Harmonic Sums and Mellin Transforms

Johannes Blümlein

a DESY Zeuthen, D-15738 Zeuthen, Germany

The finite and infinite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

1. INTRODUCTION

The splitting and coefficient functions in massless QED and QCD can be evaluated in terms of Nielsen-integrals \[ 1 \]

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite and infinite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

The finite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.
complicated functions, as e.g.

\[ F_1(x) = S_{1,2} \left( \frac{1 - x}{2} \right) + S_{1,2}(1 - x) \]  

(7)

\[ -S_{1,2} \left( \frac{1 - x}{1 + x} \right) + S_{1,2} \left( \frac{1}{1 + x} \right) \]

\[ - \log(2) \text{Li}_2 \left( \frac{1 - x}{2} \right) \]

\[ + \frac{1}{2} \log^2(2) \log \left( \frac{1 + x}{2} \right) - \log(2) \text{Li}_2 \left( \frac{1 - x}{1 + x} \right) \]

emerge. These Mellin transforms, however, turn out to be reducible using the algebraic relations given below, i.e. \( F_1(x) \) can be obtained as a linear combination of Mellin convolutions of more elementary functions and simpler argument structure. This highlights the importance of algebraic relations between the harmonic sums, to which we are turning now.

3. ALGEBRAIC RELATIONS

The finite harmonic sums of order \( k \) are related by algebraic equations. They are obtained studying sums of harmonic sums with permutations in the set of their indices. The simplest relation is due to Euler \([3]\) for two indices

\[ S_{m,n} + S_{n,m} = S_m S_n + S_m \land n, \]

where

\[ m_1 \land m_2 \land \ldots m_k = \prod_{i=1}^{k} \text{sign}(m_k) \sum_{i=1}^{k} |m_k|. \]

The corresponding relations for three- and four-fold sums are \([3]\)

\[ \sum_{\text{perm}} S_{l,m,n} = S_l S_m S_n + \sum_{\text{perm}} S_l S_m \land n + S_l m \land n, \]

(8)

and

\[ \sum_{\text{perm}} S_{k,l,m,n} = S_k S_l S_m S_n + \sum_{\text{perm}} S_k S_l S_m \land n \]

\[ + \sum_{\text{perm}} S_k \land l S_m \land n \]

\[ + 2 \sum_{\text{perm}} S_k S_l \land m \land n + 6S_k \land l \land m \land n. \]

(9)

Starting with 3-fold harmonic sums more algebraic relations can be obtained by partial permutations of the index set. For 3-fold alternating or non-alternating harmonic sums 3 relations are obtained \([6,3]\), which cover the case \([3]\) and result from the combinations of

\[ T = S_{a,b,c} + S_{a,c,b} - S_{a \land b,c} - S_{a \land c,b} - S_{a,b \land c} \]

\[ + S_{a \land b \land c} \]

(13)

\[ T = S_{c,b,a} - S_{c,a,b} + S_{c,a \land b} - S_{c,S_{a \land b}} \]

(14)

\[ T = S_{b,c,a} + S_{b,c,b} - S_{b \land c,a} - S_{c,S_{b,a}} + S_{b,S_{a,c}} - S_{b,S_{a \land c}} \]

(15)

Using these relations the number of Mellin transforms occurring in the linear representations can be reduced substantially. Moreover Mellin transforms of more complicated functional structure are recognized as transforms of convolutions of much more elementary functions. Up to 2–loop order only simple, reducible variants of harmonic sums of the type \( S_{\pm 1, \pm 1, \pm 1, \pm 1}(N) \) occur. The remaining linear representations can be represented by the Mellin-transforms of the basic functions given below.

For the analytic continuation of the Mellin moments \( \mathcal{M}[f_i(x)](N) \) in addition to the well–known relations for single harmonic sums only the Mellin transforms of 24 basic functions have to be analytically continued, see \([7]\).

In Ref. \([3]\) a systematic evaluation of the Mellin transforms of the individual functions representing the polarized and unpolarized coefficient functions and anomalous dimensions up to 2–loop order (cf. e.g. \([3]\)) are given. They can be expressed through harmonic sums recursively, which are linear functions of the Mellin transforms of the func-
transformations listed below. About 80 functions \( f_i(x) \) occur. One example is
\[
M \left[ \frac{1}{1 + z} \left[ \text{Li}_3 \left( \frac{1 - z}{1 + z} \right) - \text{Li}_3 \left( -\frac{1 - z}{1 + z} \right) \right] \right] (N) = \\
(-1)^N \left\{ S_{1,1,-2}(N - 1) - S_{1,1,2}(N - 1) \\
+ S_{-1,1,-2}(N - 1) - S_{-1,1,2}(N - 1) \\
+ 2\zeta(2) S_{1,1}(N - 1) + \frac{1}{4} \zeta(2) S_1^2(N - 1) \\
- \frac{1}{4} \zeta(2) S_{-2}(N - 1) - \left[ \frac{1}{8} \zeta(3) - \frac{3}{2} \zeta(2) \log 2 \right] \\
\times S_1(N - 1) + \left[ \frac{21}{8} \zeta(3) - \frac{3}{2} \zeta(2) \log 2 \right] S_{-1}(N - 1) \\
- 2\text{Li}_4 \left( \frac{1}{2} \right) + \frac{19}{40} \zeta^2(2) + \frac{1}{2} \zeta(2) \log^2 2 - \frac{1}{12} \log^4 2 \right\}
\]

4. CONCLUSIONS

A systematic study of the finite harmonic alternating and non-alternating sums has been performed up to the four-fold sums, which have been evaluated in explicit form in the linear representation. Algebraic relations were used to reduce this set to a representation over a much smaller set of functions. Whereas in the \( N \)-space representations these relations are truly algebraic, the corresponding relations in the \( x \)-space representation are given by sums of multiple Mellin convolutions. In this representation the algebraic relations lead to essential structural simplifications both concerning the contributing functions as well as their argument structure. The corresponding Mellin transforms were evaluated in explicit form.

Acknowledgment. The work was supported by EU contract FMRX-CT98-0194(DG 12 - MIHT).

REFERENCES

1. E. Nielsen, Der Eulersche Dilogarithmus und seine Verallgemeinerungen, Nova Acta Leopold., Vol. XC, Nr. 3, Halle, 1909, pp. 121; S. Köbling, Siam J. Math. Anal. 17 (1986) 1232.
2. H. Mellin, Acta Math. 25 (1902) 139.
3. J. Blümlein and S. Kurth, Phys. Rev. D60 (1999) 014018 and hep-ph/9708385.
4. L. Euler, Novi Comm. Acad. Sci. Petropolitanae, 1 (1775) 140.
5. R. Sita Ramachandra Rao and M.V. Subbarao, Pacific J. Math. 113 (1984) 471.
6. J.M. Borwein and R. Girgensohn, Electronic J. of Combinatorics 3 (1996), #R23; Appendix by D.J. Broadhurst.
7. J. Blümlein, DESY 98-149.
8. E.B. Zijlstra and W.L. van Neerven, Phys. Lett. B273 (1991) 476; Nucl. Phys. B383 (1992) 525.
9. List of the basic functions

\[
\begin{align*}
\log(1 + x) & \quad \frac{\log(1 + x) - \log^2(2)}{x - 1} \\
\frac{\log(1 + x)}{x + 1} & \quad \text{Li}_2(x) \\
\frac{\log(1 + x)}{x + 1} & \quad \text{Li}_2(-x) \\
\frac{\log(x) \text{Li}_2(x)}{x + 1} & \quad \frac{\log(x) \text{Li}_2(x)}{x + 1} \\
\frac{\text{Li}_3(x)}{x + 1} & \quad \frac{\text{Li}_3(x)}{x + 1} \\
\frac{\text{Li}_3(x) - \zeta(3)}{x + 1} & \quad \frac{\text{Li}_3(x) - \zeta(3)}{x + 1} \\
\frac{\text{Li}_3(-x) - 3\zeta(3)/4}{x + 1} & \quad \frac{\text{S}_{1,2}(x)}{x + 1} \\
\frac{\text{S}_{1,2}(x) - \zeta(3)}{x + 1} & \quad \frac{\text{S}_{1,2}(-x) - \zeta(3)/8}{x + 1} \\
\frac{\text{S}_{1,2}(-x)}{x + 1} & \quad \frac{\text{S}_{1,2}(x^2)}{x + 1} \\
\frac{\text{S}_{1,2}(x^2) - \zeta(3)}{x + 1} & \quad \frac{\log(1 - x) \text{Li}_2(-x)}{x + 1} \\
\frac{\log(1 + x) - \log(2) \text{Li}_2(-x)}{x - 1} & \quad \frac{\log(1 - x) \text{Li}_2(x)}{1 + x} \\
\frac{\log(1 + x) - \log(2) \text{Li}_2(x)}{x - 1} & \quad \frac{\log(1 + x) \text{Li}_2(x)}{1 + x} \\
\end{align*}
\]