ON THE ASYMPTOTIC BEHAVIOR OF THE COLORED JONES POLYNOMIAL OF THE FIGURE-EIGHT KNOT ASSOCIATED WITH A REAL NUMBER

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Dedicated to the memory of Toshie Takata

Abstract. We study the asymptotic behavior of the \( N \)-dimensional colored Jones polynomial evaluated at \( \exp(\xi/N) \) for a real number \( \xi \) greater than a certain constant. We prove that, from the asymptotic behavior, we can extract the \( \text{SL}(2; \mathbb{C}) \) Chern–Simons invariant and the Reidemeister torsion twisted by the adjoint action both associated with a representation determined by \( \xi \).

1. Introduction

Let \( N \geq 2 \) be an integer.

In [10], R. Kashaev introduced a link invariant \( \langle K \rangle_N \) for a knot \( K \) in the three-sphere \( S^3 \) by using the so-called quantum dilogarithm. In [11], he studied its asymptotic behavior when \( N \to \infty \) for several hyperbolic knots and conjectured that it would determine its hyperbolic volume for any hyperbolic knot, where a knot is called hyperbolic if its complement possesses a complete hyperbolic structure with finite volume. More precisely he conjectured

**Conjecture 1.1** (Kashaev’s conjecture). Let \( H \) be a hyperbolic knot in \( S^3 \). Then the following equality holds.

\[
\lim_{N \to \infty} \frac{\log |\langle H \rangle_N|}{N} = \frac{\text{Vol}(S^3 \setminus H)}{2\pi},
\]

where \( \text{Vol}(S^3 \setminus H) \) is the hyperbolic volume.

Let \( J_N(K; q) \) be the colored Jones polynomial associated with the \( N \)-dimensional irreducible representation of the Lie algebra \( \text{sl}(2; \mathbb{C}) \) [12, 32]. We normalize it so that \( J_N(U; q) = 1 \) for the unknot \( U \), and that \( J_2(K; q) \) is the ordinary Jones polynomial [9]. In [23], J. Murakami and the first author proved that Kashaev’s invariant \( \langle K \rangle_N \) coincides with the \( N \)-dimensional colored Jones polynomial \( J_N(K; q) \) evaluated at \( q = e^{2\pi \sqrt{-1}/N} \). They also generalized Kashaev’s conjecture for general knots:

**Conjecture 1.2** (Volume Conjecture). For any knot \( K \subset S^3 \), we have

\[
\lim_{N \to \infty} \frac{\log |J_N(K; e^{2\pi \sqrt{-1}/N})|}{N} = \frac{\text{Vol}(S^3 \setminus K)}{2\pi},
\]

Here, \( \text{Vol}(S^3 \setminus K) \) is the simplicial volume, also known as the Gromov norm [6] (see also [33]), which is normalized so that \( \text{Vol}(S^3 \setminus H) = \text{Vol}(S^3 \setminus H) \) if \( H \) is hyperbolic.
Kashaev’s conjecture was complexified by T. Takata, J. Murakami, M. Okamoto, Y. Yokota and the first author in [24] by dropping the absolute value symbol (and replacing \((H_N)\) with \(J_N\left(H; e^{2\pi \sqrt{-1}/N}\right)\)) in (1.1).

**Conjecture 1.3** (Complexification of Kashaev’s conjecture). For a hyperbolic knot \(H\), we have

\[
\lim_{N \to \infty} \frac{\log J_N\left(H; e^{2\pi \sqrt{-1}/N}\right)}{N} = \frac{V(S^3 \setminus H) + \sqrt{-1} \text{CS}(S^3 \setminus H)}{2\pi},
\]

where \(\text{CS}\) is the \(\text{SO}(3)\) Chern–Simons invariant of \(H\) associated with the Levi-Civita connection \([16]\).

More detailed asymptotic formulas are known for several knots \([1, 28, 30]\):

\[
\langle H \rangle_N \sim C \frac{\omega(H)N^{3/2}}{\sinh(\xi)} \times \tau(\xi)^{1/2} \left\{\frac{N}{\xi}\right\}^d \exp \left(\frac{N}{\xi} S(\xi)\right),
\]

where \(\text{CV}(H) := \text{Vol}(S^3 \setminus H) + \sqrt{-1} \text{CS}(S^3 \setminus H)\) is the complex volume and \(2\sqrt{-1} \omega(H)^2\) is the adjoint (cohomological) Reidemeister torsion twisted by the holonomy representation of \(\pi_1(S^3 \setminus H)\) to \(\text{SL}(2; \mathbb{C})\) \([29]\).

It was also generalized by replacing \(2\pi \sqrt{-1}\) with another complex number \(\xi\) \([2, 21]\).

**Conjecture 1.4** (Generalized Volume Conjecture). Let \(\xi \neq 2\pi \sqrt{-1}\) be a complex number close to \(2\pi \sqrt{-1}\), and \(H\) a hyperbolic knot. Then we have

\[
J_N\left(H; e^{\xi/N}\right) N \sim C \frac{\omega(H)N^{3/2}}{\sinh(\xi)} \times \tau(\xi)^{1/2} \left\{\frac{N}{\xi}\right\}^d \exp \left(\frac{N}{\xi} S(\xi)\right)
\]

with \(d\) a constant. Here \(S(\xi)\) and \(\tau(\xi)\) are related to the \(\text{SL}(2; \mathbb{C})\) Chern–Simons invariant and the adjoint Reidemeister torsion, respectively, both associated with a certain representation of \(\pi_1(S^3 \setminus H)\) to \(\text{SL}(2; \mathbb{C})\).

In this paper, we are mainly interested in the figure-eight knot. We first list known results for the asymptotic behavior of the colored Jones polynomial of the figure-eight knot \(E\).

We define a function \(\varphi\) as

\[
\varphi(\xi) := \arccosh\left(\cosh(\xi) - \frac{1}{2}\right)
\]

for a complex number \(\xi\), where we use the following branch of \(\arccosh\):

\[
\arccosh(x) := \log \left(x - \sqrt{1 - x^2}\right),
\]

and we choose the branch cut of \(\log\) as \((-\infty, 0)\). Note that \(\varphi(0) = -\pi \sqrt{-1}/3\) and \(\varphi(\kappa) = 0\), where \(\kappa := \log \left(\frac{3\pi \sqrt{3}}{4}\right)\). We also define

\[
S(\xi) := \text{Li}_2(e^{-\xi - \varphi(\xi)}) - \text{Li}_2(e^{-\xi + \varphi(\xi)}) + \xi \varphi(\xi),
\]

\[
\tilde{S}(\xi) := \text{Li}_2(e^{-\xi - \varphi(\xi)}) - \text{Li}_2(e^{-\xi + \varphi(\xi)}) + (\xi - 2\pi \sqrt{-1})(\varphi(\xi) + 2\pi \sqrt{-1}),
\]

\[
T(\xi) := \frac{2}{\sqrt{(2\cosh(\xi) + 1)(2\cosh(\xi) - 3)}},
\]

where

\[
\text{Li}_2(z) := -\int_0^z \frac{\log(1 - x)}{x} dx
\]

is the dilogarithm function, where we choose the branch cut as \((1, \infty)\). Note that

\[
S(0) = \tilde{S}(2\pi \sqrt{-1}) = \text{Li}_2(e^{\pi \sqrt{-1}/3}) - \text{Li}_2(e^{-\pi \sqrt{-1}/3}) = \sqrt{-1} \times 2.02988.
\]
The quantities $S(\xi)$, $\tilde{S}(\xi)$, and $T(\xi)$ are related to the Chern–Simons invariant and the adjoint Reidemeister torsion associated with a certain representation of $\pi_1(S^3 \setminus E)$ to $SL(2; \mathbb{C})$. See Section 5 for details.

(1) $\xi = 2\pi \sqrt{-1}$:
This corresponds to the case of the Volume Conjecture. Kashaev sketched a proof of (1.2) in [11], and then T. Ekholm gave a detailed proof of (1.2). See for example [27, Section 3.2] for Ekholm’s proof. Later, J.E. Andersen and S.K. Hansen [1, Theorem 1] followed Kashaev’s method to prove the following asymptotic formula.

$$J_N \left( E; e^{2\pi \sqrt{-1}/N} \right) \sim \frac{2\pi^{3/2}}{N} \left( \frac{N}{2\pi \sqrt{-1}} \right)^{3/2} T(2\pi \sqrt{-1})^{1/2} \exp \left( \frac{N}{2\pi \sqrt{-1}} \tilde{S}(2\pi \sqrt{-1}) \right).$$

Since $S(0) = \sqrt{-1} \text{Vol} (S^3 \setminus E)$, this refines the Volume Conjecture for the figure-eight knot.

(2) $\xi$ is close to $2\pi \sqrt{-1}$ and not purely imaginary:
Y. Yokota and the first author proved the following formula [26].

$$\lim_{N \to \infty} \frac{\log J_N(E; e^{\xi/N})}{N} = \frac{\tilde{S}(\xi)}{\xi}.$$  

(3) $\xi$ is purely imaginary with $5\pi/3 < |\xi| < 7\pi/3$, and $2\pi/|\xi|$ is irrational with finite irrationality measure:
In [17] (see [22] for correction), the first author proved the following formula.

$$\lim_{N \to \infty} \frac{\log J_N(E; e^{\xi/N})}{N} = \frac{\tilde{S}(\xi)}{\xi}.$$  

See also [19, 6.2.2].

(4) $\xi$ is of the form $2\pi \sqrt{-1} + u$ for a real number $u$ with $0 < |u| < \kappa$:
The first author proved the following asymptotic formula [21, Theorem 1.4]:

$$J_N \left( E; e^{\xi/N} \right) \sim \frac{\sqrt{-1}}{2 \sinh(u/2)} T(\xi)^{1/2} \left( \frac{N}{\xi} \right)^{1/2} \exp \left( \frac{N}{\xi} \tilde{S}(\xi) \right).$$

Note that this refines (1.5) when $\xi$ is as above.

(5) $\xi$ satisfies the inequalities $|2 \cosh \xi - 2| < 1$ and $|\text{Im} \xi| < \pi/3$:
In this case, $J_N(E; e^{\xi/N})$ converges. In fact, we have

$$\lim_{N \to \infty} J_N \left( E; e^{\xi/N} \right) = \frac{1}{\Delta(E; e^{\xi})},$$

where $\Delta(E; t) = -t + 3 - t^{-1}$ is the normalized Alexander polynomial of the figure-eight knot $E$ [18, Theorem 1.1].

See [5] for general knots.

(6) $\xi = \pm \kappa$:
In this case, $J_N(E; e^{\xi/N})$ grows polynomially. More precisely, we have

$$J_N \left( E; e^{\xi/N} \right) \sim \frac{\Gamma(1/3)}{(3\kappa)^{2/3}} N^{2/3},$$

where $\Gamma(z)$ is the Gamma function [8, Theorem 1.1].

(7) $\xi$ is real and $|\xi| > \kappa$:
The first author proved the following formula [17, Theorem 8.1] (see also [20, Theorem 3.2] and [19, Lemma 6.7]).

$$\lim_{N \to \infty} \frac{\log J_N(E; e^{\xi/N})}{N} = \frac{S(\xi)}{|\xi|}.$$
The purpose of this paper is to refine (1.7). We will show

**Theorem 1.5.** If $\xi$ is real and $|\xi| > \kappa$, then we have

$$J_N \left( E; e^{\xi/N} \right) \sim \frac{\sqrt{\pi}}{2 \sinh(|\xi|/2)} T(\xi)^{1/2} \left( \frac{N}{|\xi|} \right)^{1/2} \exp \left( \frac{N}{|\xi|} S(\xi) \right).$$

Moreover, we can show that $T(\xi)$ is the Reidemeister torsion twisted by the adjoint action of a representation $\rho$ of $\pi_1(S^3 \setminus E)$ to $\text{SL}(2; \mathbb{C})$ determined by $\xi$, and that $S(\xi) - \xi \eta/2$ is the Chern–Simons invariant of $\rho$ associated with the meridian and the preferred longitude of $E \subset S^3$. See Section 5 for details.

**Remark 1.6.** Suppose that $u := \xi - 2\pi \sqrt{-1}$ is a small complex number or a real number with $|u| < \kappa$. In this case $\text{Im} \varphi(u) < 0$. By using a well-known formula

$$\text{Li}_2(z) + \text{Li}_2(z^{-1}) + \frac{\pi^2}{6} + \frac{1}{2} (\log(-z))^2 = 0,$$

we have

$$\tilde{S}(\xi) = - \text{Li}_2(e^{\xi + \varphi(\xi)}) + \text{Li}_2(e^{\xi - \varphi(\xi)}) - \frac{1}{2} \left( \log(-e^{\xi + \varphi(\xi)}) \right)^2 + \frac{1}{2} \left( \log(-e^{\xi - \varphi(\xi)}) \right)^2 + u(\varphi(\xi) + 2\pi \sqrt{-1})$$

$$= - \text{Li}_2(e^{u + \varphi(u)}) + \text{Li}_2(e^{u - \varphi(u)}) - \frac{1}{2} (u + \varphi(u) + \pi \sqrt{-1})^2 + \frac{1}{2} (u - \varphi(u) - \pi \sqrt{-1})^2 - (2\pi \sqrt{-1} + u)\varphi(u)$$

$$= - \text{Li}_2(e^{u + \varphi(u)}) + \text{Li}_2(e^{u - \varphi(u)}) - u\varphi(u).$$

In the second equality, we use the fact that $\text{Im} \varphi(u) < 0$. Therefore (1.6) coincides with the formula appearing in [21, Theorem 1.4].

**Remark 1.7.** In [21], the first author followed [1] to obtain the asymptotic formula, but in the current paper we follow [28].

### 2. Preliminaries

In this section, we first introduce the colored Jones polynomial, and then we define a quantum dilogarithm, and variants of the logarithm and the dilogarithm. We also describe some of their properties.

For a knot $K$ in the three-sphere $S^3$, we denote by $J_N(K; q)$ the colored Jones polynomial of $K$ associated with the $N$-dimensional irreducible representation of the Lie algebra $\mathfrak{sl}(2; \mathbb{C})$ [12, 32]. We normalize it so that $J_N(U; q) = 1$ for the unknot $U \subset S^3$. Note that $J_2(K; q)$ is the original Jones polynomial [9].

Let $E$ be the figure-eight knot. We use the following formula obtained by K. Habiro [7, P. 36 (1)] and T.T.Q. Le [14, P. 129]. See also [15].

$$J_N(E; q) = \sum_{k=0}^{N-1} \prod_{l=1}^{k} \left( \frac{q^{(N-l)/2} - q^{-(N-l)/2}}{q^{(N+l)/2} - q^{(N-l)/2}} \right) \left( \frac{q^{(N+l)/2} - q^{(N+l)/2}}{1 - q^{N-l}} \right) \left( 1 - q^{N+l} \right).$$

(2.1)

Next, we define functions $T_N(z)$, $L_0(z)$, $L_1(z)$, and $L_2(z)$, which are related as

$$T_N(z) = \frac{N}{\xi} L_2(z) + O(1/N) \quad (N \to \infty),$$
\[
\frac{d L_2}{dz}(z) = -2\pi\sqrt{-1}L_1(z), \\
\frac{d L_1}{dz}(z) = -L_0(z).
\]

Let \(\xi\) be a positive real number. We define \(C_0 := (-\infty, -1] \cup \{e^{\tau\sqrt{-1}} \mid 0 \leq \tau \leq \pi\} \cup [1, \infty)\) and \(C_\theta := e^{\tau\sqrt{-1}}C_0\), where \(\theta\) is a positive real number with \(\tan \theta < \frac{\pi}{2}\) and we orient \(C_0\) from left to right.

Consider the following integrals:
\[
\int_{C_\theta} e^{(2z-1)x} dx, \quad \int_{C_\theta} x^{m} \sinh(x) dx,
\]
where \(m = 0, 1, 2\), \(\gamma := \frac{\xi}{2N\pi\sqrt{-1}}\) for an integer \(N \geq 2\), and \(z\) is a complex number with \(0 < \Re(ze^{\sqrt{-1}\theta}) < \cos \theta\). Here we follow [4] to introduce \(T_N(z)\), which plays an important role in the paper. Note that the set of the poles of the integrands are
\[
\{k\pi\sqrt{-1} \mid k \in \mathbb{Z}\} \cup \{2N\pi^2/\xi \mid l \in \mathbb{Z}\} \quad \text{and} \quad \{k\pi\sqrt{-1} \mid k \in \mathbb{Z}\},
\]
respectively. Therefore, if \(N\) is large enough, then the only pole inside the unit circle centered at the origin is \(0 \in \mathbb{C}\) and so the path of integral \(C_\theta\) avoids the poles. Proofs of their convergences (Lemmas 2.1 and 2.2 below) are given in Section 6.

**Lemma 2.1.** The integral \(\int_{C_\theta} e^{(2z-1)x} x \sinh(x) dx\) converges if \(z \in \mathbb{C}\) satisfies \(-\frac{\xi \sin \theta}{4N\pi} < \Re(ze^{\theta\sqrt{-1}}) < \cos \theta + \frac{\xi \sin \theta}{4N\pi}\).

**Lemma 2.2.** If \(z \in \mathbb{C}\) satisfies \(0 < \Re(ze^{\theta\sqrt{-1}}) < \cos \theta\), then the integral \(\int_{C_\theta} e^{(2z-1)x} x^{m} \sinh(x) dx\) converges for \(m = 0, 1, 2\).

Now define functions \(T_N(z)\) and \(L_m(z)\) \((m = 0, 1, 2)\) by using the integrals above.

**Definition 2.3.** Fix an integer \(N \geq 2\) and put \(\gamma := \frac{\xi}{2N\pi\sqrt{-1}}\). We define
\[
T_N(z) := \frac{1}{4} \int_{C_\theta} e^{(2z-1)x} x \sinh(x) dx,
\]
for a complex number \(z\) with \(-\frac{\xi \sin \theta}{4N\pi} < \Re(ze^{\theta\sqrt{-1}}) < \cos \theta + \frac{\xi \sin \theta}{4N\pi}\).

**Definition 2.4.** For a complex number \(z\) with \(0 < \Re(ze^{\sqrt{-1}\theta}) < \cos \theta\), we define
\[
L_0(z) := \int_{C_\theta} e^{(2z-1)x} \sinh(x) dx, \quad L_1(z) := -\frac{1}{2} \int_{C_\theta} e^{(2z-1)x} x \sinh(x) dx, \quad L_2(z) := \frac{\pi\sqrt{-1}}{2} \int_{C_\theta} e^{(2z-1)x} x^{2} \sinh(x) dx.
\]

**Lemma 2.5.** For \(m = 0, 1, 2\), we calculate \(L_m(z)\) as follows.

\[
L_0(z) = \frac{-2\pi\sqrt{-1}}{1 - e^{-2\pi\sqrt{-1}z}},
\]
\[
L_1(z) = \begin{cases} 
\log (1 - e^{2\pi\sqrt{-1}z}) & \text{if } \Im z \geq 0, \\
\pi\sqrt{-1}(2z - 1) + \log (1 - e^{-2\pi\sqrt{-1}z}) & \text{if } \Im z < 0,
\end{cases}
\]
(2.4) \[
    \mathcal{L}_2(z) = \begin{cases} 
    \text{Li}_2 \left( e^{2\pi \sqrt{-1}z} \right) & \text{if } \Im z \geq 0, \\
    \pi^2 \left( 2z^2 - 2z + \frac{1}{4} \right) - \text{Li}_2 \left( e^{-2\pi \sqrt{-1}z} \right) & \text{if } \Im z < 0.
    \end{cases}
\]

Proofs are also given in Section 6.

Remark 2.6. In (2.3) and (2.4), we use \( \log(1-x) \) and \( \text{Li}_2(x) \) only for \(|x| \leq 1 \) \((x \neq 1)\).

Since
\[
    T_N(z - \gamma/2) - T_N(z + \gamma/2) = \int_{C_\theta} \frac{e^{(2z-\gamma-1)x} - e^{(2z+\gamma-1)x}}{4x \sinh(x) \sinh(\gamma x)} \, dx
\]
\[
    = - \int_{C_\theta} \frac{e^{(2z-1)x}}{2x \sinh(x)} \, dx
\]
\[
    = \mathcal{L}_1(z),
\]
we have the following corollary.

Corollary 2.7. If \( 0 < \Re (ze^{\theta \sqrt{-1}}) < \cos \theta \), then we have
\[
    \frac{\exp(T_N(z - \gamma/2))}{\exp(T_N(z + \gamma/2))} = 1 - e^{2\pi \sqrt{-1}z}.
\]

We will show relations among the functions \( T_N(z) \) and \( \mathcal{L}_m(z) \) \((m = 0, 1, 2)\). We can prove that \( \frac{1}{N}T_N(z) \) uniformly converges to \( \frac{1}{x} \mathcal{L}_2(z) \). More precisely, we have the following proposition. See [28, Proposition A.1].

Proposition 2.8. For any positive real number \( M \) and a sufficiently small positive real number \( \nu \), we have
\[
    T_N(z) = \frac{N}{\xi} \mathcal{L}_2(z) + O(1/N) \quad (N \to \infty)
\]
in the region
\[
    \{ z \in \mathbb{C} \mid \nu \leq \Re (ze^{\theta \sqrt{-1}}) \leq \cos \theta - \nu, \, |\Im z| \leq M \}.
\]
In particular, the function \( \frac{1}{N}T_N(z) \) uniformly converges to \( \frac{1}{x} \mathcal{L}_2(z) \) in the region above.

A proof is given in Section 6. Since we may regard \( \mathcal{L}_2(z) \) as a variant of the dilogarithm from Lemma 2.5, the function \( T_N(z) \) is another quantum dilogarithm.

The functions \( \mathcal{L}_0(z) \), \( \mathcal{L}_1(z) \), and \( \mathcal{L}_2(z) \) are related as follows.

Lemma 2.9. The derivatives of \( \mathcal{L}_1(z) \) and \( \mathcal{L}_2(z) \) are given as follows:
\[
    \frac{d \mathcal{L}_2}{dz}(z) = -2\pi \sqrt{-1} \mathcal{L}_1(z),
\]
\[
    \frac{d \mathcal{L}_1}{dz}(z) = -\mathcal{L}_0(z).
\]

Proof. We have
\[
    \frac{d \mathcal{L}_2}{dz}(z) = \frac{\pi \sqrt{-1}}{2} \int_{C_\theta} \left( \frac{d}{dz} \frac{e^{(2z-1)x}}{x^2 \sinh(x)} \right) \, dx
\]
\[
    = \pi \sqrt{-1} \int_{C_\theta} \frac{e^{(2z-1)x}}{x \sinh(x)} \, dx
\]
\[
    = -2\pi \sqrt{-1} \mathcal{L}_1(z),
\]
and
\[
\frac{d \mathcal{L}_1}{dz}(z) = -\frac{1}{2} \int_{C_n} \left( \frac{d}{dz} e^{(2z-1)x} \right) dx
\]
\[
= -\int_{C_n} e^{(2z-1)x} \sinh(x) dx
\]
\[
= -\mathcal{L}_0(z),
\]
completing the proof. \(\square\)

3. Summation

In this section, we use the quantum dilogarithm \(T_N(z)\) to express \(J_N(E; \xi^{1/N})\) without the product of a sequence.

Since the figure-eight knot \(E\) is amphicheiral, its colored Jones polynomial is symmetric, that is, \(J_N(E; q) = J_N(E; q^{-1})\). So we have \(J_N(E; \xi^{1/N}) = J_N(E; \xi^{-1/N})\) and we do not need to consider the case where \(\xi < 0\).

Recall that we choose a positive real number \(\theta\) so that \(\tan \theta < \frac{\pi}{2}\) and that we put \(\gamma := \frac{\xi}{2\pi \sqrt{-1}}\). Putting \(z := \frac{\xi}{2\pi \sqrt{-1}}(1 - l/N)\) \((0 < l < N)\) in (2.5), we have
\[
\exp \left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}}(1 - l/N) - \frac{\xi}{4N\pi \sqrt{-1}} \right) \right) = 1 - e^{\xi(1-l/N)}
\]
since
\[
\Re \left( \frac{\xi}{2\pi \sqrt{-1}} \left( 1 - \frac{l}{N} \right) e^{\theta \sqrt{-1}} \right) = \frac{\xi}{2\pi} \left( 1 - \frac{l}{N} \right) \sin \theta < \frac{1}{2} \left( 1 - \frac{l}{N} \right) \cos \theta,
\]
which is between 0 and \(\cos \theta\). Therefore we have
\[
\prod_{l=1}^{k} (1 - e^{(N-l)\xi/N}) = \prod_{l=1}^{k} \exp \left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}}(1 - \frac{2l+1}{2N}) \right) \right)
\]
\[
= \exp \left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}}(1 - \frac{1}{2N}) \right) \right).
\]

Similarly, putting \(z := \frac{\xi}{2\pi \sqrt{-1}}(1 + l/N)\) \((0 < l < N)\) in (2.5), we have
\[
\exp \left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}}(1 + l/N) - \frac{\xi}{4N\pi \sqrt{-1}} \right) \right) = 1 - e^{\xi(1+l/N)}
\]
since
\[
\Re \left( \frac{\xi}{2\pi \sqrt{-1}} \left( 1 + \frac{l}{N} \right) e^{\theta \sqrt{-1}} \right) = \frac{\xi}{2\pi} \left( 1 + \frac{l}{N} \right) \sin \theta < \frac{1}{2} \left( 1 + \frac{l}{N} \right) \cos \theta,
\]
which is between 0 and \(\cos \theta\). So we have
\[
\prod_{l=1}^{k} (1 - e^{(N+l)\xi/N}) = \prod_{l=1}^{k} \exp \left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}}(1 + \frac{2l+1}{2N}) \right) \right)
\]
\[
= \exp \left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}}(1 + \frac{1}{2N}) \right) \right).\]
Therefore we have from (2.1)
\[ J_N(E; \exp(\xi/N)) = \frac{\exp\left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}} (1 + \frac{1}{2N}) \right) \right)}{\exp\left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}} (1 - \frac{1}{2N}) \right) \right)} \times \sum_{k=0}^{N-1} e^{-k\xi} \frac{\exp\left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}} (1 - \frac{2k+1}{2N}) \right) \right)}{\exp\left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}} (1 + \frac{2k+1}{2N}) \right) \right)} \]
We use the following lemma, a proof of which is given in Section 6.

**Lemma 3.1.** We have
\[ \frac{\exp\left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}} (1 + \frac{1}{2N}) \right) \right)}{\exp\left( T_N \left( \frac{\xi}{2\pi \sqrt{-1}} (1 - \frac{1}{2N}) \right) \right)} = \frac{1}{1 - e^{\xi}} = \frac{1}{2 \sinh(\xi/2)}. \]

Now we define
\[ f_N(z) := \frac{1}{N} \left( T_N \left( \frac{\xi (1 - z)}{2\pi \sqrt{-1}} \right) - T_N \left( \frac{\xi (1 + z)}{2\pi \sqrt{-1}} \right) \right) - \xi z + 2\pi \sqrt{-1}z \]
so that
\[ J_N(E; \exp(\xi/N)) = \frac{1}{2 \sinh(\xi/2)} \sum_{k=0}^{N-1} \exp\left( N \times f_N \left( \frac{2k + 1}{2N} \right) \right). \]

Since \( T_N(z) \) is defined for \( z \) with \( 0 < \Re(ze^{\theta \sqrt{-1}}) < \cos \theta \), the function \( f_N(z) \) is defined for \( z \) with
\[ \left| \frac{\Im z}{\tan \theta} + \Re z \right| < 1 \]
from the assumption \( \tan \theta < \frac{\pi}{\xi} \).

From Proposition 2.8, \( f_N(z) \) converges to the following function:
\[ F(z) := \frac{1}{\xi} \left( \mathcal{L}_2 \left( \frac{\xi (1 - z)}{2\pi \sqrt{-1}} \right) - \mathcal{L}_2 \left( \frac{\xi (1 + z)}{2\pi \sqrt{-1}} \right) \right) - \xi z + 2\pi \sqrt{-1}z \]
in the region
\[ \left\{ z \in \mathbb{C} \mid \left| \frac{\Im z}{\tan \theta} + \Re z \right| \leq 1 - \frac{2\pi \nu}{\xi \sin \theta} \left| \Re z \right| \leq \frac{2\pi M}{\xi} - 1 \right\}. \]

Since \( \left( \frac{\xi (1 + z)}{2\pi \sqrt{-1}} \right) = \frac{\pi}{\xi} (1 \pm \Re z) \),
\[ F(z) = \frac{1}{\xi} \left( \mathcal{L}_2 \left( e^{-\xi (1+z)} \right) - \mathcal{L}_2 \left( e^{-\xi (1-z)} \right) \right) + \xi z \]
when \( -1 < \Re z < 1 \) from (2.4).

**Remark 3.2.** When \( x \) is real and \( -1 < x < 1 \), then \( F(x) \) is also real.

From Lemma 2.9, the first and the second derivatives of \( F(z) \) are given as follows:
\[ F'(z) = \mathcal{L}_1 \left( \frac{\xi (1 - z)}{2\pi \sqrt{-1}} \right) + \mathcal{L}_1 \left( \frac{\xi (1 + z)}{2\pi \sqrt{-1}} \right) - \xi + 2\pi \sqrt{-1}, \]
\[ F''(z) = \frac{\xi}{2\pi \sqrt{-1}} \left( \mathcal{L}_0 \left( \frac{\xi (1 - z)}{2\pi \sqrt{-1}} \right) - \mathcal{L}_0 \left( \frac{\xi (1 + z)}{2\pi \sqrt{-1}} \right) \right) \]
\[ = \frac{\xi (e^{-\xi z} - e^{\xi z})}{e^{\xi} + e^{-\xi} - e^{\xi z} - e^{-\xi z}}. \]
Note that when $-1 < \Re z < 1$, we have

\[
F'(z) = \log(1 - e^{-\xi(1-z)}) + \log(1 - e^{-\xi(1+z)}) + \xi \\
= \log(e^\xi + e^{-\xi} - e^{\xi z} - e^{-\xi z})
\]

(3.2)

from (2.3), since $\Im \left( \frac{\xi(1+z)}{2\pi} \right) = -\frac{\xi}{2\pi}(1 \pm \Re z)$.

We assume that $\xi > \kappa$, where $\kappa := \text{arccosh}(3/2) = \log(\frac{3}{\sqrt{2}}) = 0.962 \ldots$. Put $\varphi(\xi) := \text{arccosh} \left( \cosh(\xi) - \frac{1}{2} \right) = \log \left( \cosh(\xi) - \frac{1}{2} + \frac{1}{2} \sqrt{2\cosh(\xi) - 1 \pm 4} \right)$.

Note that since $\xi > \kappa$, we have $\cosh(\xi) > \cosh(\varphi(\xi))$ and $\cosh(\kappa) = \frac{3}{\sqrt{2}}$. So we have $0 < \varphi(\xi) < \xi$. Note also that this definition is the same as that in Section 1.

Since $0 < \varphi(\xi < \xi$, we have

\[
F(\varphi(\xi) / \xi) = \frac{1}{\xi} \left( \text{Li}_2(e^{-\xi - \varphi(\xi)}) - \text{Li}_2(e^{-\xi + \varphi(\xi)}) \right) + \varphi(\xi),
\]

(3.3)

\[
F'(\varphi(\xi) / \xi) = 0,
\]

(3.4)

\[
F''(\varphi(\xi) / \xi) = -\xi \sqrt{2\cosh(\xi) - 1} \pm 4.
\]

(3.5)

The second equality follows since

\[
e^{\varphi(\xi)} + e^{-\varphi(\xi)} = e^\xi + e^{-\xi} - 1.
\]

Put

\[
S(\xi) := \xi F(\varphi(\xi) / \xi) = \text{Li}_2(e^{-\xi - \varphi(\xi)}) - \text{Li}_2(e^{-\xi + \varphi(\xi)}) + \xi \varphi(\xi).
\]

Since

\[
\frac{dS(\xi)}{d\xi} = (1 + \varphi'(\xi)) \log(1 - e^{-\xi - \varphi(\xi)}) - (1 - \varphi'(\xi)) \log(1 - e^{-\xi + \varphi(\xi)}) + \varphi(\xi) + \xi \varphi'(\xi) \\
= \log \frac{e^{\varphi(\xi)} - e^{-\xi}}{1 - e^{-\xi + \varphi(\xi)}} + \varphi(\xi) \log \left( e^\xi + e^{-\xi} - e^{\varphi(\xi)} - e^{-\varphi(\xi)} \right) \\
= \log \frac{e^{\varphi(\xi)} - e^{-\xi}}{1 - e^{-\xi + \varphi(\xi)}} > 0
\]

and $S(\kappa) = 0$, $S(\xi) > 0$ for $\xi > \kappa$.

Recalling that $F(x) \in \mathbb{R}$ when $x$ is real and $-1 < x < 1$, we have the following lemma.

**Lemma 3.3.** As a function of a real variable $x$, the real-valued function $F(x)$ takes its maximum $S(\xi)/\xi$ at $x = \varphi(\xi)/\xi$ in the half open interval $[0, 1)$. Moreover, it is strictly increasing (decreasing, respectively) when $x < \varphi(\xi)/\xi$ ($x > \varphi(\xi)/\xi$, respectively).

**Proof.** Since we have

\[
F'(x) = \log(e^\xi + e^{-\xi} - e^{\xi x} - e^{-\xi x}),
\]

we see that $F''(x)$ is strictly decreasing. Since $F'(0) = 0$ and $F''(x)$ becomes 0 when $x = \varphi(\xi)/\xi < 1$, the lemma follows. \(\square\)

We have the following corollary.

**Corollary 3.4.** For any $c_-$ and $c_+$ with $0 < c_- < \varphi(\xi)/\xi < c_+ < 1$, there exists $\delta > 0$ such that $F(x) < S(\xi)/\xi - \delta$ if $x < c_-$ or $x > c_+$. 
So we have $e^{N \times F(x)} < e^{N(S(\xi)/\xi - \delta)}$ if $x \notin [c_-, c_+]$. Since $f_N(x)$ uniformly converges to $F(x)$ from Proposition 2.8, we also have $|e^{N \times f_N(x)}| < e^{N(S(\xi)/\xi - \delta')}$ for some $\delta' > 0$ if $x \notin [c_-, c_+]$ and $N$ is sufficiently large.

Therefore we have

$$\left| \sum_{0<k/N<c_-} \exp \left( N \times f_N \left( \frac{2k+1}{2N} \right) \right) \right| < c_-N e^{N(S(\xi)/\xi - \delta')}$$

and

$$\left| \sum_{<c_+<k/N<1} \exp \left( N \times f_N \left( \frac{2k+1}{2N} \right) \right) \right| < (1-c_+)Ne^{N(S(\xi)/\xi - \delta')}.$$

As a result, we have

$$\sum_{k=0}^{N-1} \exp \left( N \times f_N \left( \frac{2k+1}{2N} \right) \right) \left| \sum_{c_-<k/N<c_+} \right| \exp \left( N \times f_N \left( \frac{2k+1}{2N} \right) \right) = O \left( Ne^{N(S(\xi)/\xi - \delta')} \right).$$

From (3.1), we also have

$$J_N \left( E; \exp(\xi/N) \right) = \frac{1}{2 \sinh(\xi/2)} \sum_{c_-<k/N<c_+} \exp \left( N \times f_N \left( \frac{2k+1}{2N} \right) \right) + O \left( Ne^{N(S(\xi)/\xi - \delta')} \right).$$

4. Integration

In this section, we use the Poisson summation formula (see [28, Proposition 4.2]) to change the summation in (3.6) into an integration. Then by using the saddle point method (see also [28, Proposition 3.2]) to prove Theorem 1.5.

Define

$$\psi(z) := F(z + \varphi(\xi)/\xi) - F(\varphi(\xi)/\xi)$$

in the region

$$\left\{ z \in \mathbb{C} \left| \frac{\varphi(\xi)}{\xi} \leq \text{Re} z < 1 - \frac{\varphi(\xi)}{\xi}, -1 - \frac{\varphi(\xi)}{\xi} < \frac{\text{Im} z}{\tan \theta} + \text{Re} z < 1 - \frac{\varphi(\xi)}{\xi} \right. \right\}.$$

Then since $\psi(0) = 0$, $\psi'(0) = 0$, and $\psi''(0) = -\xi \sqrt{2 \cosh(\xi) - 1}^2 - 4$ from (3.4) and (3.5), $\psi(z)$ is of the form

$$\psi(z) = -\frac{\xi}{2} \sqrt{2 \cosh(\xi) - 1}^2 - 4 \times z^2 + a_3z^3 + a_4z^4 + \cdots$$

in (4.1).

**Lemma 4.1.** If $x \neq 0$ is real and satisfies $-\varphi(\xi)/\xi \leq x < 1 - \varphi(\xi)/\xi$, then $\psi(x) < 0$.

**Proof.** Since $e^x + e^{-x} = e^{x} - e^{-x}$, we have

$$\psi'(x) = \log e^{\varphi(\xi) - e^x + \varphi(\xi)} - e^{\varphi(\xi)}$$

$$= \log (e^{\varphi(\xi)} - e^{x} + \varphi(\xi)) - e^{\varphi(\xi)}$$

$$= \log \left( e^{\varphi(\xi)}(1 - e^{x}) - e^{\varphi(\xi)} + 1 \right)$$

(4.2)
from (3.2). Therefore we see that \( \psi'(x) = 0 \) if \( x = 0 \) or \(-2\varphi(\xi)/\xi, \psi'(x) > 0 \) if \(-2\varphi(\xi)/\xi < x < 0 \), and \( \psi'(x) < 0 \) otherwise.

Therefore for \(-\varphi(\xi)/\xi < x < 0 \), \( \psi(x) \) is monotonically increasing and for \( 0 < x < 1 - \varphi(\xi)/\xi \) it is monotonically decreasing. So \( \psi(x) \) takes its unique maximum 0 at \( x = 0 \), which shows that \( \psi(x) < 0 \) when \( x \neq 0 \).

□

Now we use the following proposition.

**Proposition 4.2** ([28, Proposition 4.2]). Let \( b_- \) and \( b_+ \) be real numbers with \( b_- < 0 < b_+ \). Put

\[
\Lambda := \left\{ \frac{k}{N} \mid k \in \mathbb{Z}, b_- \leq \frac{k}{N} \leq b_+ \right\},
\]

\[
C := \{ t \in \mathbb{R} \mid b_- \leq t \leq b_+ \},
\]

\[
D := \{ z \in \mathbb{C} \mid \text{Re} \psi(z) < 0 \}.
\]

Assume that \( \psi(z) \) is a holomorphic function of the form

\[
\psi(z) = az^2 + a_3z^3 + a_4z^4 + \cdots
\]

with \( \text{Re}(a) < 0 \), defined in a neighborhood \( P \) of \( 0 \in \mathbb{C} \) that includes the \( \delta_0 \)-neighborhood \( N_{\delta_0} \) of 0 for \( \delta_0 > 0 \). We choose \( P \) so that the region \( D \cap P \) has two connected components. We also assume the following:

1. \( b_- \) and \( b_+ \) are in different components of \( D \cap P \) and moreover \( \text{Re} \psi(b_{\pm}) < -\varepsilon_0 \) for some \( \varepsilon_0 > 0 \),
2. Both \( b_- \) and \( b_+ \) are in a connected component of

\[
\{ x + y\sqrt{-1} \mid x \in [b_-, b_+], y \in [0, \delta_0], \text{Re} \psi(x + y\sqrt{-1}) < 2\pi y \}
\]

3. Both \( b_- \) and \( b_+ \) are in a connected component of

\[
\{ x - y\sqrt{-1} \mid x \in [b_-, b_+], y \in [0, \delta_0], \text{Re} \psi(x - y\sqrt{-1}) < 2\pi y \}
\]

Then there exists \( \varepsilon > 0 \), depending on \( a \) and \( \varepsilon_0 \), such that

\[
\frac{1}{N} \sum_{z \in \Lambda} e^{N\psi(z)} = \int_C e^{N\psi(z)} \, dz + O(e^{-N\varepsilon}).
\]

We will show that \( \psi(z) \) satisfies the assumptions of Proposition 4.2.

Put \( b_- := -\varphi(\xi)/(2\xi), b_+ := (1 - \varphi(\xi)/\xi)/2, \delta_0 := \frac{1}{2}(1 - \varphi(\xi)/\xi) \sin \theta \). Define the following regions:

\[
P := \left\{ z \in \mathbb{C} \mid -\frac{\varphi(\xi)}{\xi} < \text{Re} z < 1 - \frac{\varphi(\xi)}{\xi}, -1 - \frac{\varphi(\xi)}{\xi} < \frac{\text{Im} z}{\tan \theta} + \text{Re} z < 1 - \frac{\varphi(\xi)}{\xi} \right\},
\]

\[
D_- := \{ z \in \mathbb{C} \mid \text{Re} \psi(z) < 0, \text{Re} z < 0 \},
\]

\[
D_+ := \{ z \in \mathbb{C} \mid \text{Re} \psi(z) < 0, \text{Re} z > 0 \}.
\]

Note that \( P \) is just the region (4.1) and so \( \psi(z) \) is holomorphic in \( P \). See Figure 1.

Then, we show the following lemma, from which the assumptions of Proposition 4.2 hold.

**Lemma 4.3.** We assume that \( \sin \theta < \frac{\varphi(\xi)}{\xi - \varphi(\xi)} \) and that \( \tan \theta < \frac{\pi}{2\varepsilon} \). Then, the function \( \psi(z) \) satisfies the following:

1. \( \psi(z) \) is a holomorphic function of the form (4.3) defined in the \( \delta_0 \)-neighborhood \( N_{\delta_0} \) of \( 0 \in \mathbb{C} \),
2. Both \( D_+ \cap P \) and \( D_- \cap P \) are connected, and \( b_{\pm} \in D_{\pm} \cap P \). Moreover, \( \text{Re} \psi(b_{\pm}) < -\varepsilon_0 \) for some \( \varepsilon_0 > 0 \),
3. Condition (2) in Proposition 4.2 is satisfied.
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\[ \frac{1}{\xi} \sin \theta < \frac{\varphi(\xi)}{\xi}, \]

\[ -\varphi(\xi)/\xi \leq b_- < 0, \quad b_+ > 0, \quad b_\pm \in P. \]

The inequality \( \text{Re} \psi(b_\pm) < 0 \) follows from Lemma 4.1. So, we can choose \( \delta_0 > 0 \) so that \( \text{Re} \psi(b_\pm) < -\delta_0. \)

Next, we will show that for each \( x \neq 0 \) with \( -\varphi(\xi)/\xi < x < 1 - \varphi(\xi)/\xi, \) the set \( \{ y \in \mathbb{R} \mid \text{Re} \psi(x + y\sqrt{-1}) < 0 \} \cap P \) is an open interval containing 0. Then, we can see that \( D_\pm \cap P \) is connected.

If we put

\[ g(x, y) := e^\xi + e^{-\xi} - e^{\xi(x + y\sqrt{-1}) + \varphi(\xi)} - e^{-\xi(x + y\sqrt{-1}) - \varphi(\xi)}, \]

then we have

\[ \frac{d}{dy} \text{Re} \psi(x + y\sqrt{-1}) = -\arg g(x, y) \]

from (4.2). Now we have

\[ \text{Re} g(x, y) = 2 \cosh \xi - 2 \cosh \left( \xi \left( x + \frac{\varphi(\xi)}{\xi} \right) \right) \cos(\xi y), \]
Proof of Theorem 1.5. From Proposition 4.2, there exists \( \varepsilon > 0 \) such that

\[
\frac{1}{N} \sum_{-\varphi(\xi)/(2\xi) \leq k/N \leq (1-\varphi(\xi)/(2\xi))} e^{N\psi(k/N)} = \int_{-\varphi(\xi)/(2\xi)}^{(1-\varphi(\xi)/(2\xi))} e^{N\psi(z)} \, dz + O(e^{-N\varepsilon}).
\]

Since \( f_{N}(z + \varphi(\xi)/\xi + 1/(2N)) - S(\xi)/\xi \) uniformly converges to \( \psi(z) \), we have

\[
\frac{1}{N} \sum_{-\varphi(\xi)/(2\xi) \leq k/N \leq (1-\varphi(\xi)/(2\xi))} e^{N(f_{N}(k/N + \varphi(\xi)/\xi + 1/(2N)) - S(\xi)/\xi)}
\]
from [28, Remark 4.4]. Putting \( l/N := k/N + \varphi(\xi)/\xi \), we also have
\[
\sum_{-\varphi(\xi)/(2\xi) \leq k/N \leq (1-\varphi(\xi)/\xi)/2} e^{N(f_N(k/N+\varphi(\xi)/\xi+1/(2N))-S(\xi)/\xi)} = e^{-NS(\xi)/\xi} \sum_{\varphi(\xi)/(2\xi) \leq 1/N \leq (1+\varphi(\xi)/\xi)/2} \exp \left( N \times f_N \left( \frac{2l+1}{2N} \right) \right).
\]
Thus we obtain
\[
\sum_{\varphi(\xi)/(2\xi) \leq k/N \leq (1+\varphi(\xi)/\xi)/2} \exp \left( N \times f_N \left( \frac{2k+1}{2N} \right) \right)
= N \times e^{NS(\xi)/\xi} \left( \int_{-\varphi(\xi)/(2\xi)}^{(1-\varphi(\xi)/\xi)/2} e^{N\varphi(z)} dz + O \left( e^{-N\xi} \right) \right).
\]
Now by using the saddle point method [28, Proposition 3.2] (see also [28, Remark 3.3]), we have
\[
\int_{-\varphi(\xi)/(2\xi)}^{(1-\varphi(\xi)/\xi)/2} e^{N\varphi(z)} dz = \frac{\sqrt{\pi}}{\sqrt{\frac{2}{2\sinh(\xi/2)^2}} - 4\sqrt{N}} \left( 1 + O(N^{-1}) \right).
\]
Putting \( c_- := \varphi(\xi)/(2\xi) \) and \( c_+ := (1+\varphi(\xi)/\xi)/2 \) in (3.6), we finally have
\[
J_N(\varphi; \exp(\xi/N)) = \frac{N \times e^{NS(\xi)/\xi}}{2\sinh(\xi/2)} \left( \int_{-\varphi(\xi)/(2\xi)}^{(1-\varphi(\xi)/\xi)/2} e^{N\varphi(z)} dz + O \left( e^{-N \min\{\varepsilon, \delta\}} \right) \right)
= \frac{\sqrt{\pi}}{2\sinh(\xi/2) \sqrt{\frac{2}{2\sinh(\xi/2)^2}} - 4} \left( 1 + O(N^{-1}) \right).
\]
where we use (4.7) at the first equality and (4.8) at the second. Letting \( T(\xi) \) denote \( \frac{\sqrt{\pi}}{2\sinh(\xi/2)^2-4} \), we have
\[
J_N(\varphi; \exp(\xi/N)) = \frac{\sqrt{\pi}}{2\sinh(\xi/2)} \sqrt{T(\xi)} \left( \frac{N}{\xi} \times e^{\frac{N}{\xi}S(\xi)} \right) \left( 1 + O(N^{-1}) \right).
\]
Since \( J_N(\varphi; e^{-\xi/N}) = J_N(\varphi; e^{\xi/N}) \) we obtain the required formula. \( \square \)

5. Topological interpretations

In this section we give topological interpretations for \( T(\xi) \) and \( S(\xi) \).

5.1. Representations. Let \( X \) be the complement of the open tubular neighborhood of \( E \subset S^3 \). The fundamental group \( \pi_1(X) \) is presented as
\[
\langle x, y \mid wx = yw \rangle,
\]
with \( w := xy^{-1}x^{-1}y \). Let \( \rho \) be a non-Abelian representation of \( \pi_1(X) \) to \( SL(2; \mathbb{C}) \) given by
\[
\rho(x) = \begin{pmatrix} e^{\xi/2} & 1 \\ 0 & e^{-\xi/2} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} e^{\xi/2} & 0 \\ -d & e^{-\xi/2} \end{pmatrix},
\]
where \( d \) annihilates the Riley polynomial
\[
d^2 - (2\cosh(\xi) - 3)d - 2\cosh(\xi) + 3.
\]
The preferred longitude $\lambda$ is presented by $y^{-1}xy^{-2}yxy^{-1}$ and it is sent to

$$\rho(\lambda) = \begin{pmatrix} e^{\eta/2} & * \\ 0 & e^{-\eta/2} \end{pmatrix},$$

where

$$\eta := \log \left( \frac{1}{2} \left( e^{2\xi} - e^{-\xi} - e^{-\xi} + e^{2\xi} - 2 + (e^{-\xi} - e^{\xi}) \sqrt{(e^{\xi} + e^{-\xi} - 3)(e^{\xi} + e^{-\xi} + 1)} \right) \right).$$

5.2. **Adjoint Reidemeister torsion.** For a representation $\rho: \pi_1(X) \to \mathrm{SL}(2; \mathbb{C})$, one can consider the cochain complex $C^*(X; \mathfrak{sl}(2; \mathbb{C})_{\rho}) := \text{Hom}_{\mathbb{Z}[\pi_1(X)]}(C_*(X; \mathbb{Z}), \mathfrak{sl}(2; \mathbb{C}))$ twisted by the adjoint action of $\rho$. Here $\tilde{X}$ is the universal cover of $X$, $\pi_1(X)$ acts on $\tilde{X}$ as the deck transformation, and the Lie algebra $\mathfrak{sl}(2; \mathbb{C})$ is regarded as a $\mathbb{Z}[\pi_1(X)]$-module by the adjoint action of $\rho$. The Reidemeister torsion $T_\mu(\rho) \in \mathbb{C}$ associated with the meridian $\mu$, twisted by the adjoint action of $\rho$, is defined as the torsion of the cochain complex $C^*(X; \mathfrak{sl}(2; \mathbb{C})_{\rho})$. The following formula is known for the case of the figure-eight knot:

$$T_\mu(\rho) = \pm \frac{2}{\sqrt{(e^{\xi} + e^{-\xi} + 1)(e^{\xi} + e^{\xi} - 3)}}.$$

See [31, 3, 21].

5.3. **Chern–Simons invariant.** Let $M$ be a three-manifold with boundary a torus $T$, and $\{\mu, \lambda\}$ be generators of $\pi_1(T)$. For a representation $\rho: \pi_1(M) \to \mathrm{SL}(2; \mathbb{C})$, we can define the Chern–Simons invariant as follows.

Let $A$ be an $\mathfrak{sl}(2; \mathbb{C})$-valued 1-form $A$ on $M$ that defines the flat connection corresponding to $\rho$. Assume that $\rho \big|_T$ is diagonalizable for simplicity. Then by a suitable conjugation, one has

$$\rho(\mu) = \begin{pmatrix} e^{2\sqrt{-1}\alpha} & 0 \\ 0 & e^{-2\sqrt{-1}\alpha} \end{pmatrix}, \quad \rho(\lambda) = \begin{pmatrix} e^{2\sqrt{-1}\beta} & 0 \\ 0 & e^{-2\sqrt{-1}\beta} \end{pmatrix}.$$

Then up to gauge equivalence, we can assume that $A$ is of the form

$$\left( \begin{array}{cc} \sqrt{-1}\alpha & 0 \\ 0 & -\sqrt{-1}\alpha \end{array} \right) dx + \left( \begin{array}{cc} \sqrt{-1}\beta & 0 \\ 0 & -\sqrt{-1}\beta \end{array} \right) dy$$

near $T$, where $dx$ and $dy$ are the 1-forms corresponding to $\mu$ and $\lambda$ respectively. Then the Chern–Simons invariant $\text{cs}_M(\rho; \alpha, \beta)$ of $\rho$ associated with $(\alpha, \beta)$ is defined by

$$\text{CS}_M(\rho; \alpha, \beta) := -\frac{1}{8} \int_M \text{Tr} \left( dA \wedge A + \frac{2}{3} A \wedge A \wedge A \right) \in \mathbb{C}/(\pi^2 \mathbb{Z}).$$

Note that $\text{Im} \text{CS}_M(\rho_0; 0, 0)$ coincides with the hyperbolic volume if the interior of $M$ possesses a complete hyperbolic structure, where $\rho_0$ is the Levi–Civita connection. See [13] for details.

In [25], we proved the following theorem.

**Theorem 5.1** ([25]). The Chern–Simons invariant $\text{CS}_E(\rho; \xi, \eta)$ of the representation $\rho$ associated with $(\xi, \eta)$ is given by

$$\text{CS}_E(\rho; \xi, \eta) = S(\xi) - \frac{\xi \eta}{2}.$$
Here we choose $\mu$ and $\lambda$ to be the meridian and the preferred longitude of $E$ respectively, and we assume that $\rho$ sends
\[
\mu \mapsto \begin{pmatrix} e^{\xi/2} & 0 \\ 0 & e^{-\xi/2} \end{pmatrix}, \quad \lambda \mapsto \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}.
\]
up to conjugation.

6. PROOFS OF LEMMAS

In this section, we give proofs of lemmas used in this paper.

Proof of Lemma 2.1. Putting $x := ye^{\theta\sqrt{-1}}$, the integral becomes
\[
\int_{C_0} \exp \left( (2z - 1)ye^{\theta\sqrt{-1}} \right) \frac{1}{y \sinh(ye^{\theta\sqrt{-1}}) \sinh(\gamma ye^{\theta\sqrt{-1}})} \, dy.
\]
Noting that
\[
\sinh(as) \sim \begin{cases} \frac{1}{2} e^{as} & \text{as } s \to \infty \\ -\frac{1}{2} e^{-as} & \text{as } s \to -\infty \end{cases}
\]
and that
\[
\sinh(as) \sim \frac{1}{2} e^{as}
\]
for a complex number $a$ with $\Re(a) > 0$, we have
\[
\exp \left( (2z - 1)ye^{\theta\sqrt{-1}} \right) \frac{1}{y \sinh(ye^{\theta\sqrt{-1}}) \sinh(\gamma ye^{\theta\sqrt{-1}})} \sim \frac{1}{y} \exp \left( (2z - 2 - \gamma)ye^{\theta\sqrt{-1}} \right)
\]
and
\[
\exp \left( (2z - 1)ye^{\theta\sqrt{-1}} \right) \frac{1}{y \sinh(ye^{\theta\sqrt{-1}}) \sinh(\gamma ye^{\theta\sqrt{-1}})} \sim \frac{1}{y} \exp \left( (2z + \gamma)ye^{\theta\sqrt{-1}} \right)
\]
where $\Re(e^{\theta\sqrt{-1}}) = \cos \theta > 0$ and $\Re(e^{\theta\sqrt{-1}}) = \frac{\xi}{2\pi} \sin \theta > 0$. So we have
\[
\left| \exp \left( (2z - 1)ye^{\theta\sqrt{-1}} \right) \frac{1}{y \sinh(ye^{\theta\sqrt{-1}}) \sinh(\gamma ye^{\theta\sqrt{-1}})} \right| < \frac{C'}{y} \exp \left( \Re \left( (2z - 2 - \gamma)ye^{\theta\sqrt{-1}} \right) \right)
\]
when $y > 0$ for a positive constant $C'$, and
\[
\left| \exp \left( (2z - 1)ye^{\theta\sqrt{-1}} \right) \frac{1}{y \sinh(ye^{\theta\sqrt{-1}}) \sinh(\gamma ye^{\theta\sqrt{-1}})} \right| < \frac{C'}{|y|} \exp \left( -\Re \left( (2z + \gamma)ye^{\theta\sqrt{-1}} \right) \right)
\]
when $y < 0$ for a positive constant $C'$. Therefore, if $-\frac{\xi \sin \theta}{4N\pi} < \Re(ze^{\theta\sqrt{-1}}) < \cos \theta + \frac{\xi \sin \theta}{4N\pi}$, then the integrals
\[
\int_{1}^{\infty} \exp \left( (2z - 1)ye^{\theta\sqrt{-1}} \right) \frac{1}{y \sinh(ye^{\theta\sqrt{-1}}) \sinh(\gamma ye^{\theta\sqrt{-1}})} \, dy
\]
and
\[
\int_{-\infty}^{-1} \exp \left( (2z - 1)ye^{\theta\sqrt{-1}} \right) \frac{1}{y \sinh(ye^{\theta\sqrt{-1}}) \sinh(\gamma ye^{\theta\sqrt{-1}})} \, dy
\]
converge and the lemma follows. \[\square\]
Proof of Lemma 2.2. Putting \( x = ye^{\sqrt{-1} \theta} \), we have
\[
\int_{C_y} e^{(2z-1)x} \frac{x^m \sin x}{dx} = \frac{1}{e^{(m-1)\sqrt{-1} \theta}} \int_{C_0} e^{(2z-1)ye^{\sqrt{-1} \theta}} dy.
\]
So we need to show that
\[
\int_1^r e^{(2z-1)ye^{\sqrt{-1} \theta}} dy
\]
and
\[
\int_{-r}^{-1} e^{(2z-1)ye^{\sqrt{-1} \theta}} dy
\]
converge when \( r \to \infty \) for \( m = 0, 1, 2 \).

We have
\[
\left| \int_1^r e^{(2z-1)ye^{\sqrt{-1} \theta}} dy \right| \leq \int_1^r \frac{2e^{2y \text{Re}(y e^{\sqrt{-1} \theta})} - y \cos \theta}{y^m (e^{y \cos \theta} - e^{-y \cos \theta})} dy
\]
\[
= \int_1^r \frac{2e^{2y \text{Re}(y e^{\sqrt{-1} \theta})} - y \cos \theta}{y^m (1 - e^{-2y \cos \theta})} dy,
\]
which converges when \( r \to \infty \) since \( \text{Re}(ye^{\sqrt{-1} \theta}) < \cos \theta \).

We also have
\[
\left| \int_{-r}^{-1} e^{(2z-1)ye^{\sqrt{-1} \theta}} dy \right| \leq \int_{-r}^{-1} \frac{2e^{2y \text{Re}(y e^{\sqrt{-1} \theta})} - y \cos \theta}{y^m (e^{y \cos \theta} - e^{-y \cos \theta})} dy
\]
\[
= \int_{-r}^{-1} \frac{2e^{2y \text{Re}(y e^{\sqrt{-1} \theta})} - y \cos \theta}{y^m (e^{y \cos \theta} - 1)} dy,
\]
which converges when \( r \to \infty \) since \( \text{Re}(ye^{\sqrt{-1} \theta}) > 0 \). \( \square \)

Now we calculate the integrals in Lemma 2.2 to prove Lemma 2.5. The following lemma shows (2.2).

**Lemma 6.1.** If \( 0 < \text{Re}(ye^{\sqrt{-1} \theta}) < \cos \theta \), then we have
\[
\int_{C_y} e^{(2z-1)x} \frac{x}{\sinh(x)} dx = \frac{-2\pi \sqrt{-1}}{1 - e^{-2\pi \sqrt{-1} z}}.
\]

**Proof.** Put \( C_y^r := C_0 \setminus \left( (-\infty, -r) \cup (r, \infty) \right) \) for \( r > 1 \), \( C_y := e^{\sqrt{-1} \theta} C_y^r \), and \( C_y^{\pi} := \{ w + \pi \sqrt{-1} | \ w \in C_y^r \} \). We first note that
\[
\int_{C_y} e^{(2z-1)x} \frac{x}{\sinh(x)} dx = \int_{C_y^{\pi} + \pi \sqrt{-1}} e^{(2z-1)(x - \pi \sqrt{-1})} d(x - \pi \sqrt{-1})
\]
\[
= e^{-2\pi \sqrt{-1} z} \int_{C_y^{\pi} + \pi \sqrt{-1}} e^{(2z-1)x} \frac{x}{\sinh(x)} dx.
\]
Hence we have
\[
(1 - e^{-2\pi \sqrt{-1}z}) \int_{C^r_\theta} e^{(2z-1)x} \sinh(x) \, dx
= e^{-2\pi \sqrt{-1}z} \left( - \int_{C^r_\theta} e^{(2z-1)x} \sinh(x) \, dx + \int_{C^r_\theta + \pi \sqrt{-1}} e^{(2z-1)x} \sinh(x) \, dx \right)
= e^{-2\pi \sqrt{-1}z} \left( \int_{V^+_r} e^{(2z-1)x} \sinh(x) \, dx - \int_{V^-_r} e^{(2z-1)x} \sinh(x) \, dx \right)
= e^{-2\pi \sqrt{-1}z} 2\pi \sqrt{-1} \text{Res} \left( \frac{e^{(2z-1)x}}{\sinh(x)} ; x = \pi \sqrt{-1} \right),
\]
where \( V^+_r \) is the vertical segment connecting \( \pm r e^{\sqrt{-1} \theta} \) and \( \pm r e^{\sqrt{-1} \theta + \pi \sqrt{-1}} \), oriented upward. Since \( \text{Res} \left( \frac{e^{(2z-1)x}}{\sinh(x)} ; x = \pi \sqrt{-1} \right) = \lim_{x \to \pi \sqrt{-1}} e^{(2z-1)x} (x - \pi \sqrt{-1}) = -e^{(2z-1)\pi \sqrt{-1}} \), we have
\[
(1 - e^{-2\pi \sqrt{-1}z}) \int_{C^r_\theta} e^{(2z-1)x} \sinh(x) \, dx
= e^{-2\pi \sqrt{-1}z} \left( \int_{V^+_r} e^{(2z-1)x} \sinh(x) \, dx - \int_{V^-_r} e^{(2z-1)x} \sinh(x) \, dx \right) - 2\pi \sqrt{-1}.
\]
We will show that \( \lim_{r \to \infty} \int_{V^+_r} \frac{e^{(2z-1)x}}{\sinh(x)} \, dx = 0 \).

Since
\[
\int_{V^+_r} \frac{e^{(2z-1)x}}{\sinh(x)} \, dx = \pi \sqrt{-1} \int_0^1 \frac{e^{(2z-1)(\pm r e^{\sqrt{-1} \theta + \pi \sqrt{-1}s})}}{\sinh(\pm r e^{\sqrt{-1} \theta + \pi \sqrt{-1}s})} \, ds
\]
and \( |\sinh(w)| = \frac{1}{2r} |e^w - e^{-w}| \geq \frac{1}{r} |e \text{Re}w - e^{-\text{Re}w}| \) for any \( w \in \mathbb{C} \), we have
\[
\left| \int_{V^+_r} \frac{e^{(2z-1)x}}{\sinh(x)} \, dx \right| \leq \pi \int_0^1 \left| \frac{e^{(2z-1)(\pm r e^{\sqrt{-1} \theta + \pi \sqrt{-1}s})}}{\sinh(\pm r e^{\sqrt{-1} \theta + \pi \sqrt{-1}s})} \right| \, ds
\leq \pi \int_0^1 \left| \frac{2e \text{Re}((2z-1)(\pm r e^{\sqrt{-1} \theta + \pi \sqrt{-1}s}))}{e^{\text{Re}((2z-1)(\pm r e^{\sqrt{-1} \theta + \pi \sqrt{-1}s}))} - e^{-\text{Re}((2z-1)(\pm r e^{\sqrt{-1} \theta + \pi \sqrt{-1}s}))}} \right| \, ds
= \frac{2\pi e^{\pm \text{Re}(2z-1)e^{\sqrt{-1}\theta}}}{e^{r \cos \theta} - e^{-r \cos \theta}} \int_0^1 e^{-\pi s \text{Im}(2z-1)} \, ds.
\]
From the assumption \( 0 < \text{Re}(ze^{\sqrt{-1}\theta}) < \cos \theta \), we have \( |\text{Re}((2z-1)e^{\sqrt{-1}\theta})| < \cos \theta \). Therefore we see that
\[
\left| \int_{V^+_r} \frac{e^{(2z-1)x}}{\sinh(x)} \, dx \right| \to 0
\]
and so we have
\[
\int_{C^r_\theta} \frac{e^{(2z-1)x}}{\sinh(x)} \, dx = \lim_{r \to \infty} \int_{C^r_\theta} \frac{e^{(2z-1)x}}{\sinh(x)} \, dx = \frac{-2\pi \sqrt{-1}}{1 - e^{-2\pi \sqrt{-1}z}}.
\]
\[ \square \]

The following lemma shows (2.3) and (2.4).
Lemma 6.2. If $0 < \text{Re}(ze^{\theta \sqrt{-1}}) < \cos \theta$, then we have
\[
\int_{C_o} \frac{e^{(2z-1)x}}{x \sinh(x)} \, dx = \begin{cases} 
-2 \log \left( 1 - e^{2\pi \sqrt{-1} x} \right) & \text{if } \text{Im } z \geq 0, \\
-2\pi \sqrt{-1} (2z - 1) - 2 \log \left( 1 - e^{-2\pi \sqrt{-1} x} \right) & \text{if } \text{Im } z < 0.
\end{cases}
\]
and
\[
\int_{C_o} \frac{e^{(2z-1)x}}{x^2 \sinh(x)} \, dx = \begin{cases} 
-\frac{2\sqrt{-1}}{\pi} \text{Li}_2 \left( e^{2\pi \sqrt{-1} x} \right) & \text{if } \text{Im } z \geq 0, \\
-2\pi \sqrt{-1} (2z^2 - 2z + \frac{1}{2}) + \frac{2\sqrt{-1}}{\pi} \text{Li}_2 \left( e^{-2\pi \sqrt{-1} x} \right) & \text{if } \text{Im } z < 0.
\end{cases}
\]

Proof. First we assume that $\text{Im } z \geq 0$.

For a real number $r > 1$, let $U_r^\pm$ be the vertical segment connecting $\pm re^{\theta \sqrt{-1}}$ and $\pm re^{\theta \sqrt{-1}} + r \sqrt{-1}$, and $\partial f_r$ be the segment connecting $re^{\theta \sqrt{-1}} + r \sqrt{-1}$ and $re^{\theta \sqrt{-1}} + r \sqrt{-1}$. Here we assume that $r$ is not an integer multiple of $\pi$ so that $\partial f_r$ avoids the poles of $\frac{e^{(2z-1)x}}{x^{m \sinh(x)}}$ ($m = 1, 2$) as a function of $x$. We orient $U_r^\pm$ upward and $\partial f_r$ from left to right. Note that the distance between the origin and the line containing $U_r^\pm$ is $r \cos \theta$, and that the distance between the origin and the line containing $\partial f_r$ is also $r \cos \theta$.

By the residue theorem we have
\[
\int_{C_o} \frac{e^{(2z-1)x}}{x^m \sinh(x)} \, dx - \int_{\partial f_r} \frac{e^{(2z-1)x}}{x^m \sinh(x)} \, dx
\]
\[
+ \int_{U_r^+} \frac{e^{(2z-1)x}}{x^m \sinh(x)} \, dx - \int_{U_r^-} \frac{e^{(2z-1)x}}{x^m \sinh(x)} \, dx
\]

for $m = 1, 2$, where $|r|$ is the greatest integer less than or equal to $r$. Since the order of the pole $x = k\pi \sqrt{-1}$ of $\frac{e^{(2z-1)x}}{x^{m \sinh(x)}}$ for $k = 1, 2, 3, \ldots$ is one, we have
\[
\text{Res} \left( \frac{e^{(2z-1)x}}{x^m \sinh(x)} ; x = k\pi \sqrt{-1} \right) = \lim_{t \to k\pi \sqrt{-1}} \frac{(x - k\pi \sqrt{-1})e^{(2z-1)x}}{x^m \sinh(x)} = \frac{e^{2k\pi \sqrt{-1}}}{k\pi \sqrt{-1}}
\]
and
\[
\text{Res} \left( \frac{e^{(2z-1)x}}{x^m \sinh(x)} ; x = k\pi \sqrt{-1} \right) = \lim_{x \to \pm k\pi \sqrt{-1}} \frac{(x \pm k\pi \sqrt{-1})e^{(2z-1)x}}{x^m \sinh(x)} = -\frac{e^{2k\pi \sqrt{-1}}}{k^2 \pi^2}.
\]

Next, we calculate integrals along $\partial f_r$ and $U_r^\pm$. Note that since the distance between the origin and any point on these segments is greater than or equal to $r \cos \theta$, we have
\[
\left\lfloor \int_{\partial f_r} \frac{e^{(2z-1)x}}{x^m \sinh(x)} \, dx \right\rfloor \leq \frac{1}{(r \cos \theta)^m} \left\lfloor \int_{\partial f_r} \frac{e^{(2z-1)x}}{x^m \sinh(x)} \, dx \right\rfloor
\]
and
\[
\left\lfloor \int_{U_r^\pm} \frac{e^{(2z-1)x}}{x^m \sinh(x)} \, dx \right\rfloor \leq \frac{1}{(r \cos \theta)^m} \left\lfloor \int_{U_r^\pm} \frac{e^{(2z-1)x}}{x^m \sinh(x)} \, dx \right\rfloor
\]
for $m = 1, 2$. 

Putting \( x = ye^{\sqrt{-1} \theta} + r \sqrt{-1} \), we have

\[
\left| \int_{\mathbb{R}} \frac{e^{(2\pi-1)x}}{x^m \sinh(x)} \, dx \right| 
\leq \frac{1}{(r \cos \theta)^m} \left| \int_{-r}^{r} e^{(2\pi-1)(ye^{\sqrt{-1} \theta} + r \sqrt{-1})} \times e^{\sqrt{-1} \theta} \, dy \right|
\]

\[
\leq \frac{1}{(r \cos \theta)^m} \left| \int_{-r}^{r} e^{(2\pi-1)(ye^{\sqrt{-1} \theta} + r \sqrt{-1})} \, dy \right|
\]

\[
= \frac{e^{-2r \Im z}}{(r \cos \theta)^m} \int_{-r}^{r} e^{\theta \Re(z e^{\sqrt{-1} \theta})} \sinh(ye^{\sqrt{-1} \theta} + r \sqrt{-1}) \, dy
\]

\[
(M := \max_{1 \leq y \leq 1} \frac{\sinh(ye^{\sqrt{-1} \theta} + r \sqrt{-1})}{\sinh(ye^{\sqrt{-1} \theta} + r \sqrt{-1})} > 0)
\]

\[
\leq \frac{e^{-2r \Im z}}{(r \cos \theta)^m} \left(2M + \int_{-r}^{1} e^{y \cos \theta} \rho \cos \theta \, dy + \int_{-1}^{1} e^{y \cos \theta} \rho \cos \theta \, dy \right)
\]

\[
= 2 e^{-2r \Im z} \frac{(r \cos \theta)^m}{(r \cos \theta)^m} \left(M + \frac{\int_{-r}^{1} e^{y \cos \theta} \rho \cos \theta \, dy}{1 - e^{2y \cos \theta}} \right)
\]

\[
\leq \frac{e^{-2r \Im z}}{(r \cos \theta)^m} \left(M + \frac{\int_{-r}^{1} e^{y \cos \theta} \rho \cos \theta \, dy}{1 - e^{-2 \cos \theta}} \right)
\]

\[
= \frac{2e^{-2r \Im z}}{(r \cos \theta)^m} \left(M + \frac{\int_{-r}^{1} e^{y \cos \theta} \rho \cos \theta \, dy}{1 - e^{-2 \cos \theta}} \right)
\]

This converges to zero as \( r \to \infty \) since \( 0 < \Re(z e^{\sqrt{-1} \theta}) < \cos \theta \) and \( \Im z \geq 0 \). Note that \( M \) depends on \( r \) but that it is bounded because it is periodic with respect to \( r \).

Putting \( x = r(e^{\sqrt{-1} \theta} + y \sqrt{-1}) \), we have

\[
\left| \int_{\mathbb{R}} \frac{e^{(2\pi-1)x}}{x^m \sinh(x)} \, dx \right| 
\leq \frac{1}{(r \cos \theta)^m} \left| \int_{0}^{1} r e^{(2\pi-1)r(e^{\sqrt{-1} \theta} + y \sqrt{-1})} \times \sqrt{-1} \, dy \right|
\]

\[
\leq \frac{1}{(r \cos \theta)^m} \left| \int_{0}^{1} r e^{(2\pi-1)r(e^{\sqrt{-1} \theta} + y \sqrt{-1})} \, dy \right|
\]

\[
= \frac{2r e^{-2r \Im z}}{(r \cos \theta)^m} \left(M + \frac{\int_{0}^{1} e^{y \cos \theta} \rho \cos \theta \, dy}{1 - e^{-2 \cos \theta}} \right)
\]

\[
= \frac{2e^{-2r \Im z}}{(r \cos \theta)^m} \left(M + \frac{\int_{0}^{1} e^{y \cos \theta} \rho \cos \theta \, dy}{1 - e^{-2 \cos \theta}} \right)
\]

\[
= \frac{2e^{-2r \Im z}}{(r \cos \theta)^m} \left(M + \frac{\int_{0}^{1} e^{y \cos \theta} \rho \cos \theta \, dy}{1 - e^{-2 \cos \theta}} \right)
\]

\[
\to 0 \quad (r \to \infty)
\]
since \( \text{Re}(ze^{\sqrt{-1}r}) < \cos \theta \), noting that the last integral becomes either \(1\) (if \(z\) is real) or \(\frac{1-e^{-2\pi \text{Im} z}}{2\pi \text{Im} z}\) (otherwise).

Similarly, putting \(x = r(-e^{\sqrt{-1}r} + y\sqrt{-1})\), we have

\[
\left| \int_{C_{\theta}} e^{(2z-1)x} \frac{x^m \sinh(x)}{\sinh(r(e^{\sqrt{-1}r} - y\sqrt{-1}))} \right| \leq \frac{1}{(r \cos \theta)^m} \left| \int_{0}^{1} e^{-(2z-1)r(e^{\sqrt{-1}r} - y\sqrt{-1})} \sinh(r(e^{\sqrt{-1}r} - y\sqrt{-1})) \right| dy \\
\leq \frac{1}{(r \cos \theta)^m} \left| \int_{0}^{1} e^{-(2z-1)r(e^{\sqrt{-1}r} - y\sqrt{-1})} \sinh(r(e^{\sqrt{-1}r} - y\sqrt{-1})) \right| dy \\
\leq \frac{2r e^{-2r} \text{Re}(ze^{\sqrt{-1}r}) + r \cos \theta}{(r \cos \theta)^m} \left| \int_{0}^{1} e^{-2r \text{Im} z} dy \right| \\
= \frac{2r e^{-2r} \text{Re}(ze^{\sqrt{-1}r})}{(r \cos \theta)^m} \left| \int_{0}^{1} e^{-2r \text{Im} z} dy \right| \\
\to 0 \quad (r \to \infty)
\]

since \(0 < \text{Re}(ze^{\sqrt{-1}r})\). Therefore from (6.1) we have

\[
\lim_{r \to \infty} \int_{C_{\theta}} e^{(2z-1)x} \frac{x^m \sinh(x)}{x \sinh(x)} \, dx = \lim_{r \to \infty} \sum_{k=1}^{[r]} e^{2k\pi \sqrt{-1}} k,
\]

which converges to \(-2 \log(1 - e^{2\pi \sqrt{-1}})\) as \(r \to \infty\) when \(e^{2\pi \sqrt{-1}} < 1\), or \(e^{2\pi \sqrt{-1}} \neq 1\) (\(z \in \mathbb{R}\)), that is, when \(\text{Im} z \geq 0\) (Recall that we assume \(0 < \text{Re}(ze^{\sqrt{-1}r}) < \cos \theta\)).

We also have

\[
\lim_{r \to \infty} \int_{C_{\theta}} e^{(2z-1)x} \frac{x^m \sinh(x)}{x^2 \sinh(x)} \, dx = -\frac{2\sqrt{-1}}{\pi} \lim_{r \to \infty} \sum_{k=1}^{[r]} e^{2k\pi \sqrt{-1}} k^2,
\]

which converges to \(-\frac{2\sqrt{-1}}{\pi} \text{Li}_2(e^{2\pi \sqrt{-1}})\) as \(r \to \infty\) when \(\text{Im} z \geq 0\) from Lemma 6.3 below.

This completes the case where \(\text{Im} z \geq 0\).

Next we assume that \(\text{Im} z < 0\).

For \(r > 1\), let \(U^{\pm}_{r}\) be the vertical segment connecting \(\pm r(e^{\sqrt{-1}r} \text{Im} z - r\sqrt{-1})\) and \(H_r\), be the segment connecting \(-r(e^{\sqrt{-1}r} - r\sqrt{-1})\) and \(r(e^{\sqrt{-1}r} - r\sqrt{-1})\). We orient \(U^{\pm}_{r}\) upward and \(H_r\) from left to right. Note that the distance between the origin and the line containing \(U^{\pm}_{r}\) is \(r \cos \theta\), and that the distance between the origin and the line containing \(H_r\) is also \(r \cos \theta\).

For \(m = 1, 2\), we have

\[
\begin{align*}
-\int_{C_{\theta}} e^{(2z-1)x} \frac{x^m \sinh(x)}{x \sinh(x)} \, dx + \int_{U^+_{r}} e^{(2z-1)x} \frac{x^m \sinh(x)}{x \sinh(x)} \, dx \\
+ \int_{U^-_{r}} e^{(2z-1)x} \frac{x^m \sinh(x)}{x \sinh(x)} \, dx - \int_{H_r} e^{(2z-1)x} \frac{x^m \sinh(x)}{x \sinh(x)} \, dx \\
\leq 2\pi \sqrt{-1} \sum_{k=0}^{[r]} \text{Res}(e^{(2z-1)x} \frac{x^m \sinh(x)}{x \sinh(x)}; x = -k\pi \sqrt{-1}) .
\end{align*}
\]

Since the order of the pole \(x = -k\pi \sqrt{-1}\) of \(e^{(2z-1)x} \frac{x^m \sinh(x)}{x \sinh(x)} (m = 1, 2)\) for \(k = 1, 2, 3, \ldots\) is one, we have

\[
\text{Res}(e^{(2z-1)x} \frac{x^m \sinh(x)}{x \sinh(x)}; x = -k\pi \sqrt{-1}) = \lim_{x \to -k\pi \sqrt{-1}} \frac{(x + k\pi \sqrt{-1})e^{(2z-1)x}}{x \sinh(x)} = -e^{-2k\pi \sqrt{-1}} \frac{k\pi \sqrt{-1}}{k\pi \sqrt{-1}}.
\]
Since $e^{(2z-1)x} = 1 + (2z-1)x + \frac{(2z-1)^2x^2}{2} + \cdots$ and $\frac{1}{\sinh(x)} = \frac{1}{x} - \frac{x}{6} + \cdots$, we have

$$\text{Res}\left(\frac{e^{(2z-1)x}}{x\sinh(x)}; x = 0\right) = 2z - 1$$

and

$$\text{Res}\left(\frac{e^{(2z-1)x}}{x^2\sinh(x)}; x = 0\right) = 2z^2 - 2z + \frac{1}{3}.$$

We can prove the integrals along $\mathcal{H}_r^-$ and $\mathcal{U}_r^-$ converge to zero as $r \to \infty$ in similar ways to the cases of $\mathcal{H}_r^+$ and $\mathcal{U}_r^+$. Putting $x = ye^{\sqrt{-1}y} - r\sqrt{-1}$ and assuming $r > 1$, we have

$$\left|\int_{\mathcal{H}_r^-} \frac{e^{(2z-1)x}}{x^{m}\sinh(x)} \, dx\right| \leq \frac{1}{(r \cos \theta)^m} \left|\int_{-r}^{r} e^{(2z-1)(ye^{\sqrt{-1}y} - r\sqrt{-1})} \times e^{\sqrt{-1}y} \frac{dy}{\sinh(ye^{\sqrt{-1}y} - r\sqrt{-1})}\right|,$$

$$\leq \frac{e^{2r \Im z}}{(r \cos \theta)^m} \int_{-r}^{r} \left|\frac{e^{2y \Re (2z-1)e^{\sqrt{-1}y}}}{\sinh(ye^{\sqrt{-1}y} - r\sqrt{-1})}\right| \, dy,$$

$$\leq \frac{e^{2r \Im z}}{(r \cos \theta)^m} \left(2M + \int_{-r}^{r} \frac{2e^{2y \Re (ze^{\sqrt{-1}y} - y\cos \theta)}}{ie^{y \cos \theta} - e^{-y \cos \theta}} \, dy + \int_{1}^{r} \frac{2e^{2y \Re (ze^{\sqrt{-1}y} - y\cos \theta)}}{ie^{y \cos \theta} - e^{-y \cos \theta}} \, dy\right),$$

which converges to zero as $r \to \infty$.

Putting $x = r(e^{\sqrt{-1}y} - y\sqrt{-1})$, we have

$$\left|\int_{\mathcal{U}_r^+} \frac{e^{(2z-1)x}}{x^{m}\sinh(x)} \, dx\right| \leq \frac{1}{(r \cos \theta)^m} \left|\int_{0}^{1} e^{(2z-1)(r(e^{\sqrt{-1}y} - y\sqrt{-1}))} \times (-r\sqrt{-1}) \frac{dy}{\sinh(r(e^{\sqrt{-1}y} - y\sqrt{-1}))}\right|,$$

$$\leq \frac{2r e^{2r \Im z}}{(r \cos \theta)^m} \left|\int_{0}^{1} e^{-r \cos \theta} \, dy\right|,$$

$$= \frac{2e^{2r \Im z}}{r^{m-1} \cos \theta(1 - e^{-2r \cos \theta})} \times \int_{0}^{1} e^{2r \Im z} \, dy,$$

since $0 < \Re(ze^{\sqrt{-1}y}) < \cos \theta$, noting that the last integral becomes either 1 (if $z$ is real) or $\frac{e^{2r \Im z}}{2r \Im z}$ (otherwise).
Similarly, putting \( x = -r(e^{\sqrt{-1}y} + y\sqrt{-1}) \), we have
\[
\left| \int_{\mathbb{U}} \frac{e^{(2z-1)x}}{x^m \sinh(x)} \, dx \right| \leq \frac{1}{(r \cos \theta)^m} \left| \int_0^1 e^{-r(2z-1)(e^{\sqrt{-1}y} + y\sqrt{-1})} \times (-r \sqrt{-1}) \, dy \right|
\]
\[
\leq 2re^{-2r \Re(z e^{\sqrt{-1}y}) + r \cos \theta} \left| \int_0^1 e^{2ry \Im z} \, dy \right|
\]
\[
= \frac{2e^{-2r \Re(z e^{\sqrt{-1}y})}}{r^{m-1} \cos \theta (1 - e^{-2r \cos \theta})} \times \int_0^1 e^{2ry \Im z} \, dy
\]
\[
\to 0 \quad (r \to \infty)
\]
since \( 0 < \Re(ze^{\sqrt{-1}y}) \). So from (6.2) we have
\[
\lim_{r \to \infty} \int_{C_\delta} \frac{e^{(2z-1)x}}{x \sinh(x)} \, dx = -2\pi \sqrt{-1}(2z - 1) + 2 \lim_{r \to \infty} \sum_{k=1}^{\lfloor r \rfloor} e^{-2k\pi \sqrt{-1}r/k}.
\]

Since this series converges to \(-\log(1 - e^{-2\pi \sqrt{-1}})\) as \( r \to \infty \) if \( \Im z < 0 \) from Lemma 6.3, we finally have
\[
\int_{C_\delta} \frac{e^{(2z-1)x}}{x \sinh(x)} \, dx = -2\pi \sqrt{-1}(2z - 1) - 2 \log(1 - e^{-2\pi \sqrt{-1}}),
\]
completing the proof when \( \Im z < 0 \).

Similarly, we have
\[
\lim_{r \to \infty} \int_{C_\delta} \frac{e^{(2z-1)x}}{x^2 \sinh(x)} \, dx = -2\pi \sqrt{-1} \lim_{r \to \infty} \sum_{k=0}^{\lfloor r \rfloor} \text{Res} \left( \frac{e^{(2z-1)x}}{x^2 \sinh(x)} ; x = -k\pi \sqrt{-1} \right).
\]
\[
= -2\pi \sqrt{-1} \left( 2z^2 - 2z + \frac{1}{3} \right) + \frac{2\sqrt{-1}}{\pi} \lim_{r \to \infty} \sum_{k=1}^{\lfloor r \rfloor} e^{-2k\pi \sqrt{-1}r/k^2}.
\]
The series converges to \( \text{Li}_2 \left( e^{-2\pi \sqrt{-1}} \right) \) and so
\[
\int_{C_\delta} \frac{e^{(2z-1)x}}{x^2 \sinh(x)} \, dx = -2\pi \sqrt{-1} \left( 2z^2 - 2z + \frac{1}{3} \right) + \frac{2\sqrt{-1}}{\pi} \text{Li}_2 \left( e^{-2\pi \sqrt{-1}} \right),
\]
completing the proof when \( \Im z < 0 \). \( \square \)

We give a proof for the following well-known lemma.

**Lemma 6.3.** For a complex number \( w \) with \( |w| \leq 1 \), the series \( \sum_{k=1}^{\infty} \frac{w^k}{k} \) converges to \( \text{Li}_2(w) \). Here we use \( \text{Li}_2(w) := -\int_0^w \frac{\log(1-t)}{t} \, dt \) as the definition of the dilogarithm.

**Proof.** Put \( a_k := \frac{1}{k^2} \).

Then we have
\[
\left| \frac{a_{k+1}}{a_k} \right| = \left( \frac{k}{k + 1} \right)^2 \to 1
\]
as \( k \to \infty \). Therefore from d’Alembert’s ratio test, the radius of convergence of the power series \( \sum_{k=1}^{\infty} a_k w^k \) is 1. So we can differentiate it term by term if \( |w| < 1 \),
and we obtain
\[
\frac{d}{dw} \left( \frac{\sum_{k=1}^{\infty} w^k}{k^2} \right) = \sum_{k=1}^{\infty} \frac{w^{k-1}}{k} = -\frac{\log(1-w)}{w}.
\]

Therefore we conclude that \(\sum_{k=1}^{\infty} \frac{w^k}{k^2} = -\int_0^w \frac{\log(1-t)}{t} \, dt\).

If \(w = 1\), the series \(\sum_{k=1}^{\infty} a_k\) converges by the integral test.

Finally, we assume that \(|w| = 1\) with \(w \neq 1\), and apply Abel’s test. Since the sequence \(\{a_k\}\) is positive and monotonically decreasing with \(\lim_{k \to \infty} a_k = 0\), the power series \(\sum_{k=1}^{\infty} a_k w^k\) converges if \(|w| = 1\) (\(w \neq 1\)). We can also apply Abel’s theorem to conclude that \(\sum_{k=1}^{\infty} a_k z^k\) converges to \(\sum_{k=1}^{\infty} a_k w^k\) provided that \(z\) approaches \(w\) along the radius. Note that this includes the case \(w = 1\). Therefore we also conclude that \(\sum_{k=1}^{\infty} \frac{w^k}{k^2} = -\int_0^w \frac{\log(1-t)}{t} \, dt\) for the case \(|w| = 1\) by choosing the integral path as the radius connecting \(0\) and \(w\). \(\square\)

We prove that \(\frac{1}{N} T_N(z)\) uniformly converges to \(\frac{1}{\xi} L_2(z)\).

**Proof of Proposition 2.8.** We have

\[
\left| T_N(z) - \frac{N}{\xi} L_2(z) \right| = \frac{1}{4} \left| \int_{C_\theta} \left( \frac{e^{(2\xi-1)x}}{x \sinh(x) \sinh(\gamma x)} - \frac{e^{(2\xi-1)x}}{\gamma x^2 \sinh(x)} \right) \, dx \right|
\]

\[
= \frac{1}{4} \left| \int_{C_\theta} e^{(2\xi-1)x} \frac{\gamma x}{x^2 \sinh(x)} - 1 \, dx \right|
\]

\[
\leq \frac{\pi N}{2|\xi|} \int_{C_\theta} \left| \frac{e^{(2\xi-1)x}}{x^2 \sinh(x)} - 1 \right| \, dx.
\]

Since \(\frac{\gamma^2 x}{\sinh(\gamma x)} = 1 - \frac{(\gamma x)^2}{6} + O((\gamma x)^4), \left| \frac{\gamma x}{\sinh(\gamma x)} - 1 \right| < \frac{C|\gamma|^3}{N^2}\) for a positive constant \(C\) since \(\gamma = \frac{\xi}{2N^2} - 1\). So we have

\[
\left| T_N(z) - \frac{N}{\xi} L_2(z) \right| \leq \frac{C'}{N} \int_{C_\theta} \left| \frac{e^{(2\xi-1)x}}{x \sinh(x)} \right| \, dx,
\]

where \(C' := \frac{C\pi}{2N^2}\). Put

\[
I_+ := \lim_{r \to \infty} \int_{r e^{\gamma \pi i}} e^{-\gamma \pi i} \left| \frac{e^{(2\xi-1)x}}{x \sinh(x)} \right| \, dx,
\]

\[
I_- := \lim_{r \to \infty} \int_{-r e^{-\gamma \pi i}} e^{-\gamma \pi i} \left| \frac{e^{(2\xi-1)x}}{x \sinh(x)} \right| \, dx,
\]

\[
I_0 := \int_{|x|=1, \theta \leq \arg x \leq \pi + \theta} \left| \frac{e^{(2\xi-1)x}}{x \sinh(x)} \right| \, dx.
\]
Putting $x := e^{\sqrt{-1} \theta} y$, we have

$$I_+ = \int_{1}^{\infty} \frac{e^{(2z-1)ye^{\sqrt{-1} \tau}}}{\sinh(ye^{\sqrt{-1} \tau})} \, dy \leq \int_{1}^{\infty} \frac{2e^{2y \text{Re}(ze^{\sqrt{-1} \tau}) - y \cos \theta}}{e^{y \cos \theta} - e^{-y \cos \theta}} \, dy = \int_{1}^{\infty} \frac{2e^{2y \text{Re}(ze^{\sqrt{-1} \tau}) - \cos \theta}}{1 - e^{-2y \cos \theta}} \, dy \leq \frac{2}{1 - e^{-2 \cos \theta}} \int_{1}^{\infty} e^{-2\nu y} \, dy \leq \frac{e^{2\nu}}{(1 - e^{-2 \cos \theta}) \nu}.$$

Here we use the assumption $\text{Re}(ze^{\sqrt{-1} \tau}) \leq \cos \theta - \nu$.

Similarly, we have

$$I_- = \int_{-\infty}^{-1} \frac{e^{(2z-1)ye^{\sqrt{-1} \tau}}}{\sinh(ye^{\sqrt{-1} \tau})} \, dy \leq \int_{-\infty}^{-1} \frac{2e^{2y \text{Re}(ze^{\sqrt{-1} \tau}) - y \cos \theta}}{e^{-y \cos \theta} - e^{y \cos \theta}} \, dy = \int_{-\infty}^{-1} \frac{2e^{2y \text{Re}(ze^{\sqrt{-1} \tau})}}{1 - e^{2y \cos \theta}} \, dy \leq \frac{2}{1 - e^{-2 \cos \theta}} \int_{-\infty}^{-1} e^{2\nu y} \, dy \leq \frac{e^{2\nu}}{(1 - e^{-2 \cos \theta}) \nu}.$$

Here we use the assumption $\text{Re}(ze^{\sqrt{-1} \tau}) \geq \nu$.

Finally, putting $x = e^{\sqrt{-1} \tau}$ ($\theta \leq \tau \leq \theta + \pi$) and $L := \min_{\theta \leq \tau \leq \theta + \pi} |\sinh(e^{\sqrt{-1} \tau})| > 0$, we have

$$I_0 = \int_{\theta}^{\theta + \pi} \frac{e^{(2z-1)e^{\sqrt{-1} \tau}}}{|\sinh(e^{\sqrt{-1} \tau})|} |\sqrt{e^{\sqrt{-1} \tau}}| \, d\tau \leq \frac{1}{L} \int_{\theta}^{\theta + \pi} e^{2 \text{Re}(ze^{\sqrt{-1} \tau}) - \cos \tau} \, d\tau = \frac{1}{L} \int_{\theta}^{\theta + \pi} e^{(2 \text{Re} z - 1) \cos \tau - 2 \sin \tau \text{Im} z} \, d\tau,$$

which is bounded from the above because both $\text{Re} z$ and $\text{Im} z$ are bounded.

Therefore we conclude that $|T_N(z) - \frac{N}{2} L_2(z)| \leq C'' N^{-\epsilon}$ for some constant $C''$ that does not depend on $z$. \hfill \square

**Proof of Lemma 3.1.** Recall that $\gamma = \frac{\epsilon}{2N\sqrt{-1}}$. 

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By the definition of $T_N(z)$, we have
\[
T_N \left( \frac{\xi}{2\pi \sqrt{1 - 1/N}} (1 + \frac{1}{2N}) \right) = \frac{1}{4} \int_{C_\alpha} \frac{e^{\left( \frac{\xi}{2\pi \sqrt{1 - 1/N}} (1 + \frac{1}{2N}) + 1 \right) x} - e^{\left( \frac{\xi}{2\pi \sqrt{1 - 1/N}} (1 + \frac{1}{2N}) - 1 \right) x}}{x \sinh(x) \sinh(\gamma x)} \, dx
\]
\[
= \frac{1}{2} \int_{C_\alpha} \frac{e^{\left( \frac{\xi}{2\pi \sqrt{1 - 1/N}} + 1 \right) x}}{x \sinh(x)} \, dx,
\]
which equals $\pi \sqrt{1 - \xi} - \xi - \log(1 - e^{-\xi})$ from Lemma 6.2.

Remark 6.4. The proof of Lemma 2.3 in [21] is wrong, which was informed by Ka Ho Wong.

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