ON THE LOWEY LENGTH OF MODULES OF FINITE PROJECTIVE DIMENSION.

TONY J. PUTHENPURAKAL

Abstract. Let \((A, \mathfrak{m})\) be a local Gorenstein local ring and let \(M\) be an \(A\) module of finite length and finite projective dimension. We prove that the Lowey length of \(M\) is greater than or equal to order of \(A\). This generalizes a result of Avramov, Buchweitz, Iyengar and Miller [2, 1.1].

1. Introduction

Let \((A, \mathfrak{m})\) be a local Gorenstein local ring of dimension \(d \geq 0\) and embedding dimension \(c\). If \(M\) is an \(A\)-module then we let \(\lambda(M)\) denote its length. If \(A\) is singular then the order of \(A\) is given by the formula

\[
\text{ord}(A) = \min \{n \in \mathbb{N} \mid \lambda(A/\mathfrak{m}^n) < \binom{n+c}{n}\}.
\]

If \(A\) is regular we set \(\text{ord}(A) = 1\). Note that if \(A\) is singular then \(\text{ord}(A) \geq 2\). Recall that Lowey length of an \(A\)-module \(M\) is defined to be the number

\[
\ell\ell(M) = \min \{i \geq 0 \mid \mathfrak{m}^iM = 0\}.
\]

When \(M\) is finitely generated \(\ell\ell(M)\) is finite if and only if \(\lambda(M)\) is finite. Often the Lowey length of \(M\) carries more structural information than does its length.

Let \(G(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}\) be the associated graded ring of \(A\) and let \(G(A)_+\) denotes its irrelevant maximal ideal. Let \(H^i(G(A))\) be the \(i\)th-local cohomology module of \(G(A)\) with respect to \(G(A)_+\). The Castelnuovo-Mumford regularity of \(G(A)\) is

\[
\text{reg}(G(A)) = \max \{i + j \mid H^j(G(A)) \neq 0\}
\]

In the very nice paper [2, 1.1] the authors proved that if \(G(A)\) is Cohen-Macaulay then for each non-zero finitely generated \(A\)-module \(M\) of finite projective dimension

\[
\ell\ell(M) \geq \text{reg}(G(A)) + 1 \geq \text{ord}(A).
\]

We should note that the real content of their result is that \(\ell\ell(M) \geq \text{reg}(G(A)) + 1\). The fact that \(\text{reg}(G(A)) + 1 \geq \text{ord}(A)\) is elementary, see [2, 1.6]. The hypotheses \(G(A)\) is Cohen-Macaulay is quite strong, for instance \(G(A)\) need not be Cohen-Macaulay even if \(A\) is a complete intersection. In this short paper we show

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Theorem 1.1. Let $(A, \mathfrak{m})$ be a Gorenstein local ring and let $M$ be a non-zero finitely generated module of finite projective dimension. Then

$$\ell\ell(M) \geq \text{ord}(A).$$

The proof of Theorem 1.1 uses invariants of Gorenstein local rings defined by Auslander and studied by S. Ding. We also introduce a new invariant $\vartheta(A)$ which is useful in the case $G(A)$ is not Cohen-Macaulay.

We now describe the contents of this paper in brief. In section two we recall the notion of index of a local ring. In section three we introduce our invariant $\vartheta(A)$. In section four we prove Theorem 1.1.

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2. The index of a Gorenstein local ring

Let $(A, \mathfrak{m})$ be a Gorenstein local ring and let $M$ be a finitely generated $A$-module. Let $\mu(M)$ denote minimal number of generators of $M$. In this section we recall the definition of the delta invariant of $M$. Finally we recall the definition of index of $A$. A good reference for this topic is [5].

2.1. A maximal Cohen-Macaulay approximation of $M$ is a short exact sequence

$$0 \rightarrow Y_M \rightarrow X_M \xrightarrow{f} N \rightarrow 0,$$

where $X_M$ is a maximal Cohen-Macaulay $A$-module and $\text{projdim} Y_M < \infty$. If $f$ can only be factored through itself by way of an automorphism of $X_M$, then the approximation is said to be minimal. Any module has a minimal approximation and minimal approximations are unique up to non-unique isomorphisms. Suppose now that $f$ is a minimal approximation. Let $X_M = \overline{X}_M \oplus F$ where $\overline{X}_M$ has no free summands and $F$ is free. Then $\delta_A(M)$ is defined to be the rank of $F$.

2.2. Alternate definitions of the delta invariant

It can be shown that $\delta_A(M)$ is the smallest integer $n$ such that there is an epimorphism $X \oplus A^n \rightarrow M$ with $X$ a maximal Cohen-Macaulay module with no free summands, see [12, 4.2]. This definition of delta is used by Ding [3]. The stable CM-trace of $M$ is the submodule $\tau(M)$ of $M$ generated by the homomorphic images in $M$ of all MCM modules without a free summand. Then $\delta_A(M) = \mu(M/\tau(M))$, see [12, 4.8]. This is the definition of delta in [2].

2.3. We collect some properties of the delta invariant that we need. Let $M$ and $N$ be finitely generated $A$-modules.

(1) If $N$ is an epimorphic image of $M$ then $\delta_A(M) \geq \delta_A(N)$.
(2) $\delta_A(M \oplus N) = \delta_A(M) + \delta_A(N)$.
(3) $\delta_A(M) \leq \mu(M)$.
(4) If $\text{projdim} M < \infty$ then $\delta_A(M) = \mu(M)$. 
(5) Let \( x \in m \) be \( A \oplus M \) regular. Set \( B = A/(x) \). Then \( \delta_A(M) = \delta_B(M/xM) \).

(6) If \( A \) is zero-dimensional Gorenstein local ring and \( I \) is an ideal in \( A \) then
\[
\delta_A(A/I) \neq 0 \quad \text{if and only if} \quad I = 0.
\]

(7) If \( A \) is not regular then \( \delta_A(m^s) = 0 \) for all \( s \geq 1 \).

(8) \( \delta_A(k) = 1 \) if and only if \( A \) is regular.

(9) \( \delta_A(A/m^n) \geq 1 \) for all \( n \gg 0 \).

Proofs and references For (1),(2),(4),(8),(9); see \cite{2,1.2}. Notice (3) follows easily from the second definition of delta. The assertion (5) is proved in \cite{11} 5.1. For (6) note that \( A/I \) is maximal Cohen-Macaulay. The assertion (7) is due to Auslander. Unfortunately this paper of Auslander is unpublished. However there is an extension of the delta invariant to all Noetherian local rings due to Martsinkovsky \cite{6}. In a later paper he proves that \( \delta_A(m) = 0 \). see \cite{7} Theorem 6. We prove by induction that \( \delta_A(m^s) = 0 \) for all \( s \geq 1 \). For \( s = 1 \) this is true. Assume for \( s = j \). We prove it for \( s = j + 1 \). Let \( m^{j+1} = \langle a_1b_1, a_2b_2, \ldots, a_mb_m \rangle \) where \( a_i \in m \) and \( b_i \in m \). Let \( I_i = a_im \) for \( i = 1, \ldots, m \). Note \( I_i \subseteq m^{j+1} \) and the natural map \( \phi: \bigoplus_{i=1}^m I_i \to m^{j+1} \) is surjective. By (1) and (2) it is enough to show that \( \delta_A(I_i) = 0 \) for all \( i \). But this is clear as \( I_i \) is a homomorphic image of \( m \).

2.4. The index of \( A \) is defined by Auslander to be the number
\[
\text{index}(A) = \min\{n \mid \delta_A(A/m^n) \geq 1\}.
\]
It is a positive integer by \cite{2,9} and equals 1 if and only if \( A \) is regular, see \cite{2,8}.

2.5. By \cite{2,1.3} we have that if \( \text{projdim} M \) is finite then
\[
\ell(M) \geq \text{index}(A).
\]

3. The invariant \( \vartheta(A) \)

Throughout this section \( (A,m) \) is a Cohen-Macaulay local ring of dimension \( d \). We assume that \( k \), the residue field of \( A \), is infinite. In this section we define an invariant \( \vartheta(A) \). This will be useful when \( G(A) \) is not Cohen-Macaulay.

3.1. Let \( a \in A \) be non-zero. Then there exists \( n \geq 0 \) such that \( a \in m^n \setminus m^{n+1} \).

Set \( a^* = \text{image of } a \) in \( m^n/m^{n+1} \) and we consider it as an element in \( G(A) \). Also set \( 0^* = 0 \). If \( a \in m \) is such that \( a^* \) is \( G(A) \)-regular then \( G(A/(a)) = G(A)/(a^*) \).

3.2. Recall \( x \in m \) is said to be \( A \)-superficial if there exists integer \( c > 0 \) such that for \( n \gg 0 \) we have \( (m^{n+1}: x) \cap m^c = m^{n-1} \). Superficial elements exist if \( d > 0 \) as \( k \) is an infinite field. As \( A \) is Cohen-Macaulay, it is easily shown that a superficial element is a non-zero divisor of \( A \). Furthermore we have
\[
(m^{n+1}: x) = m^n \quad \text{for all } n \gg 0.
\]

This enables to define the following two invariants of \( A \) and \( x \):
\[
\vartheta(A,x) = \inf\{n \mid (m^{n+1}: x) \neq m^n\},
\]
\[ \rho(A, x) = \sup \{n \mid (m^{n+1}: x) \neq m^n \}. \]

**3.3.** Notice \((m^{n+1}: x) = m^n\) for all \(n \geq 0\) if and only if \(x^*\) is \(G(A)\)-regular. Thus in this case \(\vartheta(A, x) = +\infty\) and \(\rho(A, x) = -\infty\).

If \(\operatorname{depth} G(A) > 0\) then \(x^*\) is \(G(A)\)-regular, see [4 2.1]. Thus in this case \(\vartheta(A, x) = +\infty\) and \(\rho(A, x) = -\infty\).

If \(\operatorname{depth} G(A) = 0\) then \((m^{n+1}: x) \neq m^n\) for some \(n\). In this case we have \(\vartheta(A, x), \rho(A, x)\) are finite numbers and clearly \(\vartheta(A, x) \leq \rho(A, x)\). By [9 2.7 and 5.1] we have

\[ \rho(A, x) \leq \operatorname{reg} G(A) - 1. \]

**3.4.** A sequence \(x = x_1, \ldots, x_r\) in \(m\) with \(r \leq d\) is said to be an \(A\)-superficial sequence if \(x_i\) is \(A/(x_1, \ldots, x_{i-1})\)-superficial for \(i = 1, \ldots, r\). As the residue field of \(A\) is infinite, superficial sequences exist for all \(r \leq d\). As \(A\) is Cohen-Macaulay it can be easily shown that superficial sequences are regular sequences.

**3.5.** Let \(x = x_1, \ldots, x_d\) be a maximal \(A\)-superficial sequence. Set \(A_0 = A\) and \(A_i = A/(x_1, \ldots, x_i)\) for \(i = 1, \ldots, d\). Define

\[ \vartheta(A, x) = \inf \{\vartheta(A_i, x_{i+1}) \mid 0 \leq i \leq d - 1\}. \]

Note that \(G(A)\) is Cohen-Macaulay if and only if \(x_1^*, \ldots, x_d^*\) is a \(G(A)\)-regular sequence, see [4 2.1]. It follows from [3.3] that

\[ \vartheta(A, x) = +\infty \quad \text{if and only if} \quad G(A) \text{ is Cohen-Macaulay.} \]

We have

**Lemma 3.6.** (with hypotheses as above). If \(G(A)\) is not Cohen-Macaulay then \(\vartheta(A, x) \leq \operatorname{reg} G(A) - 1\).

**Proof.** Suppose \(\operatorname{depth} G(A) = i < d\). Then \(x_1^*, \ldots, x_i^*\) is \(G(A)\)-regular, see [4 2.1]. Furthermore \(G(A_i) = G(A)/(x_1^*, \ldots, x_i^*)\). Thus \(\operatorname{depth} G(A_i) = 0\). (Note the case \(i = 0\) is also included).

By [3.3] we have that \(\vartheta(A_i, x_{i+1}) \leq \operatorname{reg}(G(A_i)) - 1\). It remains to note that as \(x_1^*, \ldots, x_d^*\) is a regular sequence of elements of degree 1 in \(G(A)\), we have \(\operatorname{reg} G(A_i) \leq \operatorname{reg} G(A)\). \(\square\)

**3.7.** We define

\[ \vartheta(A) = \sup \{\vartheta(A, x) \mid x \text{ is a maximal superficial sequence in } A\}. \]

Note that if \(G(A)\) is not Cohen-Macaulay then \(\vartheta(A) \leq \operatorname{reg} G(A) - 1\), see [3.6] If \(G(A)\) is Cohen-Macaulay then \(\vartheta(A) = +\infty\).

**3.8.** Let \(A\) be a singular ring and Let \(x \in m\) be an \(A\)-superficial element. Let \(t = \operatorname{ord}(A)\). The following fact is well-known (for instance see [10 p. 295])

\[ (m^{i+1}: x) = m^i \quad \text{for } i = 0, \ldots, t - 1. \]
It follows that $\vartheta(A, x) \geq \operatorname{ord}(A)$ for any superficial element $x$ of $A$.

Notice $\operatorname{ord}(A/(x)) \geq \operatorname{ord}(A)$ for any superficial element $x$ of $A$ (for instance see \cite[p. 296]{10}). Thus if $x = x_1, \ldots, x_d$ is a maximal $A$-superficial sequence we have that $\vartheta(A, x_i) \geq \operatorname{ord}(A_i) \geq \operatorname{ord}(A)$ for all $i = 0, \ldots, d - 1$. It follows that

\begin{equation}
\vartheta(A, x) \geq \operatorname{ord}(A).
\end{equation}

It is possible that for some rings strict inequality in (3.8.1) can hold.

**Example 3.9.** Let $(A, m)$ be an one dimensional stretched Gorenstein local ring, i.e., there exists an $A$-superficial element $x$ such that if $n$ is the maximal ideal of $B = A/(x)$ then $n^2$ is principal. For such rings $\operatorname{ord}(B) = 2$, see \cite[1.2]{11}. So $\operatorname{ord}(A) = 2$. However for stretched Gorenstein rings $(m^3 : x) = m^2$; see \cite[2.5]{11}. (Note $(m^{i+1} : x) = m^i$ for $i \leq 1$ for any Cohen-Macaulay ring $A$). Thus $\vartheta(A, x) \geq 3$.

See \cite[Example 3]{11} for an example of a stretched one dimensional stretched Gorenstein local ring $A$ with $G(A)$ not Cohen-Macaulay.

The following result is crucial in the proof of our main result. By $e(A)$ we denote the multiplicity of $A$ with respect to $m$.

**Lemma 3.10.** Let $(A, m)$ be a $d$-dimensional Cohen-Macaulay local ring with infinite residue field. Let $x = x_1, \ldots, x_d$ be a maximal superficial sequence. Assume $G(A)$ is not Cohen-Macaulay. Let $n \leq \vartheta(A, x)$. Then

$$m^n \nsubseteq (x).$$

**Proof.** Suppose if possible $m^n \subseteq (x)$ for some $n \leq \vartheta(A, x)$. Set $A_d = A/(x)$ and $A_{d-1} = A/(x_1, \ldots, x_{d-1})$. Let $n$ be the maximal ideal of $A_{d-1}$. Notice $n \leq \vartheta(A_{d-1}, x_d)$.

We have an exact sequence

$$0 \to \frac{(n^n : x)}{m^n} \to \frac{A_{d-1}}{m^{n-1}} \to \frac{A_{d-1}}{n^n} \to \frac{A_{d-1}}{(m^n, x_d)} \to 0.$$ 

Here $\alpha(a + n^{n-1}) = ax_d + n^n$. Note that as $m^n \subseteq (x)$ we have $A_{d-1}/(n^n, x_d) = A_d$. Recall $(n^{i+1} : x_d) = n^i$ for all $i < \vartheta(A_{d-1}, x_d)$. In particular we have $(n^n : x) = n^{n-1}$.

Thus we have

$$\lambda(n^n / n^n) = \lambda(A_d).$$

Notice $e(A) = e(A_{d-1}) = e(A_d) = \lambda(A_d)$, cf., \cite[Corollary 11]{8}. Furthermore for all $i \geq 0$ we have

$$\lambda(n^i / n^{i+1}) = e(A_{d-1}) - \lambda(n^{i+1} / n^n),$$

cf., \cite[Proposition 13]{8}.

For $i = n - 1$ our result implies $n^n = x_d n^{n-1}$. It follows that $n^j = x_d n^{j-1}$ for all $j \geq n$. In particular we have $(n^j : x_d) = n^{j-1}$ for all $j \geq n$. As $n \leq \vartheta(A_{d-1}, x_d)$ we have that $(n^j : x_d) = n^{j-1}$ for all $j \leq n$. It follows that $x_d^*$ is $G(A_{d-1})$-regular. So depth $G(A_{d-1}) = 1$. By Sally descent, see \cite[Theorem 8]{8} we get $G(A)$ is Cohen-Macaulay. This is a contradiction. \qed
4. Proof of Theorem 1.1

In this section we prove our main Theorem. We will use the invariant \( \vartheta(A) \) which is defined only when the residue field of \( A \) is infinite. We first show that to prove our result we can assume that the residue field of \( A \) is infinite.

4.1. Suppose the residue field of \( A \) is finite. Consider the flat extension \( B = A[X]_{m,A,X} \). Note that \( n = mB \) is the maximal ideal of \( B \) and \( B/n = k(X) \) is an infinite field. Let \( M \) be a finitely generated \( A \)-module. The following facts can be easily proved:

1. \( \lambda_B(M \otimes A B) = \lambda_A(B) \).
2. \( m^i \otimes B = n^i \) for all \( i \geq 1 \).
3. \( \lambda_B(B/n^{i+1}) = \lambda_A(A/m^{i+1}) \) for all \( i \geq 0 \).
4. \( \text{ord}(B) = \text{ord}(A) \).
5. \( \text{projdim}_A M = \text{projdim}_B M \otimes A B \).
6. \( m^i M = 0 \) if and only if \( n^i(M \otimes A B) = 0 \).
7. \( \ell_A(M) = \ell_B(M \otimes A B) \).

We need the following result due to Ding, see [3, 2.2, 2.3, 1.5].

Lemma 4.2. Let \((A, m)\) be a Noetherian local ring and \( s \) an integer. Suppose that \( x \in m \setminus m^2 \) is \( A \)-regular and the induced map \( \mathfrak{T}: m^{i-1}/m^i \to m^i/m^{i+1} \) is injective for \( 1 \leq i \leq s \). Then

1. \( A/m^s \) is an epimorphic image of \((m^s, x)\).
2. There is an \( A \)-module decomposition

\[
\frac{(m^s, x)}{x(m^s, x)} \cong \frac{A}{(m^s, x)} \oplus \frac{(m^s, x)}{(x)}.
\]

We now give

Proof of Theorem 1.1. By 4.1 we may assume that the residue field of \( A \) is infinite.

If \( G(A) \) is Cohen-Macaulay then the result holds by Theorem 1.1 in [2]. So assume that \( G(A) \) is not Cohen-Macaulay. We prove \( \text{index}(A) \geq \vartheta(A) \). By 3.8.1 and 2.5 this will imply the result.

Let \( x = x_1, \ldots, x_d \) be an \( A \)-superficial sequence with \( \vartheta(A) = \vartheta(A, x) \). Suppose if possible \( \text{index}(A) < \vartheta(A, x) \). Say \( \delta_A(A/m^s) \geq 1 \) for some \( s < \vartheta(A, x) \). Set \( A_0 = A \) and \( A_i = A/(x_1, \ldots, x_i) \). Let \( m_i \) be the maximal ideal of \( A_i \). We prove by descending induction that

\[
\delta_{A_i}(A_i/m^s_i) \geq 1 \quad \text{for all } i.
\]

For \( i = 0 \) this is our assumption. Now assume this is true for \( i \). We prove it for \( i + 1 \). We first note that \( s < \vartheta(A, x) \leq \vartheta(A_i, x_{i+1}) \). Therefore \( (m^j_{i+1}: x_{i+1}) = m^j_i \) for all \( j \leq s \). So by 4.2 we have that \( A_i/m^s_i \) is an epimorphic image of \((m^s_i, x)\). Thus
\[ \delta_{A_i}(m^s_i, x) \geq 1. \]

We also have an \( A_i \)-module decomposition
\[ (\dagger) \quad \frac{(m^s_i, x_{i+1})}{x_{i+1}(m^s_i, x_{i+1})} \cong \frac{A_i}{(m^s_i, x_{i+1})} \oplus \frac{(m^s_i, x_{i+1})}{(x_{i+1})}. \]

By (2.3(5)) we have that
\[ \delta_{A_{i+1}} \left( \frac{(m^s_i, x_{i+1})}{x_{i+1}(m^s_i, x_{i+1})} \right) = \delta_{A_i}(m^s_i, x) \geq 1. \]

Also note that
\[ \delta_{A_{i+1}} \left( \frac{(m^s_i, x_{i+1})}{(x_{i+1})} \right) = \delta_{A_i}(m^s_{i+1}) = 0, \quad \text{by (2.3(7))}. \]

By (\dagger) and 2.3(2) it follows that
\[ 1 \leq \delta_{A_{i+1}} \left( \frac{A_i}{(m^s_i, x_{i+1})} \right) = \delta_{A_{i+1}} \left( \frac{A_{i+1}}{m^s_{i+1}} \right). \]

This proves our inductive step. So we have \( \delta_{A_d}(A_d/m^s_d) \geq 1. \) By (2.3(6)) we have that \( m^s_d = 0. \) It follows that \( m^s \subseteq (x). \) This contradicts \ref{3.10}. \( \square \)

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Department of Mathematics, IIT Bombay, Powai, Mumbai 400 076
E-mail address: tputhen\#math.iitb.ac.in