Solution of the master equation for Bak-Sneppen model of biological evolution in finite ecosystem

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ABSTRACT

The master equations describing processes of biological evolution in the framework of the random neighbor Bak-Sneppen model are studied. For the ecosystem of $N$ species they are solved exactly and asymptotical behavior of this solution for large $N$ is analyzed.
1 Introduction

The model of biological evolution proposed by Bak and Sneppen [1,2] describes mutation and natural selection of interacting species. It is the dynamical system that is defined as follows. The state of the ecosystem of $N$ species is characterized by a set \{ $x_1, ..., x_N$ \} of $N$ number, $0 \geq x_i \geq 1$. In so doing, $x_i$ represents the barrier toward father evolution of the species. Initially, each $x_i$ is set to a randomly chosen value. At each time step the barrier $x_i$ with minimal value and $K - 1$ other barriers are replaced by $K$ new random numbers. In the random neighbor model (RNM), which will be considered in this paper the $K - 1$ replaced non-minimal barriers are chosen at random.

The RNM is the simplest model describing the avalanche-like processes, which are supposed by a conception of "punctuated equilibrium" in biological evolution. These processes are the most characterizing feature for self-organized criticality recently intensively investigated both numerically and analytically [3-6] The RNM is more convenient for analytical studies. The master equations obtained in [3] for RNM are very useful for this aim. In [7] the explicit solution of master equations was found for infinite ecosystem. In this paper we solve the master equations for finite number $N$ of species in ecosystem. We restrict ourselves with the simplest case $K = 2$.

2 Master equations for RNM

The master equations for the RNM are obtained in [3]. They are of the form:

$$P_n(t + 1) = A_n P_n(t) + B_{n+1} P_{n+1}(t) + C_{n-1} P_{n-1}(t) +$$
$$D_{n+2} P_{n+2}(t) + (B_1 \delta_{n,0} + A_1 \delta_{n,1} + C_1 \delta_{n,2}) P_0(t)$$

(1)

Here, $P_n(t)$ is the probability that $n$ at the time $t$ is the number of barriers having values less than a fixed value $\lambda$ at the time $t$; $0 \leq n \leq N$, $0 \leq \lambda \leq N$, , $0 \leq t$; $P_n(0)$
are proposed to be given. For $0 < n \leq N$

$$A_n = 2\lambda(1 - \lambda) + \frac{n - 1}{N - 1}\lambda(3\lambda - 2),$$

$$B_n = (1 - \lambda)^2 + \frac{n - 1}{N - 1}(1 - \lambda)(3\lambda - 1),$$

$$C_n = \lambda^2 - \frac{n - 1}{N - 1}\lambda^2, \quad D_n = (1 - \lambda)^2\frac{n - 1}{N - 1},$$

and $A_n = B_n = C_n = D_n = 0$ for $n = 0, n > N$.

In virtue of the definition of $P_n(t)$,

$$P_n(t) \geq 0,$$  \hspace{1cm} (3)

$$\sum_{n=0}^{N} P_n(t) = 1.$$  \hspace{1cm} (4)

Making summation in (1) over $n$ and taking into account (2) it is easy to establish that

$$\sum_{n=0}^{N} P_n(t + 1) = \sum_{n=0}^{N} P_n(t).$$  \hspace{1cm} (5)

Therefore, if $P_n(0)$ are chosen in such a way that (4) is fulfilled for $t = 0$, then in virtue of (5) it is the case for the solution of (1) for $t > 0$ too. For analysis of (1) it is convenient to introduce the generating function $q(z, u)$:

$$q(z, u) \equiv \sum_{t=0}^{\infty} \sum_{n=0}^{N} P_n(t)z^n u^t.$$  \hspace{1cm} (6)

In virtue of (3),(4) $q(z, t)$ is polynomial in $z$, analytical in $u$ for $|u| < 1$ and

$$q(1, u) = \frac{1}{1 - u}.$$  \hspace{1cm} (7)

The master equations (1) can be rewritten for the generating function $q(z, u)$ as follows:

$$\frac{1}{u}(q(z, u) - q(z, 0)) = (1 - \lambda + \lambda z)^2 \left(\frac{1}{z} \left(1 - \frac{1 - z}{N - 1} \left(\frac{1}{z} - \frac{\partial}{\partial z}\right)\right)\right) \times \times (q(z, u) - q(0, u)) + q(0, u).$$  \hspace{1cm} (8)

The function $q(z, 0) = \sum_{n=0}^{N} P_n(0)z^n$ in (8) is assumed to be given.
3 Asymptotic expansion of \( q(z, u) \) for big \( N \)

If the function \( q(z, 0) \) has an asymptotic expansion in the region of big \( N \) of the form:

\[
q(z, 0) = \sum_{k=0}^{\infty} \frac{q_k(z, 0)}{(N - 1)^k}
\]

then the equation (8) enables one to obtain the similar asymptotic expansion for \( q(z, u) \):

\[
q(z, u) = \sum_{k=0}^{\infty} \frac{q_k(z, u)}{(N - 1)^k}.
\]

The main approximation of \( q_0(z, u) \), the function \( q(z, u) \) can be found from the equation

\[
(z - u(1 - \lambda + \lambda z)^2)q_0(z, u) = zq_0(z, 0) + u(1 - \lambda + \lambda z)^2(z - 1)q_0(0, u)
\]

(9)

following from (8). Since \( q_0(z, u) \) is analytical for \( |z| < 1, |u| < 1 \),

\[
0 = \alpha q_0(\alpha, 0) + u(1 - \lambda + \lambda \alpha)^2(\alpha - 1)q_0(0, u)
\]

(10)

where

\[
\alpha = \alpha(u) = \frac{1 - 2\lambda(1 - \lambda)u - (1 - 4\lambda(1 - \lambda u))^{\frac{1}{2}}}{2\lambda^2 u}
\]

(11)

is the solution of equation \( \alpha - u(1 - \lambda + \lambda \alpha)^2 = 0 \). Obviously, for sufficient little \( |u|, |\alpha| < 1 \). Thus, from equation (10) the function \( q_0(0, u) \) can be found:

\[
q_0(0, u) = \frac{q_0(\alpha, 0)}{1 - \alpha}.
\]

(12)

Substituting (12) in the right hand side of (9), one can find the solution in the following form:

\[
q_0(z, u) = \frac{zq_0(z, 0)(1 - \alpha) + (z - 1)u(1 - \lambda + \lambda z)^2q_0(\alpha, 0)}{(z - u(1 - \lambda + \lambda z)^2)(1 - \alpha)},
\]

(13)

where \( \alpha(u) \) is defined by (10).
For \( k > 0 \) the functions \( q_k(z, u) \) are defined by recurrent relations

\[
q_k(z, u) = \frac{u(1 - \lambda + \lambda z)^2(z - 1)q_k(\alpha, 0) + (1 - \alpha)zq_k(z, 0)}{(z - u(1 - \lambda + \lambda z)^2)(1 - \alpha)} + \frac{u(1 - \lambda + \lambda z)^2(r_{k-1}(z, u) + r_{k-1}(\alpha, u))}{z - u(1 - \lambda + \lambda z)^2}.
\]

(14)

Here,

\[
r_k(z, u) \equiv z \frac{\partial}{\partial z} q_k(z, u) - q_k(0, u).
\]

(15)

In virtue of (13), (14) the first correction to the lowest approximation (12) of \( q(z, u) \) has the form

\[
q_1(z, u) = \frac{u(1 - \lambda + \lambda z)^2(z - 1)q_1(\alpha, 0) + (1 - \alpha)zq_1(z, 0)}{(z - u(1 - \lambda + \lambda z)^2)(1 - \alpha)} + \frac{u(1 - \lambda + \lambda z)^2(r_0(z, u) + r_0(\alpha, u))}{z - u(1 - \lambda + \lambda z)^2}.
\]

(16)

where

\[
r_0(z, u) = z \frac{\partial}{\partial z} \frac{(1 - \alpha)q_0(z, u) + (u(1 - \lambda + \lambda z)^2 - 1)q_0(\alpha, 0)}{(z - u(1 - \lambda + \lambda z)^2)(1 - \alpha)}.
\]

(17)

4 Exact form of \( q(z, u) \)

Let us introduce the quantity

\[
Q(z, u) \equiv \frac{q(z, u) - q(0, u)}{z}.
\]

(18)

It follows from (8) that this function fulfills the relation of the form:

\[
(z - u(1 - \lambda + \lambda z)^2 - \frac{u(1 - \lambda + \lambda z)^2(1 - z)}{N - 1} \frac{\partial}{\partial z})Q(z, u) = q(z, 0) + (u(1 - \lambda + \lambda z)^2 - 1)q(0, u).
\]

(19)

This inhomogeneous differential equation for \( Q(z, u) \) has a special solution

\[
Q(z, u) = (N - 1)e^{R(z,u)} \int_{\frac{1}{z}} e^{-R(x,u)} g(x, u) dx \equiv Q_{sp}(z, u).
\]

(20)
Here,
\[ R(z, u) = \frac{N - 1}{u} \left( \ln(1 - \lambda + \lambda z) - (1 - u) \ln(1 - z) + \frac{1 - \lambda}{\lambda(1 - \lambda + \lambda z)} \right), \quad (21) \]
\[ g(x, u) = \frac{q(x, 0) + (u(1 - \lambda + \lambda z)^2 - 1)q(0, u)}{u(1 - \lambda + \lambda z)^2(1 - x)} \quad (22) \]
and the derivative of \( R(z, u) \) with respect to \( z \) has the form
\[ \frac{\partial R(z, u)}{\partial z} = \frac{(N - 1)(1 - (1 - \lambda + \lambda z)^2)}{u(1 - \lambda + \lambda z)^2(1 - z)} \quad (23) \]
General solution of the corresponding to (19) homogeneous equation
\[ (z - u(1 - \lambda + \lambda z)^2 - \frac{u(1 - \lambda + \lambda z)^2(1 - z)}{N - 1} \frac{\partial}{\partial z})S(z, u) = 0 \quad (24) \]
is of the form
\[ S(z, u) = F(u)e^{R(z,u)}. \quad (25) \]
Here, \( F(u) \) is an arbitrary function of \( u \). Hence, it follows from (19),(20),(25) that the function \( Q(z, t) \) can be represented as follows:
\[ Q(z, u) = F(u)e^{R(z,u)} + Q_{sp}(z, u). \quad (26) \]
In virtue of initial condition (7) for \( q(z, u) \)
\[ Q(1, u) = \frac{1}{1 - u} - q(0, u) \quad (27) \]
For \( 0 < u < 1, z \to 1, S(z, u) \) diverges and \( S_{sp}(z, u) \) has the finite limit:
\[ \lim_{z \to 1} Q_{sp}(z, u) = \frac{1}{1 - u} - q(0, u) \quad (28) \]
Hence, \( F(u) = 0 \) in (25) and this representation for \( Q(z, u) \) can be rewritten in the form:
\[ Q(z, u) = (N - 1)e^{R(z,u)} \int_{\lambda}^{1} e^{-R(x,u)} g(x, u) dx + \]
\[ + (N - 1)e^{R(z,u)} \int_{z}^{\lambda-1} e^{-R(x,u)} g(x, u) dx \quad (29) \]
It follows from (18) that
\[ Q\left( \frac{\lambda - 1}{\lambda}, u \right) = \frac{\lambda (q(\frac{\lambda - 1}{\lambda}, u) - q(0, u))}{\lambda - 1} \]

For the terms in the right hand side of (29) we have for \(0 < u < 1, 0 < \lambda < 1:\)
\[ \lim_{z \to \frac{\lambda - 1}{\lambda} + 0} e^{R(z, u)} = +\infty, \quad (30) \]
\[ \lim_{z \to \frac{\lambda - 1}{\lambda} + 0} (N - 1)e^{R(z, u)} \int_{z}^{\lambda - \lambda} e^{-R(x, u)} g(x, u) dx = \frac{\lambda (q(\frac{\lambda - 1}{\lambda}, u) - q(0, u))}{\lambda - 1}. \quad (31) \]

Therefore (29) can represent the function \(Q(z, u)\) with necessary analytical properties only if
\[ \int_{\frac{\lambda - 1}{\lambda}}^{1} e^{-R(x, u)} g(x, u) dx = 0 \quad (32) \]

This equation defines the function \(q(0, u)\):
\[ q(0, u) = \frac{\int_{\frac{\lambda - 1}{\lambda}}^{1} e^{-R(x, u)} \frac{q(x, 0) dx}{(1 - \lambda + \lambda x)^2(1 - x)}}{\int_{\frac{\lambda - 1}{\lambda}}^{1} e^{-R(x, u)} \frac{(1 - u(1 - \lambda + \lambda x)^2) dx}{(1 - \lambda + \lambda x)^2(1 - x)}} \quad (33) \]

Thus, we obtain from (29) the solution of equation (8) in the following form:
\[ q(z, u) = z \frac{N - 1}{u} e^{R(z, u)} \int_{z}^{\lambda - \lambda} e^{-R(x, u)} \frac{q(x, 0) dx}{(1 - \lambda + \lambda x)^2(1 - x)} + \]
\[ + q(0, u)(1 + z) \frac{N - 1}{u} e^{R(z, u)} \int_{z}^{\lambda - \lambda} e^{-R(x, u)} \frac{(1 - u(1 - \lambda + \lambda x)^2) dx}{(1 - \lambda + \lambda x)^2(1 - x)}, \quad (34) \]

where \(q(0, u)\) is defined by (33).

5 Conclusion

We constructed the solution of the master equation (8) for the finite number \(N\) of species in the ecosystem. It can be proven that the main term (13) of its asymptotic
for large $N$ coincides with the one obtained in [7]. Using (34) one can obtain all the known analytical results for RNM. One can hope that it helps to understand better the most important properties of the self-organized criticality processes.

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