Fusion procedure for cyclotomic Hecke algebras

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Abstract

A complete system of primitive pairwise orthogonal idempotents for cyclotomic Hecke algebras is constructed by consecutive evaluations of a rational function in several variables on quantum contents of multi-tableaux. This function is a product of two terms, one of which depends only on the shape of the multi-tableau and is proportional to the inverse of the corresponding Schur element.

1. Introduction

This article is a continuation of the article [10] on the fusion procedure for the complex reflection groups $G(m, 1, n)$. The cyclotomic Hecke algebra $H(m, 1, n)$, introduced in [1, 2, 3], is a natural flat deformation of the group ring of the complex reflection group $G(m, 1, n)$.

In [10], a fusion procedure, in the spirit of [8], for the complex reflection groups $G(m, 1, n)$ is suggested: a complete system of primitive pairwise orthogonal idempotents for the groups $G(m, 1, n)$ is obtained by consecutive evaluations of a rational function in several variables with values in the group ring $\mathbb{C}G(m, 1, n)$. This approach to the fusion procedure relies on the existence of a maximal commutative set of elements of $\mathbb{C}G(m, 1, n)$ formed by the Jucys–Murphy elements.

Jucys–Murphy elements for the cyclotomic Hecke algebra $H(m, 1, n)$ were introduced in [11] and were used in [9] to develop an inductive approach to the representation theory of the chain of the

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algebras $H(m,1,n)$. In the generic setting or under certain restrictions on the parameters of the algebra $H(m,1,n)$ (see Section 2 for precise definitions), the Jucys–Murphy elements form a maximal commutative set in the algebra $H(m,1,n)$.

A complete system of primitive pairwise orthogonal idempotents of the algebra $H(m,1,n)$ is indexed by the set of standard $m$-tableaux of size $n$. We formulate here the main result of the article. Let $\lambda^{(m)}$ be an $m$-partition of size $n$ and $T$ be a standard $m$-tableau of shape $\lambda^{(m)}$.

**Theorem.** The idempotent $E_T$ of $H(m,1,n)$ corresponding to the standard $m$-tableau $T$ of shape $\lambda^{(m)}$ can be obtained by the following consecutive evaluations

$$E_T = F_{\lambda^{(m)}}(u_1,\ldots,u_n)\bigg|_{u_1=c_1} \cdots \bigg|_{u_{n-1}=c_{n-1}} \bigg|_{u_n=c_n}. \tag{1}$$

Here $F_{\lambda^{(m)}}(u_1,\ldots,u_n)$ is a rational function with values in the algebra $H(m,1,n)$, $F_{\lambda^{(m)}}$ is an element of the base ring and $c_1,\ldots,c_n$ are the quantum contents of the $m$-nodes of $T$.

The classical limit of our fusion procedure for algebras $H(m,1,n)$ reproduces the fusion procedure of [10] for the complex reflection groups $G(m,1,n)$. For $\mathbb{C}G(m,1,n)$, the variables of the rational function are split into two parts, one is related to the position of the $m$-node (its place in the $m$-tuple) and the other one - to the classical content of the $m$-node. The position variables can be evaluated simultaneously while the classical content variables have then to be evaluated consequently from 1 to $n$. For the algebra $H(m,1,n)$, the information about positions and classical contents is fully contained in the quantum contents, and now the function $F_{\lambda^{(m)}}$ depends on only one set of variables.

Remarkably, the coefficient $F_{\lambda^{(m)}}$ appearing in (1) depends only on the shape $\lambda^{(m)}$ of the standard $m$-tableau $T$ (cf. with the more delicate fusion procedure for the Birman-Murakami-Wenzl algebra [3]). In the classical limit, this coefficient depends only on the usual hook length, see [10]. However, in the deformed situation, the calculation of $F_{\lambda^{(m)}}$ needs a non-trivial generalization of the hook length. It appears that the coefficient $F_{\lambda^{(m)}}$ is proportional to the inverse of the Schur element of the algebra $H(m,1,n)$ corresponding to the $m$-partition $\lambda^{(m)}$ (see [4] for an expression of the Schur elements of $H(m,1,n)$ in terms of generalized hook lengths); the proportionality factor is a unit of the ring $\mathbb{C}[q,q^{-1},v_1,\ldots,v_m]$, where $q,v_1,\ldots,v_m$ are the parameters of $H(m,1,n)$ (see Section 2 for precise definitions).

For $m=1$, the cyclotomic Hecke algebra $H(1,1,n)$ is the Hecke algebra of type A and our fusion procedure reduces to the fusion procedure for the Hecke algebra in [5]. The factors in the rational function are arranged in [5] in such a way that there is a product of “Baxterized” generators on one side and a product of non-Baxterized generators on the other side. For $m>1$ a rearrangement, as for the type A, of the rational function appearing in (1) is no more possible.

The additional, with respect to $H(1,1,n)$, generator of $H(m,1,n)$ satisfies the reflection equation whose “Baxterization” is known [7]. But - and this is maybe surprising - the full Baxterized form is not used in the construction of the rational function in (1). The rational expression involving the additional generator satisfies only a certain limit of the reflection equation with spectral parameters.

The Hecke algebra of type A is the natural quotient of the Birman-Murakami-Wenzl algebra. The fusion procedure, developed in [6], for the Birman-Murakami-Wenzl algebra provides a one-parameter family of fusion procedures for the Hecke algebra of type A. We think that for $m>1$ the fusion procedure (1) can be included into a one-parameter family as well.

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2. Definitions

2.1. Cyclotomic Hecke algebra and Baxterized elements

The cyclotomic Hecke algebra $H(m,1,n+1)$ is generated by $\tau, \sigma_1, \ldots, \sigma_n$ with the defining relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } i = 1, \ldots, n-1,$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } i, j = 1, \ldots, n \text{ such that } |i-j| > 1,$$

$$\tau\sigma_1\tau = \sigma_1\tau\tau,$$

$$\tau\sigma_i = \sigma_i\tau \quad \text{for } i > 1,$$

$$\sigma_i^2 = (q-q^{-1})\sigma_i + 1 \quad \text{for } i = 1, \ldots, n,$$

$$(\tau - v_1)\ldots(\tau - v_m) = 0 \quad \text{(2)}.$$  

The cyclotomic Hecke algebras $H(m,1,n)$ form a chain (with respect to $n$) of algebras defined by inclusions $H(m,1,n) \ni \tau, \sigma_1, \ldots, \sigma_{n-1} \mapsto \tau, \sigma_1, \ldots, \sigma_n, \sigma_{n-1} \in H(m,1,n+1)$. These inclusions allow to consider (as it will often be done in the article) elements of $H(m,1,n)$ as elements of $H(m,1,n+n')$ for any $n' = 0, 1, 2, \ldots$.

We shall work with a generic cyclotomic Hecke algebra (that is, $q, v_1, \ldots, v_m$ are indeterminates and we consider the algebra $H(m,1,n+1)$ over a certain localization of the ring $\mathbb{C}[q, q^{-1}, v_1, \ldots, v_m]$), or in a specialization such that the following conditions are satisfied

$$1 + q^2 + \cdots + q^{2N} \neq 0 \quad \text{for } N < n,$$

$$q^{2i}v_j - v_k \neq 0 \quad \text{for } i, j, k \text{ such that } j \neq k \text{ and } -n < i < n,$$

$$q \neq 0, \quad v_j \neq 0 \quad \text{for } j = 1, \ldots, m \quad \text{(3)}.$$  

Note that the restrictions (3) for the parameters of $H(m,1,n+1)$ imply the corresponding restrictions for the parameters of $H(m,1,n)$.

Define, for $i = 1, \ldots, n$, the Baxterized elements, with spectral parameters $\alpha$ and $\beta$:

$$\sigma_i(\alpha,\beta) := \sigma_i + (q-q^{-1})\frac{\beta}{\alpha - \beta}. \quad \text{(4)}$$

These Baxterized elements satisfy the Yang–Baxter equation with spectral parameters:

$$\sigma_i(\alpha,\alpha')\sigma_{i+1}(\alpha,\alpha'')\sigma_i(\alpha',\alpha'') = \sigma_{i+1}(\alpha',\alpha'')\sigma_i(\alpha,\alpha'')\sigma_{i+1}(\alpha,\alpha').$$

The following formula will be used later:

$$\sigma_i(\alpha,\beta)\sigma_i(\beta,\alpha) = \frac{(\alpha - q^2\beta)(\alpha - q^{-2}\beta)}{(\alpha - \beta)^2} \quad \text{for } i = 1, \ldots, n. \quad \text{(5)}$$

We also define the following rational function with values in $H(m,1,n+1)$:

$$\tau(\rho) := \frac{(\rho - v_1)(\rho - v_2)\ldots(\rho - v_m)}{\rho - \tau}. \quad \text{(6)}$$
Remarks. (i) One can rewrite \( \tau(\rho) \) as a polynomial, in \( \rho \), function. Indeed, let \( a_0, a_1, \ldots, a_m \) be the polynomials in \( v_1, \ldots, v_m \) defined by

\[
(X - v_1)(X - v_2) \cdots (X - v_m) = a_0 + a_1X + \cdots + a_mX^m,
\]

where \( X \) is an indeterminate. Define the polynomials \( a_i(\rho), \ i = 0, \ldots, m, \) in \( \rho \), with values in \( \mathbb{C}[v_1, \ldots, v_m] \), by

\[
a_0(\rho) = a_0 + a_1\rho + \cdots + a_m\rho^m \quad \text{and} \quad a_{i+1}(\rho) = \rho^{-1}(a_i(\rho) - a_i) \quad \text{for} \ i = 0, \ldots, m - 1.
\]

The polynomials \( a_i(\rho), \ i = 0, \ldots, m, \) are given explicitly by

\[
a_i(\rho) = \rho + a_i + a_{i+1} + \cdots + \rho^{m-i} a_m \quad \text{for} \ i = 0, \ldots, m.
\] (7)

It is straightforward to verify that

\[
(\rho - \tau)^{m-1} \sum_{i=0}^{m-1} a_{i+1}(\rho) \tau^i = a_0(\rho) = (\rho - v_1)(\rho - v_2) \cdots (\rho - v_m).
\] (8)

It follows from (3) and (6) that

\[
\tau(\rho) = a_1(\rho) + a_2(\rho) \tau + \cdots + a_m(\rho) \tau^{m-1} = \sum_{i=0}^{m-1} a_{i+1}(\rho) \tau^i.
\] (9)

For example, for \( m = 1 \), we have \( \tau(\rho) = 1 \); for \( m = 2 \), we have \( \tau(\rho) = \tau + \rho - v_1 - v_2 \); for \( m = 3 \), we have \( \tau(\rho) = \tau^2 + (\rho - v_1 - v_2 - v_3)\tau + \rho^2 - \rho(v_1 + v_2 + v_3) + v_1v_2 + v_1v_3 + v_2v_3 \).

(ii) The elements \( \tau(\rho) \) and \( \sigma_1(\alpha, \beta) \) satisfy a certain form of a reflection equation with spectral parameters:

\[
\sigma_1(\alpha, \beta) \tau(\alpha) \sigma_1^{-1}(\beta) = \tau(\beta) \sigma_1^{-1}(\alpha) \sigma_1(\alpha, \beta).
\] (10)

Indeed, due to (3) and (6), the equality (11) is equivalent to

\[
(\tau - \beta) \sigma_1(\tau - \alpha) \sigma_1(\beta, \alpha) = \sigma_1(\beta, \alpha) (\tau - \alpha) \sigma_1(\tau - \beta),
\]

which is proved by a straightforward calculation. The equation (11) is a certain (we leave the details to the reader) limit of the usual reflection equation with spectral parameters.

2.2. \( m \)-partitions, \( m \)-tableaux and generalized hook length

Let \( \lambda \vdash n + 1 \) be a partition of size \( n + 1 \), that is, \( \lambda = (\lambda_1, \ldots, \lambda_l) \), where \( \lambda_j, j = 1, \ldots, l \), are positive integers, \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \) and \( n + 1 = \lambda_1 + \cdots + \lambda_l \). We identify partitions with their Young diagrams: the Young diagram of \( \lambda \) is a left-justified array of rows of nodes containing \( \lambda_j \) nodes in the \( j \)-th row, \( j = 1, \ldots, l \); the rows are numbered from top to bottom.

An \( m \)-partition, or a Young \( m \)-diagram, of size \( n + 1 \) is an \( m \)-tuple of partitions such that the sum of their sizes equals \( n + 1 \); e. g. the Young 3-diagram (\( \square, \square, \square \)) represents the 3-partition ((2), (1), (1)) of size 4.
An $m$-node $\alpha^{(m)}$ is a pair $(\alpha, k)$ consisting of a usual node $\alpha$ and an integer $k = 1, \ldots, m$, indicating to which diagram in the $m$-tuple the node belongs. The integer $k$ will be called position of the $m$-node, and we set $\text{pos}(\alpha^{(m)}) := k$.

For an $m$-partition $\lambda^{(m)}$, an $m$-node $\alpha^{(m)}$ of $\lambda^{(m)}$ is called removable if the set of $m$-nodes obtained from $\lambda^{(m)}$ by removing $\alpha^{(m)}$ is still an $m$-partition. An $m$-node $\beta^{(m)}$ not in $\lambda^{(m)}$ is called addable if the set of $m$-nodes obtained from $\lambda^{(m)}$ by adding $\beta^{(m)}$ is still an $m$-partition. For an $m$-partition $\lambda^{(m)}$, we denote by $\mathcal{E}_-(\lambda^{(m)})$ the set of removable $m$-nodes of $\lambda^{(m)}$ and by $\mathcal{E}_+(\lambda^{(m)})$ the set of addable $m$-nodes of $\lambda^{(m)}$. For example, the removable/addable $m$-nodes (marked with -/+ for the 3-partition $(\square, \square, \square)$ are

$$
\begin{pmatrix}
+ & - & + \\
- & + & - \\
+ & + & +
\end{pmatrix}
$$

Let $\lambda^{(m)}$ be an $m$-diagram of size $n+1$. A standard $m$-tableau of shape $\lambda^{(m)}$ is obtained by placing the numbers $1, \ldots, n+1$ in the $m$-nodes of the diagrams of $\lambda^{(m)}$ in such a way that the numbers in the nodes ascend along rows and columns in every diagram.

Let $q, v_1, \ldots, v_m$ be the parameters of the cyclotomic Hecke algebra $H(m,1,n+1)$. For an $m$-node $\alpha^{(m)} = (\alpha, k)$ lying in the line $x$ and the column $y$ of the $k$-th diagram, we denote by $c(\alpha^{(m)})$ the quantum content of the node $\alpha$, $cc(\alpha^{(m)}) := v_k q^{2(y-x)}$. We denote by $cc(\alpha^{(m)})$ the classical content of the node $\alpha$, $cc(\alpha^{(m)}) := y - x$. Let $\{\xi_1, \ldots, \xi_m\}$ be the set of distinct $m$-th roots of unity, ordered arbitrarily; we define also $p(\alpha^{(m)}) := \xi_k$.

For a standard $m$-tableau $T$ of shape $\lambda^{(m)}$ let $\alpha_i^{(m)}$ be the $m$-node of $T$ occupied by the number $i$, $i = 1, \ldots, n+1$; we set $c(T|i) := c(\alpha_i^{(m)})$, $cc(T|i) := cc(\alpha_i^{(m)})$ and $p(T|i) := p(\alpha_i^{(m)})$. For example, for the standard 3-tableau $T = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}$ we have

$$
c(T|1) = v_1, \ c(T|2) = v_2, \ c(T|3) = v_1 q^2 \text{ and } c(T|4) = v_3, \\
cc(T|1) = 0, \ cc(T|2) = 0, \ cc(T|3) = 1 \text{ and } cc(T|4) = 0, \\
p(T|1) = \xi_1, \ p(T|2) = \xi_2, \ p(T|3) = \xi_1 \text{ and } p(T|4) = \xi_3,
$$

where $\{\xi_1, \xi_2, \xi_3\}$ is the set of all third roots of unity, ordered arbitrarily.

**Generalized hook length.** The hook of a node $\alpha$ of a partition $\lambda$ is the set of nodes of $\lambda$ consisting of the node $\alpha$ and the nodes which lie either under $\alpha$ in the same column or to the right of $\alpha$ in the same row; the hook length $h_\lambda(\alpha)$ of $\alpha$ is the cardinality of the hook of $\alpha$. We extend this definition to $m$-nodes. For an $m$-node $\alpha^{(m)} = (\alpha, k)$ of an $m$-partition $\lambda^{(m)}$, the hook length of $\alpha^{(m)}$ in $\lambda^{(m)}$, which we denote by $h_{\lambda^{(m)}}(\alpha^{(m)})$, is the hook length of the node $\alpha$ in the $k$-th partition of $\lambda^{(m)}$.

Let $\lambda^{(m)}$ be an $m$-partition. For $j = 1, \ldots, m$, let $I_{\lambda^{(m)},x,j}$ be the number of nodes in the line $x$ of the $j$-th diagram of $\lambda^{(m)}$, and $c_{\lambda^{(m)},y,j}$ be the number of nodes in the column $y$ of the $j$-th diagram of $\lambda^{(m)}$. The hook length of an $m$-node $\alpha^{(m)}$ lying in the line $x$ and the column $y$ of the $k$-th diagram of $\lambda^{(m)}$ can be rewritten as

$$
h_{\lambda^{(m)}}(\alpha^{(m)}) = I_{\lambda^{(m)},x,k} + c_{\lambda^{(m)},y,k} - x - y + 1.
$$
where $\alpha^{(m)}$ is the $m$-node lying in the line $x$ and the column $y$ of the $k$-th diagram of $\lambda^{(m)}$ (in particular, $h^{(k)}_{\lambda^{(m)}}(\alpha^{(m)}) = h_{\lambda^{(m)}}(\alpha^{(m)})$ is the usual hook length).

For an $m$-partition $\lambda^{(m)}$ of size $n$, we define

$$F_{\lambda^{(m)}} := (q^{-1} - q)^n \prod_{\alpha^{(m)} \in \lambda^{(m)}} \left( c(\alpha^{(m)}) \prod_{k=1}^{m} \frac{q^{-cc(\alpha^{(m)})}}{\ell_{\lambda^{(m)}}(\alpha^{(m)})^q - h^{(k)}_{\lambda^{(m)}}(\alpha^{(m)}) - v_k q h_{\lambda^{(m)}}^{(k)}(\alpha^{(m)})} \right).$$

(11)

The element $F_{\lambda^{(m)}}$ can also be written as

$$F_{\lambda^{(m)}} = \prod_{\alpha^{(m)} \in \lambda^{(m)}} \left( \frac{q^{cc(\alpha^{(m)})}}{h_{\lambda^{(m)}}(\alpha^{(m)})^q} \prod_{k=1}^{m} \frac{q^{-cc(\alpha^{(m)})}}{\ell_{\lambda^{(m)}}(\alpha^{(m)})^q - h^{(k)}_{\lambda^{(m)}}(\alpha^{(m)}) - v_k q h_{\lambda^{(m)}}^{(k)}(\alpha^{(m)})} \right),$$

(12)

where $[j]_q := q^{j-1} + q^{j-3} + ... + q^{-j+1}$ for a non-negative integer $j$.

3. Idempotents and Jucys–Murphy elements of $H(m, 1, n + 1)$

The Jucys–Murphy elements $J_i$, $i = 1, \ldots, n + 1$, of the algebra $H(m, 1, n + 1)$ are defined by the following initial condition and recursion:

$$J_1 = \tau \quad \text{and} \quad J_{i+1} = \sigma_i J_i \sigma_i, \quad i = 1, \ldots, n.$$

(13)

We recall that, under the restrictions [3], the elements $J_i$, $i = 1, \ldots, n+1$, form a maximal commutative set of $H(m, 1, n + 1)$ [1]. Recall also that

$$J_i \sigma_k = \sigma_k J_i \quad \text{for } k \neq i-1, i.$$

(14)

The irreducible representations of $H(m, 1, n + 1)$ are labelled by the $m$-partitions of size $n + 1$. The basis vectors of the irreducible representation of $H(m, 1, n + 1)$ labelled by the $m$-partition $\lambda^{(m)}$ are parameterized by the standard $m$-tableaux of shape $\lambda^{(m)}$. The Jucys–Murphy elements are diagonal in this basis. For a standard $m$-tableau $\mathcal{T}$, denote by $E_\mathcal{T}$ the corresponding primitive idempotent of $H(m, 1, n + 1)$. We have, for all $i = 1, \ldots, n + 1$,

$$J_i E_\mathcal{T} = E_\mathcal{T} J_i = c_i E_\mathcal{T},$$

(15)

where $c_i := c(\mathcal{T}|i)$, $i = 1, \ldots, n+1$. Due to the maximality of the commutative set formed by the Jucys–Murphy elements, the idempotent $E_\mathcal{T}$ can be expressed in terms of the elements $J_i$, $i = 1, \ldots, n + 1$. Let $\gamma^{(m)}$ be the $m$-node of $\mathcal{T}$ containing the number $n + 1$. As the $m$-tableau $\mathcal{T}$ is standard, the $m$-node $\gamma^{(m)}$ of $\lambda^{(m)}$ is removable. Let $\mathcal{U}$ be the standard $m$-tableau obtained from $\mathcal{T}$ by removing
the $m$-node $\gamma^{(m)}$, and let $\mu^{(m)}$ be the shape of $\mathcal{U}$. The inductive formula for $E_T$ in terms of the Jucys–Murphy elements reads:

$$E_T = E_{\mathcal{U}} \prod_{\beta^{(m)} \in \mathcal{E}^+ (\mu^{(m)})} \frac{J_{n+1} - c(\beta^{(m)})}{c(\gamma^{(m)}) - c(\beta^{(m)})},$$

(16)

with the initial condition: $E_{\mathcal{U}_0} = 1$ for the unique $m$-tableau $\mathcal{U}_0$ of size 0. Here $E_{\mathcal{U}}$ is considered as an element of the algebra $H(m, 1, n+1)$. Note that, due to the restrictions (3), we have $c(\beta^{(m)}) \neq c(\gamma^{(m)})$ for any $\beta^{(m)} \in \mathcal{E}^+ (\mu^{(m)})$ such that $\beta^{(m)} \neq \gamma^{(m)}$.

4. Fusion formula for the algebra $H(m, 1, n+1)$

Let $\lambda^{(m)}$ be an $m$-partition of size $n+1$ and $\mathcal{T}$ a standard $m$-tableau of shape $\lambda^{(m)}$. For $i = 1, \ldots, n+1$, we set $c_i := c(\mathcal{T} | i)$.

**Theorem 1.** The idempotent $E_T$ corresponding to the standard $m$-tableau $\mathcal{T}$ of shape $\lambda^{(m)}$ can be obtained by the following consecutive evaluations

$$E_T = F_{\lambda^{(m)}} \Phi(u_1, \ldots, u_{n+1}) \bigg|_{u_1 = c_1} \ldots \bigg|_{u_{n+1} = c_{n+1}} = E_{\mathcal{T}}.$$

(21)
Proof. The Theorem 1 follows, by induction on $n$, from (18) and the Propositions 2 and 5 below. \hfill \square

Till the end of the text, $\gamma^{(m)}$ and $\delta^{(m)}$ denote the $m$-nodes of $T$ containing the numbers $n+1$ and $n$ respectively; $U$ is the standard $m$-tableau obtained from $T$ by removing $\gamma^{(m)}$, and $\mu^{(m)}$ is the shape of $U$; also, $W$ is the standard $m$-tableau obtained from $U$ by removing the $m$-node $\delta^{(m)}$ and $\nu^{(m)}$ is the shape of $W$.

For a standard $m$-tableau $V$ of size $N$, we define

\[ F_V(u) := \frac{u - c(V)}{(u-v_1) \cdots (u-v_m)} \prod_{i=1}^{N-1} \frac{(u - c(V))^2}{(u - q^2c(V))} ; \] (22)

by convention, for $N = 1$, the product in the right hand side is equal to 1.

**Proposition 2.** We have

\[ F_T(u)\phi_{n+1}(c_1, \ldots, c_n, u)E_{U} = \frac{u - c_n + 1}{u - J_n + 1} E_{U}. \] (23)

**Proof.** We prove (23) by induction on $n$. As $J_1 = \tau$, we have by (20)

\[ \frac{u - c_1}{u - J_1} = \frac{u - c_1}{(u-v_1) \cdots (u-v_m)} \tau(u), \]

which verifies the basis of induction ($n = 0$).

We have: $E_WE_{U} = E_{U}$ and $E_W$ commutes with $\sigma_n$. Rewrite the left hand side of (23) as

\[ F_T(u)\sigma_n(u, c_n) \cdot \phi_{n}(c_1, \ldots, c_{n-1}, u)E_{U} \cdot \sigma_n^{-1}E_{U}. \]

By the induction hypothesis we have for the left hand side of (23):

\[ F_T(u)(F_U(u))^{-1}\sigma_n(u, c_n)\frac{u - c_n}{u - J_n} \sigma_n^{-1}E_{U}. \]

Since $J_{n+1}$ commutes with $E_{U}$, the equality (23) is equivalent to

\[ F_T(u)(F_U(u))^{-1}(u - c_n)\sigma_n^{-1}(u - J_{n+1})E_{U} = \frac{(u - c_{n+1})(u - c_n)^2}{(u - q^2c_n)(u - q^{-2}c_n)}(u - J_n)\sigma_n(c_n, u)E_{U}; \] (24)

(the inverse of $\sigma_n(u, c_n)$ is calculated with the help of (21)). By (22),

\[ F_T(u)(F_U(u))^{-1}(u - c_n) = \frac{(u - c_n)^2}{(u - q^2c_n)(u - q^{-2}c_n)}. \] (25)

Therefore, to prove (24), it remains to show that

\[ \sigma_n^{-1}(u - J_{n+1})E_{U} = (u - J_n)\sigma_n(c_n, u)E_{U}. \] (26)

Replacing $J_{n+1}$ by $\sigma_n J_n \sigma_n$, we write the left hand side of (26) in the form

\[ (u\sigma_n^{-1} - J_n\sigma_n)E_{U}. \] (27)
As \( J_n E_d = c_n E_d \), the right hand side of (26) is

\[
(u \sigma_n - J_n \sigma_n + (q - q^{-1})(u - c_n)\frac{u}{c_n - u})E_d
\]

and thus coincides with (27).

**Lemma 3.** We have

\[
F_T(u) = (u - c_{n+1}) \prod_{\beta^{(m)} \in \mathcal{E}_-(\nu^{(m)})} \left( u - c(\beta^{(m)}) \right) \prod_{\alpha^{(m)} \in \mathcal{E}_+(\mu^{(m)})} \left( u - c(\alpha^{(m)}) \right)^{-1}
\]

The proof is by induction on \( n \). For \( n = 0 \), we have

\[
F_T(u) = \frac{u - c_1}{(u - v_1) \ldots (u - v_m)},
\]

which is equal to the right hand side of (29).

Now, for \( n > 0 \), we write

\[
F_T(u) = \frac{u - c_{n+1}}{(u - v_1) \ldots (u - v_m)} \frac{(u - c_n)^2}{(u - q^2 c_n)(u - q^{-2} c_n)} \prod_{i=1}^{n-1} \frac{u - c_i)^2}{(u - q^2 c_i)(u - q^{-2} c_i)}.
\]

Using the induction hypothesis, we obtain

\[
F_T(u) = \frac{(u - c_{n+1})(u - c_n)^2}{(u - q^2 c_n)(u - q^{-2} c_n)} \prod_{\beta^{(m)} \in \mathcal{E}_-(\nu^{(m)})} \left( u - c(\beta^{(m)}) \right) \prod_{\alpha^{(m)} \in \mathcal{E}_+(\mu^{(m)})} \left( u - c(\alpha^{(m)}) \right)^{-1}
\]

Denote by \( \delta^{(m)}_l \) and \( \delta^{(m)}_b \) the \( m \)-nodes which are, respectively, just above and just below \( \delta^{(m)} \) and \( \delta^{(m)}_l \) and \( \delta^{(m)}_b \) the \( m \)-nodes which are, respectively, just on the left and just on the right of \( \delta^{(m)} \); it might happen that one of the coordinates of \( \delta^{(m)}_l \) (or \( \delta^{(m)}_b \)) is not positive, and in this situation, by definition, \( \delta^{(m)}_l \not\in \mathcal{E}_-(\nu^{(m)}) \) (or \( \delta^{(m)}_b \not\in \mathcal{E}_-(\nu^{(m)}) \)). It is straightforward to see that:

- If \( \delta^{(m)}_l \not\in \mathcal{E}_-(\nu^{(m)}) \) then
  \[
  \mathcal{E}_-(\mu^{(m)}) = \mathcal{E}_-(\nu^{(m)}) \cup \{\delta^{(m)}\}
  \]
  and 
  \[
  \mathcal{E}_+(\mu^{(m)}) = \left( \mathcal{E}_+(\nu^{(m)}) \cup \{\delta^{(m)}_b\}, \delta^{(m)}_l\right) \backslash \{\delta^{(m)}\}
  \]

- If \( \delta^{(m)}_l \in \mathcal{E}_-(\nu^{(m)}) \) and \( \delta^{(m)}_b \not\in \mathcal{E}_-(\nu^{(m)}) \) then
  \[
  \mathcal{E}_-(\mu^{(m)}) = \left( \mathcal{E}_-(\nu^{(m)}) \cup \{\delta^{(m)}_l\}\right) \backslash \{\delta^{(m)}\}
  \]
  and 
  \[
  \mathcal{E}_+(\mu^{(m)}) = \left( \mathcal{E}_+(\nu^{(m)}) \cup \{\delta^{(m)}_b\}\right) \backslash \{\delta^{(m)}\}
  \]

- If \( \delta^{(m)}_b \not\in \mathcal{E}_-(\nu^{(m)}) \) and \( \delta^{(m)}_l \in \mathcal{E}_-(\nu^{(m)}) \) then
  \[
  \mathcal{E}_-(\mu^{(m)}) = \left( \mathcal{E}_-(\nu^{(m)}) \cup \{\delta^{(m)}_l\}\right) \backslash \{\delta^{(m)}_b\}
  \]
  and 
  \[
  \mathcal{E}_+(\mu^{(m)}) = \left( \mathcal{E}_+(\nu^{(m)}) \cup \{\delta^{(m)}_l\}\right) \backslash \{\delta^{(m)}_b\}
  \]
Lemma 4. We have

\[ \mathcal{E}_-(\mu^{(m)}) = \left( \mathcal{E}_-(\nu^{(m)}) \cup \{ \delta^{(m)} \} \right) \setminus \{ \delta_i^{(m)}, \delta_j^{(m)} \} \quad \text{and} \quad \mathcal{E}_+(\mu^{(m)}) = \mathcal{E}_+(\nu^{(m)}) \setminus \{ \delta^{(m)} \} . \]

In each case, it follows that the right hand side of (30) is equal to

\[ (u - c_{n+1}) \prod_{\beta^{(m)} \in \mathcal{E}_-(\mu^{(m)})} (u - c(\beta^{(m)})) \prod_{\alpha^{(m)} \in \mathcal{E}_+(\mu^{(m)})} (u - c(\alpha^{(m)}))^{-1} , \]

which establishes the formula (29). □

Lemma 4. We have

\[ \prod_{\beta^{(m)} \in \mathcal{E}_-(\mu^{(m)})} (c_{n+1} - c(\beta^{(m)})) \prod_{\alpha^{(m)} \in \mathcal{E}_+(\mu^{(m)}) \setminus \{ \gamma^{(m)} \}} (c_{n+1} - c(\alpha^{(m)}))^{-1} = F_{\lambda^{(m)}} \frac{F_{\mu^{(m)}}}{F_{\mu^{(m)}}} . \] (31)

Proof. 1. The definition (31), for a usual partition $\lambda$, reduces to

\[ F_{\lambda} := \prod_{\alpha \in \lambda} \frac{q^{cc(\lambda)}}{\theta_{h(\lambda)}(\lambda)} . \]

The Lemma 4 for a usual partition $\lambda$ is established in [5], Lemma 3.2.

2. Set $k = \text{pos}(\gamma^{(m)})$. Define, for an $m$-partition $\theta^{(m)}$,

\[ \tilde{F}_{\theta^{(m)}} := \prod_{\alpha^{(m)} \in \theta^{(m)}} \frac{q^{cc(\alpha^{(m)})}}{h_{\theta^{(m)}}(\alpha^{(m)})} , \] (32)

and, for $j = 1, \ldots, m$ such that $j \neq k$,

\[ F_{\theta^{(m)}} = \tilde{F}_{\theta^{(m)}} \prod_{j = 1, \ldots, m} F_{\theta^{(m)}} . \] (33)

By (32), we have

\[ F_{\theta^{(m)}} = \tilde{F}_{\theta^{(m)}} \prod_{j = 1, \ldots, m} F_{\theta^{(m)}} . \] (34)

Fix $j \in \{1, \ldots, m\}$ such that $j \neq k$. We shall show that

\[ \prod_{\beta^{(m)} \in \mathcal{E}_-(\mu^{(m)}) \setminus \{ \gamma^{(m)} \}} (c_{n+1} - c(\beta^{(m)})) \prod_{\alpha^{(m)} \in \mathcal{E}_+(\mu^{(m)}) \setminus \{ \gamma^{(m)} \}} (c_{n+1} - c(\alpha^{(m)}))^{-1} = F_{\lambda^{(m)}} \frac{F_{\mu^{(m)}}}{F_{\mu^{(m)}}} . \] (35)

Let $p_1 < p_2 < \cdots < p_s$ be positive integers such that the $j$-th partition of $\mu^{(m)}$ is $(\mu_1, \ldots, \mu_s)$ with

\[ \mu_1 = \cdots = \mu_{p_1} > \mu_{p_1+1} = \cdots = \mu_{p_2} > \cdots > \mu_{p_{s-1}+1} = \cdots = \mu_{p_s} > 0 . \]
We set \( p_0 := 0 \), \( p_{s+1} := +\infty \) and \( \mu_{p_{s+1}} := 0 \). Assume that the \( m \)-node \( \gamma^{(m)} \) lies in the line \( x \) and column \( y \). The left hand side of (35) is equal to

\[
\prod_{b=1}^{s} \left( v_k q^{2(y-x)} - v_j q^{2(\mu_{p_b} - p_b)} \right) \prod_{b=1}^{s+1} \left( v_k q^{2(y-x)} - v_j q^{2(\mu_{p_{b-1}} - p_{b-1})} \right)^{-1}.
\]  

(36)

The factors in the product (33) correspond to the \( m \)-nodes of an \( m \)-partition. The \( m \)-nodes lying neither in the column \( y \) of the \( k \)-th diagrams (of \( \lambda^{(m)} \) or \( \mu^{(m)} \)) nor in the line \( x \) of the \( j \)-th diagrams do not contribute to the right hand side of (35). Let \( t \in \{0, \ldots, s\} \) be such that \( p_{t} < x \leq p_{t+1} \). The contribution from the \( m \)-nodes in the column \( y \) and lines 1, \ldots, \( p_{t} \) of the \( k \)-th diagrams is:

\[
\prod_{b=1}^{t} \left( \prod_{a=p_{b-1}+1}^{p_{b}} \frac{v_k q^{-(\mu_{p_b} - y + x - a)} - v_j q^{(\mu_{p_b} - y + x - a)}}{v_k q^{-(\mu_{p_{b-1}} - y + x - a + 1)} - v_j q^{(\mu_{p_{b-1}} - y + x - a + 1)}} \right);
\]

the contribution from the \( m \)-nodes in the column \( y \) and lines \( p_{t+1}, \ldots, x \) of the \( k \)-th diagrams is:

\[
\prod_{a=p_{t+1}}^{x-1} \left( \frac{v_k q^{-(\mu_{p_{t+1}} - y + x - a)} - v_j q^{(\mu_{p_{t+1}} - y + x - a)}}{v_k q^{-(\mu_{p_{t+1}} - y + x - a + 1)} - v_j q^{(\mu_{p_{t+1}} - y + x - a + 1)}} \right) q^{-cc(\gamma^{(m)})}.
\]

The contribution from the \( m \)-nodes lying in the line \( x \) of the \( j \)-th diagrams is:

\[
\prod_{b=t+1}^{s} \prod_{a=\mu_{p_{b-1}} + 1}^{\mu_{p_b}} \frac{v_j q^{-(y - a + p_b - x)} - v_k q^{(y - a + p_b - x)}}{v_j q^{-(y - a + p_{b-1} - x)} - v_k q^{(y - a + p_{b-1} - x)}}.
\]

After straightforward simplifications, we obtain for the right hand side of (36)

\[
x^y \prod_{b=1}^{s} \left( v_k q^{-(\mu_{p_b} - y + x - p_b)} - v_j q^{(\mu_{p_b} - y + x - p_b)} \right) \prod_{b=1}^{s+1} \left( v_k q^{-(\mu_{p_{b-1}} - y + x - p_{b-1})} - v_j q^{(\mu_{p_{b-1}} - y + x - p_{b-1})} \right)^{-1}.
\]

(37)

The comparison of (36) and (37) concludes the proof of the formula (35).

3. The assertion of the Lemma is a consequence of the formulas (31), (33) together with the part 1 of the proof.

\( \square \)

**Proposition 5.** The rational function \( F_T(u) \) is non-singular at \( u = c_{n+1} \), and moreover

\[
F_T(c_{n+1}) = F_{\lambda^{(m)}}^{-1} F_{\mu^{(m)}}^{-1}.
\]

(38)

**Proof.** The formula (29) shows that the rational function \( F_T(u) \) is non-singular at \( u = c_{n+1} \), and moreover

\[
F_T(c_{n+1}) = \prod_{\beta^{(m)} \in \mathcal{E}_-^{(\mu^{(m)})}} \left( c_{n+1} - c(\beta^{(m)}) \right) \prod_{\alpha^{(m)} \in \mathcal{E}_+^{(\mu^{(m)})}} \left( c_{n+1} - c(\alpha^{(m)}) \right)^{-1}.
\]

(39)

We use the Lemma 4 to conclude the proof of the Proposition.

\( \square \)
The rational function $a_i = \frac{\sigma_2(v_1q^2, v_2)\sigma_1(v_1q^2, v_1)\tau(v_1q^2)\sigma_1^{-1}(v_2, v_1)\tau(v_2)\sigma_1^{-1}\tau(v_1)}{(q + q^{-1})(v_1q^{-1} - v_2q)(v_1 - v_2)(v_2q^{-2} - v_1q^2)}$.

5. Remarks on the classical limit

Recall that the group ring $\mathbb{C}G(m, 1, n+1)$ of the complex reflection group $G(m, 1, n+1)$ is obtained by taking the classical limit: $q \to 1$ and $v_i \to \xi_i$, $i = 1, \ldots, m$, where $\{\xi_1, \ldots, \xi_m\}$ is the set of distinct $m$-th roots of unity. The “classical limit” of the generators $\tau$, $\sigma_1$, $\ldots$, $\sigma_n$ of $H(m, 1, n+1)$ we denote by $t$, $s_1$, $\ldots$, $s_n$.

1. Consider the Baxterized elements (41) with spectral parameters of the form $\alpha = v_\rho q^{2a}$ and $\alpha' = v_{\rho'} q^{2a'}$ with $p, p' \in \{1, \ldots, m\}$. One directly finds that

$$\lim_{q \to 1} \lim_{v_i \to \xi_i} \sigma_i(\alpha, \alpha') = s_i + \frac{\delta_{p,p'}}{a - a'} .$$

For the Artin generators $\bar{s}_1, \ldots, \bar{s}_n$ of the symmetric group $S_{n+1}$, the standard Baxterized form is:

$$\bar{s}_i(a, a') := \bar{s}_i + 1 \frac{1}{a - a'} \text{ for } i = 1, \ldots, n .$$

In view of (40), we define generalized Baxterized elements for the group $G(m, 1, n+1)$ as follows:

$$s_i(p, p', a, a') := s_i + \frac{\delta_{p, p'}}{a - a'} \text{ for } i = 1, \ldots, n .$$

These elements satisfy the following Yang–Baxter equation with spectral parameters:

$$s_i(p, p', a, a')s_{i+1}(p, p'', a, a'')s_i(p', p'', a', a'') = s_{i+1}(p', p'', a', a'')s_i(p, p'', a, a'')s_i(p, p', a, a') .$$

The Baxterized elements (41) have been used in [10] for a fusion procedure for the complex reflection group $G(m, 1, n+1)$.

2. It is immediate that

$$\lim_{v_i \to \xi_i} a_0(\rho) = \rho^m - 1 \quad \text{and} \quad \lim_{v_i \to \xi_i} a_i(\rho) = \rho^{m-i} \text{ for } i = 1, \ldots, m ,$$

where $a_i(\rho)$, $i = 0, \ldots, m$, are defined in (7). It follows from (41) that

$$\lim_{v_i \to \xi_i} \tau(\rho) = \sum_{i=0}^{m-1} \rho^{m-1-it} .$$

The rational function $t(\rho) := \frac{1}{m} \sum_{i=0}^{m-1} \rho^{m-1-it}$ with values in $\mathbb{C}G(m, 1, n+1)$ was used in [10] for a fusion procedure for the complex reflection group $G(m, 1, n+1)$.
3. Define, for an $m$-partition $\lambda^{(m)}$,

$$f_{\lambda^{(m)}} := \left( \prod_{\lambda^{(m)} \in \lambda^{(m)}} h_{\lambda^{(m)}}(\alpha^{(m)}) \right)^{-1}.$$  \hspace{1cm} (43)

The classical limit of $F_{\lambda^{(m)}}$ is proportional to $f_{\lambda^{(m)}}$. More precisely, we have

$$\lim_{q \to 1} \lim_{v_i \to \xi_i} F_{\lambda^{(m)}} = r_{\lambda^{(m)}} f_{\lambda^{(m)}}, \quad \text{where } r_{\lambda^{(m)}} = \frac{1}{m^n} \prod_{\lambda^{(m)} \in \lambda^{(m)}} p(\alpha^{(m)}). \hspace{1cm} (44)$$

The formula (44) is obtained directly from (12) since

$$\prod_{i=1}^{m} (\xi_k - \xi_i) = m/\xi_k \text{ for } k = 1, \ldots, m.$$  \hspace{1cm} (45)

4. Using formulas (40), (42) and (44), it is straightforward to check that the classical limit of the fusion procedure for $H(m, 1, n + 1)$ given by Theorem 1 leads to the fusion procedure [10] for the group $G(m, 1, n + 1)$. Also, for $m = 1$ one reobtains the fusion procedure [5] for the Hecke algebra and, in the classical limit, the fusion procedure [8] for the symmetric group.

References

[1] Ariki S. and Koike K., A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and construction of its irreducible representations, Adv. in Math. 106 (1994) 216–243.

[2] Broué M. and Malle G., Zyklotomische Heckealgebren, Asterisque 212 (1993) 119–189.

[3] Cherednik I. V., A new interpretation of Gelfand-Tzetlin bases, Duke Math. J. 54 (1987) 563–577.

[4] Chlouveraki M. and Jacon N., Schur elements for the Ariki-Koike algebra and applications, J. of Algebr. Comb. 35, 2 (2012) 291–311. ArXiv: 1105.5910

[5] Isaev A., Molev A. and Os’kin A., On the idempotents of Hecke algebras, Lett. Math. Phys. 85 (2008) 79–90. ArXiv: 0804.4214

[6] Isaev A., Molev A. and Ogievetsky O., Idempotents for Birman-Murakami-Wenzl algebras and reflection equation. arXiv:1111.2502

[7] Isaev A. P. and Ogievetsky O. V., On Baxterized solutions of reflection equation and integrable chain models, Nucl. Phys. B 760 [PM] (2007) 167–183. ArXiv: math-ph/0510078

[8] Molev A., On the fusion procedure for the symmetric group, Reports Math. Phys. 61 (2008), 181–188.

[9] Ogievetsky O. and Pouliain d’Andecy L., On representations of cyclotomic Hecke algebras, Mod. Phys. Lett. A 26 No. 11 (2011) 795-803. arXiv:1012.5844

[10] Ogievetsky O. and Pouliain d’Andecy L., Fusion formula for Coxeter groups of type B and complex reflection groups $G(m, 1, n)$. ArXiv: 1111.6293