We propose the following axioms. The formulae for these axioms are given in the appendix.

| Extensionality | Two sets with just the same members are equal. |
|----------------|-----------------------------------------------|
| Pairs          | For every two sets, there is a set that contains just them. |
| Union          | For every set of sets, there is a set with just all their members. |
| Infinity       | There are infinite ordinals $\omega^*$. |
| Replacement    | Replacing the members of a set one-for-one creates a set (i.e., bijective replacement). |
| Regularity     | Every non-empty set has a minimal member (i.e., “weak” regularity). |
| Constructibility | The subsets of $\omega^*$ are countably constructible. |

The first six axioms are the set theory of Zermelo-Frankel (ZF) without the power set axiom and with the axiom schema of subsets (a.k.a., separation) deleted from the axioms of regularity and replacement. Because of the deletion of the axiom schema of subsets, a minimal $\omega^*$, usually denoted by $\omega$ and called the set of all finite natural numbers, cannot be shown to exist in this theory; instead this set theory is uniformly dependent on $\omega^*$ and then all the finite as well as infinitely many infinite natural numbers are included in $\omega^*$. These infinite numbers are one-to-one with $\omega^*$; a finite natural number is any member of $\omega^*$ that is not infinite.

We can now adjoin to this sub-theory of ZF an axiom asserting that the subsets of $\omega^*$ are constructible. By constructible sets we mean sets that are generated sequentially by some process, one after the other, so that the process well-orders the sets. Gödel has shown that an axiom asserting that all sets are constructible can be consistently added to ZF [1], giving a theory usually called ZFC $^+$. It is well known that no more than countably many subsets of $\omega^*$ can be shown to exist in ZFC $^+$. This result will, of course, hold for the sub-theory ZFC $^+$ minus the axiom schema of subsets and the power set axiom. Thus we can now add a new axiom which says that the subsets of $\omega^*$ are countably constructible. This axiom, combined with the axiom schema of bijective replacement, creates a set of constructible subsets of $\omega^*$ and deletion of the power set axiom then assures that no other subsets of $\omega^*$ exist in this theory.

We shall refer to these seven axioms as $T$. All the sets of finite natural numbers in $T$ are finite. The general continuum hypothesis holds in this theory because all sets are countable. We now will also show that this theory is rich enough to contain functions of a real variable effectively governing physical fields.

We first show $T$ has a countable real line. Recall the definition of “rational numbers” as the set of ratios, in ZF called $Q$, of any two members of the set $\omega$. In $T$, we can likewise, using the axiom of unions, establish for $\omega^*$ the set of ratios of any two of its natural numbers, finite or infinite. This will become an “enlargement” of the rational numbers and we shall call this enlargement $Q^*$. Two members of $Q^*$ are called “identical” if their ratio is 1. We now employ the symbol “$\equiv$” for “is identical to.” Furthermore, an “infinitesimal” is a member of $Q^*$ “equal” to 0, i.e., letting $y$ signify the member and employing the symbol “$\equiv$” to signify equality, $y = 0 \leftrightarrow \forall k[y < 1/k]$, where $k$ is a finite natural number. The reciprocal of an infinitesimal is “infinite”. A member of $Q^*$ that is not an infinitesimal and not infinite is “finite”. The constructibility axiom well-orders the set of constructible subsets of $\omega^*$ and gives a metric space. These subsets then represent the binimals forming a countable real line $R^*$. In this theory $R^*$ is a subset of $Q^*$.

An equality-preserving bijective map $\phi(x, u)$ between intervals $X$ and $U$ of $R^*$ in which $x$ is in $X$ and $u \in U$ such that $\forall x_1, x_2, u_1, u_2[\phi(x_1, u_1) \land \phi(x_2, u_2) \rightarrow (x_1 - x_2 = 0 \leftrightarrow u_1 - u_2 = 0)]$ creates pieces which are biunique and homeomorphic. Note that $U = 0$ if and only if $X = 0$, i.e., the piece is inherently relational.

We can now define “functions of a real variable in $T^*$”, $u(x)$ is a function of a real variable in $T$ only if it is a constant or a sequence in $x$ of continuously connected biunique pieces such that the derivative of $u$ with respect to $x$ is also a function of a real variable in $T$. These functions are thus of bounded variation and locally homeomorphic. If some derivative is a constant, they are polynomials. If no derivative
is a constant, these functions do not per se exist in
T but can, however, always be represented as closely
as required for physics by a sum of polynomials of
sufficiently high degree obtained by an iteration of:
\[
\int_a^b p \left( \frac{du}{dx} \right)^2 - qu^2 \right) \, dx = \lambda \int_a^b ru^2 \, dx
\]
where \( \lambda \) is minimized subject to:
\[
\int_a^b ru^2 \, dx \equiv \text{const}
\]
where:
\[
a \neq b, \quad u \left( \frac{du}{dx} \right) = 0
\]
at \( a \) and \( b \); \( p, q, \) and \( r \) are functions of the real variable \( x \).

Letting \( n \) denote the \( n^{th} \) iteration, \( \forall k \exists n[\lambda_{n-1} \leq \lambda_n < 1/k] \) where \( k \) is a finite natural number. So, a
polynomial such that, say, \( 1/k < 10^{-50} \) should be su-
face for physics as it is effectively a Sturm-Liouville
eigenfunction". These can be decomposed, since they are
polynomials, into biunique "irreducible eigenfunc-
tion pieces" obeying the boundary conditions.

As a bridge to physics, let \( x_1 \) be space and \( x_2 \) be
time. We now note that in the theory \( T \) we can obtain
Hamilton’s Principle for a one-dimensional string \( \Psi =
\) just from an identity:
\[
\int \left[ \left( \frac{\partial \Psi}{\partial x_1} \right)^2 - \left( \frac{\partial \Psi}{\partial x_2} \right)^2 \right] \, dx_1 \, dx_2 = 0
\]
\[
\rightarrow \delta \int \left[ \left( \frac{\partial \Psi}{\partial x_1} \right)^2 - \left( \frac{\partial \Psi}{\partial x_2} \right)^2 \right] \, dx_1 \, dx_2 = 0
\]
\[
\rightarrow \delta \int \left[ \left( \frac{\partial \Psi}{\partial x_1} \right)^2 - \left( \frac{\partial \Psi}{\partial x_2} \right)^2 \right] \, dx_1 \, dx_2 = 0
\]

The eigenvalues \( \lambda_{1m} \) are determined by the spatial
boundary conditions. For each eigenstate \( m \), we can
use this identity to iterate the eigenfunctions \( u_{1m} \) and
\( u_{2m} \) constrained by the indicial expression \( \lambda_{1m} \equiv 2 \lambda_{2m} \).

A more general string in finitely many space-like
and time-like dimensions can likewise be produced.
Let \( u_{1mi}(x_i) \) and \( u_{2mj}(x_j) \) be eigenfunctions with non-
egative eigenvalues \( \lambda_{1mi} \) and \( \lambda_{2mj} \) respectively.

We define a "field" as a sum of eigenstates:
\[
\Psi_m = \sum_\ell \Psi_{\ell m} \Psi_{\ell m} = C \prod_i u_{1mi} \prod_j u_{2mj}
\]
with the postulate: for every eigenstate \( m \) in a
compactified space the integral of the Lagrange
density is identically null.

The physics behind this will become clear. Let \( ds \)
represent \( \prod_i r_i \, dx_i \) and \( d\tau \) represent \( \prod_j r_j \, dx_j \). Then
for all \( m \),
\[
\int \sum \frac{1}{r_i} \left[ P_{\ell mi} \left( \frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] \, ds \, d\tau
\]
\[
= \int \sum \frac{1}{r_j} \left[ P_{\ell mj} \left( \frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] \, ds \, d\tau \equiv 0
\]

In this integral expression the \( P, Q, \) and \( R \) can be
functions of any of the \( x_i \) and \( x_j \), thus of any \( \Psi_{\ell m} \) as
well.

This is a nonlinear sigma model. As we have
seen in the case of a one-dimensional string, these \( \Psi_m \)
can in principle be obtained by iterations constrained
by an indicial expression, \( \sum_i \lambda_{\ell mi} \equiv \sum_j \lambda_{\ell mj} \) for all
\( m \).

A proof in \( T \) that the integrals in the nonlinear
sigma model have only discrete values will now be
shown. Let expressions (6) and (7) both be repre-
sented by \( \alpha \), since they are identical:
\[
\sum_{\ell m} \int \frac{1}{r_i} \left[ P_{\ell mi} \left( \frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] \, ds \, d\tau
\]
\[
\sum_{\ell m} \int \frac{1}{r_j} \left[ P_{\ell mj} \left( \frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] \, ds \, d\tau
\]

I. \( \alpha \) is positive and must be closed to addition and
to the absolute value of subtraction; in the theory
\( T \), \( \alpha \) is a natural number times a unit that is
either infinitesimal or finite.

II. If the field is not present, \( \alpha \equiv 0 \); otherwise, if
the field is present, then in \( T \), \( \alpha \) must be finite
so that \( \alpha \neq 0 \); thus \( \alpha = 0 \leftrightarrow \alpha \equiv 0 \).

III. \( \cdot \alpha \equiv n \kappa \), where \( n \) is a natural number and \( \kappa \) is
some finite positive constant such that \( \alpha = 0 \leftrightarrow
n = 0 \).

If \( \pm \kappa \) is named “action,” the physical meaning of
the postulate is an assertion of symmetry of action.

With this result and without any additional physical
postulates, we can now obtain quantum mechanics
from this nonlinear sigma model in one time-like
dimension and finitely many space-like dimensions.

Let \( \ell = 1,2, r_1 = P_{1mt} = P_{2mt} = 1, Q_{1mt} = Q_{2mt} = 0, \tau = \omega_m \ell \) and we normalize \( \Psi \) as follows:
\[
\Psi_m = \sqrt{C/2\pi} \prod_i u_{1im}(x_i)[u_{1m}(\tau) + i \cdot u_{2m}(\tau)]
\]

where \( i = \sqrt{-1} \) with
\[
\int \sum \prod_i u_{1i}^2 \, ds(u_{1i}^2 + u_{2i}^2) \equiv 1
\]
then:
\[ \frac{du_{1m}}{d\tau} = -u_{2m} \quad \text{and} \quad \frac{du_{2m}}{d\tau} = u_{1m} \]
\[ (10) \]
or
\[ \frac{du_{1m}}{d\tau} = u_{2m} \quad \text{and} \quad \frac{du_{2m}}{d\tau} = -u_{1m} \]
\[ (11) \]

For the minimal non-vanishing field, the action integral \( \alpha \) over each irreducible eigenfunction piece of \( u_{1m}(\tau) \) and \( u_{2m}(\tau) \) is \( \kappa \); over a cycle it is \( 4\kappa \). Thus,
\[ \left( \frac{C}{2\pi} \right) \sum_{m} \int \left[ \left( \frac{du_{1m}}{d\tau} \right)^{2} + \left( \frac{du_{2m}}{d\tau} \right)^{2} \right] \]
\[ \prod_{i} a_{i}^{2}(x_{i})d\sigma d\tau \equiv C \equiv 4\kappa \]
\[ (12) \]

Substituting the Planck \( h \) for \( 4\kappa \), the action integral \( \alpha \) can now be put into the conventional Lagrangian form for the Schrödinger equation,
\[ \frac{1}{2i} \sum_{m} \int \left[ \Psi^{*}_{m} \left( \frac{\partial \Psi_{m}}{\partial t} \right) \right] - \left( \frac{\partial \Psi_{m}}{\partial t} \right) \Psi_{m} \] dsdt \[ (13) \]

Therefore, the Schrödinger equation is a special case of a nonlinear sigma model in one time dimension and finitely-many spatial dimensions.

In the process of this discussion, we have also shown that:

- Quantum mechanics is obtained without requiring an additional assumption of the statistical interpretation of the wave function, thereby resolving a long-standing controversy \[3\].
- Quantum mechanics is instead derived in a constructible theory using an action symmetry postulate.
- There are inherently no singularities in the physical fields obtained in this theory.

In addition, though we do not have the opportunity to discuss these points, we note that:

- Space-time is here emergent, relational and its differential properties fulfill the strict requirements of Einstein-Weyl causality \[4\], suggesting a possible foundation for quantum gravity.
- The solution to the QED divergence problem posed by Dyson \[5\] is provided, since the actual convergence or divergence of the essential perturbation series is undecidable in this theory.
- Wigner’s metaphysical question regarding the apparent unreasonable effectiveness of mathematics in physics \[6\] is directly answered, since the foundations of mathematics and physics are now linked.

Appendix: ZF - Axiom Schema of Subsets - Power Set + Constructibility

Extensionality. Two sets with just the same members are equal. \( \forall x \forall y \left( \forall z \left[ z \in x \iff z \in y \right] \rightarrow x = y \right) \)

Pairs. For every two sets, there is a set that contains just them. \( \forall x \forall y \exists z \left[ \forall w \in z \iff w = x \lor w = y \right] \)

Union. For every set of sets, there is a set with just all their members. \( \forall x \forall y \exists z \left[ \forall u \in z \iff \exists u \left[ u \in x \cup u \in x \right] \right] \)

Infinity. There are infinite ordinals \( \omega^{*} \) (i.e., sets are transitive and well-ordered by \( \in \)-relation). \( \exists \omega^{*} \left[ O \in \omega^{*} \land \forall x [x \in \omega^{*} \rightarrow x \cup \{ x \} \in \omega^{*}] \right] \)

Replacement. Replacing members of a set one-for-one creates a set (i.e., “bijective” replacement). Let \( \phi(x,y) \) a formula in which \( x \) and \( y \) are free, \( \forall x \exists y \left[ z \in y \left[ \phi(x,y) \land \forall u \exists v \left[ \phi(u,v) \rightarrow u = x \iff y = v \right] \rightarrow \exists r \forall t \left[ t \in r \iff \exists s \in z \phi(s,t) \right] \right] \right) \)

Regularity. Every non-empty set has a minimal member (i.e., “weak” regularity). \( \forall x \left[ \exists y \in x \rightarrow \exists y \left[ y \in x \land \forall z \left[ \neg(z \in x \land z \in y) \right] \right] \right) \)

Constructibility. The subsets of \( \omega^{*} \) can be countably constructible. \( \forall \omega^{*} \exists S[\left[ (\omega^{*}, O) \in S \land \forall m \exists (y,z) \in S \rightarrow \left[ \exists m_y \left[ m_y \in y \land \forall \forall u \left[ u \in t_y \iff u \in y \land u \not= m_y \right] \right] \rightarrow (t_y \cup m_y, z \cup \{ z \}) \in S \right] \right] \right) \)

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