NUMERICAL STUDY OF COSMIC RAY DIFFUSION IN MAGNETOHYDRODYNAMIC TURBULENCE

A. Beresnyak1, H. Yan2,3, and A. Lazarian1

1 Astronomy Department, University of Wisconsin, Madison, WI 53706, USA
2 Astronomy Department, University of Arizona, Tucson, AZ, USA
3 Kavli Institute of Astronomy and Astrophysics, Peking University, Beijing 100871, China

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ABSTRACT

We study the diffusion of cosmic rays (CRs) in turbulent magnetic fields using test particle simulations. Electromagnetic fields are produced in direct numerical MHD simulations of turbulence and used as an input for particle tracing, particle feedback on turbulence being ignored. Statistical transport coefficients from the test particle runs are compared with earlier analytical predictions. We find qualitative correspondence between them in various aspects of CR diffusion. In the incompressible case that we consider in this paper, the dominant scattering mechanism is the non-resonant mirror interactions with the slow-mode perturbations. Perpendicular transport roughly agrees with being produced by magnetic field wandering.

Key words: cosmic rays – magnetohydrodynamics (MHD) – scattering – turbulence

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1. INTRODUCTION

The interaction between cosmic rays (CRs), highly energetic charged particles, and astrophysical fluids is mediated by magnetic fields. As magnetic fields are usually turbulent, CRs do not freely stream along these fields but scatter (see, e.g., Schlickeiser 2002). Efficient scattering is essential for a variety of acceleration mechanisms of CRs, such as, for example, diffusive shock acceleration (Krymsky 1977; Bell 1978; Malkov & Drury 2001, and references therein).

Understanding MHD turbulence is essential for the correct description of CR propagation. One popular model has been based on the combination of slab and two-dimensional perturbations (see Bieber et al. 1988). The simplicity of this empirical model has appealed to researchers and has been used to account for the propagation of CRs in solar wind and the magnetosphere. Numerical simulations (see Cho & Vishniac 2000; Maron & Goldreich 2001; Müller & Biskamp 2000; Cho et al. 2002; Cho & Lazarian 2002, 2003), however, do not show slab modes; instead, they show Alfvénic modes that exhibit scale-dependent anisotropy consistent with predictions in Goldreich & Sridhar (1995, henceforth GS95). The scalings of compressible modes are still a subject of debate, although it is suggested that the slow mode is passively advected by the Alfvén mode (GS95; Lithwick & Goldreich 2001), which was verified by numerics; also, fast mode showed relative isotropy which was suggestive of a separate acoustic-type cascade (Cho & Lazarian 2002, 2003).

While particular aspects of the GS95 model, e.g., the value of the spectral index, have been debated (see Boldyrev 2005, 2006; Beresnyak & Lazarian 2006; Gogoberidze 2007; Beresnyak & Lazarian 2009a, 2009b), this model provides a good start for studying CR scattering. This program was realized in a number of publications such as Chandran (2000), Yan & Lazarian (2002, 2004, hereafter YL02, YL04, respectively), and Brunetti & Lazarian (2007). In the last three papers, following Cho & Lazarian (2003), MHD turbulence has been decomposed into Alfvén, slow, and fast modes.

The propagation of CRs is a mature quantitative field, which makes use of both analytical studies and numerical simulations. For example, a quasi-linear theory (QLT) was used to calculate scattering of CRs propagating in a mean magnetic field with small perturbations. However, as turbulence paradigms were changing, so were the results of CR scattering theories. The purpose of this paper is to measure CR scattering numerically, based on the best available direct numerical simulations of MHD turbulence, and to compare these results with what scattering theories predict.

QLT has demonstrated that the gyroresonance in GS95 type turbulence is substantially suppressed and negligible (Chandran 2000; YL02; YL04). However, the key assumption of QLT, that the particle’s orbit is unperturbed, significantly limits its applicability. Additionally, QLT has problems in treating scattering of particles with momentum nearly perpendicular to the magnetic field (see Jones et al. 1973, 1975; Owen 1974; Goldstein 1976; Goldstein & Kulsrud 1976) and perpendicular transport (see Kóta & Jokipii 2000; Matthaeus et al. 2003).

Various nonlinear theories have been proposed to improve the QLT (see Dupree 1966; Goldstein 1976). In the recent paper of Yan & Lazarian (2008, henceforth YL08), a nonlinear formalism (NLT) based on Goldstein (1975) was developed. The gyroresonance was found to be marginal in incompressible turbulence. However, transit time damping (TTD) was fairly efficient which is different from the QLT result. TTD due to nonlinear scattering can be understood as a scattering by large-scale magnetic compressions (magnetic bottles formed by slow mode). The ideas on perpendicular diffusion have been dominated by field line random walk (Jokipii 1966; Jokipii & Parker 1969; Forman et al. 1974). It can be justified in a situation when CRs do not scatter backward. However, in three-dimensional turbulence, parallel transport is also diffusive, and this can reduce perpendicular transport.

The difference between QLT and NLT has important astrophysical consequences. Indeed, in some phases, such as the hot interstellar medium (ISM), the fast mode is strongly damped, which leaves only Alfvén and slow modes for scattering. According to the QLT, however, these modes do not provide any significant scattering. This would predict that there are large volumes in the disk of the Galaxy where CRs do not scatter at all, which would put into question global simulations of propagation...
of CRs in the Galaxy or halo made without such an assumption. This will also somewhat contradict the isotropy of galactic CRs observed on Earth, because isotropy suggests efficient scattering. Fortunately, NLT corrects this “zero-scattering” QLT prediction, putting a lower limit on the efficiency of CR scattering, thus mitigating contradictions described above.

Diffuse γ-ray emission from the Galaxy, detected by EGRET, Fermi, and other missions, has been explained by the interaction of galactic CRs with ISM, molecular clouds, and interstellar magnetic fields. Pion production by protons and nuclei and inverse Compton (IC), bremstrahlung, and synchrotron emission by electrons and positrons all contribute to the observed γ-ray emission. Historically, however, the models of emission included a fairly detailed particle physics part and modeling of galactic matter and radiation and rather crude assumptions of CR distribution and propagation in the Galaxy. Hunter et al. (1997) used a model where CR density was a matter density smoothed on some “coupling scale” around 1.76 pc. This model gave a good spatial fit, but lacked in predicted γ-rays at energies >1 GeV. Detailed propagation and reacceleration models, including treatment of secondary particles using the GALPROP code (Strong et al. 2004), showed significant improvements in fitting the data, which included not only γ-ray observations but also CR properties detected on Earth, in particular electron to proton and boron to carbon ratios. The fitting included variations of the primary (injection) spectra of CR species which is not yet reliably known from models of CR acceleration by supernova shocks. A certain amount of residuals still remained in these models. In particular, the uncertainty in the fitting of the Galactic contribution leads to a widely varied diffuse extragalactic component (Sreekumar et al. 1998; Strong et al. 2004a).

In this paper, we study the scattering by the incompressible component of turbulence. If the fast mode is present, however, it will dominate scattering of low-energy CRs, as long as the fast mode is not effectively suppressed (YL04). Another efficient mechanism of scattering of low-energy CRs is collective scattering (see the discussion).

In what follows, we discuss numerical methods, including direct numerical simulations (DNSs) of MHD turbulence and the particle tracing technique, in Section 2. We discuss theoretical expectations for CR scattering in Section 3. We provide numerical measurements of scattering in Section 4 and measurements of space diffusion in Section 5. We discuss our results in Section 6.

2. NUMERICAL METHODS

In order to trace particle trajectories we used electromagnetic fields obtained in direct three-dimensional simulations of MHD turbulence. For the purpose of this paper we used only incompressible simulations for a variety of reasons. First, we wanted to test those predictions of the theory that pertain to incompressible case. Second, the incompressible simulations were performed with a pseudospectral code that has an explicit dissipation and, unlike finite-difference code, has no uncertainties due to numerical dissipation. Also, incompressible simulations have a larger inertial range.

2.1. DNS of Turbulence

We solved incompressible MHD equations

\[ \partial_t \mathbf{w}^\pm + \mathbf{S}(\mathbf{w}^\mp \cdot \nabla)\mathbf{w}^\pm = -\nu_b (\nabla^2)^2 \mathbf{w}^\pm + \mathbf{f}^\pm, \quad (1) \]

written in terms of Elsasser variables which are defined in terms of velocity \( \mathbf{v} \) and magnetic field in velocity units \( \mathbf{b} = B/(4\pi \rho)^{1/2} \) as \( \mathbf{w}^+ = \mathbf{v} + \mathbf{b} \) and \( \mathbf{w}^- = \mathbf{v} - \mathbf{b} \). \( \mathbf{S} \) is a solenoidal projection operator, and \( \mathbf{f}^\pm \) is Elsasser forcing. These are general equations which can be used for either turbulence with no mean magnetic field (i.e., when the average of \( \mathbf{w}^+ - \mathbf{w}^- \) is zero) or in the presence of such a mean field. In the latter case perturbations of \( \mathbf{w}^\pm \) can be seen as waves propagating opposite to the magnetic field direction. Both Alfvén and pseudo-Alfvén waves propagate with the same velocity \( \nu_A = B_0/(4\pi \rho)^{1/2} \).

We used the pseudospectral code described in more detail in Beresnyak & Lazarian 2009a, 2009b (henceforth BL09a, BL09b). The pseudospectral code solves Equation (2) as an ordinary differential equation in time for each spatial Fourier harmonic, the “pseudo” coming from the fact that the nonlinear term is calculated in real space, and then converted back to Fourier space. The dissipation and divergence-free condition for velocity and magnetic field are done with simple algebraic operations in Fourier space. For time integration we use leapfrog which is time reversible and numerical dissipation is absent, because the nonlinear term, calculated in this manner, preserves both energy and cross-helicity. Therefore, the only dissipation comes from the explicit dissipation term. The turbulence was driven by either independent Elsasser driving or by pure velocity driving (which formally corresponds to \( f^+ = f^- \)). For the purpose of this paper we used the results of 7683 balanced and imbalanced turbulent simulations from BL09a. Balanced turbulence corresponds to the well-studied limit, where the rms values of \( w^+ \) and \( w^- \) are equal. Physically, this corresponds to the situation when the flow of \( w^- \) perturbations, which propagate along the mean magnetic field direction, balances the opposing flow of \( w^+ \).
The more general case of imbalanced turbulence is less studied (see BL08, BL09a, and references therein), but is more likely to be found in nature. This is due to the fact that MHD turbulence is often driven by the strong localized source of perturbations and near the source we mostly see waves moving away from the source, such as a solar wind turbulence near the Sun, which is strongly imbalanced.\footnote{The measurement of the imbalance in the solar wind has been possible with the advent of satellites that independently measure the velocity and the magnetic field at the same point. Similar measurements for other astrophysical sources, such as ISM, are yet to be developed.} Naturally, we are also interested in particle scattering in the imbalanced turbulence, although few theoretical predictions of scattering exist in this case, if any.

Another dimension in the parameter study of BL09a was the strength of perturbations with respect to the mean field, $\delta B \sim B_0$, is called the trans-Alfvénic case, where perturbations are of the order of the mean field. We also consider the sub-Alfvénic case when perturbations were approximately 10 times weaker than the mean field $B_0$, which correspond to so-called Alfvénic Mach number $M_A \sim 0.1$.\footnote{Strong MHD turbulence appears naturally as a result of the anisotropic cascade. Even if turbulence is driven weakly with respect to the mean field, the perpendicular cascade of weak turbulence (Galtier et al. 2000) will increase the strength of interaction until it becomes strong. The realistic ISM turbulence, however, is driven strongly, such as $\delta B \sim B_0$, on the outer scale. So, turbulence is strong to begin with and continues to be strong along the cascade (GS95).} The latter case can be considered as smaller scales of trans- or super-Alfvénic turbulence that cannot be reached directly by three-dimensional simulations of the aforementioned flows. In order for sub-Alfvénic turbulence to be strong\footnote{Strong MHD turbulence appears naturally as a result of the anisotropic cascade. Even if turbulence is driven weakly with respect to the mean field, the perpendicular cascade of weak turbulence (Galtier et al. 2000) will increase the strength of interaction until it becomes strong. The realistic ISM turbulence, however, is driven strongly, such as $\delta B \sim B_0$, on the outer scale. So, turbulence is strong to begin with and continues to be strong along the cascade (GS95).} it has to be driven anisotropically on its outer scale, which was realized in BL09a and BL09b.\footnote{Here, we omit the $2\pi$ factor normally present in the size–wavevector relation, as we normalized the cube size to unity.} The computational box was also elongated in the direction of the mean field, with the parallel size 10 times larger than the perpendicular size for the $M_A = 0.1$ case. Throughout this paper, when we mention the “box size” and the “outer scale of turbulence” we mean perpendicular size; the parallel size is the same for trans-Alfvénic cubes and 10 times larger for sub-Alfvénic cubes.

A note of caution has to be said with regard to sub-Alfvénic simulations being the small scales of trans-Alfvénic turbulence. As we use periodic boundaries, the scales which are larger than the cube size are excluded from consideration. This means that we cut out a range of scales in the inertial interval of turbulence and all larger scales are represented only by the value of the mean magnetic field (the mean velocity can be excluded by the local frame of reference). This could or could not be satisfactory for simulations of particle scattering. If the resonant scattering mechanism is effective, then particles mostly interact with those scales of magnetic perturbations that are present in the simulation. In the opposite case the aforementioned interaction can be less effective than the interaction with large scale perturbations that are not present in the numerical cubes. In this case the result should not be trusted. We will return to this question below. Accidentally, the QLT consider scattering in a manner which is consistent with an approach that ignores larger scales and consider particle gyrating along a strong guiding field and interacting with small resonant perturbations.

The turbulence was driven with a specially designed quasi-stochastic driving on the outer scale (with wave numbers in the interval $k = 2 \cdots 3.5$) with a self-correlation time of 2 in code units. The driving worked until stationary state was reached. The scale of the largest coherent eddy in the simulation was around 0.2 of the cube size. This is the outer scale of turbulence $L = 0.2$. This largest correlation scale of velocity and magnetic perturbations is determined by nonlinear interaction and is typically less than the driving correlation scale. On the driving scales, i.e., $k = 2 \cdots 3.5$, turbulence is not yet fully developed and the spectrum is distorted, having a characteristic bump, and well-developed turbulence starts with $k = 5$. Another definition of the outer scale is through anisotropy. One can expect the anisotropy to follow a Goldreich–Sridhar critical balance $k_1 \sim k_2^2 \sim L^{-5/3}$. This also leads to the estimate of $L \sim 0.2$, with turbulence being approximately isotropic at $k = 5$ and approximately 1:2 anisotropic at $k = 40$ for the trans-Alfvénic case. At any given time the cube contained a large number of independent turbulent realizations ($> 40$). Spectra for one balanced and one imbalanced case are presented in Figure 1. Further details of the code and simulations can be found in BL09a and BL09b.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{spectra.png}
\caption{Sample spectra from turbulent simulations. Left—balanced case, right—imbalanced case. Shown are Elsasser energy spectra (see BL09a for more details).}
\end{figure}

### 2.2. Particle Tracing

The electric field in the laboratory frame was obtained through the $E = [-v \times B]/c$ equation, assuming $v_A/c = 10^{-5}$ (a typical value for the ISM).\footnote{The velocity was measured in the Alfvénic units and the electric field is in the same units as the magnetic field.} The particles were injected randomly through the cube and the trajectories were traced by a hybrid Runge–Kutta quality-controlled ODE solver, assuming periodic boundaries for particles as well as fields.

In particular, we solved six equations:

$$\frac{d\hat{u}}{d\tilde{s}} = \hat{\gamma} E + \hat{u} \times B \quad (2)$$

$$\frac{d\mathbf{x}}{d\tilde{s}} = r_L \hat{u}. \quad (3)$$

Here $\hat{u}$ is the normalized space component of the 4-velocity, $\hat{u} = \mathbf{u}/\gamma_0$, where $\gamma_0$ is the initial particle gamma-factor. Also, $\hat{\gamma} = \sqrt{1/\gamma_0^2 + \hat{u}^2}$, $\tilde{s} = (eB_0/mc^2)\gamma_0 t$ is self-time measured in cyclotron frequency units (a gyration frequency in the particle’s own frame). A particle with $\mu = 0$ will make a full orbit in the $B_0$ field in $2\pi$ time. Therefore, we conveniently measure scattering frequency relative to gyration frequency. The measure of the initial particle energy, the normalized Larmor radius, is expressed as $r_L = mc^2\gamma_0/eB_0$. Physically, one can think of $\gamma_0$ as a measure of the relativistic of the particle, i.e., for small $\gamma_0$ we will recover nonrelativistic equations, and for large $\gamma_0$—ultra-relativistic equations. At the same time, $r_L$ is the...
measure of energy, but with respect to the perpendicular size of the simulation box. In most simulations we took \(1/\gamma_0\) (which enters only in the equation for \(\dot{\gamma}\)) as zero or close to zero, such as \(10^{-5}\); this corresponds to ultra-relativistic particles. \(r_L\) was varied from 0.1 of the cube size to around a grid size. Figure 2 presents the ensemble-averaged square distance versus time for different \(r_L\). The square distance grows linearly with time, which is expected for diffusive motion.

3. EXPECTED CR TRANSPORT PROPERTIES IN MHD TURBULENCE

3.1. Advection–Diffusion Equation

In a complex problem of propagation and acceleration of CRs we often use the so-called diffusive approximation which assumes that the particles scatter or gain energy in small steps. In this approximation the local particle dynamics will be averaged to obtain the spatial diffusion coefficient, \(D_{xx}\), and the momentum diffusion coefficient, \(D_{pp}\), that go into the advection–diffusion equation for the evolution of quasi-isotropic CR distribution function \(f\):

\[
\frac{df}{dt} + u \frac{df}{dx} = \frac{\partial}{\partial x} \left( D_{xx} \frac{df}{dx} \right) + \frac{p}{3} \frac{d}{dp} \frac{df}{dp} + \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 D_{pp} \frac{df}{dp} \right)
\]

(e.g., Skilling 1975). The source terms have to be added to the right-hand side of this equation to account for injection from thermal particles and the proper boundary conditions should be defined. Here, we assumed for simplicity that \(f(x,p)\) depends only on one spatial coordinate \(x\) and the magnitude of CR momentum, \(p\). This equation uses a “local” system of reference, where particle momentum is measured with respect to the rest frame of the fluid. In a situation when the advection–diffusion equation is not adequate one has to fall back to more general approaches, such as Vlasov’s equation (see, e.g., Schlickeiser 2002). In this paper, we study particle dynamics assuming diffusion approximation and we monitor if this dynamics looks like diffusive dynamics or not.

3.2. Formalism for NLT

We start with explaining QLT which is the theory for resonant interactions: gyroresonance scattering and transit scattering (also called transit time damping, TTD). The resonant condition is \(\omega - k_{||} v_\mu = n\Omega\) \((n = 0, \pm 1, 2, \ldots)\), where \(\omega\) is the wave frequency, \(\Omega = \Omega_0 / \gamma\) is the relativistic gyration frequency, and \(\mu = \cos \theta\), with \(\theta\) being the pitch angle of particles. TTD corresponds to \(n = 0\) and it requires compressible perturbations. Most of the gyroresonance contribution comes from \(n = 1\).

It was demonstrated that scattering by Alfvénic turbulence is substantially suppressed due to its anisotropy (Chandran 2000; YL02). Figure 3 illustrates how interaction is suppressed. The scattering rate in GS95 turbulence with the outer scale of \(L\) and assuming that \(\theta\) is not close to 0 is given by QLT as (YL02)

\[
D_{\mu\mu} = \frac{v^{2.5} \mu^{5.5}}{\Omega^{1.5}L^{2.5}(1 - \mu^2)^{0.5}} \Gamma \left[ 6.5, k_{\text{max}}^2 k_{||,\text{res}}^{-2} L^{-4} \right]
\]

where \(\Gamma[a, z]\) is the incomplete gamma function, \(k_{\text{max}}\) corresponds to the dissipation scale of turbulence, and \(k_{||,\text{res}} v_\mu = \Omega\). The scattering frequency, therefore, is approximately Bohm-like if Larmor radius is of the order of \(L\), but then falls steeply as \(\Omega^{-1.5}\) and becomes negligible for small energies.

Contrary to QLT which assume that the magnitude of the magnetic field stays constant, NLT relaxes this assumption and allows this quantity to change gradually, adiabatically with respect to particle motion. Due to conservation of the adiabatic invariant \(p^2 / B\) (see Landau & Lifshitz 1975) the pitch angle will gradually vary, resulting in resonance broadening (Völk 1975). Nonlinear transport (NLT) formalism is based on the replacement of the sharp resonance between waves and particles \(\delta(k_{||} v_1 - \omega \pm n\Omega)\) from QLT to the “resonance function” \(R_n\) (YL08):

\[
R_n = \Re \int_0^\infty dt e^{i(k_{||} v_1 + n\Omega - \omega)t - \frac{1}{2} k_{||}^2 v_1^2 t^2} \left( \frac{-i k_{||} v_1}{\xi} \right)^n
\]

\[
= \frac{\sqrt{\pi}}{|k_{||} \Delta v_1|} \exp \left( -\frac{(k_{||} v_1 - \omega + n\Omega)^2}{k_{||}^2 \Delta v_1^2} \right)
\]

The width of the resonance function depends on the perturbation strength of the turbulence \(\Delta \mu = \Delta v_{||} / v_\perp \approx \sqrt{B^2 / B} = \sqrt{M / A}\). For gyroresonance \((n = \pm 1, 2, \ldots)\) the result depends on whether \(\mu\) is strongly or weakly perturbed by the regular field.
If \( \mu \gg \Delta \mu \), the result is similar to QLT, because the exponents in Equation (6) become close to \( \delta \) functions. For \( \mu < \Delta \mu \), however, the result is different. To demonstrate this we can consider the case of 90\( ^\circ \) scattering. Indeed, if \( \mu \to 0 \), resonance happens mostly at \( k_{||,\text{res}} \sim \Omega/\Delta \), while in QLT \( k_{||,\text{res}} \sim \Omega/v \) \( \to \infty \). Compared to TTD, however, \( D_{\mu \mu} \) for gyroresonance is still smaller in incompressible case (see Figure 4) due to anisotropy.

### 3.3. Scattering in Strong MHD Turbulence

Assuming the tensor of magnetic perturbations introduced in Cho et al. (2002), which is consistent with the GS95 model, we can calculate scattering from TTD and gyroresonance. Assuming small energies corresponding to Larmor radii much smaller than the outer scale one gets for TTD (YL08):

\[
D_{\mu \mu}^T = \frac{\sqrt{\pi} M_A^{3/2}}{16L} \left(1 - \frac{\mu^2}{\Delta \mu^2}\right)^{3/2}\left[-E_i(q \xi) - e^{-q \xi}\right]_{\xi=\max}^1 \times \exp\left[-\frac{(\mu - v_A/v)^2}{\Delta \mu^2}\right],
\]

where \( \xi = kL \), \( E_i(q \xi) = \int_0^\infty \ln(\xi t)/t \) and \( q = (\xi_{\max} M_A^2)^{-2/3} \).

Figure 4 displays the pitch angle diffusion coefficients resulting from TTD scattering and gyroresonance (YL08). We see that gyroresonance is mostly subdominant; however, at small pitch angles TTD is inefficient and gyroresonance dominates.

### 4. Measurements of Scattering

\( D_{\mu \mu} \) scattering property was measured in the tracing experiments where an ensemble of particles with the same \( r_L \) (energy) and a particular \( \mu_0 \) were traced by a certain time. This time was determined by the condition that the rms of deviations of \( \mu \) is small (i.e., 0.1–0.01). Then the curves of the ensemble-averaged \( \langle (\mu - \mu_0)^2 \rangle \) were fitted with a linear curve, and so \( D_{\mu \mu} \) was obtained. \( D_{\mu \mu} \) for the trans-Alfvénic case is presented in Figure 5.

### 5. Measurements of Space Diffusion

The measurements of space diffusion \( D_\alpha \) and \( D_\gamma \) were more straightforward than the measurements of \( D_{\mu \mu} \) because we did not limit the integration time as in the previous section. Therefore, we integrated for as long as it took for the \( \langle (x - x_0)^2 \rangle \) and \( \langle (y - y_0)^2 \rangle \) to show good diffusive linear dependence with time. Those integration times turned out to be very long, so the particles crossed the outer scale of turbulence for many times. Therefore, these measurements correspond to diffusion on the outer scale and not to “sub-diffusion” (see, e.g., YL08). Moreover, we measured diffusion with respect to some global frame of reference, determined by the global mean magnetic field. Therefore, our measurements do not necessarily correspond to theories that measure “parallel” or “perpendicular” diffusion with respect to the magnetic field lines.
Also, we were only able to obtain the lower limit of $D_{xx}$ for the sub-Alfvénic case due to the very low 90° scattering frequency. This was manifested by the fact that as we increased the precision of the code, $D_{xx}$ increased and did not show convergence. This very low scattering frequency has to do with what we discussed earlier—the lack of larger scale perturbations in sub-Alfvénic cubes. In this case, since $\Delta \mu < \mu$, the resonance function becomes narrow so that marginal interaction is available at 90°. In other words, in the absence of the large-scale perturbations, which are normally present in nature but absent in our sub-Alfvénic cubes, the 90° scattering becomes problematic, and parallel diffusion is replaced by ballistic propagation along the mean field. At the same time, this suggests that QLT (resonant scattering) cannot be used for low energy particle scattering, as large scales contribute more than the resonant scales.

As the mean magnetic field was along the “x” axis, our $D_{xx}$ coefficient corresponds to “parallel diffusion,” while $D_{yy}$ corresponds to “perpendicular diffusion.” This correspondence, however, is tentative, since most theories predict “parallel” or “perpendicular” diffusion as happening with respect to the local magnetic field lines. We nevertheless will use terms “parallel” and “perpendicular” for $D_{xx}$ and $D_{yy}$. We also claim that the measurement of the diffusion with respect to the global reference frame has more practical importance and is more easily applicable to the results of observations.

### 5.1. Parallel Diffusion

The results for parallel diffusion for the trans-Alfvénic case are presented in Figure 6. Along with the standard “balanced” MHD turbulence case (presented by the solid line) we calculated this diffusion coefficient for simulations with different degrees of imbalance, using datacubes from simulations of Beresnyak & Lazarian (2009a). As the aforementioned paper (along with the earlier study of Beresnyak & Lazarian 2008) has the first high-resolution simulations of stationary strong imbalanced turbulence, it is important to numerically study the scattering coefficient, even more so when the theory is lacking.

As we see from Figure 6 at small energies $D_{xx}/\Omega$ is linearly proportional to $r_L$, i.e., as in the case with $D_{\mu\mu}$ the scattering frequency is independent of energy. At higher energies it becomes proportional to the square of $r_L$. This is again consistent with the behavior of $D_{\mu\mu}/\Omega$ from Figure 5, i.e., that at high energies $D_{\mu\mu} \sim \Omega$ and at $D_{xx} \sim 1/D_{\mu\mu}$, $D_{xx}/\Omega \sim 1/\Omega^2 \sim r_L^2$. The transition happens at $r_L/L \approx 0.1$, same as in Figure 5. So, we conclude that the measurements of $D_{xx}$ and $D_{\mu\mu}$ are consistent with each other. A note of caution toward direct comparison of these two measurements is due, however. In the measurement of $D_{xx}$ we did not control the particle’s energy, which could undergo changes during the long integration times of $D_{xx}$ measurement. $D_{\mu\mu}$, however, was measured during short times, and as the electric field was assumed to be small (smaller than $B$ by a factor of $v_A/c \approx 10^{-5}$), there was no significant energy change during this short time. This can explain why the transition between two regimes of scattering is sharper in Figure 5 than in Figure 6. Also, as we mentioned previously, $D_{xx}$ is the diffusion coefficient measured in the global reference frame, while $D_{\mu\mu}$ defines pitch-angle scattering with respect to the local field direction.

Figure 6 also shows that the diffusion coefficient is pretty independent of the degree of imbalance, indicating that the trans-Alfvénic imbalanced turbulence has approximately as many magnetic bottles as its balanced counterpart. This is consistent with the assumption that only large-scale perturbations significantly contribute to scattering. Indeed, in the imbalance simulations of BL09a the outer-scale magnetic field was determined primarily by the stronger Elsasser component and has a similar structure and magnitude and outer-scale magnetic field to balanced simulations.

### 5.2. Perpendicular Diffusion

Perpendicular diffusion coefficients are presented in Figures 6 and 7. For the various models, regimes, and terminology of perpendicular diffusion we refer the reader to YL08 and references therein. Let us first interpret the measurements in the sub-Alfvénic case. We chose the initial pitch angle of the particle to be 45°. As we discussed earlier, due to the particular choice of data, there was not any 90° scattering in this case, i.e., particles moved ballistically along the “x” axis, but their trajectories diffused from the center due to magnetic field wandering. As suggested by Figure 7 the dependence of this plot is almost linear, i.e., $D_{yy}/\Omega \sim r/L$, or $D_{yy}$ is independent of energy.

In three-dimensional turbulence, field lines are diverging away due to shearing by Alfvén modes (see Lazarian & Vishniac 1999; Narayan & Medvedev 2001). Most recently, the diffusion in magnetic fields was considered for thermal particles in Lazarian (2006, 2007). The cross-field transport can result from the deviations of field lines at small scales, as well as field line random walk at large scale.

If we assume that the particle follow magnetic field lines and is diffused only by the outer-scale magnetic field wandering, the perpendicular diffusion can be expressed as $D_{yy}/(L^2/\Omega) \approx 2^{-1/2}(L_L/L)^2 \cdot (L/L_\|) \cdot (r_L/L)$, where $L_\|$ is the outer parallel scale (which is 10 times bigger than $L$ in our sub-Alfvénic simulation), $1/\sqrt{2}$ is the cosine of the pitch angle and $L_L$ is the distance the particle is deflected when it travels $L_L$ along the field line (see Equation (26) in YL08). We would expect $L_L$ to be close to $L$. From the fit of Figure 7 we derive $L_L/L \approx 0.92$ which is fairly close, considering the uncertainty in $L$. Using
the same argumentation we obtain $L_\parallel/L \approx 0.53$ from the fit of Figure 6 which is short of what we expected. This is an indication that the impediment to traveling in the parallel direction which is present in the trans-Alfvénic case due to $90^\circ$ scattering decreases diffusion in the perpendicular direction. We stop with this conclusion, as there is clearly not enough data for a detailed comparison with different models in YL08.

6. DISCUSSION

In this paper, we numerically measured diffusion coefficients that arise when particles propagate in turbulent magnetic fields. Unlike previous studies, we used realistic fields obtained in three-dimensional simulations of MHD turbulence. The focus of this paper was the incompressible case, where the fast magnetosonic mode is absent. The earlier QLT calculations presumed that particle scattering is negligible in this case, as the perturbations are extremely anisotropic with respect to the mean field. We figured that QLT is not applicable when the magnitude of the magnetic field is strongly perturbed, and that another approach called NLT has to be adopted (YL08). NLT allows relatively efficient scattering through TTD as the particle’s pitch angle changes adiabatically and makes $90^\circ$ scattering possible. One can interpret this as scattering through large-scale magnetic mirrors.

We confirmed this picture of mirrors by measuring scattering frequency which is independent of energy for small energies. We also studied spacial diffusion which, in the case of parallel diffusion, is related to scattering frequency. The case of perpendicular diffusion is more complicated. We showed that if particles do not scatter in the parallel direction, the perpendicular diffusion is mostly due to magnetic field line wandering. This case could be unphysical though, as the absence of parallel scattering was due to the absence of larger scales in the simulation. In the case when parallel diffusion was operating, the perpendicular diffusion was reduced. At this point, however, we do not have enough data to distinguish between different models of perpendicular diffusion.

Special attention should be brought to the astrophysical interpretation of scattering in the imbalanced turbulence. Figures 6–8 indicate that the scattering is similar to the balanced case. This qualitatively and quantitatively agrees with the picture that was presented in this paper, namely, that in the incompressible case most of the scattering will come from the outer scale of turbulence and most of the perpendicular diffusion will come from the field wandering on the outer scale. This fact, however, does not mean that the scattering in astrophysical objects will be the same regardless of the degree of imbalance. The key to understand this is to understand the nature of our MHD simulations. In these simulations, we kept the fluctuation amplitude and the anisotropy controlled on the outer scale; the physically all-important dissipation rate, however, varied greatly depending on the degree of imbalance. In astrophysics, turbulence is caused by sources of kinetic energy, such as stellar and active galactic nucleus (AGN) jets, stellar winds interacting with the ISM, supernovae, and the Sun creating perturbations in the solar wind. Turbulent dissipation will have to balance this influx of energy; however, dissipation depends greatly on the degree of imbalance. Therefore in a situation with a constant influx of energy, imbalanced turbulence will have much larger perturbation amplitudes, which will result in a much more efficient scattering. With respect to the relation between the dissipation rate and perturbation amplitude we refer to the imbalanced turbulence model presented in BL08 as the most realistic model to-date, and the simulations in BL09a and Beresnyak & Lazarian (2010, hereafter BL10). These papers predict and observe that the total dissipation rate, which will mostly come from a dissipation of a strong wave, will be reduced by a factor which is larger than the $w^+/w^-$ ratio (which is also indicated in Figure 1 where the ratio of energies is larger than the ratio of fluxes squared). The exact correspondence between total energies and fluxes, however, will depend on how turbulence is driven on the outer scale and the asymptotic power-law solution will be realized only on small scales (BL08). For the simplest case of identical driving the numerical results of BL09a can be used directly and for small imbalances the reduction factor of $w^+/w^-$ will suffice (BL10).

A word of caution toward directly using these results is due, however. The aforementioned model and the simulations in BL09a describe stationary imbalanced turbulence. However, as we learned from these studies, the time to establish stationary states greatly increases with larger imbalances. As astrophysical processes are usually transient, it is possible that in a situation with large imbalance the stationary state will not be achieved. The stationary imbalanced turbulence could still be used to infer the properties of small-scale fluctuations, as timescales are smaller, but, as we saw in this study, large scales are important for scattering. This problem will be solved by models of transient and inhomogeneous imbalanced turbulence, although at present such models are still in their infancy (see, e.g., the appendix.
in BL09a). We are optimistic, however, that the properties of CR scattering in realistic astrophysical objects that feature imbalanced turbulence, such as solar and stellar winds, AGN jets, and many others, will be figured out.

Although the NLT prediction for nonresonant mirror scattering by incompressible turbulence puts a lower limit on CR scattering, in most realistic astrophysical circumstances, there are several mechanisms that could compete with it. If the fast mode is present and has a sufficient amplitude in the range of scales corresponding to the Larmor radii of low-energy CRs, those CRs will be scattered primarily due to fast mode (YL04). Also, if CRs have large density gradients and tend to stream in a particular direction, such as CRs escaping the Galaxy or CRs streaming in front of the supernovae shock, the back-reaction of CRs will be important. Speaking of turbulence, in this paper we considered generic astrophysical turbulence driven on large scales. Other types of astrophysical turbulence are often important for scattering and acceleration. This includes MRI turbulence (see, e.g., Hawley et al. 2001) and turbulence generated by CR–MHD fluid interaction in supernova shocks (see, e.g., Beresnyak et al. 2009 and references therein).

Stochastic scattering by MHD turbulence was not studied here as the correct calculation requires time-dependent MHD fields, so that simulations of turbulence are integrated at the same time as particles propagate. This will be a matter for future research.

This paper treated fluid dynamics without back-reaction of CRs. A particular mechanism for such a process, called streaming instability, has been popular in describing CR scattering since a long time ago (see, e.g., Kulsrud & Pearce 1969). Another gyroresonance model based on the local perturbations of the magnetic field has predicted efficient scattering even without streaming (Lazarian & Beresnyak 2006). In this paper, we considered only test particle scattering. This describes an important physical limit where CR density is negligibly small and collective effects are unimportant. We took the test particle and incompressible limits to have a reliable physical model behind this regime of scattering that was not fully understood, but yet widely used in the literature. In realistic astrophysical environments CR pressure is usually dynamically important and the low-energy CR scattering will most likely be dominated by collective effects. This will be a subject of a future investigation.

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