A note on tree factorization and no particle production

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Abstract
We prove factorization of the generating functional of connected tree diagrams by exploring that it is the Legendre transform of the action. This theorem is then applied to the example of a local relativistic interacting field theory in 2D with a single massive real scalar field that has no derivative couplings and no classical tadpole. In the process we streamline the proof that the assumption of no particle production leads to either the sinh-Gordon or the Bullough–Dodd model.

Keywords: quantum field theory, scattering amplitudes, factorization, integrable systems

1. Introduction
This paper consists of two parts. In the first part (section 2), we prove the factorization of a connected tree amplitude into two subamplitudes for an arbitrary quantum field theory with a local action, a unique classical field configuration (i.e. no instanton sectors), and no classical tadpoles, cf figure 1. Here it is a non-trivial fact that the two subamplitudes are in fact amplitudes in their own rights, and not, say, proper subsets of Feynman diagrams of those. To systematically manage the diagrammatical combinatorics, we explore in a novel way the Legendre transformation between the action and the generating functional of connected tree diagrams.

The paper will focus on tree diagrams, which are the leading diagrams in a semi-classical $\hbar$-expansion. For modern reviews of factorization at loop-level and its relation to (generalized) unitarity, see e.g. [1, 2].
In the second part (section 3), we apply tree factorization to a local relativistic interacting field theory in 2D with a single massive real scalar field that has no derivative couplings and no classical tadpole (2.9). We study when the tree amplitudes have no particle production, i.e. that the connected \( n \)-point tree amplitude \( M_n = 0 \) vanishes for \( n \geq 5 \). The latter approach dates back to the seminal works of [3, 4]. For more recent works, see e.g. [5–7]. By systematically considering conditions on \( M_n \) order-by-order in \( n \), we give a streamlined complete proof that there are only two such theories:

1. The sin(h)-Gordon model.
2. The Bullough–Dodd model [8].

### 2. Connected trees and factorization

#### 2.1. Generating functional \( W_{\text{tree}}^\text{ree}[J] \) of connected trees

Let there be given a theory with an local action functional \( S[\phi] \). (This is a relatively mild assumption even if modern on-shell methods [1, 2] downplay the role of the action.) The following theorem 2.1, although not new, is included to be self-contained.

**Theorem 2.1.** If the action \( S[\phi] \) does not depend on Planck’s constant \( \hbar \), and if the Euler–Lagrange equations

\[
J_k \approx -\frac{\delta S[\phi]}{\delta \phi^k} \quad \Leftrightarrow \quad \phi^k \approx \phi^k[J]
\]

(2.1)

have a unique solution for given sources \( J_k \) (and boundary conditions), then the Legendre transform of \( S[\phi] \) is the generating functional \( W_{\text{tree}}^\text{ree}[J] \)

\[
W_{\text{tree}}^\text{ree}[J] = S[\phi] + \int J_k \phi^k
\]

(2.2)

of connected tree diagrams. In particular

\[
J_k = -\frac{\delta S[\phi]}{\delta \phi^k} \quad \Leftrightarrow \quad \phi^k = \frac{\delta W_{\text{tree}}^\text{ree}[J]}{\delta J_k}.
\]

(2.3)

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1 We use DeWitt’s condensed notation, i.e. the index \( k \) is a collection of discrete and continuous indices, and repeated indices are summed/integrated over.
Proof of theorem 2.1.

\[
\exp\left\{ \frac{i}{\hbar} W_c[J] \right\} = Z[J] = \int \mathcal{D} \phi \frac{\phi}{\sqrt{\hbar}} \exp\left\{ \frac{i}{\hbar} \left( S[\phi] + J \phi^k \right) \right\} \\
\sim \text{Det} \left( \frac{1}{i} \frac{\delta^2 S[\phi]}{\delta \phi^m \delta \phi^n} \right)^{-1/2} \exp\left\{ \frac{i}{\hbar} \left( S[\phi] + J \phi^k \right) \right\} (1 + O(\hbar)).
\]

(2.4)

In the first equality, we used that the exponential of the generating functional \( W_c[J] \) of connected diagrams yields all diagrams, i.e. the path integral \( Z[J] \). At the last step we used the stationary phase/Wentzel–Kramers–Brillouin (WKB) approximation \( \hbar \to 0 \). Finally, use the \( \hbar/\text{loop-expansion} \) to deduce the theorem: The dominating contributions on the two sides of equation (2.4) is precisely equation (2.2).

Remark 2.2. The Legendre transformation (2.2) should not be conflated with the Legendre transformation

\[
\Gamma[\phi_{cl}] = W_c[J] - J_k \phi^k_{cl}
\]

between the quantum effective action \( \Gamma[\phi_{cl}] \) and the generating functional \( W_c[J] \) of connected diagrams, which is not relevant for this paper since \( W_c[J] \) includes loop diagrams.

2.2. Perturbative expansions

Let us expand perturbatively the action

\[
S[\phi] = \sum_{n=0}^{\infty} S_n[\phi], \quad S_n[\phi] = \frac{1}{n!} (S_n)_{k_1,...,k_n} \phi^{k_1} \cdots \phi^{k_n};
\]

(2.6)

the generating functional of connected trees

\[
W_{c,\text{tree}}[J] = \sum_{n=0}^{\infty} W_{c,\text{tree},n}[J], \quad W_{c,\text{tree},n}[J] = \frac{1}{n!} (W_{c,\text{tree}})_{k_1,...,k_n} J_{k_1} \cdots J_{k_n};
\]

(2.7)

and the generating functional of rooted trees/classical solution

\[
\phi^k[J] = \sum_{n=0}^{\infty} \phi_{n}^k[J], \quad \phi_{n}^k[J] = \frac{1}{n!} (\phi_{n}^k)_{k_1,...,k_n} J_{k_1} \cdots J_{k_n}.
\]

(2.8)

In particular, note that the subscript \( n \) in equations (2.6)–(2.8) denotes all terms of order \( n \). In what follows a subscript \( \geq n \) denotes all terms of order \( \geq n \).

Assumption 2.3. We will always assume that there are no classical tadpoles

\[
(S_1)_k = 0 \quad \Leftrightarrow \quad (W_{c,\text{tree}})_{k}^k = 0 \quad \Leftrightarrow \quad \phi^k_0 = 0.
\]

(2.9)

Then there are no connected vacuum trees

\[
W_{c,v,0} = S_0,
\]

(2.10)

apart from the action constant \( S_0 \); and the quadratic terms are given by the free propagator

\[
S_2[\phi] = -\frac{1}{2} \phi^k (\Delta_0^{-1})_{k\ell} \phi^\ell \quad \Leftrightarrow \quad W_{c,\text{tree},2}[J] = \frac{1}{2} J_{k} (\Delta_0)^{k\ell} J_{\ell} \quad \Leftrightarrow \quad \phi_{1}^k[J] = (\Delta_0)^{k\ell} J_{\ell}.
\]

(2.11)
It is straightforward to generate formulas for the tree diagrams in terms of a finite number of building blocks consisting of free propagators \((\Delta_0)^k\ell\) and \(n\)-vertices \((S_n)_{k_1 \ldots k_n}\) to each order in perturbation theory.

### 2.3. Vertex expansion of a rooted tree

**Theorem 2.4.**

\[
\frac{\delta W_{\text{tree}}^c[J]}{\delta J_k} = (\Delta_0)^k\ell \left( -\frac{\delta S_2[\phi]}{\delta \phi^\ell} + \frac{\delta S[\phi]}{\delta \phi^\ell} \right) \bigg|_{\phi = \phi[J]}
\]

\[= (\Delta_0)^k\ell \frac{\delta S_{\geq 3}[\phi]}{\delta \phi^\ell} \bigg|_{\phi = \phi[J]} + \text{non-factorizable trees}.
\]

**Remark 2.5.** Theorem 2.4 can be interpreted diagrammatically as a recursion relation: the root-leg of the rooted tree \(\delta W_{\text{tree}}^c, \geq 2\) is connected via a free propagator \(\Delta_0\) to a vertex in the action \(S_{\geq 3}\). The other lines of this vertex are either (i) an external leg or (ii) an internal line that is the beginning of a new rooted tree \(\delta W_{\text{tree}}^c, \geq 2\). One should sum over possible types of vertices in the action \(S_{\geq 3}\). For an example, see figure 13.

### 2.4. Tree factorization

Imagine that there are given two copies of the sources, say \(J_k\) and \(K_k\), with not necessarily the same values. Let us consider a connected tree \(\Gamma\) with the two types of sources \(J\) and \(K\). For each line/propagator \(\Delta_0\) in \(\Gamma\) the tree \(\Gamma = \Gamma'\Delta_0\Gamma''\) divides into two subtrees \(\Gamma'\) and \(\Gamma''\).

**Definition 2.6.** If \(\Gamma'\) does not depend on \(J\) and \(\Gamma''\) does not depend on \(K\) (or vice-versa), then the line/propagator \(\Delta_0\) is called factorizable. If a tree \(\Gamma\) has no factorizable lines, it is called non-factorizable.

We can now formulate the main theorem 2.7.

**Theorem 2.7.**

\[
W_{\text{tree}}^{c,J+K} - W_{\text{tree}}^{c,J} - W_{\text{tree}}^{c,K} = \frac{\delta W_{\text{tree}}^c[J]}{\delta J_k} (\Delta_0^{-1})_k\ell \frac{\delta W_{\text{tree}}^c[K]}{\delta K_\ell} + \text{non-factorizable trees}.
\]

**Remark 2.8.** A tree with source \(J + K\) is to be viewed as a sum of trees with two types of sources \(J\) and \(K\) via the binomial formula. Examples of a term in such binomial expansion is illustrated in figure 2 and 3.

**Remark 2.9.** The main impact of theorem 2.7 is the non-trivial fact that the subdiagrams on the two sides of the (inverse) propagator \(\Delta_0^{-1}\) in equation (2.13) are the full set of rooted tree diagrams, and not, say, proper subsets thereof. Theorem 2.7 can in principle be used to recursively build tree amplitudes.


**Proof of theorem 2.7.** Define mixed terms

\[
\phi^k_{\geq 2}[J,K] := \phi^k_{\geq 2}[J+K] - \phi^k_{\geq 2}[J] - \phi^k_{\geq 2}[K].
\]  

Diagrammatically equation (2.14) consists of rooted trees with both \( J \) and \( K \) leaves. Then we may expand

\[
\phi^k[J+K] \overset{(2.9)}{=} \phi^k[J+K] + \phi^k_{\geq 2}[J+K] \\
= \phi^k[J] + \phi^k[K] + \phi^k_{\geq 2}[J+K] \\
\overset{(2.14)}{=} \phi^k[J] + \phi^k[K] + \phi^k_{\geq 2}[J,K].
\]  

Hence

\[
W_{\text{tree}}^{\text{tree}}[J+K] \overset{(2.2)}{=} S_{\geq 2} [\phi[J+K]] + (J_k + K_k) \phi^k[J+K] \\
\overset{(2.15)}{=} S_{\geq 2} [\phi[J] + \phi[K] + \phi_{\geq 2}[J,K]] + (J_k + K_k) \phi^k[J+K] \\
\overset{(2.15)}{=} S_{\geq 2} [\phi[J] + \phi[K]] + \frac{S_{\geq 2}[\phi]}{\delta \phi^k} \left|_{\phi=\phi[J]+\phi[K]} \right. \phi^k_{\geq 2}[J,K] \\
+ O \left( \phi_{\geq 2}[J,K]^2 \right) + (J_k + K_k) \left( \phi^k[J] + \phi^k[K] + \phi^k_{\geq 2}[J,K] \right) \\
\overset{(2.17)}{=} S_{\geq 2} [\phi[J]] + S_{\geq 2} [\phi[K]] + \frac{S_{\geq 2}[\phi]}{\delta \phi^k} \phi^k[J] + \phi^k[K] + \phi^k_{\geq 2}[J,K] \\
+ \phi^k[J] (\Delta_0^{-1})_{\ell \ell'} \phi^k[J] + O (\phi^k[J]^2) \\
+ \frac{S[\phi][J]}{\delta \phi^k} + \frac{S[\phi][K]}{\delta \phi^k} + O (\phi^k[\phi[K]]) + J_k + K_k \phi^k_{\geq 2}[J,K] \\
+ O \left( \phi_{\geq 2}[J,K]^2 \right) + (J_k + K_k) \left( \phi^k[J] + \phi^k[K] \right) \\
\overset{(2.3)}{=} S_{\geq 2} [\phi[J]] + S_{\geq 2} [\phi[K]] + \phi^k[J] (\Delta_0^{-1})_{\ell \ell'} \phi^k[J] \\
+ J_k \phi^k[J] + K_k \phi^k[K] + \text{non-factorizable trees} \\
\overset{(2.2)}{=} W_{\text{tree}}^{\text{tree}}[J] + W_{\text{tree}}^{\text{tree}}[K] + \frac{\delta W_{\text{tree}}^{\text{tree}}[J]}{\delta J_k} (\Delta_0^{-1})_{\ell \ell'} \frac{\delta W_{\text{tree}}^{\text{tree}}[K]}{\delta K_{\ell}} \\
+ \text{non-factorizable trees}.
\]
In equation (2.16) we have used a generalization of a binomial Taylor expansion
\[ f(x+y) = f(x) + f(y) - f(0) + O(xy), \]
\[ f(x+y) = f(x) + f(y) - f(0) + [f'(x) - f'(0)]y + x[f'(y) - f'(0)] - x^2f''(0)y + O(x^2y^2), \] (2.17)
and that if either
\[ O(\phi J^2[K]^2), \quad O(\phi J^2[K]^2), \quad O(\phi J^2[K]^2), \]
(2.18)
are attached to a vertex then the tree diagram must be non-factorizable.

2.5. On-shell tree-level scattering amplitude

Amputated/stripped amplitudes \( M_n(k_1, \ldots, k_n) \) of connected trees are obtained from the generating functional \( W^{\text{tree}}[J] \) as follows:
\[ \frac{i}{\hbar} M_n(k_1, \ldots, k_n) = \frac{1}{(2\pi)^d} \delta^d(k_1 + \ldots + k_n) = \bar{\Delta}_0^{-1}(k_1) \frac{\delta}{\delta J(k_1)} \cdots \bar{\Delta}_0^{-1}(k_n) \frac{\delta}{\delta J(k_n)} W^{\text{tree}}[J] \bigg|_{j=0}, \]
(2.19)
where \( n \geq 3 \). Spacetime translation symmetry implies total wavevector conservation
\[ k_1 + \ldots + k_n = 0. \] (2.20)
Let us analytically continue \( M_n(k_1, \ldots, k_n) \) for complex outgoing wavevectors \( k_i (= \text{momenta}/\hbar) \). The \( n \)-point tree-amplitude \( M_n \) is a rational function of \( k_i \) (before \(^4\) use of equation (2.20)). It will become crucial in what follows that poles in \( M_n \) can only come from poles in internal propagators.

On-shell condition\(^5\):
\[ k_i^2 + m_i^2 \approx 0. \] (2.21)

Theorem 2.7 now transcribes into the following tree factorization theorem 2.10, cf figure 1.

**Theorem 2.10. Tree factorization: generically\(^6\)**
\[ M_n(k_1, \ldots, k_n) \sim M_{r+1}(k_1, \ldots, k_r, \cdot) \bar{\Delta}_0(k_1 + \ldots + k_r) M_{n+1-r}(k_{r+1}, \ldots, k_n, \cdot) + \text{finite terms} \]
\[ \to \infty \]
for \( (k_1 + \ldots + k_r)^2 + m_r^2 \to 0 \)

if the factorization channel is kinematically unique.

\(^2\) The tildes denote Fourier transform.
\(^3\) The wavevector dependence of the on-shell amplitudes \( M_n(k_1, \ldots, k_n) \) and its singularities can take different function forms due to equations (2.20) and (2.21). In particular \( M_n(k_1, \ldots, k_n) \) can develop branch cuts and multivaluedness \(^7\).
\(^4\) We use the Minkowski signature \((-\,+,\,+,\ldots,\,+)\). The mass parameters \( m_i \) are assumed to have dimension of inverse length. The \( \approx \) symbol denotes in principle an on-shell equality, although we will not always use the notation.
\(^5\) It is implicitly assumed that the wavevectors \( (k_1, \ldots, k_n) \) are kept away from the poles \( \sum_i k_i^2 + m_i^2 = 0 \) of all the other possible propagators. Here \( i \subseteq \{1, \ldots, n\} \) denotes an index set different from \( \emptyset \) and \( \{1, \ldots, r\} \), and their complement sets.
Remark 2.11. Be aware that on one hand a mixed $W_e^{\text{tree}}$ correlator function has always a unique $JK$-factorization, while on the other hand the limit $(k_1 + \ldots + k_r)^2 + m^2 \to 0$ in the on-shell tree amplitude $M_n(k_1, \ldots, k_n)$ may correspond to several\footnote{After diagonalizing the mass matrix, there is in principle a sum over possible degenerate masses that could lead to non-unique factorization channels. Apart from possible degenerate masses, non-unique factorization channels are mainly an issue in low dimensions $d < 2$.} factorization channels simultaneously if the number of external legs is low. Hence the latter may not be factorizable, cf. counterexamples in sections 3.6.1 and 3.7.2.

3. Example: no particle production in 2D

We now apply the tree factorization theorems of section 2 to a scalar field theory in 2D with no particle production.

3.1. Scalar field theory in 2D

The Lagrangian density for a local relativistic interacting field theory in 2D with a single massive real scalar field $\phi$ that has no derivative couplings and no classical tadpole (2.9) is

$$\mathcal{L} = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V = \partial_{+} \phi \partial_{-} \phi - V, \quad V = \frac{1}{2} m^2 \phi^2 + \sum_{n \geq 3} \frac{\lambda_n}{n!} \phi^n,$$

$$m > 0, \quad \lambda_n = m^2 g_n, \quad \tilde{\Delta}_0(k) = \frac{1}{k^2 + m^2 - i\epsilon}. \tag{3.1}$$

The $i\epsilon$ prescription is implicitly implied from now on. We exclude interaction terms with space-time derivatives. All coupling constants $\lambda_n = m^2 g_n$ have dimension of inverse quadratic length, i.e. are relevant/renormalizable\footnote{The only divergencies in diagrams come from self-loops, which are logarithmically divergent. These can be removed in the interaction picture of the operator formalism via normal ordering [9].}. The Feynman rule for an $n$-vertex is $-\frac{i}{\hbar} \lambda_n$, while the Feynman rule for an internal propagator is $\frac{i}{\hbar} \tilde{\Delta}_0(k)$.

Since the number of vertices minus the number of propagators is always 1 in each amputated connected tree diagram, one may show that the amplitude $M_n \in \mathbb{R}$ is real for real momenta (granted that a physical sector exists and ignoring the $i\epsilon$). This is confirmed in examples below.

It will be convenient to use light-cone coordinates

$$dx^2 = -dr^2 + dx^2 = -2dx^+ dx^-, \quad x^\pm = \frac{t \pm x}{\sqrt{2}},$$

$$2k^+ k^- = -k_i^2 \approx m^2, \quad (k^+_i, k^-_i) \approx \frac{m}{\sqrt{2}} (a_i, a_i^{-1}), \quad a_i \in \mathbb{C} \setminus \{0\}. \tag{3.2}$$

Remark 3.1. No particle production here means that the outgoing particles are a permutation of the ingoing particles. This implies that $M_{n \geq 5} = 0$ must vanish. E.g. the connected $3 \to 3$ tree-amplitude must vanish because it is an analytic continuation of, say, the connected $2 \to 4$ tree-amplitude.
3.2. Propagators

3.2.1. Two-propagator. The two-propagator is

\[ \Delta_0(k_i + k_j) \approx \frac{-m^2}{(k_i + k_j)^2 + m^2} \approx \frac{1}{(a_i + a_j)(a_i^{-1} + a_j^{-1})} - 1 \]

\[ \approx \frac{a_i a_j}{(a_i + a_j)^2 - a_i a_j} = \frac{a_i a_j}{a_i^2 + a_i a_j + a_j^2} \]

\[ = \frac{a_i a_j}{(a_i - e^{2\pi i/3} a_j)(a_i - e^{-2\pi i/3} a_j)} =: \Delta_{ij}, \tag{3.3} \]

cf figure 4. Poles:

\[ a_i = e^{\pm 2\pi i/3} a_j. \tag{3.4} \]

3.2.2. Three-propagator. The three-propagator is

\[ \Delta_0(k_i + k_j + k_k) \approx \frac{-m^2}{(k_i + k_j + k_k)^2 + m^2} \]

\[ \approx \frac{1}{(a_i + a_j + a_k)(a_i^{-1} + a_j^{-1} + a_k^{-1})} - 1 \]

\[ \approx \frac{a_i a_j a_k}{(a_i + a_j + a_k)(a_i + a_j + a_k)} =: \Delta_{ijk}, \tag{3.5} \]

cf figure 5. Poles:

\[ a_i + a_j = 0. \tag{3.6} \]

Residue:

\[ \text{Res}(\Delta_{ijk}, a_i + a_j) = \frac{a_i^2 a_k}{a_i^2 - a_k^2}. \tag{3.7} \]

3.3. Three-point amplitude

The three-point tree amplitude is

\[ \frac{i}{\hbar} M_3(k_1, k_2, k_3) = -\frac{i}{\hbar} \lambda_3 \Rightarrow M_3(k_1, k_2, k_3) = -\lambda_3, \tag{3.8} \]
A two-propagator \( \Delta_0(k_i + k_j) = -m^{-2}\Delta_{ij} \) is by definition attached to two external legs.

A three-propagator \( \Delta_0(k_i + k_j + k_k) = -m^{-2}\Delta_{ijk} \) is by definition attached to three external legs.

cf figure 6. Momentum conservation (2.20) implies complex (and hence unphysical) lightcone-momenta

\[
\begin{align*}
\begin{cases}
a_1 + a_2 + a_3 = 0 \\
a_1^{-1} + a_2^{-1} + a_3^{-1} = 0
\end{cases}
\Rightarrow (a_1, a_2, a_3) \propto \left(1, e^{\pm 2\pi i/3}, e^{\mp 2\pi i/3}\right).
\end{align*}
\]

We see that the three-point function \( \mathcal{M}_3 \) is constant but unphysical because the momenta (3.9) are not real. In particular, \( \mathcal{M}_3 \) has no particle production despite being non-zero.

### 3.4. Connected on-shell tree amplitudes

An \( n \)-point function \( \mathcal{M}_n(k_1, \ldots, k_n) \) in 1+1D depends on \( 2n \) variables. We will always assume that the on-shell condition (3.2) has been applied. Then \( \mathcal{M}_n \) depends rationally on \( n \) lightcone-momenta \( (a_1, \ldots, a_n) \).

The tree amplitudes \( \mathcal{M}_n \) are Lorentz/boost invariant \( a_i \to e^\alpha a_i \) since they are built from Lorentz invariant propagators and vertices.

We can eliminate a pair of lightcone-momenta, say \( a_p \) and \( a_q \), \( 1 \leq p < q \leq n \), because of momentum conservation (2.20),

\[
\begin{align*}
a_p + a_q &= s := - \sum_{i \notin \{p,q\}} a_i, \\
a_p^{-1} + a_q^{-1} &= r := - \sum_{i \notin \{p,q\}} a_i^{-1}.
\end{align*}
\]

\( \mathcal{M}_n \) is constant but unphysical because the momenta (3.9) are not real. In particular, \( \mathcal{M}_3 \) has no particle production despite being non-zero.
Equation (3.10) leads to a second-order equation with solutions

\[ \begin{align*}
    a_p & = \frac{s}{2} \pm \sqrt{\frac{s^2}{4} - \frac{s}{r}} \quad \text{if} \quad r \neq 0; \\
    a_p + a_q & = 0 \quad \text{if} \quad r = 0 = s;
\end{align*} \tag{3.11} \]

a one-parameter solution

\[ a_p + a_q = 0 \quad \text{if} \quad r = 0 = s; \tag{3.12} \]

and

\[ \text{no solution if} \quad r = 0 \neq s. \tag{3.13} \]

The special case (3.12) is a kinematically disconnected case, which imposes two independent sets of momentum conservation laws, i.e. we can in principle eliminate an extra pair of momentum variables.

When we generically talk about the analytic dependence of \( \mathcal{M}_n \), it is implicitly understood that a pair \((a_p, a_q)\) has been eliminated via the generic case (3.11), i.e. that \( \mathcal{M}_n \) depends on \( \{a_1, \ldots, a_n\} \setminus \{a_p, a_q\} \). Let’s call this a representative of \( \mathcal{M}_n \). It is then possible to simplify the representative to the form

\[ \mathcal{M}_n = \mathcal{M}_n^{(0)} \pm \sqrt{\frac{s^2}{4} - \frac{s}{r}} \mathcal{M}_n^{(1)}, \tag{3.14} \]

where \( \mathcal{M}_n^{(0)} \) and \( \mathcal{M}_n^{(1)} \) are rational functions. Since there is only one type of particle species, the representative (3.14) is symmetric under the permutation \( a_p \leftrightarrow a_q \), so \( \mathcal{M}_n^{(1)} = 0 \) must be zero, and hence the square root (3.14) must vanishes. Therefore the representative retains its rational form [5].

Although the functional form of the amplitude depends on the representative, the values (in particular, zeroes and poles) of the amplitude do not. It will become important in what follows that if \( \mathcal{M}_n \) does not have poles, it must be constant.

3.5. Four-point amplitude

The four-point tree amplitude is

\[ \mathcal{M}_4(k_1, k_2, k_3, k_4) = -\lambda_4 + \lambda_3^2 \left( \bar{\Delta}_0(k_1 + k_2) + \bar{\Delta}_0(k_2 + k_3) + \bar{\Delta}_0(k_3 + k_1) \right) \tag{3.15} \]

\[ \Rightarrow \quad -m^{-2} \mathcal{M}_4(k_1, k_2, k_3, k_4) = g_4 + g_3^2(\Delta_{12} + \Delta_{23} + \Delta_{13}), \tag{3.16} \]

8 Each kinematically connected component satisfies a momentum conservation law. Note that the underlying Feynman diagrams do not need to be actually disconnected.

9 The reader may ponder what happens if we only demand that the absolute square \( |\mathcal{M}_4|^2 \) is symmetric under the permutation \( a_p \leftrightarrow a_q \). This would potentially allow \( \mathcal{M}_4 \) to be symmetric in a \( \pm \) double-valued sense. Let us here give an elementary argument that this is not possible. We have already discussed and can therefore exclude the single-valued case. The \( \pm \) double-valuedness would then force \( \mathcal{M}_4^{(0)} = 0 \) to be zero in equation (3.14). However, the \( \pm \) in equation (3.14) would be incompatible with a continuous limit \( r \to 0 \) with \( \frac{s}{r} \) as constant into the (possible fully) kinematically disconnected case (3.12) unless the \( \pm \) branches coincide in the limit. (A partially kinematically disconnected case could harbour extra \( \pm \) square root branches, but not the fully kinematically disconnected case.) In other words, this implies that the representative \( \mathcal{M}_4 \) should vanish in the special case (3.12), i.e. \( \mathcal{M}_4 \) should contain a factor \( (a_p + a_q) \). However, the amplitude is not expected to vanish in the disconnected limit. Contradiction. So \( \mathcal{M}_4 \) cannot be \( \pm \) double-valued after all. [7].

10 For more general Toda theories, if there is more than 1 particle species, square roots may be present, and the proof in complex function theory needs to be adapted to the global topology of momentum space [7].
Figure 6. Connected three-point tree diagrams.

cf figure 7. Momentum conservation (2.20) restricts the four lightcone-momenta \((a_1, a_2, a_3, a_4)\) to two independent parameters:

\[
\begin{align*}
\{ a_1 + a_2 + a_3 + a_4 &= 0, \\
    a_1^{-1} + a_2^{-1} + a_3^{-1} + a_4^{-1} &= 0 \} \quad \Rightarrow \quad (a_1, a_2, a_3, a_4) \text{ are 2 pairs of opposite values } a_i + a_j = 0.
\end{align*}
\]

(3.17)

Equation (3.17) implies that the four-point function \(M_4\) has no particle production. If we e.g. assume that \(a_1 + a_3 \approx 0\), then the sum over exchange diagrams/\(stu\) channels simplifies to

\[
\begin{align*}
    a_1 + a_3 \approx 0 \quad \Rightarrow \quad \Delta_{12} + \Delta_{23} + \Delta_{31} &\approx \frac{a_1 a_2}{(a_1 + a_2)^2 - a_1 a_2} - \frac{a_1 a_2}{(a_1 - a_2)^2 + a_1 a_2} - 1 \\
    &= \frac{a_1 a_2}{a_1^2 + a_1 a_2 + a_2^2} - \frac{a_1 a_2}{a_1^2 - a_1 a_2 + a_2^2} - 1 \\
    &= -\frac{2a_1^2 a_2^2}{(a_1^2 + a_2^2)^2 - a_1^2 a_2^2} - 1.
\end{align*}
\]

(3.18)

\(M_4\) is non-constant and has unphysical poles at \(a_1 = \pm e^{\pm 2\pi i/3} a_5\) caused by a two-propagator iff \(g_3 \neq 0\). \(M_4\) factorizes \(M_4 \sim M_3 \Delta_0 M_3\) on these poles, cf theorem 2.10.

3.6. Five-point amplitude

3.6.1. No single-channel \(M_5 \sim M_4 \Delta_0 M_3\) factorization. Consider e.g. a pole \(a_4 = e^{\pm 2\pi i/3} a_5\) associated with a two-propagator for the channel 45. Define

\[
a_{45} := a_4 + a_5 = -e^{\mp 2\pi i/3} a_5.
\]

(3.19)

Momentum conservation (2.20) then implies

\[
\Rightarrow \quad (a_1, a_2, a_3, a_{45}) \text{ are two pairs of opposite values } a_i + a_j = 0.
\]

(3.20)

In other words: \(\exists i \in \{1, 2, 3\} : \ a_i = e^{\mp 2\pi i/3} a_5\). This leads to a simultaneous pole for another channel \(i5\).
So factorization is not useful for the calculation of $M_5$. However, we do not have to actually calculate $M_5$ itself: It is enough to calculate the residues of the poles $[5]$.

3.6.2. Five-point amplitude. The five-point tree amplitude is

$$M_5(k_1, k_2, k_3, k_4, k_5) = -\lambda_5 + \lambda_3 \lambda_4 \sum_{i,j,k \in \{1,\ldots,5\}} \delta_0(k_i + k_j + k_k) \sum_{\ell,m \notin \{i,j,k\}} 1 \left\{ \begin{array}{c} \delta_0(k_i + k_j + k_m) \\ \delta_0(k_\ell + k_m) \\ \delta_0(k_k + k_\ell) \end{array} \right\}$$

$$\Rightarrow -m^{-2} M_5(k_1, k_2, k_3, k_4, k_5) = g_5 + g_3 g_4 \sum_{i,j,k \in \{1,\ldots,5\}} \delta_0$$

$$+ g_3^2 \sum_{i=1}^5 \sum_{j,k \notin \{i\}} \sum_{\ell,m \notin \{i,j,k\}} \left\{ \begin{array}{c} \delta_0(k_i + k_j + k_m) \\ \delta_0(k_\ell + k_m) \\ \delta_0(k_k + k_\ell) \end{array} \right\}$$

(3.21)

$$\Rightarrow$$

$$\Rightarrow (a_3, a_4, a_5) \propto (1, e^{\pm 2\pi i/3}, e^{\mp 2\pi i/3}).$$

(3.22)

3.6.3. Poles. All propagators can be seen as three-propagators, so all poles are of the form $a_1 + a_j = 0$ (possibly after using on-shell conditions). Because of $S_5$-permutation symmetry, we may w.l.o.g. consider the pole $a_1 + a_2 = 0$. Momentum conservation (2.20) then implies

$$\Rightarrow$$

$$\Rightarrow (a_3, a_4, a_5) \propto (1, e^{\pm 2\pi i/3}, e^{\mp 2\pi i/3}).$$
3.6.4. **Residue of $g_3g_4$ diagram (3.22)**. Then $(i,j) = (1,2)$ in equation (3.22).

\[
\sum_{k \not\in \{1,2\}} \text{Res}(\Delta_{12k}, a_1 + a_2) = \sum_{k \in \{3,4,5\}} \frac{a_i^2 a_k}{a_i^2 - a_k^2} \\
= \frac{a_i^2}{D} \sum_{i,j,k \in \{3,4,5\}} (a_i^2 - a_j^2) (a_i^2 - a_k^2) a_k \\
= - \frac{a_i^2}{D} \sum_{i,j,k \in \{3,4,5\}} (a_i^2 + a_j^2) a_k \\
= \frac{a_i^2}{D} \sum_{k \in \{3,4,5\}} a_k^3 \\
= 3 \frac{a_i^4}{D} a_3 a_4 a_5,
\]

where we have defined

\[
D := \prod_{k=3}^{5} (a_i^2 - a_k^2). 
\]

3.6.5. **Residue of $g_3^3$ diagram (3.22) where $i \not\in \{1,2\}$**. Then $(j,k) = (1,2)$ or $(\ell, m) = (1,2)$ in equation (3.22). They give the same contribution, so assume the first option and multiply with 2:

\[
\sum_{i \not\in \{1,2\}} \sum_{\ell \not\in \{1,2\}, \ell \not= i} \text{Res}(\Delta_{1\ell2} \Delta_{12}, a_1 + a_2) = - \sum_{i=3}^{5} \text{Res}(\Delta_{1\ell2}, a_1 + a_2) = -3 \frac{a_i^4}{D} a_3 a_4 a_5,
\]

which we already calculated in equation (3.24).

3.6.6. **Residue of $g_3^3$ diagram (3.22) where $i \in \{1,2\}$**. Then $(i,j) = \{1,2\}$ or $(i, \ell) = \{1,2\}$ in equation (3.22). They give the same contribution, so assume the first option and multiply with 2:

\[
\sum_{i=1}^{2} \sum_{k \not\in \{1,2\}} \text{Res}(\Delta_{12k} \sum_{\ell \not\in \{1,2\}} \Delta_{1\ell m}, a_1 + a_2) \\
= \sum_{k=3}^{5} \frac{a_i^2 a_k}{a_i^2 - a_k^2} \sum_{\ell \not\in \{1,2\}} \Delta_{1\ell m} a_1 + a_2 \\
= \frac{a_i^2}{D} \sum_{k=3}^{5} a_k \sum_{\ell \not\in \{1,2\}} \Delta_{1\ell m} a_1 + a_2 \\
= -2a_i(a_1 + a_2) a_3 a_4 a_5,
\]

where $a_1 = a_2 = a_3 = a_4 = a_5 = 0$. 

\[
= -6 \frac{a_i^4}{D} a_3 a_4 a_5.
\]
3.6.7. Total residue of the five-point function (3.22).

\[-m^{-2} \text{Res} \left( M_5(k_1, k_2, k_3, k_4, k_5), a_1 + a_2 \right)^{(3.24) + (3.26) + (3.27)} \frac{d^4}{D} a_3 a_4 a_5 (3g_3 g_4 - 9g_3^3).\]

Equation (3.28) yields that there are two possible ways to remove poles:

1. Case \( g_3 = 0 \).
2. Case \( g_4 = 3g_3^2 \).

We assume from now on that \( M_5 \) has no poles and hence is constant. In fact we will assume that \( g_5 \) has been adjusted such that \( M_5 = 0 \) has no particle production.

3.7. Six-point amplitude

3.7.1. \( M_6 \sim M_5 \Delta_0 M_5 \) factorization. There are no poles associated with a two-propagator since \( M_5 = 0 \), cf theorem 2.10 and figure 9.

3.7.2. No single-channel \( M_6 \sim M_4 \Delta_0 M_4 \) factorization. A single pole clearly does not uniquely specify a factorization channel. Next consider e.g. a double pole \( a_4 = -a_5 = a_6 \) associated with a three-propagator for the channel 456. Define

\[ a_{456} := a_4 + a_5 + a_6 = a_4. \]  

Momentum conservation (2.20) then implies

\[ (a_1, a_2, a_3, a_{456}) \] are two pairs of opposite values \( a_i + a_j = 0 \).

In other words: \( \exists i \in \{1, 2, 3\} : a_i = -a_4 \). This is a simultaneous double pole for another channel, e.g. 45.

3.7.3. Six-point diagrams. We cannot rely on factorization, so it is necessary to calculate the residues of the poles of \( M_6 \). The only possible poles are associated with a three-propagator (3.5), i.e. of the form \( a_i + a_j = 0 \). We therefore can exclude diagrams without three-propagators, cf figure 10. We only have to consider diagrams that contain three-propagators,
Figure 9. $M_6 \sim M_1 \Delta_0 M_3$ factorization.

Figure 10. Connected six-point tree diagrams with no three-propagator.

cf figure 11. It is straightforward to check that the six-point diagrams with three-propagators group together according to their three-propagator in an $M_4 \Delta_0 M_4$ block format. (Compare with figures 2 and 3.) In other words,

$$M_6(k_1,k_2,k_3,k_4,k_5,k_6)_{\Delta_0}$$

$$= \frac{1}{2} \sum_{i<j<k} M_4(k_i,k_j,k_k) \Delta_0(k_i+k_j+k_k) \sum_{\ell<m<n \in \{i,j,k\}} M_4(k_{\ell},k_{m},k_{n}). \quad (3.31)$$

$$\Rightarrow -m^{-2}M_6(k_1,k_2,k_3,k_4,k_5,k_6)_{\Delta_0}$$

$$= \frac{1}{2} \sum_{i<j<k} \left( M_4(k_i,k_j,k_k) \Delta_{ijk} \sum_{\ell,m,n \in \{i,j,k\}} M_4(k_{\ell},k_{m},k_{n}) \right). \quad (3.32)$$

3.74. Poles. Because of $S_6$-permutation symmetry, we may w.l.o.g. consider the pole $a_1 + a_2 = 0$. Momentum conservation then implies

$$\Rightarrow (a_3,a_4,a_5,a_6) \text{ are two pairs of opposite values } a_p + a_q = 0. \quad (3.33)$$
The dominant diagrams are, cf figure 12,

\[ -m^{-2}M_6(k_1, k_2, k_3, k_4, k_5, k_6) \sim \sum_{k=3}^{6} \mathcal{M}_4(k_1, k_2, k_k, \cdots) \Delta_{12k} \sum_{\ell, m, n \notin \{1,2,k\}}^\ell < m < n \mathcal{M}_4(k_\ell, k_m, k_n, \cdots) \]

(3.34)

\[ \Rightarrow -m^{-2}\text{Res}(M_6(k_1, k_2, k_3, k_4, k_5, k_6), a_1 + a_2) \]

\[ = \sum_{k=3}^{6} \mathcal{M}_4(k_1, k_2, k_k, \cdots) \frac{a_k^2}{a_1^2 - a_k^2} \sum_{\ell, m, n \notin \{1,2,k\}}^\ell < m < n \mathcal{M}_4(k_\ell, k_m, k_n, \cdots) \]

(3.35)

\[ = \sum_{k=3}^{6} \mathcal{M}_4(k_1, k_2, k_k, \cdots) \frac{a_k^2}{a_1^2 - a_k^2} \mathcal{M}_4(k_3, k_4, k_5, k_6) \]

\[ = \text{four terms cancel in two opposite pairs} = 0. \]

Hence \(\mathcal{M}_6\) has no poles and is constant. In fact we will assume that \(g_6\) has been adjusted such that \(\mathcal{M}_6 = 0\).
3.8. Higher-point amplitudes $\mathcal{M}_{n\geq 7} = 0$

The vanishing of higher-point amplitudes follows via induction in $n \geq 6$. Assume that $\mathcal{M}_{5\leq n \leq 6} = 0$. We want to prove that $\mathcal{M}_{n+1}$ cannot have any poles. The poles must come from internal propagators $\Delta_0$ via a single-channel factorization $\mathcal{M}_{n+1} \sim \mathcal{M}_r \Delta_0 \mathcal{M}_{n+3-r}$, where $r \in \{3, \ldots, n\}$, cf theorem 2.10. It is easy to check that at least one of the two subamplitudes already vanishes. So $\mathcal{M}_{n+1}$ is a constant. Now adjust $g_{n+1}$ so that $\mathcal{M}_{n+1} = 0$ vanishes. □

3.9. Multi-Regge limit

So far we have proven inductively in the number $n \geq 5$ of external legs that the $n$-point amplitude $\mathcal{M}_n$ is a constant (i.e. independent of the external momenta), and that there exists a unique value for the coupling constant $g_n$ such that $\mathcal{M}_n = 0$ vanishes. To determine this distinguished value $g_n$, we need to calculate $\mathcal{M}_n$ explicitly, i.e. we should sum over all possible tree diagrams. It turns out that this sum simplifies considerably in a so-called multi-Regge limit, cf statement 3.3 below.

The on-shell $n$-point amplitude $\mathcal{M}_n(k_1, \ldots, k_n)$ has in general $n-2$ independent lightcone-momenta $(a_1, \ldots, a_n)$ after we take momentum conservation (2.20) into account, cf section 3.4.

**Definition 3.2.** The **multi-Regge limit** is a one-parameter family of lightcone-momenta [6]

$$a_i = x^{-2}, \quad i \in \{2, \ldots, n-1\}, \quad n \geq 4,$$

(3.36)

where $x \gg 1$ is a free parameter.

The first and the last lightcone-momentum $a_1$ and $a_n$ are fixed (up to a two-fold ambiguity) by the momentum conservation (2.20):

$$-a_1 - a_n = \sum_{i=2}^{n-1} a_i = \sum_{i=2}^{n-1} x^{i-2} = x^{n-3}(1 + \mathcal{O}(x^{-1})), \quad (3.37)$$

and

$$-a_1^{-1} - a_n^{-1} = \sum_{i=2}^{n-1} a_i^{-1} = \sum_{i=2}^{n-1} x^{2-i} = 1 + \mathcal{O}(x^{-1}),$$

which leads to the same second-order equation for $a_1$ and $a_n$, cf equation (3.10). We fix the ambiguity of the second branches such that

$$a_1 = -1 + \mathcal{O}(x^{-1}), \quad a_n = -x^{n-3}(1 + \mathcal{O}(x^{-1})).$$

(3.38)
Consider an internal $r$-propagator in the multi-Regge limit

$$-\Delta_I := m^2 \bar{\Delta}_0 \left( \sum_{i \in I} k_i \right) = m^2 \bar{\Delta}_0 \left( - \sum_{j \in J} k_j \right)$$

where the index sets are

$$I = \{ i_1, i_2, \ldots, i_r \}, \quad J = \{ j_1, j_2, \ldots, j_{r-n} \},$$

$$r \in \{ 2, \ldots, n-2 \}.$$

Here the $n$ external lines corresponding to the index sets $I$ and $J$ are associated with opposite sides of the internal $r$-propagator.

We conclude the following statement.

**Statement 3.3.** In the multi-Regge limit the only non-vanishing internal $r$-propagator has ordered index sets

$$I = \{ 1, 2, \ldots, r \} \quad \text{and} \quad J = \{ r+1, r+2, \ldots, n \}.\quad (3.41)$$

**3.9.2. Vertex expansion of a rooted tree.** We now continue the calculation of $\mathcal{M}_n$ in the multi-Regge limit. Use a vertex expansion of a rooted tree [5, 6]

$$\mathcal{M}_n(k_1, \ldots, k_n) = -\lambda_0 \sum_{r=3}^{n-1} \lambda_r \bar{\Delta}_0 \left( \sum_{i=1}^{n-r} k_i \right) \mathcal{M}_{n+2-r}(\cdot, k_r, \ldots, k_n), \quad (3.42)$$

cf figure 13. Next use that $\lambda_r = 0$ for $r \in \{ 5, 6, \ldots, n-1 \}$ to obtain a two-step recursion relation for $g_n$:

$$0 = m^{-2} \mathcal{M}_n(k_1, \ldots, k_n) \quad (3.42) \Rightarrow \quad g_n = g_{n-1}m^{-2} \mathcal{M}_3(\cdot, k_{n-1}, k_n) - g_{n-2}m^{-2} \mathcal{M}_4(\cdot, k_{n-2}, k_{n-1}, k_n) \quad (3.43)$$

In the last line of equation (3.43) we used that only one of the $s,t,u$ channels of $\mathcal{M}_4$ is consistent with the multi-Regge limit.

**3.10. Result**

The solution to the two-step recursion relation (3.43) yields the following theorem 3.4.
Figure 13. Vertex expansion of a rooted tree, cf theorem 2.4. The root leg ‘1’ is attached to a vertex, which in turn is attached to either (i) external legs or (ii) internal lines connected to tree amplitudes. Due to the multi-Regge limit, the vertex can at most be attached to one internal line, cf statement 3.3. One should sum over possible types of vertices.

**Theorem 3.4.** Let there be given a local relativistic interacting field theory (3.1) in 2D with a single massive real scalar field $\phi$ that has no derivative couplings and no classical tadpole (2.9). The assumption of no particle production leads to two possible cases:

1. Case $g_3 = 0$:

$$g_n = \frac{1 + (-1)^n}{2} g_4^{n/2-1} \quad \Leftrightarrow \quad \mathcal{V} = \frac{m^2}{g_4} \left[ \cosh(\sqrt{g_4} \phi) - 1 \right].$$  \hspace{1cm} (3.44)

This is the sinh-Gordon model if $g_4 > 0$ and the sine-Gordon model if $g_4 < 0$. It is the $a_{1}^{(1)}$ affine Toda model.

2. Case $g_4 = 3g_3^2$:

$$g_n = \frac{2^n + 2(-1)^n}{6} g_3^{n-2} \quad \Leftrightarrow \quad \mathcal{V} = \frac{m^2}{6g_3^2} \left[ \exp(2g_3 \phi) + 2 \exp(-g_3 \phi) - 3 \right].$$  \hspace{1cm} (3.45)

This is the Bullough–Dodd model, which is the $a_{2}^{(2)}$ affine Toda model.

The above list is precisely all rank-1 affine Toda field theories with no classical tadpole (2.9) [10].

**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

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