The *B*-meson Light-Cone Distribution Amplitude in Dual Space

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Abstract

We describe a representation of the leading *B*-meson light-cone distribution amplitude in heavy-quark effective theory based on the eigenfunctions of its anomalous dimension kernel. In this representation (called the dual LCDA) different dual momenta no longer mix under renormalization. We discuss the perturbative and non-perturbative nature of different regions in this space.

Keywords: *B*-decays, Factorization, Heavy-Quark Effective Theory, Resummation

1. Introduction

Theoretical descriptions of *B*-meson decays into exclusive light final states often invoke a non-local matrix element in heavy-quark effective theory (HQET) between the *B* state and the vacuum, see e. g. [1]. Since the light final states recoil with considerable energy against the *B* meson at rest (at least in some regions of the phase space), the non-locality of the operator is light-like. The most important of such objects is called the leading light-cone distribution amplitude (LCDA), defined as [2]

\[ f_{\text{LCDA}}(t) = \langle 0| \bar{q}(t)n|m,0|\gamma\psi(0)|B \rangle, \]

where \( f_B \) is the *B*-meson decay constant in HQET, and the vector \( n^\mu \) is light-like. (There also exists another LCDA within the 2-particle Fock-state description of the *B* meson, called \( \tilde{f}_B \), which can be discussed similarly.) The Fourier transform of this function,

\[ \hat{f}_{\text{LCDA}}(\omega) = \int \frac{dt}{2\pi} e^{-i\omega t} f_{\text{LCDA}}(t), \]

is commonly used as \( \omega \) represents the *n*-projection of the light-quark’s momentum. This function is principally a non-perturbative input that enters factorization theorems and thus not calculable in perturbation theory. However, it is possible to calculate both its moments over a large enough interval using an operator product expansion (OPE) [3], as well as its dependence on the renormalization scale \( \mu \), which has been suppressed in the notation so far. Let us first consider the latter point: the \( \mu \)-dependence is governed by the integro-differential renormalization-group equation (RGE) [4]

\[ \frac{d}{d\ln \mu} \phi_B(\omega) = -\left[ \Gamma_\omega \ln \frac{\mu}{\omega} + \gamma_\omega \right] \phi_B(\omega) - \omega \int d\eta \Gamma(\omega,\eta) \phi_B(\eta). \]

The second term in this equation mixes different regions in \( \omega \) when evolving in \( \mu \). Therefore the solution to this equation requires us to integrate over the full \( \omega \) region of the LCDA at the initial scale. However, as was discovered recently the above operation possesses a continuous set of eigenfunctions [5]

\[ f_{\text{LCDA}}(\omega) = \sqrt{\frac{\omega}{\omega'}} J_1 \left( 2 \sqrt{\frac{\omega}{\omega'}} \right), \]

with the parameter \( \omega' \) of mass dimension 1, called the dual momentum. The analogon for light mesons are the Gegenbauer polynomials [6]. A suitable representation of the LCDA is therefore given by a linear combination of the eigenfunctions,

\[ \phi_B(\omega) = \int_0^\infty \frac{d\omega'}{\omega'} \rho_B(\omega') f_{\text{LCDA}}(\omega'), \]

where \( \rho_B(\omega') \) is now analogous to the Gegenbauer coefficients in the comparison with light mesons above. The relation [5] can be inverted using the orthogonality relations of Bessel functions and reads

\[ \rho_B(\omega') = \int_0^\infty \frac{d\omega}{\omega} \sqrt{\frac{\omega}{\omega'}} J_1 \left( 2 \sqrt{\frac{\omega}{\omega'}} \right) \phi_B(\omega). \]
As a result the so-defined dual LCDA renormalizes locally,
\[ \frac{d}{d \ln \mu} \rho^\omega_B(\omega') = - \left[ \Gamma \frac{\mu}{\omega'} + \gamma_\omega \right] \rho^\omega_B(\omega') , \] (7)
with the rather simple solution
\[ \rho^\omega_B(\omega', \mu) = e^{V(\mu, \mu_0)} \left( \frac{\mu_0}{\omega'} \right)^{-\gamma(\mu, \mu_0)} \rho^\omega_B(\omega', \mu_0) . \] (8)
Here the hatted quantity \( \hat{\omega}' = e^{-2\gamma_\omega \omega'} \) denotes a rescaled dual momentum, and the functions \( g \) and \( V \) involve integrals over the anomalous dimensions \( \Gamma \) and \( \gamma_\omega \), see e.g. [3] for details. Qualitatively the functions \( \phi^*_B \) and \( \rho^\omega_B \) contain the same information; constraints on one of them translate to constraints on the other.

2. Perturbative constraints and large dual momenta

It is well known that the LCDA \( \phi^*_B(\omega) \) does not fall rapidly enough to have a norm. In other words, the bare matrix element in [1] requires an extra subtraction in the local limit. Since moments over an infinite interval are therefore not defined, one must introduce an ultra-violet cutoff \( \Lambda_{\text{UV}} \), on which moments depend logarithmically. Such moments,
\[ M_\mu(\Lambda_{\text{UV}}, \mu) = \int_0^{\Lambda_{\text{UV}}} d\omega \omega^\mu \phi^*_B(\omega, \mu) , \] (9)
have been calculated in an operator product expansion in \( 1/\Lambda_{\text{UV}} \) to first order corrections as [3].

\[
\begin{align*}
M_0 &= 1 + \frac{\alpha_s(\mu) C_F}{4\pi} \left( -2 \ln^2 \frac{\Lambda_{\text{UV}}}{\mu} + 2 \ln \frac{\Lambda_{\text{UV}}}{\mu} - \pi^2 \frac{12}{4} + \frac{16\Lambda}{3\Lambda_{\text{UV}}} \right), \\
M_1 &= \Lambda_{\text{UV}} \frac{\alpha_s(\mu) C_F}{4\pi} \left( -4 \ln \frac{\Lambda_{\text{UV}}}{\mu} + 6 \right) + \frac{4\Lambda}{3} \left[ 1 + \frac{\alpha_s(\mu) C_F}{4\pi} \left( -2 \ln^2 \frac{\Lambda_{\text{UV}}}{\mu} + 8 \ln \frac{\Lambda_{\text{UV}}}{\mu} - \frac{7}{4} \pi^2 \frac{12}{4} \right) \right].
\end{align*}
\] (10)

When expressing the above in dual space we find a weighted integral over all dual momenta
\[ M_\mu(\Lambda_{\text{UV}}, \mu) = \int_0^{\infty} d\omega' N_\omega(\omega', \Lambda_{\text{UV}}) \rho^\omega_B(\omega', \mu) , \] (11)
where the first few weight functions are
\[
\begin{align*}
N_0 &= \frac{\Lambda_{\text{UV}}}{\omega'} J_1 \left( 2 \sqrt{\frac{\Lambda_{\text{UV}}}{\omega'}} \right), \\
N_1 &= \frac{\Lambda_{\text{UV}}^2}{3\omega'} \left[ 2 J_2 \left( 2 \sqrt{\frac{\Lambda_{\text{UV}}}{\omega'}} \right) - J_1 \left( 2 \sqrt{\frac{\Lambda_{\text{UV}}}{\omega'}} \right) \right].
\end{align*}
\] (12)

They primarily probe the dual LCDA in the region \( \omega' \sim \Lambda_{\text{UV}} \), and since \( \Lambda_{\text{UV}} \sim \mu \) we can infer the functional dependence of \( \rho^\omega_B \) in the region \( \omega' \sim \mu \) as an expansion in \( 1/\omega' \) modulo logarithms. We find up to second-order power corrections
\[ \rho^\omega_B(\omega', \mu) = C_0(\ln \frac{\mu}{\omega'}) \frac{1}{3} C_1(\ln \frac{\mu}{\omega'}) \left( \frac{\Lambda}{\omega'} \right)^2 , \]

3. Resummation of the tail

So far we note that the large \( \omega' \) behaviour of \( \rho^\omega_B \) is \( 1/\omega' \). This finding and the solution [8] seem incompatible and leads us to contemplate the following dilemma: suppose two model builders, \( A \) and \( B \), are given the task to create a model for \( \rho^\omega_B \) at two different scales, \( \mu_A \) and \( \mu_B \), respectively, with \( \mu_A < \mu_B \). Both feature an asymptotic \( 1/\omega' \) tail. But according to [8], the tail of the first model, \( \rho_A \), will pick up a softening contribution as we evolve from one scale to the next, and scales like \( \rho_A(\mu_B) \sim (\omega')^{-1+\gamma(\mu_B, \mu_A)} \) at the scale of the second builder. Are both models therefore incompatible?

The answer lies in the fact that we are comparing different regions in \( \omega' \). Whereas \( \rho_A \) has a \( (\omega')^{-1} \) behaviour around \( \omega' \sim \mu \) and indeed evolves to a softer

\footnote{The integral transformation [6] of the common model \( \phi^*_B(\omega) = \frac{\omega}{c_6} e^{-\omega/\omega_0} \) results in \( \rho^\omega_B(\omega') = \frac{\omega'}{c_7} e^{-\omega'/\omega_0'} \), which features a “dual” behaviour: whereas \( \phi^*_B(\omega) \) falls off exponentially for large \( \omega \) and vanishes linearly at the origin, \( \rho^\omega_B(\omega') \) vanishes exponentially for small \( \omega' \) and as a first inverse power for large \( \omega' \).}
\((\omega')^{-1+g(\mu_0, \mu_0)}\) dependence in this \(\omega' \sim \mu_A\) regime, we then compare it to the \(\omega' \sim \mu_B\) regime.

Let us therefore use standard resummation techniques by introducing an auxiliary scale \(\mu_A\) that scales like \(\omega'\) for large \(\omega' \gg \mu\) and does not become small as \(\omega'\) becomes small, for example \(\mu_{\omega'} = \sqrt{\mu^2 + \omega'^2}\). The latter aspect has no relevance to our discussion at hand, but was chosen so that we can avoid the Landau pole in \(\alpha_s(\mu_{\omega'})\) when discussing the low \(\omega'\) regime later on. From [8] it follows that

\[
\rho_B^+(\omega', \mu) = e^{-V(\mu_{\omega'}, \mu)} \left( \frac{\mu}{\mu_{\omega'}} \right)^{\nu(\mu_{\omega'}, \mu)} \rho_B^+(\omega', \mu_{\omega'}) .
\]  

(13)

This equation allows us to state that the dual LCDA is perturbatively calculable in resummed perturbation theory for the entire region \(\omega' \geq \mu\) [8]. To see how the above puzzle is resolved it helps to consider the function

\[
f(\omega', \mu) = \frac{d \ln \rho_B^+(\omega', \mu)}{d \ln \omega'} .
\]

The essence of this definition is that if \(\rho_B^+(\omega', \mu) \sim (\omega')^{-c}\) as \(\omega' \to \infty\) (with \(c\) a constant), then \(f \to -c\). We find

\[
f(\omega', \mu) = -1 - g(\mu_{\omega'}, \mu) + \Gamma_\omega(\mu_{\omega'}, L) + r \omega' .
\]

(15)

where \(L = \ln(\mu_{\omega'}/\omega')\) and \(r\) collects terms of order \(\beta_0 a_s^2\) and power corrections. Therefore \(f \approx -1 - g(\mu_{\omega'}, \mu)\) asymptotically for large \(\omega'\) as shown in Figure 1.

![Figure 1: Depiction of the function \(f(\omega', \mu_0)\) at \(\mu_0 = 1.5\) GeV and \(\omega' \geq \mu_0\).](image)

Since \(d f(\omega', \mu)/d \ln \mu = \Gamma_\omega(\mu)\) (which follows from equation [7] exactly) integrates to

\[
f(\omega', \mu_B) = f(\omega', \mu_A) + g(\mu_B, \mu_A) ,
\]

(16)

we see that \(f \approx -1\) or \(\rho_B^+(\omega', \mu_B) \sim 1/\omega'\) for \(\omega' \sim \mu_B\), irrespective of how large \(\mu_B\) becomes.

The finding that \(\rho_B^+(\omega', \mu)\) falls towards zero with an ever increasing rate since \(g(\mu_{\omega'}, \mu)\) is a monotonically growing function in \(\omega'\) leads to a new insight, to wit that the phenomenologically important first inverse moment of the LCDA, \(\lambda_B(\mu)\) see below, exists at all scales \(\mu\). (In fact, we can even state that any positive moment of \(\rho_B^+(\omega, \mu)\) exists, since \(g(\mu_{\omega'}, \mu)\) grows with no boundary.)

Previous analyses of the LCDA [3,4,9] stated solutions for the RG evolution that broke down as the spread between initial scale \(\mu_A\) and final scale \(\mu_B\) becomes so large that \(g = g(\mu_B, \mu_A) \sim 1\), as manifest in factors involving \(\Gamma(1 - g)\) and similar functions. We now understand that these perceived thresholds are not physical and disappear once a resummation for large values of \(\omega'\) is included.

4. Non-perturbative aspects and first inverse moments

In the previous narrative we have discussed the function \(\rho_B^+(\omega', \mu)\) for values of \(\omega' \geq \mu\), which is the perturbative regime. For small values of \(\omega'\), however, non-perturbative physics are dominant. In the absence of other information from non-perturbative theoretical methods and experimental input, one is forced to model \(\rho_B^+(\omega', \mu)\) in this region. To our knowledge there is no theorem stating that the original LCDA \(\phi_B^+(\omega, \mu)\) or even the dual \(\rho_B^+(\omega', \mu)\) shall be positive definite, and therefore adopted models can differ greatly. In the paper of which these proceedings report [8] we have designed a recipe that allows for the smooth merging of a given model in the low \(\omega'\) region and the perturbative findings of the previous section, while respecting the moment constraints in [10]. This construction is based on an expansion of the model in a set of basis functions with appropriate characteristics. For details we refer the reader to the original paper.

The first inverse moment of the LCDA and logarithmic modulations of it are of particular interest to phenomenology. The factorized amplitude of the exclusive \(\bar{B} \to \gamma f\bar{f}\) in the low \(q^2\) region, for example, requires knowledge of the quantities [10,11]

\[
\frac{\sigma_n(\mu)}{\lambda_B(\mu)} = \int_0^\infty d\omega' \ln\left(\frac{\omega'}{\omega} \right) \rho_B^+(\omega', \mu) ,
\]

(17)

where \(\lambda_B\) is of mass dimension 1 and \(\sigma_n\) are numbers with \(\sigma_0 = 1\). Similarly we may define inverse moments in dual space as

\[
L_n(\mu) = \int_0^\infty d\omega' \ln\left(\frac{\omega'}{\mu} \right) \rho_B^+(\omega', \mu) .
\]

(18)
It was shown that the first few of these dual moments are identical to the original ones [5], namely
\[ \lambda_0 = \frac{1}{\lambda_B}, \quad \lambda_1 = \frac{\sigma_1}{\lambda_B}, \quad \text{and} \quad \lambda_2 = \frac{\sigma_2}{\lambda_B}. \] (19)

For higher \( n > 2 \) linear combinations appear. Since we already know the integrand in (18) for \( \omega' \geq \mu \) we may separate this region out and simply calculate it. The integral over the remaining, non-perturbative part, \( \omega' \leq \mu \), can be rewritten by substituting \( z = -\ln \frac{\omega'}{\mu} \) to form
\[ L_n(\mu) = \int_0^{\infty} dz \left( -z \right)^n \rho_n^B(\mu e^{-z}, \mu). \] (20)

We may further expand the unknown function \( \rho_n^B \) in terms of Laguerre polynomials \( L_n(z) \), i.e.
\[ \rho_n^B(\mu e^{-z}, \mu) = \sum_{k=0}^{n} a_k(\mu) e^{-z} L_k(z). \] (21)

The advantage of this decomposition is that we may be able to fit the first few coefficients \( a_k(\mu) \) from precise experimental data [10][11]. The above use of Laguerre polynomials allows us to relate
\[ L_0(\mu) = a_0(\mu), \]
\[ L_1(\mu) = a_1(\mu) - a_0(\mu), \]
\[ L_2(\mu) = 2a_2(\mu) - 4a_1(\mu) + 2a_0(\mu), \]
\[ \ldots \] (22)

In conclusion we find that at this point in time the most promising way to gain further insight into the non-perturbative part of the LCDA is to increase the precision of experimental data from exclusive \( B \) decays. Whether theoretical methods like sum rules and lattice QCD can lead to more information on the LCDA is an interesting question.

5. Conclusions.

In this talk we have summarized some aspects on the recently found dual LCDA of the \( B \) meson [5]. The biggest advantage of this description results from the fact that it renormalizes locally, i.e. does not mix different regions in its argument \( \omega' \) under RG evolution. The region of large \( \omega' \) is determined by way of a short-distance operator product expansion of moments over the original LCDA, and thus perturbatively calculable. We have demonstrated this by calculating \( \rho_n^B(\omega', \mu) \) in this regime to first-order QCD corrections and first-order power corrections. We advocate the use of the dual LCDA in factorization theorems as it simplifies the RG analysis of the factorized amplitude greatly.

We paid particular attention to the tail of the dual LCDA, \( \omega' \gg \mu \), and demonstrated that resummation of large logarithms of the form \( \ln \omega'/\mu \) renders the solution valid at any (and even unphysically large) renormalization scale. Where else other formalisms break down where the RG-evolution function \( g \) assumes integer values, our representation does not.

We might be preempt in the notion that an impression exists, that an increased precision in power corrections to the moments over the original LCDA [10] leads ultimately to a prediction of the phenomenologically important quantity \( \lambda_B \), but we stress that this is not so. It would only lead to a more precise determination of the dual LCDA in the large \( \omega' \) regime, but not determine \( \lambda_B \) and its logarithmic modulations. On an intuitive level this statement can be justified by the fact that the large b-quark mass has been eliminated in HQET at leading power; on a technical level it is justified by a very slow point-wise convergence in the low-\( \omega' \) region when expanding around the large-\( \omega' \) behaviour. For more details we refer the reader to the original paper.

A similar treatment concerning the shape function of inclusive \( B \) decays is possible.

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