Non-Abelian Chiral Spin Liquid on the Kagome Lattice

Zheng-Xin Liu,1,2 Hong-Hao Tu,3 Ying-Hai Wu,3 Rong-Qiang He,2 Xiong-Jun Liu,4,5 Yi Zhou,6 and Tai-Kai Ng7

1Department of Physics, Renmin University of China, Beijing, China
2Institute for Advanced Study, Tsinghua University, Beijing, China
3Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, D-85748 Garching, Germany
4International Center for Quantum Materials and School of Physics, Peking University, Beijing 100871, China
5Collaborative Innovation Center of Quantum Matter, Beijing 100871, China
6Department of Physics, Zhejiang University, Hangzhou, China
7Department of Physics, Hong Kong University of Science and Technology, Clear Water Bay Road, Kowloon, Hong Kong

We study $S = 1$ spin liquid states on the Kagome lattice constructed by Gutzwiller projected $p_x + ip_y$ superconductors. Depending on the topology of the fermions, the obtained spin liquids can be either non-Abelian or Abelian. By calculating the modular matrices $S$ and $T$, we confirm that projected topological superconductors are non-Abelian chiral spin liquid (NACSL). The chiral central charge and the spin Hall conductance we obtained agrees very well with the $SO(3)_1$ field theory predictions. The NACSL may be stabilized by a local Hamiltonian. From a variational study we observe a topological phase transition from the NACSL to a $Z_2$ Abelian spin liquid.

Since the discovery of fractional quantum Hall (FQH) effect,1,2 topological order became a fundamental concept in condensed matter.3–5 In contrast to usual long-range orders characterized by spontaneous symmetry breaking according to Landau’s paradigm, topological orders are characterized by the topological degeneracy of ground states on a manifold with genus and the fractional bulk excitations with a nonvanishing gap. For instance, the FQH liquid with $\frac{1}{3}$ filling has three-fold degenerate ground states on a torus and its quasiparticle excitations, called anyons, carry $\frac{e}{3}$ charge. The charge-$\frac{e}{3}$ anyons respect fractional statistics, i.e., the many-body wave function obtains a Berry phase $e^{i\pi/3}$ if one anyon adiabatically exchanges its position with another one (called braiding). More interestingly, the Pfaffian state proposed by Moore and Read6 for the $\nu = \frac{5}{2}$ FQH liquid7,8 supports non-Abelian anyons in the vortex cores. The braiding of non-Abelian anyons is equivalent to operating a matrix on the degenerate states instead of multiplying a $U(1)$ phase factor.9,10 The non-Abelian topological orders have potential applications in quantum memory and quantum computation10,11.

Besides FQH systems, gapped spin liquids also exhibit nontrivial topological orders. For example, the Kalmeyer-Laughlin chiral spin liquid12 supports semionic anyons, and the short-range resonating valence bond (RVB) state on a two-dimensional (2D) non-bipartite lattice carries $Z_2$ topological order.13 In seeking of spin liquids in realistic microscopic models, antiferromagnets on the Kagome lattice have been widely studied14–18 due to its strong geometric frustrations, which is necessary for suppressing Néel order. From experimental side, promising candidates of spin liquids have been synthesized, such as the Herberitsmithite realizing the Kagome antiferromagnet.19 Recently, $S = 1$ antiferromagnets have attracted lots of interest from experimental, theoretical and numerical sides. Several exotic spin liquid phases, such as $U(1)$, $Z_2$, and chiral spin liquids, have been proposed in $S = 1$ spin systems20–26.

In this paper, we construct both Abelian and non-Abelian $S = 1$ spin liquid states on the Kagome lattice. These wave functions are constructed by Gutzwiller projection of $p_x + ip_y$-superconductors in the fermionic slave particle representation.27,28 It is shown that the topology of the fermions determines the physical properties of the spin wave function after the projection. The projected topological superconductors are non-Abelian chiral spin liquids (NACSL), which is verified by calculating their modular matrices $T$ and $S$. We show that the NACSL has quantum spin Hall effect where the spin Hall conductance is quantized to $\frac{1}{27}$.

Construction of Gutzwiller wave functions. Following Ref. 27, we introduce three species of fermionic slave particles $C_i = (c_{1i}, c_{0i}, c_{-1i})^T$, called spinons, to represent the $S = 1$ operators as $S_1^+ = \frac{1}{\sqrt{2}}(S_i^+ + iS_i^0) = c_{1i}^+ c_{0i} + c_{0i}^+ c_{-1i}$ and $S_i^0 = c_{1i}^+ c_{1i} - c_{0i}^+ c_{0i} + c_{-1i}^+ c_{-1i} = 1$. In this way, the spin-spin interactions can be represented in terms of interactions between the fermions. The spin operators and the particle number constraint are invariant under local $U(1)$ transformation $C_i \rightarrow C_i e^{i\phi_i}$, indicating an $U(1)$ gauge structure of the above fermionic representation.

We will focus on $SO(3)$ spin-rotationally invariant mean-field states on the Kagome lattice (see Fig. 1(a)), which correspond to $S = 1$ spin liquid phases. The mean-field Hamiltonian for fermions takes the following form:

$$H_{\text{mf}} = \sum_{\langle ij \rangle} [\chi_{ij} C_i^\dagger C_j + \Delta_{ij} C_i^\dagger \tilde{C}_j + \text{h.c.}] + \lambda \sum_i C_i^\dagger C_i C_{i(1)}$$

where $\tilde{C}_i = (c_{-1i}^+, -c_{0i}^+, c_{1i}^+)^T$. Both $C_i$ and $\tilde{C}_j$ are $SO(3)$ vectors since $C_i^\dagger \tilde{C}_j |\text{vac}\rangle$ is a spin singlet. Noticing the
Spin pump

Strong pairing state

liquids. On the contrary, the projected trivial superconductor (with parameters \( \lambda = -1, \Delta = 1, \lambda = 1 \)) has quantized spin Hall conductance \( 2 \pi \lambda \); (d) projected trivial superconductor (with parameters \( \chi = -1, \Delta = 1, \lambda = 10 \)) has vanishing spin Hall conductance.

pairing term has odd parity, the pairing symmetry on the Kagome lattice can be either \( p \)-wave or \( f \)-wave, depending on the angular momentum of the ‘Cooper pair’. Here we will focus on the \( (p_x + ip_y) \)-pairing states with variational parameters \( \chi_{ij} = \chi_j', \Delta_{ij} = -\Delta_{ji} = \Delta e^{i\theta_{ij}} \) (\( \theta_{ij} \) is the angle between \( x \)-axis and the bond \( ij \)) and the ‘chemical potential’ \( \lambda \). In the weak pairing region \(-2|\chi| < \lambda < 4|\chi|\), the mean-field state is a topological superconductor with nontrivial Chern number.\(^{29}\) On the other hand, in the strong pairing region \( \lambda < -2|\chi| \) or \( \lambda > 4|\chi| \), the Chern number vanishes and the mean-field state is a trivial superconductor.

To go beyond the mean-field theory, we need to consider the low energy \( U(1) \) phase fluctuations of the variational parameters, which are Higgsed into \( \mathbb{Z}_2 \) gauge fluctuations owing to the fermion pairing. Alternatively, a RVB wave function can be obtained by a Gutzwiller projection of the mean-field state,

\[
|\text{RVB} \rangle = P_G |\text{BCS} \rangle, \tag{2}
\]
where \( P_G \) is the Gutzwiller projection operator that enforces the onsite particle number constraint\(^{30}\) and \( |\text{BCS} \rangle \) is the ground state of the mean-field Hamiltonian (1). In the following we will provide evidence that the projected topological superconductors are non-Abelian chiral spin liquids. On the contrary, the projected trivial superconductors are \( \mathbb{Z}_2 \) Abelian spin liquids.

Response field theory and Integer Quantum Spin Hall effect. In the topological superconductor, each flavor of fermions has nontrivial Chern numbers, so we expect that the system has nontrivial response when it is probed by \( SO(3) \) symmetry twisting fields \( A_{ij} \), where \( A_{ij} = A_{ij}^x S^x + A_{ij}^y S^y + A_{ij}^z S^z \) behave like external \( SO(3) \) gauge fields coupling to the spins.

Owing to the \( SO(3) \) gauge invariance, we expect that, after integrating out the fermions and the \( \mathbb{Z}_2 \) gauge fluctuations, the following \( SO(3)_1 \) Chern-Simons response theory describes the low-energy physics in the hydrodynamic limit:

\[
\mathcal{L}_{\text{eff}} = i \frac{k}{4\pi} \frac{1}{2} \text{Tr} [\varepsilon^{\mu
u\lambda} (A_\mu \partial_\nu A_\lambda) - \frac{1}{3} A_\mu^3] + \mathcal{L}_{\text{Maxwell}} + \ldots,
\]

with \( k = 1 \). The \( SO(3)_1 \) Chern-Simons theory supports non-Abelian anyonic excitations, indicating that the spin wave function (2) describes a non-Abelian spin liquid. If the probe field only contains \( z \) component, \( A_\mu = A_\mu^z S^z \), then we obtain the spin Hall current, \( j^z = \frac{\delta S_{z\mu}}{\delta A_{\mu}^z} = \frac{1}{2\pi} F^z_{\mu} \), where \( F^z_{\mu} = i \varepsilon^{\mu
u\lambda} (\partial_\nu A_\lambda^z) \) is the strength of the probing field and the spin Hall conductance is quantized to \( \frac{1}{2\pi} \).

We can study the response of the spin system from Gutzwiller projected wave functions. From Laughlin’s gauge invariant argument\(^{31,32}\), the spin Hall conductance can be obtained by measuring the spin pump of the Gutzwiller projected state in a cylinder when adiabatically inserting a symmetry flux \( \Phi_\lambda \) (see Fig. 1(b)) in the mean-field Hamiltonian

\[
H_{\text{mf}} (\Phi_\lambda) = \sum_{\langle ij \rangle} [\chi_{ij} C_i^\dagger e^{i\Delta_{ij}^z S^z} C_j + \Delta_{ij} C_i^\dagger e^{i\theta_{ij}} C_j + \text{h.c.}] + \lambda \sum_i C_i^\dagger C_i, \quad \text{where } \Phi_\lambda = \oint A^z \cdot dx \text{ is the symmetry flux through the cylinder.}
\]

Our numerical data for the projected ground state of \( H_{\text{mf}} (\Phi_\lambda) \) is shown in Fig. 1(c), where the spin Hall conductance is given by \( \frac{\delta S_{z\mu}}{\delta A_{\mu}^z} = \frac{1}{2\pi} \). Alternatively, the spin Hall conductance can be obtained by calculating the many-body Chern number on a torus with twisted boundary conditions.\(^{32}\) When using the latter scheme, the many-body Chern number we obtained is also 1, which agrees with the result from Laughlin’s gauge invariant argument.

Similar to electronic quantum Hall states, the bulk of the NACSL is gapped and the boundary is gapless. This is verified by computing the spin-spin correlation functions (see Supplemental Material).

Ground-state degeneracy and modular matrices. Now we show that the projected topological superconductors are Ising-like non-Abelian spin liquids. Without lose of generality, in the following we will consider projected state with parameters \( \chi = -1, \Delta = 1, \lambda = 1.5 \). The Ising-like topological order contains three kinds of anyons \( (I, \sigma, \psi) \) with fusion rules \( \sigma \times \sigma = I + \psi, \psi \times \psi = I, \sigma \times \psi = \sigma \). Here \( I \) is a trivial anyon which stands for the vacuum, \( \psi \) is a fermion and \( \sigma \) is the Ising-like anyon (this topological order is named \( 3g/2 \) in Ref. 33).

We first figure out the ground-state degeneracy. Noticing that inserting \( \mathbb{Z}_2 \) gauge fluxes through the two holes of the torus (which is equivalent to switching
the fermion boundary conditions from periodic ones to anti-periodic ones) does not change any local physical properties of the spin system, so we have 4 different mean-field states labeled by the $\mathbb{Z}_2$ fluxes in the holes $(0,0), (0, \pi), (\pi,0), (\pi, \pi)$. However, not all these states survive after the Gutzwiller projection. In fact, the state $(0,0)$ has odd fermion parity, so if the lattice sites are even, this state vanishes after projection $|P_G(0,0)\rangle = 0$.

The three remaining states survived after Gutzwiller projection are degenerate in energy as long as the spin interactions are short-ranged. Now we further show that these three states are orthogonal to each other. To this end, we calculate the overlap between these three states on a torus with $10 \times 10$ unit cells. With $3 \times 10^8$ steps of Monte Carlo simulation, we obtain

$$|\langle P_G(0,\pi)|P_G(\pi,\pi)\rangle| = 0.0012,$$

$$|\langle P_G(\pi,0)|P_G(\pi,\pi)\rangle| = 0.001,$$

$$|\langle P_G(\pi,0)|P_G(0,\pi)\rangle| = 0.0002.$$ 

These results indicate that the three states are orthonormal up to errors of order $10^{-3}$. So we conclude that the degeneracy of ground states on a torus is three. This result supports the Ising-like topological order of the system since the degree of ground-state degeneracy on a torus should be equal to the number of simple anyon types.

To completely identify the topological order, we need to calculate the modular matrices $S$ and $T$. $S$ and $T$ are the representation matrices of the corresponding modular transformation $S$ and $T$ on a torus, where $S$ is a 90° rotation and $T$ is a Dehn twist. The two transformations $S, T$ generate the modular group of a torus. We apply the method proposed in Ref. 36 to calculate the $T_x$ and $T_y$ matrices, namely, the representations of the Dehn twist along $x$- and $y$-directions respectively. The basic idea is to adiabatically perform the Dehn twist with many sub-steps and then calculate the Berry phase in the whole process via wave function overlaps. To enhance the overlap of wave functions, each step of the Dehn twist is realized by shifting the boundary-crossing couplings by one lattice site (also see the Supplemental Material).

Generally, the size-dependence of the Berry phase (in our calculation we have set $L_x = L_y = L$, where $L_x$ and $L_y$ are the number of unit cells along $x$- and $y$-directions, respectively) is given by

$$\varphi(L^2) = \alpha + \beta L^2 + O(L^{-2}).$$

The intercept $\alpha$ is universal and only depends on the topological order.

Notice that the Dehn twist can interchange or, in general, mix the degenerate states in the ground-state subspace. For simplicity we denote $|P_G(\pi,0)\rangle$, $|P_G(0,\pi)\rangle$ and $|P_G(\pi,\pi)\rangle$ as $|x\rangle$, $|y\rangle$ and $|xy\rangle$ respectively. The Dehn twist $T_y$ keeps the sector $|x\rangle$ invariant and exchanges the sectors $|y\rangle$ and $|xy\rangle$. We denote $\varphi_{x,x}$ as the Berry phase of $|x\rangle$ during the Dehn twist. Since the Berry phase is not well defined for an open path, $\varphi_{y,xy}$ is defined as half of the Berry phase during a double $T_y$ twist (namely, $T_y^2$) for the state $|y\rangle$. The results of the scaling of $\varphi(L^2)$ are shown in Fig. 2. From the data we get $T_y = \left(\begin{array}{c c}
\alpha x_{x,y} & 0 \\
0 & 0
\end{array}\right)$, where $\alpha x_{x,y} = 0.2368\pi$ and $\alpha y_{x,y} = -0.1321\pi$. Here we omitted the overlap between different topological sectors. Owing to the equivalence between $x$- and $y$- axes, we can write out $T_x$ as $T_x = \left(\begin{array}{c c}
\alpha y_{x,xy} & 0 \\
0 & 0
\end{array}\right)$, where $\alpha y_{x,xy} = -\alpha y_{y,xy}$ and $\alpha y_{y,xy} = -\alpha x_{x,xy}$ (these relations have been numerically verified). The $S$ matrix can be obtained from the relation $S = T_y T_x^{-1} T_y$.

The $T$ and $S$ matrices can be transformed into the standard form with a unitary transformation

$$T_y = e^{i\delta_{y,xy}} \left(\begin{array}{c c}
1 & e^{i\delta_{x,x}} \\
e^{i\delta_{x,xy}} & 0
\end{array}\right), S = e^{i\delta} \left(\begin{array}{c c}
1 & \sqrt{2} \\
0 & -\sqrt{2}
\end{array}\right).$$

where $e^{i\delta} = e^{i(2\delta_{y,xy} + \alpha x_{x,xy})} = e^{-0.0274\pi i}$. In this form, the diagonal entries of $T_y^2$ stand for the self-statistics angle of anyons (the prefactor $e^{i\delta_{y,xy}}$ is chosen such that the statistics for the trivial and fermionic anyons are exact), and the entries of $S$ are proportional to the mutual statistics angle between different anyons. The phase factor $e^{i\delta}$ in $S$ is owing to numerical errors. From the $S$ matrix, we can read out that the quantum dimensions of the anyons are $1, \sqrt{2}, 1$, which are consistent with those of $I, \sigma$ and $\psi$, respectively.

From the $T_y$ and $S$ matrices, we can obtain the chiral central charge via the relations $T_y = \left(\begin{array}{c c}
1 & e^{i\delta_{x,x}} \\
e^{i\delta_{x,xy}} & 0
\end{array}\right) = e^{i\delta_{y,xy}} \left(\begin{array}{c c}
1 & e^{i\delta_{x,x}} \\
e^{i\delta_{x,xy}} & 0
\end{array}\right) \left(\begin{array}{c c}
1 & 0 \\
0 & 0
\end{array}\right)$.

FIG. 2. (Color online) Scaling of Berry phase for Dehn twist along $y$-direction. We have set $L_x = L_y$. The meaning of the subscript $x, y$ and $xy$ in $\varphi$ are explained in the main text.

### Linear Fitting

In the main text, we linearly fit the results of the scaling to obtain $\varphi_{x,x}/2\pi = -0.05598L_x \times L_y -0.06608$.

### Monte Carlo Data

In this form, the diagonal entries of $S$ are 1, $\sqrt{2}$, 1, which are consistent with the statistics for the trivial and fermionic anyons are exact, and the entries of $S$ are proportional to the mutual statistics angle between different anyons. The phase factor $e^{i\delta}$ in $S$ is owing to numerical errors. From the $S$ matrix, we can read out that the quantum dimensions of the anyons are $1, \sqrt{2}, 1$, which are consistent with those of $I, \sigma$ and $\psi$, respectively.

From the $T_y$ and $S$ matrices, we can obtain the chiral central charge via the relations $T_y = \left(\begin{array}{c c}
1 & e^{i\delta_{x,x}} \\
e^{i\delta_{x,xy}} & 0
\end{array}\right) = e^{i\delta_{y,xy}} \left(\begin{array}{c c}
1 & e^{i\delta_{x,x}} \\
e^{i\delta_{x,xy}} & 0
\end{array}\right) \left(\begin{array}{c c}
1 & 0 \\
0 & 0
\end{array}\right)$.
theory prediction

\[ \text{cors.} \] The averaged central charge is \( c_1 \) while the second relation gives \( \sum_c = 1 \).

Correlation of the 'cluster spin' \( \pi \)

FIG. 3. (Color online) (a) two \( \pi \)-vortices are connected with a string, all the mean-field couplings crossing the string reverse their sign; (b) correlation of the 'cluster spin' \( S^z_3(i) = \sum_c S^z \), which is defined as the total spin moment of the three vertices of a triangle. The red (blue) line shows the case where a \( \pi \)-vortex is present (absent) at the center of each cluster.

\[ \text{Non-Abelian anyon carries fractional spin.} \] Although \( \pi \)-flux [see Fig. 3(a)], which traps three Majorana zero modes at the mean-field level. After Gutzwiller projection, the three Majorana zero modes in the \( \pi \)-vortex correspond to a non-Abelian anyon (actually it is a composite anyon protected by the \( SO(3) \) symmetry, see Supplemental Material).

We calculate the correlation of spin momentum \( S^z_3 \) for two triangles, where \( S^z_3 \) stands for the total momentum of the three spins on the vertices of a triangle. As shown in Fig. 3(b), if each triangle contains a \( \pi \)-vortex, the correlation converges close to \(-\frac{1}{4}\); in contrast, if there is no \( \pi \)-vortex in the triangles, then the correlation converges to 0. This verifies that the non-Abelian anyon carries spin-\( 1/2 \) similar to the edge states in \( S = 1 \) Heisenberg chain.

Strictly speaking, the NACSL phase is an \( SO(3) \) symmetry-enriched topological phase, since the degeneracy according to the local spin-1/2 degree of freedom is protected by \( SO(3) \). If we break the symmetry by a weak magnetic field, then the local degeneracy will be lifted and the composite anyon corresponding to the \( \pi \)-vortex reduces to the simple anyon \( \sigma \).

A Gutzwiller projected trivial superconductor can be adiabatically connected to a nearest neighbor RVB state, which has 4-fold degenerate ground states on torus and contains \( Z_2 \) (Abelian) topological order. As shown in Fig. 1(d), this \( Z_2 \) spin liquid has no spin Hall effect, and spin moment of the anyons are not fractionalized.

Local Hamiltonian and topological phase transition.

Exact parent Hamiltonian has been proposed\(^{39,40} \) for a non-Abelian spin-1 Pfaffian state. That Hamiltonian contains three-body interactions and the interaction length is long-ranged. If the Hamiltonian is truncated to only contain short-range interactions, its ground state possibly falls in the same phase\(^{41} \). Here we consider a simple local spin model on the Kagome lattice,

\[ H = \sum_{(ij)} [J_1 S_i \cdot S_j - K (S_i \cdot S_j)^2] + J_\chi \sum_{d, s} (S_i \times S_j) \cdot S_k, (3) \]

where \( J_1, K > 0 \), and \( i, j, k \) goes counterclockwise on each equilateral triangle (generally the three-body interactions on skew triangles also exist but are ignored here for simplicity). The \( J_\chi \) term preserves \( SO(3) \) symmetry but explicitly breaks time-reversal symmetry, which may help to stabilize the NACSL state (see Supplemental Material).

Minimizing the energy of the projected \( p_x + ip_y \) superconductors with respect to the Hamiltonian (3), we obtain a tentative phase diagram with two spin liquid phases, as shown in Fig. 4. It is notable that the quantum phase transition between the two spin liquids is a topological transition since the two phases have the same symmetry and only differ by topological orders. We also use an Abelian chiral spin liquid state\(^{42,43} \) (projected Chern band insulator) as a trial wave function and find that its variational energy is generally higher than the NACSL. However, we cannot rule out the possible existence of symmetry breaking phases\(^{44,45} \) which may appear in the phase diagram when \( K \) is large enough. The full phase diagram of the Hamiltonian (3) is an interesting open issue and deserves further investigations.

The non-Abelian anyons can be used for topological quantum computation. A natural question is how to trap and control the non-Abelian anyons. Since \( \pi \)-vortices traps Ising-like anyons, the question becomes how the \( \pi \)-vortices locally change the spin interactions. Noticing that the \( \pi \)-vortex in a triangle reverses the sign of the three-body interaction (see Supplemental Material), the non-Abelian anyons can be created locally by defect triangles with a \( -J_\chi \) interaction\(^{16} \).

The method of studying the Ising-like topological order in the present work can also be applied to study other topological orders.

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Appendix A: Projected topological superconductor and its effective (response) field theory

Since the fermions are interacting, the mean-field parameters $\chi, \Delta$ and $\lambda$ are subject to fluctuations. The amplitude fluctuations are gapped, at low energy we only need to consider the phase fluctuations, which behave like gauge fields coupling to the fermionic spinons. Owing to the fermion pairing, the $U(1)$ gauge fluctuations are further Higgsed and only Z$_2$ gauge symmetry remains. So we only need to consider the Z$_2$ gauge fluctuations,

$$H_{mf}(\sigma_{ij}, \delta \lambda_i) = \sum_{\langle ij \rangle} \sigma_{ij} [\chi C_i^\dagger C_j + \Delta C_i^\dagger \tilde{C}_j + \text{h.c.}] + \sum_i (\lambda + \delta \lambda_i)(C_i^\dagger C_i - 1), \quad (A1)$$

where $\sigma_{ij}$ are spatial components of Z$_2$ gauge fields and the temporal fluctuations $\delta \lambda_i$ is still continuous. When the gauge fluctuations are integrated out, we obtain the low-energy effective field theory of the spin system.

A practical way to obtain a spin wave function from the mean-field state is to enforce the particle-number constraint by performing a Gutzwiller projection,

$$|\text{RVB}\rangle = P_G|\text{BCS}\rangle = \sum_\alpha \det A(1, -1) \text{pf} B(0, 0) |\alpha\rangle, \quad (A2)$$

where $|\alpha\rangle$ is an Ising configuration, $P_G$ is the Gutzwiller projection operator that enforces the particle-number constraint, $|\text{BCS}\rangle$ is the mean-field ground state

$$|\text{BCS}\rangle = \prod_{i>j} [1 + a_{ij}(c_{1i}^\dagger c_{-1j}^\dagger - c_{0i}^\dagger c_{0j}^\dagger + c_{-1i}^\dagger c_{1j}^\dagger)] |\text{vac}\rangle,$$

here $a_{ij} = -a_{ji}$ is the relative wave function of the spinons in a Cooper pair. The matrices $A(1, -1)$ and $B(0, 0)$ are given as

$$A(1, -1) = \begin{pmatrix} a_{m_1n_1} & a_{m_1n_2} & \cdots \\ a_{m_2n_1} & a_{m_2n_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

$$B(0, 0) = \begin{pmatrix} 0 & -a_{p_1p_2} & \cdots \\ -a_{p_2p_1} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where $m_i, n_i, p_i$ are the positions of $c_1, c_{-1}, c_0$ fermions in the configuration $|\alpha\rangle$, respectively. The particle number of $c_1$ is equal to that of $c_{-1}$ since the total spin moment $S_z$ of the ground state is 0.

If we introduce three new bases

$$c_{x}^\dagger = \frac{1}{\sqrt{2}} (c_{1}^\dagger - c_{-1}^\dagger), \quad c_{y}^\dagger = \frac{i}{\sqrt{2}} (c_{1}^\dagger + c_{-1}^\dagger), \quad c_{z}^\dagger = c_{0}^\dagger,$$

then the mean-field ground state becomes

$$|\text{BCS}\rangle = \prod_{i>j} \left[1 + a_{ij}(c_{x_1}^\dagger c_{x_j}^\dagger + c_{y_1}^\dagger c_{y_j}^\dagger + c_{z_1}^\dagger c_{z_j}^\dagger)\right] |\text{vac}\rangle = \prod_{r>s} (1 - a_{rs}c_{x_r}^\dagger c_{x_s}^\dagger) \prod_{a>v} (1 - a_{uv}c_{y_u}^\dagger c_{y_v}^\dagger) \times \prod_{p>q} (1 - a_{pq}c_{z_p}^\dagger c_{z_q}^\dagger) |\text{vac}\rangle$$

which is essentially three copies of $p_x + ip_y$ superconductors. The projected states can also be written as

$$|\text{RVB}\rangle = P_G|\text{BCS}\rangle = \sum_\alpha \text{pf} C(x, x) \text{pf} D(y, y) \text{pf} B(z, z) |\alpha'\rangle$$

where $\alpha'$ is the spin configuration created by $c_{x}^\dagger, c_{y}^\dagger, c_{z}^\dagger$. The matrices $B, C, D$ are defined as $B(z, z) = B(0, 0)$ and

$$C(x, x) = \begin{pmatrix} 0 & -a_{m_1n_2} & \cdots \\ -a_{m_2n_1} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

$$D(y, y) = \begin{pmatrix} 0 & -a_{n_1n_2} & \cdots \\ -a_{n_2n_1} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

here $m_i, n_i, p_i$ are the positions of the $c_x, c_y, c_z$ fermions in the configuration $|\alpha'\rangle$, respectively. The particle numbers of $c_x, c_y, c_z$ are arbitrary, the only constraint is that their sum is equal to the number of sites.

If the superconducting bands have nontrivial Chern numbers, then $a_{ij} \propto |r_i - r_j|^{-1}$ at large distance and the projected spin wave function resembles the pfaffian states given in Ref. 22 and 28.

In the topological superconductor, since each flavor of fermions have nontrivial Chern numbers 1/2 (the half-quantized Chern number is owing to particle-hole redundancy), we expect the system has nontrivial response when it is probed by symmetry twisting fields. Based on the mean-field theory, the spinons will couple to not only the gauge fluctuations but also the symmetry twisting fields (behaving like gauge fields)

$$H_{mf}(\sigma_{ij}, A_{ij}) = \sum_{\langle i,j \rangle} \sigma_{ij} [\chi_i C_i^\dagger C_j + \Delta_i C_i^\dagger e^{iA_i} C_j + \text{h.c.}] + \sum_i (\lambda_i + \delta \lambda)(C_i^\dagger C_i - 1), \quad (A3)$$

where $A_{ij} = A_{ij}^x S^x + A_{ij}^y S^y + A_{ij}^z S^z$ is the external $SO(3)$ probing field corresponding to spin rotation symmetry. Owing to $SO(3)$ gauge invariance, we expect the following $SO(3)$ Chern-Simons response theory (in the imaginary time formalism and in the continuum limit) after integrating out the internal gauge fluctuations and the fermions,

$$\mathcal{L}_{\text{eff}} = i - \frac{k}{4\pi} \text{Tr} [\varepsilon^{\mu\nu\lambda} (A_\mu \partial_\nu A_\lambda - \frac{1}{3} A_\lambda^3)] + \mathcal{L}_{\text{Maxwell}} + \ldots,$$
where $k = 1$ is twice of the fermion Chern-number since both the $c_1$ and $c_−1$ fermions carry symmetry charge and contribute to the response. The $SO(3)\_1$ Chern-Simons theory supports Ising-like non-Abelian anyonic excitations, indicating that the spin ground state is a non-Abelian spin liquid. The $SO(3)\_1$ Chern-Simons action is not gauge invariant under $SO(3)$ gauge transformation if space-time has a boundary. The gauge anomaly can be canceled by a $SO(3)$ chiral Wess-Zumino-Witten term, which can be represented as three species of chiral Majorana fermions at the boundary. In other words, the boundary is described by $SO(3)$ conformal field theory and carries chiral central charge $c_− = \frac{3}{2}$.

If the probe field only contains $z$-component, $A_\mu = A_\mu^z S^z$, then we obtain the spin Hall current,

$$ J^z_\mu = \frac{\delta L_{\text{eff}}}{\delta A^z_\mu} = \frac{1}{2\pi} F^z_\mu, $$

where $F^z_\mu = i\varepsilon^{\mu\nu\lambda}(\partial_\nu A^z_\lambda)$ is the strength of the probing field and the spin Hall conductance is 1 in unit of $\frac{1}{2\pi}$. Notice that the spin Hall conductance for bosonic $U(1)$ symmetry protected topological phases and $S = 1$ Abelian chiral spin liquids are even integers time $\frac{1}{2\pi}$. 43, 47–49

**Appendix B: Correlation function and zero modes in the non-Abelian chiral spin liquid**

From the discussion above, the response field theory predicts that the bulk of the NACSL is gapped and the boundary is gapless. We checked this numerically and the results are shown in Fig. 5. The spin-spin correlation in the bulk fits well to an exponential function, which confirms that the bulk is gapped; while the spin-spin correlation on the boundary fits better with a power-law function, in agreement with the fact that the boundary is gapless.

Now we focus on the $\pi$-vortex. According to Ref. 29, the $\pi$-vortex core traps Majorana zero modes in a topological superconductor. In our case since the three copies of fermions $c_x, c_p, c_z$ have the same mean-field Hamiltonian owing to the $SO(3)$ symmetry, a $\pi$-vortex will trap three Majorana zero modes $\gamma_x, \gamma_y, \gamma_z$ (where $\{\gamma_m, \gamma_n\} = 2\delta_{mn}$) in the weak pairing phase. In the following we show that these three Majorana zero modes vary as spin-1/2 under spin rotation.

The three Majorana operators form an $SO(3)$ vector. For example, under the rotation $e^{iS_z, \theta}$, the operators vary as

$$ e^{-iS_z, \theta} \begin{pmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{pmatrix} e^{iS_z, \theta} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{pmatrix}, $$

(B1)

where $S_z$ is the operator acting on the degenerate Hilbert space of degenerate of the Majorana operators. It can be checked that the following expression satisfies (B1):

$$ S_z = -\frac{i}{2} \gamma_x \gamma_y. $$

Similarly we have $S_x = -\frac{i}{2} \gamma_y \gamma_z$ and $S_y = -\frac{i}{2} \gamma_z \gamma_x$. Then we have

$$ S_x^2 + S_y^2 + S_z^2 = \frac{3}{4} = \frac{1}{2} \times \left(\frac{1}{2} + 1\right), $$

from which we can conclude that the spin quantum number of the Hilbert space for three zero modes is 1/2.

Since the Gutzwiller projection does not change the quantum number, after projection the spin-1/2 degrees of freedom remain at the $\pi$-vortex core. To verify this result, we numerically calculated the cluster spin correlation function for two $\pi$-vortices, as shown in Fig. 3 in the main text. The two vortex-cores can form a spin singlet or a spin triplet. If the distance between them is by far larger than the correlation length, then the singlet and the triplet are degenerate in energy. In Fig. 3 the calculation is performed in the singlet state.

Notice that each $\pi$-vortex not only carry a local spin-1/2 degrees of freedom, but also carry a nonlocal Hilbert space similar to the Ising anyon. Generally, braiding of the two $\pi$-vortices results in a non-local operation (which generates entanglement) together with a local spin rotation for the spin-1/2 degrees of freedom. Actually, since the $\pi$-vortex is shared by the three copies of $p_x + ip_y$ topological superconductors it is essentially a composite anyon $\sigma : \sigma \times \sigma \times \sigma = (1 + \psi) \times \sigma = \sigma + \sigma$. If the $SO(3)$ symmetry is explicitly broken by a magnetic field, the local degeneracy will be lifted and the composite anyon will reduce to the simple anyon $\sigma$.

**Appendix C: Calculation of the modular matrices**

The $T$ matrix can be obtained from the universal wave function overlap

$$ \langle \psi_n | \hat{T} | \psi_n \rangle, $$

where $\hat{T}$ is a Dehn twist operation of the torus. Generally, the overlap after a full Dehn twist is exponentially small with the increasing lattice sites. To solve this problem, we apply the trick introduced in Ref. 36 to divide the Dehn twist into many substeps such that in each substep the wave function changes adiabatically and the Berry phase can be obtained with a relatively high accuracy.

The Dehn twist is first separated into $L_y$ steps, in each step, the couplings crossing the $x$-boundary is shifted along $y$-direction by one lattice site (see Fig. 6) such that the original Hamiltonian $H[(L_x, i), (1, i + t)]$ transforms into $H[(L_x, i), (1, i + t + 1)]$. Noticing that during the shifting, some bonds cross the $y$-boundary and the sign of the coupling is affected by the $y$-boundary condition. After a full Dehn twist $T_y$, the boundary conditions $(0, \pi), (\pi, \pi)$ are shifted into $(\pi, \pi), (0, \pi)$ respectively, and the boundary condition $(\pi, 0)$ remains unchanged.

Secondly, to make the twist more smooth, each step is further divided into several sub-steps by a parameter...
The correlations are calculated on a cylinder with \( L_x \times L_y \) unit cells, where \( L_x = 20, L_y = 10 \) and \( x \)-direction has periodic boundary condition while \( y \)-direction is open. (a) The spin-spin correlation function on the (upper) boundary of the chiral spin liquid decays in power law; (b) The spin-spin correlation function in the bulk decays exponentially.

\[
\lambda \in [0, 1]; \\
H(t, \lambda) = H[(L_x, i), (1, i + t)] + \lambda H[(L_x, i), (1, i + t + 1)].
\]

In our numerical calculation, we take discrete values \( \lambda = 0, 0.25, 0.5, 0.75, 1. \)

The Berry phase is obtained from the overlap of the projected ground states of above Hamiltonians

\[
\phi(t, \lambda) = \text{Im} \left[ \ln \left( \langle t, \lambda | P_G P_C | t, \lambda + \delta \lambda \rangle \right) \right], 
\]

where \( |t, \lambda \rangle \) is the mean-field ground state of the Hamiltonian \( H(t, \lambda) \) and the total Berry phase is given by

\[
\varphi(L^2) = \sum_{t, \lambda} \phi(t, \lambda).
\]

For the sector \((\pi, 0)\), after a full Dehn twist \( T_y \) defined above, the Hamiltonian goes back to itself. So the evolution path is closed and the Berry phase is well defined.

However, for the sectors \((\pi, \pi)\) and \((0, \pi)\), since a full Dehn twist exchanges them, the path is not closed. If we act the full Dehn twist twice, the path will be closed and the Berry phase will be well defined. The Berry phase for half of the loop (a single Dehn twist) is defined as half of that of a double twist.

Finally, the Dehn twist \( \hat{T}_y \) can be transformed into \( \hat{T}_{-y} \) (the inverse of \( \hat{T}_y \)) by a global 90° clockwise rotation. The rotation does not affect the Berry phase, so we obtain the following relations:

\[
\varphi_{y,y} = -\varphi_{x,x}, \quad \varphi_{x,xy} = -\varphi_{y,yx}.
\]

**Appendix D: Decoupling the spin-spin interactions**

In this section, we explain why we can use the mean-field parameters \( \chi, \Delta \) as variational parameters. It is known that the two-body spin interactions can be written in forms of fermions as \(30, 50\):

\[
\begin{align*}
S_i \cdot S_j &= -\langle \hat{\chi}_{ij}^{\dagger} \hat{\Delta}_{ij}, \\
(S_i \cdot S_j)^2 &= \hat{\Delta}_{ij}^{\dagger} \hat{\Delta}_{ij}.
\end{align*}
\]

where \( \hat{\chi}_{ij} = C_i^{\dagger} C_j \) and \( \hat{\Delta}_{ij} = C_i^{\dagger} C_j \). So the mean-field decoupling of the \( J \) and \( K \) terms naturally give rise to the mean-field parameters \( \chi \sim \langle \hat{\chi}_{ij} \rangle \) and \( \Delta \sim \langle \hat{\Delta}_{ij} \rangle \).

Now we focus on the three-body interactions \( S_i \times S_j \cdot S_k \). In the \( c_x, c_y, c_z \) bases, the hopping and pairing op-
The spin operators can be written as

\[ S^\alpha = -i\varepsilon^{\alpha\beta\gamma} c^\dagger_\beta c_\gamma, \]

where \( \alpha, \beta, \gamma = x, y, z \) and repeated indices are summed (the same below).

Eq. (D2) can be simplified as

\[
(S_i \times S_j) \cdot S_k = i \left[ \langle \hat{\chi}_{ij} \hat{\Delta}_{jk} \hat{\Delta}_{ki} + \hat{\Delta}_{ij} \hat{\Delta}_{jk} \hat{\chi}_{ki} + \hat{\Delta}_{ij} \hat{\chi}_{jk} \hat{\chi}_{ki} \rangle - \text{h.c.} \right] \tag{D3}
\]

So using the two parameters \( \chi \) and \( \Delta \) we can decouple the three-body interaction \( J_\chi (S_i \times S_j) \cdot S_k \) in a spin liquid phase. Since the three-body term breaks time reversal symmetry, it favors a time reversal symmetry breaking state. If the ground state does not break spin rotation symmetry, then the projected \( p_x + ip_y \) superconductor will be its candidate ground state.

Furthermore, from the expression (D3), if there is a \( \pi \)-vortex in the triangle \((ijk)\), then the values of \( \langle \hat{\chi}_{ij} \rangle \) and \( \langle \hat{\Delta}_{ij} \rangle \) on one of the three bonds (namely, \( ij, jk \) and \( ki \)) reverse its sign. As a result, the value of the three-body interaction term on the triangle \( \langle J_\chi (S_i \times S_j) \cdot S_k \rangle \) also reverses its sign and the state is in a higher energy comparing with the ground state. If we reverse the sign of \( J_\chi \) on the triangles which contain \( \pi \)-vortices, then \( \langle -J_\chi (S_i \times S_j) \cdot S_k \rangle \) will still have a low energy and the state with anyons localized near the vortices becomes the ground state of the new Hamiltonian.