BETTI CATEGORIES OF GRADED MODULES AND APPLICATIONS TO MONOMIAL IDEALS AND TORIC RINGS

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ABSTRACT. We introduce the notion of Betti category for graded modules over suitably graded polynomial rings, and more generally for modules over certain small categories. Our categorical approach allows us to treat simultaneously many important cases, such as monomial ideals and toric rings. We prove that in these cases the Betti category is a finite combinatorial object that completely determines the structure of the minimal free resolution. For monomial ideals, the Betti category is the same as the Betti poset that we studied in [TV15]. We describe in detail and with examples how the theory applies to the toric case, and provide an analog for toric rings of the lcm-lattice for monomial ideals.

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1. Introduction

Toric rings are the affine coordinate rings of toric varieties. They have many applications to various areas of mathematics and a rich underlying combinatorial structure, and have long been the subject of extensive research in geometry, combinatorics, and algebra; see for example [Ful93, BG09, MS05, Pee11, Vil15] and the references there. An important open problem regarding the homological properties of toric rings is understanding the structure of their minimal free resolutions. Some of the most successful approaches to this problem (see e.g. [BS98, PS98a, PS98b]) are closely related to methods used to attack the analogous open problem of understanding the structure of minimal free resolutions of monomial ideals. However, regardless of the apparent similarities, these two open problems have always been...
treated separately, and there are important results on the monomial side that do not have toric counterparts yet. For example, the lcm-lattice and the Betti poset of a monomial ideal are shown in [GPW99] and [TV15], respectively, to be discrete combinatorial objects whose isomorphism classes completely determine (in an implicit but natural way) the structure of the minimal free resolution of the monomial ideal. In this paper, we introduce toric analogs of these objects, and prove that in a similar way they determine the minimal free resolutions of toric rings.

We achieve this by establishing a unified approach to the study of minimal free resolutions of monomial, toric, and, more generally, homogeneous ideals in suitably graded (not necessarily commutative) polynomial rings. All of these are seen to be special cases of minimal projective resolutions of modules over certain small categories $C$ where all Endomorphisms are Isomorphisms (hence called EI categories), which seem to provide the correct framework where graded syzygies over polynomial rings should be studied.

In general, given a commutative ring $k$ and a small category $C$, a $kC$-module is just a functor from $C$ to the category of $k$-modules, and a homomorphism of $kC$-modules is just a natural transformation of such functors. Modules over categories have been studied extensively in algebraic topology, in particular to investigate equivariant phenomena; see e.g. [D87, Section I.11; Lic89 Section 9; LRRV15, Section 13]. More recently they have found applications to the study of stability phenomena in representation theory [CEF15, CEFN14], and were first brought into the study of minimal free resolutions of monomial ideals by the authors in [TV15]. We review the foundations of the theory of modules over categories in Sections 2 through 5.

We expand our investigations from [TV15] by studying the properties of minimal projective resolutions of modules over a small EI category $C$. We describe a version of Nakayama’s Lemma in this context (Lemma 6.2), and establish sufficient conditions for the existence of minimal projective or free resolutions in Theorem 6.5 and Corollary 6.6.

In Section 7 we introduce the new notion of the Betti category $B(M)$ of a $kC$-module $M$. This is a full subcategory of $C$, and we show in our first main result, Theorem 8.1, that it is in a certain sense the “smallest” discrete combinatorial object that retains all essential homological information about $M$. When $M$ corresponds to a monomial ideal, the Betti category is exactly the Betti poset, and we recover all results from [TV15]. For a general graded module over a polynomial ring, the Betti category has as objects the degrees of the basis elements in the minimal graded free resolution of the module, and as morphisms the monomials of appropriate degrees; composition of morphisms is multiplication of monomials. In particular, in many important cases such as monomial ideals and toric rings, the Betti category is finite and can be readily computed without needing to compute minimal free resolutions.

As a main application, in Section 9 we analyze in greater detail the special case of toric rings over a field $k$. More specifically, let $Q$ be a submonoid of the free abelian monoid $\mathbb{N}^r$, with $M = \{a_1, \ldots, a_n\}$ the minimal generating set for $Q$. The monoid ring $T = k[Q]$ is called a toric ring. We consider $T$ as a $\mathbb{Z}^r$-graded module over the polynomial ring $R = k[x_1, \ldots, x_n]$, via the homomorphism of polynomial rings $\varphi: k[x_1, \ldots, x_n] \to k[\mathbb{N}^r]$ with image $T$ that sends $x_i$ to $a_i$, and where the $\mathbb{Z}^r$-grading of $R$ is given by $\deg x_i = a_i$. Let $B(T)$ be the Betti category of $T$. Then:
• We show in Theorem 9.2 that the bar resolution of the constant functor on $\mathcal{B}(T)$ lifts to a canonical (non minimal) finite free $\mathbb{Z}$-graded resolution $F_\bullet(T)$ of $T$ over $R$. This resolution differs from the hull resolution of $T$ constructed in [BS98].

• We show in Theorem 9.5 that the equivalence class of the Betti category $\mathcal{B}(T)$ determines completely the structure of the minimal free resolution of $T$ over $R$, in the following sense: if $T'$ is another toric ring over $k$ whose Betti category $\mathcal{B}(T')$ is equivalent to $\mathcal{B}(T)$, then the minimal free resolution of $T$ is obtained from the minimal free resolution of $T'$ in a functorial way. In particular, this shows that the Betti category is the correct toric analogue of the Betti poset of a monomial ideal.

• We introduce the lub-category $\mathcal{L}(Q)$, and we show in Theorem 9.9 that this is a toric analogue of the lcm-lattice of a monomial ideal.

We also provide an example of two well-known non-isomorphic toric rings with equivalent (in fact isomorphic) Betti categories. This raises the interesting question of characterizing those categories that are Betti categories of toric rings over a given field $k$. In the case of monomial ideals, the analogous question was answered in [TV15, Theorem 6.4 on page 5126].

2. Modules over categories

In this and the following two sections, we review the definitions of modules over a category, free modules, tensor products, and induction and restriction functors. Our presentation is self-contained, but for additional details and examples the reader may consult [TV15] or [Lüc89, Section 9]. We also describe the equivalence between graded modules and modules over the associated action category in Lemma 2.5, which is crucial to our approach, and the functorial bar resolution in Definition 3.4.

Fix a commutative, associative, and unital ring $k$, and denote by $k\text{-}\text{Mod}$ the corresponding abelian category of modules over $k$ and $k$-linear homomorphisms. Let $\mathcal{C}$ be a small category. The category $k\mathcal{C}\text{-}\text{Mod}$ of modules over $\mathcal{C}$ is defined as the functor category $\text{Fun}(\mathcal{C}, k\text{-}\text{Mod})$. Explicitly: a $k\mathcal{C}$-module is a functor from $\mathcal{C}$ to $k\text{-}\text{Mod}$; a homomorphism of $k\mathcal{C}$-modules is a natural transformation of such functors. Notice that, if $M$ is a $k\mathcal{C}$-module and $c$ and $d$ are objects of $\mathcal{C}$, then there is a $k$-linear “evaluation” homomorphism

\[
\mathbb{k}[\text{mor}_\mathcal{C}(c,d)] \otimes M(c) \longrightarrow M(d),
\]

which we call a structure map of $M$, whose adjoint is obtained by linear extension of the function $\text{mor}_\mathcal{C}(c,d) \longrightarrow \text{hom}_k(M(c), M(d))$ expressing the functoriality of $M$.

The category $k\mathcal{C}\text{-}\text{Mod}$ is an abelian category, with kernels and images computed object-wise. A sequence of $k\mathcal{C}$-modules $L \longrightarrow M \rightarrow N$ is exact if and only if $L(c) \longrightarrow M(c) \rightarrow N(c)$ is exact for each $c \in \text{obj}\mathcal{C}$.

Example 2.2 (the constant module). The constant $k\mathcal{C}$-module is the functor $\text{const}_k: \mathcal{C} \longrightarrow k\text{-}\text{Mod}$

given by $\text{const}_k(c) = k$ and $\text{const}_k(u) = \text{id}_k$ for all objects $c$ and all morphisms $u$ of the category $\mathcal{C}$.

In order to study graded modules, we are led to consider the following categories.
Definition 2.3 (action categories). Given a monoid \( \Lambda \) and a set \( S \) equipped with a left action of \( \Lambda \), the action category \( \Lambda S \) is the category with
\[
\text{obj } \Lambda S = S \quad \text{and} \quad \text{mor}_{\Lambda S}(s,t) = \{ \lambda \in \Lambda \mid \lambda s = t \},
\]
and with composition given by multiplication in \( \Lambda \).

Example 2.4. Given any monoid \( \Lambda \) we can consider the action category \( \Lambda \) of \( \Lambda \) acting on itself. More generally, if \( f: \Lambda \rightarrow \Gamma \) is a homomorphism of monoids, then \( \Lambda \) acts on \( \Gamma \) via \( f \), and we can form the action category \( \Lambda \).

Now assume that \( \Gamma \) is a group. Then \( \text{mor}_{\Lambda \Gamma}(\gamma, \gamma') = f^{-1}(\{\gamma' \gamma^{-1}\}) \). Moreover, if we also assume that \( f \) is injective, then \( \Lambda \) is just the preorder \( (\Gamma, \leq) \), considered as a category, where \( \gamma \leq \gamma' \) if and only if \( \gamma' = f(\lambda) \gamma \) for some (unique) \( \lambda \in \Lambda \).

This relation is a partial order (i.e., \((\Gamma, \leq)\) is a poset) if and only if there are no nontrivial invertible elements in \( \Lambda \).

The next result follows at once from the definitions.

Lemma 2.5 (graded modules as functors over action categories). Let \( \Lambda \) be a monoid, and let \( \deg: \Lambda \rightarrow \Gamma \) be a homomorphism of monoids. Consider the monoid ring \( k[\Lambda] \) with the induced \( \Gamma \)-grading. Then the functor
\[
k(\Lambda \Gamma) \rightarrow \text{-} \text{Mod} \longrightarrow \Gamma \text{-} \text{gr} k[\Lambda] \text{-} \text{Mod}, \quad M \mapsto \bigoplus_{\gamma \in \Gamma} M(\gamma)
\]
from the category of modules over the action category \( \Lambda \) to the category of \( \Gamma \)-graded \( k[\Lambda] \)-modules and \( \Gamma \)-graded homomorphisms is an equivalence of abelian categories.

Example 2.6 (multigraded modules over polynomial rings). Consider the monoid \( \Lambda = \mathbb{N}^m \) and the inclusion homomorphism \( \deg = \text{incl}: \mathbb{N}^m \rightarrow \mathbb{Z}^m \), where \( m \geq 1 \) is an integer. Notice that in this case the action category \( \mathbb{N}^m / \mathbb{Z}^m \) is just the poset \( \mathbb{Z}^m \), viewed as a category, with respect to the usual component-wise partial order: \((a_1, \ldots, a_m) \leq (b_1, \ldots, b_m)\) if and only if \( a_i \leq b_i \) for each \( i \); compare Example 2.4. Moreover, identifying the elements \((a_1, \ldots, a_m) \) of \( \mathbb{N}^m \) with the monomials \( x_1^{a_1} \cdots x_m^{a_m} \), we see that \( \mathbb{Z}^m \)-graded \( k[\mathbb{N}^m] \)-modules are nothing but multigraded modules over the polynomial ring \( k[x_1, \ldots, x_m] \), and Lemma 2.5 is the well-known observation that these are equivalent to functors \( \mathbb{Z}^m \rightarrow k \text{-} \text{Mod} \).

Example 2.7 (graded modules over polynomial rings). Consider \( \Lambda = \mathbb{N}^m \), where \( m \geq 1 \) is an integer, and the standard degree homomorphism \( \deg: \mathbb{N}^m \rightarrow \mathbb{Z} \) given by \((a_1, \ldots, a_m) \mapsto a_1 + \cdots + a_m \). In this case the action category \( \mathbb{N}^m / \mathbb{Z} \) is not a poset: its objects are the integers, the morphisms from \( k \) to \( l \) can be identified with the set of all monomials of degree \( l - k \) in the polynomial ring \( k[x_1, \ldots, x_m] \), and composition of morphisms is just multiplication of monomials.

3. Free modules

Here we recall the notion of free \( kC \)-modules. Any \( kC \)-module \( M \) has an underlying collection of sets \( M(c) \) indexed by the objects \( c \in \text{obj } C \), and this data can be thought of as a functor to the category of sets from the discrete category \( \text{obj } C \) (i.e., the subcategory of \( C \) whose only morphisms are the identities). We call such functors \( \text{obj } C \)-sets, and we obtain the forgetful functor
\[
U: kC \rightarrow \text{Sets}, \quad (\text{obj } C \rightarrow kC \rightarrow \text{Sets})
\]
This forgetful functor $U$ has a left adjoint

$$F: \text{obj}(C)\text{-Sets} \rightarrow kC\text{-Mod},$$

which is defined by sending an $(\text{obj} C)$-set $B$ to the $kC$-module

$$FB = \bigoplus_{c \in \text{obj} C} \bigoplus_{B(c)} k(\text{mor}_C(c, -)).$$

The unit of the adjunction is the natural transformation $\eta: B \rightarrow UF B$ that for each $c \in \text{obj} C$ sends $b \in B(c)$ to $id_c$ in the direct summand $k(\text{mor}_C(c, c) \leq FB(c)$ indexed by $b$.

**Definition 3.1.** We say that the $kC$-module $FB$ is free with basis $\eta$: $B \rightarrow UF B$, and we define a $kC$-module to be free if it is isomorphic to one in the image of the functor $F$.

It is a standard exercise to verify that free $kC$-modules are projective, and that a $kC$-module is projective if and only if it is a direct summand of a free $kC$-module.

**Example 3.2** (free module on one generator). (a) Let $c$ be an object of $C$, and let $B$ be the $\text{obj} C$-set with $B(d) = \emptyset$ for $d \neq c$ and with $B(c) = pt$. Let $\eta: B \rightarrow k(\text{mor}_C(c, -))$ be given by $\eta(pt) = id_c$. This makes the $kC$-module $k(\text{mor}_C(c, -))$ free, and we call it a free module on one generator (based at $c$).

(b) Under the equivalence of Lemma 2.5, a free $k(\Lambda/\Gamma)$-module on one generator based at $c \in \Gamma$ corresponds to a free $\Gamma$-graded $k[\Lambda]$-module on one homogeneous generator of degree $c$.

**Definition 3.3.** Given a small category $C$, we denote by $\text{cl}(\text{obj} C)$ the set of isomorphism classes of objects of $C$, and by $\text{cl}: \text{obj} C \rightarrow \text{cl}(\text{obj} C)$ the function that sends each object to its isomorphism class. We say that an $(\text{obj} C)$-set $B$ is of finite type if

$$\# \{ \text{cl}(c) \in \text{cl}(\text{obj} C) \mid B(c) \neq \emptyset \} < \infty;$$

we say that $B$ is finite if it is of finite type and moreover $\#B(c) < \infty$ for each $c \in \text{obj} C$. We say that a $kC$-module $M$ is finitely generated (or of finite type) if $M$ is a quotient of a free module with a finite (or of finite type, respectively) basis.

**Definition 3.4** (unnormalized bar resolution). Let $M$ be a $kC$-module.

(a) For each $n \geq 0$, define a $kC$-module $F_n^u$ by setting

$$F_n^u(-) = \bigoplus_{c_0 \rightarrow \cdots \rightarrow c_n} k(\text{mor}_C(c_0, -)) \otimes M(c_0),$$

where the direct sum is indexed over all $n$-tuples of composable morphisms of $C$

$$c_0 \xrightarrow{u_1} c_1 \rightarrow \cdots \rightarrow c_{n-1} \xrightarrow{u_n} c_n,$$

i.e., all $n$-simplices of the nerve of $C$. When $n = 0$ this reduces to

$$F_0^u(-) = \bigoplus_{c_0 \in \text{obj} C} k(\text{mor}_C(c_0, -)) \otimes M(c_0),$$

and the structure maps $\text{[2.1]}$ of $M$ yield a $kC$-module homomorphism $\varepsilon: F_0^u \rightarrow M$.

(b) For each $n > 0$ and $0 \leq i \leq n$, define morphisms

$$\partial_i^n: F_n^u \rightarrow F_{n-1}$$

as follows:
We then let 

\[ \gamma_{-1}: M(c) \rightarrow F_0(c) \quad \text{and} \quad \gamma_n: F_n(c) \rightarrow F_{n+1}(c) \]

\[ \text{Proposition 3.7.} \quad \text{The complexes } B_u^\bullet(M) \text{ and } B^\bullet(M) \text{ are resolutions of } M. \]

\[ \text{Proof.} \quad \text{By [Wei94] 8.3.6, 8.3.8 on pages 265–266 it is enough to show that } B_u^\bullet(M) \text{ is a resolution of } M. \text{ In order to see this, notice that we can define extra maps} \]

\[ \text{Remark 3.5.} \quad (a) \text{ The verification that } B_u^\bullet(M) \text{ is a chain complex is straightforward. In fact, } B_u^\bullet(M) \text{ is the associated (unnormalized) chain complex of the semi-simplicial } k\mathcal{C}\text{-module defined by the maps } \partial_n^u \text{ above; compare for example [Wei94] 8.1.9, 8.1.6, and 8.2.1 on pages 258–260].} \]

\[ (b) \text{ There are also degeneracy maps } \sigma_n^u: F_n \rightarrow F_{n+1}, \text{ defined by inserting identities in the obvious ways, making } B_u^\bullet(M) \text{ into a simplicial } k\mathcal{C}\text{-module.} \]

\[ \text{Definition 3.6 (normalized bar resolution). Let } M \text{ be a } k\mathcal{C}\text{-module. The degenerate chain complex } D_\bullet(M) \text{ is the subcomplex of } B_u^\bullet(M) \text{ generated by the degeneracy maps } \sigma_n^u \text{ from Remark 3.5(b). Thus in homological degree } n \text{ we have} \]

\[ D_n(M) = \sum_i \sigma_{n-1}^i(F_{n-1}). \]

\[ \text{We define the (normalized) bar complex } B^\bullet(M) \text{ as the quotient chain complex} \]

\[ B^\bullet(M) = B_u^\bullet(M)/D_u(M). \]

\[ \text{In particular, the component } F_n \text{ of } B^\bullet(M) \text{ in homological degree } n \text{ is the } k\mathcal{C}\text{-module} \]

\[ F_n(-) = \bigoplus_{c_0 \rightarrow \cdots \rightarrow c_n} k[\text{mor}_\mathcal{C}(c_n, -)] \otimes M(c_0), \]

\[ \text{where the direct sum is indexed over all } n\text{-tuples of composable non-identity morphisms of } \mathcal{C}, \text{i.e., all non-degenerate } n\text{-simplices of the nerve of } \mathcal{C}. \]
for each \( c \in \text{obj} \mathcal{C} \) and \( n \geq 0 \) as follows. The map \( \gamma_{-1} \) is induced by the map

\[
M(c) \rightarrow k[\text{mor}_\mathcal{C}(c,c)] \otimes M(c), \quad m \mapsto \text{id}_c \otimes m.
\]

To define \( \gamma_n \), given any \( c_0 \rightarrow \cdots \rightarrow c_n \) we need to construct a map

\[
k[\text{mor}_\mathcal{C}(c_n,c)] \otimes M(c_0) \rightarrow F_{n+1}(c) = \bigoplus_{d_0 \rightarrow \cdots \rightarrow d_{n+1}} k[\text{mor}_\mathcal{C}(d_{n+1},c)] \otimes M(d_0),
\]

or, equivalently, a function

\[
\text{mor}_\mathcal{C}(c_n,c) \rightarrow \text{hom}_k \left( M(c_0), \bigoplus_{d_0 \rightarrow \cdots \rightarrow d_{n+1}} k[\text{mor}_\mathcal{C}(d_{n+1},c)] \otimes M(d_0) \right).
\]

To define this latter function, we send \( u: c_n \rightarrow c \) to the homomorphism

\[
M(c_0) \rightarrow k[\text{mor}_\mathcal{C}(c,c)] \otimes M(c_0), \quad m \mapsto \text{id}_c \otimes m
\]

in the summand indexed by \( c_0 \rightarrow \cdots \rightarrow c_n \rightarrow u \rightarrow c \). It is easy to check that \( \varepsilon \gamma_{-1} = \text{id}, \partial_n^0 \gamma_0 = \gamma_{-1} \varepsilon \), and, for all \( n > 0 \) and all \( 0 \leq i \leq n \), \( \partial_n^i \gamma_{n-1} = \text{id} \) and \( \partial_{n+1}^i \gamma_n = \gamma_{n-1} \partial_n^i \).

Therefore the augmented simplicial \( k \)-module \( B_*(M)(c) \rightarrow M(c) \) is (right) contractible in the sense of [Wei94, page 275], and so the associated augmented chain complex is a resolution [Wei94, 8.4.6.1 on page 275].

**Remark 3.8.** (a) The maps \( \gamma_n \) are not natural in \( c \), i.e., they do not define homomorphisms of \( k \mathcal{C} \)-modules.

(b) Observe that we can view

\[
F_n(-) = \bigoplus_{c_n \in \text{obj} \mathcal{C}} \bigoplus_{c_0 \rightarrow \cdots \rightarrow c_n} k[\text{mor}_\mathcal{C}(c_n,-)] \otimes M(c_0),
\]

where the first direct sum is indexed over all objects \( c_n \) of \( \mathcal{C} \), and the second one is indexed over all \( n \)-tuples of composable non-identity morphisms of \( \mathcal{C} \) that end in \( c_n \). This shows that, if \( M \) is object-wise free, i.e., if the \( k \)-module \( M(c) \) is free over \( k \) for each \( c \in \text{obj} \mathcal{C} \), then each \( F_n \) is a free \( k \mathcal{C} \)-module, and so \( B_n(M) \) is a free resolution of \( M \). Obviously, this condition on \( M \) is automatically satisfied when \( k \) is a field.

4. Tensor products, induction and restriction

Now we recall the definition of tensor products of modules over categories, and then use them to study induction and restriction along functors.

Given a \( k \mathcal{C}^{\text{op}} \)-module \( N \) and a \( k \mathcal{C} \)-module \( M \), their **tensor product over** \( k \mathcal{C} \) is the \( k \)-module \( N \otimes_{k \mathcal{C}} M \) defined as the quotient

\[
\bigoplus_{c \in \text{obj} \mathcal{C}} N(c) \otimes M(c) / \left\langle nu \otimes m - n \otimes um \mid n \in N(d), m \in M(c), u \in \text{mor}_\mathcal{C}(c,d) \right\rangle,
\]

where, to underscore the analogy with the tensor product of modules over rings, we use the shorthands \( um \) and \( nu \) for the values of the homomorphisms \( M(u) \) and \( N(u) \) at the elements \( m \) and \( n \) respectively; i.e., \( um = M(u)(m) \) and \( nu = N(u)(n) \). More conceptually, \( N \otimes_{k \mathcal{C}} M \) is the coend of the functor

\[
\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow k \text{-Mod}, \quad (d,c) \mapsto N(d) \otimes M(c).
\]

Notice that if \( D \) is another small category, \( N \) is a \( k(\mathcal{C}^{\text{op}} \times D) \)-module, and \( M \) is a \( k \mathcal{C} \)-module, then \( N \otimes_{k \mathcal{C}} M \) becomes a \( kD \)-module. We think of and refer to
\(\mathbb{k}(\mathcal{C}^{\text{op}} \times \mathcal{D})\)-modules as \(\mathbb{k}\mathcal{D}\text{-}\mathbb{k}\mathcal{C}\text{-bimodules}\). Dually, given a \(\mathbb{k}\mathcal{D}\)-module \(L\), we get a \(\mathbb{k}\mathcal{C}\)-module \(\text{hom}_{\mathbb{k}\mathcal{D}}(N, L)\) of \(\mathbb{k}\mathcal{D}\)-homomorphisms; notice that \(\text{hom}_{\mathbb{k}\mathcal{D}}(N, L)\) is covariant in \(\mathcal{C}\) because \(N\) is contravariant in \(\mathcal{C}\). It is easy to see that the functors

\[
N \otimes_{\mathbb{k}\mathcal{C}} - : \mathbb{k}\mathcal{C}\text{-Mod} \longrightarrow \mathbb{k}\mathcal{D}\text{-Mod} \quad \text{and} \quad \text{hom}_{\mathbb{k}\mathcal{D}}(N, -) : \mathbb{k}\mathcal{D}\text{-Mod} \longrightarrow \mathbb{k}\mathcal{C}\text{-Mod}
\]

are adjoint, i.e., for all \(\mathbb{k}\mathcal{C}\)-modules \(M\) and all \(\mathbb{k}\mathcal{D}\)-modules \(L\), there are natural isomorphisms

\[
\text{hom}_{\mathbb{k}\mathcal{D}}(N \otimes_{\mathbb{k}\mathcal{C}} M, L) \cong \text{hom}_{\mathbb{k}\mathcal{C}}(M, \text{hom}_{\mathbb{k}\mathcal{D}}(N, L)).
\]

It follows that that \(N \otimes_{\mathbb{k}\mathcal{C}} -\) is right exact and \(\text{hom}_{\mathbb{k}\mathcal{D}}(N, -)\) is left exact.

**Definition 4.1.** Given a functor \(\alpha : \mathcal{C} \longrightarrow \mathcal{D}\), we define the **tautological** \(\mathbb{k}\mathcal{D}\text{-}\mathbb{k}\mathcal{C}\text{-bimodule} \mathbb{k}\mathcal{D}_\alpha\) as follows:

\[
\mathbb{k}\mathcal{D}_\alpha : \mathcal{C}^{\text{op}} \times \mathcal{D} \longrightarrow \mathcal{D}\text{-Mod}, \quad (c, d) \longmapsto \mathbb{k}[\text{mor}_D(\alpha(c), d)].
\]

When \(\alpha\) is the identity functor \(\text{id}_\mathcal{C} : \mathcal{C} \longrightarrow \mathcal{C}\) we omit the subscript and write simply \(\mathbb{k}\mathcal{C}\) for the tautological bimodule \(\mathbb{k}\mathcal{C}_{\text{id}_\mathcal{C}}\).

Precomposition with \(\alpha : \mathcal{C} \longrightarrow \mathcal{D}\) defines a functor

\[
\text{res}_\alpha : \mathbb{k}\mathcal{D}\text{-Mod} \longrightarrow \mathbb{k}\mathcal{C}\text{-Mod}, \quad L \longmapsto L \circ \alpha,
\]

which we call **restriction along** \(\alpha\), and which is obviously exact. Notice that, for each \(\mathbb{k}\mathcal{D}\)-module \(L\) and each \(c \in \text{obj} \mathcal{C}\), we have

\[
\text{hom}_{\mathbb{k}\mathcal{D}}(\mathbb{k}\mathcal{D}_\alpha, L)(c) = \text{hom}_{\mathbb{k}\mathcal{D}}(\mathbb{k}[\text{mor}_D(\alpha(c), -)], L(-)) \cong L(\alpha(c))
\]

by Yoneda’s Lemma, and so

\[
\text{res}_\alpha \cong \text{hom}_{\mathbb{k}\mathcal{D}}(\mathbb{k}\mathcal{D}_\alpha, -).
\]

Therefore the functor

\[
\text{ind}_\alpha = \mathbb{k}\mathcal{D}_\alpha \otimes_{\mathbb{k}\mathcal{C}} - : \mathbb{k}\mathcal{C}\text{-Mod} \longrightarrow \mathbb{k}\mathcal{D}\text{-Mod},
\]

which we call **induction along** \(\alpha\), is left adjoint to \(\text{res}_\alpha\), and hence \(\text{ind}_\alpha\) is right exact. More conceptually, \(\text{ind}_\alpha\) is the left Kan extension along \(\alpha\). Being left adjoint to an exact functor, induction preserves projective modules. Induction also preserves free modules: for any fixed \(c_0 \in \text{obj} \mathcal{C}\), and for each \(d \in \text{obj} \mathcal{D}\), we have

\[
\text{ind}_\alpha(\mathbb{k}[\text{mor}_C(c_0, -)])(d) = \mathbb{k}[\text{mor}_D(\alpha(-), d)] \otimes_{\mathbb{k}\mathcal{C}} \mathbb{k}[\text{mor}_C(c_0, -)] \cong \mathbb{k}[\text{mor}_D(\alpha(c_0), d)],
\]

where the last isomorphism is a consequence of Yoneda’s Lemma. Since induction is right exact, induction also preserves finitely generated modules. On the other hand, restriction does not preserve free nor projective nor finitely generated modules in general.

**Remark 4.2.** An equivalence of categories \(\alpha : \mathcal{C} \longrightarrow \mathcal{D}\) induces an equivalence of module categories \(\text{res}_\alpha : \mathbb{k}\mathcal{D}\text{-Mod} \longrightarrow \mathbb{k}\mathcal{C}\text{-Mod}\). More explicitly, if \(\beta : \mathcal{D} \longrightarrow \mathcal{C}\) is a functor such that the compositions \(\beta\alpha\) and \(\alpha\beta\) are naturally isomorphic to the respective identities, then the same is true for \(\text{res}_\beta\text{res}_\alpha = \text{res}_{\beta\alpha}\) and \(\text{res}_\alpha\text{res}_\beta = \text{res}_{\alpha\beta}\). Moreover, the uniqueness of adjoints implies that there are natural isomorphisms

\[
\text{ind}_\alpha \cong \text{res}_\beta \text{res}_\alpha \quad \text{and} \quad \text{ind}_\beta \cong \text{res}_\alpha \text{res}_\beta.
\]
5. EI categories, splitting functor, and supports

An EI category is a category in which all Endomorphisms are Isomorphisms. Notice that, in any EI category, if there exists an isomorphism $c \to d$, then every morphism $c \to d$ is an isomorphism.

Given a small category $C$, we denote by $\text{cl}(\text{obj } C)$ the set of isomorphism classes of objects of $C$, and by $\text{cl}: \text{obj } C \to \text{cl}(\text{obj } C)$ the function that sends each object to its isomorphism class. Define a preorder on $\text{cl}(\text{obj } C)$ by setting $\text{cl}(c) \leq \text{cl}(d)$ if and only if there exists a morphism $c \to d$. If $C$ is an EI category, then the relation $\leq$ is also antisymmetric, and so $(\text{cl}(\text{obj } C), \leq)$ is a poset; in that case $\text{cl}(c) < \text{cl}(d)$ if and only if there exists a non-isomorphism $c \to d$. When $C$ is EI we write $C_{\leq e}$ for the full subcategory of $C$ with objects all $d$ such that $\text{cl}(d) \leq \text{cl}(e)$; the category $C_{< e}$ is defined analogously.

We recall now some standard notions from order theory. Let $(P, \leq)$ be a preorder, e.g., the set of isomorphism classes of objects in a small category. A subset $A \subseteq P$ is called Artinian if every descending chain in $A$ stabilizes. Equivalently, $A$ is Artinian if and only if every non-empty subset of $A$ has a minimal element. Obviously, the property of being Artinian passes to subsets, and it is straightforward to see that an Artinian subset is a poset. A subset $A \subseteq P$ is called an upper set if $a \in A$ and $a \leq b$ imply $b \in A$. The smallest upper set containing a subset $S \subseteq P$ is denoted $\uparrow S$ and called the upper set generated by $S$. We say that an upper set $A \subseteq P$ is finitely generated if there exists a finite set $S \subseteq A$ such that $\uparrow S = A$.

We denote by $\text{iso} C$ the subcategory of $C$ with the same objects but with only the isomorphisms of $C$ as morphisms. In other words, $\text{iso} C$ is the maximal groupoid inside $C$. We write $\iota$ for the inclusion functor $\iota: \text{iso } C \to C$. Given an object $c \in \text{obj } C$, let $\text{aut}_C(c)$ denote the group of automorphisms of $c$ in $C$, and let $k[c] = k[\text{aut}_C(c)]$ be the corresponding group algebra. If $C$ is an EI category, then $\text{aut}_C(c) = \text{mor}_C(c, c) = \text{mor}_{\text{iso } C}(c, c)$. We remark that, by choosing a representative for each isomorphism class of objects in $C$, we obtain an equivalence of categories

$$\text{kiso} C\text{-Mod} \simeq \prod_{\text{cl}(c) \in \text{cl}(\text{obj } C)} k[c]\text{-Mod}.$$  

Definition 5.1. We denote by

$$\text{ind}: \text{kiso} C\text{-Mod} \to kC\text{-Mod}$$

and

$$\text{res}: kC\text{-Mod} \to \text{kiso} C\text{-Mod}$$

the adjoint induction and restriction functors along the inclusion $\iota: \text{iso } C \to C$.

Remark 5.2. (a) Let $Q$ be a $k\text{iso } C$-module. For each $c, e \in \text{obj } C$ the group $\text{aut}_C(c)$ acts on $k[\text{mor}_C(c, e)] \otimes Q(e)$, where $u \in \text{aut}_C(c)$ sends $v \otimes m$ to $vu^{-1} \otimes Q(u)m$. A routine computation using the definitions shows that

$$\text{ind } Q(e) \cong \bigoplus_{\text{cl}(c) \in \text{cl}(\text{obj } C)} \left( k[\text{mor}_C(c, e)] \otimes Q(e) \right) / \text{aut}_C(c),$$

and for $w \in \text{mor}_C(e, f)$ the morphism $\text{ind } Q(w): \text{ind } Q(e) \to \text{ind } Q(f)$ is induced by composition with $w$. Furthermore, the unit of adjunction

$$\eta_Q(e): Q(e) \to \text{res } \text{ind } Q(e)$$

corresponds to the inclusion $Q(e) \to \text{id}_e \otimes Q(e) \subseteq \left( k[\text{mor}_C(c, e)] \otimes Q(e) \right) / \text{aut}_C(e)$.

(b) Let $M$ be a $kC$-module. The counit of the adjunction

$$\varepsilon_M(e): \text{ind } \text{res } M(e) \to M(e)$$
is the morphism
\[
\bigoplus_{c \in \text{cl}(\text{obj } C)} \left( \mathbb{k}[\text{mor}_C(c, e)] \otimes M(c) \right) / \text{aut}_C(c) \to M(e)
\]
induced by the structure maps (2.1) of $M$.

**Definition 5.3** (splitting functor). Given a $\mathbb{k}C$-module $M$, for each $c \in \text{obj } C$ let

\[
\mathfrak{J}M(c) = \text{image} \left( \bigoplus_{b \to c \text{ non-iso}} M(b) \xrightarrow{\sum_{M(u)}} M(c) \right),
\]
and define $\mathfrak{S}M(c) = M(c)/\mathfrak{J}M(c)$. Given any isomorphism $c \to d$, the composition $b \to c \to d$ is an isomorphism if and only if $u$ is an isomorphism. Therefore we get two functors $\mathfrak{J}, \mathfrak{S}: \text{iso } C \to \mathbb{k}\text{-Mod}$, hence also a functor

\[
\mathfrak{J}: \mathbb{k}C\text{-Mod} \to \mathbb{k}\text{iso } C\text{-Mod},
\]
which is a category-theoretic analogue of the ring-theoretic notion of Jacobson radical, and a functor

\[
\mathfrak{S}: \mathbb{k}C\text{-Mod} \to \mathbb{k}\text{iso } C\text{-Mod},
\]
which is called the splitting functor.

**Remark 5.4.** (a) It is straightforward from the definition that both $\mathfrak{J}$ and $\mathfrak{S}$ preserve epimorphisms and direct sums.

(b) By definition, for each $\mathbb{k}C$-module $M$ there is a natural short exact sequence

\[
0 \to \mathfrak{J}M \xrightarrow{\mathfrak{J}M} \text{res } M \xrightarrow{PM} \mathfrak{S}M \to 0
\]
of $\mathbb{k}\text{iso } C$-modules. Since res is exact and $\mathfrak{J}$ and $\mathfrak{S}$ preserve epimorphisms, a routine diagram chase shows that $\mathfrak{S}$ is a right exact functor.

(c) Let $Q$ be a $\mathbb{k}\text{iso } C$-module. A direct consequence from the observations in Remark 5.2 and the definitions is that

\[
\mathfrak{S}\text{ind } Q(e) = \begin{cases} 0 & \text{if } \text{id}_e = uw \text{ for some non-isomorphisms } u, w \in \text{mor}_C(e, e); \\ Q(e) & \text{otherwise.} \end{cases}
\]
In particular $\mathfrak{S}\text{ind } Q$ is a quotient of $Q$, and it is immediate that the composition

\[
Q \xrightarrow{\eta_Q} \text{res } Q \xrightarrow{\text{Pind } Q} \mathfrak{S}\text{ind } Q
\]
is precisely the natural quotient epimorphism $\tau_Q: Q \to \mathfrak{S}\text{ind } Q$.

(d) Let $M$ be a $\mathbb{k}C$-module. Then $\mathfrak{S}M(e) = 0$ whenever $\text{id}_e = uw$ for some non-isomorphisms $u, w \in \text{mor}_C(e, e)$. Furthermore, we have

\[
\mathfrak{S}(\varepsilon_M) \circ \tau_{\text{res } M} = PM.
\]
Indeed, due to the naturality of the morphism $\eta$ and the basic properties of the unit and counit of adjunction, we have $\mathfrak{S}(\varepsilon_M) \circ \tau_{\text{res } M} = \mathfrak{S}(\varepsilon_M) \circ \text{Pind } \text{res } M \circ \eta_{\text{res } M} = PM \circ \text{res } (\varepsilon_M) \circ \eta_{\text{res } M} = PM$.
**Remark 5.5.** Let \( C \) be a small EI category.

(a) We have an equivalent description of the splitting functor as follows. Define a \( \text{kisoC}-\text{kC} \)-bimodule \( B \) by
\[
B : C^{\text{op}} \times \text{isoC} \to \text{k-mod,}
\]
\[
(c, d) \mapsto k[\text{mor}_{\text{isoC}}(c, d)].
\]
Since
\[
k[\text{mor}_{\text{isoC}}(c, d)] = \begin{cases} k[\text{mor}_{C}(c, d)] & \text{if } c \cong d, \\ 0 & \text{if } c \not\cong d, \end{cases}
\]
we obtain a natural isomorphism
\[
\mathcal{S}M \cong B \otimes_{\text{kC}} M.
\]
From this description we see that when \( C \) is EI the functor \( \mathcal{S} \) sends free, projective, and finitely generated \( \text{kC} \)-modules to \( \text{isoC} \)-modules with the same properties.

(b) For each \( \text{isoC} \)-module \( Q \), the natural epimorphism \( \tau_Q : Q \to \mathcal{S}\text{ind} Q \) from Remark 5.4(c) is given by the composition
\[
(5.6) \quad Q \cong \text{isoC} \otimes_{\text{kisoC}} Q \cong \left( B \otimes_{\text{kC}} \text{isoC} \right) \otimes_{\text{isoC}} Q \cong B \otimes_{\text{kC}} \left( \text{isoC} \otimes_{\text{kisoC}} Q \right) \cong \mathcal{S}\text{ind} Q,
\]
in particular it is an isomorphism.

We conclude this section by introducing several notions of support that we will need later.

**Definition 5.7** (support). The **support** of an obj\( C \)-set \( B \) is the following subset of \( \text{cl}(\text{objC}) \):
\[
\text{supp}(B) = \{ \text{cl}(c) \in \text{cl}(\text{objC}) \mid B(c) \neq \emptyset \}.
\]
The **support** and the **minimal support** of a \( \text{kC} \)-module \( M \) are the following subsets of \( \text{cl}(\text{objC}) \):
\[
\text{supp}(M) = \{ \text{cl}(c) \in \text{cl}(\text{objC}) \mid M(c) \neq 0 \},
\]
\[
\text{minsupp}(M) = \{ \text{cl}(c) \in \text{cl}(\text{objC}) \mid \mathcal{S}M(c) \neq 0 \}.
\]
If \( C_* \) is a chain complex of \( \text{kC} \)-modules, we let
\[
\text{minsupp}(C_*) = \bigcup_{n \in \mathbb{Z}} \text{minsupp}(C_n).
\]

**Remark 5.8.** Let \( M \) be a \( \text{kC} \)-module.

(a) We have \( \text{minsupp}(M) \subseteq \text{supp}(M) \), and \( \text{cl}(e) \notin \text{minsupp}(M) \) whenever \( \text{id}_e = uw \) for some non-isomorphisms \( u, w \in \text{mor}_{C}(e, e) \).

(b) If \( 0 \to L \to M \to N \to 0 \) is a short exact sequence of \( \text{kC} \)-modules, then \( \text{supp}(L) \cup \text{supp}(N) = \text{supp}(M) \), and, since \( \mathcal{S} \) is right exact,
\[
\text{minsupp}(N) \subseteq \text{minsupp}(M) \subseteq \text{minsupp}(N) \cup \text{minsupp}(L).
\]
However, in general \( \text{minsupp}(L) \not\subseteq \text{minsupp}(M) \).

(c) If \( C \) is EI, then any minimal element of \( \text{supp}(M) \) is contained in \( \text{minsupp}(M) \) (compare the proof of Nakayama’s Lemma 6.2). If in addition \( \text{supp}(M) \) is Artinian, then we also have \( \uparrow \text{minsupp}(M) = \uparrow \text{supp}(M) \).

**Example 5.9.** If \( F \) is a free \( \text{kC} \)-module with basis \( B \), then \( \text{minsupp}(F) \subseteq \text{supp}(B) \) and \( \text{supp}(F) = \uparrow \text{supp}(B) \). If \( C \) is EI, then also \( \text{minsupp}(F) = \text{supp}(B) \) and \( \text{supp}(F) = \uparrow \text{minsupp}(F) \).
6. Minimal projective resolutions

We begin by recalling the definitions of projective covers and minimal projective resolutions in an arbitrary abelian category $\mathcal{A}$. An epimorphism $f: L \to M$ is called \textit{superfluous} if, for any morphism $g: K \to L$, we have that $g$ is epic if and only if $fg$ is epic. A \textit{projective cover} of an object $M$ of $\mathcal{A}$ is a projective object $P$ together with a superfluous epimorphism $f: P \to M$. A \textit{minimal projective resolution} of $M$ is a resolution

$$
\varepsilon: P_\bullet \to M = \cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M
$$

such that $\varepsilon: P_0 \to M$ and $d_n: P_n \to \text{im}(d_n) = \ker(d_{n-1})$ are projective covers for each $n \geq 1$.

It is well known and easy to see that the definitions imply that projective covers and minimal projective resolutions are unique up to isomorphism, if they exist. Moreover, if $f: P \to M$ is a projective cover and $g: Q \to M$ is an epimorphism with $Q$ projective, then $Q \cong P \oplus P'$ for some $P' \leq \ker g$; if $\varepsilon: P_\bullet \to M$ is a minimal projective resolution and $\delta: Q_\bullet \to M$ is any other projective resolution, then $Q_\bullet \cong P_\bullet \oplus P'_\bullet$ for some contractible subcomplex $P'_\bullet \leq Q_\bullet$.

An abelian category $\mathcal{A}$ is called \textit{perfect} if every object has a projective cover (and therefore also a minimal projective resolution). Now suppose that $\mathcal{A} = \k \mathcal{C}\text{-Mod}$, so that we have a notion of finitely generated objects. Then $\mathcal{A}$ is called \textit{semi-perfect} if every finitely generated object has a projective cover. We say that $\mathcal{A}$ is \textit{Noetherian} if every subobject of a finitely generated object is again finitely generated. Notice that, if $\mathcal{A}$ is semi-perfect and Noetherian, then every finitely generated object has a minimal projective resolution. In the special case where $\mathcal{A} = \k \mathcal{C}\text{-Mod}$, then $\mathcal{A}$ is perfect, semi-perfect, or Noetherian if and only if the ring $\k$ has the same property.

\textbf{Example 6.1.} If $\mathcal{C}$ is a small category, then the category

$$
\k \text{iso}\mathcal{C}\text{-Mod} \simeq \prod_{c \in \text{cl(obj}\mathcal{C})} \k[c]\text{-Mod}
$$

is (semi-)perfect if and only if, for each object $c$ of $\mathcal{C}$, the group ring $\k[c]$ is (semi-)perfect. For example, if $\k$ is a field and all automorphism groups in $\mathcal{C}$ are finite, then $\k \text{iso}\mathcal{C}\text{-Mod}$ is semi-perfect.

\textbf{Nakayama’s Lemma 6.2.} Let $\mathcal{C}$ be a small EI category, and let $M$ be a $\k \mathcal{C}$-module. If $\text{supp}(M)$ has a minimal element, then $\mathcal{S}M \neq 0$.

\textit{Proof.} This is obvious: if $\text{cl}(c) \in \text{supp}(M)$ is minimal, then there are no non-isomorphisms $b \to c$ with $M(b) \neq 0$, and so by definition $\mathcal{S}M(c) = M(c) \neq 0$. \ 

\textbf{Corollary 6.3.} Let $\mathcal{C}$ be a small EI category, and let $f: L \to M$ be a homomorphism of $\k \mathcal{C}$-modules.

1. Assume that $\text{supp}(M)$ is Artinian. If $\mathcal{S}f$ is an epimorphism, then so is $f$.
2. Assume that $\text{supp}(L)$ is Artinian and that $f$ is an epimorphism. If $\mathcal{S}f$ is a superfluous epimorphism, then so is $f$.

\textit{Proof.} [1] Consider coker $f$, and notice that $\text{supp}(\text{coker} f) \subseteq \text{supp}(M)$. If $f$ is not epic, i.e., if $\text{supp}(\text{coker} f)$ is not empty, then it has a minimal element, and \textbf{Nakayama’s Lemma 6.2} implies that $\mathcal{S}\text{coker} f \neq 0$. Since $\mathcal{S}$ is right exact, we have that $\mathcal{S}\text{coker} f \cong \text{coker} \mathcal{S}f$ and can conclude that $\mathcal{S}f$ is not epic.
Given a homomorphism \( g: K \rightarrow L \) such that \( fg \) is epic, we get that \( S (fg) = \mathcal{S}f \mathcal{S}g \) is epic, since \( \mathcal{S} \) is right exact, and therefore \( \mathcal{S}g \) is epic since \( \mathcal{S}f \) is superfluous. Now apply part (1) again to conclude that \( g \) is epic. \( \square \)

**Lemma 6.4.** Let \( \mathcal{C} \) be a small EI category, and let \( M \) be a \( \mathbf{k} \mathcal{C} \)-module.

1. Assume that \( \text{supp}(M) \) is Artinian. Then \( M \) is a quotient of a free \( \mathbf{k} \mathcal{C} \)-module \( P \) with \( \text{minsupp}(P) = \text{minsupp}(M) \).

2. Assume that the upper set generated by \( \text{supp}(M) \) in \( \text{cl(obj } \mathcal{C} \) is Artinian, and assume that \( \mathcal{S}M \) has a projective cover in \( \mathbf{kisoC-Mod} \). Then \( M \) has projective cover \( P \rightarrow M \) with \( \text{minsupp}(P) = \text{minsupp}(M) \). If in addition projective \( \mathbf{kisoC} \)-modules are free, then the projective cover \( P \) of \( M \) is free.

**Proof.** We first describe a general construction. Let \( q: Q \rightarrow \mathcal{S}M \) be a homomorphism of \( \mathbf{kisoC} \)-modules, and assume that \( Q \) is projective. As in Section 5, write \( \text{ind} \) and \( \text{res} \) for the induction and restriction functors along the inclusion of categories \( \mathbf{isoC} \rightarrow \mathcal{C} \). Since \( Q \) is projective, \( q \) factors through the quotient \( p_M: \text{res} M \rightarrow \mathcal{S}M \) as follows:

\[
\begin{array}{ccc}
Q & \xrightarrow{q} & \text{res} M \\
\downarrow \text{ind} & & \downarrow p_M \\
\text{ind} Q & \xrightarrow{\text{ind} \tau} & \text{ind} \mathcal{S}M \\
\varepsilon & \searrow & \varepsilon M \\
& & M
\end{array}
\]

where \( \varepsilon \) is the counit of the adjunction and we define \( \varepsilon = \varepsilon_M \circ \text{ind} \tau \). We claim that \( q \) is equal to the composition

\[
Q \xrightarrow{\tau_Q} \mathcal{S} \text{ind} Q \xrightarrow{\mathcal{S} \varepsilon} \mathcal{S}M,
\]

where \( \tau \) is the natural isomorphism from \( \text{5.6} \). To see that this is true, use the naturality of \( \tau \) and the fact that the composition

\[
\text{res} M \xrightarrow{\tau_M} \mathcal{S} \text{ind} M \xrightarrow{\mathcal{S} \varepsilon_M} \mathcal{S}M
\]

is equal to \( p_M \); see \( \text{Remark 5.4(d)} \).

Since \( \tau_Q \) is an isomorphism and \( q = \mathcal{S} \varepsilon \circ \tau_Q \), we see that \( \mathcal{S} \varepsilon \) is a (superfluous) epimorphism if and only if \( q \) is a (superfluous) epimorphism.

Now, in order to prove (1), choose a free \( \mathbf{kisoC} \)-module \( Q \) together with an epimorphism \( q: Q \rightarrow \mathcal{S}M \) such that \( Q(c) \neq 0 \) if and only if \( \mathcal{S}M(c) \neq 0 \). Then \( \text{ind} Q \) is free and \( \text{minsupp}(\text{ind} Q) = \text{minsupp}(M) \) by construction. The assumption on \( \text{supp}(M) \) allows us to apply Corollary 6.3(1) to conclude that \( \varepsilon_M: \text{ind} Q \rightarrow M \) is an epimorphism.

For (2), let \( q: Q \rightarrow \mathcal{S}M \) be a projective cover. Then \( \text{ind} Q \) is projective, and \( \text{supp}(\text{ind} Q) = \uparrow \text{minsupp}(M) = \uparrow \text{supp}(M) \). The assumption on \( \text{supp}(M) \) allows us to apply Corollary 6.3(2) to conclude that \( q \) is a projective cover. \( \square \)

**Theorem 6.5** (Existence of projective covers and minimal projective resolutions). Let \( \mathcal{C} \) be a small EI category, and let \( M \) be a \( \mathbf{k} \mathcal{C} \)-module. Assume that the upper set generated by \( \text{supp}(M) \) in \( \text{cl(obj } \mathcal{C} \) is Artinian.
(1) If the rings \( k[c] \) are perfect for all \( \text{cl}(c) \in \uparrow \text{minsupp}(M) \), then \( M \) has a minimal projective resolution. If in addition all projective \( k[c] \)-modules are free for all \( \text{cl}(c) \in \uparrow \text{minsupp}(M) \) then the minimal projective resolution of \( M \) is a free resolution.

(2) If the rings \( k[c] \) are semi-perfect for all \( \text{cl}(c) \in \uparrow \text{minsupp}(M) \), and if \( M \) is finitely generated, then \( M \) has a projective cover. If additionally the category \( kC\text{-Mod} \) is Noetherian, then \( M \) has a minimal projective resolution.

Proof. (1) is easily proved by applying Lemma 6.4 repeatedly. The process can be iterated, because, using the notation of the proof of Lemma 6.4, we have that \( \text{supp}(\ker \tilde{q}) \subseteq \text{supp}(\text{ind} Q) = \uparrow \text{supp}(M) \) and hence \( \uparrow \text{supp}(\ker \tilde{q}) \subseteq \uparrow \text{supp}(M) \).

In order to see that the same strategy also works for (2), recall that the functor \( \mathcal{S} \) sends finitely generated \( kC \)-modules to finitely generated \( k\text{iso}C \)-modules. So the assumptions in (2) imply that \( \mathcal{S} M \) has a finitely generated projective cover \( p: P \rightarrow \mathcal{S} M \). Therefore \( \text{ind} P \) is finitely generated and, if \( kC\text{-Mod} \) is Noetherian, then \( \ker \tilde{p} \) is finitely generated, too.

Now the following result is an immediate consequence of Theorem 6.5

**Corollary 6.6.** Let \( C \) be a small EI category. Assume that every finitely generated upper set in \( \text{cl}(\text{obj} C) \) is Artinian.

(1) If the category \( k\text{iso}C\text{-Mod} \) is perfect, then any \( kC \)-module of finite type has a minimal projective resolution.

(2) If the category \( k\text{iso}C\text{-Mod} \) is semi-perfect, then also \( kC\text{-Mod} \) is semi-perfect. If additionally \( kC\text{-Mod} \) is Noetherian, then any finitely generated \( kC \)-module has a minimal projective resolution.

If in addition all projective \( k\text{iso}C \)-modules are free, then the minimal projective resolutions from parts (1) and (2) above are also free.

As a corollary of the proof of Corollary 6.6 we also have the following

**Corollary 6.7.** Let \( C \) be a small EI category, and let \( M \) be a \( kC \)-module. Assume that the upper set generated by \( \text{supp}(M) \) in \( \text{cl}(\text{obj} C) \) is Artinian. Assume that the rings \( k[c] \) are semisimple for all \( \text{cl}(c) \in \uparrow \text{minsupp}(M) \), and let \( P_\bullet \) be a minimal projective resolution of \( M \).

Then \( \mathcal{S} P_\bullet \) is a chain complex with zero differential.

Proof. As shown in the proof of Theorem 6.5, the resolution \( P_\bullet \) is constructed at stage \( n+1 \) by applying \( \text{ind} \) to a projective cover of \( \mathcal{S} \ker(P_n \rightarrow P_{n-1}) \). Because of the semisimplicity assumption \( \mathcal{S} \ker(P_n \rightarrow P_{n-1}) \) is already projective, therefore for each \( n \) we have that \( \mathcal{S}(P_{n+1} \rightarrow \ker(P_n \rightarrow P_{n-1})) \) is an isomorphism. The desired conclusion is now immediate from the right exactness of \( \mathcal{S} \).

7. **Betti categories**

**Definition 7.1** (Betti category and Betti numbers). Let \( C \) be a small EI category. Suppose that a \( kC \)-module \( M \) has a minimal projective resolution \( \varepsilon: P_\bullet \rightarrow M \).

(a) Define the **Betti category** of \( M \) to be the full subcategory \( B(M) \) of \( C \) with set of objects

\[
\text{obj } B(M) = \{ c \in \text{obj } C \mid \text{cl}(c) \in \text{minsupp}(P_\bullet) \}.
\]

(b) Suppose in addition the minimal projective resolution \( P_\bullet \) is a free resolution. Then, for each object \( c \in \text{obj } C \) and for each \( n \in \mathbb{N} \), the \( k[c] \)-module \( \mathcal{S} P_n(c) \) is free.
and has a well defined rank, which is equal to $\#B(c)$, where $B$ is a basis of $P_n$. We set

$$\beta_{n,c}(M) = \text{rank}_{k[c]} \mathcal{P}_n(c) = \#B(c),$$

and call it the $n$-th Betti number of $M$ at $c$.

Notice that the Betti category of $M$ and the Betti numbers of $M$ do not depend on $P_\bullet$, because minimal projective resolutions are unique up to isomorphism.

**Example 7.2** (Betti category and Betti numbers of the constant module). Let $C$ be an EI category where all isomorphisms are identities, and the poset $\text{cl}(\text{obj} C)$ is Artinian and finitely generated. Let $k$ be a field and let $\text{const}_k$ be the constant $kC$-module. Thus $\text{supp}(\text{const}_k) = \text{cl}(\text{obj} C)$ is generated by its finitely many minimal elements, hence $\text{const}_k$ is a finitely generated $kC$-module. Since in $\text{isoC-Mod}$ all modules are free, Corollary 6.6 shows that $\text{const}_k$ has a minimal free resolution, hence a well defined Betti category and well defined Betti numbers. Since the normalized bar resolution $B^\bullet(\text{const}_k)$ is a free resolution of $\text{const}_k$, it is a direct sum of the minimal free resolution $P_\bullet$ and a contractible complex. Furthermore, $\mathcal{P}_\bullet$ has zero differential by Corollary 6.7. It follows that for each object $c$ we have

$$\beta_{n,c}(\text{const}_k) = \dim_k H_n(\mathcal{P}_\bullet(\text{const}_k)(c)).$$

Since composition of non-identity morphisms in $C$ is not an identity, the collection $\mathcal{NC}'$ of the non-degenerate simplices of the nerve of $C'$ is a simplicial subcomplex of that nerve for any subcategory $C'$ of $C$. In particular this is true for the categories $C_{\leq c}$ and $C_{< c}$ for all objects $c$ of $C$. Therefore it is straightforward from the definitions that

$$\beta_{n,c}(\text{const}_k) = \dim_k H_n(\mathcal{NC}_{\leq c}, \mathcal{NC}_{< c}; k),$$

the $n$th relative homology of the pair $(\mathcal{NC}_{\leq c}, \mathcal{NC}_{< c})$ with coefficients in $k$. In particular, $c$ is an object of the Betti category of $\text{const}_k$ if and only if the pair $(\mathcal{NC}_{\leq c}, \mathcal{NC}_{< c})$ has nontrivial relative homology over $k$.

**Example 7.3.** Let $k$ be a field. In the polynomial ring $R = k[a,b,c,d]$ with the usual $\mathbb{Z}$-grading consider the homogeneous ideal

$$I = \langle ac - b^2, bc - ad, bd - c^2 \rangle,$$

the defining ideal of the twisted cubic curve in $\mathbb{P}^3$. The $\mathbb{Z}$-graded minimal free resolution of $R/I$ over $R$ is

$$0 \rightarrow R^2 \xrightarrow{(\begin{array}{cc} d & c \\
 & b \\
a & d \end{array})} R^3 \xrightarrow{(\begin{array}{ccc} ac-b^2 & bc-ad & bd-c^2 \\
 & n & m \end{array})} R \rightarrow R/I \rightarrow 0.$$

The degrees of the basis elements of the free modules in this resolution are 0, 2, and 3, and thus the Betti category $B$ of the $\mathbb{Z}$-graded $R$-module $R/I$, considered as a functor from the action category $\mathbb{N}^4/\mathbb{Z}$ into the category of $k$-vector spaces, is the full subcategory of $\mathbb{N}^4/\mathbb{Z}$ with set of objects $\{0,2,3\}$. Recall from Example 2.7 that morphisms in $\mathbb{N}^4/\mathbb{Z}$ are identified with monomials, in particular we have $\text{mor}_{B}(0,2) = \text{mor}_{\mathbb{N}^4/\mathbb{Z}}(0,2) = \{a^2, ab, ac, ad, b^2, bc, bd, c^2, cd, d^2\}$, and $\text{mor}_{B}(2,3) = \text{mor}_{\mathbb{N}^4/\mathbb{Z}}(2,3) = \{a, b, c, d\}$. The morphisms from 0 to 3 are given by all monomials of degree 3.

**Example 7.4.** In the polynomial ring $R = k[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}]$ with the standard $\mathbb{Z}$-grading consider the ideal

$$I = \langle x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{32} - x_{12}x_{31}, x_{21}x_{32} - x_{22}x_{31} \rangle.$$
generated by the \((2 \times 2)\)-minors of the generic \((3 \times 2)\)-matrix

\[
X = \begin{pmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22} \\
x_{31} & x_{32}
\end{pmatrix}.
\]

The \(\mathbb{Z}\)-graded minimal free resolution of \(R/I\) over \(R\) is

\[
0 \rightarrow R^2 \xrightarrow{X} R^3 \xrightarrow{(x_{21}x_{32} - x_{22}x_{31} \quad x_{12}x_{31} - x_{11}x_{32} \quad x_{11}x_{22} - x_{12}x_{21})} R \rightarrow R/I \rightarrow 0.
\]

The degrees of the basis elements of the free modules in this resolution are 0, 2, and 3, and thus the Betti category \(B\) of the \(\mathbb{Z}\)-graded \(R\)-module \(R/I\), considered as a functor from the action category \(\mathbb{N}^0/\mathbb{Z}\) into the category of \(k\)-vector spaces, is the full subcategory of \(\mathbb{N}^0/\mathbb{Z}\) with set of objects \(\{0, 2, 3\}\). Since morphisms in \(\mathbb{N}^0/\mathbb{Z}\) are identified with monomials in \(R\), we have for example that \(\text{mor}_0(2, 3) = \{x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}\}\). In particular, this Betti category is not equivalent to the Betti category from Example 7.3.

**Remark 7.5.** The quotient rings from the previous two examples will appear again as instances of toric rings in Section 9; see Examples 9.1 and 9.6. There the gradings will be different, and the corresponding Betti categories will turn out to be equivalent, and in fact even isomorphic.

### 8. Main results

We are now ready to state and prove the general version of our main result, which we specialize to toric rings in the next section. This result essentially says that the Betti category of a \(kC\)-module \(M\) is in a certain sense the “smallest” discrete combinatorial object that captures the structure of the minimal projective resolution of \(M\).

**Theorem 8.1.** Let \(C\) and \(D\) be small EI categories. Assume that:

- \(\varepsilon: P_\bullet \rightarrow M\) is a minimal projective resolution of a \(kC\)-module \(M\);
- \(\delta: Q_\bullet \rightarrow N\) is a minimal projective resolution of a \(kD\)-module \(N\);
- the upper sets \(\uparrow \text{supp}(M)\) and \(\uparrow \text{supp}(N)\) are Artinian;
- \(C'\) is a full subcategory of \(C\) such that \(\minsupp(P_\bullet) \subseteq \text{cl}(\text{obj } C')\);
- \(D'\) is a full subcategory of \(D\) such that \(\minsupp(Q_\bullet) \subseteq \text{cl}(\text{obj } D')\);
- there exist an equivalence of categories \(\alpha: C' \rightarrow D\) and an isomorphism \(f: \text{res}_\varepsilon(M) \xrightarrow{\cong} \text{res}_{\alpha\varepsilon}(N)\)

of \(kC'\)-modules, where \(C' \xrightarrow{\varepsilon} C\) and \(D' \xrightarrow{\delta} D\) are the inclusion functors.

Then \(\alpha\) induces an equivalence of Betti categories \(B(M) \rightarrow B(N)\), and \(f\) induces an isomorphism \(P_\bullet \cong \text{ind}_\varepsilon \text{res}_{\alpha\varepsilon}(Q_\bullet)\)

of chain complexes of \(kC\)-modules.

**Remark 8.2.** An important special case of this theorem is when \(C' = B(M)\) and \(D' = B(N)\), which satisfy by definition the assumptions on the minimal supports. Moreover, in many interesting cases (e.g., for monomial ideals and toric rings) the restriction of the corresponding modules to the Betti categories are the constant functors with value the ground field \(k\). In these cases, of course, the existence of the isomorphism \(f\) follows automatically from the existence of the equivalence of categories \(\alpha\).
Remark 8.3. Theorem 8.1 generalizes (and reprovos) [TV15] Theorem 5.3 on page 5124.

For the proof of this theorem, we need the following lemma and corollary.

Lemma 8.4. Let \( C \) be a small EI category and let \( P \) be a projective \( \mathbb{k}C \)-module with \( \text{supp}(P) \) Artinian. Let \( C' \) be a full subcategory of \( C \) such that \( \text{minsupp}(P) \subseteq \text{cl}(\text{obj } C') \), and let \( \varrho: C' \to C \) be the inclusion functor.

Then \( \text{res}_\varrho P \) is a projective \( \mathbb{k}C' \)-module, and the counit of the adjunction is an isomorphism \( \text{ind}_\varrho \text{res}_\varrho P \cong P \).

Proof. If \( P \) is free with basis \( B \), then \( \text{minsupp}(P) = \{ \text{cl}(c) \in \text{cl}(\text{obj } C) \mid B(c) \neq \emptyset \} \); compare Example 5.9. So, if this set is contained in \( \text{cl}(\text{obj } C') \), it is clear that \( \text{res}_\varrho P \) is a free \( \mathbb{k}C' \)-module and \( \text{ind}_\varrho \text{res}_\varrho P \cong P \).

If \( P \) is projective and \( \text{supp}(P) \) is Artinian, then \( P \) is a direct summand of a free module \( F \) with \( \text{minsupp}(F) = \text{minsupp}(P) \) by Lemma 6.4. Since induction and restriction preserve direct sums, the statements follow. \( \square \)

Corollary 8.5. Let \( C \) be a small EI category. Let \( \varepsilon: P_\bullet \to M \) be a minimal projective resolution of a \( \mathbb{k}C \)-module \( M \) with \( \uparrow \text{supp}(M) \) Artinian. Let \( C' \) be a full subcategory of \( C \) such that \( \text{minsupp}(P_\bullet) \subseteq \text{cl}(\text{obj } C') \), and let \( \varrho: C' \to C \) be the inclusion functor. Then:

1. \( \text{res}_\varrho P_\bullet \) is a projective resolution of \( \text{res}_\varrho M \), and the counit of the adjunction gives isomorphisms \( \text{ind}_\varrho \text{res}_\varrho M \cong M \) and \( \text{ind}_\varrho \text{res}_\varrho P_\bullet \cong P_\bullet \).
2. If \( \varepsilon': P'_\bullet \to \text{res}_\varrho M \) is a projective resolution in \( \mathbb{k}C' \)-Mod, then \( \text{ind}_\varrho P'_\bullet \) is a projective resolution of \( M \) in \( \mathbb{k}C \)-Mod.

Proof. 1. Since restriction is an exact functor, \( \text{res}_\varrho \varepsilon: \text{res}_\varrho P_\bullet \to \text{res}_\varrho M \) is a resolution in \( \mathbb{k}C' \)-Mod. Moreover, notice that \( \text{supp}(P_n) \subseteq \uparrow \text{supp}(M) \) for each \( n \geq 0 \), since by Lemma 6.3 the \( \mathbb{k}C \)-module \( M \) has a free resolution with the same property. Therefore, the assumptions that \( \uparrow \text{supp}(M) \) is Artinian and \( \text{supp}(P_n) \subseteq \text{cl}(\text{obj } C') \) for each \( n \geq 0 \) allow us to apply Lemma 8.4 and conclude that \( \text{res}_\varrho P_n \) is a projective \( \mathbb{k}C' \)-module, and so \( \text{res}_\varrho \varepsilon \) is a projective resolution. Moreover, the counit of the adjunction is an isomorphism \( P_\bullet \xrightarrow{\cong} \text{ind}_\varrho \text{res}_\varrho P_\bullet \), and hence also \( M \cong \text{ind}_\varrho \text{res}_\varrho M \).

2. By part 1, \( \text{res}_\varrho \varepsilon: \text{res}_\varrho P_\bullet \to \text{res}_\varrho M \) is a projective resolution in \( \mathbb{k}C' \)-Mod, hence is chain homotopy equivalent to \( \varepsilon': P'_\bullet \to \text{res}_\varrho M \). Therefore \( \text{ind}_\varrho P'_\bullet \) is chain homotopy equivalent to \( \text{ind}_\varrho \text{res}_\varrho P_\bullet \cong P_\bullet \). Since induction sends projectives to projectives, the result follows. \( \square \)

Proof of Theorem 8.1. Let \( \varrho: C' \to C \) and \( \zeta: D' \to D \) be the inclusion functors. By Corollary 8.5, \( \text{res}_\varrho \varepsilon: \text{res}_\varrho P_\bullet \to \text{res}_\varrho M \) is a projective resolution in \( \mathbb{k}C' \)-Mod, and we have isomorphisms \( P_\bullet \xrightarrow{\cong} \text{ind}_\varrho \text{res}_\varrho P_\bullet \) and \( M \cong \text{ind}_\varrho \text{res}_\varrho M \).

The same applies to \( \delta: Q_\bullet \to N \) and, since \( \text{res}_\alpha \) is an equivalence of abelian categories, we get that \( \text{res}_\alpha \delta: \text{res}_\alpha Q_\bullet \to \text{res}_\alpha N \) is a projective resolution in \( \mathbb{k}C' \)-Mod. The isomorphism \( f: \text{res}_\varrho M \to \text{res}_\alpha N \) therefore lifts to a homotopy equivalence \( f_\varrho: \text{res}_\varrho P_\bullet \to \text{res}_\alpha Q_\bullet \), and so using induction we get a homotopy equivalence \( \text{ind}_\varrho f_\varrho: \text{ind}_\varrho \text{res}_\varrho P_\bullet \to \text{ind}_\varrho \text{res}_\alpha Q_\bullet \).

Since the equivalence of categories \( \alpha \) induces a bijection \( \text{cl}(\text{obj } C') \cong \text{cl}(\text{obj } D') \), Corollary 8.5 yields that \( \text{ind}_\varrho \text{res}_\alpha Q_\bullet \) is another projective resolution of \( M \).
But $P_\bullet$ is a minimal projective resolution of $M$, and so $\text{ind}_g f_\bullet$ induces an isomorphism of chain complexes $P_\bullet \oplus P'_\bullet \cong \text{res}_{\alpha} Q_\bullet$ for some contractible subcomplex $P'_\bullet \leq \text{ind}_g \text{res}_{\alpha} Q_\bullet$. It follows that $Q_\bullet \cong \text{ind}_{\alpha} \text{res}_g \text{ind}_g \text{res}_\alpha Q_\bullet \cong \text{ind}_{\alpha} \text{res}_g P_\bullet \oplus \text{ind}_g \text{res}_\alpha Q_\bullet$ and notice that $\text{ind}_g \text{res}_g P'_\bullet$ is again contractible. Since $Q_\bullet$ is a minimal projective resolution of $N$, we conclude that $\text{ind}_{\alpha} \text{res}_g P'_\bullet \cong 0$, which forces $P'_\bullet \cong 0$, thus $P_\bullet \cong \text{ind}_g \text{res}_{\alpha} Q_\bullet$.

The claim that $\alpha$ induces an equivalence between $B(M)$ and $B(N)$ is now clear since we have $\minsupp(P_\bullet) = \minsupp(\text{ind}_g \text{res}_{\alpha} Q_\bullet) = \minsupp(\text{res}_{\alpha} Q_\bullet) \cong \minsupp(Q_\bullet)$.

\end{proof}

9. Toric rings

We now apply the theory developed so far to study homological properties of semigroup rings of the form $k[Q]$, where $k$ is a field and $Q$ is a pointed affine semigroup. These rings are called toric rings because they arise, for example, as affine coordinate rings of toric varieties, and have been studied extensively in geometry, combinatorics, and algebra; e.g., see [Ful93, BG09, MS05, Pee11, Vill15] and the references there.

Recall that a finitely generated abelian semigroup $Q$ is affine if it is isomorphic to a subsemigroup of $\mathbb{Z}^r$ for some positive integer $r$. The smallest such $r$ is called the dimension of $Q$. The affine semigroup $Q$ is pointed if the only invertible element is the identity. An $r$-dimensional affine pointed semigroup $Q$ has a canonical minimal generating set $M = \{a_1, \ldots, a_n\}$, and can be embedded as a subsemigroup of $\mathbb{N}^r$; see [MS05, Corollary 7.23 on page 140]. Because of this we will always consider the minimal generators $a_i$ as non-zero elements in $\mathbb{N}^r$; in particular, no $a_i$ can be expressed as a linear combination of the remaining $a_i$s with nonnegative integer coefficients, and the $(r \times n)$-matrix $A = (a_1 \cdots a_n)$ with columns the $a_i$s has rank $r$. The matrix $A$ gives a homomorphism of polynomial rings

$$\varphi_A : k[x_1, \ldots, x_n] \rightarrow k[t_1, \ldots, t_r], \quad x_i \mapsto t^{a_i} = t_1^{a_{i1}} \cdots t_r^{a_{ir}}.$$ 

with image exactly $k[Q]$. Let $R = k[x_1, \ldots, x_n]$. Consider the $\mathbb{Z}^r$-grading on $R$ given by $\deg x_i = a_i$, and consider the standard $\mathbb{Z}^r$-grading on $k[t_1, \ldots, t_r]$, i.e., $\deg t_i = e_i$. Then $\varphi_A$ is graded. Notice that, while the map $\varphi_A$ and the $\mathbb{Z}^r$-grading on $R$ depend on the choice of $A$, the ideal $\ker \varphi_A$ depends only on the semigroup $Q$. The ideal $\ker \varphi_A$ is called the toric ideal associated with $Q$ and is denoted $I_Q$. With respect to the $\mathbb{Z}^r$-grading on $R$ defined above, both $I_Q$ and $k[Q] \cong R/I_Q$ are $\mathbb{Z}^r$-graded $R$-modules.

Thinking of the matrix $A$ as a monoid homomorphism $A : \mathbb{N}^r \rightarrow \mathbb{Z}^r$, we can form the action category $\mathbb{N}^r / \mathbb{Z}^r$ (see [Definition 2.3] and [Example 2.4]), and the category of $\mathbb{Z}^r$-graded $R$-modules is equivalent to the category of $k[\mathbb{N}^r / \mathbb{Z}^r]$-modules by [Lemma 2.5]. As in [Example 2.6] and [Example 2.7] we identify the elements of $\mathbb{N}^r$ with the corresponding monomials in $R$. In particular, given $c, d \in \mathbb{Z}^r = \text{obj} \mathbb{N}^r / \mathbb{Z}^r$, we have that $\text{mor}_{\mathbb{N}^r / \mathbb{Z}^r}(c, d)$ is the set of all monomials in $R$ of $\mathbb{Z}^r$-degree $d - c$, which in general could contain more than one element, and so the category $\mathbb{N}^r / \mathbb{Z}^r$ is in general not a preorder. However, the assumptions on the matrix $A$ imply that all endomorphisms and all isomorphisms in the category $\mathbb{N}^r / \mathbb{Z}^r$ are identities, and so in particular we have an EI category, and it is easy to check that for the induced poset structure on $\text{cl}(\text{obj} \mathbb{N}^r / \mathbb{Z}^r) = \text{obj} \mathbb{N}^r / \mathbb{Z}^r = \mathbb{Z}^r$ every finitely generated upper set is Artinian. Since the semigroup ring $k[\mathbb{N}^r] = R$ is Noetherian, this action category is
Noetherian by Lemma 2.5. Moreover, each group ring \( k[c] \) is just \( k \), hence perfect, and projective \( k[c] \)-modules are free, thus all finitely generated \( k[N^m/Z^n] \)-modules have a minimal free resolution by Corollary 6.6, and therefore a well defined Betti category and well defined Betti numbers.

We now think of the toric ideal \( I_Q \) and the toric ring \( k[Q] \) also as functors \( N^m/Z^n \rightarrow k\text{-Mod} \). Notice that \( \text{supp}(k[Q]) = \text{im}(A: N^m \rightarrow Z^n) = Q = \uparrow 0 \).

Moreover, if \( C \) is any subcategory of \( N^m/Z^n \) with \( \text{obj}\ C \subseteq \text{supp}(k[Q]) = Q \), then 
\[
\text{res}_C k[Q] = \text{const}_k,
\]
i.e., the restriction of \( k[Q] \) to \( C \) is the constant functor \( k \). This applies in particular when \( C \) is the Betti category \( \mathcal{B}(k[Q]) \). Notice that under the equivalence of Lemma 2.5 the objects of \( \mathcal{B}(k[Q]) \) are precisely the degrees of the basis elements of the free \( Z^n \)-graded \( R \)-modules in the minimal \( Z^n \)-graded free resolution of the \( Z^n \)-graded \( R \)-module \( k[Q] \).

**Example 9.1** (twisted cubic). Consider the \((2 \times 4)\)-matrix
\[
A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}.
\]
The semigroup \( Q \) is the subsemigroup of \( Z^2 \) generated by the columns of \( A \). The associated toric ideal in \( R = k[a, b, c, d] \) is
\[
I_Q = \langle ac - b^2, bc - ad, bd - c^2 \rangle,
\]
the defining ideal of the twisted cubic curve in \( \mathbb{P}^3 \); compare the “Running Example” in [Pee11 Chapter IV]. The \( Z^2 \)-graded minimal free resolution of \( k[Q] \) over \( R \) is
\[
0 \rightarrow R^2 \xrightarrow{(d \ b \ c \ a)} R^3 \xrightarrow{(ac-b^2 \ bc-ad \ bd-c^2)} R \rightarrow k[Q] \rightarrow 0.
\]
The \( Z^2 \)-degrees of the basis elements of the free modules in this resolution are \((0, 0), (2, 4), (3, 3), (4, 2), (4, 5), \) and \((5, 4) \). Thus the Betti category of \( k[Q] \) is the following full subcategory of \( N^2/Z^4 \).
The colors are meant to indicate which “squares” commute: two compositions of two morphisms from (0, 0) to either (5, 4) or (4, 5) are equal if and only if they are labeled with the same color. For example, the “black” morphisms (0, 0) → (5, 4) given by $c \circ ac$ and $a \circ c^2$ are equal.

Since there are formulas for the $\mathbb{Z}^r$-graded Betti numbers of the $R$-module $k[Q]$ purely in terms of the topological combinatorics of $Q$ (see e.g. [MS05, Theorem 9.2 on page 192] or [Pre11 Chapter IV, Section 67]), the Betti category of $k[Q]$ can be determined without computing the minimal free resolution. This allows us to use it and produce a new canonical (but not minimal) free resolution of $k[Q]$.

**Theorem 9.2.** Let $k$ be a field, and let $Q$ be a pointed affine semigroup of rank $r$, with a given embedding into $\mathbb{N}^r$ and with canonical minimal generators $a_1, \ldots, a_n$. Consider $k[Q]$ as a $\mathbb{Z}^r$-graded module over the $\mathbb{Z}^r$-graded polynomial ring $R = k[x_1, \ldots, x_n]$, and as a module over the corresponding action category $\mathbb{N}^r[f^{\mathbb{Z}^r}]$. Consider the corresponding Betti category $\mathcal{B}(k[Q])$, and let $\varrho: \mathcal{B}(k[Q]) \to \mathbb{N}^r[f^{\mathbb{Z}^r}]$ be the inclusion functor. Let $B_\bullet(const_k)$ be the normalized bar resolution (Definition 3.6) of the constant $k\mathcal{B}(k[Q])$-module.

Then $\text{ind}_Q B_\bullet(const_k)$ is a free resolution of $k[Q]$ as a module over $\mathbb{N}^r[f^{\mathbb{Z}^r}]$. In particular, applying the equivalence of [Lemma 2.5] produces a canonical finite free $\mathbb{Z}^r$-graded resolution $F_\bullet(k[Q])$ of the $\mathbb{Z}^r$-graded $R$-module $k[Q]$.

**Proof.** Since $\text{const}_k = \text{res}_k k[Q]$, the chain complex $\text{ind}_Q B_\bullet(const_k)$ is a free resolution of $k[Q]$ by [Corollary 8.3.2].

**Example 9.3.** Consider the twisted cubic curve from Example 9.1. The nerve of that Betti category has 12 non-degenerate 2-faces, 18 non-degenerate 1-faces, and 6 (nondegenerate) 0-faces. Applying the equivalence of categories from [Lemma 2.5] to the resolution $\text{ind}_Q B_\bullet(const_k)$ produces the resolution

$$0 \to R^{12} \xrightarrow{D} R^{18} \xrightarrow{E} R^{6} \to 0$$

of the $\mathbb{Z}^2$-graded $R$-module $k[Q]$, where

$$D = \begin{pmatrix} b & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & c & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} -c^2 & -bd & -bc & -ad & -b^2 & -ad & 0 & 0 & -b^2 & -bd & -ac^2 & -abd \\ 1 & 1 & 0 & 0 & 0 & -b & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. $$

**Remark 9.4.** The resolution $F_\bullet(k[Q])$ from Theorem 9.2 differs from the hull resolution of $k[Q]$ introduced in [BS98]. For instance, the hull resolution of the twisted cubic curve $k[Q]$ from Example 9.1 is a minimal free resolution, see [MS05 Exercise 9.3 on page 189], while Example 9.3 shows that $F_\bullet(k[Q])$ is far from minimal.
The precise relationship between these two constructions of canonical finite free resolutions is not clear at this point.

Not only does the Betti category of a toric ring determine a canonical free resolution of the ring, but in fact, the equivalence class of the Betti category determines completely the structure of the minimal free resolution of the toric ring in the following sense.

**Theorem 9.5.** Let \( k \) be a field. Let \( Q \) and \( Q' \) be two pointed affine semigroups of ranks \( r \) and \( r' \) and with Betti categories \( \mathcal{B} \) and \( \mathcal{B}' \), respectively. Denote by
\[
\varrho : \mathcal{B} \rightarrow \mathbb{N}^n \int \mathbb{Z}^r \quad \text{and} \quad \varrho' : \mathcal{B}' \rightarrow \mathbb{N}^{n'} \int \mathbb{Z}^{r'}
\]
the inclusion functors in the corresponding action categories. Assume that there is an equivalence of categories
\[
\alpha : \mathcal{B} \rightarrow \mathcal{B}',
\]
and let \( P_ullet' \) be a minimal free resolution of the toric ring \( k[Q'] \). Then the chain complex \( \text{ind}_\varrho \text{res}_\alpha P_ullet' \) is a minimal free resolution of the toric ring \( k[Q] \).

**Proof.** This is a special case of Theorem 8.1. \( \square \)

**Example 9.6** (Segre threefold). Let \( Q \) be the subsemigroup of \( \mathbb{N}^4 \) generated by the columns of the \((4 \times 6)\)-matrix
\[
A = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}.
\]
The associated toric ideal in \( R = \mathbb{k}[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}] \) is the ideal
\[
I_Q = \langle x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{32} - x_{12}x_{31}, x_{21}x_{32} - x_{22}x_{31} \rangle
\]
generated by the maximal minors of the generic matrix
\[
X = \begin{pmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22} \\
x_{31} & x_{32}
\end{pmatrix}.
\]
This is the defining ideal of the Segre embedding of \( \mathbb{P}^2 \times \mathbb{P}^1 \) in \( \mathbb{P}^5 \); compare Example 66.10 on pages 264–265. It is a *generic determinantal ideal*, in the sense that it specializes via a ring homomorphism from \( R \) to any other ideal of maximal minors of a \((3 \times 2)\)-matrix. The \( \mathbb{Z}^4 \)-graded minimal free resolution of \( R/I_Q \) over \( R \) is
\[
0 \rightarrow R^2 X \rightarrow R^3 \left( x_{21}x_{32} - x_{22}x_{31}, x_{12}x_{31} - x_{11}x_{32}, x_{11}x_{22} - x_{12}x_{21} \right) \rightarrow R \rightarrow R/I_Q \rightarrow 0.
\]
The corresponding Betti category is therefore the following.
We see that it is equivalent (in fact isomorphic) to the Betti category of the twisted cubic curve from Example 9.1. Therefore, by Theorem 9.5 the minimal free resolution of the generic determinantal ideal can be recovered in a canonical way from the minimal free resolution of the twisted cubic ideal, even though there is no map on the level of rings from \( k[a, b, c, d] \) to \( R \) that can realize that “specialization”. In a sense, within the toric world, the defining ideal of the twisted cubic curve is as generic as the generic determinantal ideal.

In practice, in order to find the Betti category of a toric ring, one has to either first compute its minimal free resolution, or find out the Betti degrees using available formulas for the Betti numbers. Either way, this involves computing syzygies. We now show how to replace the Betti category with a slightly bigger canonical category that can be directly computed in practice without finding syzygies.

**Definition 9.7.** Let \( Q \) be a pointed affine semigroup of dimension \( r \), and let \( M = \{a_1, \ldots, a_n\} \) be its canonical minimal generating set. Recall that \( Q \) is a poset with \( a \leq b \) if and only if there is a \( c \in Q \) such that \( a + c = b \).

(a) The **degree** of a subset \( I \subseteq M \) is

\[
\deg(I) = \sum_{a \in I} a.
\]

(b) The **least upper bounds category** or lub-category of \( Q \) is the full subcategory \( L(Q) \) of the action category \( \mathbb{N}^n/\mathbb{Z}^r \) with set of objects

\[
\text{obj } L(Q) = \left\{ a \in Q \middle| \text{there is a set } \{I_1, \ldots, I_k\} \text{ of subsets of } M \text{ such that } a \text{ is a least upper bound in } Q \text{ for the set } \{\deg I_1, \ldots, \deg I_k\} \right\}.
\]

Note that by definition \( L(Q) \) is independent of the choice of the embedding of \( Q \) into \( \mathbb{N}^r \) and of the field \( k \).

**Lemma 9.8.** The Betti category of \( k[Q] \) is a subcategory of \( L(Q) \).

**Proof.** Let \( c \) be an object of the Betti category of \( k[Q] \). By [MS05, Theorem 9.2 on page 175], the reduced homology of the simplicial complex

\[
\Delta_c = \{I \subseteq M \mid \deg I \leq c\}
\]
is nontrivial. Let \( \{I_1, \ldots, I_k\} \) be the facets of \( \Delta_c \). It is enough to show that \( c \) is a least upper bound in \( \mathbb{Q} \) for the set \( S = \{\deg I_1, \ldots, \deg I_k\} \). Suppose that \( b < c \) and \( b \) is an upper bound in \( \mathbb{Q} \) for \( S \). Then we must have an \( a \in \mathcal{M} \) such that \( b < b + a \leq c \), and therefore \( I_j \cup \{a\} \) is a face of \( \Delta_c \) for each \( j \). Thus \( a \in I_j \) for each \( j \), and hence \( \Delta_c \) is a cone. This contradicts the fact that the reduced homology of \( \Delta_c \) is nontrivial. \( \square \)

Using Lemma 9.8, the following theorem is now an immediate consequence of Theorem 8.1, and shows that the lub-category is a finite combinatorial object that plays in the toric world a role analogous to that of the lcm-lattice \([GPW99]\) in the world of monomial ideals.

**Theorem 9.9.** Let \( Q \) and \( Q' \) be pointed affine semigroups. Assume that there is an equivalence \( \alpha: \mathcal{L}(Q) \to \mathcal{L}(Q') \) of their lub-categories. Then, for every field \( k \), the Betti categories \( \mathcal{B}(k[Q]) \) and \( \mathcal{B}(k[Q']) \) are equivalent. In particular, the minimal free resolution of \( k[Q] \) is obtained from the minimal free resolution of \( k[Q'] \) by the functorial procedure described in Theorem 9.5.

**Remark 9.10.** It is straightforward to verify that the lub-categories of the pointed affine semigroups from Examples 9.1 and 9.6 are not equivalent, even though the corresponding Betti categories are equivalent (even isomorphic) for every field \( k \).

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