The Measure of a Measurement

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Abstract
We identify a fractal scale $s$ in a family of Borel probability measures $\mu$ on the unit interval which arises independently in quantum information theory and in wavelet analysis. The scales $s$ we find satisfy $s \in \mathbb{R}_+$ and $s \neq 1$, some $s < 1$ and some $s > 1$. We identify these scales $s$ by considering the asymptotic properties of $\mu(J)/|J|^s$ where $J$ are dyadic subintervals, and $|J| \to 0$.

I Introduction

While finite non-commutative operator systems lie at the foundation of quantum measurement, they are also tools for understanding geometric iterations as used in the theory of iterated function systems (IFSs) and in wavelet analysis. Key is a certain splitting of the total Hilbert space and its recursive iterations to further iterated subdivisions. This paper explores some implications for associated probability measures (in the classical sense of measure theory), specifically their fractal components.

In quantum communication (the study of (quantum) error-correction codes), certain algebras of operators and completely positive mappings form the starting point; see especially the papers [18] and [21]. They take the form of a finite number of channels of Hilbert space operators $F_i$ which are assumed to satisfy certain compatibility conditions. The essential one is that the operators from a partition of unity, or rather a partition of the identity operator $I$ in the chosen Hilbert space. Here (Definition I.1) such a systems $(F_i)$ are known as column isometries. An extreme case of this is when a certain Cuntz relation (Definition I.1) is satisfied by. Referring back to our IFS application, the extreme case of the operator relations turn out to correspond to the limiting case of non-overlap. Using this operator theory, in
this paper we explore the fractal measures associated with the inherent self
similarity affine fractals, a subject involving both iterated function systems
(IFSs), and an aspect of quantum communication.

In the paper we aim to draw up connections between the following three
areas, quantum channels, fractal measures, and wavelets. Our understanding
of quantum measurement follows the tradition of Kraus [20]: By a quantum
measurement of a system refers to a system in some state and we want to de-
termine whether it has some property $E$, where $E$ should be thought of as an
element in a logic lattice of quantum yes-no questions. Measurement means
submitting the system to some procedure to determine whether the state sat-
ifies the property. The reference to system state in turn must be given an
operational meaning by reference to a statistical ensemble of systems. Each
measurement yields some definite value 0 or 1.

The connections to fractal measures and wavelets (see e.g., [16]) derive
from a common mathematical core based in turn on hierarchical patterns
common to them. Wavelets are computational bases in Hilbert space which
make use of the scaling law, with each step in the scaling creating a refinement
in a resolution of data, and taking advantage of similarity from one level
of resolution to the next. Hence we arrive at the characteristic feature of
fractals, see [1], [2].

Mathematically, an identification of quantum channels may be made with
a finite system of operators $(F_i)$ which together form a column isometry; see
(1) below. In the physics literature, they are known as Kraus-systems, and
mathematically they generalize the so called Cuntz relations; see [1], [2],
[7], [23], [3], [12], [20]. The systems $(F_i)$ which are the focus of our paper
in turn determine completely positive mappings $\alpha$ as recalled below. In
this paper we show that repeated application of operator system yields the
kind of refinement that underlies both wavelets and fractals, and moreover
that the statistical properties from quantum channels is of significance in
decomposition problems from wavelets. And with the scaling from wavelet
theory now corresponding to $\alpha$.

Both the quantum mechanical measurement problem and IFSs have as
starting point a finite set of operations: in the case of IFSs they are geo-
metric, and in the quantum case, they involve channels of Hilbert spaces
and associated operator systems. The particular aspects of IFSs we have in
mind are studied in [10]; and the relevant results from quantum communi-
cation in [18] and [21]. We begin the Introduction with some background
and motivation on IFSs. The operator theory, the fractal measures and their
applications are then taken up more systematically in section II below.

Let $\mathcal{H}$ be complex Hilbert space, and let $A$ be a finite set. We will be interested in an indexed set of operators $\{F_i | i \in A\}$ satisfying

$$\sum_{i \in A} F_i^* F_i = I$$

(1)

where $I$ denotes the identity operator in $\mathcal{H}$.

**Definition I.1** A finite system of operators $F_i$ in a Hilbert space $\mathcal{H}$ is said to be a column isometry if

$$\mathcal{H} \ni \xi \mapsto \begin{pmatrix} F_1 \xi \\ \vdots \\ F_N \xi \end{pmatrix} \in \mathcal{H}^\oplus \mathbb{N}$$

is isometric; and it is said to be a Cuntz-system if also

$$F_i F_j^* = S_{i,j} I.$$

Because of a certain reasoning outlined in the references below such systems are called *measurements* in quantum probability; see e.g., [16], [21, OA0404553], or [24], [25, quant-ph 101061], [22]. But they arise in other fields as well, in representation theory, in geometric measure theory, and in wavelet analysis; see e.g., [8], [14], [13], [15], and [17, Minimality, Adv. Math]. Closely related systems of Hilbert space operators play a big role in the theory of frames [11], [4] and their engineering applications. Suppose $\#A = N$. Denote by $\mathbb{N}$ the cyclic group of order $N$ viz., $\mathbb{Z}/N\mathbb{Z} \cong \{0, 1, \cdots, N - 1\}$ or $\mathbb{Z}_N$, and set

$$\Omega := \mathbb{N}^\mathbb{N} = \{\text{all functions } \mathbb{N} \to \mathbb{Z}_N\}.$$  \hspace{1cm} (2)

We shall give $\Omega$ its Tychonoff topology, and we view it as a compact Hausdorff space.

Let $\mathcal{F} = (F_i)_{i \in \mathbb{N}}$ be a measurement, and let $\psi \in \mathcal{H}$ be a unit-vector, i.e., a quantum mechanical *pure state*. Then it is immediate that

$$\mu_\psi (i) : = \|F_i \psi\|^2, \; i \in \mathbb{N}$$

(3)

is a probability distribution on $\mathbb{N}$.

The measure in the title of the present paper refers operator-valued, or scalar-valued measures on $\Omega = \Omega_N$ *induced* by (1) – (3). The induction from $\mathbb{Z}_N$ to $\Omega_N$ is based on the Kolmogorov consistency condition [19], [14], as follows:
Definition I.2
(a) Cylinder sets: Let \((i_1, \cdots, i_K) \in \mathbb{Z}_N^k\), and set
\[
C (i_1, \cdots, i_k) = \{ w \in \Omega | w (1) = i_1, \cdots, w (k) = i_k \}.
\] (4)

(b) Operator valued conditional probabilities:
\[
P (C (i_1, \cdots, i_k)) = F_{i_1}^* \cdots F_{i_k}^* F_{i_k} \cdots F_{i_1}
\] (5)

(c) Kolmogorov consistency: Formula below.
Since we have the disjoint union
\[
C (i_1, \cdots, i_k) = \bigcup_{j \in \mathbb{Z}_N} C (i_1, \cdots, i_k, j),
\] (6)
we need the formula
\[
P (C (i_1, \cdots, i_k)) = \sum_{j \in \mathbb{Z}_N} P (C (i_1, \cdots, i_k, j))
\] (7)
in order to extend (5) to a probability measure \(P\) defined on the Borel subsets of \(\Omega\). On the other hand, it is easy to see that (6) is satisfied by (5). Just use the basic formula (1) for the given measurement \(\mathcal{F}\).

Lemma I.3 There is a unique positive operator-valued probability measure \(P\) defined on \(\Omega\), and satisfying (7). For any Borel set \(B \subset \Omega\), \(P (B)\) is well defined, and \(\langle \psi | P (B) \psi \rangle \geq 0\) for all \(\psi \in \mathcal{H}\). Moreover, \(P\) is sigma-additive, or countably additive, i.e.,
\[
P \left( \bigcup_{j=1}^{\infty} B_j \right) = \sum_{j=1}^{\infty} P (B_j)
\] (8)
whenever \(B_1, B_2, \cdots\) are disjoint Borel sets;
\[
P (\Omega) = I
\] (9)

Proof. The argument for the existence and uniqueness of the extension is a standard application of Kolmogorov consistency. See [19], [14], [15], [13] for more details. ■
Examples I.4  

(a) \( N = 2, \ H = C^2, \ \psi = (1)_0 \), and

\[
F_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad F_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]  

Then

\[
\mu_\psi(C(i_1, \cdots, i_k)) = 2^{-k}, \quad i_j \in \{0, 1\}.
\]

(b) \( N = 3, \ H = C^3, \ \psi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \), and

\[
F_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then, for \( i_j \in \{0, 1, 2\}, \ k \in \mathbb{N}, \) and \( 1 \leq j \leq k, \) we have

\[
\mu_\psi(C(i_1, \cdots, i_k)) = \begin{cases} 0 & \text{if some } i_j = 1 \\ 2^{-k} & \text{otherwise.} \end{cases}
\]

Proposition I.5  

If we introduce \( N \)-adic partitions of the unit-interval \([0, 1]\) as follows

\[
C(i_1, \cdots, i_k) \rightarrow \left[ \frac{i_1}{N} + \cdots + \frac{i_k}{N^k}, \ \frac{i_1}{N} + \cdots + \frac{i_k}{N^k} + \frac{1}{N^k} \right]
\]

then the measures \( P(\cdot) \) and \( \mu_\psi(\cdot) = \langle \psi | P(\cdot) \psi \rangle \) are Borel measures, each supported on \([0, 1]\). If \( N = 2 \), the measure (11) in Example I.4 (a) turns out to be merely Lebesgue measure restricted to \([0, 1]\). If \( N = 3 \), the measure (12) in Example I.4 (b) is the middle-third Cantor measure supported in the Cantor set \( X_3 \).

Remark I.6  

Recall \( X_3 \) is the unique (compact) subset of \( \mathbb{R} \) satisfying

\[
3X_3 = X_3 \cup (X_3 + 2),
\]

and the Cantor measure \( \mu = \mu_3 \) is the unique Borel measures satisfying

\[
\int f(x) \, d\mu(x) = \frac{1}{2} \left( \int f\left(\frac{x}{3}\right) \, d\mu(x) + \int f\left(\frac{x+2}{3}\right) \, d\mu(x) \right)
\]

for all bounded Borel functions \( f \).
Proof. (of Proposition I.4) The assertions follow from standard applications of the Kolmogorov extension principle, and the reader is referred to [15], [13] for additional discussion.

The column isometries \((F_i)\) introduced above can be viewed as Kraus operators [20] from the theory of quantum channels, modeling “instruments” in Kraus’s formulation, and operating on quantum systems by producing a classical measurement (outcome.) The theory of finitely correlated states [12] has an instrument generating a classical state which takes the form of a measure on an infinite product of a finite alphabet; a construction which parallels the theme of our paper. It would be intriguing to explore how the physics of [12] reflects itself in the measures more directly associated to fractal measures, and wavelets. Part of the answer lies in how families of wavelet packets adapt to signals (or in 2D) to images, see e.g., [16]. The operators in the system \((F_i)\) are iterated in steps with each iteration step creating a subdivision of the masks in the previous more coarse resolution. A monomial of degree \(k\) in the generators \(F_i\) corresponds to a \(k\) fold subdivision. Even though this algorithm is “classical,” the generators \(F_i\) are non-commuting operators. The limit of this iterative scheme as \(k\) tends to infinity is made precise by the measures that we analyze in theorems II.3 and III.2 below.

II Fractal Scales

The authors of [9] recently adapted the discrete wavelet algorithms to fractals, and the present work extends [9].

The distinction between the two prototypical cases (a) and (b) in Example I.4 can be made precise in a number of different ways; for example, it can be checked that the fractal dimension (in this case = the Hausdorff dimension) of (a) is 1, and of (b) it is \(s = \frac{\ln 2}{\ln 3} = \log_3(2)\). For our present discussion, the following definition of the fractal dimension will suffice: If a subset \(X \subset \mathbb{R}^d\) is obtained by the iteration of a finite family of contractive and affine maps \(\mathbb{R}^d \to \mathbb{R}^d\) then the fractal dimension \(s\) of \(X\)

\[
s = \frac{\log (\text{number of replicas})}{\log (\text{magnification factor})}. \tag{17}
\]

Following Proposition I.5 especially (14), we will restrict attention in the following to subsets of \([0,1]\) and measures defined on the Borel subsets of \([0,1]\). If \(J \subset [0,1]\) is a subinterval, we denote by \(|J|\) the length of \(J\).
Definition II.1 Let $\mu$ be a probability measure on $[0, 1]$ defined on the Borel sets.

We say that $s_-$ is a lower scale of $\mu$ if

$$\liminf_{|J|\to 0} \frac{\mu(J)}{|J|^{-s_-}} > 0; \quad (18)$$

and we say that $s_+$ is an upper scale of $\mu$ if

$$\limsup_{|J|\to 0} \frac{\mu(J)}{|J|^{s_+}} < \infty. \quad (19)$$

It is easy to see that the Cantor measure $\mu = \mu_\psi$ in Example I.4 (b) has both upper and lower scale $s_+ = s_- = s = \log_3(2)$.

Our next result is motivated by examples from wavelet analysis. Before stating our general result we first recall the wavelet examples. To emphasize our point, we do not consider the wavelet examples in the widest generality.

Example II.2 Discrete Wavelet Transforms. Let $a_0, a_1, \ldots$ be a sequence of complex numbers such that

$$\sum_j a_j^2 a_{j+2k} = \delta_{0,k}. \quad (20)$$

In the summation (20), it is understood that terms are zero if the subindex is not in the range where $\neq 0$.

We define operators $F_0$ and $F_1$ on the Hilbert space $H=\ell^2$ as follows:

$$(F_0 \xi)_j := \sum_k a_{2j-k} \xi_k \quad (21)$$

and

$$(F_1 \xi)_j := \sum_k (-1)^k a_{1-2j+k} \xi_k. \quad (22)$$

Then it is easy to check that (II) holds, and so the pair $(F_0, F_1)$ defines a
measurement in the sense of the definition in Section I. In this case, more is true: The adjoint operators $F_i^*$ are isometries with orthogonal ranges, i.e.,

$$ F_i F_j^* = S_{ij} I. \quad (23) $$

If the sequence $a_0, a_1, \cdots$ from (20) is finite, then it is easy to see that the number of non-zero terms must necessarily be even. We consider $2D$ scalars,

$$ a_0, a_1, \cdots, a_{2D-1}, \quad (24) $$

and the corresponding two $(2D - 1)$ by $(2D - 1)$ matrices $F_0$ and $F_1$ defined as follows:

$$ F_0 = \begin{pmatrix}
    a_0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
    a_2 & a_1 & a_0 & & & & & \\
    \vdots & a_3 & a_2 & & & & & \\
    \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \\
    a_{2D-2} & \vdots & a_1 & 0 & 0 \\
    0 & a_{2D-1} & a_{2D-2} & a_2 & a_1 & a_0 \\
    0 & 0 & 0 & \vdots & a_3 & a_2 \\
    \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
    0 & 0 & 0 & \cdots & 0 & 0 & a_{2D-1} & a_{2D-2}
\end{pmatrix} \quad (25) $$

and $F_1$ built the same way, but using the numbers

$$ b_k := (-1)^k \frac{a_{2D-1-k}}{a_{2D-1}} \quad (26) $$

For $D = 2$, the two matrices are simply

$$ F_0 = \begin{pmatrix}
    a_0 & 0 & 0 \\
    a_2 & a_1 & a_0 \\
    0 & a_3 & a_2
\end{pmatrix} \quad \text{and} \quad F_1 = \begin{pmatrix}
    \frac{a_3}{a_1} & 0 & 0 \\
    \frac{a_1}{a_2} & -a_2 & a_3 \\
    0 & -\frac{a_0}{a_1} & \frac{a_3}{a_1}
\end{pmatrix} \quad (27) $$

Staying with $a_0, a_1, a_2, a_3$, there are practical reasons in wavelet analysis to add the following two requirements to (20):

(i) $a_i \in \mathbb{R}$

and

(ii) $\sum_{i=0}^3 a_i = \sqrt{2}$.
Taking the combined conditions together, it can be shown that \( a_0, a_1, a_2, a_3 \) are determined by a single real parameter \( \beta \) thus:

\[
\begin{align*}
    a_0 &= \frac{1}{2\sqrt{2}} (1 + \sqrt{2} \cos \beta) \\
    a_1 &= \frac{1}{2\sqrt{2}} (1 + \sqrt{2} \sin \beta) \\
    a_2 &= \frac{1}{2\sqrt{2}} (1 - \sqrt{2} \cos \beta) \\
    a_3 &= \frac{1}{2\sqrt{2}} (1 - \sqrt{2} \sin \beta).
\end{align*}
\]  

(28)

A consequence of (28) is that each of the three pairs \((a_0, a_1), (a_0, a_3),\) and \((a_1, a_2)\) lies on the circle

\[
\left( x - \frac{1}{2\sqrt{2}} \right)^2 + \left( y - \frac{1}{2\sqrt{2}} \right)^2 = \frac{1}{4};
\]  

(29)

see Fig. 1.

![Figure 1. One of the three pairs \((a_0, a_3)\)](image)

**Theorem II.3** Let the numbers \( a_0, a_n, \cdots, a_{2D-1} \) be given, and suppose (20) is satisfied. Let \( F_0 \) and \( F_1 \) be the corresponding matrices determined by (26)–(20). Suppose further that \( a_0 \cdot a_{2D-1} \neq 0 \). Let

\[
\alpha := \max \left( |a_0|^2, |a_{2D-1}|^2 \right).
\]  

(30)
Then the number
\[ s = \log_2 \alpha^{-1} = -\frac{\ln \alpha}{\ln 2} \] (31)
is a lower scale of \( \mu_0(\cdot) = \langle e_0 | P(\cdot) e_0 \rangle \) where
\[ e_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \] (32)

It follows in particular that if \( \alpha > \frac{1}{2} \), then \( \mu_0 \) has a lower scale \( s > 1 \).
Moreover, we can get the lower estimate (18) satisfied for dyadic intervals \( J \) inside any non-empty open subset of \([0, 1]\).

**Proof.** From the representation (25) of the two matrices \( F_0 \) and \( F_1 \) we conclude that
\[ F_0^* e_0 = a_0 e_0 \text{ and } F_1^* e_0 = a_{2D-1} e_0. \] (33)

With the dyadic representation (14), \( N = 2 \), set
\[ \xi = \frac{i_1}{2} + \cdots + \frac{i_k}{2^k} \] (34)
and
\[ F_\xi = F_{i_k} \cdots F_{i_1}. \] (35)
Then it follows from (15) and Lemma I.3 that
\[ \mu_0 \left( \left[ \xi, \xi + 2^{-k} \right] \right) = \langle e_0 | F_\xi^* \cdots F_{i_1}^* F_{i_k} \cdots F_{i_1} e_0 \rangle \]
\[ = \langle F_{i_k} \cdots F_{i_1} e_0 | F_{i_k} \cdots F_{i_1} e_0 \rangle \]
\[ = \| F_\xi e_0 \|^2 \]
\[ \geq | \langle e_0 | F_\xi e_0 \rangle |^2 \]
(Schwarz)
\[ = | \langle F_\xi^* e_0 | e_0 \rangle |^2 \]
\[ = | a_0 |^{2 \cdot 2^i} \cdot | a_{2D-1} |^{2 \cdot 2^i} \]
using (19)

Let \( V \) be a non-empty open subset of \([0, 1]\), and pick \( k \in \mathbb{N} \) and \( \xi \) as in (34) such that the interval \( \left[ \xi, \xi + 2^{-k} \right] \) is contained in \( V \).
We now turn to the two possibilities for the number $\alpha$ in (30).
If $\alpha = |a_0|^2$, then

$$\mu_0 \left( [\xi, \xi + 2^{-k-n}] \right) = \| F_0^n F_0 \xi \|_2^2 \geq \alpha^{#(i=0)+n} |a_{2D-1}|^{2\cdot#(i=1)}$$

and we conclude that the expression

$$\frac{\mu_0 \left( [\xi, \xi + 2^{-k-n}] \right)}{2^{-s(k+n)}}$$

is bounded below as $n \to \infty$, and hence (18) holds for $s = \log_2 (\alpha^{-1})$, see (31).

If instead $\alpha = |a_{2D-1}|^2$, then

$$\mu_0 \left( [\xi + 2^{-k} (1 - 2^{-n}), \xi + 2^{-k}] \right) \geq |a_0|^{2\cdot#(i=0)} \alpha^{#(i=1)+n}$$

by the same reasoning used in the first case. We now get the lower estimate (18) satisfied for the intervals $J = [\xi + 2^{-k} (1 - 2^{-n}), \xi + 2^{-k}]$ as $n \to \infty$. This completes the proof.

III Upper and Lower Fractal Scales for the Measure $\mu_0$

Consider the example outlined in (28) above. The two matrices $F_0$ and $F_1$ are used in wavelet analysis where they refer to low-pass and high-pass filters; terms that derive from signal processing, see [6] and [3].

Recall that when $a_0, a_1, a_2, a_3$ are given by (28) then there are solutions $\phi, \psi$ in $L^2(\mathbb{R})$ to

$$\begin{cases}
\phi(x) = \sqrt{2} \left( a_0 \phi(2x) + a_1 \phi(2x - 1) + a_2 \phi(2x - 2) + a_3 \phi(2x - 3) \right) \\
\int_{\mathbb{R}} \phi(x) \, dx = 1 \\
\psi(x) = \sqrt{2} \left( a_3 \phi(2x) - a_2 \phi(2x - 1) + a_1 \phi(2x - 2) - a_0 \phi(2x - 3) \right) \\
\int_{\mathbb{R}} \psi(x) \, dx = 0,
\end{cases}$$

and when $\beta \in \mathbb{R} \setminus \left\{ \{\pm \pi, \pm \frac{3\pi}{4}\} \cup 2\pi \right\}$, then the two functions $\phi$ (the scaling function) and $\psi$ (the wavelet) satisfy the further conditions
\[
\int_{\mathbb{R}} \phi(x) \phi(x - k) \, dx = \delta_{0,k}, \quad k \in \mathbb{Z}
\]
and
\[
\{ 2^{j/2} \psi(2^j x - k) \mid j, k \in \mathbb{Z} \}
\]
is an orthonormal basis for the Hilbert space \( L^2(\mathbb{R}) \).

The general feature of the matrices (27) is the slanted shape; and the \( F_i \)'s are called slanted Toeplitz matrices.

**Example III.1** A one-parameter family of wavelets. For the case \( D = 2 \) treated in (28), we have

\[
\left( a_0 - \frac{1}{2\sqrt{2}} \right)^2 + \left( a_3 - \frac{1}{2\sqrt{2}} \right)^2 = \frac{1}{4} \tag{37}
\]
and it follows that the number

\[
\alpha = \max \left( a_0^2, a_3^2 \right) \tag{38}
\]
satisfies \( \alpha > \frac{1}{2} \) when the parameter \( \beta \) is in one of the two intervals:

(i) \( |\beta| < \frac{\pi}{4} \) where \( \alpha = a_0^2 \);

or

(ii) \( -\frac{3\pi}{4} < \beta < -\frac{\pi}{4} \) where \( \alpha = a_3^2 \);

see Fig. 2 below. In these two regions the scale number \( s \) satisfies \( s > 1 \); hence the fractal feature of the measure \( \mu_0 \).
Figure 2. The two functions $a_0 = a_0 (\beta)$ and $a_3 = a_3 (\beta)$, (28). Two regions (i) and (ii) with scale number $s$ satisfying $s > 1$.

When the four numbers $a_0, a_1, a_2, a_3$ that make up the matrix $F_0$ are given by formulas (28), one easily computes the spectrum of $F_0$ as follows:

$$\text{spec} (F_0) = \left\{ a_0 (\beta), \frac{1}{\sqrt{2}} \frac{\sin \beta - \cos \beta}{2} \right\},$$

(39)

and we have sketched the point

$$\lambda (\beta) = \frac{\sin \beta - \cos \beta}{2}$$

(40)

in the spectrum in Figure 3. An inspection shows that $\lambda (\beta)$ is not dominant in $\text{spec} (F_0)$ in the sense that the inequalities

$$a_0 (\beta) > \frac{1}{\sqrt{2}} > |\lambda (\beta)|$$

(41)
hold in the region (i) from Figure 2.

The eigenvalue \( \lambda(\beta) = \frac{\sin \beta - \cos \beta}{2} \) is not dominant. (i): \( a_0(\beta) > \frac{1}{\sqrt{2}} > \lambda(\beta)^2 \)

We will now turn to our analysis of the two-sided scale bound for the measure \( \mu_0 \) and we show how it applies to the matrices \( F_0 \) and \( F_1 \) which are used in wavelet theory.

**Theorem III.2** Let the numbers \( a_0, a_1, \ldots, a_{2D-1} \) satisfy condition (20), and in addition

\[
\sum_{j=0}^{2D-1} a_j = \sqrt{2}. \tag{42}
\]

The two matrices \( F_0 \) and \( F_1 \) are defined as in (25)–(26). We make the following additional assumptions on the spectrum of \( F_0 \):

(i) \( a_0 \cdot a_{2D-1} \neq 0 \);

\[
\tag{43}
\]

(ii) \( |a_0| > \max \{|\lambda| : \lambda \in \text{spec} \ (F_0) \setminus \{a_0\}\} \);

\[
\tag{44}
\]

(iii) the algebraic multiplicity of \( a_0 \) in \( \text{spec} \ (F_0) \) is one.

\[
\tag{45}
\]
Then there is a unique vector $v$ such that

$$F_0v = a_0v \text{ and } \langle e_0|v \rangle = 1$$

(46)

where $e_0$ is the vector (32) with 1 is the first slot and zeros in the rest.

Moreover

$$s = \log_2 (|a_0|^{-1}) = -\frac{2\ln|a_0|}{\ln 2}$$

(47)

is both an upper scale and a lower scale in every non-empty open subset of $[0, 1]$ for the measure

$$\mu_0 (\cdot) = \langle e_0|P (\cdot) e_0 \rangle.$$  

(48)

Proof. Set $w = (1, 1, \cdots, 1)$. Then (42) implies

$$wF_0 = \frac{1}{\sqrt{2}}w \text{ or equivalently } F_0^*w^* = \frac{1}{\sqrt{2}}w^*$$

(49)

where $w^*$ denotes the column vector corresponding to $w$. So property (44), i.e., (ii) above, yields

$$|a_0|^2 > \frac{1}{2} \text{ and } |a_0| > |a_{2D-1}|.$$  

(50)

Since $F_0^*e_0 = \bar{a}_0e_0$, (45) implies that $a_0$ is not in the spectrum of the matrix $G$ arising from $F_0$ by deletion of the first row and the first column, i.e.,

$$G = \begin{pmatrix}
    a_1 & a_0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
    a_3 & a_2 & a_1 & a_0 & \vdots & \vdots & \vdots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    a_{2D-1} & a_{2D-2} & a_{2D-1} & a_{2D-1} & 0 & 0 & a_1 & a_0 \\
    0 & 0 & \cdots & \cdots & 0 & a_{2D-1} & a_{2D-2} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}$$

(51)
Now define a vector $v$ in $\mathbb{C}^{2D-1}$ by

$$v = e_0 + (a_0 I_{2D-2} - G)^{-1} \begin{pmatrix} a_2 \\
a_4 \\
\vdots \\
a_{2D-2} \\0 \\
\vdots \\
0 \end{pmatrix}.$$ \hfill (52)

To better visualize (51) the reader may check that, if $D = 2$,

$$F_0 \begin{pmatrix} a_0 & 0 & 0 \\
a_2 & a_1 & a_0 \\
0 & a_3 & a_2 \end{pmatrix}, \quad G = \begin{pmatrix} a_1 & a_0 \\
a_3 & a_2 \end{pmatrix},$$ \hfill (53)

and

$$v = \left( a_0 I_2 - \begin{pmatrix} a_1 & a_0 \\
a_3 & a_2 \end{pmatrix} \right)^{-1} \begin{pmatrix} a_2 \\0 \end{pmatrix} = \begin{pmatrix} \frac{1}{p(a_0)} \frac{a_0-a_2}{a_3a_2} \\
\frac{a_0-a_2}{a_3a_2} \frac{p(a_0)}{a_3a_2} \end{pmatrix}.$$ \hfill (54)

where

$$p(a_0) = a_0^2 - (a_1 + a_2) a_0 + a_1 a_2 - a_0 a_3.$$ 

Recall that the characteristic polynomial of $G$ is

$$p(\lambda) = \lambda^2 - (\text{trace } G) \lambda + \det G,$$ \hfill (55)

and that $p(a_0)$ in the fraction of (54) is evaluation of (55) at $\lambda = a_0$. Hence, assumption (45) comes into play.

Returning to the general case, we claim that $v$ satisfies (46). Indeed, let
be given by (52). then

\[ Fv = a_0 e_0 + \left( I + G (a_0 I - G)^{-1} \right) \begin{pmatrix} a_2 \\ a_4 \\ \vdots \\ a_{2D-2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a_0 e_0 + a_0 (a_0 I - G)^{-1} \begin{pmatrix} a_2 \\ a_4 \\ \vdots \\ a_{2D-2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a_0 v, \]

and

\[ \langle e_0 | v \rangle = 1. \]

This proves (46). Conversely, using (44)–(45), one shows that the solution \( v \) to (46) is unique.

Now let \( V \) be a non-empty open subset of \([0, 1]\). Then pick \( k \in \mathbb{N} \) and a dyadic fraction

\[ \xi = \frac{i_1}{2} + \cdots + \frac{i_k}{2^k} \] (56)

such that \([\xi, \xi + 2^{-k}) \subset V \).

We now wish to estimate \( \mu_0 \left( [\xi, \xi + 2^{-k-n}) \right) \) and get the asymptotic scaling rate as \( n \to \infty \).

To that end, we prove in the next section (in a separate lemma; see especially (59) that \( \lim_{n \to \infty} a_0^{-n} F_0 F_\xi e_0 = \langle e_0 | F_\xi e_0 \rangle \)\( v = \langle F_\xi e_0 | e_0 \rangle \)\( v = a_0^{-\#(i=0)} a_{2D-1}^{-\#(i=1)} v \),

and as a result \( \lim |a_0|^{-2n} \mu_0 \left( [\xi, \xi + 2^{-k-n}) \right) = |a_0|^{-2\#(i=0)} |a_{2D-1}|^{-2\#(i=1)} \|v\|^{2} n \to \infty \). Since \( \|v\|^2 \geq 1 \), the desired conclusion follows. ■

In applications to wavelets, the measures in the title of the paper are used in the computation of transition matrices for transformation between two orthogonal families in the Hilbert space \( L^2(\mathbb{R}) \):
(i) a wavelet basis \( (2^{p/2} \psi (2^p x - k))_{p,k \in \mathbb{Z}} \);

and

(ii) a wavelet packet \((\phi_n)_{n \in \mathbb{N}_0}, \mathbb{N}_0 = \{0, 1, 2, \cdots \};\)
\[
\phi_0 := \phi, \phi_1 := \psi,
\phi_{2n} (x) = \sqrt{2} \sum_{k \in \mathbb{Z}} a_k \phi_n (2x - k),
\phi_{2n+1} (x) = \sqrt{2} \sum_{k \in \mathbb{Z}} b_k \phi_n (2x - k).
\]

The adjustment of dyadic scaling in (i) is made with variations in \( p \in \mathbb{Z}; \)
and hence with the size of the dyadic intervals \( J (k, p) = [k2^{-p}, (k + 1)2^{-p}) \).

The concentration of mass at each \( J (k, p) \) is determined by the measure.

**IV A TechnicalLemma**

In the proof of Theorem [III.2] above, we relied on the following lemma regarding operators in a finite-dimensional Hilbert space. While it is analogous to the classical Perren-Frobenius theorem, our present result makes no mention of positivity. In fact, our matrix entries will typically be complex.

**Notation IV.1** If \( \mathcal{M} \) is a complex Hilbert space, we denote by \( L (\mathcal{M}) \) the algebra of all bounded linear operators on \( \mathcal{M} \). If \( \mathcal{M} \) is also finite-dimensional, we will pick suitable matrix representations for operators \( F: \mathcal{M} \rightarrow \mathcal{M} \).

Suppose \( \mathcal{M} \) contains two subspaces \( \mathcal{M}_i, i = 1, 2 \) such that \( \mathcal{M}_1 \perp \mathcal{M}_2 \) and \( \mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \), then we get a block-matrix representation
\[
F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]  
(57)

where the entries are linear operators specified as follows.

\( A: \mathcal{M}_1 \rightarrow \mathcal{M}_1, \, B: \mathcal{M}_2 \rightarrow \mathcal{M}_1; \)

and

\( C: \mathcal{M}_1 \rightarrow \mathcal{M}_2, \, D: \mathcal{M}_2 \rightarrow \mathcal{M}_2. \)

If \( \dim \mathcal{M}_1 = 1 \), and \( \mathcal{M}_1 = \mathbb{C} w \) for some \( w \in \mathcal{M} \), then we will identify the operators \( \mathcal{M}_1 \rightarrow \mathcal{M} \) with \( \mathcal{M} \) via \( T_\eta: \mathbb{C} \ni \eta \mapsto z \eta, \) where \( \eta \in \mathcal{M} \). the adjoint operator is \( T_\eta^* x = \langle \eta | x \rangle w, \) \( x \in \mathcal{M}. \)
Lemma IV.2 Let $\mathcal{M}$ be a finite-dimensional complex Hilbert space, with $d = \dim \mathcal{M}$. Let $F \in L(\mathcal{M})$, and let $a \in \mathbb{C}$ satisfy the following four conditions:

(i) $a \in \text{spec} (F)$;
(ii) $|a| > \max \{|\lambda| | \lambda \in \text{spec} (F) \setminus \{a\}\}$;
(iii) the algebraic multiplicity of $a$ is one;
(iv) there is a $w \in \mathcal{M}$, $\|w\| = 1$, such that $F^*w = \bar{a}w$.

Then there is a unique $\xi \in \mathcal{M}$ such that

$$\langle w|\xi \rangle = 1 \text{ and } F\xi = a\xi.$$ 

Moreover,

$$\lim_{n \to \infty} a^{-n}F^n x = \langle w|x \rangle \xi \text{ for all } x \in \mathcal{M}.$$ 

Remark IV.3 There is a constant $C$ independent of $d = \dim \mathcal{M}$ and $x$, such that

$$\|a^{-n}F^n x - \langle w|x \rangle \xi \| \leq Cn^{d-1} \max \left\{ \left| \frac{s}{a} \right| | s \in \text{spec} (F) \setminus \{a\} \right\}$$

Proof. (Lemma IV.2.) Set $\mathcal{M}^\perp := \mathcal{M} \ominus \mathbb{C}w = \{x \in \mathcal{M}|\langle w|x \rangle = 0\}$. Then

$$\mathcal{M} = \mathbb{C}w \oplus \mathcal{M},$$

and we get the resulting block-matrix representation of $F$,

$$F = \begin{pmatrix} a & 0 & \cdots & 0 \\ \eta & 0 & \cdots & 0 \\ & \eta & 0 & \cdots \\ & & \ddots & \eta \\ & & & 0 \end{pmatrix}$$

where $a$ is the number in (i), the vector $\eta \in \mathcal{M}^\perp$, and operator $G \in L(\mathcal{M}^\perp)$, are uniquely determined.

As a result, we get the factorization

$$\det (\lambda - F) = (\lambda - a) \det (\lambda - G)$$

for the characteristic polynomial. Assumptions (ii) and (iii) imply

$$\text{spec} (F) \setminus \{a\} = \text{spec} (G) ;$$

and in particular, we note that $a$ is not in the spectrum of $G$. Hence the inverse $(a - G)^{-1}$ is well defined, and $(a - G)^{-1} \in L(\mathcal{M}^\perp)$. 

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We claim that the vector
\[ \xi = w + (a - G)^{-1} \eta \] (65)
satisfies the conditions in (58).

First note that \((a - G)^{-1} \eta \in \mathcal{M}^\perp\), so \(\langle w|\xi \rangle = \langle w|w \rangle = \|w\|^2 = 1\).
Moreover,
\[
F\xi = aw + \eta + G(a - G)^{-1} \eta \\
= aw + a(a - G)^{-1} \eta \\
= a\xi,
\]
which proves the second condition in (58). Uniqueness of the vector \(\xi\) in (58) follows from (64).

Using the matrix representation (62), we get
\[
F^2 = \begin{pmatrix}
a^2 & 00 \cdots 0 \\
0 & G^n \\
a\eta + G\eta & 00 \cdots 0
\end{pmatrix}
\]
and by induction,
\[
F^2 = \begin{pmatrix}
a^n & 00 \cdots 0 \\
a^{n-1}\eta + a^{n-2}G\eta + \cdots + G^{n-1}\eta & G^n \\
0 & 00 \cdots 0
\end{pmatrix}
\]
(66)
\[
= \begin{pmatrix}
a_n & 00 \cdots 0 \\
(a^n - G^n)(a - G)^{-1} \eta & G^n \\
0 & 00 \cdots 0
\end{pmatrix}.
\]

Hence, if we show that
\[
\lim_{n \to \infty} a^{-n}G^n = 0, \tag{67}
\]
then the desired conclusion (59) will follow. Using the matrix form (66), the conclusion (59) reads
\[
\lim_{n \to \infty} a^{-n}F^n = \begin{pmatrix}
1 & 00 \cdots 0 \\
(a - G)^{-1} \eta & 0
\end{pmatrix}
\]
(68)

In proving (67), we will make use of the Jordan-form representation for \(G\). Jordan’s theorem applied to \(G\) yields three operators \(D, V, N \in L(\mathcal{M}^\perp)\) with the following properties:
(1) $D$ is a diagonal matrix with the numbers $\text{spec} (F) \setminus \{a\}$ down the diagonals;
(2) $V$ is invertible;
(3) $N$ is nilpotent: If $d - 1 = \dim \left( \mathcal{M}^\perp \right)$ then $N^{d-1} = 0$;
(4) $[N, D] = ND - DN = 0$;
(5) $G = V (D + N) V^{-1}$.

Let $x \in \mathcal{M}^\perp$, and let $n \geq d$. Using (2)–(5), we get

$$a^{-n} F^n x = Va^n (D + N)^n V^{-1} x = \sum_{i=0}^{d-2} \binom{n}{i} V a^{-n} D^{n-i} N^i v^{-1} x.$$ But the matrix $a^{-n} D^{n-i}$ is diagonal with entries $\{a^{-n} s^{n-i} \mid s \in \text{spec} (F) \setminus \{a\}\}, 0 \leq i < d - 1$. Using finally assumption (ii), we conclude that

$$\lim_{n \to \infty} \binom{n}{i} a^{-n} s^{n-i} = 0,$$

and the proof of (67) is completed. □

**Proof.** (Remark IV.3) Let the conditions be as stated in the Remark. From the arguments in the proof of Lemma IV.2, we see that the two vectors on the left-hand side in (60) may be decomposed as follows:

$$a^{-n} F^n x = \langle w | x \rangle w + \left( 1 - a^{-n} G^n \right) (a - G)^{-1} \eta + a^{-n} G^n P_{\mathcal{M}^\perp} x \quad (70)$$

and

$$\langle w | x \rangle \xi = \langle w | x \rangle w + (a - G)^{-1} \eta. \quad (71)$$

Hence, the difference is in $\mathcal{M}^\perp$, and

$$\|a^{-n} F^n x - \langle w | x \rangle \xi\| = \|a^{-n} G^n (P_{\mathcal{M}^\perp} x - (a - G)^{-1} \eta)\| \leq C n^{d-1} \max \left\{ \frac{|s|^n}{a} \mid s \in \text{spec} (F) \setminus \{a\} \right\}$$

which is the desired conclusion. □
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