ON THE CONE OF CURVES OF AN ABELIAN VARIETY

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Abstract. Let $X$ be a smooth projective variety over the complex numbers and let $N_1(X)$ be the real vector space of 1-cycles on $X$ modulo numerical equivalence. As usual denote by $NE(X)$ the cone of curves on $X$, i.e. the convex cone in $N_1(X)$ generated by the effective 1-cycles. One knows by the Cone Theorem [4] that the closed cone of curves $\overline{NE}(X)$ is rational polyhedral whenever $c_1(X)$ is ample. For varieties $X$ such that $c_1(X)$ is not ample, however, it is in general difficult to determine the structure of $\overline{NE}(X)$. The purpose of this paper is to study the cone of curves of abelian varieties. Specifically, the abelian varieties $X$ are determined such that the closed cone $\overline{NE}(X)$ is rational polyhedral. The result can also be formulated in terms of the nef cone of $X$ or in terms of the semi-group of effective classes in the Néron-Severi group of $X$.

0. Introduction. Let $X$ be a smooth projective variety over the complex numbers and let $N_1(X)$ be the real vector space

$$N_1(X) = \{1\text{-cycles on } X \text{ modulo numerical equivalence}\} \otimes \mathbb{R}.$$ 

As usual denote by $NE(X)$ the cone of curves on $X$, i.e. the convex cone in $N_1(X)$ generated by the effective 1-cycles. The closed cone of curves $\overline{NE}(X)$ is the closure of $NE(X)$ in $N_1(X)$. One knows by the Cone Theorem [4] that it is rational polyhedral whenever $c_1(X)$ is ample. For varieties $X$ such that $c_1(X)$ is not ample, however, it is in general difficult to determine the structure of $\overline{NE}(X)$, since it may depend in a subtle way on the geometry of $X$ (cf. [1, §4]). This becomes already apparent in the surface case, as work of Kovács on K3 surfaces shows (see [2]).

The purpose of this paper is to study the cone of curves of abelian varieties. Specifically, we focus on the problem of determining the abelian varieties $X$ such that the closed cone $\overline{NE}(X)$ is rational polyhedral. Attacking the question from the dual point of view, one is lead to consider the nef cone $Nef(X)$ or the semi-group $NS^+(X)$ of homology classes of effective line bundles, i.e. the subset

$$NS^+(X) = \{\lambda \in NS(X) \mid \lambda = c_1(L) \text{ for some } L \in \text{Pic}(X) \text{ with } h^0(X, L) > 0\}$$

of the Néron-Severi group of $X$. In fact, Rossoff has studied this semi-group in

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[7], where he gives examples of abelian varieties for which $NS^+(X)$ is finitely generated, as well as examples where finite generation fails. He shows:

1. If $X$ is a singular abelian variety, i.e. if $\text{rk}NS(X) = (\dim X)^2$, and if $\dim X \geq 2$, then $NS^+(X)$ is not finitely generated.

2. For elliptic curves $E_1$ and $E_2$, $NS^+(E_1 \times E_2)$ is finitely generated if and only if $\text{rk}NS(E_1 \times E_2) = 2$.

Considering these examples it is natural to ask if the abelian varieties $X$ for which $NS^+(X)$ is finitely generated or, equivalently, for which $\overline{NE}(X)$ is rational polyhedral, can be characterized in a simple way. Our main result shows that this is in fact the case:

**Theorem.** Let $X$ be an abelian variety over the field of complex numbers. Then the following conditions are equivalent:

1. The closed cone of curves $\overline{NE}(X)$ is rational polyhedral.
2. The nef cone $\text{Nef}(X)$ is rational polyhedral.
3. The semi-group $NS^+(X)$ is finitely generated.
4. $X$ is isogenous to a product

$$X_1 \times \cdots \times X_r$$

of mutually non-isogenous abelian varieties $X_i$ with $NS(X_i) \cong \mathbb{Z}$ for $1 \leq i \leq r$.

Note that, since on abelian varieties the nef cone coincides with the effective cone, the equivalence of (ia), (ib) and (ic) follows from elementary properties of cones and is stated here merely for the sake of completeness (see Section 3). Observe that the theorem of course contains statement (2) above, while statement (1) follows from the theorem plus the fact that by [8] a singular abelian variety is isogenous to a product $E^n$ for some elliptic curve $E$.

**Notation and conventions.** We work throughout over the field $\mathbb{C}$ of complex numbers. We will always use additive notation for the tensor product of line bundles, since this is more convenient for our purposes (for example when working with $\mathbb{Q}$- or $\mathbb{R}$-line bundles). Numerical equivalence of divisors or line bundles, which for abelian varieties coincides with algebraic equivalence, will be denoted by $\equiv$.

If $X$ is an abelian variety and $L$ is a line bundle on $X$, then $\phi_L$ denotes the homomorphism $X \longrightarrow \hat{X}$, $x \mapsto t_x^*L-L$, where $t_x$ is the translation map $y \mapsto x+y$ and $\hat{X} = \text{Pic}^0(X)$ is the dual abelian variety. Recall that $\phi_L$ depends only on the algebraic equivalence class of $L$.

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1. Effective classes on simple abelian varieties. In this section we consider the semi-group of effective divisor classes on simple abelian varieties. We start by stating alternative characterizations of $NS^+(X)$ which we will use in the sequel. While these are at least implicitly well-known, we include a proof for the convenience of the reader.

**Lemma 1.1.** Let $X$ be an abelian variety of dimension $n$ and let $A$ be an ample line bundle on $X$. Then the following conditions on a line bundle $L$ are equivalent:

(i) $L$ is algebraically equivalent to some effective line bundle.

(ii) $L$ is nef.

(iii) $L^i A^{n-i} \geq 0$ for $1 \leq i \leq n$.

**Proof.** Condition (i) certainly implies (ii), since if $L \equiv L'$ for some effective line bundle $L'$, then a suitable translate of an effective divisor in $|L'|$ will intersect any given curve properly. The implication (ii) $\Rightarrow$ (iii) is clear, since an intersection product of nef line bundles is nonnegative. For (iii) $\Rightarrow$ (ii) it is enough to show that the line bundle $A + mL$ is ample for all $m \geq 0$. But this follows from

$$(A+mL)^i A^{n-i} = A^n + \sum_{k=0}^{i-1} \binom{i}{k} A^{n+k-i} L^{i-k} m^{i-k} > 0,$$

and the version of the Nakai-Moishezon Criterion given in [3, Corollary 4.3.3]. Finally, for the implication (ii) $\Rightarrow$ (i), suppose that $L$ is nef. Then $A + mL$ is ample for all $m \geq 0$, so that the first Chern class of $L$, viewed as a hermitian form on $T_0X$, cannot have negative eigenvalues. But this implies that there is a line bundle $P \in \text{Pic}^0(X)$ such that $L + P$ descends to an ample line bundle on a quotient of $X$ and is therefore effective (cf. [3, Section 3.3] and [5, p. 95]).

We show next that on a simple abelian variety the existence of two algebraically independent line bundles already prevents $NS^+(X)$ from being finitely generated:

**Proposition 1.2.** Let $X$ be a simple abelian variety such that $NS^+(X)$ is finitely generated. Then $NS(X) \cong \mathbb{Z}$.

**Proof.** Assume to the contrary that $\text{rk} NS(X) > 1$ and choose ample line bundles $L_1$ and $L_2$ whose classes are not proportional in $NS_{\mathbb{Q}}(X)$. Consider then
the positive real number

\[ s = \inf \left\{ t \in \mathbb{R} \mid tL_1 - L_2 \text{ is nef} \right\}. \]

Here \( tL_1 - L_2 \) is considered as an \( \mathbb{R} \)-line bundle and nefness means that \( tL_1 C \geq L_2 C \) for every irreducible curve \( C \) in \( X \). We assert that

(1.2.1) \[ s \notin \mathbb{Q}. \]

Suppose to the contrary that \( s \) is rational and consider the line bundle

\[ L = sL_1 - L_2 \in \text{Pic}_\mathbb{Q}(X). \]

We choose an integer \( n \) such that \( ns \in \mathbb{Z} \). The line bundle \( nL \) is then algebraically equivalent to an effective (and integral) line bundle. But \( L \), and hence \( nL \), is certainly not ample, so that the kernel \( K(nL) \) of \( \phi_{nL} \) is of positive dimension. On the other hand, since \( L_1 \) and \( L_2 \) are not proportional, \( nL \) is not algebraically equivalent to 0, and hence \( K(nL) \) cannot be the whole of \( X \). So we find that the neutral component of \( K(nL) \) is a nontrivial abelian subvariety of \( X \), contradicting the simplicity assumption on \( X \). This establishes the assertion (1.2.1).

One checks next that, since \( NS^+(X) \) is finitely generated, its intersection with \( \mathbb{Z}[L_1] \oplus \mathbb{Z}[L_2] \) is finitely generated as well (cf. for example [9, Section 1.3]). Choose generators \( N_1, \ldots, N_k \) for this intersection. Let now \( 0 < \varepsilon \ll 1 \) and fix large integers \( p_1, p_2 \) such that

(1.2.2) \[ s < \frac{p_1}{p_2} < s + \varepsilon. \]

The line bundle

\[ A = p_1L_1 - p_2L_2 = p_2L + p_2 \left( \frac{p_1}{p_2} - s \right) L_1 \]

is then ample and therefore effective, so that we have \( A \equiv \sum_{i=1}^k \ell_i N_i \) with integers \( \ell_i \geq 0 \). Thus, writing \( N_i \equiv a_i L_1 - b_i L_2 \) with \( a_i, b_i \geq 0 \), we get

\[ \frac{p_1}{p_2} = \frac{\sum_{i=1}^k \ell_i a_i}{\sum_{i=1}^k \ell_i b_i}, \]

which, upon letting

\[ q = \inf \left\{ \frac{a_i}{b_i} \mid 1 \leq i \leq k \right\}, \]
yields the lower bound
\[ \frac{p_1}{p_2} \geq q. \]

But, due to the fact that \( s \) is irrational, which implies \( q > s \), and since \( \varepsilon \) can be taken arbitrarily small, this is incompatible with (1.2.2).

2. Classes on products. We study in this section effective divisor classes on the self-product \( X \times X \) of an abelian variety \( X \). Suppose for a moment that \( X \) is an elliptic curve. Then, since \( NS(X \times X) \) is of rank \( \geq 3 \), statement (2) of the introduction says that \( NS^+(X \times X) \) is not finitely generated. The argument given in [7] revolves around the alternating matrices associated with effective line bundles. Our aim here is to prove by different methods that the analogous statement holds in any dimension. To simplify the proof, we only consider abelian varieties of Picard number 1 for now, as the general case will follow with no effort from the proof of the theorem in Section 3.

**Proposition 2.1.** Let \( X \) be an abelian variety with \( NS(X) \cong \mathbb{Z}. \) Then \( NS^+(X \times X) \) is not finitely generated.

**Proof.** We denote by \( \iota_1, \iota_2, \iota_3 \) the closed embeddings of \( X \) in \( X \times X \) given by
\[
\iota_1: x \mapsto (x, 0), \quad \iota_2: x \mapsto (0, x), \quad \iota_3: x \mapsto (x, x).
\]

Further, fix an ample line bundle \( M \) whose algebraic equivalence class generates \( NS(X) \) and let \( n \) denote the dimension of \( X \). Supposing to the contrary that \( NS^+(X \times X) \) is finitely generated, our first claim is then the following boundedness statement:

\[ (*) \text{ There is an integer } c > 0 \text{ such that for all effective line bundles } B \text{ on } X \times X \text{ with } \iota_1^* B \equiv M \text{ the inequality } \]
\[
(\iota_2^* B - \iota_3^* B)^n \leq c
\]

holds.

To prove \( (*) \), choose a finite set of generators \( N_1, \ldots, N_k \) of \( NS^+(X \times X) \) and write
\[
B \equiv \sum_{i=1}^k b_i N_i
\]

with integers \( b_i \geq 0 \). Because of \( NS(X) = \mathbb{Z} \cdot [M] \) we have \( \iota_1^* N_i \equiv n_i M \) with
integers $n_i \geq 0$ for $1 \leq i \leq k$. The equivalences

$$M \equiv \nu_1^* B \equiv \sum_{i=1}^{k} b_i \nu_1^* N_i \equiv \left( \sum_{i=1}^{k} b_i n_i \right) M$$

show that there is a subscript $i_0$ with the property

$$b_i n_i = \begin{cases} 1, & \text{if } i = i_0 \\ 0, & \text{if } i \neq i_0. \end{cases}$$

If now $N$ is any effective line bundle on $X \times X$ with $\nu_1^* N = 0$, then it follows (for instance using the Seesaw Principle) that $N$ is a multiple of $pr_2^* M$, where $pr_2: X \times X \to X$ is the second projection, so that $\nu_2^* N = \nu_3^* N$. In particular we therefore obtain

$$\nu_2^* B - \nu_3^* B \equiv \sum_{i=1}^{k} b_i \left( \nu_2^* N_i - \nu_3^* N_i \right) \equiv \nu_2^* N_{i_0} - \nu_3^* N_{i_0},$$

so that (*) will hold, if we take the integer constant $c$ to be

$$c = \max \left\{ \left( \nu_2^* N_i - \nu_3^* N_i \right)^n \mid 1 \leq i \leq k \right\}.$$

Having established (*), the idea is now to construct a contradiction by exhibiting a sequence of nef line bundles $B_m, m \geq 1$, satisfying

$$(2.1.1) \quad \nu_1^* B_m \equiv M \quad \text{and} \quad \lim_{m \to \infty} \left( \nu_2^* B_m - \nu_3^* B_m \right)^n = \infty.$$

To this end we set

$$L_1 = pr_1^* M, \quad L_2 = pr_2^* M, \quad L_3 = \mu^* M,$$

where $pr_1, pr_2$ are the projections and $\mu$ is the addition map $X \times X \to X$. We then consider the line bundles

$$B_m = \nu_1^* (1 - m) L_1 + (m^2 - m) L_2 + m L_3.$$

One checks that with this choice of the bundles $B_m$ the conditions (2.1.1) are satisfied. So we will be done as soon as have shown that $B_m$ is nef for $m \geq 1$.

Now recall that an ample line bundle $A$ on $X \times X$ defines an injective homomorphism of vector spaces

$$NS_Q \left( X \times X \right) \to \operatorname{End}_Q \left( X \times X \right), \quad L \mapsto \phi_A^{-1} \phi_L,$$
whose image consists of the elements of \( \text{End}_\mathbb{Q} (X \times X) \) which are symmetric with respect to the Rosati involution \( f \mapsto f^* = \phi_A^{-1} f \phi_A \). In particular, for an endomorphism \( f \) of \( X \times X \), the pullback \( f^* A \) corresponds to the symmetric endomorphism \( f^* f \). Let now \( A = L_1 + L_2 \). One checks that, thanks to the fact that \( A \) is a product polarization, an endomorphism

\[
 f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} : X \times X \longrightarrow X \times X
\]

is symmetric if and only if both \( f_1 \) and \( f_4 \) are symmetric and \( f_2^* = f_3 \). Therefore the endomorphisms \( \alpha_1, \alpha_2, \alpha_3 \), which are defined by

\[
\begin{align*}
\alpha_1 &: (x,y) \mapsto (x,0) \\
\alpha_2 &: (x,y) \mapsto (0,x) \\
\alpha_3 &: (x,y) \mapsto (x+y,x+y)
\end{align*}
\]

are symmetric and, upon using \( \alpha_1^2 = \alpha_1, \alpha_2^2 = \alpha_2 \) and \( \alpha_3^2 = 2\alpha_3 \), one finds that they correspond to the line bundles \( L_1, L_2, L_3 \). This in turn shows that the line bundle \( B_m \) corresponds to the endomorphism

\[
\beta_m = (1-m)\alpha_1 + (m^2-m)\alpha_2 + m\alpha_3.
\]

The point is now that \( \beta_m^2 = (m^2+1)\beta_m \), so that \( \beta_m \mid \text{im} \beta_m \) is just multiplication by \( m^2+1 \). Therefore, if we denote by \( Y_m \) the complementary abelian subvariety of \( \text{im} \beta_m \), then the differential of \( \beta_m \) at the point 0 is the map

\[
d_0 \beta_m : T_0 \text{im} \beta_m \oplus T_0 Y_m \longrightarrow T_0 \text{im} \beta_m \oplus T_0 Y_m, \quad (u,v) \mapsto ((m^2+1)u,0),
\]

so that the analytic characteristic polynomial of \( \beta_m \) is

\[
P_m(t) = t^n \left( t - (m^2 + 1) \right)^n,
\]

But the alternating coefficients of \( P_m(t) \) are positive multiples of the intersection numbers \( A^i B_m^{2n-i} \), so that \( B_m \) is nef, as claimed. This completes the proof of the proposition.

3. The cone of curves and the nef cone of an abelian variety. Finally, we give in this section the proof of the theorem stated in the introduction. So let \( X \) be an abelian variety and denote by \( N_1(X) \) the vector space of numerical equivalence classes of real-valued 1-cycles on \( X \), and by \( NE(X) \) the convex cone in \( N_1(X) \) generated by irreducible curves. Through the intersection product the vector space \( N_1(X) \) is dual to the Néron-Severi vector space \( NS_\mathbb{R} (X) = NS(X) \otimes \mathbb{R} \).
The dual cone of $NE(X)$ is the nef cone

$$\text{Nef}(X) = \{ \lambda \in NS_R(X) \mid \lambda \xi \geq 0 \text{ for all } \xi \in NE(X) \},$$

which in the case of abelian varieties coincides with the effective cone (cf. Lemma 1.1). The dual of $\text{Nef}(X)$ in turn is the closed cone $\overline{NE}(X)$, so that one of these two cones is rational polyhedral if and only if the other is. By Gordon’s Lemma this is equivalent to the semi-group $NS^+(X)$ being finitely generated. (See e.g. [6, Theorem 14.1 and 19,20] for the elementary properties of cones used here.)

The idea is now, given an abelian variety, to first apply Poincaré’s Complete Reducibility Theorem, i.e., to decompose it up to isogenies into a product of powers of non-isogenous simple abelian varieties, and to apply Proposition 1.2 and Proposition 2.1 subsequently. One needs then that finite generation of $NS^+(X)$ is a property which is invariant under isogenies:

**Lemma 3.1.** [7] Let $X$ and $Y$ be isogenous abelian varieties. Then $NS^+(X)$ is finitely generated if and only if $NS^+(Y)$ is.

Since this observation is crucial for our approach, let us briefly indicate a proof, before we proceed to the proof of the theorem. So suppose that $NS^+(X)$ is finitely generated and that there is an isogeny $f: X \to Y$. Thanks to the fact that $f^*$ embeds $NS^+(Y)$ into $NS^+(X)$ and to the symmetry of the situation, it is enough to show that $f^* NS^+(Y)$ is finitely generated. Let then $N_1, \ldots, N_k$ be generators for $NS^+(X)$ and put for $1 \leq i \leq k$

$$n_i = \min \{ n \in \mathbb{Z} \mid nN_i \in f^* NS^+(Y) \}.$$

(The set on the right-hand side is nonempty, since $f$ is an isogeny.) Then $f^* NS^+(Y)$ is generated by the elements $n_1N_1, \ldots, n_rN_r$ together with those elements $\sum_{i=1}^k m_iN_i$, $0 \leq m_i < n_i$, which belong to $f^* NS^+(Y)$.

Consider now for an abelian variety $X$ its decomposition

$$X^{n_1} \times \cdots \times X^{n_r}$$

up to isogenies, where $X_1, \ldots, X_r$ are mutually nonisogenous simple abelian varieties and $n_1, \ldots, n_r$ are positive integers. In view of the remarks made at the beginning of this section, the theorem stated in the introduction will follow from:

**Theorem 3.2.** The semi-group $NS^+(X)$ is finitely generated if and only if $NS(X_i) \cong \mathbb{Z}$ and $n_i = 1$ for $1 \leq i \leq r$.

**Proof.** Suppose first that the conditions on the factors $X_i$ and the exponents $n_i$ are satisfied for $X$. By Lemma 3.1 we may assume that $X$ is the product
$X_1 \times \cdots \times X_r$. Fix for $1 \leq i \leq r$ an ample generator $N_i$ of $\text{Pic}(X_i)$ and let $A$ be the product polarization $A = \sum_{i=1}^r \text{pr}_i^* N_i$. Due to the fact that the $X_i$ are nonisogenous, we have

$$NS_Q\left( \prod_{i=1}^r X_i \right) \cong \text{End}_Q^s \left( \prod_{i=1}^r X_i \right) \cong \bigoplus_{i=1}^r \text{End}_Q^s(X_i) \cong \bigoplus_{i=1}^r NS_Q(X_i),$$

where $\text{End}_Q^s(\prod_{i=1}^r X_i)$ and $\text{End}_Q^s(X_i)$ denote the subgroups of symmetric endomorphisms with respect to the Rosati involutions associated with $A$ and $N_i$ respectively. Therefore

$$NS^+(X) = \bigoplus_{i=1}^r \mathbb{Z}^+ \cdot [N_i]$$

is finitely generated.

Now suppose conversely that $NS^+(X)$ is finitely generated. By Lemma 3.1 again, we may assume that $X$ is the product $X_1^{n_1} \times \cdots \times X_r^{n_r}$. Note that if $V_1$ and $V_2$ are varieties such that $NS^+(V_1 \times V_2)$ is finitely generated, then $NS^+(V_1)$ and $NS^+(V_2)$ are finitely generated as well. So in particular Proposition 1.2 applies to the factors $X_i$ and shows that we have $NS(X_i) \cong \mathbb{Z}$ for all $i$. Further, if we had $n_i > 1$ for some $i$, i.e. if a multiple factor $X_i$ appeared in the product decomposition of $X$, then $NS^+(X_1 \times X_i)$ would be finitely generated, which however is impossible according to Proposition 2.1. This completes the proof of the theorem. 

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