Ch. VI (1.3) and that, if \( k \in \mathbb{N}\setminus\{0\} \) and \( 2k|I| > x_I \), then \( 2k+1I \subset (0,2k+1|I|) \) and
\[ \int_{2k+1} f(y) - f_{2k+1I} |dy| \leq \int_{0}^{2k+1|I|} |f(y) - f_{2k+1I}| |dy| \leq 2k+1|I| \|f\|_{BMO}. \]

We conclude that
\[ \frac{1}{|I|} \int_{I} |D_{\lambda,x} P_t^\lambda(f_2)(x)|^2 dx dt \leq C \frac{1}{|I|} \|f\|_{BMO}^2, \]
with \( C \) independent of \( I \) and \( f \).
2.3. Proof of (4) for \( j = 3 \). Note firstly that
\[
|f_{2I}| \leq \frac{1}{|I|} \int_{2I} |f(y)|dy \leq \frac{x_j + |I|}{|I|} \|f\|_{BMO_m}. 
\]
Then, estimation (4) for \( j = 3 \) will be proved once we show the following.

**Lemma 2.3.** Let \( \lambda > 0 \). There exists \( C > 0 \) such that
\begin{equation}
\frac{(x_j + |J|)^2}{|J|^3} \int_0^{|J|} \int_J |tD_{\lambda,x}P_t^{\lambda}(1)(x)|^2 \frac{dxdt}{t} \leq C, 
\end{equation}
for every bounded interval \( J \) on \((0, \infty)\).

**Proof.** We take in mind the decomposition (5). As in (6) we get
\[
(8) \quad \int_0^\infty P_t^{\lambda,2}(x,y)dy \leq C \int_0^\infty \frac{dy}{x^2 + y^2 + t^2} \leq \frac{C}{x + t}, \quad x, t \in (0, \infty).
\]

Then, (9) \[
\int_0^\infty P_t^{\lambda,1}(x,y)dy = \left( \int_{\mathbb{R}^2} Q_1^1(x) + Q_2^1(x) + Q_3^1(x), \quad x \in (0, \infty). \right)
\]

According to (7) we get
\[
|Q_1^1(x)| \leq C \int_0^{x/2} \frac{dy}{x^2 + y^2 + t^2} \leq \frac{C}{x + t}, \quad x, t \in (0, \infty),
\]
and
\[
|Q_2^1(x)| \leq C \int_{3x/2}^{\infty} \frac{dy}{(y - x)^2 + t^2} \leq \frac{C}{x + t}, \quad x, t \in (0, \infty).
\]

We decompose \( Q_3^1(x) \), \( t, x \in (0, \infty) \), in the following way.
\[
Q_3^1(x) = -\frac{4\lambda + 1}{\pi} t \int_{x/2}^{3x/2} (xy)^{\lambda}(x - y) \left( \frac{1 - \cos \theta}{((x - y)^2 + t^2 + 2xy(1 - \cos \theta))^{\lambda + 1}} \right) d\theta dy 
\]
\[
- \frac{4\lambda + 1}{\pi} t \int_{x/2}^{3x/2} (xy)^{\lambda}(x - y) \left( \frac{1 - \cos \theta}{((x - y)^2 + t^2 + 2xy(1 - \cos \theta))^{\lambda + 1}} \right) d\theta \int_0^{\pi/2} \frac{d\theta}{\sin \theta^{2\lambda - 1}}
\]
\[
\times \int_{x/2}^{3x/2} (xy)^{\lambda}(x - y) \int_{\pi/2}^{\infty} \frac{d\theta}{\sin \theta^{2\lambda - 1}}
\]
\[
+ \frac{4\lambda + 1}{\pi} t \int_{x/2}^{3x/2} (xy)^{\lambda}(x - y) \int_{\pi/2}^{\infty} \frac{d\theta}{\sin \theta^{2\lambda - 1}}
\]
\[
- \frac{4\lambda + 1}{\pi} t \int_{x/2}^{3x/2} (xy)^{\lambda}(x - y) \int_0^{\pi/2} \frac{d\theta}{\sin \theta^{2\lambda - 1}}
\]
\[
= \sum_{j=1}^4 I_j(x,t), \quad x, t \in (0, \infty).
\]

Observe firstly that \( I_4(x,t) = 0 \), \( t, x \in (0, \infty) \). Indeed, we have that
\[
I_4(x,t) = -\frac{4\lambda + 1}{\pi} t \int_0^{\pi/2} \frac{u^{2\lambda - 1}}{ (1 + u^2)^{\lambda + 2} } du \int_{x/2}^{3x/2} \frac{(xy)^{\lambda}(x - y)}{(x - y)^2 + t^2 + xy(1 - \cos \theta)^{\lambda + 2}} d\theta dy 
\]
\[
= -\frac{2t}{\pi} \int_{-x/2}^{x/2} \frac{z}{(z^2 + t^2)^{\lambda + 2}} dz = 0, \quad x, t \in (0, \infty).
\]

We are going to see that
\[
|I_j(x,t)| \leq \frac{C}{x + t}, \quad x, t \in (0, \infty) \text{ and } j = 1, 2, 3.
\]
Since $2(1 - \cos \theta) \sim \theta^2$ and $\sin \theta \sim \theta$, when $\theta \in [0, \pi/2]$, we can write
\[
|I_1(x, t)| \leq C t x^{2\lambda + 1} \int_{x/2}^{3x/2} \int_0^{\pi/2} \frac{\theta^{2\lambda + 1}}{((x - y)^2 + t^2 + (x\theta)^2)^{\lambda + 2}} \, d\theta \, dy
\]
\[
\leq C t \int_{x/2}^{3x/2} \int_0^{\pi/2} \frac{\theta^{2\lambda - 1}}{((x - y)^2 + t^2 + (x\theta)^2)^{\lambda + 3/2}} \, d\theta \, dy
\]
\[
\leq C \frac{t}{x} \int_{x/2}^{3x/2} \frac{dy}{(x - y)^2 + t^2} \leq C \frac{t}{x + t}, \quad x, t \in (0, \infty).
\]

Also,
\[
|I_3(x, t)| \leq C t x^{2\lambda} \int_{x/2}^{3x/2} \int_0^{\pi/2} \frac{\theta^{2\lambda - 1}}{((x - y)^2 + t^2 + (x\theta)^2)^{\lambda + 3/2}} \, d\theta \, dy
\]
\[
\leq C \frac{t}{x} \int_{x/2}^{3x/2} \int_0^{\pi/2} \frac{\sqrt{(x - y)^2 + t^2} \, dy}{(x - y)^2 + t^2} \, du \leq C \frac{t}{x + t}, \quad x, t \in (0, \infty).
\]

By using that $|(\sin \theta)^{2\lambda - 1} - \theta^{2\lambda - 1}| \leq C \theta^{2\lambda + 1}, \theta \in (0, \pi/2)$, and that
\[
\frac{1}{((x - y)^2 + t^2 + 2xy(1 - \cos \theta))^\lambda + 2} - \frac{1}{((x - y)^2 + t^2 + x\theta^2)^\lambda + 2} \leq C \frac{x^y\theta^4}{((x - y)^2 + t^2 + x\theta^2)^\lambda + 3},
\]
for each $\theta \in (0, \pi/2)$ and $t, x, y \in (0, \infty)$, we obtain
\[
\left|\frac{1}{((x - y)^2 + t^2 + 2xy(1 - \cos \theta))^\lambda + 2} - \frac{\theta^{2\lambda - 1}}{((x - y)^2 + t^2 + x\theta^2)^\lambda + 2}\right|
\leq C \frac{x^y\theta^4}{((x - y)^2 + t^2 + x\theta^2)^\lambda + 3}, \quad \theta \in (0, \pi/2), \quad x, y, t \in (0, \infty).
\]

Then,
\[
|I_2(x, t)| \leq C t x^{2\lambda} \int_{x/2}^{3x/2} \int_{x/2}^{3x/2} \frac{\theta^{2\lambda - 1}}{((x - y)^2 + t^2 + (x\theta)^2)^{\lambda + 2}} \, d\theta \, dy
\]
\[
\leq C t x^{2\lambda} \int_{x/2}^{3x/2} \int_{x/2}^{3x/2} \frac{\theta^{2\lambda + 1}}{((x - y)^2 + t^2 + (x\theta)^2)^{\lambda + 3/2}} \, d\theta \, dy
\]
\[
\leq C \frac{t}{x} \int_{x/2}^{3x/2} \frac{dy}{(x - y)^2 + t^2} \leq C \frac{t}{x + t}, \quad x, t \in (0, \infty).
\]

We conclude that
\[
|Q^i_t(x)| \leq \frac{C}{x + t}, \quad x, t \in (0, \infty).
\]

Hence,
\[
(10) \quad \left|\int_0^\infty P^1_{x, t}(x, y) \, dy\right| \leq \frac{C}{x + t}, \quad x, t \in (0, \infty).
\]

Let $J$ a bounded interval in $(0, \infty)$. If $x < |J|$, we obtain by (10) and (10)
\[
\frac{(x + |J|)^2}{|J|^3} \int_0^{|J|} \int_0^\infty \int_0^\infty |D_{x, |J|^2} P^1_{x, t}(x, y) dy|^2 \, dx \, dt \leq C \frac{|J|^2}{|J|^2} \int_0^{|J|} \int_0^{|J|^2} \frac{t}{(x + t)^2} \, dx \, dt
\]
Note that the constant $C$ does not depend on $J$. Thus, (8) is established. 

By considering Lemma 2.3 and the estimate for $|f_{2J}|$ we deduce that

$$
\frac{1}{|J|} \int_{0}^{1} \int_{J} \int_{0}^{t} |D_{\lambda x} P_{t}^{\lambda}(f_{x})(x) y|^{2} dy dx dt \leq C \frac{(x_{J} + |J|)^{2}}{|J|^{3}} \int_{0}^{1} t \int_{x_{J} - |J|/2}^{x_{J} + |J|/2} \frac{dx}{x} \leq C \frac{(x_{J} + |J|)^{2}}{|J|^{3}} \left( \frac{1}{x_{J} - |J|/2} - \frac{1}{x_{J} + |J|/2} \right) \leq C \frac{(x_{J} + |J|)^{2}}{(x_{J} + |J|/2)(x_{J} - |J|/2)} \leq C \frac{2x_{J} + |J|}{2x_{J} - |J|} \leq C.
$$

Thus, (9) is established.

We start this section showing the following characterization of $BMO_{o}(R)$ which we need later. Its proof follows the arguments in [6, Theorem 1.1] with minor modifications.

**Lemma 3.1.** Let $\lambda > 0$. Suppose $f \in L^{1}_{loc}(0, \infty)$. Then, the following assertions are equivalent.

(i) $f \in BMO_{o}(R)$,

(ii) $x^{\lambda}(1 + x^{2})^{-\lambda-1} f \in L^{1}(0, \infty)$ and

$$
d_{\gamma f}(x,t) = |t \partial_{t} P_{t}^{\lambda}(f)(x)|^{2} \frac{dx dt}{t}
$$

is a Carleson measure on $(0, \infty) \times (0, \infty)$.

Moreover, the quantities $\|f\|^{2}_{BMO_{o}(R)}$ and $\|\gamma f\|_{\infty}$ are equivalent.

**Proof.** (i) $\Rightarrow$ (ii). It follows from Theorem 1.

(ii) $\Rightarrow$ (i). We can proceed as in [6, Section 4] by establishing the result in [6, Proposition 4.4] for the new conditions on $f$. Actually, we only have to take into account the following estimations.

Let be an (odd)-atom, that is, a measurable function satisfying one of the next properties:

(a) $a = \delta^{-1} \chi_{(0,\delta)}$, for some $\delta > 0$;

(b) there exists a bounded interval $I \subset (0, \infty)$ such that $supp a \subset I$, $\int_{I} a(x) dx = 0$ and $\|a\|_{L^{\infty}(0,\infty)} \leq |I|^{-1}$.

We have that

$$
\int_{0}^{\infty} |t \partial_{t} P_{t}^{\lambda}(y,z) \partial_{t} P_{t}^{\lambda}(a)(y)| dy \leq C \frac{z^{\lambda}}{(1 + z^{2})^{\lambda+1}}, \quad z,t \in (0, \infty),
$$

with $C$ independent of $z$.

Indeed, since

$$
\partial_{t} P_{t}^{\lambda}(x,y) = \frac{2\lambda}{\pi} (xy)^{\lambda} \left[ \int_{0}^{\pi} \frac{(\sin \theta)^{2\lambda-1}}{((x-y)^{2} + t^{2} + 2xy(1 - \cos \theta))^{\lambda+2}} d\theta \right. \\
- 2(\lambda + 1)t^{2} \left. \int_{0}^{\pi} \frac{(\sin \theta)^{2\lambda-1}}{((x-y)^{2} + t^{2} + 2xy(1 - \cos \theta))^{\lambda+2}} d\theta \right], \quad x,y,t \in (0, \infty),
$$

we get

$$
|\partial_{t} P_{t}^{\lambda}(x,y)| \leq C \int_{0}^{\pi} \frac{(xy)^{\lambda}(\sin \theta)^{2\lambda-1}}{((x-y)^{2} + t^{2} + 2xy(1 - \cos \theta))^{\lambda+1}} d\theta \leq C \frac{(xy)^{\lambda}}{((x-y)^{2} + t^{2})^{\lambda+1}}, \quad x,y,t \in (0, \infty),
$$

and also

$$
|\partial_{t} P_{t}^{\lambda}(x,y)| \leq \frac{C}{(x-y)^{2} + t^{2}}, \quad x,y,t \in (0, \infty).
$$
Assume that \( \text{supp } a \subset (0, \alpha) \) for certain \( \alpha > 0 \). Then,

\[
|\partial_t P_t^\lambda(a)(y)| \leq ||a||_{L^\infty(0, \alpha)} \int_0^\alpha \frac{(yz)^\lambda}{(yz)^2 + t^2)^{\lambda+1}} dz \\
\leq C y^\lambda \begin{cases} 
\frac{t}{z^{\lambda-2}} \int_0^\alpha z^\lambda dz, & 0 < y \leq 2\alpha, \\
(y^2 + t^2)^{-\lambda-1} \int_0^\alpha z^\lambda dz, & y \geq 2\alpha,
\end{cases}
\leq C \frac{y^\lambda}{(1 + y^2)^{\lambda+1}}, \quad y, t \in (0, \infty),
\]

where \( C > 0 \) does not depend on \( y \). Hence, by using (12) and (13) it follows that

\[
\int_0^\infty |t\partial_t P_t^\lambda(y, z)\partial_t P_t^\lambda(a)(y)| dy \\
\leq C \left[ \left( \int_0^{z/2} + \int_2^{\infty} \right) \frac{t(yz)^\lambda}{(yz)^2 + t^2)^{\lambda+1}} \left( \frac{y^\lambda}{(1 + y^2)^{\lambda+1}} + \frac{t}{z^{\lambda-2}} \int_0^\alpha z^\lambda dz \right) dy + \int_2^{z/2} \frac{t}{z^{\lambda-2}} \int_0^\alpha z^\lambda dz \right] \\
\leq Ct^\lambda \left[ \frac{1}{(z^2 + t^2)^{\lambda+1}} \int_0^{z/2} \frac{y^{2\lambda}}{(1 + y^2)^{\lambda+1}} dy + \frac{1}{(1 + z^2)^{\lambda+1}} \int_2^{z/2} \frac{dy}{(x-y)^2 + t^2} \right] \leq C \frac{z^\lambda}{(1 + z^2)^{\lambda+1}}, \quad z, t \in (0, \infty).
\]

Here, the constant \( C \) can depend on \( t \), but is independent of \( z \).

On the other hand we need to estimate \( \sup_{t > 0} |M_t^\lambda(a)(z)|, z \in (0, \infty) \), where

\[
M_t^\lambda(a) = \frac{1}{4} \left[ t\partial_t P_{2t}^\lambda(a)_{|v=1} - P_{2t}^\lambda(a) \right], \quad t \in (0, \infty).
\]

According to [4] p. 492 we have that

\[
\sup_{t > 0} |M_t^\lambda(a)(z)| \leq C \begin{cases} 
1, \quad 0 < z \leq 2\alpha, \quad \leq C \quad \frac{1}{z^\lambda}, \quad 0 < z \leq 2\alpha, \quad \leq C \quad \frac{z^\lambda}{(1 + z^2)^{\lambda+1}}, \quad z \geq 2\alpha,
\end{cases}
\]

which allows us to obtain

\[
(14) \quad \int_0^\infty |f(z)| \sup_{t > 0} |M_t^\lambda(a)(z)| dz \leq \int_0^{2\alpha} |f(z)| dz + \int_{2\alpha}^\infty \frac{z^\lambda |f(z)|}{(1 + z^2)^{\lambda+1}} dz < \infty.
\]

By using (11) and (14) and proceeding as in [6] Section 4 we conclude our result. \( \square \)

Assume that \( u \) is a \( \lambda \)-harmonic function on \( (0, \infty) \times (0, \infty) \) such that \( x^{-\lambda} u(x, t) \in C^\infty(\mathbb{R} \times (0, \infty)) \) is even in the \( x \)-variable and that the measure

\[
d\mu_\lambda(x, t) = |t\nabla u(x, t)|^2 dt dx
\]

is Carleson on \( (0, \infty) \times (0, \infty) \).

The function \( u \) satisfies the equation

\[
\partial_t^2 u + \partial_x^2 u - \frac{\lambda(\lambda - 1)}{x^2} u = 0,
\]

in a weak sense on \( \mathbb{R} \times (0, \infty) \), that is, for every \( \phi \in C_c^\infty(\mathbb{R} \times (0, \infty)) \), the space of smooth functions having compact support on \( \mathbb{R} \times (0, \infty) \),

\[
0 = \int_{\mathbb{R} \times (0, \infty)} \left( \partial_t u(x, t) \partial_t \phi(x, t) + \partial_x u(x, t) \partial_x \phi(x, t) + \frac{\lambda(\lambda - 1)}{x^2} u(x, t) \phi(x, t) \right) dx dt.
\]

Indeed, let \( \phi \in C_c^\infty(\mathbb{R} \times (0, \infty)) \). We choose \( 0 < a < \infty \) and \( 0 < b_1 < b_2 < \infty \) such that \( \text{supp}(\phi) \subset [-a, a] \times [b_1, b_2] \) and define \( v(x, t) = x^{-\lambda} u(x, t), \) \( (x, t) \in \mathbb{R} \times (0, \infty) \).

Since \( v \in C^2(\mathbb{R} \times (0, \infty)) \) and \( \lambda > 1 \), we have that \( \frac{\partial^2}{x^2}, \partial_x u \) and \( \partial_t u \) are in \( L^1_{\text{loc}}(\mathbb{R} \times (0, \infty)) \), and \( \lim_{x \to 0} \partial_x u(x, t) = 0 \), for every \( t \in (0, \infty) \). Moreover,

\[
\partial_t^2 u(x, t) + \partial_x^2 u(x, t) - \frac{\lambda(\lambda - 1)}{x^2} u(x, t) = 0, \quad (x, t) \in (\mathbb{R} \setminus \{0\}) \times (0, \infty).
\]
Then, we can write
\[
\int_{\mathbb{R} \times (0, \infty)} \left( \partial_t u(x,t) \partial_t \phi(x,t) + \partial_x u(x,t) \partial_x \phi(x,t) + \frac{\lambda(\lambda - 1)}{x^2} u(x,t) \phi(x,t) \right) \, dx \, dt \\
= \lim_{\varepsilon \to 0^+} \int_{b_1} \left( \int_{a - \varepsilon}^{b_1} \left( \partial_t u(x,t) \partial_t \phi(x,t) + \partial_x u(x,t) \partial_x \phi(x,t) + \frac{\lambda(\lambda - 1)}{x^2} u(x,t) \phi(x,t) \right) \, dx \right) \, dt \\
= \lim_{\varepsilon \to 0^+} \int_{b_1} \left( \int_{a - \varepsilon}^{b_1} \left( -\partial^2 u(x,t) - \partial^2 u(x,t) + \frac{\lambda(\lambda - 1)}{x^2} u(x,t) \phi(x,t) \right) \, dx \right) \, dt = 0.
\]

Since (15) holds, by proceeding as in [23, Lemma 2.6] (see also [49, Lemma 2.1]) we can prove that the function \( u^2 \) is subharmonic in \( \mathbb{R} \times (0, \infty) \). Hence, for every \( x_0 \in \mathbb{R}, t_0 \in (0, \infty) \) and \( 0 < r < t_0 \),
\[
|u(x_0, t_0)| \leq \left( \frac{1}{\pi r^2} \int_{B(x_0, t_0)} |u(x, t)|^2 \, dx \, dt \right)^{1/2}.
\]
It is clear that \( \partial_t u \) satisfies the same properties than \( u \). Then, for every \( x_0 \in \mathbb{R}, t_0 \in (0, \infty) \) and \( 0 < r < t_0 \),
\[
|\partial_t u(x_0, t_0)| \leq \left( \frac{1}{\pi r^2} \int_{B(x_0, t_0), r} |\partial_t u(x, t)|^2 \, dx \, dt \right)^{1/2}.
\]

Since the measure \( t|\partial_t u(x, t)|^2 \, dx \, dt \) is Carleson on \( (0, \infty) \times (0, \infty) \) we have that, for every \( x_0, t_0 \in (0, \infty) \),
\[
|\partial_t u(x_0, t_0)| \leq C \left( \frac{1}{l_0} \int_{B(x_0, t_0), t_0/2} |\partial_t u(x, t)|^2 \, dx \, dt \right)^{1/2} \leq C \left( \frac{1}{l_0} \int_{t_0/2}^{3t_0/2} \int_{x_0-t_0/2}^{x_0+t_0/2} |\partial_t u(x, t)|^2 \, dx \, dt \right)^{1/2}
\]
\[
\leq C \left( \frac{1}{l_0} \int_{t_0}^{3t_0/2} \int_{I(x_0, t_0)} t |\partial_t u(x, t)|^2 \, dx \, dt \right)^{1/2} \leq \frac{C}{l_0} \mu(t |\partial_t u(x, t)|^2 \, dx \, dt)^{1/2},
\]
where \( I(x_0, t_0) = (x_0 - \frac{3t_0}{4}, x_0 + \frac{3t_0}{4}) \cap (0, \infty) \). We have used that \( |\partial_t u(x, t)| = |\partial_t u(-x, t)|, x \in \mathbb{R} \) and \( t \in (0, \infty) \).

From (16) we deduce that, for every \( t_0 > 0 \), there exists \( C > 0 \) such that
\[
|\partial_t u(x, t)| \leq C, \quad x \in \mathbb{R} \text{ and } t \geq t_0.
\]

Our next objective is to show that, for every \( t_0 > 0 \),
\[
\partial_t u(x, t + t_0) = P_t^\lambda((\partial_t u(\cdot, s))_{s=t_0})(x), \quad x, t \in (0, \infty).
\]

In order to see this property we establish previously some results.

**Lemma 3.2.** Let \( \lambda > 0 \). Suppose that \( f \) is a continuous function on \( (0, \infty) \) such that
\[
\int_0^\infty \frac{y^\lambda |f(y)|}{(1 + y^2)^{\lambda + 1}} \, dy < \infty.
\]
Then, the function
\[
v(x, t) = \begin{cases} P_t^\lambda(f)(x), & x, t \in (0, \infty), \\ f(x), & x \in (0, \infty), \ t = 0 \end{cases}
\]
is \( \lambda \)-harmonic in \( (0, \infty) \times (0, \infty) \) and continuous in \( (0, \infty) \times [0, \infty) \).

**Proof.** Differentiating under the integral sign and using [42 (16.1)] it is not hard to see that \( v \) is \( \lambda \)-harmonic function on \( (0, \infty) \times (0, \infty) \).

Suppose firstly that \( f \) is bounded in \( (0, \infty) \). Let \( x_0 \in (0, \infty) \). We write the following decomposition
\[
P_t^\lambda(f)(x) - f(x_0) = \int_0^\infty P_t^\lambda(x, y) [f(y) - f(x_0)] \, dy + \left( \int_0^\infty P_t^\lambda(x, y) \, dy - 1 \right) f(x_0)
\]
\[
= : I_1(x, t) + I_2(x, t), \quad x, t \in (0, \infty).
\]
Assume that \( \varepsilon > 0 \). There exists \( \delta \in (0, x_0/2) \) such that \( |f(y) - f(x_0)| < \varepsilon \) provided that \( |y - x_0| < \delta \), because \( f \) is continuous in \( x_0 \). Since \( f \) is bounded in \( (0, \infty) \) we get
\[
|I_1(x, t)| \leq \left( \int_{|y - x_0| < \delta} + \int_{|y - x_0| \geq \delta} \right) P_t^\lambda(x, y) |f(y) - f(x_0)| \, dy
\]
Let \( x \in (0, \infty) \) and \( t > 0 \). From [42, p. 86, (b)] we obtain
\[
\int_{[y-x_0] \leq \delta} P_t^\lambda(x, y) \, dy \leq C \int_{-\infty}^{+\infty} \frac{t}{(x-y)^2 + t^2} \, dy \leq C, \quad x, t \in (0, \infty),
\]
and
\[
\int_{[y-x_0] \geq \delta} P_t^\lambda(x, y) \, dy \leq C \int_{[y-x_0] \geq \delta} \frac{t}{(x-y)^2 + t^2} \, dy \leq Ct \int_{[y-x_0] \geq \delta} \frac{dy}{(x-y)^2} \leq Ct, \quad |x-x_0| < \frac{\delta}{2} \text{ and } t > 0.
\]

Hence,
\[
|I_1(x, t)| \leq C \left( \varepsilon + \frac{t}{\delta} \right), \quad |x-x_0| < \frac{\delta}{2} \text{ and } t > 0.
\]

On the other hand, by taking into account that \( \int_0^\infty x^{-\lambda} y^\lambda P_t^\lambda(x, y) \, dy = 1, \ x, t \in (0, \infty) \), (see, [26 p. 29 (4)], [34 §2 (1), (2)] and [42 (16.1)]), we get
\[
\left| \int_0^\infty P_t^\lambda(x, y) \, dy - 1 \right| \leq \int_0^\infty \left| 1 - \left( \frac{y}{x} \right)^\lambda \right| P_t^\lambda(x, y) \, dy.
\]

We choose \( \eta \in (0, 1) \) such that \( |1 - z^\lambda| < \varepsilon \) provided that \( |1 - z| < \eta \). From [42, p. 86, (b)] we deduce that
\[
\int_0^\infty P_t^\lambda(x, y) \, dy - 1 \leq \left( \int_0^{(1-\eta)x} + \int_0^{(1+\eta)x} + \int_{(1+\eta)x}^\infty \right) \left| 1 - \left( \frac{y}{x} \right)^\lambda \right| \, dy
\]
\[
\leq C \left( \int_0^{(1-\eta)x} \frac{(1 + (1-\eta)^\lambda) t}{(x-y)^2 + t^2} \, dy + \varepsilon \int_0^{(1+\eta)x} \frac{t}{(x-y)^2 + t^2} \, dy + \int_0^\infty \frac{t(x-y)^\lambda}{(1+\eta)^2} \, dy \right)
\]
\[
\leq C \left( \frac{(1 + (1-\eta)^\lambda) t}{(1+\eta)^2} \int_0^{(1-\eta)x} \, dy + \varepsilon + t \int_0^\infty \frac{y^\lambda}{(1+\eta)^{2\lambda+2} y^2} \, dy \right)
\]
\[
\leq C \left( t \frac{1 + \varepsilon + \frac{t}{(1+\eta)^{2\lambda+2}} \int_0^\infty \, dy}{\eta^{2\lambda+2} x^2} \right) \leq C \left( \varepsilon + \frac{t}{\eta^{2\lambda+2} x^2} \right)
\]
\[
\leq C \left( \varepsilon + \frac{t}{\eta^{2\lambda+2} x^2} \right), \quad |x-x_0| < \frac{x_0}{2} \text{ and } t > 0.
\]

Then
\[
|I_2(x, t)| \leq C \left( \varepsilon + \frac{t}{\eta^{2\lambda+2} x^2} \right), \quad |x-x_0| < \frac{x_0}{2} \text{ and } t > 0.
\]

Putting together (19) and (20) we conclude that
\[
\lim_{\substack{x \to (x_0, 0) \\
(\varepsilon, t) \to (x_0, 0)}} P_t^\lambda(f)(x) = f(x_0).
\]

We now study the general case, that is, consider \( f \) a continuous function such that
\[
\int_0^\infty y^\lambda |f(y)| \left( 1 + y^2 \right)^\lambda+1 \, dy < \infty.
\]

Let \( x_0 \in (0, \infty) \). For every \( n \in \mathbb{N} \) we denote by \( \phi_n \) a smooth function on \( (0, \infty) \) such that \( \phi_n(x) = 1, \ x \in (1/n, n) \), and \( \phi_n(x) = 0, \ x \in (0, \infty) \setminus (1/(n + 1), n + 1) \).

Suppose that \( \varepsilon > 0 \) and let \( n_0 \in \mathbb{N} \) such that \( x_0 \in (1/n_0, n_0) \). We can write
\[
|P_t^\lambda(f)(x) - f(x_0)| \leq |P_t^\lambda(f - f\phi_n)(x)| + |P_t^\lambda(f\phi_n)(x) - (f\phi_n)(x_0)| + |(f\phi_n)(x_0) - f(x_0)|
\]
\[
= |P_t^\lambda(f - f\phi_n)(x)| + |P_t^\lambda(f\phi_n)(x) - (f\phi_n)(x_0)|, \quad n \geq n_0.
\]
According to [42, p. 86 (b)] we have that, for each \( |x - x_0| < x_0/2, t \in (0, 1) \) and \( n \in \mathbb{N}, n \geq 4n_0, \)

\[
|P_t^\lambda (f - f \phi_n)(x)| \leq C t x^\lambda \left( \int_0^{1/n} + \int_n^{\infty} \right) \frac{y^\lambda |f(y)|}{((x - y)^2 + t^2)^{\lambda + 1}} dy
\]

\[
\leq C x_0^\lambda \left( \int_0^{1/n} y^\lambda |f(y)| \frac{dy}{(x_0^2 + t^2)^{\lambda + 1}} \right) + \int_n^{\infty} \frac{y^\lambda |f(y)|}{y^{2\lambda + 2}} dy + x_0^{\lambda + 1} \int_n^{\infty} \frac{y^\lambda |f(y)|}{(1 + y^2)^{\lambda + 1}} dy,
\]

with \( C \) independent of \( x, t \) and \( n \).

Then, we can find \( n_1 \in \mathbb{N}, n_1 \geq 4n_0, \) such that

\[
|P_t^\lambda (f - f \phi_n)(x)| < \varepsilon, \quad |x - x_0| < x_0/2, \quad t \in (0, 1), \quad n \in \mathbb{N}, \quad n \geq n_1.
\]

On the other hand, for each \( n \in \mathbb{N}, \) since \( f \phi_n \) is continuous and bounded on \((0, \infty), \)

\[
\lim_{(x,t) \to (x_0,0)} P_t^\lambda (f \phi_n)(x) = (f \phi_n)(x_0).
\]

By considering (21), (22) and (23) we conclude that

\[
\lim_{(x,t) \to (x_0,0)} P_t^\lambda (f)(x) = f(x_0).
\]

The space of \( \lambda \)-harmonic functions on \((0, \infty) \times \mathbb{R}\) form a Breton harmonic space. Then, it is well-known that \( \lambda \)-harmonic functions on \((0, \infty) \times \mathbb{R}\) satisfy the mean value properties with respect to the \( \lambda \)-harmonic measures. Recently, Eriksson and Orelma [27] have established explicit mean value properties for solutions of Weinstein operators. We recall some results in [27] specified for our particular case and that will be useful.

We consider on \((0, \infty) \times \mathbb{R}\) the hyperbolic metric \( d_h \) defined by

\[
d_h(a, b) = \text{arcosh} \sigma(a, b), \quad a, b \in (0, \infty) \times \mathbb{R},
\]

where

\[
\sigma(a, b) = \frac{(a_1 - b_1)^2 + (a_2 - b_2)^2 + 2a_1 b_1}{2 a_1 b_1}, \quad a = (a_1, b_1), b = (b_1, b_2) \in (0, \infty) \times \mathbb{R}.
\]

The hyperbolic ball \( B_h(a, r) \) with center \( a \in (0, \infty) \times \mathbb{R} \) and radius \( r > 0 \) is defined as usual by

\[
B_h(a, r) = \{ b \in (0, \infty) \times \mathbb{R} : d_h(a, b) < r \}.
\]

For every \( a \in (0, \infty) \times \mathbb{R} \) and \( r > 0, B_h(a, r) \) is actually an Euclidean ball. We have that, for each \( a = (a_1, a_2) \in (0, \infty) \times \mathbb{R} \) and \( r > 0 \)

\[
B_h(a, r) = \{ b \in (0, \infty) \times \mathbb{R} : |\tilde{a} - b| < a_1 \sinh r \},
\]

where \( \tilde{a} = (a_1 \cosh r, a_2). \)

In [11] Akin and Leutwiler introduced the function

\[
\varphi_\alpha(r) = \frac{(1 - r^2)^\alpha}{2} \int_{-1}^{1} \frac{dy}{|r - y|^{2\alpha}}, \quad 0 < r < 1,
\]

in their investigations about Weinstein equations.

From [27, Theorem 3.3] it follows the following mean value property for \( \lambda \)-harmonic functions.

\begin{lemma}
Let \( \lambda > 0 \). Assume that \( U \) is an open subset of \((0, \infty) \times \mathbb{R}\). If \( v \) is a \( \lambda \)-harmonic function in \( U \) then, for every \( a \in U \) and \( r > 0 \) such that \( B_h(a, r) \subset U, \)

\[
v(a) = \frac{1}{2 \sinh(r) \varphi_\alpha(\tanh(r/2))} \int_{\partial B_h(a, r)} v(b_1, b_2) \frac{d\tau(b_1, b_2)}{b_1},
\]

where \( \alpha = (1 + |2\lambda - 1|)/2 \) and \( \tau \) denotes the length measure on \( \partial B_h(a, r) \).
\end{lemma}

We now prove the converse of Lemma 3.3.
Lemma 3.4. Let \( \lambda > 0 \) and let \( U \) be an open subset of \( (0, \infty) \times \mathbb{R} \). Suppose that \( v \) is a continuous function on \( U \) such that the mean value property \( [24] \) holds for every \( a \in U \) and \( r > 0 \) such that \( B_h(a, r) \subset U \). Then, \( v \) is \( \lambda \)-harmonic in \( U \).

Proof. In order to show this property we follow a procedure similar to the classical one used to establish the corresponding result for harmonic functions.

In a first step we prove a maximum principle in this context. Let \( a \in U \) and \( r > 0 \) such that \( B_h(a, r) \subset U \). Since \( v \) is continuous in \( B_h(a, r) \), the set

\[
A = \{ b \in B_h(a, r) : v(b) \geq v(c), \ c \in B_h(a, r) \} \neq \emptyset.
\]

Suppose that \( A \cap \partial B_h(a, r) = \emptyset \). Since \( A \) is closed,

\[
d(A, \partial B_h(a, r)) = \min\{|c - z| : c \in A, z \in \partial B_h(a, r)\} > 0.
\]

We choose \( b \in A \) such that

\[
d(b, \partial B_h(a, r)) = \inf\{|b - z| : z \in \partial B_h(a, r)\} = d(A, \partial B_h(a, r))
\]

and \( R > 0 \) such that \( B_h(b, R) \subset B_h(a, r) \). We consider the sets

\[
M_+ = A \cap \partial B_h(b, R) \quad \text{and} \quad M_- = A^c \cap \partial B_h(b, R).
\]

Since \( \tau(M_-) > 0 \) we deduce that

\[
\frac{1}{2 \sinh(R) \varphi(\tanh(R/2))} \int_{\partial B_h(b, R)} v(z_1, z_2) \frac{d\tau(z_1, z_2)}{z_1} \leq \frac{1}{2 \sinh(R) \varphi(\tanh(R/2))} \left( \int_{M_+} + \int_{M_-} \right) v(z_1, z_2) \frac{d\tau(z_1, z_2)}{z_1} < v(b).
\]

We have taken into account that

\[
\int_{\partial B_h(b, R)} \frac{d\tau(z_1, z_2)}{z_1} = 2 \sinh(R) \varphi(\tanh R/2).
\]

Hence, since \( v \) satisfies \( [24] \) for every \( a \in U \) and \( r > 0 \) such that \( B_h(a, r) \subset U \), \( A \cap \partial B_h(a, r) \neq \emptyset \).

Then,

\[
\max_{b \in B_h(a, r)} v(b) = \max_{b \in \partial B_h(a, r)} v(b).
\]

We now observe that the operator

\[
\mathcal{L}_\lambda = \partial_x^2 + \partial_y^2 - \frac{\lambda(\lambda - 1)}{x^2},
\]

is uniformly elliptic on every bounded domain \( \Omega \) such that \( \Omega \subset (0, \infty) \times \mathbb{R} \). Then, for every \( b \in U \) and \( R > 0 \) such that \( B_h(b, R) \subset U \) and every continuous function \( f \) on \( \partial B_h(b, R) \), there exists a continuous function \( w \) in \( \overline{B_h(b, R)} \) such that \( w_{|\partial B_h(b, R)} = f \) and \( w \) is \( \lambda \)-harmonic in \( B_h(b, R) \). Hence, according to Lemma 3.3, this function \( w \) satisfies the mean value property \( [24] \) for every \( a \in B_h(b, r) \) and \( r > 0 \) such that \( B_h(a, r) \subset B_h(b, R) \).

Let \( b \in U \) and \( R > 0 \) such that \( B_h(b, R) \subset U \). We define \( f = v_{|\partial B_h(b, R)} \) and denote by \( w \) the continuous function in \( \overline{B_h(b, R)} \) such that \( w_{|\partial B_h(b, R)} = f \) and \( w \) is \( \lambda \)-harmonic in \( B_h(b, R) \). We consider the function \( F = v - w \) in \( B_h(b, R) \). It is clear that \( F_{|\partial B_h(b, R)} = 0 \) and \( F \) satisfies the mean value property \( [24] \) for every \( a \in B_h(b, R) \) and \( r > 0 \) such that \( B_h(a, r) \subset B_h(b, R) \). The maximum (minimum) property allows us to conclude that \( v = w \) in \( B_h(b, R) \). Thus, we prove that \( v \) is \( \lambda \)-harmonic in \( U \).

Remark. As it can be deduced from the proof of Lemma 3.4, in order to see that a function \( v \) continuous in an open subset \( U \) of \( (0, \infty) \times (0, \infty) \) is \( \lambda \)-harmonic in \( U \), it is sufficient to show that, for every \( a \in U \), there exists a sequence \( (r_n)_{n \in \mathbb{N}} \subset (0, \infty) \) such that \( r_n \to 0 \), as \( n \to \infty \), that \( B_h(a, r_n) \subset U \), \( n \in \mathbb{N} \), and

\[
v(a) = \frac{1}{2 \sinh(r_n) \varphi(\tanh(r_n/2))} \int_{\partial B_h(a, r_n)} v(b_1, b_2) \frac{d\tau(b_1, b_2)}{b_1},
\]

with \( \alpha = (1 + |2\lambda - 1|)/2 \).

Now we establish a uniqueness result for \( \lambda \)-harmonic functions in \( (0, \infty) \times (0, \infty) \).
Lemma 3.5. Let \( \lambda > 1 \). Suppose that \( v \) is a bounded and continuous function on \( (0, \infty) \times (0, \infty) \) such that \( v \) is \( \lambda \)-harmonic in \( (0, \infty) \times (0, \infty) \) and \( v(x, 0) = 0, x \in (0, \infty) \). Then, \( v = 0 \) in \( (0, \infty) \times (0, \infty) \).

Proof. We define
\[
w(x, t) = \begin{cases} v(x, t), & x \in (0, \infty), \ t \in [0, \infty) \\
-v(x, -t), & x \in (0, \infty), \ t \in (-\infty, 0),
\end{cases}
\]
w is a continuous function in \( (0, \infty) \times \mathbb{R} \). Moreover, \( w \) is \( \lambda \)-harmonic in \( (0, \infty) \times \mathbb{R} \setminus \{0\} \). According to Theorem 2.2 in order to see that \( w \) is \( \lambda \)-harmonic in \( (0, \infty) \times \mathbb{R} \) it is sufficient to observe that, for every \( x \in (0, \infty) \) and \( r > 0 \) such that \( B_r((x,0),r) \subset (0, \infty) \times \mathbb{R} \),
\[
0 = \int_{\partial B_r((x,0),r)} w(b_1,b_2) \frac{d\gamma(b_1,b_2)}{b_1}.
\]
Note that this property holds because \( w \) is odd in the second variable and every hyperbolic ball centered in the line \((0, \infty) \times \{0\}\) is actually an Euclidean ball with center in the same line.

Since \( v \) is bounded in \( (0, \infty) \times (0, \infty) \), \( w \) is also bounded in \( (0, \infty) \times \mathbb{R} \). Then, there exists \( M > 0 \) such that \( |w(x, t)| \leq M, x \in (0, \infty) \) and \( t \in \mathbb{R} \). The function \( g(x, t) = x^\lambda + x^{1-\lambda}, x \in (0, \infty) \) and \( t \in \mathbb{R} \), is \( \lambda \)-harmonic in \( (0, \infty) \times \mathbb{R} \). We define the function
\[
\tilde{w}(x, t) = w(x, t) + M(x^\lambda + x^{1-\lambda}), \ x \in (0, \infty) \text{ and } t \in \mathbb{R}.
\]
Thus, \( \tilde{w}(x, t) \geq 0, x \in (0, \infty) \) and \( t \in \mathbb{R} \), and \( \tilde{w} \) is \( \lambda \)-harmonic in \( (0, \infty) \times \mathbb{R} \). According to [39] Theorem 2.2 there exists a positive \( \sigma \)-finite measure \( \gamma \) on \( \mathbb{R} \) and \( m \geq 0 \) such that
\[
\tilde{w}(x, t) = x^\lambda \left( m + \int_{-\infty}^{\infty} \frac{d\gamma(s)}{(t-s)^2 + x^2} \right), \ x \in (0, \infty) \text{ and } t \in \mathbb{R}.
\]
Then,
\[
w(x, t) = -Mx^{1-2\lambda} + M + \int_{-\infty}^{\infty} \frac{d\gamma(s)}{(s^2 + x^2)^\lambda} = 0, \ x \in (0, \infty).
\]
By letting \( x \to +\infty \) and by dominated convergence theorem we deduce that \( m = M \). Hence,
\[
(25) \quad w(x, t) = x^{1-\lambda} \left( -M + \int_{-\infty}^{\infty} \frac{x^{2\lambda-1}}{(t-s)^2 + x^2} d\gamma(s) \right), \ x \in (0, \infty) \text{ and } t \in \mathbb{R},
\]
and again, since \( w(x, 0) = 0, x \in (0, \infty) \), we deduce that
\[
(26) \quad M = \int_{-\infty}^{\infty} \frac{x^{2\lambda-1}}{(s^2 + x^2)^\lambda} d\gamma(s), \ x \in (0, \infty).
\]
By using Radon-Nikodym theorem we can write \( d\gamma(s) = hds + d\mu(s) \), where \( 0 \leq h \in L^1_{\text{loc}}(\mathbb{R}) \) and \( \mu \) is a positive measure that is orthogonal to the Lebesgue measure on \( \mathbb{R} \).

It can be seen that
\[
(27) \quad \lim_{x \to 0^+} \int_{-\infty}^{\infty} \frac{x^{2\lambda-1}}{(t-s)^2 + x^2} d\gamma(s) = Ah(t), \ \text{a.e. } t \in \mathbb{R}.
\]
Here, a.e. is understood with respect to the Lebesgue measure on \( \mathbb{R} \) and
\[
A = \int_{-\infty}^{\infty} \frac{1}{(s^2 + 1)^\lambda} ds = \frac{\sqrt{\pi} \Gamma(\lambda - 1/2)}{\Gamma(\lambda)}.
\]
Indeed, fix \( N \in \mathbb{N} \). It is sufficient to see [27] for a.e. \( |t| \leq N \). Denote by \( K_x, x \in (0, \infty) \), the kernel
\[
K_x(t, s) = \frac{x^{2\lambda-1}}{(t-s)^2 + x^2}^\lambda, \ t, s \in \mathbb{R}.
\]
For every \( n \in \mathbb{N} \), let us define \( h_n(t) = b(t)\chi(-n,n)(t) \), \( t \in \mathbb{R} \). Then, since \( \int_{-\infty}^{\infty} K_x(t, s) ds = A, \ x \in (0, \infty), t \in \mathbb{R} \), it follows that, for each \( n \in \mathbb{N} \), \( n \geq N \), we can write
\[
\int_{-\infty}^{\infty} K_x(t, s) d\gamma(s) - Ah(t) = \int_{-\infty}^{\infty} K_x(t, s) [h(s) - h_n(s)] ds + \int_{-\infty}^{\infty} K_x(t, s) h_n(s) ds - Ah_n(t)
\]+ \int_{-\infty}^{\infty} K_x(t, s) d\mu(s), \ x \in (0, \infty), |t| \leq N.
\]
When \( n \geq 2N \), the first term can be bounded as follows,
\[
\left| \int_{-\infty}^{+\infty} K_x(t, s)[h(s) - h_n(s)]ds \right| \leq \int_{|s| > n} \frac{|h(s)|}{((t-s)^2 + x^2)^\lambda} ds \leq C \int_{|s| > n} \frac{|h(s)|}{(s^2 + 1)^\lambda} ds, \quad x \in (0, 1), \ |t| \leq N.
\]

Thus, for every \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N}, n_0 \geq 2N \), independent of \( x \in (0, 1) \) and \( |t| \leq N \), such that
\[
(29) \quad \left| \int_{-\infty}^{+\infty} K_x(t, s)[h(s) - h_{n_0}(s)]ds \right| < \varepsilon, \quad x \in (0, 1), \ |t| \leq N.
\]

On the other hand, we observe that
\[
|K_x(t, s)| \leq C \begin{cases} 
\frac{1}{x}, & |t-s| < x, \\
\frac{1}{2(2\lambda - 1)k^2k_x}, & 2^{k-1}x \leq |t-s| < 2^kx,
\end{cases} \quad x \in (0, \infty), \ t, s \in \mathbb{R}, \ k \in \mathbb{N}.
\]

Then, since \( \lambda > 1 \), it is not difficult to see that
\[
\sup_{x \in (0, \infty)} \int_{-\infty}^{+\infty} K_x(t, s)h_{n_0}(s)ds \leq C.\mathcal{M}(|h_{n_0}|)(t), \quad t \in \mathbb{R},
\]
and
\[
\sup_{x \in (0, \infty)} \int_{-\infty}^{+\infty} K_x(t, s)d\mu(s)ds \leq C.\mathcal{M}(\mu)(t), \quad t \in \mathbb{R},
\]
where \( \mathcal{M} \) represents the classical Hardy-Littlewood maximal function defined on \( L^1(\mathbb{R}) \) and on the set of the Borel measures on \( \mathbb{R} \).

By following standard arguments (see [2] Theorems 6.39 and 6.42, for instance) we obtain that
\[
(30) \quad \lim_{x \to 0^+} \int_{-\infty}^{+\infty} K_x(t, s)h_{n_0}(s)ds = Ah_{n_0}(t), \quad a.e. \ t \in \mathbb{R},
\]
and
\[
(31) \quad \lim_{x \to 0^+} \int_{-\infty}^{+\infty} K_x(t, s)d\mu(s) = 0, \quad a.e. \ t \in \mathbb{R}.
\]

Putting together (25), (29), (30), and (31) we obtain (27) for a.e. \( |t| \leq N \).

By taking into account that \( w \) is a bounded function in \((0, \infty) \times \mathbb{R} \) and \( \lambda > 1 \), from (25) we deduce that
\[
-M + Ah(t) = 0, \quad a.e. \ t \in \mathbb{R},
\]
and by (26), it follows that
\[
\int_{-\infty}^{+\infty} \frac{d\mu(s)}{(s^2 + x^2)^\lambda} = 0, \quad x \in (0, \infty).
\]

Hence, \( \mu = 0 \). By using again (25) we obtain
\[
w(x, t) = x^{1-\lambda} \left( -M + \frac{M}{A} \int_{-\infty}^{+\infty} \frac{x^{2\lambda - 1}}{(t-s)^2 + x^2} ds \right) = 0, \quad x \in (0, \infty) \text{ and } t \in \mathbb{R}.
\]

Then \( v(x, t) = 0, \ x \in (0, \infty) \) and \( t \geq 0 \).

**Proof of (18).** Let \( t_0 > 0 \). We define the function \( v(x, t) = \partial_t u(x, t + t_0), \ x \in (0, \infty) \) and \( t \in [0, \infty) \).

We have that \( v \) is bounded (see (17)), continuous in \((0, \infty) \times [0, \infty) \) and \( \lambda \)-harmonic in \((0, \infty) \times (0, \infty) \).

We consider \( f(x) = v(x, 0), \ x \in (0, \infty) \), and define
\[
V(x, t) = \begin{cases} 
P_t^\lambda(f)(x), & x, t \in (0, \infty), \\
f(x), & x \in (0, \infty) \text{ and } t = 0.
\end{cases}
\]

Since \( f \) is bounded and continuous in \((0, \infty) \), by Lemma 3.2 the function \( V \) is continuous and bounded in \((0, \infty) \times [0, \infty) \) and \( \lambda \)-harmonic in \((0, \infty) \times (0, \infty) \). The function \( V - v \) is bounded and continuous in \((0, \infty) \times [0, \infty) \), and \( \lambda \)-harmonic in \((0, \infty) \times (0, \infty) \). Moreover, \( V(x, 0) = v(x, 0), \ x \in (0, \infty) \). According to Lemma 3.5 \( V(x, t) = v(x, t), \ x \in (0, \infty) \) and \( t \in [0, \infty) \). Thus, (18) is established. \( \square \)
Our next objective is to establish that
\begin{equation}
(32) \quad u(x, t + r) = P_t^\lambda(u(\cdot, r))(x), \quad x, t, r \in (0, \infty).
\end{equation}

We have that, for every \( r > 0 \),
\begin{equation}
(33) \quad \int_0^\infty \frac{y^\lambda |u(y, r)|}{(1 + y^\lambda)^{\lambda+1}} dy < \infty,
\end{equation}
and then the integral defining \( P_t^\lambda(u(\cdot, r))(x) \) is absolutely convergent, for every \( x, t \in (0, \infty) \).

In order to show \((34)\) we see previously that
\begin{equation}
(34) \quad \lim_{r \to \infty} \int_0^\infty \partial_t P_t^\lambda(x, y)u(y, r)dy = 0, \quad x, t \in (0, \infty).
\end{equation}

We note that the arguments that we will use to prove \((34)\) also allow us to obtain \((33)\).

**Proof of \((34)\).** Since, for every \( x, t \in (0, \infty), \int_0^\infty P_t^\lambda(x, y)y^\lambda dy = x^\lambda \) (\cite[p. 84]{44} we can write
\begin{equation}
\begin{split}
\partial_t \int_0^\infty P_t^\lambda(x, y)u(y, r)dy &= \partial_t \int_0^\infty P_t^\lambda(x, y)y^\lambda y^{-\lambda}u(y, r)dy \\
&= \partial_t \int_0^\infty P_t^\lambda(x, y)y^\lambda [y^{-\lambda}u(y, r) - x^{-\lambda}u(x, r)]dy \\
&= \partial_t \int_0^\infty P_t^\lambda(x, y)y^\lambda \int_x^y \partial_z[z^{-\lambda}u(z, r)]dzdy, \quad x, t, r \in (0, \infty).
\end{split}
\end{equation}

Moreover, we have that
\begin{equation}
\begin{split}
\left| \int_x^y \partial_z[z^{-\lambda}u(z, r)]dz \right| &\leq \int_x^y |D_{\lambda, z}u(z, r)|z^{-\lambda}dz \\
&\leq C |y^{1-\lambda} - x^{1-\lambda}| \sup_{z \in I_{x,y}} |D_{\lambda, z}u(z, r)|, \quad x, y, r \in (0, \infty).
\end{split}
\end{equation}

Here, \( I_{x,y} = [\min\{x, y\}, \max\{x, y\}], x, y \in (0, \infty) \).

Since \( u \) is \( \lambda \)-harmonic in \((0, \infty) \times (0, \infty) \), we get
\begin{equation}
(\partial_t^2 - D_{\lambda,x} D_{\lambda,x}^*) D_{\lambda,x} u(x, t) = D_{\lambda,x} (\partial_t^2 - D_{\lambda,x}^*) D_{\lambda,x} u(x, t) = 0, \quad x, t \in (0, \infty).
\end{equation}

Note that
\begin{equation}
-D_{\lambda,x} D_{\lambda,x}^* = x^\lambda D_x e^{-2\lambda} D_x x^\lambda = u'' - \frac{(\lambda + 1)\lambda}{x^2} u = B_{\lambda+1}.
\end{equation}

Then, \( D_{\lambda,x} u \) is \((\lambda+1)\)-harmonic in \((0, \infty) \times (0, \infty) \). Moreover, \( x^{-\lambda-1}D_{\lambda,x} u = \frac{1}{\lambda} \partial_x (x^{-\lambda} u) \) is regular in \( \mathbb{R} \times (0, \infty) \) and even in the \( x \)-variable. By proceeding as in the beginning of Section 3 after Lemma \ref{3.3} we can see that \( (D_{\lambda,x} u)^2 \) is subharmonic in \( \mathbb{R} \times (0, \infty) \).

Let \( x, t \in (0, \infty) \). The subharmonicity of \( (D_{\lambda,x} u)^2 \) allows us to write
\begin{equation}
\begin{split}
\sup_{z \in I_{x,y}} |D_{\lambda, z}u(z, r)| &\leq C \sup_{z \in I_{x,y}} \left( \frac{1}{r^2} \int_{B[(z,r), r/4]} |D_{\lambda,a}u(a, b)|^2 dadb \right)^{1/2} \\
&\leq C \left( \frac{1}{r^2} \int_{\frac{r}{4}}^{\frac{r}{4}} \int_{x - \frac{5r}{4}}^{x + \frac{5r}{4}} |D_{\lambda,a}u(a, b)|^2 dadb \right)^{1/2} \\
&\leq C \|b|D_{\lambda,a}u(a, b)|^2 dadb\|_r^{1/2}, \quad |x - y| \leq r, y, r \in (0, \infty).
\end{split}
\end{equation}

Also, we have that (see \cite[Lemma 3.2]{13})
\begin{equation}
(36) \quad |y^{1-\lambda} - x^{1-\lambda}| \leq C |x - y| \min\{x, y\}^{2 - \lambda}, \quad y \in (0, \infty).
\end{equation}

Then, by using \cite{12} we obtain
\begin{equation}
(37) \quad \left| \int_{0, |x - y| \leq r} \partial_t P_t^\lambda(x, y)y^\lambda \int_x^y \partial_z[z^{-\lambda}u(z, r)]dzdy \right| \\
\leq C \frac{\|b|D_{\lambda,a}u(a, b)|^2 dadb\|_r^{1/2}}{r} \int_0^\infty \frac{x^{\lambda-1}y^{\lambda+1}|x - y| \min\{x, y\}^{2 - \lambda}}{((x - y)^2 + t^2)^{\lambda+1}} dy \\
\leq C \left[ \int_{|x - y| \leq r} \frac{x^{\lambda-1}y^{\lambda+1}|x - y|}{((x - y)^2 + t^2)^{\lambda+1}} dy + \int_x^{x+r} \frac{xy^{\lambda-1}|x - y|}{((x - y)^2 + t^2)^{\lambda+1}} dy \right]
By combining (35), (38) and (39) we deduce that (34) holds.

By combining (35), (38) and (39) we deduce that (34) holds.
Proof of (32). According to (15) we have that
\[ \partial_t u(x, t + r) = \partial_t u(x, t + r) = P_t^X \left[ \partial_t u(\cdot, r) \right](x), \quad x, t, r \in (0, \infty). \]
Since the differentiation under the integral sign is justified by the properties of the function \( u \), we obtain
\[ \partial_t |u(x, t + r) - P_t^X(u(\cdot, r))(x)| = 0, \quad x, t, r \in (0, \infty), \]
and
\[ \partial_t \left[ \partial_t u(x, t + r) - \partial_t P_t^X(u(\cdot, r))(x) \right] = 0, \quad x, t, r \in (0, \infty). \]
From (16) and (34) it follows that
\[ \lim_{r \to \infty} \left| \partial_t u(x, t + r) - \partial_t P_t^X(u(\cdot, r))(x) \right| = 0, \quad x, t \in (0, \infty). \]
Then,
\[ \partial_t |u(x, t + r) - P_t^X(u(\cdot, r))(x)| = 0, \quad x, t, r \in (0, \infty). \]
Also, (33) and Lemma 3.2 lead to
\[ \lim_{t \to 0^+} (u(x, t + r) - P_t^X(u(\cdot, r))(x)) = 0, \quad x, r \in (0, \infty). \]
We conclude that
\[ u(x, t + r) = P_t^X(u(\cdot, r))(x), \quad x, t, r \in (0, \infty), \]
and (32) is proved. \( \square \)

For every \( k \in \mathbb{N} \), we define
\[ u_k(x, t) = u \left( x, t + \frac{1}{k} \right), \quad x \in (0, \infty), \quad t \in [0, \infty). \]
We now establish that there exists \( C > 0 \) such that
\[ \sup_I \frac{1}{|I|} \int_I \int_I t|\partial_t u_k(x, t)|^2 dx dt \leq C \| t|\partial_t u(x, t)|^2 dx dt \|_\varphi, \]
where the supremum is taken over all bounded intervals \( I \subset (0, \infty) \).

Proof of (40). Let \( k \in \mathbb{N} \) and let \( I \) be a bounded interval in \( (0, \infty) \). Suppose that \( |I| \geq 1/k \). We obtain
\[ \frac{1}{|I|} \int_I \int_I t|\partial_t u_k(x, t)|^2 dx dt \leq \frac{1}{|I|} \int_I \int_I \left( t + \frac{1}{k} \right) \left| \partial_t u \right|(x, t + \frac{1}{k})^2 dx dt \]
\[ \leq \frac{1}{|I|} \int_I \int_I |s| \partial_s u(x, s)|^2 dx ds \leq 2 \| s| \partial_s u(x, s)|^2 dx ds \|_\varphi, \]
where \( I = (a, 2b - a) \) when \( I = (a, b) \) with \( 0 \leq a < b < \infty \).
Assume now that \( |I| < 1/k \). According to (16) we deduce that
\[ \left| \partial_t u \left( x, t + \frac{1}{k} \right) \right| \leq \frac{C}{k + 1/k} \| s| \partial_s u(x, s)|^2 dx ds \|_\varphi^{1/2}, \quad x, t \in (0, \infty). \]
Then
\[ \frac{1}{|I|} \int_I \int_I t|\partial_t u \left( x, t + \frac{1}{k} \right) |^2 dx dt \leq \frac{C}{|I|} \| s| \partial_s u(x, s)|^2 dx ds \|_\varphi \int_I \int_I \frac{t}{(t + 1/k)^2} dx ds \]
\[ \leq C \| s| \partial_s u(x, s)|^2 dx ds \|_\varphi k^2 \int_I \int_I t dt \leq C \| s| \partial_s u(x, s)|^2 dx ds \|_\varphi. \]
Putting together (41) and (42) we prove (40). \( \square \)

We define, for every \( k \in \mathbb{N} \), \( f_k(x) = u_k(x, 0), \ x \in (0, \infty) \). By (32), (33), (40) and Lemma 3.1 we obtain that, for every \( k \in \mathbb{N} \), \( f_k \in BMO_o(\mathbb{R}) \) and
\[ \| f_k \|_{BMO_o(\mathbb{R})} \leq C \| s| \partial_s u(x, s)|^2 dx ds \|_\varphi^{1/2}. \]
Hardy spaces associated with Bessel operators have been studied in [7] and [24]. A function \( f \in L^1(0, \infty) \) is in the Hardy space \( H^1_2(\mathbb{R}) \) provided that
\[ \sup_{t \geq 0} |P_t^X(f)| \in L^1(0, \infty), \]
for some (equivalently, for every) $\nu > 1$. For every $\nu > 1$, we define
\[ ||f||_{H^1_\nu} := \sup_{t>0} |P^\nu_t(f)||_{L^1(0,\infty)}, \quad f \in H^1_\nu(\mathbb{R}). \]

For each $\nu, \mu > 1$, the norms $\| \cdot \|_{H^1_\nu}$ and $\| \cdot \|_{H^1_\mu}$ are equivalent on $H^1_\nu(\mathbb{R})$. The space $H^1_\nu(\mathbb{R})$ endowed with the norm $\| \cdot \|_{H^1_\nu}$ ($\nu > 1$) is a Banach space. The dual space of $H^1_\nu(\mathbb{R})$ is $BMO_\nu(\mathbb{R})$ ([20 Theorem 1]).

To finish the proof the following results will be useful.

**Lemma 3.6.** Let $\nu > 0$. For every $x, t \in (0, \infty)$, $P^\nu_t(x, \cdot) \in H^1_\nu(\mathbb{R})$.

**Proof.** Let $x, t \in (0, \infty)$. From the semigroup property it follows that
\[ P^\nu_t[P^\nu_s(x, \cdot)](z) = \int_0^\infty P^\nu_s(z, y)P^\nu_t(x, y)dy = P^\nu_{t+s}(x, z), \quad z, s \in (0, \infty). \]

According to [42, p. 86, (b)] we have that
\[ |P^\nu_s[|P^\nu_t(x, \cdot)|](z)| \leq C \frac{(t+s)(xz)^\nu}{(t+s)^2 + (x-z)^2} \leq C \frac{(xz)^\nu}{(t+s + |x-z|)^{2\nu+1}}, \quad z, s \in (0, \infty). \]

Then,
\[ \int_0^\infty \sup_{s > 0} |P^\nu_s[|P^\nu_t(x, \cdot)|](z)|dz \leq C \left( \int_0^{2x} + \int_2^\infty \right) \frac{(xz)^\nu}{(t+z)^{2\nu+1}}dz \leq C \left( \frac{2x}{t} \frac{(xz)^\nu}{(t+z)^{2\nu+1}} + \frac{x}{t} \nu \right). \]

Thus, we prove that $P^\nu_t(x, \cdot) \in H^1_\nu(\mathbb{R})$. 

**Lemma 3.7.** Assume that $g \in BMO_\nu(\mathbb{R})$ and $G \in H^1_\nu(\mathbb{R})$ satisfying that $gG \in L^1(0, \infty)$. Then,

(44) \[ \langle g, G \rangle_{BMO_\nu(\mathbb{R}), H^1_\nu(\mathbb{R})} = \int_0^\infty g(x)G(x)dx \]

**Proof.** According to the atomic characterization of $H^1(\mathbb{R})$ ([7, Theorem 1.10]) we can find a sequence of measurable functions of compact support $(G_j)_{j \in \mathbb{N}}$ such that, for every $j \in \mathbb{N}$, $G_j$ is a linear combination of $H^1(\mathbb{R})$-atoms, and $G_j \to G$, as $j \to \infty$, in $H^1_\nu(\mathbb{R})$. Then,

\[ \langle g, G \rangle_{BMO_\nu(\mathbb{R}), H^1_\nu(\mathbb{R})} = \lim_{j \to \infty} \langle g, G_j \rangle_{BMO_\nu(\mathbb{R}), H^1_\nu(\mathbb{R})} = \lim_{j \to \infty} \int_0^\infty g(x)G_j(x)dx. \]

On the other hand, since $gG \in L^1(0, \infty)$ and $gG_j \in L^1(0, \infty)$, $j \in \mathbb{N}$, by [5] p. 25, we have that

\[ \left| \int_0^\infty g(x)(G(x) - G_j(x))dx \right| \leq \|g\|_{BMO_\nu(\mathbb{R})} \|G - G_j\|_{H^1_\nu(\mathbb{R})}, \quad j \in \mathbb{N}. \]

By letting $j \to \infty$, we conclude (44). 

By using Banach-Alaoglu theorem and by taking into account (43) there exists $f \in BMO_\nu(\mathbb{R})$ and a strictly increasing $\phi : \mathbb{N} \to \mathbb{N}$ such that $f_{\phi(k)} \to f$, as $k \to \infty$, in the weak star topology of $BMO_\nu(\mathbb{R})$, that is, for every $g \in H^1_\nu(\mathbb{R})$,

(45) \[ \langle f_{\phi(k)}, g \rangle_{BMO_\nu(\mathbb{R}), H^1_\nu(\mathbb{R})} \to \langle f, g \rangle_{BMO_\nu(\mathbb{R}), H^1_\nu(\mathbb{R})}, \quad as \, k \to \infty. \]

Moreover,
\[ \|f\|_{BMO_\nu(\mathbb{R})} \leq C \|s|\partial_s u(s, s)|^2ds\|_o^{1/2}. \]

By using (43) and Lemma 3.6 we obtain, for every $x, t \in (0, \infty)$,

\[ \langle f_{\phi(k)}, P^\nu_t(x, \cdot) \rangle_{BMO_\nu(\mathbb{R}), H^1_\nu(\mathbb{R})} \to \langle f, P^\nu_t(x, \cdot) \rangle_{BMO_\nu(\mathbb{R}), H^1_\nu(\mathbb{R})}, \quad as \, k \to \infty. \]

Since $BMO_\nu(\mathbb{R}) \subset BMO(\mathbb{R})$, by [42, p. 86, (b)] and [50, p. 141], for every $g \in BMO_\nu(\mathbb{R})$, we get
\[ \int_0^\infty |g(y)||P^\nu_t(x, y)|dy \leq C \int_0^\infty \frac{|g(y)|}{1 + y^2} dy \sup_{y \in (0, \infty)} \frac{1 + y^2}{t^2 + (x - y)^2} \leq C \left( \frac{1}{t^2} + \frac{1}{t^2 + (x - y)^2} \right) \leq C \left( \frac{1 + x^2}{t^2} + 1 \right), \quad x, t \in (0, \infty). \]
For every \( x,t \in (0,\infty) \), Lemma 3.7 leads to

\[
\int_0^\infty f(\phi_k(y)) P_k^\Lambda(x,y) dy \to \int_0^\infty f(y) P_k^\Lambda(x,y) dy, \quad \text{as } k \to \infty.
\]

By (32) we conclude that

\[
u(x,t) = \int_0^\infty f(y) P_k^\Lambda(x,y) dy, \quad x,t \in (0,\infty).
\]

Thus the proof is finished.

**References**

[1] Ö. Akin and H. Leutwiler, *On the invariance of the solutions of the Weinstein equation under Möbius transformations*, in Classical and modern potential theory and applications (Chateau de Bonas, 1993), vol. 430 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer Acad. Publ., Dordrecht, 1994, pp. 19–29.

[2] S. Axler, P. Bourdon, and W. Ramey, *Harmonic function theory*, vol. 137 of Graduate Texts in Mathematics, Springer-Verlag, New York, second ed., 2001.

[3] J. J. Betancor, D. Buraczewski, J. C. Fariña, T. Martínez, and J. L. Torrea, *Riesz transforms related to Bessel operators*, Proc. Roy. Soc. Edinburgh Sect. A, 137 (2007), pp. 701–725.

[4] J. J. Betancor, A. J. Castro, and J. Curbelo, *Harmonic analysis operators associated with multidimensional Bessel operators*, Proc. Roy. Soc. Edinburgh Sect. A, 142 (2012), pp. 945–974.

[5] J. J. Betancor, A. J. Castro, and L. Rodríguez-Mesa, *UMD-valued square functions associated with Bessel operators in Hardy and BMO spaces*, arXiv:1303.5571v1, 2013.

[6] J. J. Betancor, A. Chicco Ruiz, J. C. Fariña, and L. Rodríguez-Mesa, *Odd BMO(\mathbb{R}) functions and Carleson measures in the Bessel setting*, Integral Equations Operator Theory, 66 (2010), pp. 463–494.

[7] J. J. Betancor, J. Dziubański, and J. L. Torrea, *On Hardy spaces associated with Bessel operators*, J. Anal. Math., 107 (2009), pp. 195–219.

[8] J. J. Betancor, J. C. Fariña, T. Martínez, and L. Rodríguez-Mesa, *Higher order Riesz transforms associated with Bessel operators*, Ark. Mat., 46 (2008), pp. 219–250.

[9] J. J. Betancor, J. C. Fariña, and A. Sanabria, *On Littlewood-Paley functions associated with Bessel operators*, Glasg. Math. J., 51 (2009), pp. 55–70.

[10] J. J. Betancor, T. Martínez, and L. Rodríguez-Mesa, *Laplace transform type multipliers for Hankel transforms*, Canad. Math. Bull., 51 (2008), pp. 487–496.

[11] J. J. Betancor and K. Stempak, *On Hankel conjugate functions*, Studia Sci. Math. Hungar., 41 (2004), pp. 59–91.

[12] G. Bourdaud, *Remarques sur certains sous-espaces de BMO(\mathbb{R}^n) et de bmo(\mathbb{R}^n)*, Ann. Inst. Fourier (Grenoble), 52 (2002), pp. 1187–1218.

[13] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. (2), 76 (1962), pp. 547–559.

[14] S. Chaabi and S. Rigat, *Decomposition theorem and Riesz basis for axisymmetric potentials in the right half-plane*, arXiv:1402.0473v1, 2014.

[15] S.-Y. A. Chang and R. Fefferman, *A continuous version of duality of H^1 with BMO on the bidisc*, Ann. of Math. (2), 112 (1980), pp. 179–201.

[16] J. Chen, *Boundary behavior of harmonic functions on manifolds*, J. Math. Anal. Appl., 267 (2002), pp. 310–328.

[17] J. C. Chen and C. Luo, *Duality of H^1 and BMO on positively curved manifolds and their characterizations*, in Harmonic analysis (Tianjin, 1988), vol. 1494 of Lecture Notes in Math., Springer, Berlin, 1991, pp. 23–38.

[18] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. (9), 72 (1993), pp. 247–286.

[19] M. Cotlar and C. Sadosky, *Two distinguished subspaces of product BMO and Nehari-AAK theory for Hankel operators on the torus*, Integral Equations Operator Theory, 26 (1996), pp. 273–304.

[20] J. E. Daly and S. Fridli, *The dual spaces of certain Hardy spaces on \( \mathbb{R}^n \) and on \( \mathbb{N} \)*, Ann. Univ. Ferrara, Sez. VII, Sc. Mat., 25 (1979), pp. 147–157.

[21] Spaces of harmonic functions representable by Poisson integrals of functions in BMO and \( L_{p,\lambda} \), Indiana Univ. Math. J., 25 (1976), pp. 159–170.
E. B. Fabes and U. Neri, Dirichlet problem in Lipschitz domains with BMO data, Proc. Amer. Math. Soc., 78 (1980), pp. 33–39.

C. Fefferman and E. M. Stein, $H^p$ spaces of several variables, Acta Math., 129 (1972), pp. 137–193.

S. H. Ferguson and M. T. Lacey, A characterization of product BMO by commutators, Acta Math., 189 (2002), pp. 143–160.

J. B. Garnett, Bounded analytic functions, Springer, New York, 2007.

I. I. Hirschman, Jr., Variation diminishing Hankel transform, J. d’Analyse Math., 8 (1960-61), pp. 307–336.

S. Hofmann and S. Mayboroda, Hardy and BMO spaces associated to divergence form elliptic operators, Math. Ann., 344 (2009), pp. 37–116.

T. Hytönen and L. Weis, The Banach space-valued BMO, Carleson’s condition, and paraproducts, J. Fourier Anal. Appl., 16 (2010), pp. 495–513.

F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math., 14 (1961), pp. 415–426.

M. Lacey and E. Terwilleger, Hankel operators in several complex variables and product BMO, Houston J. Math., 35 (2009), pp. 159–183.

H. Leutwiler, Best constants in Harnack inequality for the Weinstein equation, Aequ. Math., 34 (1987), pp. 304–315.

H. Liu, L. Tang, and H. Zhu, Weighted Hardy spaces and BMO spaces associated with Schrödinger operators, Math. Nachr., 285 (2012), pp. 2173–2207.

J. Mateu, P. Mattila, A. Nicolau, and J. Orobitg, BMO for nondoubling measures, Duke Math. J., 102 (2000), pp. 533–565.

B. Muckenhoupt and E. M. Stein, Classical expansions and their relation to conjugate harmonic functions, Trans. Amer. Math. Soc., 118 (1965), pp. 17–92.

F. Nazarov, S. Treil, and A. Volberg, The $Tb$-theorem on non-homogeneous spaces, Acta Math., 190 (2003), pp. 151–239.

A. Nowak and L. Roncal, Potential operators associated with Jacobi and Fourier-Bessel expansions, arXiv:1212.6342v2, 2013.

A. Nowak and K. Stempak, Weighted estimates for the Hankel transform transplantation operator, Tohoku Math. J. (2), 58 (2006), pp. 277–301.

S. Pott and C. Sadosky, $L^p$-contractivity of Laguerre semigroups, Illinois J. Math., 56 (2012), pp. 433–452.

S. Polo and C. Sadosky, Bounded mean oscillation on the bidisk and operator BMO, J. Funct. Anal., 189 (2002), pp. 475–495.

R. Rochberg and S. Seoames, A decomposition theorem for BMO and applications, J. Funct. Anal., 67 (1986), pp. 228–263.

Z. W. Shen, $L^p$ estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier (Grenoble), 45 (1995), pp. 513–546.

E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, vol. 43 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1993.

E. C. Titchmarsh, Introduction to the theory of Fourier integrals, Chelsea Publishing Co., New York, third ed., 1986.

X. Tolsa, BMO, $H^1$, and Calderón-Zygmund operators for non doubling measures, Math. Ann., 319 (2001), pp. 89–149.

M. Varapoulos, BMO function and the $\overline{\partial}$ equation, Pacific J. Math., 71 (1977), pp. 221–273.

M. Villani, Riesz transforms associated to Bessel operators, Illinois J. Math., 52 (2008), pp. 77–89.

Jorge J. Betancor, Alejandro J. Castro, Juan C. Fariña and Lourdes Rodríguez-Mesa
Departamento de Análisis Matemático, Universidad de La Laguna, Campus de Anchieta, Avenida Astrofísico Francisco Sánchez, s/n, 38271, La Laguna (Sta. Cruz de Tenerife), Spain
E-mail address: jbetanco@ull.es, ajcastro@ull.es, jcfarina@ull.es, lrguez@ull.es