Horizons and singularity in Clifton’s spherical solution of $f(R)$ vacuum

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Abstract

Due to the failure of Birkhoff’s theorem, black holes in $f(R)$ gravity theories in which an effective time-varying cosmological “constant” is present are, in general, dynamical. Clifton’s exact spherical solution of $R^{1+\delta}$ gravity, which is dynamical and describes a central object embedded in a spatially flat universe, is studied. It is shown that apparent black hole horizons disappear and a naked singularity emerges at late times.
1 Introduction

The study of Type Ia supernovae \cite{43, 41, 44, 42, 43, 61, 30, 46, 3} revealed that the universe is currently accelerating its expansion. This discovery has generated an enormous amount of activity and theoretical models in order to find an explanation of this phenomenon. The most common models are based on General Relativity (GR) and invoke mysterious forms of dark energy (see \cite{32} for a list of references). However, dark energy, possibly even phantom energy, is too exotic and \textit{ad hoc} and attempts have been made to model the cosmic acceleration without dark energy. \(f(R)\) theories of gravity reminiscent of the quadratic corrections to the Einstein-Hilbert action introduced by renormalization have been re-introduced in the metric \cite{5, 10}, Palatini \cite{63}, and metric-affine \cite{51, 52, 53, 56, 57} formulations and have received much attention in recent years (see \cite{55, 16} for reviews and \cite{58, 54, 6, 20, 7, 39, 48} for introductions). Emilio Elizalde has given many contributions to the development of \(f(R)\) gravity and cosmology.

While many cosmological and other aspects of \(f(R)\) gravity (stability, weak-field limit, ghost content) have been discussed in recent years, it is important to understand spherical solutions (both vacuum and interior) in these theories \cite{28, 11, 50, 49, 31, 2, 15, 37, 62}. Metric \(f(R)\) gravity is described by the action

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S^{(\text{matter})},
\]  

(1.1)

where \(f(R)\) is a non-linear function of its argument and \(S^{(\text{matter})}\) is the matter part of the action. \(R\) denotes the Ricci scalar of the metric \(g_{ab}\) with determinant \(g\), \(\kappa \equiv 8\pi G\) where \(G\) is Newton’s constant, and we adopt the notations of Ref. \cite{64}.

The Jebsen-Birkhoff theorem of GR fails in these theories, adding to the variety of spherical solutions \cite{23}. Of particular interest are black holes in generalized gravity, which have been studied especially in relation to their thermodynamics \cite{24} and references therein. Since \(f(R)\) theories are designed to produce an effective dynamical cosmological constant, physically relevant spherically symmetric and black hole solutions are likely to describe central objects embedded in cosmological backgrounds. This kind of solution is poorly understood even in GR, although a few examples are available there \cite{59, 21, 9, 8, 35, 36, 25, 47, 29, 40, 34, 26}. Even less is known about \(f(R)\) black holes, which are certainly worth exploring. Here we consider a specific solution of vacuum \(f(R) = R^{1+\delta}\) gravity discovered in \cite{11}. Solar System experiments set the limits \(\delta = (-1.1 \pm 1.2) \cdot 10^{-5}\) on the parameter \(\delta\) \cite{11, 14, 12, 13, 65}, while local stability requires \(f''(R) \geq 0\) \cite{17, 18, 19, 38}, hence we restrict to the range \(0 < \delta < 10^{-5}\).

\(^1\)See also \cite{14} for this specific form of the function \(f(R)\).
The solution of \cite{11} is dynamical and describes a time-varying central object embedded in a spatially flat universe in vacuum $R^{1+\delta}$ gravity. This solution is made possible by the fact that the fourth order field equations of metric $f(R)$ gravity

$$f'(R)R_{ab} - \frac{f(R)}{2} g_{ab} = \nabla_a \nabla_b f'(R) - g_{ab} \Box f'(R)$$

in vacuo can be rewritten as effective Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R = \frac{1}{f'(R)} \left[ \nabla_a \nabla_b f' - g_{ab} \Box f' + g_{ab} \frac{(f - R f')}{2} \right]$$

with geometric terms acting as effective matter on the right hand side. This time-varying effective matter invalidates the Jebsen-Birkhoff theorem and can propel the acceleration of the universe. An equivalent representation of metric $f(R)$ gravity as an $\omega = 0$ Brans-Dicke theory with a special scalar field potential reveals explicitly the presence of a massive scalar degree of freedom $f'(R)$ responsible for these effects \cite{55}. Since analytical spherical and dynamical solutions of $f(R)$ gravity in asymptotically Friedmann-Lemaitre-Robertson-Walker (FLRW) backgrounds are harder to find than in GR (where only a few are known anyway), Clifton’s solution is particularly valuable.

## 2 Clifton’s solution and its horizons

In this section we describe the Clifton solution \cite{11} and the work \cite{22} locating the horizons of this solution.

The spherically symmetric and time-dependent solution of vacuum $R^{1+\delta}$ gravity of \cite{11} is given by

$$ds^2 = -A_2(r) dt^2 + a^2(t) B_2(r) \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right],$$

and (using the isotropic radius and the notations of \cite{11})

$$A_2(r) = \left( \frac{1 - C_2/r}{1 + C_2/r} \right)^{2/q},$$

$$B_2(r) = \left( 1 + \frac{C_2}{r} \right)^4 A_2(r)^{q+2\delta-1},$$

$$a(t) = t^{\frac{\delta(1+2\delta)}{1-\delta}},$$

$$q^2 = 1 - 2\delta + 4\delta^2.$$
Once $\delta$ is fixed, two classes of solutions exist, corresponding to the sign of $C_2 qr$. The line element (2.1) reduces to the FLRW one if $C_2 \rightarrow 0$. In the limit $\delta \rightarrow 0$ in which the theory reduces to GR, eq. (2.1) reduces to the Schwarzschild metric in isotropic coordinates provided that $C_2 qr > 0$, hence positive and negative values of $r$ are possible according to the sign of $C_2$, but we assume $r > 0, C_2 > 0$ and take the positive root in the expression $q = \pm \sqrt{1 - 2\delta + 4\delta^2}$, so that $q \simeq 1 - \delta$ as $\delta \rightarrow 0$. The solution (2.1)-(2.5) is conformal to the Fonarev solution [27] which is conformally static [33], and therefore is also conformally static, similar to the Sultana-Dyer [59, 21, 9, 8] and certain generalized McVittie solutions [26] of GR.

In order to identify possible apparent horizons, it is convenient to cast the metric (2.1) in the Nolan gauge. Using first the Schwarzschild-like radius

$$\tilde{r} \equiv r \left(1 + \frac{C_2}{r}\right)^2,$$

(2.6)

giving $dr = \left(1 - \frac{C_2^2}{r^2}\right)^{-1} d\tilde{r}$ and then the areal radius

$$\rho \equiv a(t) \sqrt{B_2(r)} \tilde{r} = a(t) \tilde{r} A_2(r) \frac{q^2 + 2\delta - 1}{2},$$

(2.7)

the line element (2.1) takes the form

$$ds^2 = -A_2 dt^2 + a^2 A_2^{2\delta - 1} d\tilde{r}^2 + \rho^2 d\Omega^2.$$ 

(2.8)

Denoting the differentiation with respect to time with an overdot and using the identities

$$d\tilde{r} = \frac{dp - A_2^{q + 2\delta - 1} \dot{a} \tilde{r} dt}{a \left[ A_2^{q + 2\delta - 1} + \frac{2(q + 2\delta - 1) C_2}{q} A_2^{\frac{3q - 1}{2}} \right]} \equiv \frac{dp - A_2^{q + 2\delta - 1} \dot{a} \tilde{r} dt}{a A_2^{q + 2\delta - 1} C(r)},$$

(2.9)

one obtains

$$C(r) = 1 + \frac{2(q + 2\delta - 1)}{q} \frac{C_2}{r} A_2^q = 1 + \frac{2(q + 2\delta - 1)}{q} \frac{C_2 a}{\rho} A_2^{\frac{3q - 1}{2}},$$

(2.10)

which turns the metric into the Painlevé-Gullstrand-like form

$$ds^2 = -A_2 \left[ 1 - \frac{A_2^{2(\delta - 1)} \dot{a}^2 \tilde{r}^2}{C^2} \right] dt^2 - \frac{2A_2^{-\frac{q + 2\delta - 1}{2}}}{C^2} \dot{a} \tilde{r} dtd\rho$$

$$+ \frac{d\rho^2}{A_2^2 C^2} + \rho^2 d\Omega^2.$$ 

(2.11)
Now we introduce a new time coordinate $\bar{t}$ defined by

$$d\bar{t} = \frac{1}{F(t, \rho)} [dt + \beta(t, \rho)d\rho] \quad (2.12)$$

in order to eliminate the cross-term $dt\,d\rho$. Here $F(t, \rho)$ is an integrating factor which guarantees that $d\bar{t}$ is an exact differential and is determined by

$$\frac{\partial}{\partial \rho} \left( \frac{1}{F} \right) = \frac{\partial}{\partial t} \left( \frac{\beta}{F} \right). \quad (2.13)$$

The line element becomes

$$ds^2 = -A_2 \left[ 1 - \frac{A_2^{2(\delta-1)}}{C^2} \dot{a}^2 \bar{r}^2 \right] F^2 d\bar{t}^2 + 2F \left\{ A_2 \beta \left[ 1 - \frac{A_2^{2(\delta-1)}}{C^2} \dot{a}^2 \bar{r}^2 \right] - \frac{A_2^{-q+2\delta-1}}{C^2} \dot{a} \right\} d\bar{t}d\rho$$

$$+ \left\{ -A_2 \left[ 1 - \frac{A_2^{2(\delta-1)}}{C^2} \dot{a}^2 \bar{r}^2 \right] \beta^2 + \frac{2A_2^{-q+2\delta-1}}{C^2} \dot{a} \beta + \frac{1}{A_2^3} \right\} d\rho^2 + \rho^2 d\Omega^2. \quad (2.14)$$

The choice

$$\beta = \frac{A_2^{-q+2\delta-1}}{C^2} \frac{\dot{\bar{r}}}{1 - \frac{A_2^{2(\delta-1)}}{C^2} \dot{a}^2 \bar{r}^2} \quad (2.15)$$

eliminates the $dt\,d\rho$ term and casts the metric in the Nolan gauge

$$ds^2 = -A_2 DF^2 d\bar{t}^2 + \frac{1}{A_2^3} \left[ 1 + \frac{A_2^{-q-1}H^2}{C^2 D} \right] d\rho^2$$

$$+ \rho^2 d\Omega^2, \quad (2.16)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter of the background universe and

$$D \equiv 1 - \frac{A_2^{2(\delta-1)}}{C^2} \dot{a}^2 \bar{r}^2 = 1 - \frac{A_2^{-q-1}}{C^2} H^2 \rho^2. \quad (2.17)$$
Using the second of these equations, the line element (2.16) assumes the simple form
\[ ds^2 = -A_2^2 dF^2 dt^2 + \frac{d\rho^2}{A_2^2 C^2 D} + \rho^2 d\Omega^2. \] (2.18)

The apparent horizons, if they exist, are located at \( g^{\rho\rho} = 0 \), which yields \( A_2^2 C^2 D = 0 \) and \( A_2^2 \left( C^2 - H^2 R^2 A_2^{-q-1} \right) = 0 \). Therefore, \( g^{\rho\rho} \) vanishes if \( A_2 = 0 \) or \( H^2 R^2 = C^2 A_2^{q+1} \). \( A_2 \) vanishes at \( r = C_2 \), which describes the Schwarzschild horizon when \( \delta \to 0 \) (the GR limit). This locus corresponds to a spacetime singularity because the Ricci scalar \( R = \frac{6(\dot{H}+2H^2)}{A_2(r)} \) diverges as \( r \to C_2 \) (it reduces to the usual FLRW value \( 6 \left( \dot{H} + 2H^2 \right) \) as \( C_2 \to 0 \)). This singularity is strong according to Tipler’s classification [60] because the areal radius \( \rho = a\tilde{r} A_2^{q+1} \) vanishes when \( r = C_2 \) for \( \delta > 0 \), in contrast with the Schwarzschild metric corresponding to \( \delta = 0 \) in which \( \rho = \tilde{r} = 4C_2 \) at \( r = C_2 \).

The second possibility \( H^2 \rho^2 = C_2 A_2^{q+1} \) yields
\[ H\rho = \pm \left[ 1 + \frac{2(q + 2\delta - 1)}{q} C_2 a A_2^{\frac{q+2\delta-1}{q-1}} \right] A_2^{\frac{q+1}{q}}, \] (2.19)

with the positive sign corresponding to an expanding universe. When \( \delta \to 0 \), this equation reduces to \( H\rho = \left[ 1 + \frac{2\delta C_2 a}{\rho} A_2^{\frac{q+2\delta-1}{q-1}} \right] A_2^{1-\delta} \).

To gain some insight, consider the following two limits. As \( C_2 \to 0 \) (the central object disappears and the solution is FLRW space), \( r = \tilde{r} \) and \( \rho \) become a comoving and a proper radius, respectively, while eq. (2.19) reduces to \( H\rho = 1 \) with solution \( \rho_c = 1/H \), the radius of the cosmological horizon. In the limit \( \delta \to 0 \) in which the theory reduces to GR, eq. (2.19) reduces to \( A_2 = 0 \) or \( r = C_2 \) with \( H \equiv 0 \).

Using eqs. (2.4) and (2.7), the left hand side of eq. (2.19) is expressed as
\[ HR = \frac{\delta (1 + 2\delta)}{1 - \delta} \frac{x^{\frac{q+2\delta-1}{q-1}}}{(1-x)^{1-q}(1+x)^{-q+2\delta-1}}, \] (2.20)

where \( x \equiv C_2/r \), while the right hand side of (2.19) is
\[ \left( \frac{1-x}{1+x} \right)^{\frac{q+1}{q}} \left[ 1 + \frac{2(q + 2\delta - 1)}{q} \frac{x}{(1-x)^2} \right]. \] (2.21)
Eq. (2.19) then becomes

\[
\frac{1}{t^{\frac{1-2\delta-2\delta^2}{1-\delta}}} = \frac{(1-\delta)}{(1+2\delta)C_2} \frac{x (1-x)^{2q+2\delta-2}}{(1-x)^{2(\delta-1)\frac{q}{4}}} \\
\cdot \left[1 + 2 \left( \frac{q+2\delta-1}{q} \frac{x}{(1-x)^2} \right) \right]
\]  
(2.22)

(note that \(\frac{1-2\delta-2\delta^2}{1-\delta}\) is positive for \(0 < \delta < \frac{\sqrt{3}-1}{2} \approx 0.366\)).

At late times \(t\), the left hand side of eq. (2.22) vanishes, \(x \approx 0\), and there exists a unique root of the equation locating the apparent horizons, which corresponds to a cosmological horizon, consistently with the fact that \(r \to \infty\) as \(x = C_2/\rho \to 0\). The limit \(x \to 0\) can also be obtained when the parameter \(C_2 \to 0\), in which case \(H \rho \to 1\) and \(r \approx \rho \approx H^{-1} = \frac{1-\delta}{\delta (1+2\delta)} t\) is the radius of the cosmological horizon of the FLRW space without a central object. Hence, there is only a cosmological apparent horizon and no black hole apparent horizons at late times: the central singularity at \(\rho = 0\) becomes naked.

The radii \(\rho\) of the apparent horizons and the time \(t\) can be expressed in the parametric form

\[
\rho(x) = t(x) \frac{\delta (1+2\delta)}{1-\delta} \frac{C_2}{x} (1-x)^{\frac{2q+2\delta-1}{q}} (1+x)^{\frac{2-2\delta+1}{q}}, \quad (2.23)
\]

\[
t(x) = \left\{ \frac{(1-\delta)}{\delta (1+2\delta)C_2} \frac{x (1+x)^{2q+2\delta-1}}{(1-x)^{\frac{2(\delta-1)}{q}}} \left[1 + 2 \left( \frac{q+2\delta-1}{q} \frac{x}{(1-x)^2} \right) \right] \right\}^{\frac{1-\delta}{2q+2\delta-1}}, \quad (2.24)
\]

using \(x\) as a parameter. Fig. 1 reports \(\rho\) versus \(t\) for the parameter values \(C_2 = 1\) and \(\delta = 0.13\), showing that two inner horizons develop after the Big Bang covering the central singularity \(\rho = 0\), then they approach each other, merge, and disappear, while a third, cosmological horizon keeps expanding. The \(\rho = 0\) singularity becomes naked after this merging event.

### 3 Discussion and conclusions

Cosmologists may be detecting deviations from GR and therefore it is necessary to understand spherical solutions of \(f(R)\) gravity, which has been proposed as a simple alternative to the mysterious dark energy. Since the Jebsen-Birkhoff theorem fails in these theories, spherical solutions do not have to be static. \(f(R)\) theories are designed
with a built-in dynamical cosmological constant to model the present acceleration of
the universe, hence analytical spherical solutions describing a central object embedded
in a FLRW background are the relevant ones. Unfortunately, such solutions are poorly
understood even in GR [59, 21, 9, 5, 33, 8, 36, 25, 17, 29, 40, 34, 26]. It seems difficult to
find generic solutions describing black holes embedded in FLRW backgrounds. Finding
numerically spherical interior solutions of $f(R)$ gravity is also an active area of research
[28, 21, 50, 49, 31, 2, 15, 37, 62]. All these issues deserve further attention in the future.

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Figure 1: Radii of the apparent horizons of Clifton’s solution (vertical axis) versus time (horizontal axis) for the parameter values $C_2 = 1$ and $\delta = 0.13$. 