SOME RESULTS ON LOCAL COHOMOLOGY OF POLYNOMIAL AND
FORMAL POWER SERIES RINGS: THE ONE DIMENSIONAL CASE

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Abstract. In this paper, we prove several results on the finiteness of local cohomology of
polynomial and formal power series rings. In particular, we give a partial affirmative answer
for a question of L. Núñez-Betancourt in [J. Algebra 399 (2014), 770–781].

1. Introduction

The motivation of this paper is the following conjecture of G. Lyubeznik: If $R$ is a regular
ring, then each local cohomology module $H^i_I(R)$ has finitely many associated prime ideals.
The Lyubeznik conjecture has affirmative answers in several cases: for regular rings of prime
characteristic (cf. [7, 8]); for regular local and affine rings of characteristic zero (cf. [5]); for
unramified regular local rings of mixed characteristic (cf. [11, 13]) and for smooth $\mathbb{Z}$-algebras
(cf. [2]). The method of the proof of these results is considering the module structure of local
cohomology over non-commutative rings, $D$-modules (resp. $F$-modules). The finiteness of
these module structures (for example, finite length) yields the finiteness of $\text{Ass}_{S}H^i_I(S)$.

Motivated by the above finiteness results, M. Hochster raised the following related question
(cf. [14, Question 1.1]):

Question 1. Let $(R, \mathfrak{m}, k)$ be a local ring and $S$ a flat extension of $R$ with regular closed
fiber. Then is

$$\text{Ass}_{S}H^0_{\mathfrak{m}S}(H^i_I(S)) = V(\mathfrak{m}S) \cap \text{Ass}_{S}H^i_I(S)$$

finite for every ideal $I \subset S$ and for every integer $i \geq 0$?

Suppose $S$ is a flat extension of $R$ with regular fibers. It is worth to note that if Question
1 has an affirmative answer, then the finiteness conditions of $\text{Ass}_{S}H^i_I(S)$ and $\text{Ass}_{R}H^i_I(S)$
are equivalent. In [14], L. Núñez-Betancourt gave a positive answer for Question 1 when $S$ is
either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$ and $\dim R/(I \cap R) \leq 1$. In that paper, he introduced
the notion of $\Sigma$-finite $D$-modules. It should be noted that $\Sigma$-finite $D$-modules maybe not
have finite length but they have finitely many associated primes. Núñez-Betancourt asked
the following question (cf. [14 Question 5.1]).

Question 2. Let $(R, \mathfrak{m}, k)$ be a local ring and $S$ either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. Then is $H^i_mH^j_J(S)$ $\Sigma$-finite for every ideal $J \subset S$ and $i, j \geq 0$?

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Throughout this paper, let \( R \) be a commutative Noetherian ring and \( S \) be either \( R[X_1, \ldots, X_n] \) or \( R[[X_1, \ldots, X_n]] \). In Section 3 we modify the definition of \( \Sigma \)-finite \( D \)-modules for rings that not necessarily local rings. We prove that \( H^j_j(S) \) is \( \Sigma \)-finite for every ideal \( J \subseteq S \) satisfying \( \dim R/(J \cap R) = 0 \) (cf. Proposition 3.7). Applying this result we give a positive answer for Question 2 when \( \dim R/(J \cap R) \leq 1 \) (cf. Theorem 3.8). Moreover, a finiteness result of associated primes of local cohomology is given (cf. Corollary 3.9).

In Section 4 we consider the following problem.

**Question 3.** Suppose that \( \dim R = 1 \) and \( S \) is either \( R[X_1, \ldots, X_n] \) or \( R[[X_1, \ldots, X_n]] \). Is it true that \( H^j_j(S) \) has only finitely many associated primes for all ideals \( J \) of \( S \) and all \( i \geq 0 \)?

By the work of B. Bhatt et al. [2] Question 3 has a positive answer when \( S = \mathbb{Z}[x_1, \ldots, x_n] \). The next interesting case of Lyubeznik’s conjecture is seem to be the case \( S = R[x_1, \ldots, x_n] \) with \( R \) is a Dedekind domain (containing the field of rational numbers). This is a special case of Question 3. In this section we will give a partial affirmative answer of Question 3 in the case \( R \) contains a field of positive characteristic (cf. Proposition 4.4). It should be noted that H. Dao and the author showed that local cohomology of Stanley-Reisner rings over a field of positive characteristic have only finitely many associated primes, see [4] for a more general result (see also [6]). Finally, the readers are encouraged to [15, 16] for some results about the finiteness of associated primes of local cohomology of polynomial and power series rings over a normal domain containing a field of zero characteristic.

## 2. Preliminary

In this section we collect some basic facts on rings of differential operators and \( D \)-modules. Let \( R \) be a Noetherian ring and \( S = R[X_1, \ldots, X_n] \) or \( S = R[[X_1, \ldots, X_n]] \).

**Rings of differential operators.** Let \( D(S, R) \) (or \( D \) if there is no confusion) be the ring of \( R \)-linear differential operators of \( S \). The ring \( D(S, R) \) is defined by recursion as follows. The differential operators of order zero are the morphisms induced by multiplying by elements in \( S \). An element \( \delta \in \text{Hom}_R(S, S) \) is a differential operator of order less than of equal to \( k + 1 \) if \( [\delta, s] := \delta \circ s - s \circ \delta \) is a differential operator of order less than or equal to \( k \) for every \( s \in S = \text{Hom}_S(S, S) \). Notice that \( D(S, R) \) is not a commutative ring, but \( R \) is contained in the center of \( D(S, R) \). In our cases \( S = R[X_1, \ldots, X_n] \) or \( S = R[[X_1, \ldots, X_n]] \), it is well known that (see [5] Theorem 16.12.1)

\[
D(S, R) = S \left[ \frac{1}{t!} \frac{\partial^t}{\partial x_i^t} \right]_{t \in \mathbb{N}, 1 \leq i \leq n} \subseteq \text{Hom}_R(S, S).
\]

**Homomorphic.** Let \( R' \) be another ring with \( \phi : R \to R' \) a homomorphism of rings. Let \( S' \) be either \( R'[x_1, \ldots, x_n] \) or \( R'[[x_1, \ldots, x_n]] \), respectively. Then \( \phi \) induces a homomorphism between rings of differential operators \( \Phi : D(S, R) \to D(S', R') \). In particular, we have a natural surjection \( D(S, R) \to D(S/IS, R/I) \) for every ideal \( I \subseteq R \).

**Example 2.1 (of \( D \)-modules).** (i) It is well known that \( S \) is a \( D \)-module.

(ii) Let \( M \) be an \( R \)-module. Then \( M[x_1, \ldots, x_n] \cong R[x_1, \ldots, x_n] \otimes_R M \) (resp. \( R[[x_1, \ldots, x_n]] \otimes_R M \) and \( M[[x_1, \ldots, x_n]] \)) are \( D \)-modules. In particular for each \( m \in \text{Max}(R) \) we have \((R/m)[x_1, \ldots, x_n] \) (resp. \((R/m)[[x_1, \ldots, x_n]] \)) are \( D \)-modules of finite length.
Definition 3.1. Let $\Sigma$ be a set of factors. Let $D$ be a short exact sequence of $\Sigma$-finite $R$-modules. If $M$ is a $D$-module then its localization and local cohomology of $M$ are $D$-modules.

Remark 3.2. In [10], Lyubeznik defined the subcategory of the category of $D(S, R)$-modules, says $C(S, R)$, is the smallest subcategory of $D(S, R)$-modules that contains $S_f$ for all $f \in S$ and that is closed under taking submodules, quotients and extensions. In particular, the kernel, image and cokernel of a morphism of $D(S, R)$-modules that belongs to $C(S, R)$ are also objects in $C(S, R)$. Notice that $H^i_f \cdots H^n_f(S)$ is an object in $C(S, R)$. The critical fact for the study of the finiteness of local cohomology is that every module in $C(S, R)$ has finite length as a $D$-module provided $R$ is a field (see [10, Corollary 6]).

3. $\Sigma$-finite $D$-modules

First, we give the definition of $\Sigma$-finite $D$-modules. Notice that we do not assume $R$ is local as [14]. Let $M$ be a $D$-module, we denote by $\text{Fin}(M)$ the set of all $D$-submodules of $M$ that have finite length. Let $N$ be a $D$-module of finite length. There is a filtration of submodules $0 = N_0 \subset N_1 \subset \cdots \subset N_h = N$ such that $N_i/N_{i-1}$ is a nonzero simple $D$-module for all $i = 1, \ldots, h$. The factors, $N_i/N_{i-1}$, are the same, up to permutation and isomorphism, for every filtration. We denote that set of factors by $\mathcal{C}(N)$.

Definition 3.1. Let $M$ be a $D$-module such that $\text{Supp}_R(M) \subseteq \text{Max}(R)$. We say that $M$ is $\Sigma$-finite if

(i) $\bigcup_{N \in \text{Fin}(M)} N = M$,
(ii) $\bigcup_{N \in \text{Fin}(M)} \mathcal{C}(N)$ is finite, and
(iii) For every $N \in \text{Fin}(M)$ and $L \in \mathcal{C}(N)$, $L \in \mathcal{C}(S/mS, R/m)$ for some $m \in \text{Max}(M)$.

If $M$ is $\Sigma$-finite, we denote $\mathcal{C}(M) := \bigcup_{N \in \text{Fin}(M)} \mathcal{C}(N)$. It is easy to see that if
\[0 \to M' \to M \to M'' \to 0\]
is a short exact sequence of $\Sigma$-finite $D$-modules, then $\mathcal{C}(M) = \mathcal{C}(M') \cup \mathcal{C}(M'')$.

Remark 3.2. If $M$ is $\Sigma$-finite then $\text{Supp}_R(M)$ is a finite subset of $\text{Max}(R)$. If $\text{Supp}_R(M) = \{m_1, \ldots, m_r\} \subseteq \text{Max}(R)$, then $M \cong \Gamma_{m_1}(M) \oplus \cdots \oplus \Gamma_{m_r}(M)$. Therefore all results proved in Section 3 of [14] (in the case $R$ is a local ring) can be extended for our notion of $\Sigma$-finite. For example, if $M$ is a $\Sigma$-finite $D$-module, then $H^i_f(M)$ is also a $\Sigma$-finite $D$-module for every ideal $J \subseteq S$ and integer $i \geq 0$.

The following give us examples of $\Sigma$-finite $D$-modules.

Lemma 3.3. Let $A$ be an Artinian $R$-module. Then $M = A \otimes_R R[x_1, \ldots, x_n]$ (resp. $M = A \otimes_R R[[x_1, \ldots, x_n]]$) is a $\Sigma$-finite $D$-module.

Proof. It is easy to see that $\text{Supp}_R(M) = \text{Supp}_R(A)$ is a finite subset of $\text{Max}(R)$. Since $A$ is Artinian, it is union of all submodules of finite length. Moreover if $L$ is an $R$-module of finite length, then $L \otimes_R S$ is a $D$-module of finite length. The assertion now follows. \(\square\)

Remark 3.4. Suppose that $S = R[[x_1, \ldots, x_n]]$. In general $A \otimes_R R[[x_1, \ldots, x_n]] \not\cong A[[x_1, \ldots, x_n]]$ and $A[[x_1, \ldots, x_n]]$ may not be $\Sigma$-finite. For example, let $R = k[t]$, where $k$ is a field and $t$ an indeterminate. Let $A = E_k(k)$ be the injective hull of $k$. Then $A \cong k[t^{-1}]$. Choose the element $a = \sum_{i=0}^{\infty} t^{-i} x_i \in S$ we have $\text{Ann}_R(a) = 0 \not\subseteq \text{Max}(R)$. 
Lemma 3.5. Let $I$ be an ideal of $R$ such that $\dim R/I = 0$. Then $H^i_{JS}(S)$ is a $\Sigma$-finite $D$-module.

Proof. We have $\dim R/I = 0$ so $\sqrt{I} = m_1 \cap \cdots \cap m_r$ with $m_i \in \text{Max}(R)$ for all $i = 0, \ldots, r$. By the Mayer-Vietoris sequence we have $H^i_I(R) \cong H^i_{m_1}(R) \oplus \cdots \oplus H^i_{m_r}(R)$. So $H^i_I(R)$ is Artinian for all $i \geq 0$ by [3 Theorem 7.1.3]. By Lemma 3.3, we have $H^i_{JS}(S) \cong H^i_I(R) \otimes_R S$ is $\Sigma$-finite.

The following is very useful in the sequel.

Lemma 3.6. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $D$-modules. Then

(i) If $M$ is $\Sigma$-finite then $M'$ and $M''$ are $\Sigma$-finite.

(ii) Conversely, if $M'$ and $M''$ are $\Sigma$-finite and $M'$ has finite length as a $D$-module, then $M$ is $\Sigma$-finite.

Proof. (i) This part is [14 Proposition 3.6].

(ii) Since $M''$ is $\Sigma$-finite we have $M'' = \cup_{N'' \in \text{Fin}(M'')} N''$. For each $N'' \in \text{Fin}(M'')$, let $N$ be the preimage of $N''$. One can check that $N$ admits a $D$-module structure. We have the following short exact sequence of $D$-modules.

$$0 \to M' \to N \to N'' \to 0.$$

Since $M'$ has finite length as a $D$-module we have $N$ has finite length as a $D$-module. Hence $M = \cup_{N \in \text{Fin}(M)} N$. The two last conditions of Definition 3.1 are not difficult to prove. □

Recalling that a Serre’s category is a category that closes under taking submodules, quotients and extensions. If $R$ contains the rational numbers, then the category of $\Sigma$-finite $D$-modules is a Serre’s subcategory of the category of $D$-module (cf. [14 Proposition 3.7]). At the time of writing, we do not know whether the condition $\mathbb{Q} \subseteq R$ can be removed. Fortunately, the statement of Lemma 3.6 (ii) is enough for our purpose. In the following we prove the global case of [14 Proposition 4.3]. While the proof of [14] is based on spectral sequences, our proof is elementary.

Proposition 3.7. Let $R$ be a (not necessary local) Noetherian ring and $S = R[[x_1, \ldots, x_n]]$ or $S = R[[x_1, \ldots, x_n]]$. Let $J$ be an ideal of $S$ such that $\dim R/J \cap R = 0$. Then $H^i_J(S)$ is $\Sigma$-finite for every $i \in \mathbb{N}$.

Proof. We can assume that $J$ is a radical ideal, so $J \cap R = m_1 \cap \cdots \cap m_r$ where $m_k \in \text{Max}(R)$ for all $k = 1, \ldots, r$. Set $J_k = m_k S + J$, $k = 1, \ldots, r$, we have $J = J_1 \cap \cdots \cap J_r$. Since $m_k + m_h = R$ for all $k \neq h$, we have $J_k + J_h = S$ for all $k \neq h$. By using Mayer-Vietoris’s sequence one can prove that

$$H^i_J(S) \cong H^i_{J_1}(S) \oplus \cdots \oplus H^i_{J_r}(S)$$

for all $i \geq 0$. Therefore, it is enough to prove the assertion in the case $J \cap R = m \in \text{Max}(R)$ (cf. [14 Lemma 3.9]). We proceed by induction of $t = \text{ht}(m)$.

The case $t = 0$, we have that $m$ is a minimal prime of $R$. Let $U = H^0_{m}(R)$ and $\overline{R} = R/U$. We have $U$ is an $R$-module of finite length so $U \otimes_R S$ is a $\Sigma$-finite $D$-module by Lemma 3.3.
By [14, Corollary 3.10], $H^i_j(U \otimes_R S)$ is $\Sigma$-finite for all $i \geq 0$. Applying local cohomology functor for the short exact sequence

$$0 \to U \otimes_R S \to S \to \overline{S} \to 0,$$

where $\overline{S} = R \otimes_R S$, we get the following exact sequence

$$\cdots \to H^{-1}_j(\overline{S}) \to H^i_j(U \otimes_R S) \to H^i_j(S) \to H^i_j(\overline{S}) \to \cdots.$$  

On the other hand, we have $\text{Ass}_R \overline{R} = \text{Ass}_R R \setminus V(\mathfrak{m})$. Notice that $\text{ht}(\mathfrak{m}) = 0$ so $\mathfrak{p} \not\subseteq \mathfrak{m}$ for all $\mathfrak{p} \in \text{Ass}_R \overline{R}$, and hence $\text{Ann}_R(\overline{R}) \not\subseteq \mathfrak{m}$. Moreover $\mathfrak{m} \in \text{Max}(R)$ we have $\text{Ann}_R(\overline{R}) + \mathfrak{m} = R$. Therefore $1 \in \text{Ann}_R(\overline{R})S + J$ because $J \cap R = \mathfrak{m}$. Thus $\text{Ann}_S(\overline{S}) + J = S$ since $\text{Ann}_S(\overline{S}) = \text{Ann}_R(\overline{R})S$. So $H^i_j(\overline{S}) = 0$ for all $i \geq 0$ and hence $H^i_j(S) \cong H^i_j(U \otimes_R S)$ is $\Sigma$-finite for all $i \geq 0$.

For $t > 0$, set $U = H^0_t(R)$ and $\overline{R} = R/H^0_t(R)$. Let $\overline{S} = \overline{R} \otimes_R S$. The short exact sequence

$$0 \to U \otimes_R S \to S \to \overline{S} \to 0$$

induces the exact sequence of local cohomology modules

$$\cdots \to H^i_j(U \otimes_R S) \xrightarrow{\alpha} H^i_j(S) \xrightarrow{\beta} H^i_j(\overline{S}) \to \cdots.$$  

We have the short exact sequence

$$0 \to \text{im}(\alpha) \to H^i_j(S) \to \text{im}(\beta) \to 0.$$  

Since $U$ has finite length as an $R$-module, $U \otimes_R S$ and hence $H^i_j(U \otimes_R S)$ have finite length as a $D$-module by Example 24 (iv) (see also [12, Proposition 3.3]). Thus $\text{im}(\alpha)$ is a $D$-module of finite length. Suppose $H^i_j(\overline{S})$ is $\Sigma$-finite we have $\text{im}(\beta)$ is also a $\Sigma$-finite $D$-module by Lemma 3.0 (i). Lemma 3.0 (ii) implies that $H^i_j(S)$ is $\Sigma$-finite for all $i \geq 0$. Therefore we can assume henceforth that $H^0_t(R) = 0$. Choose an $R$-regular element $a \in \mathfrak{m} = J \cap R$, we have $a$ is also $S$-regular and $a \in J$. So $H^0_t(S) = 0$. For $i \geq 1$ we consider the following short exact sequence

$$0 \to S \to S_a \to S_a/S \to 0.$$  

This sequence induces the exact sequence of local cohomology

$$\cdots \to H^{-1}_j(S_a) \to H^{-1}_j(S_a/S) \to H^1_j(S) \to H^1_j(S_a) \to \cdots.$$  

Notice that $a \in J$, so $H^1_j(S_a) = 0$ for all $i \geq 0$. Thus

$$H^i_j(S) \cong H^1_j(S_a/S) \cong H^{-1}_j(\text{lim}(S/a^n S)) \cong \text{lim}_n H^{-1}_j(S/a^n S).$$  

By inductive hypothesis we have $H^{-1}_j(S/a^n S)$ is a $\Sigma$-finite $D(S/a^n S, R/a^n R)$-module for all $n$ and $i \geq 1$. So $H^{-1}_j(S/a^n S)$ is a $\Sigma$-finite $D(S, R)$-module for all $n$ and $i \geq 1$. By [14, Proposition 3.11] we need only to prove that $\cup_n \mathcal{C}(H^i_j(S/a^n S))$ is finite for all $i \geq 0$. We shall prove that $\mathcal{C}(H^i_j(S/a^n S)) \subseteq \mathcal{C}(H^i_j(S/a S))$ for all $n \geq 1$. The case $n = 1$ is trivial. For $n > 1$, the short exact sequence

$$0 \to S/a S \xrightarrow{a^{-n-1}} S/a^n S \to S/a^{n-1} S \to 0$$

induces the exact sequence

$$\cdots \to H^i_j(S/a S) \xrightarrow{H^i_j(S/a^n S)} H^i_j(S/a^{n-1} S) \to \cdots.$$
Hence \( C(H^i_j(S/a^nS)) \subseteq C(H^i_j(S/aS)) \cup C(H^i_j(S/a^{n-1}S)) \subseteq C(H^i_j(S/aS)) \) by inductive hypothesis. The proof is complete. \( \Box \)

We are ready to prove the main result of this section, it gives a partial positive answer for [14, Question 5.1].

**Theorem 3.8.** Let \((R, m)\) be a local ring and \(S = R[x_1, \ldots, x_n]\) or \(S = R[[x_1, \ldots, x_n]]\). Let \(J\) be an ideal of \(S\) such that \(\dim R/(J \cap R) \leq 1\). Then \(H^i_{mS}H^j_S(S)\) is \(\Sigma\)-finite for every \(i, j \in \mathbb{N}\). In particular \(\text{Ass}_SH^j_{mS}H^i_j(S)\) is \(\Sigma\)-finite for all \(i, j \in \mathbb{N}\).

**Proof.** Since \(\dim R/(J \cap R) \leq 1\), there exists \(f \in m\) such that \(mS \subseteq \sqrt{(J + fS)}\). Thus \(\sqrt{J + mS} = \sqrt{J + fS}\). Notice that \(H^i_j(S)\) is \(J\)-torsion. So

\[
H^j_{mS}H^i_j(S) \cong H^j_{(J + mS)}H^i_j(S) \cong H^j_{(J + fS)}H^i_j(S) \cong H^j_{fS}H^i_j(S)
\]

for all \(i, j \geq 0\). Therefore \(H^j_{mS}H^i_j(S) = 0\) for all \(j > 1\). Hence we need only to prove that \(H^1_{fS}H^i_j(S)\) and \(H^0_{fS}H^i_j(S)\) are \(\Sigma\)-finite for all \(i \geq 0\). By [13, Proposition 8.1.2] we have the following exact sequence

\[
\cdots \rightarrow H^1_{fS}(S) \rightarrow H^1_{(J + fS)}(S) \rightarrow H^i_j(S) \rightarrow H^j_f(S) \rightarrow \cdots.
\]

On the other hand we have the following exact sequence (cf. [13, Remark 2.2.17])

\[
0 \rightarrow H^0_{fS}H^i_j(S) \rightarrow H^1_j(S) \rightarrow H^j_f(S) \rightarrow H^1_{fS}H^1_j(S) \rightarrow 0
\]

for all \(i \geq 0\). Therefore for each \(i \geq 0\) we have the following short exact sequence

\[
0 \rightarrow H^1_{fS}H^{i-1}_j(S) \rightarrow H^{i}_j(S) \rightarrow H^1_{fS}H^i_j(S) \rightarrow 0.
\]

Since \(\dim R/((J + fS) \cap R) = 0\), we have \(H^i_j(S)\) is \(\Sigma\)-finite for all \(i \geq 0\) by Proposition 3.7. Hence \(H^0_{fS}H^i_j(S)\) and \(H^1_{fS}H^1_j(S)\) are \(\Sigma\)-finite for all \(i \geq 0\) by Lemma 3.6. The last assertion follows from the property of \(\Sigma\)-finite \(D\)-modules. The proof is complete. \( \Box \)

We get a result of on the finiteness of associated primes of local cohomology of polynomial rings.

**Corollary 3.9.** Let \((R, m)\) be a local ring and \(S = R[x_1, \ldots, x_n]\). Let \(J\) be an ideal of \(S\) such that \(\dim R/(J \cap R) \leq 1\). Then \(\text{Ass}_SH^i_j(S)\) is \(\Sigma\)-finite for all \(i \geq 0\).

**Proof.** Similarly the proof of Theorem 3.8 we have an element \(f \in m\) such that \(mS \subseteq \sqrt{(J + fS)}\). Consider the exact sequence

\[
\cdots \rightarrow H^{i}_j(J + fS)(S) \rightarrow H^i_j(S) \rightarrow H^j_f(S) \rightarrow \cdots.
\]

We have \(\text{Ass}_SH^i_j(S) \subseteq \text{Ass}_S(\text{im}(\alpha)) \cup \text{Ass}_SH^i_j(S_f)\). Since \(H^i_j(J + fS)(S)\) is \(\Sigma\)-finite, so is \(\text{im}(\alpha)\). Hence \(\text{Ass}_S(\text{im}(\alpha))\) is a finite set. On the other hand we have \(H^i_j(S_f) \cong H^i_{(JS_f)}(S_f)\). Notice that \(S_f \cong R_f[x_1, \ldots, x_n]\) and \(\dim R_f/(JS_f \cap R_f) = 0\), we have \(H^i_j(S_f)\) is a \(\Sigma\)-finite \(D(S_f, R_f)\)-module by Proposition 3.7. So \(\text{Ass}_SH^i_j(S_f)\) is \(\Sigma\)-finite. The proof is complete. \( \Box \)
4. Rings of dimension one

In this section $R$ is a Noetherian ring of dimension one and $S = R[x_1, \ldots, x_n]$ or $S = R[[x_1, \ldots, x_n]]$. We recall our question.

**Question 3.** Is it true that $H^i_J(S)$ has only finitely many associated primes for all ideals $J$ of $S$ and all $i \geq 0$?

The following is an immediate consequence of Corollary 3.9 which was shown before by Núñez-Betancourt in [12, Corollary 3.7].

**Corollary 4.1.** Suppose that $R$ is local and $S = R[x_1, \ldots, x_n]$. Then $\text{Ass}_S H^i_J(S)$ is finite for all ideal $J$ and all $i \geq 0$.

We shall consider the question when $R$ contains a field of characteristic $p > 0$. We start with the following.

**Lemma 4.2.** Let $W$ is the largest ideal of finite length of $R$ and $\overline{R} = R/W$. Let $\overline{S} = \overline{R} \otimes_R S$. Suppose $\text{Ass}_S H^i_J(S)$ is finite for all $i \geq 0$. Then $\text{Ass}_S H^i_J(S)$ is finite for all $i \geq 0$.

**Proof.** The short exact sequence

$$0 \to W \otimes_R S \to S \to \overline{S} \to 0$$

induces the exact sequence of local cohomology

$$\cdots \to H^i_J(W \otimes_R S) \to H^i_J(S) \to H^i_J(\overline{S}) \to \cdots.$$ 

Since $W$ has finite length we have $H^i_J(W \otimes_R S)$ is a $\Sigma$-finite $D$-module by Lemma 3.3 and Remark 3.2. Hence so is $\text{im}(\alpha)$. Moreover $\text{Ass}_S H^i_J(S) \subseteq \text{Ass}_S(\text{im}(\alpha)) \cup \text{Ass}_S H^i_J(\overline{S})$. Therefore if $\text{Ass}_S H^i_J(\overline{S})$ is finite, then so is $\text{Ass}_S H^i_J(S)$. \hfill $\square$

**Proposition 4.3.** Let $R$ be an excellent domain of dimension one and of characteristic $p > 0$. Then $\text{Ass}_S H^i_J(S)$ is finite for all ideal $J$ and all $i \geq 0$.

**Proof.** Let $T$ be the integral closure of $R$. We have $T$ is a finitely generated $R$-module. Since $\dim R = 1$ we have $T/R$ is an $R$-module of finite length. Set $V = T \otimes_R S$. Then $V$ is either $T[x_1, \ldots, x_n]$ or $T[[x_1, \ldots, x_n]]$. The short exact sequence

$$0 \to S \to V \to V/S \to 0$$

induces the exact sequence

$$\cdots \to H^{i-1}_J(V/S) \to H^i_J(S) \to H^i_J(V) \to \cdots.$$ 

Notice that $V/S$ is a $\Sigma$-finite $D$-module of finite length and so is $H^{i-1}_J(V/S)$. Therefore $\text{Ass}_S(\text{im}(\alpha))$ is finite. Since $T$ is Dedekind we have $V$ is a regular ring of characteristic $p > 0$. So $\text{Ass}_V H^i_J(V)$ is finite by [7] or [9]. By the independent theorem we have $H^i_J(V) \cong H^i_{JV}(V)$. Thus $\text{Ass}_S H^i_J(V)$ is finite. The proof is complete. \hfill $\square$

The following is the main result of this section.

**Proposition 4.4.** Let $R$ be an excellent reduced ring of dimension one and of characteristic $p > 0$. Let $S$ is either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. Then $\text{Ass}_S H^i_J(S)$ is finite for all ideal $J$ and all $i \geq 0$. 
Proof. By Lemma 4.2 we can assume that \( \dim R/p = 1 \) for all \( p \in \text{Ass}_R R \). Since \( R \) is reduced, \( 0 = p_1 \cap \cdots \cap p_r \). We proceed by induction on \( r \). The case \( r = 1 \) follows from Proposition 4.3. For \( r > 1 \), the following exact sequence

\[
0 \to S \to (S/(p_1 \cap \cdots \cap p_{r-1})S) \oplus S/p_r S \to S/(p_1 \cap \cdots \cap p_{r-1} + p_r)S \to 0
\]

induces the exact sequence

\[
\cdots \to H_{i-1}^j(S/(p_1 \cap \cdots \cap p_{r-1} + p_r)S) \to H_i^j(S) \to H_i^j(S/(p_1 \cap \cdots \cap p_{r-1})S) \oplus H_i^j(S/p_r S) \to \cdots.
\]

Since \( p_1 \cap \cdots \cap p_{r-1} + p_r \) is not contained in any minimal prime and \( \dim R = 1 \), we have \( R/(p_1 \cap \cdots \cap p_{r-1} + p_r) \) has finite length. Thus

\[
H_{i-1}^j(S/(p_1 \cap \cdots \cap p_{r-1} + p_r)S)
\]

is \( \Sigma \)-finite for all \( i \geq 1 \). Thus \( \text{Ass}_S(\text{im}(\alpha)) \) is finite. Combining with the inductive hypothesis we obtain the assertion. \( \square \)

Inspired by [1] and [2] we raise the following question.

**Question 4.** Let \( R \) be a Noetherian ring of dimension zero and of characteristic \( p > 0 \). Let \( S = R[x_1, \ldots, x_n] \) or \( S = R[[x_1, \ldots, x_n]] \). For each ideal \( J = (a_1, \ldots, a_t) \) of \( S \), is it true that the image of the canonical map

\[
\varphi : H^i(a_1, \ldots, a_t; S) \to H^i_J(S)
\]

generates \( H^i_J(S) \) as a \( D \)-module.

If the above question has a positive answer, then by the same method used in [2] we can extend the result of Proposition 4.4 in the case \( S = R[x_1, \ldots, x_n] \) for any ring of dimension one and of characteristic \( p > 0 \).

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