Stochastic Logarithmic Lipschitz Constants:
A Tool to Analyze Contractivity of Stochastic Differential Equations

Zahra Aminzare

Abstract—We introduce the notion of stochastic logarithmic Lipschitz constants and use these constants to characterize stochastic contractivity of Itô stochastic differential equations (SDEs) with multiplicative noise. We find an upper bound for stochastic logarithmic Lipschitz constants based on known logarithmic norms (matrix measures) of the Jacobian of the drift and diffusion terms of the SDEs. We discuss noise-induced contractivity in SDEs and common noise-induced synchronization in network of SDEs and illustrate the theoretical results on a noisy Van der Pol oscillator. We show that a deterministic Van der Pol oscillator is not contractive. But, adding a multiplicative noise makes the underlying SDE stochastically contractive.

I. INTRODUCTION

Contraction theory is a methodology for assessing the global convergence of trajectories of a dynamical system to each other instead of convergence to a pre-specified attractor. Given a vector norm with its induced matrix norm, the Logarithmic norm of a linear operator \( A \) is defined as the directional derivative of the matrix norm in the direction of \( A \) and evaluated at the identity matrix, \([1], [2]\). This definition can be extended to any nonlinear operator using the notion of logarithmic Lipschitz constant \([3]\). Logarithmic Lipschitz constants of a nonlinear vector field or the logarithmic norm of the Jacobian of the vector field can characterize the contraction property of a nonlinear system.

Studying contractivity of ordinary differential equations (ODEs) and reaction-diffusion partial differential equations using logarithmic norms and logarithmic Lipschitz constants is a well-established research topic (see e.g. \([4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]\)). However, there are not too many attempts to study contractivity of non-deterministic systems and in particular, Itô stochastic differential equations (SDEs). In \([21], [22]\) contraction theory is studied using stochastic Lyapunov function and incremental stability. In \([23], [24], [25]\) contraction theory is studied for random dynamical systems. In \([26], [27]\) contractivity is generalized to Riemannian metrics and Wasserstein norms, respectively. In \([28]\), stochastic contraction is studied for Poisson shot noise and finite-measure Lévy noise. This work takes a step forward and extends contraction theory to SDEs using generalized forms of logarithmic norms and logarithmic Lipschitz constants.

Stochastic contraction theory can be used to study the stability of SDEs and to characterize the synchronization behavior of networks of nonlinear and noisy systems. Synchronization induced by common noise has been observed experimentally and confirmed theoretically in many networks of nonlinear dynamical systems without mutual coupling. Indeed, this kind of synchronization is equivalent to the stochastic contraction of SDEs that we study in Section \( V \) below. Therefore, extending contraction theory to SDEs can be beneficial for characterizing networks’ synchronization.

In \([29]\), the authors introduced stochastic logarithmic norms and used them to study the stability properties of linear SDEs. Analog to the deterministic version, stochastic logarithmic norms are proper tools for characterizing the contractivity of linear SDEs, but they are not directly applicable to nonlinear SDEs. Our main contributions are to generalize the notion of logarithmic Lipschitz constants, which generalize the logarithmic norms to nonlinear operators, and use them to study the contractivity of nonlinear SDEs.

The remainder of the paper is organized as follows. Section \( II \) reviews logarithmic Lipschitz constants of deterministic nonlinear operators and contraction properties of ODEs. Sections \( III \) and \( IV \) contain the definition of stochastic logarithmic Lipschitz constants and main results on characterizing the stochastic contractivity of SDEs. Section \( V \) discusses how noise can induce stochastic contractivity and synchronization and illustrates the results in a numerical example. Section \( VI \) is the conclusion and discussion. Some of the proofs are given in an Appendix, Section \( VII \).

II. BACKGROUND

In this section we review the definitions of logarithmic norms and logarithmic Lipschitz constants and explain how they are helpful to study contraction properties of ODEs.

Definition 1: (Logarithmic norm) Let \((\mathcal{X}, \|\cdot\|_\mathcal{X})\) be a finite dimensional normed vector space over \( \mathbb{R} \) or \( \mathbb{C} \). The space \( \mathcal{L}(\mathcal{X}, \mathcal{X}) \) of linear transformations \( A: \mathcal{X} \to \mathcal{X} \) is also a normed vector space with the induced operator norm \( \|A\|_{\mathcal{X} \to \mathcal{X}} = \sup_{\|x\|_\mathcal{X}=1} \|Ax\|_\mathcal{X} \). The logarithmic norm (or matrix measure) of \( A \) induced by \( \|\cdot\|_\mathcal{X} \) is defined as the directional derivative of the matrix norm,

\[
\mu_\mathcal{X}[A] = \lim_{h \to 0^+} \frac{1}{h} \left( \| I + hA \|_{\mathcal{X} \to \mathcal{X}} - 1 \right),
\]

where \( I \) is the identity operator on \( \mathcal{X} \).

Definition 2: ([3], Logarithmic Lipschitz constants) Assume \( F: \mathcal{Y} \subseteq \mathcal{X} \to \mathcal{X} \) is an arbitrary function. Two generalizations of the logarithmic norms are the strong least upper bound (s-lub) and least upper bound (lub) logarithmic Lipschitz constants, which are respectively defined by
where $x \in \mathcal{Y} \subset \mathbb{R}^n$ is an $n$-dim vector describing the state of the system, $t \in [0, \infty)$ is the time, and $F$ is an $n$-dim nonlinear vector field. Assume that $\mathcal{Y}$ is convex and $F$ is continuously differentiable on $x$ and continuous on $t$. The system (1) is called contractive if there exists $c > 0$ such that for any two solutions $X$ and $Y$ that remain in $\mathcal{Y}$, and $t \geq 0$, $\|X(t) - Y(t)\| \leq e^{-ct}\|X(0) - Y(0)\|$. In the following theorem and corollary, we find a value for the contraction rate $c$ using logarithmic Lipschitz constant of the vector field $F$ and the logarithmic norm of the Jacobian of $F$ induced by a norm $\|\cdot\|_X$ on $\mathbb{R}^n$. Theorem 1: ([14, Proposition 3], Contrativity of ODES using logarithmic Lipschitz constants) For a given norm $\|\cdot\|_X$ and any two trajectories $X(t)$ and $Y(t)$ of (1) and any $t \geq 0$ the following inequality holds

$$\|X(t) - Y(t)\|_X \leq e^{M_X[F]t}\|X(0) - Y(0)\|_X.$$ 

In particular, if $M_X[F] < 0$, then (1) is contractive.

Corollary 1: (Contrativity of ODES using logarithmic norms) Under the conditions of Theorem 1 if $\mathcal{Y}$ is connected and $\sup_{(x,t) \in \mathcal{Y} \times [0,t]} \mu_X[J_F(x,t)] \leq -c$, for some constant $c > 0$ and some norm $\|\cdot\|_X$, then (1) is contractive.

III. STOCHASTIC LOGARITHMIC LIPSCHITZ CONSTANTS

In this section we generalize the definition of logarithmic Lipschitz constants given in Definition 2.

Notation 1: We will use the following notations for the rest of the paper.

- $(\mathcal{X}, \|\cdot\|_X)$ is a normed space over $\mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathcal{X}$.
- $F: \mathcal{Y} \to \mathbb{R}^n$ is a vector field with components $F_i$.
- $G$ is an $n \times d$ matrix of continuously differentiable column vectors $G_j: \mathcal{Y} \to \mathbb{R}^d$, for $j = 1, \ldots, d$.
- $W(t)$ is a $d$-dim Wiener process with components $W_j$.
- $\Delta W_j = W_j(t+h) - W_j(t) = \int_t^{t+h} dW_j(s)$ and $\Delta W = (\Delta W_1, \ldots, \Delta W_d)^T$.
- $\Delta W_{i,j} = \int_t^{t+h} dW_j(s) \cdot \int_s^t dW_i'(s')$ and $\Delta W^2$ is a $d \times d$ matrix with components $\Delta W^2_{i,j}$.
- $M_{F,G}^{(h,W)}$ is an $n$-dim function on $\mathcal{Y}$ with components:

$$hF_i + \sum_{j=1}^d G_{ij} \Delta W_j + \sum_{j,k=1}^d L_{ik} G_{ij} \Delta W^2_{j,k}.$$ 

Let $\Delta _{jk} = \Delta W_j \Delta W_k + \Delta W^2_{j,k} - \Delta W^2_{k,j}$. Using [31, Equation (15.5.26)], $\Delta W^2_{j,k} = \frac{1}{2}(\Delta_{jk} - h\delta_{jk})$, where $\delta_{jk}$ is the Kronecker delta. A straightforward calculation shows that

$$M_{F,G}^{(h,W)} = h\left(F - \frac{1}{2} \sum_{i=1}^d J_G G_{i} \right) + \sum_{i=1}^d G_i \Delta W + R, $$

where $R = \frac{1}{2} \sum_{i=1}^d L_{ik} G_{ij} \Delta_{jk}$.

Definition 4: (Stochastic logarithmic Lipschitz constants) The s-lub and lub stochastic logarithmic Lipschitz constants of $F$ and $G$ in the $l$-th mean and induced by $\|\cdot\|_X$ are respectively:

$$M_{X,l}^{[+,s]}(F,G) = \sup_{\mathbf{t} = (t_k, k=1,\ldots,l)} \lim_{h \to 0^+} \frac{1}{h} \left\{ \mathbb{E} \sup_{u,v \in \mathcal{Y} \cap \mathbf{t}} \frac{1}{h} \left| \frac{\|u - v + M_{F,G}^{(h,W)}(u) - M_{F,G}^{(h,W)}(v)\|_X}{\|u - v\|_X} - 1 \right| \right\},$$

$$M_{X,l}^{(+)}(F,G) = \sup_{\mathbf{t} = (t_k, k=1,\ldots,l)} \lim_{h \to 0^+} \frac{1}{h} \left\{ \mathbb{E} \sup_{u,v \in \mathcal{Y} \cap \mathbf{t}} \frac{1}{h} \left| \frac{\|u - v + M_{F,G}^{(h,W)}(u) - M_{F,G}^{(h,W)}(v)\|_X}{\|u - v\|_X} - 1 \right| \right\},$$

where $\mathbb{E}$ denotes the expected value.

In [29] the authors introduced the notion of stochastic logarithmic norm which is a special case of $M_{X,l}^{(+)}(F,G)$ with linear $F$ and $G_j$, i.e., $F(u) = Au$ and $G_j(u) = B_j u$ for square matrices $A, B_j$.

Proposition 3: (Some properties of stochastic logarithmic Lipschitz constants) Let $\alpha > 0$ be a constant, $F, F^1$, and $F^2$ be vector functions as described in Notation 1 and $G, G^1$, and $G^2$ be matrices as described in Notation 1. The following statements hold.

1. For a zero matrix $G$, $M_{X,l}^{(+)}(F,0) = lM_{X,l}^{(+)}(F)$.
2. $M_{X,l}^{(+)}(F,G) \leq M_{X,l}^{(+)}(F,G)$.
3. Unlike the deterministic ones, the stochastic logarithmic Lipschitz constants are not sub-linear. However, they satisfy:

$$M_{X,l}^{(+)}(F, \alpha F, \sqrt{\alpha} G) = \alpha M_{X,l}^{(+)}(F,G),$$

$$M_{X,l}^{(+)}(F^1 + F^2, G^1 + G^2) \leq M_{X,l}^{(+)}(F^1, G^1 + G^2) + M_{X,l}^{(+)}(F^2, G^2 + G^2).$$

Similar properties hold for $M_{X,l}^{(+)}$. A proof is given in Appendix, Section VII.
IV. Contraction Properties of SDEs

In this section we first define stochastic contractivity and then provide conditions that guarantee contractivity in SDEs. Consider

$$dX(t) = F(X(t))dt + G(X(t))dW(t),$$  

(3)

where all the terms are as defined in Notation I. Furthermore, we assume that $F$ and $G$ satisfy the Lipschitz and growth conditions: \( \exists K_{1}, K_{2} > 0 \) such that \( \forall x, y \):

$$\|F(x) - F(y)\| + \|G(x) - G(y)\| \leq K_{1}\|x - y\|, \text{ and}$$

$$\|F(x)\|^2 + \|G(x)\|^2 \leq K_{2}(1 + \|x\|^2),$$

where \( \| \cdot \| \) denotes the Euclidean norm. Note that for the matrix $G$, \( \|G\|^2 = \sum_{i,j} |G_{ij}|^2 \). Under these conditions, for any given initial condition \( X(0) \) (with probability one) the SDE has a unique non-anticipating solution, i.e., the solution is independent of the Wiener process, see [31, Chapter 4].

**Definition 5 (Stochastic contraction):** An SDE described by (3) is \( l \)-th moment contractive if there exists a constant \( c > 0 \) such that for any solutions \( X(t) \) and \( Y(t) \) with initial conditions \( X(0) \) and \( Y(0) \), and \( \forall t \geq 0 \),

$$\mathbb{E}\|X(t) - Y(t)\|_{X}^{l} \leq \mathbb{E}\|X(0) - Y(0)\|_{X}^{l} \cdot e^{-ct}. \tag{4}$$

**Theorem 2:** (Stochastic contraction based on stochastic logarithmic Lipschitz constants) For any two solutions \( X(t) \) and \( Y(t) \) of (3) and \( \forall t \geq 0 \),

$$\mathbb{E}\|X(t) - Y(t)\|_{X} \leq \mathbb{E}\|X(0) - Y(0)\|_{X} e^{M_{X,l}[F,G]t}. \tag{5}$$

Moreover, if \( M_{X,l}[F,G] \leq -c \) for some \( c > 0 \), (3) becomes \( l \)-th moment stochastically contractive.

**Proof:** If \( \mathbb{E}\|X(t) - Y(t)\|_{X} = 0 \), then (5) holds. Therefore, we assume that \( \mathbb{E}\|X(t) - Y(t)\|_{X} \neq 0 \). Using Milstein algorithm [31, Chapter 15] any solution \( X(t) \) can be approximated by

$$X(t + h) = X(t) + M^{(h,W)}_{F,G}(X(t)).$$

By subtracting Milstein approximations of \( X \) and \( Y \), we get

$$X(t + h) - Y(t + h) = X(t) - Y(t) + M^{(h,W)}_{F,G}(X(t)) - M^{(h,W)}_{F,G}(Y(t)). \tag{6}$$

Taking \( \| \cdot \|_{X} \), raising to the power \( l \), taking expected value \( \mathbb{E} \), subtracting \( \mathbb{E}\|X(t) - Y(t)\|_{X} \), and dividing by \( h \), taking limit as \( h \to 0^{+} \), we get (to fit the equations, we dropped some of \( l \) arguments):

$$\lim_{h \to 0^{+}} \frac{1}{h} \left\{ \mathbb{E}\|X(t + h) - Y(t + h)\|_{X} - \mathbb{E}\|X(t) - Y(t)\|_{X} \right\} \leq \lim_{h \to 0^{+}} \frac{1}{h} \left\{ \mathbb{E}\|X(t) - Y(t) + M^{(h,W)}_{F,G}(X(t)) - M^{(h,W)}_{F,G}(Y(t))\|_{X} \right\} \leq \lim_{h \to 0^{+}} \frac{1}{h} \left\{ \mathbb{E}\|X(t) - Y(t)\|_{X} \right\} \leq \lim_{h \to 0^{+}} \frac{1}{h} \left\{ \mathbb{E}\|X(t) - Y(t)\|_{X} \right\} \leq M_{X,l}[F,G] \mathbb{E}\|X(t) - Y(t)\|_{X},$$

where the last inequality holds by the definition of \( M_{X,l}[F,G] \) and the fact that \( X(t) - Y(t) \) is non-anticipating, and hence, independent of the Wiener increment \( dW \). The first term of the above relationships is the upper Dini derivative of \( \mathbb{E}\|X(t) - Y(t)\|_{X} \). Hence,

$$D^{+} \mathbb{E}\|X(t) - Y(t)\|_{X} \leq M_{X,l}[F,G] \mathbb{E}\|X(t) - Y(t)\|_{X}. \tag{7}$$

Applying comparison lemma [32, Lemma 3.4], \( \forall t \geq 0 \):

$$\mathbb{E}\|X(t) - Y(t)\|_{X} \leq \mathbb{E}\|X(0) - Y(0)\|_{X} e^{M_{X,l}[F,G]t},$$

which is the desired result.

In this work, inspired by common noise-induced synchronization, we assume that all the trajectories realize the same Wiener process \( W \) and therefore (6) is a valid equation. Studying stochastic contractivity for the trajectories driven by distinct Wiener processes is a topic of future investigations.

Next proposition gives an upper bound for \( M_{X,l}[F,G] \) based on the deterministic logarithmic norms of \( J_{F} \) and \( J_{G} \), \( j = 1, \ldots, d \). The upper bound makes the result of Theorem 4 more applicable, since computing deterministic logarithmic norms induced by some norms, such as \( L^{p} \) norms and weighted \( L^{p} \) norms for \( p = 1, 2, \infty \) are straightforward.

**Proposition 4:** (Relationship between deterministic and stochastic logarithmic Lipschitz constants) Let \( F, G, \) and \( W \) be as described in Notation I. Then

$$M_{X,l}[F,G] \leq lM_{X,l}^{\frac{1}{2}} \left[ F - \frac{1}{2} \sum_{j} J_{G_{j}} G_{j} \right] + \frac{l}{\sqrt{2\pi}} \sum_{j} \left( M_{X,l}[G_{j}] + M_{X,l}[-G_{j}] \right). \tag{8}$$

Furthermore, if \( F \) and \( G_{j} \)s are continuously differentiable and \( \mathcal{Y} \) is convex, then the following inequality holds.

$$M_{X,l}[F,G] \leq l \sup_{x} \mu_{X} \left( J_{F} - \frac{1}{2} \sum_{j} J_{G_{j}} G_{j} (x) \right) + \frac{l}{\sqrt{2\pi}} \sum_{j} \left( \sup_{x} \mu_{X} [J_{G_{j}} (x)] + \sup_{x} \mu_{X} [-J_{G_{j}} (x)] \right).$$

See Appendix VII for a proof.

**Corollary 1:** (Stochastic contraction based on deterministic logarithmic norms) Under the conditions of Proposition 4 if there exists \( c > 0 \) such that the right hand side of (3) is upper bounded by \( -c \), then (3) is \( l \)-th moment stochastically contractive.

**Proof:** Since the right hand side of (3) is bounded by \( -c \), so is its left hand side, i.e., \( M_{X,l}[F,G] \leq -c \). Therefore, by Theorem 2 system (3) is stochastically contractive.

V. Noise-induced contractivity and synchronization

In this section we show how a multiplicative noise can be beneficial for a system and make it contractive. Suppose \( \dot{x} = F(x) \) is not contractive, that is, for any given norm \( \| \cdot \|_{X} \), \( \sup_{x} \mu_{X}[J_{F}(x)] \geq 0 \). Corollary 2 suggests that for appropriate choices of the noise term \( G \) and norm \( \| \cdot \|_{X} \), the underlying stochastic system \( dx = F(x)dt + G(x)dW \) may become stochastically contractive. The reason is that there might exist \( G \) and \( \| \cdot \|_{X} \) such that for any \( x, \mu_{X}[J_{F} - \frac{1}{2} \sum_{j} J_{G_{j}} G_{j} (x)] \) becomes a small enough negative
number. Note that by sub-additivity of the logarithmic norms, 
\[ 0 = \mu_X[J_G_i - J_G_j] \leq \mu_X[J_G_i] + \mu_X[-J_G_j]. \]
Hence, the last sum on the right hand side of (8) is always non-negative. 
Therefore, the first term must be small enough such that the sum becomes negative. For example, for a linear diffusion term, i.e., \( G_j(x) = \sigma_jx, \sigma_j > 0 \):
\[
\mu_X[J_G_i] + \mu_X[-J_G_j] = \sigma(\mu_X[I] + \mu_X[-I]) = 0, 
\]
and by sub-additivity of logarithmic norms:
\[
\mu_X \left[ J_F - \frac{1}{2} \sum_j \sigma_j^2 I \right] = \mu_X \left[ J_F - \frac{1}{2} \sum_j \sigma_j^2 \right. 
\leq \mu_X \left[ J_F - \frac{1}{2} \sum_j \sigma_j^2 \right. 
\]
(9)
For some large \( \sigma_j \), \( \mu_X[J_F] - \frac{1}{2} \sum_j \sigma_j^2 \) becomes negative, and hence, the SDE becomes stochastically contractive. 
Intuitively, since we assumed all the trajectories sense the same Wiener process, the noise plays the role of a common external force to all the trajectories. Therefore, for a strong enough noise, the trajectories converge to each other. See Example 1 below.

Equation (9) guarantees that linear multiplicative stochastic terms do not destroy the contraction properties of contractive systems, no matter how large the perturbations are.

Note that Corollary 2 argues that multiplicative noise may aid contractivity. For an additive noise, i.e., for a state-independent noise term \( G_j(x) \equiv \sigma_j, \mu_X[J_G_i] + \mu_X[-J_G_j] = 0 \) and \( \mu_X[J_F - \frac{1}{2} \sum_j \sigma_j^2 I] = \mu_X[J_F]. \) Therefore, \( \mathcal{M}_{\mathcal{X}_1}^+[F,G] \leq \mu_X[J_F] \) and \( \mu_X[J_F] \geq 0 \) do not give any information on the sign of \( \mathcal{M}_{\mathcal{X}_1} \), and hence, on the contractivity of the SDE.

Example 1: We consider the Van der Pol oscillator subject to a multiplicative noise
\[
dx = \left( x - \frac{1}{3}x^3 - y \right) dt + \sigma g_1(x)dw, 
\]
\[
dy = xdt + \sigma g_2(y)dw, 
\]
(10)
where we assume that the Wiener process is one dimensional, \( d = 1 \). The state of the oscillator is denoted by \( X = (x,y)^T \) which its change of rate is described by \( F = (x - \frac{1}{3}x^3 - y, x)^T \). The noise of the system is described by the column vector \( G(x,y) = (\sigma g_1(x), \sigma g_2(y))^T \).

A simple calculation shows that the Jacobian of \( F \) evaluated at the origin is not Hurwitz, i.e., the eigenvalues are not negative. Therefore, the deterministic Van der Pol is not contractive with respect to any norm. Figure 1(a) depicts two trajectories \( (x_1, y_1)^T \) and \( (x_2, y_2)^T \) of (10) in the absence of noise which do not converge.

In Figure 1(b) an additive noise \( g_1(x) = g_2(y) = 1 \) with intensity \( \sigma = 0.35 \) is added. We observe that the trajectories still do not converge. As discussed above, our result in Corollary 2 does not guarantee noise-induced contractivity in the case of additive noise.

In Figure 1(c) a state-dependent multiplicative noise \((g_1(x), g_2(y)) = (1 + 4x, 1 + 4y)\) with noise intensity \( \sigma = 0.35 \) is added and two trajectories with initial conditions \((1, -1)^T\) and \((2, -2)^T\) are plotted. We observe that the two trajectories converge to each other. A simple calculation shows that \( \mu_2[J_G_i] + \mu_2[-J_G_j] = 4\sigma - 4\sigma = 0 \), where \( \mu_2[A] = \frac{1}{2} \max \lambda(A + A^T) \) is the logarithmic norm induced by \( L^2 \) norm and \( \max \lambda \) denotes the largest eigenvalue. Also,
\[
\sup_{(x,y)} \mu_2[J_F - \frac{1}{2} J_G (x,y)] = \sup_{(x,y)} \{ 1 - x^2 - 8\sigma^2, -8\sigma^2 \} = 1 - 8\sigma^2. 
\]
By Corollary 2 \( \mathcal{M}_{\mathcal{X}_1}^+[F,G] \leq 2(1 - 8\sigma^2) \). Therefore, for \( \sigma \geq \frac{1}{\sqrt{8}} \approx 0.35 \), \( \mathcal{M}_{\mathcal{X}_1}^+[F,G] < 0 \) and the system becomes \( l \)-th moment stochastically contractive for any \( l \geq 1 \).

Figure 1(d) shows the mean square difference of the two solutions plotted in Figure 1(c) over 5000 simulations, which converges to zero, as expected.

Now consider a network of \( N \) isolated nonlinear systems which are driven by a multiplicative common noise, i.e., the only interaction between the systems is through the common noise. The dynamics of such a network can be described by the following SDEs. For \( i = 1, \ldots, N \),
\[
dX_i = F(X_i)dt + \sigma G(X_i)dw, 
\]
(11)
with initial conditions \( X_i(0) = X_{i0} \). Then (11) stochastically synchronizes if for any \( i, j, \mathbb{E}[\|X_i(t) - X_j(t)\|^2] \to 0 \) as \( t \to \infty \), which can be concluded from \( dX = F(X)dt + \sigma(G(X)dw) \) being contractive.

VI. CONCLUSIONS

Deterministic logarithmic Lipschitz constants generalize classical logarithmic norms to nonlinear operators. These constants are proper tools to characterize the contraction properties of ODEs. In this work, we introduced the notions of stochastic logarithmic Lipschitz constants and used them to extend contraction theory to a class of SDEs. Unlike some logarithmic norms, computing stochastic (or deterministic) logarithmic Lipschitz constants is not straightforward. Therefore, to make our theory more applicable, we found some relationships between stochastic logarithmic Lipschitz constants and logarithmic norms. We discussed how multiplicative noise could aid contractivity and foster stochastic synchronization in nonlinear dynamical systems.

In this paper, we assumed that a common Wiener process drives all the trajectories. Studying contractivity (respectively, network synchronization) in the case that distinct and independent Wiener processes drive the trajectories (respectively, nonlinear dynamical systems) is a topic of future investigations. In this case, we need to define an “approximate” contraction in the sense that the trajectories exponentially enter a tube and stay there but do not necessarily converge. See [21] (respectively, [33]) for this type of contractivity (respectively, synchronization) which are based on stochastic Lyapunov function. Proposition 3 provides a mechanism to characterize stochastic contractivity in a class of nonlinear SDEs and understand stochastic synchronization in networks driven by common noise. Generalizing this result to the case of independent Wiener increments is another topic of future investigations.
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VII. APPENDIX

Proof of Proposition 3

1. For \( h > 0 \), let \( \Omega(h) = \|u - v + hF(u) - F(v)\|_X \). Using the equality \( \Omega(1) = (\Omega - 1)(\Omega^{1-1} + \cdots + 1) \) and the fact that \( \lim_{h \to 0} \Omega(h) = 1 \), we have,

\[
M^+_{X,t}[F,0] = \sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{1}{h} \left( \frac{\|u - v + h(F(u) - F(v))\|_X}{\|u - v\|_X} - 1 \right)
\]

\[
= \sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{1}{h} \left( \frac{\|u - v + h(F(u) - F(v))\|_X}{\|u - v\|_X} - 1 \right)
\]

\[
= \sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{1}{h} \left( \frac{\|u - v + h(F(u) - F(v))\|_X}{\|u - v\|_X} - 1 \right)
\]

\[
= 1M^+_{X}[F].
\]

2. The proof is straightforward from the definition of \( M^+_{X,t} \) and \( M_{X,t} \).

3. By the definition of \( M^{(h,W)}_{F,G} \) given in Notation 1

\[
M^{(h,W)}_{\alpha F,\sqrt{\alpha} G} = M^{(\alpha h,\sqrt{\alpha} W)}_{F,G}.
\]

Using the fact that \( W \) is of order \( \sqrt{h} \), and therefore, \( \sqrt{\alpha} W \) is of order \( \sqrt{\alpha h} \), we have:

\[
M^+_{X,t}[\alpha F,\sqrt{\alpha} G] = \sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{1}{h} \times \left( \frac{\|u - v + M^{(\alpha h,\sqrt{\alpha} W)}_{F,G}(u) - M^{(\alpha h,\sqrt{\alpha} W)}_{F,G}(v)\|_X}{\|u - v\|_X} - 1 \right)
\]

\[
= \sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{\alpha}{\alpha h} \times \left( \frac{\|u - v + M^{(\alpha h,\sqrt{\alpha} W)}_{F,G}(u) - M^{(\alpha h,\sqrt{\alpha} W)}_{F,G}(v)\|_X}{\|u - v\|_X} - 1 \right)
\]

\[
= \alpha M^+_{X,t}[F,G].
\]

The second inequality in part 3 can be obtained by the definition of \( M^+_{X,t} \) and the following equality.

\[
2M^{(h,W)}_{F^1,F^2,G^1,G^2} = M^{(2h,\sqrt{2}W)}_{F^1,G^1,F^2,G^2} + M^{(2h,\sqrt{2}W)}_{F^2,G^1,F^1,G^2}.
\]

Proof of Proposition 4

For fixed \( u, v, F, G \), and \( h > 0 \), define \( K(h) \) and \( K_l(h) \) as follows:

\[
K(h) := \left[ \frac{\|u - v + M^{(h,W)}_{F,G}(u) - M^{(h,W)}_{F,G}(v)\|_X}{\|u - v\|_X} \right]
\]

\[
K_l(h) := \left[ \frac{\|u - v + M^{(h,W)}_{F,G}(u) - M^{(h,W)}_{F,G}(v)\|_X^l}{\|u - v\|_X^l} \right]
\]

\[
= \frac{1}{E} K(h)^l.
\]

Note that \( M^+_{X,t}[F,G] = \sup_{u \neq v} D^+ K_l(0) \). A simple calculation shows that the derivative of \( K_l \) evaluated at \( h = 0 \) is equal to \( l \frac{1}{E} K_l(h)^l \frac{d}{dh} \frac{K_l(h)}{h} \), since \( K_l(0) = 1 \) and by Dominated Convergence Theorem, the limit in the definition of \( D^+ \) and the expected value can
be exchanged. Therefore, by the definition of upper Dini derivative:

\[
D^+ K(0) = \lim_{h \to 0^+} \frac{1}{h} \left\{ \frac{\|u - v + M_{F,G}^{(h,W)}(u) - M_{F,G}^{(h,W)}(v)\|_X}{\|u - v\|_X} - 1 \right\}
\]

\[
\leq \lim_{h \to 0^+} \frac{1}{2h} \left\{ \frac{\|u - v + M_{F,G}^{(2,W)}(u) - M_{F,G}^{(2,W)}(v)\|_X}{\|u - v\|_X} - 1 \right\}
\]

\[
+ \lim_{h \to 0^+} \frac{1}{2h} \left\{ \frac{\|u - v + \frac{1}{2} \sum_j J_G G_j \|_X}{\|u - v\|_X} - 1 \right\}
\]

Therefore, by taking \( E \) from both sides, we get

\[
l \mathbb{E}\left( D^+ K(0) \right) \leq l \sum_j M_X^j \left[ F - \frac{1}{2} \sum_j J_G G_j \right]
\]

\[
+ \frac{1}{2} \sum_j E[\xi_j] \left( M_X^j[G_j] + M_X^j[-G_j] \right)
\]

Equation (7) is obtained by plugging \( E[\xi_j] = \sqrt{\frac{2}{\pi}} \).

Equation (8) holds by Proposition 2.

REFERENCES

[1] D. C. Lewis, “Metric properties of differential equations,” Amer. J. Math., vol. 71, pp. 294–312, 1949.

[2] P. Hartman, “On stability in the large for systems of ordinary differential equations,” Canad. J. Math., vol. 13, pp. 480–492, 1961.

[3] G. Soderlind, “The logarithmic norm. history and modern theory,” BIT, vol. 46, no. 3, pp. 631–652, 2006.

[4] G. Dahlquist, Stability and error bounds in the numerical integration of ordinary differential equations. Inaugural dissertation, University of Stockholm, Almqvist & Wikells Boktryckeri AB, Uppsala, 1958.

[5] B. P. Demidovî, “On the dissipativity of a certain non-linear system of differential equations. I,” Vestnik Moskov. Univ. Ser. I Mat. Meh., vol. 1961, no. 6, pp. 19–27, 1961.

[6] B. P. Demidovî, Lektsiî po matematichskoi teorii ustoichivosti. Izdat. “Nauka”, Moscow, 1967.

[7] T. Yoshizawa, Stability theory by Liapunov’s second method. Publications of the Mathematical Society of Japan, No. 9, The Mathematical Society of Japan, Tokyo, 1966.

[8] T. Yoshizawa, Stability theory and the existence of periodic solutions and almost periodic solutions. Springer-Verlag, New York-Heidelberg, 1975. Applied Mathematical Sciences, vol. 14.

[9] G. Soderlind, “Bounds on nonlinear operators in finite-dimensional Banach spaces,” Numer. Numer. v. 50, no. 1, pp. 27–44, 1986.

[10] W. Lohmiller and J. J. E. Slotine, “On contraction analysis for nonlinear systems,” Automatica, vol. 34, pp. 683–696, 1998.

[11] M. Arcak, “Certifying spatially uniform behavior in reaction-diffusion PDE and compartmental ODE systems,” Automatica, vol. 47, no. 6, pp. 1219–1229, 2011.

[12] J. W. Simpson-Porco and F. Bullo, “Contraction theory on Riemannian manifolds,” Systems Control Lett., vol. 65, pp. 74–80, 2014.

[13] G. Russo, M. di Bernardo, and E. Sontag, “A contraction approach to the hierarchical analysis and design of networked systems,” IEEE Transactions on Autom. Control, vol. 58, pp. 1328–1331, 2013.

[14] Z. Aminzare and E. Sontag, “Contraction methods for nonlinear systems: A brief introduction and some open problems,” in Proc. IEEE Conf. Decision and Control, Los Angeles, pp. 3835–3847, 2014.

[15] S. Coogan and M. Arcak, “A compartmental model for traffic networks and its dynamical behavior,” IEEE Transactions on Automatic Control, vol. 60, no. 10, pp. 2708–2703, 2015.

[16] M. Margaliot, E. D. Sontag, and T. Tuller, “Contraction after small transients,” Automatica, vol. 67, pp. 178–184, 2016.

[17] Z. Aminzare and E. D. Sontag, “Some remarks on spatial uniformity of solutions of reaction–diffusion pdes,” Nonlinear Analysis: Theory, Methods & Applications, vol. 147, pp. 125–144, 2016.

[18] S. Coogan, “A contractive approach to separable lyapunov functions for monotone systems,” Automatica, vol. 106, pp. 349–357, 2019.

[19] P. Cisneros-Velarde, S. Jafarpour, and F. Bullo, “Contraction theory for dynamical systems on Hilbert spaces,” Oct. 2020.

[20] A. Davydov, S. Jafarpour, and F. Bullo, “Non-Euclidean contraction theory for robust nonlinear stability,” July 2021. Submitted.

[21] Q. C. Pham, N. Tabareau, and J. Slotine, “A contraction theory approach to stochastic incremental stability,” IEEE Transactions on Automatic Control, vol. 54, no. 4, pp. 816–820, 2009.

[22] S. Han and S.-J. Chung, “Incremental nonlinear stability analysis for stochastic systems perturbed by Lévy noise” arXiv:2103.13338, 2021.

[23] N. Tabareau and J. Slotine, “Contraction analysis of nonlinear random dynamical systems,” arXiv:1309.5317 v2, 2013.

[24] J. Bouvrie and J. Slotine, “Random attraction in the tasep model,” SIAM Journal on Applied Dynamical Systems, vol. 20, no. 1, pp. 65–93, 2021.

[25] J. Newman, “ Necessary and sufficient conditions for stable synchronization in random dynamical systems,” Ergodic Theory and Dynamical Systems, vol. 38, no. 5, pp. 1857–1875, 2018.

[26] Q. C. Pham and J. Slotine, “Stochastic contraction in Riemannian metrics,” arXiv:1304.0340, 2013.

[27] J. Bouvrie and J. Slotine, “Wasserstein contraction of stochastic nonlinear systems,” arXiv:1902.08567v2, 2019.

[28] A. Dani, P. Ashwin, S. J. Chung, and S. Hutchinson, “Observer Design for Stochastic Nonlinear Systems via Contraction-Based Incremental Stability,” IEEE Transactions on Automatic Control, vol. 60, no. 3, pp. 1700–1714, 2015.

[29] S. S. Ahmad and S. Raha, “On estimation of transient stochastic stability of linear systems,” Stochastics and Dynamics, vol. 10, no. 03, pp. 385–405, 2010.

[30] Z. Aminzare and E. D. Sontag, “Logarithmic Lipschitz norms and diffusion-induced instability,” Nonlinear Analysis: Theory, Methods & Applications, vol. 83, pp. 31–49, 2013.

[31] C. Gardiner, Stochastic Methods: A Handbook for the Natural and Social Sciences. Springer, fourth ed., 2009.

[32] H. K. Khalil, Nonlinear systems. Prentice Hall, third ed., 2002.

[33] Z. Aminzare and V. Srivastava, “Stochastic synchronization in nonlinear network systems driven by intrinsic and coupling noise,” under review, 2021.