Uncertain Graph Sparsification

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ABSTRACT

Uncertain graphs are prevalent in several applications including communications systems, biological databases and social networks. The ever increasing size of the underlying data renders both graph storage and query processing extremely expensive. Sparsification has often been used to reduce the size of deterministic graphs by maintaining only the important edges. However, adaptation of deterministic sparsification methods fails in the uncertain setting. To overcome this problem, we introduce the first sparsification techniques aimed explicitly at uncertain graphs. The proposed methods reduce the number of edges and redistribute their probabilities in order to decrease the graph size, while preserving its underlying structure. The resulting graph can be used to efficiently and accurately approximate any query and mining tasks on the original graph. An extensive experimental evaluation with real and synthetic datasets illustrates the effectiveness of our techniques on several common graph tasks, including clustering coefficient, page rank, reliability and shortest path distance.

1. INTRODUCTION

Uncertain graphs, where edges are associated with a probability of existence, have been used widely in numerous applications. For instance, in communication systems, each edge \((u, v)\) is often associated with a reliability value that represents the probability that the channel from \(u\) to \(v\) will not fail. In biological databases, uncertain edges between vertices representing proteins are due to error-prone laboratory measurements. In social networks, edge probabilities can model the influence between friends, or the likelihood that two users will become friends in the future.

Several techniques have been proposed for diverse query processing and mining tasks on uncertain graphs (e.g. \cite{13, 12, 24}), most of which assume possible-world semantics. Specifically, let \(G = (V, E, p)\) be an uncertain (also called probabilistic) graph\textsuperscript{1} where \(p : E \rightarrow (0, 1]\) assigns a probability to each edge. \(G\) is interpreted as a set \(\{G = (V, E_G) \mid E_G \subseteq E\}\) of \(2^{|E|}\) possible deterministic graphs, each defined on a subset of \(E\). For example, since the uncertain graph of Figure 1(a) consists of 6 edges, there are \(2^6\) possible worlds. Under this interpretation, exact processing requires query evaluation on all possible worlds and aggregation of the partial results. In general, the probability of a query predicate \(Q\) is derived by the sum of probabilities of all possible worlds \(G\) for which \(Q(G) = \text{true}\):

\[
Q(G) = \sum_{Q(G)=\text{true}} \Pr(G)
\]

(A) uncertain graph \(G\)  (B) sparsified graph \(G'\)

Figure 1: Uncertain graph sparsification example

Applying Equation 1, the probability that the uncertain graph of Figure 1(a) contains a single connected component is \(\Pr(G\text{ is connected}) = 0.219\). This is obtained by generating the \(2^6\) deterministic graphs, and adding the probabilities of the connected ones.

Consequently, exact processing is prohibitive even for uncertain graphs of moderate size due to the exponential number of worlds. Thus, most techniques provide approximate results by applying Monte-Carlo (MC) sampling on a random subset of possible worlds. However, even MC may be very expensive for large uncertain graphs because generating a sample is time consuming as it involves sampling each edge. Moreover, due to the high entropy\textsuperscript{2} of the uncertain graphs, there is significant variance among the possible worlds, which implies the need of numerous samples for accurate query estimation. This imposes huge overhead at query processing cost, because the query must be executed at every sample.

\textsuperscript{1}We assume that \(G\) is unweighted, undirected and connected.

\textsuperscript{2}The entropy \(H(G)\) of an uncertain graph \(G\) is defined as the joint entropy of its edges \(H(e)\) for all \(e \in E\). Due to the edge independence, \(H(G) = \sum_{e \in E} H(e) = \sum_{e \in E} (-p_e \log p_e - (1 - p_e) \log(1 - p_e))\).
In order to tackle the high cost, we develop techniques for uncertain graph sparsification. Specifically, given $G$ and a parameter $\alpha \in (0, 1)$, the proposed methods generate a sparsified probabilistic subgraph $G' = (V, E', p')$, which contains a fraction of the edges, i.e., $E' \subset E, |E'| = \alpha|E|$. $G'$ preserves the structural properties of $G$, has less entropy, and can be used to approximate the result of a wide range of queries on $G$. Sparsification yields significant benefits in execution time because the cost of sampling is linear to the number of edges. Moreover, the required number of samples is proportional to the graph’s entropy ($H(G)$), which is lower in the sparcified graph. Finally, similar to the case of deterministic graphs, sparsification reduces the storage cost, and facilitates visualization of complex networks.

Figure 1(b) illustrates $G'$, a sparsified subgraph of $G$ that contains half the original edges. Observe that the edges of $G'$ have higher probabilities than those in $G$, in order to compensate for the missing ones. Assume, for instance, a query that asks for the probability that $G$ consists of a single connected component. Since $\Pr[G]$ is connected $= 0.216$ and $\Pr[G']$ is connected $= 0.219$, $G'$ can be used to effectively approximate the result of the query. Furthermore, the entropy decreases from 0.94 to 0.4. Since, $G'$ has fewer edges, it is more efficient to sample and store. Additionally, since it has less entropy, it requires fewer samples for accurate estimation. Our goal is to generate sparsified graphs that can be used for a variety of queries and tasks.

To the best of our knowledge, this is the first work on uncertain graph sparsification. On the contrary, sparsification has received considerable attention in the deterministic graph literature. In that context, most techniques aim at approximating all shortest path distances up to a multiplicative or additive factor, or preserving all cuts up to an arbitrarily small multiplicative error. As we demonstrate in our experimental evaluation, the adaptation of such methods to uncertain graphs yields poor results. On the other hand, our sparsification techniques achieve high accuracy and small variance for common graph tasks by capturing the expected node degrees, or the expected cut sizes up to a certain value. Summarizing, the contributions of the paper are:

- We propose a novel framework of uncertain graph sparsification with entropy reduction.
- We design algorithms that reduce the number of edges and tune the probability of the remaining ones to preserve crucial properties.
- We experimentally demonstrate that the sparsified graphs are effective for a variety of common tasks including shortest path distance, reliability, pagerank and clustering coefficient.

The rest of the paper is organized as follows. Section 2 surveys the related work. Section 3 defines the problem, introduces our uncertain sparsification framework and presents baseline solutions motivated by the deterministic graph literature. Section 4 proposes sparsification algorithms that capture the expected vertex degrees. Section 5 analyzes rules to preserve cut sizes and modifies our algorithms for this case. Section 6 contains an extensive experimental evaluation on real and synthetic datasets, and Section 7 concludes the paper.

2. RELATED WORK

Existing sparsification methods focus exclusively on deterministic graphs. Section 2.1 and 2.2 present sparsification techniques that preserve the graph cuts and the shortest path distances, respectively. Section 2.3 discusses other related methods.

2.1 Cut-based sparsifiers

Let a deterministic, undirected, weighted graph $G = (V, E, w)$, where $|V| = n$. Given a vertex set $S \subseteq V$, the cut $E_G(S)$ is the set of edges with exactly one endpoint in $S$, i.e., $E_G(S) = \{(u, v) \in E | (u \in S, v \notin S)\}$. The size $C_G(S)$ of the cut is the sum of weights of the edges in $E_G(S)$, i.e., $C_G(S) = \sum_{e \in E_G(S)} w(e)$. Cut-based sparsification preserves the size of all cuts of the graph within a multiplicative error. Formally, given a dense weighted graph $(|E| = \Theta(n^2))$ and an approximation error $\epsilon \in (0, 1)$, the output is $G' = (V, E', w')$, where $E' \subseteq E$ and $|E'| = O(n \log n/\epsilon^2)$, such that for any set $S \subseteq V$, $C_G'(S)$ is within $\epsilon$ of the original cut size $C_G(S)$, i.e., $C_G'(S) \in (1 \pm \epsilon)C_G(S)$ with high probability.

Algorithms for cut-based sparsification follow a two step approach. The first assigns a probability $p_e$ to each edge based on the topology of the graph. The second step samples each edge with probability $p_e$, and assigns weight $w'_e \propto \frac{1}{p_e}$ to the sampled edges. Intuitively, sparsification through this framework relies on the following observations:

- Edges in dense areas are not crucial for maintaining the graph connectivity. Thus, they have low sampling probability $p_e$.
- Edges sampled with low $p_e$ are assigned large weights in order to compensate for their missing neighbouring edges.

The existing methods assume integer weights and differ mainly in the first step, i.e., that of choosing $p_e$ for each edge $e \in (u, v)$. Spielman and Srivastava [23] generate the electrical equivalent of the graph by assuming resistors of $\Omega$ at each link. The sampling probability of edge $(u, v)$ is proportional to the amount of current that flows through $e$ when a unit voltage difference is applied to $u, v$. This approach also preserves every eigenvalue of the original graph. Benczúr and Karger [3] assign probabilities inversely proportional to the $k$-strong connectivity of $u, v$. Fung et al. [10] simplify analysis by utilizing sampling probabilities inversely proportional to the size of the minimum cut separating $u$ and $v$.

Nagamochi and Ibaraki [19] estimate the edge connectivities using the NI index $\lambda_e$ of an edge $e$. The NI index is generated by iteratively constructing spanning forests of the initial graph, while reducing the weights of the selected edges. In essence, the NI index is the last spanning forest that contains $e$, given that an edge with weight $w_e$ needs to participate in $w_e$ contiguous forests. To ensure that the sparse graph has $O(n \log n/\epsilon^2)$ edges in expectation, the sampling probability is set to $p_e = \frac{\lambda_e}{\rho}$, where $\rho = O(\log n/\epsilon^2)$. To see this, let $\mathbb{E}(E')$ denote the expected number of edges of $G'$. According to [4], $\sum_{e} \frac{1}{\lambda_e} = O(n \log n)$; thus, $\mathbb{E}(E') = \sum_{e} p_e = \rho \sum_{e} \frac{1}{\lambda_e} = O(n \log^2 n/\epsilon^2)$. A more refined analysis [10] reduces this to $O(n \log n/\epsilon^2)$.

\footnote{\textsuperscript{3}A maximal $k$-strong connected component is a maximal induced subgraph of $V$ that remains connected after removing up to $k$ edges. The $k$-strong connectivity of an edge $(u, v)$ is the maximum $k$ such that $u$ and $v$ belong to the same $k$-strong connected component.}
2.2 Spanners

Spanners aimed at preserving the shortest path distances between all node pairs is based on the concept of a $t$-spanner [23]. A $t$-spanner of a weighted deterministic graph $G = (V, E, w)$ is a subgraph $G' = (V, E', w)$ such that, for any pair of vertices $u, v \in V$, their distance in $G'$ is at most $t \in \mathbb{N}^+$ times their distance in $G$, i.e.,

$$\text{dist}_G(u, v) \leq t \cdot \text{dist}_{G'}(u, v).$$

The parameter $t$ is the stretch factor of $G'$. Computing a $t$-spanner with the minimum number of edges has received considerable attention in the literature [26]. Peleg and Schäffer [23] prove that a spanner number of edges has received considerable attention in the literature [26].

Based on this lower bound, a simple technique [1] to generate $(2t - 1)$-spanners processes the edges in increasing order of their weights. Initially, the spanner is $E' = \emptyset$. An edge $e = (u, v)$ is included to $E'$, if the distance between $u$ and $v$ in $E'$ exceeds $(2t - 1)w_e$. The algorithm has $O(|E|n^{1+1/t})$ time complexity due to the shortest path distance computations. Roditty and Zwick [25] propose an improved $O(tn^{2+1/t})$ time method, which incrementally maintains a single source shortest path tree up to a given distance. Finally, Baswana et al. [2] design a randomized algorithm to compute a spanner of $(2t - 1)$-stretch and $O(|E|n^{1+1/t})$ size in $O(t|E|)$ expected time, which uses a novel clustering approach and avoids distance computations. An adaptation of this method provides a benchmark for our experimental evaluation.

2.3 Other related techniques

Some deterministic graph sparsification techniques focus on approximating the result of particular queries, as opposed to structural graph properties. For instance, [16] propose algorithms for generating deterministic representative instances that approximate the expected node degrees of the uncertain graph. Queries can then be processed by applying conventional algorithms on these instances. Since the representatives have fewer edges than the original uncertain graph, this could be viewed as a special case of sparsification. However, a representative (i.e., deterministic graph) cannot be used to answer queries whose output is uncertain, e.g., return the probability that the graph consists of a single connected component, or the probability that two vertices are reachable from each other. On the other hand, our techniques generate uncertain graphs that can be used for the same query and mining tasks as the original graph. Moreover, the methods of [21, 22] do not provide control over the number of edges in the representative graphs. Intuitively, while [21, 22] aim at eliminating uncertainty by extracting a representative instance (i.e., zero entropy), in this work we aim at decreasing the uncertainty of the input graph (i.e., reducing its entropy).

3. PROBLEM DEFINITION AND FRAMEWORK

Let $G = (V, E, p)$ be a probabilistic undirected graph, where $p : E \rightarrow (0, 1)$ is a function that assigns a probability $p(u, v)$ to each edge $(u, v) \in E$. Given a sparsification ratio $\alpha \in (0, 1)$, we extract from $G$ a sparsified graph $G' = (V, E', p')$ such that $E' \subseteq E$ and $|E'| = \alpha|E|$. $G'$ should preserve the structural properties of $G$, so that it can be used to accurately approximate the result of diverse queries on $G$. Moreover, $G'$ should reduce the entropy of $G$ in order to decrease the variance of the queries. In addition to diminishing the storage overhead, sparsification yields significant benefits in terms of query processing because the cost of sampling is proportional to the number of edges. Moreover, through entropy reduction, the resulting graph is less uncertain, thus it requires fewer samples for accurate query estimation.

3.1 Uncertain sparsification

As stated in Section 2, a prevalent goal of deterministic graph sparsification is preservation of the cut sizes. The notion of a cut can be extended naturally to uncertain graphs. In this case, due to the linearity of expectation, the expected size of a cut is the sum of the probabilities of the edges involved in the cut.

**Definition 1 (Expected cut size).** Given an uncertain graph $G = (V, E, p)$ and a subset $S \subseteq V$, the expected cut size of $S$ in $G$ is the summation of the probabilities of the edges with exactly one endpoint in $S$:

$$C_G(S) = \sum_{e=(u,v) \in E, u \in S, v \in V \setminus S} p_e$$

We define the absolute discrepancy $\delta_A(S)$ of a vertex set $S$ in a sparsified graph $G'$ as the difference of $S$’s expected cut size in $G'$ to its expected cut size in $G$, i.e.,

$$\delta_A(S) = C_G(S) - C_{G'}(S)$$

Accordingly, the relative discrepancy $\delta_R(S)$ is the absolute discrepancy of $S$ divided by the original cut size:

$$\delta_R(S) = \frac{C_G(S) - C_{G'}(S)}{C_G(S)}$$

To simplify notation, we collectively refer to $\delta_A$ and $\delta_R$ as $\delta$, and we only differentiate when required. In addition, we use the term cut to also refer to the expected cut size. Motivated by the work in deterministic graph sparsification, we aim at cut-preserving sparsified graphs, or, using our notation, at minimizing discrepancy $\delta$. The exponential number of cuts renders their exhaustive enumeration intractable. To overcome this, we target cuts of sets $S$ with specific cardinality $k$.

Formally, given an integer $k$, we define the $k$-discrepancy $\Delta_k$ of a graph $G'$ as the sum of the absolute values of the discrepancies for all sets with cardinality $k$:

$\Delta_k$ Sampling techniques have complexity $O(E)$ [16].
Here, $\delta(S)$ is the absolute or relative discrepancy of $S$. The absolute discrepancy emphasizes vertices with high degree because they are more likely to yield large absolute errors. On the other hand, the relative discrepancy targets all node degrees equally, by considering the relative error. We aim at minimizing the sum of $\Delta_i$ for $1 \leq i \leq k$, or equivalently at preserving the size of all cuts up to $k$. Accordingly, the problem we tackle in this work is:

**Problem 1.** Given an uncertain graph $G = (V, E, p)$, and a sparsification ratio $\alpha \in (0, 1)$, find an uncertain graph $G' = (V, E', p')$, with $|E'| = \alpha |E|$ that minimizes the sum of discrepancies $\sum_{i=1}^{k} \Delta_i(G')$ up to a given $k \geq 1$ and the entropy $H(G')$.

For the special case of $k = 1$, minimizing $\Delta_1$ is equivalent to preserving the expected degrees for all vertices, which has been shown to be effective when generating deterministic representatives of uncertain graphs [11].

### 3.2 Framework

The proposed framework involves two phases. The first (backbone generation), is independent of $k$, and creates an unweighted backbone graph $G_b = (V, E_b)$, with $|E_b| = \alpha |E|$. The second, (probability assignment), depends on $k$ and generates probabilities for the edges of $G_b$ that capture the expected degrees or cuts, while avoiding the probability values around 0.5 that incur high entropy.

For the first phase, a simple approach could sample the edges of $G$ in random order according to their probabilities, until it obtains $\alpha |E|$ edges. However, this would not ensure the connectivity of $G_b$, especially for small $\alpha$. Disconnected graphs can introduce large errors on various queries such as shortest distances between vertices of different connected components. Moreover, individual vertices may become entirely disconnected, which would lead to high total discrepancy.

To overcome this problem, we generate connected backbone graphs using the following method. We first compute a maximum spanning tree of $G$, where the probabilities act as weights. Then, we remove the tree edges from $G$ and insert them to $G_b$. This process is repeated until $G_b$ contains $\alpha' |E|$ edges, where $\alpha' < \alpha$. Note that after the first application of the maximum spanning tree, $G_b$ may become disconnected; thus, subsequent applications may return spanning forests instead of trees. Finally, the remaining $(\alpha - \alpha') |E|$ edges of $G_b$ are generated by random sampling based on the edge probabilities. Algorithm 1 illustrates backbone graph generation.

Parameter $\alpha'$ tunes the number of edges obtained through spanning forests. Intuitively, generating all $\alpha |E|$ edges using spanning forests is not desirable, because all vertices would be treated equally, independently of their degrees. In our experiments, we set the value of $\alpha'$ so that it is the minimum of 0.5$\alpha$ and the number of edges in the first six maximum spanning forests.

For the second phase, we propose heuristic algorithms that aim at minimizing $\Delta_1$ (Section 3.1), and $\Delta_k$ for $k \geq 1$ (Section 3.4). In both cases probability values that would incur high entropy are avoided. Next, we introduce two benchmark solutions, adapted from the deterministic sparsification literature.

### 3.3 Benchmark solutions

Deterministic graph sparsification techniques are theoretical, and to the best of our knowledge, they have not been implemented or evaluated in practice. Furthermore, our uncertain graph sparsification setting differs from that of spanners and cut sparsifiers. All spanners focus on selecting a subset of edges without changing their weights. On the other hand, we modify the probabilities of edges in the sparsified graph, in order to compensate for the eliminated edges. Moreover, probabilities of the uncertain graphs, unlike weights of deterministic graphs are bounded by 1, inhibiting the direct application of cut-based sparsification methods, all of which assume unbounded weights. Lastly, uncertain sparsification explicitly targets to reduce the entropy of the input graph in order to minimize the variance of query estimators. Deterministic sparsifiers do not consider entropy reduction. Nevertheless, in the following, we extend two state-of-the-art methods, one based on cut sparsifiers and the other on spanners, to uncertain graphs. The resulting algorithms are used as benchmarks in our experiments.

As a representative of cut sparsifiers, we adopt the NI method of [10] by transforming the uncertain graph $G$ to a weighted deterministic $G_w$. Recall that NI requires integer weights and an approximation parameter $\epsilon \in (0, 1)$ to produce a sparsified graph $G'_w$ with $O(n \log n/\epsilon^2)$ edges on expectation. Intuitively, the probabilities of $G$ are directly analogous to the weights of $G_w$ in terms of (expected) cut size. To maintain this analogy while ensuring $w_e \in \mathbb{N}^+$, our transformation divides each probability of $G$ by the smallest value $p_{min}$, and rounds the result to the closest integer, i.e., $w_e = \lfloor p_e / p_{min} \rfloor$. To relate $\epsilon$ to our sparsification ratio $\alpha$, we set $\epsilon = \sqrt{n \log n/\alpha |E|}$. Next, we use NI as a black box to sparsify $G_w$ into $G'_w$. However, since the number of edges of $G_w$ in [10] is given on expectation with asymptotic notation, it is not guaranteed to equal $\alpha |E|$. If the resulting graph has more (fewer) edges than $\alpha |E|$, we iteratively execute NI after increasing (decreasing) $\epsilon$ by a small factor $\theta$, until the first (last) graph for which $|E'| \leq \alpha |E|$. The remaining $\alpha |E| - |E'|$ edges are randomly sampled from $E \setminus E'$ using the initial probabilities $p$. Intuitively, this calibration process approximates the minimum $\epsilon$, which ensures $|E'| \leq \alpha |E|$. Finally, for each edge $e$, we convert $w'_e$ [9]Any method of Section 2.1 can be applied similarly.
4. PRESERVING EXPECTED DEGREES

We first focus on Problem 1 for \( k = 1 \), and describe methods to generate a sparsified graph \( G' \) that preserves the expected degrees of all vertices in \( G \). Section 4.1 describes probability assignment that minimizes \( \Delta_i \) using linear programming (LP). Due to the inefficiency of LP on large graphs, Sections 4.2 and 4.3 propose GDB and EMD, which assign probabilities inspired by gradient descent and expectation maximization, respectively.

4.1 Optimal probability assignment for minimizing \( \Delta_i \)

Given the backbone graph \( G_b \), we wish to compute the edge probabilities that minimize the discrepancy for \( k = 1 \). Let \( d \) be a vector of size \(|V|\) that contains the expected degrees of the original graph \( G \). We represent \( G_b \) by an incidence matrix \( A_b \) of size \(|V| \times |E_b|\). Using this notation, an equivalent formulation for minimizing the sum of absolute discrepancies \( \Delta \) is

\[
\min_{p'} \quad |d - A_b p'|
\]

s.t. \( p' \in (0,1)^{|E_b|} \)  

LEMMA 1. For an incidence matrix \( A_b \), there is a probability assignment \( p^* \) that minimizes \( \Delta_i \) for which the expected degree \( d'_u \geq d_u, \forall u \in V \).

Proof: Consider a probability assignment \( p^* \) that minimizes \( |d - A_b p^*| \) and let \( d' = A_b p^* \) be the new vector of expected degrees. For the sake of contradiction, assume also that \( d' \) contains illegal vertices, i.e., vertices with \( d'_u < d_u \). Each illegal vertex \( u \) is adjacent to at least one legal vertex \( v \), with \( d'_v > d_v \), otherwise the assignment is not optimal (setting \( p^*(u,v) = p(u,v) - \epsilon \), for a small \( \epsilon > 0 \) yields a better result) . Let \( \theta = d'_u - d_u > 0 \). We prove that, if we subtract \( \theta \) from \( p(u,v) \), then the resulting probability assignment \( p' \):

\[
p'_e = \begin{cases} 
    p_e - \theta, & \text{if } e = (u,v), \\
    p_e, & \text{otherwise}
\end{cases}
\]

is also optimal. Let \( d' = A_b p' \) be the corresponding vector of expected degrees. Then,

\[
d'_i = \begin{cases} 
    d^*_i - \theta, & \text{if } i = u, \\
    d^*_i, & \text{if } i = v, \\
    d^*_i, & \text{otherwise}
\end{cases}
\]

Since \( p^* \) is optimal, \( \sum_{i \in V} |d_i - d^*_i| \) is minimum. Then,

\[
\sum_{i \in V} |d_i - d^*_i| = \sum_{i \in V \setminus \{u,v\}} |d_i - d^*_i| + |d_u - d^*_u| + |d_v - d^*_v| \\
= \sum_{i \in V \setminus \{u,v\}} |d_i - d^*_i| + |d_u - d_v - \theta| + |d_v - d_u - \theta| \tag{4}
\]

Given that \( v \) is legal, \( d_u - d_v \geq 0 \), \( d_v - d_u - \theta \geq 0 \). Thus, Equation (4) becomes:

\[
\sum_{i \in V} |d_i - d^*_i| = \sum_{i \in V \setminus \{u,v\}} |d_i - d^*_i| + \theta + |d_v - d^*_v| - \theta = \sum_{i \in V} |d_i - d^*_i|
\]

Since \( \sum_{i \in V} |d_i - d^*_i| \) is minimum, \( \sum_{i \in V} |d_i - d^*_i| \) is also minimum. Thus, \( p^* \) is optimal. By applying the above argument to all illegal vertices we construct an optimal instance that contains only legal vertices. \( \square \)

Through Lemma 1 we prove the following theorem.

THEOREM 1. Given a backbone incidence matrix \( A_b \) and an expected degree vector \( d \), the optimal probability distribution \( p^* \) for degree discrepancy \( \delta_A \), is the solution of the following LP:

\[
\max_{p'} \quad |p'| \\
\text{s.t.} \quad A_b p' \leq d \\
p' \in (0,1)^{|E|} \tag{5}
\]

Proof: According to Lemma 1 there exists an optimal assignment \( p^* \) for which \( A_b p^* \leq d \). We only need to prove that the objective function is equivalent to Equation (2). Let \( A_b = [a_{u_1},a_{u_2},...a_{u_n}]^T \), i.e., the row vectors of matrix \( A_b \). Then

\[
\min_{p'} |d - A_b p'| = \min_{u \in V} \sum_{u \in V} |d_u - a_u p'| \geq \min_{u \in V} \sum_{u \in V} (d_u - a_u p') = \max_{u \in V} \sum_{u \in V} a_u p' = \max_{u \in V} \sum_{u \in V} p_u = \max |p'| \quad \square
\]

Theorem 1 states that the probability assignment step can be performed optimally by any linear programming solver, e.g., simplex. However, the running time of such solvers is prohibitive for large graphs, which is also confirmed in our experimental evaluation. Moreover, LP does not explicitly reduce entropy. The following method closely approximates the optimal probability assignment at a small fraction of the time, while also decreasing the entropy compared to \( G \).

4.2 Gradient descent backbone

Given the backbone graph \( G_b = (V,E_b) \), Gradient Descent Backbone (GDB) initially generates a seed uncertain graph \( \tilde{G} = (V,E,\tilde{p}) \), \( \tilde{p} = p \), and proceeds in iterations. Let \( \tilde{p} \) and \( p' \) be the probabilities of the previous and the current iteration, respectively. At each iteration, GDB optimizes the probability \( p'_e \) of each edge \( e = (u,v) \), considering the remaining probabilities fixed. To this end, we need to calculate the derivative of the objective function \( \sum_{u \in V} |\delta(u)| \). Since
At each iteration, the edges of the backbone graph are depicted with bold, and in $[0,1]$ because it is convex (i.e., $G$ is injective function is below a threshold $G$). Then, the probability that sets the first derivative to zero, is:

$$p'_e = p_e + stp\text{, where } stp = \frac{\pi(v_0)\delta_A(u_0; \pi) + \pi(u_0)\delta_A(v_0; \pi)}{\pi(u_0) + \pi(v_0)}$$

Equation (5) raises two concerns: First, probability $p'_e$ may fall outside the range $[0,1]$. In this case, $D_1$ is monotonic in $[0,1]$ because it is convex (i.e., $\frac{\partial D_1}{\partial p_e} > 0$). Second, the probability increase $stp$ may result to higher entropy for $e$, which is not desirable. GDB overcomes these concerns by assigning probabilities using the following rule:

$$p'_e = 0$$

where $[0, x]^1 = \max\{0, \min\{x, 1\}\}$ and $h \in [0,1]$.

In essence, GDB performs gradient descent which is guaranteed to reach a local minimum of the objective function $G$. Parameter $h$ relates the step size of gradient descent to the entropy. Intuitively, if the optimal assignment of Equation (5) results in entropy increase, GDB adds only a fraction $h$ of $stp$ to $p_e$, attenuating the negative side effect. This is a common practice in gradient decent techniques: instead of moving directly to the goal, move in smaller steps towards the correct direction [7]. In addition to limiting entropy increase, this allows other neighbouring edges to update their probabilities, decoupling the local minimum from the edge ordering.

Algorithm 2 contains the pseudocode of GDB. Lines 13 initialize $G'$ with all edges of the backbone graph $G_b$ using their corresponding probabilities in $G$. Then, the algorithm iteratively examines every edge of $E'$ and decides its probability: if the optimal assignment of Equation (5) leads to entropy increase, then $stp$ is capped by parameter $h$ (line 10). Otherwise, both the discrepancy and the entropy are reduced by the optimal assignment, and no limit is applied. The algorithm terminates when the improvement of the objective function is below a threshold $\tau$.

Figure 2 illustrates the execution of GDB for $\delta^4$ with $\alpha = 0.6$ and $h = 1$ in the uncertain graph of Figure 2 (a). The edges of the backbone graph are depicted with bold, and the absolute discrepancies are shown next to each vertex. At each iteration, GDB examines edges $(u_1, u_4)$, $(u_2, u_3)$, $(u_3, u_4)$, and decides their best probabilities. For example, for edge $(u_1, u_4)$, $p'_e(u_1, u_4) = p(u_1, u_4) + \frac{\delta_{(u_1, u_4)}}{2} = 0.2 + \frac{0.6 + 0.0}{2} = 0.5$. Figure 2 (b) contains the output of GDB. Note that at this point, Equation (2) cannot further modify the probability of any edge in the output graph. GDB improved the objective function $D_1 = \sum_{u \in V} \delta^4(u)$ from 0.56 to 0.36 and reduced the entropy from 3.85 to 2.60.

![GDB example](image)

**Figure 2:** GDB example

### 4.3 Expectation maximization degree

Since GDB only updates the edge probabilities of the backbone graph $G_b = (V, E_b)$ (without inserting or removing edges), it is sensitive to the choice of $G_b$. On the other hand, Expectation-Maximization Degree (EMD) modifies both $E_b$ and the edge probabilities. EMD is inspired by Expectation-Maximization [9], which is an iterative optimization framework that estimates two sets of interdependent unknown parameters. In our case, EMD estimates the following sets of parameters: i) the set of edges in the sparsified graph and ii) their probabilities.

Similarly to GDB, the objective function is $D_1 = \sum_{u \in V} \delta^4(u)$. EMD starts with the input backbone graph, and the corresponding probabilities $p$ of $G$. Then, it enters the iterative process, which consists of two phases. $E$-phase replaces edges of $E_b$ with edges from $E'/E_b$ considering the edge probabilities fixed. The new graph is denoted by $G'_e = (V, E'_e)$. $M$-phase calls GDB to optimize the edge probabilities considering $G'_e = (V, E'_e)$ as fixed. We denote with $\hat{p}$ and $\tilde{p}$ the probabilities of the previous and the current iteration respectively.

Equation (9) provides a rule to estimate the values of $\hat{p}'$ with respect to some backbone graph $G_b$ and entropy parameter $h$. Accordingly, we need a rule to generate the graph $G'_e$ that minimizes the objective function with respect to a fixed set of probabilities $\hat{p}$. To this end, we define the gain...
of an edge \( e = (u_0, v_0) \) as follows:

\[
g(e)\big|_{n'_e} = \delta^2(u_0)\big|_0 - \delta^2(u_0)\big|_{n'_e} + \delta^2(v_0)\big|_0 - \delta^2(v_0)\big|_{n'_e} \tag{10}
\]

where \( n'_e \) is the degree of Equation \( \theta \) and \( \delta(u_0) \) is the degree discrepancy of vertex \( u_0 \) for \( n'_e \). Intuitively, \( g(e) \) quantifies the maximum improvement to \( D_1 \), incurred by including \( e \) with the best probability \( n'_e \).

Our goal is to swap the edges of the current backbone \( E_b \), with edges from \( E \backslash E_b \) that have higher gain. An intuitive approach stores the edges of \( E \backslash E_b \) in a dynamic max-heap \( H \) based on \( g(e) \). At each iteration, it removes each edge \( e = (u, v) \in E_b \) from \( E_b \), and adds it to \( H \). Consequently, it recomputes the gains of all edges incident to \( u \) or \( v \) that have been affected by \( e \)’s removal, and updates \( H \) with the new gains. Then, it includes the top of \( H \), \( e_i \), to the new backbone \( E_b' \). Lastly, it updates \( H \) after recalculating the gain of edges incident to \( e_i \). Unfortunately, this approach yields high computational cost for the following reasons:

- The size of \( H \) is \( O((1 - \alpha)|E|) \). For small sparsification ratio, \( H \) is prohibitively large.
- For every edge, \( O(|E|/|V|) \) heap operations must be performed because each edge affects on average \( 2(1 - \alpha)|E|/|V| \) other edges of \( H \). Thus, the total heap overhead of every \( E \)-phase is \( O(\alpha(1 - \alpha)|E|^2 \log |V|/|V|) \).

EMD alleviates this overhead by maintaining a max-heap \( H_v \) of the vertices \( V \), based on their discrepancy \( \delta \). The method follows an approach similar to the above framework. This time however, an edge \( e \in E_b \) is swapped with the edge \( e_v \) that is incident to the top of \( H_v \), and has the highest gain. This greatly reduces the running time of EMD compared to the above framework:

- \( H_v \) has size \( O(|V|) \), which is much smaller than \( H \).
- For every edge, EMD performs only \( O(1) \) heap operations because each edge affects the discrepancy of only two vertices. Thus, at every \( E \)-phase, the total heap overhead is \( O(\alpha|E| \log |V|) \).

Algorithm 3 illustrates EMD. Lines 1-5 initialize \( \delta_A \), \( E' \) and \( p \). Lines 6-20 contain \( E \)-phase. Initially, EMD builds a max-heap \( H_v \) of the vertices \( V \) based on \( \delta_A \). At the iterative step, for each edge \( e = (u, v) \in E_b \), EMD excludes \( e \) from \( E_b \), calculates its gain \( g(e) \) and updates the corresponding entries of \( H_v \) for the affected vertices \( u \) and \( v \) (line 12). Then, it retrieves the top of \( H_v \), namely \( e_V \), without removing it from the heap. For all edges \( \epsilon \in E \backslash E_b \) adjacent to \( e_V \), EMD calculates its best probability according to Equation 9. Using this probability, it computes the gain \( g(\epsilon) \). Let \( e_{max} = \arg \max \{ g(\epsilon_v), g(\epsilon) \} \) be the edge with the maximum gain among edges \( \epsilon \) in the original edge \( e \). EMD includes \( e_{max} \) to the backbone graph and updates the incident vertices \( u_{max}, v_{max} \) in \( H_v \) (lines 19-20). This process repeats until all edges have been examined. Based on the new backbone graph, \( M \)-phase further optimizes the probability assignment of \( G_b' \) by calling GDB (line 21). The procedure terminates when the improvement of an iteration is below a threshold \( \tau \).

Figure 3 illustrates the execution of \( E \)-phase of EMD, for \( \delta_A \), in the uncertain graph of Figure 2(a), with the same backbone \( E_b \) (in bold) and entropy parameter \( h = 1 \). At the iterative phase, edge \( (u_1, u_4) \) is removed from \( E_b \) and vertices \( u_1, u_4 \) update their discrepancy to the values of the left table (3(a)). The top of \( H_v \) is vertex \( u_1 \) and its adjacent edges are \( (u_1, u_3), (u_1, u_2) \) and \( (u_1, u_4) \). Equations 9 and 10 compute their best probability and gain respectively (right table of Figure 2(a)). Edge \( (u_1, u_2) \) has the highest gain 0.78, therefore it is included in the backbone \( E_b' \). Figure 2(b) demonstrates \( E_b' \) with the corresponding probabilities (discrepancies) next to the edges (vertices).

![Figure 3: EMD example](image-url)

Let \( (u_2, u_4) \) be the second edge to be examined. The left table of Figure 2(c) shows \( H_v \) after the exclusion of \( (u_2, u_4) \). Again, the top of \( H_v \) is vertex \( u_1 \). This time the relevant edges are \( (u_1, u_3) \) and \( (u_1, u_2) \), along with the edge under consideration \( (u_2, u_4) \). The right table contains their corresponding best probability and gain. Edge \( (u_1, u_3) \) has the
highest gain, thus EMD includes it to $E'$. Figure 2(d) contains $E'_k$. It can be easily verified that the remaining edge $(u_3, u_4)$ does not affect the backbone graph structure (it is removed and reinserted), thus the $E$-phase is complete. A subsequent $M$-phase on the updated backbone graph, calculates the new $p'$ probabilities for the respective edges $(u_1, u_2), 0.55$; $(u_1, u_4), 0.2$; and $(u_3, u_4), 0.55$. Note that the resulting discrepancy $D_1 = 0.01$ has improved considerably.

The original objective function $\Delta_1$ has also decreased to 0.2, from 1.2 in the backbone graph of Figure 2(a). Similarly, entropy has decreased to 2.7 from 3.85 in the original graph.

5. PRESERVING EXPECTED CUTS

EMD cannot be applied for $k > 1$ because our gain definition requires the computation of the discrepancy for all $k$-cuts that contain an edge, whose number is exponential. In the following we design a new rule that enables the application of GDB to arbitrary values of $k \geq 1$. Since $\delta(S)$ is not differentiable, we again utilize the squared discrepancies $\delta^2(S)$, focusing on the absolute discrepancy $\delta_A$. Accordingly, the objective function is:

$$D_k = \sum_{i=1}^{k} \sum_{S \subseteq V, |S| = k} \delta_A(S)$$

Its derivative with respect to $p'_c$ for an edge $e = (u_0, v_0)$ is:

$$\frac{\partial D_k}{\partial p'_c} = -2 \sum_{i=1}^{k} \sum_{S \subseteq V, |S| = k} \delta_A(S) - 2 \sum_{i=1}^{k} \sum_{S \subseteq V, |S| = k} \delta_A'(S)$$

Changing the probability of edge $e$ from $\hat{p}_c$ to $p'_c$, the discrepancies of all cuts that contain $e$ become:

$$\delta_A(S) = \hat{\delta}_A(S) + \hat{p}_c - p'_c$$

Setting the derivative equal to zero and solving with respect to $p'_c$, using Equation 11, we obtain the best probability for $e$:

$$p'_c = \hat{p}_c + \frac{\sum_{i=1}^{k} \left( \sum_{S \subseteq V, |S| = k} \hat{\delta}_A(S) \right) + \sum_{i=1}^{k} \sum_{S \subseteq V, |S| = k} \hat{\delta}_A(S)}{\sum_{i=1}^{k} \sum_{S \subseteq V, |S| = k} \frac{1}{\sum_{S \subseteq V, |S| = k}}}$$

(12)

The computation of the above equation is not tractable due to the fact that we need to enumerate all $k$-cuts that contain edge $e = (u_0, v_0)$. To avoid this we introduce the following enumeration function:

$$\binom{n}{k} = \begin{cases} 0 & \text{if } k < 0 \\ \sum_{i=0}^{k} \binom{n}{i} & \text{if } k > 0 \end{cases}$$

We count how many times the discrepancy of an edge is present in the sum of the numerator. If the edge is incident to $u_0$ or $v_0$, it will be counted $\binom{n-1}{k-1}$ times because we restrict its other vertex from entering $S$. All other edges, not incident to $u_0$ or $v_0$, will be counted $2\binom{n-4}{k-2}$ times in the sum, since only one of their incident vertices belongs to $S$.

$$\sum_{i=1}^{k} \sum_{S \subseteq V, |S| = k} \delta_A(S) = \binom{n-3}{k-1} \hat{\delta}_A(u_0) + 2 \binom{n-4}{k-2} \hat{\delta}_A(v_0) + (n-3) \sum_{S \subseteq V, |S| = k} \hat{\delta}_A(S)$$

where:

$$\hat{\Delta}(e) = \sum_{u_1, v_1 \in E, u_1 \neq u_0, v_1 \neq v_0} p_{u_1, v_1} - \hat{p}_{u_1, v_1}$$

Therefore Equation 12 reduces to:

$$p'_c = \hat{p}_c + \frac{\left( \binom{n-3}{k-1} \hat{\delta}_A(u_0) + \hat{\delta}_A(v_0) \right) + 4 \binom{n-4}{k-2} \hat{\delta}_A(S)}{2 \binom{n-4}{k-2}}$$

(13)

Equation 13 proposes a probability change that weights the degree discrepancies $\hat{\delta}(u_0)$ and $\hat{\delta}(v_0)$ versus the discrepancy of the edges that are not incident to $u_0$ and $v_0$, $\hat{\Delta}(e)$. The best probability $p'_c$ can exceed $[0,1]$. However, since the objective function is again convex (i.e., $\frac{\partial^2 D_k}{\partial p'_c^2} > 0$), the optimal probability is:

$$\hat{p}_c' = \left[ \hat{p}_c + \hat{h} \left( \binom{n-3}{k-1} \hat{\delta}_A(u_0) + \hat{\delta}_A(v_0) \right) + 4 \binom{n-4}{k-2} \hat{\delta}_A(S) \right]^{1}$$

(14)

where the entropy parameter $h \in [0, 1]$ tunes the step size of gradient descent.

The only modification of GDB, is that in line 7 of Algorithm 2, the optimal step $\Delta$ is replaced by the ratio in Equation 13. Special cases of the general rule include $k = 1$, $k = 2$ and $k = n$. For $k = 1$ and absolute discrepancies $\delta_A$, the above equation reduces to Equation 3, and thus takes into account only the degree discrepancies. For $k = 2$, the best probability is:

$$\hat{p}_c' = \left[ \hat{p}_c + \hat{h} \left( \binom{n-3}{k-1} \hat{\delta}_A(u_0) + \hat{\delta}_A(v_0) \right) + 4 \binom{n-4}{k-2} \hat{\delta}_A(S) \right]^{1}$$

(15)

For $k = n$, the update rule changes to the following formula, which distributes the cumulative probability of eliminated edges to all the remaining ones. This corresponds to random probability reassignment.

$$\hat{p}_c' = \left[ \hat{p}_c + \hat{h} \left( \binom{n-3}{k-1} \hat{\delta}_A(u_0) + \hat{\delta}_A(v_0) \right) + 4 \binom{n-4}{k-2} \hat{\delta}_A(S) \right]^{1}$$

(16)

The general rule of Equation 14 is analytic and does not require any enumeration of cuts. Consequently, the running time of GDB is insensitive to $k$ and depends only on the convergence speed of gradient descent.

6. EXPERIMENTS

In our evaluation, we use two real undirected uncertain graphs with various sizes, densities, and edge probabilities, summarized in Table 1. Flickr 24 is a social network, where edge probabilities are based on the principle that similar interests indicate social ties. This is the densest dataset; a vertex has on average about 130 neighbours. Twitter 5 is extracted from the popular online micro-blogging service. Probabilities denote the influence that the associated users exert on each other. Although sparser than Flickr, Twitter has higher average probability on the edges.

In order to assess the behaviour of the methods in graphs with increasing density, we also use 4 synthetic undirected datasets, whose characteristics are summarized in Table 1. They all stem from an induced subgraph of Flickr with 1000 vertices, where edges have been added between random pairs of vertices, until the density becomes 15, 30, 50, 90 % of the complete graph. The additional edge probabilities follow the same distribution as the original Flickr.
Table 1: Characteristics of datasets

|    | dataset | vertices | edges  | $|E|/|V| | $|E|_{p_{\alpha}} | $|E|_{d_{\text{un}}} |
|----|---------|----------|--------|-----------|----------------|----------------|
| Flickr | 78,322  | 10,171,099 | 129.89 | 0.09      | 22.93          |
| Twitter | 20,362  | 661,706   | 25.17  | 0.15      | 7.71           |
| Synthetic | 1,000  | 322,679   | 44.3   | 0.09      | 24.3           |

Table 2: Mean Absolute Error (MAE) of absolute degree discrepancy $\delta_A(u)$ (Flickr reduced)

|     | 8% | 10% | 32% | 64% |
|-----|----|-----|-----|-----|
| LP  | 1.04 | 2.56E-02 | 4.22E-03 | 2.70E-04 |
| GDB$^A$ | 1.19 | 2.68E-02 | 4.38E-03 | 3.92E-04 |
| GDB$^R$ | 1.21 | 2.67E-02 | 4.38E-03 | 3.92E-04 |
| GDB$^R_{2}$ | 1.73 | 4.25E-02 | 7.46E-03 | 6.04E-04 |
| EMD$^A$ | 1.33 | 2.56E-02 | 1.22E-02 | 7.98E-13 |
| EMD$^R$ | 1.35 | 2.18E-02 | 1.43E-03 | 1.79E-12 |
| GDB$^{A,t}$ | 2.27 | 2.95E-04 | 2.99E-05 | 2.62E-12 |
| GDB$^{R,t}$ | 3.54 | 1.78E-03 | 1.82E-04 | 1.66E-04 |
| EMD$^{A,t}$ | 2.47 | 4.11E-04 | 2.99E-05 | 2.62E-12 |
| EMD$^{R,t}$ | 2.55 | 9.23E-05 | 8.17E-13 | 7.34E-13 |

6.1 Assessment of proposed techniques

We evaluate the proposed methods GDB (Gradient Descent Backbone) and EMD (Expectation Maximization Degree) on structural properties using absolute degree discrepancy. As a benchmark, we use LP, the linear programming technique of Section 4.1, which, given a backbone graph, yields the optimal probability assignment that minimizes $\Delta_1$ with entropy parameter $h = 0$. Because LP fails to terminate within reasonable time in the real datasets, the experiments are performed on an induced subgraph of Flickr that consists of 5,000 vertices and 655,275 edges, and was extracted using Forest Fire [15].

We use the following notation to differentiate among variants of each method:

- $A$ and $R$ superscripts denote the variants that aim at minimization of the absolute $\delta_A$ and relative $\delta_R$ discrepancy, respectively. The first type favors nodes with high degree by targeting the absolute error, whereas the second treats all degrees equally.
- $t$ suffix signifies that the backbone graph is generated by Algorithm 1 which, given a backbone graph, yields the optimal probability assignment that minimizes $\Delta_1$ with entropy parameter $h = 0$. Because LP fails to terminate within reasonable time in the real datasets, the experiments are performed on an induced subgraph of Flickr that consists of 5,000 vertices and 655,275 edges, and was extracted using Forest Fire [15].
- $k = \{2, n\}$ subscript denotes GDB preserving $k$-cuts. Absence of this subscript implies that $k = 1$ (i.e., preservation of the expected degrees).

Table 2 shows the mean absolute error (MAE) of the absolute degree discrepancy $\delta_A$ of variants of GDB, EMD and LP for sparsification ratio $\alpha$ ranging from 8% to 64%. $GDB^A$ has by far the worst performance as it randomly distributes the missing probabilities to all edges of $G'$. The accuracy of $GDB^R$, $GDB^R_{2}$ and $GDB^R_{2}$ is similar and comparable to LP for the corresponding backbone graph. The backbone graphs generated by Algorithm 1 benefit all variants (compared to random backbones). However, for $\alpha = 8\%$, the spanning nature of the graph increases the discrepancy of some vertices that would be otherwise disconnected. In this case, LP and GDB perform better using random backbones.

The respective $\delta^R$ results are similar and omitted.

as input. In general, EMD improves the accuracy compared to the respective GDB versions by re-structuring the backbone. Methods that preserve the relative discrepancy have similar behaviour to those aiming at absolute discrepancy. The variant with the best overall performance is $EMD^{R,t}$, which achieves the highest accuracy for all values of sparsification ratio $\alpha > 8\%$, shown in bold in Table 2.

Figure (a) shows the MAE of the cut discrepancies versus $\alpha$. Since it is intractable to measure all cuts, we randomly select 1000 $k$-cuts for $k = 1$ up to $|V|$ for each value of $\alpha$, and we compute the average absolute discrepancy. LP is excluded because it explicitly aims at $\Delta_1$. Similar to Table 2 and for the same reasons $GDB^R_{2}$ under-performs the other variants for $\alpha > 8\%$. The superior performance of $GDB^A$ for $\alpha = 8\%$ is explained as follows. In Flickr (also in the reduced graph), the average probability is 0.09; thus, the expected number of edges is approximately 0.09$|E|$. For $\alpha < 9\%$ the sparsified graph does not contain enough edges to reach the expected number under any probability assignment. $GDB^A$ assigns the maximum probability $p = 1$ to all available edges. On the other hand, the rest of the methods, respecting the constraints on degree and 2-cuts do not entirely redistribute the missing probabilities. For $\alpha > 8\%$, this ceases to be the case and they yield high accuracy.

(d) MAE of $\delta_A(S)$

Figure 4: (a) MAE of cut size discrepancy $\delta_A(S)$ and (b) execution time (Flickr reduced)

Figure (b) illustrates the running time as a function of $\alpha$. For both GDB and EMD the running time is independent of the discrepancy (absolute or relative), and the structure of the backbone graph. In addition, GDB optimizes cut sizes using analytic equations; thus its performance is insensitive to $k$. Accordingly, the plots of GDB and EMD capture all variants of each method. Both techniques are significantly faster than LP, which cannot be applied for large graphs.
EMD is slower than GDB since it invokes it as a module for the M-phase. However, the overhead is small, which confirms the efficiency of the vertex (as opposed to edge) heap.

In order to compare against the benchmark methods, we select as representative variants of our techniques EMD<sup>R</sup>-<i>t</i> and GDB<sup>A</sup>. EMD<sup>R</sup>-<i>t</i> has the most balanced performance for the settings of Table 2 and Figure 4(a). GDB<sup>A</sup> is in general inferior, but it outperforms EMD<sup>R</sup>-<i>t</i> for sparsification ratio \( \alpha = 8\% \). Moreover, the two variants collectively cover all combinations of discrepancy type and backbone structure. Thus, in the following, the terms EMD and GDB refer to these variants. Since in the remaining we use the real graphs, LP is excluded due its high cost.

A final remark concerns the fine-tuning of entropy parameter \( h \in [0,1] \) in GDB (and EMD, since GDB constitutes one of its modules). Recall from Section 4.2 that \( h \) reduces the gradient descent step size when the optimal probability assignment increases entropy. Figure 5(a) plots the MAE of the absolute degree discrepancy \( \delta_A \) versus the sparsification ratio, for various values of \( h \). In the extreme case of \( h = 0 \), GDB yields poor performance for \( \delta_A \) because it discards any probability assignment that increases the edge entropy. On the other hand, for \( h = 1 \) GDB yields the best result on \( \delta_A \), but the worst entropy values as shown in Figure 5(b), which plots the relative entropy \( \frac{H(G''_t)}{H(G)} \) versus \( \alpha \). Intermediate values of \( h \) span between these two extremes. In the remaining, we set \( h = 0.05 \) as it has the most balanced performance.

![Figure 5: Effect of entropy parameter \( h \) on GDB (Flickr reduced)](image)

**6.2 Comparison with benchmarks on structural properties**

We compare EMD and GDB against the benchmarks NI and SS. Recall that NI constitutes the adaptation of a cut-based deterministic sparsification method, whereas SS extends a spanner-based technique to the uncertain setting. Figure 6 plots the MAE of the absolute degree discrepancy \( \delta_A(u) \) and the MAE of the cut discrepancy \( \delta_A(S) \) (objective function for \( k = 1 \) and \( k \geq 1 \), respectively) versus \( \alpha \). The proposed methods consistently outperform the benchmarks for both structural properties in all datasets. The low accuracy of SS can be explained by the fact that it was designed to capture shortest path distances, instead of cuts or degrees. NI is comparable to the proposed techniques for small values of \( \alpha \) in Twitter that have high edge probabilities. In these cases, the backbone graph is almost deterministic (most edges have probability 1) and there is little space for improvement by EMD and GDB. For the other settings, NI fails because it assumes unbounded weights. Bounding the maximum weight to 1 seriously affects both its performance and its theoretical guarantees: NI yields a mild probability redistribution that fails to preserve degrees and cuts in practice. Moreover, NI is designed for dense graphs with \( E = \Theta(n^2) \), whereas the evaluated datasets are much sparser. As expected from Table 2 and Figure 4(a), in most settings EMD outperforms GDB.

![Figure 6: MAE of absolute degree discrepancy \( \delta_A(u) \) and absolute cut size discrepancy \( \delta_A(S) \) (real datasets)](image)

Figure 7 plots MAE of \( \delta_A(u) \) and \( \delta_A(S) \) as a function of the graph density (percentage of the complete graph), on the synthetic datasets, which are denser than the real ones. The sparsification ratio \( \alpha \) is fixed to 16%. As the graph density increases, all methods yield increasing error. To see why this happens, consider SS that does not perform probability redistribution. In the simplified case of uniform edge distribution on the vertices with mean probability \( \bar{p} \), MAE(\( \delta_A(u) \)) = \( \frac{|E|}{\bar{p}} \). Since all other factors are constant, MAE(\( \delta_A(u) \)) increases linearly with \( |E| \). NI that applies limited probability redistribution yields smaller error. Clearly the winner again is EMD with much smoother increase, verifying its robustness for dense graphs.
entropy for small $\alpha$ compared to NI and SS which overall perform similarly. This is expected since our methods aim at reducing entropy, unlike the competitors that are designed for deterministic graphs (zero entropy). Relative entropy increases with $\alpha$, always remaining less than 1. Figure 3(c) plots the entropy of the synthetic graphs with $\alpha = 16\%$. The relative entropy is constant because the percentage of edges in the sparsified graph remains the same.

Figure 8: Graph entropy $H$ (real and synthetic datasets)

The last experiment measures the running time of GDB, EMD and NI in the real graphs. As shown in Figure 9, the proposed methods usually terminate within a minute, whereas NI is more than an order of magnitude slower. SS is omitted from the diagrams because it requires several hours to terminate.

Figure 9: Execution time (real graphs)

6.3 Comparison with benchmarks on queries

We evaluate the following graph queries. (i) Pagerank (PR) is a measure of the node’s influence in the graph and has been widely used to rank Web page search results according to their links [20]. (ii) Shortest path distance (SP) is the average shortest distance between a pair of vertices in all worlds excluding the ones that disconnect them. SP is essential for any task involving shortest path computations [21]. (iii) Reliability (RL) is the probability that a vertex is reachable from another in the graph. It is a common metric for the resilience of router networks. (iv) Clustering coefficient (CC) is the ratio of the number of edges between the neighbours of a vertex to the maximum number of such links. CC constitutes an important metric for search strategies and social networks [11]. Although we evaluate CC and PR on all vertices of $G$, we choose 1000 random vertex pairs for the evaluation of SP and RL, because the evaluation on all pairs would be too expensive to terminate for our datasets.

Quality results. Let $G'$ be a sparsified subgraph of $G$, and a query $Q$. $Q$ is evaluated through Monte-Carlo sampling on 500 possible worlds of both $G'$ and $G$. The various outcomes of $Q$ in different samples of $G'$ form a cumulative distribution $F_{G',Q}(x)$ of results. To quantify the similarity of $G'$ to $G$ with respect to $Q$, we have to measure the difference between $F_{G',Q}(x)$ and $F_{G,Q}(x)$. To this end, a robust metric is the earth mover’s distance $D_{em}$ [27]. Intuitively, $D_{em}$ measures the minimum change that aligns the two distributions. Formally, let $\{x_0, x_1, \cdots, x_M\}$ be the ordered set of all observed results of $Q$ in $G$ and $G'$. To compute $D_{em}$, we apply the following equation:

$$D_{em}(G, G', Q) = \sum_{i = 1}^{M} |x_i - x_{i-1}| (x_i - x_{i-1})$$

Figure 10 plots $D_{em}$ versus the sparsification ratio. Each row of diagrams corresponds to a dataset and each column to a query. With few exceptions, GDB and EMD outperform the benchmarks for all settings, usually by a wide margin. Moreover, the diagrams are consistent with those on structural properties, confirming the correlation of our objective functions with the performance of the sparsified graphs for diverse queries. SS yields the highest error even on the SP metric, which constitutes its focus. The main cause for its poor performance is that it does not involve any probability redistribution. Although NI achieves good approximation
for CC, it usually introduces large error for the rest of the queries. EMD is the winner for high sparsification ratio, while GDB is preferable for small α in most queries. This is in accordance with Figure 6 where GDB preserves better the structural properties for α = 8%.

Figure 11(a) (resp. Figure 11(b)) illustrates D_{em} of PR (resp. SP) on the synthetic datasets, as a function of density for α = 16%. The proposed techniques clearly yield smaller error than the benchmarks. Observe that PR is node centric and highly correlated with the degree; thus, the diagram of Figure 11(a) is similar to that of Figure 4(a). The plots of CC are similar to those of PR and omitted. On the other hand, the error of SP decreases with increasing density because more alternative short paths are available, due to the abundance of edges. RL has practically zero error for all methods, since the dense graph has reliability almost 1 for all pairs of vertices.

Figure 11: Earth mover’s distance D_{em} for Pagerank (PR) and Shortest Path distance (SP) (synthetic datasets)

**Variance results.** We assess the performance of the various sparsifiers with respect to the variance of an MC estimator on the above queries. Specifically, due to the randomized nature of MC estimators, different executions of the same experiment may yield different results. The variance quantifies the deviation of results from the mean. Let Φ(G) denote the result of an MC simulation on an uncertain graph G.

A sparsified graph with low variance on MC estimator

implies the need of fewer samples for accurate estimation. Specifically, according to the theory of MC simulations, the unknown expected value of a query in G belongs to the confidence interval CI = [Φ(G) − 1.96σ(G)/√N, Φ(G) + 1.96σ(G)/√N] with probability 95%, where σ(G) is the variance of the query in G and N is the number of samples. Let the confidence width CW be the length of the confidence interval, i.e., CW = 2CI = 3.92σ(G)/√N. In order to achieve the same level of accuracy CW’ between the original graph G and the sparsified G’, we require CW = CW’ → σ(G)/√N = σ(G’)/√N’ → N’/N = (σ(G’)/σ(G))^2. Intuitively, small relative variance σ(G’)/σ(G) implies the need of fewer samples for accurate estimation. This is why variance is among the most important metrics for the quality of MC simulation (see [11, 19]). However, calculating the actual variance is intractable. Thus, we follow a strategy similar to [18] for an unbiased estimator of the variance of Φ(G), denoted as σ(G). Specifically, we run each estimator (Φ(G)) 100 times. Then the unbiased estimator of the variance of Φ(G) is σ(G) = ∑_{i=1}^{100} (Φ_i(G) – Φ(G))^2/99, where Φ_i(G) denotes the mean of Φ_i(G) for (i = 1, ..., 100).

Figure 12 illustrates the relative variance of the queries versus the sparsification ratio α. Let σ(G) and σ(G’) denote the variance of the MC estimator on the original and the sparsified graph respectively. The y-axis represents the relative variance, i.e., the ratio σ(G’)/σ(G). Each row of diagrams corresponds to a dataset and each column to a query. Consistently, EMD and GDB drop the variance of the original graph up to several orders of magnitude. On the other hand, NI and SS have, in most cases, higher variance than the original graph. The justification of this result is based on the fact that NI and SS perform limited (if any) probability redistribution. Assume for instance the SP query. During evaluation we only consider reliable possible worlds for which the distance of the query points is less than infinite. Sparsification without probability redistribution reduces the number of possible worlds for which a path exists. Consequently, a bigger proportion of possible worlds is discarded, increasing the variance of the estimator, which now depends on fewer samples. Moreover, according to the small world phenomenon, the majority of shortest paths are small (i.e., less

Figure 12: Relative variance for Pagerank (PR), Shortest Path distance (SP), Reliability (RL) and Clustering Coefficient (CC)
than 10). Thus, the removal of edges is more likely to affect the the probability of short paths. This in turn increases the probability of larger paths, inflating the variance.

Our techniques alleviate the above short-comings by applying aggressive probability redistribution that preserves the expected number of edges, reducing the entropy of the sparsified graphs. This results in many edges having probability one. For instance, in Twitter with \( \alpha = 8\% \), 75% of the edges of GDB have probability 1. In comparison, in NI only 25% of the edges are deterministic. As \( \alpha \) increases, fewer edges reach probability 1; thus, the variance of EMD and GDB increases. This result is very important and highlights one of the core differences of our methods compared to the competitors: the goal for entropy reduction.

Summarizing the experiments, as shown in Table 2 and Figure 2, the proposed techniques capture well the structural properties of the input uncertain graph, even if they do not constitute their explicit optimization criterion. For instance, variants that aim at the relative discrepancy \( \delta_R \), e.g. EMD\( ^R \)-t, also preserve the absolute one \( \delta_A \). According to Figures 10-11, the preservation of structural properties leads to accurate results for various queries with different characteristics. Moreover, reducing the entropy of the uncertain graph (Figure 8), our methods decrease the variance of the MC estimator of all evaluated queries (Figure 12).

This has huge effect in processing time, as considerably less samples are required for accurate query estimation. As opposed to the proposed methods, techniques based on deterministic sparsification usually fail, both in terms of result quality and variance. Finally, our algorithms are efficient and applicable to large uncertain graphs.

7. CONCLUSION

Sparsification has often been used to reduce the size of deterministic graphs and facilitate efficient query processing. However, it has not been applied previously to uncertain graphs, despite the fact that they incur significantly higher cost for common query and mining tasks. This paper introduces novel sparsification techniques that, given an uncertain graph \( \mathcal{G} = (V, E, p) \) and a parameter \( \alpha \in (0, 1) \), they return a subgraph \( \mathcal{G}' = (V, E', p') \), such that \( E' : E' \subseteq E, |E'| = \alpha |E| \). \( \mathcal{G}' \) preserves the structural properties of \( \mathcal{G} \), has less entropy than \( \mathcal{G} \), and can approximate the result of various queries on \( \mathcal{G} \).

The proposed methods, GDB (Gradient Descent Backbone) and EMD (Expectation Maximization Degree), involve a two-step framework. First, a backbone deterministic graph \( G_b \) with \( \alpha |E| \) edges is generated. In order to obtain \( \mathcal{G}' \), GDB assigns probabilities to the edges of \( G_b \), aiming at preserving the expected vertex degrees or cut sizes, while reducing the entropy. In addition to assigning probabilities, EMD also changes the structure of \( G_b \) by adding or removing edges. An extensive experimental evaluation with real and synthetic uncertain graphs confirms that GDB and EMD consistently outperform benchmarks adapted from the deterministic graph literature, on several graph queries and metrics.

8. REFERENCES

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APPENDIX

Algorithm 4 Nagamochi Ibaraki (NI)

Input: graph \(G_w = (V, E, w)\), approximation parameter \(\epsilon\)
Output: sparse graph \(G'_w = (V, E', w')\)
1: \(E' \leftarrow \emptyset; \quad E_c \leftarrow E; \quad F_0 = \emptyset\)
2: \(r = 0\)
3: while \(E_c \neq \emptyset\) do
4: \(r \leftarrow r + 1\)
5: compute a spanning forest \(F_r\) of \(E_c\) such that \((F_r \setminus F_{r-1}) \cap E_c = \emptyset\)
6: for each edge \(e \in F_r\) do
7: \(w_e \leftarrow w_e - 1\)
8: if \(w_e = 0\) then
9: sample \(e\) with probability 
   \(\ell_e = \min \{ \log |V|/e^2 \cdot r, 1\}\)
10: if \(e\) is selected then
11: \(E' \leftarrow E' \cup \{e\}\) with \(w'_e \leftarrow w_e/\ell_e\)
12: else
13: discard \(e\)
14: \(E_c \leftarrow E_c \setminus \{e\}\)

This section provides details of the benchmark methods NI \([19]\) and SS \([2]\). Algorithm 4 contains the core iterative process of NI. The method requires as input an approximation parameter \(\epsilon\), which is initially tuned depending on our parameter \(\alpha; \epsilon = \sqrt{|V| \log^2 |V|/|E|}\). The result set \(E'\) is initially empty and the set of available edges \(E_c \leftarrow E\). At each iteration \(r\), a spanning forest \(F_r\) is computed on the set \(E_c\) with the requirement that if an edge \(e\) appears in \(F_{r-1}\), then it must also appear in \(F_r\) (contiguous spanning forests). Each time an edge \(e\) is added by a spanning forest, \(w_e\) is reduced by one. When \(w_e\) becomes 0, \(e\) is sampled with probability 
\(\ell_e = \min \{ \log |V|/e^2 \cdot r, 1\}\). If \(e\) is selected, then line 9 adds it to the result set. Otherwise, \(e\) is discarded. The iterative process stops when all edges of \(E\) have been examined. Intuitively, the sampling probability \(\ell_e\) approximates the connectivity of edge \(e\); if \(e\) belongs to a sparse subgraph, it is covered by spanning trees of early iterations, therefore it is sampled with high probability. On the other hand, an edge in a dense component is covered after several iterations \(r\), thus it’s sampling probability is significantly smaller.

Algorithm 5 presents the main process of SS \([2]\), which computes a \((2t - 1)\)-spanner of \(O(t \cdot m^{1+1/t})\) expected size. The algorithm performs \(t - 1\) iterations, incrementally forming clusters of vertices with the minimum edge that connects them. \(C_i\) maintains the set of vertex clusters for iteration \(i\). Initially, \(C_i\) contains \(|V|\) sets of individual vertices (line 3) and the spanner \(E'\) is empty. At each iteration, a set of clusters, \(R_i\), is selected with probability \(n^{-1/t}\) for each cluster. For each vertex \(u \notin R_i\) that is not yet connected to the spanner, SS examines its neighbours. If none of them is in \(R_i\), then for each adjacent cluster \(c \in C_{i-1}\), SS adds the least weight edge to \(E'\), and updates the clusters (line 22). If \(u\) has an adjacent node in \(R_i\), the least weight edge \(e = (u, v \in R_i)\) is added to the spanner (line 10). Then, SS iterates over all edges of \(u\), adding for each adjacent cluster \(c\), the minimum weight edge \(e'\), if \(w(e') < w(e)\) (line 17). To ensure that the resulting spanner is connected, the algorithm joins all remaining clusters using the connecting edge with the minimum weight (lines 23-26).

Recall from Section 3.3 that, in both methods, the expected number of edges is given in \(O\)-notation, thus the sparsified graph is not guaranteed to reach \(\alpha|E|\) edges. To ensure this, we run Algorithms 4 and 5 with modified parameters to approximate the smallest \(\epsilon\) and \(t\) that contains \(E' < \alpha|E|\). The remaining \(\alpha|E| - |E'|\) edges are sampled using the original probabilities.

Algorithm 5 Spanner Sparsification (SS)

Input: uncertain graph \(G = (V, E, w)\), \(t\)
Output: sparse spanner \(G' = (V, E', w)\)
1: \(E' \leftarrow \emptyset\) spanner edges
2: \(V_S \leftarrow \emptyset\) spanner vertices
3: \(C_0 = \{\{u\}|u \in V\}\) clusters
4: for \(i=1\) to \(t-1\) do
5: \(R_i \leftarrow \text{sample } C_{i-1}\) with probability \(n^{-1/t}\)
6: \(C_i \leftarrow R_i\)
7: for each \(u \in V \setminus V_S\) and \(u \notin R_i\) do
8: \(N_R \leftarrow N(u) \cap R_i\) neighbours of \(u\) in \(R_i\)
9: if \(N_R \neq \emptyset\) then
10: \(e \leftarrow \text{minimum weight edge } u, v \in N_R\)
11: \(E' \leftarrow E' \cup e; \quad V_S \leftarrow V_S \cup u\)
12: \(E \leftarrow E \setminus E(u, N_R)\)
13: merge clusters \(C_i(u)\)
14: for each cluster \(c \in C_{i-1}\) and \(c \notin R_i\) do
15: \(c' \leftarrow \text{minimum weight edge} \quad \text{if } w(e') < w(e)\) then
16: \(E' \leftarrow E' \cup e'\)
17: \(E \leftarrow E \setminus E(u, c)\)
18: merge clusters \(C_i(u)\)
20: else
21: for each cluster \(c \in C_{i-1}\) do
22: \(e \leftarrow \text{minimum weight edge } u, N(u) \cap c\)
23: \(E' \leftarrow E' \cup e; \quad V_S \leftarrow V_S \cup u\)
24: \(E \leftarrow E \setminus E(u, c)\)
25: merge clusters \(C_i(u)\)
26: for each \(c \in C_{i-1}\) do
27: \(e \leftarrow \text{minimum weight edge } c, v \in N(c)\)
28: \(E' \leftarrow E' \cup e\)