Semiring Programming:
A Declarative Framework for Generalized Sum Product Problems

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Abstract
To solve hard problems, AI relies on a variety of disciplines such as logic, probabilistic reasoning, machine learning and mathematical programming. Although it is widely accepted that solving real-world problems requires an integration amongst these, contemporary representation methodologies offer little support for this.

In an attempt to alleviate this situation, we introduce a new declarative programming framework that provides abstractions of well-known problems such as SAT, Bayesian inference, generative models, learning and convex optimization. The semantics of programs is defined in terms of first-order logic structures with semiring labels, which allows us to freely combine and integrate problems from different AI disciplines and represent non-standard problems over unbounded domains.

Keywords: Weighted model counting, Declarative Languages, Semantic abstractions, Semiring frameworks

1. Introduction
AI applications, such as robotics and logistics, rely on a variety of disciplines such as logic, probabilistic reasoning, machine learning and mathematical programming. These applications are often described in a combination of natural and mathematical language, and need to be engineered for the individual application.

Declarative formalisms and methods are ubiquitous in AI as they enable re-use and descriptive clarity. Initial approaches, such as that by Kowalski \cite{51}, were rooted in logic, but they have eventually engendered an impressive family of languages. In knowledge representation and constraint programming, for example, languages such ASP \cite{17} and Essence \cite{36} are prominent, which use SAT, SMT and MIP technology \cite{8,64}. In machine learning and probabilistic reasoning, statistical relational learning systems and probabilistic programming languages such as Markov Logic \cite{61}, Church \cite{39} and Problog \cite{21} are increasingly used to codify intricate inference and learning tasks. In mathematical programming and optimization, disciplined programming \cite{40} and AMPL \cite{31} have been developed. Finally, the DARPA project Probabilistic Programming for Advancing Machine Learning is motivated in the same declarative spirit. Across disciplines in AI, it has become increasingly clear that taming the model building process, admitting reusable descriptions in expressive languages, and providing general but powerful inference engines is essential.

Be that as it may, it is widely accepted that solving real-world AI problems requires an integration of different disciplines. Consider, for example, that a robot may decide its course of action using a SAT-based planner, learn about the world using Kalman filters, and grasp objects using geometric optimization technology. But contemporary declarative frameworks offer little support for such universality: knowledge representation and constraint formalisms mostly focus on model generation for discrete problems, probabilistic programming languages do not handle linear and arithmetic constraints, and finally, optimization frameworks work with linear algebra and algebraic constraints to...
specify the problem and thus are quite different from the high-level descriptions used in the other disciplines and do not support probabilistic or logical reasoning.

Of course, hybrid approaches that treat different computations as independent but communicating processes is an option to address this challenge, but these integrations may not be transparent. So what is lacking here is a universal modeling framework that allows us to declaratively specify problems involving logic and constraints, mathematical programs, as well as discrete and continuous probability distributions in a simple, uniform, modular and transparent manner. Such a framework, together with a generic inference mechanism, would greatly simplify the development and understanding of AI systems with integrated capabilities, and would tame the model building process.

There has been recent progress on this front. A key observation made in [49] is that reasoning about possible worlds is fundamental to many tasks in computer science, including dynamic programming [24], constraint programming [13], database theory [41], probabilistic inference [4], probabilistic logic programming [21], and network analysis [6]. In fact, these tasks essentially invoke a version of the sum product problem [4], but differ in the exact operations carried out over the possible worlds, which can then be recast in the very same way via semirings. The resulting framework, referred to as algebraic model counting (AMC) [49], shows that the computation underlying all these tasks can be defined over a certain class of arithmetic circuits, which then implies that they can be solved via a single algorithm that obtains local solutions and composes them to yield a global solution. The main limitation of the AMC proposal is that the underlying language is propositional logic, and so the semantics is that of classical logic and computations are essentially defined over discrete spaces.

In this paper, we propose a new declarative framework called semiring programming (SP) that attempts to generalize AMC. Our main thesis is to still formulate computations as the sum product problem, but we will rigorously define a semantics that not only defines AMC in unbounded domains, but also non-standard (e.g., non-monotonic) ones. To put the proposal in perspective, consider that Eugene Freuder [32] famously quipped: “constraint programming represents one of the closest approaches computer science has yet made to the Holy Grail of programming: the user states the problem, the computer solves it.” The underlying idea was summarized in the slogan:

(constraint) program = model + solver

The vision of SP builds upon this equation in that:

(semiring) program = logical theory + semiring + solver

Thus a model is expressed in a logical system in tandem with a semiring and a weight function. As a framework, SP is set up to allow the modeler to freely choose the logical theory (syntax and semantics) and so everything from non-classical logical consequence to real-arithmetic is fair game. Together with a semiring and these weights, a program computes the count. Our task will be to show that the usual suspects from AI disciplines, such as SAT, CSP, Bayesian inference, and convex optimization can be: (a) expressed as a program, and (b) count is a solution to the problem, that is, the count can be a \{yes, no\} answer in SAT, a probability in Bayesian inference, or a bound in convex optimization. In other words, the count is shown to semantically abstract challenging AI tasks. We will also demonstrate a variety of more complex problems, such as matrix factorization, and one on compositionality.

Informally, from an expressiveness viewpoint, SP is designed to be:

• universal, in that it can represent classical problems from disciplines ranging from logic to mathematical programming, and it inherits the strengths of both camps;
• declarative, which means that domain knowledge can be expressed in a program in a natural, human-readable way, and that it is possible to easily cope with changes in requirements and information in a principled manner;
• generic, in that it permits instantiations to a particular language – including real arithmetic (as needed in machine learning), quantification and interpreted symbols (as in satisfiability modulo theory, or SMT), and non-classical consequence (as in answer set programming, or ASP) – since these often correspond closely to the kind of problems they attempt to formalize and solve;
• solver-independent, in that inference assumes the role of algorithms, and the formulation of the inference problems is separate from the solution strategies;
usual inductive rules. Then, we obtain a propositional theory

Example 2.

order logic, as well as non-classical logical consequence.

$M$ connectives,

models

set called the

$L$ ∈

written as

Definition 1.

familiarity with) terminology from predicate logic [25].

The formulation in the sequel will provide insights on how such a language could be obtained.

in Section 3; we introduce the semiring programming framework and illustrate it on a number of examples; in Section

4, we discuss how different semirings can be combined; in Section 5 different strategies for solvers are introduced;

and finally, in Sections 6 and 7 we discuss related work and conclude.

The set $L$ is implicitly assumed to be defined over a vocabulary $\text{vocabulary}(L)$ of relation and function symbols, each

with an associated arity. Constant symbols are 0-ary function symbols. Every $L$-model $M \in M$ is a tuple containing a

universe $\text{domain}(M)$, and a relation (function) for each relation (function) symbol of $\text{vocabulary}(L)$. For relation (constant)

symbol $p$, the relation (universe element) corresponding to $p$ in a model $M$ is denoted $p_M$. For $\phi \in L$ and $M \in M$, we write

$M \models \phi$ to say that $M$ satisfies $\phi$. We say a formula $\phi$ is valid iff $\phi$ is satisfied at every model, which is then

written as $\models \phi$. We let $M(\phi)$ denote $\{M \in M \mid M \models \phi\}$. Finally, the set $\text{sentences}(L)$ denotes the literals in $L$, and we write

$l \in M$ to refer to the $L$-literals that are satisfied at $M$.

This formulation is henceforth used to instantiate a particular logical system, such as fragments/extensions of first-

order logic, as well as non-classical logical consequence.

Example 2. Suppose $T = (L, M, \models)$ is as follows: $L$ is obtained using a set of 0-ary predicates $P$ and Boolean

connectives, $M$ are $\{0, 1\}$ assignments to the elements of $P$, and $\models$ denotes satisfaction in propositional logic via the

usual inductive rules. Then, we obtain a propositional theory.
Example 3. Define a theory \((\mathcal{L}, M, \triangleright)\) where \(\mathcal{L}\) is the positive Horn fragment from a finite vocabulary, \(M\) the set of propositional interpretations, and for \(M \in M, \phi \in \mathcal{L}\), define \(M \triangleright \phi\) iff \(M \triangleright_{PL} \phi\) (i.e., satisfaction in propositional logic) and for all \(M' \subseteq M\), it is not the case that \(M' \triangleright_{PL} \phi\). This theory can be used to reason about the minimal models of a formula.

Example 4. Define a theory \((\mathcal{L}, M, \triangleright)\) where \(\mathcal{L}\) is a first-order language involving 0-ary functions \([c,\ldots,d]\), inequalities \(\leq,\geq,\prec,\succ,=\neq\). Let \(M\) be the set of mappings from \([c,\ldots,d]\) to \(\mathbb{R}\). We define \(M \triangleright \phi\) for \(M \in M\) and \(\phi \in \mathcal{L}\) as in first-order logic, assuming in particular that \(=,\prec,\succ,0,1,+,\cdot,\geq,\leq\), exponentiation and logarithms have their usual interpretations \([3]\). (That is,”1 + 0 = 1” is true in all models, as is “\(x > y \equiv \neg(y > x)\),” and so on \([31]\).) This theory can be used to reason about linear arithmetic (i.e., allowing formulas such as \(c + d \leq 5\) and \(d \leq e\), where \(c,d,e\) are integers or reals) and non-linear arithmetic (i.e., allowing formulas such as \(c + d^2 \geq e\)).

2.2. Weighted Model Counting

Semiring programming draws from the conceptual simplicity of weighted model counting (WMC), which we briefly recap here. WMC is an extension of \#SAT, where one simply counts the number of models of a propositional formula \([37]\). In WMC, one accords a weight to a model in terms of the literals true at the model, and computes the sum of the weights of all models.

Definition 5. Suppose \(\phi\) is a formula from a propositional language \(\mathcal{L}\) with a finite vocabulary, and suppose \(M\) is the set of \(\mathcal{L}\)-models. Suppose \(w\) : lits(\(\mathcal{L}\)) \(\rightarrow \mathbb{R}^{\geq 0}\) is a weight function. Then:

\[
\text{WMC}(\phi, w) = \sum_{M \models \phi, \mathcal{L}} w(l)
\]

is called the weighted model count (WMC) of \(\phi\).

Here, in the context of \(\mathcal{L}\)-models \(M\), we simply write \(t = \sum_{M \models \phi} u\) to mean \(t = \sum_{(M \models \phi)} u\).

The formulation elegantly decouples the logical sentence from the weight function. In this sense, it is clearly agnostic about how weights are specified in the modeling language, and thus, has emerged as an assembly language for Bayesian networks \([19]\), and probabilistic programs \([29]\), among others.

2.3. Commutative Semirings

Our programming model is based on algebraic structures called semirings \([52]\): the essentials are as follows:

Definition 6. A (commutative) semiring \(S\) is a structure \((S, \oplus, \otimes, 0, 1)\) where \(S\) is a set called the elements of the semiring, \(\oplus\) and \(\otimes\) are associative and commutative, \(0\) is the identity for \(\oplus\), and \(1\) is the identity for \(\otimes\).

Abusing notation, when the multiplication operator is not used, we simply refer to the triple \((S, \oplus, 0)\) as a semiring.

Example 7. The structure \((\mathbb{N}, +, \times, 0, 1)\) is a commutative semiring in that for every \(a, b \in \mathbb{N}\), \(a + 0 = a, a \times 1 = a, a + b = b + a\), and so on.

3. Semiring Programming

In essence, the semiring programming scheme is as follows:

- **Input**: a theory \(T = (\mathcal{L}, M, \triangleright)\), a sentence \(\phi \in \mathcal{L}\), a commutative semiring \(S\), and a weight function \(w\).

- **Output**: the count, denoted \(#(\phi, w)\).

The scope of these programs is broad, and so we will need different kinds of generality. Roughly, the distinction boils down to: (a) whether the set of models for a formula is finite or infinite; (b) whether the weight function can be factorized over the literals or not (in which case the weight function directly labels the models of a theory); and (c) whether \(\triangleright\) is defined in a classical (monotonic) manner or not. The thrust of this section is: (i) to show how these distinctions subsume important model generation notions in the literature, and (ii) providing rigorous definitions for the count operator. In terms of organization, we begin with the finite case, before turning to the infinite ones. We present an early preview of some of the models considered in Figure \([1]\).
factorized | non-factorized | non-classical
finite | WMC | logical-integer programming | ASP
infinite | polyhedron volume | convex optimization | first-order ASP

Figure 1: scope of programs

(set-logic PL)
(set-algebra [NAT,max,*,0,1])
declare-predicate p ()
declare-predicate q ()
F = (p or q)
declare-weight (p 1)
declare-weight ((neg p) 2)
declare-weight (q 3)
declare-weight ((neg q) 4)
(count F)

Figure 2: most probable explanation

3.1. Finite
Here, we generalize the WMC formulation to semiring labels, but also go beyond classical propositional logic.

Definition 8. We say the theory \( T = (\mathcal{L}, \mathcal{M}, \triangleright) \) is finite if for every \( \phi \in \mathcal{L} \), \( \{ M \in \mathcal{M} \mid M \triangleright \phi \} \) is finite.

So, a propositional language with a finite vocabulary is a finite theory, regardless of (say) standard or minimal models. Similarly, a first-order language with a finite Herbrand base is also a finite theory.

Definition 9. Suppose \( T = (\mathcal{L}, \mathcal{M}, \triangleright) \) is a finite theory. Suppose \( S = (\mathbb{S}, \oplus, \otimes, 0, 1) \) is a commutative semiring. Suppose \( w: \mathcal{M} \rightarrow \mathbb{S} \). For any \( \phi \in \mathcal{L} \), we define:

\[
#(\phi, w) = \bigoplus_{M \triangleright \phi} w(M)
\]

If \( w: \text{lits}(\mathcal{L}) \rightarrow \mathbb{S} \), then the problem is factorized, where:

\[
w(M) = \bigotimes_{l \in M} w(l).
\]

Essentially, as in WMC, we sum over models and take products of the weights on literals but w.r.t. a particular semiring. Needless to say, we immediately subsume the framework of algebraic model counting (AMC) [49]:

Proposition 10. Suppose \( T = (\mathcal{L}, \mathcal{M}, \triangleright) \) is a finite propositional theory, \( S \) a commutative semiring, and \( w \) a factorized weight function. Suppose \( \triangleright \) is the standard satisfaction relation in propositional logic. Then SP is equivalent to AMC, that is, the computation of every SP instance can be defined as an AMC task and vice versa.

Let us consider a few examples.

Example 11. We demonstrate SAT and \( \#\text{SAT} \). Consider a propositional theory \( T = (\mathcal{L}, \mathcal{M}, \triangleright) \) where \( \text{vocab}(\mathcal{L}) = \{ p, q \} \), and suppose \( \phi = (p \lor q) \). Letting \( \mathcal{M} \) be standard \( \mathcal{L} \)-models, clearly \( |\mathcal{M}| = 4 \) and \( |\{ M \in \mathcal{M} \mid M \triangleright \phi \}| = 3 \). Suppose for every \( M \in \mathcal{M} \), \( w \) is function such that \( w(M) = 1 \). For the semiring \((\{0, 1\}, \lor, 0)\):

\[
#(\phi, w) = \bigvee_{M \triangleright \phi} w(M) = 1 \lor 1 \lor 1 = 1.
\]
Figure 3: logical-integer programming

Consider the semiring \((\mathbb{N}, +, 0)\) instead. Then

\[
#(\varphi, w) = \sum_{M\in\varphi} w(M) = 1 + 1 + 1 = 3.
\]

As with Definition 5, the framework is agnostic about the modeling language. But for presentation purposes, programs are sometimes described using a notation inspired by the SMT-LIB standard [7].

Example 12. We demonstrate MPE (most probable explanation) and WMC. Consider the theory, semiring and weight function \(w\) from Figure 2 which specifies, for example, a vocabulary of two propositions \(p\) and \(q\), \(w(p) = 1\) and \(w(\neg p) = 2\). In accordance with that semiring, the weight of a model, say \(\{p, \neg q\}\), of the formula \(F\) is \(1 \times 4 = 4\). Thus, for the semiring \((\mathbb{N}, \max, \times, 0, 1)\), we have:

\[
#(F, w) = \max \{6, 3, 4\} = 6
\]

which finds the most probable assignment. Consider the semiring \((\mathbb{N}, +, \times, 0, 1)\) instead. Then:

\[
#(F, w) = 6 + 3 + 4 = 13
\]

which gives us the weighted model count.

Example 13. Extending the discussion in [45], we consider a class of mathematical programs where linear constraints and propositional formulas can be combined freely. See Figure 3 for an example with non-linear objectives. Formally, quantifier-free linear integer arithmetic and propositional logic are specified as the underlying logical systems, and the domains of constants are typed. The program declares formulas \(F\), \(G\), and \(H\).

The counting task is non-factorized, and our convention for assigning weights to models is by letting the 

\textit{declare-weight} directive also take arbitrary formulas as arguments. Of course, TRUE holds in every model, and so, the weight of every model is determined by the evaluation of \(x_1 \times x_2\) at the model, that is, for any \(M\), its weight is \(x_1^M \times x_2^M\). For example, a model that assigns 1 to \(x_1\) and 1 to \(x_2\) is accorded the weight 1+1. Computing the count over \((\mathbb{N}, \max, \times, 0, 1)\) then yields a model of \(H\) with the highest value for \(x_1 \times x_2\).

To see this program in action, consider that every model of \(H\) must satisfy \(p_2\), and so must admit \(3 \times x_1 \leq 4\) and \(2 \times x_2 \leq 5\). Since \(\{x_1, x_2\}\) can only take values \(\{1, \ldots, 10\}\), given the constraints, the desired model must assign \(x_1\) and \(x_2\) to 1 and 2 respectively. Then, its weight is 1+2.

Encoding finite domain constraint satisfaction problems as propositional satisfiability is well-known. The benefit, then, of appealing to our framework is the ability to easily formulate counting instances:
Proposition 14. Suppose $Q$ is a CSP over variables $X$, domains $D$ and constraints $C$. There is a first-order theory $(\mathcal{L}, M, \triangleright)$, a $\mathcal{L}$-sentence $\phi$ and a weight function $w$ such that $\#(\phi, w) = 1$ over $([0, 1], \lor, 0)$ iff $Q$ has a solution. Furthermore, $Q$ has $n$ solutions iff $\#(\phi, w) = n$ over $(\mathbb{N}, +, 0)$.

Constraints are Boolean-valued functions \(^{[3]}\), and so constraints over $X$ can be encoded as $\mathcal{L}$-sentences, as in the example below:

Example 15. See Figure 4 for a counting instance of graph coloring: $\text{edge}(x, y)$ determines there is an edge between $x$ and $y$, $\text{node}(x)$ says that $x$ is a node, and $\text{color}(x, y)$ says that node $x$ is assigned the color $y$. The actual graph is provided using the formula $\text{DATA}$, which declares a fully connected 3-node graph. Also, $\text{CONS}$ is a conjunction of the usual coloring constraints, e.g., an edge between nodes $x$ and $y$ means that they cannot be assigned the same color.

Let $M$ be a set of first-order structures for the vocabulary $\{\text{edge}, \text{node}, \text{color}\}$, respecting types from Figure 4. The interpretation of $\{\text{edge}, \text{node}\}$ is assumed to be the same for all the models in $M$ and is as given by $\text{DATA}$. Basically, then, the models differ in their interpretation of $\text{color}$. One model of $\phi = (\text{DATA} \text{ and } \text{CONS})$, for example, is $\{\text{color}(1, r), \text{color}(2, g), \text{color}(3, b)\}$. The weights of all models is 1, and so, for $(\mathbb{N}, +, 0)$ we get:\(^2\)

$$\#(\phi, w) = 6.$$ 

To summarize, the following result is easily shown for semiring programs:\(^3\)

Proposition 16. Suppose $\theta \in \{\text{SAT, #SAT, WMC, MPE, CSP, #CSP}\}$. Suppose $\theta^\circ$ is a solution to $\theta$, in that $\theta^\circ \in [0, 1]$ for SAT and CSP, and $\theta^\circ \in \mathbb{R}$ for the rest. Then for any $\theta$, there is a $T = (\mathcal{L}, M, \triangleright)$, $S$, $w$ and $\phi \in \mathcal{L}$ such that $\#(\phi, w) = \theta^\circ$.

3.2. Non-standard

A particular advantage of defining a logical theory in the way we did is that the framework is immediately applicable to model-level operations with non-standard semantics. We give a notable example that is simple to capture but which to the best of our knowledge has not been previously formulated in a semiring framework like ours.

\(^2\)In an analogous fashion, soft and weighted CSPs can be expressed using semirings \(^{[15]}\).

\(^3\)We remark that although the count operator in itself does not provide the variable assignments for SAT, MPE and CSP, we assume this can be retrieved from the satisfying interpretations.
Definition 17. A stable model environment is defined as follows. Let $T = (\mathcal{L}, \mathcal{M}, \triangleright)$ be a logical theory, where $\mathcal{L}$ is defined over a set of propositions $\mathcal{P}$ and $\mathcal{M}$ is the set of mappings from $\mathcal{P}$ to $[0, 1]$. Let us split $\mathcal{P}$ into two disjoint sets of variables founded variables $\mathcal{P}_f$ and standard variables $\mathcal{P}_s$. An answer set program $\delta$ is a tuple $(\mathcal{P}, \mathcal{R}, C)$ where $\mathcal{R}$ is a set of rules of form: $a \leftarrow b_1 \land \ldots \land b_k \land \lnot c_1 \land \ldots \land \lnot c_m$ such that $a \in \mathcal{P}_f$, the body variables $\{b_1, \ldots, c_m\} \subseteq \mathcal{P}_s$, and $C$ is a set of constraints over the propositions (specified as rules with an empty head). A rule is positive if its body only contains positive founded literals. The least assignment of a set of positive rules $\mathcal{R}$, written $L(\mathcal{R})$ is the one that satisfies all the rules and contains the least number of positive literals. Given an assignment $M \in \mathcal{M}$ and a program $\delta$, the reduct of $M$ wrt $\delta$, written, $\delta^M$ is a set of positive rules that is obtained as follows: for every rule $r$, if any $c_i \in M$, or $\lnot b_j \in M$ for any standard positive literal, then $r$ is discarded, otherwise, all negative literals and standard variables are removed from $r$ and it is included in the reduct. An assignment $M \in \mathcal{M}$ is a stable model of a program $\delta$ iff it satisfies all its constraints and $M$ agrees with $L(\delta^M)$ wrt $\mathcal{P}_f$. We say two assignments $M$ and $M'$ agree wrt $\mathcal{P}$ iff the set of positive literals (restricted to $\mathcal{P}$) in $M$ is identical to the set in $M'$, and the set of negative literals (restricted to $\mathcal{P}$) in $M$ is identical to the set in $M'$. For a program $\delta$, we let $M \triangleright M$ if $M$ is a stable model of $\delta$.

This is basically an adaptation of the answer set programming by SAT formulation in [3]. By way of the semirings considered in Example 11, it immediately follows that:

Proposition 18. Suppose $T = (\mathcal{L}, \mathcal{M}, \triangleright)$ is a stable model environment, $\delta \in \mathcal{L}$ is a program (as defined above). Suppose $\mathcal{S}$ is the semiring $\{(0, 1), \lor, 0\}$ and $w(M) = 1$ for every $M \in \mathcal{M}$. Then $\#(\delta, w)$ tells us whether there is a stable model. For the semiring $(\mathbb{N}, +, 0)$, $\#(\delta, w)$ yields the number of stable models.

It is important to note that the formulation only semantically characterizes the task of finding stable models as well as stable model counting. Algorithmic solutions for the task may very well involve other ideas [3].

3.3. Infinite: Non-factorized

Defining measures [42] on the predicate calculus is central to logical characterizations of probability theory [43]. We adapt this notion for semirings to introduce a general form of counting. For technical reasons, we assume that the universe of the semirings is $\mathbb{R}$. (If required, the range of the weight function can always be restricted to any subset of $\mathbb{R}$.)

Definition 19. Let $\mathcal{S} = (\mathbb{R}, \land, 0)$ be any semiring and $\mathcal{T} = (\mathcal{L}, \mathcal{M}, \triangleright)$ a theory. Let $\Sigma$ be a $\sigma$-algebra over $\mathcal{M}$ in that $(\mathcal{M}, \Sigma)$ is an $\sigma$-finite measurable space wrt the measure $\mu : \Sigma \rightarrow \mathbb{R}$ respecting $\land$. That is, for all $E \in \Sigma$, $\mu(E) \geq 0$, $\mu(\emptyset) = 0$ and $\mu$ is closed under complement and countable unions: for all pairwise disjoint countable sets $E_1, \ldots \in \Sigma$, we have

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j).$$

Moreover, because the spaces are $\sigma$-finite, $\mathcal{M}$ is the countable union of measurable sets with finite measure. Then for any $\phi \in \mathcal{L}$, $\#(\phi, \mu) = \mu(\mathcal{M}(\phi))$.

Example 20. We demonstrate that convex optimization can be cast as a semiring programming problem. Suppose $\mathcal{T} = (\mathcal{L}, \mathcal{M}, \triangleright)$ is a first-order theory only containing constants $X = \{x_1, \ldots, x_k\}$ with domains $D_i = \mathbb{R}$. Suppose $\phi(x_1, \ldots, x_k) \in \mathcal{L}$ is a conjunction of formulas of the form $c_1 x_1 + \ldots + c_k x_k \leq d$, for real numbers $c_1, \ldots, c_k, d$, and thus describing a polyhedron. In other words, for every $M \in \mathcal{M}$, $\text{dom}(M) = \mathbb{R}$, and so, $M$ is a real-valued assignment to $X$. In particular, if $M \triangleright \phi$, then $x_1^M, \ldots, x_k^M$ is a point inside the polyhedron $\phi$. Let $\Sigma$ be a $\sigma$-algebra over $\mathcal{M}$ and so every $E \in \Sigma$ is a measurable set of points.

Suppose $f(x_1, \ldots, x_k) : \mathbb{R}^k \rightarrow \mathbb{R}$ is a convex function that we are to minimize. Consider the semiring $(\mathbb{R}, \inf, 0)$ and a measure $\mu$ such that for any $E \in \Sigma$:

$$\mu(E) = \inf \left\{ f(x_1^M, \ldots, x_k^M) \mid M \in E \right\}$$

which finds the infimum of the $f$-values across the assignments in $E$. Then, $\mu(\mathcal{M}(\phi)) = \#(\phi, \mu)$ gives the minimum of the convex function in the feasible region determined by $\phi$.

Suppose $f$ is a concave function that is to be maximized. We would then use $(\mathbb{R}, \sup, 0)$ instead, which finds the supremum of the $f$-values across assignments in $E \in \Sigma$. 

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longer the case. For example, suppose \( x \) independently, and construct a measure for the entire space by spaces are common [67]. The main idea then is to apply our definition for counting by measures to each variable separately, prompting a factorized formulation of counting. More generally, in many robotic applications, such hybrid a joint distribution on the probability space \( \mathbb{R} \times \mathbb{R} \). Consider, for example, a matrix factorization, a fundamental concern in in-

Proposition 21. Suppose \( \{x_1, \ldots, x_k\} \) is a set of real-valued variables. Suppose \( P \subseteq \mathbb{R}^k \) is the feasible region of an optimization problem of the form \( g_i(x_1, \ldots, x_k) \leq d_i \) for \( i \in \mathbb{N} \), where \( g_i \in \{<, \leq, \geq, >\} \) and \( d_i : \mathbb{R}^k \rightarrow \mathbb{R} \). Suppose \( f : \mathbb{R}^k \rightarrow \mathbb{R} \) is a function to be maximized (minimized). Then there is a \( T = (\mathcal{L}, \mathcal{M}, \triangleright) \), \( S, \mu \) and \( \phi \in \mathcal{L} \) such that \( \#(\phi, \mu) \) is the maximum (minimum) value for \( f \) in the feasible region \( P \).

Example 22. For a non-trivial example, consider the problem of matrix factorization, a fundamental concern in in-

Figure 5: matrix factorization

More generally, the same construction is easily shown to be applicable for other families of mathematical pro-

\[
\begin{align*}
\text{(set-logic LRA)} \\
\text{(set-algebra [REAL,inf,0])} \\
\text{(declare-function input (INT,INT) REAL)} \\
\text{... /* declare left, right, app */} \\
\text{(declare-function err () REAL)} \\
\text{DATA = /* entries in input matrix (omitted) */} \\
\text{F = app(x,y) == sum(e) left(x,e) * right(e,y)} \\
\text{G = err == norm(sum(x,y) input(x,y) - app(x,y))} \\
\text{(declare-weight TRUE err)} \\
\text{(count (DATA and F and G))}
\end{align*}
\]

\[app(x,y). Here, \text{app computes the product of the matrices left and right, and err computes the Frobenius Norm wrt app and input.}\]

Let \( \Sigma \) be a \( \sigma \)-algebra over \( \mathcal{M} \). The weight function in Figure 5 determines a measure \( \mu \) such that for any \( E \in \Sigma \):

\[\mu(E) = \inf \{ \text{err}^M | M \in E \} \).

Therefore, \( \#(\phi, \mu) \) yields the lowest \text{err} value; the model \( M \) such that \text{err}^M = \#(\phi, \mu) \) is one with the best factorization of matrix \( I \).

3.4. Infinite: Factorized

Despite the generality of the above definition, we would like to address the factorized setting for a number of applications, the most prominent being probabilistic inference in hybrid graphical models [10]. Consider, for example, a joint distribution on the probability space \( \mathbb{R} \times [0, 1] \). Here, it is natural to define weights for each random variable separately, prompting a factorized formulation of counting. More generally, in many robotic applications, such hybrid spaces are common [67]. The main idea then is to apply our definition for counting by measures to each variable independently, and construct a measure for the entire space by \textit{product measures} [42].

A second technicality is that in the finite case, the set of literals true at a model was finite by definition. This is no longer the case. For example, suppose \( x \) is a real-valued variable in a language \( \mathcal{L} \), and \( M \) is a \( \mathcal{L} \)-model that assigns
3 to \( x \). Then, \( M \triangleright (x = 3) \) but also \( M \triangleright (x \neq 3.1) \), \( M \triangleright (x \neq 3.11) \), and so on. Thus, for technical reasons, we assume that \( \mathcal{L} \) only consists of constant symbols \( X = \{x_1, \ldots, x_k\} \) with fixed (possibly infinite) domains \( \{D_1, \ldots, D_k\} \); the measures are defined for these domains.

**Definition 23.** Let \( S = ([R, \otimes, 0, 1]) \) be any semiring, \( \mathcal{T} = (\mathcal{L}, M, \triangleright) \) any theory where \( \text{vocab}(\mathcal{L}) = \{x_1, \ldots, x_k\} \) over fixed domains \( D_i \). Suppose \( \phi \in \mathcal{L} \). Suppose \( \Sigma_i \) is a \( \sigma \)-algebra over \( D_i \) in that \( (D_i, \Sigma_i) \) is a \( \sigma \)-finite measurable space with respect to the measure \( \mu_i : \Sigma_i \rightarrow R \) respecting \( \otimes \) (as in Definition 19). Define the product measure \( \mu^* = \mu_1 \times \cdots \times \mu_k \) on the measurable space \( (D_1 \times \cdots \times D_k, \Sigma_1 \times \cdots \times \Sigma_k) \) satisfying

\[
\mu^*(E_1 \times \cdots \times E_k) = \mu_1(E_1) \otimes \cdots \otimes \mu_k(E_k)
\]

for all \( E_i \in \Sigma_i \). Finally, define

\[
\#(\phi, \mu^*) = \mu^*([\phi])
\]

where \([\phi] = \{x^M_1 \times \cdots \times x^M_k \mid M \in M(\phi)\}\).

Intuitively, \( E_i \in \Sigma_1, \ldots, E_k \in \Sigma_k \) capture sets of assignments, and the product measure considers the algebraic product of the weights on assignments to terms. As before, for \( E_i, E'_i \in \Sigma_i, \mu_i(E_i \cup E'_i) = \mu_i(E_i) \oplus \mu_i(E'_i) \). Finally, precisely because the measures are defined on the domains of the terms, we obtain these for all of the satisfying interpretations using the construction \([\phi] \).

**Example 24.** We demonstrate the problem of finding the volume of a polyhedron, needed in the static analysis of probabilistic programs [62, 20]. Suppose \( \mathcal{T} = (\mathcal{L}, M, \triangleright) \) and \( \phi \in \mathcal{L} \) is as in Example 20 that is, \( \phi \) defines a polyhedron. For every 0-ary function symbol \( x_i \in \{x_1, \ldots, x_k\} \) with domain \( D_i = R \), let \( \Sigma_i \) be the set of all Borel subsets of \( R \), and let \( \mu_i \) be the Lebesgue measure. Thus, for any \( E \in \Sigma_i, \mu_i(E) \) gives the length of this line. Then, for the semiring \( (R, +, \times, 0, 1) \), the \( + \) operator sums the lengths of lines for each variable, and \( \times \) computes the products of these lengths. Thus, \( \#(\phi, \mu^*) \) is the volume of \( \phi \).

To see this in action, suppose \( \phi = (2x \leq 5) \land (x \geq 1) \land (0 \leq y \leq 2) \). Then \( M(\phi) = \{ (x \rightarrow n, y \rightarrow m) \mid n, m \in R, (x, y) \in \phi_{n,m}^{x,y} \} \), that is, all assignments to \( x \) and \( y \) such that \( \phi_{n,m}^{x,y} \) is a valid expression in arithmetic. Therefore,

\[
[\phi] = \{(n, m) \mid n \in [1, 2.5], m \in [0, 2], n \in R, m \in R\}.
\]

Assuming \( \mu_x \) is the Lebesgue measure for all Borel subsets of the domain of \( x \), we have

\[
\mu_x(\{n \mid n \in [1, 2.5], n \in \mathbb{R}\}) = 1.5.
\]

Analogously, \( \mu_x(\{m \mid m \in [0, 2], m \in \mathbb{R}\}) = 2 \). Then, the volume is \( \#(\phi, \mu^*) = \mu^*([\phi]) = 1.5 \times 2 = 3 \).

**Example 25.** We demonstrate probabilistic inference in hybrid models [10] by extending Example 24. Consider a probabilistic program:

\[
X \sim \text{UNIFORM}(0, 1) \quad Y \sim \text{FLIP}(0, 1)
\]

\[
\text{if } (X > .6 \text{ AND } Y \neq 1) \text{ then return DONE}
\]

In English: \( X \) is drawn uniformly from \([0,1]\) and \( Y \in [0, 1] \) is the outcome of a coin toss. If \( X > .6 \) and \( Y \) is not 1, the program terminates successfully. Suppose we are interested in the probability of DONE, which is expressed as the formula:

\[
\phi = (0 \leq X \leq 1) \land (Y = 0 \lor Y = 1) \land (X > .6 \land Y \neq 1).
\]

Suppose \( \mathcal{T} = (\mathcal{L}, M, \triangleright) \) is the theory of linear real arithmetic, with \( \text{vocab}(\mathcal{L}) = \{X, Y\}, D_X = \mathbb{R} \) and \( D_Y = \{0, 1\} \). As in Example 24 let \( \Sigma_X \) be the set of all Borel subsets of \( R \) and \( \mu_X \) the Lebesgue measure. Let \( \Sigma_Y \) be the set of

---

4The product measure is unique owing to the \( \sigma \)-finite assumption via the Hahn-Kolmogorov theorem [42].
all subsets of \([0, 1]\) and \(\mu_Y\) be the counting measure, e.g., \(\mu_Y([1]) = 1\), and \(\mu_Y((0, 1]) = 2\). So \(M(\phi) = \{(X \to n, Y \to m) \mid n \in \mathbb{R}, m \in [0, 1], \triangleright \phi_{X,Y}^{n,m}\}\) and:

\[
\{\phi\} = \{(n, m) \mid n > .6, 0 \leq n \leq 1, n \in \mathbb{R}, m \in [0, 1]\} \cap \\
\{(n, m) \mid 0 \leq n \leq 1, n \in \mathbb{R}, m \neq 1, m \in [0, 1]\}.
\]

This means that \(\mu_Y([n \mid \exists m \ (n, m) \in [\phi]]) = .4\) and also \(\mu_Y([m \mid \exists n \ (n, m) \in \phi]) = 1\). Thus, \(\mu'([\phi]) = .4 \times 1\). Analogously, \(\mu'([\phi \lor \neg \phi]) = 2\). Therefore, the probability of \(\text{DONE}\) is \(.4/2 = .2\).

In general, we have a variant of Proposition \[21\] 44):

**Proposition 26.** Suppose \(\{x_1, \ldots, x_k\}\) is any set of real-valued variables. Suppose \(D\) is any countable set. Suppose \(P \subset \mathbb{R}^m \times D^n\), where \(m + n = k\), is any region given by conjunctions of expressions of the form \(c_1 x_1 + \ldots + c_k x_k \triangleright e\) and \(x_i \triangleright d\), where \(\triangleright \in \{\leq, <, >, \geq\}\), \(\triangleright \in \{\neq, =\}\), \(c_i, e \in \mathbb{R}\) and \(d \in D\). Then there is a \(T = (L, M, \triangleright), S, \mu\) and \(\phi \in L\) such that \(#(\phi, \mu)\) is the volume of \(P\).

4. Towards Compositionality

A noteworthy feature of many logic-based knowledge representation formalisms is their compositional nature. In semiring programming, using the expressiveness of predicate logic, it is fairly straightforward to combine theories over possibly different signatures (e.g., propositional logic and linear arithmetic), as seen, for example, in SMT solvers [8].

A more intricate flavor of compositionality is when the new specification becomes difficult (or impossible) to define using the original components. This is a common occurrence in large software repositories, and has received a lot of attention in the AI community [53].

In this section, we do not attempt to duplicate such efforts, but propose a different account of compositionality that is closer in spirit to semiring programming. It builds on similar ideas for CSPs [13], and is motivated by machine learning problems where learning (i.e., optimization) and inference (i.e., model counting) need to be addressed in tandem. More generally, the contribution here allows us to combine two semiring programs, possibly involving different semirings. For simplicity of presentation, we consider non-factorized and finite problems over distinct vocabularies.

**Definition 27.** Suppose \(S_1 = (\mathbb{S}_1, \ominus_1, 0_1)\) and \(S_2 = (\mathbb{S}_2, \ominus_2, 0_2)\) are any two semirings. We define their composition \(S = (\mathbb{S}, \ominus, 0)\) as:

- \(\mathbb{S} = \{(a, b) \mid a \in \mathbb{S}_1, b \in \mathbb{S}_2\}\);
- \(0 = (0_1, 0_2)\);
- for every \(c = (a, b) \in \mathbb{S}\) and \(c' = (a', b') \in \mathbb{S}\), let \(c \ominus c' = (a \ominus_1 a', b \ominus_2 b')\).

That is, the composition of the semirings is formed from the Cartesian product, respecting the summation operator for the individual structures.

**Definition 28.** Suppose \(T_1 = (L_1, M_1, \triangleright_1)\) and \(T_2 = (L_2, M_2, \triangleright_2)\) are theories, where \(L_1\) and \(L_2\) do not share atoms, \(S_1\) and \(S_2\) semirings, and \(w_1\) and \(w_2\) weight functions for \(M_1\) and \(M_2\) respectively. Given the environments \((T_1, S_1, w_1)\) and \((T_2, S_2, w_2)\), we define its composition as \((T, S, w)\), where \(S\) is a composition of \(S_1\) and \(S_2\) and \(T = (L, M, \triangleright)\):

- \(\phi \in L\) is obtained over Boolean connectives from \(L_1 \cup L_2\).
- \(M = \{(M_1, M_2) \mid M_i \in M_i\}\).
- The meaning of \(\phi \in L\) is defined inductively:

  - \((M_1, M_2) \triangleright p\) for atom \(p\) iff \(M_1 \triangleright_1 p\) if \(p \in L_1\) and \(M_2 \triangleright_2 p\) otherwise;
\[
\begin{align*}
\text{Example 29.} & \quad \text{We demonstrate a (simple) instance of combined learning and inference. Imagine a robot navigating a world by performing move actions, and believes its actuators need repairs. But before it alerts the technician, it would like to test this belief. A reasonable test, then, is to inspect its trajectory so far, and check whether the expected outcome of a move action in the current state matches the behavior of the very first move action. More precisely, the robot needs to appeal to linear regression to estimate its expected outcome, and query its beliefs based on the regression model. We proceed as follows.}

\text{Environment 1}

\text{Let } \mathcal{T}_1 = (\mathcal{L}_1, M_1, \triangleright_1) \text{ be the theory of real arithmetic with } \text{vocab}(\mathcal{L}_1) = \{s_0, s_1, \ldots, s_k, a, b, e\}. \text{ Suppose that by performing a move action, the robot’s position changes from } s_i \text{ to } s_{i+1}. \text{ Let } \phi_1 \in \mathcal{L}_1 \text{ be as follows:}

\[
(s_0 = 1 \land s_1 = 2 \land s_2 = 3) \land e = \sum (s_{i+1} - b - a \cdot s_i)^2
\]

\text{The idea is that the values of } s_i \text{ are the explanatory variables in the regression model and } s_{i+1} \text{ are the response variables, that is, the trajectory data is of the form } [(s_0, s_1), (s_1, s_2)].

\text{Consider a weight function } w_1: M \rightarrow e^M. \text{ That is, the weight of a model } M \text{ is the universe element corresponding to the constant } e \text{ in } M, \text{ analogous to Figure 3. (For simplicity, we assume that the coefficients of the regression model are natural numbers.) For the semiring } S_1 = (\mathbb{N}, \text{min}, \infty):

\[
\#(\phi_1, w_1) = \min \left\{ e^M \mid M \triangleright_1 \phi_1, M \in M_1 \right\}.
\]

\text{In other words, models in } M_1 \text{ interpret } s_i \text{ as given by the data, and models differ in their interpretation of } a, b \text{ and thus, } e. \text{ For the data in } \phi_1, \text{ we would have a model } M \text{ where } e^M = 0, a^M = 1 \text{ and } b^M = 1, \text{ and so } \#(\phi_1, w_1) = 0.

\text{Environment 2}

\text{Let } \mathcal{T}_2 = (\mathcal{L}_2, M_2, \triangleright_2) \text{ be a propositional theory, where } \text{vocab}(\mathcal{L}_2) = \{\text{repair}\}. \text{ Imagine a weight function } w_2 \text{ as follows: } w_2(\text{repair}) = .7, \text{ and } w_2(\neg\text{repair}) = .3. \text{ That is, the robot believes that repairs are needed with a higher probability. Indeed, for the semiring } (\mathbb{R}, +, 0):

\[
\#(\text{repair}, w_2) = .7 \text{ versus } \#(\neg\text{repair}, w_2) = .3.
\]

---

\footnote{This is analogous to SMT solvers for combinations of theories. However, see \S \S \S 24 for accounts of modularity based on first-order structures sharing vocabularies (and thus, atoms).}
Composition

Let \((T, S, w)\) be the composition of the two environments with \(T = (\mathcal{L}, M, \triangleright)\) and \(S = (\mathcal{S}, \oplus, 0)\). Suppose \(\phi \in \mathcal{L}\) is as follows:

\[
\phi \land (s_3 = a \cdot s_2 + b) \land (s_3 - s_2) \neq (s_1 - s_0) \equiv \text{repair}.
\]

The final conjunct basically checks whether the expected change in position matches what was happening initially, and if not, \text{repair} should be true.

To see Definition 28 in action, observe that for any \(M_1 \in M_1, M_2 \in M_2, (M_1, M_2) \in M\), the formulas \(\phi_1\) and those involving \(s_i\) are interpreted in \(M_1\), but \text{repair} in \(M_2\). The weight function is as follows. Given \((M_1, M_2) \in M\) and \((M'_1, M'_2) \in M\), we have:

\[
w((M_1, M_2)) \oplus w((M'_1, M'_2)) = (\min(w_1(M_1), w_1(M'_1)), w_2(M_2) + w_2(M'_2))
\]

Then the robot can obtain the weight of \text{repair} and \(\phi\) using:

\[
\#(\phi \land \text{repair}, w) = (0, 0)
\]

where, of course, the first argument is the error of the regression model and the second is 0 because \(\phi \land \text{repair}\) is inconsistent. That can be contrasted to the count below:

\[
\#(\phi \land \neg \text{repair}, w) = (0, .3).
\]

It is also easy to see that \(#(\phi, w) = (0, .3)\).

As in WMC [19], suppose the robot obtains the probability of a query \(q\) given \(\phi\) using:

\[
\frac{\#(\phi \land q, w)}{\#(\phi, w)}
\]

where the division is carried out by ignoring the regression error. Then the probability of \(\neg \text{repair}\) given \(\phi\) is 1. Thus, no repairs are needed.

Example 30. We show that by combining theories, a natural semantics can be given to hybrid problems such as task and motion planning. Here, the concern is to integrate a symbolic high-level planner, described in logic, and geometric constraints; see [6] for terminology and notation. Suppose \(Z\) is the theory of integer arithmetic that interprets a motion planning space \((C, N, p)\) where \(C = \mathbb{Z}\) is the configuration space, \(N \subset C\) is an obstacle region, and \(p \in C\) is the initial pose of a robot’s gripper. (For simplicity, we consider a one-dimensional setting.) The result of doing a motion \(t\) is the pose \(p + t\). We axiomatize that \(x\) is reachable from \(y\) by following \(t\) without touching obstacles as:

\[\text{IsReachable}(x, y, t) \equiv (x = y + t) \land \forall z (\exists t \geq (y + z \notin N)).\]

Next, suppose \(T\) is a propositional theory that interprets a task planning space \((S, A, i, g)\), where \(A\) is a set of actions, \(S\) a set of states, and \(i, g \in S\) are the initial and goal states. The task is to synthesize action sequences reaching \(g\). However, in robotic applications, actions are predicated on geometric constraints; e.g., \(A, g\) the precondition for \text{pickup}(a, p, t) – the action of picking up an object \(o\) wrt a trajectory \(t\) and the gripper’s pose \(p\) – is the sentence:

\[\text{IsReachable}(o, p, t) \land \text{IsGripperFree}.\]

Interestingly, \text{IsGripperFree} is from the language of \(T\) but \text{IsReachable}(x, y, z) is from that of \(Z\), and using our semantic setup, such complex systems can be easily interpreted.

To describe an illustrative counting problem, suppose that for every \(Z\)-model \(M\) and \(T\)-model \(M', (M, M')\) determines a combined task-motion plan of some length. A plan is valid iff \(g\) can be reached from \(i\) by following this plan wrt the domain’s axioms (e.g., avoiding obstacle regions and satisfying action preconditions). Suppose plans are of the form \(t_1, a_1, \ldots, t_m, a_n\), where \(a_i \in A\) and \(t_j\) are motions. Suppose actions and motions incur costs, and the weight of a model is \(\sum \text{cost}(a_i) + \sum \text{cost}(t_j)\) wrt the plan it determines. Assuming \(Z \cup T\) is a finite theory (for simplicity), the semiring \((\mathbb{R}, \min, +, 0, 1)\) yields a cost optimal plan.\(^6\)

---

\(^6\)Prior work on semiring composition [13] can further allow us to minimize one aspect (e.g., cost) and maximize another (e.g., number of rooms cleaned).
5. Solver Construction

The upshot of semiring programming is that it encourages us to inspect strategies for a unified inferential mechanism [47]. This has to be done carefully, as we would like to build on scalable methodologies in the literature, by restricting logical theories where necessary. In this section, we discuss whether our programming model can be made to work well in practice.

Let us consider two extremes:

- **Option 1:** At one extreme is a solver strategy based on a single computational technique. Probabilistic programming languages, such as Church [39], have made significant progress in that respect for generative stochastic processes by appealing to Markov Chain Monte Carlo sampling techniques. Unfortunately, such sampling techniques do not scale well on large problems and have little support for linear and logical constraints.

- **Option 2:** At the other extreme is a solver strategy that is arbitrarily heterogeneous, where we develop unique solvers for specific environments, that is, \((T, S)\) pairs.

**Option 3:** We believe the most interesting option is in between these two extremes. In other words, to identify the smallest set of computational techniques, and effectively integrate them is both challenging and insightful. This may mean that such a strategy is less optimal than Option 2 for the environment, but we would obtain a simpler and more compact execution model. To that end, let us make the following observations from our inventory of examples:

- **Finite versus infinite:** variable assignments are taken from finite sets versus infinite or uncountable sets.

- **Non-factorized versus factorized:** the former is usually an optimization problem with an objective function that is to be maximized or minimized. The latter is usually a counting problem, where we would need to identify one or all solutions.

- **Compositionality:** locally consistent solutions (i.e., in each environment \((T, S)\)) need to be tested iteratively for global consistency.

Thus, Option 3 would be realized as follows:

- Factorized problems need a methodology for effective enumeration, and therefore, advances in model counting [37], such as knowledge compilation, are the most relevant. For finite theories, we take our cue from the Problog family of languages [29, 48], that have effectively applied arithmetic circuits for tasks such as WMC and MPE. In particular, it is shown in [49] how arbitrary semiring labels can be propagated in the circuit. See [27, 28] for progress on knowledge compilation in CSP-like environments.

  For infinite theories, there is growing interest in effective model counting for linear arithmetic using SMT technology [20, 11]. Like in [48], however, we would need to extend these counting approaches to arbitrary semirings.

  For non-factorized problems, a natural candidate for handling semirings does not immediately present itself, making this a worthwhile research direction [1]. Appealing to off-the-shelf optimization software [1] is always an option, but they embody diverse techniques and the absence of a simple high-level solver strategy makes adapting them for our purposes less obvious. In that regard, solvers for optimization modulo theories (OMT) [64] are perhaps the most promising. OMT technology extends SMT technology in additionally including a cost function that is be maximized (minimized). In terms of expressiveness, CSPs [58] and certain classes of mathematical programs can be expressed, even in the presence of logical connectives. In terms of a solver strategy, they use binary search in tandem with lower and upper bounds to find the maximum (minimum). This is not unlike DPLL traces in knowledge compilation, which makes that technology the most accessible for propagating arbitrary semiring labels.

7 Approaches like [13, 28] on knowledge compilation for optimization and [63] on circuits for linear constraints are compelling, but their applicability to non-trivial arithmetic and optimization problems remains to be explored.
### 6. Related Work

Semiring programming is related to efforts from different disciplines within AI, and we discuss representative camps. In a nutshell, SP can be seen as a very general semantical framework, as noted in Figure 6.

**6.1. Statistical modeling**

Formal languages for generative stochastic processes, such as Church [39] and BLOG [56], have received a lot of attention in the learning community. Such languages provide mechanisms to compactly specify complex probability distributions, and appeal to sampling for inference.

Closely related to such proposals are probabilistic logic programming languages such as Problog [21] that extends Prolog with probabilistic choices and uses WMC for inference [29]. In particular, a semiring generalization of Problog, called aProbLog [48], was the starting point for our work and employs the semiring variation of WMC for inference [49]. A recent extension of aProbLog, called kProbLog, by [59] is able to further combine several semirings and does not require factorized weights as it uses meta-functions \( w(a) = f(w(a_1), ..., w(a_n)) \) to compute the weight \( w(a) \) of an atom from the weights of the atoms \( a_1, ..., a_n \) appearing in its proofs. But the kind of weight function and factorization used in kProbLog differs from the unfactorized weight function over the models used in the present paper. Nevertheless, kProbLog is able to represent tensors, compute kernel functions and perform algorithmic differentiation. It will be interesting to investigate whether kProbLog can be combined with semiring programming.

As discussed before, SP generalized the formulation of algebraic model counting (AMC) [49], and in that sense, provides a semantic characterization for the sum product problem [4] to richer class of languages and models. But by giving up the propositional language, and in particular, the use of arithmetic circuits, we lose the tractability results.

---

|   | FT | FN | IT | IN | C   | S   | R   |
|---|----|----|----|----|-----|-----|-----|
| AProbLog | ✓  | ×  | ×  | ×  | ✓   | ✓   | ✓   |
| Church   | ✓  | ×  | ✓  | ×  | ✓   | ×   | ×   |
| Essence  | ×  | ✓  | ×  | ×  | ✓   | ×   | ×   |
| CSP      | ×  | ✓  | ×  | ✓  | ✓   | ×   | ×   |
| Semiring CSP | ×  | ✓  | ×  | ×  | ✓   | ×   | ×   |
| AMPL     | ×  | ✓  | ×  | ✓  | ✓   | ×   | ×   |
| (O)SMT   | ✓  | ×  | ✓  | ×  | ✓   | ×   | ×   |
| SP       | ✓  | ✓  | ✓  | ✓  | ✓   | ✓   | ✓   |

Figure 6: A comparison, where F = finite, T = factorized, N = non-factorized, I = infinite, C = logical connectives and quantifiers, R = non-standard, S = semiring apparatus, and • denotes that a feature is available in a restricted sense.

- Compositional settings are, of course, more intricate. Along with OMT, and classical iterative methods like expectation maximization [50], there are a number of recent approaches employing branch-and-bound search strategies to navigate between local and global consistency [54]. Which of these can be made amenable to compositions of SP programs remains to be seen however.

Overall, we believe the most promising first step is to limit the vocabulary of the logical language to propositions and constants (i.e., 0-ary functions), which make appealing to knowledge compilation and OMT technology straightforward. It will also help us better characterize the complexity of the problems that SP attempts to solve. At first glance, SP is seen to naturally capture \#P-complete problems in the factorized setting, both in the finite case [37] and the infinite one [23]. In the non-factorized setting, many results from OMT and mathematical programming are inherited depending on the nature of the objective function and the domains of the program variables [46, 8, 64]. By restricting the language as suggested, the applicability of these results can be explored more thoroughly.

---

*Richer fragments have to be considered carefully to avoid undecidable properties [16].*
and unified evaluation scheme offered by AMC. Of course, it is always possible to restrict and/or otherwise map infinitary languages to finite and decomposable grammars via abstraction. This has been investigated in the case of a recent continuous extension to WMC called weighted model integration \cite{10,11}, where by interpreting the pieces of a density function as propositions, propositional circuits are leveraged for inference. In an effort independent to ours\footnote{A preliminary version of this work was online on September 2016: https://arxiv.org/abs/1609.06954}, it is shown how an algebraic extension to sum product networks \cite{60} enable tractable problem solving \cite{35}, closely following the observations in \cite{49}. In that work, a continuous extension is considered as well, but under the assumption that the specification of the weight function as well as the computation of the count can be factorized.

The use of semirings in machine learning is not new to aProbLog, see e.g., \cite{38}, and programming languages such as Dyna \cite{24}. Dyna is based on Datalog; our logical setting is strictly more expressive than Datalog and its extensions (e.g., non-Horn fragment, constraints over reals). Dyna also labels proofs but not interpretations, as would SP (thus capturing weighted model counting, for example).

6.2. Constraints

The constraints literature boasts a variety of modeling languages, such as Essence \cite{36}, among others \cite{55,68}. (See \cite{30}, for example, for a proposal on combining heterogeneous solvers.) On the one hand, SP is more expressive from a logical viewpoint as constraints can be described using arbitrary formulas from predicate logic, and we address many problems beyond constraints, such as probabilistic reasoning. On the other hand, such constraint languages make it easier for non-experts to specify problems while SP, in its current form, assumes a background in logic. Such languages, then, would be of interest for extending SP’s modeling features.

A notable line of CSP research is by Bistarelli \cite{13} and his colleagues \cite{15,13}. Here, semirings are used for diverse CSP specifications, which has also been realized in a CLP framework \cite{14}. In particular, our account of compositionality is influenced by \cite{13}. Under some representational assumptions, SP and such accounts are related, but as noted, SP can formulate problems such as probabilistic inference in hybrid domains that does not have an obvious analogue in these accounts.

6.3. Optimization

Closely related to the constraints literature are the techniques embodied in mathematical programming more generally. There are three major traditions in this literature that are related to SP. Modeling languages such as AMPL \cite{31} are fairly close to constraint modeling languages, and even allow parametrized constraints, which are ground at the time of search. The field of disciplined programming \cite{40} supports features such as object-oriented constraints. Finally, relational mathematical programming \cite{2} attempts to exploit symmetries in parametrized constraints.

From a solver construction perspective, these languages present interesting possibilities. From a framework point of view, however, there is little support for logical reasoning in a general way.

6.4. Knowledge representation

Declarative problem solving is a focus of many proposals, including ASP \cite{17}, model expansion \cite{57,66}, among others \cite{18}. These proposals are (mostly) for problems in NP, and so do not capture \#P-hard problems like model counting and WMC. Indeed, the most glaring difference is the absence of weight functions over possible worlds, which is central to the formulation of statistical models. Weighted extensions of these formalisms, e.g., \cite{5,54}, are thus closer in spirit.

The generality of SP also allows us to instantiate many such proposals, including formalisms using linear arithmetic fragments \cite{11,64}. Consider OMT for example. OMT can be used to express quantifier-free linear arithmetic sentences with a linear cost function, and a first-order structure that minimizes the cost function is sought. From a specification point of view, SP does not limit the logical language, does not require that objective functions be linear, and a variety of model comparisons, including counting, are possible via semirings. Compositionality in SP, moreover, goes quite beyond this technology.

Finally, there is a longstanding interest in combining different (logical) environments in a single logical framework, as seen, for example, in modular and multi-context systems \cite{53,26}. In such frameworks, it would be possible
to get an ILP program and ASP program to communicate their solutions, often by sharing atoms. In our view, Section 4 and these frameworks emphasize different aspects of compositionality. The SP scheme assumes the modeler will formalize a convex optimization problem and a SAT problem in the same programming language since they presumably arise in a single application (e.g., a task and motion planner); this allows model reuse and enables transparency. In contrast, modular systems essentially treat diverse environments as black-boxes, which is perhaps easier to realize. On the one hand, it would be interesting to see whether modular systems can address problems such as combined inference and learning. On the other hand, some applications may require that different environments share atoms, for which our account on compositionality could be extended by borrowing ideas from modular systems.

7. Conclusions

In a nutshell, SP is a framework to declaratively specify four major concerns in AI applications:

- logical reasoning;
- non-standard models;
- discrete and continuous probabilistic inference;
- discrete and continuous optimization.

To its strengths, we find that SP is universal (in the above sense) and generic (in terms of allowing instantiations to particular logical languages and semirings). Thus, we believe SP represents a simple, uniform, modular and transparent approach to the model building process of complex AI applications.

SP comes with a rigorous semantics to give meaning to its programs. In that sense, we imagine future developments of SP would follow constraint programming languages and probabilistic programming languages in providing more intricate modeling features which, in the end, resort to the proposed semantics in the paper.

Perhaps the most significant aspect of SP is that it also allows us to go beyond existing paradigms as the richness of the framework admits novel formulations that combine theories from these different fields, as illustrated by means of a combined regression and probabilistic inference example. In the long term, we hope SP will contribute to the bridge between learning and reasoning.

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