Touching multifunctions on a Hilbert space

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Abstract
We introduce the concept of the touching of two multifunctions on a real Hilbert space, and deduce that certain multifunctions on the space have a unique fixed point. These results are applied to the theory of generalized cycles and generalized gap vectors for the composition of the projections onto a finite number of closed convex space in a real Hilbert space.

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1 Introduction
The analysis in this paper was originally motivated by [1], in which the geometry conjecture (originally formulated in 1977, about the fixed point set of the composition of projections onto a finite number of nonempty closed convex subsets) was resolved. The initial part of [1] contained some technical Hilbert space results, and the later part of [1] discussed some special cases and contained results on numerical computation. [5] contained a streamlining of the Hilbert space results in [1]. The techniques introduced in [5] were used in [2] to obtain further results on (classical and phantom) cycles and gap vectors. This paper contains Hilbert space results that extend the main results in [2].

Here is a brief plan of this paper. The analysis in Section 2 is about multifunctions on a real Hilbert space, which we will denote by \( Y \). In Definition 5, we define the concepts of touching multifunctions, and give results in Theorem 6 and Corollary 8 on certain multifunctions that touch every maximally monotone multifunction. Corollary 8 leads rapidly to Lemma 9 and Corollary 11. Lemma 9 is couched in terms of the fixed points of certain multifunctions on \( Y \). Corollary 11 is a restatement of [2, Lemma 3.1] in the notation of this paper.

In Section 3, \( Y \) is a closed subspace of a Hilbert space \( X \). Theorem 13 is a (not altogether immediate) consequence of Lemma 9. Theorem 13(b) is a restatement of [2, Theorem 4.10] in the notation of this paper and is generalized in Theorem 13(a).

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2 Very unmonotone multifunctions

If $M$ is a multifunction, we write $G(M)$ for the graph of $M$.

**Definition 1.** Let $\mu > 0$ and $Q : Y \rightrightarrows Y$. We say that $Q$ is $\mu$–unmonotone if 
\[(y_1, q_1), (y_2, q_2) \in G(Q) \implies \langle y_1 - y_2, q_1 - q_2 \rangle + \mu \| (y_1 - y_2, q_1 - q_2) \|^2 \leq 0.\]

We write $\mathcal{B}(Y)$ for the set of all bounded linear operators from $Y$ into $Y$, and $\mathbb{I}_Y \in \mathcal{B}(Y)$ for the identity map on $Y$.

**Lemma 2.** Let $\mu > 0$ and $Q : Y \rightrightarrows Y$ be $\mu$–unmonotone. Then $-Q - \mu \mathbb{I}_Y$ is monotone.

**Proof.** For all $(y_1, q_1), (y_2, q_2) \in G(Q)$,
\[
\langle y_1 - y_2, (-q_1 - \mu y_1) - (-q_2 - \mu y_2) \rangle = \langle y_1 - y_2, -(q_1 - q_2) - \mu (y_1 - y_2) \rangle
\]
\[
= -\langle y_1 - y_2, q_1 - q_2 \rangle - \mu \| y_1 - y_2 \|^2
\]
\[
\geq -\langle y_1 - y_2, q_1 - q_2 \rangle - \mu \| (y_1 - y_2, q_1 - q_2) \|^2 \geq 0.
\]

This completes the proof of Lemma 2.

Theorem 6 depends on two results from the theory of maximally monotone multifunctions, Fact 4 and Fact 3. Fact 3 follows from the sum theorem, [3] Corollary 24.4(i), p. 353.

If $M$ is a multifunction, we write $D(M)$ for the domain of $M$.

**Fact 3.** Let $M_1$ and $M_2$ be maximally monotone multifunctions on $Y$ and $D(M_1) = Y$. Then $M_1 + M_2$ is maximally monotone.

Fact 4 is Minty’s theorem, [11], or [3] Theorem 21.1, pp. 311.

**Fact 4.** Let $N$ be a maximally monotone multifunction on $Y$ and $\mu > 0$. Then there exists $y \in Y$ such that $0 \in \mu y + Ny$.

**Definition 5.** Let $M, Q : Y \rightrightarrows Y$. We say that $M$ and $Q$ touch if $G(M) \cap G(Q)$ is a singleton in $Y \times Y$.

**Theorem 6.** Let $\mu > 0$, $Q : Y \rightrightarrows Y$ be $\mu$–unmonotone and $D(Q) = Y$. Suppose also that $-Q - \mu \mathbb{I}_Y$ is maximally monotone. Then $Q$ touches every maximally monotone multifunction on $Y$.

**Proof.** Let $M : Y \rightrightarrows Y$ be maximally monotone. We start off by proving that $G(M) \cap G(Q)$ contains at most one element of $Y \times Y$. To this end, let $(y_1, q_1), (y_2, q_2) \in G(M) \cap G(Q)$. Since $(y_1, q_1), (y_2, q_2) \in G(M)$,
\[
\langle y_1 - y_2, q_1 - q_2 \rangle \geq 0.
\]

(1)

On the other hand, since $(y_1, q_1), (y_2, q_2) \in G(Q)$, from Definition 1 and 11,
\[
\mu \| (y_1 - y_2, q_1 - q_2) \|^2 \leq -\langle y_1 - y_2, q_1 - q_2 \rangle \leq 0.
\]
and so \((y_1, q_1) = (y_2, q_2)\). Thus \(G(M) \cap G(Q)\) contains at most one point.

On the other hand, \(M\) and \(\mu y\) are both maximally monotone and \(D(\mu y) = Y\). From Fact \[3\] \(M - \mu y\) is maximally monotone, and Fact \[4\] provides \(y \in Y\) such that \(0 \in \mu y + (M - \mu y)y\), that is, \(0 \in (M - Q)y\). It follows easily from this that \(G(M) \cap G(Q) \neq \emptyset\), which completes the proof of Theorem \[6\].

We conclude this section with applications to bounded linear operators. Our analysis depends on the following result about monotone functions. See [3, Corollary 20.25, p. 298]

**Fact 7.** Any continuous (single-valued) monotone function from \(Y\) into \(Y\) is maximally monotone.

**Corollary 8.** Let \(Q \in B(Y)\), \(\lambda > 0\) and,

\[
\text{for all } y \in Y, \quad \langle y, Qy \rangle + \lambda \|y\|^2 \leq 0.
\]

Then \(Q\) touches every maximally monotone multifunction on \(Y\).

**Proof.** Let \(\mu := \lambda/(1 + \|Q\|^2)\). Let \((y_1, q_1), (y_2, q_2) \in G(Q)\). Then \(q_1 = Qy_1\) and \(q_2 = Qy_2\). Consequently,

\[
\langle y_1 - y_2, q_1 - q_2 \rangle + \mu \|y_1 - y_2\|^2 \\
\leq \langle y_1 - y_2, Q(y_1 - y_2) \rangle + \mu \|y_1 - y_2\|^2 \\
\leq \langle y_1 - y_2, Q(y_1 - y_2) \rangle + \mu (0 + \|Q\|^2) \|y_1 - y_2\|^2 \\
\leq \langle y_1 - y_2, Q(y_1 - y_2) \rangle + \lambda \|y_1 - y_2\|^2 \leq 0.
\]

Thus \(Q\) is \(\mu\)-unmonotone. From Lemma \[2\] and Fact \[4\] \(-Q - \mu I_Y\) is maximally monotone. The result now follows from Theorem \[6\].

**Lemma 9.** Let \(T \in B(Y)\) be surjective and bijective, \(\lambda > 0\) and

\[
\text{for all } x \in Y, \quad \langle x, Tx \rangle + \lambda \|Tx\|^2 \leq 0.
\]

Then, whenever \(M\) is a maximally monotone multifunction on \(Y\), the multifunction \(MT\) has a unique fixed point.

**Proof.** From the open mapping theorem, there exists \(Q \in B(Y)\) such that

\[
\text{for all } y \in Y, \quad y = T(Qy) \quad \text{and} \quad y = Q(Ty).
\]

Let \(M\) be maximally monotone. If \(y \in Y\), let \(x := Qy \in Y\). From \[1\], \(y = T(Qy) = Tx\). Thus, from \[3\], \(\langle y, Qy \rangle + \lambda \|y\|^2 = \langle Tx, x \rangle + \lambda \|Tx\|^2 \leq 0\), and Corollary \[8\] implies that \(M\) and \(Q\) touch. Let \(G(M) \cap G(Q) = \{(d, e)\} \subset Y \times Y\).

If \(y\) is a fixed point of \(MT\) then \(y \in M(Ty)\) and so \((Ty, y) \in G(M)\). From \[1\] again, \((Ty, y) \in G(Q)\), and so \((Ty, y) = (d, e)\), from which \(y = e\). Thus \(e\) is the only possibility as a fixed point of \(MT\).

On the other hand, since \((d, e) \in G(M) \cap G(Q)\), \(e = Qd\) and \(e \in Md\). From \[1\], \(d = T(Qd) = Te\), and so \(e \in M(Te) = (MT)e\). Thus \(e\) is, in fact, a fixed point of \(MT\).
The final result of this section, Corollary 11, is a restatement of [2, Lemma 3.1] which, in turn, generalizes [5, Lemma 16]. We will use Fact 10 below, see [3, Theorem 20.40, p. 304] and [3, Theorem 21.2, p. 312] for proofs.

If \( H \) is a real Hilbert space, we write \( \Gamma_0(H) \) for the set of all proper, convex, lower semicontinuous functions from \( H \) into \( ]-\infty, \infty[ \).

**Fact 10.** Let \( g \in \Gamma_0(Y) \). Then the subdifferential of \( g \), \( \partial g \), is maximally monotone.

The three quantities \( X, Y \) and \( Q \) used in Corollary 11 were defined in terms of three other quantities \( R, M \) and \( S \) in [2, Section 2]. The statement of Corollary 11 shows that this is unnecessary. We also note that [5] uses an equality, while [2] uses an inequality, which suffices for Corollary 8.

We write \( B(X,Y) \) for the set of all bounded linear operators from \( X \) into \( Y \).

**Corollary 11.** Let \( Y \) be a closed subspace of a Hilbert space \( X \), \( Q \in B(X,Y) \),

\[
\text{for all } y \in Y, \quad (y, Qy) + \frac{1}{2}\|y\|^2 = 0, \quad (5)
\]

\( f \in \Gamma_0(X) \) and \( f^*|_Y \in \Gamma_0(Y) \). Then there exists \( (d, e) \in Y \times Y \) such that

\[
G(\partial(f^*|_Y)) \cap G(Q) = \{(d, e)\}.
\]

**Proof.** This follows from Fact 10, Corollary 8 and Definition 5 with the maximally monotone multifunction \( \partial(f^*|_Y) \).

**Remark 12.** In [2, Remark 3.6], \( e \) is called the generalized cycle of \( f \), and \( d \) is called the generalized gap vector of \( f \).

A comparison of the statements of Corollary 8 and Corollary 11 shows that \( X \) does not play a fundamental role in Corollary 11.

On the other hand, \( X \) does play a fundamental role in the results of the next section.

### 3 Hilbert subspaces

Theorem 13 is a restatement of [2, Theorem 4.10] which, in turn, generalizes [5, Lemma 16]. Theorem 13(b) follows easily from Theorem 13(a). We note that Theorem 13(a), does not require \( f \) to attain a minimum on \( X \), whereas Theorem 13(b) does.

**Theorem 13.** Let \( Y \) be a closed subspace of a Hilbert space \( X \), \( S \in B(X,Y) \), \( S|_Y \in B(Y) \) be surjective and injective and,

\[
\text{for all } x \in X, \quad (x, Sx) + \frac{1}{2}\|Sx\|^2 = 0. \quad (6)
\]

Let \( f \in \Gamma_0(X) \) and \( f^*|_Y \in \Gamma_0(Y) \). Then it follows that the multifunction \( [\partial(f^*|_Y)] \circ S|_Y : Y \rightrightarrows Y \) has a unique fixed point, \( e \), and

\[
x \in X \text{ and } Sx \in \partial f(x) \quad \Rightarrow \quad Sx = Se. \quad (7)
\]
Furthermore:
(a) If \( x \in X \), \( f(x) \in \mathbb{R} \) and \( f(x) \leq (f^*|_{\mathcal{Y}})^*(e) \) then

\[
Sx \in \partial f(x) \iff f^*(Sx) + \frac{1}{2}\|Sx\|^2 + f(x) = 0 \iff Sx = Se. \tag{8}
\]

(b) If \( x \in X \) and \( f(x) = \min_X f \) then \([3]\) holds.

Proof. From Fact 10, \( \partial(f^*|_{\mathcal{Y}}) \) is maximally monotone, and so Lemma 3 (with \( \lambda := \frac{1}{2} \), \( T := S|_{\mathcal{Y}} \) and \( M := \partial(f^*|_{\mathcal{Y}}) \)) implies that \( \partial(f^*|_{\mathcal{Y}}) \circ S|_{\mathcal{Y}} \) has a unique fixed point, \( e \). Since \( e \in \partial(f^*|_{\mathcal{Y}})(S|_{\mathcal{Y}}) \), \( f^*(Se) = f^*|_{\mathcal{Y}}(Se) < \infty \), \( (f^*|_{\mathcal{Y}})^*(e) < \infty \) and,

\[
(e, Se) = f^*(Se) + (f^*|_{\mathcal{Y}})^*(e). \tag{9}
\]

We now establish \([7]\). To this end, suppose that \( x \in X \) and \( Sx \in \partial f(x) \). Then \( f(x) < \infty \), \( f^*(Sx) < \infty \) and

\[
\langle x, Sx \rangle = f(x) + f^*(Sx). \tag{10}
\]

From the Fenchel–Young inequality:

\[
- \langle x, Se \rangle \geq - f(x) - f^*(Se) \quad \text{and} \quad - \langle e, Sx \rangle \geq - (f^*|_{\mathcal{Y}})^*(e) - f^*(Sx). \tag{11}
\]

Now \( \langle x - e, S(x - e) \rangle = \langle e, Se \rangle + \langle x, Sx \rangle - \langle x, Se \rangle - \langle e, Sx \rangle \). Consequently, by adding \([9] - [11]\), we see that \( \langle x - e, S(x - e) \rangle \geq 0 \). From \([6]\),

\[
- \frac{1}{2}\|Sx - Se\|^2 = - \frac{1}{2}\|S(x - e)\|^2 = \langle x - e, S(x - e) \rangle \geq 0.
\]

Thus \( Sx = Se \), which completes the proof of \([7]\).

(a) Suppose that \( x \in X \), \( f(x) \in \mathbb{R} \), \( f(x) \leq (f^*|_{\mathcal{Y}})^*(e) \) and \( Sx = Se \). Then, using \([6]\) twice and \([9]\),

\[
\langle x, Sx \rangle = - \frac{1}{2}\|Sx\|^2 = - \frac{1}{2}\|Se\|^2 = \langle e, Se \rangle = f^*(Sx) + (f^*|_{\mathcal{Y}})^*(e) \geq f^*(Sx) + f(x).
\]

Consequently, \( Sx \in \partial f(x) \), and \([8]\) follows from \([6]\) and \([7]\).

(b) We first note that \( (f^*|_{\mathcal{Y}})^*(e) \geq (0, e) - (f^*|_{\mathcal{Y}})(0) = - f^*(0) = \inf_X f \). So if \( x \in X \), and \( f(x) = \min_X f \) then \([8]\) follows from (a).

**Remark 14.** The significance of \([4]\) is not only that it makes Theorem 13 possible. Since \( \langle x, Sx \rangle + \frac{1}{2}\|Sx\|^2 = 0 \iff \|x\|^2 + 2\langle x, Sx \rangle + \|Sx\|^2 = \|x\|^2 \) and \( \|x\|^2 + 2\langle x, Sx \rangle + \|Sx\|^2 = \|Sx + x\|^2 \), \([4]\) is equivalent to the statement that the linear map \( S + I_X \) is an isometry. Historically, this is backwards: in \([5]\) and \([2]\), the parameters of the geometry conjecture provided us with an isometry \( R \), and \( S \) was defined by \( S := R - I_X \).
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